SHARP TIME DECAY ESTIMATES FOR THE DISCRETE KLEIN-GORDON EQUATION

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ABSTRACT. We establish sharp time decay estimates for the the Klein-Gordon equation on the cubic lattice in dimensions \( d = 2, 3, 4 \). The \( \ell^1 \to \ell^\infty \) dispersive decay rate is \( |t|^{-3/4} \) for \( d = 2 \), \( |t|^{-7/6} \) for \( d = 3 \) and \( |t|^{-3/2} \log |t| \) for \( d = 4 \). These decay rates are faster than conjectured by Kevrekidis and Stefanov (2005). The proof relies on oscillatory integral estimates and proceeds by a detailed analysis of the the singularities of the associated phase function. We also prove new Strichartz estimates and discuss applications to nonlinear PDEs and spectral theory.

1. INTRODUCTION AND MAIN RESULTS

Dispersive estimates play a crucial role in the study of evolution equations. Proving such estimates often boils down to establishing decay estimates for the \( \ell^\infty \) norm of the solution at time \( t \) in terms of the \( \ell^1 \) norm of its initial data. It is by now well-established that the \( \ell^1 \to \ell^\infty \) decay estimates give rise to a whole family of mixed space-time norm estimates, called Strichartz estimates \([29, 12, 18]\). For the continuous Klein-Gordon equation such estimates have been established e.g. by Brenner \([6]\), Pecher \([26]\) and Ginibre and Velo \([10, 11]\); let us also mention the textbook exposition by Nakanishi–Schlag \([24]\). The dispersive estimates are halfway between those for the Schrödinger equation (for low frequencies) and the wave equation (for high frequencies), see \([24, 2.5]\). In the discrete case the frequencies are bounded, and one might expect the same decay rate for the discrete Klein-Gordon equation (DKG) as for the discrete Schrödinger equation (DS). In fact, this was conjectured by Kevrekidis and Stefanov \([19]\), who proved the sharp \( |t|^{-d/3} \) decay rate for the DS in any dimension \( d \geq 1 \) and for the DKG in one dimension. The obstruction to proving similar estimates for DKG in higher dimensions was that, contrary to the DS, the fundamental solution does not separate variables. Apparently unbeknownst to the authors, earlier, Schultz \([27]\) had already made the striking observation that the decay rate for the discrete wave equation (to which the DKG reduces in the zero mass limit) in dimensions \( d = 2, 3 \) is \( |t|^{-3/4} \) and \( |t|^{-7/6} \), respectively, better than the conjectured estimates. The same decay rate for the \( d = 2 \) DKG was established by Borovyk and Goldberg \([4]\), who also proved that the fundamental solution decays exponentially outside the light cone and that the decay rate is independent of additional parameters (mass and wave speeds in the coordinate directions).
1.1. Discrete Klein–Gordon equation (DKG). Consider the Cauchy problem for the DKG:

\[
\begin{align*}
\left\{ \begin{array}{c}
u_{tt}(t,x) - \Delta_x u(t,x) + u(t,x) &= F(t,x), \quad (t,x) \in \mathbb{R} \times \mathbb{Z}^d, \\
u(0,x) &= f(x), \quad u_t(0,x) = g(x),
\end{array} \right.
\end{align*}
\]

where \(\Delta_x\) is the discrete Laplacian,

\[\Delta_x u(t,x) = \sum_{|x-y|=1} u(y,t) - 2du(t,x).\]

For the plane waves \(e_\xi(x) = e^{ix \cdot \xi}\) we have \((1 - \Delta_x)e_\xi(x) = \omega(\xi)^2 e_\xi(x)\), where \(\omega: \mathbb{T}^d \to \mathbb{R}\) is the dispersion relation, given by

\[\omega(\xi) := \sqrt{1 + \sum_{j=1}^d 2(1 - \cos(\xi_j))}.
\]

We will identify the torus \(\mathbb{T}^d = (\mathbb{R}/2\pi\mathbb{Z})^d\) with the fundamental domain \([0, 2\pi]^d\).

The solution to (1) is given (for sufficiently regular data \(f, g, F\)) by Duhamel’s formula,

\[
u(t,x) = \cos(t\sqrt{1 - \Delta_x})f(x) + \frac{\sin(t\sqrt{1 - \Delta_x})}{\sqrt{1 - \Delta_x}}g(x)
\]

\[+ \int_0^t \sin((t-s)\sqrt{1 - \Delta_x}) F(s,x) ds.
\]

Here, \(\cos(t\sqrt{1 - \Delta_x})e_\xi = \cos(t\omega(\xi))e_\xi\), \(\frac{\sin(t\sqrt{1 - \Delta_x})}{\sqrt{1 - \Delta_x}} e_\xi = \frac{\sin(t\omega(\xi))}{\omega(\xi)}e_\xi\), and the action on Schwartz functions is defined by using the Fourier inversion formula.

1.2. Decay estimates. The solution \(\nu(t,x)\) is thus a sum of oscillatory integrals of the form

\[I(t,x) := \int_{[0,2\pi]^d} e^{i(x \cdot \xi - t\omega(\xi))}a(\xi)d\xi,
\]

where \(a: [0,2\pi]^d \to \mathbb{C}\) is a smooth function; in fact, \(a(\xi) = 1\) or \(a(\xi) = \omega(\xi)^{-1}\). We will be interested in obtaining time decay estimates on \(I(t,x)\), uniformly in \(x \in \mathbb{Z}^d\).

In other words, we want to find (the largest possible) \(\sigma > 0\) such that

\[\|I(t,\cdot)\|_{L^\infty(\mathbb{Z}^d)} \leq C(1 + |t|)^{-\sigma}, \quad t \in \mathbb{R},
\]

for some constant \(C\) independent of \(t\). The following theorem is our main result.

**Theorem 1.** For the oscillatory integrals \(I\) the following estimates hold for all \(t \in \mathbb{R}\):

\[\|I(t,\cdot)\|_{L^\infty(\mathbb{Z}^d)} \leq C(1 + |t|)^{-\frac{d}{2}},
\]

\[\|I(t,\cdot)\|_{L^\infty(\mathbb{Z}^d)} \leq C(1 + |t|)^{-\frac{d}{2}},
\]

\[\|I(t,\cdot)\|_{L^\infty(\mathbb{Z}^d)} \leq C(1 + |t|)^{-\frac{d}{2} \log(2 + |t|)},
\]

where \(C\) is a constant independent of \(t\).
Remark 1. (i) The exponent in the estimates $[15]$, $[16]$, $[17]$ is sharp. In fact, there are vectors $v \in \mathbb{R}^d$ and non-zero constants $c_d$ ($d = 2, 3, 4$) such that

\[
\lim_{t \to +\infty} t^d |I(t, tv)| = c_d \quad \text{for} \quad d = 2,
\]

\[
\lim_{t \to +\infty} t^{2d} |I(t, tv)| = c_d \quad \text{for} \quad d = 3,
\]

\[
\lim_{t \to +\infty} t^{3d} \frac{d}{\log t} |I(t, tv)| = c_d \quad \text{for} \quad d = 4,
\]

see $[17]$ for the case $d = 4$ and $[15]$ for similar observations.

(ii) We conjecture that for $d \geq 5$, the following estimate is true,

\[
\|I(t, \cdot)\|_{L^\infty(\mathbb{R}^d)} \leq C_{a,d}(1 + |t|)^{-\frac{2d+1}{6}} \log(2 + |t|)^{d-4}. \tag{8}
\]

We remark here that the original conjecture of Kevrekidis and Stefanov $[19]$ can be proved by relatively simple stationary phase arguments (see Remark $[5]$). However, we expect that the better estimates $[5]$ hold.

(iii) The estimates $[5]–[7]$ continue to hold if $L^\infty(\mathbb{R}^d)$ is replaced by $L^\infty(\mathbb{Z}^d)$.

(iv) It will be necessary to prove Theorem $[1]$ in each case $d = 2, 3, 4$ separately (see Propositions $[9]$ $[10]$ $[12]$). This is in contrast to the Schrödinger case in $[19]$, where the same proof works in all dimensions.

If $x = vt$, where $v \in \mathbb{R}^d$ is the velocity, then the oscillatory integral $[5]$ takes the form

\[
J(t, v) := \int_{[0,2\pi]^d} e^{it\Phi(v, \xi)} a(\xi) d\xi, \quad \Phi(v, \xi) = v \cdot \xi - \omega(\xi). \tag{9}
\]

In fact, the proof of Theorem $[1]$ gives more precise information on the set of velocities for which the indicated decay occurs. These velocities are the images, under the map $\xi \mapsto \nabla \omega(\xi)$, of the critical points of the phase function $\Phi(\cdot, v)$. The time-decay of $J(t, v)$ is governed by the degeneracy of the phase function at the critical points lying within the support of $a$. For a fixed value of the parameter $v = v_0$ a critical point of $\Phi(\cdot, v) = \Phi(v_0, \cdot)$ is a point $\xi = \xi_0$ where the gradient $\nabla \Phi(\xi)$ vanishes; in the case of the phase function in $[11]$ this happens if $\nabla \omega(\xi_0) = 0$. If there are no critical points, then $J(t, v_0)$ decays faster than any polynomial in $t$. Since the Klein-Gordon equation has finite speed of propagation, $|\nabla \omega(\xi)| \leq 1$, there are no critical points if $|v_0| > 1$; this follows from the alternative formula

\[
\omega(\xi)^2 = 1 + \sum_{j=1}^d \sin^2(\xi_j).
\]

The region $|v| \leq 1$ in velocity space is called the light cone. Hence, the solution $u(t, x)$ decays rapidly outside the light cone. In fact, since $\omega(\cdot)$ is analytic, the decay is exponential; this can be established by the method of steepest descent, like in $[27]$ $[4]$, but we will not pursue the issue here. Inside the light cone, there are critical points. Generically (i.e. for most values of $v_0$) these critical points are non-degenerate, that is $\det \text{Hess } \phi(\xi_0) \neq 0$. We note in passing that the Hessian is invariantly defined (i.e. invariant under changes of coordinates) at a critical point. The stationary phase method (see e.g. $[28]$) yields a $|t|^{-d/2}$ decay at such points. However, unlike in the continuous case, there are caustics, i.e. regions inside the light cone $|v| < 1$ where the solution decays slower, at a rate $|t|^{-\sigma}$ (possibly with
an additional logarithmic loss $\log^k |t|$), where $d/2 - \sigma > 0$ is called the order of the caustic [9] or the singular index [2]. In the simplest case,

$$\text{rank Hess } \phi(\xi_0) = d - 1, \quad \nabla \det \text{Hess } \phi(\xi_0) \neq 0. \quad (10)$$

