An end-to-end construction for compact constant mean curvature surfaces

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1 Introduction

The theory of constant mean curvature surfaces in Euclidean space has been the object of intensive study in the past years. In the case of complete noncompact constant mean curvature surfaces, the moduli space of such surfaces is now fairly well understood (at least in the genus 0 case) [13], [10], [11] and many technics have been developed to produce examples of such surfaces [8], [5], [14], [16].

By contrast, the set of compact constant mean curvature surfaces is not so well understood. In the early 80’s, H. Wente has constructed the first examples of genus 1 constant mean curvature surfaces [20]. These genus 1 surfaces have then been thoughtfully studied by U. Pinkall and I. Sterling [18]. Examples of compact constant mean curvature surface of higher genus are due to N. Kapouleas. In the genus 2 case [7], these surfaces are obtained by ”fusing” Wente tori while in the case where the genus is greater than or equal to 3, these surfaces are obtain by connecting together large number of mutually tangent unit spheres, using small catenoid necks [6].

In this paper, we would like to explain how the current knowledge on the set of complete noncompact constant mean curvature surfaces can be exploited to produce new examples of compact constant mean curvature surfaces of genus greater than or equal to 3.

Our construction is based on three important tools which have been developed for the understanding of complete noncompact constant mean curvature surfaces :

(i) The moduli space theory as developed by R. Kusner, R. Mazzeo and D. Pollack [13].

(ii) The end-addition result which has been developed by R. Mazzeo, F. Pacard and D. Pollack [15], [16] to produce complete noncompact constant mean curvature surfaces with prescribed ends.

(iii) The end-to-end construction which was developed by J. Ratzkin [19] to connect two constant mean curvature surfaces along their ends.

This ideas behind our construction can be described as follow : One can use the end-addition theory developed in [15], [16] to produce complete constant mean curvature surfaces with prescribed Delaunay type ends. This addition of ends procedure is quite

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flexible and one can arrange so that the ends of these surfaces can be "plugged" together to produce sequences (indexed by a discrete parameter \( n \in \mathbb{N} \)) of compact surfaces which have mean curvature one except in finitely many annular regions where their mean curvature can be estimated by \( 1 + O(e^{-\gamma n}) \) for some \( \gamma > 0 \). Next, one studies the mapping properties of the Jacobi operator about this (almost) constant mean curvature surface. To perform this analysis, we rely on the fact that parametrices for the Jacobi operators on each complete noncompact summand have been obtained in [13] and we explain how these can be glued together. Finally, it will remain to use a standard perturbation argument to produce sequences of compact constant mean curvature surfaces of arbitrary genus, greater than or equal to 3.

The main advantage of our construction versus the one developed by N. Kapouleas is that it is technically simple (once the above mentioned techniques are understood!), paralleling the fact that the end-to-end construction of J. Ratzkin is simpler than the previous constructions of complete noncompact surfaces. We obtain a very precise description of the surfaces we produce. In particular, our construction sheds light on the structure of the set of compact constant mean curvature surfaces, showing that these surfaces are isolated. Though this is probably a minor point, the example of compact constant mean curvature surfaces we obtain are geometrically different from the one obtained by N. Kapouleas and cannot be obtained using his technic (roughly speaking all the surfaces constructed by N. Kapouleas have close to sequences of unit spheres linked by small catenoids and hence have small injectivity radius while our examples do not necessarily have small necks and hence have injectivity radius uniformly bonded from below).

Maybe a more important issue is the fact that our construction points out interesting directions toward which the theory of complete noncompact constant mean curvature surfaces should be developed. Some properties of complete noncompact constant mean curvature surfaces have been neglected and turn out to be extremely important. This is for example the case of the notion of "regular end" of a constant mean curvature surface (which is also important in the construction of J. Ratzkin).

Although our method can be applied to produce non symmetric surfaces, the complete description of the set of compact surfaces is far beyond our understanding, this is the reason why we have chosen not to look for the utmost generality but to focus on the construction of symmetric surfaces. In order to explain the ideas in our construction (keeping the technicalities as low as possible and the notations as simple as possible), we will construct constant mean curvature surfaces of arbitrary genus (\( \geq 3 \)) which have dihedral symmetry.

Final remark, our construction generalizes to any dimension [4].

2 Delaunay surfaces

In this section we recall some well known facts concerning the family of Delaunay surfaces \( D_\tau \) which are rotationally invariant constant mean curvature surfaces in \( \mathbb{R}^3 \) [2]. We refer to [14] for further details.

2.1 Isothermal parametrization

Delaunay surfaces can be parameterized, in isothermal coordinates, by

\[
X_\tau(s, \theta) = \frac{1}{2} \left( \tau e^{\sigma(s)} \cos \theta, \tau e^{\sigma(s)} \sin \theta, \kappa(s) \right),
\]

(1)
for \((s, \theta) \in \mathbb{R} \times S^1\), where the functions \(\sigma\) and \(\kappa\) are described as follows: For any \(\tau \in (0, 1]\), the function \(\sigma\) is defined to be the unique smooth nonconstant solution of the ordinary differential equation

\[
(\partial_s \sigma)^2 + \tau^2 \cosh^2 \sigma = 1, \quad \partial_s \sigma(0) = 0, \quad \sigma(0) < 0,
\]

while, for any \(\tau \in (-\infty, 0)\), the function \(\sigma\) is defined to be the unique smooth nonconstant solution of the ordinary differential equation

\[
(\partial_s \sigma)^2 + \tau^2 \sinh^2 \sigma = 1, \quad \partial_s \sigma(0) = 0, \quad \sigma(0) < 0.
\]

Again, the definition of \(\kappa\) differs according to the sign of \(\tau\). When \(\tau \in (0, 1]\), we define the function \(\kappa\) by

\[
\partial_s \kappa = \tau^2 e^\sigma \cosh \sigma, \quad \kappa(0) = 0,
\]

while when \(\tau < 0\), we define the function \(\kappa\) by

\[
\partial_s \kappa = \tau^2 e^\sigma \sinh \sigma, \quad \kappa(0) = 0.
\]

Observe that when \(\tau > 0\), \(\kappa\) is monotone increasing, and hence \(X_\tau\) is an embedding, whereas when \(\tau < 0\), this is no longer true and the surfaces are only immersed. The embedded (resp. immersed) Delaunay surfaces \(D_\tau\) are known as unduloids (resp. nodoids). The parameter \(\tau\) will be referred to as the Delaunay parameter.

As noted above, these surfaces are all periodic because the functions \(\sigma\) are. When \(\tau \neq 1\), we define \(s_\tau\) to be equal to the half of the least period of \(\sigma\). The physical least period of the Delaunay surface \(D_\tau\) is given by

\[
2T_\tau := \frac{1}{2} \kappa_\tau(2s_\tau)
\]

**Warning**: We agree that \(2s_\tau\) is equal to the least period of \(\sigma\) and \(2T_\tau\) is the least period of \(D_\tau\).

We claim that

**Lemma 1** For all \(\tau \in (-\infty, 0) \cup (0, 1]\), we have \(\partial_\tau T_\tau > 0\).

**Proof**: Observe that \(\partial_\tau \sigma > 0\) on \((0, s_\tau)\) and \(\partial_\sigma \sigma < 0\) on \((s_\tau, 2s_\tau)\). Hence, for \(s \in (0, s_\tau)\), we can use \(\sigma\) as a change of variable and express \(\kappa\) as a function of \(\sigma \in (-\sigma_*, \sigma_*)\) where \(\sigma_* > 0\) satisfies \(\tau^2 \cosh^2 \sigma_* = 1\) when \(\tau \in (0, 1]\) and \(\sigma_* > 0\) satisfies \(\tau^2 \sinh^2 \sigma_* = 1\) when \(\tau < 0\).

When \(\tau < 0\), we get

\[
2T_\tau = \int_{-\sigma_*}^{\sigma_*} \frac{\tau^2 e^\sigma \sinh \sigma}{\sqrt{1 - \tau^2 \sinh^2 \sigma}} \, d\sigma = \int_{-\sigma_*}^{\sigma_*} \frac{\tau^2 \sinh^2 \sigma}{\sqrt{1 - \tau^2 \sinh^2 \sigma}} \, d\sigma
\]

Performing the change of variable \(\tau \sinh \sigma = \sin x\) we conclude that

\[
2T_\tau = \int_{-\pi/2}^{\pi/2} \frac{\sin^2 x}{\sqrt{\tau^2 + \sin^2 x}} \, dx
\]

which clearly implies that \(\partial_\tau T_\tau > 0\) when \(\tau < 0\).
When $\tau > 0$, we have

$$2T_{\tau} = \int_{-\sigma_0}^{\sigma_0} \frac{\tau^2 e^{\sigma} \cosh \sigma}{\sqrt{1 - \tau^2 \cosh^2 \sigma}} d\sigma = \int_{-\sigma_0}^{\sigma_0} \frac{\tau^2 \cosh^2 \sigma}{\sqrt{1 - \tau^2 \cosh^2 \sigma}} d\sigma$$

Performing the change of variable $\tau \sinh \sigma = \sqrt{1 - \tau^2 \cosh^2 \sigma}$ we conclude that

$$2T_{\tau} = \int_{-\pi/2}^{\pi/2} \sqrt{1 - (1 - \tau^2) \cos^2 x} dx$$

which again implies that $\partial_\tau T_{\tau} > 0$. \hfill $\square$

It will be convenient to denote by $\vec{a}_{\tau}$ (resp. $X_{\tau}$) the Delaunay surface (resp. the parameterization of the Delaunay surface) whose Delaunay parameter is equal to $\tau$, whose axis is the line directed by $\vec{a}$ passing through the origin, and which has a neck passing through the plane $\vec{x} \cdot \vec{a} = 0$. In particular, $D_{\tau}^\vec{a}$ is invariant under the symmetry with respect to the plane whose normal is $\vec{a}$. Granted this notation, the Delaunay surface $D_{\tau}$ described (1) is equal to $D_{\tau}^\vec{e}_3$ and $D_{\tau}^\vec{a}$ is obtained from $D_{\tau}^\vec{e}_3$ by applying a rotation which sends $\vec{e}_3$ to $\vec{a}$. Given a vector $\vec{b}$, the surface $D_{\tau}^\vec{a} + \vec{b}$ is the surface $D_{\tau}^\vec{a}$ which has been translated by $\vec{b}$. It is parameterized by $X_{\tau}^\vec{a} + \vec{b}$ and, when $\vec{b} = 0$, the unit normal vector field compatible with the orientation will be denoted by $\vec{N}_{\tau}^\vec{a}$.

