Jack polynomials with prescribed symmetry
and hole propagator of spin Calogero-Sutherland model

Yusuke Kato\[1\] and Takashi Yamamoto\[2\]

\[1\] Institut f"{u}r Theoretische Physik, Universit"{a}t zu K"{o}ln, Z"{u}lpicher str. 77, D-50937, K"{o}ln, Germany

\[2\] Department of Physics, Tohoku University, Sendai 980-77, Japan

We study the hole propagator of the Calogero-Sutherland model with SU(2) internal symmetry. We obtain the exact expression for arbitrary non-negative integer coupling parameter $\beta$ and prove the conjecture proposed by one of the authors. Our method is based on the theory of the Jack polynomials with a prescribed symmetry.

I. INTRODUCTION

One of the goals in physics of interacting particle systems is to understand the dynamics. In particular, dynamical properties in one-dimensional systems are anticipated to be intriguing, because interaction effects are crucial and naive pictures based on perturbations lose their validity. As a non-perturbative approach, conformal field theory is a powerful method to study the low energy physics of the Tomonaga-Luttinger liquid. Beyond the conformal limit, on the other hand, integrable systems give us opportunities for analytical study of the dynamics. Among them, the Calogero-Sutherland (CS) model\[1,2\] of particles interacting with the two-body inverse square interaction provides the simplest example of interacting particle systems with nontrivial dynamics. For the spinless CS model, the density-density correlation function\[3–6\], hole propagator\[4,5,7\] and particle propagator\[8,9\] have been obtained analytically.

The spinless CS model has a number of variants. One of them is the spin CS model\[10,11\]. This model describes $n$ particles with coordinates $X = (X_1, \ldots, X_n)$ moving along a circle of length $L$ and each particle possesses a spin with $p$ possible values. The Hamiltonian of the model is given by

$$
\hat{H}_n = -\sum_{i=1}^{n} \frac{\partial^2}{\partial X_i^2} + 2 \left( \frac{\pi}{L} \right)^2 \sum_{1 \leq i < j \leq n} \frac{\beta (\beta + P_{ij})}{\sin^2 \frac{\pi}{L} (X_i - X_j)},
$$

(1)

where $\beta$ is a coupling parameter and $P_{ij}$ is the spin exchange operator. In this paper, we take $\beta$ to be a non-negative integer.

The spin CS model with $p = 2$ is particularly relevant to the condensed matter physics; most one-dimensional systems are realized experimentally in electron systems and hence we should take account of spin degrees of freedom of each particle. Further we can regard the spin CS model with $p = 2$ as a one-dimensional variant of singlet fractional quantum Hall (FQH) system; the ground state of\[12\] can be derived from the Halperin wavefunction\[12\] for the singlet FQH state by the restriction of particle coordinates on a ring in two-dimensional plane.

For the dynamics of the spin CS model with $p = 2$, the expression for the hole part of Green function has been proposed\[13\] relying on the finite size calculations. Subsequently the dynamical density-density and spin-spin correlation functions have been derived by Uglov\[14\] in an exact treatment. Since the paper\[14\] appeared, however, exact derivations of the hole propagator have been still missing. Recently, in\[15\], Dunkl has developed the theory of the Jack polynomials with a prescribed symmetry; those polynomials are symmetric or alternating with respect to the interchange of certain subsets of variables. Dunkl’s results allow us to derive ‘the binomial formula’, which is directly related to the matrix element of the local field operator in the spin CS model.

The aim of this paper is to prove the earlier conjecture on the hole propagator\[13\] utilizing Dunkl’s results\[15\]. Though the method in this paper is applicable also to the general SU($p$) case, we concentrate on the SU(2) case for simplicity.

Here we shall recall the conjecture in\[13\] on the hole propagator. In the thermodynamic limit, the expression for the hole propagator $G(r, t)$ is expected to be

$$
G(r, t) = c(\beta) \prod_{k=1}^{\beta} \int_{-1}^{1} \frac{\beta}{2} \prod_{l=1}^{\beta+1} \int_{-1}^{1} dv_l |F(u, v)|^2 \exp \left[ -i \left( E(u, v)t - Q(u, v)r \right) \right].
$$

(2)
Here $c(\beta)$ is a constant factor and $u = (u_1, \cdots, u_\beta)$, $v = (v_1, \cdots, v_{\beta+1})$ represent normalized velocities of the quasiholes. In the expression (3), $F$, $Q$ and $E$ represent the form factor, momentum and energy, respectively. The explicit forms of them are as follows: The form factor $F$ is given by

$$F(u,v) = \frac{\prod_{1 \leq k < l \leq \beta} (u_k - u_l)^{g_d} \prod_{1 \leq k < l \leq \beta+1} (v_k - v_l)^{g_d} \prod_{k=1}^{\beta} \prod_{l=1}^{\beta+1} (u_k - v_l)^{g_o}}{\prod_{k=1}^{\beta} (1 - u_k^2)^{(1-g_d)/2} \prod_{l=1}^{\beta+1} (1 - v_l^2)^{(1-g_d)/2}},$$

where $g_d = (\beta + 1)/(2\beta + 1)$ and $g_o = -\beta/(2\beta + 1)$. The momentum $Q$ and energy $E$ are respectively given by

$$Q(u,v) = \frac{\pi \rho_0}{2} \left( \sum_{k=1}^{\beta} u_k + \sum_{l=1}^{\beta+1} v_l \right),$$

$$E(u,v) = -(2\beta + 1) \left( \frac{\pi \rho_0}{2} \right)^2 \left( \sum_{k=1}^{\beta} u_k^2 + \sum_{l=1}^{\beta+1} v_l^2 \right).$$

Here $\rho_0 = 2M/L$ is the mean density of particles. In what follows, we prove this conjecture and show that the constant $c(\beta)$ is given by

$$c(\beta) = \frac{\rho_0}{4(2\beta + 1)^3} \frac{2^{\beta+1}}{\Gamma((\beta + 1)/(2\beta + 1))} \prod_{k=1}^{\beta} \Gamma(k/(2\beta + 1))^2,$$

where $\Gamma(z)$ is the gamma function. The physical implication of these expressions (2), (3), (4) and (5) has been discussed in [13,16].

This paper is organized as follows. In next section, we define the Jack polynomials with a prescribed symmetry and discuss their basis properties. Particularly, we derive the binomial formula of the Jack polynomials with the prescribed symmetry, the norm, Cauchy formula, and evaluation formulae. Last, from the Cauchy and evaluation formulae, we derive the binomial formula, which gives the matrix element of the local field operator in the calculation of the hole propagator.

II. JACK POLYNOMIALS WITH PRESCRIBED SYMMETRY

In this section, we present mathematical results necessary to derive the expression for the hole propagator. First, as a preliminary, we fix our notations. Second, we define the non-symmetric Jack polynomials. Third, we define the Jack polynomials with a prescribed symmetry in terms of the non-symmetric Jack polynomials. Fourth, we present the basic formulae of the Jack polynomials with the prescribed symmetry; the norm, Cauchy formula, and evaluation formulae are discussed. Last, from the Cauchy and evaluation formulae, we derive the binomial formula, which gives the matrix element of the local field operator in the calculation of the hole propagator.

