SOME RESULTS RELATING TO \((p, q)\)-TH RELATIVE GOL’DBERG ORDER AND \((p, q)\)-RELATIVE GOL’DBERG TYPE OF ENTIRE FUNCTIONS OF SEVERAL VARIABLES

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Abstract. In this paper we introduce the notions of \((p, q)\)-th relative Gol’dberg order and \((p, q)\)-th relative Gol’dberg type of entire functions of several complex variables where \(p, q\) are any positive integers. Then we study some growth properties of entire functions of several complex variables on the basis of their \((p, q)\)-th relative Gol’dberg order and \((p, q)\)-th relative Gol’dberg type.

1. Introduction and Definitions.

Let \(\mathbb{C}^n\) and \(\mathbb{R}^n\) respectively denote the complex and real \(n\)-space. Also let us indicate the point \((z_1, z_2, \ldots, z_n)\), \((m_1, m_2, \ldots, m_n)\) of \(\mathbb{C}^n\) or \(\mathbb{R}^n\) by their corresponding unsuffixed symbols \(z, m\) respectively where \(I\) denotes the set of non-negative integers.

The modulus of \(z\), denoted by \(|z|\), is defined as \(|z| = (|z_1|^2 + \cdots + |z_n|^2)^{\frac{1}{2}}\). If the coordinates of the vector \(m\) are non-negative integers, then \(z^m\) will denote \(z_1^{m_1} \cdots z_n^{m_n}\) and \(\|m\| = m_1 + \cdots + m_n\).

If \(D \subseteq \mathbb{C}^n\) (\(\mathbb{C}^n\) denote the \(n\)-dimensional complex space) be an arbitrary bounded complex \(n\)-circular domain with center at the origin of coordinates then for any entire function \(f(z)\) of \(n\) complex variables and \(R > 0\), \(M_{f,D}(R)\) may be define as \(M_{f,D}(R) = \sup_{z \in D_R} |f(z)|\) where a point \(z \in D_R\) if and only if \(\frac{z}{R} \in D\). If \(f(z)\) is non-constant, then \(M_{f,D}(R)\) is strictly increasing and its inverse \(M_{f,D}^{-1}: (|f(0)|, \infty) \rightarrow (0, \infty)\) exists such that \(\lim_{R \to \infty} M_{f,D}^{-1}(R) = \infty\).

Considering this, the Gol’dberg order (resp. Gol’dberg lower order) \(\{\text{cf. [4], [5]}\}\) of an entire function \(f(z)\) with respect to any bounded complete \(n\)-circular domain \(D\) is given by

\[
\rho_{f,D} = \lim_{R \to +\infty} \frac{\log^2 M_{f,D}(R)}{\log R} \quad \text{(resp.} \quad \lambda_{f,D} = \lim_{R \to +\infty} \frac{\log^2 M_{f,D}(R)}{\log R} \text{)}.
\]

where \(\log^k R = \log(\log^{k-1} R)\) for \(k = 1, 2, 3, \ldots\); \(\log^{[0]} R = R\) and \(\exp^k R = \exp(\exp^{k-1} R)\) for \(k = 1, 2, 3, \ldots\); \(\exp^{[0]} R = R\).

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It is well known that $\rho_{f,D}$ is independent of the choice of the domain $D$, and therefore we write $\rho_f$ instead of $\rho_{f,D}$ (resp. $\lambda_f$ instead of $\lambda_{f,D}$)\{cf. [4], [5]\}. For any bounded complete $n$-circular domain $D$, an entire function of $n$-complex variables for which Gol’dberg order and Gol’dberg lower order are the same is said to be of regular growth. Functions which are not of regular growth are said to be of irregular growth.

To compare the relative growth of two entire functions of $n$-complex variables having same non zero finite Gol’dberg order, one may introduce the definition of Gol’dberg type and Gol’dberg lower type in the following manner:

**Definition 1.** \{cf. [4], [5]\} The Gol’dberg type and Gol’dberg lower type respectively denoted by $\Delta_{f,D}$ and $\nabla_{f,D}$ of an entire function $f(z)$ of $n$-complex variables with respect to any bounded complete $n$-circular domain $D$ are defined as follows:

$$\Delta_{f,D} = \lim_{R \to +\infty} \frac{\log M_{f,D}(R)}{(R)_{\rho_f}} \quad \text{and} \quad \nabla_{f,D} = \lim_{R \to +\infty} \frac{\log M_{f,D}(R)}{(R)_{\lambda_f}} , \quad 0 < \rho_f < +\infty .$$

Analogously to determine the relative growth of two entire functions of $n$-complex variables having same non zero finite Gol’dberg lower order, one may introduce the definition of Gol’dberg weak type in the following way:

**Definition 2.** The Gol’dberg weak type denoted by $\tau_{f,D}$ of an entire function $f(z)$ of $n$-complex variables with respect to any bounded complete $n$-circular domain $D$ is defined as follows:

$$\tau_{f,D} = \lim_{R \to +\infty} \frac{\log M_{f,D}(R)}{(R)_{\lambda_f}} , \quad 0 < \lambda_f < +\infty .$$

Also one may define the growth indicator $\mathfrak{T}_{f,D}$ in the following manner:

$$\mathfrak{T}_{f,D} = \lim_{R \to +\infty} \frac{\log M_{f,D}(R)}{(R)_{\lambda_f}} , \quad 0 < \lambda_f < +\infty \ .$$

Gol’dberg has shown that [5] Gol’dberg type depends on the domain $D$. Hence all the growth indicators define in Definition 1 and Definition 2 are also depend on $D$.

However, extending the notion of Gol’dberg order, Datta and Maji [1] defined the concept of $(p,q)$-th Gol’dberg order (resp. $(p,q)$-th Gol’dberg lower order) of an entire function $f(z)$ for any bounded complete $n$-circular domain $D$ where $p \geq q \geq 1$ in the following way:

$$\rho_{f,D}(p,q) = \lim_{R \to +\infty} \frac{\log^{[p]} \ M_{f,D}(R)}{\log^{[q]} R} = \lim_{R \to +\infty} \frac{\log^{[p]} R}{\log^{[q]} M_{f,D}^{-1}(R)} \ .$$

(resp. $\lambda_{f,D}(p,q) = \lim_{R \to +\infty} \frac{\log^{[p]} \ M_{f,D}(R)}{\log^{[q]} R} = \lim_{R \to +\infty} \frac{\log^{[p]} R}{\log^{[q]} M_{f,D}^{-1}(R)} \ .$$

These definitions extended the generalized Gol’dberg order $\rho_{f,D}^{[l]}$ (resp. generalized Gol’dberg lower order $\lambda_{f,D}^{[l]}$) of an entire function $f(z)$ for any bounded complete $n$-circular domain $D$ for each integer $l \geq 2$ since these correspond to the particular case
where $\rho_{f,D}^{[0]} = \rho_{f,D} (l, 1)$ (resp. $\lambda_{f,D}^{[0]} = \lambda_{f,D} (l, 1)$). Clearly $\rho_{f,D} (2, 1) = \rho_{f,D} (2, 1) = \lambda_{f,D} (2, 1)$. Further in the line of Gol'dberg (cf. [4], [5]), one can easily verify that $\rho_{f,D} (p, q)$ (resp. $\lambda_{f,D} (p, q)$) is independent of the choice of the domain $D$, and therefore one can write $\rho_{f,D} (p, q)$ (resp. $\lambda_{f,D} (p, q)$).

In this connection let us recall that if $0 < \rho_{f,D} (p, q) < \infty$, then the following properties hold

$\rho_{f,D} (p - n, q) = \infty$ for $n < p$, $\rho_{f,D} (p, q - n) = 0$ for $n < q$, and

$\rho_{f,D} (p + n, q + n) = 1$ for $n = 1, 2, ...$

Similarly for $0 < \lambda_{f,D} (p, q) < \infty$, one can easily verify that

$\lambda_{f,D} (p - n, q) = \infty$ for $n < p$, $\lambda_{f,D} (p, q - n) = 0$ for $n < q$, and

$\lambda_{f,D} (p + n, q + n) = 1$ for $n = 1, 2, ...$

Recalling that for any pair of integer numbers $m, n$ the Kroenecker function is defined by $\delta_{m,n} = 1$ for $m = n$ and $\delta_{m,n} = 0$ for $m \neq n$, the aforementioned properties provide the following definition.

**Definition 3.** For any bounded complete $n$-circular domain $D$, an entire function $f(z)$ of $n$-complex variables is said to have index-pair $(1, 1)$ if $0 < \rho_{f,D} (1, 1) < \infty$. Otherwise, $f(z)$ is said to have index-pair $(p, q) \neq (1, 1)$, $p \geq q \geq 1$, if $\delta_{p,q,0} < \rho_{f,D} (p, q) < \infty$ and $\rho_{f,D} (p - 1, q - 1) \notin R^+$.

**Definition 4.** For any bounded complete $n$-circular domain $D$, an entire function $f(z)$ of $n$-complex variables is said to have lower index-pair $(1, 1)$ if $0 < \lambda_{f,D} (1, 1) < \infty$. Otherwise, $f(z)$ is said to have lower index-pair $(p, q) \neq (1, 1)$, $p \geq q \geq 1$, if $\delta_{p,q,0} < \lambda_{f,D} (p, q) < \infty$ and $\lambda_{f,D} (p - 1, q - 1) \notin R^+$.

For any bounded complete $n$-circular domain $D$, an entire function $f(z)$ of $n$-complex variables of index-pair $(p, q)$ is said to be of regular $(p, q)$-Gol'dberg growth if its $(p, q)$-th Gol'dberg order coincides with its $(p, q)$-th Gol'dberg lower order, otherwise $f(z)$ is said to be of irregular $(p, q)$-Gol'dberg growth.

To compare the relative growth of two entire functions having same non zero finite $(p, q)$-Gol’dberg order, one may introduce the definition of $(p, q)$-Gol’dberg type and $(p, q)$-Gol’dberg lower type in the following manner:

**Definition 5.** The $(p,q)$-th Gol’dberg type and $(p,q)$-th Gol’dberg lower type respectively denoted by $\Delta_{f,D} (p, q)$ and $\overline{\Delta}_{f,D} (p, q)$ of an entire function $f(z)$ of $n$-complex variables with respect to any bounded complete $n$-circular domain $D$ are defined as follows:

$\Delta_{f,D} (p, q) = \lim_{R \to +\infty} \frac{\log_{\rho_{f,D} (p, q)} \log_{[p-1]} M_{f,D} (R)}{\log_{[q-1]} R}$

and $\overline{\Delta}_{f,D} (p, q) = \lim_{R \to +\infty} \frac{\log_{\rho_{f,D} (p, q)} \log_{[p-1]} M_{f,D} (R)}{\log_{[q-1]} R}$

where $p \geq q \geq 1$. 

An entire function \( f(z) \) of \( n \)-complex variables of index-pair \( (p,q) \) is said to be of perfectly regular \( (p,q) \)-Goldberg growth if its \( (p,q) \)-th Goldberg order coincides with its \( (p,q) \)-th Goldberg lower order as well as its \( (p,q) \)-th Goldberg type coincides with its \( (p,q) \)-th Goldberg lower type.

Analogously to determine the relative growth of two entire functions of \( n \)-complex variables having same non zero finite \( (p,q) \)-Goldberg lower order, one may introduce the definition of \( (p,q) \)-Goldberg weak type in the following way:

**Definition 6.** The \( (p,q) \)-th Goldberg weak type denoted by \( \tau_{f,D}(p,q) \) of an entire function \( f(z) \) of \( n \)-complex variables with respect to any bounded complete \( n \)-circular domain \( D \) is defined as follows:

\[
\tau_{f,D}(p,q) = \lim_{R \to +\infty} \frac{\log^{[p-1]} M_{f,D}(R)}{\log^{[q-1]} R} \lambda_f(p,q), \quad 0 < \lambda_f(p,q) < +\infty.
\]

Also one may define the growth indicator \( \varpi_{f,D}(p,q) \) in the following manner:

\[
\varpi_{f,D}(p,q) = \lim_{R \to +\infty} \frac{\log^{[p-1]} M_f(R)}{\log^{[q-1]} R} \lambda_f(p,q), \quad 0 < \lambda_f(p,q) < +\infty,
\]

where \( p \geq q \geq 1 \).

