Low-momentum dynamic structure factor of a strongly interacting Fermi gas at finite temperature: A two-fluid hydrodynamic description

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We provide a description of the dynamic structure factor of a homogeneous unitary Fermi gas at low momentum and low frequency, based on the dissipative two-fluid hydrodynamic theory. The viscous relaxation time is estimated and is used to determine the regime where the hydrodynamic theory is applicable and to understand the nature of sound waves in the density response near the superfluid phase transition. By collecting the best knowledge on the shear viscosity and thermal conductivity known so far, we calculate the various diffusion coefficients and obtain the damping width of the (first and second) sound waves. We find that the damping width of the first sound is greatly enhanced across the superfluid transition and very close to the transition the second sound might be resolved in the density response for the transferred momentum up to the half of Fermi momentum. Our work is motivated by the recent measurement of the local dynamic structure factor at low momentum at Swinburne University of Technology and the on-going experiment on sound attenuation of a homogeneous unitary Fermi gas at Massachusetts Institute of Technology. We discuss how the measurement of the velocity and damping width of the sound modes in low-momentum dynamic structure factor may lead to an improved determination of the universal superfluid density, shear viscosity and thermal conductivity of a unitary Fermi gas.

I. INTRODUCTION

Dynamic structure factor, which determines the response of the system with respect to an external density perturbation, provides valuable information on the low-energy elementary excitations of strongly correlated many-body systems [1, 2]. In superfluid $^4$He, it was measured by Brillouin and Raman light scattering (at long wavelength) [3] and inelastic neutron scattering [2] and played a central role in consolidating the modern ideas of quasiparticles (i.e., phonons, maxons and rotons) [4] and of two-fluid hydrodynamics of superfluids [4, 5].

A strongly interacting Fermi gas of ultracold atoms with infinitely large scattering length (i.e., $a_s = \pm \infty$, the so-called unitary limit) is a new type of Fermi superfluids [6], which has received increasing attention from different branches of physics since its first realization in 2002 [7]. To date, a number of intriguing properties of a unitary Fermi gas [9], in particular the universal equations of state [10], have been measured to a great accuracy. However, less is known about its low-energy elementary excitation spectrum [11-12]. This is largely due to the external harmonic trapping potential that is necessary to keep atoms from escaping. Only very recently, two-photon Bragg spectroscopy has been applied with low transferred momentum to determine the local dynamic structure factor of the central part of a trapped unitary Fermi gas at low energy [13], and hence visualize the Bogoliubov-Anderson phonon mode in the deep superfluid phase at $T = 0.09(1)T_F \simeq 0.54 T_c$, where $T_F$ is the Fermi temperature and $T_c \simeq 0.167 T_F$ is the superfluid transition temperature [10]. Analogous to inelastic neutron scattering in superfluid $^4$He [2], it is natural to anticipate that further measurements of the temperature dependence of the local low-momentum Bragg spectroscopy may give a full plot of the dynamic structure factor and elementary excitation spectrum across the superfluid transition. At this point, it is worth noting that the measurement of sound propagation of a unitary Fermi gas trapped in a uniform box potential is also undergoing [14]. Analogous to sound attenuation experiments in superfluid $^4$He [2], the determination of sound velocity and attenuation of a uniform unitary Fermi gas provides a useful alternative way to characterize its elementary excitations.

These on-going experiments on density response at Swinburne University Technology (SUT) and at Massachusetts Institute of Technology (MIT) urge us to look for new theoretical development to describe the dynamic structure factor of a unitary Fermi gas at low momentum and low energy. This turns out to be a notoriously difficult task, since for a unitary Fermigas there is no small parameter to control the accuracy of the theory [9]. Fortunately, under certain conditions at low energy the density response of a unitary Fermi gas may be well described by the seminal Landau two-fluid hydrodynamic theory [4]. The requirement for hydrodynamics is usually summarized as $\omega \tau \ll 1$, where $\omega$ is the frequency of a collective excitation and $\tau$ is the appropriate relaxation time. The short relaxation time or collision time for excitations ensures the establishment of local hydrodynamic equilibrium and the dynamics of the system is thus governed by a set of equations that satisfy conservation laws [4]. In the previous theoretical investigations [15, 16], the non-dissipative two-fluid hydrodynamic theory has been applied to illustrate the sizable coupling between first and second sound in a unitary Fermi gas and to show the promising opportunity of exciting second sound with density probes.

The purpose of the present work is to update such a
two-fluid hydrodynamic calculation by including the crucial dissipation terms. In doing so, we are able to obtain quantitative predictions of the dynamic structure factor in the hydrodynamic regime, once the relevant transport coefficients in the dissipation terms are known. In greater detail, we estimate the realistic experimental condition for reaching the hydrodynamic regime and clarify the nature of sound waves of a unitary Fermi gas near the superfluid transition. We then calculate the hydrodynamic dynamic structure factor based on the existing knowledge on transport coefficients such as shear viscosity and thermal conductivity. We also discuss the interesting features of the dynamic structure factor that may arise in the on-going experiments at SUT, with emphasis on the optimal experimental condition for probing the second sound.

It should be emphasized that the attempt to calculate the dynamic structure factor with the use of the dissipative two-fluid hydrodynamic theory, as the one carried out in the present work, can only be made possible very recently, since the key inputs to the theory - the transport coefficients of the unitary Fermi gas such as the shear viscosity - are determined with reasonable precision only recently. Interestingly, we may reverse this procedure and consider the accurate measurement of the dynamic structure factor as the input. The transport coefficients can then be extracted from the measured velocity and damping width of sounds. For this purpose, in the end of this paper, we will discuss the sensitivity of the velocity and damping width of sounds on the transport coefficients.

It is also worth noting that the sound attenuation of a strongly interacting Fermi gas - which is more relevant to the on-going sound attenuation experiment at MIT - has been theoretically investigated by Braby, Chao and Schäfer, by taking a high-temperature approximation for both shear viscosity and thermal conductivity (see Fig. 6 in Ref. [19]). In this work, we work with more accurate shear viscosity (as recently measured) and focus on the more fundamental property of dynamic structure factor.

