MARTIN BOUNDARY OF A FINELY OPEN SET AND A
FATOU-NAÎM-DOOB THEOREM FOR FINELY
SUPERHARMONIC FUNCTIONS

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Abstract. We construct the Martin compactification $\overline{U}$ of a fine domain $U$ in $\mathbb{R}^n$, $n \geq 2$, and the Riesz-Martin kernel $K$ on $U \times \overline{U}$. We obtain the integral representation of finely superharmonic functions $\geq 0$ on $U$ in terms of $K$ and establish the Fatou-Naim-Doob theorem in this setting.

1. Introduction

The fine topology on an open set $\Omega \subset \mathbb{R}^n$ was introduced by H. Cartan in classical potential theory. It is defined as the smallest topology on $\Omega$ making every superharmonic function on $\Omega$ continuous. This topology is neither locally compact nor metrizable. The fine topology has, however, other good properties which allowed the development in the 1970’s of a ‘fine’ potential theory on a finely open set $U \subset \Omega$, starting with the book [11] of the second named author. The harmonic and superharmonic functions and the potentials in this theory are termed finely [super]harmonic functions and fine potentials. Generally one distinguishes by the prefix ‘fine(ly)’ notions in fine potential theory from those in classical potential theory on a usual (Euclidean) open set. Very many results from classical potential theory have been extended to fine potential theory. In this article we study the invariant functions, generalizing the non-negative harmonic functions in the classical Riesz decomposition theorem; and the integral representation of finely superharmonic functions in terms of the ‘fine’ Riesz-Martin kernel. We close by establishing the Fatou-Naim-Doob theorem on the fine limit of finely superharmonic functions at the fine Martin boundary, inspired in particular by the axiomatic approach of Taylor [21].

In a forthcoming continuation [10] we study sweeping on a subset of the Riesz-Martin space, and the Dirichlet problem at the Martin boundary.

Speaking in slightly more detail we consider the standard $H$-cone $\mathcal{S}(U)$ of all finely superharmonic functions $\geq 0$ on a given fine domain $U$ (that is, a finely connected finely open subset of a Green domain $\Omega \subset \mathbb{R}^n$). Generalizing the classical Riesz representation theorem it was shown in [15], [16] that every
function $u \in \mathcal{S}(U)$ has a unique integral representation of the form

$$u(x) = \int G_U(x, y)d\mu(y) + h(x), \quad x \in \Omega,$$

where $\mu$ is a (positive) Borel measure on $U$, $G_U$ is the (fine) Green kernel for $U$, and $h$ is an invariant function on $U$. The term ‘invariant’ reflects a property established in [15, Theorem 4.4] and generalizing fine harmonicity.

An interesting problem is whether every extreme invariant function is finite valued and hence finely harmonic on $U$, or equivalently whether every invariant function is the sum of a sequence of finely harmonic functions. A negative answer to this question was recently obtained (in dimension $n > 2$) by Gardiner and Hansen [17].

In [8] the first named author has defined a topology on the cone $\mathcal{S}(U)$, generalizing the topology of R.-M. Hervé in classical potential theory. By identifying this topology with the natural topology, now on the standard $H$-cone $\mathcal{S}(U)$, it was shown in [8] that $\mathcal{S}(U)$ has a compact base $B$, and by Choquet’s theorem that every function $u \in \mathcal{S}(U)$ admits a unique integral representation of the form

$$u(x) = \int_B s(x)d\mu(s), \quad x \in U,$$

where $\mu$ is a finite measure on $B$ carried by the (Borel) set of all extreme finely superharmonic functions belonging to $B$.

In the present article we define the Martin compactification $\overline{U}$ and the Martin boundary $\Delta(U)$ of $U$. While the Martin boundary of a usual open set is closed and hence compact, all we can say in the present setup is that $\Delta(U)$ is a $G_δ$ subset of the compact Riesz-Martin space $\overline{U} = U \cup \Delta U$ endowed with the natural topology. Nevertheless we can define a good lower semicontinuous Riesz-Martin kernel $K : U \times \overline{U} \rightarrow [0, +\infty]$. Every function $u \in \mathcal{S}(U)$ has an integral representation $u(x) = \int_{\overline{U}} K(x, Y)d\mu(Y)$ in terms of a Radon measure $\mu$ on $\overline{U}$. This representation is unique if it is required that $\mu$ be carried by $U \cup \Delta_1(U)$ where $\Delta_1(U)$ denotes the minimal Martin boundary of $U$, which is likewise a $G_δ$ in $\overline{U}$. In that case we write $\mu = \mu_u$. It is shown that, for any Radon measure $\mu$ on $\overline{U}$, the associated function $u = \int K(., Y)d\mu(Y) \in \mathcal{S}(U)$ is a fine potential, resp. an invariant function, if and only if $\mu$ is carried by $U$, resp. $\Delta(U)$.

There is a notion of minimal thinness of a set $E \subset U$ at a point $Y \in \Delta_1(U)$, and an associated minimal-fine filter $\mathcal{F}(Y)$. As a generalization of the classical Fatou-Naim-Doob theorem we show that for any finely superharmonic function $u \geq 0$ on $U$ and for $\mu_1$-almost every point $Y \in \Delta_1(U)$, $u(x)$ has the limit $(d\mu_u/d\mu_1)(Y)$ as $x \to Y$ along the minimal-fine filter $\mathcal{F}(Y)$. Here
\( \frac{d\mu_u}{d\mu_1} \) denotes the Radon-Nikodým derivative of the absolutely continuous component of \( \mu_u \) with respect to the absolutely continuous component of the measure \( \mu_1 \) representing the constant function 1, which is finely harmonic and hence invariant. Actually, we establish for any given invariant function \( h > 0 \) a more general \( h \)-relative version of this result.

**Notations:** For a Green domain \( \Omega \subset \mathbb{R}^n \) we denote by \( G_\Omega \) the Green kernel for \( \Omega \). If \( U \) is a fine domain in \( \Omega \) we denote by \( S(U) \) the convex cone of finely superharmonic functions \( \geq 0 \) on \( U \) in the sense of [[11]]. The convex cone of fine potentials on \( U \) (that is, the functions in \( S(U) \) for which every finely subharmonic minorant is \( \leq 0 \)) is denoted by \( \mathcal{P}(U) \). The cone of invariant functions on \( U \) is denoted by \( \mathcal{H}_u(U) \); it is the orthogonal band to \( \mathcal{P}(U) \) relative to \( S(U) \). By \( G_U \) we denote the (fine) Green kernel for \( U \), cf. [[12]]. If \( A \subset U \) and \( f : A \rightarrow [0, +\infty] \) one denotes by \( R^A_f \), resp. \( \tilde{R}^A_f \), the reduced function, resp. the swept function, of \( f \) on \( A \) relative to \( U \), cf. [[11], Section 11]. For any set \( A \subset \Omega \) we denote by \( \tilde{A} \) the fine closure of \( A \) in \( \Omega \), and by \( b(A) \) the base of \( A \) in \( \Omega \), that is, the \( G_\delta \) set of points of \( \Omega \) at which \( A \) is not thin, in other words the set of all fine limit points of \( A \) in \( \Omega \). We define \( r(A) = \mathcal{C}(b(\mathcal{C}A)) \) (complements and bases relative to \( \Omega \)). Thus \( r(A) \) is a \( K_\delta \) set, the least regular finely open subset of \( \Omega \) containing the fine interior \( A' \) of \( A \), and \( r(A) \setminus A \) is polar.

**2. The natural topology on the cone \( S(U) \)**

Throughout this article (in the absence of other indication) \( U \) is a regular fine domain in a Green domain \( \Omega \subset \mathbb{R}^n, n \geq 2 \). We begin by establishing some basic properties of invariant functions on \( \bar{U} \) which play a crucial role in the sequel.

Until Theorem [[2.6]] it is, however, not required that \( U \) be regular.

**Lemma 2.1.** If \( h \) is invariant on \( U \), if \( u \in S(U) \), and if \( h \leq u \), then \( h \preceq u \).

**Proof.** There is a polar set \( E \subset U \) such that \( h \) is finely harmonic on \( U \setminus E \), and hence \( u - h \) is finely superharmonic on \( U \setminus E \). According to [[11], Theorem 9.14], \( u - h \) extends by fine continuity to a function \( s \in S(U) \) such that \( h + s = u \) on \( U \setminus E \) and hence on all of \( U \), whence \( h \preceq u \). \( \Box \)

**Lemma 2.2.** If \( u \in S(U) \) and \( A \subset U \) then \( \tilde{R}_u^A(x) = \int_U u \, d\varepsilon_x^{A \cup \mathcal{C}U} \) for \( x \in U \).

**Proof.** \( \tilde{R}_u^A \) and \( \int_U u \, d\varepsilon_x^{A \cup \mathcal{C}U} \) remain unchanged if \( A \) is replaced by its base \( b(A) \cap U \) relative to \( U \). We may therefore assume that \( A = b(A) \cap U \), and so the set \( V := U \setminus A \) is finely open. Let \( u_0 \) denote the extension of \( u \) to \( \hat{U} = U \cup \partial f U \) by the value 0 on \( \partial f U \). Every function \( s \in S(U) \) with \( s \geq u \) on \( A \) is an upper function (superfunction) for \( u_0 \) relative to \( V \), see [[11], §§14.3–14.6] concerning
the (generalized) fine Dirichlet problem. It follows that

\[ \hat{R}_u^A(x) = R_u^A(x) \geq \|\mathcal{F}_{u_0}(x) = \int_{\Omega}^* u_0 d\varepsilon_x^U_\Omega = \int_{U}^* u d\varepsilon_{x}^{A \cup \hat{U}} \]

for \( x \in V \). For the reverse inequality, consider any upper function \( v \) for \( u_0 \) relative to \( V \). In particular, \( v \geq -p \) on \( V \) for some finite and hence semibounded potential \( p \) on \( \Omega \). Define \( w = u \wedge v \) on \( V \) and \( w = u \) on \( A = U \setminus V \). Then

\[
\operatorname{fine \lim \inf}_{y \to x} w(y) \geq u(x) \wedge \operatorname{fine \lim \inf}_{y \to x} v(y) = u(x)
\]

for every \( x \in U \cap \partial_1 V \). It therefore follows by [11, Lemma 10.1] that \( w \) is finely hyperharmonic on \( U \). Moreover, \( w \) is an upper function for \( u \) relative to \( V \) because \( w = u \wedge v \geq 0 \wedge (-p) = -p \) on \( V \). Since \( w = u \) on \( A \) we have \( w \geq \hat{R}_u^A \) on \( U \), and in particular \( v \geq w \geq \hat{R}_u^A \) on \( V \). By varying \( v \) we obtain \( \|\mathcal{F}_{u_0} \geq \hat{R}_u^A \) on \( V \). Altogether we have established the asserted equality for \( x \in V \). It also holds for \( x \in U \setminus V = A = b(A) \cup U \) because \( \hat{R}_u^A(x) = R_u^A(x) = u(x) \) and because \( \varepsilon_{x}^{A \cup \hat{U}} = \varepsilon_x \), noting that \( x \in b(A) \subset b(A \cup \hat{U}) \).

\[ \square \]

**Lemma 2.3.** Let \( u \in \mathcal{S}(U) \) and let \( A \) be a subset of \( U \). The restriction of \( \hat{R}_u^A \) to any finely open subset \( V \) of \( U \setminus \hat{A} \) is invariant.

