Quasiconvexity preserving property for fully nonlinear nonlocal parabolic equations

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Abstract. This paper is concerned with a general class of fully nonlinear parabolic equations with monotone nonlocal terms. We investigate the quasiconvexity preserving property of positive, spatially coercive viscosity solutions. We prove that if the initial value is quasiconvex, the viscosity solution to the Cauchy problem stays quasiconvex in space for all time. Our proof can be regarded as a limit version of that for power convexity preservation as the exponent tends to infinity. We also present several concrete examples to show applications of our result.

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1. Introduction

In this paper, we study a class of fully nonlinear nonlocal parabolic equations:

\[
\begin{aligned}
&u_t + F(u, \nabla u, \nabla^2 u, K \cap \{u(\cdot, t) < u(x, t)\}) = 0 \quad \text{in } \mathbb{R}^n \times (0, \infty), \\
u(\cdot, 0) = u_0 \quad \text{in } \mathbb{R}^n,
\end{aligned}
\]

where \( u : \mathbb{R}^n \times [0, \infty) \rightarrow \mathbb{R} \) is an unknown function, and \( u_t, \nabla u \) and \( \nabla^2 u \) denote the time derivative, the spatial gradient and Hessian of \( u \), respectively. Here \( K \subset \mathbb{R}^n \) is a compact set, the initial condition \( u_0 : \mathbb{R}^n \rightarrow \mathbb{R} \) is in \( UC(\mathbb{R}^n) \), where \( UC(\mathbb{R}^n) \) is the set of uniformly continuous functions on \( \mathbb{R}^n \), and \( F : \mathbb{R} \times (\mathbb{R}^n \setminus \{0\}) \times \mathbb{S}^n \times \mathcal{B}_K \rightarrow \mathbb{R} \) is a given continuous function, where \( \mathbb{S}^n \) denotes the space of \( n \times n \) real symmetric matrices and \( \mathcal{B}_K \) represents the collection of all measurable subsets of \( K \).

The goal of this paper is to investigate the preservation of spatial quasiconvexity of viscosity solutions to (1.1) and (1.2). Here, a function \( u \in \)
$C(\mathbb{R}^n \times [0, \infty))$ is said to be \textit{spatially quasiconvex} if all sublevel sets of $u(\cdot, t)$ are convex in $\mathbb{R}^n$, or equivalently,

$$u(\lambda y + (1 - \lambda)z, t) \leq \max\{u(y, t), u(z, t)\}$$

holds for all $y, z \in \mathbb{R}^n$, $t \geq 0$ and $\lambda \in (0, 1)$.

Our work is closely related to [8], where a general class of set evolutions with nonlocal terms is shown to preserve the convexity of the initial set. A typical example is the level set equation of the surface evolution equation

$$V = a + bm(K \cap \Omega_t) - c \text{div}_t \xi(n(x)),$$

(1.3)

where $a \in \mathbb{R}$, $b \geq 0$ and $c \geq 0$ are given constants, $V$ is the outward normal velocity of an evolving compact hypersurface $\Gamma_t$, $\Omega_t$ is the set enclosed by $\Gamma_t$, $\xi$ denotes the so-called Cahn-Hoffman vector, and $n(x)$ denotes the unit normal to $\Gamma_t$ at $x$ pointing outward $\Omega_t$. Here, $m(A)$ represents the $n$-dimensional Lebesgue measure of $A \subset \mathbb{R}^n$ and $\text{div}_t \xi(n(x))$ stands for the anisotropic mean curvature of $\Gamma_t$ at $x$. We shall give precise assumptions on $\xi$ in Sect. 5.1.

Such nonlocal evolutions can be reformulated via the so-called level set method as geometric parabolic equations, which in our context requires $F$ to satisfy

$$F(r_1, c_1p, c_1X + c_2p \otimes p, A) = c_1 F(r_2, p, X, A)$$

for all $c_1 > 0, c_2 \in \mathbb{R}, r_1, r_2 \in \mathbb{R}, p \in \mathbb{R}^n \setminus \{0\}, X \in S^n, A \in \mathcal{B}_K$; (1.4)

in particular, $F$ is independent of the unknown. We refer to [17] for more details about the level set formulation. Such equations also appear in the singular limit of some nonlocal reaction diffusion equations [11].

It is natural to ask whether it is possible to obtain the same result for nonlocal evolution equations without assuming (1.4) in order to allow broader applications. In this paper we give an affirmative answer to this question. Under a set of relaxed assumptions, we show the convexity preserving property for all sublevel sets of solutions and present several concrete examples of applications. See [13,14] for quasiconvexity results for classical solutions of elliptic problems without nonlocal terms.

Let us begin with our assumptions on the operator $F$.

(F1) $F$ is proper and degenerate elliptic; namely, for any $p \in \mathbb{R}^n \setminus \{0\}$ and $A \in \mathcal{B}_K$,

$$F(r_1, p, X, A) \leq F(r_2, p, X, A)$$

holds for all $r_2 \geq r_1$ and $X \in S^n$, and

$$F(r, p, X_1, A) \leq F(r, p, X_2, A)$$

(1.5)

holds for all $r \in \mathbb{R}, X_1, X_2 \in S^n$ satisfying $X_1 \geq X_2$.

(F2) $F$ is locally bounded in the sense that for each $R > 0$, there holds

$$\sup\{|F(r, p, X, A)| : r \in \mathbb{R}, |p| \leq R \text{ with } p \neq 0, |X| \leq R, A \in \mathcal{B}_K\} < \infty.$$}

(F3) $F$ is continuous in $\mathbb{R} \times (\mathbb{R}^n \setminus \{0\}) \times S^n \times \mathcal{B}_K$ with the topology of $\mathcal{B}_K$ given by $d(A_1, A_2) = m(A_1 \triangle A_2)$, where $A_1 \triangle A_2$ stands for the symmetric difference of $A_1$ and $A_2$, that is $A_1 \triangle A_2 := (A_1 \cup A_2) \setminus (A_1 \cap A_2)$ for
all $A_1, A_2 \in B_K$. Moreover, for any $R > 0$, there exists a modulus of continuity $\omega_R$ such that
\[
F(r,p,X,A_1) - F(r,p,X,A_2) \leq \omega_R \left( m(A_1 \triangle A_2) \right)
\] (1.6)
for all $r \in \mathbb{R}$, $p \in \mathbb{R}^n \setminus \{0\}$ with $|p| \leq R$, $X \in \mathbb{S}^n$ and $A_1, A_2 \in B_K$.

(F4) $F$ is monotone with respect to the set argument; namely,
\[
F(r,p,X,A_1) \leq F(r,p,X,A_2)
\]
holds for all $r \in \mathbb{R}$, $p \in \mathbb{R}^n \setminus \{0\}$, $X \in \mathbb{S}^n$, $A_1, A_2 \in B_K$ with $A_1 \subset A_2$.

(F5) $F$ satisfies the structure condition for the comparison principle: There exists a modulus of continuity $\omega$ such that
\[
F(r,p_1,X_1,A) - F(r,p_2,X_2,A) \leq \omega \left( \frac{|Z||p_1 - p_2|}{\min\{|p_1|,|p_2|\}} + |p_1 - p_2| + \alpha \right)
\]
for all $\alpha \geq 0$, $r \in \mathbb{R}$, $p_1, p_2 \in \mathbb{R}^n \setminus \{0\}$, $A \in B_K$, and $X_1, X_2, Z \in \mathbb{S}^n$ satisfying
\[
\begin{pmatrix} X_1 & 0 \\ 0 & -X_2 \end{pmatrix} \leq \begin{pmatrix} Z & -Z \\ -Z & Z \end{pmatrix} + \alpha \begin{pmatrix} I & I \\ I & I \end{pmatrix},
\]
where we denote by $I$ the $n \times n$ identity matrix.

(F6) There exists $\mu \in C(\mathbb{R})$ such that
\[
\sup_{r \in \mathbb{R}, A \in B_K} |F(r,p,X,A) - \mu(r)| \to 0 \quad \text{as} \quad (p,X) \to (0,0).
\]

The assumption (F6) immediately implies that $F^*(r,0,0,A) = F_*(r,0,0,A) = \mu(r)$ for all $r > 0$ and $A \in B_K$, where we denote by $F_*, F^*$ the upper and lower semicontinuous envelope of $F$, respectively (see [12,17] for definitions).

The ellipticity (1.5) of $F$ yields
\[
F_*(r,0,X_1,A) \leq F_*(r,0,X_2,A), \quad F^*(r,0,X_1,A) \leq F^*(r,0,X_2,A),
\]
for all $r \in \mathbb{R}$, $X_1, X_2 \in \mathbb{S}^n$ satisfying $X_1 \geq X_2$. In particular, we have
\[
F^*(r,0,X_1,A) \leq \mu(r) \leq F_*(r,0,X_2,B) \quad \text{for all} \quad r \in \mathbb{R}, X_1 \geq 0 \geq X_2 \quad \text{and} \quad A, B \in B_K.
\]

We stress that the monotonicity (F4) is crucial for validity of the comparison principle, which facilitates our analysis. See [8,15,35,36] for comparison results for monotone evolution equations in different settings. On the other hand, in the non-monotone case, one cannot expect the comparison principle to hold and alternative methods are needed to prove uniqueness of solutions and other related properties (see [1,4,5,32] for instance).

In order to clearly demonstrate our arguments for the convexity property, throughout this paper we shall only consider viscosity solutions to (1.1) and (1.2) that are uniformly positive and coercive in space, that is,
\[
u \geq c_0 \quad \text{in} \quad \mathbb{R}^n \times [0,\infty), \quad \text{for some} \quad c_0 > 0
\] (1.7)
and
\[
\inf_{|x| \geq R, t \leq T} u(x,t) \to \infty \quad \text{as} \quad R \to \infty \quad \text{for any} \quad T \geq 0.
\] (1.8)
We actually include further assumptions on $u_0 \in UC(R^n)$ to guarantee the existence and uniqueness of solutions $u \in C(R^n \times [0, \infty))$ of (1.1) and (1.2), satisfying (1.7) and (1.8). Moreover, we can show that

$$|u(x, t) - u(y, t)| \leq \omega_0(|x - y|) \quad \text{for all } x, y \in R^n \text{ and } t \geq 0,$$

where $\omega_0$ denotes the modulus of continuity of $u_0$. See Theorem 3.3 for details.

In addition, we impose a key concavity condition on $F$ via a transformed operator $G_\beta$ defined by

$$G_\beta(r, p, X, A) = \frac{1}{1 - \beta} r^\beta F \left( r^{1-\beta}, (1 - \beta)r^{-\beta}p, (1 - \beta)r^{-\beta}X + (\beta^2 - \beta)r^{-\beta-1}p \otimes p, A \right)$$

for $0 < \beta < 1$, $r > 0$, $p \in R^n \setminus \{0\}$, $X \in S^n$ and $A \in B_K$.

(F7) For any $\beta < 1$ close to 1,

$$(r, X) \mapsto G_\beta(r, p, X, A) \quad \text{is concave in } [c_0, \infty) \times S^n$$

for any $p \in R^n \setminus \{0\}$ and $A \in B_K$, and

$$r \mapsto r^\beta \mu(r^{1-\beta}) \quad \text{is concave in } [c_0, \infty),$$

where $\mu$ is given by (F6).