The rest of the paper is set out as follows. In the next section, we briefly review the well-known dissipative two-fluid hydrodynamic theory and present the expression of the dynamic structure factor. The various input parameters to the theory are discussed. In Sec. III, we estimate the viscous relaxation time and determine the conditions for the application of the hydrodynamic theory. The nature of sound waves at different temperatures near the superfluid transition is sketched. In Sec. IV, we report in detail the hydrodynamic dynamic structure factor at two typical transferred momenta, considering the realistic experimental situations at SUT and MIT, respectively. We focus particularly on the dependence of the second sound on the transferred momentum. In Sec. V, we discuss the dependence of the sound velocity and sound attenuation on the input parameters of superfluid density and thermal conductivity. Finally, we conclude in Sec. VI.

II. TWO-FLUID HYDRODYNAMIC THEORY

The dynamic structure factor \( S(k, \omega) \) measures the scattering rate of a density probe that imports a momentum \( h \mathbf{k} \) and energy \( \hbar \omega \) to the system \([1,2]\). It is formally related to the imaginary part of the density response function \( \chi_{nn}(k, \omega) = \chi_{nn}(k, \omega) \),

\[
S(k, \omega) = -\frac{1}{n \pi} \frac{1}{(1 - e^{-\beta \omega})} \text{Im} \chi_{nn}(k, \omega + i 0^+) ,
\]

where \( n \) is the total number density and \( \beta = 1/(k_B T) \) is the inverse temperature. The expression of the density response function within the two-fluid hydrodynamic theory was first derived by Hohenberg and Martin in their seminal work \([19]\) and takes the following form for an isotropic superfluid (such that \( \chi_{nn}(k, \omega) = \chi_{nn}(k, \omega) \)),

\[
\chi_{nn} = \frac{(nk^2/m) \left( \omega^2 - v^2 k^2 + iD_k k^2 \omega \right)}{\left( \omega^2 - c_1^2 k^2 + iD_1 k^2 \omega \right) \left( \omega^2 - c_2^2 k^2 + iD_2 k^2 \omega \right)}.
\]

Here, \( m \) is the mass of atoms,

\[
v^2 \equiv T \frac{s^2}{c_v} \frac{\rho_s}{\rho_n}
\]

is a velocity determined by the equilibrium entropy per unit mass \( s \equiv S/(N m) \), the specific heat per unit mass \( c_v \equiv T(\partial s/\partial T)_p \) and the superfluid and normal fluid mass densities, \( \rho_s \) and \( \rho_n \) (the total mass density is \( \rho = m \rho_s + \rho_n \)). \( c_1 \) and \( c_2 \) are the well-known exact first and second sound velocities that satisfy the relations,

\[
c_1^2 + c_2^2 = v^2 + v_s^2 ,
\]

\[
c_1^2 c_2^2 = v_s^2 v_T^2 = \frac{v_s^2}{\gamma} ,
\]

where \( v_s^2 = (\partial P/\partial \rho)_s \) and \( v_T^2 = (\partial P/\partial \rho)_T \) are the adiabatic and isothermal sound velocities, respectively. According to standard thermodynamic relations, the ratio of these two velocities is related to the ratio of two specific heats,

\[
\gamma \equiv \frac{c_p}{c_v} = \frac{v_s^2}{v_T^2} .
\]

Quite generally, \( v_s \) differs from \( v_T \) due to the finite thermal expansion of the system, implying that \( \gamma > 1 \). It will become clear later that this difference, as measured by the so-called Landau-Placzek (LP) ratio \( \epsilon_{LP} = \gamma - 1 \), determines the coupling strength between first and second sound. In Eq. (2), the damping rate or sound attenuation of the density response is characterized by three diffusion coefficients \( D_1, D_2 \) and \( D_s \), which are determined...
by solving [20],

\[
D_1 + D_2 = \frac{4}{3} \frac{\eta}{\rho} + \frac{\kappa}{\rho c_v},
\]

\[
\frac{c_d^2 D_2 + c_s^2 D_1}{v_s^2} = \frac{4}{3} \frac{\eta}{\rho} \left[ \frac{v_s^2}{\rho_s} - \frac{2}{\rho_s} \frac{(\partial P/\partial T)_s}{v_s^2} + \frac{\rho_s}{\rho_n} \right] + \frac{\kappa}{\rho c_v},
\]

\[
D_s = \frac{4}{3} \frac{\eta}{\rho} \frac{\rho_s}{\rho_n} + \frac{\kappa}{\rho c_v},
\]

where \( \kappa \) is the thermal conductivity and \( \eta \) is the shear viscosity. In the above expressions, we do not include the various second viscosities \( \zeta_i \) \((i = 1, 2, 3, 4)\), since for a unitary Fermi gas, it is known that only \( \zeta_3 \) can be nonzero but its value is too small to have sizable contribution [21][22].

Eq. (2) gives the density response for any weakly or strongly interacting superfluids that satisfy the Galilean invariance in the hydrodynamic regime. This universal description is very powerful, considering the lack of microscopic theoretical treatments in the strongly correlated regime. The only inputs to the theory are the knowledge on the equations of state, superfluid density and some transport coefficients like shear viscosity and thermal conductivity. For a strongly correlated unitary Fermi gas, all of them can be expressed in terms of some universal functions that depend on a single parameter \( T/T_F \).

where \( T/T_F \) is the reduced temperature, and as the first-order approximation, we assume that this expression is applicable at all temperatures.

