Dirac’s and Generalized Faddeev-Jackiw brackets for Einstein’s theory in the $G \to 0$ limit

Alberto Escalante

Instituto de Física Luis Rivera Terrazas, Benemérita Universidad Autónoma de Puebla, (IFUAP). Apartado postal J-48 72570 Puebla, Pue., México and
LUTH, Observatoire de Paris, Meudon, France

Omar Rodríguez Tzompantzi

Facultad de Ciencias Físico-Matemáticas, Benemérita Universidad Autónoma de Puebla, Apartado postal 1152, 72001 Puebla, Pue., México

(Dated: October 28, 2015)

In this paper the Dirac and Faddeev-Jackiw formulation for Einstein’s theory in the $G \to 0$ limit is performed; the fundamental Dirac’s and Faddeev-Jackiw brackets for the theory are obtained. First, the Dirac brackets are constructed by eliminating the second class constraints remaining the first class ones, then we fix the gauge and we convert the first class constraints into second class constraints and the new fundamental Dirac’s brackets are computed. Alternatively, we reproduce all relevant Dirac’s results by means of the symplectic method. We identify the Faddeev-Jackiw constraints and we prove that the Dirac and the Faddeev-Jackiw brackets coincide to each other.

PACS numbers:

I. INTRODUCTION

The study of gauge theories by means of the canonical analysis developed by Dirac is an important step that should be performed. In fact, it is well-known that the correct identification of the constraints and the symplectic structure of a singular system is the best guideline for performing its quantization [1,2]. In this respect, Dirac’s formalism is an elegant framework for studying singular systems, which allow us to know the constraints, the fundamental Poisson brackets, the extended Hamiltonian and the Dirac brackets [3,4]. However, in some cases, to develop the Dirac method is large and tedious task and some times if all the Dirac steps are not applied correctly or some of them are omitted [5,6], then the obtained results could be incorrect [7–9]. Hence, because of these complications, it is necessary to use alternative formulations that could give us a complete description of the theory, in this sense, there is a different approach for studying gauge theories, the so-called the Faddeev-Jackiw [FJ] formalism [10]. The [FJ] method is a symplectic approach, namely, all the relevant information of the theory can be obtained through an invertible symplectic tensor, it is

*Electronic address: aescalan@ifuap.buap.mx
constructed by means the symplectic variables that are identified as the degrees of freedom. Since the theory under interest is singular there will be constraints, and [FJ] has the advantage that all the constraints of the theory are treated at the same footing, namely, it is not necessary to perform the classification of the constraints in primary, secondary, first class or second class such as in Dirac’s method is done \[11\]. Furthermore, in [FJ] approach it is possible to obtain the gauge transformations of the theory and the generalized [FJ] brackets coincide with the Dirac ones, basically in [FJ] we only choose the symplectic variables either the configuration space or the phase space and by fixing the appropriated gauge, we can invert the symplectic matrix in order to obtain a complete analysis. In this manner, it is possible to obtain all the Dirac results, say, in Dirac’s approach we can construct the Dirac brackets by means two ways; eliminating only the second class constraints remaining the first class ones or we can fix the gauge and convert the first class constraints into second class ones, in any case, we can reproduce these Dirac’s results by means of the [FJ] framework. In fact, for the former we use the configuration space as symplectic variables, for the later we use the phase space \[12, 13\].

With these motivations we study the Einstein theory in the $G \to 0$ limit. Einstein’s theory in the $G \to 0$ limit is an interesting theory, it has as scenario to Minkowski spacetime, lacks of physical degrees of freedom with reducible constraints; in certain sense, it is a copy of a four dimensional $BF$ theory. The theory has been analyzed in the context of Dirac by introducing a kind of ADM variables \[14\], the analysis shows that there are only reducible first class constraints and the algebra of the constraints is similar to the algebra between the first class constraints found in General Relativity [GR]. On the other side, the theory has been also analyzed in \[15\] by using a pure Dirac’s approach, namely, the analysis was developed by following all the Dirac steps and using the full phase space as degrees of freedom without introducing extra variables. In that paper was reported that the theory has reducible first class constraints and irreducible second class constraints, the algebra of the constraints is closed and it has the required structure that Dirac’s framework demands \[8\], namely, the Poisson brackets between first class constraints are linear in first class constraints and quadratic in second class constraints etc., \[2\]. Nevertheless, in spite of those analysis the Dirac brackets were not reported. In fact, if we use the results given in \[14\], it is difficult develop the construction of the Dirac brackets in terms of ADM type variables. On the other hand, in the paper reported in \[15\], there are first class and second class constraints, thus, in order to construct the Dirac brackets we can choose fixing or not fixing the gauge, but the reducibility among the constraints complicate the computation and it is necessary to expand the phase space with canonical auxiliary fields for obtaining that aim. Thus, in this paper we use the results reported in \[15\] and we construct the Dirac brackets by fixing or not fixing the gauge, then we perform the [FJ] analysis and we obtain by a different way all relevant Dirac’s results.

