We investigate domain wall junctions in a generalized Wess-Zumino model with a $Z_N$ symmetry. We present a method to identify the junctions that are potentially BPS saturated. We then use a numerical simulation to show that those junctions indeed saturate the BPS bound for $N = 4$. In addition, we study the decay of unstable non-BPS junctions.

I. INTRODUCTION.

In field theories with discrete, degenerate vacua, domain walls – field configurations that interpolate between different vacua – may occur. Supersymmetric theories, for which the vacuum energy is positive semi-definite, frequently have degenerate vacua. In many instances, multiple vacua exist in which the vacuum energy vanishes and supersymmetry remains unbroken. The degenerate vacua may or may not be related by a spontaneously broken discrete symmetry.

General considerations concerning domain walls in supersymmetric theories are presented in [1–3]. A lower bound on the tension, the BPS bound, can be calculated without explicitly solving for the profile of the wall. Domain walls break the translation symmetry in one direction, and supersymmetry is either completely broken, or $1/2$ of it is preserved. The degenerate vacua may or may not be related by a spontaneously broken discrete symmetry.

In [4,5] it was noted that more complicated field configurations with axial geometry, domain wall junctions, may also occur. These configurations, of the “hub and spoke” type, are a natural generalization of domain walls, and interpolate between more than two degenerate vacua. Far away from the center of the junction, the fields are approximately constant – at the vacuum expectation values of the various vacua – in sectors. These sectors are separated by “spokes”, where the field interpolates between one vacuum to the next following approximate domain wall profiles. In the center of the junction, “the hub”, the domain walls meet and the field configuration resembles a string. If these junctions are BPS saturated, $1/4$ of the original supersymmetry is preserved and both the $(1,0)$ and $(1/2,1/2)$ central charges appear in the supersymmetry algebra. The junction tension is determined by a combination of the $(1/2,1/2)$ and $(1,0)$ central charges $\mathfrak{C}$. Although the two central charges contain (related) ambiguities, it is shown in [4,7] that the ambiguities cancel in physical quantities like the tension.

It is worthwhile to emphasize that domain walls in supersymmetric models are not necessarily BPS saturated. For example, in the model of [8] the BPS saturation of a domain wall depends on the values of the model parameters. In many models it is not possible to obtain explicit analytical solutions to the first order BPS equations. In those models the question whether solutions exist at all can still be answered. In [8] a necessary and sufficient condition which indicates whether two vacua are connected by a PBS saturated domain wall was developed.

The issue of BPS saturation in junctions is more complicated, because a similar condition does not exist. At this time, there is only one known model [10] in which a domain wall junction was found as an explicit analytical solution to the BPS equations. In the absence of analytical expression for the field configuration, alternative methods must be employed to study junctions. Some information can be obtained from analytical results for specific limiting values of parameters or coordinates. Complementary results can be obtained from numerical simulation of the equations of motion.

For any domain wall junction, a BPS bound on the junction tension can be calculated. In [7] the general

*It is not obvious that this method works also for theories with more than one chiral superfield.
method to obtain this bound from the domain walls surrounding the junction is presented. In this work, it was also established that the junction tension is negative in general, although “exotic” models in which domain wall junctions have positive tension may exist.

In this paper we will study domain wall junctions in a class of generalized Wess-Zumino (WZ) models with $\mathbb{Z}_N$ symmetry. The Lagrangian is given by

$$\mathcal{L} = \int d^2 \theta \int d^2 \bar{\theta} K(\Phi, \Phi^\dagger) + \left\{ \int d^2 \theta \mathcal{W}(\Phi) + h.c. \right\},$$

(1.1)

where the Kähler potential has the canonical form $K(\Phi, \Phi^\dagger) = \Phi \Phi^\dagger$, and the superpotential is given by

$$\mathcal{W}(\Phi) = \Phi - \frac{1}{N+1} \Phi^{N+1}.$$  

(1.2)

