On piecewise pluriharmonic functions*

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We extend some results on piecewise linear functions on \( \mathbb{C}^n \) to piecewise pluriharmonic functions on any complex manifold. We construct a ring generated by currents \( h \) and \( dd^c h \), where \( \{ h \} \) is a finite set of piecewise pluriharmonic functions. We prove that, with some restrictions on the set \( \{ h \} \), the map \( \{ h \mapsto dd^c h, \ dd^c h \mapsto 0 \} \) can be continued to the derivation on the ring. As a corollary, the current \( dd^c g_1 \land \cdots \land dd^c g_k \) depends on the product of piecewise pluriharmonic functions \( g_1, \cdots, g_k \) only.

1 Results.

A function \( g: M \to \mathbb{R} \) on a complex manifold \( M \) is called pluriharmonic if \( dd^c g = 0 \) (recall that \( d^c g(x_t) = dg(\sqrt{-1}x_t) \)). Another definition: the function \( g \) is a real part of some holomorphic function in some neighborhood of any point \( x \in M \).

Definition 1. Continuous function \( g: M \to \mathbb{R} \) on \( n \)-dimensional complex manifold \( M \) is called piecewise pluriharmonic (or PPH-function) if \( g \) is pluriharmonic on any closed \( 2n \)-dimensional simplex of some locally finite triangulation of the manifold \( M \) (see subsection 2.1).

Piecewise linear functions on the space \( \mathbb{C}^n \) are the simplest examples of PPH-functions. Piecewise linear functions are used in convex geometry, algebraic geometry, and complex analysis [2]-[9]. We extend some results on piecewise linear functions on \( \mathbb{C}^n \) [2] to PPH-functions on any complex manifold.

There exists a nonconstant PPH-function on any complex manifold. Indeed, let \( h \) be a piecewise linear function on the space \( \mathbb{C}^n \) and \( h = 0 \) outside some small neighborhood of zero. Let \( H \) be a function on \( M \) such that \( H = h \) on the coordinate neighborhood of some point \( x \in M \) and \( H = 0 \) outside this neighborhood. The function \( H \) is piecewise pluriharmonic.

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**Definition 2.** Let \( A \) be some vector space consisting of PPH-functions on an \( n \)-dimensional complex manifold \( M \). The vector space \( A \) is called the **constructive space** if for any finite set of elements \( H = \{ h_i \in A \} \) there exists a triangulation \( \mathcal{P}_H \) of \( M \) such that each function \( h_i \) is pluriharmonic on every closed \( 2n \)-dimensional simplex \( \Delta \in \mathcal{P}_H \).

In the sequel, all PPH-functions are the elements of some fixed constructive space \( A \).

**Example 1.** The space of piecewise linear functions on \( \mathbb{C}^n \) is the constructive space.

**Question 1.** Is the space of all PPH-functions constructive?

Let \( h_i: M \to \mathbb{R} \) be a function on a complex \( n \)-dimensional manifold \( M \). The mixed Monge-Ampère operator on \( M \) of degree \( k \) is (by definition) the map \( (h_1, \ldots, h_k) \mapsto dd^c h_1 \wedge \cdots \wedge dd^c h_k \). If \( h_1, \ldots, h_k \) are continuous plurisubharmonic functions, then [10] the Monge-Ampère operator value \( dd^c h_1 \wedge \cdots \wedge dd^c h_k \) is well defined as a current (that is a functional on the space of smooth compactly supported differential \((2n - 2k)\)-forms). This means that if the sequence of smooth plurisubharmonic functions \((f_i)\) converges locally uniformly to \( h \), then the sequence of currents \( dd^c (f_i) \wedge \cdots \wedge dd^c (f_k) \) converges to the limit current, independent of the choice of approximation. This limit current is the current of measure type, i.e., may be continued to a functional on the space of continuous compactly supported forms. It follows that any polynomial in the variables \( h_1, \ldots, h_p, dd^c g_1, \ldots, dd^c g_q \) with continuous plurisubharmonic functions \( h_i \) and \( g_i \) gives the well-defined current on \( M \).

It is easy to prove that any PPH-function can be locally written as a difference of two continuous plurisubharmonic functions. It follows that above defined currents are well defined for PPH-functions \( h_1, \ldots, h_p, g_1, \ldots, g_q \) also.

