The Emergence of a Heavy Quark Family on a Lattice

Giuliano Preparata and She-Sheng Xue

INFN - Section of Milan, Via Celoria 16, Milan, Italy

Abstract

Within the framework of the “Rome approach” for a lattice chiral gauge theory, the four-quark interaction with flavour symmetry is included. We analyse spontaneous symmetry breaking and compute composite modes and their contributions to the ground state energy. As a result, it is shown that the emergence of a heavy quark family is the energetically favoured solution.

June, 1995
PACS 11.15Ha, 11.30.Rd, 11.30.Qc

\(^{a)}\) E-mail address: xue@milano.infn.it
1. The problem of how mass gets generated in the Standard Model (SM) of fundamental interactions, $SU_c(3) \otimes SU_L(2) \otimes U_Y(1)$, is perhaps the most important that is now facing both theoretical and experimental high-energy physics. As well known, the mechanism of mass generation now generally considered, based on the Anderson-Higgs-Kibble mechanism associated to a fundamental local scalar isodoublet Yukawa-coupled to the basic Fermi-fields (quarks and leptons), is also generally believed to be "too ugly" to be really fundamental, leading to the conviction that it must be but the simple surrogate of a deeper, yet to be found and understood, layer of particle interactions. Of particular interest in this direction is the phenomenologically proposed $\bar{t}t$-condensate model\cite{1}\cite{2}, in the light of the experimental observation that the quark family $(t, b)$ is much heavier than others. This model revives the Nambu-Jona Lasinio (NJL) proposal\cite{3} of a 4-fermion interaction. However, it cannot be denied that the addition of an NJL-interaction of the quark family $(t, b)$ only to the usual gauge-invariant Lagrangian density gravely lacks compelling motivation. In this paper, it is shown that the $\bar{t}t$-condensate model emerges as an energetically favoured solution in a lattice-regularized SM with the extension of a four-quark interaction.

The fermion “doubling” phenomenon is a well-known problem arising when fermion fields are defined on a lattice. In fact, the “no-go” theorem of Nielsen-Ninomiya\cite{4}, which stipulates that no simple “lattice transcription” of the bilinear fermion Lagrangian of SM exists, indicates that the SM on a lattice may include extra gauge-symmetric quadrilinear interactions ($S_4$)\cite{5}\cite{6}. Thus, we ask whether the physical incompleteness (lack of mass-generation) of the SM, as formulated in continuous space-time, could not be the symptom of a basic lattice structure of space-time: the arena of physical reality\cite{7}.

In order to remove doublers, for each quark we add Wilson terms ($S_w$)\footnote{Here we are not claiming to solve the problem of chiral gauge theories on a lattice by adding quadrilinear terms} that explicitly breaks chiral gauge symmetries of the SM. For the purpose of obtaining the SM in the low-energy region (target theory), we adopt the “Rome approach”\cite{9} by adding all necessary counterterms ($S_{ct}$) to allow tuning to impose the satisfying of the Ward identities associated with gauge symmetries of the SM\footnote{Since we shall consider only computation of gauge invariant quantities, gauge fixing and ghosts fields and BRST symmetry are not introduced.}. Thus, we have the following Lagrangian for the quark sector:

$$S = S_d + S_4 + S_w + S_{ct},$$

where $S_d$ is the naive lattice transcription of the SM. The Wilson term and the
four-quark interaction are

\[ S_w = \frac{r}{a} \sum_{x, \mu} \bar{\psi}(x) \left( \psi(x + a\mu) + \psi(x - a\mu) - 2\psi(x) \right); \quad (2) \]

\[ S_4 = -G_1 \sum_x \bar{\psi}_L(x) \cdot \psi_R^i(x) \bar{\psi}_R^j(x) \cdot \psi_L^i(x), \quad (3) \]

where “a” is the lattice spacing and “i, j” are indices of quark family and weak isospin.

Due to the fact that the gauge-variant regulator (the Wilson term \( S_w \)) is compensated by the gauge-variant counterterms \( (S_{ct}) \) by forcing satisfying of Ward identities associated with gauge symmetries of the SM, the total action, eq. (1), is symmetric at the cutoff in the sense that it possesses the same symmetries as that of the continuum massless SM. In addition, the total action (1) has global \( U(N_g) \) \( (N_g \) is the number of quark families) flavour symmetry and the four-quark interaction \( S_4 \) introduces the interactions between quarks in different families.

In the light of consideration that gauge interactions should not play an essential role in the mass generation of the heaviest quark family, we approximately eliminate gauge degree of freedoms in the action (1). It turns out to be a much simpler system of Wilson quarks, the four-quark interaction and the simplest mass counterterm that must cancel the hard symmetry breaking term induced by the Wilson term so that the action (1) has chiral symmetry at the cutoff.

