Brück Conjecture with hyper-order less than one

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Abstract

In this paper we affirm Brück conjecture provided $f$ is of hyper-order less than one by studying the infinite hyper-order of solutions of a complex differential equation.

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1 Introduction and main results

In this article, we assume the reader is familiar with standard notations and basic results of Nevanlinna’s value distribution theory in the complex plane $\mathbb{C}$, see [13, 23]. The order and hyper-order of an entire function $f$ are defined as

$$
\rho(f) = \limsup_{r \to +\infty} \frac{\log^+ T(r, f)}{\log r},
$$
$$
\rho_2(f) = \limsup_{r \to +\infty} \frac{\log^+ \log^+ T(r, f)}{\log r},
$$

respectively, where $M(r, f)$ denotes the maximum modulus of $f$ on the circle $|z| = r$.

If $f, g$ are two meromorphic functions in the complex plane, we say $f, g$ share a constant $a$ CM if $f - a$ and $g - a$ have the same zeros with the same multiplicities. Rubel and Yang [16] proved for a nonconstant entire function, if $f$ and its derivative $f'$ share two finite distinct values CM, then $f \equiv f'$. Later on, Brück [1] constructed entire functions with integer or infinite hyper-order to show that $f$ and $f'$ share 1 CM fails to obtain $f \equiv f'$. Therefore, Brück proposed the following conjecture.

Brück Conjecture[1] Let $f$ be a nonconstant entire function such that its hyper-order is finite but not a positive integer. If $f$ and $f'$ share one finite value a CM, then $f' - a = c(f - a)$, where $c$ is a nonzero constant.

The conjecture is false in general for meromorphic function $f$, see a counterpart in [10]. Brück [1] showed the conjecture is right for the case $a = 0$. Later, Gundersen and Yang [10] proved the conjecture is true for the case that $f$ is of finite order. Further on, Chen and Shon [5] showed that the conjecture is also true under the condition $f$ is of hyper-order strictly less than $1/2$. Recently, Cao [3]
gave an affirmative answer to this conjecture under hypothesis that the hyper-order of $f$ is equal to $1/2$.

There are many results closely related to Brück conjecture, mainly in two directions. One is generalizing the shared value $a$ to a nonconstant function, such as polynomial, entire small function respect to $f$, or entire functions with lower order than $f$, (e.g. see [2, 4, 14, 15, 19]) and another is improving the first derivative of $f$ to arbitrary $k$-th derivative (e.g. see [2, 6, 7, 15, 20]). The main purpose of this paper is to confirm Brück conjecture provided the hyper-order of $f$ is less than one. In fact, we obtain the following result.

**Theorem 1.1.** Let $f$ be a nonconstant entire function with hyper-order $\rho_2(f) < 1$. If $f$ shares one finite value $a$ CM with its $k$-th derivative, then $f^{(k)} - a = c(f - a)$, where $c$ is a nonzero constant.

Thus, for the final solving of this conjecture one should consider the remaining case that the hyper-order of $f$ lying in $(1, +\infty) \setminus \mathbb{N}$. In order to study this conjecture, many authors paid attention to the nonhomogeneous linear complex differential equation

$$ f^{(k)} - e^{p(z)} f = Q(z), \quad k \in \mathbb{N}, $$

where $p(z)$ is an entire function and $Q(z)$ is a constant or an entire function. Under suitable conditions on $p(z)$ and $Q(z)$ thought of ways to prove the nontrivial solution of this equation is of infinite order or infinite hyper-order, such as [2, 5, 6, 10, 15, 20, 21, 22]. For the proof of Theorem 1.1, we need one of such results as follows.

**Theorem 1.2.** [15, Theorem 1.1] Let $p(z)$ be a nonconstant polynomial and $Q(z)$ be a nonzero polynomial, then the hyper-order of $f$ is just equal to the degree of $p(z)$.

In the above theorem, $Q(z)$ can be a constant. In order to achieve our goal, we shall also study the hyper-order of solutions of a complex differential equation firstly. Following is the statement.