**Proof.** We have \( \hat{R}_u^A = \sup_{k \in \mathbb{N}} \hat{R}_{u \wedge k}^A \). Each of the functions \( \hat{R}_{u \wedge k}^A \) is finely harmonic on \( U \setminus \hat{A} \) by [11, Corollary 11.13], and hence \( \hat{R}_u^A \) is invariant on \( U \setminus \hat{A} \) in view of Lemma 2.4 because the invariant functions on \( V \) form a band (the orthogonal band to \( \mathcal{P}(V) \)).

\[ \square \]

While the invariant functions on finely open subsets of \( U \) do not form a sheaf, they nevertheless have a kind of countable sheaf property.

**Example 2.4.** Let \( \mu \) be the one-dimensional Lebesgue measure on a line segment \( E \in \Omega := U := \mathbb{R}^3 \). Then \( E \) is polar and hence everywhere thin in \( \mathbb{R}^3 \). Every point \( x \in \mathbb{R}^3 \) therefore has a fine neighborhood \( V_x \) with \( \mu(V_x) = 0 \). (For \( x \in E \) take \( V_x = \{x\} \cup \mathcal{C}L \), where \( L \) denotes the whole line extending \( E \).) Thus \( \mu \) does not have a (minimal) fine support (unlike measures which do not charge any polar set). The Green potential \( G_{\Omega \mu} \) is invariant on each \( V_x \) (but of course not on \( \Omega \)). For if \( p \) denotes a non-zero fine potential on \( V_x \), \( x \in E \), such that \( p \leq G_{\Omega \mu} \) on \( V_x \), then \( p \) is finely harmonic on \( V_x \setminus \{x\} = \mathcal{C}L \) along with \( G_{\Omega \mu} \). Hence \( p \) behaves on \( V_x \) near \( x \) like a constant times \( G_{V_x \varepsilon_x} \) and is therefore of the order of magnitude \( 1/r \), where \( r \) denotes the distance from \( x \), cf. [12, Théorème]. But on the plane through \( x \) orthogonal to \( L \), \( G_{\Omega \mu} \) behaves like a constant times \( \log(1/r) \), in contradiction with \( p \leq G_{\Omega \mu} \).

**Theorem 2.5.** (a) Let \( u \in \mathcal{S}(U) \) be invariant and let \( V \) be a finely open subset of \( U \). Then \( u|_V \) is invariant.
(b) Let \( u \in \mathcal{S}(U) \), and let \( (U_j) \) be a countable cover of \( U \) by finely open subsets of \( U \). If each \( u|_{U_j} \) is invariant then \( u \) is invariant.

(c) Let \( (u_\alpha) \) be a decreasing net of invariant functions in \( \mathcal{S}(U) \). Then \( \hat{\inf}_\alpha u_\alpha \) is invariant. Moreover, the set \( V = \{ \inf_\alpha u_\alpha < +\infty \} \) is finely open, and \( \hat{\inf}_\alpha u_\alpha = \inf_\alpha u_\alpha \) on \( V \).

\[\text{Proof.}\]

(a) Let \( p \in \mathcal{P}(V) \) satisfy \( p \ll u \) on \( V \). Write \( p = G_V \mu \) (considered on \( V \)) in terms of the associated Borel measure \( \mu \) on the topological subspace \( V \) of \( \Omega \). We shall prove that \( p = 0 \). An extension of \( \mu \) from \( V \) to a larger subspace of \( \Omega \) by \( 0 \) off \( U \) will also be denoted by \( \mu \). Thus \( p = G_V \mu \) extends to the fine potential \( G_r(V) \mu \) on \( r(V) \) and further extends to a finely continuous function \( f : U \rightarrow [0, +\infty] \) which equals 0 on \( U \setminus r(V) \).

Suppose to begin with that \( \mu \) is finite and carried by a compact set \( K \subset r(V) \). Let \( q := \hat{R}_f \). Then \( f \leq G_V \mu \leq G_{\Omega} \mu \) and hence \( q \leq G_{\Omega} \mu < +\infty \) q.e. on \( U \), and so \( q \) is a fine potential on \( U \) along with \( G_U \mu \). On the other hand, \( q \leq u \) because \( f \leq u \) on \( U \), noting that \( f = p \leq u \) on \( V \) and \( f = 0 \) on \( U \setminus V \). Since \( p \ll u \) on \( V \) we have \( G_r(V) \mu \ll u \) on \( r(V) \). Thus there is a unique \( s \in \mathcal{S}(r(V)) \) with \( u = G_r(V) \mu + s \) on \( r(V) \). Since fine \( \lim G_r(V) \mu = 0 \) at \( \partial f r(V) \) we have fine \( \lim s = u \) at \( \partial f r(V) \). It follows by \[11\] Lemma 10.1 that the extension of \( s \) to \( U \) by \( u \) on \( U \setminus r(V) \) is of class \( \mathcal{S}(U) \). Denoting this extension by \( s \) we have \( u = s + f \) on \( U \). By Mokobodzki’s inequality in our setting, see \[11\] Lemma 11.14, we infer that \( q = \hat{R}_f \ll u \). Since \( u \in \mathcal{H}_\iota(U) \) and \( q \in \mathcal{P}(U) \) it follows that \( q = 0 \) and hence \( p \ll q = 0 \) on \( V \), showing that \( u|_V \) is invariant.

Dropping the above temporary hypothesis that \( \mu \) be finite and carried by a compact subset of \( r(V) \) we decompose \( \mu \) in accordance with \[15\] Lemma 2.3] into the sum of a sequence of finite measures \( \mu_j \) with compact supports \( K_j \subset r(V) \). Since \( G_V \mu_j \ll G_V \mu = p \ll u|_V \), the result of the above paragraph applies with \( \mu \) replaced by \( \mu_j \). It follows that \( G_V \mu_j = 0 \) and hence \( p = G_V \mu = \sum_j G_V \mu_j = 0 \). Thus \( \hat{R}_u \) is indeed invariant on \( V \).

(b) For each index \( j \) there is by \[15\] Theorem 4.4] a countable finely open cover \( (V_{jk})_k \) of \( U_j \) such that \( V_{jk} \subset U_j \) and (with sweeping relative to \( U_j \))

\[ \hat{R}_u|_{U_j}^{U_j \setminus V_{jk}} = u|_{U_j} \]

for each \( k \). It follows that (with sweeping relative to \( U \), resp. \( U_j \))

\[ u \geq \hat{R}_u^{U \setminus V_{jk}} \geq \hat{R}_u|_{U_j}^{U_j \setminus V_{jk}} = u \]

on each \( U_j \), so equality prevails here. In particular, \( u = \hat{R}_u^{U \setminus V_{jk}} \) on \( V_{jk} \) for each \( j, k \). Consequently, \( u \) is invariant according to the quoted theorem applied to the countable cover \( (V_{jk})_{jk} \) of \( U \).

(c) For indices \( \alpha, \beta \) with \( \alpha < \beta \) we have \( u_\alpha \leq u_\beta \) and hence \( u_\alpha \ll u_\beta \) by Lemma \[22\]. The claim therefore follows from \[11\] c), p. 132]. \qed
We proceed to introduce and study the natural topology on the $H$-cone $S(U)$ of non-negative finely superharmonic functions on the fine domain $U$, which is henceforth assumed to be regular.

**Theorem 2.6.** There exists a resolvent family $(W_\lambda)$ of kernels on $U$ which are absolutely continuous with respect to a measure $\sigma$ on $U$ such that $S(U)$ is the cone of excessive functions which are finite $\sigma$-a.e.

**Proof.** Let $p = G_\Omega \tau$ be a strict bounded continuous potential on the Greenian domain $\Omega$ in $\mathbb{R}^n$. Then the measure $\tau$ doesn’t charge the polar sets and we have $\tau(\omega) > 0$ for any fine open subset of $\Omega$. Denote by $V$ the Borel measurable kernel on $\Omega$ defined by

$$V f(x) = \int G_\Omega(x, y)f(y)d\tau(y)$$

for $x \in \Omega$, and by $(V_\lambda)$ the resolvent family of kernels whose kernel potential is $V$. According to [2, Proposition 10.2.2, p. 248], the cone of excessive functions of the resolvent $(V_\lambda)$ is the cone $S(\Omega) \cup \{+\infty\}$ (and hence $S(\Omega)$ is the cone cone of excessive functions of $(V_\lambda)$ which are finite $\tau$-a.s). Define a kernel $W$ on $U$ by

$$W f = V \tilde{f} - \widehat{R}_{V \tilde{f}}$$

(restricted to $U$) for any Borel measurable function $f \geq 0$ in $U$, where $\tilde{f}$ denotes the extension of $f$ to $\Omega$ by 0 in $\Omega \setminus U$. We begin by showing that the kernel $W$ satisfies the complete maximum principle. Let $f, g : \Omega \rightarrow [0, +\infty[$ be two Borel measurable functions such that $a + W g \geq W f$ on $\{f > 0\}$. We may suppose that $f$ is bounded. Consider the function

$$u = \begin{cases} (W g + \widehat{R}_{V \tilde{f}}) \land V \tilde{f} & \text{in } U \\ V \tilde{f} & \text{in } \partial U. \end{cases}$$

We have

$$\text{fine lim inf}_{x \rightarrow y, x \in U} (W g(x) + \widehat{R}_{V \tilde{f}}(x)) \geq V \tilde{f}(y)$$

for every $y \in \partial f U$. By [11, Lemma 10.1] $u$ therefore is finely superharmonic on $\Omega$ and hence supermedian with respect to the resolvent $(V_\lambda)$. On the other hand we have $a + u \geq V \tilde{f}$ on $\{f > 0\} = \{\tilde{f} > 0\}$, and so $a + u \geq V \tilde{f}$ on $\Omega$ according to [7, Théorème 27, p. 16]. It follows that $a + W g \geq W f$ on $U$, showing that $W$ indeed satisfies the complete maximum principle. This implies by [7, Théorème 88, p. 75] that there exists a unique resolvent family $(W_\lambda)$ of Borel measurable kernels having the potential kernel $W$.

