A comparative study of analytical solutions of space-time fractional hyperbolic-like equations with two reliable methods

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ABSTRACT
This paper deals with a comparative study of analytical solutions of \((n+1)\)-dimensional space-time fractional hyperbolic-like equations (with Caputo type fractional derivatives) using two reliable semi-analytical methods: “new integral projected differential transform method (NIPDTM)” and “fractional reduced differential transform method (FRDTM)”. Three test problems are carried out in order to illustrate the efficiency of these methods. The computed results are also compared with the results from various schemes, in the literature. The computed series solutions from either method converge to the exact solutions. FRDTM solutions are easy to compute without using any transformation as compared to NIPDTM, but FRDTM needs more iterations to obtain the solutions with the same accuracy.

1. Introduction

Fractional partial differential equations (FPDE) occur more frequently in different research fields. For instance, some of the applications have been identified in image processing, plasma physics, life sciences, electrochemistry, biological modelling, non-linear control theory, and astrophysics etc. (Atangana & Baleanu, 2016; Baleanu, Machado, & Luo, 2012; Carpinteri & Mainardi, 1997; El-Sayed, 1996; He, 1998, 1999; Herzallah, El-Sayed, & Baleanu, 2010; Jesus & Machado, 2008; Magin, 2006; Mainardi, 1971; Miller & Ross, 1993; Tarasov, 2008).

In FPDEs, many models are associated with ecology, water waves, vibrations of strings or membranes, propagation of sound waves and electromagnetic waves or transmissions of electric signals in cables are generally described as wave equations. Space and time fractional wave and heat equations appear mainly in conventional diffusion or wave equations, with fractional derivative of arbitrary order instead of conventional derivative (Mainardi, 1971). These space-time fractional wave equations have many applications such as: to investigate Brownian diffusion, unification of diffusion and propagation phenomena of a wave, sub-diffusion systems, and random walk (Agrawal, 2002; Andrezei, 2002; Klafter, Blumen, & Zumofen, 1984).

We consider multi-dimensional space-time fractional hyperbolic-like equation of the form

\[ D_{X}^\beta u(X,t) = f_1(X)D_{x_1}^\delta u(X,t) + f_2(X)D_{x_2}^\delta u(X,t) + \ldots + f_n(X)D_{x_n}^\delta u(X,t) + h(X,t), \]

\[ u(X,0) = \psi_0(X), \frac{\partial u(X,t)}{\partial t} \bigg|_{t=0} = \psi_1(X), \quad 1 < \alpha, \beta \leq 2, \]

where \(D_{X}^\beta u = \frac{\partial^\beta u}{\partial t^\beta}\) represents Caputo fractional derivative of \(u, u(X,t)\) is the density of the diffusing material at point \(X = (x_1, \ldots, x_n)\) at time \(t\), and \(f_1, \ldots, f_n\) are the diffusion coefficients for \(u\) at \(X, h\) is a smooth function. If \(h = 0\), diffusion coefficients are independent of the density (i.e. \(f_1 = f_2 = \ldots = f_n = \sigma^2\), a constant) and \(\beta = 2\), then Eq. (1) reduces to time-fractional wave equation, i.e. \(D_{t}^\alpha u = \sigma^2 \nabla^2 u\), and for constant diffusion coefficient, Eq. (1) reduces to classical wave equation \(u_{tt} = \sigma^2 \nabla^2 u\) when \(\alpha = 2\).

The fundamental solution in 1D was computed first in (Mainardi, 1996), later in (Hanyga, 2002) for a multi-dimensional case, and in a simpler form (Mentrelli & Pagnini, 2015). The hyperbolic-like model describes many physical problems in different fields of science and engineering such as earthquake stresses (Holliday, Rundle, Tiampo, Klein, & Donnellan, 2006) and non-homogeneous elastic waves in soils (Manolis & Rangelov, 2006).

In recent years, a great deal of effort has been expanded to develop techniques for the computation of the approximate solution behavior of the fractional differential equations, see (Singh & Kumar, 2016; Baleanu, Machado, & Khan, 2016; Tarasov, 2008).
2018, 2017a, 2016) and the references therein. Fu et al. studied numerically the space-fractional parabolic-like models by implementing the Kansa method (Pang, Chen, & Fu, 2015), the time-fractional parabolic-like models by implementing the Boundary particle method (Fu, Chen, & Yang, 2013) while constant- and variable-order time-fractional parabolic-like models were studied by implementing a domain-type meshless method (Fu, Chen, & Ling, 2015). Time-fractional parabolic-like and hyperbolic-like models have been studied by using many techniques, among them the homotopy perturbation techniques, among them the homotopy perturbation method (Momani, 2005; Wazwaz & Goruris, 2004), the variational iteration method (Shou & He, 2008), the homotopy perturbation method (Ozis & Agirseven, 2008), the reduced differential transforms method (Arshad, Lu, & Wang, 2017; Singh & Srivastava, 2015; Taghizadeh & Noori, 2017) while fractional parabolic-like (or hyperbolic-like) equations have been studied using the differential transforms method (Secer, 2012), the modified homotopy perturbation method (Jafari & Momani, 2007), the homotopy analysis method (Xu & Cang, 2008; Zhang, Zhao, Liu, & Tang, 2014), the variational iteration method (Mollq, Noorani, & Hashim, 2009; Yin, Song, & Cao, 2013), the modified homotopy analysis method (Yin, Kumar, & Kumar, 2015), the fractional homotopy analysis transforms method (Khader, Kumar, & Abbasbandy, 2016), the fractional variational iteration method with He’s polynomials (Tang, Wang, Wei, & Zhang, 2014), the homotopy decomposition method (HDM) (Atangana & Alabaraoye, 2013), the homotopy analysis fractional Sumudu transforms method (Pandey and Mishra, 2017), and the variable separation method (Zhang, Zhu, & Zhang, 2016).

In present paper my main goal is to compute approximate series solutions of (n + 1)D space-time fractional hyperbolic (STFH)-like equation with appropriate initial conditions using FRDTM (Arshad et al., 2017; Singh & Srivastava, 2015) and NIPDTM (Kunjan, Twinkle, & Kilicman, 2017).

2. Basic definitions and notations

The definitions of Caputo order fractional integration and differentiation and its properties as in (Miller & Ross, 1993; Singh and Kumar, 2017a) which are used throughout the paper are listed below:

((Miller & Ross, 1993)) Let \( \mu \in \mathbb{R} \) and \( m \in \mathbb{N} \). A function \( f : \mathbb{R}^+ \to \mathbb{R} \) belongs to \( C_{\mu} \) if there exists \( k \in \mathbb{R}, k > \mu \) and \( g \in C[0, \infty) \) such that \( f(x) = x^k g(x) \), \( \forall x \in \mathbb{R}^+ \). Moreover, \( f \in C_m^{\mu} \) if \( f^{(m)} \in C_{\mu} \). (\( C_{\mu} \))

Let \( f \in C_{\mu} \), then R-L fractional derivative, \( \mathcal{J}_{\alpha}^r f(t) \) \( (\alpha \geq 0) \), of \( f \) is defined by \( \mathcal{J}_{\alpha}^r f(t) = f(t) \) and

\[
\mathcal{J}_{\alpha}^r f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \quad \text{if} \ \alpha > 0,
\]

where \( \Gamma(\alpha) := \int_0^\infty e^{-t} t^{\alpha-1} dt \).

((Miller & Ross, 1993; Singh and Kumar, 2017a)) Let \( f \in C_{\mu} \), \( \mu \geq -1 \) and \( m-1 < \alpha \leq m, m \in \mathbb{N} \). Then \( \mathcal{D}^m_{\alpha} f(t) = \mathcal{J}_{\alpha-m}^{m-\alpha} D^m f(t) \).

Some basic properties of the operator \( \mathcal{D}^m_{\alpha} \) are as follows:

\( a) \ \mathcal{D}^m_{\alpha} \mathcal{J}_{\alpha}^r f(t) = f(t); \quad (b) \ \mathcal{D}^m_{\alpha} t^\nu = \frac{\Gamma(1+\nu)}{\Gamma(1+\nu-\alpha)} t^{\nu-\alpha}; \quad (c) \ \mathcal{D}^m_{\alpha} C = 0; \quad (d) \ \mathcal{J}_{\alpha}^r \mathcal{D}^m_{\alpha} f(t) = f(t) - \sum_{k=0}^{m-1} \frac{f^{(k)}(0^+)}{k!} t^k. \)

The Caputo fractional derivative deals with traditional initial and boundary conditions in the formulation of the physical problems, see (Atangana & Baleanu, 2016; Baleanu et al., 2012; Carpinteri & Mainardi, 1997; Mainardi, 1997; Miller & Ross, 1993).

