Gaps between classes of matrix monotone functions

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1 Introduction

Almost seventy years have passed since K. Löwner [8] proposed the notion of operator monotone functions. A real, continuous function $f : I \to \mathbb{R}$ defined on an (non trivial) interval $I$ is said to be matrix monotone of order $n$ if

$$x \leq y \implies f(x) \leq f(y) \quad (1)$$

for any pair of self-adjoint $n \times n$ matrices $x, y$ with eigenvalues in $I$. We denote by $P_n(I)$ the set of such functions. A function $f : I \to \mathbb{R}$ is said to be operator monotone if it is matrix monotone of arbitrary orders. We evidently have $P_{n+1}(I) \subseteq P_n(I)$ for each natural number $n$, and

$$P(I) = \bigcap_{n=1}^{\infty} P_n(I)$$

is the set of operator monotone functions defined on $I$. If (1) holds for any pair of self-adjoint elements $x, y$ in a $C^*$-algebra $A$ with spectra contained in $I$, then we say that $f$ is $A$-monotone.

Löwner characterized the set of matrix monotone functions of order $n$ in terms of positivity of certain determinants (the so called Löwner determinants and the related Pick determinants) and proved that a function is operator monotone if and only if it allows an analytic continuation to a Pick function, that is an analytic function defined in the complex upper half plane.

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with non-negative imaginary part. Dobsch continued Löwner’s investigation and gave an alternative characterization of matrix monotonicity which we shall use in this paper.

Forty years after Löwner’s work W. Donoghue published a comprehensive book on the subject in which he refined Dobsch’ necessary and sufficient condition for a function on an interval to be matrix monotone of order $n$ ([3, Chapter 7, Theorem VI and Chapter 8, Theorem V]. Donoghue then asserted ([3, p. 84] that with this insight one may recognize that the classes $P_n(I)$ are all distinct for different values of $n$. We shall denote this as the (asserted) existence of gaps between the different classes of matrix monotone functions.

However, both Löwner’s and Dobsch’ conditions for matrix monotonicity of order $n$ are very hard to check even for $n = 3$, and explicit examples of functions showing such gaps are given by Donoghue only for $n = 1$ and $n = 2$. Now another almost thirty years have passed after Donoghue’s work and there are still, to our knowledge, no examples in the literature showing the gaps between $P_n(I)$ and $P_{n+1}(I)$ for arbitrary natural numbers $n$. The purpose of this article is to prove exactly the existence of such gaps for every $n$. We also characterize, for any natural number $n$, the C*-algebras $A$ with the property that any function $f \in P_n(I)$ is $A$-monotone. It is interesting to notice that this question is closely connected to the problem of matricial structure of operator algebras with respect to positive linear maps.

2 The gap between $P_n(I)$ and $P_{n+1}(I)$

For a positive integer $n$ let $g_n(t)$ be the polynomial defined by

$$g_n(t) = t + \frac{1}{3}t^3 + \cdots + \frac{1}{2n-1}t^{2n-1}.$$  \hspace{1cm} (2)

Following the notations in ([3) we consider the matrix valued function associated with $g_n(t)$ and given by

$$M_n(g_n; t) = \left( \frac{g_n^{(i+j-1)}(t)}{(i+j-1)!} \right)_{i,j=1}^n.$$  \hspace{1cm} (2)

The following lemma is an application of standard arguments from the theory of moment problems for Hankel matrices.

**Lemma 1** The matrix $M_n(g_n; 0)$ is positive definite.
Proof. We set
\[ b_k = \frac{1}{2} \int_{-1}^{1} t^k dt \quad \text{for} \quad k = 0, 1, 2, \ldots \]
and calculate
\[ b_k = \begin{cases} \frac{1}{k+1} & \text{if} \ k \ \text{is even}, \\ 0 & \text{if} \ k \ \text{is odd}. \end{cases} \]
Hence, we can write \( g_n \) as
\[ g_n(t) = b_0 t + b_1 t^2 + \cdots + b_{2n-2} t^{2n-1} \]
and therefore obtain
\[ g_n^{(i+j-1)}(0) = (i+j-1)! \cdot b_{i+j-2} \quad i, j = 1, \ldots, n. \]
Consequently
\[ M_n(g_n; 0) = \left( b_{i+j-2} \right)_{i,j=1}^{n}. \]
Now take a vector \( c = (c_1, c_2, \ldots, c_n) \in \mathbb{C}^n \) and calculate
\[
\left( M_n(g_n; 0) c \mid c \right) = \sum_{i=1}^{n} \sum_{j=1}^{n} b_{i+j-2} c_j \overline{c_i} \\
= \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \int_{-1}^{1} t^{i+j-2} c_j \overline{c_i} dt \\
= \frac{1}{2} \int_{-1}^{1} \left| \sum_{i=1}^{n} c_i t^{i-1} \right|^2 dt.
\]
It follows that the matrix \( M_n(g_n; 0) \) is positive semidefinite. Moreover, if \( M_n(g_n; 0)c = 0 \) we see that
\[ \sum_{i=1}^{n} c_i t^{i-1} = 0 \quad \text{a.e.} \]
Since this is a polynomial, it is identically zero on the interval \([-1, 1]\). All entries of the vector \( c \) are therefore zero and the matrix \( M_n(g_n; 0) \) is positive definite. QED