Then $\Sigma := \{ \xi \in T^d : \det \text{Hess } \phi(\xi) = 0 \}$ is a smooth $d - 1$ dimensional manifold, and there exists a (non-unique) kernel vector field, i.e. a smooth non-zero vector field $V$ along $\Sigma$ such that $\text{Hess } \phi(\xi)(V(\xi)) = 0$ for all $\xi \in \Sigma$. A phase function $\Phi$ satisfying the condition (10) is said to have a corank one singularity. The simplest corank one singularity occurs when, for every $\xi \in \Sigma$, $\ker \text{Hess } \phi(\xi)$ intersects $T_\xi \Sigma$ transversally, or equivalently, $(\nabla_\xi \det \text{Hess } \phi) \cdot V \neq 0$ on $\Sigma$. This singularity is called the (Whitney) “fold”. In the classification of Arnol’d [3] the fold is called an $A_2$ singularity [1], and the oscillatory integral (9) reduces to an Airy integral in one variable [22, 14] (after integrating out the other $d - 1$ variables by stationary phase). The next more complicated singularity, called the “cusp” or $A_3$ singularity, occurs at points in $\Sigma$ where $(\nabla_\xi \det \text{Hess } \phi) \cdot V$ vanishes to first order. A systematicatization of these ideas gives rise to the Thom-Boardman classes. We refer the interested reader to [7, 13, 1] for an introduction to singularity theory. The situation becomes more complicated if we allow the parameter $v$ to vary. Then we have to consider a family of functions as opposed to a single function. This introduces (topological) notions of “typicality” (or transversality) and “stability” in the space of functions depending on a given number of parameters. In our case, the number of parameters equals the dimension $d$ of the underlying space $\mathbb{Z}^d$. Since we are considering $d \leq 4$ here, the only stable singularities are Thom’s seven elementary catastrophes [15.1]; in the terminology of Arnol’d these are the $A_2$, $A_3$, $A_4$, $A_5$, $D_4^-$, $D_4^+$, $D_5$ singularities. For the specific phase in (9), we will show in Section 2 that all these, except $D_5$, appear in $d \leq 4$ dimensions. However, an additional (unstable) singularity appears in $d = 4$. More precisely, in $d = 2$, there are only $A_k$ ($k \leq 3$) singularities. In $d = 3$, the phase function has a more degenerate $D_4^+$ type singularity. In $d = 4$, the phase function has only critical points with finite multiplicity, and in the most degenerate case is similar to a hyperbolic singularity ($T_{4,4,4}$ in the classification of Arnol’d [15.1]). The critical point of a hyperbolic singularity is (complex) isolated; however, our uniform estimates hold true for more general phase functions having non-isolated critical points. From the knowledge of the type of singularity, we can determine the singular index. For the $A_k$, $D_k$ singularities (and hence in $d = 2, 3$) this follows from the work of Duistermaat [9]. For the unstable singularity in $d = 4$ we use a result of Karpushkin [16].

For ease of reference, Table 1.2 lists the normal forms and singularity indices for $A_k$, $D_k$ and $T_{p,q,r}$ type singularities (compare [11 15.1], [26 6.1.10]). The normal form contains the “active variables” only. By this we mean the following: The Splitting Lemma [7, 14.12] says that, near a critical point, a smooth function $\phi : \mathbb{R}^d \to \mathbb{R}$ can be expressed in local coordinates as

$$\phi(x_1, \ldots, x_d) = f(x_1, \ldots, x_r) + Q(x_{r+1}, \ldots, x_d),$$

where $r$ is the corank of $\phi$ (or the number of active variables) at the critical point. Table 1.2 lists the normal forms of $f$, i.e. after a change of coordinates, $f$ reduces to one of the tabulated normal forms in the cases encountered here (at least in $d \leq 3$; the $T_{4,4,4}$ singularity is tabulated for comparison only).
Table 1. Normal forms and singularity index for \( A_k \), \( D_k \) and \( T_{4,4,4} \) type singularities.

| Type         | Normal form | Singular index \(d/2 - \sigma\) |
|--------------|-------------|----------------------------------|
| \( A_k, k \geq 1 \) | \( x_4^{k+1} \) | \( k-1 \) \( 2k+2 \) |
| \( D_k, k \geq 4 \) | \( x_2^2x_2 + x_2^{k-1} \) | \( k-2 \) \( 2k-2 \) |
| \( T_{4,4,4} \) | \( x_4^4 + x_3^4 + x_2^4 + ax_1x_2x_3, a \neq 0 \) | \( 1/2 \) |

1.3. Strichartz estimates. As a consequence of Theorem 1, we obtain Strichartz estimates. The proof follows from the (by now) standard argument of Keel–Tao [18]. More precisely, we apply [18 Theorem 1.2] to the operator \( U(t) = e^{-it\sqrt{1-\Delta}} \). For fixed \( t \), \( U(t) \) is a unitary operator on the Hilbert space \( H = \ell^2_x \). Here and henceforth, \( \ell^2_x = \ell^r(Z^d) \) denote the spatial Lebesgue spaces. The mixed space-time Lebesgue spaces \( L^q_t L^r_x \) are endowed with the norms

\[
\|F\|_{L^q_t \ell^r_x} = \left( \int_\mathbb{R} \left( \sum_{x \in \mathbb{Z}^d} |F(t,x)|^r \right)^{\frac{q}{r}} \right)^{\frac{1}{q}}
\]

for \( 1 \leq q, r < \infty \), with obvious modifications for \( q = \infty \) or \( r = \infty \). We recall that a pair of exponents \((q, r)\) is called \( \sigma \)-admissible [18] Definition 1.1] if

\[
q, r \geq 2, \quad (q, r, \sigma) \neq (2, \infty, 1), \quad \frac{1}{q} + \frac{\sigma}{r} \leq \frac{\sigma}{2}.
\]

If equality holds in the last condition, then \((q, r)\) is said to be sharp \( \sigma \)-admissible. Taking \( \sigma \) as the decay rate in (4–7), the combination of Duhamel’s formula (2) with Theorem 1 and [18 Theorem 1.2] yields the following family of Strichartz estimates.

Theorem 2. Let \( u \) be a solution of the Cauchy problem (1). Let \( q, r, \eta, \tau \geq 2 \),

\[
\begin{align*}
\frac{1}{q} &< \frac{4}{d} \left( \frac{4}{d} - \frac{1}{2} \right) & \text{if } d = 2, \\
\frac{1}{q} &< \frac{2}{d} \left( \frac{2}{d} - \frac{1}{2} \right) & \text{if } d = 3, \\
\frac{1}{q} &< \frac{2}{d} \left( \frac{2}{d} - \frac{1}{2} \right) & \text{if } d = 4,
\end{align*}
\]

and similarly for \( \eta, \tau \). Then \( u \) satisfies the estimate

\[
\|u\|_{L^q_t \ell^r_x} \leq C_{q, r, \eta, \tau}(\|f\|_{\ell^2} + \|g\|_{\ell^2} + \|F\|_{L^q_t \ell^r_x}),
\]

(12)

Remark 2. (i) We call \((q, r)\) a Strichartz pair if it satisfies (11). For any such pair there exists \( r_0 \in [2, r] \) such that \((q, r_0)\) satisfies (11) with equality (in \( d = 4 \) we subtract a fixed, arbitrarily small \( \epsilon > 0 \) from the right hand side). Thus there is a \( \tau \in [0, 1] \) such that

\[
(1/q, 1/r_0) = \tau(1/q_0, 0) + (1-\tau)(0, 1/2),
\]

where \( q_0 = 8/3 \) if \( d = 2 \), \( q_0 = 12/7 \) if \( d = 3 \) and \( q_0 = 4/3 + \epsilon' \) if \( d = 4 \) (here \( \epsilon' > 0 \) can be made arbitrarily small by choosing \( \epsilon \) sufficiently small). Note that the Lebesgue spaces over \( \mathbb{Z}^d \) form a filtration, i.e. \( \ell^{p_1}(\mathbb{Z}^d) \subset \ell^{p_2}(\mathbb{Z}^d) \) for \( p_1 \leq p_2 \).
This fact, together with the Gagliardo-Nirenberg and Young’s inequality (compare Remark 5 in [20]) yields
\[ \|u\|_{L_t^q L_x^r} \leq \|u\|_{L_t^{q_0} L_x^{r_0}} \leq \|u\|_{L_t^{q_0} L_x^{r_0}}^{1-\tau} \|u\|_{L_t^\infty L_x^\infty}^{\tau} =: \|u\|_{L_t^{q_0} L_x^{r_0} \cap L_t^\infty L_x^\infty}. \]
Hence, all the Strichartz estimates in Theorem 2 can be subsumed in the inequality
\[ \|u\|_{L_t^q L_x^r \cap L_t^\infty L_x^\infty} \leq C_{\mathbf{F}}(\|f\|_{\mathbf{F}}^2 + \|g\|_{\mathbf{F}}^2 + \inf \|F\|_{L_t^q L_x^r}), \]
where the infimum is taken over all Strichartz pairs \((q, r)\). (ii) The Strichartz estimates are also commonly expressed in terms of mapping properties for the operator \(U(t) = e^{-it\sqrt{1-\Delta_x}}\),
\[ \|U(t)f\|_{L_t^q L_x^r} \leq C\|f\|_{\mathbf{F}}^2, \]
\[ \| \int_{\mathbb{R}} U(-t)F(s)\|_{\mathbf{F}}^2 \leq C\|F\|_{L_t^q L_x^r}, \]
\[ \| \int_{s \leq t} U(t-s)F(s)\|_{L_t^q L_x^r} \leq C\|F\|_{L_t^q L_x^r}. \]
Note that \((1 - \Delta_x)^{-1/2}\) is a bounded operator on \(L_x^r\) for every \(r \in [1, \infty]\), see [19, Lemma 1]; hence \((14)\) follows from \((13)\) and \((2)\).
(iii) By [24, Lemma 3.9] the following sharp \(\sigma\)-admissible estimates are best possible,
\[ \left\{ \begin{array}{ll}
(q, r) = \left(\frac{q}{4}, \infty\right) & \text{if } d = 2, \\
(q, r) = \left(\frac{12}{7}, \infty\right) & \text{if } d = 3, \\
(q, r) = \left(\frac{6}{5} +, \infty\right) & \text{if } d = 4,
\end{array} \right. \]
in the sense that Strichartz estimates cannot hold for a pair \((\tilde{q}, \infty)\) with \(\tilde{q} < q\).