We proceed to determine the excessive functions for the resolvent $(W_\lambda)$. Every superharmonic function $s \geq 0$ on $S(\Omega)$ is excessive for the resolvent $V_\lambda$, and hence there exists by [7, Théorème 17, p. 11] an increasing sequence $(f_j)$ of bounded Borel measurable functions $\geq 0$ such that $s = \sup V f_j$. It follows
that \( s - \hat{R}^U_s = \sup_j W(g_j) \), where \( g_j \) denotes the restriction of \( f_j \) to \( U \). This shows that \( s - \hat{R}^U_s \) is excessive for \( (W_\lambda) \). For any \( u \in S(U) \cup \{+\infty\} \) there exists by [14, Theorem 3] an increasing sequence of superharmonic functions \( s_j \geq 0 \) on \( \Omega \) such that
\[
  u = \sup_j (s_j - \hat{R}^U_{s_j}),
\]
and hence \( u \) is excessive for \( (W_\lambda) \). Conversely, let \( u \) be excessive for \( (W_\lambda) \). According to [7, Théorème 17, p. 11] there exists an increasing sequence \( (f_j) \) of bounded Borel measurable functions \( \geq 0 \) such that \( u = \sup_j Wf_j \). For each \( j \) we have \( W(f_j) = V(\bar{f}_j) - \hat{R}^U_{V(\bar{f}_j)} \). But \( V(\bar{f}_j) \) is finite and continuous on \( U \), and so \( \hat{R}^U_{V(\bar{f}_j)} \) is finely harmonic on \( U \). It follows that \( W(f_j) \in S(U) \). Consequently, \( u \) is finely hyperharmonic on \( U \), that is, \( u \in S(U) \cup \{+\infty\} \).

Let \( \sigma \) be the restriction of the measure \( \tau \) to \( U \). Then for any \( A \in \mathcal{B}(U) \) (the finely Borel \( \sigma \)-algebra on \( U \)) such that \( \sigma(A) = 0 \) we have \( W1_A = (V1_A)|_U = 0 \), hence the resolvent \( (W_\lambda) \) is absolutely continuous with respect to \( \sigma \). Since \( \tau \) does not charge the polar sets, we see that \( S(U) \) is the cone of excessive functions which are finite \( \sigma \)-a.e. This completes the proof of Theorem 2.6 \( \square \)

It follows from Theorem 2.6 by [4, Theorem 4.4.6] that \( S(U) \) is a standard \( H \)-cone of functions on \( U \). Following [4, Section 4.3] we give \( S(U) \) the natural topology. This topology on \( S(U) \) is metrizable and induced by the weak topology on a locally convex topological vector space in which \( S(U) \) is embedded as a proper convex cone. This cone is well-capped with compact caps, but we show that the cone \( S(U) \) even has a compact base, and that is crucial for our investigation. We shall need the following results from [8]:

**Theorem 2.7.** [8, Lemme 3.5]. There exists a sequence \( (K_j) \) of compact subsets of \( \Omega \) (in the Euclidean topology) contained in \( U \) and a polar set \( P \subset U \) such that
1. \( U = P \cup \bigcup_j K_j^\prime \), where \( K_j^\prime \) denotes the fine interior of \( K_j \).
2. For any \( j \) the restriction of any function from \( S(U) \) to \( K_j \) is l.s.c. in the Euclidean topology.

**Corollary 2.8.** There exists a sequence \( (H_j) \) of compact subsets of \( U \), each non-thin at any of its points, and a polar set \( P \) such that
1. \( U = P \cup \bigcup_j H_j \).
2. For any \( j \) the restriction of any function from \( S(U) \) to \( H_j \) is l.s.c. in the Euclidean topology.

**Proof.** Write \( U = \bigcup_j K_j^\prime \cup P \) as in Theorem 2.7. For each \( j \) let \( (U_j^m) \) denote the fine components of \( K_j^\prime \). For each couple \( (j, m) \) let \( y_{j,m} \) be a point of \( U_j^m \). And for each integer \( n > 0 \) put \( H_{j,m,n} = \{ x \in U_j^m : G_{U_j^m}(x, y_{j,m}) \geq \frac{1}{n} \} \). The sets
$H_{j,m,n}$ are compact and non-thin at any of its points (in view of [11, Theorem 12.6]), and we have $U_j^n = \bigcup_n H_{j,m,n}$. The sequence $(H_{j,m,n})$ and the polar set $P$ have the stated properties.

We shall now use the sequence $(H_j)$ from this corollary to define in analogy with [20] a locally compact topology on the cone $S(U)$. For each $j$ let $C_j(H_j)$ denote the space of $\text{l.s.c.}$ functions on $H_j$ with values in $\mathbb{R}_+$, and provide this space with the topology of convergence in graph (cf. [20]). It is known that $C_j(H_j)$ is a compact metrizable space in this topology. Let $d_j$ denote a distance compatible with this topology. We define a pseudo-distance $d$ on $S(U) \cup \{+\infty\}$ by

$$d(u,v) = \sum_j \frac{1}{2\delta(C_j(H_j))} d_j(u|_{H_j}, v|_{H_j})$$

for each couple $(u,v)$ of functions from $S(U) \cup \{+\infty\}$, where $\delta(C_j(H_j))$ denotes the diameter of $C_j(H_j)$. Since two finely hyperharmonic functions are identical if the coincide quasi-everywhere it follows that $d$ is a true distance on $S(U) \cup \{+\infty\}$. We denote by $\mathcal{T}$ the topology on $S(U) \cup \{+\infty\}$ defined by the distance $d$.

For any filter $\mathcal{F}$ on $S(U) \cup \{+\infty\}$ we write

$$\liminf_{\mathcal{F}} \sup_{M \in \mathcal{F}} \inf_{u \in M} u,$$

where the $\text{l.s.c.}$ regularized $\inf_{u \in M} u$ is taken with respect to the fine topology.

**Theorem 2.9.** [8, Théorème 3.6]. The cone $S(U) \cup \{+\infty\}$ is compact in the topology $\mathcal{T}$. For any convergent filter $\mathcal{F}$ on $S(U) \cup \{+\infty\}$ we have

$$\lim_{\mathcal{F}} = \liminf_{\mathcal{F}}.$$

**Proof.** Let $\mathcal{U}$ be an ultrafilter on $S(U) \cup \{+\infty\}$. For any $M \in \mathcal{U}$ put $u_M = \inf_{u \in M} u$. For each $j$ the ultrafilter base $\mathcal{U}_j$ obtained from $\mathcal{U}$ by taking restrictions to the compact $H_j$, converges in the compact space $C_j(H_j)$ to the function $u_j := \sup_{M \in \mathcal{U}_j} \overset{\sim}{u}_M^j$ (where $\overset{\sim}{v}$ for $v \in S(U) \cup \{+\infty\}$ denotes the finely $\text{l.s.c.}$ regularized of the restriction of $v$ to $H_j$). The finely $\text{l.s.c.}$ regularized $\overset{\sim}{u}_M$ of $u_M$ in $U$ is $\text{l.s.c.}$ in $H_j$ by Theorem 2.4, and minorizes $u_M$, whence $\overset{\sim}{u}_M \leq \overset{\sim}{u}_M^j$. On the other hand there exists a polar set $A \subset \Omega$ such that $u_M = \overset{\sim}{u}_M$ in $U \setminus A$, and so $\overset{\sim}{u}_M^j \leq \overset{\sim}{u}_M$ in $H_j \setminus A$. But for $x \in A$ we have $\overset{\sim}{u}_M^j(x) \leq \overset{\sim}{u}_M(x)$ because $\overset{\sim}{u}_M$ is finely continuous on $U$ and $x$ is in the fine closure of $H_j \setminus A$ since $H_j$ is non-thin at $x$. We conclude that $u_j = \lim\overset{\sim}{u}_M$ in $H_j$ for each $j$. Since the function $u := \lim\overset{\sim}{u}_M$ belongs to $S(U) \cup \{+\infty\}$ according to [11, §12.9] it follows that the filter $\mathcal{U}$ converges to $u$ in the topology $\mathcal{T}$. This proves that $S(U) \cup \{+\infty\}$ is compact in the topology $\mathcal{T}$. \qed
Corollary 2.10. The topology of convergence in graph coincides with the natural topology on $S(U)$.

Proof. The corollary follows immediately from Theorem 2.9 and [4, Theorem 4.5.8]. □

Corollary 2.11. The cone $S(U)$ endowed with the natural topology has a compact base.

Proof. From Theorem 2.9 and Corollary 2.10 it follows that the natural topology on $S(U)$ is locally compact, and we infer by a theorem of Klee [2, Theorem II.2.6], that indeed $S(U)$ has a compact base. □

Corollary 2.12. For given $x \in U$ the affine forms $u \mapsto u(x)$ and $u \mapsto \widehat{R}_u^A(x)$ ($A \subset U$) are l.s.c. in the natural topology on $S(U)$.

Proof. Clearly, the map $u \mapsto \widehat{R}_u^A(x)$ is affine for fixed $x \in U$. Let $(u_j)$ a sequence in $S(U)$ converging naturally to $u \in S(U)$. For any index $k$ we have

$$\inf_{j \geq k} \widehat{R}_{u_j}^A(x) \geq \widehat{R}_{\inf_{j \geq k} u_j}^A(x).$$

Either member of this inequality increases with $k$, and we get for $k \to \infty$

$$\lim_{k} \inf_{u_k} \widehat{R}_u^A(x) \geq \lim_{k} \inf_{u_k} \widehat{R}_u^A(x) \geq \widehat{R}_{\lim_{k} u_k}^A(x) \geq \widehat{R}_u^A(x),$$

and so the map $u \mapsto \widehat{R}_u^A(x)$ is indeed l.s.c. on $S(U)$. For the map $u \mapsto u(x)$ take $A = U$. □

In the rest of the present Section we denote by $B$ a fixed compact base of $S(U)$. As shown by Choquet (cf. [2, Corollary I.4.4]) the set $\text{Ext}(B)$ of extreme elements of $B$ is a $G_\delta$ subset of $B$. On the other hand it follows by the fine Riesz decomposition theorem that every element of $\text{Ext}(B)$ is either a fine potential or an extreme invariant function. We denote by $\text{Ext}_p(B)$ (resp. $\text{Ext}_i(B)$) the cone of all extreme fine potentials (resp. all extreme invariant functions) in $B$. According to the theorem on integral representation of fine potentials [16], any element of $\text{Ext}_p(B)$ has the form $\alpha G_{U}(., y)$, where $\alpha$ is a constant $> 0$ and $y \in U$.

Proposition 2.13. [8, Proposition 4.3]. $\text{Ext}_p(B)$ and $\text{Ext}_i(B)$ are Borel sub-sets of $B$.

When $\mu$ is a non-zero Radon measure on $B$ there exists a unique element $s$ of $B$ such that

$$l(s) = \int_B l(u) d\mu(u)$$

for every continuous affine form $l : B \to [0, +\infty]$ on $B$ (in other words, $s$ is the barycenter of the unit measure $\frac{1}{\mu(B)} \mu$). For any l.s.c. affine form
In particular, for fixed $x \in U$ and $A \subset U$, the affine forms $u \mapsto u(x)$ and $u \mapsto \hat{R}_s^A(x)$ are l.s.c. according to Corollary 2.12, and hence
$$s(x) = \int_B u(x) d\mu(u) \quad \text{and} \quad \hat{R}_s^A(x) = \int_B \hat{R}_s^A(x) d\mu(u).$$

The following theorem is an immediate consequence of Theorems 2.15 and 2.16.