The action is invariant under the transformation $\Phi(x, \theta) \to e^{2\pi i/N} \Phi(x, \theta^{-\pi/N})$. Moreover, the model has $N$ physically equivalent vacua given by

$$\phi = e^{2\pi ik/N}, \quad k = 0, 1, \ldots, N - 1,$$

(1.3)

where $\phi$ is the scalar component of the superfield $\Phi$. Despite its apparent simplicity, the model is of physical interest because it is related to the Veneziano-Yankielowicz action [1], which is a low energy effective action for supersymmetric QCD. The parameter $N$ is related to the number of colors in the SU(N) gauge group.

For $N = 2$, the model reduces to the renormalizable Wess-Zumino model. It is well known that there is an analytical expression for the domain wall in this case (see [2], for example). In the large $N$ limit, an analytical expression for the “basic” domain wall – the wall that connects vacua for consecutive values of $k$ – is presented in [3,4] to next to leading order in $1/N$. In this reference it is also noted that all vacua in the model are connected by BPS saturated domain walls at finite $N$.

The “basic” domain wall junction has also been studied. We define the “basic” junction as the field configuration that cycles through all $N$ vacua, consecutively from $k = 0$ to $k = N - 1$, counter-clockwise around the center of the junction. In [4] an analytical solution to the BPS equation for the “basic” junction in the $N \to \infty$ limit was presented. In [5] it was confirmed that the “basic” junction is also BPS saturated for finite values of $N$ by numerical simulations. The tension of the “basic” junction was also calculated, analytically to next to leading order in $1/N$, and numerically also for finite values of $N$.

Beside the “basic” junction, the vacua of the model allow for a multitude of other types of junctions. As discussed further in the paper, we devise a general method to identify all potential BPS saturated junction in this model. There is a very stringent (asymptotic) consistency condition which allows only for a well defined set of intersection angles between the walls. The set of potential BPS junctions we identify contains the junctions that appear in [6,7,8], where tilings of domains and networks of domain walls in this model are studied.

We consider four types of domain wall junctions in the model with $N = 4$, two of which belong to the class of potential BPS saturated, and two more that are not BPS saturated. For each junction we perform a numerical simulation of the second order equations of motion to obtain the junction profile. For the potential BPS junctions, we compare the energy of the junction to the BPS bound (which we evaluated from the domain walls surrounding the junction by the methods in [5]) to determine whether they are indeed BPS saturated. For the other two junctions, we investigate their stability and decay. Although we perform our numerical calculations for the value $N = 4$, we expect our results to apply to other values as well. Moreover, the methods we use are quite general, and can be applied to other models.

The remainder of the paper is organized as follows. In Sec. [A] we describe the possible domain walls of the model. Then in Sec. [B1] and [B2] we describe in detail the two types of domain walls which exist in the case $N = 4$. These walls appear far away from the center of the junctions that we consider. In Sec. [B3] we first discuss the potential BPS junctions in the models for generic $N$, which satisfy a very restrictive consistency condition. We then focus on the $N = 4$ case and describe in detail the two types of BPS junctions (Sec. [B1] and [B2]) and the non-BPS types (Sec. [B3]) which occur. In Sec. [C] we give a brief outline of the computational scheme used for obtaining the numerical representation of the junctions; the paper ends with some comments in Sec. [D].

A. Domain walls.

BPS saturated domain walls are solutions to the equation

$$\partial_x \phi = e^{i\delta} (1 - \phi^N),$$

(1.4)

with the boundary condition that $\phi$ approaches vacuum values at $x \to -\infty$ and $x \to \infty$. Such solutions exist for particular values of the phase $\delta$, which are determined by the equation

$$\text{Im} \left\{ e^{-i\delta} \mathcal{W}(\phi(x \to -\infty)) \right\} = \text{Im} \left\{ e^{-i\delta} \mathcal{W}(\phi(x \to +\infty)) \right\}. $$

(1.5)
For the model with $N = 4$, there are two types of domain walls. The first type connects adjacent vacua, for which $k$ differs by one. The second type connects opposite vacua, for which $k$ differs by two. We will discuss each of these types in turn.

