BACH-FLAT ASYMPTOTICALLY LOCALLY EUCLIDEAN METRICS

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Abstract. We obtain a volume growth and curvature decay result for various classes of complete, noncompact Riemannian metrics in dimension 4; in particular our method applies to anti-self-dual or Kähler metrics with zero scalar curvature, and metrics with harmonic curvature. Similar results were obtained for Einstein metrics in [And89], [BKN89], [Tia90], but our analysis differs from the Einstein case in that (1) we consider more generally a fourth order system in the metric, and (2) we do not assume any pointwise Ricci curvature bound.

1. Introduction

In dimension 4, the Euler-Lagrange equations of the functional

\[ W : g \mapsto \int_X |W_g|^2 \, dV_g, \]

where \( W_g \) is the Weyl curvature tensor, are given by

\[ B_{ij} \equiv \nabla^k \nabla^l W_{ikjl} + \frac{1}{2} R^{kl} W_{ikjl} = 0, \]

where \( W_{ijkl} \) and \( R^{kl} \) are the components in of the Weyl and Ricci tensors, respectively [Bes87], [Der83]. The tensor \( B_{ij} \) is referred to as the Bach tensor [Bac21]. Note that, in particular, metrics which are locally conformal to an Einstein metric are Bach-flat, and half conformally flat metrics are Bach-flat [Bes87]. Half conformally flat metrics are also known as self-dual or anti-self-dual if \( W^- = 0 \) or \( W^+ = 0 \), respectively.

A smooth Riemannian manifold \((X, g)\) is called an asymptotically locally Euclidean (ALE) end of order \( \tau \) if there exists a finite subgroup \( \Gamma \subset SO(4) \) acting freely on \( \mathbb{R}^4 \setminus B(0, R) \) and a \( C^\infty \) diffeomorphism \( \Psi : X \to (\mathbb{R}^4 \setminus B(0, R))/\Gamma \) such that under this identification,

\[ g_{ij} = \delta_{ij} + O(r^{-\tau}), \]

\[ \partial^k g_{ij} = O(r^{-\tau-k}), \]

for any partial derivative of order \( k \) as \( r \to \infty \). We say an end is ALE of order 0 if we can find a coordinate system as above with \( g_{ij} = \delta_{ij} + o(1) \), and \( \partial^k g_{ij} = o(r^{-k}) \) as \( r \to \infty \). A complete, noncompact manifold \((X, g)\) is called ALE if \( X \) can be written as the disjoint union of a compact set and finitely many ALE ends.

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We note that ALE spaces have been studied often in the literature, for example, they arise naturally in the positive mass theorem and the Yamabe problem [Bar86], [LP87], [Sch84], [SY81]; in orbifold compactness of Einstein metrics and ALE Ricci-flat metrics [And89], [BKN89], [CT94], [Kro89], [Tia90], [Nak90]; and in gluing theorems for scalar-flat Kähler metrics [CS02], [KS01]. There has been a considerable amount of research on the existence of anti-self-dual metrics on compact manifolds, we mention [Poo86], [LeB91], [Flo91], [DF89], [Tau92], also see [LeB95] for a nice survey and further references. We note that on any such manifold, by using the Green’s function for the conformal Laplacian as a conformal factor, we obtain a complete non-compact scalar-flat anti-self-dual ALE metric. By taking a finite sum of Green’s functions based at different points as a conformal factor, we obtain examples with several ends.

We recall that for a complete, noncompact manifold, the Sobolev constant $C_S$ is defined as the best constant $C_S$ so that for all $f \in C^1_0(X)$, we have

$$\|f\|_{L^{2^n}} \leq C_S \|\nabla f\|_{L^2}.$$ 

(1.5)

**Theorem 1.1.** Let $(X, g)$ be a complete, noncompact 4-dimensional Riemannian manifold which is Bach-flat and has zero scalar curvature. Assume that

$$\int_X |Rm_g|^2 dV_g < \infty, \text{ and } C_S < \infty,$$

where $Rm_g$ denotes the Riemannian curvature tensor. Fix a base point $p \in X$, then

$$\sup_{S(r)} |\nabla^k Rm_g| = o(r^{-2-k}),$$

as $r \to \infty$, where $S(r)$ denotes sphere of radius $r$ centered at $p$.

Assume furthermore that

$$b_1(X) < \infty,$$

(1.8)

where $b_1(X)$ denotes the first Betti number. Then there exists a constant $C$ (depending on $g$) such that

$$Vol(B(p, r)) \leq Cr^4,$$

(1.9)

$(X, g)$ has finitely many ends, and each end is ALE of order 0.

**Remark.** From [Car98, Theorem 1], there is a bound on the number of ends depending only upon the Sobolev constant and the $L^2$-norm of curvature (moreover, all of the the $L^2$-Betti numbers are bounded).

We emphasize that, in the Einstein case, the upper volume growth bound in (1.9) follows from the Bishop volume comparison theorem [BC64], but since we are not assuming any pointwise Ricci curvature bound, this estimate is non-trivial. We state the volume comparison theorem separately here, since it is of independent geometric interest:

**Theorem 1.2.** Let $(X, g)$ be a complete, noncompact, $n$-dimensional Riemannian manifold with base point $p$. Assume that there exists a constant $C_1 > 0$ so that

$$Vol(B(q, s)) \geq C_1 s^n,$$

(1.10)
for any \( q \in X \), and all \( s \geq 0 \). Assume furthermore that as \( r \to \infty \),
\[
\sup_{S(r)} |Rm_g| = o(r^{-2}),
\]
(1.11)
where \( S(r) \) denotes the sphere of radius \( r \) centered at \( p \). If \( b_1(X) < \infty \), then \( (X, g) \) has finitely many ends, and there exists a constant \( C_2 \) (depending on \( g \)) so that
\[
Vol(B(p, r)) \leq C_2 r^n.
\]
(1.12)
Furthermore, each end is ALE of order 0.

An important problem is to find geometric conditions so that each end of a complete space will be ALE of order \( \tau > 0 \), and to determine the optimal order of decay. We examine this problem for the following cases:

a. Self-dual or anti-self-dual metrics with zero scalar curvature.
b. Scalar-flat metrics with harmonic curvature.

In class (a) we have scalar-flat Kähler metrics, which are an important class of extremal Kähler metrics \cite{Cal82}. Class (b) is equivalent to scalar-flat and \( \delta W = 0 \), so in both classes we have scalar-flat locally conformally flat metrics \cite{SY88}, \cite{Sch91}.

**Theorem 1.3.** Let \( (X, g) \) be a complete, noncompact 4-dimensional Riemannian manifold with \( g \) of class (a) or (b). Assume that
\[
\int_X |Rm_g|^2 dV_g < \infty, \ C_S < \infty, \ and \ b_1(X) < \infty.
\]
(1.13)
Then \( (X, g) \) has finitely many ends, and each end is ALE of order \( \tau \) for any \( \tau < 2 \).

**Remark.** It is unclear if this theorem can be improved exactly to ALE of order 2, since this corresponds to an exceptional value, but we remark that ALE order of 2 would be the best possible. When \( |Rm| = O(r^{-4}) \), in particular the mass is finite, but it is not necessarily zero. The examples of Claude LeBrun in \cite{LeB83} are a particular case of (a), which have negative mass, and \( |Rm| = O(r^{-4}) \). Another example is given by the Schwarzschild metric; on \( \mathbb{R}^4 \setminus \{0\} \), let \( g = (1 + m/(3r^2))^2 g_0 \). This is scalar-flat, and is by definition locally conformally flat, therefore it is Bach-flat, and also of harmonic curvature. For \( m > 0 \), the curvature decays like \( C/r^4 \), but the ADM mass is \( m \neq 0 \). Since \( g \) has two ends, this example also shows that it is possible to have more than one end; this is in contrast to the Einstein case where the Cheeger-Gromoll splitting theorem rules out this possibility. We remark also that in the Einstein case, it was proved in \cite{BKN89} that \( (X, g) \) is moreover ALE of order 4.

**Remark.** An important application of the results in this paper is to the convergence of sequences of Bach-flat metrics on compact manifolds to orbifold metrics, this will be discussed in detail in a forthcoming paper.

We end the introduction with a brief outline of the paper. In Section 2 we will discuss the systems satisfied by Bach-flat metrics and metrics with harmonic curvature. The decay rate (1.7) will be proved using a Moser iteration process in Section 3.
The crucial volume growth estimate in Theorem 1.2 will be proved in Section 4.

To prove Theorem 1.3 in Section 5 we will derive various improved Kato inequalities, which will allow us to improve the order of decay of the Ricci tensor to \(|Ric| = O(r^{-\alpha})|\) for any \(\alpha < 4\). In Section 6 we use a Yang-Mills type argument as in \([Uhl82b]\), \([Tia90]\), to show that the full curvature tensor satisfies \(|Rm| = O(r^{-\alpha})|\) for any \(\alpha < 4\). The result in \([BKN89]\) then implies that \((X, g)\) is ALE of order \(\tau\) for any \(\tau < 2\). In Section 7 we remark that the Gauss-Bonnet and signature formulas put constraints on the ends of the ALE spaces which arise in Theorem 1.3.

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2. Critical metrics

In this section, we briefly discuss the systems of equations satisfied by Bach-flat metrics and metrics with harmonic curvature.

2.1. Bach-flat metrics with constant scalar curvature. As stated in the introduction, the Euler-Lagrange equations of the functional

\[ W : g \mapsto \int_X |W_g|^2 dV_g, \]  

in dimension 4, are

\[ B_{ij} = \nabla^k \nabla^l W_{ikjl} + \frac{1}{2} R^{kl} W_{ikjl} = 0. \]  

Since the Bach tensor arises in the Euler-Lagrange equations of a Riemannian functional, it is symmetric, and since the functional (2.1) is conformally invariant, it follows that the Bach-flat equation (2.2) is conformally invariant.

We note that (see \([ACG03]\))

\[ B_{ij} = 2\nabla^k \nabla^l W^+_{ikjl} + R^{kl} W^+_{ikjl} = 2\nabla^k \nabla^l W^-_{ikjl} + R^{kl} W^-_{ikjl}, \]  

so that both self-dual and anti-self-dual metrics are Bach-flat. If \(g\) is Kähler and has zero scalar curvature then \(W^+ \equiv 0\) \([Der83]\), so these metrics are in particular anti-self-dual. Using the following Bianchi identities

\[ \nabla^i R_{ijkl} = \nabla_k R_{ijl} - \nabla_l R_{ijk}, \]  

and

\[ \nabla^i W_{ijkl} = (n - 3)(\nabla_k A_{jl} - \nabla_l A_{jk}), \]
where \( A_{ij} \) are the components of the Weyl-Schouten tensor
\[
A = \frac{1}{n-2} \left( \text{Ric} - \frac{1}{2(n-1)} R \cdot g \right),
\]
and \( R \) denotes the scalar curvature, a computation shows that we may rewrite the Bach-flat equation (in dimension 4) as
\[
B_{ij} = \Delta A_{ij} - \nabla^k \nabla^i A_{jk} + \frac{1}{2} R^{kl} W_{ikjl} = 0.
\]
To simplify notation, in the following we will work in an orthonormal frame, and write all indices as lower. We recall a standard formulas for commuting covariant derivatives: if \( S \) is any 2-tensor, we have
\[
S_{ij;kl} - S_{ij;lk} = R^{km} S_{mj} + R^{jm} S_{im}.
\]
(2.6)
Using (2.6), we have the formula
\[
\nabla_k \nabla_i A_{jk} = \nabla_i \nabla_k A_{jk} + R_{ikjp} A_{pk} + R_{ikpp} A_{jp}.
\]
So we may write the Bach tensor as
\[
B_{ij} = \Delta A_{ij} - \nabla_i \nabla_k A_{jk} + R_{ikjp} A_{pk} - R_{ip} A_{jp} + \frac{1}{2} R^{kl} W_{ikjl}.
\]
If \( g \) also has constant scalar curvature, then
\[
\nabla_k A_{jk} = \frac{1}{6} \nabla_k R = 0,
\]
so we have
\[
B_{ij} = \frac{1}{2} (\Delta \text{Ric})_{ij} + R_{ikjl} A_{kl} - R_{il} A_{jl} + A_{kl} W_{ikjl}.
\]
Therefore we may write the Bach-flat equations as
\[
(\Delta \text{Ric})_{ij} = 2(R_{il} g_{jk} - R_{ikjl} - W_{ikjl}) A_{kl}.
\]
Introducing a convenient shorthand, we write this as
\[
(\Delta \text{Ric})_{ij} = \text{Rm} * \text{Ric}.
\]
2.2. Metrics with harmonic curvature tensor. The condition for harmonic curvature is that
\[
\delta \text{Rm} = -R_{ijkl} = 0.
\]
This condition was studied in \([\text{Bou81}], [\text{Der85}], [\text{Bes87}]\), and is the Riemannian analogue of a Yang-Mills connection. From the Bianchi identity (2.4), an equivalent condition is that \( R_{ij;k} = R_{ik;j} \). The Riemannian identity \( R_{ik;j} = (1/2) R_k \) implies that the scalar curvature is constant. An equivalent condition for harmonic curvature
is therefore that $\delta W = 0$ and $R = \text{constant}$. In particular, locally conformally flat metrics with constant scalar curvature have harmonic curvature. We compute

$$(\Delta Rm)_{ijkl} = R_{ijkl;m;m}$$

$= (-R_{ijlm;k} - R_{ijmk;l})_{;m}$$

$= -R_{ijlm;mk} - R_{ijmk;ml} + Q(Rm)_{ijkl} = Q(Rm)_{ijkl}.$$\tag{2.10}

where $Q(Rm)$ denotes a quadratic expression in the curvature tensor. In the shorthand, we write this as

$$\Delta Rm = Rm * Rm.$$\tag{2.10}

3. LOCAL REGULARITY

We can consider more generally any system of the type

$$(\Delta Ric) = Rm * Ric.$$\tag{3.1}

Any Riemannian metric satisfies

$$\Delta Rm = L(\nabla^2 Ric) + Rm * Rm,$$\tag{3.2}

where $L(\nabla^2 Ric)$ denotes a linear expression in second derivatives of the Ricci tensor, and $Rm * Rm$ denotes a term which is quadratic in the curvature tensor (see [Ham82, Lemma 7.2]).

For $X$ compact, we define the Sobolev constant $C_S$ as the best constant $C_S$ so that for all $f \in C^{0,1}(X)$ (Lipschitz) we have

$$\|f\|_{L^2} \leq C_S (\|\nabla f\|_{L^2} + \|f\|_{L^2}).$$\tag{3.3}

If $X$ is a complete, noncompact manifold, the Sobolev constant $C_S$ is defined as the best constant $C_S$ so that for all $f \in C^{0,1}_c(X)$ (Lipschitz with compact support), we have

$$\|f\|_{L^{\frac{2n}{n-2}}} \leq C_S \|\nabla f\|_{L^2}.$$\tag{3.4}

Even though second derivatives of the Ricci occur in (3.2), overall the principal symbol of the system (3.1) and (3.2) in triangular form. The equations (3.1) and (3.2), when viewed as an elliptic system, together with the bound on the Sobolev constant, enable us to prove an $\epsilon$-regularity theorem (see [Uhl82a, Uhl82b, Nak88, Cha93a]):

**Theorem 3.1.** Assume that (3.1) is satisfied, let $r < \text{diam}(X)/2$, and $B(p,r)$ be a geodesic ball around the point $p$, and $k \geq 0$. Then there exist constants $\epsilon_0, C_k$ (depending upon $C_S$) so that if

$$\|Rm\|_{L^2(B(p,r))} = \left\{ \int_{B(p,r)} |Rm|^2 dV_g \right\}^{1/2} \leq \epsilon_0,$$\tag{3.4}
then

\[ \sup_{B(p,r/2)} |\nabla^k Rm| \leq \frac{C_k}{r^{2+k}} \left\{ \int_{B(p,r)} |Rm|^2 dV_g \right\}^{1/2} \leq \frac{C_k \epsilon_0}{r^{2+k}}. \]

Remark. In the harmonic curvature case, the theorem is much easier, since we have an equation on the full curvature tensor \( \{2.10\} \), not just an equation on the Ricci; see [Aku94, And89, Nak88, Tia90].

Proof. We will consider the case where the Sobolev inequality \( \{3.3\} \) is satisfied, the case of \( \{3.4\} \) is similar. The proof will involve several lemmas. First,

**Lemma 3.2.** There exist constants \( \epsilon, C \) so that if \( \|Rm\|_{L^2(B(p,r))} \leq \epsilon \), then

\[ \left\{ \int_{B(p,r/2)} |Ric|^4 dV_g \right\}^{1/2} \leq \frac{C}{r^2} \int_{B(p,r)} |Ric|^2 dV_g. \]  

Proof. First, from \( \{3.1\} \), it follows that

\[ \Delta |Ric| \geq -|Rm||Ric|. \]  

Without loss of generality, we may assume that \( r = 1 \). The lemma then follows by scaling the metric. Let \( 0 \leq \phi \leq 1 \) be a function supported in \( B(p,1) \), then

\[ \int_{B(p,1)} \phi^2 |Ric|^2 |Rm| \geq \int_{B(p,1)} \phi^2 |Ric| (-\Delta |Ric|) = \int \nabla (\phi^2 |Ric|) \cdot \nabla |Ric| \]

\[ = \int 2\phi |Ric| \nabla \phi \cdot \nabla |Ric| + \int \phi^2 |\nabla |Ric||^2 \]

\[ \geq - \int (\delta^{-1} |\nabla \phi|^2 |Ric|^2 + \delta \phi^2 |\nabla |Ric||^2) + \int |\phi \nabla |Ric||^2 \]

\[ = -\delta^{-1} \int |\nabla \phi|^2 |Ric|^2 + (1 - \delta) \int |\phi \nabla |Ric||^2. \]

Next, using the Sobolev constant bound, we have

\[ \left\{ \int (\phi |Ric|)^4 \right\}^{1/2} \leq C \int |\nabla (\phi |Ric|)|^2 + C \int \phi^2 |Ric|^2 \]

\[ = C \int |\nabla \phi|^2 |Ric|^2 + C \int |\phi \nabla |Ric||^2 + C \int \phi^2 |Ric|^2. \]

so choosing \( \delta \) sufficiently small

\[ \left\{ \int \phi^2 |Ric|^4 \right\}^{1/2} \leq C \int \phi^2 |Ric|^2 |Rm| + C \int (\phi^2 + |\nabla \phi|^2) |Ric|^2 \]

\[ \leq C \left\{ \int \phi^2 |Rm|^2 \right\}^{1/2} \left\{ \int \phi^2 |Ric|^4 \right\}^{1/2} + C \int (\phi^2 + |\nabla \phi|^2) |Ric|^2. \]

Therefore for \( \epsilon \) sufficiently small we have

\[ \left\{ \int \phi^2 |Ric|^4 \right\}^{1/2} \leq C \int (\phi^2 + |\nabla \phi|^2) |Ric|^2. \]  

(3.7)
We then choose the cutoff function $\phi$ such that $\phi \equiv 1$ in $B(p, 1/2)$, $\phi = 0$ for $r = 1$, $|\nabla \phi| \leq C$, and we have

$$\left\{ \int_{B(p, 1/2)} |Ric|^4 \right\}^{1/2} \leq C \int_{B(p, 1)} |Ric|^2.$$  

Scaling the metric, we obtain (3.3).

