Minimal Gromov–Witten rings

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Abstract. We construct an abstract theory of Gromov–Witten invariants of genus 0 for quantum minimal Fano varieties (a minimal class of varieties which is natural from the quantum cohomological viewpoint). Namely, we consider the minimal Gromov–Witten ring: a commutative algebra whose generators and relations are of the form used in the Gromov–Witten theory of Fano varieties (of unspecified dimension). The Gromov–Witten theory of any quantum minimal variety is a homomorphism from this ring to \( \mathbb{C} \).

We prove an abstract reconstruction theorem which says that this ring is isomorphic to the free commutative ring generated by ‘prime two-pointed invariants’. We also find solutions of the differential equation of type \( DN \) for a Fano variety of dimension \( N \) in terms of the generating series of one-pointed Gromov–Witten invariants.

§ 1. Introduction

Consider a smooth Fano variety \( V \) of dimension \( N \). Let \( H^*_H(V, \mathbb{Q}) \subset H^*(V, \mathbb{Q}) \) be the subspace multiplicatively generated by the anticanonical class \( H \in H^2(V, \mathbb{Q}) \). It is tautologically closed under multiplication in the cohomology ring. Gromov–Witten theory (or rather the set of three-pointed Gromov–Witten invariants of genus 0) enables one to ‘deform’ the cohomology ring, that is, to define a quantum multiplication on the graded space \( QH^*(V) = H^*(V, \mathbb{Q}) \otimes \mathbb{C}[q] \). This multiplication becomes the usual multiplication in \( H^*(V, \mathbb{Q}) \) if we put \( q = 0 \). The subspace \( QH^*_H(V) = H^*_H(V) \otimes \mathbb{C}[q] \) is not generally closed under quantum multiplication.

If it is closed, then the variety is said to be \textit{quantum minimal}. Examples of quantum minimal varieties are three-dimensional Fano varieties with Picard group \( \mathbb{Z} \) or complete intersections in projective spaces.

Let \( V \) be quantum minimal. Then its quantum \( \mathcal{D} \)-module contains a natural submodule corresponding to the subring \( QH^*_H(V) \). The connection in this \( \mathcal{D} \)-module is given by the matrix of quantum multiplication by the anticanonical class (in other words, by the prime two-pointed Gromov–Witten invariants of genus 0). By regularizing the module, we get the determinantal operator (or operator of type \( DN \)) for \( V \). Such operators are studied in detail in [1]. One of the conjectures of mirror symmetry states that this operator coincides with the Picard–Fuchs operator for the Ginzburg–Landau model dual to \( V \) (see, for example, [2]). This is known, for example, for threefolds (see [3] and [4]).

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We shall prove the following theorem. Let $\langle \tau_i H^j \rangle_d$ be the one-pointed Gromov–Witten invariant for a curve of anticanonical degree $d$ and consider the regularized $I$-series (the generating series for one-pointed Gromov–Witten invariants)

$$\tilde{I}^V = 1 + \sum_{0 \leq j < N, d > 0} \langle \tau_{d+j-2} H^{N-j} \rangle_d q^d h^j / H^N (h+1) \cdots (h+d) \in \mathbb{C}[[q]][h]/h^N.$$ 

**Theorem 1.1.** Write $\tilde{I}^V = \sum_{0 \leq k < N} \tilde{I}^k h^k$, where $\tilde{I}^i$ is a series in $q$ for every $i$. Then the $N$ functions

$$\tilde{I}^0, \quad \tilde{I}^0 \log(q) + \tilde{I}^1, \quad \tilde{I}^0 \log(q)^2 / 2! + \tilde{I}^1 \log(q) + \tilde{I}^2, \quad \ldots$$

form a basis of the kernel of the operator of type $DN$ for $V$.

The idea of expressing the solutions of equations of type $DN$ in terms of Gromov–Witten invariants goes back to Witten, Dijkgraaf and Dubrovin. The proof of Theorem 1.1 is based on the relations between Gromov–Witten invariants of genus 0. Thus the theorem holds for all sets of numbers that satisfy these relations. This enables us to generalize the Gromov–Witten theories of quantum minimal varieties and their determinantal operators and obtain a formal theory where analogous relations hold. These generalizations, being independent of any particular varieties, depend only on their ‘dimension’ $N$. However, Poincaré duality enables one to generalize the invariants further and make them independent of the dimension. Such a universal theory is studied in this paper. In this theory, numerical invariants become universal polynomials that are unique for all quantum minimal Fano varieties of any dimension.

We fix $N \in \mathbb{N}$ and put $A_N = \mathbb{C}[a_{ij}], \ 0 \leq i, j \leq N, \ D = \mathbb{C}[q, \frac{d}{dq}]$ and $D = q \frac{d}{dq}$.

Consider the matrix

$$M = \begin{pmatrix}
(a_{00}(Dq) & a_{01}(Dq)^2 & \cdots & a_{0,N-1}(Dq)^N & a_{0N}(Dq)^{N+1}) \\
1 & a_{11}(Dq) & \cdots & a_{1,N-1}(Dq)^N & a_{1N}(Dq)^N \\
& \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & 1 & a_{NN}(Dq)
\end{pmatrix}$$

with entries in $A_N \otimes D$. Define an operator\(^1\) $L_N \in A_N \otimes D$ by putting

$$\det_{\text{right}}(D - M) = DL_N.$$

(The algebra $A_N \otimes D$ is non-commutative, and the determinant is taken by expanding with respect to the last column.)

**Example 1.2.** Regarding the operators $L_N, \ N \in \mathbb{N}$, as (non-commutative) polynomials in $q$ and $D$, it is easy to check that the degree of such an operator in $q$ and $D$ is independent of its polynomial representation. As a polynomial, $L_2$ (resp. $L_3$) is

\(^1\)The operators corresponding to matrices that are symmetric with respect to the anti-diagonal (that is, matrices with $a_{ij} = a_{N-j,N-i}$) are called operators of type $DN$. 
of degree 2 (resp. 3) in $D$ and degree 3 (resp. 4) in $q$. The analytic solutions of the equations $L_2\Phi = 0$ and $L_3\Phi = 0$ in terms of the variables $a_{ij}$ are as follows:

$$1 + a_{00}q + \left(\frac{1}{2}a_{01} + a_{00}^2\right)q^2 + \left(\frac{7}{6}a_{01}a_{00} + \frac{1}{3}a_{01}a_{11} + \frac{2}{9}a_{02}\right)q^3$$

$$+ \left(\frac{23}{12}a_{01}a_{00}^2 + \frac{1}{4}a_{01}a_{11} + \frac{3}{8}a_{01} + \frac{1}{8}a_{22}a_{02} + \frac{5}{6}a_{11}a_{00}a_{01} + \frac{3}{16}a_{01}a_{12}\right)q^4 + O(q^5),$$

$$1 + a_{00}q + \left(\frac{1}{2}a_{01} + a_{00}^2\right)q^2 + \left(\frac{7}{6}a_{01}a_{00} + a_{00}^3 + \frac{1}{3}a_{01}a_{11} + \frac{2}{9}a_{02}\right)q^3$$

$$+ \left(\frac{23}{12}a_{01}a_{00} + \frac{1}{4}a_{01}a_{11} + \frac{3}{8}a_{01} + \frac{1}{8}a_{22}a_{02} + \frac{5}{6}a_{11}a_{00}a_{01} + \frac{3}{16}a_{01}a_{12}\right)q^4 + O(q^5).$$

The first few terms of these solutions coincide. Moreover, the solutions become ‘the same’ if the variables $a_{ij}$ vanish whenever they ‘do not occur in $L_2$’. In other words, if we put $a_{i3} = 0$ in the solution of $L_3\Phi = 0$, then we get the solution of $L_2\Phi = 0$.

The same turns out to be true in the general case. We fix positive integers $N_1 < N_2$. Let $\Phi_1$ and $\Phi_2$ be analytic solutions of $L_{N_1}\Phi = 0$ and $L_{N_2}\Phi = 0$, normalized by putting $\Phi_i(0) = 1$. We put $a_{ij} = 0$ for $N_1 < i, j \leq N_2$ in the series $\Phi_2$ and denote the resulting series by $\Phi'_2$. Then $\Phi_1 = \Phi'_2$. Moreover, the following result is easily proved for every $n$ (see Lemma 2.3 and Proposition 3.4 below): if $N_1 \gg n$, then

$$\Phi_1 \equiv \Phi_2 (\text{mod } q^n).$$

A similar assertion holds for the logarithmic solution of $L_N\Phi = 0$.