In greater detail, we express the pressure of a unitary Fermi gas in terms of the universal energy function \( f_E(\vartheta) \equiv E/(NE_F) \), the entropy in terms of \( f_s(\vartheta) \equiv S/(Nk_B) \), and the shear viscosity in terms of \( f_\eta(\vartheta) \equiv \eta/(\hbar) \). These dimensionless universal functions can be directly read from the experimental data. For the thermal conductivity, we instead use the dimensionless Prandtl ratio [19],

\[
Pr(\vartheta) \equiv \frac{\eta c_v}{\kappa}.
\]

Using the universal relation \( P = 2E/(3V) \) [23][27], it is straightforward to obtain,

\[
v_s^2 = \left( \frac{\partial P}{\partial \rho} \right)_s = \frac{5}{9} f_E(\vartheta) v_F^2,
\]

\[
v_T^2 = \left( \frac{\partial P}{\partial T} \right)_T = \left[ \frac{5}{9} f_E(\vartheta) - \frac{2}{9} \vartheta f'_E(\vartheta) \right] v_F^2.
\]

where \( f'(\vartheta) \equiv df(\vartheta)/d\vartheta \) and \( v_F \) is the Fermi velocity. Therefore, \( \gamma \) and the LP ratio \( \epsilon_{LP} \equiv \gamma - 1 \) are given by,

\[
\gamma = \frac{v_T^2}{v_s^2} = 1 - 2 \frac{\vartheta f'_E(\vartheta)}{f_E(\vartheta)},
\]

\[
\epsilon_{LP} = \frac{\vartheta f'_E(\vartheta)}{(5/2) f_E(\vartheta) - \vartheta f'_E(\vartheta)}.
\]

Using \( s = (k_B/m)f_s(\vartheta) \), the specific heat \( c_v \) takes the form,

\[
c_v = \frac{k_B}{m} \vartheta f'_s(\vartheta).
\]

We then obtain,

\[
v_s^2 = \frac{1}{2} \frac{n_s f_s^2(\vartheta)}{n_n f_n^2(\vartheta)} v_F^2,
\]

where \( n_s \equiv \rho_s/\rho \) and \( n_n \equiv \rho_n/\rho = 1 - n_s \) are the superfluid fraction and normal fluid fraction, respectively.

By plugging the expressions [12][17] into Eqs. [4] and [5] and solving the coupled equations, we obtain the first and second sound velocities of a unitary Fermi gas, as reported in Fig. [1] together with the LP ratio in the inset. In the previous work, the two speeds of sounds were already calculated [15][16], based on the GPF equations of state [24] and the GPF superfluid density [25][26]. The current update in Fig. [1] improve the results near the transition temperature \( T_c \approx 0.167T_F \), because of the use of more accurate equations of state. In particular, close to \( T_c \) the improved LP ratio is significantly smaller than the old result (see Fig. 3 in Ref. [19]) and exhibits an

A. The sound velocities and diffusion coefficients

In this work, we use the MIT equations of state [10] and the NCSU shear viscosity [17][18]. For the superfluid density, unless specified otherwise we adopt the prediction from a gaussian pair fluctuation (GPF) theory [24][26], which agrees well with the Innsbruck data (see the inset of Fig. [10]). The thermal conductivity of a unitary Fermi gas, on the other hand, is less investigated both experimentally and theoretically. We take the known expression of the thermal conductivity at high temperature [19],

\[
\frac{\kappa}{\hbar} = \frac{k_B}{m} \frac{675 \sqrt{2 \pi^{3/2}}}{512} \vartheta^{3/2},
\]

where \( \vartheta \equiv T/T_F \), because of the use of more accurate equations of state. In particular, close to \( T_c \) the improved LP ratio is significantly smaller than the old result (see Fig. 3 in Ref. [19]) and exhibits an

where \( \vartheta \equiv T/T_F \) is the reduced temperature, and as the first-order approximation, we assume that this expression is applicable at all temperatures.
interesting peak structure at $T \sim 0.15 T_F \simeq 0.9 T_c$. Due to the smallness of the LP ratio $\epsilon_{\text{LP}}$, the sound velocities are well approximated by the expansion in terms of $\epsilon_{\text{LP}}$

$$c_1^2 = v_s^2 (1 + \epsilon_{\text{LP}} x + \cdots),$$  
(18)

$$c_2^2 = \frac{v^2}{\gamma} (1 - \epsilon_{\text{LP}} x + \cdots),$$  
(19)

where

$$x \equiv \frac{v^2}{\gamma v_s^2}.$$  
(20)

As shown in Fig. 1, the approximate sound velocities $c_1 = v_s$ and $c_2 = v/\sqrt{\gamma}$ are almost identical to the exact results.

To calculate the diffusion coefficients $D_1$, $D_2$ and $D_s$, we note that the quantity $(\partial P/\partial T)_{\rho}$ can be obtained by using the identity,

$$\left(\frac{\partial P}{\partial \rho}\right)_s = \left(\frac{\partial P}{\partial \rho}\right)_T + \left(\frac{\partial P}{\partial T}\right)_\rho \left(\frac{\partial T}{\partial \rho}\right)_s.$$  
(21)

For a unitary Fermi gas, a fixed entropy per unit mass $s$ means $T/\rho^{2/3}$ is a constant and hence

$$\left(\frac{\partial T}{\partial \rho}\right)_s = \frac{2 T}{3 \rho}.$$  
(22)

Thus, it is easy to find that,

$$\frac{v^2}{v_s^2} \left(\frac{\partial P}{\partial \rho}\right)_s = \frac{3}{2} \left(\frac{v^2}{v_s^2} - \frac{v^2}{\rho^2}\right) \left(\frac{\partial T}{\partial \rho}\right)_s = \frac{n_s 3 \epsilon_{\text{LP}} f_s}{\gamma}.$$  
(23)

By substituting this expression into the right hand side of Eq. (8) and defining the two dimensionless variables,

$$A = f_\gamma \left[\frac{4}{3} n_s + \gamma \frac{1}{\Pr}\right],$$  
(24)

$$B = f_\gamma \left[\frac{4}{3} x \gamma + \frac{n_s}{n} \left(1 - \frac{3 \epsilon_{\text{LP}} f_s}{\gamma} \frac{1}{\partial f_s}\right) \right].$$  
(25)

we then obtain,

$$D_1 = \frac{\hbar}{m} \left(\frac{A c_1^2 - B v_s^2}{c_1^2 - c_s^2}\right),$$  
(26)

$$D_2 = \frac{\hbar}{m} \left(\frac{B v^2_s - A c_2^2}{c_1^2 - c_2^2}\right).$$  
(27)

