Investigation of Commuting Graphs for Elements of Order 3 in Certain Leech Lattice Groups

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Abstract
Assume that G is a finite group and X is a subset of G. The commuting graph is denoted by C(G,X) and has a set of vertices X with two distinct vertices x, y ∈ X, being connected together on the condition of xy = yx. In this paper, we investigate the structure of C(G,X) when G is a particular type of Leech lattice groups, namely Higman–Sims group HS and Janko group J2, along with X as a G-conjugacy class of elements of order 3. We will pay particular attention to analyze the discs’ structure and determinate the diameters, girths, and clique number for these graphs.

Keywords: Sporadic groups; commuting graph; diameter, cliques.

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1. Introduction and Preliminaries
It is believed that studying the action of a group on a graph is one of the best comprehensible ways of analyzing the structure of the group. Suppose that G is a group and X is a subset of G; the commuting graph denoted by C(G,X) has the set of vertices X with two vertices x, y ∈ X, which are connected if x ≠ y, where xy = yx. The commuting graphs were first illustrated by Fowler and Brauer in a seminal paper [1]. They were eminent for giving evidence of a prescribed isomorphism of an involution centralizer, where there is a limited number of non-abelian groups capable of containing it. These graphs were extremely vital for the works of the Margulis-Platanov conjecture [2], as the graphs mentioned in [1] have X = G(1) where 1 is the identity element of G). When X is a conjugacy class of involution (conjugacy class X of G means that for any two elements x, y ∈ X there are an element

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g ∈ G such that x^2 = y, whereas involution means that all elements of X have order 2), then the commuting graph is known as the commuting involution graph. Rowley, Hart (née Perkins), Bates, and Bundy put their efforts into investigating the commuting involution graphs and supplying the diameters and disc sizes [3,4,5,6]. Suppose that X is a conjugacy class of elements of order 3. Nawawi and Rowley [7] analyzed the C(G,X) when G is either a symmetric group Sn or a sporadic group McL. The aim of this paper is to investigate the commuting graphs when G is a particular type of Leech lattice groups, such as the Higman–Sims group HS and Janko group J2, along with X as a G-conjugacy class for elements of order 3. The research involves scrutinizing the discs’ structures and calculating the diameters, girths, and clique number for these graphs. From now, we shall assume that G is one of the aforementioned groups. Also, let t be an element of order 3 in G and X = t^G. It is clear that G, acting by conjugation, induces graph automorphisms of C(G,X) and is transitive on its vertices. For x ∈ X and i ∈ Z^+, Δ_i(x) denotes the set of vertices of C(G,X) which has a distance i from x. Using the usual distance function for graphs, this distance function will be denoted by d( , ). We use G_t(= C_G(x)) to denote the stabilizer in G of x. Obviously, Δ_i(x) will be a union of certain Gx-orbits. Therefore, we are looking for finding the Gx-orbits of X. In the computational group theory of Magma [8] and GAP [9], packages are considered most commonly utilized. In most steps of the algorithm, Gap will be dominant in the implementation, while the permutation character of the centralizer of t in G (C_G(t)) may be verified using Magma and, hence, the number of C_G(t)-orbits (Permutation Rank on X) under the action of X on C_G(t) is calculated. Finally, we will use the online Atlas of Group Representations [10] to get a class name of the groups and we refer to it as The Online Atlas.

For the aforesaid groups, the sizes of conjugacy classes for elements of order 3 and the permutation ranks on each class, as well as the structure of the centralizer of t in G, which can be seen in [11], are given in the next table.

Table 1-Class size and Permutation Rank of X

| Group | Class | Size of Class | Permutation Rank | C_G(t) |
|-------|-------|---------------|------------------|--------|
| HS    | 3A    | 12300         | 399              | C_3 x S_3 |
| J_2   | 3A    | 560           | 10               | C_3 . A_6 |
|       | 3B    | 16800         | 531              | C_3 x A_4 |

2. Computational Method

Let x ∈ Δ_i(t) and z ∈ C_G(t), then one can see immediately that x^i ∈ Δ_i(t). Thus, for a finite group G, each disc Δ_i(t) of the commuting graph C(G,X) is a union of specific C_G(t)-orbits. The size of C_G(t)-orbits under the action of conjugation on t^G can be calculated by using the character table of the group, as we can see in the following result:

Proposition 2.1. [12]. Let G be a group acting transitively on a finite set Ω, with a permutation character χ. Suppose that α ∈ Ω and G_α has exactly k orbits on Ω. Then <χ,χ> = k.