**Theorem 1.** Let \( A \) be a ring of currents, generated by currents \( h \) and \( dd^c h \) with \( h \in A \). There exists a derivation \( \delta \) on the ring \( A \) such that \( \delta(h) = dd^c h \) and \( \delta(dd^c h) = 0 \) for any PPH-function \( h \in A \).

**Remark 1.** Let \( h_i \) be PPH-functions and \( F = h_0 dd^c h_1 \wedge \cdots \wedge dd^c h_k \); then \( \delta(F) = dd^c F \). But \( \delta(h^2) = 0 \) and \( dd^c (h^2) \neq 0 \) for any nonconstant pluriharmonic function \( h \).

**Corollary 1.** \( \delta^k(h_1 \cdots h_k) = k! dd^c h_1 \wedge \cdots \wedge dd^c h_k \) for any PPH-functions \( h_1, \ldots, h_k \).
Corollary 2. The current $dd^c h_1 \wedge \cdots \wedge dd^c h_k$ depends on the product of PPH-functions $h_1, \cdots, h_k$ only.

Corollary 3. Let the space $A$, generated by the PPH-functions $h_1, \cdots, h_k$, be a constructive space and let $h = \max(h_1, \cdots, h_k)$. Suppose that $h \in A$. Then $dd^c(h - h_1) \wedge \cdots dd^c(h - h_k) = 0$.

The proof of theorem 1 uses (as the proofs of similar theorems for piecewise linear functions in [1, 2]) a construction of $k$-th corner locus. Such corner loci (in some more special situation) were constructed in [1].

2 PPH-cycles and corner loci.

2.1 P-cycles and operator $D_c$.

A simplex $\Delta$ in a complex $n$-dimensional manifold $M$ is the image of a smooth nonsingular embedding $\Delta^{2n} \to M$, where $\Delta^{2n}$ is the standard closed $2n$-dimensional simplex. A locally finite set $\mathcal{P}$ of simplices in $M$ is called a triangulation if $\bigcup_{\Delta \in \mathcal{P}} \Delta = M$ and intersection of any two simplices from $\mathcal{P}$ is their common (may be empty) face. Any face of any simplex $\Delta \in \mathcal{P}$ is called a cell of triangulation $\mathcal{P}$. An odd form $\omega$ on the cell $\Delta$ is called the frame of $\Delta$ (recall that the odd form on a manifold with orientations $\alpha, \beta$ is a pair of forms $\omega_\alpha, \omega_\beta$ such that $\omega_\alpha = -\omega_\beta$). Let $k$-dimensional chain (or $k$-chain) be a map $X: \Delta \mapsto X_\Delta$ of the set of all $k$-dimensional cells to their frames. Let the map $\partial$ take each $k$-chain $X$ to $(k - 1)$-chain $\partial X$, where

$$(\partial X)_\Lambda = \sum_{\Delta \supset \Lambda, \dim \Delta = k} X_\Delta,$$

where the orientations of the cells $\Lambda$ and $\Delta$ agreed as usual. A $k$-chain $X$ is called a $k$-cycle if $\partial X = 0$.

Corollary 4. For any $k$-chain $X$ the $(k - 1)$-chain $\partial X$ is a cycle.

In the sequel we assume that $k \geq n$.

Let $\Delta, T^x_\Delta$, and $C^x_\Delta$ be (respectively) a $k$-dimensional cell, the tangent space of $\Delta$ at the point $x \in \Delta$, and the maximal complex subspace of $T^x_\Delta$. Say that the point $x \in \Delta$ is nondegenerate if $\text{codim}_C C^x_\Delta = \text{codim} T^x_\Delta$. For $k = 2n, 2n - 1$ any point of $k$-dimensional cell is nondegenerate. If $x$ is nondegenerate, then $\dim T^x_\Delta - \dim_R C^x_\Delta = 2n - k$. If $x$ is degenerate, then the form $X_\Delta$ (from definition 3) is zero at $x$. 