See [14, (3.2)] for a related condition for fully nonlinear local elliptic problems.

Let us state our main result.

**Theorem 1.1.** (Quasiconvexity preserving property) Assume (F1)--(F7). Let $u_0 \in UC(R^n)$. Let $u \in C(R^n \times [0, \infty))$ be the unique viscosity solution of (1.1) and (1.2) satisfying (1.7), (1.8) and (1.9). If $u_0$ is quasiconvex in $R^n$, that is, $\{u_0 < h\}$ is convex for all $h \in R$, then $u(\cdot, t)$ is quasiconvex in $R^n$ for all $t \geq 0$.

Our result is applicable to a general class of nonlinear parabolic equations including the level set equations for nonlocal geometric evolutions studied in [8]. See Sect. 5 for applications of Theorem 1.1 to some concrete equations. We emphasize that these equations do not need to be geometric. The assumptions in Theorem 1.1 actually allow the operator $F$ to additionally depend on the unknown $u$.

It is also worth pointing out that in general quasiconvexity breaking may occur if (F7) does not hold. It is known that the heat equation (without gradient terms) fails to preserve quasiconvexity of solutions in the space variable as shown in [22,27] (see [9,20] for counterexamples in other settings). Note that the Laplace operator does not fulfill (F7), since in this case

$$G_\beta(r, p, X) = -\text{tr}X + \beta r^{-1}|p|^2$$

is not concave with respect to $r \in (0, \infty)$.

The novelty of our work lies not only at the more relaxed setting and wider applications, but also at an improved method in our proof. Instead of the set-theoretic arguments in [8], we develop a PDE-based approach. Convexity/concavity of solutions is a classical topic of geometric properties of elliptic and parabolic equations. A non-exhaustive list of references includes [6,7,30,31,33] for classical solutions and [2,18,28,34] for viscosity solutions. A
generalized type, called power convexity/concavity, is investigated in [21,24–26] for various equations. One major idea applied in [2,21,30] is to show the corresponding convex envelope of a solution is a supersolution of the equation and then use the comparison principle to conclude the proof. This machinery can be implemented also for quasiconvexity or quasiconcavity of classical solutions to the Dirichlet boundary problems for semilinear elliptic or parabolic equations [13,14,16,23]. For viscosity solutions of our fully nonlinear nonlocal problem, we essentially adopt the same strategy but incorporate an additional limit process, based on the fact that quasiconvexity can be regarded as a limit notion of the power convexity as the exponent tends to infinity.

For a fixed $\lambda \in (0,1)$ and a given positive viscosity solution $u$ of (1.1), in order to prove that the spatially quasiconvex envelope $u_{*,\lambda}$, defined by

$$u_{*,\lambda}(x,t) = \inf\left\{ \max\{u(y,t),u(z,t)\} : x = \lambda y + (1 - \lambda)z \right\}$$

for $(x,t) \in \mathbb{R}^n \times (0,\infty)$, is a viscosity supersolution, we approximate $u_{*,\lambda}$ by the spatially power convex envelope $u_{q,\lambda}$, given by

$$u_{q,\lambda}(x,t) = \inf\left\{ \left(\lambda u(y,t)^q + (1 - \lambda)u(z,t)^q\right)^{\frac{1}{q}} : \lambda y + (1 - \lambda)z = x \right\}$$

as the exponent $q \to \infty$. The concavity condition (F7), with the choice $\beta = 1 - 1/q$, plays an important role of relating $u_{q,\lambda}$ to the supersolution property of (1.1) for $q > 1$ arbitrarily large. Also, thanks to the coercivity of $u$ in (1.8), the large exponent approximation by $u_{q,\lambda}$ can be conducted in the sense of locally uniform convergence and thus the supersolution property can be passed on to $u_{*,\lambda}$. More details will be given in Sect. 4.

It is worth pointing out that although the power convex envelope is used as a key ingredient in our proof, in general one cannot expect the preservation of power convexity in the space variable, due to the nonlocal structure of the equation. We will clarify this via Example 4.1 in Sect. 4. Since quasiconvexity is regarded as the weakest notion in the category of any power convexity, in this sense our generalized convexity preserving result in Theorem 1.1 can be considered to be optimal.

As mentioned above, we need a comparison principle to complete the whole proof. Since the comparison results in the literature concerning nonlocal equations are only for bounded solutions, for our own purpose we include a comparison theorem, Theorem 3.1, for possibly unbounded solutions satisfying growth condition (3.1). Our proof is an adaptation of that in [18] to nonlocal problems. It would be interesting to obtain a similar quasiconvexity result for non-monotone evolution equations, for which the comparison principle fails to hold in general.

We remark that the presence of the compact set $K$ in (1.1) largely facilitates our arguments in this work. It enables us to show the unique existence
of solutions that are coercive in space. The case when \( K \) is unbounded seems more challenging.

Our interests in (1.1) are not restricted to the quasiconvexity of solutions. In the upcoming work [29], we give an optimal control interpretation for a class of first order nonlocal evolution equations and use it to study the large time behavior of solutions, which is not studied much.

The rest of the paper is organized in the following way. In Sect. 2 we recall the definition and some basic properties of viscosity solutions of (1.1). We provide a comparison principle for our later applications in Sect. 3. Section 4 is devoted to the proof of our main result, Theorem 1.1. In Sect. 5 we present several concrete examples with further discussions on the assumptions, especially (F7).

2. Preliminaries

In this section, we recall some basic results on (1.1). We first recall the definition of viscosity solutions to (1.1). For a set \( Q \subset \mathbb{R}^n \times [0, \infty) \), we denote by \( USC(Q) \) and \( LSC(Q) \), respectively, the set of the upper and lower semicontinuous functions in \( Q \).

**Definition 1.** (Viscosity solutions) (i) A function \( u \in USC(\mathbb{R}^n \times (0, \infty)) \) is called a viscosity subsolution of (1.1) if whenever there exist \( (x_0, t_0) \in \mathbb{R}^n \times (0, \infty) \) and \( \varphi \in C^2(\mathbb{R}^n \times (0, \infty)) \) such that \( u - \varphi \) attains a local maximum at \( (x_0, t_0) \),

\[
\varphi_t(x_0, t_0) + F_*(u(x_0, t_0), \nabla \varphi(x_0, t_0), \nabla^2 \varphi(x_0, t_0), K \cap \{ u(\cdot, t) < u(x_0, t_0) \}) \leq 0.
\]

(ii) A function \( u \in LSC(\mathbb{R}^n \times (0, \infty)) \) is called a viscosity supersolution of (1.1) if whenever there exist \( (x_0, t_0) \in \mathbb{R}^n \times (0, \infty) \) and \( \varphi \in C^2(\mathbb{R}^n \times (0, \infty)) \) such that \( u - \varphi \) attains a minimum at \( (x_0, t_0) \),

\[
\varphi_t(x_0, t_0) + F^*(u(x_0, t_0), \nabla \varphi(x_0, t_0), \nabla^2 \varphi(x_0, t_0), K \cap \{ u(\cdot, t) \leq u(x_0, t_0) \}) \geq 0.
\]

(iii) A function \( u \in C(\mathbb{R}^n \times (0, \infty)) \) is called a viscosity solution of (1.1) if it is both a viscosity subsolution and a viscosity supersolution.

We are always concerned with viscosity solutions in this paper, and the term “viscosity” is omitted henceforth.

**Proposition 2.1.** (Refined test functions) Assume (F1)–(F4) and (F6) hold. Let \( u \in LSC(\mathbb{R}^n \times (0, \infty)) \) (resp., \( u \in USC(\mathbb{R}^n \times (0, \infty)) \)) be a locally bounded function in \( \mathbb{R}^n \times (0, \infty) \). Then \( u \) is a supersolution (resp., subsolution) of (1.1) if whenever there exist \( (x_0, t_0) \in \mathbb{R}^n \times (0, \infty) \) and \( \varphi \in C^2(\mathbb{R}^n \times (0, \infty)) \) such that \( u - \varphi \) attains a local minimum (resp., maximum) at \( (x_0, t_0) \), the following conditions hold:

- If \( \nabla \varphi(x_0, t_0) \neq 0 \), then

\[
\varphi_t(x_0, t_0) + F(u(x_0, t_0), \nabla \varphi(x_0, t_0), \nabla^2 \varphi(x_0, t_0), K \cap \{ u(\cdot, t) \leq u(x_0, t_0) \}) \geq 0
\]

(2.1)
\( (\text{resp.}, \varphi_t(x_0, t_0) + F(u(x_0, t_0), \nabla \varphi(x_0, t_0), \nabla^2 \varphi(x_0, t_0), K \cap \{u(\cdot, t_0) < u(x_0, t_0)\}) \leq 0). \)

- If \( \nabla \varphi(x_0, t_0) = 0 \) and \( \nabla^2 \varphi(x_0, t_0) = 0 \), then
  \[
  \varphi_t(x_0, t_0) + \mu(u(x_0, t_0)) \geq 0 \quad (\text{resp.}, \varphi_t(x_0, t_0) + \mu(u(x_0, t_0)) \leq 0). \tag{2.2}
  \]

We refer to [17, Proposition 2.2.8] for a similar property in the case of local singular equations. In order to prove Proposition 2.1, let us first give an elementary result (see also [35, Equation (5)]).

**Lemma 2.2.** (Continuity of measures) Let \( a \in \mathbb{R} \) and \( \{a_\mu\}_{\mu > 0} \subset \mathbb{R} \). Let \( \{f_\mu\}_{\mu > 0} \) be a family of measurable functions in \( \mathbb{R}^n \). If \( \lim \inf_{\mu \to 0} a_\mu \geq a \) and \( f \) is a measurable function such that \( f \geq \limsup_{\mu \to 0} f_\mu \), then

\[
  m(K \cap (\{f < a\} \setminus \{f_\mu < a_\mu\})) \to 0 \quad \text{as} \quad \mu \to 0. \tag{2.3}
\]

If \( \lim \sup_{\mu \to 0} a_\mu \leq a \) and \( f \) is a measurable function such that \( f \leq \liminf_{\mu \to 0} f_\mu \), then

\[
  m(K \cap (\{f_\mu \leq a_\mu\} \setminus \{f \leq a\})) \to 0 \quad \text{as} \quad \mu \to 0. \tag{2.4}
\]

Here, we denote by \( \limsup_{\mu \to 0} f_\mu \) and \( \liminf_{\mu \to 0} f_\mu \) the half relaxed limits of \( \{f_\mu\} \) (see [12,17]).

