Field-dependent BRST transformations in Yang-Mills theory

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Abstract

We find an explicit form for the Jacobian of arbitrary field-dependent BRST transformations in Yang-Mills theory. For the functional-integral representation of the (gauge-fixed) Yang-Mills vacuum functional, such transformations merely amount to a precise change in the gauge-fixing functional. This proves the independence of the vacuum functional under any field-dependent BRST transformation. We also give a formula for the transformation parameter functional which generates a prescribed change of gauge and evaluate it for connecting two arbitrary $R_\xi$ gauges.

Keywords: Yang-Mills theory, field-dependent BRST transformation, gauge-fixing procedure

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1 Introduction and summary

In quantum field theory, changing variables in the functional-integral representation of generating functionals for Green's functions is a major calculational tool. In particular, the derivation of Slavnov-Taylor identities \[1, 2\] in Yang-Mills theory and of Ward identities in general gauge theories \[3\] utilize field substitutions related to BRST symmetry \[4, 5\]. In the quantization of dynamical systems with constraints, the gauge independence of S-matrix elements in the Batalin-Fradkin-Vilkovisky formalism is proven via a canonical change of variables pertaining to the Hamiltonian version of BRST symmetry, with constant as well as with field-dependent parameter \[6, 7\]. The Batalin-Vilkovisky formalism for the covariant quantization of general gauge theories \[3\] also employs a change of functional variables in a form of field-dependent BRST transformations.

In present paper, we investigate field-dependent BRST transformations in Yang-Mills theory. They are obtained from standard BRST transformations by replacing the Grassmann-odd constant parameter \(\lambda\) with a Grassmann-odd functional \(\Lambda(\phi)\) of the fields \(\phi\) in the theory. For a change of variables of such type in a functional integral, we derive a simple explicit form of its Jacobian in terms of the Slavnov variation of \(\Lambda(\phi)\). By ‘inverting’ this variation, we absorb any such field-dependent BRST transformation inside the Yang-Mills vacuum functional into a modification of its gauge-fixing functional. This proves the independence of the vacuum functional under arbitrary field-dependent BRST transformations. Let us turn the question around: Given two different gauges, can one construct a field-dependent BRST transformation which brings us from one to the other? Our result gives a simple recipe for the answer. We demonstrate its use by computing the parameter functional \(\Lambda(\phi)\) which connects any two \(R_\xi\) gauges (including the Landau gauge).

The paper is organized as follows. In Section 2, we briefly review the salient features of the Faddeev-Popov quantization of Yang-Mills fields coupled to arbitrary matter. Section 3 introduces field-dependent BRST transformations and computes their Jacobian (the supertrace of the Jacobian supermatrix). In Section 4, we write (the log of) the Jacobian as a Slavnov variation of a shift in the gauge-fixing functional and solve for \(\Lambda(\phi)\) in terms of the latter, before giving the explicit solution for the case of two \(R_\xi\) gauges.

We employ the condensed notation of DeWitt \[8\]. Derivatives with respect to fields are taken from the right. Left functional derivatives are labeled by a subscript \(l\).

2 Yang-Mills theory in Faddeev-Popov quantization

Our framework in this paper is Yang-Mills theory of gauge potentials \(A^{\alpha\mu}(x)\) (with Lorentz index \(\mu\) and color index \(\alpha\)), coupled to some matter fields, such as scalars \(\varphi^r(x)\) or spinors \(\psi^s(x)\) (where \(r\) and \(s\) are gauge group representation indices). In this section, we introduce our notation and remind the reader of the basics of the Faddeev-Popov quantization method \[9\].
For convenience of notation, let us introduce joint (discrete and continuous) indices

\[ i = (x, \mu, a, r, s, \ldots) \quad \text{and} \quad \alpha = (x, a) \]  

(2.1)

and group the above (physical) fields as

\[ \{A^i\} = \{A^{a\mu}(x), \varphi^r(x), \psi^s(x), \ldots\} \]  

(2.2)

with Grassmann parities \( \varepsilon(A^i) \equiv \varepsilon_i \). We often abbreviate functional derivatives as \( \frac{\delta X}{\delta A^i} \equiv X_{,i} \). The starting point is a classical action \( S_0(A) \).