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Definition 3. A $k$-cycle $X$ is called $P$-cycle if the following conditions hold:

1. $\deg X_\Delta = 2n - k$
2. $X_\Delta = \sum_{1 \leq j < \infty} g^j_\Delta W^j_\Delta$, where $(2n-k)$-forms $W^j_\Delta$ are closed
3. $W_\Delta(\xi_1, \ldots, \xi_{2n-k}) = 0$ for any $x \in \Delta$ and any $(\xi_1, \ldots, \xi_{2n-k}) \subset T^\xi_\Delta$ if

Let $X$ be a $k$-dimensional $P$-cycle. Define the $(k-1)$-cycle $D_cX = \partial Y$, where $Y$ is a $k$-chain such that

$$Y_\Delta = \sum_{1 \leq j < \infty} d^c G^j_\Delta \wedge W^j_\Delta,$$

where $G^j_\Delta$ are smooth functions in some neighborhood of $\Delta$ such that $G^j_\Delta(x) = g^j_\Delta(x)$ for any $x \in \Delta$.

Corollary 5. If $k = n$ then $D_cX = 0$.

Proposition 1. (a) The $(k-1)$-cycle $D_cX$ does not depend on the choice of functions $G^j_\Delta$ in (1) and on the choice of functions $g^j_\Delta$ in decomposition $X_\Delta = \sum_{1 \leq j < \infty} g^j_\Delta W^j_\Delta$ from definition 3.

(b) $D_cX_\Lambda(\xi_1, \ldots, \xi_{2n-k+1}) = 0$ for $x \in \Lambda$ and $(\xi_1, \ldots, \xi_{2n-k+1}) \subset T^\xi_\Lambda$ if

Proof. If $F(x) = 0$ for any $x \in \Delta$, then the form $d^c F$ is zero on the subspace $C^\xi_\Delta \subset T^\xi_\Delta$. Using definition 3 (3) we get that the form $d^c F \wedge W^j_\Delta$ is zero on $\Delta$. It follows that the form $(D_cX)_\Lambda$ does not depend on the choice of functions $G^j_\Delta$.

Now we prove that the form $(D_cX)_\Lambda$ does not depend on the choice of functions $g^j_\Delta$. Let $x_1 \wedge \cdots \wedge x_{2n-k+1} \neq 0$, where $x_i \in T^\xi_\Lambda$ and $x_1 \in C^\xi_\Lambda$. If $(\sum_j g^j_\Delta W^j_\Delta) = 0$, then

$$\sum_j d^c G^j_\Delta \wedge W^j_\Delta(x_1, x_2, \cdots) = \sum_j dG^j_\Delta \wedge W^j_\Delta(ix_1, x_2, \cdots) =$$

$$\sum_j d g^j_\Delta \wedge W^j_\Delta(ix_1, x_2, \cdots) = d \left( \sum_j g^j_\Delta \wedge W^j_\Delta \right)(ix_1, x_2, \cdots) = 0.$$

The first equality follows from the definition of operator $d^c$ and from definition 3, the third – from the closedness of forms $W^j_\Delta$. Assertion (a) is proved.

Let $\dim \Lambda = k - 1$ and let $x_1 \wedge \cdots \wedge x_{2n-k+1} \neq 0$, where $x_i \in T^\xi_\Lambda$ and
We prove that \((DcX)_\Lambda(x_1, \cdots, x_{2n-k+1}) = 0\).

\[
(DcX)_\Lambda(x_1, x_2, \cdots) = \sum_{\Delta \supset \Lambda, \dim \Delta = k} \sum_{j} dG^j_{\Delta} \wedge W^j_{\Delta}(x_1, x_2, \cdots) = \sum_{\Delta \supset \Lambda, \dim \Delta = k} \sum_{j} dG^j_{\Delta} \wedge W^j_{\Delta}(ix_1, x_2, \cdots) = d \left( \sum_{\Delta \supset \Lambda, \dim \Delta = k} X_{\Delta}(ix_1, x_2, \cdots) \right) = d(\partial X)_\Lambda = 0.
\]

The first equality is a definition of \((k-1)\)-chain \(\partial Y\). Other equalities follow from the definition of operator \(d^c\), the closedness of forms \(W^j_{\Delta}\), and the closedness of the \(k\)-chain \(X\).

