Generalized Morse and Pöschl-Teller potentials: The connection via Schrödinger equation

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Abstract

We present here a systematic and unified treatment to connect the Schrödinger equation corresponding to generalized Morse and Poschl-Teller potentials. We then show that the wave functions and generalized potentials are linked through the Fourier and Hankel transforms, respectively.

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1 Introduction

Group theory provides us with efficient algebraic techniques which are used for the description of the energy spectra of quantum mechanical potentials [1-5]. One of the important aspects of algebraic techniques is how to construct the Hamiltonian from the Casimir operator(s) related to the group-algebraic structures. To this end, the most used technique is the potential group approach [1,4,5]. It’s based on two approaches, the former, called the algebraic approach [1], is considered as the convenient way to construct the spectrum-generating algebra for the system and this by introducing a set of boson creation and annihilation operators. Therefore, it can be connected to the exactly soluble Schrödinger equations with certain potentials, whereas the latter called potential approach [1] can be connected to the first one, i.e. algebraic approach, by introducing the boson creation and annihilation operators as differential operators in two-dimensional harmonic oscillator space and sphere.

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The supersymmetric quantum mechanics is an another algebraic approach based on the generalized version of the creation and annihilation operators provided by the factorization method and investigates Hamiltonians which split into two components whose spectra are the same, with the exception of the ground-state [6-14]. These operators are often called the bosonic and fermionic parts of supersymmetric system.

We concentrate our attention in this paper on the Morse and the Pöschl-Teller one-dimensional potentials. The former can be realized on $SU(2)$ two-dimensional harmonic oscillator space [15], whereas the latter can be realized on $SU(2)$-sphere [16]. It has been proved that the energy spectra of both potentials are related to the same representation of the $su(2) \cong so(3)$ Lie algebra ($\cong$ means isomorphic to) [1]; it plays the role of dynamical algebra. Therefore, the $su(2)$ Lie algebra can gives unitary representation of the $SU(2)$ Lie group.

The connection between the Schrödinger equations with Morse and Pöschl-Teller potentials was already established by Alhassid et al. [1]. This present paper would be an extension of algebraic treatment of $su(2)$ Lie Algebra to their generalized potentials, through the supersymmetric quantum mechanics which allows us to generalize any given potential, with a view to establishing a causal connection underlying the relationship between both generalized Morse and Pöschl-Teller potentials. We then show that the Schrödinger equation for generalized Pöschl-Teller potentials can be constructed mathematically starting off with a Schrödinger equation for generalized Morse potentials, proving that each equation can be interpreted as the Fourier transform of the other (see the appendix). As a consequence of this, the algebra associated with $SU(2)$ group provides, in a straightforward way, the relation connecting both energy spectra.

The arrangement of this paper is as follows. First we recall a brief overview of $su(2)$ Lie algebra and the basic relations involved in supersymmetric quantum mechanics, in order to construct the generalized Morse [13] and Pöschl-Teller potentials. Section 3 deals with different steps of transformation connecting their generalized potentials via the Schrödinger equation. The final section will be devoted to discussions and in the appendix, mathematical details of transformation connecting generalized potentials will be presented.
2 Classes of $SU(2)$ realizations and Supersymmetry

The generators of the compact $SU(2)$ Lie group obey to commutation relations [1]

\[ [\mathcal{J}_z, \mathcal{J}_\pm] = \pm \mathcal{J}_\pm, \quad (1.a) \]
\[ [\mathcal{J}_+, \mathcal{J}_-] = 2\mathcal{J}_z. \quad (1.b) \]

with $\mathcal{J}_\pm = \mathcal{J}_x \pm i\mathcal{J}_y$. Here, the operators $\mathcal{J}_\alpha$ with $\alpha = +, -, z$ can be obtained by using the set of boson operators as

\[ \mathcal{J}_+ = a^\dagger b, \quad (2.a) \]
\[ \mathcal{J}_- = b^\dagger a, \quad (2.b) \]
\[ \mathcal{J}_z = \frac{1}{2}(a^\dagger a - b^\dagger b). \quad (2.c) \]

In order to complete the $SU(2)$ algebra, one needs a fourth operator namely the total boson number operator $\mathcal{N} = a^\dagger a + b^\dagger b$ which belongs to $U(2)$ group, and is related to the Casimir operator $\mathcal{C}_2$ of $U(2)$ by

\[ \mathcal{C}_2 = \frac{1}{4}\mathcal{N}(\mathcal{N} + 2) \]
\[ = \mathcal{J}_+ \mathcal{J}_+ + \mathcal{J}_z (\mathcal{J}_z - 1). \quad (3) \]

