Exact and Approximate Heterogeneous Bayesian Decentralized Data Fusion

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Abstract—In Bayesian peer-to-peer decentralized data fusion, the underlying distributions held locally by autonomous agents are frequently assumed to be over the same set of variables (homogeneous). This requires each agent to process and communicate the full global joint distribution, and thus leads to high computation and communication costs irrespective of relevancy to specific local objectives. This work formulates and studies heterogeneous decentralized fusion problems, defined as the set of problems in which either the communicated or the processed distributions describe different, but overlapping, random states of interest that are subsets of a larger full global joint state. We exploit the conditional independence structure of such problems and provide a rigorous derivation of novel exact and approximate conditionally factorized heterogeneous fusion rules. We further develop a new version of the homogeneous Channel Filter algorithm to enable conservative heterogeneous fusion for smoothing and filtering scenarios in dynamic problems. Numerical examples show more than 99.5% potential communication reduction for heterogeneous channel filter fusion, and a multi-target tracking simulation shows that these methods provide consistent estimates while remaining computationally scalable.

Index Terms—Bayesian decentralized data fusion (DDF), distributed robot systems, multi-robot systems, sensor fusion.

I. INTRODUCTION

Bayesian decentralized data fusion (DDF) has a wide range of applications, such as cooperative localization [1], multi-target tracking [2], multi-robot localization and mapping [3], and more. DDF, while generally less accurate compared to centralized data fusion, offers advantages in terms of scalability, flexibility and robustness. One of the challenges of decentralized data fusion stems from the difficulty of accounting for common data and dependencies between communicating agents and avoiding ‘rumor propagation’, where dependencies between sources are incorrectly ignored, causing data or prior information (e.g. process noise) to be counted more than once.

In decentralized multi-agent systems aiming at some joint mission, such as autonomous cooperative robot localization [1], [4], [5], or tracking [2], the optimal solution can be achieved only if each agent recursively updates and communicates a global full joint posterior probability distribution function (pdf) over an identical (homogeneous) set of random variables (rvs), such that all dependencies in the data can be accounted for. This leads to large overhead in local processing and in communication bandwidth. It is therefore desirable to enable processing, communication and fusion of a posterior pdf over a subset of different but overlapping rvs; we name such a process heterogeneous fusion. Consider for example the 30 robot cooperative localization scenario given in [1]. If each agent has 4 unknown random position states, the full joint distribution has 120 variables, and requires processing a $120 \times 120$ covariance matrix (assuming Gaussian pdf). This includes states of agents ‘far away’ from each other in the network, which has negligible effect on the local position estimate and might be considered ‘irrelevant’. But, if each agent includes in its estimate a subset of only immediate ‘relevant’ neighbors’ states, then the local heterogeneous joint distribution shrinks, e.g. to 16 states for a 3-neighbor topology. This has a clear computation and communication gain over the homogeneous alternative. However it might lead to indirect dependencies between variables not mutually monitored by both agents and result in an overconfident estimate. While there are existing methods that allow for heterogeneous fusion, as detailed later, they are application-specific and do not explore or give general insights for a solution to the full heterogeneous fusion problem, as described in this paper.

DDF algorithms can be considered exact or approximate, depending on how they account for dependencies in the data shared between agents in order to guarantee that every piece of data is counted only once. In exact methods, these dependencies are explicitly accounted for either by pedigree-tracking, which can be cumbersome and impractical in large ad hoc networks [6], or by adding a ‘Channel Filter’ (CF) [7]. Approximate methods assume different levels of dependency between the communicating agents and fuse them in such a way that the common data is promised not to be double counted. This is a necessary condition to ensure conservativeness of the fused posterior pdf, where conservative means that the approximate posterior does not underestimate the uncertainty relative to the true pdf. The most commonly used approximate method is covariance intersection (CI) [8], where agents share only the first two moments (mean and covariance) of their underlying distributions (often representing fusion of estimates), or the geometric mean density (GMD) for general pdf fusion [9]. Critically, it is usually assumed in both exact and approximate fusion that the process takes place over homogeneous states, i.e. that all underlying posteriors describe
the same full set of rvs. Thus, these methods cannot be directly applied to heterogeneous fusion, where pdfs with rvs from different overlapping parts of the full joint pdf are fused. This work’s main insight is to understand how conditional independence properties of the problem can be leveraged and maintained in such cases to achieve correct and beneficial fusion at scale.

The goal of this paper is to define and explore the heterogeneous DDF problem, suggest new fusion rules, and understand their limitations. The paper builds upon the work presented in [10] and further develops a rigorous Bayesian probabilistic approach for fusion of heterogeneous pdfs. In developing heterogeneous fusion rules for robotics applications, challenges arise from: (i) non-Gaussianity of the true underlying system; and (ii) dynamics. To gain a better fundamental understanding of the issues and requirements for corresponding fusion rules applicable to robotics, the analysis is constructed as follows. First, in Sec. [II] the general probabilistic formulation of the homogeneous fusion rule is described. We initially assume a static system and develop two fusion rules by exploiting and maintaining conditional independence properties between subsets of random variables (e.g., states). These fusion rules, namely the Bi-directional factorized fusion (BDF-fusion) and the Heterogeneous state fusion (HS-fusion), allow the communication of only marginal pdfs corresponding to new and relevant data. Next, in Sec. [IV] the static assumption is relaxed and the problem of conditional independence in dynamic systems is discussed with its solution approach. Finally, Sec. [V] completes the theoretical analysis, gives further intuition into the heterogeneous fusion problem, and suggests a practical heterogeneous fusion rules for robotics applications, challenges and the problem of conditional independence in dynamic multi-agent multi-target tracking scenarios (Sec. [VI]). Sec. [VII] then draws conclusions and suggests avenues for additional work.

II. PROBLEM STATEMENT AND RELATED WORK

To motivate the problem, consider a simple static target tracking problem with one common target and two tracking agents as a running example (Fig. 1a). Both agents $i$ and $j$ track the position of the common target, described by the random state vector $x$, and are assumed to have perfect knowledge of their own position, but unknown constant, non-zero, biases in the agent-target relative measurement position measurement described by the local random state vectors $s^i$ and $s^j$, respectively (similar to the bias in [11]).

As shown in Fig. 1a, at each time step $k$, the agents take two types of sensor measurements: (i) relative agent-target, described by $y^i_k$ and $y^j_k$ and (ii) $m^i_k$ and $m^j_k$ to different landmarks to locally collect data on their biases $s^i$ and $s^j$, respectively. It can be seen that the agents’ local biases $s^i$ and $s^j$ become coupled due to measurements $y^i_k$ and $y^j_k$ of the common target $x$. Note that while we choose to use biases in the above example, the local, non-mutual, random states can be any other type of state, e.g., another target not mutually monitored by both agents.

In the case of homogeneous data fusion, the two agents preform inference over and communicate the full joint pdfs describing all rvs $p(x, s^i, s^j)$, including each other’s local biases. But in the heterogeneous fusion case, agents might hold a pdf over only a subset of the rvs $p(x, s^i)$, over the common rv (target $x$) and the agent’s non-mutual rv (local bias $s^j$), making the dependencies between the non-mutual rvs hidden. Therefore an agent might not be aware of the existence of another local bias random state vector (e.g., $s^j$) held by the other agent. These dependencies are key to the problem, and the main challenge in heterogeneous fusion compared to homogeneous fusion is to account for them during fusion, where they are not explicitly represented in the local posterior pdfs.

A. Problem Statement

Assume a network of $n_a$ autonomous agents, performing recursive decentralized Bayesian updates to their prior pdf, with the goal of inferring the states of some global set of rvs $\chi_k \in \mathbb{R}^N$ at time $k$. Each agent $i \in \{1, ..., n_a\}$ is tasked with the inference of a local subset of rvs $\chi_k^i \subseteq \chi_k$, which can be represented by an $n_i$-dimensional vector of random variables. An agent can update its local prior pdf for $\chi_k^i$, by (i) using Bayes’ rule to recursively update a posterior pdf for $\chi_k^i$ with local sensor data $Y_k^i$ (e.g., $Y_k^i = (y^i_k)^T, (m^i_k)^T$ in Fig. 1a) described by the conditional likelihood $p(Y_k^i|\chi_k^i)$.
and (ii) performing peer-to-peer fusion of external data $Z_{k}^{j,-}$ relevant to $\chi_{k}^{i}$ from any neighboring agent $j \in N_{a}^{i}$, where $Z_{k}^{j,-}$ is the local data agent $j$ has at time step $k$, prior to fusion with agent $i$ (i.e. from local sensor data $Y_{k}^{j, i}$ and from information received via fusion with other neighboring agents up to time $k - 1$), and $N_{a}^{i}$ is the set of agents communicating with $i$.