1.4. Discrete nonlinear Klein–Gordon equation. Strichartz estimates can be used in conjunction with a contraction mapping argument to prove global well-posedness for certain nonlinear equations with small initial data. Here we consider the discrete nonlinear Klein–Gordon equation
\[ u_{tt} - \Delta_x u + u \pm |u|^{2s} u = 0, \quad \text{in } \mathbb{R} \times \mathbb{Z}^d, \]
where \((u(0), u_t(0)) \in L_x^2 \times L_x^2\) and \(s\) satisfies
\[ \left\{ \begin{array}{ll}
s \geq \frac{4}{3} & \text{if } d = 2, \\
s \geq \frac{9}{7} & \text{if } d = 3, \\
s \geq \frac{8}{3} & \text{if } d = 4.
\end{array} \right. \]
Given the Strichartz estimates, the proof of the following theorem is standard (see e.g. [19, Theorem 6]), but we will give proofs of the PDE applications in Section 3.

**Theorem 3** (Global well-posedness for small data). Assume that \(s\) satisfies \((16)\). There exists \(\epsilon > 0\) and a constant \(C\) so that, whenever \(\|u(0)\|_{\mathbf{F}} + \|u_t(0)\|_{\mathbf{F}} \leq \epsilon\), then \((15)\) has a unique global solution. Moreover, the solution satisfies
\[ \|u\|_{L_t^q L_x^r} \leq C\epsilon \]
for any Strichartz pair \((q, r)\).
Interpolating between the bounds of Theorem 4 and energy conservation for the linear DKG ([11] (without forcing term, i.e. $F = 0$) we get the following decay estimates for the $\ell^p$-norm of the solution. For $2 \leq p \leq \infty$,
\begin{equation}
\|u(t)\|_{\ell^p} \leq C_p(1 + |t|)^{-\sigma(p-2)/p}\|\delta(u(0), u_t(0))\|_{\ell^p \times \ell^p'},
\end{equation}
where $\sigma = \sigma_d$ is the decay rate for $p = \infty$, i.e.
\begin{equation}
\sigma_d := \begin{cases} 
\frac{3}{2} & \text{if } d = 2, \\
\frac{7}{6} & \text{if } d = 3, \\
\frac{3}{4} & \text{if } d = 4,
\end{cases}
\end{equation}
where $3/2 - \epsilon$ means $3/2 - \epsilon$ for arbitrary fixed $\epsilon > 0$. Before stating the next theorem we define
\begin{equation}
p_2 := \frac{1}{6}(13 + \sqrt{97}) \approx 3.8, \quad p_3 := \frac{1}{14}(27 + \sqrt{337}) \approx 3.2, \quad p_4 := 3,
\end{equation}
\begin{equation}
s_2 := \frac{1}{12}(1 + \sqrt{97}) \approx 0.9, \quad s_3 := \frac{1}{28}(\sqrt{337} - 1) \approx 0.6, \quad s_4 := \frac{1}{2}.
\end{equation}

The next theorem is related to a conjecture by Weinstein [31].

**Theorem 4** (Decay of small solutions). Let $d = 2, 3, 4$ and $s > s_d$. Assume that $u$ satisfies the discrete nonlinear Klein–Gordon equation ([13]). There exists $\epsilon > 0$ such that, whenever $\|u(0)\|_{\ell^p_d} + \|u_t(0)\|_{\ell^p_d} \leq \epsilon$, then for every $p \in [2, p_d]$, we have
\begin{equation}
\|u(t)\|_{\ell^p} \leq C_p(1 + |t|)^{-\sigma(p-2)/p}\|\delta(u(0), u_t(0))\|_{\ell^p \times \ell^p'},
\end{equation}
where $\sigma = \sigma_d$.

**Remark 3.** Theorem 4 implies that no standing wave solutions $u(t, x) = e^{i\lambda t}\phi(x)$ are possible under the stated smallness assumption. Weinstein [31] proved the existence of an excitation threshold for the nonlinear Schrödinger equation in the continuum and conjectured that, for $s \geq 2/d$, solutions with sufficiently small initial conditions satisfy $\lim_{|t| \to \infty} \|u(t)\|_{\ell^p} = 0$ for all $p \in [1, \infty]$. On the lattice, Kevrekidis and Stefanov [13] proved that, for $s > d/2$, suitably small solutions of the nonlinear Schrödinger equation actually decay like the free solution in $\ell^p$. They also obtained analogues for the nonlinear Klein–Gordon equation in one space dimension. Theorem 4 establishes analogues of this result in dimensions $d = 2, 3, 4$.

**1.5. Resolvent estimates and spectral consequences.** Here we consider a stationary version of the discrete Klein–Gordon equation, namely
\begin{equation}
\sqrt{1} - \Delta \psi + V \psi = \lambda \psi, \quad \psi \in \ell^2(\mathbb{Z}^d).
\end{equation}
We start with a resolvent estimate for the unperturbed operator $\sqrt{1} - \Delta$. The idea of using Strichartz estimates to prove resolvent estimates is not new. It has appeared e.g. in [19] [21] [22] [3] [4]; the authors of [4] attribute the argument to T. Duyckaerts. The following results are analogues to [30] Proposition 3.3 and [19] Theorem 4) for the stationary Schrödinger equation. Again, in the Klein–Gordon case, better estimates are possible: in particular, our resolvent estimates hold in $d = 3$, whereas the estimates in [30] [19] require $d \geq 4$.

**Theorem 5.** Let $d = 3, 4$. There exists a constant $C$ such that for all $\lambda \in \mathbb{C}$ we have the estimate
\begin{equation}
\|\psi\|_{\ell^2} \leq C\|(\sqrt{1} - \Delta - \lambda)\psi\|_{\ell^2},
\end{equation}
where \( \sigma = \sigma_d \) is given by (18).

**Corollary 6.** Let \( d = 3, 4 \). There exists \( \epsilon > 0 \) such that, whenever \( \| V \|_{L^\sigma} \leq \epsilon \), then the eigenvalue problem (21) has no nontrivial solution.

1.6. **Organization of the paper.** Section 2 contains the main body of the paper. After some preliminary reductions, we perform a case-by-case study of the singularity structure of the phase function in dimensions \( d = 2, 3, 4 \). This yields the proof of Theorem 1. Section 3 contains proofs of the PDE applications. Numerical solutions to a system of equations that appears in the proof of the decay estimates are listed in an appendix.

### 2. Singularities of the phase function

In this section we consider the oscillatory integrals (9). To conform with standard notation we use (in this section only) the parameters \((\lambda, s)\) instead of \((t, v)\) and the integration variable \(x\) instead of \(\xi\). That is, we consider

\[
J(\lambda, s) = \int_{[0,2\pi]^d} e^{i\lambda \Phi(x,s)} dx,
\]

where

\[
\Phi(x, s) := \omega(x) - sx,
\]

\[
\omega(x) := \sqrt{1 + 2d - 2 \sum_{j=1}^{d} \cos(x_j)}.
\]

Here and in the following \(sx\) means \(s \cdot x\). Suppose \(s = s^0\) is a fixed vector and \(\Phi(x, s^0)\) has a critical point \(x^0\) (perhaps non-unique, but we consider one of them).

Note that \(\omega(x) \geq 1\) for all \(x \in [0, 2\pi]\). From now on we use the abbreviations \(c_j := \cos(x^0_j), s_j := \sin(x^0_j)\) for \(j = 1, \ldots, d\). Consider the functions

\[
\phi_1(x) := \Phi(x, s^0) - \Phi(x^0, s^0),
\]

\[
\phi_2(x) := \omega(x) + s^0x + \Phi(x^0, s^0).
\]

Obviously, \(\phi_1(x^0) = 0\) and \(\nabla \phi_1(x^0) = 0\). Moreover, \(\phi_2(x^0) \neq 0\); otherwise, we would have \(0 = \phi_1(x^0) + \phi_2(x^0) = 2\omega(x^0) \geq 2\). Instead of \(\Phi\) we will consider the new function

\[
\phi(x) := \phi_1(x)\phi_2(x)
\]

and investigate the type of singularities of the critical point \(x^0\). By the properties of \(\phi_1\) and \(\phi_2\) just mentioned, the singularity type (for the so-called weighted homogeneous cases) of the functions \(\Phi\) and \(\phi\) at \(x^0\) is the same (and the critical value is zero for the latter). But, as we will see, singularities of \(\phi\) are easier to study since we can avoid radicals. The critical point equation yields \(s^0 = \nabla \omega(x^0)\). In order to avoid possible confusion between \(s_j\) and the components of \(s^0\) we will not use the latter notation any more.

We introduce the new variables by \(\eta = x - x^0\) and, by using (22), we have

\[
\phi(\eta) = 1 + 2d - 2 \sum_{j=1}^{d} \cos(x^0_j + \eta_j) - \frac{(\sum_{j=1}^{d} s_j \eta_j + 1 + 2d - 2 \sum_{j=1}^{d} c_j)^2}{1 + 2d - 2 \sum_{j=1}^{d} c_j}.
\]
For the Hessian matrix we have

\[(\text{Hess } \phi(0))_{kj} := \partial_k \partial_j \phi(0) := 2 \left( c_j \delta_{kj} - \frac{s_k s_j}{1 + 2d - 2 \sum_{l=1}^d c_l} \right), \quad (24)\]

where \(\delta_{kj}\) is the Kronecker “delta”, i.e. \(\delta_{kj} = 1\) whenever \(k = j\) and otherwise \(\delta_{kj} = 0\). Moreover, for the determinant of Hessian matrix the following relation

\[\det \text{Hess } \phi(0) = 2^d \prod_{k=1}^d c_k \left( 1 - \sum_{j=1}^d \frac{s_j^2}{c_j (1 + 2d - 2 \sum_{l=1}^d c_l)} \right) \quad (25)\]

holds true, whenever \(c_j \neq 0, j = 1, \ldots, d\). Note that \(\det \text{Hess } \phi(0)\) is a rational function of \(\{c_j\}_{j=1}^d\) because \(s_j^2 = 1 - c_j^2\). A standard limiting argument then yields

\[\det \text{Hess } \phi(0) = 2^d \left( \prod_{k=1}^d c_k - \sum_{j=1}^d \frac{s_j^2 \prod_{m \neq j} c_m}{(1 + 2d - 2 \sum_{l=1}^d c_l)} \right)\]

for any \(c_j\), not only for \(c_j \neq 0\). The following observation about (24) will be useful later: Hess \(\phi(0)\) is a rank one perturbation of a diagonal matrix. By rank subadditivity we then have the following result.