**Theorem 1.3.** Suppose $A(z) = -e^{p(z)}$, where $p(z)$ is a transcendental entire function with nonzero finite order and $B(z)$ is an entire function with nonzero finite hyper-order. If

$$ \frac{1}{\rho_2(A)} + \frac{1}{\rho_2(B)} > 2, $$

then every solution $f(\neq 0)$ of equation

$$ f^{(k)} + A(z) f' + B(z) f = 0, \quad k \in \mathbb{N}, $$

is of infinite hyper-order.

The method of proving this theorem is originally from Rossi [17]. It was also used by Cao [3] to affirm the Brück conjecture when $f(z)$ is of hyper-order $1/2$.

## 2 Preliminary lemmas

**Lemma 2.1.** [2] Let $f$ be a transcendental meromorphic function. Let $\alpha > 1$ be a constant, and $k, j$ be integers satisfying $k > j \geq 0$. There exists a set $E \subset [0, 2\pi)$ which has zero linear measure, such that if $\theta \in [0, 2\pi) \setminus E$, then there is a constant $R(= R(\theta)) > 0$ such that

$$ \left| \frac{f^{(k)}(z)}{f^{(j)}(z)} \right| \leq K \left[ \frac{T(\alpha r, f)}{r} (\log r)^{\alpha} \log T(\alpha r, f) \right]^{k-j} $$

holds for all $z$ satisfying $\arg z = \theta$ and $|z| \geq R$. 

\[ \text{Page 2} \]
The following Lemma is proved in [17] by using [18, Theorem III.68]. Some notations are needed to state it. Suppose \( D \) is a domain in \( \mathbb{C} \). For each \( r \in \mathbb{R}^+ \) set \( \theta^*_D(r) = \theta^*(r) = +\infty \) if the entire circle \( |z| = r \) lies in \( D \). Otherwise, let \( \theta^*_D(r) = \theta^*(r) \) be the measure of all \( \theta \) in \([0, 2\pi)\) such that \( re^{i\theta} \in D \). As usual, we define the order \( \rho(u) \) of a function \( u \) subharmonic in the plane as

\[
\rho(u) := \limsup_{R \to +\infty} \frac{\log M(r, u)}{\log r},
\]

here \( M(r, u) \) denote the maximum modulus of subharmonic function \( u \) on a circle of radius \( r \).

**Lemma 2.2.** [17] Let \( u \) be a subharmonic function in \( \mathbb{C} \) and let \( D \) be an open component of \( \{z : u(z) > 0\} \). Then

\[
\rho(u) \geq \limsup_{R \to +\infty} \frac{\pi}{\log R} \int_1^R \frac{dt}{t \theta^*_D(t)}.
\]

Furthermore, given \( \varepsilon > 0 \), define \( F = \{r : \theta^*_D \leq \varepsilon \pi\} \). Then

\[
\limsup_{R \to +\infty} \frac{1}{\log R} \int_{F \cap [1, R]} \frac{dt}{t} \leq \varepsilon \rho(u).
\]

**Lemma 2.3.** [5, 17] Let \( l_1(t) > 0 \), \( l_2(t) > 0(t \geq t_0) \) be two measurable functions on \((0, +\infty)\) with \( l_1(t) + l_2(t) \leq (2 + \varepsilon) \pi \), where \( \varepsilon > 0 \). If \( G \subseteq (0, +\infty) \) is any measurable set and

\[
\pi \int_G \frac{dt}{tl_1(t)} \leq \alpha \int_G \frac{dt}{t}, \quad \alpha \geq \frac{1}{2},
\]

then

\[
\pi \int_G \frac{dt}{tl_2(t)} \geq \frac{\alpha}{(2 + \varepsilon)\alpha - 1} \int_G \frac{dt}{t}.
\]