Definition \[5\] and Definition \[6\] are extended the generalized Goldberg type \( \Delta_{f,D}^{[l]} \) (resp. generalized Goldberg lower type \( \overline{\Delta}_{f,D}^{[l]} \)) and generalized Goldberg weak order \( \tau_{f,D}^{[l]} \) of an entire function \( f(z) \) of \( n \)-complex variables with respect to any bounded complete \( n \)-circular domain \( D \) for each integer \( l \geq 2 \) since these correspond to the particular case \( \Delta_{f,D}^{[1]} = \Delta_{f,D}(1,1) \) (resp. \( \overline{\Delta}_{f,D}^{[1]} = \overline{\Delta}_{f,D}(1,1) \)) and \( \tau_{f,D}^{[1]} = \tau_{f,D}(1,1) \) (resp. \( \overline{\tau}_{f,D}^{[1]} = \overline{\tau}_{f,D}(1,1) \)). Clearly \( \Delta_{f,D}(2,1) = \Delta_{f,D}(2,1) \) (resp. \( \overline{\Delta}_{f,D}(2,1) = \overline{\Delta}_{f,D}(2,1) \)) and \( \tau_{f,D}(2,1) = \tau_{f,D}(2,1) \) (resp. \( \overline{\tau}_{f,D}(2,1) = \overline{\tau}_{f,D}(2,1) \)).

Since Goldberg has shown that \[5\] Goldberg type depends on the domain \( D \), therefore all the growth indicators define in Definition \[5\] and Definition \[6\] are also depend on \( D \).

For any two entire functions \( f(z) \) and \( g(z) \) of \( n \)-complex variables and for any bounded complete \( n \)-circular domain \( D \) with center at all the origin \( \mathbb{C}^n \), Mondal and Roy \[6\] introduced the concept relative Goldberg order which is as follows:

\[
\rho_{g,D}(f) = \inf \{\mu > 0 : M_{f,D}(R) < M_{g,D}(R^\mu) \text{ for all } R > R_0(\mu) > 0\}
= \lim_{R \to +\infty} \frac{\log M_{g,D}^{-1}(M_{f,D}(R))}{\log R}.
\]

In \[6\], Mandal and Roy also proved that the relative Goldberg order of \( f(z) \) with respect to \( g(z) \) is independent of the choice of the domain \( D \). So the relative Goldberg order of \( f(z) \) with respect to \( g(z) \) may be denoted as \( \rho_g(f) \) instead of \( \rho_{g,D}(f) \).
Likewise, one can define the relative Gol'dberg lower order $\lambda_{g,D}(f)$ in the following manner:
\[
\lambda_{g,D}(f) = \lim_{R \to +\infty} \frac{\log M_{g,D}^{-1}M_{f,D}(R)}{\log R}.
\]

In the line of Mandal and Roy \{cf. \cite{6}\}, one can also verify that $\lambda_{g,D}(f)$ is independent of the choice of the domain $D$, and therefore one can write $\lambda_g(f)$ instead of $\lambda_{g,D}(f)$.

In the case of relative Gol'dberg order, it therefore seems reasonable to define suitably the $(p, q)$-th relative Gol'dberg order of entire function of $n$-complex variables and for any bounded complete $n$-circular domain $D$ with center at the origin in $\mathbb{C}^n$. With this in view one may introduce the definition of $(p, q)$-th relative Gol'dberg order $\rho_{g,D}^{(p, q)}(f)$ of an entire function $f(z)$ with respect to another entire function $g(z)$ where both $f(z)$ and $g(z)$ are of $n$-complex variables and $D$ be any bounded complete $n$-circular domain with center at the origin in $\mathbb{C}^n$, in the light of index-pair. Our next definition avoids the restriction $p > q$ and gives the more natural particular case of Generalized Gol'dberg order i.e, $\rho_{g,D}^{[1, 1]}(f) = \rho_{g,D}(f)$.

**Definition 7.** Let $f(z)$ and $g(z)$ be any two entire functions of $n$-complex variables with index-pair $(m, q)$ and $(m, p)$, respectively, where $p, q, m$ are positive integers such that $m \geq q \geq 1$ and $m \geq p \geq 1$ and $D$ be any bounded complete $n$-circular domain with center at the origin in $\mathbb{C}^n$. Then the $(p, q)$-th relative Gol'dberg order of $f(z)$ with respect to $g(z)$ is defined as
\[
\rho_{g,D}^{(p, q)}(f) = \inf \left\{ \mu > 0 : M_{f,D}(r) < M_{g,D} \left( \exp^{[p]} \left( \mu \log^{[q]} R \right) \right) \right\}
\]
for all $R > R_0(\mu) > 0$
\[
= \lim_{R \to +\infty} \frac{\log^{[p]} M_{g,D}^{-1}M_{f,D}(R)}{\log^{[q]} R} = \lim_{R \to +\infty} \frac{\log^{[p]} M_{g,D}^{-1}(R)}{\log^{[q]} M_{f,D}^{-1}(R)}.
\]

Similarly, the $(p, q)$-th relative Gol'dberg lower order of $f(z)$ with respect to $g(z)$ is defined by:
\[
\lambda_{g,D}^{(p, q)}(f) = \lim_{R \to +\infty} \frac{\log^{[p]} M_{g,D}^{-1}M_{f,D}(R)}{\log^{[q]} R} = \lim_{R \to +\infty} \frac{\log^{[p]} M_{g,D}^{-1}(R)}{\log^{[q]} M_{f,D}^{-1}(R)}.
\]

In this connection, one may introduce the definition of relative index-pair of an entire function with respect to another entire function (both of $n$-complex variables) which is relevant in the sequel:

**Definition 8.** \cite{6} Let $f(z)$ and $g(z)$ be any two entire functions (both of $n$-complex variables) with index-pairs $(m, q)$ and $(m, p)$ respectively where $m \geq q \geq 1$, $m \geq p \geq 1$ and $D$ be any bounded complete $n$-circular domain. Then the entire function $f(z)$ is said to have relative index-pair $(p, q)$ with respect to another entire function $g(z)$, if $b < \rho_{g,D}^{(p, q)}(f) < \infty$ and $\rho_{g,D}^{(p-1, q-1)}(f)$ is not a nonzero finite number, where $b = 1$ if $p = q = m$ and $b = 0$ for otherwise. Moreover if $0 < \rho_{g,D}^{(p, q)}(f) < \infty$, then $\rho_{g,D}^{(p-n, q)}(f) = \infty$ for $n < p$, $\rho_{g,D}^{(p, q-n)}(f) = 0$ for $n < q$ and
Similarly for \(0 < \lambda_{g,D}^{(p,q)}(f) < \infty\), one can easily verify that

\[
\lambda_{g,D}^{(p-n,q)}(f) = \infty \text{ for } n < p, \quad \lambda_{g,D}^{(p,q-n)}(f) = 0 \text{ for } n < q \quad \text{and} \quad \lambda_{g,D}^{(p+n,q+n)}(f) = 1 \text{ for } n = 1, 2, \ldots .
\]

Further an entire function \(f(z)\) for which \((p,q)\)-th relative Gol’dberg order and \((p,q)\)-th relative Gol’dberg lower order with respect to another entire function \(g(z)\) are the same is called a function of regular relative \((p,q)\)-Gol’dberg growth with respect to \(g(z)\). Otherwise, \(f(z)\) is said to be irregular relative \((p,q)\)-Gol’dberg growth with respect to \(g(z)\).

Next we introduce the definition of \((p,q)\)-th relative Gol’dberg type and \((p,q)\)-th relative Gol’dberg lower type in order to compare the relative growth of two entire functions of \(n\)-complex variables having same non zero finite \((p,q)\)-th relative Gol’dberg order with respect to another entire function of \(n\)-complex variables.

**Definition 9.** Let \(f(z)\) and \(g(z)\) be any two entire functions of \(n\)-complex variables with index-pair \((m,q)\) and \((m,p)\), respectively, where \(p,q,m\) are positive integers such that \(m \geq q \geq 1\) and \(m \geq p \geq 1\) and \(D\) be any bounded complete \(n\)-circular domain with center at the origin in \(\mathbb{C}^n\). Then the \((p,q)\)-th relative Gol’dberg type and \((p,q)\)-th relative Gol’dberg lower type of \(f(z)\) with respect to \(g(z)\) are defined as

\[
\Delta_{g,D}^{(p,q)}(f) = \lim_{R \to +\infty} \frac{\log^{[p-1]} M_{g,D}^{-1} M_{f,D}(R)}{[\log^{[q-1]} R]^{\rho_{g,D}^{(p,q)}(f)}} \quad \text{and} \quad \Delta_{g,D}^{(p,q)}(f) = \lim_{R \to +\infty} \frac{\log^{[p-1]} M_{g,D}^{-1} M_{f,D}(R)}{[\log^{[q-1]} R]^{\rho_{g,D}^{(p,q)}(f)}} , \quad 0 < \rho_{g,D}^{(p,q)}(f) < +\infty .
\]

Analogously to determine the relative growth of two entire functions of \(n\)-complex variables having same non zero finite \(p,q\)-th relative Gol’dberg lower order with respect to another entire function of \(n\)-complex variables, one may introduce the definition of \((p,q)\)-th relative Gol’dberg weak type in the following way:

**Definition 10.** Let \(f(z)\) and \(g(z)\) be any two entire functions of \(n\)-complex variables with index-pair \((m,q)\) and \((m,p)\), respectively, where \(p,q,m\) are positive integers such that \(m \geq q \geq 1\) and \(m \geq p \geq 1\) and \(D\) be any bounded complete \(n\)-circular domain with center at the origin in \(\mathbb{C}^n\). Then \((p,q)\)-th relative Gol’dberg weak type denoted by \(\tau_{g,D}^{(p,q)}(f)\) of an entire function \(f(z)\) with respect to another entire function \(g(z)\) is defined as follows:

\[
\tau_{g,D}^{(p,q)}(f) = \lim_{R \to +\infty} \frac{\log^{[p-1]} M_{g,D}^{-1} M_{f,D}(R)}{[\log^{[q-1]} R]^{\lambda_{g,D}^{(p,q)}(f)}} , \quad 0 < \lambda_{g,D}^{(p,q)}(f) < +\infty .
\]
Similarly the growth indicator \( \varphi_{g,D}^{(p,q)} (f) \) of an entire function \( f(z) \) with respect to another entire function \( g(z) \) both of \( n \)-complex variables in the following manner:

\[
\varphi_{g,D}^{(p,q)} (f) = \lim_{R \to +\infty} \frac{\log^{[p-1]} M_{g,D}^{-1} M_{f,D} (R)}{\log^{[q-1]} R} \lambda_{g,D}^{(p,q)} (f), \quad 0 < \lambda_{g,D}^{(p,q)} (f) < +\infty.
\]

Therefore, for any two entire functions \( f(z) \) and \( g(z) \) both of \( n \)-complex variables, we note that

\[
\rho_{g,D}^{(p,q)} (f) \neq \lambda_{g,D}^{(p,q)} (f), \Delta_{g,D}^{(p,q)} (f) > 0 \Rightarrow \tau_{g,D}^{(p,q)} (f) = +\infty \quad \text{and} \quad \rho_{g,D}^{(p,q)} (f) \neq \lambda_{g,D}^{(p,q)} (f), \Sigma_{g,D}^{(p,q)} (f) > 0 \Rightarrow \tau_{g,D}^{(p,q)} (f) = +\infty.
\]

Since Gol'dberg has shown that \([5]\) Gol’dberg type depends on the domain \( D \). Hence all the growth indicators define in Definition \( 9 \) and Definition \( 10 \) are also depend on \( D \).