The paper is organized as follows: In section I, we develop a review of the results obtained in \[15\], then we construct the Dirac brackets for the theory under study by eliminating only the second class constraints. In section II, we reproduce the Dirac results obtained in the previous section by using the [FJ] approach. We will work with the configuration space as symplectic variables, in order to
invert the symplectic tensor we fixing the temporal gauge, then we prove that the generalized [FJ] and the Dirac brackets are the same. In section III, we use the fact that the scenario of the theory under study corresponds to Minkowski spacetime background, the Dirac brackets are constructed by fixing the gauge; we convert to the first class constraints into second class constraints. In order to invert the matrix whose entries are given by the Poisson brackets between the second class constraints, we expand the phase space by introducing auxiliary canonical fields that will be useful for constructing the Dirac brackets, then the fundamental Dirac’s brackets are calculated. In section IV, we reproduce the results obtained in the Section III by using the [FJ] analysis. In fact, now we use the phase space as symplectic variables, we fix the gauge by using a Coulomb gauge, because of the reducibility among the first class constraints we also introduce a reducible gauge. Then we show that the generalized [FJ] and Dirac’s brackets coincide to each other. Finally, in Section V we present some remarks and conclusions.

II. HAMILTONIAN ANALYSIS

The action that we will study in this section is given by Einstein’s theory of gravity written in the first order formalism expressed by \[S[A,e] = \frac{1}{8} \int \epsilon^{\alpha\beta\gamma\delta} \epsilon^{IJKL} (e_{\alpha I} e_{\beta J} R_{\gamma\delta KL}) dx^4,\] (1)

where \(\epsilon^{IJKL}\) is the completely antisymmetric object with \(\epsilon^{0123} = 1\), \(e^I_{\alpha}\) is the tetrad field and \(R_{\gamma\delta KL} = \partial_\gamma A_{\delta KL} - \partial_\delta A_{\gamma KL} + G (A_{\gamma K} A_{\delta J} - A_{\delta K} A_{\gamma J})\) is the curvature of the \(SO(3,1)\) connection \(A^{IJ}_\alpha\). Here, \(G\) is the gravitational coupling constant, \(\mu, \nu = 0, 1, ..., 3\) are space-time indices, \(x^\mu\) are the coordinates that label the points for the 4-dimensional manifold \(M\) and \(I, J = 0, 1, ..., 3\) are internal indices that can be raised and lowered by the internal Lorentzian metric \(\eta^{IJ} = (1, -1, -1, -1)\). In the reference [14] is reported that by setting the \(G \to 0\) limit and performing a change of variables, the action (1) is reduced to a copy of BF-like theory

\[S[e, B] = \frac{1}{2} \int \epsilon^{\alpha\beta\gamma\delta} B_{\alpha\beta}^I (\partial_\gamma e_{\delta I} - \partial_\delta e_{\gamma I}) dx^4,\] (2)

where \(B_{\alpha\beta}^I = -\frac{1}{2} \epsilon^{IJKL} e_{[\alpha J} A_{\beta]KL}\) and the fields \(e^I_{\alpha}\) are a collection of four gauge invariant \(U(1)\) vector fields. It is important to comment that one of the equations of motion obtained from (2) given by \(\epsilon^{\alpha\beta\gamma\delta} \partial_\gamma e^I_{\delta} = 0\), implies that \(e^I_{\alpha} = \partial_\alpha f^I\), so the metric \(g_{\mu\nu} = \partial_\mu f^I \partial_\nu f^J \eta_{IJ}\) corresponds locally to Minkowski spacetime, this fact will be used in the following sections. Furthermore, in [14] it was performed the Hamiltonian analysis of the action (2) by using a kind of ADM variables and working on a reduced phase space. In that paper only first class constraints were identified and the construction of the Dirac brackets of theory was not reported. On the other hand, in [15] it was reported a pure Hamiltonian analysis, using the full phase space, and were identified reducible first class constraints and irreducible second class constraints. Furthermore, the extended action and the complete Poisson algebra among the first class constraints were reported. In this manner, the structure of the constraints presented in [15] is more suitable to work than the structure of the
constraints reported in [14], in fact, by using the results reported in [15] there are two possibilities for constructing the Dirac brackets, by fixing or not fixing the gauge; in this work, we will use the results reported in [15] for constructing the Dirac brackets and then we perform the [FJ] analysis, we shall prove that the Dirac brackets and the generalized [FJ] brackets are the same.

Therefore, by working a pure Dirac’s analysis of the action (2) [15], the following extended action was reported

\[ S[e^I_\mu, \Pi^I_\mu, B^I_{\mu\nu}, u_0^I, u^I, u^I_0, u^a_I, v^I_{ab}] = \int \left\{ \dot{e^I_\mu} \Pi^I_\mu - H_E - v^I_a \chi^a_I - v^I_{ab} \chi^{ab}_I \right\}, \quad (3) \]

where \( v^I_a, v^I_{ab} \) are Lagrange multipliers enforcing the second class constraints \( \chi^a_I \) and \( \chi^{ab}_I \) given by

\[
\begin{align*}
\chi^a_I &= \Pi^a_I - \eta^{abc} B_{Ibc} \approx 0, \\
\chi^{ab}_I &= \Pi^{ab}_I \approx 0,
\end{align*}
\]