The quantity k in Proposition 2.1 is called the permutation rank of G on Ω. Therefore, the permutation rank of C_G(t) on X is the number of C_G(t)-orbits under the conjugation action on X.

Now, let C be a G-conjugacy class. It is obvious that the set X_C = {x ∈ X : tx ∈ C} under the conjugation action of C_G(t) breaks up into sub orbits. Thus, to find all the sub orbits of X_C, we have to identify the C_G(t) orbits of X_C, for all those C, such that X_C ≠ Ø. The next definition gives us the size of the set X_C and, therefore, the size of sub orbits of X_C:

Definition 2.2. [13]. Let C_1, C_2 and C_n be the conjugacy classes of a finite group G. Then, for a fixed element g ∈ C_k, the following set is defined:

\[ a_{ik} = |\{(g_a, g) \in C_i \times C_k | g.g.a = g\}| \]

Then, for all possible i, j, and k, the value a_{ik} is called a class structure constant for G.

The next lemma will be used to compute the class structure constants for G.

Lemma 2.3. [13]. Let G be a finite group with n conjugacy classes C_1, C_2, ..., C_n. Then for all i, j, and k, we have g as follows:

\[ a_{jk} = \frac{1}{|C_G(g)| |C_G(g)|} \sum_{\chi \in Irr(G)} \chi(g)^i \chi(g)^j \chi(g)^k \]

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where \( g_i, g_j, \) and \( g_k \) are respectively in \( C_i, C_j \) and \( C_k \), and \( \chi \in \text{Irr}(G) \) is the irreducible character table in \( G \).

For the proof of the above lemma, see [13], page 128, Lemma 2.12. Now, since \(|X_c| = |\{(c, x) \in C \times X \mid cx = t\}| = |a_{jk}|\), then by employing Lemma 2.3, we get

\[
|X_c| = \frac{|G|}{|C_G(t)||C_G(0)|} \sum_{\chi \in \text{Irr}(G)} \frac{\chi(g)\chi(t)^2}{\chi(t)}
\]

Therefore, from the complex character table of \( G \), which is available in the GAP character table library, and using the GAP function of "Class Multiplication Coefficient", we immediately obtain the size of \( X_c \).

3. Diameters, Girths, and Clique Number

To determine the diameters, girths, and clique number for the \( C(G,X) \), we will generated the graph by using the gap package YAGS [14] (specifically, Graph By Relation) in the next algorithm, which can be realized by using the definition of the commuting graph. The algorithm is given as follows:

**Algorithm 1**

**Input:** The group \( G, t \in G \) (the elements of order 3);

i: \( X = t^e \) the \( G \)-conjugacy class of \( t \).

ii: \( \text{Relation} = \{ \text{the set of elements satisfies the condition} : x \neq y \text{ and } x*y = y*x \} \)

iii: \( C(G,X) \) is the graph generated by \( X \) and relation.

v: Calculate Diameter\((C(G,X))\), Girth\((C(G,X))\) and Clique number \((C(G,X))\).

**Output:** The diameter, girth and the clique number of \( C(G,X) \).

The results in table 2 are obtained by applying Algorithm 1 on the groups specified in the table.

| Table 2-Diameter, Girth, and clique number of commuting graphs |
|-------------------------------------------------------------|
| **Graph** | **Diameter** | **Girth** | **clique number** | **group** |
| \( C(HS,3A) \) | 4 | 3 | 8 | \( C_3 \times C_3 \) |
| \( C(J_2,3A) \) | disconnected | no girth (forest) | 2 | \( C_3 \) |
| \( C(J_2,3B) \) | 8 | 3 | 6 | \( C_3 \times C_3 \) |

The last column of Table 2 represents the maximum elementary abelian group generated by the maximum clique.

4. Analyzing the Discs Structures

This section is dedicated to analyze the structures of the \( \Delta_i(t) \) of the commuting graph \( C(G,X) \).

4.1. \( C_G(t) \)-Orbits of \( \Delta_i(t) \)

The next algorithm is employed to break \( \Delta_i(t) \) into \( C_G(t) \)-sub orbits of \( X_c \), for all those \( C \) (\( G \)-Conjugacy class), such that \( X_c \neq \emptyset \), where their sizes are also provided.
Algorithm 2.

