Universal formula for the energy–momentum tensor via a flow equation in the Gross–Neveu model

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For the fermion field in the two-dimensional Gross–Neveu model, we introduce a flow equation that allows a simple $1/N$ expansion. By employing the $1/N$ expansion, we examine the validity of a universal formula for the energy–momentum tensor which is based on the small flow-time expansion. We confirm that the formula reproduces a correct normalization and the conservation law of the energy–momentum tensor by computing the translation Ward–Takahashi relation in the leading non-trivial order in the $1/N$ expansion. Also, we confirm that the expectation value at finite temperature correctly reproduces thermodynamic quantities. These observations support the validity of a similar construction of the energy–momentum tensor via the gradient/Wilson flow in lattice gauge theory.
1. Introduction

It has been well recognized \cite{1, 2} that the construction of the energy–momentum tensor—the Noether current associated with the translational invariance—is quite involved in lattice field theory. This is because lattice regularization explicitly breaks the translational invariance and the energy–momentum tensor is a composite operator containing local products of field variables. Because of radiative corrections, a naive discretization as it stands cannot reproduce a correct normalization and the conservation law of the energy–momentum tensor in the continuum limit. Recently, a completely new approach to this problem, on the basis of the gradient/Wilson flow \cite{3–5} and the small flow-time expansion \cite{6}, has been proposed \cite{7–9} in the context of lattice gauge theory. In that approach, especially that in Refs. \cite{7, 9}, one constructs a “universal formula” for the energy–momentum tensor using a perturbative solution of the gradient flow. This construction relies on the UV finiteness of the gradient flow in gauge theory \cite{4, 6} such that renormalization of any composite operator of flowed fields is very simple. The universal formula is supposed to provide a regularization-independent expression for the energy–momentum tensor and thus is expected to be usable even with lattice regularization.

The above approach is based on natural assumptions such as the existence of the energy–momentum tensor and the renormalizability of the gradient flow in the non-perturbative level. Also, the formula in Ref. \cite{7} has been numerically tested for quenched QCD at finite temperature \cite{10, 11}. However, it still remains important to investigate the validity of the approach in various possible ways. In particular, it is of great interest whether and how the universal formulas in Refs. \cite{7, 9}, which are constructed by using perturbation theory, can capture non-perturbative low-energy physics or not.

As shown in Ref. \cite{12}, the gradient flow in the two-dimensional $O(N)$ non-linear sigma model \cite{13} possesses a UV finiteness quite similar to that of four-dimensional gauge theory. By utilizing this UV finiteness, one can imitate the above construction of the universal formula \cite{12}. For the two-dimensional $O(N)$ non-linear sigma model, the $1/N$ expansion is available and, to some extent, the gradient flow can also be solved in the large $N$ limit \cite{14, 15}.

In Ref. \cite{14}, using this non-perturbative solution, the universal formula for the energy–momentum tensor has been analytically tested by computing the expectation value at finite temperature. The expectation value correctly reproduces thermodynamic quantities obtained by the conventional $1/N$ expansion. This study demonstrates that the universal formula reproduces a correct normalization at least for those quantities.

Another interesting issue is whether the conservation law (and more general Ward–Takahashi relations associated with the translational invariance) is correctly reproduced by the universal formula. This analysis for the two-dimensional $O(N)$ non-linear sigma model has not been carried out, because in Ref. \cite{14} the gradient flow was solved only in the leading order in the $1/N$ expansion with which any correlation function is factorized into one-point functions.

In the present paper, with the above motivations, we consider a similar universal formula for the energy–momentum tensor in the two-dimensional Gross–Neveu model \cite{16}. The point is that, in this non-gauge, unconstrained system, one can introduce a very simple flow

\footnote{It might be possible to use the large $N$ solution given in Ref. \cite{13} to investigate this issue.}
equation that does not contain any interaction. Although this is not the gradient flow in the sense that the flow is defined with respect to the equation of motion of the original system, such a choice is perfectly legitimate from the perspective of the UV finiteness of the flow. Similar simplification has also been adopted for the flow of the fermion field in gauge theory [4]. Because of this simplification in the flow equation, the conventional $1/N$ expansion [16, 17] directly provides the solution of the flowed fields. We can then readily examine, in the leading non-trivial order in the $1/N$ expansion, if the universal formula correctly reproduces the translation Ward–Takahashi relation.

This paper is organized as follows. In Sect. 2, we introduce a flow equation for the fermion field in the Gross–Neveu model. In Sect. 3, along the line of reasoning in Refs. [7, 9, 12], we construct a universal formula for the energy–momentum tensor in the present system. This construction itself is based on one-loop matching with the expression with dimensional regularization. In Sect. 4, we recapitulate the conventional $1/N$ expansion of the present system. Section 5 is the main part of the paper and, in the leading non-trivial order in the $1/N$ expansion, we examine if the universal formula correctly reproduces (some particular cases of) the translation Ward–Takahashi relation. Here, we observe that the universal formula precisely reproduces expected relations with the presence of the non-perturbative mass gap, although the construction of the universal formula itself uses one-loop perturbation theory. As another support for the universal formula, in Sect. 6, we compute the expectation value of the energy–momentum tensor defined by the universal formula at finite temperature as Ref. [14]. It reproduces the correct results. The last section is devoted to conclusions.

2. Flow equation in the Gross–Neveu model

The Euclidean action of the Gross–Neveu model [16] is given by
\begin{equation}
S = \int d^Dx \left\{ \bar{\psi}^i(x) \partial^\mu \psi^i(x) - \frac{\lambda_0}{2N} \bar{\psi}^i(x) \psi^i(x) \right\},
\end{equation}
where $D = 2$ for our target theory and the fermion field has $N$ components ($i = 1, 2, \ldots, N$). In this system, we introduce a flow equation. That is, we introduce a fictitious time $t$ and suppose that the fermion field evolves according to
\begin{align}
\partial_t \chi(t, x) &= \partial_\mu \partial_\mu \chi(x), \quad \chi(t = 0, x) = \psi(x), \\
\partial_t \bar{\chi}(t, x) &= \partial_\mu \partial_\mu \bar{\chi}(x), \quad \bar{\chi}(t = 0, x) = \bar{\psi}(x),
\end{align}
where the initial value for the evolution is given by the original fermion field which is the subject of the functional integral (with the distribution defined by Eq. (2.1)). Note that Eqs. (2.2) and (2.3) are very simple; the right-hand sides are defined by the free Laplacian without any interaction. Although the above flow is not the gradient flow in the sense that the flow is defined by the equation of motion for the original action (2.1), such a choice is completely legitimate as far as a UV finiteness of the flow is concerned—see the following discussions.