**Proof.** Let us first show (2.3). Since \( f \geq \limsup_{\mu \to 0} f_\mu \) and \( \lim \inf_{\mu \to 0} a_\mu \geq a \) hold, for any \( x \in \mathbb{R}^n \) satisfying \( f(x) < a \), there exists \( \mu > 0 \) such that \( f_\delta(x) < a_\delta \) for all \( \delta \leq \mu \); that is to say

\[
  \{f < a\} \subset \lim \inf_{\delta \to 0} \{f_\delta < a_\delta\} := \bigcup_{\mu > 0} \bigcap_{\delta \leq \mu} \{f_\delta < a_\delta\}.
\]

It follows that

\[
  m\left( \bigcap_{\mu > 0} \bigcup_{\delta \leq \mu} K \cap (\{f < a\} \setminus \{f_\delta < a_\delta\}) \right) = 0.
\]

Then for any \( \varepsilon > 0 \), there exists \( \mu > 0 \) small such that

\[
  m\left( \bigcup_{\delta \leq \mu} K \cap (\{f < a\} \setminus \{f_\delta < a_\delta\}) \right) < \varepsilon,
\]

which immediately yields

\[
  m(K \cap (\{f < a\} \setminus \{f_\mu < a_\mu\})) < \varepsilon.
\]

We thus complete the proof of (2.3).

One can show (2.4) similarly. In fact, this time our assumptions enable us to get

\[
  \limsup_{\delta \to 0} \{f_\delta \leq a_\delta\} := \bigcap_{\mu > 0} \bigcup_{\delta \leq \mu} \{f_\delta \leq a_\delta\} \subset \{f \leq a\}.
\]
Then, for any $\varepsilon > 0$,

$$m \left( \bigcup_{\delta \leq \mu} K \cap \{ f_\delta \leq a_\delta \} \setminus \{ f \leq a \} \right) < \varepsilon$$

when $\mu > 0$ is taken small. We thus have

$$m(K \cap \{ f_\mu \leq a_\mu \} \setminus \{ f \leq a \}) < \varepsilon$$

for any $\mu > 0$ small, which concludes the proof of (2.4).

Proof. (Proof of Proposition 2.1) We only prove the statement for supersolutions. Suppose that there exist $\varphi \in C^2(\mathbb{R}^n \times (0, \infty))$ and $(x_0, t_0)$ such that $u - \varphi$ attains a strict local minimum at $(x_0, t_0)$. Without loss of generality we may assume $\| \nabla \varphi \|_{L^\infty} \leq R$ for some $R > 0$. It is clear that $u$ satisfies the supersolution property if $\nabla \varphi(x_0, t_0) \neq 0$. We thus only consider the case when $\nabla \varphi(x_0, t_0) = 0$.

Let

$$\Phi(x, y, t) := u(x, t) - \varphi(y, t) - \frac{|x - y|^4}{\varepsilon}.$$ 

By a standard argument, we see that, for any $\varepsilon > 0$ small,

$$\Phi$$

attains a local minimum at some $(x_\varepsilon, y_\varepsilon, t_\varepsilon) \in \mathbb{R}^n \times \mathbb{R}^n \times (0, \infty)$ satisfying $(x_\varepsilon, y_\varepsilon, t_\varepsilon) \to (x_0, t_0), \ u(x_\varepsilon, t_\varepsilon) \to u(x_0, t_0)$ as $\varepsilon \to 0$. \hspace{1cm} (2.5)

Noting that $y \mapsto \Phi(x_\varepsilon, y, t_\varepsilon)$ takes a local minimum at $y = y_\varepsilon$, we have

$$\nabla \varphi(y_\varepsilon, t_\varepsilon) = \frac{4}{\varepsilon} |x_\varepsilon - y_\varepsilon|^2 (y_\varepsilon - x_\varepsilon),$$

$$\nabla^2 \varphi(y_\varepsilon, t_\varepsilon) \leq \frac{4}{\varepsilon} |x_\varepsilon - y_\varepsilon|^2 I + 8 \varepsilon (x_\varepsilon - y_\varepsilon) \otimes (x_\varepsilon - y_\varepsilon).$$ \hspace{1cm} (2.6)

We here divide into two cases:

- Case 1. $x_\varepsilon \neq y_\varepsilon$ along a sequence $\varepsilon = \varepsilon_j \to 0$;
- Case 2. $x_\varepsilon = y_\varepsilon$ for all small $\varepsilon > 0$.

In Case 1, by (2.5),

$$(x, t) \mapsto u(x, t) - \varphi(x - x_\varepsilon + y_\varepsilon, t) - \frac{|x_\varepsilon - y_\varepsilon|^4}{\varepsilon}$$

attains a local minimum at $(x_\varepsilon, t_\varepsilon)$. Thus we can adopt (2.1) to obtain

$$\varphi_t(y_\varepsilon, t_\varepsilon) + F(u(x_\varepsilon, t_\varepsilon), \nabla \varphi(y_\varepsilon, t_\varepsilon), \nabla^2 \varphi(y_\varepsilon, t_\varepsilon), K \cap \{ u(\cdot, t_\varepsilon) \leq u(x_\varepsilon, t_\varepsilon) \}) \geq 0.$$ \hspace{1cm} (2.7)

Due to (2.5) and the lower semicontinuity of $u$, by Lemma 2.2 we have

$$m(K \cap \{ u(\cdot, t_\varepsilon) \leq u(x_\varepsilon, t_\varepsilon) \} \setminus \{ u(\cdot, t_0) \leq u(x_0, t_0) \}) \to 0 \quad \text{as} \ \varepsilon \to 0.$$ \hspace{1cm} (2.8)

Set $a_\varepsilon := u(x_\varepsilon, t_\varepsilon), \ \eta_\varepsilon := \nabla \varphi(y_\varepsilon, t_\varepsilon) \neq 0, \ Y_\varepsilon := \nabla^2 \varphi(y_\varepsilon, t_\varepsilon), \ A_\varepsilon := K \cap \{ u(\cdot, t_\varepsilon) \leq u(x_\varepsilon, t_\varepsilon) \}, \text{ and } B := K \cap \{ u(\cdot, t_0) \leq u(x_0, t_0) \}$. By (2.7), (F3), and (F4),
\[0 \leq \varphi_t(y_\varepsilon, t_\varepsilon) + F(a_\varepsilon, \eta_\varepsilon, Y_\varepsilon, A_\varepsilon)\]
\[\leq \varphi_t(y_\varepsilon, t_\varepsilon) + F(a_\varepsilon, \eta_\varepsilon, Y_\varepsilon, B) + F(a_\varepsilon, \eta_\varepsilon, Y_\varepsilon, A_\varepsilon \cap B)\]
\[= \varphi_t(y_\varepsilon, t_\varepsilon) + F(a_\varepsilon, \eta_\varepsilon, Y_\varepsilon, B) + \omega_R(m(A_\varepsilon \cap (A_\varepsilon \cap B)))\]

By (2.8), sending \(\varepsilon \to 0\) yields
\[\varphi_t(x_0, t_0) + F^*(u(x_0, t_0), 0, \nabla^2 \varphi(x_0, t_0), B) \geq 0. \quad (2.9)\]

In Case 2, we see that \(u - \psi_\varepsilon\) attains a local minimum at \((x_\varepsilon, t_\varepsilon)\), where \(\psi_\varepsilon\) is given by
\[\psi_\varepsilon(x, t) := \varphi(y_\varepsilon, t) + \frac{|x - y_\varepsilon|^4}{\varepsilon}. \quad (3.1)\]

Since \(x_\varepsilon = y_\varepsilon\), it is clear that
\[\nabla \psi_\varepsilon(x_\varepsilon, t_\varepsilon) = 0, \quad \nabla^2 \psi_\varepsilon(x_\varepsilon, t_\varepsilon) = 0. \quad (3.2)\]

It then follows from (2.2) that
\[\varphi_t(y_\varepsilon, t_\varepsilon) + \mu(u(x_\varepsilon, t_\varepsilon)) = (\psi_\varepsilon)_t(x_\varepsilon, t_\varepsilon) + \mu(u(x_\varepsilon, t_\varepsilon)) \geq 0. \quad (2.10)\]

By (2.6) we have
\[\nabla \varphi(y_\varepsilon, t_\varepsilon) = 0, \quad \nabla^2 \varphi(y_\varepsilon, t_\varepsilon) \leq 0. \quad (3.3)\]

Sending \(\varepsilon \to 0\), we obtain \(\nabla \varphi(x_0, t_0) = 0\), and \(\nabla^2 \varphi(x_0, t_0) \leq 0\). Combining these with (2.10) and applying (F6), we are led to (2.9) again. \(\square\)

3. Comparison principle

In this section, we present a comparison principle for (1.1).

**Theorem 3.1.** (Comparison principle) Assume that (F1)–(F6) hold. Let \(u \in USC(\mathbb{R}^n \times [0, \infty))\) and \(v \in LSC(\mathbb{R}^n \times [0, \infty))\) be, respectively, a subsolution and a supersolution to (1.1). Assume in addition that for any \(T > 0\), there exists \(L_T > 0\) such that
\[u(x, t) \leq L_T(|x| + 1), \quad v(x, t) \geq -L_T(|x| + 1) \quad \text{for all} \ (x, t) \in \mathbb{R}^n \times [0, T]. \quad (3.1)\]

If there exists a modulus of continuity \(\omega_0\) such that
\[u(x, 0) - v(y, 0) \leq \omega_0(|x - y|) \quad \text{for all} \ x, y \in \mathbb{R}^n, \quad (3.2)\]
then \(u \leq v\) holds in \(\mathbb{R}^n \times [0, \infty)\).

The following result is an adaptation of [18, Proposition 2.3].
Proposition 3.2. (Growth estimate) Assume the same assumptions as in Theorem 3.1. For any fixed \( T > 0 \) and any \( L > L_T \) large, there exists \( M > 0 \) such that
\[
u(x, t) - v(y, t) \leq L|x - y| + M(1 + t) \quad \text{for all } x, y \in \mathbb{R}^n \text{ and } t \in [0, T).
\]

Proof. Note that (3.2) yields the existence of \( C_0 > 0 \) such that
\[
u(x, 0) - v(y, 0) \leq C_0(|x - y| + 1) \quad \text{for all } x, y \in \mathbb{R}^n. \tag{3.3}
\]
Take \( L > \max\{L_T, C_0\} \). Our goal is to show that
\[
u(x, t) - v(y, t) - \psi(x, y) - M(1 + t) \leq 0 \quad \text{for } x, y \in \mathbb{R}^n, t \in [0, T) \tag{3.4}
\]
for \( M > 0 \) sufficiently large, where
\[
\psi(x, y) = L(|x - y|^2 + 1)^{\frac{1}{2}}.
\]
Suppose that (3.4) fails to hold for any arbitrarily large \( M > 0 \). Then, we may assume that \( M > C_0 \), and there exist \( \hat{x}, \hat{y} \in \mathbb{R}^n \), \( \hat{t} \in [0, T) \) such that
\[
u(\hat{x}, \hat{t}) - v(\hat{y}, \hat{t}) - \psi(\hat{x}, \hat{y}) - M(1 + \hat{t}) > 0. \tag{3.5}
\]
Set, for \( \varepsilon, \lambda > 0 \) small and \( R > 0 \) large,
\[
\Psi_\varepsilon(x, y, t, s) = \nu(x, t) - v(y, s) - \psi(x, y) - L(g_R(x) + g_R(y)) - \frac{(t - s)^2}{2\varepsilon} - M(1 + t) - \frac{\lambda}{T - \hat{t}},
\]
where \( g_R \in C^2(\mathbb{R}^n) \) is a nonnegative function satisfying
\[
g_R(x) = 0 \quad \text{for } |x| < R, \quad \frac{g_R(x)}{|x|} \to 1 \quad \text{as } |x| \to \infty,
\]
\[
\sup\{|\nabla g_R(x)| + |\nabla^2 g_R(x)| : x \in \mathbb{R}^n, R > 0\} < \infty.
\]
It follows from (3.5) that \( \Psi_\varepsilon(\hat{x}, \hat{t}, \hat{y}, \hat{t}) > 0 \) if we take \( R > |\hat{x}|, |\hat{y}| \) and \( \lambda > 0 \) small depending only on \( M \).

