Matching for probabilistic entanglement swapping

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The basic entanglement swapping protocol allows to achieve the main aim of deterministically projecting two qubits, which have never interacted, onto a maximally entangled state. For deterministic swapping the key ingredient is the maximal entanglement initially contained in two pairs of qubits and the ability to measure onto a Bell basis. In other words, the basic and deterministic entanglement swapping scheme involves three maximal level of entanglement. In this work we address probabilistic entanglement swapping processes performed with different amounts of initial entanglement. Besides, we suggest a non Bell measuring-basis, thus we introduce a third entanglement level in the process. Additionally, we propose the unambiguous state extraction scheme as the local mechanism for probabilistically achieving the main aim. Amalgamating these three elements allows us to design four strategies for performing probabilistic entanglement swapping. Surprisingly, we encountered a twofold entanglement matching effect related to the concurrence of the measuring-basis. Specifically, the maximal probability of accomplishing the main aim becomes a constant for concurrences higher than or equal to the matching entanglement values. Thus, we show that maximal entanglement in the measuring-basis is not required for attaining the main aim with optimal probability.

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I. INTRODUCTION

At the beginning of quantum mechanics Erwin Schrödinger noticed the existence of a special correlation, the entanglement, present in a superposition of tensorial product states of two or more systems [1]. Currently, it is known that the entanglement is a purely quantum ingredient, which introduces non-local effects in protocols for processing information on atomic and molecule scales [2–5]. Thus, almost instantly the efforts focussed on finding well defined measurements of entanglement. The most known of these measurements is the entanglement of formation, which quantifies the resources needed to create a given entangled state [6]. Specifically, W. K. Wootters found a closed analytical formula for the concurrence of an arbitrary state of two qubits, which is a monotone function of the entanglement of formation [7,8]. In consequence, for two qubits, the concurrence can be used as a measurement of entanglement in its own merit, which is what we consider here. For a 2⊗2 pure state, its concurrence can be evaluated at a glance, if it is represented in the Schmidt decomposition; it is two times the product of its Schmidt coefficients [7,8]. So, a factorized state lacks concurrence, whereas a maximal entangled state (EPR) has a concurrence equal to 1.

In this context, the generation, manipulation, control, and the practical scope of this quantum correlation become important research areas. In particular, the capacity to distribute entanglement between distant systems has potential applications in designing innovative protocols of quantum information without classical counterpart [9,22].

The entanglement swapping, as a mechanism for distributing the entanglement correlation, has been extensively studied both theoretically and experimentally [10,23–38].

In this article, we analyze four schemes for performing probabilistic entanglement swapping, in which the main aim is to project two qubits, that have never interacted, onto a Bell state. We consider, that the two pairs of qubits can be in different partially entangled pure states. Besides, we consider that the measuring-basis can consist of non-Bell states. Additionally, we propose the so called unambiguous state extraction (USE) protocol [39,42] as the mechanism for implementing the local operations.

The article is organized as follows. In Sec. II we briefly recall the well known, deterministic and basic entanglement swapping scheme. In Sec. III we propose four strategies for achieving the aim of probabilistically projecting onto an EPR state, shared by two qubits which have never interacted. In the last Sec. IV we summarize the principal results. Additionally, we included the Appendix A where we show that the unambiguous state extraction (USE) protocol [39,42] can be locally applied to extract an EPR state with optimal probability. In particular, we introduce an explicit joint unitary transformation, which allows to accomplish the main aim, additionally finding the optimal probability of success as a function of the initial concurrence [43–47]. The results of the appendix are directly applied onto the first, second and fourth strategy.
II. DETERMINISTIC AND BASIC ENTANGLEMENT SWAPPING

The basic entanglement swapping scheme considers four qubits $A$, $C_1$, $B$, $C_2$, the pairs $AC_1$ and $BC_2$ previously prepared in Bell states. The qubits $C_1$ and $C_2$ remain in the same laboratory (Lab-$C$), whereas the qubits $A$ and $B$ are carried to two different laboratories, away from each other and away from Lab-$C$. Therefore, joint operations are only allowed between qubits $C_1$ and $C_2$. Local operations can be applied onto each of the four qubits and classical communication is enabled among them all. The aim of this scheme is to maximally entangle the pair $AB$ by using classical communication, local and joint operations. Figure 1 shows the different locations of the four qubits.