**Input:** the group G, t ∈ G, (the elements of order 3), C (G-Conjugacy Class)

i: X → t the G-conjugacy class of t
ii: C_G(t) → centralizer in G of t.
iii: O → the orbits in C_G(t) of X.
iv: |X_C| → “Class Multiplication Coefficient” of C in X.
v: For i → 1 to size (O) Do
vi: If t * O[i][1] Conjugate to C → Y_C ∪ O[i]
vii: Repeat the steps vi, vii until |X_C| = |Y_C|.
viii: X_C → Y_C = Y_C. ix: For j → 1 to size O Do
x: If t in O[j] → t = O_j (there is only one)
xii: For i in O[j] Do
xiii: For h → 1 to size (O) Do
xiv: If d(O[h][1], i) = 1 → {Y_1 ∪ {h}} \ Y_0
xv: For x in Y_1 → ∆_1(t) ∪ O[x]
xvi: for j in Y_1 Do
xvii: Repeat the steps x1, x2 and
xviii: If d(O[h][1], i) = 1 → {Y_2 ∪ {h}} \ Y_1 ∪ Y_0
xix: For x in Y_2 → ∆_2(t) U O[x]
xx: Repeat the above steps and replace the Y_i+1 with Y_i and ∆_i+1(t) with ∆_i(t)

**Output:** The positions of the sets X_C in the ∆_i(t) with their sizes.

For each graph in Table-2, we provide information about the discs structure. We should note that, in the next tables, the value n * m means the number and the size of C_G(t)-orbits in certain ∆_i(t), respectively. The exceptional case is C(J_2, 3A) as the graph is disconnected. The cases are:

1- C(J_2, 3A): Form Table- 1, the permutation rank of 3A is 10 and C_G(t) = C_3 ∪ A_6. In Table- 2, one can see that the graph is disconnected and, by Algorithm 1, there are 280 connected 2-components of C(J_2, 3A).

2- C(J_2, 3B): Form Table- 1, the permutation rank of 3B is 531. Form Table- 2, the graph is connected with diameter 8. The centralizer of t ∈ 3B is isomorphic to C_3 x A_6. The disc structure of the C(J_2, 3B) is described in Table-3.

Table 3-Discs structure of C(J_2, 3B)

| Class Name | ∆_1(t) | ∆_2(t) | ∆_3(t) | ∆_4(t) | ∆_5(t) | ∆_6(t) | ∆_7(t) | ∆_8(t) |
|------------|--------|--------|--------|--------|--------|--------|--------|--------|
| 1A         | 1      |        |        |        |        |        |        |        |
| 2A         |        |        | 3*2    | 12     |        |        |        |        |
| 2B         |        | 12*2   |        | 36*2   |        |        |        |        |
| 3A         | 4*2    |        | 12*2   |        |        |        |        |        |
| 3B         | 4*2,1  | 12*2   | 36*2,  | 36*4,  | 36*2,  | 36*4,  | 12*6   |        |
|            |        |        | 12*2   | 12*4,  | 12*4,  | 12*3   |        |        |
| 4A         |        | 36*2   | 36*3   |        | 12*4   | 36     |        |        |
| 5A         |        |        |        |        | 12*2   | 36     | 12*2,3 | 12     |
| 5B         |        |        |        |        | 12*2   | 36     | 12*2,3 | 12     |
| 5C         |        |        |        |        | 36*2   | 36     | 36*4   | 36     |
| 5D         |        |        |        |        | 36*2   | 36     | 36*4   | 36     |
| 6A | 36*4 | 12*2 | 12*4 | 36*12 | 36*4 |
|----|------|------|------|-------|------|
| 6B | 12*2 | 36*4 | 36*4 | 36*20 |
| 7A | 36*3 | 36*36| 36*24| 36*8  |
| 8A | 36*4 | 36*28| 36*32|
| 10A| 36*6 | 36*12|
| 10B| 36*6 |
| 10C| 36*10| 36*12| 36*26| 36*2 |
| 10D| 36*10| 36*12| 36*26| 36*2 |
| 12A| 36*6 | 36*16| 36*4 | 36*2, 12*12 |
| 15A| 36*4 | 36*10, 12*2 |
| 15B| 36*4 |

3- C(HS, 3A): Form Table-1, the permutation rank of 3A is 399. Form Table-2, the graph is connected with diameter 4. The centralizer of \( t \in 3A \) is isomorphic to \( C_3 \times S_5 \). The discs structure of C(HS, 3B) is described in Table-4.

