A WEIGHTED SOBOLEV SPACE THEORY FOR THE
DIFFUSION-WAVE EQUATIONS WITH TIME-FRACTIONAL
DERIVATIVES ON $C^1$ DOMAINS

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(Communicated by Enrico Valdinoci)

Abstract. We introduce a weighted $L^p$-theory ($p > 1$) for the time-fractional
diffusion-wave equation of the type

$$\partial_t^\alpha u(t, x) = a^{ij}(t, x) u_{x_i x_j}(t, x) + f(t, x), \quad t > 0, x \in \Omega,$$

where $\alpha \in (0, 2)$, $\partial_t^\alpha$ denotes the Caputo fractional derivative of order $\alpha$, and
$\Omega$ is a $C^1$ domain in $\mathbb{R}^d$. We prove existence and uniqueness results in Sobolev
spaces with weights which allow derivatives of solutions to blow up near the
boundary. The order of derivatives of solutions can be any real number, and
in particular it can be fractional or negative.

1. Introduction. In this article we deal with the weighted Sobolev space theory
for the time-fractional partial differential equation

$$\partial_t^\alpha u = a^{ij} u_{x_i x_j} + b^i u_{x_i} + c u + f, \quad t > 0, x \in \Omega$$

(1.1)
on $C^1$ domains in $\mathbb{R}^d$. Here $\alpha \in (0, 2)$ and $\partial_t^\alpha$ is the Caputo fractional derivative of
order $\alpha$. The coefficients depend on $(t, x)$, and Einstein’s summation convention on
$i, j \in \{1, 2, \ldots, d\}$ is assumed.

The equation describes different phenomena according to the range of $\alpha$. If
$\alpha = 1$ then the equation becomes the classical heat equation which represents the
heat propagation in homogeneous media. If $\alpha \neq 1$ then the equation describes
anomalous diffusion. In particular, if $\alpha \in (0, 1)$, then it describes subdiffusive
aspect of anomalous diffusion, caused by particle sticking and trapping effects (see e.g. [21, 22]), and if $\alpha \in (1, 2)$, the fractional wave equation gives information on
wave propagation in viscoelastic media (see e.g. [19, 20]).

Since the boundary of $\Omega$ is not supposed to be regular enough, in general second
and higher order derivatives of solutions are not square integrable, and therefore one
has to look for solutions in appropriate weighted Sobolev spaces allowing derivatives
of solutions to blow up near the boundary. The weighted Sobolev space we use in this
article is the one introduced by Krylov in [11, 16]. Our interest in such weighted

2020 Mathematics Subject Classification. 45D05, 45K05, 45N05, 35B65, 26A33.

Key words and phrases. Time-fractional equation, Caputo fractional derivative, Sobolev space
with weights, variable coefficients, $C^1$ domains.

The authors were supported by the National Research Foundation of Korea(NRF) grant funded
by the Korea government(MSIT) (No. NRF-2019R1A5A1028324).

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Sobolev space comes from the theory of stochastic partial differential equations (SPDEs in short). Even on $C^\infty$ domains, Hölder space and Sobolev space without weights are inappropriate to obtain general regularity results for SPDEs because the second and higher order derivatives of solutions to SPDEs blow up substantially fast near the boundary of domains. On the other hand, it turns out that Krylov’s weighted Sobolev space perfectly fits for SPDEs on $C^1$ domains (see e.g. [10, 12, 14]). In the near future we plan to extend our work to the stochastic counterpart of equation (1.1) on $C^1$ domains.

In this article we prove the unique solvability result of equation (1.1) in weighted Sobolev spaces. We estimate arbitrary-order derivatives of solutions with the help of appropriate weights. In other words, the order of derivatives of solutions can be any real number and it varies according to the regularity condition of the free term $f$. In particular, we prove that for the solution to equation (1.1) with zero initial and zero boundary data we have

$$\int_0^T \int_\Omega \left( |\rho u_{xx}|^p + |ux|^p + |\rho^{-1} u|^p \right) \rho^{\theta-d}dxdt \leq C \int_0^T \int_\Omega |\rho f|^p \rho^{\theta-d}dxdt \quad (1.2)$$

provided that $p > 1$ and $\theta \in (d-1, d-1 + p)$. Here $\rho(x) = \text{dist}(x, \partial \Omega)$.

We remark that if $\alpha = 1$, then estimate (1.2) is not new and was proved in [16] (on a half space) and [11] (on general $C^1$ domains). Our approach is quite different from the case $\alpha = 1$ and many new difficulties arise due to the presence of the nonlocal operator. For instance, if $\alpha = 1$ then one can estimate $\rho^{-1} u$ in (1.2) from the chain rule $\partial_1^t (|u|^p) = p|u|^{p-2}uu_t$ and appropriate integration by parts (see [16, Lemma 6.3]). However if $\alpha \neq 1$ we do not have such fancy formula, and estimation of $u$, not to mention its derivatives, becomes quite nontrivial.

As far as we know, related works for the case $\alpha \neq 1$ have been done only on the entire spaces, half space and $C^2$ domains, and derivatives of solutions up to the second order were estimated in classical $L_p$ spaces without weights. Below we list some of mostly related works. In [28] an $L_p$-theory of the divergence-type equation was proved on bounded $C^2$ domain with the restriction $\alpha \in (0, 1)$, $d \geq 2$, and $p > d + 2/\alpha$. Also, in [27] more general abstract parabolic Volterra equation was studied in $L_p$ space on entire space and in a half line under the condition $\alpha \notin \{1/p, 1 + 1/p\}$ on entire space and $\alpha \notin \{1/p, 2/(2p-1), 1 + 1/p, 1 + 3/(2p-1)\}$ in a half line. Those algebraic conditions on $\alpha, p$ are dropped in [8] for equation (1.1) on $\mathbb{R}^d$ under the assumption that the coefficients $a^{ij}(t, x)$ are uniformly continuous in $(t, x)$. The continuity condition assumed in [8] is considerably weakened in [8] (on entire space) and [2] (on half space). However in [3, 2] only the case $\alpha \in (0, 1)$ is covered.

The novelty of this article is that it provides a general weighted Sobolev space theory on non-smooth domains, that is on $C^1$ domains, without any restriction on $p$ and $\alpha$, and arbitrary order derivatives of solutions are estimated in appropriate $L_p$ spaces. Moreover, we allow the coefficients of the equation to substantially oscillate or blow up near the boundary of the domain (see Assumption 2.7).

This article is organized as follows. In Section 2, we introduce some definitions and facts related to fractional calculus and weighted Sobolev spaces, and we also present our main results, Theorem 2.10 and Theorem 2.12. In Section 3, we prove our main results when the coefficients are constants, and the general cases are proved in Section 4 and Section 5.
Finally we introduce some notations used in this article. We use "\(\triangleq\)" to denote a definition. As usual, \(\mathbb{R}^d\) and \(\mathbb{N}\) stand for the \(d\)-dimensional Euclidean space of points \(x = (x_1, \ldots, x_d) := (x_1, x')\) and set of natural numbers respectively. \(\mathbb{R}_+^d := \{x = (x_1, \ldots, x_d) \in \mathbb{R}^d : x^1 > 0\}, \quad B_r(x) := \{y \in \mathbb{R}^d : |x - y| < r\}, \quad \text{and } B_r := B_r(0)\). For \(i = 1, \ldots, d\), multi-indices \(\beta = (\beta_1, \ldots, \beta_d)\), and functions \(u(t, x)\) we set
\[
\partial_i u = \frac{\partial u}{\partial t}, \quad u_{x_i} = \frac{\partial u}{\partial x_i} = D_i u, \quad D^\beta u = D_1^{\beta_1} \cdots D_d^{\beta_d} u, \quad |\beta| = \beta_1 + \cdots + \beta_d.
\]
We also use the notation \(D^m\) (or \(D^m\)) for a partial derivative of order \(m\) with respect to \(x\). By \(C^\infty_c(\mathcal{O})\), we denote the collection of infinitely differentiable functions with compact support in \(\mathcal{O}\), where \(\mathcal{O}\) is an open set in \(\mathbb{R}^d\) or \(\mathbb{R}^d + 1\). For \(p > 1\) and a normed space \(F\) by \(L_p(\mathcal{O}; F)\) we denote the set of \(F\)-valued Lebesgue measurable function \(u\) on \(\mathcal{O}\) satisfying
\[
\|u\|_{L_p(\mathcal{O}; F)} := \left(\int_\mathcal{O} \|u(x)\|_F^p \, dx\right)^{1/p} < \infty.
\]
Generally, for a given measure space \((X, \mathcal{M}, \mu), \; L_p(X, \mathcal{M}, \mu; F)\) denotes the space of all \(F\)-valued \(\mathcal{M}\)-measurable functions \(u\) so that
\[
\|u\|_{L_p(X, \mathcal{M}, \mu; F)} := \left(\int_X \|u(x)\|_F^p \, \mu(dx)\right)^{1/p} < \infty,
\]
where \(\mathcal{M}\) denotes the completion of \(\mathcal{M}\) with respect to the measure \(\mu\). If there is no confusion for the given measure and \(\sigma\)-algebra, we usually omit the measure and the \(\sigma\)-algebra. We denote by
\[
\mathcal{F}(f)(\xi) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-i\xi \cdot x} f(x) \, dx, \quad \mathcal{F}^{-1}(g)(x) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{i\xi \cdot x} g(\xi) \, d\xi,
\]
the Fourier and the inverse Fourier transforms of \(f\) and \(g\) in \(\mathbb{R}^d\) respectively. We use the notation \([a^{ij}]_{m \times n}\) for a \(m \times n\) matrix with entries \(a^{ij}\), \(a \wedge b := \min\{a, b\}\), and we write \(a \sim b\) if there are constants \(N_1\) and \(N_2\) independent of \(a, b\) such that \(N_1 a \leq b \leq N_2 a\). If we write \(N = N(a, b, \cdots)\), this means that the constant \(N\) depends only on \(a, b, \cdots\).

2. Main result. First, we introduce some definitions and facts related to the fractional calculus. For more details, see e.g. \([1, 23, 24, 25]\). For \(\alpha > 0\) and \(\varphi \in L_1((0, T))\), the Riemann-Liouville fractional integral is defined as follows.
\[
I^{\alpha} \varphi(t) = (I^{\alpha}_t \varphi)(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} \varphi(s) \, ds, \quad t \leq T,
\]
where \(\Gamma(\alpha)\) is the gamma function.

One can easily check that
\[
I^{\alpha + \beta} \varphi(t) = I^{\alpha} I^{\beta} \varphi(t), \quad \alpha, \beta > 0,
\]
and for any \(p \geq 1\),
\[
\|I^{\alpha} \varphi\|_{L_p((0, T))} \leq N(\alpha, p, T)\|\varphi\|_{L_p((0, T))}.
\]
Let \(n\) be an integer such that \(n - 1 < \alpha < n\). If \(\varphi\) is \((n - 1)\)-times differentiable, and \((\frac{d}{dt})^{n-1} I^{\alpha-n} \varphi\) is absolutely continuous on \([0, T]\), then the Riemann-Liouville fractional derivative \(D^\alpha_t \varphi\) and the Caputo fractional derivative \(\partial^\alpha_t \varphi\) are defined by
\[
D^\alpha_t \varphi := \left(\frac{d}{dt}\right)^n (I^{\alpha-n}_t \varphi), \quad \partial^\alpha_t \varphi := D^\alpha_t \left(\varphi(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} \varphi^{(k)}(0)\right).
\]
In particular, if \( \varphi(0) = \varphi^{(1)}(0) = \cdots = \varphi^{(n-1)}(0) = 0 \), then \( D_0^n \varphi = \partial_t^n \varphi \). Also it is easy to show that, for any \( \alpha, \beta \geq 0 \), \( D_0^n D^{\beta} \varphi = D^{\alpha+\beta} \varphi \) and

\[
D_0^n D^{\beta} \varphi = \begin{cases} 
D_0^{\alpha-\beta} \varphi & \text{if } \alpha > \beta \\
I_0^{\beta-\alpha} \varphi & \text{if } \alpha \leq \beta.
\end{cases}
\]

In particular, if \( \varphi \in C^2([0, T]) \) and \( \alpha \in (0, 2) \), then \( \partial_t^\alpha \varphi = f \) if and only if

\[
\varphi(t) - \varphi(0) - \int_0^t \varphi'(0) s \, ds = \int_0^t f(s) \, ds, \quad \forall t \in [0, T].
\]  

Next we introduce some function spaces and their properties. For \( p > 1 \) and \( \gamma \in \mathbb{R} \), let \( H_p^\gamma \) denote the class of all tempered distributions \( u \) on \( \mathbb{R}^d \) such that

\[
\|u\|_{H_p^\gamma} := \|(1 - \Delta)^{\gamma/2} u\|_{L_p} < \infty,
\]

where

\[
(1 - \Delta)^{\gamma/2} u = \mathcal{F}^{-1} \left((1 + |\xi|^2)^{\gamma/2} \mathcal{F}(u)\right).
\]

It is well known that if \( \gamma \in \{0, 1, 2, \ldots\} \), then

\[
H_p^\gamma = W_p^\gamma := \{u : D^\beta u \in L_p(\mathbb{R}^d), \forall |\beta| \leq \gamma\}.
\]

To define weighted Sobolev spaces, we set \( \psi(x) := x^1 \) if \( \Omega = \mathbb{R}_+^d \), and if \( \Omega \) is a bounded \( C^1 \) domain then we choose an infinitely differentiable function \( \psi(x) \) which is comparable to \( \rho(x) := \text{dist}(x, \partial \Omega) \), that is \( \psi \sim \rho \), and for any multi-index \( \beta \),

\[
\sup_{\Omega} \rho^{(\beta)} |D^\beta \psi| \leq N(\beta) < \infty.
\]

(See e.g. [17]). We also fix a nonnegative function \( \zeta \in C_c^\infty(\mathbb{R}_+) \) such that

\[
\sum_{n \in \mathbb{Z}} \zeta^p(e^n t) > c > 0, \quad \forall t \in \mathbb{R}_+,
\]

where \( c \) is a constant. It is easy to check that any non-negative function \( \zeta \) satisfies (2.2) if \( \zeta > 0 \) on \([1, e]\). For \( x \in \Omega \) and \( n \in \mathbb{Z} \), define

\[
\zeta_n(x) = \begin{cases} 
\zeta(e^n \psi(x)) & : x \in \Omega \\
0 & : x \notin \Omega.
\end{cases}
\]

Note that \( \zeta_n(x) = 0 \) if \( x \in \partial \Omega \) and is sufficiently close to \( \partial \Omega \). Thus,

\[
\zeta_n \in C_c^\infty(\Omega), \quad \sup_{\Omega} |D^m \zeta_n(x)| \leq N e^{mn}.
\]

For any distribution \( u \) on \( \Omega \), since \( \zeta_n \in C_c^\infty(\Omega) \), we can consider \( u \zeta_n \) as a distribution on \( \mathbb{R}^d \). For \( \gamma, \theta \in \mathbb{R} \) and \( p > 1 \), let \( H_{p, \theta}^\gamma(\Omega) \) denote the set of all distributions \( u \) on \( \Omega \) such that

\[
\|u\|_{H_{p, \theta}^\gamma(\Omega)} := \sum_{n \in \mathbb{Z}} e^{n\theta} \|\zeta_n(e^n \cdot) u(e^n \cdot)\|_{H_p^\gamma} < \infty.
\]

Note that if \( \Omega = \mathbb{R}_+^d \), then \( \psi(x) = x^1 \) and \( \zeta_n(x) = \zeta(e^{-n} x^1) \), and therefore (2.3) becomes

\[
\|u\|_{H_{p, \theta}^\gamma(\mathbb{R}_+^d)} := \sum_{n \in \mathbb{Z}} e^{n\theta} \|\zeta(\cdot) u(e^n \cdot)\|_{H_p^\gamma}.
\]

The spaces \( H_{p, \theta}^\gamma(\Omega) \) are independent of the choice of \( \psi \) and \( \zeta \), and the norms introduced by other choices of \( \psi \) and \( \zeta \) are all equivalent (see [16, 18] for details). In particular, for any \( \eta \in C_c^\infty(\mathbb{R}_+) \) and \( \gamma \in \mathbb{R} \),

\[
\sum_{n \in \mathbb{Z}} e^{n\theta} \|\eta(\cdot) u(e^n \cdot)\|_{H_p^\gamma} \leq N(\eta, \zeta, \gamma, p) \|u\|_{H_{p, \theta}^\gamma(\mathbb{R}_+^d)}.
\]
Furthermore, if \( u \) is a non-negative integer then
\[
\|u\|_{H_{p,\theta}^\gamma(\Omega)} \sim \sum_{|\beta| \leq \gamma} \int_\Omega |\rho^{|\beta|}(x)D^\beta u(x)|^p \rho^{-\theta}(x)dx.
\]
(2.6)

Furthermore, if \( u \) has a support in a strip \( A_{a,b} := \{ x \in \Omega : a < \psi(x) < b \} \), where \( 0 < a < b \), then \( u \in H_{p,\theta}^\gamma(\Omega) \) if and only if \( u \in H_{p,\theta}^\gamma(\Omega) \). Moreover \( \|u\|_{H_{p,\theta}^\gamma(\Omega)} \) and \( \|u\|_{H_{p,\theta}^\gamma(\Omega)} \) are comparable, that is
\[
N^{-1}\|u\|_{H_{p,\theta}^\gamma(\Omega)} \leq \|u\|_{H_{p,\theta}^\gamma(\Omega)} \leq N\|u\|_{H_{p,\theta}^\gamma(\Omega)},
\]
(2.7)

where \( N = N(\theta, p, a, b) \).

Fix \( \kappa_0 \in (0, 1) \). For \( 0 < \delta < 1 \), \( k = 0, 1, \ldots \), and \( \gamma \in \mathbb{R} \), set
\[
\rho(x, y) = \rho(x) \wedge \rho(y), \quad [f]_0^{(0)} = \sup_{|\beta| = k, x \in \Omega} \sup_{y \in \Omega} \rho(x, y)^k |D^\beta f(x)|,
\]
\[
[f]_{k+\delta}^{(0)} = \sum_{j=0}^k [f]_j^{(0)}, \quad |f]_{k+\delta}^{(0)} = |f]_{k}^{(0)} + [f]_{k+\delta}^{(0)}.
\]

We also use the norm \( |f|_{|\gamma|+} \) for functions \( f \) defined on \( \mathbb{R}^d \), which is defined by formally taking \( \rho(x) = \rho(x, y) = 1 \) above (see e.g. [4]).

It is well known that if \( 0 \leq \mu < \nu < \infty \), then
\[
|f]_{\mu}^{(0)} \leq N(d, \mu, \nu) \left( \sup_{x \in \Omega} |f(x)| \right)^{1-\mu/\nu} \left( |f]_{0}^{(0)} \right)^{\mu/\nu}.
\]
(2.8)

Also, for any \( \mu \geq 0 \), and \( k = 1, 2, \ldots \),
\[
|\psi^k Df]_{\mu}^{(0)} \leq N|\psi^{k-1}f]_{\mu+1}^{(0)}.
\]
(2.9)

See e.g. [11, Lemma 3.5] for (2.8) and [11, Lemma 2.8, Lemma 3.4] for (2.9).