Thus the solutions of such equations are obtained from each other by ‘restriction’, and the expansions of solutions of different equations have the same initial terms. This enables us to define a ‘universal series’ whose ‘restrictions’ to $A_N$ are solutions of $L_N\Phi = 0$. This series is the ‘generating series for abstract one-pointed Gromov–Witten invariants’ in the following sense.

Below we define the minimal Gromov–Witten ring $GW$ as a commutative algebra whose generators and relations are of the form used in Gromov–Witten theory. This definition is similar to Dubrovin’s definition of formal Frobenius manifolds and to the treatment of Gromov–Witten invariants by Kontsevich and Manin. The difference is that we do not fix the dimension and consider ‘abstract Gromov–Witten invariants for Fano varieties of unspecified dimension’. This is done using ‘invariants with one class replaced by its Poincaré dual’ and restating the Kontsevich–Manin axioms in terms of such ‘invariants’.

**Definition 1.3.** Consider formal symbols of the form

$$\langle \tau_{d_1}H^{i_1}, \ldots, \tau_{d_{n-1}}H^{i_{n-1}}, \tau_{d_n}H_r \rangle,$$
where \( n \geq 1, \ i_1, \ldots, i_{n-1}, \ r, d_1, \ldots, d_n \in \mathbb{Z}_{\geq 0} \) (the last term is indexed by a subscript). For the sake of simplicity, we write \( H^1, H_j \) instead of \( \tau_0 H^1, \tau_0 H_j \). We define the degree of such a symbol as the number \( \sum d_s + \sum i_s - r + (3 - n) \). Let \( F \) be the set of symbols of non-negative degree. The \textit{minimal Gromov–Witten ring} is the graded ring

\[
GW = \mathbb{C}[F]/\text{Rel},
\]

where Rel is the ideal generated by the following relations.

GW1 (the \( S_n \)-covariance axiom; compare [2], § 2.2.1). Consider any permutation \( \sigma \in S_{n-1} \). Put \( j_k = i_{\sigma(k)} \) and \( f_k = d_{\sigma(k)} \). Then

\[
\langle \tau_{d_1} H^{i_1}, \ldots, \tau_{d_{n-1}} H^{i_{n-1}}, \tau_{d_n} H_r \rangle = \langle \tau_{f_1} H^{j_1}, \ldots, \tau_{f_{n-1}} H^{j_{n-1}}, \tau_{f_n} H_r \rangle.
\]

GW2 (normalization; compare [5], Proposition 1.4.1). Let \( r = \sum d_s + \sum i_s + (3 - n) \). Then

\[
\langle \tau_{d_1} H^{i_1}, \ldots, \tau_{d_{n-1}} H^{i_{n-1}}, \tau_{d_n} H_r \rangle = \frac{(d_1 + \ldots + d_n)!}{d_1! \cdots d_n!} \cdot M,
\]

where \( M = 1 \) if \( \sum d_j = n - 3 \), and \( M = 0 \) otherwise.

GW3 (the fundamental class axiom, or string equation; compare [6], Ch. VI, Proposition 5.1). We have

\[
\langle H^0, \tau_{d_1} H^{i_1}, \ldots, \tau_{d_{n-1}} H^{i_{n-1}}, \tau_{d_n} H_r \rangle = \sum_{j=1}^{n} \langle \tau_{d_1} H^{i_1}, \ldots, \tau_{d_{j-1}} H^{i_{j-1}}, \tau_{d_j-1} H_{i_j}, \tau_{d_{j+1}} H^{i_{j+1}}, \ldots, \tau_{d_n} H_r \rangle
\]

except in the case of \( \langle H^0, H^1, H_i \rangle \), which is given by GW2.

GW4 (the divisor axiom; compare [6], Ch. VI, Proposition 5.4). We have

\[
\langle H^1, \tau_{d_1} H^{i_1}, \ldots, \tau_{d_{n-1}} H^{i_{n-1}}, \tau_{d_n} H_r \rangle = d \langle \tau_{d_1} H^{i_1}, \ldots, \tau_{d_n} H_r \rangle + \sum_{s=1}^{n-1} \langle \tau_{d_1} H^{i_1}, \ldots, \tau_{d_s-1} H^{i_{s+1}}, \ldots, \tau_{d_n} H_r \rangle + \langle \tau_{d_1} H^{i_1}, \ldots, \tau_{d_{n-1}} H_{r-1} \rangle,
\]

where \( d > 0 \) is the degree of the left-hand side.

GW5 (topological recursion; compare [7], formula (6)). Given any numbers \( c_1, \ldots, c_n, i_1, \ldots, i_n \) and any set \( S \subset \{1, \ldots, n\} \), we denote the sequence \( \tau_{c_1} H^{i_{s_1}}, \ldots, \tau_{c_k} H^{i_{s_k}} \) (where \( s_1, \ldots, s_k \) are distinct elements of \( S \)) by \( \prod_S \). Then the following equations hold for any \( n \geq 0 \):

\[
\langle \prod_{\{1, \ldots, n\}} \tau_{d_1} H^{j_1}, \tau_{d_2} H^{j_2}, \tau_{d_3} H_r \rangle = \sum \langle \tau_{d_1-1} H^{j_1}, \prod_{S_1} H_a \rangle \langle H^a, \prod_{S_2} \tau_{d_2} H^{j_2}, \tau_{d_3} H_r \rangle,
\]

\[
\langle \prod_{\{1, \ldots, n\}} \tau_{d_1} H^{j_1}, \tau_{d_2} H^{j_2}, \tau_{d_3} H_r \rangle = \sum \langle \prod_{S_1} \tau_{d_1} H^{j_1}, \tau_{d_2} H^{j_2}, H_a \rangle \langle H^a, \prod_{S_2} \tau_{d_3-1} H_r \rangle,
\]
where the sums are taken over all partitions $S_1 \sqcup S_2 = \{1, \ldots, n\}$ and all $a \in \mathbb{Z}_{\geq 0}$ such that the degrees of the symbols occurring are non-negative. (Note that this sum is finite.)

It turns out that the ring $GW$ has a convenient multiplicative basis. We consider the ring $A = \mathbb{C}[a_{ij}]$, $0 \leq i \leq j$, $j > 0$ and define a map $r: A \rightarrow GW$ by the formula $a_{ij} \mapsto \langle H^1, H^j, H_i \rangle$.

**Theorem 1.4** (the abstract reconstruction theorem). The map $r$ is an isomorphism.

This is an abstract version of the First Reconstruction Theorem of Kontsevich and Manin (see [2], Theorem 3.1).

Suppose that $a_{00} = 0$ (the 'geometric case'; see Remark 4.4 for the general case). Using the map $r$, one can regard the coefficients of the equations $L_N \Phi = 0$ and of their solutions as elements of $GW$.

Consider the series

$$
\tilde{I} = 1 + \sum_{j \geq 0, i > j - 2} \langle \tau_i H_j \rangle q^{i-j+2} h^i (h+1) \cdots (h+i-j+2) \in A \otimes \mathbb{C}[[q]][[h]].
$$

It may be rewritten in terms of $a_{ij}$ as

$$
\tilde{I} = 1 + (a_{11} h + (a_{22} - a_{11}) h^2)q + \left( \frac{a_{01}}{2} + \frac{a_{21} h}{4} + \frac{a_{12} h^2}{4} \right) h
$$

$$
+ \left( - \frac{a_{01}}{8} + \frac{a_{23}}{8} + \frac{a_{11} a_{22}}{2} - \frac{a_{11}^2 h}{4} + \frac{a_{22}^2 h^2}{4} \right) q^2 + O(q^3, h^3).
$$

This universal generating series of one-pointed Gromov–Witten invariants determines the solutions of $L_N \Phi = 0$ for all $N$. Namely, the following theorem is a direct corollary of Theorem 1.4 and Theorem 1.1. For any $\mathbb{C}$-algebra $R$ we define a map $r_N: A \otimes R \rightarrow A_N \otimes R$ by putting $a_{ij} \mapsto a_{ij}$ if $0 \leq i, j \leq N$, and $a_{ij} \mapsto 0$ otherwise. We define $\tilde{I}^s \in A \otimes \mathbb{C}[[q]]$ and $S_i$ by putting $\tilde{I} = \sum \tilde{I}^s h^s$, $S_0 = \tilde{I}^0$, $S_1 = \tilde{I}^0 \log(q) + \tilde{I}^1$, $S_2 = \tilde{I}^0 \log(q)^2/2! + \tilde{I}^1 \log(q) + \tilde{I}^2$, and so on.