2. The action (1) containing the four-quark interaction \( (S_4) \), although it is forced to satisfy Ward identities at the cutoff, is not prevented from developing quark mass terms \( (m\bar{\psi}\psi \text{ dimension-3 operators}) \) that are soft spontaneous symmetry breakings in the sense that the deviation of imposed Ward identities is \( O(a) \). The massive continuum SM \( (\text{with } ma \simeq 0, m \neq 0) \) should be achieved by careful tuning only one “free” parameter\(^\S\) in our lattice action (1) \( (\text{that is } G_1 \text{ to be seen later}) \), at the same time, the anomaly of the theory is restored\(^\S\).

Since the total action enjoys the flavour and weak isospin symmetry, we are allowed to chose a particular basis where the quark self-energy function \( \Sigma^{ij}(p) = \delta^{ij}\Sigma(p) \) is diagonal in the flavour and weak isospin space. In the planar approximation for the large \( N_c \) \( (N_c \gg 1, N_cG_1 \text{ fixed}) \), one has the following Dyson equation for \( \Sigma(p) \),

\[ \Sigma(p) = -M + \frac{r}{a} \omega(p) + 2g_1 \int \frac{\Sigma(l)}{\text{den}(l)} \quad (4) \]

\(^1\text{It should be noted that not only the four-quark interaction } S_4 \text{ but also Wilson term } (S_w) \text{ can induce these dimension-3 operators.}\)

\(^\S\text{The “free” parameter stands for the parameter free from being tuned to satisfy Ward identities.}\)
where \( g_1 a^2 = N_c G_1; l_\mu = q_\mu a, \int_\mu = \int_{-\pi}^\pi \frac{d^4 q}{(2\pi)^4}; w(l) = \sum_\mu (1 - \cos l_\mu) \) and \( \text{den}(l) = \sin^2 l_\mu + (a \Sigma(l))^2 \). We can write \( \Sigma(p) = \Sigma(0) + \frac{w(p)}{a} \) and get

\[
\Sigma(0) = 2 g_1 \int_\mu \frac{\Sigma(0)}{a \text{den}(l)}, \quad (5)
\]

\[
M = 2 g_1 \int_\mu \frac{a w(l)}{\text{den}(l)}; \quad (6)
\]

The first equation is a gap equation of the NJL-type, which has non-trivial solution: \( \Sigma(0) \neq 0 \) for \( g_1 > g_1^c \) (the critical value). The second equation indicates that the mass counterterm “M” completely cancel the hard breaking “\( \frac{1}{a} \)” term contributed by the doublers (seeing there is a factor \( w(l) \) inside the integrand \( (5) \)), so as to preserve the chiral symmetry at the high energy region. The correspond Ward identity that guarantees this cancelation is\( (3) \),

\[
\langle \bar{\psi}_L(0) \psi_R(x) \rangle = 0 \quad (x \gg a) \quad (7)
\]

which must be obeyed up to powers of the lattice spacing \( O(a) \). This means that \( a \Sigma(0) \) must be of the order of the lattice spacing \( \sim O(a) \) and gives rise to a soft breaking \( \Sigma(0) \bar{\psi} \psi \) that is totally irrelevant in the high energy region. Thus, we make a consistent fine tuning on \( g_1 \) around its critical line \( g_1^c(r) \) (Fig.(1)) so that \( a \Sigma(0) \sim O(a) \) at the same time as forcing cancellation \( (3) \) to be obeyed. As a result, owing to the symmetric action \( (1) \), we have obtained a soft spontaneous symmetry breaking \( \Sigma(0) = m \) to generate quark mass term \( m \bar{\psi} \psi \), at the same time doublers are removed from the low energy spectrum and the anomaly should be reinstated\( (3) \). The feature of the fine tuning of \( g_1 \), which is very unnatural due to there being no symmetry protection, will not be discussed in this paper.

3. Composite modes are bound to be produced once the spontaneous symmetry breakdown occurs \( m \neq 0 \), is evident from the non-trivial solutions of the gap equation \( (3) \). In order to see these modes, we calculate the four-quark scattering amplitudes associated with the vertex \( S_4 \) within the planar approximation. This calculation is straightforward and we just present the results. The composite modes in the pseudo-scalar channel \( \Gamma_p(q^2) \) and the scalar channel \( \Gamma_s(q^2) \) are

\[
\Gamma_p(q^2) = \frac{1}{2} \frac{1}{I_p(q^2) A(q^2)}; \quad (8)
\]

\[
\Gamma_s(q^2) = \frac{1}{2} \frac{1}{4 N_c \int k \frac{(ma + rw(k))^2}{D(k,qa)} + I_s(q^2) A(q^2)}; \quad (9)
\]

where

\[
A(q^2) = \sum_\mu \left( \frac{2}{a} \sin \frac{q_\mu a}{2} \right)^2,
\]

3
\[ I_p(q) = \frac{N_c}{4} \int_k \frac{c^2(k) + (r)^2 s^2(k)}{D(k, qa)}, \]
\[ I_s(q) = \frac{N_c}{4} \int_k \frac{c^2(k)}{D(k, qa)}. \]

where \( c^2 = \sum_\mu \cos^2 k_\mu \), \( s^2 = \sum_\mu \sin^2 k_\mu \) and \( D(k, qa) = \text{den}(k + \frac{qa}{2})\text{den}(k - \frac{qa}{2}) \).

