A steady-state dynamical model for the COBE-detected Galactic bar

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ABSTRACT

A 3D steady-state stellar dynamical model for the Galactic bar is constructed with 485 orbit building blocks using an extension of the Schwarzschild technique. The model is smooth. It fits the density profile of the COBE light distribution, the observed solid-body stellar rotation curve, the fall-off of minor-axis velocity dispersion and the velocity ellipsoid at Baade’s window. Maps and tables of observable velocity moments are made for easy comparisons with observation. The model has also been used to set up equilibrium initial conditions for an N-body simulation to show that it is stable. The technique used here can be applied to interpret high-quality velocity data of external bulges/bars and galactic nuclei.

Key words: methods: numerical – celestial mechanics, stellar dynamics – Galaxy: kinematics and dynamics – Galaxy: structure.

1 INTRODUCTION

Numerous evidence has convincingly shown that our Galaxy has a central bar with its near end on the positive Galactic longitude side (see the review by Gerhard 1995). The interesting theoretical question at this point is no longer proving whether such a bar exists, but rather in building a comprehensive stellar dynamical model of the bar. The most interesting models are those that can fit the high-quality COBE infrared surface brightness maps of the Galaxy (Weiland et al. 1994), which are by far the most comprehensive observational constraints on the spatial distribution of the bar. Such a model will also form a basis for us to interpret the stellar velocity data, the microlensing observations and the gas kinematics; comparisons with those observations require us to make inferences of the radial velocity at the proper motions of the bar stars as well as the potential of the bar (Binney et al. 1991; Wada et al. 1994; Zhao, Spergel & Rich 1994). These cannot be done satisfactorily with existing models – some deliver information on only the density or the potential of the bar (e.g., Blitz & Spergel 1991, Binney et al. 1991 and Dwek et al. 1995), others are stellar dynamical models but do not fit the COBE maps in detail (Kent 1992; Kuijken 1994; Zhao et al. 1994).

Building fully consistent steady-state 3D bar models has been a challenging theoretical problem without a satisfactory solution for many years (see Binney & Tremaine 1987 and Sellwood & Wilkinson 1993 for a review of the subject). Bar models that also fit specific observations of the stellar light and/or velocity distribution are, as far as we know, not existent except for Zhao (1994). A few attempts to build 3D steady-state bar models have been made in the past based mainly on three different approaches.

One approach, which is very much similar to the conventional approach to spherical or oblate models, is to build a bar with its phase-space distribution function written explicitly as a function of analytical integrals of motions. Unfortunately the only directly available integral of motion in a bar potential is the Jacobi integral\( E \equiv E - \Omega L_z \); \( \Omega \) is the pattern speed of the triaxial potential. In models with a \( f(E) \) distribution function (e.g., the polytropic Jacobi ellipsoids by Vandervoort 1980), surfaces of equal density and equal effective potential \( \Phi_{\text{eff}}(x,y,z) \) coincide. As surfaces of equal potential \( \Phi(x,y,z) \) are always nearly spherical, and surfaces of equal \( \Phi_{\text{eff}}(x,y,z) \) are only mildly triaxial, the models are limited to fat bars with a flat density profile. Also the models have a rigorously solid-body rotation and an isotropic velocity distribution – none of these resemble real bars.

The second approach, known as the Schwarzschild (1979, 1982) technique, one views a collisionless stellar system as a realization of a distribution of stars on different orbits inside a common potential. Since the two-body relaxation time-scale is so long that orbits are completely decoupled from each other, each orbit – computed by integrating...
the equations of motion numerically inside a fixed common potential – is a time-invariant building block. By linearly superimposing the building blocks and adjusting the relative weighting of them, one can hope to match observations and to reach theoretical consistency. Using regular orbits of a variety of shapes and sizes in a triaxial potential, Schwarzschild (1979, 1982) was able to build 3D triaxial systems resembling elliptical galaxies. Extending the technique to include irregular orbits explicitly and using the non-negative least squares method, Pfenniger (1984) was also able to build 2D rapidly rotating bar models that fit observed velocity fields of a bar galaxy. However, possibly because of the complexity of the irregular orbits and some practical limitations of the method, the Schwarzschild method has not been tried again in making 3D rapidly rotating bars.

A third, popular approach is to evolve an initially unstable disc with N-body simulations. The idea is straightforward and has the added advantage of giving a stable bar as well. However, although some models (e.g. Fux et al. 1995a,b, Sellwood 1993) can even match observations qualitatively, the approach does not have the built-in freedom to fit observations in detail.

This paper presents a 3D bar model that fits the COBE maps of the Galactic bar as well as a handful of radial-velocity, radial-dispersion, proper-motion-dispersion and metallicity measurements of K and M giants in the bulge (see de Zeeuw 1993). The work here is a continuation of our efforts to link these data in a dynamical model (Zhao et al. 1994; Zhao 1994).

The technique used here is a hybrid one, combining the advantages of the above three techniques with the Schwarzschild method being the primary one. The bar is modelled by 485 orbit building blocks (see Fig. 1). The mass of the bar is partly allocated to direct regular orbits, which give the model both enough quadrupole-moment amplitude to support the bar potential and an anisotropic velocity distribution. The rest of the mass of the bar is allocated to steady-state building blocks with the distribution function of each being a different delta function of the integral $E_j$. Each delta function – which we give the name a ‘composite orbit’ – consists of several different orbits of the same energy implicitly, including irregular orbits. These building blocks form a roundish component in the volume space which smoothly fills up the gaps between the regular orbits. By adjusting the weights of the composite orbits and the weights of the regular orbits but within the bounds of a positive and acceptably smooth orbit distribution, we were able to fit the COBE bar. Finally the stability of the model is tested with an N-body simulation.

The structure of the paper is as follows. Following an abstract formulation for steady-state models, we explain composite orbits, smoothing and other modifications to the original Schwarzschild technique in Section 2. Section 3 gives the key data used in this analysis: the COBE Galactic plane map and the stellar kinematic data. The fits of the model to observations are shown in Section 4. Section 5 discusses the fraction of retrograde and irregular orbits and their implication on bar formation. The stability of the model is tested with N-body simulations in Secton 6. Section 7 summarizes the technique and the results. Finally, we point out other applications of the technique in Section 8. Some technical details on the formulation and on efficient orbit integration and orbit library are given in the Appendices A and B. Tables 1 to 6 also summarize the main input quantities and results of the model.

2 MODELLING TECHNIQUE

2.1 Formulation

In this section we formulate the problem of building a steady-state model to fit certain observations in abstract terms. Throughout this section, when the difference is not critical we shall loosely refer to both the composite orbits and the regular orbits as orbits, and loosely refer to both theoretical input quantities (such as the volume density of the bar) and quantities from observation (such as projected surface brightness and velocity dispersions) as observables.