2.1. Fractional reduced differential transform method

This section describes the basic properties of FRDTM (Arshad et al., 2017; Singh & Srivastava, 2015; Singh & Kumar, 2017).

Using the basic properties of 1-dim DTM, a function \( \psi \) of two variables with the separation property \( \psi(x, t) = f(x)g(t) \) can be embodied as

\[
\psi(x, t) = \sum_{l=0}^{\infty} F_l(t) t^l \sum_{j=0}^{\infty} G(j) x^j = \sum_{l=0}^{\infty} \Psi_l(x) t^l,
\]

where \( 0 < \alpha \leq 1 \), \( \Psi_l(x) := G(j) F_l(t) x^j \) is the spectrum of \( \psi(x, t) \). Throughout the paper \( \Psi_l(x) \) (upper case) is used for FRDT of function \( \psi(x, t) \) (lower case). Some basic properties of FRDTM are described in the following:

((Arshad et al., 2017; Singh & Srivastava, 2015; Singh & Kumar, 2017)) Let \( \psi(x, t) \) be an analytic and continuously differentiable, then

\( a) \ \text{FRDT or the spectrum of} \ \psi \ \text{is given by} \)

\[
\Psi_l^k(x) = \frac{1}{\Gamma(kx+1)} \left[ D_{t}^k (\psi(x, t)) \right]_{t=b}, \quad k = 0, 1, 2, \ldots
\]

where \( D_{t}^k \equiv \frac{d^k}{dt^k} \), \( x \) describes the order of time-fractional derivative.
(b) The inverse FRDT of $\Psi(x)$ is defined by
\[
\psi(x,t) = \lim_{m \to \infty} s_m = \sum_{k=0}^{m} \psi_k(x)(t - t_0)^k x^k,
\]
where $s_m = \sum s_{m-k} \Psi^k(x)(t - t_0)^k x^k$ denotes the $m$th order approximate solution.

In particular for $t_0 = 0$, we get $\psi(x,t) = \sum \Psi_k(x)x^k$.

Let $X = (x_1, x_2, ..., x_n)$ be a vector of $n$ variables. Some basis properties of FRDTM are listed below.

**Theorem 2.6.** (Arshad et al., 2017; Singh & Kumar, 2017) Let $\Psi(x)$ and $\Phi(x)$ be the spectrums of the analytic and continuously differentiable functions $\psi(x,t)$ and $\phi(x,t)$, respectively, and

1. If $\theta(x,t) = \ell_1 \psi(x,t) \pm \ell_2 \phi(x,t)$, then
   \[ \Theta(x) = \ell_1 \Psi(x) \pm \ell_2 \Phi(x). \]
2. If $\theta(x,t) = \psi(x,t) \psi(x,t)$, then $\Theta(x) = \sum \psi(x,t) \psi(-x) \psi(x,t)$.
3. If $\theta(x,t) = f(x,t)$, then $\Theta(x) = f(x) \Psi(x)$.
4. If $\theta(x,t) = x^n \psi(x,t)$, then $\Theta(x) = \left\{ \begin{array}{ll} x^n \psi_{x^n}(x) & \text{if } k \geq n \\ \psi_{x^n}(x) & \text{if } k < n \\ 0 & \text{else.} \end{array} \right.$
5. If $\theta(x,t) = D_x^n \psi(x,t)$, then $\Theta(x) = x^n \psi_{x^n}(x)$.
6. If $\theta(x,t) = D_x^n \psi(x,t)$, then $\Theta(x) = \frac{1}{\Gamma(1 + k + n)} D_x^n \Psi(x)$. In particular,
   a. If $\theta(x,t) = D_x^n \psi(x,t)$, then $\Theta(x) = D_x^n \Psi(x)$.
   b. If $\theta(x,t) = D_x^n \psi(x,t)$, then $\Theta(x) = \frac{1}{\Gamma(1 + k + n + 1)} D_x^n \Psi(x)$.

**Definition 2.7.** The new integral transform $K$ of $u(t)$ is defined by
\[
K(u(t)) = \frac{1}{\mu^{n+1}} \int_0^\infty e^{-\frac{t}{\mu}} u(t) dt,
\]
for all $u(t) \in \mathcal{F}$.

The following are some properties of new integral transform:

**Theorem 2.8.** (Kunjian et al., 2017) a) The new integral transform of $\frac{d^n}{dt^n}$ is given by
\[
K(\frac{d^n}{dt^n} u(t)) = \frac{1}{\mu^{n+1}} \frac{d^n}{dt^n} K(u(t)),
\]
for all $u(t) \in \mathcal{F}$.

**2.2. Implementation of NIPDTM on fractional hyperbolic-like equations**

The new integral transform on Eq. (1) with Theorem 2.8(a) yields
\[
U(x, \mu) = \mu \psi_0(x) + \mu^3 \psi_1(x) + \mu^{2 \ell} \psi_2(x),
\]
for $\mu \geq 0$.

Inverse new integral transform on (7) yields
\[
u(x, t) = \psi_0(x) + t \psi_1(x) + \psi_2(x) \psi_2(x) + \psi_3(x) \psi_2(x) + \psi_4(x) \psi_2(x) + \psi_5(x) \psi_2(x)
\]
for $\mu \geq 0$.

The projected differential transform (PDT) on (8) yields

\[
\begin{aligned}
\psi(x, t) &= \sum_{i=0}^{\infty} \psi_i(x)(t - t_0)^i x^i,
\end{aligned}
\]

where a given $f \in \mathcal{F}$ the constant $M$ must be finite while $k_1, k_2$ may be finite or infinite.
\[ U(X, 0) = g_1(X, t), \]
\[ U(X, \ell + 1) = K^{-1} \left\{ \mu^{2 \nu} \{ f_1(X) D_{x}^{\mu} U(X, \ell) \} + f_2(X) D_{x}^{\mu} U(X, \ell) + \ldots + f_n(X) D_{x}^{\mu} U(X, \ell) \right\}, \ell \geq 0. \]

(9)

where \( g_1(X, t) = \psi_0(X) + t \psi_1(X) + K^{-1} \{ \mu^{2 \nu} \Phi(X, \mu) \} \).

The expression (9) is referred to as “new integral-projected differential transform (NIPDT)” of problem (1). The \( m \)th order series solution of Eq. (1) is given by

\[ U(X, 0) = x + tx^2, \quad U(x, 1) = \frac{x^{4-\beta}}{\Gamma(3-\beta)} \frac{t^{2+1}}{\Gamma(2+\beta)} \frac{t^{2+1}}{2}, \quad U(x, 2) = \frac{\Gamma(5-\beta)x^{6-2\beta}}{2\Gamma(3-\beta)\Gamma(5-2\beta)} \frac{t^{2+1}}{\Gamma(2+2\beta)}. \]

(12)

\[ U(x, 3) = \frac{4\Gamma(3-\beta)\Gamma(5-2\beta)\Gamma(7-3\beta)}{\Gamma(3-\beta)\Gamma(7-2\beta)\Gamma(9-3\beta)} \frac{t^{2+1}}{\Gamma(2+4\beta)} \frac{t^{2+1}}{2}, \]

\[ U(x, 4) = \frac{8\Gamma(3-\beta)\Gamma(5-2\beta)\Gamma(7-3\beta)\Gamma(9-4\beta)}{\Gamma(3-\beta)\Gamma(7-2\beta)\Gamma(9-3\beta)\Gamma(11-4\beta)} \frac{t^{2+1}}{\Gamma(2+4\beta)} \frac{t^{2+1}}{5}, \]

\[ U(x, 5) = \frac{16\Gamma(5-\beta)\Gamma(7-2\beta)\Gamma(7-3\beta)\Gamma(9-4\beta)\Gamma(11-5\beta)}{\Gamma(3-\beta)\Gamma(7-2\beta)\Gamma(7-3\beta)\Gamma(9-4\beta)\Gamma(11-5\beta)} \frac{t^{2+1}}{\Gamma(2+5\beta)} \frac{t^{2+1}}{7}. \]