A. Notations

First of all, we fix notations (see Refs. [17,18,19]). For a fixed non-negative integer $n$, let $\Lambda_n = \{ \eta = (\eta_1, \eta_2, \cdots, \eta_n) \mid \eta_i \in \mathbb{Z}_{\geq 0}, 1 \leq i \leq n \}$ be the set of all compositions with length less than or equal to $n$. The weight $|\eta|$ of a composition $\eta = (\eta_1, \eta_2, \cdots, \eta_n) \in \Lambda_n$ is defined by $|\eta| = \sum_{i=1}^{n} \eta_i$. The length $l(\eta)$ of $\eta$ is defined as the number of non-zero $\eta_i$ in $\eta$. The set of all partitions with length less than or equal to $n$ is defined by $\Lambda_n^+ = \{ \lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n) \in \Lambda_n \mid \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0 \}$. The dominance order $<$ on partitions is defined as follows: for $\lambda, \mu \in \Lambda_n^+$, $\lambda < \mu$ if $|\lambda| = |\mu|$ and $\sum_{i=1}^{k} \lambda_i \leq \sum_{i=1}^{k} \mu_i$ for all $k = 1, \cdots, n$. For a composition $\eta \in \Lambda_n$, we denote by $\eta^+$ the (unique) partition which is a rearrangement of the composition $\eta$. Now we define a partial order $\prec$ on compositions as follows: for $\nu, \eta \in \Lambda_n$, $\nu \prec \eta$ if $\nu^+ < \eta^+$ with dominance ordering on partitions or if $\nu^+ = \eta^+$ and $\sum_{i=1}^{k} \nu_i \leq \sum_{i=1}^{k} \eta_i$ for all $k = 1, \cdots, n$.

For a given composition $\eta = (\eta_1, \eta_2, \cdots, \eta_n) \in \Lambda_n$ and pairs of integers $s = (i,j)$ satisfying $1 \leq i \leq l(\eta)$ and $1 \leq j \leq \eta_i$, we define the following quantities:
\[ a(s) = \eta_i - j, \]
\[ a'(s) = j - 1, \]
\[ l(s) = \# \{ k \in \{1, \ldots, i - 1\} | j \leq \eta_k + 1 \leq \eta_i \} + \# \{ k \in \{ i + 1, \ldots, n \} | j \leq \eta_k \leq \eta_i \}, \]
\[ l'(s) = \# \{ k \in \{1, \ldots, i - 1\} | \eta_k \geq \eta_i \} + \# \{ k \in \{ i + 1, \ldots, n \} | \eta_k > \eta_i \}. \]

Here, for a set \( A \), \( \# A \) denotes the number of elements. In the above expressions, \( a(s), a'(s), l(s) \) and \( l'(s) \) are called arm-length, coarm-length, leg-length, and coleg-length, respectively. Further, for a composition \( \eta \in \Lambda_n \) and a parameter \( \beta \), we define the following four quantities:

\[ d_\eta = \prod_{s \in \eta} ((a(s) + 1)/\beta + l(s) + 1), \]
\[ d'_\eta = \prod_{s \in \eta} ((a(s) + 1)/\beta + l(s)), \]
\[ e_\eta = \prod_{s \in \eta} ((a'(s) + 1)/\beta + n - l'(s)), \]
\[ e'_\eta = \prod_{s \in \eta} ((a'(s) + 1)/\beta + n - l'(s) - 1). \]

**B. Non-symmetric Jack polynomials**

Now we define the non-symmetric Jack polynomials \([23, 24]\). For this purpose, we define the Cherednik-Dunkl operators \([22, 23]\) as

\[ \hat{d}_i = x_i \frac{\partial}{\partial x_i} + \beta \sum_{k=1}^{i-1} \frac{x_i}{x_i - x_k}(1 - s_{ik}) + \beta \sum_{k=i+1}^{n} \frac{x_k}{x_i - x_k}(1 - s_{ik}) + \beta(1 - i) \]

for \( 1 \leq i \leq n \). The operator \( s_{ij} = (i, j) \) is the transposition which interchanges coordinates \( x_i \) and \( x_j \) (\( s_{ij} \) is called the coordinate exchange operator \([24]\)). The operator \( \hat{d}_i \) is a mapping in homogeneous polynomials of \( x = (x_1, \ldots, x_n) \). Further all the operators \( \{ \hat{d}_i \} \) commute with each other, and hence these operators can be diagonalized simultaneously.

For a given composition \( \eta \), the non-symmetric Jack polynomial \( E_\eta(x; 1/\beta) \) is defined as the \( |\eta| \)-th order homogeneous polynomial satisfying the following two conditions:

1. The polynomial \( E_\eta(x; 1/\beta) \) has the form

\[ E_\eta(x; 1/\beta) = x^\eta + \sum_{\nu \in \Lambda_n, \nu > \eta} c_{\nu\eta} x^\nu, \]

in terms of the monomials \( x^\nu = x_1^{\nu_1} \cdots x_n^{\nu_n} \).

2. \( E_\eta(x; 1/\beta) \) is a simultaneous eigenfunction of \( \hat{d}_i \) for \( 1 \leq i \leq n \).

Here we remark on two important properties of the non-symmetric Jack polynomial \( E_\eta \). One is the eigenvalue of \( E_\eta \) for \( \hat{d}_i \), which is given by

\[ \bar{\eta}_i = \eta_i - \beta (\# \{ k \in \{1, \ldots, i - 1\} | \eta_k \geq \eta_i \} + \# \{ k \in \{ i + 1, \ldots, n \} | \eta_k > \eta_i \}). \]

The other is the action of the transposition \( s_i := s_{ii+1} = (i, i + 1) \) on \( E_\eta \) \([18, 19]\),

\[ s_i E_\eta = \begin{cases} 
\xi_i E_\eta + (1 - \xi_i^2) E_{s_i \eta}, & \eta_i > \eta_{i+1}, \\
E_{s_i \eta}, & \eta_i = \eta_{i+1}, \\
\xi_i E_\eta + E_{s_i \eta}, & \eta_i < \eta_{i+1}, 
\end{cases} \]

where \( \xi_i = \beta / (\bar{\eta}_i - \bar{\eta}_{i+1}) \). In particular, the property \([18]\) plays an important role in the proof of the basic properties of the Jack polynomials with a prescribed symmetry.
C. Jack polynomials with prescribed symmetry