The heterogeneous fusion problem now seeks a peer-to-peer fusion rule $F$ which, given the local prior distribution $P(\chi_{k}^{i} | Z_{k}^{i,+})$ and a distribution $P(\chi_{r,k}^{i,j} | Z_{k}^{i,-})$ over a subset of 'relevant' random states from a neighboring agent $j$, returns a local fused conservative posterior distribution,

$$P_{f}^{j}(\chi_{k}^{i} | Z_{k}^{i,+}) = F[P(\chi_{k}^{i} | Z_{k}^{i,+}), P(\chi_{r,k}^{i,j} | Z_{k}^{j,-})],$$

where $Z_{k}^{i,+} = Z_{k}^{i,-} \cup Z_{k}^{j,-}$ is the combined data after fusion and $\chi_{r,k}^{i,j}$ is the subset of random states at agent $j$ for which it has data to contribute to agent $i$, i.e., are relevant to agent $i$, and is assumed to be a non-empty set. For instance, in the target tracking example, if $\chi_{k}^{i} = [x^{T}, (s^{i})^{T}]^{T}$ and $\chi_{r,k}^{i,j} = [x^{T}, (s^{j})^{T}]^{T}$, then the relevant random states in $j$ are $\chi_{r,k}^{i,j} = x$. However, if $\chi_{k}^{i} = [x_{k}^{i} = [x^{T}, (s^{i})^{T}]^{T}, (s^{i})^{T}]^{T}$, but the agents collect data from local measurements (as in Fig. [1](a)), agent $j$ might only have relevant data regarding the common target $x$ and its local bias $s^{j}$, thus $\chi_{r,k}^{i,j} = [x^{T}, (s^{i})^{T}]^{T}$.

Both of the above examples can be represented by the heterogeneous fusion rule [1]. Thus heterogeneous fusion encompasses the set of problems where the set of relevant rvs $\chi_{r,k}^{i,j}$ is a subset of either agent $i$'s rvs $\chi_{r,k}^{i,j} \subset \chi_{r,k}^{i,j}$ or agent $j$'s rvs $\chi_{r,k}^{i,j} \subset \chi_{r,k}^{i,j}$ or both. As a side note, the case where $\chi_{k}^{i} = \chi_{r,k}^{i,j}$ simplifies to homogeneous fusion.

It is not immediately obvious how to to assess the validity of the sought fusion rule [1], especially for heterogeneous fusion. Briefly, we look for a fusion rule which provides a conservative posterior pdf, i.e., does not underestimate the uncertainty relative to the true pdf. In Gaussian pdfs for example, this means that the difference between the estimated covariance at agent $i$, $\Sigma_{i}$ and a centralized estimator (having all the data from all agents), $\Sigma_{Cen}$, is positive semi-definite (PSD), i.e., $\Sigma_{i} - \Sigma_{Cen} \succeq 0$. The reader is referred to Appendix A for a more detailed discussion on the criteria for a ‘good fusion rule’.

### III. HETEROGENEOUS FUSION

The analysis starts with a general probabilistic formulation of the homogeneous fusion rule. Then as seen in Fig. 2 different fusion problems of interest are discussed. Starting from the homogeneous fusion rule, where all agents keep and share a posterior pdf of the full global joint random state vector
\(\chi\)' cases are then considered where an agent is only interested in and/or observes a subset of the full set of rvs. The key idea is to exploit the structure of the problem, specifically, conditional independence between local sets of rvs. Assuming the dependencies in the data are explicitly tracked, an exact heterogeneous fusion rule, the Bi-directional factorized fusion (BDF-fusion) is developed, in which an agent still infers the global set of rvs, but needs to only communicate a subset of them. By discarding dependencies to non-local rvs, at the cost of approximation of the full global joint pdf, agents are able to hold pdfs over overlapping subsets of rvs. This Heterogeneous state fusion (HS-fusion) rule further improves the scalability of the system, as agents only infer a subset of the data are explicitly tracked, an exact conditional independence between local sets of rvs. Assuming the idea is to exploit the structure of the problem, specifically, in and/or observes a subset of the full set of rvs. The key \[\chi\] problem is exploited to conditionally factorize into relevant \(s_i\) and \(s_j\) \(\chi_C = \chi_C^{ij}\) are conditionally independent from the local states and data at agent \(j\) \(s_j = s^j\) and \(Z_k^{-j}\),

\[\chi_L, Z_k^{-j} \perp \chi_C^i | \chi_C^j \Rightarrow s^j, y_k, m_k^i \perp s^j, y_k, m_k^j|x.\] (4)

Therefore, the factorization in (3) can be further simplified for agents \(i\) and \(j\),

\[p^f(\chi | Z_k^i) = p^f(\chi L | Z_k^i) \cdot p^f(\chi C | Z_k^i) \cdot p^f(\chi C | Z_k^j)^{-1}.\] (5)

The homogeneous fusion rule in (7) can then be conditionally factorized as,

\[p_f(\chi | Z_k^j) = \frac{p_f(\chi C^j | Z_k^j)^{-1}}{p_f(\chi C^j | Z_k^j)} \cdot p_f(\chi L | Z_k^j)^{-1} \cdot p_f(\chi C^j | Z_k^j)^{-1} \cdot p_f(\chi L | Z_k^j)^{-1},\] (6)

where to simplify the pdf in the denominator we used the fact that if \(\chi_L^i\) is conditionally independent from \(Z_k^i\), it is also conditionally independent from the intersection of \(Z_k^i\) and \(Z_k^{j}, i.e., \chi_L^i \perp Z_k^i \cap Z_k^{j} | \chi_{C}^{ij}\). The expressions for the local states fusion \(p_f(\chi L | \chi C^j, Z_k^j)\) and \(p_f(\chi L | \chi C, Z_k^j)\) can be further simplified by recognizing:

\[p_f(\chi L | \chi C^j) = p_f(\chi L | \chi C^j) \text{ and } p_f(\chi L | \chi C) = p_f(\chi L | \chi C).

The heterogeneous BDF-fusion rule can then be written as,

\[p_f(\chi | Z_k^j) = \frac{p_f(\chi L | Z_k^j)^{-1}}{p_f(\chi L | Z_k^j)} \cdot p_f(\chi C^j | Z_k^j)^{-1} \cdot p_f(\chi L | Z_k^j)^{-1} \cdot p_f(\chi C^j | Z_k^j)^{-1}.\] (7)

where \(p_f(\chi L | Z_k^j)\) is defined in (6). The BDF-fusion equation agrees with the intuition described before: if, when conditioning on the common states \(\chi^{ij}_C\), agent \(j\) does not gain any new local data regarding agent \(i\)’s (\(i\)’s) local state \(\chi_L^i\), it need not communicate its respective local conditional pdf \(p_f(\chi L | \chi C^j) (p_f(\chi L | \chi C^i)).\)
Homogeneous Fusion | BDF-Fusion | Heterogeneous Fusion
---|---|---
\[
\begin{align*}
\chi_C ^{ij} & \quad \mathcal{p}^i (\chi) \quad \chi_C ^{ij} \\
\chi_L ^{i} & \quad \mathcal{p}^i (\chi) \quad \chi_L ^{i} \\
\chi_L ^{j} & \quad \mathcal{p}^j (\chi) \quad \chi_L ^{j}
\end{align*}
\]
Inference Task | Full set of rvs $\chi^i = \chi^j = \chi$ | Full set of rvs $\chi^i = \chi^j = \chi$ | Subset of rvs $\chi^i \neq \chi^j \subseteq \chi$

Fig. 2: Progression of fusion rules derived in this paper, describing the set of rvs which needs to be: (i) communicated between any two agents in an undirected acyclic graph; (ii) inferred (estimated) by each agent, corresponding to computation load.

