SOME NEW INTEGRAL INEQUALITIES FOR N-TIMES DIFFERENTIABLE R-CONVEX AND R-CONCAVE FUNCTIONS

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Abstract. In this work, by using an integral identity together with both the Hölder and the Power-Mean integral inequality we establish several new inequalities for n-time differentiable r-convex and concave functions.

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1. INTRODUCTION

A function $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is said to be convex if the inequality

$$f((tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

is valid for all $x, y \in I$ and $t \in [0,1]$. If this inequality reverses, then $f$ is said to be concave on interval $I \neq \emptyset$. This definition is well known in the literature. Convexity theory has appeared as a powerful technique to study a wide class of unrelated problems in pure and applied sciences. Many articles have been written by a number of mathematicians on convex functions and inequalities for their different classes, using, for example, the last articles [3, 8–15] and the references in these papers.

$f : [a, b] \rightarrow \mathbb{R}$ be a convex function, then the inequality

$$f\left(\frac{a + b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a) + f(b)}{2}$$

is known as the Hermite-Hadamard inequality (see [6] for more information). Since then, some refinements of the Hermite-Hadamard inequality on convex functions have been extensively investigated by a number of authors (e.g., [3, 4, 15]). In [20], the first author obtained a new refinement of the Hermite-Hadamard inequality for convex functions. The Hermite-Hadamard inequality was generalized in [17] to an r-convex positive function which is defined on an interval $[a, b]$.

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Definition 1. A positive function \( f : I \subseteq \mathbb{R} \rightarrow \mathbb{R} \) is called \( r \)-convex function on \([a, b]\), if for each the \( x, y \in [a, b] \) and \( t \in [0, 1] \),
\[
f(tx + (1-t)y) \leq \begin{cases} 
[tf'(x) + (1-t)f'(y)]^\frac{1}{r} , & r \neq 0, \\
[f(x)]^t [f(y)]^{1-t} , & r = 0.
\end{cases}
\]
If the equality is reversed, then the function \( f \) is said to be \( r \)-concave.

It is obvious 0-convex functions are simply log-convex functions, 1-convex functions are ordinary convex functions and \(-1\)-convex functions are arithmetically harmonically convex. One should note that if \( f \) is \( r \)-convex on \([a, b]\), then the function \( f^r \) is a convex function for \( r > 0 \) and \( f^r \) is a concave function for \( r < 0 \). We note that if \( f \) and \( g \) are convex and \( g \) is increasing, then \( gof \) is convex; moreover, since \( f = \exp(\log f) \), it follows that a log-convex function is convex.

The definition of \( r \)-convexity naturally complements the concept of \( r \)-concavity, in which the inequality is reversed [18] and which plays an important role in statistics.

It is easily seen that if \( f \) is \( r \)-convex on \([a, b]\),
\[
f^r\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f^r(x) \, dx \leq \frac{f^r(a) + f^r(b)}{2}, \quad r > 0 \tag{1.1}
\]
\[
f^r\left(\frac{a+b}{2}\right) \geq \frac{1}{b-a} \int_a^b f^r(x) \, dx \geq \frac{f^r(a) + f^r(b)}{2}, \quad r < 0 \tag{1.2}
\]

Some refinements of the Hadamard inequality for \( r \)-convex functions could be found in [2,7,16,19,21]. In [1], Bessenyei studied Hermite-Hadamard-type inequalities for generalized 3-convex functions. In [16], the authors showed that if \( f \) is \( r \)-convex in \([a, b]\) and \( 0 < r \leq 1 \), then
\[
\frac{1}{b-a} \int_a^b f(x) \, dx \leq \frac{r}{r+1} \left[ f^r(a) + f^r(b) \right]^{\frac{1}{r}}. \tag{1.3}
\]

Theorem 1 ([5]). Suppose that \( f \) is a positive \( r \)-convex function on \([a, b]\). Then
\[
\frac{1}{b-a} \int_a^b f(t) \, dt \leq L_r (f(a), f(b)).
\]
If \( f \) is a positive \( r \)-concave function, then the inequality is reversed, where
\[
L_r (f(a), f(b)) = \begin{cases} 
\frac{r+1}{r+1} \frac{f^{r+1}(a)-f^{r+1}(b)}{f(a)-f(b)}, & r \neq 0, -1, \quad f(a) \neq f(b) \\
\frac{f(a)-f(b)}{\inf f(a)-\inf f(b)}, & r = 0, \quad f(a) \neq f(b) \\
f(a) f(b) \frac{\ln f(a)-\ln f(b)}{f(a)-f(b)}, & r = -1, \quad f(a) \neq f(b) \\
f(a), & f(a) = f(b).
\end{cases}
\]
Theorem 2. Let \( f : [a, b] \to (0, \infty) \) be \( r \)-convex function and \( r \geq 1 \). Then the following inequality holds:
\[
\frac{1}{b-a} \int_a^b f(x) \, dx \leq \left[ \frac{f^r(a) + f^r(b)}{2} \right]^\frac{1}{r}.
\]

Lemma 1. Let \( a \geq 0, b \geq 0 \). Then \( (a+b)^\lambda \leq a^\lambda + b^\lambda, \ 0 < \lambda \leq 1 \).

Let \( 0 < a < b \), throughout this paper we will use
\[
A(a,b) = \frac{a+b}{2}, \quad G(a,b) = \sqrt{ab}
\]
\[
L_p(a,b) = \left( \frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)} \right)^\frac{1}{p}, \quad a \neq b, \ p \in \mathbb{R}, \ p \neq -1, 0
\]
for the arithmetic, geometric, generalized logarithmic mean, respectively. Also for shortness we will use the following notation:
\[
I(a,b,n,f) = \sum_{k=0}^{n-1} (-1)^k \left( \frac{f^{(k)}(b)b^{k+1} - f^{(k)}(a)a^{k+1}}{(k+1)!} \right) - \int_a^b f(x) \, dx
\]
where an empty sum is understood to be nil.