and \( H_E \) is the extended Hamiltonian given by

\[
H_E = -B^I_{0a} \left( \eta^{abc} (\partial_0 e_{Ic} - \partial_c e_{Ib}) - 2 \partial_0 B^I_{ab} \right) - e^I_0 \partial_0 \Pi^a_I - u^I_0 \gamma^I_0 - u^I_a \gamma^I_a - u^I_{0a} \gamma^I_{0a}, \quad (5)
\]

with \( u^I_0, u^I, u^I_0, u^a_I \) being Lagrange multipliers enforcing the following first class constraints

\[
\begin{align*}
\gamma^0_I &= \Pi^0_I \approx 0, \\
\gamma^{0a}_I &= \Pi^{0a}_I \approx 0, \\
\gamma^I &= \partial_0 \Pi^a_I, \\
\gamma^a_I &= \eta^{abc} (\partial_0 e_{Ic} - \partial_c e_{Ib}) - 2 \partial_0 B^I_{ab} \approx 0.
\end{align*}
\]

We can observe that the price to pay for working with the complete phase space, is that there are second class constraints, which makes a difference with respect to the results reported in [14] where only first class constraints were found. Moreover, we also observe that these constraints are not independent because there exist reducibility among the constraints given by \( \partial_0 \gamma^a_I = 0 \), which complicates the construction of the Dirac brackets, however, that trouble can be fixed by enlarging the phase space as we will see below. Thus, in order to construct the Dirac brackets without fixing the gauge, we calculate the Poisson brackets among the second class constraints

\[
\begin{align*}
\{ \chi^a_I(x), \chi^b_J(y) \} &= 0, \\
\{ \chi^a_I(x), \chi^{bc}_J(y) \} &= -\eta^{aij} \delta^{bc}_{ij} \eta_{IJ}, \\
\{ \chi^{af}_I, \chi^{ab}_J(y) \} &= 0.
\end{align*}
\]

Furthermore, the matrix, namely \( C_{\alpha\beta} \), whose entries are given by the Poisson brackets among the second class constraints takes the form

\[
C_{\alpha\beta} = \begin{pmatrix} 0 & -\eta^{abc} \eta_{IJ} \\ \eta^{abc} \eta_{IJ} & 0 \end{pmatrix}.
\]
and its inverse is given by

\[
(C^{\alpha\beta})^{-1} = \frac{1}{2} \begin{pmatrix}
0 & \eta_{abc}\eta^{IJ} \\
-\eta_{abc}\eta^{IJ} & 0
\end{pmatrix}.
\]

In this manner, the Dirac bracket of two functionals \(F, G\) defined on the phase space, is expressed by

\[
\{F(x), G(z)\}_D \equiv \{F(x), G(z)\} - \int d^2u d^2w \{F(x), \xi_\alpha(u)\} (C^{\alpha\beta})^{-1}\{\xi_\beta(w), G(z)\},
\]

where \(\{F(x), G(z)\}\) is the Poisson bracket between two functionals \(F, G\), and \(\xi_\alpha = (\chi^a_I, \chi_I^{ab})\) represent the set of second class constraints. By using this fact, we obtain the following Dirac’s brackets

\[
\begin{align*}
\{e^I_{\ a}(x), \Pi^a_I(y)\}_D &= \delta^b_a \delta^I_J \delta^3(x-y), \\
\{\Pi^a_I(x), \Pi^a_I(y)\}_D &= 0, \\
\{e^I_{\ a}(x), e^J_{\ b}(y)\}_D &= 0, \\
\{e^I_{\ a}(x), \Pi^{cd}_I(y)\}_D &= 0, \\
\{\Pi^a_I(x), B^I_{\ cd}(y)\}_D &= 0, \\
\{e^I_{\ a}(x), B^J_{\ cd}(y)\}_D &= \frac{\eta^{agd}}{2} \eta^{IJ} \delta^3(x-y), \\
\{B^I_{\ ab}(x), \Pi^{gd}_I(y)\}_D &= 0.
\end{align*}
\] (7)

Now, by using these Dirac’s brackets we can show that the algebra among the constraints vanish identically. Hence, there is a difference respect the results obtained in [14] because in that work it was used a kind of ADM variables and the Poisson algebra among the constraints is linear combinations of constraints. In this section, we have worked with the full phase space without resorting to extra variables and the algebra among the constraints is more suitable for working with them. Furthermore, we have eliminated only the second class constraints, and at this step it was not necessary to take into account the reducibility conditions; however, if we fix the gauge and we convert the first class constraints into second class constraints, then the reducibility conditions will be taken into account, we shall explain this fact in later sections.

