Small Time Convergence of Subordinators
with Regularly or Slowly Varying
Canonical Measure

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Abstract

We consider subordinators $X_\alpha = (X_\alpha(t))_{t \geq 0}$ in the domain of attraction at 0 of a stable subordinator $(S_\alpha(t))_{t \geq 0}$ (where $\alpha \in (0, 1)$); thus, with the property that $\Pi_\alpha$, the tail function of the canonical measure of $X_\alpha$, is regularly varying of index $-\alpha \in (-1, 0)$ as $x \downarrow 0$. We also analyse the boundary case, $\alpha = 0$, when $\Pi_\alpha$ is slowly varying at 0. When $\alpha \in (0, 1)$, we show that $(t \Pi_\alpha(X_\alpha(t)))^{-1}$ converges in distribution, as $t \downarrow 0$, to the random variable $(S_\alpha(1))^{-\alpha}$. This latter random variable, as a function of $\alpha$, converges in distribution as $\alpha \downarrow 0$ to the inverse of an exponential random variable. We prove these convergences, also generalised to functional versions (convergence in $\mathbb{D}[0, 1]$), and to trimmed versions, whereby a fixed number of its largest jumps up to a specified time are subtracted from a process. The $\alpha = 0$ case produces convergence to an extremal process constructed from ordered jumps of a Cauchy subordinator. Our results generalise random walk and stable process results of Darling, Cressie, Kasahara, Kotani and Watanabe.

1 Introduction

A classic result of Lévy (1937) is that stable laws with index $\alpha \in (0, 2)$ constitute the entire class of possible non-normal limit laws of a normed and centered random walk in $\mathbb{R}$. Random walks with such behaviour are said to be in the domain of attraction of the corresponding stable distribution.

A significant connection, going back to Doeblin (1940), and expanded on by Feller (1967, 1971), was to use Karamata’s regular variation theory to characterise random walks in domains of attraction by regularly varying conditions on the tail of the distribution of the increments of the random walk. With an appropriate interpretation, the boundary case $\alpha = 2$ also

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corresponds to a stable law, namely the normal distribution, and the corresponding domain of attraction can be characterised with regular variation-related results.

What of the other boundary case, $\alpha = 0$? Cressie (1975) showed that if $S_\alpha$ is a Stable($\alpha$) random variable with index $\alpha \in (0, 2)$ and shift constant $\gamma$, then $|S_\alpha - \gamma|^\alpha$ converges in distribution as $\alpha \downarrow 0$ to the reciprocal of an exponential random variable. Kasahara (1986), in a result he attributes to Kotani, extended this in the following way: let $(S_\alpha(t))_{t \geq 0}$ be a positive stable process of index $\alpha \in (0, 1)$, i.e., a subordinator with Lévy triplet $(0, 0, x^{-\alpha-1}dx1_{\{x>0\}})$, having Laplace transform

$$E e^{-\lambda S_\alpha(t)} = e^{-t\Gamma(1-\alpha)\lambda^\alpha}, \lambda > 0, t > 0.$$  

Then Kasahara’s result states that

$$((S_\alpha(t))^\alpha)_{t \geq 0} \xrightarrow{D} (e_t)_{t \geq 0}, \text{ as } \alpha \downarrow 0,$$  

(1.1)

where $\xrightarrow{D}$ denotes convergence in the Skorohod $J_1$ topology, and $(e_t)$ is an extremal process with marginal distributions

$$P(e_{t_1} \leq x_1, \ldots, e_{t_n} \leq x_n) = \prod_{i=1}^{n} e^{-t_i/x_i},$$

for $0 \leq t_1 < \cdots < t_n$ and $0 < x_1 < \cdots < x_n$. We refer to Resnick (1987) for background information on extremal processes.

For each $t > 0$, $e_t$ has the distribution of the reciprocal of an exponential random variable, so (1.1) represents an extension of the Cressie (1975) result. The identity

$$e_t = \sup_{0 < s \leq t} \Delta \xi_s,$$  

(1.2)

for each $t > 0$, also holds, where $(\xi_t)_{t \geq 0}$ is a Cauchy subordinator, i.e., a Lévy process with triplet $(0, 0, x^{-2}dx1_{\{x>0\}})$, and jump process $\Delta \xi_t := \xi_t - \xi_{t-}$, $t > 0$.

When $0 < \alpha < 2$, the tail of the increment distribution of a random walk in the domain of attraction of a Stable($\alpha$) distribution is regularly varying at $\infty$ with index $-\alpha$. So for the boundary case, $\alpha = 0$, it is natural to consider a slowly varying tail. In this case affine norming and centering of the random walk cannot lead to a finite nondegenerate limit random variable, but a transformation, whereby the tail of the increment distribution is applied as a function to the random walk, and norming is by the sample size, produces as a limiting random variable the reciprocal of an exponential random variable. This was proved by Darling (1952) in a 1-dimensional version, and, subsequent to this, in Watanabe (1980), the random walk is interpolated to a function in $D[0, 1]$, and finite dimensional convergence of
the resulting process is proved. In a later paper, Kasahara (1986) proved $J_1$ convergence of the functions in $\mathbb{D}[0, 1]$.

In view of this background, the continuous time environment is a natural one in which to consider results like these, and the aim of the present paper is, firstly to transfer from random walk versions to Lévy processes, in which the convergence is for small time parameter, rather than large time, and, secondly, to generalise the results to trimmed versions of Lévy processes. By “trimming” we mean removing a fixed number of large jumps of the processes. This is natural in the random walk context, because the slowly varying, heavy tails are associated with large jumps (“outliers”) in the random walk, and it is interesting in the process context as the effect of a slowly varying measure near 0 is previously little explored. Apart from these aspects, some quite interesting analytical differences occur between the small and large time situations.

Thus our basic assumption will be of the kind that a generic Lévy process $(Y(t))_{t \geq 0}$ with triplet $(\gamma_Y, \sigma^2_Y, \Pi_Y(dy))$, is in a non-normal domain of attraction at small times, by which we mean there exist non-stochastic functions $a_t \in \mathbb{R}$ and $b_t > 0$ such that

$$\frac{Y(t) - a_t}{b_t} \overset{D}{\to} S, \quad \text{as } t \downarrow 0, \quad (1.3)$$

where $S$ is an almost surely (a.s.) finite, non-degenerate, non-normal random variable.

Conditions on the Lévy measure for (1.3) to hold (in small time) can be deduced from Theorem 2.3 of Maller and Mason (2008), whose result can also be used to show that (1.3) can be extended to convergence in $\mathbb{D}[0, 1]$; that is,

$$\left( \frac{Y(\lambda t) - \lambda a_t}{b_t} \right)_{0 < \lambda t \leq 1} \overset{\text{weakly}}{\to} (S(\lambda))_{0 < \lambda \leq 1}, \quad \text{as } t \downarrow 0, \quad (1.4)$$

weakly with respect to the Skorohod $J_1$ topology. Then (1.3) is equivalent to the two-sided tail $\Pi_Y$ of $Y$ being regularly varying at 0 with index $\alpha \in (0, 2)$, together with a balance condition on the right and left tails of the Lévy measure $\Pi_Y$. The limit random variable $S$ in (1.3) has the distribution of $S_\alpha(1)$, where $(S_\alpha(\lambda))_{0 < \lambda \leq 1}$ is a Stable($\alpha$) Lévy process.

In Buchmann, Ipsen and Maller (2017) (1.4) was extended to a functional theorem for a trimmed version of $Y$, which result will be quoted below (see the proof of Theorem 3.1). The case of a slowly varying tail for $\Pi$ seems not to have been considered before, in our context (but see Kevei and Mason (2014) and Ipsen, Maller and Resnick (2018) for limits of ratios of large jumps of subordinators in this case). Although stated in (1.3) and (1.4) for general Lévy processes, from now on we restrict ourselves to subordinators. Some discussion relevant to this is given at the end of the next section.
2 Notation and Statement of Results

All processes will be defined on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\). Since the index \(\alpha\) will be a variable in our results, we have to indicate its presence in the notation. We have tried to come up with a notation that is minimal but clear and conveys the necessary information.

For each \(\alpha \geq 0\) let \((X_\alpha(t))_{t \geq 0}\) be a driftless subordinator with canonical measure \(\Pi_\alpha(dx)\), where \(\Pi_\alpha\) has tail \(\Pi_\alpha(x) := \Pi_\alpha((x, \infty))\), \(x > 0\), satisfying \(\Pi_\alpha(x) = x^{-\alpha}L(x)\mathbf{1}_{\{x > 0\}}\), with \(0 \leq \alpha < 1\), and \(L(x)\) a function slowly varying as \(x \downarrow 0\). For the \(\alpha = 0\) case, simply write \(X(t) := X_0(t)\) and \(\Pi := \Pi_0\). In this case, \(L(x)\) is assumed to be nonincreasing with \(L(0+) = \infty\). Since the processes \(X_\alpha(t)\) are subordinators, \(\alpha\) is necessarily restricted to \([0, 1)\).

Our development goes as follows. For each \(\alpha \in (0, 1)\), \(X_\alpha(t)\) is in the domain of attraction of a positive Stable(\(\alpha\)) distribution as \(t \downarrow 0\); in fact, the process \((X_\alpha(\lambda t))_{0 < \lambda < 1}\), converges in \(\mathbb{D}[0, 1]\), as \(t \downarrow 0\), after norming, to a Stable(\(\alpha\)) process \((S_\alpha(\lambda))_{\lambda \geq 0}\). This implies that \((t\Pi_\alpha(X_\alpha(\lambda t)))^{-1}\) converges to \((S_\alpha(\lambda))^{\alpha}\) in \(\mathbb{D}[0, 1]\). In turn, this latter process itself converges in distribution, as \(\alpha \downarrow 0\), to the largest jump up till time \(\lambda\) of a Stable(1) (Cauchy) process with measure \(x^{-2}dx\mathbf{1}_{\{x > 0\}}\). We denote this process as \((\xi_t)_{t \geq 0}\), consistent with the notation in (\([12]\)).