**Lemma 3.3.** There exist constants $\epsilon, C$ so that if $\|Rm\|_{L^2(B(p, r))} \leq \epsilon$, then

$$\int_{B(p, r/2)} |\nabla Ric|^2 dV \leq \frac{C}{r^2} \int_{B(p, r)} |Ric|^2 dV_g,$$

and

$$\left\{ \int_{B(p, r/4)} |Rm|^4 dV \right\}^{1/2} \leq \frac{C}{r^2} \int_{B(p, r)} |Rm|^2 dV_g.$$  

**Proof.** Again we may assume that $r = 1$, and the lemma will follow by scaling the metric. Let $\phi$ be a cutoff function in $B(p, 1)$, such that $\phi \equiv 1$ in $B(p, 1/2)$ and $|\nabla \phi| \leq C$. We have

$$\int_{B(p, 1)} \phi^2 |\nabla Ric|^2 = - \int \phi^2 <\Delta Ric, Ric> - 2 \int \phi <\nabla Ric, \nabla \phi \cdot Ric>$$

$$= - \int \phi^2 <Rm \ast Ric, Ric> - 2 \int \phi <\nabla Ric, \nabla \phi \cdot Ric>$$

$$\leq C \int \phi^2 |Rm||Ric|^2 + \frac{C}{\delta} \int |\nabla \phi|^2 |Ric|^2 + C\delta \int \phi^2 |\nabla Ric|^2$$

$$\leq C \left\{ \int \phi^2 |Rm|^2 \right\}^{1/2} \left\{ \int \phi^2 |Ric|^4 \right\}^{1/2} + C \int_{B(p, 1)} |Ric|^2 + C\delta \int \phi^2 |\nabla Ric|^2$$

$$\leq C \left\{ \int \phi^2 |Rm|^2 \right\}^{1/2} \cdot C \int_{B(p, 1)} |Ric|^2 + C \int_{B(p, 1)} |Ric|^2 + C\delta \int \phi^2 |\nabla Ric|^2,$$

where we have used (3.7). By choosing $\delta$ small, and $\epsilon < 1$, we obtain

$$\int_{B(p, 1/2)} |\nabla Ric|^2 \leq (1 + \epsilon) C \int_{B(p, 1)} |Ric|^2 \leq 2C \int_{B(p, 1)} |Ric|^2.$$  

Scaling the metric, we obtain (3.8). Next, let $\phi$ be a cutoff function in $B(p, 1/2)$, such that $\phi \equiv 1$ in $B(p, 1/4)$ and $|\nabla \phi| \leq C$,

$$\int_{B(p, 1/2)} <\Delta Rm, \phi^2 Rm> = \int <\nabla^2 Ric + Rm \ast Rm, \phi^2 Rm>$$

$$= \int <\nabla^2 Ric, \phi^2 Rm> + \int <Rm \ast Rm, \phi^2 Rm>$$

$$= - \int <2\phi \nabla Ric, \nabla Rm> - \int \phi^2 <\nabla Ric, Rm> + \int <Rm \ast Rm, \phi^2 Rm>.$$
This yields
\[
\left| \int_{B(p,1/2)} \langle \Delta Rm, \phi^2 Rm \rangle \right| \leq C \int \phi^2 |\nabla Ric|^2 + C \int |\nabla \phi|^2 |Rm|^2 \\
+ \frac{C}{\delta} \int \phi^2 |\nabla Ric|^2 + C \delta \int \phi^2 |\nabla Rm|^2 + C \int \phi^2 |Rm|^3.
\]
Integrating by parts,
\[
\int_{B(p,1/2)} \phi^2 |\nabla Rm|^2 = - \int \langle 2\phi \nabla Rm, \nabla \phi \cdot Rm \rangle - \int \phi^2 \langle \Delta Rm, Rm \rangle \\
\leq \frac{C}{\delta} \int |\nabla \phi|^2 |Rm|^2 + 2C\delta \int \phi^2 |\nabla Rm|^2 + C \int |\nabla \phi|^2 |Rm|^2 \\
+ \frac{C'}{\delta} \int \phi^2 |\nabla Ric|^2 + C \int \phi^2 |Rm|^3.
\]
Choosing $\delta$ sufficiently small and using (3.8), we obtain
\[
\int \phi^2 |\nabla Rm|^2 \leq C \int |\nabla \phi|^2 |Rm|^2 + C \int_{B(p,1)} |Rm|^2 + C \int \phi^2 |Rm|^3.
\]
Using the Sobolev inequality,
\[
\left\{ \int \phi^2 Rm \right\}^{1/2} \leq C \int |\nabla \phi Rm|^2 + C \int \phi^2 |Rm|^2 \\
\leq C \int |\nabla \phi|^2 |Rm|^2 + C \int \phi^2 |\nabla |Rm||^2 + C \int \phi^2 |Rm|^2 \\
\leq C \int_{B(p,1)} |Rm|^2 + C \int \phi^2 |Rm|^3 \\
\leq C \int_{B(p,1)} |Rm|^2 + C \left\{ \int_{B(p,1/2)} |Rm|^2 \right\}^{1/2} \left\{ \int_{B(p,1/2)} \phi^4 |Rm|^4 \right\}^{1/2}.
\]
Therefore by choosing $\epsilon$ small, we obtain
\[
\left\{ \int_{B(p,1/4)} |Rm|^4 \right\}^{1/2} \leq C \int_{B(p,1)} |Rm|^2.
\]
Scaling the metric, we obtain (3.9). \hfill \Box

**Lemma 3.4.** Suppose $0 \leq u \in L^2(B_r)$ satisfies the inequality
\[
(3.10) \quad \Delta u \geq -u \cdot f
\]
with
\[
(3.11) \quad \int_{B(p,r)} |f|^4 dV_g \leq \frac{C_1}{r^4}.
\]
Then there exists a constant $C$ depending only upon $C_1, C_S$ so that

$$\sup_{B(p, r/2)} u \leq \frac{C}{r^2} \| u \|_{L^2(B(p, r))}.$$  

Proof. This is a standard Moser iteration argument, see [BKN89, Lemma 4.6]. \qed

Lemma 3.5. There exist constants $\epsilon, C$ so that if $\| Rm \|_{L^2(B(p, r))} \leq \epsilon$, then

$$\sup_{B(p, r/2)} |Ric| \leq \frac{C}{r^2} \| Ric \|_{L^2(B(p, r))}.$$  

Proof. From (3.1), there exists a constant $C$ so that

$$\Delta |Ric| \geq -C |Rm| |Ric|.$$  

Also, using (3.9), we have

$$\int_{B(p, r/2)} |Rm|^4 dV_g \leq \frac{C\epsilon}{r^4}.$$  

Therefore, we may apply Lemma 3.4 with $u = |Ric|, f = |Rm|$.

An important consequence is the following

Corollary 3.6. There exist constants $\epsilon, C$ so that if $\| Rm \|_{L^2(B(p, r))} \leq \epsilon$, then

$$\text{Vol}(B(p, r/2)) \leq C r^4.$$  

Proof. Taking $\epsilon$ as in Lemma 3.5 and scaling to unit size, we have

$$\sup_{B(p, 1/2)} |Ric| \leq C \cdot \epsilon.$$  

Therefore by the Bishop volume comparison theorem, $\text{Vol}(B(p, 1/2)) \leq C$. \qed

Lemma 3.7. There exist constants $\epsilon, C$ so that if $\| Rm \|_{L^2(B(p, r))} \leq \epsilon$, then

$$\int_{B(p, r/2)} |\nabla Ric|^2 dV_g \leq C \int_{B(p, r)} |Ric|^2 dV_g.$$  

Remark. This lemma will be crucial later in Section 6.

Proof. Again we may assume that $r = 1$, and the lemma will follow by scaling the metric. Let $\phi$ be a cutoff function in $B(p, 1)$, such that $\phi \equiv 1$ in $B(p, 1/2)$ and
\[|\nabla \phi| \leq C. \] We have
\[\int_{B(p,1)} \phi^2 |\nabla \nabla R|^2 = - \int \phi^2 \langle \Delta \nabla R, \nabla R \rangle - 2 \int \langle \phi \nabla R, \nabla \phi \rangle R \]
\[\leq C \int \phi^2 |Rm* R|^2 + \frac{C}{\delta} \int |\nabla \phi|^2 |R|^2 + C \delta \int \phi^2 |\nabla R|^2 \]
\[\leq C \left\{ \int \phi^2 |R|^2 \right\}^{1/2} \left\{ \int \phi^2 |R|^4 \right\}^{1/2} + C \int |R|^2 + C \delta \int \phi^2 |\nabla R|^2 \]
where we have used (3.5). By choosing \( \delta \) small, and \( \epsilon < 1 \), we obtain
\[\int_{B(p,1/2)} |\nabla R|^2 \leq (1 + \epsilon) C \int |R|^2 \leq C \int |R|^2.\]

\[\square\]

**Lemma 3.8.** For any \( k \geq 0 \), there exist constants \( \epsilon, C \) so that if \( \|Rm\|_{L^2(B(p,r))} \leq \epsilon \), then
\[
\left\{ \int_{B(p,r/2)} |\nabla^k R|^4 dV_g \right\}^{1/2} \leq \frac{C}{r^{2(k+1)}} \int_{B(p,r)} |Rm|^2 dV_g,
\]
\[
\int_{B(p,r/4)} |\nabla^{k+1} R|^2 dV_g \leq \frac{C}{r^{2(k+1)}} \int_{B(p,r)} |Rm|^2 dV_g,
\]
\[
\int_{B(p,r/8)} |\nabla^k R|^2 dV_g \leq \frac{C}{r^{2k}} \int_{B(p,r)} |Rm|^2 dV_g,
\]
\[
\left\{ \int_{B(p,r/16)} |\nabla^k Rm|^4 dV_g \right\}^{1/2} \leq \frac{C}{r^{2(k+1)}} \int_{B(p,r)} |Rm|^2 dV_g.
\]

**Proof.** We use induction on \( k \). The case \( k = 0 \) has been proved in the previous lemmas. Assume the lemma is true for \( i = 1 \ldots k - 1 \). In the following proof, one must shrink the ball further after each step of the iteration. For simplicity, we will do this automatically without mention, so that the final step shrinks by the appropriate factor.

From the equation \( \Delta R = Rm \ast R \), it follows that \([\text{Cao93a, Theorem 2.4}]\)
\[
\Delta(\nabla^k R) = \sum_{l=0}^{k} \nabla^l Rm \ast \nabla^{k-l} R.
\]
Similarly, from the equation $\Delta Rm = Rm \ast Rm + \nabla^2 Ric$, it follows that ([Cha93a, Corollary 3.11])

$$\Delta(\nabla^k Rm) = \sum_{l=0}^{k} \nabla^l Rm \ast \nabla^{k-l} Rm + \nabla^{k+2} Ric.$$  

**Proof of (3.15).** We let $\phi$ be a cutoff function as before, and consider the following expression

$$\int_{B(p,1)} \langle \Delta(\nabla^k Ric), \phi^2 \nabla^k Ric \rangle = \int \langle \sum_{l=0}^{k} \nabla^l Rm \ast \nabla^{k-l} Ric, \phi^2 \nabla^k Ric \rangle$$

$$= \int \langle \nabla^k Rm \ast Ric, \phi^2 \nabla^k Ric \rangle + \int \langle Rm \ast \nabla^k Ric, \phi^2 \nabla^k Ric \rangle$$

$$+ \sum_{l=1}^{k-1} \int \langle \nabla^l Rm \ast \nabla^{k-l} Ric, \phi^2 \nabla^k Ric \rangle.$$  

Integrate the first term in (3.21) by parts

$$\int \langle \nabla^k Rm \ast Ric, \phi^2 \nabla^k Ric \rangle = -2 \int \langle \nabla \phi \cdot \nabla^{k-1} Rm \ast Ric, \phi \nabla^k Ric \rangle$$

$$- \int \langle \nabla^{k-1} Rm \ast \nabla Ric, \phi^2 \nabla^k Ric \rangle - \int \langle \nabla^{k-1} Rm \ast Ric, \phi^2 \nabla^{k+1} Ric \rangle.$$  

We estimate

$$\left| \int \langle \nabla^k Rm \ast Ric, \phi^2 \nabla^k Ric \rangle \right| \leq C \int \phi |\nabla \phi| |\nabla^{k-1} Rm||Ric||\nabla^k Ric| + C \int \phi^2 |\nabla^{k-1} Rm||\nabla Ric||\nabla^k Ric|$$

$$+ C \int \phi^2 |\nabla^{k-1} Rm||Ric||\nabla^{k+1} Ric|$$

$$\leq C \int \phi^2 |\nabla^{k-1} Rm|^4 + C \int \phi^2 |Ric|^4 + C \int \phi^2 |\nabla^k Ric|^2$$

$$+ C \int \phi^2 |\nabla^{k-1} Rm||\nabla Ric||\nabla^k Ric|$$

$$+ \frac{C}{\delta} \left( \int \phi^2 |\nabla^{k-1} Rm|^4 + \int \phi^2 |Ric|^4 \right) + C \delta \int \phi^2 |\nabla^{k+1} Ric|^2.$$  

Using the inductive hypothesis, we obtain

$$\left| \int \langle \nabla^k Rm \ast Ric, \phi^2 \nabla^k Ric \rangle \right| \leq C \delta \int \phi^2 |\nabla^{k+1} Ric|^2 + C \int |Rm|^2 + C \left\{ \int |Rm|^2 \right\}^2$$

$$+ C \int \phi^2 |\nabla^{k-1} Rm||\nabla Ric||\nabla^k Ric|$$
Therefore when $\| Rm \|_{L^2(B(p,1))} < \epsilon < 1$, we have

$$\left| \int \langle \nabla^k Rm \ast \text{Ric}, \phi^2 \nabla^k \text{Ric} \rangle \right| \leq C\delta \int \phi^2 |\nabla^{k+1} \text{Ric}|^2 + C \int |Rm|^2$$

$$+ C \int \phi^2 |\nabla^{k-1} Rm| |\nabla \text{Ric}| |\nabla^k \text{Ric}|.$$

Next we estimate the second term in (3.24)

$$\int \langle Rm \ast \nabla^k \text{Ric}, \phi^2 \nabla^k \text{Ric} \rangle \leq \left\{ \int \phi^2 |Rm|^2 \right\}^{1/2} \left\{ \int \phi^2 |\nabla^k \text{Ric}|^4 \right\}^{1/2}$$

$$\leq \epsilon \left\{ \int \phi^2 |\nabla^k \text{Ric}|^4 \right\}^{1/2}$$

Next we estimate the sum term in (3.24) (for $k \geq 2$)

$$\sum_{l=1}^{k-1} \int \langle \nabla^l Rm \ast \nabla^{k-l} \text{Ric}, \phi^2 \nabla^k \text{Ric} \rangle$$

$$\leq \sum_{l=1}^{k-1} \left( \int \phi^2 |\nabla^l Rm|^4 + \int \phi^2 |\nabla^{k-l} \text{Ric}|^4 + \int \phi^2 |\nabla^k \text{Ric}|^2 \right) \leq C \int |Rm|^2.$$

by the inductive hypothesis, and since $\| Rm \|_{L^2(B(p,1))} < \epsilon < 1$. Adding these estimates we obtain

$$\left| \int_{B(p,1)} \langle \Delta (\nabla^k \text{Ric}), \phi^2 \nabla^k \text{Ric} \rangle \right|$$

$$\leq C\delta \int \phi^2 |\nabla^{k+1} \text{Ric}|^2 + C \int |Rm|^2$$

$$+ C \int \phi^2 |\nabla^{k-1} Rm| |\nabla \text{Ric}| |\nabla^k \text{Ric}| + \epsilon \left\{ \int \phi^2 |\nabla^k \text{Ric}|^4 \right\}^{1/2}.$$

On the other hand, we have

$$\int_{B(p,1)} \langle \Delta \nabla^k \text{Ric}, \phi^2 \nabla^k \text{Ric} \rangle = -2 \int \langle \phi \nabla^{k+1} \text{Ric}, \nabla \phi \cdot \nabla^k \text{Ric} \rangle$$

$$- \int \langle \nabla^{k+1} \text{Ric}, \phi^2 \nabla^{k+1} \text{Ric} \rangle,$$
which implies that
\[
\int \phi^2 |\nabla^{k+1} \text{Ric}|^2 \leq \left| \int \langle \Delta \nabla^k \text{Ric}, \phi^2 \nabla^k \text{Ric} \rangle \right| + 2 \int \phi |\nabla \phi||\nabla^k \text{Ric}||\nabla^{k+1} \text{Ric}|
\]
\[
\leq \left| \int \langle \Delta \nabla^k \text{Ric}, \phi^2 \nabla^k \text{Ric} \rangle \right| + \frac{2}{\delta} \int |\nabla \phi|^2 |\nabla^k \text{Ric}|^2 + 2 \delta \int \phi^2 |\nabla^{k+1} \text{Ric}|^2
\]
\[
\leq C \delta \int \phi^2 |\nabla^{k+1} \text{Ric}|^2 + C \int |\text{Rm}|^2
\]
\[
+ C \int \phi^2 |\nabla^{k-1} \text{Rm}||\nabla \text{Ric}||\nabla^k \text{Ric}|^2 + \epsilon \left\{ \int \phi^2 |\nabla^k \text{Ric}|^4 \right\}^{1/2}
\]
\[
\leq \frac{2}{\delta} \int |\nabla \phi|^2 |\nabla^k \text{Ric}|^2 + 2 \delta \int \phi^2 |\nabla^{k+1} \text{Ric}|^2.
\]
So by choosing \(\delta\) small,
\[
\int \phi^2 |\nabla^{k+1} \text{Ric}|^2 \leq C \int |\text{Rm}|^2 + C \int \phi^2 |\nabla^{k-1} \text{Rm}||\nabla \text{Ric}||\nabla^k \text{Ric}|
\]
\[
+ \epsilon \left\{ \int \phi^2 |\nabla^k \text{Ric}|^4 \right\}^{1/2} + C \int |\nabla \phi|^2 |\nabla^k \text{Ric}|^2.
\]
Next from the Sobolev inequality and Lemma (3.5),
\[
\left\{ \int (\phi |\nabla^k \text{Ric}|^4) \right\}^{1/2} \leq C \int |\nabla (\phi |\nabla^k \text{Ric}|)|^2 + C \int \phi^2 |\nabla^k \text{Ric}|^2
\]
\[
\leq C \int (\phi^2 + |\nabla \phi|^2) |\nabla^k \text{Ric}|^2 + C \int \phi^2 |\nabla^{k+1} \text{Ric}|^2
\]
\[
\leq C \int (\phi^2 + |\nabla \phi|^2) |\nabla^k \text{Ric}|^2 + C \int |\text{Rm}|^2
\]
\[
+ C \int \phi^2 |\nabla^{k-1} \text{Rm}||\nabla \text{Ric}||\nabla^k \text{Ric}| + \epsilon \left\{ \int \phi^2 |\nabla^k \text{Ric}|^4 \right\}^{1/2}.
\]
Choosing \(\epsilon\) small, and using the inductive hypothesis, we obtain
\[
\left\{ \int (\phi |\nabla^k \text{Ric}|^4) \right\}^{1/2} \leq C \int |\text{Rm}|^2 + C \int \phi^2 |\nabla^{k-1} \text{Rm}||\nabla \text{Ric}||\nabla^k \text{Ric}|.
\]
Now if \(k = 1\), the last term is
\[
\int \phi^2 |\text{Rm}||\nabla \text{Ric}|^2 \leq \left\{ \int_{B(p,1)} |\text{Rm}|^2 \right\}^{1/2} \left\{ \int \phi^4 |\nabla \text{Ric}|^4 \right\}^{1/2} \leq \epsilon \left\{ \int \phi^4 |\nabla \text{Ric}|^4 \right\}^{1/2},
\]
so this term may be absorbed into the left, and we obtain
\[
(3.22) \quad \left\{ \int (\phi |\nabla \text{Ric}|)^4 \right\}^{1/2} \leq C \int |\text{Rm}|^2.
\]
For $k \geq 2$, the last term is

$$
\int \phi^2 |\nabla^{k-1} Rm| |\nabla Ric| |\nabla Ric| \leq C \int \phi^2 |\nabla^{k-1} Rm|^2 |\nabla Ric|^2 + C \int \phi^2 |\nabla k_{Ric}|^2 
$$

$$
\leq C \int \phi^2 |\nabla^{k-1} Rm|^4 + C \int \phi^2 |\nabla Ric|^4 + C \int \phi^2 |\nabla k_{Ric}|^2,
$$

so by the inductive hypothesis and $\|Rm\|_{L^2(B(p,1))} < \epsilon < 1$, we obtain

$$
(3.23) \quad \left\{ \int (\phi |\nabla k_{Ric}|)^4 \right\}^{1/2} \leq C \int |Rm|^2.
$$

Inequality (3.16) follows by choosing $\phi$ similar to before and then scaling the metric.