2. MAIN RESULTS

We will use the following Lemma for obtain our main results.

Lemma 2 ([14]). Let \( f : I \subseteq \mathbb{R} \to \mathbb{R} \) be \( n \)-times differentiable mapping on \( I^o \) for \( n \in \mathbb{N} \) and \( f^{(n)} \in L[a, b] \), where \( a, b \in I^o \) with \( a < b \), we have the identity
\[
I(a,b,n,f) = \sum_{k=0}^{n-1} (-1)^k \left( \frac{f^{(k)}(b)b^{k+1} - f^{(k)}(a)a^{k+1}}{(k+1)!} \right) - \int_a^b f(x) \, dx
\]
where an empty sum is understood to be nil.

Theorem 3. For \( n \in \mathbb{N} \); let \( f : I \subseteq (0, \infty) \to \mathbb{R} \) be \( n \)-times differentiable function on \( I^o \), \( r > 0 \) and \( a, b \in I^o \) with \( a < b \). If \( f^{(n)} \in L[a, b] \) and \( \left| f^{(n)} \right|^q \) for \( q > 1 \) is \( r \)-convex function on \( [a, b] \), then the following inequality holds:
\[
|I(a,b,n,f)| \leq \frac{b-a}{n!} L_{n,p}(a,b) L_p^\frac{1}{r} \left( \left| f^{(n)}(a) \right|^{qr}, \left| f^{(n)}(b) \right|^{qr} \right)
\]

Proof. If \( \left| f^{(n)} \right|^q \) for \( q > 1 \) is \( r \)-convex function on \([a, b]\) and \( r > 0 \), using Lemma 2, the Hölder integral inequality and
\[
\left| f^{(n)}(x) \right|^q = \left| f(n) \left( \frac{x-a}{b-a} b + \frac{b-x}{b-a} a \right) \right|^q
\]
we have

$$|I(a, b, n, f)| \leq \frac{1}{n!} \int_a^b x^n |f^{(n)}(x)| \, dx$$

$$\leq \frac{1}{n!} \left( \int_a^b x^{np} \, dx \right)^{\frac{1}{p}} \left( \int_a^b \left| f^{(n)}(x) \right|^q \, dx \right)^{\frac{1}{q}}$$

$$\leq \frac{1}{n!} \left( \int_a^b x^{np} \, dx \right)^{\frac{1}{p}} \left( \int_a^b \left[ \frac{x-a}{b-a} f^{(n)}(b) \right]^{qr} + \left[ \frac{b-x}{b-a} f^{(n)}(a) \right]^{qr} \right)^{\frac{1}{r}} \left( \int_a^b \left[ \frac{b-a}{b-a} \right]^{qr} \, dx \right)^{\frac{1}{q}}$$

$$= \frac{1}{n!} \left( \int_a^b x^{np} \, dx \right)^{\frac{1}{p}} \left( \int_a^b \left[ f^{(n)}(b) \right]^{qr} + \left[ f^{(n)}(a) \right]^{qr} \right)^{\frac{1}{r}} \left( \int_a^b \left[ \frac{b-a}{b-a} \right]^{qr} \, dx \right)^{\frac{1}{q}}$$

$$= \frac{1}{n!} (b-a) \left( \frac{b^{np+1} - a^{np+1}}{(np+1)(b-a)} \right)^{\frac{1}{p}} \left( \frac{f^{(n)}(b)^{qr(\frac{1}{r}+1)} - f^{(n)}(a)^{qr(\frac{1}{r}+1)}}{(\frac{1}{r} + 1) \left( \frac{f^{(n)}(b)^{qr} - f^{(n)}(a)^{qr}}{f^{(n)}(b)^{qr} - f^{(n)}(a)^{qr}} \right)} \right)^{\frac{1}{q}}$$

$$= \frac{1}{n!} (b-a) L_{np}^n (a, b) \left( \frac{f^{(n)}(b)^{qr(\frac{1}{r}+1)} - f^{(n)}(a)^{qr(\frac{1}{r}+1)}}{(\frac{1}{r} + 1) \left( \frac{f^{(n)}(b)^{qr} - f^{(n)}(a)^{qr}}{f^{(n)}(b)^{qr} - f^{(n)}(a)^{qr}} \right)} \right)^{\frac{1}{q}}$$

$$= \frac{1}{n!} (b-a) L_{np}^n (a, b) L_{\frac{1}{r}}^\frac{1}{p} \left( \frac{f^{(n)}(a)^{qr}, f^{(n)}(b)^{qr}}{f^{(n)}(b)^{qr} - f^{(n)}(a)^{qr}} \right).$$

This completes the proof of theorem. □

Remark 1. The results obtained in this paper reduces to the results of [14] in case of \( r = 1 \).

Corollary 1. Under the conditions Theorem 3 for \( n = 1 \) we have the following inequality:

$$\left| \frac{f(b) - f(a)}{b-a} - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq L_p(a, b) L_{\frac{1}{r}}^\frac{1}{p} \left( \left| f'(a)^{qr}, f'(b)^{qr} \right| \right).$$

Proposition 1. Let \( a, b \in (0, \infty) \) with \( a < b \), \( q > 1 \) and \( m \geq 1 \), \( r \geq 1 \) we have

$$L_{\frac{mq}{r}+1}^{\frac{mq}{r}+1} (a, b) \leq L_p(a, b) L_{\frac{1}{r}}^\frac{1}{p} (a^{mr}, b^{mr}).$$
Proof. Under the assumption of the Proposition, let \( f(x) = \frac{q}{m+q} x^{\frac{m}{q}+1}, \ x \in (0, \infty) \). Then \(|f'(x)|^q = x^m\) is \(r\)-convex on \((0, \infty)\) and the result follows directly from Corollary 1. \(\square\)

Remark 2. Under the assumption of the Proposition 2.1, if \(r = 1, \ m = 1\), then the results obtained in this paper reduces to the results of [14].