**Table 4**-Discs structure of C(HS,3A)

| Class Name | \( \Delta_1(t) \) | \( \Delta_2(t) \) | \( \Delta_3(t) \) | \( \Delta_4(t) \) |
|-----------|-----------------|-----------------|-----------------|-----------------|
| 1A        | 1               |                 |                 |                 |
| 2A        |                 | 120,15          | 90              |                 |
| 2B        |                 |                 | 18*2            |                 |
| 3A        | 20*3,1          | 120*4,15        | 90              | 360*4,90*4,18*2 |
| 4B        |                 | 360             |                 | 360*3,90*5     |
| 4C        |                 | 360*8,180       |                 | 360*6,90*2     |
| 5A        |                 | 18*2            |                 |                 |
| 5B        |                 | 60,120          | 360,180         |                 |
| 5C        |                 | 360*18          | 360*27          |                 |
| 6B        | 120*3           | 360*10          | 360*4,90*3      |                 |
| 7A        |                 | 360*11          | 360*70          |                 |
| 8A        |                 | 360*4           | 360*16,90*8     |                 |
| 8B        |                 | 360*2           | 360*12          |                 |
| 8C        |                 | 360*2           | 360*12          |                 |
| 10A       |                 | 360*6           | 360*18,90*6     |                 |
| 10B       |                 |                 | 360*2           |                 |
| 11A       |                 | 360*3           | 360*25          |                 |
| 11B       |                 | 360*3           | 360*25          |                 |
| 12A       |                 |                 | 360*16,90*8     |                 |
| 15A       | 120*3           |                 |                 | 360*8          |
| 20A       |                 | 360*2,90*2      | 360*2,90*2      |                 |
| 20B       |                 | 360*2,90*2      | 360*2,90*2      |                 |

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4.2. The subgroup \(< t, x >\)

For any \(x, y\) in the arbitrary \(C_G(t)\)-orbit, the subgroup generated by \(t\) and \(x\), i.e. \(< t, x >\), is conjugated to the subgroup generated by \(t\) and \(y\). This is obvious because if we conjugate one of them by \(t\), we get the other. For each \(C_G(t)\)-sub orbit of \(X_C\), the next algorithm provides the algebraic structure of the subgroup generated by \(t\) and random elements of \(C_G(t)\)-sub orbits.

We give the algebraic structure of each subgroup generated by \(< t, x >\) for a random element \(x\) in a \(C_G(t)\)-suborbits of \(X_C\), with their numbers and sizes, for each connected graph mentioned in Tables- 5 and 6. For more information about these subgroups, we refer to several previous works [11, 13, 15].

**Algorithm 3**.

**Input**: The Sub orbits \(O_C\) in the Set \(X_C\);

\[i: \text{For } i \rightarrow 1 \text{ to Size } O_C \text{ Do}\]
\[ii: \text{If } \text{Subgroup} (t, O_C [i][1])) \text{ equal Subgroup} (t, O_C [1][1])) \text{ Then}\]
\[iii: \text{Subgroup} \rightarrow \text{Subgroup} (t, O_C [i][1]));\]
\[iv: \text{Set}_C \rightarrow \text{Set}_C \cup O_C [i];\]
\[v: \text{If Size } \text{Set}_C \text{ equal Size } X_C \text{ Stop}\]
\[vi: \text{Else } X_C \rightarrow X_C \setminus \text{Set}_C\]
\[vii: \text{repeat the steps } i \text{ to } iv \text{ until the step } v \text{ is true}\]

**Output**: The group algebraic structure of the subgroup generated by \(t\) and the suborbits of the sets \(X_C\) in the \(\Delta_i(t)\).