III. FADDEEV-JACKIW FORMALISM BY USING A TEMPORAL GAUGE

Now we will reproduce the above results by using the [FJ] formalism. We can see that the action (2) can be written in the following form

\[
L^{(0)} = \eta^{abc} B_{Ibc} e^I_a - V^{(0)},
\] (8)

where the symplectic potential is given by \(V^{(0)} = -\eta^{abc} B_{Iab}(\partial_1 e^I_a - \partial_2 e^I_b) - \partial_3 (\eta^{abc} B_{Iab}) e^I_0\). The corresponding symplectic equations of motion are given by

\[
f^{(0)}_{ij} \ddot{\xi}^j = \frac{\partial V^{(0)}(\xi)}{\partial \xi^i},
\] (9)
where the symplectic matrix \( f_{ij}^{(0)} \) takes the form

\[
f_{ij}^{(0)}(x, y) = \frac{\delta a_j(y)}{\delta \xi^i(x)} - \frac{\delta a_i(x)}{\delta \xi^j(y)},
\]

with \( \xi^{(0)i} \) and \( a^{(0)} \), representing a set of symplectic variables. Hence, in order to reproduce by means of the [FJ] method the results obtained in previous section we will work by using the configuration space as symplectic variables. In fact, it has been showed in [13] that if we construct the Dirac brackets by eliminating only the second class constraints, then in the [FJ] scheme it is necessary to work with the configuration space. We know that introducing the Dirac brackets by eliminating the second class constraints, in particular eliminating the primary second class constraints, then the momenta become to be a label because the momenta can be expressed in terms of the fields. Therefore, from (8) we identify the following symplectic variables

\[
\begin{pmatrix}
\varepsilon_i^{(0)} = (e^I a^a, B^{I 0 a}, e^{I 0 a}, B^{I ab}) \\
\end{pmatrix}
\]

and the 1-forms

\[
\begin{pmatrix}
a_i^{(0)} = (\eta^{abc} B^{I bc}, 0, 0, 0) \\
\end{pmatrix}
\]

In this manner, by using these symplectic variables, we construct the following symplectic matrix

\[
\begin{pmatrix}
0 & 0 & 0 & -\eta^{abc} \eta_{IJ} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\eta^{abc} \eta_{IJ} & 0 & 0 & 0
\end{pmatrix} \delta^3(x - y).
\]

We observe that the matrix (11) is not invertible, hence, there exist constraints. It is easy to observe that there are the following modes

\[
\begin{pmatrix}
v_1^{(0)} = (0, v^{B_{0 a}}, 0, 0) \\
v_2^{(0)} = (0, 0, u^{e_{I 0}}, 0)
\end{pmatrix}
\]

where \( v^{B_{0 a}} \) and \( u^{e_{I 0}} \) are arbitrary functions. By using these modes we obtain the following [FJ] constraints

\[
\begin{align*}
\Omega^{aI} &= \int d^2 x (v^{(0)}_i)^T (x) \frac{\delta}{\delta \xi^{(0)i}(x)} \int d^2 y V^{(0)}(\xi) \\
&= \eta^{abc} (\partial_b e^I c - \partial_c e^I b) = 0,
\end{align*}
\]

\[
\begin{align*}
\Omega^I &= \int d^2 x (v^{(0)}_i)^T (x) \frac{\delta}{\delta \xi^{(0)i}(x)} \int d^2 y V^{(0)}(\xi) \\
&= \partial_c (\eta^{abc} B^{I ab}) = 0.
\end{align*}
\]

We can observe that (12) is a reducible constraint because of \( \partial_a \Omega^{aI} = 0 \). Now, we will observe if there are present more constraints in the [FJ] context. For this aim, we write in matrix form the following system

\[
f_{kj} \dot{\xi}^j = Z_k(\xi),
\]

where

\[
Z_k(\xi) = \begin{pmatrix}
\frac{\partial V^{(0)}(\xi)}{\partial \xi^k} \\
0 \\
0
\end{pmatrix},
\]
and

\[ f_{ij} = \left( \frac{\partial L^{(1)}}{\partial \dot{\phi}_i} \right) = \begin{pmatrix} 0 & 0 & 0 & -\eta^{abc} \eta_{IJ} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \eta^{abc} \eta_{IJ} & 0 & 0 & 0 \\ 2\eta^{abc} \delta^I \partial_b & 0 & 0 & 0 \\ 0 & 0 & 0 & \eta^{abc} \delta^I \partial_b \end{pmatrix} \delta^3(x - y), \]  

(16)

we can observe that (16) is not a square matrix as expected, however, it has linearly independent modes. It is straightforward calculate the modes, say \((\nu^{(1)})_k^T\), and from the contraction \((\nu^{(1)})_k^T Z_k = 0\) we can prove that there are not more [FJ] constraints. Now, we introduce all that information by constructing a new symplectic Lagrangian. For this aim, we use the Lagrange multipliers \(\lambda_{aI}\) and \(\rho^{I}\) associated to the constraints, the new symplectic Lagrangian is given by

\[ \mathcal{L}^{(1)} = \eta^{abc} B_{1bc} \dot{\epsilon}^I_a - \dot{\Omega}^{aI} \lambda_{aI} - \dot{\Omega}_I \rho^I - V^{(1)}, \]  

(17)

where \(V^{(1)} = V^{(0)} \mid_{(1)}_{(2)} = 0\). In this manner, the symplectic Lagrangian is given by

\[ \mathcal{L}^{(1)} = \eta^{abc} B_{1bc} \dot{\epsilon}^I_a - \dot{\Omega}^{aI} \lambda_{aI} - \dot{\Omega}_I \rho^I, \]  