The transition from the homogeneous fusion rule to the heterogeneous BDF-fusion rule is shown in Fig. 2 and is enabled by using the underlying conditional independence structure of the problem to separate common and local variables. This fusion rule provides a possible $\mathcal{F}$ sought in \([1]\): here, the communicated distributions are over different sets of random states, as agent $i$ sends the marginal pdf $\mathcal{p}^i(\chi_C ^{ij}, \chi_L ^{i}|Z_k ^{i,-})$ and receives from $j$ the marginal $\mathcal{p}^j(\chi_C ^{ij}, \chi_L ^{j}|Z_k ^{j,-})$. The sets of relevant states are then $\chi^{ij}_r = \chi_C ^{ij} \cup \chi_L ^{i}$ and $\chi^{ij}_r = \chi_C ^{ij} \cup \chi_L ^{j}$ when fusing at agent $i$ and $j$, respectively.

Note that while the fused posterior pdf in (7) is equivalent to the one achieved by the homogeneous fusion rule (2) for static systems, or when the full time history is maintained), there are two key distinctions between them. First, the math behind this heterogeneous fusion rule is fundamentally different. In the latter, the conditional prior pdf regarding the non-local states $\chi_L ^{j}$ at agent $i$ (and similarly at $j$) is simply replaced by the pdf received from $j$, $\mathcal{p}^j(\chi_C ^{ij}, \chi_L ^{j}|Z_k ^{j,-})$, which treats the common random state $\chi_C ^{ij}$ as a function parameter. Then, the fused marginal is recombined with the two conditional pdfs in the joint distribution via the law of total probability [3] at each agent. This weights the conditional pdf differently as a function of $\chi_C ^{ij}$ and changes the overall joint pdf, thus implicitly/indirectly updating the marginal pdf over $\chi_L ^{j}$. For example, in the target tracking example, that means that while agent $i$ does not directly receive new data from $j$ regarding its local bias $s^i$, its marginal pdf over $s^i$ gets updated due to the fusion of data over $x$ and $s^j$. Second, the BDF-fusion rule offers considerable reduction in communication requirements achieved by sending only new and relevant data (given that the above assumptions are met), as shown later in Sec. V-F.

### C. Approximate Heterogeneous Fusion

So far we considered a set of problems where all agents across the network hold a local pdf over the same global random state vector $\chi$, which includes all locally relevant random states. This requires each agent to hold a local pdf over the full global random state vector and to communicate the conditional pdfs regarding its local random states. Despite its potential communication reduction relative to the homogeneous fusion rule, it might still require considerable communication volume, for example, when $\chi_L ^{i}$ are agent $i$'s local states for a navigation filter, which typically has 16 or more states [23]. By allowing each agent to only reason about a subset of states, e.g., only states within its inference task, significant reductions in both computation and communication requirements can be gained. This motivates the development of an approximate heterogenous fusion rule that scales with the agent tasks and not the global network tasks (or number of agents), we dub it Heterogeneous state (HS) fusion.

More formally, the set of problems where each agent only holds a pdf over its locally relevant subsets of states $\chi^i \subseteq \chi$ is considered. Here the heterogeneous subsets of random states are defined as $\chi^i = \chi_C ^{ij} \cup \chi_L ^{i}$ and $\chi^j = \chi_C ^{ij} \cup \chi_L ^{j}$ for agents $i$ and $j$, respectively. By marginalizing out the set of ‘irrelevant’ states, $\chi_L ^{i} (\chi_L ^{j})$, the fusion rule for each agent, over their locally relevant random states, can be written as

\[
\begin{align*}
\mathcal{p}^i_f (\chi^i | Z_k ^{i,+}) & \propto \mathcal{p}^i_{f} (\chi_C ^{ij} | Z_k ^{j,-}) \cdot \mathcal{p}^j (\chi_C ^{ij}, \chi_L ^{i} | \chi_L ^{j}, Z_k ^{j,-}) \\
\mathcal{p}^j_f (\chi^j | Z_k ^{j,+}) & \propto \mathcal{p}^f (\chi_L ^{i} | Z_k ^{i,-}) \cdot \mathcal{p}^j (\chi_C ^{ij}, \chi_L ^{j} | \chi_L ^{i}, Z_k ^{i,-}).
\end{align*}
\]

HS-fusion gives another fusion rule $\mathcal{F}$ for the problem statement in [1], where the sets or relevant states are now $\chi^{ij}_r = \chi_C ^{ij}$. Notice that while the two pdfs are over different sets of random states, the marginal pdfs over the common random state vector $\chi_C ^{ij}$ held by both agents will be equal, i.e., $\mathcal{p}^i_f (\chi_C ^{ij} | Z_k ^{j,-}) = \mathcal{p}^j_f (\chi_C ^{ij} | Z_k ^{i,-})$. The advantages of this fusion rule in scalability is demonstrated in Sec. V-F.
D. Fusion Algorithm

Decentralized fusion algorithms, in general, are built out of two main steps: sending out a message and fusion of incoming messages. In homogeneous fusion rules, messages are over the same full state vector $\chi$. In heterogeneous fusion, on the other hand, either the communicated or the local distributions are over different random state vectors. Thus there is a need to clarify what are the step that an agent $i$ needs to perform to locally construct and fuse messages to or from its $N_a^i \geq 2$ neighboring agents.

1) Constructing Messages: In BDF-fusion, an agent $i$ holds a posterior distribution over the full global random state vector $\chi$. Assuming the communication topology for the network of agents is an acyclic undirected graph, agent $i$ needs to communicate to any of its neighboring agents $j \in N_a^i$ a distribution over the set of local states $\chi^i = \chi_L^j \cup \chi_C^j$ and the set $\chi_{ij}^i$ (see Sec.III-D3 for definition and explanation). On the other hand, in HS-fusion, agent $i$’s local distribution is only over the set of local relevant states $\chi^i$, and only communicates the common subset $\chi_C^i$. Agent $i$ thus sends agent $j$ the following marginal distributions,

\[
\begin{align*}
\text{BDF-CF:} & \quad p^i_{ij}(\chi^i \cup \chi_{ij}^i) = \int p^i(\chi)d\chi_{\sim i} \quad \forall j \in N_a^i \\
\text{HS-CF:} & \quad p^i_{ij}(\chi_C^i) = \int p^i(\chi^i \setminus \chi_C^i)d\chi_{\sim i} \quad \forall j \in N_a^i,
\end{align*}
\]

where $\chi_{\sim i} = \chi \setminus \{\chi^i \cup \chi_{ij}^i\}$ and the dependency on the data $Z^i$ is implied from here on for brevity and will be explicitly shown if needed.

2) Fusing Messages: Since, in general, agent $i$ has a different sets of random states in common, $\chi_C^i$, with any of its neighboring agents $j \in N_a^i$, the local fusion equations requires multiplying (and dividing) pdfs over different sets of random states. The following equations detail the heterogeneous fusion operation from the perspective of an agent $i$, communicating with its $n_a^i$ neighboring agents,

\[
\begin{align*}
\text{BDF-CF:} & \quad p^i_j(\chi) = p^i(\chi^i \cup \chi_{ij}^i) \cdot \prod_{j \in N_a^i} \frac{p^i_{ij}(\chi^i \cup \chi_{ij}^i)}{p^i_{C,i}(\chi_C^i)}, \\
\text{HS-CF:} & \quad p^i_j(\chi^i) = p^i(\chi^i) \cdot \prod_{j \in N_a^i} \frac{p^i_{ij}(\chi_{ij}^i)}{p^i_{C,i}(\chi_C^i)}.
\end{align*}
\]

Note that for these equations to be valid, the local subsets must be conditionally independent given the common subsets $\chi_C^i$. However, in problems corresponding to dynamic or partially-dynamic Bayesian networks, e.g., in Fig. 1(b), it is generally not possible to claim conditional independence of local states (or local data) based on the filtered dynamic state. In Fig. 1(b), $s_i \neq s_j | x_k$, i.e., $s_i$ and $s_j$ are not conditionally independent given $x_k$ when it is successively marginalized over time. Thus there is a need to regain conditional independence in dynamic stochastic systems for the above fusion rules to work correctly.

There are two approaches to solve this problem: (i) keeping a distribution over the full time history $p(\chi_{C,k:1}, \chi_L|Z_k)$, where $\chi_{C,k:1}$ denotes all common dynamic rvs from $k = 1$ until current time step $k$; (ii) enforcing conditional independence after marginalization by conservative approximations which disconnects the dependencies between the relevant random states. In the following, these solutions for the case of general pdfs are discussed. Then, Sec. IV derives specific closed-form representations for Gaussian distributions, and a pseudo-code, summarizing the different steps each agent takes to perform heterogeneous fusion is given in Algorithm 1.