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2 The summation over repeated “flavor” indices $i, j, \ldots$, and Lorentz indices $\mu, \nu, \ldots$, is always understood in this paper.
Since flow equations (2.2) and (2.3) do not contain any interaction, the flowed fermion field becomes a simple linear functional of the fermion field at the zero flow time. That is,
\[ \chi(t, x) = \int d^D y K_t(x - y) \psi(y), \quad \bar{\chi}(t, x) = \int d^D y K_t(x - y) \bar{\psi}(y), \] (2.4)
where
\[ K_t(x) = \int_p e^{ipx} e^{-tp^2} = \frac{e^{-x^2/4t}}{(4\pi t)^{D/2}} \] (2.6)
is the heat kernel for the free Laplacian. Using Eq. (2.4), correlation functions of the flowed fermion field can directly be obtained in terms of correlation functions of the original fermion field. For example, since the tree-level propagator of the original fermion field is given by
\[ \langle \psi^i(x) \bar{\psi}^j(y) \rangle_0 = \delta^{ij} \int_p \frac{1}{i\hat{p}}, \] (2.7)
the tree-level propagator of the flowed field is
\[ \langle \chi^i(t, x) \bar{\chi}^j(s, y) \rangle_0 = \delta^{ij} \int_p e^{i\hat{p}(x-y)} \frac{e^{-(t+s)p^2}}{i\hat{p}}. \] (2.8)
Also, the renormalization property of the unflowed fermion field is directly inherited by the flowed fermion field. In particular, their wave function renormalization constants are identical. This is quite different from the flowed fermion field in gauge theory \[4\] in which the wave function renormalization constant for the flowed fermion field is independent of that of the original fermion field, due to interaction in the flow equation.

3. Universal formula for the energy–momentum tensor

In this section, following the idea of Refs. \[7, 9, 12\], we construct a universal formula for the energy–momentum tensor in the Gross–Neveu model (2.1) using the small flow-time expansion \[6\]. We first assume dimensional regularization with \( D = 2 - \epsilon \) and derive the explicit form of the energy–momentum tensor. Since dimensional regularization preserves the translational invariance, that energy–momentum tensor fulfills the Ward–Takahashi relation associated with the translational invariance; this implies that the energy–momentum tensor is correctly normalized and is conserved. However, the energy–momentum tensor with dimensional regularization is useful only in perturbation theory. Our universal formula below is intended to provide a regularization-independent expression for the energy–momentum tensor. This universal formula is thus also expected to be usable with lattice regularization for example, with which non-perturbative calculations are possible.

\[ \int_p \equiv \int d^D p \left( \frac{(2\pi)^D}{2\pi} \right). \] (2.5)
Assuming dimensional regularization, the energy–momentum tensor can be obtained from the variation of the action

$$\delta S = -\int d^D x \xi_{\nu} \partial_{\mu} T_{\mu\nu}(x)$$

under the transformations

$$\delta \psi(x) = \xi_{\mu}(x) \partial_{\mu} \psi(x), \quad \delta \bar{\psi}(x) = \xi_{\mu}(x) \partial_{\mu} \bar{\psi}(x).$$

The explicit form is given by

$$T_{\mu\nu}(x) = \frac{1}{4} \bar{\psi}^i(x) \left( \gamma_{\mu} \overset{\leftrightarrow}{\partial}_{\nu} + \gamma_{\nu} \overset{\leftrightarrow}{\partial}_{\mu} \right) \psi^i(x) - \delta_{\mu\nu} \left\{ \bar{\psi}^i(x) \frac{1}{2} \gamma^{\nu} \psi^i(x) - \frac{\lambda_0}{2N} \left[ \bar{\psi}^i(x) \psi^i(x) \right]^2 \right\},$$

where $\overset{\leftrightarrow}{\partial}_{\mu} \equiv \partial_{\mu} - \overleftarrow{\partial}_{\mu}$. This operator does not receive the multiplicative renormalization, because of the translation Ward–Takahashi relation

$$\langle \mathcal{O}_{\text{ext}} \int_{\mathcal{D}} d^D x \partial_{\mu} T_{\mu\nu}(x) \mathcal{O}_{\text{int}} \rangle = - \langle \mathcal{O}_{\text{ext}} \partial_{\nu} \mathcal{O}_{\text{int}} \rangle,$$

where $\mathcal{D}$ is a bounded integration region, $\mathcal{O}_{\text{ext}}$ is an operator outside the region and $\mathcal{O}_{\text{int}}$ is an operator inside the region. We define a renormalized energy–momentum tensor by subtracting the (potentially UV-divergent) vacuum expectation value as

$$\{T_{\mu\nu}\}_R(x) \equiv T_{\mu\nu}(x) - \langle T_{\mu\nu}(x) \rangle.$$

Now, to derive the universal formula, we express the composite operator (3.3) in terms of the composite operator of the flowed fermion field. This can be archived by the so-called small flow-time expansion in Ref. [6]. By a one-loop perturbative calculation similar to that of Refs. [7, 9, 12], we find