The following identity gives account of the basic entanglement swapping procedure,

$$
|\psi_{AC_1}^+|\psi_{BC_2}^+ = \frac{1}{2} (|\phi_{C_1C_2}^+\rangle|\phi_{AB}^+\rangle + |\psi_{C_1C_2}^+\rangle|\psi_{AB}^+\rangle - |\psi_{C_1C_2}^-\rangle|\phi_{AB}^-\rangle - |\phi_{C_1C_2}^-\rangle|\psi_{AB}^-\rangle),
$$

where

$$
|\psi_{ij}^+\rangle = (|0_i\rangle|1_j\rangle ± |1_i\rangle|0_j\rangle)/\sqrt{2},
$$

$$
|\phi_{ij}^\pm\rangle = (|0_i\rangle|0_j\rangle ± |1_i\rangle|1_j\rangle)/\sqrt{2},
$$

are the Bell-basis of a pair of qubits $i \otimes j$ [8]. The right hand side of Eq. (1) clearly shows the one-to-one correlation among the Bell states of the pairs $C_1C_2$ and $AB$. From this, one realizes that by measuring the Bell states in the Lab-$C$ the pair $AB$ is projected onto a Bell state, thus deterministically achieving the main aim. It is worth noting, that determinism demands the key ingredient of maximal entanglement in the two initial states and also in the measuring-basis.

III. PROBABILISTIC ENTANGLEMENT SWAPPING

We analyze four strategies for probabilistically obtaining an EPR state in the bipartite system $AB$, when the pairs $AC_1$ and $BC_2$ initially are in the following partially entangled pure states,

$$
|\tilde{\psi}_{AC_1}\rangle = \alpha|0_A\rangle|1_{C_1}\rangle + \beta|1_A\rangle|0_{C_1}\rangle, \quad (2a)
$$

$$
|\tilde{\psi}_{BC_2}\rangle = \gamma|0_B\rangle|1_{C_2}\rangle + \delta|1_B\rangle|0_{C_2}\rangle, \quad (2b)
$$

where, without loss of generality, we assume the amplitudes $\alpha$, $\beta$, $\gamma$, $\delta$ to be real and non-negative numbers,

$$
\alpha \leq \beta, \quad \text{and} \quad \gamma \leq \delta. \quad (3)
$$

For normalization $\alpha^2 + \beta^2 = 1$ and $\gamma^2 + \delta^2 = 1$.

The initial concurrences of the states (2) are $C_{AC_1} = 2\alpha\beta$ and $C_{BC_2} = 2\gamma\delta$ respectively.

A. First strategy

The first, simplest strategy is to extract an EPR state from each bipartite state (2) with the unitary-reduction local operations described in Appendix A.

If both processes are successful, the basic swapping scheme can be carried out. Accordingly, by considering Eqs. (A5), (3), the success probability of obtaining an EPR in the pair $AB$ is equal to the product of the probabilities of extracting an EPR from each state (2), which becomes,

$$
P_{s_1} = 4\alpha^2\gamma^2,
$$

$$
= \left(1 - \sqrt{1-C_{AC_1}^2}\right) \left(1 - \sqrt{1-C_{BC_2}^2}\right). \quad (4)
$$

Note, that $P_{s_1}$ is an increasing function of the initial concurrences, $C_{AC_1}$ and $C_{BC_2}$, besides, the entanglement of both initial states is necessary and sufficient for having a success probability different from zero.

B. Second strategy

The second strategy is suggested by writing the initial tensorial product state $|\tilde{\psi}_{AC_1}\rangle|\tilde{\psi}_{BC_2}\rangle$ in the representation of the Bell-basis of $C_1C_2$, i.e.,

$$
|\tilde{\psi}_{AC_1}\rangle|\tilde{\psi}_{BC_2}\rangle = \sqrt{\frac{p}{2}}|\phi_{C_1C_2}^+\rangle|\bar{\phi}_{AB}^+\rangle
$$

$$
+ \sqrt{\frac{1-p}{2}}|\phi_{C_1C_2}^-\rangle|\bar{\phi}_{AB}^-\rangle
$$

$$
- \sqrt{\frac{1-p}{2}}|\psi_{C_1C_2}^-\rangle|\bar{\psi}_{AB}^+\rangle
$$

$$
- \sqrt{\frac{p}{2}}|\psi_{C_1C_2}^+\rangle|\bar{\psi}_{AB}^-\rangle, \quad (5)
$$
where we have defined the states of the pair \( AB \) as follows:

\[
\begin{align*}
|\tilde{\phi}^\pm_{AB}\rangle &= \frac{\alpha\gamma|0\rangle|0\rangle \pm \beta\delta|1\rangle|1\rangle}{\sqrt{p}}, \\
|\tilde{\psi}^\pm_{AB}\rangle &= \frac{\alpha\delta|0\rangle|1\rangle \pm \beta\gamma|1\rangle|0\rangle}{\sqrt{1-p}},
\end{align*}
\]

and the probability

\[ p = \alpha^2\gamma^2 + \beta^2\delta^2. \]

The identity (5) shows a one-to-one correlation among the Bell states of \( C_1C_2 \) and the states (6) of the pair \( AB \). Thus we realize, that by measuring Bell states in Lab-\( C \), the bipartite system \( AB \) is projected onto one of the partially entangled states (6) with respective probabilities \( p/2 \) and \((1 - p)/2\). Notice, that the minus sign in \( |\tilde{\phi}^\pm_{AB}\rangle \) and \( |\tilde{\psi}^\pm_{AB}\rangle \) can be removed by locally applying the Pauli operator \( \sigma_z \) on \( A \) or \( B \).

Therefore, in practice, the system \( AB \) has two outcomes \( |\tilde{\phi}^+_{AB}\rangle \) with probability \( p \) and \( |\tilde{\psi}^+_{AB}\rangle \) with probability \( 1 - p \).

From each of these states, \( |\tilde{\phi}^+_{AB}\rangle \) and \( |\tilde{\psi}^+_{AB}\rangle \), one of the receivers, \( A \) or \( B \), can probabilistically extract a Bell state by means of the local USE scheme. If the outcome is \( |\tilde{\phi}^+_{AB}\rangle \), the conditional probability of extracting a Bell state becomes \( p_{ext,\phi} = 2\alpha^2\gamma^2/p \); otherwise, if the outcome is \( |\tilde{\psi}^+_{AB}\rangle \), there are two cases for evaluating the conditional probability:

(i) when \( \alpha\delta \leq \beta\gamma \), being equivalent to \( \alpha \leq \gamma \), the conditional probability of succesful extraction is \( p_{ext,\psi} = 2\alpha^2\delta^2/(1 - p) \);

(ii) when \( \alpha\delta \geq \beta\gamma \), which is equivalent to \( \alpha \geq \gamma \), the conditional success probability is given by \( p_{ext,\psi} = 2\beta^2\gamma^2/(1 - p) \).

Accordingly, the total success probability of achieving an EPR state in the pair \( AB \) becomes,

\[
P_{s_2} = pp_{ext,\phi} + (1-p)p_{ext,\psi} = 2 \min\left\{ \alpha^2, \gamma^2 \right\},
\]

\[
= \min\left\{ 1 - \sqrt{1-C^2_{AC_1}}, 1 - \sqrt{1-C^2_{BC_2}} \right\}.
\]

Note, that this expression of \( P_{s_2} \) exhibits the so called entanglement matching effect, since it is determined by the smallest entanglement of the initial states \( |\tilde{\psi}_{AC_1}\rangle \) and \( |\tilde{\psi}_{BC_2}\rangle \); It is worth emphasizing, that both initial entanglements are necessary and sufficient for having a success probability different from zero. Besides, by comparing the success probabilities of both strategies we find that,

\[
P_{s_2} \geq P_{s_1}.
\]

The equality \( P_{s_2} = P_{s_1} \) holds true, if at least one of the two initial concurrences is equal to 1 or equal to zero. For instance, by considering \( C_{AC_1} \) fixed, \( P_{s_2} \) achieves its maximal value for \( C_{BC_2} = C_{AC_1} \), whereas \( P_{s_1} \) reaches the same maximal value at \( C_{BC_2} = 1 \). In other words, for a fixed \( C_{AC_1} \), the maximal \( P_{s_2} \) demands \( C_{BC_2} = C_{AC_1} \), while the same maximal value for \( P_{s_1} \) demands \( C_{BC_2} = 1 \). Figure 2 illustrates the behavior of the probabilities (4) and (7) for different values of \( C_{AC_1} \).