We find massless Goldstone modes that should be candidates for the longitudinal modes of massive gauge bosons, and scalar modes that should be a candidate for Higgs particles.

4. So far, we know that the gap equation (5) can have the following three possible solutions for the quark mass matrix in the quark family space (electrical charges \( Q = \frac{2}{3}, -\frac{1}{3} \))

\[
\begin{pmatrix}
 m & 0 & 0 \\
 0 & m & 0 \\
 0 & 0 & m \\
\end{pmatrix}_{n=36} (i); \quad
\begin{pmatrix}
 0 & 0 & 0 \\
 0 & m & 0 \\
 0 & 0 & m \\
\end{pmatrix}_{n=16} (ii); \quad
\begin{pmatrix}
 0 & 0 & 0 \\
 0 & 0 & 0 \\
 0 & 0 & 0 \\
\end{pmatrix}_{n=4} (iii),
\]

where \( n \) stands for the number of Goldstone modes or scalar modes associated with each solution. In order to ascertain which solution is physically realizable, we turn to the computation of the ground state energy. In the one-loop approximation \( (O(N_c)) \), the effective potential upon the occurrence of this soft spontaneous symmetry breaking is given by

\[
V(m, r) = \frac{m^2}{G_1} - N_c tr \int l \ell n \left\{ \frac{\gamma_\mu \sin l_\mu}{a} + (m + \frac{r}{a} w(l)) \right\} + \cdots,
\]

and the difference between the energy of the symmetric vacuum and broken vacuum \( \Delta E_o = V(m, r) - V(0, 0) \) is given by

\[
\Delta E_o = -\frac{2N_g^m}{a^4} \int \sum_{k=1}^\infty \frac{2N_c}{k+1} \left[ \frac{(ma + rw(l))^2}{s^2(l) + (ma + rw(l))^2} \right]^{k+1},
\]

which is obtained from (12) by considering the gap equation (5,6). The negative \( \Delta E_o \) shows that the non-trivial solutions of the gap equations characterize a chirally asymmetric vacuum that has an energy density lower than that of the symmetric vacuum. However, it shows that more quark families \( (N^m_g) \) acquire masses the lower ground state energy is, which leads us to select the first quark mass matrix (i) in eq.(11). A phenomenological disaster occurs, for quarks get equally massive and, furthermore, 36 Goldstone modes appear.

On the other hand, noticing that composite bosons give a positive energy density to such broken “vacua” and this positive contribution certainly increases as the
number of composite modes increases, we turn to the computation of the total vacuum energy \( \Delta E \) (vacuum bubble diagrams) containing both quark and composite mode contributions on the basis of gap equations (5, 6),

\[
\Delta E = - \left[ \ell n \int f \exp(-S_{\text{eff}}(m, r)) - \ell n \int f \exp(-S_{\text{eff}}(0, 0)) \right],
\]

where \( S_{\text{eff}}(m, r) \) is the effective Wilson action over the ground state. The details of the calculation are lengthy and will not be reported in this letter, we just present the result (\( \Delta E^\circ \) is \( O(N_c) \) and the second term \( O(N_c^0) \)):

\[
\Delta E \simeq \Delta E^\circ - (2N_g m)^2 \left[ 1 - e^{-\Delta E^\circ} \right]
\]

\[
\cdot \left[ -4 + \int_t (4g_1 \tilde{\Gamma}_s(l) + \frac{1}{4g_1 \tilde{\Gamma}_s(l)}) + \int_t (4g_1 \tilde{\Gamma}_p(l) + \frac{1}{4g_1 \tilde{\Gamma}_p(l)}) \right],
\]

where

\[
\tilde{\Gamma}_p(l) = \frac{a^2 I_p(l) A(l^2)}{4N_c},
\]

\[
\tilde{\Gamma}_s(l) = \frac{a^2}{4N_c} \left( \frac{4N_c}{a^2} \int_k \frac{[ma + rw(k)]^2}{D(k, l)} + I_s(l) A(l^2) \right).
\]

Combining positive and negative contributions in eq. (13) and putting \( ma \simeq 0 \), \( N_c = 3 \) and \( g_1 \simeq g_1(r) \) obtained from eq. (5), we find (Fig. (2)) that the solution \( (iii)N_g^m = 1 \) in eq. (11) is the energetically favoured solution. This shows that, through this mechanism, only one quark family acquires mass and the other quark families remain massless. We thus give the names of top and bottom to this massive quark family. The three Goldstone modes \( \langle \bar{t} \gamma_5 b \rangle, \langle \bar{b} \gamma_5 t \rangle \) and \( \frac{1}{\sqrt{2}}(\langle \bar{t} \gamma_5 t \rangle - \langle \bar{b} \gamma_5 b \rangle) \) should become the longitudinal modes of the intermediate gauge bosons.