**Remark 2.** \(DcX\) is not necessarily a P-cycle.

### 2.2 Corner loci of PPH-polynomials.

**Definition 4.** Let \(H = \{h_1, \cdots, h_q\}\) be a basis of the constructive space \(A\), \(P(x_1, \cdots, x_q)\) be a polynomial of degree \(m\) in the variables \(x_1, \cdots, x_q\). The function \(P(h_1, \cdots, h_q)\) is called a PPH-polynomial of degree \(m\).

The degree of PPH-polynomial is not uniquely defined. But PPH-polynomials of degree 0 are constants and PPH-polynomials of degree 1 are PPH-functions.

Below we fix the set \(H\) and the triangulation \(\mathcal{P}_H\). The restriction of PPH-polynomial \(P\) to standardly oriented \(2n\)-cells of triangulation \(\mathcal{P}_H\) give the \(2n\)-dimensional P-cycle \(X^P\). Say that \((2n-1)\)-cycle \(DcX^P\) is the corner locus of PPH-polynomial \(P\).

**Example 2.** Corner locus of PPH-function. Any \((2n-1)\)-dimensional cell \(\Delta\) of triangulation \(\mathcal{P}_H\) is a common face of \(2n\)-dimensional cells \(B^+\) and \(B^-\). Let \(h \in A\) and \(h^+_\Delta = h|_{B^+}\), \(h^-_\Delta = h|_{B^-}\). The ordering of the pair \((B^+, B^-)\) sets the coorientation of \(\Delta\). The standard orientation of \(M\) and the orientation of \(\Delta\) together set the orientation of the cell \(\Delta\). Using this orientation put

\[
(DcX^h)_\Delta = d^c h^+_\Delta - d^c h^-_\Delta.
\]

**Corollary 6.** For any PPH-polynomial \(P\)

\[
(DcX^P)_\Delta = \sum_{1 \leq i \leq q} \frac{\partial P}{\partial x_i}(h_1, \cdots, h_q)(d^c(h_i))^+_\Delta - d^c(h_i)^-\Delta),
\]

where 1-forms \(d^c(h_i)^\pm\Delta\) on the \((2n-1)\)-dimensional oriented cell \(\Delta\) are defined in the text of example 2.
Corollary 7. The corner locus $D_c X^P$ is a P-cycle.

Definition 5. The P-cycle $D_c^k X^P$ is called a $k$-th corner locus of PPH-polynomial $P$.

The validity of definition 5 is based on the assertion (1) of lemma 1.

Below we use the following notation:

1. $I = \{i_1, \ldots, i_q; i_j \geq 0\}, |I| = i_1 + \cdots + i_q, I! = i_1! \cdots i_q!, x^I = x_1^{i_1} \cdots x_q^{i_q}$.
2. If $i_p > 0$ then $I \setminus p = \{i_1, \ldots, i_p - 1, \ldots, i_q\};$ else $I \setminus p = \emptyset$.
3. If $I \neq \emptyset$ then $P_I(h_1, \ldots, h_q) = \frac{\partial^{|I|}P}{\partial x_1^{i_1} \cdots \partial x_q^{i_q}}(h_1, \ldots, h_q);$ else $P_I(h_1, \ldots, h_q) = 0$.
4. $\Delta \mapsto \tilde{\Delta}$ is some fixed mapping of the set of cells of triangulation $\mathcal{P}_H$ into itself such that
   (a) $\Delta \subset \tilde{\Delta}$
   (b) $\dim \tilde{\Delta} = 2n$
   (c) if $\dim \Delta = 2n$ then $\tilde{\Delta} = \Delta$.
5. $H^I_\Delta$ is the restriction of pluriharmonic function $(h_i)_{\tilde{\Delta}}$ to some neighborhood of the cell $\Delta$
6. $G^i_{\Delta, \Gamma}$ is the restriction of pluriharmonic function $(h_i)_{\Gamma}$ to some neighborhood of the cell $\Delta$, where $\Gamma$ ranges over the set of $2n$-dimensional cells containing $\Delta$.

Lemma 1. For any PPH-polynomial $P$ and any $k \geq 1$

(1) $D_c^k X^P$ is a P-cycle.