1. Type I domain walls.

Although it is not possible to find an analytical solution for the domain wall profile, the characteristic features can be readily uncovered. For definiteness, we focus on the wall that connects the vacua labeled by $k = 0$ and $k = 1$. From Eq. (1.5) it follows that for this wall the value of $\delta$ is either $-\pi/4$ or $3\pi/4$. For $\delta = -\pi/4$ ($3\pi/4$) the wall connects the vacuum with $k = 1$ ($k = 0$) at $x = -\infty$ to the vacuum with $k = 0$ ($k = 1$) at $x = \infty$.

A general solution to the BPS equation contains one integration constant, indicating the freedom to shift the center of the wall. We fix this constant by choosing the center of the wall to be at the origin: The wall is then invariant under the transformation $\phi(x) = -\phi(-x)$. The field vanishes at the origin, its derivative being (for $\delta = 0$)

$$\partial_x \phi|_{x=0} = 1,$$

while the wall’s tension in this case is

$$T_1^{(I)} = 2 |\mathcal{W}(\phi(x \to \infty)) - \mathcal{W}(\phi(x \to -\infty))| = \frac{16}{5}.$$

Table II lists all domain walls of this type with their corresponding value of $\delta$.

| $\delta$ | $\phi(-\infty)$ | $\phi(\infty)$ |
|---|---|---|
| $-3\pi/4$ | $1$ | $-i$ |
| $-\pi/4$ | $i$ | $-1$ |
| $\pi/4$ | $i$ | $1$ |
| $3\pi/4$ | $-i$ | $1$ |

2. Type II domain walls.

For definiteness, we focus on the wall connecting the vacua labeled by $k = 2$ and $k = 0$. For this wall, the value of $\delta$ is either 0 or $\pi$. For $\delta = 0$ ($\delta = \pi$) the wall connects the vacuum with $k = 2$ ($k = 0$) at $x \to -\infty$ to the vacuum with $k = 0$ ($k = 2$) at $x \to \infty$. The wall is purely real. Again, we fix the integration constant so that the wall is centered at the origin; with this choice, the wall is invariant under the transformation $\phi(x) = -\phi(-x)$.

The field vanishes at the origin, its derivative being (for $\delta = 0$)

$$\partial_x \phi|_{x=0} = 1,$$

while the wall’s tension in this case is

$$T_1^{(II)} = 2 |\mathcal{W}(\phi(x \to \infty)) - \mathcal{W}(\phi(x \to -\infty))| = \frac{16}{5}.$$

Table II lists all domain walls of this type with their corresponding value of $\delta$.

| $\delta$ | $\phi(-\infty)$ | $\phi(\infty)$ |
|---|---|---|
| 0 | $1$ | $-i$ |
| $\pi/2$ | $i$ | 1 |
| $\pi$ | $-1$ | $i$ |
| $3\pi/2$ | $-i$ | $-1$ |

B. Junctions.

BPS junctions are solutions to the equation

$$\partial_z \phi = \frac{e^{i\delta}}{2} (1 - \tilde{\phi}^N).$$

Here $z = x + iy$ and $\partial_z = 1/2 (\partial_x - i \partial_y)$. Junctions are static configurations that have the geometry of a hub and spokes system. In the sectors between the spokes, the field $\phi$ approaches the vacuum values; far away from the hub, $\phi$ follows domain wall profiles from one sector to the next on trajectories perpendicular to the spokes. Each spoke is therefore associated with a domain wall. The hub is the center of the junction where the domain walls meet.

A few considerations are important to consider what junctions are possible, and which of those junctions are BPS saturated.