**Theorem 1.5.** The set $\{r_N(S_0), \ldots, r_N(S_{N-1})\}$ is a basis in the space of solutions of the differential equation $L_N \Phi = 0$.

Theorem 1.1 is a special case of Theorem 4.2 but their proofs are almost identical. Therefore we devote the first part of the paper (§§ 2, 3) to a proof (without invoking abstract Gromov–Witten theory) of Theorem 1.1, which can be applied directly to the study of the geometry of Fano varieties. In the second part (§ 4) we prove Theorem 1.4. Then we use it formally to deduce Theorem 4.2 from Theorem 1.1.

In § 2 we consider a quantum minimal Fano variety $V$ of dimension $N$. The two-pointed Gromov–Witten invariants determine the quantum connection and the corresponding differential operator. Elements of the kernel of this quantum differential operator are given by the $I$-series of $V$ (that is, the generating series of one-pointed Gromov–Witten invariants of $V$). In § 3 we study the regularization of quantum differential operators, which leads to the operator of type $DN$. 
The method of Frobenius yields explicit expressions for the solutions of equations of type $DN$ (associated with $V$) in terms of the $I$-series of $V$. All the proofs in §§2, 3 are based on the fundamental class axiom, the divisor axiom and the topological recursion relations for $V$. In §4 we prove the abstract reconstruction theorem and abstract versions of the theorems in §§2, 3 using the same arguments as in the abstract setup. In §5 we find explicit recursion relations for all solutions of the differential equations under consideration.

§2. Quantum operators

2.1. Non-commutative determinants. Let $R$ be an associative $C$-algebra (not necessarily commutative) and consider matrices with entries in $R$. The elements of a matrix $M$ of order $N + 1$ are labelled by indices ranging from 0 to $N$. The submatrix of order $i$ in the north-west corner of $M$ is called the $i$th leading principal submatrix.

Definition 2.1 ([1], Definition 1.3). A matrix $M$ with entries in $R$ is said to be almost triangular if $M_{ij} = 0$ for $i + 1 > j$ and $M_{i+1,i} = -1$.

Definition 2.2 ([1], Definition 1.2). Let $M$ be a matrix with entries in $R$. The right determinant of $M$ is defined by expanding with respect to the last column:

$$\det_{\text{right}}(M) = \sum_{i=0}^{N} M_{iN}C_{iN},$$

where $C_{iN}$ are the cofactors taken as right determinants.

For every $(N + 1) \times (N + 1)$ matrix $M = (M_{ij})_{0 \leq i,j \leq N}$ we define a matrix $M^\tau$ by putting $M^\tau_{ij} = M_{N-j,N-i}$ (that is, $M^\tau$ is the ‘transpose of $M$ with respect to the anti-diagonal’).

Lemma 2.3 ([1], §1.4). Let $M$ be an almost triangular $(N + 1) \times (N + 1)$ matrix. We put

$$P_0 = 1, \quad P_{i+1} = \sum_{j=0}^{i} M_{ji}P_j.$$

Then $P_i$ is the right determinant of the $i$th leading principal submatrix of $M$. In particular, $P_{N+1} = \det_{\text{right}}(M)$.

Proof. The proof is by induction on the order of the submatrices. The case $i = 1$ is trivial. We denote the $(i + 1)$th leading principal submatrix of $M$ by $M_{i+1}$. Let $M^j_{i+1}$ be the matrix obtained from $M_{i+1}$ by deleting the last column and the $j$th row. Then the right determinant of $M^j_{i+1}$ is given by

$$\det_{\text{right}} M^j_{i+1} = (-1)^{i-j}P_j.$$

Thus we have

$$\det_{\text{right}} M_{i+1} = \sum_{j=0}^{i} (-1)^{j+i}M_{ji} \det_{\text{right}} M^j_{i+1} = \sum_{j=0}^{i} M_{ji}P_j = P_{i+1}.$$
Lemma 2.4 (compare [1], Proposition 1.7). Let $M$ be an almost triangular matrix and let $\xi = (\xi_0, \ldots, \xi_N)^T$. If $M\xi = 0$, then
\[
\det_{\text{right}}(M^\tau)\xi_N = 0.
\]

Proof. We have the following system of equations in the coordinates $\{\xi_i\}$:
\[
\begin{align*}
M_{00}\xi_0 + \cdots + M_{0N}\xi_N &= 0, \\
-\xi_0 + M_{11}\xi_1 + \cdots + M_{1N}\xi_N &= 0, \\
\cdots \\
-\xi_{i-1} + M_{ii}\xi_i + \cdots + M_{iN}\xi_N &= 0, \\
\cdots \\
-\xi_{N-1} + M_{NN}\xi_N &= 0.
\end{align*}
\]

Let $P_i$ be given by Lemma 2.3 applied to $M^\tau$. We solve the system by moving step-by-step in the reverse direction. This yields $P_i\xi_N = \xi_{N-i}$. Thus,
\[
\det_{\text{right}}(M^\tau)\xi_N = \left(\sum_{i=0}^{N} M_{iN}^T P_i\right)\xi_N = \sum_{i=0}^{N} M_{0i}\xi_i = 0.
\]

2.2. Quantum operators. Let $V$ be a smooth Fano variety of dimension $N$ with $\text{Pic}(V) \cong \mathbb{Z}$ and put $H = -K_V$ and $H^*(V) = H^*(V, \mathbb{Q})$. (Since the natural map $\text{Pic}(V) \to H^2(V, \mathbb{Z})$ is an isomorphism for smooth Fano varieties, we use the same notation for an element of $\text{Pic}(V) \otimes \mathbb{Q}$ and its class in $H^2(V)$.)

Let $H_H^*(V) \subset H^*(V)$ be the divisorial subspace, that is, the subspace generated by the powers of $H$.

Suppose that $\gamma_1, \ldots, \gamma_n \in H^*(V)$ and let $d$ be the anticanonical degree of an effective algebraic curve $\beta \in H_2(V)$. The corresponding Gromov–Witten invariant (of genus 0) with descendants of degrees $d_1, \ldots, d_n \in \mathbb{Z}_{\geq 0}$ (see [6], Ch. VI, §2.1) will be denoted by $\langle \tau_{d_1}\gamma_1, \ldots, \tau_{d_n}\gamma_n \rangle_d$.

The subspace $H_H^*(V) \subset H^*(V)$ is tautologically closed under multiplication: the product $\gamma_1 \cdot \gamma_2$ of any classes $\gamma_1, \gamma_2 \in H_H^*(V)$ also lies in $H_H^*(V)$. One can deform the multiplicative structure on the cohomology ring by considering the quantum cohomology ring $QH^*(V) = H^*(V) \otimes \mathbb{C}[q]$ (see [6], Definition 0.0.2) with the quantum multiplication $\star : QH^*(V) \times QH^*(V) \to QH^*(V)$, that is, the bilinear map given by
\[
\gamma_1 \star \gamma_2 = \sum_{\gamma} q^d \langle \gamma_1, \gamma_2, \gamma^\vee \rangle_d \gamma
\]
for all $\gamma_1, \gamma_2, \gamma \in H^*(V)$, where the class $\gamma^\vee$ is the Poincaré dual of $\gamma$ (we identify the elements $\gamma \in H^*(V)$ and $\gamma \otimes 1 \in QH^*(V)$). The constant term of $\gamma_1 \star \gamma_2$ (with respect to $q$) is $\gamma_1 \cdot \gamma_2$. The subspace $QH_H^*(V) = H_H^*(V) \otimes \mathbb{C}[q]$ is not generally closed under $\star$. Examples of varieties $V$ with non-closed subspaces $QH_H^*(V)$ are given by the Grassmannians $G(k, n)$, $k, n-k > 1$, of dimension $> 4$ (for instance, $G(2, 5)$) and their hyperplane sections of dimension at least 4.
Definition 2.5. A variety \( V \) is said to be quantum minimal if \( QH^*_H(V) \) is quantum closed, that is, if the Gromov–Witten invariant \( \langle \gamma_1, \gamma_2, \mu \rangle_d \) vanishes\(^2\) for all elements \( \gamma_1, \gamma_2 \in H^*_H(V), \mu \in H^*_H(V) \).\(^1\)

In other words, \( V \) is quantum minimal if and only if \( QH^*_H(V) \) is a subring of \( QH^*(V) \).