The fields are subject to gauge transformations,

\[ \delta A^i = R^i_\alpha(A)\xi^\alpha \quad \text{such that} \quad \delta S_0(A) = 0 \iff S_{0,i}(A)R^i_\alpha(A) = 0, \]  

(2.3)

where \( \xi^\alpha \) are arbitrary functions with Grassmann parities \( \varepsilon(\xi^\alpha) \equiv \varepsilon_\alpha \), and \( R^i_\alpha(A) \) are the generators of the gauge transformations. The latter’s algebra is inherited from the Lie algebra of the gauge group,

\[ R^i_{\alpha,j}(A)R^j_\beta(A) - (-1)^{\varepsilon_\alpha\varepsilon_\beta}R^i_{\beta,j}(A)R^j_\alpha(A) = -R^i_\gamma(A)f^\gamma_{\alpha\beta}, \]  

(2.4)

where \( f^\gamma_{\alpha\beta} = -(-1)^{\varepsilon_\alpha\varepsilon_\beta}f^\gamma_{\beta\alpha} \) are the structure constants. Faddeev-Popov quantization can be applied to the algebra (2.4) if, in addition, the generators \( R^i_\alpha \) are linearly independent with respect to \( \{\alpha\} \).

Let us introduce the extended configuration space of fields as follows:

\[ \{\phi^A\} = \{A^i, B^\alpha, C^\alpha, \bar{C}^\alpha\}, \]

\( \varepsilon(A^i) = \varepsilon_i \), \( \varepsilon(B^\alpha) = \varepsilon_\alpha \), \( \varepsilon(C^\alpha) = \varepsilon(\bar{C}^\alpha) = \varepsilon_\alpha + 1 \),

\( gh(A^i) = gh(B^\alpha) = 0 \), \( gh(C^\alpha) = 1 \), \( gh(\bar{C}^\alpha) = -1 \),

where \( B^\alpha \) are Nakanishi-Lautrup auxiliary fields, and \( C^\alpha \) and \( \bar{C}^\alpha \) are the Faddeev-Popov ghost and anti-ghost fields, respectively. We also have introduced the ghost-number grading \( gh \).

Then, the total action is constructed according to the Faddeev-Popov rule,

\[ S(\phi) = S_0(A) + \bar{C}^\alpha \chi_{\alpha,i}(A)R^i_\beta(A)C^\beta + \chi_\alpha(A)B^\alpha \]  

(2.5)

where \( \chi_\alpha(A) \) with \( \varepsilon(\chi_\alpha) = \varepsilon_\alpha \) are some gauge functionals which lift the degeneracy of the classical gauge-invariant action \( S_0(A) \).

The generating functional of the Greens functions is written in the form of a functional integral,

\[ Z(J) = \int \mathcal{D}\phi \ \exp \left\{ \frac{i}{\hbar} \left( S(\phi) + J_A\phi^A \right) \right\}. \]  

(2.6)
If, in addition,

\[ (-1)^{\varepsilon_\alpha} f^{\beta\gamma}_{\beta\alpha} = 0 \quad \text{and} \quad (-1)^{\varepsilon_i} \frac{\delta I_i^\alpha}{\delta A_i^\alpha} = 0, \tag{2.7} \]

then one can prove the gauge independence of the vacuum functional \( Z(0) \) and of the \( S \)-matrix. For pure Yang-Mills theories the relations (2.7) are valid due to the antisymmetry of the structure constants \( f_{bc}^a \).

The action (2.5) is invariant under the BRST transformation \( \delta \lambda A^i = R^i_\alpha (A) C^\alpha \lambda, \delta \lambda C^\alpha = -\frac{1}{2} (-1)^{\varepsilon_\alpha} f^{\alpha\beta\gamma}_{\beta\gamma} C^\beta C^\gamma \lambda, \delta \lambda B^\alpha = 0. \tag{2.8} \)

Here, \( \lambda \) is a constant Grassmann parameter \( (\varepsilon(\lambda) = 1) \). Due to the gauge invariance of \( S_0 \) and the Jacobi identity for the structure constants, the BRST transformation is nilpotent. It is quite useful to introduce the Slavnov variation \( sX \) of any functional \( X \) by writing

\[ \delta \lambda X(\phi) = (sX(\phi)) \lambda \quad \implies \quad sX(\phi) = \frac{\delta X(\phi)}{\delta \phi^A} R^A(\phi) \equiv X_A(\phi) R^A(\phi) \tag{2.10} \]

with the notation

\[ R^A(\phi) = (R^i_\alpha (A) C^\alpha, 0, -\frac{1}{2} (-1)^{\varepsilon_\alpha} f^{\alpha\beta\gamma}_{\beta\gamma} C^\beta C^\gamma, B^\alpha), \quad (\varepsilon(R^A(\phi))) = \varepsilon_A + 1. \tag{2.11} \]

In particular, from (2.8) and (2.9) we read off \( s \phi^A \) for all the fields.