Therefore, it is clear that the eigenstates of $\mathcal{C}_2$ and $\mathcal{J}_z$ serve as basis for the irreducible representation of $su(2)$ algebra. Let us derive now the standard supersymmetric quantum mechanics (SUSY-QM) relations. Considering SUSY-QM with $\mathcal{D} = 2$ [6-14], we define the supercharges $\mathcal{Q} = d\sigma_-$ and $\mathcal{Q}^\dagger = d^\dagger \sigma_+$ where $d$ ($d^\dagger$) and $\sigma_-$ ($\sigma_+$) are, respectively, the bosonic operators and Pauli matrices

\[ d = \partial_x + x, \quad (4.a) \]
\[ d^\dagger = -\partial_x + x. \quad (4.b) \]

\[ \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} ; \quad \sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \quad (5) \]

where we have used the abbreviation $\partial_x = \frac{d}{dx}$.
We define the supersymmetric Hamiltonian [11,13]

\[ \mathcal{H}_{SUSY} = \{Q, Q^\dagger\}, \]

where \( \mathcal{H}_+ \) is called the supersymmetric partner of \( \mathcal{H}_- \). Both \( \mathcal{H}_+ \) and \( \mathcal{H}_- \) have the same spectra except for the ground-state, which belongs to \( \mathcal{H}_+ \).

In order to find the generalized potentials, we shall first generalize the bosonic operators \( d \) and \( d^\dagger \) [6,11] as

\[
D = \partial_x + f(x), \quad (7.a)
\]
\[
D^\dagger = -\partial_x + f(x). \quad (7.b)
\]

By imposing that \( \mathcal{H}_- = dd^\dagger \), we obtain the Ricatti differential equation from the Schrödinger equation

\[
\partial_x f(x) + f^2(x) = (\partial_x W(x))^2 + \partial_x^2 W(x), \quad (8)
\]

where \( f(x) \) is the derivative of \( W(x) \), this later is called superpotential and is associated with \( \mathcal{H}_+ \) ground-state eigenfunctions \( \psi_{+,0} \).

Therefore, we can define a new Hamiltonian using (7)

\[
\tilde{\mathcal{H}}_+ = D^\dagger D = DD^\dagger - [D, D^\dagger] = DD^\dagger - 2 \partial_x W(x). \quad (9)
\]

Following supersymmetry, it seems that the spectrum of \( \tilde{\mathcal{H}}_+ \) is the same as the spectrum of \( \mathcal{H}_- \) [8,11].

2.1 The first class \( SU(2) \) realization: Morse potential

The algebraic approach based on (1)-(3) can be connected to the Schrödinger equation of the Morse potential into a two-dimensional harmonic space by introducing the following realizations of the boson operators [1]

\[
a = \frac{1}{\sqrt{2}} (x + \partial_x), \quad (10.a)
\]
\[
a^\dagger = \frac{1}{\sqrt{2}} (x - \partial_x), \quad (10.b)
\]
\[
b = \frac{1}{\sqrt{2}} (y + \partial_y), \quad (10.c)
\]
\[
b^\dagger = \frac{1}{\sqrt{2}} (y - \partial_y). \quad (10.d)
\]

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\(^1\)Here \( \{\cdots,\cdots\} \) stands for the anticommutation relations, i.e. \( \{A,B\} = AB + BA \).
In terms of the variables $x$, $y$, the two operators $N$ and $J_z$ become

\begin{align}
N &= \frac{1}{2} (x^2 + y^2 - \partial_x^2 - \partial_y^2 - 2), \\
J_z &= -\frac{i}{2} (x \partial_x - y \partial_y).
\end{align}

A change of variables $x = r \cos \varphi$, $y = r \sin \varphi$, with $0 \leq r < \infty$ and $0 \leq \varphi < 2\pi$ would be helpful, leading to re-express (11.a) and (11.b) by

\begin{align}
N &= \frac{1}{2} \left( -\frac{1}{r} \partial_r (r \partial_r) - \frac{1}{r^2} \partial_\varphi^2 + r^2 \right) - 1, \\
J_z &= -\frac{i}{2} \partial_\varphi.
\end{align}