First, the sum of the forces exerted on the junction by the domain walls that are attached to it has to vanish for a static configuration.

\[ \sum \text{forces} = 0. \]
Second, the value of the phase $\delta$ in the BPS equation for the junction merely determines the orientation of the junction. The effective value of the phase for the BPS domain wall equation for a wall associated with a spoke that makes an angle $\alpha$ with the $\hat{y}$ axis is equal to $\delta - \alpha$. Once the direction of one spoke is fixed, for example to point in the $\hat{y}$ direction, the value of $\delta$ is fixed. For the remainder of the spokes, only specific values of $\alpha$ are then possible. This restriction on the possible values of $\alpha$ turns out to be a very stringent consistency condition. In the case of the model with $N = 4$, inspection of Tables I and II shows that this condition allows only two types of junctions as potential BPS candidates. We call these two types A and B respectively, and we investigate them below.

Once it has been established that a static junction exists, another interesting issue to determine is whether or not it is stable. If a junction is BPS saturated, it is guaranteed to be stable (also see the discussion in [7]). For non-BPS junctions, the issue is not so clear; if the boundary conditions also allow a BPS saturated configuration, then the non-BPS junction is at most meta-stable.

One can actually devise a general method for generating all the potential BPS saturated junctions in our class of models. In fact, given the presence of the $N$ degenerate vacua (1.3), one has that for $N$ even (odd) there are $N/2 ((N - 1)/2)$ types of domain walls. These different types of walls can be labeled by the integer $l = |k_1 - k_2|$ or $l = N - |k_1 - k_2|$, whichever is smaller (here $k_1$ and $k_2$ indicate the vacua that are connected by the wall); moreover the tension of a wall, labeled by $l$, is equal to

$$T^{(l)} = \frac{2N}{N + 1} \sqrt{2 - 2 \cos \frac{2\pi l}{N}}. \quad (1.12)$$

Now, we can represent each type of wall by a vector with length equal to its tension; thus a junction will be represented by the vectors of the walls that it contains far away from the origin, with the direction of the vectors chosen so that they point in the direction where the corresponding wall is located. In a sense, the vectors play the role of the spokes in the hub and spoke system, but they have additional information because of their length. Crossing a spoke counter clockwise means that the value of $k$ increases by $l$ from one domain to the next along the corresponding domain wall: Thus, two junctions will be equivalent if they can be mapped onto each other by inversions and rotations.

All of the potential BPS saturated junctions can then be obtained as follows. We start with the basic $N$-junction, which consists of $N$ vectors representing walls with $l = 1$ (type I walls) at angles of $2\pi/N$. Then we add any two neighboring vectors in the diagram representing the junction, repeating this process until all inequivalent diagrams have been generated. For $N = 4$ this procedure gives the type A and B junctions of above (Fig. 1). In Fig. 2 all the potential BPS junctions are shown for the case $N = 6$.

Here we study the junction of the type shown in Fig. 1a). Far away from the center of the junction, there are four sectors in which the field consecutively takes on the four possible vacuum expectation values. These sectors are separated by BPS domain walls of type I, which were discussed before.

Near the center of the junction, the field can be expanded in a power series in $z$ and $\bar{z}$. Accordingly one has

$$\phi(z, \bar{z}) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_{mn} z^n \bar{z}^m, \quad (1.13)$$

with $a_{mn}$ some complex coefficients.'
The differential equation \[1.11\] and the boundary conditions are invariant under the transformations \((\gamma = -\pi/2 + \pi/N)\)

\[
\begin{align*}
Z_2 &: \phi(z, \bar{z}) \rightarrow \phi^i(e^{-2i\gamma \bar{z}}e^{2i\gamma z}); \\
Z_N &: \phi(z, \bar{z}) \rightarrow -e^{-2i\gamma} \phi(-e^{2i\gamma z}, -e^{-2i\gamma}).
\end{align*}
\]

(1.14) (1.15)