Thus, the solution of the problem (12),

\[ u(x, t) = \sum_{\ell=0}^{\infty} U(X, \ell) = x + tx^2 + \frac{x^{4-\beta}}{\Gamma(3-\beta)} \frac{t^{2+1}}{\Gamma(2+\beta)} \frac{t^{2+1}}{2} + \frac{\Gamma(5-\beta)x^{6-2\beta}}{2\Gamma(3-\beta)\Gamma(5-2\beta)} \frac{t^{2+1}}{\Gamma(2+2\beta)} \frac{t^{2+1}}{5} \]

\[ + \frac{4\Gamma(3-\beta)\Gamma(5-2\beta)\Gamma(7-3\beta)}{\Gamma(3-\beta)\Gamma(7-2\beta)\Gamma(9-3\beta)} \frac{t^{2+1}}{\Gamma(2+4\beta)} \frac{t^{2+1}}{5} + \frac{8\Gamma(3-\beta)\Gamma(5-2\beta)\Gamma(7-3\beta)\Gamma(9-4\beta)}{\Gamma(3-\beta)\Gamma(7-2\beta)\Gamma(9-3\beta)\Gamma(11-4\beta)} \frac{t^{2+1}}{\Gamma(2+4\beta)} \frac{t^{2+1}}{7} + \ldots \]

(14)

\[ s_m = \sum_{\ell=0}^{m} U(X, \ell), \]

and so, the series solution of Eq. (1) is given by

\[ u(X, t) = \lim_{m \to \infty} s_m. \]

(11)

3. Solutions of \((n+1)D\) STFH-like equation via FRDTM and NIPDTM

This section deals with the main goal of the paper, to present numerical study of three test problems of \((n+1)D\) STFH-like diffusion equations using FRDTM and NIPDTM to validate their reliability and efficiency.

**Example 3.1.** Consider initial values system of \((1+1)D\) STFH-like equation

\[ D_{t}^{\sigma} u(x, t) = \frac{\partial^\sigma u(x, t)}{\partial x^\sigma}, \]

\[ u(x, 0) = x, \quad \frac{\partial u(x, 0)}{\partial t} = x^2, \quad 1 < \alpha, \beta \leq 2 \]
Eq. (16) is solution of associated classical hyperbolic wave equation, which is same as obtained in (Momanli, 2005; Taghizadeh & Noori, 2017). Hence, the computed results approach towards the exact results and are in good agreement with the results due to earlier schemes.

(b) FRDTM: For $\alpha = N \eta$, the FRDTM of (12) yields

\[
\begin{align*}
U_{i}^{0}(x) &= x, \\
U_{i}^{1}(x) &= 0, \\
U_{i}^{2}(x) &= x^2, \\
U_{i}^{3}(x) &= 0, \\
U_{i}^{4}(x) &= 0, \\
U_{i}^{5}(x) &= 0, \\
U_{i}^{6}(x) &= x, \\
U_{i}^{7}(x) &= 0, \\
U_{i}^{8}(x) &= 0, \\
U_{i}^{9}(x) &= 0,
\end{align*}
\]

It is worth mentioning that the recurrence relation (17) can be solved for any given $\alpha$, which confirms that FRDTM solution can be obtained for any given alpha. But a general solution in terms of arbitrary value of alpha cannot be obtained. More precisely, for any two different values of $\alpha$ one has to solve the recurrence relation (17), separately.

(i) If $\alpha = 1.5$ ($\eta = \frac{1}{2}, N = 3$): Then the recursive values of $U_{i}^{k}(x)$'s are obtained from (17) as follows:

\[
\begin{align*}
U_{i}^{0}(x) &= x, \\
U_{i}^{1}(x) &= 0, \\
U_{i}^{2}(x) &= x^2, \\
U_{i}^{3}(x) &= 0, \\
U_{i}^{4}(x) &= 0, \\
U_{i}^{5}(x) &= x^4, \\
U_{i}^{6}(x) &= \frac{1}{3}, \\
U_{i}^{7}(x) &= \frac{4}{3}, \\
U_{i}^{8}(x) &= 0, \\
U_{i}^{9}(x) &= 0.
\end{align*}
\]

Thus, inverse FRDTM leads the solution of the problem (12) as follows:

\[
\begin{align*}
u(x, t) &= \sum_{i=0}^{\infty} U_{i}^{k}(x) t^{i} = x + t^2 \frac{x^4}{3} + \frac{x^8}{720} + \frac{x^{12}}{120} + \ldots
\end{align*}
\]

In particular, for $\alpha = 2, \beta = 2$, we get solution of classical wave equation

\[
u(x, t) = x + x^3 \left( t + \frac{t^3}{6} + \frac{t^5}{120} + \ldots \right) = x + x^2 \sinh(t)
\]

Example 3.2. Consider the initial value system of (2+1)D STFH-like equation

\[
\begin{align*}
D_u u &= x^2 \frac{\partial^2 u}{\partial \eta^2} + y^2 \frac{\partial^2 u}{\partial \eta^2}, 0 < x, y < 1, 1 < \alpha, \beta \leq 2, t > 0 \\
u(x, y, 0) &= x^4, \\
u_t (x, y, 0) &= y^4
\end{align*}
\]

(a) NIPDTM: The NIPDTM on Eq. (22) yields

\[
\begin{align*}
U_{i}(x, y, \ell + 1) &= \kappa^{-1} \left\{ \mu^2 \kappa \left[ \frac{x^2}{12} \frac{\partial^2 U(x, y, t)}{\partial x^2} + \frac{y^2}{12} \frac{\partial^2 U(x, y, t)}{\partial y^2} \right] \right\}, \ell > 0
\end{align*}
\]
On solving relation (23), we get

\[
U(x, y, 0) = x^4 + ty^4, \quad U(x, y, 1) = \frac{2t^x}{\Gamma(5 - \beta)} \left( \frac{x^{6-\beta}}{\Gamma(1 + x)} + \frac{ty^{6-\beta}}{\Gamma(2 + x)} \right),
\]

\[
U(x, y, 2) = \frac{(7 - \beta)t^{2x}}{6\Gamma(5 - \beta)\Gamma(7 - 2\beta)} \left( \frac{x^{8-2\beta}}{\Gamma(1 + 2x)} + \frac{ty^{8-2\beta}}{\Gamma(2 + 2x)} \right),
\]

\[
U(x, y, 3) = \frac{(7 - \beta)\Gamma(9 - 2\beta)t^{3x}}{72\Gamma(5 - \beta)\Gamma(7 - 2\beta)\Gamma(9 - 3\beta)} \left( \frac{x^{10-3\beta}}{\Gamma(1 + 3x)} + \frac{ty^{10-3\beta}}{\Gamma(2 + 3x)} \right),
\]

\[
U(x, y, 4) = \frac{(7 - \beta)\Gamma(9 - 2\beta)\Gamma(11 - 3\beta)t^{4x}}{864\Gamma(5 - \beta)\Gamma(7 - 2\beta)\Gamma(9 - 3\beta)\Gamma(11 - 4\beta)} \left( \frac{x^{12-4\beta}}{\Gamma(1 + 4x)} + \frac{ty^{12-4\beta}}{\Gamma(2 + 4x)} \right),
\]

\[
U(x, y, 5) = \frac{(7 - \beta)\Gamma(9 - 2\beta)\Gamma(11 - 3\beta)\Gamma(13 - 4\beta)t^{5x}}{864\Gamma(5 - \beta)\Gamma(7 - 2\beta)\Gamma(9 - 3\beta)\Gamma(11 - 4\beta)\Gamma(13 - 5\beta)} \left( \frac{x^{14-5\beta}}{\Gamma(1 + 5x)} + \frac{ty^{14-5\beta}}{\Gamma(2 + 5x)} \right),
\]

Thus, the solution of the problem (22),

\[
u(x, y, t) = \sum_{\ell=0}^{\infty} U(x, y, \ell) = x^4 + ty^4 + \frac{2t^x}{\Gamma(5 - \beta)} \left( \frac{x^{6-\beta}}{\Gamma(1 + x)} + \frac{ty^{6-\beta}}{\Gamma(2 + x)} \right)
\]

\[
+ \frac{(7 - \beta)t^{2x}}{6\Gamma(5 - \beta)\Gamma(7 - 2\beta)} \left( \frac{x^{8-2\beta}}{\Gamma(1 + 2x)} + \frac{ty^{8-2\beta}}{\Gamma(2 + 2x)} \right)
\]

\[
+ \frac{(7 - \beta)\Gamma(9 - 2\beta)t^{3x}}{72\Gamma(5 - \beta)\Gamma(7 - 2\beta)\Gamma(9 - 3\beta)} \left( \frac{x^{10-3\beta}}{\Gamma(1 + 3x)} + \frac{ty^{10-3\beta}}{\Gamma(2 + 3x)} \right)
\]

\[
+ \frac{(7 - \beta)\Gamma(9 - 2\beta)\Gamma(11 - 3\beta)t^{4x}}{864\Gamma(5 - \beta)\Gamma(7 - 2\beta)\Gamma(9 - 3\beta)\Gamma(11 - 4\beta)} \left( \frac{x^{12-4\beta}}{\Gamma(1 + 4x)} + \frac{ty^{12-4\beta}}{\Gamma(2 + 4x)} \right)
\]

\[
+ \frac{(7 - \beta)\Gamma(9 - 2\beta)\Gamma(11 - 3\beta)\Gamma(13 - 4\beta)t^{5x}}{864\Gamma(5 - \beta)\Gamma(7 - 2\beta)\Gamma(9 - 3\beta)\Gamma(11 - 4\beta)\Gamma(13 - 5\beta)} \left( \frac{x^{14-5\beta}}{\Gamma(1 + 5x)} + \frac{ty^{14-5\beta}}{\Gamma(2 + 5x)} \right)
\]

