Markovianity criteria for quantum evolution

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Abstract
We characterize a class of Markovian dynamics using the concept of a divisible dynamical map. Moreover, we provide a family of criteria which can distinguish Markovian and non-Markovian dynamics. These Markovianity criteria are based on a simple observation that Markovian dynamics implies monotonic behaviour of several well-known quantities such as distinguishability of states, fidelity, relative entropy and genuine entanglement measures.

1. Introduction
The dynamics of open quantum systems attracts nowadays increasing attention [1–4]. It is relevant not only for the better understanding of quantum theory, but also it is fundamental in various modern applications of quantum mechanics. Since the system–environment interaction causes dissipation, decay and decoherence, it is clear that the dynamics of open systems is fundamental in modern quantum technologies, such as quantum communication, cryptography and computation [5].

The usual approach to the dynamics of an open quantum system consists in applying an appropriate Markovian approximation that leads to the following local master equation:

\[ \dot{\rho}_t = L \rho_t, \]

where \( \rho_t \) is the density matrix of the system investigated and \( L \) is the time-independent generator of the dynamical semigroup. It is well known that under certain conditions on \( L \) [6, 7], the corresponding dynamics \( \rho \to \rho_t := \Lambda_t \rho \) gives rise to completely positive and trace preserving map \( \Lambda_t \) [3, 1] (one usually calls \( \Lambda_t \) a dynamical map). The characteristic feature of the Markovian approximation leading to a dynamical semigroup \( \Lambda_t = e^{Lt} \) is that it neglects all memory effects caused by the nontrivial interaction of the system with an external world. However, recent theoretical studies and technological progress call for a more refined approach based on the non-Markovian evolution.

Non-Markovian systems appear in many branches of physics, such as quantum optics [1, 8], solid state physics [9], quantum chemistry [10] and quantum information processing [11]. Since non-Markovian dynamics modifies the monotonic decay of quantum coherence, it turns out that when applied to composite systems it may protect quantum entanglement for a longer time than the standard Markovian evolution [12]. In particular, it may protect the system against the sudden death of entanglement [13]. It is therefore not surprising that non-Markovian dynamics was intensively studied over the last few years [14–25].

The standard approach to the dynamics of an open system uses the Nakajima–Zwanzig projection operator technique [26], which shows that under fairly general conditions, the master equation for the reduced system density matrix \( \rho_t \) takes the form of the following non-local equation:

\[ \dot{\rho}_t = \int_0^t K_{t-\tau} \rho_\tau \, d\tau, \]

in which quantum memory effects are taken into account through the introduction of the memory kernel \( K_t \); this simply means that the rate of change of the state \( \rho_t \) at time \( t \) depends on its history (starting at \( t = 0 \)). It should be stressed that there is a perfectly equivalent approach (usually called time convolutionless (TCL)) [1, 27], which describes quantum dynamics by local in time equation:

\[ \dot{\rho}_t = L_t \rho_t, \]

where \( L_t \) denotes the local generator. It is clear that if \( L_t = L \) does not depend on time and \( K_t = 2\delta(t)L \), then both (2) and (3) reduce to the standard master equation (1). These equations provide, therefore, natural generalization of the standard Markovian semigroup. One of the open problems in this theory is to characterize properties of \( K_t \) and \( L_t \) which guarantee that the corresponding solution \( \rho_t = \Lambda_t \rho \) defines the legitimate dynamical map \( \Lambda_t \).
Interestingly, the concept of (non)Markovianity is not uniquely defined. One approach is based on the idea of the composition law which is essentially equivalent to the idea of divisibility [33]. This approach was used recently [35] to construct the corresponding measure of non-Markovianity. A different approach is presented in [34] where non-Markovian dynamics corresponds to a time evolution for the open system characterized by a temporary flow of information from the environment back into the system and manifests itself as an increase in the distinguishability of pairs of evolving quantum states. The aim of this paper is to characterize a class of Markovian dynamics using the concept of a divisible dynamical map and to provide a family of criteria which can distinguish Markovian and non-Markovian dynamics, i.e. these criteria are satisfied if the dynamics is Markovian and hence their violation is a clear sign of non-Markovianity.

The paper is organized as follows: in the next section we recall the standard reduced dynamics of an open system. We stress that the problem of characterizing the properties of $K_t$ and/or $L_t$ is in general untractable. It considerably simplifies in the case of commutative dynamics, i.e. if the dynamical map $\Lambda_t$ commutes in different times $[\Lambda_t, \Lambda_u] = 0$ for arbitrary $t, u \geq 0$ (see section 3). In section 4, we characterize Markovian dynamics using the concept of the divisible dynamical map. Interestingly, we provide a simple method which enables one to go beyond Markovian dynamics fully controlling the corresponding local generator. This method is illustrated by pure decoherence. Necessary criteria for Markovianity are discussed in section 6. Final conclusions are collected in section 7.

2. Reduced dynamics of an open system

Consider an $N$-level quantum system living in $H$ coupled to a reservoir with the corresponding (usually infinite dimensional) Hilbert space $H_R$. Throughout the paper we use standard notation: $B(H)$ and $T(H)$ denote the class of bounded and trace class operators in $H$, respectively. Let $H$ denote the Hamiltonian of the total composed system living in $H \otimes H_R$ and $\omega$ be a fixed state of the reservoir. One defines the reduced dynamics $\Lambda_t : T(H) \to T(H)$ by the following formula:

$$\Lambda_t \rho := \text{Tr}_R [e^{-iHt} (\rho \otimes \omega_R) e^{iHt}],$$

where $\text{Tr}_R$ denotes the partial trace over the reservoir degrees of freedom. Note that $\Lambda_t$ is completely positive and trace preserving for all $t \geq 0$ and it satisfies $\Lambda_0 = 1$. Therefore, it provides a legitimate quantum evolution of the system living in $H$. Actually, it is well known that any legitimate $\Lambda_t$ may be defined as a reduced dynamics for appropriate $H_R$ and the total Hamiltonian $H$. The standard Nakajima–Zwanzig projection operator technique [26, 1] shows that reduced dynamics $\Lambda_t$ satisfies the following non-local equation:

$$\dot{\Lambda}_t = \int_0^t K_{t\tau} \Lambda_{\tau} d\tau, \quad \Lambda_0 = 1.$$  