IV. CONDITIONAL INDEPENDENCE IN DYNAMIC SYSTEMS

To describe and analyze the problem of conditional independence for heterogeneous fusion in dynamic systems, we assume (without loss of generality) that the local subset $\chi_L$ describe static rvs, and that the common subsets $\chi_C$ describes dynamic rvs. Note that the opposite case, where the local subset is dynamic and common subset is static, is simpler, since filtering preserves the required conditional independence assumption.

For the heterogeneous fusion algorithms in (10) to be valid, the local subsets must be conditionally independent given the common subsets $\chi_C$. However, in problems corresponding to dynamic or partially-dynamic Bayesian networks, e.g., in Fig. 1(b), it is generally not possible to claim conditional independence of local states (or local data) based on the filtered dynamic state. In Fig. 1(b), $s_i \neq s_j | x_k$, i.e., $s_i$ and $s_j$ are not conditionally independent given $x_k$ when it is successively marginalized over time. Thus there is a need to regain conditional independence in dynamic stochastic systems for the above fusion rules to work correctly.

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A. Augmented State

The distribution over the full augmented random state $\chi_{k:1} = \chi_{C,k:1} \cup \chi_L$, given the data $Z_k$, can be recursively
updated using the following formula [24]:

\[
p^i(\chi_{k:1}|Z_k^{i,+}) \propto p(\chi_{k-1:1}|Z_k^{i,+})p(\chi_{C,k}|\chi_{C,k-1})p(Y_k^i|\chi_k),
\]

where here \(Z_k^{i,+}\) is used to indicate an agent’s data after the previous fusion step, \(k-1\), \(Y_k^i\) for the local sensor data gained at the current time step \(k\), and \(Z_k^{i,-}\) is the data at time step \(k\) prior to fusion.

Keeping the full time history of the dynamic common states maintains the conditional independence assumption, as can be seen for example in Fig. 1(b). Here \(\chi_{C,k-1} = x_{k-1}, \chi_k = [(s^i)^T, (s^j)^T]^T\) and the conditional independence assumption \(s^i \perp s^j | x_{k-1}\) holds.

While the augmented state approach leads to an increase in the communication and computation requirements as the size of the state vector \(\chi_{k:1}\) increases, efficient inference algorithms that exploit the Markovian property of the dynamic system and its structure can be designed. For example, for Gaussian distributions, the information matrix structure is close to block-diagonal (see Sec. V-C), i.e., marginalization can be done efficiently. Furthermore, the communication requirement can be bounded, as in practice agents do not need to communicate the full state time history. Instead the communicated pdf can be reduced to be on a time window relative to the size of the network; this guarantees information can propagate to the ‘far’ end of the network graph. For example, in the chain network in Fig. 3, the data that agent 1 gathers at time step 1 will reach agent 6 after 5 communication steps. Thus after 5 time steps, there is no new data to communicate over the target position at time step 1.

B. Conservative Filtering

Full knowledge over past system states has been assumed thus far, which enables conditional independence between two agents local states and the derivation of new heterogeneous fusion rules. However, in many distributed fusion applications it is desirable to maintain only a limited time window of recent state history. Thus, marginalizing out past states into a smaller sliding window of recent time steps might be favored, as maintaining the full accumulated state densities results in rapid state dimension growth and yields computation and communication burden. While marginalization is rather trivial for homogeneous fusion problems, in heterogeneous fusion extra care must be taken to maintain conditional independence. Without loss of generality, for the rest of the paper, a small sliding window of only the current time step (as done in the Kalman-filter (KF) for example) will be used.

Given a joint distribution \(p(\chi_{C:2:1}, \chi^L_i, \chi^L_j)\), described by the PGM of Fig. 1(b), for example, marginalizing out \(\chi_{C,1} = x_1\), as done in filtering, results in coupling of all the variables in its Markov blanket, \(\chi_{C,2} = x_2, \chi^L_i = s^i\) and \(\chi^L_j = s^j\). Since conditional independence between \(\chi^L_i\) and \(\chi^L_j\) is an imperative assumption in the basis of the proposed fusion rules, it is necessary to retain it after marginalization. Thus, the goal is to approximate the dense distribution \(p(\chi_{C:2}, \chi^L_i, \chi^L_j)\) by a conservative sparse distribution,

\[
\tilde{p}(\chi_{C:2}, \chi^L_i, \chi^L_j) \propto p(\chi_{C:2})p(\chi^L_i|\chi_{C:2})p(\chi^L_j|\chi_{C:2}).
\]

For the pdf to be consistent, the approximation \(\tilde{p}(\chi_{C:2}, \chi^L_i, \chi^L_j)\) has to be conservative w.r.t. \(p(\chi_{C:2}, \chi^L_i, \chi^L_j)\). Loosely speaking, this means the approximate distribution \(\tilde{p}(\cdot)\) does not underestimate the uncertainty of the true distribution \(p(\cdot)\). A more detailed discussion and definitions of consistency and conservativeness as treated in this paper is given in Appendix A.

The above discussion is brought here for completeness, but the treatment of conservative sparse marginalization of general pdfs is out of scope of this paper and is left for future work. However, for Gaussian pdfs, where there is agreement of the definition of conservativeness, a conservative sparse marginalization solution is derived in the next section (Sec. V-D).

V. HETEROGENEOUS CHANNEL FILTER - A CLOSED FORM ALGORITHM

Thus far, the problem and derivation are stated in general pdf terms. From here, to further the understanding and increase the intuition of the heterogeneous fusion problem, we explicitly track the information flow and dependencies in heterogeneous DDF problems by: (i) extending the (homogeneous) channel filter (CF) algorithm to heterogeneous DDF; (ii) focus our attention to the information (canonical) form of the Gaussian distribution, and linear models.

The homogeneous CF: The CF is a simple method to track dependencies in the data for a network of agents performing homogeneous DDF [7]. Over the past two decades, the CF algorithm has been used in many practically fielded robotics applications, such as search tasks in multi-UAV [25] and human-robot teams [26], terrain estimation [27], and multi-target tracking [28] where it has been used to fuse information between particle filters. The core idea of the CF is to add a filter on the communication channel between any pair of agents. This filter explicitly tracks \(p_{ij}(\cdot)\) for example, the posterior pdf conditioned on the common data shared (over the channel) by agents \(i\) and \(j\). For homogeneous fusion, [7] shows that each agent is able to recover the optimal centralized state estimate if: (i) the communication graph between the \(n_a\) agents is undirected and acyclic (such that data does not circle back to its sender), e.g., tree or chain communication topology; (ii) there is full rate communication and sequential processing of incoming data, i.e., an agent sends a message at each time step \(k\) without delays, then processes incoming messages one after the other; and (iii) the dynamic system and measurement models are linear with additive white Gaussian noise (AWGN). Note that we choose to use the CF for its conceptual simplicity, as it allows us to directly compute the fused marginal pdf, \(p_f(\chi^{ij})\), in (7) and (8), and as a way to gain fundamental understanding into the heterogeneous fusion problem. However, the algorithm presented is general in the sense that different methods to explicitly track or approximate \(p_{ij}(\cdot)\) can be used when assumptions (i)-(iii) don’t necessarily hold, e.g., by pedigree tracking [6] or GMD approximation [9], respectively.

The Gaussian information form: The information form of the Gaussian distribution is particularly convenient for
deriving and describing key steps in data fusion processing, for example in the information filter, the CF [7], the CI algorithm [8], and more. This allows to directly ‘read’ conditional independence from the information matrix, develop closed-form heterogeneous fusion rules, namely summing and subtracting the sufficient statistics (information vector and information matrix), and to suggest approximations for conservative marginalization, where the term ‘conservative’ is defined in clear terms.