Next we define the Jack polynomials with a prescribed symmetry. For this purpose, we introduce some notations. The interval \( I = [1, n] \) denotes \( \{ i \in \mathbb{Z} | 1 \leq i \leq n \} \) for a positive integer \( n \). For an integer \( m \in I \), we define \( I_{\downarrow} = [1, m] \) and \( I_{\uparrow} = [m + 1, n] \). In addition, we introduce the notations \( n_{\downarrow} = \# I_{\downarrow} = m \) and \( n_{\uparrow} = \# I_{\uparrow} = n - m \). We consider the subgroup \( S_{n_{\downarrow}} \times S_{n_{\uparrow}} \) of the symmetric group \( S_n \) which leaves \( \{1, \cdots, n_{\downarrow}\} \) and \( \{n_{\downarrow} + 1, \cdots, n\} \) invariant. Further we define \( \Lambda^{\uparrow}_{n_{\uparrow}} \subset \Lambda_n \) as a set of ‘partial partitions’

\[
\Lambda^{\uparrow}_{n_{\uparrow}} = \{ \mu = (\mu^{\downarrow}, \mu^{\uparrow}) = (\mu^{\downarrow}_{n_{\downarrow}}, \cdots, \mu^{\downarrow}_{n_{\downarrow} - 1}, \mu^{\uparrow}_{n_{\uparrow}}, \mu^{\uparrow}_{n_{\uparrow} - 1}, \cdots, \mu^{\uparrow}_{n_{\uparrow} - 1}, \mu^{\uparrow}_{n_{\uparrow} - 2}, \cdots, \mu^{\uparrow}_1) \in \Lambda_n | \mu^{\downarrow}_{n_{\downarrow}} > \cdots > \mu^{\downarrow}_{n_{\downarrow} - 1}, \mu^{\uparrow}_{n_{\uparrow}} > \cdots > \mu^{\uparrow}_{n_{\uparrow} - 1} \}.
\]

Now we consider polynomials which are alternating under the action of \( S_{n_{\downarrow}} \times S_{n_{\uparrow}} \). We define the (alternating) Jack polynomial with the prescribed symmetry \( K_{\mu}(x; \beta) \) for \( \mu = (\mu^{\downarrow}, \mu^{\uparrow}) \in \Lambda^{\uparrow}_{n_{\uparrow}} \) by the following two conditions:

1. The polynomial \( K_{\mu}(x) \) has the form

\[
K_{\mu}(x; \beta) = \sum_{\eta = (\eta^{\downarrow}, \eta^{\uparrow})} a_{\eta} E_{\eta}(x; 1/\beta)
\]

with the normalization \( a_{\mu} = 1 \). Here the sums over \( \eta^{\downarrow} \) and \( \eta^{\uparrow} \) are taken on the whole rearrangements of \( \mu^{\downarrow} \) and \( \mu^{\uparrow} \), respectively.

2. Under the action of the transposition \( s_i \), the polynomial \( K_{\mu}(x) \) is transformed as \( s_i K_{\mu}(x) = -K_{\mu}(x) \) for \( i \in I_{\downarrow} \setminus \{n_{\downarrow}\} \) or \( i \in I_{\uparrow} \setminus \{n\} \). (Here, for a set \( A \) and its subset \( B \), we denote by \( A \setminus B \) the complementary set of \( B \) in \( A \).)

From the above definition and \([18]\), we can derive the recursion relation for the coefficient \( a_{\eta} \)

\[
a_{s_i, \eta} = -\frac{n_i - n_{i+1} - \beta}{n_i - n_{i+1}} a_{\eta},
\]

where \( i \in I_{\downarrow} \setminus \{n_{\downarrow}\} \) or \( i \in I_{\uparrow} \setminus \{n\} \). From the alternating property of \( K_{\mu} \), we can also write as

\[
\rho_{\mu} K_{\mu}(x) = \sum_{\sigma \in S_{n_{\downarrow}} \times S_{n_{\uparrow}}} \text{sgn}(\sigma) E_{\mu}(x),
\]

where the symbol \( \text{sgn}(\sigma) \) denotes the sign of the permutation \( \sigma \). The expression for the factor \( \rho_{\mu} \) can be obtained from the normalization \( a_{\mu} = 1 \) and the relation \([18]\) as

\[
\rho_{\mu} = \prod_{\sigma = \mu^{\downarrow}, \mu^{\uparrow}} \prod_{i < j} \frac{\mu_i - \mu_j - \beta}{\mu_i - \mu_j}
\]

for \( \mu \in \Lambda^{\uparrow}_{n_{\uparrow}} \).

D. Basic properties

Next we discuss the basic properties of the Jack polynomials with the prescribed symmetry. The combinatorial norm, integral norm, Cauchy formula and evaluation formula are discussed.

a) combinatorial norm: Define the polynomials \( \{q_{\eta}(x)\}_{\eta \in \Lambda_n} \) by

\[
\Omega(x|y) = \prod_{i=1}^{n} (1 - x_i y_i)^{-1} = \prod_{i,j=1}^{n} (1 - x_i y_j)^{-\beta} = \sum_{\eta \in \Lambda_n} q_{\eta}(x) y^{\eta}.
\]

The (combinatorial) inner product \( \langle \bullet, \bullet \rangle^n \) is then defined by \( \langle q_{\eta}, x^\eta \rangle^n = \delta_{\eta \mu} \). The Jack polynomials with the prescribed symmetry are orthogonal with respect to the inner product \( \langle \bullet, \bullet \rangle^n \). Using the norm formula \( \langle E^\eta \rangle^n_c^2 = \langle E^\eta \rangle E^\eta \rangle^n_c = d^\eta / d_n \) for \( \eta \in \Lambda_n \) \([14]\) and the transformation properties \([18]\) of the non-symmetric Jack polynomials, we can prove...
\[ (|K_{\mu}|_n^c)^2 = \langle K_{\mu}, K_{\mu} \rangle_n^c = \frac{n! n_\mu^1! d^\mu_{n\mu}}{\rho_{\mu} (\beta + 1)^n} \]  

for \( \mu \in \Lambda_n^+. \)

b) integral norm: For functions \( f(x) \) and \( g(x) \) in complex variables \( x = (x_1, \cdots, x_n) \), we define the inner product \( \langle \bullet, \bullet \rangle_n^c \) by the following formula:

\[ \langle f, g \rangle_n^c = \prod_{i=1}^n \int_{|x_i|=1} \frac{dx_i}{2\pi i x_i} f(x) \overline{g(x)} \Delta(x)^{2\beta}. \]  

Here \( \Delta(x) = \prod_{i,j \in I} (x_i - x_j) \) is the van der Monde determinant and \( \overline{g(x)} \) denotes the complex conjugation of \( g(x) \). The Jack polynomials with the prescribed symmetry are orthogonal with respect to the inner product \( \langle \bullet, \bullet \rangle_n^c \) \[25\]. It is known \[27\] that

\[ \langle E_\eta, E_\eta \rangle_n^c / \langle E_\eta, E_\eta \rangle_n^c = \frac{\Gamma(n\beta + 1) \epsilon_{\eta}}{\Gamma(\beta + 1)^n e_{\eta}^!}. \]  

for \( \eta \in \Lambda_n \). We note that the right-hand side of (27) depends on \( \eta \) through \( \eta^+ \). Thus the relation (27) immediately leads to

\[ \langle K_{\mu}, K_{\mu} \rangle_n^c / \langle K_{\mu}, K_{\mu} \rangle_n^c = \frac{\Gamma(n\beta + 1) \epsilon_{\mu}}{\Gamma(\beta + 1)^n e_{\mu}^!}. \]  