$$\bar{\chi}^i(t, x) \left( \gamma_{\mu} \overset{\leftrightarrow}{\partial}_{\nu} + \gamma_{\nu} \overset{\leftrightarrow}{\partial}_{\mu} \right) \chi^i(t, x) - \left\langle \bar{\chi}^i(t, x) \left( \gamma_{\mu} \overset{\leftrightarrow}{\partial}_{\nu} + \gamma_{\nu} \overset{\leftrightarrow}{\partial}_{\mu} \right) \chi^i(t, x) \right\rangle = \bar{\psi}^i(x) \left( \gamma_{\mu} \overset{\leftrightarrow}{\partial}_{\nu} + \gamma_{\nu} \overset{\leftrightarrow}{\partial}_{\mu} \right) \psi^i(x) - \frac{\lambda_0 \lambda_0}{N \pi} \left[ \frac{2}{\epsilon} + \ln(8\pi t) + 1 \right] \delta_{\mu\nu} \left[ \bar{\psi}^i(x) \psi^i(x) \right]^2 + O(t)$$

and

$$\left[ \bar{\chi}^i(t, x) \chi^i(t, x) \right]^2 - \left\langle \left[ \bar{\chi}^i(t, x) \chi^i(t, x) \right]^2 \right\rangle$$

$$= \left\{ 1 - \frac{\lambda_0}{\pi} \left[ \frac{2}{\epsilon} + \ln(8\pi t) \right] \right\} \left[ \bar{\psi}^i(x) \psi^i(x) \right]^2 + O(t).$$

In this and following one-loop computations, we retain only terms leading in the large $N$ limit, because only leading terms are relevant in the analyses in the following sections.

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4 Here, we have taken only the part of the expression appearing in Eq. (3.1) being symmetric under $\mu \leftrightarrow \nu$; the anti-symmetric part generates the Lorentz transformation and is not explicitly considered in what follows.
From Eq. (3.6), we also have
\[
\bar{\chi}^i(t, x) \overset{\to}{\partial} \chi^i(t, x) - \langle \bar{\chi}^i(t, x) \overset{\to}{\partial} \chi^i(t, x) \rangle = \bar{\psi}^i(x) \overset{\to}{\partial} \psi^i(x) - \lambda_0 \frac{2}{N} \left[ \frac{2}{\pi} + \ln(8\pi t) \right] [\bar{\psi}^i(x)\psi^i(x)]^2 + O(t).
\] (3.8)

The relations (3.6), (3.7), and (3.8) may be inverted for composite operators of the unflowed fermion field. We then substitute those expressions in Eq. (3.3) to yield
\[
\{T_{\mu\nu}\}_R(x) = \frac{1}{4} \bar{\chi}^i(t, x) \left( \gamma_\mu \overset{\to}{\partial}_\nu + \gamma_\nu \overset{\to}{\partial}_\mu \right) \chi^i(t, x) - \frac{1}{2} \delta_{\mu\nu} \bar{\chi}^i(t, x) \overset{\to}{\partial} \chi^i(t, x) + \frac{\lambda_0}{2N} \left[ 1 + \lambda_0 \frac{2}{\pi} \left[ \frac{2}{\pi} + \ln(8\pi t) + 1 \right] \right] \delta_{\mu\nu} [\bar{\chi}^i(t, x)\chi^i(t, x)]^2 - VEV
+ O(t),
\] (3.9)

where VEV denotes the vacuum expectation value of the composite operator appearing in the right-hand side. In the one-loop order, the coupling constant is renormalized in the minimal subtraction (MS) scheme as
\[
\lambda_0 = \mu^\epsilon \lambda \left( 1 - \frac{\lambda}{\pi} \right).
\] (3.10)

Then in terms of the renormalized coupling \( \lambda \), we have
\[
\{T_{\mu\nu}\}_R(x) = c_1(\lambda; \mu) \frac{1}{4} \bar{\chi}^i(t, x) \left( \gamma_\mu \overset{\to}{\partial}_\nu + \gamma_\nu \overset{\to}{\partial}_\mu \right) \chi^i(t, x) - c_2(\lambda; \mu) \frac{1}{2} \delta_{\mu\nu} \bar{\chi}^i(t, x) \overset{\to}{\partial} \chi^i(t, x) + c_3(\lambda; \mu) \delta_{\mu\nu} [\bar{\chi}^i(t, x)\chi^i(t, x)]^2 - VEV
+ O(t),
\] (3.11)

where
\[
c_1(\lambda; \mu) = c_2(\lambda; \mu) = 1 + O(\lambda^2),
\] (3.12)
\[
c_3(\lambda; \mu) = \frac{\lambda}{2N} \left[ 1 + \frac{\lambda}{2\pi} \left[ \ln(8\pi \mu^2 t) + 1 \right] \right].
\] (3.13)

As we have noted, in the present system, the flowed fermion field receives the wave function renormalization common to the unflowed fermion field. Since the fermion field does not receive the wave function renormalization to the one-loop order in the present system, even composite operators of the bare flowed fermion field are UV finite without multiplicative renormalization; the flow ensures this UV finiteness. Then Eqs. (3.12) and (3.13) show that the right-hand side of Eq. (3.11) is UV finite. This should be so, because the energy–momentum tensor (after subtracting the vacuum expectation value) in the left-hand side must be UV finite.

