Higher Level $q$-Oscillator Representations for $U_q(C_n^{(1)}), U_q(C^{(2)}(n + 1))$ and $U_q(B^{(1)}(0, n))$

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Abstract: We introduce higher level $q$-oscillator representations for the quantum affine (super)algebras of type $C_n^{(1)}, C^{(2)}(n + 1)$ and $B^{(1)}(0, n)$. These representations are constructed by applying the fusion procedure to the level one $q$-oscillator representations which were obtained through the studies of the tetrahedron equation. We prove that these higher level $q$-oscillator representations are irreducible. For type $C_n^{(1)}$ and $C^{(2)}(n + 1)$, we compute their characters explicitly in terms of Schur polynomials.

1. Introduction

Let $\mathfrak{g}$ be an affine Lie algebra and $U_q(\mathfrak{g})$ the Drinfeld-Jimbo quantum group (without derivation) associated to it. For a node $r$ of the Dynkin diagram of $\mathfrak{g}$ except 0 and a positive integer $s$, there exists a family of finite-dimensional $U_q(\mathfrak{g})$-modules $W^{r,s}$ called Kirillov–Reshetikhin modules. They have distinguished properties. One of them is the existence of crystal bases in Kashiwara’s sense (see [1,6,21] and references therein).

Consider the affine Lie algebras $\mathfrak{g} = B_n^{(1)}, D_n^{(1)}, D_{n+1}^{(2)}$, whose Dynkin diagrams are given in the left side of Table 1. The Kirillov–Reshetikhin modules $W^{n,1}$ corresponding to the node $n$ and the integer 1 have a particularly simple structure. Let $V$ be a two-dimensional vector space. Then $W^{n,1}$ can be realized as $V^\otimes n$ with an easy description of the $U_q(\mathfrak{g})$-action. It is irreducible when $\mathfrak{g} = B_n^{(1)}, D_{n+1}^{(2)}$, but for $\mathfrak{g} = D_n^{(1)}$ it decomposes into two components; $V^\otimes n = W^{n,1} \oplus W^{n-1,1}$.

We can consider the quantum $R$ matrix on the tensor product of Kirillov–Reshetikhin modules. We introduce a spectral parameter $x$ to the representation $W^{n,1}$, and denote the
associated representation by $W^{n,1}(x)$. Let $\Delta$ be the coproduct and $\Delta^{\text{op}}$ its opposite. Then the quantum $R$ matrix $R(x/y)$ is defined as an intertwiner of $\Delta$ and $\Delta^{\text{op}}$, namely, linear operator satisfying $R(x/y)\Delta(u) = \Delta^{\text{op}}(u)R(x/y)$ for any $u \in U_q(g)$ on $W^{n,1}(x) \otimes W^{n,1}(y)$. ($R$ is found to depend only on $x/y$.)

In [17], Kuniba and Sergeev initiated an attempt to obtain quantum $R$ matrices from the solution to the tetrahedron equation, a three-dimensional analogue of the Yang–Baxter equation [24]. Let $\mathcal{L}$ be a solution of the tetrahedron equation. It is a linear operator on $F \otimes V \otimes V$ where $F$ is an infinite-dimensional vector space spanned by $\{ |m\rangle | m \in \mathbb{Z}_{\geq 0} \}$. By composing this $\mathcal{L}$ $n$ times and applying suitable boundary vectors in $F$ and $F^\ast$, they obtained linear operators on $(V \otimes^n) \otimes (V \otimes^n)$ satisfying the Yang–Baxter equation. The commuting symmetry algebras were found to be $U_q(g)$ for $g$ in Table 1. The reason they have variations is that there are two choices of boundary vectors in each $F$ and $F^\ast$ corresponding to the shapes of the Dynkin diagrams at each end.

Associated to the tetrahedron equation, there is yet another solution $\mathcal{R}$, which is a linear operator on $F^{\otimes 3}$. In [15], Kuniba and the second author performed the same scheme to $\mathcal{R}$ and constructed linear operators on $(F^{\otimes n}) \otimes (F^{\otimes n})$. As the symmetry algebra this time, they found $U_q(C_n^{(1)})$, $U_q(D_{n+1}^{(2)})$ and $U_q(A_{2n}^{(2)})$, and they called these representations $\mathcal{W} = F^{\otimes n}$ $q$-oscillator representations (of level one). To be precise, there are two irreducible components $\mathcal{W}_+$ and $\mathcal{W}_-$ for type $C_n^{(1)}$, and one can think of $\mathcal{W}$ as either $\mathcal{W}_+$ or $\mathcal{W}_-$. By construction, $\mathcal{W}$ is a bosonic analogue of $W^{n,1}$.

The purpose of this paper is to introduce and study a higher level $q$-oscillator representation corresponding to $W^{n,s}$ for $s \geq 1$. To construct a higher level $q$-oscillator representation, we apply the fusion construction [11] to $\mathcal{W}$ as in the case of $W^{n,s}$. There is, however, a difficulty in understanding $\mathcal{W}$ and its tensor power since $\mathcal{W}$ does not have a suitable classical limit ($q \to 1$) for $D_{n+1}^{(2)}$ and $A_{2n}^{(2)}$. So we first resolve this difficulty by considering $\mathcal{W}$ for these two types as $q$-oscillator representations over the quantum affine superalgebras $U_q(\mathcal{g})$ with $\mathcal{g} = C_2^{(2)}(n+1)$ and $B_1^{(1)}(0,n)$ given in the right side of Table 1, respectively. Note that the filled nodes in the Dynkin diagrams signify anisotropic odd simple roots. If they were not filled, then the Dynkin diagrams for $C_2^{(2)}(n+1)$ and $B_1^{(1)}(0,n)$ would be the ones for $D_{n+1}^{(2)}$ and $A_{2n}^{(2)}$, respectively, where the diagram for $A_{2n}^{(2)}$ is the same as $A_{2n}^{(2)}$ with the opposite labeling of nodes. This step is important for our construction of higher level representations. We use the twistor on quantum covering groups [5], which provides explicit connection between representations of $U_q(D_{n+1}^{(2)}$) (resp. $U_q(A_{2n}^{(2)})$) and $U_q(C_2^{(2)}(n+1)$) (resp. $B_1^{(1)}(0,n)$).

We then investigate the quantum $R$ matrices for $\mathcal{W}(x) \otimes \mathcal{W}(y)$ and apply the fusion construction [11] to obtain a higher level representation $\mathcal{W}^{(s)}$ for $s \in \mathbb{Z}_{\geq 0}$ and $\mathcal{g} = C_2^{(2)}(n+1)$ and $B_1^{(1)}(0,n)$.
Table 2. Dynkin diagrams of $(G_\infty, \overline{G}_\infty)$

\begin{center}
\begin{tabular}{c|c|c}

\hline

$B_\infty$ & $B(0, \infty)$ & \\
\hline

$D_\infty$ & $C_\infty$ & \\
\hline

\end{tabular}
\end{center}

$C_n^{(1)}$, $C_n^{(2)}(n+1)$, $B_n^{(1)}(0,n)$ in Table 1. We prove that $\mathcal{W}^{(s)}$ is an irreducible $U_q(\overline{g})$-module. Moreover, we have a stronger result when $\overline{g} = C_n^{(1)}$, $C_n^{(2)}(n+1)$. In this case, we show that $\mathcal{W}^{(s)}$ is classically irreducible, that is, irreducible as a representation of the subalgebra corresponding to the nodes $\{1, \ldots, n\}$ in Table 1, and show that its classical limit is isomorphic to an irreducible highest weight oscillator representation of type $C_n$ and $B(n,0)$ (see [2,19] and references therein). We compute the character of $\mathcal{W}^{(s)}$ explicitly, which is given by a multiplicity-free sum of Schur polynomials. We also give a conjectural character formula when $\overline{g} = B_n^{(1)}(0,n)$. In Appendix A, we explain how to construct a level one $q$-oscillator representation

It is rather surprising that the $U_q(g)$-module $W^{n,s}$ corresponding to the $U_q(\overline{g})$-module $\mathcal{W}^{(s)}$ is also classically irreducible when $(g, \overline{g}) = (D_n^{(1)}, C_n^{(1)})$ and $(D_{n+1}^{(2)}, C_n^{(2)}(n+1))$. We would like to point out that this correspondence between $W^{n,s}$ and $\mathcal{W}^{(s)}$ also occurs in the context of super duality [4] as representations of finite-dimensional simple Lie (super)algebras after a classical limit. One can also mix $L$ and $R$ to construct a new solution to the Yang-Baxter equation. This procedure gives rise to a new quantum group [16].

The theory of super duality is an equivalence between certain parabolic Bernstein–Gelfand–Gelfand categories of classical Lie (super)algebras of infinite-rank. As a special case of this duality, this yields an equivalence between the categories for $G_\infty$ and $\overline{G}_\infty$, where $(G_\infty, \overline{G}_\infty) = (B_\infty, B(0, \infty))$, $(D_\infty, C_\infty)$ with Dynkin diagrams given in Table 2 [3,4]. Let $G_n$ and $\overline{G}_n$ denote the subalgebras of $G_\infty$ and $\overline{G}_\infty$ of finite rank $n$, respectively. Let $V_\infty$ be a given integrable highest weight $G_\infty$-module. Under this equivalence, it corresponds to an irreducible highest weight $\overline{G}_\infty$-module, say $W_\infty$, called an oscillator representation. By applying a truncation functor to $V_\infty$ and $W_\infty$, we also obtain irreducible representations $V_n$ and $W_n$ of $G_n$ and $\overline{G}_n$, respectively. Let $(g, \overline{g})$ be one of the pairs in Table 1, and suppose that $G_n$ and $\overline{G}_n$ be the subalgebra of $g$ and $\overline{g}$ corresponding to $\{1, \ldots, n\}$, respectively. Then we observe that when $V_n$ is the classical limit of $W^{n,s}$, then the corresponding $W_n$ is the classical limit of $\mathcal{W}^{(s)}$ when $(g, \overline{g}) = (D_n^{(1)}, C_n^{(1)})$ and $(D_{n+1}^{(2)}, C_n^{(2)}(n+1))$. Conjecture 5.22 on $\mathcal{W}^{(s)}$ for $\overline{g} = B_n^{(1)}(0,n)$ is based on this observation in case of $(g, \overline{g}) = (B_n^{(1)}, B_n^{(1)}(0,n))$, which is true for $s = 2$. We strongly expect that there is a quantum affine analogue of super duality which relates the category of finite-dimensional $U_q(g)$-modules and a suitable category of infinite-dimensional $U_q(\overline{g})$-modules including the $q$-oscillator representations, and hence explains the correspondence in this paper (cf. [20]).

The paper is organized as follows: In Section 2, we briefly review the notion of quantum superalgebras. In Section 3, we construct a level one $q$-oscillator representation $\mathcal{W}$ of $U_q(\overline{g})$ and study some of its properties including the crystal base. In Section 4, we consider the quantum $R$ matrix on $\mathcal{W}(x) \otimes \mathcal{W}(y)$ and apply the fusion construction to define $\mathcal{W}^{(s)}$. In Section 5, we prove the classically irreducibility of $\mathcal{W}^{(s)}$ and give its character formula when $\overline{g} = C_n^{(1)}$, $C_n^{(2)}(n+1)$. A conjecture when $\overline{g} = B_n^{(1)}(0,n)$ is also given.
of $U_q(\mathfrak{g})$ when $\mathfrak{g} = C(2)(n + 1)$ and $B(1)(0, n)$ from the one for $D_{n+1}^{(2)}$ and $A_{2n}^{(2)}$ in [15], respectively, by using the quantum covering groups and twistor [5]. In Appendices B and C, we construct the quantum $R$ matrix on $\mathcal{W}(x) \otimes \mathcal{W}(y)$ for $U_q(\mathfrak{g})$ from the one in [15]. In Appendix D, we prove the irreducibility of $\mathcal{W}(s)$.

2. Quantum Superalgebras

2.1. Variant of $q$-integer. Throughout the paper, we let $q$ be an indeterminate. Following [5], we introduce variants of $q$-integer, $q$-factorial and $q$-binomial coefficient. Let $\epsilon = \pm 1$. For $m \in \mathbb{Z}_{\geq 0}$, we set

$$[m]_{q, \epsilon} = \frac{(\epsilon q)^m - q^{-m}}{\epsilon q - q^{-1}}.$$ 

For $m \in \mathbb{Z}_{\geq 0}$, set

$$[m]_{q, \epsilon}! = [m]_{q, \epsilon}[m - 1]_{q, \epsilon} \cdots [1]_{q, \epsilon} \quad (m \geq 1), \quad [0]_{q, \epsilon}! = 1.$$

For integers $m, n$ such that $0 \leq n \leq m$, we define

$$\begin{bmatrix} m \\ n \end{bmatrix}_{q, \epsilon} = \frac{[m]_{q, \epsilon}!}{[n]_{q, \epsilon}[m - n]_{q, \epsilon}!}.$$ 

They all belong to $\mathbb{Z}[q, q^{-1}]$. Let $A_0$ be the subring of $\mathbb{Q}(q)$ consisting of rational functions without a pole at $q = 0$. Then we have

$$[m]_{q, \epsilon} \in q^{-1-m}(1 + q A_0), \quad [m]_{q, \epsilon}! \in q^{-m(m-1)/2}(1 + A_0), \quad \begin{bmatrix} m \\ n \end{bmatrix}_{q, \epsilon} \in q^{-n(n-1)}(1 + q A_0).$$

We simply write $[m] = [m]_{q, 1}$, $[m]! = [m]_{q, 1}$ and $\begin{bmatrix} m \\ n \end{bmatrix} = \begin{bmatrix} m \\ n \end{bmatrix}_{q, 1}$.

2.2. Quantum (super)algebra $U_q(sl_2)$ and $U_q(osp_{1|2})$. The quantum (super)algebras $U_q(sl_2)$ ($\epsilon = 1$) and $U_q(osp_{1|2})$ ($\epsilon = -1$) are defined as a $\mathbb{Q}(q)$-algebra generated by $e, f, k^\pm$ satisfying the following relations:

$$kk^{-1} = k^{-1}k = 1, \quad kek^{-1} = q^2e, \quad kfk^{-1} = q^{-2}f, \quad ef - fe = \frac{k - k^{-1}}{q - q^{-1}}.$$ 

Set $e^{(m)} = e^m/[m]_{q, \epsilon}!$ and $f^{(m)} = f^m/[m]_{q, \epsilon}!$. We will use the following formula.

**Proposition 2.1.**

$$e^{(m)} f^{(n)} = \sum_{j \geq 0} \frac{e^{mn-j(j+1)/2}}{[j]_{q, \epsilon}!} f^{(n-j)} \left( \prod_{l=0}^{j-1} \frac{(\epsilon q)^{2j-m-n-l}k - q^{-2j+m+n+l}k^{-1}}{q - q^{-1}} \right) e^{(m-j)}.$$ 

**Proof.** The $U_q(sl_2)$ ($\epsilon = 1$) case is derived easily from [13, (1.1.23)]. The $U_q(osp_{1|2})$ ($\epsilon = -1$) case can be shown by induction. $\square$
2.3. Quantum affine (super)algebras $U_q(C^{(1)}_n)$, $U_q(C^{(2)}(n + 1))$, $U_q(B^{(1)}(0, n))$. Set $I = [0, 1, \ldots, n]$. In this paper, we consider the following three Cartan data $(a_{ij})_{i,j \in I}$, or Dynkin diagrams (cf. [10]), and $(d_i)_{i \in I}$ such that $d_i a_{ij} = d_j a_{ji}$ for $i, j \in I$.

- $C^{(1)}_n$:

  
  \[
  (a_{ij})_{i,j \in I} = \begin{pmatrix}
  2 & -1 & -1 & \ldots & -1 \\
  -2 & 2 & 2 & \ldots & 2 \\
  -1 & -1 & -1 & \ldots & -1 \\
  \end{pmatrix}
  \]

  
  $(d_i)_{i \in I} = (2, 1, \ldots, 1, 2)$

- $C^{(2)}(n + 1)$:

  
  \[
  (a_{ij})_{i,j \in I} = \begin{pmatrix}
  2 & -2 & -1 & -1 & \ldots & -1 \\
  -2 & 2 & 2 & 2 & \ldots & 2 \\
  -1 & -1 & -1 & -1 & \ldots & -1 \\
  \end{pmatrix}
  \]

  
  $(d_i)_{i \in I} = (\frac{1}{2}, 1, \ldots, 1, \frac{1}{2})$

- $B^{(1)}(0, n)$:

  
  \[
  (a_{ij})_{i,j \in I} = \begin{pmatrix}
  2 & -1 & -1 & \ldots & -1 \\
  -2 & 2 & 2 & \ldots & 2 \\
  -1 & -1 & -1 & \ldots & -1 \\
  \end{pmatrix}
  \]

  
  $(d_i)_{i \in I} = (2, 1, \ldots, 1, \frac{1}{2})$.

Let $d = \min\{d_i \mid i \in I\}$. For $i \in I$, let $q_i = q^{d_i}$, and let $p(i) = 0, 1$ such that $p(i) \equiv 2d_i \pmod{2}$. Set

\[
[m]_i = [m]_{q_i, (-1)^{p(i)}}, \quad [m]_i! = [m]_{q_i, (-1)^{p(i)}}, \quad \begin{pmatrix} m \\ k \end{pmatrix}_{q_i, (-1)^{p(i)}} = \begin{pmatrix} m \\ k \end{pmatrix},
\]

for $0 \leq k \leq m$ and $i \in I$. 


For a Cartan datum $X = C_n^{(1)}, C^{(2)}(n + 1), B^{(1)}(0, n)$, the quantum affine (super)algebra $U_q(X)$ is defined to be the $\mathbb{Q}(q^d)$-algebra generated by $e_i, f_i, k_i^{\pm 1}$ ($i \in I$) with the following relations:

$$k_i k_j = k_j k_i, \quad k_i k_i^{-1} = k_i^{-1} k_i = 1, \quad k_i e_j k_i^{-1} = q_i^{a_{ij}} e_j, \quad k_i f_j k_i^{-1} = q_i^{-a_{ij}} f_j,$$

$$e_i f_j - (-1)^{p(i)p(j)} f_j e_i = \delta_{ij} \frac{k_i - k_i^{-1}}{q_i - q_i^{-1}},$$

$$\sum_{m=0}^{1-a_{ij}} (-1)^{m+p(i)m(m-1)/2+m p(i)p(j)} e_i^{(1-a_{ij}-m)} e_j e_i^{(m)} = 0 \quad (i \neq j),$$

$$\sum_{m=0}^{1-a_{ij}} (-1)^{m+p(i)m(m-1)/2+m p(i)p(j)} f_i^{(1-a_{ij}-m)} f_j f_i^{(m)} = 0 \quad (i \neq j),$$

where

$$e_i^{(m)} = \frac{e_i^m}{[m]_i!}, \quad f_i^{(m)} = \frac{f_i^m}{[m]_i!}.$$

We define the automorphism $\tau$ of $U_q(X)$ for $X = C_n^{(1)}, C^{(2)}(n + 1)$ by

$$\tau(k_i) = k_i^{-1}, \quad \tau(e_i) = f_{n-i}, \quad \tau(f_i) = e_{n-i}, \quad \text{if } X = C_n^{(1)};$$

$$\tau(k_i) = k_i^{-1}, \quad \tau(e_i) = (-1)^{\delta_{i0}} n f_{n-i}, \quad \tau(f_i) = (-1)^{\delta_{i0}} e_{n-i}, \quad \text{if } X = C^{(2)}(n + 1),$$

for $i \in I$ and define the anti-automorphism $\eta$ of $U_q(X)$ by

$$\begin{cases}
\eta(k_i) = k_i \\
\eta(e_i) = (-1)^{\delta_{i0}+\delta_{in}} q_i^{-1} k_i^{-1} f_i & \text{if } X = C_n^{(1)}, B^{(1)}(0, n), \\
\eta(f_i) = (-1)^{\delta_{i0}+\delta_{in}} q_i^{-1} k_i e_i \\
\eta(k_i) = k_i \\
\eta(e_i) = (-1)^{\delta_{in}} q_i^{-1} k_i^{-1} f_i & \text{if } X = C^{(2)}(n + 1), \\
\eta(f_i) = (-1)^{\delta_{in}} q_i^{-1} k_i e_i
\end{cases}$$

for $i \in I$. Both $\tau$ and $\eta$ are involutions.