Below we collect some well known properties of the weighted Sobolev space \( H_{p,\theta}^\gamma(\Omega) \). For \( \delta \in \mathbb{R} \), we write \( u \in \psi^\delta H_{p,\theta}^\gamma(\Omega) \) if \( \psi^{-\delta}u \in H_{p,\theta}^\gamma(\Omega) \).

**Lemma 2.1.** Let \( 1 < p < \infty \), \( \gamma, \theta \in \mathbb{R} \).

(i) \( C_c^\infty(\Omega) \) is dense in \( H_{p,\theta}^\gamma(\Omega) \).

(ii) The dual space of \( H_{p,\theta}^\gamma(\Omega) \) is \( H_{p',\theta'}^{-\gamma}(\Omega) \), where
\[
1/p + 1/p' = 1, \quad \theta/p + \theta'/p' = d.
\]

(iii) For any \( \delta \in \mathbb{R} \), \( H_{p,\theta}^\gamma(\Omega) = \psi^\delta H_{p,\theta+\delta p}^\gamma(\Omega) \). In other words, \( u \in H_{p,\theta}^\gamma(\Omega) \) if and only if \( \psi^{-\delta}u \in H_{p,\theta+\delta p}^\gamma(\Omega) \). Moreover,
\[
\|u\|_{H_{p,\theta}^\gamma(\Omega)} \leq N\|\psi^{-\delta}u\|_{H_{p,\theta+\delta p}^\gamma(\Omega)} \leq N\|u\|_{H_{p,\theta}^\gamma(\Omega)},
\]
(2.10)

where \( N \) is independent of \( u \).
(iv) The operators $\psi D, D\psi : H^{\gamma}_{p,\theta} (\Omega) \rightarrow H^{\gamma - 1}_{p,\theta} (\Omega)$ are bounded, that is, for any $u \in H^{\gamma}_{p,\theta} (\Omega)$,
\[ \| \psi D u \|_{H^{\gamma - 1}_{p,\theta} (\Omega)} + \| D (\psi u) \|_{H^{\gamma - 1}_{p,\theta} (\Omega)} \leq N (d, p, \theta, \gamma, \Omega) \| u \|_{H^{\gamma}_{p,\theta} (\Omega)}. \]
It also holds that if $\theta \in (d - 1, d - 1 + p)$ then for any $u \in \mathbb{H}^{\gamma}_{p,\theta-p} (\mathbb{R}^d_+)$,
\[ \| u \|_{H^{\gamma}_{p,\theta-p} (\mathbb{R}^d_+)} \leq N \| D u \|_{H^{\gamma - 1}_{p,\theta-p} (\mathbb{R}^d_+)} \leq N \| u \|_{H^{\gamma}_{p,\theta-p} (\mathbb{R}^d_+)}. \tag{2.11} \]
(v) There exists a constant $N = N (d, p, \gamma, |\gamma|_+) \rightarrow \infty$ such that for any $f \in H^{\gamma}_{p,\theta} (\Omega)$,
\[ \| a f \|_{H^{\gamma}_{p,\theta} (\Omega)} \leq N \| a \|_{|\gamma|_+} \| f \|_{H^{\gamma}_{p,\theta} (\Omega)}. \tag{2.12} \]
In addition, if $\gamma$ is a nonnegative integer, then
\[ \| a f \|_{H^{\gamma}_{p,\theta} (\Omega)} \leq N \sup_{\Omega} |a| \| f \|_{H^{\gamma}_{p,\theta} (\Omega)} + N_1 \gamma_1 |a|_{|\gamma|_+} \| f \|_{H^{\gamma - 1}_{p,\theta} (\Omega)}. \tag{2.13} \]

**Remark 2.2.** (i) By Lemma 2.1 (iii) and (iv), for any $u \in H^{\gamma + 2}_{p,\theta-p} (\Omega)$ we have
\[ \| D u \|_{H^{\gamma + 1}_{p,\theta} (\Omega)} + \| D^2 u \|_{H^{\gamma + 2}_{p,\theta-p} (\Omega)} \leq N (d, p, \theta, \gamma, \Omega) \| u \|_{H^{\gamma + 2}_{p,\theta-p} (\Omega)}. \tag{2.14} \]
(ii) If $\Omega$ is bounded, then $|\psi|_{|\gamma|_+} < \infty$ for any $\delta > 0$ and positive integer $n$. Therefore, if $\theta_1 \leq \theta_2$, then by (2.10) and (2.12)
\[ \| u \|_{H^{\gamma}_{p,\theta_2} (\Omega)} \leq N \| \psi (\theta_2 - \theta_1) / H \|_{H^{\gamma}_{p,\theta_1} (\Omega)} \leq N \| u \|_{H^{\gamma}_{p,\theta_1} (\Omega)}. \tag{2.15} \]

**Definition 2.3** (Solution space). Let $\alpha \in (0, 2)$, $\theta \in \mathbb{R}$, and $p > 1$.
(i) The space $\mathcal{S}^{\alpha, \gamma+2}_{p,\theta} (\Omega, T)$ is the closure of $C^{\infty}_{c,x} ((0, T) \times \Omega)$ with respect to the norm
\[ \| u \|_{\mathcal{S}^{\alpha, \gamma+2}_{p,\theta} (\Omega, T)} := \| \partial_t^\alpha u \|_{H^{\gamma+2}_{p,\theta-p} (\Omega, T)} + \| u \|_{H^{\gamma+2}_{p,\theta-p} (\Omega, T)}. \]
In other words, $u \in \mathcal{S}^{\alpha, \gamma+2}_{p,\theta} (\Omega, T)$ if $u \in H^{\gamma+2}_{p,\theta-p} (\Omega, T)$ and there exists a (defining) sequence $u_n \in C^{\infty}_{c,x} ((0, T) \times \Omega)$, $n = 1, 2, \cdots$, so that
\[ \| u - u_n \|_{H^{\gamma+2}_{p,\theta-p} (\Omega, T)} \to 0 \quad \text{and} \quad \| \partial_t^\alpha u_n - \partial_t^\alpha u_m \|_{H^{\gamma+2}_{p,\theta-p} (\Omega, T)} \to 0 \]
as $n$ and $m$ go to infinity. In this case, we write $\partial_t^\alpha u = f$ if
\[ f = \lim_{n \to \infty} \partial_t^\alpha u_n \quad \text{in} \quad H^{\gamma+2}_{p,\theta+p} (\Omega, T). \]
(ii) For $u \in \mathcal{S}^{\alpha, \gamma+2}_{p,\theta} (\Omega, T)$, we write $u (0, \cdot) = 0$ (resp. $\partial_t u (0, \cdot) = 0$) if there is a defining sequence $u_n \in C^{\infty}_{c,x} ((0, T) \times \Omega)$ such that $u_n (0, \cdot) = 0$ (resp. $\partial_t u_n (0, \cdot) = 0$) for all $n = 1, 2, \cdots$. We write $u \in \mathcal{S}^{\alpha, \gamma+2}_{p,\theta} (\Omega, T)$ if $u \in \mathcal{S}^{\alpha, \gamma+2}_{p,\theta} (\Omega, T)$ and
\[ u (0, \cdot) = 0, \quad 1_{\alpha > 1} \partial_t u (0, \cdot) = 0. \]

**Theorem 2.4.** Let $p > 1$, $\alpha \in (0, 2)$, and $\gamma, \theta \in \mathbb{R}$.
(i) $\mathcal{S}^{\alpha, \gamma+2}_{p,\theta} (\Omega, T)$ is a Banach space with the norm
\[ \| u \|_{\mathcal{S}^{\alpha, \gamma+2}_{p,\theta} (\Omega, T)} := \| \partial_t^\alpha u \|_{H^{\gamma+2}_{p,\theta-p} (\Omega, T)} + \| u \|_{H^{\gamma+2}_{p,\theta-p} (\Omega, T)}. \]
Therefore, we conclude $u$ is dense in the space $w$.

Proof. (i) This can be readily proved by following a straightforward argument.

(ii) Suppose that $u_n \in \mathcal{S}_{p,\theta}^{\gamma,\gamma+2}(\Omega, T)$, $u \in \mathcal{S}_{p,\theta}^{\gamma,\gamma+2}(\Omega, T)$ and $u_n \rightarrow u$ in $\mathcal{S}_{p,\theta}^{\gamma,\gamma+2}(\Omega, T)$. Since $u_n \in \mathcal{S}_{p,\theta}^{\gamma,\gamma+2}(\Omega, T)$, there exist $v_n \in C_c(\mathbb{R}^d)$ such that $v_n(0, x) = 0, 1_{\alpha > 1} \partial_\xi v_n(0, x) = 0$.

Therefore, we conclude $u \in \mathcal{S}_{p,\theta}^{\gamma,\gamma+2}(\Omega, T)$ since $\{v_n\}$ is a defining sequence for $u$.

(iii) We use the result of [8, Theorem 2.7], which is proved on $\mathbb{R}^d$ without any weights. By the definition of $\mathcal{S}_{p,\theta}^{\gamma,\gamma+2}(\Omega, T)$, it suffices to show that $C_c^{\infty}((0, \infty) \times \Omega)$ is dense in the space

$$C_c^{\infty}([0, T] \times \Omega) \cap \{u : u(0) = 0, 1_{\alpha > 1} \partial_\xi u(0, \cdot) = 0\}.$$

Take $u$ in the above set. Then,

$$u \in H_p^{\gamma+2}(T), \quad \partial_\xi^\alpha u \in H_p^\gamma(T).$$

Choose $\zeta \in C_c^\infty(\Omega)$ such that $u = u\zeta$. Since $\zeta$ has compact support, by (2.7) for any $w \in H_p^{\gamma'}$, where $\gamma' = \gamma$ or $\gamma' = \gamma + 2$, and $\theta' = \theta - p$ or $\theta' = \theta + p$

$$|||w|||_{H_p^{\gamma'}} \leq N(p, \theta', \gamma', \zeta)|||w|||_{H_p^{\gamma'}} \leq N'(p, \theta', \gamma', \zeta)|||w|||_{H_p^{\gamma'}}.$$

Let $\varepsilon > 0$ be given. Then by [8, Theorem 2.7], there exists $v \in C_c^{\infty}((0, T) \times \mathbb{R}^d)$ such that

$$||u - v|||_{H_p^{\gamma+2}(T)} + ||\partial_\xi^\alpha u - \partial_\xi^\alpha v|||_{H_p^\gamma(T)} \leq \frac{\varepsilon}{N'(d, \zeta)}.$$

Thus,

$$||u - \zeta v|||_{H_p^{\gamma+2}(\Omega, T)} + ||\partial_\xi^\alpha u - \partial_\xi^\alpha (\zeta v)|||_{H_p^{\gamma+2}(\Omega, T)}$$

$$= ||\zeta(u - v)|||_{H_p^{\gamma+2}(\Omega, T)} + ||\zeta(\partial_\xi^\alpha u - \partial_\xi^\alpha v)|||_{H_p^{\gamma+2}(\Omega, T)} \leq \varepsilon.$$}

This certainly proves (iii).

(iv) Since $u \in \mathcal{S}_{p,\theta}^{\gamma,\gamma+2}(\Omega, T)$, by (iii), there exists $u_n \in C_c^{\infty}((0, \infty) \times \Omega)$ so that $u_n \rightarrow u$ in $H_p^{\gamma+2}(\Omega, T)$ and $\partial_\xi^\alpha u_n \rightarrow \partial_\xi^\alpha u$ in $H_p^{\gamma}(\Omega, T)$. By (2.1) and Minkowski’s inequality, we have

$$||\psi u_n(s, \cdot)|||_{H_p^\gamma(\Omega)} \leq N \int_0^s (s - r)^{\alpha - 1} ||\psi \partial_\xi^\alpha u_n(r, \cdot)|||_{H_p^\gamma(\Omega)} dr.$$ (2.17)

Taking integral to (2.17) with respect to $s$, we have

$$\int_0^t ||\psi u_n(s, \cdot)|||_{H_p^\gamma(\Omega)} ds \leq N \int_0^t \int_0^s (s - r)^{\alpha - 1} ||\psi \partial_\xi^\alpha u_n(r, \cdot)|||_{H_p^\gamma(\Omega)} dr ds.$$ (2.18)
Assumption 2.6. (i) The coefficients $\psi \partial^3 u_n$ is a Cauchy sequence in $\mathbb{H}^\infty_{p,\theta}(\Omega, T)$ due to Fubini’s theorem. We conclude that the right hand side of (2.18) converges. The left hand side also converges by (2.15). Therefore, by taking the limit we get (2.16).

Let $\mathcal{H}^{\alpha,\gamma+2}_p(T) = \mathcal{H}^{\alpha,\gamma+2}_p(\mathbb{R}^d, T)$ be the closure of $C^\infty((0, \infty) \times \mathbb{R}^d)$ with respect to the norm $\| \cdot \|_{H^{\alpha,\gamma+2}_p(T)} := \| \cdot \|_{H^{\alpha,\gamma+2}_p(\mathbb{R}^d, T)} + \| \partial_t \cdot \|_{H^{\alpha,\gamma+2}_p(T)}$.

Lemma 2.5. Let $u(t, \cdot)$ have a support in a strip $A_{a,b} = \{ x \in \Omega : a < \psi(x) < b \}$ for each $t > 0$, where $a, b > 0$. Then, $u \in \mathcal{H}^{\alpha,\gamma+2}_p(\Omega, T)$ if and only if $u \in \mathcal{S}_{\alpha,\gamma+2}(\Omega, T)$. 

Proof. Suppose $u \in \mathcal{H}^{\alpha,\gamma+2}_p(\Omega, T)$. Take a $C^\infty$ function $\zeta$ with a support in $A_{a,b}$ such that $u \zeta = u$. Then, by definition, there exists $u_n \in C^\infty((0, \infty) \times \mathbb{R}^d)$ such that

$$\| u_n - u \|_{\mathcal{H}^{\alpha,\gamma+2}_p(T)} + \| \partial_t^\alpha u_n - \partial_t^\alpha u \|_{\mathcal{H}^{\alpha,\gamma+2}_p(T)} \to 0$$

as $n \to \infty$. Then $\partial_t^\alpha u = \zeta \partial_t^\alpha u$ and $u_n \zeta \in C^\infty((0, \infty) \times \Omega)$. Thus, by (2.7) we have

$$\| u - u_n \|_{\mathcal{H}^{\alpha,\gamma+2}_p(\Omega, T)} + \| \partial_t^\alpha u - \partial_t^\alpha u_n \|_{\mathcal{H}^{\alpha,\gamma+2}_p(\Omega, T)} \leq \| u - u_n \|_{\mathcal{H}^{\alpha,\gamma+2}_p(\Omega, T)} + \| (\partial_t^\alpha u - (\partial_t^\alpha u_n)) \zeta \|_{\mathcal{H}^{\alpha,\gamma+2}_p(\Omega, T)}$$

$$\leq N(\theta, \gamma, \zeta) \left( \| \zeta(u - u_n) \|_{\mathcal{H}^{\alpha,\gamma+2}_p(T)} + \| \partial_t^\alpha u - \partial_t^\alpha u_n \|_{\mathcal{H}^{\alpha,\gamma+2}_p(T)} \right)$$

as $n \to \infty$. Thus $u_n \zeta$ becomes a defining sequence and $u \in \mathcal{S}_{\alpha,\gamma+2}(\Omega, T)$. Similarly, one can prove the other direction.

Now we introduce our assumptions on the coefficients.

Assumption 2.6. (i) The coefficients $a^{ij}, b^i$ and $c$ are Borel measurable in $(t, x)$, and $a^{ij} = a^{ji}$.

(ii) There exist constants $\delta, K > 0$ such that for any $t > 0, x \in \Omega$, and $\lambda \in \mathbb{R}^d$,

$$\delta |\lambda|^2 \leq a^{ij}(t, x) \lambda^i \lambda^j \leq K |\lambda|^2. \quad (2.19)$$

(iii) (Point-wise continuity in $x$). The coefficients $a^{ij}$ are point-wise continuous in $x$, uniformly in $t$. That is, for any $\varepsilon > 0$ and $x \in \Omega$, there exists a constant $\delta' = \delta'(x) > 0$ such that

$$|a^{ij}(t, x) - a^{ij}(t, y)| \leq \varepsilon \quad \text{if} \quad |x - y| \leq \delta', \quad \forall t.$$

(iv) (Uniform continuity in $t$). The coefficients $a^{ij}$ are uniformly continuous in $t$, uniformly with respect to $x$.

Assumption 2.7. There is a control on the behavior of $a$, $b$, and $c$ near $\partial \Omega$, i.e.,

$$\lim_{p(x) \to 0} \sup_{x \in \Omega} \sup_{y \in \Omega} \frac{|a^{ij}(t, x) - a^{ij}(t, y)|}{|x - y|} = 0. \quad (2.20)$$

$$\lim_{p(x) \to 0} \sup_{x \in \Omega} \sup_{|x - y| \leq p(x, y)} \left( \frac{p(x)}{|b^i(t, x)|} + p(x)|c(t, x)| \right) = 0. \quad (2.21)$$

Remark 2.8. (i) Obviously, condition (2.20) is much weaker than uniform continuity in $x$. For instance, if $\delta \in (0, 1)$, $d = 1$, and $\Omega = \mathbb{R}^+$, then the function $a(x)$ equal to $2 + \sin(|\ln x|^\delta)$ for $0 < x \leq 1/2$ satisfies (2.20). See [11, Remark 2.5] for detail.
(ii) Note that condition (2.21) allows $b^i$ and $c$ to blow up near the boundary. This condition holds e.g. if
\[ |b^i(t, x)| \leq N_1 \rho^{\alpha-1}(x), \quad |c(t, x)| \leq N_2 \rho^{\gamma-2}(x), \]
where $\varepsilon > 0$.

The following assumption is used for the regularity of the solutions.

**Assumption 2.9 (γ).** For each $t > 0$,
\[ |a^{ij}(t, \cdot)|_{|\gamma|+}^{(0)} + |\psi b^i(t, \cdot)|_{|\gamma|+}^{(0)} + |\psi^2 c(t, \cdot)|_{|\gamma|+}^{(0)} \leq K. \]

Here is our main result on bounded $C^1$ domains.

