Extrinsic homogeneity of parallel submanifolds

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Abstract

We consider parallel submanifolds $M$ of a Riemannian symmetric space $N$ and study the question whether $M$ is extrinsically homogeneous in $N$, i.e. whether there exists a subgroup of the isometry group of $N$ which acts transitively on $M$. First, given a “2-jet” $(W,b)$ at some point $p \in N$ (i.e. $W \subset T_pN$ is a linear space and $b : W \times W \to W^\perp$ is a symmetric bilinear form), we derive sufficient – and, up to a certain degree, also necessary – conditions for the existence of an extrinsically homogeneous parallel submanifold which passes through $p$ and whose 2-jet at $p$ is given by $(W,b)$. Second, we focus our attention on complete, (intrinsically) irreducible parallel submanifolds of $N$. Provided that $N$ is of compact type or of non-compact type, we establish the extrinsic homogeneity of every complete, irreducible parallel submanifold of $N$ whose dimension is at least 3 and which is not contained in any flat of $N$.

1 Introduction

In this article, always $N$ is a Riemannian symmetric space and $f : M \to N$ an isometric immersion; $TM$ denotes the tangent bundle of $M$, $\perp f$ the normal bundle of $f$, $h : TM \times TM \to \perp f$ the second fundamental form and $S : TM \times \perp f \to TM, (x,\xi) \mapsto S_\xi(x)$ the shape operator. Let $\nabla^M$ and $\nabla^N$ denote the Levi-Civita connection of $M$ and $N$, respectively, and let $\nabla^\perp$ be the induced connection on $\perp f$ (obtained by projection). The equations of Gauß and Weingarten state for $X,Y \in \Gamma(TM), \xi \in \Gamma(\perp f)$

\[ \nabla^N_X T f Y = Tf(\nabla^M X Y) + h(X,Y) \quad \text{and} \quad \nabla^N_X \xi = -T f(S_\xi(X)) + \nabla^\perp_X \xi. \]  

On the vector bundle $L^2(TM, \perp f)$ there is a connection induced by $\nabla^M$ and $\nabla^\perp$ in the usual way, often called “Van der Waerden-Bortolotti connection”.

Definition 1. $f$ is called parallel if its second fundamental form $h$ is a parallel section of the vector bundle $L^2(TM, \perp f)$.

In a similar fashion, we define parallel submanifolds of $N$ (via the isometric immersion given by the inclusion map $\iota^N : M \hookrightarrow N$).

Example 1. 1-dimensional parallel isometric immersions $c : \mathbb{R} \to N$ are either geodesics or (extrinsic) circles (in the sense of [NY]); they are given by the Frenet curves of osculating rank 1 resp. 2, parameterized by arc-length.

Let $I(N)$ denote the Lie group of isometries of $N$ (see [He], Ch. IV, § 2 and § 3), $I^0(N)$ be its connected component and $i(N)$ the corresponding Lie algebra. For each $X \in i(N)$ we have the one-parameter subgroup $\psi^X_t := \exp(tX)$ of isometries on $N$; the corresponding “fundamental vector field” $X^*$ on $N$ (in the sense of [KN]) defined by

\[ X^*(p) := \left. \frac{d}{dt} \right|_{t=0} \psi^X_t(p) \]  

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is a Killing vector field on $N$ such that $\psi^X(t \in \mathbb{R})$ is the flow of $X^\ast$. The isotropy subgroup of $I^0(N)$ resp. the isotropy representation with respect to some fixed origin $o \in N$ are given by

$$K := \{ g \in I^0(N) \mid g(o) = o \},$$

(3)

$$K \to SO(T_oN), \ g \mapsto T_o g.$$  

(4)

Let $\mathfrak{t}$ denote the Lie algebra of $K$, $i(N) = \mathfrak{t} \oplus \mathfrak{p}$ denote the Cartan decomposition with respect to some base point $o \in M$ and $\pi_1 : i(N) \to T_o N$ denote the canonical projection, given by $X \mapsto X^\ast(o)$; then

$$\mathfrak{t} = \{ X \in i(N) \mid \pi_1(X) = 0 \},$$

(5)

and $\pi_1 \mid \mathfrak{p}$ induces the usual isomorphism $\mathfrak{p} \to T_o N$, whose inverse $\Gamma : T_o N \to \mathfrak{p}$ is usually called the transvection map of $N$ at $o$. Let $\pi_2 : \mathfrak{t} \to SO(T_oN)$ denote the linearized isotropy representation, i.e.

$$\forall X \in \mathfrak{t}, \forall u \in T_o N : \pi_2(X) = \frac{d}{dt}_{t=0} T_o \psi^X_t(u).$$

(6)

Then one knows that $\pi_1$ is an equivariant map of $\mathfrak{t}$-modules, in the following sense:

$$\forall X \in \mathfrak{t}, Y \in \mathfrak{p} : \pi_1(\text{ad}(X)Y) = \pi_2(X) \pi_1(Y).$$

(7)

Given a submanifold $M \subset N$, there may exist a connected Lie subgroup $G \subset I(N)$ with $g(M) = M$ for each $g \in G$ and $M = \{ g(p) \mid g \in G \}$ for some point $p \in M$, in which case $M$ will be called a homogeneous submanifold (likewise we say that $M$ is extrinsically homogeneous). Note that a homogeneous submanifold is always a complete Riemannian space; however its topology is not necessarily the subspace topology, for more details see [Var], p. 17.

Since the parallelity of $h$ can be seen as the extrinsic analogue of the infinitesimal characterization of a symmetric space, $\nabla R = 0$, one should intuitively expect that a complete parallel submanifold of $N$ is a homogeneous submanifold. In fact, if $N$ is a Euclidean space and $M$ is a complete parallel submanifold of $N$, then it was observed by Ferus in [F1] that $M$ is a symmetric submanifold (i.e. $M$ is invariant under the reflections at the various normal spaces) and hence, in particular, $M$ is a homogeneous submanifold; a geometric proof of this observation was given in [St]. This simple relation between parallel and symmetric submanifolds remains true if the ambient space is a space form, but no longer in more general ambient spaces:

**Definition 2.** (a) An intrinsically flat, totally geodesic submanifold of $N$ is shortly called a flat of $N$.

(b) The rank of $N$ is the maximal dimension of a flat of $N$, cf. [He], Ch. V, § 6.

If the ambient space $N$ is a rank-1 symmetric space, then, however, as a result of the explicit classification of parallel submanifolds in $N$, every complete parallel submanifold is extrinsically homogeneous (cf. the results presented in [Eco], Ch. 9 and Theorem 2.1 of [MT]). But, as already observed in [MT], this is definitely no longer true if $N$ is of “higher rank”, because of the following fact:

**Proposition 1.** Suppose that $N$ is of compact type or of non-compact type*. If $M \subset N$ is a (not necessarily parallel) homogeneous submanifold which is contained in some flat of $N$, then $M$ is a flat of $N$, too.

**Proof.** Let $\tilde{N}$ be a flat of $N$ with $o \in \tilde{N}$. Since $\tilde{N}$ is a totally geodesic submanifold of $N$, it is well known that then $\mathfrak{m} := \Gamma(T_o \tilde{N})$ is a “Lie triple system”, i.e. $[\mathfrak{m}, [\mathfrak{m}, \mathfrak{m}]] \subset \mathfrak{m}$; (cf. [He], Ch. IV, § 7). I claim that the Lie subalgebra $\mathfrak{g} := \{ X \in i(N) \mid X^\ast(\tilde{N}) \subset T\tilde{N} \}$ (which corresponds to the Killing vector fields which are tangent to $\tilde{N}$) is equal to $[\mathfrak{m}, \mathfrak{m}] \oplus \mathfrak{m}$. For this:

Since $\tilde{N}$ is totally geodesic, $[\mathfrak{m}, \mathfrak{m}] \oplus \mathfrak{m} \subset \mathfrak{g}$ is obvious. In the other direction, let $\mathfrak{o}$ denote the orthogonal complement of $[\mathfrak{m}, \mathfrak{m}] \oplus \mathfrak{m}$ in $\mathfrak{g}$ with respect to the Killing form $B$ of $i(N)$. Given $X \in \mathfrak{o}$, we have (by means of (7))

$$\text{ad}(X) \mathfrak{m} \subset \mathfrak{m}.$$  

*Recall that $N$ is of compact resp. of non-compact type if the Killing form of $i(N)$ restricted to $\mathfrak{p}$ is strictly negative resp. strictly positive; see [He], Ch. V, § 1.
On the other hand, by means of the invariance of $B$, we have for all $Y, Z \in \mathfrak{m}$:

$$0 = B(X, [Y, Z]) = B(\text{ad}(X)Y, Z)$$

and hence $\text{ad}(X) = 0$, since $B|\mathfrak{m}$ is negative or positive definite. Therefore, $X = 0$ follows from the faithfulness of the linearized isotropy representation (and using again (7)).

Thus $\mathfrak{g} = [\mathfrak{m}, \mathfrak{m}] \oplus \mathfrak{m}$ holds. Moreover, since $\bar{N}$ is a flat of $N$, we have $[\mathfrak{m}, \mathfrak{m}] = \{0\}$ (cf. [He], Ch. V, Prop. 6.1); hence $\mathfrak{g} = \mathfrak{m}$. Now suppose that $M$ is a homogeneous submanifold of $N$ which passes through $o$, say $M = \{g(o) | g \in G\}$ for some subgroup $G \subset I(N)^0$, and that there exists a flat $\bar{N}$ of $N$ with $M \subset \bar{N}$; where, without loss of generality, we may assume that $\bar{N}$ is the smallest flat of $N$ with this property. Therefore, $g(\bar{N}) = \bar{N}$ for each $g \in G$; and thus, by the previous, $\mathfrak{g} = \mathfrak{m} \subset \mathfrak{p}$ (where $\mathfrak{g}$ denotes the Lie algebra of $G$). Hence $M = \exp^N(T_o M)$; the result follows. \hfill \Box

Therefore, if $N$ is a symmetric space whose rank is at least $2$ and which is moreover of compact type or of non-compact type, then there always exist parallel immersed submanifolds of $N$ which are not extrinsically homogeneous: Let $k := \text{rank}(N)$, $\bar{N}^k$ be a maximal flat of $N$, $f : \mathbb{R}^k \to \bar{N}$ be the universal covering and $S^l \subset \mathbb{R}^k$ ($1 \leq l \leq k - 1$) be a totally umbilical sphere. Then $f|S^l : S^l \to N$ is a parallel isometric immersion and hence $f(S^l)$ is a parallel immersed submanifold of $N$ which is not a homogeneous submanifold of $N$ according to Proposition 1.

In this article we will study the extrinsic homogeneity of parallel submanifolds of $N$; thereby the case that $N$ is of higher rank is always implicitly included. In this case there seems to be “not much known about parallel submanifolds of $N$” (cited from [BCO]); hence our results might serve as a first step towards a better understanding of parallel submanifolds in ambient symmetric spaces of higher rank. In particular, we consider parallel isometric immersions $f : M \to N$ defined on an irreducible symmetric space $M$; another article [J2] is planned in order to extend our results to parallel isometric immersions defined on a symmetric space $M$ without Euclidian factor.

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1.1 Overview

This section is meant to provide a thorough overview on the results presented in this article.

Analogous to the notation from [OS], we make:

Definition 3. Let $M$ be a submanifold of $N$. We say that $M$ has extrinsically homogeneous tangent holonomy bundle if there exists a connected Lie subgroup $G \subset I(N)$ with the following properties:

- $g(M) = M$ for all $g \in G$.
- For every $p \in M$ and every curve $c : [0, 1] \to M$ with $c(0) = p$ there exists some $g \in G$ such that $g(p) = c(1)$ and
  \begin{equation}
  (||c||')^u = T_p g | T_p M : T_p M \to T_{c(1)} M .
  \end{equation}

It follows that a submanifold with extrinsically homogeneous tangent holonomy bundle is a homogeneous submanifold of $N$.

Example 2. A 1-dimensional parallel isometric immersion $c : \mathbb{R} \to N$ is the orbit of a one-parameter subgroup of $I(N)$ if and only if $c$ is a covering onto a parallel submanifold with extrinsically homogeneous tangent holonomy bundle.

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According to a well known result of [34], a complete parallel submanifold \( M \subset N \) is uniquely determined by the “2-jet” \( (T_pM, h_p) \) at one point \( p \in M \). Conversely, let a point \( p \in N \), a linear subspace \( W \subset T_pN \) and a symmetric bilinear map \( b : W \times W \to W^\perp \) be given (in the following called a (formal) 2-jet at \( p \)). The following question is somehow more delicate: Does there exist a parallel submanifold \( M \subset N \) with extrinsically homogeneous tangent holonomy bundle which passes through \( p \) and whose 2-jet at \( p \) is given by \( (W, b) \) \( ^{\dagger} \) The answer will be given by Theorem \( ^{\dagger} \) below.

With this aim, given a 2-jet \( (W, b) \) at \( p \) as above, we introduce its (formal) “first normal space” \( \perp (b) := \{ b(x, y) \mid x, y \in W \} \). In this situation, given a linear map \( A \in \mathfrak{so}(T_pN) \) with \( A(O(b)) \subset O(b) \), we put

\[
A^O := A|O(b) : O(b) \to O(b). \tag{9}
\]

Let \( \sigma^+ \in O(b) \) denote the linear reflection in \( \perp (b) \), and \( \operatorname{Ad}(\sigma^+) : \mathfrak{so}(O(b)) \to \mathfrak{so}(O(b)), A \mapsto \sigma^+ \circ A \circ \sigma^+ \) the induced involution on \( \mathfrak{so}(O(b)) \). Let \( \mathfrak{so}(O(b))_{\pm} \) resp. \( \mathfrak{so}(O(b))_{\pm} \) denote the +1- resp. −1-eigenspaces of \( \operatorname{Ad}(\sigma^+) \), i.e.

\[
\mathfrak{so}(O(b))_{\pm} := \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \bigg| A \in \mathfrak{so}(W), B \in \mathfrak{so}(\perp (b)) \right\}, \tag{10}
\]

\[
\mathfrak{so}(O(b))_{\pm} := \left\{ \begin{pmatrix} 0 & -C^* \\ C & 0 \end{pmatrix} \bigg| C \in L(W, \perp (b)) \right\}. \tag{11}
\]

According to \( ^{\dagger} \), there exists a unique linear map \( b : W \to \mathfrak{so}(O(b))_{\pm} \) characterized by

\[
\forall x, y \in W : b(x) y = b(x, y). \tag{12}
\]

Inspired by [Co], we make:

**Definition 4.** Let a 2-jet \( (W, b) \) at \( o \) be given. \( (W, b) \) is called an *infinitesimal model* if the following properties hold:

- \( W \) is a curvature invariant subspace of \( T_oN \), i.e. we have \( R^N(x, y)z \in W \) for all \( x, y, z \in W \).
- \( b \) is “\( R^N_o \)-semiparallel” (in the sense of [JR], Definition 1), i.e.

\[
\forall x, y, z \in W, v \in T_pN : b(R^N(x, y)z - [b(x), b(y)]z) v = [R^N(x, y) - [b(x), b(y)], b(z)] v. \tag{13}
\]

- For every \( x \in W \) there exists some \( X \in \mathfrak{f} \) such that the linear map \( A := \pi_2(X) \) satisfies

\[
A(O(b)) \subset O(b) \quad \text{and} \quad A^O = b(x). \tag{14}
\]

Note that we have \( \pi_2(\mathfrak{f}) = \mathfrak{so}(T_oN) \) if and only if \( N \) is a space form.

**Theorem 1.** In the situation of Definition \( ^{\dagger} \), the 2-jet \( (W, b) \) is an infinitesimal model if and only if there exists a parallel submanifold with extrinsically homogeneous tangent holonomy bundle which passes through \( o \) and whose 2-jet at \( o \) is given by \( (W, b) \).

In Sections \( ^{\dagger} \) we will give the proof of the “only-if” direction of the above theorem, which is suitably done in the context of canonical connections. In the other direction, given an infinitesimal model \( (W, b) \), in Section \( ^{\dagger} \) we conduct an “extrinsic analogue of the Nomizu construction” \( ^{\dagger} \) (cited from [Co]) for arbitrary ambient symmetric space \( N \), as will be explained there. As an application of Theorem \( ^{\dagger} \) Proposition \( ^{\dagger} \) gives an idea how parallel submanifolds of \( N \) are possibly related to certain extrinsically symmetric submanifolds.

\(^{\dagger}\) In [JR] the analogous problem was solved for arbitrary parallel submanifolds.