(5)

The memory kernel $K_t$ encodes all dynamical properties of the system and depends upon the total Hamiltonian of the ‘system + reservoir’ and the reservoir reference state $\omega_R$. This equation is exact but in general very difficult to analyse. This is due to the fact that the memory kernel $K_t$ depends upon all reservoir correlation functions. To simplify analysis, one usually tries to perform a suitable Markovian approximation to neglect all unwanted memory effects. The validity of such approximation is based on the existence of two characteristic timescales: the characteristic time $\tau_S$ of variation of $\rho$, and the decay time $\tau_R$ of the reservoir correlation functions. The Markovian approximation assumes that $\tau_S \gg \tau_R$. Basically, there are two ways of rigorous treatment of the limit $\tau_S/\tau_R \to \infty$. One assumes that $\omega_R$ is invariant under the free evolution of the reservoir. Representing the Hamiltonian $H$ as

$$H = H_S \otimes 1_R + 1_S \otimes H_R + \lambda H_{int},$$

(6)

one performs the weak-coupling limit $\lambda \to 0$ with the rescaled time $\tau = \lambda^2 t$. In this scheme, $\tau_R$ remains constant, while $\tau_S \to \infty$. This approach was analysed in great detail by Davies [28, 29] (see also [30]).

On the other hand, in the singular coupling limit, one has $\tau_R \to 0$. It is achieved by considering the following Hamiltonian:

$$H = H_S \otimes 1_R + e^{-2\delta S} \otimes H_R + e^{-1} H_{int},$$

(7)

and performing the limit $\epsilon \to 0$ [31]. As a result, reservoir correlation functions become $\delta$ function. In both scenarios, the limiting dynamics is governed by the well-known master equation

$$\dot{\Lambda}_t = L\Lambda_t, \quad \Lambda_0 = 1,$$

(8)

where $L$ denotes the identity map and the Markovian generator $L$ is given by

$$L\rho = -i[H, \rho] + \frac{1}{2} \sum_{\alpha \beta \gamma} \left( \{V_{\alpha}, \rho V_{\beta}^\dagger\} + \{V_{\beta}, \rho V_{\gamma}^\dagger\} \right).$$

(9)

In what follows, we call $L$ represented by (9) the GKS L generator. In the above formula, $H$ represents the effective system Hamiltonian and $\{V_{\alpha}\}$ is the collection of arbitrary operators encoding the interaction between the system and environment. Equation (8) gives rise to the Markovian semigroup $\Lambda_t = e^{Lt}$, satisfying the following homogeneous composition law:

$$L\Lambda_t \Lambda_u = \Lambda_{t+u},$$

(10)

for any $t, u \geq 0$.

It should be stressed that one obtains the Markovian master equation (8) from the general Nakajima–Zwanzig equation (5) only if the total Hamiltonian $H$ enables one to perform the suitable Markovian approximation and hence it covers only the limited number of physically interesting systems. In general, the Markovian approximation is not suitable, and one has to deal with the much more involved non-local equation (5). One of the main problems is to characterize the properties of the corresponding memory kernel $K_t$ which guarantees that the corresponding solution $\Lambda_t$ represents a legitimate dynamical map.

Note that instead of the non-local equation (5) one may equivalently describe dynamics using the local equation. This approach (usually called TCL [1]) leads to the following local in time equation:

$$\dot{\Lambda}_t = L_t\Lambda_t, \quad \Lambda_0 = 1,$$

(11)
with the time-dependent local generator $L_t$. We stress that these two approaches are equivalent. Assuming that $\Lambda_t$ is differentiable it always satisfies the local in time master equation. Indeed, formally one has $\Lambda_t = \Lambda_t \Lambda_t^{-1} \Lambda_t = L_t \Lambda_t$, where we assumed the existence of the inverse map $\Lambda_t^{-1}$. Note that the inverse, even if it exists, needs not be completely positive. Again, one would like to perform the characterization of time-dependent generators $L_t$, giving rise to legitimate dynamical maps $\Lambda_t$. This problem seems to be intractable in full generality. Note that the formal solution to (11) has the following form:

$$\Lambda_t = T \exp \left( \int_0^t L_u \, du \right),$$

(12)

where ‘$T$’ stands for chronological product. It is clear that the above expression has rather a formal meaning.

### 3. Commutative class of dynamical maps

As we have already stressed, the general solution to the local in time master equation (11) has only a formal meaning and in general we do not control the properties of $L_t$ which guarantee that the T-product exponential formula $T \exp \left( \int_0^t L_u \, du \right)$ defines the dynamical map. Note, however, that if $L_t$ defines a commutative family of generators, that is,

$$[L_t, L_u] = 0,$$

(13)

for any $t, u \geq 0$, formula (12) considerably simplifies: the ‘$T$’ product drops out and the solution is fully controlled by the integral $\int_0^t L_u \, du$.

**Theorem 3.1.** If $L_t$ defines a commutative family, then $L_t$ is a legitimate generator of a quantum dynamical map if and only if $\int_0^t L_u \, du$ defines a legitimate GKSL generator for all $t \geq 0$.