(18)

from (18) we identify the following symplectic variables given by \((\dot{\epsilon}^i = (\epsilon^I_a, \lambda_{aI}, B_{1bc}, \rho^I)\) and the 1-forms \((\dot{a}_i = (\eta^{abc} B_{1bc}, \Omega^{aI}, 0, \Omega_I))\). By using these symplectic variables we obtain the following symplectic matrix

\[ f^{(1)}_{ij} = \begin{pmatrix} 0 & -2\delta^I \delta^J \eta^{abc} \partial_b & -\eta^{abc} \delta^I J & 0 \\ 2\delta^I J \eta^{abc} \partial_b & 0 & 0 & 0 \\ \eta^{abc} \delta^I J & 0 & 0 & -\delta^I J \eta^{abc} \partial_c \\ 0 & 0 & \eta^{abc} \delta^I J \partial_c & 0 \end{pmatrix} \delta^3(x - y), \]  

(19)

were we can see that \(f^{(1)}_{ij}\) is a singular matrix. In fact, this matrix has 40 null vectors, thus, it is not invertible, however, we have showed that there are not more constraints. Therefore, in the scheme this means that the theory has a gauge symetry. In order to invert the symplectic matrix (19), we fix the following temporal gauge \(\epsilon^I_0 = 0\) and \(B_{I0a} = 0\), this fact means that \(\rho^I = \text{constant}\) and \(\lambda_{aI} = \text{constant}\). In this manner, we introduce this information in a new symplectic Lagrangian given by

\[ \mathcal{L}^{(2)} = \eta^{abc} B_{1bc} \dot{\epsilon}^I_a - (\dot{\Omega}^{aI} - \delta^{aI}) \lambda_{aI} - (\Omega_I - \rho_I) \dot{\rho}^I, \]  

(20)

where we identify the following symplectic variables \(\dot{\epsilon}^i = (\epsilon^I_a, \lambda_{aI}, \phi^I, B_{1bc}, \alpha^{aI}, \rho_I)\) and the 1-forms \(a_i = (\eta^{abc} B_{1bc}, -\dot{\Omega}^{aI} - \delta^{aI}), -\dot{\Omega}_I, 0, 0, 0\). By using these symplectic variables, the
symplectic matrix is given by

\[
(\text{symplectic matrix}) = \begin{pmatrix}
0 & -2\delta^I_J\eta^{abc}\partial_b & 0 & -\eta^{abc}\delta^I_J & 0 & 0 \\
2\delta^I_J\eta^{abc}\partial_b & 0 & 0 & 0 & -\delta^a_b\delta^I_J & 0 \\
0 & 0 & 0 & -\delta^I_J\eta^{abc}\partial_c & 0 & \delta^I_J \\
\eta^{abc}\delta^I_J & 0 & \delta^I_J\eta^{abc}\partial_c & 0 & 0 & 0 \\
0 & \delta^a_b\delta^I_J & 0 & 0 & 0 & 0 \\
0 & 0 & -\delta^I_J & 0 & 0 & 0
\end{pmatrix}
\]

We can observe that \( f^{(2)}_{ij} \) is not singular, therefore we can construct its inverse. The inverse of \((21)\) is called the symplectic tensor

\[
( f^{(2)}_{ij} )^{-1} = \begin{pmatrix}
0 & 0 & 0 & \frac{\eta^{abc}}{2}\delta^I_J & 0 & \delta^I_J\partial_a \\
0 & 0 & 0 & 0 & \delta^I_J\delta^a_b & 0 \\
0 & 0 & 0 & 0 & 0 & -\delta^I_J \\
-\frac{\eta^{abc}}{2}\delta^I_J & 0 & 0 & 0 & -\delta^I_J\delta^{ad}\partial_d & 0 \\
0 & -\delta^I_J\delta^a_b & 0 & \delta^I_J\delta^{cd}\partial_d & 0 & 0 \\
-\delta^I_J\partial_a & 0 & \delta^I_J & 0 & 0 & 0
\end{pmatrix}
\]

In this manner, we identify the following [FJ] generalized brackets

\[
\begin{align*}
\{ e^I_a, B_{Jbc} \} & = \delta^I_J \frac{\eta^{abc}}{2}\delta^3(x-y), \\
\{ e^I_a, \rho_{J} \} & = \delta^I_J \partial_a \delta^3(x-y), \\
\{ \lambda_{aI}, \alpha^{bJ} \} & = \delta^b_a \delta^I_J \delta^3(x-y), \\
\{ \phi^I, \rho_J \} & = -\delta^I_J \delta^3(x-y), \\
\{ B_{tab}, \alpha^{cJ} \} & = -\delta^I_J \delta^{cd}_{ab} \partial_d \delta^3(x-y),
\end{align*}
\]

where we can see that these [FJ] brackets coincide with the Dirac brackets found in previous section. Furthermore, we have not taken into account the reducibility conditions just like it was done in Dirac’s method, but in latter sections we will.