### 2.2 The Jacobi operator

Let $\Sigma$ be a constant mean curvature surface. Any surface which is close to $\Sigma$ may be represented as a normal graph over $\Sigma$

$$\Sigma_w = \{x + w(x) \vec{N}(x) : x \in \Sigma\},$$

where $\vec{N}$ is the unit normal vector field compatible with the orientation of $\Sigma$ and $w$ is a (small) scalar function. The mean curvature of $\Sigma_w$ is denoted by $H(w)$. By definition, the Jacobi operator about $\Sigma$ is the differential of the mapping $w \rightarrow 2H(w)$ at $w = 0$. It is given by

$$\mathcal{L}_\Sigma := \Delta_\Sigma + |A_\Sigma|^2.$$

A solution $w$ of the homogeneous problem $\mathcal{L}_\Sigma w = 0$ is called a Jacobi field.

We denote by $\mathcal{L}_{D_{\tau}}$ the Jacobi operator associated to the Delaunay surface $D_{\tau}$. In terms of the isothermal parametrization given in the previous paragraph, it is given by

$$\mathcal{L}_{D_{\tau}} = \frac{4}{\tau^2 e^{2\sigma}} \left( \partial^2_s + \partial^2_\theta + \tau^2 \cosh(2\sigma) \right).$$

For the sake of simplicity, we shall now assume that $\tau \neq 1$, namely that $D_{\tau}$ is not a cylinder. There is no loss of generality in doing so since our construction, which is based on the end-to-end construction, does not work for surfaces which have ends asymptotic to cylinders. Some Jacobi fields are easy to describe since they correspond to explicit geometric deformations of the Delaunay surfaces [14]. We briefly describe these now since they will play a key role in the subsequent analysis.

The Jacobi fields corresponding to an infinitesimal translations of $D_{\tau}$ will be denoted by $\Phi_{T,\vec{e}}^{\tau}$, where $|\vec{e}| = 1$. They are obtained by projecting the constant vector field $\vec{e}$ on the normal vector field $\vec{N}_{\tau}$ on $D_{\tau}$.

$$\Phi_{T,\vec{e}}^{\tau} := \vec{e} \cdot \vec{N}_{\tau}$$
It is geometrically obvious that there are 3 linearly independent such Jacobi fields (this is where we use the fact that \( \tau \neq 1 \) and hence \( D_\tau \) is not a cylinder) which only depend only on \( s \) and are periodic, hence which are bounded as \( s \to \pm \infty \).

The two Jacobi fields corresponding to infinitesimal rotations of the axis of \( D_\tau \) will be denoted by \( \Phi_\tau^R, e \), where \( |e| = 1 \). They obtained by projecting the Killing vector fields

\[
(x_1, x_2, x_3) \longrightarrow (\bar{x} \cdot e') \bar{e}'' - (\bar{x} \cdot e'') \bar{e}'
\]

where \((\bar{x}, e', e'')\) is a direct orthonormal basis, on the vector field \( \vec{N}_\tau \).

It is geometrically obvious that there are 2 linearly independent such Jacobi fields which only depend only on \( s \) and grow linearly in \( s \).

So far all the Jacobi fields we have defined can be explicitly computed in terms of the function \( \sigma \) and its derivatives \([14]\), even though we will not need these expressions.

There is a last Jacobi field, whose geometric meaning is obvious but whose analytical expression is more intricate, which will be denoted by \( \Phi_\tau^D \) and which corresponds to the one parameter family \( D_\tau \) obtained by varying the Delaunay parameter \( \tau \). Since \( D_\tau \) are surfaces of revolution, this Jacobi field depends only on \( s \). The fact that \( \partial_\tau T_\tau \neq 0 \) when \( \tau \neq 1 \) implies that this Jacobi field is linearly growing in \( s \). Observe that, there exists \( p_\tau \in \mathbb{R} - \{0\} \) such that

\[
\Phi_\tau^D(s + 2 s_\tau) = \Phi_\tau^D + p_\tau \Phi_\tau^T
\]

This follows from the fact that \( s \rightarrow \Phi_\tau^D(s + 2 s_\tau) - \Phi_\tau^D(s) \) is a bounded Jacobi field which only depends on \( s \), hence is proportional to \( \Phi_\tau^T \). The constant \( p_\tau \) is not equal to 0 since \( \Phi_\tau^D \) is linearly growing.

The Jacobi operator \( L_{D_\tau} \) being invariant with respect to rotations about the Delaunay axis, we can perform the eigenfunction decomposition of any function \((s, \theta) \rightarrow w(s, \theta)\) in the \( \theta \) variable and the analysis of \( L_{D_\tau} \) reduces to the analysis of the sequence of operators

\[
L_{\tau,j} := \frac{4}{\tau^2 e^{2\sigma}} \left( \partial_s^2 + \tau^2 \cosh(2\sigma) - j^2 \right)
\]

for \( j \in \mathbb{N} \). The potential in \( L_{\tau,j} \) being periodic of period \( s_\tau \) (observe that \( \cosh(2\sigma) \) is \( s_\tau \) periodic since \( \sigma \) is 2 \( s_\tau \) periodic and odd), it follows from Bloch wave theory \([16]\) that the following alternative holds :

(i) Either the homogeneous problem \( L_{\tau,j} w = 0 \) has two independent solutions \( w^\pm \) (depending on \( \tau \) and \( j \)) such that

\[
w^\pm(s + s_\tau) = e^{\pm \zeta_{\tau,j} s_\tau} w^\pm(s).
\]

for some complex number \( \zeta_{\tau,j} \), with \( \Re \zeta_{\tau,j} \geq 0 \).

(ii) Or the homogeneous problem \( L_{\tau,j} w = 0 \) has one periodic solution and one linearly growing solution. In which case, we set \( \zeta_{\tau,j} := 0 \).

For each \( j \), we define the indicial roots associated to the operator \( L_{\tau,j} \) to be the real numbers \( \pm \gamma_{\tau,j} \) where

\[
\gamma_{\tau,j} := \Re \zeta_{\tau,j} \geq 0.
\]

It is proven in \([16]\) that :
Proposition 1 \textit{The indicial roots of }\( L_\tau \text{ satisfy the following properties :}\\

(i) For any \( \tau \in (-\infty, 0) \cup (0, 1], \gamma_{\tau, 0} = \gamma_{\tau, 1} = 0.\\

(ii) There exists \( \tau_\ast < 0 \text{ such that, for all } j \geq 2 \text{ and } \tau \in (\tau_\ast, 0) \cup (0, 1], \gamma_{\tau,j} > 0.\\

The first property is a consequence of the fact that the Jacobi fields \( \Phi_{D_\tau}, \Phi_{T,\vec{e}_\tau} \) and \( \Phi_{R,\vec{e}_\tau} \) are either bounded or linearly growing.

3 Moduli space theory

We now briefly describe the moduli space theory for \( k \)-ended complete noncompact constant mean curvature surfaces as developed in [13] and extended in [16]. We define \( M_{\tau_\ast}^{g,k} \) to be the set of all complete, noncompact constant mean curvature surfaces which have genus \( g \) and \( k \) ends asymptotic to Delaunay surfaces whose Delaunay parameter belongs to \( (\tau_\ast, 0) \cup (0, 1] \). Observe that we do not mod out by the group of rigid motions.

We can decompose a surface \( \Sigma \in M_{\tau_\ast}^{g,k} \) into overlapping connected pieces: A compact component \( K \) and the ends \( E_\ell, \ell = 1, \ldots, k \) and we can require that each \( K \cap E_\ell \) is homeomorphic to an annulus \([0, 1] \times S^1\). For each \( \ell \), we choose standard isothermal coordinates \((s, \theta)\) for the model Delaunay end \( D_{\vec{a}_\ell, \tau_\ell} \) so that the end \( E_\ell \) is parametrized by \( Y_\ell := X_{\vec{a}_\ell} + w_\ell \vec{N}_{\vec{a}_\ell} + \vec{b}_\ell, \) for \((s, \theta) \in [0, +\infty) \times S^1\). Since we have assumed that the end \( E_\ell \) is asymptotic to \( D_{\vec{a}_\ell, \tau_\ell} + \vec{b}_\ell \), this means that the function \( w_\ell \) is exponentially decreasing. To be more specific, we need the:

Definition 1 \textit{Given } r \in \mathbb{N}, \alpha \in (0, 1) \text{ and } \mu \in \mathbb{R}, \text{ the space } \mathcal{E}_{r,\alpha}^{\tau,\mu}(\Sigma) \text{ is the space of functions } v \in C_{\text{loc}}^r([0, +\infty) \times S^1) \text{ for which}\\
\|v\|_{\mathcal{E}_{r,\alpha}^{\tau,\mu}(\Sigma)} := \sup_{s \geq 0} e^{-\mu s} |v|_{C_{r,\alpha}([s,s+1] \times S^1)}\\
\text{is finite.}\\

Granted this definition, it is known that \( w_\ell \in \mathcal{E}_{-\gamma_{\tau_\ast,2}}^{2,\alpha}([0, \infty) \times S^1) \). In other words the rate of decay of the function \( w_\ell \) is dictated by the indicial root \( \gamma_{\tau_\ast,2} \). We refer to [14] for a proof of this fact. The moduli space theory is based on the:

Definition 2 \textit{For } r \in \mathbb{N}, \alpha \in (0, 1) \text{ and } \mu \in \mathbb{R}, \text{ let } \mathcal{D}_{r,\alpha}^\mu(\Sigma) \text{ be the space of functions } v \in C_{\tau,\alpha}(\Sigma) \text{ for which}\\
\|v\|_{\mathcal{D}_{r,\alpha}^\mu(\Sigma)} := \|v|_K \|_{C_{r,\alpha}} + \sum_{\ell=1}^k \|v \circ Y_\ell|_{E_\ell} \|_{\mathcal{E}_{r,\alpha}^{\tau,\mu}}\\
\text{is finite.}\\

We can now give the precise definition of a nondegenerate constant mean curvature surface.
Definition 3 The surface \( \Sigma \in \mathcal{M}_{g,k} \) is nondegenerate if
\[
\mathcal{L}_\Sigma : D^{2,0}_\mu(\Sigma) \longrightarrow D^{0,0}_\mu(\Sigma)
\]
is injective for all \( \mu < 0 \).