These results are included in our main theorem, Theorem 2.1 set out in diagrammatic form below. It deals, not just with the processes mentioned, but also with “trimmed” versions of them. To introduce trimmed processes, write \((\Delta X_\alpha(t) := X_\alpha(t) - X_\alpha(t^{-}) )_{t > 0}\), with \(\Delta X_\alpha(0) = 0\), for the jump process of \(X_\alpha\), and \(\Delta X_\alpha^{(1)}(t) \geq \Delta X_\alpha^{(2)}(t) \geq \cdots\) for the ordered jumps at \(t > 0\). Since \(\Pi\{0, \infty\} = \infty\), there are infinitely many positive jumps, a.s., in any finite time interval \([0, t], t > 0\), the \(\Delta X_\alpha^{(r)}(t)\) are positive a.s. for all \(t > 0\), and \(\lim_{t \downarrow 0} \Delta X_\alpha^{(r)}(t) = 0\) a.s. for all \(r \in \mathbb{N}\). (Throughout, let \(\mathbb{N} := \{1, 2, \ldots\}\) and \(\mathbb{N}_0 := \{0, 1, 2, \ldots\}\).) The \(r\)-trimmed process is defined to be \(X_\alpha(t)\) minus its \(r\) largest jumps, at a given time \(t\):

\[
(r)X_\alpha(t) := X_\alpha(t) - \sum_{i=1}^{r} \Delta X_\alpha^{(i)}(t), \quad r \in \mathbb{N}, \quad t > 0
\]  

(2.1)

(and we set \((0)X_\alpha(t) \equiv X_\alpha(t)\)). Detailed definitions and properties of this kind of ordering and trimming are given in Buchmann, Ipsen and Maller (2016), where the (positive) \(\Delta X_\alpha(t)\) are identified with the points of a Poisson point process on \([0, \infty)\).

We similarly denote the ordered jumps up till time \(\lambda\) of the Cauchy process \((\xi_\lambda)_{\lambda \geq 0}\) with jump process \((\Delta \xi_\lambda)_{\lambda \geq 0}\) as \(\Delta \xi_\lambda^{(1)} \geq \Delta \xi_\lambda^{(2)} \geq \cdots\).

**Theorem 2.1.** For each \(\alpha \in [0, 1)\) let \((X_\alpha(t))\) be a driftless subordinator whose tail measure \(\Pi_\alpha\) is regularly varying at zero with exponent \(-\alpha\) and
satisfies $\Pi_\alpha(0+) = \infty$; and for each $r \in \mathbb{N}_0$ let $(^rX_\alpha(t))$ be the trimmed version of $(X_\alpha(t))$ defined in \eqref{2.1}. When $\alpha = 0$ assume in addition

$$\lim_{t \to 0} \frac{(^rX_0(t))}{\Delta X_0^{(r+1)}(t)} = 1, \text{ a.s.},$$

(2.2)

where “a.s.” denotes almost sure convergence. Then for all $r \in \mathbb{N}_0$ we have the following convergences in distribution, as $t \downarrow 0$, and as $\alpha \downarrow 0$, with respect to the Skorohod $J_1$-topology and the parameter $\lambda \in (0, 1]$

\[
\begin{array}{c}
\begin{pmatrix}
(^rS_\alpha(\lambda))^{-1} & (\pi^{(r)}(0))^{-1} \\
0 & 0
\end{pmatrix}
\end{array}
\begin{array}{c}
\begin{pmatrix}
\alpha \to 0 & \Delta \xi_{\lambda}^{(r+1)} \\
\downarrow & \downarrow
\end{pmatrix}
\end{array}
\]

Figure 1: Main Convergence Diagram. The upper nodes represent processes in $0 < \lambda \leq 1$, indexed by $t > 0$. The lower nodes represent processes in $0 < \lambda \leq 1$. The index $r \in \mathbb{N}_0$ indicates the order of trimming. The vertical arrows indicate process convergence of the upper node processes as $t \downarrow 0$ to the lower node processes for each $\alpha \in (0, 1)$ on the left, and with $\alpha$ set equal to 0 on the right. The horizontal arrow indicates process convergence of the left lower node process as $\alpha \downarrow 0$ to the right lower node process.

Remarks. (i) Some comment on Figure 1 is in order. Since $\Pi_\alpha(0+) = \infty$ (i.e., $\Pi$ is of “infinite activity”) for each $\alpha \geq 0$, and $\lim_{t \downarrow 0} (^rX_\alpha(t)) = 0$ a.s., we have $\lim_{t \downarrow 0} \Pi_\alpha(^rX_\alpha(t)) = \infty$ a.s., and under the regularly varying (at 0) assumption we impose on $\Pi_\alpha$, it turns out that multiplying by $t$ is the correct scaling to get a nondegenerate limit law for $\Pi_\alpha(^rX_\alpha(t))$ as $t \downarrow 0$. It is then convenient to consider the limit of the reciprocal of $t\Pi_\alpha(^rX_\alpha(t))$ as we do in the topmost entries of Figure 1 because it produces the trimmed stable in the upright orientation as we see in the bottom left entry of the figure, thereby providing a direct generalisation of the Kotani result in \eqref{1.1}. Taking the function $\Pi_\alpha(^rX_\alpha(t))$ of $(^rX_\alpha(t))$ is a natural way of generalising the Darling (1952) result for random walks, but it’s clear that some quite different considerations enter in; note for example that $\Pi$ slowly varying at zero reflects a mild singularity, while $\alpha \in (0, 1)$ is steeper – whereas, at infinity, a slowly varying $\Pi$ betokens a very heavy tailed random walk.

(ii) The appearance of the almost sure condition (2.2) among the other weak convergence results is at first surprising. We discuss this in more detail after the proof of Theorem 4.1.
Given the exposition in (1.3) and (1.4), it is logical to ask if there are versions of the convergences in Theorem 2.1 for (necessarily centered) general Lévy processes, other than subordinators. We have not investigated in detail whether this can be done, but the results for subordinators are certainly of interest in themselves, (i) as being generalisations of non-negative random walk versions which have appeared in the literature discussed in Section II and, (ii) because subordinators and their jumps play a prominent role for example in the theory of Poisson-Dirichlet distributions initiated by Kingman (1975), which is not geared to the application of general Lévy processes. A further interesting point is that the Kingman Poisson-Dirichlet development relates at its heart to the *small time* behaviour of the stable subordinators, such as we consider here.

3 Convergence of $X_\alpha(t)$ as $t \downarrow 0$, for fixed $\alpha \in (0, 1)$

In this section we prove the lefthand vertical convergence in Figure 1. Here the parameter $\alpha$ does not vary; the convergence is as $t \downarrow 0$, for fixed $\alpha$.

**Theorem 3.1.** Fix $\alpha \in (0, 1)$ and let $(X_\alpha(t))$ be a driftless subordinator whose tail measure $\Pi_\alpha$ is regularly varying at zero with exponent $-\alpha$. For each $r \in \mathbb{N}$ let $(r)X_\alpha(t))$ be the trimmed process defined in (2.1). Then

$$\left(\frac{1}{t\Pi_\alpha((r)X_\alpha(t\lambda))}\right)_{0<\lambda\leq 1} \overset{D}{\longrightarrow} \left((r)S_\alpha(\lambda)^\alpha\right)_{0<\lambda\leq 1}, \text{ as } t \downarrow 0,$$

with respect to the $J_1$-topology.

In what follows, define the generalized inverse function of a monotonically decreasing function $g$ by $g^{-1}(x) := \inf \{y > 0: g(y) \leq x\}$, for $x > 0$.

**Proof of Theorem 3.1** Fix $\alpha \in (0, 1)$ and assume $\Pi_\alpha$ is regularly varying at zero with exponent $-\alpha$. Then

$$\left(\frac{X_\alpha(t\lambda)}{\Pi_\alpha(1/t)}\right)_{0<\lambda\leq 1} \overset{D}{\longrightarrow} (S_\alpha(\lambda))_{0<\lambda\leq 1}, \text{ as } t \downarrow 0,$$

with respect to the $J_1$-topology. This result follows from Theorem 2.3 of Maller and Mason (2008) (see also their references for antecedents) quoted as the generic version in (1.4). Maller and Mason do not mention that the norming function $b_t$ in (1.4) can be taken as the inverse function to the tail measure of the process, or that, in the driftless subordinator case, the centering function $a_t$ can be taken as 0, as we have done in (3.2); but these facts are easily checked.
Taking \((3.2)\) as given, it further implies the trimmed version

\[
\left( \begin{array}{c}
\left( r \right) X_\alpha(t\lambda) \\
\Pi_\alpha \left( 1/t \right) \\
\end{array} \right)
\xrightarrow{D_+} \left( \begin{array}{c}
\left( r \right) S_\alpha(\lambda) \\
0 < \lambda \leq 1 \\
\end{array} \right), \text{ as } t \downarrow 0, \text{ for } r \in \mathbb{N}_0,
\]

(3.3)

with respect to the \(J_1\)-topology, as shown in Theorem 3 of Buchmann, Ipsen and Maller (2017). The convergence in \((3.3)\) additionally implies

\[
\left( \frac{\Pi_\alpha \left( 1/t \right)}{\Pi_\alpha \left( r \right) X_\alpha(t\lambda) } \right)_{0 < \lambda \leq 1} \xrightarrow{D} \left( \begin{array}{c}
\left( r \right) S_\alpha(\lambda) \alpha \\
0 < \lambda \leq 1 \\
\end{array} \right),
\]

(3.4)

by application of the following Lemma \(3.2\) and \((3.4)\) implies \((3.1)\), thereby completing the proof of Theorem \(3.1\).

\[\square\]

**Lemma 3.2.** Suppose \(\Pi_\alpha\) is regularly varying at zero with exponent \(-\alpha\), \(\alpha > 0\). Then for two functions \(f_t > 0\) and \(g_t > 0\) on \([0, \infty)\) with \(\lim_{t \downarrow 0} f_t = \lim_{t \downarrow 0} g_t = 0\), we have \(\lim_{t \downarrow 0} \frac{\Pi_\alpha \left( g_t(\lambda) \right) }{\Pi_\alpha \left( f_t(\lambda) \right) } = c^\alpha\) if and only if \(\lim_{t \downarrow 0} \frac{f_t}{g_t} = c \in (0, \infty)\).