Given the potential of the system, one tries to match the distribution of $N$ different orbits with several observables $\mu_j$, where $j = 1, \ldots, n_c$. Generally $\mu_j$ is related to the orbits by the following equation:

$$
\mu_j = \frac{\sum_{k=1,N} \gamma_{j,k} \beta_{j,k} w_k}{\sum_{k=1,N} \beta_{j,k} w_k},
$$

where both $\beta_{j,k}$ and $\gamma_{j,k}$ are known quantities, computed by orbit integration and projection into observable space and $w_k$ is the set of unknown weight of the $N$ orbits to be solved. More practically, equation (1) can be rewritten into a set of linear equations which will be solved by standard numerical routines.

$$
\sum_{k=1,N} (\mu_j - \gamma_{j,k}) \beta_{j,k} w_k = 0, \quad \text{for } j = 1, n_c.
$$

The mathematical form of equations (1) and (2) holds if the observable $\mu_j$ is the volume density, the surface density, or the velocity first moment, velocity tensor moments or higher velocity moments. However, the meanings of $\beta_{j,k}$ and $\gamma_{j,k}$ vary. See Appendix A for the details. In the case of velocity first moment, equation (1) follows directly by combining equations (2.4) and (2.5) of Pfenniger (1984).

There is a slight complication when the observable is a velocity dispersion $\sigma$. It is necessary to first combine $\sigma$ with the first moment $\eta_1 = V$ to form the second moment $\eta_2 = V^2 + \sigma^2$. Then $\eta_2$ satisfies equation (1). Similarly one can build higher moments $\eta_n$ with $n = 3, 4$ using skewness and kurtosis.

Unfortunately inverting equation (2) is not a mathematically well-defined problem; there are often many exact but arbitrary noisy solutions or only approximate solutions which satisfy the equation about equally well. To reduce the numerical noise of direct inversion, and to pick a physically most-meaningful solution from these, one needs to impose additional physically plausible constraints to regularize the

$^3$Pfenniger (1984) argued that the velocity dispersion $\sigma$ cannot be programed as a linear constraint because it is a quadratic function of the orbit weights. But as we show above, this problem is easily circumvented if one uses $V^2 + \sigma^2$ as a constraint.
Figure 1. Dots with plus and diamond symbols of increasing size indicate composite orbits and regular direct \( \langle J_z \rangle \geq \text{Min}[J_z(t) > 0] \) orbits assigned with increasing weights in the model (an isolated dot indicates a very low weight). The upper panel shows the distribution of the orbits in the energy versus angular momentum plane inside the dynamical boundary (dashed line). The middle panel shows the axial-ratio distribution. The lower panel shows that the time variation of the numerically integrated orbits is required to be below the 1 per cent level. Note that the composite orbits rotate as fast as the bar potential in time-average so \( \langle J_z \rangle > 0 \).
solution. Obviously mathematical solutions where some of the weights of the orbits are assigned negative values should be excluded. Somewhat less obviously, mathematical solutions where the mass distribution of the orbits is a wildly oscillating function in phase space should be excluded as well (Merritt 1993).\(^3\)

To put the positivity and smoothness constraints into mathematical terms, and combine with the equation (2), the dynamical modelling problem can be formulated as solving the following non-negative least squares (NNLS hereafter) problem. Having integrated \(N\) orbits in a fixed potential, one populates these orbits with certain unknown positive fractions \(w_k\)

\[
w_k \geq 0 \quad \text{for} \quad k = 1, N,
\]

(3)
to minimize the following \(\chi^2\),

\[
\chi^2 = \sum_{j=1,n} \frac{[P(w_j)]^2}{\sigma_j^2} + \lambda \sum_{k=1,n} |S(w_k)|^2,
\]

(4)
where \(P(w_j)\) is the dimensional residual of fitting the observable \(\mu_j\)

\[
P(w_j) = \sum_{k=1,N} (\mu_j - \gamma_{j,k}) \beta_{j,k} w_k,
\]

(5)
and \(\sigma_j\) is a to-be-defined scaling quantity with the same dimension as the observable \(\mu_j\) so that \(\chi^2\) is dimensionless. \(\lambda\) is a small non-zero positive parameter tuned to yield solutions of the desired smoothness, and we measure the non-smoothness indicator \(S(w_k)\) by the difference between the weight \(w_k\) and an average weight of its \(n\) nearest neighbouring orbits,

\[
S(w_k) = w_k - \frac{1}{n} \sum_{j=1,n} s_{k,j} w_{k+j},
\]

(6)
so that the fit to observation is penalized in the \(\chi^2\) sense when neighbouring orbits are populated with a drastically different number of stars. The kernel \(s_{k,k+s}\) is a smooth function of a to-be-defined ‘distance’ between the \(k\)-th orbit and its \(i\)-th neighbour.

The above NNLS problem is solvable with standard programs such as the OPROG of IMSL and EBBNAP of NAG.\(^4\)

### 2.2 Smoothing in the effective integral space

In principle one can use the three integrals of motion as an indicator of the ‘distance’ between two regular orbits because two regular orbits with nearly equal integrals of motion are often close in real space. If \(I=(I_1, I_2, I_3)\) are three dimensionless integrals of order unity, then \((I-I')^2\) defines a distance between two orbits.

In practice it is difficult to compute explicitly the three integrals of motion of a general bar potential except for the analytical \(E, J\), and irregular orbits and the composite orbits only have one integral \(E\). Instead we label orbits and measure their relative distances by a variety of characteristic quantities which we call effective integrals. If \(E, J\), and \(I\), are an orbit’s instantaneous-energy and angular-momentum components in the short \(z\)-axis and in the long \(x\)-axis, then the time-averaged quantities \(\langle E\rangle, \langle J\rangle\) and \(\langle J^2\rangle\) can be used to describe the most important properties of the orbit, namely, its radial extent, its sense of rotation along the minor axis and its vertical extent. Since \(\langle E\rangle, \langle J\rangle, \langle J^2\rangle\) reduce to exact integrals in oblate or prolate potentials, which are two extremes of bar potential, they seem to be a convenient choice for effective integrals. Other quantities, such as the axis ratio \((y/x, z/R)\) defined by the square root of the ratios of the time-averaged principal axes of the moment of inertia tensor for an orbit, are also useful effective integrals. We rescale all these effective integrals to the range of 0 to 1 to define the distance between orbits.

The kernel \(s\) in equation (6) is chosen to be a Gaussian function of the distance which peaks at zero. Typically the number of nearest neighbours \(n\) is about \(3^2 - 1 = 26\). We fix the smoothness measure at \(\lambda = N^{-2}\), where \(N\) is the number of orbits. We also fix the scaling quantities \(\sigma_j\) in equation (4) for the volume density, the surface density and the velocities etc. to be the respective r.m.s. averages over the whole system. These choices here are certainly not the only one. For our model, however, we find various other choices of smoothing and weighting give largely equivalent and mathematically stable results.

### 2.3 Composite orbits, irregular orbits and time dependency

Before implementing the technique, we now discuss the meaning of and the advantage of using the composite orbits, and why both the regular orbits and the composite orbits are needed for the model.