In consequence, if we consider entanglement as a resource, we can conclude that the second strategy is more efficient than the first one.

**C. Strategies without Bell-basis measurement**

Now, instead of projecting onto Bell states in \( C_1C_2 \), we propose to measure an observable that has the following eigenstates:

\[
|\mu^+_{C_1C_2}\rangle = x|00\rangle + y|11\rangle,
\]

\[
|\mu^-_{C_1C_2}\rangle = y|00\rangle - x|11\rangle,
\]

\[
|\nu^+_{C_1C_2}\rangle = x|01\rangle + y|10\rangle,
\]

\[
|\nu^-_{C_1C_2}\rangle = y|01\rangle - x|10\rangle,
\]

where, without loss of generality, we can consider the amplitudes \( x \) and \( y \) to be real and non-negative numbers and

\[
x \leq y.
\]

For normalization \( x^2 + y^2 = 1 \). Each state (9) has the same concurrence \( C_{C_1C_2}(x) = 2xy \), which is a function of the basis determining parameter \( x \).

So the natural question, if there are *special values* for the concurrence \( C_{C_1C_2}(x) \), with regard to the initial ones \( C_{AC_1} \) and \( C_{BC_2} \), for which the main aim can be accomplished, arises. Obviously, those special values should be different from 1.

The answer is affirmative, we encountered two *special values* of the concurrence \( C_{C_1C_2}(x) \) given by

\[
C_{C_1C_2}^\pm = \frac{C_{AC_1}C_{BC_2}}{1 \pm \sqrt{(1-C^2_{AC_1})(1-C^2_{BC_2})}}.
\]
In the search for it we found two schemes, which we will describe below.

Those two schemes are suggested by representing the initial state $|\psi_{AC}\rangle|\psi_{BC}\rangle$ in the basis $|00\rangle$, which leads us to analyze the identity,

$$
|\psi_{AC}\rangle|\psi_{BC}\rangle = \sqrt{p_{\mu-}}|\phi_{C1,C2}\rangle|\phi_{AB}\rangle + \sqrt{p_{\mu+}}|\phi_{C1,C2}\rangle|\phi_{AB}\rangle - \sqrt{p_{\mu-}}|\phi_{C1,C2}\rangle|\phi_{AB}\rangle - \sqrt{p_{\mu+}}|\phi_{C1,C2}\rangle|\phi_{AB}\rangle.
$$

Here we have defined the possible outcome states of the qubits $AB$,

$$
|\phi_{AB}\rangle = \frac{\alpha \gamma y |00\rangle + \beta \delta x |11\rangle}{\sqrt{p_{\mu-}}}, \quad (13a)
$$

$$
|\psi_{AB}\rangle = \frac{\alpha \delta y |01\rangle + \gamma \beta x |10\rangle}{\sqrt{p_{\mu+}}}, \quad (13b)
$$

$$
\psi_{AB} = \frac{\alpha \delta x |01\rangle - \gamma \beta y |10\rangle}{\sqrt{p_{\mu-}}}, \quad (13c)
$$

$$
|\phi_{AB}\rangle = \frac{\alpha \gamma x |00\rangle - \beta \delta y |11\rangle}{\sqrt{p_{\mu+}}}, \quad (13d)
$$

and their respective probabilities,

$$
p_{\mu+} = \alpha^2 \gamma^2 y^2 + \beta^2 \delta^2 x^2, \quad (14a)
$$

$$
p_{\mu-} = \alpha^2 \delta^2 y^2 + \beta^2 \gamma^2 x^2, \quad (14b)
$$

$$
p_{\nu-} = \alpha^2 \beta \gamma y x^2 + \beta^2 \gamma x y^2, \quad (14c)
$$

$$
p_{\nu-} = \alpha^2 \gamma \beta y x^2 + \beta^2 \delta x y^2. \quad (14d)
$$

It is important to take into account that $x$ is a key measuring-basis parameter, to be strategically set in each of the following schemes. By this, we fix the required amount of entanglement of the measuring-basis.

