Feshbach resonance induced shock waves in Bose-Einstein condensates

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We propose a method for generating shock waves in Bose-Einstein condensates by rapidly increasing the value of the nonlinear coefficient using Feshbach resonances. We show that in a cigar-shaped condensate there exist primary (transverse) and secondary (longitudinal) shock waves. We analyze how the shocks are generated in multidimensional scenarios and describe the phenomenology associated with the phenomenon.

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Shock waves have been thoroughly investigated during the last century in several branches of physics including fluid mechanics, plasma physics, astrophysics, optics and solid state physics (see e.g. [1]). Very recently it has been recognized that there exists one more type of media – ultracold dilute alkali gases in a condensed state, i.e. Bose-Einstein condensates (BEC’s) – where existence of shock waves is possible [2, 3, 4, 5, 6].

In general, BEC’s represent a universal laboratory for nonlinear phenomena. In the last few years there have been theoretical predictions and experimental realizations of many nonlinear structures in BEC’s such as vortices, vortex lattices and vortex rings [7], bright [8] and dark [9] solitons, collapsing waves [10] and stabilized solitons [11], to cite a few examples.

Theoretical studies of shock waves in BEC’s so far have been restricted to effectively one-dimensional (1D) models and explored regimes of free expansion of the condensate. Moreover, the initial conditions considered in some of the previous studies seem to be not experimentally feasible. In this Letter we propose and investigate a new mechanism for the generation of matter shock waves in BEC’s which could be easy to implement experimentally. As a key ingredient of our analysis we keep the multidimensional character of the condensate, something which is missing in previous works [2, 3, 4, 5, 6].

Estimates for shock wave generation.- In order a shock could develop from an initially smooth pulse, the quantum pressure must be negligible at the initial stages of evolution. In this situation the hydrodynamical approach (see e.g. [12]) holds and exploiting the analogy one can expect the appearance of a breaking point and the subsequent development of a shock wave in a BEC. In other words, for the breaking point to be reached one has to require $|(\nabla^2 \sqrt{\rho})/\sqrt{\rho}| \ll 8\pi a_s n$ where $n$ is the density of the condensate and $a_s$ is the $s$-wave scattering length (considered positive hereafter). The quantum pressure can be estimated as $1/L_0^2$ where $L_0$ is a characteristic scale of the condensate to be estimated as the minimal scale between all the characteristic spatial scales of the order parameter and its first derivative. This scale must be larger than the healing length, $\xi = (8\pi n a_s)^{-1/2}$. The hydrodynamic approximation and thus the previous estimates fail when discontinuities appear since when a shock starts to develop $L_0 \rightarrow 0$ because of growth of the first derivative near the edge of the shock (although the typical size of the condensate is preserved). Due to the role of the kinetic energy near the shock edge one should not expect formation of real discontinuities (like ones analyzed, for example in [4]) in a matter-wave system. This is similar to what it happens in real fluids where the existence of dissipation smoothes out the shock waves. When the characteristic size of the wave front becomes of order of the healing length, thus making the dispersive term relevant, we expect the appearance of oscillations of the wave front near the quasi-discontinuity. This behavior, which differs from the fluid case, is due to the essential differences between the regularization mechanisms of fluids (diffusion) and matter waves (dispersion).

The condition $L_0^2 a_s n \gg 1$ is not sufficient for the generation of shock waves since there are two essential factors not accounted for by this estimate. First, the density, and thus the sound velocity rapidly decrease during the condensate expansion. Second, the largest spatial derivative of the condensate is not achieved where the sound velocity has its maximum. As an example in the well known Thomas-Fermi approximation this point is reached at the minimum of the sound velocity. This requires special, with finite support [2] or non-smooth [4], initial states of the condensate, in order to produce shock waves.

Our shock-wave generation method is based on the use of Feshbach resonances (FR) [13] to control the nonlinear interactions. The idea is to make $a_s$ increase with time thus increasing the time domain in which the quantum pressure is negligible. In order to make qualitative estimates of the effects expected in this scenario we assume that initially the product $a_s n_{\text{max}}$ is negligible so that the initial spatial distribution of the atoms is well approx-
be identified as the healing length. For simplicity, we consider a cylindrically symmetric trap, with transversal and longitudinal frequencies $\omega_\perp$ and $\omega_\parallel$, respectively. Then the characteristic spatial scales of the problem can be identified as the healing length $\xi$, and the transverse $L_\perp = \sqrt{\hbar/m\omega_\perp}$ and longitudinal $L_\parallel = \sqrt{\hbar/m\omega_\parallel}$ linear oscillator lengths.