A composite orbit is not an orbit in the usual sense, because it involves no orbit integration to compute it. Rather it is a building block in which several orbits\(^5\) with equal analytical integral(s) of motion are implied and are combined together in certain fixed relative weighting. For a general axisymmetric model a composite orbit can have a distribution function \(\delta(E-E_0)\delta(L_z-L_{0z})\) where some number of irregular orbits and some number of regular orbits of a different third integral \(I_3\) but the same energy and angular momentum are put into one mass component, so that overall the component does not depend on the third integral. In this sense a two-integral model \(f(E, L_z)\) for oblate systems with isotropic dispersions is made entirely of these composite orbits. For a general triaxial or bar model, a composite orbit has a \(\delta(E-E_0)\) distribution function which includes a combination of all regular orbits and irregular orbits with the same Jacobi integral \(E\) so that the

\(^3\)One expects that phase-space fine structures of the initial state are erased (Lynden-Bell 1967; Spergel & Hernquist 1992) – efficiently if not completely – by direct scattering between stars and molecular clouds, by violent relaxation (scattering between stars and potential temporal fluctuations when the system forms or when certain instability sets in), or by scattering off resonances during secular evolution of the potential.

\(^4\)A source code is in Hanson & Lawson (1995). Also an excellent general discussion of the inverse problem and the NNLS method is in ‘Numerical Recipes in FORTRAN’ (Press et al. 1992, p. 779–809).

\(^5\)An exception is that in special potentials with three analytical integrals of motion (Stackel potentials), a composite orbit always corresponds to a regular orbit (say a box or a loop orbit).
distribution does not depend on unknown integrals. An example of bar models made entirely of such composite orbits would be a polytropic Jacobi ellipsoid with a distribution function \( f(E_J) \propto (E_{max} - E_J)^{-\gamma} \).

Our bar model is a hybrid model which consists of both composite orbits and (direct) regular orbits. Composite orbits form a dynamical composite with a distribution function \( f_c(E_J) \) as a function of one integral \( E_J \) (the subscript \( c \) stands for composite). Regular orbits form a different component with a distribution function \( f_r(I) \) \( [I \) is a short-hand for the three integrals \( E_1, E_2, E_3 \)], and the subscript \( n \) stands for numerically integrated orbits]. Together the model has a distribution function \( f = f_c(I) + f_r(E_J) \).

The model distribution function \( f \) is rigorously time-independent. It satisfies the collisionless Boltzmann equation (Vlasov equation)

\[
\frac{d}{dt} f - \frac{d}{dt} f_c(I) = \frac{d}{dt} f_r(E_J) = 0.
\]

This follows directly from the definition of integrals of motion

\[
\frac{d}{dt} I = \frac{d}{dt} E_J = 0.
\]

This also shows that both composite orbits and regular orbits are legitimate building blocks for steady-state models.

Neither composite orbits nor regular orbits alone are enough to build a consistent Galactic bar. As explained in the Introduction, a \( f_c(E_J) \) distribution function alone cannot model the anisotropy and streaming motion of real bars. On the other hand gaps are ubiquitous in the phase space and/or volume space of regular orbits in 2D or 3D triaxial potentials, which do not resemble the observed systems with often smooth projected density. The empty regions need to be filled up with stars on other orbits. In the case of the Galactic bar with a nucleus and strong pattern rotation, the minor axis of the bar is cleared out of regular orbits because the minor-axis orbits are unstable. Only orbits without a fixed sense of rotation can reach the minor axis (Zhao et al. 1994). If the COBE bar does not have an intrinsic hole around the minor axis, as inferred from the best-fitting volume density models of Dwek et al. (1995), some irregular orbits are necessary for a consistent model.

Composite orbits are used in our model to substitute mostly the role of irregular orbits in bar dynamics. We estimate that a typical composite orbit in our bar model has roughly 2/3 of its mass in irregular orbit (the remaining 1/3 is mostly in retrograde orbits; very little is in direct regular orbits, which occupy a very small region of the phase space).

Roughly speaking the irregular orbits occupy a similar 5D surface of the phase space bounded by the Jacobi integral, and a similar featureless roundish component in the volume space bounded by the effective potential \( \Phi_{eff}(x, y, z) \) as the composite orbits. However, irregular orbits are much more complex in structure and much more difficult to compute precisely. This is because an irregular orbit cannot intersect with the phase space already taken by regular orbits, and must avoid other irregular orbits (if they exist) of the same Jacobi integral. So their structure is likely to be a complex ‘web’ in the 5D surface of constant \( E_J \). As a result of this both stars in galaxies and computer simulations never complete one round-trip of this ‘web’ in the relevant time-scales. Merritt (1984) finds that the spatial density of an irregular orbit averaged over \( t \) (say a Hubble time) or \( t/2 \) differs by a few per cent, and the fluctuation does not become much smaller when \( t \) is increased.

A model explicitly using irregular orbits is both extremely time-consuming to compute and is not guaranteed to be steady state. Although a combination of long integration and a clever grouping of the irregular orbits can make timescales for system evolution much longer than a Hubble time (Merritt & Fridman 1996), it is much easier to use the time-dependent composite orbits instead, which require no numerical orbit integration; the computation involved is only a few trivial projections to know the volume density and velocity distribution of each orbit.

### 2.4 Constructing the orbit library

A complete and efficient orbit library is essential in building a steady-state model. To be consistent with a smooth and strongly ellipsoidal bar density profile, one needs the direct regular orbits, particularly \( x_c \)-type orbits and the banana-type orbits (see, for example, Binney & Tremaine 1987 or Sellwood & Wilkinson 1993) to provide the quadrupole moments of the bar, and the composite orbits to fill the bar smoothly, particularly to occupy the region around the minor axis of the bar. For many preliminary runs, we came to realize that these two types are the only critical orbits in making the bar. Other orbits (irregular or retrograde orbits) are not used explicitly other than those implied in the composite orbits.

For composite orbits, we assign their semimajor axis (which is an implicit function of \( E_J \)) uniformly between zero and the last bounded surface of equal effective potential. This includes all orbits bounded by the corotation of the bar.

More care is needed to generate direct regular orbits, which occupy very small regions of the phase space but are critical for supporting the bar potential. Basically if one populates the orbits with a fully random set of initial conditions to cover all possible orbits (Zhao et al. 1994; Zhao 1994), the library will be dominated by irregular orbits and retrograde orbits, which together with escaping orbits occupy most of the phase case, and the bar model (e.g. Zhao 1994) will be too round to be consistent with the COBE map.

More details of technique are given in Appendix B, where we discuss the algorithm to penalize generating too many irregular orbits, retrograde orbits, and escaping orbits, as well as the algorithm for efficient orbit integration and storage.

