An Exploration of a New Group of Channel Symmetries

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Abstract—We study a certain symmetry group associated to any given communication channel, which can informally be viewed as the set of transformations of the set of inputs that “commute” with the action of the channel. In a general setting, we show that the distribution over the inputs that maximizes the mutual information between the input and output of a given channel is a “fixed point” of the action of the channel’s group. We consider as examples the groups associated with the binary symmetric channel and the binary deletion channel. We show that the group of the binary symmetric channel is extremely large (it contains a number of elements that grows faster than any exponential function of \( n \)), and we give empirical evidence that the group of the binary deletion channel is extremely small (it contains a number of elements constant in \( n \)). This serves as some formal justification for why the analysis of the binary deletion channel has proved much more difficult than its memoryless counterparts.

Index Terms—channel symmetries, groups.

I. INTRODUCTION

Many natural models of communication errors, like those captured by the class of discrete memoryless channels, are by now well understood. Their capacity has been known since Shannon’s original paper \([1]\), and codes with efficient encoding and decoding algorithms have been proved to achieve the capacity (e.g., \([2]\)). By contrast, other similarly natural error models, like those captured by the binary deletion channel or other synchronization channels, are much less well understood. For example, the capacity of the deletion channel is unknown, although several lower and upper bounds have been proved \([3]–[6]\). In this paper, we work towards an answer of the following question: can we give a formal justification for why the binary deletion channel is so much more difficult to analyze than the binary symmetric channel? In a talk in 2008, Mitzenmacher \([7]\) gave the following example to illustrate the difference between the two: consider the strings

\[
s_1 = 00000, \quad s_2 = 01010.
\]

From the point of view of the binary deletion channel, these two strings seem quite different: for example, deleting any one bit from \( s_1 \) will produce the same output, while deleting different single bits from \( s_2 \) will produce all different outputs. On the other hand, from the point of view of the binary symmetric channel, these two strings seem “equivalent,” in the sense that there is no clear formal way to distinguish them in terms of the consequences that bit flips have on them. Mitzenmacher went on to say that “erasure and error channels have pleasant symmetries; deletion channels do not” and that “understanding this asymmetry seems fundamental” \([7]\).

In this paper, we study a certain symmetry group associated to any given communication channel, which can be informally viewed as the set of transformations of the set of inputs that “commute” with the action of the channel. In a general setting, we show that the distribution over the inputs that maximizes the mutual information between the input and output of a given channel is a “fixed point” of the action of the channel’s group. We then consider as examples the groups associated with the binary symmetric channel and the binary deletion channel. We show that the group of the binary symmetric channel is extremely large (it contains a number of elements that grows faster than any exponential function of \( n \)), and we give empirical evidence that the group of the binary deletion channel is extremely small (it contains a number of elements constant in \( n \)). Given the group of a channel, we define a notion of equivalence between strings: we say strings \( s_1 \) and \( s_2 \) are equivalent if there exists a group element mapping \( s_1 \) to \( s_2 \). In this formal sense, the two example strings given by Mitzenmacher are equivalent with respect to the binary symmetric channel, but not with respect to the binary deletion channel. More generally, given any code \( C \) for a channel, applying any group element to all its codewords yields a new code \( C' \) that is “equivalent” to \( C \) in a formal sense: \( C \) and \( C' \) have the same number of codewords, and the existence of a decoder for \( C \) implies the existence of a decoder for \( C' \) with the same error probability, and vice versa.

The invariance of specific deterministic codes with respect to symmetry groups has been studied in the literature, and several results have shown that “sufficient symmetry” can often imply good code performance. Specifically, Kudekar et al \([8]\) showed that Reed-Muller codes achieve capacity on erasure channels via the general result that any linear code whose rate is in \((0,1)\) and which is invariant under a group of permutations of its indices that is doubly transitive must achieve capacity on erasure channels. Very recently, Reeves and Pfister \([9]\) extended this result, showing that Reed-Muller codes achieve capacity over general binary memoryless channels, by similarly appealing to the doubly transitive nature of the permutation group of such codes. Establishing a formal relationship between the results of \([8]\) and \([9]\) and the results presented in this paper, if one exists, is an open problem.

