Various versions of analytic QCD and skeleton-motivated evaluation of observables

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We present skeleton-motivated evaluation of QCD observables. The approach can be applied in analytic versions of QCD in certain classes of renormalization schemes. We present two versions of analytic QCD which can be regarded as low-energy modifications of the “minimal” analytic QCD and which reproduce the measured value of the semihadronic τ decay ratio $r_\tau$. Further, we describe an approach of calculating the higher order analytic couplings $A_k$ ($k = 2, 3, \ldots$) on the basis of logarithmic derivatives of the analytic coupling $A_1(Q^2)$. This approach can be applied in any version of analytic QCD. We adjust the free parameters of the afore-mentioned two analytic models in such a way that the skeleton-motivated evaluation reproduces the correct known values of $r_\tau$ and of the Bjorken polarized sum rule (BjPSR) $d_b(Q^2)$ at a given point (e.g., at $Q^2 = 2$ GeV$^2$). We then evaluate the low-energy behavior of the Adler function $d_s(Q^2)$ and the BjPSR $d_b(Q^2)$ in the afore-mentioned evaluation approach, in the three analytic versions of QCD. We compare with the results obtained in the “minimal” analytic QCD and with the evaluation approach of Milton et al. and Shirkov.

Changes in v3: the values of parameters of analytic QCD models M1 and M2 were refined and the numerical results modified accordingly; the penultimate paragraph of Sec. II and the ultimate paragraph of Sec. III are new; discussion of Figs. 4 was extended; new references were added.

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I. INTRODUCTION

In perturbative QCD (pQCD), the coupling parameter $a(Q^2) \equiv \alpha_s(Q^2)/\pi$ [where: $Q^2 = -q^2 = -(q^0)^2 + q^2$] is obtained on the basis of the perturbative β-function which is a (truncated) polynomial of $a$. As a consequence, $a(Q^2)$ has Landau singularities in an infrared space-like zone ($Q^2 > 0$), and therefore these singularities are unphysical. This problem was fully recognized and a solution found about ten years ago by Shirkov and Solovtsov [1]. The solution found was minimal in the sense that the analyticization $a(Q^2) \rightarrow A_1(Q^2)$ was performed by removing the Landau-cut singularities, while keeping the singularities on the time-like axis unchanged. Further, completely analogous minimal analyticization was performed for the higher powers $a^k \rightarrow A_k$ ($k \geq 2$) and this replacement was performed term-by-term in the simple truncated perturbation series (STPS – in powers of $a$) of observables by Milton, Solovtsov, Solovtsova, and Shirkov [2, 3, 4] (“Analytic Perturbation Theory” – APT). The resulting series have in general better convergence behavior and much less sensitivity under the variation of the renormalization scale (RScl) and scheme (RSch). We will call the analytic QCD model based on the afore-mentioned analytic coupling the “minimal analytic” (MA) model $[\rightarrow A_1^{(MA)}(Q^2)]$, and the afore-mentioned evaluation approach (involving the truncated analytic series) the APT-evaluation approach.

The MA coupling $A_1^{(MA)}(Q^2)$ contains just one adjustable parameter, the QCD scale Λ. Reproduction of the measured values of the higher energy QCD observables ($|q^2| > 10$ GeV$^2$) fixes the scale parameter to the value $\Lambda_{(n_f=5)} \approx 0.26$ GeV, corresponding to $\Lambda_{(n_f=3)} \approx 0.4$ GeV. However, then the well-measured value of the massless strangeless semihadronic τ-decay ratio $r_\tau(\Delta S = 0, m_q = 0) = 0.204 \pm 0.005$ [8, 9] (cf. Appendix E) cannot be reproduced [4] unless large values of the $u$, $d$ and $s$ quark masses are introduced ($m_q \approx 0.25-0.45$ GeV) and the threshold effects become very important. One may want to avoid introduction of such large quark masses, by modifying the MA model at low energies while keeping the analyticity of $A_1(1/Q^2)$ in the non-time-like region. In this work we introduce two somewhat different modifications $\Delta A_1(Q^2)$ ($A_1 \equiv A_1^{(MA)} + \Delta A_1$), both having power-like behaviors. We construct in a systematic way the higher order couplings $A_k(Q^2)$ based on the logarithmic derivatives of $A_1(Q^2)$. Further, we construct a skeleton-expansion-motivated algorithm of evaluation of QCD observables, which can be applied in any analytic version of QCD and in a large class of renormalization schemes. For such an evaluation,
we have to know the first few coefficients of STPS and all the leading-$\beta_0$ coefficients of the full perturbation series. We believe that the inclusion in this evaluation of the light-by-light contributions, if they contribute, should be avoided. Such contributions have a different topological structure and their evaluation should be performed separately in most evaluation (resummation) methods – see, for example, Ref. [11]. Some of the main results of the present work were published by us in a summarized form in Ref. [12].

In Sec. II, we explain the main features of the analytic versions of QCD (anQCD), we present the known MA model, and propose two versions of modified MA – the models ‘M1’ and ‘M2’ [$\rightarrow A_{1}^{(M1)}(Q^2), A_{1}^{(M2)}(Q^2)$]. In Sec. III, we introduce the higher order couplings $A_k(Q^2)$ ($k \geq 2$) in a way that can be applied in any version of anQCD, by imposing on them specific natural behavior under the change of scale $Q^2$ and of RSch. In Sec. IV we then present an algorithm which allows us to evaluate any QCD observable in any version of anQCD, an algorithm motivated by the skeleton expansion. In Sec. V we fix the free parameters in the M1 and M2 anQCD couplings $A_{1}^{(M1)}(Q^2)$ and $A_{1}^{(M2)}(Q^2)$ in such a way that the afore-mentioned skeleton-motivated approach gives us the measured values of $r_T$ and of the Bjorken polarized sum rule (BjPSR) $d_0(Q^2)$ at $Q^2 = 2 \text{ GeV}^2$. We then present the resulting low-energy curves for the V-channel Adler function $d_0(Q^2)$ and of the BjPSR $d_0(Q^2)$ in the skeleton-motivated approach, in the anQCD versions MA, M1, M2. We investigate the RSch- and RSch-dependence of the numerical curves, and in the MA-case we compare the results of $d_0(Q^2)$ obtained by our skeleton-motivated evaluation approach with those of the APT approach of Refs. [2, 3, 4]. Numerical calculations were performed using Mathematica [13]. In Sec. VI we present our conclusions and prospects for further work in this direction. Appendix [A] contains details of the coefficients appearing in the evaluation method. In Appendix [B] we present another evaluation method that is even more closely related to the skeleton expansion. Appendix [C] contains a derivation of the leading skeleton (LS) characteristic function of the BjPSR, and relations between the space-like and time-like formulations for the LS-term. Appendix [D] is a compilation of expressions of some coefficients used in this work, and Appendix [E] describes an extraction of the experimental value of $r_T(\Delta S = 0, m_q = 0)$.

II. MINIMAL ANALYTIC QCD AND TWO EXTENSIONS OF IT

The perturbative QCD coupling $a(Q^2) \equiv \alpha_s(Q^2)/\pi$ in the space-like region $|Q^2|$ not in $(-\infty, 0]$ has the scale dependence governed by the renormalization group equation (RGE)

$$\frac{\partial a(\ln Q^2; \beta_2, \ldots)}{\partial \ln Q^2} = - \sum_{j=2}^{j_{\text{max}}} \beta_{j-2} a^j(\ln Q^2; \beta_2, \ldots),$$

(1)

where the first two coefficients $\beta_0 = (1/4)(11 - 2n_f/3)$ and $\beta_1 = (1/16)(102 - 38n_f/3)$ are scheme-independent in mass-independent schemes, and the other coefficients $\beta_j$ ($j \geq 2$) characterize the RSch. In practice, the above sum is truncated at a certain $j_{\text{max}}$ where $j_{\text{max}} - 1$ is the loop level. The perturbative RGE (1) has a standard iterative solution in the form

$$a(Q^2) = \sum_{k=1}^{\infty} \sum_{\ell=0}^{k-1} K_{k\ell} \frac{(\ln L)^{\ell}}{L^k},$$

(2)

where $L = \ln(Q^2/\Lambda^2)$ and $K_{k\ell}$ are constants depending on the $\beta_j$ coefficients and on the choice of the scale $\Lambda$. If the conventional (\text{MS}\bar{\text{R}}) scale $\Lambda = \Lambda$ [14, 15] is used, the coefficients $K_{k\ell}$ are

$$K_{10} = 1/\beta_0; \quad K_{20} = 0; \quad K_{21} = -\beta_1/\beta_0^2; \quad K_{30} = -\beta_2^2/\beta_0^3 + \beta_2/\beta_0^4; \quad K_{31} = -\beta_3 = -\beta_1^2/\beta_0^5; \quad \ldots$$

(3)

Further coefficients $K_{k\ell}$, up to $k = 6$, are given in Appendix [D]. The coupling $a(Q^2)$, Eq. (2), has nonanalytic structure along the time-like axis $Q^2(\equiv -q^2) < 0$. In addition, it has singularities in the space-like region $0 < Q^2 \leq \Lambda^2$, which are formally the consequence of the (truncated) power expansion structure of the beta-function on the RHS of Eq. (1). Application of the Cauchy theorem to function $a(Q^2)$ in the $Q^2$-plane gives then the following dispersion relation for $a$:

$$a(Q^2) = \frac{1}{\pi} \int_{\sigma = -\Lambda^2 - \eta}^{\infty} \frac{da_{\text{pt}}(\rho_1)(\sigma)}{(\sigma + Q^2)},$$

(4)
where \( \rho_1^{(pt)}(\sigma) \) is the (pQCD) discontinuity function of \( a \) along the cut axis in the \( Q^2 \)-plane: \( \rho_1^{(pt)}(\sigma) = \text{Im} a(-\sigma - i\epsilon) \). In the integration, \( \eta \) is positive (\( \eta \to +0 \) can be taken), reflecting the fact that the corresponding contour integration path avoids entirely the singularities of \( a(z) \) in the complex plane, including the singularity at \( z = -\sigma = \Lambda^2 \) [cf. Eq. (2)].

By special relativity and causality, observables are analytic functions of the associated physical momentum squared \( q^2 \equiv -Q^2 \) in the \( Q^2 \)-plane with the time-like axis \( (Q^2 < 0) \) excluded. Since QCD observables are functions of the invariant coupling \( a(Q^2) \), both should have the same analyticity properties. The singularity sector \( 0 < Q^2 \leq \Lambda^2 \) in a\( (Q^2) \), Eqs. (2) and (4), is therefore nonphysical. The most straightforward rectification of this problem is to eliminate that sector from the dispersion relation (4) while keeping the pQCD discontinuity function \( \rho^{(pt)}(\sigma; \beta_2, \ldots) \) unchanged on the time-like axis \( \sigma > 0 \) [1], thus leading to the specific “minimal analytic” (MA) coupling

\[
A_1^{(MA)}(Q^2) = \frac{1}{\pi} \int_{\sigma=0}^{\infty} \frac{d\rho_1^{(pt)}(\sigma)}{(\sigma + Q^2)} .
\]  

(5)

In practice, truncated series (2) can be used to obtain the discontinuity function \( \rho_1^{(pt)}(\sigma) \) and thus the coupling (5). Prescription (5) was investigated from calculational viewpoints in Refs. [16, 17, 18]. There exists a practical iterative solution [16, 17] to RGE (1) based on the Lambert function [19]. This is a solution of a different form than (2). When the number of terms in the Lambert-based expansion and in expansion (2) increases, the two solutions for \( A_1^{(MA)} \) converge to the exact numerical solution rapidly for all \( Q^2 \), but the Lambert-based expansion converges faster. When \( k_{\text{max}} \geq 4 \) in (2), the corresponding solution \( A_1^{(MA)}(Q^2) \) differs in \( \overline{\text{MS}} \) from the exact numerical solution by less than one per cent for all \( Q^2 > 0 \) [17]. In the present work, we will use expansion (2) with \( k_{\text{max}} = 5 \) or 6.

Other types of analytization of \( a \) can be performed by focussing on the analyticity properties of the beta function \( [20, 21] \), or by subtracting certain power correction terms \( \sigma \approx q^2 > 4 \) GeV) [10] and the mass threshold effects become central.

The above consideration motivates us to introduce low-energy modifications of the MA coupling. Modifications, although simple, introduce additional parameters which have to be fixed by requiring reproduction of the measured...
values of low-energy QCD observables, including of $r_v$. One possible modification is inspired by the well measured \cite{6} \cite{8} isovector hadronic spectral function $R_v(s)$. At low energies ($s < 1$ GeV$^2$) it is dominated by the $\rho$-resonance ($M_\rho = 0.776$ GeV), which, in the narrow width approximation, can be represented as a delta function $\delta(s-M_\rho^2)$ \cite{27}. This is in the spirit of the Vector Meson Dominance (VMD). If we assume that the $s$-dependence of the time-like quantity $R_v(s)$ is at least qualitatively described by the first order time-like coupling $A_1(s)$, Eq. \cite{8}, then the afore-mentioned delta-like structure should appear in it. This then leads to the following ansatz (model 'M1'):

\[
A_1^{(M1)}(s) = c_f \frac{\Lambda^2}{\sqrt{M_\rho^2}} \delta(s-M_\rho^2) + k_0 \Theta(M_0^2-s) + \Theta(s-M_0^2) A_1^{(MA)}(s),
\]

where $c_f$, $k_0$, $c_r = \frac{\Lambda^2}{\sqrt{M_\rho^2}}$, $c_0 = \frac{M_0^2}{\Lambda^2}$ are four dimensionless parameters of the model; $\Theta(x)$ is the Heaviside step function ($+1$ for $x > 0$, zero otherwise). In this model, the MA behaviour of $A_1(s)$ at low energies $s < \frac{M_0^2}{\Lambda^2}$ has been replaced by a constant ($k_0$) plus a delta function (at $s = \frac{M_\rho^2}{\Lambda^2}$). The more literal application of the VMD approach results in $k_0 = -1$ \cite{28}. This is so because $R_v(s) = 1 + A_1(s) + O(\Lambda^2)$, and $R_v(s) \to 0$ when $s \to 0$, implying $A_1(s) \to -1$. However, such a model appears to restrict the low energy behavior of $A_1(s)$ and of $A_1(Q^2)$ too severely, especially if we want to impose the condition of merging $A_1(Q^2)$ of the model with $A_1^{(MA)}(Q^2)$ at high $Q^2$. As a consequence, values of various unrelated low energy observables, such as Adler function (or: $r_v$) and Bjorken polarized sum rule, cannot be reproduced simultaneously in such a model. Therefore, unlike the choice $k_0 = -1$ in Ref. \cite{28}, we keep here the constant $k_0$ in Eq. \cite{11} free. Applying transformation \cite{21} to expression \cite{11} gives the space-like analytic coupling of the model:

\[
A_1^{(M1)}(Q^2) = A_1^{(MA)}(Q^2) + \Delta A_1^{(M1)}(Q^2),
\]

\[
\Delta A_1^{(M1)}(Q^2) = -\frac{1}{\pi} \int_{\sigma=0}^{\infty} d\sigma \rho_1^{(pt)}(\sigma) = c_f \frac{\Lambda^2}{\sqrt{M_\rho^2}} \frac{M_0^2 Q^2}{(Q^2+M_0^2)^2} - k_f \frac{\Lambda^2}{\sqrt{M_\rho^2}} \frac{M_0^2}{(Q^2+M_0^2)^2}.
\]

where the constant $d_f$ is

\[
d_f = -k_0 + \frac{1}{\pi} \int_{\sigma=0}^{\infty} d\sigma \rho_1^{(pt)}(\sigma).
\]

The coupling \cite{12} \cite{13} can also be rewritten in a somewhat different, but equivalent, form

\[
A_1^{(M1)}(Q^2) = c_f \frac{\Lambda^2}{\sqrt{M_\rho^2}} \frac{M_0^2 Q^2}{(Q^2+M_0^2)^2} + k_0 \frac{M_0^2}{(Q^2+M_0^2)} + \frac{Q^2}{(Q^2+M_0^2)} \frac{1}{\pi} \int_{\sigma=\pi_0^2}^{\infty} d\sigma \rho_1^{(pt)}(\sigma) \equiv \frac{1}{\pi} \int_{\sigma=\pi_0^2}^{\infty} d\sigma \rho_1^{(pt)}(\sigma).
\]

In general, this coupling differs from the MA coupling \cite{5} by terms $\Delta A_1^{(M1)} \sim \frac{\Lambda^2}{Q^4}$. However, we will choose to require $\Delta A_1^{(M1)} \sim \frac{\Lambda^2}{Q^4}$, i.e., that M1 effectively merge into MA at higher energies, as we did in Ref. \cite{28}. This condition eliminates one of the four new parameters, for example $k_0$:

\[
k_0 = -\frac{c_r c_f}{c_0} + \frac{1}{\pi} \frac{1}{\Lambda^2} \int_0^{\Lambda^2} d\alpha \rho_1^{(pt)}(\sigma) + \frac{1}{\pi} \int_{\Lambda^2}^{\infty} d\alpha \rho_1^{(pt)}(\sigma).
\]

Since the presented version of M1 merges with MA at higher energies, the value of the scale parameter $\Lambda$ remains practically unchanged, $\Lambda_{(n_f=3)} = 0.4$ GeV, and the model contains only three dimensionless parameters $c_f$, $c_r$, and $c_0$.

Another, somewhat simpler, modification of the MA coupling consists in adding a constant value ($c_v$) in the low-energy region of the MA time-like coupling (model 'M2'):

\[
A_1^{(M2)}(s) = A_1^{(MA)}(s) + c_v \Theta(M_p^2-s),
\]

\[
A_1^{(M2)}(Q^2) = A_1^{(MA)}(Q^2) + c_v \frac{M_p^2}{Q^2+M_p^2},
\]

where $c_v$ and $c_p = \frac{M_p^2}{\Lambda^2}$ are two dimensionless parameters of the model. For simplicity, we will assume that the scale parameter is unchanged: $\Lambda_{(n_f=3)} = 0.4$ GeV. The resulting additional term $\propto 1/(Q^2+M_p^2)$ in $A_1(Q^2)$ can be
interpreted, or motivated, as the leading power-like modification ($\propto 1/Q^2$) of the MA coupling such that the condition $|A_1(Q^2 = 0)| < \infty$ is preserved. The latter condition is regarded as desirable in our approach developed in Sec. IV because the so called leading-skeleton resummation of observables remains finite in such a model.