**Lemma 7.** Let \(D = 2 \text{ diag}(c_1, \ldots, c_d)\) and \(S_{kj} = 2s_k s_j / \omega(x^0)^2\). Then \(\text{Hess } \phi(0) = D - S\) and

\[\text{rank}(D) - 1 \leq \text{rank}(\text{Hess } \phi(0)) \leq \text{rank}(D) + 1.\]

In the following, we will consider each of the cases \(d = 2, 3, 4\) separately. In view of (7) we distinguish each case into \(d + 1\) sub-cases according to how many of the \(c_j\) are zero.

We will need the following notion of homogeneity.

**Definition 1.** Let \(\kappa = (\kappa_1, \ldots, \kappa_d)\) with \(0 < \kappa_j \leq 1\) for all \(j = 1, \ldots, d\). We say that a function \(\phi : \mathbb{R}^d \to \mathbb{R}\) is homogeneous of degree \(r \geq 0\) with respect to the weight \(\kappa\) if for any \(\lambda > 0\),

\[\phi(\lambda^{\kappa_1} x_1, \ldots, \lambda^{\kappa_d} x_d) = \lambda^r \phi(x_1, \ldots, x_d).\]

For a given weight \(\kappa\) and degree \(r\) we will indicate terms of degree \(> r\) by \(\ldots\). If \(\kappa\) and \(r\) are clear from the context, we will not comment this further. For example, in (26) the meaning of “\(\ldots\)” is the standard one (\(\kappa_j = 1, r = 5\)). We will usually normalize \(\kappa\) such that \(r = 1\) (e.g. \(\kappa_j = 1/5, r = 1\) in the previous example).

If \(\phi\) is analytic (as will be the case here), then it has an expansion in \(\kappa\)-homogeneous polynomials of increasing degrees. The polynomial with the lowest degree will be called the “principal part”, \(\phi_{pr}\).

### 2.1. Two dimensions.

If \(d = 2\), then (24) can be written as

\[\det \text{Hess } \phi(0) = 4 \left( c_1 c_2 - \frac{c_1 s_1^2 + c_2 s_2^2}{5 - 2(c_1 + c_2)} \right).\]

**Case 1:** If \(c_1 = 0, c_2 = 0\) then \(s_j = \pm 1\). Note that \(s_j^3 = s_j\), so we have

\[\phi(\eta) = -\frac{1}{5}(s_1 \eta_1 + s_2 \eta_2)^2 - \frac{2}{3!}((s_1 \eta_1)^3 + (s_2 \eta_2)^3) + \frac{2}{5!}((s_1 \eta_1)^5 + (s_2 \eta_2)^5) + \ldots. \quad (26)\]
We will show that the function $\phi$ has an $A_3$ type singularity at the point $\eta = 0$. Indeed, by using the change of variables $y_1 = s_1 \eta_1 + s_2 \eta_2, y_2 = s_2 \eta_2$ we can see that

$$
\phi(\eta(y)) = -\frac{1}{5} y_1^2 - \frac{2}{3!} y_1 y_2^2 + \ldots,$$

where 


denotes a sum of homogeneous polynomials of degree strictly bigger than $r = 1$ with respect to the weight $\kappa = (1/2, 1/4)$. Hence, the principal part of the function has the form

$$
\phi_{pr}(y) = -\frac{1}{5} y_1^2 - y_1 y_2^2.
$$

Changing variables again, $u_1 = (5/2)^{1/2} y_2, u_2 = \frac{1}{5} y_1 + \frac{5}{2} y_2^2$, we can write this as

$$
\phi_{pr}(u) = u_1^4 - u_2^2.
$$

From Table 1.2 we infer (dropping the quadratic part, as we may) that this is an $A_3$ type singularity.

**Case 2:** Assume that exactly one of the $c_j$ is zero. Without loss of generality we assume that $c_1 = 0$ and $c_2 \neq 0$. Then since $s_1 = \pm 1$, it is easy to see that $\det \Hess \phi(0) \neq 0$. Hence the function $\phi$ has a non-degenerate critical point at the origin, i.e. an $A_1$ type singularity.

**Case 3:** If $c_1 \neq 0$ and $c_2 \neq 0$ and $\det \Hess \phi(0) = 0$, then the condition $\det \Hess \phi(0) = 0$ can be written as (using $s_j = 1 - c_j^2$)

$$
5 - (c_1 + c_2) - \frac{1}{c_1} + \frac{1}{c_2} = 0. \tag{27}
$$

**Case 3a:** If $c_1 - \frac{s_1^2}{5 - 2(c_1 + c_2)} = 0$, then the condition $\det \Hess \phi(0) = 0$ yields $s_2 = 0$, so $c_2 = \pm 1$. We will argue that $\phi$ has an $A_2$ type singularity at the origin $\eta = 0$. Indeed, we have

$$
\phi(\eta) = c_2 \eta_2^2 - \frac{2}{3!} s_1 \eta_1^3 + \ldots,
$$

where $s_1 \neq 0$ (otherwise we would have $c_1 = \pm 1$ and hence $\det \Hess \phi(0) \neq 0$), and 

... means sum of homogeneous polynomials of degree $> 1$ with respect to the weight $(1/3, 1/2)$. Consequently, the principal part of the function $\phi$ has the form

$$
\phi_{pr} = c_2 \eta_2^2 - \frac{2}{3!} s_1 \eta_1^3.
$$

Therefore, the phase function has an $A_2$ type singularity.

**Case 3b:** The case $c_2 - \frac{s_2^2}{5 - 2(c_1 + c_2)} = 0$ (and $\det \Hess \phi(0) = 0$) is analogous.

**Case 3c:** Lastly, assume that $c_j \delta_{j,k} - \frac{s_j s_k}{5 - 2(c_1 + c_2)} \neq 0$ for $j, k = 1, 2$ (this covers all remaining cases). Then under the condition $\det \Hess \phi(0) = 0$ (or equivalently \[27\]) the function $\phi$ can be written as

$$
\phi(\eta) = \left( c_1 - \frac{s_1^2}{5 - 2(c_1 + c_2)} \right) \left( \eta_1 - \frac{s_1 s_2 \eta_2}{5c_1 - 2c_1 c_2 - c_1^2 - 1} \right)^2 - \frac{2}{3!} (s_1 \eta_1^3 + s_2 \eta_2^3) + \ldots
$$

We claim that $\phi$ has only an $A_k$ type singularity with $k \leq 2$. Indeed, a straightforward calculation shows that $\phi$ has $A_k$ type singularities with $k \geq 3$ if and only if

$$
\frac{s_1^2 s_2^2}{(5c_1 - 2c_1 c_2 - c_1^2 - 1)^3} + s_2 = 0 \tag{28}
$$
and
\[ 5 - (c_1 + c_2) - \left( \frac{1}{c_1} + \frac{1}{c_2} \right) = 0. \]  
\[ (29) \]

Since \( s_2 \neq 0 \), and \( s_1 \neq 0 \) the \( (28) \) under the condition \( (29) \) can be written as
\[ c_2 s_1^3 + c_1 s_2^3 = 0 \]
or equivalently
\[ (1 - c_1^2) c_2^3 + (1 - c_2^2) c_1^3 = 0. \]  
\[ (30) \]

We claim that the system of equations \( (29) \) and \( (30) \) has no solution satisfying \(|c_j| < 1, j = 1, 2\). Indeed, if the pair \((c_1, c_2)\) is a solution to that system, then \( c_1 \) and \( c_2 \) have opposite signs. Without loss of generality, assume \( 1 \geq c_1 > 0 \) and \(-1 \leq c_2 < 0 \).

**Lemma 8.** Under the conditions \( 1 \geq c_1 > 0 \) and \(-1 \leq c_2 < 0 \), the system of equations \( (29) - (30) \) has no solutions.

**Proof.** Assume that \( (29) \) holds. The left hand side of \( (30) \) can be written as
\[
\text{LHS of } (30) = (1 - c_1^2) c_2^3 + (1 - c_2^2) c_1^3 = c_1^3 + c_2^3 - 2(c_1^2 c_2^3 + c_1^3 c_2) + c_1 c_2^3 + c_1^3 c_2^2
\]
\[
= (c_1 + c_2)(c_1^2 - c_1 c_2 + c_2^2 - 2c_1^2 c_2 + c_1 c_2^2)
\]

Since the signatures of \( c_1 \) and \( c_2 \) are different and since \( 0 < |c_j| < 1 \), we have
\[
c_1^2 - c_1 c_2 + c_2^2 - 2c_1^2 c_2 + c_1 c_2^2 = c_2^3(1 - c_1^2) + c_1^3(1 - c_2^2) - c_1 c_2(1 - c_1 c_2^2) > 0.
\]

Moreover, since \( c_1 + c_2 \neq 0 \) by \( (29) \), it follows that \( (30) \) cannot hold. \( \square \)

In summary, we have proved the following.

**Proposition 9.** If \( d = 2 \), then the phase function \( \phi \) has only singularities of type \( A_k \) with \( k \leq 3 \). More, precisely, if \( c_1 = c_2 = 0 \) then \( \phi \) has \( A_3 \) type singularity; if \( c_1 \neq 0 \) or \( c_2 \neq 0 \) then \( \phi \) has \( A_k \) type singularities with \( k \leq 2 \). For the corresponding oscillatory integral \( J(\lambda, s) \) the estimate
\[ |J(\lambda, s)| \leq C(1 + |\lambda|)^{-\frac{3}{4}} \]
holds, where the \( C \) constant does not depend on \( \lambda, s \).

**Remark 4.** The following comment appears in [19] (page no. 1853) : “Numerical simulations seem to confirm the validity of (17) since in a two-dimensional numerical experiment for a DKG lattice, the best fit to the decay was found to be \( O(t^{-0.675}) \).” Note that (17) in [19] is their conjectured bound \(|J(\lambda, s)| \leq C(1+|\lambda|)^{-\frac{3}{4}} \) (for \( d = 2 \)). Proposition 4 actually shows that the decay rate is better than the numerical bound. We speculate that this is related to the “constant problem”. As mentioned in the introduction, the bound of Proposition 4 was already proved by Borovyk–Goldberg [4], without reference to the conjecture of Kevrekidis and Stefanov [19].