Proof of (3.16). Integrating by parts and using (3.19),

$$
\int_{B(p,r)} \phi^2 |\nabla^{k+1} Ric|^2 = -\int \langle \phi^2 \nabla k_{Ric}, \Delta \nabla k_{Ric} \rangle - 2 \int \langle \nabla k_{Ric}, \phi \nabla^{k+1} Ric \rangle
$$

$$
(3.24) = -\int \langle \phi^2 \nabla k_{Ric}, \sum_{l=0}^k \nabla^l Rm * \nabla^{k-l} Ric \rangle - 2 \int \langle \nabla k_{Ric}, \phi \nabla^{k+1} Ric \rangle.
$$

For the first term in (3.24),

$$
-\sum_{l=0}^k \int \langle \phi^2 \nabla k_{Ric}, \nabla^l Rm * \nabla^{k-l} Ric \rangle
$$

$$
= \int \langle \phi^2 \nabla k_{Ric}, \nabla^k Rm * Ric \rangle - \sum_{l=0}^{k-1} \int \langle \phi^2 \nabla k_{Ric}, \nabla^l Rm * \nabla^{k-l} Ric \rangle.
$$

Integrating by parts

$$
\int \langle \phi^2 \nabla k_{Ric}, \nabla^k Rm * Ric \rangle = -2 \int \langle \phi \nabla k_{Ric}, \nabla \phi \cdot \nabla^{k-1} Rm * Ric \rangle
$$

$$
- \int \langle \phi^2 \nabla^{k+1} Ric, \nabla^{k-1} Rm * Ric \rangle - \int \langle \phi^2 \nabla k_{Ric}, \nabla^{k-1} Rm * \nabla Ric \rangle,
$$
consequently

\[
\left| \int \langle \phi^2 \nabla^k \text{Ric}, \nabla^k \text{Rm} \ast \text{Ric} \rangle \right| \leq C \int |\phi||\nabla \phi||\nabla^k \text{Ric}||\nabla^{k-1} \text{Rm}||\text{Ric}|
\]

\[
+ C \int \phi^2 |\nabla^{k+1} \text{Ric}||\nabla^{k-1} \text{Rm}||\text{Ric}| + C \int \phi^2 |\nabla^k \text{Ric}||\nabla^{k-1} \text{Rm}||\nabla \text{Ric}|
\]

\[
\leq \int \phi^2 |\nabla^k \text{Ric}|^2 + \int |\nabla \phi|^2 |\nabla^{k-1} \text{Rm}|^2 |\text{Ric}|^2
\]

\[
+ C \delta \int \phi^2 |\nabla^{k+1} \text{Ric}|^2 + \frac{C}{\delta} \int \phi^2 |\nabla^{k-1} \text{Rm}|^2 |\text{Ric}|^2
\]

\[
+ C \int \phi^2 |\nabla^k \text{Ric}|^2 + C \int \phi^2 |\nabla^{k-1} \text{Rm}|^2 |\nabla \text{Ric}|^2
\]

\[
\leq C \int |\text{Rm}|^2 + C \delta \int \phi^2 |\nabla^{k+1} \text{Ric}|^2,
\]

by the inductive hypothesis and (3.15). Also by induction, we have

\[
\left| \sum_{l=0}^{k-1} \int \langle \phi^2 \nabla^l \text{Ric}, \nabla^l \text{Rm} \ast \nabla^{k-l} \text{Ric} \rangle \right|
\]

\[
\leq \sum_{l=0}^{k-1} \left( \int |\nabla^l \text{Rm}|^4 + \int |\nabla^{k-l} \text{Ric}|^4 + \int |\nabla^k \text{Ric}|^2 \right) \leq C \int |\text{Rm}|^2.
\]

We estimate the last term in (3.22)

\[
\left| 2 \int \langle \nabla \phi \nabla^k \text{Ric}, \phi \nabla^{k+1} \text{Ric} \rangle \right| \leq C \int |\nabla \phi| |\phi| |\nabla^k \text{Ric}| |\nabla^{k+1} \text{Ric}|
\]

\[
\leq C \delta \int \phi^2 |\nabla^{k+1} \text{Ric}|^2 + \frac{C}{\delta} \int |\nabla \phi|^2 |\nabla^k \text{Ric}|^2
\]

\[
\leq C \delta \int \phi^2 |\nabla^{k+1} \text{Ric}|^2 + C \int |\text{Rm}|^2.
\]

Combining the above estimates, we obtain

\[
\int_{B(p,r)} \phi^2 |\nabla^{k+1} \text{Ric}|^2 \leq C \delta \int \phi^2 |\nabla^{k+1} \text{Ric}|^2 + C \int |\text{Rm}|^2,
\]

So by choosing \( \delta \) small, and \( \phi \) as before, we obtain (3.16).

**Proof of (3.17).** Integrating by parts and using (3.20),

\[
\int_{B(p,1)} \phi^2 |\nabla^k \text{Rm}|^2 = - \int_{B(p,1)} \langle \phi^2 \nabla^{k-1} \text{Rm}, \Delta \nabla^{k-1} \text{Rm} \rangle - 2 \int \langle \nabla \phi \nabla^{k-1} \text{Rm}, \phi \nabla^k \text{Rm} \rangle
\]

\[
= - \int \langle \phi^2 \nabla^{k-1} \text{Rm}, \sum_{l=0}^{k-1} \nabla^l \text{Rm} \ast \nabla^{k-1-l} \text{Rm} + \nabla^{k+1} \text{Ric} \rangle
\]

\[
- 2 \int \langle \nabla \phi \nabla^{k-1} \text{Rm}, \phi \nabla^k \text{Rm} \rangle,
\]

We estimate the last term in (3.22)
which implies the estimate

\[
\begin{align*}
\int_{B(p,1)} \phi^2 |\nabla^k Rm|^2 &\leq \sum_{l=0}^{k-1} \int \phi^2 |\nabla^{k-1} Rm||\nabla^l Rm||\nabla^{k-1-l} Rm| \\
+ \int \phi^2 |\nabla^{k-1} Rm||\nabla^{k+1} Ric| + 2 \int \phi |\nabla \phi||\nabla^{k-1} Rm||\nabla^k Rm| \\
&\leq C \sum_{l=0}^{k-1} \left( \int \phi^2 |\nabla^{k-1} Rm|^4 + \int \phi^2 |\nabla^l Rm|^4 + \int \phi^2 |\nabla^{k-1-l} Rm|^2 \right) \\
+ \left\{ \int \phi^2 |\nabla^{k-1} Rm|^2 \right\}^{1/2} \left\{ \int \phi^2 |\nabla^{k+1} Ric|^2 \right\}^{1/2} \\
+ \frac{C}{\delta} \int |\nabla \phi|^2 |\nabla^{k-1} Rm|^2 + C\delta \int \phi^2 |\nabla^k Rm|^2.
\end{align*}
\]

Using the inductive hypothesis, (3.16), and choosing \(\delta\) sufficiently small, we obtain

\[
\int_{B(p,1)} \phi^2 |\nabla^k Rm|^2 \leq C \int |Rm|^2.
\]

**Proof of (3.18).** We let \(\phi\) be a cutoff function as before, and consider the following expression

\[
\begin{align*}
\int_{B(p,1)} &\langle \Delta (\nabla^k Rm), \phi^2 \nabla^k Rm \rangle \\
= &\sum_{l=0}^{k} \int \langle \nabla^l Rm \ast \nabla^{k-l} Rm + \nabla^{k+2} Ric, \phi^2 \nabla^k Rm \rangle \\
= &\int \langle \nabla^k Rm \ast Rm, \phi^2 \nabla^k Rm \rangle + \int \langle \nabla^{k+2} Ric, \phi^2 \nabla^k Rm \rangle \\
+ &\sum_{l=1}^{k-1} \int \langle \nabla^l Rm \ast \nabla^{k-l} Rm, \phi^2 \nabla^k Rm \rangle.
\end{align*}
\]

(3.25)

For the first term in (3.25),

\[
\begin{align*}
\left| \int \langle \nabla^k Rm \ast Rm, \phi^2 \nabla^k Rm \rangle \right| &\leq C \int \phi^2 |Rm||\nabla^k Rm|^2 \\
&\leq \left\{ \int \phi^2 |Rm|^2 \right\}^{1/2} \left\{ \int \phi^2 |\nabla^k Rm|^4 \right\}^{1/2} \leq C \left\{ \int \phi^2 |\nabla^k Rm|^4 \right\}^{1/2}.
\end{align*}
\]

Next we estimate the second term in (3.25)

\[
\begin{align*}
\int \langle \nabla^{k+2} Ric, \phi^2 \nabla^k Rm \rangle &= -\int \langle \nabla^{k+1} Ric, \phi^2 \nabla^{k+1} Rm \rangle \\
- &\int \langle \nabla^{k+1} Ric, 2\phi \nabla \phi \nabla^k Rm \rangle.
\end{align*}
\]
So we have
\[
\left| \int \langle \nabla^{k+2} \text{Ric}, \phi^2 \nabla^k \text{Rm} \rangle \right| \leq C \int |\nabla^{k+1} \text{Ric}| |\phi^2| |\nabla^{k+1} \text{Rm}| \\
+ C \int |\nabla^{k+1} \text{Ric}| |\phi|||\nabla^k \text{Rm}| \\
\leq \frac{C}{\delta} \int \phi^2 |\nabla^{k+1} \text{Ric}|^2 + C \delta \int \phi^2 |\nabla^{k+1} \text{Rm}|^2 \\
+ C \int \phi^2 |\nabla^{k+1} \text{Ric}|^2 + C \int |\nabla^k|^2 |\nabla^k \text{Rm}|^2.
\]

Using induction and (3.16) we obtain
\[
\left| \int \langle \nabla^{k+2} \text{Ric}, \phi^2 \nabla^k \text{Rm} \rangle \right| \leq C \delta \int \phi^2 |\nabla^{k+1} \text{Rm}|^2 + C \int |\text{Rm}|^2.
\]

Next we estimate the sum term in (3.25) (for \( k \geq 2 \))
\[
\sum_{l=1}^{k-1} \int \langle \nabla^l \text{Rm} \ast \nabla^{k-l} \text{Rm}, \phi^2 \nabla^k \text{Rm} \rangle \\
\leq \sum_{l=1}^{k-1} \left( \int \phi^2 |\nabla^l \text{Rm}|^4 + \int \phi^2 |\nabla^{k-l} \text{Rm}|^4 + \int \phi^2 |\nabla^k \text{Rm}|^2 \right) \leq C \int |\text{Rm}|^2.
\]

by (3.17), the inductive hypothesis, and since \( \|\text{Rm}\|_{L^2(B(p,1))} < \epsilon < 1 \). Adding these estimate we obtain
\[
\left| \int_{B(p,1)} \langle \Delta (\nabla^k \text{Rm}), \phi^2 \nabla^k \text{Rm} \rangle \right| \\
\leq C \delta \int \phi^2 |\nabla^{k+1} \text{Rm}|^2 + C \int |\text{Rm}|^2 + \epsilon \left\{ \int \phi^2 |\nabla^k \text{Rm}|^4 \right\}^{1/2}.
\]

On the other hand, we have
\[
\int_{B(p,1)} \langle \Delta \nabla^k \text{Rm}, \phi^2 \nabla^k \text{Rm} \rangle = - \int \langle \nabla^{k+1} \text{Rm}, 2 \phi \nabla \phi \nabla^k \text{Rm} \rangle \\
- \int \langle \nabla^{k+1} \text{Rm}, \phi^2 \nabla^{k+1} \text{Rm} \rangle.
\]
which implies that
\[
\int \phi^2 |\nabla^{k+1} Rm|^2 \leq \left| \int \langle \Delta \nabla^{k} Rm, \phi^2 \nabla^{k} Rm \rangle \right| + 2 \int \phi |\nabla \phi||\nabla^{k+1} Rm||\nabla^{k+1} Rm| \\
\leq \left| \int \langle \Delta \nabla^{k} Rm, \phi^2 \nabla^{k} Rm \rangle \right| + \frac{2}{\delta} \int |\nabla \phi|^2 |\nabla^{k} Rm|^2 + 2 \delta \int \phi^2 |\nabla^{k+1} Rm|^2 \\
\leq C \delta \int \phi^2 |\nabla^{k+1} Rm|^2 + C \int |Rm|^2 + \epsilon \left\{ \int \phi^2 |\nabla^{k} Rm|^4 \right\}^{1/2} \\
+ \frac{2}{\delta} \int |\nabla \phi|^2 |\nabla^{k} Rm|^2 + 2 \delta \int \phi^2 |\nabla^{k+1} Rm|^2. 
\]

So by choosing \( \delta \) small, we have
\[
\int \phi^2 |\nabla^{k+1} Rm|^2 \leq C \int |Rm|^2 + \epsilon \left\{ \int \phi^2 |\nabla^{k} Rm|^4 \right\}^{1/2} + C \int |\nabla \phi|^2 |\nabla^{k} Rm|^2. 
\]

Next from the Sobolev inequality we have
\[
\left\{ \int (\phi |\nabla^{k} Rm|)^4 \right\}^{1/2} \leq C \int (\nabla (\phi |\nabla^{k} Rm|))^2 + C \int \phi^2 |\nabla^{k} Rm|^2 \\
\leq C \int (\phi^2 + |\nabla \phi|^2) |\nabla^{k} Rm|^2 + C \int \phi^2 |\nabla^{k+1} Rm|^2 \\
\leq C \int (\phi^2 + |\nabla \phi|^2) |\nabla^{k} Rm|^2 + C \int |Rm|^2 + \epsilon \left\{ \int \phi^2 |\nabla^{k} Rm|^4 \right\}^{1/2}. 
\]

Using the inductive hypothesis and choosing \( \epsilon \) sufficiently small, we obtain
\[
(3.26) \quad \left\{ \int (\phi |\nabla^{k} Rm|)^4 \right\}^{1/2} \leq C \int |Rm|^2.
\]

Inequality (3.18) follows by scaling the metric.

The following is the main Moser iteration lemma

**Lemma 3.9.** Suppose \( 0 \leq u \in L^2(B(p, r)) \) satisfies the inequality
\[
(3.27) \quad \Delta u \geq -u \cdot f - h,
\]
with \( h \geq 0 \) and \( f \geq 0 \) satisfying
\[
(3.28) \quad \int_{B(p,r)} f^4 dV_g \leq \frac{C_1}{r^4}.
\]

If \( \text{Vol}(B(p, r)) \leq C_2 r^4 \), then there exists a constant \( C \) depending only upon \( C_1, C_2, C_S \) so that
\[
\sup_{B(p,r/2)} u \leq \frac{C}{r^2} \|u\|_{L^2(B(p,r))} + C r \|h\|_{L^4(B(p,r))}.
\]
Proof. Let \( k = r \| h \|_{L^4(B(p,r))} \), and \( \bar{u} = u + k \). Assume \( k \neq 0 \), then \( \bar{u} \) satisfies
\[
\Delta \bar{u} = \Delta u \geq -uf - h \geq -\bar{u}f - h \geq -\bar{u} f + \frac{h}{k} = -\bar{uf}.
\]
We have
\[
\left\{ \int_{B(p,r)} \bar{f}^4 dV \right\}^{1/4} \leq \left\{ \int_{B(p,r)} f^4 dV \right\}^{1/4} + \left\{ \int_{B(p,r)} \left( \frac{h}{k} \right)^4 dV \right\}^{1/4} \leq \frac{C_1}{r} + \frac{1}{k} \left\{ \int_{B(p,r)} h^4 dV \right\}^{1/4} \leq \frac{(C_1 + 1)}{r}.
\]
Lemma 3.4 implies that
\[
\sup_{B(p,r/2)} \bar{u} \leq \frac{C_1}{r^2} \| \bar{u} \|_{L^2(B(p,r))}.
\]
In terms of \( u \), we have
\[
\sup_{B(p,r/2)} u \leq \frac{C_1}{r^2} \| u \|_{L^2(B(p,r))} \leq \frac{C_1}{r^2} \left( \| u \|_{L^2(B(p,r))} + k \| L^2(B(p,r)) \| \right) \leq \frac{C_1}{r^2} \left( \| u \|_{L^2(B(p,r))} + k (Vol(B(p,r)))^{1/2} \right) \leq \frac{C_1}{r^2} \| u \|_{L^2(B(p,r))} + C r \| h \|_{L^4(B(p,r))}.
\]
We now complete the proof of the Theorem. We may assume that \( r = 1 \). First consider the case \( k = 0 \). From (3.29), it follows there exists constants \( C_1 \) and \( C_2 \) so that
\[
(3.29) \quad \Delta |Rm| \geq -C_1 |Rm|^2 - C_2 |\nabla^2 Ric|.
\]
This case \( k = 0 \) follows by applying Lemma 3.9 to (3.29), using the \( L^4 \) bound on \( \nabla^2 Ric \) from Lemma 3.8 and the volume bound from Corollary 3.6.

Assume the theorem is true for \( i = 0 \ldots k - 1 \), so that we have pointwise bounds on \( |\nabla^i Rm| \) for \( i = 0 \ldots k - 1 \). From Lemma 3.8 we have bounds on \( \| \nabla^{k+2} Ric \|_{L^4} \), Using (3.20), we obtain
\[
\Delta |\nabla^k Rm| \geq -C \sum_{l=0}^{k} |\nabla^l Rm||\nabla^{k-l} Rm| - C |\nabla^{k+2} Ric|
\]
\[
= -C |\nabla^k Rm||Rm| - C \sum_{l=1}^{k-1} |\nabla^l Rm||\nabla^{k-l} Rm| - C |\nabla^{k+2} Ric|.
\]
Lemma 3.9 and Corollary 3.6 then yield the pointwise bounds on \( |\nabla^k Rm| \). \( \square \)
We next apply Theorem 3.1 to noncompact spaces to give a rate of curvature decay at infinity.

**Theorem 3.10.** Assume that \((X, g)\) is a complete, noncompact space satisfying (3.1),
\[
\int_X |Rm_g|^2 dV_g < \infty, \quad \text{and} \quad C_S < \infty.
\]
(3.30)

Fix a basepoint \(p\), and let \(D(r) = X \setminus B(p, r)\) be the complement of a geodesic ball around the point \(p\), and \(k \geq 0\). There exist constants \(\epsilon_0, C_k\) (depending upon \(C_S\)) so that if \(\|Rm\|_{L^2(D(r))} \leq \epsilon_0\), then
\[
\sup_{D(2r)} |\nabla^k Rm| \leq \frac{C_k}{r^{2+k}} \left\{ \int_{D(r)} |Rm|^2 dV_g \right\}^{1/2} \leq \frac{C_k \epsilon_0}{r^{2+k}}.
\]

Proof. Given \(\epsilon < \epsilon_0\) from Theorem 3.1, there exists an \(R\) large so that
\[
\int_{D(R)} |Rm|^2 dV_g < \epsilon < \epsilon_0.
\]
Choose any \(x \in X\) with \(d(x, p) = r(x) > 2R\), then \(B(x, r) \subset D(R)\). From Theorem 3.1 we have
\[
\sup_{B(x, r/2)} |\nabla^k Rm| \leq \frac{C_k}{r^{2+k}} \left\{ \int_{B(x, r)} |Rm|^2 dV_g \right\}^{1/2} \leq \frac{C_k \epsilon}{r^{2+k}},
\]
which implies
\[
|\nabla^k Rm|(x) \leq \frac{C_k \epsilon}{r^{2+k}}.
\]

Clearly, as we take \(R\) larger, we may choose \(\epsilon\) smaller, and we see that
\[
\sup_{S(r)} |\nabla^k Rm| = o(r^{-2-k}),
\]
(3.32)
as \(r \to \infty\), where \(S(r)\) denotes the sphere of radius \(r\) centered at \(p\).

4. **Volume Growth**

This section will be devoted to proving the following

**Theorem 4.1.** Let \((X, g)\) be a complete, noncompact, 4-dimensional Riemannian manifold with base point \(p\). Assume that there exists a constant \(C_1 > 0\) so that
\[
Vol(B(q, s)) \geq C_1 s^4,
\]
(4.1)
for any \(q \in X\), and all \(s \geq 0\). Assume furthermore that as \(r \to \infty\),
\[
\sup_{S(r)} |Rm_q| = o(r^{-2}),
\]
(4.2)
where $S(r)$ is sphere of radius $r$ centered at $p$. If $b_1(X) < \infty$, then $(X, g)$ has finitely many ends, and there exists a constant $C_2$ (depending on $g$) so that
\[
(4.3) \quad \text{Vol}(B(p, r)) \leq C_2 r^4.
\]
Furthermore, each end is ALE of order 0.