When $X = C^{(2)}(n + 1), B^{(1)}(0, n)$, let

$$U_q(X)^\sigma = U_q(X) \oplus U_q(X)\sigma$$

denote the $\mathbb{Q}(q^d)$-algebra generated by $U_q(X)$ and $\sigma$, where

$$\sigma^2 = 1, \quad \sigma k_i = k_i \sigma, \quad \sigma e_i = (-1)^{p(i)} e_i \sigma, \quad \sigma f_i = (-1)^{p(i)} f_i \sigma \quad (i \in I).$$

We extend $\tau$ and $\eta$ to $U_q(X)^\sigma$ by $\tau(\sigma) = \eta(\sigma) = \sigma$.

The algebras $U_q(C_n^{(1)}), U_q(C^{(2)}(n + 1))^\sigma$ and $U_q(B^{(1)}(0, n))^\sigma$ have a Hopf algebra structure. In particular, the coproduct $\Delta$ is given by

$$\begin{align*}
\Delta(k_i) &= k_i \otimes k_i, \quad \Delta(\sigma) = \sigma \otimes \sigma, \\
\Delta(e_i) &= e_i \otimes \sigma^{p(i)\delta_{i0}} k_i^{-1} + \sigma^{p(i)\delta_{in}} \otimes e_i, \\
\Delta(f_i) &= f_i \otimes \sigma^{p(i)\delta_{i0}} + \sigma^{p(i)\delta_{in}} k_i \otimes f_i,
\end{align*}$$

for $i \in I$. 
3. Level One $q$-Oscillator Representation

Let $\mathcal{W}$ be an infinite-dimensional vector space over $\mathbb{Q}(q^d)$ defined by
\[ \mathcal{W} = \bigoplus_m \mathbb{Q}(q^d) |m\rangle, \]
where $|m\rangle = |m_1, \ldots, m_n\rangle$ is a basis vector parametrized by $m = (m_1, \ldots, m_n) \in \mathbb{Z}^n_{\geq 0}$. Let $|m\rangle = \sum_{j=1}^n m_j$ and let $e_j$ be the $j$-th standard vector in $\mathbb{Z}^n$ for $1 \leq j \leq n$. Let $|0\rangle = |0, \ldots, 0\rangle$. In this section, we introduce the so-called $q$-oscillator representation of level one for each algebra.

3.1. Type $C_n^{(1)}$. 

3.1.1. $U_q(C_n^{(1)})$-module $\mathcal{W}_\pm$ Consider the quantum affine algebra $U_q(C_n^{(1)})$. Let $U_q(C_n)$ and $U_q(A_{n-1})$ be the subalgebras generated by $e_i$, $f_i$, $k^\pm_1$ for $i \in I \setminus \{0\}$ and $i \in I \setminus \{0, n\}$, respectively.

**Proposition 3.1.** For a non-zero $x \in \mathbb{Q}(q)$, the space $\mathcal{W}$ admits a $U_q(C_n^{(1)})$-module structure given as follows:
\[
\begin{align*}
e_0|m\rangle &= xq^{-1} \frac{[m_1 + 1][m_1 + 2]}{[2]} |m + 2e_1\rangle, \\
f_0|m\rangle &= -x^{-1} q^{m_1} \frac{1}{[2]} |m - 2e_1\rangle, \\
k_0|m\rangle &= q^{2m_1+1} |m\rangle, \\
e_j|m\rangle &= [m_{j+1} + 1] |m - e_j + e_{j+1}\rangle, \\
f_j|m\rangle &= [m_{j} + 1] |m + e_j - e_{j+1}\rangle, \\
k_j|m\rangle &= q^{-m_j} |m\rangle, \\
e_n|m\rangle &= -q^{m_n} \frac{1}{[2]} |m - 2e_n\rangle, \\
f_n|m\rangle &= q^{-1} \frac{[m_n + 1][m_n + 2]}{[2]} |m + 2e_n\rangle, \\
k_n|m\rangle &= q^{-2m_n-1} |m\rangle,
\end{align*}
\]
where $1 \leq j \leq n - 1$. Here we understand the vector on the right-hand side is zero when any of its components does not belong to $\mathbb{Z}_{\geq 0}$.

**Remark 3.2.** For $|m\rangle = |m_1, \ldots, m_n\rangle \in \mathcal{W}$, set $\tau(|m\rangle) = |m_n, \ldots, m_1\rangle$, and extend linearly to any vector of $\mathcal{W}$. Then, when $x = 1$ we have the following symmetry
\[ \tau(u|m\rangle) = \tau(u) \tau(|m\rangle), \]
for $u \in U_q(C_n^{(1)})$. Here the automorphism $\tau$ on $U_q(C_n^{(1)})$ is given in (2.1).
Lemma 3.5. The space \( |m|_{\text{new}} = \frac{(q[2])^{|m|/2}}{\prod_{i=1}^{n}|m_{i}|!} |m|_{\text{old}}, \)
and the automorphism of \( U_q(C^{(1)}_n) \) sending \( f_0 \mapsto -f_0, e_n \mapsto -e_n, k_i \mapsto -k_i \) for \( i = 0, n \) with the other generators fixed.

We assume that \( \varepsilon \) denotes + or −. Set \( \zeta(\varepsilon) = 0 \) and \( \varepsilon = + \) and −, respectively. For \( m \in \mathbb{Z}_{\geq 0} \), let \( \text{sgn}(m) \) be + and − if \( m \) is even and odd, respectively. Define the subspace \( \mathcal{W}_\varepsilon \) of \( \mathcal{W} \) by
\[
\mathcal{W}_\varepsilon = \bigoplus_{\text{sgn}(|m|) = \varepsilon} \mathbb{Q}(q)|m\rangle.
\]

Proposition 3.4. For a non-zero \( x \in \mathbb{Q}(q) \), \( \mathcal{W}_\varepsilon \) is an irreducible \( U_q(C^{(1)}_n) \)-module.

We denote this module by \( \mathcal{W}_\varepsilon(x) \), and call it a (level one) \( q \)-oscillator representation. We simply write \( \mathcal{W}_\varepsilon = \mathcal{W}_\varepsilon(1) \) as a \( U_q(C^{(1)}_n) \)-module. Then as \( U_q(A_{n-1}) \)-modules, the characters of \( \mathcal{W}_\pm \) are given by
\[
\text{ch} \mathcal{W}_+ = \sum_{l \in 2\mathbb{Z}_{\geq 0}} s(l)(x_1, \ldots, x_n) = \frac{1}{\prod_{i=1}^{n}(1-x_i^2)},
\]
\[
\text{ch} \mathcal{W}_- = \sum_{l \in 1+2\mathbb{Z}_{\geq 0}} s(l)(x_1, \ldots, x_n) = \frac{\prod_{i=1}^{n}(1+x_i) - 1}{\prod_{i=1}^{n}(1-x_i^2)},
\]
where \( s(l)(x_1, \ldots, x_n) \) is the Schur polynomial corresponding to the partition \( (l) \). Here the weight lattice for \( A_{n-1} \) is identified with the \( \mathbb{Z} \)-lattice spanned by \( e_i \) for \( 1 \leq i \leq n \), and hence the variable \( x_i \) corresponds to the weight of \( e_i \).

3.1.2. Classical limit Let \( A \) be the localization of \( \mathbb{Z}[q, q^{-1}] \) at \([2] = q + q^{-1} \). Let
\[
\mathcal{W}_\varepsilon(x)_A = \sum_{\text{sgn}(|m|) = \varepsilon} A|m\rangle.
\]
Then \( \mathcal{W}_\varepsilon(x)_A \) is invariant under \( e_i, f_i, k_i \) and \( \{k_i\} := \frac{k_i - k_i^{-1}}{q_i - q_i^{-1}} \) for \( i \in I \setminus \{0\} \). Let
\[
\overline{\mathcal{W}_\varepsilon(x)} = \mathcal{W}_\varepsilon(x)_A \otimes_A \mathbb{C},
\]
where \( \mathbb{C} \) is an \( A \)-module such that \( f(q) \cdot c = f(1)c \) for \( f(q) \in A \) and \( c \in \mathbb{C} \).

Let \( E_i, F_i \) and \( H_i \) be the \( \mathbb{C} \)-linear endomorphisms on \( \overline{\mathcal{W}_\varepsilon(x)} \) induced from \( e_i, f_i \) and \( \{k_i\} \) for \( i \in I \setminus \{0\} \). We can check that they satisfy the defining relations for the universal enveloping algebra \( U(C_n) \) of type \( C_n \) (cf. [8, Chapter 5]). Hence \( \overline{\mathcal{W}_\varepsilon(x)} \) becomes a \( U(C_n) \)-module.

Lemma 3.5. The space \( \overline{\mathcal{W}_\varepsilon(x)} \) is isomorphic to the irreducible highest weight \( U(C_n) \)-module with highest weight \( -(\frac{1}{2} + \zeta(\varepsilon))\varpi_n \), where \( \varpi_n \) is the \( n \)-th fundamental weight for \( C_n \).
Proof. It is clear that $E_i(|0\rangle \otimes 1) = 0$ for all $i \in I \setminus \{0\}$. Since

$$H_n(|0\rangle \otimes 1) = \left(\frac{k_n - k_n^{-1}}{q^n - q_n^{-1}}|0\rangle\right) \otimes 1 = \left(-\frac{1}{q + q^{-1}}|0\rangle\right) \otimes 1 = -\frac{1}{2}|0\rangle \otimes 1,$$

and $H_i(|0\rangle \otimes 1) = 0$ for $1 \leq i \leq n - 1$, $\mathcal{W}_+(x)$ is a highest weight $U(C_n)$-module with highest weight $-\frac{1}{2}\omega_n$. It follows from the actions of $E_i$ for $i \in I \setminus \{0\}$ that any submodule of $\mathcal{W}_+(x)$ contains $|0\rangle \otimes 1$. This implies that $\mathcal{W}_+(x)$ is irreducible. The proof for $\mathcal{W}_-(x)$ is similar. \qed

3.1.3. Polarization Define a symmetric bilinear form on $\mathcal{W}_c$ by

$$(|\mathbf{m}\rangle, |\mathbf{m}'\rangle) = \delta_{\mathbf{m},\mathbf{m}'} q^{-\frac{1}{2} \sum_{i=1}^n m_i(m_i-1)} \prod_{i=1}^n [m_i]!,$$ (3.1)

for $|\mathbf{m}\rangle, |\mathbf{m}'\rangle$ with $\mathbf{m} = (m_1, \ldots, m_n)$. Note that $(|\mathbf{m}\rangle, |\mathbf{m}\rangle) \in 1 + qA_0$.

Lemma 3.6. The bilinear form in (3.1) is a polarization on $\mathcal{W}_c$, that is,

$$(uv, v') = (v, \eta(u)v'),$$

for $u \in U_q(C_n^{(1)})$ and $v, v' \in \mathcal{W}_c$.

Proof. It suffices to show this when $u$ is one of the generators. If $u = k_i$, it is trivial. Let us show that

$$(e_i|\mathbf{m}\rangle, |\mathbf{m}'\rangle) = (|\mathbf{m}\rangle, \eta(e_i)|\mathbf{m}'\rangle),$$ (3.2)

for $i \in I$ and $|\mathbf{m}\rangle, |\mathbf{m}'\rangle \in \mathcal{W}_c$. The proof for $f_i$ is almost identical since (3.1) is symmetric.

Case 1. Suppose that $1 \leq i \leq n - 1$. We may assume $\mathbf{m}' = \mathbf{m} - e_i + e_{i+1}$. The right-hand side is

$$(|\mathbf{m}\rangle, \eta(e_i)|\mathbf{m} - e_i + e_{i+1}\rangle) = (|\mathbf{m}\rangle, q_i^{-1}k_i^{-1}f_i|\mathbf{m} - e_i + e_{i+1}\rangle) = [m_i]q^{-1+m_i-m_{i+1}}(|\mathbf{m}\rangle, |\mathbf{m}\rangle),$$

and the left-hand side is

$$(e_i|\mathbf{m}\rangle, |\mathbf{m} - e_i + e_{i+1}\rangle) = [m_{i+1} + 1](|\mathbf{m} - e_i + e_{i+1}\rangle, |\mathbf{m} - e_i + e_{i+1}\rangle)$$

$$= q^{X [m_{i+1} + 1]} [m_{i+1} + 1]! \prod_{j \neq i, i+1} [m_j]!$$

$$= q^{m_i - m_{i+1} - 1}[m_i](|\mathbf{m}\rangle, |\mathbf{m}\rangle),$$

since

$$X = -\frac{1}{2} \sum_{j \neq i, i+1} m_j(m_j - 1) - \frac{1}{2} (m_i - 1)(m_i - 2) - \frac{1}{2} (m_{i+1} + 1)m_{i+1}$$

$$= -\frac{1}{2} \sum_{1 \leq j \leq n} m_j(m_j - 1) + m_i - m_{i+1} - 1.$$ 

Hence (3.2) holds.
Case 2. Suppose that \( i = n \). We may assume \( m' = m - 2e_n \). The right-hand side is

\[
(|m|, \eta(e_n)|m - 2e_n|) = (|m|, -q_n^{-1}k_n^{-1}f_n|m - 2e_n|) = -q^{2m_n-2} \frac{[m_n - 1][m_n]}{[2]} (|m|, |m|),
\]

and the left-hand side is

\[
(e_n|m|, |m - 2e_n|) = -\frac{q}{[2]} (|m - 2e_n|, |m - 2e_n|) = -\frac{q}{[2]} \frac{q^Y}{(m_n - 2)! \prod_{j \neq n} [m_j]!} (|m|, |m|)
\]

\[
= -q^{2m_n-2} \frac{[m_n - 1][m_n]}{[2]} (|m|, |m|),
\]

since

\[
Y = -\frac{1}{2} \sum_{j \neq n} m_j(m_j - 1) - \frac{1}{2} (m_n - 2)(m_n - 3) = -\frac{1}{2} \sum_{1 \leq j \leq n} m_j(m_j - 1) + 2m_n - 3.
\]

Hence (3.2) holds.

Case 3. Suppose that \( i = 0 \). We have to show \((e_0v, v') = (v, -q^{-2}k_0^{-1}f_0v')\).

By Remark 3.2 and the property \((\tau(|m|), \tau(|m'|)) = (|m|, |m'|)\), it is equivalent to \((f_n \tau(v), \tau(v')) = (\tau(v), -q^{-2}k_ne_n\tau(v'))\). However, it is equivalent to the one proved in Case 1. \( \Box \)

3.1.4. Crystal base Let \( M \) be a \( U_q(C_n^{(1)}) \)-module. For \( 1 \leq j \leq n - 1 \), we assume that \( e_j \) and \( f_j \) are locally nilpotent on \( M \), and define \( \tilde{e}_j, \tilde{f}_j \) to be the usual lower crystal operators [13]. For \( i = 0, n \), we introduce new operators \( \tilde{e}_i \) and \( \tilde{f}_i \) as follows:

Case 1. Let \( u \in M \) be a weight vector such that \( e_nu = 0 \) and \( k_nu = q_n^{-l}u \) for some \( l \in \mathbb{Z}_{>0} \). Put

\[
uk := q_n^{\frac{k(k+2l)}{2}} f_n^{(k)} u \quad (k \geq 0).
\] (3.3)

Then we define

\[
\tilde{f}_nu_k = u_{k+1}, \quad \tilde{e}_nu_{k+1} = u_k \quad (k \geq 0).
\] (3.4)

Case 2. Let \( u \in M \) be a weight vector such that \( f_0u = 0 \) and \( k_0u = q_0^{-l}u \) for some \( l \in \mathbb{Z}_{>0} \). Put

\[
uk := q_0^{\frac{k(k+2l)}{2}} e_0^{(k)} u \quad (k \geq 0).
\] (3.5)

Then we define

\[
\tilde{e}_0u_k = u_{k+1}, \quad \tilde{f}_0u_{k+1} = u_k \quad (k \geq 0).
\] (3.6)

Remark 3.7. The definitions of \( \tilde{e}_i \) and \( \tilde{f}_i \) (\( i = 0, n \)) are based on the idea that

\[
(f_n^ku, f_n^ku) \in 1 + qA_0 \quad (e_0^ku', e_0^ku') \in 1 + qA_0 \quad (k \geq 0),
\] (3.7)

for \( u, u' \in \mathcal{W}_e \) such that \( e_nu = 0 \) and \( f_0u' = 0 \) (use Proposition 2.1).
Let $A_0$ be the subring of $\mathbb{Q}(q)$ consisting of functions which are regular at $q = 0$. We define an $A_0$-lattice $\mathcal{L}_\varepsilon$ of $\mathcal{W}_\varepsilon$ and a $\mathbb{Q}$-basis $B_\varepsilon$ of $\mathcal{L}_\varepsilon/q\mathcal{L}_\varepsilon$ by

$$\mathcal{L}_\varepsilon = \bigoplus_{\text{sgn}(m) = \varepsilon} A_0|m\rangle, \quad B_\varepsilon = \{ |m\rangle \pmod{q\mathcal{L}_\varepsilon} | \text{sgn}(m) = \varepsilon \}.$$ 

It is clear from (3.1) that $(\mathcal{L}_\varepsilon, \mathcal{L}_\varepsilon) \subset A_0$, and $B_\varepsilon$ is an orthonormal basis of $\mathcal{L}_\varepsilon/q\mathcal{L}_\varepsilon$ with respect to $(\ , \ )_{q=0}$. The following shows that $(\mathcal{L}_\varepsilon, B_\varepsilon)$ is a crystal base of $\mathcal{W}_\varepsilon$ in the sense of [13] even though the action of $e_i, f_i$ on $\mathcal{W}_\varepsilon$ for $i = 0, n$ is not locally nilpotent.