2.2. **Three dimensions.** If \( d = 3 \), then \( (24) \) becomes
\[
\det \text{Hess} \phi(0) = 8c_1 c_2 c_3 - \frac{8(c_1 c_2 s_1^2 + c_1 c_3 s_2^2 + c_2 c_3 s_1^2)}{1 + 2d - 2(c_1 + c_2 + c_3)}.
\]

**Case 1:** If \( c_1 = c_2 = c_3 = 0 \), then we have \( s_j = \pm 1 \) and
\[
\phi(\eta) = -\frac{(s_1 \eta_1 + s_2 \eta_2 + s_3 \eta_3)^2}{7} - \frac{2}{3!}((s_1 \eta_1)^3 + (s_2 \eta_2)^3 + (s_3 \eta_3)^3) + \ldots
\]
Change of variables $y_1 = s_1 \eta_1 + s_2 \eta_2 + s_3 \eta_3, y_2 = s_2 \eta_2, y_3 = s_3 \eta_3$ yields

$$\phi(\eta(y)) = -\frac{y_1^2}{7} - 2\frac{1}{3!}((y_1^3 - y_1 y_2 + y_3)^3 + 3y_1(y_2 + y_3)^2 - 3y_2y_3(y_2 + y_3)) + \ldots$$

The principal part of $\phi$ with respect to the weight $(1/2, 1/3, 1/3)$ has the form

$$\phi_{pr}(y) = -\frac{y_1^2}{7} + y_2 y_3(y_2 + y_3).$$

By a change of variables this can be reduced to $\phi_{pr}(z) = -z_1^2 + z_2^3 z_3 - z_3^3$. Table 1.2 tells us that this is a $D_7^-$ type singularity.

**Case 2:** Assume that exactly one of the $c_j$ is zero. Without loss of generality, we suppose that $c_1 = 0$ and $c_2 \neq 0, c_3 \neq 0$. Then

$$\det \text{Hess} \phi(0) = -\frac{8c_2c_3}{1 + 2d - 2(c_2 + c_3)} \neq 0,$$

hence we have a non-degenerate critical point at $\eta = 0$ (an $A_1$ singularity).

**Case 3:** Assume that exactly two of the $c_j$ are zero; without loss of generality, $c_1 = 0 = c_2$ and $c_3 \neq 0$. Then $s_1 = \pm 1$ and $s_2 = \pm 1$. Hence,

$$\phi(\eta) = c_3 \eta_3^2 \left( \frac{(s_1 \eta_1 + s_2 \eta_2 + s_3 \eta_3)^2}{7 - 2c_3} - \frac{2}{3!}((s_1 \eta_1)^3 + (s_2 \eta_2)^3 + s_3 \eta_3^3) - \frac{2}{4!} c_3 \eta_3^4 \right)
+ \frac{2}{5!}((s_1 \eta_1)^5 + (s_2 \eta_2)^5 + s_3 \eta_3^5) + \ldots$$

We change variables $y_1 = s_1 \eta_1 + s_2 \eta_2 + s_3 \eta_3, y_2 = s_2 \eta_2, y_3 = \eta_3$ to find

$$\phi(\eta(y)) = c_3 y_3^2 - \frac{y_1^2}{7 - 2c_3} - \frac{2}{3!}((y_1 - y_2 - s_3 y_3)^3 + y_2^3 + s_3 y_3^3) - \frac{2}{4!} c_3 y_3^4
+ \frac{2}{5!}((y_1 - y_2 - s_3 y_3)^5 + y_2^5 + s_3 y_3^5) + \ldots$$

$$= c_3 y_3^2 - \frac{y_1^2}{7 - 2c_3} - \frac{2}{3!}((y_1 - s_3 y_3)^3 - 3(y_1 - s_3 y_3)^2 y_2 + 3(y_1 - s_3 y_3) y_2^2 + s_3 y_3^3)
- \frac{2}{4!} c_3 y_3^4 + \frac{2}{5!}((y_1 - y_2 - s_3 y_3)^5 + y_2^5 + s_3 y_3^5) + \ldots$$

The principal part with respect to the weight $(1/2, 1/4, 1/2)$ is given by

$$\phi_{pr}(\eta(y)) = c_3 y_3^2 - \frac{y_1^2}{7 - 2c_3} - \frac{2}{3!}((y_1 - s_3 y_3) y_2^2) = c_3 y_3^2 - \frac{y_1^2}{7 - 2c_3} - (y_1 - s_3 y_3) y_2^2.$$

We can rewrite $\phi(\eta(y))$ as

$$\phi(\eta(y)) = c_3 (y_3 + s_3 y_3^2 y_2^2) - \frac{1}{7 - 2c_3} (y_1 + \frac{(7 - 2c_3) y_2^2}{2})^2 - \frac{2}{4!} c_3 y_3^4
- \frac{2}{3!}((y_1 - s_3 y_3)^3 - 3(y_1 - s_3 y_3)^2 y_2 + s_3 y_3^3)
- \frac{2}{5!}((y_1 - y_2 - s_3 y_3)^5 + y_2^5 + s_3 y_3^5) + \ldots$$

From this we see that if $s_3^2 + 2c_3^2 - 7c_3 \equiv 1 + c_3^2 - 7c_3 \neq 0$, then we have an $A_3$ type singularity at $y = \eta = 0$. On the other hand, if $1 + c_3^2 - 7c_3 = 0$ (i.e.
\[ c_3 = 2 / (7 + 3\sqrt{3}) \], then we have
\[
\phi(\eta(y)) = c_3(y_3 + s_3 y_3^2) - \frac{1}{2} \frac{(7 - 2c_3)y_3^2}{2c_3} - \frac{2}{4!} c_3 y_3^4 + \frac{2}{5!} (5(y_1 - s_3 y_3)y_3^2 + \ldots.
\]

Note that the Taylor series of the function \(\phi(\eta(y))\) has no term \(cy_2^7\) with non-zero coefficient \(c\). By using the change of variables
\[
z_2 = y_2, z_1 = y_1 + \frac{(7 - 2c_3)y_3^2}{2}, z_3 = y_3 + s_3 y_3^2,
\]
we obtain (under the condition \(1 + c_3^2 - 7c_3 = 0\) we have \(y_1 - s_3 y_3 = z_1 - s_3 z_3\))
\[
\phi(\eta(y(z))) = c_3 z_3^2 - \frac{1}{2} \frac{z_3^2}{2} + \frac{2}{5!} (5(z_1 - s_3 z_3)z_3^4 + \frac{s_3^4 z_3^6}{24c_3^3} + \ldots,
\]
where “\ldots” means sum of homogeneous polynomials of degree \(> 1\) with respect to the weight \((1/2, 1/6, 1/2)\). Note that the degree of the polynomial \((z_1 - s_3 z_3)z_3^4\) is \(7/6 > 1\) with respect to that weight. Hence, the principal part is
\[
\phi_{pr}(z) = c_3 z_3^2 - \frac{1}{2} \frac{z_3^2}{2} + \frac{s_3^4 z_3^6}{24c_3^3}.
\]

This is an \(A_5\) type singularity at \(z = y = \eta = 0\), because if \(1 + c_3^2 - 7c_3 = 0\) (as we assumed), then \(s_3 \neq 0\).

**Case 4**: Finally, assume \(c_1 \neq 0, c_2 \neq 0, c_3 \neq 0\) and \(\det \text{Hess } \phi(0) = 0\). This will be the longest and most delicate case. We claim that the function has an \(A_k\) type singularity with \(k \leq 3\). Indeed, due to Lemma 7, the rank of the matrix \(\text{Hess } \phi(0)\) is at least 2. Since we are assuming \(\det \text{Hess } \phi(0) = 0\), the rank of that Hessian matrix is exactly 2.

**Case 4a**: Suppose that one of the \(s_j\) vanishes. Without loss of generality, we will assume that \(s_1 = 0\). Then \(c_1 = \pm 1\). Since \(c_2 \neq 0, c_3 \neq 0\), arguing as in case 3a in \(d = 2\), we see that \(\phi\) has an \(A_2\) type singularity (we exclude the \(A_1\) case since we assume \(\det \text{Hess } \phi(0) = 0\)).

**Case 4b**: Further, assume \(c_j \neq 0, s_j \neq 0, j = 1, 2, 3\). Then for some \(j\) we have \(c_j - \frac{s_j^2}{7 - 2(c_1 + c_2 + c_3)} \neq 0\); otherwise we would have
\[
\det \text{Hess } \phi(0) = -\frac{8 s_1^2 s_2^2 s_3^2}{(7 - 2(c_1 + c_2 + c_3))^3} \neq 0,
\]
i.e. \(\phi\) would have an \(A_1\) type (or non-degenerate) critical point at \(\eta = 0\). Without loss of generality, we will assume that \(c_1 - \frac{s_1^2}{7 - 2(c_1 + c_2 + c_3)} \neq 0\). The quadratic part of \(\phi\) is given by
\[
p_2(\eta) := c_1 \eta_1^2 + c_2 \eta_2^2 + c_3 \eta_3^2 - \frac{(s_1 \eta_1 + s_2 \eta_2 + s_3 \eta_3)^2}{7 - 2(c_1 + c_2 + c_3)}.
\]

For the kernel \(\nabla p_2(\eta) = 0\) we have the relation
\[
\eta_1 = \frac{s_1 c_3}{c_1 s_3} \eta_3, \quad \eta_2 = \frac{s_2 c_3}{c_2 s_3} \eta_3.
\]
Moreover, using the identity

\[ -\frac{(s_2\eta_2 + s_3\eta_3)^2}{c_2^2 + c_3^2} + c_2\eta_2^2 + c_3\eta_3^2 = \frac{c_2^2s_2^2\eta_2^2 + c_3^2s_3^2\eta_3^2 - 2c_2c_3s_2s_3\eta_2\eta_3}{c_1s_2^2 + c_2s_3^2}, \]

we find that

\[ p_2(\eta) - \left( c_1 - \frac{s_1^2}{7 - 2(c_1 + c_2 + c_3)} \right) \left( \eta_1 - \frac{s_1(s_2\eta_2 + s_3\eta_3)}{c_1(7 - 2(c_1 + c_2 + c_3)) - s_1^2} \right)^2 = -\frac{(s_2\eta_2 + s_3\eta_3)^2}{c_2^2 + c_3^2} + c_2\eta_2^2 + c_3\eta_3^2 = \frac{(c_2s_2\eta_2 - c_3s_3\eta_3)^2}{c_1s_2^2 + c_2s_3^2}. \]