\(^{\dagger}\) Following an idea due to K. Nomizu, given a pair \( (V, R) \) where \( V \) is a Euclidian vector space and \( R : V \times V \to \mathfrak{so}(V) \) is a “curvature-like tensor” which generates a symmetric holonomy system, in [BCO], Ch. 4.3, there is constructed a symmetric space \( N \) with \( T_pN \cong V \) and \( R^N_p \cong R \) at some point \( p \in N \).
It is well known that a parallel submanifold of \(N\) is (intrinsically) a locally symmetric space (see Parts (e) and (f) of Proposition \([\mathbb{2}]\)). Furthermore, according to Theorem 7 of \([\mathbb{1}R]\), for every (possibly not complete) parallel submanifold \(M_{\text{loc}} \subset N\) there exists a simply connected symmetric space \(M\), an open subset \(U \subset M\) and a parallel isometric immersion \(f : M \to N\) such that \(f|U : U \to M_{\text{loc}}\) is a covering. Therefore, in order to study parallel submanifolds of \(N\), it suffices to consider parallel isometric immersions \(f : M \to N\) defined on a simply connected (globally) symmetric space \(M\). In this situation, in order to keep the notation as simple as possible, we implicitly identify \(T_pM\) with the “first osculating space” \(T_f(T_pM)\) by means of the injective linear map \(T_pf\) for each \(p \in M\). Then the \(2\)-jet of \(f\) at \(p\) is given by \((T_pM, h_p)\), and we have the first normal space \(\perp^1_p f = \{h(x,y) \mid x,y \in T_pM\}_\mathbb{R}\) and the second osculating space \(\mathcal{O}_p f = T_pM \oplus \perp^1_p f\). If \(M \subset N\) is actually a (smooth) submanifold, then the first normal space \(\perp^1_p M\) and the second osculating space \(\mathcal{O}_p M\) are defined as before via the isometric immersion \(\pi^M : M \to N\). Furthermore, in analogy with \([\mathbb{1}2]\) we make:

**Definition 5.** For each \(p \in M\) and \(x \in T_pM\) let \(h_p : T_pM \to \mathfrak{so}(\mathcal{O}_p f)_-\) be the linear map characterized by

\[
\forall x,y \in T_pM : h(x,y) = h(x,y). \tag{15}\]

The following criterion seems to be new, although weaker versions \(^3\) and special cases \(^4\) of it are well known:

**Theorem 2.** Let a simply connected symmetric space \(M\), a parallel isometric immersion \(f : M \to N\) and some origin \(o \in M\) be given. Then the following two assertions are equivalent:

(a) \(f(M)\) is a parallel submanifold with extrinsically homogeneous tangent holonomy bundle and \(f : M \to f(M)\) is a covering.

(b) For every \(x \in T_oM\) there exists some \(X \in \mathfrak{X}\) such that the linear map \(A := \pi_2(X)\) satisfies

\[
A(\mathcal{O}_o f) \subset \mathcal{O}_o f \quad \text{and} \quad A^\mathcal{O} = h(x) \quad \text{(in the sense of \([\mathbb{11}]\)).} \tag{16}\]

The proof of Theorem 2 can be found in Section 2.3.

**Example 3.** (a) If \(N\) is a space form, then the compliance of Assertion (b) is implicitly assured and hence every complete parallel submanifold of a space form has extrinsically homogeneous tangent holonomy bundle.

(b) More generally, according to Theorem 4 of \([\mathbb{1}E]\), for every symmetric submanifold \(M \subset N\) (in the sense of \([\mathbb{2}2], \mathbb{1}5, \mathbb{1}1\)), Assertion (b) is implicitly guaranteed; thus parallel submanifolds with extrinsically homogeneous tangent holonomy bundle can be seen as a natural generalization of symmetric submanifolds.

In order to continue with our investigations, we use the following convention:

**Definition 6 (BCO, A.1).** A Riemannian manifold \(M\) is called “reducible” if its universal covering is the Riemannian product of two Riemannian spaces of dimension at least 1, respectively; otherwise \(M\) is called “irreducible”.

Now let \(f : M \to N\) be a parallel isometric immersion defined on a simply connected, irreducible symmetric space \(M\). According to Proposition \([\mathbb{1}2]\), \(T_oM\) is a curvature invariant subspace of \(T_{f(o)} N\) (remember that we always use \(T_o M \cong T_f(T_o M)\)); hence, by virtue of a result due to E. Cartan, the submanifold

\[
\bar{M} := \exp^N(T_o M) \subset N \tag{17}\]

is a totally geodesically embedded symmetric space. Furthermore, according to Proposition \([\mathbb{6}]\) either \(\bar{M}\) is a flat of \(N\) or \(\bar{M}\) is an irreducible symmetric space, too. In case \(\bar{M}\) is a flat of \(N\), we have the following:

\(^3\) For a weaker version of “(a) \(\Rightarrow\) (b)” see Theorem 3 of \([\mathbb{1}E]\).

\(^4\) For a symmetric submanifold “(b) \(\Rightarrow\) (a)” follows from Theorem 4 of \([\mathbb{1}E]\). For a circle (i.e. \(M \cong \mathbb{R}\)), Theorem \([\mathbb{1}2]\) can easily be derived from Corollary 1.4 of \([\mathbb{1}M\mathbb{T}]\), combined with Example \([\mathbb{2}]\) of this article.
Theorem 3. Suppose that \( N \) is of compact type or of non-compact type and let a parallel isometric immersion \( f : M \to N \) defined on a simply connected, irreducible symmetric space \( M^m \) be given. If \( M \) is a flat of \( N \), then \( f(M) \) is contained in some flat of \( N \).

The proof of Theorem 3 can be found in Section 3.1. Theorem 2 essentially uses the classification of parallel submanifolds in a Euclidian space achieved by D. Ferus in [F1].

In Section 3.2 we will deal with the case that \( M \) is irreducible. Thereby, in order to make effectively use of Theorem 2 we pursue the following strategy: It is well known that the second osculating bundle

\[
\mathcal{O}_f := \bigcup_{p \in M} \mathcal{O}_p f \quad \text{resp.} \quad \mathcal{O}_M := \mathcal{O}_t M \quad \text{if} \quad M \subset N \text{ is a submanifold}
\]

is a \( \nabla^N \)-parallel subbundle of the pull-back bundle \( f^*TN \) (see Proposition 2). Thus there exist the following linear connections on \( TN \), \( f^*TN \) and \( \mathcal{O}_f \), respectively: \( TN \) is equipped with \( \nabla^N \), on \( f^*TN \) we have the connection induced by \( \nabla^N \), and on the parallel subbundle \( \mathcal{O}_f \subset f^*TN \) the connection obtained by restriction.

Definition 7 (cf. [J1]). Let \( \mathfrak{hol}(N) \), \( \mathfrak{hol}(f^*TN) \) and \( \mathfrak{hol}(\mathcal{O}_f) \) denote the Lie algebras of the holonomy groups with respect to the base point \( o \) belonging to the linear connections introduced above, respectively.

According to Lemma 11 the availability of the relation

\[
\mathfrak{h}(T_o M) \subset \mathfrak{hol}(\mathcal{O}_f)
\]

implies the compliance of Assertion (b) from Theorem 2. Moreover, we have:

Theorem 4. Let a parallel isometric immersion \( f : M \to N \) defined on a simply connected, irreducible symmetric space \( M^m \) be given. If also \( M \) is an irreducible symmetric space and \( m \geq 3 \), then the availability of Equation (19) is assured.

Summing up the previous, we obtain our main result:

Theorem 5. Suppose that \( N \) is of compact type or of non-compact type and let a parallel isometric immersion \( f : M \to N \) defined on a simply connected, irreducible symmetric space \( M^m \) with \( m \geq 3 \) be given. If \( f(M) \) is not contained in any flat of \( N \), then the following assertions are valid:

(a) Also \( M \) is an irreducible symmetric space.

(b) Equation (19) holds.

(c) \( f(M) \) is a parallel submanifold with extrinsically homogeneous tangent holonomy bundle and \( f : M \to f(M) \) is a covering.

Definition 8 ([BCO], Ch. 2.5). An isometric immersion \( f : M \to N \) is called full if \( f(M) \) is not contained in any proper, totally geodesic submanifold of \( N \).

The following two examples should be seen against the background of Theorem 5.

Example 4. Consider the Riemannian product space \( N := \mathbb{C}P^1 \times \mathbb{C}P^1 \), which is a Hermitian symmetric space of compact type and whose rank equals 2. Let \( \pi_i : \mathbb{C}P^1 \times \mathbb{C}P^1 \to \mathbb{C}P^1 \), \((p_1, p_2) \mapsto p_i\) denote the canonical projections \((i = 1, 2)\). Choose a point \( o \in N := \mathbb{C}P^1 \times \mathbb{C}P^1 \), a unit vector \( x \in T_o N \) and some \( y \in T_o N \) with \( \langle x, y \rangle = 0 \). Then there exists a unique circle \( c : \mathbb{R} \to N \) with \( c(0) = o \), \( \dot{c}(0) = x \) and \( \nabla^N_x \dot{c}(0) = y \). Suppose moreover that the linear span of \( \{x, y\} \) belongs to the open and dense subset of the real Grassmannian \( G_2(T_o N) \) formed by the 2-planes \( W \subset T_o N \) with the following properties:

- \( T_o \pi_i W : W \to T_o \mathbb{C}P^1 \) is an isomorphism for each \( i \).
- \( W \) is neither a complex nor a totally real subspace of \( T_o N \).
• The real numbers \( \kappa_i = \|y_i\|/\|x_i\| \) (\( i = 1, 2 \)) are different (with \( x_i := T_0\pi_i(x) \) and \( y_i := T_0\pi_i(y) \)). Then \( c \) is a full parallel isometric immersion, but \( c \) is not the orbit of a 1-parameter subgroup of \( I(N) \).

A proof of Example 4 is added at the end of Section 2.3.

**Example 5.** Consider \( N := SU(2n)/S(U(n) \times U(n)) \), which is an irreducible Hermitian symmetric space of compact type and whose rank equals \( 2n \). According to [BCO], Table A.7, there exists some \( X \in \mathfrak{p} \) with \( X \neq 0 \) and \( \text{ad}(X)^3 = -\text{ad}(X) \), i.e. \( \text{Ad}(K)X \subset \mathfrak{p} \) is an “irreducible symmetric R-space” (here \( \text{Ad} \) resp. \( \text{ad} \) denote the adjoint representations of \( I(N) \) resp. \( i(N) \)). Moreover, there is associated with this symmetric R-space a family \( M_c \) of symmetric submanifolds of \( N \) (where \( c \) ranges over \( \mathbb{R} \)). Thereby, \( M_0 \) is totally geodesic and \( M_c \) is full in \( N \) for \( c \neq 0 \). One can show that the universal covering space of \( M_c \) splits off a 1-dimensional factor. In case \( c \neq 0 \), \( M_c \) is a parallel submanifold with \( OM_c = TN|M_c \), in accordance with Theorem 1 of [J1]; however (19) fails for \( M := M_c \), as was mentioned in Theorem 6 of [J1].

Finally, we will show that certain full parallel submanifolds of \( N \) are “2-symmetric” (in the sense of [BCO], Ch. 7.2): For every symmetric space \( M \) let \( \text{Sym}(M) \) denote the subgroup of \( I(M) \) generated by its geodesic symmetries. Then one can show that \( \text{Sym}(M) \) is a Lie subgroup of \( I(M) \) (see Lemma 21 and its proof).

**Definition 9.** A submanifold \( M \subset N \) will be called 2-symmetrically embedded if \( M \) is a symmetric space and there exists a Lie group homomorphism \( \hat{f} : \text{Sym}(M) \to I(N) \) such that

\[
\forall g \in \text{Sym}(M) : \hat{f}(g)|M = \iota^M \circ g \tag{20}
\]

**Example 6.** Every symmetric submanifold of \( N \) is a 2-symmetrically embedded parallel submanifold.

**Theorem 6.** Suppose that \( N \) is of compact type or of non-compact type and let a full parallel isometric immersion \( f : M \to N \) defined on a simply connected, irreducible symmetric space \( M \) be given. If additionally \( M \) is isometric to one of

- the 2-fold coverings of the real Grassmannians \( G_{2k}(\mathbb{R}^{2m}) \) \((k \geq 2, k \leq m)\) or \( G_k(\mathbb{R}^{2m+1}) \) \((m \geq 1 \text{ and } k \leq 2m + 1)\), in particular the Euclidean spheres \( S^{2m} \) with \( m \geq 2 \),
- the Grassmannians over the quaternions,
- the Hermitian symmetric spaces of compact type,
- \( G_2/\text{SO}(4) \), \( E_6/\text{SO}(16) \), \( E_8/\text{E}_7 \times \text{SU}(2) \), \( E_7/\text{SO}(12) \times \text{SU}(2) \), \( F_4/\text{Sp}(3) \times \text{SU}(2) \),
- or the non-compact duals of these compact symmetric spaces,

then \( M := f(M) \) is 2-symmetrically embedded in \( N \).

The proof of this theorem can be found in Section 2.3.

## 2 Parallel submanifolds with extrinsically homogeneous tangent holonomy bundle

Let \( N \) be a symmetric space and \( f : M \to N \) be a parallel isometric immersion. In the following, we keep to our convention from Section 1 that we implicitly identify \( T_pM \) with the first osculating space \( Tf(T_pM) \) for each \( p \in M \); for convenience, the reader may assume that \( M \subset N \) is a submanifold and \( f = \iota^M \).

Then we have:

**Proposition 2.** (a) The tangent spaces \( T_pM \) are curvature invariant subspaces of \( T_{f(p)}N \).

(b) The first normal spaces \( \perp^p \) are curvature invariant subspaces of \( T_{f(p)}N \), too.
(c) We have for all \( p \in M \) and \( x_1, x_2, y_1, y_2 \in T_p M \):
\[
R^1(x_1, x_2) h(y_1, y_2) = h(R^M(x_1, x_2) y_1, y_2) + h(y_1, R^M(x_1, x_2) y_2) \quad \text{and} \quad (21)
\]
\[
h(R^M(x_1, x_2) y_1) = [R^N(x_1, x_2) - [h(x_1), h(x_2)]], h(y_1)]. \quad (22)
\]

(d) \( \mathcal{O}_f \) is a \( \nabla^N \)-parallel vector subbundle of \( f^*TN \). Hence we have \( R^N(x_1, x_2)(\mathcal{O}_p f) \subset \mathcal{O}_p f \) for all \( x_1, x_2 \in T_p M \), and the curvature tensor of \( \mathcal{O}_f \) is given by
\[
R^{\mathcal{O}_f}(x, y) = (R^N(x, y))^\mathcal{O}_f. \quad (23)
\]

(e) \( M \) is locally symmetric, i.e., \( R^M \) is parallel.

(f) If \( M \) is a complete and simply connected parallel submanifold of \( N \), then \( M \) is a symmetric space.

**Proof.** (a) follows from the Codazzi Equation. For (b) see Theorem 1 of [J1]. For (c) see [J1], Proposition 1. (d) is an immediate consequence of the parallelity of \( h \). For the proof of (e) one needs Assertion (a) and the curvature equation of Gauß. If \( M \) is simply connected and complete, then it is even globally symmetric (cf. [H1], Ch.IV, Theorem 5.6).

In accordance with (5), let \( \pi_2 : \mathfrak{t} \to \mathfrak{so}(T_o N) \) denote the linearized isotropy representation. This representation is well known to be a faithful representation, thus we conclude by means of (5):
\[
(\pi_1(X) = 0 \quad \text{and} \quad \pi_2(X) = 0) \Rightarrow X = 0. \quad (24)
\]

If \( M \) is a full parallel submanifold of \( N \), then Lemma 1 below states an improvement version of Equation (23), which will become useful in the following. For its proof, we need a well known result on the “reduction of the codimension” in the sense of Erbacher [12] (cf. Theorem 2 of [J1] or Lemma 2.2 of [Ts1]):

**Theorem 7** (Dombrowski). If \( f : M \to N \) is a parallel isometric immersion and for some point \( p \in M \) the second osculating space \( \mathcal{O}_p f \) is contained in a curvature invariant subspace \( V \) of \( T_p N \), then \( f(M) \subset \bar{N} \), where \( \bar{N} \) denotes the totally geodesic submanifold \( \exp^N_p(V) \).

**Lemma 1.** Suppose that \( V \) is a linear subspace of \( T_o N \) which is not contained in any proper, curvature invariant subspace of \( T_o N \).

(a) If \( g \) is an isometry of \( N \) with
\[
g(o) = o \quad \text{and} \quad T_o g|V = \text{Id},
\]
then \( g = \text{Id}_N \).

(b) If \( \pi_1(X) = 0 \) and \( \pi_2(X)|V = 0 \) holds for some \( X \in i(N) \), then \( X = 0 \).

Let \( M \subset N \) be a full parallel submanifold.

(c) If \( X^*|M = 0 \) holds for some \( X \in i(N) \), then \( X = 0 \).

According to Theorem 7 a parallel submanifold \( M \) is full in \( N \) if and only if \( V := \mathcal{O}_o M \) satisfies the hypothesis of this Lemma.

**Proof.** For (a): Since \( g \) is an isometry on \( N \), \( V := \{ v \in T_o N \mid T_o g(v) = v \} \) is a curvature invariant linear subspace with \( \mathcal{O}_o M \subset V \). Thus \( T_o g = \text{Id}_{T_o N} \) besides \( g(o) = o \), and therefore we have \( g = \text{Id}_N \) since an isometry is determined by its value and differential at one point.

For (b): Of course it suffices to prove hat \( \psi^N_1 = \text{Id}_N \). By assumption we have counter \( \psi^N_1(o) = o \), hence \( \pi_1(X) = 0 \) and thus \( \pi_2(X)|V = 0 \) implies \( T_o \psi^N_1(V) \subset V \) and \( T_o \psi^N_1|V = \text{Id} \) as a consequence of (24); therefore \( \psi^N = \text{Id}_N \) by Part (a).