As a result of (27) and (28), we obtain

\[ (|K_{\mu}|_n^c)^2 = \langle K_{\mu}, K_{\mu} \rangle_n^c = \frac{n! n_\mu^1! \Gamma(n\beta + 1) \epsilon_{\mu} d^\mu_{n\mu}}{\Gamma(\beta + 1)^n e_{\mu}^! d_{n\mu}^!}. \]  

for \( \mu \in \Lambda_n^+. \)

c) Cauchy formula: For the coordinate \( x = (x_1, \cdots, x_n) \) and a fixed integer \( m \in I \), we define \( x^\downarrow = (x_1, \cdots, x_m) \) and \( x^\uparrow = (x_{m+1}, \cdots, x_n) \). The Cauchy formula for the Jack polynomials with the prescribed symmetry is given by

\[ \prod_{s=\downarrow, \uparrow i,j \in I} (1 - x_i y_j)^{-1} \prod_{i,j \in I} (1 - x_i y_j)^{-\beta} = n_\downarrow! n_\uparrow! \sum_{\mu \in \Lambda_n^+} (|K_{\mu}|_n^c)^{-2} \tilde{K}_{\mu}(x) \tilde{K}_{\mu}(y), \]  

where \( \tilde{K}_{\mu}(x) = K_{\mu}(x)/(\Delta(x^\downarrow) \Delta(x^\uparrow)) \) with \( \Delta(x^\downarrow) = \prod_{i,j \in I} (x_i - x_j), \, (s = \downarrow, \uparrow) \). The proof of the Cauchy formula (30) is based on the Cauchy formula for the non-symmetric polynomials \( \Omega(x|y) = \sum_{\eta \in \Lambda_n} (|E_\eta|_n^c)^{-2} E_{\eta}(x) E_{\eta}(y) \) \[19\] and the Cauchy’s determinant identity (see also \[25\]). The proof also requires the transformation properties \[18\] and \[21\] together with those for \( d_\eta \) and \( d_{\mu}^! \) \[18\].

d) evaluation formula: Using the evaluation formula for the non-symmetric Jack polynomials \( E_\eta(1, \cdots, 1) = e_\eta/d_\eta \) \[19\] and new skew operators \[13\], Dunkl obtained the evaluation formula for the Jack polynomials with the prescribed symmetry\[22\]

\[ \tilde{K}_\mu(1, \cdots, 1) = \beta^{-|\delta|} \frac{1}{e_\delta} \frac{\epsilon_{\mu} \pi_{\mu}}{d_{\mu}^! \rho_{\mu}}, \]  

where

\[ \pi_{\mu} = \prod_{s=\downarrow, \uparrow i,j \in I} (\tilde{\mu}_i - \tilde{\mu}_j - \beta) \]  

for \( \mu \in \Lambda_n^+ \). In \[22\], the composition \( \delta \in \Lambda_n^+ \) is introduced as \( \delta = \delta(n^\downarrow, n^\uparrow) := (\delta^\downarrow, \delta^\uparrow) \) with \( \delta^\downarrow = (n^\downarrow - 1, \cdots, 1) \) and \( \delta^\uparrow = (n^\uparrow - 1, \cdots, 1, 0) \).

\[ ^1 \text{The polynomial } E_\eta \text{ for a composition } \eta \text{ in this paper is different from } \zeta_\eta \text{ in } \[15\] \text{ by the constant factor } d_\eta/d_\eta; \text{ } E_\eta = (d_\eta/d_\eta) \zeta_\eta. \]
E. Binomial formula

In this subsection, we derive the binomial formula with the use of the Cauchy formula (30) and evaluation formula (31). For \( a \in \mathbb{C} \), the binomial formula for the Jack polynomials with the prescribed symmetry is given by

\[
\prod_{s=1}^{n} \prod_{i \in I_s} (1 - x_i)^{\alpha - n_s} = \sum_{\mu \in \Lambda_n^\uparrow} \chi_\mu(a) \tilde{K}_\mu(x),
\]

(33)

where

\[
\chi_\mu(a) = \beta^{-|\mu|} \frac{(1 - a)^{\mu_+} \pi_{\mu}}{(1 - a)^{\beta^+} d_\mu}
\]

(34)

Here, for an indeterminate \( t \), a partition \( \lambda \) and a parameter \( \beta \), the generalized shifted factorial is defined by

\[
(t)_\lambda = \prod_{i=1}^{n} \frac{\Gamma(t - \beta(i - 1) + \lambda_i)}{\Gamma(1 - \beta(i - 1))}.
\]

(35)

We can prove the formula (33) in a way similar to that used in Lemma 5.2 in [19] and Proposition 2.4 in [27].

Let \( n' \) be \( n' = n'_i + n'_j (\geq n) \) for that pair \( (n'_i, n'_j) \) of positive integers \( n'_i \) and \( n'_j \) which satisfies \( n'_i - n'_j = n_i - n_j \). We consider the Cauchy formula (30) in \( n' \) variables. First we discuss the left-hand side; the left-hand side of (30) becomes

\[
\prod_{s=1}^{n} \prod_{i \in I_s} \prod_{j \in I'} (1 - x_i y_j)^{-1} \prod_{i,j \in I'} (1 - x_i y_j)^{-\beta},
\]

(36)

with \( I'_i = [1, n'_i], I'_j = [n'_i + 1, n'] \) and \( I' = [1, n'] \). Now we set \( x_{n_i+1} = \cdots = x_{n'_i} = 0, x_{n'_i + n_i+1} = \cdots = x_{n'_i + n'_i} = 0 \) and \( y_{n_i} = \cdots = y_{n'_i + n'_i} = 1 \). Further we replace \( (x_{n'_i + 1}, \cdots, x_{n'_i + n'_i}) \) by \( (x_{n_i+1}, \cdots, x_{n_i+n_i}) \). The expression (36) then turns into

\[
\prod_{i \in I = [1, n_i]} (1 - x_i)^{n' - n'_i} \prod_{j \in I' = [n_i + 1, n]} (1 - x_j)^{-n'_i + n_i + n' - n_i}.
\]

(37)

Next we discuss the right-hand side of the Cauchy formula in \( n' \) variables. We set \( y_i = 1 \) for \( 1 \leq i \leq n' \). With the use of the evaluation formula (31), we immediately see that right-hand side of the Cauchy formula becomes

\[
\sum_{\nu} \beta^{-|\nu'|} \frac{e_{\nu, \pi_\nu}}{e_{\delta^+} d_{\nu}} \tilde{K}_\nu(x).
\]

(38)

Here the sum is taken over \( \nu \in \Lambda_{n'}^\uparrow \). The symbol \( \delta' = (\delta'^{\downarrow}, \delta'^{\uparrow}) \) denotes the composition \( \delta(n'_i, n'_j) \in \Lambda_{n'_i}^\uparrow \).