(2) If $\dim \Delta = 2n - k$ then

$$(D_c^k X^P)_\Delta = \sum_{|I|=k} P_I(h_1, \ldots, h_q)Q^k_{I, \Delta},$$

where $Q^k_{I, \Delta}$ is a polynomial of degree $k$ in variables $\{d^{i}G^i_{\Delta, \Gamma}\}$ and is independent of the choice of a polynomial $P$.

(3) If $|I| = k$ and $P = x^I$ then $(D_c^k X^P)_\Delta = I!Q^k_{I, \Delta}$.
Proof. The proof is by induction on $k$. For $k = 1$, all the assertions follow from corollary 6. If the assertion is true for $k - 1$ then

$$
(D^k_cX^P)_\Lambda = \sum_{\Delta \supset \Lambda, \dim \Delta = 2n-k+1} \sum_{|l| = k-1} d^cP_l(H^1_\Delta, \cdots, H^q_\Delta)Q_{I,l,\Delta}^{k-1} = \\
\sum_{\Delta \supset \Lambda, \dim \Delta = 2n-k+1} \sum_{|l| = k} \left( \sum_{1 \leq i \leq q} \frac{\partial P_l}{\partial x_i}(h_1, \cdots, h_q)d^cH^i_\Delta \right) \wedge Q_{I,l,\Delta}^{k-1} = \\
\sum_{\Delta \supset \Lambda, \dim \Delta = 2n-k+1} \sum_{|l| = k} P_l(h_1, \cdots, h_q) \sum_{1 \leq i \leq q} d^cH^i_\Delta \wedge Q_{I,l,\Delta}^{k-1} = \\
\sum_{|l| = k} P_l(h_1, \cdots, h_q) \sum_{\Delta \supset \Lambda, 1 \leq i \leq q} d^cH^i_\Delta \wedge Q_{I,l,\Delta}^{k-1},
$$

where the product $d^cH^i_\Delta \wedge Q_{I,l,\Delta}^{k-1}$ is an odd form (as a product of even and odd forms) restricted to the cell $\Lambda$. So we can put

$$
Q^k_{I,\Lambda} = \sum_{\Delta \supset \Lambda, 1 \leq i \leq q} d^cH^i_\Delta \wedge Q_{I,l,\Delta}^{k-1}.
$$

Assertion (2) is proved.

Applying assertion (2) to $P = x^I$ we get assertion (3).

Combining assertion (3) and proposition 1 (b) we obtain assertion (1).

Corollary 8. If $k \geq n$ then $D^k_cX^P = 0$.

2.3 Corner loci of PPH-cycles.

Let $i = 1, \cdots, p$ and $X_i$ be a $k$-dimensional P-cycle (definition 3). For PPH-polynomials $P^1, \cdots, P^p$ we define the $k$-cycle $X = P^1X_1 + \cdots + P^pX_p$ as $X_\Delta = P^1(X_1)_\Delta + \cdots + P^p(X_p)_\Delta$.

Definition 6. The $k$-cycle $X$ is called a PPH-cycle if the forms $(X_i)_\Delta$ are closed. Put $\deg X = \max_i \deg P^i$.

Corollary 9. Any PPH-cycle is a P-cycle.

Corollary 10. PPH-cycles form the module over the ring of PPH-polynomials.

Definition 7. The cycle $D_cX$ is called a corner locus of PPH-cycle $X$.

Proposition 2. The corner locus $D_cX$ of any $k$-dimensional PPH-cycle $X$ is a $(k-1)$-dimensional PPH-cycle and $\deg D_cX = \deg X - 1$. 