1. **Third strategy**

It is worth noting that the concurrences of the outcomes $|\phi_{AB}\rangle$ can attain the maximal value 1, but at different values for $x$. Specifically, the state $|\phi_{AB}\rangle$ is maximally entangled at $x = x_1$, the outcome $|\psi_{AB}\rangle$ at $x = x_2$, the state $|\psi_{AB}\rangle$ at $x_3$, and $|\phi_{AB}\rangle$ at $x = x_4$, where

$$
x_1 = \frac{\alpha \gamma}{\sqrt{\alpha^2 \gamma^2 + \beta^2 \delta^2}}, \quad (15a)
$$

$$
x_2 = \frac{\alpha \delta}{\sqrt{\alpha^2 \delta^2 + \beta^2 \gamma^2}}, \quad (15b)
$$

$$
x_3 = \frac{\beta \gamma}{\sqrt{\alpha^2 \beta \gamma + \beta^2 \gamma \delta}}, \quad (15c)
$$

$$
x_4 = \frac{\beta \delta}{\sqrt{\alpha^2 \beta \gamma + \beta^2 \delta^2}}. \quad (15d)
$$

In general, these $x_i$ are different and, according to (3), they are ordered as follows,

$$
x_1 \leq x_2 \leq \frac{1}{\sqrt{2}} \leq x_3 \leq x_4, \quad (16a)
$$

$$
x_1 \leq x_3 \leq \frac{1}{\sqrt{2}} \leq x_2 \leq x_4, \quad (16b)
$$

This means that the possible EPR outcomes are displaced in the measuring-basis parameter $x$. Besides, we observe by considering $|\phi_{AB}\rangle$ with $\alpha \neq \beta$ and $\gamma \neq \delta$, that $|\phi_{AB}\rangle$ does not exhibit maximal entanglement, since $x_4 \notin [0,1/\sqrt{2}]$ (see condition (10)). On the other hand, the maximal entanglement outcome can be at $|\psi_{AB}\rangle$ or $|\phi_{AB}\rangle$ depending on the relation between $\alpha$ and $\gamma$. In addition, note that if (3) is not satisfied, then it is $|\phi_{AB}\rangle$ which exhibits maximal entanglement instead of $|\phi_{AB}\rangle$.

Therefore, to obtain the optimal result, we choose the $x_i$ corresponding to the greatest probability as the measuring-basis parameter. By inserting the $x_i$ in their respective probability (14), we realize that the greatest probability $P_{s_3}$ is given by

$$
P_{s_3} = \begin{cases} 
 p_{\mu+}(x_2), & \text{if } \alpha \leq \gamma, \\
 p_{\mu-}(x_3), & \text{if } \alpha \geq \gamma,
\end{cases}
$$

$$
= \frac{2\alpha^2 \beta^2 \delta \gamma^2}{\alpha^2 \delta^2 + \beta^2 \gamma^2}, \quad (17)
$$

This probability is smaller than, or equal to the ones found in the first and second strategy, i.e.,

$$
P_{s_3} \leq P_{s_1} \leq P_{s_2}. \quad (18)
$$

Note also, that $P_{s_3} \leq 1/4$ and the equality only holds true for the limit of the basic and deterministic scheme, i.e., $C_{AC_1} = C_{BC_2} = 1$ and $x_1 = x_2 = x_3 = x_4 = 1/\sqrt{2}$, meaning each outcome is maximally entangled and each one can be obtained with probability 1/4.

In spite of (18), it is worth realizing that in general $x_2$ and $x_3$ are different from $1/\sqrt{2}$, which stands for not requiring maximal entanglement of the measuring-basis (9) in order to have a probability different from zero of obtaining an EPR state in the outcome $|\phi_{AB}\rangle$ or $|\psi_{AB}\rangle$. Specifically, the values $x_2$ and $x_3$ are associated with the same special value $C_{C_1C_2}$ of the concurrence,

$$
C_{C_1C_2} = C_{C_1C_2}(x_2) = C_{C_1C_2}(x_3).
$$

From Eq. (11) we can note that if one of the two initial concurrences $C_{AC_1}$ or $C_{BC_2}$ is equal to 1, to say $C_{AC_1} = 1$, then $C_{C_1C_2} = C_{BC_2}$. If $C_{AC_1} = C_{BC_2}$, then we can conclude $C_{C_1C_2} = 1$. Provided, one of the two initial concurrences, $C_{AC_1}$ or $C_{BC_2}$, is equal to 0, then $C_{C_1C_2} = 0$ and $P_{s_3} = 0$.