With the fermionic gauge-fixing functional

\[ \psi(\phi) = \bar{C}^\alpha \chi_\alpha(A) \tag{2.12} \]

we can present the action (2.5) in the form

\[ S(\phi) = S_0(A) + \frac{\delta \psi(\phi)}{\delta A^i} R^i_\alpha C^\alpha + \frac{\delta \psi(\phi)}{\delta C^\alpha} B^\alpha = S_0(A) + s \psi(\phi) \tag{2.13} \]

where its BRST invariance is obvious. The nilpotency of the Slavnov variation, \( s^2 = 0 \), implies that

\[ 0 = sR^A(\phi) = \frac{\delta R^A(\phi)}{\delta \phi^B} R^B(\phi) \equiv R^A_{,B}(\phi) R^B(\phi). \tag{2.14} \]

Let us indicate the choice of the gauge-fixing functional \( \psi \) in the definition of the generating functional by denoting it as \( Z_\psi(J) \). It is well known that the vacuum functional of Yang-Mills theory does not depend on the gauge. It has been proven that, under an infinitesimal change of the gauge-fixing functional, one has

\[ Z_\psi(0) = Z_{\psi + \delta \psi}(0). \tag{2.15} \]
3 Field-dependent BRST transformations

In this section we are going to consider a more general class of BRST transformations by allowing its Grassmann parameter to depend on the fields of the theory. Such transformations are admissible in the functional formulation of quantum field theory. So let us generalize (2.10) to

\[ \delta \Lambda X(\phi) = (sX(\phi)) \Lambda(\phi) = X_A R^A \Lambda(\phi), \quad \varepsilon(\Lambda(\phi)) = 1, \quad \Lambda^2(\phi) = 0. \]  

(3.1)

On the field themselves, a shift by a Slavnov variation amounts to a change of variables,

\[ \phi^A = \phi^A(\phi) = \phi^A + \delta \Lambda \phi^A = \phi^A + (s\phi^A) \Lambda(\phi) = \phi^A + R^A(\phi) \Lambda(\phi), \]

(3.2)

with a Jacobian supermatrix

\[ M^A_B(\phi) = \frac{\delta \phi^A(\phi)}{\delta \phi^B} = \delta^A_B + \frac{\delta R^A(\phi)}{\delta \phi^B} \Lambda(\phi)(-1) \varepsilon_B + R^A(\phi) \frac{\delta \Lambda(\phi)}{\delta \phi^B} \]

\[ \equiv \delta^A_B + R^A_{\cdot B}(\phi)(-1) \varepsilon_B + R^A \Lambda_{\cdot B}(\phi), \]

(3.3)

where \( \varepsilon(M^A_B(\phi)) = \varepsilon_A + \varepsilon_B. \)

Now consider a functional integral

\[ I = \int \mathcal{D}\phi \exp \left\{ \frac{i}{\hbar} W(\phi) \right\} \]

(3.4)

with some functional \( W(\phi). \) Changing the field variables according to (3.2) then yields

\[ I = \int \mathcal{D}\phi \ \text{sDet} M(\phi) \exp \left\{ \frac{i}{\hbar} W(\phi(\phi)) \right\} \]

\[ = \int \mathcal{D}\phi \ \exp \left\{ \frac{i}{\hbar} \left[ W(\phi(\phi)) - i\hbar \text{sTr} \ln M(\phi) \right] \right\} \]

(3.5)

where sDet and sTr denote the functional superdeterminant and supertrace, respectively. Due to \( \Lambda^2 = 0 \) and (2.14), the computation of sTr \ln M simplifies considerably:

\[ \text{sTr} \ln M(\phi) = - \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \text{sTr} \left( R^A_{\cdot B} \Lambda(-1) \varepsilon_B + R^A \Lambda_{\cdot B} \right)^n \]

\[ = - \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \text{sTr} \left( R^A \Lambda_{\cdot B} \right)^n \]

\[ = + \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \left( \Lambda_{\cdot A} R^A \right)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} (s\Lambda)^n \]

\[ = - \ln \left( 1 + s\Lambda(\phi) \right). \]

(3.6)

Hence, we can give an explicit formula for the Jacobian of an arbitrary field-dependent BRST transformation,

\[ \text{sDet} M(\phi) = \frac{1}{1 + s\Lambda(\phi)}. \]

(3.7)
Employing $W(\phi + \delta \Lambda \phi) = W(\phi) + \delta \Lambda W(\phi)$, the functional integral \((3.5)\) can be expressed as

$$I = \int D\phi \exp\left\{ \frac{i}{\hbar} [W(\phi) + s(W(\phi))\Lambda(\phi) + i\hbar \ln (1 + s\Lambda(\phi))] \right\} \quad (3.8)$$

being valid for arbitrary functionals $W(\phi)$ and $\Lambda(\phi)$.