**Proposition 3.8.** The pair $(\mathcal{L}_\varepsilon, B_\varepsilon)$ is a crystal base of $\mathcal{W}_\varepsilon$, that is,

1. $\mathcal{L}_\varepsilon$ is invariant under $\tilde{e}_i$ and $\tilde{f}_i$ for $i \in I$,
2. $\tilde{e}_i B_\varepsilon \subset B_\varepsilon \cup \{0\}$ and $\tilde{f}_i B_\varepsilon \subset B_\varepsilon \cup \{0\}$ for $i \in I$, where we have

$$\tilde{f}_i|m\rangle = \begin{cases} |m + 2e_i\rangle & \text{if } i = n, \\ |m + e_i - e_{i+1}\rangle & \text{if } m_{i+1} \geq 1 \text{ and } 1 \leq i \leq n - 1, \\ |m - 2e_i\rangle & \text{if } m_1 \geq 2 \text{ and } i = 0, \\ 0 & \text{otherwise}, \end{cases} \pmod{q \mathcal{L}_\varepsilon}.$$ 

**Proof.** It is enough to prove (2).

**Case 1.** Suppose that $1 \leq i \leq n - 1$. Let $|m\rangle = |m_1, \ldots, m_n\rangle \in \mathcal{L}_\varepsilon$ be given with $m_{i+1} \geq 1$. Since $e_i|m - m_i e_i + m_i e_{i+1}\rangle = 0$, we have

$$\tilde{f}_i|m\rangle = \tilde{f}_i^{m_{i+1}}|m - m_i e_i + m_i e_{i+1}\rangle = |m + e_i - e_{i+1}\rangle,$$

and hence $\tilde{f}_i|m\rangle = \tilde{f}_i^{m_{i+1}}|m - m_i e_i + m_i e_{i+1}\rangle = |m + e_i - e_{i+1}\rangle$.

**Case 2.** Suppose that $i = n$. First, suppose that $m_n$ is even. Since $e_n|m - m_n e_n\rangle = 0$ and $k_n|m - m_n e_n\rangle = q^{-1}|m - m_n e_n\rangle$, we have

$$\tilde{f}_n^{m_n}|m - m_n e_n\rangle = q^{\frac{m_n}{2}} \cdot \tilde{f}_n^{m_n}|m - m_n e_n\rangle = (1 + q^2)^{-\frac{m_n}{2}} q^{\frac{m_n}{2}} \cdot \frac{[m_n]!}{[\frac{m_n}{2}]_n!} |m - m_n e_n\rangle,$$

and hence

$$\tilde{f}_n|m\rangle = (1 + q^2)^{-\frac{m_n}{2}} q^{-\frac{m_n}{2}} \cdot \frac{[m_n]!}{[\frac{m_n}{2}]_n!} \tilde{f}_n^{m_n}|m - m_n e_n\rangle = (1 + q^2)^{-\frac{m_n}{2}} q^{-\frac{m_n}{2}} \cdot \frac{[m_n]!}{[\frac{m_n}{2}]_n!} |m + 2e_n\rangle,$$

and

$$\tilde{f}_n|m\rangle = |m + 2e_n\rangle \pmod{q \mathcal{L}_\varepsilon}.$$ 

Since

$$q^{\frac{m_n+1}{2}} - q^{\frac{m_n}{2}} \frac{[m_n + 2][m_n + 1]}{[\frac{m_n}{2} + 1]_n} = q^{m_n+1} \frac{[m_n + 2][m_n + 1]}{[\frac{m_n}{2} + 1]_n} \in (1 + q A_0).$$
Next, suppose that \( m_n \) is odd. Since \( e_n | m - (m_n - 1)e_n \rangle = 0 \) and \( k_n | m - (m_n - 1)e_n \rangle = q^{-3} | m - (m_n - 1)e_n \rangle \), we have

\[
\tilde{f}_n^{mn-1} | m - (m_n - 1)e_n \rangle = q^{\left(\frac{m_n-1}{2}\right)\left(\frac{m_n+3}{2}\right)} f_n^{mn-1} | m - (m_n - 1)e_n \rangle \bigg/ \left(\frac{m_n+1}{2}\right)_n \]

\[
= (1 + q^2)^{-\frac{mn-1}{2}} q^{\left(\frac{m_n+1}{2}\right)\left(\frac{m_n+5}{2}\right)} [m_n]_n | m \rangle \equiv | m + 2e_n \rangle \pmod{q L_0}.
\]

and hence

\[
\tilde{f}_n | m \rangle
\]

\[
= (1 + q^2)^{-\frac{mn-1}{2}} q^{\left(\frac{m_n-1}{2}\right)\left(\frac{m_n+3}{2}\right)} f_n^{mn-1} | m - (m_n - 1)e_n \rangle \bigg/ \left(\frac{m_n+1}{2}\right)_n \]

\[
= (1 + q^2)^{-\frac{mn-1}{2}} q^{\left(\frac{m_n+1}{2}\right)\left(\frac{m_n+5}{2}\right)} (1 + q^2)^{-\frac{mn+1}{2}} q^{\left(\frac{m_n+1}{2}\right)\left(\frac{m_n+5}{2}\right)} [m_n + 2]_n | m + 2e_n \rangle \]

\[
= (1 + q^2)^{-1} q^{\left(\frac{m_n+1}{2}\right)\left(\frac{m_n+5}{2}\right)} - \frac{\left(\frac{m_n-1}{2}\right)\left(\frac{m_n+3}{2}\right)}{\left(\frac{m_n+1}{2}\right)_n} [m_n + 2][m_n + 1]_n | m + 2e_n \rangle \]

since

\[
q^{\left(\frac{m_n+1}{2}\right)\left(\frac{m_n+5}{2}\right)} - \frac{\left(\frac{m_n-1}{2}\right)\left(\frac{m_n+3}{2}\right)}{\left(\frac{m_n+1}{2}\right)_n} [m_n + 2][m_n + 1]_n = q^{m_n+1} \left(\frac{m_n+1}{2}\right)_n \in (1 + q A_0).
\]

**Case 3.** Suppose that \( i = 0 \). We can prove this case by the same arguments as in **Case 2** by using the automorphism \( \tau \) in (2.1). \( \Box \)

### 3.2. Type \( C^{(2)}(n + 1) \)

#### 3.2.1. \( U_q(C^{(2)}(n+1))-\text{module } W \)

Consider the quantum affine superalgebra \( U_q(C^{(2)}(n+1)) \). Let \( U_q(B(0, n)) \) and \( U_q(A_{n-1}) \) be the subalgebras of \( U_q(C^{(2)}(n + 1)) \) generated by \( e_i, f_i, k^\pm_i \) for \( i \in I \setminus \{0\} \) and \( i \in I \setminus \{0, n\} \), respectively. We also write \( U_q(B(0, n)) = U_q(osp_{1|2n}) \), where \( osp_{1|2n} \) is the orthosymplectic Lie superalgebra corresponding to the Dynkin diagram:

![Dynkin Diagram](image)

**Proposition 3.9.** For a non-zero \( x \in \mathbb{Q}(q^{\frac{1}{2}}) \), the space \( W \) admits an irreducible \( U_q(C^{(2)}(n + 1))^{\sigma} \)-module structure given as follows:

\[
e_0 | m \rangle = x q^{-\frac{1}{2}} [m_1 + 1]_n | m + e_1 \rangle,
\]

\[
f_0 | m \rangle = x^{-1} q^{-\frac{1}{2}} | m - e_1 \rangle,
\]
where $1 \leq j \leq n - 1$.

Proof. See Appendix A.2 

We denote this module by $\mathcal{W}(x)$ and call it a (level one) $q$-oscillator representation. We simply write $\mathcal{W} = \mathcal{W}(1)$ as a $U_q(C^{(2)}(n + 1))$-module. Note that as a $U_q(A_{n-1})$-module, the character of $\mathcal{W}$ is given by

$$\text{ch}\mathcal{W} = \sum_{l \in \mathbb{Z}_{\geq 0}} s_{(l)}(x_1, \ldots, x_n) = \frac{1}{\prod_{i=1}^{n}(1 - x_i)}.$$ 

Remark 3.10. When $x = 1$ we also have the following symmetry

$$\tau(u|m) = \tau(u)\tau(m),$$

for $u \in U_q(C^{(2)}(n + 1))$ and $m \in \mathcal{W}$ where $\tau$ is the automorphism in (2.2) (cf. Remark 3.2).

3.2.2. Classical limit

Let

$$\mathcal{W}(x)_{A} = \sum_{m} A|m|, \quad \overline{\mathcal{W}(x)} = \mathcal{W}(x)_{A} \otimes_{\mathbb{C}} \mathbb{C},$$

where $A = \mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$ and $\mathbb{C}$ is an $A$-module such that $f(q^{\frac{1}{2}})c = f(1)c$ for $f(q^{\frac{1}{2}}) \in A$ and $c \in \mathbb{C}$.

One can check directly that $\mathcal{W}(x)_A$ is invariant under $e_i$, $f_i$, and $\{k_i\}$ for $i \in I \setminus \{0\}$, and the induced operators $E_i$, $F_i$, and $H_i$ on $\overline{\mathcal{W}(x)}$, respectively, satisfy the defining relations of $U(osp_{1|2n})$.

Lemma 3.11. The space $\overline{\mathcal{W}(x)}$ is isomorphic to the irreducible highest weight $U(osp_{1|2n})$-module with highest weight $-\varpi_n$, where $\varpi_n$ is the $n$-th fundamental weight for $osp_{1|2n}$.

Proof. We have

$$H_n(|0\rangle \otimes 1) = \left( \frac{k_n - k_n^{-1}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} |0\rangle \right) \otimes 1 = \left( \frac{q^{-\frac{1}{2}} - q^{\frac{1}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} |0\rangle \right) \otimes 1 = -|0\rangle \otimes 1,$$

and $H_i(|0\rangle \otimes 1) = 0$ for $1 \leq i \leq n - 1$. By the same argument as in Lemma 3.5, $\overline{\mathcal{W}(x)}$ is an irreducible highest weight $U_q(osp_{1|2n})$-module with highest weight $-\varpi_n$. 

□
3.2.3. Polarization Define a symmetric bilinear form on \( \mathcal{W} \) by (3.1).

**Lemma 3.12.** The bilinear form in (3.1) is a polarization on \( \mathcal{W} \), that is,

\[
(uv, v') = (v, \eta(u)v'),
\]

for \( u \in U_q(\mathfrak{g}(2)(n+1)) \) and \( v, v' \in \mathcal{W} \).

**Proof.** Let us show (3.1) for \( e_i \) with \( 1 \leq i \leq n-1 \) and for \( e_n \) in (3.3). By Proposition 3.8, we prove the case of \( e_n \). Let us prove the case of \( e_0 \). By using \( \frac{1}{n-1} \) and \( \frac{1}{2} \) for (3.3)–(3.6) for \( u, v \), we define an \( A \) that \( e_j \) is identical to Lemma 3.1, and the proof for \( e_0 \) is obtained by using \( \tau \). We may assume \( m' = m - e_n \). The right-hand side is

\[
(|m|, \eta(e_n)|m - e_n|) = (|m|, -q^{-1}_n k^{-1}_n f_n |m - e_n|) = -q^{2m_n - 1}_n [m_n]|(m|, |m)|,
\]

and the left-hand side is

\[
(e_n|m|, |m - e_n|) = -q^{\frac{1}{2}}(|m - e_n|, |m - e_n|)
\]

\[
= -q^{\frac{1}{2}} q^{-\frac{1}{2}} \sum_{m_i(m_i-1)} [m_n] q^{m_n - 1} = -q^{m_n - \frac{1}{2}} [m_n]|(m|, |m)|.
\]

Hence the equality holds. □

3.2.4. Crystal base Let \( M \) be a \( U_q(\mathfrak{g}(2)(n+1)) \)-module. For \( 1 \leq j \leq n-1 \), assume that \( e_j \) and \( f_j \) are locally nilpotent on \( M \), and define \( \tilde{e}_j, \tilde{f}_j \) to be the usual lower crystal operators. For \( i = 0, n \), we consider the operators \( \tilde{e}_i \) and \( \tilde{f}_i \) defined in the same way as in (3.3)–(3.6) for \( U_q(\mathfrak{g}(n+1)) \), which also satisfy (3.7).

Let \( A_0 \) be the subring of \( \mathbb{Q}[q^\frac{1}{2}] \) consisting of functions which are regular at \( q^{\frac{1}{2}} = 0 \). We define an \( A_0 \)-lattice \( \mathcal{L} \) of \( \mathcal{W} \) and a \( \mathbb{Q} \)-basis \( \mathcal{B} \) of \( \mathcal{L}/q^{\frac{1}{2}} \mathcal{L} \) by

\[
\mathcal{L} = \bigoplus_m A_0|m|, \quad \mathcal{B} = \{ |m| \mod q^{\frac{1}{2}} \mathcal{L} \}.
\]

It is clear from (3.1) that \( (\mathcal{L}, \mathcal{L}) \subset A_0 \), and \( \mathcal{B} \) is an orthonormal basis of \( \mathcal{L}/q^{\frac{1}{2}} \mathcal{L} \) with respect to \( (\ , \)|\( q^{\frac{1}{2}} = 0 \).

**Proposition 3.13.** The pair \( (\mathcal{L}, \mathcal{B}) \) is a crystal base of \( \mathcal{W} \) in the sense of Proposition 3.8, where

\[
\tilde{f}_i|m| \equiv \begin{cases} 
|m + e_n| & \text{if } i = n, \\
|m + e_i - e_{i+1}| & \text{if } m_{i+1} \geq 1 \text{ and } 1 \leq i \leq n - 1, \\
|m - e_1| & \text{if } m_1 \geq 1 \text{ and } i = 0, \\
0 & \text{otherwise},
\end{cases} \quad \text{(mod } q^{\frac{1}{2}} \mathcal{L}).
\]

**Proof.** It suffices to prove the above formula for \( \tilde{f}_i \) when \( i = 0, n \) since the other cases are proved in Proposition 3.8. Let us prove the case of \( \tilde{f}_n \) only. Recall that \( [m]_n = [m]_{q^{\frac{1}{2}}-1} \) for \( m \in \mathbb{Z}_{\geq 0} \).

Let \( |m| \) be given. Since \( e_n|m - m_ne_n| = 0 \) and \( k_n|m - m_ne_n| = q^{-\frac{1}{2}}|m - m_ne_n| \), we have

\[
\tilde{f}_n^m|m - m_ne_n| = q_n^{-\frac{1}{2}} [m_n]_{q^{2n-1}} |m - m_ne_n| = q_n^{\frac{m_n(m_n-1)}{2}} [m_n]_{q^{2n-1}} |m|.
\]
and hence

\[
\tilde{f}_n |m\rangle = q_n^{-\frac{m_n(m_n-1)}{2}} [m_n]_{q_n}^{-1} \tilde{f}_{n+1}^m |m - mn e_n\rangle
\]

\[
= q_n^{-\frac{m_n(m_n-1)}{2}} [m_n]_{q_n}^{-1} q_n^{-\frac{m_n(m_n+1)}{2}} [m_n+1]_{q_n}^{-1} |m + e_n\rangle
\]

\[
= q_n^{m_n} [m_n+1]_{q_n} |m + e_n\rangle = |m + e_n\rangle \quad (\text{mod } q^\frac{1}{2} L).
\]

\[\Box\]

3.3. Type B\((1)(0, n)\).

3.3.1. \(U_q(B^{(1)}(0, n))\)-module \(W\) Consider the quantum affine superalgebra \(U_q(B^{(1)}(0, n))\).

Let \(U_q(B(0, n))\) (or \(U_q(osp_{1|2n})\)) and \(U_q(A_{n-1})\) be the subalgebras of \(U_q(B^{(1)}(0, n))\) generated by \(k_i, e_i, f_i\) for \(i \in I \setminus \{0\}\) and \(i \in I \setminus \{0, n\}\), respectively.

**Proposition 3.14.** For a non-zero \(x \in \mathbb{Q}(q^{\frac{1}{2}})\), the space \(W\) admits an irreducible \(U_q(B^{(1)}(0, n))^\sigma\)-module structure given as follows:

\[e_0|m\rangle = x q^{-1} [m + 1][m + 2]|m + 2e_1\rangle,\]

\[f_0|m\rangle = -x q^{-1}|m - 2e_1\rangle,\]

\[k_0|m\rangle = q^{2m+1}|m\rangle,\]

\[e_j|m\rangle = [m_{j+1} + 1]|m - e_j + e_{j+1}\rangle,\]

\[f_j|m\rangle = [m_{j+1} + 1]|m + e_j - e_{j+1}\rangle,\]

\[k_j|m\rangle = q^{-m_j+m_{j+1}}|m\rangle,\]

\[e_n|m\rangle = -q^\frac{1}{2}|m - e_n\rangle,\]

\[f_n|m\rangle = q^{-\frac{1}{2}}[m + 1]|m + e_n\rangle,\]

\[k_n|m\rangle = q^{-m_{n-1}}|m\rangle,\]

\[\sigma|m\rangle = (-1)^{|m|}|m\rangle,
\]

where \(1 \leq j \leq n - 1\).

**Proof.** See Appendix A.2. \(\Box\)

We also denote this module by \(W(x)\) and call it a (level one) \(q\)-oscillator representation. Note that the classical limit of \(W(x)\) as a \(U_q(osp_{1|2n})\)-module is the same as in Lemma 3.11.

3.3.2. Polarization and crystal base

**Lemma 3.15.** The bilinear form in (3.1) is a polarization on \(W\), that is,

\[(uv, v') = (v, \eta(u)v'),\]

for \(u \in U_q(B^{(1)}(0, n))\) and \(v, v' \in W\).
Proof. All the cases are already shown in Lemmas 3.6 and 3.12 since the action of $e_i$ for $0 \leq i < n$ (resp. $i = n$) is the same as the one for $C_n^{(1)}$ (resp. $C^{(2)}(n + 1)$). □

We define an $A_0$-lattice $\mathcal{L}$ of $\mathcal{W}$ and a $\mathbb{Q}$-basis $\mathcal{B}$ of $\mathcal{L}/q^{1/2}\mathcal{L}$ as in (3.8). We also define the operators $\tilde{e}_i$ and $\tilde{f}_i$ in the same way as in $U_q(C_n^{(1)})$ and $U_q(C^{(2)}(n + 1))$.

**Proposition 3.16.** The pair $(\mathcal{L}, \mathcal{B})$ is a crystal base of $\mathcal{W}$ in the sense of Proposition 3.8, where

$$\tilde{f}_i|\mathbf{m}\rangle \equiv \begin{cases} |\mathbf{m} + e_n\rangle & \text{if } i = n, \\ |\mathbf{m} + e_i - e_{i+1}\rangle & \text{if } m_{i+1} \geq 1 \text{ and } 1 \leq i \leq n - 1, \\ |\mathbf{m} - 2e_1\rangle & \text{if } m_1 \geq 2 \text{ and } i = 0, \\ 0 & \text{otherwise}, \end{cases} \pmod{q^{1/2}\mathcal{L}}.$$

Proof. It follows from Propositions 3.8 and 3.13. □

4. Quantum $R$ Matrix and Fusion Construction

In this section, we review the quantum $R$ matrix and its spectral decomposition for $U_q(X)$ and introduce higher level $q$-oscillator representations by the so-called fusion construction [11].