**Theorem 2.10.** Let $\Omega$ be a bounded $C^1$ domain, $\gamma \in \mathbb{R}$, $p > 1$, and $\theta \in (d-1, d-1+p)$. Suppose Assumption 2.6, Assumption 2.7, and Assumption 2.9(γ) hold. Then for any $f \in \mathcal{H}^{\alpha,\gamma,p+\theta}_0(\Omega, T)$, the equation
\[ \partial_t^\alpha u = a^{ij} u_{x^j x^i} + b^i(t) u + c(t) + f, \quad t > 0; \quad u(0) = 0, 1_{\alpha > 1} \partial_t u(0) = 0 \tag{2.22} \]
has a unique solution $u \in \mathcal{D}^{\alpha,\gamma+2}_{\beta,\delta,p}(\Omega, T)$. Furthermore, for this solution we have
\[ \|u\|_{\mathcal{D}^{\alpha,\gamma+2}_{\beta,\delta,p}(\Omega, T)} \leq N \|f\|_{\mathcal{H}^{\alpha,\gamma,p+\theta}_0(\Omega, T)}, \tag{2.23} \]
where $N = N(\alpha, d, p, \theta, \gamma, \delta, K, \Omega, T)$.

**Remark 2.11.** (i) Note that $\gamma + 2$, which is the number of differentiability of the solution, can be any real number.

(ii) One can obtain interior Hölder regularities of solutions with respect to $x$ based on [18, Theorem 4.3] (see also [16, Theorem 3.1]). That is, if $\gamma + 2 > d/p$, and $\gamma + 2 - d/p = k + \nu$, where $k$ is an integer and $\nu \in (0, 1)$, then
\[ \sum_{m=0}^k \left[ \psi^{m+\nu+p} \partial^m D^\nu u(0) + \psi^{k+\nu+p} D^k u(0) \right] \leq N(d, p, \theta, \gamma) \|u\|_{\mathcal{H}^{\alpha,\gamma+2}_{\beta,\delta,p}(\Omega)}. \]

(iii) The restriction $\theta \in (d-1, d-1+p)$ is necessary even for the heat equation $u_t = \Delta u + f$ (see [16]).

The proof of Theorem 2.10 is highly based on the following result on $\mathbb{R}^d_+$.

**Theorem 2.12.** Let $\Omega = \mathbb{R}^d_+$, and $\omega > 0$ be a constant. Suppose Assumption 2.6 and Assumption 2.9(γ) hold. Assume that
\[ |a^{ij}(t, x) - a^{ij}(t, y)| + x^1 |b^i(t, x)| + (x^1)^2 |c(t, x)| \leq \omega \tag{2.24} \]
whenever $t > 0, x, y \in \mathbb{R}^d_+$, and $|x - y| \leq x^1 \wedge y^1$. Then there exists a constant $\omega_0 \in (0, 1)$ depending only on $\alpha, d, p, \theta, \gamma, \delta$ and $K$ such that if $\omega \leq \omega_0$, then for any $f \in \mathcal{H}^{\alpha,\gamma,p+\theta}_0(\mathbb{R}^d_+, T)$, equation (2.22) with zero initial condition has a unique solution $u \in \mathcal{D}^{\alpha,\gamma+2}_{\beta,\delta,0}(\mathbb{R}^d_+, T)$. Moreover, the solution $u$ satisfies
\[ \|u\|_{\mathcal{D}^{\alpha,\gamma+2}_{\beta,\delta,0}(\mathbb{R}^d_+, T)} \leq N \|f\|_{\mathcal{H}^{\alpha,\gamma,p+\theta}_0(\mathbb{R}^d_+, T)}, \tag{2.25} \]
where $N = N(\alpha, d, p, \theta, \gamma, \delta, K, T)$. Furthermore, if $a^{ij}$ are independent of $t$, then $N$ does not depend on $T$. 
3. **Equation with constant coefficients on the half space** $\mathbb{R}^d_+$. In this section, we study the equation

$$\partial_t^\alpha u(t, x) = a^{ij} D_{ij} u(t, x) + f(t, x), \quad (t, x) \in (0, T) \times \mathbb{R}^d_+$$  \hspace{1cm} (3.1)

with zero initial and zero boundary data. We assume that the coefficients $a^{ij}$ are independent of $(t, x)$, and satisfy the uniform ellipticity condition, that is

$$\delta |\lambda|^2 \leq a^{ij} \lambda^i \lambda^j \leq K |\lambda|^2, \quad \forall \lambda \in \mathbb{R}^d.$$

Let $p(t, x)$ be the fundamental solution to the following equation on $\mathbb{R}^d$:

$$\partial_t^\alpha u(t, x) = \Delta u(t, x), \quad u(0) = u_0, \quad 1_{\alpha > 1} \partial_t u(0) = 0.$$ \hspace{1cm} (3.2)

In other words, $p(t, x)$ is the function such that the solution of (3.2) is given by

$$u(t, x) = p(t, \cdot) * u_0 = \int_{\mathbb{R}^d} p(t, y) u_0(x - y) dy$$ \hspace{1cm} (3.3)

if $u_0$ is smooth enough. It is known that such $p$ exists and absolutely continuous in $t$. Define

$$q(t, x) = \begin{cases} I_{t}^{\alpha - 1} p(t, x) & \alpha \in (1, 2), \\ p(t, x) & \alpha = 1, \\ D_{t}^{1-\alpha} p(t, x) & \alpha \in (0, 1). \end{cases}$$

Below we collect some well known properties of $p(t, x)$ and $q(t, x)$. See [8, 13] for details.

**Lemma 3.1.**  
(i) There exists a fundamental solution $p(t, x)$ satisfying (3.3). Moreover, for all $t \neq 0$ and $x \neq 0$, we have

$$\partial_t^\alpha p(t, x) = \Delta p(t, x), \quad \frac{\partial p(t, x)}{\partial t} = \Delta q(t, x).$$

Also, for each $x \neq 0$, $\frac{\partial p(t, x)}{\partial t} \to 0$ as $t \downarrow 0$. Furthermore, $\frac{\partial p(t, \cdot)}{\partial t}$ is integrable in $\mathbb{R}^d$ uniformly on $t \in [\epsilon, T]$ for any $\epsilon > 0$.

(ii) For each $n \in \mathbb{N}$,

$$|D_{x}^n q(t, x)| \leq N(d, \alpha, n) t^{-\frac{d+1}{2}+\alpha-\frac{n}{2}}(|t^{-\alpha/2} x|^{-d+2-n} \wedge |t^{-\alpha/2} x|^{-d-n}).$$

(iii) There is a scaling property of $q(t, x)$, i.e.,

$$q(t, x) = t^{-\frac{d+1}{2}+\alpha-1} q(1, xt^{-\frac{1}{2}}).$$

Proof. See e.g. [8, Lemma 3.2] and equality (5.2) of [13].

Recall that $H_{p,0}^{\alpha, \gamma+2}(T) = H_{p,0}^{\alpha, \gamma+2}(\mathbb{R}^d, T)$ is the closure of $C_c^\infty((0, \infty) \times \mathbb{R}^d)$ with respect to the norm $\|\cdot\|_{H_{p,0}^{\alpha, \gamma+2}(T)} := \|\cdot\|_{L^2_{0}^p(T)} + \|\partial_t \cdot\|_{L_{0}^p(T)}$.

**Lemma 3.2.**  
(i) Let $u \in C_c^\infty((0, \infty) \times \mathbb{R}^d)$, and let $f := \partial_t^\alpha u - \Delta u$. Then

$$u(t, x) = \int_0^t \int_{\mathbb{R}^d} q(t - s, x - y) f(s, y) dy ds.$$ \hspace{1cm} (3.4)

(ii) Let $p > 1$, $\gamma \in \mathbb{R}$, $T < \infty$, and $f \in H_{p}^{\gamma}(T)$. Then there is a unique solution $u$ to the equation

$$\partial_t^\alpha u = \Delta u + f, \quad 0 < t < T, x \in \mathbb{R}^d; \quad u(0) = 1_{\alpha > 1} \partial_t u(0) = 0,$$

in the class $H_{p,0}^{\alpha, \gamma+2}(T)$, and for this solution we have

$$\|D^2 u\|_{L^2_{0}^p(T)} \leq N(\alpha, d, p, \gamma) \|f\|_{L^p_{0}(T)}.$$
Moreover if \( f \in C^\infty_c((0, \infty) \times \mathbb{R}^d) \), then the solution is given by formula (3.4).

**Proof.** Considering \( (1-\Delta)^{\gamma/2} \), which is an isometry from \( H^\gamma_{p_0} \) to \( L_{p_0} \), we may assume \( \gamma = 0 \). In this case the claims follow from [5, Lemma 3.2], which actually is a collection of results proved in [8]. \( \square \)

Below is a weighted norm estimate of solutions on the entire space.

**Theorem 3.3.** Let \( f \in C^\infty_c((0, \infty) \times \mathbb{R}^d) \) and \( u \) be defined as in (3.4). If \( 1 < p < \infty \), and \( d-1 < \theta < d-1+p \), then we have

\[
\int_0^T \int_{\mathbb{R}^d} |D^2 u(t,x)|^p |x|^{\theta-d} dx dt \leq N \int_0^T \int_{\mathbb{R}^d} |f(t,x)|^p |x|^{\theta-d} dx dt,
\]

where the constant \( N \) depends only on \( \alpha, d, p, \) and \( \theta \).

**Proof.** The claim is a consequence of [5, Theorem 3.4] with \( w_2(t) = 1 \) and \( w_1(x) = |x|^\theta d \) there. We remark that \( |x|^\theta d \in A_p(\mathbb{R}^d) \) if and only if \( \theta \in (d-1, d-1+p) \). Here \( A_p(\mathbb{R}^d) \) is the class of Muckenhoupt \( A_p \) weights (e.g. [6, Remark 4.4]). \( \square \)

**Lemma 3.4.** Let \( p > 1 \) and \( \theta \in (d-1, d-1+p) \).

(i) Let \( f^i \in C^\infty_c((0, \infty) \times \mathbb{R}^d_+) \). Define

\[
u(t,x) := \int_0^t \int_{\mathbb{R}^d} (q(t-s,x,y) - q(t-s,x,y^*)) D_i f^i(s,y) dy ds,
\]

where \( y^* = (-y^1, y^j) \). Then \( u \in \mathcal{H}^{\alpha,1}_{p,\theta,0}(\mathbb{R}^d_+,T) \) for any \( T > 0 \) and satisfies

\[
\partial^\nu_t u(t,x) = Du(t,x) + D_i f^i(t,x), \quad (t,x) \in (0,T) \times \mathbb{R}^d_+.
\]

Furthermore,

\[
\|u\|_{\mathcal{H}^{\alpha,1}_{p,\theta,0}(\mathbb{R}^d_+,T)} \leq N(\alpha, d, p, \theta) \sum_{i=1}^d \|f^i\|_{L^p_{p,\theta}(\mathbb{R}^d_+,T)}.
\]

(ii) For any \( f^i \in L^p_{p,\theta}(\mathbb{R}^d_+,T) \), equation (3.7) has a solution \( u \in \mathcal{H}^{\alpha,1}_{p,\theta,0}(\mathbb{R}^d_+,T) \) satisfying (3.8).

**Proof.** (i) Let \( \tilde{f}^i \) be the even extension of \( f^i \), and \( \bar{f}^i \) be the odd extension of \( f^i \) for \( i = 2, \ldots, d \) with respect to \( x^1 = 0 \).

By the integration by parts, for each \( j = 1, 2, \ldots, d \),

\[
D_j u = \sum_{i=1}^d \int_0^t \int_{\mathbb{R}^d} D_{ij} q(t-s,x,y) \bar{f}^i(s,y) dy ds.
\]

Thus by Theorem 3.3 and (2.6),

\[
\|D u\|_{L^p_{p,\theta}(\mathbb{R}^d_+,T)} \leq N \sum_{i=1}^d \int_0^T \int_{\mathbb{R}^d} |\bar{f}^i(s,x)|^p |x|^\theta d dx dt \leq N \sum_{i=1}^d \|f^i\|_{L^p_{p,\theta}(\mathbb{R}^d_+,T)}.
\]

Therefore, we have \( D u \in L^p_{p,\theta}(\mathbb{R}^d_+,T) \). To prove \( u \in H^{\alpha,\gamma}_{p,\theta-\gamma}(\mathbb{R}^d_+,T) \), by (2.6), it suffices to check \( u \in L^{p,\theta-\gamma}_{p,\theta}(\mathbb{R}^d_+,T) \). By Lemma 3.2 (ii), \( u \in H^{\alpha,\gamma}_{p,\theta}(\mathbb{R}^d_+,T) \) is the unique solution to the equation

\[
\partial^\nu_t u = \Delta u + \sum_{i=1}^d \bar{f}^i, \quad t > 0, \ x \in \mathbb{R}^d; \quad u(0) = 1_{\alpha \geq 1} \partial_t u(0) = 0.
\]
It follows that since \(-u(t, -x^1, x')\) also satisfies this equation, \(u\) is an odd function with respect to \(x^1 = 0\), and \(u(t, x) = 0\) when \(x^1 = 0\). Thus, by the fundamental theorem of calculus and the change of variable, we have \((x^1)^{-1}u(t, x) = \int_0^1 u_{x^1}(t, rx^1, x')dr\). Therefore, by Hölder inequality

\[
\left( \int_0^T \int_{\mathbb{R}^d_+} \left| (x^1)^{-1}u(t, x) \right|^p (x^1)^{\theta - d} dxdt \right)^{1/p} 
\leq \int_0^1 \left( \int_0^T \int_{\mathbb{R}^d_+} |Du(t, rx^1, x')|^p (x^1)^{\theta - d} dxdt \right)^{1/p} dr
\]

\[
= \int_0^1 r^{-(1 - \theta + d)/p} dr \left( \int_0^T \int_{\mathbb{R}^d_+} |Du(t, x')|^p (x^1)^{\theta - d} dxdt \right)^{1/p} \leq \|Du\|_{L^p_0(\mathbb{R}^d_+, T)}.
\]

For the last inequality we used \(\theta < d - 1 + p\). Thus, by (3.9), we have \(u \in L_p_{0, -p}(\mathbb{R}^d_+, T)\), and consequently \(u \in \mathbb{H}^{1}_{p, \theta - p}(\mathbb{R}^d_+, T)\).

Next we prove \(\partial_t^n u \in \mathbb{H}^{-1}_{p, \theta + p}(T)\). By (3.9) again, we have

\[
||\partial_t^n u||_{\mathbb{H}^{-1}_{p, \theta + p}(\mathbb{R}^d_+, T)} = \||\Delta u + D_i f^i||_{\mathbb{H}^{-1}_{p, \theta + p}(\mathbb{R}^d_+, T)}
\]

\[
\leq N\|Du\|_{L^p_0(\mathbb{R}^d_+, T)} + N \sum_{i=1}^d \|f^i\|_{L^p_{\theta, \theta + p}(\mathbb{R}^d_+, T)} \leq N \sum_{i=1}^d \|f^i\|_{L^p_{\theta, \theta + p}(\mathbb{R}^d_+, T)}.
\]

By Lemma 3.2, we have \(u(0) = 1_{\alpha > 0} u_0(0) = 0\), and therefore we conclude \(u \in \mathcal{S}_{p, \theta, 0}^{\alpha, 1}(\mathbb{R}^d_+, T)\). Estimate (3.8) is a consequence of (3.9), (3.10), and (2.11).

(ii) This follows from (i) and the denseness of \(C^\infty_c((0, \infty) \times \mathbb{R}^d_+)\) in \(L^p_0(\mathbb{R}^d_+, T)\). Indeed, let \(f^i_n\) be a sequence of functions in \(C^\infty_c((0, \infty) \times \mathbb{R}^d_+)\) which converges to \(f^i\) in \(L^p_0(\mathbb{R}^d_+, T)\). For each \(n\), define \(u_n\) as in (3.6). By (3.8), \(u_n\) is a Cauchy sequence in \(\mathcal{S}_{p, \theta, 0}^{\alpha, 1}(T)\). Now take \(v\) as the limit in \(\mathcal{S}_{p, \theta, 0}^{\alpha, 1}(T)\). Then \(v\) satisfies equation (3.7) and (3.8).

**Lemma 3.5.** Let \(\theta > d - 1\) and \(p > d + \theta + 2/\alpha\), and \(f^i \in \mathbb{H}^{1}_{p, \theta}(\mathbb{R}^d_+, T)\). Define \(u(t, x)\) as in (3.6). Then \(u(t, x)\) is well defined for all \((t, x) \in (0, T) \times \mathbb{R}^d_+\) in the sense of Lebesgue integral, and

\[
|u(t, x)| \leq N \sum_{i=1}^d \|f^i\|_{\mathbb{H}^{1}_{p, \theta}(\mathbb{R}^d_+, T)},
\]

where \(N = N(\alpha, d, \theta, x^1, t)\).

**Proof.** By assumption we have \(d - 1 < \theta < p - d - 2/\alpha < p + d - 1\). Choose \(p'\) and \(\theta'\) such that

\[
1/p + 1/p' = 1, \quad \theta/p + \theta'/p' = d.
\]

Then one can check

\[
d - 1 < \theta' < d - 1 + p,\]

\[
(d - 1)(1 - p') + \theta' - d > -1,
\]

\[
\frac{\alpha d}{2}(1 - p') + \left(\frac{\alpha}{2} - 1\right)p' + \frac{\alpha}{2}(\theta' - d) > -1,
\]

where \(\alpha > 0\).
and
\[(d-1)(1-p') > -1 \quad \text{(equivalently, } (d-1)p' < d),\]
\[
\frac{\alpha d}{2}(1-p') + \left(\frac{\alpha}{2} - 1\right) p' > -1.
\]
(3.14)

By the fundamental theorem of calculus and Hölder’s inequality,
\[|u(t,x)| \leq \int_0^t \int_{\mathbb{R}^d} |q(t-s,x-y) - q(t-s,x-y_0)||D_1 f(s,y)|dxdy\]
\[\leq \int_0^t \int_{\mathbb{R}^d} |q_x(t-s,x',y')||y_1|D_1 f(s,y)|dxdy\]
\[\leq N I(\theta,t,x') \sum_{i=1}^d \|x|^d_i f_i\|_{L_p(s,y_1,T)} ,\]

where
\[I(\theta,t,x_1) := \int_{-1}^1 \left( \int_0^t \int_{\mathbb{R}^d} |q_x(s,x^1 - ry^1,y')||y^1|^{|\theta|'-d}dy'ds \right)^{1/p'} dr\]
\[= \int_{-1}^1 r^{(d-\theta'-1)/p'} dr \left( \int_0^t \int_{\mathbb{R}^d} |q_x(s,x^1 - ry^1,y')||y^1|^{|\theta|'-d}dy'ds \right)^{1/p'}\]
\[\leq N \left( \int_0^t \int_{\mathbb{R}^d} |q_x(s,x^1 - ry^1,y')||y^1|^{|\theta|'-d}dy'ds \right)^{1/p'} .
\]

For the inequality above we used (3.12).