The remaining diffusion coefficient $D_s$ is given by,

$$D_s = \frac{\hbar}{m} f_\gamma \left[\frac{4}{3} n_s + \gamma \frac{1}{\Pr}\right].$$  
(28)

In Figs. 2a and 2b, we show the universal function $f_\gamma(\delta)$ and the Prandtl number $\Pr(\delta)$, respectively, by

![Figure 1](image-url)  
FIG. 1. (color online). First and second sound velocities of a unitary Fermi gas as a function of the reduced temperature $\theta = T/T_F$. The blue dash-dotted line and the red dashed line are the approximate results $c_1 = v_s$ and $c_2 = v/\sqrt{\gamma}$, which are nearly indistinguishable with the exact results (black curves) due to the small LP ratio. The latter is shown in the inset. Here, we adopt the MIT equations of state [10] and the GPF superfluid density [26]. In this figure, and some figures below, the (vertical) thin blue dot-dashed line indicates the superfluid transition temperature $T_c \simeq 0.167 T_F$.

![Figure 2](image-url)  
FIG. 2. (color online). (a) Temperature dependence of the shear viscosity of a unitary Fermi gas measured at NCSU [17]. Above the superfluid transition, we use the refined experimental data by Bluhm, Hou, and Schäfer [18]. The blue dashed line shows the shear viscosity in the deep superfluid phase, contributed by the dominant four-phonon process (see Eq. (19)). (b) Temperature dependence of the Prandtl number calculated by using the high-temperature approximate expression for thermal conductivity (see Eq. (10)). The red circles are the result of helium-II, with temperature rescaled according to $(T/T_F^{He}) \times 0.167 T_F$, where $T_F^{He} \sim 2.1768$ K is the transition temperature of helium-II. The arrow to the right shows the expected Prandtl number in the high-temperature limit, $\Pr = 2/3$. (c) The various diffusion coefficients as a function of the reduced temperature $\theta = T/T_F$. 

In Figs. 2a and 2b, we show the universal function $f_\gamma(\delta)$ and the Prandtl number $\Pr(\delta)$, respectively, by
using the NCSU shear viscosity data \[17, 18\] and the high-temperature approximation for thermal conductivity \[19\]. The resulting diffusion coefficients are reported in Fig. 2. Near the superfluid transition, all these coefficients are about $h/m$, indicating the strongly interacting nature of a unitary Fermi gas.

**B. The general structure of hydrodynamic dynamic structure factor**

Before we present the results of hydrodynamic dynamic structure factor, it is useful to briefly discuss its general behavior, which is well known in the literature \[15, 16, 20, 28, 29\]. In the superfluid phase, the LP ratio $\epsilon_{\text{LP}}$ is small, we may expand $D_1$ and $D_2$ in powers of $\epsilon_{\text{LP}}$ \[20\]:

\[
D_1 = \frac{4\eta}{3\rho} + O(\epsilon_{\text{LP}}),
\]

\[
D_2 = \frac{4\eta}{3\rho} \frac{\rho_s}{\rho_p} + \frac{\kappa}{\rho_p} + O(\epsilon_{\text{LP}}).
\]

At the leading order of $\epsilon_{\text{LP}}$, we obtain,

\[
\frac{\chi_{nn}}{nk^2/m} \simeq \frac{\rho c_s^2}{\omega^2 - (\epsilon_{n1}^2/k^2) + i\Gamma_1\omega} + \frac{\rho c_s^2}{\omega^2 - (\epsilon_{n2}^2/k^2) + i\Gamma_2\omega},
\]

where $\epsilon_{n1} = (\epsilon_{11}^2 - \omega^2)/(\epsilon_{12}^2 - \omega^2)$ and $\epsilon_{n2} = (\epsilon_{21}^2 - \omega^2)/(\epsilon_{22}^2 - \omega^2)$.

The corresponding hydrodynamic dynamic structure factor then takes the form,

\[
S(k, \omega) = \frac{k^2}{2m(1 - e^{-\beta\hbar\omega})} \left[ \frac{Z_1}{\omega^2 - (\epsilon_{11}^2/k^2) + (\Gamma_1/2)^2} + \frac{Z_2}{\omega^2 - (\epsilon_{12}^2/k^2) + (\Gamma_2/2)^2} \right] + O(\epsilon_{\text{LP}}),
\]

\[
\simeq \frac{k^2}{2m(1 - e^{-\beta\hbar\omega})} \omega \left[ \frac{Z_1}{\omega^2 - (\epsilon_{11}^2/k^2) + (\Gamma_1/2)^2} + \frac{Z_2}{\omega^2 - (\epsilon_{12}^2/k^2) + (\Gamma_2/2)^2} \right] + O(\epsilon_{\text{LP}}),
\]

\[
\omega S(k, \omega) \simeq \frac{h}{2m} k^2/\omega.
\]

It is straightforward to calculate the static structure factor $S(k) = h \int S(k, \omega) d\omega$. In the low-energy limit, where $h\omega \ll k_B T$, the static structure factor is approximated by,

\[
S(k) \simeq \frac{k_B T}{2m} \left[ \frac{Z_1}{c_1^2} + \frac{Z_2}{c_2^2} \right] \equiv S_1(k) + S_2(k).
\]

Thus, the relative weight of second and first sound in $S(k, \omega)$ is given by \[16\],

\[
\frac{S_2}{S_1} = \frac{Z_2/c_2^2}{Z_1/c_1^2} = \frac{\omega^2 - c_2^2}{\omega^2 - c_1^2} \simeq \frac{\epsilon_{\text{LP}}}{1 - \omega^2/c_s^2}.
\]