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II. DEFINITIONS

A. Notation and Elementary Definitions

Throughout this paper, for $\mathcal{X}$ a set (alphabet), $\mathcal{X}^n$ denotes the set of strings of $n$ symbols from $\mathcal{X}$; we also let $\mathcal{X}^* = \bigcup_{j=0}^{\infty} \mathcal{X}^j$ and $\mathcal{X}^{\leq n} = \bigcup_{j=0}^{n} \mathcal{X}^j$. For $x \in \mathcal{X}^n$, we let $x_i \in \mathcal{X}^{j-i+1}$ denote the substring of $x$ starting at index $i$ and ending at $j$; inclusive; unless otherwise specified, we let $x_i := x_1$. If $\mathcal{X}$ can naturally be viewed as a field, we use $x \in \mathcal{X}^n$ to refer to the vector space element or the string interchangeably. For $x, y \in \mathcal{X}^*$, we let $xy$ denote their concatenation. For $x \in \mathcal{X}^n$, we let $|x| = n$ denote the string length. All logs (hence entropies, etc.) in this paper are of base equal to the alphabet size unless otherwise specified.

To treat symmetry groups of general channels, it will be useful to view a channel as acting on strings of arbitrary length.

**Definition II.1.** For $\Omega$ a probability space and $\mathcal{X}$ a set (alphabet), a channel is a map $\text{Ch} : \mathcal{X}^* \times \Omega \to \mathcal{X}^*$. For $x \in \mathcal{X}^*$, we write $\text{Ch} x$ for the random variable $\omega \mapsto \text{Ch}(x, \omega)$.

We are fixing the input and output alphabets to be the same —this constraint is only superficial, but will allow for a cleaner presentation; in Section [V] we suggest a natural way to overcome this constraint. For completeness, we give definitions of memoryless channels, the binary symmetric channel and the binary deletion channel in this notation.

**Definition II.2.** A channel $\text{Ch} : \mathcal{X}^* \times \Omega \to \mathcal{X}^*$ is memoryless if $x \in \mathcal{X}$ implies $\text{Ch} x \in \mathcal{X}$ with probability 1, and for $x \in \mathcal{X}^n$ we have $\text{Ch} x = \text{Ch}^1 x_1 \ldots \text{Ch}^n x_n$, where equality in distribution and the $\text{Ch}^i$ are independent copies of $\text{Ch}$.

**Definition II.3.** Let $\mathcal{X} = \{0, 1\}$, $p \in [0, 1]$, and $\Omega = (0, 1)^\infty$ (the infinite product space) with a Bernoulli$(p)$ measure (the infinite product measure). The **binary symmetric channel** acts on an input $x \in \mathcal{X}^n$ as $\text{BSC}_p(x, \omega) = x + \omega^n$, where addition is elementwise and mod 2. The **binary deletion channel** acts on an input $x \in \{0, 1\}^n$ as $\text{BDC}_p(x, \omega) = x_1, x_2 \ldots x_{i_k}$, where $|x| - k$ is the hamming weight of (number of ones in) $\omega^n$, and $i_j$ is the index of the $j$th zero in $\omega$.

It will sometimes be useful to consider the action of the channel only on strings of a particular length.

**Definition II.4.** Let $\text{Ch} : \mathcal{X}^* \times \Omega \to \mathcal{X}^*$ be a channel, $n \in \mathbb{N}$, and let $\text{Ch}|_n : \mathcal{X}^n \times \Omega \to \mathcal{X}^*$ be the restriction of $\text{Ch}$ to the strings of length $n$. Suppose there exists $m = m(n) \in \mathbb{N}$ such that the image of $\text{Ch}|_n$ is contained in $\mathcal{X}^m$. Then the $n$th **transition matrix** of $\text{Ch}$ is the linear map $M_n : \mathbb{R}^{\mathcal{X}^n} \to \mathbb{R}^{\mathcal{X}^n}$ giving the transition probabilities of $\text{Ch}|_n$.