If \( f(z) \) and \( g(z) \) both of \( n \)-complex variables have got index-pair \((m,1)\) and \((m,l)\), respectively, then the above two definitions reduces to the definition of generalized relative Gol’dberg type \( \Delta_{g,D}^{[l]} (f) \) (resp generalized relative Gol’dberg lower type \( \Sigma_{g,D}^{[l]} (f) \)) and generalized relative Gol’dberg weak type \( \tau_{g,D}^{[l]} (f) \). If the entire functions \( f(z) \) and \( g(z) \) (both of \( n \)-complex variables) have the same index-pair \((p,1)\) where \( p \) is any positive integer, we get the definitions of relative Gol’dberg type as introduced by Roy \([7]\) (resp relative Gol’dberg lower type) and relative Gol’dberg weak type.

During the past decades, several authors \([1],[2],[3],[6],[7],[8]\) made closed investigations on the properties of entire functions of several complex variables using different growth indicator such as Gol’dberg order, \((p,q)\)-th Gol’dberg order etc. In this paper we wish to measure some properties of entire functions relative to another entire function of several complex variables and \( D \) will represent a bounded complete \( n \)-circular domain. Actually in this paper we wish to study some relative growth properties of entire functions of \( n \)-complex variables (all the entire functions under consideration will be transcendental unless otherwise stated) using \((p,q)\)-th relative Gol’dberg order, \((p,q)\)-th relative Gol’dberg type and \((p,q)\)-th relative Gol’dberg weak type.

2. Main Results.

In this section we state the main results of the paper.

**Theorem 1.** Let \( f(z) \) and \( g(z) \) be any two entire functions of \( n \)-complex variables and \( D \) be a bounded complete \( n \)-circular domain with center at origin in \( \mathbb{C}^n \). Then \((p,q)\)-th relative Gol’dberg order \( \rho_{g,D}^{(p,q)} (f) \) and \((p,q)\)-th relative Gol’dberg lower order \( \lambda_{g,D}^{(p,q)} (f) \) of \( f(z) \) with respect to \( g(z) \) is independent of the choice of the domain \( D \) where \( p \) and \( q \) are any positive integers.

**Proof.** Let us consider \( D_1 \) and \( D_2 \) any two bounded complete \( n \)-circular domains. Then there exist two real numbers \( \alpha, \beta > 0 \) such that \( \alpha D_1 \subset D_2 \subset \beta D_1 \) and therefore

\[
M_{f,\alpha D_1} (R) \leq M_{f,D_2} (R) \leq M_{f,\beta D_1} (R).
\]
Hence for any bounded complete $n$-circular domain $D$,
\[ M_{g,D}^{-1}(M_{f,D_1}(R)) \leq M_{g,D}^{-1}(M_{f,D_2}(R)) \leq M_{g,D}^{-1}(M_{f,D_1}(R)). \tag{2.1} \]
Now for any $\theta > 0$ and any $D$, we get that
\[ M_{f,\theta D}(R) = M_{f,D}(\theta R). \]
Therefore
\[
\lim_{R \to \infty} \frac{\log^{[p]} M_{g,D}^{-1}M_{f,\theta D}(R)}{\log^{[q]} R} = \lim_{R \to \infty} \frac{\log^{[p]} M_{g,D}^{-1}M_{f,D}(\theta R)}{\log^{[q]} R}.
\]
Hence by (2.1) we obtain that
\[
\lim_{R \to \infty} \frac{\log^{[p]} M_{g,D_1}^{-1}M_{f,D_1}(R)}{\log^{[q]} R} = \lim_{R \to \infty} \frac{\log^{[p]} M_{g,D_2}^{-1}M_{f,D_2}(R)}{\log^{[q]} R}.
\]
Similarly one can easily verify that
\[
\lim_{R \to \infty} \frac{\log^{[p]} M_{g,D_1}^{-1}M_{f,D_1}(R)}{\log^{[q]} R} = \lim_{R \to \infty} \frac{\log^{[p]} M_{g,D_2}^{-1}M_{f,D_2}(R)}{\log^{[q]} R}.
\]
Hence the theorem follows. \qed

Since $\rho_{g,D}^{(p,q)}(f)$ and $\lambda_{g,D}^{(p,q)}(f)$ are independent of the choice of the domain $D$, and therefore we write $\rho_{g}^{(p,q)}(f)$ and $\lambda_{g}^{(p,q)}(f)$ instead of $\rho_{g,D}^{(p,q)}(f)$ and $\lambda_{g,D}^{(p,q)}(f)$ respectively.

**Theorem 2.** Let $f(z)$ and $g(z)$ be any two entire functions of $n$-complex variables with index-pair $(m, q)$ and $(m, p)$, respectively, where $m \geq q \geq 1$ and $m \geq p \geq 1$ and $D$ be a bounded complete $n$-circular domain with center at origin in $\mathbb{C}^n$. Then
\[
\frac{\lambda_f(m, q)}{\rho_g(m, p)} \leq \lambda_{g}^{(p,q)}(f) \leq \min \left\{ \frac{\lambda_f(m, q)}{\lambda_g(m, p)}, \frac{\rho_f(m, q)}{\rho_g(m, p)} \right\} \leq \max \left\{ \frac{\lambda_f(m, q)}{\lambda_g(m, p)}, \frac{\rho_f(m, q)}{\rho_g(m, p)} \right\} \leq \rho_{g}^{(p,q)}(f) \leq \frac{\rho_f(m, q)}{\lambda_g(m, p)}.
\]

**Proof.** From the definitions of $\rho_{g}^{(p,q)}(f)$ and $\lambda_{g}^{(p,q)}(f)$ we get that
\[
\log \rho_{g}^{(p,q)}(f) = \lim_{R \to +\infty} \left[ \log^{[p+1]} M_{g,D}^{-1}(R) - \log^{[q+1]} M_{f,D}^{-1}(R) \right], \tag{2.2}
\]
and
\[
\log \lambda_{g,D}^{(p,q)}(f) = \lim_{R \to +\infty} \left[ \log^{[p+1]} M_{g,D}^{-1}(R) - \log^{[q+1]} M_{f,D}^{-1}(R) \right]. \tag{2.3}
\]
Now from the definitions of \( \rho_f(m, q) \) and \( \lambda_f(m, q) \), it follows that
\[
\log \rho_f(m, q) = \lim_{R \to +\infty} \left[ \log^{[m+1]} R - \log^{[q+1]} M_{f,D}^{-1}(R) \right] \quad \text{and} \quad (2.4)
\]
\[
\log \lambda_f(m, q) = \lim_{R \to +\infty} \left[ \log^{[m+1]} R - \log^{[q+1]} M_{f,D}^{-1}(R) \right] . \quad (2.5)
\]

Similarly, from the definitions of \( \rho_g(m, p) \) and \( \lambda_g(m, p) \), we obtain that
\[
\log \rho_g(m, p) = \lim_{R \to +\infty} \left[ \log^{[m+1]} R - \log^{[p+1]} M_{g,D}^{-1}(R) \right] \quad \text{and} \quad (2.6)
\]
\[
\log \lambda_g(m, p) = \lim_{R \to +\infty} \left[ \log^{[m+1]} R - \log^{[p+1]} M_{g,D}^{-1}(R) \right] . \quad (2.7)
\]

Therefore from (2.3), (2.5) and (2.6), we get that
\[
\log \lambda_g^{(p,q)}(f) = \lim_{R \to +\infty} \left[ \log^{[m+1]} R - \log^{[q+1]} M_{f,D}^{-1}(R) - \left( \log^{[m+1]} R - \log^{[p+1]} M_{g,D}^{-1}(R) \right) \right]
\]
i.e., \( \log \lambda_g^{(p,q)}(f) \geq \left[ \lim_{R \to +\infty} \left( \log^{[m+1]} R - \log^{[q+1]} M_{f,D}^{-1}(R) \right) \right]
\]
\[
- \lim_{R \to +\infty} \left( \log^{[m+1]} R - \log^{[p+1]} M_{g,D}^{-1}(R) \right) \right] \quad \text{i.e.,} \quad (2.8)
\]

Similarly, from (2.2), (2.4) and (2.7), it follows that
\[
\log \rho_g^{(p,q)}(f) = \lim_{R \to +\infty} \left[ \log^{[m+1]} R - \log^{[q+1]} M_{f,D}^{-1}(R) - \left( \log^{[m+1]} R - \log^{[p+1]} M_{g,D}^{-1}(R) \right) \right]
\]
i.e., \( \log \rho_g^{(p,q)}(f) \leq \left[ \lim_{R \to +\infty} \left( \log^{[m+1]} R - \log^{[q+1]} M_{f,D}^{-1}(R) \right) \right]
\]
\[
- \lim_{R \to +\infty} \left( \log^{[m+1]} R - \log^{[p+1]} M_{g,D}^{-1}(R) \right) \right] \quad \text{i.e.,} \quad (2.9)
\]

Again, in view of (2.3), (2.4), (2.5), (2.6) and (2.7), we obtain that
\[
\log \lambda_g^{(p,q)}(f) = \lim_{R \to +\infty} \left[ \log^{[m+1]} R - \log^{[q+1]} M_{f,D}^{-1}(R) - \left( \log^{[m+1]} R - \log^{[p+1]} M_{g,D}^{-1}(R) \right) \right]
\]
i.e., \( \log \lambda_g^{(p,q)}(f) \leq \)
\[
\min \left[ \lim_{R \to +\infty} \left( \log^{[m+1]} R - \log^{[q+1]} M_{f,D}^{-1}(R) \right) + \lim_{R \to +\infty} \left( \log^{[m+1]} R - \log^{[q+1]} M_{g,D}^{-1}(R) \right), \right]
\]
\[
\lim_{R \to +\infty} \left( \log^{[m+1]} R - \log^{[q+1]} M_{f,D}^{-1}(R) \right) + \lim_{R \to +\infty} \left( \log^{[m+1]} R - \log^{[q+1]} M_{g,D}^{-1}(R) \right) \right] \quad \text{i.e.,} \quad (2.10)
\]
\[
\min \left[ \lim_{R \to +\infty} \left( \log^{[m+1]} R - \log^{[q+1]} M^{-1}_{f,D}(R) \right) - \lim_{R \to +\infty} \left( \log^{[m+1]} R - \log^{[p+1]} M^{-1}_{g,D}(R) \right) \right],
\]
\[
\lim_{R \to +\infty} \left( \log^{[m+1]} R - \log^{[q+1]} M^{-1}_{f,D}(R) \right) - \lim_{R \to +\infty} \left( \log^{[m+1]} R - \log^{[p+1]} M^{-1}_{g,D}(R) \right)
\]

i.e., \( \log \lambda^{(p,q)}_g(f) \leq \min \{ \log \lambda_f(m,q) - \log \lambda_g(m,p), \log \rho_f(m,q) - \log \rho_g(m,p) \} \).

Further from (2.2), (2.4), (2.5), (2.6) and (2.7), it follows that
\[
\log \rho^{(p,q)}_g(f) = \lim_{R \to +\infty} \left[ \log^{[m+1]} R - \log^{[q+1]} M^{-1}_{f,D}(R) - \left( \log^{[m+1]} R - \log^{[p+1]} M^{-1}_{g,D}(R) \right) \right]
\]

i.e., \( \log \rho^{(p,q)}_g(f) \geq \)
\[
\max \left[ \lim_{R \to +\infty} \left( \log^{[m+1]} R - \log^{[q+1]} M^{-1}_{f,D}(R) \right) + \lim_{R \to +\infty} - \left( \log^{[m+1]} R - \log^{[p+1]} M^{-1}_{g,D}(R) \right) \right],
\]
\[
\lim_{R \to +\infty} \left( \log^{[m+1]} R - \log^{[q+1]} M^{-1}_{f,D}(R) \right) + \lim_{R \to +\infty} - \left( \log^{[m+1]} R - \log^{[p+1]} M^{-1}_{g,D}(R) \right)
\]

i.e., \( \log \rho^{(p,q)}_g(f) \geq \max \{ \log \lambda_f(m,q) - \log \lambda_g(m,p), \log \rho_f(m,q) - \log \rho_g(m,p) \} \).

Thus the theorem follows from (2.8), (2.9), (2.10) and (2.11).