5. As has been seen, this research provides evidence and a “raison d’être” for the hierarchy structure of the quark spectrum and thus a theoretical motivation for the \( \bar{t}t \)-condensate model. However, composite scalar modes disappear from the low-energy spectrum since their masses are proportional to \( \frac{r}{a} (4m_s^2 = 4m^2 + 0.8r\frac{m_s}{a} + 0.9\frac{r^2}{a^2}) \) obtained from eq. (9)), since in this study, there is no symmetry to protect their masses from being contributed to by doublers. Whether Higgs masses are pinned down by gauge interactions or other reasons will be the subject of future work. If we turn on gauge fields and couple them to quarks in the action (1), the “Rome approach” should be very powerful to establish the link between heavy quark masses and intermediate gauge bosons (\( W^\pm, Z^0 \)) masses[11]. As for splitting the degeneracy of top and bottom quark masses [12] and the mass generation of other light quark masses, we suspect that this should eventually occur owing to gauge interactions and mixing between quark families.
References

[1] Y. Nambu, in New theories in physics, Proc. XI Int. Symp. on elementary
particle physics, eds. Z. Ajduk, S. Pokorski and A. Trautman (World Scientific,
Singapore, 1989).

[2] W.A. Bardeen, C.T. Hill and M. Linder, *Phys. Rev.* D41 (1990) 1647;
V.A. Miranski, M. Tanabashi and K. Yamawaki, *Mod. Phys. Lett.* A4 (1989)
1043; *Phys. Lett.* B221 (1989) 117;
W.J. Marciano, *Phys. Rev. Lett.* 62 (1989) 2793; *Phys. Rev.* D41 (1990) 219;
W.A. Bardeen, C.T. Hill and S. Love, *Nucl. Phys.* B323 (1989) 493,
W.A. Bardeen, S. T. Love and V.A. Miransky, *Phys. Rev.* D42 (1990) 3514;
M. Lindner, *Int. Mod. Phys.* A8 (1993) 2167.

[3] Y. Nambu and G. Jona-Lasinio, *Phys. Rev.* 122 (1961) 345.

[4] H.B. Nielsen and M. Ninomiya, *Nucl. Phys.* B185 (1981) 20, *ibid.* B193 (1981)
173, *Phys. Lett.* B105 (1981) 219.

[5] E. Eichten and J. Preskill, *Nucl. Phys.* B268 (1986) 179.

[6] G. Preparata and S.-S. Xue, *Phys. Lett.* B264 (1991) 35; *Nucl. Phys.* B26
Proc. Suppl.) (1992) 501; *Nucl. Phys.* B30 (Proc. Suppl.) (1993) 647.

[7] We also recall that proposals by C.W. Misner, K.S. Thorne and J.A. Wheeler,
*Gravitation* (Freeman, San Fransisco, 1973) exist, based on the violent quantum
fluctuations of the metric field at space-time distances of the order of the Planck-
length $a_p \sim 10^{-33}$cm, that endow space-time with a “foam-like” structure of
grain-size $a_p$, thus making it equivalent to a 4-dimensional random lattice of
lattice constant $a_p$, which we may call a Planck lattice.

[8] K. Wilson, in *New phenomena in subnuclear physics* (Erice, 1975) ed.
A. Zichichi (Plenum, New York, 1977).

[9] A. Borrelli, L. Maiani, G.C. Rossi, R. Sisto and M. Testa, *Nucl. Phys.* B333
(1990) 335; L. Maiani, G.C. Rossi, R. Sisto and M. Testa, *Phys. Lett.* B221
(1989) 360; *ibid* 261 (1991) 479.

[10] L. H. Karsten and J. Smit, *Nucl. Phys.* B144 (1978) 536.

[11] G. Preparata and S.-S. Xue, *Phys. Lett.* B329 (1994) 87.

[12] G. Preparata and S.-S. Xue, *Phys. Lett.* B325 (1994) 161.
Figure Captions

**Figure 1:** The critical line $g_1^c(r)$, where $m = 0$, in terms of $r$.

**Figure 2:** The vacuum energy $\Delta E(r)$ for different massive quark families $N_g^m$. 