**Theorem 4.** For \(n \in \mathbb{N}; \) let \(f : I \subset (0, \infty) \to \mathbb{R}\) be \(n\)-times differentiable function on \(I^o\), \(r > 0\) and \(a, b \in I^o\) with \(a < b\). If \(f^{(n)} \in L[a, b]\) and \(|f^{(n)}|^q\) for \(q \geq 1\) is \(r\)-convex function on \([a, b]\), then the following inequality holds:

\[
|I(a, b, n, f)| \leq \left\{ \begin{array}{ll}
\frac{1}{n!} (b-a)^{1-\frac{1}{q}} \frac{1}{\sigma r} \int_a^b x^n \left( f^{(n)}(x) \right)^q dx 
+ C_1 \left( f^{(n)}(b) \right)^q + C_2 \left( f^{(n)}(a) \right)^q \right]^\frac{1}{q} & , \ r \geq 1 \\
\frac{1}{n!} (b-a)^{1-\frac{1}{q}} \frac{1}{\sigma r} \int_a^b x^n \left( f^{(n)}(x) \right)^q r^{1-\frac{1}{q}} dx 
+ C_1 \left( f^{(n)}(b) \right)^q r^{\frac{1}{q}} + C_2 \left( f^{(n)}(a) \right)^q r^{\frac{1}{q}} \right]^\frac{1}{q} & , \ r \leq 1
\end{array} \right.
\]

where

\[
C_1 = C_1(a, b, r, n) = \int_a^b x^n (x-a)^{\frac{1}{r}} dx, \ C_2 = C_2(a, b, r, n) = \int_a^b x^n (b-x)^{\frac{1}{r}} dx.
\]

Proof. From Lemma 2 and Power-Mean integral inequality, we get

\[
|I(a, b, n, f)| \\
\leq \frac{1}{n!} \int_a^b x^n \left( f^{(n)}(x) \right) dx \\
\leq \frac{1}{n!} \left( \int_a^b x^n dx \right)^{1-\frac{1}{q}} \left( \int_a^b x^n \left( f^{(n)}(x) \right)^q dx \right)^{\frac{1}{q}} \\
\leq \frac{1}{n!} \left( \int_a^b x^n dx \right)^{1-\frac{1}{q}} \left( \int_a^b x^n \left[ \frac{x-a}{b-a} f^{(n)}(b) \right]^{q r} + \left[ \frac{b-x}{b-a} f^{(n)}(a) \right]^{q r} \right)^{\frac{1}{q}} dx \right. \\
\left. \right) \frac{1}{q}.
\]

Here, using Lemma 1 we obtain respectively.

For \(r \geq 1\)

\[
|I(a, b, n, f)| \\
\leq \frac{1}{n!} \left( \int_a^b x^n dx \right)^{1-\frac{1}{q}} \times \left( \int_a^b x^n \left[ \frac{x-a}{b-a} f^{(n)}(b) \right]^{q r} + \left[ \frac{b-x}{b-a} f^{(n)}(a) \right]^{q r} \right)^{\frac{1}{q}} dx \right) \frac{1}{q} \\
= \frac{1}{n!} \left( \frac{1}{b-a} \left( \frac{b^{n+1} - a^{n+1}}{n+1} \right) \right)^{1-\frac{1}{q}}.
\]
\[
\times \left( \int_a^b x^n(x-a)^{\frac{1}{2}} \left| f^{(n)}(b) \right|^q dx + \int_a^b x^n(b-x)^{\frac{1}{2}} \left| f^{(n)}(a) \right|^q dx \right)^{\frac{1}{q}}
\]

\[
= \frac{1}{n!} \left( \frac{1}{b-a} \right)^{\frac{1}{q}} \left( \frac{b^{n+1} - a^{n+1}}{n+1} \right)^{\frac{1}{q}}
\times \left[ C_1(a,b,r,n) \left| f^{(n)}(b) \right|^q + C_2(a,b,q,n) \left| f^{(n)}(a) \right|^q \right]^{\frac{1}{q}}
\]

\[
= \frac{1}{n!} \left( \frac{b^{n+1} - a^{n+1}}{n+1} \right)^{\frac{1}{q}} L_n^{(\frac{q-1}{q})} (a,b) \left[ C_1 \left| f^{(n)}(b) \right|^q + C_2 \left| f^{(n)}(a) \right|^q \right]^{\frac{1}{q}}.
\]

For \( r \leq 1 \), using Minkowski inequality, we have

\[
|I(a,b,n,f)| 
\leq \frac{1}{n!} \left( \int_a^b x^n dx \right)^{\frac{1}{q}} \left( \frac{1}{b-a} \right)^{\frac{1}{q}}
\times \left( \int_a^b \left| x^n r (x-a) \left| f^{(n)}(b) \right|^q + x^n (b-x) \left| f^{(n)}(a) \right|^q \right|^{\frac{1}{q}} dx \right)^{\frac{1}{q}}
\]

\[
= \frac{1}{n!} \left( \frac{b^{n+1} - a^{n+1}}{n+1} \right)^{\frac{1}{q}}
\times \left( \left\{ \left[ \int_a^b x^n(x-a)^{\frac{1}{2}} \left| f^{(n)}(b) \right|^q dx \right]^r + \left[ \int_a^b x^n(b-x)^{\frac{1}{2}} \left| f^{(n)}(a) \right|^q dx \right]^r \right\} \right)^{\frac{1}{q}}
\]

\[
= \frac{1}{n!} \left( \frac{b^{n+1} - a^{n+1}}{n+1} \right)^{\frac{1}{q}} L_n^{(\frac{q-1}{q})} (a,b) \left[ C_1^r \left| f^{(n)}(b) \right|^qr + C_2^r \left| f^{(n)}(a) \right|^qr \right]^{\frac{1}{q}}.
\]