We assume throughout that \( V \) is quantum minimal.

Consider the ring \( B = \mathbb{C}[q, q^{-1}] \) and the basis \( \{H^i\}, i = 0, \ldots, N, \) of \( H^*_H(V) \), where \( H^0 \) is the identity of the ring and \( H = H^1 \). Let \( H_i \) be the Poincaré dual of \( H^i \) and \( HQ \) the (trivial) vector bundle over \( \text{Spec}(B) \) with fibre \( H^*_H(V) \). Put \( h^i = H^i \otimes 1 \in H^*_H(V) \otimes B \), and also \( h = h^1, k_V = K_V \otimes 1 = -h \) and \( S = H^0(HQ) \). Since \( S \cong H^*_H(V) \otimes B \), we can regard quantum multiplication as a map \( *: S \times S \to S \).

Let \( D = \frac{d}{dq} \in \mathcal{D} = \mathbb{C}[q, q^{-1}, \frac{d}{dq}] \). Consider a (flat) connection \( \nabla \) on \( HQ \) whose action on the sections \( h^i \) is defined by

\[
\left( \nabla(h^i), q \frac{d}{dq} \right) = k_V * h^i
\]

(this is the natural pairing between differential forms and vector fields). This connection endows \( S \) with the structure of a \( \mathcal{D} \)-module by the formula \( D(h^i) = (\nabla(h^i), D) \).

Clearly,

\[
D \left( \sum_{i=0}^{N} f_i(q) h^i \right) = \sum_{i=0}^{N} q \frac{\partial f_i(q)}{\partial q} h^i - h * \left( \sum_{i=0}^{N} f_i(q) h^i \right).
\]

Definition 2.6. The \( \mathbb{C} \)-linear operator \( D: S \to S \) is called the quantum operator.

We define an operator \( D_B: S \to S \) by

\[
D_B \left( \sum_{i=0}^{N} f_i(q) h^i \right) = \sum_{i=0}^{N} q \frac{\partial f_i(q)}{\partial q} h^i.
\]

Let \( h * h^j = \sum_j \alpha_{ij} h^j, \alpha_{ij} \in B \). We define a matrix \( M \) by putting\(^3\) \( M_{ij} = -\alpha_{ij} \in \mathcal{D} \) for \( i \neq j \) and \( M_{ii} = D - \alpha_{ii} \in \mathcal{D} \).

Definition 2.7. The differential operator \( L^Q_V = \det_{\text{right}}(M) \in \mathcal{D} \) is called the quantum differential operator for \( V \).

In what follows we study solutions of the equations corresponding to these differential operators. These solutions are ‘formal series with logarithms’ and do not lie in \( S \). Hence we must change the base. We put \( T = \mathbb{C}[[q]][t]/t^{N+1}, \) \( B^{\text{form}} = B \otimes_{\mathbb{C}[q]} T \) and \( S^{\text{form}} = S \otimes_{\mathbb{C}[q]} T \). Let \( \mathcal{D} \) act on \( T \) by \( D^t = 1 \). Thus

\(^2\)A Fano variety is said to be minimal if its cohomology is as simple as it can be (just \( \mathbb{Z} \) at every even dimension and zero at every odd dimension). A quantum minimal variety has ‘quantum anticanonical part’ as small as it can be: this part is similar to the quantum cohomology of a minimal variety. Hence quantum minimal varieties are natural analogues of classical minimal varieties.

\(^3\)We identify any matrix \( A \) with entries in \( \mathcal{D} \) with the operator \( S \to S \) given by \( A(\sum f_i h^i) = \sum_i (\sum_j A_{ij} f_j) h^i \). Then \( M \) is the matrix of the operator \( D = D_B - h * \).
the informal meaning of \( t \) is \( \log(q) \). In what follows we regard \( D, D_B, h\star \) and so on as \( \mathbb{C} \)-operators \( S^{\text{form}} \to S^{\text{form}} \), and \( L^{Q}_V \) as a \( \mathbb{C} \)-operator \( B^{\text{form}} \to B^{\text{form}} \).

2.3. Relations. For simplicity we shall also use (formally undefined) Gromov–Witten invariants for curves of negative degree or invariants with negative descendants. We define these invariants to be equal to zero.

**Theorem 2.8** (topological recursion; see, for example, [6], Ch. VI, Corollary 6.2.1). Suppose that \( \gamma_1, \gamma_2, \gamma_3 \in H^*(V) \), \( a_1 \in \mathbb{Z}_{\geq 0} \), \( a_2, a_3, d \in \mathbb{Z}_{\geq 0} \). Then

\[
\langle \tau_{a_1} \gamma_1, \tau_{a_2} \gamma_2, \tau_{a_3} \gamma_3 \rangle_d = \sum_{d_1 + d_2 = d, a = 0, \ldots, N} \langle \tau_{a_1-1} \gamma_1, H^a \rangle_{d_1} \langle H_a, \tau_{a_2} \gamma_2, \tau_{a_3} \gamma_3 \rangle_{d_2}.
\]

**Theorem 2.9** (the divisor axiom; see, for example, [6], Ch. VI, Proposition 5.4). Suppose that \( \gamma_1, \ldots, \gamma_n \in H^*(V) \), \( \gamma_0 = rH \in H^2(V, \mathbb{Q}) \) is an ample divisor and \( a_1, \ldots, a_m \in \mathbb{Z}_{\geq 0} \). Then

\[
\langle \gamma_0, \tau_{a_1} \gamma_1, \ldots, \tau_{a_m} \gamma_m \rangle_d = r d \langle \tau_{a_1} \gamma_1, \ldots, \tau_{a_m} \gamma_m \rangle_d + \sum_{s=1}^m \langle \tau_{a_1} \gamma_1, \ldots, \tau_{a_s-1} \gamma_0 \cdot H^s, \ldots, \tau_{a_m} \gamma_m \rangle_d.
\]

**Theorem 2.10** (the fundamental class axiom; see, for example, [6], Ch. VI, Proposition 5.1). Suppose that \( \gamma_1, \ldots, \gamma_k \in H^*(V) \), \( a_1, \ldots, a_k \in \mathbb{Z}_{\geq 0} \). Then

\[
\langle \tau_{a_1} \gamma_1, \ldots, \tau_{a_k} \gamma_k, H^0 \rangle_d = \sum_{i=1}^k \langle \tau_{a_1} \gamma_1, \ldots, \tau_{a_i-1} \gamma_{i-1}, \tau_{a_i-1} \gamma_i, \tau_{a_{i+1}} \gamma_{i+1}, \ldots, \tau_{a_k} \gamma_k \rangle_d.
\]

2.4. The fundamental solution. We put

\[
e^{Ht} = \sum_{r=0}^{\infty} \frac{H^r t^r}{r!} \in H^*_H(V) \otimes B^{\text{form}}
\]

(this sum is finite). We also put \( \langle \tau_{d_1} t^{a_1} \gamma_1, \ldots, \tau_{d_s} t^{a_s} \gamma_s \rangle_d = t^{\sum \alpha_i} \langle \tau_{d_1} \gamma_1, \ldots, \tau_{d_s} \gamma_s \rangle_d \).

Consider a matrix \( \Phi \) with entries

\[
\Phi_a^b = \sum_{d \geq 0} q^d \left( \langle \tau_{d+a-b-1} H^b, H_a \rangle_d + \langle \tau_{d+a-b} H^{b+1}, H_a \rangle_d + \langle \tau_{d+a-b+1} \frac{t^2}{2!} H^{b+2}, H_a \rangle_d + \cdots \right) = \sum_{d \geq 0} q^d \langle \tau_a e^{Ht} H^b, H_a \rangle_d,
\]

where \( 0 \leq a, b \leq N \). Here and in what follows, \( \bullet \) stands for the number

\[
N + d - 3 - S,
\]

where \( S \) is the difference between the sum of the codimensions of the cohomology classes occurring in the invariant and the number of these classes.
We denote \(\langle \tau_{d+a-2}H^1, H_a \rangle_d + \langle \tau_{d+a-3}H^2, H_a \rangle_d\) by \(\langle \tau_* (H^1 + H^2), H_a \rangle_d\) and so on. Since two-pointed Gromov–Witten invariants are not defined for curves of degree zero, we put

\[
\langle \tau \ast e^{Ht}H^b, H_a \rangle_0 = \langle H^0 \ast e^{Ht}H^b, H_a \rangle_0.
\]

**Proposition 2.11** ([7], Proposition 2). Let \(\phi^i = \sum_{a=0}^N \Phi_i^a h^a \in S_{\text{form}}\) be the sections corresponding to the columns of \(\Phi\). Then the following assertions hold.