Let $x, y \in \mathbb{Q}(q^d)$ be generic, and let $\mathcal{W}(x)$ be a level one $q$-oscillator representation of $U_q(X)$ including $\mathcal{W}_\varepsilon(x)$ ($\varepsilon = \pm$) for type $C_n^{(1)}$. The quantum $R$ matrix $R(x, y)$ on $\mathcal{W}(x) \otimes \mathcal{W}(y)$ is defined as a linear operator satisfying

$$R(x, y) \Delta(a) = \Delta^{\text{op}}(a) R(x, y)$$

for $a \in U_q(X)$, where $\Delta^{\text{op}}$ denotes the opposite coproduct, namely, the coproduct obtained by interchanging the first and second components in $\Delta$. If $\mathcal{W}(x) \otimes \mathcal{W}(y)$ is irreducible, then $R(x, y)$ is unique up to a scalar function of $x, y$ and depends only on $z = x/y$. Let $P$ be the linear operator on $\mathcal{W}(x) \otimes \mathcal{W}(y)$ such that $P(u \otimes v) = v \otimes u$ and set $\check{R}(x, y) = P R(x, y)$. Then $\check{R}(x, y)$ maps $\mathcal{W}(x) \otimes \mathcal{W}(y)$ to $\mathcal{W}(y) \otimes \mathcal{W}(x)$.

The spectral decomposition of $\check{R}(x, y)$ can be obtained from the results in [15] and Appendix C. For this, we only need to care about the difference between the coproduct (2.4) and that of [15] and Appendix C, say $\Delta$. More precisely, we have $\Delta = (\Delta^\varepsilon)^{\text{op}}$, where $\Delta^\varepsilon = (\zeta \otimes \zeta) \circ \Delta \circ \zeta$ and $\zeta$ is the automorphism given by $\zeta(e_i) = e_i k_i^{-1}$, $\zeta(f_i) = k_i f_i$, and $\zeta(k_i) = k_i$ for $i \in I$.

Hence, to translate the results in [15] and Appendix C, we replace $\check{R}(x, y)$ with $\check{R}_\varepsilon(x, y)$. Let $V^\varepsilon_i$ and $V_i$ denote the irreducible submodules of $\mathcal{W} \otimes$ over $U_q(C_n)$ and $U_q(osp_{1|2n})$ considered in [15] and Appendix C to compute the spectral decomposition. Here we replace them with $PV^\varepsilon_i$ and $PV_i$, respectively. Thus, we obtain the spectral decomposition of $\check{R}_\varepsilon(x, y)$ as follows.

For $U_q(C_n^{(1)})$, we have

$$\check{R}_\varepsilon(x, y) = \sum_{l \in 2\mathbb{Z}_{\geq 0}} \prod_{j=1}^{l/2} \frac{1 - q^{4j-2}z}{z - q^{4j-2}z} P^\varepsilon_l,$$

where $\check{R}_\varepsilon(x, y) : \mathcal{W}_\varepsilon(x) \otimes \mathcal{W}_\varepsilon(y) \rightarrow \mathcal{W}_\varepsilon(y) \otimes \mathcal{W}_\varepsilon(x)$ for $\varepsilon = +, -$ and $P^\varepsilon_l$ is the projection onto $V^\varepsilon_l$.
For $U_q(C(2)(n + 1))$, we have from Proposition C.4 and the spectral decomposition for $U_q(D^{(2)}_{n+1})$ in [15, Proposition 7] that

$$\tilde{R}(x, y) = \sum_{l \in \mathbb{Z}_{\geq 0}} \prod_{j=1}^{l} \frac{1 + (-q)^j z}{z + (-q)^j} P_l,$$

where $P_l$ is the projection onto $V_l$.

Finally for $U_q(B^{(1)}(0, n))$, we have from Proposition C.4 and the spectral decomposition for $U_q(A^{(2)\dagger}_{2n})$ in Appendix B that

$$\tilde{R}(x, y) = \sum_{l \in 2\mathbb{Z}_{\geq 0}} \prod_{j=1}^{l/2} \frac{1 - q^{4j-1} z}{z - q^{4j-1}} P_l + \sum_{l \in 1 + 2\mathbb{Z}_{\geq 0}} \prod_{j=0}^{(l-1)/2} \frac{1 - q^{4j+1} z}{z - q^{4j+1}} P_l. \quad (4.3)$$

Next, we explain the fusion construction. For $s \geq 2$, let $\mathcal{S}_s$ denote the group of permutations on $s$ letters generated by $s_i = (i, i + 1)$ for $1 \leq i \leq s - 1$. We have $U_q(X)$-linear maps

$$\tilde{R}_w(x_1, \ldots, x_s) : \mathcal{W}(x_1) \otimes \cdots \otimes \mathcal{W}(x_s) \longrightarrow \mathcal{W}(x_{w(1)}) \otimes \cdots \otimes \mathcal{W}(x_{w(s)}),$$

for $w \in \mathcal{S}_s$ and generic $x_1, \ldots, x_s \in \mathbb{Q}(q^d)$ satisfying

$$\tilde{R}_1(x_1, \ldots, x_s) = \text{id}_{\mathcal{W}(x_1) \otimes \cdots \otimes \mathcal{W}(x_s)},$$

$$\tilde{R}_{s_i}(x_1, \ldots, x_s) = \left( \otimes_{j < i} \text{id}_{\mathcal{W}(x_j)} \right) \otimes \tilde{R}(x_i/x_{i+1}) \otimes \left( \otimes_{j > i+1} \text{id}_{\mathcal{W}(x_j)} \right),$$

$$\tilde{R}_{ww'}(x_1, \ldots, x_s) = \tilde{R}_w(x_{w(1)}, \ldots, x_{w(s)}) \tilde{R}_w(x_1, \ldots, x_s),$$

for $1 \leq i \leq s - 1$ and $w, w' \in \mathcal{S}_s$ with $\ell(ww') = \ell(w) + \ell(w')$, where $\ell(w)$ denotes the length of $w$. Hence we have a $U_q(X)$-linear map $\tilde{R}_s = \tilde{R}_{w_0}(x_1, \ldots, x_s)$ with $x_i = q^{d(2i - s - 1)}$:

$$\tilde{R}_s : \mathcal{W}(q^{d(1-s)}) \otimes \cdots \otimes \mathcal{W}(q^{d(s-1)}) \longrightarrow \mathcal{W}(q^{d(s-1)}) \otimes \cdots \otimes \mathcal{W}(q^{d(1-s)}),$$

where $w_0$ is the longest element in $\mathcal{S}_s$. Now we define a $U_q(X)$-module

$$\mathcal{W}^{(s)} = \text{Im}\tilde{R}_s. \quad (4.4)$$

**Theorem 4.1.** For $s \geq 2$, $\mathcal{W}^{(s)}$ is an irreducible $U_q(X)$-module.

**Proof.** See Appendix D. □

### 5. Character of Higher Level $q$-Oscillator Representation

In this section, we discuss the character of higher level $q$-oscillator representation. Let $\mathcal{P}$ be denote the set of partitions $\lambda = (\lambda_i)_{i \geq 1}$, and $\mathcal{P}_t$ the subset of $\lambda$ such that $\text{sgn}(\lambda_i) = \varepsilon$ for all $i$ with $\lambda_i \neq 0$. Recall that $\text{sgn}(m) = +$ and $-\text{sgn}(m) = -$ if $m \in \mathbb{Z}_{\geq 0}$ is even and odd, respectively. We denote by $\ell(\lambda)$ the length of $\lambda \in \mathcal{P}$. Let $s_{\lambda}(x_1, \ldots, x_n)$ denote the Schur polynomial in $x_1, \ldots, x_n$ corresponding to a partition $\lambda \in \mathcal{P}$. 
5.1. Type $C_n^{(1)}$. For $s \geq 2$ and $\varepsilon = \pm$, let $\mathcal{W}_\varepsilon^{(s)}$ denote the higher level $q$-oscillator module in (4.4) corresponding to $\mathcal{W}_\varepsilon$. The following is the main result in this subsection.

**Theorem 5.1.** For $s \geq 2$, $\mathcal{W}_\varepsilon^{(s)}$ is irreducible as a $U_q(C_n)$-module, and its character is given by

$$\text{ch} \mathcal{W}_\varepsilon^{(s)} = \sum_{\lambda \in P_\varepsilon, \ell(\lambda) \leq s} s_\lambda(x_1, \ldots, x_n).$$

**Corollary 5.2.** The character of $\mathcal{W}_\varepsilon^{(s)}$ has a stable limit for $s \geq n$ as follows:

$$\text{ch} \mathcal{W}_\varepsilon^{(s)} = \sum_{\lambda \in P_\varepsilon, \ell(\lambda) \leq n} s_\lambda(x_1, \ldots, x_n) = \frac{1}{\prod_{1 \leq i \leq j \leq n}(1 - x_i x_j)} \quad (\varepsilon = +).$$

The rest of this subsection is devoted to proving Theorem 5.1. We construct a $\mathbb{Q}(q)$-basis of $\mathcal{W}_\varepsilon^{(2)}$, which is compatible with the action of $\hat{R}(z)$, and which plays an important role in the proof of Theorem 5.1.

We note from (4.1) that

$$\mathcal{W}_\varepsilon^{(2)} = V_\varepsilon = U_q(C_n)((|\varsigma(\varepsilon)e_n| \otimes |\varsigma(\varepsilon)e_n|),$$

and hence it is irreducible. Moreover, we have the following character formula for $\mathcal{W}_\varepsilon^{(2)}$.

**Proposition 5.3.** We have

$$\text{ch} \mathcal{W}_\varepsilon^{(2)} = \text{ch} V_\varepsilon^{(2)} = \sum_{\lambda \in P_\varepsilon, \ell(\lambda) \leq 2} s_\lambda(x_1, \ldots, x_n).$$

**Proof.** Write $\mathcal{W}_\varepsilon = \mathcal{W}_\varepsilon(q^{\pm 1})$ for short since we may consider the action of $U_q(C_n)$ only. Let $(\mathcal{W}_\varepsilon \otimes \mathcal{W}_\varepsilon)_A$ be the $A$-span of $|m\rangle \otimes |m'\rangle$ in $\mathcal{W}_\varepsilon \otimes \mathcal{W}_\varepsilon$. Then $(\mathcal{W}_\varepsilon \otimes \mathcal{W}_\varepsilon)_A$ is also invariant under $e_i, f_i, k_i$ and $\{k_i\}$ for $i \in I \setminus \{0\}$. This yields its classical limit $\mathcal{W}_\varepsilon \otimes \mathcal{W}_\varepsilon := (\mathcal{W}_\varepsilon \otimes \mathcal{W}_\varepsilon)_A \otimes \mathbb{C}$, which is a $U(C_n)$-module. Also, we have as a $U(C_n)$-module

$$\mathcal{W}_\varepsilon \otimes \mathcal{W}_\varepsilon \cong \mathcal{W}_\varepsilon \otimes \mathcal{W}_\varepsilon.$$

By Lemma 3.5, $\mathcal{W}_\varepsilon$ is an irreducible highest weight module. By the theory of super duality [4], it belongs to a semisimple category of $U(C_n)$-module which is closed under tensor product (see [18, Section 5.4] for more details, where we put $m = 0$ there). Hence $\overline{\mathcal{W}_\varepsilon} \otimes \overline{\mathcal{W}_\varepsilon}$ is semisimple, and the classical limit $\overline{V_\varepsilon}$, the submodule generated by $(|\varsigma(\varepsilon)e_n| \otimes |\varsigma(\varepsilon)e_n|) \otimes 1$, is an irreducible highest weight $U(C_n)$-module with highest weight $-(1 + 2\varsigma(\varepsilon))\varpi_n$. The character of $\overline{V_\varepsilon^{(2)}}$ and hence $\overline{V_\varepsilon^{(2)}}$ follows from [19, Theorem 6.1]. □
Let us construct a $\mathbb{Q}(q)$-basis of $\mathcal{W}_\varepsilon^{(2)}$ which is compatible with its $U_q(\mathfrak{A}_{n-1})$-crystal base. For this, we find all the $U_q(\mathfrak{A}_{n-1})$-highest weight vectors in $\mathcal{W}_\varepsilon^{(2)}$.

For $l \in \mathbb{Z}_{\geq 0}$, let

$$v_l = \sum_{k=0}^{l} (-1)^k q^{k(k-l+1)} \left[ \begin{array}{c} l \\ k \end{array} \right]^{-1} |k \mathbf{e}_{n-1} + (l-k) \mathbf{e}_n \rangle \otimes |(l-k) \mathbf{e}_{n-1} + k \mathbf{e}_n \rangle.$$

(5.1)

Lemma 5.4. For $l \in \mathbb{Z}_{\geq 0}$, $v_l$ is a $U_q(\mathfrak{A}_{n-1})$-highest weight vector in $\mathcal{W}_\varepsilon^{(2)}$, and

$$v_l \equiv |l \mathbf{e}_n \rangle \otimes |l \mathbf{e}_{n-1} \rangle \quad (\text{mod } q \mathcal{L}_\varepsilon^{\otimes 2}),$$

where $\text{sgn}(l) = \varepsilon$.

Proof. It is straightforward to check that $e_i v_l = 0$ for $1 \leq i \leq n-1$. Next we claim that $v_l \in \mathcal{W}_\varepsilon^{(2)}$. Note that

$$\text{ch} \mathcal{W}_\varepsilon = \sum_{l \in \varepsilon(\varepsilon)+2\mathbb{Z}_{\geq 0}} s_{(l)}(x_1, \ldots, x_n),$$

and hence

$$\text{ch} \mathcal{W}_\varepsilon^{\otimes 2} = (\text{ch} \mathcal{W}_\varepsilon)^2 = \sum_{\text{sgn}(\lambda) = +} a_{\lambda} s_{\lambda}(x_1, \ldots, x_n),$$

(5.2)

where for $\lambda = (\lambda_1, \lambda_2) \in \mathcal{P}$,

$$a_{\lambda} = \begin{cases} \frac{\lambda_1 - \lambda_2}{2} - \varepsilon(\varepsilon) & \text{if } \lambda_1 > \lambda_2, \\ 1 & \text{if } \lambda_1 = \lambda_2. \end{cases}$$

Let $S_l$ be the $U_q(\mathfrak{A}_{n-1})$-submodule of $\mathcal{W}_\varepsilon^{\otimes 2}$ generated by $v_l$. Since the character of $S_l$ is $s_{(l^2)}(x_1, \ldots, x_n)$, and the multiplicity of $s_{(l^2)}(x_1, \ldots, x_n)$ in (5.2) is one, it follows from Proposition 5.3 that $S_l \subset \mathcal{W}_\varepsilon^{(2)}$. This shows that $v_l \in \mathcal{W}_\varepsilon^{(2)}$. The lemma follows from

$$q^{k(k-l+1)} \left[ \begin{array}{c} l \\ k \end{array} \right]^{-1} \in q^k (1 + qA_0).$$

One can prove more directly that $v_l \in \mathcal{W}_\varepsilon^{(2)}$ using the following lemma.

Lemma 5.5. Set $\mathcal{E} = e^{(2)}_{n-2} \cdots e^{(2)}_1 e_0$, where it should be understood as $e_0$ when $n = 2$. Then for $l \in \mathbb{Z}_{\geq 0}$ we have

$$(\mathcal{E} e^{(2)}_1 \mathcal{E} - \frac{1}{[3]!} (e_1 \mathcal{E})^2) v_l = q^{-2} \frac{[2]}{[3]} ([l+1][l+2])^2 v_{l+2}.$$
linear map defined by \( \pi(\text{m}) = [m_{n-1}, m_n] \), where \( \text{m} = (m_1, \ldots, m_n) \). Then we can show by direct calculation that the following diagram commutes.

\[
\begin{array}{c}
(\mathcal{W}_{k,n}^0)^{\otimes 2} \xrightarrow{\pi^{\otimes 2}} \mathcal{W}_{k,2}^{\otimes 2} \\
\downarrow \varepsilon \quad \quad \quad \downarrow e_0 \\
(\mathcal{W}_{k,n}^0)^{\otimes 2} \xrightarrow{\pi^{\otimes 2}} \mathcal{W}_{k,2}^{\otimes 2}
\end{array}
\]

This fact reduces the proof of the lemma to the case of \( n = 2 \). When \( n = 2 \), one calculates

\[
e_0 e_1^{(2)} e_0 v_l = \sum_k c_k [l - k + 1][l - k + 2] \times \\
q^{-2l+2k-4}[k + 1][k + 2][l + 2, l - k + 2] \otimes |l - k, k) \\
+ q^{-2l-7}[k - 1][k][l, l - k + 2) \otimes |l - k + 2, k) \\
+ q^{-2k}[l - k + 3][l - k + 4][l - k, l - k + 2) \otimes |l - k + 4, k) \\
\]

Here \( c_k = (-1)^k q^{k(k-l+1)} \left( \begin{array}{c} l \\ k \end{array} \right)^{-1} \) and we have used the relation \( q^{l-2k-2}[l - k]c_{k+1} + [k + 1]c_k = 0 \). On the other hand, we also get

\[
(e_1 e_0)^2 v_l = [2] \sum_k c_k [l - k + 1][l - k + 2] \times \\
[q^{-2l+2k-4}[k + 1][k + 2][l + 2, l - k + 2] \otimes |l - k, k) \\
+ A_k[k, l - k + 2) \otimes |l - k + 2, k) \\
+ [3]q^{-2k}[l - k + 3][l - k + 4][l - k, l - k + 2) \otimes |l - k + 4, k) \\
\]

where

\[
A_k = \frac{q^{l-2k}}{q - q^{-1}} \left\{ (1 + q^{-2l-6})(q^{2}[k + 1][l - k + 2] - [k][l - k + 3]) \\
- q^{-2l+2k}(1 + q^{-4})([k + 1][l - k + 2] - q^{-4}[k][l - k + 3]) \right\}.
\]

Combining these results, we obtain

\[
(e_0 e_1^{(2)} e_0 - \frac{1}{[3]} (e_1 e_0)^2) v_l \\
= [2][l + 1][l + 2] \sum_k c_k q^{-2k-2}[l - k + 1][l - k + 2][l, l - k + 2) \otimes |l - k + 2, k) \\
= q^{-2} [2][l + 1][l + 2]^2 v_{l+2}.
\]
For \( l \in \mathbb{Z}_{\geq 0} \) and \( l' = 2m \in 2\mathbb{Z}_{\geq 0} \), set

\[
\mathbf{v}_{l, l'} = q^n \frac{m(m+2l+1)}{2} f_n^{(m)} \mathbf{v}_l.
\]

Note that \( \mathbf{v}_{l, l'} \) may not be equal to \( f_n^m \mathbf{v}_l \) in the sense of (3.4) since \( e_n \mathbf{v}_{l, l'} \neq 0 \) in general.

