SEPARATEDNESS OF MODULI OF K-STABLE VARIETIES

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Dedicated with admiration to Professor Mukai on his sixtieth birthday

Abstract. Given a one parameter flat family of polarized algebraic varieties, we show that any K-stable limit is unique. In particular, moduli spaces of K-stable polarized varieties are automatically Hausdorff when they exist.

We also give a characterization of K-stable limits in terms of the CM line bundle, and some applications to moduli. Our methods work for arbitrary projective schemes in any characteristic.

1. Introduction

We work over a field $k$ of arbitrary characteristic. By a polarized family over $B$ we mean a flat projective morphism of algebraic schemes $\mathcal{X} \to B$ with connected equidimensional fibers, endowed with a relatively ample line bundle $L$. Our main result is then the following.

Theorem 1.1 (Valuative criterion for separatedness). Suppose $(\mathcal{X}, L)$, $(\mathcal{Y}, M)$ are polarized families over a smooth curve $C \ni 0$ which are isomorphic (as polarized families) away from their central fibers:

$$(\mathcal{X}, L)|_{C \setminus \{0\}} \cong (\mathcal{Y}, M)|_{C \setminus \{0\}}.$$

If $(\mathcal{X}_0, L_0)$ is K-stable and $(\mathcal{Y}_0, M_0)$ is K-semistable, then $(\mathcal{X}, L) \cong (\mathcal{Y}, M \otimes L)$ for some line bundle $L$ pulled back from $B$.

With stronger assumptions, it is sufficient to ask only that $L$ and $M$ are algebraically equivalent on the punctured families; see Section 3.

K-stability was introduced for Fano manifolds in [Ti]. We use Donaldson’s reformulation [Do2], which works for polarized projective schemes (over any field). It is a certain limiting version of Geometric Invariant Theory stability which was conjectured by Yau, Tian and Donaldson to be equivalent to the existence of constant scalar curvature Kähler metrics. This differential geometric viewpoint suggests the existence of a Hausdorff moduli space of K-stable polarized manifolds.

1For instance: the existence of moduli spaces of Einstein metrics, the Donaldson-Fujiki moment map picture [Do1, Fu] expressing moduli spaces of complex structures as an infinite dimensional symplectic quotient, and the algebraicity of Gromov-Hausdorff limits of Kähler-Einstein Fano manifolds [DS].
This indicates that K-stable varieties or schemes might form good moduli.\footnote{This is despite the fact that asymptotic Hilbert or Chow stability do not produce good moduli \[WX\] due to the counterexamples with high multiplicity singularities of Shepherd-Barron \[SB\].} Precise conjectures are made in \[Od1, 5.2\], \[Od3, 3.1\]; see also \[OSS, 6.2\] in the $\mathbb{Q}$-Fano case. Evidence is provided by the recent construction of projective moduli of semi log canonical models by the theory of Kollár-Shepherd-Barron-Alexeev; cf. \[Ko\]. By \[Od2, Od 5\] this is precisely a moduli space of K-stable varieties. And \[BG\] links this back to the differential geometric motivation.

So assuming that quasiprojective moduli of K-stable varieties (or schemes) do exist, Theorem \[1.1\] implies that they are separated. Note that as a special case, combined with \[Od2\] it reproves the standard uniqueness of relative log canonical models.

If the fibers of the families are generically K-stable then we can characterize the central fiber as follows.

**Theorem 1.2.** Suppose $(\mathcal{X}, \mathcal{L}) \to C$ is a polarized family of K-stable varieties, and $(\mathcal{Y}, \mathcal{M}) \to C$ is a polarized family which is isomorphic away from a finite number of fibers. Then either $(\mathcal{X}, \mathcal{L}) = (\mathcal{Y}, \mathcal{M} \otimes L)$ for a line bundle $L$ pulled back from $C$, or

$$CM(\mathcal{X}, \mathcal{L}) < CM(\mathcal{Y}, \mathcal{M}).$$

If we replace K-stable with K-semistable then $CM(\mathcal{X}, \mathcal{L}) \leq CM(\mathcal{Y}, \mathcal{M})$.

Here $CM$ denotes the degree the CM line bundle which lives on the base of any polarized family \[PT\]. In short, K-stable families minimize the CM degree. \[WX, Theorem 6\] proved a version of this result in the canonically polarized case.