In particular for \( \beta \to 2 \), the solution (24) reduces to

\[
u(x, y, t) = x^4 \left\{ 1 + \frac{t^x}{\Gamma(x + 1)} + \frac{t^{2x}}{\Gamma(2x + 1)} + \frac{t^{3x}}{\Gamma(3x + 1)} + \ldots \right\}
\]

\[
+ y^4 \left\{ 1 + \frac{t^x}{\Gamma(x + 2)} + \frac{t^{2x}}{\Gamma(2x + 2)} + \frac{t^{3x}}{\Gamma(3x + 2)} + \ldots \right\}
\]

This solution is the same as the analytical solution obtained by using VIM (Molliq et al., 2009; Yin et al., 2013), modified HAM (Yin et al., 2015) and LHAM (Gupta and Kumar, 2012).

Here \( X = (x, y) \). For \( x = N\eta \), the FRDTM of (22) yields

\[
U^\xi(x) = x^\xi, \quad \text{and} \quad U^\eta(x) = \begin{cases} y^\eta, & \text{if } r = 1, \\ 0, & \text{if } r \neq 1, \end{cases} \quad r \in \{0, 1, \ldots, N-1\}
\]

\[
U^\xi(x) = \frac{\Gamma(k+1)}{12} \left( x^2 \frac{\partial^k U^\eta(x)}{\partial x^k} + y^2 \frac{\partial^k U^\eta(x)}{\partial y^k} \right), \quad k \geq 0.
\]

The solution of recurrence relation (26) for different values of \( \eta \) is as follows:

(i) If \( \eta = 1.25 \) (i.e., \( \frac{1}{2}, N = 5 \)): Similar to the previous problem, the recursive values of \( U^\xi(x) \)'s are obtained from (26), and so, the inverse FRDTM leads to the solution of the problem (22),
\[ u(X, t) = x^4 + ty^4 + \frac{2t^\beta}{\Gamma(5 - \beta)} \left( \frac{x^{6-\beta}}{\Gamma\left(\frac{11}{4}\right)} + \frac{ty^{6-\beta}}{\Gamma\left(\frac{15}{4}\right)} \right) + \frac{\Gamma(7 - \beta)t^{4\beta}}{6\Gamma(5 - \beta)\Gamma(7 - 2\beta)} \left( \frac{x^{8-2\beta}}{\Gamma\left(\frac{13}{4}\right)} + \frac{ty^{8-2\beta}}{\Gamma\left(\frac{17}{4}\right)} \right) \]

\[ + \frac{\Gamma(7 - \beta)\Gamma(9 - 2\beta)t^{12\beta}}{72\Gamma(5 - \beta)\Gamma(7 - 2\beta)\Gamma(9 - 3\beta)} \left( \frac{x^{10-3\beta}}{\Gamma\left(\frac{19}{4}\right)} + \frac{ty^{10-3\beta}}{\Gamma\left(\frac{23}{4}\right)} \right) \]

\[ + \frac{\Gamma(7 - \beta)\Gamma(9 - 3\beta)t^6}{864\Gamma(5 - \beta)\Gamma(7 - 2\beta)\Gamma(9 - 3\beta)\Gamma(13 - 4\beta)\Gamma(11 - 3\beta)} \left( \frac{x^{12-4\beta}}{\Gamma\left(\frac{17}{4}\right)} + \frac{ty^{12-4\beta}}{\Gamma\left(\frac{21}{4}\right)} \right) \]

\[ + \frac{10368\Gamma(5 - \beta)\Gamma(7 - 2\beta)\Gamma(9 - 3\beta)\Gamma(11 - 4\beta)\Gamma(13 - 5\beta)}{10368\Gamma(5 - \beta)\Gamma(7 - 2\beta)\Gamma(9 - 3\beta)\Gamma(11 - 4\beta)\Gamma(13 - 5\beta)} \left( \frac{x^{14-5\beta}}{\Gamma\left(\frac{25}{4}\right)} + \frac{ty^{14-5\beta}}{\Gamma\left(\frac{29}{4}\right)} \right) + \ldots \]

(ii) If \( \alpha = 1.5 \) (i.e., \( \eta = \frac{1}{2} \), \( N = 3 \)): Similar to previous case, the recursive values of \( U_n^i(X) \)'s are obtained from (26), and so, the inverse FRDTM leads to the solution of the problem (22) for \( \eta = \frac{1}{2} \) as follows:

\[ u(X, t) = x^4 + ty^4 + \frac{2t^\beta}{\Gamma(5 - \beta)} \left( \frac{x^{6-\beta}}{\Gamma\left(\frac{11}{4}\right)} + \frac{ty^{6-\beta}}{\Gamma\left(\frac{15}{4}\right)} \right) + \frac{\Gamma(7 - \beta)t^{4\beta}}{6\Gamma(5 - \beta)\Gamma(7 - 2\beta)} \left( \frac{x^{8-2\beta}}{\Gamma\left(\frac{13}{4}\right)} + \frac{ty^{8-2\beta}}{\Gamma\left(\frac{17}{4}\right)} \right) \]

\[ + \frac{\Gamma(7 - \beta)\Gamma(9 - 2\beta)t^{12\beta}}{72\Gamma(5 - \beta)\Gamma(7 - 2\beta)\Gamma(9 - 3\beta)} \left( \frac{x^{10-3\beta}}{\Gamma\left(\frac{19}{4}\right)} + \frac{ty^{10-3\beta}}{\Gamma\left(\frac{23}{4}\right)} \right) \]

\[ + \frac{\Gamma(7 - \beta)\Gamma(9 - 3\beta)t^6}{864\Gamma(5 - \beta)\Gamma(7 - 2\beta)\Gamma(9 - 3\beta)\Gamma(13 - 4\beta)\Gamma(11 - 3\beta)} \left( \frac{x^{12-4\beta}}{\Gamma\left(\frac{17}{4}\right)} + \frac{ty^{12-4\beta}}{\Gamma\left(\frac{21}{4}\right)} \right) \]

\[ + \frac{10368\Gamma(5 - \beta)\Gamma(7 - 2\beta)\Gamma(9 - 3\beta)\Gamma(11 - 4\beta)\Gamma(13 - 5\beta)}{10368\Gamma(5 - \beta)\Gamma(7 - 2\beta)\Gamma(9 - 3\beta)\Gamma(11 - 4\beta)\Gamma(13 - 5\beta)} \left( \frac{x^{14-5\beta}}{\Gamma\left(\frac{25}{4}\right)} + \frac{ty^{14-5\beta}}{\Gamma\left(\frac{29}{4}\right)} \right) + \ldots \]