With this lemma we can show the existence of a gap between \( P_n(I) \) and \( P_{n+1}(I) \) for any positive integer \( n \) and any nontrivial interval \( I \) different from the whole real line.
**Theorem 2** For any natural number \( n \) there exists a real number \( \alpha_n > 0 \) and a function \( g_n : [0, \alpha_n] \to \mathbb{R} \) such that

1. \( g_n \) is matrix monotone of order \( n \) on \([0, \alpha_n]\).
2. \( g_n \) is not matrix monotone of order \( n + 1 \) on \([0, \alpha_n]\), nor is it matrix monotone of order \( n + 1 \) on any subinterval.

**Proof.** Consider the polynomial \( g_n(t) \) introduced in the proof of Lemma 1. By the continuous dependence of eigenvalues of matrices as a function of their entries, there exists by Lemma 1 a positive number \( \alpha_n \) such that the matrix function \( M_n(g_n; t) \) is positive definite for \( t \in [0, \alpha_n] \). Since \( g_n(t) = \frac{2n - 3}{2n - 1} + (n - 1)t^2 \) is positive and convex on \([0, \alpha_n]\) we conclude, cf. [3, Chap. 8, Theorem V], that the function \( g_n \) is matrix monotone of order \( n \) on the interval \([0, \alpha_n]\).

The last principal matrix of order 3 of the matrix \( M_{n+1}(g_n; t) \) is given by

\[
\begin{pmatrix}
\frac{1}{2n - 3} + (n - 1)t^2 & t & \frac{1}{2n - 1} \\
t & \frac{1}{2n - 1} & 0 \\
\frac{1}{2n - 1} & 0 & 0
\end{pmatrix}
\]

and this matrix has determinant \(-(2n - 1)^{-3}\) regardless of the value of \( t \). The matrix \( M_{n+1}(g_n; t) \) is thus not positive semi-definite and the function \( g_n \) is not matrix monotone of order \( n + 1 \) on any subinterval \( J \subseteq [0, \alpha_n] \). This completes the proof. QED

Consider the concrete function \( g_n \) defined in equation 2. A calculation shows that the largest possible value of \( \alpha_2 \) is 1. It is exceedingly difficult to calculate the largest possible value for \( n \geq 3 \).

**Proposition 3** Let either \( I = [a, b] \) or \( I = [a, \infty[ \) for real numbers \( a < b \) and take \( \alpha > 0 \). Then there exists a bijection \( h : [0, \alpha[ \to I \) such that both \( h \) and the inverse map are operator monotone. Likewise, with \( J = ]a, b[ \) or \( J = ]-\infty, b] \), there exists a bijection \( g : ]0, \alpha] \to J \) such that both \( g \) and the inverse map are operator monotone.

**Proof.** An affine map of the form \( t \to ct + d \) with \( c > 0 \) is operator monotone, and so is the inverse map \( t \to c^{-1}(t-d) \). We may therefore assume that \( \alpha = 1 \), \( I = [0, 1[ \) or \( I = [0, \infty[ \), and \( J = ]0, 1] \) or \( J = ]-\infty, 0] \). The function

\[
h(t) = t(1+t)^{-1} \quad \text{with inverse} \quad h^{-1}(t) = t(1-t)^{-1}
\]
is a bijection of $[0, \infty] \to [0, 1]$. Likewise is the function

$$g(t) = (1 - t)^{-1} \quad \text{with inverse} \quad h^{-1}(t) = 1 - t^{-1}$$

a bijection of $[\infty, 0]$ to $[0, 1]$. The assertion follows since $h, h^{-1}, g, g^{-1}$ are all operator monotone functions, cf. [1, 6]. QED

Notice that we cannot find a bijection $h : [0, 1] \to \mathbb{R}$ such that both $h$ and $h^{-1}$ are operator monotone. An operator monotone function defined on the whole real line is necessarily affine, cf. [3]. Its range is therefore either a constant or the whole real line.

**Corollary 4** Let $I = [a, b]$ or $I = [a, \infty[$ for real numbers $a < b$. For any natural number $n$ there exists a function $f_n : I \to \mathbb{R}$ such that

1. $f_n$ is matrix monotone of order $n$ on $I$.

2. $f_n$ is not matrix monotone of order $n+1$ on $I$, nor is it matrix monotone of order $n + 1$ on any subinterval.

Let $I$ be any open real interval and take $t_0 \in I$. Bendat and Sherman proved in [1, Theorem 3.2] that a function $f : I \to \mathbb{R}$ is matrix convex of order $n$, if and only if the function

$$F(t) = \frac{f(t) - f(t_0)}{t - t_0}$$

is matrix monotone of order $n$. Notice that $f$, for $n \geq 2$, automatically is differentiable and $F(t_0) = f'(t_0)$. One may set $F(t_0) = (f(t_0)_+ + f(t_0)_-)/2$ for $n = 1$. We also have $f(t) = f(t_0) + F(t)(t - t_0)$.