**Table 5-Subgroup Structure of \(< t, x >\) in C(HS, 3A)**

| \(\Delta_i(t)\) | Class of \(t \ast x\) | Number of Orbits | Size of Orbits | Subgroup \(< t, x >\) |
|-----------------|----------------------|-----------------|----------------|------------------------|
| \(\Delta_1(t)\) | 1A                   | 1               | 1              | \(C_3\)               |
|                  | 3A                   | 3               | 60             | \(C_3 \times C_3\)    |
|                  | 3A                   | 1               | 1              | \(C_3\)               |
| \(\Delta_2(t)\) | 2A                   | 2               | 135            | \(A_4\)               |
|                  | 3A                   | 3               | 360            | \(C_3 \times A_4\)    |
|                  | 3A                   | 2               | 135            | \(A_4\)               |
|                  | 5B                   | 1               | 60             | \(A_5\)               |
|                  | 5B                   | 1               | 120            | \(GL(2,4)\)           |
|                  | 6B                   | 3               | 360            | \(C_3 \times A_4\)    |
|                  | 15A                  | 3               | 360            | \(GL(2,4)\)           |
| \(\Delta_3(t)\) | 2A                   | 1               | 90             | \(A_4\)               |
|                  | 3A                   | 1               | 90             | \(A_4\)               |
|                  | 4B                   | 1               | 360            | \(PSL(3,2)\)          |
|                  | 4C                   | 5               | 1620           | \(A_6\)               |
|                  | 4C                   | 4               | 1440           | \(PSL(3,2)\)          |
|                  | 5A                   | 2               | 36             | \(A_5\)               |
|                  | 5B                   | 1               | 360            | \(A_7\)               |
|                  | 5B                   | 1               | 180            | \(A_6\)               |
|                  | 5C                   | 4               | 1440           | \((C_2 \times C_2 \times C_2 \times C_2) : A_6\) |
|                  | 5C                   | 4               | 1440           | \(A_7\)               |
| \( \Delta_0(t) \) | \( \Delta_2(t) \) |
|---|---|
| \( 5C \) | 2 | 720 | \( A_5 \) |
| \( 5C \) | 4 | 1440 | \( A_6 \) |
| \( 5C \) | 4 | 1440 | \( (C_2 \times C_2 \times C_2 \times C_2) : A_5 \) |
| \( 6B \) | 4 | 1440 | \( (C_2 \times C_2 \times C_2) : (C_7 : C_3) \) |
| \( 6B \) | 5 | 1800 | \( A_7 \) |
| \( 6B \) | 1 | 360 | \( A_4 \times A_4 \) |
| \( 7A \) | 4 | 1440 | \( (C_2 \times C_2 \times C_2) : (C_7 : C_3) \) |
| \( 7A \) | 2 | 720 | \( A_7 \) |
| \( 7A \) | 4 | 1440 | \( \text{PSL}(3,4) \) |
| \( 7A \) | 1 | 360 | \( \text{PSL}(3,2) \) |
| \( 8B \) | 2 | 720 | \( (C_2 \times C_2 \times C_2) : \text{PSL}(3,2) \) |