Remark. As stated in the introduction, this theorem holds in dimension $n$. For simplicity, we consider the 4 dimensional case - the same proof applies in dimension $n$ with appropriate modification of constants.

Remark. We emphasize that our proof requires a weaker condition than $b_1(X) < \infty$. The condition is that there are only finitely many disjoint “bad” annuli components in $X$. We say a component $A_0(r_1, r_2)$ of an annulus $A(r_1, r_2) = \{ q \in X \mid r_1 < d(p, q) < r_2 \}$ is bad if $S(r_1) \cap A_0(r_1, r_2)$ has more than 1 component, where $S(r_1)$ is the sphere of radius $r_1$ centered at $p$. If $b_1(X) < \infty$ then $X$ may contain only finitely many disjoint bad annuli (see the Mayer-Vietoris argument in Lemma 4.7 below). We expect that this assumption can be removed, but at the moment substantial technical difficulties remain.

4.1. Area Comparison. For notation, we let $\rho(x) = d(p, x)$ denote the distance function from $p$, and recall that $\Delta \rho(x)$ is the mean curvature of the level set of $\rho$, at any smooth point $x$ of $\rho$. We note that the distance function $\rho(x)$ is only Lipschitz, but $\Delta \rho(x)$ is well-defined almost everywhere. Letting $C(p)$ denote the cut locus of $p$, it is well-known that $C(p)$ has measure zero, and $\rho(x)$ is smooth on $X \setminus C(p)$ (see [Cha93b, Chapter 3]).

**Lemma 4.2.** There exists a decreasing function $\epsilon(r) \geq 0$ with $\epsilon(r) \to 0$ as $r \to \infty$, such that
\[
(4.4) \quad \Delta \rho(x) \leq \frac{3 + \epsilon(\rho(x))}{\rho(x)},
\]
at any point $x$ where the distance function $\rho(\cdot) = d(p, \cdot)$ is smooth.

**Proof.** First we construct a radial comparison metric as follows. Let $\tilde{g} = dr^2 + h(r)^2 d\theta^2$. The Jacobi equation is
\[
(4.5) \quad h'' = -K(r)h,
\]
where $K(r)$ is the radial curvature. We choose a smooth radial curvature function $K(r)$ so that
\[
3K(r) \leq \min\{\text{Ric}_g, 0\}
\]
and
\[
3K(r) = -Ar^{-2} \quad \text{for} \quad r > r_0 \quad \text{large}.
\]
The existence of a radial metric $\tilde{g}$ satisfying $h(0) = 0, h'(0) = 1$, and with $K(r)$ as its radial curvature function is obtained easily (see [GW79, Proposition 4.2]). We see that $r^q$ solves the Jacobi equation on $(r_0, \infty)$ where $q(q - 1) = \frac{4}{3}$. We thus have two linearly independent solutions, and therefore the general solution is given by $h(r) = c_1 r^{q_+} + c_2 r^{q_-}$ on $(r_0, \infty)$ for some constants $c_1, c_2$, where $q_{\pm} = \frac{1}{2} \left( 1 \pm \sqrt{1 + \frac{44}{3}} \right)$. We
may arrange so that our comparison metric satisfies $c_2 = 0$, and we then have for $r > r_0$,

$$
\frac{h'}{h} = \frac{c_1 q_+ r^{q_- - 1}}{c_1 r^{q_+}} = \frac{q_+}{r}.
$$

Using this radial metric and the decay condition $[4.2]$, the lemma follows from the Laplacian comparison theorem (see $[GW79]$).

**Definition 4.3.** Let $S(r)$ denote the geodesic sphere of radius $r$. Since the distance function is Lipschitz, by the coarea formula, for almost every $r$, $S(r)$ is $\mathcal{H}^3$-measurable, where $\mathcal{H}^3$ denotes $3-$Hausdorff measure, and we define

$$
\mathcal{H}(r) = \mathcal{H}^3(S(r)).
$$

For a set $K$ we let

$$
\mathcal{H}_K(r) = \mathcal{H}^3(S(r) \cap K).
$$

Next we define the lower area of distance spheres.

**Definition 4.4.** Let $S_p$ denote the unit sphere in $T_p(X)$, and $D_p$ denote the maximal star-shaped region on which $\exp_p$ is a diffeomorphism. Also, let $D_p(r)$ be the subset of $S_p$ of directions $\xi$ such that $r \xi \in D_p$, that is,

$$
r \cdot D_p(r) = S(r) \cap D_p.
$$

We define the lower area of $S(r)$ by

$$
\mathcal{A}(p, r) = \int_{D_p(r)} \sqrt{g}(r; \xi) d\sigma,
$$

where $\sqrt{g}(r; \xi)$ is given by $\exp^* dV_g(r; \xi) = \sqrt{g}(r; \xi) dr d\sigma$, and $d\sigma$ is the spherical area element of $S_p$, induced by Lebesgue measure on $T_p(X)$.

The following proposition gives the relation between the lower area and the 3-Hausdorff measure of distance spheres:

**Proposition 4.5.** For almost every $r$,

$$
\mathcal{H}(r) = \mathcal{A}(r).
$$

**Proof.** This is an easy consequence of the coarea formula, see $[Cha93b]$ Proposition 3.4.

**Theorem 4.6** (Area Comparison). There exists a function $\epsilon(r)$ with $\epsilon(r) \to 0$ as $r \to \infty$ such that for almost every $r_1$ and $r_2$,

$$
\mathcal{H}(r_2) \leq \mathcal{H}(r_1) \left( \frac{r_2}{r_1} \right)^{3+\epsilon(r_1)}.
$$
Proof. In spherical exponential coordinates, we have the formula [Pet98],

\[
\partial_r \sqrt{g}(r; \theta) = \Delta \rho \cdot \sqrt{g}(r; \theta).
\]

Integrating, we obtain

\[
\sqrt{g}(r_2; \theta) = \sqrt{g}(r_1; \theta) \cdot \exp \left( \int_{r_1}^{r_2} \Delta \rho(t; \theta) dt \right).
\]

Using Lemma 1.2, we obtain

\[
\sqrt{g}(r_2; \theta) \leq \sqrt{g}(r_1; \theta) \cdot \exp \left( \int_{r_1}^{r_2} \frac{3 + \epsilon}{t} dt \right) \leq \sqrt{g}(r_1; \theta) \left( \frac{r_2}{r_1} \right)^{3+\epsilon}.
\]

Recalling the notation introduced in Definition 4.4, we have that for \( r_1 < r_2 \),

\[
D_p(r_2) \subseteq D_p(r_1),
\]

that is, if a point \( x \in \exp_p \{ (r_2, D_p(r_2)) \} \), then there exists a minimal geodesic from \( p \) to \( x \), and this geodesic will necessarily hit the distance sphere of radius \( r_1 \).

Next using Proposition 4.5, for almost every \( r_1, r_2 \),

\[
\mathcal{H}(r_2) = \mathcal{A}(p, r_2) = \int_{D_p(r_2)} \sqrt{g}(r_2, \theta) d\sigma
\]

\[
\leq \int_{D_p(r_2)} \sqrt{g}(r_1, \theta) \left( \frac{r_2}{r_1} \right)^{3+\epsilon} d\sigma
\]

\[
\leq \left( \frac{r_2}{r_1} \right)^{3+\epsilon(r_1)} \int_{D_p(r_1)} \sqrt{g}(r_1, \theta) d\sigma = \mathcal{H}(r_1) \left( \frac{r_2}{r_1} \right)^{3+\epsilon(r_1)}.
\]

\[
\square
\]

4.2. Selection of Annuli. Fix \( s > 1 \), and we now choose a sequence of components of annuli in the following manner. Start with \( A_0(1, s) = \) any component of \( A(1, s) \). The outer boundary \( \partial A_0(1, s) \), which we denote by \( S_{outer,0} \), may have several components. Choose any component, and call this \( S_{inner,1} \). Next consider the annulus \( A(s, s^2) \). This may have several components, but we define \( A_1(s, s^2) \) to be the component which has inner boundary portion \( S_{inner,1} \). We repeat this procedure: given \( A_j(s^j, s^{j+1}) \), choose any component of the outer boundary, \( S_{outer,j} \), and call this component \( S_{inner,j+1} \). Choose \( A_{j+1}(s^{j+1}, s^{j+2}) \) to be the component of \( A(s^{j+1}, s^{j+2}) \) which has inner boundary portion \( S_{inner,j+1} \).

Lemma 4.7. Let \( A_j(s^j, s^{j+1}) \) be any sequence of components of the metric annulus \( A(s^j, s^{j+1}) \). Then \( S(s^j) \cap \overline{A_j(s^j, s^{j+1})} \) has only 1 component, except for finitely many \( j \). That is, there are only finitely many \( A_j(s^j, s^{j+1}) \) with the initial boundary sphere having more than 1 component. Therefore, there exists an \( N \) such that for all \( j > N \), \( S(s^j) \cap \overline{A_j(s^j, s^{j+1})} \) has only 1 component.

Proof. From the assumption, \( H_1(X) \) is finitely generated, say there are at most \( k \) generators. Assume that there are \( k+1 \) annuli with \( S(s^j) \cap \overline{A_j(s^j, s^{j+1})} \) having more than 1 component, index these by \( j_i, i = 1 \ldots k+1 \). For simplicity, assume that
number of components of the initial boundary of $\overline{A}_j$ is 2, and that the number of components of the outer boundary of $\overline{A}_j$ is 1. Let $U = \bigsqcup A_j(s^j, s^{j+1})$, and write $X = U \cup V$, where $V$ is a open set which intersects $A_j$ in an $\varepsilon$-neighborhood of each boundary component of $A_j$. Notice that $U$ has $k + 1$ components, and by choosing $\varepsilon$ sufficiently small, $U \cap V$ has $3(k + 1)$ components. The important observation is that $V$ has at most $k + 2$ components. This is because all of the initial spheres are connected to the base point by a geodesic, so this gives 1 connected component of $V$, and there are at most $k + 1$ more components of $V$ which touch each outer boundary sphere. We then consider the following portion of the Mayer-Vietoris sequence in homology:

$$H_1(X) \rightarrow H_0(U \cap V) \rightarrow H_0(u) \oplus H_0(V) \rightarrow H_0(X) \rightarrow 0,$$

By the observation above on the number of components of $U, V,$ and $U \cap V$, this sequence is

$$H_1(X) \rightarrow \mathbb{Z}^{3k} \rightarrow \mathbb{Z}^k \oplus \mathbb{Z} \oplus \mathbb{Z}^{\leq k+1} \rightarrow \mathbb{Z} \rightarrow 0.$$

It is then easy to see this forces $H_1(X)$ to contain a $\mathbb{Z}^{k+1}$, which contradicts the fact that $H_1(X)$ has at most $k$ generators. For simplicity we have restricted to this simple case, but a similar argument shows that if we have $k + 1$ disjoint annuli with the initial boundary of each having more than 1 component, then this forces $H_1(X)$ to contain at least a $\mathbb{Z}^{k+1}$.

**Proposition 4.8.** There exists a subsequence $\{ j \} \subset \{ i \}$ satisfying

$$H^3(S_{\text{inner},j+1}) \geq (1 - \eta_j)H^3(S_{\text{inner},j})s^3,$$

where $\eta_j \rightarrow 0$ as $j \rightarrow \infty$.

**Proof.** Consider the sequence of components of annuli $\{ A_i(s^i, s^{i+1}) \}$ for $i = 0 \ldots \infty$. If there does not exists such a subsequence satisfying (4.14), then for all $j \geq N$, we have

$$H^3(S_{\text{inner},j+1}) \leq (1 - \eta')H^3(S_{\text{inner},j})s^3,$$

and $\eta' > 0$. Given any $r \geq s^N$, choose $j$ so that

$$s^j \leq r \leq s^{j+1}.$$

Theorem 4.6 implies that there exists a function $\varepsilon(r)$ with $\varepsilon(r) \rightarrow 0$ as $r \rightarrow \infty$ such that

$$H^3(S(r) \cap A_j) \leq H^3(S_{\text{inner},j})s^{3+\varepsilon}$$

Note we are using Lemma 4.7, since the initial boundary has only 1 component for all $j$ large, we can apply the area comparison. Note also that Theorem 4.6 holds for almost every $r_1, r_2$, but in the above and what follows, we are free to change the sequence chosen by arbitrarily small changes, and we will do this automatically.

Since $s$ is bounded, and $\varepsilon \rightarrow 0$ as $r \rightarrow \infty$, we have

$$H^3(S(r) \cap A_j) \leq H^3(S_{\text{inner},j})s^{3+\varepsilon} \leq H^3(S_{\text{inner},j})s^3(1 + \varepsilon).$$
Therefore
\[
\mathcal{H}^3(S(r) \cap A_j) \leq (1 + \epsilon)s^3(1 - \eta')(1 - \eta')^{-N}\mathcal{H}^3(S_{\text{inner},j-1})s^3 \leq (1 + \epsilon)(1 - \eta')^{-N}s^{3-J}H^3(S_{\text{inner},N})s^{3j} \leq C(1 - \eta')^{-N}s^{3j} \leq C(1 - \eta')^{-N}r^3.
\]

From the coarea formula, we have for \(j >> N\),
\[
Vol(A_j(s^j, s^{j+1})) = \int_{s^j}^{s^{j+1}} H(S(t) \cap A_j) dt \leq C(1 - \eta')^{-N}(s^{4(j+1)} - s^{4j}).
\]

But the condition (4.11) implies that there exists \(C' > 0\) so that
\[
Vol(A_j(s^j, s^{j+1})) \geq C's^{4(j+1)},
\]
a contradiction.

In the following we will take our subsequence \(\{j\}\) to be maximal, that is, so that
\[
(4.18) \quad \mathcal{H}^3(S_{\text{inner},i+1}) < \mathcal{H}^3(S_{\text{inner},i})s^3,
\]
for \(i\) not in our subsequence \(\{j\} \subset \{i\}\). This is possible since if
\[
\mathcal{H}^3(S_{\text{inner},i+1}) \geq \mathcal{H}^3(S_{\text{inner},i})s^3,
\]
then we may obviously include \(i\) in the subsequence \(\{j\}\). In the following we will reserve the index \(j\) for this subsequence, while the index \(i\) will index all annuli.

4.3. **Laplacian of the distance function.** Some of the material in this section in well-known to experts, but we for completeness, we include the proof.

**Lemma 4.9.** For almost every \(r_1, r_2\),
\[
(4.19) \quad \int_{A(r_1, r_2)} (-\Delta \rho) dV_g \leq \mathcal{H}(r_1) - \mathcal{H}(r_2)
\]

**Remark.** In the above, we are viewing \(\Delta \rho\) as a measurable function, defined almost everywhere.

**Proof.** We begin with an approximation process as in [SY94 Proposition 1.1]. As before, we let \(D_p\) denote the maximal star shaped domain inside of the \(T_p(X)\) such that \(\exp : D_p \mapsto \exp_p(E_p) = D_p\) is a diffeomorphism, and \(C(p) = \partial D_p\), where \(C(p)\) is the cut locus. We note that \(C(p)\) has measure zero and \(X = D_p \cup C(p)\). The distance function \(\rho(x) = d(p, x)\) is Lipschitz, smooth on \(X \setminus C(p)\) and satisfies \(|\nabla \rho|^2 = 1\) on \(X \setminus \{C(p) \cup p\}\). We claim that for any \(\phi \in C^\infty_c(X)\), \(\phi \geq 0\), with support of \(\phi\) not containing \(p\), we have
\[
(4.20) \quad -\int_X \phi \Delta \rho \leq \int_X \nabla \phi \cdot \nabla r.
\]
To prove this, let $D_p = exp_p(E_p)$, and since the cut locus has measure zero, we have
\[
\int_X \phi \Delta \rho = \int_{D_p} \phi \Delta \rho.
\]
Since $D_p$ is star shaped, we may construct a family of smooth star-shaped domains $D_\epsilon \subset D_p$ with $\lim_{\epsilon \to 0} D_\epsilon = D_p$. Since $D_\epsilon$ is star-shaped, we have
\[
\frac{\partial r}{\partial \nu_\epsilon} > 0 \text{ on } \partial D_\epsilon.
\]
On $D_\epsilon$, $r$ is smooth, so applying Green’s first identity, we have
\[
\int_{D_\epsilon} \phi \Delta \rho = - \int_{D_\epsilon} \nabla \phi \cdot \nabla \rho + \int_{\partial D_\epsilon} \phi \frac{\partial r}{\partial \nu_\epsilon} \geq - \int_X \nabla \phi \cdot \nabla \rho.
\]
Since $|\nabla \rho| = 1$ almost everywhere, by dominated convergence we have
\[
\limsup_{\epsilon \to 0} \int_{D_\epsilon} \phi \Delta \rho \geq - \int_{D_p} \nabla \phi \cdot \nabla \rho = - \int_X \nabla \phi \cdot \nabla \rho
\]
Next, from Lemma 4.2, $\Delta \rho \leq C$ almost everywhere on the complement of any compact subset containing $p$, so from the assumption on the support of $\phi$, we have $C - \Delta \rho \geq 0$. By Fatou’s Lemma, we conclude that
\[
(4.21) \quad \int_{D_p} \phi (C - \Delta \rho) \leq \liminf_{\epsilon \to 0} \int_{D_\epsilon} \phi (C - \Delta \rho).
\]
This yields
\[
\int_X \phi \Delta \rho = \int_{D_p} \phi \Delta \rho \leq \limsup_{\epsilon \to 0} \int_{D_\epsilon} \phi \Delta \rho \geq - \int_X \nabla \phi \cdot \nabla \rho.
\]
which proves (4.20). Note that by an approximation argument, (4.20) holds for $\phi \in C^{0,1}_c(X)$, that is Lipschitz functions with compact support.

Next, define Lipschitz cutoff functions $a_\epsilon \in C^{0,1}(r_1, r_2)$ such that
\[
(4.22) \quad a_\epsilon = \begin{cases} 
1 & \text{if } r_1 + \epsilon \leq t \leq r_2 - \epsilon \\
0 & \text{if } t \leq r_1 \text{ or } t \geq r_2
\end{cases}
\]
and such that $|\nabla a_\epsilon| = \epsilon^{-1}$ for $r_1 \leq t \leq r_1 + \epsilon$ and $r_2 - \epsilon \leq t \leq r_2$. Let $\phi_\epsilon(x) = a_\epsilon(\rho(x))$, which is also a Lipschitz function, therefore we may apply (4.20) to obtain
\[
- \int_{A(r_1, r_2)} \phi_\epsilon \Delta \rho = - \int_{A(r_1 + \epsilon, r_2)} \phi_\epsilon \Delta \rho \leq \int_X \nabla \phi_\epsilon \cdot \nabla \rho
\]
\[
= \int_{A(r_1, r_1 + \epsilon)} \frac{1}{\epsilon} - \int_{A(r_2 - \epsilon, r_2)} \frac{1}{\epsilon}
\]
\[
= Vol(A(r_1, r_1 + \epsilon)) - \frac{\epsilon}{\epsilon} V ol(A(r_2 - \epsilon, r_2)).
\]
Since $\Delta \rho$ is bounded from above away from $p$, an application of Fatou’s Lemma as in (4.21) above, yields the inequality
\begin{equation}
- \int_{A(r_1, r_2)} \Delta \rho \leq \liminf_{\epsilon \to 0} - \int \phi_\epsilon \Delta \rho.
\end{equation}

From the coarea formula
\begin{equation}
\text{Vol}(A(0, t)) = \int_0^t \mathcal{H}(s) ds,
\end{equation}
therefore $\text{Vol}(A(0, t))$ is differentiable almost everywhere with derivative $\mathcal{H}(t)$. Since $\text{Vol}(A(a, t)) = \text{Vol}(A(0, t)) - \text{Vol}(A(0, a))$, we have that $\text{Vol}(A(a, t))$ is differentiable at $t = a$ for almost every $a$ with derivative $\mathcal{H}(a)$. Similarly, $\text{Vol}(A(t, a))$ is differentiable at $t = a$ for almost every $a$ with derivative $\mathcal{H}(a)$. Taking limits in the above, we have
\begin{equation}
- \int_{A(r_1, r_2)} \Delta \rho \leq \mathcal{H}(r_1) - \mathcal{H}(r_2),
\end{equation}
for almost every $r_1$ and $r_2$. \hfill \square

**Corollary 4.10.**
\begin{equation}
\Delta \rho \in L^1_{\text{loc}}.
\end{equation}

**Proof.** As $x \to p$, $\Delta \rho \sim \frac{3}{r}$, so the result is true for any small enough ball around $p$. On any annulus $A(r_1, r_2)$, $r_1 > 0$, by Lemma 4.2, $\Delta \rho \leq C$ almost everywhere. Therefore for almost every $r_1, r_2$
\begin{align*}
\int_{A(r_1, r_2)} |\Delta \rho| &= \int_{A(r_1, r_2)} |\Delta \rho - C + C| \\
&\leq CVol(A(r_1, r_2)) + \int_{A(r_1, r_2)} |\Delta \rho - C| \\
&= CVol(A(r_1, r_2)) + \int_{A(r_1, r_2)} (C - \Delta \rho) \\
&\leq 2CVol(A(r_1, r_2)) + \mathcal{H}(r_1) - \mathcal{H}(r_2).
\end{align*}
\hfill \square

Next we want the explicit error term in Lemma 4.9. We recall the following from [H98], [O74], [MM03]. The cutlocus is $(n - 1)$-rectifiable and can be decomposed as the disjoint union $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$, where $\mathcal{C}_2$ has finite $\mathcal{H}^{n-2}$-measure, and $\mathcal{C}_1$ is a union of smooth hypersurfaces. The set $\mathcal{C}_1$ is characterized by the following: If $x \in \mathcal{C}_1$ then (i) $x$ is not conjugate to $p$, and (ii) there are exactly 2 geodesics joining $p$ to $x$. The set $\mathcal{C}_2$ is characterized by: if $x \in \mathcal{C}_1$ then $x$ is either a conjugate point, or there are more than 2 minimal geodesics from $p$ to $x$.