Finally we define the automorphism group of a set; the channel symmetry groups we will study will be subgroups of the automorphism group of the message set.

**Definition II.5.** Given a set $A$, the **automorphism group** of $A$, denoted $\text{Aut}(A)$, is the set of bijections from $A$ to itself.

In the case where $A$ is finite, we have $\text{Aut}(A) \cong S_{|A|}$, the group of permutations of $|A|$ elements.

B. Channel Symmetry Groups

Given a channel $\text{Ch}$ over an alphabet $\mathcal{X}$, we consider the subgroup of elements of $\text{Aut}(\mathcal{X}^*)$ which "commute with $\text{Ch}".

**Definition II.6.** Given a channel $\text{Ch} : \mathcal{X}^* \times \Omega \to \mathcal{X}^*$, we let the channel group of $\text{Ch}$ be defined as $G_{\text{Ch}} = \{g \in \text{Aut}(\mathcal{X}^*) : (g^{-1} \text{Ch} g)x \overset{D}{=} \text{Ch} x, |g(x)| = |x| \forall x \in \mathcal{X}^*\}$, where $D$ denotes equality in distribution, and by the conjugation $(g^{-1} \text{Ch} g)x$ we mean the random variable $\Omega \ni \omega \mapsto g^{-1}(\text{Ch}(g(x), \omega))$.

It’s an elementary exercise to check that this is indeed a group in the formal sense. The requirement that the group elements preserve string length will be automatically true for most channels of interest, and it will be a convenient assumption when dealing with general channels in Section [III].

Given the channel group $G_{\text{Ch}}$, we can define a notion of equivalence between strings.

**Definition II.7.** Given two strings $x, y \in \mathcal{X}^*$ and a channel $\text{Ch}$ over the alphabet $\mathcal{X}$, we say $x$ and $y$ are **equivalent with respect to $\text{Ch}$**, and write $x \sim y$, if there exists $g \in G_{\text{Ch}}$ such that $gx = y$. We then define the equivalence class of $x \in \mathcal{X}^*$ as $[x] = \{y \in \mathcal{X}^* : x \sim y\}$.

It’s again an elementary exercise to check that this defines an equivalence relation in the formal sense, and hence partitions the space of messages $\mathcal{X}^*$ into a new set of disjoint equivalent classes $(\mathcal{X}^*/\sim) = \{[x] : x \in \mathcal{X}^*\}$, called the **quotient space** of $\mathcal{X}^*$ by $\sim$. We remark that, given a code $C = \{c_n\}_{n \in \mathbb{N}}$ for $\text{Ch}$, where $c_n \subseteq \mathcal{X}^n$, and a group element $g \in G_{\text{Ch}}$, we can define a new code $gC = \{gc_n\}_{n \in \mathbb{N}}$, where by $gC_n$, we mean $\{gc : c \in c_n\}$. The code $gC$ can be decoded by applying $g^{-1}$ to the received message, and then using any decoder for $C$; by the definition of $G_{\text{Ch}}$, this new decoder will have the same probability of error as the decoder for $C$.

III. GENERAL RESULTS

As our first result, we show that, for a wide class of channels, the distribution over the input strings that achieves the maximum mutual information between input and output is a fixed point of the action of the channel group. As we illustrate in Section [IV] the conditions of the following theorem apply broadly.