Now we set \( x_{n_i+1} = \cdots = x_{n'_i} = 0 \) and \( x_{n'_i + n_i+1} = \cdots = x_{n'_i + n'_i} = 0 \) in (38). Non-vanishing contributions in the sum (38) then come from only the compositions \( \nu = (\nu^\downarrow, \nu^\uparrow) \in \Lambda_{n'_i}^\uparrow \) satisfying \( l(\nu^\downarrow - \delta'^{\downarrow}) \leq n_i \) and \( l(\nu^\uparrow - \delta'^{\uparrow}) \leq n'_i \) where \( \nu^\downarrow - \delta'^{\downarrow} = (\nu^\downarrow_s - \delta'^{\downarrow}_s, \cdots, \nu^\downarrow_{n'_i} - \delta'^{\downarrow}_{n'_i}) \in \Lambda_{n'_i}^\uparrow \) \( (s = 0, 1) \). The reason is as follows: If either \( l(\nu^\downarrow - \delta'^{\downarrow}) > n_i \) or \( l(\nu^\uparrow - \delta'^{\uparrow}) > n'_i \) holds for the composition \( \nu = (\nu^\downarrow, \nu^\uparrow) \in \Lambda_{n'_i}^\uparrow \), then, in each monomial of the polynomials \( K_\nu \), the minimum power of \( x_{n_i+1} \) is 1 or that of \( x_{n'_i + n_i+1} \) is 1. Therefore those \( \nu \) do not contribute to the sum in (38) when both \( x_{n_i+1} \) and \( x_{n'_i + n_i+1} \) are set to be zero.

For a composition \( \nu = (\nu^\downarrow, \nu^\uparrow) \in \Lambda_{n'_i}^\uparrow \) satisfying \( l(\nu^\downarrow - \delta'^{\downarrow}) \leq n_i \) and \( l(\nu^\uparrow - \delta'^{\uparrow}) \leq n'_i \), the composition \( \mu = (\mu^\downarrow, \mu^\uparrow) \in \Lambda_{n}^\uparrow \) can be defined as the composition satisfying the relation

\[
\mu^\downarrow - \delta'^{\downarrow} = \nu^\downarrow - \delta'^{\downarrow} \quad \text{and} \quad \mu^\uparrow - \delta'^{\uparrow} = \nu^\uparrow - \delta'^{\uparrow}.
\]

(39)

When the relation (39) holds for compositions \( \mu \in \Lambda_{n}^\uparrow \) and \( \nu \in \Lambda_{n'}^\uparrow \), we can then find the following consequences: First, the two polynomials

\[
\tilde{K}_\nu(x_1, \cdots, x_{n_i}, 0, \cdots, 0, x_{n'_i+1}, \cdots, x_{n'_i+n_i+1}, 0, \cdots, 0)
\]

(40)
and
\[ \tilde{K}_\mu(x_1, \cdots, x_{n_1}, x_{n_1'}+1, \cdots, x_{n_\gamma} ) \] (41)

are equal. From now on, we replace \( x_{n_1'}+1, \cdots, x_{n_\gamma} \) by \( x_{n_1}+1, \cdots, x_n \). Second, \( \pi_\nu/d'_\nu = \beta^{\nu-|\mu|} \pi_\mu/d'_\mu \). Third, the expression \( e_\nu/e_\delta \) can be rewritten as
\[
\prod_{s \in \nu} \frac{(a'(s) + 1)/\beta + n' - l'(s)}{\prod_{s \in \delta} (a'(s) + 1)/\beta + n' - l'(s)} \\
= \prod_{s \in \nu} \frac{(a'(s) + n'_l - n_\nu + 1)/\beta + n' - l'(s)}{\prod_{s \in \delta} (a'(s) + n'_l - n_\nu + 1)/\beta + n' - l'(s)} \\
= \beta^{\delta-|\mu|} \frac{(1 + n'_\beta + n'_l - n_\nu)_{\mu+}}{(1 + n'_\beta + n'_l - n_\nu)_{\delta+}}. \tag{42}
\]

(Notice the relation \( \prod_{s \in \eta} (a'(s) + k)/\beta + k' - l'(s) = \beta^{-|\eta|}(k'/\beta + k)_{\eta^+} \) for \( \eta \in \Lambda_n \) and integers \( j, k \).) As a result of these three relations, the expression (38) can be rewritten as
\[
\sum_{\mu} \beta^{-|\nu|} \frac{(1 + n'_\beta + n'_l - n_\nu)_{\mu+}}{(1 + n'_\beta + n'_l - n_\nu)_{\delta+}} d'_\mu \tilde{K}_\mu(x), \tag{43}
\]
where the sum is taken over \( \mu \in \Lambda_n^{\uparrow} \).

Now we obtain the relation
\[
\prod_{i \in I_\alpha} (1 - x_i)^{-n'_\beta - n'_l} \prod_{j \in I_\gamma} (1 - x_j)^{-n'_\beta - n'_l + n_\nu - n_\gamma} = \sum_{\mu} \beta^{-|\nu|} \frac{(1 + n'_\beta + n'_l - n_\nu)_{\mu+}}{(1 + n'_\beta + n'_l - n_\nu)_{\delta+}} d'_\mu \tilde{K}_\mu(x), \tag{44}
\]
from the expressions (37) and (43). We notice that the right-hand side of (44) is a polynomial of \( n'_\beta + n'_l \) and hence the relation (44) also holds for arbitrary complex values of \( n'_\beta + n'_l \). Further we replace \( -n'_\beta - n'_l + n_\nu \) by a complex variable \( a \). Consequently we obtain the binomial formula (33).

III. HOLE PROPAGATOR FOR THE SPIN CALOGERO-SUTHERLAND MODEL

A. Result for a finite number of particles

In this section, for arbitrary non-negative integer \( \beta \), we exactly compute the hole propagator of the SU(2) spin CS model with a finite number of particles.

We consider the 2\( M \)-particle system whose Hamiltonian is given by (1) with \( p = 2 \). We assume that \( M \) is an odd integer. The hole propagator of the model is given by
\[
G(r, t) = 2M \langle 0 | \hat{\psi}_\uparrow^\dagger (r, t) \hat{\psi}_\uparrow (0, 0) | 0 \rangle_{2M}/2M \langle 0 | 0 \rangle_{2M}, \tag{45}
\]
where \( | 0 \rangle_{2M} \) represents the singlet ground state for 2\( M \)-particle system. The operator \( \hat{\psi}_\uparrow (r, t) = \exp(iH_{2M-1}t)\hat{\psi}_\uparrow (r) \exp(-iH_{2M}t) \) is the Heisenberg representation of the annihilation operator \( \hat{\psi}_\uparrow (r) \) of particles with spin up which acts on the 2\( M \)-particle states.