**Lemma 4.11.** For almost every $r_1, r_2$,
\begin{equation}
\int_{A(r_1, r_2)} (-\Delta \rho) dV_g = \mathcal{H}(r_1) - \mathcal{H}(r_2) - \int_{\mathcal{C}} \left( \langle (\nabla \rho)^+, \partial_+^C \rangle + \langle (\nabla \rho)^-, \partial_-^C \rangle \right) d\mathcal{H}^3,
\end{equation}
Since $C_2$ has $H^3$-measure zero, the error term integration is only over $C_1$, and there are 2 terms corresponding to the 2 radial directions, with $\partial_C^\pm$ denoting the corresponding outward normals to $C$.

**Proof.** We first claim that

$$-\int_X \phi \Delta \rho = \int_X \nabla \phi \cdot \nabla \rho - \int_{D_\epsilon \cap A(r_1,r_2)} \phi \left( \langle (\nabla \rho)^+ , \partial_C^+ \rangle + \langle (\nabla \rho)^- , \partial_C^- \rangle \right) dH^3, \quad (4.28)$$

To prove this, we follow the proof of Lemma 4.9, applying Green’s identity, we have

$$\int_{D_\epsilon} \phi \Delta \rho = - \int_{D_\epsilon} \nabla \phi \cdot \nabla \rho + \int_{\partial D_\epsilon} \phi \frac{\partial r}{\partial \nu_\epsilon}.$$  

From Lemma 4.10, $\Delta \rho \in L^1_{loc}$ so we may apply dominated convergence to show that

$$-\int_X \phi \Delta \rho = \lim_{\epsilon \to 0} \int_{D_\epsilon} \phi \Delta \rho = - \int_{D_\epsilon} \phi \Delta \rho$$

$$= \lim_{\epsilon \to 0} \left( - \int_{D_\epsilon} \nabla \phi \cdot \nabla \rho + \int_{\partial D_\epsilon} \phi \frac{\partial r}{\partial \nu_\epsilon} \right)$$

$$= - \int_X \nabla \phi \cdot \nabla \rho - \lim_{\epsilon \to 0} \left( \int_{\partial D_\epsilon} \phi \frac{\partial r}{\partial \nu_\epsilon} \right).$$

From the remarks above, away from a set of finite $H^2$-measure, the cut locus is a union of smooth hypersurfaces. It is then clear that

$$\lim_{\epsilon \to 0} \frac{\partial r}{\partial \nu_\epsilon} = \langle \partial_C, \nabla \rho \rangle,$$

except on a set of $H^3$-measure zero. So we have

$$\lim_{\epsilon \to 0} \left( \int_{\partial D_\epsilon} \phi \frac{\partial r}{\partial \nu_\epsilon} \right) = - \int_C \phi \left( \langle (\nabla \rho)^+ , \partial_C^+ \rangle + \langle (\nabla \rho)^- , \partial_C^- \rangle \right) dH^3, \quad (4.29)$$

which proves (4.28). Finally we may imitate the argument in Lemma 4.9 using the sequence of cutoff functions $a_\epsilon$, and the error term is

$$- \lim_{\epsilon \to 0} \int_C a_\epsilon \left( \langle (\nabla \rho)^+ , \partial_C^+ \rangle + \langle (\nabla \rho)^- , \partial_C^- \rangle \right) dH^3$$

$$= - \int_{C \cap A(r_1,r_2)} \left( \langle (\nabla \rho)^+ , \partial_C^+ \rangle + \langle (\nabla \rho)^- , \partial_C^- \rangle \right) dH^3,$$

the limit being justified since by the choice of $r_1$ and $r_2$, the cut locus hits $S(r_1)$ and $S(r_2)$ in a set of $H^3$-measure zero. $\square$

**Lemma 4.12.** For almost every $r_1, r_2$,

$$- \int_{A(r_1,r_2)} \Delta (\rho^2) dV_g = 2 \left( r_1 \mathcal{H}(r_1) - r_2 \mathcal{H}(r_2) \right)$$

$$- 2 \int_{C \cap A(r_1,r_2)} \left( \langle \rho(\nabla \rho)^+ , \partial_C^+ \rangle + \langle \rho(\nabla \rho)^- , \partial_C^- \rangle \right) dH^3, \quad (4.30)$$
Proof. The formula (4.28) is replaced with
\[
\int_X \phi \Delta \rho^2 = 2 \int_X \nabla \phi \cdot \nabla \rho - 2 \int_{C \cap \mathcal{A}(r_1, r_2)} \phi \left( \rho(\nabla \rho)^+, \partial_C^+ \right) + \left( \rho(\nabla \rho)^-, \partial_C^- \right) dH^3,
\]
and the proof proceeds as in Lemma 4.11. □

Remark. Our proof does not require such an explicit expression for the error term. An examination of the proof below shows we just require 2 properties: (i) that the error term is additive under finite disjoint unions, and that (ii) the error term with \( \phi \) can be estimated by \( \| \phi \|_{L^\infty} \) times the error term with \( \phi \equiv 1 \). Both of these properties are clear from the definition of the error term as the limit in (4.29).

4.4. \( L^1 \) convergence. For any \( s > 0 \), let \((\tilde{\mathcal{A}}^r(1, s), \tilde{g}) = (A(r, sr), g/r^2)\) denote the annulus \( \{ x \in X : r < d(p, x) < sr \} \), with the rescaled metric \( g/r^2 \). From (4.2) we have
\[
\sup_{\tilde{\mathcal{A}}^r} |Rm| \leq \epsilon_1(r),
\]
where \( \epsilon_1(r) \to 0 \) as \( r \to \infty \).

Let \((\tilde{\mathcal{A}}_i(1, s), \tilde{g}) = (A_i(s^i, s^{i+1}), g/s^{2i})\) denote our previously chosen sequence of annular components, but with the rescaled metric \( g/s^{2i} \), and let \( \tilde{\rho}_i = s^{-i} \cdot \rho \) denote the rescaled distance function.

From (4.2) we have
\[
\sup_{\tilde{\mathcal{A}}_i(1, s)} |Rm| \leq \epsilon_2(i),
\]
where \( \epsilon_2(i) \to 0 \) as \( i \to \infty \).

**Proposition 4.13.** For the subsequence \( \{ j \} \subset \{ i \} \),
\[
\frac{1}{Vol(\tilde{\mathcal{A}}_j(1, s))} \int_{\tilde{\mathcal{A}}_j(1, s)} \left| \Delta(\tilde{\rho}_j)^2 - 8 \right| d\tilde{V}_j \to 0,
\]
and
\[
\frac{1}{Vol(\tilde{\mathcal{A}}_j(1, s))} \int_{C \cap \tilde{\mathcal{A}}_j(1, s)} \left( \langle \tilde{\rho}_j(\nabla \tilde{\rho}_j)^+, \partial_C^+ \rangle + \langle \tilde{\rho}_j(\nabla \tilde{\rho}_j)^-, \partial_C^- \rangle \right) dH^3 \to 0,
\]
as \( j \to \infty \).

**Proof.** For notation, we set
\[
E_j = \int_{C \cap \tilde{\mathcal{A}}_j(1, s)} \left( \langle \tilde{\rho}_j(\nabla \tilde{\rho}_j)^+, \partial_C^+ \rangle + \langle \tilde{\rho}_j(\nabla \tilde{\rho}_j)^-, \partial_C^- \rangle \right) dH^3
\]
If the proposition is not true, then for some subsequence (which for simplicity we continue to index by \( j \)),
\[
\frac{1}{Vol(\tilde{\mathcal{A}}_j(1, s))} \left( \int_{\tilde{\mathcal{A}}_j(1, s)} |\Delta(\tilde{\rho}_j)^2| - 8 |d\tilde{V}_j + 2E_j| \right) \geq C > 0,
\]
which in the original regions is
\[
\frac{1}{Vol(A_j)} \left( \int_{A_j} \left| \Delta (\rho^2) - 8 \right| dV_g + 2E_j \right) \geq C > 0.
\]

Note from (4.2), we have
\[
\Delta (\rho^2) = 2(\rho \Delta \rho + |\nabla \rho|^2) \leq 2(\rho \frac{3+\epsilon_j}{\rho} + 1) = 8 + \epsilon_j',
\]
at any smooth point of \( \rho \). Letting \( C \) denote the cut locus, recall that \( C \) is a set of
measure zero, and the distance function is smooth on \( X \setminus C \), so in all the following
integrals we may disregard the cut locus. Using (4.36), we have
\[
\frac{1}{Vol(A_j)} \int_{A_j} \left| \Delta (\rho^2) - 8 \right| dV_g = \frac{1}{Vol(A_j)} \int_{A_j} \left| \Delta (\rho^2) - (8 + \epsilon_j) \right| dV_g + \epsilon_j dV_g
\]
\[
\leq \frac{1}{Vol(A_j)} \left( \int_{A_j} \left| \Delta (\rho^2) - (8 + \epsilon_j) \right| dV_g \right) + \epsilon_j
\]
\[
= \frac{1}{Vol(A_j)} \left( \int_{A_j} (8 + \epsilon_j - \Delta (\rho^2)) dV_g \right) + \epsilon_j
\]
\[
= -\frac{1}{Vol(A_j)} \left( \int_{A_j} \Delta (\rho^2) dV_g \right) + 8 + 2\epsilon_j.
\]

where \( \epsilon_j \to 0 \) as \( j \to \infty \). Using Lemma \( \text{[41.2]} \) we obtain
\[
C \leq \frac{2}{Vol(A_j)} \left( s^j \cdot \mathcal{H}_{A_j}(s^j) - s^{j+1} \cdot \mathcal{H}_{A_j}(s^{j+1}) \right) + 8 + 2\epsilon_j.
\]

In the rescaled form, this is
\[
\frac{2}{Vol(A_j)} \left( s \cdot \mathcal{H}_{\tilde{A}_j}(s) - \mathcal{H}_{\tilde{A}_j}(1) \right) \leq -C + 8 + 2\epsilon_j.
\]

Since \( \tilde{A}_j \) are chosen to satisfy \( \mathcal{H}_{\tilde{A}_j}(s) \geq (1 - \eta_j)\mathcal{H}_{\tilde{A}_j}(1)s^3 \), we have
\[
\frac{2}{Vol(\tilde{A}_j)} \cdot \mathcal{H}_{\tilde{A}_j}(1) \cdot \left( (1 - \eta_j)s^4 - 1 \right) \leq -C + 8 + 2\epsilon_j.
\]

From Theorem \( \text{[4.6]} \) we have the estimate
\[
\mathcal{H}_{\tilde{A}_j}(t) \leq \mathcal{H}_{\tilde{A}_j}(1)t^{3+\epsilon_j}.
\]

Since \( s \) is bounded, we write this as
\[
\mathcal{H}_{\tilde{A}_j}(t) \leq \mathcal{H}_{\tilde{A}_j}(1)t^{3}(1 + \epsilon_j'),
\]

for \( 1 \leq t \leq s \), where \( \epsilon_j' \to 0 \) as \( i \to \infty \).

From the coarea formula
\[
Vol(\tilde{A}_j) = \int_1^s \mathcal{H}_{\tilde{A}_j}(t)dt \leq \int_1^s \mathcal{H}_{\tilde{A}_j}(1)t^{3}(1 + \epsilon_j')dt = \frac{1 + \epsilon_j'}{4} \mathcal{H}_{\tilde{A}_j}(1)(s^4 - 1).
\]
Substituting (4.38) into (4.37), we obtain
\[
\frac{8}{(1 + \epsilon_j')} \cdot \frac{(1 - \eta_j) s^4 - 1}{s^4 - 1} - 8 - 2\epsilon_j \leq -C < 0.
\]
The left hand side approaches 0 as \( j \to \infty \). Since the right hand side is a strictly negative constant, this is a contradiction. \(\square\)

4.5. **Completion of proof.** We next claim that there exists a constant \( C \) so that \( \mathrm{Vol}(\tilde{A}_j) < C \). To prove this, assume by contradiction that \( \mathrm{Vol}(\tilde{A}_j) \) is not bounded. Then there exists a subsequence \( \{j\} \subset \{i\} \) such that \( \mathrm{Vol}(\tilde{A}_j) \to \infty \) as \( j \to \infty \). We next find connected subsets with large, but bounded volume. Intuitively, it is obvious that if the volume of an annulus is very large, we may cut the annulus into several subsets with large, but bounded volume. But we need the cutting to have certain non-collapsing properties, so the cutting is rather delicate.

**Proposition 4.14.** We can find disjoint sets \( \tilde{K}_{j,i} \subset \tilde{A}_j, i = 1 \ldots N_j \) such that

1. \( \tilde{K}_{j,i} \) is connected.
2. \( \tilde{K}_{j,i} \) is the union of balls of radius 1/4 centered at points in \( \tilde{A}_j(3/2, s - 1/2) \).
3. \( \pi^2(s^4 - 1) + 1 < \mathrm{Vol}(\tilde{K}_{j,i}) < C_2 \), with \( C_2 \) a bounded constant.
4. \( \mathrm{Vol}(\tilde{A}_j \setminus \{\bigcup_{l=1}^{N_j} \tilde{K}_{j,i}\}) \leq C \mathrm{Vol}(\{\bigcup_{l=1}^{N_j} \tilde{K}_{j,i}\}) \) with \( C \) a uniformly bounded constant.

**Proof.** We first need to shrink the annuli, so consider \( \tilde{A}_j(3/2, s - 1/2) \). This set may have several components, this will be dealt with below. The sets we will choose below will be unions of balls of radius 1/4 centered at points in \( \tilde{A}_j(3/2, s - 1/2) \). The following lemma says that the volume of the annulus cannot be concentrated near the beginning portion or end portion of the annulus.

**Lemma 4.15.**
\[
\mathrm{Vol}(\tilde{A}_j(1, s) \setminus \tilde{A}_j(3/2, s - 1/2)) \leq C \cdot \mathrm{Vol}(\tilde{A}_j(3/2, s - 1/2)).
\]

**Proof.** From the coarea formula and area comparison (Theorem 4.6), we have
\[
\begin{align*}
\mathrm{Vol}(\tilde{A}_j(1, 3/2)) &= \int_{1}^{3/2} \mathcal{H}^3_{A_j}(t) dt \\
&\leq \int_{1}^{3/2} \mathcal{H}^3_{A_j}(1) t^{3+\epsilon} dt \\
&\leq C \cdot \mathcal{H}^3(\tilde{A}_j \cap S(1)).
\end{align*}
\]
Using the coarea formula, we may estimate
\[
\begin{align*}
\mathrm{Vol}(\tilde{A}_j(3/2, s - 1/2)) &= \int_{3/2}^{s-1/2} \mathcal{H}^3_{A_j}(t) dt \\
&= \int_{3/2}^{s-1/2} \frac{\mathcal{H}^3_{A_j}(t)}{t^3} \cdot t^3 dt \\
&\geq \frac{\mathcal{H}^3_{A_j}(t_0)}{t_0^3} \int_{3/2}^{s-1/2} t^3 dt \\
&\geq C \cdot \mathcal{H}^3(\tilde{A}_j \cap S(t_0)),
\end{align*}
\]
where \( t_0 \) is chosen so that
\[
\underset{3/2 \leq t \leq s-1/2}{\text{ess. inf.}} \frac{\mathcal{H}_{A_j}(t)}{t^3} = \frac{\mathcal{H}_{A_j}(t_0)}{t_0^3}.
\]
That is, \( t_0 \) is the minimal area sphere (when rescaled to unit size) for \( 3/2 \leq t \leq s - 1/2 \). If the minimum value is not actually achieved (i.e., in case \( H_{\tilde{A}_j}(t) t^{-3} \) has a discontinuity), then we approximate by a sequence approaching the minimum, we omit the details.

From area comparison and our choice of annulus, we have
\[
(1 - \eta_j)H^3(\tilde{A}_j \cap S(1)) s^3 \leq H^3(S_{\text{inner},j+1}) \leq H^3(\tilde{A}_j \cap S(s)) \leq C \cdot H^3(\tilde{A}_j \cap S(t_0)).
\]
(4.43)

Inequalities (4.41), (4.42), and (4.43) yield
\[
Vol(\tilde{A}_j(1,3/2)) \leq CVol(\tilde{A}_j(3/2, s - 1/2)).
\]
(4.44)

Next, from coarea and area comparison,
\[
Vol(\tilde{A}_j(s - 1/2, s)) \leq CH^3(\tilde{A}_j \cap S(s - 1/2)).
\]
(4.45)

Using area comparison,
\[
H^3(\tilde{A}_j \cap S(s - 1/2)) \leq CH^3(\tilde{A}_j \cap S(t_0)).
\]
(4.46)

Using the inequalities (4.42), (4.45), and (4.46), we obtain
\[
Vol(\tilde{A}_j(s - 1/2, s)) \leq CH^3(\tilde{A}_j \cap S(s - 1/2)) \leq CH^3(\tilde{A}_j \cap S(t_0)) \leq CVol(\tilde{A}_j(3/2, s - 1/2)).
\]
(4.47)

Inequalities (4.44) and (4.47) together yield
\[
Vol(\tilde{A}_j(1,3/2) \cup \tilde{A}_j(s - 1/2, s)) \leq C \cdot Vol(\tilde{A}_j(3/2, s - 1/2)).
\]
(4.48)

A technical point, we started with a connected component \( \tilde{A}_j(1, s) \), but after shrinking, \( \tilde{A}_j(3/2, s - 1/2) \) may have several components. The next lemma says that we only need consider the component which goes “in the direction” of \( S_{\text{inner},j+1} \).

**Lemma 4.16.** \( \tilde{A}_j(3/2, s - 1/2) \) has exactly 1 component \( \tilde{A}_j^0(3/2, s - 1/2) \) in the direction of \( S_{\text{inner},j+1} \), in the sense that any radial geodesic which hits this component, and lasts until \( r = s \), must hit the outer spherical portion \( S_{\text{inner},j+1} \). For this component we have the estimate
\[
Vol(\tilde{A}_j(3/2, s - 1/2) \setminus \tilde{A}_j^0(3/2, s - 1/2)) \leq C \cdot Vol(\tilde{A}_j^0(3/2, s - 1/2)).
\]
(4.49)

**Proof.** From Lemma 4.17 our selection of annuli have the property that \( S_{\text{inner},j} \) has one component, but \( S_{\text{outer},j} \) may have several components. The component in the direction of the next annulus we have labeled \( S_{\text{inner},j+1} \). When we shrink the annulus, we may have many components. If 2 of these components have radial geodesics which extend to \( S_{\text{inner},j+1} \), then in an argument similar to the proof of Lemma 4.17 this would yield another generator of \( H_1(X) \). But since there are only finitely many generators of \( H_1(X) \), this cannot happen for \( R \) sufficiently large.