Following the analysis of the Jacobi fields we have done in §2.2 and using the parameterization \( (3) \) together with \( (4) \), it is easy to see that, on each end \( E_\ell \) of \( \Sigma \), there exists 5 (globally defined) independent Jacobi fields \( \Phi_{E_\ell}^{T,\varepsilon} \) and \( \Phi_{E_\ell}^{R,\varepsilon} \) which satisfy
\[
\begin{align*}
\Phi_{E_\ell}^{T,\varepsilon} \circ Y_\ell - \Phi_{\tau_\ell}^{T,\varepsilon} & \in \mathcal{C}^{2,0}_\mu([0, +\infty) \times S^1), \\
\Phi_{E_\ell}^{R,\varepsilon} \circ Y_\ell - \Phi_{\tau_\ell}^{R,\varepsilon} & \in \mathcal{C}^{2,0}_\mu([0, +\infty) \times S^1)),
\end{align*}
\]
(5)
where \( |\varepsilon| = 1 \).

The existence of a Jacobi field \( \Phi_{E_\ell}^D \) (only defined on \( E_\ell \)) which is asymptotic to \( \Phi_{\tau}^D \) is not a trivial fact. This follows from a perturbation argument \( [14] \) and, in general, this Jacobi field is only defined on \( E_\ell \) away from a compact set in \( \Sigma \) and is not globally defined. This motivates the :

Definition 4 The end \( E_\ell \) of \( \Sigma \) is said to be regular if there exists a globally defined Jacobi field \( \Phi_{E_\ell}^D \) satisfying
\[
\Phi_{E_\ell}^D \circ Y_\ell - \Phi_{\tau_\ell}^D \in \mathcal{C}^{2,0}_\mu([0, +\infty) \times S^1)),
\]
(6)
for all \( \mu \in (-\gamma_{\tau_\ell,2}, 0) \).

The fact that such a globally defined Jacobi field exists is usually a consequence of the existence of a one parameter family of constant mean curvature surfaces \( \Sigma(\varepsilon) \), for \( \varepsilon \in (-\varepsilon_0, \varepsilon_0) \), which have \( k \) ends, are close to \( \Sigma \) (in a suitable sense), satisfy \( \Sigma_0 = \Sigma \) and whose \( \ell \)-th end \( E_\ell(\varepsilon) \) is asymptotic to a Delaunay surface of parameter \( \tau_\ell + \varepsilon \).

As in \( [13] \), we define the 6\( k \)-dimensional deficiency space
\[
\mathcal{W}_\Sigma := \oplus_{\ell=1}^k \text{Span} \left\{ \chi_{E_\ell} \Phi_{E_\ell}^D, \chi_{E_\ell} \Phi_{E_\ell}^{T,\varepsilon}, \chi_{E_\ell} \Phi_{E_\ell}^{R,\varepsilon} : |\varepsilon| = 1 \right\},
\]
where \( \chi_{E_\ell} \) is a cutoff function equal to 0 on \( \Sigma - E_\ell \) and equal to 1 on \( Y_\ell([1, \infty) \times S^1) \).

The following Proposition is the key result for the study of the structure of \( \mathcal{M}_{g,k}^{\tau_*} \).

Proposition 2 \( [13] \) Assume that \( \Sigma \in \mathcal{M}_{g,k}^{\tau_*} \) is nondegenerate and fix \( \mu \in (-\inf_{\ell} \gamma_{\tau_\ell,2}, 0) \).

Then the mapping
\[
\mathcal{L}_\Sigma : D^{2,0}_\mu(\Sigma) \oplus \mathcal{W}_\Sigma \longrightarrow D^{0,0}_\mu(\Sigma)
\]
is surjective and has a kernel of dimension 3\( k \). Moreover, there exists a 3\( k \)-dimensional subspace \( \mathcal{N}_\Sigma \subset \mathcal{W}_\Sigma \) such that
\[
\text{Ker} \mathcal{L}_\Sigma \subset D^{2,0}_\mu(\Sigma) \oplus \mathcal{N}_\Sigma.
\]
Finally, given any 3\( k \)-dimensional subspace \( \mathcal{K}_\Sigma \subset \mathcal{W}_\Sigma \) such that \( \mathcal{K}_\Sigma \oplus \mathcal{N}_\Sigma = \mathcal{W}_\Sigma \), the mapping
\[
\mathcal{L}_\Sigma : D^{2,0}_\mu(\Sigma) \oplus \mathcal{K}_\Sigma \longrightarrow D^{0,0}_\mu(\Sigma)
\]
is an isomorphism.

It follows from this result that \( \mathcal{M}_{g,k}^{\tau_*} \) is locally a 3\( k \)-dimensional smooth manifold near any nondegenerate element \( [13] \) (observe that we have not taken the quotient by the group of rigid motions of \( \mathbb{R}^3 \)).
4 Building blocks

We describe two families of complete noncompact constant mean curvature surfaces which will be used in the construction. The members of the first family are 3-ended surfaces while the members of the second family are \( k \)-ended surfaces. We give a fairly precise description of the elements of each family and explain how these families can be obtained using already known constructions of complete noncompact constant mean curvature surfaces. In this paper we do not give a proof of the existence of these families but rather to rely on their existence. We hope that the reader will either be convinced by the explanations below or take the existence of these families for granted.

We start by recalling the well known balancing formula \([12]\). Given a constant mean curvature surfaces \( \Sigma \subset \mathbb{R}^3 \) with finitely many ends \( E_\ell \), for \( \ell = 1, \ldots, k \), which are asymptotic to Delaunay surfaces \( D_{\bar{\alpha}}^\tau + b_\ell \), the balancing formula reads:

\[
\sum_{\ell=1}^k \tau_\ell |\tau_\ell| \vec{a}_\ell = 0 \tag{9}
\]

where \( \vec{a}_\ell \) is the direction of the axis of \( E_\ell \), which is normalized by \( |\vec{a}_\ell| = 1 \) and points toward the end of \( E_\ell \).

We fix \((\vec{e}_1, \vec{e}_2, \vec{e}_3)\) a direct orthonormal basis of \( \mathbb{R}^3 \).

4.1 Type-1 surfaces

The members of the first family are denoted by \( \Sigma_{\tau,\alpha} \), where \( \tau \) and \( \alpha \) are parameters. These surfaces are assumed to enjoy the following properties:

(i) Each \( \Sigma_{\tau,\alpha} \) is a complete noncompact constant mean curvature surface with 3 ends which are denoted by \( E_{\tau,\alpha}^{-1}, E_{\tau,\alpha}^0 \) and \( E_{\tau,\alpha}^1 \).

(ii) The surface \( \Sigma_{\tau,\alpha} \) is invariant under the action of the group

\[
G := \{ I, S_1, S_3 \}
\]

where \( S_i \) is the symmetry with respect to the plane \( x_i = 0 \).

(iii) Each \( \Sigma_{\tau,\alpha} \) is nondegenerate and the parameters \((\tau, \alpha)\) are local parameters on the moduli space of constant mean curvature surfaces with 3 ends, which are invariant under the action of the group \( G \).

(iv) The end \( E_{\tau,\alpha}^0 \) is asymptotic to a Delaunay surface of parameter \( \tau \) and axis the \( x_2 \)-axis. The vector \( -\vec{e}_2 \) is directed toward the end of \( E_{\tau,\alpha}^0 \). In particular, there exists a smooth function \( \tau \rightarrow d_{\tau,\alpha}^0 \) such that \( E_{\tau,\alpha}^0 \) is a graph (for an exponentially decaying function) over the Delaunay surface \( D_\tau^2 - d_{\tau,\alpha}^0 \vec{e}_2 \).

(v) The end \( E_{\tau,\alpha}^1 \) is asymptotic to the Delaunay surface of parameter \( \bar{\tau} \) and axis passing through the origin and of direction

\[
\vec{a}_\alpha := -\sin \alpha \vec{e}_1 - \cos \alpha \vec{e}_2.
\]

The vector \( \vec{a}_\alpha \) is directed toward the end of \( E_{\tau,\alpha}^1 \). In particular, there exists a smooth function \( \tau \rightarrow d_{\tau,\alpha}^1 \) such that \( E_{\tau,\alpha}^1 \) is a graph (for an exponentially decaying function) over the Delaunay surface \( D_\tau^2 + d_{\tau,\alpha}^1 \vec{a}_\alpha \).
(vi) The ends of $\Sigma_{\tau,\alpha}$ are regular.

Observe that the image of $E^1_{\tau,\alpha}$ by $S_1$ is $E^{-1}_{\tau,\alpha}$ and that $E^0_{\tau,\alpha}$ remains globally fixed under the action of $S_1$. Also each end remains globally fixed under the action of $S_3$. Applying the balancing formula (9), we conclude that the Delaunay parameters $\bar{\tau}$ and $\tau$ are related by the formula

$$\tau |\tau| + 2 \cos \alpha \bar{\tau} |\bar{\tau}| = 0.$$  

(10)

In particular, if $\alpha \in (0, \pi/2)$, the signs of $\tau$ and $\bar{\tau}$ are different and this implies that the surface $\Sigma_{\tau,\alpha}$ has always an end which is not embedded (asymptotic to a nodoid) in this case.