The action of (12.a) on the wave function $\psi (r, \varphi) = R (r) e^{2\sqrt{E_M} \varphi}$, taking into account the transformation $r^2 = 2\lambda e^{-\rho}$, leads to the Schrödinger equation for a particle constrained to move in one-dimensional Morse potential

\begin{equation}
[-\partial_\rho^2 + \lambda^2 \left( e^{-2\rho} - 2 e^{-\rho} \right) ] R (r) = E_M R (r).
\end{equation}

In order to make the eigenvalues of the ground-state equal to zero, i.e. $E_{+,0}^{(M)} = 0$, one displaces the potential given in (13) by the quantity $(\lambda - \frac{1}{2})^2$, i.e.

\begin{equation}
\lambda^2 \left( e^{-2\rho} - 2 e^{-\rho} \right) \rightarrow \lambda^2 \left( e^{-2\rho} - 2 e^{-\rho} \right) + \left( \lambda - \frac{1}{2} \right)^2 \\
= \lambda^2 \left( 1 - e^{-\rho} \right)^2 - \lambda + \frac{1}{4}.
\end{equation}

As a consequence of this, the Hamiltonian $H_+$ given in (6) becomes [13]

\begin{equation}
H_+ \equiv d^\dagger d = -\partial_\rho^2 + \lambda^2 \left( 1 - e^{-\rho} \right)^2 - \lambda + \frac{1}{4},
\end{equation}

and then the supersymmetric partner of the Hamiltonian (15) reads

\begin{equation}
H_- \equiv d \, d^\dagger = H_+ + 2 \lambda e^{-\rho}.
\end{equation}

Both Hamiltonians have the same spectra, except that (15) has a ground-state with zero eigenvalue.
In order to generalize (15), we define the operators given by (7) and by solving the corresponding Ricatti differential equation, we obtain

$$f_M(\rho) = \lambda \left(1 - e^{-\rho}\right) - \frac{1}{2} + \frac{\exp[-(2\lambda - 1)\rho - 2\lambda e^{-\rho}]}{\Gamma + \int_0^\rho \exp[-(2\lambda - 1)\tilde{\rho} - 2\lambda e^{-\tilde{\rho}}]}. \quad (17)$$

where $\Gamma$ is an arbitrary constant and is chosen $\Gamma > 0$ in order to avoid singularities.

We can define a new Hamiltonian $\tilde{H}_+$ where the corresponding potential is

$$\tilde{H}_+ (\rho) = -\partial_\rho^2 + \lambda^2 \left(1 - e^{-\rho}\right)^2 - \lambda + \frac{1}{4} - 2\partial_\rho \left[\frac{\exp[-(2\lambda - 1)\rho - 2\lambda e^{-\rho}]}{\Gamma + \int_0^\rho \exp[-(2\lambda - 1)\tilde{\rho} - 2\lambda e^{-\tilde{\rho}}]}\right]. \quad (18)$$

The Schrödinger equation corresponding to generalized Morse potential (18) is

$$\left[-\partial_r^2 + \lambda^2 \left(1 - e^{-\rho}\right)^2 - \lambda + \frac{1}{4} - 2\partial_r q_{\lambda,\Gamma}^{(M)} (\rho)\right] \mathcal{R} (\rho) = E_M \mathcal{R} (\rho), \quad (19)$$

with

$$q_{\lambda,\Gamma}^{(M)} (\rho) = \frac{\exp[-\rho (2\lambda - 1) - 2\lambda e^{-\rho}]}{\Gamma + \int_0^\rho \exp[-\tilde{\rho} (2\lambda - 1) - 2\lambda e^{-\tilde{\rho}}]}. \quad (20)$$

The change of variable $\rho = \ln \frac{2\lambda}{r^2}$ and the introduction of a parameter $a = \lambda - \frac{1}{2}$ bring (19) and (20) to

$$\left[-\frac{1}{r} \partial_r (r \partial_r) + r^2 + \frac{4(a^2 - E_M)}{r^2} + \frac{4}{r} \partial_r q_{a,\Gamma}^{(M)} (r)\right] \mathcal{R} (r) = (4a + 2) \mathcal{R} (r), \quad (21)$$
with

\[ q_{a,\Gamma}^{(M)}(r) = \frac{\left( \frac{r^2}{2a+1} \right)^{2a} e^{-r^2}}{\Gamma + \int_0^r \frac{d\tilde{r}}{\tilde{r}^2} \left( \frac{r^2}{2} - \frac{2}{\tilde{r}^2} \right)^{2a} e^{-\tilde{r}^2}} \] (22)