1) The sections \(\phi^i\) are flat, that is, \(D\phi^i = 0\).

2) If \(D\phi = 0\), then \(\phi = \sum_{i=0}^N \alpha_i \phi^i, \alpha_i \in \mathbb{C}\).

In other words, we can say informally that \(\Phi\) is the ‘matrix of fundamental solutions of the equation given by the quantum operator in the standard basis’.

**Proof** (Pandharipande). 1) We must prove that \(D_B \phi^i = h \ast \phi^i\).

Transforming the left-hand side, we obtain

\[
D_B \left( \sum_a \Phi_i^a h^a \right) = \sum_a \sum_{d \geq 0} \left( dq^d \langle \tau \ast e^{Ht}H^i, H_a \rangle_d + q^d \langle \tau_\ast H e^{Ht}H^i, H_a \rangle_d \right) h^a
\]

\[
= \sum_a \sum_{d \geq 0} q^d \langle \tau_\ast e^{Ht}H^i, H, H_a \rangle_d h^a
\]

by the divisor axiom (Theorem 2.9). We also transform the right-hand side:

\[
h \ast \left( \sum_a \Phi_i^a h^a \right) = \sum_s \sum_{d_1, d_2 \geq 0} \sum_a q^{d_1} \langle \tau_\ast e^{Ht}H^i, H_a \rangle_{d_1} q^{d_2} \langle H^a, H, H_s \rangle_{d_2} h^s
\]

\[
= \sum_s \sum_d q^d \langle \tau_\ast e^{Ht}H^i, H, H_s \rangle_d h^s
\]

by the topological recursion (Theorem 2.8). Thus the sections \(\phi^i\) satisfy the desired equation.

2) The constant term of \(\Phi\) (with respect to \(t\) and \(q\)) is the identity matrix. Thus the columns of \(\Phi\) are linearly independent. Since a differential operator of order \(N+1\) has at most an \((N+1)\)-dimensional space of solutions, we see that this space is spanned by the \(N+1\) functions \(\phi^i\).

**Remark 2.12.** Consider the matrix \(M\) (see Definition 2.7). Proposition 2.11 states that \(M \Phi^i = 0\), where \(\Phi^i = (\Phi^i_0, \ldots, \Phi^i_N)^T\) are the columns corresponding to the sections \(\phi^i\).

**Corollary 2.13.** Define a matrix \(\Psi\) by putting

\[
\Psi_i^j = \sum_{d \geq 0} q^d \langle \tau_\ast e^{Ht}H^i, H^{N-j} \rangle_{d}, \quad 0 \leq i, j \leq N,
\]

and let \(\Psi_i = (\Psi_i^0, \ldots, \Psi_i^N)^T\) be its columns. Then \(M^T \Psi_i = 0\) for \(0 \leq i \leq N\).

**Proof.** This is proved in the same way as Proposition 2.11.
2.5. Solutions. Consider any series
\[ I = \sum_{i=0}^{N} I^i(q) h^i \in \mathbb{C}[q][h]/h^N + 1, \quad I^0(q), \ldots, I^N(q) \in \mathbb{C}[q], \]
and put \( I_r = \sum_0^r (I^r - i(q) t_i i!) \in T. \)

Definition 2.14. The series \( I \) is called a perturbed solution of the equation \( PI = 0 \) (or just of the operator \( P \in \mathcal{D} \)) if \( PI_r = 0 \) for all \( r \leq N \).

In other words, given an operator \( P = P(q, D) \), we consider the operator \( P_H = P(q, D_B) \) with \( D \) replaced by \( D_B \). The series \( I \) is a perturbed solution of \( P \) if and only if \( P_H(e^{ht} \cdot I) = 0. \)

We recall that the \( I \)-series of a variety \( V \) is given by
\[ I^V = 1 + \sum_{i,j,d>0} \langle \tau_i H_j \rangle q^d h^j q^d \in S_{\text{form}}. \]

Theorem 2.15. 1) The series \( I^V \) is a perturbed solution of the equation \( L^Q V I = 0. \)
2) If \( L^Q V I = 0 \), then \( I = \sum a_i I^V_i \) for some \( a_0, \ldots, a_N \in \mathbb{C}. \)

Proof. 1) We have
\[ L^Q V I^V = L^Q V \left( \frac{t_i}{t_i^2} + \sum_{d>0} q^d \langle \tau \cdot e^{Ht} H_i \rangle_d \right) = L^Q V \left( \sum_{d>0} q^d \langle \tau \cdot e^{Ht} H_i, H^0 \rangle_d \right) = \text{det}_{\text{right}}(M) \Psi_i^N = \text{det}_{\text{right}}(M^T \tau) \Psi_i^N = 0 \]
by Lemma 2.4 and Corollary 2.13.
2) The solutions \( I^V_s \) are linearly independent (by Proposition 2.11) and hence form a basis of the \((N+1)\)-dimensional space of solutions of the differential equation of order \( N + 1 \) corresponding to the operator \( L^Q V. \)

§ 3. Operators of type \( DN \)

3.1. Regularization. We consider the quantum differential operator
\[ L^Q_V = P_{V,0}(D) + qP_{V,1}(D) + \cdots + q^n P_{V,n}(D) \in \mathcal{D} \]
of a smooth Fano variety of dimension \( N \) (usually \( n = N + 1 \)). Its singularities are generally irregular.

Definition 3.1 ([8], § 1.9). The operator
\[ \tilde{L}_V = P_{V,0}(D) + qP_{V,1}(D)(D+1) + \cdots + q^n P_{V,n}(D)(D+1) \cdots (D+n) \]
is called the regularization of \( L^Q_V. \)

The singular points of all known operators \( \tilde{L}_V \) are regular.\(^4\) Clearly, \( \tilde{L}_V \) is divisible on the left by \( D. \)

\(^4\)They are also regular if the matrix of quantum multiplication by the anticanonical class is diagonalizable (see [1], Remark 3.6).
Definition 3.2 ([8], Definition 2.10). The operator \( L_V \) with \( DL_V = \tilde{L}_V \) is called the (geometric) operator of type \( DN \).

It is conjectured that the solutions of equations associated with geometric operators of type \( DN \) are \( G \)-series.\(^5\)

Consider an arbitrary differential operator
\[
P = P_0(D) + qP_1(D) + \cdots + q^n P_n(D) \in \mathbb{C}[q, q \frac{d}{dq}].
\]

As above, we define its regularization by putting
\[
\tilde{P} = P_0(D) + qP_1(D)(D + 1) + \cdots + q^n P_n(D)(D + 1) \cdots (D + n).
\]

3.2. The Frobenius method. We describe an ‘algebraic’ interpretation of the Frobenius method of solving differential equations. The standard version is in [9], Ch. IV, § 8.

Put \( R = \mathbb{C}[\varepsilon]/\varepsilon^{N+1}, \ N + 1 \in \mathbb{N} \), and consider the differential operator
\[
P_\varepsilon = P_0(D + \varepsilon) + qP_1(D + \varepsilon) + \cdots + q^n P_n(D + \varepsilon) \in \mathcal{D} \otimes R.
\]

Definition 3.3. Consider a sequence \( \{\tilde{c}_i\}, \ i \geq 0, \ \tilde{c}_i \in R \). It is called a Newton solution of the operator \( P_\varepsilon \) if, for every \( m \in \mathbb{Z} \),
\[
\tilde{c}_m P_n(m + \varepsilon) + \tilde{c}_{m+1} P_{n-1}(m + 1 + \varepsilon) + \cdots + \tilde{c}_{m+n} P_0(m + n + \varepsilon) = 0
\]

(we declare \( \tilde{c}_i \) to be zero when \( i \) is negative).

Proposition 3.4. A sequence \( \{\tilde{c}_i\} \) is a Newton solution of \( P_\varepsilon \) if and only if the series
\[
I = \tilde{c}_0 + q\tilde{c}_1 + \cdots \in \mathbb{C}[[q]] \otimes R
\]
is a perturbed solution of \( P \).