**Proof of Lemma 3.2.** This is a straightforward application of Potter’s bounds, see for example Theorem 1.5.6 of Bingham, Goldie and Teugels (1987). We omit the details. \[\square\]

### 4 Convergence of \(X_t = X_0(t)\) as \(t \downarrow 0\), Case \(\alpha = 0\)

Next we prove the righthand vertical convergence in Figure 1. The process \(X(t) = X_0(t)\) is now assumed to have tail \(\Pi(x)\) slowly varying as \(x \downarrow 0\), and the results in this section formally correspond to the case \(\alpha = 0\). So we drop the subscript \(\alpha\) and write \(X_t\) rather than \(X(t)\) throughout this section. Keep \(r \in \mathbb{N}_0\) fixed. Recall that \(\Delta \xi^{(1)}_\lambda \geq \Delta \xi^{(2)}_\lambda \geq \cdots\) are the ordered jumps, up till time \(\lambda\), of \(\xi\). The main result for this section is:

**Theorem 4.1.** Suppose \(X_t\) is a driftless subordinator whose Lévy measure \(\Pi\) has tail \(\Pi\) slowly varying at zero. Assume \((2.2)\) in addition. Then

\[
\left( \frac{1}{\Pi\left( r \right) X_\lambda(t\lambda) } \right)_{0 < \lambda \leq 1} \xrightarrow{D} \left( \Delta \xi^{(r+1)}_\lambda \right)_{0 < \lambda \leq 1}, \text{ as } t \downarrow 0,
\]

with respect to the \(J_1\)-topology.

Proof of Theorem 4.1 proceeds by way of some lemmas and propositions. The first lemma proves convergence in the supremum norm of the difference of two quantities to 0, stronger than proving \(J_1\) convergence.
Lemma 4.2. Assume the conditions of Theorem 4.1 including \((2.2)\). Then for each \(r \in \mathbb{N}_0\)

\[
\sup_{0 < \lambda \leq 1} \left| \frac{1}{t \Pi(\Delta X_{t \lambda}^{(r)})} - \frac{1}{t \Pi(\Delta X_{t \lambda}^{(r+1)})} \right| \to 0, \quad \text{as } t \downarrow 0. \quad (4.1)
\]

Proof of Lemma 4.2. Hold \(r \in \mathbb{N}_0\) fixed throughout. Since \(X_t\) is a subordinator, its jumps are positive, and so \((r)X_{t \lambda} \geq \Delta X_{t \lambda}^{(r+1)}\) for \(t > 0\) and \(\lambda \in (0, 1]\). Thus \(1/\Pi((r)X_{t \lambda}) \geq 1/\Pi(\Delta X_{t \lambda}^{(r+1)})\), and for \((4.1)\) it suffices to prove that for all \(y > 0\) and \(\eta > 0\) there exists \(t_0 = t_0(y, \eta) > 0\) such that

\[
P(\sup_{0 < \lambda \leq 1} \left| \frac{1}{t \Pi(\Delta X_{t \lambda}^{(r)})} - \frac{1}{t \Pi(\Delta X_{t \lambda}^{(r+1)})} \right| \geq y) < \eta. \quad (4.2)
\]

Take \(K > 0\). The left hand side of \((4.2)\) equals

\[
P\left( \sup_{0 < \lambda \leq 1} \left( \frac{1}{t \Pi(\Delta X_{t \lambda}^{(r)})} - \frac{1}{t \Pi(\Delta X_{t \lambda}^{(r+1)})} \right) > y \right) < \eta.
\]

and this is bounded above by

\[
P\left( \sup_{0 < \lambda \leq 1} \frac{1}{t \Pi(\Delta X_{t \lambda}^{(r+1)})} \left( \frac{\Pi(\Delta X_{t \lambda}^{(r+1)})}{\Pi((r)X_{t \lambda})} - 1 \right) > \frac{y}{K} \right) \]

\[
+ \mathbb{P}\left( \sup_{0 < \lambda \leq 1} \frac{1}{t \Pi(\Delta X_{t \lambda}^{(r+1)})} > K \right). \quad (4.3)
\]

We bound the first probability in \((4.3)\) by ignoring the first supremum in it. To deal with the remaining part of that term, we need to invoke \((2.2)\). This condition implies that there is an event \(\Omega_1 \subseteq \Omega\) with \(P(\Omega_1) = 1\) such that, for \(\omega \in \Omega_1\) and \(\delta > 0\), there exists \(t_1 \in (0, t_0)\) such that for \(t \in (0, t_1)\) we have \(W_t^{(r)} := (r)X_t/\Delta X_t^{(r+1)} < 1 + \delta\), and thus \(\sup_{0 < \lambda \leq 1} W_{t \lambda}^{(r)} < 1 + \delta\). Hence, we can find \(t_2 \in (0, t_1)\) such that, for \(t \in (0, t_2)\),

\[
P\left( \sup_{0 < \lambda \leq 1} W_{t \lambda}^{(r)} > 2 \right) < \frac{\eta}{3}.
\]
Then for $t \in (0, t_2)$ we have
\[
\mathbb{P} \left( \sup_{0 < \lambda \leq 1} \left( \frac{\Pi(\Delta X_{t\lambda}^{(r+1)})}{\Pi(2\Delta X_{t\lambda}^{(r+1)})} - 1 \right) > \frac{y}{K} \right) 
\leq \mathbb{P} \left( \sup_{0 < \lambda \leq 1} \left( \frac{\Pi(\Delta X_{t\lambda}^{(r+1)})}{\Pi(2\Delta X_{t\lambda}^{(r+1)})} - 1 \right) > \frac{y}{K}, \sup_{0 < \lambda \leq 1} W_{t\lambda}^{(r)} \leq 2 \right) 
+ \mathbb{P} \left( \sup_{0 < \lambda \leq 1} W_{t\lambda}^{(r)} > 2 \right).
\] (4.4)

The slow variation of $\Pi$ implies there exists $x_0 > 0$ such that, for all $x \in (0, x_0]$, $\Pi(x)/\Pi(2x) - 1 \leq y/K$. Further, notice that $\{\Delta X_{t\lambda}^{(r+1)} \leq x_0\}$ implies $\{\sup_{0 < \lambda \leq 1} \Delta X_{t\lambda}^{(r+1)} \leq x_0\}$, and thus, when $\Delta X_{t\lambda}^{(r+1)} \leq x_0$,
\[
\left\{ \sup_{0 < \lambda \leq 1} \left( \frac{\Pi(\Delta X_{t\lambda}^{(r+1)})}{\Pi(2\Delta X_{t\lambda}^{(r+1)})} - 1 \right) \leq \frac{y}{K} \right\}.
\]

Hence, the probability on the righthand side of (4.4) can be estimated as
\[
\mathbb{P} \left( \sup_{0 < \lambda \leq 1} \left( \frac{\Pi(\Delta X_{t\lambda}^{(r+1)})}{\Pi(2\Delta X_{t\lambda}^{(r+1)})} - 1 \right) > \frac{y}{K} \right) \leq \mathbb{P} \left( \sup_{0 < \lambda \leq 1} \Delta X_{t\lambda}^{(r+1)} > x_0 \right) 
= \mathbb{P} \left( \Delta X_t^{(r+1)} > x_0 \right). \tag{4.5}
\]

Since $\lim_{t \downarrow 0} \Delta X_t^{(r)} = 0$ a.s., there exists $t_3 \in (0, t_2)$ such that the righthand side of (4.5) does not exceed $\eta/3$, for $t \in (0, t_3)$.

To estimate the second probability on the righthand side of (4.3), we will use that there exists $K > 0$ and $t_4 \in (0, t_3)$ such that, for $t \in (0, t_4)$,
\[
\mathbb{P} \left( \sup_{0 < \lambda \leq 1} \frac{1}{t\Pi(\Delta X_{t\lambda}^{(r+1)})} > K \right) \leq \mathbb{P} \left( \frac{1}{t\Pi(\Delta X_t^{(r+1)})} > K \right) \leq \frac{\eta}{3}. \tag{4.6}
\]

This holds because, as a special case of the convergence in Proposition 4.3 below, $1/t\Pi(\Delta X_t^{(r+1)})$ converges to a finite positive random variable; we defer proof of (4.6) till then.

Accepting (4.6), then, we can combine (4.3) with (4.4), (4.5) and (4.6) to get, for $t \in (0, t_4)$,
\[
\mathbb{P} \left( \sup_{0 < \lambda \leq 1} \left( \frac{1}{t\Pi^{(r)}(X_{t\lambda})} - \frac{1}{t\Pi(\Delta X_{t\lambda}^{(r+1)})} \right) > y \right) \leq 3 \left( \frac{\eta}{3} \right) = \eta.
\]
Since $\eta$ is arbitrary this completes the proof of (4.2), and of Lemma 4.2.