In view of Theorem 2, one can easily verify the following corollaries:

**Corollary 1.** Let \( f(z) \) be an entire function of \( n \)-complex variables with index-pair \((m,q)\) and \( g(z) \) be any two entire functions of \( n \)-complex variables of regular \((m,p)\)-th Gol’derberg growth where \( m \geq q \geq 1 \) and \( m \geq p \geq 1 \) and \( D \) be a bounded complete \( n \)-circular domain with center at origin in \( \mathbb{C}^n \). Then
\[
\lambda^{(p,q)}_g(f) = \frac{\lambda_f(m,q)}{\rho_g(m,p)} \quad \text{and} \quad \rho^{(p,q)}_g(f) = \frac{\rho_f(m,q)}{\rho_g(m,p)}.
\]

Moreover, if \( \rho_f(m,q) = \rho_g(m,p) \), then
\[
\rho^{(p,q)}_g(f) = \lambda^{(p,q)}_f(g) = 1.
\]

**Corollary 2.** Let \( f \) and \( g \) be any two entire functions of \( n \)-complex variables and with regular \((m,q)\)-th Gol’derberg growth and regular \((m,p)\)-th Gol’derberg growth, respectively,
where \( m \geq q \geq 1 \) and \( m \geq p \geq 1 \). Also and \( D \) be a bounded complete \( n \)-circular domain with center at origin in \( \mathbb{C}^n \). Then

\[
\lambda_g^{(p,q)}(f) = \rho_g^{(p,q)}(f) = \frac{\rho_f(m,q)}{\rho_g(m,p)}.
\]

**Corollary 3.** Let \( f \) and \( g \) be any two entire functions of \( n \)-complex variables and with regular \((m,q)\)-th Gol’dberg growth and regular \((m,p)\)-th Gol’dberg growth, respectively, where \( m \geq q \geq 1 \) and \( m \geq p \geq 1 \). Also and \( D \) be a bounded complete \( n \)-circular domain with center at origin in \( \mathbb{C}^n \) and \( \rho_f(m,q) = \rho_g(m,p) \). Then

\[
\lambda_g^{(p,q)}(f) = \rho_g^{(p,q)}(f) = \lambda_f^{(q,p)}(g) = \rho_f^{(q,p)}(g) = 1.
\]

**Corollary 4.** Let \( f \) and \( g \) be any two entire functions of \( n \)-complex variables and with regular \((m,q)\)-th Gol’dberg growth and regular \((m,p)\)-th Gol’dberg growth, respectively, where \( m \geq q \geq 1 \) and \( m \geq p \geq 1 \). Also and \( D \) be a bounded complete \( n \)-circular domain with center at origin in \( \mathbb{C}^n \). Then

\[
\rho_g^{(p,q)}(f).\rho_f^{(q,p)}(g) = \lambda_g^{(p,q)}(f).\lambda_f^{(q,p)}(g) = 1.
\]

**Corollary 5.** Let \( f(z) \) and \( g(z) \) be any two entire functions of \( n \)-complex variables with index-pair \((m,q)\) and \((m,p)\), respectively, where \( m \geq q \geq 1 \) and \( m \geq p \geq 1 \) and \( D \) be a bounded complete \( n \)-circular domain with center at origin in \( \mathbb{C}^n \). If either \( f \) is not of regular \((m,q)\)-th Gol’dberg growth or \( g \) is not of regular \((m,p)\)-th Gol’dberg growth, then

\[
\lambda_g^{(p,q)}(f).\lambda_f^{(q,p)}(g) < 1 < \rho_g^{(p,q)}(f).\rho_f^{(q,p)}(g).
\]

**Corollary 6.** Let \( f(z) \) be an entire function of \( n \)-complex variables with index-pair \((m,q)\) where \( m \geq q \geq 1 \) and \( D \) be a bounded complete \( n \)-circular domain with center at origin in \( \mathbb{C}^n \). Then for any entire function \( g \) of \( n \)-complex variables

(i) \( \lambda_g^{(p,q)}(f) = \infty \) when \( \rho_g(m,p) = 0 \),

(ii) \( \rho_g^{(p,q)}(f) = \infty \) when \( \lambda_g(m,p) = 0 \),

(iii) \( \lambda_g^{(p,q)}(f) = 0 \) when \( \rho_g(m,p) = \infty \)

and

(iv) \( \rho_g^{(p,q)}(f) = 0 \) when \( \lambda_g(m,p) = \infty \),

where \( m \geq p \geq 1 \).

**Corollary 7.** Let \( g(z) \) be an entire function of \( n \)-complex variables with index-pair \((m,p)\) where \( m \geq p \geq 1 \) and \( D \) be a bounded complete \( n \)-circular domain with center at origin in \( \mathbb{C}^n \). Then for any entire function \( f \) of \( n \)-complex variables

(i) \( \rho_f^{(p,q)}(f) = 0 \) when \( \rho_f(m,q) = 0 \),

(ii) \( \lambda_f^{(p,q)}(f) = 0 \) when \( \lambda_f(m,q) = 0 \),

(iii) \( \rho_f^{(p,q)}(f) = \infty \) when \( \rho_f(m,q) = \infty \),

and

(iv) \( \lambda_f^{(p,q)}(f) = \infty \) when \( \lambda_f(m,q) = \infty \),
where \( m \geq q \geq 1 \).

**Remark 1.** From the conclusion of Theorem 2, one may write \( \rho_g^{(p,q)}(f) = \frac{\rho_f(m,q)}{\rho_g(m,p)} \) and \( \lambda_g^{(p,q)}(f) = \frac{\lambda_f(m,q)}{\lambda_g(m,p)} \) when \( g(z) \) be an entire function of \( n \)-complex variables with regular \((m,p)\)-Goldberg growth. Similarly \( \rho_g^{(p,q)}(f) = \frac{\lambda_f(m,q)}{\lambda_g(m,p)} \) and \( \lambda_g^{(p,q)}(f) = \frac{\rho_f(m,q)}{\rho_g(m,p)} \) when \( f(z) \) be an entire function of \( n \)-complex variables with regular \((m,q)\)-Goldberg growth.

When \( f(z) \) and \( g(z) \) are any two entire functions both of \( n \)-complex variables and with index-pair \((m,q)\) and \((n,p)\), respectively, where \( m \geq q + 1 \geq 1 \) and \( n \geq p + 1 \geq 1 \), but \( m \neq n \), the next definition enables us to study their relative order for any bounded complete \( n \)-circular domain \( D \) with center at origin in \( \mathbb{C}^n \).

**Definition 11.** Let \( f(z) \) and \( g(z) \) be any two entire functions of \( n \)-complex variables with index-pair \((m,q)\) and \((n,p)\), respectively, where \( m \geq q \geq 1 \) and \( n \geq p \geq 1 \) and \( D \) be a bounded complete \( n \)-circular domain with center at origin in \( \mathbb{C}^n \). then the \((p + m - n, q)\)-th relative Goldberg order (resp. \((p + m - n, q)\)-th relative Goldberg lower order) of \( f(z) \) with respect to \( g(z) \) is defined as

\[
(i) \quad \rho_g^{(p+m-n,q)}(f) = \lim_{{R \to +\infty}} \frac{\log^{[p+m-n]} M^{-1}_{g,D} M^{1}_{f,D}(R)}{\log^{[q]} R},
\]

\[
(resp. \quad \lambda_g^{(p+m-n,q)}(f) = \lim_{{R \to +\infty}} \frac{\log^{[p+m-n]} M^{-1}_{g,D} M^{1}_{f,D}(R)}{\log^{[q]} R}).
\]

If \( m < n \), then the \((p,q + n - m)\)-th relative Goldberg order (resp. \((p,q + n - m)\)-th relative Goldberg lower order) of \( f(z) \) with respect to \( g(z) \) is defined as

\[
(ii) \quad \rho_g^{(p,q+n-m)}(f) = \lim_{{R \to +\infty}} \frac{\log^{[p]} M^{-1}_{g,D} M^{1}_{f,D}(R)}{\log^{[q+n-m]} R},
\]

\[
(resp. \quad \lambda_g^{(p,q+n-m)}(f) = \lim_{{R \to +\infty}} \frac{\log^{[p]} M^{-1}_{g,D} M^{1}_{f,D}(R)}{\log^{[q+n-m]} R}).
\]

Move to the left.

**Theorem 3.** Under the hypothesis of Definition 11 for \( m > n \):

\[
(i) \quad \rho_g^{(p+m-n,q)}(f) = \lim_{{R \to +\infty}} \frac{\log^{[m]} M_{f,D}(R)}{\log^{[q]} R},
\]

\[
\lambda_g^{(p+m-n,q)}(f) = \lim_{{R \to +\infty}} \frac{\log^{[m]} M_{f,D}(R)}{\log^{[q]} R}.
\]

and for \( m < n \):

\[
(ii) \quad \rho_g^{(p,q+n-m)}(f) = \lim_{{R \to +\infty}} \frac{\log^{[p]} R}{\log^{[n]} M_{g,D}(R)},
\]

\[
\lambda_g^{(p,q+n-m)}(f) = \lim_{{R \to +\infty}} \frac{\log^{[p]} R}{\log^{[n]} M_{g,D}(R)}.
\]
In the next theorem we intend to find out \((p, q)\)-th relative Gol'dberg order (resp. \((p, q)\)-th relative Gol'dberg lower order) of an entire function \(f(z)\) with respect to another entire function \(g(z)\) (both \(f(z)\) and \(g(z)\) are of \(n\)-complex variables) when \((m, q)\)-th relative Gol'dberg order (resp. \((m, q)\)-th relative Gol'dberg lower order) of \(f(z)\) and \((m, p)\)-th relative Gol'dberg order (resp. \((m, p)\)-th relative Gol'dberg lower order) of \(g(z)\) with respect to another entire function \(h(z)\) (\(h(z)\) is also of \(n\)-complex variables) are given where \(p, q\) and \(m\) are any positive integers.

**Theorem 4.** Let \(f(z)\), \(g(z)\) and \(h(z)\) be any three entire functions of \(n\)-complex variables and \(D\) be a bounded complete \(n\)-circular domain with center at origin in \(\mathbb{C}^n\). Also let \(m, p, q\) are any three positive integers. If \((m, q)\)-th relative Gol'dberg order (resp. \((m, q)\)-th relative Gol'dberg lower order) of \(f(z)\) with respect to \(h(z)\) and \((m, p)\)-th relative Gol'dberg order (resp. \((m, p)\)-th relative Gol'dberg lower order) of \(g(z)\) with respect to \(h(z)\) are respectively denoted by \(\lambda_h^{(m,q)}(f)\) (resp. \(\lambda_h^{(m,q)}(f)\)) and \(\rho_h^{(m,p)}(g)\) (resp. \(\lambda_h^{(m,p)}(g)\)), then

\[
\frac{\lambda_h^{(m,q)}(f)}{\rho_h^{(m,p)}(g)} \leq \frac{\lambda_g^{(p,q)}(f)}{\rho_h^{(m,p)}(g)} \leq \min \left\{ \frac{\lambda_h^{(m,q)}(f)}{\lambda_h^{(m,p)}(g)}, \frac{\rho_h^{(m,q)}(f)}{\rho_h^{(m,p)}(g)} \right\} \\
\leq \max \left\{ \frac{\lambda_h^{(m,q)}(f)}{\lambda_h^{(m,p)}(g)}, \frac{\rho_h^{(m,q)}(f)}{\rho_h^{(m,p)}(g)} \right\} \leq \frac{\rho_h^{(m,q)}(f)}{\lambda_h^{(m,p)}(g)}.
\]

The conclusion of the above theorem can be carried out after applying the same technique of Theorem 2 and therefore its proof is omitted.