\footnote{This persists also in the leading order \( 1/N \) expansion that is relevant to our analyses below. Thus, in this paper, we do not need to consider the wave function renormalization of the flowed fermion field. In gauge theory, on the other hand, renormalization of the fermion field has to be taken into account; see Ref. \[9\].}
Finally, we utilize a renormalization group argument. We apply the operation
\[
\left( \mu \frac{\partial}{\partial \mu} \right)_0
\]
(3.14)
to both sides of Eq. (3.11), where the subscript 0 implies that the bare quantities are kept fixed under the derivative. Since the energy–momentum tensor [3.3] is entirely given by bare quantities and the composite operators in the right-hand side of Eq. (3.11) are also bare, we infer that \((\mu \partial/\partial \mu)_0 c_i(\lambda; \mu) = 0\) for \(i = 1, 2,\) and \(3\). These equations say that \(c_i(\lambda; \mu) = c_i(\lambda(q); q)\) for arbitrary \(q\), where \(\lambda(q)\) is the running coupling in the MS scheme with the renormalization scale \(q\). Since the renormalization scale \(q \) in \(c_i(\lambda(q); q)\) is arbitrary, we may take \(q = 1/\sqrt{8\ell}\) by using the flow time \(t\). Then, since \(\lambda(1/\sqrt{8\ell}) \to 0\) for \(t \to 0\) by the asymptotic freedom, the above perturbative computation is justified for \(t \to 0\). In this way, we arrive at
\[
\{ T_{\mu \nu} \}_R (x) = \lim_{t \to 0} \left[ \hat{T}_{\mu \nu}(t, x) - \langle \hat{T}_{\mu \nu}(t, x) \rangle \right],
\]
(3.15)
where
\[
\hat{T}_{\mu \nu}(t, x) \equiv \frac{1}{4} \chi^i(t, x) \left( \gamma_\mu \frac{\partial}{\partial \nu} + \gamma_\nu \frac{\partial}{\partial \mu} \right) \chi^i(t, x) - \frac{1}{2} \delta_{\mu \nu} \chi^i(t, x) \frac{\partial}{\partial \mu} \chi^i(t, x) + \frac{\lambda(1/\sqrt{8\ell})}{2N} \left[ 1 + \frac{\lambda(1/\sqrt{8\ell})}{2\pi} (\ln \pi + 1) \right] \delta_{\mu \nu} \left[ \chi^i(t, x) \chi^i(t, x) \right]^2.
\]
(3.16)
This is our universal formula for the energy–momentum tensor. This is universal in the sense that it does not refer to any specific regularization; the composite operator in the right-hand side is a renormalized quantity that must be independent of regularization as far as the parameters are properly renormalized.

We stress that our computation which led to Eq. (3.16) is purely one-loop. Although we retained only large \(N\) leading terms in one-loop coefficients, no non-perturbative \(1/N\) expansion is invoked at this stage. In particular, the fermion is treated as massless. We stress this point because the intention of the present paper is to see how the formula (3.16) that is obtained by one-loop perturbation theory can capture non-perturbative physics. More specifically, we want to see if the idea that coefficients in the universal formula can be determined by perturbation theory while low-energy non-perturbative physics is contained in matrix elements of composite operators works or not. This is the idea for the construction of the lattice energy–momentum tensor in Refs. [16, 17].

4. 1/N expansion in the Gross–Neveu model

Now, for the analyses in subsequent sections, we briefly recapitulate the well-known non-perturbative solution in the present system (2.1), an expansion in powers of \(1/N\) [16, 17].

For a systematic \(1/N\) expansion, it is convenient to introduce an auxiliary field \(\sigma(x)\) and rewrite the action (2.1) as
\[
S = \int d^D x \left[ \bar{\psi}^i(x) \slashed{D} \psi^i(x) + \sigma(x) \bar{\psi}^i(x) \psi^i(x) + \frac{N}{2\lambda_0} \sigma(x)^2 \right].
\]
(4.1)
If we first integrate over the fermion field, the partition function becomes
\[
Z = \int \left[ \prod_x d\sigma(x) \right] \exp \left\{ -\frac{N}{2\lambda_0} \int d^D x \sigma(x)^2 + N \text{Tr} \ln [\slashed{D} + \sigma(x)] \right\}.
\]
(4.2)
Since the exponent is proportional to \(N\) in this expression, in the leading order of the \(1/N\) expansion, the integral over the auxiliary field can be approximated by the value at the
saddle point. The saddle point is specified by the stationary condition for the exponent, i.e., by the gap equation,

$$\frac{1}{\lambda_0} = \text{tr} \int \frac{1}{i\ell + \sigma}. \quad (4.3)$$

The momentum integration in the right-hand side requires regularization. If we use dimensional regularization with \( D = 2 - \epsilon \), we have

$$\frac{1}{\lambda_0} = \frac{1}{\pi} \left[ \frac{1}{\epsilon} - \frac{1}{2} \ln \left( \frac{e^\gamma \sigma^2}{4\pi} \right) \right] \sigma, \quad (4.4)$$

where \( \gamma \) is the Euler constant. This tells us that, setting

$$\lambda_0 = \mu \epsilon \lambda Z, \quad (4.5)$$

the renormalization factor is given by

$$Z^{-1} = 1 + \frac{\lambda_0}{\pi \epsilon} \quad (4.6)$$

in the MS scheme. In terms of the renormalized coupling \( \lambda \), the saddle point is expressed as

$$\sigma^2 = 4\pi e^{-\gamma} \Lambda^2, \quad \Lambda \equiv \mu e^{-\pi/\lambda}. \quad (4.7)$$

As Eq. (4.1) shows, this saddle point provides a non-perturbative mass gap for the (originally massless) fermion. Corresponding to Eq. (4.6), the beta function is given by

$$\beta \equiv \left( \frac{\partial}{\partial \mu} \right)_0 \lambda = -\epsilon \lambda - \frac{\lambda^2}{\pi} \quad (4.8)$$

and thus the running coupling in the MS scheme is

$$\tilde{\lambda}(q) = -\frac{2\pi}{\ln(\Lambda^2/q^2)} = -\frac{2\pi}{\ln[e^\gamma \sigma^2/(4\pi q^2)]}.$$ \( \quad (4.9) \)

To obtain the next-to-leading order corrections in the \( 1/N \) expansion, we have to consider the integration over the fluctuation around the saddle point in Eq. (4.2). So we set

$$\sigma(x) = \sigma + \delta\sigma(x). \quad (4.10)$$

The expansion of the exponent in Eq. (4.2) is then

$$-\frac{N}{2\lambda_0} \int d^D x \sigma(x)^2 + N \text{Tr} \ln [\partial + \sigma(x)]$$

$$= -\frac{N}{2\lambda_0} \int d^D x \sigma^2 + N \int d^D x \text{ tr} \int_p \ln(i\ell + \sigma)$$

$$- \frac{N}{2\lambda_0} \int d^D x \delta\sigma(x)^2$$

$$- \frac{N}{2} \int d^D x \int d^D y \delta\sigma(x) \delta\sigma(y) \int_p e^{ip(x-y)} \text{ tr} \int_\ell \frac{1}{i\ell + \sigma} + \frac{1}{i(\ell - \hat{p}) + \sigma} + \mathcal{O}(\delta\sigma^3). \quad (4.11)$$