Proof. We recall that \( T = \mathbb{C}[[q]][t]/t^{N+1} \). One can regard \( T \) as a \( \mathbb{C} \)-vector space. Consider the linear space (over \( \mathbb{C} \))
\[
C = \{ (\tilde{a}_0, \tilde{a}_1, \ldots, \tilde{a}_i) \in R \}
\]
with basis \( \{ b_{ij} = (0, \ldots, 0, \varepsilon^{N-j}, 0, \ldots), \ i \geq 0, \ 0 \leq j \leq N \} \) (where \( \varepsilon^{N-j} \) is in the \( i \)th place) and define an isomorphism \( l: C \rightarrow T \) by the formula \( b_{ij} \mapsto q^i t^j / j! \). It is easy to see that the formulae \( q \cdot b_{ij} = b_{i+1,j} \) and \( D \cdot b_{ij} = (i + \varepsilon) b_{ij} \) determine an action of the ring \( \mathcal{D} \) on \( C \). We clearly have \( l(q \cdot b_{ij}) = q \cdot l(b_{ij}) \) and \( l(D \cdot b_{ij}) = D \cdot l(b_{ij}) \), that is, the actions of \( \mathcal{D} \) on \( C \) and \( T \) commute. Thus \( P(\{\tilde{c}_i\}) = 0 \) if and only if \( P(I_N) = 0 \) (we follow the notation in Definition 2.14 with \( h = \varepsilon \)). Similarly, if \( P(\{\tilde{c}_i\}) = 0 \), then \( P(I_r) = 0 \) for \( 0 \leq r \leq N \).

\(^5\)This means that the following conditions hold for every solution of the form \( I = \sum a_i q^i \), \( a_n \in \mathbb{Q} \). Suppose that \( a_n = \frac{p_n}{q_n} \), \( (p_n, q_n) = 1, \ q_n \geq 1 \). Then \( I \) has positive radii of convergence in \( \mathbb{C} \) and \( \overline{\mathbb{Q}}_p \) for any prime \( p \) and there is a constant \( C < \infty \) such that \( \text{LCM}(q_1, q_2, \ldots, q_n) < C^n \) for all \( n \).
**Remark 3.5.** A Newton solution of $P_\varepsilon$ exists in $R$ if and only if $\text{mult}_0 P_0 \geq N + 1$.

**Remark 3.6.** We have treated the case $P_0(0) = 0$. The cases of other roots of $P_0$ reduce to this case by shifting the coordinate.

**Corollary 3.7.** Let $I = \sum_{i,j} a_{ij} q^{i} \varepsilon^{j} \in \mathbb{C}[[q]] \otimes R$ be a perturbed solution of $P$. Then the series
\[ \tilde{I} = \sum_{i,j} a_{ij} q^{i} \varepsilon^{j} (\varepsilon + 1) \cdots (\varepsilon + i) \]
is a perturbed solution of $\tilde{P}$.

**Proof.** This follows from the formula for $\tilde{P}$ and Proposition 3.4.

Let $V$ be a smooth quantum minimal Fano variety of dimension $N$, $L_V^Q$ the corresponding quantum differential operator, and $L_V$ the corresponding operator of type $DN$. We define polynomials $I^i(h)$ in $h$ by putting $I^V = \sum_{i=0}^{\infty} I^i(h) q^i$. Let
\[ \tilde{I}^V = \sum_{i=0}^{\infty} I^i(h)(h+1) \cdots (h+i) q^i. \]

Then Corollary 3.7 yields Theorem 1.1.

**Proof of Theorem 1.1.** By Theorem 2.15 the series $I^V$ is a perturbed solution of $L_V^Q$. Let $\tilde{L}_V$ be the regularization of $L_V^Q$. Then $\tilde{I}^V$ is a perturbed solution of $\tilde{L}_V$ by Corollary 3.7. The relations for the Newton solution of $L_{V,\varepsilon}$ are proportional to the corresponding relations for $\tilde{L}_{V,\varepsilon}$ modulo $\varepsilon^N$ (we identify the parameter $\varepsilon$ in the Frobenius method with $h$). Hence the Newton solutions of these operators also coincide modulo $\varepsilon^N$. Moreover, a standard argument on linear independence (see the proof of Theorem 2.15) shows that the $N$ functions in the statement of the theorem form a basis of the kernel of $L_V$.

**Example 3.8.** The matrix of quantum multiplication for $\mathbb{P}^N$ is
\[
\begin{pmatrix}
0 & 0 & \ldots & 0 & (N + 1)^{N+1} q^{N+1} \\
1 & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 1 & 0 
\end{pmatrix}.
\]
The corresponding quantum differential operator is given by
\[ L_{\mathbb{P}^N}^Q = D^{N+1} - (N + 1)^{N+1} q^{N+1}. \]

Let $F$ be the class dual to a hyperplane in $\mathbb{P}^N$ (that is, $-K_{\mathbb{P}^N} = (N + 1)F$) and put $f = F \otimes 1 \in S^\text{form}$. It is easy to see that the series
\[ I_{\mathbb{P}^N} = \sum_{d \geq 0} \frac{q^{(N+1)d}}{(f+1)^{N+1} \cdots (f+d)^{N+1}} \]
is a perturbed solution of the equation $L_{\mathbb{P}^N}^Q I = 0$. 

The operator of type $DN$ for $\mathbb{P}^N$ is given by
\[ L_{\mathbb{P}^N} = D^N - (N + 1)^{N+1} q^{N+1} (D + 1) \cdots (D + N), \]
and its perturbed solution is the series
\[ \tilde{I}_{\mathbb{P}^N} = \sum_{d \geq 0} q^{(N+1)d} (h + 1) \cdots (h + (N + 1)d) \frac{(f + 1)^{N+1} \cdots (f + d)^{N+1}}{(f + 1)^{N+1} \cdots (f + d)^{N+1}}. \]

§ 4. Universality of operators of type $DN$ and their solutions

All the formulae above are formal corollaries of the formulae in Theorems 2.8–2.10. Hence it is natural to define an ‘abstract Gromov–Witten theory’, that is, to regard the Gromov–Witten invariants as formal variables with natural relations. Moreover, if we consider the ‘invariants’ corresponding to several classes of type $H^i$ and one Poincaré dual class of type $H_r$, then we can develop a universal abstract Gromov–Witten theory which is independent of the dimension $N$.

A convenient multiplicative basis for the ring $GW$ is given by Theorem 1.4, which goes back to the First Reconstruction Theorem in [2].

Proof of Theorem 1.4. We follow the notation introduced in Definition 1.3.

The following relation is a formal corollary of the relations of type $GW5$.

$GW6$. For every finite set $S \subset \mathbb{N}$ we denote $H^{i_1}, \ldots, H^{i_k}$ (where the $s_j$ are distinct elements of $S$) by $\coprod_S$. Then the following formula holds for any $n \geq 0$:
\[ \sum \langle \coprod_{S_1} H^{i_1}, H^{i_2}, H_a \rangle \mathbf{H} \langle \coprod_{S_2} H^{i_3}, H_r \rangle = \sum \langle \coprod_{T_1} H^{i_1}, H^{i_2}, H_b \rangle \mathbf{H} \langle \coprod_{T_2} H^{i_3}, H_r \rangle, \]
where the sums are taken over all partitions $S_1 \sqcup S_2 = \{4, \ldots, n\}$, $T_1 \sqcup T_2 = \{4, \ldots, n\}$ and all $a, b$ such that the degrees of all the symbols occurring are nonnegative (note that both sums are finite).

In the geometric case, the formulae $GW6$ are referred to as quadratic relations (see [2], § 3.2.2). If $S$ is empty, then these formulae are called associativity equations or WDVV equations. Although such relations follow from $GW5$, it is convenient to include them in the set of generators of the ideal of relations in the ring $GW$.

We claim that $r$ is an epimorphism. Our proof is an abstract version of that in [2]. For simplicity we again denote $r(a_{ij})$ by $a_{ij}$. We must prove that all the ‘invariants’ $\langle \tau_{d_1} H^{i_1}, \ldots, \tau_{d_n} H_r \rangle$ can be expressed in terms of the $a_{ij}$.

Applying the relation $GW4$ to one-pointed and two-pointed invariants (abstract symbols), we can assume that $n \geq 3$ (see the proof of Proposition 5.2 in [3]). Using the relations $GW5$ (and $GW2$), we can assume that all the $d_i$ are equal to zero.