Observe that (iv) implies that the end $E^0_{\tau,\alpha}$ can be parameterized by

$$X^0_{\tau,\alpha}(s, \theta) := X^0_\tau(s, \theta) - d^0_{\tau,\alpha} \vec{e}_2 + w^0_{\tau,\alpha}(s, \theta) \vec{N}^\tau_\tau(s, \theta)$$  

(11)

with $(s, \theta) \in [0, \infty) \times S^1$, for some function $w^0_{\tau,\alpha} \in \mathcal{E}^{2,\alpha}_{-\gamma,\tau}(0, +\infty) \times S^1)$. (In general the function $w^0_{\tau,\alpha}$ is only defined on $[c, +\infty) \times S^1$ for some $c > 0$ large enough. However increasing the value of $d_{\tau,\alpha}$ by a $2mT_\tau$ for some $m \in \mathbb{N}$, if this is necessary, we can assume that the function $w^0_{\tau,\alpha}$ is defined on $[0, +\infty) \times S^1$).

Similarly (v) implies that the end $E^1_{\tau,\alpha}$ can be parameterized by

$$X^1_{\tau,\alpha}(s, \theta) := X^1_\tau(s, \theta) + d^1_{\tau,\alpha} \vec{a}_\alpha + w^1_{\tau,\alpha}(s, \theta) \vec{N}^\alpha_{\tau}(s, \theta)$$  

(12)

with $(s, \theta) \in [0, +\infty) \times S^1$, for some function $w^1_{\alpha,\tau} \in \mathcal{E}^{2,\alpha}_{-\gamma,\tau}(0, +\infty) \times S^1$.

**Definition 5** Given $s_0, s_1 > 0$, we define the compact surface with 3 boundaries

$$\Sigma_{\tau,\alpha}(s_0, s_1) := \Sigma_{\tau,\alpha} - (X^0_{\tau,\alpha}((s_0, +\infty) \times S^1) \cup X^1_{\tau,\alpha}((s_1, +\infty) \times S^1)$$

$$\cup S_1 X^1_{\tau,\alpha}((s_1, +\infty) \times S^1))$$

In the case where the surfaces are Alexandrov embedded the surfaces described above have been classified in [9]. However, it does not follow from this description that the surfaces are nondegenerate and have regular ends. This is the reason why we give now two examples of construction of such a family which rely on connected sum constructions and for which it is possible to check that the surfaces constructed are both nondegenerate and have regular ends:

**Example 1** A first family can be obtained by gluing on the unit sphere $S^2 \subset \mathbb{R}^3$, three half Delaunay surfaces of parameters $\bar{\tau}, \tau$ and $\bar{\tau}$ respectively at the points of coordinates

$$(-\sin \alpha, -\cos \alpha, 0), \quad (0, -1, 0) \quad \text{and} \quad (\sin \alpha, -\cos \alpha, 0)$$

respectively, using a modified version of the connected sum result of [15], [16] and [4]. The construction works if one imposes the surfaces to be invariant under the action of the group $G$. Given the symmetries of the surfaces constructed, there remains only two degrees of freedom which are: The Delaunay parameter $\tau$ and the angle $\alpha$ between the ends. The construction works for any $\alpha \in (0, \pi/2) \cup (\pi/2, \pi)$ and any $\tau \neq 0$ close enough to 0. The fact that the ends are regular follows from the construction itself since $\tau$ can be used to parameterize this family of surfaces and differentiation with respect
to this parameter yields a Jacobi field whose asymptotic along any end has a nontrivial component on $\chi_{E_{\ell}} \Phi^{\ell}_{E_{\ell}}$, for $\ell = 0, \pm 1$.

**Example 2** A second family can be obtained by gluing on a Delaunay surface of parameter $\tau$ and axis $x_1$ which is translated so that it is invariant under the action of the symmetry $S_1$ (namely either $D_{\tau}^1$ or $D_{\tau}^1 + T \tau \hat{e}_1$), a half Delaunay surface of axis $x_2$ and small Delaunay parameter $\tau$. Again, the construction works if one imposes the surfaces to be invariant under the action of the group $G$. Given the symmetries of the surfaces constructed, there remains only two degrees of freedom which are: The Delaunay parameters $\tau$ and $\bar{\tau}$. The construction works for any small value of the parameter $\tau \neq 0$ [15], [16] and [4] and provides a surface with an angle $\alpha$ close, but not equal, to $\pi/2$ which is determined by the equation $\tau |\tau| + 2 \cos \alpha \bar{\tau} |\bar{\tau}| = 0$. This shows that $(\alpha, \tau)$ are local parameters on the corresponding moduli space and, as in the previous example, the ends of the surfaces are regular.

In both cases, the surfaces are seen to be nondegenerate, when $\tau$ is close enough to 0, using the strategy developed in [14].

### 4.2 Type-2 surfaces

We fix $k \geq 3$. The members of the second family are denoted by $\bar{\Sigma}_\tau$, where $\tau$ is a parameter. These surfaces are assumed to enjoy the following properties:

(i) Each $\bar{\Sigma}_\tau$ is a complete noncompact constant mean curvature surface with $k$ ends which are denoted by $\bar{E}_0^\tau, \ldots, \bar{E}_{k-1}^\tau$.

(ii) The surface in invariant under the action of the group

$$G_k := \{ R_{2\pi j/k} : j \in \mathbb{Z} \}$$

where $R_\theta$ is the rotation of angle $\theta$ in the $x_1, x_2$ plane.

(iii) Each $\bar{\Sigma}_\tau$ is nondegenerate and the parameter $\tau$ is local parameter on the moduli space of constant mean curvature surfaces with $k$ ends, which are invariant under the action of the group $G_k$.

(iv) The end $\bar{E}_0^\tau$ is asymptotic to a Delaunay surface of parameter $\tau$ and axis the $x_2$-axis. The vector $\bar{e}_2$ being directed toward the end of $\bar{E}_0^\tau$. In particular, there exists a smooth function $\tau \rightarrow \bar{d}_0^\tau$ such that $\bar{E}_0^\tau$ is a graph (for an exponentially decaying function) over the Delaunay surface $D_{\tau}^{\bar{e}_2} + \bar{d}_0^\tau \bar{e}_2$.

(v) The ends of $\bar{\Sigma}_\tau$ are regular.

Observe that, for $\ell = 1, \ldots, k - 1$ the image of $E_\tau^0$ by $R_{2\pi \ell/k}$ is the end $\bar{E}_\ell^\tau$. Hence the angle between two consecutive ends is given by $2 \pi/k$ and, to check that the ends of $\bar{\Sigma}_\tau$ are regular it is enough to check that $\bar{E}_0^\tau$ is regular.

As in the case of Type-1 surfaces, (iv) implies that the end $\bar{E}_0^\tau$ can be parameterized by

$$X_\tau^0(s, \theta) := X_{\tau}^{\bar{e}_2}(s, \theta) + \bar{d}_0^\tau \bar{e}_2 + \tilde{w}_0^\tau(s, \theta) \tilde{N}_{\tau}^{\bar{e}_2}(s, \theta)$$

with $(s, \theta) \in (0, +\infty) \times S^1$, for some function $\tilde{w}_0^\tau \in \mathcal{C}^{2, \alpha}_{x_2}(0, +\infty)$. 

\[10\]
Definition 6 Given $s_0 > 0$, we define the compact surface with $k$ boundaries

$$\overline{\Sigma}(s_0) := \overline{\Sigma} - \bigcup_{\ell=0}^{k-1} R_{2\pi\ell/k} X^0_r((s_0, +\infty) \times S^1)$$

We now give two examples of such a family.

Example 1 A first family can be obtained by gluing on the unit sphere $S^2 \subset \mathbb{R}^3$, $k$ copies of a half Delaunay surface with small Delaunay parameter $\tau \neq 0$ in such a way that the surface remains invariant under the action of $G_k$. Again this is a byproduct of the end addition result proved in [15], [16] or this is also a byproduct of the result of N. Kapouleas in [5]. These surfaces have also been constructed and described by K. Grosse-Brauckmann [8].

Example 2 A second family can be obtained by gluing on a $k$-noid (a minimal surface with $k$ ends of catenoidal type [3], [1]) which is invariant under the action of $G_k$, $k$ copies of a half Delaunay surface with small Delaunay parameter $\tau \neq 0$ in such a way that the symmetries are preserved. This construction is the one described in [14].

In either case, given the symmetries of the surfaces constructed there remains only one degree of freedom which is $\tau$, the Delaunay parameter of the ends. Either construction works for any $\tau \neq 0$ close enough to 0. The fact that (v) holds follows at once from the construction itself since $\tau$ can be used to parameterize this family of surfaces and differentiation with respect to this parameter yields a Jacobi field whose asymptotic has a nontrivial component on $\chi_{E^0} \Phi^{D}_{E^0}$. The fact that the surfaces constructed are nondegenerate follows from [14].

4.3 Jacobi fields

We give a precise description of the Jacobi fields on both $\Sigma_{\tau,\alpha}$ and on $\overline{\Sigma}_\tau$. This description yields a description of the spaces $K_{\Sigma_{\tau,\alpha}}$ and $K_{\overline{\Sigma}_\tau}$ which have been introduced in Proposition 2.