Model M1 was motivated by simulating roughly the $\rho$-resonance contribution in the one-loop expression for $R_V(s)$, via a VMD narrow width approximation ansatz in $A_1(s)$. However, this was only a motivation for the construction of an explicit form of $A_1(s)$ as the starting point of the model, and the higher-loop contributions $A_k(Q^2)$ ($k \geq 2$) are then constructed on the basis of this $A_1(s)$ (see the next Section). The approximation of the $\rho$-resonance is then expected to get worse at higher loop level. Another possible approach, which we will not follow here, would be to refine (retroactively) $A_1(s)$ so that higher-loop evaluations of $R_V(s)$ give us a given specified approximation of the $\rho$-resonance at low energies. A similar approach could possibly be followed also in M2. In general, reproduction of the correct low-energy behavior of time-like observables such as $R_V(s)$ represents a difficult problem. In this work, we will follow a more modest approach – in Sec. V we will fix the free parameters of models M1 and M2 by requiring, at loop-level three or four, the reproduction of the central experimental values for the Bjorken polarized sum rule $d_0(Q^2)$ at two (in M1) or one (in M2) values of scale $Q$ ($\geq 1$ GeV), and the reproduction of the measured value of $r_\tau(\Delta S = 0)$.

All the versions of anQCD presented here are infrared finite, i.e., the zero momentum limits $A_1(0) = A_1(0)$ are finite.

III. ANALYTIZATION OF HIGHER POWERS OF THE COUPLING PARAMETER

In the previous Section, a few of the possibilities of constructing the analytic version $A_1(Q^2)$ of $a(Q^2)$ were presented. For evaluation of QCD observables, the analytic versions of higher powers $a^k(Q^2)$ are needed as well. For that, there is no unique way of constructing the correspondence $a^k \leftrightarrow A_k$. In the MA QCD, one possibility is to apply the MA procedure \cite{2} to each power of $a$ \cite{2}:

$$a^k(Q^2) \rightarrow A^{(MA)}_k(Q^2) = \frac{1}{\pi} \int_0^\infty \frac{d\sigma}{\sigma + Q^2} \rho^{(pt)}_k(\sigma) \quad (k = 1, 2, \ldots),$$

where $\rho^{(pt)}_k = \text{Im}[a^k(-\sigma - i\epsilon)]$, and $a$ is given, e.g., by Eq. (2). Other choices would be, e.g., $a^k \rightarrow A_1^k, A_1^{k-2}A_2$, etc. With construction \cite{19}, it was shown \cite{18} that the RGE’s governing the evolution of $A_k$’s are identical to those governing the evolution of $a^k$’s in pQCD when the replacements $a^j \rightarrow A_j^{(MA)}$ are made [cf. Eq. (1)]

$$\frac{\partial A^{(MA)}_k(\mu^2)}{\partial \ln \mu^2} = -k \sum_{j=2}^{j_{max}} \beta_{j-2} A^{(MA)}_{j+k-1}(\mu^2) = -k \beta_0 A^{(MA)}_{k+1}(\mu^2) - \cdots,$$

$$\frac{\partial^2 A^{(MA)}_k(\mu^2)}{\partial (\ln \mu^2)^2} = k \sum_{\ell,k=2}^{j_{max}} \beta_{\ell-2} \beta_{k-2} (\ell + k - 1) A^{(MA)}_{j+k-2}(\mu^2) = k(k+1) \beta_0^2 \beta_2 A^{(MA)}_{k+2}(\mu^2) + \cdots, \quad \text{etc.} \quad (20)$$

The reason for this lies in the fact that $a^k$, and consequently $\rho^{(pt)}_k(\sigma)$, fulfill analogous RGE’s. Further, the renormalization scheme (RSch) dependence in pQCD, i.e., dependence of $a^k$ of $\beta_j$ ($j \geq 2$), is known \cite{22} (cf. also \cite{30}), the same dependence holds for the discontinuity functions $\rho^{(pt)}_k(\sigma, \beta_2, \ldots)$ and thus for the MA couplings $A_k$ \cite{19} the analogous dependence via $a^j \leftrightarrow A_j^{(MA)}$ is obtained ($k = 1, 2, \ldots$):

$$\frac{\partial A^{(MA)}_k(\mu^2)}{\partial \beta_2} = \frac{k}{\beta_0} A^{(MA)}_{k+2}(\mu^2) + \frac{k \beta_2}{3 \beta_0^2} A^{(MA)}_{k+4}(\mu^2) + O(A^{(MA)}_{k+5}), \quad (21)$$

$$\frac{\partial A^{(MA)}_k(\mu^2)}{\partial \beta_3} = \frac{k}{2 \beta_0} A^{(MA)}_{k+3}(\mu^2) - \frac{k \beta_1}{6 \beta_0^2} A^{(MA)}_{k+4}(\mu^2) + O(A^{(MA)}_{k+5}), \quad (22)$$

$$\frac{\partial A^{(MA)}_k(\mu^2)}{\partial \beta_4} = \frac{k}{3 \beta_0} A^{(MA)}_{k+4}(\mu^2) + O(A^{(MA)}_{k+5}). \quad (23)$$

The RGE-type relations \cite{20}-\cite{28}, valid in the MA QCD, imply the following important property: If the evaluation of a space-like QCD observable quantity $D(Q^2)$ is based on the analytization of STPS of that quantity according to the rule $a^k(\mu^2) \rightarrow A_k^{(MA)}(\mu^2)$ ($k \geq 1$), then the evaluated value of $D(Q^2)$ has a dependence on RSch $\mu$ and on RSch
In these definitions \((26)\), as well as in \(\beta\) (nonanalytic functions of \(A\) (truncated) evolution equations \((26)\), any higher order quantity \(\beta_{RSch} (\text{momenta})\) where we have altogether \(Eqs. (26)\) represent definitions \(\textit{of} A_{n}\) on the RHS. Then the analytized evaluated values \(D_{\text{an}}(Q^{2}); (Q^{2})\) will have the RSc- and RSc-independence precision \(\partial D_{\text{an}}(Q^{2}); (Q^{2})/\partial X \sim A_{n+1} \) \((X = \ln \mu^{2}, \beta_{j})\) which has its perturbative analog \(\partial D_{\text{STPS}}(Q^{2}); (Q^{2})/\partial X \sim a^{n+1}\).

In view of these considerations, we propose to maintain evolution relations \((20)\) (for \(k = 1\) only) terms of up to \(A_{n}\) on the changes in pQCD. The precision \(\mathcal{O}(A_{n}^{\text{MA}})\) corresponds in pQCD to the precision \(\mathcal{O}(\alpha^{n})\).

Having the STPS with terms up to \(\sim a^{n_{\text{max}}} \) \((n_{\text{max}} \equiv n_{m})\), as well as its analytized analog

\[
P_{\text{STPS}}^{(m)}(Q^{2}) = a(\mu^{2}; \beta_{2}, \ldots) + \sum_{n=2}^{n_{m}} d_{n-1}a^{n}(\mu^{2}; \beta_{2}, \ldots),
\]

\[
P_{\text{an}}^{(m)}(Q^{2}) = A_{1}(\mu^{2}; \beta_{2}, \ldots) + \sum_{n=2}^{n_{m}} d_{n-1}A_{n}(\mu^{2}; \beta_{2}, \ldots),
\]

it is then enough to include in the evolution rules \((20)\) and \((21)-(23)\) (for \(k = 1\) only) terms of up to \(A_{n}\) on the evolution \(A_{2}\). The quantities \((\beta_{2}, \beta_{2}, \ldots)\) in any chosen anQCD version in a given chosen RSch \((\text{momenta})\), can be constructed, in the given RSch.

In our approach, the basic space-like quantities are \(A_{1}(\mu^{2})\) of a given anQCD model (e.g., MA, M1, M2) and its logarithmic derivatives

\[
\frac{\partial A_{1}(\mu^{2}; \beta_{2}, \ldots)}{\partial \ln \mu^{2}} \approx -\beta_{0}A_{2} - \cdots - \beta_{n_{m}+2}A_{n_{m}},
\]

\[
\frac{\partial^{2} A_{1}(\mu^{2}; \beta_{2}, \ldots)}{\partial (\ln \mu^{2})^{2}} = 2\beta_{0}^{2}A_{3} + 5\beta_{0}\beta_{1}A_{4} + \cdots + k_{n_{m}}^{(2)}A_{n_{m}}, \quad \text{etc.},
\]

where we have altogether \(n_{m} - 1\) equations, and \(k_{n_{m}}^{(\ell)}\) are the corresponding coefficients of the pQCD evolution equations. Eqs. \((26)\) represent \textit{definitions} of \(\beta_{k}\)'s \((2 \leq k \leq n_{m})\) via combinations of derivatives \(\partial^{n} A_{1}/\partial(\ln \mu^{2})^{n}\).

On the other hand, evolution equations \((21)-(23)\) (for \(k = 1\)) for the change of RSch remain of the same form, but with aforementioned truncation

\[
\frac{\partial A_{1}(\mu^{2}; \beta_{2}, \ldots)}{\partial \ln \mu^{2}} \approx \sum_{n=2}^{n_{m}} A_{n} \beta_{n}^{(2)} A_{n+1},
\]

\[
\frac{\partial^{2} A_{1}(\mu^{2}; \beta_{2}, \ldots)}{\partial (\ln \mu^{2})^{2}} \approx \sum_{n=2}^{n_{m}} A_{n} \beta_{n}^{(3)} A_{n+1}, \quad \text{etc.},
\]

where we have altogether \(n_{m} - 2\) equations, and \(k_{n_{m}}^{(\ell)}\) are the corresponding coefficients of the pQCD evolution equations. Eqs. \((27)\) are, in contrast to Eqs. \((26)\), not definitions, but in general approximations for the evolution under RSch-changes. The RSch-dependence of \(A_{1}(\mu^{2})\) is treated in more detail later in this work.

On the basis of Eqs. \((26)-(27)\), expressions for the (truncated) derivatives \(\partial A_{k}/\partial X\), for \(k \geq 2\) \((X = \ln \mu^{2}, \beta_{j})\), can be obtained.

In our approach, the basic space-like quantites are \(A_{1}(\mu^{2})\) of a given anQCD model (e.g., MA, M1, M2) and its logarithmic derivatives

\[
\tilde{A}_{n}(\mu^{2}) \equiv \frac{(1-\beta_{0})^{n-1}}{\beta_{0}^{(n-1)}(n-1)!} \partial^{n-1} A_{1}(\mu^{2}), \quad (n = 1, 2, 3, \ldots),
\]

whose pQCD analogs are

\[
\tilde{a}_{n}(\mu^{2}) \equiv \frac{(1-\beta_{0})^{n-1}}{\beta_{0}^{(n-1)}(n-1)!} \partial^{n-1} a(\mu^{2}), \quad (n = 1, 2, 3, \ldots).
\]

The quantities \((\tilde{A}_{1}(\mu^{2}), \tilde{A}_{2}(\mu^{2}), \tilde{A}_{3}(\mu^{2}), \ldots)\), all derived from \(A_{1}(\mu^{2}) \equiv \tilde{A}_{1}(\mu^{2})\), are known functions of the space-like momenta \(\mu\) in any chosen anQCD version in a given chosen RSch \((\beta_{2}, \beta_{3}, \ldots)\). On the basis of these quantities and the (truncated) evolution equations \((26)\), any higher order quantity \(A_{k}(\mu^{2})\) \((k \geq 2)\) can be constructed, in the given RSch. Further, (truncated) equations \((26)-(27)\) then give us the values of \(\tilde{A}_{k}(\mu^{2})\) and of \(A_{k}(\mu^{2})\) \((k \geq 1)\) in any other chosen RSch \((\beta_{2}, \beta_{3}, \ldots)\). We emphasize that in this approach, the higher order quantities \(A_{k}(\mu^{2})\) \((k \geq 2)\) are not basic, they are defined via Eqs. \((20)\) for convenience of having closer notational analogy with pQCD formulas (and \(a^{k} \leftrightarrow A_{k}\)). In these definitions \((26)\), as well as in \(\beta_{j}\)-running Eqs. \((27)\), we could have kept one more term \((\sim A_{n_{m}+1})\), in order to come closer to the exact analogy \(A_{k} = a^{k} + \text{NP}\) for \(k \geq 2\), where NP stands for nonperturbative contributions (nonanalytic functions of \(a\) at \(a = 0\)). However, this is not necessary, as argued below.

\(^{2}\) \(A_{k} = a^{k} + \text{NP}\) holds exactly for the construction Eq. \((19)\), i.e., the construction by Milton et al. [2] [3] [4] in MA.
The basic analytization rule we adopt will thus be
\[ \tilde{a}_n \rightarrow \tilde{A}_n \quad (n = 1, 2, \ldots), \] (30)
where \( \tilde{A}_n \) and \( \tilde{a}_n \) are defined in Eqs. (28) and (29), respectively.

At loop level \( n_{\text{max}} = n_m \), and in a chosen 'starting' RSch \((\beta_2, \beta_3, \ldots)\), the truncation ('tr') of the RGE-running of the pQCD coupling \( a(\mu^2) \) is in principle via Eq. (11) with \( j_{\text{max}} = n_{\text{max}} + 1 \) (\( a = a_{\text{tr}}, \tilde{a}_n = a_{n,\text{tr}} \)). The corresponding truncated \( \tilde{A}_n = \tilde{A}_{n,\text{tr}} \) are then
\[ \tilde{A}_n(\mu^2) = \tilde{a}_n + NP = \tilde{a}_n(\mu^2)(\infty) + NP + \mathcal{O}(\beta_0^{-n-1}a^{n+m}n) \quad (n = 1, 2, \ldots), \] (31)
and we assumed that we are in the class of the RSch's where \( \beta_j \sim \beta_0 \) in the large-\( \beta_0 \) limit. We recall that \( A_1 \equiv \tilde{A}_1 \) and \( \tilde{a}_1 = a \). The subscript \( (\infty) \) in Eq. (31) means that this is the quantity obtained by not truncating RGE beta-function (11), i.e., for \( j_{\text{max}} = \infty \) and keeping the same value of \( \Lambda \) in expansion (2) as in the case of the truncated beta-function (i.e., \( j_{\text{max}} = n_{\text{max}} + 1 \)). The second identity in Eq. (31) thus shows, as an additional reference, the magnitude of error committed due to the truncation of the beta-function. Definitions (26) of \( A_n \)'s then imply
\[ A_n(\mu^2) = a^n(\mu^2) + NP + \mathcal{O}(\beta_0^{-n-1}a^{n+m}n) \quad (n = 2, \ldots, n_m), \] (32)
Since the RGE-running (11) of \( a \) is truncated, we have \( a^n = a^n(\infty) + \mathcal{O}(\beta_0^{-n-1}a^{n+m}n) \), and relations (32) remain unchanged when \( a^n(\mu^2) \) there is replaced by \( a^n(\infty) \).

The \( \beta_j \)-running Eqs. (27) are also truncated, i.e., the RHS's there have errors \( \sim A_{n_{\text{max}}+1} \), so that the changes of RSch entail additional errors. It can be verified that this effect, when going from a chosen 'starting' RSch \((\beta_2, \beta_3, \ldots)\) to another RSch \((\beta_2', \beta_3', \ldots)\), modifies relations (34) to
\[ \tilde{A}_1(\equiv A_1(\mu^2)) = a(\mu^2) + \mathcal{O}(\beta_0^{-2}a^{n+m}) + NP, \]
\[ \tilde{A}_n(\mu^2) = \tilde{a}_n + \mathcal{O}(\beta_0^{-n}a^{n+m}) + NP \quad (n = 2, \ldots, n_m), \] (33)
while relations (32) do not get modified. We should keep in mind that there is yet another truncation involved, namely in the solution (2) of RGE (11) the sum over index \( k \) has in the calculational practice finite number of terms. In our calculations, we will take there \( k_{\text{max}} = n_{\text{max}} + 2 \) (= \( j_{\text{max}} + 1 \)), which is so high that it does not affect "precision estimate" relations (33) and (32).

For example, at loop level three \( (n_{\text{max}} = 3) \), where we include in RGE (11) term with \( j_{\text{max}} = 4 \) (thus \( \beta_2 \)), relations (34) are
\[ \tilde{A}_2(\mu^2) = A_2(\mu^2) + \frac{\beta_3}{\beta_0}A_3(\mu^2), \quad \tilde{A}_3(\mu^2) = A_3(\mu^2), \] (34)
implying
\[ A_2(\mu^2) = \tilde{A}_2(\mu^2) - \frac{\beta_3}{\beta_0}\tilde{A}_3(\mu^2), \quad A_3(\mu^2) = \tilde{A}_3(\mu^2). \] (35)
The RSch \((\beta_2)\) dependence is obtained from the truncated Eqs. (28) and (29)
\[ \frac{\partial\tilde{A}_j(\mu^2; \beta_2)}{\partial \beta_2} = \frac{1}{2\beta_0^3} \frac{\partial^2\tilde{A}_j(\mu^2; \beta_2)}{\partial(\ln \mu^2)^2} \left( \equiv \frac{1}{\beta_0} \tilde{A}_3(\mu^2; \beta_2) \right) \quad (j = 1, 2, \ldots), \] (36)
where \( \tilde{A}_1 \equiv A_1 \). These are second order approximate partial differential equations for \( A_1(\mu^2; \beta_2), \tilde{A}_2(\mu^2; \beta_2), \tilde{A}_3(\mu^2; \beta_2) \). Higher order terms \( \sim \tilde{A}_4 \) are neglected on the right-hand side of the RSch-evolution equation (36).