In contrast to the standard BRST transformation $\delta\lambda$, the field-dependent generalization $\delta \Lambda$ fails to be nilpotent since, for $sX \neq 0$,

$$\delta \Lambda^2 X(\phi) = \delta \Lambda \left[ (sX(\phi))\Lambda(\phi) \right] = (sX(\phi))(s\Lambda(\phi))\Lambda(\phi) \quad (3.9)$$

vanishes only if

$$0 = s\Lambda(\phi) = \Lambda_A(\phi)R^A(\phi) \quad \implies \quad \Lambda(\phi) = \lambda = \text{constant}. \quad (3.10)$$

In this case, however, $s\text{Det} M(\phi) = 1$, and the transformation should be considered as trivial.

### 4 Relating different gauges

Let us apply the results of the previous section to the Yang-Mills vacuum functional

$$Z_\psi(0) = \int D\phi \exp \left\{ \frac{i}{\hbar} S(\phi) \right\}. \quad (4.1)$$

Since the action \((2.13)\) is invariant under the field-dependent BRST transformation \((3.2)\), i.e. $S(\phi(\phi)) = S(\phi)$, using the formula \((3.8)\) yields

$$Z_\psi(0) = \int D\phi \exp \left\{ \frac{i}{\hbar} \left[ S(\phi) + i\hbar \ln (1 + s\Lambda(\phi)) \right] \right\}. \quad (4.2)$$

It may seem strange that \((3.7)\) may be inserted into the vacuum functional without any cost, but this clarifies by writing

$$i\hbar \ln (1 + s\Lambda(\phi)) = s \delta \psi(\phi) \quad \text{with} \quad \delta \psi(\phi) = i\hbar \Lambda(\phi)(s\Lambda(\phi))^{-1} \ln (1 + s\Lambda(\phi)). \quad (4.3)$$

It follows that the insertion of the Jacobian \((3.7)\) amounts to adding another BRST-exact piece to the action,

$$Z_\psi(0) = \int D\phi \exp \left\{ \frac{i}{\hbar} \left[ S_0(A) + s \psi(\phi) + s \delta \psi(\phi) \right] \right\} = Z_{\psi + \delta \psi}(0). \quad (4.4)$$

We see that an arbitrary field-dependent BRST transformation in the vacuum functional can be transformed into a modification of the gauge-fixing functional, which keeps the vacuum functional unchanged. Let us note that this reasoning does not require $\delta \psi$ to be infinitesimal. It holds not only on a $\delta \psi$-linearized level but exactly, in particular to all orders in a power series expansion in $\delta \psi$. 


It is instructive to turn the above result around: Any change of gauge, \( \psi \to \psi + \delta \psi \), may be effected by a field-dependent BRST transformation, whose parameter \( \Lambda(\phi) \) can be found by inverting (4.3), i.e. by solving

\[
s \Lambda(\phi) = \exp \left\{ \frac{1}{\hbar} s \delta \psi \right\} - 1.
\]

(4.5)

Let us try to achieve this for changing the parameter \( \xi \) in the covariant class of \( R_\xi \) gauges, given by the gauge-fixing functional

\[
\psi(\phi) = \bar{C}^a \left( \partial^\mu A^a_\mu + \frac{\xi}{2} B^a \right).
\]

(4.6)

The field-dependent BRST transformation which connects an \( R_\xi \) gauge to an \( R_{\xi + \delta \xi} \) gauge is given by (4.5) with

\[
\delta \psi = \frac{1}{2} s \delta \xi \bar{C}^a B^a \quad \Longrightarrow \quad s \delta \psi = \frac{1}{2} s \delta \xi B^2 \quad \text{with} \quad B^2 = B^a B^a.
\]

(4.7)

In this simple case the solution, up to BRST-exact terms, is easy to find:

\[
s \Lambda(\phi) = \exp \left\{ \frac{\delta \xi}{2 \hbar} B^2 \right\} - 1 \quad \Longrightarrow \quad \Lambda(\phi) = \bar{C}^a (B^2)^{-1} \left( \exp \left\{ \frac{\delta \xi}{2 \hbar} B^2 \right\} - 1 \right),
\]

(4.8)

which gives a power series expansion in \( \delta \xi \),

\[
\Lambda(\phi) = \frac{\delta \xi}{2 \hbar} \bar{C}^a B^a \left\{ 1 + \frac{\delta \xi}{4 \hbar} B^2 + \frac{1}{4!} \left( \frac{\delta \xi}{2 \hbar} B^2 \right)^2 + \frac{1}{5!} \left( \frac{\delta \xi}{2 \hbar} B^2 \right)^3 + \ldots \right\}.
\]

(4.9)

By taking \( \delta \xi = \xi \), we may connect in particular the Landau gauge (at \( \xi=0 \)) to any of the \( R_\xi \) gauges with the help of a field-dependent BRST transformation.

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