We have also made two ‘full-system’ tests of the procedure of constructing equilibrium models using known models. One is with the Hernquist (1990) spherical-bulge model, which is known analytically. The other is the axisymmetric isotropic model of Kent (1992), which has been constructed by solving the Jeans equation. Both models are tested without making explicit use of the spherical or the axial symmetry. The details of these test runs can be found.
3 INPUTS OF THE MODEL

3.1 Density model of the bar

In addition to the cosmological achievements of COBE, its DIRBE experiment has mapped the Galactic bulge and the Galactic plane with 0.7 × 0.7 resolution in 10 infrared bands, where the extinction is less severe than in the optical. After correction for the dust reddening which shows up as pixel-to-pixel colour variations in the four infrared maps (Arendt et al. 1994), Weiland et al. (1994) presented maps of the high-latitude Galactic plane with 0°7 × 0°7 resolution in 10 infrared wavelengths 1.25, 2.2, 3.5, 4.9 μm. These maps clearly show a flattened boxy (rather than peanut-shaped) bulge with axis ratio ~0.6 and the asymmetry in light distribution that is qualitatively consistent with a Galactic bar with its near end in the first Galactic quadrant.

As a basic input to our bar-orbit model, a volume-density model for the Galactic bar and nucleus region is needed. We assume that the density of both components together is given by the following form,

\[ \rho(x, y, z) = \rho_0 \left[ \exp \left( -\frac{s_x^2}{2} \right) + s_a^{-1.45} \exp(-s_a) \right]. \]  

(7)

The first term prescribes a bar with a Gaussian radial profile and boxy intrinsic shape, where

\[ s_x^2 = \left( \frac{x}{a} \right)^2 + \left( \frac{y}{b} \right)^2, \]

\[ s_a = \left( \frac{z}{c} \right)^2, \]

\[ x, y, z \] are the three principal axes of the bar, and the three scalelengths of the bar \( a = 1.49 \pm 0.05 \), \( b = 0.58 \pm 0.01 \) and \( c = 0.40 \pm 0.01 \) kpc for Galactocentric distance \( R_0 = 8 \) kpc.

The long axis of the bar, the \( x \)-axis, points parallel to the direction of \( l = -20° \) and \( b = 0° \).

The second term prescribes an oblate spheroidal nucleus with a steep inner power law and an exponential outer profile, where

\[ s'_a = q_a^2(x^2 + y^2) + z^2, \]

\[ q_a = 0.6. \]

The density of the bar component comes from Dwek et al.’s (1995) G2 model – a boxy Gaussian photometric model which gives the best fit in the least-squares sense. They have fitted a set of parametrized luminosity models with different shape and radial profile to the COBE map within 3° < \( |b| < 10° \) and \( |l| < 10° \). Because this region excludes the central \( r \sim 3° \sim 400 \) pc, the G2 model with its finite core misses the light from the central nucleus of the bulge.

The photometric nuclear component is also dynamically important. It has observable effect on the kinematics of the bulge: the turn-over of radial velocity dispersion at a few degrees in the Kent (1992) axisymmetric bulge model is an example. For bars, including such a nucleus also makes it theoretically more challenging to find axial orbits supporting the bar.

We model the density of the nucleus similar to Kent (1992). We extrapolate the observed steep power law \( \rho(r) \sim r^{-1.85} \) of the inner few arcsec (Becklin & Neugebauer 1968) to the inner bulge (Matsumoto et al. 1982; Sellwood & Sanders 1988), and smoothly turn it over to an exponential bulge (Kent, Dame & Fazio 1991; Kent 1992). We use a spheroidal nucleus for the lack of strong evidence for a corotating triaxial boxy component in the very centre and the expectation that the boxlet orbits may not be sufficient to support strong triaxiality in the centre.

The model density is smooth and monotonically decreasing in all radial directions except that the density is truncated beyond 3.3 kpc. The reason for the truncation is both because the Dwek et al. model is unconstrained beyond 10° ~ 1.4 kpc, and that an elongated bar ends before corotation, which is at 3.3 kpc in our model. The constant \( \rho_0 \) will be determined by normalizing the total mass of the bar plus nucleus to fit the velocity dispersion at Baade’s window.

Fig. 2 compares the density model with the COBE map. It shows the projected density of the model and a dereddened COBE map. A Miyamoto–Nagai disc as described below is also included in the model. The smoothness of the dereddened contours shows that dust subtraction is sufficient for the bulge region. One can see that the modified model matches the COBE K-band map within 10° of the Galactic Centre to a similar accuracy as the Dwek et al. model. Particularly, the model matches the boxyness and the longitude asymmetry in the dereddened COBE map reasonably well. We shall later on call our density model of the bar the modified Dwek model. Although it is still boxy, the modified model is significantly rounder in the xy-plane than the G2 model as a result of the added axisymmetric component, and it fits the COBE map roughly as well. The effective axis ratio is about 1:0.6:0.4.

3.2 Potential models of the bar and the disc

The model potential is a sum of two components, a bar and a disc. The isothermal halo is neglected because we are interested in dynamics in the inner 2 kpc, where a halo with
reasonably large core radius contributes little. The bar density is based on the Dwek et al. G2-model, but is modified to account for the inner nucleus.

The disc potential is modelled as a Miyamoto–Nagai (MN) disc with an analytical potential of the following form,

$$\Phi(x, y, z) = -\frac{GM_d}{D^2},$$

where

$$D^2 = x^2 + y^2 + [a_{MN} + (z^2 + b_{MN}^2)^{1/2}]^2,$$

$$a_{MN} = 6.5 \text{kpc}, b_{MN} = 0.26 \text{kpc} \text{ and total disc mass } M_d = 8 M_{\text{bar}}.$$  

The disc parameters are chosen to have a vertical height of 0.2 kpc and together with the bar produces a flat rotation curve up to 3 kpc. Such a disc is only a very rough approximation to the conventional double-exponential disc. It does not fit the COBE map in the region outside 10° of the centre in any detail (see Fig. 2). However, since our primary interest is the bulge, which is roughly self-gravitating anyway, a simple parametrization of the disc potential is acceptable and useful. The analytical MN disc potential helps to increase the speed of the orbit integration. Scaling the disc mass with the bar mass and neglecting the halo potential for the inner 3 kpc simplifies our fitting procedure, as the bar mass is scaled out of most of the calculations, and can be obtained at the end by renormalizing with the velocity dispersion of Baade’s window.

Figure 3. The circular speed of the model as a function of radius along the intermediate y-axis of the bar. The model potential is from our input bar model plus a Miyamoto–Nagai disc, with the bar potential computed using 10 terms (solid line) and 84 terms (dashed line) of the Hernquist–Ostriker expansions. The halo potential is neglected for the inner 3 kpc. Note different truncations give only less than 20 km s⁻¹ difference in velocity and both produce the flat rotation curve, characteristic of the Galaxy.

