The twisted conjugacy problem for pairs of endomorphisms in nilpotent groups

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Abstract

An algorithm is constructed that, when given an explicit presentation of a finitely generated nilpotent group \( G \), decides for any pair of endomorphisms \( \phi, \psi : G \to G \) and any pair of elements \( u, v \in G \), whether or not the equation \( (x\phi)u = v(x\psi) \) has a solution \( x \in G \). Thus it is shown that the problem of the title is decidable. Also we present an algorithm that produces a finite set of generators of the subgroup (equalizer) \( Eq_{\phi, \psi}(G) \leq G \) of all elements \( u \in G \) such that \( u\phi = u\psi \).

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1 Introduction

Let \( G \) be a group, and \( u, v \in G \). Given an endomorphism \( \phi \in \text{End}(G) \), one says that \( u \) and \( v \) are \( \phi \)-twisted conjugate, and one writes \( u \sim_{\phi} v \), if and only if there exists \( x \in G \) such that \( u = (x\phi)^{-1}vx \), or equivalently \( (x\phi)u = vx \). More generally, given a pair of endomorphisms \( \phi, \psi \in \text{End}(G) \), one says that

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the elements $u, v$ are $\varphi, \psi$--twisted conjugate, and one writes $u \sim_{\varphi, \psi} v$, if and only if there exists an element $x \in G$ such that

$$(x \varphi)u = v(x \psi).$$  \hfill (1)

The recognition of twisted conjugacy classes with respect to a given pair of endomorphisms $\varphi, \psi \in \text{End}(G)$ in the case of any finitely generated nilpotent group $G$ is the main concern of this paper.

### 2 Preliminary results

It is well known (see, for example, [2], [4]) that most of algorithmic problems for finitely generated nilpotent groups are decidable. In particular, the standard conjugacy problem is decidable in any finitely generated nilpotent group [1]. Some undecidable problems exist too. The endomorphism (see [5]) and the epimorphism (see [3]) problems are among them, as well as impossibility to decide if a given equation has a solution in a given finitely generated nilpotent group (see [5]).

For the purpose of this paper we need in the following result of special interest.

**Proposition 2.1.** Let $G$ be a finitely generated nilpotent group. Let $\varphi, \psi \in \text{End}(G)$ be a pair of endomorphisms of $G$. Then there is an algorithm which finds a finite set of generators of the subgroup (equalizer)

$$\text{Eq}_{\varphi, \psi}(G) = \{ x \in G | x\varphi = x\psi \}. \hfill (2)$$

**Proof.** Let $G$ be abelian. Then $\text{Eq}_{\varphi, \psi}(G) = \ker(\varphi - \psi)$, thus there is a standard procedure to find a generating set for it.

Let $G$ be a finitely generated nilpotent group of class $c + 1 \geq 2$. Suppose by induction that there is an algorithm which finds for any finitely generated nilpotent group $H$ of class $\leq c$ and any pair of endomorphisms $\alpha, \beta \in \text{End}(H)$ a finite set of generators of the equalizer $\text{Eq}_{\alpha, \beta}(H)$.

Let $C = \gamma_c(G)$ be the last non trivial member of the lower central series of $G$. Then the quotient $H = G/C$ has class $c$. Since $C$ is invariant for every endomorphism of $G$ we can consider the induced by $\varphi, \psi \in \text{End}(G)$ endomorphisms $\bar{\varphi}, \bar{\psi} \in \text{End}(H)$. By the assumption we can construct a finite set of generators of the equalizer $\text{Eq}_{\bar{\varphi}, \bar{\psi}}(H) \leq H$. Let $G_1$ be the full preimage of $\text{Eq}_{\bar{\varphi}, \bar{\psi}}(H)$ in $G$. We call $G_1$ a $C$–equalizer of $\varphi, \psi$, and we write $G_1 = \text{Eq}_{C, \varphi, \psi}(G)$. By definition

$$\text{Eq}_{C, \varphi, \psi}(G) = \{ g \in G | g\varphi = c_g(g\psi), \text{ where } c_g \in C \}. \hfill (3)$$

Obviously, $\text{Eq}_{C, \varphi, \psi}(G) \leq \text{Eq}_{C, \bar{\varphi}, \bar{\psi}}(G)$, and $C \leq \text{Eq}_{C, \varphi, \psi}(G)$.

Now we define a map

$$\mu : \text{Eq}_{C, \varphi, \psi}(G) \rightarrow C \text{ by } \mu(g) = c_g. \hfill (4)$$
Easily to see that this map $\mu$ is a homomorphism, and that the derived subgroup $(Eq_{\phi,\psi}(G))'$ lies in $\ker(\mu)$.