The significance of these results is that there is a probability different from zero of accomplishing the main
of success

We want to mention here that in Ref. [36] the authors focus on the study of the behaviour and the relation among the concurrences of the outcome states, the parameter of the measuring-basis, and the respective probabilities of the outcome states.

2. Fourth strategy

Once knowing the measurement result [9], one of the receivers, A or B, can probabilistically extract an EPR state by means of the local USE procedure from the respective outcome [13].

According to Appendix A the conditional probabilities of successfully extracting depend on the relation between \( \alpha \) and \( \gamma \) and read as follows. If the outcome is \( |\varphi_{AB}\rangle \), the conditional probability \( p_{\text{ext},\varphi} \) of extracting a Bell state becomes,

\[
p_{\text{ext},\varphi} = \begin{cases} \frac{2\beta^2\gamma^2 x^2}{p_{\mu+}}, & \text{if } x \leq x_1, \\ \frac{2\alpha^2\gamma^2 y^2}{p_{\mu-}}, & \text{if } x \geq x_1. \end{cases}
\]

(19a)

When the outcome is \( |\psi_{AB}\rangle \), the conditional probability of success \( p_{\text{ext},\psi} \) is,

\[
p_{\text{ext},\psi} = \begin{cases} \frac{2\beta^2\gamma^2 x^2}{p_{\nu+}}, & \text{if } x \leq x_2, \\ \frac{2\alpha^2\gamma^2 y^2}{p_{\nu-}}, & \text{if } x \geq x_2. \end{cases}
\]

(19b)

While, if the outcome is \( |\psi_{AB}\rangle \), then the conditional success probability \( p_{\text{ext},\psi} \) reads,

\[
p_{\text{ext},\psi} = \begin{cases} \frac{2a^2\beta^2 x^2}{p_{\mu+}}, & \text{if } x \leq x_3, \\ \frac{2\beta^2\gamma^2 x^2}{p_{\mu-}}, & \text{if } x \geq x_3. \end{cases}
\]

(19c)

If the outcome is \( |\varphi_{AB}\rangle \), the conditional probability \( p_{\text{ext},\varphi} \) of extracting an EPR state is given by,

\[
p_{\text{ext},\varphi} = \frac{2\alpha^2\gamma^2 x^2}{p_{\mu-}}.
\]

(19d)

Since from each possible outcome [13] an EPR state can be probabilistically extracted, the total success probability is given by the sum of the probabilities [14], each one multiplied by its respective conditional probability of extracting [19], i.e.,

\[
P_{s_4}(x) = p_{\mu+}p_{\text{ext},\varphi} + p_{\nu+}p_{\text{ext},\psi} + p_{\mu-}p_{\text{ext},\psi} + p_{\mu-}p_{\text{ext},\varphi}.
\]

Similarly, for evaluating \( P_{s_4}(x) \), we must take the relation between \( \alpha \) and \( \gamma \) into account. In consequence, the total success probability becomes,

\[
P_{s_4}(x) = \begin{cases} 2x^2, & \text{if } x \leq x_1, \\ 2\alpha^2\gamma^2 + 2\left(\alpha^2\delta^2 + \beta^2\gamma^2\right)x^2, & \text{if } x_1 \leq x \leq \min\{x_2, x_3\}, \\ 2\min\{\alpha^2, \gamma^2\}, & \text{if } \min\{x_2, x_3\} \leq x \leq 1/\sqrt{2}. \end{cases}
\]

(20)

From the expression [20] we notice the following effects:

- The slope of \( P_{s_4}(x) \) exhibits two discontinuities, at \( x_1 \) and \( \min\{x_2, x_3\} \).
- The probability increases for \( x \in [0, \min\{x_2, x_3\}] \), hence for \( x \in [\min\{x_2, x_3\}, 1/\sqrt{2}] \) the probability \( P_{s_4}(x) \) becomes constant and equal to the maximal value obtained in the second strategy.
- There is the same entanglement matching, as found in the second strategy, between \( C_{AC_1} \) and \( C_{BC_2} \) for all \( x \in [\min\{x_2, x_3\}, 1/\sqrt{2}] \).
- The maximal probability \( 2\min\{\alpha^2, \gamma^2\} \) is achieved at \( x = \min\{x_2, x_3\} \), which in general is smaller than \( 1/\sqrt{2} \). Therefore, the maximal entanglement in the measuring-basis [49] is not required for reaching the optimal success probability.