Given an invariant $C = \langle H^{i_1+1}, \ldots, H^{i_n}, H_r \rangle$ with $n \geq 2$ and $i_1 > 1$, we consider the relations $GW6$ for the classes $H, H^{i_1}, \ldots, H^{i_n}, H_r$. It is easy to see that, modulo invariants with a smaller number of terms and relations of types $GW2$ and $GW4$,
the invariant $C$ is equal to the sum of the invariants for $H^{i_1}, \ldots, H^{i_n}, H_r$. Therefore, using GW1–GW6, we can express any invariant in terms of three-pointed invariants with first term $H^1$, that is, in terms of the $a_{i,j}$. Thus $r$ is an epimorphism.

We now claim that $r$ is a monomorphism.

**Step 1.** Let GW3$_p$ and GW4$_p$ be the relations of type GW3 and GW4 for invariants without descendants. Then the ideal Rel is generated by relations of types GW1, GW2, GW3$_p$, GW4$_p$, GW5, GW6 (the relations ‘commute’). Note that GW3$_p$ is a particular type of GW2.

**Step 2.** Put

$$\text{GW}' = \mathbb{C}[F']/(\text{GW4}_p, \text{GW5}, \text{GW6}),$$

where $F' \subset F$ are the invariants of positive degree of the type

$$\langle \tau_{d_1}H^{i_1}, \ldots, \tau_{d_{n-1}}H^{i_{n-1}}, \tau_{d_n}H_r \rangle$$

with $i_k \geq i_l$ for $k > l$ and $d_k \geq d_l$ if $i_k = i_l$. (Thus the left-hand side of every relation of type GW2 becomes just a symbol for the number on the right-hand side.) Clearly, GW' \cong GW.

**Step 3.** Put $A_p = \mathbb{C}[F'_p]/(\text{GW4}_p, \text{GW6})$, where $F'_p \subset F'$ is the subset of invariants without descendants. We claim that the natural map $A_p \to \text{GW}'$ is a monomorphism. Indeed, consider the ordering on the invariants defined by the function

$$w((\tau_{d_1}H^{i_1}, \ldots, \tau_{d_{n-1}}H^{i_{n-1}}, \tau_{d_n}H_r)) = \left( \sum d_j, d_1, i_1, \ldots, d_n, r \right)$$
on $F'_p$ by saying that $C_1 > C_2$ if $w(C_1) > w(C_2)$ (with respect to the natural lexicographic ordering). We define a lexicographic ordering on the monomials in $F'$: given any monomials $M_1 = \alpha C_1^{a_1} \cdots C_n^{a_n}$ and $M_2 = \beta C_1^{b_1} \cdots C_n^{b_n}$ (where $\alpha, \beta \in \mathbb{C}$ and $C_1 > C_2 > \cdots > C_n$), we put $M_1 > M_2$ if either $a_1 > b_1$, or $a_1 = b_1$ and $a_2 > b_2$, and so on. We denote the leading term of $E \in \mathbb{C}[F']$ with respect to this ordering by $L(E)$. We also denote a relation (which is not uniquely determined) of type GW5 with the invariant $C$ on the left-hand side by GW5$(C)$. If $C$ is a prime invariant (that is, $C$ has no descendants), then we put GW5$(C) = C$. Consider any element $P \neq 0$ in $\mathbb{C}[F'_p]$ such that $r(P) \in (\text{GW4}_p, \text{GW5}, \text{GW6}) \subset \mathbb{C}[F']$ for the natural map $r: \mathbb{C}[F'_p] \to \mathbb{C}[F']$. For simplicity we again denote $r(P)$ by $P$.

Suppose that $P = \sum_{j \in J} \beta_j \cdot \prod_{i \in I} C_i^{b_{ij}} \text{GW5}(C_j)$ modulo $\text{GW4}_p, \text{GW6}$, where $\beta_j \in \mathbb{C}$. By applying relations of type GW4 if necessary, we can assume that the invariants $C_i$ contain at least three ‘elements’ inside the brackets. Hence we can apply a relation of type GW5 to them. We denote the largest of the leading terms of all summands of type $\prod_{i \in I} C_i^{b_{ij}} \text{GW5}(C_j)$ by $L$. Let $J_0 \subset J$ be the subset of indices such that $L(\prod_{i \in I} C_i^{b_{ij}} \text{GW5}(C_j)) = L$ for $j \in J_0$. Suppose that $L$ has a factor with descendants. Then

$$P = \sum_{j \in J_0} \beta_j \cdot \prod_{i \in I} C_i^{b_{ij}} \text{GW5}(C_j) + S,$$

where $S$ stands for summands with smaller leading terms. Clearly, $L(\text{GW5}(C)) = C$. One can check that the difference between two relations of type GW5$(C)$ is expressible in terms of relations of type GW5 with smaller leading terms and relations of
type GW6. Thus for all $i$, the expressions of type $GW5(C_i)$ in the sum on the
right-hand side coincide modulo summands with smaller leading terms. Then

$$P = \sum_{j \in J_0} \beta_j \cdot \prod_{i \in I} GW5(C_i)^{b_{ij}} GW5(C_j) + S_1 = \sum_{j \in J_0} \beta_j \cdot \prod_{i \in I \cup J_0} GW5(C_i)^{c_i} + S_2,$$

where $S_1, S_2$ stand for summands with smaller leading terms. Since $L(P) < L$, we have $\sum_{j \in J_0} \beta_j = 0$. This yields an expression for $P$ with smaller leading term $L$. Repeating this procedure, we obtain an expression for $P$ with $L(P) = L$, that is, without relations of type GW5. Thus we have $P \in (GW4_p, GW6)$ and $A_p \cong GW' \cong GW$, that is, any invariant is uniquely expressible in terms of prime invariants, as claimed.

Step 4. We claim that any prime invariant may be expressed uniquely in terms of
generators of type $a_{ij}$. The invariants of type $\langle H^k, H^1, \ldots, H^1, H_r \rangle$ are said to be
trivial because relations of type GW6 are trivial for them. Let $F_t \subset F'_p$ be the subset
of trivial invariants. Clearly, $A \cong A_t = C[F_t]/(GW4_p) \cong C[F_t]/(GW4_p, GW6)$. Let
$F'_t \subset F'_p$ be the subset of invariants not containing terms of the form $H^1$, and let
GW6' be the set of relations of type GW6 in which invariants containing $H^1$ are
replaced by invariants without such terms in accordance with GW4_p. We claim
that $A_t \cong C[F'_t]/(GW6') \cong A_p$.

Indeed, we define a function $w'$ on the elements of $F'_t$ by putting

$$w'((H^{i_1}, \ldots, H^{i_{n-1}}, H_r)) = (n, i_1, \ldots, i_{n-1}, r).$$

We also define an ordering on the monomials in $F'$ and the leading term $L'(E)$ of
any $E \in C[F'_p]$ in the same way as before. A direct computation shows that the
difference between two relations of type GW6' with the same leading term may be
expressed in terms of relations of type GW6' with smaller leading terms (‘relations
of type GW6 commute’). Suppose that $P = P(a_{ij}) \in (GW6') \subset C[F'_t]$. As before,
we obtain an expression for $P$ in terms of relations of type GW6' that contains only
trivial invariants. Since these relations vanish, we have $P = 0$ and $A \cong A_p \cong GW$.

Remark 4.1. Thus the Gromov–Witten theory of a quantum minimal Fano vari-
ety $V$ of dimension $N$ is a particular function from $GW_N = r(i_N(A_N))$ to $C$, where
$i_N : A_N \rightarrow GW$ is given by $i_N(a_{ij}) = a_{ij}$ if $(i, j) \neq (0, 0)$, and
$i_N(a_{00}) = 0$ (the def-
inition of $A_N$ is given in § 1).