**Lemma 5.6.** For \( l \in \mathbb{Z}_{\geq 0} \) and \( l' \in 2\mathbb{Z}_{\geq 0} \) with \( \text{sgn}(l) = \varepsilon \), \( \mathbf{v}_{l, l'} \) is a \( U_q(A_{n-1}) \)-highest weight vector in \( \mathcal{W}_\varepsilon^{(2)} \), and

\[
\mathbf{v}_{l, l'} \equiv |l\mathbf{e}_n \rangle \otimes |l\mathbf{e}_{n-1} + l'\mathbf{e}_n \rangle \pmod{q \mathcal{L}_\varepsilon^{(2)}}.
\]

**Proof.** Let us assume that \( l \) is even, and hence \( \varepsilon = + \), since the proof for odd \( l \) is almost identical. Since \( e_j (1 \leq j \leq n-1) \) commutes with \( f_n \), it is clear that \( \mathbf{v}_{l, l'} \) is a \( U_q(A_{n-1}) \)-highest weight vector in \( \mathcal{W}_\varepsilon^{(2)} \).

Let \( l' = 2m \). For \( 0 \leq c \leq l \), we have

\[
|c\mathbf{e}_{n-1} + (l-c)\mathbf{e}_n \rangle \equiv \begin{cases} f_n^{\lfloor \frac{l-c}{2} \rfloor} |c\mathbf{e}_{n-1} \rangle & \text{if } c \text{ is even}, \\ f_n^{\lfloor \frac{l-c}{2} \rfloor} |c\mathbf{e}_{n-1} + \mathbf{e}_n \rangle & \text{if } c \text{ is odd}, \end{cases} \pmod{q \mathcal{L}_+}.
\]

Put \( a = \lfloor \frac{l-c}{2} \rfloor \) and \( b = \lfloor \frac{c}{2} \rfloor \).

**Case 1.** Suppose that \( c \) is even. Let

\[
u_1 = |c\mathbf{e}_{n-1} \rangle, \quad u_2 = |(l-c)\mathbf{e}_{n-1} \rangle.
\]

We have

\[
\Delta (f_n^{(m)})(\tilde{f}_n^a u_1 \otimes \tilde{f}_n^b u_2)
\]

\[
= \sum_{k=0}^{m} q_n^{-k(m-k)} f_n^{(m-k)} f_n^k (\tilde{f}_n^a u_1) \otimes f_n^k (f_n^b u_2)
\]

\[
= \sum_{k=0}^{m} q_n^{-k(m-k)} f_n^{(m-k)} f_n^k a \otimes f_n^k b \tilde{f}_n^a u_1 \otimes f_n^k f_n^b u_2
\]

\[
= \sum_{k=0}^{m} q_n^{-k(m-k)} (\tilde{f}_n^a)^{2k} (\tilde{f}_n^b)^{2k} f_n^{(m-k)} f_n^k (m-k+a) \otimes f_n^k f_n^b u_1 \otimes f_n^k f_n^b u_2
\]

\[
= \sum_{k=0}^{m} q_n^{-k(m-k)} (\tilde{f}_n^a)^{2k} (\tilde{f}_n^b)^{2k} f_n^{(m-k)} f_n^k (m-k+a) \otimes f_n^k f_n^b u_1 \otimes f_n^k f_n^b u_2
\]

\[
= \sum_{k=0}^{m} q_n^{-k(m-k)} (\tilde{f}_n^a)^{2k} (\tilde{f}_n^b)^{2k} \times f_n^{(m-k+a)} u_1 \otimes f_n^{(k+b)} u_2
\]

\[
= \sum_{k=0}^{m} f_{a, b}(q) \tilde{f}_n^{m-k+a} u_1 \otimes \tilde{f}_n^{k+b} u_2.
\]
Multiplying $q_n^{m(m+2l+1)}$ on both sides, we have $q_n^{m(m+2l+1)} f_{a,b}(q) \in q^d (1 + q A_0)$, where

$$
d = m(m + 2l + 1) - 2k(m - k) - 2\left(\frac{1}{2} + 2a\right)k$$

$$+ a^2 + b^2 - (m - k + a)^2 - (k + b)^2 - 2(m - k)a - 2kb$$

$$= 2lm + (m - k) - 4ma - 4kb = 2lm + (m - k) - 4m\left(\frac{l - c}{2}\right) - 4k\left(\frac{c}{2}\right)
$$

(5.3)

since $a = \frac{l - c}{2}$ and $b = \frac{c}{2}$.

**Case 2.** Suppose that $c$ is odd. Let

$$u_1 = |ce_{n-1} + e_n|, \quad u_2 = |(l - c)e_{n-1} + e_n|.$$

We have

$$\Delta(f_n^{(m)})(\tilde{f}_n^a \otimes \tilde{f}_n^b)u_2$$

$$= \sum_{k=0}^{m} q_n^{m(m-k)} f_n^{(m-k)} k_n^{(a)} (f_n u_1) \otimes f_n^{(k)} (f_n u_2)$$

$$= \sum_{k=0}^{m} q_n^{m(m-k)}(\frac{3}{2} + 2a) k_n^{(a)} u_1 \otimes f_n^{(k)} (f_n u_2)$$

$$= \sum_{k=0}^{m} q_n^{m(m-k)}(\frac{3}{2} + 2a) k_n^{(a)} \left[\begin{array}{c} m - k + a \\ a \end{array}\right] \left[\begin{array}{c} k + b \\ b \end{array}\right] n^{m-k+a} u_1 \otimes f_n^{(k)} (f_n u_2)$$

$$= \sum_{k=0}^{m} q_n^{m(m-k)} a^{(a+2)k} + b^{(b+2)k} - (m-k+a)(m-k+a+2) - (k+b)(k+b+2)$$

$$\times \left[\begin{array}{c} m - k + a \\ a \end{array}\right] \left[\begin{array}{c} k + b \\ b \end{array}\right] n^{m-k+a} u_1 \otimes f_n^{(k+b)} u_2$$

$$= \sum_{k=0}^{m} q_n^{m(m-k+a)} (f_n u_1) \otimes f_n^{(m-k+a)} (f_n u_2)$$

Multiplying $q_n^{m(m+2l+1)}$ on both sides, we have $q_n^{m(m+2l+1)} g_{a,b}(q) \in q^{d'} (1 + q A_0)$, where

$$d' = d - 2k + 2a + 2b - 2(m - k + a) - 2(k + b)$$

$$= d - 2k - 2m$$

$$= 2lm + (m - k) - 4ma - 4kb - 2k - 2m$$

$$= 2lm + (m - k) - 4m\left(\frac{l - c - 1}{2}\right) - 4k\left(\frac{c - 1}{2}\right) - 2k - 2m$$

(5.4)

$$= (m - k) + 2c(m - k) = (2c + 1)(m - k)$$
by putting $a = \frac{l-c-1}{2}$ and $b = \frac{c-1}{2}$. By (5.3), (5.4), and Lemma 5.4, we have
\[ q_n^{m(m+2j+1)/2} f_n^{(m)} v_l \equiv |l e_n | \otimes |l e_{n-1} + 2m e_n | \pmod {q \mathcal{L}^\otimes_+}. \]

\[ \square \]

**Corollary 5.7.** The set \( \{ v_{l,l'} | l \in \mathbb{Z}_{\geq 0}, l' \in 2\mathbb{Z}_{\geq 0}, \text{sgn}(l) = \varepsilon \} \) is the set of \( U_q(A_{n-1}) \)-highest weight vectors in \( \mathcal{W}_e^{(2)} \).

**Proof.** The character of the \( U_q(A_{n-1}) \)-submodule of \( \mathcal{W}_e \) generated by \( v_{l,l'} \) is \( s_{\lambda}(x_1, \ldots, x_n) \) where \( \lambda = (l' + l, l) \). Hence it follows from Proposition 5.3 that there is no other \( U_q(A_{n-1}) \)-highest weight vectors in \( \mathcal{W}_e^{(2)} \). \( \square \)

Now we define the pair \( (\mathcal{L}_e^{(2)}, \mathcal{B}_e^{(2)}) \) as follows:

\[ \mathcal{L}_e^{(2)} = \sum_{l_1 \in \mathbb{Z}_{\geq 0}} \sum_{l_2 \in 2\mathbb{Z}_{\geq 0}} \sum_{r \geq 0} A_0 \tilde{f}_{l_1} \cdots \tilde{f}_{l_r} v_{l_1,l_2}, \]

\[ \mathcal{B}_e^{(2)} = \left\{ \tilde{f}_{l_1} \cdots \tilde{f}_{l_r} v_{l_1,l_2} \pmod {q \mathcal{L}_e^{(2)}} \right\} \]

where \( l_1 \in \mathbb{Z}_{\geq 0}, \text{sgn}(l_1) = \varepsilon, l_2 \in 2\mathbb{Z}_{\geq 0}, r \geq 0, 1 \leq i_1, \ldots, i_r \leq n-1 \) \( \setminus \{0\} \).

**Proposition 5.8.** We have

1. \( \mathcal{L}_e^{(2)} \subset \mathcal{L}_e^{\otimes 2} \) and \( \mathcal{B}_e^{(2)} \subset \mathcal{B}_e^{\otimes 2} \),
2. \( (\mathcal{L}_e^{(2)}, \mathcal{B}_e^{(2)}) \) is a \( U_q(A_{n-1}) \)-crystal base of \( \mathcal{W}_e^{(2)} \).

**Proof.** (1) By Proposition 3.8, \( \mathcal{L}_e^{\otimes 2} \) is a crystal base of \( \mathcal{W}_e \) as a \( U_q(A_{n-1}) \)-module, hence it is invariant under \( \tilde{f}_i \) for \( 1 \leq i \leq n-1 \). Since \( v_{l_1,l_2} \in \mathcal{L}_e^{\otimes 2} \) by Lemma 5.6, we have \( \tilde{f}_{l_1} \cdots \tilde{f}_{l_r} v_{l_1,l_2} \in \mathcal{L}_e^{\otimes 2} \) and hence \( \tilde{f}_{l_1} \cdots \tilde{f}_{l_r} v_{l_1,l_2} \in \mathcal{B}_e^{\otimes 2} \pmod {q \mathcal{L}_e^{(2)}} \).

(2) By definition of \( (\mathcal{L}_e^{(2)}, \mathcal{B}_e^{(2)}) \) and Lemma 5.6, \( (\mathcal{L}_e^{(2)}, \mathcal{B}_e^{(2)}) \) is a \( U_q(A_{n-1}) \)-crystal base of the submodule \( V \) of \( \mathcal{W}_e^{(2)} \) generated by \( v_{l_1,l_2} \) for \( l_1, l_2 \). On the other hand, we have \( V = \mathcal{W}_e^{(2)} \) by Proposition 5.3. Hence \( (\mathcal{L}_e^{(2)}, \mathcal{B}_e^{(2)}) \) is a \( U_q(A_{n-1}) \)-crystal base of \( \mathcal{W}_e^{(2)} \). \( \square \)

For \( |m| = |m_1, \ldots, m_n| \) \( \in \mathcal{W}_e \), let \( T(m) \) denote the semistandard tableau of shape \( (|m|) \), a single row of length \( |m| \), with letters in \( \{ \overline{n} < \cdots < \overline{1} \} \) such that the number of occurrences of \( \overline{i} \) is \( m_i \) for \( 1 \leq i \leq n \).

Suppose that \( |m_1|, \ldots, |m_s| \) are given such that \( |m_1| \leq \cdots \leq |m_s| \). Let \( \lambda = (|m_s| \geq \cdots \geq |m_1|) \), which is a partition or its Young diagram, and \( \lambda^\pi \) denote the Young diagram obtained by \( 180^\circ \)-rotation of \( \lambda \). We denote by \( T(m_1, \ldots, m_s) \) the row-semistandard tableau of shape \( \lambda^\pi \), whose \( j \)-th row from the top is equal to \( T(m_j) \) for \( 1 \leq j \leq s \).

**Example 5.9.** Suppose that \( n = 5 \). If \( |m_1| = |2, 1, 0, 0, 2| \) and \( |m_2| = |0, 1, 2, 3, 1| \), then

\[ T(m_1, m_2) = \begin{array}{ccccc}
5 & 5 & 2 & \overline{1} & \overline{1} \\
5 & 4 & 4 & 3 & 3 & 2
\end{array} \]
Proposition 5.10. We have

\[ B_{\kappa}^{(2)} = \left\{ |m_1| \otimes |m_2| \ (\mod qL_{\kappa}^{(2)}) \left| |m_1| \leq |m_2|, \ T(m_1, m_2) \text{ is semistandard} \right. \right\}. \]

Proof. For \( l_1 \in \mathbb{Z}_{\geq 0} \) and \( l_2 \in 2\mathbb{Z}_{\geq 0} \) with \( \text{sgn}(l_1) = \varepsilon \), let us identify \( v_{l_1, l_2} = |l_1 e_n| \otimes |l_1 e_{n-1} + l_2 e_n| \) in \( B_{\kappa}^{(2)} \) with the pair \( (l_1 e_n, l_1 e_{n-1} + l_2 e_n) \) and the connected component of \( v_{l_1, l_2} \) as a \( U_q(A_{n-1}) \)-crystal with the set of corresponding set of pairs \( (m_1, m_2)'s \). Then \( T(v_{l_1, l_2}) \) is the semistandard tableau of shape \( (l_1 + l_2, l_1)^{\pi} \). Since \( \partial v_{l_1, l_2} = 0 \) for \( 1 \leq j \leq n \) and \( T(v_{l_1, l_2}) \) is the tableau of highest weight and the set

\[ \left\{ T(\tilde{f}_{i_1} \cdots \tilde{f}_{i_r} v_{l_1, l_2}) \left| r \geq 0, \ 1 \leq i_1, \ldots, i_r \leq n - 1 \right. \right\} \setminus \{0\} \quad (5.5) \]

is equal to the set of semistandard tableau of shape \( (l_1 + l_2, l_1)^{\pi} \) with letters in \( \{\overline{n} < \cdots < \overline{1}\} \).

Let \( |m_1|, |m_2| \in B_{\kappa} \) be given with \( |m_1| = d_1 \) and \( |m_2| = d_2 \), let \( P(m_1, m_2) \) denote a unique semistandard tableau of shape \( \mu^{\pi} \) for some partition \( \mu \), which is equivalent to \( |m_1| \otimes |m_2| \) as an element of \( U_q(A_{n-1}) \)-crystals. This is equivalent to saying that if we read the row word of \( T(m_1) \) from left to right, and then apply the Schensted’s column insertion to \( T(m_2) \) in a reverse way starting from the right-most column, then the resulting tableau is \( P(m_1, m_2) \). So \( P(m_1, m_2) \) is of shape \( (d'_2, d'_1)^{\pi} \) for some \( d'_1 \leq d_1, d'_2 \geq d_2 \), and \( d'_1 + d'_2 = d_1 + d_2 \). In particular, \( P(m_1, m_2) = T(m_1, m_2) \) if \( d_1 \leq d_2 \) and \(|m_1| \otimes |m_2| \in B_{\kappa}^{(2)} \).

Example 5.11. Let \( |m_1|, |m_2| \) be as in Example 5.9. Then

\[ P(m_1, m_2) = \begin{bmatrix} \overline{3} & \overline{3} \\ \overline{5} & \overline{4} & \overline{4} & \overline{4} & \overline{3} & \overline{3} & \overline{2} & \overline{2} & \overline{1} & \overline{1} \end{bmatrix}. \]

Let \( l_1 \in \mathbb{Z}_{\geq 0} \) and \( l_2 \in 2\mathbb{Z}_{\geq 0} \) be given with \( \text{sgn}(l_1) = \varepsilon \). Put \( \lambda = (\lambda_1, \lambda_2) = (l_1 + l_2, l_1) \). Let \( \text{SST}(\lambda^{\pi}) \) be the set of semistandard tableaux of shape \( \lambda^{\pi} \) with letters in \( \{\overline{n} < \cdots < \overline{1}\} \). For each \( T \in \text{SST}(\lambda^{\pi}) \), we choose \( i_1, \ldots, i_r \in I \setminus \{0, n\} \) such that \( T = T(\tilde{f}_{i_1} \cdots \tilde{f}_{i_r} v_{l_1, l_2}) \) (see (5.5)), and define

\[ v_T = \tilde{f}_{i_1} \cdots \tilde{f}_{i_r} v_{l_1, l_2} \in L_{\kappa}^{(2)}. \quad (5.6) \]

By Proposition 5.10, we have a \( \mathbb{Q}(q) \)-basis of \( \mathcal{W}_{\kappa}^{(2)} \)

\[ \bigcup_{\lambda \in \mathcal{P}_{\kappa}, \ell(\lambda) \leq 2} \left\{ v_T \left| T \in \text{SST}(\lambda^{\pi}) \right. \right\}. \quad (5.7) \]

Lemma 5.12. For \( T \in \text{SST}(\lambda^{\pi}) \), we have

\[ v_T = |m_1| \otimes |m_2| + \sum_{m_i', m_2'} c_{m_i', m_2'} |m_1'| \otimes |m_2'|, \]

where \( P(m_1, m_2) = T, P(m_1', m_2') \) is of shape \( \mu^{\pi} \) with \( \mu \triangleright \lambda \) and \( \mu \neq \lambda \), and \( c_{m_i', m_2'} \in qA_0 \). Here \( \triangleright \) denotes a dominance order on partitions, that is, \( \mu_1 > \lambda_1 \), and \( \mu_1 + \mu_2 = \lambda_1 + \lambda_2 \).
Proof. By Lemmas 5.4 and 5.6 (see also their proofs), we observe that
\[ v_{l_1,l_2} = |l_1 e_n \rangle \otimes |l_1 e_{n-1} + l_2 e_n \rangle + \sum c_{x,y,z,w} |x e_{n-1} + y e_n \rangle \otimes |z e_{n-1} + w e_n \rangle, \]  
\hspace{1cm} (5.8)
where the sum is over \((x, y, z, w)\) such that
\begin{enumerate}
  \item \(0 < x \leq l_1\) with \(x + z = l_1\),
  \item \(y \geq z, w \geq x\) with \(y + w = l_1 + l_2\),
  \item \(c_{x,y,z,w} \in A_0\).
\end{enumerate}
We may regard \(|l_1 e_n \rangle \otimes |l_1 e_{n-1} + l_2 e_n \rangle\) as the case when \((x, y, z, w) = (0, l_1, l_1, l_2)\). Then it is not difficult to see that if the shape of \(P(x e_{n-1} + y e_n, z e_{n-1} + w e_n)\) is \(\mu^\pi = (\mu_1, \mu_2)^\pi\), then \(\mu_2 = z = l_1 - x \leq l_1\) and hence \(\mu \triangleright \lambda\), and \(\mu \neq \lambda\) when \(x > 0\).