By (3.1), we see that \( \Psi_\varepsilon \) attains a positive maximum in \((\mathbb{R}^2 \times [0, T])^2\) at \((x_\varepsilon, t_\varepsilon, y_\varepsilon, s_\varepsilon)\) for \( R > 0 \) and \( M > 0 \) large and for \( \varepsilon > 0 \) small. In fact, \( x_\varepsilon \) and \( y_\varepsilon \) are bounded uniformly with respect to \( \varepsilon \). Moreover, we have \( t_\varepsilon, s_\varepsilon \to t_0 \) for some \( t_0 \in [0, T) \) as \( \varepsilon \to 0 \). In view of the upper semicontinuity of \( \nu \) and lower semicontinuity of \( v \) as well as (3.3), we deduce that \( t_0 \neq 0 \) and thus \( t_\varepsilon, s_\varepsilon > 0 \) for all \( \varepsilon > 0 \) small.

Since \( u \) and \( v \) are, respectively, a subsolution and a supersolution of (1.1), we obtain
\[
\frac{t_\varepsilon - s_\varepsilon}{\varepsilon} + M + \frac{\lambda}{(T - t_\varepsilon)^2} + F_*(u(x_\varepsilon, t_\varepsilon), p_1, X_1, K, \{u(\cdot, t_\varepsilon) < u(x_\varepsilon, t_\varepsilon)\}) \leq 0,
\]
\[
\frac{t_\varepsilon - s_\varepsilon}{\varepsilon} + F^*(v(y_\varepsilon, s_\varepsilon), p_2, X_2, K, \{v(\cdot, t_\varepsilon) \leq v(y_\varepsilon, t_\varepsilon)\}) \geq 0,
\]
where
\[
p_1 = L(|x_\varepsilon - y_\varepsilon|^2 + 1)^{-\frac{1}{2}}(x_\varepsilon - y_\varepsilon) + L\nabla g_R(x_\varepsilon),
\]
\[
p_2 = L(|x_\varepsilon - y_\varepsilon|^2 + 1)^{-\frac{1}{2}}(x_\varepsilon - y_\varepsilon) - L\nabla g_R(y_\varepsilon),
\]
\[
X_1 = L(|x_\varepsilon - y_\varepsilon|^2 + 1)^{-\frac{1}{2}} I - L(|x_\varepsilon - y_\varepsilon|^2 + 1)^{-\frac{3}{2}} (x_\varepsilon - y_\varepsilon) \otimes (x_\varepsilon - y_\varepsilon) \\
+ L \nabla^2 g_R(x_\varepsilon),
\]
\[
X_2 = -L(|x_\varepsilon - y_\varepsilon|^2 + 1)^{-\frac{1}{2}} I + L(|x_\varepsilon - y_\varepsilon|^2 + 1)^{-\frac{3}{2}} (x_\varepsilon - y_\varepsilon) \otimes (x_\varepsilon - y_\varepsilon) \\
- L \nabla^2 g_R(y_\varepsilon).
\]

Since the boundedness of \(p_1, p_2, X_1, X_2\) depends only on \(L\), taking the difference between the viscosity inequalities above and applying (F2), we have \(C_L > 0\) such that \(M \leq C_L\), which is a contradiction to the arbitrariness of \(M > 0\). \(\square\)

Let us now proceed to the proof of Theorem 3.1.

**Proof.** (Proof of Theorem 3.1) Assume by contradiction that \(\sup_{\mathbb{R}^n \times [0,T]}(u - v) =: \theta > 0\). Then, there exists \(\lambda > 0\) such that
\[
\sup_{(x,t) \in \mathbb{R}^n \times [0,T]} \left\{ u(x,t) - v(x,t) - \frac{\lambda}{T - t} \right\} > \frac{3\theta}{4}.
\]

There exists \((x_1, t_1) \in \mathbb{R}^n \times [0, T)\) such that \(u(x_1, t_1) - v(x_1, t_1) - \lambda/(T - t_1) > \theta/2\). Noting that \(\sup_{\mathbb{R}^n} (u(\cdot, 0) - v(\cdot, 0)) \leq 0\), we have \(t_1 > 0\).

Define
\[
\Phi(x, y, t) := u(x, t) - v(y, t) - \frac{|x - y|^4}{\varepsilon^4} - \alpha(|x|^2 + |y|^2) - \frac{\lambda}{T - t}
\]
for \(\varepsilon, \alpha > 0, \lambda > 0\). It is then clear that there exists \(\alpha_0 > 0\) such that
\[
\sup_{(x,y,t) \in \mathbb{R}^n \times [0,T]} \Phi(x,y,t) > \frac{\theta}{4}
\]
for all \(0 < \alpha < \alpha_0\) and \(\varepsilon > 0\) small. The growth condition (3.1) implies that \(\Phi\) attains a maximum at some \((x_{\varepsilon, \alpha}, y_{\varepsilon, \alpha}, t_{\varepsilon, \alpha}) \in \mathbb{R}^n \times [0, T)\). We write \((\tilde{x}, \tilde{y}, \tilde{t})\) for \((x_{\varepsilon, \alpha}, y_{\varepsilon, \alpha}, t_{\varepsilon, \alpha})\) to simplify our notations. It follows that
\[
\frac{|\tilde{x} - \tilde{y}|^4}{\varepsilon^4} + \alpha(|\tilde{x}|^2 + |\tilde{y}|^2) \\
\leq u(\tilde{x}, \tilde{t}) - v(\tilde{y}, \tilde{t}) - u(x_1, t_1) + v(x_1, t_1) + 2\alpha|x_1|^2 + \frac{\lambda}{T - t_1} - \frac{\lambda}{T - \tilde{t}}.
\]

In view of Proposition 3.2, we have
\[
\frac{|\tilde{x} - \tilde{y}|^4}{\varepsilon^4} + \alpha(|\tilde{x}|^2 + |\tilde{y}|^2) \leq L(|\tilde{x} - \tilde{y}| + 1) + M(\tilde{t} + 1) - u(x_1, t_1) + v(x_1, t_1) + 2\alpha|x_1|^2 + \frac{\lambda}{T - t_1} - \frac{\lambda}{T - \tilde{t}}.
\]

It follows that
\[
\frac{|\tilde{x} - \tilde{y}|^4}{\varepsilon^4} - L|\tilde{x} - \tilde{y}| + \alpha(|\tilde{x}|^2 + |\tilde{y}|^2) \leq C
\]
for some \(C \geq 0\) which is independent of \(\varepsilon, \alpha\), which implies that \(\alpha(|\tilde{x}| + |\tilde{y}|) \to 0\) as \(\alpha \to 0\)
for any \(\varepsilon > 0\), and \(\sup_{0 < \alpha < \alpha_0} |\tilde{x} - \tilde{y}| \to 0\) as \(\varepsilon \to 0\). (3.6)
Hence, there exists $\varepsilon_0 > 0$ such that

$$\omega_0(|\tilde{x} - \tilde{y}|) \leq \theta/4$$

uniformly for all $0 < \varepsilon < \varepsilon_0$ and $0 < \alpha < \alpha_0$, where $\omega_0$ is the modulus of continuity appearing in (3.2).

On the other hand, we have

$$u(\tilde{x}, \tilde{t}) - v(\tilde{y}, \tilde{t}) \geq \Phi(\tilde{x}, \tilde{y}, \tilde{t}) > \frac{\theta}{4}.$$ 

It follows that $\tilde{t} > 0$ for any $0 < \alpha < \alpha_0$ and $0 < \varepsilon < \varepsilon_0$. In what follows, we fix $0 < \varepsilon < \varepsilon_0$.

By the Crandall-Ishii lemma [12], for all $\rho > 0$, there exists $(h, p + 2\alpha \tilde{x}, X) \in P^{2,+} u(\tilde{x}, \tilde{t})$, $(k, p - 2\alpha \tilde{y}, Y) \in P^{2,-} v(\tilde{y}, \tilde{t})$ such that

$$h - k = \frac{\lambda}{(T - \tilde{t})^2}, \quad p = \frac{4|\tilde{x} - \tilde{y}|^2 (\tilde{x} - \tilde{y})}{\varepsilon^4},$$

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq \begin{pmatrix} Z & -Z \\ -Z & Z \end{pmatrix} + \rho \begin{pmatrix} Z & -Z \\ -Z & Z \end{pmatrix}^2,$$

where we denote by $P^{\pm} u(x, t)$ the semijets of $u$ at $(x, t)$ (see [12,17] for the definition), and

$$Z := \frac{4}{\varepsilon^4} (|\tilde{x} - \tilde{y}|^2 I + 2(\tilde{x} - \tilde{y}) \otimes (\tilde{x} - \tilde{y})) + 2\alpha \begin{pmatrix} I \\ I \end{pmatrix}.$$ 

Here, fix $\varepsilon > 0$ small enough so that $\tilde{t} > 0$. We divide into two cases:

Case 1. $\liminf_{\alpha \to 0} |\tilde{x} - \tilde{y}| > 0$,

Case 2. $\liminf_{\alpha \to 0} |\tilde{x} - \tilde{y}| = 0$, i.e., $\exists \alpha_i \to 0$ such that $\lim_{\alpha_i \to 0} |\tilde{x} - \tilde{y}| = 0$.