As the s-wave scattering length is ramped up to a larger value we get an expansion of the initial state with a velocity gradient strengthened by the parabolic potential. If the characteristic time, $\tau_0$, of the growth of $a_s$ is small enough and the increase of $a_s$ is large enough then at the initial stages one can consider free expansion of the condensate. In our case there are two characteristic velocities of the expansion \[ t = \frac{\hbar}{2mL_\alpha} \] were $\alpha$ stands for $\perp$ or $\parallel$ depending which direction is considered. There are four characteristic times in this problem: the two breaking times $t_\alpha = L_\alpha/c_{\max}$ (i.e. the times at which the breaking would occur if expansion would be absent) and the expansion times $T_\alpha = L_\alpha/v_\alpha$. We notice that these estimates involve $c_{\max}$ which is the maximal sound velocity after the growth of the scattering length is terminated. In order the breaking point to be reached experimentally in a given direction one should have $t_\alpha \ll T_\alpha$ what can be rewritten as the condition $R_\alpha = L_\alpha^2 a_s n_{\max} \gg 1$ where $n_{\max}$ is the BEC’s density in the center of the trap after the increase of the scattering length.\[ t_\alpha = \frac{\hbar}{2mL_\alpha} \]

At this point it is important to emphasize that due to existence of the trapping potential only the regions of large density undergo a quasi-free expansion while the expansion of the low-density regions located near to the periphery of the atomic cloud is suppressed by the potential. This is in a sharp contrast to the truly free expansion considered elsewhere.

In what follows we choose $a_s = 1$ without loss of generality and study the dependence of the formation of shock waves on the parameters $T_0, \tau_0, \nu$, and grid parameters $\Delta t = 0.001, N_x = 2048, N_z = 1024$ on $(x, y) \in [-30, 30] \times [-60, 60]$.

The adimensional quantities used are related with the physical spatial $(r_j)$ and temporal $(\tau)$ variables through $x_j = r_j/L_\perp$, $t = \omega_\perp \tau$. The new wavefunction $\psi$ is related to the physical one $\Psi$ by $\psi(x, t) = \Psi(r, \tau)\sqrt{L_\perp^3/N}$, where $N$ is the number of particles in the condensate. Finally $U = 4\pi Na_s/L_\perp$, $a_s$ being the scattering length of the cold collisions within the condensate and $\nu = \omega_\parallel/\omega_\perp$. The normalization for $\psi$ is $\int |\psi|^2 d^3x = 1$.

We ramp up the scattering length according to

\[
U(t) = \begin{cases} 
\alpha \left( e^{t/\tau_0} - 1 \right), & t < T_0, \\
\alpha \left( e^{t/\tau_0} - 1 \right), & t \geq T_0.
\end{cases}
\]

In what follows we choose $\alpha = 1$ without loss of generality and study the dependence of the formation of shock waves on the parameters $T_0, \tau_0, \nu$.\[ S_\perp = \frac{\partial^2 \psi}{\partial x^2} \]

FIG. 1: (a,b) Initial ($t = 0$, thin lines) and final ($t = t_f$, thick lines) profiles of a quasi-1D condensate for $\nu = 0.05$ ($t_f = 0.45$) and $\nu = 1$ ($t_f = 0.21$). (c) Dependence of the breaking time, $t_b$, of the 1D shock wave on the aspect ratio $\nu$ for $T_0 = 0.16$ and $\tau_0 = 0.02$.

FIG. 2: Results of 2D simulations for $\tau_0 = 0.02, \nu = 0.05$, and grid parameters $\Delta t = 0.001, N_x = 2048, N_z = 1024$ on $(x, y) \in [-30, 30] \times [-60, 60]$. (a) Short and (b) long time evolution of the derivatives $S_\perp(t) = \max_{x,z} |\partial \psi/\partial x|$. (c) Short and (d) long time evolution of the derivatives $S_\parallel(t) = \max_{x,z} |\partial \psi/\partial z|$. Dotted lines correspond to $T_0 = 0.16$ while solid lines are for $T_0 = 0.14$.\[ x \approx \frac{\hbar}{mL_\alpha} \]

\[ S_\parallel = \frac{\partial^2 \psi}{\partial z^2} \]
FIG. 3: Formation of the first shock along the $x$ direction for parameters $T_0 = 0.16, \tau_0 = 0.02, \nu = 0.05$. (a) $|\psi(x,z)|$ for $t = 0$ and $t = 0.26$ just after the first shock has formed. (b) Detail of the region $x \in [1,3]$ where the irregularities in which the shock degenerates are more evident. (c-d) Surface plots of $|\psi(x,z)|$ for (c) $t = 0$ and (d) $t = 0.26$. In all cases the arrows mark the location of the shock wave.