To compute the potential of the bar, we have used the rather efficient Hernquist & Ostriker (1992) expansion technique: the density (as in equation 7) is expanded on an orthogonal analytical basis set the lowest order term of which is a Hernquist spherical model with a scalelength of 1 kpc. If each expansion term is denoted with three quantum numbers \((n, l, m)\), then for a triaxial model, only the even \((l, m)\) quantum number terms are non-zero. We keep only the 10 dominant terms of the bar potential, and set other terms to zero. We check the effect of this truncation on accuracy by comparing the circular velocity curve of the potential model with that computed from the first 84 terms.

Fig. 3 plots the circular velocity along the intermediate axis of the bar. A bar mass of \(2 \times 10^{10} M_{\odot}\) is used. The difference owing to truncations is less than 20 km s⁻¹ for the region between 0.3–3 kpc. Also note that the rotation curve is in agreement with the observed flat rotation curve within 3 kpc (Clemens 1985), although a more rigorous comparison involves computing the velocity of the closed orbits of the bar rather than the circular velocity (Binney et al. 1991). The result of the closed orbit calculation and its comparison with the \(\text{H} \text{I}\) longitude–velocity map are shown in fig. 1 of Zhao et al. (1996). Overall, the model potential appears to be reasonable for the inner bulge.

3.3 Kinematic data

The low extinction fields of the Galactic bulge, particularly Baade’s window (BW) \((l, b) = (1°, -4°)\), have been the
target of many spectroscopic studies. Among the largest kinematic and abundance samples at Baade’s window are the radial velocities of 300 M giants by Sharples, Walker & Cropper (1990), proper motions of 400 K and M giants by Spahnauer, Jones & Whitford (1992), and the published kinematics and metallicities of 88 K giants by Rich (1988, 1990). Of these, there are 62 K giants for which metallicity, radial velocity and proper motions are all measured (Zhao et al. 1994). A larger overlap sample has also been obtained recently by Terndrup, Sadler & Rich (1995).

To constrain our bar model, we use the following kinematic data: the radial velocity dispersion at Baade’s window, and the proper motion data of Spaenhauer et al., the radial velocity and dispersion at the (8°, 7°) field (Minniti et al. 1994), a larger overlap sample has also been obtained from the bulge stellar tracers like Miras, SiO maser stars, overall solid-body rotation curve of slope 80 km s⁻¹ kpc⁻¹ for the radial velocities of 300 M giants by Sharples, Walker & Cropper (1990), the (−1°, 2°) field (Blum et al. 1994, 1995), the overall line-of-sight dispersion of 113 ± 6 km s⁻¹ for all stars at Baade’s window is used to normalize the mass of the bar; the number is mostly based on the 200 M giants from Sharples et al. The proper motion and rotation data help to constrain the amount of anisotropy in the model. Other bulge kinematic data (see the data compiled in Kent 1992 and de Zeeuw 1993) are not used as model constraints, but serve as reference values to compare with our model. In particular, we compare our model with the observed minor-axis drop-off of velocity dispersion for the M giants (Terndrup et al., in preparation).

4 RESULTS: FITTING STELLAR LIGHT AND KINEMATICS

We have undertaken to build dynamical bar models that are consistent with the COBE map. The input parameters of the model are summarized in Table 1. We fix the angle of the bar at an often-quoted value of 20°, close to the value of 13.4° found by Dwek et al. We set the pattern speed at 60 ξ⁷¹₂ km s⁻¹ kpc⁻¹, where ξ = Mₚbar / (2 × 10¹⁰ M⊙). This corresponds to a corotation of 3.3 kpc. The pattern speed here is slightly smaller than used by Binney et al. (1991) if the bar mass is 1–2 × 10¹⁰ M⊙.

The orbit library consists of 1000 orbits, which are piped into the NNLS routines to fit a 10 × 10 × 10 density cell in the first quadrant, and some 500 constraints from the projected density and velocity. Note that the projected densities and the volume density are not completely redundant owing to a different boundary and grid. Still there are more constraints than there are unknowns and the problem is over-constrained. We also require that nearby orbits have comparable weights to exclude very noisy solutions. As a result of fitting these constraints, the NNLS routine assigns 325 regular direct boxy orbits and 160 composite orbits with non-zero unequal weights. The rest of the orbits have zero weight. The 485 orbits with non-zero weight form our best-fitting model. The basic results of the model are summarized in Tables 2, 3, 4.

The model is theoretically consistent with the input volume density model to good extent. Fig. 4 shows the volume density slices in the xy- and yz-planes for the orbits and the input model. The differences between the two densities is relatively small, ~7 per cent in r.m.s. The orbits fit the density profile both along the minor axis and in shell-average. They also fit the elongated bar shape in the xy-plane as well as the boxy contours in the yz-plane from 100 pc to 1 kpc on the major axis. Beyond 2 kpc on the major axis, the model contours are somewhat flatter and less barred than the input model.

The model should also be consistent with the input potential. In terms of the expansion coefficients, the potential of the Dwek et al. model and that of the orbits differ by ~3 per cent in r.m.s.

Table 1. Parameters of the bar model.

| Quantities | Values |
|------------|--------|
| Density of a boxy Gaussian bar with a power-law oblate nucleus | \( \rho(x, y, z) = \rho_0 \left[ \exp \left( -\frac{s_x^2}{2} \right) + s_a^{-1.85} \exp(-s_a) \right] \), where \( s_x^2 = \sqrt{\left( \frac{x}{1.49} \right)^2 + \left( \frac{y}{0.58} \right)^2 + \left( \frac{z}{0.4} \right)^2} \), and \( s_a = \sqrt{\left( \frac{x}{0.57} \right)^2 + \left( \frac{y}{0.57} \right)^2 + \left( \frac{z}{0.4} \right)^2} \). |
| Direction of bar’s long axis | \( l = -20°, b = 0° \) |
| Mass of the bar with nucleus | \( M_{\text{bar}} = (2.2 \pm 0.2) \times 10^{10} M_{\odot} \left( \frac{\sigma_{\text{BW}}}{113} \right)^2 \) |
| Pattern speed | \( \Omega = 60 \left( \frac{M_{\text{bar}}}{2 \times 10^{10} M_{\odot}} \right)^{\frac{1}{2}} \) km s⁻¹ kpc⁻¹ |
| Corotation Radius | \( R_{\text{cor}} \approx 3.3 \) kpc |
| A Miyamoto-Nagai disk potential | \( \Phi(x, y, z) = -\frac{G M_{\text{bar}}}{\sqrt{x^2+y^2+(6.8+0.2y^2+z^2)^2}} \) |

Note: Galactocentric distance \( R_0 = 8 \) kpc is assumed. The volume density of the bar is truncated outside corotation.
Our bar model has not yet reached mathematical consistency. It may mean that our orbit library still misses a few important orbits, or the Dwek et al. volume density model is not physically possible to every detail. Since it is a rather arbitrarily parametrized 3D model obtained by fitting 2D maps, and it still has systematic residuals in fitting the COBE light distribution (see fig. 3 of Dwek et al. and our Fig. 2), there is no reason to believe that the volume density model has to be realizable to realize every detail. More meaningful deprojected models should use orbits as basis functions.