**Theorem 2.17.** Let $A \subset \text{Ext}_p(B)$ (resp. $A \subset \text{Ext}_i(B)$), and let $\mu$ be a Radon measure on $B$. Then $\int_A u d\mu(u)$ is a fine potential (resp. an invariant function).

**Remark 2.18.** In view of [11, Section 11.16] the set $\mathcal{H}_s(U)$ of all invariant functions on $U$ is clearly a lower complete and conditionally upper complete sublattice of $\mathcal{S}(U)$ in the specific order. According to Lemma 2.1 the specific order on $\mathcal{H}_s(U)$ coincides with the pointwise order.

**Remark 2.19.** On a Euclidean domain the invariant functions are the same as the harmonic functions $\geq 0$. It is well known in view of Harnack’s principle that the set of these functions is closed in $\mathcal{H}(U)$ with the natural topology (which coincides with the topology of R.-M. Hervé [19]). However, when $U$ is just a fine domain in a Green space $\Omega$, the set $\mathcal{H}_s(U)$ of invariant functions on $U$ need not be closed in the induced natural topology on $U$, as shown by
the following example: Let \( y \in U \) be a Euclidean non-inner point of \( U \), and let \( (y_k) \) be a sequence of points of \( \Omega \setminus U \) which converges Euclidean to \( y \). The sequence \( (G_{\Omega}(., y_k)|_{U}) \) then converges naturally in \( S(U) \) to \( G_{\Omega}(., y)|_{U} \), which does not belong to \( H_1(U) \) because its fine potential part \( G_U(., y) \) is non-zero.

3. **Martin compactification of \( U \) and integral representation in \( S(U) \)**

We continue considering a regular fine domain \( u \) in a Green domain \( \Omega \subset \mathbb{R}^n \), \( n \geq 2 \). Let \( B \) be a compact base of the cone \( S(U) \) (\( U \) a regular fine domain in \( \Omega \)) and let \( \Phi \) be a continuous affine form \( \geq 0 \) on \( S(U) \) such that

\[
B = \{ u \in S(U) : \Phi(u) = 1 \}.
\]

Then \( \Phi(u) > 0 \) except at \( u = 0 \). Consider the mapping \( \varphi : U \to B \) defined by

\[
\varphi(y) = P_y = \frac{G_U(., y)}{\Phi(G_U(., y))},
\]

and identify \( y \in U \) with \( \varphi(y) = P_y \in B \) and hence \( U \) with \( \varphi(U) \).

We denote by \( \overline{U} \) the closure of \( U \) in \( B \) (with the natural topology), and write \( \Delta(U) = \overline{U} \setminus U \). Then \( \overline{U} \) is compact in \( B \) and is called the Martin compactification of \( U \), and \( \Delta(U) \) is called the Martin boundary of \( U \).

If \( B \) and \( B' \) are two bases of \( S(U) \) the Martin compactifications of \( U \) relative to \( B \) and to \( B' \) are clearly homeomorphic.

Throughout the rest of this article we fix a compact base \( B \) of the cone \( S(U) \) and a continuous affine form \( \Phi : S(U) \to [0, +\infty[ \) defining this base, that is such that \( B = \{ u \in S(U) : \Phi(u) = 1 \} \). It is with respect to this base and affine form that we consider the Martin compactification \( \overline{U} \subset B \) and the Martin boundary \( \Delta(U) = \overline{U} \setminus U \) of \( U \).

For any \( y \in \overline{U} \) consider the function \( K(., y) \in B \subset S(U) \) defined on \( U \) by \( K(x, y) = \varphi(y)(x) \) if \( y \in U \) and \( K(., Y) = Y \) if \( Y \in \Delta(U) \).

**Definition 3.1.** The function \( K : U \times \overline{U} \to \mathbb{R}_+ \) defined by \( K(x, y) = K(., y)(x) \) is called the (fine) Riesz-Martin kernel for \( U \), and its restriction to \( U \times \Delta(U) \) is called the (fine) Martin kernel for \( U \).

**Lemma 3.2.** The mapping \( \varphi : U \ni y \mapsto P_y \in S(U) \) is continuous from \( U \) with the fine topology to \( S(U) \cup \{+\infty\} \) with the natural topology. In other words, the topology on \( U \) induced by the natural topology on \( \overline{U} \) is coarser than the fine topology on \( U \).

**Proof.** Consider a net \( (y_\alpha)_{\alpha \in I} \) on \( U \) which converges finely to a point \( y \), that is, \( s(y_\alpha) \to s(y) \) for every \( s \in S(U) \). Taking \( s = G_U(x, .) \) we have \( G_U(x, y_\alpha) \to \)
$G_U(x, y)$ in $[0, +\infty]$ for every $x \in U$. Writing $\Phi(G_U(., y) = c$ and $\Phi(G_U(., y)) = c$ we shall prove that

$$\frac{G_U(., y)}{c} \rightarrow \frac{G_U(., y)}{c}$$

in $B$ (with the natural topology). We may assume that the net $(y_\alpha)$ is universal. By compactness of $B$ and $[0, +\infty]$ there exist the limits

$$\lim_{\alpha \in I} \frac{G_U(., y_\alpha)}{c_\alpha} = z(\cdot) \in B \quad \text{and} \quad \lim_{\alpha \in I} c_\alpha = d \in [0, +\infty].$$

For any $x \in U$ with $G_U(x, y) < +\infty$ (that is $x \neq y$) we have $G_U(x, y_\alpha)/c_\alpha \rightarrow G_U(x, y)/d$ because the quotient $G_U(x, y)/d$ is well defined in $[0, +\infty]$. From $y_\alpha \rightarrow z$ naturally it follows by Theorem 2.9 that $\lim \inf_{\alpha} G_U(., y_\alpha)/c_\alpha = z(\cdot)$ q.e. on $U$. We infer that

$$z(\cdot) = \frac{G_U(., y)}{d} \quad \text{q.e. on } U$$

and hence everywhere on $U$ by fine continuity of $z(\cdot), G_U(., y) \in \mathcal{S}(U) \cup \{+\infty\}$. Furthermore, $d = \Phi(G_U(., y)) = c$ because $z \in B$. □

**Proposition 3.3.** (i) The Riesz-Martin kernel $K : U \times \overline{U} \rightarrow \mathbb{R}_+$ has the following properties, $\overline{U}$ being given the natural topology:

(i) For any $x \in U$, $K(x, \cdot)$ is l.s.c. on $U$.

(ii) For any $Y \in \overline{U}$, $K(\cdot, Y) \in \mathcal{S}(U)$ is finely continuous on $U$.

(iii) $K$ is l.s.c. on $U \times \overline{U}$ when $U$ is given the fine topology and $\overline{U}$ the natural topology.

**Proof.** (i) follows from Corollary 2.12 applied to $u = K(., Y)$ while identifying $K(., Y)$ with $Y$.

(ii) is obvious.

(iii) Let $x_0 \in U$, $Z \in \overline{U}$, and let $(V_j)$ be a fundamental system of open neighborhoods of $Z$ in $\overline{U}$ such that $V_{j+1} \subset V_j$ for any $j$. For a given constant $c > 0$ consider the increasing sequence of functions

$$k_j := \inf_{Y \in V_j} K(., Y) \wedge c$$

and their finely l.s.c. regularizations $\hat{k}_j \in \mathcal{S}(U)$. By [13] there exists a fine neighborhood $H$ of $x_0$ in $U$ such that $H$ is compact (in the Euclidean topology) and that the restrictions of the functions $\hat{k}_j \in \mathcal{S}(U)$ and of $K(., Z) \wedge c \in \mathcal{S}(U)$ to $H$ are continuous (again in the induced Euclidean topology). By Theorem 2.9 and Dini’s theorem there exists for given $\epsilon > 0$ an integer $j_0 > 0$ such that

$$K(., Z) \wedge c = \sup_j \hat{k}_j < \hat{k}_i + \epsilon$$
on $H$ for any $i \geq j_0$. Since $K(., Z)$ is finely continuous there is a fine
neighborhood $W$ of $x_0$ with $W \subset H$ such that
$$\inf_{x \in W} K(x, Z) \land c > K(x_0, Z) \land c - \varepsilon,$$
and hence altogether
$$\inf_{x \in W, Y \in V_j} K(x, Y) \land c = \inf_{x \in W} k_j(x) \geq \inf_{x \in W} \hat{k}_j(x) \geq \inf_{x \in W} K(x, Z) \land c - \varepsilon \geq K(x_0, Z) - 2\varepsilon.$$
This completes the proof of Proposition 3.3. □

Remark 3.4. The Riesz-Martin kernel $K$ is in general not l.s.c. in the product
of the induced natural topology on $U$ and the natural topology on $\overline{U}$. Not even
the function $K(., y) = G_U(., y)/\Phi(G_U(., y))$, or equivalently $G_U(., y)$ itself, is
l.s.c. on $U$ with the induced natural topology for fixed $y \in U$. For if the set
$V := \{x \in U : G_U(x, y) > 1\}$ were open for every $y \in U$ then $U$ would be a
natural neighborhood of $y$ for every $y \in U$, that is, $U$ would be naturally open
in $\overline{U}$. But that is in general not the case because $\overline{U} \setminus U = \Delta(U)$ need not be
naturally compact, see Example 3.9 below.

Remark 3.5. A set $A \subset U$ is termed a Euclidean nearly Borel set if it differs
by a polar set from a Euclidean Borel set. We denote by $B(U)$, resp. $B^*(U)$,
the $\sigma$-algebra of all Euclidean Borel, resp. nearly Borel subsets of $U$. Every
finely open set $V \subset U$ is Euclidean nearly Borel because its regularized $r(V)$ is
a Euclidean $F_\sigma$-set and $r(V) \setminus V$ is polar. It follows that every open subset $W$
of $U \times \overline{U}$, now with the fine topology on $U$ (and of course the natural topology
on $\overline{U}$), belongs to the $\sigma$-algebra $B^*(U) \times B(\overline{U})$ generated by all sets $A_1 \times A_2$
where $A_1 \in B^*(U)$ and where $A_2 \in B(\overline{U})$, that is, $A_2$ is a Borel subset of $\overline{U}$. In
view of Proposition 3.3 (iii) every set $\{(x, Y) \in U \times \overline{U} : K(x, Y) > \alpha\}$ ($\alpha \in \mathbb{R}$)
is such an open set $W$ and therefore belongs to $B^*(U) \times B(\overline{U})$. This means
that the Riesz-Martin kernel $K$ is measurable with respect to $B^*(U) \times B(\overline{U})$.

Definition 3.6. An invariant function $h \in \mathcal{S}(U)$ is termed minimal if it belongs to an extreme generator of the cone $\mathcal{S}(U)$.

Recall that $\text{Ext}(B)$ denotes the set of extreme points of $B$, and $\text{Ext}_i(B)$ the
subset of extreme invariant functions in $B$.