Another consequence of Theorem \[1.2\] is the following.

**Theorem 1.3.** Let $(\mathcal{X}, \mathcal{L}) \to C$ be a polarized family over a smooth curve $C \ni 0$ with K-semistable fibers. If $\mathcal{X}$ is $\mathbb{Q}$-Gorenstein then

$$\mathcal{L}^{\otimes a}|_{C\setminus\{0\}} \sim_{\mathbb{Q}} K_{\mathcal{X}/C}|_{C\setminus\{0\}} \implies \mathcal{L}^{\otimes a} \sim_{\mathbb{Q}} K_{\mathcal{X}/C} \otimes L,$$

where $L$ is pulled back from $C$.

This corresponds to the differential-geometric fact that the (Kähler)-Einstein condition is preserved in Gromov-Hausdorff limits. Combined with \[Od5, Theorems 1.2, 1.3\], we have the following Corollaries for the cases $a > 0$, $a = 0$ and $a < 0$ respectively.

**Corollary 1.4** (Limits of KE varieties). The following classes of varieties are closed under passing to K-semistable limits:

(i) Canonically polarized semi log canonical models,
(ii) Polarized semi log canonical Calabi-Yau varieties,
(iii) Anticanonically polarized log terminal $\mathbb{Q}$-Fano varieties.

Moreover, if there is a log terminal limit of a family in case (ii) then there is no other semi log canonical Calabi-Yau limit.

On the other hand, it is expected that to actually construct the limit should involve MMP techniques, as is the case for $a > 0$ (cf. [Ko]).

We note that – given the results of [Od2] – Corollary 1.4(i) extends the results of [WX] Theorem 2] from asymptotic Chow stability to the K-semistability case. Corollary 1.4(iii) can be thought of as the algebro-geometric counterpart of a result of [DS] that the Gromov-Hausdorff limits of Kähler-Einstein varieties are (log terminal) $\mathbb{Q}$-Fano varieties.

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2. Proof of Theorem 1.1

It is sufficient to prove Theorem 1.1 with $L$ replaced by $L^m$ for all $m \gg 0$: K-stability is unaffected by scaling the line bundle, and $L$ is determined by $L^m$ and $L^{m+1}$. We can also tensor $L$ by multiples of the divisor $\mathcal{X}_0 \subset \mathcal{X}$. So we may assume that $L$ and $\mathcal{M}$ are very ample.

2.1. Relating the two families. Adding multiples of the divisor $\mathcal{Y}_0 \subset \mathcal{Y}$ to $\mathcal{M}$, we can assume the regular sections of $\mathcal{M}$ pull back to regular sections of $L$ under the birational map

$$(\mathcal{X}, L) \dasharrow (\mathcal{Y}, \mathcal{M}).$$

These sections generate a subsheaf

$$(2) \quad L \otimes I_Z \subset L.$$

The subscheme $Z \subset \mathcal{X}$ is supported set theoretically on the central fiber because (1) is an isomorphism away from the central fibers. Thus we get morphisms

$$Z := \text{Bl}_Z \mathcal{X}$$

such that $p_2^* \mathcal{M} = p_1^* L(-E)$, where $E \subset Z$ is the exceptional divisor.

In particular, setting $\mathcal{N} := p_1^* L(-E) = p_2^* \mathcal{M}$ we have

$$H^0(\mathcal{Y}, \mathcal{M}^k) \subset H^0(Z, \mathcal{N}^k).$$
Morally speaking, we think of \((Y, \mathcal{M})\) and \((Z, \mathcal{N})\) as more-or-less the same, because the cokernel of (3) is “small”,

\[
h^0((p_2)_*\mathcal{O}_Z/\mathcal{O}_Y) \otimes \mathcal{M}^k = ck^n + O(k^{n-1}),
\]
and is zero if \(Y\) is normal. Here \(n = \dim \mathcal{X}_0\), and \(h^0(Y, \mathcal{M}^k) = O(k^{n+1})\).

