On the correspondence between a screw dislocation in gradient elasticity and a regularized vortex

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Abstract

We show the correspondence between a screw dislocation in gradient elasticity and a regularized vortex. The effective Burgers vector, nonsingular distortion and stress fields of a screw dislocation and the effective circulation, smoothed velocity and momentum of a vortex are given and discussed.

Keywords: Dislocation; Vortex; Gradient theory

1 Introduction

The close relationship between vortices in uid mechanics and dislocations in elasticity theory has been known for many years (see, e.g., [1, 2]). Both the dislocations and the vortices have singularities at the dislocation line and vortex line, respectively. Nowadays it is very popular to regularize the stress and/or strain fields of a dislocation by means of nonlocal elasticity [3] and gradient elasticity [4,5]. In such a theory an additional parameter called gradient coefficient is introduced. On the other hand, in uid mechanics a so-called 'alpha model' is used to smooth out the velocity of a vortex and an arbitrary parameter appears [11,13].

The main purpose of the present letter will be to give a one to one correspondence between the uid quantities of a nonsingular screw dislocation in gradient elasticity and a smoothed vortex in uid mechanics. Some similarities between gradient elasticity and the so-called 'alpha model' will be discussed. For instance, we will see that the parameter is a kind of a gradient coefficient in such a model.

2 Screw dislocation in gradient elasticity

Dislocations are fundamental line-like defects in solids. We consider a single screw dislocation contained within an infinitely extended solid body. Its dislocation line and Burgers vector...
are directed along the z-direction. The Burgers vector is the measure of strength (topological charge) of a dislocation and may be expressed in elasticity as follows

\[ b_i = \int_{\Omega} \varepsilon \, dx_j; \]  

(1)

where \( \varepsilon \) is the elastic distortion tensor in elasticity. In cylindrical coordinates, the elastic distortion of a single screw dislocation is given by [14]

\[ \varepsilon = \frac{b}{2\pi} \frac{1}{r}; \]  

(2)

where \( r^2 = x^2 + y^2 \). Therefore, we obtain \( b_z = b \). In gradient elasticity, one may define a smoothed distortion tensor \( \varepsilon \) which has to satisfy the following inhomogeneous Helmholtz equation

\[ \varepsilon = \int_{\Omega} \varepsilon \, dx_j; \]  

(3)

where the right hand side is given in terms of \( \varepsilon \). It is obvious that Eq. (3) may be rewritten in the form of an integral relation as a convolution integral

\[ \varepsilon = \int_{\Omega} G(r, r') \varepsilon(r') \, dV; \]  

(4)

where the two-dimensional Green function is given by

\[ G(r) = \frac{2}{2\pi} K_0(r); \]  

(5)

Here \( K_n \) denotes the modified Bessel function of the second kind and of order \( n \). In Eq. (5) a parameter with the dimension of an inverse length is introduced. Eventually, one may define a characteristic length scale \( \lambda = \frac{1}{r} \).

Substituting (2) into Eq. (3), we find for the distortion in gradient elasticity [12,14,15]

\[ \varepsilon = \frac{b}{2\pi} \frac{1}{r} \int_{\Omega} G(r, r') \varepsilon(r') \, dV; \]  

(6)

This distortion is smoothed within the dislocation core and decays like \( r^{-1} \) for large \( r \). It is zero at \( r = 0 \) and has a maximum of \( \frac{b}{2\pi} \cdot 0.399 \) at \( r' = 1 \). The distortion (6) satisfies the condition, \( \varepsilon_i = 0 \).

Using this smoothed distortion, the effective Burgers vector is calculated as

\[ b_z(r) = \int_{\Omega} G(r, r') \varepsilon(r') \, dV; \]  

(7)

It differs from the constant value \( b \) in the region from \( r = 0 \) up to \( r' = 6 \). In fact, we find \( b_z(0) = 0 \) and \( b_z(1) = b \).

The dislocation density tensor is defined by

\[ \varepsilon_{ij} = \frac{1}{2\pi} \int_{\Omega} \varepsilon \, dx_k \varepsilon_{ik}; \]  

(8)

and fulfills the Bianchi identity for the torsion

\[ \varepsilon_i = 0; \]  

(9)
which means that dislocations cannot end inside the solid body. With (6) and (8), the dislocation density of a single screw dislocation is obtained as

$$\chi = \frac{b^2}{2} K_0(\gamma)$$

(10)

In the limit as $$\gamma \rightarrow 0$$, Eq. (10) converts to the classical dislocation density

$$\chi = \frac{b}{2}(\gamma)$$

By means of the smoothened distortion and the Hooke law

$$\sigma_{ij} = 2\frac{\epsilon_{ij} + \epsilon_{ij}}{r}$$

(11)

where and are the Lamé constants, the smoothened stress tensor of the single screw dislocation has the following form

$$\chi = \frac{b}{2} r K_0(\gamma)$$

(12)

which does not possess a singularity.