Imposing the \(Z_N\) symmetry the expansion \[1.13\] takes the form

\[
\phi(z, \bar{z}) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \left[ b_{kl} z^{1+Nk}(\bar{z})^l + c_{kl} \bar{z}^{N(k+1)-1}(z)^l \right],
\]

(1.16)

with \(b_{kl}\) and \(c_{kl}\) again complex coefficients. Imposing in addition the \(Z_2\) symmetry, we find the following constraints on these coefficients

\[
\begin{align*}
b_{kl} &= (-1)^{Nk} e^{2ir} b_{kl}^†, \\
c_{kl} &= (-1)^{N(k+1)} e^{2i\gamma} c_{kl}^†.
\end{align*}
\]

(1.17) (1.18)

Thus, if \(N\) is even, one can introduce the real coefficients \(d_{kl}\) and \(e_{kl}\), obtaining the expansion

\[
\phi(z, \bar{z}) = e^{i\gamma} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \left[ d_{kl} z^{1+Nk}(\bar{z})^l + e_{kl} \bar{z}^{N(k+1)-1}(z)^l \right].
\]

(1.19)

If on the other hand \(N\) is odd, one has

\[
\begin{align*}
b_{kl} &= (-1)^{k} e^{2ir} b_{kl}^†, \\
c_{kl} &= (-1)^{(k+1)} e^{2i\gamma} c_{kl}^†,
\end{align*}
\]

(1.20) (1.21)

so that, according to the value of the index \(k\), half of the coefficients is real, the other half being purely imaginary. In the case \(N = 4\) one has \(\gamma = -\pi/4\), so that

\[
\phi(z, \bar{z}) = e^{-i\pi/4} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \left[ d_{kl} z^{1+4k}(\bar{z})^l + e_{kl} \bar{z}^{3+4k}(z)^l \right].
\]

(1.22)

We can now calculate the coefficients \(d_{kl}\) and \(e_{kl}\) using the BPS equation; it turns out that all coefficients are determined except for the ones which multiply terms that contain only \(\bar{z}\), that is, the \(e_{k0}\). These coefficients have to be adjusted to obtain the correct domain walls far from the center of the junction\[n\].

Close to the origin, the configuration looks like a string. We have proved then that there are sensible solutions to the BPS equations both near the origin and far away from it. In order to determine whether those asymptotic solutions can be connected consistently and thus to investigate whether the junction is BPS saturated we performed a numerical simulation of the second order equations of motions for the field \(\phi\), while imposing the domain walls as the boundary conditions at large distance from the origin. In order to allow the field to relax to its minimum energy configuration, we also introduced a damping term (see Sec. III C for details). In Fig. 3 we show the modulus of \(\phi\) after the configuration has come to rest.

FIG. 3. Modulus of \(\phi\) for the type A junction.

In order to determine whether the junction is BPS saturated, we will first calculate the BPS bound on the energy inside a square of “radius” \(R\), with the center of the junction located at the center of the square. Assuming the junction to be BPS saturated the energy inside a square with radius \(R\) can be calculated entirely in terms of quantities related to the domain walls surrounding the junction when the radius \(R\) is much larger than the size of the junction. Explicitly, for large \(R\), the bound on the energy takes the form

\[
\frac{E_{BPS}}{L} = T_2(A) + 4T_1(1)R.
\]

(1.23)

Here \(L\) is a distance in the direction perpendicular to the \(x-y\) plane. The second term in \[1.23\] is the contribution to the energy from the four type I domain walls connected to the junction; the first term represents instead the energy of the junction, which is equal to

\[
T_2(A) = \int_{\text{square}}^\infty a_i \cdot \bar{a}\, dl_i,
\]

(1.24)

where

\[
a_i = \text{Im} \left\{ \phi \partial \overline{\phi} \right\}.
\]

(1.25)

Now, each wall on the contour of the square contributes the same amount to the tension. The BPS bound on the junction tension is therefore equal to

---

\[n\] Note that the expansion up to terms of eleventh order in \(z\) and \(\bar{z}\) contains only two adjustable parameters.
\[ T_2^{(A)} = 4 \int_{-\infty}^{\infty} a^{(I)} dx, \]  

(1.26)

where \( a^{(I)} = \text{Im} \{ \phi^{(I)} \partial_x \phi^{(I)} \} \) and \( \phi^{(I)} \) is the profile of a type I wall along the \( \hat{x} \) axis. Using a numerical representation of the type I domain wall, we determined that \( T_2^{(A)} = -2.8614 \).