Let us consider Case 1 first. Since $\Phi(x, x, \tilde{t}) \leq \Phi(\tilde{x}, \tilde{y}, \tilde{t})$ for all $x \in \mathbb{R}^n$, we have

$$u(x, \tilde{t}) - u(\tilde{x}, \tilde{t}) \leq v(x, \tilde{t}) - v(\tilde{y}, \tilde{t}) - \frac{|\tilde{x} - \tilde{y}|^4}{\varepsilon^4} + 2\alpha|x|^2 - \alpha(|\tilde{x}|^2 + |\tilde{y}|^2).$$

(3.7)

Since $K$ is bounded, we have

$$\liminf_{\alpha \to 0} \left( - \frac{|\tilde{x} - \tilde{y}|^4}{\varepsilon^4} + 2\alpha \max_{x \in K} |x|^2 - \alpha(|\tilde{x}|^2 + |\tilde{y}|^2) \right)$$

$$\leq \liminf_{\alpha \to 0} \left( - \frac{|\tilde{x} - \tilde{y}|^4}{\varepsilon^4} + 2\alpha \max_{x \in K} |x|^2 \right) < 0,$$

which implies that, for all $x \in K$ and $\alpha > 0$ small,

$$- \frac{|\tilde{x} - \tilde{y}|^4}{\varepsilon^4} + 2\alpha |x|^2 - \alpha(|\tilde{x}|^2 + |\tilde{y}|^2) < 0.$$ 

By (3.7), we thus have $u(x, \tilde{t}) - u(\tilde{x}, \tilde{t}) < v(x, \tilde{t}) - v(\tilde{y}, \tilde{t})$ for all $x \in K$, which implies

$$K \cap \{v(\cdot, \tilde{t}) \leq v(\tilde{y}, \tilde{t})\} \subset K \cap \{u(\cdot, \tilde{t}) < u(\tilde{x}, \tilde{t})\}. $$

(3.8)
Moreover, noticing that $|p|$ is bounded away from 0 uniformly in $\alpha$, we have
\begin{align*}
p_1 := p + 2\alpha \tilde{y} \neq 0, \quad p_2 := p - 2\alpha \tilde{x} \neq 0 \quad \text{for all } \alpha > 0 \text{ small.} \quad (3.9)
\end{align*}
Since $u$ and $v$, respectively, are a viscosity subsolution and supersolution to (1.1), we get
\begin{align*}
&h + F(u(\tilde{x}, \tilde{t}), p_1, X, K \cap \{u(\cdot, \tilde{t}) < u(\tilde{x}, \tilde{t})\}) \leq 0, \\
&k + F(v(\tilde{y}, \tilde{t}), p_2, Y, K \cap \{v(\cdot, \tilde{t}) \leq v(\tilde{y}, \tilde{t})\}) \geq 0.
\end{align*}
Since (3.8) and $v(\tilde{y}, \tilde{t}) < u(\tilde{x}, \tilde{t})$ hold, applying (F1) and (F4) to the second inequality above yields
\begin{align*}
k + F(u(\tilde{x}, \tilde{t}), p_2, Y, K \cap \{u(\cdot, \tilde{t}) < u(\tilde{x}, \tilde{t})\}) \geq 0.
\end{align*}
Using (F5), we then obtain
\begin{align*}
\frac{\lambda}{T^2} - \frac{\lambda}{(T - \tilde{t})^2} &= h - k \\
&\leq F(v(\tilde{y}, \tilde{t}), p_2, Y, K \cap \{u(\cdot, \tilde{t}) < u(\tilde{x}, \tilde{t})\}) - F(u(\tilde{x}, \tilde{t}), p_1, X, K \cap \{u(\cdot, \tilde{t}) < u(\tilde{x}, \tilde{t})\}) \\
&\leq \omega \left( \frac{|Z||p_1 - p_2|}{\min\{|p_1|, |p_2|\}} + |p_1 - p_2| + 2\alpha + O(\rho) \right) \\
&\leq \omega \left( 2\alpha|Z|(|\tilde{x}| + |\tilde{y}|) + 2\alpha(2|\tilde{x}| + |\tilde{y}| + 1) + O(\rho) \right).
\end{align*}
Note that
\begin{align*}
|Z| \leq c, \quad \min\{|p_1|, |p_2|\} \geq \frac{1}{c}
\end{align*}
for some $c > 0$ which is independent of $\alpha$. Sending $\rho \to 0$, $\alpha \to 0$ in this order and using (3.6) yield $\lambda/T^2 \leq 0$, which is a contradiction.

Let us turn to Case 2. In this case, fixing $\rho > 0$, we have
\begin{align*}
|p| \to 0, \quad X \to X^\varepsilon, \quad Y \to Y^\varepsilon, \quad \text{as } \alpha_i \to 0,
\end{align*}
and
\begin{align*}
\begin{pmatrix}
X^\varepsilon & 0 \\
0 & -Y^\varepsilon
\end{pmatrix} \leq 0,
\end{align*}
which implies $X^\varepsilon \leq 0 \leq Y^\varepsilon$. Let us consider the subsequence along $\alpha_i$. Let $p_1, p_2$ be as in (3.9) above. Then we can adopt the definition of subsolutions and supersolutions to get
\begin{align*}
0 \geq h + F_*(u(\tilde{x}, \tilde{t}), p_1, X, K \cap \{u(\cdot, \tilde{t}) \leq u(\tilde{x}, \tilde{t})\}) \geq h + F_*(u(\tilde{x}, \tilde{t}), p_1, X, \emptyset), \\
0 \leq k + F^*(v(\tilde{y}, \tilde{t}), p_2, Y, K \cap \{v(\cdot, \tilde{t}) < v(\tilde{y}, \tilde{t})\}) \leq k + F^*(v(\tilde{y}, \tilde{t}), p_2, Y, K).
\end{align*}
Thus, we are led to
\begin{align*}
\frac{\lambda}{T^2} \leq \frac{\lambda}{(T - \tilde{t})^2} = h - k \leq F^*(v(\tilde{y}, \tilde{t}), p_2, Y, K) - F_*(u(\tilde{x}, \tilde{t}), p_1, X, \emptyset).
\end{align*}
Sending $\alpha \to 0$ and applying (F1), we get

\[
\frac{\lambda}{T^2} \leq \limsup_{\alpha_i \to 0} \left( F^{*}(v(\tilde{y}, \tilde{t}), 0, Y, K) - F^{*}(u(\tilde{x}, \tilde{t}), 0, X, 0) \right) \\
\leq \limsup_{\alpha_i \to 0} \left( F^{*}(u(\tilde{x}, \tilde{t}), 0, 0, K) - F^{*}(u(\tilde{x}, \tilde{t}), 0, 0, \emptyset) \right)
\]

Using (F6) yields $\lambda/T^2 \leq 0$, which is a contradiction. \[\square\]

By using the stability result [35, (P2)], we can adapt the standard “bump-up” argument in Perron’s method (see [12,17] for instance) to (1.1). Assuming that $u_0$ is uniformly continuous, we can prove the existence of the unique viscosity solution. Comparing the solution with its spatial translations, we can further prove its uniform continuity in space as in (1.9); see similar arguments in the proofs of [18, Corollary 2.11] and [17, Theorem 3.5.1]. In addition, if there exists an appropriate subsolution as below, then by the comparison principle, we obtain the existence of a unique viscosity solution that satisfies the positivity and coercivity conditions (1.7) and (1.8).

(I) There exists a function $\phi \in C(R^n \times [0, \infty))$ such that

(i) $\phi(\cdot, t) \in UC(R^n)$ for any $t \geq 0$;
(ii) $u_0 \geq \phi(\cdot, 0)$ in $R^n$;
(iii) $\phi \geq c_0$ in $R^n \times [0, \infty)$ for some $c_0 > 0$;
(iv) $\phi$ is coercive in space, that is,

\[
\inf_{|x| \geq R, \ t \leq T} \phi(x, t) \to \infty \ \text{as} \ R \to \infty \ \text{for any} \ T \geq 0;
\]

(v) $\phi$ is a viscosity subsolution of (1.1).

The results described above can be summarized as follows.

**Theorem 3.3.** (Existence) Assume that (F1)–(F6) hold. Let $u_0 \in UC(R^n)$. Then there exists a unique solution $u$ of (1.1) and (1.2) that satisfies (1.9). Moreover, if the additional assumption (I) holds, then the unique solution $u$ also satisfies (1.7) and (1.8).

4. Quasiconvexity preserving

This section is devoted to proving our main result, Theorem 1.1. Fix arbitrarily $\lambda \in (0, 1)$. For $u \in C(R^n \times [0, \infty))$, let $u_{*, \lambda}$ be given by (1.13). Our goal is to show that

\[
u_{*, \lambda}(x, t) = u(x, t) \ \text{for all} \ (x, t) \in R^n \times [0, \infty).
\]

Theorem 1.1 follows immediately, since $u$ is quasiconvex in space if and only if (4.1) holds for all $\lambda \in (0, 1)$. By the definition of $u_{*, \lambda}$, it is clear that $u_{*, \lambda} \leq u$ in $R^n \times [0, \infty)$. It thus suffices to prove the reverse inequality. To this end, we first approximate $u_{*, \lambda}$ via the power convex envelope function $u_{q, \lambda}$ ($q > 1$) given by (1.14).
**Proposition 4.1.** (Approximation by power convex envelope) Let \( u \in C(\mathbb{R}^n \times [0, \infty)) \) satisfy (1.7) for some \( c_0 > 0 \). Fix \( \lambda \in (0, 1) \). Let \( u_{*, \lambda} \) and \( u_{q, \lambda} \) be given respectively by (1.13) and (1.14). Then \( u_{q, \lambda} \to u_{*, \lambda} \) locally uniformly in \( \mathbb{R}^n \times [0, \infty) \) as \( q \to \infty \).

**Proof.** By definition, it is clear that \( u_{q, \lambda} \leq u_{*, \lambda} \) in \( \mathbb{R}^n \times [0, \infty) \). For any \( j \in \mathbb{N} \), let \( (x, t) \) be given respectively by (1.13) and (1.14). Then

\[
x = \lambda y_j + (1 - \lambda)z_j, \quad (\lambda u(y_j, t)^q + (1 - \lambda)u(z_j, t)^q)^{\frac{1}{q}} \leq u_{q, \lambda}(x, t) + \frac{1}{j}.
\]

It follows from the positiveness of \( u \) that

\[
\max \{u(y_j, t), u(z_j, t)\} \leq \left( \frac{1}{\min\{\lambda, 1 - \lambda\}} \right)^{\frac{1}{q}} (u_{q, \lambda}(x, t) + 1) \tag{4.2}
\]

\[
\leq \left( \frac{1}{\min\{\lambda, 1 - \lambda\}} \right)^{\frac{1}{q}} (u_{*, \lambda}(x, t) + 1)
\]

Moreover, we have

\[
u_{*, \lambda}(x, t) - u_{q, \lambda}(x, t) \leq \max \{u(y_j, t), u(z_j, t)\} - (\lambda u(y_j, t)^q + (1 - \lambda)u(z_j, t)^q)^{\frac{1}{q}} + \frac{1}{j}.
\]

This yields

\[
u_{*, \lambda}(x, t) - u_{q, \lambda}(x, t) \leq \frac{1}{q}
\]

\[
\max \{u(y_j, t), u(z_j, t)\} \max\{\lambda^{\frac{1}{q} - 1}(1 - \lambda), (1 - \lambda)^{\frac{1}{q} - 1}\}\leq (1 - r^q) + \frac{1}{j}, \tag{4.3}
\]

where

\[
r = \frac{\min \{u(y_j, t), u(z_j, t)\}}{\max \{u(y_j, t), u(z_j, t)\}} \in (0, 1].
\]

Indeed, for any \( 0 < a \leq b \), we have

\[
b - (\lambda a^q + (1 - \lambda)b^q)^{\frac{1}{q}} = b(g(1) - g(r^q)),
\]

where \( g(s) = (\lambda s + 1 - \lambda)^{\frac{1}{q}} \) and \( r = a/b \in (0, 1] \). Since \( g \) is concave, we get

\[
g(1) - g(r^q) \leq g'(r^q)(1 - r^q) = \frac{1}{q}\lambda(\lambda r^q + 1 - \lambda)^{\frac{1}{q} - 1}(1 - r^q)
\]

\[
\leq \frac{1}{q}\lambda(1 - \lambda)^{\frac{1}{q} - 1}(1 - r^q).
\]

This implies that

\[
b - (\lambda a^q + (1 - \lambda)b^q)^{\frac{1}{q}} \leq \frac{1}{q}b\lambda(1 - \lambda)^{\frac{1}{q} - 1}(1 - r^q).
\]

If \( a \geq b > 0 \), then we can similarly obtain

\[
a - (\lambda a^q + (1 - \lambda)b^q)^{\frac{1}{q}} \leq \frac{1}{q}a(1 - \lambda)^{\frac{1}{q} - 1}(1 - r^q)
\]
with \( r = b/a \). In general, for any \( a, b > 0 \), we have
\[
\max\{a, b\} - (\lambda a^q + (1 - \lambda)b^q) \frac{1}{q} \\
\leq \frac{1}{q} \max\{a, b\} \max\{\lambda^{\frac{1}{q} - 1}(1 - \lambda), (1 - \lambda)^{\frac{1}{q} - 1}\lambda\}(1 - r^q)
\]
with \( r = \min\{a, b\}/\max\{a, b\} \), which leads us to (4.3).