**Example 3.1.** As an example of commutative dynamics consider the following evolution of the qubit:

$$L_t \rho = -\frac{i}{2} [\sigma_z, \rho] + \frac{\gamma_t}{2} (\sigma_z \rho \sigma_z - \rho),$$

(14)

where $\omega_t, \gamma_t : \mathbb{R}^+ \rightarrow \mathbb{R}$. The corresponding density matrix evolves as follows:

$$\rho_t = \begin{pmatrix} \rho_{00} & \rho_{01} e^{(-i \Omega_t - i \Gamma_t)} \\ \rho_{10} e^{(i \Omega_t - i \Gamma_t)} & \rho_{11} \end{pmatrix},$$

(15)

where

$$\Omega_t = \int_0^t \omega_u \, du, \quad \Gamma_t = \int_0^t \gamma_u \, du.$$  

$L_t$ gives rise to legitimate quantum evolution if and only if $\Gamma_t \geq 0$ for all $t \geq 0$ and hence the evolution corresponds to simple decoherence. The corresponding dynamics is Markovian iff $\gamma_t \geq 0$.

This example may easily be generalized for the $d$-level system. Consider the following class of generators: let $\lambda = e^{2\pi i / 4}$ and define

$$V_u = \sum_{\beta=0}^{d-1} \lambda^{u \beta} P_{\beta},$$

(16)

where $P_{\beta} = |e_{\beta}\rangle \langle e_{\beta}|$ and $|e_0, \ldots, e_{d-1}\rangle$ stands for an arbitrary orthonormal basis in $\mathbb{C}^d$. Now, let $c_{u \beta}(t)$ be a time-dependent Hermitian matrix and define

$$L_t \rho = -i [H_t, \rho] + \sum_{k,l=1}^{d-1} c_{kl}(t) \left( [V_k, \rho V_l^\dagger] + [V_k^\dagger, \rho V_l] \right),$$

(17)

where the time-dependent Hamiltonian $H_t$ reads as follows:

$$H_t = \sum_{k=1}^{d-1} (h_k(t)V_k + \bar{h}_k(t)V_k^\dagger),$$

and $h_k : \mathbb{R}^+ \rightarrow \mathbb{C}$. $L_t$ generates legitimate dynamics if and only if the matrix

$$C_{u \beta}(t) = \int_0^t c_{u \beta}(u) \, du$$

is positive definite. Dynamics is Markovian if and only if $c_{u \beta}(t)$ is itself positive definite.

### 4. Markovian dynamics

In this section, we characterize an important class of dynamical maps representing the Markovian evolution. It is important to clarify these issues since there are a few definitions used in the literature recently. We call a dynamical map $\Lambda_t$ divisible if for any $t \geq s \geq 0$ one has the following decomposition:

$$\Lambda_t = V_{t,s} \Lambda_s,$$

(18)

with a completely positive propagator $V_{t,s}$. Note that $V_{t,s}$ satisfies the inhomogeneous composition law

$$V_{t,s} V_{s,u} = V_{t,u},$$

(19)

for any $t \geq s \geq u$. In this paper following [35] we accept the following.

**Definition 4.1.** The dynamical map $\Lambda_t$ corresponds to the Markovian evolution if and only if it is divisible.

Interestingly, the property of being Markovian (or divisible) is fully characterized in terms of the local generator $L_t$. Note that if $\Lambda_t$ satisfies (11), then $V_{t,s}$ satisfies

$$\partial_t V_{t,s} = L_t V_{t,s}, \quad V_{t,s} = \mathbb{I},$$

(20)

and the corresponding solution reads

$$V_{t,s} = T \exp \left( \int_s^t L_u \, du \right).$$

(21)

The central result consists in the following.

**Theorem 4.1.** The map $\Lambda_t$ is divisible if and only if $L_t$ has the GKSL form for all $t$.

**Proof.** (The proof goes similarly as for the time-independent case (cf [1, 3].) Assume that $\Lambda_t$ is divisible, that is, $V_{t,s}$ satisfies (20). Then, one has

$$L_t = \lim_{\epsilon \to 0} \frac{V_{t+s,\epsilon} - \mathbb{I}}{\epsilon},$$

(22)
for any \( t \). Now, to compute \( L_t \) let \( F_0 \) be the orthonormal basis in \( M_d(\mathbb{C}) \) such that \( F_0 = \frac{1}{\sqrt{d}} \). Now, since \( V_{t,s} \) is completely positive, one has the corresponding Kraus representation
\[
V_{t,s} \rho = \sum_{a, \beta = 0}^{d^2-1} c_{a \beta} \rho \rho F_{a}^{\dagger},
\]
where the matrix \( c_{a \beta} \) is positive definite for all \( t \). Thus \( V_{t,s} \) implies \( c_{a \beta} = \delta_{a \beta} \rho \) for all \( a, \beta = 0, 1, \ldots, d^2 - 1 \). Now, one finds
\[
L_t \rho = \lim_{\epsilon \to 0} \left\{ \frac{c_{00}(t + \epsilon, t) - d}{\epsilon} F_0 \rho F_0 + \sum_{k=1}^{d^2-1} \frac{c_{0k}(t + \epsilon, t)}{\epsilon} F_0 \rho F_k + \sum_{k=1}^{d^2-1} \frac{c_{k0}(t + \epsilon, t)}{\epsilon} F_k \rho F_0 + \sum_{k, l=1}^{d^2-1} \frac{c_{kl}(t + \epsilon, t)}{\epsilon} F_k \rho F_l^{\dagger} \right\}.
\]
where
\[
\begin{align*}
a_{00}(t) &= \lim_{\epsilon \to 0} \frac{c_{00}(t + \epsilon, t) - d}{\epsilon}, \
a_{0k}(t) &= \lim_{\epsilon \to 0} \frac{c_{0k}(t + \epsilon, t)}{\epsilon}, \
a_{kl}(t) &= \lim_{\epsilon \to 0} \frac{c_{kl}(t + \epsilon, t)}{\epsilon}.
\end{align*}
\]
Note that the Hermitian time-dependent matrix \( a_{kl}(t) \) is positive definite. Moreover, let
\[
G_t = \frac{1}{d^2} \sum_{k=1}^{d^2-1} a_{0k}(t) F_k,
\]
with \( A_t \) being defined by
\[
A_t = \frac{1}{\sqrt{d}} \sum_{k=1}^{d^2-1} a_{0k}(t) F_k.
\]
Finally, one obtains the following formula for the local generator:
\[
L_t \rho = -i[H_t, \rho] + \{G_t, \rho\} + \sum_{k=1}^{d^2-1} a_{k0}(t) F_k \rho F_k^{\dagger}.
\]
Taking into account that \( A_t \) is trace preserving, one has \( \text{Tr}(L_t \rho) = 0 \) for all \( \rho \) which implies
\[
G_t = -\frac{1}{2} \sum_{k,l=1}^{d^2-1} a_{k0}(t) F_k \rho F_k^{\dagger},
\]
and hence
\[
L_t \rho = -i[H_t, \rho] + \sum_{k,l=1}^{d^2-1} a_{k0}(t) \left( F_k \rho F_k^{\dagger} - \frac{1}{2} F_k^{\dagger} F_k \rho \right).
\]
reproduces the standard GKSL form of \( L_t \) (recall that \( a_{k0}(t) \) is positive definite).