There is no \( \mathcal{O}(\delta\sigma) \) term because \( \sigma \) is the saddle point. After the momentum integration and the parameter renormalization (4.5), we have

$$-\frac{N}{2\lambda_0} \int d^D x \sigma(x)^2 + N \text{Tr} \ln [\partial + \sigma(x)]$$

$$= \frac{N}{4\pi} \int d^D x \sigma^2 - \frac{N}{4\pi} \int d^D x \int d^D y \delta\sigma(x) \delta\sigma(y) \int_p e^{ip(x-y)} B(p^2, \sigma^2) + \mathcal{O}(\delta\sigma^3), \quad (4.12)$$
\begin{align}
B(p^2, \sigma^2) & \equiv \sqrt{\frac{p^2 + 4\sigma^2}{p^2}} \ln \left( \frac{\sqrt{p^2 + 4\sigma^2} + \sqrt{p^2}}{\sqrt{p^2 + 4\sigma^2} - \sqrt{p^2}} \right). 
\end{align}

From this, the propagator of the fluctuating field \( \delta \sigma(x) \) is given by
\begin{align}
\langle \delta \sigma(x) \delta \sigma(y) \rangle &= \frac{2\pi}{N} \int_p e^{ip(x-y)} B(p^2, a^2)^{-1} + O(1/N^2). 
\end{align}

In the above computation, we may use lattice regularization as well [18, 19]. With the lattice spacing \( a \), one may discretize the action (4.1) by replacing the Dirac operator \( \partial \) by the Wilson Dirac operator for example,
\begin{align}
\frac{1}{2} \left[ \gamma_\mu \left( \partial_\mu + \partial_\mu^* \right) - a \partial_\mu^* \partial_\mu \right] + m_0, 
\end{align}
where \( \partial_\mu \) and \( \partial_\mu^* \) are forward and backward difference operators, respectively; \( m_0 \) is the bare mass parameter to be tuned to restore the chiral symmetry explicitly broken by the Wilson term. Then setting
\begin{align}
\tilde{\sigma} & \equiv \sigma + m_0, 
\end{align}
the gap equation with lattice regularization reads
\begin{align}
\frac{1}{\lambda_0^{\text{LAT}}(\tilde{\sigma} - m_0)} = \text{tr} \int_\mathcal{B} \frac{d^2p}{(2\pi)^2} \frac{1}{ip + \frac{1}{2} a \hat{p}^2 + \tilde{\sigma}} 
& = 0.7698 \frac{1}{a} - \frac{1}{2\pi} \left[ \ln(a^2 \tilde{\sigma}^2) + 1.11861 \right] \tilde{\sigma}, 
\end{align}
where \( \lambda_0^{\text{LAT}} \) is the bare coupling with lattice regularization, \( \mathcal{B} \) is the Brillouin zone \( \mathcal{B} \equiv \{ p_\mu \mid -\pi/a < p_\mu \leq \pi/a \} \) and
\begin{align}
\hat{p}_\mu & \equiv \frac{2}{a} \sin \left( \frac{ap_\mu}{2} \right), \\
\hat{p}_\mu & \equiv \frac{1}{a} \sin \left( ap_\mu \right).
\end{align}
We choose the bare mass parameter \( m_0 \) so that the gap equation possesses a “symmetric solution” \( \tilde{\sigma} = 0 \); this corresponds to a massless fermion because \( \tilde{\sigma} \) provides the fermion mass in the leading order of the \( 1/N \) expansion. This requirement leads to
\begin{align}
m_0 = -0.7698 \frac{\lambda_0^{\text{LAT}}}{a}. 
\end{align}
Under this choice, Eq. (4.17) says that
\begin{align}
\tilde{\sigma}^2 = e^{-1.11861} e^{-2\pi/\lambda_0^{\text{LAT}} \frac{1}{a^2}}.
\end{align}
\( \tilde{\sigma} \) has the same physical meaning as \( \sigma \) in Eq. (4.7). Thus, by choosing \( \lambda_0^{\text{LAT}} \) in Eq. (4.20) so that \( \tilde{\sigma} = \sigma \) as a function of \( a \), and rewriting everything in terms of this renormalized quantity, the dependence of physical quantities on adopted regularization disappears. In particular, it can be directly seen that the expression (4.12) also remains the same for lattice regularization (with \( \sigma = \tilde{\sigma} \)). In what follows, we assume that this sort of parameter renormalization is made.
5. Restoration of the translation Ward–Takahashi relation

By using the large $N$ solution in the previous section, we now consider correlation functions which contain the composite operator (3.16). Then, by studying the small flow-time limit of the correlation functions, we examine if the energy–momentum tensor defined by our universal formula, Eq. (3.15) with Eq. (3.16), fulfills (some particular cases of) the translation Ward–Takahashi relation, Eq. (3.4).

We first note that the fermion propagator in the $1/N$ expansion is, from Eqs. (4.1) and (4.10),

$$\langle \psi^i(x) \bar{\psi}^j(y) \rangle = \delta^{ij} \int_p e^{ip(x-y)} \frac{1}{ip + \sigma} + O(1/N)$$

and thus the propagator of the flowed fermion field is given by

$$\langle \chi^i(t, x) \bar{\chi}^j(s, y) \rangle = \delta^{ij} \int_p e^{ip(x-y)} e^{-(t+s)p^2} \frac{1}{ip + \sigma} + O(1/N)$$

by Eq. (2.4). The propagator between the flowed and unflowed fermion fields, $\langle \chi^i(t, x) \bar{\psi}^j(y) \rangle$ for example, is given by simply setting the corresponding flow time zero ($s = 0$ in this example) in Eq. (5.2).