The relation (4.3), together with (4.2), immediately yields
\[
0 \leq u_{\ast, \lambda}(x, t) - u_{q, \lambda}(x, t) \leq \frac{C}{q} (u_{\ast, \lambda}(x, t) + 1) + \frac{1}{j}
\]
for some \( C > 0 \) independent of \((x, t)\) and \( j \geq 1 \). Letting \( j \to \infty \), we thus deduce the uniform convergence of \( u_{q, \lambda} \) to \( u_{\ast, \lambda} \) in any compact subset of \( \mathbb{R}^n \times [0, \infty) \) as \( q \to \infty \).

\[ \Box \]

**Remark 4.2.** Under the coercivity condition on \( u \), the infimum in (1.14) can be obtained and the points \( y_j, z_j \) in the proof above can be taken as minimizers. The convergence result certainly still holds.

We show a key ingredient to prove (4.1), which stems from the idea in [2] to prove convexity of solutions to fully nonlinear equations by using its convex envelope. Such an idea is later developed in [21] to show a power-type convexity or concavity with a finite exponent. We here makes a further step, studying the limit case as the exponent tends to \( \infty \).

**Lemma 4.3.** Assume that (F1)–(F7) hold. Let \( u \in C(\mathbb{R}^n \times [0, \infty)) \) be a supersolution of (1.1) satisfying (1.7) and (1.8). Let \( \lambda \in (0, 1) \) and \( u_{\ast, \lambda} \) be the function defined by (1.13). Then \( u_{\ast, \lambda} \) is a supersolution of (1.1).

**Proof.** For simplicity of notation, we write \( w_\ast = u_{\ast, \lambda} \) and \( w_q = u_{q, \lambda} \). Let us first show that \( w_\ast \in \text{LSC}(\mathbb{R}^n \times [0, \infty)) \). For an arbitrary \((x_0, t_0) \in \mathbb{R}^n \times [0, \infty)\), let \((x_j, t_j)\) be a sequence satisfying
\[
(x_j, t_j) \to (x_0, t_0), \quad w_\ast(x_j, t_j) \to \liminf_{(x, t) \to (x_0, t_0)} w_\ast(x, t) \quad \text{as} \quad j \to \infty.
\]
Due to the coercivity (1.8), there exist \( y_j, z_j \in \mathbb{R}^n \) such that
\[
x_j = \lambda y_j + (1 - \lambda)z_j, \quad w_\ast(x_j, t_j) = \min\{u(y_j, t_j), u(z_j, t_j)\}.
\]
Since \( x_j \) is a bounded sequence, if either of the sequences \( y_j, z_j \) is unbounded, so does the other. Thus we can choose a subsequence such that \(|y_j|, |z_j| \to \infty\), for which by (1.8) again we have
\[
u(y_j, t_j) \to \infty, \quad u(z_j, t_j) \to \infty \quad \text{as} \quad j \to \infty.
\]
It follows from (4.4) that
\[
w_\ast(x_j, t_j) \to \infty \quad \text{as} \quad j \to \infty,
\]
which is a contradiction to the fact that \( w_\ast \leq u \) in \( \mathbb{R}^n \times [0, \infty) \). Therefore, it is sufficient to assume that \( y_j \) and \( z_j \) are bounded sequences. Choosing subsequences \( y_j \) and \( z_j \) converging to \( y_0 \) and \( z_0 \) in \( \mathbb{R}^n \) respectively, we can take the limit of (4.4) to obtain
\[
\liminf_{(x, t) \to (x_0, t_0)} w_\ast(x, t) = \min\{u(y_0, t_0), u(z_0, t_0)\} \geq w_\ast(x_0, t_0).
\]
Hence, \( w_* \in \text{LSC}(\mathbb{R}^n \times [0, \infty)) \). The lower semicontinuity of \( w_q \) can be proved similarly.

Let us next proceed to show that \( w_* \) satisfies the supersolution property. Suppose that there exist \((x_0, t_0) \in \mathbb{R}^n \times (0, \infty) \) and \( \varphi \in C^2(\mathbb{R}^n \times (0, \infty)) \) such that \( w_* - \varphi \) attains a strict minimum at \((x_0, t_0) \). Without loss of generality we may assume \( \varphi > 0 \) in \( \mathbb{R}^n \times (0, \infty) \).

In light of Proposition 4.1, there exists a sequence, indexed by \( q \), of \((x_q, t_q) \in \mathbb{R}^n \times (0, \infty) \) such that \( w_q - \varphi \) attains a strict minimum at \((x_q, t_q) \) and
\[
(x_q, t_q) \to (x_0, t_0), \quad w_q(x_q, t_q) \to w_*(x_0, t_0) \quad \text{as} \quad q \to \infty.
\]
Due to (1.8), there exist \( y_q, z_q \in \mathbb{R}^n \) such that
\[
x_q = \lambda y_q + (1 - \lambda)z_q, \quad w_q(x_q, t_q) = (\lambda u(y_q, t_q)^q + (1 - \lambda)u(z_q, t_q)^q)^{\frac{1}{q}}.
\]
(4.5)

Shifting \( \varphi \) so that \( \varphi(x_q, t_q) = w_q(x_q, t_q) \) and letting \( v := u^q \) and \( \psi := \varphi^q \), we see that
\[
(y, z, t) \mapsto \lambda v(y, t) + (1 - \lambda)v(z, t) - \psi(y) + (1 - \lambda)z(t)
\]
takes a minimum at \((y_q, z_q, t_q) \in \mathbb{R}^n \times \mathbb{R}^n \times (0, \infty) \).

By the Crandall-Ishii lemma \([12]\), for any \( \varepsilon > 0 \) small we have \((h_q, \eta_q, Y_q) \in \mathcal{P}^2(v(y_q, t_q)) \) and \((k_q, \zeta_q, Z_q) \in \mathcal{P}^2(v(z_q, t_q)) \) such that
\[
\lambda h_q + (1 - \lambda)k_q = \psi_t(x_q, t_q), \quad \eta_q = \zeta_q = \nabla \psi(x_q, t_q),
\]
(4.6)

and
\[
\begin{pmatrix} \lambda Y_q & 0 \\ 0 & (1 - \lambda)Z_q \end{pmatrix} \geq \begin{pmatrix} \lambda^2 X_q & \lambda(1 - \lambda)X_q \\ \lambda(1 - \lambda)X_q & (1 - \lambda)^2 X_q \end{pmatrix} - \varepsilon \begin{pmatrix} \lambda^2 X_q & \lambda(1 - \lambda)X_q \\ \lambda(1 - \lambda)X_q & (1 - \lambda)^2 X_q \end{pmatrix}^2,
\]
(4.7)

where \( X_q = \nabla^2 \psi(x_q, t_q) \). It follows that
\[
\lambda \xi^T Y_q \xi + (1 - \lambda)\xi^T Z_q \xi \geq \xi^T X_q \xi - C \varepsilon |\xi|^2 \quad \text{for all} \quad \xi \in \mathbb{R}^n,
\]
where \( C > 0 \) depends on the uniform bound of \(|\nabla^2 \psi(x_q, t_q)|\). In other words, we get
\[
\lambda Y_q + (1 - \lambda)Z_q \geq X_q - C \varepsilon I.
\]
(4.8)

Since \( u \) is a supersolution of (1.1), it is not difficult to show that \( v \) is a supersolution of
\[
v_t + G_\beta(v, \nabla v, \nabla^2 v, K \cap \{v(-, t) \leq v(x, t)\}) = 0
\]
with \( \beta = 1 - \frac{1}{q} \), where \( G_\beta \) is the transformed operator given by (1.10). Adopting the definition of supersolutions at \((y_q, t_q) \) and \((z_q, t_q) \), we have
\[
h_q + G_\beta(a_q, \eta_q, Y_q, K \cap \{v(-, t_q) \leq v(y_q, t_q)\}) \geq 0,
\]
\[
k_q + G_\beta(b_q, \zeta_q, Z_q, K \cap \{v(-, t_q) \leq v(z_q, t_q)\}) \geq 0,
\]
(4.9)

where \( a_q = v(y_q, t_q) \), \( b_q = v(z_q, t_q) \).
Applying Proposition 2.1, we can divide our argument into the following two cases:

- Case 1. $\nabla \varphi(x_0, t_0) \neq 0$,
- Case 2. $\nabla \varphi(x_0, t_0) = 0$, and $\nabla^2 \varphi(x_0, t_0) = 0$.

We write $\xi_q = \nabla \psi(x_q, t_q)$ for simplicity of notation.

In Case 1, by (4.6), we have $\nabla \psi(x_q, t_q) = q \hat{\eta}_q = \zeta_q \neq 0$ when $q > 1$ is sufficiently large. Multiplying the first inequality in (4.9) by $\lambda$ and the second by $1 - \lambda$ and then adding them up, by (4.6) we are led to

$$
\psi_t(x_q, t_q) + \lambda \beta(b_q, \xi_q, Z_q, W_q[x_0, t_0]) + (1 - \lambda) G_{\beta}(b_q, \xi_q, Z_q, W_q[x_0, t_0]) - G_{\beta}(a_q, \xi_q, Y_q, U[y_q, t_q])
$$

where we denote, for any $(s, t) \in \mathbb{R}^n \times (0, \infty),

$$
W_q[x, t] := K \cap \{w_q(\cdot, t) \leq w_q(x, t)\},
$$

$$
U[x, t] := K \cap \{u(\cdot, t) \leq u(x, t)\} = K \cap \{v(\cdot, t) \leq v(x, t)\}.
$$

Applying (F1), (F7) and (4.8) and noticing that

$$
\lambda a_q + (1 - \lambda)b_q = w_q(x_q, t_q)^q,
$$

we then get

$$
\psi_t(x_q, t_q) + G_{\beta}(w_q(x_q, t_q)^q, \nabla \psi(x_q, t_q), \nabla^2 \psi(x_q, t_q) - C \varepsilon I, W_q[x_0, t_0])
$$

$$
\geq \lambda (G_{\beta}(a_q, \xi_q, Y_q, W_q[x_0, t_0]) - G_{\beta}(a_q, \xi_q, Y_q, U[y_q, t_q]))
$$

$$
+ (1 - \lambda) (G_{\beta}(b_q, \xi_q, Z_q, W_q[x_0, t_0]) - G_{\beta}(b_q, \xi_q, Z_q, U[z_q, t_q])).
$$