At loop level four \( (n_{\text{max}} = 4) \), where we include in RGE (11) term with \( j_{\text{max}} = 5 \) (thus \( \beta_3 \)), relations analogous to (35) are
\[ A_2(\mu^2) = \tilde{A}_2(\mu^2) - \frac{\beta_3}{\beta_0}A_3(\mu^2) + \left( \frac{5}{2} \frac{\beta_1}{\beta_0^2} - \frac{\beta_2}{\beta_0} \right) \tilde{A}_4(\mu^2), \]
\[ A_3(\mu^2) = \tilde{A}_3(\mu^2) - \frac{5}{2} \frac{\beta_1}{\beta_0}A_4(\mu^2), \quad A_4(\mu^2) = \tilde{A}_4(\mu^2), \] (37)
while the changes of the RSch are governed by (approximate) relations

$$\frac{\partial \tilde{A}_j(\mu^2)}{\partial \beta_2} = \left[ \frac{1}{21 \beta_0^0} \frac{\partial^2}{\partial (\ln \mu^2)^2} + \frac{5}{3!} \frac{\beta_1}{\beta_0^0} \frac{\partial^3}{\partial (\ln \mu^2)^3} \right] \tilde{A}_j(\mu^2),$$

$$\frac{\partial \tilde{A}_j(\mu^2)}{\partial \beta_3} = -\frac{1}{3!} \frac{\partial^3 \tilde{A}_j(\mu^2)}{\partial (\ln \mu^2)^3} \quad (j = 1, 2, \ldots).$$

(38)

Our approach is in a sense maximally truncating. Namely, the evolution under the changes of the RSch is truncated in such a way that $\partial \mathcal{D}_{\text{an.r}}(Q^2)/\partial \beta_j \sim A_{n_{\text{max}}+1}$. Further, our definition of $A_k$’s ($k \geq 2$) via Eqs. (34) [cf. Eqs. (35) and (37)] involve short (“truncated”) series which, however, still ensure the correct RScl-dependence $\partial \mathcal{D}_{\text{an.r}}(Q^2)/\partial \mu^2 \sim A_{n_{\text{max}}+1}$. Furthermore, it may seem that, for loop level three ($n_{\text{max}} = 3$), the RHS of the first of Eqs. (34) represents only two perturbative terms $[a^2 + (\beta_1/\beta_0)a^3]$ plus nonperturbative terms (NP). However, since taking $j_{\text{max}} = n_{\text{max}} + 1 = 4$ in RGE (1) as the basis for calculation of $A_1(\mu^2)$, it is straightforward to show that the following holds:

$$\left( \tilde{A}_2(\mu^2) = \frac{\beta_1}{\beta_0} A_3(\mu^2) = a^2(\mu^2) + \frac{\beta_1}{\beta_0} a^3(\mu^2) + \frac{\beta_2}{\beta_0} a^4(\mu^2) + \mathcal{O}(\beta_0^2 a^5) + \text{NP} \right).$$

(39)

Completely analogous result holds at loop level 4 ($n_{\text{max}} = 4$ and $j_{\text{max}} = 5$).

In the MA QCD, in the approach of [2], here Eq. (19) for $A_k$, a truncation is performed only in expansion (2) for $a \leftarrow \rho^{(pt)}(\sigma)$, apparently with $k_{\text{max}} = n_{\text{max}}$, and then powers of this truncated $a$ are used to define $\rho^{(pt)}$ and thus $A_k$ ($k \geq 2$). This implies that in the MA QCD our $A_k$’s ($k = 2, \ldots$), on the one hand, and those of the approach of Milton, Solovtsov, Solovtsova, and Shirkov (MSSSh) [2, 3, 4], on the other hand, are not the same, although they must gradually merge when the loop level is increased. This is illustrated in Figs. 1 and 2 where the MA-coupling

![Diagram](https://via.placeholder.com/150)

**FIG. 1:** The coupling parameters $A_2(Q^2)$ and $A_3(Q^2)$ in MA in $\overline{\text{MS}}$ RSch, with $n_f = 3$ and $\overline{\Sigma}(n_f=3) = 0.4$ GeV, calculated at (a) loop-level=3 (and $k_{\text{max}} = 5$), and (b) loop-level=4 (and $k_{\text{max}} = 6$). Presented are results of construction of Milton et al. (MSSSh) [2, 3, 4], and of our construction.

parameters $A_2(Q^2)$ and $A_3(Q^2)$ of both approaches are compared, for $n_f = 3$, at loop-level ($= n_{\text{max}}$) three and four, in $\overline{\text{MS}}$ and in RSch A, respectively. The Adler (A) RSch is defined later in Eqs. (35) [cf. Eq. (34)]. For both $A_2$ and $A_3$ one can see a decrease in the absolute difference between our and MSSSh methods when going from loop-level=3 to 4, Fig. 1 in $\overline{\text{MS}}$ RSch, and Fig. 2 in RSch A. The decrease can be understood as coming largely from the fact that the perturbative part of this difference is $\mathcal{O}(a^3)$ when loop-level=3, and $\mathcal{O}(a^5)$ when loop-level=4. Further, inspection

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3 When the anQCD is not MA, but rather M1 or M2, RGE and the (truncated) expansion still remain the basis for calculation of the MA-part of $A_1(\mu^2)$, the difference between $A_1(\mu^2)$ and $A_1^{(\text{MA})}(\mu^2)$ being purely nonperturbative, cf. Eqs. (12), (13), (18).
of Figs. 1(a) and 2(a) reveals that the $A_2$-curves practically merge already at loop-level=3 if RSch is MS, but less so if RSch is A. An indication towards understanding this resides in the fact that the coefficient at $a^4$ of the difference between the two curves is proportional to $(2\beta_0\beta_2 - 5\beta_1^2)$, this being in MS about one fifth of the corresponding value in RSch A (when $n_f = 3$). In Fig. 3 the coupling parameters $A_1(Q^2)$ and $A_2(Q^2)$ of anQCD models M1, M2 and MA are presented as functions of the scale $Q$, for specific chosen fixed parameters of the models M1 and M2 (see Sec. V) and in the aforementioned specific RSch A. Note that we used $k_{\text{max}} = n_{\text{max}} + 2$ in the calculation of $\rho^{(pt)}_1$ via Eq. (2) in all cases, i.e., also in the MSSSh cases. In Fig. 3 loop-level=4 and $k_{\text{max}} = 6$ was taken (using our described approach). In Figs. 1, 2, 3 the basis for calculation was the $k_{\text{max}}$-truncated series (2) in the corresponding RSch.

Even when already having anQCD coupling $A_1(Q^2)$, there is no unique way to merge analyticity requirements with the perturbative results at higher orders, i.e., Eq. (32) for $A_k(Q^2)$ ($k \geq 2$). The latter relations are ensured by our definitions of $A_k(Q^2)$ for $k \geq 2$ via relations (26), but this is just one of the possibilities of addressing the problem. In MA the construction of $A_1(Q^2)$ is very closely related to the perturbative solution $a(Q^2)$ via dispersion relation
Therefore, it is very natural to keep that close analogy at higher orders, via dispersion relations \[19\]. As a consequence, the RGE-type of relations \[20\] are fulfilled in MA \[16\]. For a general an\(n\)QCD model, this approach does not apply. Deviations of \(A_1(Q^2)\) and \(A_1(s)\) from their MA values imply that the discontinuity function \(\rho_k(\sigma)\) deviates from its MA analog \(\rho^{(pt)}(\sigma) = \text{Im}(-\sigma - i\epsilon)\) at low values of \(\sigma\), cf. Eqs. \[10\], \[11\] and \[17\]. Therefore, there is no direct natural way of prescribing the low-\(\sigma\) behavior of the higher order discontinuity functions \(\rho_k(\sigma)\) appearing in the dispersion relations of the type of Eq. \[19\] for \(A_k\), i.e., prescribing their deviations from \(\rho^{(pt)}_k(\sigma) = \text{Im}a^k(-\sigma - i\epsilon)\) for \(k \geq 2\). We define \(A_k(Q^2)\) for \(k \geq 2\) by forcing them to obey the truncated RGE-type relations \[20\]. We emphasize that these relations define, in our approach, the couplings \(A_k(Q^2)\) for \(k \geq 2\). Thus, we indirectly define the corresponding discontinuity functions \(\rho_k\). This construction of \(A_k\)’s is motivated also by the skeleton approach as discussed in Ref. \[12\]. Furthermore, as we will see later, this construction of \(A_k\)’s allows us to suppress systematically the RScl- and RSch-dependence in the evaluated observables with the increasing order, because an RGE-type of analogy with pQCD is being preserved.

IV. SKELETON-MOTIVATED EXPANSION

Consider an observable \(D(Q^2)\) depending on a single space-like physical scale \(Q^2(\equiv -q^2) > 0\). Its perturbation expansion has the form

\[
D(Q^2)_{\text{pt}} = a + d_1 a^2 + d_2 a^3 + \cdots ,
\]

where \(a = a(\mu^2; \beta_2, \ldots)\) is taken at a given RScl (\(\mu\)) and RSch (\(\beta_2, \ldots\)). As mentioned before, we will take the convention \(\Lambda = \vec{\Lambda}\), i.e., the \(\overline{\text{MS}}\) QCD scale as the reference scale for \(\mu\) [cf. Eq. \[2\]-\[3\]]. Further, we will work in the following classes of RSch: each \(\beta_k (k \geq 2)\) is a polynomial in \(n_f\) of order \(k\); equivalently, it is a polynomial in \(\beta_0\):

\[
\beta_k = \sum_{j=0}^{k} b_{kj} \beta_0^j , \quad k = 2, 3, \ldots
\]

The \(\overline{\text{MS}}\) clearly belongs to this class of schemes. In such schemes, the coefficients \(d_n\) of expansion \[40\] have the following specific form in terms of \(\beta_0\), as can be deduced from the scheme independence of observable \(D(Q^2)\), e.g. by using relations of Ref. \[29\]:

\[
d_1 = c_{11}^{(1)} \beta_0 + c_{10}^{(1)} , \quad d_n = \sum_{k=-1}^{n} c_{nk}^{(1)} \beta_0^k ,
\]

i.e., each \(d_n\) is a polynomial of order \(n\) in \(\beta_0\) and includes in general, in addition, a term with the negative power \(1/\beta_0\) (\(d_1\) does not have it). In the \(\overline{\text{MS}}\) scheme, the negative powers do not occur.

We will now construct a separation of the series \[40\] into a sum of RScl-independent subseries

\[
D(Q^2)_{\text{pt}} = D^{(1)}(Q^2)_{\text{pt}} + \sum_{n=2}^{\infty} k_n D^{(n)}(Q^2)_{\text{pt}} ,
\]

with the following properties: (a) each dimensionless constant \(k_n\) is RScl-independent; (b) each subseries \(D^{(n)}_{\text{pt}} (n \geq 1)\) is RScl-independent, and it is normalized so that \(D^{(n)}_{\text{pt}} = a^n + \mathcal{O}(a^{n+1})\); (c) the subseries \(D^{(n)}(Q^2)_{\text{pt}}\) contains all the leading-\(\beta_0\) coefficients of the following “rest”:

\[
\frac{1}{k_n} \left[ D(Q^2)_{\text{pt}} - D^{(1)}(Q^2)_{\text{pt}} - \cdots - k_{n-1} D^{(n-1)}(Q^2)_{\text{pt}} \right] .
\]

We will show that these conditions uniquely determine factors \(k_n\) and perturbation expansions of all \(D^{(n)}(Q^2)\). Further, we show in Appendix \[14\] that the above subseries, which always exist, would coincide with the expansions of the corresponding skeleton terms in the skeleton expansion of the observable if such an expansion existed in the considered RSch.

We consider first the leading-\(\beta_0\) part of expansion \[40\]

\[
D^{(1)}_{\text{pt}}(Q^2) = a + \sum_{j=2}^{\infty} a^j \left[ c_{jj}^{(1)} \beta_0^j \right] .
\]
Under the change of RScl from $\mu^2$ to $\mu^2_{\ast}$, using the notation $L_{\ast} \equiv \ln(\mu^2_\ast/\mu^2)$, we have by RGE (1)

$$a = a_{\ast} + \sum_{n=1}^{\infty} a_{s+n+1} a^n_{s} L_{\ast}^n$$

$$= a_{\ast} + a_{s}^2 \beta L_{\ast} + a_{s}^3(\beta^2 L_{\ast}^2 + \beta L_{\ast}) + a_{s}^4 \left(\beta^3 L_{\ast}^3 + \frac{5}{2} \beta \beta_1 L_{\ast}^2 + \beta_2 L_{\ast}\right) + \cdots \,,$$

where $a \equiv a(\mu^2)$ and $a_{\ast} \equiv a(\mu^2_{\ast})$. Inserting this into expansion (49) we obtain the transformation rules for the coefficients $c^{(1)}_{ij}$ under the change of RScl. Specifically, for the diagonal coefficients the transformations are

$$c^{(1)}_{\ast kk} = \sum_{s=0}^{k} \left( \begin{array}{c} k \\ s \end{array} \right) L_{\ast}^s c^{(1)}_{kk} \,,$$

where we use the notations $c^{(1)}_{\ast ij} \equiv c^{(1)}_{ij}(\mu^2)$ and $c^{(1)}_{\ast ij} \equiv c^{(1)}_{ij}(\mu^2_{\ast})$ (and $c^{(1)}_{\ast 00} = 1$ by definition). Inserting expansion (46) into expansion (45) we obtain

$$D^{(1)}(Q^{2})_{pt} = a_{\ast} + a_{s}^2 \left[ \beta_0 c^{(1)}_{11} \right] + a_{s}^3 \left[ \beta_0^2 c^{(1)}_{22} + \beta_1 c^{(1)}_{11} \right] + a_{s}^4 \left[ \beta_0^3 c^{(1)}_{33} + O(\beta_0^4 a^5) \right] .$$

This implies that the leading-$\beta_0$ series (45) does not maintain its form under the change of RScl, since a new term $a_{s}^3 \beta_1(c^{(1)}_{11} - c^{(1)}_{00})$ appears at $\sim a^3$. The RScl"covariant" form, up to $\sim a^3$, is then

$$D^{(1)}(Q^{2})_{pt} = a + a_{s}^2 \left[ \beta_0 c^{(1)}_{11} \right] + a_{s}^3 \left[ \beta_0^2 c^{(1)}_{22} + \beta_1 c^{(1)}_{11} \right] + a_{s}^4 \left[ \beta_0^3 c^{(1)}_{33} + O(\beta_0^4 a^5) \right] .$$

We now iteratively repeat the procedure: we insert expansion (46) into expansion (49) and, after some algebra and using relations (47), obtain

$$D^{(1)}(Q^{2})_{pt} = a + a_{s}^2 \left[ \beta_0 c^{(1)}_{11} \right] + a_{s}^3 \left[ \beta_0^2 c^{(1)}_{22} + \beta_1 c^{(1)}_{11} \right] + a_{s}^4 \left[ \beta_0^3 c^{(1)}_{33} + \beta_0 \beta_1 (c^{(1)}_{22} - c^{(1)}_{22}) + \beta_2 (c^{(1)}_{11} - c^{(1)}_{11}) \right] + O(\beta_0^4 a^5) .$$

The new terms appearing at $\sim a_{s}^2$ here require the following restoration of the RScl"covariance" up to order $\sim a_{s}^5$.

$$D^{(1)}(Q^{2})_{pt} = a + a_{s}^2 \left[ \beta_0 c^{(1)}_{11} \right] + a_{s}^3 \left[ \beta_0^2 c^{(1)}_{22} + \beta_1 c^{(1)}_{11} \right] + a_{s}^4 \left[ \beta_0^3 c^{(1)}_{33} + \beta_0 \beta_1 (c^{(1)}_{22} - c^{(1)}_{22}) + \beta_2 (c^{(1)}_{11} - c^{(1)}_{11}) \right] + O(\beta_0^4 a^5) .$$

This procedure can be continued to any required order. Expression (51) is now the RScl"covariant" leading-$\beta_0$ part of the full perturbation expansion (49). This means that it keeps its form (51) under any change of RScl $\mu^2$. Variations of $a = a(\mu^2)$ and of $c_{kk} = c_{kk}(\mu^2)$ under the RScl variation are governed by the RScl-invariance of the entire observable $D$ and of its perturbation expansion (49), as reflected by relations (46) and (47). The additional terms appearing in expansion (51), in comparison with the original leading-$\beta_0$ series (45), are subleading in $\beta_0$ and represent effects beyond one loop involving diagonal coefficients $c^{(1)}_{kk}$. As shown in Appendix B the covariant leading-$\beta_0$ expansion (51) is the expansion of the leading skeleton (LS) term in an assumed skeleton expansion of the observable $D$.