We start with the analysis of the Jacobi fields on $\overline{\Sigma}_\tau$ since this is the simplest. Since the surface $\overline{\Sigma}_\tau$ is assumed to be nondegenerate, the deficiency space $D_{\Sigma_{\tau}}$ is 6$k$-dimensional. However, since we are working in the space of surfaces which are invariant under the action of the group $G_k$ and this reduces the dimension of the corresponding moduli space to 1 and the deficiency space is now spanned by the 2 functions

$$\bar{\psi}_T^T := \sum_{\ell=0}^{k-1} \chi_{E^\ell} \Phi^{T}_{E^\ell} \tilde{a}_\ell$$

and

$$\bar{\psi}_D^T := \sum_{\ell=0}^{k-1} \chi_{E^\ell} \Phi^{D}_{E^\ell}$$

where $\tilde{a}_\ell = R_{2\pi\ell/k} \vec{e}_2$ is the direction of the end $E^\ell$. Observe that the symmetries of $\Sigma_{\tau}$ imply that

$$\Phi^{T}_{E^\ell} \tilde{a}_\ell = \Phi^{T}_{E^0_\tau} \circ (R_{2\pi\ell/k})^{-1}$$

Since the end $E^0_\tau$ is assumed to be regular, there exists a globally Jacobi field (which is invariant under the action of $G_k$) whose asymptotic on $E^0_\tau$ has a nontrivial component on $\Phi^{D}_{E^0_\tau}$. In fact this Jacobi field is obtained by moving the parameter $\tau$. Multiplying this Jacobi field by a suitable constant, we can assume that it is asymptotic to $\bar{\psi}_D^T + \bar{c} \bar{\psi}_T^T$.
on each $\tilde{E}_\tau^\ell$, were the constant $\bar{c}$ depends on $\tau$. This implies that the space $K_{\Sigma_\tau}$ can be chosen to be

$$K_{\Sigma_\tau} = \text{Span}\{\bar{\psi}_\tau^T\}$$

We now analyze the Jacobi fields on $\Sigma_{\tau,\alpha}$. By assumption, $\Sigma_{\tau,\alpha}$ is nondegenerate and has 3 ends, therefore the deficiency space $D_{\Sigma_{\tau,\alpha}}$ is 18-dimensional. Now, recall that we are working in the space of surfaces which are invariant under the action of the group $G$ and this reduces the dimension of the corresponding moduli space to 3 and the deficiency space is spanned by the 6 functions we now describe:

$$\psi_{\tau,E_1}^T := \chi_{E_{\tau,\alpha}}^0 \Phi_{E_{\tau,\alpha}}^T \tilde{e}_2,$$

$$\psi_{\tau,E_0}^D := \chi_{E_{\tau,\alpha}}^0 \Phi_{E_{\tau,\alpha}}^D,$$

$$\psi_{\tau,E_1}^D := \chi_{E_{\tau,\alpha}}^0 \Phi_{E_{\tau,\alpha}}^D + (\chi_{E_{\tau,\alpha}}^0 \Phi_{E_{\tau,\alpha}}^D) \circ S_1,$$

$$\psi_{\tau,E_1}^T := \chi_{E_{\tau,\alpha}}^0 \Phi_{E_{\tau,\alpha}}^T \tilde{a},$$

$$\psi_{\tau,E_1}^R := \chi_{E_{\tau,\alpha}}^0 \Phi_{E_{\tau,\alpha}}^R \tilde{e}_3.$$
5 The construction

We fix $k \geq 3$ and define

$$\alpha_k := \frac{\pi}{2} - \frac{\pi}{k}$$

We assume that, for $\tau$ in some closed (nonempty) interval $I \subset (\tau_*, 0) \cup (0, 1)$, we are given a family of surfaces $\Sigma_{\tau,\alpha_k}$ of Type 1 and a family of surfaces $\bar{\Sigma}_\tau$ of Type 2. For the sake of simplicity we now drop the dependence on $\alpha_k$ in all the quantities related to $\Sigma_{\tau,\alpha_k}$ and simply write $\Sigma_{\tau}$, $E^\ell_{\tau}$, $d^0_{\tau}$, $d^1_{\tau}$, ... The parameter $\tau$ being chosen in $I$, we recall that $\bar{\tau}$ is given by

$$\tau |\tau| + 2 \cos \alpha_k \bar{\tau} |\bar{\tau}| = 0.$$  \hspace{1cm} (14)

Given $n, m \in \mathbb{N}$, we set

$$\delta_{n,\tau} := d^0_{\tau} + \bar{d}^0_{\tau} + 2nT_{\tau}$$

We agree on the notation

$$\Sigma_{n,\tau} := \Sigma_{\tau} + \delta_{n,\tau} \bar{e}_2$$

and the ends of this surface are denoted by

$$E_j^1_{n,\tau} := E_j^1 + \delta_{n,\tau} \bar{e}_2$$

and are parameterized by

$$X_j^1_{n,\tau} := X_j^1 + \delta_{n,\tau} \bar{e}_2.$$  

Also we define the truncated surface (see Definition 5)

$$\Sigma_{n,\tau}(s_0, s_1) := \Sigma_{\tau}(s_0, s_1) + \delta_{n,\tau} \bar{e}_2$$

With these notations in mind, we consider the truncated surface $\Sigma_{n,\tau}(n s_\tau, m s_\tau)$ together with the images of this surface by $R_{2\ell\pi/k}$, for $\ell = 1, \ldots, k - 1$ and also the truncated surface $\bar{\Sigma}(n s_\tau)$ (see Definition 6). These surfaces with boundaries are now connected together using appropriate cutoff functions, to produce a compact surface which is invariant under the action of $G_k$. More precisely, for each $\ell = 0, \ldots, k - 1$ : The end $E^\ell_{\tau}$ of $\Sigma_{\tau}$ can be connected with the image of $E^0_{n,\tau}$ by $R_{2\ell\pi/k}$ since they are graphs over the same Delaunay surface. And, provided $n$ and $\tau$ are suitably chosen, the image of the end $E^1_{n,\tau}$ by $R_{2\pi(\ell+1)/k}$ can be connected with the image of $E^{\ell+1}_{n,\tau}$ by $R_{2\pi(\ell+1)/k}$. We now describe analytically this procedure. Given the fact that the surface we want to construct should be invariant under the action of $G_k$ it is enough to describe the :

5.1 Connection of $E^0_{n,\tau}$ with $\bar{E}^0_{\tau}$.

By construction the ends $E^0_{n,\tau}$ and $\bar{E}^0_{\tau}$ are normal graphs over the same Delaunay surface. Given the parameterizations defined in (11) and (13) we can connect the two pieces together by considering the parameterization

$$Y^0_{n,\tau}(s, \theta) = X^0_{\tau}(s + n s_\tau, \theta) + d^0_{\tau} \bar{e}_2 + \bar{w}_{\tau}(s, \theta) N^0_{\tau}(s + n s_\tau, \theta)$$

for $(s, \theta) \in (-n s_\tau, n s_\tau) \times S^1$ where

$$\bar{w}_{\tau}(s, \theta) := \xi(s) \bar{w}^0_{\tau}(s + n s_\tau, \theta) + (1 - \xi(s)) w^0_{\tau}(n s_\tau - s, \theta)$$

Here $\xi$ is a cutoff function identically equal to 1 for $s \leq -1$ and identically equal to 0 for $s \geq 1$ and which satisfies

$$\xi(-s) = 1 - \xi(s).$$
We will denote by $A_{n,\tau}^0$ the image of $(-1,1) \times S^1$ by $Y_\rho^0$. We define

$$Y_{n,\tau}^\ell = R_{2\pi \ell/k} Y_{n,\tau}^0$$

for $\ell = 1, \ldots, k-1$ which describes the connection of $E_\tau^\ell$ with the image of $E_{n,\tau}^0$ by $R_{2\pi \ell/k}$.

5.2 Connection of $E_{n,\tau}^1$ with the image of $E_{n,\tau}^{-1}$ by $R_{2\pi/k}$.

We define the plane

$$\Pi_k := \{ x \in \mathbb{R}^3 : \tan(2\pi/k) x_2 = -x_1 \}$$

Observe that the image of $E_{n,\tau}^1$ by the symmetry with respect to $\Pi_k$ is equal to the image of $E_{n,\tau}^{-1}$ by $R_{2\pi/k}$. By definition, the end $E_{n,\tau}^1$ is a graph over the Delaunay surface $D_\tau^2 + d_1^j \bar{a} + \delta_{n,\tau} \bar{e}_2$. Therefore the end $E_{n,\tau}^1$ and its image by the symmetry with respect to the plane $\Pi_k$ are normal graphs over the same Delaunay surface if and only if the Delaunay surface $D_\tau^2 + d_1^j \bar{a} + \delta_{n,\tau} \bar{e}_2$ is invariant under the symmetry with respect to the plane $\Pi_k$. This condition is translated into the fact that there exists an integer $m \in \mathbb{N}$ such that

$$\sin(\pi/k) (d_0^0 + \delta_{n,\tau}^0 + 2n T_\tau) = d_1^1 + m T_\tau$$

(15)

If this condition is fulfilled we can connect the end $E_{n,\tau}^1$ and its image by $R_{2\pi/k}$, using the parameterization

$$Z_{n,\tau}^0(s, \theta) = X_\tau^2(s + m s_\tau, \theta) + d_1^j \bar{a} + \delta_{n,\tau} \bar{e}_2 + \bar{w}_\tau(s, \theta) \bar{N}_\tau^0(s + m s_\tau, \theta)$$

where

$$\bar{w}_\tau(s, \theta) := \xi(s) w_{\tau}^1(s + m s_\tau, \theta) + (1 - \xi(s)) w_{\tau}^1(m s_\tau - s, \theta)$$

We will denote by $A_{n,\tau}$ the image of $(-1,1) \times S^1$ by $Z_{n,\tau}^0$. We set

$$Z_{n,\tau}^\ell := R_{2\pi \ell/k} Z_{n,\tau}^0$$

for $\ell = 1, \ldots, k-1$ which describes the connection of the image of $E_{n,\tau}^1$ by $R_{2\pi \ell/k}$ with the image of $E_{n,\tau}^{-1}$ by $R_{2\pi(k+1)/k}$.

5.3 Estimate of the mean curvature of the connected surface

The compact surface which is obtained through these connections will be denoted by $S_{n,\tau}$. It is an immersed compact surface of genus $k$. By construction, the mean curvature of the surface $S_{n,\tau}$ is equal to 1 except in annular regions $A_{n,\tau}^0$, $A_{n,\tau}^1$ and in their images by the elements of $G_k$. The following estimates follow at once from the fact that the functions $w_{\tau}^0$, $\bar{w}_{\tau}^0$ and $w_{\tau}^1$ are exponentially decaying, as explained in §4.

Lemma 2 We have

$$\|H_{S_{n,\tau}} - 1\|_{C^{0,\alpha}(A_{n,\tau}^0)} \leq c e^{-n \gamma_2 s_\tau}$$

and, provided (15) is satisfied, we have

$$\|H_{S_{n,\tau}} - 1\|_{C^{0,\alpha}(A_{n,\tau}^1)} \leq c e^{-m \gamma_2 s_\tau}$$

where the constant $c > 0$ does not depend on $\tau \in I$ nor on $n \in \mathbb{N}$. 