Assume now that \( L_t \) is defined by (27). It may be rewritten as follows:
\[
L_t = \Phi_t - \Psi_t,
\]
where \( \Phi_t \) is a family of completely positive maps
\[
\Phi_t \rho = \sum_{k,l=1}^{d^2-1} a_{k0}(t) F_k \rho F_l^{\dagger},
\]
and
\[
\Psi_t \rho = C_t \rho - \rho C_t^{\dagger},
\]
where
\[
C_t = iH_t + G_t.
\]
Actually, due to (26) one has \( G_t = -\frac{1}{2} \Phi_t \). Note that by construction \( \text{Tr}(L_t \rho) = 0 \), and hence the corresponding solution \( V_{t,s} \) is trace preserving. It remains to show that \( V_{t,s} \) is completely positive for all \( t \). Considering the following equation
\[
\partial_s N_{t,s} = -\Psi_s N_{t,s}, \quad N_{t,s} = \mathbb{I},
\]
on one easily finds
\[
N_{t,s} \rho = Y_{t,s} \rho Y_{t,s}^{\dagger},
\]
where \( Y_{t,s} \) itself satisfies \( \partial_s Y_{t,s} = C_t Y_{t,s} \), and hence
\[
Y_{t,s} = T \exp \left( \int_{s}^{t} C_u du \right).
\]
It is clear that \( N_{t,s} \) is completely positive. Moreover, it is invertible and the inverse \( N_{s,t} \) reads
\[
N_{s,t} \rho = Y_{s,t} \rho Y_{s,t}^{\dagger},
\]
where
\[
Y_{s,t} = T_0 \exp \left( -\int_{t}^{s} C_u du \right),
\]
with \( T_0 \) denoting anti-chronological operator. Hence, \( N_{s,t}^{-1} \) is completely positive as well. To solve the original equation (26), let us pass to the ‘interaction’ picture and define
\[
V_{t,s} = N_{t,s} V_{t,s}^{\text{(int)}},
\]
One finds
\[
\partial_s V_{t,s}^{\text{(int)}} = \Phi_{t,s}^{\text{(int)}} V_{t,s}^{\text{(int)}}, \quad V_{t,s}^{\text{(int)}} = \mathbb{I},
\]
where
\[
\Phi_{t,s}^{\text{(int)}} = N_{t,s}^{-1} \circ \Phi_t \circ N_{t,s}^{-1}.
\]
The above formula shows that \( \Phi_{t,s}^{\text{(int)}} \) is completely positive being the composition of three completely positive maps: \( N_{t,s} \), \( \Phi_t \) and \( N_{t,s}^{-1} \). One easily solves (38) and obtains
\[
V_{t,s}^{\text{(int)}} = T \exp \left( \int_{t}^{s} \Phi_{t,s}^{\text{(int)}} du \right).
\]
It is therefore clear that \( V_{t,s}^{\text{(int)}} \) can be represented as the following series:
\[
V_{t,s}^{\text{(int)}} = 1 + \int_{t}^{s} \partial_1 \Phi_{t,s}^{\text{(int)}} + \int_{t}^{s} \partial_1 \Phi_{t,s}^{\text{(int)}} \circ \Phi_{t,s}^{\text{(int)}} + \cdots.
\]
It shows that \( V_{t,s}^{\text{(int)}} \) is completely positive being a sum of completely positive maps. Hence, taking into account formula (37) it finally shows that \( V_{t,s} \) is completely positive. □

It is clear from (41) that the complete positivity of \( \Phi_{t,s}^{\text{(int)}} \) is sufficient for the complete positivity of \( V_{t,s}^{\text{(int)}} \). Note that formula (39) implies
\[
\Phi_t^\# = N_t^\# \circ \Phi_{t,s}^{\text{(int)}\#} \circ N_{t,s}^{-1},
\]
where $\Lambda^\# : B(\mathcal{H}) \to B(\mathcal{H})$ denotes a dual map defined by
\[ \text{Tr}(\rho \cdot \Lambda^\# a) = \text{Tr}(a \cdot \Lambda \rho), \] (43)
for all $\rho \in T(\mathcal{H})$ and $a \in B(\mathcal{H})$. Now, if $\Phi^\text{(init)}_t$ is completely positive, then formula (42) proves that $\Phi_t$ is completely positive as well. Hence, the complete positivity of $\Phi_t$ cannot be relaxed.