Now write

$$\frac{1}{t\Pi ((r)X_{t\lambda})} = \left( \frac{1}{t\Pi ((r)X_{t\lambda})} - \frac{1}{t\Pi (\Delta X^{(r+1)}_{t\lambda})} \right) + \frac{1}{t\Pi (\Delta X^{(r+1)}_{t\lambda})}. $$

By Lemma 4.2 the first summand converges to zero in probability uniformly in $0 < \lambda \leq 1$. Thus, the processes

$$\left( \frac{1}{t\Pi ((r)X_{t\lambda})} \right)_{0 < \lambda \leq 1} \quad \text{and} \quad \left( \frac{1}{t\Pi (\Delta X^{(r+1)}_{t\lambda})} \right)_{0 < \lambda \leq 1}$$

have the same limit in distribution as $t \downarrow 0$. So to complete the proof of Theorem 4.1 it remains only to prove the following proposition.

**Proposition 4.3.** Assume the conditions of Theorem 4.1, including (2.2). Then, for all $r \in \mathbb{N}$, as $t \downarrow 0$,

$$\left( \frac{1}{t\Pi (\Delta X^{(r)}_{t\lambda})} \right)_{0 < \lambda \leq 1} \xrightarrow{D} (\Delta \xi^{(r)}_{\lambda})_{0 < \lambda \leq 1}, \text{ in } D[0, 1]. \quad (4.7)$$

We prove this in a classical way, first establishing finite dimensional ("fidi") convergence, then tightness of the process on the left of (4.7). This is done in the next two subsections.

### 4.1 Proof of fidi convergence in Proposition 4.3

Define the following random variables

$$Z_{r,t,\lambda} := \frac{1}{t\Pi (\Delta X^{(r)}_{t\lambda})}, \quad r \in \mathbb{N}, \ t > 0, \ \lambda > 0, \quad (4.8)$$

and note that $Z_{r,t,\lambda}$ is nondecreasing in $\lambda$. Recall that $\Delta \xi^{(1)}_{\lambda} \geq \Delta \xi^{(2)}_{\lambda} \geq \cdots$ are the ordered jumps, at time $\lambda$, of the Cauchy process $(\xi_{\lambda})_{\lambda \geq 0}$ having Lévy measure $x^{-2}dx 1_{\{x > 0\}}$. Let $\lambda_1 < \cdots < \lambda_n$. We aim to show

$$\lim_{t \downarrow 0} \mathbb{P}(Z_{r,t,\lambda_1} \leq y_1, \ldots, Z_{r,t,\lambda_n} \leq y_n) = \mathbb{P}(\Delta \xi^{(r)}_{\lambda_1} \leq y_1, \ldots, \Delta \xi^{(r)}_{\lambda_n} \leq y_n), \ n, r \in \mathbb{N}, \quad (4.9)$$

wherein it is sufficient to restrict ourselves to values $0 < y_1 < \cdots < y_n$, since $\{Z_{r,t,\lambda_i} \leq y_i\} \supseteq \{Z_{r,t,\lambda_j} \leq y_j\}$ whenever $i < j$ and $y_i \geq y_j$.

For formal reasons let $\lambda_1 := 0$ and $\lambda_{n+1} := \infty$, and introduce triangular arrays of random variables $(V_{t,j})_{1 \leq t \leq j \leq n}$ and $(\tilde{V}_{t,j})_{1 \leq t \leq j \leq n, t \geq 0}$ by setting

$$V_{t,j} := \# \{s \in (\lambda_{t-1}, \lambda_t) : \Delta \xi_s \in (y_j, y_{j+1})\} \quad (4.10)$$

and note that $V_{t,j}$ is nondecreasing in $\lambda_t$. Recall that $\Delta \xi^{(1)}_{\lambda_t} \geq \Delta \xi^{(2)}_{\lambda_t} \geq \cdots$ are the ordered jumps, at time $\lambda_t$, of the Cauchy process $(\xi_{\lambda_t})_{\lambda_t \geq 0}$ having Lévy measure $x^{-2}dx 1_{\{x > 0\}}$. Let $\lambda_1 < \cdots < \lambda_n$. We aim to show

$$\lim_{t \downarrow 0} \mathbb{P}(Z_{t,j,\lambda_t} \leq y_1, \ldots, Z_{t,j,\lambda_n} \leq y_n) = \mathbb{P}(\Delta \xi^{(r)}_{\lambda_t} \leq y_1, \ldots, \Delta \xi^{(r)}_{\lambda_n} \leq y_n), \ n, r \in \mathbb{N}. \quad (4.9)$$

wherein it is sufficient to restrict ourselves to values $0 < y_1 < \cdots < y_n$, since $\{Z_{t,j,\lambda_i} \leq y_i\} \supseteq \{Z_{t,j,\lambda_j} \leq y_j\}$ whenever $i < j$ and $y_i \geq y_j$.
and
\[ \tilde{V}_{\ell,j,t} := \# \{ s \in (t\lambda_{\ell-1}, t\lambda_{\ell}] : \Delta X_s \in (\prod^\ell \left( (ty_j)^{-1} \right), \prod^{\ell+1} \left( ty_{j+1} \right)^{-1}) \} , \]
for \( t > 0 \) and pairs \( \ell, j \) fulfilling \( 1 \leq \ell \leq j \leq n \). The events \( \{ \Delta \xi^{(r)}_{\lambda_1} \leq y_i \} \) and \( \{ \sum_{\ell=1}^i \sum_{j=1}^n V_{\ell,j} \leq r - 1 \} \) are equal. This can be seen as follows. By the definition of \( V_{\ell,j} \) we have that \( \sum_{j=1}^n V_{\ell,j} = \# \{ s \in (\lambda_{\ell-1}, \lambda_{\ell}] : \Delta \xi_s > y_i \} \). Thus,
\[
\sum_{\ell=1}^i \sum_{j=1}^n V_{\ell,j} = \sum_{\ell=1}^i \# \{ s \in (\lambda_{\ell-1}, \lambda_{\ell}] : \Delta \xi_s > y_i \} = \# \{ s \in (0, \lambda_1] : \Delta \xi_s > y_i \} .
\]
Hence, \( \sum_{\ell=1}^i \sum_{j=1}^n V_{\ell,j} \leq r - 1 \) holds if and only if \( \# \{ s \in (0, \lambda_1] : \Delta \xi_s > y_i \} \leq r - 1 \), which is equivalent to \( \{ \Delta \xi^{(r)}_{\lambda_1} \leq y_i \} \).

We assert that the event on the right hand side of (4.9) can be written as a finite union of disjoint events, each of which is the intersection of a finite number of events of the form \( \{ V_{\ell,j} = \kappa_{\ell,j} \} \). Here the \( (\kappa_{\ell,j})_{1 \leq \ell \leq j \leq n} \) are triangular arrays of non-negative integers in which the \( V_{\ell,j} \) and \( \tilde{V}_{\ell,j,t} \) take values. To verify that assertion, define
\[
B_{r,n,i} := \{ \kappa = (\kappa_{\ell,j})_{1 \leq \ell \leq j \leq n} : \sum_{\ell=1}^i \sum_{j=1}^n \kappa_{\ell,j} \leq r - 1 \} .
\]
Assume that for a given tuple \( \kappa = (\kappa_{\ell,j}) \) we have that \( \{ V_{\ell,j} = \kappa_{\ell,j} \} \) for all pairs \( \ell, j \) with \( 1 \leq \ell \leq j \leq n \). Then \( \sum_{\ell=1}^i \sum_{j=1}^n V_{\ell,j} \leq r - 1 \) holds if and only if \( \kappa \in B_{r,n,i} \). On the other hand, that the event \( \{ V_{\ell,j} = \kappa_{\ell,j} \} \) holds simultaneously for all pairs \( \ell, j \) with \( 1 \leq \ell \leq j \leq n \), can also be written as \( \bigcap_{1 \leq \ell \leq j \leq n} \{ V_{\ell,j} = \kappa_{\ell,j} \} \). This implies
\[
\{ \Delta \xi^{(r)}_{\lambda_1} \leq y_i \} = \bigcup_{\kappa = (\kappa_{\ell,j}) \in B_{r,n,i}} \bigcap_{1 \leq \ell \leq j \leq n} \{ V_{\ell,j} = \kappa_{\ell,j} \} .
\]
Now let \( A_{r,n} := \bigcap_{i=1}^n B_{r,n,i} \), so that \( A_{r,n} \) denotes the set of tuples \( \kappa = (\kappa_{\ell,j}) \) whose components satisfy \( \sum_{\ell=1}^i \sum_{j=1}^n \kappa_{\ell,j} \leq r - 1 \) for \( 1 \leq \ell \leq n \). Then
\[
\{ \Delta \xi^{(r)}_{\lambda_1} \leq y_1, \ldots, \Delta \xi^{(r)}_{\lambda_n} \leq y_n \} = \bigcap_{i=1}^n \{ \Delta \xi^{(r)}_{\lambda_i} \leq y_i \} = \bigcap_{i=1}^n \left( \bigcup_{\kappa = (\kappa_{\ell,j}) \in A_{r,n}} \bigcap_{1 \leq \ell \leq j \leq n} \{ V_{\ell,j} = \kappa_{\ell,j} \} \right) .
\]
(4.11)
The same construction holds with $\tilde{V}_{\ell,j,t}$ in place of $V_{\ell,j}$, which means we can relate $\{Z_{r,t,\lambda_1} \leq y_1, \ldots, Z_{r,t,\lambda_n} \leq y_n\}$ to $\{\sum_{\ell=1}^{r} \sum_{j=1}^{n} \tilde{V}_{\ell,j,t} \leq r - 1\}$ using the same sets $B_{r,n,i}$. Thus

$$\{Z_{r,t,\lambda_1} \leq y_1, \ldots, Z_{r,t,\lambda_n} \leq y_n\} = \bigcup_{\kappa=(\kappa_{\ell,j}) \in A_{r,n}} \bigcap_{1 \leq \ell \leq j \leq n} \left\{ \tilde{V}_{\ell,j,t} = \kappa_{\ell,j} \right\}$$

(4.12)

for the same sets $A_{r,n}$.