### 3 Proof of Theorems

**Proof of Theorem 1.3**

Suppose that \( \rho_z(f) < +\infty \), and would obtain the assertion by reduction to a contradiction. From Lemma 2.1 and the definition of growth order, there exist constants \( K > 0 \), \( \beta > 1 \) and \( C = C(\varepsilon) \) (depending on \( \varepsilon \)) such that

\[
\left| \frac{f^{(j)}(z)}{f(z)} \right| \leq K \left( \frac{T(\beta r, f)}{\log r} \right) \log r \log T(\beta r, f) \right)^j \leq \exp\{rC\}, \quad j = 1, k
\]

holds for all \( r > r_0 = R(\theta) \) and \( \theta \not\in J(r) \), where \( J(r) \) is a zero linear measure set. For any given positive small \( \varepsilon \), we give \( m(J(r)) \leq \varepsilon . \)

Fix \( \varepsilon > 0 \) and take a positive integer \( N \) such that \( N > C = C(\varepsilon) \). Since \( B(z) \) is an entire function with infinite order and \( \log \log |B(z)| \) is \( o(1) \) as \( |B(z)| \) sufficiently large, it’s feasible to define the set \( \bar{D} := \{z : \log \log |B(z)| = N \log |z| > 0\} \cap \{z : \log \log |B(z)| - N \log |z| > 0\} \) (here, and in the sequel, taking the principal value of complex logarithm). From [11], we know that if \( u(z) \) is analytic in a domain \( D \), then \( \log |u(z)| \) is subharmonic in \( D \). Since \( \frac{\log B(z)}{z^\alpha} \) is analytic in \( \bar{D} \),
the function \( \log \left[ \frac{\log B(z)}{z^N} \right] = \log \log B(z) - N \log |z| \) is subharmonic in the open set \( \bar{D} \). Choose one unbounded component of \( \bar{D} \), called \( D_1 \), such that if we define

\[
u(z) = \begin{cases} 
\log |\log B(z)| - N \log |z|, & z \in D_1, \\
0, & z \in \mathbb{C} \setminus D_1,
\end{cases}
\]

then \( \nu(z) \) is subharmonic in \( \mathbb{C} \) with

\[
\rho(u) \leq \rho_2(B) \tag{3.2}
\]

Let \( D_2 \) be an unbounded component of the set \( \{ z : \log |\log(-e^{-p(z)})| > 0 \} \cap \{ z : \log \log |e^{-p(z)}| > 0 \} \), such that if we define

\[
u(z) = \begin{cases} 
\log |\log(-e^{-p(z)})|, & z \in D_2, \\
0, & z \in \mathbb{C} \setminus D_2,
\end{cases}
\]

then \( \nu(z) \) is subharmonic in \( \mathbb{C} \) with \( \rho(v) = \rho_2(A) \). Moreover, define \( D_3 := \{ re^{i\theta} : \theta \in J(r) \} \). For the above given \( \epsilon \), if \( (D_1 \cap D_2) \setminus D_3 \) contains an unbounded sequence \( \{ r_n e^{i\theta_n} \} \), by (1.2) and (3.1) and together with the properties the sets \( D_1, D_2 \) have we get

\[
\exp \{ r_n^N \} < |B(r_n e^{i\theta_n})| \leq \left| \frac{f^{(k)}(r_n e^{i\theta_n})}{f(r_n e^{i\theta_n})} \right| + \left| e^{\rho(r_n e^{i\theta_n})} \right| \left| \frac{f'(r_n e^{i\theta_n})}{f(r_n e^{i\theta_n})} \right| \leq 2 \exp \{ r_n^C \},
\]

this clearly contradicts \( N > C \) for \( n \) large enough. Thus we could assume that \( (D_1 \cap D_2) \setminus D_3 \) is bounded for arbitrary \( \epsilon \), this implies that for \( r \geq r_1 \geq r_0 \) (\( r_0 \) is from the bottom of (3.1))

\[
K_r := \{ \theta : re^{i\theta} \in D_1 \cap D_2 \} \subseteq J(r).
\]

Obviously,

\[
m(K_r) \leq \epsilon \pi. \tag{3.3}
\]