IV. DIRAC’S BRACKETS BY FIXING THE GAUGE

It is well-known that in [GR] the coordinates of the space and time lacks of physical meaning, the physical relevance in [GR] is giving by the relation of fields respect to other fields [17]. In this respect, we can not localise the gravitational field in the spacetime (gauge fixing) because the spacetime is a dynamical system, however, for the theory under study we have commented above that one of the equations of motion obtained from the action (2) implies that the spacetime is locally Minkowski. Therefore, we can fix the gauge and we will construct the Dirac brackets in order to perform the quantization of the theory. Hence, for this aim, we fix the following gauge \( e^I_0 \approx 0, \)
$B_{t_{0a}} \approx 0$, $\partial^a e^i_{a} \approx 0$ and $-\partial^b B_{ab}^t \approx 0$. We can observe that the gauge $-\partial^b B_{ab}^t$ is also reducible, and this fact does not allow us to calculate the Dirac's brackets, however, we will introduce auxiliary fields in order to convert the reducible constraints in irreducible ones \[18, 19\]. Thus we obtain the following set of second class constraints

$$\begin{align*}
\chi_1 &= e^i_{0} \approx 0, \\
\chi_2 &= \Pi^0_{t} \approx 0, \\
\chi_3 &= B_{t_{0a}} \approx 0, \\
\chi_4 &= \Pi^{0a}_{t} \approx 0, \\
\chi_5 &= \partial^a e^i_{a} \approx 0,
\end{align*}$$

and this fact does not allow us calculate the Dirac's brackets, however, we will introduce auxiliary fields to convert the reducible constraints in irreducible ones. Thus we obtain the following set of second class constraints

$$\begin{align*}
\chi_6 &= \partial_a \Pi^a_{t} \approx 0, \\
\chi_7 &= -\partial^b B_{ab}^t + \partial_a q^{f} \approx 0, \\
\chi_8 &= 2\eta^{abc}\partial_b e_{1c} - 2\partial_b \Pi^{ab}_{t} + \partial^a P_l \approx 0, \\
\chi_9 &= \Pi^a_{t} - \eta^{abc} B_{t_{bc}} \approx 0, \\
\chi_{10} &= \Pi^{ab}_{t} \approx 0,
\end{align*}$$

(23)

where $q^I$ and $P_J$ are auxiliary fields satisfying \{$q^I(x), P_J(y)$\} = $\delta^I_J \delta^3(x - y)$. In this manner, the matrix whose entries are the Poisson brackets among the constraints (23) is given by

$$C_{\alpha\beta} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\delta^I_J & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\delta^a_b \delta^I_J & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \eta^{abc} \delta^I_J & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} \delta^3(x - y),$$

and its inverse

$$\left(C_{\alpha\beta}\right)^{-1} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\delta^I_J & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \delta^I_J & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} \times \delta^3(x - y).$$

It is worth to mention, that the auxiliary fields $q^I$ and $P_J$ have been added because they allow us to
find the inverse matrix $C_{\alpha\beta}^{-1}$. In fact, without these auxiliary fields it is not possible to calculate the inverse matrix, this is a common problem in reducible theories \[18, 19\]. However, these auxiliary fields do not contribute to the dynamics of the system because the Dirac brackets among the auxiliary fields and the dynamical variables vanish, this is

\[
\begin{align*}
\{q^I(x), P_J(y)\}_D &= 0, & \{q^I(x), e^J_a(y)\}_D &= 0, & \{q^I(x), \Pi^0_J(y)\}_D &= 0, \\
\{q^I(x), B^J_{0a}(y)\}_D &= 0, & \{q^I(x), \Pi^{0a}_J(y)\}_D &= 0, & \{q^I(x), e^J_a(y)\}_D &= 0, \\
\{q^I(x), \Pi^{ab}_J(y)\}_D &= 0, & \{q^I(x), \Pi^{0}_J(y)\}_D &= 0, & \{q^I(x), B^J_{ab}(y)\}_D &= 0, \\
\{P_I(x), e^J_a(y)\}_D &= 0, & \{P_I(x), \Pi^0_J(y)\}_D &= 0, & \{P_I(x), B^J_{0a}(y)\}_D &= 0, \\
\{P_I(x), \Pi^{0a}_J(y)\}_D &= 0, & \{P_I(x), e^J_a(y)\}_D &= 0, & \{P_I(x), \Pi^a_J(y)\}_D &= 0, \\
\{P_I(x), \Pi^{ab}_J(y)\}_D &= 0.
\end{align*}
\]

After a long computation, we can obtain the following non-zero Dirac’s brackets among dynamical variables given by

\[
\begin{align*}
\{e^I_a, \Pi^{b}_J\}_D &= \delta^I_J \left( \delta^b_a - \frac{\partial_a \partial_b}{\nabla^2} \right) \delta^3(x - y), \\
\{e^I_a, B_{Jbc}\}_D &= \frac{\delta^I_J}{2} \left( \eta^{abc} - \eta_{abc} \frac{\partial_a \partial^b}{\nabla^2} \right) \delta^3(x - y).
\end{align*}
\]  

(24)

It is important to comment, that all these results were not reported in previous works \[14, 15\]. On the other hand, with all these results obtained along this section we have at hand all the necessary tools for comparing at Hamiltonian level the theory under study and BF versions of [GR]. In fact, we have comment that the action (2) is a copy of a BF theory, however, the context of these two theories is not the same. For the former we have to Minkowski spacetime as scenario, for the latter there is not scenario at all because BF theory is a background independent theory.

In the following lines we will reproduce the results obtained in this section by means the [FJ] framework.