Due to the Poisson nature of the jumps of the processes $Z$ and $\xi$ in (4.11) and (4.12), counts of the numbers of points falling in disjoint subrectangles are independent; in particular, the events $\{V_{\ell,j} = \kappa_{\ell,j}\}$ are independent for all pairs $\ell, j$, $1 \leq \ell \leq j \leq n$, and the same is true for the events $\{\tilde{V}_{\ell,j,t} = \kappa_{\ell,j}\}$. Furthermore, the events

$$\bigcap_{1 \leq \ell \leq j \leq n} \left\{ V_{\ell,j} = \kappa_{\ell,j} \right\} \quad \text{and} \quad \bigcap_{1 \leq \ell \leq j \leq n} \left\{ \tilde{V}_{\ell,j,t} = \kappa_{\ell,j} \right\}$$

are disjoint if $\kappa_{\ell,j} \neq \kappa_{\ell,j}$ for at least one tuple $(\ell, j)$, and the same is true for the tilde version also.

Thus, (4.11) and (4.12) imply

$$\mathbb{P}(\Delta \xi^{(r)}_{\lambda_1} \leq y_1, \ldots, \Delta \xi^{(r)}_{\lambda_n} \leq y_n) = \sum_{\kappa=(\kappa_{\ell,j}) \in A_{r,n}} \prod_{1 \leq \ell \leq j \leq n} \mathbb{P}\{V_{\ell,j} = \kappa_{\ell,j}\}$$

(4.13)

and

$$\mathbb{P}(Z_{r,t,\lambda_1} \leq y_1, \ldots, Z_{r,t,\lambda_n} \leq y_n) = \sum_{\kappa=(\kappa_{\ell,j}) \in A_{r,n}} \prod_{1 \leq \ell \leq j \leq n} \mathbb{P}\{\tilde{V}_{\ell,j,t} = \kappa_{\ell,j}\}.$$

Hence, to prove (4.9), it remains only to show that for all $m \in \mathbb{N}_0$ the probabilities of the elementary events $\{\tilde{V}_{\ell,j,t} = m\}$ converge to the probabilities of the events $\{V_{\ell,j} = m\}$ as $t \downarrow 0$. If we define $N_I : \mathbb{R}^+ \to \mathbb{N}$ by

$$N_I(x) := \# \{ s \in I : \Delta X_s > x \},$$

(4.14)

where $I$ is any subinterval of $(0, \infty)$, and set

$$\gamma_{j,t} := \Pi(\Pi^-((ty_j)^{-1})) - \Pi(\Pi^-((ty_{j+1})^{-1})),$$

then we can write

$$\tilde{V}_{\ell,j,t} = N_t(\lambda_{\ell-1,\lambda_{\ell}}) \Pi(\Pi^-((ty_{j+1})^{-1})) - N_t(\lambda_{\ell-1,\lambda_{\ell}}) \Pi(\Pi^-((ty_j)^{-1}))$$

$$\sim \text{Poiss}(t(\lambda_{\ell} - \lambda_{\ell-1}) \gamma_{j,t}).$$
Noting further that
\[ \lim_{t \downarrow 0} \gamma_{j,t} = \frac{1}{y_j} - \frac{1}{y_{j+1}}, \]
which follows easily from the slow variation of \( \Pi(x) \) at 0 and the relation
\[ \Pi\left( \Pi^{-}(x) \right) \leq x < \Pi\left( \Pi^{-}(x-) \right), \quad x > 0, \]
the convergence of the probabilities of the elementary events finally follows from
\[ \lim_{t \downarrow 0} \mathbb{P} ( \tilde{V}_{\ell,j,t} = m ) = \lim_{t \downarrow 0} e^{-t(\lambda_{\ell}-\lambda_{\ell-1})} \frac{(\lambda_{\ell}-\lambda_{\ell-1})^m}{m!} \cdot \left( \frac{1}{y_j} - \frac{1}{y_{j+1}} \right)^m \]
for all pairs \( \ell, j \) fulfilling \( 1 \leq \ell \leq j \leq n \). With this, we have completed the proof of finite dimensional convergence in Proposition 4.3.

\[ \blacksquare \]

### 4.2 Proof of tightness in Proposition 4.3

Recall the \( Z_{r,t,\lambda} \) defined in (4.8), which are positive and nondecreasing in \( \lambda \) for each \( r \in \mathbb{N} \) and \( t > 0 \), and have the convergence behaviour described in Proposition 4.3. In this subsection we show:

**Proposition 4.4.** Assume \( \Pi \) has tail \( \Pi \) slowly varying at zero. Then for all \( r \in \mathbb{N} \) the process \( \left( (\delta \Pi(\Delta X_{t,\lambda}^{(r)}))^{-1} \right)_{0 < \lambda \leq 1} \) is tight in \( D[0,1] \) as \( t \downarrow 0 \).

**Proof of Proposition 4.4:** We use Theorem 15.3 of Billingsley (1968), where the result is only stated for discrete time but can immediately be generalised to continuous time as in the next lemma.

**Lemma 4.5.** For each \( r \in \mathbb{N} \) the process \( (Z_{r,t,\lambda})_{0 < \lambda \leq 1} \) indexed by \( t > 0 \) is tight in \( D[0,1] \) as \( t \downarrow 0 \) if and only if the following conditions hold:

(i) \[ \lim_{y \to \infty} \limsup_{t \downarrow 0} \mathbb{P} \left( \sup_{0 < \lambda \leq 1} Z_{r,t,\lambda} > y \right) = 0; \] (4.15)

(ii) for all \( y > 0 \),
\[ \limsup_{\delta \downarrow 0} \lim\sup_{t \downarrow 0} \mathbb{P} \left( \sup_{\lambda_1,\lambda_2 \in A_\delta} \min \{ Z_{r,t,\lambda} - Z_{r,t,\lambda_1}, Z_{r,t,\lambda_2} - Z_{r,t,\lambda} \} > y \right) = 0, \] (4.16)

where
\[ A_\delta := \{ \lambda_1, \lambda_2 \in (0,1) : \lambda_1 \leq \lambda_2, \lambda_2 - \lambda_1 \leq \delta \} \] (4.17)
(iii) for all $y > 0$,
\[
\lim_{\delta \downarrow 0} \lim_{t \downarrow 0} \sup_{\lambda_1, \lambda_2 \in [0, \delta]} \mathbb{P}(\sup_{\lambda_1, \lambda_2 \in [0, \delta]} |Z_{r,t,\lambda_1} - Z_{r,t,\lambda_2}| > y) = 0; \tag{4.18}
\]

(iv) for all $y > 0$,
\[
\lim_{\delta \downarrow 0} \lim_{t \downarrow 0} \sup_{\lambda_1, \lambda_2 \in [1-\delta,1)} \mathbb{P}(\sup_{\lambda_1, \lambda_2 \in [1-\delta,1)} |Z_{r,t,\lambda_2} - Z_{r,t,\lambda_1}| > y) = 0. \tag{4.19}
\]

In what follows we prove (4.15), (4.16), (4.18) and (4.19) in sequence, keeping $r \in \mathbb{N}$ fixed.

**Proof of Condition (i):** The probability in the lefthand side of (4.15) is
\[
\mathbb{P}(\sup_{0 < \lambda \leq 1} Z_{r,t,\lambda} > y) = \mathbb{P}(\Delta X^{(r)} > \overline{\Pi}^-((ty)^{-1}))
\]
\[
\leq \mathbb{P}(\Delta X^{(1)} > \overline{\Pi}^-((ty)^{-1}))
\]
\[
= 1 - \mathbb{P}(N_{(0,t)}(\Pi^-((ty)^{-1})) = 0)
\]
\[
= 1 - \exp\left(-t \overline{\Pi}^-((ty)^{-1})\right)
\]
\[
\leq 1 - \exp\left(-1/y\right). \tag{4.20}
\]
(Recall the definition of $N_I$ in (4.14)). The last inequality in (4.20) follows from the fact that $\overline{\Pi}^-((x)) \leq x$, $x > 0$. Letting $y \to \infty$ in (4.20) gives (4.15).

**Proof of Condition (ii):** In the following, keep $y > 0$ and $\eta > 0$ fixed, and take $\lambda_0 \in (0,1)$ such that $1 - e^{-2\lambda_0/y} < \eta/2$. Recall $A_\delta$ in (4.17) and define $A^{\leq}_\delta(\lambda_0) := \{\lambda_1, \lambda_2 \in A_\delta : \lambda_1 \leq \lambda_0\}$ and $A^\geq_\delta(\lambda_0) := \{\lambda_1, \lambda_2 \in A_\delta : \lambda_1 > \lambda_0\}$.

Decompose the probability in the lefthand side of (4.16) as
\[
\mathbb{P}\left(\sup_{\lambda_1, \lambda_2 \in A_\delta} \min\{Z_{r,t,\lambda} - Z_{r,t,\lambda_1}, Z_{r,t,\lambda_2} - Z_{r,t,\lambda}\} > y\right)
\]
\[
\leq \mathbb{P}\left(\sup_{\lambda_1, \lambda_2 \in A^\leq_\delta(\lambda_0)} \min\{Z_{r,t,\lambda} - Z_{r,t,\lambda_1}, Z_{r,t,\lambda_2} - Z_{r,t,\lambda}\} > y\right)
\]
\[
+ \mathbb{P}\left(\sup_{\lambda_1, \lambda_2 \in A^\geq_\delta(\lambda_0)} \min\{Z_{r,t,\lambda} - Z_{r,t,\lambda_1}, Z_{r,t,\lambda_2} - Z_{r,t,\lambda}\} > y\right). \tag{4.21}
\]
In the first summand on the righthand side of (4.21), $\lambda_1 \leq \lambda \leq \lambda_2 \leq \lambda_1 + \delta \leq \lambda_0 + \delta$, so the probability is bounded above by
\[
\mathbb{P}\left(\sup_{\lambda \in [\lambda_1, \lambda_0 + \delta]} Z_{r,t,\lambda} > y\right) \leq 1 - e^{-(\lambda_0 + \delta)/y}, \tag{4.22}
\]
just as in (4.20). When $\delta$ is chosen less than $\lambda_0$, the righthand side is less than $1 - e^{-2\lambda_0/y} < \eta/2$.  