2.2 The second class SU(2) realization: Pöschl-Teller potential

A realization of \( su(2) \) algebra on the sphere leads to a connection between \( su(2) \) algebra and the Schrödinger equation with Pöschl-Teller potential. This last is obtained by using the following realizations [1]

\[ J_\pm = e^{\pm i\varphi} (\pm \partial_\theta + i \cot \varphi \partial_\varphi), \quad (23.a) \]
\[ J_z = -i \partial_\varphi, \quad (23.b) \]
\[ N = -\left[ \frac{1}{\sin \theta} \partial_\theta (\sin \theta \partial_\theta) + \frac{1}{\sin^2 \theta} \partial^2_\varphi \right]. \quad (23.c) \]

The substitution \( \cos \theta = \tanh \rho \) with \( -\infty < \rho < \infty \) brings the eigenstate equation \( N\xi(\theta, \varphi) = \mu(\mu + 1)\xi(\theta, \varphi) \), with \( \xi(\theta, \varphi) = U(\theta) e^{\sqrt{E_{PT}} \varphi} \), to a dimensionless Schrödinger equation with Pöschl-Teller potential [1,16]

\[ \left[ -\partial^2_\rho - \frac{\mu(\mu + 1)}{\cosh^2 \rho} \right] U(\rho) = E_{PT} U(\rho). \quad (24) \]

In order to displace the ground-state energy to zero, we add a constant term \( \mu^2 \) to the potential in (24) such that the Hamiltonian becomes

\[ H_+ = -\partial^2_\rho - \frac{\mu(\mu + 1)}{\cosh^2 \rho} + \mu^2. \quad (25) \]

The supersymmetric partner of \( H_+ \), following (6), reads as

\[ H_- = H_+ + \frac{2\mu}{\cosh^2 \rho}. \quad (26) \]

As in case of Morse potential, \( H_+ \) and \( H_- \) share the same spectrum except that (25) has ground-state energy equal to zero [1-7,17,18].
Performing similar transformations to (7), we define the corresponding Ricatti differential equation having the solution

\[ f_{PT}(\rho) = \mu \tanh \rho + \frac{\cosh^{-2\mu} \rho}{\Gamma + \int_0^\rho d\tilde{\rho} \cosh^{-2\mu} \tilde{\rho}}, \]  

(27)

where \( \Gamma > 0 \) is added in order to avoid contingent singularities.

Therefore, we can define a new Hamiltonian

\[ \tilde{\mathcal{H}}_+ = -\partial^2_\rho - \frac{\mu (\mu + 1)}{\cosh^2 \rho} + \mu^2 - 2\partial_\rho \left[ \frac{\cosh^{-2\mu} \rho}{\Gamma + \int_0^\rho d\tilde{\rho} \cosh^{-2\mu} \tilde{\rho}} \right]. \]

(28)

We can verify that \( \tilde{\mathcal{H}}_+ \) and \( \mathcal{H}_- \) have the same spectra except, once again, for the ground-state energy.

From (28), the Schrödinger equation for generalized Pöschl-Teller potential reads

\[ \begin{bmatrix} -\partial^2_\rho - \frac{\mu (\mu + 1)}{\cosh^2 \rho} + \mu^2 - 2\partial_\rho \left( \frac{\cosh^{-2\mu} \rho}{\Gamma + \int_0^\rho d\tilde{\rho} \cosh^{-2\mu} \tilde{\rho}} \right) \\ \end{bmatrix} \mathcal{U}(\rho) = E_{PT} \mathcal{U}(\rho). \]

(29)

The following substitutions

\[ \begin{align*}
\zeta &= \sinh \rho, \\
\sigma &= \cot^{-1} \zeta, \\
t &= \tan \frac{\sigma}{2},
\end{align*} \]

(30.a, b, c)

bring (29) to the form

\[ \begin{bmatrix} -t\partial_t (t\partial_t) - 4\mu (\mu + 1) \frac{t^2}{(1 + t^2)^2} + \mu^2 + 2t\partial_t q_{\mu, \Gamma}^{(PT)}(t) \end{bmatrix} \mathcal{U}(t) = E_{PT} \mathcal{U}(t), \]