Next, we prove \(I(\theta,t,x') < \infty\). By the change of variables \(x^1 - y^1 \to s^{\alpha/2}z_1^1\) and \(y' \to s^{\alpha/2}z_1'\), and by Lemma 3.1(ii), we have
\[\int_0^t \int_{\mathbb{R}^d} |q_x(s,x^1 - y^1,y')||y^1|^{|\theta|'-d}dy'ds\]
\[= \int_0^t \int_{\mathbb{R}^d} |q_x(s,s^{\alpha/2}z_1^1)||x^1 - s^{\alpha/2}z_1^1|^{|\theta|'-d}s^{\alpha/2}dz'ds\]
\[\leq N \int_0^t \int_{\mathbb{R}^d} s^{-(\alpha/2 + \alpha/2 - 1)}|z_1^1|^{|\theta|'-d}dz'ds .
\]

Put \(z_1^1 = x^1 - s^{\alpha/2}z_1^1\), and fix \(n\) such that \(t^{\alpha/2}/n \leq 1/2\). Note that
\[|x^1 - s^{\alpha/2}z_1^1|^{|\theta|'-d} \leq N_1^{\theta' \geq d} \left( |x^1|^{|\theta|'-d} + s^{\alpha(\theta'-d)/2}|z_1^1|^{|\theta|'-d} \right)\]
\[+ N_1^{\theta' \leq d} \left( 1_{|z_1^1| \geq s^{\alpha/2}z_1^1} s^{\alpha/2}z_1^1 |x^1|^{|\theta|'-d} + 1_{|z_1^1| \leq s^{\alpha/2}z_1^1/2} |z_1^1|^{|\theta|'-d} \right)\]
\[\leq N \left( |x^1|^{|\theta|'-d} + s^{\alpha(\theta'-d)/2}|z_1^1|^{|\theta|'-d} + |s^{\alpha/2}z_1^1|^{|\theta|'-d} \right)\]
\[+ N_1^{\theta' \leq 1} \left( 1_{|z_1^1| \leq s^{\alpha/2}z_1^1} |z_1^1|^{|\theta|'-d} = N(F_1 + F_2) .
\]

Note also that, since \(|z| \geq 1\) whenever \(|z'| \geq 1\),
\[|z|^{-d+1} \leq 1_{|z'| \leq 1} (|z|^{-d+1} + |z|^{|\theta|'-d-1}) + 1_{|z'| \geq 1} |z|^{-d-1}
\]
(if \(d = 1\), then \(z' : = 1\)). Also if \(|z_1^1| \leq x^1 s^{\alpha/2}/n\), then \(s^{\alpha/2}z_1^1 \geq x^1/2\). This implies
\[|z_1^1|^{-d+1} + |z_1^1|^{-d-1} \leq N(x^1)(s^{d\alpha/2-\alpha/2} + s^{d\alpha/2+\alpha/2}) \leq N(x^1,t)s^{d\alpha/2-\alpha/2} .
\]
Thus it follows that
\[
\int_{\mathbb{R}^d} \left( |z|^{-d+1} \wedge |z|^{-d-1} \right) p' F_2 dz \\
\leq N \int_{\mathbb{R}^d} \left( |z| \leq 1 \right) \left( s^{\alpha d/2 - \alpha/2} + 1 \right) |z|^{-d-1} p' F_2 dz \\
\leq N \left( s^{\alpha d/2 - \alpha/2} + 1 \right) \int_{|z| \leq 2} \left( |z|^{-1} - s^{\alpha/2} |z|^{-\theta'} dz \right) \\
\leq N \left( s^{\alpha d/2 - \alpha/2} + 1 \right) \int_{|z| \leq 2} \left( |z|^{-1} - s^{\alpha/2} |z|^{-\theta'} dz \right)
\]
Since \( p > d + \theta + 2/\alpha \), it holds that \(-p' + \alpha d/2 + \alpha(\theta' - d)/2 = -1\). Using this and (3.13), we have
\[
\int_{0}^{t} \int_{\mathbb{R}^d} s^{-(\alpha d/2 + \alpha/2 - 1) p' + \alpha d/2} |z|^{-d+1} \wedge |z|^{-d-1} p' F_2 dz ds \\
\leq N \int_{0}^{t} s^{-p' + \alpha d/2 + \alpha(\theta' - d)/2} ds \\
+ N \int_{0}^{t} s^{-(\alpha d/2 + \alpha/2 - 1) p' + \alpha d/2 + \alpha(\theta' - d)/2} ds \leq N(\alpha, d, p, x^1, t) < \infty.
\]
Similarly (actually much easily), by using (3.12), (3.13), and (3.14), one can check that the integral
\[
\int_{0}^{t} \int_{\mathbb{R}^d} s^{-(\alpha d/2 + \alpha/2 - 1) p' + \alpha d/2} |z|^{-d+1} \wedge |z|^{-d-1} p' F_1 dz ds
\]
is bounded by a constant depending only on \( \alpha, d, p, \theta, x^1, t \). Therefore, \( I(\theta, t, x^1) < \infty \), and we get (3.11) using Lemma 2.1(iv). The lemma is proved.

**Remark 3.6.** Let all the assumptions in Lemma 3.5 hold and \( u \) be taken from the lemma. Suppose \( f^i_n \in C^\infty_c((0, \infty) \times \mathbb{R}^d) \) is a sequence of functions such that \( f^i_n \rightarrow f^i \) in \( H^1_\theta(\mathbb{R}^d, T) \). Then, by (3.11), \( u(t, x) \) is the point-wise limit of
\[
u_n(t, x) := \int_{0}^{t} \int_{\mathbb{R}^d} \left( q(t-s, x-y) - q(t-s, x-y') \right) D_1 f^i_n(s, y) ds dy.
\]
It follows from the proof of the Lemma 3.4 (ii) that \( u \in S^{\alpha, \mu}_{\gamma, \theta, 0}(\mathbb{R}^d, T) \) is a solution to equation (3.7), and it also satisfies estimate (3.8). The issue of the uniqueness still remains and will be handled later.

**Lemma 3.7.** Let \( \theta \in \mathbb{R}, \mu \leq \gamma \), and \( f^i \in H^\gamma_\theta(\mathbb{R}^d, T) \). Suppose \( u \in S^{\gamma, \theta}_{\mu, \mu, 0}(\mathbb{R}^d, T) \) is a solution to (3.7) with zero initial condition. Then \( u \in S^{\gamma, \theta}_{\mu, \mu, 0}(\mathbb{R}^d, T) \) and
\[
\| u \|_{S^{\gamma, \theta}_{\mu, \mu, 0}(\mathbb{R}^d, T)} \leq N \sum_{i=1}^{d} \| f^i \|_{S^{\gamma, \theta}_{\mu, \mu, 0}(\mathbb{R}^d, T)} + N \| u \|_{S^{\gamma, \theta}_{\mu, \mu, 0}(\mathbb{R}^d, T)} \tag{3.15}
\]
where \( N = N(d, p, \theta, \gamma, \mu) \).

**Proof.** We repeat the argument used in the proof of [14, Lemma 3.6].

First, assume that \( \gamma = \mu + 1 \). For each \( n \in \mathbb{Z} \), denote \( u_n(t, x) := u(t, e^{2n}t, e^n x) \). Then by the formula
\[
\left( \partial_t^\mu u(\cdot) \right)(t) = e^\alpha \partial_t^\mu u(\alpha t),
\]
$u_n$ satisfies the equation
\[ \partial_t^n u_n = \Delta u_n + f_n, \]
where (with Einstein’s summation convention in force)
\[ f_n = e^{2n(D_t f)^i}(e^{\frac{2n}{2}} t, e^n x). \]
Take $\zeta(x) = \zeta(x^1) \in C_c^\infty(\mathbb{R}_+)$ satisfying (2.2). Then, we have
\[ \partial_t^n (\zeta u_n) = \Delta (u_n \zeta) + \tilde{f}_n, \]
where
\[ \tilde{f}_n = \zeta f_n - 2D_t u_n D_t \zeta - u_n \Delta \zeta. \]
Since $u \in \mathcal{H}^\alpha_{p, \theta}(\mathbb{R}^d)$, $f \in \mathbb{H}^\gamma(\mathbb{R}^d, T)$, and $\zeta$ has compact support in $\mathbb{R}_+$, by (2.7) and Lemma 2.5 we have
\[ \zeta u_n \in \mathcal{H}^\alpha_{p, \theta}(e^{-2n/\alpha} T), \quad \tilde{f}_n \in \mathbb{H}^\gamma(e^{-2n/\alpha} T). \]
By Lemma 3.2, we conclude $\zeta u_n \in \mathcal{H}^\alpha_{p, \theta} (e^{-2n/\alpha} T)$ and
\[ \| (\zeta u_n)_{xx} \|_{\mathbb{H}^\gamma(e^{-2n/\alpha} T)} \leq N \| \tilde{f}_n \|_{\mathbb{H}^\gamma(e^{-2n/\alpha} T)}, \quad (3.16) \]
where $N$ is independent of $T$ and $n$.

By definition of the norm $\| \cdot \|_{\mathbb{H}^\gamma(e^{-2n/\alpha} T)}$,
\[ \| u \|_{\mathbb{H}^\gamma(e^{-2n/\alpha} T)} = \sum_{n \in \mathbb{Z}} e^{n(\theta - p)} \| \zeta(\cdot) u^\gamma(\cdot, e^n) \|_{\mathbb{H}^\gamma(e^{-2n/\alpha} T)} \]
\[ = \sum_{n \in \mathbb{Z}} e^{n(\theta - p + \frac{2}{\alpha})} \| \zeta(\cdot) u_n(\cdot, \cdot) \|_{\mathbb{H}^\gamma(e^{-2n/\alpha} T)} \]
\[ \leq N (\zeta) \sum_{n \in \mathbb{Z}} e^{n(\theta - p + \frac{2}{\alpha})} \| (\zeta u_n)_{xx} \|_{\mathbb{H}^\gamma(e^{-2n/\alpha} T)} \]

where the inequality above holds because $\zeta u_n$ is supported in a strip $\{ x \in \mathbb{R}^d : a < x^1 < b \}$ for some $a, b > 0$ (see e.g. [16, Remark 1.13]). By (3.17) and (3.16),
\[ \| u \|_{\mathbb{H}^\gamma(e^{-2n/\alpha} T)} \]
\[ \leq N \sum_{n \in \mathbb{Z}} e^{n(\theta - p + \frac{2}{\alpha})} \| \tilde{f}_n \|_{\mathbb{H}^\gamma(e^{-2n/\alpha} T)} \]
\[ \leq N \sum_{n \in \mathbb{Z}} e^{n(\theta - p)} \| \zeta(\cdot) e^{2n f_i^i(\cdot, e^n)} - 2e^{n} \zeta x(\cdot) u_{xx}(\cdot, e^n) - u(\cdot, e^n) \Delta \zeta(\cdot) \|_{\mathbb{H}^\gamma(e^{-2n/\alpha} T)} \]
\[ \leq N \sum_{n \in \mathbb{Z}} e^{n(\theta + p)} \| \zeta f_i^i(\cdot, e^n) \|_{\mathbb{H}^\gamma(e^{-2n/\alpha} T)} + N \sum_{n \in \mathbb{Z}} e^{n(\theta - p)} \| \zeta u_{xx}(\cdot, e^n) \|_{\mathbb{H}^\gamma(e^{-2n/\alpha} T)} + \Delta \zeta(\cdot) \|_{\mathbb{H}^\gamma(e^{-2n/\alpha} T)} \]
\[ \leq N \| \zeta f_i^i \|_{\mathbb{H}^\gamma(e^{-2n/\alpha} T)} + N \| u_{xx} \|_{\mathbb{H}^\gamma(e^{-2n/\alpha} T)} + N \| u \|_{\mathbb{H}^\gamma(e^{-2n/\alpha} T)} \]

For the last inequality above we used (2.5). This with the relations (see Lemma 2.1)
\[ \| u_x \|_{\mathbb{H}^\gamma(e^{-2n/\alpha} T)} \leq N \| u \|_{\mathbb{H}^\gamma(e^{-2n/\alpha} T)} \]
\[ \| f_i^i \|_{\mathbb{H}^\gamma(e^{-2n/\alpha} T)} \leq N \| f_i^i \|_{\mathbb{H}^\gamma(e^{-2n/\alpha} T)} \]
yields (3.15) for $\gamma = \mu + 1$. If $\gamma = \mu + k$, where $k \geq 2$ is an integer, then use the induction. For general case, let $\gamma = \mu + k + c$, where $k = 0, 1, 2, \ldots$, and $0 < c < 1$. 
Then for $\mu' = \mu - 1 + c$, $\gamma - \mu' = k + 1$ is an integer. Therefore, we obtain (3.15) with $\mu'$. Since $\mu' \leq \mu$, we finally have (3.15) for $\mu$. The lemma is proved.

**Remark 3.8.** Let $\theta \in (d - 1, d - 1 + p)$, $\gamma \in \mathbb{R}$, and $f \in \mathbb{H}_{\mu', \theta + p}^{\gamma}(\mathbb{R}_+^d, T)$, or equivalently $x^1 f \in \mathbb{H}_{\mu', \theta + p}^{\gamma}(\mathbb{R}_+^d, T)$. Then, by the second assertion of [16, Corollary 2.12 (i)] applied to $x^1 f$, there exist $f^i \in \mathbb{H}_{\mu', \theta + p}^{\gamma + 1}(\mathbb{R}_+^d, T)$ ($i = 1, \ldots, d$) such that

$$f = \sum_{i=1}^d D_i f^i \quad \text{and} \quad \|f\|_{\mathbb{H}_{\mu', \theta + p}^{\gamma + 1}(\mathbb{R}_+^d, T)} \sim \sum_{i=1}^d \|f^i\|_{\mathbb{H}_{\mu', \theta + p}^{\gamma + 1}(\mathbb{R}_+^d, T)}. \quad (3.18)$$

Moreover, by inspecting the proof of [16, Corollary 2.12 (i)], one can check that if $f$ is also contained in $\mathbb{H}_{\mu', \theta + q}^{\gamma + 1}(\mathbb{R}_+^d, T)$ for other $q \geq p$, then $f^i$ can be chosen such that $f^i \in \mathbb{H}_{\mu', \theta + q}^{\gamma + 1}(\mathbb{R}_+^d, T) \cap \mathbb{H}_{\mu', \theta + q}^{\gamma + 1}(\mathbb{R}_+^d, T)$ and (3.18) also holds for $q$.

Now we prove the existence and the uniqueness of equation (3.1).

**Theorem 3.9.** Let $p > 1$, $\gamma \geq 0$, and $d - 1 < \theta < d - 1 + p$. Suppose $f \in \mathbb{H}_{\mu', \theta + p}^{\gamma}(\mathbb{R}_+^d, T)$ and $a^{ij}$ are independent of $(t, x)$. Then, equation (3.1) with zero initial condition has a unique solution $u \in \mathcal{S}_{\mu, \theta, 0}^{\alpha, \gamma + 2}(\mathbb{R}_+^d, T)$. Moreover, $u$ satisfies

$$\|u\|_{\mathcal{S}_{\mu, \theta, 0}^{\alpha, \gamma + 2}(\mathbb{R}_+^d, T)} \leq N\|f\|_{\mathbb{H}_{\mu, \theta + p}^{\gamma}(\mathbb{R}_+^d, T)}, \quad (3.19)$$

where $N = N(\alpha, d, p, \theta, \delta, K)$.

**Proof.** **Step 1.** Let $\gamma = 0$ and $a^{ij} = \delta^{ij}$.

By Remark 3.8, there exist $f^i \in \mathbb{H}_{\mu, \theta + p}^{\gamma}(\mathbb{R}_+^d, T)$ such that (3.18) holds. By Lemma 3.4 (ii), there exists a solution $u \in \mathcal{S}_{\mu, \theta, 0}^{\alpha, \gamma + 2}(\mathbb{R}_+^d, T)$ to equation (3.1) satisfying

$$\|u\|_{\mathcal{S}_{\mu, \theta, 0}^{\alpha, \gamma + 2}(\mathbb{R}_+^d, T)} \leq N \sum_{i=1}^d \|f^i\|_{\mathcal{S}_{\mu, \theta, 0}^{\alpha, \gamma + 2}(\mathbb{R}_+^d, T)}. \quad (3.19)$$

Applying Lemma 3.7 with $\gamma = 1$ and $\mu = 0$, we get

$$\|u\|_{\mathbb{H}_{\mu, \theta + p}^{\gamma}(\mathbb{R}_+^d, T)} \leq N\|u\|_{\mathbb{H}_{\mu, \theta + p}^{\gamma}(\mathbb{R}_+^d, T)} + N \sum_{i=1}^d \|f^i\|_{\mathbb{H}_{\mu, \theta + p}^{\gamma}(\mathbb{R}_+^d, T)} \leq N\|f\|_{\mathbb{H}_{\mu, \theta + p}^{\gamma}(\mathbb{R}_+^d, T)} \leq N\|f\|_{\mathbb{H}_{\mu, \theta + p}^{\gamma}(\mathbb{R}_+^d, T)}.$$

For the last inequality above we used (3.18). Moreover, since $\partial_t \alpha u = \Delta u + f$, we have $\|\partial_t \alpha u\|_{\mathbb{H}_{\mu, \theta + p}^{\gamma}(\mathbb{R}_+^d, T)} \leq N\|f\|_{\mathbb{H}_{\mu, \theta + p}^{\gamma}(\mathbb{R}_+^d, T)}$. This proves the existence of a solution satisfying (3.19).

To prove the uniqueness, let $u \in \mathcal{S}_{\mu, \theta, 0}^{\alpha, \gamma + 2}(\mathbb{R}_+^d, T)$ satisfy $\partial_t \alpha u - \Delta u = 0$ with zero initial condition. Take a sequence $u_n$ in $C_{c}^{\infty}(\mathbb{R}_+ \times \mathbb{R}_+^d)$ which converges to $u$ in $\mathcal{S}_{\mu, \theta, 0}^{\alpha, \gamma + 2}(T)$. Let $f_n := \partial_t \alpha u_n - \Delta u_n$. We will show that the inequality

$$\|u_n\|_{\mathcal{S}_{\mu, \theta, 0}^{\alpha, \gamma + 2}(\mathbb{R}_+^d, T)} \leq N(\alpha, d, p, \theta)\|f_n\|_{\mathbb{H}_{\mu, \theta + p}^{\gamma}(\mathbb{R}_+^d, T)} \quad (3.20)$$

holds for each $n$. This certainly proves $u_n = 0$, because the right hand side above goes to zero as $n \to \infty$ by the assumption that $u_n \to u$ in $\mathcal{S}_{\mu, \theta, 0}^{\alpha, \gamma + 2}(\mathbb{R}_+^d, T)$.