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Next we estimate the second summand on the righthand side of (4.21). In it, \( \lambda_1 > \lambda_0 \). Take \( \lambda \in [\lambda_1, \lambda_2] \) and \( \gamma_1, \gamma_2 > 0 \), and set
\[
\Gamma_t := \{ \gamma_1 \leq Z_{r,t,\lambda_0} \leq Z_{r,t,\lambda} \leq \gamma_2 \}.
\]
Now (4.7) implies \( Z_{r,t,\lambda} \xrightarrow{D} \Delta \xi^{(r)}_\lambda \) for each \( \lambda \in (0,1] \) as \( t \downarrow 0 \). The Cauchy process \( (\xi_\lambda)_{\lambda \geq 0} \) has Lévy measure \( x^{-2}dx1_{\{x > 0\}} \), so the number of jumps exceeding \( x > 0 \) up till time \( \lambda \) is Poisson with expectation \( \lambda/x \).

This defines a proper distribution with no mass at 0:
\[
P(\Delta \xi^{(r)}_\lambda \leq x) = P(\#\{ s \in (0, \lambda) : \Delta \xi_s > x \} \leq r - 1) = e^{-\lambda/x} \sum_{j=0}^{r-1} \frac{(\lambda/x)^j}{j!}.
\]

Choosing \( 0 < t \leq 1/(\gamma_1 \Pi(T)) \), we have \( \Pi^{-1}(1/(\gamma_1 t)) \leq T \), so by (4.25),
\[
\frac{\Pi(\Delta X^{(r)}_{t\mu})}{\Pi(\Delta X^{(r)}_{t\kappa})} \geq \frac{\Delta X^{(r)}_{t\mu}}{\Delta X^{(r)}_{t\kappa}}.
\]
This yields
\[
\frac{1}{\Pi(\Delta X_{t\mu}^{(r)})} - \frac{1}{\Pi(\Delta X_{t\nu}^{(r)})} \leq \frac{1}{\Pi(\Delta X_{t\mu}^{(r)})} - \left( \frac{\Delta X_{t\mu}^{(r)} - \Delta X_{t\nu}^{(r)}}{\Delta X_{t\mu}^{(r)} \Pi(\Delta X_{t\mu}^{(r)})} \right) \frac{1}{\Pi(\Delta X_{t\nu}^{(r)})}
\]
\[= \frac{\Delta X_{t\mu}^{(r)} - \Delta X_{t\nu}^{(r)}}{\Delta X_{t\mu}^{(r)} \Pi(\Delta X_{t\mu}^{(r)})} \leq \frac{\Delta X_{t\mu}^{(r)} - \Delta X_{t\nu}^{(r)}}{\Delta X_{t\nu}^{(r)} \Pi(\Delta X_{t\nu}^{(r)})}.
\]

on the event $\Gamma_t$, when $0 < t \leq 1/(\gamma_1 \Pi(T))$. With this inequality we have proved the inclusion in (4.24).

Continuing from (4.24), argue from (4.26) that, on $\Gamma_t$, (here note too that $1/\Pi(\Delta X_{t}^{(r)}) = t Z_{r,t,1} \leq t \gamma_2$ on $\Gamma_t$). For the following, set $a_t := (y/\gamma_2) \Pi^-(1/(t \gamma_1))$. Applying (4.27) once for $\mu := \lambda$ and $\kappa := \lambda_1$, and once for $\mu := \lambda_2$ and $\kappa := \lambda$, we obtain

\[
P\left( \sup_{\lambda_1, \lambda_2 \in A_t^\gamma} \sup_{\lambda \in [\lambda_1, \lambda_2]} \min \{ Z_{r,t,\lambda} - Z_{r,t,\lambda_1}, Z_{r,t,\lambda_2} - Z_{r,t,\lambda} \} > y; \Gamma_t \right)
\]
\[\leq P\left( \sup_{\lambda_1, \lambda_2 \in A_t^\gamma} \sup_{\lambda \in [\lambda_1, \lambda_2]} (\Delta X_{t\lambda}^{(r)} - \Delta X_{t\lambda_1}^{(r)}, \Delta X_{t\lambda_2}^{(r)} - \Delta X_{t\lambda}^{(r)}) > a_t; \Gamma_t \right).
\]

For given $\lambda_1, \lambda_2$, the event
\[
\left\{ \sup_{\lambda \in [\lambda_1, \lambda_2]} (\Delta X_{t\lambda}^{(r)} - \Delta X_{t\lambda_1}^{(r)}, \Delta X_{t\lambda_2}^{(r)} - \Delta X_{t\lambda}^{(r)}) > a_t \right\}
\]
requires that there exist at least two points $s_1, s_2 \in (\lambda_1, \lambda_2]$ such that $\Delta X_{ts_1} > a_t$ and $\Delta X_{ts_2} > a_t$. To see this, assume there is no point $s \in (\lambda_1, \lambda_2]$ with $\Delta X_{ts} > a_t$. Then $\Delta X_{t\lambda_2}^{(r)} - \Delta X_{t\lambda_1}^{(r)} \leq a_t$ and thus $\Delta X_{t\lambda}^{(r)} - \Delta X_{t\lambda_1}^{(r)} \leq a_t$ hold for any $\lambda \in (\lambda_1, \lambda_2]$. This is not possible under (4.28). If there is only one point $s \in (\lambda_1, \lambda_2]$ with $\Delta X_{ts} > a_t$, then for any $\lambda \in (\lambda_1, \lambda_2]$ we have that either $\Delta X_{t\lambda}^{(r)} - \Delta X_{t\lambda_1}^{(r)} \leq a_t$ or $\Delta X_{t\lambda_2}^{(r)} - \Delta X_{t\lambda}^{(r)} \leq a_t$, also not possible under (4.28). Hence we deduce

\[
P\left( \sup_{\lambda_1, \lambda_2 \in A_t^\gamma} \sup_{\lambda \in [\lambda_1, \lambda_2]} \min \{ Z_{r,t,\lambda} - Z_{r,t,\lambda_1}, Z_{r,t,\lambda_2} - Z_{r,t,\lambda} \} > y; \Gamma_t \right)
\]
\[\leq P\left( \exists \lambda \in [\lambda_0, 1 - \delta] : N_{t\lambda, t(\lambda+\delta)}(a_t) \geq 2; \Gamma_t \right),
\]
Now define intervals \( I_{k,t,\delta} := [t(\lambda_0 + k\delta), t(\lambda_0 + (k + 2)\delta)) \), for \( t > 0 \), \( \delta > 0 \) and \( k \in \mathbb{N} \). Note that the length of each of these intervals is \( 2t\delta \). Further, define the integers \( k_\delta := \lceil (1 - \lambda_0) / \delta \rceil \).

For given \( \delta > 0 \) and \( \lambda \in [\lambda_0, 1 - \delta] \) there exists \( k \in [0, k_\delta] \cap \mathbb{N} \) such that \( \lambda \in [\lambda_0 + k\delta, \lambda_0 + (k + 1)\delta) \), hence \( t\lambda \in [t(\lambda_0 + k\delta), t(\lambda_0 + (k + 1)\delta)) \). This implies \([t\lambda, t(\lambda + \delta)) \subseteq [t(\lambda_0 + k\delta), t(\lambda_0 + (k + 1)\delta)) = I_{k,t,\delta} \) for the same \( k \), so for each interval \([t\lambda, t(\lambda + \delta)) \) there exists \( k \in [0, k_\delta] \cap \mathbb{N} \) such that \([t\lambda, t(\lambda + \delta)) \subset I_{k,t,\delta} \).

Thus,

\[
\{ \exists \lambda \in [\lambda_0, 1 - \delta] : N_{[t\lambda,t(\lambda + \delta))} (a_t) \geq 2 \} \subseteq \bigcup_{k=0}^{k_\delta} \{ N_{I_{k,t,\delta}} (a_t) \geq 2 \} .
\] (4.30)

The intervals \( I_{k,t,\delta} \) are constructed in such a way that every second interval is disjoint from the preceding one. Thus the events \( \{ N_{I_{2k-1,t,\delta}} (a_t) > 2 \} \) for \( k \in [1, [k_\delta / 2]] \cap \mathbb{N} \) are mutually independent, as are the events \( \{ N_{I_{2k,t,\delta}} (a_t) > 2 \} \) for \( k \in [0, [k_\delta / 2]] \cap \mathbb{N}_0 \). Accordingly, write the righthand side of (4.30) as

\[
\bigcup_{k=0}^{k_\delta} \{ N_{I_{k,t,\delta}} (a_t) \geq 2 \} = \bigcup_{k=0}^{[k_\delta / 2]} \{ N_{I_{2k,t,\delta}} (a_t) \geq 2 \} \cup \bigcup_{k=1}^{[k_\delta / 2]} \{ N_{I_{2k-1,t,\delta}} (a_t) \geq 2 \},
\]

and combine this with (4.29) and (4.30) to get

\[
\mathbb{P} \left( \sup_{\lambda_1,\lambda_2 \in A_\nu^k(\lambda_0)} \sup_{\lambda \in [\lambda_1,\lambda_2]} \min \{ Z_{r,t,\lambda} - Z_{r,t,\lambda_1}, Z_{r,t,\lambda_2} - Z_{r,t,\lambda} \} \geq y; \Gamma_t \right)
\]

\[
\leq \mathbb{P} \left( \bigcup_{k=0}^{[k_\delta / 2]} \{ N_{I_{2k,t,\delta}} (a_t) \geq 2 \} \right) + \mathbb{P} \left( \bigcup_{k=1}^{[k_\delta / 2]} \{ N_{I_{2k-1,t,\delta}} (a_t) \geq 2 \} \right)
\]

\[
= 2 - \mathbb{P} \left( \bigcup_{k=0}^{[k_\delta / 2]} \{ N_{I_{2k,t,\delta}} (a_t) \leq 1 \} \right) - \mathbb{P} \left( \bigcup_{k=1}^{[k_\delta / 2]} \{ N_{I_{2k-1,t,\delta}} (a_t) \leq 1 \} \right).
\]