Since \( c_1 - \frac{s_1^2}{7 - 2(c_1 + c_2 + c_3)} \neq 0 \) the function \( \phi \) can be written as

\[ \phi(\eta) = r_1(\eta_1 + b_2\eta_2 + b_3\eta_3)^2 + r_2(\eta_2 - \frac{s_2c_3}{c_2s_3}\eta_3)^2 - \frac{2}{3!}(s_1\eta_1^3 + s_2\eta_2^3 + s_3\eta_3^3) + \ldots, \]

where \( r_1, r_2, b_2, b_3 \) are nonzero real numbers satisfying the conditions:

\[ r_1 = c_1 - \frac{s_1^2}{7 - 2(c_1 + c_2 + c_3)}, \quad b_2 = \frac{s_1c_3}{c_2s_3}, \quad b_3 = \frac{s_1c_3}{c_1s_3}. \]

**Case 4bi:** Assume that

\[ 7 - \sum_{j=1}^{3} (c_j + 1/c_j) = 0 \quad \text{and} \quad \sum_{j=1}^{3} \frac{s_j^4}{c_j^4} \neq 0. \quad (31) \]

We claim that \( \phi \) has an \( A_2 \) type singularity at \( \eta = 0 \). Indeed, the first condition in (31) is equivalent to \( \det \text{Hess} \phi(0) = 0 \). Under the second condition we can use change of variables

\[ y_1 = \eta_1 + b_2\eta_2 + b_3\eta_3, \quad y_2 = \eta_2 - \frac{s_2c_3}{c_2s_3}\eta_3, \quad y_3 = \eta_3 \]

to get

\[ \phi(\eta) = r_1y_1^2 + r_2y_2^2 - \frac{2}{3!} \left( \sum_{j=1}^{3} \frac{s_j^4}{c_j^4} \right) y_3^3 + \ldots, \]

where “…” means sum of homogeneous polynomials of degree \( >1 \) with respect to the weight \((1/2, 1/2, 1/3)\). Hence, \( \phi \) has an \( A_2 \) type singularity at \( \eta = 0 \).

**Case 4bii:** Assume

\[ 7 - \sum_{j=1}^{3} (c_j + 1/c_j) = 0 \quad \text{and} \quad \sum_{j=1}^{3} \frac{s_j^4}{c_j^4} = 0, \quad (32) \]

and recall that the first condition simply means \( \det \text{Hess} \phi(0) = 0 \). We then have the following relation,

\[ 7 - 2(c_1 + c_2 + c_3) - \frac{s_1^2}{c_1} = \left( \frac{s_2}{c_2} + \frac{s_3}{c_3} \right)^2. \]

Changing variables

\[ y_1 = \eta_1 - \frac{s_1(s_2\eta_2 + s_3\eta_3)}{c_1(7 - 2(c_1 + c_2 + c_3)) - s_1^2}, \quad y_2 = \eta_2 - \frac{c_3s_2}{c_2s_3}\eta_3, \quad y_3 = s_2\eta_2 + s_3\eta_3, \]

we arrive at

\[ p_2(\eta(y)) = \left( c_1 - \frac{s_1^2}{7 - 2(c_1 + c_2 + c_3)} \right) y_1^2 + \frac{c_2^2 s_3^2 y_2^2}{c_3 s_2^2 + c_2 s_3^2} \]
\[ = \frac{c_1}{7 - 2(c_1 + c_2 + c_3)} \left( \frac{s_2^2}{c_2} + \frac{s_3^2}{c_3} \right) y_1^2 + \frac{c_2^2 s_3^2 y_2^2}{c_3 s_2^2 + c_2 s_3^2} \]
\[ = \frac{c_1 (c_3 s_2^2 + c_2 s_3^2)}{c_2 c_3 (7 - 2(c_1 + c_2 + c_3))} y_1^2 + \frac{c_2^2 s_3^2 y_2^2}{c_3 s_2^2 + c_2 s_3^2}. \]

Under the conditions \[ (32) \] we can write

\[ \eta_1 = y_1 + \frac{s_1 y_3}{c_1 (7 - 2(c_1 + c_2 + c_3)) - s_1^2}, \]
\[ \eta_2 = \frac{s_2 y_2}{c_2} + \frac{s_2 y_3}{c_3}, \]
\[ \eta_3 = -\frac{s_2 y_2}{c_3} + \frac{s_2 y_3}{c_3}. \]

or equivalently,

\[ \eta_1 = y_1 + \frac{c_2 s_3 y_3}{c_1 s_2^2 + c_2 s_3^2}, \]
\[ \eta_2 = \frac{c_2 s_3^2 y_2}{c_3 s_2^2 + c_2 s_3^2} + \frac{c_3 s_2 y_3}{c_3 s_2^2 + c_2 s_3^2}, \]
\[ \eta_3 = -\frac{c_2 s_3^2 y_2}{c_3 s_2^2 + c_2 s_3^2} + \frac{c_3 s_2 y_3}{c_3 s_2^2 + c_2 s_3^2}. \]

A straightforward but tedious computation yields

\[ s_1 \eta_1^3 + s_2 \eta_2^3 + s_3 \eta_3^3 = s_1 \left( y_1 + \frac{c_2 s_3 y_3}{c_1 s_2^2 + c_2 s_3^2} \right)^3 + s_2 \left( \frac{c_2 s_3^2 y_2}{c_3 s_2^2 + c_2 s_3^2} + \frac{c_3 s_2 y_3}{c_3 s_2^2 + c_2 s_3^2} \right)^3 \]
\[ + s_3 \left( -\frac{s_3 s_2 c_2 y_2}{c_3 s_2^2 + c_2 s_3^2} + \frac{c_2 s_3 y_3}{c_3 s_2^2 + c_2 s_3^2} \right)^3 \]
\[ = \frac{3 s_1^3 c_2^2 c_3^2 y_1^3}{c_1^3 (c_3 s_2^2 + c_2 s_3^2)^2} + \frac{3 y_2^3 c_2 s_3 y_3^3 (c_3 s_2^2 - c_2 s_3^2)}{(c_3 s_2^2 + c_2 s_3^2)^3} + \ldots \]
\[ = \frac{3 s_1^3 c_2^2 c_3^2 y_1^3}{c_1^3 (c_3 s_2^2 + c_2 s_3^2)^2} + \frac{3 y_2^3 c_2 s_3 y_3^3 (c_1^2 - c_2^2)}{(c_3 s_2^2 + c_2 s_3^2)^3} + \ldots, \]

where “...” comprises a sum of terms with degree > 1 with respect to the weight \((1/2, 1/2, 1/4)\). Under the condition

\[ \frac{s_1^4}{c_1^4} + \frac{s_2^4}{c_2^4} + \frac{s_3^4}{c_3^4} = 0, \]

there is no term \( c y_3^3 \) with a non-zero coefficient \( c \). Hence all other terms which are not indicated have degree > 1 with respect to the weight \((1/2, 1/2, 1/4)\). We claim that if the following condition (see \[ (34) \])

\[ s_1 c_2 c_3 (7 - 2(c_1 + c_2 + c_3)) + c_1^5 s_2^2 s_3^2 (c_1^2 - c_2^2)^2 \neq 0, \]

(33)
is satisfied, then the phase function has $A_3$ type singularity at \( \eta = 0 \). Indeed, under the conditions (32), we have

\[
\phi(\eta(y)) = \left( \frac{c_1(c_3s_2^2 + c_2s_3^2)}{c_2c_3(7 - 2(c_1 + c_2 + c_3))} \right) y_1^2 + \frac{c_2^2s_3^2y_2^2}{c_3s_2^2 + c_2s_3^2} \\
- \frac{s_1^2c_2c_3y_1y_2}{c_1(c_3s_2^2 + c_2s_3^2)^2} - \frac{y_2y_3^2c_2s_3s_2(c_3^2 - c_2^2)}{(c_3s_2^2 + c_2s_3^2)^3} + \ldots
\]

\[
= \left( \frac{c_1(c_3s_2^2 + c_2s_3^2)}{c_2c_3(7 - 2(c_1 + c_2 + c_3))} \right) \left( y_1 - \frac{s_1^2c_3^2(7 - 2(c_1 + c_2 + c_3))y_2^2}{2c_1(c_3s_2^2 + c_2s_3^2)^3} \right)^2
\]

\[
+ \frac{c_2^2s_3^2}{c_3s_2^2 + c_2s_3^2} \left( y_2 - \frac{y_2^2s_2(c_3^2 - c_2^2)}{2c_2(c_3s_2^2 + c_2s_3^2)^2} \right)^2
\]

\[
- \frac{1}{4} \frac{s_1^2c_2^5(7 - 2(c_1 + c_2 + c_3)) + c_5s_3^2s_2^2(c_3^2 - c_2^2)^2}{c_1(c_3s_2^2 + c_2s_3^2)^3} y_3^4 + \ldots,
\]

where “...” comprises a sum of terms with degree > 1 with respect to the weight (1/2, 1/2, 1/4). If we use change of variables

\[
z_1 = y_1 - \frac{s_1^2c_2c_3(7 - 2(c_1 + c_2 + c_3))y_2}{2c_1(c_3s_2^2 + c_2s_3^2)^3}, \quad z_2 = y_2 - \frac{y_2^2s_2(c_3^2 - c_2^2)}{2c_2(c_3s_2^2 + c_2s_3^2)^2}, \quad z_3 = y_3,
\]

then we can write the last expression in the following equivalent form,\n
\[
\phi(\eta(z)) = \left( c_1 - \frac{s_1^2}{7 - 2(c_1 + c_2 + c_3)} \right) z_1^2 + \frac{c_2^2s_3^2z_2^2}{c_3s_2^2 + c_2s_3^2}
\]

\[
- \frac{1}{4} \frac{s_1^2c_2^5c_3(7 - 2(c_1 + c_2 + c_3)) + c_5s_3^2s_2^2(c_3^2 - c_2^2)^2}{c_1(c_3s_2^2 + c_2s_3^2)^3} z_3^4 + \ldots,
\]

where again “...” denotes a sum of homogeneous polynomials of degree > 1 with respect to the weight (1/2, 1/2, 1/4). Actually, we have to show that, under the conditions (32), the terms of the Taylor expansion of order \( \geq 4 \) do not contribute to the principal part

\[
\phi_{pr}(\eta(y(z))) := \left( c_1 - \frac{s_1^2}{7 - 2(c_1 + c_2 + c_3)} \right) z_1^2 + \frac{c_2^2s_3^2z_2^2}{c_3s_2^2 + c_2s_3^2}
\]

\[
- \frac{1}{4} \frac{s_1^2c_2^5c_3(7 - 2(c_1 + c_2 + c_3)) + c_5s_3^2s_2^2(c_3^2 - c_2^2)^2}{c_1(c_3s_2^2 + c_2s_3^2)^3} z_3^4.
\]