| \( 8A \) | 4 | 1440 | \( (C_2 \times C_2 \times C_2 \times C_2) : A_6 \) |
| \( 8C \) | 2 | 720 | \( (C_2 \times C_2 \times C_2) : \text{PSL}(3,2) \) |
| \( 10A \) | 6 | 2160 | \( \text{HS} \) |
| \( 11A \) | 2 | 720 | \( M_{22} \) |
| \( 11A \) | 1 | 360 | \( \text{HS} \) |
| \( 11B \) | 1 | 360 | \( \text{HS} \) |
| \( 11B \) | 2 | 720 | \( M_{22} \) |
| \( 20A \) | 2 | 720 | \( \text{HS} \) |
| \( 20A \) | 2 | 180 | \( (((C_2 \times C_2 \times C_2) : (C_2 \times C_2)) : C_2) : A_5 \) |
| \( 20B \) | 2 | 720 | \( \text{HS} \) |
| \( 20B \) | 2 | 180 | \( (((C_2 \times C_2 \times C_2) : (C_2 \times C_2)) : C_2) : A_5 \) |
| \( 2B \) | 2 | 36 | \( A_4 \) |
| \( 3A \) | 6 | 1080 | \( (C_4 \times C_4) : C_3 \) |
| \( 3A \) | 2 | 720 | \( C_7 : C_3 \) |
| \( 3A \) | 2 | 36 | \( A_4 \) |
| \( 4B \) | 2 | 720 | \( A_6 \) |
| \( 4B \) | 4 | 360 | \( (C_4 \times C_4) : C_3 \) |
| \( 4B \) | 1 | 360 | \( \text{PSL}(3,2) \) |
| \( 4B \) | 1 | 90 | \( \text{SL}(2,3) \) |
| \( 4C \) | 4 | 1440 | \( \text{PSL}(3,2) \) |
| \( 4C \) | 2 | 180 | \( \text{SL}(2,3) \) |
| \( 4C \) | 2 | 720 | \( (C_4 \times C_4) : C_3 \) |
| \( 5C \) | 8 | 2880 | \( (C_2 \times C_2 \times C_2 \times C_2) : A_5 \) |
| \( 5C \) | 2 | 720 | \( A_6 \) |
| \( 5C \) | 12 | 4320 | \( \text{PSL}(3,4) \) |
| \( 5C \) | 4 | 1440 | \( \text{PSL}(2,11) \) |
| \( 5C \) | 1 | 360 | \( A_5 \) |
| \( 6B \) | 4 | 1440 | \( \text{PSL}(2,11) \) |
|    |     |     |          |                           |
|----|-----|-----|----------|--------------------------|
| 6B | 3   | 270 | SL(2,3)  |
| 7A | 24  | 8640| M_{22}   |
| 7A | 4   | 1440| (C_2 x C_2 x C_2) . PSL(3,2) |
| 7A | 20  | 7200| PSL(3,4) |
| 7A | 8   | 2880| A_7      |
| 7A | 8   | 2880| (C_4 x C_4 x C_4) : (C_7 : C_3) |
| 7A | 4   | 1440| PSL(3,2) |
| 7A | 2   | 720 | C_7 : C_3 |
| 8B | 6   | 2160| PSU(3,5) |
| 8B | 4   | 1440| M_{11}   |
| 8B | 2   | 720 | (C_2 x C_2 x C_2) . PSL(3,2) |
| 8A | 16  | 5760| M_{22}   |
| 8A | 8   | 720 | (C_4 . (C_4 x C_4)) : C_3 |
| 8C | 6   | 2160| PSU(3,5) |
| 8C | 4   | 1440| M_{11}   |
| 8C | 2   | 720 | (C_2 x C_2 x C_2) . PSL(3,2) |
| 10A| 12  | 4320| PSU(3,5) |
| 10A| 6   | 2160| HS       |
| 10A| 2   | 180 | ((C_2 x ((C_4 x C_2) : C_2)) : C_2) : A_5 |
| 10A| 2   | 180 | (((C_2 x C_2 x C_2) : (C_2 x C_2)) : C_2) : A_5 |
| 10A| 2   | 180 | SL(2,5)  |
| 10B| 2   | 720 | HS       |
| 11A| 16  | 5760| M_{22}   |
| 11A| 8   | 2880| HS       |
| 11A| 1   | 360 | PSL(2,11) |
| 11B| 8   | 2880| HS       |
| 11B| 16  | 5760| M_{22}   |
| 11B| 1   | 360 | PSL(2,11) |
| 12A| 8   | 2880| HS       |
| 12A| 8   | 2880| (C_4 x C_4 x C_4) : (C_7 : C_3) |
| 12A| 8   | 720 | (C_4 . (C_4 x C_4)) : C_3 |
| 15A| 6   | 2160| HS       |
| 15A| 2   | 720 | A_8      |
| 20A| 2   | 720 | HS       |
| 20A| 2   | 180 | ((C_2 x ((C_4 x C_2) : C_2)) : C_2) : A_5 |
| 20B| 2   | 720 | HS       |
| 20B| 2   | 180 | ((C_2 x ((C_4 x C_2) : C_2)) : C_2) : A_5 |
Table-6 illustrates the subgroup structure of \(< t, x >\) in the graph \(C(J_2, 3B)\).