- There are two special values, \( C_{C_1C_2}^{\pm} \), of the measuring-basis concurrence, for which \( P_{s_4}(x) \) changes its behavior; at \( C_{C_1C_2}^{\pm} = C_{C_1C_2}^{\pm} \), the probability [20] abruptly changes its slope, but for all \( C_{C_1C_2}(x) \geq C_{sv} \) the probability \( P_{s_4}(x) \) remains constant at its maximum value.

Accordingly, although \( C_{C_1C_2}^{\pm} \) differs from \( C_{AC_1} \) and \( C_{BC_2} \), it plays the role of an entanglement matching level, which is a function of the initial concurrences.

Finally, we would like to point out that the application of the USE procedure allows us to increase the probability of achieving an EPR state in the pair \( AB \), with respect to \( P_{s_3} \), while demanding the same entanglement \( C_{C_1C_2}^{\pm} \) in the measuring-basis. In other words, this strategy combines the best characteristics of the second and third strategy, i.e., it has the highest probability, obtained in the second strategy, and demands the special
value for the concurrence of the measuring-basis, found for the third strategy. Figure 3 illustrates the behavior of \(C_{BC2}\) as a function of \(C_{BC2} = C_{C1,C2}\) for \(C_{AC1} = 0.7\); the increasing surface clearly shows the abrupt change of the slope, and the plateau in the surface corresponds to the twofold matching effect.

Therefore, by considering the greatest probability of success and the entanglement as a resource, the fourth strategy is more efficient than the three others.

IV. CONCLUSIONS

In summary, we have addressed unambiguous and indeterministic entanglement swapping, having projection of two qubits onto an EPR state, which have never interacted, as main aim. In our analysis, a second aim becomes the maximization of the probability of success and the minimization of the required entanglement in the measuring-basis.

We have introduced three levels of entanglement, two of them in the initial states and another one by considering non-Bell state in the measuring-basis. Additionally, we proposed the unambiguous state extraction scheme as a mechanism for implementing the unitary-reduction local operator.

These considerations allow us to design four strategies for achieving the main aim. The first one enables to accomplish the main aim, but with non-optimal probability of success. In the second strategy, we found the optimal probability of success, related to an entanglement matching effect between the two concurrences of the initial states, i.e., the maximal success probability is determined by the smallest entanglement of the initial states. The third strategy has a success probability smaller than the first strategy, but gives account that the main aim can be accomplished without maximal entanglement in the measuring-basis. Thus, the three schemes lead us to suggest the fourth strategy. We consider the fourth strategy as the optimal one, because it combines the best characteristics of the second and third strategy, i.e., it has the highest probability, founded in the second strategy, and demands the non-maximal special value for the concurrence of the measuring-basis, found for the third strategy. We realized that the concurrences special value \(C_{C1,C2}\) plays the role of a matching effect between it and the initial concurrences, showing thus that there is a scheme with optimal success probability, which does not require maximal entanglement in the measurement process. In other words, the entanglement matching effect between the two initial concurrences matches with \(C_{C1,C2}\); therefore, any concurrence value greater than \(C_{C1,C2}\) does not affect the success probability, i.e., it remains constant.

Besides, we found another special value \(C_{C1,C2}\), for which the increasing success probability changes its slope abruptly.

Accordingly, here we have shown that, in order to accomplish the main aim of the entanglement swapping with maximal probability, the maximal entanglement in the measuring-basis is not required, equivalently, the special value \(C_{C1,C2}\) is necessary and sufficient.

Appendix A: Optimal EPR extraction

Here we succinctly show that the locally applied USE protocol [39] allows us to extract an EPR state, with optimal probability, from a partially entangled state \(|\psi_{ij}\rangle\), shared by the qubits \(i\) and \(j\), i.e.,

\[
|\tilde{\psi}_{ij}\rangle \rightarrow \text{local USE} \frac{|0_i\rangle|1_j\rangle + |1_i\rangle|0_j\rangle}{\sqrt{2}},
\]

with

\[
|\tilde{\psi}_{ij}\rangle = u|0_i\rangle|1_j\rangle + v|1_i\rangle|0_j\rangle.
\]

The \(|0,1\rangle\) are eigenstates of the Pauli operator \(\sigma_z\) of the qubit labeled by the subindex. Without loss of generality, we assume that \(u\) and \(v\) are real and non-negative numbers and due to normalization \(u^2 + v^2 = 1\). The entanglement of the initial state (A2) can be valued by the concurrence \(C_{\psi} = 2uv\).