In view of Theorem 4 one can easily verify the following corollaries:

**Corollary 8.** Let \(f(z)\), \(g(z)\) and \(h(z)\) be any three entire functions of \(n\)-complex variables and \(D\) be a bounded complete \(n\)-circular domain with center at origin in \(\mathbb{C}^n\). Also let \(f(z)\) be an entire function with regular relative \((m, q)\)-Gol'dberg growth with respect to entire function \(h(z)\) and \(g(z)\) be entire having relative index-pair \((m, p)\) with respect to another entire function \(h(z)\) where \(m, p, q\) are any three positive integers. Then

\[
\lambda_g^{(p,q)}(f) = \frac{\rho_h^{(m,q)}(f)}{\rho_h^{(m,p)}(g)} \quad \text{and} \quad \rho_g^{(p,q)}(f) = \frac{\rho_h^{(m,q)}(f)}{\lambda_h^{(m,p)}(g)}.
\]

In addition, if \(\rho_h^{(m,q)}(f) = \rho_h^{(m,p)}(g)\), then

\[
\lambda_g^{(p,q)}(f) = \rho_f^{(p,q)}(g) = 1.
\]

**Corollary 9.** Let \(f(z)\), \(g(z)\) and \(h(z)\) be any three entire functions of \(n\)-complex variables and \(D\) be a bounded complete \(n\)-circular domain with center at origin in \(\mathbb{C}^n\). Also let \(f(z)\) be an entire function with relative index-pair \((m, q)\) with respect to entire function \(h(z)\) and \(g(z)\) be entire of regular relative \((m, p)\)-Gol’dberg growth with respect to another entire function \(h(z)\) where \(m, p, q\) are any three positive integers. Then

\[
\lambda_g^{(p,q)}(f) = \frac{\lambda_h^{(m,q)}(f)}{\rho_h^{(m,p)}(g)} \quad \text{and} \quad \rho_g^{(p,q)}(f) = \frac{\rho_h^{(m,q)}(f)}{\rho_h^{(m,p)}(g)}.
\]
In addition, if $\rho_{h}^{(m,q)} (f) = \rho_{h}^{(m,p)} (g)$, then
\[
\rho_{g}^{(p,q)} (f) = \lambda_{f}^{(q,p)} (g) = 1.
\]

**Corollary 10.** Let $f (z)$, $g (z)$ and $h (z)$ be any three entire functions of $n$-complex variables and $D$ be a bounded complete $n$-circular domain with center at origin in $\mathbb{C}^{n}$. Also let $f (z)$ and $g (z)$ be any two entire functions with regular relative $(m,q)$-Gol’dberg growth and regular relative $(m,p)$-Gol’dberg growth with respect to entire function $h (z)$ respectively where $m,p,q$ are any three positive integers. Then
\[
\lambda_{g}^{(p,q)} (f) = \rho_{g}^{(p,q)} (f) = \lambda_{f}^{(q,p)} (g) = \rho_{f}^{(q,p)} (g) = 1
\]
if $\rho_{h}^{(m,q)} (f) = \rho_{h}^{(m,p)} (g)$.

**Corollary 11.** Let $f (z)$, $g (z)$ and $h (z)$ be any three entire functions of $n$-complex variables and $D$ be a bounded complete $n$-circular domain with center at origin in $\mathbb{C}^{n}$. Also let $f (z)$ and $g (z)$ be any two entire functions with relative index-pairs $(m,q)$ and $(m,p)$ with respect to entire function $h (z)$ respectively where $m,p,q$ are any three positive integers and either $f (z)$ is not of regular relative $(m,q)$ - Gol’dberg growth or $g (z)$ is not of regular relative $(m,p)$ - Gol’dberg growth, then
\[
\rho_{g}^{(p,q)} (f) . \rho_{f}^{(q,p)} (g) \geq 1.
\]
If $f (z)$ and $g (z)$ are both of regular relative $(m,q)$ - Gol’dberg growth and regular relative $(m,p)$ - Gol’dberg growth with respect to entire function $h (z)$ respectively, then
\[
\rho_{g}^{(p,q)} (f) . \rho_{f}^{(q,p)} (g) = 1.
\]

**Corollary 12.** Let $f (z)$, $g (z)$ and $h (z)$ be any three entire functions of $n$-complex variables and $D$ be a bounded complete $n$-circular domain with center at origin in $\mathbb{C}^{n}$. Also let $f (z)$ and $g (z)$ be any two entire functions with relative index-pairs $(m,q)$ and $(m,p)$ with respect to entire function $h (z)$ respectively where $m,p,q$ are any three positive integers and either $f (z)$ is not of regular relative $(m,q)$ - Gol’dberg growth or $g (z)$ is not of regular relative $(m,p)$ - Gol’dberg growth, then
\[
\lambda_{g}^{(p,q)} (f) . \lambda_{f}^{(q,p)} (g) \leq 1.
\]
If $f (z)$ and $g (z)$ are both of regular relative $(m,q)$ - Gol’dberg growth and regular relative $(m,p)$ - Gol’dberg growth with respect to entire function $h (z)$ respectively, then
\[
\lambda_{g}^{(p,q)} (f) . \lambda_{f}^{(q,p)} (g) = 1.
\]
Corollary 14. Let \( f (z) \), \( g(z) \) and \( h(z) \) be any three entire functions of \( n \)-complex variables and \( D \) be a bounded complete \( n \)-circular domain with center at origin in \( \mathbb{C}^n \). Also let \( f (z) \) be an entire function with relative index-pair \((m, q)\), Then

\[
\begin{align*}
(i) \quad & \lambda_g^{(p,q)}(f) = \infty \text{ when } \rho_h^{(m,p)}(g) = 0 , \\
(ii) \quad & \rho_g^{(p,q)}(f) = \infty \text{ when } \lambda_h^{(m,p)}(g) = 0 , \\
(iii) \quad & \lambda_g^{(p,q)}(f) = 0 \text{ when } \rho_h^{(m,p)}(g) = \infty
\end{align*}
\]

and

\[
(iv) \quad \rho_g^{(p,q)}(f) = 0 \text{ when } \lambda_h^{(m,p)}(g) = \infty,
\]

where \( m, p, q \) are any three positive integers.

Corollary 15. Let \( f (z) \), \( g(z) \) and \( h(z) \) be any three entire functions of \( n \)-complex variables and \( D \) be a bounded complete \( n \)-circular domain with center at origin in \( \mathbb{C}^n \). Also let \( g(z) \) be an entire function with relative index-pair \((m, p)\), Then

\[
\begin{align*}
(i) \quad & \rho_g^{(p,q)}(f) = 0 \text{ when } \rho_h^{(m,q)}(f) = 0 , \\
(ii) \quad & \lambda_g^{(p,q)}(f) = 0 \text{ when } \lambda_h^{(m,q)}(f) = 0 , \\
(iii) \quad & \rho_g^{(p,q)}(f) = \infty \text{ when } \rho_h^{(m,q)}(f) = \infty
\end{align*}
\]

and

\[
(iv) \quad \lambda_g^{(p,q)}(f) = \infty \text{ when } \lambda_h^{(m,q)}(f) = \infty,
\]

where \( m, p, q \) are any three positive integers.

Remark 2. Under the same conditions of Theorem 4, one may write \( \rho_g^{(p,q)}(f) \) and \( \lambda_g^{(p,q)}(f) \) when \( \lambda_h^{(m,p)}(g) = \rho_h^{(m,p)}(g) \). Similarly \( \rho_g^{(p,q)}(f) = \rho_h^{(m,p)}(f) \) and \( \lambda_g^{(p,q)}(f) = \rho_h^{(m,p)}(f) \) when \( \lambda_h^{(m,q)}(f) = \rho_h^{(m,q)}(f) \).

Next we prove our theorem based on \((p, q)\)-th relative Gol'dberg type and \((p, q)\)-th relative Gol'dberg weak type of entire functions of \( n \)-complex variables

Theorem 5. Let \( f (z) \) and \( g(z) \) be any two entire functions of \( n \)-complex variables with index-pair \((m, q)\) and \((m, p)\), respectively, where \( m \geq q \geq 1 \) and \( m \geq p \geq 1 \) and \( D \) be a bounded complete \( n \)-circular domain with center at origin in \( \mathbb{C}^n \). Then

\[
\max \left\{ \frac{\Delta_{f,D}(m,q)}{\tau_{g,D}(m,p)} \right\} \lambda_g^{(m,p)} = \left[ \frac{\Delta_{f,D}(m,q)}{\tau_{g,D}(m,p)} \right] \lambda_g^{(m,p)} \leq \Delta_{f,D}(m,q) \rho_h^{(m,q)} \leq \left[ \frac{\Delta_{f,D}(m,q)}{\tau_{g,D}(m,p)} \right] \rho_h^{(m,p)}
\]

Proof. From the definitions of \( \Delta_{f,D}(m,q) \) and \( \overline{\Delta}_{f,D}(m,q) \), we have for all sufficiently large values of \( R \) that

\[
\begin{align*}
M_{f,D}(R) & \leq \exp^{[m-1]} \left[ (\Delta_{f,D}(m,q) + \varepsilon) \left[ \log^{[q-1]} R \right]^{\rho_f^{(m,q)}} \right], \\
M_{f,D}(R) & \geq \exp^{[m-1]} \left[ (\overline{\Delta}_{f,D}(m,q) - \varepsilon) \left[ \log^{[q-1]} R \right]^{\rho_f^{(m,q)}} \right]
\end{align*}
\]
and also for a sequence of values of $R$ tending to infinity, we get that
\begin{align}
M_{f,D}(R) & \geq \exp^{[m-1]} \left[ (\Delta_{f,D}(m,q) - \varepsilon) \left[ \log^{[q-1]} R \right]^{\rho_f(m,q)} \right], \quad (2.14) \\
M_{f,D}(R) & \leq \exp^{[m-1]} \left[ (\Delta_{f,D}(m,q) + \varepsilon) \left[ \log^{[q-1]} R \right]^{\rho_f(m,q)} \right]. \quad (2.15)
\end{align}

Similarly from the definitions of $\Delta_{g,D}(m,p)$ and $\Delta_{g,D}(m,p)$, it follows for all sufficiently large values of $R$ that
\begin{align}
M_{g,D}(R) & \leq \exp^{[m-1]} \left[ (\Delta_{g,D}(m,p) + \varepsilon) \left[ \log^{[p-1]} R \right]^{\rho_g(m,p)} \right] \quad i.e., \quad R \leq M_{g,D}^{-1} \left[ \exp^{[m-1]} \left[ (\Delta_{g,D}(m,p) + \varepsilon) \left[ \log^{[p-1]} R \right]^{\rho_g(m,p)} \right] \right] \quad (2.16) \\
M_{g,D}(R) & \leq \exp^{[p-1]} \left[ \log^{[m-1]} R \left( \frac{1}{\Delta_{g,D}(m,p) - \varepsilon} \right) \right] \quad (2.17)
\end{align}

Also for a sequence of values of $R$ tending to infinity, we obtain that
\begin{align}
M_{g,D}(R) & \leq \exp^{[p-1]} \left[ \log^{[m-1]} R \left( \frac{1}{\Delta_{g,D}(m,p) - \varepsilon} \right) \right] \quad (2.18) \\
M_{g,D}(R) & \geq \exp^{[p-1]} \left[ \log^{[m-1]} R \left( \frac{1}{\Delta_{g,D}(m,p) + \varepsilon} \right) \right] \quad (2.19)
\end{align}