Let \(i_1, \ldots, i_r \in I \setminus \{0, n\}\) be the sequence in (5.6). By the tensor product rule of crystals, we have
\[ \tilde{f}_{i_1} \cdots \tilde{f}_{i_r} (|x e_{n-1} + y e_n \rangle \otimes |z e_{n-1} + w e_n \rangle) = \sum_{m_1, m_2} c_{m_1,m_2} |m_1 \rangle \otimes |m_2 \rangle, \]  
\hspace{1cm} (5.9)
where the sum is over \(m_1, m_2\) such that
\begin{enumerate}
  \item \(c_{m_1,m_2}(q) \in A_0\) such that
    \[ c_{m_1,m_2}(0) = \begin{cases} 1 & \text{if } |m_1 \rangle \otimes |m_2 \rangle = \tilde{f}_{i_1} \cdots \tilde{f}_{i_r} (|x e_{n-1} + y e_n \rangle \otimes |z e_{n-1} + w e_n \rangle), \\ 0 & \text{otherwise.} \end{cases} \]
  \item \(v \triangleright \lambda\) and \(v \neq \lambda\), where \(v^\pi\) is the shape of \(P(m_1, m_2)\).
\end{enumerate}
Therefore, we obtain the result by (5.8) and (5.9). \(\square\)

Corollary 5.13. We have \(\mathcal{L}_e^{(2)} = \mathcal{L}_e^{\otimes 2} \cap \mathcal{W}_e^{(2)}\).

Proof. It is clear that \(\mathcal{L}_e^{(2)} \subset \mathcal{L}_e^{\otimes 2} \cap \mathcal{W}_e^{(2)}\) by Proposition 5.8. Conversely, suppose that \(v \in \mathcal{L}_e^{\otimes 2} \cap \mathcal{W}_e^{(2)}\) is given. By (5.7), we have
\[ v = \sum_T c_T v_T, \]  
\hspace{1cm} (5.10)
for some \(c_T \in \mathbb{Q}(q)\). We may assume that all the shape of \(T\) in (5.10) is the same. Fix \(T\) with \(c_T \neq 0\). Let \(|m_1 \rangle \otimes |m_2 \rangle\) be such that \(|m_1 \rangle \otimes |m_2 \rangle\) appears in (5.10) with non-zero coefficient, and \(P(m_1, m_2) = T\). By Lemma 5.12, the coefficient of \(|m_1 \rangle \otimes |m_2 \rangle\) is \(c_T\). Hence \(c_T \in A_0\), and \(v \in \mathcal{L}_e^{(2)}\). \(\square\)

Proof of Theorem 5.1. Let \(\mathcal{W}_e^{\otimes 2} = \mathcal{W}_e^{(2)} \oplus W\), where \(W\) is the complement of \(\mathcal{W}_e^{(2)}\) in \(\mathcal{W}_e^{\otimes 2}\) as a \(U_q(A_{n-1})\)-module since it is completely reducible. By Corollary 5.13, we have
\[ \mathcal{L}_e^{\otimes 2} = \mathcal{L}_e^{(2)} \oplus \mathcal{M}^{(2)}, \]  
\hspace{1cm} (11.1)
where \(\mathcal{M}^{(2)} = \mathcal{L}_e^{\otimes 2} \cap W\) is the crystal lattice of \(W\) as a \(U_q(A_{n-1})\)-module. Then we have
\[ \tilde{R}_2(\mathcal{L}_e^{\otimes 2}) \subset \mathcal{L}_e^{(2)}, \quad \tilde{R}_2|_{q=0}(\mathcal{B}_e^{\otimes 2}) \subset \mathcal{B}_e^{(2)}. \]  
\hspace{1cm} (11.2)
More generally, by (4.1) and (5.11), we have for $a \in \mathbb{Z}_{>0}$

$$\tilde{R}(q^{-2a})(L_q^2) \subset L_q^2. \quad (5.13)$$

For each $1 \leq i \leq s - 1$, we have

$$\tilde{R}_s = \tilde{R}_{s_i}(q^{s-2}, q^{s-2i-1}, \ldots, q^{s-2i+1}) \tilde{R}_{w_0s_i}(q^{1-s}, \ldots, q^{s-1}).$$

We have $\tilde{R}_{w_0s_i}(q^{1-s}, \ldots, q^{s-1})(L_q^s) \subset L_q^s$ by (5.13), and hence by (5.12)

$$\tilde{R}_s(L_q^s) \subset L_q^{s-1} \otimes L_q^2 \otimes L_q^{s-i-1},$$

$$\tilde{R}_s(B_q^s) \subset B_q^{s-1} \otimes B_q^2 \otimes B_q^{s-i-1}.$$ 

Therefore $\tilde{R}_s(B_q^s)$ is spanned by $B_q^s$, where

$$B_q^s = \left\{ |m_1 \rangle \otimes \ldots \otimes |m_s \rangle \pmod{qL_q^s} \mid |m_j \rangle \otimes |m_{j+1} \rangle \in B_q^2 (1 \leq j \leq s - 2) \right\}.$$ 

By Proposition 5.10, the set

$$\left\{ T(m_1, \ldots, m_s) \mid |m_1 \rangle \otimes \ldots \otimes |m_s \rangle \in B_q^s \right\}$$

is equal to the set of semistandard tableau of shape $\lambda^T$ where $\lambda = (|m_s| \geq \cdots \geq |m_1|)$. Hence

$$\text{ch} \mathcal{W}_q^{(s)} = \sum_{\lambda \in \mathcal{P}_q, \ell(\lambda) \leq s} s_\lambda(x_1, \ldots, x_n). \quad (5.14)$$

Let $V_0^{(s)}$ be the $U_q(C_n)$-submodule of $\mathcal{W}_q^{(s)}$ generated $|z(\epsilon)e_n \rangle^\otimes s$. The classical limit $V^{(s)}$ of $V_0^{(s)}$ is a highest weight $U(C_n)$-module with highest weight

$$\Lambda^{(s)} := -s\left(\frac{1}{2} + \zeta(\epsilon)\right)\sigma_n.$$

On the other hand, the $(U(A_{n-1}))$-character of the irreducible highest weight $U(C_n)$-module with highest weight $\Lambda^{(s)}$, say $V(\Lambda^{(s)})$, is also equal to (5.14) by [19, Theorem 6.1]. Since $V(\Lambda^{(s)})$ is a quotient of $V_0^{(s)}$, we conclude that

$$\text{ch} \mathcal{W}_q^{(s)} = \text{ch} V_0^{(s)} = \text{ch} V_0^{(s)} = \text{ch} V(\Lambda^{(s)}).$$

In particular, $V_0^{(s)}$ is an irreducible $U_q(C_n)$-module and hence so is $\mathcal{W}_q^{(s)} = V_0^{(s)}$. This completes the proof. □
5.2. Type $C^{(2)}(n+1)$. Next, let us prove that $\mathcal{W}^{(5)}$ is an irreducible $U_q(osp_{1|2n})$-module and compute its character. The proof is similar to that of Theorem 5.1 for $U_q(C^{(1)}_n)$. So we give a sketch of the proof and leave the details to the reader.

As usual, we realize the weight lattice for $U_q(osp_{1|2n})$ in $\bigoplus_{l=1}^n \mathbb{R}e_i$ equipped with the standard symmetric bilinear form such that $(e_i, e_j) = \delta_{ij}$. The simple roots $\alpha_i$ ($i \in I \setminus \{0\}$) are given by $\alpha_i = e_{i+1} - e_i$ for $1 \leq i \leq n - 1$ and $\alpha_n = -e_n$, and $\varpi_n = -\frac{1}{2}(e_1 + \cdots + e_n)$.

We first consider $\mathcal{W}^{(2)}$. By (4.2), we have

$$\mathcal{W}^{(2)} = V_0 = U_q(osp_{1|2n})(\{0\} \otimes \{0\}),$$

(5.15)

which is an irreducible representation of $U_q(osp_{1|2n})$ and hence of $U_q(C^{(1)}_n(n+1))$. By similar arguments as in Proposition 5.3, we have the following.

**Proposition 5.14.** We have

$$\text{ch} \mathcal{W}^{(2)} = \text{ch} V_0 = \sum_{\lambda \in \mathfrak{p}, \ell(\lambda) \leq 2} s_{\lambda}(x_1, \ldots, x_n).$$

**Lemma 5.15.** For $l \in \mathbb{Z}_{\geq 0}$, let $v_l$ be the vector in (5.1). Then $v_l$ is a $U_q(A_{n-1})$-highest weight vector in $\mathcal{W}^{(2)}$, and $v_l = |e_n\rangle \otimes |e_{n-1}\rangle \ (\text{mod } q^{\frac{1}{2}}L^{(2)})$.

**Proof.** Since the actions of $e_i, f_i, k_i^{\pm 1}$ for $1 \leq i \leq n - 1$ are the same as in the case of $C^{(1)}_n$, it follows from Lemma 5.4 that $v_l$ is a $U_q(A_{n-1})$-highest weight vector. Note that

$$\text{ch} \mathcal{W}^{(2)} = (\text{ch} V)^2 = \sum_{\ell(\lambda) \leq 2} a_{\lambda}s_{\lambda}(x_1, \ldots, x_n),$$

(5.16)

where $a_{\lambda} = \lambda_1 - \lambda_2$. Then we have $v_l \in \mathcal{W}^{(2)}$ by the same argument as in Lemma 5.4. 

We have an analogue of Lemma 5.5, which also proves that $v_l \in \mathcal{W}^{(2)}$.

**Lemma 5.16.** Set $E = e_{n-2} \cdots e_1 e_0$, where it is understood as $e_0$ when $n = 2$. Then for $l \geq 0$ we have

$$(Ee_{n-1}E - \frac{1}{2}e_{n-1}E^2)v_l = (-1)^l q^{-5/2} \cdot \frac{1 + q}{2}[l + 1]v_{l+1}.$$
Proof. Since $e_j$ for $1 \leq j \leq n - 1$ commutes with $f_n$, $v_{l,m}$ is a $U_q(A_{n-1})$-highest weight vector in $W^{(2)}_\epsilon$.

For $0 \leq c \leq l$, put $a = l - c$ and $b = c$. Let

$$u_1 = |c e_{n-1}>, \quad u_2 = |(l-c)e_{n-1}>.$$ 

By (2.4), we have

$$\Delta(f^{(m)}_n)(f^a_n u_1 \otimes f^b_n u_2)$$

$$= \sum_{k=0}^m \alpha^k q_n^{-k(m-k)} f^{(m-k)}_n f^a_n u_1 \otimes f^b_n u_2$$

$$= \sum_{k=0}^m \alpha^k q_n^{-k(m-k)-(1+2a)k} f^{(m-k)}_n f^a_n u_1 \otimes f^b_n u_2$$

$$= \sum_{k=0}^m \alpha^k q_n^{-k(m-k)-(1+2a)k + \frac{a(a+1)}{2} + \frac{b(b+1)}{2}} f^{(m-k)}_n f^a_n u_1 \otimes f^b_n u_2$$

$$= \sum_{k=0}^m \alpha^k f_{a,b}(q) f^{m-k+a}_n u_1 \otimes f^{b}_n u_2.$$ 

Multiplying $\frac{m(m+4l+3)}{2}$ on both sides, it is straightforward to see that

$$q_n^{\frac{m(m+4l+3)}{2}} f_{a,b}(q) \in q_n^{(2c+1)(m-k)} (1 + q^{\frac{1}{2}} A_0).$$

This implies that $v_{l,m} \equiv |e_n> \otimes |e_{n-1} + m e_n> \pmod{q^{\frac{1}{2}} \mathcal{L} \otimes 2}$. \hfill \Box

Now we define the pair $(\mathcal{L}^{(2)}, \mathcal{B}^{(2)})$ as follows:

$$\mathcal{L}^{(2)} = \sum_{l_1, l_2 \in \mathbb{Z}_{\geq 0}} \sum_{1 \leq l_i \leq n-1} A_0 \tilde{f}_{l_1} \ldots \tilde{f}_{l_r} v_{l_1, l_2},$$

$$\mathcal{B}^{(2)} = \left\{ \tilde{f}_{l_1} \ldots \tilde{f}_{l_r} v_{l_1, l_2} \pmod{q^{\frac{1}{2}} \mathcal{L}^{(2)}} \mid l_1, l_2 \in \mathbb{Z}_{\geq 0}, r \geq 0, 1 \leq l_1, \ldots, l_r \leq n-1 \right\} \setminus \{0\}.$$

Proposition 5.18. We have

1. $\mathcal{L}^{(2)} \subset \mathcal{L} \otimes 2$ and $\mathcal{B}^{(2)} \subset \mathcal{B} \otimes 2$,
2. $(\mathcal{L}^{(2)}, \mathcal{B}^{(2)})$ is a $U_q(A_{n-1})$-crystal base of $W^{(2)}$, where

$$\mathcal{B}^{(2)} = \left\{ |m_1> \otimes |m_2> \pmod{q^{\frac{1}{2}} \mathcal{L}^{(2)}} \mid |m_1| \leq |m_2|, T(m_1, m_2) \text{ is semistandard} \right\}.$$
Proof. It follows from the same arguments as in Propositions 5.8 and 5.10. □

Corollary 5.19. We have \( \mathcal{L}^{(2)} = \mathcal{L}^\otimes 2 \cap \mathcal{W}^{(2)}. \)

Proof. By Proposition 5.18, one can check that Lemma 5.12 also holds for \( \mathcal{W}^{(2)} \), which
implies \( \mathcal{L}^{(2)} = \mathcal{L}^\otimes 2 \cap \mathcal{W}^{(2)}. \) □

Theorem 5.20. For \( s \geq 2 \), \( \mathcal{W}^{(s)} \) is irreducible as a \( U_q(\mathfrak{osp}_{1|2n}) \)-module, and its character is given by

\[
\text{ch} \mathcal{W}^{(s)} = \sum_{\lambda \in \mathcal{P} \atop \ell(\lambda) \leq s} s_{\lambda}(x_1, \ldots, x_n).
\]

Proof. We may apply the same arguments as in Theorem 5.1 and the result in [19, Theorem 6.1] by using Proposition 5.18 and Corollary 5.19. □

Corollary 5.21. The character of \( \mathcal{W}_e^{(s)} \) has a stable limit for \( s \geq n \) as follows:

\[
\text{ch} \mathcal{W}_e^{(s)} = \sum_{\lambda \in \mathcal{P} \atop \ell(\lambda) \leq n} s_{\lambda}(x_1, \ldots, x_n) = \frac{1}{\prod_{1 \leq i \leq n} (1 - x_i) \prod_{1 \leq i < j \leq n} (1 - x_i x_j)}.
\]

5.3. Type \( B^{(1)}(0, n) \). For \( \lambda \in \mathcal{P}_+ \) with \( \ell(\lambda) \leq \min\{n, s/2\} \), we put

\[
\Lambda_{(s)}^{(e)} = -s \varpi_n + \sum_{i=1}^n \lambda_i e_{n-i+1}.
\]

Let \( V(\Lambda_{(s)}^{(e)}) \) be the irreducible highest weight \( U(\mathfrak{osp}_{1|2n}) \)-module with highest weight \( \Lambda_{(s)}^{(e)} \). Note that \( \Lambda_{(2)}^{(j)} \) is the weight of the maximal vector \( v_l \) and \( V(\Lambda_{(2)}^{(j)}) = V_l \) for \( l \geq 0 \). Generalizing the decomposition of \( \mathcal{W}^{(2)} \) into \( U_q(\mathfrak{osp}_{1|2n}) \)-modules, we have the following conjecture on \( \mathcal{W}^{(s)} \).

Conjecture 5.22. For \( s \geq 2 \), the character of \( \mathcal{W}^{(s)} \) is given by

\[
\text{ch} \mathcal{W}^{(s)} = \sum_{\lambda \in \mathcal{P}_+ \atop \ell(\lambda) \leq \min\{n, s/2\}} \text{ch} V(\Lambda_{(s)}^{(e)}).
\]

Remark 5.23. The family of infinite-dimensional \( U(\mathfrak{osp}_{1|2n}) \)-modules \( V(\Lambda_{(s)}^{(e)}) \) have been introduced in [2] in connection with Howe duality. They are unitarizable and form a semisimple tensor category. The Weyl-Kac type character formula for \( V(\Lambda_{(s)}^{(e)}) \) can be found in [2, Theorem 6.13], while a combinatorial formula is given in [19, Corollary 6.6].

Corollary 5.24. For \( s \geq 2n \), we have

\[
\text{ch} \mathcal{W}^{(s)} = \frac{\sum_{\lambda \in \mathcal{P}_+} s_{\lambda}(x_1, \ldots, x_n)}{\prod_{1 \leq i \leq n} (1 - x_i) \prod_{1 \leq i < j \leq n} (1 - x_i x_j)} \cdot \frac{1}{\prod_{1 \leq i \leq n} (1 - x_i)(1 - x_i^2) \prod_{1 \leq i < j \leq n} (1 - x_i x_j)^2}.
\]
Proof. The first equality follows from the fact [19, Corollary 6.6] that if \( \lambda \in \mathcal{P}_q \) with \( \ell(\lambda) \leq n \), then
\[
\text{ch} V(\Lambda^{(x)}_\lambda) = \frac{s_\lambda(x_1, \ldots, x_n)}{\prod_{1 \leq i \leq n} (1 - x_i) \prod_{1 \leq i < j \leq n} (1 - x_i x_j)}.
\]
The second one follows from the well-known Littlewood identity. \( \square \)

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Appendix A. Twistor

In this appendix, we prove Propositions 3.9 and 3.14. We first review the twistor introduced in [5] that relate quantum groups to quantum supergroups. Then we use it to relate to published maps and institutional affiliations.

Appendix A.1. The twistor of the covering quantum group.

We review the covering quantum group \( \hat{U}_q(D_{n+1}^2) \) in [15] to a representation of \( U_q(C^{(2)}(n+1)) \). An advantage to do so is that in the latter case we can take a classical limit \( q \to 1 \). We also obtain a representation of \( U_q(B^{(1)}(0, n)) \) from the \( q \)-oscillator representation of \( U_q(A^{(2)}_n) \), where \( A^{(2)}_n \) has the same Dynkin diagram as \( A^{(2)}_{2n} \) in [9] but the labeling of nodes are opposite.