Now we subtract the LS expansion (51) from expansion (49), and the difference now involves only subleading-$\beta_0$ terms

$$\left[ D(Q^{2})_{pt} - D^{(1)}(Q^{2})_{pt} \right] = k_2 \left[ a_{2} + \sum_{n = 1}^{\infty} a_{n}^2 d^{(2)}_{n} \right] \,,$$

where the coefficients $d^{(2)}_{n}$ have a structure similar to that of $d_{n}$'s (42)

$$d^{(2)}_{n} = \sum_{k = -1}^{n} c_{nk}^{(2)} a_{0}^{k} \, \beta_{0}^{k} \, (n = 1, 2, \ldots) .$$
Coefficients $c^{(2)}_{ij}$ are related to the original coefficients $c^{(1)}_{ij}$ by relations

\[
\begin{align*}
c_{10}^{(1)}c_{ij}^{(2)} &= c_{ij}^{(1)} - b_{ij}c_{11}^{(1)} \quad (j = 1, 0, -1), \\
c_{10}^{(1)}c_{2j}^{(2)} &= c_{3j}^{(1)} - \frac{5}{2} b_{1j} - c_{22}^{(1)} - b_{2j}c_{11}^{(1)} \quad (j = 2, 1, 0, -1),
\end{align*}
\]

(54)

and coefficients $b_{kj}$ are those of the expansion of $\beta_k$ coefficients (11) in powers of $\beta_0$ (including the case $k = 1$). Specifically, we have $b_{k,-1} = 0$ ($k = 1, 2, \ldots$). For $k = 1$, we have: $b_{11} = 19/4$ and $b_{10} = -107/16$, both numbers being RScl-independent. Now we repeat the previous construction, but now for the (canonically normalized) rest (1/k2)/D - D(1)) of Eq. (52) instead of $D(0)$. Its RScl-covariant leading-$\beta_0$ part $D^{(2)}$ then turns out to give

\[
k_2D^{(2)}(Q^2)_{pt} = k_2\{a^2 + a^3 \left[\beta_0c_{11}^{(2)}\right] + a^4 \left[\beta_0^2c_{22}^{(2)} + \beta_1c_{11}^{(2)}\right] + O(\beta_0^3a^5)\}.
\]

(55)

Subtracting this from the rest (52), we obtain

\[
[\mathcal{D}(Q^2)_{pt} - \mathcal{D}^{(1)}(Q^2)_{pt} - k_2\mathcal{D}^{(2)}(Q^2)_{pt}] = k_3 \left[ a^3 + \sum_{n \geq 1} a^{n+3d^{(3)}_n} \right],
\]

(56)

\[
k_3 = c_{10}^{(1)} \left(c_{10}^{(2)} + \frac{1}{\beta_0} c_{11}^{(2)}\right),
\]

(57)

\[
d^{(3)}_1 = \beta_0(c_{21}^{(2)} - b_{11}c_{11}^{(2)})/c_{11}^{(2)} + k_4/k_3,
\]

(58)

where $k_4/k_3$ is a number $\sim \beta_0^2$ which will be given explicitly below. The (RScl-covariant) leading-$\beta_0$ part $D^{(3)}$ of the canonically normalized expression (1/k3)/(D - D(1)) - k2D(2)) gives

\[
k_3D^{(3)}(Q^2)_{pt} = k_3 \left\{a^3 + a^4 \left[\beta_0c_{11}^{(3)}\right] + O(\beta_0^3a^5)\right\},
\]

(59)

\[
c_{10}^{(2)}c_{11}^{(3)} = (c_{21}^{(2)} - b_{11}c_{11}^{(2)}).
\]

(60)

Defining

\[
\mathcal{D}^{(4)}(Q^2)_{pt} = a^4 + O(\beta_0a^5),
\]

(61)

and following the procedure pattern, we subtract expression (59) from expression (60) and obtain

\[
\mathcal{D}(Q^2)_{pt} = \mathcal{D}^{(1)}(Q^2)_{pt} + k_2\mathcal{D}^{(2)}(Q^2)_{pt} + k_3\mathcal{D}^{(3)}(Q^2)_{pt} + k_4\mathcal{D}^{(4)}(Q^2)_{pt} + O(\beta_0^3a^5),
\]

(62)

where perturbation expansions for $\mathcal{D}^{(j)}$'s are given by (51), (55), (59), (61); coefficients $k_2$ and $k_3$ are given by Eqs. (62) and (57); coefficients $c_{ij}^{(1)}$, $c_{ij}^{(2)}$, $c_{ij}^{(3)}$ are given by Eqs. (42), (51), (60); and an explicit expression for the coefficient $k_4$ is

\[
k_4 = c_{10}^{(1)} \left[c_{20}^{(2)} - b_{10}c_{11}^{(2)} - c_{11}^{(2)}(c_{21}^{(2)} - b_{11}c_{11}^{(2)})/c_{11}^{(2)} + \frac{1}{\beta_0} c_{2,-1}^{(2)}\right].
\]

(63)

It is straightforward to check that all the coefficients $k_2$, $k_3$, $k_4$ are RScl-independent [as are the subseries $\mathcal{D}^{(j)}(Q^2)$]. Thus, identity (62), obtained by our construction, represents identity (45) to order $n = 4$. This construction can be continued to any order.

In practice, we know only all the leading-$\beta_0$ parts of the coefficients $d_j$ of observable $\mathcal{D}(Q^2)$ Eq. (40), i.e., all the coefficients $c_{ij}^{(1)}$; and in addition, we usually know only one, two or three full coefficients ($d_1$, $d_2$, and possibly $d_3$). This implies that the first term $\mathcal{D}^{(1)}$ on the RHS of identity (52) is known to all orders, while the other terms ($\mathcal{D}^{(2)}$, $\mathcal{D}^{(3)}$, and possibly $\mathcal{D}^{(4)}$) are known only in their truncated version. This means that the rest term in Eq. (52) is, in such a case, $O(\beta_0^3a^5)$, not $O(\beta_0^6a^5)$.

The perturbation expansion $\mathcal{D}^{(1)}_{pt}$ of the “leading-skeleton” (LS) term can be written in a resummed form

\[
\mathcal{D}^{(1)}(Q^2)_{pt} = \int_0^\infty \frac{dt}{t} F_0^E(t) a(tE^2Q^2),
\]

(64)
where \( F_D^E(t) \) is the LS-characteristic function\(^4\) which often can be written in a closed explicit form\(^31\). In principle, \( F_D^E(t) \) can be obtained for any space-like observable whose leading-\( \beta_0 \) parts \( c_{nk}^{(1)} \) of all coefficients are known. The value of \( C \) in \( (61) \) depends on the value of the reference scale \( \Lambda \) used in the RGE-running; in our convention, as mentioned before, we use \( \Lambda = \tilde{\Lambda} \) which corresponds to \( C = \tilde{C} \equiv -5/3 \).

At this point, we will turn to the question of the RSch-dependence of the (truncated) perturbation series \( (62) \). The RSch independence of the series \( (40) \) implies specific transformation rules of the expansion coefficients \( d_j \) under the change of \( \beta_j \)'s \((j \geq 2)\) \(29\)

\[
\begin{align*}
  d_1 &= \overline{d}_1, \\
  d_2 &= \overline{d}_2 - \frac{1}{\beta_0} (\beta_2 - \overline{\beta}_2), \\
  d_3 &= \overline{d}_3 - 2\overline{d}_1 \frac{1}{\beta_0} (\beta_2 - \overline{\beta}_2) - \frac{1}{2\beta_0} (\beta_3 - \overline{\beta}_3), \ldots
\end{align*}
\]

(65)

where the bars denote the values with \( \overline{\text{MS}} \) RSch parameters \( \beta_k = \overline{\beta}_k = \sum k_j \beta_j^\mu \), and unchanged RScl. This implies, in view of relations \( (42), (54), (60) \), specific transformation rules for \( c_{nk}^{(3)} \) coefficients. We will consider that the first term in skeleton-motivated expansion \( (62) \) has a known characteristic function, cf. Eq. \( (61) \), and that at most the first three nonleading coefficients of the perturbation expansion \( (40) \) of observable \( D \) are known: \( \overline{d}_1, \overline{d}_2, \) and \( \overline{d}_3 - \overline{\text{MS}} \) RSch and at RScl \( \mu^2 = Q^2 \). Each term in expansion \( (62) \) is RScl-independent, we can re-expand each \( D^{(j)}(Q^2)_{\text{pt}} \) \((j \geq 2)\) in powers of \( a(Q^2) \), i.e., at different chosen RScl’s \( Q_j \), in a chosen common RSch \( (\beta_2, \beta_3, \ldots) \). The resulting subseries, however, will now be truncated since \( d_j \)'s for \( j \geq 4 \) are not known. This leads to the following form of the skeleton-motivated expansion \( (62) \):

\[
D(Q^2)_{\text{pt}} = D(Q^2)_{\text{TPS}} + O(\beta_0^3 a^5),
\]

(66)

\[
D(Q^2)_{\text{TPS}} = D^{(1)}(Q^2) + t_2^{(2)} a^2(Q^2) + \sum_{j=2}^{3} t_j^{(j)} a^3(Q^2) + \sum_{j=2}^{4} t_j^{(j)} a^4(Q^2),
\]

(67)

where the coefficients \( t_j^{(j)} \) depend on the scale ratios \( Q_j^2/Q^2 \) and the RSch parameters \( \beta_k \) \( (41) \), and are written explicitly in Appendix \( A \) in terms of the coefficients \( \tilde{\sigma}_j^{(1)} \), the latter comprising via Eq. \( (42) \) the coefficients \( \tilde{\sigma}_n \) of the original perturbation series \( (40) \) in \( \overline{\text{MS}} \) RSch and at the RScl \( \mu^2 = Q^2 \).

We now turn to the question of analytization of the perturbation series \( (67) \), within a given anQCD model with known analytic couplings \( A_k \), Eqs. \( (5), (28)\).\(^{[37]}\). For the first (LS) term, the natural analytization procedure is to replace the perturbative coupling \( a(t e^C Q^2) \) by its anQCD counterpart \( A_1(t e^C Q^2).\)\(^5\)

\[
D^{(1)}(Q^2)_{\text{an}} = D^{(\text{LS})}(Q^2) = \int_0^\infty \frac{dt}{t} \ F_D^E(t) A_1(t e^C Q^2).
\]

(68)

In contrast to expression \( (64) \) which is an ill-defined integral due to the Landau singularities of \( a \), expression \( (68) \) is a well-defined integral in any given anQCD model \{unless \( A_1(Q^2) \) diverges too strongly when \( Q^2 \to 0 \). We can adopt the viewpoint that any anQCD model is defined: (a) by a specific expression for \( A_1(Q^2) \), and (b) by prescription \( (65) \) for calculation of the LS-terms of any space-like observable. The analytization of the other terms in \( (67) \), after the choice of an anQCD model, i.e., of \( A_3(Q^2) \), can be performed in different ways. For example, the replacements \( a^k(Q^2_j) \to A_k^E(Q^2_j), A_k^{k-2}(Q^2_j), A_2(Q^2_j), \ldots, A_k(Q^2_j) \) all appear equally natural at first, since the perturbative parts of these expressions are all the same to the order considered – cf. relations \( (62) \) and \( (63) \). However, construction of the higher order couplings \( A_k (k \geq 2) \) on the basis of the anQCD coupling \( A_1 \), as presented in Sec. \( III \), suggests that it is the replacement

\[
\left[ \tilde{a}_k(Q^2_j) \to \tilde{A}_k(Q^2_j) \Rightarrow a^k(Q^2_j) \to A_k(Q^2_j) \right. \quad (k \geq 1)
\]

(69)

---

\(^4\) The superscript \( E \) means “Euclidean”, since the scales involved \( (Q^2, t e^C Q^2) \) are space-like.

\(^5\) A different approach to considering the perturbative LS term \( (62) \) was developed by the authors of Ref. \( [33] \). They present a novel version of the leading-\( \beta_0 \) renormalon calculus, and consider that an OPE-term exists whose \( Q^2 \)-dependence is the same as that of the renormalon ambiguity of the perturbative LS term and that the ambiguity cancels in the sum (“PT+NP”). This sum can be presented in the LS form \( (64) \) with the perturbative coupling \( a(t e^C Q^2) \) there replaced by a modified (but nonanalytic) coupling with one parameter. Since they work in the OPE framework, the latter parameter is observable-dependent.
that appears to be the most natural from the point of view of the requirement of the RScL- and RScCh-invariance of the observables. Namely, \( A_k(\mu^2; \beta_2, \ldots) \)'s fulfill, to the order considered, the same evolution equations under the changes of the RScL and of RScCh as \( a_k(\mu^2; \beta_2, \ldots) \)'s when the replacements (69) are performed everywhere. Further, the LS-analytization (68) of the first term \( D^{(1)}_{pt} \) of (67) is also equivalent to the term-by-term analytization (69) of the perturbation expansion of \( D^{(1)}_{pt} \), as is explicitly shown in Appendix B. The analytization (69) of the TPS (67), which results in the “truncated analytic series” (TAS)

\[
D(Q^2) = D(Q^2)_{TAS} + O(\beta^5_0 A_5)
\]

\[
D(Q^2)_{TAS} = D^{(LS)}(Q^2) + t_2^{(2)} A_2(Q^2) + \sum_{j=2}^3 t_j^{(j)} A_j(Q^2) + \sum_{j=2}^4 t_j^{(j)} A_j(Q^2)
\]

has, as a consequence, the suppression of the RScL- and RScCh-dependence just as is known for the corresponding TPS in pQCD, but with \( a_k \rightarrow A_k \):

\[
\frac{\partial D(Q^2)_{TAS}}{\partial \ln Q^2_j} = O(\beta^{5-j}_0 A_5) \quad (j = 2, 3, 4),
\]

\[
\frac{\partial D(Q^2)_{TAS}}{\partial \beta_k} \leq O(\beta^{3-k}_0 A_5) \quad (k = 2, 3).
\]

We are allowed, in principle, to vary in the TAS series (71) three different RScL’s \( Q_j \) and 3 + 4 RScCh parameters \( b_{2j} \) and \( b_3 \) appearing in \( \beta_2 \) and \( \beta_3 \). One may want to have, for given chosen RScL’s \( Q_j \), such a RScCh that effectively only the first coefficient \( t_2^{(2)} \) in the beyond-the-LS contribution is nonzero. This implies various conditions involving the other five \( t_j^{(j)} \)'s [Eqs. (A4)-(AS)]:

\[
t_3^{(2)} = t_3^{(3)} = 0; \quad \sum_{j=2}^4 t_j^{(j)} = 0,
\]

\[
\Rightarrow D(Q^2) = D^{(LS)}(Q^2) + t_2^{(2)} A_2(Q^2) + O(\beta^3_0 A_5).
\]

Specifically, if we choose for all three \( D^{(j)}(Q^2; \mu^2 = Q^2_j)_{TAS} \) \( (j = 2, 3, 4) \) the same RScL

\[
Q_2^2 = Q_3^2 = Q_4^2 = Q^2 \exp(C),
\]

the corresponding \( \beta_k = b_{kj} \beta^j_0 \) \( (k = 2, 3) \) have the following \( \delta b_{kj} \equiv b_{kj} - \overline{b}_{kj} \):

\[
\delta b_{22} = \pi^{(1)}_{10}(\pi^{(2)}_{11} + 2C),
\]

\[
\delta b_{23} = \pi^{(1)}_{10}c_{10}, \quad \delta b_{20} = 0,
\]

\[
\frac{1}{2} \delta b_{33} = \pi^{(1)}_{10}(\pi^{(2)}_{22} + 3\pi^{(1)}_{12}c_{12} + C) - \delta b_{22}3\pi^{(1)}_{11} + C),
\]

\[
\frac{1}{2} \delta b_{32} = \pi^{(1)}_{10}(\pi^{12}_{21} + C(3\pi^{(1)}_{10}\pi^{(2)}_{10} + 2b_{11}c_{11}) - \delta b_{22}2\pi^{(1)}_{10} - \delta b_{21}3\pi^{(1)}_{11} + C),
\]

\[
\frac{1}{2} \delta b_{31} = \pi^{(1)}_{10}(\pi^{(2)}_{20} + 2b_{10}\pi^{(1)}_{10} - \delta b_{21}2\pi^{(1)}_{10} - \delta b_{20}3\pi^{(1)}_{11} + C),
\]

\[
\frac{1}{2} \delta b_{30} = -\delta b_{20}2\pi^{(1)}_{10}(= 0).
\]

Here, \( c_{ij}^{(k)} \equiv c_{ij}^{(k)}(\mu^2 = Q^2; \overline{\text{MS}}) \). Results (77)-(82) are obtained by using explicit expressions (A4)-(AS) obtained in Appendix A, applying to them conditions (74) for the RScL choice (76). Specifically, result (77) is obtained by the requirement \( t_3^{(2)} = 0 \); results (78) from the requirement \( t_3^{(3)} = 0 \), being zero both the coefficient at \( \beta_0^3 \) and at \( 1/\beta_0 \), respectively; results (79)-(82) are obtained from requirement \( \sum t_j^{(j)} = 0 \), being zero all the coefficients at the \( \beta_0 \)-powers \( \beta_0^3, \beta_0^4, \beta_0^5, 1/\beta_0 \), respectively.