(31)
with
\[ q^{(PT)}_{\mu, \Gamma} (t) = \frac{\left( \frac{2t}{1 + t^2} \right)^{2\mu}}{\Gamma + \int_0^t \frac{dt}{1 + t^2} \left( \frac{-2}{1 + t^2} \right) \left( \frac{2t}{1 + t^2} \right)^{2\mu - 1}}. \] (32)

Dividing now the whole of (31) by \( t^2 \), and by transposing the energy term to the left-hand side and the second term of the left member to the right-hand side, we get the final Schrödinger equation
\[ \left[ -\frac{1}{t} \partial_t (t \partial_t) - \frac{E_{PT} - \mu^2}{t^2} + \frac{2}{t} \partial_r q^{(PT)}_{\mu, \Gamma} (t) \right] \mathcal{U} (t) = \frac{4\mu (\mu + 1)}{(1 + t^2)^2} \mathcal{U} (t). \] (33)

Since both the Morse and the Pöschl-Teller potentials appear to be related to the same representations of \( SU(2) \), we conjecture that, as was made for potentials [1], there must be a transformation connecting their generalized potentials (18) and (28), and then their corresponding Schrödinger equation (21) and (33). It is what we are going to prove in section 3.

3 Connection between generalized Morse and Pöschl-Teller potentials

Before proceeding, let us recall that the solutions of the Schrödinger equations with the generalized Morse potentials, namely \( \psi (r, \varphi) \), should be periodic in \( \varphi \) with period \( 2\pi \), and it can be written in the form
\[ \psi (r, \varphi) = e^{2im\varphi} \mathcal{R} (r), \] (34)
where \( m \) is an integer carrying information about energy.

By performing the second derivative of \( \psi (r, \varphi) \) with respect to \( \varphi \), and identifying the result with the third term in (21), we get
\[ \partial^2_{\varphi} \equiv -4m^2 = -4 (a^2 - E_M), \] (35)
and which permits to re-express (21) in the form
\[ \left[ -\frac{1}{r} \partial_r (r \partial_r) + r^2 - \frac{1}{r^2} \partial^2_{\varphi} + \frac{4}{r} \partial_r q^{(M)}_{\mu, \Gamma} (r) \right] \psi (r, \varphi) = (4a + 2) \psi (r, \varphi). \] (36)
Let us make a change of variables by introducing the bidimensional vector \( t \equiv (t_x, t_y) \) through \([1]\)

\[
\begin{align*}
  t_x &= t \cos \Phi ; \\
  t_y &= t \sin \Phi. \\
  t &= \frac{r^2}{2} ; \\
  \Phi &= 2 \varphi.
\end{align*}
\]

By performing the first and second derivatives with respect to \( r \) and \( \varphi \), taking into account (37), the Schrödinger equation (36) becomes

\[
\begin{equation}
  t \left[ -\frac{1}{t} \partial_t (t \partial_t) - \frac{1}{t^2} \partial^2_\Phi + \frac{2}{t} \partial_t q_a^{(M)} (t) \right] \psi (t, \Phi) = (2a + 1) \psi (t, \Phi). \tag{38}
\end{equation}
\]

The canonically conjugate momenta vector, \( \tau \equiv (\tau_x, \tau_y) \) associated to \( t \equiv (t_x, t_y) \), is introduced \([1]\). Consequently, (38) can be written as

\[
\begin{equation}
  t \left( 1 + \tau^2 \right) \psi (t, \Phi) = (2a + 1) \psi (t, \Phi), \tag{39}
\end{equation}
\]

with

\[
\tau^2 \equiv -\frac{1}{t} \partial_t (t \partial_t) - \frac{1}{t^2} \partial^2_\Phi + \frac{2}{t} \partial_t q_a^{(M)} (t). \tag{40}
\]

Proceeding now to square (39). This amounts to multiply this later on the left-hand side by \( t \left( 1 + \tau^2 \right) \), i.e.

\[
\begin{equation}
  t \left( 1 + \tau^2 \right) t \left( 1 + \tau^2 \right) \psi (t, \Phi) = (2a + 1) t \left( 1 + \tau^2 \right) \psi (t, \Phi) = (2a + 1)^2 \psi (t, \Phi). \tag{41}
\end{equation}
\]