The first correlation function we consider is

$$\langle \partial_\mu \tilde{T}_{\mu\nu}(t, x) \psi^i(y) \bar{\psi}^i(z) \rangle.$$  (5.3)

In the leading non-trivial order of the $1/N$ expansion, there are two types of connected diagrams which contribute to this correlation function; both are of $O(N)$. These two types of diagrams are depicted in Figs. 1 and 2 respectively. The contribution of type I diagrams

Fig. 1  Type I diagrams which contribute to Eq. (5.3). The solid line is the fermion propagator and the blob denotes the composite operator (3.16).

in Fig. 1 is, for small $t$,

$$\langle \partial_\mu \tilde{T}_{\mu\nu}(t, x) \psi^i(y) \bar{\psi}^i(z) \rangle_I$$

$$= \int_q \int_p e^{ip(x-y)} e^{iq(x-z)} N \frac{e^{-tp^2}}{ip + \sigma}$$

$$\times i(-p + q)_\mu \left\{ \frac{1}{4} \left[ \gamma_\mu i(p + q)_\nu + \gamma_\nu i(p + q)_\mu \right] - \frac{1}{2} \delta_\mu_\nu \delta(p + q) \right\}$$

$$\times \frac{\lambda(1/\sqrt{8t})}{2\pi} \ln(2e^{\gamma^2t}2\pi) \left[ 1 + \frac{\lambda(1/\sqrt{8t})}{2\pi} (\ln \pi + 1) \right] \delta_{\mu\nu} \sigma \frac{e^{-tq^2}}{iq + \sigma}$$.
The second diagram in Fig. 1 contains a loop integral arising from the self-contraction in the last four-fermi term of Eq. (3.16). The loop integral is finite, however, because of the Gaussian damping factor in the propagator (5.2). We can rewrite the integrand in Eq. (5.4) as

\[
\left(\frac{i}{p} + \sigma\right) \left\{ \frac{1}{4} \left[ \gamma_{\mu} i(p + q)_{\nu} + \gamma_{\nu} i(p + q)_{\mu} \right] - \frac{1}{2} \delta_{\mu\nu} i(\not{p} + \not{q}) \right. \\
+ \frac{\lambda(1/\sqrt{8t})}{2\pi} \ln(2e^{\gamma\sigma^2 t}) \left[ \ln(\pi + 1) \right] \delta_{\mu\nu} \left. \right\}
\]

\[
= \left( i\not{p} + \sigma \right) \left[ -iq_{\nu} + \frac{1}{8} [i(\not{p} + \not{q}), \gamma_{\nu}] + \left[ i\nu_{\nu} - \frac{1}{8} \left[ i(\not{p} + \not{q}), \gamma_{\nu} \right] \right] (i\not{q} + \sigma) \right. \\
+ \left\{ \frac{i\lambda(1/\sqrt{8t})}{2\pi} \ln(2e^{\gamma\sigma^2 t}) \left[ 1 + \frac{i\lambda(1/\sqrt{8t})}{2\pi} (\ln(\pi + 1)) + 1 \right] \right. i(-p + q)_{\nu}, \sigma.
\]

(5.5)

On the other hand, from Eq. (4.9), we have

\[
\frac{\lambda(1/\sqrt{8t})}{2\pi} = -\frac{1}{\ln(2e^{\gamma\sigma^2 t}/\pi)}
\]

(5.6)

and we find the following \( t \to 0 \) limits:

\[
\lim_{t\to0} \frac{\lambda(1/\sqrt{8t})}{2\pi} = 0,
\]

(5.7)

\[
\lim_{t\to0} \frac{\lambda(1/\sqrt{8t})}{2\pi} \ln(2e^{\gamma\sigma^2 t}) = -1,
\]

(5.8)

\[
\lim_{t\to0} \ln(2e^{\gamma\sigma^2 t}) \left[ 1 + \frac{\lambda(1/\sqrt{8t})}{2\pi} \ln(2e^{\gamma\sigma^2 t}) \right] = -\ln(\pi),
\]

(5.9)

\[
\lim_{t\to0} \left[ \frac{\lambda(1/\sqrt{8t})}{2\pi} \ln(2e^{\gamma\sigma^2 t}) \right]^2 = 1.
\]

(5.10)

From these, we see that the last line of Eq. (5.5) vanishes for \( t \to 0 \). Then when Eq. (5.5) is substituted in Eq. (5.4), the factor \( (i\not{p} + \sigma) \) in Eq. (5.5) cancels the external propagator \( 1/(i\not{p} + \sigma) \) and then the integration over \( p \) produces the delta function \( \delta^2(x - y) \) for \( t \to 0 \).
The situation is similar for the factor \((i\slashed{q} + \sigma)\) in \((5.5)\). In this way, we have
\[
\lim_{t \to 0} \left\langle \partial_\mu \hat{T}^\mu_{\nu}(t, x) \psi^i(y) \bar{\psi}^i(z) \right\rangle_1
= -\delta^2(x - y) \left\langle \partial_\nu \psi^i(y) \bar{\psi}^i(z) \right\rangle
- \delta^2(x - z) \left\langle \psi^i(y) \partial_\nu \bar{\psi}^i(z) \right\rangle
+ \partial_\mu \left\{ \frac{1}{8} \left[ \gamma_\mu, \gamma_\nu \right] \right\}
\left[ \frac{1}{8} \left[ \gamma_\mu, \gamma_\nu \right] \right] + O(N^0).
\]
\((5.11)\)

This is precisely the expected form of the Ward–Takahashi relation associated with the translational invariance. In fact, by considering the integration over the position \(x\) over the region that contains the points \(y\) and \(z\), we observe that Eq. \((3.4)\) with \(O_{\text{int}} = \psi^i(y) \bar{\psi}^i(z)\) (and \(O_{\text{ext}} = 1\)) holds.