Theorem 1.4 enables us to define a universal $I$-series $I \in A \otimes C[[q]][[h]]$ such
that for every $N$ the abstract $I$-series for dimension $N$,

$$I^N = 1 + \sum_{i,j} \langle r_i H_j \rangle q^d h^j \in GW_N \otimes C[[q]][h]/h^{N+1},$$

is a restriction of $I$, that is, $I^N = r_N(I \bmod h^{N+1})$. One can similarly define
a universal ‘regularized $I$-series’ $\tilde{I}$ such that the series

$$\tilde{I}^N = 1 + \sum_{i,j} \langle r_i H_j \rangle q^d h^j (h + 1) \cdots (h + d) \in GW_N \otimes C[[q]][h]/h^{N+1}$$

satisfies $\tilde{I}^N = r_N(\tilde{I} \bmod h^{N+1})$. 
Consider the torus \( \mathbb{T} = \text{Spec} \mathbb{C}[q, q^{-1}] \) and the trivial vector bundle \( HQ^N \) with fibre \( GW_N \otimes \langle H^0, H^1, \ldots, H^N \rangle \) (where the \( H^i \) denote basis vectors). Put \( h^i = 1 \otimes H^i \) and

\[
A^N = \begin{pmatrix}
a_{00} q & a_{01} q^2 & \cdots & a_{0,N-1} q^N & a_{0N} q^{N+1} \\
1 & a_{11} q & \cdots & a_{1,N-1} q^{N-1} & a_{1N} q^N \\
0 & 0 & \cdots & 1 & a_{NN} q
\end{pmatrix},
\]

where \( a_{00} = 0 \). Define the abstract quantum connection \( \nabla^N \) by the formula

\[
\left( \nabla^N(h^i), q \frac{d}{dq} \right) = A^N h^i
\]

(this connection commutes with \( a_{ij} \)). All the above arguments can be repeated in the abstract case. In particular, we define the abstract quantum differential operator \( L^Q_N \in GW_N \otimes D \) and the operator \( L_N \in GW_N \otimes D \) (we recall that this operator becomes the geometric operator of type \( DN \) if we specialize the abstract Gromov–Witten invariants to geometric invariants). (It is easy to see that this operator coincides with the operator defined in the introduction.) Then we obtain the following theorem.

**Theorem 4.2.** 1) The series \( I^N \) is a perturbed solution of \( L^Q_N I = 0 \).

2) The series \( \tilde{I}^N \mod h^N \) is a perturbed solution of \( L_N I = 0 \), that is, Theorem 1.5 holds.

In other words, putting \( a_{ij} = 0 \) for \( N_0 < i, j \leq N \) in the series \( \tilde{I}^N \) (resp. in the operator \( L_N \)), we get the series \( \tilde{I}^{N_0} \) (resp. the operator \( D^{N-N_0} L_{N_0} \)).

**Remark 4.3.** The same holds for operators of type \( DN \). We recall that such operators are obtained from the operators \( L_N \) by identifying \( a_{ij} \) and \( a_{N-j,N-i} \). Let \( J^N \) be a perturbed solution of such an operator. If \( n \ll N \) and \( N < N_0 \), then

\[
J^N \equiv J^{N_0}(\mod q^n, h^N).
\]

**Remark 4.4.** If we define \( L^Q_N \) and \( L_N \) as operators in \( \mathbb{C}[a_{ij}] \), \( 0 \leq i \leq j \) (that is, if we let \( a_{00} \) be arbitrary), then the universality still holds for their solutions. The universal series for \( L^Q_N \) is \( e^{a_{00} q} I' \), where \( I' \) is obtained from \( I \) by the shift \( a_{ii} \mapsto a_{ii} - a_{00} \). The universal series for \( L_N \) is the regularization of \( e^{a_{00} q} I' \).

§ 5. Appendix

Consider a differential operator \( P = \sum_{i=0}^N q^i P_i(D) \in D \) and denote its \( r \)-th formal derivative with respect to \( D \) by \( P^{(r)} \).

**Theorem 5.1.** A series \( I = \sum_{i=0}^N I^i h^i \) (where the \( I^i \) are series in \( q \)) is a perturbed solution of \( P \) if and only if the following equation holds for all \( s \leq N \):

\[
\frac{P^{(s)}(I^0)}{s!} + \frac{P^{(s-1)}(I^1)}{(s-1)!} + \cdots + P(I^s) = 0.
\]
Proof. We note that

\[ P(tJ(q)) = tP(J(q)) + P^{(1)}(J(q)) \]

for all \( J(q) \in \mathbb{C}[[q]] \) (see \[10\], Proposition 4.3.1). Thus,

\[ P(t^rJ) = \sum_{i=0}^{r} \binom{r}{i} t^i P^{(r-i)}(J). \]

For any \( s \leq N \) we have

\[ P(I_s) = P\left( \frac{t^s}{s!} I^0 + \frac{t^{s-1}}{(s-1)!} I^1 + \cdots + I^s \right) = \sum_{\alpha=0}^{s} P\left( \frac{t^{\alpha}}{\alpha!} I^{s-\alpha} \right) \]

\[ = \sum_{\alpha=0}^{s} \sum_{\beta=0}^{\alpha} \binom{\alpha}{\beta} t^{\beta} P^{(\alpha-\beta)}(I^{s-\alpha}) \]

\[ = \sum_{\alpha=0}^{s} \sum_{\beta=0}^{\alpha} \left( \frac{t^{\beta} P^{(\alpha-\beta)}(I^{s-\alpha})}{\beta! (\alpha - \beta)!} \right) = \sum_{\alpha=0}^{s} \sum_{\beta=0}^{\alpha} R_{\alpha,\beta}, \]

where

\[ R_{\alpha,\beta} = \frac{t^{\beta} P^{(\alpha-\beta)}(I^{s-\alpha})}{\beta! (\alpha - \beta)!}. \]

We prove the theorem by induction on \( s \). Suppose that it holds for all \( s_0 < s \). Then

\[ \frac{P^{(s)}(I^0)}{s!} + \frac{P^{(s-1)}(I^1)}{(s-1)!} + \cdots + P(I^s) = \sum_{a=0}^{s} t^{a} \left( \frac{P^{(s-a)}(I^0)}{(s-a)!} + \cdots + P(I^{s-a}) \right) \]

\[ = \sum_{a=0}^{s} \sum_{b=0}^{s-a} t^{a} P^{(b)}(I^{s-a-b}) \]

\[ = \sum_{a=0}^{s} \sum_{b=0}^{s-a} S_{a,b}, \]

where

\[ S_{a,b} = \frac{t^{a} P^{(b)}(I^{s-a-b})}{a! b!}. \]

Clearly, \( S_{a,b} = R_{a+b,a} \). Therefore,

\[ \sum_{a=0}^{s} \sum_{b=0}^{s-a} S_{a,b} = \sum_{a=0}^{s} \sum_{b=0}^{s-a} R_{a+b,a} = \sum_{a=0}^{s} \sum_{\beta=0}^{s-a} R_{\beta,a} = \sum_{0 \leq \alpha \leq \beta \leq s} R_{\beta,\alpha} = \sum_{a=0}^{s} \sum_{b=0}^{a} R_{a,b}. \]

Thus,

\[ \frac{P^{(s)}(I^0)}{s!} + \frac{P^{(s-1)}(I^1)}{(s-1)!} + \cdots + P(I^s) = P(I_s), \]

as required.
Remark 5.2. Theorem 5.1 is proved for \( s = 1 \) in [10], Proposition 4.3.2 and for \( s \leq 2 \) in [11], Appendix B.

**Proposition 5.3** (Newton’s method). The series
\[
\Phi = a_0 + a_1q + a_2q^2 + \cdots \in B, \quad a_i \in \mathbb{C},
\]
is a solution of \( P\Phi = 0 \) (as a formal series) if and only if the following equation holds for all \( m \in \mathbb{Z} \):
\[
a_m P_N(m) + a_{m+1} P_{N-1}(m+1) + \cdots + a_{m+N} P_0(m+N) = 0,
\]
where \( a_i = 0 \) for \( i < 0 \).

**Proof.** This is proved by substituting the series \( \Phi \) in the equation \( P\Phi = 0 \).

Theorem 5.1 and Proposition 5.3 enable us to find relations for the solutions of \( P\Phi = 0 \).

**Corollary 5.4.** A series \( I = \sum_{i,j} a_{ij} h^i q^j \) is a perturbed solution of \( P \) if and only if the following equation holds for all \( s \leq N \) and all \( m \in \mathbb{N} \):
\[
a_{0m} P_N^{(s)}(m) + a_{0,m+1} P_{N-1}^{(s)}(m+1) + \cdots + a_{0,m+N} P_0^{(s)}(m+N) + \cdots
\]
\[
+ a_{1m} P_N^{(s-1)}(m) + a_{1,m+1} P_{N-1}^{(s-1)}(m+1) + \cdots + a_{1,m+N} P_0^{(s-1)}(m+N) + \cdots
\]
\[
+ \cdots + a_{sm} P_N(m) + a_{s,m+1} P_{N-1}(m+1) + \cdots + a_{s,m+N} P_0(m+N) = 0.
\]

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