To compare with observation, we show the observables projected on to the sky plane. The upper panel of Fig. 5 compares the projected density of the orbits with that of our input model. The agreements in the boxyness and asymmetry are good, considering also that the Dwek et al. model is not constrained by the COBE map beyond 10°. The r.m.s. residual is 5 per cent. The model is also compared with the COBE map in Fig. 2 after adding a disc. The dynamical model fits the Dwek et al. model in projected density as well as the COBE map in the range of |l| < 10° and |b| < 6°.

More interesting predictions are the rotation field and dispersion field shown in the middle two panels of Fig. 5. Both maps are smooth and regular. The mass of the bar, (2.2 ± 0.2) × 10^10 M☉, is normalized by Baade's window dispersion 113 ± 6 km s⁻¹. The rotation field is shown to have nearly evenly spaced contours and is indicative of a solid body rotation field with a mean slope of (100 ± 10) km s⁻¹ kpc⁻¹. This is somewhat higher than the observed rotation rate of various bulge tracers. The model velocity dispersion declines away from the centre. A detailed prediction is given in Table 5.

To inspect the model's predictions of the dispersion more closely, we also plot the velocity dispersion along the minor axis and other slices of the bulge for the model and observations in Fig. 6. The model fits the minor-axis data, including the dispersion at Baade's window and the general trend of drop-off along the minor axis. The data points are from M

| Table 2. Relative r.m.s. residuals of the bar model. |
|---|---|
| Fitting quantities | RMS  |
| volume density | 7%  |
| potential | 3%  |
| surface density | 5%  |

| Table 3. Kinematics of the COBE bar and the model. |
|---|---|---|---|
| Fields | Observed Velocity (km s⁻¹) | Ref. | Model Velocity (km s⁻¹) |
| l = 1°, b = -4° | σ_r = 113 ± 6 | (1) | σ_r = 113(2.2×10^10 M☉) 1/2 |
| | σ_l > σ_r > σ_b | (2) | σ_l > σ_r > σ_b |
| | σ_r̂ ≠ 0 | (2) | σ_r̂ < 0 |
| | V_r = -75 ± 24, σ_r = 127 ± 17 | (3) | V_r = -40, σ_r = 128 |
| l = 8°, b = 2° | V_r = 45 ± 10, σ_r = 85 ± 7 | (4) | V_r = 80, σ_r = 57 |
| l = 10°, b = -10° | V_r = -82 ± 8, σ_r = 67 ± 6 | (5) | V_r = -89, σ_r = 91 |
| l ≈ b ≈ 0° | σ_r̂ ≈ 110 | (6) | σ_r̂ ≈ 117 |
| l ≈ 0°, | σ_r̂ ≈ (120 - 70) | (7) | σ_r̂ ≈ (120 - 70) |
| | V_r = (5 - 10) × l | (8) | V_r ≈ 14 × l |

Note: ‘Ref.’ refers to (1) Sharples et al. (1991), Terndrup et al. (1995); (2) Spaelenauer et al. (1992), Rich (1988, 1990), Zhao et al. (1994); (3) Blum et al. (1994, 1995); (4) Minniti et al. (1992); (5) Morrison & Harding (1993); (6) Lindqvist (1992a,b); (7) Terndrup et al. (1996); (8) Izumiura et al. (1995), plus references in Dejonghe & Habing (1993).
Figure 4. A comparison of the mass distribution of the orbits (solid line) with our input volume density (dashed line) in the yz- and xy-planes. Contours are spaced with a factor of 2 interval. The r.m.s. difference between the two models is only 7 per cent of the mean density.

giants at Baade's window (Sharples et al. 1990) and several other fields on the minor axis (Terndrup et al., in preparation), and one data point representing a typical 110 km s$^{-1}$ dispersion at 100 pc for the central OH/IR stars (Lindqvist 1992a,b). Note that we did not impose the fall off along the minor axis as a constraint to our bar model, just as in the oblate rotator model of Kent (1992). It appears to be an inevitable prediction of models without strong intrinsic velocity anisotropies.

The upper panel of Fig. 6 also shows the run of the velocities along the $b = -4^\circ$ longitude slice, and the $b = -7^\circ$ slice. These two slices and their positive counterparts set the boundaries of observable low-extinction region where the bulge light still dominates the disc light. This intermediate range has been the target for velocity surveys of the bulge (e.g. Izumiura et al. 1995). In addition to the nearly cylindrical rotation of the bulge, the model shows almost no dependence of velocity dispersion on longitude.

A summary of the bar's observed kinematic data and model predictions is given in Table 3. Overall, the model fits the observed line-of-sight velocities and dispersions reasonably well, except that it predicts too much rotation for the bulge at large radii. The following three simplifications may have contributed to the rapid rotation: (1) we did not explicitly use any retrograde orbits, (2) the observed stellar populations in the bulge do not exactly follow a ubiquitous solid-body rotation law (Izumiura et al. 1995; Minniti et al. 1992; Lindqvist et al. 1992a), and (3) the Dwek et al. model may be too boxy and barred near corotation. It is likely that one needs some extra retrograde orbits to fit the kinematics at large radii.

The model also shows several characteristic signatures of a bar in its proper motions. Bars have anisotropic velocity ellipsoids. If $\sigma_1$ and $\sigma_2$ are the proper motion dispersions for
Table 5. Predicted distributions of the line-of-sight velocity \( V \) and dispersion \((\sigma)\) over the inner 10° of the Galactic bulge.

| \( b \) | \( l = \pm 1° \) | \( l = \pm 3° \) | \( l = \pm 5° \) | \( l = \pm 7° \) | \( l = \pm 9° \) |
|---|---|---|---|---|---|
| 0° | 24(117) | 42(134) | 74(118) | 88(107) | 97(88) |
| | -25(117) | -43(132) | -80(115) | -101(100) | -113(88) |
| 1° | 27(131) | 45(133) | 76(118) | 96(105) | 103(87) |
| | -28(129) | -47(131) | -83(113) | -107(99) | -110(86) |
| 2° | 37(130) | 55(126) | 78(112) | 99(101) | 106(88) |
| | -39(128) | -61(122) | -88(109) | -115(95) | -125(82) |
| 3° | 20(122) | 51(120) | 76(104) | 100(96) | 108(84) |
| | -23(120) | -58(115) | -92(100) | -120(89) | -136(79) |
| 4° | 15(113) | 52(110) | 70(97) | 99(86) | 117(77) |
| | -25(111) | -64(108) | -94(93) | -112(80) | -128(76) |
| 5° | 12(98) | 44(98) | 68(93) | 87(74) | 108(68) |
| | -22(102) | -61(95) | -96(83) | -115(74) | -132(68) |
| 6° | 9(84) | 30(84) | 66(82) | 88(68) | 97(60) |
| | -30(81) | -60(78) | -90(76) | -106(79) | -97(82) |
| 7° | 0(72) | 30(74) | 62(75) | 84(62) | 94(57) |
| | -36(69) | -65(69) | -77(81) | -68(80) | -91(69) |
| 8° | 3(67) | 36(64) | 76(74) | 79(57) | 82(57) |
| | -31(69) | -58(84) | -100(88) | -113(60) | -92(67) |
| 9° | 22(83) | 43(67) | 86(63) | 103(60) | 71(32) |
| | -35(83) | -57(75) | -94(79) | -98(66) | -65(75) |
| 10° | 14(63) | 33(60) | 58(64) | 90(59) | 68(41) |
| | -43(71) | -53(58) | -81(65) | -98(75) | -96(61) |

Note: at each latitude \( b \), the upper and lower rows are for the positive and negative longitude \( l \) fields respectively. Both \( V \) and \( \sigma \) are in units of km s\(^{-1}\) and in the galactocentric frame; the latter is bracketed in the table. The model is not able to reliable predictions beyond 10° from the centre, where the surface density of the bar is low and better disc and halo models are necessary.