Let us observe that a class of Markovian evolution may easily be generalized as follows: we are looking for the solution of (11) with $L_t$ being represented as in (28). Our aim is to find $\Phi_t$ and $\Psi_t$ such that $L_t$ defines a legitimate generator. Suppose that we have given a completely positive map $N_t$ satisfying $N_0 = \mathbb{I}$ and that $N_t$ is not trace preserving (if it were then it would represent legitimate dynamics and hence we are done). Now, let us define $\Phi_t$ as a local in the time generator such that
\[ \dot{N}_t = -\Psi_t N_t, \quad N_0 = 1. \] (44)
One obviously has $\Psi_t = -N_t^{-1} N_{-t}$. Now, the question is: does there exist $\Phi_t$ such that $L_t = \Phi_t - \Psi_t$ provides a legitimate generator? Note that the solution of (11) has the following form $\Lambda^\text{(init)}_t = 1 + \int_0^t dt \Phi^\text{(init)}_t$, where $\Lambda^\text{(init)}_t$ is defined in terms of the following series:
\[ \Lambda^\text{(init)}_t = 1 + \int_0^t dt \Phi^\text{(init)}_t + \int_0^t dt \int_0^t dt_1 \Phi^\text{(init)}_t \circ \Phi^\text{(init)}_t + \cdots, \] (45)
where $\Phi^\text{(init)} = N_t^{-1} \circ \Phi_t \circ N_t$. It is clear that if $\Phi^\text{(init)}_t$ is completely positive, so is $\Lambda_t$. Now, $\Phi_t = N_t \circ \Phi_t^\text{(init)} \circ N_t^{-1} = \Theta_t N_t^{-1}$,
(46)
with $\Theta_t := N_t \Phi_t^\text{(init)}$ being a completely positive map. Note that $\Phi_t$ need not be completely positive. However, if $N_t^{-1}$ is completely positive, then necessarily $\Phi_t$ is completely positive as well. Hence, if $\Phi_t$ is constructed by (46), then $\Lambda_t$ is completely positive. It remains to check for trace preservation. Note that $\Lambda_t$ is trace preserving if $L^\# t \mathbb{I} = 0$. Therefore, one has
\[ L^\# t \mathbb{I} = N_t^{-1} \mathbb{I} (\Theta_t^\# + N_t^\#) \mathbb{I}, \] (47)
and hence if
\[ \Theta_t^\# \mathbb{I} + N_t^\# \mathbb{I} = 0, \] (48)
then $\Lambda_t$ defines the legitimate dynamical map. It is therefore clear that if
\[ -N^\# t \mathbb{I} \geq 0, \] (49)
then one can always find a completely positive $\Theta_t$ such that the normalization condition (48) holds. Clearly, the choice of $\Theta_t$ is highly non-unique. If $\Theta_t$ satisfies (48) and $M_t$ is an arbitrary family of quantum channels, then the following ‘gauge transformation’ $\Theta_t \to \Theta^\text{M}_t := M_t \Theta_t$ gives rise to another admissible $\Theta^\text{M}_t$, satisfying (48).

**Proposition 4.1.** If $N_t$ with $N_0 = \mathbb{I}$ is a family of completely positive maps satisfying (49), then there exists a completely positive $\Theta_t$ satisfying (48) such that $L_t = (\Theta_t + N_t)N_t^{-1}$ gives rise to the legitimate local generator.

Actually, condition (49) is always satisfied for the Markovian evolution. Indeed, taking into account (33) one finds
\[ \partial_s N^\# t_{s,t} \mathbb{I} = (\partial_s X^\#_s, X^\# s, + X^\#_s, (\partial_s X^\#_s)), \] (50)
and hence using $\partial_s X^\#_{s,t} = C_s X^\#_{s,t}$ one finds
\[ \partial_s N^\# t_{s,t} \mathbb{I} = X^\#_{s,t} (C_s + C^\#_s) X^\#_{s,t} = 2X^\#_{s,t} G X^\#_{s,t}, \] (51)
which shows that $\partial_s N^\# t_{s,t} \mathbb{I} \leq 0$ due to $G_t \leq 0$. Therefore, the presented method generalizes the Markovian generator keeping $N_t$ completely positive and satisfying (49), but admitting $\Phi_t$ to be not completely positive. It should be stressed that this construction provides a local analogue of semi-Markov dynamics constructed recently in [38].

**Example 4.1.** Consider the following family of completely positive maps:
\[ N_t \rho = \sum_{k,l=1}^N n_{kl}(t) e_{kk} \rho e_{ll}, \] (52)
where $e_{ij} = [i] [j]$, and the matrix $n_{kl}(t)$ is positive definite with $n_{kl}(0) = 1$. Then, one easily finds for $\Psi_t$:
\[ \Psi_t \rho := -N_t N_t^{-1} \rho = -\sum_{k,l=1}^N n_{kl}(t) n_{lk}(t) e_{kk} \rho e_{ll}. \] (53)
Note that condition (49) is equivalent to the following condition for the diagonal elements:
\[ n_{kk}(t) \leq 0, \] (54)
which implies $n_{kk}(t) \leq n_{kk}(0) = 1$. Now, let $\Theta_t$ be a family of completely positive maps
\[ \Theta_t \rho = \sum_{k,l=1}^N \theta_{kl}(t) e_{kk} \rho e_{ll}, \] (55)
where the matrix $\theta_{kl}(t)$ is positive definite. The normalization condition (48) shows that the diagonal elements of the matrix $\theta_{kl}(t)$ are uniquely determined by $\theta_{kl}(t) = -\hat{n}_{kk}(t)$. The off-diagonal elements $\theta_{kl}(t)$ are arbitrary provided that $\theta_{kl}(t)$ is positive definite. The simplest choice corresponds to $\theta_{kl}(t) = 0$ for $k \neq l$. It finally gives
\[ \Theta_t \rho = \Theta_t N_t^{-1} \rho = \sum_{k=1}^N \hat{n}_{kk}(t) e_{kk} \rho e_{kk}, \] (56)
and hence
\[ L_t \rho = (\Phi_t - \Psi_t) \rho = \sum_{k,l=1}^N \hat{n}_{kl}(t) e_{kk} \rho e_{ll}, \] (57)
provides the pure decoherence dynamics: $L_t \rho = \sum_{k,l=1}^N c_{kl}(t) e_{kk} \rho e_{ll}$ with
\[ c_{kl}(t) = n_{kl}(t), \quad (k \neq l) \quad \text{and} \quad c_{kk} = 1. \] (58)
By construction the matrix $c_{kl}(t)$ is positive definite.
5. Characterizing Markovian dynamics