We conclude that $Eq_{\phi,\psi}(G) = \ker(\mu)$. So, a generating set for $Eq_{\phi,\psi}(G)$ can be derived by the standard procedure.

\[ \square \]

3 The twisted conjugacy problem for pairs of endomorphisms

Let $G$ be a finitely generated group, and $\phi, \psi \in \text{End}(G)$ be any pair of endomorphisms. Let $u \sim_{\phi,\psi} v$ be a pair of $\phi, \psi$−twisted conjugate elements of $G$. We write $\{u\}_{\phi,\psi}$ for the $\phi, \psi$−twisted conjugacy class of element $u \in G$.

The question about $\phi, \psi$−twisted conjugacy of given elements $u, v \in G$ can be reduced to the case where one of the elements is trivial. To do this we change $\phi$ to $\phi' = \phi \circ \sigma_u$, where $\sigma_u \in \text{Aut}(G)$ is the inner automorphism $h \mapsto u^{-1}hu$. Hence $(x\phi)u = v(x\psi)$ if and only if

\[ x\phi' = u(x\psi), \tag{5} \]

where $w = u^{-1}v$.

Now we are ready to prove our main result about twisted conjugacy in finitely generated nilpotent groups.

**Theorem 3.1.** Let $G$ be a finitely generated nilpotent group of class $c \geq 1$. Then there exists an algorithm which decides the twisted conjugacy problem for any pair of endomorphisms $\phi, \psi \in \text{End}(G)$.

**Proof.** Induction by $c$. For $c = 1$ (abelian case) the statement is obviously true.

Suppose that the statement is true in the case of any finitely generated nilpotent group $N$ of class $\leq c - 1$.

Let $u, v \in G$ be any pair of elements, and $\phi, \psi \in \text{End}(G)$ be any pair of endomorphisms. We change $\phi$ to $\phi' = \phi \circ \sigma_u$, and write the equation (5) with $w = u^{-1}v$, as were explained above. Since the last non trivial member $C$ of the lower central series of $G$ is invariant for every endomorphism, we decide the twisted conjugacy problem in $G/C$ with respect to the induced by $\phi', \psi$ endomorphisms $\bar{\phi}, \bar{\psi} \in \text{End}(G/C)$ and the induced by $w$ element $\bar{w} \in G/C$.

More exactly, we decide if there exists an element $\bar{x} \in G/C$ for which

\[ \bar{x}\bar{x'} = \bar{w}(\bar{x}\bar{\psi}). \tag{6} \]

By our assumption we can decide this problem effectively. If such element $\bar{x}$ does not exist the element $x$ does not exist too.

Suppose that $\bar{x}$ exists. Then there is an element $x_1 \in G$ for which

\[ x_1\phi' = cg(x_1\psi), \tag{7} \]
Twisted conjugacy

where $c \in C$. If $x_2 \varphi' = c' g(x_2 \psi)$ for some element $x_2 \in G$ and $c' \in C$, we derive that

$$(x_2^{-1} x_1) \varphi' = c''((x_2^{-1} x_1) \psi),$$

where $c'' = (c')^{-1} c \in C$. Thus $x_2^{-1} x_1 \in E_{q_{C}, \varphi', \psi}(G)$. In the case when $x_2 = x$ is a solution of [4] we have $c'' = c$. Conversely, the equality $c'' = c$ means that there is a solution $x$ of [4].

By Proposition [2.1] we construct a finite generating set of $E_{q_{C}, \varphi', \psi}(G/C)$, and so we can construct a finite generating set of its full preimage:

$$E_{q_{C}, \varphi', \psi}(G) = gp(g_1, ..., g_l).$$

Thus we have

$$g_i \varphi = c_i(g_i \psi),$$

where $c_i \in C$ for $i = 1, ..., l$.

Then we apply a the homomorphism $\mu$ defined by the map $g_i \mapsto c_i$ for $i = 1, ..., l$. As we noted above, $1 \sim_{C, \psi} w$ if and only if the element $c$ belongs to the image $(E_{q_{C}, \varphi', \psi}(G))\mu = gp(c_1, ..., c_l)$. So, our problem is reduced to the membership problem in a finitely generated abelian group $C$. It is well known that this problem is decidable.

References

[1] N. Blackburn, *Conjugacy in nilpotent groups*, Proc. Amer. Math. Soc., 16 (1965), 143-148.

[2] M.I. Kargapolov, V.N. Remeslenikov, N.S. Romanovskii, V.A. Roman’kov, V.A. Churkin, *Algorithmic questions for $\alpha$-powered groups*, Algebra and Logic, 8 (1969), 364-373.

[3] V.N. Remeslenikov, *An algorithmic problem for nilpotent groups and rings*, Siberian Math. J, 20 (1979), 761-764.

[4] V.N. Remeslenikov, V.A. Roman’kov, *Algorithmic and model theoretic problems in groups*, J. Soviet Math. 21 (1983), 2887-2939.

[5] V.A. Roman’kov, *About unsolvability of problem of endomorphic reducibility in free nilpotent groups and free rings*, Algebra and Logic, 16 (1977), 310-320.