![Figure 6](https://academic.oup.com/mnras/article-abstract/283/1/149/960571/1531c16b96c571.png)

**Figure 6.** The upper panel plots the radial velocity dispersion \((\sigma_r)\) and rotation velocity \((V_r)\) along the \( b = -4° \) slice, the \( b = -7° \) slice from the model. The lower panel plots minor-axis runs of the radial-velocity dispersion for the model (solid line) and observations (diamond symbols), the longitude and latitude proper-motion dispersion \( \sigma_\alpha \) and \( \sigma_\beta \) and the cross term \( \sigma_{\alpha\beta} \) and its measurement at Baade’s window (asterisk). Note the fall-off of dispersion on the minor axis. See text for the references to the data points.

The Galactic bar integrated over a line of sight, one expects that \( \sigma_v \geq \sigma_r \) and \( \sigma_v \geq \sigma_\beta \) due to both the intrinsic anisotropy and rotation broadening in the \( l \)-direction. One also expects that the cross term of the velocity ellipsoid \( \sigma_{\alpha\beta} \neq 0 \) due to triaxiality (Zhao et al. 1994), where \( \sigma_{\alpha\beta} = \text{sign}(u) \sigma_\alpha \sigma_\beta \), and \( u = \langle v_\alpha v_\beta \rangle \) is the sign of the cross term tells the orientation of the velocity ellipsoid in the \( v_\alpha \) versus \( v_\beta \) plane.

These signatures are clearly seen in our bar model. Tables 5 and 6 give detailed predictions of the velocity ellipsoid for fields within 10° of the centre. The lower panel of Fig. 6 also shows the four moments of the velocity ellipsoid along the minor axis, \( \sigma_\alpha \) is clearly systematically larger than \( \sigma_r \) and \( \sigma_\beta \), and the cross term \( \sigma_{\alpha\beta} \neq 0 \) for almost the entire minor axis. At Baade’s window, the model predicts \( \sigma_{\alpha\beta} / \sigma_\alpha = 1.3 \pm 0.1 \), larger than the Spacynkaerz et al. (1992) observation of \( \sigma_{\alpha\beta} / \sigma_\alpha = 1.15 \pm 0.06 \) for all the K and M giants, but consistent with that of their metal-rich subsample. On the other hand, the vertex deviation shown by the cross term is a more definitive means of showing the triaxiality of the bulge. At Baade’s window, we predict \( \sigma_{\alpha\beta} / (\sigma_\alpha \sigma_\beta)^{1/2} = -0.4 \). The result confirms the vertex deviation seen in a small overlap sample with complete velocity information and in the previous semiconsistent model (Zhao et al. 1994), and strengthens the argument that the metal-rich bulge is triaxial.

In terms of planning future observations, we find the combined proper-motion and radial-velocity data is more sensitive to the triaxiality of the bar than separate samples. Although the observable velocity moments show asymmetry with respect to the \( l = 0 \) axis as a result of perspective effects of the bar (see Tables 5 and 6), the typical difference of only 20 km s\(^{-1}\) in velocity \( V \) and dispersions \( \sigma_v \) and \( \sigma_r \) can be difficult to observe with a typical sample size of 100 stars per field. On the other hand, our model predicts that \( \sigma_{\alpha\beta} < \sigma_\alpha \) on the minor axis and it will change sign (become positive) for fields at negative longitude and high latitude. As it is easier to distinguish between two perpendicular velocity ellipsoids, it may be worth the effort to measure radial velocities of a proper-motion sample with about 100 stars on the minor axis or two proper-motion samples at opposite longitude fields of the bulge.

In summary our bar model fits the observations of the Galactic bulge in the light distribution and in the kinematics and is theoretically consistent. The combination of
Table 6. Predicted distributions of the cross term $\sigma_x$ and proper motion dispersions $[\sigma_y, \sigma_z]$ for the Galactic bulge.

| $|\alpha|$ | $l = \pm 1^\circ$ | $l = \pm 3^\circ$ | $l = \pm 5^\circ$ | $l = \pm 7^\circ$ | $l = \pm 9^\circ$ |
|---------|-----------------|-----------------|-----------------|-----------------|-----------------|
| $b = 0^\circ$ | -34[133 127] | -41[134 132] | -51[127 131] | -46[116 124] | -42[101 119] |
| $b = 1^\circ$ | -45[143 134] | -47[134 129] | -50[125 128] | -48[115 122] | -42[103 117] |
| $b = 2^\circ$ | -44[143 134] | -49[135 128] | -37[128 124] | -20[118 120] | 21[109 114] |
| $b = 3^\circ$ | -57[141 122] | -55[135 120] | -52[122 119] | -49[115 115] | -46[97 109] |
| $b = 4^\circ$ | -53[140 123] | -46[136 119] | -29[128 116] | -25[114 111] | 18[105 107] |
| $b = 5^\circ$ | -59[146 118] | -55[134 112] | -47[121 111] | -44[112 106] | -45[97 102] |
| $b = 6^\circ$ | -57[148 117] | -40[137 110] | -23[128 105] | 8[112 100] | 31[96 99] |
| $b = 7^\circ$ | -58[140 106] | -54[133 102] | -36[116 103] | -36[105 97] | -41[92 89] |
| $b = 8^\circ$ | -57[141 106] | -38[135 103] | -23[125 100] | 17[109 93] | 43[91 93] |
| $b = 9^\circ$ | -53[141 97] | -57[130 94] | -55[114 94] | -37[91 89] | -37[87 83] |
| $b = 10^\circ$ | -46[137 97] | -40[129 93] | -25[115 88] | 45[105 91] | 46[103 90] |

Note: see the note for previous table.