In this section we analyse special properties of divisible (and hence Markovian) dynamical maps. Let us recall that if a linear map \( \Lambda : T(\mathcal{H}) \to T(\mathcal{H}) \) is trace preserving, then \( \Lambda \) is positive if and only if
\[
||\Lambda a|| \leq ||a||,
\]
for all Hermitian \( a \). Note that \( \Lambda \) need not be contractive for non-Hermitian elements. However, if \( \Lambda \) is completely positive, then (59) holds for all \( a \in \mathcal{B}(\mathcal{H}) \). Actually, it turns out [39] that if \( \Lambda \) is 2-positive and trace preserving, then
\[
||\Lambda|| := \sup_{||a||=1} ||\Lambda a|| = 1.
\]

**Corollary 5.1.** If \( \Lambda \) is a dynamical map, then \( ||\Lambda|| = 1 \), that is, \( \Lambda \) is contractive in the trace norm
\[
||\Lambda a|| \leq ||a||.
\]
Moreover, if \( \Lambda \) is a divisible map, then
\[
\frac{d}{dt} ||\Lambda a|| \leq 0,
\]
for an arbitrary \( a \in T(\mathcal{H}) \).

A similar property holds for dynamical maps in the Heisenberg picture. Recall that if \( \Lambda \) is a dynamical map in the Schrödinger picture, then its dual \( \Lambda^\dagger : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}) \) corresponds to the Heisenberg picture. It is clear that \( \Lambda^\dagger \) is a unital completely positive map for all \( t \geq 0 \). If \( \Lambda^\dagger \) is unital and completely positive, then its operator norm satisfies
\[
||\Lambda^\dagger|| := \sup_{||a||=1} ||\Lambda^\dagger a|| = 1.
\]

**Corollary 5.2.** If \( \Lambda^\dagger \) is a dynamical map in the Heisenberg picture, then \( ||\Lambda^\dagger|| = 1 \), that is, \( \Lambda^\dagger \) is contractive in the operator norm
\[
||\Lambda^\dagger a|| \leq ||a||.
\]
Moreover, if \( \Lambda \) is a divisible map, then
\[
\frac{d}{dt} ||\Lambda^\dagger a|| \leq 0,
\]
for an arbitrary \( a \in \mathcal{B}(\mathcal{H}) \).

**Example 5.1.** Considering once more the generator defined in (14), one has
\[
||\Lambda^\dagger a\sigma_+|| \leq ||a\sigma_+|| = e^{-\Gamma t} ||\sigma_+|| = e^{-\Gamma t} - \Gamma t = e^{-\Gamma t} - \gamma_t \Gamma_t,
\]
where \( \gamma_t = |\langle 0 | \rangle \). It implies
\[
\frac{d}{dt} ||\Lambda^\dagger a\sigma_+|| = -\gamma_t \Gamma_t,
\]
which shows that Markovianity of \( \Lambda_t \) implies \( \gamma_t \geq 0 \).

Let us observe that if the total Hamiltonian \( H \) of the 'system + reservoir' has a discrete spectrum
\[
H = \sum_a \epsilon_a P_a,
\]
then the dynamical map \( \Lambda_t \) defined by
\[
\Lambda_t \rho = \text{Tr}_R[e^{-iHt} (\rho \otimes \omega_R) e^{iHt}]
\]
has the following form:
\[
\Lambda_t \rho = \sum_{\alpha,\beta} e^{-i(t - t') \epsilon} \Lambda_{\alpha \beta} \rho,
\]
where \( \Lambda_{\alpha \beta} = \text{Tr}_R (P_\beta \rho \otimes \omega_R P_\alpha) \). It is clear that due to the presence of the oscillatory terms \( e^{-i(t - t') \epsilon} \) the corresponding trace norm \( ||\Lambda_t a|| \) is an almost quasi-periodic function and hence cannot be monotonically decreasing. It proves that in such a case one obtains a genuine non-Markovian dynamics.

If \( \Lambda : T(\mathcal{H}) \to T(\mathcal{H}) \) is a linear map, then one defines the so-called diamond norm
\[
||\Lambda|| = \sup_{||W||=1} ||(1 \otimes \Lambda) W||_1.
\]

**Theorem 5.1.** Let \( \Lambda_t \) be a dynamical map. Then, the following conditions are equivalent:

1. \( \Lambda_t \) is divisible,
2. \( ||V_{t,s}|| = 1 \) for all \( t \geq s \) and
3. one has
\[
\frac{d}{dt} ||(1 \otimes \Lambda_t) W||_1 \leq 0,
\]
for all Hermitian \( W \in T(\mathcal{H} \otimes \mathcal{H}) \).

The corresponding theorem in the Heisenberg picture can be formulated as follows: recall that \( \Lambda^\dagger : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}) \) is completely bounded if
\[
||\Lambda^\dagger||_{cb} := ||(1 \otimes \Lambda^\dagger)||_\infty < \infty,
\]
and it is completely contractive if \( ||\Lambda^\dagger||_{cb} \leq 1 \).