| \(\Delta(t)\) | Class of \(t^x\) | Number of Orbits | Size of Orbits | Subgroup \(< t, x >\) |
|-------------|------------------|------------------|----------------|------------------|
| \(\Delta_1(t)\) | 1A | 1 | 1 | \(C_1\) |
| | 3A | 2 | 8 | \(C_1 \times C_1\) |
| | 3B | 2 | 8 | \(C_1 \times C_1\) |
| | 3B | 1 | 1 | \(C_3\) |
| \(\Delta_2(t)\) | 2B | 2 | 24 | \(A_4\) |
| | 3A | 2 | 24 | \(C_1 \times A_4\) |
| | 3B | 2 | 24 | \(A_4\) |
| | 6B | 2 | 24 | \(C_1 \times A_4\) |
| | 3B | 2 | 24 | \(C_1 \times A_4\) |
| | 3B | 2 | 72 | \((C_4 \times C_4) : C_3\) |
| | 4A | 2 | 72 | \((C_4 \times C_4) : C_3\) |
| | 4A | 2 | 72 | \((C_4 \times C_4) : C_3\) |
| \(\Delta_3(t)\) | 2A | 2 | 6 | \(A_4\) |
| | 2B | 2 | 72 | \(A_4\) |
| | 3B | 4 | 78 | \(A_4\) |
| | 3B | 2 | 72 | \((C_4 \times C_4) : C_3\) |
| | 3B | 4 | 48 | \(C_1 \times A_4\) |
| | 4A | 2 | 72 | \((C_4 \times C_4) : C_3\) |
| | 4A | 1 | 36 | \(PSL(3,2)\) |
| | 5C | 2 | 72 | \(HJ\) |
| | 5D | 2 | 72 | \(HJ\) |
| | 6A | 4 | 48 | \(C_1 \times A_4\) |
| | 7A | 1 | 36 | \(PSL(3,2)\) |
| | 7A | 2 | 72 | \(HJ\) |
| | 8A | 4 | 144 | \(PSU(3,3)\) |
| | 10C | 10 | 360 | \(HJ\) |
| | 10D | 10 | 360 | \(HJ\) |
| | 12A | 4 | 144 | \(PSU(3,3)\) |
| | 12A | 2 | 72 | \(HJ\) |
| | 15A | 4 | 144 | \(HJ\) |
| | 15B | 4 | 144 | \(HJ\) |
| \(\Delta_4(t)\) | 2A | 1 | 12 | \(A_4\) |
| | 3B | 2 | 72 | \((C_5 \times C_5) : C_3\) |
| | 3B | 1 | 12 | \(A_4\) |
| $\Delta_d(t)$ |   |   |   |
|---|---|---|---|
| 5A | 2 | 24 | GL(2,4) |
| 5B | 2 | 24 | GL(2,4) |
| 5C | 1 | 36 | $(C_5 \times C_5) : C_3$ |
| 5D | 1 | 36 | $(C_5 \times C_5) : C_3$ |
| 6B | 4 | 144 | HJ |
| 7A | 28 | 1008 | HJ |
| 7A | 8 | 288 | PSU(3,3) |
| 8A | 20 | 720 | HJ |
| 8A | 8 | 288 | PSU(3,3) |
| 10A | 2 | 72 | $A_5 \times A_4$ |
| 10A | 4 | 144 | HJ |
| 10B | 4 | 144 | HJ |
| 10B | 2 | 72 | $A_5 \times A_4$ |
| 10C | 12 | 432 | HJ |
| 10D | 12 | 432 | HJ |
| 12A | 16 | 576 | HJ |
| 15A | 8 | 288 | HJ |
| 15A | 2 | 72 | $A_5 \times A_4$ |
| 15B | 2 | 24 | GL(2,4) |
| 15B | 2 | 72 | $A_5 \times A_4$ |
| 15B | 8 | 288 | HJ |
| 15B | 2 | 24 | GL(2,4) |
| 3B | 2 | 72 | $(C_5 \times C_5) : C_3$ |
| 3B | 2 | 72 | $C_7 : C_3$ |
| 4A | 4 | 48 | $C_3 \cdot A_6$ |
| 5A | 1 | 36 | $(C_3 \times C_3) : C_3$ |
| 5B | 1 | 36 | $(C_5 \times C_5) : C_3$ |
| 5C | 4 | 144 | HJ |
| 5D | 4 | 144 | HJ |
| 6A | 8 | 288 | PSU(3,3) |
| 6A | 4 | 144 | HJ |
| 6B | 20 | 720 | HJ |
| 7A | 22 | 792 | HJ |
| 7A | 2 | 72 | $C_7 : C_3$ |
| 8A | 28 | 1008 | HJ |
| 8A | 4 | 144 | PSU(3,3) |
| 10A | 12 | 432 | HJ |
| 10B | 12 | 432 | HJ |
| 10C | 26 | 936 | HJ |
| 10D | 26 | 936 | HJ |
| 12A | 4 | 144 | PSU(3,3) |
| 15A | 9 | 324 | HJ |
| 15A | 4 | 48 | GL(2,4) |
4.3. Collapsed Adjacency Matrix