Proposition 3.7. For any point $Y \in \Delta(U)$, if the function $K(., Y)$ is an
extreme point of $B$, then $K(., Y)$ is a minimal invariant function.

Proof. By the Riesz decomposition of functions from $\mathcal{S}(U)$, $K(., Y)$ is either
an extreme fine potential on $U$ or else an extreme invariant function on $U$.
In the former case it follows by the integral representation of fine potentials
that $K(., Y)$ must be equal to $P_y$ for some $y \in U$, which contradicts $Y \in$
where \( \Delta(U) \). Thus \( K(.,Y) \) is indeed an extreme invariant function. The converse is obvious.

**Definition 3.8.** A point \( Y \in \Delta(U) \) is termed minimal if the function \( K(.,Y) \) belongs to an extreme generator of the cone \( S(U) \).

We denote by \( \Delta_1(U) \) the set of all minimal points of \( \Delta(U) \).

Contrary to the case where \( U \) is Euclidean open in \( \Omega \), \( \Delta(U) \) is in general not compact (in the natural topology), as shown by the following example.

**Example 3.9.** Let \( \omega \) be a Hölder domain in \( \mathbb{R}^n \) (\( n \geq 2 \)) such that \( \omega \) is irregular with a single irregular boundary point \( z \) (for example a Lebesgue spine), and take \( U = \omega \cup \{z\} \). According to [1] Theorems 1 and 3.1 the Euclidean boundary \( \partial \omega \) of \( \omega \) is contained in \( \Delta(\omega) \). It follows that \( z \) belongs to the Euclidean closure of \( \Delta(\omega) \setminus \{x\} \). But \( \Delta(U) = \Delta(\omega) \setminus \{x\} \), where \( x \) is identified with \( P_z \), and since \( \Delta(\omega) \) is compact we infer that \( \Delta(U) \) is noncompact.

However, we have the following

**Proposition 3.10.** \( \Delta(U) \) is a \( G_\delta \) of \( \overline{U} \).

**Proof.** Let \( (B_i) \) be a sequence of open balls in \( \mathbb{R}^n \) such that \( \Omega = \bigcup_i B_i \). For integers \( k > 0 \) and \( l \) put

\[ A_{kl} = \{y \in U : \Phi(G_U(.,y)) \geq 1/k \} \cap \overline{B_l}, \]

where \( \overline{B_l} \) denotes the Euclidean closure of \( B_l \). The sets \( A_{kk} \) cover \( U \) because \( G_U(.,y) > 0 \) and hence \( \Phi(G_U(.,y)) > 0 \). We first show that each \( A_{kl} \) is compact in \( \overline{U} \) with the natural topology. Let \( (y_j) \) be a sequence of points of \( A_{kl} \). After passing to a subsequence of \( (y_j) \) we may suppose that \( (y_j) \) converges Euclidean to a point \( y \in \overline{B_l} \), and that the sequences \( (G_U(.,y_j)) \) and \( (\widehat{R}_{G_G(\cdot,y_j)}) \) (restricted to \( U \)) converge in \( S(U) \cup \{+\infty\} \). It follows that (with \( G_G(\cdot,y_j) \) restricted to \( U \))

\[ \Phi(G_G(\cdot,y_j)) = \Phi(G_U(.,y)) + \Phi(\widehat{R}_{G_G(\cdot,y_j)}), \]

and hence by passing to the limit in \( S(U) \cup \{+\infty\} \) as \( j \to +\infty \)

\[ \Phi(G_G(\cdot,y)) = \lim_j \Phi(G_U(.,y_j)) + \lim_j \Phi(\widehat{R}_{G_G(\cdot,y_j)}). \]

On the other hand,

\[ \widehat{R}_{G_G(\cdot,y)} = \widehat{R}_{\lim \inf_j G_G(\cdot,y_j)} \leq \lim_j \widehat{R}_{G_G(\cdot,y_j)}. \]

The restriction of the function \( \widehat{R}_{G_G(\cdot,y)} \) to \( U \) being invariant according to Lemma 2.3 it follows by Lemma 2.1 that \( \widehat{R}_{G_G(\cdot,y)} \leq \lim \inf_j \widehat{R}_{G_G(\cdot,y_j)} \) (after restriction to \( U \) here, and often in the rest of the proof), and hence

\[ \Phi(\widehat{R}_{G_G(\cdot,y_j)}) \leq \Phi(\lim_j \inf \widehat{R}_{G_G(\cdot,y_j)}) = \lim_j \Phi(\widehat{R}_{G_G(\cdot,y_j)}). \]
We infer from (3.1) by the definition of $A_{kl}$ that
\[
\Phi(G_{U}(\cdot, y)) \geq \frac{1}{k} + \Phi(\hat{R}^{u}_{A_{kl}(\cdot, y)}),
\]
and consequently $y \in U$ and $\Phi(G_{U}(\cdot, y)) \geq 1/k$, whence $y \in A_{kl}$. Now put $s = \lim_{j} G_{U}(\cdot, y_{j})$. We have $s > 0$ because $y_{j} \in A_{kl} \subset U$ and hence
\[
\Phi(s) = \lim_{j} \Phi(G_{U}(\cdot, y_{j})) \geq 1/k.
\]
Hence
\[
\varphi(y_{j}) = \frac{G_{U}(\cdot, y_{j})}{\Phi(G_{U}(\cdot, y_{j}))} \rightarrow \frac{s}{\Phi(s)} \in S(U).
\]
This shows that $A_{kl}$ is compact in the natural topology on $\overline{U}$, and consequently $\Delta(U) = \overline{U} \setminus U = \bigcap_{k,l}(\overline{U} \setminus A_{kl})$ is indeed a $G_{\delta}$ in $\overline{U}$.

**Corollary 3.11.** $\Delta_{1}(U)$ is a $G_{\delta}$ in $\overline{U}$.

**Proof.** Because $\Delta_{1}(U) = \Delta(U) \cap \text{Ext}_{i}(B)$ it suffices according to Proposition 3.10 to use the well-known fact that $\text{Ext}(B)$ is a $G_{\delta}$ in $B$. □

The proof of Proposition 3.10 also establishes

**Corollary 3.12.** For any integers $k, l > 0$ the set $C_{kl} := \{P_{y} : y \in A_{kl}\}$ is compact in $B$, and the mapping $\varphi : y \mapsto P_{y}$ is a homeomorphism of $A_{kl}$ onto $C_{kl}$.

**Corollary 3.13.** Let $K \subset U$ be compact in the Euclidean topology. Then the set $C_{K} := \{P_{y} : y \in K\}$ is a (natural) $K_{\sigma}$ in $B$, and so is therefore $\text{Ext}_{p}(B) = \{P_{y} : y \in U\}$.

**Proof.** Since $C_{K} = \cup_{k} \varphi(K \cap A_{k})$ the assertion follows by the preceding corollary. □

**Proposition 3.14.** Let $u \in S(U)$ and let $\overline{A} \subset U$ be overline{A} in $U$, where $\overline{A}$ denotes the closure of $A$ in $\overline{U}$. The measure $\mu$ on $B$ carried by the extreme points of $B$ and representing $R^{A}_{u}$ is then carried by $\overline{A}$.

**Proof.** Let $p$ be a finite fine potential $> 0$ on $U$. For any pair $(k, l)$ and any integer $j > 0$ the function $R^{A}_{u \wedge jp}$ is a fine potential on $U$ and finely harmonic on $U \setminus \overline{A}$. The measure $\mu_{j}$ on $B$ carried by $\text{Ext}(B)$ and representing $R^{A}_{u \wedge jp}$ is carried by $\overline{A}$. The sequence of probability measures $\frac{1}{\Phi(R^{A}_{u \wedge jp})} \mu_{j}$ has a subsequence $(\mu_{jk})$ which converges to a probability measure $\mu$ on $B$ carried by $\overline{A}$. We thus have
\[
R^{A}_{u} = \lim_{k \rightarrow \infty} R^{A}_{u \wedge jkp} = \Phi(R^{A}_{u}) \lim_{k \rightarrow \infty} \int q \, d\mu_{jk}(q) = \Phi(R^{A}_{u}) \int q \, d\mu(q).
\]
The assertion now follows by the fact that $\overline{A} \subset U \subset \text{Ext}(B)$ and from the uniqueness of the integral representation in Choquet’s theorem. □
Corollary 3.15. Let \( u \in S(U) \) et \( A \subset \overline{A} \subset U \), where \( \overline{A} \) denotes the closure of \( A \) in \( U \). Then \( \widehat{R}_u^A \) is a fine potential on \( U \).

Theorem 3.16. Every extreme element of the base \( B \) of the cone \( S(U) \) belongs to \( U \). In particular, any extreme invariant function \( h \) in \( B \) has the form \( h = K(., Y) \) where \( Y \in \Delta_1(U) \).

Proof. Let \( p \) be an extreme element of \( B \). By Riesz decomposition, either \( p \) is the fine potential of a measure supported by a single point \( y \in U \), or else \( p \) is an invariant function on \( U \). In the former case we have \( p = P_y \) and hence \( p \in U \). In the latter case it follows by the proof of Proposition 3.10 that there exists an increasing sequence of compact subsets \( (K_j) \) of \( U \) (of the form \( A_{kl} \)) such that \( \cup_j K_j = U \) and that, for each \( j \), \( \widehat{R}_{K_j}^P \) is a fine potential on \( U \). For any \( j \) there exists a Radon measure \( \mu_j \) on \( B \) such that \( \widehat{R}_{K_j}^P p = \int q \, d\mu_j(q) \). The measure \( \mu_j \) is carried by \( U \). Because \( p \) is invariant it follows by Lemma 2.1 that the sequence \( \widehat{R}_{K_j}^P \) increases specifically to \( p \) as \( j \to \infty \), and the sequence \( (\mu_j) \) is therefore increasing. Consequently, \( \int d\mu_j = \Phi(\widehat{R}_{K_j}^P) \to \Phi(p) \) as \( j \to \infty \), and the sequence \( (\mu_j) \) converges vaguely to a measure \( \mu \) on \( U \). It follows that \( p = \int_U q \, d\mu(q) \), and since \( p \) is extreme we conclude that \( p \in U \). □

Corollary 3.17. \( \text{Ext}(B) = \Delta_1(U) \cup U \).

Proof. Every extreme element of \( B \) is either the fine potential of a measure supported by a single point \( y \in U \), hence of the form \( P_y \), or else a minimal invariant functions, hence of the form \( K(., Y) \) with \( Y \in \Delta_1(U) \), according to Proposition 3.7. This establishes the inclusion \( \text{Ext}(B) \subset \Delta_1(U) \cup U \). The opposite inclusion is evident. □

Theorem 3.18. For any invariant function \( h \) on \( U \) there exists a unique Radon measure \( \mu \) on \( U \) carried by \( \Delta_1(U) \) such that

\[
    h(x) = \int_{\Delta(U)} K(x, Y) \, d\mu(Y), \quad x \in U.
\]

Proof. The theorem follows immediately from Theorem 2.15 and Corollaries 3.12 and 3.17. □

Proposition 3.19. Let \( u \in S(U) \) and let \( V \) be a finely open Borel subset of \( U \). Let \( \mu \) be the measure on \( B \) carried by \( \text{Ext}(B) \) and representing \( u \). Then \( \mu(V) = 0 \) if and only if the restriction \( u|_V \) is invariant.