For (c): By assumption we have \( \psi^N_1|M = \text{Id} \); in particular \( T_o \psi^N_1|T_o M = \text{Id} \) and (since \( \psi^N_1 \) is an isometry of \( N \))
\[
(T_o \psi^N_1) h(x, y) = h(T_o \psi^N_1 x, T_o \psi^N_1 y) = h(x, y).
\]
Therefore \( T_o \psi^N_1|\mathcal{O}_o M = \text{Id} \) and thus \( \psi^N_1 = \text{Id}_N \) by Part (b). Because by assumption \( M \subset N \) is full, the subspace \( V := \mathcal{O}_o M \) of \( T_o N \) can not be contained in any proper curvature invariant subspace \( V \subset T_o N \); since otherwise \( M \) would be contained in the totally geodesic submanifold \( \exp_o(V) \subset N \) by means of Theorem 7.
2.1 Homogeneous vector bundles and canonical connections

In this section, we will prove the “only if” direction of Theorem 1. First, let $G$ be an arbitrary connected Lie group and $M$ be some homogeneous $G$-manifold; i.e. we only assume that there exists a transitive action $G \times M \to M$, $(g, p) \mapsto g \cdot p$. For an arbitrary origin $o \in M$ let $H \subset G$ denote the isotropy subgroup at $o$ and $\mathfrak{g}$ resp. $\mathfrak{h}$ denote the Lie algebras of $G$ resp. $H$.

**Definition 10.**

(a) $M$ is called a reductive homogeneous space if there exists a “reductive decomposition”

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} \quad \text{with} \quad \forall h \in H : \text{Ad}(h)(\mathfrak{m}) \subset \mathfrak{m},$$

where $\text{Ad}$ denotes the adjoint representation of $G$ (cf. [BCO], A.3).

(b) A vector bundle $E \to M$ is called a homogeneous vector bundle if there exists an action $\alpha : G \times \mathfrak{g} \to \mathfrak{g}$ by vector bundle isomorphisms such that the bundle projection of $E$ is equivariant.

If $M$ is a reductive homogeneous space and $E \to M$ is a homogeneous vector bundle over $M$, then it induces a distinguished connection $\nabla^E$ on $E$, called the canonical connection. In the framework of [KN], it can be obtained as follows:

$$\tau : G \to M, g \mapsto g \cdot o,$$

is a principal fiber bundle,

$$\mathcal{H}_g := \{ X_g \mid X \in \mathfrak{m} \}$$

defines a $G$-invariant connection $\mathcal{H}$ on it, where the elements of $\mathfrak{m}$ are also considered as left-invariant vector fields on $G$ (see [KN], p. 239). Since $E$ is a vector bundle associated with $\tau$ via

$$G \times \mathfrak{g}_o \to \mathfrak{g}, (g, v) \mapsto \alpha(g, v),$$

the connection $\mathcal{H}$ induces a linear connection $\nabla^c$ on $E$, see [KN], p. 87 or [Po], p. 290. One knows that $\nabla^c$ does not depend on the special choice of the base point $o$; therefore it is called the canonical connection.

In order to relate the parallel displacement in $E$ with respect to $\nabla^c$ to the horizontal structure $\mathcal{H}$, let a curve $c : \mathbb{R} \to M$ with $c(0) = p$ and some $g \in G$ with $g \cdot o = p$ be given; then

$$\forall v \in \mathbb{E}_p : \left(\begin{array}{c} 1 \\ 0 \end{array}\right) v = \alpha(\dot{c}(1) g^{-1}, v),$$

where $\dot{c} : [0, 1] \to G$ denotes the $\mathcal{H}$-lift of $c$ with $\dot{c}(0) = g$.

**Example 7.** If $\mathfrak{m}$ is a reductive complement in the sense of (25), then:

(a) For each $X \in \mathfrak{m}$ the 1-parameter subgroup $: \mathbb{R} \to G, t \mapsto \exp(t X)$ is the integral curve of $X$ and hence, in accordance with (27), this is a horizontal curve. Therefore, in view of (29), if $E \to M$ is a homogeneous vector bundle, then we have for all $X \in \mathfrak{m}, v \in \mathbb{E}_o$:

$$t \mapsto \alpha(\exp(t X), v) \text{ is a } \nabla^c\text{-parallel section of } E \text{ along the curve } t \mapsto \exp(t X) \cdot o.$$

(b) The induced action $\alpha^M : G \times TM \to TM$ equips $E := TM$ with the structure of a homogeneous vector bundle over $M$, in accordance with Definition 10. Let $\nabla^c$ denote the canonical connection of $TM$ induced by the reductive decomposition (25). In accordance with [KN], Ch. X, Corollary 2.5, the $\nabla^c$-geodesics $\gamma : \mathbb{R} \to M$ with $\gamma(0) = o$ are given by

$$\gamma(t) = \exp(t X) \cdot o \quad (X \in \mathfrak{m}).$$

Now we assume that $M$ is a Riemannian manifold, whose Levi Civita connection is denoted by $\nabla^M$, and $G$ acts transitively through isometries on $M$. Then $TM$ is a homogeneous vector bundle over $M$, as explained in the previous example.

**Proposition 3 ([KN], Ch. X, Theorem 2.8).** Suppose that:
• \( G \) acts effectively on \( M \) through \( \nabla^M \)-parallel vector bundle isomorphisms, 
• and for every curve \( c : [0, 1] \to M \) with \( c(0) = o \) there exists some \( g \in G \) with

\[
\forall y \in T_oM : (\frac{1}{0}c)_M y = \alpha(g, y).
\]

Then there exists a unique reductive decomposition \((25)\) such that \( \nabla^M \) is the corresponding canonical connection.

Let \( N \) be a symmetric space, \( o \in N \) a base point and \( \iota(N) = \mathfrak{k} \oplus \mathfrak{p} \) denote the Cartan decomposition. In order to give a characterization of full parallel submanifolds \( M \subset N \) with extrinsically homogeneous tangent holonomy bundle in terms of canonical connections, for each \( X \in \iota(N) \) we introduce the covariant derivative \( \nabla^N \), which is a skew-symmetric tensor field of type \((1, 1)\) on \( M \); hence we may define \( \pi_2 : \iota(N) \to \mathfrak{so}(T_oN), \ X \mapsto \nabla^N X^*(o) \). Then we have the well known formula \((cf. \ [KN], p. 245)\):

\[
\forall u \in T_oN : \pi_2(X) u = \frac{\nabla^N}{dt} \bigg|_{t=0} T_o\Psi^X_t(u).
\]

In particular, \( \pi_2 \mathfrak{k} \) coincides with the linearized isotropy representation introduced in \((10)\).

**Proof for \((33)\).** Let \( Y \in \iota(N) \) with \( \pi_1(Y) = u \) and consider the “variation” \( F(t, s) := F_t(s) := \Psi^X_t \circ \Psi^Y_s(o) \) of \( c = F(\cdot, 0) \). We have \( X \circ F = \frac{\partial F}{\partial s} + \frac{\partial F}{\partial t} = TF(\frac{\partial}{\partial t}|_{(t, 0)}) \) and \( T_0\Psi^X_t(u) = \frac{\partial F}{\partial t}|_{(t, 0)} = TF(\frac{\partial}{\partial t}|_{(t, 0)}) \). Therefore, using the structure equation for the torsion due to E. Cartan,

\[
\nabla^N\psi=\frac{\nabla^N}{ds} \bigg|_{s=0} (X \circ F_t) = \frac{\nabla^N}{ds} \frac{\partial F}{\partial t} \bigg|_{(t, 0)} = \frac{\nabla^N}{ds} \frac{\partial F}{\partial s} \bigg|_{(t, 0)} = \frac{\nabla^N}{dt} T_0\Psi^X_t(u).
\]

Furthermore, we note that the Cartan decomposition of \( \iota(N) \) is a reductive decomposition in the sense of Section \((2.1)\). Let \( \nabla^c \) denote the corresponding canonical connection of \( TN \). According to \([KN], \text{Ch. X.2}\), \( \nabla^c \) is a metric and torsion-free connection on \( TN \), hence \( \nabla^c = \nabla^N \). Therefore, by means of \( \text{Example} \((7)\) \) combined with \((33)\), we have \( \pi_2(X) = 0 \) for all \( X \in \mathfrak{p} \); moreover, by the previous, \( \pi_2 \mathfrak{k} \) is the linearized isotropy representation (which is a faithful representation). Therefore, we obtain the well known characterization (see \([BCO], \text{A.4})\):

\[
X \in \mathfrak{p} \text{ if and only if } \pi_2(X) = 0.
\]

**Proposition 4.** Let a full parallel submanifold \( M \subset N \) with \( o \in M \) be given. Then \( M \) has extrinsically homogeneous tangent holonomy bundle if and only if there exists:

• A connected Lie subgroup \( G \subset \iota(N) \) with \( \mathfrak{g}(M) = \mathfrak{m} \) for all \( g \in G \) such that the natural action \( G \times M \to M \) is transitive,

• and a reductive decomposition \( \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} \) of its Lie algebra with respect to this action and the base point \( o \), such that the Levi-Civita connection of \( M \) is the canonical connection on \( TM \) as described in \( \text{Example} \((7)\) \).

Then the reductive complement \( \mathfrak{m} \) is uniquely determined; in fact, we always have \( \mathfrak{m} = \mathfrak{m}_0 \) with

\[
\mathfrak{m}_0 := \{ X \in \iota(N) | \pi_1(X) \in T_oM, \pi_2(X)|\mathcal{O}_oM \subset \mathcal{O}_oM \text{ and } (\pi_2(X))^\mathcal{O} = \mathfrak{h}(\pi_1(X)) \}, \tag{35}
\]

where \((\pi_2(X))^\mathcal{O} := \pi_2(X)|\mathcal{O}_oM : \mathcal{O}_oM \to \mathcal{O}_oM \).

**Proof.** Using also Lemma \((1)\) the first part of the proposition follows from our previous discussion. Thus it remains to show that always \( \mathfrak{m} = \mathfrak{m}_0 \) holds; for which purpose it suffices to establish \( \mathfrak{m} \subset \mathfrak{m}_0 \), because (as a consequence of Lemma \((1)\) and Equation \((35)\)) the restriction of \( \pi_1 \) to \( \mathfrak{m}_0 \) is injective and thus
Given $x \in \mathfrak{m}$ put $x := \pi_1(X)$. Then, in accordance with Example 7, the geodesic $\gamma$ of $M$ with $\dot{\gamma}(0) = x$ is given by

$$\gamma(t) = \exp(t \cdot X)(o)$$
and we have

$$\forall y \in T_o M : \left( \frac{\partial}{\partial t} \gamma \right)(0) = T_o \exp(t \cdot X)y .$$

Therefore, since $M$ is a symmetric space, by virtue of (33) (for $M$) we have:

$$\forall y \in T_o M : \nabla^M_o (X|M) = 0 .$$

Thus, on the one hand, the Gauß equation yields

$$\pi_2(X)|T_o M = h(x, .) \in L(T_o M, \perp^1_o M) .$$

On the other hand, if $x_1, \ldots, x_k$ is a basis of $T_o M$, then according to (36) the sections $T_o \exp(t \cdot X)x_i$ define a parallel frame of $TM$ along $\gamma$; hence by $\xi_{i,j}(t) := h(x_i(t), x_j(t))$ are also defined parallel sections of $\perp M$ along $\gamma$, since $h^M$ is parallel. Because $\psi_t := \exp(t \cdot X)$ is an isometry of $N$ with $\psi_t^N(M) = M$, we have (with $\xi_{i,j} := h(x_i, x_j)$)

$$T_o \psi_t \xi_{i,j} = h(T_o \psi_t x_i, T_o \psi_t x_j) = \xi_{i,j}(t) ,$$
from which we conclude

$$\pi_2(X) \xi_{i,j} = \left. \frac{\nabla^N}{dt} \right|_{t=0} \xi_{i,j}(t) = -S_{\xi_{i,j}}x \in T_o M .$$

Since the vectors $\xi_{i,j}$ span $\perp^1_o M$, the last line combined with (38) implies that $X \in \mathfrak{m}_0$ holds; and therefore we finally conclude that $\mathfrak{m}$ is contained in $\mathfrak{m}_0$.

Proof of the “only if” direction of Theorem 1. Let $M \subset N$ be a parallel submanifold with extrinsically homogeneous tangent holonomy bundle which passes through $o$ and whose 2-jet at $o$ is given by $(W, b)$. In the following, we assume that $M$ is full in $N$, the general case is left to the reader. We notice that:

- $W := T_o M$ is a curvature invariant subspace of $T_o N$, according to Proposition 2.
- Equation (13) holds by means of (22) combined with the curvature Equation of Gauß.

Furthermore, according to Proposition 4, there exists a connected Lie subgroup $G$ of $\text{I}(N)$ with $g(M) = M$ for all $g \in G$ whose Lie algebra is denoted by $\mathfrak{g}$ such that the linear space $\mathfrak{m}_0$ defined in (35) is a reductive complement in $\mathfrak{g}$ with respect to the natural action $G \times M \rightarrow M$ and the base point $o$. In particular, given $x \in W$, there exists some $Z \in \mathfrak{m}_0$ with $\pi_1(Z) = x$ and hence, by definition of $\mathfrak{m}_0$, $(\pi_2(Z))^{\mathfrak{o}} = h(x)$. Now we decompose $Z = X + Y \in \mathfrak{t} \oplus \mathfrak{p}$; then one verifies from (34) that

- $x$ together with $A := \pi_2(X)$ solves (14).

2.2 An extrinsic analogue of the Nomizu construction

In this section, we will give the proof of the “if direction” of Theorem 1. Given an infinitesimal model $(W, b)$, we aim to find a connected Lie subgroup $G \subset \text{I}(N)$ such that the orbit $M := \{ g(o) | g \in G \}$ is a full parallel submanifold of $N$ with extrinsically homogeneous tangent holonomy bundle whose 2-jet at $o$ is given by $(W, b)$; where the corresponding Lie subalgebra $\mathfrak{g} \subset \text{i}(N)$ gets constructed from the infinitesimal model in an explicit way. To this end, we consider the following “bracket” defined on the linear space $T_o N \oplus \pi_2(t)$,

$$\forall x, y \in T_o N : [x, y] := -R^N(x, y) ,$$

$$\forall A \in \pi_2(t), x \in T_o N : [A, x] := -[x, A] := A x ,$$

$$\forall A, B \in \pi_2(t) : [A, B] := A \circ B - B \circ A .$$
Lemma 2. (39) - (11) equips $T_{o}N \oplus \pi_{2}(\mathfrak{t})$ with the structure of a Lie algebra such that the linear map
\[ \iota: i(N) \rightarrow T_{o}N \oplus \pi_{2}(\mathfrak{t}), \quad X \mapsto \pi_{1}(X) + \pi_{2}(X) \] (42)
becomes a Lie algebra isomorphism.

Proof. Injectivity of $\iota$ follows from (22) combined with (43). Since $\pi_{1}$ induces the usual isomorphism $\mathfrak{p} \cong T_{o}N$, $\iota$ is a linear isomorphism. For the bracket relations cf. [BCO], Ch. 4.3.