The events \( \{ N_{I_{2k,t,\delta}} (a_t) \leq 1 \} \) with \( k \in [0, [k_\delta / 2]] \cap \mathbb{N}_0 \) are mutually independent as well as the events \( \{ N_{I_{2k-1,t,\delta}} (a_t) \leq 1 \} \) with \( k \in [0, [k_\delta / 2]] \cap \mathbb{N} \). This implies

\[
\mathbb{P} \left( \sup_{\lambda_1,\lambda_2 \in A_\nu^k(\lambda_0)} \sup_{\lambda \in [\lambda_1,\lambda_2]} \min \{ Z_{r,t,\lambda} - Z_{r,t,\lambda_1}, Z_{r,t,\lambda_2} - Z_{r,t,\lambda} \} \geq y; \Gamma_t \right)
\]

\[
\leq 2 - \left( \prod_{k=0}^{[k_\delta / 2]} \mathbb{P} \left( N_{I_{2k,t,\delta}} (a_t) \leq 1 \right) \right) - \left( \prod_{k=1}^{[k_\delta / 2]} \mathbb{P} \left( N_{I_{2k-1,t,\delta}} (a_t) \leq 1 \right) \right)
\]

\[
= 2 \left( 1 - \mathbb{P}^{[k_\delta / 2]+1} (N_{I_0,t,\delta} (a_t) \leq 1) \right).
\] (4.31)
Here the last equality follows from the fact that each of the intervals has the same length and thus the probabilities \( P(N_{I_k,t,\delta}(a_t) \leq 1) \) are equal for all \( k \in [0, k_0] \cap N_0 \). Furthermore
\[
P(N_{I_0,t,\delta}(a_t) \leq 1) = \left(1 + 2t\Pi(a_t)\right) e^{-2t\Pi(a_t)}
\]
thus,
\[
P^{[k_0/2]+1}(N_{I_0,t,\delta}(a_t) \leq 1) = \left(1 + \delta c_t\right)^{[k_0/2]+1} e^{-\delta([k_0/2]+1)c_t},
\]  
(4.32)
where \( c_t := 2t\Pi(a_t) \).
Letting \( t \downarrow 0 \), so that
\[
c_t = 2t\Pi(a_t) = 2t\Pi((t\gamma_1)/(y/\gamma_2)) \to 2/\gamma_1,
\]
followed by \( \delta \downarrow 0 \), so that \( ([k_0/2] + 1)\delta \to (1 - \lambda_0)/2 \), shows that the righthand side of (4.32) tends to \( e^{((1 - \lambda_0)/\gamma_1) - (1 - \lambda_0)/\gamma_1} = 1 \). Then we deduce from (4.31) that
\[
\limsup_{t \downarrow 0} P\left( \sup_{\lambda_1,\lambda_2 \in A_0^k(\lambda_0)} \min \{ Z_{r,t,\lambda} - Z_{r,t,\lambda_1}, Z_{r,t,\lambda_2} - Z_{r,t,\lambda} \} > y; \Gamma_t \right)
\]
tends to 0 as \( \delta \downarrow 0 \). Combining this with (4.22) and (4.23) yields that
\[
\lim_{\delta \downarrow 0} \limsup_{t \downarrow 0} P\left( \sup_{\lambda_1,\lambda_2 \in A_0^k(\lambda_0)} \min \{ Z_{r,t,\lambda} - Z_{r,t,\lambda_1}, Z_{r,t,\lambda_2} - Z_{r,t,\lambda} \} > y \right)
\]
is less than \( \eta \), and since \( \eta \) is arbitrary this proves (4.16).

**Proof of Condition (iii):** The probability in the lefthand side of (4.18) can be written as
\[
P\left( \sup_{\lambda_1,\lambda_2 \in [0,\delta]} |Z_{r,t,\lambda_2} - Z_{r,t,\lambda_1}| > y \right),
\]
and this is no larger than \( P(Z_{r,t,\delta/2} > y) \). Using a similar calculation as in (4.20), there exists \( t_5 > 0 \) such that for all \( t \in (0, t_5) \)
\[
P(Z_{r,t,\delta} > y) \leq 1 - e^{-\delta/y}.
\]
For fixed \( y > 0 \) and \( \delta > 0 \) small enough this is no larger than \( \eta \).

**Proof of Condition (iv):** The probability in the lefthand side of (4.19) is no larger than \( P(Z_{r,t,1} - Z_{r,t,1-\delta} > y) \). Just as in the proof of Condition (ii) there exists \( t_6 > 0 \) such that for \( t \in (0, t_6) \)
\[
P\left( Z_{r,t,1} - Z_{r,t,1-\delta} > y \right) \leq P(N_{I(1-\delta),t}(a_t) \geq 1) + P(\Gamma_t^c)
\leq 1 - e^{-\delta \Pi(a_t)} + \eta
\leq 1 - e^{-2\delta/\gamma_1} + \eta.
For $\delta > 0$ small enough this is no larger than $2\eta$, so the proof of Condition (iv) is complete, and this finally completes the proof of Proposition 4.4.

**Remarks.** The almost sure condition (2.2) may seem anomalous in the midst of the other weak convergence conditions, but it is not excessive in context. It follows from the proof of Lemma 4.2, via (4.4) that, with

$$R_{t\lambda} := \frac{\Pi(\Delta X_{\lambda}^{(r+1)})}{\Pi((r)X_{t\lambda})} \geq 1,$$

we have

$$\lim_{t \downarrow 0} \mathbb{P}\left( \sup_{0 < \lambda \leq 1} R_{t\lambda} > 1 + \varepsilon \right) = 0,$$

for all $\varepsilon > 0$. Hence, for given $\varepsilon > 0$, $\delta > 0$, and $t$ small enough, $t \leq t_0(\varepsilon, \delta)$,

$$\mathbb{P}\left( R_{t\lambda} > 1 + \varepsilon \text{ for some } \lambda \in (0, 1) \right) \leq \delta.$$

But this implies

$$\mathbb{P}(R_s > 1 + \varepsilon \text{ for some } s \leq t) \leq \delta$$

whenever $t \leq t_0$. Hence

$$\lim_{t \downarrow 0} \frac{\Pi(\Delta X_{t\lambda}^{(r+1)})}{\Pi((r)X_{t\lambda})} = 1, \text{ a.s.}$$

This is close to (2.2) but does not imply it in general because the converse part of Lemma 3.2 is not true for $\alpha = 0$ in general (take, for example, $\Pi(x) = |\log x|$, $f_t = t|\log t|$, $g_t = t$, for $0 < x, t < 1$). So we have to impose (2.2) as a side condition.

We remark incidentally that the slow variation of $\Pi(x)$ at 0 is equivalent to a weak version of (2.2), namely that $(r)X_t/\Delta X_t^{(r+1)} \overset{p}{\to} 1$ as $t \downarrow 0$ for $r \in \mathbb{N}_0$ (see Buchmann, Ipsen, Maller (2016). A necessary and sufficient condition for (2.2) itself in the case $r = 1$ is in Maller (2016).

5 Convergence of the Trimmed Stable as $\alpha \downarrow 0$

In this section, to complete Figure 1 we prove that $(^{(r)}S_\alpha(\lambda))^\alpha$ converges to $(\Delta \xi_{\lambda}^{(r+1)})$ in $\mathbb{D}[0, 1]$ as $\alpha \downarrow 0$, for each $r \in \mathbb{N}_0$. First suppose $r = 0$. As in the proof of Kasahara (1986), we obtain that $S_\alpha(\lambda)$ can be written as

$$S_\alpha(\lambda) = \int_{u \in (0, \lambda]} \int_{x > 0} x^{1/\alpha} N(du, dx),$$

where $N(du, dx)$ is a Poisson random measure with intensity measure $du \times x^{-2}dx$. This is the Poisson random measure governing the jumps of the
Cauchy process \((\xi_\lambda)_{0<\lambda \leq 1}\), so we can write
\[
(S_\alpha(\lambda))_{0<\lambda \leq 1} = \left( \sum_{0<s \leq \lambda} (\Delta \xi_\lambda)^{1/\alpha} \right)_{0<\lambda \leq 1}.
\]
The jumps up till time \(\lambda\) of \(\xi_\lambda\) can be ordered as \(\Delta \xi_\lambda^{(1)} \geq \Delta \xi_\lambda^{(2)} \geq \cdots\), and then since raising to the power \(1/\alpha\) does not change the order of the jumps,
\[
((r) S_\alpha(\lambda))^\alpha = \left( \sum_{i \geq r+1} (\Delta \xi_\lambda^{(i)})^{1/\alpha} \right)^\alpha. \tag{5.1}
\]
Using a classical argument\(^4\) we can show that when \(\alpha \downarrow 0\) each term in the process on the righthand side of (5.1) converges surely (i.e., for each \(\omega \in \Omega\)) to
\[
\sup_{i \geq r+1} \Delta \xi_\lambda^{(i)} = \Delta \xi_\lambda^{(r+1)}.
\]
Consequently, also the process on the righthand side of (5.1) converges surely to the process \((\Delta \xi_\lambda^{(r+1)})\). This of course also implies convergence in distribution. So we obtain the required result. \(\square\)