Proof. Write \( u = p + h \) with \( p \in \mathcal{P}(U) \) and \( h \) invariant on \( U \). Let \( \lambda \) and \( \nu \) be the measures on \( \text{Ext}(B) \) representing \( p \) and \( h \), respectively. Then \( \mu = \lambda + \nu \) with \( \nu(U) = 0 \) according to Corollary 3.17 or 2.16 and Theorem 3.18. Writing \( \Phi(G_U; \mu) = \alpha \) we have by [15, Lemma 2.6]

\[
    \alpha p = G_U \lambda = G_V \lambda|_V + \widehat{R}^{U/V}_{G_U \lambda} \preceq \alpha u \quad \text{on } V,
\]
and hence $G_V\lambda_V \leq \alpha u$ on $V$. If $u|_V$ is invariant the fine potential $G_V\lambda_V$ must therefore be 0, whence $\lambda(V) = 0$ and finally $\mu(V) = \lambda(V) + \nu(V) = 0$ because $\nu(V) \leq \mu(V) = 0$. Conversely, suppose that $\mu(V) = 0$ and hence $\lambda(V) = 0$. In the above display $\hat{R}_{G_V\lambda}$ is invariant on $V$ by Lemma 2.3 and so is therefore $p$. It follows that $u = p + h$ is invariant on $V$, $h$ being invariant on $U$ and hence on $V$ by Theorem 2.5 (a).

For any (positive) Borel measure $\mu$ on $U$ define a function $K \mu : U \mapsto [0, +\infty]$ by
\[ K \mu = \int K(.,Y) d\mu(Y), \quad x \in U. \]
This integral exists in view of Proposition 3.3 (i).

**Theorem 3.20.** 1. For any Borel measure $\mu$ on $U$, $K \mu$ is finely hyperharmonic on $U$, that is, $K \mu \in S(U) \cup \{+\infty\}$.

2. Every function $u \in S(U)$ has a unique integral representation $u = K \mu$ in terms of a Borel measure $\mu$ on $U \cup \Delta_1(U)$.

**Proof.** 1. The kernel $K$ is measurable with respect to the product $\sigma$-algebra $B^*(U) \times B(U)$ in view of the conclusion in Remark 3.5. It follows that the function $K \mu$ is nearly Euclidean Borel measurable on $U$. We begin by showing that $K \mu$ is nearly finely hyperharmonic, cf. [11, Definition 11.1]. Let $V \subset U$ be finely open with $\bar{V} \subset U$. For any $x \in V$ the swept measure $\epsilon_{x}^{\bar{V}}$ is carried by the fine boundary $\partial_f V \subset U$ and does not charge any polar set. Hence $\epsilon_{x}^{\bar{V}}$ may be regarded as a measure on the $\sigma$-algebra $B^*(U)$ of all Euclidean nearly Borel subsets of $U$, cf. again Remark 3.5. Altogether, Fubini’s theorem applies, and we obtain for any $x \in U$:
\[
\hat{R}_{K \mu}^{U \setminus V}(x) = \int_U K \mu(y) d\epsilon_{x}^{\bar{V}}(y) \\
= \int_U \left( \int_U K(y,Y) d\epsilon_{x}^{\bar{V}}(y) \right) d\mu(Y) \\
\leq \int_U K(x,Y) d\mu(Y) = K \mu(x),
\]
the inequality because $K(.,Y) \in S(U)$, cf. [11, Definition 8.1]. Thus $K \mu$ is nearly finely hyperharmonic on $U$. To show that $K \mu$ is actually finely hyperharmonic we shall prove that the finely l.s.c. envelope $\hat{K} \mu$ of $K \mu$ equals $K \mu$, cf. [11] Lemma 11.2 and Definition 8.4] according to which
\[
\hat{K} \mu(x) = \sup_{V \in \mathcal{V}} \int_U u d\epsilon_{x}^{\bar{V}},
\]
where $\mathcal{V}$ denotes the lower directed family of all finely open sets $V \subset U$ of Euclidean compact closure in $\Omega$ contained in $U$. For each $V \in \mathcal{V}$ and $x \in V$
we have, again by Fubini in view of Remark 3.5
\[
\int_{U} K_\mu(y) d\varepsilon^U_x(y) = \int_{U} \left( \int_{U} K(y, Y) d\varepsilon^U_x(y) \right) d\mu(Y) = \int_{U} \hat{R}^U_{K(.,Y)}(x) d\mu(Y).
\]
Taking supremum over all \( V \in \mathcal{V} \) leads to \( \hat{K}_\mu = K_\mu \) as desired. In fact, the increasing net of finely superharmonic functions \( (\hat{R}^U_{K(.,Y)})_{V \in \mathcal{V}} \) admits an increasing subsequence with the same pointwise supremum, by [11, Remark (p. 91)], and this supremum equals \( \hat{R}^U_{K(.,Y)} = \hat{R}^U_{K(.,Y)} = K(.,Y) \). It follows that
\[
\int_{U} K_\mu d\varepsilon^U_x = \int_{U} \hat{R}_{K(.,Y)}(x) d\mu(Y) \supseteq \int_{U} K(x, Y) d\mu(Y) = K_\mu(x)
\]
according to [11, Theorem 11.12]. We conclude that \( \hat{K}_\mu = K_\mu \), and so \( K_\mu \) is indeed finely hyperharmonic on \( U \).

2. As noted in [3, Theorem 4.1] this follows immediately from Choquet’s integral representation theorem applied to the cone \( \mathcal{S}(U) \) with the base \( B \).

**Lemma 3.21.** For any set \( A \subset U \) and any Radon measure \( \mu \) on \( \overline{U} \) we have \( \hat{R}^A_{K_\mu} = \int \hat{R}^A_{K(.,Y)} d\mu(Y) \).

**Proof.** As in the proof of Theorem 3.20 the kernel \( K \) on \( U \times \overline{U} \) is measurable with respect to \( \mathcal{B}^*(U) \times \mathcal{B}(\overline{U}) \). Furthermore, \( K_\mu \) is finely hyperharmonic on \( U \), and we have again by Lemma 2.2 and Fubini for any \( x \in U \)
\[
\hat{R}^A_{K_\mu}(x) = \int_{U} K_\mu d\varepsilon^A_{x} d\mu(Y) = \int_{U} K(.,Y) d\varepsilon^A_{x} d\mu(Y) = \int_{U} \hat{R}^A_{K(.,Y)}(x) d\mu(Y).
\]

We close this section with the following characterizations of the invariant functions and the fine potentials on \( U \). These characterizations are analogous (but only partly comparable) to [13, Theorems 4.4 and 4.5], respectively, where the present condition \( \overline{V} \subset U \) was replaced by the weaker condition \( \overline{V} \subset U \).

**Theorem 3.22.** Let \( u \in \mathcal{S}(U) \). Then \( u \) is invariant if and only \( \hat{R}^U_{u|V} = u \) for any regular finely open set \( V \subset U \) such that the closure \( V \) of \( V \) in the natural topology on \( B \) is contained in \( U \).
Proof. Suppose that $u$ is invariant. For any $V$ as stated we have $u \leq \widehat{R}_u^V + \widehat{R}_u^{U \setminus V}$. By the Riesz decomposition property \cite[p. 129]{11} there are functions $u_1, u_2 \in \mathcal{S}(U)$ such that $u = u_1 + u_2$ with $u_1 \leq \widehat{R}_u^V$ and $u_2 \leq \widehat{R}_u^{U \setminus V}$. But $\widehat{R}_u^V$ is a fine potential by Corollary 3.15 and so is therefore $u_1$. It follows that $u_1 = 0$ because $u_1 \leq u$ and $u$ is invariant. Consequently, $u \leq \widehat{R}_u^{U \setminus V}$, and so indeed $\widehat{R}_u^{U \setminus V} = u$. Conversely, suppose that $\widehat{R}_u^{U \setminus V} = u$ for any regular finely open set $V \subset U$ with $\overline{V} \subset U$. Let $\mu$ be the (finite) measure on $U$ which represents $u$. Then

$$u = \int_U P_y d\mu(y) + \int_{\Delta_1(U)} K(\cdot, Y) d\mu(Y).$$

For any regular finely open set $V \subset U$ with $\overline{V} \subset U$ we have by hypothesis

$$u = \widehat{R}_u^{U \setminus V} = \int_U \widehat{R}_u^{U \setminus V} d\mu(y) + \int_{\Delta_1(U)} \widehat{R}_u^{U \setminus V} d\mu(Y),$$

and hence

$$\int_U (P_y - \widehat{R}_u^{U \setminus V}) d\mu(y) = 0.$$ 

Since $V$ is regular and $P_y - \widehat{R}_u^{U \setminus V} = G_V(\cdot) / \Phi(G_V(\cdot, y)) > 0$ on $V$ it follows that $\mu(V) = 0$ and hence $\mu$ is carried by $\Delta_1(U)$. Consequently, $u$ is invariant according to Theorem 2.17.

**Corollary 3.23.** Let $u \in \mathcal{S}(U)$. Then $u$ is a fine potential if and only if $\inf_j \widehat{R}_u^{U \setminus V_j} = 0$ for any cover of $U$ by a sequence $(V_j)$ of regular finely open sets such that $\overline{V}_j \subset V_{j+1} \subset U$ for any $j$.

Proof. Suppose that $u$ is a fine potential, and put $v = \inf_j \widehat{R}_u^{U \setminus V_j}$. Because $V_j$ is regular the measure $\mu_j$ on $U$ representing $\widehat{R}_u^{U \setminus V_j}$ is proportional to the restriction of the swept measure $\mu^{V_j}$ to $U$ and hence carried by $U \setminus V_j$, cf. Lemma 2.2. Since the decreasing sequence $\widehat{R}_u^{U \setminus V_j}$ converges to $v$ in the topology on $\mathcal{S}(U)$, cf. Theorem 2.9 the sequence of measures $(\mu_j)$ converges to the measure $\mu$ representing $v$. For any $j$ and any $k > j$ we have $\mu_k(\overline{V_j}) = 0$ and hence $\mu(\overline{V_j}) = 0$. It follows that $\mu$ is carried by $\Delta_1(U)$ and hence is $0$ because $\mu$ is a fine potential and so $v = 0$. Conversely, suppose that $\inf_j \widehat{R}_u^{U \setminus V_j} = 0$. We have $u = p + h$, where $p$ is a fine potential and $h$ is an invariant function. By the preceding theorem, $h = \widehat{R}_h^{U \setminus V_j} \leq \widehat{R}_u^{U \setminus V_j}$ for every $j$, and hence $h = 0$, showing that $u$ indeed is a fine potential.