**Theorem III.1.** Let $\text{Ch}$ be a channel over an alphabet $\mathcal{X}$. Suppose the $n$th transition matrix $M_n$ of $\text{Ch}$ exists and is full-rank. If we have

$$D \in \arg \max_{D'} I(X'; Y'),$$

where $X' \sim D'$ and $Y' = \text{Ch} X'$, then $gX \overset{D}{=} X$ for $X \sim D$ for all $g \in G_{\text{Ch}}$. Above, the maximum is taken over all probability distributions $D'$ supported on $\mathcal{X}^n$. 
We remark that, by Shannon’s Theorem [1], for memoryless channels Ch, if D is the mutual-information-maximizing distribution from the theorem above and \( X \sim D, Y = Ch X \), then \( \frac{1}{n} I(X; Y) \) is the capacity of Ch, for every n. Even for non-memoryless channels like the deletion channel or other synchronization channels, Dobrushin [10] proved that the capacity is given by the limit of \( \frac{1}{n} I(X; Y) \) as \( n \to \infty \).

Informally, Theorem III.1 shows that, when looking for the distribution that achieves capacity, we can restrict attention to the distributions which “respect the symmetries of the channel.” The following corollary, which follows immediately from Theorem III.1, makes this more concrete.

**Corollary III.2.** If D is the mutual-information-maximizing distribution of Theorem III.1, then D is uniform when restricted to the subsets of \( X^n \) that are equivalence classes with respect to Ch.

In other words, if \( x, y \in X^n \), \( x \sim y \) with respect to Ch, and \( X \sim D \), then \( P(X = x) = P(X = y) \). Hence maximizing the mutual information over all distributions in \( X^n \) is equivalent to maximizing it over the smaller quotient space (\( X^n / \sim \)).

**Proof of Theorem III.1** Fix a channel Ch over an alphabet \( \mathcal{X} \) and a group element \( g \in G_{Ch} \). Suppose D is the mutual-information-maximizing distribution of the statement of the theorem, \( X \sim D \) and \( Y = Ch X \). Then let \( X' = gX \) and \( Y' = Ch X' \). We will prove two claims: (1) that \( I(X; Y) = I(X'; Y') \), and (2) that, under the assumptions of the theorem, the mutual information \( I(X; Y) \) is maximized by a unique distribution of \( X \). The combination of these two claims yields \( X' \overset{D}{\sim} X \), proving the theorem.

For the first claim, we show the stronger statement that \((g^{-1}X', g^{-1}Y') \overset{D}{=} (X, Y)\); since \( g^{-1} \) is a bijection, this then immediately gives (1). Indeed, we have

\[
P((g^{-1}X', g^{-1}Y') = (x, y)) \\
= P(g^{-1}gX = x, g^{-1}Ch g X = y) \\
= P(X = x, g^{-1}Ch g X = y) \\
= P(X = x)P(g^{-1}Ch g X = y | X = x) \\
= P(X = x)P(g^{-1}Ch g x = y) \\
= P(X = x)P(Ch x = y) \\
= P(X = x)P(Ch X = y | X = x) \\
= P(X = x, Ch X = y) \\
= P((X, Y) = (x, y)),
\]

as desired.

For the second claim, if \( X \sim D_X \), it suffices to show that the function \( D_X \mapsto I(X; Y) \) is strictly concave. But writing

\[
I(X; Y) = H(Y) - H(Y | X),
\]

the second term is a linear function of \( D_X \) (it can be written \( H(Y | X) = \mathbb{E}_{x \sim D_X} H(Y | X = x) \)), and the first is a strictly concave function (the entropy) of the distribution of \( Y \) (call it \( D_Y \)). Now we may obtain \( D_Y \) as a linear function of the distribution of \( X \), i.e. \( D_Y = M_n D_X \) (where \( M_n \) is the \( n \)th transition matrix of \( Ch \)), and since \( M_n \) is full-rank, \( D_X \mapsto D_Y \) is an injective linear function, hence \( D_X \mapsto H(Y) \) is strictly concave. This proves (2) and hence the theorem.