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**Proof.** Let $Y$ be a PPH-cycle of degree 0. Using the definition of operator $D_c$, we have

$$(D_c(h_j Y))_\Lambda = \sum_{\Delta \supset \Lambda, \dim \Delta = k} d^c H^j_\Delta \wedge Y_\Delta,$$

where the functions $H^j_\Delta$ defined in subsection 2.2. The forms $d^c H^j_\Delta \wedge Y_\Delta$ are closed. It follows that $D_c(h_j Y)$ is PHP-cycle of degree 0. Now it remains to observe that if $X = P^1 X_1 + \cdots + P^p X_p$ then

$$D_c X = \sum_{1 \leq i \leq p, 1 \leq j \leq q} \frac{\partial P_i}{\partial x_j}(h_1, \cdots, h_q) D_c(h_j X_i).$$

**Corollary 11.** If $h \in A$ then $D_c(h^k X) = kh^{k-1} D_c(h X)$.

### 2.4 PPH-cycles as currents.

Let $X$ be a $k$-chain. Now suppose $\bar{X}$ is a current such that

$$\bar{X}(\phi) = \sum_{\Delta \in \mathcal{P}_H, \dim \Delta = k} \int_\Delta X_\Delta \wedge \phi.$$ 

**Lemma 2.** Let $X$ be a $k$-cycle such that the forms $X_\Delta$ are closed. Then the current $\bar{X}$ is closed.

**Proof.**

$$d\bar{X}(\psi) = \bar{X}(d\psi) = \sum_{\Delta \in \mathcal{P}_H, \dim \Delta = k} \int_\Delta X_\Delta \wedge d\psi = \sum_{\Delta \in \mathcal{P}_H, \dim \Delta = k} \int_\Delta d(X_\Delta \wedge \psi) =$$

$$\sum_{\Lambda \in \mathcal{P}_H, \dim \Lambda = k-1} \sum_{\Delta \supset \Lambda, \dim \Delta = k} \int_\Lambda X_\Delta \wedge \psi = \sum_{\Lambda} \int_\Lambda (\partial X)_\Lambda \wedge \psi = (\partial X)(\psi) = 0$$

**Corollary 12.** Let $X$ be a $k$-dimensional PPH-cycle of degree 0. Then the current $\bar{X}$ is closed.

**Proposition 3.** Let $h \in A$, $X$ be a $k$-dimensional PPH-cycle of degree 0. Then $D_c(h X) = d d^c (h X)$. 

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Proof.

\[ dd^c (hX)(\psi) = - \sum_{\Delta \in \mathcal{P}_H, \dim \Delta = k} \int_{\Delta} hX_{\Delta} \wedge dd^c \psi = \]

\[ (-1)^{k+1} \sum_{\Delta} \int_{\Delta} (d(hX_{\Delta} \wedge d^c \psi) - dh \wedge X_{\Delta} \wedge d^c \psi) = \]

\[ (-1)^k \sum_{\Delta} \int_{\Delta} dh \wedge X_{\Delta} \wedge d^c \psi = (-1)^{k+1} \sum_{\Delta} \int_{\Delta} d^c h \wedge X_{\Delta} \wedge d \psi = \]

\[ \sum_{\Delta} \int_{\Delta} d(d^c h \wedge X_{\Delta} \wedge \psi) = \sum_{\Lambda \subseteq \Delta, \dim \Lambda = k-1} \int_{\Lambda} d^c h \wedge X_{\Delta} \wedge \psi = \overline{D_c(hX)}(\psi). \]

(3)

The formula (3) consists of seven equalities. We shall comment to each one.

1. Determination of the current derivative and the identity \( dd^c = -d^c d \).
2. The closedness of the form \( X_{\Delta} \).
3. The Stokes formula and the \( k \)-chain \( hX \) closedness.
4. Let the form \( \psi \) bidegree is \( (k - n - 1, k - n - 1) \) (the values of the current \( dd^c (hX) \) on homogeneous components of other degrees are zero). Suppose \( x \in \Delta \) and \( \xi_1, \cdots, \xi_{2k-2n} \) is a basis of the (real) vector space \( \mathbb{C}^x_{\Delta} \). It is easy to prove that

\[ (dh \wedge d^c \psi)(\xi_1, \cdots, \xi_{2k-2n}) = -(d^c h \wedge d \psi)(\xi_1, \cdots, \xi_{2k-2n}). \]

Hence \( (dh \wedge X_{\Delta} \wedge d^c \psi) = -(d^c h \wedge X_{\Delta} \wedge d \psi) \).
5. The closedness of the forms \( d^c h \) and \( X_{\Delta} \).
6. The Stokes formula.
7. The determination of \( D_c \).