In the following, the statistics of particles are chosen as boson (fermion) for odd (even) \( \beta \), so that we can set \( P_{ij} = (-1)^{\beta+1} s_{ij} \). (Notice that \( P_{ij} s_{ij} \) is nothing but the particle exchange operator for a pair \((i, j)\).)

Along the calculations in (33), the hole propagator reduces to the following expression:
\[
G(r, t) = c_0 \prod_{i=1}^{2M-1} \int_{|x_i|=1} \frac{dx_i}{2\pi i x_i} \Delta^\beta(x) \Theta(x; \beta) \exp \left[ -i \left( \hat{\mathcal{H}} - E^0_{2M} \right) t + i\hat{P}_r \right] \Delta^\beta(x) \Theta(x; \beta), \tag{46}
\]
where complex coordinates \( x = (x_1, \cdots, x_{2M-1}) \) are related to original coordinates \( X \) in (1) by the formulae \( x_i = \exp(2\pi i X_i/L) \) for \( 1 \leq i \leq 2M - 1 \). The constant factor \( c_0 \) is given by \( \rho_0/(2D(M)) \), in terms of the mean density of particles \( \rho_0 = 2M/L \) and
\[
D(M) = \frac{2^M}{M!} \prod_{i=1}^{2M} \frac{dx_i}{2\pi i x_i} \prod_{1 \leq i < j \leq 2M} |x_i - x_j|^{2\beta} \prod_{1 \leq i < j \leq M} |z_i - z_j|^2 \prod_{M+1 \leq i < j \leq 2M} |z_i - z_j|^2
= \frac{M!}{(2\beta + 1)^M} \frac{\Gamma((2\beta + 1)M + 1)}{\Gamma((\beta + 1)2M)}. \tag{47}
\]

In (47), the function \(\Theta(x; \beta)\) has the form
\[
\Theta(x; \beta) = \prod_{i=1}^{M} (1 - x_i)^\beta \prod_{i=M+1}^{2M-1} (1 - x_i)^{\beta+1} \prod_{1 \leq i < j \leq M} (x_i - x_j) \prod_{M+1 \leq i < j \leq 2M-1} (x_i - x_j). \tag{48}
\]

Further the symbols \(\hat{H}, E_{2M}^0\) and \(\hat{P}\) denote the Hamiltonian, the ground state energy of \(2M\)-particle system and total momentum, respectively. In terms of the complex variables \(x\), the expressions for \(\hat{H}\) and \(\hat{P}\) are given by
\[
\hat{H} = \left(\frac{2\pi}{L}\right)^2 \left[ \sum_{i=1}^{2M-1} \left( x_i \frac{\partial}{\partial x_i} - \Delta P \right) \right] - \sum_{1 \leq i < j \leq 2M-1} \frac{2\beta (\beta - (-1)^{\beta} s_{ij}) x_i x_j}{(x_i - x_j)^2}, \tag{49}
\]
\[
\hat{P} = \frac{2\pi}{L} \sum_{i=1}^{2M-1} \left( x_i \frac{\partial}{\partial x_i} - \Delta P \right), \tag{50}
\]
respectively. Here \(\Delta P\) denotes \((\beta(2M - 1) + M - 1)/2\).

Now we introduce a transformed Hamiltonian \(\hat{H}\) and momentum \(\hat{P}\) as
\[
\hat{H} = \Delta^{-\beta} \hat{H} \Delta^\beta - E_{2M}^0, \tag{51}
\]
\[
\hat{P} = \Delta^{-\beta} \hat{P} \Delta^\beta. \tag{52}
\]

Using the notations in the section referred to with \(n = 2M - 1, n_\downarrow = M,\) and \(n_\uparrow = M - 1,\) the expression (46) turns into
\[
G(r, t) = c_0(\Theta, \exp \left( -i\hat{H}t + i\hat{P}r \right) \Theta)^{0}_{2M-1}. \tag{53}
\]

In (53), our problem has reduced to the spectral decomposition of \(\Theta(x; \beta)\) in terms of the joint eigenfunctions of \(\hat{H}\) and \(\hat{P}\). From the following two observations, we can see that the Jack polynomials with the prescribed symmetry are proper bases of the decomposition: First, both \(\Theta(x)\) and \(K_\mu(x)\) are polynomials with a common symmetric property; \(s_i \Theta(x) = -\Theta(x)\) and \(s_i K_\mu(x) = -K_\mu(x)\) for \(i \in I_\downarrow \{ n_\downarrow \}\) or \(\in I_\uparrow \{ n_\uparrow \}\). Second, the polynomials \(K_\mu\) are joint eigenfunctions of \(\hat{H}\) and \(\hat{P}\) with the eigenvalues
\[
\omega(\mu) = \left( \frac{2\pi}{L} \right)^2 \sum_{i=1}^{2M-1} \left( \bar{\mu}_i - \frac{M - 1}{2} + \beta \right)^2, \tag{54}
\]
\[
q(\mu) = \frac{2\pi}{L} \sum_{i=1}^{2M-1} \left( \mu_i - \frac{\beta(2M - 1) + M - 1}{2} \right). \tag{55}
\]
respectively. In the following paragraphs, we discuss the second issue.

From the expression (43), we can see that \(E_\eta(x)\) for \(\eta \in \Lambda_n\) and \(x = (x_1, \cdots, x_n)\) is an eigenfunction of the operators \(\sum_{i=1}^{n} d_i^k\) with the eigenvalue \(\sum_{i=1}^{n} \bar{d}_i^k\) for \(k = 1, \cdots, n\). Further the operators \(\sum_{i=1}^{n} \bar{d}_i^k\) for \(k = 1, \cdots, n\) commute with permutations \(\sigma \in S_{n_\downarrow} \times S_{n_\uparrow}\). In addition, \(K_\mu\) can be constructed by a ‘symmetrization’ of \(E_\mu\) (see the expression (23)). Therefore the Jack polynomials with the prescribed symmetry \(K_\mu\) for \(\mu \in \Lambda_n^{\downarrow}\) are eigenfunctions of \(\sum_{i=1}^{n} d_i^k\) with the eigenvalue \(\sum_{i=1}^{n} \bar{\mu}_i^k\) for \(k = 1, \cdots, n\).

On the other hand, in terms of the Cherednik-Dunkl operators (15), the Hamiltonian \(\hat{H}\) and total momentum \(\hat{P}\) can be written as
\[
\hat{H} = \left( \frac{2\pi}{L} \right)^2 \sum_{i=1}^{2M-1} \left( d_i - \frac{M - 1}{2} + \beta \right)^2, \tag{56}
\]
and

\[
\hat{P} = \frac{2\pi}{L} \left[ 2^{M-1} \sum_{i=1}^{2M-1} \left( \hat{d}_i - \frac{M-1}{2} \right) + \frac{\beta}{2} (2M-1) \right].
\]  

(57)

These expressions lead the fact that the joint eigenfunctions of \( \sum_{i=1}^{2M-1} \hat{d}_i \) for \( k = 1, 2 \) are those of \( \hat{H} \) and \( \hat{P} \). From this fact, we find that the polynomials \( K_\mu \) are joint eigenfunctions of \( \hat{H} \) and \( \hat{P} \). The eigenvalues (54) and (55) follow from (13), (34) and (57).