As the temperature increases across the superfluid transition, it seems that the second sound ceases to exist, as $Z_2$ becomes zero. This is not fully correct, as in Eq. \[33\] we neglect the terms at the order of $O(\epsilon_{\text{LP}})$. What actually happens is that the propagating second sound mode turns into a diffusive relaxation mode. To see this, we note that in the normal state \[20\],

\[
D_1 = \frac{4\eta}{3\rho} + \frac{\kappa}{\rho c_s^2} \epsilon_{\text{LP}},
\]

\[
D_2 = \frac{4\eta}{3\rho} \frac{\rho_s}{\rho_p} + \frac{\kappa}{\rho_p} + O(\epsilon_{\text{LP}}),
\]

and

\[
\chi_{nn} = \frac{nk^2/m}{\omega^2 - (\epsilon_{n1}^2/k^2) + i\Gamma_1\omega} + \chi_{nn}^{(2)},
\]

\[
\chi_{nn}^{(2)} = \frac{nk^2/m}{\omega^2 - (\epsilon_{n2}^2/k^2) + i\Gamma_2\omega}.
\]

It is clear that $\chi_{nn}^{(2)}(k, \omega)$ peaks at $\omega = 0$. Thus, we may approximate

\[
\chi_{nn}^{(2)} \simeq -\frac{n}{mc_s^2} \frac{i(D_s - D_2)k^2}{\omega + i\Gamma_2}.
\]

The corresponding dynamic structure factor $S^{(2)}(k, \omega) = -1/\pi n (1 - e^{-\beta\hbar\omega}) \text{Im} \chi_{nn}^{(2)}$ takes the form ($\omega \sim 0$),

\[
S^{(2)}(k, \omega) = \epsilon_{\text{LP}} \left( \frac{k_B T}{mc_s^2} \right) \frac{\Gamma_2/\pi}{\omega^2 + \Gamma_2^2},
\]

which describes a thermally diffusive mode of width $2\Gamma_2 = 2D_2k^2 = 2[k/(\rho c_s^2)]k^2$ \[28, 29\]. The factor of 2 in the width comes from the fact that the second sound doublet below $T_c$, each of width $\Gamma_2$, merges into a single central peak at $\omega = 0$. 

III. COLLISIONLESS VS HYDRO_DYNAMIC

In this section, we present an estimation of the viscous relaxation time (Fig. 2) and sketch out a sort of “phase diagram”, which at a given temperature determines the boundary between the collisionless and hydrodynamic regimes in the plane of the transferred momentum $k$ and the energy $\omega$ (Fig. 1).

We note that, for a trapped unitary Fermi gas, collective density oscillations such as the breathing mode have been thoroughly studied in the literature $^{30, 32}$, by using the hydrodynamic theory. For these oscillations, the effective momentum and energy are often much smaller than Fermi momentum and Fermi energy, and hence the hydrodynamic description is applicable without question $^{36}$. In our case, the characteristic momentum and energy of interest are about $0.1 \sim 1k_F$ and $0.1 \sim 1E_F$, respectively. The condition of using the two-fluid hydrodynamic theory then should be carefully examined.

A. Viscous relaxation time

In many cases, the boundary between the collisionless and hydrodynamic domains might be determined by finding the average lifetime $\tau$ of the elementary quasi-particles that make the dominant contribution to the thermodynamic and transport properties of the system considered. For a unitary Fermi gas, this is difficult as the picture of well-established quasi-particles may break down due to the inherent strong correlations. In this work, we are lured into considering that the viscous relaxation time related to the shear viscosity may be regarded as a characteristic relaxation time scale.

This consideration is inspired by the recent studies on the viscosity spectral function $\eta(\omega)$, which is found to exhibit a clear Drude peak of width $\hbar/\tau_n$ at zero frequency and a $\omega^{-1/2}$ tail at large $\omega$ $^{37}$, and satisfies an interesting shear viscosity sum-rule at unitarity $^{38}$,

\[
\frac{2}{\pi} \int_0^\infty d\omega \left[ \eta(\omega) - \frac{\hbar^3/2C}{15\pi\sqrt{m\omega}} \right] = P, \tag{45}
\]

where $C$ is Tan’s contact density and $P$ is the pressure. As a result, it was shown that in the normal state the viscosity spectral function assumes the following form $^{37}$,

\[
\eta(\omega) \simeq \frac{P\tau_n}{1 + (\omega\tau_n)^2} + \frac{\hbar^3/2C}{15\pi\sqrt{m\omega}} \omega\tau_n \frac{1 + \omega\tau_n}{1 + (\omega\tau_n)^2}, \tag{46}
\]

where the Drude weight in the first term on the right-hand-side of the expression has to be the pressure $P$, in order to fulfill the sum-rule. As $\omega \to 0$, we thus obtain immediately the useful viscosity-pressure relation,

\[
\tau_n = \frac{\eta(\omega \to 0)}{P} \equiv \frac{\eta}{P}. \tag{47}
\]

Indeed, this relation holds in the high-temperature limit, where quasi-particles are well-defined and hence $\tau_n$ can be unambiguously calculated using the kinetic theory $^{39}$. The argument presented here indicates that the relation is applicable down to the superfluid transition, near which quasi-particles may not be well-defined.

The viscosity-pressure relation can alternatively be understood from the Einstein relation $\eta = \rho_n D_n$ that was first derived by Hohenberg and Martin $^{20}$. Here, $\rho_n$ is the normal fluid density and $D_n \sim v_{c,ff}^2 / \tau_n$ is the diffusion coefficient related to shear viscosity. In the normal state, $\rho_n = \rho = mn$ and the effective velocity $v_{c,ff} \sim v_s$. By using the fact that $mv_s^2n \sim E/V \sim P$, we find the desired relation $\eta \sim \tau_n P$.