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5.4 Partition of unity on $S_{n,\tau}$

Subordinate to the above construction is a partition of unity we now describe.

(i) The function $\chi_{n,\tau}$ is a smooth function defined on $S_{n,\tau}$ which is equal to 1 on

$$\Sigma_{n,\tau}(n s_\tau - 1, m s_\bar{\tau} - 1) \subset S_{n,\tau}$$

and which is equal to 0 on the complement of

$$\Sigma_{n,\tau}(n s_\tau - 1, m s_\bar{\tau} - 1) \cup Y^0_{n,\tau}((-1, 1) \times S^1) \cup Z^0_{n,\tau}((-1, 1) \times S^1)$$

$$\cup Z^{k-1}_{n,\tau}((-1, 1) \times S^1)$$

in $S_{n,\tau}$. To be more precise, on the part of $S_{n,\tau}$ parameterized by $Y^0_{n,\tau}$, the function $\chi_{n,\tau}$ is equal to 1 for $s \geq 1$ and equal to 0 for $s \leq -1$ and on the part of $S_{n,\tau}$ parameterized by $Z^0_{n,\tau}$, the function $\chi_{n,\tau}$ is equal to 1 for $s \leq -1$ and equal to 0 for $s \geq 1$. This function is assumed to be invariant under the action of $S_1$.

(ii) The function $\bar{\chi}_{n,\tau}$ is a smooth function defined on $S_{n,\tau}$ which is equal to 1 on

$$\bar{\Sigma}_{\tau}(n s_\tau - 1) \subset S_{n,\tau}$$

and which is equal to 0 on the complement of

$$\bar{\Sigma}_{\tau}(n s_\tau - 1) \cup^{k-1}_{\ell=0} Y^\ell_{n,\tau}((-1, 1) \times S^1)$$

in $S_{n,\tau}$. To be more precise, on the part of $S_{n,\tau}$ parameterized by $Y^0_{n,\tau}$, the function $\bar{\chi}_{n,\tau}$ is equal to 1 for $s \leq -1$ and equal to 0 for $s \geq 1$. This function is assumed to be invariant under the action of $G_k$.

(iii) We also ask that

$$\bar{\chi}_{n,\tau} + \sum^{k-1}_{\ell=0} \chi_{n,\tau} \circ (R_{2\pi\ell/k})^{-1} = 1$$

on $S_{n,\tau}$.

There is another set of cutoff functions which will be needed. They can be described as follows:

(i) The function $\chi^0_{n,\tau}$ is a smooth function defined on $S_{n,\tau}$ which is equal to 1 on

$$\Sigma_{n,\tau}(n s_\tau - 1, m s_\bar{\tau} - 1) \cup Y^0_{n,\tau}((-n s_\tau + 2, 1) \times S^1) \cup Z^0_{n,\tau}((-1, m s_\bar{\tau} - 2) \times S^1)$$

$$\cup Z^{k-1}_{n,\tau}((-1, m s_\bar{\tau} - 2) \times S^1)$$

and which is equal to 0 on the complement of

$$\Sigma_{n,\tau}(n s_\tau - 1, m s_\bar{\tau} - 1) \cup Y^0_{n,\tau}((-n s_\tau + 2, 1) \times S^1) \cup Z^0_{n,\tau}((-1, m s_\bar{\tau} - 1) \times S^1)$$

$$\cup Z^{k-1}_{n,\tau}((-1, m s_\bar{\tau} - 1) \times S^1)$$

To be more precise, on the part of $S_{n,\tau}$ parameterized by $Y^0_{n,\tau}$, the function $\chi_{n,\tau}$ is equal to 1 for $s \geq -n s_\tau + 2$ and equal to 0 for $s \leq -n s_\tau + 1$ and on the part of $S_{n,\tau}$ parameterized by $Z^0_{n,\tau}$, the function $\chi_{n,\tau}$ is equal to 1 for $s \leq m s_\bar{\tau} - 2$ and equal to 0 for $s \geq m s_\bar{\tau} - 1$. This function is assumed to be invariant under the action of $S_1$. 

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(ii) The function $\bar{\chi}^{e}_{n,\tau}$ is a smooth function which is equal to 1 on

$$\bar{\Sigma}_{\tau}(n s_{\tau} - 1) \cup_{\ell=0}^{k-1} Y_{n,\tau}^{\ell}((0, n s_{\tau} - 2) \times S^1)$$

and which is equal to 0 on the complement of

$$\bar{\Sigma}_{\tau}(n s_{\tau} - 1) \cup_{\ell=0}^{k-1} Y_{n,\tau}^{\ell}((0, n s_{\tau} - 1) \times S^1)$$

To be more precise, on the part of $S_{n,\tau}$ parameterized by $Y_{n,\tau}^{0}$, the function $\chi_{n,\tau}$ is equal to 1 for $s \leq n s_{\tau} - 2$ and equal to 0 for $s \geq n s_{\tau} - 1$. This function is assumed to be invariant under the action of $G_k$.

5.5 Extension of the elements of $\mathcal{K}_{\Sigma_{\tau}}$ and $\mathcal{K}_{\bar{\Sigma}_{\tau}}$

Building on the analysis of §4.3, we explain how the restriction of the elements of $\mathcal{K}_{\Sigma_{n,\tau}}$ to $\Sigma_{n,\tau}(n s_{\tau}-1, m s_{\tau}-1) \subset S_{n,\tau}$ and the restriction of the elements of $\mathcal{K}_{\bar{\Sigma}_{\tau}}$ to $\bar{\Sigma}_{\tau}(n s_{\tau}-1) \subset S_{n,\tau}$ can be extended to functions which are defined on $S_{n,\tau}$. By ”extension” we mean that these restrictions are first connected with restrictions of the elements of $\mathcal{N}_{\Sigma_{n,\tau}}$ to $\Sigma_{n,\tau}(n s_{\tau}-1, m s_{\tau}-1)$ and the restriction of the elements of $\mathcal{N}_{\bar{\Sigma}_{\tau}}$ to $\bar{\Sigma}_{\tau}(n s_{\tau}-1)$ and then extended to $S_{n,\tau}$ by using the action of $G_k$. The fact that these extensions are meaningful (see Lemma 3) relies on (15). We keep the same notations for the elements of $\mathcal{K}_{\Sigma_{n,\tau}}$ and $\mathcal{K}_{\bar{\Sigma}_{\tau}}$.

(i) The restriction of $\psi^{T,\bar{b}}_{\tau,E_1}$ and $\psi^{T,\bar{a}}_{\tau,E_1}$ to $\Sigma_{n,\tau}(n s_{\tau}, m s_{\tau})$ can be easily extended to $S_{n,\tau}$ using the fact that the ends $E^1_{\tau}$ and $E_{\tau}^1$ are symmetric with respect to $\Pi_k$. For example, for $\bar{b} = \bar{a}, \bar{a}^\perp$, we can first define a function $\Psi^{T,\bar{b}}_{n,\tau}$ on the part of $S_{n,\tau}$ which is parameterized by $Z_{n,\tau}^{1}$,

$$\Psi^{T,\bar{b}}_{n,\tau} = \chi_{n,\tau} \psi^{T,\bar{b}}_{\tau,E_1} + (1 - \chi_{n,\tau}) \psi^{T,\bar{b}}_{\tau,K_1} \circ (R_{2\pi/k})^{-1}$$

and then use the action of $G_k$ to extend this function to the other components of $S_{n,\tau}$.

(ii) The restriction of the element $\bar{\psi}_{\tau}$ of $\mathcal{K}_{\bar{\Sigma}_{\tau}}$ to $\bar{\Sigma}_{\tau}(n s_{\tau}-1)$ can be extended to $S_{n,\tau}$ using the restriction to $\Sigma_{n,\tau}(n, s_{\tau}-1, m, s_{\tau}-1)$ of $\Phi^{T,\bar{e}}_{\Sigma_{n,\tau}}$, the (unique) element of $\mathcal{N}_{\Sigma_{n,\tau}}$ which is asymptotic to $\Phi^{T,\bar{e}}_{\Sigma_{n,\tau}}$ on $E_{n,\tau}^{0}$ (i.e. the globally defined Jacobi field which corresponds to translation of $\Sigma_{n,\tau}$ along the $x_2$ axis). We define a function $\bar{\psi}^{T}_{n,\tau}$ first by writing

$$\bar{\psi}^{T}_{n,\tau} = \bar{\chi}_{n,\tau} \bar{\psi}_{\tau} + (1 - \bar{\chi}_{n,\tau}) \Phi^{T,\bar{e}}_{\Sigma_{\tau}}$$

on the part of $S_{n,\tau}$ which is parameterized by $Y_{n,\tau}^{0}$. Observe that $\Phi^{T,\bar{e}}_{\Sigma_{\tau}}$ is asymptotic to a linear combination of $\psi^{T,\bar{a}}_{\tau,E_1}$ and $\psi^{T,\bar{a}}_{\tau,E_1}$ on the other ends of $\Sigma_{\tau}$ and we can use the extension described in (i) to extend the function to $S_{n,\tau}$. For example,

$$\bar{\psi}^{T}_{n,\tau} = \chi_{n,\tau} \Phi^{T,\bar{e}}_{\Sigma_{\tau}} + (1 - \chi_{n,\tau}) \Phi^{T,\bar{e}}_{\Sigma_{\tau}} \circ (R_{2\pi/k})^{-1}$$

on the part of $S_{n,\tau}$ which is parameterized by $Z_{n,\tau}^{0}$, and then we use the action of $G_k$ to extend this function to the other components of $S_{n,\tau}$.
(iii) One can choose the parameter $t$ in such a way that the element $\psi_{\tau,E^0}^D + t \psi_{\tau,E^0}^T$ of $K_{S_{n,\tau}}$ is asymptotic to $\Phi_{E^0}^D$, the generator of $\mathcal{N}_{\Sigma_{\tau}}$. Indeed, $(\psi_{\tau,E^0}^D + t \psi_{\tau,E^0}^T) \circ X^0_\tau$ is asymptotic to $\Phi_{D_\tau}^D + t \Phi_{D_\tau}^T$ and $\Phi_{E^0}^D \circ X^0_\tau$ is asymptotic to $\Phi_{D_\tau}^D$. Granted the definition of $Y^0_{n,\tau}$ (in terms of $X^0_\tau$ and $X^0_\tau$) together with (2), we choose

$$t = n p_\tau s_\tau.$$ 

These two functions are then connected, as in (i) or (ii), to define the function $\Psi_{n,\tau}^D$.