Since the vectors \( t \) and \( \tau \) verify the Hyllerras commutation relations \([17]\)

\[
\begin{align*}
  [t, t \cdot \tau] &= it, \tag{42.a} \\
  [t, \tau^2] &= (2i \tau \cdot t - 1) \frac{1}{t}, \tag{42.b}
\end{align*}
\]

then, taking into account (42.b), the left-side of (41) is expanded following

\[
\begin{align*}
  t \left( 1 + \tau^2 \right) t \left( 1 + \tau^2 \right) \psi (t, \Phi) &= \left( t^2 + t \tau^2 t \right) \left( 1 + \tau^2 \right) \psi (t, \Phi) \\
  &= \left[ t^2 + t^2 \tau^2 - 2i t \tau \cdot t - 1 \right] \left( 1 + \tau^2 \right) \psi (t, \Phi)
\end{align*}
\]

and using (42.a), we obtain

\[
\begin{equation}
\left[ t^2 \left( 1 + \tau^2 \right) - 2i \tau \cdot t + 3 \right] \left( 1 + \tau^2 \right) \psi (t, \Phi) = (2a + 1)^2 \psi (t, \Phi). \tag{44}
\end{equation}
\]
Proceeding now to canonical transformations, which permit to exchange the coordinates to momenta through
\[
\begin{pmatrix}
  t' \\
  \tau'
\end{pmatrix} = \begin{pmatrix}
  0 & 1 \\
  -1 & 0
\end{pmatrix} \begin{pmatrix}
  t \\
  \tau
\end{pmatrix}.
\] (45)

It is clear that the transformation (45) has the following properties \( t'^2 = \tau'^2 \), \( \tau'^2 = t^2 \) and \( \tau \cdot t = -t' \cdot \tau' \). These transformations amount to describe a two-dimensional Fourier transform [1]. In terms of these, (44) becomes
\[
\left[ \tau'^2 (1 + t'^2) + 2it' \cdot \tau' + 3 \right] (1 + \tau'^2) \Psi (t', \Phi') = (2a + 1)^2 \Psi (t', \Phi'),
\] (46)
where, as a consequence of this, the wave function \( \Psi (t', \Phi') \) is the Fourier transform of \( \psi (t, \Phi) \).

Using coordinates \( t' \) and \( \Phi' \), the Fourier transform of (40) reads
\[
\tau'^2 \equiv -\frac{1}{v'} \partial_{t'} (t' \partial_{t'}) - \frac{1}{v'^2} \partial_{\Phi'}^2 + \frac{2}{v'} \partial_{t'} Q_{a,\Gamma}^{(M)} (t') .
\] (47)
where, as was established for the wave functions above, the function \( \frac{1}{t} \partial_t Q_{a,\Gamma}^{(M)} (t) \) is the Fourier transform of \( \frac{1}{t} \partial_t Q_{a,\Gamma}^{(M)} (t) \) (see appendix).

Inserting both (47) and definition of scalar product \( t' \cdot \tau' = -it' \partial_{t'} \) into (46), we obtain
\[
\left[ \left( -\frac{1}{v'} \partial_{t'} (t' \partial_{t'}) - \frac{1}{v'^2} \partial_{\Phi'}^2 + \frac{2}{v'} \partial_{t'} Q_{a,\Gamma}^{(M)} (t') \right) (1 + t'^2) + 2t' \partial_{t'} + 3 \right] (1 + \tau'^2) \Psi (t', \Phi') = (2a + 1)^2 \Psi (t', \Phi').
\] (48)

By introducing a new wave function \( \xi (t', \Phi')^2 \) according to [1]
\[
\xi (t', \Phi') = (1 + t'^2)^3/2 \Psi (t', \Phi'),
\] (49)
we obtained from (48), after derivation and a long calculation
\[
\left[ -\frac{1}{v'} \partial_{t'} (t' \partial_{t'}) - \frac{1}{v'^2} \partial_{\Phi'}^2 + \frac{2}{v'} \partial_{t'} Q_{a,\Gamma}^{(M)} (t') \right] \xi (t', \Phi') = \frac{4a (a + 1)}{(1 + t'^2)^2} \xi (t', \Phi').
\] (50)