In the superfluid phase, the situation becomes complicated, since the viscosity spectral function near zero frequency may deviate the Drude form shown in Eq. (46), although the viscosity sum-rule should still be applicable. For simplicity, however, we assume that this derivation is small and continue to use the relation Eq. (17). It is also worth noting that, experimentally it becomes difficult to measure the shear viscosity in the low temperature regime (i.e., $T < 0.1T_F$). Theoretically, the only result of low-temperature shear viscosity relies on the conjecture that at that low temperature, phonons make the dominant contribution to thermodynamic and transport properties. The consideration of a four-phonons scattering process leads to a viscosity-entropy ratio $^{37} 40$,

\[
\frac{\eta_{ph}}{s_{ph}} = \frac{\hbar}{k_B} \times 10^{-5} \xi^{5/2} \varphi^{-8}, \tag{48}
\]
where $\xi \approx 0.376$ is the Bertsch parameter $^{[10]}$. Using the standard low-temperature expression $s_{ph} = (2\pi^2 k_B/45)(k_B T/hc)^3$ and $c_1(T = 0) = \sqrt{3}v_F$, we find that
\[
\frac{\eta_{ph}}{nh} \simeq 1.81 \times 10^{-4} \xi^{7/2} g^{-5},
\]
which is shown in Fig. 2 by a blue dashed line.

In Fig. 3, we report the crossover frequency $\omega_\eta \equiv \tau_\eta^{-1} = P/\eta$ as a function of temperature, calculated by using the MIT pressure equation of state and the NCSU data for shear viscosity (black solid line). Near zero temperature, where the NCSU data becomes very noisy, we adopt the phonon expression Eq. (49) for the shear viscosity and plot the result with a purple dashed line. The dynamics of the unitary Fermi gas is collisionless (hydrodynamic) when $\omega \gg \omega_\eta$ ($\omega \ll \omega_\eta$). It is not surprising that quite generally $\omega_\eta \sim O(\epsilon_\eta/h)$, since the unitary Fermi gas is strongly correlated and the Fermi energy sets the characteristic energy scale close to the superfluid transition. We observe a wide parameter window for the application of the two-fluid hydrodynamic theory near the transition. In particular, at low temperature, if the thermodynamics and dynamics of the unitary Fermi gas are dominated by phonon excitations (which is to be confirmed experimentally yet), we anticipate a crossover from the collisionless regime to the hydrodynamic regime at $T \sim 0.10T_F \simeq 0.6T_c$, analogous to superfluid $^4$He. For the latter, the crossover occurs at about $T \sim 0.8K \simeq 0.4T_c$ $^4$He $^{[3]}$. Of course, if the phonon assumption is not valid at low temperature, the hydrodynamic region may extend down to zero temperature.

To close this subsection, it is worth noting the second sound has been observed at Innsbruck at $T = 0.11 - 0.155T_F$ $^{[33]}$ under a sinusoidally modulation of repulsive laser beam at the trap center $^{[12]}$. The modulation frequency $\omega_2$ is about 1720 Hz or 0.092$T_F/h$, as shown in Fig. 3 by an empty star. The existence of the second sound implies that $\omega_2 \tau \ll 1$. Thus, we must have $\tau^{-1}(T = 0.135T_F) \gg \omega_2 \sim 0.1\epsilon_F/h$, where we approximate $T_F^{trap} \approx T_F$. Naïvely, we may estimate a low bound for the crossover frequency
\[
\omega_\eta(T = 0.135T_F) \simeq 3\omega_2 \sim 0.3\frac{\epsilon_F}{h}.
\]
This low bound is illustrated in Fig. 3 by a solid star. On the other hand, if we estimate the second sound velocity $c_2(T = 0.135T_F) \sim 0.08v_F$, we find that the wavevector of the experimentally excited second sound is typically about $k = \omega_2/c_2 \sim 0.5k_F$.

### B. Nature of sounds at two typical temperatures

Using the estimated crossover frequency $\omega_\eta = \tau_\eta^{-1}$, we may qualitatively determine the collisionless-hydrodynamic boundary as a function of the transferred momentum at a given temperature. This is sketched in Fig. 4 for two typical temperatures below and above the superfluid transition.

In the superfluid phase ($a$, $T = 0.15T_F$), the crossover frequency is large (i.e., $\omega_\eta \simeq 0.7\epsilon_F/h$), leading to a significant characteristic momentum $k_\eta \simeq 0.9k_F$. Typically we find two propagating sound modes at low momentum $k \ll k_\eta$. The hydrodynamic condition $\omega \ll \omega_\eta$ is always well maintained for the second sound, due to its small frequency. However, as the momentum increases, the rapidly increasing damping rate of the second sound finally turns it into a diffusive mode, before reaching $k_\eta$. For the first sound, it instead turns into a collisionless zero sound once $k > k_\eta$. Here, we anticipate a large sound damping due to the hydrodynamic to collisionless crossover, as sketched in Fig. 4 at around $k \sim k_\eta$ or $\omega \sim \omega_\eta$. By further increasing momentum, the collision-
less zero sound enters the two-particle continuum and is again damped via breaking Cooper pairs or scattering off fermionic quasi-particles.

In the normal phase (b, $T = 0.33 T_F$), the crossover frequency is also sizable (i.e., $\omega_0 \simeq 0.4 t_F / \hbar$). At this temperature, the second sound is already a thermal diffusive mode, as we discuss earlier. The first sound or the ordinary sound becomes the collisionless zero sound at $k \sim k \eta \simeq 0.4 t_F$.

IV. DYNAMIC STRUCTURE FACTOR AND SOUND ATTENUATION AT LOW MOMENTUM

We are now ready to present the hydrodynamic dynamic structure factor and understand the sound waves in the experimentally relevant parameter space. In the following, we first consider the sound attenuation measurement at MIT (where $k \leq 0.1 t_F$) and then the Bragg scattering experiment at SUT (where $k \sim 0.5 t_F$). At the end of this section, we finally discuss the momentum dependence of the dynamic structure factor slightly below the superfluid transition (with $T = 0.15 t_F$).