For example, we define

$$\Psi_{n,\tau}^D = \bar{\chi}_{n,\tau} \Phi_{E^0}^D + (1 - \bar{\chi}_{n,\tau}) (\psi_{\tau,E^0}^D + n p_\tau s_\tau \psi_{\tau,E^0}^T)$$

on the part of $S_{n,\tau}$ which is parameterized by $Y^0_{n,\tau}$ and then extend this function to all $S_{n,\tau}$ using the action of $G_k$.

We define $\mathcal{L}_{S_{n,\tau}}$ to be the Jacobi operator about the surface $S_{n,\tau}$. The following result again follows from the fact that the functions $w^0_\tau$, $\bar{w}^0_\tau$ and $w_\tau^1$ are exponentially decaying.

**Lemma 3** There exists a constant $c > 0$ which does not depend on $\tau \in I$ nor on $n$ such that

$$\| \mathcal{L}_{S_{n,\tau}} \Psi_{n,\tau}^D \psi^{T,\bar{b}} \|_{C^{0,\alpha}(A_{n,\tau}^1)} \leq c e^{-\gamma_{\tau,2} m s_\tau}$$

for $\bar{b} = \bar{a}, \bar{a}^\perp$ and

$$\| \mathcal{L}_{S_{n,\tau}} \Phi_{n,\tau}^D \|_{C^{0,\alpha}(A_{n,\tau}^0)} \leq c e^{-\gamma_{\tau,2} n s_\tau}$$

and

$$\| \mathcal{L}_{S_{n,\tau}} \bar{\psi}_{n,\tau}^D \|_{C^{0,\alpha}(A_{n,\tau}^1)} \leq c e^{-\gamma_{\tau,2} m s_\tau}$$

Finally, given $\mu \in (-\gamma_{\tau,2}, 0)$, there exists a constant $c_\mu > 0$ which does not depend on $\tau \in I$ nor on $n$ such that

$$\| \mathcal{L}_{S_{n,\tau}} \Psi_{n,\tau}^D \psi^{D,\bar{a}} \|_{C^{0,\alpha}(A_{n,\tau}^0)} \leq c_\mu e^{-\mu n s_\tau}$$

6 Perturbation of $S_{n,\tau}$

6.1 Mapping properties

We define the weighted spaces on $S_{n,\tau}$. Roughly speaking, to evaluate the norm in this space, we restrict a function to each summand constituting $S_{n,\tau}$ and then evaluate each term using the norm defined in Definition 2.

**Definition 7** Given $r \in \mathbb{N}$, $\alpha \in (0,1)$ and $\mu \in \mathbb{R}$, we define $C^{r,\alpha}_\mu(S_{n,\tau})$ to be the space of functions $w \in C^{r,\alpha}(S_{n,\tau})$ which are invariant under the action of $G_k$. This space is endowed with the norm

$$\| w \|_{C^{r,\alpha}_\mu(S_{n,\tau})} := \| \chi_{n,\tau} w \|_{C^{r,\alpha}(\Sigma_\tau)} + \| \bar{\chi}_{n,\tau} w \|_{C^{r,\alpha}(\bar{\Sigma}_\tau)}$$

We also define the 4 dimensional space

$$K_{S_{n,\tau}} = \text{Span}\{ \bar{\psi}_{n,\tau}^T, \Psi_{n,\tau}^D, \bar{\psi}_{n,\tau}^T \bar{a}, \Psi_{n,\tau}^T \bar{a}^\perp \}$$

In the following result we glue together the parametrices for $\mathcal{L}_{\Sigma_\tau}$ and $\mathcal{L}_{\bar{\Sigma}_\tau}$ to obtain a parametrix for $\mathcal{L}_{S_{n,\tau}}$.
Proposition 3 Assume that $\mu \in (-\inf(\gamma_\tau,2,\gamma_\tau,2),0)$ is fixed. There exist $n_0 > 0$ and $c > 0$ and, for all $n \geq n_0$ and $\tau \in I$ for which (13) holds, one can find an operator

$$G_{n,\tau} : C^{0,\alpha}_\mu(S_{n,\tau}) \to C^{2,\alpha}_\mu(S_{n,\tau}) \oplus \mathcal{K}_{S_{n,\tau}},$$

such that $w := G_{n,\tau}(f)$ solves $\mathcal{L}_{S_{n,\tau}} w = f$ on $S_{n,\tau}$ and

$$\|w\|_{C^{2,\alpha}_\mu(S_{n,\tau}) \oplus \mathcal{K}_{S_{n,\tau}}} \leq c \|f\|_{C^{0,\alpha}_\mu(S_{n,\tau})},$$

for some constant which does not depend on $\tau \in I$ nor on $n \geq n_0$.

Proof: Given a function $g$ defined on $S_{n,\tau}$, it will be convenient to identify the function $\chi_{n,\tau} g$ (resp. $\bar{\chi}_{n,\tau} g$) with a function which is defined on $\Sigma_{n,\tau}$ (resp. $\bar{\Sigma}_{\tau}$). This identification is done in the natural way on the common parts of the surfaces and by identifying $(\chi_{n,\tau} g) \circ Z^{0}_{n,\tau}$ with $(\chi_{n,\tau} g) \circ X^{1}_{n,\tau}$, $(\chi_{n,\tau} g) \circ Y^{0}_{n,\tau}$ with $(\chi_{n,\tau} g) \circ X^{0}_{n,\tau}$ and so on. . . on the ends of the surfaces.

Conversely, given a function $g$ define in $\Sigma_{n,\tau}$ (resp. $\bar{\Sigma}_{\tau}$) we will identify the function $\chi_{n,\tau} g$ (resp. $\bar{\chi}_{n,\tau} g$) with a function which is defined on $\Sigma_{\tau}$ (resp. $\bar{\Sigma}_{\tau}$).

Given $f \in C^{0,\alpha}_\mu(S_{n,\tau})$ we want to solve the equation

$$\mathcal{L}_{S_{n,\tau}} w = f$$

on $S_{n,\tau}$. We solve

$$\mathcal{L}_{\Sigma_{n,\tau}} w_1 = \chi_{n,\tau} f$$

on $\Sigma_{n,\tau}$ and

$$\mathcal{L}_{\bar{\Sigma}_{\tau}} w_2 = \bar{\chi}_{n,\tau} f$$

on $\bar{\Sigma}_{\tau}$.

The existence of $w_1$ follows at once from the analysis described in §3 and we have the estimate

$$\|w_1\|_{C^{2,\alpha}_\mu(\Sigma_{n,\tau}) \oplus \mathcal{K}(\Sigma_{n,\tau})} + \|w_2\|_{C^{2,\alpha}_\mu(\bar{\Sigma}_{\tau}) \oplus \mathcal{K}(\bar{\Sigma}_{\tau})} \leq c \|f\|_{C^{0,\alpha}_\mu(S_{n,\tau})}$$

(16)

where the constant $c > 0$ does not depend on $n$ nor on $\tau \in I$. Observe that the function $w_1$ can be decomposed as

$$w_1 := v_1 + a_1 (\psi^{D}_{n,\tau} + t \psi^{T}_{n,\tau}) + b_1 \psi^{T,\bar{a}}_{n,\tau} + c_1 \psi^{T,\bar{a}}_{n,\tau},$$

and the function $w_2$ can be decomposed as

$$w_2 := v_2 + a_2 \bar{\psi}^{T}_{n,\tau},$$

This being understood, we define the function $w$ on $S_{n,\tau}$ by

$$w = \chi_{n,\tau} v_1 + a_1 \Psi^{D}_{n,\tau} + b_1 \Psi^{T,\bar{a}}_{n,\tau} + c_1 \Psi^{T,\bar{a}}_{n,\tau} + \bar{\chi}_{\tau} v_2 + a_2 \bar{\Psi}^{T}_{n,\tau}.$$

Observe that

$$\|w\|_{C^{2,\alpha}_\mu(S_{n,\tau}) \oplus \mathcal{K}_{S_{n,\tau}}} \leq c \|f\|_{C^{0,\alpha}_\mu(S_{n,\tau})}$$

for some constant which does not depend on $n$ nor on $\tau \in I$. We claim that

$$\|\mathcal{L}_{S_{n,\tau}} w - f\|_{C^{0,\alpha}_\mu(S_{n,\tau})} \leq c (e^{2\nu_2 s_{\tau}^2} + e^{\mu n s_{\tau}^2} + e^{2\mu n s_{\tau}^2} + e^{2\mu m s_{\tau}^2}) \|f\|_{C^{0,\alpha}_\mu(S_{n,\tau})},$$

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Since our problem is invariant under the action of $G_k$, it is enough to evaluate this quantity on $Y^0_{n,\tau}((-n \sigma, n \sigma) \times \mathbb{S}^1)$ and on $Z^0_{n,\tau}((-m \sigma, m \sigma) \times \mathbb{S}^1)$. We focus our attention on the estimate of $\mathcal{L}_{S_{n,\tau}} w - f$ on $Y^0_{n,\tau}((-n \sigma, 0) \times \mathbb{S}^1)$, since the estimates on the other parts can be obtained similarly.

In $Y^0_{n,\tau}((-n \sigma + 2, -1) \times \mathbb{S}^1)$, all the elements of $\mathcal{K}_{S_{n,\tau}}$ are pieces of Jacobi fields in the sense that, for all $W \in \mathcal{K}_{S_{n,\tau}}$

$$\mathcal{L}_{S_{n,\tau}} W = 0$$

in this set. Therefore,

$$\mathcal{L}_{S_{n,\tau}} w - f = \mathcal{L}_{S_{n,\tau}} (v_1 + v_2) - f = \mathcal{L}_{S_{n,\tau}} v_1$$

in this set since.