Let \( q_2, \pi \) be indeterminates and \( i = \sqrt{-1} \). For a ring \( R \) with 1, we set \( R^{\pi} = R[\pi]/(\pi^2 - 1) \). The covering quantum group \( \hat{U} \) associated to a Cartan datum is the \( \mathbb{Q}[q, i] \)-algebra with generators \( E_i, F_i, K_i^{\pm 1}, J_i^{\pm 1} \) for \( i \in I \) subject to the following relations:

\[
\begin{align*}
J_i J_j &= J_j J_i, & K_i K_j &= K_j K_i, & J_i K_j &= K_j J_i, \\
K_i K_i^{-1} &= K_i^{-1} K_i = J_i J_i^{-1} = J_i^{-1} J_i = J_i^2 = 1, \\
J_i E_j &= \pi^{a_{ij}} E_j J_i, & J_i F_j &= \pi^{-a_{ij}} F_j J_i, \\
K_i E_j &= q^{a_{ij}} E_j K_i, & K_i F_j &= q^{-a_{ij}} F_j K_i, \\
E_i F_j - \pi^{p(i)p(j)}/(\pi_i q_i - q_i^{-1}) &= \delta_{ij} J_i K_i - K_i^{-1}, \\
\sum_{l=0}^{1-a_{ij}} (-1)^l \pi^{(l-1)p(i)/2} l p(i)p(j) \left[ 1 - a_{ij} \right]_{q_i, \pi_i} E_i^{1-a_{ij}-l} E_j E_i^l &= 0 \quad (i \neq j),
\end{align*}
\]
Higher Level $q$-Oscillator Representations

\[ \sum_{l=0}^{1-a_{ij}} (-1)^l \pi^l (l-1)p(i)/2 + l(p(i)p(j)) \left[ \frac{1-a_{ij}}{l} \right] q_i \pi^l F_{ij} F_j = 0 \quad (i \neq j). \]

Remark A.1. We changed the notations from [5]. We replaced $v$ with $q$, $t$ with $i$, and $J_{d,i}$, $K_{d,i}$, $T_{d,i}$ with $J_i$, $K_i$, $T_i$.

We extend $U$ by introducing the generators $T_i$, $\Upsilon_i$ for $i \in I$. They commute with each other and with $J_i$, $K_i$. They also have the commutation relations with $E_i$, $F_i$ as

\[
T_i E_j = i^{d_{ij}} E_j T_i, \quad T_i F_j = i^{-d_{ij}} F_j T_i, \quad \Upsilon_i E_j = i^{\phi_{ij}} E_j \Upsilon_i, \quad \Upsilon_i F_j = i^{-\phi_{ij}} F_j \Upsilon_i,
\]

where

\[
\phi_{ij} = \begin{cases} 
d_{ij} & \text{if } i > j, \\
d_i & \text{if } i = j, \\
-2p(i)p(j) & \text{if } i < j.
\end{cases}
\]

We denote this extended algebra by $\hat{U}$.

Theorem A.2. [5] There is a $\mathbb{Q}(i)$-algebra automorphism $\hat{\Psi}$ on $\hat{U}$ such that

\[
E_i \mapsto i^{-d_i} \Upsilon_i^{-1} T_i E_i, \quad F_i \mapsto F_i \Upsilon_i, \quad K_i \mapsto T_i^{-1} K_i,
\]

\[
J_i \mapsto T_i^2 J_i, \quad T_i \mapsto T_i, \quad \Upsilon_i \mapsto \Upsilon_i,
\]

\[
q \mapsto i^{-1} q, \quad \pi \mapsto -\pi.
\]

A.2. Image of the twistor $\hat{\Psi}$. We apply the twistor $\hat{\Psi}$ given in the previous subsection for the Cartan datum corresponding to $B_n$, namely, $I = \{1, 2, \ldots, n\}$ and the Cartan matrix is given by

\[
(a_{ij}) = \begin{pmatrix} 
2 & -1 & \cdots & \cdots & -1 \\
-1 & 2 & -1 & \cdots & \cdots & -1 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
-1 & \cdots & \cdots & \cdots & 2 & -1 \\
-2 & \cdots & \cdots & \cdots & 2 & 1
\end{pmatrix}
\]

Through it, we are to regard the $q$-oscillator representation $\mathcal{W} = \bigoplus_m \mathbb{Q}(q^{1/2^m})(m)$ of $U_q(B_n)$, the subalgebra of $U_q(D_{n+1}^{(2)})$ generated by $e_i, f_i, k_i$ for $i \in I \setminus \{0\}$, given in [15, Proposition 1] as a representation of $U_q(osp_{1|2n})$. Although we normalized the symmetric bilinear form on the weight lattice so that $(\alpha_i, \alpha_i) \in \mathbb{Z}$ for any $i \in I$ in the previous subsection, we renormalize it so that $(\alpha_n, \alpha_n) = 1$ to adjust it to the notations in [15]. The generators $T_i$, $\Upsilon_i$ are represented on $\mathcal{W}$ as

\[
T_i |m\rangle = \begin{cases} 
i^{2(m_i+1-m_i)} |m\rangle & (1 \leq i < n) \\
i^{-2m_i} |m\rangle & (i = n)
\end{cases}, \quad \Upsilon_i |m\rangle = \begin{cases} 
i^{-2m_i} |m\rangle & (1 \leq i < n) \\
i^{|m|-2m_n} |m\rangle & (i = n)
\end{cases}.
\]
Let $u_i$ ($i \in I$, $u = e, f, k$) be the generators of $U_q(B_n)$ ($\pi = 1$) and $\tilde{u}_i = \tilde{\Psi}(u_i)$ be the image ($\pi = -1$) of the twistor $\tilde{\Psi}$. Then $\tilde{u}_i$ satisfy the relations for $U_q(osp1|2n)$ where $\tilde{q}^{\frac{1}{2}} = i^{-1}q^{\frac{1}{2}}$. On the space $\mathcal{W}$, they act as follows:

$$
\tilde{e}_i|m\rangle = i^{2m_{i+1}}|m_i|\langle m - e_i + e_{i+1},
\tilde{f}_i|m\rangle = i^{-2m_i}|m_{i+1}|\langle m + e_i - e_{i+1},
\tilde{k}_i|m\rangle = i^{2m_i-2m_{i+1}}q^{-m_i+m_{i+1}}|m\rangle,
\tilde{\epsilon}_n|m\rangle = \kappa i^{-1-m_m|m_n|}\langle m - e_n,
\tilde{f}_n|m\rangle = i^{m_{m+2n}}|m + e_n|,
\tilde{k}_n|m\rangle = i^{2m_{n+1}}q^{-m_n-\frac{1}{2}}|m\rangle,
$$

where $1 \leq i < n$, $\kappa = (q+1)/(q-1)$.

By introducing the actions of $\tilde{e}_0$, $\tilde{f}_0$, $\tilde{k}_0$, we want to make $\mathcal{W}$ a representation of the quantum affine superalgebra associated to $C(2)(n+1)$ or $B(1)(0, n)$. For the former, we set

$$
\tilde{e}_0|m\rangle = x i^{2m_{1-|m|}}|m + e_1|,
\tilde{f}_0|m\rangle = x^{-1}\kappa i^{m_{|m+1|}}|m - e_1|,
\tilde{k}_0|m\rangle = i^{2m_{1-1}}q^{m_{1-1}+\frac{1}{2}}|m\rangle,
$$

and for the latter

$$
\tilde{e}_0|m\rangle = x(-1)^{|m|}|m + 2e_1|,
\tilde{f}_0|m\rangle = x^{-1}(-1)^{|m|}|m + 1 - 1|^{2}|m - 2e_1|,
\tilde{k}_0|m\rangle = -q^{m_{1+1}+1}|m\rangle,
$$

where $x$ is the so-called spectral parameter. We also note that the quantum parameter is still $\tilde{q}^{\frac{1}{2}} = i^{-1}q^{\frac{1}{2}}$.

To obtain the representation for the quantum parameter $q$, we need to we switch $\tilde{q}^{\frac{1}{2}}$ to $i\tilde{q}^{\frac{1}{2}}$ ($\tilde{q}^{\frac{1}{2}}$ to $q^{\frac{1}{2}}$). Also, the relations in Section A.1 and those in Section 2.3 are different. For the node $i$ that is signified as $\bullet$ in the Dynkin diagram, there is a relation

$$
e_i f_i + f_i e_i = \frac{k_i - k_i^{-1}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}
$$

in Section 2.3 rather than

$$
e_i f_i + f_i e_i = \frac{k_i - k_i^{-1}}{-q^{\frac{1}{2}} - q^{-\frac{1}{2}}}
$$

in Section A.1. The former relation is realized by deleting $\kappa$ from the action of $\tilde{e}_i$ or $\tilde{f}_i$ in the formulas of the $q$-oscillator representation above. By doing so, we obtain

$$
\tilde{e}_0|m\rangle = \begin{cases} x i^{2m_{1-|m|}}|m + e_1| & \text{for } U_q(C(2)(n+1)) \\
x(-1)^{|m|}|m + 2e_1| & \text{for } U_q(B(1)(1, n)) \end{cases},
$$

where $1 \leq i < n$, $\kappa = (q+1)/(q-1)$. 
In this appendix, we consider the quantum $q$-oscillator representation of $U_q(C(2)(n + 1))$ and $U_q(B(1)(0, n))$ in Propositions 3.9 and 3.14, respectively, we perform the basis change where we also apply $e_i \mapsto -e_i$ and the other generators fixed. For $U_q(C(2)(n + 1))$, we also apply $f_0 \mapsto -f_0$. Accordingly, the coproduct also changes. For $U_q(B(1)(0, n))$, we alternatively apply $e_0 \mapsto i[2]e_0$, $f_0 \mapsto 1/2^1 f_0$. This completes the proof.

Appendix B. Quantum R Matrix for $U_q(A_{2n}^{(2)})$

In this appendix, we consider the quantum R matrix for the $q$-oscillator representation of $U_q(A_{2n}^{(2)})$ where $A_{2n}^{(2)}$ is the Dynkin diagram whose nodes have the opposite labelings to $A_{2n}^{(2)}$. This will be used in Appendix C to derive the quantum R matrix for $U_q(B(1)(0, n))$.

B.1. $q$-oscillator representation for $U_q(A_{2n}^{(2)})$. By $A_{2n}^{(2)}$ we denote the following Dynkin diagram.

![Dynkin Diagram](image)

Although we did not deal with the $q$-oscillator representation for $U_q(A_{2n}^{(2)})$ in [15], it is easy to guess from other cases given there. On the space $\mathcal{V}$, the actions are given as follows.

$$e_0 |m\rangle = x |m + 2e_1\rangle,$$

$$f_0 |m\rangle = x^{-1} \frac{[m_1][m_1 - 1]}{[2]^2} |m - 2e_1\rangle,$$
\[ k_0|\mathbf{m}\rangle = -q^{2m_1+1}|\mathbf{m}\rangle, \]
\[ e_i|\mathbf{m}\rangle = [m_i]|\mathbf{m} - e_i + e_{i+1}\rangle, \]
\[ f_i|\mathbf{m}\rangle = [m_{i+1}]|\mathbf{m} + e_i - e_{i+1}\rangle, \]
\[ k_i|\mathbf{m}\rangle = q^{-2m_i+2m_{i+1}}|\mathbf{m}\rangle, \]
\[ e_n|\mathbf{m}\rangle = i\kappa [m_n]|\mathbf{m} - e_n\rangle, \]
\[ f_n|\mathbf{m}\rangle = |\mathbf{m} + e_n\rangle, \]
\[ k_n|\mathbf{m}\rangle = i q^{-m_n-1/2}|\mathbf{m}\rangle, \]

where \(0 < i < n\) and \(\kappa = (q + 1)/(q - 1)\). Denote this representation map by \(\pi_x\).

Recall that the \(U_q(B_n)\)-highest weight vectors \(|v_l \mid l \in \mathbb{Z}_{\geq 0}\rangle\) are calculated in [15, Proposition 4]. We take the coproduct (C.1) with \(\pi = 1\).

**Lemma B.1.** For \(x, y \in \mathbb{Q}(q)\) we have

1. \((\pi_x \otimes \pi_y)\Delta(f_{0}f_{1}^{(2)} \cdots f_{l-1}^{(2)})v_l = -\frac{[l][l-1]}{[2]^{2}}(q^{2l-2}x^{-1} + q^{-1}y^{-1})v_{l-2} \quad (l \geq 2),\)
2. \((\pi_x \otimes \pi_y)\Delta(e_ne_{n-1}^{(2)} \cdots e_{1}^{(2)}e_{0})v_{0} = \frac{\kappa [2]}{1-q}((y + qx)v_{1} - q(y + x)\Delta(f_{n})v_{n}).\)

Define the quantum \(R\) matrix \(\tilde{R}_{KO}(z, q)\) for \(U_q(A_{2n}^{(2)}\dagger)\) as in Proposition C.4. The existence of such \(\tilde{R}_{KO}(z, q)\) is essentially given in [15, Theorem 13]. Namely, although \(A_{2n}^{(2)}\dagger\) is not listed there, the corresponding gauge transformed quantum \(R\) matrix is \(S_{2,1}^{(z)}(z)\) and the proof has been done as the cases (i),(iv) and (v). By using Lemma B.1, we have the following.

**Proposition B.2.** We have the following spectral decomposition

\[
\tilde{R}_{KO}(z) = \sum_{l \in \mathbb{Z}_{+}} \prod_{j=1}^{l/2} \frac{z + q^{4j-1}}{1 + q^{4j-1}} P_l + \sum_{l \in 1+2\mathbb{Z}_{+}} \prod_{j=0}^{(l-1)/2} \frac{z + q^{4j+1}}{1 + q^{4j+1}} P_l,
\]

where \(P_l\) is the projector on the subspace generated by the \(U_q(B_n)\)-highest weight vector \(v_l \quad (l \geq 0)\).

**Appendix C. Quantum R Matrix for \(U_q(C^{(2)}(n + 1))\) and \(U_q(B^{(1)}(0, n))\)**

In this appendix, we compare the quantum \(R\) matrix for the \(q\)-oscillator representation for \(U_q(C^{(2)}(n + 1))\) with the one for \(U_q(D_{n+1}^{(2)})\) given in [15]. We also compare the quantum \(R\) matrix for \(U_q(B^{(1)}(0, n))\) with the one for \(U_q(A_{2n}^{(2)}\dagger)\) in Appendix B. We keep the notations in Appendix A.

**C.1. Gauge transformation.** We take the following coproduct

\[
\Delta(k_i) = k_i \otimes k_i, \]
\[
\Delta(e_i) = 1 \otimes e_i + e_i \otimes \sigma^{\frac{1+2}{2}} p(i) k_i, \quad (C.1)\]
\[
\Delta(f_i) = f_i \otimes \sigma^{-\frac{1+2}{2}} p(i) + k_i^{-1} \otimes f_i,
\]
for \( i \in I \), where \( \sigma \) satisfies (2.3). We also take the same coproduct (C.1) for \( \bar{u}_i \). Let \( \Gamma \) be an operator acting on \( \mathcal{V}^@2 \) by
\[
\Gamma |m\rangle \otimes |m'\rangle = i^{\sum_{k,i} \varphi_{ki} m_k m'_i} |m\rangle \otimes |m'\rangle,
\]
(C.2)
for \( m = (m_1, \ldots, m_n) \) and \( m' = (m'_1, \ldots, m'_n) \). Here we have the constraint \( \varphi_{kl} + \varphi_{lk} = 0 \). Then by [23] (see also [22]),
\[
\Delta \Gamma(u) = \Gamma^{-1} \Delta(u) \Gamma
\]
gives another coproduct of \( U_q(B_n) \) acting on \( \mathcal{V}^@2 \). Take \( \varphi_{kl} \) to be 1 for \( k < l \). We also set
\[
K |m\rangle = i^{c(m)} |m\rangle,
\]
(C.3)
where
\[
c(m) = -\frac{1}{2} \sum_k m_k^2 + \sum_k \left( k - n - \frac{1}{2} \right) m_k.
\]
Set
\[
\gamma_i(m) = \begin{cases} 
-|m| + m_1 & (i = 0 \text{ and for } U_q(C^{(2)}(n + 1))) \\
-2|m| + 2m_1 & (i = 0 \text{ and for } U_q(B^{(1)}(0, n))) \\
m_i + m_{i+1} & (0 < i < n) \\
-m_1 + m_{i+1} & (0 < i < n) \\
-m_i + m_{i+1} & (0 < i < n) \end{cases},
\]
\[
\beta_i(m) = \begin{cases} 
-m_1 + n & (i = 0 \text{ and for } U_q(C^{(2)}(n + 1))) \\
2m_1 + 2n + 1 & (i = 0 \text{ and for } U_q(B^{(1)}(0, n))) \\
-m_n & (i = n) \end{cases}.
\]

Let \( \alpha_0 = e_1 \) for \( U_q(C^{(2)}(n + 1)) \), \( 2e_1 \) for \( U_q(B^{(1)}(0, n)) \), \( \alpha_i = -e_i + e_{i+1} \) \( (0 < i < n) \), and \( \alpha_n = -e_n \).

**Lemma C.1.** The following formulas hold for \( m, m' \), and \( i \in I \):

1. \( \Gamma^{-1}(1 \otimes e_i) \Gamma |m\rangle \otimes |m'\rangle = i^{\gamma_i(m)} |m\rangle \otimes e_i |m'\rangle, \)
2. \( \Gamma^{-1}(e_i \otimes 1) \Gamma |m\rangle \otimes |m'\rangle = i^{\gamma_i(m')} e_i |m\rangle \otimes |m'\rangle, \)
3. \( \Gamma^{-1}(1 \otimes f_i) \Gamma |m\rangle \otimes |m'\rangle = i^{\gamma_i(m)} f_i |m\rangle \otimes |m'\rangle, \)
4. \( \Gamma^{-1}(f_i \otimes 1) \Gamma |m\rangle \otimes |m'\rangle = i^{\gamma_i(m')} f_i |m\rangle \otimes |m'\rangle. \)

**Lemma C.2.** The following formulas hold for \( m \) and \( i \in I \):

1. \( K^{-1} e_i K |m\rangle = i^{\beta_i(m)} e_i |m\rangle, \)
2. \( K^{-1} f_i K |m\rangle = i^{-\beta_i(m)} f_i |m\rangle. \)

**Proposition C.3.** For \( u_i \) \( (i \in I, u = e, f, k) \), we have
\[
\Delta(\bar{u}_i) |m\rangle \otimes |m'\rangle = i^{\Lambda_i(m+m')} (K \otimes K)^{-1} \Delta \Gamma(u_i)(K \otimes K) |m\rangle \otimes |m'\rangle,
\]
on $\mathcal{W}^\otimes 2$. Here
\[
\Lambda_i(m) = \begin{cases} 
    m_i + m_{i+1} - (\delta_{i0} + \delta_{in})|m| - n\delta_{i0} & (u = e) \\
    m_i + m_{i+1} + (\delta_{i0} + \delta_{in})(|m| + 1) - 2 & (u = f) \\
    2m_i - 2m_{i+1} & (u = k)
\end{cases},
\]
except when $i = 0$ and for $U_q(B^{(1)}(0, n))$, where
\[
\Lambda_0(m) = \begin{cases} 
    2m_1 - 2|m| - 2n + 1 & (u = e) \\
    2m_1 - 2|m| - 2n + 3 & (u = f) \\
    0 & (u = k)
\end{cases}.
\]
Here we should understand $m_0 = m_{n+1} = 0$.