**Corollary 3.24.** Let $\mu$ be a Borel measure on $\Delta_1(U)$ and let $h = K\mu$. If $h$ is finite q.e. then $h \in \mathcal{S}(U)$ and $h$ is invariant.
Proof. Let $V$ be open in $U$ with $\overline{V} \subset U$. In particular, $V$ is finely open in $U$ in view of Lemma 3.2. For any $x \in r(V)$ the measure $\varepsilon_x^{\mathcal{B}_V} = \varepsilon_x^{\mathcal{K}_V} = \varepsilon_x^{\mathcal{B}(V)}$ (bases and complements relative to $\Omega$) is carried by $x$ in view of Lemma 3.2. For any theorem and Lemma 2.2 to obtain $K\nu$ it is also shown that $\sigma$-charge any polar set. This measure may therefore be regarded as a measure on the $\sigma$-algebra $\mathcal{B}^*(U)$ of all nearly Borel subsets of $U$, cf. Remark 3.5, where it is also shown that $K$ is measurable with respect to the product $\sigma$-algebra $\mathcal{B}^*(U) \times \mathcal{B}(U)$. Supposing that $h < +\infty$ q.e. on $U$ we may hence apply Fubini’s theorem and Lemma 2.2 to obtain

$$\hat{R}_h^{U \setminus V}(x) = \int_{U} h(y)d\varepsilon_x^{\mathcal{B}_V}(y)$$

$$= \int_{\Delta_1(U)} \left( \int_{U} K(y,Y)d\varepsilon_x^{\mathcal{B}_V}(y) \right) d\mu(Y)$$

$$= \int_{\Delta_1(U)} \hat{R}_h^{U \setminus V}(x)d\mu(Y)$$

$$= \int_{\Delta_1(U)} K(x,Y)d\mu(Y) = h(x).$$

In the remaining case $x \in U \setminus r(V)$ the resulting equation $\hat{R}_h^{U \setminus V}(x) = h(x)$ holds because $x \in U \cap b(\mathcal{C}_V) \subset U \cap b(U \setminus V)$. We conclude that $h$ is invariant according to Theorem 3.22 or [15, Theorem 4.4]. \hfill \Box

Remark 3.25. The finiteness condition on $h$ in Corollary 3.24 is equivalent with $h \neq +\infty$, that is $h \in \mathcal{S}(U)$.

Corollary 3.26. For any finite measure $\mu$ on $\Delta_1(U)$ the function $h = \int_{\Delta_1(U)} K(.,Y)d\mu(Y)$ is an invariant function.

Proof. Let $\nu$ be the measure on $B$ defined by $\nu(A) = \mu(\Delta_1(U) \cap A)$ for any Borel set $A \subset B$. Then $\nu$ is a finite measure on $B$, and we may suppose that $|\nu| = 1$. Let $h \in B$ be the barycenter of $\nu$. Then $k = \int_{\Delta_1(U)} K(.,Y)d\mu(Y) = h$, and hence $h$ is invariant according to Theorem 3.22 and Corollary 3.17. \hfill \Box

Corollary 3.27. Let $\mu$ be a Borel measure on $U \cup \Delta_1(U)$. A function $u = K\mu \neq +\infty$ is a fine potential, resp. an invariant function, if and only $\mu$ is carried by $U$, resp. by $\Delta_1(U)$.

Proof. For $y \in U$ write $\alpha(y) := \Phi(G_U(.,y))$. If $\mu$ is carried by $U$ then $K\mu(y) = G_U(\alpha(y)^{-1}\mu)$ is a fine potential on $U$. Conversely, if $u$ is a fine potential on $U$ then there is a measure $\nu$ on $U$ such that $u(y) = G_U \nu(y) = K(\alpha(y)\nu)$. On the other hand, if $\mu$ is carried by $\Delta_1(U)$ then $u = p + h$ with $p = K\mu$ for some $\mu$ on $U$ and $h = K\lambda$ for some $\lambda$ carried by $\Delta_1(U)$. By uniqueness in Theorem 3.20 $\mu = \nu + \lambda$, where $\nu$ is carried by $U$ and hence $\nu = 0$, so that $K\mu = K\lambda$ is invariant according to Corollary 3.24. Conversely, if $K\mu$ is invariant then
\[ u = p + h = K\nu + K\lambda \text{ as above, and here } h \text{ is invariant according to Corollary 3.26.} \]
It follows that \( p = 0 \), that is \( \nu = 0 \), and so \( \mu = \lambda \) is carried by \( \Delta_1(U) \).  

\section{The Fatou-Naïm-Doob theorem for finely superharmonic functions}

As mentioned in the Introduction, this section is inspired by the axiomatic approach to the Fatou-Na"	ext{im}-Doob theorem given in [21]. These axioms are, however, only partially fulfilled in our setting. In particular, our invariant functions, which play the role of positive harmonic functions, may take the value \(+\infty\). We therefore choose to give the proof of the Fatou-Na"	ext{im}-Doob theorem without reference to the proofs in [21].

We continue considering a regular fine domain \( U \) in a Green domain \( \Omega \subset \mathbb{R}^n \), \( n \geq 2 \). Recall that \( P(U) \) denotes the band in \( S(U) \) consisting of all fine potentials on \( U \), and that the orthogonal band \( P(U)^\perp = H_i(U) \) relative to \( S(U) \) consists of all invariant functions \( h \) on \( U \); these are characterized within \( S(U) \) by their integral representation
\[ h(x) = K\mu(x) = \int K(x,Y)d\mu(Y) \]
in terms of a unique measure \( \mu \) on \( \overline{U} \) carried by the minimal Martin boundary \( \Delta_1(U) \) (briefly: a measure on \( \Delta_1(U) \)), see Theorem 3.20 and Corollary 3.27. In the present section we shall not consider the whole Riesz-Martin space \( U \) and the full Riesz-Martin kernel \( K : U \times \overline{U} \to [0, +\infty] \) (Definition 3.1), but only the Martin boundary \( \Delta(U) \) and the Martin kernel, the restriction of the Riesz-Martin kernel to \( U \times \Delta(U) \), and \( K \) will henceforth stand for this restriction. It is understood that \( U \) and \( \Delta(U) \) are given the natural topology (induced by the natural topology on the Riesz-Martin space \( U \)) and that \( \Delta(U) \) is given the natural topology (induced by the natural topology on the Riesz-Martin space \( U \)). In particular, the proof of Proposition 3.10 shows that \( U \) is a \( K_\sigma \) subset of \( U \), and we know that \( \Delta(U) \) and the minimal Martin boundary \( \Delta_1(U) \) are \( G_\delta \) subsets of \( U \) (Proposition 3.10 and Corollary 3.11). We shall need the following three preparatory lemmas.

\begin{lemma}
\begin{enumerate}[(a)]
\item If \( h_1, h_2 \in \mathcal{H}_i(U) \), \( p \in P(U) \), and if \( h_1 \leq h_2 + p \), then \( h_1 \preceq h_2 \) and \( h_2 - h_1 \in \mathcal{H}_i(U) \).
\item If \( (h_j) \) is an increasing sequence of functions \( h_j \in \mathcal{H}_i(U) \) majorized by some \( u \in S(U) \) then \( \sup_j h_j \in \mathcal{H}_i(U) \).
\end{enumerate}
\end{lemma}

It is understood in (a) (and similarly elsewhere) that \( h_2 - h_1 \) is defined to be the extension by fine continuity from \( \{x \in U : h_1(x) < +\infty\} \) to \( U \), cf. [11, Theorem 9.14]. (Equivalently, \( h_2 - h_1 \) is well defined because \( S(U) \) is an \( H \)-cone.) For any set \( A \subset U \), \( R_u^A \) and \( \hat{R}_u^A \) are understood as reduction and
sweeping of a function $u$ on $U$ relative to $U$, whereas $\varepsilon_x^A$ stands for sweeping of $\varepsilon_x$ on $A$ relative to all of $\Omega$.

**Proof.** (a) There exists $s \in \mathcal{S}(U)$ such that $h_1 + s = h_2 + p$. We have $s = h + q$ with $h \in \mathcal{H}_i(U)$ and $q \in \mathcal{P}(U)$. It follows that $(h_1 + q) + q = h_2 + p$ with $h_1 + h \in \mathcal{H}_i(U)$, and hence $h_1 + h = h_2$ (and $q = p$).

(b) According to (a) the sequence $(h_j)$ is even specifically increasing. Because $\sup_j u_j \in \mathcal{S}(U)$ along with $u$ we have $\sup_j u_j = \gamma_j$ [11, (b), p. 132], which belongs to the band $\mathcal{H}_i(U)$ along with each $h_j$.

**Lemma 4.2.** For any set $E \subset U$ and any $Y \in \Delta_1(U)$ we have $\widehat{R}^E_{K(\cdot,Y)} \neq K(\cdot,Y)$ if and only if $\widehat{R}^E_{K(\cdot,Y)} \in \mathcal{P}(U)$.

**Proof.** Note that, for any $u \in \mathcal{S}(U)$ and $E \subset U$, we have $R^E_u \neq u \iff \widehat{R}^E_u \neq u$. Proceeding much as in [21, proof of G]), see also [18], we first suppose that $\widehat{R}^E_{K(\cdot,Y)} \neq K(\cdot,Y)$. Since $\widehat{R}^E_{K(\cdot,Y)} \leq K(\cdot,Y)$ there exists $x_0 \in U$ with $\widehat{R}^E_{K(\cdot,Y)}(x_0) < K(x_0, Y)$. Thus there exists $u \in \mathcal{S}(U)$ such that $u \geq K(\cdot,Y)$ on $E$ and $u(x_0) < K(x_0, Y)$. Replacing $u$ by $u \wedge K(\cdot,Y) \in \mathcal{S}(U)$ we arrange that $u \leq K(\cdot,Y)$ on all of $U$. Writing $u = q + h$ with $q \in \mathcal{P}(U)$ and $h \in \mathcal{U}(U)$ we have $h \leq u \leq K(\cdot,Y)$ on $U$. Because $K(\cdot,Y) \in \mathcal{H}_i$ is minimal invariant and $h \not\preceq K(\cdot,Y)$ by Lemma 4.1 (with $p = 0$), there exists $\alpha \in [0, 1]$ such that $h = \alpha K(\cdot,Y)$, and hence $\preceq K(\cdot,Y)$. On $E$ we have $q = u - h = K(\cdot,Y) - h = (1 - \alpha)K(\cdot,Y)$, and hence

$$K(\cdot,Y) = p := \frac{1}{1 - \alpha} q$$

on $E$.

Thus $p \in \mathcal{P}(U)$ and $\widehat{R}^E_{K(\cdot,Y)} = \widehat{R}_p \leq p$ on $U$, so indeed $\widehat{R}^E_{K(\cdot,Y)} \in \mathcal{P}(U)$. Conversely, suppose that $\widehat{R}^E_{K(\cdot,Y)} = K(\cdot,Y)$ and (by contradiction) that $\widehat{R}^E_{K(\cdot,Y)} \in \mathcal{P}(U)$. Being thus both a fine potential and invariant, $K(\cdot,Y)$ must equal 0, which is false.