A. Preliminaries

Assume the full joint distribution over the random state vector \( \chi \), is a multivariate Gaussian with mean \( \mu \) and covariance matrix \( \Sigma \). The pdf in information form for the normally distributed state \( \chi \), with information vector \( \zeta \) and information matrix \( \Lambda \) is [29]:

\[
p(\chi; \zeta, \Lambda) = \frac{\exp(-\frac{1}{2} \zeta^T \Lambda^{-1} \zeta)}{\sqrt{\det(2\pi \Lambda)}} \exp \left( -\frac{1}{2} \chi^T \Lambda^{-1} \chi \right),
\]

(13)

with

\[
\zeta = \Sigma^{-1} \mu = \left( \zeta_{XC}, \zeta_{XL} \right), \quad \Lambda = \Sigma^{-1} = \begin{pmatrix} \Lambda_{XCXC} & \Lambda_{XCLC} \\ \Lambda_{XLCX} & \Lambda_{XLXL} \end{pmatrix}.
\]

(14)

Here \( \chi_C \) and \( \chi_L \) are the common and local subsets of random states, respectively. This pdf can also be expressed using factorization [3], where the marginal and conditional distributions of a Gaussian are also Gaussian,

\[
p(\chi_C) = N^{-1}(\chi_C; \zeta_{XC}, \Lambda_{XCXC}),
p(\chi_L|\chi_C) = N^{-1}(\chi_L|\chi_C; \zeta_{XL|XC}, \Lambda_{XLCX}).
\]

(15)

with \( N^{-1} \) representing the information form of the Gaussian distribution \( N \) and \( (\zeta_{XC}, \Lambda_{XCXC}) \) and \( (\zeta_{XL|XC}, \Lambda_{XLCX}) \) are the sufficient statistics for the marginal and conditional pdfs in information form, respectively, defined as [30]:

\[
\tilde{\zeta}_{XC} = \zeta_{XC} - \Lambda_{XCXL} \Lambda_{XLXL}^{-1} \zeta_{XL},
\Lambda_{XCXC} = \Lambda_{XCXC} - \Lambda_{XCXL} \Lambda_{XLXL}^{-1} \Lambda_{XLXC},
\zeta_{XL|XC} = \zeta_{XL} - \Lambda_{XLXC} \chi_C, \quad \Lambda_{XL|XC} = \Lambda_{XLXL}.
\]

(16)

B. Fusion

To develop the closed form heterogeneous CF algorithms we start with the original homogeneous fusion rule [2]. By substituting linear Gaussian distributions, taking logs and differentiating once for the fused information vector \( \zeta_f \) and twice for the fused information matrix \( \Lambda_f \), [21] the following fusion equations can be obtained,

\[
\zeta_f = \zeta_i + \zeta_j - \zeta_{ij}, \quad \Lambda_f = \Lambda_i + \Lambda_j - \Lambda_{ij}.
\]

(17)

These equations are the basis of the original linear-Gaussian CF [7], which explicitly tracks the ‘common information’ vector and matrix \( (\zeta_{ij}, \Lambda_{ij}) \), describing the pdf over \( \chi \) conditioned on the common data between communicating pairs of agents \( i \) and \( j \), \( p_f(\chi_i| \chi_j^{t-1} \cap \chi_j^{t-1}) \) in an undirected acyclic communication graph.

Define \( \tilde{\zeta}_{\chi_C^i,j,f} \) and \( \tilde{\Lambda}_{\chi_C^i,j,f} \) to be the fused marginal information vector and matrix, respectively, over the common random state \( \chi_C^{ij} \) between agents \( i \) and \( j \), corresponding to \( p_f(\chi_C^{ij}|Z^+) \), represented in information form. Without loss of generality, the fused marginal information vector and matrix can be achieved using different fusion methods, exact and approximate (CI [8] for example). This paper restricts attention to the CF for exact fusion. Then by using (17), the fused marginal information vector and matrix for Gaussian distributions are given by

\[
\tilde{\zeta}_{\chi_C^{ij},f} = \tilde{\zeta}_{\chi_C^{i},f} + \tilde{\zeta}_{\chi_C^{j},f} - \tilde{\zeta}_{\chi_C^{ij},c},
\tilde{\Lambda}_{\chi_C^{ij},f} = \tilde{\Lambda}_{\chi_C^{i},f} + \tilde{\Lambda}_{\chi_C^{j},f} - \tilde{\Lambda}_{\chi_C^{ij},c}.
\]

(18)

The HS-fusion rule (8) for Gaussian pdfs is now dubbed HS-CF and is represented by the simple closed form expression,

\[
\zeta_f = \begin{pmatrix} \zeta_{\chi_C^{i},f} \\ 0 \end{pmatrix} + \begin{pmatrix} \Lambda_{\chi_C^{i},f}^{-1} C_{\chi_C^{i}Z} X_i \\ \Lambda_{\chi_C^{j},f}^{-1} C_{\chi_C^{j}Z} X_j \end{pmatrix},
\Lambda_f = \begin{pmatrix} \tilde{\Lambda}_{\chi_C^{i},f} & \tilde{\Lambda}_{\chi_C^{i},f} \Lambda_{\chi_C^{j},f}^{-1} C_{\chi_C^{i}Z} X_j \\ \Lambda_{\chi_C^{j},f}^{-1} C_{\chi_C^{j}Z} X_i & \tilde{\Lambda}_{\chi_C^{j},f} \end{pmatrix}.
\]

(19)

where \( \tilde{\zeta}_{\chi_C^{ij},f} \) and \( \tilde{\Lambda}_{\chi_C^{ij},f} \) are given in [18] and \( \chi_L^j \in \chi_L \) is the subset of \( i \)'s local random states. An equivalent expression for the fused information vector and matrix at agent \( j \) is achieved by switching \( i \) with \( j \).

It is important to note, as seen from (18), that the fused marginal pdf \( p(\chi_C^{ij}|Z^+) = N^{-1}(\tilde{\zeta}_{\chi_C^{ij},f}, \tilde{\Lambda}_{\chi_C^{ij},f}) \) is the same for agents \( i \) and \( j \). However, the conditional part is kept local (the right part of (19)), which means that after fusion, the local joint distributions in \( i \) (w.r.t. \( \chi_i = \chi_C^{i} \cup \chi_L^i \) and \( j \) (w.r.t. \( \chi_j = \chi_C^{j} \cup \chi_L^j \) are different. While agents only update the information vector and matrix of the marginal pdf, over the common random state \( \chi_C^{ij} \), the local joint distribution in moment representation (e.g., \( \mu_j^i, \Sigma_j^i \)) will be updated, thus also updating the local states \( \chi_L^i \) (\( \chi_L^j \)).
linear-Gaussian version of (32), which we dub the Information Augmented State (iAS) smoother.

Consider a dynamic system, described by the linear discrete time equations
\[
\begin{align*}
x_k &= F_k x_{k-1} + G u_k + \omega_k, \quad \omega_k \sim \mathcal{N}(0, Q_k) \\
y_k &= H_k x_k + v_k, \quad v_k \sim \mathcal{N}(0, R_k),
\end{align*}
\]
where \(F_k\) is the state transition matrix, \(G\) is the control effect matrix and \(H_k\) is the sensing matrix. \(\omega_k\) and \(v_k\) are the zero mean white Gaussian process and measurement noise, respectively.

The predicted information vector and matrix for the time step \(k - 1\) are given by
\[
\begin{align*}
\zeta_{k-1|k-1} &= \begin{pmatrix} Q_k^{-1} & 0 \\ 0 & Q_k^{-1} \end{pmatrix} \begin{pmatrix} G u_k \\ F^T Q_k^{-1} G u_k \end{pmatrix} \\
\Lambda_{k-1|k-1} &= \begin{pmatrix} Q_k^{-1} & -F^T Q_k^{-1} \\ -F Q_k^{-1} & Q_k^{-1} F \end{pmatrix} \begin{pmatrix} A_{k-1|k-1} & 1 \end{pmatrix} + F^T Q_k^{-1} F,
\end{align*}
\]
where \(F = [F_k - 1 \ 0_{m \times (k-n-2)}]\) and \(m\) is the size of the (not augmented) state vector.

For completeness, the measurement update in Gaussian information space is (31)
\[
\begin{align*}
\zeta_{k|n} &= \zeta_{k-1|k-1} + J_k i_k \\
\Lambda_{k|n} &= \Lambda_{k-1|k-1} + J_k I_k J_k^T,
\end{align*}
\]
where \(J_k = [I_m \ 0_{m \times (k-n-1)}]^T\), \(i_k = H_k^T R_k^{-1} y_k\) and \(I_k = H_k^T R_k^{-1} H_k\).

In linear-Gaussian problems, the above iAS can be used to resolve the communication load problem which scales with the size of the network - as messages need to propagate through from one end of the network graph to the other. Instead, a filtering approach is taken to marginalize past states to process only a sliding window \(k : n\) \((n > 1)\) while maintaining conditional independence. This requires conservative filtering, discussed in the next section.

D. Conservative Filtering

In the graphSLAM literature the idea of disconnecting dependencies between landmarks due to marginalization of past robot states is known as conservative sparsification [33], [44]. A dense information matrix is approximated by a sparse one, with the goal of reducing the computational complexity when reasoning over the map. The approach of Vial et al. [33] is adopted here with the goal of enforcing conditional independence after marginalization of past states.