3 Theorem 1 (proof).

Below we use the following notation (with the notation of subsection 2.2):

1. \( \mathcal{B} \) is the ring of PPH-polynomials.
2. \( \mathcal{B}' \) is the symmetric algebra of the space \( A \).
3. $T$ is the ring (with the unity) generated by currents $ddc h$, where $h \in A$.

4. $T' = T''/I$, where $T''$ is the symmetric algebra of the space of currents $ddc h$, where $h \in A$, and $I$ is the ideal generated by elements of degree $> n$.

5. $A' = B' \otimes T'$ is the tensor product of the rings.

6. $\delta'$ is the derivation on the ring $A'$ such that
   \[
   \delta'(h \otimes 1) = 1 \otimes ddc h, \quad \delta'(1 \otimes ddc h) = 0
   \]
   for any $h \in A$.

7. $\pi : A' \to A$ is the ring homomorphism such that
   
   \[
   h_i \otimes 1 \mapsto h_i, \quad 1 \otimes ddc h_i \mapsto ddc h_i
   \]

Theorem 1 follows from proposition 4. Proposition 4 says that the derivation $\delta'$ survives on the quotient ring $A$ of the ring $A'$.

**Proposition 4.** Let $\pi(F) = 0$; then $\pi\delta'(F) = 0$.

Proposition 4 follows from (see below) proposition 5.

**Theorem 2.** (1) For any $\tau \in A$ there exists a unique PPH-cycle $X = \iota(\tau)$ such that $\tau = \overline{X}$.

(2) If $\tau \in T$, $\deg \tau = 2k$ then $\iota(\tau)$ is a $(2n - k)$-dimensional PPH-cycle of degree 0.

**Proof.** The uniqueness of PPH-cycle is obvious.

First we prove the existence of $\iota(\tau)$ for $\tau \in T$ and the assertion (2). The proof is by induction on $\deg(\tau)$.

If $\deg(\tau) = 0$ then $\tau$ is a constant function $f(x) = c$ and $\iota(\tau)$ is the 2n-cycle $X_\Delta = c$.

Let $\deg(\nu) = 2k - 2$ and $\tau = ddc h \wedge \nu$, where $h \in A$. By the inductive assumption, $\iota(\nu)$ is a $(2n - k + 1)$-dimensional PPH-cycle of degree 0.

By proposition 3, it follows that $\tau = D_c(h\iota(\nu))$. The degree of $(2n - k)$-dimensional PPH-cycle $D_c(h\iota(\nu))$ is 0. So we can put $\iota(\tau) = D_c(h\iota(\nu))$. The theorem for $\tau \in T$ is proved.

The $A$ as $B$-module is generated by the elements of the ring $T$. Similarly the $B$-module of PPH-cycles is generated by PPH-cycles of degree 0. So the map $\iota$ can be continued as the homomorphism of $B$-modules.

**Corollary 13.** If $\tau \in T$ and $h \in A$ then $\iota(h\tau) = D_c(h\iota(\tau))$
Proposition 5. $\nu \delta' = D_c \nu$

Proof. On elements of the ring $\mathcal{T}'$ both parts of the required equality are zeroes. Let $\Upsilon \in \mathcal{T}'$ and $H \in \mathcal{B}'$ be an element of the first degree. If $\pi(1 \otimes \Upsilon) = \nu$ and $\pi(H \otimes 1) = h$, then $\nu \in \mathcal{T}$ and $h \in A$. Any element of the ring $\mathcal{A}'$ is a linear combination of elements of the form $H^k \otimes \Upsilon$ ($k$ is not fixed). So we must prove that $\nu \delta'(H^k \otimes \Upsilon) = D_c \nu(h^k \nu)$.

Now using the notation from the beginning of subsection, the Leibniz product rule, and the corollaries 13 and 11, we get

$$\nu \delta'(H^k \otimes \Upsilon) = \nu \delta' \left((H \otimes 1)^k(1 \otimes \Upsilon)\right) = \nu \left(k(H \otimes 1)^{k-1} \delta'(H \otimes 1)(1 \otimes \Upsilon)\right) = \nu \left(k(h^k \otimes 1)(d\delta'h \otimes 1)\right) = \nu \left(\nu d\delta'h\right) = \nu D_c(h \nu) = D_c(h^k \nu).$$

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