We now use the fact that $Y^0_{n,\tau}((-n \sigma + 2, -1) \times \mathbb{S}^1)$ can be considered as a normal graph over $E^0_{n,\tau}$ for some function bounded and whose derivatives are bounded by a constant times $e^{-\gamma_{\sigma,2}(s+n \sigma)}$ in $(-n \sigma, 0) \times \mathbb{S}^1$. In particular, this implies that

$$\mathcal{L}_{S_{n,\tau}} - \mathcal{L}_{\Sigma_{n,\tau}}$$

is a second order partial differential operator whose coefficients as well as their derivatives are bounded by a constant times $e^{-\gamma_{\sigma,2}(s+n \sigma)}$ in $(-n \sigma, 0) \times \mathbb{S}^1$. Since $\mathcal{L}_{\Sigma_{n,\tau}} v_1 = 0$ in this set, we conclude that

$$\|e^{\mu_s} (\mathcal{L}_{S_{n,\tau}} w - f)\|_{C^0(\alpha)(Y^0_{n,\tau},((-n \sigma+1,-1)\times\mathbb{S}^1))} \leq c(e^{2n \mu_s} + e^{\gamma_{\sigma,2} n \sigma}) \|f\|_{C^0(\alpha)(S_{n,\tau})}$$

In $Y^0_{n,\tau}((-n \sigma + 1, -n \sigma + 2) \times \mathbb{S}^1)$, we obtain, using similar arguments and taking into account the influence of the cutoff function $\chi^\epsilon_{n,\tau}$

$$\|e^{\mu_s} (\mathcal{L}_{S_{n,\tau}} w - f)\|_{C^0(\alpha)(Y^0_{n,\tau},((-n \sigma+1,-n \sigma+2)\times\mathbb{S}^1))} \leq c e^{2n \mu_s} \|f\|_{C^0(\alpha)(S_{n,\tau})}$$

and in $Y^0_{n,\tau}((-1,0) \times \mathbb{S}^1)$, we obtain, using similar arguments

$$\|e^{\mu_s} (\mathcal{L}_{S_{n,\tau}} w - f)\|_{C^0(\alpha)(Y^0_{n,\tau},(-1,0)\times\mathbb{S}^1))} \leq c e^{-\gamma_{\sigma,2} n \sigma} \|f\|_{C^0(\alpha)(S_{n,\tau})}$$

So far, we have produced a linear operator

$$\tilde{G}_{n,\tau} : \mathcal{C}^0(\alpha)(S_{n,\tau}) \rightarrow \mathcal{C}^2(\alpha)(S_{n,\tau}) \oplus \mathcal{K}_{S_{n,\tau}},$$

defined by $\tilde{G}_{n,\tau}(f) := w$, which is uniformly bounded (with respect to $n \in \mathbb{N}$ and $\tau \in I$) and which satisfies

$$\|\mathcal{L}_{S_{n,\tau}} \circ \tilde{G}_{n,\tau} - I\| \leq c(e^{-\gamma_{\sigma,2} n \sigma} + e^{2 \mu_n \sigma} + e^{-\gamma_{\sigma,2} m \sigma} + e^{2 \mu m \sigma}).$$

for some constant independent of $n \in \mathbb{N}$ and $\tau \in I$. The result then follows from a simple perturbation argument, provided $n$ is chosen large enough. \[\square\]
6.2 The nonlinear argument

We define the functions

\[ \Lambda(\tau) := \frac{1}{T_\tau} \left( \sin(\pi/k) \left( d_0^0 \tau + d_0^1 \right) - d_1^1 \right) \]

and

\[ \Gamma(\tau) := 2 \sin(\pi/k) \frac{T_\tau}{T_\bar{\tau}} \]

Recall that \( \tau \) and \( \bar{\tau} \) are related through (14). We now prove the main result of the paper:

**Theorem 1** There exists \( n_0 > 0 \) such that, for all \( n \geq n_0 \) and all \( \tau \in I \) satisfying

\[ \Lambda(\tau) + n \Gamma(\tau) \in \mathbb{N} \]

the surface \( \Sigma_{n,\tau} \) can be perturbed into a constant mean curvature 1 surface.

**Proof**: We consider surfaces which can be written as a normal graph over \( S_{n,\tau} \), for some function \( w \in C^2_{\mu,\alpha}(S_{n,\tau}) \oplus K_{S_{n,\tau}} \). The equation which guarantees that this surface has constant mean curvature equal to 1 can be written as

\[ L_{S_{n,\tau}} w + Q_{n,\tau}(w) = 1 - H_{S_{n,\tau}}, \]

where \( L_{S_{n,\tau}} \) is the Jacobi operator about \( S_{n,\tau} \), \( H_{S_{n,\tau}} \) is the mean curvature of \( S_{n,\tau} \) and \( Q_{n,\tau} \) collects all the nonlinear terms. It should be clear from the construction of \( S_{n,\tau} \) that, given \( r \in \mathbb{N} \) there exists \( c_r > 0 \) (independent of \( \tau \in I \) and of \( n \in \mathbb{N} \)) such that the following pointwise bound holds

\[ |Q_{n,\tau}(w_2) - Q_{n,\tau}(w_1)|c^r \leq c_r \left( |w_2|c^{r+2} + |w_1|c^{r+2} \right) |w_2 - w_1|c^{r+2} \]

provided \( |w_1|c^1 + |w_2|c^1 \leq 1 \), where

\[ |w|c^r = \sum_{j=0}^r |\nabla^j w| \]

and partial derivatives are computed using the vector fields \( \partial_s \) and \( \partial_\theta \) along the pieces of \( S_{n,\tau} \) parameterized by \( Y_{n,\tau}^s \) and \( Z_{n,\tau}^s \) and using a fixed set of vector fields (independent of \( n \)) away from these pieces.

We fix \( \mu \in (-\inf(\gamma_{\tau,2},\gamma_{\bar{\tau},2}), 0) \). Using the result of Proposition 3, our problem reduces to finding a fixed point for:

\[ F_{n,\tau} : w \rightarrow G_{n,\tau} \left( 1 - H_{S_{n,\tau}} - Q_{n,\tau}(w) \right). \]

which belongs to \( C^2_{\mu,\alpha}(S_{n,\tau}) \oplus K_{S_{n,\tau}} \). It follows from the result of Lemma 2 that

\[ \|1 - H_{S_{n,\tau}}\|_{C^0_{\mu,\alpha}(S_{n,\tau})} \leq c \left( e^{-(\gamma_{\tau,2}+\mu)n_s \tau} + e^{-(\gamma_{\tau,2}+\mu)m_s \bar{\tau}} \right). \]

We set

\[ \rho_{n,\tau} := \left( e^{-(\gamma_{\tau,2}+\mu)n_s \tau} + e^{-(\gamma_{\tau,2}+\mu)m_s \bar{\tau}} \right). \]
Applying the result of Proposition 3, we conclude that

$$||G_{n,\tau}(1-H_{S_n,\tau})||_{C^{2,\alpha}(S_{n,\tau})\oplus K_{S_{n,\tau}}} \leq \bar{c}\rho_{n,\tau}. \quad (21)$$

for some constant $\bar{c} > 0$ which does not depend on $\tau \in I$ nor on $n \in \mathbb{N}$, for which (17) holds.

Now, it follows from (19) that there exists a constant $c > 0$ which does not depend on $\tau \in I$ nor on $n \in \mathbb{N}$ such that

$$||Q_{n,\tau}(w_2) - Q_{n,\tau}(w_1)||_{C^{2,\alpha}(S_{n,\tau})} \leq c\left(n^2 + e^{-\mu n s \tau} + e^{-\mu m s \bar{\tau}}\right)\rho_{n,\tau}||w_2 - w_1||_{C^{2,\alpha}(S_{n,\tau})\oplus K_{S_{n,\tau}}}, \quad (22)$$

provided $||w_2||_{C^{2,\alpha}(S_{n,\tau})\oplus K_{S_{n,\tau}}} + ||w_1||_{C^{2,\alpha}(S_{n,\tau})\oplus K_{S_{n,\tau}}} \leq 2\bar{c}\rho_{n,\tau}$. The $n^2$ which appears in this estimate arises from the fact that the element $\Psi^D_{n,\tau}$ of $K_{S_{n,\tau}}$ is not bounded uniformly in $n$, but is bounded, as well as its derivatives, by a constant (independent of $\tau$ and $n$) times $n$.

We choose $\mu$ close enough to 0 (but still negative !) so that

$$\lim_{n \to +\infty} (e^{-\mu n s \tau} + e^{-\mu m s \bar{\tau}})\rho_{n,\tau} = 0$$

uniformly for $\tau \in I$ (Recall that $n$ and $m$ are related by (17), in particular there exists $c > 0$, independent of $\tau \in I$, such that $n \leq cm$ and $m \leq cn$). The fact that, provided $n$ is chosen large enough, the mapping $F_{n,\tau}$ has a fixed point in the ball of radius $2\bar{c}\rho_{n,\tau}$ in $C^{2,\alpha}(S_{n,\tau})\oplus K_{S_{n,\tau}}$ follows directly from (21) and (22).

The surfaces we have obtained are immersed, compact surfaces with genus $k$ (these surfaces are not embedded since the Type-1 elements which have been used for their construction are never embedded). The surfaces obtained for different values of $\tau$ and $n$ satisfying (17) are geometrically different (i.e. are not congruent modulo a rigid motion), provided $n_0$ is chosen large enough. Hence, the set solutions of (17) give a local picture of the set of compact constant mean curvature surfaces of genus $k$ with symmetry group $G_k$.

Finally, observe that the result of Lemma 1 together with the fact that $\tau$ and $\bar{\tau}$ are related by (14) implies that

$$\partial_\tau \left(\frac{T_\tau}{T_{\bar{\tau}}}\right) > 0$$

Therefore (17) has nontrivial solutions $\tau \in I$, for any $n$ large enough.

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