The sparse structure of the marginalized approximate information matrix is enforced by removing the links between \(\chi^L_i\) and \(\chi^L_j\). In other words, given a true dense Gaussian distribution \(\mathcal{N}(\zeta_{tr}, \Lambda_{tr})\), a sparse approximate distribution \(\mathcal{N}(\zeta_{sp}, \Lambda_{sp})\) is sought such that the mean is unchanged and the approximation is conservative in the PSD sense,
\[
\Lambda_{tr}^{-1} \zeta_{tr} - \Lambda_{sp}^{-1} \zeta_{sp}, \quad \Lambda_{tr} - \Lambda_{sp} \succeq 0,
\]
where again the information form of the Gaussian distribution is used. A visualization of the information matrix is given in Fig. 5 where bulleted-filled cells mean non-zero terms and empty cells mean zero terms in the information matrix, with the latter indicating conditional independence. Fig. 5(b) shows that filtering causes direct dependencies between local rv subsets, indicated by the filling of corresponding cells in the information matrix. Then, we wish to regain conditional independence by setting matrix terms to zero, or by ’emptying’ cells as shown in Fig. 5(c). To ensure conservativeness the information matrix has to be deflated, shown by faded gray color of the cells. Note that since an agent \(i\) in HS-fusion only estimate their own local states \((\chi^L_i)\) and not their neighbor’s \((\chi^L_j)\), the suggested conservative marginalization method only applies to BDF-fusion.

Reference [33] minimizes the Kullback-Leibler Divergence (KLD) to find a lower bound on the dense true information matrix \(\Lambda_{tr}\). Along similar lines, [35] suggests a method named ‘Uniform Pre-Transmission Eigenvalue-Based Scaling’ to conservatively approximate a covariance matrix \(\Sigma\) by inflating a diagonal matrix \(D\) built out of the diagonal entries of the full matrix \(\Sigma\). To achieve a conservative approximation \(D_c\), \(D\) is inflated by multiplying by the largest eigenvalue of \(Q = D^{-\frac{1}{2}} \Sigma D^{-\frac{1}{2}}\). This results in \(D_c = \lambda_{max} D\) such that \(D_c - \Sigma \succeq 0\).

This method is generalized here to find a lower bound sparse information matrix \(\Lambda_{sp}\) and regain conditional independence between \(\chi^L_i\) and \(\chi^L_j\). This new generalized method differs from the one suggested in [35] in two ways. Firstly, the approximation, \(\Lambda_{sp}\), is allowed to be any information matrix achieved by setting any off-diagonal elements of the true dense information matrix \(\Lambda_{tr}\) to zero, i.e., the resulting matrix is in general not diagonal or even block-diagonal (e.g., as in Fig. 5(c)). Note that since the information matrix (i.e. not the covariance) is changed, setting off-diagonal elements to zero directly controls the conditional independence structure of the underlying distribution. Specifically for the purpose of this paper, terms relating local random states (e.g., \(\chi^L_i\) and \(\chi^L_j\)) in \(\Lambda_{tr}\) are set to zero to regain conditional independence given common states (e.g., \(\chi_{C,k:n}\)).

The second change from the original method is that the information matrix is approximated, as opposed to the covariance matrix. This means that a lower bound is sought and not an upper bound, i.e. ‘information’ must be deflated, instead of uncertainty being inflated. This is achieved by choosing the minimal eigenvalue of \(\tilde{Q} = \Lambda_{sp}^{-2} \Lambda_{tr} \Lambda_{sp}^{-2}\), resulting in
\[
\Lambda_{tr} - \lambda_{min} \Lambda_{sp} \succeq 0,
\]
where $\lambda_{\text{min}}\Lambda_{\text{sp}}$ is the sought of conservative marginal sparse approximation of the dense information matrix $\Lambda_{tr}$. The new information vector is computed to such that (24) holds

$$\zeta_{\text{sp}} = (\lambda_{\text{min}}\Lambda_{\text{sp}})\Lambda_{\text{tr}}^{-1}\zeta_{\text{tr}},$$  

(26)

### E. Closed-Form Algorithm

This subsection provides a full summary description of the different steps each agent $i$ takes to locally process, communicate and fuse data with its neighbor $j$. While the steps described in the pseudo code in Algorithm 1 are general in the sense that it can be applied with any pdf, it assumes linear-Gaussian distributions and uses the theory developed above to provide equation references (green arrows) for the closed-form expressions of the different operations. The algorithms are named BDF-CF and HS-CF, as the agents use the CF to track dependencies in the data and are used as an example to detail the algorithm from the perspective of one agent $i$ communicating with a neighbor $j$.

**Algorithm 1** BDF-CF / HS-CF algorithm

1. Define: $x_i$, $x_{ij}$, $x_{ij}^C$, $x_{ij}^L$, prior pdfs
2. for All time steps do
3. Propagate state local states
4. Propagate common states in the CF
5. if BDF-CF then
6. Conservative filtering
7. else if HS-CF then
8. Marginalize out past state
9. end if
10. Measurement update
11. Send message
12. Fuse received message
13. Update common information
14. end for
15. return

F. Calculation of Communication and Computation Savings

To highlight the potential gain of the proposed heterogeneous fusion rules with respect to communication and computation complexity and how they change with scale, three numerical examples (small, medium and large) of a multi-agent multi-target tracking problem are presented. Consider the problem introduced earlier of tracking $n_t$ ground targets by $n_a$ agents (trackers), where each agent computes a local KF estimate, i.e., the system dynamics are assumed to be linear with additive Gaussian white noise (20). Each agent $i \in N_a$ has 6 unknown local position states described by the random vector $x_i = s_i$ and takes measurements to $n_t$ targets, each having 4 position/velocity states described by the random vector $x_t$ (e.g., east and north coordinates and velocities). The full state vector then has $6n_a + 4n_t$ random states. Assume a tree topology in ascending order, where each agent tracks 1/2/3 targets, corresponding to the small/medium/large examples, respectively, but has only one target in common with its neighbor. Now, using the same logic as before, assume that each agent is only concerned with the targets it takes measurements of, and its own position states. Each agent has only 10/14/18 local ‘relevant’ random states for tracking 1/2/3 targets, respectively.

Table 1 presents a comparison between the different heterogeneous fusion algorithms, when a channel filter is used (Algorithm 1), for the three different scale problems, in a filtering scenario. The baseline for comparison is the original (homogeneous) CF, with each agent estimating the full state vector. For the communication data requirement, double precision (8 bytes per element) is assumed. Since the matrices are symmetric covariances, agents only need to communicate $n(n + 1)/2$ upper diagonal elements, where $n$ is the number of random states. Each agent’s computation complexity is determined by the cost of inverting an $n \times n$ matrix. It can be seen from the table that even for the small scale problem, the communication data reduction is significant: the BDF-CF requires about 42.7% of the original CF, while the HS-CF requires only 9.2% as agents only communicate common targets states information vectors and matrices. These gains then increase with scale, for the medium and large problems the BDF-CF communication is about 33% of the original CF and for the HS-CF is less than 1%.

Another important aspect is computation complexity. As seen from the table, while the BDF-CF algorithm requires each agent to process the full random state vector, the HS-CF offers significant computational reduction. Since each agent...
only processes the locally relevant random states, the HS-CF scales with subset of states and not with number of agents and targets. In the medium and large scale problems, as the size of the full system random vector states increase, the HS-CF computation complexity is less than 1% of the other algorithms, which can be critical in terms of computing power for resource-constrained platforms.

VI. SIMULATION STUDIES

Multi-agent multi-target tracking simulation scenarios were performed to compare and validate the proposed algorithms. Since the dynamics and measurement models are assumed to be linear with Gaussian noise, Algorithm [II] is used together with the iAS as the inference engine, i.e., agents estimate the sufficient statistics (information vector and matrix) of the random state vector. First, the algorithms are tested on a static target case, where conditional independence of the local states can be easily guaranteed. This is followed by a dynamic target test case with only two agents and one target, to validate and compare the smoothing and the conservative filtering approaches. Lastly, the conservative filtering approach is used for a more challenging 4-agents 5-target scenario. Results for all the different scenarios are based on Monte Carlo simulations and compare the new algorithms to an optimal centralized estimator.