Rewriting this relation in terms of the operator $F$, we are led to

$$
\varphi_t(x_q, t_q) + F(w_q(x_q, t_q)), \nabla \varphi(x_q, t_q), \nabla^2 \varphi(x_q, t_q) - C \varepsilon I, W_q[x_0, t_0])
$$

$$
\geq \lambda u(y_q, t_q)\varphi(x_q, t_q)^{q-1} D_{1,q} + (1 - \lambda) u(z_q, t_q)\varphi(x_q, t_q)^{q-1} D_{2,q},
$$

where

$$
D_{1,q} = F(u(y_q, t_q), \nabla \varphi(x_q, t_q), Y_q', W_q[x_0, t_0]) - F(u(y_q, t_q), \nabla \varphi(x_q, t_q), Y_q', U[y_q, t_q]),
$$

$$
D_{2,q} = F(u(z_q, t_q), \nabla \varphi(x_q, t_q), Z_q', W_q[x_0, t_0]) - F(u(z_q, t_q), \nabla \varphi(x_q, t_q), Z_q', U[z_q, t_q])
$$

with

$$
Y_q' = \frac{1}{q} \varphi(x_q, t_q)^{1-q} Y_q - \frac{q - 1}{q^2} \varphi(x_q, t_q)^{1-2q} \nabla \varphi(x_q, t_q) \otimes \nabla \varphi(x_q, t_q),
$$

$$
Z_q' = \frac{1}{q} \varphi(x_q, t_q)^{1-q} Z_q - \frac{q - 1}{q^2} \varphi(x_q, t_q)^{1-2q} \nabla \varphi(x_q, t_q) \otimes \nabla \varphi(x_q, t_q).
$$
Let us proceed to estimate $D_{1,q}, D_{2,q}$ in (4.10). Note that by (4.5)

$$
\lim_{q \to \infty} \sup u(y_q, t_q) \leq \limsup_{q \to \infty} w_q(x_q, t_q) = w_*(x_0, t_0).
$$

Also, it is easily seen that

$$
w_*(\cdot, t_0) = u_*, \lambda(\cdot, t_0) \leq u(\cdot, t_0) \leq \liminf_{q \to \infty} u(\cdot, t_q) \quad \text{in } \mathbb{R}^n.
$$

Thus, in light of Lemma 2.2 we have

$$
m(U[y_q, t_q] \setminus W_*[x_0, t_0]) \to 0 \quad \text{as } q \to \infty.
$$

Since (F4) implies

$$
F(u(y_q, t_q), \nabla \varphi(x_q, t_q), Y'_q, W_*[x_0, t_0]) \\
\geq F(u(y_q, t_q), \nabla \varphi(x_q, t_q), Y'_q, W_*[x_0, t_0] \cap U[y_q, t_q]),
$$

in view of (1.6) in (F3), we deduce that

$$
D_{1,q} \geq -\omega_R \left( m(U[y_q, t_q] \setminus W_*[x_0, t_0]) \right)
$$

for $q > 1$ sufficiently large, where $R = |\nabla \varphi(x_0, t_0)| + 1$. Similarly, we have

$$
D_{2,q} \geq -\omega_R \left( m(U[z_q, t_q] \setminus W_*[x_0, t_0]) \right)
$$

with

$$
m(U[z_q, t_q] \setminus W_*[x_0, t_0]) \to 0 \quad \text{as } q \to \infty.
$$

Hence, thanks to (4.11), sending $\varepsilon \to 0$ and then $q \to \infty$ in (4.10), we get

$$
\varphi_t(x_0, t_0) + F(w_*(x_0, t_0), \nabla \varphi(x_0, t_0), \nabla^2 \varphi(x_0, t_0), K \cap \{u_*, \lambda(\cdot, t_0) \leq u_*, \lambda(x_0, t_0)\}) \geq 0.
$$

The proof in Case 1 is complete. Let us next discuss Case 2. If $\nabla \varphi(x_q, t_q) \neq 0$ along a subsequence, then passing to the limit of (4.10) as $\varepsilon \to 0$ and $q \to \infty$ via the subsequence, by (F6) we get

$$
\varphi_t(x_0, t_0) + \mu(w_*(x_0, t_0)) \geq 0,
$$

as desired.

It remains to consider the case when $\nabla \varphi(x_q, t_q) = 0$ for all $q > 1$ large. By (4.7), we deduce that

$$
Y_q \geq \lambda(1 - \varepsilon)X_q, \quad Z_q \geq (1 - \lambda)(1 - \varepsilon)X_q.
$$
where $X_q = \nabla^2 \varphi(x_q, t_q) \to 0$ as $q \to \infty$. As in Case 1, we take $\beta = 1 - \frac{1}{q}$ with $q > 0$ large. Adopting the definition of supersolutions in this case, we have
\[
\begin{align*}
&h_q + G^*_\beta(v(y_q, t_q), 0, Y_q, K \cap \{v(\cdot, t_q) \leq v(y_q, t_q)\}) \geq 0, \\
k_q + G^*_\beta(v(z_q, t_q), 0, Z_q, K \cap \{v(\cdot, t_q) \leq v(z_q, t_q)\}) \geq 0,
\end{align*}
\]
which by (F4) yields
\[
\begin{align*}
&h_q + G^*_\beta(v(y_q, t_q), 0, Y_q, K) \geq 0, \\
k_q + G^*_\beta(v(z_q, t_q), 0, Z_q, K) \geq 0.
\end{align*}
\]
In view of (4.13) and (F6), we have
\[
\begin{align*}
&(1 - \beta)h_q + v(y_q, t_q)^{\beta} \mu(v(y_q, t_q)^{1-\beta}) \geq -v(y_q, t_q)^{\beta} \omega(|X_q|), \\
&(1 - \beta)k_q + v(z_q, t_q)^{\beta} \mu(v(z_q, t_q)^{1-\beta}) \geq -v(z_q, t_q)^{\beta} \omega(|X_q|).
\end{align*}
\]
It follows from (F7) that
\[
\frac{1}{q}(\lambda h_q + (1 - \lambda)k_q) + w_q(x_q, t_q)^{q-1} \mu(w_q(x_q, t_q)) \geq -\lambda v(y_q, t_q)^{1-\frac{1}{q}} \omega(|X_q|) - (1 - \lambda)v(z_q, t_q)^{1-\frac{1}{q}} \omega(|X_q|),
\]
which, together with the relation $w_q(x_q, t_q) = \varphi(x_q, t_q)$, yields,
\[
\frac{1}{q}\varphi(x_q, t_q)^{1-q}(\lambda h_q + (1 - \lambda)k_q) + \mu(w_q(x_q, t_q)) \geq -w_q(x_q, t_q)^{1-q} \left(\lambda u(y_q, t_q)^{q-1} + (1 - \lambda)u(z_q, t_q)^{q-1}\right) \omega(|X_q|).
\]
Noticing that, due to (4.6),
\[
\varphi_t(x_q, t_q) = \frac{1}{q}\varphi(x_q, t_q)^{1-q}\psi_t(x_q, t_q) = \frac{1}{q}\varphi(x_q, t_q)^{1-q}(\lambda h_q + (1 - \lambda)k_q),
\]
we obtain
\[
\varphi_t(x_q, t_q) + \mu(w_q(x_q, t_q)) \geq -\frac{\lambda u(y_q, t_q)^{q-1} + (1 - \lambda)u(z_q, t_q)^{q-1}}{w_q(x_q, t_q)^{q-1}} \omega(|X_q|).
\]
(4.14)
In view of (4.11), letting $q \to \infty$ in (4.14), we again end up with (4.12). \(\square\)

**Proof.** (Proof of Theorem 1.1) By the quasiconvexity of $u_0$, we have $u_{*,\lambda}(\cdot, t) = u_0$ in $\mathbb{R}^n$. Moreover, since $c_0 \leq u_{*,\lambda} \leq u$ holds in $\mathbb{R}^n \times [0, \infty)$, $u_{*,\lambda}$ obviously satisfies the growth condition (3.1). The relation (4.1) is then an immediate consequence of Lemma 4.3 and the comparison principle, Theorem 3.1. Noticing that (4.1) implies the quasiconvexity of $u$ in space, we complete the proof of Theorem 1.1, \(\square\)

**Remark 4.4.** We can weaken the assumption (F7) if the solution $u$ satisfies additional regularity assumptions. If $x \mapsto u(x, t)$ is assumed to be $L$-Lipschitz uniformly for all $t \geq 0$, we can show Lemma 4.3 and thus Theorem 1.1 under the following relaxed version of (1.11) in (F7): for any $\beta < 1$ close to 1, $(r, X) \to G_\beta(r, p, X, A)$ is concave in $[c_0, \infty) \times S^n$ for any $p \in \mathbb{R}^n \setminus \{0\}$ with $|p| \leq L$ and $A \in B_K$. 

Although our method relies on the power convex envelope $w_g$, in general one cannot expect the preservation of spatial power convexity to hold for any finite exponent $q$, as indicated by the simple example below.

**Example 4.1.** Consider the first order equation in two space dimensions:

\[
  u_t + \left| \nabla u \right| m(K \cap \{ u(\cdot, t) < u(x, t) \}) = 0 \quad \text{in } \mathbb{R}^2 \times (0, \infty),
\]

where $K \subset \mathbb{R}^2$ is taken to be the closed ball $B_R(0)$ centered at 0 with radius $R > 0$. Let the initial value be

\[
  u_0(x) = |x| + 1, \quad x \in \mathbb{R}^2.
\]

Since $u_0$ is radially symmetric and attains a minimum at $x = 0$, we can consider radially symmetric solutions $u(x, t) = \varphi(|x|, t)$ (see [19]), where $\varphi$ satisfies

\[
  \varphi_t + \left| \varphi_t \right| \min \{ R^2, r^2 \} = 0 \quad \text{for } r > 0, t > 0,
\]

\[
  \varphi(0, t) = 1 \quad \text{for } t > 0,
\]

\[
  \varphi(r, 0) = r + 1 \quad \text{for } r \geq 0.
\]

Using the optimal control formula (see [3,37] for instance), we can express the viscosity solution to (4.17) as

\[
  \varphi(r, t) = \inf \{ \varphi(\gamma(t), 0) \mid \gamma(0) = r, |\gamma(s)| \leq \pi \min \{ R^2, \gamma(s)^2 \} \text{ for all } s \in [0, t] \}.
\]

The optimal control $\gamma^*$ in this setting is rather simple. Indeed, for any given $r \geq 0$ and $t \geq 0$, if $r \leq R$ then $\gamma^*$ solves

\[
  \begin{cases}
    \dot{\gamma}^*(s) = -\pi \gamma^*(s)^2 & \text{for } s > 0, \\
    \gamma^*(0) = r,
  \end{cases}
\]

and thus $\gamma^*(t) = r/(1 + \pi rt)$. If $r > R + \pi R^2t$, then $\gamma^*$ is the solution to

\[
  \begin{cases}
    \dot{\gamma}^*(s) = -\pi R^2 & \text{for } s > 0, \\
    \gamma^*(0) = r,
  \end{cases}
\]

which is $\gamma^*(t) = r - \pi R^2t$. Finally, if $R < r < R + \pi R^2t$, then letting $\tilde{t} := (r - R)/\pi R^2$, we see that $\gamma^*$ satisfies

\[
  \begin{cases}
    \dot{\gamma}^*(s) = -\pi R^2 & \text{for } 0 < s < \tilde{t}, \\
    \gamma^*(0) = r, \quad \text{and} \quad \dot{\gamma}^*(s) = -\pi \gamma^*(s)^2 & \text{for } \tilde{t} < s < t,
  \end{cases}
\]

By elementary computations, we obtain

\[
  \gamma^*(t) = \frac{R^2}{\pi R^2t - r + 2R}.
\]

Hence, the unique viscosity solution of (4.15) can be written as

\[
  u(x, t) = \begin{cases}
    \frac{|x|}{1 + \pi |x|t} + 1 & \text{if } |x| \leq R, \\
    \frac{R^2}{\pi R^2t - |x| + 2R} + 1 & \text{if } R < |x| \leq R + \pi R^2t, \\
    |x| - \pi R^2t + 1 & \text{if } |x| > R + \pi R^2t.
  \end{cases}
\]

It is not difficult to see that $x \mapsto u(x, t)$ is not convex (for $|x| < R$) for any $t > 0$ in spite of the convexity of $u_0$.