Let $n$ be fixed. Since $u_n \in C_{c}^{\infty}(\mathbb{R}_+ \times \mathbb{R}_+^d)$, we easily have $f_n \in \mathbb{L}_{p', \theta + p}^{\alpha, \gamma + 2}(\mathbb{R}_+^d, T)$ for any $p' > \max\{d + \theta + 2/\alpha, p\}$. Therefore, by Remark 3.8, there exist $f_n^i \in \mathbb{H}_{\mu, \theta}^{\alpha, \gamma + 2}(\mathbb{R}_+^d, T)$ such that the claims of Remark 3.8 holds with $f^i, f_n, \gamma =$
0, \rho, p', \theta in place of f, f', \gamma, p, q, \theta respectively. Let \bar{u}_n and \bar{f}_n denote the odd extensions of \bar{u}_n and \bar{f}_n respectively with respect to \bar{x} = 0. Since \bar{u}_n \in C^\infty_c((0, \infty) \times \mathbb{R}^d) and \bar{f}_n = \partial_t^2 \bar{u} - \Delta \bar{u} on \mathbb{R}^d, by Lemma 3.2 and the relation \bar{f}_n = \sum_i D_i f_{n,i}^a, we have
\[
\bar{u}_n(t, x) = \int_0^t \int_{\mathbb{R}^d} q(t-s, x-y) \bar{f}_n(s, y) dy ds
= \int_0^t \int_{\mathbb{R}^d_+} (q(t-s, x-y) - q(t-s, x-y^*)) \bar{f}_n(s, y) dy ds
= \int_0^t \int_{\mathbb{R}^d_+} (q(t-s, x-y) - q(t-s, x-y^*)) D_i f_{n,i}^a(s, y) dy ds,
\]
where \( y^* = (-y^1, y^i) \). By [16, Remark 1.21], there exists a sequence \( f_{n,k}^i \in C^\infty_c((0, \infty) \times \mathbb{R}^d) \) which converges to \( f_n^i \) in \( H^{1}_{p, \rho}(\mathbb{R}^d_+, T) \). Let \( u_{n,k} \in \mathcal{S}_{p, \rho, 0}(\mathbb{R}^d_+, T) \) be defined by (3.6) with \( f_{n,k}^i \). Then, by (3.8) and Lemma 3.7 with \( \gamma = 1 \) and \( \mu = 0 \), we have \( u_{n,k} \in \mathcal{S}_{p, \rho, 0}(\mathbb{R}^d_+, T) \),
\[
\| u_{n,k} \|_{\mathcal{S}_{p, \rho, 0}(\mathbb{R}^d_+, T)} \leq N \sum_{i=1}^d \| f_{n,k}^i \|_{H^{1}_{p, \rho}(\mathbb{R}^d_+, T)},
\]
and
\[
\| u_{n,k} - u_{n,k'} \|_{\mathcal{S}_{p, \rho, 0}(\mathbb{R}^d_+, T)} \leq N \sum_{i=1}^d \| f_{n,k}^i - f_{n,k'}^i \|_{H^{1}_{p, \rho}(\mathbb{R}^d_+, T)},
\]
where the constant \( N \) is independent of \( n, k \). This implies that there exists \( u'_n \in \mathcal{S}_{p, \rho, 0}(\mathbb{R}^d_+, T) \), the limit of \( u_{n,k} \), satisfying
\[
\| u'_n \|_{\mathcal{S}_{p, \rho, 0}(\mathbb{R}^d_+, T)} \leq N \sum_{i=1}^d \| f_{n,k}^i \|_{H^{1}_{p, \rho}(\mathbb{R}^d_+, T)} \leq N \| f_n \|_{L^{p, \rho}_{p, \rho}(\mathbb{R}_+, T)}. \]
Also by Remark 3.6, \( u_{n,k} \) converges to \( \bar{u}_n \) pointwise in \( (0, T) \times \mathbb{R}^d_+ \). Therefore, \( u_n = u'_n \in L^{p, \rho}_{p, \rho}(\mathbb{R}^d_+, T) \). Since \( u'_n \in L^{p, \rho}_{p, \rho}(\mathbb{R}^d_+, T) \) we have \( u_n \in L^{p, \rho}_{p, \rho}(\mathbb{R}^d_+, T) \). Moreover, since \( \partial_t^2 u_n = \Delta u_n + f_n \), (3.20) also follows.

**Step 2.** Let \( \gamma = 0 \) and \( A = [a^{ij}]_{d \times d} \).

By (2.19), there exists an invertible symmetric matrix \( \sigma \) such that \( A = \sigma^2 \). Let \( Q \) be an orthogonal transform from \( \sigma(\mathbb{R}^d_+) \to \mathbb{R}^d_+ \). Then \( Q \sigma : \mathbb{R}^d_+ \to \mathbb{R}^d_+ \) is one-to-one, onto, and there exist constants \( c, c' > 0 \) such that
\[
c x^1 \leq (Q \sigma x)^1 \leq c^{-1} x^1, \quad c' x^1 \leq (\sigma^{-1} Q T x)^1 \leq c' x^1, \quad \forall x \in \mathbb{R}^d_+.
\]
It follows that for any \( n = 0, 1, \ldots \), and \( \theta \in \mathbb{R} \),
\[
\| h \|_{H^{1}_{p, \sigma}(\mathbb{R}^d_+)} \sim \| h(\sigma) \|_{H^{1}_{p, \sigma}(\mathbb{R}^d_+)} \| h(\sigma^{-1} Q T) \|_{H^{1}_{p, \sigma}(\mathbb{R}^d_+)} \| (3.22)
\]
due to (2.6). Consider the equation
\[
\partial_t^2 v(t, x) = \Delta v(t, x) + f(t, Q \sigma x) \quad (t, x) \in (0, T) \times \mathbb{R}^d_+ \quad (3.23)
\]
Define
\[
u(t, x) = v(t, \sigma^{-1} Q T x).
\]
Then
\[
\Delta v(t, x) = (Q \sigma)^{ik}(Q \sigma)^{il} D_{kl} u(t, Q \sigma x)
= (\sigma Q^T)^{kl}(Q \sigma)^{il} D_{kl} u(t, Q \sigma x) = a^{kl} D_{kl} u(t, Q \sigma x).
\]
Thus, \( u \in \mathcal{S}^{p,\gamma,0}_{p,\theta,0}(\mathbb{R}^d_+, T) \) is a solution to (3.1) if and only if \( v \in \mathcal{S}^{p,\gamma,0}_{p,\theta,0}(\mathbb{R}^d_+, T) \) is a solution to (3.23). Uniqueness, existence and estimate (3.19) follow from Step 1 and (3.22).

**Step 3.** Let \( \gamma \geq 0 \).

The uniqueness easily follows from Step 2 since \( \mathcal{S}^{p,\gamma+1,0}_{p,\theta,0}(\mathbb{R}^d_+, T) \subset \mathcal{S}^{p,\gamma,0}_{p,\theta,0}(\mathbb{R}^d_+, T) \).

Also, since \( \mathbb{H}^{\gamma+1}_{p,\theta+p}(\mathbb{R}^d_+, T) \subset \mathbb{L}_{p,\theta+p}(T) \), by Step 2, there is a solution \( u \in \mathcal{S}^{p,\gamma,0}_{p,\theta,0}(\mathbb{R}^d_+, T) \) to equation (3.19). Take \( f^i \in \mathbb{H}^{\gamma+1}_{p,\theta}(\mathbb{R}^d_+, T) \) from Lemma 3.8 such that \( f = \sum_i D_i f^i \) and \( \|f^i\|_{\mathbb{H}^{\gamma+1}_{p,\theta+p}(\mathbb{R}^d_+, T)} \leq N \|f\|_{\mathbb{H}^{\gamma+1}_{p,\theta+p}(\mathbb{R}^d_+, T)} \).

Then by Lemma 3.7 with \( \mu = 1 \), we have \( u \in \mathcal{S}^{p,\gamma+2,0}_{p,\theta,0}(\mathbb{R}^d_+, T) \), and

\[
\|u\|_{\mathbb{H}^{\gamma+2}_{p,\theta-p}(\mathbb{R}^d_+, T)} \leq N \|u\|_{\mathbb{H}^{\gamma}_{p,\theta-p}(\mathbb{R}^d_+, T)} + N \sum_{i=1}^{d} \|f^i\|_{\mathbb{H}^{\gamma+1}_{p,\theta+p}(\mathbb{R}^d_+, T)} \leq N \|f\|_{\mathbb{H}^{\gamma}_{p,\theta+p}(\mathbb{R}^d_+, T)}.
\]

From the equality \( \partial_t^\alpha u = \Delta u + f \), we also get the estimate of \( \partial_t^\alpha u \), and therefore the theorem is proved. \( \square \)

4. **Proof of Theorem 2.12.** First, we introduce some results related to perturbation argument, which are from [11] (see Lemma 3.3 and Lemma 3.1 therein).

**Lemma 4.1.** For each \( \varepsilon > 0 \) and \( q = 1, 2, \ldots \) there exist nonnegative functions \( \eta_k \in C^\infty_c(\mathbb{R}^d_+) \), \( k = 1, 2, \ldots \), such that

(i) on \( \mathbb{R}^d_+ \) for each multi-index \( \beta \) with \( 1 \leq |\beta| \leq q \) we have

\[
\sum_k \eta_k^\beta \geq 1, \quad \sum_k \eta_k \leq N(d), \quad \sum_k (x^{1})^{|\beta|} |D^\beta \eta_k| \leq \varepsilon;
\]

(ii) for any \( k \) and \( x, y \) support of \( \eta_k \) we have \( |x - y| \leq N(x^{1} \wedge y^{1}) \), where \( N = N(d, q, \varepsilon) \).

**Lemma 4.2.** Let constants \( C, \delta \in (0, \infty) \), a function \( u \in H^\gamma_{p,\theta}(\mathbb{R}^d_+) \), and \( q \) be the smallest integer such that \( |\gamma| + 2 \leq q \).

(i) Let \( \eta_k \in C^\infty(\mathbb{R}^d_+) \), \( k = 1, 2, \ldots \), satisfy

\[
\sum_k (x^{1})^{|\beta|} |D^\beta \eta_k| \leq C \quad \text{in} \quad \mathbb{R}^d_+
\]

for any multi-index \( \beta \) such that \( 0 \leq |\beta| \leq q \). Then

\[
\sum_k \|\eta_k u\|_{H^\gamma_{p,\theta}(\mathbb{R}^d_+)} \leq N C \|u\|_{H^\gamma_{p,\theta}(\mathbb{R}^d_+)},
\]

where \( N = N(d, p, \gamma) \) is independent of \( u \) and \( C \).

(ii) If in addition to the condition (i)

\[
\sum_k \eta_k^2 \geq \delta \quad \text{on} \quad \mathbb{R}^d_+,
\]

then

\[
\|u\|_{H^\gamma_{p,\theta}(\mathbb{R}^d_+)} \leq N \sum_k \|\eta_k u\|_{H^\gamma_{p,\theta}(\mathbb{R}^d_+)},
\]

where \( N = N(d, p, \theta, C, \delta, \gamma) \) is independent of \( u \) and \( \theta \).
Now we divide the proof of the theorem into two cases.

**Step 1.** Suppose that \( a^{ij} \) are independent of \( t \). In this case we use a perturbation argument with respect to \( x \). This argument is taken from the proof of Theorem 2.14 of [11], where the case \( \alpha = 1 \) is handled.

**Case 1.** \( \gamma = 0, 1, 2, \cdots \).

Due to the method of the continuity (see e.g. [7, Lemma 5.1]) and the solvability result for equations with constant coefficients (Theorem 3.9), it is enough to show that there exists \( \omega_0 \in (0, 1) \) such that a priori estimate (2.25) holds given that a solution \( u \in S_p, g^{\cdot +1} (\mathbb{R}^d_+, T) \) already exists and (2.24) holds for \( \omega < \omega_0 \).

Take the least integer \( q \) satisfying \( q \geq |\gamma| + 4 \). Let \( \varepsilon \in (0, 1) \) which will be specified later, and take a sequence of functions \( \eta_m \) from Lemma 4.1 corresponding to \( q \), and \( \varepsilon \). Then, since \( \varepsilon < 1 \), by Lemma 4.2(ii) with \( \delta = 1 \),

\[
\|u\|_{S_p, g^{\cdot +1} (\mathbb{R}^d_+, T)} \leq N(d, p, \theta, \gamma) \sum_{m=1}^{\infty} \|\eta_m^2 u\|_{S_p, g^{\cdot +1} (\mathbb{R}^d_+, T)}. \tag{4.1}
\]

Denote \( a_m^{ij} = a^{ij}(x_m) \). Then \( u\eta_m^2 \) satisfies

\[\partial^\alpha (u\eta_m^2) = a_m^{ij} D_{ij}(u\eta_m^2) + f_m,\]

where

\[f_m = -2a_m^{ij} D_i D_j (\eta_m^2) - a_m^{ij} u D_j (\eta_m^2) + (a^{ij} - a_m^{ij}) (D_i u)(\eta_m^2) + f u_m^2 + \eta_m^2 b^i D_i u + \eta_m^2 c u.\]

Let \( M^\alpha \) denote the operator of multiplying by \((x_1)^\alpha\) and \( M = M^1 \). Note that \( f_m \in H^p_{\theta + p} (\mathbb{R}^d_+, T) \), which is equivalent to \( M f_m \in H^p_{\theta + p} (\mathbb{R}^d_+, T) \). By Theorem 3.9,

\[
\|u\|_{H^p_{\theta + p} (\mathbb{R}^d_+)} \leq N ||M f_m||_{H^p_{\theta + p} (\mathbb{R}^d_+)} \tag{4.2}
\]

Note that for any \( x, y \in \text{supp} \eta_m \), by the construction of \( \eta_m \), we have \( |x - y| \leq N(\varepsilon)(x_1 \wedge y_1) \), where the constant \( N(\varepsilon) \) is independent of \( m \). From this, one can find at most \( N(\varepsilon) + 2 < 3N(\varepsilon) \) points \( x_i \), in the line segment connecting \( x \) and \( y \), such that \( |x_i - x_{i+1}| \leq (x_1 \wedge y_1) \). Therefore by assumption (2.24),

\[
\sup_{x \in \mathbb{R}^d_+} \| (a^{ij} - a_m^{ij}) \eta_m \| \leq N(\varepsilon) \omega.
\]

Therefore, by (2.13),

\[
\|M (a^{ij} - a_m^{ij}) \eta_m^2 D_{ij} u\|_{H^p_{\theta + p} (\mathbb{R}^d_+)} \leq N \sup_{x \in \mathbb{R}^d_+} \| (a^{ij} - a_m^{ij}) \eta_m M D_{ij} u\|_{H^p_{\theta + p} (\mathbb{R}^d_+)} + N 1_{\gamma \neq 0} \| \eta_m M D_{ij} u\|_{H^{\cdot -1} (\mathbb{R}^d_+)} + N 1_{\gamma \neq 0} \| \eta_m M D_{ij} u\|_{H^{\cdot -1} (\mathbb{R}^d_+)};
\]

(4.3)

where \( N = N(d, p, \theta, \gamma, K) \). Similarly,

\[
\|M M^2 b^i D_i u\|_{H^p_{\theta + p} (\mathbb{R}^d_+)} + \|M^2 \eta_m^2 c u\|_{H^p_{\theta + p} (\mathbb{R}^d_+)}
\]

\[
= \| \eta_m M b^i \eta_m D_i u\|_{H^p_{\theta + p} (\mathbb{R}^d_+)} + \| \eta_m M^2 c \eta_m M^{-1} u\|_{H^p_{\theta + p} (\mathbb{R}^d_+)} \leq N(\varepsilon) \omega \| \eta_m M D u\|_{H^p_{\theta + p} (\mathbb{R}^d_+)} + \| \eta_m M^{-1} u\|_{H^p_{\theta + p} (\mathbb{R}^d_+)}
\]

\[
+ N 1_{\gamma \neq 0} \| \eta_m M D u\|_{H^{\cdot -1} (\mathbb{R}^d_+)} + \| \eta_m M^{-1} u\|_{H^{\cdot -1} (\mathbb{R}^d_+)}.
\]
Coming back to (4.1) and (4.2), and then using above calculations and Lemma 4.2, we conclude
\[
\|u\|_{H^{\gamma+2}_{p,\theta-\rho}(\mathbb{R}^d_+, T)}^p \leq N(N(\varepsilon)\omega^p + C\varepsilon^p)\|u\|_{H^{\gamma+2}_{p,\theta-\rho}(\mathbb{R}^d_+, T)}^p + N\|Mf\|_{H^{\gamma}_{p,\theta}(\mathbb{R}^d_+, T)} + \frac{\varepsilon}{\omega - \omega^0}\|u\|_{H^{\gamma+1}_{p,\theta-\rho}(\mathbb{R}^d_+, T)}^p,
\]
where \(N = N(d, p, \theta, \gamma, K)\), and
\[
C := \sup_{x \in \mathbb{R}^d_+} \sup_{|\beta| \leq q-2} \sum_{m=1}^{\infty} (x^1)^{|\beta|} |D^\beta \eta_m| \leq N\varepsilon.
\]
It follows that if \(\gamma = 0\), then
\[
\|u\|_{H^{\gamma+2}_{p,\theta-\rho}(\mathbb{R}^d_+, T)}^p \leq N\|Mf\|_{H^{\gamma}_{p,\theta}(\mathbb{R}^d_+, T)} + N\|u\|_{H^{\gamma+1}_{p,\theta-\rho}(\mathbb{R}^d_+, T)}^p \leq N\|Mf\|_{H^{\gamma}_{p,\theta}(\mathbb{R}^d_+, T)}.
\]
Now we choose \(\varepsilon\) and take \(\omega_0\) in order such that \(NN(\varepsilon)(\omega_0^p + \varepsilon^p) \leq 1/2\). If \(\omega < \omega_0\), then the a priori estimate holds.

If \(\gamma = 1, 2, \cdots\), then we use the induction argument. Note that if Assumption 2.9 holds for \(\gamma\) then it also holds for \(\gamma - 1\). Assume that Assumption 2.9(\(\gamma\)) holds and there exists a constant \(\omega_0 > 0\) such that the a priori estimate (2.25) holds for \(\gamma - 1\) (not for \(\gamma\)) given that \(\omega < \omega_0\). Using (4.4), and chosing \(\varepsilon\) and \(\omega\) in order we conclude that if \(\omega\) is sufficiently small, say \(\omega < \omega_0 \leq \omega_0\) then
\[
\|u\|_{H^{\gamma+2}_{p,\theta-\rho}(\mathbb{R}^d_+, T)}^p \leq N\|Mf\|_{H^{\gamma}_{p,\theta}(\mathbb{R}^d_+, T)} + N\|u\|_{H^{\gamma+1}_{p,\theta-\rho}(\mathbb{R}^d_+, T)} \leq N\|Mf\|_{H^{\gamma}_{p,\theta}(\mathbb{R}^d_+, T)}.
\]
The second inequality above holds by the assumption that the a priori estimate holds for \(\gamma - 1\) given that \(\omega < \omega_0\). Thus the induction goes through and the a priori estimate holds for \(\gamma = 0, 1, 2, \cdots\).