\( ^2 \)It is evident that the function \( \xi (t', \Phi') \) is the same wave function introduced in subsection 2.2.
We introduce the wave function $\xi(t', \Phi') = U(t') e^{i m \Phi'}$ in (50). At this point, it is interesting to re-express the second term in the left-hand side of (50) in term of the second-order differential operator, as was done in (35), accordingly to

$$\partial^2_{\Phi'} \equiv E_M - a^2,$$

which finally brings (50) to

$$\left[- \frac{1}{t} \partial_{\nu'} (t' \partial_{\nu'}) - \frac{E_M - a^2}{t^2} + \frac{2}{t'} \partial_{\nu'} Q_{a, \Gamma}^{(M)} (t') \right] U(t') = \frac{4a(a + 1)}{(1 + t'^2)^2} U(t'),$$

Since the Schrödinger equation deduced in (52) is the same as previously derived in (33), then

$$F \left[ \frac{1}{t} \partial_{\nu} q_{a, \Gamma}^{(M)} (t) \right] = \frac{1}{t'} \partial_{\nu'} Q_{a, \Gamma}^{(M)} (t') \\
\equiv \frac{1}{t'} \partial_{\nu'} q_{\mu, \Gamma}^{(PT)} (t'),$$

where $F$ is the Fourier transform operator. It should be clear that both generalized potentials are connected by means of the Hankel transform which is one in a large number of ways in which the Fourier transform can be written (see Appendix, (A.14)).

Identifying (33) to (52), term by term, is considered as an alternative way able to reproduce the relationship connecting both energy spectra. Then, comparing both second terms and right-hand sides of (33) and (52), respectively, and taking into consideration $a = \lambda - \frac{1}{2}$, we obtain

$$E_{PT} = E_M + \lambda - \mu - \frac{1}{2}.$$  

We note that the energy spectrum of the generalized Pöschl-Teller potential is related to the energy spectrum of the generalized Morse potential by shifting the $E_M$ value by $\lambda - \mu - \frac{1}{2}$.

4 Conclusion

The main purpose of the present paper is to extend the procedure of [1] for generalized Morse and Pöschl-Teller potentials in order to connect them with the Schrödinger equation. These generalized potentials are determined
by applying the generalized version of creation and annihilation operators provided by the factorization method. Our primary concern is to construct mathematically the Schrödinger equation for generalized Pöschl-Teller potential starting off with the Schrödinger equation for Morse potential. It is found that both Schrödinger equations are related by the Fourier transform. Our secondary purpose consists to offer an explanation that there exist an intimate correlation between generalized potentials with a view to establishing a causal connection underlying the relationship which connects them. This establishment of the correspondence between the generalized potentials means that the known generalized Morse potential automatically provides us with the generalized Pöschl-Teller potential and vice versa, through the Hankel transform (see (A.14)). As a consequence of this, energy spectrum of generalized Pöschl-Teller potential is related to the energy spectrum of generalized Morse potential by shifting the $E_M$ value by $\lambda - \mu - \frac{1}{2}$.

It will be interesting to analyze the connection between generalized Morse and Pöschl-Teller potentials using path-integrals formalism rather than Schrödinger equation, where Refs. [18,19] suggests a close connection between the eigenfunctions and sum-over-paths representation of the propagators.

5 Appendix

In this appendix, we add some mathematical details to the discussion of section 3 in order to prove that both generalized potentials are linked by Hankel transform. Comparing (33) and (50), and taking into account (49), we can write that

$$\partial_t' Q_{a, \Gamma}^{(M)} (t') \xi (t', \Phi') = (1 + t'^2)^{3/2} \partial_t' Q_{a, \Gamma}^{(M)} (t') \Psi (t', \Phi') = (1 + t'^2)^{3/2} F \left[ \partial_q^{\mu, \Gamma} (P_T) \psi (t, \Phi) \right]. \quad (A.1)$$

The Fourier transform for the function of two variables, e.g. $t_x$ and $t_y$ of section 3 is

$$\Psi (t_x, t_y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dt_x dt_y \psi (t_x, t_y) \exp [-i (t_x t_u + t_y t_v)]. \quad (A.2)$$
Let us make a change of variables taken above in (37), (A.2) becomes

\[ \Psi (t', \Phi') = \int_0^{2\pi} d\Phi \int_0^\infty d\tau \psi (t, \Phi) \exp [-i (t t' \cos \Phi \cos \Phi' + t t' \sin \Phi \sin \Phi')] \]

\[ = \int_0^{2\pi} d\Phi \int_0^\infty d\tau \psi (t, \Phi) e^{-i t t' \cos(\Phi - \Phi')} \] (A.3)