**One-dimensional shocks.** We first consider the cross-section of the condensate along one of the directions (radial or axial) as an effectively 1D case. By solving numerically the 1D version of the GPE we have found the dependence of the time of the shock wave formation $t_b$ on the size of the condensate characterized by the trap asymmetry. As it clear from Fig. 4 the shock waves develop faster in time in systems with smaller aspect ratio in agreement with our previous qualitative arguments.

**Two-dimensional shocks.** We have studied the formation of shock waves from gaussian ground states under variation of the parameters $\nu, \tau_0,$ and $T_0$. Results of typical numerical simulations of Eq. 11 are summarized in Figs. 2 and 3 corresponding to $T_0 = 0.14, \tau_0 = 0.02$ thus $U_{\text{max}} = U(T_0) \approx 1100$ [Fig. 2 solid lines] and $T_0 = 0.16, \tau_0 = 0.02, U_{\text{max}} \approx 3 \times 10^3$ [Fig. 3a,c] dotted lines and Figs. 3 and 4. The increase of the nonlinearity leads to an expansion of the condensate which is faster along the transverse direction due to the higher compression and thus the increased non-stationarity along this direction (notice that this is the case $t_{\perp} \ll t_{\parallel}$ qualitatively described above, see also 2). On long time scales the effect of the potential is to confine the wavepacket thus generating recurrent oscillations.

As it has been predicted above qualitatively and on the basis of the numerical simulations of the 1D model, the faster transversal expansion implies that the shock wave is first generated along the transverse direction and not along the longitudinal one as one could naively think because of the geometry of the condensate. The formation of this shock is seen as an increase of the maximum value of the spatial derivatives along $x$ [Fig. 2a]. This phenomenon is accompanied by a small sympathetic increase on the derivatives along $z$ which does not correspond to a true shock. The true longitudinal shock appears later due to the slower expansion along this direction [Fig. 2c].

The structures of the longitudinal and transverse shocks are shown in Figs. 3 and 4. It can be seen how the longitudinal shocks lead to a very oscillating behavior near the shock edge while for the transverse shocks the behavior is smoother, although small amplitude oscillations are also present.

For longer times and due to the effect of the trapping potential we observe the recurrent formation of very strong shocks (see the peaks in $S_x, S_z$ for $t \simeq \pi, 2\pi$ in Fig. 2). In Fig. 5 we observe that the wave becomes filamented due to the strongly compression induced by the combined effect of nonlinear forces and the harmonic potential, a phenomenon which could be used as a very efficient matter wave compressor.

The amplitude of the shocks increases as the nonlin-
crity is increased. In Fig. 6 we study the dependence of the shocks amplitude and breaking times as a function of the trap asymmetry $\nu$. It can be seen that as the trap becomes more symmetric the breaking time for the transverse and longitudinal shocks become more similar. This is a reasonable result since in the case $\nu = 1$ the shocks should develop at the same time along all the spatial directions starting from symmetric initial data. We have also studied the dependence of the shock formation on the rising time of the nonlinearity $\tau_0$ and seen that it plays an essential role in the transverse shock formation. For instance, taking $\tau_0 = 0.1$ and $T_0 = 0.70$, which give the same values of $U_{\text{max}}$ as those shown in Figs. 2(a) and 3 we have found that there are no transverse shocks (only those corresponding to the revival of the oscillation in the trap appear). This effect has a simple explanation: since $L_\perp \ll L_\parallel$, for a definite region of densities $n_{\text{max}}$ one may have $R_\parallel \gg R_\perp \sim 1$, what means that it is possible to reach the breaking point only longitudinally.

Three-dimensional condensates.- We have also studied the formation of shock waves in three-dimensional condensates using the same type of numerical methods on grids of up to $256 \times 256 \times 200$ points. With this resolution we have been able at least to detect the formation of the first shock waves. We have found features similar to those described for the two-dimensional case with the shocks first generating transversely and later longitudinally if the nonlinearity rises fast enough.

To conclude, we have proposed a mechanism for generating shock waves and described their dynamics. In doing so we have kept the multidimensional nature of the system, something which is missing in previous works and could lead to flawed results. We hope that our method could guide experimentalists in the field of matter waves to observe the formation of shock waves with Bose-Einstein condensates.

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