Now, as in the cases of the binary symmetric channel or the binary deletion channel, we often deal with not just one channel Ch but a parameterized family of channels \( \{Ch_t\}_{t \in I} \) over some index set I. As we illustrate in Section IV after applying a transformation in the parameter space I, these families may often be viewed as a continuous-time homogeneous Markov chains over the index set \( I = [0, \infty) \). In these cases, in order for the channel group to be useful, we would want it to be independent of the parameter. This is in fact true in a general setting. For simplicity, we will consider only channels \( Ch : \mathcal{X}^* \times \Omega \to \mathcal{X}^* \) which restrict to \( Ch : \mathcal{X}^{\leq n} \times \Omega \to \mathcal{X}^{\leq n} \), i.e. they can only decrease the length of an input string, but not increase it.

While this restriction is unnecessary, it will allow us to deal with distributions over finite sets, simplifying the proofs.

**Theorem III.3.** Let \( \{Ch_t\}_{t \in [0, \infty)} \) be a family of channels over a common alphabet \( \mathcal{X} \) forming a homogeneous Markov chain (assumed to be right-continuous), i.e. for \( t_1, t_2 \in [0, \infty) \) we have \( (Ch_{t_1} \circ Ch_{t_2}) x \overset{D}{=} Ch_{t_1 + t_2} x \) for all \( x \in \mathcal{X}^* \). Suppose further that \( Ch_t \) restricts to a map \( Ch_t : \mathcal{X}^{\leq n} \times \Omega \to \mathcal{X}^{\leq n} \) for all \( n \) and \( t \). Then \( G_{Ch_t} \approx G_{Ch_{t'}} \) for all \( t, t' \in [0, \infty) \).

By the composition \( (Ch_{t_1} \circ Ch_{t_2}) x \) we mean the random variable \( \Omega \times \Omega \ni (\omega, \omega') \mapsto Ch_t(Ch_{t_2}(x, \omega), \omega') \). To prove the theorem above, it’s useful to deal with the transition matrix \( M_{\leq n}(t) : \mathbb{R}^{\mathcal{X}^{\leq n}} \to \mathbb{R}^{\mathcal{X}^{\leq n}} \) that gives the transition probabilities of the restriction \( Ch_t : \mathcal{X}^{\leq n} \times \Omega \to \mathcal{X}^{\leq n} \); this is obtained by stacking the columns of the \( j \)th transition matrices \( M_j(t) : \mathbb{R}^n \to \mathbb{R}^{\mathcal{X}^{\leq n}} \) of \( Ch_t \), for \( 1 \leq j \leq n \). The conditions of the theorem ensure that \( M_{\leq n}(t_1)M_{\leq n}(t_2) = M_{\leq n}(t_1 + t_2) \) for all \( t_1, t_2 \in [0, \infty) \). In this language, and given that the elements of \( G_{Ch_t} \) are assumed to preserve string length, the restriction of some \( g \in G_{Ch_t} \) to the set \( \mathcal{X}^{\leq n} \) may be viewed as a permutation matrix \( P_g : \mathbb{R}^{\mathcal{X}^{\leq n}} \to \mathbb{R}^{\mathcal{X}^{\leq n}} \). Then the condition for inclusion of \( g \) in \( G_{Ch_t} \), is precisely that \( P_g \) commutes with \( M_{\leq n}(t) \), i.e. for \( x \in \mathcal{X}^{\leq n} \),

\[
(g^{-1}Ch_t g)x \overset{D}{=} Ch_t x \quad \forall x \in \mathcal{X}^{\leq n} \\
\iff P_g^{-1}M_{\leq n}(t)P_g = M_{\leq n}(t).
\]

We can now prove the theorem.

**Proof of Theorem III.3** Fixing \( t \in (0, \infty) \), by the discussion above it suffices to show that a permutation matrix \( P : \mathbb{R}^{\mathcal{X}^{\leq n}} \to \mathbb{R}^{\mathcal{X}^{\leq n}} \) commutes with \( M_{\leq n}(t) \) if and only if it commutes with \( M_{\leq n}(1) \). First we consider \( t \in \mathbb{Q} \), so that we have \( t = k/l \) for \( k, l \in \mathbb{N} \). Suppose first that \( P \) commutes with

1Note that the BSC and BDC satisfy this condition.