A. Example 1 - Static Case

A chain network, consisting of five agents connected bidirectionally in ascending order (1 ← 2 ← 3 ← 4 ← 5), as depicted in (Fig. 3), attempts to estimate the position of six stationary targets in a 2D space. Assume each tracking agent \( i = 1, ..., 5 \) has perfect self position knowledge, but with a constant agent-target relative position measurement bias vector in the east and north directions \( s^i = [b_e^i, b_n^i]^T \). In every time step \( k \), each agent takes two kinds of measurements: one for the target and one to collect data on the local sensor bias random vector, which can be transformed into the linear pseudo-measurements,

\[
\begin{align*}
\hat{y}_{k}^{i,t} &= x^t + s^i + v_k^{i,1} \sim \mathcal{N}(0, R_k^{i,1}), \\
m_i^t &= s^i + v_k^{i,2} \sim \mathcal{N}(0, R_k^{i,2}),
\end{align*}
\]

where \( \hat{y}_{k}^{i,t} \) is agent \( i \)'s relative measurement to target \( t \) at time step \( k \) and \( m_i^t \) is a measurement to a known landmark at time step \( k \) for bias estimation. \( x^t = [e^t, n^t]^T \) is the east and north position of target \( t = 1, ..., 6 \). The tracking assignments for each agent, along with the measurements noise error covariances for the relative target \( (R_k^{i,1}) \) and landmark \( (R_k^{i,2}) \) measurements are given in Table II and illustrated in Fig. 3. The relative target measurement noise characteristics for different targets measured by the same agent are taken to be equal. For example, agent 1 takes noisy measurements to targets 1 and 2 with \( 1 m^2 \) and \( 10 m^2 \) variances in the east and north directions, respectively, and \( 3 m^2 \) in both directions for the landmark.

Following the definitions from Sec. III the full state vector includes 22 random states

\[
\chi = [(x^1)^T, ..., (x^6)^T, (s^1)^T, ..., (s^5)^T]^T,
\]

where for the HS-CF fusion, define the local random state vector at agent \( i \)

\[
\chi_i = [(X_i)^T, (s_i)^T]^T.
\]

Here \( T_i \) is the set of targets observed by agent \( i \) and \( X_i \) includes all target random state vectors \( x^t, s, t \in T_i \). In other words, the local random state vector at each agent includes only locally relevant targets and the local biases. In the HS-CF, where two agents \( i \) and \( j \) only share the marginal statistics regarding common states, messages should only consist data regarding targets \( t \in T_i \cap T_j \). For example, according to Table II and the network tree topology, for agents 1 and 2: \( T_1 \cap T_2 = T_2 \), i.e. the common random state is \( x^2 \). Notice that the local subset of rvs is not limited to only one type of state. For example, agents 1 and 5 has target position state vectors \( x^1 \) and \( x^6 \), respectively, in their local subset, since they are the only agents tasked with the corresponding targets.

The data communication requirements for this relatively small system were calculated: similar to the results from Sec. III the BDF-CF and the HS-CF require about 38% and 2.6% of the original CF communication data requirements, respectively.

The BDF-CF and the HS-CF performance was tested with 500 Monte Carlo simulations and compared to a centralized estimator. As mentioned before, in the BDF-CF each platform processes the full random state vector [25], while in the HS-CF each platform processes only the locally relevant random states [29]. In the simulations fusion occurs in every time step.

Fig. 6 shows a NEES chi-square consistency test [36], [37] results for agents 1 and 4 and a centralized estimator. Results

| Agent | Targets | \( R_k^{i,1}[m^2] \) | \( R_k^{i,2}[m^2] \) |
|-------|---------|---------------------|---------------------|
| 1     | \( T_1, T_2 \) | \( x^2 \) | \( s^1, x^1 \) | diag[1,10] | diag[3,3] |
| 2     | \( T_2, T_3 \) | \( x^2, x^4 \) | \( s^2 \) | diag[3,3] | diag[3,3] |
| 3     | \( T_3, T_4, T_5 \) | \( x^1, x^4, x^5 \) | \( s^3 \) | diag[4,4] | diag[2,2] |
| 4     | \( T_4, T_5 \) | \( x^4, x^5 \) | \( s^4 \) | diag[10,1] | diag[4,4] |
| 5     | \( T_5, T_6 \) | \( x^5 \) | \( s^5, x^6 \) | diag[2,2] | diag[5,5] |

TABLE II: Local platform target assignments, common and local rv sets and sensor measurement error covariances.

TABLE I: Data communication requirements and computational complexity for different fusion algorithms, for different problem scales.
where with time-varying acceleration control, simulation. Here the target follows a linear dynamics model. This is shown using a two agent, one (dynamic) target tracking scenario. Shown is a comparison between iAS and filtering. The plots in figure (a) show the NEES chi-square consistency tests (75 simulations, 95% confidence level) for agent 1, where the centralized (filtering) results are suggested, the first using the iAS approach, the BDF-CF was conservative with minimal eigenvalues of much higher computation and communication load. 

**B. Example 2 - Dynamic Case**

In dynamic systems, as discussed in Sec. [IV] there is a challenge in maintaining conditional independence. Two ways to solution are suggested, the first using the iAS, thus keeping a distribution over the full time history over target random states, which is costly in both communication and computation requirements. The second, more efficient solution, is to perform conservative filtering by enforcing conditional independence in the marginalization step and deflating the information matrix (Algorithm [I]). Since this process loses information due to deflation, the BDF-CF becomes an approximate solution and becomes pessimistic relative to the centralized estimator (see appendix [A]). To validate, the local covariance matrix must be checked to see whether it is consistent relative to the centralized covariance. One simple test is by checking if the eigenvalues of \( \Sigma_{\chi} - \Sigma_{\chi,\text{cent}} \) are equal or larger than zero, where \( \Sigma_{\chi} \) is the agent’s covariance and \( \Sigma_{\chi,\text{cent}} \) is the centralized marginal covariance over \( \chi \). In the above simulations the minimal eigenvalues between all agents and all simulations were 0, for both the BDF-CF and the HS-CF, thus they are conservative in the PSD sense.

### Example 2 - Dynamic Case

In dynamic systems, as discussed in Sec. [IV], there is a challenge in maintaining conditional independence. Two ways to solution are suggested, the first using the iAS, thus keeping a distribution over the full time history over target random states, which is costly in both communication and computation requirements. The second, more efficient solution, is to perform conservative filtering by enforcing conditional independence in the marginalization step and deflating the information matrix (Algorithm [I]). Since this process loses information due to deflation, the BDF-CF becomes an approximate solution and is expected to be less accurate than the iAS implementation. This is shown using a two agent, one (dynamic) target tracking simulation. Here the target follows a linear dynamics model with time-varying acceleration control,

\[
x_{k+1} = Fx_k + Gu_k + \omega_k, \quad \omega_k \sim N(0, 0.08 \cdot I_{n_x \times n_x}),
\]

where

\[
F = \begin{bmatrix} 1 & \Delta t & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \Delta t \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad G = \begin{bmatrix} \frac{1}{2} \Delta t^2 & 0 \\ 0 & \Delta t \\ 0 & \frac{1}{2} \Delta t^2 \\ 0 & \Delta t \end{bmatrix}.
\]

The acceleration input in the east and north directions is given by \( u_k = [a_e \cdot \cos(d_e \cdot k \Delta t), a_n \cdot \sin(d_n \cdot k \Delta t)]^T \), where \( a_e/a_n \) and \( d_e/d_n \) define the east and north amplitude and frequency, respectively. The measurement model is as in the

![Figure 6: Example 1 (static) - NEES chi-square test based on 500 Monte Carlo simulations, where the dashed lines show bounds for 95% confidence level. Shown are test results of agents 1 and 4 using for different fusion methods, the results indicate all methods produce a consistent estimate.](image)

![Figure 7: Results from a 75 Monte Carlo simulation of a 2 agent, 1 target dynamic target tracking scenario. Shown is a comparison between iAS and filtering. (a) NEES chi-square test with the upper figure showing the BDF-CF compared to the centralized (filtering) and the lower showing the HS-CF. Here the dashed black lines show bounds for 95% confidence level. (b) RMSE comparisons between the BDF-CF and the HS-CF for iAS and filtering approaches for agent 1 (upper) and agent 2 (lower).](image)
measurement parameters defined by the first 4 agents in Table II. The advantages of the BDF-CF and HS-CF regarding communication and computation costs are highlighted again, as the BDF-CF saves 58% in communication costs relative to the original CF, and the HS-CF saves 94.5% in communication and 87.5% in computation complexity.