This completes the proof of theorem. \( \Box \)

**Corollary 2.** Under the conditions Theorem 4 for \( n = 1 \) we have the following inequalities:

\[
|J(a,b,f)| 
\leq A^\frac{1}{q} \left( a,b \right) \left\{ \left( \frac{r(b-a)}{2r+1} \right)^r \left| f'(b) \right|^q - \left| f'(a) \right|^q \right\}^{\frac{1}{q}}
\leq A^\frac{1}{q} \left( a,b \right) \left\{ \left( \frac{r^2(a+b)+br}{(r+1)(2r+1)} \right)^r \left| f'(b) \right|^q r + \left( \frac{r^2(a+b)+ar}{(r+1)(2r+1)} \right)^r \left| f'(a) \right|^q r \right\}^{\frac{1}{q}}, \quad r \geq 1
\]

\[
= A^\frac{1}{q} \left( a,b \right) \left\{ \left( \frac{r^2(a+b)+br}{(r+1)(2r+1)} \right)^r \left| f'(b) \right|^q r + \left( \frac{r^2(a+b)+ar}{(r+1)(2r+1)} \right)^r \left| f'(a) \right|^q r \right\}^{\frac{1}{q}}, \quad r \leq 1
\]

where \( J(a,b,f) = \frac{I(a,b,f)}{b-a} \).
Proposition 2. Let $a, b \in (0, \infty)$ with $a < b$, $q \geq 1$ and $m \geq 1$, we have the following inequalities:

$$L^m_{m+1} \left( a, b \right) \leq \begin{cases} \left. A^{1-\frac{q}{q+1}} \left( a, b \right) \right[ \frac{2rA(a^{m+1}b^{m+1})}{2r+1} + \frac{2r^2G^2(a,b)A(a^{m-1}b^{m-1})}{(r+1)(2r+1)} \right]^{\frac{1}{q}}, \quad r \geq 1 \\ \left. A^{1-\frac{q}{q+1}} \left( a, b \right) \right[ \left( \frac{r^2(a+b)+br}{r+1} \right)^r b^r + \left( \frac{r^2(a+b)+ar}{r+1} \right)^r a^r \right]^{\frac{1}{q}}, \quad r \leq 1. \end{cases}$$

Proof. The result follows directly from Corollary 2 for function $f(x) = \frac{q}{m+q} x^{\frac{m}{q+1}}$, $x \in (0, \infty)$.

Corollary 3. Using Proposition 2 for $m = 1$, we have the following inequalities:

$$L^{\frac{1}{q+1}} \left( a, b \right) \leq \begin{cases} \left. A^{1-\frac{q}{q+1}} \left( a, b \right) \right[ \frac{2rA(a^{m+1}b^{m+1})}{2r+1} + \frac{2r^2G^2(a,b)A(a^{m-1}b^{m-1})}{(r+1)(2r+1)} \right]^{\frac{1}{q}}, \quad r \geq 1 \\ \left. A^{1-\frac{q}{q+1}} \left( a, b \right) \right[ \left( \frac{r^2(a+b)+br}{r+1} \right)^r b^r + \left( \frac{r^2(a+b)+ar}{r+1} \right)^r a^r \right]^{\frac{1}{q}}, \quad r \leq 1. \end{cases}$$

Corollary 4. Using Proposition 2 for $q = 1$, we have following inequalities:

$$L^{m+1} \left( a, b \right) \leq \begin{cases} \left. \frac{2rA(a^{m+1}b^{m+1})}{2r+1} + \frac{2r^2G^2(a,b)A(a^{m-1}b^{m-1})}{(r+1)(2r+1)} \right], \quad r \geq 1 \\ \left. \left( \frac{r^2(a+b)+br}{r+1} \right)^r b^r + \left( \frac{r^2(a+b)+ar}{r+1} \right)^r a^r \right. \right\}, \quad r \leq 1. \end{cases}$$

Corollary 5. Using Corollary 4. for $m = 1$, we have following inequalities:

$$L^{q} \left( a, b \right) \leq \begin{cases} \frac{2rA(a^{q+1}b^{q+1})}{2r+1} + \frac{2r^2G^2(a,b)A(a^{q-1}b^{q-1})}{(r+1)(2r+1)} \right], \quad r \geq 1 \\ \left. \left( \frac{r^2(a+b)+br}{r+1} \right)^r b^r + \left( \frac{r^2(a+b)+ar}{r+1} \right)^r a^r \right. \right\}, \quad r \leq 1. \end{cases}$$

Corollary 6. Under the conditions Theorem 4 for $q = 1$ we have the following inequalities:

$$|I(a,b,n,f)| \leq \begin{cases} \frac{n!}{r!} (b-a)^{-\frac{q}{q+1}} \left[ C_1 \left| f^{(n)}(b) \right| + C_2 \left| f^{(n)}(a) \right| \right], \quad r \geq 1 \\ \frac{n!(b-a)^{-\frac{q}{q+1}}}{r!} \left[ C_1^f \left| f^{(n)}(b) \right|^q + C_2^f \left| f^{(n)}(a) \right|^q \right]^\frac{1}{q}, \quad r \leq 1. \end{cases}$$