2.2. The induced test configuration. The main idea is to produce a test configuration from the data of \(Z \subset (\mathcal{X}, \mathcal{L})\). We first degenerate \(\mathcal{X}\) to the normal cone of its central fiber \(\mathcal{X}_0 \subset \mathcal{X}\). That is, we consider

\[
\text{Bl}_{\mathcal{X}_0 \times \{0\}}(\mathcal{X} \times \mathbb{A}^1) \longrightarrow \mathbb{A}^1
\]

to be a flat family over \(\mathbb{A}^1\) with general fiber \(\mathcal{X}\) and special fiber

\[
\mathcal{X} \cup \mathcal{X}_0(\mathcal{X}_0 \times \mathbb{P}^1).
\]

(The union glues \(\mathcal{X}_0 \subset \mathcal{X}\) to \(\mathcal{X}_0 \times \{0\} \subset \mathcal{X}_0 \times \mathbb{P}^1\).) We use the polarization given by pulling back \(\mathcal{L}\) and subtracting the exceptional divisor.

The proper transform \(\overline{Z} \times \mathbb{A}^1\) of \(Z \times \mathbb{A}^1 \subset \mathcal{X} \times \mathbb{A}^1\) defines a flat degeneration of \(Z \subset \mathcal{X}\) to a new subscheme

\[
Z_0 \subset \mathcal{X}_0 \times \mathbb{P}^1
\]
in the central fiber (6), supported set theoretically on \(\mathcal{X}_0 \times \{\infty\}\).

Blowing up (5) in \(\overline{Z} \times \mathbb{A}^1\), we get a degeneration of \((Z, \mathcal{N})\) to

\[
(\mathcal{X} \cup \mathcal{X}_0 \text{Bl}_{Z_0}(\mathcal{X}_0 \times \mathbb{P}^1), \mathcal{L}')
\]

Here \(\mathcal{L}'\) is the gluing of \(\mathcal{L}(-\mathcal{X}_0)\) on \(\mathcal{X}\) to \((\mathcal{L}_0 \boxtimes \mathcal{O}(1))(-E_0)\) on \(\text{Bl}_{Z_0}(\mathcal{X}_0 \times \mathbb{P}^1)\), where \(E_0\) is the exceptional divisor over \(Z_0\).

The standard \(\mathbb{G}_m\)-invariant we get an induced action on the blow up and so on the central fiber (7). This fixes \(\mathcal{X}\) and acts on the second component

\[
T := \text{Bl}_{Z_0}(\mathcal{X}_0 \times \mathbb{P}^1)
\]
covering the usual \(\mathbb{G}_m\) action on \(\mathbb{P}^1\).

In particular \((T, \mathcal{L}')\) is a semi test configuration\(^3\) for \((\mathcal{X}_0, \mathcal{L}_0)\) in the sense of [Od4] (but back-to-front, with the roles of 0 and \(\infty\) reversed). Next we will calculate its Donaldson-Futaki invariant [Do2].

\(^3\)This is the same as a test configuration – compactified as a product over \(\infty\) – except the line bundle is only required to be relatively semiample rather than ample. Via its space of sections it contracts to genuine test configurations with the same sections and so the same Donaldson-Futaki invariant. In particular we can therefore test K-stability with semi test configurations.
2.3. **Numerical invariants.** Fix any polarized family \((V, L) \to C\) over a curve \(C\) with \(n\)-dimensional central fiber \((V_0, L_0)\). We can also allow \(L\) to be semi ample. Writing
\[
\chi(V_0, L_0^k) = a_0k^n + a_1k^{n-1} + \ldots ,
\]
\[
\chi(V, L^k) = b_0k^{n+1} + b_1k^n + \ldots ,
\]
we set
\[
CM(V, L) := a_1b_0 - a_0b_1 + a_0^2.
\]
Replacing \(L\) by \(L \otimes L\), where \(L\) is the pull back of a degree \(d\) line bundle on \(C\), changes \(b_i\) to \(b_i + da_i\) and so leaves \(CM\) unchanged.