**Definition 4.3.** A set $E \subset U$ is said to be minimal-thin at a point $Y \in \Delta_1(U)$ if $\widehat{R}^E_{K(\cdot,Y)} \neq K(\cdot,Y)$, or equivalently (by the preceding lemma) if $\widehat{R}^E_{K(\cdot,Y)}$ is a fine potential on $U$.

**Corollary 4.4.** For any $Y \in \Delta_1(U)$ the sets $W \subset U$ for which $U \setminus W$ is minimal-thin at $Y$ form a filter on $U$.

This follows from Lemma 4.2 which easily implies that for any $W_1, W_2 \subset U$ such that $\widehat{R}^E_{K(\cdot,Y)} \neq K(\cdot,Y)$ for $i = 1, 2$, we have $\widehat{R}^E_{K(\cdot,Y)} \neq K(\cdot,Y)$.

The filter from Corollary 4.4 is called the minimal-fine filter at $Y$ and will be denoted by $\mathcal{F}(Y)$. A limit along that filter will be called a minimal-fine limit and will be denoted by $\lim_{\mathcal{F}(Y)}$. 
For any two functions \( u, v \in S(U) \) with \( v \neq 0 \) the quotient \( u/v \) is assigned some arbitrary value, say 0, on the polar set of points at which both functions take the value \(+\infty\). The choice of such a value does not affect a possible minimal-fine limit of \( u/v \) at a point \( Y \in \Delta_1(U) \) because every polar set \( E \) clearly is minimal-thin at any point of \( \Delta_1(U) \).

For any two measures \( \mu, \nu \) on a measurable space we denote by \( d\nu/d\mu \) the Radon-Nikodym derivative of the absolutely continuous component of \( \nu \) relative to that of \( \mu \), cf. e.g. [6, p. 773], [3, p. 305f].

For any function \( u \in S(U) \) with Riesz decomposition \( u = p + h \), where \( p \in P(U) \) and \( h \in H_1(U) \), we denote by \( \mu_u \) the unique measure on \( \Delta_1(U) \) which represents the invariant part \( h \) of \( u \), that is, \( h = \int K(., Y)d\mu_u(Y) \).

We may now formulate the Fatou-Naïm-Doob theorem in the present setting of finely superharmonic functions. It clearly contains the classical Fatou-Naïm-Doob theorem for which we refer to [6, 1.XII.19][3, 9.4].

**Theorem 4.5.** Let \( u, v \in S(U) \), where \( v \neq 0 \). Then \( u/v \) has minimal-fine limit \( d\mu_u/d\mu_v \) at \( \mu_v \)-a.e. point \( Y \) of \( \Delta_1(U) \).

For the proof of Theorem 4.5 we begin by establishing the following important particular case, cf. [21, Theorem 1.2].

**Proposition 4.6.** (21.) Let \( u \in S(U) \) and \( h \in H_1(U) \setminus \{0\} \), and suppose that \( u \wedge h \in P(U) \). Then \( u/h \) has minimal-fine limit \( d\mu_u/d\mu_h = 0 \) at \( \mu_h \)-a.e. point \( Y \) of \( \Delta_1(U) \).

**Proof.** Write \( u \wedge h = p \). Given \( \alpha \in ]0,1[ \), consider any point \( Y \in \Delta_1(U) \) such that \( \lim_{F(Y)} \frac{u}{h} > \alpha \). Then \( \{u \leq \alpha h\} \notin F(Y) \), that is, \( \{u > \alpha h\} \) is not minimal-thin at \( Y \):

(4.1) \[ \widehat{R}_{K(.,Y)}^{\{u > \alpha h\}} = K(., Y). \]

It follows that (always with \( Y \) ranging over \( \Delta_1(U) \))

(4.2) \[ \{Y : \limsup_{F(Y)} \frac{u}{h} > \alpha\} \subset A_\alpha := \{Y : \widehat{R}_{K(.,Y)}^{\{u > \alpha h\}} = K(., Y)\}. \]

We show that \( \mu_h(A_\alpha) = 0 \). Consider the measure \( \nu = 1_{A_\alpha} \mu_h \) on \( \Delta_1(U) \) and the corresponding function

\[ v = \int K(., Y)d\nu(Y) = \int \widehat{R}_{K(.,Y)}^{\{u > \alpha h\}} d\nu(Y) = \widehat{R}_{K\nu}^{\{u > \alpha h\}} = \widehat{R}_v^{\{u > \alpha h\}}, \]

the second equality by (7.1) and the third equality by Lemma 3.21. Since \( v \leq \int K(., Y)d\mu_h = h \) and \( 0 < \alpha < 1 \) it follows that

\[ v = \widehat{R}_v^{\{u > \alpha h\}} \leq \widehat{R}_h^{\{u > \alpha h\}} \leq \frac{u}{\alpha} \wedge h \leq \frac{u}{\alpha} \wedge \frac{h}{\alpha} = \frac{p}{\alpha}. \]
Because \( v = \int K(., Y)dv(Y) \in \mathcal{H}_i(U) \) and \( p/\alpha \in \mathcal{P}(U) \) we find by Lemma 4.1 (applied to \( h_1 = v, h_2 = 0 \)) that \( v = 0 \), that is,

\[
v = \int K(., Y)dv(Y) = \int_{A_\alpha} K(., Y)d\mu_h(Y) = 0,
\]

and since \( K(., Y) > 0 \) it follows that \( \mu_h(A_\alpha) = 0 \). By varying \( \alpha \) through a decreasing sequence tending to 0 we conclude from (7.2) that indeed \( \mu_h(\{Y : \limsup_{\mathcal{F}(Y)} u/h > 0\}) = 0 \).

The rest of the proof of Theorem 4.5 proceeds much as in the classical case. For the convenience of the reader we bring most of the details, following in part [3, Section 9.4].

**Corollary 4.7.** Let \( u, h \in \mathcal{H}_i(U) \), where \( h \neq 0 \) and \( \mu_u, \mu_h \) are mutually singular. Then \( u \wedge h \in \mathcal{P}(U) \), and \( u/h \) has minimal-fine limit \( 0 \) \( \mu_h \)-a.e. on \( \Delta_1(U) \).

**Proof.** There are Borel subsets \( A_1, A_2 \) of \( U \) such that \( A_1 \cup A_2 = \Delta_1(U) \) and \( \mu_u(A_1) = \mu_h(A_2) = 0 \). Write \( u \wedge h = p + k \) with \( p \in \mathcal{P}(U), k \in \mathcal{H}_i(U) \). Then \( k \leq u \), hence \( k \ll u \), and so \( \mu_k \leq \mu_u \). Similarly, \( \mu_k \leq \mu_h \). It follows that \( \mu_k(\Delta_1(U)) \leq \mu_u(A_1) + \mu_h(A_2) = 0 \) and hence \( k = 0 \), and so \( u \wedge h = p \in \mathcal{P}(U) \).

The remaining assertion now follows from Proposition 4.6.

**Corollary 4.8.** Let \( h \in \mathcal{H}_i(U) \setminus \{0\} \), let \( A \) be a Borel subset of \( \Delta(U) \), and let \( h_A = K(1_A \mu_h) = \int_A K(., Y)d\mu_h(Y) \). Then \( h_A/h \) has minimal-fine limit \( 1_A(Y) \) at \( Y \) \( \mu_h \)-a.e. for \( Y \in \Delta_1(U) \).

**Proof.** Write \( u = h - h_A \in \mathcal{H}_i(U) \), which is invariant because \( h_A \leq h \). Since \( h_A = K(1_A \mu_h) \) and \( 1_A \mu_h \) is carried by \( \Delta_1(U) \) along with \( \mu_h \) we have \( \mu_{h_A} = 1_A \mu_h \), which is carried by \( A \cap \Delta_1(U) \). Similarly, \( \mu_u = \mu_h - \mu_{h_A} = 1_{U \setminus A} \mu_h \) is carried by \( \Delta_1(U) \setminus A \). In particular, \( \mu \) and \( \mu_h \) are mutually singular. It follows by Corollary 4.7 that \( h_A/h \) and \( u/h = 1 - h_A/h \) have minimal fine limit \( 0 \) \( \mu_h \)-a.e. on \( A \) and on \( \Delta_1(U) \setminus A \), respectively, whence the assertion.

**Definition 4.9.** For any function \( h \in \mathcal{H}_i(U) \setminus \{0\} \) and any \( \mu_h \)-integrable function \( f \) on \( \Delta_1(U) \) we define

\[
u_{f,h}(x) = \int K(x, Y)f(Y)d\mu_h(Y) \quad \text{for} \quad x \in U.
\]

**Proposition 4.10.** Let \( h \in \mathcal{H}_i(U) \setminus \{0\} \) and let \( f \) be a \( \mu_h \)-integrable function on \( \Delta(U) \). Then \( u_{f,h}/h \) has minimal-fine limit \( f(Y) \) at \( \mu_h \)-a.e. point \( Y \) of \( \Delta_1(U) \).

**Proof.** We may assume that \( f \geq 0 \). The case where \( f \) is a (Borel) step function follows easily from Corollary 4.8. For the general case we refer to the proof of [3, Theorem 9.4.5], which carries over entirely.
We are now prepared to prove Theorem 4.5, the Fatou-Naïm-Doob theorem in our setting.

Proof of Theorem 4.5: Write \( v = p + h \) with \( p \in \mathcal{P}(U) \) and \( h \in \mathcal{H}_i(U) \). By our definition of \( \mu_v \) we then have \( \mu_v = \mu_h \). We may assume that \( h \neq 0 \), for otherwise \( \mu_v = 0 \) and the assertion becomes trivial. Let \( \nu \) be the singular component of \( \mu_u \) with respect to \( \mu_v = \mu_h \). Write \( u = q + k \) with \( q \in \mathcal{P}(U) \) and \( k \in \mathcal{H}_i(U) \). Then \( \mu_u = \mu_k = f \mu_h + \nu \), and hence in view of Definition 4.9

\[ u = q + u_{f,h} + \int K(\cdot, Y) d\nu(Y). \]

By applying Proposition 4.6 with \( u \) replaced by \( q \) (hence \( \mu_u \) by \( \mu_q = 0 \)), next Corollary 4.7 with \( u \) replaced by \( K\nu \) (hence \( \mu_h \) replaced by \( \nu \) and \( d\mu_u/d\mu_h \) by \( d\nu/d\mu_h = 0 \)), and finally by applying Proposition 4.10 to the present \( u_{f,h} \), we see that \( u/h \) has minimal-fine limit \( f(Y) \) at \( \mu_v \)-a.e. point \( Y \) of \( \Delta_1(U) \). Since \( u/v \) is defined quasi-everywhere in \( U \) and

\[ \frac{u}{v} = \frac{u/h}{1 + p/h}, \]

the theorem now follows by applying Proposition 4.6 with \( u \) there replaced by \( p \) (and hence \( \mu_u \) by the present \( \mu_p = 0 \)). \( \square \)

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