Theorem 5. For $n \in \mathbb{N}$, let $f : I \subset (0, \infty) \to \mathbb{R}$ be $n$-times differentiable function on $I^r$, $r > 0$ and $a, b \in I^r$ with $a < b$. If $f^{(n)} \in L[a,b]$ and $\left| f^{(n)} \right|^q$ for $q > 1$ is $r$-convex function on $[a,b]$, then the following inequality holds:

$$|I(a,b,n,f)| \leq \begin{cases} \frac{n!}{r!} (b-a)^{-\frac{q}{q+1}} \left( \left| f^{(n)}(b) \right|^q D_1 + \left| f^{(n)}(a) \right|^q D_2 \right)^\frac{1}{q}, \quad r \geq 1 \\ \frac{n!(b-a)^{-\frac{q}{q+1}}}{r!} \left( \left| f^{(n)}(b) \right|^q D_1^f + \left| f^{(n)}(a) \right|^q D_2^f \right)^\frac{1}{q}, \quad r \leq 1. \end{cases}$$

where

$$D_1 = D_1(a,b,r,n,q) = \int_a^b x^n q (x-a)^{\frac{1}{2}} dx$$
\[ D_2 = D_2(a, b, r, n, q) = \int_a^b x^{nq} (b - x)^{\frac{1}{r}} \, dx. \]

Proof. Since \( |f^{(n)}|^q \) for \( q > 1 \) is \( r \)-convex function on \([a, b]\), using Lemma 2 and the Hölder integral inequality, we have the following inequality:

\[
|I(a, b, n, f)| \leq \frac{1}{n!} \left( \int_a^b 1 \, dx \right)^{\frac{1}{p}} \left( \int_a^b x^{nq} \left| f^{(n)}(x) \right|^q \, dx \right)^{\frac{1}{q}}.
\]

Here, using Lemma 1 we obtain respectively.

For \( r \leq 1 \), using Minkowski inequality, we have

\[
|I(a, b, n, f)| \leq \frac{1}{n!} \left( \int_a^b 1 \, dx \right)^{\frac{1}{p}} \left( \int_a^b x^{nq} \left[ \frac{x-a}{b-a} \right]^q \left| f^{(n)}(b) \right|^{qr} + \frac{b-x}{b-a} \left| f^{(n)}(a) \right|^{qr} \right)^{\frac{1}{q}}.
\]
This completes the proof of the theorem. \hfill \Box

**Corollary 7.** Under the conditions Theorem 5 for \( n = 1 \) we have the following inequalities:

\[
\left| \frac{f(b) - f(a)}{b - a} - \frac{1}{b - a} \int_a^b f(x) \, dx \right| \\
\leq \begin{cases} 
(b - a)^{\frac{1}{p} - \frac{1}{q} - 1} \left( |f'(b)|^q D_1 + |f'(a)|^q D_2 \right)^{\frac{1}{q}}, & r \geq 1 \\
(b - a)^{\frac{1}{p} - \frac{1}{q} - 1} \left( |f'(b)|^{qr} D_1^r + |f'(a)|^{qr} D_2^r \right)^{\frac{1}{qr}}, & r \leq 1.
\end{cases}
\]

**Proposition 3.** Let \( a, b \in (0, \infty) \) with \( a < b \), \( q > 1 \) and \( m \geq 1 \), we have

\[
L_{\frac{m}{q} + 1}(a, b) \leq \begin{cases} 
(b - a)^{\frac{1}{p} - \frac{1}{q} - 1} \left( b^m D_1 + a^m D_2 \right)^{\frac{1}{q}}, & r \geq 1 \\
(b - a)^{\frac{1}{p} - \frac{1}{q} - 1} \left( b^m r D_1^r + a^m r D_2^r \right)^{\frac{1}{qr}}, & r \leq 1.
\end{cases}
\]

**Proof.** The result follows directly from Corollary 7 for \( f(x) = \frac{q}{m+q} x^{\frac{m}{q}+1}, x \in (0, \infty) \). \hfill \Box

**Corollary 8.** For \( m = 1 \) from Proposition 3, we obtain the following inequality:

\[
L_{\frac{1}{q} + 1}(a, b) \leq \begin{cases} 
(b - a)^{\frac{1}{p} - \frac{1}{q} - 1} (b D_1 + a D_2)^{\frac{1}{q}}, & r \geq 1 \\
(b - a)^{\frac{1}{p} - \frac{1}{q} - 1} (b^r D_1^r + a^r D_2^r)^{\frac{1}{qr}}, & r \leq 1.
\end{cases}
\]

**Theorem 6.** For \( n \in \mathbb{N} \), let \( f : I \subset (0, \infty) \rightarrow \mathbb{R} \) be \( n \) times differentiable function on \( I^\circ \) (interior of \( I \)), \( r > 0 \) and \( a, b \in I^\circ \) with \( a < b \). If \( f^{(n)} \in L[a, b] \) and \( \left| f^{(n)} \right|^q \) for \( q > 1 \) is \( r \)-convex function on \( [a, b] \), then the following inequalities hold:

\[
|I(a, b, n, f)| \\
\leq \begin{cases} 
2\frac{r-a}{n!} \left( \frac{r}{r+1} \right)^{\frac{1}{2}} L^n_{np}(a, b) A^{\frac{1}{\sigma}} \left( \left| f^{(n)}(a) \right|^q, \left| f^{(n)}(b) \right|^q \right), & 0 < r \leq 1, \\
\frac{b-a}{n!} L^n_{np}(a, b) A^{\frac{1}{\sigma}} \left( \left| f^{(n)}(a) \right|^q, \left| f^{(n)}(b) \right|^q \right), & r \geq 0, \\
\frac{1}{n!} (b-a) L^n_{np}(a, b) \left( L_r \left( \left| f^{(n)}(a) \right|^q, \left| f^{(n)}(b) \right|^q \right) \right)^{\frac{1}{2}}, & r > 0.
\end{cases}
\]

**Proof.** For \( 0 < r \leq 1 \), since \( \left| f^{(n)} \right|^q \) for \( q > 1 \) is \( r \)-convex function on \( [a, b] \), with respect to Hermite-Hadamard inequality we have

\[
\int_a^b \left| f^{(n)}(x) \right|^q \, dx \leq (b-a) \frac{r}{r+1} \left[ \left| f^{(n)}(a) \right|^q + \left| f^{(n)}(b) \right|^q \right]^{\frac{1}{r}}.
\]