Our evaluation method (71), with the choice of the scheme described above [Eqs. (75), (76), (77)-(82)], emphasizes in the beyond-the-LS parts the role of the analytic couplings \( A_k(\mu^2) \) \( (k \geq 2) \) constructed in Sec. III from the couplings \( \overline{A}_n(\mu^2) \), Eq. (28) [see Eqs. (37)]. The couplings \( A_k(\mu^2) \) \( (k \geq 2) \) were constructed in such a way as to have, at
perturbative level, their equivalence with $a^n(\mu^2)$. However, the construction in Sec. III strongly suggests that the couplings $\tilde{A}_n(\mu^2)$ ($n \geq 2$) are more basic since they are constructed as derivatives of $A_1(\mu^2)$ which is the basic quantity in any anQCD model. Further, the skeleton-expansion arguments presented in Appendix B show that $\tilde{A}_n(\mu^2)$ are the basic elements for the expansion of each term in the skeleton expansion. Therefore, a more natural choice for RSch $(\beta_2, \beta_3)$ in the evaluation method (71), with RScl's (70), would be such that the resulting TAS expression is

$$D(Q^2) = D(Q^2)_{(TAS)} + O(\beta_0^3 A_5) ,$$  (83)

$$D(Q^2)_{(TAS)} = D^{(LS)}(Q^2) + \tilde{t}_3 \tilde{A}_2(Q^2 e^C) .$$  (84)

To obtain the $β_k$'s $(k = 2, 3)$ necessary for this result, we first re-express all $A_k$'s $(k \geq 2)$ in TAS (71) in terms of $\tilde{A}_n$'s, Eqs. (37). Keeping the RScl's according to (70), this implies that, in a general RSch $(β_2, β_3)$ expression (71) can be re-expressed as

$$D(Q^2)_{(TAS)} = D^{(LS)}(Q^2) + \tilde{t}_3 \tilde{A}_2(Q^2 e^C) + \tilde{t}_3 \tilde{A}_3(Q^2 e^C) + \tilde{t}_4 \tilde{A}_4(Q^2 e^C) ,$$  (85)

where the coefficients $\tilde{t}_3$ are certain combinations of $t_s^{(k)}$, and are written explicitly in Appendix A, Eqs. (A16)-(A21). Requiring the form (84), i.e.,

$$\tilde{t}_3 = \tilde{t}_4 = 0 ,$$  (86)

implies, by Eqs. (A16)-(A21), that the corresponding $β_k = b_{kj}β_0^2$ $(k = 2, 3)$ have the following $\delta b_{kj} = b_{kj} - 1$:

$$\delta b_{22} = \frac{1}{2} \tilde{c}_{10}^{(1)} (\tilde{c}_{10}^{(2)} + 2C) ,$$  (87)

$$\delta b_{21} = \frac{1}{2} \tilde{b}_{11} (\tilde{c}_{10}^{(2)} - b_{11} \tilde{c}_{10}^{(1)}) , \quad \delta b_{20} = -b_{11} \tilde{c}_{10}^{(1)} ,$$  (88)

$$\delta b_{33} = \frac{1}{2} \tilde{c}_{10}^{(1)} (\tilde{c}_{10}^{(2)} + 3\tilde{c}_{10}^{(1)} C (\tilde{c}_{10}^{(2)} + C) - \delta b_{23} \tilde{c}_{10}^{(1)} + C) ,$$  (89)

$$\frac{1}{2} \delta b_{32} = \frac{1}{2} \tilde{c}_{10}^{(1)} (\tilde{c}_{10}^{(2)} - 5 \tilde{b}_{11} \tilde{c}_{10}^{(1)} - \tilde{c}_{10}^{(2)} \tilde{b}_{11} + 3 \tilde{c}_{10}^{(1)} \tilde{c}_{10}^{(2)} - 3 \tilde{b}_{11} + 3 \tilde{b}_{11} + 3 \tilde{b}_{11}) ,$$  (90)

$$\frac{1}{2} \delta b_{31} = \frac{1}{2} \tilde{c}_{10}^{(1)} (\tilde{c}_{10}^{(2)} - 5 \tilde{b}_{11} \tilde{c}_{10}^{(1)} - \tilde{c}_{10}^{(2)} \tilde{b}_{11} + 3 \tilde{c}_{10}^{(1)} \tilde{c}_{10}^{(2)} + \tilde{c}_{10}^{(1)} (\tilde{b}_{11} - \tilde{b}_{11} + 3 \tilde{b}_{11}))$$

$$\delta b_{30} = -\frac{5}{2} \tilde{b}_{11} \tilde{c}_{10}^{(1)} \tilde{c}_{10}^{(2)} + 5 \tilde{b}_{11} \tilde{c}_{10}^{(1)} \tilde{c}_{10}^{(2)} - \tilde{b}_{20} \tilde{c}_{10}^{(2)} + \tilde{b}_{20} \tilde{c}_{10}^{(2)} - \tilde{b}_{20} \tilde{c}_{10}^{(2)} + \tilde{b}_{20} \tilde{c}_{10}^{(2)} .$$  (92)

In these expressions, $\tilde{b}_{2j}$ are the coefficients $b_{2j}$ in MS: $\tilde{b}_{22} = 325/96$, $\tilde{b}_{21} = 243/32$, $\tilde{b}_{20} = -37117/1536$ (and $b_{11} = 19/4$, $b_{10} = -107/16$). We will apply, as a rule, our evaluation approach in the RSch (87)-(92), i.e, where the resulting formula is (83)-(84), and will use the RScl's (70) with $C = C = -5/3$. The RSch evidently depends on the observable. Our starting point will be this RSch for the massless Adler function $D(Q^2) = d_0(Q^2)$, where the STPS is known to a large degree of accuracy up to $a^4$ (up to $a^3$ it’s known exactly) – we will call this RSch $A$ (‘A’ for Adler).6 If an observable is known in STPS only up to $a^3$, only formulas (87)-(88) are to be applied, as $\tilde{t}_4$ is not known; in that case, in Eq. (83) the unknown rest term is $O(\beta_0^3 A_4)$. For example, Bjorken polarized sum rule $d_0(Q^2)$ is such an observable.

In Appendix B a different method of evaluation is presented, which would be an evaluation of the skeleton expansion itself if such an expansion existed in the considered RSch. The RSch-dependence of that method is numerically stronger, which may be a reflection of the fact that this expansion, if it exists, is valid only in a specific (‘skeleton’) RSch that is hitherto unknown.

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6 The difference between this RSch A and the RSch $A'$ (83)-(84) for the Adler function is small. For example, for $n_f = 3$, the values are $\beta_2(A) = -18.92$, $\beta_2(A') = -18.59$; $\beta_4(A) = -33.84$, $\beta_4(A') = -32.72$. In Ref. 12 we used RSch $A'$ (77)-(82) (with RScl's (70) with $C = C = -5/3$), and denoted this approach as 'v2'.

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V. NUMERICAL RESULTS

In this Section, we take the position that the anQCD models M1 and M2, introduced in Sec. III, the form of \( A_1(Q^2) \) there, Eqs. (18) and (19), is achieved in the aforementioned “optimal” RSch (82)-(92) for the massless Adler function \( d_v(Q^2) \) – RSch A. We must keep in mind that models M1 and M2 change the form of \( A_1(Q^2) \) when the RSch \((\beta_2, \beta_3, \ldots)\) is changed.\(^7\)

We will calculate numerically various low-energy QCD observables in the anQCD models MA, M1 and M2, with \( n_f = 3 \), by using the skeleton-motivated evaluation method presented in the previous Section, Eq. (85). One such quantity is the massless Adler function \( d_v(Q^2) \) whose pQCD expansion coefficients \( d_1 \) and \( d_2 \) (in \( \overline{\text{MS}} \) RSch and at RScl \( \mu^2 = Q^2 \)) are known exactly (cf. Appendix E, Eq. (E6)). The heavy quarks \((c, b)\) do not contribute when \( n_f = 3 \). Further, the LS characteristic function \( F_\varepsilon^T(t) \) for \( d_v(Q^2) \) was obtained in (31), and is given in Appendix C in Eqs. (C9)-(C11).

Evaluation method (85) can thus be applied by including terms \( \sim A_1 \) in the case of the massless Adler function (for a different approach to evaluating Adler function, see Ref. [39]). The optimal RSch for the massless Adler function \( d_v(Q^2) \) is then obtained by requiring disappearance of \( \sim A_3 \) and \( \sim A_4 \) terms, Eq. (86), where we choose RScl according to (79) with \( \zeta = -5/3 \). We call this RSch Adler (A), and it can be obtained from \( \overline{\text{MS}} \) RSch by applying relations<br>(87)-(92), resulting in

\[
\begin{align*}
\beta_2^{(A)} &= -23.6074 - 16.0248\beta_0 + 8.04784\beta_0^2, \\
\beta_3^{(A)} &= 127.38 - 35.8577\beta_0 - 12.8734\beta_0^2 - 1.34926\beta_0^3.
\end{align*}
\]

(93)

The values for \( n_f = 3 \) are \( \beta_2 = -18.9211 \) and \( \beta_3 = -33.8404 \) (in \( \overline{\text{MS}} \) RSch, at \( n_f = 3 \), the values are 10.0599 and 47.2281, respectively). In RSch A, the evaluated massless \( d_v(Q^2) \) is thus

\[
d_v(Q^2)_{\text{TAS}} = \int_0^\infty \frac{dt}{t} F_\varepsilon^T(t) A_1(te^{-tQ^2}; \beta_2^{(A)}, \beta_3^{(A)}) + \frac{1}{12} \tilde{A}_2(e^{-tQ^2}),
\]

(94)

and the difference between the (massless) true \( d_v(Q^2) \) and \( d_v(Q^2)_{\text{TAS}} \) is formally \( O(\beta^3_0, \tilde{A}_5) \).

The \((V + A)-\)channel semihadronic \( \tau \) decay rate ratio \( r_\tau \) is one of the best measured low-energy QCD quantities, its massless part for non-strange hadron production has the value \( r_\tau(\Delta S = 0, m_q = 0) = 0.204 \pm 0.005 \) (cf. Appendix E, Eq. (E6)). The heavy quarks \((c \text{ and } b)\) do not contribute, since \( r_\tau \) is a Minkowskian observable, and the \( \tau \) particle cannot decay to charged mesons because their masses are larger than \( m_\tau \).\(^8\) Our evaluation approach for \( r_\tau(\Delta S = 0, m_q = 0) \) uses the aforementioned evaluation (91) of the (massless) Adler function \( d_v(Q^2) \) which is then inserted in the contour integral (C5). The LS-part can then be written in the form (C9) with the time-like LS characteristic function (C10)-(C11). The beyond-the-LS (bLS) contribution is the contour integral

\[
r_\tau(\Delta S = 0, m_q = 0)^{\text{(bLS)}} = \frac{1}{2\pi} \int_{-\pi}^{+\pi} d\phi \left(1 + e^{i\phi}\right) \left(1 - e^{i\phi}\right) \frac{1}{12} \tilde{A}_2(e^{-tQ^2} m_\tau^2 e^{i\phi}).
\]

(95)

Yet another low-energy QCD observable that we will consider is the Bjorken polarized sum rule (BjPSR) \( d_b(Q^2) \). Its LS-characteristic function is obtained in Appendix C on the basis of the known leading-\(\beta_0\) coefficients (40) using the technique of (31). The full perturbation coefficients \( d_1 \) and \( d_2 \) for the massless \( d_b(Q^2) \), in \( \overline{\text{MS}} \) RSch and at RScl \( \mu^2 = Q^2 \), were obtained in Refs. (41) (see Appendix D for explicit expressions for \( d_1 \) and \( d_2 \)). For the coefficient \( d_3 \), only the leading-\(n_f \) part \((x n_f^2)\) is known exactly (40); based on this, estimates of \( d_3 \) as a polynomial in \( \beta_0 \) were performed in Ref. (42) using naive nonabelianization (NNA) \( n_f \mapsto -6\beta_0 \) (33). For the evaluation of (the massless part of) \( d_b(Q^2) \) we will not use estimates of the full \( d_3 \), i.e., we will use method (85) with terms up to \( \tilde{A}_3 \) included, in any chosen RSch and with RScl's (76) with \( \zeta = -5/3 \). The formal difference between the evaluated and the true value is then \( O(\beta^2_0, \tilde{A}_4) \). The experimental values of \( d_b(Q^2) \) at low \( Q^2 \) are much less precise than those of \( r_\tau(\Delta S = 0) \). At \( Q^2 = 2 \) and 1 GeV\(^2\) they are \( d_b(2\text{GeV}^2) = 0.16 \pm 0.11 \) and \( d_b(1\text{GeV}^2) = 0.17 \pm 0.07 \) (for an application,

\(^7\) When \( \beta_j \)'s \((j \geq 2)\) change, the change of \( A_1(Q^2) \) in general cannot be described just by running of the parameters of the model with \( \beta_j \)'s, since new terms appear that depend on those parameters.

\(^8\) The contributions of heavy quarks in Euclidean observables \( D(Q^2) \), such as the Adler function, can be more important, even though \( Q^2 < m_c^2 \) – see the discussion later in this Section.
TABLE I: Results of evaluation of the semihadronic tau decay ratio \( r_\tau(\Delta S = 0, m_q = 0) \) and of BjPSR \( d_b(Q^2 = 2 \text{ GeV}^2) \), in various anQCD models, using evaluation method [58] in RSch A [93]. The basis for calculation of \( \rho^{(pt)}_\tau(\sigma) \) is expansion (2) at loop-level=4 (i.e., when \( \beta^{(A)}_3 \) included) and with \( k_{\text{max}} = 6 \). In parentheses are the results at loop-level=3 and \( k_{\text{max}} = 5 \) (in that case, the \( d_s \)-term of the Adler function is not included). Presented are the results of the full evaluation (leading-skeleton and beyond: LS+bLS), Eq. [59], and for \( r_\tau(\Delta S = 0, m_q = 0) \) also the results of LS. The experimental values are \( r_\tau(\Delta S = 0, m_q = 0) = 0.204 \pm 0.005, d_b(Q^2 = 2 \text{ GeV}^2) = 0.16 \pm 0.11 \) and \( d_b(Q^2 = 1 \text{ GeV}^2) = 0.17 \pm 0.07 \). See the text for further details.

| Model | \( r_\tau(\Delta S = 0, m_q = 0) \) | \( r_\tau(\Delta S = 0, m_q = 0) \) [LS] | \( d_b(Q^2 = 2 \text{ GeV}^2) \) | \( d_b(Q^2 = 1 \text{ GeV}^2) \) |
|-------|----------------------------------|------------------------------------|-----------------------------------|-----------------------------------|
| MA    | 0.141 (0.142)                    | 0.139 (0.141)                      | 0.137 (0.138)                     | 0.155 (0.155)                     |
| M1    | 0.204 (0.205)                    | 0.197 (0.198)                      | 0.160 (0.161)                     | 0.170 (0.171)                     |
| M2    | 0.204 (0.206)                    | 0.203 (0.204)                      | 0.189 (0.190)                     | 0.219 (0.220)                     |

cf. Ref. [45]). The contributions of massive quarks \((m_c, m_u)\) are \(|d_b(Q^2; m_q \neq 0)| < 10^{-3} \) for \( Q^2 \leq 2 \text{ GeV}^2 \) [46], thus negligible. We recall that both \( d_s \) and \( d_b \) are massless observables which are normalized here according to the convention [10] for \( n_f = 3 \). Although the uncertainty of the measured values of \( d_b(Q^2) \) is significantly lower at \( Q^2 = 1 \text{ GeV}^2 \) than at \( Q^2 = 2 \text{ GeV}^2 \), we will use both central values. We expect the theoretical predictions of our evaluations in general to be more reliable at higher momenta \( Q^2 > 1 \text{ GeV}^2 \).

Now we will fix the parameters of models M1 and M2. Model M1 [11, 19] has three independent parameters \( c_f, c_v, c_0 \) (and \( \Lambda = 0.4 \text{ GeV} \) as in MA). Requiring the reproduction of the aforementioned experimental central values \( r_\tau(\Delta S = 0, m_q = 0) = 0.204, d_b(2\text{GeV}^2) = 0.16 \) and \( d_b(1\text{GeV}^2) = 0.17 \), we obtain a solution for the three parameters, with the following values: \( c_f = 1.08, c_v = 0.45, c_0 = 2.94 \). We will use these parameter values in M1 (in RSch A). In general, the predicted values of observables do not change a lot when \( c_0 \) is varied in the regime \( 1 \); they change more when \( c_v \) and/or \( c_f \) are varied. The experimental values of various higher-energy QCD observables \( D(Q^2), \mathcal{R}(s) \) \((Q^2, s \gtrsim 10 \text{ GeV}^2)\) should be well reproduced in M1, because condition [10] ensures that M1 and MA merge at higher energies \( Q^2, s \gg \Lambda^2 \), and it has been demonstrated that MA with \( \Lambda_{(n_f=3)} = 0.4 \text{ GeV} \) \((\Rightarrow \Lambda_{(n_f=5)} = 0.26 \text{ GeV})\) reproduces well those values [58]. We note that model MA (with \( \Lambda = 0.4 \text{ GeV} \)) predicts \( r_\tau(\Delta S = 0, m_q = 0) \approx 0.14 \), which is significantly too low.

Model M2 [17, 18] has two free parameters \( c_v \) and \( c_p \), both assumed to be \( \approx 1 \). Requiring reproduction of the central value of \( r_\tau(\Delta S = 0, m_q = 0) = 0.204 \), and requiring \(|c_p|, |c_v| \approx 0.1\), it turns out that the model then always predicts values \( d_b(2\text{GeV}^2) > 0.19 \). Requiring the minimal possible value \( d_b(2\text{GeV}^2) \approx 0.19 \) gives us the parameter values \( c_v = 0.1 \) and \( c_p = 3.4 \). We will use these parameter values in M2 (in RSch A).

In Table I we present results of calculations of \( r_\tau(\Delta S = 0, m_q = 0) \) and \( d_b(Q^2 = 2 \text{ GeV}^2) \) with our evaluation method [58], in the aforementioned RSch A [93, 94] and at loop-level=4 and 3, in various anQCD models: M1, M2, and MA. When loop-level=4 (and \( k_{\text{max}} = 6 \)), we used in the calculation of \( r_\tau(\Delta S = 0, m_q = 0) \) the estimated N3LO perturbation coefficient \( d_{\Lambda,3} \) of Ref. [38] for the Adler function (cf. Appendix D) as mentioned earlier. In the case of \( d_b(Q^2 = 2 \text{ GeV}^2) \), when loop-level=3 or 4, evaluation formula [58] was used in RSch A by inclusion of terms up to \( \mathcal{A}_3 \) only, as the N3LO coefficient \( d_{\Lambda,3} \) is not known there. We note that MA (with \( \Lambda_{(n_f=3)} = 0.4 \text{ GeV} \)), with light quark masses \( m_u, m_d, m_s \ll \Lambda \) \((m_u, m_d, m_s \approx 0)\), does not reproduce the well-measured experimental value \( r_\tau(\Delta S = 0, m_q = 0) = 0.204 \pm 0.005 \), as already mentioned in the Introduction. This fact led us to suggest alternative versions of anQCD (e.g., M1, M2).