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\(^4\)When \(a_r \geq a_{r+1} \geq \cdots > 0\) and \(\sum_{i \geq r} a_i < \infty\), then
\[
\alpha \log \left( \sum_{i \geq r} a_i^{1/\alpha} \right) = \log a_r + \alpha (1 + \sum_{i > r} (a_i/a_r)^{1/\alpha}).
\]
Take \(\alpha < 1\) and choose \(i_0(r) \geq r\) so that \((a_{i_0}/a_r)^{1/\alpha-1} < 1\). Then the second term on the righthand is less than \(\alpha (i_0 - r + \sum_{i > i_0} a_i/a_r) \to 0\) as \(\alpha \downarrow 0\).
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6 Appendix

To give a concrete formula for the finite dimensional distribution for the \(r\)th jump of a subordinator \((Y_t)\) without drift is tedious in general. However, in case \(r = 1\) this is a classical result and can be found for example in Chapter 4.1 of Resnick (2008). Let \(\Lambda\) be the Lévy measure of \(Y\), with tail \(\Lambda\). Then we have, for \(\lambda_1 < \cdots < \lambda_n\) and \(y_1 < \cdots < y_n\),

\[
P(\Delta Y_{\lambda_1}^{(1)} \leq y_1, \ldots, \Delta Y_{\lambda_n}^{(1)} \leq y_n) = e^{-\lambda_1 \Lambda(y_1)} e^{-(\lambda_2 - \lambda_1) \Lambda(y_2)} \cdots e^{-(\lambda_n - \lambda_{n-1}) \Lambda(y_n)}.
\]

In case \((Y_t) = (\xi_t)\) is a Cauchy process, this simplifies to

\[
P(\Delta \xi_{\lambda_1}^{(1)} \leq y_1, \ldots, \Delta \xi_{\lambda_n}^{(1)} \leq y_n) = e^{-\lambda_1/y_1} e^{-(\lambda_2 - \lambda_1)/y_2} \cdots e^{-(\lambda_n - \lambda_{n-1})/y_n}.
\] (6.1)

We can also get this as a calculation from (4.13). In case \(r = 1\) we take \(A_{1,n} = \{ (\kappa_{\ell,j})_{1 \leq \ell \leq j \leq n} : \kappa_{\ell,j} = 0 \}\), and the formula simplifies to

\[
P(\Delta \xi_{\lambda_1}^{(1)} \leq y_1, \ldots, \Delta \xi_{\lambda_n}^{(1)} \leq y_n) = \prod_{1 \leq \ell \leq j \leq n} P(V_{\ell,j} = 0)
\]

\[
= \prod_{1 \leq \ell \leq j \leq n} e^{-(\lambda_r - \lambda_{r-1}) (1/y_j - 1/y_{j+1})},
\]

which is the same as the righthand side of (6.1) (recall \(\lambda_0 = 0\) and \(y_{n+1} = \infty\)).

6.1 A formula for the fidi distribution of the 2nd largest jump

In the following we derive an implicit formula for \(r = 2\). For larger \(r\) the formula could be derived in a similar way. Let \(Y = (Y_t)\) be any subordinator without drift and with Lévy measure \(\Lambda\). We aim to give a formula for

\[
P(\Delta Y_{\lambda_1}^{(2)} < y_1, \ldots, \Delta Y_{\lambda_n}^{(2)} < y_n),
\]

where \(0 < \lambda_1 < \cdots < \lambda_n\) and \(0 < y_1 < \cdots < y_n\).

Analogously to (4.13), we will set \(\lambda_0 = 0\) and \(y_{n+1} = \infty\) and then

\[
V_{\ell,j} := \# \{ s \in [\lambda_{\ell-1}, \lambda_{\ell}) : \Delta Y_s \in [y_j, y_{j+1}) \}.
\]
One way to calculate the finite dimensional distribution would be to construct the set $A_{r,n}$ given in Section 4.1. However, this would require constructing the set of triangular arrays fulfilling $\sum_{i=1}^{n} \sum_{j=i}^{n} \kappa_{i,j} \leq r$ for all $i \in \{1, \ldots, n\}$ at the same time. To our knowledge there is no simple way to do that. So we choose a slightly different approach.

To start, we set $D_{n+1,n} = \Omega$ and, for $2 \leq i \leq n$,

$$D_{i,n} := \{ \Delta Y_{\lambda_i,\lambda_{i-1}}^{(2)} < y_i, \ldots, \Delta Y_{\lambda_n,\lambda_{n-1}}^{(2)} < y_n \}.$$  

Then note that

$$\mathbb{P}(\Delta Y_{\lambda_i}^{(2)} < y_i, \ldots, \Delta Y_{\lambda_n}^{(2)} < y_n | \Delta Y_{\lambda_{i-1}}^{(1)} < y_i) = \mathbb{P}(D_{i,n}). \quad (6.2)$$

This follows from the fact that the numbers of jumps in different intervals are independent. Given there are no jumps exceeding $y_i$ in the interval $[0, \lambda_{i-1}]$, there are in particular no jumps exceeding $y_i, \ldots, y_n$. Hence, under the condition $\Delta Y_{\lambda_{i-1}}^{(1)} < y_i$, the number of jumps exceeding $y_i, \ldots, y_n$ on the intervals $[0, \lambda_i], \ldots, [0, \lambda_n]$ is the same as the number of jumps exceeding $y_i, \ldots, y_n$ on the intervals $(\lambda_{i-1}, \lambda_i], \ldots, (\lambda_{n-1}, \lambda_n]$. Since the increments are stationary, we obtain the formula in (6.2).

Next we state our recursive formula and give an explanation following it. The formula is

$$\mathbb{P}(\Delta Y_{\lambda_1}^{(2)} < y_1, \ldots, \Delta Y_{\lambda_n}^{(2)} < y_n) = \prod_{j=1}^{n} \mathbb{P}(V_{1,j} = 0) \cdot \mathbb{P}(D_{2,n}) +$$

$$\sum_{i=1}^{n} \mathbb{P}(V_{1,i} = 1) \left( \prod_{i \neq j} \mathbb{P}(V_{1,j} = 0) \right) \left( \prod_{\ell=2}^{i} \prod_{j=\ell}^{n} \mathbb{P}(V_{\ell,j} = 0) \right) \cdot \mathbb{P}(D_{i+1,n}), \quad (6.3)$$

where by convention we set $\prod_{k=2}^{1} = 1$. Note that the formula is recursive in that sense that the probability of the elementary events $D_{n,n}$ can immediately be calculated by noticing that $D_{n,n} = \{V_{n,n} \leq 1\}$ and

$$\mathbb{P}(V_{\ell,j} = k) = e^{-(\lambda_{\ell} - \lambda_{\ell-1}) \Lambda([y_{j-1}, y_j])} \cdot \frac{(\lambda_{\ell} - \lambda_{\ell-1})^k \cdot \Lambda([y_{j-1}, y_j])^k}{k!}.$$  

For $i < n$ the events $D_{i,n}$ are of the form of the lefthand side of (6.3) with smaller $n$ which specifies the recursion.

Notice also that $\Delta Y_{\lambda_1}^{(2)} < y_1$ if and only if $\sum_{j=1}^{n} V_{1,j} \leq 1$. First assume $\sum_{j=1}^{n} V_{1,j} = 0$. Then it suffices to have $\sum_{\ell=2}^{k} \sum_{j=\ell}^{n} V_{\ell,j} \leq 1$ for all $k \in \{2, \ldots, n\}$. This is equivalent to $D_{2,n}$ and gives the first summand of (6.3).

To obtain the second summand of (6.3) let us assume that $\sum_{j=1}^{n} V_{1,j} = 1$, which is equivalent to the statement that there exists $i \in \{1, \ldots, n\}$ such that $V_{1,i} = 1$ and $V_{1,\ell} = 0$ for all $\ell \neq i$ which are represented in the sum in (6.3).
Assume that this is the case and remember from Section 4.1 that \( \{ \Delta Y_{\lambda_k}^{(2)} < y_k \} = \{ \sum_{\ell=1}^{k} \sum_{j=k}^{n} V_{\ell,j} \leq 1 \} \) holds for all \( k \in \{ 2, \ldots, i \} \). Then in order that \( \bigcap_{k=1}^{i} \{ \Delta Y_{\lambda_k}^{(2)} < y_k \} \) holds it is necessary and sufficient that \( \sum_{\ell=2}^{k} \sum_{j=k}^{n} V_{\ell,j} = 0 \) for all \( k \in \{ 2, \ldots, i \} \). This in turn is equivalent to \( V_{\ell,j} = 0 \) for all pairs \( \ell, j \) with \( \ell \in \{ 2, \ldots, i \} \) and \( j \in \{ \ell, \ldots, n \} \).

Given this is the case, then for each of the events \( \{ \Delta Y_{\lambda_k}^{(2)} < y_k \} = \{ \sum_{\ell=1}^{k} \sum_{j=k}^{n} V_{\ell,j} \leq 1 \} \) with \( k \in \{ i+1, \ldots, n \} \) to hold it is additionally necessary and sufficient that \( \sum_{\ell=i+1}^{k} \sum_{j=k}^{n} V_{\ell,j} \leq 1 \) for all \( k \in \{ i+1, \ldots, n \} \). The intersection over the last events with indices \( k \in \{ i+1, \ldots, n \} \) is equivalent to \( D_{i+1,n} \).

Combining all these argumentations gives the formula in (6.3). \( \square \)