Using Lemma 2 and the Hölder integral inequality we have

\[
|I(a, b, n, f)|
\]
\[
\begin{align*}
&\leq \frac{1}{n!} \int_a^b x^n |f^{(n)}(x)| \, dx \\
&\leq \frac{1}{n!} \left( \int_a^b x^{np} \, dx \right)^\frac{1}{p} \left( \int_a^b |f^{(n)}(x)|^q \, dx \right)^\frac{1}{q} \\
&\leq \frac{1}{n!} \left( \int_a^b x^{np} \, dx \right)^\frac{1}{p} \left( b - a \frac{r}{r+1} \left[ |f^{(n)}(a)|^{qr} + |f^{(n)}(b)|^{qr} \right] \right)^\frac{1}{q} \\
&= 2^{\frac{1}{q}} \frac{b - a}{n!} \left( \frac{r}{r+1} \right) \frac{1}{q} \left[ \frac{b^{np+1} - a^{np+1}}{(np + 1)(b - a)} \right] \left[ \frac{1}{2} \left[ |f^{(n)}(a)|^{qr} + |f^{(n)}(b)|^{qr} \right] \right]^\frac{1}{q}.
\end{align*}
\]

For \( r \geq 1 \), since \( |f^{(n)}|^q \) for \( q > 1 \) is \( r \)-convex function on \([a, b] \), with respect to Theorem 2 we get

\[
\int_a^b |f^{(n)}(x)|^q \, dx \leq (b - a) \left[ \frac{1}{2} \left[ |f^{(n)}(a)|^{qr} + |f^{(n)}(b)|^{qr} \right] \right]^\frac{1}{q}.
\]

Using Lemma 2 and the Hölder integral inequality we have

\[
|I(a, b, n, f)| \leq \frac{1}{n!} \int_a^b x^n |f^{(n)}(x)| \, dx \\
\leq \frac{1}{n!} \left( \int_a^b x^{np} \, dx \right)^\frac{1}{p} \left( \int_a^b |f^{(n)}(x)|^q \, dx \right)^\frac{1}{q} \\
= \frac{b - a}{n!} \left[ \frac{b^{np+1} - a^{np+1}}{(np + 1)(b - a)} \right] \left[ \frac{1}{2} \left[ |f^{(n)}(a)|^{qr} + |f^{(n)}(b)|^{qr} \right] \right]^\frac{1}{q}.
\]

For \( r > 0 \), using Lemma 2, Theorem 1 and the Hölder integral inequality we have

\[
|I(a, b, n, f)| \leq \frac{1}{n!} \int_a^b x^n |f^{(n)}(x)| \, dx.
\]
\[
\frac{1}{n!} \left( \int_a^b x^n p \, dx \right)^{\frac{1}{p}} \left( \int_a^b f^{(n)}(x) \, dx \right)^{\frac{1}{q}}
\leq \frac{1}{n!} \left( \int_a^b x^n p \, dx \right)^{\frac{1}{p}} \left( (b-a) L_r \left( \left| f^{(n)}(a) \right|^q, \left| f^{(n)}(b) \right|^q \right) \right)^{\frac{1}{q}}
\]

This completes the proof of theorem. \(\square\)

**Corollary 9.** Under the conditions Theorem 6 for \(n = 1\) we have the following inequalities:

\[
\left| \frac{f(b) - f(a)}{b-a} - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq \begin{cases} 
2^\frac{1}{r} A \left( f'(a) \right)^q, & 0 < r \leq 1 \\
L_p(a,b) \left( f'(a) \right)^q, & r \geq 1 \\
L_p(a,b) \left( L_r \left( \left| f'(a) \right|^q, \left| f'(b) \right|^q \right) \right)^{\frac{1}{q}}, & r > 0
\end{cases}
\]

**Proposition 4.** Let \(a, b \in (0, \infty)\) with \(a < b\), \(q > 1\) and \(m \in [0, 1]\), we have

\[
L_{\frac{m}{q} + 1}(a,b) \leq \begin{cases} 
2^\frac{1}{q} A \left( f'(a) \right)^q, & 0 < r \leq 1 \\
L_p(a,b) \left( f'(a) \right)^q, & r \geq 1 \\
L_p(a,b) \left( L_r \left( \left| f'(a) \right|^q, \left| f'(b) \right|^q \right) \right)^{\frac{1}{q}}, & r > 0
\end{cases}
\]

Under the assumption of the Proposition, let \(f(x) = \frac{q}{m+q} x^{\frac{m}{q} + 1}, x \in (0, \infty)\). Then \(f'(x))^q = x^m\) is \(r\)-convex on \((0, \infty)\) and the result follows directly from Corollary 9.

**Corollary 10.** For \(m = 1\) from Proposition 4, we obtain the following inequalities:

\[
L_{\frac{1}{q} + 1}(a,b) \leq \begin{cases} 
2^\frac{1}{q} A \left( f'(a) \right)^q, & 0 < r \leq 1 \\
L_p(a,b) \left( f'(a) \right)^q, & r \geq 1 \\
L_p(a,b) \left( L_r \left( \left| f'(a) \right|^q, \left| f'(b) \right|^q \right) \right)^{\frac{1}{q}}, & r > 0
\end{cases}
\]