We notice that the binomial formula (33) is useful in rewriting (53), because the spectral decomposition of \( \Theta(x; \beta) \) in terms of \( K_\mu(x) \) is a special case of the formula (33). Now we can express \( \Theta \) in terms of \( K_\mu \) as

\[
\Theta(x; \beta) = \sum_{\mu \in \Lambda^M_{2M-1}} \chi_\mu(\beta + M)K_\mu(x; \beta),
\]

(58)

with the coefficients \( \chi_\mu(\beta + M) \) (34).

Using the orthogonal properties of the Jack polynomials with the prescribed symmetry and the spectral decomposition (58), we have

\[
G(r, t) = \rho_0 \frac{1}{2} \sum_{\mu \in \Lambda^M_{2M-1}} \left( \chi_\mu(\beta + M) \right)^2 \left( \frac{||K_\mu||^{(2M-1)}_2}{||\delta(M, M)||^{(2M)}_2} \right)^2 \exp\left[-i(\omega(\mu)t - q(\mu)r)\right].
\]

(59)

(Notice that \( K_{(M, M)}(x) = \Delta(x^\dagger)\Delta(x^\uparrow) \) and then \( D(M) = (||K_M(M, M)||^{(2M)}_2)^2 \) Every factor in (59) is available from the expressions (24), (34), (44) and (53). The expression (59) is our main result in this subsection.

It is important to consider the condition that intermediate states \( \mu \) can contribute to the sum in (59). By a close examination of the expression (59), we find the relation:

\[
\chi_\mu(\beta + M) \neq 0 \iff (1, \beta + M) \not\in \mu^{\dagger}.
\]

(60)

This relation leads a selection rule; only those states \( \mu \) the largest entry of which is less than \( \beta + M \) contribute to the sum in (59). Here we remark on the parametrization of the relevant intermediate state. For a composition \( \mu \in \Lambda^M_n \), there is a unique composition \( \hat{\mu} = (\hat{\mu}^\dagger, \hat{\mu}^\uparrow) \in \Lambda_n \) such that \( \mu = \hat{\mu} + \delta \). Notice that \( \hat{\mu}^\dagger \in \Lambda^M_{n^\dagger} \) for \( s = \downarrow, \uparrow \). Using these notations, we have

\[
\chi_\mu(\beta + M) \neq 0 \iff (1, \beta + 1) \not\in \hat{\mu}^\dagger \text{ and } (1, \beta + 2) \not\in \hat{\mu}^\uparrow,
\]

(61)

where \( \hat{\mu}^\uparrow \) and \( \hat{\mu}^\dagger \) are regarded as partitions. In the decomposition \( \mu = \hat{\mu} + \delta \), \( \delta \) represents the condensate or the pseudo Fermi sea and \( \hat{\mu} \) represents the excitations. From above conditions, we see that such partitions \( \hat{\mu}^\dagger \) and \( \hat{\mu}^\uparrow \) are parametrized by \( \beta \) and \( \beta + 1 \) integers, respectively. Therefore, the relevant intermediate states \( \mu \) can be parametrized by \( 2\beta + 1 \) integers.

An example of those states is shown in Fig. 1, where we take \( \beta = 2 \), \( M = 7 \), and \( \mu = (2, 2, 1, 1, 0, 0, 3, 2, 2, 1, 0) + \delta(n_\downarrow = 7, n_\uparrow = 6) \). The open blocks correspond to \( \delta(n_\downarrow = 7, n_\uparrow = 6) \) and the shaded ones represent the excitations.

In next subsection, we will see that another but equivalent parametrization leads \( (2\beta+1) \)-fold integral representation for \( G(r, t) \) in the thermodynamic limit.

B. Thermodynamic limit

In this subsection, we derive the expression (14) for the hole propagator in the thermodynamic limit. First of all, to take the thermodynamic limit, we parametrize the intermediate states in the following way. Let \( \mu = (\mu^\dagger, \mu^\uparrow) \in \Lambda^M_{2M-1} \) be a composition which satisfies the condition (33). From the consideration on the intermediate states, the complementary set of \{\( \mu^\dagger_i \}_{i=1}^{M} \} in \{M + \beta - 1, M + \beta - 2, \ldots, 1, 0 \} is well-defined. We define \{\( p_k \)\}_{k=1}^{\beta} so that \( p_1 > \cdots > p_\beta \) and \( \{M + \beta - 1, M + \beta - 2, \ldots, 1, 0\} \setminus \{\mu^\dagger_i \}_{i=1}^{M} = \{M + \beta - 1 - p_k\}_{k=1}^{\beta} \). Similarly, we define \{\( q_l \)\}_{l=1}^{\beta + 1} so that \( q_1 > \cdots > q_{\beta + 1} \) and \( \{M + \beta - 1, M + \beta - 2, \ldots, 1, 0\} \setminus \{\mu^\uparrow_i \}_{i=1}^{M-1} = \{M + \beta - 1 - q_l\}_{l=1}^{\beta + 1} \). The resultant set of \{\( p_k, q_l \)\} exhausts the relevant intermediate state in (59). In Fig. 1, the new parameters \{\( p_1, p_2, q_1, q_2, q_3 \)\} are also shown.