since the correct Ward–Takahashi relation is already saturated by type I diagrams, Eq. \((5.11)\), the type II diagrams in Fig. 2 should not contribute to the translation Ward–Takahashi identity. To see this, and for a later use, it is useful to compute first the left-hand side parts of the type II diagrams depicted in Fig. 3. An explicit computation of the diagrams in Fig. 3 yields
\[
\int e^{i(r(x-y))} \frac{N}{2\pi} \sigma \left( \delta_{\mu\nu} - \frac{r_{\mu} r_\nu}{r^2} \right) [B(\gamma^2, \sigma^2) - \frac{r_{\mu} r_\nu}{r^2}]
- \delta_{\mu\nu} \left( 1 + \frac{\lambda(1/\sqrt{8t})}{2\pi} \ln(2e^{\gamma \sigma^2 t}) + \frac{\lambda(1/\sqrt{8t})}{2\pi} (\ln \pi + 1) \right) \right] \\left( \ln(2e^{\gamma \sigma^2 t}) + B(\gamma^2, \sigma^2) \right).
\]
\((5.12)\)

Using Eqs. \((5.7)\)–\((5.10)\), we then have
\[
\lim_{t \to 0} \text{Eq. } \((5.12)\) = \int e^{i(r(x-y))} \frac{N}{2\pi} \sigma \left( \delta_{\mu\nu} - \frac{r_{\mu} r_\nu}{r^2} \right) [B(\gamma^2, \sigma^2) - 2].
\]
\((5.13)\)

From this, for the type II diagrams in Fig. 2
\[
\lim_{t \to 0} \left\langle \hat{T}^\mu_{\nu}(t, x) \psi^i(y) \bar{\psi}^i(z) \right\rangle_{\text{II}}
= \int \int e^{i(p(x-y))} e^{i(q(x-z))} N\sigma \left( \delta_{\mu\nu} - \frac{r_{\mu} r_\nu}{r^2} \right) \left[ 1 - 2B(\gamma^2, \sigma^2) \right] \frac{1}{i\slashed{p} + \sigma} \frac{1}{i\slashed{q} + \sigma}.
\]
\((5.14)\)

\footnote{The energy–momentum tensor always possesses the ambiguity that results in the total divergence in the (unintegrated) Ward–Takahashi relation associated with the translational invariance. The second line of Eq. \((5.11)\), which corresponds to the Lorentz rotation generated by the anti-symmetric part of the canonical energy–momentum tensor, being the total divergence, does not contribute to the integrated form of the translation Ward–Takahashi relation, Eq. \((3.4)\).}
where
\[ r \equiv -p + q, \]  
(5.15)
and we have the desired result
\[
\lim_{t \to 0} \left\langle \partial_{\mu} \hat{T}_{\mu\nu}(t, x) \psi^i(y) \bar{\psi}^i(z) \right\rangle_{II} = 0.
\]  
(5.16)

Thus, in the leading non-trivial order of the $1/N$ expansion, we have confirmed that the universal formula, Eq. (3.15) with Eq. (3.16), reproduces the translation Ward–Takahashi relation (3.4) for the product of elementary fields, $O_{\text{int}} = \psi^i(y) \bar{\psi}^i(z)$ (and $O_{\text{ext}} = 1$). This shows that the universal formula reproduces the correct normalization and the conservation law for the energy–momentum tensor, at least in the correlation function with elementary fields.

The above computation in fact demonstrates that Eq. (3.4) is also reproduced for the scalar density operator, that is,
\[
O_{\text{int}} = Z_S \psi^i(y) \psi^i(y), \quad O_{\text{ext}} = 1,
\]  
(5.17)
where $Z_S$ is an appropriate renormalization factor for the scalar density. In the leading non-trivial order of the $1/N$ expansion, there are two types of diagrams which contribute to
\[
\left\langle \partial_{\mu} \hat{T}_{\mu\nu}(t, x) \psi^i(y) \bar{\psi}^i(y) \right\rangle,
\]  
(5.18)
as depicted in Figs. 4 and 5.

**Fig. 4** Type I diagrams which contribute to Eq. (5.18). In each diagram, the left blob denotes the composite operator (3.16) and the right blob denotes the scalar density operator in Eq. (5.17).

**Fig. 5** Type II diagrams which contribute to Eq. (5.18). In each diagram, the left blob denotes the composite operator (3.16) and the right blob denotes the scalar density operator in Eq. (5.17).

For type I diagrams, the computation is identical to that for Eqs. (5.12) and (5.13). As is clear from Eq. (5.13), we have $\lim_{t \to 0} \langle \partial_{\mu} \hat{T}_{\mu\nu}(t, x) \bar{\psi}^i(y) \psi^i(y) \rangle_1 = 0$. For type II diagrams also, we do not need a new calculation because Eqs. (5.12) and (5.13) (which correspond to Fig. 3) give the parts of the diagrams in Fig. 5. Thus, again from Eq. (5.13), we
have \( \lim_{t \to 0} \langle \partial_\mu \hat{T}_{\mu\nu}(t, x) \hat{\psi}^j(y) \hat{\psi}^j(y) \rangle_{II} = 0. \) These reproduce Eq. (3.4) with Eq. (5.17) because \( \langle \partial_\nu [\hat{\psi}^j(y) \hat{\psi}^j(y)] \rangle = 0 \) by the translational invariance.

We can further argue that Eq. (3.4) is reproduced when \( \mathcal{O}_{\text{ext}} \) is a collection of renormalized composite operators of the fermion field and \( \mathcal{O}_{\text{int}} = 1. \) That is, for this situation, we can argue that

\[
\lim_{t \to 0} \left\langle \mathcal{O}_{\text{ext}} \int d^D x \partial_\mu \hat{T}_{\mu\nu}(t, x) \right\rangle = 0. \tag{5.19}
\]

This shows that the conservation law of the energy–momentum tensor is reproduced in the correlation function with generic composite operators. The argument is simple: There exist two types of diagrams which contribute to Eq. (5.19). For type I diagrams in Fig. 6, we can use the identity (5.5) for fermion lines starting from the vertex of the composite operator (3.16). Then, as Eq. (5.11), we have

\[
\lim_{t \to 0} \left\langle \psi^i(y) \bar{\psi}^i(z) \cdots \partial_\mu \hat{T}_{\mu\nu}(t, x) \right\rangle_{I} = -\delta^2(x - y) \left\langle \partial_\nu \psi^i(y) \bar{\psi}^i(z) \cdots \right\rangle - \delta^2(x - z) \left\langle \psi^i(y) \partial_\nu \bar{\psi}^i(z) \cdots \right\rangle - \cdots + [\delta^2(x - y), \delta^2(x - z), \ldots, \text{inside the total divergence in } x]. \tag{5.20}
\]

Then for \( x \neq y, x \neq z, \ldots, \) the right-hand side vanishes. For the type II diagrams in Fig. 7, from Eq. (5.13), we simply have \( \lim_{t \to 0} \langle \mathcal{O}_{\text{ext}} \partial_\mu \hat{T}_{\mu\nu}(t, x) \rangle_{II} = 0. \) These imply Eq. (5.19).