2This will usually not hold for \( t \) or \( t' \) equal to zero.
\[ M_{\leq n}(t), \text{i.e. we have } P^{-1}M_{\leq n}(k/l)P = M_{\leq n}(k/l). \] Then by the Markov property,
\[ M_{\leq n}(k) = M_{\leq n}(k/l)^t \]
\[ = (P^{-1}M_{\leq n}(k/l)P)^t \]
\[ = P^{-1}M_{\leq n}(k). \]

Now since \( M_{\leq n}(k) \) is a stochastic matrix, it’s positive semi-definite, as is \( P^{-1}M_{\leq n}(k/l)P \), and hence they each have a unique \( k \)th root. Since \( M_{\leq n}(1) \) is a \( k \)th root of the left-hand side and \( P^{-1}M_{\leq n}(1)P \) is a \( k \)th root of the right-hand side, we must have
\[ M_{\leq n}(1) = P^{-1}M_{\leq n}(1)P, \]
as desired. The other direction (for \( t \) rational) follows by an identical argument.

We extend to \( t \) irrational by right-continuity of \( \{M_t\}_{t \geq 0} \). Indeed, we take \( q_m \geq 0 \) such that \( t + q_m \in \mathbb{Q} \) and \( q_m \to 0 \). Then we have
\[ M_{\leq n}(t)M_{\leq n}(q_m) = M_{\leq n}(t + q_m) \]
\[ = P^{-1}M_{\leq n}(t + q_m)P \]
\[ = (P^{-1}M_{\leq n}(t)P)(P^{-1}M_{\leq n}(q_m)P) \]
and letting \( m \to \infty \) by right-continuity we get \( M_{\leq n}(q_m) \to I \) and \( P^{-1}M_{\leq n}(q_m)P \to I \), so \( M_t \) commutes with \( P \). Again, the other direction can be proved by the same argument. \( \square \)

**IV. Examples**

**A. Binary Symmetric Channel**

First, we check that the channel group is independent of the error probability. We note that the binary symmetric channels \( \{\text{BSC}_{p}\}_{p \in [0,1/2]} \) form an inhomogeneous Markov chain in the sense that for \( p_1, p_2 \in [0,1/2], \) we have
\[ \text{BSC}_{p_1} \circ \text{BSC}_{p_2} = \text{BSC}_p, \]
\[ p = p_1(1 - p_2) + (1 - p_1)p_2 = p_1 + p_2 - 2p_1p_2 \in [0,1/2]. \]

Luckily it can be made homogeneous by a transformation in the parameter space. We seek a bijection \( f : [0, \infty) \to [0,1/2] \) such that \( f(0) = 0 \) and for all \( t_1, t_2 \in [0, \infty) \) we have \( \text{BSC}_{f(t_1)} \circ \text{BSC}_{f(t_2)} = \text{BSC}_{f(t_1 + t_2)}, \) i.e. \( f(t_1 + t_2) = f(t_1) + f(t_2) - f(t_1)t_2(1 - 2f(t_1)). \) Dividing by \( t_2 \) and sending \( t_2 \to 0 \) yields the differential equation \( f'(t_1) = f'(0)(1 - f(t_1)), \) with a particular solution \( f(t) = \frac{1}{2}(1 - e^{-2t}), \) which can be verified to satisfy the required properties. Hence defining \( \text{BSC}^t = \text{BSC}_{f(t)} \) now \( \{\text{BSC}^t\}_{t \in [0,\infty)} \) is a homogeneous Markov chain and hence by Theorem [III.1], it has a channel group independent of the parameter \( t \in (0, \infty), \) or equivalently \( p \in (0,1/2). \) We denote this group by \( \mathcal{G}_{\text{BSC}}. \)

In order for Theorem [III.1] to apply, we need to show that the BSC’s \( n \)th transition matrix \( M_n \) is full-rank.