Results of 500 Monte Carlo simulation with filtering for agents 1 and 4 are presented in Fig. 8 (similar results for agents 2-3 are not presented to avoid clutttering the figure). The plots in (a) show the NEES chi-square consistency test with 95% confidence bound marked with black dashed lines. The upper plot shows the centralized (black circles) and the BDF-CF for agents 1 (blue squares) and 4 (red x) NEES statistics for the 28-state vector. The lower plot shows the HS-CF results, which has a smaller 10-state random vector. (b) shows the corresponding RMSE for agent 1 (upper plot) and 4 (lower plot). Note that the RMSE results for the centralized estimate and BDF-CF, which hold distributions over the full 28-state vector, are marginalized and computed only over relevant local 10 agent random states for this comparison.

It is seen from the NEES statistics plots that, as expected, the centralized estimator produces a consistent MMSE estimate, and the BDF-CF overestimates the uncertainty due to the information matrix deflation (covariance inflation) in the conservative filtering step. The BDF-CF also produces a conservative MMSE estimate relative to the centralized in the PSD sense for all agents, since the minimum eigenvalue between the agents is positive ($3e-04$). The HS-CF is slightly overconfident for both the consistency test and the PSD test, with negative minimal eigenvalue of $-0.26$. However, the degree of non-conservativeness in the HS-CF will in general be highly problem- and topology dependent. Hence, the choice of whether to task agents with the full random state vector, with either homogeneous DDF methods (e.g., classical CF and conventional CI) or heterogeneous fusion with the BDF-CF, or to task them with only a subset of relevant random states using the HS-CF, will hinge on the desired tradeoff in communication/computation complexity vs. resulting overconfidence in state MMSE estimates, provided that the HS-CF allows for stable convergence.

The HS-CF overconfidence is attributed to inaccurate removal of implicit and hidden correlations due to marginalization in the filtering step (line 8 in Algorithm I). Correctly accounting for these dependencies is not in the scope of this paper, but is the focus of ongoing work [38].

VII. CONCLUSIONS

Heterogeneous fusion defines a key family of problems for Bayesian DDF, as it enables flexibility for large scale autonomous sensing networks. As shown in this work, separating the global joint distribution into smaller subsets of local distributions significantly reduces local agent computation and communication requirements. The heterogeneous fusion rules, analysis and derivations presented in this paper, assume acyclic networks for the purposes of exact fusion via the BDF- and HS-fusion algorithms, thus offer a basis for developing and analyzing similar algorithms. These assumptions can be gradually relaxed to solve more general heterogeneous scenarios involving exact or approximate fusion in more complex networked fusion settings.

Probabilistic graphical models (PGMs) were used here to develop Bayesian DDF algorithms. PGMs provided insight into the origin of the dependencies between random states not mutually tracked by two agents and enabled exploitation of the conditional independence structure embedded in these graphs. This led to two novel heterogeneous fusion rules for general probabilistic and Gaussian pdfs that were demonstrated on (but not limited to) static and dynamic target tracking problems. The latter motivated the development and use of the linear information augmented state (iAS) smoother to regain conditional independence, on the expense of increasing computation and communication costs. To overcome this problem a conservative filtering approach was demonstrated to maintain conditional independence over a small time window, without the need of the full time history.

The DDF framework naturally enables sparse distributed estimation for high-dimensional state estimation and the heterogeneous fusion rules represent a practical and theoretical shift in the state of the art, subject to usual provisos and limitations of DDF. From a practical standpoint, the algorithms developed in this paper can already be used to improve scalability in a variety of decentralized Bayesian estimation problems, such as cooperative localization and navigation [23], multi-agent SLAM [3] and terrain height mapping [27], where a height distribution is estimated on a grid map. In this case, for example, the BDF-CF can be used to reduce communication in the network by dividing the the map into several overlapping regions of interest, allowing agents to communicate only regarding those cells in which they have
new data to contribute. This scales the communication with the number of locally observed grid cells instead of the entire map.

Indeed, some works are already leveraging heterogeneous DDF ideas for robotics, [1], [2], despite the gap in theoretical guarantees and understanding on the full nature of the problem and its limitations. This paper makes progress by building theoretical foundations for future research and surfacing a discussion on the assumptions and definitions of homogeneous DDF, as they appear to be inadequate for real world robotics problems of heterogeneous fusion. Heterogeneous fusion, as defined in this paper, requires a careful revisit of the idea of ‘ideal’ centralized/decentralized Bayesian estimation as well as the definitions of consistency and conservativeness for general (non-Gaussian) pdfs and more specifically in the case of heterogeneous pdfs in dynamic systems.

APPENDIX A
WHAT IS A GOOD FUSION RULE?

In the problem formulation (Sec. II) it is left to define a good fusion rule $F$ and how to evaluate it. In a recent paper, Lubold and Taylor [39] claim that a fusion rule should provide a posterior pdf which is conservative, i.e., “overestimates the uncertainty of a system”. They suggest new definitions for conservativeness, but to the best of our knowledge it is not widely used. In the context of homogeneous fusion, common definitions use the terms consistent [4] and conservative interchangeably [8], [40] and assume that the uncertainty of a point estimate can be described by its mean and covariance. In the following, the intuition regarding conservativeness from [41] is combined with the common definitions of consistency from homogeneous fusion to define a ‘good’ heterogeneous fusion rule $F$, firstly in terms of pdfs and then in the case of Bayesian point estimation.

From the standpoint of pdf fusion, a good heterogeneous fusion rule $F$ results in an updated local posterior pdf that: (i) does not underestimate the uncertainty relative to the true pdf, where Bar-Shalom et al. dub this ‘dynamic (filter) consistency’ [37]; and (ii) is conservative over the agent’s random states of interest $\chi_i^k \subseteq \chi_k$ relative to a consistent centralized estimators’ marginal pdf over $\chi_i^k$,

$$p_i(\chi_i^k) \geq \int p_{\text{cent}}(\chi_k) d\chi_i^k,$$  

(32)

where ‘$\geq$’ denotes conservative and $\chi_i^k = \chi_k \setminus \chi_i^k$ is the set of variables not included in agent $i$’s random states of interest.

The centralized pdf refers to the posterior pdf over the full random state vector $\chi_k$ conditioned on all the available data from all the agents up and including time step $k$:

$$p_{\text{cent}}(\chi_k | \bigcup_{i \in N_k} Z_i^{k-1}).$$

Since consistency and conservativeness are often defined by the first two moments of the pdf, i.e., the mean and covariance, the above definition can be further narrowed in the context of Bayesian point estimation. A good heterogeneous fusion rule $F$ in this case is then one that when forming a point estimate from its resulting local posterior $\tilde{p}(\chi_i^k | Z_i^{k+})$, for example by finding the minimum mean squared error (MMSE) estimate, the estimate: (i) does not underestimate the uncertainty relative to the true state error statistics, and (ii) is conservative relative to the marginal error estimate of a consistent centralized point estimator.

For example, assume the means of the Gaussian random state vectors $\chi_i$ and $\chi$ are $\mu_i$ and $\mu$, and the covariances, describing the mean squared error, are $\Sigma_i = E[(\chi_i - \mu_i)(\chi_i - \mu_i)^T]$ and $\Sigma = E[(\chi - \mu)(\chi - \mu)^T]$, respectively, where $E[\cdot]$ is the expectation operator. The actual values are unknown, and the approximate estimate of them is given by $\Sigma_i \preceq \Sigma$, $\Sigma_i \succeq \Sigma_i$, and $\Sigma_i \preceq \Sigma_i \preceq \Sigma_i$. The definitions above then translate to the following:

1) Not underestimating of the uncertainty relative to the true error statistics implies that $\Sigma_i \preceq \Sigma_i \preceq 0$, i.e., the resulting matrix difference is PSD.

2) Conservativeness relative to the marginal estimate of the centralized estimator implies that $\Sigma_i \succeq \Sigma_i \succeq 0$, where $\Sigma_i$ is the marginal covariance over $\chi_i$, from the joint centralized covariance over $\chi$ (32).

The centralized estimate in this case can be considered consistent if, for example, it passes the NEES chi-square test [36], [37]. Note that since a consistent centralized estimate neither overestimates nor underestimates the uncertainty, a conservative (higher uncertainty) local estimate is expected to not underestimate the uncertainty. Thus, requiring the local estimate to be conservative relative to a consistent centralized estimate implicitly requires it to not underestimate the uncertainty of the true error statistics.

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