Now let an infinitesimal model $(W, b)$ be given; where we additionally assume that $\mathcal{O}(b)$ is not contained in any proper curvature invariant subspace of $T_{o}N$. Then Definition 4 in combination with Part (b) of Lemma 4 exhibits the existence of a unique (and hence linear) map $\hat{\Gamma}: W \rightarrow \mathfrak{t}$ such that for each $x \in W$ its value $A := \pi_{2}(b(x))$ solves (14). We introduce the linear map
\[ \hat{\Gamma}: W \rightarrow i(N), \quad x \mapsto \hat{\Gamma}_{x} := \Gamma_{x} + \hat{b}(x), \] (43)
where $\Gamma : T_{o}N \rightarrow \mathfrak{p}$ is the inverse of $\pi_{1}|_{\mathfrak{p}}$. We obtain:

Lemma 3. $\hat{\Gamma}_{x}$ is uniquely characterized by the following properties:
\[ \iota(\hat{\Gamma}_{x}) = x + A \in W \oplus so(T_{o}N), \quad \text{where } A \text{ satisfies } \] (44)
\[ A(\mathcal{O}(b)) \subset \mathcal{O}(b) \quad \text{and} \quad A^{O} = b(x). \] (45)

Lemma 4. For all $x, y, z \in W$ we have:
\[ [\hat{\Gamma}_{x}, \hat{\Gamma}_{y}] = [\Gamma_{x}, \Gamma_{y}] + [\hat{b}(x), \hat{b}(y)] \in \mathfrak{t}, \] (46)
\[ B := \pi_{2}(\hat{\Gamma}_{x}, \hat{\Gamma}_{y}) \text{ maps } W \text{ to } W, \quad \text{and} \quad [[\hat{\Gamma}_{x}, \hat{\Gamma}_{y}], \hat{\Gamma}_{z}] = \hat{\Gamma}_{B(z)}. \] (47)

Proof. First let us calculate the Lie brackets $[\hat{\Gamma}_{x}, \hat{\Gamma}_{y}]$ and $[[\hat{\Gamma}_{x}, \hat{\Gamma}_{y}], \hat{\Gamma}_{z}]$ for all $x, y, z \in W$. This will suitably be done by means of the linear map $\iota$ introduced in Lemma 2, moreover, in the following we will make use of Lemma 2 several times without further reference. The symmetry of $\hat{b}$ implies that
\[ \iota([\Gamma_{x}, \hat{b}(y)]) = \hat{b}(y)(x) = -b(y, x) = -b(x, y) = -\iota([\hat{b}(x), \Gamma_{y}]); \] (48)
thus, by means of the distributive law for the Lie bracket, (49) follows immediately from (13). Let us write
\[ \iota([[\hat{\Gamma}_{x}, \hat{\Gamma}_{y}], \hat{\Gamma}_{z}]) = \hat{x} + A \] (49)
for certain $\hat{x} \in T_{o}N$ and $A \in so(T_{o}N)$. Because $\hat{b}$ takes its values in $\mathfrak{t}$, we obtain from (49) that $[[\hat{\Gamma}_{x}, \hat{\Gamma}_{y}], \hat{b}(z)]$ again is an element of $\mathfrak{t}$, and thus according to (43) combined with (48):
\[ \hat{x} = \iota([[\hat{\Gamma}_{x}, \hat{\Gamma}_{y}], \Gamma_{z}]) \quad \text{and} \quad B(z); \] (50)

furthermore we conclude from (39), (11) and (46) in combination with the definition of $\hat{b}$:
\[ \forall v \in \mathcal{O}(b): \quad B v = - R^{N}(x, y) v + [b(x), b(y)] v, \] (51)
therefore $B(W) \subset W$ by means of the curvature invariance of $W$ and moreover (using similar arguments as already used before), (48) implies that
\[ A = \iota([[\hat{\Gamma}_{x}, \hat{\Gamma}_{y}], \hat{b}(z)]) \quad \text{and} \quad \forall v \in \mathcal{O}(b): \quad A v = [B, \hat{b}(z)] v \] (52)
As a consequence of Lemma 31 (17) now follows from (18), (49) and (51).
We define a linear subspace of i(N) via
\[ m := \{ \hat{\Gamma}(x) \mid x \in W \}. \] (52)

As an analogue of [Co], Proposition 2.3, we have:

**Lemma 5.** (a) The linear space
\[ g := [m, m] \oplus m \text{ is a direct sum,} \] (53)
\[ [m, m] = g \cap \mathfrak{t} \text{ and} \]
\[ [m, m], m] \subset m. \] (54) (55)

According to (53) and using the Jacobi identity, \( g \) is even a Lie subalgebra of i(N): let \( G \) denote the corresponding connected Lie subgroup of I(N). Hence we may consider the orbit \( M := \{ g(o) \mid g \in G \}. \)

(b) The 2-jet of \( M \) is given at \( o \) by \((W, b)\).

Let \( H \) denote the isotropy subgroup in \( G \) at \( o \).

(c) The Lie algebra of \( H \) is given by \( h := [m, m] \) and we have \( m = m_0 \); in particular, \( m \) is an Ad(H)-invariant subspace of \( g \).

As a consequence of (c), the natural transitive action \( G \times M \to M \) and the decomposition (53) equips \( M \) with the structure of a reductive homogeneous space.

**Proof.** For (a): Equation (40) implies that we have \([m, m] \subset \mathfrak{t}\) and therefore \([m, m] = g \cap \mathfrak{t}\), because \( \mathfrak{t} = \text{Kern}(\pi_1) \) and \( \pi_1|m \) is a linear isomorphism. Furthermore, (47) yields the relation \([[[m, m], m] \subset m\),
which already implies that \( g \) is a Lie algebra.

For (b): We have \( T_oM = \pi_1(g) = W \) as a consequence of (5), (44), (52)-(53). In order to give evidence to the equality \( b = h_o \), we consider for each \( x \in W \) the skew symmetric tensor field \( \nabla^N \hat{\Gamma}_x^* \) of type \((1,1)\) on \( TN \). Since moreover for each \( x \in T_oM \) the Killing vector field \( \hat{\Gamma}_x^* \) of \( N \) is tangent to \( M \) with \( \pi_1(\hat{\Gamma}_x) = x \), the Gauß equation gives
\[ \forall y \in T_oM : h(x, y) = h(x, y) = h(y, \pi_1(\hat{\Gamma}_x)) = (\pi_2(\hat{\Gamma}_x) y) \perp b(x, y). \]

For (c): The Lie algebra of \( H \) is given by \( g \cap \mathfrak{t} \) and hence is equal to \([m, m]\) in accordance with Part (a).

Using Part (b), and comparing (53) with (44), (45) and (52), we immediately verify the inclusion \( m \subset m_0 \).
Furthermore, because of (53) combined with (54), \( \pi_1|m \) is injective and hence \( \dim(M) = \dim(m) \); we also note that \( \dim(M) \) is an upper bound for the dimension of \( m_0 \), since the projection \( m_0 \to T_oM, X \mapsto \pi_1(X) \) is injective by virtue of Lemma 1. Therefore, actually \( m = m_0 \) holds. Since \( H \) is a subgroup of I(N) with \( h(M) = M \) for each \( h \in H \), the Ad(H)-invariance of \( m_0 \) is obvious. Now the proof is finished.

In the situation of Lemma 5, the pullback bundle \( TN|M \) can be seen as a homogeneous vector bundle over \( M \) via the induced action \( \alpha : G \times TN|M \to TN|M \), and the reductive decomposition (53) induces a canonical connection \( \nabla^c \) on \( TN|M \). Furthermore, the direct sum \( \nabla^M \oplus \nabla^\perp \) defines another connection on \( TN|M = TM \oplus \perp M \); hence we have the difference tensor \( \Delta := \nabla^M \oplus \nabla^\perp - \nabla^c \in L(TM, \text{End}(TN|M)). \)

Analogously to Lemma 2.2 of [Co] we have:

**Lemma 6.** (a) \( TM, \perp M \), the first normal bundle \( \perp^1M \) and the second osculating bundle \( OM = TM \oplus \perp^1M \) are \( \nabla^c \)-parallel vector subbundles of \( TN|M \). Moreover, the connection obtained from \( \nabla^c \) on \( TM \) (by restriction) is the connection described in Example 7.

(b) Both \( \Delta \) and \( h \) are parallel sections of \( L(TM, \text{End}(TN|M)) \) resp. of \( L^2(TM, \perp M) \) with respect to the linear connection induced by \( \nabla^c \) on the two bundles, respectively.
Proof. For (a): We have $Tg(TM) \subset TM$ , $Tg(\perp M) \subset \perp M$ and $h(Tg(x), Tg(y)) = Tg(h(x, y))$ (since $G$ is a subgroup of $I(N)$ with $g(M) = M$ for each $g \in G$), hence the vector bundles listed in (a) are invariant under the action of $G$. Thus it suffices to show that every $G$-invariant vector subbundle $F \subset TN|M$ is parallel with respect to $\nabla^c$ and the corresponding connection on $F$ (obtained by restriction) is the canonical connection induced by the action $\alpha G \times F : G \times F \to F$.

For this: Let $c : [0, 1] \to M$ be a curve with $c(0) = p$ and $v \in \mathbb{F}_p$. Then there exists $g \in G$, $\hat{v} \in \mathbb{F}_o$ with $g(o) = p$ and $\alpha(g, \hat{v}) = v$, hence $v = \alpha(g^{-1}, \tilde{v}) \in \mathbb{F}_o$ by the $G$-invariance of $\mathbb{F}$. Let $\hat{c} : [0, 1] \to G$ be the $\mathcal{H}$-lift of $c$ with $\hat{c}(0) = g$ (see (27)), then according to Equation (29)

$$\left\{ \frac{\partial}{\partial v} \hat{c} \right\} = \alpha(\hat{c}(1), v) \in \mathbb{F}_c(1) ,$$

again because of the $G$-invariance of $\mathbb{F}$; hence $\mathbb{F}$ is parallel along $c$, and moreover $\nabla^c$ coincides on $\mathbb{F}$ with the canonical connection of $\mathbb{F}$ (since both connections have the same parallel displacement, according to (29)).

For (b): Let us first verify the statement for $h$. Because $G$ is a subgroup of $I(N)$, we have for each $g \in G$:

$$\forall x, y \in T_pM : h(T_pgx, T_py) = T_pgh(x, y).$$

This implies that $h$ is $G$-invariant; thus we can use similar arguments as in (a) to show its $\nabla^c$-parallelity.

To see that also $\Delta$ is $G$-invariant (and hence is $\nabla^c$-parallel), note that $G$ acts on $TN|M$ by vector bundle isomorphisms which are parallel with respect to $\nabla^c$ by construction of the canonical connection (see (29)) and also with respect to $\nabla^M \oplus \nabla^\perp$ (because $G$ is a subgroup of the isometries of $N$). Being the difference of two $G$-invariant linear connections, $\Delta$ is $G$-invariant, too.

Proof of the “if-direction” of Theorem 1. Let an infinitesimal model $(W, b)$ be given. In the following, we assume that $O(b)$ is not contained in any proper curvature invariant subspace of $T_oN$, the general case is left to the reader. Let $G$ denote the Lie subgroup of $I(N)$ constructed in Lemma 5 then the 2-jet of $M := \{ g(o) \mid g \in G \}$ is given at $o$ by $(W, b)$. I claim that $M$ is a full parallel submanifold of $N$ with extrinsically homogeneous tangent holonomy bundle.

For “fullness”: If $\tilde{N} \subset N$ is a totally geodesic submanifold with $M \subset \tilde{N}$, then $O(b)$ is contained in the curvature invariant subspace $T_o\tilde{N}$; and hence $T_oN = T_o\tilde{N}$, by assumption. Thus $N = \tilde{N}$, and therefore $M$ is full in $N$.

For “parallelity”: Since the second fundamental form of $M$ is a parallel section of $L^2(TM, \perp M)$ with respect to $\nabla^c$ according to Part (b) of Lemma 6 the parallelity of $M$ will be established by showing that $\nabla^c$ coincides with $\nabla^M \oplus \nabla^\perp$ on $O_M$.

Moreover, $\Delta$ (which is the difference tensor of these two connections) is a $\nabla^c$-parallel section of $L(TM, \text{End}(TN|M))$ and $O_M$ is a $\nabla^c$-parallel vector subbundle of $TN|M$, too, in accordance with Lemma 6 thus for the compliance of (56) it suffices to prove that we have $\Delta(x)v = 0$ for each $x \in T_oM$ and $v \in O\_oM$.

To this end, for each $x \in T_oM$ and each $\xi \in \perp^1 M$ the curves defined by $y(t) := T_oexp(\alpha(\xi)\partial \Gamma)_x y$ resp. $\xi(t) := T_oexp(\alpha(\xi)\partial \Gamma)_x \xi$ are $\nabla^c$-parallel sections of $TN$ along the curve $\gamma(t) = exp(\alpha(\xi)\partial \Gamma)(o)$, according to (30) combined with (32); therefore, we have

$$\Delta(x)y \equiv \left( \frac{\partial}{\partial t} - \nabla^c_t \right) |_{t=0} y(t) = \left( \frac{\partial}{\partial t} \right) |_{t=0} y(t) = \left( \frac{\partial}{\partial t} \right) |_{t=0} \left( \frac{\partial}{\partial t} \right) |_{t=0} y(t) = 0 ,$$

and for similar reasons $\Delta(x)\xi = 0$. Thus (56) is established.

For “extrinsically homogeneous tangent holonomy bundle”: In order to apply Proposition 11 we notice that $G$ is a Lie subgroup of $I(N)$ which acts transitively on $M$, by definition. Furthermore, pursuant to Part (a) of Lemma 6 and Equation (56), the canonical connection on $TM$ induced by the reductive decomposition (53) coincides with the Levi Civita connection of $M$. Thus, as a consequence of Proposition 11 $M$ has extrinsically homogeneous tangent holonomy bundle.

As an application of Theorem 1 let me point out the following possibility how to obtain parallel submanifolds of $N$ which are not extrinsically symmetric:
Proposition 5. Let a second symmetric space $\tilde{N}$, a full symmetric submanifold $\tilde{M} \subset \tilde{N}$ and some origin $p \in M$ be given. Let $\iota(\tilde{N}) = \mathfrak{k} \oplus \mathfrak{p}$ denote the corresponding Cartan decomposition, $\pi_2^N : \mathfrak{k} \to \mathfrak{so}(T_oN)$ and $\tilde{\pi}_2^N : \mathfrak{k} \to \mathfrak{so}(T_o\tilde{N})$ denote the linear isotropy representations of $N$ and $\tilde{N}$, respectively. If there exists a proper linear subspace $V \subset T_oN$ which is not contained in any proper curvature invariant subspace of $T_oN$, a linear isometry $F : T_p\tilde{N} \to V$ and a Lie algebra homomorphism $F : \mathfrak{k} \to \mathfrak{k}$ such that
\begin{align}
\forall x, y \in T_p\tilde{M}, v \in T_p\tilde{N} : F(R^N(x, y) v) &= R^N(F(x), F(y)) (F(v)) \\
\forall x \in \mathfrak{k}, v \in T_p\tilde{N} : F(\tilde{\pi}_2^N(x) v) &= \pi_2^N(F(X)(v)) (F(v)),
\end{align}
then there exists a full parallel submanifold $M \subset N$ with extrinsically homogeneous tangent holonomy bundle which is not extrinsically symmetric in $N$. More precisely, the 2-jet of $M$ at $o$ is given by $(W, b)$, with $W := F(T_p\tilde{M})$ and where $b : W \times W \to W$ is the bilinear map characterized by $\forall x, y \in T_p\tilde{M} : b(F(x), F(y)) = F(h^M(x, y))$.

Proof. I claim that $(W, b)$ is an infinitesimal model of $N$ in the sense of Definition 1.

Because $T_p\tilde{M}$ is curvature invariant, $(57)$ implies that $W$ is curvature invariant, too; moreover the symmetry of $h^M_p$ implies by means of $(57)$ that $b$ is symmetric, and for the same reason $(13)$ holds, too. Given $x \in W$ there exists $\tilde{x} \in T_p\tilde{M}$ with $x = F(\tilde{x})$; moreover, according to Theorem 7 of [J1], there exists some $\tilde{X} \in \mathfrak{k}$ with $\tilde{\pi}_2^N(\tilde{X}) = \tilde{h}(\tilde{x})$; then by means of $(58)$, $x$ together with $X := F(\tilde{X})$ provides a solution to $(14)$. Thus Theorem 1 exhibits the existence of a full parallel submanifold $M \subset N$ with extrinsically homogeneous tangent holonomy bundle such that $o \in M$, $T_oM = W$ and $\tilde{h}_o = b$. Since $V$ is strictly contained in $T_oN$, the submanifold $M$ is not 1-full and hence not extrinsically symmetric in $N$ according to [J1], Theorem 1.

2.3 Characterization by means of the 2-jet

Proof of Theorem 4. "(a) $\Rightarrow$ (b)" follows immediately from Theorem 1. For "(b) $\Rightarrow$ (a)" consider the 2-jet of $f$ at $o$, given by $(W, b) := (T_oM, h_o)$. In order to apply Theorem 1 we establish the following observations:

- $T_oM$ is a curvature invariant subspace of $T_f(o)N$ according to Proposition 2.
- Equation $(13)$ holds because of $(22)$ combined with the curvature Equation of Gauß.
- By assumption, for each $x \in T_oM$ there exists $X \in \mathfrak{k}$ such that Equation $(14)$ (and hence also Equation $(11)$) holds.

Thus $(W, b)$ is an infinitesimal model, and therefore Theorem 1 exhibits the existence of a parallel submanifold $M \subset N$ with extrinsically homogeneous tangent holonomy bundle such that $f(o) \in M$, $T_f(o)M = W$ and $\tilde{h}_f(o) = b$. Let $\tau : \tilde{M} \to \tilde{M}$ denote the universal covering and $p \in \tilde{\tau}^{-1}(f(o))$. Since both $f$ and $\tilde{f} := \tilde{\tau} \circ f$ are parallel isometric immersions into $N$ defined on a complete and simply connected Riemannian manifold with the same 2-jet at $o$ and $p$, respectively, Theorem 6 of [J1] (which stands in accordance with the ideas from [St]) implies the existence of a unique isometry $g : \tilde{M} \to M$ with $g(p) = o$ and $f = f \circ g$. The result follows.

Proof of Example 4. Put $\kappa := \|y\|$ and let $c : \mathbb{R} \to N$ be the solution to the ordinary differential equation
\[ \nabla^N_0 \nabla^N_\beta \dot{c}(t) = -\kappa^2 \dot{c}(t) \quad \text{with} \quad c(0) = o, \quad \dot{c}(0) = x \quad \text{and} \quad \nabla^N_\beta \dot{c}(0) = y, \]
hence $c$ is a circle of curvature $\kappa$ (see [MT], p.1); thus, in particular, $c$ is a parallel isometric immersion.