We also calculated the energy in a square with radius \( R \) as a function of \( R \) directly from our numerical representation of the junction using the equation

\[
\frac{E}{L} = \int_{\text{square}} \{ \partial_x \phi \partial_x \phi + \partial_y \phi \partial_y \phi + V \} \, dx \, dy,
\]  

(1.27)

where \( V = \partial_y \phi \partial_y \phi \) is the scalar potential. In Fig. 4 we compare this energy with the BPS bound. It is clear that the energy converges to the bound for large \( R \) (compared to the size of the junction) and therefore the junction does indeed saturate the BPS bound.

![FIG. 4. Energy inside a square of radius \( R \) as a function of \( R \) (dots) compared to the BPS bound (solid line) for the type A junction.](image)

2. **Type B junction.**

The second type of junction that is potentially BPS saturated has two spokes corresponding to type I walls at an angle of \( \pi/2 \). A third spoke associated with a type II domain wall intersects each of the other spokes at an angle of \( 3\pi/4 \). The configuration is sketched in Fig. 4 (b).

This type of junction may be obtained from a junction of type A by pinching two neighboring spokes together in a symmetric fashion, thereby eliminating one domain. We performed a numerical simulation of the second order equations of motion, with the appropriate domain wall profiles as boundary conditions, and including a damping term (see again Sec. 1C for details). In Fig. 5 we show the modulus of the field \( \phi \) after the configuration has come to rest.

![FIG. 5. Modulus of \( \phi \) for the type B junction.](image)

To analyze whether this type of junction is BPS saturated we follow a procedure similar to the one used for the type A junction. We first calculate the bound on the energy in a square of radius \( R \) for large \( R \), assuming the junction to be BPS saturated. This bound takes the form

\[
\frac{E_{\text{BPS}}}{L} = T_2^{(B)} + 2T_1^{(I)}R + \sqrt{2}T_1^{(II)}R - \Delta T. \]  

(1.28)

The first term represents the BPS bound on the tension of the junction. The second and the third term represent the contribution to the energy from the two type I walls and the type II wall (the factor of \( \sqrt{2} \) appears because the spoke representing the type II wall is in the diagonal direction). Special care must be taken because the spoke representing the type II wall intersects the sides of the square diagonally, so that the energy contributed by two triangular areas needs to be subtracted. This is the origin of the fourth term, \( \Delta T \), which takes the form

\[
\Delta T = 4 \int_0^{\infty} \partial_x \phi^{(II)} \partial_x \phi^{(II)} \, dx,
\]  

(1.29)

where \( \phi^{(II)} \) is the profile of a type II wall along the \( \hat{x} \) axis.

The field \( \phi \) on the type II wall connecting the \( k = 0 \) sector to the \( k = 2 \) sector of this junction is purely real, so that it does not contribute to the BPS bound on the tension of the junction. On the other hand the tension receives equal contributions from each of the type I walls. The BPS bound on the tension of type B junctions is therefore half of the tension of a type A junction

\[
T_2^{(B)} = \frac{1}{2} T_2^{(A)}. \]  

(1.30)

We can finally evaluate \( \Delta T \) using the numerical representation of the type II wall, finding in this way

\[
\Delta T = -\frac{1}{2} T_2^{(A)}. \]  

(1.31)

This is a curious equality, as \( \Delta T \) is a quantity calculated from a type II wall, and \( T_2^{(A)} \) can be calculated from a
type I wall profile. We have no further comments on the nature of the equality here, but as a consequence the total energy in a square with large radius is equal for type A and type B junctions.