The local operation can be indistinctly applied onto qubit \(i\) or \(j\). For instance, let us consider the qubit \(i\) and an auxiliary qubit \(a\), initially in the state \(|0_a\rangle\). Because the probability amplitude must have a module smaller than or equal to 1, we have to consider two cases:

(i) If \(u \leq v\), we apply the joint unitary \(U_{ia}\) onto the tensorial product state \(|\tilde{\psi}_{ij}\rangle|0_a\rangle\), with

\[
U_{ia} = |0_i\rangle\langle 0_i| \otimes I_a + |1_i\rangle\langle 1_i| \otimes U_a,
\]

\[
U_a|0_a\rangle = \frac{u}{v}|0_a\rangle + \sqrt{1 - \frac{u^2}{v^2}}|1_a\rangle,
\]

where \(I_a\) is the identity operator of the auxiliary qubit \(a\). Thus, \(U_{i,a}\) transforms \(|\tilde{\psi}_{ij}\rangle|0_a\rangle\) as follows:

\[
U_{ia}|\tilde{\psi}_{ij}\rangle|0_a\rangle = \sqrt{2u} \left( \frac{|0_i\rangle|1_j\rangle + |1_i\rangle|0_j\rangle}{\sqrt{2}} \right) |0_a\rangle
+ \sqrt{v^2 - u^2}|1_i\rangle|0_j\rangle|1_a\rangle,
\]

FIG. 3: Success probability \(P_{s4}\) as a function of \(C_{BC2}\) and \(C_{C1,C2}\) for \(C_{AC1} = 0.7\).
from where we realize, that by measuring $\sigma_z$ of the auxiliary qubit, the pair $ij$ is projected onto an EPR state with probability $p_{ext} = 2u^2$, otherwise the correlation is lost.

(ii) If $u \geq v$, we must apply the joint unitary

$$U_{ia} = |0_i\rangle\langle 0_i| \otimes U_a + |1_i\rangle\langle 1_i| \otimes I_a,$$  \hspace{1cm} (A4)

$$U_a|0_a\rangle = \frac{v}{u}|0_a\rangle + \sqrt{1 - \frac{v^2}{u^2}} |1_a\rangle,$$

to obtain

$$U_{ia}|\psi_{ij}\rangle|0_a\rangle = \sqrt{2v} |0_i\rangle|1_j\rangle + |1_i\rangle|0_j\rangle \frac{1}{\sqrt{2}} |0_a\rangle + \sqrt{u^2 - v^2} |0_i\rangle|1_j\rangle|1_a\rangle.$$  

Similarly, by measuring $\sigma_z$ of the auxiliary qubit, the pair $ij$ is projected onto an EPR state with probability $p_{ext} = 2v^2$, otherwise the correlation is lost.

Therefore, for an given bipartite pure state $|\psi_{ij}\rangle$ the probability $p_{ext}$ of extracting an EPR state by means of local operators and one way classical communication becomes

$$p_{ext} = 2 \text{min}\{u^2, v^2\},$$  

$$= 1 - \sqrt{1 - c^2_{\psi}},$$  \hspace{1cm} (A5)

where $u$ and $v$ are its Schmidt coefficients.

Noteworthy, that the probability $[A5]$ agrees with the optimal ones found in Refs. [43–45, 47] and the results does not depend on which of the two qubits, 1 or 2, the USE scheme was applied. It is worth to mention here that, the joint unitaries $[A3]$ and $[A4]$ can be implemented experimentally in different physical systems [48–51] by composition of local unitaries and CNOT gates.

Additionally, if the initial state is $|\bar{\psi}_{ij}\rangle = u|0_i\rangle|0_j\rangle + v|1_i\rangle|1_j\rangle$, then it can be transformed to $|\psi_{ij}\rangle$ by applying the local unitary rotation $U_j = e^{-i(1-\sigma_x)\pi/2}$ onto qubit $j$, and the above describe scheme can be implemented.

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