**Theorem 5.2.** Let \( \Lambda^\dagger \) be a dynamical map in the Heisenberg picture. Then, the following conditions are equivalent:

1. \( \Lambda^\dagger \) is divisible,
2. \( ||V_{t,s}^\dagger||_{cb} = 1 \) for all \( t \geq s \), and hence \( V_{t,s}^\dagger \) is completely contractive and
3. one has
\[
\frac{d}{dt} ||(1 \otimes \Lambda^\dagger) A||_1 \leq 0,
\]
for all \( A \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H}) \).

The Markovian evolution may be characterized in a slightly different way: we know that \( \Lambda_t \) corresponds to Markovian evolution iff the two-parameter family of propagators \( V_{t,s} \) is completely positive for \( t \geq s \). Denote by \( \psi^+ \) the maximally entangled state in \( \mathcal{H} \otimes \mathcal{H} \) and let
\[
P^+ = |\psi^+\rangle \langle \psi^+|.
\]
Note that \( V_{t,s} \) is completely positive if and only if \( (1 \otimes V_{t,s}) P^+ \geq 0 \) which is equivalent to the following simple condition:
\[
V_{t,s} := ||(1 \otimes V_{t,s}) P^+||_1 = 1.
\]
Let us define
\[
g_t := \frac{\text{d}V_{t,s}}{\text{d}t} ||_{us = t},
\]
that is,
\[
g_t = \lim_{\epsilon \to 0^+} \frac{V_{t,s+\epsilon,t} - 1}{\epsilon},
\]
where we have used \( V_{t,t} = 1 \). Taking into account that
\[
V_{t,s+\epsilon,t} = 1 + \epsilon L_t + O(\epsilon^2),
\]
one finds
\[
g_t = \lim_{\epsilon \to 0^+} \frac{||P^+ + \epsilon (1 \otimes L_t) P^+||_1 - 1}{\epsilon}.
\]
Corollary 5.3 ([35]). A map $\Lambda_t$ is divisible if and only if $g_t = 0$ for all $t \geq 0$.

Finally, let us provide the characterization of the corresponding generator in the Heisenberg picture. Let us recall that if $\Lambda^\# : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is unital and 2-positive the following Kadison inequality holds:

$$\Lambda^\# (aa^*) \geq \Lambda^\# (a) \Lambda^\# (a^*).$$

This inequality may be used to characterize Markovian generators. Note that Markovian dynamics $\Lambda^\#_t$ satisfies

$$\partial_t V^\#_{t,s} = V^\#_{t,s} L^\#_{t,s}, \quad V^\#_{t,s} = 1,$$

where $V^\#_{t,s}$ denotes the dual propagator. Now, differentiating the Kadison inequality

$$V^\#_{t,s} (aa^*) \geq V^\#_{t,s} (a) V^\#_{t,s} (a^*),$$

one finds

$$V^\#_{t,s} L^\#_{t,s} (aa^*) \geq V^\#_{t,s} L^\#_{t,s} (a) \cdot V^\#_{t,s} (a^*) + V^\#_{t,s} (a) \cdot V^\#_{t,s} (a^*).$$

Taking $t = s$ and using $V^\#_{t,s} = 1$ one obtains

$$L^\#_{s} (aa^*) \geq L^\#_{s} (a) \cdot a^* + a \cdot L^\#_{s} (a^*).$$

**Definition 5.1.** A Hermitian map $\Psi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is dissipative iff

$$\Psi(aa^*) \geq \Psi(a) \cdot a^* + a \cdot \Psi(a^*),$$

for all $a \in \mathcal{B}(\mathcal{H})$. $\Psi$ is completely dissipative if $1 \otimes \Psi$ is dissipative.

Now, one has

**Proposition 5.1.** $\Lambda^\#_t$ defines Markovian dynamics in the Heisenberg picture if and only if $L^\#_t$ is completely dissipative.

6. Simple criteria for Markovianity

In this section we develop a series of necessary criteria for Markovian dynamics. It turns out that dynamics represented by a divisible map displays characteristic monotonic behaviour for several interesting quantities. Breaking monotonicity reveals the non-Markovian character of the corresponding quantum evolution. Therefore, the violation of any of Markovianity criterion provides a clear sign for non-Markovian behaviour. Hence each Markovianity criterion may be equivalently called non-Markovianity witnesses in analogy with entanglement witnesses and separability criteria.

6.1. Distinguishability

The trace norm defines a natural distance between quantum states represented by density operators: given two density operators $\rho$ and $\sigma$ one defines

$$D(\rho, \sigma) = \frac{1}{2} \|\rho - \sigma\|_1.$$  

The quantity $D(\rho, \sigma)$ is usually interpreted as a measure of distinguishability of the quantum states $\rho$ and $\sigma$. It is well known that if $\Lambda$ is a positive trace-preserving map, then

$$D(\Lambda \rho, \Lambda \sigma) \leq D(\rho, \sigma).$$

**Corollary 6.1.** If $\Lambda_t$ is a divisible map, then

$$\frac{d}{dt} D(\Lambda_t \rho, \Lambda_t \sigma) \leq 0,$$

that is, for the Markovian evolution, distinguishability of any pair of initial states monotonically decreases.

It is well known that if $\Lambda_t$ corresponds to the unitary dynamics $\Lambda_t \rho = U_t \rho U^+_t$, with unitary $U_t$, then $D(\Lambda_t \rho, \Lambda_t \sigma) = 0$. Moreover, if $\Lambda_t = e^{\mathcal{L} t}$ represents the dynamical semigroup, then $D(\Lambda_t \rho, \Lambda_t \sigma) < 0$. The above property was used by Breuer et al [34] as another definition of Markovianity. This criterion identifies non-Markovian dynamics with certain physical features of the system–reservoir interaction. They define non-Markovian dynamics as a time evolution for the open system characterized by a temporary flow of information from the environment back into the system. This backflow of information may manifest itself as an increase in the distinguishability of pairs of evolving quantum states. It turns out that these two concepts of Markovianity do not agree (see e.g. [37, 36]). Clearly, divisibility implies (87), but the converse need not be true.