Let us consider the sum of monomials of degree 4 in the original coordinate system,

\[
c_1\eta_1^4 + c_2\eta_2^4 + c_3\eta_3^4 = c_1 \left( y_1 + \frac{c_2s_3y_3}{c_3s_2 + c_2s_3} \right)^4 + c_2 \left( \frac{c_2s_3^2y_2}{c_3s_2^2 + c_2s_3^2} + \frac{c_3s_2y_3}{c_3s_2 + c_2s_3} \right)^4
\]

\[
+ c_3 \left( \frac{s_3^2y_2}{c_1s_2 + c_2s_3} + \frac{c_2s_3y_3}{c_3s_2^2 + c_2s_3^2} \right)^4
\]

\[
= \frac{c_4^4}{(c_3s_2^2 + c_2s_3^2)^4} \sum_{j=1}^{3} \frac{s_3^j}{c_3^j} y_3^j + y_1P_1(y) + y_2P_2(y),
\]

where \( P_1, P_2 \) are homogeneous polynomials of degree 3. The degree (with respect to the weight (1/2, 1/2, 1/4)) of any monomial of \( y_jP_j(y) \), \( j = 1, 2 \), is at least 1/2 + 3/4 = 5/4. Similarly one shows that higher order terms in the Taylor expansion also have degree at least 5/4.
We now assume that the expression in (33) vanishes, i.e.
\[ s_1^5c_2^5c_3^5(7 - 2(c_1 + c_2 + c_3)) + c_1^5s_2^2c_2^2(c_3^2 - c_2^2)^2 = 0. \] (35)
If the function \( \phi(\eta) \) had a singularity of type \( A_k \) with \( k > 3 \), then (33) would have to hold. However, it can be checked numerically that under the condition (32), the latter equation has no solution satisfying \( |c_j| < 1 \) (see Appendix A).

We conclude that the phase function has \( A_k \) \( (k \leq 3) \) type singularities at \( \eta = 0 \) whenever \( c_j \neq 0, j = 1, 2, 3 \).

In summary, we have the following result for \( d = 3 \).

**Proposition 10.** If \( d = 3 \), then the phase function can have \( A_k \) \( (k \leq 5) \) and \( D_4 \) type singularities. The corresponding oscillatory integral is estimated by
\[ |J(\lambda, s)| \leq C(1 + |\lambda|)^{-\frac{7}{2}}, \]
where \( C \) does not depend on \( \lambda, s \).

**Remark 5.** (i) Note that the singularity index for \( A_5 \) and \( D_4 \) is \( \frac{1}{3} \).
(ii) For \( d = 2, 3 \) the most delicate case is \( c_j \neq 0 \) for all \( j \). Incidentally, the estimate \( O(|\tau|^{-\frac{1}{3}}) \), conjectured in (10), can obtained by simpler arguments, even in higher dimensions. Indeed, if \( d \geq 3 \) and \( c_j \neq 0 \) for all \( j \) then the rank of Hessian matrix is at least \( d - 1 \). Hence, by stationary phase, we get \( J(\lambda, s) = O(|\lambda|^{-(d-1)/2}) \). Note that \( (d-1)/2 \geq d/3 \) for \( d \geq 3 \). Moreover if \( d \geq 4 \) then \( (d-1)/2 \geq (2d+1)/6 \). For these estimates, conditions such as \( |c_j| < 1 \) and \( c_j^2 + s_j^2 = 1 \) are not needed; it is enough to assume \( c_j^2 + s_j^2 \neq 0 \) \( (j = 1, \ldots, d) \).

2.3. **Four dimensions.** Let \( d = 4 \). Again we first consider the most degenerate case.

**Case 1:** \( c_1 = c_2 = c_3 = c_4 = 0 \). Then \( s_j = \pm 1 \). Then we have
\[
\phi(\eta) = -\frac{(s_1\eta_1 + s_2\eta_2 + s_3\eta_3 + s_4\eta_4)^2}{9} - \frac{2}{3!}((s_1\eta_1)^3 + (s_2\eta_2)^3 + (s_3\eta_3)^3 + (s_4\eta_4)^3) + \frac{2}{5!}((s_1\eta_1)^5 + (s_2\eta_2)^5 + (s_3\eta_3)^5 + (s_4\eta_4)^5) + \ldots
\]
where \( \ldots \) denotes higher order terms in the Taylor expansion that can be considered as a small perturbation of the principal part. We will show that this small perturbation can be removed by smooth change of variables, leading to a \( T_{4,4,4} \) type singularity.

**Lemma 11.** The function \( \phi \) has \( T_{4,4,4} \) type singularity at \( \eta = 0 \).

**Proof.** We use the change of variables \( y_4 = s_1\eta_1 + s_2\eta_2 + s_3\eta_3 + s_4\eta_4, y_j = s_j\eta_j \), for \( j = 1, 2, 3 \). Then we have
\[
\phi(\eta(y)) = -\frac{y_4^2}{9} - \sigma_1(y)\sigma_2(y) + \sigma_3(y) + \ldots,
\]
where \( \sigma_j \) are elementary symmetric polynomials,
\[
\sigma_1(y) := y_1 + \cdots + y_n,
\sigma_2(y) := y_1y_2 + y_1y_3 + \cdots + y_{n-1}y_n,
\sigma_3(y) := y_1y_2y_3 + \cdots + y_{n-2}y_{n-1}y_n,
\]
and \( n = d - 1 \) is the number of “active variables” (so \( n = 3 \) in the present case). Here we used the relation
\[
y_1^3 + \cdots + y_n^3 - \sigma_1^3 = 3\sigma_3 - 3\sigma_1\sigma_2.
\]
If \( n = 3 \), then it can be shown that \( \sigma_3 - \sigma_1\sigma_2 \) is linearly equivalent (by linear change of variables) to \( z_1z_2z_3 \). Oscillatory integrals with such phase functions have been investigated by Karpushkin [16]. They satisfy estimates of the form \( O(|\lambda|^{-\frac{4}{3}} \log |\lambda|) \) as \(|\lambda| \to \infty \). It follows from the arguments of [15] that the same estimate holds when we add a small linear perturbation, i.e. the phase function is \( z_1z_2z_3 + O(|z|^4) \).

In fact, Karpushkin’s theorem holds for arbitrary small perturbations, but we only need this weaker result here. We explain this now more precisely: The function \( \phi(z) \) can be written as
\[
\phi(z) = b_1(y_1 + y_2 + y_3, y_4)(y_4 - (y_1 + y_2 + y_3)^2(\omega(y_1 + y_2 + y_3))^2 + y_1y_2y_3 + \ldots,
\]
where \( b_1 \) and \( \omega \) are smooth functions of two and one variables respectively, with \( b_1(0,0) \neq 0 \) and \( \omega(0) \neq 0 \). Hence, it is easy to see that the function
\[
\phi_2(y) = \frac{\phi_2(y)}{\phi_1(y)}
\]
can be written as
\[
\phi_2(y) = \tilde{b}_1(y_1, y_2, y_3, y_4)(y_4 - (y_1 + y_2 + y_3)^2(\omega(y_1 + y_2 + y_3)) + \omega_1(y_1, y_2, y_3))^2
\]
+ \( y_1y_2y_3 + \ldots \),
where \( \tilde{b}_1, \omega_1(y) \) are smooth functions satisfying \( b_1(0,0,0,0) \neq 0 \) and \( \omega_1 \) satisfies the condition that for any nonnegative integer triples \( (k_1, k_2, k_3) \), with \( k_1 + k_2 + k_3 \leq 3 \), one has \( \partial^{k_1} \partial^{k_2} \partial^{k_3} \omega_1(0,0,0) = 0 \); moreover, “\( + \ldots \)” means \( O(|y|^4) \) with \( y = (y_1, y_2, y_3, y_4) \). Then the total phase function can be written as \( \Phi(y, s) = \phi_2(y) - sy \) up to an irrelevant additive constant. By using the stationary phase method in the \( y_1 \) variable, we obtain a perturbation of a function of three variables having the form \( \phi_1(y_1, y_2, y_3) = y_1y_2y_3 + \ldots \). Thus, the total phase function \( \Phi \) of three variables and four parameters can be written as
\[
\Phi(y, s) = \phi_1(y_1, y_2, y_3) - (s_1 + g_1(s_1))y_1 - (s_2 + g_2(s_4))y_2 - (s_3 + g_3(s_4))y_3
\]
+ \( s_4(y_1 + y_2 + y_3)^2(\omega(y_1 + y_2 + y_3)) + s_4\omega_2(y, s_4) + s_4\omega_3(y, s_4) \),
where \( g_l \) \((l = 1, 2, 3) \) are smooth functions, with \( g_0(0) = 0 \), and \( \omega_j(y, s_4) \) \((j = 2, 3) \) are smooth functions such that for any nonnegative integer triples \( (k_1, k_2, k_3) \), with \( k_1 + k_2 + k_3 \leq j \), we have the relation \( \partial^{k_1} \partial^{k_2} \partial^{k_3} \omega_j(0,0,0,0) = 0 \). Now we can apply the classical theorem of Karpushkin [16] to the function \( \Phi \) and obtain the required bound.  

Case 2: Suppose that only one of the \( c_j \) is zero; without loss of generality, \( c_1 = 0 \) and \( c_2 \neq 0, c_3 \neq 0, c_4 \neq 0 \). Then by [23] we have \( \det \text{Hess} \phi(0) \neq 0 \). Thus, the phase function has an \( A_1 \) (non-degenerate) singularity.