Since \( \psi (t, \Phi) = R(t) e^{i m \Phi} \) and \( \xi (t', \Phi') = U(t') e^{i m \Phi'} \), (A.1) becomes

\[ \partial_{t'} Q_{a, \Gamma}^{(M)} (t') \xi (t', \Phi') = (1 + t'^2)^{3/2} \int_0^{2\pi} d\Phi \int_0^\infty d\tau R(t) \partial_{t} q_{\mu, \Gamma}^{(PT)} (t) e^{-i t t' \cos(\Phi - \Phi') + i m \Phi}. \] (A.4)

We perform now the \( \Phi \)-integration. To this end, the useful way of treating this integration is to employ the integral representation of the cylindrical Bessel functions [20,21]

\[ J_v (x) = \frac{1}{2\pi i} \oint dz \ z^{-1-v} \exp \left[ \frac{x}{2} (z - \frac{1}{z}) \right]. \] (A.5)

Let \( z = e^{i \Phi} \) where the path of integration is the unit circle, the integral (A.5) becomes, by setting \( v = m \) and \( x = t t' \)

\[ J_m (t t') = \frac{1}{2\pi} \int_0^{2\pi} d\Phi e^{i t t' \sin \Phi - i m \Phi}. \] (A.6)

In order to recover the exponent in (A.4), a new change of variable, \( \Phi \rightarrow \Phi - \Phi' - \pi/2 \), is introduced

\[ J_m (t t') = \frac{1}{2\pi} i^m e^{i m \Phi'} \int_0^{2\pi} d\Phi e^{-i t t' \cos(\Phi - \Phi') - i m \Phi}, \] (A.7)

and by replacing \( m \rightarrow -m \) and taking into account the identity \( J_{-m} (x) = (-1)^m J_m (x) \) [22,23], we find that

\[ \int_0^{2\pi} d\Phi e^{-i t t' \cos(\Phi - \Phi') + i m \Phi} = 2\pi (-i)^m e^{i m \Phi'} J_m (t t'). \] (A.8)
Inserting (A.8) into (A.4), we obtain

\[
\partial_t Q^{(M)}_{a,\Gamma} (t') U (t') = 2\pi (-i)^m (1 + t'^2)^{3/2} \int_0^\infty t dt R (t) \partial_t q^{(PT)}_{\mu,\Gamma} (t) J_m (tt').
\]  

(A.9)

It was already established by Alhassid et al. [1] that the connection between the "radial" solutions of the Schrödinger equation with Morse and Pöschl-Teller potentials is

\[
U (t') = 2\pi (-i)^m (1 + t'^2)^{3/2} \int_0^\infty t dt R (t) J_m (tt').
\]  

(A.10)

Substituting (A.10) into (A.9), we obtain after simplification

\[
\partial_t Q^{(M)}_{a,\Gamma} (t') \int_0^\infty t dt R (t) J_m (tt') = \int_0^\infty t dt R (t) \partial_t q^{(PT)}_{\mu,\Gamma} (t) J_m (tt').
\]  

(A.11)

Performing now the \(t'\)-integration and by interchanging the order of integration, we find that

\[
\int_0^\infty t dt R (t) \int_0^\infty t dt' t' \frac{1}{t'} \partial_t Q^{(M)}_{a,\Gamma} (t') J_m (tt') = \int_0^\infty t dt R (t) \partial_t q^{(PT)}_{\mu,\Gamma} (t) \int_0^\infty dt' J (tt')
\]

\[
= \int_0^\infty t dt R (t) \partial_t q^{(PT)}_{\mu,\Gamma} (t) \frac{1}{t},
\]  

(A.12)

where the integral of Bessel function depending on a parameter \(p\) yields [20,21]

\[
\int_0^\infty dx J_v (px) = \frac{1}{p},
\]  

(A.13)

Therefore, by identifying terms in (A.12) and using the Hankel transform (Fourier-Bessel integral) [20,21] which is one in a large number of ways in which the Fourier transform can be written, we obtain
\[
\frac{1}{t} \partial_t q_{\mu, \Gamma}^{(PT)}(t) = \int_0^\infty t' dt' \frac{1}{t'} \partial_{t'} Q_{a, \Gamma}^{(M)}(t') J_m(tt') = F \left[ \frac{1}{t'} \partial_{t'} Q_{a, \Gamma}^{(M)}(t') \right],
\]

which is a result already established above in (53).

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