A. $k = 0.1 t_F$

In Fig. 5, we show the temperature evolution of the dynamic structure factor at $k = 0.1 t_F$. At such a small transferred momentum, we anticipate that the hydrodynamic condition may be well satisfied for temperature $T \geq 0.12 T_F$. In the previous study, the dynamic structure factor at the same transferred momentum was calculated, using the non-dissipative two-fluid hydrodynamic theory (see Fig. 4 in Ref. [10]) and a temperature-independent artificial spectral broadening $\Gamma = 0.002 t_F / \hbar$. Our updated results in Fig. 5 include the damping effect and therefore remove the uncertainty in theoretical predictions. The results can be directly compared with any experimental data once they are available.

In the superfluid phase ($T < 0.167 T_F$), the two sound modes are clearly visible. In particular, around zero frequency, we observe a second sound doublet. With increasing temperature towards the superfluid transition, the second sound becomes increasingly pronounced, as the height of the sound peak increases. The width of the second sound, however, is less dependent on temperature. As the width is roughly given by $D_2 k^2$, the temperature insensitivity of the width may be understood from the fact that $D_2$ does not vary too much with temperature, as can be seen from Fig. 2b. Above the critical temperature, the second sound doublet merges into a single broad Drude peak with width doubled (i.e., $2D_2 k^2$), as mentioned earlier.

The first sound, on the other hand, has a damping width that depends critically on the temperature, as highlighted in Fig. 6. The width follows closely the expression $D_1 k^2$ as we anticipate. Therefore, the temperature dependence of the width is a direct reflection of the temperature dependence of the diffusion coefficient $D_1$, which exhibits a sharp increase across the superfluid transition (see again Fig. 2a). As a result, the peak height of the first sound decreases rapidly across the critical temperature, as shown in Fig. 6.
At this momentum, from Fig. 4a we find that $\omega \tau_\eta \simeq 0.6 \sim 1$ for the first sound and $\omega \tau_\eta \simeq 0.1 \ll 1$ for the second sound at $T = 0.15T_F$. Thus, the hydrodynamic condition is marginally satisfied for the first sound and the relevant results should be considered as qualitatively reliable only. On the other hand, although the hydrodynamic condition is well fulfilled for the second sound due to its small sound velocity, the damping rate of $D_2k^2$ increases rapidly with $k$ and it may already be comparable to the frequency $\omega = \epsilon_2k$ at $k \sim 0.5k_F$. As a consequence, the second sound may barely be seen in the dynamic structure factor, although it was observed from the sound wave propagation experiment at similar wavevector as we have discussed at the end of Sec. IIIA. Indeed, around the zero frequency we do not find interesting feature at most temperatures, except at the temperature very close to the transition temperature (i.e., at $T = 0.16T_F$, the blue solid line), where a very broad shoulder is observed. This broad shoulder should be viewed as a remnant of the second sound.

We note that, the damping width for the first sound is also significant. Actually, it is so significant at $T > 0.14T_F$ that the width can not be described by the expression $D_1k^2$ any more (see Fig. 8), which is applicable only for small damping rates. The peak height of the first sound decreases rapidly across the superfluid transition (see Fig. 9), similar to what happens in the case of $k = 0.1k_F$.

C. Momentum dependence of first and second sounds close to the transition

It is encouraging to find theoretically a remnant of the second sound in dynamic structure factor slightly below the superfluid transition, at the transferred momentum $k$ as large as $0.5k_F$. However, we should bear in mind that, in Bragg scattering experiments, there is an additional source for the spectral width of the sound modes, the so-called instrumental broadening, due to the finite duration of the Bragg pulses. It is about $0.1k_F$ in the latest Bragg scattering experiment [13]. This additional broadening may completely wash out the signal of the broad shoulder near $\omega = 0$. Therefore, experimentally it may be preferable to take a smaller transferred momentum, although the small momentum may significantly reduce the density response and hence make experimental data much more noisy.

In Fig. 9 we show the hydrodynamic dynamic structure factor as a function of the transferred momentum at $T = 0.15T_F$. It turns out that $k = 0.3k_F$ could be an optimal choice for the transferred momentum. On one hand, the damping width of the second sound is reduced by a factor of $(5/3)^2 \sim 3$ and hence the second sound can manifest itself clearly in the dynamic structure factor (see the blue solid line). On the other hand, the hydrodynamic condition is improved, as $\omega \tau_\eta \sim 0.3$ becomes much smaller for the first sound, compared with that in

**FIG. 7.** (color online) Temperature dependence of the hydrodynamic dynamic structure factor at the transferred momentum $k = 0.5k_F$. We plot the dynamic structure factor at positive frequency only. The dynamic structure factor at negative frequency can be obtained by using the “detailed balance” relation, $S(k, -\omega) = e^{-\beta\omega}S(k, \omega)$ [1, 2]. The dynamic structure factor is measured in units of $E_F^{-1}$.

**FIG. 8.** (color online) (a) The damping width of the first sound as a function of temperature at $k = 0.5k_F$. The black solid line corresponds to the FWHM of the first sound mode that is numerically determined. The red dashed line is the anticipated damping rate $\Gamma = \Gamma_1 = D_1k^2$, in the limit of small damping $\Gamma \rightarrow 0$. (b) The peak value of the first sound (in units of $E_F^{-1}$) as a function of temperature at $k = 0.5k_F$.

**B. $k = 0.5k_F$**

We consider now a moderately large transferred momentum $k = 0.5k_F$, which is of relevance to the SUT experiment. The hydrodynamic dynamic structure factor is reported in Fig. 7 at several temperatures across the superfluid transition. The corresponding damping width and peak height of the first sound mode are shown in Figs. 8 and 9, respectively.
A thorough discussion on the theoretical predictions of the superfluid density in the unitary limit has been given in Ref. [26]. Naïvely, we anticipate that the GPF theory provides so far the best prediction. Indeed, it gives the superfluid density in the unitary limit has been given by the BCS mean-field theory. The resulting two sound velocities are shown in the main figure of Fig. 10.