Case 3: Suppose that two of the \( c_j \) are zero; without loss of generality, \( c_1 = 0 = c_2 \) and \( c_3 \neq 0 \), \( c_4 \neq 0 \). Since \( |s_1| = |s_2| = 1 \), the matrix \((S_{kj})_{j=1}^3\) (see Lemma 7 for the notation) has rank 1. Hence the rank of the matrix \( \text{Hess} \phi(0) \) equals 3. Therefore, the corresponding integral decays as \( O(|\lambda|^{-3/2}) \). This is sufficient for our result.
Case 4: Suppose that three of the $c_j$ are zero; without loss of generality, $c_1 = c_2 = c_3 = 0$ and $c_4 \neq 0$. Then we have

$$\phi(\eta) = c_4 \eta_4^2 - \frac{\sum_{j=1}^{4} s_j \eta_j}{9 - 2c_4} - \frac{2}{3!}((s_1 \eta_1)^3 + (s_2 \eta_2)^3 + (s_3 \eta_3)^3 + s_4 \eta_4^3) + \ldots$$

By a linear change of variables $y_1 = s_1 \eta_1$, $y_2 = s_2 \eta_2$, $y_3 = \sum_{j=1}^{4} s_j \eta_j$, $y_4 = \eta_4$,

$$\phi(\eta) = c_4 y_4^2 - \frac{y_2^2}{9 - 2c_4} - \frac{2}{3!}(y_3^2 + y_2^3 - (y_1 + y_2)^3) + \ldots$$

$$= c_4 y_4^2 - \frac{y_2^2}{9 - 2c_4} - \frac{2}{3!}(y_3^2 + y_2^3 - (y_1 + y_2)^3)$$

$$= c_4 y_4^2 - \frac{y_2^2}{9 - 2c_4} + y_1 y_2 (y_1 + y_2).$$

where “…” consists of sum of terms with degree $> 1$ with respect to the weight $(1/3, 1/3, 1/2, 1/2)$, and the principal part is given by the expression without “…”.

In this case the phase function has a $D_4^-$ type singularity. The corresponding oscillatory integral decays as $O(1/|\lambda|^{5/3})$ (see Table 1.2).

Case 5: Finally, we consider the case $c_1, c_2, c_3, c_4 \neq 0$. Then by Lemma 4, the rank of the quadratic part is at least 3, and hence the corresponding phase function has $A_k$ type singularities. In this case we do not need to investigate the multiplicity of the corresponding phase function, because the corresponding oscillatory integral decays at least as $O(1/|\lambda|^{-3/2})$, by stationary phase.

We record our findings for the $d = 4$ case.

**Proposition 12.** If $d = 4$, then the phase function has $T_{4,4,4}$, $D_4^-$ or $A_k$ type singularities. The corresponding oscillatory integral is estimated by

$$|J(\lambda, s)| \leq C(1 + |\lambda|)^{-\frac{3}{2}} \log(2 + |\lambda|),$$

where $C$ does not depend on $\lambda, s$.

3. Proofs of the PDE applications

**Proof of Theorem 3.** We are going to use a contraction mapping argument (Banach’s fix point theorem) in the metric space

$$X = \{ u \in L^\infty_t \ell^2_x : \|u\|_{L^q \ell^r, L^{q_0} \ell^{r_0}} \leq 2C_{1,2}\|/(f,g)\|_{\ell^2 \times \ell^2} \},$$

where $C_{1,2}$ denotes the constant in the Strichartz estimate [13] with $(\eta, \tau) = (1, 2)$ and $q_0$ as in Remark 2. For convenience, we also introduced the notation $(f,g) := (u(0), u_t(0))$ and

$$\|(f,g)\|_{\ell^2 \times \ell^2} := \|f\|_{\ell^2} + \|g\|_{\ell^2}.$$

Consider the solution map

$$Au(t) := U_0(t)f + U_1(t)g \pm \int_0^t U_1(t - \tau)|u(\tau)|^{2s}u(\tau)d\tau,$$

where

$$U_0(t) := \cos(t\sqrt{1 - \Delta_x}), \quad U_1(t) := \frac{\sin(t\sqrt{1 - \Delta_x})}{\sqrt{1 - \Delta_x}}.$$
A solution to the discrete nonlinear Klein–Gordon equation (15) is a fixed point of \( \Lambda \). To apply the contraction mapping argument, we first check that \( \Lambda X \subset X \). Indeed, applying the Strichartz estimate (13) with \((\overline{q}, \overline{r}) = (1, 2)\), we have, for \( u \in X \),

\[
\| \Lambda u \|_X \leq C_{1,2} \left( \| (f, g) \|_{\ell^q_x \times \ell^r_t} + \| u \|_{L^{2s+1}_{t} \ell^{2s+1}_{x}} \right)
\]

\[
\leq C_{1,2} \left( \| (f, g) \|_{\ell^q_x \times \ell^r_t} + \| u \|_X \right)
\]

\[
\leq C_{1,2} \left( \| (f, g) \|_{\ell^q_x \times \ell^r_t} + (2C_{1,2} \epsilon)^{2s+1} \right).
\]

In the second inequality we used that the pair \((2s+1, 2(2s+1))\) is a Strichartz pair and is thus controlled by the \(X\)-norm; in the last inequality we used the assumption on the initial data. For \( \epsilon \) sufficiently small, the last expression is bounded by \( 2C_{1,2} \| (f, g) \|_{\ell^q_x \times \ell^r_t} \), and hence \( \Lambda u \in X \). Similarly, using

\[
\| |u|^2 u - |v|^2 v|_{L^2_x} \leq C_s \| u - v \|_{L^{2s+1}_{t} \ell^{2s+1}_{x}} \left( \| u \|_{L^{2s+1}_{t} \ell^{2s+1}_{x}} + \| v \|_{L^{2s+1}_{t} \ell^{2s+1}_{x}} \right),
\]

one verifies that \( \Lambda : X \to X \) is a contraction.

**Proof of Theorem 4.** Here we are use the contraction mapping argument in the metric space

\[
X = \{ u \in L^\infty_t \ell^2_x : \| u(t) \|_{\ell^2_x} \leq 2C_p \| (t)^{-\sigma(p-2)/p} \| (u(0), u_t(0)) \|_{\ell^p_x \times \ell^p_t} \},
\]

where \((t) := (1 + |t|), 2 \leq p \leq p_d, \sigma = \sigma_d, \) and \( C_p \) denotes the constant in the \( \ell^p \) decay estimate for the linear equation (14). By the latter, we have, for \( u \in X \),

\[
\| \Lambda u \|_X \leq C_p \| (t)^{-\sigma(p-2)/p} \| (u(0), u_t(0)) \|_{\ell^p_x \times \ell^p_t} \]

\[
+ C_p \int_0^t \langle t - \tau \rangle^{-\sigma(p-2)/p} \| u(\tau) \|_{\ell^{2s+1}_{t} \ell^{2s+1}_{x}} \, d\tau.
\]

It suffices to show that the second term is bounded by the first. The assumptions on \( p, s \) and (19) imply that

\[
(2s+1)p' \geq p, \quad \sigma(p-2)(2s+1)/p > 1,
\]

from which it follows that the second term is bounded by (using that \( u \in X \))

\[
C_p(2C_p \| (u(0), u_t(0)) \|_{\ell^p_x \times \ell^p_t})^{2s+1} \int_0^\infty (\tau)^{-\sigma(p-2)/p} \| u(\tau) \|_{\ell^{2s+1}_{t} \ell^{2s+1}_{x}} \, d\tau.
\]

By the second inequality in (13), the integral is convergent and bounded by a constant times \((t)^{-\sigma(p-2)/p} \). Since \( \| (u(0), u_t(0)) \|_{\ell^p_x \times \ell^p_t} \leq \epsilon^{2s} \), we may choose \( \epsilon \) so small that \( \Lambda u \in X \). The contractvity of \( \Lambda \) again follows in a similar manner. Banach’s fix point theorem then gives the existence of a unique solution, together with the a priori bound (20).

**Proof of Theorem 5.** We first remark that, by standard arguments, the Strichartz estimates (14) can be localized in time to an interval \([0,T]\). We then apply the localized estimates with \((q, r) = (\overline{q}, \overline{r}) = (2, \frac{2s}{\sigma-1})\) to the function \( u(t) := e^{it\lambda} \psi \), which satisfies the equation

\[
i \partial_t u + H_0 u = (H_0 - \lambda) u, \quad u(0) = \psi,
\]

where \( H_0 := \sqrt{1-\Delta} \). Here we assume without loss of generality that \( \lambda \) is in the lower half plane; otherwise we consider \( e^{-it\lambda} \psi \). The result is that

\[
c(\lambda, T) \| \psi \|_{\ell^{2s}} \leq C(\| \psi \|_{\ell^2} + c(\lambda, T) \| (H_0 - z) \psi \|_{\ell^{2s}}),
\]
where \( c(z, T) := \| e^{it\hat{z}} \|_{L^2_t(0, T)} \geq T^{1/2} \) by the assumption that \( z \) is in the lower half plane. Dividing by \( c(z, T) \) and letting \( T \to \infty \) yields the claimed bound. \(\square\)

Proof of Corollary 6. Since \((H_0 - \lambda)\psi = -V\psi\), we have, by H"older’s inequality,

\[
\|\psi\|_{L^\sigma} \leq C\|V\psi\|_{L^{\sigma+1}} \leq C\|V\|_{L^\sigma} \|\psi\|_{L^\sigma}.
\]

If \( \epsilon \) is so small that \( C\epsilon < 1 \) we get a contradiction, unless \( \psi = 0 \). \(\square\)

Appendix A. Numerical results

Here we list the numerical solutions of the system (32), (35). These were input in Wolfram alpha (access Oct 25, 2020) in the form

\[
\begin{align*}
7 - (x + y + z) - 1/x - 1/y - 1/z &= 0, \\
(1 - x^2)^2/x^3 + (1 - y^2)^2/y^3 + (1 - z^2)^2/z^3 &= 0, \\
(1 - x^2)y^5z^5(7 - 2(x + y + z)) + x^5(1 - y^2)(1 - z^2)(z^2 - y^2)^2 &= 0.
\end{align*}
\]

Real solutions:

\[
\begin{align*}
x &\approx -0.143939, y \approx 0.144912, z \approx 6.90076, \\
x &\approx -0.143939, y \approx 6.90076, z \approx 0.144912, \\
x &\approx 12.6977, y \approx -2.486, z \approx -0.402253, \\
x &\approx 12.6977, y \approx -0.402253, z \approx -2.486.
\end{align*}
\]

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