For the volume estimate, we look more closely at the proof of the Area Comparison Theorem 4.6. In the last step, we can make an improvement, namely instead of
integrating over $D_p(r_1)$, we need only integrate over the directions in $D_p(r_1)$ whose corresponding geodesics do not hit the cut locus before reaching $r_2$, which is exactly $D_p(r_2)$ from the inclusion $D_p(r_2) \subset D_p(r_1)$. This yields the improved estimate

\[
H(r_2) \leq H^3 \left\{ \exp \left( r_1 \cdot D_p(r_2) \right) \right\} \left( \frac{r_2}{r_1} \right)^{3+\epsilon(r_1)}.
\]

(4.50)

Our choice of annulus satisfies $H^3(S_{inner,j+1}) \geq (1 - \eta_j)H^3(S_{inner,j})s^3$, so together with the above, after rescaling we have

\[
(1 - \eta_j)H^3(S_{inner,j}) \leq H^3 \left\{ S_{inner,j} \cap \exp(1 \cdot D_p(s)) \right\} s^\epsilon.
\]

(4.51)

In other words, most of the directions at $S_{inner,j}$, make it to $S_{inner,j} + 1$ before hitting the cut locus.

Above we have shown that only 1 component $\tilde{A}_j \circ (3/2, s - 1/2)$ has geodesics which make it to $S_{inner,j+1}$. Therefore all the radial geodesics in the other components must either hit the cut locus, or hit a different outer boundary component of $S_{outer,j}$. Let us call this set

\[
S_{bad,j} = \{ S_{inner,j} \setminus \exp(1 \cdot D_p(s)) \}.
\]

(4.52)

Therefore the other components are contained in the set

\[
\tilde{A}_{bad,j} = \{ \gamma(t), 1 \leq t \leq s : \gamma(t) \text{ is a radial geodesic with } \gamma(1) \in S_{bad,j} \}.
\]

By the coarea formula and area comparison, similar to (4.41) we have

\[
Vol(\tilde{A}_{bad,j}) \leq CH^3(S_{bad,j})s^4.
\]

(4.54)

Next rewrite (4.51)

\[
H^3(S_{inner,j}) \leq H^3 \left\{ S_{inner,j} \cap \exp(s \cdot D_p(s)) \right\} \frac{s^\epsilon}{1 - \eta_j}
\]

\[
= (H^3(S_{inner,j}) - H^3(S_{bad,j})) \frac{s^\epsilon}{1 - \eta_j}
\]

which implies

\[
H^3(S_{bad,j}) \leq H^3(S_{inner,j}) \left( 1 - \frac{1 - \eta_j}{s^\epsilon} \right).
\]

Substituting into (4.54), we have

\[
Vol(\tilde{A}_{bad,j}) \leq CH^3(S_{inner,j}).
\]

From the coarea formula, arguing as in (4.42), we have

\[
Vol(\tilde{A}_j(3/2, s - 1/2)) \geq C \cdot H^3(S(t_0) \cap \tilde{A}_j(3/2, s - 1/2))(s - 1),
\]

where $t_0$ is the minimal area sphere (when rescaled to unit size) and $3/2 \leq t_0 \leq s - 1/2$.

Also, from comparison

\[
H^3(S_{inner,j+1}) \leq H^3(S(t_0) \cap \tilde{A}_j(3/2, s - 1/2)))s^{3+\epsilon},
\]
so from the previous 2 lemmas, we have

\begin{align*}
\text{Lemma 4.17. We have} \\
\text{(4.55)} & \quad \text{Vol}(\tilde{A}_j(1, s) \setminus \tilde{A}_j^c(3/2, s - 1/2)) \leq 3C \cdot \text{Vol}(\tilde{A}_j^c(3/2, s - 1/2)).
\end{align*}

\begin{proof}
We have the inclusion

\[
\{\tilde{A}_j(1, s) \setminus \tilde{A}_j^c(3/2, s - 1/2)\} \\
\subset \{\tilde{A}_j(3/2, s - 1/2) \setminus \tilde{A}_j^c(3/2, s - 1/2)\} \cup \{\tilde{A}_j(1, s) \setminus \tilde{A}_j(3/2, s - 1/2)\}.
\]

So from the previous 2 lemmas, we have

\[
\text{Vol}(\{\tilde{A}_j(1, s) \setminus \tilde{A}_j^c(3/2, s - 1/2)\}) \\
\leq \text{Vol}(\{\tilde{A}_j^c(3/2, s - 1/2)\}) + \text{Vol}(\{\tilde{A}_j(3/2, s - 1/2)\}) \\
\leq 3C \cdot \text{Vol}(\{\tilde{A}_j^c(3/2, s - 1/2)\}).
\]

For \(\tilde{A}_j^c(3/2, s - 1/2)\), chose a maximal 1/4-separated set, that is, choose points \(p_{j,l} \in \tilde{A}_j^c(3/2, s - 1/2), l = 1 \ldots Q_j\) such that

\[
B(p_{j,l}, 1/8) \cap B(p_{j,l'}, 1/8) = \emptyset, \text{ for } l' \neq l,
\]

and such that

\begin{align*}
\text{(4.56)} & \quad \tilde{A}_j^c(3/2, s - 1/2) \subset \bigcup_{l=1}^{Q_j} B(p_{j,l}, 1/4).
\end{align*}

(For simplicity, we are assuming \(s\) is very large, and we take 1/4-separated sets. In general, we can take the separation to be some small multiple of \(s\).)

We define the set \(\tilde{K}_j = \bigcup_{l=1}^{Q_j} B(p_{j,l}, 1/4)\). From \(\text{(4.56)}\), and Lemma \((4.17)\), we have

\begin{align*}
\text{(4.57)} & \quad \text{Vol}(\tilde{A}_j(1, s) \setminus \tilde{K}_j) \leq 3C \cdot \text{Vol}(\tilde{K}_j).
\end{align*}
For $i$ sufficiently large, from the curvature decay condition (4.33), the curvature on $\tilde{A}_j$ will be arbitrarily small. Using Bishop’s volume comparison theorem, this means there exists a constant $C_2$ so that

$$
(4.58) \quad \text{Vol}(B(x, t)) \leq C_2 t^4,
$$

for any $x \in \tilde{A}_j$, and $t < 1/2$. Using the volume growth assumption (4.1), we therefore have constants $C_1$ and $C_2$ so that (for $r$ sufficiently large)

$$
(4.59) \quad C_1 t^4 \leq \text{Vol}(B(x, t)) \leq C_2 t^4.
$$

This implies that $B(p_{j,t}, 1/4)$ can only hit a uniformly bounded number of neighbors $B(p_{j,t'}, 1/4)$, since if

$$
(4.60) \quad B(p_{j,t}, 1/4) \cap B(p_{j,t'}, 1/4) \neq \emptyset,
$$

then $B(p_{j,t'}, 1/4) \subset B(p_{j,t}, 1/2)$ and the latter must contain a bounded number of $1/4$-balls by (4.59).

So we define graph $G_j$ with vertices $p_{j,t}$, $l = 1 \ldots Q_j$, and $p_{j,t}$ is connected to $p_{j,t'}$ only if $B(p_{j,t}, 1/4) \cap B(p_{j,t'}, 1/4) \neq \emptyset$. By the above observation $G_j$ has a uniformly bounded number of edges at each vertex. Define a distance on the graph $d(p_l, p_{l'}) = \text{minimal number of edges to traverse from } p_l \text{ to } p_{l'}$.

We begin to choose our sets, start with $p_1$, and choose the minimal integer $I$ such that

$$
(4.61) \quad I \ast \text{Vol}(B(p_1, 1/4)) > 10\pi^2(s^4 - 1).
$$

We then consider the union $\tilde{K}_{j,1} = \bigcup B(p_{j,t}, 1/4)$ where the union is taken over all points $p_{j,t}$ such that $d(p_{j,t}, p_{j,l}) \leq I$. That is, we add all points $p_{j,t}$ at a bounded graph distance to $p_{j,1}$ until the union of the $1/4$-balls centered at these points has large enough volume. From the volume bounds (4.59), $\text{Vol}(\tilde{K}_{j,1})$ is uniformly bounded, and (iii) of the Proposition is satisfied.

If there is no point $p_l$ such that $d(p_1, p_l) > 2I$ then we stop. All remaining points are within bounded graph distance, so the remaining portion has comparable volume, that is we must have

$$
(4.62) \quad \text{Vol}(\tilde{K}_j \setminus \tilde{K}_{j,1}) \leq C\text{Vol}(\tilde{K}_{j,1}).
$$

Otherwise, we choose $p_2$ such that $d(p_1, p_2) > 2I$, and repeat the process, our second set will be $\tilde{K}_{j,2} = \bigcup B(p_{j,t}, 1/4)$ where the union is taken over all points $p_{j,t}$ such that $d(p_{j,t}, p_{j,l}) \leq I$.

Next, if there is no point $p_{j,t}$ such that $d(p_{j,1}, p_{j,t}) > 2I$ and $d(p_{j,2}, p_{j,t}) > 2I$, then all the remaining balls are within bounded graph distance of $p_{j,1}$ and $p_{j,2}$. So what is remaining has small relative volume, and we must have the inequality

$$
(4.63) \quad \text{Vol}(\tilde{K}_j \setminus \{\tilde{K}_{j,1} \cup \tilde{K}_{j,2}\}) \leq C\text{Vol}(\{\tilde{K}_{j,1} \cup \tilde{K}_{j,2}\}).
$$

We continue this procedure until it stops, and we are left with disjoint sets

$$
\tilde{K}_{j,1}, \ldots, \tilde{K}_{j,N_j} \subset \tilde{K}_j,
$$
and such that we have
\[(4.64) \quad \text{Vol}(\tilde{K}_j \setminus \{\cup_{l=1}^{N_j} \tilde{K}_{j,l}\}) \leq CVol(\{\cup_{l=1}^{N_j} \tilde{K}_{j,l}\}).\]
We have the inclusion
\[
\{\tilde{A}_j(1, s) \setminus \{\cup_{l=1}^{N_j} \tilde{K}_{j,l}\} \subset \{\tilde{A}_j(1, s) \setminus \tilde{K}_j\} \cup \{\tilde{K}_j \setminus \{\cup_{l=1}^{N_j} \tilde{K}_{j,l}\}\}.
\]
From the inequalities (4.57) and (4.64), this implies
\[(4.65) \quad \text{Vol}(\tilde{A}_j(1, s) \setminus \{\cup_{l=1}^{N_j} \tilde{K}_{j,l}\}) \leq 5C \cdot \text{Vol}(\{\cup_{l=1}^{N_j} \tilde{K}_{j,l}\}),\]
which is (iv). □

**Lemma 4.18.** Assume that
\[(4.66) \quad \frac{a_1 + a_2}{b_1 + b_2} < \delta,
\]
with \(a_1, a_2, b_1, \) and \(b_2\) nonnegative. If \(b_2 < Cb_1\) then
\[(4.67) \quad \frac{a_1}{b_1} < (1 + C)\delta.
\]

**Proof.** Since \(a_2 > 0\), we have \(\frac{a_1}{b_1 + b_2} < \delta\), then \(\frac{a_1}{(1 + C)b_1} < \delta\). □

From Proposition 4.13 there exists a sequence \(\delta_j \to 0\) as \(j \to \infty\) such that
\[
\frac{1}{\text{Vol}(\tilde{A}_j)} \left( \int_{\tilde{A}_j} \left| \Delta (\tilde{\rho}_j) \right|^2 - 8 \left| \tilde{d}V_j + 2E_j \right| \right) < \delta_j.
\]
To simplify notation, let \(\tilde{K}_j = \cup_{l=1}^{N_j} \tilde{K}_{j,l}\). We decompose \(\tilde{A}_j(1, s)\) into disjoint sets
\[
\tilde{A}_j(1, s) = \{\tilde{A}_j(1, s) \setminus \tilde{K}_j\} \cup \tilde{K}_j.
\]
Using inequality (4.65), and applying Lemma 4.18, we have that
\[(4.68) \quad \frac{1}{\text{Vol}(\tilde{K}_j)} \left( \int_{\tilde{K}_j} \left| \Delta (\tilde{\rho}_j) \right|^2 - 8 \left| \tilde{d}V_j + 2E_j \right| \right) < (1 + 5C)\delta_j.
\]

**Proposition 4.19.** For each \(j\), there exists \(l'\) (depending on \(j\)) so that on \(K_{j,l'} \subset \tilde{A}_j\), we have
\[
\int_{K_{j,l'}} \left| \Delta (\tilde{\rho}_j) \right|^2 - 8 \left| \tilde{d}V_j + 2E_{j,l'} \right| \to 0,
\]
as \(j \to \infty\), where \(E_{j,l}\) denotes the error term, but only integrated over \(C \cap  \tilde{K}_{j,l'}\).

**Proof.** We write (4.68) as
\[
\frac{\int_{\tilde{K}_j} \left| \Delta (\tilde{\rho}_j) \right|^2 - 8 \left| \tilde{d}V_j + 2E_j \right|}{\text{Vol}(\tilde{K}_j)} = \frac{\sum_t \left( \int_{\tilde{K}_{j,l}} \left| \Delta (\tilde{\rho}_j) \right|^2 - 8 \left| \tilde{d}V_j + 2E_{j,l} \right| \right)}{\sum_t \text{Vol}(\tilde{K}_{j,l})} < \delta_j,
\]
since the error term is clearly additive under disjoint unions. This implies that
\[
\min_t \frac{\int_{\tilde{K}_{j,l}} \left| \Delta (\tilde{\rho}_j) \right|^2 - 8 \left| \tilde{d}V_j + 2E_{j,l} \right|}{\text{Vol}(\tilde{K}_{j,l})} < \delta'_j
\]
therefore for some $l'$,
\[
\int_{\tilde{K}_{j,l'}} \left| \Delta (\tilde{\rho}_j)^2 - 8 \right| d\tilde{V}_j + 2E_{j,l'} < \delta' \text{Vol}(\tilde{K}_{j,l'}) .
\]
From (iii) of Proposition 4.14, Vol($\tilde{K}_{j,l'}$) is uniformly bounded, and the proposition follows. \hfill \Box

To simplify notation, we will now write $\tilde{K}_{j,l}$ as just $\tilde{K}_j$, and $E_{j,l}$ as $E_j$.

**Lemma 4.20 (Cheeger–Gromov compactness).** Let $\tilde{K}^\circ_j$ denote the interior of $\tilde{K}_j$. Then $(\tilde{K}^\circ_j, g/s^{2j})$ has a subsequence which converges uniformly in the $C^{1,\alpha}$ topology on compact subsets to a flat Riemannian manifold $\tilde{K}^\circ_\infty$.

**Proof.** From the assumption on the curvature decay, on $\tilde{K}_j$ we have $|Rm| \leq \epsilon_1 \to 0$ as $j \to \infty$. The assumption (4.1) implies that there exists a constant $C_1 > 0$ so that
\[
\text{Vol}(B(q, s)) \geq C_1 s^4 ,
\]
for any $q \in \tilde{K}^\circ_j$, with $B(q, s) \subset \tilde{K}^\circ_j$. Together with the curvature bounds, by [CGT82], this implies that there exists a constant $C_2$ so that for any $x \in \tilde{K}^\circ_j$,
\[
\text{inj}(x) \geq C_2 \rho(x),
\]
where inj($x$) denotes the injectivity radius at $x$. Furthermore, the particular way we have chosen the sets (as union of balls), there is no collapsing. More precisely, for any $\epsilon > 0$, consider
\[
\tilde{K}^\epsilon_j = \{ x \in \tilde{K}_j : \text{dist}(x, \partial \tilde{K}_j) > \epsilon \} .
\]
From the definition of $\tilde{K}_j$ as unions of balls, for $\epsilon$ sufficiently small, $\tilde{K}^\epsilon_j$ is nonempty, and also connected. We may then apply a suitable version of the Cheeger-Gromov convergence theorem, see [And89], [Tia90]. \hfill \Box

It follows there exist $C^{1,\alpha}$ diffeomorphisms $\Phi_j : \tilde{K}^\circ_\infty \to \tilde{K}^\circ_j$ such that the metrics $(\Phi_j)^* g_j$ converge in the $C^{1,\alpha}$ topology on compact subsets of $\tilde{K}^\circ_\infty$. By passing to a subsequence, we may assume that the rescaled distance functions $\tilde{\rho}_j = (\rho/s^j) \circ \Phi_j$ converge to a function $\rho_\infty$ in the $C^\alpha$ topology on compact subsets of $\tilde{K}^\circ_\infty$, since the distance function is Lipschitz with Lipschitz constant 1. Proposition 4.19 then implies
\[
\int_{\tilde{K}_\infty} \left| \Delta_j (\tilde{\rho}_j)^2 - 8 \right| d\tilde{V}_j + 2E_{j} \to 0,
\]
as $j \to \infty$, where $\tilde{\rho}_j = (\rho/s^j) \circ \Phi_j$ is the rescaled distance function on $\tilde{K}_j$, and $\Delta_j$ is the Laplacian with respect to $\tilde{g}_j = (\Phi_j)^*(g/s^{2j})$. Similarly, we pull-back the error term $E_j$ to $\tilde{K}_\infty$ under the diffeomorphism $\Phi_j$.

**Proposition 4.21.** On $\tilde{K}^\circ_\infty$, $\rho_\infty$ is a weak solution of the equation
\[
\Delta_\infty (\rho_\infty)^2 = 8 ,
\]
Proof. For a test function \( \phi \in C^\infty_c(\tilde{K}^\infty) \) we have
\[
\int_{\tilde{K}^\infty} \rho^2_\infty(\Delta_\infty \phi) \, dV_\infty = \int_{\tilde{K}^\infty} (\Delta_\infty \phi)(\lim_{j \to \infty} \tilde{\rho}^2_j d\tilde{V}_j) = \lim_{j \to \infty} \int_{\tilde{K}^\infty} \tilde{\rho}^2_j(\Delta_j \phi) \, d\tilde{V}_j,
\]
since \( \tilde{g}_j \to g_\infty \) in \( C^{1,\alpha} \) as \( j \to \infty \). Using the formula (4.31), we integrate by parts:
\[
\int_{\tilde{K}^\infty} \rho^2_\infty(\Delta_\infty \phi) \, dV_\infty = \lim_{j \to \infty} \left( \int_{\tilde{K}^\infty} (\Delta_j \tilde{\rho}^2_j \phi) \, d\tilde{V}_j + 2E_j \right),
\]
where
\[
E_j = \int_{C_j \cap \tilde{K}^\infty} \phi \left( \langle \tilde{\rho}_j(\nabla \tilde{\rho}_j)^+, \partial^+_C \rangle + \langle \tilde{\rho}_j(\nabla \tilde{\rho}_j)^-, \partial^-_C \rangle \right) d\mathcal{H}^3.
\]
From (4.71), we have
\[
\lim_{j \to \infty} \left| \int_{\tilde{K}^\infty} (\Delta_j \tilde{\rho}^2_j - 8) \phi \, d\tilde{V}_j \right| \leq C \sup |\phi| \lim_{j \to \infty} \|\Delta_j \tilde{\rho}^2_j - 8\|_{L^1(\tilde{K}^\infty)} = 0.
\]
Also, the error term above \( E_j \) can be estimated in the same manner by (4.35), so the error term also vanishes in the limit. Consequently, as \( j \to \infty \), we have
\[
\int_{\tilde{K}^\infty} \rho^2_\infty(\Delta_\infty \phi) \, dV_\infty = \int_{\tilde{K}^\infty} 8\phi \, dV_\infty
\]
for any test function \( \phi \in C^\infty_c(\tilde{K}^\infty) \).

In particular, from elliptic regularity, we conclude that \( \rho_\infty \) is smooth in \( \tilde{K}^\infty \), and the \( \rho_\infty \) is a smooth solution of (4.72).