**Theorem 7.** For \(n \in \mathbb{N}\); let \(f : I \subset (0, \infty) \to \mathbb{R}\) be \(n\)-times differentiable function on \(I^\circ\), \(r > 0\) and \(a, b \in I^\circ\) with \(a < b\). If \(f^{(n)} \in L[a,b]\) and \(f^{(n)})^{\frac{1}{q}}\) for \(q > 1\) is \(r\)-convex function on \([a,b]\), then the following inequality holds:

\[
|I(a,b,n,f)| \leq \frac{1}{n!} (b-a) L_{\frac{n}{q}p}(a,b) A^{\frac{1}{q}} \left( \left| f^{(n)}(a) \right|, \left| f^{(n)}(b) \right| \right)^{\frac{1}{q}}.
\]
Proof. If \( \left| f^{(n)} \right|^q \) for \( q > 1 \) is \( r \)-convex function on \([a, b]\) and \( r > 0 \), using (1.1) inequality, Lemma 2 and the Hölder integral inequality respectively, we have

\[
\int_a^b \left| f^{(n)}(x) \right|^q \, dx = \int_a^b \left( \left| f^{(n)}(x) \right|^\frac{q}{r} \right)^r \, dx \leq (b-a) \left( \frac{\left| f^{(n)}(a) \right|^q + \left| f^{(n)}(b) \right|^q}{2} \right) \]

and

\[
|I(a, b, n, f)| \leq \frac{1}{n!} \int_a^b x^n \left| f^{(n)}(x) \right| \, dx \leq \frac{1}{n!} \left( \int_a^b x^{np} \, dx \right)^\frac{1}{p} \left( \int_a^b \left| f^{(n)}(x) \right|^q \, dx \right)^\frac{1}{q} = \frac{1}{n!} \left( \frac{b^{np+1} - a^{np+1}}{np+1} \right)^\frac{1}{p} \left( b-a \right)^{\frac{1}{p}} \left( \frac{\left| f^{(n)}(a) \right|^q + \left| f^{(n)}(b) \right|^q}{2} \right)^\frac{1}{q} = \frac{1}{n!} (b-a) L_{np}^{n}(a,b) A_\frac{1}{q} \left( \left| f^{(n)}(a) \right|^q, \left| f^{(n)}(b) \right|^q \right).
\]

This completes the proof of theorem. \( \square \)

Corollary 11. Under the conditions Theorem 7 for \( n = 1 \) we have the following inequality:

\[
\left| \frac{f(b) - f(a)}{b-a} \right| - \frac{1}{b-a} \int_a^b f(x) \, dx \leq L_p(a, b) A_\frac{1}{q} \left( \left| f'(a) \right|^q, \left| f'(b) \right|^q \right).
\]

Proposition 5. Let \( a, b \in (0, \infty) \) with \( a < b \), \( q > 1 \) and \( m \in [0, 1] \), we have

\[
\left| \frac{f(b) - f(a)}{b-a} - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq L_p(a, b) A_\frac{1}{q} \left( a^{mr}, b^{mr} \right).
\]

Proof. Under the assumption of the Proposition, let \( f(x) = \frac{q}{mr+q} x^{\frac{mr}{q}+1} \), \( x \in (0, \infty) \). Then \( |f'(x)|^q = x^m \) is \( r \)-convex on \((0, \infty)\) and the result follows directly from Corollary 11. \( \square \)

Corollary 12. For \( m = 1 \) from Proposition 5, we obtain the following inequality:

\[
\left| \frac{f(b) - f(a)}{b-a} - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq L_p(a, b) A_\frac{1}{q} \left( a^r, b^r \right).
\]

Theorem 8. For \( n \in \mathbb{N} \); let \( f : I \subset (0, \infty) \to \mathbb{R} \) be \( n \)-times differentiable function on \( I^r \), \( r > 0 \) and \( a, b \in I^r \) with \( a < b \). If \( f^{(n)} \in L[a, b] \) and \( \left| f^{(n)} \right|^\frac{q}{r} \) for \( q > 1 \) is
$r$-concave function on $[a, b]$, then the following inequality holds:

$$|I(a, b, n, f)| \leq \frac{b-a}{n!} L^n_{np} (a, b) \left| f^{(n)} \left( \frac{a+b}{2} \right) \right|.$$

**Proof.** If $\left| f^{(n)} \right|^\frac{q}{r}$ for $q > 1$ is $r$-concave function on $[a, b]$ and $r > 0$, using Lemma 2, the Hölder integral inequality and

$$\int_a^b \left| f^{(n)}(x) \right|^q \, dx = \int_a^b \left( \left| f^{(n)}(x) \right|^\frac{q}{r} \right)^r \, dx \leq (b-a) \left| f^{(n)} \left( \frac{a+b}{2} \right) \right|^q$$

we have

$$|I(a, b, n, f)| \leq \frac{1}{n!} \int_a^b x^n \left| f^{(n)}(x) \right| \, dx$$

$$\leq \frac{1}{n!} \left( \int_a^b x^{np} \, dx \right)^\frac{1}{p} \left( \int_a^b \left| f^{(n)}(x) \right|^q \, dx \right)^\frac{1}{q}$$

$$\leq \frac{1}{n!} \left( \int_a^b x^{np} \, dx \right)^\frac{1}{p} \left( (b-a) \left| f^{(n)} \left( \frac{a+b}{2} \right) \right|^q \right)^\frac{1}{q}$$

$$= \frac{1}{n!} (b-a) L^n_{np} (a, b) \left| f^{(n)} \left( \frac{a+b}{2} \right) \right|^q.$$

This completes the proof of theorem. □

**Corollary 13.** Under the conditions Theorem 8 for $n = 1$ we have the following inequality:

$$\left| f(b) b - f(a) a \right| \leq \frac{1}{b-a} \left( \int_a^b f(x) \, dx \right) \leq L_p(a, b) \left| f' \left( \frac{a+b}{2} \right) \right|.$$

**Proposition 6.** Let $a, b \in (0, \infty)$ with $a < b$, $q > 1$ and $m \in [0, 1]$, we have

$$\left| f(b) b - f(a) a \right| \leq \frac{1}{b-a} \int_a^b f(x) \, dx \leq L_p(a, b) A_{\frac{mr}{q}} (a, b).$$