I claim that $\kappa$ is full. By contradiction, suppose that there exists a proper, totally geodesic submanifold $M$ through $o$ with $c(\mathbb{R}) \subset M$; thus $V := T_oM$ is a curvature invariant subspace of $T_oN$ whose dimension is 2 or 3. Without loss of generality we may assume that $M$ is maximal in $N$ with this property, and hence $V$ is a maximal proper, curvature invariant subspace of $T_oN$ with $\{x, y\} \subset V$. It is well known that $N$ is isometric to the complex hypersurface $Q^2(\mathbb{C}) := \{z_0 : \ldots : z_3 \in \mathbb{C}^3 \mid z_0^2 + \cdots + z_3^2 = 0\}$ of $\mathbb{C}^3$, usually called the (2-dimensional) "complex quadric". Using the classification of totally geodesic submanifolds of the complex quadric $Q^n(\mathbb{C})$ (for arbitrary $n$) from [K1] (cf. Theorem 4.1 and Section 5 there), we infer that $M$ belongs to exactly one of the following "types" (thereby, we keep to the notation from [K1]):
• “Type (G3)”: $M \cong \mathbb{C}P^1 \times S^1$; thus there exists a totally geodesic embedding $S^1 \hookrightarrow \mathbb{C}P^1$ and either $M \cong \mathbb{C}P^1 \times S^1$ or $M \cong S^1 \times \mathbb{C}P^1$ such that the product structure of $M$ is compatible with the product structure of $N$.

• “Type (P1, 2)”: $\dim(M) = 2$ and $M$ is holomorphic congruent to the graph $\{(p, \bar{p}) \mid p \in \mathbb{C}P^1\}$ of the “complex conjugation” $(\cdot) : \mathbb{C}P^1 \to \mathbb{C}P^1$, $[z_0 : z] \mapsto [\bar{z}_0 : \bar{z}]$.

• “Type (P2)”: $\dim(V) = 2$ and $M$ is holomorphic congruent to the diagonal $\{(p, p) \mid p \in \mathbb{C}P^1\}$.

We now see: If $\dim(M) = 3$ and hence $M$ is of Type (G3), then one of the linear maps $T_o \pi : W : W \to T_o \mathbb{C}P^1$ is not surjective, which is excluded by assumption. If $\dim(M) = 2$ and hence $M$ is of Type (P1, 2) or (P2), then $T_o M$ is either a complex or a totally real subspace of $T_o N$ which is also not possible by assumption. Thus $c$ is full.

I claim that Assertion (b) of Theorem 2 is not valid here: Let $W$ denote the second osculating space of $c$ at $0$; this is the 2-dimensional linear subspace of $T_o N$ spanned by $\{x, y\}$ (since $c$ is a Frenet curve of osculating rank 2, see [MT], p.2), and suppose by contradiction that there exists some $X \in \mathfrak{T}^N$ such that linear map $A := \pi_2^N(X)$ satisfies $A(W) \subset W$ and $A^O = h(x)$. Hence we have $A(x) = y$ and thus also $A(y) = -\kappa^2 x$ (because $A$ is skew adjoint and $\dim(W) = 2$). It is well known that $i(N) \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ and this splitting of $i(N)$ reflects the product structure of $N$; hence there exist $X_i \in i(\mathbb{C}P^1)$ such that

$$\forall p \in N \quad \text{and} \quad i = 1, 2 : T_p \pi_i(X^*(p)) = X^*_i(p_i).$$

Put $A_i := \pi_2^N(X_i) \in \mathfrak{so}(T_o \mathbb{C}P^1)$; then, by the previous, $A = A_1 \oplus A_2$ and

$$A_i(x_i) = y_i \quad \text{and} \quad A_i(y_i) = -\kappa^2 x_i \quad \text{for} \quad i = 1, 2.$$

Thus $A_i = \pm \kappa J_i$, where $J_i$ denotes the complex structure of $T_o \mathbb{C}P^1$, and therefore $\kappa_i = ||y_i||/||x_i|| = ||\kappa J_i(x_i)||/||x_i|| = \kappa$ for each $i = 1, 2$, which again is not possible.

Thus Assertion (b) of Theorem 2 is not valid here; therefore, $c$ is not the orbit of a 1-parameter subgroup of $\text{I}(N)$, in accordance with Example 2.

### 3 Geometry of irreducible parallel submanifolds

Throughout this section, we assume that $N$ is a symmetric space which is of compact type or of non-compact type and $f : M \to N$ is a parallel isometric immersion defined on a simply connected, irreducible symmetric space $M$. Let $o \in M$ be some origin, $\mathfrak{ho}(M)$, $\mathfrak{ho}(N)$ and $\mathfrak{ho}(M)$ denote the holonomy Lie algebras of $M$, $N$ and the totally geodesic submanifold $M \subset N$ (see (17)) with respect to the base points $o$ and $f(o)$, respectively. Again, we implicitly use $T_o M \cong T_f(T_o M)$; likewise, the reader may assume for simplicity that $M \subset N$ is a submanifold with $f = i^M$. Since the curvature tensors of $N$, $M$ and $M$ are parallel, respectively, and since moreover $M$ is totally geodesically embedded in $N$, the Theorem of Ambrose/Singer implies that

$$\mathfrak{ho}(M) = \{ R^M(x, y) \mid x, y \in T_o M \}_{\mathbb{R}} \subset \mathfrak{so}(T_o M),$$

$$\mathfrak{ho}(N) = \{ R^N(u, v) \mid u, v \in T_{f(o)} N \}_{\mathbb{R}} \subset \mathfrak{so}(T_{f(o)} N),$$

$$\mathfrak{ho}(\tilde{M}) = \{ (R^N(x, y))^{T_o M} \mid x, y \in T_o M \}_{\mathbb{R}} \subset \mathfrak{so}(T_o M),$$

with $\forall x, y \in T_o M : (R^N(x, y))^{T_o M} := R^N(x, y) |_{T_o M} : T_o M \to T_o M$.

Furthermore, there exists a decomposition $T_o M = \bigoplus_{i=0}^{k} W_i$ such that $W_i$ is an irreducible $\mathfrak{ho}(\tilde{M})$-module for $i \geq 1$, and $W_0$ is the largest vector subspace on which $\mathfrak{ho}(\tilde{M})$ acts trivially. By virtue of the “de Rham decomposition theorem” (see [BCO], p.290), there exists a Euclidean space $E$, irreducible Riemannian manifolds $M_i$ ($i = 1, \ldots, k$) such that $\tilde{M} \cong E \times M_1 \times \cdots \times M_k$ with $W_0 \cong T_E E$ and $W_i \cong T_{M_i} M_i$ for each $i \geq 1$. This “de Rham decomposition” of $\tilde{M}$ (and thus also the linear spaces $W_i$) are unique up to a permutation of the index set $\{1, \ldots, k\}$. Of course, similar considerations apply also for $M$. Therefore:

**Lemma 7.** (a) $\mathfrak{ho}(M)$ acts irreducible on $T_o M$.
(b) \( \bar{M} \) is an irreducible Riemannian space if and only if \( \mathfrak{hol}(\bar{M}) \) acts irreducible on \( T_oM \), too.

(c) \( \bar{M} \) is a flat if and only if \( W_o = T_oM \).

With the intent to describe certain dependencies between the holonomy representations of \( M \) and \( \bar{M} \), we use Proposition 2 to define a tensor field \( T \) of type \((1,3)\) on \( M \),

\[
\forall p \in M, x, y, z \in T_pM : T(x, y) z = R^N(x, y) z .
\]

For a proof of the following Lemma see Proposition 7 of [11] or Lemma 2.3 of [Ts1]:

**Lemma 8.** \( T \) is a parallel tensor field.

In accordance with [Ts1], Theorem 2.4, (1) & (3), we have:

**Proposition 6.** (a) \( W_i \) is also \( \mathfrak{hol}(M) \)-invariant for each \( i \).

(b) Either \( \bar{M} \) is a flat of \( N \), or \( \bar{M} \) is an irreducible symmetric space, too.

**Proof.** For (a): For arbitrary \( A \in \mathfrak{hol}(M) \) let \( g_t := \exp(tA) \) denote the one parameter subgroup of \( \mathfrak{so}(T_oM) \) generated by \( A \). It suffices to prove \( g_t(W_i) = W_i \) for each \( t \in \mathbb{R} \).

For this: Since \( A \in \mathfrak{hol}(M) \), for each \( t \in \mathbb{R} \) there exists a loop \( c_t : [0,1] \to M \) centered at \( o \) with \( g_t = ([c_t])^M \). From Lemma 8 we infer that

\[
\forall x, y, z \in T_oM : R^N(g(t)x, g(t)y) g(t)z = g(t)(R^N(x, y) z) ,
\]

which implies (by means of (61)) that for each \( i \geq 1 \) the linear space \( g_i(W_i) \) is an irreducible \( \mathfrak{hol}(\bar{M}) \)-module, too, and \( g_i(W_0) = W_0 \). Since the decomposition of \( T_oM \) above is unique up to permutations of the \( W_i \)'s with \( i \geq 1 \), we hence conclude from a continuity argument that \( g_i(W_i) = g_0(W_i) = W_i \) for each \( i \); the result follows.

For (b): If \( M \) is an irreducible Riemannian space, then \( T_oM \) is an irreducible \( \mathfrak{hol}(M) \)-module and hence either \( T_oM = W_0 \) or \( T_oM = W_1 \), as a consequence of (a). In the first case, \( \bar{M} \) is a flat of \( N \); in the second case, \( \bar{M} \) is an irreducible Riemannian space, in accordance with Lemma 7. \( \square \)

### 3.1 The case that \( \bar{M} \) is flat

Besides the conventions made at the beginning of Section 3, in this section we will additionally assume that \( \bar{M} \) is a flat of \( N \). The proof of Theorem 3 will be given at the end of this section. Recall the definition of the transvection map \( \Gamma \) of \( N \) at \( o \), see Section 1. Then we have:

**Proposition 7.** Let a linear subspace \( W \subset T_oN \) be given. The following is equivalent:

(a) \( W \) is a curvature isotropic subspace of \( T_oN \).

(b) \( [\Gamma_u, \Gamma_v] = 0 \) for all \( u, v \in W \).

(c) \( \exp^N(W) \) is a flat of \( N \).

(d) The sectional curvature of \( N \) vanishes on every 2-plane of \( W \), i.e. \( (R^N(u,v)v,u) = 0 \) for all \( u, v \in W \).

In particular, \( T_oM \) is a curvature isotropic subspace of \( T_oN \).

**Proof.** (a) \( \iff \) (b) is an immediate consequence of Lemma 2 applied to Equation (30). For (b) \( \iff \) (c) cf. [He], Ch. V, Prop. 6.1. While (c) \( \implies \) (d) is obvious, let us give a proof of (d) \( \implies \) (b) in case \( N \) is irreducible: Then, by means of the canonical isomorphism \( \pi_1 : p \to T_oN \), the metric is given at \( o \) by a multiple \( c \neq 0 \) of the Killing form \( B \) of \( i(N) \) restricted to \( p \); where without loss of generality we may assume that \( c \in \{-1,1\} \). Let two orthonormal vectors \( u, v \in T_oN \) be given and denote by \( K_{u,v} \) the sectional curvature of the 2-plane spanned by \( \{u,v\} \). Then, according to [He], Ch. V, § 3, Equation (2),

\[
K_{u,v} = c B([\Gamma_u, \Gamma_v], [\Gamma_u, \Gamma_v]) ;
\]
hence $K_{u,v} = 0$ forces $[T_u, \Gamma_v] = 0$ by the (positive or negative) definiteness of $B$. Now $(d) \Rightarrow (b)$ follows immediately. For the general case, cf. \cite{He}, Ch. V, § 3, and use Equation (1) there instead of Equation (2).  

Recall the following result from \cite{He} (Proposition 7 and Corollary 2 there): We have for all $x, y \in T_oM$ and $v \in \mathcal{O}_o f$:

\[
R^N(h(x,x), h(y,y)) v = [h(x), [h(y), R^N(x,y)]] v
\]

and furthermore, using similar arguments,

\[
I \text{ claim that fixed, and } v \in \mathcal{O}_o f:
\]

\[
R^N(h(x), h(y))) v = R^N(h(x) h(y)) v.
\]

\[\text{(62)}\]

**Lemma 9.** Also the first normal space $\perp_o f$ is curvature isotropic.

**Proof.** Let $\xi, \eta \in \perp_o f$ be given; then, without loss of generality, we may assume that there exist $x, y \in T_oM$ with $\xi = h(x,x), \eta = h(y,y)$ (since $h$ is a symmetric bilinear map). Moreover, we have $h(x) h(y) = -S_{h(y,x)}(x) \in T_oM$ for all $x, y \in T_oM$, and hence r.h.s. of (63) vanishes (since $T_oM$ is curvature isotropic); and so $R^N(\xi, \eta)$ vanishes on $\mathcal{O}_o f$, too. In particular, $(R^N(\xi, \eta)) \in o$; therefore, $\perp_o f$ is a curvature isotropic subspace of $T_f(o,N)$, according to Proposition 9.

It was shown in \cite{He}, Corollary 3, that the tensor of type $(0,4)$ on $\mathcal{O} f$ defined by

\[
R^b(v_1, v_2, v_3, v_4) := (R^N(v_1, v_2) v_3, v_4) \text{ for } v_1, \ldots, v_4 \in \mathcal{O}_o f
\]

satisfies

\[
\forall x \in T_pM, v_1, \ldots, v_4 \in \mathcal{O}_o f : \sum_{i=1}^4 R^b(v_1, \ldots, h(x) v_i, \ldots, v_4) = 0.
\]

Furthermore, for every subspace $V \subset so(\mathcal{O}_o f)$ we introduce its centralizer in $so(\mathcal{O}_o f)$,

\[
c(V) := \{ A \in so(\mathcal{O}_o f) \mid \forall B \in V : A \circ B = B \circ A \}.
\]

**Lemma 10.** If $c(h(T_oM)) \cap so(\mathcal{O}_o f)_\perp = \{0\}$, then $\mathcal{O}_o f$ is a curvature isotropic subspace of $T_f(o,N)$, too.

**Proof.** By virtue of Proposition 7 it is enough to show that $(R^N(v_1, v_2) v_3, v_4) = 0$ for all $v_1, v_2, v_3, v_4 \in \mathcal{O}_o f$. Furthermore, according to Lemma 9 we have $(R^N(x_1, x_2) v_1, v_2) = 0$ and $(R^N(\xi, \eta)) v_1, v_2) = 0$ for all $x_1, x_2 \in T_oM$, $\xi, \eta \in \perp_o f$ and $v_1, v_2 \in \mathcal{O}_o f$; and hence it remains to prove that $(R^N(y, \xi) v_1, v_2) = 0$ for all $y \in T_oM$, $\xi \in \perp_o f$, and $v_1, v_2 \in \mathcal{O}_o f$. To this end, let $y \in T_oM$ and $\xi \in \perp_o f$ be arbitrary, but fixed, and $A \in so(\mathcal{O}_o f)$ be the linear map characterized by

\[
\forall v_1, v_2 \in \mathcal{O}_o f : (A v_1, v_2) = (R^N(y, \xi) v_1, v_2).
\]

I claim that $A$ belongs to $c(h(T_oM)) \cap so(\mathcal{O}_o f)_\perp$.

In fact, using the symmetries of $R^N$, we have

\[
\forall x_1, x_2 \in T_oM : (A x_1, x_2) = (R^N(y, \xi) x_1, x_2) = (R^N(x_1, x_2) y, \xi) = 0;
\]

and furthermore, using similar arguments, $(A \xi, \xi) = 0$ for all $\xi, \xi \in \perp_o f$. Hence $A \in so(\mathcal{O}_o f)_\perp$, in accordance with Equation (63). Moreover,

\[
\forall v_1, v_2 \in \mathcal{O}_o f, x \in T_oM : \langle h(x), A \rangle v_1, v_2) = -\langle A v_1, h(x) v_2 \rangle - \langle A(h(x) v_1), v_2 \rangle
\]

\[
= -\langle R^N(y, \xi) v_1, h(x) v_2 \rangle - \langle R^N(\xi, h(x)) v_1, v_2 \rangle = \sum_{i=1}^{\perp_o f} \langle R^N(\xi, h(x)) v_1, v_2 \rangle.
\]

Thus $A \in c(h(T_oM)) \cap so(\mathcal{O}_o f)_\perp$, and therefore, by assumption, $A = 0$; the result follows.
In order to make effectively use of the above lemma, a better understanding of the linear space \( c(h(T_0M)) \cap so(C_{o}f) \) is necessary.

**Proposition 8.** There exists a complete and full parallel submanifold \( \tilde{M} \subset O_{o}f \) with \( 0 \in \tilde{M} \), \( T_{0}\tilde{M} = T_{o}M \) and \( h_{0} = h_{o} \). Moreover, \( \tilde{M} \) is an irreducible symmetric space, too.

**Proof.** We consider the Euclidian vector space \( V := O_{o}f \) and aim to apply Theorem 1 in order to establish the existence of a parallel submanifold \( M \subset V \) with \( 0 \in M \) and whose 2-jet at 0 is given by \( (W,b) \), with \( W := T_{o}M \) and \( b := h_{o} \). For this, we have to check that \( (W,b) \) is an infinitesimal model of \( V \) in the sense of Definition \( \mathbf{4} \) for which purpose it suffices to establish \( \mathbf{13} \) (since \( V \) is a Euclidian space). Let \( b : W \rightarrow so(V) \) be the linear map defined by \( \mathbf{12} \). Since \( T_{o}M \) is a curvature isotropic subspace of \( T_{f(o)N} \), the equation of Gauß yields

\[
\forall x, y, z \in T_{o}M : R^{M}(x, y) z = -[b(x), b(y)] z ;
\]

furthermore, we have

\[
\forall x, y, z \in T_{o}M, v \in O_{o}f : b(R^{M}(x, y) z) v = -[[b(x), b(y)], b(z)] v ,
\]

by virtue of \( \mathbf{22} \). Combining the previous two equations, \( \mathbf{13} \) follows. Therefore, \( (W,b) \) is an infinitesimal model of \( V \) and hence, according to Theorem \( \mathbf{1} \) there exists a homogeneous parallel submanifold \( \tilde{M} \subset V \) whose 2-jet at 0 is given by \( (W,b) \); in particular, \( M \) is a symmetric space, too. Using \( \mathbf{66} \) and again the equation of Gauß, we notice that \( R^{M}(x, y) z = R^{\tilde{M}}(x, y) z \); hence the universal covering space of \( \tilde{M} \) is isometric to \( M \), by virtue of the "Theorem of Cartan/Ambrose/Hicks"; therefore, \( \tilde{M} \) is an irreducible symmetric space, too.