**Case 2.** \(\gamma = -1, -2, \cdots\)

We again use the induction argument for \(\gamma \in \{0, -1, -2, \cdots\}\). Recall that all the claims of the theorem is proved in Case 1 for \(\gamma = 0\). Since, \(|\gamma| \geq |\gamma + 1|\), if Assumption 2.9(\(\gamma\)) holds then Assumption 2.9(\(\gamma + 1\)) also holds. Now suppose Assumption 2.9(\(\gamma\)) holds and there exists \(\omega_0 > 0\) such that the theorem holds for \(\gamma + 1\) provided \(\omega < \omega_0\).

Assume \(\omega \leq \omega_0\), and let \(T\) denote the map sending \(f \in H^{\gamma+1}_{p,\theta-\rho}(\mathbb{R}^d_+, T)\) to the solution \(u \in \mathbb{D}^{\alpha,\gamma+3}\). By [16, Corollary 2.12], there exist \(f^k \in H^{\gamma+1}_{p,\theta-\rho}(\mathbb{R}^d_+, T)\), \(k = 1, 2, \ldots, d\) such that
\[
f = \sum_k MD_k f^k \quad \text{and} \quad \sum_{k=1}^{d} \|Mf^k\|_{H^{\gamma+1}_{p,\theta-\rho}(\mathbb{R}^d_+, T)} \leq N\|Mf\|_{H^{\gamma}_{p,\theta}(\mathbb{R}^d_+, T)}.
\]
Let \(w^k = T f^k\) for \(k = 1, 2, \ldots, d\) and \(v = \sum_k MD_k w^k\). By Lemma 2.1 (iv) and the induction hypothesis,
\[
\|v\|_{H^{\gamma+2}_{p,\theta-\rho}(\mathbb{R}^d_+, T)} \leq N \sum_{k=1}^{d} \|w^k\|_{H^{\gamma+3}_{p,\theta-\rho}(\mathbb{R}^d_+, T)} \leq N \sum_{k=1}^{d} \|Mf^k\|_{H^{\gamma+1}_{p,\theta}(\mathbb{R}^d_+, T)} \leq N\|Mf\|_{H^{\gamma}_{p,\theta}(\mathbb{R}^d_+, T)}.
\]
Moreover, \(v\) satisfies the equation
\[
\partial_\alpha^\alpha v = a^{ij} D_{ij} v + b^i D_i v + c v + f + \bar{f}, \quad v(0) = \partial_\alpha v(0) (\alpha > 1) = 0.
\]
However, all the results in \([7]\) hold for $p > p$ under the condition

$$
\text{Consequently, using a regularity result on } R
$$

Note that

$$
\gamma
$$

Case 3.

Observe that

$$
\|MD_k a^{(0)}_{\gamma+1}\| = \|MD_k a^{(0)}_{\gamma-1}\| \leq N|a^{(0)}_{\gamma}|
$$

$$
\|M^2 D_k b^{(0)}_{\gamma+1}\| = \|M^2 D_k b^{(0)}_{\gamma-1}\| \leq N|b^{(0)}_{\gamma}|
$$

$$
\|M^3 D_k c^{(0)}_{\gamma+11}\| = \|M^3 D_k c^{(0)}_{\gamma-1}\| \leq N|M^2 c^{(0)}_{\gamma}|
$$

due to (2.9). By (4.6), (4.5), and Lemma 2.1 we have

$$
\|M\bar{f}\|_{H^{\gamma+1}(\mathbb{R}^d_+; T)} \leq N\|Mf\|_{H^{\gamma}_{p,p}(\mathbb{R}^d_+; T)}.
$$

This implies that one can define $\bar{u} := T\bar{f} \in S^{\alpha,\gamma+3}_{p,\theta,0}(\mathbb{R}^d_+; T)$. Take $u = v - \bar{u}$. Note that $v \in S^{\alpha,\gamma+2}_{p,\theta,0}(\mathbb{R}^d_+; T)$. Since $S^{\alpha,\gamma+2}_{p,\theta,0}(\mathbb{R}^d_+; T) \subset S^{\alpha,\gamma+2}_{p,\theta,0}(\mathbb{R}^d_+; T)$, we conclude $u \in S^{\alpha,\gamma+2}_{p,\theta,0}(\mathbb{R}^d_+; T)$ is a solution to equation (2.22), and (2.25) also follows.

Now it remains to prove the uniqueness result for $\gamma$. Suppose $u \in S^{\alpha,\gamma+2}_{p,\theta,0}(\mathbb{R}^d_+; T)$ is a solution to equation (2.22) with $f = 0$. Take $\eta_m$ from Case 1 corresponding to $\varepsilon = 1$. Then,

$$
\partial_t^\alpha (\eta_m u) = a^{ij} D_{ij} (\eta_m u) + b^i D_i (\eta_m u) + c \eta_m u + f_m,
$$

where $f_m := -2a^{ij} D_{ij} \eta_m - \alpha^i u D_i \eta_m - b^i u D_i \eta_m$. Since $u \in H^{\gamma+2}_{p,\theta,p}(T)$ and $f_m$ has compact support in $\mathbb{R}^d_+$, we conclude $f_m \in H^{\gamma+1}_p(T)$. Furthermore, since $\eta_m u$ has compact support in $\mathbb{R}^d_+$, without hurting the above equation we can replace $a^{ij}, b^i, c$ by $\bar{a}^{ij}, \bar{b}^i, \bar{c}$ respectively such that all of them are defined on $\mathbb{R}^d_+$, $\bar{a}^{ij}$ are uniformly elliptic and uniformly continuous in $\mathbb{R}^d$, and

$$
|\bar{a}^{ij}|_{\gamma} + |\bar{b}^i|_{\gamma} + |\bar{c}|_{\gamma} < \infty.
$$

Consequently, using a regularity result on $\mathbb{R}^d$, [7, Theorem 2.3], we conclude $u \eta_m \in H^{\alpha,\gamma+3}_{p,\theta,0}(T)$, which yields $\eta_m u \in S^{\alpha,\gamma+3}_{p,\theta,0}(\mathbb{R}^d_+; T)$ due to Lemma 2.5. We remark that the condition $p \geq 2$ is assumed in [7, Theorem 2.3] to handle stochastic PDEs. However all the results in [7] holds for $p > 1$ for the deterministic case.

By Lemma 4.2 and the induction hypothesis,

$$
\|u\|_{H^{\gamma+3}_{p,\theta,p}(\mathbb{R}^d_+; T)} \leq N \sum_{m=1}^\infty \|u \eta_m\|_{H^{\gamma+3}_{p,\theta,p}(\mathbb{R}^d_+; T)} \leq N \sum_{m=1}^\infty \|M f_m\|_{H^{\gamma+1}_{p,p}(\mathbb{R}^d_+; T)},
$$

Using Lemma 4.2(i), we find that the last term above is controlled by a constant times of $\|u\|_{H^{\gamma+2}_{p,\theta,p}(\mathbb{R}^d_+; T)}$. Therefore, $u \in S^{\alpha,\gamma+3}_{p,\theta,0}(\mathbb{R}^d_+; T)$, and consequently $u = 0$ due to the uniqueness result for $\gamma + 1$.

**Case 3.** $\gamma$ is not an integer.

We follows the proof of Case 1 and prove the a priori estimate.

Proceed as in Case 1, and then in place of (4.3) we use

$$
\|M(a^{ij} - a^{ij}_m)\eta_m^2 D_{ij} u\|_{H^{\gamma}_{p,p}(\mathbb{R}^d_+; T)} \leq N\|\eta_m |a^{ij} - a^{ij}_m| \eta_m\|_{\gamma}\|\eta_m MD_{ij} u\|_{H^{\gamma}_{p,p}(\mathbb{R}^d_+; T)}
$$

$$
\leq N \sup_{\mathbb{R}^d_+} \|a^{ij} - a^{ij}_m\eta_m\|_{\gamma}\|\eta_m D_{ij} u\|_{H^{\gamma}_{p,p}(\mathbb{R}^d_+; T)},
$$

where

$$
\tilde{f} = D_{ij} u^M D_k a^{ij} + D_{ij} u^M D_k b^i + u^M D_k c - 2a^{ij} D_{ij} u^M - D_k u^M b^i.
$$

(4.7)
where $\gamma' = |\gamma| + \kappa_0(\gamma)/2$, and $\kappa_1 = 1 - (|\gamma| + \gamma')$. For the inequalities above we used Lemma 2.1 (v) and (2.8). By the construction of $\eta_m$ and (2.24),

$$\sup_{x \in \mathbb{R}^d_+} |(a^i_j - a_m^j)|\eta_m^{\kappa_1} \leq 3N(\varepsilon)\omega^{\kappa_1}.$$

Similarly,

$$\|M\eta_m^2 b^i D_j u\|_{\mathcal{H}^0_{p,\theta}(\mathbb{R}^d_+, T)} + \|M\eta_m^2 cu\|_{\mathcal{H}^0_{p,\theta}(\mathbb{R}^d_+, T)} \leq NN(\varepsilon)\omega^{\kappa_1} \left( \|\eta_m D_j u\|_{\mathcal{H}^0_{p,\theta}(\mathbb{R}^d_+, T)} + \|\eta_m M^{-1} u\|_{\mathcal{H}^0_{p,\theta}(\mathbb{R}^d_+, T)} \right).$$

Following the proof of Case 1, we conclude

$$\|u\|_{\mathcal{H}^{p,\theta,+2}_{p,\theta}(\mathbb{R}^d_+, T)} \leq N(\varepsilon)\omega^{\kappa_1 + \varepsilon} \|u\|_{\mathcal{H}^{p,\theta,+2}_{p,\theta}(\mathbb{R}^d_+, T)} + N\|f\|_{\mathcal{H}^{p,\theta,+2}_{p,\theta}(\mathbb{R}^d_+, T)}.$$

Choose $\varepsilon$ and take $\omega_0$ in order such that $N(\varepsilon)(\omega^{\kappa_1 + \varepsilon}) \leq 1/2$. Then, we have the a priori estimate if $\omega < \omega_0$.

**Step 2.** General case. We use the perturbation argument with respect to $t$. This argument is based on the following two results.

**Lemma 4.3.** Let $\gamma \notin \{-1, -2, -3, \cdots \}$. Take $\omega_0 = \omega_0(\gamma)$ from Step 1. Then there exists $T_0 \in (0, 1)$ depending only on $\alpha, d, p, \theta, \gamma, K$, the the modulus of continuity of $a^i_j$ in $t$ such that if (2.24) holds for $\omega < \omega_0$ and $T \leq T_0$ then a priori estimate (2.25) holds with a constant $N$ independent of $T$.

**Proof.** Let $u \in \mathcal{S}_{p,\theta,0}^{\gamma, \gamma+2}(\mathbb{R}^d_+, T)$ be a solution to equation (2.22). Denote $a_0^{ij}(x) = a^{ij}(0, x)$ and assume $\omega < \omega_0$. Then, we have

$$\partial_t^\gamma u = a_0^{ij} u_{x^i x^j} + b^i D_i u + cu + f + (a^{ij} - a_0^{ij})u_{x^i x^j}.$$

Since $a_0^{ij}$ is independent of $t$, by the result of Step 1, for any $\gamma \in \mathbb{R}$,

$$\|u\|_{\mathcal{H}^{p,\theta,+2}_{p,\theta}(\mathbb{R}^d_+, T)} \leq N_0 \|M f\|_{\mathcal{H}^0_{p,\theta}(\mathbb{R}^d_+, T)} + N_0 \|\|a^{ij} - a_0^{ij}\|\|M u_{x^i x^j}\|_{\mathcal{H}^0_{p,\theta}(\mathbb{R}^d_+, T)}, \quad (4.9)$$

where $N_0$ is independent of $T$. By (2.8) and Lemma 2.1(v),

$$\|a^{ij} - a_0^{ij}\|u_{x^i x^j}\|_{\mathcal{H}^{p,\theta,+2}_{p,\theta}(\mathbb{R}^d_+, T)} \leq N_1(K, d, p, \gamma) \sup_{[0,T] \times \mathbb{R}^d_+} \|a^{ij} - a_0^{ij}\|^{\kappa_1} \|D_{ij} u\|_{\mathcal{H}^{p,\theta,+2}_{p,\theta}(\mathbb{R}^d_+, T)}, \quad (4.10)$$

where $\gamma' = |\gamma| + \kappa_0(\gamma)/2$, and $\kappa_1 = 1 - (|\gamma| + \gamma')$. Also it is obvious that if $\gamma = 0$, then one can choose $\kappa_1 = 1$. By Assumption 2.6 (iv), one can find $T_0$ such that if $T \leq T_0$ then

$$N_0N_1 \sup_{[0,T] \times \mathbb{R}^d_+} \|a^{ij} - a_0^{ij}\|^{\kappa_1} \leq 1/2,$$

and this certainly yields our claim.

If $\gamma = 1, 2, \ldots$, we use the induction. Note that by reducing $\omega_0(\gamma)$ if necessary we may assume $\omega_0(\gamma) \leq \omega_0(\gamma - 1)$. Suppose that there exists $T_0$ such that the claim of the lemma holds for $\gamma - 1$ if $\omega \leq \omega_0(\gamma - 1)$ and $T \leq T_0$.

Now suppose $\omega < \omega_0(\gamma)$ and $T \leq T_0$. Then using (4.9) and the inequalities (see (2.13))

$$\|a^{ij} - a_0^{ij}\|u_{x^i x^j}\|_{\mathcal{H}^{p,\theta,+2}_{p,\theta}(\mathbb{R}^d_+, T)} \leq N \sup_{[0,T] \times \mathbb{R}^d_+} \|a^{ij} - a_0^{ij}\| \|D_{ij} u\|_{\mathcal{H}^{p,\theta,+2}_{p,\theta}(\mathbb{R}^d_+, T)} + N\|MD^2 u\|_{\mathcal{H}^{p,\theta,+2}_{p,\theta}(\mathbb{R}^d_+, T)};$$

and this certainly yields our claim.
we find that if $T$ is sufficiently small, say $T \leq T_1 \leq T_0$, then (4.9) yields
\[
\|u\|_{\mathcal{H}_{p,\theta}^{\gamma+2}(\mathbb{R}^d_+)} \leq N_0\|f\|_{\mathcal{H}_{p,\theta}^{\gamma}(\mathbb{R}^d_+)} + N\|u\|_{\mathcal{H}_{p,\theta}^{\gamma+1}(\mathbb{R}^d_+)}.
\]
The last term above can be estimated due to the induction hypothesis. The lemma is proved.

Lemma 4.4. Let $\bar{T} \leq T$.

(i) For any $u \in \mathcal{S}^{\alpha,\gamma+2}_{p,\theta,0}(\mathbb{R}^d_+, \bar{T})$, there exists $\bar{u} \in \mathcal{S}^{\alpha,\gamma+2}_{p,\theta,0}(\mathbb{R}^d_+, T)$ such that $\bar{u}(t) = u(t)$ for all $t \leq \bar{T}$ and
\[
\|\bar{u}\|_{\mathcal{S}^{\alpha,\gamma+2}_{p,\theta,0}(\mathbb{R}^d_+, \bar{T})} \leq N\|u\|_{\mathcal{S}^{\alpha,\gamma+2}_{p,\theta,0}(\mathbb{R}^d_+, \bar{T})},
\]
where $N$ is independent of $\bar{T}$.

(ii) Let $u, v \in \mathcal{S}^{\alpha,\gamma+2}_{p,\theta,0}(\mathbb{R}^d_+, T)$ and $u(t) = v(t)$ for $t \leq \bar{T}$. Then $\bar{u}(t) := u(\bar{T} + t) - v(\bar{T} + t) \in \mathcal{S}^{\alpha,\gamma+2}_{p,\theta,0}(\mathbb{R}^d_+, T - \bar{T})$.

Proof. (i) By Theorem 3.9, the equation
\[
\partial_t^\alpha \bar{u} = \Delta \bar{u} + (\partial_t^\alpha u - \Delta u)1_{t \leq \bar{T}}, \quad t \leq T,
\]
has a solution $\bar{u} \in \mathcal{S}^{\alpha,\gamma+2}_{p,\theta,0}(\mathbb{R}^d_+, T)$, and (4.11) also follows. The fact $\bar{u}(t) = u(t)$ for $t \leq \bar{T}$ follows from the uniqueness result of Theorem 3.9 and the fact that $\bar{u} := u - \bar{u} \in \mathcal{S}^{\alpha,\gamma+2}_{p,\theta,0}(\mathbb{R}^d_+, T)$ is a solution to the equation $\partial_t^\alpha \bar{u} = \Delta \bar{u}$ for $t \leq \bar{T}$.

(ii) By [16, Corollary 1.20], one can find $\zeta_n \in C^\infty(\mathbb{R}^d_+)$ whose support is in a strip $A_n := \{ x : 1/n < x^i < n \}$ so that $u\zeta_n$ and $v\zeta_n$ converge to $u$ and $v$ in $\mathcal{S}^{\alpha,\gamma+2}_{p,\theta,0}(\mathbb{R}^d_+, T)$ respectively. By [8, Lemma 5.3] one get $\zeta_n(u - v)(\bar{T} + t) \in \mathcal{H}^{\alpha,\gamma+2}_{p,\theta,0}(T - \bar{T})$. By Lemma 2.5, $\zeta_n(u - v)(\bar{T} + t) \in \mathcal{S}^{\alpha,\gamma+2}_{p,\theta,0}(\mathbb{R}^d_+, T - \bar{T})$. Since it converges to $(u - v)(\bar{T} + t)$ in $\mathcal{S}^{\alpha,\gamma+2}_{p,\theta,0}(\mathbb{R}^d_+, T - \bar{T})$, the proof is done.

Now we continue the proof for Step 2.

Assume firstly that $\gamma$ is not a negative integer.

We prove the a priori estimate given that a solution already exists and $\omega < \omega_0$, where $\omega_0 = \omega(\gamma)$ is taken from Step 1.