**Proof.** Under the assumption of the Proposition, let $f(x) = \frac{q}{mr+q} x^{\frac{mr}{q}+1}$, $x \in (0, \infty)$. Then $|f'(x)|^\frac{q}{r} = x^m$ is $r$-concave on $(0, \infty)$ and the result follows directly from Corollary 13. □

**Corollary 14.** For $m = 1$ from Proposition 6, we obtain the following inequality:

$$\left| f(b) b - f(a) a \right| \leq \frac{1}{b-a} \int_a^b f(x) \, dx \leq L_p(a, b) A^r (a, b).$$
REFERENCES

[1] M. Bessenyei, “Hermite-Hadamard-type inequalities for generalized 3-convex functions.” *Publ. Math. (Debr.),* vol. 65, no. 15, pp. 223–232, 2004.

[2] F. Chen and X. Liu, “Refinements on the Hermite-Hadamard Inequalities for r-Convex Functions.” *J. Appl. Math.*, vol. 2013, no. 1-2, p. 5, 2013, doi: 10.1155/2013/978493.

[3] S. Dragomir, “Refinements of the Hermite-Hadamard integral inequality for log-convex functions.” *Aust. Math. Soc. Gaz.*, vol. 28, no. 3, pp. 129–134, 2001.

[4] S. Dragomir and C. Pearce, *Selected Topics on Hermite-Hadamard Inequalities and Its Applications.* Victoria University: RGMIA Monograph, 2002.

[5] P. Gill, C. Pearce, and J. Pečarić, “Hadamard’s Inequality for r-Convex Functions.” *J. Math. Anal. Appl.*, vol. 215, no. AY975645, pp. 461–470, 1997.

[6] J. Hadamard, “Étude sur les propriétés des fonctions entières en particulier d’une fonction considérée par Riemann.” *J. Math. Pures Appl.*, no. 58, pp. 171–216, 1893.

[7] L. Han and L. Liu, “Integral Inequalities of Hermite-Hadamard Type for r-Convex Functions.” *Appl. Math.*, no. 3, pp. 1967–1971, 2012, doi: 10.4236/am.2012.312270.

[8] I. Işcan, “Ostrowski type inequalities for p-convex functions.” *New Trends in Mathematical Sciences (Ntmsci).* vol. 4, no. 3, pp. 129–134, 2001.

[9] I. Işcan, H. Kadakal, and M. Kadakal, “Some new integral inequalities for n-times differentiable log-convex functions.” *New Trends in Mathematical Sciences (Ntmsci)*, vol. 5, no. 2, pp. 10–15, 2017, doi: 10.20852/ntmsci.2017.150.

[10] I. Işcan and M. Kurt, “Hermite-Hadamard-Fejer type inequalities for quasi-geometrically convex functions via fractional integrals.” *J. Math.*, vol. 2016, no. 6523041, p. 7, 2016, doi: 10.1155/2016/6523041.

[11] I. Işcan and S. Turhan, “Generalized Hermite-Hadamard-Fejer type inequalities for GA-convex functions via Fractional integral.” *Moroccan J. Pure and Appl. Anal. (MJPAA)*, vol. 2, no. 1, pp. 34–46, 2016, doi: 10.7603/s40956-016-0004-2.

[12] H. Kadakal, M. Kadakal, and I. Işcan, “Some new integral inequalities for n-times differentiable s-convex and s-concave functions in the second sense.” *Mathematics and Statistic.,* vol. 5, no. 2, pp. 94–98, 2017, doi: 10.13189/ms.2017.050207.

[13] M. Kadakal, H. Kadakal, and I. Işcan, “Some new integral inequalities for n-times differentiable s-convex functions in the first sense.” *Turkish Journal of Analysis and Number Theory (Tjant)*, vol. 5, no. 2, pp. 63–68, 2017, doi: 10.12691/tjant-5-2-4.

[14] M. Maden, H. Kadakal, M. Kadakal, and I. Işcan, “Some new integral inequalities for n-times differentiable convex and concave functions.” *J. Nonlinear Sci. Appl.*, vol. 10, no. 12, pp. 6141–6148, 2017, doi: 10.22436/jnsa.010.12.01.

[15] B. Mihaly, “Hermite-Hadamard-type inequalities for generalized convex functions.” *J. inequal. pure and appl. math.*, vol. 9, no. 3, p. 51, 2008.

[16] N. Ngoc, N. Vinh, and P. Hien, “Integral inequalities of Hadamard type for r-convex functions.” *Int. Math. Forum*, vol. 4, pp. 1723–1728, 2009.

[17] C. Pearce, J. Pečaric, and V. Šimić, “Stolarsky means and Hadamard’s inequality.” *J. Math. Anal. Appl.*, vol. 220, no. 1, pp. 99–109, 1998, doi: 10.1006/jmaa.1997.5822.

[18] B. Uhrin, “Some remarks about the convolution of unimodal functions.” *Ann. Probab.*, vol. 12, no. 2, pp. 640–645, 1984, doi: 10.1214/aop/1176993312.

[19] G. Yang and D. Hwang, “Refinements of Hadamard inequality for r-convex functions.” *Indian J. Pure Appl. Math.*, vol. 32, no. 10, pp. 1571–1579, 2001.

[20] G. Zabandan, “A new refinement of the Hermite-Hadamard inequality for convex functions.” *J. Inequal. Pure Appl. Math.*, vol. 10, no. 2, p. 7, 2009.
[21] G. Zabandan, A. Bodagh, and A. Kilicman, “The Hermite-Hadamard inequality for r-convex functions.” *Journal of Inequalities and Applications*, vol. 2012, no. 1, p. 215, 2012, doi: 10.1186/1029-242X-2012-215.

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