(iii) If \( \alpha = 1.75 \) (i.e., \( \eta = \frac{1}{4} \), \( N = 7 \)): the recursive values of \( U_n^i(X) \)'s are obtained from (26), and so, the inverse FRDTM leads to the solution of the problem (22) for \( \eta = \frac{1}{4} \) as follows:

\[ u(X, t) = x^4 + ty^4 + \frac{2t^\beta}{\Gamma(5 - \beta)} \left( \frac{x^{6-\beta}}{\Gamma\left(\frac{11}{4}\right)} + \frac{ty^{6-\beta}}{\Gamma\left(\frac{15}{4}\right)} \right) + \frac{\Gamma(7 - \beta)t^{4\beta}}{6\Gamma(5 - \beta)\Gamma(7 - 2\beta)} \left( \frac{x^{8-2\beta}}{\Gamma\left(\frac{13}{4}\right)} + \frac{ty^{8-2\beta}}{\Gamma\left(\frac{17}{4}\right)} \right) \]

\[ + \frac{\Gamma(7 - \beta)\Gamma(9 - 2\beta)t^{12\beta}}{72\Gamma(5 - \beta)\Gamma(7 - 2\beta)\Gamma(9 - 3\beta)} \left( \frac{x^{10-3\beta}}{\Gamma\left(\frac{19}{4}\right)} + \frac{ty^{10-3\beta}}{\Gamma\left(\frac{23}{4}\right)} \right) \]

\[ + \frac{\Gamma(7 - \beta)\Gamma(9 - 3\beta)t^6}{864\Gamma(5 - \beta)\Gamma(7 - 2\beta)\Gamma(9 - 3\beta)\Gamma(13 - 4\beta)\Gamma(11 - 3\beta)} \left( \frac{x^{12-4\beta}}{\Gamma\left(\frac{17}{4}\right)} + \frac{ty^{12-4\beta}}{\Gamma\left(\frac{21}{4}\right)} \right) \]

\[ + \frac{10368\Gamma(5 - \beta)\Gamma(7 - 2\beta)\Gamma(9 - 3\beta)\Gamma(11 - 4\beta)\Gamma(13 - 5\beta)}{10368\Gamma(5 - \beta)\Gamma(7 - 2\beta)\Gamma(9 - 3\beta)\Gamma(11 - 4\beta)\Gamma(13 - 5\beta)} \left( \frac{x^{14-5\beta}}{\Gamma\left(\frac{25}{4}\right)} + \frac{ty^{14-5\beta}}{\Gamma\left(\frac{29}{4}\right)} \right) + \ldots \]

(iv) If \( \alpha = 2 \) (i.e., \( \eta = 1 \), \( N = 2 \)): the recursive values of \( U_n^i(X) \)'s are obtained from (26), and so, the inverse FRDTM leads to the solution of the problem (22) for \( \eta = 1 \) as follows:
\[ u(X, t) = x^4 + ty^4 + \frac{2t^2}{\Gamma(5 - \beta)} \left( x^{6-\beta} + ty^{6-\beta} \right) \frac{\Gamma(7-\beta) t^4}{6\Gamma(5 - \beta)\Gamma(7 - 2\beta)} \left( x^{8-2\beta} + ty^{8-2\beta} \right) + \frac{\Gamma(7-\beta) t^4}{\Gamma(5)\Gamma(6)} \left( \frac{x^{10-3\beta} + ty^{10-3\beta}}{\Gamma(7)} \right) + \frac{\Gamma(7-\beta) t^4}{\Gamma(8)} \left( \frac{x^{12-4\beta} + ty^{12-4\beta}}{\Gamma(9) + \Gamma(10)} \right) \]

\[ \frac{\Gamma(7-\beta) t^4}{\Gamma(11) + \Gamma(12)} \left( \frac{x^{14-5\beta} + ty^{14-5\beta}}{\Gamma(13-4\beta)\Gamma(11-4\beta)\Gamma(11-3\beta)\Gamma(9-3\beta)\Gamma(7-2\beta)\Gamma(5-\beta)} \right) + \ldots \]  

\[ (29) \]

The solution for \( \alpha = \beta = 2 \) can be obtained from solution (29) as

\[ u(x, y, t) = x^4 \left\{ 1 + \frac{t^2}{\Gamma(3)} + \frac{t^4}{\Gamma(5)} + \frac{t^6}{\Gamma(7)} + \ldots \right\} + y^4 \left\{ t + \frac{t^3}{\Gamma(4)} + \frac{t^5}{\Gamma(6)} + \frac{t^7}{\Gamma(8)} + \ldots \right\} \]

(30)

which is the same as the solution (25) for \( \alpha = 2 \), and is the closed form of the solution is \( u(x, t) = \)

\[ \begin{cases} 
U(X, 0) = (x^2 + y^2 - z^2) t + (x^2 + y^2 + z^2) \frac{t^4}{\Gamma(1 + \alpha)} \\
U(X, \ell + 1) = K^{-1} \left\{ \mu^{2\alpha} K \left( \frac{x^2 \partial^{\alpha} U(X, \ell)}{2 \partial x^{\alpha}} + \frac{y^2 \partial^{\alpha} U(X, \ell)}{2 \partial y^{\alpha}} + \frac{z^2 \partial^{\alpha} U(X, \ell)}{2 \partial z^{\alpha}} \right) \right\}, \ell > 0 
\end{cases} \]

(32)

\[ x^4 \cosh(t) + y^4 \sinh(t) \] which is the same as obtained in Wazwaz & Goruis, 2004.

**Example 3.3.** Consider the initial value system of (3+1)D STFH-like equation

\[ D^\alpha_t u = x^2 + y^2 + z^2 \frac{\partial^{\alpha} u}{\partial x^{\alpha}} + \frac{y^2 \partial^{\alpha} u}{2 \partial y^{\alpha}} + \frac{z^2 \partial^{\alpha} u}{2 \partial z^{\alpha}}, \]

\[ 0 < x, y, z < 1, 1 < \alpha, \beta \leq 2, t > 0 \]

\[ u(X, 0) = 0, \quad u_t(X, 0) = x^2 + y^2 - z^2, \quad X = (x, y, z). \]

(31)

(a) NIPDTM: The NIPDT on Eq. (31) yields

\[ U(X, 0) = (x^2 + y^2 - z^2) t + (x^2 + y^2 + z^2) \frac{t^4}{\Gamma(1 + \alpha)} \]

\[ U(X, 1) = \left( \frac{x^{6-\beta} + y^{6-\beta}}{\Gamma(3 - \beta)} \right) \left( \frac{t^4}{\Gamma(2x + 1)} + \frac{t^{e+1}}{\Gamma(x + 2)} \right) + \left( \frac{z^{6-\beta}}{\Gamma(3 - \beta)} \right) \left( \frac{t^4}{\Gamma(2x + 1)} - \frac{t^{e+1}}{\Gamma(x + 2)} \right), \]

\[ U(X, 2) = \frac{\Gamma(5-\beta)}{\Gamma(5 - 2\beta)} \left( \frac{t^{4x}}{\Gamma(3x + 1)} + \frac{t^{3x+1}}{\Gamma(4x + 1)} \right) + \frac{z^{6-2\beta}}{\Gamma(3 - \beta)} \left( \frac{t^4}{\Gamma(2x + 1)} - \frac{t^{e+1}}{\Gamma(x + 2)} \right), \]

\[ U(X, 3) = \left( \frac{x^{6-3\beta} + y^{6-3\beta}}{\Gamma(3 - \beta)} \right) \left( \frac{t^{4x}}{\Gamma(4x + 1)} + \frac{t^{3x+1}}{\Gamma(3x + 2)} \right) + \left( \frac{z^{8-3\beta}}{\Gamma(3 - \beta)} \right) \left( \frac{t^4}{\Gamma(2x + 1)} - \frac{t^{e+1}}{\Gamma(x + 2)} \right), \]

\[ U(X, 4) = \frac{\Gamma(5-\beta)}{\Gamma(5 - 2\beta)} \Gamma(9-3\beta) \Gamma(9 - 4\beta) \left( \frac{t^{4x}}{\Gamma(5x + 1)} + \frac{t^{4x+1}}{\Gamma(4x + 2)} \right) + \frac{z^{10-4\beta}}{\Gamma(3 - \beta)} \left( \frac{t^5}{\Gamma(5x + 1)} - \frac{t^{4x+1}}{\Gamma(4x + 2)} \right), \]

\[ U(X, 5) = \left( \frac{x^{12-5\beta} + y^{12-5\beta}}{\Gamma(3 - \beta)} \right) \left( \frac{t^{5x}}{\Gamma(5x + 1)} + \frac{t^{5x+1}}{\Gamma(4x + 2)} \right) + \frac{z^{12-5\beta}}{\Gamma(3 - \beta)} \left( \frac{t^5}{\Gamma(5x + 1)} - \frac{t^{5x+1}}{\Gamma(4x + 2)} \right), \]

\[ \ldots \]
Thus, the solution of the problem (31),

\[
u(X,t) = \left( x^2 + y^2 - z^2 \right) t + \left( x^2 + y^2 + z^2 \right) \frac{t^2}{\Gamma(1+\beta)} + \left( x^2 + y^2 + z^2 \right) \frac{t^2}{\Gamma(3-\beta)} \left( \frac{t^2}{\Gamma(2x+1)} + \frac{t^2}{\Gamma(2x+2)} + \frac{t^2}{\Gamma(3-\beta)} \left( \frac{t^2}{\Gamma(2x+1)} - \frac{t^2}{\Gamma(2x+2)} \right) \right) + \left( x^2 + y^2 + z^2 \right) \frac{t^2}{\Gamma(7-3\beta)} \left( \frac{t^2}{\Gamma(5x+1)} + \frac{t^2}{\Gamma(5x+2)} + \frac{t^2}{\Gamma(7-3\beta)} \left( \frac{t^2}{\Gamma(5x+1)} - \frac{t^2}{\Gamma(5x+2)} \right) \right) + \left( x^{10-4\beta} + y^{10-4\beta} + z^{10-4\beta} \right) \frac{t^{10-4\beta}}{\Gamma(5x+1)} + \frac{t^{10-4\beta}}{\Gamma(5x+2)} + \left( x^{12-5\beta} + y^{12-5\beta} + z^{12-5\beta} \right) \frac{t^{12-5\beta}}{\Gamma(5x+1)} + \frac{t^{12-5\beta}}{\Gamma(5x+2)} \right) + \ldots
\]