Because of the previous proposition, now we will study full, intrinsically irreducible parallel submanifolds of a Euclidian space.

**Definition 11** ([BCO], Example 3.7). Let \( \tilde{N} \) be a simply connected, irreducible symmetric space of compact type whose isotropy subgroup at \( \tilde{o} \) is denoted by \( \tilde{K} \) and whose Cartan decomposition is given by \( i(\tilde{N}) = \mathfrak{k} \oplus \mathfrak{p} \). Suppose that there exists some \( X \in \mathfrak{p} \) with \( \text{ad}(X)^{3} = -\text{ad}(X) \) and \( X \neq 0 \). Then \( \text{Ad}(K)X \subset \mathfrak{p} \) is called a standard embedded irreducible symmetric R-space (here \( \text{Ad} \) and \( \text{ad} \) denote the adjoint representations of \( I(\tilde{N}) \) and \( i(\tilde{N}) \), respectively).

Let \( \tilde{B} \) denote the Killing form of \( i(\tilde{N}) \); then, since \( i(\tilde{N}) \) is a compact, semisimple Lie algebra (cf. [He], Ch. V, § 1), \( B \) is a negative definite, invariant form (cf. [He], Ch. II, § 6). It is well known that every standard embedded irreducible symmetric R-space \( \tilde{M} := \text{Ad}(\tilde{K})X \subset \mathfrak{p} \) is a parallel submanifold (where \( \tilde{p} \) is seen as a Euclidian vector space by means of the positive definite symmetric bilinear form \( -\tilde{B}(\tilde{p} \times \tilde{p}) \)), cf. [BCO], Prop. 3.7.7. In particular, \( \tilde{M} \) is a symmetric space; however, note that \( \tilde{M} \) is not necessarily intrinsically irreducible.

In the other direction, we obtain as a consequence of Theorem 3.7.8 of [BCO]:

**Theorem 8** (Ferus). If \( \tilde{M} \subset E \) is a full, complete, (intrinsically) irreducible parallel submanifold of a Euclidian space \( E \), then there exists a simply connected, irreducible symmetric space \( \tilde{N} \) of compact type which admits a standard embedded irreducible symmetric R-space \( \text{Ad}(K)X \), some \( c < 0 \) and an isometry \( F : V \rightarrow \mathfrak{p} \) such that \( F(\tilde{M}) = \text{Ad}(K)X \) (where \( \mathfrak{p} \) is seen as a Euclidian vector space by means of the positive definite symmetric bilinear form \( c\tilde{B}(\tilde{p} \times \tilde{p}) \)).

From now on, we consider the following setting:

- \( \tilde{N} \) is a simply connected, irreducible symmetric space of compact type whose isotropy subgroup at \( \tilde{o} \) is denoted by \( \tilde{K} \) and whose Cartan decomposition is given by \( i(\tilde{N}) = \mathfrak{k} \oplus \mathfrak{p} \),
- there exists \( X \in \mathfrak{p} \) with \( \text{ad}(X)^{3} = -\text{ad}(X) \) and \( X \neq 0 \),
- \( b \) is the second fundamental form at \( X \) of the standard embedded irreducible symmetric R-space \( \tilde{M} := \text{Ad}(\tilde{K})X \subset \mathfrak{p} \),

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• and \( b : T_X \bar{M} \to so(p) \) is the 1-form which is associated with \( b \) in the sense of Definition 5.

In this situation, we introduce \( \hat{\kappa}_0 := \{ Y \in \hat{\kappa} \mid [X,Y] = 0 \} \) and \( \hat{\kappa}_- := \{ Y \in \hat{\kappa} \mid ad(X)^2 Y = -Y \} \); then we have

\[
ad(\kappa_0) \subset so(p)_+ \quad \text{and} \quad ad(\kappa_-) \subset so(p)_- ;
\]

(68)

cf. Lemma 1 of [JJ]. Furthermore, put \( \hat{K}_X := \{ k \in \hat{K} \mid Ad(k) X = X \} \). According to Theorem 2 of [EH], \((\hat{K}, \hat{K}_X)\) is a Riemannian symmetric pair (in the sense of [He], Ch. IV, Definition preceeding Prop. 3.4), which is associated with the orthogonal symmetric Lie algebra given by

\[
\hat{\kappa} = \hat{\kappa}_0 \oplus \hat{\kappa}_-.
\]

(69)

(in the sense of [He], Ch. IV, Remark preceeding Prop. 3.6).

For each \( Z \in \hat{\kappa} \) let \( ad(Z)_p : \hat{p} \to \hat{p} \) denote the induced endomorphism of \( \hat{p} \). According to [JJ], Lemma 1 and Proposition 4 (see in particular Equations (36) and (40) there), one knows the following:

**Proposition 9.** \( \hat{M} \) is a full parallel submanifold of \( \hat{p} \) such that \( b(T_X \hat{M}) = \{ ad(Z)_p \mid Z \in \hat{\kappa}_- \} \).

**Lemma 11.** \( \hat{K} \) acts effectively through isometries on \( \hat{M} \) via \( \Ad[\hat{K} \times \hat{M} : \hat{K} \times \hat{M} \to \hat{M} \). Furthermore, we have \( \kappa_0 = [\hat{\kappa}, \hat{\kappa}^-] \).

**Proof.** It is clear that the action of \( \hat{K} \) on \( \hat{M} \) is isometric. To see that this action is effective, suppose that \( Ad(k) \hat{M} = Id \). Since \( \hat{M} \) is full in \( \hat{p} \), we hence have \( Ad(k)\hat{p} = Id \); thus \( k = Id_{\hat{K}} \), because \( K \) acts effectively on \( T_0 N \cong \hat{p} \). Therefore, \( \hat{K} \) acts effectively on \( \hat{M} \). In particular, the induced action of \( \hat{K}_X \) on \( T_X \hat{M} \cong \hat{\kappa} \) by means of \( \hat{K}_X \times \hat{\kappa} \to \hat{\kappa} \) is effective, too (since \( \hat{K} \) acts through isometries on \( \hat{M} \)).

Now I claim that \( \kappa_0 = [\hat{\kappa}, \hat{\kappa}^-] \) holds. For this:

Let \( \hat{B} \) denote the Killing form of \( i(\hat{N}) \) and \( [\hat{\kappa}, \hat{\kappa}^-] \) denote the orthogonal complement of \( [\hat{\kappa}, \hat{\kappa}^-] \) in \( \hat{\kappa} \) with respect to \( \hat{B} \). Then we have for each \( Z_1 \in [\hat{\kappa}, \hat{\kappa}^-] \) and \( Z_2, Z_3 \in \hat{\kappa} \)

\[
\hat{B}((Z_1, Z_2), Z_3) = \hat{B}(Z_1, [Z_2, Z_3]) = 0,
\]

hence \( ad(Z_1) \hat{\kappa}^- = 0 \) (since \( \hat{B} \) is negative definite) and thus \( Z_1 = 0 \), because \( \kappa_0 \) acts effectively on \( T_0 M \cong \hat{\kappa} \) by means of \( ad : \kappa_0 \times \hat{\kappa} \to \hat{\kappa} \).

Let \( \hat{c} \) denote the center of \( \hat{\kappa} \).

**Lemma 12.** For each \( A \in so(p) \) we have

\[
\forall Z \in \hat{\kappa} : A \circ ad(Z)_p = ad(Z)_p \circ A
\]

(70)

if and only if there exists \( Z^* \in \hat{\kappa} \) with \( A = ad(Z^*)_p \).

**Proof.** We have \( ad(Z_1)_p \circ ad(Z_2)_p = ad(Z_2)_p \circ ad(Z_1)_p \) for all \( Z_1 \in \hat{\kappa}, Z_2 \in \hat{\kappa} \), since \( ad_p : \hat{\kappa} \to so(p), Z \mapsto ad(Z)_p \) is a representation.

In the other direction, let \( \tilde{\pi}_2 : \hat{\kappa} \to so(T_0 \hat{N}) \) denote the linearized isotropy representation (as in (9)) and \( \tilde{R} \) denote the curvature tensor of \( \hat{N} \); since \( \hat{N} \) is irreducible, we have (see [He], Ch. V, Theorem 4.1, (i) & (iii))

\[
\tilde{\pi}_2(\hat{\kappa}) = \{ \tilde{R}(u, v) \mid u, v \in T_0 \hat{N} \} \mathbb{R}.
\]

Furthermore, in the following the well known relation \( \forall Z \in \hat{\kappa} : \tilde{\pi}_1 \circ ad(Z)_p = \tilde{\pi}_2(Z) \circ \tilde{\pi}_1 \) (see (7)) will be used implicitly; moreover, it is convenient to suppress the canonical isomorphism \( \tilde{\pi}_1 : \hat{p} \to T_0 \hat{N} \).

Now suppose that \( A \in so(T_0 \hat{N}) \) satisfies (70); then, by means of the previous,

\[
\forall u, v \in T_0 \hat{N} : A \circ \tilde{R}(u, v) = \tilde{R}(u, v) \circ A.
\]

Therefore, using the symmetry of \( \tilde{R} \), we conclude that

\[
\forall u, v \in T_0 \hat{N} : [A, \tilde{R}(u, v)] = 0 = \tilde{R}(Au, v) + \tilde{R}(u, Av).
\]

Hence, according to [He], Ch. V, Theorem 4.1 (ii), \( A \) even belongs to the center of \( \tilde{\pi}_2(\hat{\kappa}) \) which is equal to \( \tilde{\pi}_2(\hat{c}) \), since \( \tilde{\pi}_2 \) is a faithful representation. This finishes the proof. \( \square \)
Lemma 13. If $\tilde{M}$ is an irreducible Riemannian space, then $\tilde{c} \cap \tilde{k} = \{0\}$.

Proof. Let $\tilde{\mathfrak{o}}$ denote the orthogonal complement of $\tilde{c} \cap \tilde{k}$ in $\tilde{k}$ with respect to $\tilde{B}$, and consider the following two orthogonal symmetric Lie algebras:

- $\tilde{\mathfrak{k}}_0 \oplus \tilde{\mathfrak{o}}$ (which corresponds to a simply connected symmetric space of compact type whose dimension is equal to the dimension of $\tilde{\mathfrak{o}}$), and
- $\tilde{c} \cap \tilde{k}$ (which corresponds to a Euclidean space whose dimension is equal to the dimension of $\tilde{c} \cap \tilde{k}$).

This is a decomposition of the orthogonal symmetric Lie algebra $\tilde{\mathfrak{k}}$ (see [69]) into ideals as described in [He], Ch. V, Theorem 1.1; hence, the universal covering space of $\tilde{M}$ splits off a Euclidean factor as described in the proof of [He], Ch. 4, Prop. 4.2. □

Proof of Theorem 3. I claim that $\mathfrak{c}(\mathfrak{h}(T_oM)) \cap \mathfrak{so}(\mathcal{O}_o f)_- = \{0\}$ holds. For this:

By the strength of Proposition 8 combined with Theorem 8, there exists a simply connected, irreducible symmetric space $\tilde{N}$ of compact type and (in the previous notation) some $X \in \tilde{\mathfrak{p}}$ with $\mathfrak{ad}(X)^3 = -\mathfrak{ad}(X)$ such that $h_o : T_oM \times T_oM \to \mathfrak{t}_o$ is algebraically equivalent to the second fundamental form $b$ at $X$ of an (intrinsically) irreducible, standard embedded irreducible symmetric R-space $M := \text{Ad}(\tilde{K}) \times \mathfrak{p}$; therefore, it suffices to show that $\mathfrak{c}(\mathfrak{b}(T_oM)) \cap \mathfrak{so}(\mathfrak{p})_- = \{0\}$ holds, as follows:

Let $A \in \mathfrak{c}(\mathfrak{b}(T_oM)) \cap \mathfrak{so}(\mathfrak{p})_- \text{ be given, hence } A \circ \mathfrak{ad}(Z)_{\tilde{\mathfrak{p}}} = \mathfrak{ad}(Z)_{\tilde{\mathfrak{p}}} \circ A \text{ for all } Z \in \tilde{\mathfrak{k}}_-, \text{ according to Proposition 8, thus}$

$$A \circ \mathfrak{ad}(Z)_{\tilde{\mathfrak{p}}} = \mathfrak{ad}(Z)_{\tilde{\mathfrak{p}}} \circ A \text{ for all } Z \in \tilde{\mathfrak{k}}_-, \tilde{\mathfrak{k}}_- \oplus \tilde{\mathfrak{k}}_- = \tilde{\mathfrak{k}}_0 \oplus \tilde{\mathfrak{k}}_- \overset{\text{[11]}}{=} \tilde{\mathfrak{k}},$$

since $\mathfrak{ad}_{\tilde{\mathfrak{p}}}$ is a representation and were the last equality follows from Lemma 11. Hence there exists $Z^* \in \tilde{c}$ with $A = \mathfrak{ad}_{\tilde{\mathfrak{p}}}(Z^*)$, by means of Lemma 12 moreover, then we even have $Z^* \in \tilde{c} \cap \tilde{k}$, by virtue of (68); hence $A = 0$, according to Lemma 13 (because $\tilde{M}$ is intrinsically irreducible), which establishes our claim.

Thus $\mathfrak{c}(\mathfrak{h}(T_oM)) \cap \mathfrak{so}(\mathcal{O}_o f)_- = \{0\}$ holds and therefore $\mathcal{O}_o f$ is a curvature isotropic subspace of $Tf_{f(0)}N$, according to Lemma 10. Let $\bar{N}$ denote the totally geodesic submanifold $\exp(\mathcal{O}_o f)$, which is a flat of $N$, as a consequence of Proposition 7. Then $f(M)$ is contained in $\bar{N}$, by virtue of Theorem 7. □

3.2 The case that $\tilde{M}$ is irreducible

Besides the conventions made at the beginning of Section 3, in this section we will additionally assume that $\tilde{M}$ is an irreducible symmetric space; furthermore, here we do not require that $N$ is of compact type or of non-compact type. At the end of this section we will give the proof of Theorem 2.

Part (b) of the following Lemma 14 should be seen in the context of Theorem 2.

Lemma 14. (a) We have

$$\mathfrak{hol}(N) \subset \pi_2(\mathfrak{k})$$

with equality if $N$ is of compact type or of non-compact type.

(b) For each $A \in \mathfrak{hol}(\mathcal{O}_f)$ there exists some $X \in \mathfrak{k}$ with

$$\pi_2(X)(\mathcal{O}_o f) \subset \mathcal{O}_o f \text{ and } (\pi_2(X))^0 = A.$$  

Hence Equation (11) implies Assertion (b) of Theorem 2.

Proof. For (a): In case that $N$ is of compact type or of non-compact type, [71] holds with equality, according to [He], Ch. V, Theorem 4.1 (iii) in combination with [60]. Let me give a quick proof of [71] which applies for arbitrary $N$, as follows:

Given $A \in \mathfrak{hol}(N)$, by $g_t := \exp(tA)$ is defined a one-parameter subgroup of the Holonomy group $\text{Hol}(N) \subset \text{SO}(Tf_{f(0)}N)$. Thus we have $g_t = (\mathfrak{c}_t)^N$ for some loop $\mathfrak{c}_t : [0, 1] \rightarrow N$ with $\mathfrak{c}_t(0) = o$. Hence, since the curvature tensor of $N$ is parallel, $g(t)$ satisfies

$$\forall u, v \in Tf_{f(0)}N, t \in \mathbb{R} : g(t) \circ R^N(u, v) \circ g(t)^{-1} = R^N(g(t)u, g(t)v).$$  

(73)
Thus there exists an isometry $G_t$ of $N$ with $G_t(o) = o$ and $T_oG_t = g(t)$, as a consequence of the “Theorem of Cartan/Ambrose/Hicks”. The result follows.

For (b): Remember that $Of \subset f^{*}TN$ is a $\nabla^N$-parallel vector subbundle, according to Proposition 2. Therefore, using an argument on the level of the corresponding Holonomy groups, we conclude that for each $A \in \text{hol}(f^{*}TN)$ we have $A(O_{o,f}) \subset O_{o,f}$, $A^O \in \text{hol}(Of)$ and the canonical map $\text{hol}(f^{*}TN) \rightarrow \text{hol}(Of)$, $A \mapsto A^O$ is surjective. Since $\text{Hol}(f^{*}TN) \subset \text{hol}(N)$ is a Lie subalgebra, the result follows immediately from (a).