**Lemma IV.1.** For every \( n \) and \( p \in (0,1/2), \) the \( n \)th transition matrix of the BSC\(_p\) is full-rank.

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**Proof.** For \( x \in \{0,1\}^n, \) let \( \bar{x} \in \mathbb{R}^{0,1}^n \) be the probability vector with a 1 in the coordinate corresponding to \( x, \) and zeros in all other coordinates. It suffices to show that \( \{M_n\}_{x \in \{0,1\}^n} \) are linearly independent. Suppose for contradiction there exist \( x_0, \ldots, x_m \in \{0,1\}^m \) distinct and \( \alpha_1, \ldots, \alpha_m \in \mathbb{R} \) such that \( M_n\bar{x}_0 = \alpha_1 M_n\bar{x}_1 + \cdots + \alpha_m M_n\bar{x}_m. \) Note that since \( M_n\bar{x}_0 \) is a probability vector, so must be \( \alpha_1 M_n\bar{x}_1 + \cdots + \alpha_m M_n\bar{x}_m \), and hence so must be \( \bar{x}_1 + \cdots + M_n\bar{x}_m. \) But it’s easy to see that a point mass at \( x_0 \) is the unique distribution at the input which maximizes the probability of \( x_0 \) at the output, and hence the value of the \( x_0 \)th index of \( M_n\bar{x}_0 \) is greater than the value of the \( x_0 \)th index of \( M_n\bar{x}_1 + \cdots + M_n\bar{x}_m, \) yielding the desired contradiction. \( \square \)

We now give two families of examples of elements in \( \mathcal{G}_{\text{BSC}}. \) In each case, we only specify the action of the group element on strings of a particular length \( n. \) A general group element may be formed by any choice a fixed-length transformation per string length.

1) Any permutation of the indices. This is in fact a subgroup of the channel group of any memoryless channel, as is easily verified.

2) Addition of any element. Fix an element \( x \in \{0,1\}^n \) and consider the transformation \( y \mapsto y + x, \) where addition is elementwise and mod 2. This is clearly an element of \( \mathcal{G}_{\text{BSC}} \) with inverse equal to itself.

While there may be other families of transformations, just the transformations of type (2) suffice to show that any two strings of the same length are equivalent with respect to the BSC. Namely, for \( y, z \in \{0,1\}^n, \) let \( x = z - y, \) the map \( w \mapsto w + x \) maps \( y \) to \( z, \) and hence \( y \sim z. \) Then \( |x| = \{0,1\}^n \) for every \( x \in \{0,1\}^n \) and from Corollary [III.2] we recover the classical result that the maximum mutual information in the BSC is achieved by a uniform distribution over the input.

Finally, we note that, just by considering transformations of type (1), the length \( n \) channel subgroup of the BSC, given by the restrictions to the domain \( \{0,1\}^n \) of all elements in \( \mathcal{G}_{\text{BSC}}, \) has size at least \( n!, \) and hence grows faster than any exponential function. This will be in stark contrast with the case of the BDC, which we now consider.

**B. Binary Deletion Channel**

As in the case of the BSC, the binary deletion channels form an inhomogeneous Markov chain \( \{\text{BDC}_p\}_{p \in [0,1]} \). Here the composition rule is
\[ \text{BDC}_{p_1} \circ \text{BDC}_{p_2} = \text{BDC}_p \]
for \( p = p_2 + (1 - p_2)p_1 = p_1 + p_2 - p_1p_2 \) which may be made homogeneous by the same method as in the case of the BSC, arriving at the transformation \( f : [0, \infty) \to [0,1] \) given by \( f(t) = 1 - e^{-t}. \) It’s easily explicitly verified that this satisfies the required relation \( f(t_1 + t_2) = f(t_1) + (1 - f(t_1))f(t_2). \) Hence by Theorem [III.1] there is a unique channel group \( \mathcal{G}_{\text{BDC}} \) for the BDC.
Next we check the applicability of Theorem \textbf{III.1} by showing that the $n$th transition matrix of the BDC is full-rank.