Moreover, since the operator $\tilde{F}$ in this case is geometric, satisfying $\tilde{F}(cp, A) = cF(p, A)$ for all $c \leq 0$, we can follow [17, Theorem 4.2.1] to prove that $g \circ u$ is...
still a solution of (4.15) for any nondecreasing continuous function \( g : \mathbb{R} \to \mathbb{R} \). In particular, we deduce that \( u^{1/2} \) (with \( u \) given by (4.18)) is the unique solution of (4.15) with initial value \( u_0^{1/2} \) for any \( q \geq 1 \). This means that in general solutions of (4.15) may fail to preserve \( q \)-convexity for every \( q \geq 1 \).

In the example above, we obtained the solution of (4.15) and (4.16) by means of a control-based interpretation. It is not straightforward at all to find a representation formula for solutions of more general problems. If we slightly change the equation and consider

\[
    u_t + \left\{ m(K \cap \{ u(\cdot, t) < u(x, t) \}) - 1 \right\} |\nabla u| = 0 \quad \text{in} \quad \mathbb{R}^n \times (0, \infty),
\]

then the Hamiltonian, depending in a nonlocal manner on \( u \), is neither coercive nor convex in general. The optimal control interpretation in this case becomes rather complicated. We study this problem in our upcoming work [29].

5. Applications

To conclude our paper, let us provide several applications of our quasiconvexity preserving result. The parabolic operators in the examples below satisfy all of the assumptions in Theorem 1.1 including the assumption (F7). We also verify the assumption (I) for our existence result, Theorem 3.3.

5.1. Nonlocal geometric flows

Let us study the anisotropic surface evolution given by (4.17). For this type of evolution, \( \xi \) is the Cahn-Hoffman vector associated to an (extended) surface energy density \( \gamma \in C(\mathbb{R}^n) \cap C^2(\mathbb{R}^n \setminus \{0\}) \), which is positively homogeneous of degree 1 in \( \mathbb{R}^n \), and satisfies \( \gamma > 0 \) and \( \nabla \gamma = \xi \) in \( \mathbb{R}^n \setminus \{0\} \). In this case, the level set formulation leads us to the following PDE:

\[
    u_t + a|\nabla u| + b|\nabla u|m(K \cap \{ u(\cdot, t) < u(x, t) \}) - c|\nabla u|\tr(\nabla^2\gamma(-\nabla u)\nabla^2u) = 0
\]

(5.1)

in \( \mathbb{R}^n \times (0, \infty) \), where \( a \in \mathbb{R} \), \( b \geq 0 \) and \( c \geq 0 \) are given constants. The operator \( F \) can be expressed as

\[
    F(p, X, A) = a|p| + b|p|m(A) - |p|\tr(\nabla^2\gamma(-p)X).
\]

We can show that \( F \) satisfies the condition (F7). In fact, we can verify this property for a more general class of geometric operators \( F \) and thus recover the results on convexity preserving property in [8]. Suppose that (1.4) holds. It is clear that the transformed operator

\[
    G_{\beta}(r, p, X, A) = F(p, X, A)
\]

for all \( 0 < \beta < 1 \). Then (1.11) holds provided that \( X \mapsto F(p, X, A) \) is concave, i.e.,

\[
    \lambda F(p, X, A) + (1 - \lambda)F(p, Y, A) \leq F(p, \lambda X + (1 - \lambda)Y, A)
\]

for all \( \lambda \in (0, 1) \), \( p \in \mathbb{R}^n \setminus \{0\} \), \( X, Y \in \mathbb{S}^n \) and \( A \in \mathcal{B}_K \). In addition, since \( \mu = 0 \) in this case, we have (1.12) as well. Hence, (F7) is verified.
Furthermore, for the anisotropic surface evolution equation (5.1) with \( \gamma \) convex, we can construct a function \( \phi \) satisfying (I) provided that \( u_0 \) is uniformly continuous, positive and coercive in \( \mathbb{R}^n \). Since (5.1) is geometric, without loss of generality we may assume that \( u_0 \geq c_0 \) for some \( c_0 > 0 \). We can choose an increasing and coercive function \( \sigma \in C^\infty([0, \infty)) \cap UC([0, \infty)) \) such that

\[
\inf_{\gamma(x) \geq R} u_0(x) \geq \sigma(R) \geq c_0 \quad \text{for any } R \in [0, \infty).
\]

Take

\[
\phi(x, t) := \sigma(\max\{\gamma(x) - Ct, 0\}),
\]

where

\[
C := \left( \sup_{p \in \mathbb{R}^n \setminus \{0\}} |\nabla \gamma(x)| \right) \{|a| + bm(K)\}.
\]

Note that the supremum of \( |\nabla \gamma(x)| \) can be attained since \( \nabla \gamma(x) \) is positively homogeneous of degree 0. Since any level set \( \{x \in \mathbb{R}^n : \gamma(x) = h\} \) is a scaled Wulff shape and its anisotropic mean curvature \( \text{tr}\left( \nabla^2 \gamma(-\nabla \gamma(x)) \nabla^2 \gamma(x) \right) \geq 0 \) if \( \gamma \) is convex, by (1.4) we have

\[
\text{tr}\left( \nabla^2 \gamma(-\nabla \phi(x, t)) \nabla^2 \phi(x, t) \right) = \sigma'(\gamma(x) - Ct) \text{tr}\left( \nabla^2 \gamma(-\nabla \gamma(x)) \nabla^2 \gamma(x) \right) \geq 0
\]

if \( \gamma(x) > Ct \). Therefore, we have

\[
\phi_t(x, t) + F(\nabla \phi(x, t), \nabla^2 \phi(x, t), K \cap \{\phi(\cdot, t) < \phi(x, t)\})
\]

\[
\leq \sigma'(\gamma(x) - Ct) \left\{ -C + |\nabla \gamma(x)| \left( -a + bm(K) \right) \right\} \leq 0,
\]

which ensures (v) in (I). The other conditions in (I) hold obviously due to the choice of \( \sigma \) and the definition of \( \phi \).

### 5.2. Nonlocal evolution equations depending on \( u \)

We now consider a slightly different type of nonlocal curvature flows, which resembles those local ones arising in the study of crystal growth [38]. A typical equation reads

\[
u_t + V(u) + |\nabla u|m(K \cap \{u(\cdot, t) < u(x, t)\}) - |\nabla u| \text{div} \left( \frac{\nabla u}{|\nabla u|} \right) = 0 \quad (5.4)
\]

in \( \mathbb{R}^n \times (0, \infty) \), where \( V \in C^2(\mathbb{R}) \) is a given function. In contrast to the example in Sect. 5.1, this equation in general does not satisfy the geometricity condition (1.4). The operator \( F \) is

\[
F(r, p, X, A) = V(r) + |p|m(A) - \text{tr} \left( \left( I - \frac{p \otimes p}{|p|^2} \right) X \right).
\]

We see that \( F \) satisfies (F7) under appropriate assumptions on \( V \). Indeed, for \( p \neq 0 \), the operator \( G_\beta \) as in (1.10) is

\[
G_\beta(r, p, X, A) = \frac{1}{1 - \beta} r^\beta V(r^{1-\beta}) + |p|m(A) - \text{tr} \left( \left( I - \frac{p \otimes p}{|p|^2} \right) X \right).
\]
It is clear that, for $0 < \beta < 1$ close to 1, the condition (1.11) holds in this case if $r \mapsto r^\beta V(r^{1-\beta})$ is concave in $(0, \infty)$. By direct computations, we get

$$(r^\beta V(r^{1-\beta}))'' = \beta(1-\beta)r^{-1}(V'(r^{1-\beta}) - r^{\beta-1}V(r^{1-\beta}) - (1-\beta)^2 r^{-\beta}V''(r^{1-\beta}).$$

Thus, a sufficient condition to guarantee the concavity of $r \mapsto r^\beta V(r^{1-\beta})$ is that $V'' \leq 0$ in $(0, \infty)$ and $V(0) \geq 0$. In fact, these conditions yield

$$V'(r^{1-\beta}) - \frac{1}{r^{1-\beta}}V(r^{1-\beta}) = - \frac{1}{r^{1-\beta}}\left(\int_0^{r^{1-\beta}} sV''(s) \, ds - V(0)\right) \leq 0$$

and therefore $(r^\beta V(r^{1-\beta}))'' \leq 0$ for $r > 0$. In order for (F1)(F2) to hold, we also need to assume that $V$ is nondecreasing and bounded.

In this case, we can construct $\phi$ satisfying (I) in a similar way as in (5.3) if $u_0 \in UC(\mathbb{R}^n)$ satisfies

$$\lim_{R \to \infty} \inf_{|x| \geq R} \frac{u_0(x)}{|x|} > 0, \quad \inf_{x \in \mathbb{R}^n} u_0(x) \geq c_0$$

for some $c_0 > 0$. We may choose an increasing function $\sigma \in C^\infty([0, \infty)) \cap UC([0, \infty))$ such that (5.2) holds and $\sigma'(s) > 0$ when $s > 0$ is large. Note that the second assumption in (5.5) is needed because of the $u$-dependent term $V(u)$ in the equation (5.4). Using this function $\sigma$, the subsolution $\phi$ can be constructed as in (5.3) by replacing $\gamma$ and $C$ respectively by

$$\gamma(x) := |x|, \quad C := \left(\sup_{r \in \mathbb{R}} V(r)\right) / \left(\inf_{R \in [0, \infty)} \sigma'(R)\right) + m(K).$$

We finally remark that it is possible to extend the application of our results to a more general class of non-geometric equations such as

$$u_t + V(u) + a|\nabla u|m(K \cap \{u(\cdot, t) < u(x, t)\}) - b|\nabla u|\text{div} \left(\frac{\nabla u}{|\nabla u|}\right) + c|\nabla u|^\delta = 0$$

with $a, b, c \geq 0$ and $0 < \delta < 1$ given. We omit the detailed verification of the assumptions here.

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