**Fig. 6** Type I diagrams which contribute to Eq. (5.19). In each diagram, the leftmost blob denotes the composite operator (3.16) and other blobs denote the fermion composite operators contained in \( \mathcal{O}_{\text{ext}} \) in Eq. (5.19).

**Fig. 7** Type II diagrams which contribute to Eq. (5.19). In each diagram, the leftmost blob denotes the composite operator (3.16) and other blobs denote the fermion composite operators contained in \( \mathcal{O}_{\text{ext}} \) in Eq. (5.19).
6. Expectation value at finite temperature

The Ward–Takahashi relation (5.11) shows that our universal formula for the energy–momentum tensor gives rise to the correct normalization at least within the correlation function with elementary fields. To give a further support on the correct normalization, in this section we compute the expectation value of the composite operator (3.16) at finite temperature and compare it with thermodynamic quantities directly obtained in the conventional $1/N$ expansion. A similar analysis for the two-dimensional $O(N)$ non-linear sigma model has been carried out in Ref. [14].

At finite temperature with inverse temperature $\beta$, the propagator is given by

$$\langle \chi^i(t,x)\bar{\chi}^j(s,y) \rangle_\beta = \delta^{ij} \frac{1}{\beta} \sum_{n=-\infty}^{\infty} \int \frac{dp_1}{2\pi} e^{ip_1(x-y)} \frac{e^{-(t+s)(\omega_n^2 + p_1^2)}}{i\gamma_0 \omega_n + i\gamma_1 p_1 + \sigma_\beta} + O(1/N), \quad (6.1)$$

where $\omega_n$ is the Matsubara frequency

$$\omega_n \equiv \frac{2\pi n}{\beta}, \quad (6.2)$$

and $\sigma_\beta$ is the large $N$ saddle point at finite temperature; $\sigma_\beta$ is given by a finite-temperature counterpart of the gap equation (4.3):

$$\frac{1}{\lambda_0} = \text{tr} \frac{1}{\beta} \sum_{n=-\infty}^{\infty} \int \frac{dp_1}{2\pi} \frac{1}{i\gamma_0 \omega_n + i\gamma_1 p_1 + \sigma_\beta}. \quad (6.3)$$

The required computation is almost the same as that in Ref. [14], because of the similarity of expressions. Using Eq. (3.15) with Eq. (3.16), for the energy density $\varepsilon$ we have

$$\varepsilon = -\langle \{T_{00}\}_R(x) \rangle_\beta = \frac{N}{4\pi} (\sigma_\beta^2 - \sigma^2) - \frac{N}{\pi} \sigma_\beta^2 \sum_{n=1}^{\infty} K_2(\beta \sigma_\beta n) + O(N^0) \quad (6.4)$$

and, for the pressure $P$,

$$P = \langle \{T_{11}\}_R(x) \rangle_\beta = \frac{N}{4\pi} (\sigma_\beta^2 - \sigma^2) - \frac{N}{\pi} \sigma_\beta^2 \sum_{n=1}^{\infty} K_2(\beta \sigma_\beta n) + O(N^0). \quad (6.5)$$

These are the results of the universal formula.

On the other hand, the free energy density of the present system is given by

$$f(\beta) = \frac{N}{2\lambda_0} \beta \sigma_\beta^2 - N \frac{1}{\beta} \sum_{n=-\infty}^{\infty} \int \frac{dp_1}{2\pi} \ln \left( \omega_n^2 + p_1^2 + \sigma_\beta^2 \right), \quad (6.6)$$

and the energy-density and the pressure are given by $\varepsilon = \partial f(\beta)/\partial \beta$ and $P = -f(\beta)/\beta$, respectively. From comparison of Eq. (6.6) with Eq. (A1) of Ref. [14], we see that these quantities can be obtained by making the substitutions $f(\beta) \rightarrow -f(\beta)$, $N \rightarrow 2N$, $\lambda_0 \rightarrow 2\lambda_0$, $\sigma_\beta \rightarrow \sigma_\beta^2$, and $\sigma \rightarrow \sigma^2$ in Eqs. (A11) and (A9) of Ref. [14]. We then observe a complete agreement with Eqs. (6.4) and (6.5). This result again supports the validity of our universal formula for the energy–momentum tensor.
7. Conclusion

The flow in quantum field theory and the small flow-time expansion can give rise to a regularization-independent expression for composite operators. In this paper, we examined the validity of a universal formula for the energy–momentum tensor by using the Gross–Neveu model and the non-perturbative $1/N$ expansion. In the leading non-trivial order in the $1/N$ expansion, we have observed that (some particular cases of) the Ward–Takahashi relation associated with the translational invariance is correctly reproduced by the universal formula even with the non-perturbative mass gap. This is interesting because the construction of the universal formula itself requires only (one-loop) perturbation theory. We have also observed that the formula reproduces thermodynamic quantities correctly. These observations support the validity of a similar construction of the energy–momentum tensor via the gradient/Wilson flow in lattice gauge theory.

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Note added

I learned that a strikingly analogous idea to define the energy–momentum tensor to ours has been considered on the basis of the operator product expansion (instead of the small flow-time expansion) in Ref. [20]. I would like to thank Jan Holland for the information.

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