(33)

In particular for \( \beta \to 2 \), the series solution (33) reduces to

\[
u(X,t) = \left( x^2 + y^2 - z^2 \right) t + \left( x^2 + y^2 + z^2 \right) \frac{t^2}{\Gamma(1+\beta)} + \left( x^2 + y^2 + z^2 \right) \frac{t^2}{\Gamma(3-\beta)} \left( \frac{t^2}{\Gamma(2x+1)} + \frac{t^2}{\Gamma(2x+2)} + \frac{t^2}{\Gamma(3-\beta)} \left( \frac{t^2}{\Gamma(2x+1)} - \frac{t^2}{\Gamma(2x+2)} \right) \right) + \left( x^2 + y^2 + z^2 \right) \frac{t^2}{\Gamma(7-3\beta)} \left( \frac{t^2}{\Gamma(5x+1)} + \frac{t^2}{\Gamma(5x+2)} + \frac{t^2}{\Gamma(7-3\beta)} \left( \frac{t^2}{\Gamma(5x+1)} - \frac{t^2}{\Gamma(5x+2)} \right) \right) + \left( x^{10-4\beta} + y^{10-4\beta} + z^{10-4\beta} \right) \frac{t^{10-4\beta}}{\Gamma(5x+1)} + \frac{t^{10-4\beta}}{\Gamma(5x+2)} + \left( x^{12-5\beta} + y^{12-5\beta} + z^{12-5\beta} \right) \frac{t^{12-5\beta}}{\Gamma(5x+1)} + \frac{t^{12-5\beta}}{\Gamma(5x+2)} \right) + \ldots
\]

(34)

It is worth noting that solution (34) is the same as obtained using the homotopy decomposition method (Atangana & Alabaraoye, 2013), DTM (Seker, 2012), LHAM (Gupta and Kumar, 2012), ADM (Momani, 2005), VIM (Mollig et al., 2009; Yin et al., 2013), FRDTM (Arshad et al., 2017) and variable separation method (Zhang et al., 2016). Moreover, for \( x \to 2, \beta \to 2 \), solution (33) reduces to

\[
u(X,t) = \left( x^2 + y^2 - z^2 \right) \sinh(t) + \left( x^2 + y^2 + z^2 \right) \cosh(t) - 1.
\]
(b) FRDTM: Here $X = (x, y, z)$. For $\alpha = N\eta$, the FRDTM of (31) yields

$$
\begin{align*}
U^0_{\eta}(X) &= 0, \quad U^1_{\eta}(X) = \begin{cases} x^2 + y^2 - z^2, & \text{if } r = 1, \\
0, & \text{if } r \neq 1, \end{cases} \\
U^{k+N}_{\eta}(X) &= \frac{\Gamma(k\eta + 1)}{\Gamma((k+N)\eta + 1)} \\
& \quad \left( x^2 + y^2 + z^2 \right)^k \cdot \left( x^2 \frac{\partial^2 U^k_{\eta}(X)}{\partial x^2} + y^2 \frac{\partial^2 U^k_{\eta}(X)}{\partial y^2} + z^2 \frac{\partial^2 U^k_{\eta}(X)}{\partial z^2} \right),
\end{align*}
$$

(iii) The solution of recurrence relation (31) for different values of $x$ is as follows:

i. If $x = 1.5$ (i.e., $\eta = \frac{1}{2}$, $N = 3$), then on solving (35) for recursive values of $U^k_{\frac{1}{2}}(X)$:

The inverse FRDTM leads to the solution of the equation (32):

$$
\begin{align*}
U^0_{\frac{1}{2}}(X) &= 0, \quad U^1_{\frac{1}{2}}(X) = 0, \quad U^2_{\frac{1}{2}}(X) = x^2 + y^2 - z^2, \quad U^3_{\frac{1}{2}}(X) = \frac{x^2 + y^2 + z^2}{\Gamma\left(\frac{3}{2}\right)}, \\
U^4_{\frac{1}{2}}(X) &= \frac{\Gamma\left(5 - \beta\right) \left(x^6 - 2y^6 + y^6 - 2z^6 - z^6 \right)}{2\Gamma(5)\Gamma(3 - \beta)\Gamma(5 - 2\beta)}, \quad U^6_{\frac{1}{2}}(X) = \frac{\Gamma\left(5 - \beta\right) \left(x^6 - 2y^6 + y^6 - 2z^6 + z^6 \right)}{2\Gamma(5)\Gamma(3 - \beta)\Gamma(5 - 2\beta)}, \\
U^8_{\frac{1}{2}}(X) &= \frac{\Gamma\left(5 - \beta\right) \left(x^8 - 3y^8 + y^8 - 3z^8 - z^8 \right)}{4\Gamma\left(\frac{7}{2}\right)\Gamma(3 - \beta)}, \\
U^{10}_{\frac{1}{2}}(X) &= 0, \quad U^{11}_{\frac{1}{2}}(X) = \frac{\Gamma\left(5 - \beta\right) \left(x^8 - 3y^8 + y^8 - 3z^8 + z^8 \right)}{4\Gamma\left(\frac{7}{2}\right)\Gamma(3 - \beta)}, \quad U^{12}_{\frac{1}{2}}(X) = 0, \\
U^{14}_{\frac{1}{2}}(X) &= \frac{\Gamma\left(5 - \beta\right) \left(x^{10} - 4y^{10} + y^{10} - 4z^{10} - z^{10} \right)}{8\Gamma(8)\Gamma(3 - \beta)}, \\
U^{15}_{\frac{1}{2}}(X) &= \frac{\Gamma\left(5 - \beta\right) \left(x^{10} - 4y^{10} + y^{10} - 4z^{10} + z^{10} \right)}{8\Gamma\left(\frac{7}{2}\right)\Gamma(3 - \beta)}, \ldots
\end{align*}
$$

| Table 1. Comparison of absolute errors for $x = \beta = 2$ at different time levels $t \leq 1$. |
|---|---|---|---|---|
| $x$ | $t$ | $|u - s_1|$ | $|u - s_1|$ | $|u - s_1|$ | $|u - s_1|$ |
| 0.25 | 0.50 | 1.22818E-15 | 0.00000E + 0 | 9.72184E-08 | 3.37159E-10 |
| 0.75 | 0.50 | 2.39093E-13 | 6.93889E-18 | 1.66830E-06 | 1.29984E-08 |
| 1.00 | 0.50 | 1.00849E-11 | 1.66533E-16 | 1.25746E-05 | 1.73809E-07 |
| 0.75 | 1.00 | 4.91274E-15 | 0.00000E + 0 | 3.38873E-07 | 1.38464E-09 |
| 1.00 | 1.00 | 9.56374E-13 | 2.77556E-17 | 6.67322E-06 | 5.19938E-08 |
| 0.75 | 0.50 | 4.03395E-11 | 6.66134E-16 | 5.02984E-05 | 6.95236E-07 |
| 1.00 | 0.50 | 1.10467E-14 | 5.55112E-17 | 8.74965E-07 | 3.03433E-09 |
| 0.75 | 0.75 | 2.15183E-12 | 0.00000E + 0 | 5.10474E-05 | 1.16986E-07 |
| 1.00 | 0.75 | 9.07638E-11 | 1.55431E-15 | 1.13171E-04 | 1.56428E-06 |
Table 2. mth order approximate solutions $S_m$ ($m = 5, 7$) of example 3.1 for $\alpha = \beta = 1.5$ at different time levels $t \leq 1$.