Consider the semi test configuration \((T, L') \to \mathbb{P}^1\) whose fibers have the same Hilbert polynomial as \((X_0, L_0)\). By an observation made many times ([Mu, 2.14], [Do3, 5.1], [RT2, 2.8.1], [Wa], [Od4, 3.3]), the total weight of the \(\mathbb{G}_m\) action on the space of sections \(H^0(T_0, L'_0^k)\) on the central fiber is
\[
h^0(T, L'^k) - h^0(T_0, L'^k).
\]
Since \(h^i(T, L'^k) = O(k^{n-1})\) for \(i > 0\) ([Od4, Lemma 3.4]), approximating \(h^0\) by \(\chi\) we find that, in the notation of (8, 9), the total weight is
\[
b_0k^{n+1} + (b_1 - a_0)k^n + O(k^{n-1}),
\]
with Donaldson-Futaki invariant
\[
DF(T, L') := a_1b_0 - a_0(b_1 - a_0) = CM(T, L').
\]

We calculate \(\chi(T, L'^k)\) as
\[
\chi(X \cup T, L'^k) - \chi(X, L'^k) + \chi(X_0, L_0^k).
\]
By the flatness of our degeneration of \((Z, N)\) to \((X \cup X_0, T, L')\), this is
\[
\chi(Z, N^k) - \chi(X, L^k(-kX_0)) + \chi(X_0, L_0^k).
\]
Equating the coefficients of \(k^{n+1}\) and \(k^n\) and using both (11) and the invariance of \(CM\) under twisting by line bundles pulled back from the base, this implies the key formula
\[
DF(T, L') = CM(T, L') = CM(Z, N) - CM(X, L)
\]
\[
= CM(Y, M) - a_0c - CM(X, L)
\]
\[
\leq CM(Y, M) - CM(X, L).
\]

\(^4\)The notation anticipates Section\[4\] where we will see it is the degree of the CM line bundle on \(C\) induced by \((V, L)\).
Here $c \geq 0$ is the constant given in [4], and we have again used [Od4, Lemma 3.4] to approximate $h^0$ by $\chi$ to $O(k^{n-1})$. Therefore the K-stability of $(X_0, L_0)$ implies that

$$CM(\mathcal{V}, \mathcal{M}) \geq CM(\mathcal{X}, \mathcal{L}),$$

with equality only if $\mathcal{T}$ is a trivial configuration. By symmetry, the K-semistability of $(Y_0, M_0)$ similarly implies the opposite inequality

$$CM(\mathcal{X}, \mathcal{L}) \geq CM(\mathcal{V}, \mathcal{M}).$$

Thus $\mathcal{T}$ must indeed be a trivial test configuration:

$$T, \mathcal{L} \cong (X_0 \times \mathbb{P}^1, L_0 \boxtimes \mathcal{O}_{\mathbb{P}^1}(i)).$$

In particular, the blow up of (5) in $Z \times A_1$ is an isomorphism over the central fiber, so is an isomorphism everywhere. Thus $Z \subset X$ is a Cartier divisor supported on the central fiber. If it is not a multiple of the central fiber then the polarization $\mathcal{L}'$ on the test configuration $\mathcal{T}$ is nontrivial, again contradicting the K-stability of $(X_0, L_0)$. So $Z = jX_0$ for some $j$, and $(\mathcal{Y}, \mathcal{M}) \cong (X, \mathcal{L}(-jX_0)).$ $\square$

3. Cohomological polarizations

Instead of forming moduli of polarized varieties it is often preferable to form moduli of varieties marked only by an ample class in $H^2$. This ties in well with the differential geometric motivation: the conjectural link between K-stability and cscK metrics uses only the first Chern class of the polarization. We can extend the separatedness result to this setting, under stronger conditions.

**Theorem 3.1.** Suppose $(\mathcal{X}, \mathcal{L}), (\mathcal{Y}, \mathcal{M})$ are polarized families over a smooth curve $C \ni 0$ with an isomorphism

$$\mathcal{X}|_{C \setminus \{0\}} \cong \mathcal{Y}|_{C \setminus \{0\}}$$

which makes $\mathcal{L}$ and $\mathcal{M}$ fiberwise algebraically equivalent over the preimage of $C \setminus \{0\}$. We make the additional assumptions that

- $\text{char}(k) = 0$,
- the fibers of $\mathcal{Y}$ are geometrically normal, and
- the total space of $\mathcal{Y}$ is $\mathbb{Q}$-Gorenstein.