From the definitions of $\tau_{f,D}(m,q)$ and $\tau_{f,D}(m,q)$, we have for all sufficiently large values of $R$ that
\begin{align}
M_{f,D}(R) & \leq \exp^{[m-1]} \left[ (\tau_{f,D}(m,q) - \varepsilon) \left[ \log^{[q-1]} R \right]^{\lambda_f(m,q)} \right], \quad (2.20) \\
M_{f,D}(R) & \geq \exp^{[m-1]} \left[ (\tau_{f,D}(m,q) + \varepsilon) \left[ \log^{[q-1]} R \right]^{\lambda_f(m,q)} \right] \quad (2.21)
\end{align}

and also for a sequence of values of $R$ tending to infinity, we get that
\begin{align}
M_{f,D}(R) & \geq \exp^{[m-1]} \left[ (\tau_{f,D}(m,q) - \varepsilon) \left[ \log^{[q-1]} R \right]^{\lambda_f(m,q)} \right], \quad (2.22) \\
M_{f,D}(R) & \leq \exp^{[m-1]} \left[ (\tau_{f,D}(m,q) + \varepsilon) \left[ \log^{[q-1]} R \right]^{\lambda_f(m,q)} \right]. \quad (2.23)
\end{align}
Similarly from the definitions of $\tau_{g,D}(m,p)$ and $\tau_{g,D}(m,p)$, it follows for all sufficiently large values of $R$ that

$$M_{g,D}(R) \leq \exp^{[m-1]} \left[ (\tau_{g,D}(m,p) + \epsilon) \left[ \log^{[p-1]} \frac{R}{\lambda_g(m,p)} \right] \right]$$

i.e.,

$$R \leq M_{g,D}^{-1} \left( \exp^{[m-1]} \left[ (\tau_{g,D}(m,p) + \epsilon) \left[ \log^{[p-1]} \frac{R}{\lambda_g(m,p)} \right] \right] \right)$$

i.e., $M_{g,D}^{-1}(R) \geq \left[ \exp^{[p-1]} \frac{\log^{[m-1]} R}{\tau_{g,D}(m,p) + \epsilon} \right] \frac{1}{\lambda_g(m,p)}$ and (2.24)

$$M_{g,D}(R) \leq \left[ \exp^{[p-1]} \frac{\log^{[m-1]} R}{\tau_{g,D}(m,p) - \epsilon} \right] \frac{1}{\lambda_g(m,p)}.$$  (2.25)

Also for a sequence of values of $R$ tending to infinity, we obtain that

$$M_{g,D}^{-1}(R) \leq \left[ \exp^{[p-1]} \frac{\log^{[m-1]} R}{\tau_{g,D}(m,p) - \epsilon} \right] \frac{1}{\lambda_g(m,p)}$$

and (2.26)

$$M_{g,D}^{-1}(R) \geq \left[ \exp^{[p-1]} \frac{\log^{[m-1]} R}{\tau_{g,D}(m,p) + \epsilon} \right] \frac{1}{\lambda_g(m,p)}.$$  (2.27)

Now from (2.14) and in view of (2.24), we get for a sequence of values of $R$ tending to infinity that

$$M_{g,D}^{-1}M_{f,D}(R) \geq M_{g,D}^{-1} \left[ \exp^{[m-1]} \left[ (\Delta_{f,D}(m,q) - \epsilon) \left[ \log^{[q-1]} \frac{R}{\lambda_f(m,q)} \right] \right] \right]$$

i.e.,

$$M_{g,D}^{-1}M_{f,D}(R) \geq \left[ \exp^{[p-1]} \frac{\log^{[m-1]} \left( \Delta_{f,D}(m,q) - \epsilon \right) \left[ \log^{[q-1]} \frac{R}{\lambda_f(m,q)} \right]}{\tau_{g,D}(m,p) + \epsilon} \right] \frac{1}{\lambda_g(m,p)} \cdot \left[ \log^{[q-1]} \frac{R}{\lambda_f(m,q)} \right] \frac{1}{\lambda_g(m,p)}.$$

i.e.,

$$\log^{[p-1]} M_{g,D}^{-1}M_{f,D}(R) \geq \left[ \frac{\Delta_{f,D}(m,q) - \epsilon}{\tau_{g,D}(m,p) + \epsilon} \right] \frac{1}{\lambda_g(m,p)} \cdot \left[ \log^{[q-1]} \frac{R}{\lambda_f(m,q)} \right] \frac{1}{\lambda_g(m,p)}.$$
Since in view of Theorem 2 \( \frac{\rho_f(m,q)}{\chi_g(m,p)} \geq \frac{\rho_g(p,q)}{\chi_f(p,q)}(f) \) and as \( \varepsilon > 0 \) is arbitrary, therefore it follows from above that
\[
\lim_{R \to +\infty} \frac{\log^{[p-1]} M_{g,D}^{-1} M_{f,D}(R)}{\log^{[q-1]} R} \geq \left[ \Delta_{f,D}(m,q) \frac{1}{\tau_{g,D}(m,p)} \right] \frac{1}{\chi_g(m,p)}
\]
\( i.e., \Delta_{g,D}^{(p,q)}(f) \geq \left[ \Delta_{f,D}(m,q) \frac{1}{\tau_{g,D}(m,p)} \right] \frac{1}{\chi_g(m,p)}. \quad (2.28) \)

Similarly from (2.13) and in view of (2.27), it follows for a sequence of values of \( R \) tending to infinity that
\[
M_{g,D}^{-1} M_{f,D}(R) \geq M_{g,D}^{-1} \left[ \exp^{[m-1]} \left( \Delta_{f,D}(m,q) - \varepsilon \right) \frac{1}{\log^{[q-1]} R} \rho_f(m,q) \right] \]
\( i.e., M_{g,D}^{-1} M_{f,D}(R) \geq \left[ \exp^{[p-1]} \left( \log^{[m-1]} \exp^{[m-1]} \left( \frac{1}{\tau_{g,D}(m,p)} \right) \right) \right] \frac{1}{\chi_g(m,p)} \)
\( i.e., \log^{[p-1]} M_{g,D}^{-1} M_{f,D}(R) \geq \left[ \frac{\Delta_{f,D}(m,q) - \varepsilon}{\tau_{g,D}(m,p) + \varepsilon} \right] \frac{1}{\chi_g(m,p)} \cdot \log^{[q-1]} R \frac{\rho_f(m,q)}{\chi_g(m,p)}. \quad (2.29) \)

Since in view of Theorem 2 it follows that \( \frac{\rho_f(m,q)}{\chi_g(m,p)} \geq \frac{\rho_g(p,q)}{\chi_f(p,q)}(f) \). Also \( \varepsilon > 0 \) is arbitrary, so we get from above that
\[
\lim_{R \to +\infty} \frac{\log^{[p-1]} M_{g,D}^{-1} M_{f,D}(R)}{\log^{[q-1]} R} \geq \left[ \Delta_{f,D}(m,q) \frac{1}{\tau_{g,D}(m,p)} \right] \frac{1}{\chi_g(m,p)}
\]
\( i.e., \Delta_{g,D}^{(p,q)}(f) \geq \left[ \Delta_{f,D}(m,q) \frac{1}{\tau_{g,D}(m,p)} \right] \frac{1}{\chi_g(m,p)}. \quad (2.29) \)

Again in view of (2.17), we have from (2.12) for all sufficiently large values of \( R \) that
\[
M_{g,D}^{-1} M_{f,D}(R) \leq M_{g,D}^{-1} \left[ \exp^{[m-1]} \left( \Delta_{f,D}(m,q) + \varepsilon \right) \frac{1}{\log^{[q-1]} R} \rho_f(m,q) \right] \]
\( i.e., M_{g,D}^{-1} M_{f,D}(R) \leq \left[ \exp^{[p-1]} \left( \log^{[m-1]} \exp^{[m-1]} \left( \frac{1}{\tau_{g,D}(m,p)} \right) \right) \right] \frac{1}{\chi_g(m,p)} \)
\( i.e., \Delta_{g,D}^{(p,q)}(f) \leq \left[ \frac{\Delta_{f,D}(m,q) + \varepsilon}{\tau_{g,D}(m,p)} \right] \frac{1}{\chi_g(m,p)}. \quad (2.29) \)
i.e., \( \log^{[p]} M_{g,D}^{-1} M_{f,D}(R) \leq \left[ \frac{\Delta_{f,D}(m,q) + \varepsilon}{\Delta_{g,D}(m,p) - \varepsilon} \right] \rho_{g(m,p)}^{1} \cdot \left[ \log^{[q]} R \right] \frac{\rho_{f(m,q)}}{\rho_{g(m,p)}} \). (2.30)

As in view of Theorem 2 it follows that \( \frac{\rho_{f(m,q)}}{\rho_{g(m,p)}} \leq \rho_{g(m,p)}^{(p,q)}(f) \) Since \( \varepsilon (>0) \) is arbitrary, we get from (2.30) that

\[
\lim_{R \to +\infty} \frac{\log^{[p]} M_{g,D}^{-1} M_{f,D}(R)}{\left[ \log^{[q]} R \right] \rho_{g(m,p)}^{(p,q)}(f)} \leq \left[ \frac{\Delta_{f,D}(m,q)}{\Delta_{g,D}(m,p)} \right] \frac{\rho_{g(m,p)}}{\rho_{g(m,p)}} \]

i.e., \( \Delta_{g,D}^{(p,q)}(f) \leq \left[ \frac{\Delta_{f,D}(m,q)}{\Delta_{g,D}(m,p)} \right] \frac{\rho_{g(m,p)}}{\rho_{g(m,p)}} \). (2.31)

Thus the theorem follows from (2.28), (2.29) and (2.31).

The conclusion of the following corollary can be carried out from (2.17) and (2.20); (2.20) and (2.25) respectively after applying the same technique of Theorem 5 and with the help of Theorem 2 Therefore its proof is omitted.

**Corollary 16.** Let \( f(z) \) and \( g(z) \) be any two entire functions of \( n \) complex variables with index-pair \( (m,q) \) and \( (m,p) \), respectively, where \( m \geq q \geq 1 \) and \( m \geq p \geq 1 \) and \( D \) be a bounded complete \( n \)-circular domain with center at origin in \( \mathbb{C}^{n} \). Then

\[
\Delta_{g,D}^{(p,q)}(f) \leq \min \left\{ \left[ \frac{\tau_{f,D}(m,q)}{\tau_{g,D}(m,p)} \right] \lambda_{g(m,p)}^{1} , \left[ \frac{\tau_{f,D}(m,q)}{\Delta_{g,D}(m,p)} \right] \frac{\rho_{g(m,p)}}{\rho_{g(m,p)}} \right\} .
\]

Similarly in the line of Theorem 5 and with the help of Theorem 2 one may easily carried out the following theorem from pairwise inequalities numbers (2.21) and (2.24); (2.18) and (2.20), (2.17) and (2.23) respectively and therefore its proofs is omitted:

**Theorem 6.** Let \( f(z) \) and \( g(z) \) be any two entire functions of \( n \) complex variables with index-pair \( (m,q) \) and \( (m,p) \), respectively, where \( m \geq q \geq 1 \) and \( m \geq p \geq 1 \) and \( D \) be a bounded complete \( n \)-circular domain with center at origin in \( \mathbb{C}^{n} \). Then

\[
\frac{\tau_{f,D}(m,q)}{\tau_{g,D}(m,p)} \left[ \frac{\rho_{g(m,p)}}{\rho_{g(m,p)}} \right] \leq \tau_{g,D}^{(p,q)}(f) \leq \min \left\{ \left[ \frac{\tau_{f,D}(m,q)}{\Delta_{g,D}(m,p)} \right] \frac{\rho_{g(m,p)}}{\rho_{g(m,p)}} , \left[ \frac{\tau_{f,D}(m,q)}{\Delta_{g,D}(m,p)} \right] \frac{\rho_{g(m,p)}}{\rho_{g(m,p)}} \right\} .
\]

**Corollary 17.** Let \( f(z) \) and \( g(z) \) be any two entire functions of \( n \) complex variables with index-pair \( (m,q) \) and \( (m,p) \), respectively, where \( m \geq q \geq 1 \) and \( m \geq p \geq 1 \) and \( D \) be a bounded complete \( n \)-circular domain with center at origin in \( \mathbb{C}^{n} \). Then

\[
\tau_{g,D}^{(p,q)}(f) \geq \max \left\{ \left[ \frac{\Delta_{f,D}(m,q)}{\Delta_{g,D}(m,p)} \right] \frac{\rho_{g(m,p)}}{\rho_{g(m,p)}} , \left[ \frac{\Delta_{f,D}(m,q)}{\Delta_{g,D}(m,p)} \right] \frac{\rho_{g(m,p)}}{\rho_{g(m,p)}} \right\} .
\]

With the help of Theorem 2, the conclusion of the above corollary can be carry out from (2.13), (2.16) and (2.13), (2.21) respectively after applying the same technique of Theorem 5 and therefore its proof is omitted.
Theorem 7. Let \( f(z) \) and \( g(z) \) be any two entire functions of \( n \)-complex variables with index-pair \((m, q)\) and \((m, p)\), respectively, where \( m \geq q \geq 1 \) and \( m \geq p \geq 1 \) and \( D \) be a bounded complete \( n \)-circular domain with center at origin in \( \mathbb{C}^n \). Then

\[
\frac{\Delta_{f,D}(m, q)}{\Delta_{g,D}(m, p)} \leq \mathfrak{g}_{(p,q)}(f) \leq \min \left\{ \frac{\Delta_{f,D}(m, q)}{\Delta_{g,D}(m, p)}, \frac{\Delta_{f,D}(m, q)}{\mathfrak{g}_{(p,q)}(f)} \right\}.
\]

Proof. From (2.13) and in view of (2.24), we get for all sufficiently large values of \( R \)

\[
M_{g,D}^{-1}M_{f,D}(R) \geq M_{g,D}^{-1}\left[ \exp^{m-1}\left[ \Delta_{f,D}(m, q) \right] \log^{[q-1]} R \right]^{\rho_f(m,q)}
\]

i.e., \( M_{g,D}^{-1}M_{f,D}(R) \geq \left[ \frac{\Delta_{f,D}(m, q)}{\mathfrak{g}_{(p,q)}(f)} \right]^{\frac{1}{\mathfrak{g}_{(p,q)}(f)}} \left[ \log^{[q-1]} R \right]^{\rho_f(m,q)} \).