Furthermore we shall introduce a set of notations. We respectively define \( \tilde{p}_k \) \((1 \leq k \leq \beta)\) and \( \tilde{q}_l \) \((1 \leq l \leq \beta + 1)\) by
\[ \hat{p}_k = p_k - \gamma \beta (\beta - k + \frac{1}{2} \{ l' \in \{ 1, \cdots, \beta + 1 \} \vert q_{l'} < p_k \}) \]
\[ \hat{q}_l = q_l - \gamma \beta (\beta + 1 - l + \frac{1}{2} \{ k' \in \{ 1, \cdots, \beta \} \vert p_{k'} < q_l \}) \]
where \( \gamma = 1/(2\beta + 1) \). We can regard \( \hat{p}_k \) and \( \hat{q}_l \) as 'rapidities' of quasiholes with spin up and down. Also, \( \bar{\Delta}_k (1 \leq k \leq \beta) \) and \( \bar{\Delta}_l (1 \leq l \leq \beta + 1) \) are defined as \( \gamma \beta \{ l' \in \{ 1, \cdots, \beta + 1 \} \vert q_{l'} = p_k - 1 \} \) and \( \gamma \beta \{ k' \in \{ 1, \cdots, \beta \} \vert p_{k'} = q_l - 1 \} \), respectively. Moreover, we introduce \( U_{kl} = \bar{\Delta}_k - \bar{\Delta}_l \) for \( 1 \leq k < l \leq \beta \) and \( V_{kl} = \bar{q}_l - \bar{q}_l - \bar{\Delta}_l \) for \( 1 \leq k < l \leq \beta + 1 \). They describe the interplay between the quasiholes with a same spin. In order to describe the interaction between the quasiholes with an opposite spin, we introduce \( W_{kl} \) for \( 1 \leq k \leq \beta \) and \( 1 \leq l \leq \beta + 1 \) as
\[
W_{kl} = \begin{cases} \hat{p}_k - \hat{q}_l - \gamma (\beta + 1) \frac{1}{2} \{ k' \in \{ 1, \cdots, \beta \} \vert p_{k'} = q_l + 1 \} & , \text{for } q_l + 2 \leq p_k, \\ 1 & , \text{for } q_l \leq p_k \leq q_l + 1, \\ \hat{q}_l - \hat{p}_k + \gamma (\beta + 1) \frac{1}{2} \{ l' \in \{ 1, \cdots, \beta + 1 \} \vert q_{l'} = p_k \} & , \text{for } p_k \leq q_l - 1, \end{cases}
\]
and
\[
\bar{\Delta}_{kl} = \begin{cases} \gamma \beta & , \text{for } p_k = q_l \text{ or } p_k = q_l + 1, \\ 0 & , \text{otherwise}. \end{cases}
\]
Here \( \bar{\Delta}_{kl} \) is introduced to describe exceptional configurations of rapidities.

Using these new notations, we can rewrite the norm and matrix element more explicitly. The expression for \( \chi_\mu (\beta + M) \) in the new notation is given by
\[
\chi_\mu (\beta + M) = (-1) \sum_{k=1}^\beta p_k + \sum_{l=1}^{\beta + 1} q_l - \beta^2 \prod_{k=1}^{2\beta + 1} \Gamma (\gamma k)^{-1} \times \prod_{k=1}^\beta \Gamma (\gamma (\beta + 1) - \bar{\Delta}_k) \times \prod_{1 \leq k < l \leq \beta} \frac{\Gamma (U_{kl} + \gamma (\beta + 1))}{\Gamma (U_{kl})} \times \prod_{1 \leq k < l \leq \beta + 1} \frac{\Gamma (V_{kl} + \gamma (\beta + 1))}{\Gamma (V_{kl})} \times \beta^{\beta + 1} \frac{\Gamma (W_{kl} + \bar{\Delta}_{kl} - \gamma \beta)}{\Gamma (W_{kl})}.
\]
The expression for the norm is given by
\[
\left( \frac{\|K_\mu\|_2^0}{\|K_\mu (M, M)\|_2^0} \right)^2 = \frac{\Gamma (\beta + 1) \Gamma (M (2\beta + 1) - \beta)}{M \Gamma (M (2\beta + 1))} \prod_{k=1}^{2\beta + 1} \frac{\Gamma (M + 1 - \gamma k)}{\Gamma (M + 1 - \gamma k - \gamma \beta)} \times \prod_{k=1}^\beta \frac{\Gamma (1 - \bar{\Delta}_k) \Gamma (\hat{p}_k + \gamma (\beta + 1)) \Gamma (M - \hat{p}_k - \gamma \beta)}{\Gamma (\gamma (\beta + 1) - \bar{\Delta}_k) \Gamma (\hat{p}_k + 1) \Gamma (M - \hat{p}_k)} \times \prod_{l=1}^{\beta + 1} \frac{\Gamma (1 - \bar{\Delta}_l) \Gamma (\hat{q}_l + \gamma (\beta + 1)) \Gamma (M - \hat{q}_l - \gamma \beta)}{\Gamma (\gamma (\beta + 1) - \bar{\Delta}_l) \Gamma (\hat{q}_l + 1) \Gamma (M - \hat{q}_l)} \times \prod_{1 \leq k < l \leq \beta} \frac{\Gamma (U_{kl} + 1) \Gamma (U_{kl})}{\Gamma (U_{kl} + \gamma (\beta + 1)) \Gamma (U_{kl} + \gamma \beta)} \times \prod_{1 \leq k < l \leq \beta + 1} \frac{\Gamma (V_{kl} + 1) \Gamma (V_{kl})}{\Gamma (V_{kl} + \gamma (\beta + 1)) \Gamma (V_{kl} + \gamma \beta)} \times \prod_{k=1}^\beta \frac{\Gamma (W_{kl}) \Gamma (W_{kl} + \bar{\Delta}_{kl})}{\Gamma (W_{kl} + \gamma \beta) \Gamma (W_{kl} + \bar{\Delta}_{kl} - \gamma \beta)}.
\]
Now we consider the thermodynamic limit, i.e., \( M \to \infty, L \to \infty \) with \( \rho_0 = 2M/L \) fixed. In this limit, only the configurations with \( p_k, q_l, |p_k - p_l|, |q_k - q_l| \) and \( |p_k - q_l| \sim \mathcal{O}(M) \) give finite contributions to the hole propagator. Let
us introduce the normalized velocities $u_k$ and $v_l$ of the holes with up and down spin, respectively. These are defined by

$$\lim_{M \to \infty} \frac{2p_k}{M} = 1 - u_k, \quad (68)$$
$$\lim_{M \to \infty} \frac{2q_l}{M} = 1 - v_l. \quad (69)$$

In the thermodynamic limit, we can use the Stirling formula: $\Gamma(z + 1) \to \sqrt{2\pi z} \frac{z^z}{e^z}$ for $|z| \to \infty$. Taking the symmetry of the integrand into consideration, the sum in (59) over $\{p_k, q_l\}$ reduces to the integral over $\{u_k, v_l\}$ as

$$\sum_{0 \leq p_0 < \cdots < p_{\beta} \leq M + \beta - 1} \sum_{0 \leq q_0 < \cdots < q_{\beta - 1} \leq M + \beta - 1} \to \frac{1}{\beta!(\beta + 1)!} \left( \frac{M}{2} \right)^{2\beta + 1} \prod_{k=1}^{\beta} \int_{-1}^{1} du_k \prod_{l=1}^{\beta} \int_{-1}^{1} dv_l. \quad (70)$$

Combining the expressions (59), (66) and (67) with the above limiting procedure, we arrive at the final expression (2) for the hole propagator in the thermodynamic limit.

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* Permanent Address: Department of Applied Physics, University of Tokyo, Tokyo 113-8656, Japan.
** CREST researcher: Japan Science and Technology Corporation (JST), Kawaguchi 332-0012, Japan.

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FIG. 1. A diagram that contributes to the hole propagator. This figure corresponds to the case with $\beta = 2$, $M = 7$ and $\mu = (2, 2, 1, 1, 0, 0, 0, 3, 3, 2, 2, 1, 0) + \delta(n_\downarrow = 7, n_\uparrow = 6)$. The open and shaded blocks represent the pseudo Fermi sea and the excitation, respectively. The parameters \{p_k, q_l\} adopted in III B are also shown.