(We remark here that if \( D_1 \) and \( D_2 \) were disjoint, the proof of Theorem 1.3 would follow easily from (2.2) and Lemma 2.3) In fact, from (2.2) and (3.3) we can deduce that the sets are disjoint in some sense.) Define

\[
l_j(t) = \begin{cases} 
2\pi, & \text{if } \theta^*_j(t) = \infty, \\
\theta^*_j(t), & \text{otherwise},
\end{cases}
\]

for \( j = 1, 2 \). Since \( D_1 \) and \( D_2 \) are unbounded open sets we have that \( l_1(t) > 0, l_2(t) > 0 \) for \( t \) sufficiently large, and (3.3) gives

\[
l_1(t) + l_2(t) \leq 2\pi + \epsilon \pi. \tag{3.4}
\]

Set

\[
\alpha := \limsup_{R \to \infty} \frac{\pi}{\log R} \int_1^R \frac{dt}{tl_1(t)}. \tag{3.5}
\]
From (3.5) and the fact $l_1(t) \leq 2\pi$, it's clear that
\[ \alpha \geq \frac{1}{2} \log R \int_1^R \frac{dt}{t} = \frac{1}{2}. \]
Also by (3.5) we have
\[ \pi \int_1^R \frac{dt}{tl_1(t)} \leq \alpha \log R = \alpha \int_1^R \frac{dt}{t}, \]
Then the conditions of Lemma 2.3 are satisfied, we obtain
\[ \pi \int_1^R \frac{dt}{tl_2(t)} \geq \frac{\alpha}{(2+\varepsilon)\alpha - 1} \int_1^R \frac{dt}{t} = \frac{\alpha}{(2+\varepsilon)\alpha - 1} \log R, \]
this means,
\[ \limsup_{R \to +\infty} \frac{\pi}{\log R} \int_1^R \frac{dt}{tl_2(t)} \geq \frac{\alpha}{(2+\varepsilon)\alpha - 1}. \quad (3.6) \]
Define the sets
\[ B_j := \{ r : \theta^*_D(r) = +\infty \} \]
for $j = 1, 2$. If $r \in B_1$ and $r \geq r_1$, then $\theta^*_D(r) \leq \varepsilon \pi$ by (3.4). Thus $B_1 \subseteq \{ r : \theta^*_D(r) \leq \varepsilon \pi \}$. By Lemma 2.2 we have
\[ \limsup_{R \to \infty} \frac{1}{\log R} \int_{B_1 \cap [1, R]} \frac{dt}{t} \leq \varepsilon \rho_2(-e^{-\rho(z)}) = \varepsilon \rho_2(-e^{\rho(z)}) = \varepsilon \rho_2(A). \quad (3.7) \]
The equality follows by the first Nevanlinna theorem. Set $\tilde{B}_j = R^+ \setminus B_j, j = 1, 2$. Then (2.2), (3.5) and (3.7) give
\[ \rho(u) \geq \limsup_{R \to \infty} \frac{\pi}{\log R} \int_1^R \frac{dt}{t \theta^*_{D_1}(t)} \]
\[ = \limsup_{R \to \infty} \frac{\pi}{\log R} \int_{B_1 \cap [1, R]} \frac{dt}{t \theta^*_{D_1}(t)} \]
\[ = \limsup_{R \to \infty} \frac{1}{\log R} \left[ \pi \int_1^R \frac{dt}{tl_1(t)} - \frac{1}{2} \int_{B_1 \cap [1, R]} \frac{dt}{t} \right] \]
\[ \geq \alpha - \frac{\varepsilon \rho_2(A)}{2}, \]
which together with (3.2) show
\[ \rho_2(B) \geq \alpha - \frac{\varepsilon \rho_2(A)}{2}. \quad (3.8) \]
For the set $B_2$, we have the similar result as follows by the above arguments for $B_1$. If $r \in B_2$ and $r \geq r_1$, then $\theta^*_D(r) \leq \varepsilon \pi$. Thus $B_2 \subseteq \{ r : \theta^*_D(r) \leq \varepsilon \pi \}$. Also from Lemma 2.2 we get
\[ \limsup_{R \to \infty} \frac{1}{\log R} \int_{B_2 \cap [1, R]} \frac{dt}{t} \leq \varepsilon \rho(u). \quad (3.9) \]
Combining (2.2), (3.6) with (3.9) we obtain