V. GENERALIZED FADDEEV-JACKIW BRACKETS BY USING THE PHASE SPACE AS SYMPLECTIC VARIABLES

Now, in this section we will reproduce by means of the [FJ] formulation the results obtained in the previous section where the Dirac brackets have been obtained by fixing the gauge. In particular by fixing the gauge we have choosen a particular configuration of the fields, in this manner, in [FJ] we should to work with the phase space as symplectic variables \[13\]. In order to perform our analysis, from \(8\) we identify the symplectic Lagrangian

\[
\mathcal{L}^{(0)} = \Pi^a_I e^I_a - V^{(0)},
\]

(25)

where $\Pi^a_I = \eta^{abc} B_{Ibc}$ and $V^{(0)} = -\eta^{abc} B_{I0a} (\partial_b e^I_c - \partial_c e^I_b) - \partial_a \Pi^a_I e^I_0$. Thus, the symplectic coordinates $\xi^{(0)} = (e^I_a, \Pi^a_I, e^I_0, B_{I0a})$ and $\bar{a}^{(0)} = (\Pi^a_I, 0, 0, 0)$, hence the symplectic matrix is
given by

\[
(0) \quad f_{ij} = \begin{pmatrix}
0 & \delta^b_a \delta^I_J & 0 & 0 \\
-\delta^b_a \delta^I_J & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} \delta^3(x-y),
\]

this matrix has two null vectors given by \(V^{(1)} = (0,0,V^{\epsilon^j_0},0), \quad V^{(2)} = (0,0,0,V^{B^{10a}}),\) where \(V^{B^{10a}}, \quad V^{\epsilon^j_0}\) are arbitrary functions. In this manner, we need calculate the following

\[
\frac{\partial}{\partial \varepsilon} V^{(0)} = 0,
\]

this implies that

\[
(1) \quad \Omega = \partial_a \Pi^a_1 = 0, \quad (2) \quad \Omega = \eta^{abc} \partial_b \epsilon_{1c} = 0.
\]

We can observe that these constraints are the secondary constraints found in Dirac’s method. Furthermore, \(\Omega\) is a reducible constraint, because \(\partial_a \Omega^a = 0\), and we will take into account this fact in the following computations. On the other hand, we need calculate the following

\[
\frac{\partial}{\partial \varepsilon} f_{ij} = \begin{pmatrix}
0 & \delta^b_a \delta^I_J & 0 & 0 \\
-\delta^b_a \delta^I_J & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} \delta^3(x-y).
\]

We can observe, this matrix is not squared, however, it has two modes given by \((V^{(1)})^T_i = (\partial_b V^\lambda a, 0, V^{A_0}, v^{B^{10a}}, 0, -V^\lambda I)\) and \((V^{(2)})^T_i = (0, \eta^{abc} \partial_b V_{1c}, V^{A_0}, v^{B^{10a}}, 0, 0)\). By performing the contraction with the modes, we find that \((V)_i^T Z_h = 0,\) is an identity. Therefore, there are not more constraints for the theory under study. In this manner, we will add all that information by constructing a new symplectic Lagrangian given by

\[
L = \Pi^a_1 \dot{c}^I_a - 2\dot{\lambda}^I_a (\eta^{abc} \partial_b \epsilon_{1c} - \rho^I (\partial_a \Pi^a_1) - \dot{\theta}_I \partial^a \lambda^I_a, \quad \text{(27)}
\]

because of \(\Omega\) is a reducible constraint in \((27)\) was necessary to add a Lagrange multiplier \(\theta_I\) of the Lagrange multiplier \(\lambda^I\) \([20]\). Thus, from \((27)\) we identify the new set of symplectic variables are given by \(\varepsilon_1 = (c^I, \Pi^a_1, \lambda^I_a, \rho^I, \theta_I)\) and the 1-forms \(a_i = (\Pi^a_1, 0, -2\eta^{abc} \partial_b \epsilon_{1c}, -\partial_a \Pi^a_1, \partial^a \lambda^I_a)\) in this manner the symplectic matrix takes the form

\[
(1) \quad f_{ij} = \begin{pmatrix}
0 & -\delta^b_a \delta^I_J & -2\eta^{abc} \eta_{IJ} \partial_b & 0 & 0 \\
\delta^a_b \delta^I_J & 0 & 0 & -\delta^I_J \partial_a & 0 \\
2\eta_{IJ} \eta^{abc} \partial_b & 0 & 0 & 0 & -\delta^I_J \partial_a \\
0 & \delta^I_J \partial_a & 0 & 0 & 0 \\
0 & 0 & \delta^I_J \partial^a & 0 & 0
\end{pmatrix} \delta^3(x-y), \quad \text{(28)}
\]
We can observe that $f_{ij}$ is singular, however, we have showed that there are not more constraints, the noninvertibility of $f_{ij}$ means that the theory has a gauge symmetry. In this manner, we will take the following fixing gauge $\Omega = \partial^a e^I_a = 0$ and $\Omega = \frac{1}{2} \eta_{abc} \partial b \Pi_c I$, this information needs to be added to the symplectic Lagrangian through one new Lagrange multipliers, namely, $\eta_I$ and $\alpha^I$. We observe, however, that $\Omega$ is a reducible constraint (gauge fixing), in this manner in the symplectic Lagrangian we will add one more Lagrange multiplier, $\beta_I$, again, the Lagrange multiplier of the Lagrange multiplier [20]

\[
(2) \quad \mathcal{L} = \Pi^a I^a I - 2 \dot{\lambda}^I_a (\eta_{abc} \partial_b e^c_I) - \rho^I (\partial_a \Pi^a I) - \dot{\theta}^I \partial^a \lambda^I_a - \left( \frac{1}{2} \eta_{abc} \partial b \Pi_c I \right) \partial a^I
\]

\[-(\partial^a e^I a) \eta_I - (\partial^a \alpha^I a) \beta_I.\]