We also calculated the energy in a square with radius $R$ as a function of $R$ directly from our numerical representation of the type B junction using Eq. (1.27). In Fig. 6 we compare this energy with the BPS bound. It is clear that for this junction too, the energy in a square converges to the bound for large $R$ (compared to the size of the junction). The type B junctions therefore also saturate the BPS bound.

![FIG. 6. Energy inside a square of radius $R$ as a function of $R$ (dots) compared to the BPS bound (solid line) for the type B junction.](image)

3. Non-BPS junctions.

Here we discuss static junctions that are not BPS saturated. For such junctions the values of $\delta$ associated with the various spokes are not consistent. However, the total force on the junction vanishes. Moreover, they are invariant under an extended symmetry group, and therefore extremize the energy.

The first junction of this type that we discuss is displayed in Fig. 7 a). It consists out of four sectors separated by spokes which intersect at $\pi/2$ angles. However, in contrast to the BPS four-junction, the spokes are associated with type II walls.

![FIG. 7. Non-BPS saturated junctions in the case $N = 4$.](image)

We found by a numerical simulation of the second order equations of motion that a static configuration of this type does indeed exist. In Fig. 8 we show the modulus of $\phi$ for the final configuration at rest.

![FIG. 8. Modulus of $\phi$ for the non-BPS four-junction.](image)

However, this junction has a negative mode. It is unstable against local perturbations that break its symmetry, and it decays into a configuration with three domains separated by two domain walls, as shown in Fig. 9. Two domains with the same vacuum expectation value connect, and the resulting two domain walls are pushed out of the center.

![FIG. 9. Modulus of $\phi$ after the non-BPS four-junction has decayed into a configuration with three domains separated by two walls.](image)

The second type of non-BPS walls we discuss is the eight-junction shown schematically in Fig. 7 b). Compared to the BPS four-junction, this junction has winding number two. We again found by numerical simulation of the second order equations of motion that a static configuration of this type exists. We show the modulus of the field $\phi$ for this configuration in Fig. 10.

![FIG. 10. Modulus of $\phi$ for the non-BPS eight-junction.](image)
We also observed that this junction can decay into two four-junctions with winding number one when a symmetry breaking perturbation is introduced. The two resulting junctions repel each other, as shown in Fig. 11. We noticed that the eight-junction only decays when the perturbation is sufficiently large. For now, we leave open the question whether this indicates that the junction is meta-stable, or that this behavior is an artifact of the finite size of the lattice we used.

C. Numerical methods.

We simulated the second order equations of motion on a lattice using a forward predicting algorithm. The lattice spacing was chosen to be much smaller than the size of the junctions, and, at the same time, the size of the lattice was much larger than the size of the junctions. For our purposes, a lattice of 251 by 251 points offered sufficient resolution.

We added a damping term to the equations of motion which allows the field to relax to a minimum energy configuration. At the same time, this damping stabilizes the forward predicting scheme.

In order to generate unstable junctions, we started our simulation with an initial configuration that was invariant under the same symmetry transformations as the boundary condition. As these symmetry are not broken by the equations of motion, the final configuration is the lowest energy configuration consistent with the symmetries. Such a configuration may be unstable against local perturbations which break the symmetry.

II. SUMMARY

In this paper we have studied in detail domain wall junctions in a generalized Wess-Zumino model. We have presented a method to identify all potentially BPS saturated junctions, and we have described a procedure to determine whether these junctions indeed satisfy the BPS bound. We showed that in the case $N = 4$ (the lowest value of $N$ for which there is more than one potential BPS junctions), these junctions are in fact BPS saturated.

On the basis of our results, in conjunction with the results for the basic junction for generic value of $N [7]$ and the large $N$ results for the basic junction [6], we speculate that every potential BPS saturated junction for any $N$ in this model indeed saturates the BPS bound.

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