\[ \rho_2(A) = \rho(v) \geq \limsup_{R \to \infty} \frac{1}{\log R} \pi \int_1^R \frac{dt}{t \theta_{D_2}(t)} \]

\[ = \limsup_{R \to \infty} \frac{1}{\log R} \int_{B_2 \cap [1,R]} \frac{dt}{t \theta_{D_2}(t)} \]

\[ = \limsup_{R \to \infty} \frac{1}{\log R} \left[ \pi \int_1^R \frac{dt}{t \mu(t)} - \frac{1}{2} \int_{B_2 \cap [1,R]} \frac{dt}{t^\alpha} \right] \]

\[ \geq \frac{\alpha}{(2 + \varepsilon)\alpha - 1} - \frac{\varepsilon \rho(u)}{2}. \tag{3.10} \]

Since \( \frac{\alpha}{(2 + \varepsilon)\alpha - 1} \) is a monotone decreasing function of \( \alpha \), inequalities (3.2), (3.8) and (3.10) give

\[ \rho_2(A) \geq \frac{\rho_2(B) + \frac{\varepsilon \rho_2(A)}{2}}{(2 + \varepsilon)(\rho_2(B) + \frac{\varepsilon \rho_2(A)}{2}) - 1} - \frac{\varepsilon \rho_2(B)}{2}. \]

Note that \( \varepsilon \) is arbitrary positive small and \( \rho_2(A), \rho_2(B) \) are finite, we obtain

\[ \rho_2(A) \geq \frac{\rho_2(B)}{2\rho_2(B) - 1}. \]

It can be transformed into

\[ \frac{1}{\rho_2(A)} + \frac{1}{\rho_2(B)} \leq 2, \]

which contradicts the assumption. Thus, every solution \( f(\neq 0) \) of equation (1.2) is of infinite hyper-order.

**Proof of Theorem 1.1**

By the assumption that \( f \) is a nonconstant entire function with hyper-order \( \rho_2(f) < 1 \), obviously we have

\[ \rho_2 \left( \frac{f^{(k)} - a}{f - a} \right) < 1. \]

Noting that \( f^{(k)} - a \) and \( f - a \) share 0 CM and by the result of the essential part of the factorization theorem for meromorphic function of finite iterated order [12 Satz12.4], we have

\[ \frac{f^{(k)} - a}{f - a} = e^{p(z)}, \tag{3.11} \]

where \( p(z) \) is an entire function with \( \rho(p(z)) = \rho_2(e^{p(z)}) < 1 \). Suppose that \( p(z) \) is not a constant. Set \( F := f - a \), clearly it’s not identically equal to zero, then \( f^{(k)} = F^{(k)} \). Equation (3.11) becomes

\[ F^{(k)} - e^{p(z)} F = a. \tag{3.12} \]

Differential both sides we obtain

\[ F^{(k+1)} - e^{p(z)} F' - p'(z)e^{p(z)} F = 0. \tag{3.13} \]
Set $A(z) = -e^{p(z)}, B(z) = -p'(z)e^{p(z)}$, thus $\rho(p) = \rho_2(A) = \rho_2(B) < 1$. If $p(z)$ is a nonconstant polynomial, applying Theorem 1.2 to equation (3.12) we deduce that $\rho_2(F) = \rho_2(f)$ is equal to a positive integer, which contradicts the assumption $\rho_2(f) < 1$. If $p(z)$ is transcendental with $\rho(p) < 1/2$, by the proof of Theorem 2 Case (3) in [6], it follows a contradiction. If $p(z)$ is transcendental with $1/2 \leq \rho(p) < 1$, applying Theorem 1.3 to equation (3.13), we have $\rho_2(F) = \rho_2(f)$ is infinite, also contradicts the assumption $\rho_2(f) < 1$. Therefore, $p(z)$ must be a constant, and thus $e^{p(z)}$ is just a nonzero constant. Then, we complete the proof.

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