**Proposition 4.22.** For the subsequence \( \{j\} \subset \{i\} \), as \( j \to \infty \),
\[
(4.73) \quad \int_{K_j} \left| \Delta \tilde{\rho}_j - \frac{3}{\tilde{\rho}_j} \right| d\tilde{V}_j + E'_j \to 0,
\]
where
\[
E'_j = \int_{C_j \cap K_j} \left( (\nabla \tilde{\rho}_j)^+, \partial^+_C \right) + (\nabla \tilde{\rho}_j)^-, \partial^-_C ) d\mathcal{H}^3.
\]

**Proof.** At any point where the distance function is smooth we have,
\[
\Delta(\rho^2) = 2\rho \Delta \rho + 2|\nabla \rho|^2,
\]
Since \( |\nabla \rho|^2 = 1 \) almost everywhere, we have that the equation
\[
\Delta \rho^2 = 2\rho \Delta \rho + 2,
\]
holds almost everywhere. We then have
\[
\int_{\tilde{K}_j} |\Delta \tilde{\rho}_j - \frac{3}{\tilde{\rho}_j}| d\tilde{V}_j = \int_{\tilde{K}_j} \frac{1}{2\tilde{\rho}_j} |2\tilde{\rho}_j \Delta \tilde{\rho}_j - 6| d\tilde{V}_j \\
= \int_{\tilde{K}_j} \frac{1}{2\tilde{\rho}_j} |\Delta (\tilde{\rho}_j)^2 - 8| d\tilde{V}_j \\
\leq \frac{1}{2} \int_{\tilde{K}_j} |\Delta (\tilde{\rho}_j)^2 - 8| d\tilde{V}_j,
\]
and the right hand side goes to zero as \( j \to \infty \) by Proposition 4.19. Clearly \( E_j' \) is dominated by \( E_j \) from Proposition 4.19. □

**Proposition 4.23.** On \( \tilde{K}_\infty \), \( \rho_\infty \) is a strong solution of the equation
\[
(4.74) \quad \Delta_\infty \rho_\infty = \frac{3}{\rho_\infty}.
\]

*Proof.* Using Proposition 4.22, the proof is identical to the proof of Proposition 4.21. □

**Proposition 4.24.** On \( \tilde{K}_\infty \), \( \rho_\infty \) is a strong solution of the equation
\[
(4.75) \quad \nabla^2_\infty \rho_\infty = \frac{1}{\rho_\infty} (g_\infty - d\rho_\infty \otimes d\rho_\infty).
\]

*Proof.* By (4.72) and (4.74), we have
\[
8 = \Delta (\rho_\infty)^2 = 2(\rho_\infty \Delta \rho_\infty + |\nabla \rho_\infty|^2) \\
= 2(3 + |\nabla \rho_\infty|^2),
\]
therefore
\[
(4.76) \quad |\nabla \rho_\infty| \equiv 1.
\]

From the Bochner formula, since \( \tilde{K}_\infty \) is flat, we have
\[
0 = \Delta |\nabla \rho_\infty|^2 = (\nabla \rho_\infty, \nabla \Delta \rho_\infty) + |\nabla^2 \rho_\infty|^2 \\
= \left( \nabla \rho_\infty, \frac{3}{\rho_\infty} \right) + |\nabla^2 \rho_\infty|^2 \\
= -\frac{3}{\rho_\infty^2} (\nabla \rho_\infty, \nabla \rho_\infty) + |\nabla^2 \rho_\infty|^2 \\
= -\frac{3}{\rho_\infty^2} + |\nabla^2 \rho_\infty|^2 \\
= \left| \nabla^2 \rho_\infty - \frac{1}{\rho_\infty} (g_\infty - d\rho_\infty \otimes d\rho_\infty) \right|^2,
\]
and we are done.
An alternative proof of this proposition is as follows: from (4.76), it follows that \( \rho_\infty \) is a distance function. Since \( \tilde{K}_\infty \) is flat, the Hessian comparison theorem implies
\[
\nabla^2_\infty \rho_\infty \leq \frac{1}{\rho_\infty} (g^\infty - d\rho_\infty \otimes d\rho_\infty).
\]
Together with (4.72), this implies the proposition. \( \square \)

Since \( \tilde{K}_\infty \) is flat, equation (4.73) and the Gauss equations imply that level sets of \( \rho_\infty \) have constant positive sectional curvature, and therefore are locally isometric to portions of Euclidean spheres. It follows that \( \tilde{K}_\infty \) is contained in a Euclidean annulus (or quotient thereof), \( 1 \leq r \leq s \), and therefore \( \text{Vol}(\tilde{K}_\infty) \leq \frac{\pi^2}{2} (s^4 - 1) \). But from (iv) of Lemma 4.14 we chose \( \tilde{K}_j \) so that \( \text{Vol}(\tilde{K}_j) > \frac{\pi^2}{2} (s^4 - 1) + 1 \). Since the convergence is smooth, we have \( \text{Vol}(\tilde{K}_\infty) \geq \frac{\pi^2}{2} (s^4 - 1) + 1 \), a contradiction. Thus there exists a constant \( C \) so that \( \text{Vol}(\tilde{A}_j(1, s)) \leq C \), for each annulus in the subsequence chosen in Proposition 4.18. Moreover, we have proved that the distance function converges to the Euclidean distance function on this subsequence, so not only is the volume bounded, the area of level sets is bounded. Therefore, for our original subsequence \( s^j \), there exists a constant \( C \) so that for all \( j \) we have \( \mathcal{H}_{A_j}(s^j) \leq C s^{3j} \). We conclude that for all \( i \),
\[
(4.77) \quad \mathcal{H}_{A_i}(s^i) \leq C s^{3i},
\]
(since for \( i \) not in our subsequence, we will have \( \mathcal{H}_{A_i}(s^{i+1}) \leq \mathcal{H}_{A_i}(s^i)s^{3j} \)). Finally, given any \( r \) sufficiently large, choose \( i \) so that
\[
s^i \leq r \leq s^{i+1}.
\]
From Lemma 4.12 and (4.77) and Lemma 4.17 we have
\[
(4.78) \quad \mathcal{H}_{A_i}(r) \leq \mathcal{H}_{A_i}(s^i)s^{3} \leq \mathcal{H}_{A_i}(s^i)s^3(1 + \epsilon) \leq Cs^{3i} \leq Cr^3,
\]
therefore by the coarea formula there exists a constant \( C \) so that for \( s^i \leq r < s^{i+1} \),
\[
(4.79) \quad \text{Vol}(B(p, r) \cap A_i(s^i, s^{i+1})) \leq Cr^4.
\]
An examination of the proof shows that for \( r \) large, we may take the constant \( C \) in (4.79) to be close to the corresponding Euclidean constant, that is
\[
(4.80) \quad \text{Vol}(A_i(s^i, s^{i+1})) \leq (\omega_4 + \alpha_i)(s^{4(i+1)} - s^{4i}),
\]
where \( \alpha_i \to 0 \) as \( i \to \infty \) (by Euclidean constant we mean \( \text{Vol}(B(0, r)) = \omega_4 r^4 = (\pi^2/2)r^4 \) in \( \mathbb{R}^4 \)). This is because for \( j \) in the subsequence, our proof above shows that \( \Delta \tilde{\rho}_j \to (3/\tilde{\rho}_j) \) as \( j \to \infty \), so for any subsequence \( \{j'\} \subset \{j\} \), there will be a subsequence \( \{k\} \subset \{j'\} \) so that \( \tilde{A}_k(1, s) \) will converge to a Euclidean annulus or quotient thereof (note that when repeating our above \( L^1 \)-convergence argument, we do not need to cut as in Proposition 4.14, since we have already proved the estimate (4.79)). We conclude that for \( j \) sufficiently large,
\[
(4.81) \quad \mathcal{H}_{A_j}(s^j) \leq (4\omega_4 + \alpha'_j)s^{3j},
\]
where \( \alpha'_j \to 0 \) as \( j \to \infty \). Recall again that for \( i \) not in the subsequence \( \{ j \} \), we have \( \mathcal{H}_{A'_i}(s_i^{j+1}) \leq \mathcal{H}_{A'_i}(s_i) s^3 \). This fact, inequality \( \text{(4.81)} \), area comparison and the coarea formula then imply \( \text{(4.80)} \).

We have derived a bound for the volume of the original sequence of annular components \( A'_i(s_i^{j}, s_i^{j+1}) \), but we are free to change the starting radius of the dyadic sequence. The above argument therefore shows that we have volume bounds on any annular components of fixed ratio of inner and outer radius, that is for any \( 1 \leq r_0 < s \),

\[
\text{(4.82)} \quad \text{Vol}(A_i(r_0 s^i, r_0 s^{i+1})) \leq (\omega_4 + \alpha_i) \left( (r_0 s)^{4(i+1)} - (r_0 s)^{4i} \right),
\]

where \( \alpha_i \to 0 \) as \( i \to \infty \). Since \( s \) was arbitrary, the above will hold for all \( s \), although the sequence \( \alpha_i \) might depend on \( s \).

We next consider the cone at infinity in the direction of our chosen sequence of annuli. For any \( n > 0 \), and \( t > 0 \), let \( (A^t(1/n, n), \bar{g}) = (A(t/n, nt), g/r^2) \) denote the component of the annulus \( \{ x : t/n < \rho(x) < nt \} \), which touches some annular component in our chosen sequence, with the rescaled metric \( g/r^2 \). From \( \text{(4.2)} \) we have

\[
\text{(4.83)} \quad \sup_{A^t} |\bar{R}m| \leq \epsilon_3(t),
\]

where \( \epsilon_3(t) \to 0 \) as \( t \to \infty \). As in Lemma \( \text{(4.20)} \) it follows that some subsequence \( (A^t_i, \bar{g}^t_i) \) converges in the \( C^{1,\alpha} \) topology on compact subsets to a flat limit space \( (A^\infty(1/n, n), \bar{g}_\infty) \). By taking subsequences, we construct limit spaces with natural inclusions \( A^\infty(1/m, m) \subset A^\infty(1/n, n) \) for \( m < n \). We may then take a subsequence \( n_i \) and obtain in the limit a flat connected Riemannian manifold \( A^\infty(0, \infty) \) with \( A^\infty(1/n, n) \subset A^\infty(0, \infty) \) for any \( n \). Note there is a distance function \( \rho_\infty \) obtained as a limit of the (rescaled) original distance function \( \rho(p, \cdot) \). From \( \text{(4.82)} \), follows the estimate

\[
\text{(4.84)} \quad \text{Vol}(A^\infty(r_1, r_2)) \leq \omega_4 (r_2^4 - r_1^4),
\]

for any \( 0 < r_1 < r_2 \).

We claim that level sets \( S^\infty_\infty(s) \equiv \{ \rho_\infty = s \} \subset A^\infty(0, \infty) \) are connected. First, \( A^\infty(0, \infty) \) must be connected, since we have constructed this tangent cone at infinity by taking sequences of connected components of annuli which touch, that is, these sequences correspond to an end of the manifold. If \( S^\infty_\infty(s) \) was disconnected for \( s \) small, then we would find a connected annulus in \( A^\infty(0, \infty) \) with more than one component in the beginning portion, or a “bad” annulus in the above terminology. This would correspond to an infinite sequence of bad annuli in \( X \), which contradicts the Betti number estimate \( b_1(X) < \infty \).

Since \( S^\infty_\infty(s) \) is connected for \( s \) small, \( \text{(4.84)} \) and \( \text{(4.1)} \) then imply an estimate

\[
\text{(4.85)} \quad \text{diam}\{S^\infty_\infty(s)\} \leq C s.
\]

A standard analysis of the developing map for flat structures (see \( \text{ES87}, \text{Tia90}, \text{And89} \)), implies that we may add a point to \( A^\infty(0, \infty) \) to obtain a complete length space \( A^{\infty}_{\text{finite}} \), and we conclude that \( A^{\infty}_{\text{finite}} \) is isometric to a cone on \( S^3/\Gamma \) where \( \Gamma \subset SO(4) \) is a finite subgroup of isometries of \( S^3 \).
An alternative proof of this step, not relying on the Betti number estimate, is as follows. The estimate (4.85) will hold for each component of $\mathcal{S}_\infty$. Using an analysis of the developing map, and the estimate (4.84), the universal cover of $A_\infty(0, \infty)$ must be $\mathbb{R}^n \setminus \{p_0, \ldots, p_k\}$, with distance function being the distance to this finite set, that is, $\rho_\infty(\cdot) = \min_i d(p_i, \cdot)$ (see [ES87]). This implies that $A_\infty$ is a flat cone $C(S^3/\Gamma)$ with finitely many points $\{p_1, \ldots, p_k\}$ identified to the vertex $p_0$, and distance function $\rho_\infty(\cdot) = \min_i d(p_i, \cdot)$. The volume growth estimate (4.84) then implies that, $\rho_\infty(\cdot) = d(p_0, \cdot)$, therefore $A_\infty$ is isometric to the flat cone $C(S^3/\Gamma)$.

We have shown that every tangent cone, in the direction of our initial sequence of annuli, is isometric $C(S^3/\Gamma)$. It is standard to conclude that $\Gamma$ is unique and to construct an ALE coordinate system of order zero, we refer the reader to [Tia90], [And89] for details.

This implies that our sequence of annuli must eventually meet a Euclidean end of the manifold. Since that the initial sequence of annuli can be chosen arbitrarily, there must be finitely many ends. That is, assuming there are infinitely many ends, then we can choose a sequence of components of annuli which touch, and for which there are always infinitely many ends past any given annular component in the sequence. This is a contradiction, since such a sequence could not lead to a Euclidean end. This finishes the proof of Theorem 4.1.

5. Kato inequalities and Ricci decay

In this section, we discuss some improved Kato inequalities, which we use to improve Ricci curvature decay.

**Lemma 5.1.** If $(X, g)$ satisfies $\delta W^+ = 0$ and has zero scalar curvature, then

\begin{equation}
|\nabla |Ric| |^2 \leq \frac{2}{3} |\nabla Ric| |^2,
\end{equation}

at any point where $|Ric| \neq 0$.

**Remark.** This is rather surprising, because if one assumes moreover that $\delta W = 0$, the best constant is still $2/3$. In an earlier version of this paper, we had an improved Kato inequality in this case, but our constant was not optimal. The authors are indebted to Tom Branson for showing us how the best constant in this case follows easily from his general theory of Kato constants in [Bra00].

**Proof.** We assume familiarity with the paper of Branson [Bra00]. We are in dimension 4, so weight labels have only 2 entries. Since $Ric$ is traceless, $Ric$ is a section of the $(2,0)$ bundle, which we denote by $TFS^2$. The covariant derivative $\nabla Ric$ is then a section of $T^*X \otimes TFS^2$. This decomposes into irreducible pieces under the action of the orthogonal group, and the selection rule targets are $(3,0)$, $(2,1)$, $(2,-1)$, $(1,0)$. These correspond to trace-free symmetric 3-tensors, the self-dual and anti-self-dual parts of Codazzi tensors, and $\Lambda^1$. The assumption is that $Ric$ is in the kernel of the gradients targeted at $(1,0)$ and $(2,1)$.
A calculation shows that
\[ s(2, 1) = s(2, -1) = -\frac{1}{2}, \]
\[ s(3, 0) = -\frac{7}{2}, \quad s(1, 0) = \frac{5}{2}. \]

We next compute some of the \( \tilde{c} \) numbers. Note that \( t(\lambda) = 2 \) for \( \lambda = (2, 0) \); substituting and using the \( s \) numbers above,
\[ \tilde{c}(2, 1) = \tilde{c}(2, -1) = \frac{1}{18}, \]
\[ \tilde{c}(1, 0) = -\frac{1}{18}. \]

The Kato constant is expressed as a minimum over the branching products of \((2, 0)\). These branching products are the \( \text{SO}(3) \)-modules \((2), (1), \text{ and } (0)\), which have dimensions 5, 3, and 1, respectively. The minimum works out as
\[
\min_{\beta=0,1,2} \left( \tilde{c}(2, 1) \left( \tilde{\beta}^2 - \left( \frac{1}{2} \right)^2 \right) + \tilde{c}(1, 0) \left( \tilde{\beta}^2 - \left( \frac{5}{2} \right)^2 \right) \right)
\]
where \( \tilde{\beta} = \beta + \frac{1}{2} \). This is
\[
\min \left( \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right) = \frac{1}{3}.
\]

Therefore the Kato constant is 1/3, which is
\[
|\nabla|\text{Ric}|^2 \leq (1 - \frac{1}{3})|\nabla \text{Ric}|^2 = \frac{2}{3} |\nabla \text{Ric}|^2.
\]

We remark that if one takes the stronger assumption that \( \delta W = 0 \), the one must add in \((2, -1)\) terms above, equal to the \((2, 1)\) terms. The result is
\[
\min \left( \frac{1}{3}, \frac{4}{9}, \frac{2}{3} \right) = \frac{1}{3},
\]
which is still the same Kato constant.

\[ \square \]

Remark. We would like to thank the referees for pointing out that the same constant follows from the methods in [CGH00]. The case considered in Lemma 5.1 is exactly the case \( r = s = 2 \) in the last line of the table on [CGH00 page 253], giving immediately the best constant 2/3.

**Proposition 5.2.** Under the assumptions of Theorem 1.5, in case (a) we have
\[
(5.2) \quad \sup_{S(r)} |\text{Ric}| = O(r^{-\alpha}), \text{ for any } \alpha < 4.
\]

**Proof.** From (3.1), there exists a constant \( C \) so that
\[
|\Delta \text{Ric}| \leq C|Rm||\text{Ric}|.
\]
Noting that
\[
\frac{1}{2} |\Delta \text{Ric}|^2 = \langle \text{Ric}, \Delta \text{Ric} \rangle + |D \text{Ric}|^2 = |\text{Ric}|\Delta |\text{Ric}| + |d|\text{Ric}|^2,
\]
and using the Kato inequality (5.1), we obtain
\[ \langle \text{Ric}, \Delta \text{Ric} \rangle \leq |\text{Ric}| |\Delta \text{Ric}| - \frac{1}{3} |D \text{Ric}|^2 \]
Next we have
\[ -|\text{Ric}| |\Delta \text{Ric}| \leq -|\langle \text{Ric}, \Delta \text{Ric} \rangle| \leq |\langle \text{Ric}, \Delta \text{Ric} \rangle| \leq |\text{Ric}| |\Delta \text{Ric}| - \frac{1}{3} |D \text{Ric}|^2, \]
so that when $|\text{Ric}| > 0$,
\[ (5.3) \quad \Delta |\text{Ric}| \geq -C|\text{Rm}||\text{Ric}| + \frac{1}{3} |D \text{Ric}|^2. \]
We obtain
\[ \Delta |\text{Ric}|^{1/2} = -\frac{1}{4} |\text{Ric}|^{-3/2} |d \text{Ric}|^2 + \frac{1}{2} |\text{Ric}|^{-1/2} \Delta |\text{Ric}| \]
\[ \geq -\frac{1}{6} |\text{Ric}|^{-3/2} |D \text{Ric}|^2 + \frac{1}{2} |\text{Ric}|^{-1/2} \left( -C|\text{Rm}||\text{Ric}| + \frac{1}{3} |D \text{Ric}|^2 \right) \]
\[ = -\frac{1}{2} C|\text{Ric}|^{1/2} |\text{Rm}|. \]
Therefore $|\text{Ric}|$ satisfies weakly the differential inequality
\[ (5.4) \quad \Delta |\text{Ric}|^{1/2} \geq -C|\text{Ric}|^{1/2} |\text{Rm}|. \]
Recall that $D(R) = X \setminus B(p, R)$.

**Lemma 5.3.** There exist constants $C, R$, depending upon $C_S$ so that for $r > R$ we have
\[ (5.5) \quad \left\{ \int_{D(2r)} |\text{Rm}|^4 dV_g \right\}^{1/2} \leq \frac{C}{r^2} \int_{D(r)} |\text{Rm}|^2 dV_g. \]

**Proof.** This is proved as in (5.9) above, except now we choose the cutoff function $\phi$ such that $\phi \equiv 1$ on $D(2r) \setminus D(r')$, $\phi = 0$ on $B(r) \cup D(2r')$, and $|\nabla \phi| < C(r^{-1} + r'^{-1})$, and then let $r' \to \infty$. \[ \Box \]

From Lemma 5.3 and equation (5.4), we may apply [BKN89, Proposition 4.8 (1)] to conclude $|\text{Ric}| = O(r^{-\alpha})$ for any $\alpha < 4$ in the self-dual or anti-self-dual case. \[ \Box \]

Next, we consider the case of harmonic curvature in dimension 4. The following lemma is standard, since harmonic curvature is the Riemannian analogue of Yang-Mills (see [Rad93]).