3 Regularized vortex

Vortices are fundamental line-like excitations of a uid. We consider a vortex in an infinite extended uid for which the conditions of static current and of incompressibility are fulfilled

$$\Theta_{ij} = (\Theta_{ij})\chi = 0; \quad \text{with} \quad \Theta_{ij}\chi = 0$$

(13)

Here \(\chi\) denotes the uid velocity and

$$\chi = \frac{b}{r} \frac{1}{r^2} K_0(\gamma)$$

(14)

which is infinite at \(r = 0\). Here, \(\chi\) denotes the circulation which is a measure of strength of the singular vortex and

$$\chi = \int \frac{1}{r} \chi(\gamma)$$

(15)

On the other hand, one may define a locally smoothened velocity \(\chi\) by the following inhomogeneous Helmholtz equation (see, e.g., [12,13])

$$\chi = \frac{b}{2} \frac{1}{r^2} K_0(\gamma)$$

(16)

and the corresponding integral relation

$$\chi(\gamma) = \int G(\gamma, \chi) \frac{1}{r^2} \chi(\gamma) dV(\gamma);$$

(17)

which is introduced in the \"alpha-model\". It is interesting to note that Eqs. (15) and (17) are similar in form as Eqs. (9) and (10), respectively. For two-dimensional problems \(G\) is given by

$$G(\gamma) = \frac{1}{2} \frac{1}{r} K_0(\gamma)$$

(18)

and the three-dimensional \(G\) green function reads

$$G(\gamma) = \frac{1}{4} \frac{1}{r} \exp(\gamma)$$

(19)
Figure 1: Smoothed velocity $u_r$ of a vortex (solid curve) is given in units of $\mu$. The dashed curve represents the classical solution.

In our case the problem has cylindrical symmetry around the vortex line and, thus, it is two-dimensional. Using (14) with (12), the smoothed velocity for a single vortex is found as

$$u_r = \sum_{n=1}^{\infty} \frac{1}{n} \frac{1}{r} r = K_1 r = 0.$$  \hspace{1cm} (20)

This smoothed velocity is displayed graphically in Fig. 1. It has no singularity. In fact, it is zero at $r = 0$ and the maximum value of $u_r^{\text{max}} \approx 0.399$ occurs at $r' = 1.14$. Thus, the position and the value of the maximum velocity depend on the parameter. On the other hand, these quantities should be measured in an experiment or in a related simulation. Therefore, in this way one can determine the parameter for such a single vortex.

Using Eq. (20), the effective circulation may be calculated

$$I = \int u_r r d' = \sum_{n=1}^{\infty} \frac{1}{n} 1 1 r = K_1 r = 0.$$  \hspace{1cm} (21)

The effective circulation is plotted in Fig. 2. One can see that it differs from the constant value in the region from $r = 0$ up to $r' = 6$. It has $I(0) = 0$ and $I(1) \approx 0$. Thus, one might take $r_c' \approx 6$ as the core radius of the vortex. Again, the parameter can be determined by the measurement of the profile for the effective circulation.

In analogy to the dislocation tensor (13), one may define a smoothed vorticity or vortex density as follows

$$\zeta_z = \frac{1}{2} \sum_{j,k} \epsilon_{ijk} \epsilon_{j}.$$  \hspace{1cm} (22)

and it fulfills the Bianchi identity in uid mechanics

$$\Phi_1 \zeta_z = 0.$$  \hspace{1cm} (23)

Thus, vortices cannot end inside the uid. With (20) the vortex density is given by

$$\zeta_z = \frac{1}{2} \sum_{j,k} \epsilon_{ijk} \epsilon_{j} K_0 r = 0.$$  \hspace{1cm} (24)

which is smeared out around the vortex line. It is interesting to note that in the limit as $r \to 0$, $\zeta_z$ approaches $\zeta$ which is given by Eq. (25).
Eventually, the smoothed momentum vector is defined as
\[ \mathbf{p}_i = \%u_i; \]  
where \% denotes the density. For the single vortex we find the smoothed momentum
\[ \mathbf{p} = \% \frac{k}{2} \frac{1}{r} \quad r = K_1 r^o. \]  
(26)

It does not possess any singularity.

4 Summary

We have shown a one to one relationship between a screw dislocation in gradient elasticity and a smoothed vortex. A review of the correspondence between a regularized vortex in uid mechanics and a screw dislocation in gradient elasticity is given in Table 1. One mathematical

| Vortex                          | Screw Dislocation |
|---------------------------------|-------------------|
| Smoothered velocity             | \( u_z \)         |
| Effective circulation \((r)\)   | \( \varepsilon(r) \) |
| Smoothered circulation \( t_z \) | \( b_z(r) \)      |
| Density \( \% \)                | \( \% \) shear modulus |
| Smoothered momentum \( p^s \)   | \( \varepsilon_z \) |
| Characteristic length scale     | Characteristic length scale |

Table 1: The correspondence between a vortex and a screw dislocation

distinction, however, is that the characteristic uid quantities are vectors or scalars and their counterparts in solids are second rank tensors or vectors. We have discussed the similarities...
between gradient elasticity and the basic equations in the \textit{alpha-model}. It turns out that the \textit{alpha-model} may be considered as a gradient or nonlocal theory for the vortex in uid mechanics. The \(-\) parameter is the gradient parameter in the model. In addition, we have given some hints to determine the \(-\) parameter for a vortex in an experiment or simulation.

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