Now, from (24) it is possible to identify the new set of symplectic variables \((\varepsilon^I)^{\alpha I} = (e^I_a, \Pi^a I, \lambda^I_a, \rho^I, \theta_I, \alpha^a I, \eta_I, \beta_I)\) and the 1-form \((\tilde{a}^I_i) = (\Pi^a I, 0, -2 \eta_{abc} \partial_b e^c_I, -\partial_a \Pi^a I, -\partial^a \lambda^I_a, \frac{1}{2} \eta_{abc} \partial b \Pi_c I, -\partial^a e^I_a, \partial^a \alpha^I a)\). In this manner, the symplectic matrix is given by

\[
(2) \quad f_{ij} =\begin{pmatrix}
0 & -\delta^a_I \delta^I_J & -2 \eta_{abc} \eta_{IJ} \partial_b & 0 & 0 & 0 & \cdots & 0 \\
-\delta^a_I \delta^I_J & 0 & 0 & \cdots & 0 & \cdots & \cdots & 0 \\
2 \eta_{IJ} \eta_{abc} \partial_b & 0 & 0 & \cdots & 0 & \cdots & \cdots & 0 \\
0 & \delta^I_J \partial_a & 0 & \cdots & 0 & \cdots & \cdots & 0 \\
0 & 0 & \delta^I_J \partial_a & 0 & \cdots & 0 & \cdots & 0 \\
0 & \frac{\delta^a_I}{\delta^I_J} \eta_{abc} \partial^c & 0 & \cdots & 0 & \cdots & \cdots & 0 \\
\delta^I_J \partial^a & 0 & 0 & \cdots & 0 & \cdots & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 & \cdots & \cdots & 0 \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\end{pmatrix} \times \delta^3(x - y),
\]

we can observe that this matrix is not singular, thus, there exists its inverse. The inverse of the symplectic matrix $f_{ij}$ is given by

\[
(2) \quad (f_{ij})^{-1} =\begin{pmatrix}
0 & \delta^I_J \left( \frac{\delta^a_I}{\delta^I_J} \partial_a \right) & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\end{pmatrix} \times \delta^3(x - y),
\]
Therefore, from (30) it is possible to identify the following [FJ] generalized brackets
\[ \{ \xi^{(2)}_i(x), \xi^{(2)}_j(y) \}_{FD} = [f^{(2)}_{ij}(x, y)]^{-1}, \] (31)
thus
\[ \{ e_I^a(x), \Pi^I_b(y) \}_{FD} = [f^{(2)}_{12}(x, y)]^{-1} = \delta^I_J \left( \delta^a_b - \frac{\partial^n \partial_b}{\nabla^2} \right) \delta^3(x - y), \] (32)
that correspond to the Dirac brackets found in previous section. It is important to comment that in [FJ] method we did not use auxiliary fields but we used the Lagrange multiplier of the Lagrange multiplier [20], in this sense, the [FJ] framework is more economic than Dirac’s method.

VI. CONCLUSIONS

In this paper a detailed Hamiltonian and a [FJ] analysis for Einstein’s theory in the \( G \rightarrow 0 \) limit have been performed. We have worked with the full phase space and we have constructed the Dirac brackets by means two ways, fixing and without fixing the gauge. In this respect, a complete Hamiltonian study has been performed, in order to construct the Dirac brackets by fixing the gauge, we have showed that it is necessary to extend the phase space by means auxiliary variables, however, this is a large task and for this reason we worked with the [FJ] method. In this manner, we have performed a complete [FJ] framework for the theory under study, we found all the [FJ] constraints and we have constructed the generalized [FJ] brackets; we have worked with both the configuration space and the phase space. In both cases we showed the equivalence between the Dirac and the generalized [FJ] brackets. It is important to comment, that in [FJ] is not necessary extend the phase space by introducing canonical auxiliary variables as Dirac’s method requires; in the [FJ] scheme the symplectic matrix is inverted by constraining the Lagrange multipliers. In this sense, the [FJ] framework is more economic than Dirac’s method. Finally, with the results obtained in this paper we can extend our study to \( BF \) theories [6] and pure gravity [8]. In fact, in these papers a pure Dirac’s analysis has been performed by working with the complete phase space, but the Dirac brackets were not calculated; thus, the learned in this paper could be useful for that aim. However, all these ideas are in progress and will be reported in forthcoming works [21].

Acknowledgements

This work was supported by CONACyT under Grant No. CB-2014-01/ 240781. Alberto Escalante
wishes to thank Eric Gourgoulhon and the observatoire de Paris (LUTH) for the hospitality.

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