**Lemma 5.4.** If $(X, g)$ satisfies $\delta \text{Rm} = 0$ and has zero scalar curvature, then
\[ (5.6) \quad |\nabla |\text{Rm}||^2 \leq \frac{2}{3} |\nabla \text{Rm}|^2, \]
at any point where $|\text{Rm}| \neq 0$.\[ \Box \]
Proposition 5.5. Under the assumptions of Theorem 1.3, in case (b) we have
\[ \sup_{s(r)} |Rm| = O(r^{-\alpha}), \text{ for any } \alpha < 4. \]

Proof. A computation similar to above shows we have the improved equation
\[ \Delta |Rm|^{1/2} \geq -C |Rm|^{1/2} |Rm|. \]
Applying the method in Section 4 of [BKN89] we find that
\[ |Rm| = O(r^{-\alpha}) \text{ for any } \alpha < 4. \]

This completes the proof of Theorem 1.3 for the case of harmonic curvature, since by [BKN89, Theorem 1.1], \((X, g)\) is ALE of order \(\tau<2\).

Remark. We remark that a version of Theorem 1.3 in the locally conformally flat case was considered in [CH02] (however, in that work the volume growth estimate (1.12) was insufficiently justified).

6. Asymptotically Locally Euclidean

In this section, we complete the proof of Theorems 1.1 and 1.3. We first have a lemma on lower volume growth

Lemma 6.1. Let \((X, g)\) be complete, noncompact, and assume that the Sobolev inequality (1.3) is satisfied for all \(f \in C^{0,1}_c(X)\). Then there exists a constant \(C_1 > 0\) depending only upon \(C_S\) so that
\[ \text{Vol}(B(q, s)) \geq C_1 s^4, \]
for any \(q \in X\), and all \(s \geq 0\).

Proof. Fix \(r > 0\), and consider the function
\[ f(x) = \begin{cases} d(q, x) - r & d(q, x) \leq r \\ 0 & \text{otherwise}. \end{cases} \]
Note \(f\) is Lipschitz, with \(|\nabla f| = 1\) almost everywhere in \(B(q, r)\), so in particular \(f \in C^{0,1}_c(X)\). From the Sobolev inequality (1.3), we obtain
\[ \frac{r}{2} \text{Vol}(B(q, r/2))^{1/4} \leq \text{Vol}(B(q, r))^{1/2}. \]
The lemma follows by iterating this inequality (see [Heb96, Lemma 3.2]).

Remark. The connection between Sobolev constant and lower volume growth has appeared many times in the literature, see for example [Aku94, Proposition 2.1], [Car96].

Proposition 6.2. Let \((X, g)\) satisfy the assumptions of Theorem 1.1. Then \((X, g)\) has finitely many ends, and each end of \((X, g)\) is ALE of order zero.

Proof. The volume growth condition (1.1) follows from Lemma 6.1. The curvature decay condition (1.2) follows from Theorem 3.10. The proposition then follows from Theorem 4.1.
Proposition 6.2 completes the proof of Theorem 1.1. For Theorem 1.3 we assume furthermore that \((X, g)\) is of type (a), (b), or (c). We begin by listing some of the properties of \((X, g)\) that we have proved: \((X, g)\) is a complete noncompact 4-manifold with finitely many ends with base point \(p\), satisfying

\[
\text{(6.2)} \quad \text{Each end of } X \text{ is ALE of order zero,}
\]

\[
\text{(6.3)} \quad \| Rm \|_{L^2(X)} < \infty,
\]

\[
\text{(6.4)} \quad \sup_{S(r)} |Ric| = O(r^{-(2+\alpha)}), \text{ for any } \alpha < 2,
\]

\[
\text{(6.5)} \quad \sup_{D(2r)} |Rm| \leq C \frac{\int_{D(r)} |Rm|^2 dV_g}{r^2},
\]

\[
\text{(6.6)} \quad \int_{D(r)} |\nabla Ric|^2 dV_g \leq C \frac{r^{2+2\alpha}}{r^{2+2\alpha}}.
\]

Note that (6.4) was proved in Section 5, (6.5) is the \(\epsilon\)-regularity Theorem 3.10. Note that (6.6) is proved as in (3.14), but now taking a cutoff function supported in the complement of a ball (see the proof of Lemma 5.3).

We will show that conditions (6.2)-(6.6) imply that \(|Rm| = O(r^{-(2+\alpha)})\). For simplicity, assume \((X, g)\) has only one end (in general do the same argument for each end). The proof is very similar to the argument given in Section 4 of [Tia90], and that argument was inspired by the proof of Uhlenbeck for Yang-Mills connections in [Uhl82b]. Let \(\tilde{A}\) be the Levi-Civita connection form of the metric \(g\), where the covariant derivative is given by \(\tilde{D} = d + \tilde{A}\). Note that \(\tilde{A} \in C^{1,\alpha}(X, so(4) \times R^4)\). Since \(g\) is ALE of order zero, we transform from Euclidean to spherical coordinates, and any one-form \(A = (A_r, A_\psi)\) splits into radial and spherical parts. The following lemma, which is proved in [Tia90 Lemma 4.1], shows we can choose a good gauge, and is essentially an application of the Implicit Function Theorem.

**Lemma 6.3.** Let \(r\) be sufficiently large. Then there is a gauge transformation \(u\) in \(C^\infty(A(r, 2r), so(4))\) with the property that if \(D = e^{-u} \cdot \tilde{D} \cdot e^u = d + A\), then

\[
\text{(6.7)} \quad D^* A = 0 \text{ on } A(r, 2r)
\]

\[
\text{(6.8)} \quad d^*_\psi A_\psi = 0 \text{ on } \partial A(r, 2r)
\]

\[
\text{(6.9)} \quad \int_{A(r, 2r)} A(\nabla_F r(x)) dV_g = 0,
\]

where \(d^*, d^*_\psi\) are the adjoint operators of the exterior differentials on \(A(r, 2r), \partial A(r, 2r)\) with respect to \(g\), respectively, \(\nabla_F\) denotes the standard gradient, and \(dV_g\) is the volume form with respect to \(g\). Moreover, we have

\[
\text{(6.10)} \quad \sup_{A(r, 2r)} (\|A\|_g(x)) \leq \frac{\epsilon_2(r)}{r},
\]

where \(\epsilon_2(r)\) is a decreasing function of \(r\) with \(\lim_{r \to 0} \epsilon_2(r) = 0\).
The next lemma shows how the connection form decays in the Hodge gauge, and the proof is exactly as in [Tia90, Lemma 4.2].

**Lemma 6.4.** Let \( A \) be the connection form given in Lemma 6.3. For \( r \) sufficiently large, we have

\[
\sup_{A(r,2r)} \| A \|_g(x) \leq Cr \sup_{A(r,2r)} \| R_A \|_g(x) \tag{6.11}
\]

\[
\int_{A(r,2r)} \| A \|_g^2(x) dV_g \leq Cr^2 \int_{A(r,2r)} \| R_A \|_g^2(x) dV_g. \tag{6.12}
\]

The following lemma gives an improvement on the decay of the full curvature tensor, given the Ricci decay. This was proved in [Tia90, Lemma 4.3] for the Einstein case, but with some extra work we prove here that the only improved Ricci decay is necessary.

**Lemma 6.5.** There exists \( \beta > 0 \) such that for \( r \) sufficiently large, we have

\[
\sup_{D(2r)} \| Rm \|_g \leq \frac{C}{r^{2+\beta}}. \tag{6.13}
\]

**Proof.** Choose \( r_0 \) large, and let \( r_i = 2r_{i-1} \). Let \( A_i \) be the connection on \( A(r_{i-1}, r_i) \) from Lemma 6.3. Then

\[
d^* A_i |_{\partial A(r_{i-1}, r_i)} = 0,
\]

\[
d^* A_{(i-1)} |_{\partial A(r_{i-2}, r_{i-1})} = 0,
\]

so the restrictions \( A_{i\psi} \) and \( A_{(i-1)\psi} \) differ by a constant gauge on \( \partial B(r_{i-1}) \), and we may therefore assume that

\[
A_{i\psi}|_{\partial B(r_{i-1})} = A_{(i-1)\psi}|_{\partial B(r_{i-1})}.
\]

Letting \( \Omega_i = A(r_{i-1}, r_i) \), we compute

\[
\int_{\Omega_i} \| R_{A_i} \|_g^2 dV_g = \int_{\Omega_i} \langle dA_i + A_i \wedge A_i, R_{A_i} \rangle_g dV_g = \int_{\Omega_i} \langle D_i A_i - [A_i, A_i], R_{A_i} \rangle_g dV_g
\]

\[
= - \int_{\Omega_i} \langle [A_i, A_i], R_{A_i} \rangle_g dV_g - \int_{\Omega_i} \langle A_i, D^* R_{A_i} \rangle_g dV_g - \int_{S_i} \langle A_{i\psi}, (R_{A_i})_{r_i\psi} \rangle_g d\sigma_g + \int_{S_{i-1}} \langle A_{i\psi}, (R_{A_i})_{r_{i-1}\psi} \rangle_g d\sigma_g,
\]

where \( D_i = d + A_i, S_i = \partial B(r_i) \), and \( d\sigma \) is the induced area form.
Next we sum over \( i \), and since \( (R_{A_i})_{r\psi} = (R_{A_{i+1}})_{r\psi} \) on \( S_i \), we obtain

\[
\int_{D(r_0)} \| Rm_g \|_g^2 dV_g = \sum_{i=1}^{\infty} \int_{\Omega_i} \| Rm_g \|_g^2 dV_g = \sum_{i=1}^{\infty} \int_{\Omega_i} \| R_{A_i} \|_g^2 dV_g = -\sum_{i=1}^{\infty} \int_{\Omega_i} \langle [A_i, A_i], R_{A_i} \rangle_g dV_g + \int_{\partial B(r_0)} \langle A_1\psi, (R_{A_1})_{r\psi} \rangle_g d\sigma_g - \sum_{i=1}^{\infty} \int_{\Omega_i} \langle A_i, D_t^* R_{A_i} \rangle_g dV_g.
\]

Since \( A_i \) is equivalent to \( \tilde{A} \) by a gauge transformation, we have \( D^* R_{A_i} = D^* \tilde{R}_{\tilde{A}} \). Using the inequality \( ab \leq \delta r_i^{-2}a + \delta^{-1}r_i^{-2}b \), assumption (6.6), Lemma 6.4, and the Bianchi identity \( D^* Rm = d^\nabla Ric \), we estimate the last term as

\[
\int_{\Omega_i} \langle A_i, D_t^* R_{\tilde{A}} \rangle_g dV_g \leq \delta r_i^{-2} \int_{\Omega_i} \| A_i \|_g^2 dV_g + C r_i^{-2} \int_{\Omega_i} \| \nabla Ric \|_g^2 dV_g \\
\leq \delta \int_{\Omega_i} \| R_{A_i} \|_g^2 dV_g + C r_i^{-2\alpha}.
\]

Summing this over \( i \), we obtain

\[
\sum_{i=1}^{\infty} \int_{\Omega_i} \langle A_i, D_t^* R_{\tilde{A}} \rangle_g dV_g \leq \delta \int_{D(r_0)} \| Rm_g \|_g^2 dV_g + C \sum_{i=1}^{\infty} r_i^{-2\alpha} \\
\leq \delta \int_{D(r_0)} \| Rm_g \|_g^2 dV_g + C r_0^{-2\alpha}.
\]

Also we have

\[
\left| \sum_{i=1}^{\infty} \int_{\Omega_i} \langle [A_i, A_i], R_{A_i} \rangle_g dV_g \right| \leq \sum_{i=1}^{\infty} C \sup_{x \in \Omega_i} (\| R_{A_i} \|_g(x)) \int_{\Omega_i} \| A_i \|_g^2 dV_g \\
\leq \sum_{i=1}^{\infty} C \epsilon(r_{i-1}) \int_{\Omega_i} \| R_{A_i} \|_g^2 dV_g \\
\leq C \epsilon(r_0) \int_{D(r_0)} \| Rm_g \|_g^2 dV_g.
\]

Combining the above we obtain

\[
(1 - C \epsilon(r_0) - \delta) \int_{D(r_0)} \| Rm_g \|_g^2 dV_g \leq \int_{\partial B(r_0)} \langle A_1\psi, (R_{A_1})_{r\psi} \rangle_g d\sigma_g + C r_0^{-2\alpha}.
\]

We have shown in Section 4 that \( (\partial B(r), 1/g) \) converges uniformly to \( (S^3/\Gamma, g_0) \), a quotient of the unit sphere with the standard metric, in the Gromov-Hausdorff topology. Therefore, by the proof of Corollary 2.6 in [Uhl82b], we may find a decreasing
function $\epsilon'(r)$ of $r$ with $\lim_{r \to 0} \epsilon'(r) = 0$ such that
\begin{equation}
(6.14) \quad \int_{\partial B(r)} \| A_1 \|_g^2 dV_g \leq (2 - \epsilon'(r))^{-2} r^2 \int_{\partial B(r)} \| (F_{A_1})_{\psi\psi} \|_g^2 dV_g.
\end{equation}
Combining the above, and letting $r_0 = r$, we have
\begin{align*}
\int_{\bar{D}(r)} \| Rm_g \|_g^2 dV_g & \leq c(r) \cdot r \left\{ \int_{\partial B(r)} \| (R_{A_1})_{\psi\psi} \|_g^2 dV_g \right\}^{1/2} \cdot \left\{ \int_{\partial B(r)} \| (R_{A_1})_{r\psi} \|_g^2 d\sigma_g \right\}^{1/2} \\
& \leq \frac{c(r)}{2} \cdot r \int_{\partial B(r)} \| Rm_g \|_g^2 dV_g + C r^{-2\alpha},
\end{align*}
where $c(r) = (2 - \epsilon'(r))^{-1} (1 - C \epsilon(r) - \delta)^{-1}$. Therefore for all $r$ sufficiently large, choosing $\delta$ sufficiently small, there exists a small constant $\delta' > 0$
\begin{equation}
(6.15) \quad f(r) \leq c_1 r f'(r) + c_2 r^{-2\alpha},
\end{equation}
where $c_1 = -\frac{1}{4} (1 + \delta')$. Since $\alpha < 2$, we have $0 < 1 + 2c_1 \alpha$, so let
\begin{equation}
(6.16) \quad c_3 = \frac{c_2}{1 + 2c_1 \alpha},
\end{equation}
and consider the function $h(r) = \max\{ f(r) - c_3 r^{-2\alpha}, 0 \}$. When $h > 0$ we have
\begin{align*}
c_1 r h'(r) = c_1 r (f'(r) + 2 \alpha c_3 r^{-2\alpha - 1}) & \geq f(r) - c_2 r^{-2\alpha} + 2c_1 c_3 r^{-2\alpha} \\
& = f(r) - C r^{-2\alpha},
\end{align*}
and therefore $h(r)$ satisfies the inequality
\begin{equation}
(6.17) \quad h(r) \leq c_1 r h'(r).
\end{equation}
Integrating this inequality, we obtain $h(r) \leq h(1) \cdot r^{1/c_1}$, and therefore $f(r) \leq c_3 r^{-2\beta}$, where $2\beta = \max\{-2\alpha, 1/c_1\}$. The pointwise decay (6.13) follows from (6.17).

To finish the proof, Lemma 6.5 and Theorem 3.10 give improved pointwise decay on the full curvature tensor and its covariant derivatives, and the ALE property then follows by the result in [BKN89, Theorem 1.1].

7. Constraints

For an ALE space $X$ with several ends, we have the signature formula
\begin{equation}
(7.1) \quad \tau(X) = \frac{1}{12\pi^2} \left( \int_X |W_g^+|^2 dV_g - \int_X |W_g^-|^2 dV_g \right) - \sum_i \eta(S^i / \Gamma_i),
\end{equation}
where $\Gamma_i \subset SO(4)$ is the group corresponding to the $i$th end, and $\eta(S^3/\Gamma_i)$ is the $\eta$-invariant. The Gauss-Bonnet formula in this context is

\begin{equation}
\chi(X) = \frac{1}{8\pi^2} \left( \int_X |W_g|^2 dV_g - \frac{1}{2} \int_X |Ric_g|^2 dV_g + \frac{1}{6} \int_X R_g^2 dV_g \right) + \sum_i \frac{1}{|\Gamma_i|}.
\end{equation}

(7.2)

See [Hit97] for a nice discussion of these formulas. We remark that these put constraints on the ends of the ALE spaces that can arise in Theorem 1.3. For example, if $X$ is locally conformally flat and scalar flat, then

\begin{equation}
\tau(X) = -\sum_i \eta(S^3/\Gamma_i),
\end{equation}

(7.3)

\begin{equation}
\chi(X) = -\frac{1}{16\pi^2} \int_X |Ric_g|^2 dV_g + \sum_i \frac{1}{|\Gamma_i|}.
\end{equation}

(7.4)

Equation (7.3) says that the sum of the $\eta$-invariants must be integral, and (7.4) gives the inequality,

\begin{equation}
\chi(X) \leq \sum_i \frac{1}{|\Gamma_i|} \leq \# \{\text{ends of } X\}.
\end{equation}

(7.5)

If we assume that $X$ is scalar-flat and anti-self-dual we obtain

\begin{equation}
3\tau(X) = -\frac{1}{4\pi^2} \int_X |W_g|^2 dV_g - 3 \sum_i \eta(S^3/\Gamma_i),
\end{equation}

(7.6)

\begin{equation}
2\chi(X) = \frac{1}{4\pi^2} \left( \int_X |W_g|^2 dV_g - \frac{1}{2} \int_X |Ric_g|^2 dV_g \right) + 2 \sum_i \frac{1}{|\Gamma_i|}.
\end{equation}

(7.7)

Adding these together,

\begin{equation}
2\chi(X) + 3\tau(X) = -\frac{1}{8\pi^2} \int_X |Ric_g|^2 dV_g + \sum_i \left( \frac{2}{|\Gamma_i|} - 3\eta(S^3/\Gamma_i) \right),
\end{equation}

(7.8)

which implies the inequality

\begin{equation}
2\chi(X) + 3\tau(X) \leq \sum_i \left( \frac{2}{|\Gamma_i|} - 3\eta(S^3/\Gamma_i) \right),
\end{equation}

(7.9)

with equality if and only if $g$ is Ricci-flat.

We conclude this section with a pinching theorem.

**Theorem 7.1.** Let $(M_1, g)$ be a complete, oriented, noncompact 4-dimensional Riemannian manifold with $g$ scalar-flat and anti-self-dual. Assume that

\begin{equation}
C_S < \infty, \text{ and } b_1(M) = 0.
\end{equation}

(7.10)

Then there exists an $\epsilon_1$ depending upon $C_S$ such that if $\|Rm\|_{L^2} < \epsilon_1$ then $(M_1, g)$ is isometric to $\mathbb{R}^4$. 
Proof. From [Car99, Proposition 1]), there exists $\epsilon_1 > 0$ (depending on the Sobolev constant), so that if $\|Rm\|_{L^2} < \epsilon_1$, then the first $L^2$ Betti number vanishes, which implies that $(M_1, g)$ has only one end. From Theorem [1.3] this end is ALE of order $\tau < 2$. Poincare duality gives $b_3(M_1) = b_1(M_1)$, $b_4(M_1) = 0$, and the Gauss-Bonnet formula is

$$1 + b_2(M_1) - \frac{1}{|\Gamma|} = \frac{1}{8\pi^2} \int_{M_1} \left( -\frac{1}{2} |Ric|^2 + |W|^2 \right) dV_g. \tag{7.11}$$

If the right hand side is sufficiently small, we conclude that $\tilde{\Gamma} = \{e\}$, and $b_2(M_1) = 0$. Since $(M_1, g)$ has exactly one asymptotically Euclidean end and since $(M_1, g)$ is half-conformally flat, from the signature formula we conclude that $(M_1, g)$ is conformally flat. From (7.11), $(M_1, g)$ is therefore also Ricci-flat. By Bishop’s relative volume comparison theorem [BC64], $(M_1, g)$ is isometric to Euclidean space.

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