| $x$  | $t$  | NIPDTM |          | FRDTM |
|------|------|--------|----------|-------|
|      |      | $S_5$  | $S_7$    | $S_5$ | $S_7$    |
| 0.25 | 0.50 | 2.8320456689E-01 | 2.8320456692E-01 | 2.8312565891E-01 | 2.8312565891E-01 |
|      | 0.75 | 3.0245881576E-01 | 3.0245884643E-01 | 3.0204370839E-01 | 3.0204370839E-01 |
| 0.5  | 1.00 | 3.2445616976E-01 | 3.2445618296E-01 | 3.2311032953E-01 | 3.2311032953E-01 |
|      | 0.50 | 6.3625062794E-01 | 6.3625062794E-01 | 6.3561032953E-01 | 6.3561032953E-01 |
|      | 0.75 | 7.2011742184E-01 | 7.2011742184E-01 | 7.1673863004E-01 | 7.1673863004E-01 |
| 0.75 | 1.00 | 8.2125491892E-01 | 8.2125491892E-01 | 8.1002108774E-01 | 8.1002108774E-01 |
|      | 0.50 | 1.0626735467E+00 | 1.0626735596E+00 | 1.0604886300E+00 | 1.0604886300E+00 |
|      | 0.75 | 1.2609119664E+00 | 1.2609119664E+00 | 1.2524471899E+00 | 1.2524471899E+00 |
| 1.00 | 1.5171027178E+00 | 1.5171027178E+00 | 1.4778986686E+00 | 1.4778986686E+00 |

Figure 1. The solution behavior of Example 3.1 obtained by using NIPDTM. (a) Absolute error in 5th order solution for $\alpha = \beta = 2$. (b) Absolute error in 7th order solution $\alpha = \beta = 2$. (c) Different order solutions for $\alpha = \beta = 2$ for large time interval. (d) Sixth order solution for different values of $\alpha$, $\beta$. (e) Eighth order solution for $\alpha = \beta = 2$ with $x = 1, t \in (0, 1)$. (f) Eighth order solution for $\alpha = 1.75$, $\beta = 1.85$ with $x = 1, t \in (0, 1)$.
Table 3. Comparison of absolute errors in of example 3.2 for $\alpha = \beta = 2$ at different time levels $t \leq 1$. 

| $x$   | $y$   | $t$   | $|u - s_1|$    | $|u - s_2|$    | $|u - s_1|$    | $|u - s_2|$    |
|-------|-------|-------|----------------|----------------|----------------|----------------|
| 0.25  | 0.25  | 0.50  | 2.07126E-15    | 0.00000E + 00  | 2.84922E-04    | 2.84831E-04    |
| 0.25  | 0.25  | 0.75  | 2.74064E-13    | 3.46945E-18    | 6.87725E-04    | 6.86656E-04    |
| 0.50  | 0.50  | 0.50  | 3.21960E-15    | 0.00000E + 00  | 8.95365E-04    | 8.95183E-04    |
| 0.75  | 0.75  | 0.75  | 4.98206E-13    | 6.93889E-18    | 2.74923E-03    | 2.74660E-03    |
| 1.00  | 1.00  | 0.50  | 1.82848E-11    | 3.60822E-16    | 6.20993E-03    | 6.18517E-03    |
| 0.75  | 0.75  | 0.75  | 8.21565E-15    | 2.77556E-17    | 3.54062E-03    | 3.54004E-03    |
| 1.00  | 1.00  | 0.50  | 5.92546E-11    | 1.05471E-15    | 1.16824E-02    | 1.16740E-02    |
| 0.50  | 0.50  | 0.50  | 3.19744E-14    | 0.00000E + 00  | 3.94831E-03    | 3.94965E-03    |
| 0.75  | 0.75  | 0.50  | 6.04907E-12    | 2.77556E-17    | 8.94210E-03    | 8.92665E-03    |
| 1.00  | 1.00  | 0.50  | 1.31830E-10    | 3.01148E-15    | 1.60397E-02    | 1.59521E-02    |
| 0.50  | 0.50  | 0.50  | 3.31402E-14    | 0.00000E + 00  | 4.57503E-03    | 4.55730E-03    |
| 0.75  | 0.75  | 0.50  | 4.38302E-12    | 5.55112E-17    | 1.10036E-02    | 1.09665E-02    |
| 1.00  | 1.00  | 0.50  | 1.41285E-10    | 3.16414E-15    | 2.09343E-02    | 2.08351E-02    |

(i) If $\alpha = 1.75$ (i.e., $\eta = \frac{1}{3}$, $N = 7$): the recursive values of $u(N)(X)'s$ can be computed, and so, the inverse FRDTM leads to the solution of the problem (31) for $\alpha = 1.75$ as follows:

$$u(X, t) = (x^2 + y^2 - z^2) t + \left(\frac{11}{4}\right)^\alpha \Gamma(3 - \beta) \Gamma\left(\frac{15}{4}\right) + \frac{\left(x^4 - y^4 - z^4\right) t^2}{\Gamma\left(\frac{18}{4}\right)}$$

$$+ \frac{\Gamma(7-2\beta)}{\Gamma(7-\beta)} \left(\frac{\left(x^8 - y^8 - z^8\right) t^2}{\Gamma\left(\frac{29}{4}\right)}\right) + \cdots$$
Figure 2. The solution behavior of Example 3.2 obtained by using NIPDTM. (a) Comparison of different order solutions with exact solutions for $a = b = 2, x = y = 1, t \leq 16$. (b) Absolute errors in different order solutions for $a = b = 2, x = y = 1, t \leq 1$. (c) Absolute errors in 5th order solutions for $a = b = 2, y = 1, t \leq 1$. (d) Absolute errors in 7th order solutions for $a = b = 2, y = 1, t \leq 1$. (e) Seventh order solution behavior for $a = 1.5, b = 1.75$ at $t = 1$. (f) Seventh order solution behavior for $a = 1.5, b = 1.75$ at $t = 1$. (g) Seventh order solution behavior for $a = 2, b = 2$ at $t = 1$. (h) Plots of seventh order solutions for different $a$, $b$, $x = y = 1$. 
Table 4. mth order approximate solutions \( S_m \) \((m = 5, 7)\) of example 3.2 for \( \alpha = \beta = 1.5 \) at different time levels \( t \leq 1 \).

| \( x \) | \( y \) | \( t \) | \( s_5 \) | \( s_7 \) | \( s_5 \) | \( s_7 \) |
|-----|-----|-----|-----|-----|-----|-----|
| \( \frac{1}{2} \) | \( \frac{1}{2} \) | \( 0.5 \) | 6.246916961E-03 | 6.246691696E-03 | 6.235067984E-03 | 6.243047758E-03 |
| 0.75 | 0.75 | 1.00 | 7.619320231E-03 | 7.619320239E-03 | 7.582587126E-03 | 7.612594533E-03 |
| \( \frac{1}{2} \) | \( \frac{1}{2} \) | 1.50 | 9.143679295E-03 | 9.143679803E-03 | 9.050377196E-03 | 9.121579469E-03 |
| 0.75 | 0.75 | 2.00 | 5.741396040E-02 | 5.741396084E-02 | 5.525403609E-02 | 5.528304580E-02 |
| \( \frac{1}{2} \) | \( \frac{1}{2} \) | 2.50 | 7.596984061E-02 | 7.596841936E-02 | 7.592352827E-02 | 7.536466351E-02 |
| 0.75 | 0.75 | 3.00 | 1.719703968E-01 | 1.714903700E-01 | 1.711935738E-01 | 1.712024748E-01 |
| \( \frac{1}{2} \) | \( \frac{1}{2} \) | 3.50 | 2.674136535E-01 | 2.674136545E-01 | 2.659586954E-01 | 2.658953616E-01 |
| 0.75 | 0.75 | 4.00 | 3.754993471E-01 | 3.754993855E-01 | 3.708169472E-01 | 3.708839240E-01 |
| \( \frac{1}{2} \) | \( \frac{1}{2} \) | 4.50 | 7.188220435E-02 | 7.188220426E-02 | 7.158919998E-02 | 7.187403072E-02 |
| 0.75 | 0.75 | 5.00 | 7.960899564E-02 | 7.960899569E-02 | 7.950282187E-02 | 7.950282187E-02 |

Table 5. Comparison of absolute errors in of example 3.3 for \( \alpha = \beta = 2 \) at different time levels \( t \leq 1 \).