**Corollary 5** Let $I = [a, b]$ or $I = [a, \infty[$ for real numbers $a < b$. For any natural number $n$ there exists a function $f_n : I \to \mathbb{R}$ such that

1. $f_n$ is matrix convex of order $n$ on $I$.

2. $f_n$ is not matrix convex of order $n+1$ on $I$, nor is it matrix convex of order $n + 1$ on any subinterval.

The statement follows by combining Bendat and Sherman’s result with Corollary [4].
3 Characterization of $C^*$-algebras in terms of matrix monotone functions

As we have discussed in [7], we may regard the question of monotonicity of functions as a kind of nonlinear version of the problem of matricial structure of operator algebras. Recall that a positive linear map $\tau$ from a $C^*$-algebra $A$ to a $C^*$-algebra $B$ is said to be $n$-positive if the map

$$\tau_n : (a_{ij})_{i,j=1}^n \rightarrow (\tau(a_{ij}))_{i,j=1}^n$$

is a positive map from $M_n(A)$ to $M_n(B)$. If $\tau$ is $n$-positive for all positive integers, then it is said to be completely positive.

Although the introduction of these notions by Stinespring [9] is of a much later date than the work of Löwner, they have turned out to be very important notions for the matricial structure of operator algebras i.e. $C^*$-algebras and von Neumann algebras. One may simply recognize this aspect by the recent publication [4] by Effros and Ruan. Meanwhile examples of $n$-positive maps which are not $(n+1)$-positive had been investigated, and it had been discussed for which types of $C^*$-algebras $A$ every $n$-positive map from or to $A$ for an another $C^*$-algebra $B$ is also $(n+1)$-positive. In this sense, gaps between $P_{n+1}(I)$ and $P_n(I)$ are nonlinear versions of the above sort of problems. We are thus naturally led to the problem of the characterization of those $C^*$-algebras $A$ on which every matrix monotone functions of order $n$ is $A$-monotone. The following theorem is a generalization of a previous result [1], Theorem 1] where the two last authors essentially treated the gap between $P_1(I)$ and $P_2(I)$. In this investigation we reach the same kind of $C^*$-algebras as in the study of positive linear maps by the third author [11].

**Theorem 6** Let $A$ be a $C^*$-algebra, and let $I$ be an interval of the form $I = [a, b]$ or $I = [0, \infty]$ for real numbers $a < b$. The following assertions are equivalent:

1. Every matrix monotone function of order $n$ defined on $I$ is $A$-monotone.

2. The dimension of every irreducible representation of $A$ is less or equal to $n$.

3. Every $n$-positive linear map from/to $A$ for another $C^*$-algebra $B$ is completely positive.

**Proof.** (1) $\Rightarrow$ (2): We first notice that we, without loss of generality, may choose the interval $I = [0, \infty]$. Suppose that $A$ had an irreducible representation $\pi$ on a Hilbert space $H$ whose dimension is greater than $n$. Take an
(n + 1)-dimensional projector $e$ in $H$. We then have $\pi(A)e = B(H)e$ by [11, Theorem 4.18], hence

$$e\pi(A)e = eB(H)e = B(eH) \cong M_{n+1}.$$  

Let $B$ be the $C^*$-subalgebra of $A$ defined by setting

$$B = \{ a \in A \mid \pi(a)e = e\pi(a) \}.$$  

By the theorem cited above, the restriction of $\pi$ to $B$ is a $*$-homomorphism onto $eB(H)e$. We choose a function $f$ in $P_n(I)$ which is not matrix monotone of order $n + 1$, cf. Corollary [11]. Let $c$ and $d$ be arbitrary positive elements in $eB(H)e$ with $c \leq d$. It is easily verified that we can find positive elements $a$ and $b$ in $B$ such that $a \leq b$, $\pi(a) = c$ and $\pi(b) = d$. Since $a \leq b$ we obtain $f(a) \leq f(b)$ by the assumption, whence

$$f(c) = f(\pi(a)) = \pi(f(a)) \leq \pi(f(b)) = f(\pi(b)) = f(d).$$  

Therefore, $f$ is matrix monotone of order $n + 1$ on $I$, a contradiction.

(2) $\Rightarrow$ (1) : Take a function $f$ in $P_n(I)$ and let $a$ and $b$ be self-adjoint elements in $A$ with spectra contained in $I$ such that $a \leq b$. We consider an irreducible representation $\pi$ of $A$. Since also the spectra of $\pi(a)$ and $\pi(b)$ are contained in $I$, we obtain by the assumptions that

$$\pi(f(a)) = f(\pi(a)) \leq f(\pi(b)) = \pi(f(a)).$$  

It follows that $f(a) \leq f(b)$, thus $f$ is $A$-monotone.

(2) $\Leftrightarrow$ (3) : The assertion is proved in [11]. QED

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