6.2. Fidelity

Given two density operators $\rho$ and $\sigma$ one defines the Uhlmann fidelity

$$F(\rho, \sigma) = \langle \sqrt{\rho} \sigma \sqrt{\rho} \rangle. $$

Equivalently, one has

$$F(\rho, \sigma) = ||\sqrt{\rho} \sqrt{\sigma}||_1^2,$$

which shows that $F(\rho, \sigma) = F(\sigma, \rho)$. One proves

$$1 - F(\rho, \sigma) \leq D(\rho, \sigma) \leq \sqrt{1 - F(\rho, \sigma)^2}. $$

Moreover, If $\Lambda$ is a quantum channel, then

$$F(\rho, \sigma) \leq F(\Lambda \rho, \Lambda \sigma).$$

**Corollary 6.2.** If $\Lambda_t$ is a divisible map, then

$$\frac{d}{dt} F(\Lambda_t \rho, \Lambda_t \sigma) \geq 0.$$  

6.3. Entropic quantities

Let us recall the definition of Renyi $S_\alpha$ and Tsallis $T_q$ relative entropies

$$S_\alpha(\rho || \sigma) = \frac{1}{\alpha - 1} \log[\text{Tr} \rho^\alpha \sigma^{1-\alpha}],$$

for $\alpha \in [0, 1) \cup (1, \infty)$ and

$$T_q(\rho || \sigma) = \frac{1}{1-q} [1 - \text{Tr} \rho^q \sigma^{1-q}],$$

for $q \in [0, 1)$. Note that in the limit,

$$\lim_{\alpha \to 1} S_\alpha(\rho || \sigma) = \lim_{q \to 1} T_q(\rho || \sigma) = S(\rho || \sigma),$$

then one recovers the well-known formula for the relative entropy

$$S(\rho || \sigma) = \text{Tr} (\rho \log \rho - \log \sigma).$$
It turns out [5, 32] that if $\Lambda$ is a quantum channel, then $S_\alpha$ and $T_\alpha$ satisfy

$$S_\alpha(\Lambda \rho \parallel \Lambda \sigma) \leq S_\alpha(\rho \parallel \sigma), \quad T_\alpha(\Lambda \rho \parallel \Lambda \sigma) \leq T_\alpha(\rho \parallel \sigma), \quad (96)$$

for $\alpha \in [0, 1) \cup (1, 2]$, and $q \in [0, 1)$. Clearly, the same property holds for the relative entropy.

**Corollary 6.3.** If $\Lambda_t$ is a divisible map, then

$$\frac{d}{dt} S_{\alpha}(\Lambda_t \rho \parallel \Lambda_t \sigma) \leq 0, \quad \frac{d}{dt} T_{\alpha}(\Lambda_t \rho \parallel \Lambda_t \sigma) \leq 0, \quad (97)$$

for $\alpha \in [0, 1) \cup (1, 2]$, $q \in [0, 1)$, and

$$\frac{d}{dt} S(\Lambda_t \rho \parallel \Lambda_t \sigma) \leq 0. \quad (98)$$

6.4. **Entanglement measures**

Consider a composed system living in $\mathcal{H} \otimes \mathcal{H}'$ and let $W$ be an arbitrary density matrix in $\mathcal{H} \otimes \mathcal{H}'$. It is well known [5] that for an arbitrary genuine entanglement measure $E$ one has

$$E[(\Phi \otimes \Phi')(W)] \leq E(W), \quad (99)$$

where $\Phi : T(\mathcal{H}) \rightarrow T(\mathcal{H})$ and $\Phi' : T(\mathcal{H}') \rightarrow T(\mathcal{H}')$ are the quantum channels. Denote by $W_t$ the trajectory $W_t = (\Lambda_t \otimes 1)W$ starting at $W$. Now, if $E$ is an entanglement measure, then

$$E(W_t) \leq E(W), \quad (100)$$

for an arbitrary dynamical map $\Lambda_t$. It is, therefore, clear that if $\Lambda_t$ is divisible, then

$$\frac{d}{dt} E(W_t) \leq 0, \quad (101)$$

for each initial state $W$. Of course, the above relation is nontrivial only if the initial state $W$ is entangled. In particular, if $W = P^+$ (maximally entangled state), then the entanglement measure by $E$ monotonically decreases from the maximal value $E(P^+)$ [35].

7. **Conclusions**

We characterized a class of Markovian dynamics using the concept of the divisible dynamical map. Interestingly, Markovian dynamics is fully controlled in the local approach via the properties of the corresponding local generator. Characterization of Markovianity in terms of the memory kernel $K_t$ is an open problem. It should be stressed that the standard Markovian master equation (1) is defined by the corresponding macroscopic model and a suitable Markovian approximation. Note that for the general Markovian evolution characterized by a time-dependent local generator the construction of the corresponding macroscopic model is not known.

Moreover, we provided a family of criteria which can distinguish Markovian and non-Markovian dynamics. These Markovianity criteria are based on a simple observation that Markovian dynamics implies the monotonic behaviour of several well-known quantities such as distinguishability of states, fidelity, relative entropy and genuine entanglement measures.

We stress that the problem of characterization of admissible $L_t$ and $K_t$ giving rise to legitimate dynamical map $\Lambda_t$ is rather untractable. Only the commutative case is fully controlled. One may wonder if there is another way to describe dynamics of an open system. In a forthcoming paper we propose a new approach to this problem.

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