Now in view of Theorem 2, it follows that \( \frac{\rho_f(m,q)}{\mathfrak{g}_{(p,q)}(f)} \geq \frac{\rho_f(m,q)}{\mathfrak{g}_{(p,q)}(f)} \). Since \( \varepsilon > 0 \) is arbitrary, we get from above that

\[
\lim_{R \to +\infty} \frac{\log^{[p-1]} M_{g,D}^{-1}M_{f,D}(R)}{\log^{[q-1]} R} \geq \left[ \frac{\Delta_{f,D}(m, q)}{\mathfrak{g}_{(p,q)}(f)} \right]^{\frac{1}{\mathfrak{g}_{(p,q)}(f)}} \left[ \log^{[q-1]} R \right]^{\rho_f(m,q)} \]

i.e., \( \Delta_{g,D}(f) \geq \left[ \frac{\Delta_{f,D}(m, q)}{\mathfrak{g}_{(p,q)}(f)} \right]^{\frac{1}{\mathfrak{g}_{(p,q)}(f)}} \left[ \log^{[q-1]} R \right]^{\rho_f(m,q)} \). (2.32)

Further in view of (2.18), we get from (2.12) for a sequence of values of \( R \) tending to infinity that

\[
M_{g,D}^{-1}M_{f,D}(R) \leq M_{g,D}^{-1}\left[ \exp^{m-1}\left[ \Delta_{f,D}(m, q) + \varepsilon \right] \log^{[q-1]} R \right]^{\rho_f(m,q)}
\]

i.e., \( M_{g,D}^{-1}M_{f,D}(R) \leq \left[ \frac{\Delta_{f,D}(m, q) + \varepsilon}{\mathfrak{g}_{(p,q)}(f) \left( \Delta_{g,D}(m, p) - \varepsilon \right)} \right]^{\frac{1}{\mathfrak{g}_{(p,q)}(f)}} \left[ \log^{[q-1]} R \right]^{\rho_f(m,q)} \). (2.33)
Again as in view of Theorem\textsuperscript{2} \(\rho_{f}^{(m,q)} \leq \rho_{g}^{(p,q)} (f)\) and \(\varepsilon (>0)\) is arbitrary, therefore we get from (2.33) that

\[
\lim_{R \to +\infty} \frac{\log^{[p-1]} M_{g,D}^{1} M_{f,D} (R)}{[\log^{[q-1]} R]} \leq \left[ \frac{\Delta_{f,D} (m, q)}{\Delta_{g,D} (m, p)} \right]^{\rho_{g}(m,p)}^{1/\rho_{g}(m,p)}
\]

i.e., \(\Delta_{g,D}^{(p,q)} (f) \leq \left[ \frac{\Delta_{f,D} (m, q)}{\Delta_{g,D} (m, p)} \right]^{\rho_{g}(m,p)}^{1/\rho_{g}(m,p)} \)  \hspace{1cm} (2.34)

Likewise from (2.15) and in view of (2.17), it follows for a sequence of values of \(R\) tending to infinity that

\[
M_{g,D}^{-1} M_{f,D} (R) \leq \exp^{[m-1]} \left[ (\Delta_{f,D} (m, q) + \varepsilon) \right] \left[ \log^{[q-1]} R \right]^{\rho_{f}(m,q)}
\]

i.e., \(M_{g,D}^{-1} M_{f,D} (R) \leq \left[ \frac{\Delta_{f,D} (m, q) + \varepsilon}{\Delta_{g,D} (m, p) - \varepsilon} \right]^{\rho_{g}(m,p)}^{1/\rho_{g}(m,p)} \exp^{[m-1]} \left[ \log^{[q-1]} R \right]^{\rho_{f}(m,q)} \) \hspace{1cm} (2.35)

Analogously, we get from (2.35) that

\[
\lim_{r \to +\infty} \frac{\log^{[p-1]} M_{g,D}^{1} M_{f,D} (R)}{[\log^{[q-1]} R]} \leq \left[ \frac{\Delta_{f,D} (m, q)}{\Delta_{g,D} (m, p)} \right]^{\rho_{g}(m,p)}^{1/\rho_{g}(m,p)}
\]

i.e., \(\Delta_{g,D}^{(p,q)} (f) \leq \left[ \frac{\Delta_{f,D} (m, q)}{\Delta_{g,D} (m, p)} \right]^{\rho_{g}(m,p)}^{1/\rho_{g}(m,p)} \) \hspace{1cm} (2.36)

since in view of Theorem\textsuperscript{2} \(\rho_{f}^{(m,q)} \leq \rho_{g}^{(p,q)} (f)\) and \(\varepsilon (>0)\) is arbitrary.

Thus the theorem follows from (2.32), (2.34) and (2.36). \(\Box\)

**Corollary 18.** Let \(f (z)\) and \(g (z)\) be any two entire functions of \(n\)-complex variables with index-pair \((m, q)\) and \((m, p)\), respectively, where \(m \geq q \geq 1\) and \(m \geq p \geq 1\) and \(D\) be a bounded complete \(n\)-circular domain with center at origin in \(\mathbb{C}^n\). Then

\[
\Delta_{g,D}^{(p,q)} (f) \leq \min \left\{ \left[ \frac{\tau_{f,D} (m, q)}{\tau_{g,D} (m, p)} \right]^{\rho_{g}(m,p)}^{1/\rho_{g}(m,p)}, \left[ \frac{\varphi_{f,D} (m, q)}{\varphi_{g,D} (m, p)} \right]^{\rho_{g}(m,p)}^{1/\rho_{g}(m,p)}, \left[ \frac{\sigma_{f,D} (m, q)}{\sigma_{g,D} (m, p)} \right]^{\rho_{g}(m,p)}^{1/\rho_{g}(m,p)}, \left[ \frac{\tau_{f,D} (m, q)}{\tau_{g,D} (m, p)} \right]^{\rho_{g}(m,p)}^{1/\rho_{g}(m,p)} \right\}.
\]
The conclusion of the above corollary can be carried out from pairwise inequalities no (2.17) and (2.23); (2.18) and (2.20); (2.22) and (2.25); (2.21) and (2.26) respectively after applying the same technique of Theorem 7 and with the help of Theorem 2. Therefore its proof is omitted.

Similarly in the line of Theorem 5 and with the help of Theorem 2 one may easily carried out the following theorem from pairwise inequalities no (2.22) and (2.24); (2.21) and (2.27); (2.17) and (2.20) respectively and therefore its proofs is omitted:

**Theorem 8.** Let \( f(z) \) and \( g(z) \) be any two entire functions of \( n \)-complex variables with index-pair \( (m, q) \) and \( (m, p) \), respectively, where \( m \geq q \geq 1 \) and \( m \geq p \geq 1 \) and \( D \) be a bounded complete n-circular domain with center at origin in \( \mathbb{C}^n \). Then

\[
\max \left\{ \frac{\Delta f,D (m, q)}{\Delta g,D (m, p)} \rho_g (m, p)^{-\frac{1}{2}}, \frac{\Delta f,D (m, q)}{\Delta g,D (m, p)} \rho_g (m, p)^{-\frac{1}{2}} \right\} \leq \frac{\Delta (p,q)}{\Delta g,D (m, p)} (f) \leq \frac{\Delta f,D (m, q)}{\Delta g,D (m, p)} \rho_g (m, p)^{-\frac{1}{2}}.
\]

**Corollary 19.** Let \( f(z) \) and \( g(z) \) be any two entire functions of \( n \)-complex variables with index-pair \( (m, q) \) and \( (m, p) \), respectively, where \( m \geq q \geq 1 \) and \( m \geq p \geq 1 \) and \( D \) be a bounded complete n-circular domain with center at origin in \( \mathbb{C}^n \). Then

\[
\frac{\Delta (p,q)}{\Delta g,D (m, p)} (f) \leq 
\max \left\{ \frac{\Delta f,D (m, q)}{\Delta g,D (m, p)} \rho_g (m, p)^{-\frac{1}{2}}, \frac{\Delta f,D (m, q)}{\Delta g,D (m, p)} \rho_g (m, p)^{-\frac{1}{2}} \right\} \cdot
\]

The conclusion of the above corollary can be carried out from pairwise inequalities no (2.14) and (2.16); (2.19) and (2.24) and (2.21) and (2.27) respectively after applying the same technique of Theorem 7 and with the help of Theorem 2. Therefore its proof is omitted.

Now we state the following theorems without their proofs as because they can be derived easily using the same technique or with some easy reasoning with the help of Remark 1 and therefore left to the readers.

**Theorem 9.** Let \( f(z) \) and \( g(z) \) be any two entire functions of \( n \)-complex variables with index-pair \( (m, q) \) and \( (m, p) \), respectively, where \( m \geq q \geq 1 \) and \( m \geq p \geq 1 \) and \( D \) be a bounded complete n-circular domain with center at origin in \( \mathbb{C}^n \). Also let \( g(z) \) is of regular \( (m, p) \)-Gol’dberg growth. Then

\[
\frac{\Delta f,D (m, q)}{\Delta g,D (m, p)} \rho_g (m, p)^{-\frac{1}{2}} \leq \frac{\Delta (p,q)}{\Delta g,D (m, p)} (f) \leq \min \left\{ \frac{\Delta f,D (m, q)}{\Delta g,D (m, p)} \rho_g (m, p)^{-\frac{1}{2}}, \frac{\Delta f,D (m, q)}{\Delta g,D (m, p)} \rho_g (m, p)^{-\frac{1}{2}} \right\} 
\leq \max \left\{ \frac{\Delta f,D (m, q)}{\Delta g,D (m, p)} \rho_g (m, p)^{-\frac{1}{2}}, \frac{\Delta f,D (m, q)}{\Delta g,D (m, p)} \rho_g (m, p)^{-\frac{1}{2}} \right\} \leq \frac{\Delta (p,q)}{\Delta g,D (m, p)} (f) \leq \frac{\Delta f,D (m, q)}{\Delta g,D (m, p)} \rho_g (m, p)^{-\frac{1}{2}}.
\]
and
\[
\left[ \frac{\tau_{f,D}(m,q)}{\tau_{g,D}(m,p)} \right]^{\frac{1}{\lambda g(m,p)}} \leq \tau_{g,D}^{(p,q)}(f) \leq \min \left\{ \left[ \frac{\tau_{f,D}(m,q)}{\tau_{g,D}(m,p)} \right]^{\frac{1}{\lambda g(m,p)}}, \left[ \frac{\tau_{f,D}(m,q)}{\tau_{g,D}(m,p)} \right]^{\frac{1}{\lambda g(m,p)}} \right\}
\]
\[
\leq \max \left\{ \left[ \frac{\tau_{f,D}(m,q)}{\tau_{g,D}(m,p)} \right]^{\frac{1}{\lambda g(m,p)}}, \left[ \frac{\tau_{f,D}(m,q)}{\tau_{g,D}(m,p)} \right]^{\frac{1}{\lambda g(m,p)}} \right\} \leq \tau_{g,D}^{(p,q)}(f) \leq \left[ \frac{\tau_{f,D}(m,q)}{\tau_{g,D}(m,p)} \right]^{\frac{1}{\lambda g(m,p)}}.
\]