In the following, ad and Ad will denote the adjoint representations of $\mathfrak{so}(O_{o,f})$ and $\mathfrak{SO}(O_{o,f})$, respectively. As a consequence of the Jacobi identity, if $A \in \mathfrak{so}(O_{o,f})$, then the linear map $\text{ad}(A)$ is a derivation of $\mathfrak{so}(O_{o,f})$; i.e. for all $B,C \in \mathfrak{so}(O_{o,f})$ we have:

$$\text{ad}(A)[B,C] = [\text{ad}(A)B,C] + [B,\text{ad}(A)C].$$

(74)

Recall the splitting $\mathfrak{so}(O_{o,f}) = \mathfrak{so}(O_{o,f})_+ \oplus \mathfrak{so}(O_{o,f})_-$ defined according to (10) and (11) for the 2-jet of $f$ at $o$ (as explained in Section 1.1). The following is proved in a straightforward manner:

**Lemma 15.**  (a) We have $A \in \mathfrak{so}(O_{o,f})_+$ if and only if $A(T_oM) \subset T_oM$.

(b) $\mathfrak{so}(O_{o,f})_- \rightarrow \mathcal{L}(T_oM, \perp_o f)$, $A \mapsto A|T_oM : T_oM \rightarrow \perp_o f$ is a linear isomorphism.

The next Proposition prepares a purely algebraic approach towards the availability of $\mathfrak{so}(T_{f(o)}N)$:

**Proposition 10.**  (a) Let $\sigma^\perp : O_{o,f} \rightarrow O_{o,f}$ denote the linear reflection in $\perp_o f$. Then we have

$$\text{Ad}(\sigma^\perp)(\text{hol}(Of)) = \text{hol}(Of).$$

Consequently, we obtain the decomposition

$$\text{hol}(Of) = \text{hol}(Of)_+ \oplus \text{hol}(Of)_-$$

(76)

with $\text{hol}(Of)_+ := \mathfrak{so}(O_{o,f})_+ \cap \text{hol}(Of)$ and $\text{hol}(Of)_- := \mathfrak{so}(O_{o,f})_- \cap \text{hol}(Of)$.

(b) For each $x \in T_oM$, $\text{ad}(h(x))$ defines an outer derivation of $\text{hol}(Of)$, i.e. we have

$$[h(x),\text{hol}(Of)] \subset \text{hol}(Of)$$

(77)

(c) The vector space

$$\mathfrak{h} := \{R^N(x,y) | x,y \in T_oM\}_\mathbb{R}$$

(78)

is a Lie subalgebra of $\mathfrak{so}(T_{f(o)}N)$. For each $A \in \mathfrak{h}$ we have $A(T_oM) \subset T_oM$, $A(\perp_o f) \subset \perp_o f$ and moreover $A^O \in \text{hol}(Of)_+$.

**Proof.** For Parts (a) and (b) see Theorem 3 of [11].

For (c): The fact that $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{so}(T_{f(o)}N)$ follows from the curvature invariance of $T_oM$ (Proposition 2) combined with the well known relation $R^N \cdot R^N = 0$, i.e.

$$\forall u_1,u_2,v_1,v_2 \in T_{f(o)}N : [R^N(u_1,u_2),R^N(v_1,v_2)] = R^N(R^N(u_1,v_2)v_1,v_2) + R^N(v_1,R^N(u_1,u_2)v_2).$$

Furthermore, $R^N(x,y)(O_{o,f}) \subset O_{o,f}$ and $(R^N(x,y))^O$ is the corresponding curvature endomorphism of $Of$ at $o$, pursuant to Part (d) of Proposition 2 thus $A^O \in \text{hol}(Of)$ for each $A \in \mathfrak{h}$, by virtue of the Theorem of Ambrose/Singer. Since $R^N(x,y)(T_oM) \subset T_oM$ as a consequence of the curvature invariance of $T_oM$, we also have $A^O \in \mathfrak{so}(O_{o,f})_+$ for each $A \in \mathfrak{h}$, in accordance with Part (a) of Lemma 15. The result follows.

\[\text{Note that in case [10] holds, (77) is obvious (since } \text{hol}(Of) \text{ is a Lie subalgebra of } \mathfrak{so}(O_{o,f}).\]
With the intent to show that the “outer derivations” mentioned in Part (b) of Proposition 10 are in fact “inner derivations” of $\mathfrak{hol}(\mathcal{O}f)$, we consider the usual positive definite scalar product on $\mathfrak{so}(\mathcal{O}_o f)$ given by

$$\langle A, B \rangle := -\text{trace}(A \circ B) .$$

It satisfies for all $A, B, C \in \mathfrak{so}(\mathcal{O}_o f)$

$$\langle [A, B], C \rangle = \langle A, [B, C] \rangle .$$

(79)

In other words, $\text{ad}(A)$ is skew-symmetric for each $A \in \mathfrak{so}(\mathcal{O}_o f)$.

Example 8. If $\sigma^\perp$ denotes the linear reflection in $\perp_\mathcal{O}_o f$ (which is an orthogonal map), then $\text{Ad}(\sigma^\perp) : \mathfrak{so}(\mathcal{O}_o f) \to \mathfrak{so}(\mathcal{O}_o f)$ is an orthogonal map, too.

Definition 12. Let $P : \mathfrak{so}(\mathcal{O}_o f) \to \mathfrak{hol}(\mathcal{O}f)$ denote the orthogonal projection onto $\mathfrak{hol}(\mathcal{O}f)$ with respect to the metric introduced above.

Proposition 11. (a) For each $x \in T_o M$ the outer derivation of $\mathfrak{hol}(\mathcal{O}f)$ induced by $\text{ad}(\mathfrak{h}(x))$ is actually an inner derivation of $\mathfrak{hol}(\mathcal{O}f)$; more precisely we have $\forall A \in \mathfrak{hol}(\mathcal{O}M) : [\mathfrak{h}(x), A] = [P(\mathfrak{h}(x)), A]$, i.e.

$$\mathfrak{h}(x) - P(\mathfrak{h}(x)) \in c(\mathfrak{hol}(\mathcal{O}f)) \; \text{(see (65))} .$$

(80)

(b) We have

$$P(\mathfrak{h}(x)) \in \mathfrak{hol}(\mathcal{O}f)_{-} .$$

(81)

(c) The linear map $\mathfrak{h} - P \circ h : T_o M \to \mathfrak{so}(\mathcal{O}_o f)_-$ is injective or identically equal to 0.

Proof of Proposition 11. For (a): Equation (65) is seen as follows: We can write $\mathfrak{h}(x) = P(\mathfrak{h}(x)) + \mathfrak{h}(x)^\perp$ with $\mathfrak{h}(x)^\perp \in \mathfrak{hol}(\mathcal{O}f)^\perp$. For each $A \in \mathfrak{hol}(\mathcal{O}f)$ we have:

$$\left[ \mathfrak{h}(x), A \right] = \left[ P(\mathfrak{h}(x)), A \right] + \left[ \mathfrak{h}(x)^\perp, A \right] ,$$

(82)

from which we see that $[\mathfrak{h}(x)^\perp, A] \in \mathfrak{hol}(\mathcal{O}f)$ . I claim that $[\mathfrak{h}(x)^\perp, A] = 0$ (and therefore (82) yields (80)): In fact, for each $B \in \mathfrak{hol}(\mathcal{O}f)$ we have

$$\langle B, [\mathfrak{h}(x)^\perp, A] \rangle = -\langle B, [A, \mathfrak{h}(x)^\perp] \rangle = -\langle [B, A], \mathfrak{h}(x)^\perp \rangle = 0 ;$$

$$\in \mathfrak{hol}(\mathcal{O}f)$$

which implies that $[\mathfrak{h}(x)^\perp, A] = 0$, since $\langle \cdot, \cdot \rangle$ is non-degenerate.

For (b): From (79) and Example 8 we conclude that $P \circ \text{Ad}(\sigma^\perp) = \text{Ad}(\sigma^\perp) |\mathfrak{hol}(\mathcal{O}f) \circ P$, hence $\text{Ad}(\sigma^\perp) P(\mathfrak{h}(x)) = P(\text{Ad}(\sigma^\perp) \mathfrak{h}(x)) = -P(\mathfrak{h}(x))$ by virtue of Definition 5 in this way Equation (81) has been proved.

For (c): Since $T_o M$ is an irreducible $\mathfrak{hol}(M)$-module (Lemma 7), it suffices to show that $\text{Kern}(\mathfrak{h} - P \circ h)$ is invariant under the natural action of $\mathfrak{hol}(M)$ on $T_o M$ . For this, let $y \in T_o M$ with $\mathfrak{h}(y) \in \mathfrak{hol}(\mathcal{O}f)$ and $A \in \mathfrak{hol}(M)$ be given. Thereby, according to (59), without loss of generality we can assume that there exist $x_1, x_2 \in T_o M$ with $A = R^M(x_1, x_2)$; then

$$\mathfrak{h}(R^M(x_1, x_2) y) \in [R^N(x_1, x_2), \mathfrak{h}(y)] - [\mathfrak{h}(x_1), \mathfrak{h}(x_2)], \mathfrak{h}(y)] .$$

The second term of the r.h.s. of the last line is contained in $\mathfrak{hol}(\mathcal{O}f)$, in accordance with (64). For the first term of the r.h.s. above, note that

$$(R^N(x_1, x_2))^C \in \mathfrak{hol}(\mathcal{O}M) ,$$

according to Part (c) of Proposition 10; the result now follows. □
Corollary 1. If \( \dim(c(\mathfrak{hol}(\mathcal{O})) \cap \mathfrak{so}(\mathcal{O})_-) < m \), then the availability of \( (19) \) is assured.

Proof. We note that \((h - P \circ h)(T_o M) \subset c(\mathfrak{hol}(\mathcal{O})) \cap \mathfrak{so}(\mathcal{O})_-\) according to Part (a) and (b) of Proposition\( [11] \). Therefore, since \( \dim(c(\mathfrak{hol}(\mathcal{O})) \cap \mathfrak{so}(\mathcal{O})_-) < m \), the linear map \( h - P \circ h \) is not injective, thus \( h - P \circ h \) vanishes identically, consequently to Part (c) of Proposition\( [11] \). Hence \( h(x) = P(h(x)) \in \mathfrak{hol}(\mathcal{O}) \) for each \( x \in T_o M \).

Therefore, in the following we wish to find an appropriate upper bound for the dimension of the linear space \( c(\mathfrak{hol}(\mathcal{O})) \cap \mathfrak{so}(\mathcal{O})_-\). At least, one knows that the condition \( c(\mathfrak{hol}(\mathcal{O})) \cap \mathfrak{so}(\mathcal{O})_- = \{0\} \) is not always satisfied, even if also \( N \) is an irreducible symmetric space:

Example 9. Let \( M := \text{SU}(n) \) and \( N := \mathbb{CP}^n \); here \( M \) is a compact simple Lie Group (seen as a Riemannian manifold by means of a bi-invariant metric) and hence \( M \) is an irreducible symmetric space. According to [BCO], p. 261, Table 9.2, there exists an isometric embedding \( f : M \to N \) which is onto a full Lagrangian symmetric submanifold of \( N \). Moreover, \( f(M) \) is a 1-full parallel submanifold according to Theorem 1 of [11] (hence \( O_{\alpha} f = T_{f(\alpha)} N \)); furthermore, if \( j \) denotes the complex structure of \( \mathbb{CP}^n \) at \( f(\alpha) \), then, according to [BCO], A.4, \( j \) belongs to the center of the Lie algebra \( \pi_2(t) = \mathfrak{hol}(N) \) (the last equality uses Lemma\( [13](a) \)) and, since \( M \) is Lagrangian, \( j \in \mathfrak{so}(T_oN)_-\); in particular, \( j \in c(\mathfrak{hol}(f^*TN)) \cap \mathfrak{so}(T_oN)_-\).

In order to use Proposition\( [11] \) as an effective tool to establish the availability of \( (19) \), we need certain basic concepts which are well known from the representation theory of Lie algebras over the real numbers.

Let \( W, U \) be vector spaces over a field \( K \in \{\mathbb{R}, \mathbb{C}\} \) and \( \rho_W : \mathfrak{h} \to \mathfrak{gl}(W), \rho_U : \mathfrak{h} \to \mathfrak{gl}(U) \) be \( \mathbb{R} \)-linear representations of a real Lie algebra \( \mathfrak{h} \).

Definition 13. Put

\[
\text{Hom}_\mathfrak{h}(W,U) := \{ \lambda \in \text{L}(W,U) | \forall h \in \mathfrak{h} : \lambda \circ \rho_W(h) = \rho_U(h) \circ \lambda \};
\]

then \( \text{Hom}_\mathfrak{h}(W,U) \) is a vector space over \( K \).

Lemma 16. In the above situation, suppose that \( W \) is an irreducible \( \mathfrak{h} \)-module. Then we have:

(a) Each \( \lambda \in \text{Hom}_\mathfrak{h}(W,U) \) is either an injective map or identically equal to 0; in case \( \lambda \neq 0 \) its image \( \lambda(W) \) is an irreducible \( \mathfrak{h} \)-submodule of \( U \) and \( \lambda^{-1} : \lambda(W) \to W \) is an \( \mathfrak{h} \)-homomorphism, too.

(b) If \( \lambda \in \text{Hom}_\mathfrak{h}(W,W) \) has at least one eigenvalue, then \( \lambda = c \text{Id}_W \) for some \( c \in K \). In particular, if \( K = \mathbb{C} \), then \( \text{Hom}_\mathfrak{h}(W,W) = \mathbb{C} \text{Id}_W \).

Now suppose that \( W, U \) are Euclidian vector spaces and that \( \rho_W(h) \subset \mathfrak{so}(W) \) and \( \rho_U(h) \subset \mathfrak{so}(U) \). Put \( d := \dim(\text{Hom}_\mathfrak{h}(W,W)) \).

(c) We have \( d \in \{1, 2, 4\} \), depending on the following: \( d \geq 2 \) if and only if \( W \) has the underlying structure of a Hermitian vector space such that \( \rho_W(h) \subset \mathfrak{u}(W) \); \( d = 4 \) if and only if there is the underlying structure of a quaternionic Hermitian vector space (in the sense of \( \mathbb{T}s\mathbb{S}2; (2.1) \)) such that \( \rho_W(h) \subset \mathfrak{sp}(W) \).

(d) We have \( \dim(U) \geq \dim(W) \cdot \dim(\text{Hom}_\mathfrak{h}(W,U))/d \).

Proof. For (a) and (b): This is usually known as “Schur’s Lemma”.

For (c): Obviously, if \( W \) is Hermitian equipped with the underlying structure of a Hermitian (resp. quaternionic Hermitian) vector space such that \( \rho_W(h) \subset \mathfrak{u}(W) \) (resp. \( \rho_W(h) \subset \mathfrak{sp}(W) \)), then \( d \geq 2 \) (resp. \( d \geq 4 \)).

In the other direction, we first aim to prove that \( d \in \{1, 2, 4\} \) holds. For this:

We consider the complexification \( W^C := W \oplus iW \), seen as a Hermitian vector space in the usual way and the induced representation \( \rho_W^C : \mathfrak{h} \to \mathfrak{su}(W^C) \); i.e. \( \rho_W^C(h, \cdot) \) is the \( \mathbb{C} \)-linear endomorphism of \( W^C \) which is equal to \( \rho_W(h, \cdot) \) on \( W \) for each \( A \in \mathfrak{h} \). Then, in accordance with a result of [GT], p. 17, either \( W^C \) is an irreducible \( \mathfrak{h} \)-module, too, or there exists a proper complex subspace \( V \subset W^C \) such that \( W^C = V \oplus \overline{V} \) is a decomposition into two irreducible \( \mathfrak{h} \)-submodules (where \( \overline{\cdot} \) means “complex conjugation”). Furthermore, we observe that the natural complex linear isomorphism \( (\mathfrak{gl}(W))^C \to \mathfrak{gl}(W^C, W^C) \)
induces an isomorphism \((\text{Hom}_h(W,W))^\mathbb{C} \rightarrow \text{Hom}_h(W^c,W^c)\) of vector spaces over \(\mathbb{C}\); in particular, \(d = \dim_{\mathbb{C}}(\text{Hom}_h(W^c,W^c))\). Because of Schur's Lemma (applied “over \(\mathbb{C}\)”), it now easily follows that \(d \in \{1, 2, 4\}\).

In the following, we assume that \(d \in \{2, 4\}\). In order to construct an underlying Hermitian (resp. quaternionic Hermitian) structure on \(W\), we proceed as follows:

For each \(A \in \text{Hom}_h(W,W)\) we have the decomposition \(A = R(A) + I(A) := \frac{1}{2}(A + A^*) + \frac{1}{2}(A - A^*)\) (where \((\cdot)^*\) means “adjoint”); hence \(R(A)\) is self adjoint, \(I(A)\) is skew adjoint, and both \(R(A), I(A)\) again are elements of \(\text{Hom}_h(W,W)\). Furthermore, because of Schur’s Lemma (and since every self adjoint endomorphism of \(W\) has at least one eigenvalue), \(R(A)\) is a multiple of the identity; thus we obtain the decomposition

\[
\text{Hom}_h(W,W) = R \text{Id}_W \oplus V \quad \text{with} \quad V := \{ A \in \text{Hom}_h(W,W) \mid A^* = -A \}.
\]

Then \(V\) is a Euclidian vector space by means of the inner product given by \(\langle A, B \rangle := -\text{trace}(A \circ B)\). We notice that \(A^2\) is selfadjoint with strictly negative eigenvalues for each \(A \in V\) with \(A \neq 0\); hence \(A^2 = -\kappa \text{Id}_W\) for some \(\kappa \in \mathbb{R}_+\), again by means of Schur’s Lemma. By the previous, there exists \(i \in V\) with \(i^2 = -\text{Id}_W\). Then \(i^* = -i = i^{-1}\) and thus \(i\) equips \(W\) with the structure of a Hermitian vector space such that \(\rho(h) \subset u(W)\).