Then if $(X_0, L_0)$ is K-stable and $(Y_0, M_0)$ is K-semistable we have

$$(\mathcal{X}, \mathcal{L}) \cong (\mathcal{Y}, \mathcal{M} \otimes L)$$

for some line bundle $L$ on $\mathcal{Y}$ fiberwise algebraically equivalent to zero.
Proof. By [Od5, Theorem 1.2] and our assumptions, \( Y_0 \) is log canonical. Therefore \( Y_t \) is also log canonical for \( t \) in a neighborhood \( U \) of \( 0 \in C \). Since it is also normal, \( Pic_0(Y_t) \) is projective. Log canonical implies Du Bois, so \( U \ni t \mapsto h_1(O_{Y_t}) \) is locally constant on \( U \) [KK Corollary 1.2]. Therefore by [FGA, 9.4.8, 9.5.20] the relative Picard scheme \( Pic_0(Y/U) \) is projective with fibers \( Pic_0(Y_t) \).

So the line bundle \( L_t \otimes M_t^{-1} \) on \( Y \setminus Y_0 \) extends to a line bundle \( L \) on \( Y \), fiberwise algebraically equivalent to zero. We can now apply Theorem 1.1 to \( (X, L) \) and \( (Y, M \otimes L) \), since K-semistability is unaffected by tensoring by numerically trivial line bundles. □

If we assume that \( Y/C \) is smooth then using [FGA, 9.6.18] we can replace algebraic equivalence by numerical equivalence throughout.

4. Characterization of K-stable limits

Any polarized family \( (\mathcal{V}, \mathcal{L}) \rightarrow C \) induces a line bundle \( \lambda_{CM} \) on its base \( C \) [FS, PT]. A good reference is [PR]. By Riemann-Roch applied to the definition of \( \lambda_{CM} \) [PT] its degree on a curve \( C \) is, in the notation of (8, 9, 10),

\[
\deg(\lambda_{CM}) = a_1b_0 - a_0b_1 + a_0^2 = CM(\mathcal{V}, \mathcal{L}),
\]

cf. [Wa, 3.2]. In particular, if \( (\mathcal{V}, \mathcal{L}) \) is a semi test configuration, then by comparison with (11),

\[
(14) \quad DF(\mathcal{V}, \mathcal{L}) = \deg(\lambda_{CM}),
\]

as in [PT, Theorem 1]; cf. [Wa, Proposition 17], [Od4, Theorem 3.2].

Proof of Theorem 1.2. Relating \( (\mathcal{X}, \mathcal{L}) \) and \( (\mathcal{Y}, \mathcal{M}) \) by a third family \( (\mathcal{Z}, \mathcal{N}) \) as in Section 2, the key formula (12) gives

\[
\sum_i DF(T_i, L_i') \leq CM(\mathcal{Y}, \mathcal{M}) - CM(\mathcal{X}, \mathcal{L}).
\]

Here the \( (T_i, L_i') \) are the test configurations constructed in Section 2.2 at the points of \( C \) where \( \mathcal{X} \) and \( \mathcal{Y} \) differ. By stability the left hand side is strictly positive unless the test configurations are trivial, in which case \( (\mathcal{X}, \mathcal{L}) = (\mathcal{Y}, \mathcal{M} \otimes L) \) just as in (13). □

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5 In testing for strict K-instability we can perturb the polarization on a test configuration by a small ample \( \mathbb{Q} \)-line bundle without affecting the strict inequality \( DF < 0 \). Thus we can work with nef line bundles, which are defined cohomologically. (A priori, however, K-stability appears to depend on the precise polarization. Assuming the Yau-Tian-Donaldson conjecture this dependence must disappear.)
Proof of Theorem 1.3. Consider the $\mathbb{Q}$-divisor class $E := K_{X/C} - aL$. Then the key argument of [LX] (cf. the proof of Proposition 4) gives
\[
\frac{d}{dt} \left. C(M, L(tE)) \right|_{t=0} = (n+1)(L^{n-1}.E^2).
\]
By standard arguments using Zariski’s lemma or the Hodge index theorem (cf. [LX, Lemma 3]), this is $\leq 0$ with equality if and only if $E$ is proportional to a fiber.

The strict inequality would contradict the final statement of Theorem 1.2, so $K_{X/C} - aL$ must be pulled back from $C$. \qed

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