Now that the parameters of the presented anQCD models have been fixed, we can present various results of these models, evaluated with the method [58]. In Fig. 1(a) we present curves for the massless Adler function \( d_{\Lambda,0}(Q^2) \) (with \( n_f = 3 \)) as functions of energy \( Q \), in models M1, M2, and MA. The RSch used is RSch A [93, 94]. Loop-level is four, i.e., we include the value \( \beta^{(A)}_3 \) in our calculation for \( \rho^{(pt)}_\tau \), with \( k_{\text{max}} = 6 \) [cf. Eq. (2)], and use the estimated N3LO perturbation coefficient \( d_{\Lambda,3} \) of Ref. [38] (cf. Appendix D). The light-by-light contributions, which have a different topology of diagrams and should probably be resummed separately (cf. Ref. [11]), appear for the first time at \( \sim a^2 \) and are proportional to the square of the sum of the quark charges \((\sum f_j)^2 \) [77]. This sum is zero in the case \( n_f = 3 \) considered here. Fig. 1(b) represents the results for the full Adler function, i.e., the V-channel heavy quark corrections \( \delta d_{\Lambda,0}(Q^2; m_c, m_u) \) have been added there. For the calculation of the latter, we follow the procedure of [47], including the \( a^2 \)-contributions (note that \( d_{\Lambda,0}(Q^2) \equiv (1/2)D(Q^2) - 1 \), where \( D \) is defined in [47]). The first seven coefficients of the low-momentum Taylor expansion for the heavy quark \( a^2 \)-contributions are calculated in [48]. Through a conformal mapping together with Padé improvement, as proposed in [49], an approximant is obtained. The approximant reproduces the low-momentum behavior and fits very well the large-momentum expansion [50] for this quantity up to energies \( Q^2 \approx 16m_c^2 \) (see also Fig. 4 of Ref. [47]). Thus, this method can be safely used for
FIG. 4: Adler function as predicted by pQCD, and by our approach in several analytic QCD models (see the text): (a) the massless part ($n_f = 3$); (b) the full quantity, with the contribution of massive quarks included.

the $q = c, b$ quarks in the energy range we are interested in. In the heavy quark contributions, we simply replaced $a(Q^2)$ and $a^2(Q^2)$ by $A_1(Q^2)$ and $A_2(Q^2)$ (using $\Lambda = \overline{\Lambda} = 0.4$ GeV). The indicated ± uncertainties in the full Adler function curves are those $c$ quark contributions which are proportional to $A_2$. In Figs. 4(a), (b) we included the STPS’s [truncated forms of Eq. (40)] in $\overline{M_S}$ RSch and with RScl $\mu^2 = Q^2$. In Fig. 4(b) we included experimental values, for comparison. The experimental values of $d_\nu(Q^2)$ are taken from Ref. [47] where the integral expression for $d_\nu(Q^2)$ in terms of the $e^+ e^-$ QCD ratio $R_{e^+ e^-}(s)$ is evaluated. All the values of $R_{e^+ e^-}(s)$ are needed – from the two-pion threshold to infinity. The evaluation is based on the data compilation of Ref. [51]. The pQCD result for $R_{e^+ e^-}(s)$ is used in the integral where it can be trusted, and data in the rest of the energy interval. Resonances are included separately. In Fig. 4(b) we can see that various anQCD models predict at low energies ($Q < 1.2$ GeV) values which are significantly closer to the experimental values than STPS’s. Further, STPS’s lose any predictability at $Q < 1.2$ GeV, mainly because of the vicinity of the unphysical Landau pole in the pQCD coupling $a(Q^2)$.

FIG. 5: Bjorken polarized sum rule (BjPSR) $d_b(Q^2)$ in (a) model M1, and (b) comparison of M1 and MA; in various RSch’s and at various RScl’s. The vertical lines in (a) represent experimental data, with errorbars in general covering the entire depicted range of values.

In Fig. 5(a), we present results of calculation of BjPSR $d_b(Q^2)$ at low energies in model M1, at loop-level=3 (and

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9 Some contributions from heavy quarks are not considered here as we base our analysis on the expressions of Ref. 15. The relevant diagrams are shown in Fig. 2 of Ref. 15; the contributions with internal heavy and external light quarks are not included. These type of $a^2$-contributions have been obtained for the $R_{e^+ e^-}(s)$ function in Refs. 51, 52, 53. We checked that these contributions, when translated into the corresponding contributions for $d_\nu(Q^2)$ via the usual integral transformation relating $R$ and $d_\nu$, result in $a^2$-contributions which are of order of magnitude smaller than the heavy quark $a^2$-contributions included in our curves.
\( k_{\text{max}} = 5 \), in two different RSch’s: RSch A \(^{(53)}\), and RSch B which is the "optimal" RSch for \( d_b(Q^2) \), i.e., \( \beta_2^{(B)} \) is obtained from the requirement \( t_3 = 0 \) for \( d_b \), Eqs. \(^{(77)}-^{(85)}\)

\[
\beta_2^{(B)} = -30.2949 - 10.4415 \beta_0 + 7.44582 \beta_0^2.
\]

(96)

At \( n_f = 3 \) we have \( \beta_2^{(B)}(n_f = 3) = -16.0938 \). The analytic couplings in RSch B are obtained from those in RSch A by applying the loop-level=3 RSch-evolution equations \(^{(53)}\). In addition, we present in Fig. 5(a) results when the RScl in the beyond-the-LS terms \( (Q_2^2, Q_3^2) \) is increased from \( Q^2 \exp(-5/3) \) to \( Q^2 \) (note that coefficients \( t_2 \) and \( t_3 \) then change accordingly). We see that at low energies \( Q < 2 \text{ GeV} \) the results in M1 change moderately but not insignificantly under the variation of RSch and RScl. For comparison, we included the curve obtained from the skeleton evaluation \(^{(121)}\) in RSch A [with \( Q_2^2 = Q_3^2 = Q^2 \exp(-5/3) \)], assuming that the skeleton expansion exists in RSch A (which is probably not true). We include the present experimental data, with the crosses representing the central values; the errorbars extend in general over the entire depicted range of values, most of the experimental uncertainties are of the order of \( \pm 0.1 \). The experimental data were deduced from Fig. 2 of Ref. \(^{(44)}\), with the neutron decay parameter value \( |g_A| = 0.21158 \pm 0.00048 \) taken from \(^{(55)}\). The present experimental errors are too high to discriminate between various evaluation methods. In Fig. 5(b) we compare the results for MA and M1. The RSch- and RScl-dependence of MA-results remains very weak in all the shown region.

![FIG. 6: BjPSR \( d_b(Q^2) \) in (a) model M2, and (b) comparison of M2 and M1; at various RScl’s (a,b) and in various RSch’s (a). The vertical lines in (a) represent the experimental data.](image)

In Fig. 5(a) we present the same type of curves for M2 model. We see that the RSch- and RScl-dependence in M2 remains quite weak down to low energies. In Fig. 5(b) we compare the results of M2 and M1 models. Only the curves in RSch A are presented in Fig. 5(b).

Up until now, we applied the (skeleton-motivated) method \(^{(85)}\) for the evaluation of QCD observables, in various anQCD models for \( A_1(\mu^2) \), with the higher-order couplings \( \mathcal{A}_k \) \((k \geq 2)\) constructed by Eqs. \(^{(28)}\) in a certain RSch (usually RSch A) and equivalently the higher-order couplings \( \mathcal{A}_k \) by Eqs. \(^{(55)}\) [Eqs. \(^{(37)}\) if loop-level=4]. There remains a question of how this method of evaluation compares with the APT evaluation approach of Milton et al. \(^{(2,3)}\) and Shirkov \(^{(2,3)}\). We recall that the APT approach was defined for the MA anQCD model, and it consists of using the available (NLO or N\(^2\)LO) STPS of an observable \(^{(10)}\) and replacing there \( d^k(Q^2) \rightarrow \mathcal{A}_k(Q^2)^{(MA)} \) \((k \geq 1)\), where the higher-order MA couplings \( \mathcal{A}_k(Q^2)^{(MA)} \) were constructed according to formula \(^{(10)}\). In the N\(^2\)LO STPS case \( \text{e.g.}, \) for \( d_b(Q^2) \), this reads

\[
P_{\text{APT}}(Q^2) = A_1(Q^2)^{(MA)} + d_1A_2(Q^2)^{(MA)} + d_2A_3(Q^2)^{(MA)}.
\]

(97)

The RSch is usually taken to be \( \overline{\text{MS}} \), but could in principle be any RSch. One of the differences between our and APT evaluation method here is the construction of the higher-order couplings \( \mathcal{A}_k(Q^2)^{(MA)} \) of the model MA, where comparison with our construction has been made in Figs. 1 and 2 in Sec. III. Another difference is that our evaluation method \(^{(85)}\) includes, in addition, the leading-\( \beta_0 \) contributions to all orders. We compare in Figs. 7(a), (b) the results of our method \(^{(85)}\) and APT-method \(^{(77)}\) for BjPSR \( d_b(Q^2) \), in MA model, in two different RSch’s: A \(^{(23)}\), and \( \overline{\text{MS}} \) (relevant for \( \beta_2 \) coefficient only). In Figs. 7(a) and (b) the renormalization scale was taken to be \( \mu^2 = Q^2 \exp(-5/3) \) and \( \mu^2 = Q^2 \), respectively, in both beyond-the-LS terms of our approach \( (\mathcal{A}_2, \mathcal{A}_3) \), and in APT approach. The RSch-change from RSch A to \( \overline{\text{MS}} \) was performed in our approach according to the loop-level=3 evolution equations.
In this work we suggested various models of analytic QCD (anQCD), i.e., models for construction of the anQCD coupling \( A_1(Q^2) \) which is an analytic analog of the perturbative QCD coupling \( \alpha_s(Q^2) \equiv \frac{\alpha_s}{\pi} \). The main reason why we suggest alternatives to the minimal analytic (MA) model, i.e., to the coupling \( A_1^{(MA)}(Q^2) \) of Shirkov and Solovtsov \(^6\), is that it cannot correctly reproduce simultaneously various higher-energy QCD observables on the one hand and the low-energy observable \( r_\tau \) (semihadronic \( \tau \) decay rate ratio) on the other hand, unless large masses of \( u, d \) and \( s \) quarks are introduced \(^4\). The described alternative models (M1 and M2) have \( A_1(Q^2) \) with additional dimensionless parameters in it, which can be adjusted in order to modify the behavior at low \( Q^2 \). Furthermore, we presented, for any anQCD model, an algorithm which allows construction of higher-order analytic couplings \( A_k(Q^2) \) \((k \geq 2)\) which are the analytic analogs of \( a^k(Q^2) \). In addition, we presented a method of evaluation of Euclidean QCD observables in anQCD models, a method which is (partly) motivated by the so-called skeleton expansion structure but does not depend on the existence of such a skeleton expansion. The evaluation method sums up all the leading-\(\beta_0\) contributions (LS: leading-skeleton) and adds those contributions beyond the LS which are known by the knowledge of a first few perturbation expansion coefficients of the considered observable. We tested this evaluation method, for three anQCD models, in the case of the Adler function, semihadronic \( \tau \) decay ratio, and the Bjorken polarized sum rule (BjPSR) at low energies. The results show in general good stability under variation of the renormalization scale and scheme down to low energies \( Q \sim 1 \) GeV. We further carried out comparison of our evaluation method with that of Milton et al. (APT) \(^2, 3, 4\), for the BjPSR, in the MA model where the latter method can be applied. The two methods give results which at low energies differ in general by only a few per cent for this observable.

**VI. SUMMARY AND CONCLUSIONS**

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APPENDIX A: RELEVANT COEFFICIENTS OF THE SKELETON-MOTIVATED EXPANSION

In this Appendix, we present explicit formulas for the coefficients \( t_i^{(j)} \) appearing in the skeleton-motivated expansion \((67)\), which is a slightly reorganized form of expansion \((62)\). We consider the case when, in \(\overline{\text{MS}}\) RSch \((\beta_k = \bar{b}_k = \sum b_{k_j} \beta_{0j}^k, k \geq 2)\) and at RSc \(\mu^2 = Q^2\), the first three coefficients in expansion \((10)\) are explicitly known \((\mathcal{D}_j, j = 1, 2, 3)\), and all the leading-\(\beta_0\) parts \(\tau_n^{(1)}\) of coefficients \(\mathcal{D}_n\) \((n \geq 1)\) in expansion \((12)\) are known (we note that \(\tau_{n-1}^{(1)} = 0\) in \(\overline{\text{MS}}\)). The RSch \((\beta_2, \beta_3, \ldots)\) is chosen and common to all terms \(\mathcal{D}^{(j)} \((j \geq 1)\), and belongs to the class of RSch’s of Eq. \((11)\). The RSc’s used in the resulting truncated versions of \(\mathcal{D}^{(j)}(Q^2) \((j \geq 2)\) are \(Q_j^2\), they may be mutually different as each \(\mathcal{D}^{(j)}(Q^2)\) \((and k_j)\) is RSc-independent. For the RSh and the RSc’s we will use notations

\[
\delta b_{kj} = b_{kj} - \overline{b}_{kj}, \quad Q_j^2 = Q^2 \exp(C_j). \tag{A1}
\]

We then obtain for the coefficients \(t_i^{(j)}\) of expansion \((67)\) the following expressions, on the basis of relations \((51)-(53)\), as well as \((11)-(12)\) and \((63)\):

\[
t_2^{(2)} = \tau_2^{(2)} = \tau_1^{(1)}, \tag{A3}
\]

\[
t_3^{(2)} = \tau_3^{(2)} - \beta_0 \delta b_{22} + \beta_0 \tau_1^{(1)} \tau_2^{(1)} C_2, \tag{A4}
\]

\[
t_3^{(3)} = \tau_3^{(3)} - \delta b_{21} - \frac{1}{\beta_0} \delta b_{20}, \tag{A5}
\]

\[
t_4^{(2)} = \tau_4^{(2)} + (b_{11} \beta_0 + b_{10}) \left( -\delta b_{22} + 2 \tau_1^{(1)} C_2 \right)
+ \beta_0 \left( -\delta b_{22} \tau_1^{(1)} - \frac{1}{2} \delta b_{33} + 3 \tau_1^{(1)} \tau_2^{(1)} C_2 - \delta b_{22} \tau_1^{(1)} C_2 + 3 \tau_2^{(1)} C_2^2 \right), \tag{A6}
\]

\[
t_4^{(3)} = \tau_4^{(3)} + \beta_0 \left( -\delta b_{22} \tau_1^{(1)} - \delta b_{21} \tau_1^{(1)} - \frac{1}{2} \delta b_{32} + b_{11} \delta b_{22} \right)
- \delta b_{20} (\tau_1^{(1)} \tau_2^{(1)} - \delta b_{21})^{-1} \left( \tau_1^{(1)} \tau_2^{(1)} - \delta b_{22} \tau_1^{(1)} - \delta b_{21} \tau_1^{(1)} - \frac{1}{2} \delta b_{32} - b_{11} \tau_1^{(1)} \tau_2^{(1)} + b_{11} \delta b_{22} \right)
+ \beta_0 \left( \tau_1^{(1)} \tau_2^{(1)} - \delta b_{21} - \frac{1}{\beta_0} \delta b_{20} \right) 3 C_3, \tag{A7}
\]

\[
t_4^{(4)} = \tau_4^{(4)} + \left( -\delta b_{21} \tau_1^{(1)} - \delta b_{20} \tau_1^{(1)} - \frac{1}{2} \delta b_{31} + b_{10} \delta b_{22} \right)
+ \delta b_{20} (\tau_1^{(1)} \tau_2^{(1)} - \delta b_{21})^{-1} \left( \tau_1^{(1)} \tau_2^{(1)} - \delta b_{22} \tau_1^{(1)} - \delta b_{21} \tau_1^{(1)} - \frac{1}{2} \delta b_{32} - b_{11} \tau_1^{(1)} \tau_2^{(1)} + b_{11} \delta b_{22} \right)
\]

\[
+ \frac{1}{\beta_0} \left( -\delta b_{20} \tau_1^{(1)} - \frac{1}{2} \delta b_{30} \right). \tag{A8}
\]

Here, \(\tau_i^{(j)}\) are the values of the \(t_i^{(j)}\) in \(\overline{\text{MS}}\) RSch and with RSc \(\mu^2 = Q^2\):

\[
\tau_2^{(2)} = \tau_1^{(1)}, \tag{A9}
\]

\[
\tau_3^{(2)} = \beta_0 \tau_1^{(1)} \tau_2^{(1)}, \tag{A10}
\]

\[
\tau_3^{(3)} = \tau_1^{(1)} \tau_2^{(1)}, \tag{A11}
\]

\[
\tau_4^{(2)} = (b_{11} \beta_0 + b_{10} \tau_1^{(1)} \tau_2^{(1)} + \beta_0 \tau_1^{(1)} \tau_2^{(1)}), \tag{A12}
\]

\[
\tau_4^{(3)} = \beta_0 \tau_1^{(1)} \tau_2^{(1)} - b_{11} \tau_1^{(1)} \tau_2^{(1)}, \tag{A13}
\]

\[
\tau_4^{(4)} = \tau_1^{(1)} \tau_2^{(1)} - b_{10} \tau_1^{(1)} \tau_2^{(1)}. \tag{A14}
\]
Coefficients $\bar{c}^{(j)}_i$ appearing in the above formulas can be obtained directly from coefficients $\bar{c}^{(1)}_i$ by using relations (51) in $\overline{\text{MS}}$ scheme (with $\text{RScl} \mu^2 = Q^2$):

$$
\bar{c}^{(1)}_{10} \bar{c}^{(2)}_{ij} = \bar{c}^{(1)}_{ij} - b_{1j} \bar{c}^{(1)}_{11} \quad (j = 1, 0, -1),
$$

$$
\bar{c}^{(1)}_{10} \bar{c}^{(2)}_{ij} = \bar{c}^{(1)}_{ij} - \frac{5}{2} b_{1j} \bar{c}^{(1)}_{22} - b_{2j} \bar{c}^{(1)}_{11} \quad (j = 2, 1, 0, -1).
$$

Formulas (A15), (A9), (A14), (A3)–(A8), with notations (A1) and (A2), allow us to obtain all the coefficients $\bar{t}^{(j)}_i$ of the skeleton-motivated expansion (67) in any chosen RScl and with chosen RScl’s $Q^2$, if we know in $\overline{\text{MS}}$ RScl at $\text{RScl} \mu^2 = Q^2$ all the leading-$\beta_0$ parts $\bar{c}^{(1)}_n \beta_0^{(1)}$ of the expansion coefficients $\bar{d}_n = \sum_n \bar{c}^{(1)}_n \beta_0^{(1)}$ of observable $D(Q^2)$ Eq. (10), and we know exactly the full coefficients $\bar{d}_j$ for $j = 1, 2, 3$, i.e., we know $\bar{c}^{(1)}_j$ for $j = 1, 2, 3$ and $k = 0, \ldots, j$. If, on the other hand, we do not know $\bar{d}_3$, the above formulas can be applied for $\bar{t}^{(j)}_i$ for $i = 2, 3$ only.