Assume $\omega < \omega_0$ and take $T_0$ from Lemma 4.3. Let $l$ be an integer so that $T/l < \frac{1}{2}T_0$, and let $T_k = kT/l$. Then by Lemma 4.3, if $u \in \mathcal{H}^{\alpha,\gamma+2}_{p,\theta,0}(T)$ is a solution of (1.1), then we have a priori estimate (2.25) if $T \leq T_1$. Suppose that (2.25) holds for $T_k$, $k < l$, with a constant $N(k)$. Take $\bar{u}$ from Lemma 4.4 corresponding to $\bar{T} := T_k$. Set $\bar{u}(t, x) = (u - \bar{u})(\bar{T} + t, x)$. Then $\bar{u}$ satisfies the equation
\[
\partial_t^\alpha \bar{u} = a^j(\bar{T} + t)D_j\bar{u} + b^j(\bar{T} + t, x)D_j\bar{u} + c(\bar{T} + t, x)\bar{u} + \bar{f}, \quad t \leq T - \bar{T},
\]
where
\[
\bar{f} = f(\bar{T} + t, x) - \partial_t^\alpha \bar{u}(\bar{T} + t) + a^j(\bar{T} + t)\bar{u}(\bar{T} + t, x) + b(\bar{T} + t, x)\bar{u}(\bar{T} + t, x).
\]
Therefore, by Lemma 4.3 and (4.11),
\[
\|\bar{u}\|_{\mathcal{H}^{\alpha,\gamma+2}_{p,\theta,0}(\mathbb{R}^d_+, T_1)} \leq N\|f\|_{\mathcal{H}^{\alpha,\gamma+2}_{p,\theta,0}(T_1)} + N\|\bar{f}\|_{\mathcal{H}^{\alpha,\gamma+2}_{p,\theta,0}(T - \bar{T})} \leq N\|f\|_{\mathcal{H}^{\alpha,\gamma+2}_{p,\theta,0}(T_0)} + N\|\bar{f}\|_{\mathcal{H}^{\alpha,\gamma+2}_{p,\theta,0}(T)} \leq N\|f\|_{\mathcal{H}^{\alpha,\gamma+2}_{p,\theta,0}(T_0)} + N\|\bar{f}\|_{\mathcal{H}^{\alpha,\gamma+2}_{p,\theta,0}(T)} \leq N\|f\|_{\mathcal{H}^{\alpha,\gamma+2}_{p,\theta,0}(T_0)} + N\|\bar{f}\|_{\mathcal{H}^{\alpha,\gamma+2}_{p,\theta,0}(T)}.
\]
We say that $\Omega$ is a Definition 5.1. properties.

Proof of Theorem 2.10. 5. We prove the a priori estimate following the argument in the proof of [11, Step 1.

Theorem 2.10].

\[
\|u\|_{L^{2,\gamma+2}_{p,\theta}(\mathbb{R}^d,T)} \leq \|\tilde{u}\|_{L^{2,\gamma+2}_{p,\theta}((\mathbb{R}^d,T))} + \|\tilde{u}\|_{L^{2,\gamma+2}_{p,\theta}(\mathbb{R}^d,T)} \leq N(k+1)\|f\|_{L^{2,\gamma+2}_{p,\theta}(\mathbb{R}^d,T)}.
\]

Now we see that the induction goes through, and therefore Case 2 is proved if $\gamma$ is not negative integer.

If $\gamma$ is a negative integer, then it is enough to follow the proof of Step 1 word for word.

5. Proof of Theorem 2.10. We first recall the definition of $C^1$ domain and its properties.

Definition 5.1. We say that $\Omega$ is a $C^1$ domain if there exist $K_0, r_0 > 0$ such that for any $x_0 \in \partial \Omega$, there exists a one-to-one continuously differentiable mapping

\[
\Psi = \Psi_{x_0} : B_{r_0}(x_0) \to G,
\]

where $G$ is a domain in $\mathbb{R}^d$, such that

(i) $G_+ := \Psi(B_{r_0}(x_0) \cap \Omega) \subset \mathbb{R}^d_+$ and $\Psi(x_0) = 0.$

(ii) $\Psi(B_{r_0}(x_0) \cap \partial \Omega) = G \cap \{y \in \mathbb{R}^d : y^1 = 0\}$.

(iii) $\|\Psi\|_{C^1(B_{r_0}(x_0))} \leq K_0$, and $|\Psi^{-1}(y_1) - \Psi^{-1}(y_2)| \leq K_0|y_1 - y_2|$ for any $y_1, y_2$ in $G$.

(iv) $D\Psi$ is uniformly continuous on $B_{r_0}(x_0)$.

Lemma 5.2. Let $\Omega$ be of class $C^1_{\alpha}$. Then the function $\Psi$ can be chosen in such a way that $\Psi$ is infinitely differentiable “in” $B_{r_0}(x_0) \cap \Omega$ (not up to the boundary), and for any nonnegative integer $n$

\[
|D\Psi|^{(0)}_{n,B_{r_0}(x_0)\cap\Omega} + |D\Psi^{-1}|^{(0)}_{n,G_+} < N(n) < \infty
\]

and

\[
\rho(x)D^2\Psi(x) \to 0 \quad \text{as} \quad x \in B_{r_0}(x_0) \cap \Omega, \quad \text{and} \quad \rho(x) \to 0, \quad (5.1)
\]

where the constant $N(n)$ and the convergence in (5.1) are independent of $x_0$.

Proof. See e.g. [9, Lemma 2.5 (ii)] for the proof. We remark that since $\partial \Omega$ is only $C^1$, the derivatives of $\Psi$ are not regular enough up to the boundary. However, $\Psi$ can be chosen such interior Hölder norms of its derivatives are finite as above. □

Now we start the proof.

Step 1. We prove the a priori estimate following the argument in the proof of [11, Theorem 2.10].

Suppose that $u \in H_{p,\theta,0}^{\alpha,\gamma+2}(\Omega, T)$ satisfies the equation (2.22). Denote $r = r_0/K_0$. Take $\eta \in C_{\infty}(B_r)$ and $\varphi \in C_{\infty}(\mathbb{R})$ so that $\eta = 1$ in $B_{r/2}$, $\varphi(t) = 1$ for $t \leq -3$, $\varphi(t) = 0$ if $t \geq -1$, and $-1 \leq \varphi \leq 0$. For $k = 1, 2, \ldots$, and $x \in \mathbb{R}^d_+$, define $\varphi_k = \varphi(k^{-1}\log x^1)$. Also, define

\[
\hat{a}_k = \hat{a}(x_0)\varphi_k + (1 - \eta\varphi_k)I, \quad \hat{b}_k = \hat{b}\eta\varphi_k, \quad \hat{c}_k = \hat{c}\eta\varphi_k,
\]

where $I$ is $d \times d$ identity matrix.

\[
\hat{a}(t,x) = \hat{a}(t,\Psi^{-1}(x)), \quad \hat{b}(t,x) = \hat{b}(t,\Psi^{-1}(x)),
\]

\[
\hat{a}^j = a^j_k D_k(\Psi^1) D_i(\Psi^j), \quad \hat{b}^j = a^j_k D_k(\Psi^1) + b^j_k D_k(\Psi^j), \quad (5.2)
\]

\[
\hat{c}(t,x) = c(t,\Psi^{-1}(x)).
\]
Using Lemma 3.4 and Lemma 3.7 in [11], one can check that \( \hat{a}_k, \hat{b}_k, \) and \( \hat{c}_k \) satisfy Assumption 2.6 and Assumption 2.9 on \( \mathbb{R}^d_+ \) with new constants \( \delta, K > 0 \) independent of \( x_0 \) and \( k \).

Take \( \omega_0 \) from Theorem 2.12 corresponding to \( \alpha, d, p, \theta, \delta, \gamma, K \). Observe that \( \varphi(k^{-1} \ln x^1) = 0 \) for \( x^1 \geq e^{-k} \) and \( |\varphi(k^{-1} \ln y^1) - \varphi(k^{-1} \ln y^1)| \leq k^{-1} \) if \( |x^1 - y^1| \leq x^1 \wedge y^1 \). Also we easily see that (5.1) implies \( x^1 \Psi_{xx}(\Psi^{-1}(x)) \to 0 \) as \( x^1 \to 0 \). Using these facts, and Assumption 2.7, one can find sufficiently large \( k > 0 \) independent of \( x_0 \) such that

\[
|\hat{a}_k(t, x) - \hat{a}_k(t, y)| + x^1|\hat{b}_k(t, x) + (x^1)^2|\hat{c}_k(t, x)| \leq \omega_0,
\]

whenever \( t > 0, x, y \in \mathbb{R}^d_+ \) and \( |x - y| \leq x^1 \wedge y^1 \). Now we fix a \( \rho_0 < r_0 \) such that

\[
\Psi(B_{\rho_0}(x_0)) \subset B_{r/2} \cap \{x : x^1 \leq e^{-3k}\}.
\]

By [18, Theorem 3.2], for any \( \nu, \alpha \in \mathbb{R} \) and \( g \in \psi^{-\alpha}H^\nu_{p,\theta}(\Omega) \) with support in \( B_{\rho_0}(x_0) \),

\[
\|\psi^{-\alpha}g\|_{H^{\nu}_{p,\theta}(\Omega)} \sim \|M^\alpha g(\Psi^{-1})\|_{H^\nu_{p,\theta}(\mathbb{R}^d_+)}.
\]

(5.3)

Let \( \zeta \) be a smooth function with support in \( B_{\rho_0}(x_0) \) and denote \( v := (u\zeta)(\Psi^{-1}) \) and continue \( v \) as zero in \( \mathbb{R}^d_+ \setminus \Psi(B_{\rho_0}(x_0)) \). Since \( \eta \varphi_m = 1 \) on \( \Psi(B_{\rho_0}(x_0)) \), the function \( v \) satisfies the equation

\[
\partial^\mu_t v = \hat{a}^{ij}_k D_{ij} v + \hat{b}^i_k D_i v + \hat{c}_k v + \hat{f}
\]
on \( (0, T) \times \mathbb{R}^d_+ \), where

\[
f = \hat{f}(\Psi^{-1}), \quad \hat{f} = -2a^{ij}D_{ij}uD_j\zeta - a^{ij}uD_{ij}\zeta - b^i uD_i\zeta + f\zeta.
\]

By Theorem 2.12 and (5.3), it follows that \( v \in \mathcal{H}^{\alpha, \gamma+2}_{p,\theta,0}(\mathbb{R}^d_+, T) \), and

\[
\|v\|_{\mathcal{H}^{\alpha, \gamma+2}_{p,\theta,0}(\mathbb{R}^d_+, T)} \leq N\|M\hat{f}\|_{\mathcal{H}^{\gamma}_{p,\theta}(\mathbb{R}^d_+, T)}.
\]

(5.3)

By (5.3), for \( t \leq T \)

\[
\|u\zeta\|_{\mathcal{H}^{\gamma+2}_{p,\theta-\rho}(\Omega, t)} \leq N\|v\|\mathcal{H}^{\gamma+2}_{p,\theta-\rho}(\mathbb{R}^d_+, t)
\]

\[
\leq N\|M\hat{f}\|_{\mathcal{H}^{\gamma}_{p,\theta}(\mathbb{R}^d_+, t)} \leq N\|uD^2\zeta\|_{\mathcal{H}^{\gamma+2}_{p,\theta-\rho}(\Omega, t)} + N\|aD\zeta U\|_{\mathcal{H}^{\gamma}_{p,\theta-\rho}(\Omega, t)}
\]

\[
+ N\|uD\zeta\|_{\mathcal{H}^{\gamma}_{p,\theta-\rho}(\Omega, t)} + N\|\zeta Mf\|_{\mathcal{H}^{\gamma}_{p,\theta}(\Omega, t)}.
\]

By Lemma 2.1 (iv) and Assumption 2.9,

\[
\|u\zeta\|_{\mathcal{H}^{\gamma+2}_{p,\theta-\rho}(\Omega, t)} \leq N\|Du\|_{\mathcal{H}^{\gamma}_{p,\theta-\rho}(\Omega, t)} + N\|u\|_{\mathcal{H}^{\gamma}_{p,\theta-\rho}(\Omega, t)} + N\|f\|_{\mathcal{H}^{\gamma}_{p,\theta-\rho}(\Omega, t)}.
\]

(5.4)

Observe that all the constants \( N \) are independent of \( x_0 \). Take a partition of unity \( \zeta_0, \zeta_1, \ldots, \zeta_N \) of \( \Omega \) such that \( \zeta_0 \in C^\infty_c(\Omega) \) and \( \zeta_i \in C^\infty_c(B_{\rho_0}(x_i)) \) for \( x_i \in \partial^\Omega \). Since \( u\zeta_0 \) has a compact support in \( \Omega \), we have \( u\zeta_0 \in \mathcal{H}^{\alpha, \gamma+2}_{p,\theta}(T) \), and \( \|u\zeta_0\|_{\mathcal{H}^{\gamma+2}_{p,\theta-\rho}(\Omega, t)} \sim \|u\zeta_0\|_{\mathcal{H}^{\gamma+2}(\Omega)} \). Therefore one can estimate \( u\zeta_0 \) based on a result on \( \mathbb{R}^d \) ([7, Theorem 2.3]) and Lemma 2.5. The other \( u\zeta_m \) \( (m = 1, 2, \ldots, N') \) can be obtained in (5.4).

Summing up these estimates we get

\[
\|u\|_{\mathcal{H}^{\gamma+2}_{p,\theta-\rho}(\Omega, t)} \leq N\|Du\|_{\mathcal{H}^{\gamma}_{p,\theta-\rho}(\Omega, t)} + N\|u\|_{\mathcal{H}^{\gamma}_{p,\theta-\rho}(\Omega, t)} + N\|f\|_{\mathcal{H}^{\gamma}_{p,\theta-\rho}(\Omega, t)}.
\]

\[
\leq N\|u\|_{\mathcal{H}^{\gamma+1}_{p,\theta}(\Omega, t)} + N\|f\|_{\mathcal{H}^{\gamma}_{p,\theta}(\Omega, t)}
\]

\[
\leq N\|u\|_{\mathcal{H}^{\gamma+2}_{p,\theta-\rho}(\Omega, t)} + N\|\zeta_0\|_{\mathcal{H}^{\gamma}_{p,\theta-\rho}(\Omega, t)} + N\|f\|_{\mathcal{H}^{\gamma}_{p,\theta-\rho}(\Omega, t)}.
\]
By (2.10), (2.16), and Fubini’s theorem, for any $t$, certainly implies the last inequality is due to an interpolation inequality ([18, Proposition 2.4]). This certainly implies

$$
\|u\|_{H^{\gamma+2}_{p,\theta-p}(\Omega,t)} \leq N\|u\|_{H^{\gamma}_{p,\theta}(\Omega,t)} + N\|f\|_{H^{\gamma}_{p,\theta}(\Omega,t)}.
$$

By (2.10), (2.16), and Fubini’s theorem, for any $t \leq T$,

$$
\|u\|_{H^{\gamma+2}_{p,\theta-p}(\Omega,t)} \leq \int_0^t \int_0^s (s-r)^{\alpha-1} \|\psi^{-1} u(r,\cdot)\|_{H^{\gamma+2}_{p,\theta}(\Omega)} + \|\psi f(r,\cdot)\|_{H^{\gamma}_{p,\theta}(\Omega)} dr ds 
+ N\|\psi f\|_{H^{\gamma}_{p,\theta}(\Omega,T)}
\leq N \int_0^t (t-s)^{\alpha-1} \|u\|_{H^{\gamma+2}_{p,\theta}(\Omega,s)} \|\psi f\|_{H^{\gamma}_{p,\theta}(\Omega,s)} ds + N T \|f\|_{H^{\gamma}_{p,\theta}(\Omega,T)}.
$$

By a version of Gronwall’s lemma (see, for example, [26, Corollary 2]),

$$
\|u\|_{H^{\gamma+2}_{p,\theta-p}(\Omega,T)} \leq N(T)\|f\|_{H^{\gamma}_{p,\theta}(\Omega,T)}.
$$

Thus we obtain a priori estimate (2.23).

**Step 2.** The existence for heat equation.

By the method of continuity and the a priori estimate in Step 1, to finish the proof it suffices to prove the existence result for the equation

$$
\partial_t^\alpha u = \Delta u + f \quad (t,x) \in (0,T) \times \Omega, \quad u(0) = \partial_t u(0)_{1_{a>1}} = 0.
$$

(5.5)

Our proof is based on the argument in [15, Chapter 8]. Take points $x_l \in \partial \Omega$, and the partition of unity $\zeta_0, \zeta_1, \ldots, \zeta_{N'}$ from Step 1, which satisfy $\zeta_0 \in C_0^\infty(\Omega)$, $\sum_{l=0}^{N'} \zeta_l^2 = 1$ on $\Omega$, and the support of $\zeta_l$ is contained in $B_{\rho_l}(x_l)$ for $l = 1, 2, \ldots, N'$, where $\rho_l$ is taken from Step 1. Also, take $\hat{a}_l = \hat{a}_{k,l}, \hat{b}_l = \hat{b}_{k,l}$, and $\hat{c}_l = \hat{c}_{k,l}$ as in (5.2) with $a^{ij} = \delta^{ij}, b = c = 0$ corresponding to $x_l$. Define

$$
\hat{L}^l := \hat{a}_{ij} D_{ij} + \hat{b}_l D_l + \hat{c}_l.
$$

Let $\Psi_l$ be from Definition 5.1 corresponding to $x_l \in \partial \Omega$. Note that if $w$ has support in $\Omega \cap B_{\rho_l}(x_l)$, then for $\tilde{w} := w(\Psi_l^{-1})$ we have

$$
(\Delta w)(\Psi_l^{-1}) = \hat{L}^l \tilde{w},
$$

and therefore

$$
(\partial^{\alpha}_t - \Delta) w = f \quad \text{if and only if} \quad (\partial^{\alpha}_t - \hat{L}^l) \tilde{w} = f(\Psi_l^{-1}).
$$

Let $h \in H^{\gamma+2}_{p,\theta}(\mathbb{R}^d, T)$, and let $\mathcal{R}_l h$ be the solution to the equation

$$
\partial_t^\alpha u = \hat{L}^l u + h, \quad t > 0, x \in \mathbb{R}^d; \quad u(0) = 1_{a} u(0) = 0.
$$

The map $h \rightarrow \mathcal{R}_l h \in L^{\alpha+\gamma+2}_{p,\theta}(\mathbb{R}^d, T)$ is well defined and bounded due to Theorem 2.12. Define a map $R_l^\alpha$ on $H^{\gamma+2}_{p,\theta}(\Omega, T)$ as follows.

$$
R_l^\alpha f := (\mathcal{R}_l^\alpha \hat{f}_l)(\Psi_l),
$$

(5.6)

where $\hat{f}_l = (\zeta_l f)(\Psi_l^{-1})$. Note that for $l = 1, 2, \ldots, N'$

$$
(\partial^{\alpha}_t - \Delta) R_l^\alpha f = \zeta_l f, \quad \zeta_l u = R_l^\alpha (\partial^{\alpha}_t - \Delta)(\zeta_l u) \quad \text{on} \quad B_{\rho_l}(x_l) \cap \Omega.
$$

(5.7)

Define $R_l^0$ as the inverse of $\partial^{\alpha}_t - \Delta$ on $\mathbb{R}^d$ induced by [7, Theorem 2.3].

To proceed further we need the following Lemma.
Lemma 5.3. Suppose that $f \in \mathbb{H}^\gamma_{p,\theta+p}(\Omega, T)$. If $u \in S^{\alpha,\gamma+2}_{p,\theta,0}(\Omega, T)$ is a solution to the equation

$$u = \sum_{l=0}^{N'} \zeta_l R^{l}_{\alpha}(\zeta_l f - A^l u), \quad A^l u = u(\Delta \zeta_l) + 2 \sum_i D_i u D_i \zeta_l$$

then $\partial_t^\alpha u = \Delta u + f$ in $(0, T) \times \Omega$.