It is apparent that the second sound velocity depends critically on the superfluid fraction. This is easy to understand, since the square of the second sound velocity can be accurately approximated by (see Eq. (19) and the red dashed line in the main figure of Fig. 11).

\[ c_2^2 \simeq T s^2 \frac{\rho_s/\rho}{c_p \left(1 - \rho_s/\rho\right)} \]  

As the entropy \( s \) and the heat capacity \( c_p \) have already been accurately determined at MIT [10], the measurement of the second sound velocity may provide a direct way to calibrate the superfluid fraction of a unitary Fermi gas [22].

The first sound velocity, on the other hand, is only weakly affected by the superfluid fraction via the coupling to the second sound. The weak dependence is clear from the approximate sound velocity in Eq. (18),

\[ c_1^2 \simeq v_s^2 + \epsilon_{LP} c_2^2 \simeq v_s^2 + \epsilon_{LP} T \frac{s^2}{c_p} \frac{\rho_s/\rho}{1 - \rho_s/\rho} \]  

In Fig. 11, we present the dependence of the hydrodynamic dynamic structure factor on the superfluid fraction, calculated with \( k = 0.1 k_F \) at \( T = 0.15 T_F \). By changing the superfluid fraction, the movement of the peak position of the two sound modes can be understood from the change in the sound velocities. The damping width of the two sounds becomes larger if we use a larger superfluid fraction (such as that of superfluid \( ^4\)He). In turn, it results in a smaller peak height.
B. Dependence on the thermal conductivity

To understand the dependence of hydrodynamic dynamic structure factor on the thermal conductivity, we consider a constant Prandtl number \( \Pr = 2/3 \), in addition to the high-temperature approximation for thermal conductivity that we have already used. The choice of the factor of 2/3 is inspired by the fact that the strongly interacting helium-II system has a Prandtl number \( \Pr \sim 2/3 \) at \( T_{c1} \leq T \leq 2T_{c1} \) (see, red circles in Fig. 11). We note also that in the high temperature limit the Prandtl number is exactly 2/3 [19]. Another extreme limit that we may choose is to simply set \( \Pr = \infty \). This is equivalent to considering a vanishingly small thermal conductivity, \( \kappa = 0 \).

The hydrodynamic dynamic structure factors with \( k = 0.3k_F \) and \( T = 0.15T_F \) and at different choices of thermal conductivity are reported in Fig. 12. We do observe a strong dependence of the dynamic structure factor on thermal conductivity. In particular, if we do not take into account the effect of the thermal conductivity (\( \kappa = 0 \)), the second sound peak becomes much narrower and higher. This may be understood from the approximate expression for the diffusion coefficient \( D_2 \) in the superfluid phase (see Eq. (30)). The first sound seems to be less sensitive to the thermal conductivity than the second sound, since the effect of thermal conductivity to the diffusion coefficient \( D_1 \) is weakened by a factor of the LP ratio \( \epsilon_{\text{LP}} \), according to Eq. (29).

C. Transport coefficients from sound attenuation

Ideally, if the damping width of the two sound modes, \( W_1 \) and \( W_2 \), can experimentally measured at low momentum, we may directly determine the shear viscosity and thermal conductivity of a unitary Fermi gas, based on the known superfluid density and equations of state.

In the normal phase, the first sound width is \( W_1 = D_1 k^2 \). The second sound is a thermally diffusive mode at \( \omega = 0 \) and its Drude width \( W_2 = 2D_2 k^2 \). Using Eq. (38) and Eq. (39), we find that,

\[
\frac{\eta}{\hbar} = \frac{3}{4} \frac{(W_1 - \epsilon_{\text{LP}} W_2/2)}{(\hbar k^2/m)},
\]

\[
\frac{\kappa}{\hbar} = \frac{1}{2} \frac{W_2}{(\hbar k^2/m) \epsilon_{\text{LP}}},
\]

In the superfluid phase, the situation is a bit complicated, due to the sound mode coupling. The damping widths of the first and second sound are given by \( W_1 = D_1 k^2 \) and \( W_2 = D_2 k^2 \), to the leading order of \( O(\epsilon_{\text{LP}}) \), respectively. From the measured \( W_1 \) and \( W_2 \), we may calculate the two dimensionless parameters \( A \) and \( B \) (see Eq. (7) and Eq. (8)),

\[
A = \frac{W_1 + W_2}{(\hbar k^2/m)},
\]

\[
B = \left( \frac{c_2}{v_s} \right)^2 \frac{W_1}{(\hbar k^2/m)} + \left( \frac{c_1}{v_s} \right)^2 \frac{W_2}{(\hbar k^2/m)}.
\]

The shear viscosity \( \eta_s = \eta/\hbar \) and the Prandtl number \( \Pr \equiv \eta c_p/\kappa \) can then be obtained, by solving the linearly coupled equations in Eq. (24) and (25).

VI. CONCLUSIONS

In summary, we have investigated the low-momentum dynamic structure factor of a homogeneous unitary Fermi
gas, from the viewpoint of a dissipative two-fluid hydrodynamic theory. To this aim, we have estimated the viscous relaxation time and have determined the characteristic crossover frequency that distinguishes the collisionless region and hydrodynamic region. Our estimation suggests that the dynamics of the unitary Fermi gas is well described by the hydrodynamic theory near the superfluid transition at the transferred momentum as low as \(0.5k_F\), where \(k_F\) is the Fermi wavevector.

By collecting the best knowledge on the superfluid density, shear viscosity and thermal conductivity, we have painted a general picture of the hydrodynamic density response and have discussed in detail the sensitive dependence of the two sound modes on the superfluid density and thermal conductivity. The condition for observing the second sound has been specifically addressed, in relation to the on-going experiments at Swinburne University of Technology.

We have shown that the measurements of the velocity and damping width of both first and second sound at sufficiently small momentum may lead to an accurate determination of the superfluid density, shear viscosity and thermal conductivity of a unitary Fermi gas. In this respect, the on-going experiment at Swinburne University of Technology, if the Bragg scattering experiment can be carried out with \(k \leq 0.3k_F\) and the second sound can be successfully observed, then we will have a very promising opportunity to improve the precision of the measured superfluid density and shear viscosity and to determine the thermal conductivity, which remains largely unknown both experimentally and theoretically.

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