In case \(d = 4\), let \(j \in V\) with \(\langle i, j \rangle = 0\) and such that \(j^2 = -\text{Id}_W\) be given and put \(k := i \circ j\). Then \(\langle i, j \rangle = 0\) forces \(R(k) = 0\), hence \(k \in V\). Therefore,

\[
j \circ i = j^* \circ i^* = k^* = -k = -i \circ j.
\]

Thus we observe that \(\{i, j, k\}\) is an orthonormal basis of \(V\) such that the usual quaternionic relations \(i^2 = j^2 = -\text{Id}_W\), \(i \circ j = -j \circ i = k\) hold; therefore \(W\) is equipped with the structure of a quaternionic Hermitian vector space such that \(\rho(h) \subset \mathfrak{sp}(W)\). This finishes the proof of (c).

For (d): Decompose \(U\) into irreducible submodules and then use (a).

We now consider the Lie algebra \(\mathfrak{h}\) defined by (78) and its linear representations

\[
\rho_1 : \mathfrak{h} \rightarrow \text{so}(T_oM), \quad A \mapsto A|T_oM : T_oM \rightarrow T_oM, \quad (84)
\]

\[
\rho_2 : \mathfrak{h} \rightarrow \text{so}(\perp^1_0 f), \quad A \mapsto A|\perp^1_0 f : \perp^1_0 f \rightarrow \perp^1_0 f, \quad (85)
\]

proposed by Part (c) of Proposition 10.

**Lemma 17.** The isomorphism \(\text{so}(\mathcal{O}_f)_- \rightarrow L(T_oM, \perp^1_0 f)\) provided by Lemma 15 induces an inclusion

\[
\mathfrak{c}(\mathfrak{ho}(\mathcal{O}f)) \cap \mathfrak{so}(\mathcal{O}f)_- \hookrightarrow \text{Hom}_h(T_oM, \perp^1_0 f).
\]

**Proof.** Let \(A \in \mathfrak{c}(\mathfrak{ho}(\mathcal{O}f)) \cap \mathfrak{so}(\mathcal{O}f)_-\) be given. Using Part (c) of Proposition 10 it is straightforward to show that \(A|T_oM : T_oM \rightarrow \perp^1_0 f\) belongs to \(\text{Hom}_h(T_oM, \perp^1_0 f)\). \(\square\)

From (61) combined with (78) and (84) and Lemma 7 we obtain:

**Lemma 18.** We have

\[
\mathfrak{ho}(\tilde{M}) = \rho_1(h),
\]

and \(T_oM\) is an irreducible \(\mathfrak{h}\)-module.

We now define the integer

\[
d := \dim(\text{Hom}_h(T_oM, T_oM))\]

**Lemma 19.** We have \(d \leq 2\).
Proof. We have \( d \in \{1, 2, 4\} \), according to Lemma \[16\] in combination with Lemma \[18\]. I claim that \( d = 4 \) is not possible:

By contradiction, assume that \( d = 4 \); then, according to Lemma \[16\] \( T_o M \) has the underlying structure of a quaternionic Hermitian vector space such that \( \rho_1(\mathfrak{h}) \subset \mathfrak{sp}(T_o M) \) and hence, by means of Lemma \[18\] the orthogonal frame bundle of \( \tilde{M} \) is locally reducible to a subbundle whose holonomy group is \( \mathfrak{sp}(n) \subset \mathfrak{su}(n) \). Thus \( M \) is a Kähler manifold which locally admits a non vanishing, parallel, complex holomorphic volume form and therefore the Ricci form (or likewise the Ricci tensor) of \( \tilde{M} \) vanishes, according to \[He\]. Corollary 2.97. On the other hand, \( \tilde{M} \) is an irreducible symmetric space by assumption, hence of compact type or non compact type; which implies that its sectional curvature is non-negative or non-positive, according to \[He\], Ch. V, Theorem 3.1. Therefore, if the Ricci tensor of \( \tilde{M} \) vanishes, then \( \tilde{M} \) is flat, thus of Euclidian type, a contradiction. We hence conclude that \( d \in \{1, 2\} \).

\[\Box\]

Lemma 20. If \( \dim(\perp f) > d \), then \( \mathfrak{hol}(\mathcal{O}f)_- \neq \{0\} \); note that we always have \( d \in \{1, 2\} \) according to Lemma \[14\].

Proof. By contradiction, assume that \( \mathfrak{hol}(\mathcal{O}f)_- = \{0\} \). Using Parts (a) and (b) of Proposition \[10\] Definition \[5\] and the rules for \( \mathbb{Z}/2\mathbb{Z} \) graded Lie algebras, we conclude that

\[
\forall x \in T_o M : [h(x), \mathfrak{hol}(\mathcal{O}f)_+] \subset \mathfrak{hol}(\mathcal{O}f)_- = \{0\},
\]

and therefore also

\[
\forall A \in \mathfrak{hol}(\mathcal{O}f)_+, x \in T_o M : [h(x), A] = 0.
\]

Let \( A \in \mathfrak{h} \) be given and put \( A_1 := \rho_1(A) \in \mathfrak{so}(T_o M) \) and \( A_2 := \rho_2(A) \in \mathfrak{so}(\perp f) \). Consequently to Part (c) of Proposition \[10\] the endomorphism \( A^O = A_1 \oplus A_2 \) belongs to \( \mathfrak{hol}(\mathcal{O}f)_+ \). The previous implies that for all \( x \in T_o M \)

\[
[A, h(x)] = 0; \text{ therefore } [A, h(x)]|T_o M = 0, \text{ i.e. } \forall y \in T_o M : A_2 h(x) y = h(x) A_1 y;
\]

hence, for all \( x, y \in T_o M \)

\[
h(x, A_1 y) = A_2 h(x, y) = A_2 h(y, x) = h(y, A_1 x).
\]

Multiplication of the last equation with \( \xi \in \perp f \) yields

\[
\langle x, S_\xi A_1 y \rangle = \langle h(x, A_1 y), \xi \rangle = \langle h(y, A_1 x), \xi \rangle = \langle y, S_\xi A_1 x \rangle.
\]

Since \( A_1 \) is skew-symmetric, whereas \( S_\xi \) is symmetric, it follows that

\[
A_1 \circ S_\xi = -S_\xi \circ A_1;
\]

and therefore

\[
\forall \xi, \eta \in \perp f : A_1 \circ S_\xi \circ S_\eta = -S_\xi \circ A_1 \circ S_\eta = S_\xi \circ S_\eta \circ A_1.
\]

We now conclude: \( S_\xi \circ S_\eta \in \text{Hom}_h(T_o M, T_o M) \). Let \( \xi \in \perp f \) be an element different from 0. Because \( S_\xi^2 \) is self adjoint and strictly positive, there exists \( \kappa > 0 \) such that \( V := \ker(S_\xi^2 - \kappa \cdot \text{Id}) \neq \{0\} \) and, by the previous, \( V \) is an \( \mathfrak{h} \)-invariant subspace of \( T_o M \). Since \( \mathfrak{h} \) acts irreducible on \( T_o M \), according to Lemma \[18\] Schur’s Lemma implies that \( V = T_o M ; \text{ thus } S_\xi^2 = \kappa \cdot \text{Id}_{T_o M} \), in particular \( S_\xi \) is invertible and the following map is injective:

\[
\perp f \to \text{Hom}_h(T_o M, T_o M), \eta \mapsto S_\xi \circ S_\eta.
\]

Therefore, the inequality \( \dim(\perp f) \leq d \) is established.

\[\Box\]

Proposition 12. If \( m \geq 3 \), then \( \dim(\mathcal{c}(\mathfrak{hol}(\mathcal{O}f)) \cap \mathfrak{so}(\mathcal{O}f)_-) < 3 \).

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Lemma 21. (a) $G$ (see [KN] p. 275) that $i$ in particular, $\Theta$ early independent elements of Hom$_g(T_oM, T_oM)$ are linearly independent elements of Hom$_g(T_oM, T_oM)$, consequently to Lemma 17. Put $U_j := \lambda_j|T_oM)$, then $\lambda_j|U_j$ is an isomorphism onto $U_j$ according to Lemma 15. It is not possible that $U_1 = U_2 = U_3$, since otherwise $\text{Id}_{T_oM} = \lambda_1^2 \cdot \lambda_2 \cdot \lambda_3$ were three linearly independent elements of the vector space Hom$_g(T_oM, T_oM)$, but $d \leq 2$ according to Lemma 18. Therefore, without loss of generality we may assume that $U_1 \neq U_2$; then even $U_1 \cap U_2 = \{0\}$, since $U_1$ and $U_2$ are irreducible $\mathfrak{g}$-modules. I claim that this already implies that $\mathfrak{hol}(\mathcal{O})_\gamma = \{0\}$. For this:

By the above, for $j = 1, 2$ the linear maps

$$\lambda_j = A_j|T_oM : T_oM \rightarrow U_j \quad \text{and} \quad \lambda_j^* = -A_j|U_j : U_j \rightarrow T_oM$$

are linear isomorphisms; therefore, for every $x \in T_oM$ and $j = 1, 2$ there exists $\xi_j \in U_j$ with $A_j(\xi_j) = x$.

Hence, given $A \in \mathfrak{hol}(\mathcal{O})_\gamma$, we have $[A_1, A] = [A_2, A] = 0$, according to (80); thus

$$Ax = A(A_j\xi_j) = A_j(A_j\xi_j) \in U_j \quad \text{for } j = 1, 2,$$

and therefore $A \in U_1 \cap U_2 = \{0\}$. We obtain $A|T_oM = 0$ and because of Lemma 15 even $A = 0$.

Therefore, $\mathfrak{hol}(\mathcal{O})_\gamma = \{0\}$, thus $\dim(\mathfrak{l}^f) \leq 2$, according to Lemma 20. On the other hand, $\dim(\mathfrak{hol}) = \dim(U_1) = m \geq 3$, a contradiction. □

Proof of Theorem 6. Use Corollary 1 in combination with Proposition 12.

3.3 2-symmetric submanifolds

In this section, $M$ denotes an irreducible symmetric space, whose geodesic symmetries at the various points $p \in M$ are denoted by $\sigma_p$, and whose Cartan decomposition is given by $\text{I}(M) = \mathfrak{t}^M \oplus \mathfrak{p}^M$. At the end of this section we will give the proof of Theorem 6.

For each smooth geodesic line $\gamma$ of $M$ with $\gamma(0) = p$ we have the family of “transvections along $\gamma$”, given by

$$\forall t \in \mathbb{R} : \Theta^\gamma(t) := \sigma^\gamma_M(t) \circ \sigma^\gamma_M(p). \quad (89)$$

It is elementary to show that

$$\Theta^\gamma(t)(\gamma(p)) = \gamma(t) \quad \text{and} \quad T_p\Theta^\gamma(t) = \left(\left[t, \gamma\right]\right)^\gamma ; \quad (90)$$

in particular, $\Theta^\gamma(t)$ is a differentiable one-parameter subgroup of $\text{I}(M)$.

Lemma 21. (a) $\text{I}(M)^0$ is generated (as an abstract group) by the set

$$\left\{ \Theta^\gamma(t) \mid \gamma : [0, 1] \rightarrow M \text{ is a smooth geodesic line, } t \in \mathbb{R} \right\} . \quad (91)$$

(b) The group $\text{Sym}(M)$ which is generated (as an abstract group) by the geodesic symmetries of $M$ is a Lie subgroup of $\text{I}(M)$ with at most two components, and $\text{I}(M)^0$ is its connected component.

(c) If $M$ is isometric to one of the symmetric spaces listed in Theorem 6, then $\text{I}(M)^0 = \text{Sym}(M)$.

Proof. For (a): Let $G$ denote the group generated by the set (91). Then, as a consequence of (90), every element of $G$ can be joined with $\text{Id}_M$ by a $C^\infty$-path in $\text{I}(M)$; thus it follows from a result of Freudenthal (see [KN] p. 275) that $G$ is already a connected Lie subgroup of $\text{I}(M)$. Let $\mathfrak{g}$ denote the Lie algebra of $G$; I claim that $\mathfrak{p}^M = \mathfrak{g}$ holds: Since $\nabla^M$ is the canonical of $TM$ induced by the Cartan decomposition of $\text{I}(M)$ (by means of the arguments given in Section 2.1), for each $X \in \mathfrak{p}^M$ the curve $\gamma(t) := \exp(tX)(0)$ is a geodesic of $M$ and $T_t\exp(tX)y$ is a parallel section of $TM$ along $\gamma$ for each $y \in T_oM$, in accordance with Example 7. Thus $\exp(tX) = \Theta^\gamma_M(t)$ for all $t \in \mathbb{R}$, consequently to (90); hence $X \in \mathfrak{g}$. Therefore, we actually have $\mathfrak{p}^M = \mathfrak{g}$; moreover, since $\mathfrak{t}^M = [\mathfrak{p}^M, \mathfrak{p}^M]$ holds, according to [He], Ch. V, § 4, we even have $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$, which finishes the proof of (a).

For (b): This follows from (89) and Part (a).
For (c): Recall the following result of [Hc], Ch. IX, Corollary 5.8: For a symmetric space \( M \) of non-compact type, \( \mathfrak{p}^0(M) \) contains the geodesic reflections of \( M \) if and only if \( \mathfrak{t}^k \) contains a maximal Abelian subalgebra of \( \mathfrak{i}(M) \). Furthermore, one carefully verifies that this result remains true if \( M \) is replaced by \( M^* \), the (simply connected) compact dual space. Recalling the classification of symmetric spaces (cf. [BCO], A.4), we find that in Theorem \( \text{[9]} \) there are listed all those simply connected, irreducible symmetric spaces \( M \) for which \( \mathfrak{t}^k \) contains a maximal Abelian subalgebra of \( \mathfrak{i}(M) \); note that for every Hermitian symmetric space \( M \) the geodesic symmetries of \( M \) are contained in \( \mathfrak{p}^0(M) \) (see [Hc], Ch. VIII, Theorem 4.5). Now the result follows.

Proof of Theorem \( \text{[9]} \). Let \( M \) be one of the symmetric spaces listed in Theorem \( \text{[9]} \) and \( f : M \to N \) be a full parallel isometric immersion; in particular, then \( \tilde{M} := f(M) \) is not contained in any flat of \( N \), \( M \) is simply connected and irreducible, we have \( \dim(M) \geq 3 \), and \( I(M)^0 = \text{Sym}(M) \) by virtue of Lemma \( \text{[2]} \). Thus, according to Theorem \( \text{[9]} \) \( \tilde{M} \) is a parallel submanifold with extrinsically homogeneous tangent holonomy bundle and \( f : M \to \tilde{M} \) is a covering. Let \( G \) be a subgroup of \( I(N) \) as described in Definition \( \text{[8]} \). I claim that already \( \tilde{M} \) is a symmetric space with \( \text{Sym}(\tilde{M}) = I(M)^0 \) and that the natural group homomorphism \( \pi : G \to I(M)^0 \) given by \( g \mapsto g|\tilde{M} \) is onto.

Let \( p \in \tilde{M} \) be given and \( q \in M \) with \( f(q) = p \). Then, by the previous, \( \sigma_q^M \in I(M)^0 \); hence, in accordance with \( \text{[1]} \), there exist certain smooth geodesic lines \( \gamma_i : [0, 1] \to M \) with \( \gamma_i(0) = q_i \) (\( i = 1, \ldots, n \)) such that

\[
\sigma_q^M = \Theta_{\gamma_1}(1) \circ \cdots \circ \Theta_{\gamma_n}(1).
\]

Furthermore, \( f \circ \gamma_i \) is a curve into \( \tilde{M} \); hence, in accordance with Definition \( \text{[8]} \) for each \( i = 1, \ldots, n \) there exists \( g_i \in G \) with \( g_i(M) = \tilde{M} \) and \( g_i(q_i) = f(\gamma_i(1)) \) and

\[
Tg_i|_{q_i} \tilde{M} = (\| f \circ \gamma_i \|^2)_{\tilde{M}}.
\]

Put \( g := g_1 \circ \cdots \circ g_n \); then, since \( f : M \to \tilde{M} \) is a Riemannian covering, \( \text{[1]} \), \( \text{[2]} \) and \( \text{[3]} \) imply that \( Tg|_{p} \tilde{M} = -\text{Id} \) holds. Thus \( g^2|\tilde{M} = \text{Id} \) and hence \( g|M \) is the geodesic symmetry of \( M \) at \( p \). This proves our claim.

Since moreover \( \pi \) is injective (because \( \tilde{M} \) is full in \( N \) and by means of Lemma \( \text{[1]} \), its inverse is a Lie group homomorphism \( \tilde{f} : \text{Sym}(M) \to I(N) \) which has the properties described in Definition \( \text{[8]} \).

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