**Proof.** It follows from Lemmas C.1 and C.2, and the following calculations. For instance, for $i = n$
\[
\Delta(\tilde{e}_n)|m\rangle \otimes |m'\rangle = (1 \otimes \tilde{e}_n + \tilde{e}_n \otimes \sigma \tilde{k}_n)|m\rangle \otimes |m'\rangle \\
= \kappa(i^{1-|m'|}[m']|m\rangle \otimes |m'\rangle - e_n) \\
+ (-1)^{|m'|}i^{2-|m|+2m'_n}q^{-2m'_n-1}[m_n]|m - e_n\rangle \otimes |m'\rangle),
\]
\[
\Delta^\Gamma(\tilde{e}_n)|m\rangle \otimes |m'\rangle = (\Gamma^{-1}(1 \otimes e_n)\Gamma + \Gamma^{-1}(e_n \otimes 1)\Gamma \cdot (1 \otimes k_n))|m\rangle \otimes |m'\rangle \\
= \kappa(i^{1-|m'|+m'_n}[m']|m\rangle \otimes |m'\rangle - e_n) \\
+ i^{-|m'|+m'_n+2}q^{-2m'_n-1}[m_n]|m - e_n\rangle \otimes |m'\rangle),
\]
and for $i \neq n$
\[
\Delta(\tilde{e}_i)|m\rangle \otimes |m'\rangle = (1 \otimes \tilde{e}_i + \tilde{e}_i \otimes \tilde{k}_i)|m\rangle \otimes |m'\rangle \\
= i^{2m'_i}[m'_i]|m\rangle \otimes |m'\rangle - e_i + e_{i+1}) \\
+ i^{2m_i+2m'_i-2m_{i+1}}q^{-2m'_i+2m_{i+1}}[m_i]|m - e_i + e_{i+1}\rangle \otimes |m'\rangle,
\]
\[
\Delta^\Gamma(\tilde{e}_i)|m\rangle \otimes |m'\rangle = (\Gamma^{-1}(1 \otimes e_i)\Gamma + \Gamma^{-1}(e_i \otimes 1)\Gamma \cdot (1 \otimes k_i))|m\rangle \otimes |m'\rangle \\
= i^{-m_i-m_{i+1}}[m'_i]|m\rangle \otimes |m'\rangle - e_i + e_{i+1}) \\
+ i^{m'_i+m_{i+1}}q^{-2m'_i+2m_{i+1}}[m_i]|m - e_i + e_{i+1}\rangle \otimes |m'\rangle.
\]
\]

\[
\square
\]

For a quantum affine superalgebras such as $U_q(D^{(2)}_{n+1})$, $U_q(A^{(2)}_{2n}^\dagger)$, $U_q(C^{(2)}(n + 1))$, and $U_q(B^{(1)}(0, n))$, a quantum $R$ matrix $R(z)$ is defined, if it exists, as an intertwiner satisfying
\[
\check{R}(z)(\pi_x \otimes \pi_y)\Delta(u) = (\pi_y \otimes \pi_x)\Delta(u)\check{R}(z),
\]
where $\check{R}(z) = PR(z)$, $P$ is the transposition of the tensor components and $z = x/y$. We also note that the coprodct we use here is (C.1). For $U_q(D^{(2)}_{n+1})$ or $U_q(A^{(2)}_{2n}^\dagger)$, the existence of quantum $R$ matrices are proved in [15] or Appendix B. We denote them by $\check{R}_{KO}(z)$. Let $\check{R}_{new}(z)$ be the quantum $R$ matrices for the quantum groups $U_q(C^{(2)}(n+1))$ or $U_q(B^{(1)}(0, n))$. From Proposition C.3, we have
Lemma D.2. Let us consider Proof.

In this appendix, we prove Theorem 4.1. We adopt the arguments used in the finite-dimensional representations of the quantum affine algebras [12]. We assume that $X = C_n^{(1)}$ and $\mathcal{W} = \mathcal{W}_+$ since the proof for the other two cases are similar.

D.1. Normalized R matrix. Let us use the following notations.

- $k$: the base field, which is the algebraic closure of $\mathbb{Q}(q)$ in $\bigcup_n \mathbb{C}(q^{-1/n})$,
- $\mathcal{W}[z] = k[z^\pm 1] \otimes_k \mathcal{W}(1)$: the affinization of $\mathcal{W}(1)$, where $z$ is a formal variable (see [14, Section 4.2]),
- $\mathcal{W}[z_1] \otimes \mathcal{W}[z_2]$, $\mathcal{W}[z_2] \otimes \mathcal{W}[z_1]$: the completions of $\mathcal{W}[z_1] \otimes \mathcal{W}[z_2]$ and $\mathcal{W}[z_2] \otimes \mathcal{W}[z_1]$, respectively, where $z_1, z_2$ are formal commuting variables (see [14, Section 7]),
- $k[z_1^{\pm 1}]$ : the ring of formal power series in $z_1^{\pm 1}$ over $k$, where we regard $k[z_1^{\pm 1}] \subset k[z_1]$ under the identification $\frac{1}{1 - c z_1^{\pm 1}} = \sum_{k \geq 0} c^k (z_1^{\pm 1})^k \in k[z_1^{\pm 1}]$ for $c \in k$.

Let $R^{univ}$ be the universal $R$ matrix for $U_q(C_n^{(1)})$. Then we have (cf. [14, (7.6)])

$$\mathcal{W}[z_1] \otimes \mathcal{W}[z_2] \xrightarrow{R^{univ}} \mathcal{W}[z_2] \otimes \mathcal{W}[z_1].$$

Lemma D.1. $k[z_1^{\pm 1}]$ is an irreducible module over $k[z_1^{\pm 1}] \otimes U_q(C_n^{(1)})[z_1^{\pm 1}, z_2^{\pm 1}]$.

Proof. See [15, Proposition 12].

Lemma D.2. The spaces $\mathcal{W}[z_1] \otimes \mathcal{W}[z_2]$ and $\mathcal{W}[z_2] \otimes \mathcal{W}[z_1]$ are invariant under the action of $k[z_1^{\pm 1}]$.

Proof. Let us consider $\mathcal{W}[z_2] \otimes \mathcal{W}[z_1]$. The proof for $\mathcal{W}[z_1] \otimes \mathcal{W}[z_2]$ is the same. It suffices to check that $\mathcal{W}[z_2] \otimes \mathcal{W}[z_1]$ is invariant under multiplication by $k[z_1^{\pm 1}]$. Let $Q$ be the root lattice for $C_n^{(1)}$ and $Q_+$ the set of non-negative integral linear combinations of simple roots. By definition of $\mathcal{W}[z_2] \otimes \mathcal{W}[z_1]$, we have

$$\mathcal{W}[z_2] \otimes \mathcal{W}[z_1] = \sum_{(\lambda, \mu)} F_{(\lambda, \mu)}(\mathcal{W}[z_2] \otimes \mathcal{W}[z_1]),$$

where $F_{(\lambda, \mu)}(\mathcal{W}[z_2] \otimes \mathcal{W}[z_1]) = \prod_{\beta \in Q_+} \mathcal{W}[z_2]_{\lambda - \beta} \otimes \mathcal{W}[z_1]_{\mu + \beta}$. Here we understand the weights $\lambda - \beta$ and $\mu + \beta$ of $\mathcal{W}[z_i]$ ($i = 1, 2$) as elements in the affine weight lattice, say $P$ in [14, Section 2.1]. We have

$$(z_1^{\pm 1})^k F_{(\lambda, \mu)}(\mathcal{W}[z_2] \otimes \mathcal{W}[z_1]) \in F_{(\lambda, \mu)}(\mathcal{W}[z_2] \otimes \mathcal{W}[z_1]),$$

for $k \in \mathbb{Z}_{\geq 0}$, since for a given $\mu + \beta$ ($\beta \in Q_+$), there exist only finitely many $k \in \mathbb{Z}_{\geq 0}$ and $\beta' \in Q_+$ such that $\mu + \beta = \mu + \beta' + k\delta$, where $\delta$ is the null root of $C_n^{(1)}$. This proves the lemma. □
By Lemma D.2, we may regard
\[ k(\frac{z_1}{z_2}) \otimes k(\frac{z_1}{z_2})^\pm \mathbb{W}[z_1] \otimes \mathbb{W}[z_2] \subset k(\frac{z_1}{z_2}) \otimes k(\frac{z_1}{z_2})^\pm \mathbb{W}[z_1] \otimes \mathbb{W}[z_2], \]
\[ k(\frac{z_1}{z_2}) \otimes k(\frac{z_1}{z_2})^\pm \mathbb{W}[z_2] \otimes \mathbb{W}[z_1] \subset k(\frac{z_1}{z_2}) \otimes k(\frac{z_1}{z_2})^\pm \mathbb{W}[z_2] \otimes \mathbb{W}[z_1]. \]
Note that the weight of \( z_1^k \otimes |m\) is in \( -\frac{1}{2} \sigma_n + \mathbb{Z}_{\geq 0} \delta \) if and only if \(|m\) = |0\). Hence one can check without difficulty that
\[ R(00) = a(\frac{z_1}{z_2}) (00), \]
for some non-zero \( a(\frac{z_1}{z_2}) \in k(\frac{z_1}{z_2}) \), which is invertible. Now, we define the normalized \( R \) matrix by
\[ R_{z_1, z_2} = a(\frac{z_1}{z_2})^{-1} R(00). \]

Lemma D.3. The normalized \( R \) matrix \( R_{z_1, z_2} \) gives a map \( k(\frac{z_1}{z_2})^\pm \mathbb{W}[z_1] \otimes \mathbb{W}[z_2] \)
linear map
\[ k(\frac{z_1}{z_2}) \otimes k(\frac{z_1}{z_2})^\pm \mathbb{W}[z_1] \otimes \mathbb{W}[z_2] \]
where \( R_{z_1, z_2} (00) = (00) \). Moreover, \( R_{z_1, z_2} \) is a unique such map and hence
\[ \tilde{R}_{z_1, z_2} = R_{z_1, z_2}, \]
where \( \tilde{R}_{z_1, z_2} \) is the quantum \( R \) matrix in (4.1).

Proof. The well-definedness and uniqueness of \( R_{z_1, z_2} \) follows from Lemma D.1 and \( R_{z_1, z_2} (00) = (00) \). In particular, we have \( R_{z_1, z_2} = \tilde{R}_{z_1, z_2}. \)

D.2. Irreducibility of \( \mathbb{W}^{(2)} \). Let us prove first that \( \mathbb{W}^{(2)} \) is irreducible. Recall that \( \mathbb{W}^{(2)} \) is the image of
\[ \mathbb{W}(q^{-1}) \otimes \mathbb{W}(q) \]
which is well-defined by (4.1) and Lemma D.3. Let
\[ K = k(\frac{z_1}{z_3}) \otimes k(\frac{z_2}{z_3}), \quad D = k[(\frac{z_1}{z_3})^{\pm 1}, (\frac{z_2}{z_3})^{\pm 1}]. \]

Consider the following maps
\[ K \otimes D \mathbb{W}[z_1] \otimes \mathbb{W}[z_2] \otimes \mathbb{W}[z_3] \]
\[ K \otimes D \mathbb{W}[z_1] \otimes \mathbb{W}[z_3] \otimes \mathbb{W}[z_2]. \]
By the hexagon property of $R^{\text{univ}}$, we have
\[
(R^{\text{norm}}_{z_1, z_3} \otimes \text{id}) \circ (\text{id} \otimes R^{\text{norm}}_{z_2, z_3}) = a(z_1 z_3^{-1}) R^{\text{univ}}_{1, 3} \otimes \text{id}) \circ (a(z_2 z_3^{-1})^{-1} \text{id} \otimes R^{\text{univ}}_{2, 3})
= a(z_1 z_3^{-1})^{-1} a(z_2 z_3^{-1})^{-1} (R^{\text{univ}}_{1, 3} \otimes \text{id}) \circ (\text{id} \otimes R^{\text{univ}}_{2, 3}) \quad (D.1)
= a(z_1 z_3^{-1})^{-1} a(z_2 z_3^{-1})^{-1} R^{\text{univ}}_{12, 3},
\]
where the second equality follows from the fact that the action of $z_i$ commutes with those of $e_j$ and $f_j$, and $R^{\text{univ}}_{12, 3}$ in the last equality is understood as the map
\[
(W[z_1] \otimes W[z_2]) \otimes W[z_3] \xrightarrow{R^{\text{univ}}} W[z_3] \otimes (W[z_1] \otimes W[z_2]).
\]
Put
\[
R = a(z_1 z_3^{-1})^{-1} a(z_2 z_3^{-1})^{-1} R^{\text{univ}}_{12, 3} = (R^{\text{norm}}_{z_1, z_3} \otimes \text{id}) \circ (\text{id} \otimes R^{\text{norm}}_{z_2, z_3}).
\]

By (4.1), we have a well-defined non-zero $U_q(C_n^{(1)})$-linear map $r := R|_{z_1 z_3^{-1} = z_3 = q}$:
\[
W(q) \otimes W(q^{-1}) \otimes W(q) \xrightarrow{r} W(q) \otimes W(q) \otimes W(q^{-1}).
\]

By the hexagon property (D.1), we obtain the following.

**Lemma D.4.** Let $S \subset W(q) \otimes W(q^{-1})$ be a $U_q(C_n^{(1)})$-submodule. Then we have
\[
r(S \otimes W(q^{-1})) \subset W(q^{-1}) \otimes S,
\]
and the following diagram commutes:
\[
\begin{array}{c}
S \otimes W(q^{-1}) \xrightarrow{r|_{S \otimes W(q^{-1})}} W(q^{-1}) \otimes S \\
\downarrow \quad \downarrow \\
W(q) \otimes W(q^{-1}) \otimes W(q) \xrightarrow{r} W(q) \otimes W(q) \otimes W(q^{-1})
\end{array}
\]

**Proposition D.5.** $W^{(2)}$ is irreducible.

**Proof.** Note that $W(q^{\pm 1})$ is irreducible, and $r = (r_{1, 3} \otimes \text{id}) \otimes (\text{id} \otimes r_{2, 3})$, where $r_{2, 3} = R^{\text{norm}}_{z_2, z_3} |_{z_3 = q}$ and $r_{1, 3} = R^{\text{norm}}_{z_1, z_3} |_{z_3 = q} = \text{id}$. Hence, we may apply the same arguments as in [12, Theorem 3.12] to show that if $S$ is a non-zero submodule of $W(q) \otimes W(q^{-1})$, then $S$ includes $\text{Im}(R^{\text{norm}}_{q^{-1}, q})$. This implies that $W^{(2)}$ is irreducible. □
D.3. Proof of Theorem 4.1. Fix \( s \geq 2 \). Let \( z_1, \ldots, z_s \) be formal commuting variables. Consider

\[ W[z_1] \otimes \cdots \otimes W[z_s]. \]

For \( a \in \mathbb{Q}(q)^\times \), let \( x_i = aq^{2i-1-s} \) and \( W_i = W(x_i) \) for \( 1 \leq i \leq s \). We have a map \( r_{i,j} := R_{x_i,x_j}^{\text{norm}} : W_i \otimes W_j \to W_j \otimes W_i \) for \( 1 \leq i < j \leq s \) and

\[ W_1 \otimes \cdots \otimes W_s \xrightarrow{r} W_s \otimes \cdots \otimes W_1, \]

which is the composition of \( r_{i,j} \) associated to a reduced expression of \( w_0 \in S_s \).

Let us prove that \( W^{(s)} = \text{Im}(r) \) is irreducible. Use induction on \( s \). It is true for \( s = 2 \) by Proposition D.5. Suppose that \( s \geq 3 \). Let \( r = r_s \) be the map in the statement and let \( r_{s-1} \) be the map corresponding to the first \( s - 1 \) factors. We have the following commutative diagram:

\[
\begin{array}{ccc}
W_1 \otimes \cdots \otimes W_s & \xrightarrow{r_s} & W_s \otimes \cdots \otimes W_1 \\
\downarrow r_{s-1} \otimes \text{id}_{W_s} & & \downarrow r_{s-1} \circ \hat{r}_1 \\
\text{Im}(r_{s-1}) \otimes W_s & & W_{s-1} \otimes \cdots \otimes W_1 \otimes W_s
\end{array}
\]

where \( \hat{r}_i = \text{id} \otimes s_i^{-1} \otimes r_{i,s} \otimes \text{id}^{-1} \) for \( 1 \leq i \leq s - 1 \). Note that \( r_s \neq 0 \) since \( r((0)^s) = (0)^s \). Thus \( r_{s-1} \) has a nonzero image \( W^{(s-1)} \), which is irreducible by the induction hypothesis. Applying the hexagon property repeatedly, we have the following commutative diagram:

\[
\begin{array}{ccc}
W^{(s-1)} \otimes W_s & \xrightarrow{r_{W^{(s-1)}},W_s} & W_s \otimes W^{(s-1)} \\
\downarrow W_{s-1} \otimes \cdots \otimes W_1 \otimes W_s & & \downarrow W_{s-1} \circ \cdots \circ \hat{r}_1 \\
W_{s-1} \otimes \cdots \otimes W_1 \otimes W_s & & W_s \otimes \cdots \otimes W_2 \otimes W_1
\end{array}
\]

Here the map \( r_{W^{(s-1)}},W_s \) is given by

\[
r_{W^{(s-1)}},W_s = c R^{\text{univ}}_{1,\ldots,s-1,s} |_{z_i = aq^{2i-1-s}},
\]

where \( c \) is an element in \( \bigotimes_{i<j} k(\frac{z_i}{z_j}) \), and \( R^{\text{univ}}_{1,\ldots,s-1,s} \) is the universal \( R \) matrix

\[
(W[z_1] \otimes \cdots \otimes W[z_{s-1}]) \otimes W[z_s] \xrightarrow{R^{\text{univ}}} W[z_s] \otimes (W[z_1] \otimes \cdots \otimes W[z_{s-1}]).
\]

Thus the image of \( r_s \) is equal to that of \( r_{W^{(s-1)}},W_s \).
Now, let $S$ be a non-zero submodule of $\mathcal{W}_s \otimes \mathcal{W}^{(s-1)}$. Put $r' = r_{s,s} \circ r_{\mathcal{W}^{(s-1)}, \mathcal{W}_s}$. Then as in Lemma D.4, we can check the following commutative diagram:

\[
\begin{array}{ccc}
S \otimes \mathcal{W}_s & \xrightarrow{r' |_{S \otimes \mathcal{W}_s}} & \mathcal{W}_s \otimes S \\
\downarrow & & \downarrow \\
\mathcal{W}_s \otimes \mathcal{W}^{(s-1)} \otimes \mathcal{W}_s & \xrightarrow{r'} & \mathcal{W}_s \otimes \mathcal{W}_s \otimes \mathcal{W}^{(s-1)}
\end{array}
\]

Again by the same arguments as in [12, Theorem 3.12], we conclude that $\mathcal{W}^{(s)} \subset S$, which implies that $\mathcal{W}^{(s)}$ is irreducible. This completes the proof.

**Remark D.6.** The universal $R$ matrix for $C^{(2)}(n + 1)$ and $B^{(1)}(0, n)$ can be found in [7].

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