When the beyond-the-LS contributions in our approach (71) are expressed in terms of $\bar{d}_j$ by relations (37) between $A_k$’s and $\bar{A}_n$’s. After some straightforward algebra, we obtain

$$
\bar{t}_2 = \bar{t}_2 = c^{(1)}_{10},
$$

$$
\bar{t}_3 = \bar{t}_3 = \beta_0 \delta b_{22} + \beta_0 \delta t^{(1)}_{10} C - \delta b_{21} - \frac{1}{\beta_0} \delta b_{20},
$$

$$
\bar{t}_4 = \bar{t}_4 + \beta_0^2 \left[ -\frac{1}{2} \delta b_{33} - \delta b_{22} C \left( \bar{c}^{(1)}_{11} + C \right) + 3 \bar{c}^{(1)}_{10} \left( \bar{c}^{(2)}_{11} + C \right) \right]
$$
$$
+ \beta_0 \left[ -\frac{1}{2} \delta b_{32} + \delta b_{22} \left( -3 \bar{c}^{(1)}_{10} + \frac{5}{2} b_{11} \right) - \delta b_{21} \bar{c}^{(1)}_{11} + 3 \bar{c}^{(1)}_{10} \left( \bar{c}^{(2)}_{10} - b_{11} \right) \right]
$$
$$
+ \left[ -\frac{1}{2} \delta b_{31} + \frac{5}{2} b_{10} \delta b_{21} + \delta b_{21} \left( -3 \bar{c}^{(1)}_{10} + \frac{5}{2} b_{11} \right) - 3 \delta b_{20} \left( \bar{c}^{(1)}_{11} + C \right) - 3 b_{10} \bar{c}^{(1)}_{10} C \right]
$$
$$
+ \left[ -\frac{1}{2} \delta b_{30} + \frac{5}{2} b_{10} \delta b_{21} + \delta b_{20} \left( -3 \bar{c}^{(1)}_{10} + \frac{5}{2} b_{11} \right) \right] + \frac{1}{\beta_0^2} \frac{5}{2} b_{10} \delta b_{20},
$$

where $\bar{t}_i$ are the values of $\bar{t}_i$ in $\overline{\text{MS}}$ and with $\text{RScl} \mu^2 = Q^2$:

$$
\bar{t}_2 = \bar{c}^{(1)}_{10},
$$

$$
\bar{t}_3 = \beta_0 \bar{c}^{(1)}_{10} C - \bar{t}^{(1)}_{10} b_{11} - \frac{1}{\beta_0} \bar{c}^{(1)}_{10} b_{10},
$$

$$
\bar{t}_4 = \beta_0 \bar{c}^{(1)}_{10} \bar{t}^{(1)}_{10} - \frac{1}{\beta_0} \bar{c}^{(1)}_{10} \bar{t}^{(1)}_{10} \quad \text{(A18)}
$$

APPENDIX B: SKELETON EXPANSION

In this Appendix we will construct an expression for evaluation of QCD space-like observables $D(Q^2)$ (for any anQCD model) which will be derived directly from the QCD skeleton expansion. Here we will take the position that such an expansion exists in the class of schemes with the QCD scale $\Lambda_{\text{QCD}}^2 = \Lambda_{\text{QCD}}^2 \exp(C)$ where $\Lambda_{\text{QCD}}$ is the so-called $V$-scheme scale and $C$ is an arbitrary $n_f$-independent constant, and with $\beta_k$ of Eq. (11) where $b_{kj}$ are arbitrary constants. In this context, choosing the $\overline{\text{MS}}$ scale parameter $C = \bar{C} = -5/3 \Lambda = \overline{\Lambda}$ for scaling the RScl $\mu^2$ represents no additional restriction. This expansion involves in the integrands the characteristic functions $F_0^2(t_1, \ldots, t_n)$, which are considered $n_f$-independent, and the (singular) pQCD coupling $a(\mu^2)$. We replace $a(\mu^2)$ by an anQCD coupling $A_1(\mu^2)$ in the skeleton integrals which makes the integrals unambiguous

$$
D(Q^2)_{\text{skeleton}} = \int_0^\infty \mathcal{D}\left(t_{\text{skeleton}}(t) A_1(tQ^2 e^x) \right) \prod_{n=2}^{\infty} s_n \int_0^\infty \mathcal{D}\left[t_j A_1(t_j Q^2 e^x) \right] \mathcal{D}\left(t_1, \ldots, t_n\right)
$$

$$
= D^{(\text{LS})}(Q^2) + s_1 D^{(\text{NLS})(Q^2)} + s_2 D^{(N^2\text{LS})(Q^2)} + s_3 D^{(N^3\text{LS})(Q^2)} + \cdots.
$$
Here, $F_2^\varepsilon(t_1, \ldots, t_n)$ are the characteristic functions and have the normalizations
\[
\int_0^\infty \frac{dt}{t} F_2^\varepsilon(t) = 1, \quad \int_{t_1}^{t_2} \frac{dt_1}{t_1} \frac{dt_2}{t_2} F_2^\varepsilon(t_1, t_2) = 1, \ldots, \tag{B3}
\]
implying for the perturbative parts
\[
D^{(\kappa)}(Q^2)_{pt} = a^{n_\kappa} [1 + O(a)], \tag{B4}
\]
where $n_\kappa = 1$ for $\kappa = \text{LS}$, $n_\kappa = 2$ for $\kappa = \text{NLS}$, etc. The perturbative part of $A_1(\mu^2)$ is $a(\mu^2)$ $[A_1(\mu^2) = a(\mu^2) + \text{NP}$, where NP involves non-analytic in $a = 0$ functions of $a$, cf. Eq. (31)]. We will use RGE evolution series (46) for expansion of $a(\text{te}^\varepsilon Q^2)$ around $a(\mu^2) \equiv a$
\[
a(\text{te}^\varepsilon Q^2) = a + \sum_{n=1}^{\infty} a_{n+1} \beta_0^n (-\ln \mathcal{T})^n
\]
\[
= a + a^2 \beta_0 (-\ln \mathcal{T}) + a^3 \left[ \beta_0^2 \ln^2 \mathcal{T} - \beta_1 \ln \mathcal{T} \right] + a^4 \left[ - \beta_0^3 \ln^3 \mathcal{T} + \frac{5}{2} \beta_0 \beta_1 \ln^2 \mathcal{T} - \beta_2 \ln \mathcal{T} \right] + \cdots, \tag{B5}
\]
where $\mathcal{T} \equiv tQ^2 e^\varepsilon/\mu^2$, and $a_n$ are defined in Eq. (29). Using expansion (B5) in the leading-skeleton (LS) term in (B1), this term can be shown to have the following expansion for its perturbative part:
\[
D^{(\text{LS})}(Q^2)_{pt} = a + a^2 \beta_0 (-\ln \mathcal{T})_{(1)} + a^3 \left[ \beta_0^2 \ln^2 \mathcal{T} + \beta_1 (-\ln \mathcal{T})_{(1)} \right]
\]
\[
+ a^4 \left[ - \beta_0^3 \ln^3 \mathcal{T} + \frac{5}{2} \beta_0 \beta_1 \ln^2 \mathcal{T} - \beta_2 \ln \mathcal{T} \right] + O(\beta_0^4 a^5), \tag{B6}
\]
where we adhere to notations summarized in the following:
\[
\mathcal{T} = \frac{tQ^2 e^\varepsilon}{\mu^2}, \quad a \equiv a(\mu^2), \tag{B7}
\]
\[
\langle f(t_1, \ldots, t_n) \rangle_{(n)} = \prod_{j=1}^{n} \int_{t_j}^{\infty} \frac{dt_j}{t_j} F_2^\varepsilon(t_1, \ldots, t_n) f(t_1, \ldots, t_n). \tag{B8}
\]
Requiring that the perturbative part of the LS-term absorb all the leading-$\beta_0$ parts of $D(Q^2)_{pt}$ [see Eqs. (40)-(42)] implies that
\[
\left\langle (-\ln \mathcal{T})^n \right\rangle_{(1)} = c_{n0} \quad (n = 0, 1, 2, \ldots) \tag{B9}
\]
This, in conjunction with expansion (B5), implies that $D^{(\text{LS})}(Q^2)_{pt}$ is precisely $D^{(1)}(Q^2)_{pt}$ of construction in Sec. IV, i.e., we really have for $D^{(1)}(Q^2)_{pt}$ the resummed form (41). Taylor expansion of $A_1(\text{te}^\varepsilon Q^2)$ around $Q^2$ is completely analogous to expansion (B5) a non-power analytic expansion
\[
A_1(\text{te}^\varepsilon Q^2) = A_1 + \sum_{n=1}^{\infty} \tilde{A}_{n+1} \beta_0^n (-\ln \mathcal{T})^n
\]
\[
= A_1 + A_2 \beta_0 (-\ln \mathcal{T}) + A_3 \left[ \beta_0^2 \ln^2 \mathcal{T} - \beta_1 \ln \mathcal{T} \right] + A_4 \left[ - \beta_0^3 \ln^3 \mathcal{T} + \frac{5}{2} \beta_0 \beta_1 \ln^2 \mathcal{T} - \beta_2 \ln \mathcal{T} \right] + \cdots, \tag{B10}
\]
where $\tilde{A}_{k} \equiv \tilde{A}_{k}(\mu^2)$ and $A_k \equiv A_k(\mu^2)$. In the last identity we used the fact that the $A_n$’s appear on the left-hand side of RGE’s (29), which are analogous to pQCD RGE’s with $a_n$’s on the left-hand side [analogy valid up to terms $O(\tilde{A}_{n_m})$ where $n_m =$ loop-level] when the correspondence $\alpha^k \leftrightarrow A_k$ is made. Eq. (B10) implies for the (full analytic) LS-term of the skeleton expansion (B1) a non-power analytic expansion
\[
D^{(\text{LS})}(Q^2) \equiv \int_0^\infty \frac{dt}{t} F_2^\varepsilon(t) A(tQ^2 e^\varepsilon)
\]
\[
= A_1 + \sum_{n=1}^{\infty} \tilde{A}_{n+1} \beta_0^n (-\ln \mathcal{T})_{(1)}
\]
\[
= A_1 + A_2 \beta_0 (-\ln \mathcal{T})_{(1)} + A_3 \left[ \beta_0^2 (-\ln \mathcal{T})^2_{(1)} + \beta_1 (-\ln \mathcal{T})_{(1)} \right]
\]
\[
+ A_4 \left[ - \beta_0^3 (-\ln \mathcal{T})^3_{(1)} + \frac{5}{2} \beta_0 \beta_1 (-\ln \mathcal{T})^2_{(1)} + \beta_2 (-\ln \mathcal{T})_{(1)} \right] + O(A_5), \tag{B11}
\]
which is just the analyticized analog [according to the rule (69)] of perturbation expansion (B6) and (51).

Now we will investigate the beyond-the-LS contributions of the skeleton expansion (B1). In view of normalization conditions (B4), it follows immediately that

\[ s_1^D = c_{10}^{(1)}, \] (B12)

which is just the coefficient \( k_2 \) (52) in the approach of Sec. IV. In analogy with the LS-part, we now require that \( \mathcal{D}^{(N\text{LS})}(Q^2)_{\text{pt}} \) be such as to absorb all the leading-\( \beta_0 \) parts of the difference \( (1/s_1^D)[\mathcal{D}(Q^2) - \mathcal{D}^{(\text{LS})}(Q^2)]_{\text{pt}} \). In completely analogous way as before, we can show that this is equivalent to

\[ (-1)^n \langle \ln^n T_1 + \ln^{n-1} T_1 \ln T_2 + \cdots + \ln^n T_2 \rangle_{(2)} = c_{m_n}^{(2)} \quad (n = 1, 2, \ldots), \] (B13)

where \( T_j = t_j Q^2 e^{-c}/\mu^2 \), and coefficients \( c_{m_n}^{(2)} \) are defined in Eqs. (22) and (24). These coefficients are known if the perturbative coefficients \( d_j \) (40) are known. The (nonpower) expansion in \( A_k \equiv A_k(\mu^2) \) of the NLS-term is then

\[ s_1^D \mathcal{D}^{(N\text{LS})}(Q^2) = c_{10}^{(1)} \left\{ A_1 + A_1 A_2 \beta_0 c_{11}^{(2)} + A_1 A_3 \left[ \beta_0^2 c_{22}^{(2)} + \beta_1 c_{11}^{(2)} \right] \right\} + \left[ A_2^2 - A_1 A_3 \right] \beta_0^2 \langle \ln T_1 \ln T_2 \rangle_{(2)} + \mathcal{O}(A_1 A_4, A_2 A_3, \ldots) \right\}. \] (B14)

The last term in brackets has a coefficient \( \propto \langle \ln T_1 \ln T_2 \rangle_{(2)} \) which cannot be obtained on the basis of the perturbative coefficients \( d_j \) (40). The perturbative part of this last term is zero. We know \( c_{10}^{(1)} \) if we know the NLO coefficient \( d_1 \) of the perturbation expansion (40) of observable \( \mathcal{D}(Q^2) \); for the knowledge of \( c_{11}^{(2)} \) we need, in addition, the knowledge of \( d_2 \), and for \( c_{22}^{(2)} \) the knowledge of \( d_4 \).

We now continue analogously one step further. In view of the normalization conditions (B4), it follows immediately

\[ s_2^D = c_{10}^{(1)} \left( c_{10}^{(2)} + \frac{1}{\beta_0} c_{12}^{(2)} \right), \] (B16)

which is identical to the coefficient \( k_3 \) (57) in the approach of Sec. IV. We require that the third (\( N^2\text{LS} \)) term \( s_2^D \mathcal{D}^{(N^2\text{LS})}(Q^2) \) in skeleton expansion (32) satisfy the condition: \( \mathcal{D}^{(N^2\text{LS})}(Q^2)_{\text{pt}} \) be such as to absorb all the leading-\( \beta_0 \) parts of the difference \( (1/s_2^D)[\mathcal{D}(Q^2) - \mathcal{D}^{(\text{LS})}(Q^2)] - s_2^D \mathcal{D}^{(N\text{LS})}(Q^2)_{\text{pt}} \). This then implies

\[ \langle - \ln T_1 - \ln T_2 - \ln T_3 \rangle_{(3)} = c_{11}^{(3)}, \] (B17)

where \( c_{11}^{(3)} \) is given in Eq. (60); and similarly for higher terms (\( c_{22}^{(3)}, \) etc.). The (nonpower) expansion in \( A_k \equiv A_k(\mu^2) \) of the \( N^2\text{LS} \)-term is then

\[ s_2^D \mathcal{D}^{(N^2\text{LS})}(Q^2) = s_2^D \left\{ A_1^3 + A_1^2 A_2 \beta_0 c_{11}^{(3)} + \mathcal{O}(A_1^3 A_3, A_1 A_2^2, \ldots) \right\}. \] (B18)

We know the quantity \( s_2^D \) if we know the coefficients \( d_1 \) and \( d_2 \) in the perturbation expansion (40) of observable \( \mathcal{D}(Q^2) \); for the knowledge of \( c_{11}^{(3)} \) we need, in addition, the knowledge of \( d_3 \).

Normalization conditions (B4) now imply that the coefficient \( s_3^D \) of the \( N^3\text{LS} \)-term in the skeleton expansion (B1-B2) is

\[ s_3^D = c_{10}^{(1)} \left[ c_{20}^{(2)} - b_{10} c_{11}^{(2)} - \frac{1}{\beta_0} c_{21}^{(2)} - b_{11} c_{11}^{(2)} \right], \] (B19)

which is identical to the coefficient \( k_4 \) (63) in the approach of Sec. IV. The (nonpower) expansion in \( A_k \equiv A_k(\mu^2) \) of the \( N^3\text{LS} \)-term is then

\[ s_3^D \mathcal{D}^{(N^3\text{LS})}(Q^2) = s_3^D A_1^4 + \mathcal{O}(A_1^3 A_2, A_1^2 A_3, \ldots). \] (B20)

We know the quantity \( s_3^D \) if we know the coefficients \( d_1, d_2 \) and \( d_3 \) in the perturbation expansion (40) of observable \( \mathcal{D}(Q^2) \).

Finally, we can combine the LS-term (68), whose characteristic function is usually known, with all the beyond-the-LS terms written hitherto (B1-B20) which are known if \( d_1, d_2 \) and \( d_3 \) are known; since each of these terms
is RScl-independent, we can use in the most general case various (space-like) RScl’s \( Q_j^2 = Q^2 \exp(C_j) \) as Eq. (A2) \((j = 2, 3, 4)\) for the NLS, \( N^3 \)LS and \( N^3 \)LS terms, respectively. This then results in

\[
\mathcal{D} = D^{(LS)}(Q^2) + t_2^{(2)} \left[ A_1(Q^2) \right]^2 + \left\{ t_3^{(2)} A_1(Q^2) A_2(Q^2) + t_3^{(3)} [A_1(Q^2)]^3 \right\} + \left\{ t_4^{(2)} A_1(Q^2) A_3(Q^2) + t_4^{(3)} [A_1(Q^2)]^2 A_2(Q^2) + t_4^{(4)} [A_1(Q^2)]^4 \right\} + \left\{ [A_2(Q^2)]^2 - A_1(Q^2) A_3(Q^2) \right\} c_{10}^{(1)} \beta_0^2 (\ln T_1 \ln T_2)_2 + O(A_1^5, A_1^3 A_2, \ldots),
\]

where the coefficients \( t_{ij}^{(n)} \) are precisely those given in Appendix A Eqs. (A3)–(A14). Therefore, the evaluation method presented in the present Appendix, which is a representation of an assumed skeleton expansion [B13]–[B22], reduces to the evaluation method presented in Sec. XIV when the following replacements are made:

\[
[A_1(\mu^2)]^{k_1} [A_2(\mu^2)]^{k_2} \cdots [A_s(\mu^2)]^{k_s} \mapsto A_{k_1 + 2k_2 + \cdots + sk_s}(\mu^2).
\]

In the present method, the coefficient at the last term in brackets in expression (B21) can be evaluated only if certain assumptions about the NLS characteristic function \( F_\beta^\phi(t_1, t_2) \) are made. For simplicity, we will make the factorization assumption

\[
F_\beta^\phi(t_1, t_2) = w_\beta^\phi(t_1) w_\beta^\phi(t_2) \Rightarrow \langle \ln T_1 \ln T_2 \rangle_2 = \frac{1}{4} \left( c_{11}^{(2)} \right)^2,
\]

where the last identity is obtained on the basis of identity (B13) for \( n = 1 \).