Proof. Since $u \in \mathbb{H}^{\gamma+1}_{p,\theta+p}(\Omega, T)$, we have $A^l u \in \mathbb{H}^\gamma_{p,\theta+p}(\Omega, T)$ due to (2.15). Therefore, $\zeta_l R^{l}_{\alpha}(\zeta_l f - A^l u) \in S^{\alpha,\gamma+2}_{p,\theta,0}(\Omega, T)$ for $l = 1, 2, \ldots, N'$ by (5.3). Since $\zeta_0$ has support in $\Omega$, $\zeta_0 R^{0}_{\alpha}(\zeta_0 f - A^0 u) \in \mathbb{H}^{\gamma+2}_{p,\theta+p}(\Omega, T)$ by Lemma 2.5.

Let $g := \partial_t^\alpha u - \Delta u$. Observe that

$$u = \sum_{l=0}^{N'} \zeta_l^2 u = \sum_{l=0}^{N'} \zeta_l u \zeta_l = \sum_{l=0}^{N'} \zeta_l R^{l}_{\alpha}(\partial_t^\alpha \zeta_l u - \Delta (\zeta_l u)) = \sum_{l=0}^{N'} \zeta_l R^{l}_{\alpha}(\zeta_l g - A^l u),$$

due to (5.7). By the linearity of $R^{l}_{\alpha}$ and the assumption on $u$,

$$0 = \sum_{l=0}^{N'} \zeta_l R^{l}_{\alpha}(\zeta_l (f - g)).$$

Therefore,

$$0 = (\partial_t^\alpha - \Delta) \sum_{l=0}^{N'} \zeta_l R^{l}_{\alpha}(\zeta_l (f - g))$$

$$= \sum_{l=0}^{N'} \zeta_l (\partial_t^\alpha - \Delta) R^{l}_{\alpha}(\zeta_l (f - g)) - \sum_{l=0}^{N'} A^l R^{l}_{\alpha}(\zeta_l (f - g))$$

$$= \sum_{l=0}^{N'} \zeta_l^2 (f - g) - T_\alpha (f - g) = (f - g) - T_\alpha (f - g),$$

where $T_\alpha h := \sum_{l=0}^{N'} A^l R^{l}_{\alpha}(\zeta_l h)$ for $h \in \mathbb{H}^\gamma_{p,\theta+p}(\Omega, T)$, and the third equality above is due to the first relation in (5.7). Note that

$$T_\alpha h = \sum_{l=0}^{N'} \left( (\Delta \zeta_l) R^{l}_{\alpha}(\zeta_l h) + 2 \sum_i D_i \zeta_l D_i R^{l}_{\alpha}(\zeta_l h) \right).$$

By (5.6) and (5.3), for any $t \leq T$ we have

$$\|T_\alpha h\|_{\mathbb{H}^\gamma_{p,\theta+p}(\Omega, t)}^p \leq N(p, N') \sum_{l=0}^{N'} p(\Delta \zeta_l) R^{l}_{\alpha}(\zeta_l h) + 2 \sum_i D_i \zeta_l D_i R^{l}_{\alpha}(\zeta_l h)) \|_{\mathbb{H}^\gamma_{p,\theta+p}(\Omega, t)}^p$$

$$\leq N N' \sum_{l=0}^{N'} \left( B_l + \sum_i C_{l,i} \right),$$

where $N$ is independent of $h$,

$$B_l = \|((\Delta \zeta_l)(\Psi_l^{-1})R^{l}_{\alpha}(\hat{h}_l))\|_{\mathbb{H}^\gamma_{p,\theta+p}(\mathbb{R}^d, t)}^p, \quad C_{l,i} = \|(D_i \zeta_l)(\Psi_l^{-1})D_i R^{l}_{\alpha}(\hat{h}_l))\|_{\mathbb{H}^\gamma_{p,\theta+p}(\mathbb{R}^d, t)}^p.$$
for \( l = 1, 2, \ldots, N' \), and
\[
B_0 = \| (\Delta \zeta_0) R^0_\alpha (\zeta_0 h) \|_{p, \theta + p}^{p, \gamma} (\Omega),
C_{0,i} = \| D_i \zeta_0 D_i R^0_\alpha (\zeta_0 h) \|_{p, \theta + p}^{p, \gamma} (\Omega).
\]

By Theorem 2.12, Lemma 2.1(v) and [18, Proposition 2.4] (interpolation inequality), for \( \varepsilon > 0 \) and \( l = 1, 2, \ldots, N' \)
\[
B_l \leq \| R^l_\alpha (\hat{h}_l) \|_{p, \theta + p}^{p, \gamma} (\Omega, \varepsilon) \leq \varepsilon \| R^l_\alpha (\hat{h}_l) \|_{p, \theta + p}^{p, \gamma} (\Omega, \varepsilon) + N(\varepsilon) \| R^l_\alpha (\hat{h}_l) \|_{p, \theta + p}^{p, \gamma} (\Omega, \varepsilon)
\leq \varepsilon N(\varepsilon) \| R^l_\alpha (\hat{h}_l) \|_{p, \theta + p}^{p, \gamma} (\Omega, \varepsilon) + N(\varepsilon) \| R^l_\alpha (\hat{h}_l) \|_{p, \theta + p}^{p, \gamma} (\Omega, \varepsilon)
\leq \varepsilon N(\varepsilon) \| R^l_\alpha (\hat{h}_l) \|_{p, \theta + p}^{p, \gamma} (\Omega, \varepsilon) + N(\varepsilon) \| R^l_\alpha (\hat{h}_l) \|_{p, \theta + p}^{p, \gamma} (\Omega, \varepsilon)
\leq \varepsilon N(\varepsilon) \| R^l_\alpha (\hat{h}_l) \|_{p, \theta + p}^{p, \gamma} (\Omega, \varepsilon) + N(\varepsilon) \| R^l_\alpha (\hat{h}_l) \|_{p, \theta + p}^{p, \gamma} (\Omega, \varepsilon)
\]
and
\[
C_{l,i} \leq \varepsilon \| R^l_\alpha (\hat{h}_l) \|_{p, \theta + p}^{p, \gamma} (\Omega, \varepsilon) + N(\varepsilon) \| R^l_\alpha (\hat{h}_l) \|_{p, \theta + p}^{p, \gamma} (\Omega, \varepsilon)
\leq \varepsilon N(\varepsilon) \| R^l_\alpha (\hat{h}_l) \|_{p, \theta + p}^{p, \gamma} (\Omega, \varepsilon) + N(\varepsilon) \| R^l_\alpha (\hat{h}_l) \|_{p, \theta + p}^{p, \gamma} (\Omega, \varepsilon)
\leq \varepsilon N(\varepsilon) \| R^l_\alpha (\hat{h}_l) \|_{p, \theta + p}^{p, \gamma} (\Omega, \varepsilon) + N(\varepsilon) \| R^l_\alpha (\hat{h}_l) \|_{p, \theta + p}^{p, \gamma} (\Omega, \varepsilon)
\leq \varepsilon N(\varepsilon) \| R^l_\alpha (\hat{h}_l) \|_{p, \theta + p}^{p, \gamma} (\Omega, \varepsilon) + N(\varepsilon) \| R^l_\alpha (\hat{h}_l) \|_{p, \theta + p}^{p, \gamma} (\Omega, \varepsilon)
\]
Since \( \zeta_0 \in C_\infty^\infty (\Omega) \), by [7, Theorem 2.3] and (2.7),
\[
B_0 + C_{0,i} \leq \varepsilon \| R^0_\alpha (\zeta_0 h) \|_{p, \theta + p}^{p, \gamma} (\Omega, \varepsilon) + N(\varepsilon) \| R^0_\alpha (\zeta_0 h) \|_{p, \theta + p}^{p, \gamma} (\Omega, \varepsilon)
\leq \varepsilon N(\varepsilon) \| R^0_\alpha (\zeta_0 h) \|_{p, \theta + p}^{p, \gamma} (\Omega, \varepsilon) + N(\varepsilon) \| R^0_\alpha (\zeta_0 h) \|_{p, \theta + p}^{p, \gamma} (\Omega, \varepsilon)
\leq \varepsilon N(\varepsilon) \| R^0_\alpha (\zeta_0 h) \|_{p, \theta + p}^{p, \gamma} (\Omega, \varepsilon) + N(\varepsilon) \| R^0_\alpha (\zeta_0 h) \|_{p, \theta + p}^{p, \gamma} (\Omega, \varepsilon)
\]
Using (2.16), Fubini's theorem, and (5.3), we get
\[
\| R^0_\alpha (\hat{h}_l) \|_{p, \theta + p}^{p, \gamma} (\Omega, \varepsilon) \leq N \int_0^t (t-s)^{\alpha-1} \| \partial_\alpha^\gamma (R^0_\alpha (\hat{h}_l)) \|_{p, \theta + p}^{p, \gamma} (\Omega, \varepsilon) \] \[ \leq N \int_0^t (t-s)^{\alpha-1} \| \hat{h}_l \|_{p, \theta + p}^{p, \gamma} (\Omega, \varepsilon) \] \[ \leq N \int_0^t (t-s)^{\alpha-1} \| \zeta_0 h \|_{p, \theta + p}^{p, \gamma} (\Omega, \varepsilon) \] \[ \leq N \int_0^t (t-s)^{\alpha-1} \| \zeta_0 h \|_{p, \theta + p}^{p, \gamma} (\Omega, \varepsilon) \] Similarly, using [7, Theorem 2.1 (iv)] we also get
\[
\| R^0_\alpha (\zeta_0 h) \|_{p, \theta + p}^{p, \gamma} (\Omega, \varepsilon) \leq N \int_0^t (t-s)^{\alpha-1} \| \partial_\alpha^\gamma (R^0_\alpha (\zeta_0 h)) \|_{p, \theta + p}^{p, \gamma} (\Omega, \varepsilon) \] \[ \leq N \int_0^t (t-s)^{\alpha-1} \| \zeta_0 h \|_{p, \theta + p}^{p, \gamma} (\Omega, \varepsilon) \] \[ \leq N \int_0^t (t-s)^{\alpha-1} \| \zeta_0 h \|_{p, \theta + p}^{p, \gamma} (\Omega, \varepsilon) \] where the last inequality holds since \( \zeta_0 \in C_\infty^\infty (\Omega) \). Therefore we get
\[
\| T_\alpha h \|_{p, \theta + p}^{p, \gamma} (\Omega, \varepsilon) \leq \sum_{l=1}^{N'} \left( B_l + \sum_i C_{l,i} \right)
\leq \sum_{l=1}^{N'} N(\varepsilon) \int_0^t (t-s)^{\alpha-1} \| \zeta_0 h \|_{p, \theta + p}^{p, \gamma} (\Omega, \varepsilon) \] \[ \leq N(\varepsilon) \int_0^t (t-s)^{\alpha-1} \| h \|_{p, \theta + p}^{p, \gamma} (\Omega, \varepsilon) \] \[ + \varepsilon N \sum_{l=1}^{N'} \| \zeta_0 h \|_{p, \theta + p}^{p, \gamma} (\Omega, \varepsilon) \] \[ \leq N(\varepsilon) \int_0^t (t-s)^{\alpha-1} \| h \|_{p, \theta + p}^{p, \gamma} (\Omega, \varepsilon) \] \[ + \varepsilon N \int_0^t \| h \|_{p, \theta + p}^{p, \gamma} (\Omega, \varepsilon) \]
Thus, for any \( m = 1, 2, \ldots \), using the identity
\[
\int_0^t (t-s_1)^{\alpha-1} \cdots \int_0^{s_m-1} (s_m-1-s_{m-1})^{\alpha-1} ds_m \cdots ds_1 = \frac{\Gamma(\alpha)^m}{\Gamma(m\alpha+1)} t^{m\alpha},
\]
we obtain
\[
\|T^m_{\alpha} h\|^p_{\mathcal{H}_{p,\theta+1}^\gamma(\Omega, T)} \\
\leq \sum_{k=0}^{\infty} \left( \frac{m}{k} \right) (N(\varepsilon))^k \frac{\Gamma(\alpha)^k}{\Gamma(\alpha+1)^k} T^{k\alpha} \varepsilon^{m-k} \|h\|^p_{\mathcal{H}_{p,\theta+1}^\gamma(\Omega, T)} \\
\leq 2^m \varepsilon^m \max_k \{ (N(\varepsilon))^k \frac{\Gamma(\alpha)^k}{\Gamma(\alpha+1)^k} T^{k\alpha}) \} \|h\|^p_{\mathcal{H}_{p,\theta+1}^\gamma(\Omega, T)}.
\]

Note that if we fix \( \varepsilon > 0 \), then \( \max_k \{ (N(\varepsilon))^k \frac{\Gamma(\alpha)^k}{\Gamma(\alpha+1)^k} T^{k\alpha}) \} \) is bounded and independent of \( m \). Taking \( \varepsilon < \frac{1}{8} \), we get
\[
\|T^m_{\alpha} h\|^p_{\mathcal{H}_{p,\theta+1}^\gamma(\Omega, T)} \leq \left( \frac{1}{2} \right)^m N(T) \|h\|^p_{\mathcal{H}_{p,\theta+1}^\gamma(\Omega, T)}.
\]

If \( m \) is large enough, then the map \( T^m_{\alpha} \) on \( \mathcal{H}_{p,\theta+1}^\gamma(\Omega, T) \) is a contraction. Since \( T_{\alpha}(f-g) = f-g \), we have \( f-g = T^m_{\alpha}(f-g) \), and thus \( f = g \). Therefore \( g := \partial_t^\alpha u - \Delta u = f \). The lemma is proved.

We continue our proof for the existence result.

For \( n = 0, 1, \ldots \), we define \( u_n \in \mathcal{S}_{p,\theta,0}^{\alpha,\gamma+2} (\Omega, T) \) as follows.
\[
u_n = \sum_{l=0}^{N'} \zeta_l R^l_{\alpha}(\zeta_l f - A^l u_{n-1}) \quad (n \geq 1).
\]

Note that \( u_n \) are well defined since \( f \in \mathcal{H}_{p,\theta+1}^\gamma(\Omega, T) \). By (2.12), (5.3), [7, Theorem 2.3], and Theorem 2.12
\[
\|u_0\|_{\mathcal{H}_{p,\theta}^{\alpha,\gamma+2}(\Omega, T)} \leq N \sum_{l=0}^{N'} \|\zeta_l (\Psi_{l}^{-1})^l R^l_{\alpha}(\zeta_l f)\|_{\mathcal{H}_{p,\theta}^{\alpha,\gamma+2}(\mathbb{R}^d, T)} \leq N \|f\|_{\mathcal{H}_{p,\theta+1}^\gamma(\Omega, T)},
\]
where \( \Psi_0(x) := x \). Observe that
\[
\sum_{l=0}^{N'} \|\zeta_l R^l_{\alpha}(A^l u_{n-1})\|_{\mathcal{H}_{p,\theta}^{\alpha,\gamma+2}(\Omega, t)} \leq N \sum_{l=0}^{N'} \|\zeta_l (\Psi_{l}^{-1})^l R^l_{\alpha}(\zeta_l A^l u_{n-1})(\Psi_{l}^{-1})\|_{\mathcal{H}_{p,\theta}^{\alpha,\gamma+2}(\mathbb{R}^d, t)} \leq N \sum_{l=0}^{N'} \|\zeta_l A^l u_{n-1} - 1\|_{\mathcal{H}_{p,\theta}^{\alpha,\gamma+2}(\mathbb{R}^d, t)} \|\Psi_{l}^{-1}\|_{\mathcal{H}_{p,\theta+1}^\gamma(\Omega, T)},
\]
and by definition of \( A^l \) one can check that the last term above is bounded by a constant multiple of
\[
\sum_{l=0}^{N'} \left( \|\zeta_l (\Delta \zeta_l u_{n-1})\|_{\mathcal{H}_{p,\theta+1}^\gamma(\mathbb{R}^d, t)} + \sum_{l=0}^{N'} \|\zeta_l D_l (\zeta_l D_l u_{n-1})\|_{\mathcal{H}_{p,\theta+1}^\gamma(\mathbb{R}^d, t)} \right).
\]

Since \( u_{n-1} \in \mathcal{S}_{p,\theta,0}^{\alpha,\gamma+2}(\Omega, T) \), by following the argument in (5.9), one can obtain
\[
\|\zeta_l (\Delta \zeta_l u_{n-1})\|_{\mathcal{H}_{p,\theta+1}^\gamma(\mathbb{R}^d, t)} \leq N \int_0^t (t-s)^{\alpha-1} \|\partial_s^\alpha u_{n-1}\|_{\mathcal{H}_{p,\theta+1}^\gamma(\Omega, s)} ds.
\]
Similarly for any \( t \leq T \) and \( \varepsilon > 0 \), as in (5.8)
\[
\| (\zeta D_t \zeta D_t u_n - 1)(\Psi^{-1}) \|_{L^p_{\gamma, p} + p}(\mathbb{R}^d, t) \leq N \| u_{n-1} \|_{L^p_{\gamma, p} + p}(\Omega, t) + \varepsilon \| u_{n-1} \|_{L^p_{\gamma, p} + p}(\Omega, t)^2.
\]
Therefore, we have for \( n = 1, 2, \ldots \),
\[
\sum_{l=0}^{N'} \| \zeta R^l_\alpha(A^l u_{n-1}) \|_{L^p_{\gamma, p} + p}(\Omega, t)
\leq N(\varepsilon) \int_0^t (t - s)^{\alpha-1} \| \partial_t^\alpha v_{n-1}(s, \cdot) \|_{L^p_{\gamma, p} + p}(\Omega, s) ds + \varepsilon N \| u_{n-1} \|_{L^p_{\gamma, p} + p}(\Omega, t).
\]
Recall that \( u_n - u_{n-1} = \sum_{l=0}^{N'} \zeta R^l_\alpha(A^l (u_{n-2} - u_{n-1})). \) By (5.10)
\[
\| u_n - u_{n-1} \|_{L^p_{\gamma, p} + p}(\Omega, T) \leq 2^n \varepsilon^n \max_k \Gamma(\alpha)^k \Gamma(k \alpha + 1) \varepsilon^{k} T^{k \alpha} \| f \|_{L^p_{\gamma, p} + p}(\Omega, T).
\]
Take \( \varepsilon < \frac{1}{2} \). If \( n \) is large enough, then
\[
\| u_n - u_{n-1} \|_{L^p_{\gamma, p} + p}(\Omega, T) \leq \left( \frac{1}{2} \right)^n.
\]
This implies that \( u_n \) is a Cauchy in \( \mathcal{D}_{p, \gamma, p}^{\alpha, \gamma + 2}(\Omega, T) \). Let \( u \) denote the limit of \( u_n \) in \( \mathcal{D}_{p, \gamma, p}^{\alpha, \gamma + 2}(\Omega, T) \). Then \( u \) satisfies
\[
u = \sum_{l=0}^{N'} \zeta R^l_\alpha(\zeta \xi - A^l u),
\]
and therefore by Lemma 5.3, \( u \) becomes a solution to (5.5). The theorem is proved.

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Received February 2020; revised November 2020.

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