Δ_i(t) is a union of certain C_G(t)-orbits. For any Δ_i(t) of C(G, X), let Δ^1_i, Δ^2_i, ..., Δ^r_i be the set of the C_G(t)-orbits in Δ_i(t) of size r. If n is the number of the C_G(t)-orbits of C(G, X), then the collapsed adjacency matrix for C(G, X) is n×n with entry (s, h)^th. The number of vertices in the row is indexed by Δ_h^i, which is connected to a single vertex in the column, indexed by Δ_s^j. For this purpose, we create the following algorithm.

**Algorithm 4**

**Input:** The suborbits Δ^i_h and Δ^i_s;

i: For x → Random in Δ^i_h Do
ii: For y → in Δ^i_s Do
iii: If x commute with y Then
iv: entry entry_s,h → entry_s,h ∪ {y}
v: (s, h)^th → size of entry_s,h.

**Output:** entry (s, h)^th the number of vertices in row, indexed by Δ^i_h, is connected to a single vertex in column, indexed by Δ^i_s.
The collapsed adjacency matrix is an \( n \times n \) matrix, such that \( n \) is the permutation rank given in the Table-1. The matrix is too big to fit in the paper. For this reason, we only give an example for applying Algorithm 4 to find the entry of the collapsed adjacency matrix. Now, we calculate some entries of the \( 399 \times 399 \) collapsed adjacency matrix of \( \text{C}(\text{HS},3\text{B}) \). We provide the entries for such matrix since the technique we use is similar for any matrix size. Therefore, we just give \( 10 \times 10 \) parts of \( 399 \times 399 \) the collapsed adjacency matrix of \( \text{C}(\text{HS},3\text{B}) \), as showing below:

**Table 7-Part of Collapsed Adjacency Matrix \( \text{C}(\text{HS},3\text{A}) \)**

| \( C^t(t) \)-orbits | \( t = \Delta_0 \) | \( \Delta^1_0 \) | \( \Delta^2_0 \) | \( \Delta^1_1 \) | \( \Delta^2_1 \) | \( \Delta^1_2 \) | \( \Delta^2_2 \) | \( \Delta^1_3 \) | \( \Delta^2_3 \) | \( \Delta^1_4 \) | \( \Delta^2_4 \) |
|---------------------|-----------------|---------------|---------------|---------------|---------------|---------------|---------------|---------------|---------------|---------------|---------------|
| \( \Delta^1_0 \)    | 1 1             | 20            | 0             | 0             | 0             | 0             | 0             | 0             | 0             | 0             | 0             |
| \( \Delta^2_0 \)    | 1 1             | 20            | 0             | 0             | 0             | 0             | 0             | 0             | 0             | 0             | 0             |
| \( \Delta^1_1 \)    | 1 1             | 20            | 0             | 0             | 0             | 0             | 0             | 0             | 0             | 0             | 0             |
| \( \Delta^2_1 \)    | 0 0             | 0             | 1             | 0             | 0             | 0             | 0             | 0             | 0             | 0             | 0             |
| \( \Delta^1_2 \)    | 0 0             | 6             | 0             | 1             | 0             | 0             | 0             | 0             | 0             | 0             | 0             |
| \( \Delta^2_2 \)    | 0 0             | 0             | 0             | 0             | 1             | 1             | 0             | 0             | 0             | 0             | 0             |
| \( \Delta^1_3 \)    | 0 0             | 0             | 0             | 0             | 0             | 1             | 1             | 0             | 0             | 0             | 0             |
| \( \Delta^2_3 \)    | 0 0             | 0             | 0             | 0             | 0             | 1             | 1             | 0             | 0             | 0             | 0             |
| \( \Delta^1_4 \)    | 0 0             | 0             | 0             | 0             | 0             | 0             | 1             | 0             | 0             | 0             | 0             |
| \( \Delta^2_4 \)    | 0 0             | 0             | 0             | 0             | 0             | 0             | 0             | 0             | 0             | 0             | 1             |

5. Main Results

The graph \( C(J_2,3\text{A}) \) is disconnected with 280 connected 2-components, as seen above. For a connected commuting involution graph \( C(G,X) \), given in Table-1, the graph structure is described in the following theorem.

**Corollary 5.1.** For \( G \) is one of the groups of Table-2, we have the following results:

- \( \text{Diam } C(J_2,3\text{B}) = 8 \) and \( |\Delta_1| = 18 \), \( |\Delta_2| = 96 \), \( |\Delta_3| = 480 \), \( |\Delta_4| = 2052 \), \( |\Delta_5| = 5304 \), \( |\Delta_6| = 7392 \), \( |\Delta_7| = 1338 \), \( |\Delta_8| = 120 \).
- \( \text{Diam } C(\text{HS},3\text{A}) = 6 \) and \( |\Delta_1| = 62 \), \( |\Delta_2| = 1530 \), \( |\Delta_3| = 27216 \), \( |\Delta_4| = 94392 \).

**Proof.** Each of \( \Delta(t) \) of the commuting graph \( C(G,X) \) is a union of specific \( C^t(t) \)-orbits. Thus, using the previous tables, we obtain the proof.

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