Similarly as the skeleton-motivated evaluation method (70)–(71), the skeleton evaluation method (B21) can be performed in principle at any chosen RScl’s \( Q_j \), and in any RScl of the class (11). This method was denoted as ‘v1’ in Ref. [12]. However, skeleton method (B21) makes sense only if the skeleton expansion (B2) really exists. If the latter exists, it probably does so only in a specific (‘skeleton’) scheme [34, 35]. In contrast, the skeleton-motivated evaluation method (70)–(71) does not rely on the existence of the skeleton expansion.

**APPENDIX C: LEADING-SKELETON CHARACTERISTIC FUNCTIONS IN THE SPACE-LIKE AND TIME-LIKE FORM**

In this Appendix we summarize the knowledge of the LS characteristic functions for the space-like observables \( D(Q^2) \). In the space-like formulation (65), which involves the space-like coupling \( A_1 \), the characteristic function can be obtained from the knowledge of the leading-\( \beta_0 \) coefficients \( c_{nn}^{(1)} \) – cf. Eqs. (40) and (42), following the formalism of Neubert [31].

For example, in the case of the Bjorken polarized sum rule (BjPSR) \( d_b(Q^2) \), the leading-\( \beta_0 \) coefficients were obtained in Ref. [40]. In \( \overline{\text{MS}} \) RSch and at RScl \( \mu^2 = Q^2 \exp(C) \) (we use \( \Lambda = \Lambda \) throughout, i.e., \( C = \overline{C} \equiv -5/3 \)), they are

\[
c_{nn}^{(1)} = n! \left[ \frac{8}{9} + \frac{4}{9} (-1)^n - \frac{5}{18} 2^n - \frac{1}{18} \frac{1}{2^n} (-1)^n \right] (n = 0, 1, \ldots).
\]

This implies that the (leading-\( \beta_0 \)) Borel transform is

\[
\hat{S}_b(u; Q^2; \mu^2 = Q^2 e^C) = \sum_{n=0}^{\infty} \frac{1}{n!} c_{nn}^{(1)} u^n = \frac{1}{3} \frac{(3 + u)}{(1 - u^2)(1 - u^2/4)}.
\]

The renormalon poles are at \( u = \pm 1, \pm 2 \). The LS characteristic function appearing in (68) is then obtained by the general formula

\[
F_\beta^\phi(\tau) = \frac{1}{2\pi^2} \int_{u_0 - i\infty}^{u_0 + i\infty} du \hat{S}_b(u) \tau^u,
\]

where \( u_0 \) is any real number closer to the origin than the leading renormalon \( (-1 < u_0 < 1) \). We can choose \( u_0 = 0 \) and introduce a new integration variable \( r = -iu \). The integral, with \( \hat{S}(u) \) of Eq. (C2), then reduces to

\[
F_\beta^\phi(\tau) = \frac{2}{3\pi} \int_{-\infty}^{+\infty} dr e^{ir \ln \tau} \frac{(3 + ir)}{(r + i)(r - i)(r + 2i)(r - 2i)},
\]
which can be calculated by the use of the Cauchy theorem in the complex $r$-plane: when $\tau > 1$, we close the path with a large semicircle in the upper half plane; when $\tau < 1$, in the lower half plane. This gives us the result

$$F^E_b(\tau) = \begin{cases} \frac{8}{9} \tau (1 - \frac{8}{9} \tau) & \tau \leq 1 \\ \frac{4}{9} \tau (1 - \frac{4}{9} \tau) & \tau \geq 1 \end{cases},$$

(C5)

which we already used in \[12\].

The LS characteristic function for the Adler function $d_v(Q^2)$ was obtained in Ref. \[31\], on the basis of the large-$\beta_0$ expansion of the Borel transform of $d_v$ obtained in Refs. \[56, 57\].

$$F^E_v(t) = 2C_F t \left[ \left( \frac{7}{4} - \ln t \right) t + (1 + t) \left( \text{PolyLog}_2(-t) + \ln t \ln(1 + t) \right) \right] \quad (t \leq 1)$$

(C6)

$$= 2C_F \left[ t(1 + \ln t) + t(1 + t)(\text{PolyLog}_2(-1/t) - \ln t \ln(1 + 1/t)) \right] \quad (t \geq 1),$$

(C7)

where $C_F = (N_c^2 - 1)/(2N_c) = 4/3$.

The semihadronic $\tau$ decay ratio $r_\tau$ is a time-like observable. The LS term of $r_\tau$ can be obtained from the LS-term of the Adler function on the basis of the relation

$$r_\tau(\Delta S = 0, m_q = 0) = \frac{1}{2\pi} \int_{-\pi}^{+\pi} d\phi (1 + e^{i\phi})^3 (1 - e^{i\phi}) d_v(Q^2 = m^2, e^{i\phi}).$$

(C8)

This implies for the LS term of $r_\tau(\Delta S = 0, m_q = 0)$

$$r_\tau(\Delta S = 0, m_q = 0)^{(LS)} = \int_0^{\infty} \frac{dt}{t} F^M_r(t) A_1(t e^Q m^2),$$

(C9)

where $A_1$ is the time-like coupling appearing in Eqs. (7)-(10), and superscript $M$ in the characteristic function means that it is Minkowskian (time-like). The latter was obtained in Ref. \[32\].

$$F^M_r(t) = 4C_F \left[ \begin{array}{l} \left( \frac{73}{12} - 23 t^2 - \frac{259}{432} t^3 - 2 \text{PolyLog}_3(-t) - 3\zeta(3) \\
+ \left( \frac{11}{6} + \frac{1}{3} t^2 + \text{PolyLog}_2(-t) \right) \ln t + t \left( \frac{3}{4} t^2 + \frac{1}{6} t^3 \right) \right] \end{array} \right] \quad (t \leq 1),$$

(C10)

$$F^M_r(t) = 4C_F \left[ \begin{array}{l} \frac{575}{216} - \frac{37}{48} t^2 - \frac{17}{12} t^3 - 2 t \text{PolyLog}_3(-1/t) \\
- \left( \frac{85}{36} t - \frac{1}{4} + \frac{4}{3} t^2 + \frac{1}{3} t^3 \right) t \left( \ln t \ln(1 + t) + \text{PolyLog}_2(-1/t) \right) \ln t \\
+ \frac{11}{6} t^2 + 3 + 3 \frac{t^2}{3} + \frac{1}{3} t^3 \right] \ln t \\
\quad (t \geq 1).$$

(C11)

Here, PolyLog$_n$ is the polylogarithm function of $n$’th order (using notation of \[13\]).

The LS part of any space-like observable $D(Q^2)$ can be written in two equivalent forms – the form involving the space-like coupling $A_1$, Eq. \[65\], and the form involving the time-like $A_1$ of Eqs. (7)-(10)

$$D^{(LS)}(Q^2) = \int_0^{\infty} \frac{dt}{t} F^E_D(t) A_1(t e^Q Q^2) = \int_0^{\infty} \frac{dt}{t} F^M_D(t) A_1(t e^Q Q^2),$$

(C12)

where the superscript $M$ stands for the “Minkowskian” (time-like) formulation, and the two characteristic functions are related via relations

$$F^M_D(t) = -\pi \frac{d}{dt} F^E_D(t) = t \int_0^{\infty} \frac{dt'}{(t' + t)^2} F^E_D(t')$$

(C13)

$$F^E_D(t) = \text{Im} F_D(-t - i\varepsilon) \quad \text{where:} \quad F_D(t) \equiv \frac{1}{t} \int_0^{\infty} \frac{d\tau}{\tau + t} F^E_D(\tau).$$

(C14)

\[10\] We use a different normalization, so an additional factor of $t/4$ appears, in comparison to \[32\].
Identity \((C14)\) is a direct consequence of the definition of \(F_D\) there. On the other hand, relation \((C12)\) is a direct consequence of identity \((C13)\) and of the following identity in the complex \(\sigma\)-plane (where \(\sigma = k^2\) is square of a four-vector):

\[
\int_{C_1+C_2} \frac{d\sigma}{\sigma} A_1(K^2 = -\sigma e^C) \left[ F_D(\sigma/Q^2) - F_D(0) \right] = 0 ,
\]

where function \(F_D\) is defined by identity \((C14)\), and the path \(C_1 + C_2\) is depicted in Fig. 8. In the \(\sigma\)-plane, the only singularities of the integrand in Eq. \((C15)\) are the cut of \(A_1(-\sigma e^C)\) along the positive semiaxis, and the cut of \([F_D(\sigma/Q^2) - F_D(0)]\) along the negative semiaxis. Identity \((C15)\) thus follows from the Cauchy theorem.

When applying relation \((C13)\) to the characteristic function \((C5)\) of BjPSR, we obtain for the time-like characteristic function of that observable

\[
F^M_b(t) = t \left[ -\frac{10}{9} - \frac{1}{3t} - \frac{2}{9t^2} - \frac{2}{9} (5t + 4) \ln t + \frac{2}{9} \left( 5t + 4 + \frac{2}{t^2} + \frac{1}{t^3} \right) \ln(1 + t) \right] ,
\]

which agrees with the corresponding expression in Ref. [58] after identifying in their Eq. (4.57): \(\dot{F}_3(\epsilon, N = 1) \equiv (-3/2)F^M_b(t)\), and \(\epsilon \equiv \mu^2/Q^2\) (Ref. [58] uses apparently \(C = 0\)).

**APPENDIX D: EXPLICIT EXPRESSIONS FOR VARIOUS COEFFICIENTS**

Expansion \((2)\) is solution of the perturbative RGE equation \((1)\). If the conventional ("MS") reference scale \(\Lambda\) \([14, 15]\) is adopted, and RGE \((1)\) is iteratively solved for large \(Q^2/\Lambda^2 \gg 1\) in an arbitrary RSch \((\beta_2, \beta_3, \ldots)\), this results in expansion \((2)\) with coefficients \(K_{k\ell}\) given in Eqs. \((3)\) for \(k \leq 3\), and for \(k = 4, 5, 6\) given below (notations: \(c_j \equiv \beta_j/\beta_0\)).

For \(k = 4\):

\[
\begin{align*}
K_{40} &= -\frac{1}{2} \left( \frac{c_3^4}{\beta_0^3} \right) \left( 1 - \frac{c_3}{c_1} \right) , \\
K_{41} &= - \left( \frac{c_1^3}{\beta_0} \right) \left( -2 + 3 \frac{c_2}{c_1} \right) , \\
K_{42} &= \frac{5}{2} \left( \frac{c_1^4}{\beta_0^3} \right) , \\
K_{43} &= - \left( \frac{c_1^4}{\beta_0^3} \right) .
\end{align*}
\]

For \(k = 5\):

\[
\begin{align*}
K_{50} &= \frac{1}{6\beta_0} \left( 7c_4^4 - 18c_1^3c_2 + 10c_2^2 - c_1c_3 + 2c_4 \right) , \\
K_{51} &= \frac{c_1^4}{\beta_0} \left( 4c_1^3 - 3c_1c_2 - 2c_3 \right) , \\
K_{52} &= -\frac{3}{2\beta_0} \left( c_1^4 - 4c_1^3c_2 \right) , \\
K_{53} &= - \frac{13c_4^4}{3\beta_0} , \\
K_{54} &= \frac{c_4^4}{\beta_0} .
\end{align*}
\]
For $k = 6$:

$$K_{60} = \frac{1}{12\beta_0^6} \left( 17c_1^2 - 18c_1^2c_2 - c_1c_2^2 - 23c_2^2c_3 + 24c_2c_3 - 2c_1c_4 + 3c_5 \right),$$

$$K_{61} = \frac{1}{6\beta_0^6} \left( -11c_1^2 + 72c_1^2c_2 - 50c_1^2c_3 - 7c_2^2c_3 - 10c_1c_4 \right),$$

$$K_{62} = \frac{1}{2\beta_0^6} \left( -23c_1^2 + 27c_1^2c_2 + 10c_2^2c_3 \right),$$

$$K_{63} = \frac{1}{6\beta_0^6} \left( -11c_1^2 - 60c_1^2c_2 \right),$$

$$K_{64} = \frac{77c_1^2}{12\beta_0^6}, \quad K_{65} = -\frac{c_1^2}{\beta_0^6}. \quad \text{(D3)}$$

In practical calculations, we use: (a) at loop-level=3: $c_3 = c_4 = c_5 = 0$ and we include in expansion (2) terms $K_{6t}$ up to $k_{\text{max}} = 5$; (b) at loop-level=4: $c_4 = c_5 = 0$ and we include terms up to $k_{\text{max}} = 6$.

The perturbation coefficients $d_j$ ($j = 1, 2$) of the perturbation expansion for the massless Adler function $d_\nu(Q^2)$, cf. Eq. (40), in MS RSch and at RScl $\mu^2 = Q^2$, are known exactly, Refs. [36, 37], respectively

$$d_1^{(\text{Adl.})} = \frac{1}{12} + 0.691772\beta_0, \quad d_2^{(\text{Adl.})} = -27.8059 + 9.26212\beta_0 + 3.10345\beta_0^2. \quad \text{(D4)}$$

The N$^3$LO coefficient $d_3$ in the aforementioned RSch and RScl, was obtained in an approximate form in Ref. [38] [(Eqs. (20) and (12) in [38])]:

$$d_3^{(\text{Adl.})} = 46.1992 - 131.04\beta_0 + 49.5237\beta_0^2 + 2.18004\beta_0^3, \quad \text{(D5)}$$

where the coefficients at $\beta_0^3$ and at $\beta_0^2$ are known exactly ([56, 57, 13]), and the other two coefficients were estimated in Ref. [38] by using the methods of the principle of minimal sensitivity (PMS) [29], and of the effective charge (ECH) [59, 60].

The light-by-light contributions are not included in the coefficients [D4] and [D5]. They have a different topology of diagrams and should probably be resummed separately (cf. Ref. [11]), and they appear for the first time at $\sim a^3$ [37]. They are proportional to the sum of the charges $\sum Q_f$. This sum is zero in the case $n_f = 3$ considered here.

Coefficients $d_1$ and $d_2$ for BjPSR $d_\nu(Q^2)$, in the aforementioned RSch and RScl, were obtained in Ref. [41] and are

$$d_1^{(\text{Bj.})} = \frac{11}{12} + 2\beta_0, \quad d_2^{(\text{Bj.})} = -35.7644 + 10.5048\beta_0 + 6.38889\beta_0^2. \quad \text{(D6)}$$

In the coefficient $d_3^{(\text{Bj.})}$, only the leading-$n_f$ part ($\propto n_f^3$) is known exactly [40] (($\propto$ the leading-$\beta_0$ part, $\propto \beta_0^3$)). On this basis, the authors of Ref. [42] obtained estimates of $d_3^{(\text{Bj.})}$ as a polynomial in $\beta_0$ by using naive nonabelianization (NNA) [43]: $n_f \mapsto -6\beta_0$. Several relations between BjPSR, Bjorken unpolarized sum rule, and Gross-Llewellyn Smith sum rule were found out and investigated in Refs. [61].

**APPENDIX E: MASSLESS PART OF THE STRANGELESS TAU DECAY RATIO**

In this Appendix we extract the measured value of the massless part of the QCD-canonical strangeless ratio $r_\tau(\Delta S = 0, m_\tau = 0)$ for the semihadronic decay, on the basis of the results of the final ALEPH data analysis [8, 41].\footnote{For an extraction of $r_\tau(\Delta S = 0, m_\tau = 0)$ based on the older set of measured results [7], see for example Ref. [62].}

This quantity is related to the ALEPH-measured [8, 41] (V+A)-decay ratio

$$R_\tau(\Delta S = 0) = \frac{\Gamma(\tau^+ \rightarrow n_\tau \text{hadrons}(\gamma))}{\Gamma(\tau^- \rightarrow \nu_\tau e^- \bar{\nu}_e(\gamma))} - R_\tau(\Delta S \neq 0) \quad \text{(E1)}$$

$$= \frac{(1 - B_e - B_\mu)}{B_e} - R_\tau(\Delta S \neq 0) = 3.482 \pm 0.014. \quad \text{(E2)}$$
These values were obtained in Ref. [9] from the measured leptonic branching ratio $B_{\ell} = B(\tau^{-} \to \nu_{\tau} e^{-} \mu^{-} \nu_{\mu}) = (17.810 \pm 0.039)\%$ (ALEPH, [8]), from $B_{u} \equiv B(\tau^{-} \to \nu_{\tau} e^{-} \mu^{-} \nu_{\mu}) = (17.332 \pm 0.049)\%$ (world average, [9]), and from the strangeness-changing branching ratio $B_{s} = (2.85 \pm 0.11)\%$ (ALEPH, [8]). The relation between the canonical massless quantity $r_{\tau}(\Delta S = 0, m_{q} = 0)$ and quantity $[E2]$ is

$$r_{\tau}(\Delta S = 0, m_{q} = 0) = r_{\tau}(\Delta S = 0, m_{q}) - \delta r_{\tau}(\Delta S = 0, m_{u,d} \neq 0)$$

where $\delta r_{\tau}(\Delta S = 0, m_{u,d} \neq 0)$ due to the nonzero quark masses are about $0.006 \pm 0.002$ and the sum of the $D = 2, 4, 6,$ and $8$-dimensional corrections $\frac{\delta_{u,d}^{D} + \delta_{u,d}^{A}}{2}$ and their value is $0.006 \pm 0.004$. The gluon condensate contribution is included, and $(-5.0 \pm 1.7) \times 10^{-3}$ if the gluon condensate contribution is not included (using for the gluon condensate the ALEPH-value $\langle aGG \rangle = (-0.5 \pm 0.3) \times 10^{-2}$). Inserting all the aforementioned values in relation [E4] and taking into account the value [E2], we extract the experimental prediction for $r_{\tau}(\Delta S = 0, m_{q} = 0)$ based on the most recent ALEPH data

$$r_{\tau}(\Delta S = 0, m_{q} = 0)_{\text{exp.}} = 0.204 \pm 0.005$$

where the uncertainties have been added in quadrature. The uncertainty in Eq. [E6] is dominated by the experimental uncertainty $\delta R_{\tau} = \pm 0.014$ [E2]. The value [E6] remains unaffected up to the displayed digits when we either include or exclude from the above quantity the gluon condensate contribution.

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