| \( x \) | \( y \) | \( t \) | \( |u - s_1| \) | \( |u - s_2| \) | \( |u - s_1| \) | \( |u - s_2| \) |
|-----|-----|-----|-----|-----|-----|-----|
| 0.25 | 0.25 | 0.50 | 1.76240E-15 | 1.38778E-17 | 3.72898E-06 | 1.64884E-08 |
| 0.75 | 0.75 | 1.00 | 3.58977E-13 | 3.81778E-17 | 2.50406E-04 | 2.59988E-04 |
| 0.50 | 0.50 | 0.50 | 5.55112E-12 | 2.77556E-17 | 5.5112E-17 | 5.5112E-17 |
| 0.75 | 0.75 | 0.75 | 1.11472E-12 | 7.77156E-16 | 1.11022E-16 | 1.11022E-16 |
| 0.75 | 0.75 | 1.00 | 4.00304E-11 | 7.77156E-16 | 1.02054E-10 | 1.05341E-15 |
| 0.75 | 0.75 | 0.75 | 1.19349E-10 | 9.52899E-09 | 1.54060E-05 | 6.7711E-08 |
| 0.50 | 0.50 | 0.50 | 5.55112E-11 | 2.77556E-16 | 5.5112E-12 | 5.5112E-12 |
| 0.75 | 0.75 | 0.75 | 1.11472E-12 | 5.53249E-04 | 1.8166E-04 | 1.78312E-04 |
| 0.50 | 0.50 | 0.50 | 9.38138E-15 | 1.11022E-16 | 9.38138E-15 | 1.11022E-16 |
| 0.75 | 0.75 | 1.00 | 8.04447E-11 | 1.4647E-05 | 1.47112E-04 | 1.44888E-04 |
| 0.75 | 0.75 | 1.00 | 1.56541E-10 | 0.0000E+00 | 0.0000E+00 | 0.0000E+00 |
| 0.75 | 0.75 | 1.00 | 3.12994E-11 | 2.22045E-16 | 3.12994E-11 | 2.22045E-16 |

3.1. Results and discussion

In Example 3.1, the absolute errors of order \( m \) \((m = 5, 7)\) for \( \alpha = \beta = 2 \) obtained by both FRDTM and NIDTPM is reported in Table 1 whereas the mth order \((m = 5, 7)\) solutions computed by these two methods are reported in Table 2, for \( \alpha = \beta = 1.5 \). Table 1 and solution expressions confirm that FRDT solutions are easier in comparison to NIDTP solutions but NIDTP solutions converge to the exact solutions faster than the FRDTM solutions. Note that FRDT solutions of Example 3.1, in either case, can be obtained directly from NIDTP solution (14) by using the respective value of \( x \).
The absolute errors in the 5th and 7th order NIPDT solutions of \((1+1)D\) STFH-like equation (12) for \(\alpha = \beta = 2, x, t \in (0, 1)\) is depicted graphically in Figure 1(a) and Figure 1(b), respectively. The two-dimensional plots of different order solutions are depicted in Figure 1(c) for \(x = 1\). It is evident from the above figures and Table 1 that the accuracy in the computed results increases rapidly with increasing order of approximation. The solution behavior of \((1+1)D\) STFH-like equation (12) for \(\alpha = \beta = 2\) is depicted in Figure 1(e) and that for \(\alpha = 1.75, \beta = 1.85\) is depicted in Figure 1(f). The two-dimensional

**Table 6.** \(m\)th order approximate solutions \(S_m\) \((m = 5, 7)\) of example 3.3 for \(z = 0.2, \alpha = \beta = 1.5\) at different time levels \(t \leq 1\).

| \(x\) | \(y\) | \(t\) | \(S_5\) | \(S_7\) |
|------|------|------|-------|-------|
| 0    | 0    | 0    | 9.11325516424E-02 | 9.11325517558E-02 |
| 0.5  | 0.5  | 0.5  | 1.59146564622E-01 | 1.59146544602E-01 |
| 1    | 1    | 1    | 2.4292442296E-01  | 2.4292442296E-01  |
| 1    | 1    | 1    | 2.47648270106E-01 | 2.47648270106E-01 |
| 1    | 1    | 1    | 4.3113142222E-01  | 4.3113142222E-01  |
| 1    | 1    | 1    | 6.6258920868E-01  | 6.6258920868E-01  |
| 1    | 1    | 1    | 9.1497653858E-01  | 9.1497653858E-01  |
| 1    | 1    | 1    | 9.05649827897E-01 | 9.05649827897E-01 |
| 1    | 1    | 1    | 1.4144121948E-01  | 1.4144121948E-01  |
| 1    | 1    | 1    | 2.47648270106E-01 | 2.47648270106E-01 |
| 1    | 1    | 1    | 4.3113142222E-01  | 4.3113142222E-01  |
| 1    | 1    | 1    | 6.6258920868E-01  | 6.6258920868E-01  |
| 1    | 1    | 1    | 4.0413994373E-01  | 4.0413994373E-01  |
| 1    | 1    | 1    | 7.03116786229E-01 | 7.03116786229E-01 |
| 1    | 1    | 1    | 1.0822439773E-01  | 1.0822439773E-01  |
| 1    | 1    | 1    | 6.7149255405E-01  | 6.7149255405E-01  |
| 1    | 1    | 1    | 1.1776355245E-01  | 1.1776355245E-01  |
| 1    | 1    | 1    | 1.8340563258E-01  | 1.8340563258E-01  |

The solution behavior of Example 3.3 obtained by using NIPDTM. (a) Comparison of different order solutions with exact solutions for \(x = \beta = 2, x = y = z = 1, t \leq 16\). (b) Absolute errors in different order solutions for \(x = \beta = 2, x = y = z = 1, t \leq 1\). (c) Absolute errors in 5th order solutions for \(x = \beta = 2 y = z = 1, t \leq 1\). (d) Absolute errors in 7th order solutions for \(x = \beta = 2, y = z = 1, t \leq 1\).
plots of sixth order solutions for different values of $a, \beta$ are depicted graphically in Figure 1(d) for $x = 1, t \in (0, 1)$.

In Example 3.2, the absolute errors in $m$th order ($m = 5, 7$) solutions for $a = \beta = 2$ computed by both FRDTM and NIDTPM are reported in Table 3. The same order computed solutions by these methods for $a = \beta = 1.5$ are reported in Table 4. The conclusion from these two tables is analogous to Example 3.1. The two-dimensional plots, for $a = \beta = 2$ with $x = y = 1$, of NIDTPM solutions of a different order ($m = 3, 5, 7$) are depicted graphically in Figure 2(a) for $t \leq 16$ while the absolute errors in the $m$th order ($m = 4, 5, 7$) solutions are depicted graphically in Figure 2(b) for $t \leq 1$. The absolute errors in the $5$th/7th order NIDTPM solutions of $(2+1)$D STH-like equation (23) for $a = \beta = 2, x, t \in (0, 1), y = 1$ are depicted graphically in Figure 2(c,d). The findings from the above figures and Table 3 show that the accuracy in the computed results increases rapidly with increasing order of approximation. The surface solution behaviors of Example 3.2 for $a = 1.25, \beta = 1.5, a = 1.5, \beta = 1.75$ and $a = \beta = 2$ are depicted graphically in Figure 2(e,f,g). The two-dimensional plots of seventh order solutions for different values of $a, \beta$ for $x = y = 1, t \in (0, 1)$ are depicted graphically in Figure 2(h).

In Example 3.3, it is evident from Table 5 and Figure 3 that the computed results for $(3+1)$D STH-like equation using NIDPTM are more accurate in comparison to FRDT solutions for the same order of approximation. Table 6 reports fifth/seventh order solutions computed by using both of these methods.

4. Concluding remark

In this paper, a comparative study of analytical solutions of $(n+1)$-dimensional space-time fractional hyperbolic-like equations (with Caputo fractional derivatives) has been made using two reliable semi-analytical methods: “new integral projected differential transform method (NIDPTM)” “and fractional reduced differential transform method (FRDTM)” Three test problems are used in order to illustrate the efficiency of these methods.

It is noted from Section 3 that the computed approximate series solutions for each problem with $\beta = 2$ are in excellent agreement with those obtained by modified HAM (Yin et al., 2015), DTM (Seker, 2012), ADM (Momani, 2005), HDM (Atangana & Alabaraoye, 2013), LHAM (Gupta and Kumar, 2012), FRDTM (Singh & Srivastava, 2015), VIM (Molliq et al., 2009; Yin et al., 2013), variable separation method (Zhang et al., 2016), and approaches to the exact solutions. The computed solutions from either method are obtained in terms of an infinite power series that converges to the exact solutions. The application of FRDTM is very easy, efficient, valuable and suitable for getting approximate solutions of (non)linear fractional PDEs. The NIDPTM is more general and it provides better accuracy in the results in compression to FRDTM, for the same order of approximations, i.e. NIDPT solutions converge to the exact solutions faster than the FRDT solutions.

Disclosure statement

No potential conflict of interest was reported by the authors.

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