**Theorem 10.** Let \(f(z)\) and \(g(z)\) be any two entire functions of \(n\)-complex variables with index-pair \((m, q)\) and \((m, p)\), respectively, where \(m \geq q \geq 1\) and \(m \geq p \geq 1\) and \(D\) be a bounded complete \(n\)-circular domain with center at origin in \(\mathbb{C}^n\). Also let \(f(z)\) is of regular \((m, q)\)-Goldberg growth. Then
\[
\left[ \frac{\Delta_{f,D}(m,q)}{\Delta_{g,D}(m,p)} \right]^{\frac{1}{\rho g(m,p)}} \leq \tau_{g,D}^{(p,q)}(f) \leq \min \left\{ \left[ \frac{\Delta_{f,D}(m,q)}{\Delta_{g,D}(m,p)} \right]^{\frac{1}{\rho g(m,p)}}, \left[ \frac{\Delta_{f,D}(m,q)}{\Delta_{g,D}(m,p)} \right]^{\frac{1}{\rho g(m,p)}} \right\}
\]
\[
\leq \max \left\{ \left[ \frac{\Delta_{f,D}(m,q)}{\Delta_{g,D}(m,p)} \right]^{\frac{1}{\rho g(m,p)}}, \left[ \frac{\Delta_{f,D}(m,q)}{\Delta_{g,D}(m,p)} \right]^{\frac{1}{\rho g(m,p)}} \right\} \leq \tau_{g,D}^{(p,q)}(f) \leq \left[ \frac{\Delta_{f,D}(m,q)}{\Delta_{g,D}(m,p)} \right]^{\frac{1}{\rho g(m,p)}}
\]

and
\[
\left[ \frac{\tau_{f,D}(m,q)}{\tau_{g,D}(m,p)} \right]^{\frac{1}{\lambda g(m,p)}} \leq \Delta_{g,D}^{(p,q)}(f) \leq \min \left\{ \left[ \frac{\tau_{f,D}(m,q)}{\tau_{g,D}(m,p)} \right]^{\frac{1}{\lambda g(m,p)}}, \left[ \frac{\tau_{f,D}(m,q)}{\tau_{g,D}(m,p)} \right]^{\frac{1}{\lambda g(m,p)}} \right\}
\]
\[
\leq \max \left\{ \left[ \frac{\tau_{f,D}(m,q)}{\tau_{g,D}(m,p)} \right]^{\frac{1}{\lambda g(m,p)}}, \left[ \frac{\tau_{f,D}(m,q)}{\tau_{g,D}(m,p)} \right]^{\frac{1}{\lambda g(m,p)}} \right\} \leq \Delta_{g,D}^{(p,q)}(f) \leq \left[ \frac{\tau_{f,D}(m,q)}{\tau_{g,D}(m,p)} \right]^{\frac{1}{\lambda g(m,p)}}.
\]

In the next theorems we intend to find out \((p,q)\)-th relative Goldberg type (resp. \((p,q)\)-th relative Goldberg lower type, \((p,q)\)-th relative Goldberg weak type) of an entire function \(f(z)\) with respect to another entire function \(g(z)\) (both \(f(z)\) and \(g(z)\) are of \(n\)-complex variables) when \((m, q)\)-th relative Goldberg type (resp. \((m, q)\)-th relative Goldberg lower type, \((m, q)\)-th relative Goldberg weak type) of \(f(z)\) and \((m, p)\)-th relative Goldberg type (resp. \((m, p)\)-th relative Goldberg lower type, \((m, p)\)-th relative Goldberg weak type) of \(g(z)\) with respect to another entire function \(h(z)\) (\(h(z)\) is also of \(n\)-complex variables) are given where \(m \geq p \geq 1\) and \(m \geq q \geq 1\). Basically we state the theorems without their proofs as those can easily be carried out after applying the same technique our previous discussion and with the help of Theorem 4 and Remark 2.

**Theorem 11.** Let \(f(z)\), \(g(z)\) and \(h(z)\) be any three entire functions of \(n\)-complex variables and \(D\) be a bounded complete \(n\)-circular domain with center at origin in \(\mathbb{C}^n\). Also let \(f(z)\) and \(g(z)\) be entire functions with relative index-pairs \((m, q)\) and \((m, p)\) with respect to \(h(z)\) respectively where \(p, q, m\) are all positive integers. If \(\lambda_h^{(m,p)}(g) = \)
\[\rho_h^{(m,p)}(g)\], then

\[
\left[ \frac{\sigma_{h,D}^{(m,q)}(f)}{\sigma_{h,D}^{(m,p)}(g)} \right]^{1/\rho_h^{(m,p)}(g)} \leq \sigma_{g,D}^{(p,q)}(f) \leq \min \left\{ \left[ \frac{\sigma_{h,D}^{(m,q)}(f)}{\sigma_{h,D}^{(m,p)}(g)} \right]^{1/\rho_h^{(m,p)}(g)}, \left[ \frac{\sigma_{h,D}^{(m,q)}(f)}{\sigma_{h,D}^{(m,p)}(g)} \right]^{1/\rho_h^{(m,p)}(g)} \right\}
\]

\[
\leq \max \left\{ \left[ \frac{\sigma_{h,D}^{(m,q)}(f)}{\sigma_{h,D}^{(m,p)}(g)} \right]^{1/\rho_h^{(m,p)}(g)}, \left[ \frac{\sigma_{h,D}^{(m,q)}(f)}{\sigma_{h,D}^{(m,p)}(g)} \right]^{1/\rho_h^{(m,p)}(g)} \right\}
\]

and

\[
\left[ \frac{\tau_{h,D}^{(m,q)}(f)}{\tau_{h,D}^{(m,p)}(g)} \right]^{1/\lambda_h^{(m,p)}(g)} \leq \tau_{g,D}^{(p,q)}(f) \leq \min \left\{ \left[ \frac{\tau_{h,D}^{(m,q)}(f)}{\tau_{h,D}^{(m,p)}(g)} \right]^{1/\lambda_h^{(m,p)}(g)}, \left[ \tau_{h,D}^{(m,q)}(f) \right]^{1/\lambda_h^{(m,p)}(g)} \right\}
\]

\[
\leq \max \left\{ \left[ \frac{\tau_{h,D}^{(m,q)}(f)}{\tau_{h,D}^{(m,p)}(g)} \right]^{1/\lambda_h^{(m,p)}(g)}, \left[ \tau_{h,D}^{(m,q)}(f) \right]^{1/\lambda_h^{(m,p)}(g)} \right\}
\]

**Theorem 12.** Let \(f(z), g(z)\) and \(h(z)\) be any three entire functions of \(n\)-complex variables and \(D\) be a bounded complete \(n\)-circular domain with center at origin in \(\mathbb{C}^n\). Also let \(f(z)\) and \(g(z)\) be entire functions with relative index-pairs \((m,q)\) and \((m,p)\) with respect to \(h(z)\) respectively where \(p, q, m\) are all positive integers. If \(f(z)\) is of regular relative \((m,q)\)-Goldberg growth with respect to entire function \(h(z)\), then

\[
\left[ \frac{\sigma_{h,D}^{(m,q)}(f)}{\sigma_{h,D}^{(m,p)}(g)} \right]^{1/\rho_h^{(m,p)}(g)} \leq \tau_{g,D}^{(p,q)}(f) \leq \min \left\{ \left[ \frac{\sigma_{h,D}^{(m,q)}(f)}{\sigma_{h,D}^{(m,p)}(g)} \right]^{1/\rho_h^{(m,p)}(g)}, \left[ \sigma_{h,D}^{(m,q)}(f) \right]^{1/\rho_h^{(m,p)}(g)} \right\}
\]

\[
\leq \max \left\{ \left[ \frac{\sigma_{h,D}^{(m,q)}(f)}{\sigma_{h,D}^{(m,p)}(g)} \right]^{1/\rho_h^{(m,p)}(g)}, \left[ \sigma_{h,D}^{(m,q)}(f) \right]^{1/\rho_h^{(m,p)}(g)} \right\}
\]

and

\[
\left[ \frac{\tau_{h,D}^{(m,q)}(f)}{\tau_{h,D}^{(m,p)}(g)} \right]^{1/\lambda_h^{(m,p)}(g)} \leq \sigma_{g,D}^{(p,q)}(f) \leq \min \left\{ \left[ \frac{\tau_{h,D}^{(m,q)}(f)}{\tau_{h,D}^{(m,p)}(g)} \right]^{1/\lambda_h^{(m,p)}(g)}, \left[ \tau_{h,D}^{(m,q)}(f) \right]^{1/\lambda_h^{(m,p)}(g)} \right\}
\]

\[
\leq \max \left\{ \left[ \frac{\tau_{h,D}^{(m,q)}(f)}{\tau_{h,D}^{(m,p)}(g)} \right]^{1/\lambda_h^{(m,p)}(g)}, \left[ \tau_{h,D}^{(m,q)}(f) \right]^{1/\lambda_h^{(m,p)}(g)} \right\}
\]

**Theorem 13.** Let \(f(z), g(z)\) and \(h(z)\) be any three entire functions of \(n\)-complex variables and \(D\) be a bounded complete \(n\)-circular domain with center at origin in \(\mathbb{C}^n\).
Also let $f(z)$ and $g(z)$ be entire functions with relative index-pairs $(m, q)$ and $(m, p)$ with respect to $h(z)$ respectively where $p, q, m$ are all positive integers. Then

$$
\max \left\{ \left[ \frac{\sigma_{h,D}^{(m,q)}(f)}{\tau_{h,D}^{(m,p)}(g)} \right]^{\frac{1}{\lambda_h^{(m,q)}(g)}}, \left[ \frac{\sigma_{h,D}^{(m,q)}(f)}{\tau_{h,D}^{(m,p)}(g)} \right]^{\frac{1}{\rho_h^{(m,p)}(g)}} \right\} \leq \sigma_{g,D}^{(p,q)}(f)
$$

and

$$
\max \left\{ \left[ \frac{\sigma_{h,D}^{(m,q)}(f)}{\tau_{h,D}^{(m,p)}(g)} \right]^{\frac{1}{\lambda_h^{(m,q)}(g)}}, \left[ \frac{\sigma_{h,D}^{(m,q)}(f)}{\tau_{h,D}^{(m,p)}(g)} \right]^{\frac{1}{\rho_h^{(m,p)}(g)}} \right\} \leq \tau_{g,D}^{(p,q)}(f)
$$

\textbf{Theorem 14.} Let $f(z)$, $g(z)$ and $h(z)$ be any three entire functions of $n$-complex variables and $D$ be a bounded complete $n$-circular domain with center at origin in $\mathbb{C}^n$. Also let $f(z)$ and $g(z)$ be entire functions with relative index-pairs $(m, q)$ and $(m, p)$ with respect to $h(z)$ respectively where $p, q, m$ are all positive integers. Then

$$
\max \left\{ \left[ \frac{\sigma_{h,D}^{(m,q)}(f)}{\tau_{h,D}^{(m,p)}(g)} \right]^{\frac{1}{\lambda_h^{(m,q)}(g)}}, \left[ \frac{\sigma_{h,D}^{(m,q)}(f)}{\tau_{h,D}^{(m,p)}(g)} \right]^{\frac{1}{\rho_h^{(m,p)}(g)}} \right\} \leq \tau_{g,D}^{(p,q)}(f)
$$

and

$$
\max \left\{ \left[ \frac{\sigma_{h,D}^{(m,q)}(f)}{\tau_{h,D}^{(m,p)}(g)} \right]^{\frac{1}{\lambda_h^{(m,q)}(g)}}, \left[ \frac{\sigma_{h,D}^{(m,q)}(f)}{\tau_{h,D}^{(m,p)}(g)} \right]^{\frac{1}{\rho_h^{(m,p)}(g)}} \right\} \leq \tau_{g,D}^{(p,q)}(f).
$$
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