**Lemma IV.2.** For every $n$ and $p \in (0,1)$, the $n$th transition matrix of the BDC$_p$ is full-rank.

**Proof.** Let $\{M_n,x\}$ be exactly as in the proof of Lemma \textbf{V.1} It again suffices to show that they are linearly independent. But note that the $y$th coordinate of $M_n,x$ is nonzero if and only if $x = y$, and hence they are clearly linearly independent. \hfill $\square$

As before, we now give two examples of elements of $G_{\text{BDC}}$; it’s trivial to verify that these are indeed in the channel group.

1) A flip of all bits. This is the same as addition by the all-ones string.

2) A rotation of the bits about the center. This is the transformation $x_1 x_2 \ldots x_n \mapsto x_n x_{n-1} \ldots x_1$.

Each of these operations are of order 2 (they are their own inverses), and they commute. Hence the group they generate is of size 4. Amazingly, we conjecture that these are essentially all the symmetries of the BDC! More precisely, these symmetries certainly generate a subgroup of $G_{\text{BDC}}$; call this subgroup $G_{\text{BDC}}^\text{gen}$. Then two strings $x$ and $y$ are equivalent with respect to $G_{\text{BDC}}$ if there is $g \in G_{\text{BDC}}$ (i.e. either a flip of all bits or a rotation around the center, or their composition) such that $gx = y$. We conjecture that any two strings $x,y \in \{0,1\}^n$ are equivalent with respect to $G_{\text{BDC}}$ if and only if they are equivalent with respect to $G_{\text{BDC}}^\text{gen}$. This in particular would imply that the maximum size of any equivalence class with respect to the BDC is 4. In other words, when searching for the distribution that achieves capacity, while in the case of the BSC symmetry suffices to solve the problem, in the case of the BDC it essentially buys us nothing.

To justify our conjecture, we note that if two strings $x,y \in \{0,1\}^n$ are equivalent with respect to the BDC, then since there is a group element such that $g^{-1} \text{BDC}_{1/2} y = \text{BDC}_{1/2} x$, in particular we must have $H(\text{BDC}_{1/2} y) = H(\text{BDC}_{1/2} x)$. In the data files available in Arvix together with this paper, we give numerical evidence showing that for string lengths $n$ up to 14, the equivalence classes under $G_{\text{BDC}}^\text{gen}$ coincide exactly with the sets of strings of equal entropy when passed through the BDC$_{1/2}$.

**Definition V.1.** Given $\text{Ch} : \mathcal{X}^* \times \mathcal{Y} \to \mathcal{Y}^*$ a channel, for $\mathcal{X}$ and $\mathcal{Y}$ the input and output alphabets, respectively, the generalized channel group of $\text{Ch}$ is defined as

\[ G_{\text{Ch}}^\text{gen} = \{ g \in \text{Aut}(\mathcal{X}^*) : \exists h \in \text{Aut}(\mathcal{Y}^*), (h \text{ Ch } g) x \overset{D}{=} \text{Ch } x, \quad |g(x)| = |x| \forall x \in \mathcal{X}^* \}. \]

The proof of Theorem \textbf{III.1} goes through in this more general setting, but it’s not as clear how to extend Theorem \textbf{III.3} We consider it an interesting open problem to determine, for the channels for which $G_{\text{Ch}}$ is well-defined, whether $G_{\text{Ch}}^\text{gen}$ can be much larger than $G_{\text{Ch}}$.

As a further direction, while Theorem \textbf{III.1} shows that the single distribution that attains the maximum mutual information must be invariant under the group action, it says nothing about the symmetry properties of distributions which are “close” to achieving the maximum mutual information. If a relation between the results of this paper and deterministic code constructions is to be established, this seems like an important first step.

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