5 RESULTS: RELATIVE FRACTIONS OF THE MAJOR ORBIT FAMILIES

In this section, we look into the phase space of the model to get a quantitative understanding of the different roles of the major orbit families in the Galactic bar. Fig. 1 shows the distributions of the orbits in our model. No irregular orbits, retrograde orbits or orbits without fixed sense of rotation are explicitly used to build the model (see Section 2.4). The direct regular orbits and the composite orbits of increasing weight are indicated with diamond and plus symbols of increasing size. The plus symbols trace a one-parameter sequence of composite orbits as a function $E$, except for a gap between $E, = -2.5$ and $E, = -2.2$. In the nucleus, there are some composite orbits and direct $x_5$ orbits (Binney & Tremaine 1987) ($y/x \sim 1$ and $E, < -2.2$). The direct banana orbits are in the region ($y/x \sim 0.5, z/R > 0.6$). The axial ratio of the Dwek et al. model would be at the coordinate ($y/x = 0.4, z/R = 0.25$) in the middle panel. The lower panel also shows the amount of time dependency for the regular direct orbits at the end of the integration, which is less than 1 per cent after 100 epicycles.

The roles of various major orbit families in the model are shown more quantitatively in Fig. 7 by their cumulative fractions as a function of the Jacobi integral of an orbit. One can see that the mass is roughly equally divided between the direct regular orbits and the composite orbits. The direct regular orbits are mostly $x_3$ orbits with about 1/5 in banana (2:1:1) family, and very little in $x_5$ orbits. Comparing typical surface-of-sections for a 2D bar potential (see, for example, fig. 13 in Sellwood & Wilkinson 1993), we estimate that the composite orbits here imply roughly 2/3 in irregular orbits, and 1/3 in retrograde $x_4$ orbits, and virtually zero in direct orbits, which are concentrated to small regions of the phase space. When we run other simulations without composite orbits but with explicit retrograde orbits and irregular orbits (Zhao 1994), we find similar fractions, namely, about 15 per cent of the mass of the bar in the retrograde orbits and 30 per cent in the irregular orbits.

The large fraction of composite orbits in our model (~45 per cent) comes from consistency with the minor-axis density profile of the Dwek et al. bar. Both the volume density model of Dwek et al. and the dust-corrected boxy...
Figure 7. This figure shows the fractions of various types of orbits in the model with the energy lower than $E_J$. The Jacobi energy $E_J$ measures the radial extent of an orbit; at $E_J = -2.2$ and $-1.8$, the typical radius of an orbit is 0.5 kpc and 1 kpc respectively. While the regular direct boxy orbits are the dominant orbits, 40 per cent of the mass of the bar is in the composite orbits (labelled collective-orbit here).

COBE maps imply that stars do not clear out a region near the rotation axis of the bar. Only irregular orbits which are lumped in composite orbits can go through the rotation axis; they are parented by the unstable closed orbits on the short axis of a rotating triaxial potential with a central nucleus. A stationary nearly prolate bar with a finite core and an axis ratio 1.9:1:0.7 could have many regular box orbits (Schwarzschild 1979) which go though the minor axis and the centre, and some long-axis tube orbits which go through the rotation axis. Apparently in our bar the entire rotation axis is destabilized by the pattern rotation (Heisler, Merritt & Schwarzschild 1982) and the nucleus. The box orbits and loop orbits end up mostly in irregular orbits or some in the retrograde $x_4$ orbits or the tilted-loop orbits (anomalous retrograde orbits).

The significant fraction of retrograde orbits (15 per cent) and irregular orbits (30 per cent) in our model, as implied by composite orbits, may shed some light on the formation history of the bar. A working scenario would be that a bar develops from instability of a thin disc. It bends out of the plane a few rotations later and resymmetrizes to a thickened bulge. According to the KAM theory (e.g. Moser 1983), regular orbits far from the resonances of an integrable system can survive adiabatic changes in potential. As virtually all disc stars (except those in the rare counter-rotating discs) have a direct sense of rotation by definition, the retrograde or irregular orbits in the developed bar must come from scattering by the resonances and the rapid fluctuations in potential in the bending phase. Gravity is weak away from the plane. Many orbits scattered to high latitude can easily switch their sense of rotation, and some stars can populate the retrograde and irregular orbits during this phase. It would be interesting to follow how orbits change the sense of rotation in $N$-body simulations and test if the changed fraction can be used as an indicator of the strength of relaxation in the bending phase.

6 RESULTS: $N$-BODY TEST OF STABILITY

An orbit model is not guaranteed to be stable. A stability study of our bar model is important for two main reasons. The model would be of less interest if it fitted the COBE
map but was unstable. Also we need to check whether the power-law nucleus and/or the 7 per cent r.m.s. residual in reaching theoretical consistency with the input volume density leads to a strong evolution of the system.

To study stability and evolution of the model, we convert our orbit model into an N-body model. To generate an N-body initial condition, particles of equal mass are sampled at a random phase of an orbit, and the number of particles on an orbit is proportional to the weight of the orbit. The particle system is then evolved with the Self-Consistent Field (SCF) code (Hernquist & Ostriker 1992). We evolve \( N = 30 \) K particles with a time-step of 1 Myr \((1/100\) of the rotation period). We keep the SCF expansion terms with quantum numbers up to \( n = 6 \) and \( m = l = 4 \), including the odd \((l, m)\) terms. A rigid MN-disc potential is added on top of the bar potential.

The N-body simulation with a rigid disc shows that the residual is not significant enough to cause disruption of the bar or strong dynamical evolution of the bar; the conclusion seems to be also supported by simulations with a live disc run with a different code (private communication with Roger Fux). The system relaxes to a configuration close to the initial one in one rotation \((0.2 \) Gyr) and remains in nearly steady state for at least another 9 rotations. Fig. 8 shows snap shots of the model at \( t = 0 \) and \( t = 1 \) Gyr. Because we sample particles from a small library of orbits, the initial condition still shows the discreteness of the orbits, e.g. the faint loop structures seen in the initial snap shot. But after one rotation these features are washed out. The overall shape and density at the initial and final states are similar except that the final state is somewhat rounder than the initial state; the axis ratio changed from \( 1:0.6:0.4 \) to \( 1:0.7:0.4 \). A more detailed look at the variation is shown in Fig. 9.

Figure 8. A plot of an N-body realization of the steady-state model (the left panels), and the configuration after evolving for 10 rotation periods (the right panels). The solid line indicates our line of sight to the centre. Note the elongated bar shape in the face-on view (the lower panels) and the boxyness in the edge-on view (the upper panels) are similar at two epochs. The final bar has settled down to dynamical equilibrium.

It is well-known that equilibrium systems satisfy the (steady-state) Virial theorem, namely \( W + 2K = 0 \), where \( W \) is the Clausius Virial of the system and \( K \) is the total kinetic energy of the system. This allows us to measure how close the model is to equilibrium by computing \(-2K/W\). Fig. 9 shows \(-2K/W\) versus \( t \) for our N-body model. At \( t = 0 \), \(-2K/W = 0.98\) for the model. It stays close to unity for the next ten rotations. The initial deviation from unity is because the potential is slightly different from that which we used in orbit calculation. The figure also shows the moments of inertia of the bar as a function of time. The axial ratio of the bar \((I_x/I_y, I_x/I_z)\), measured by its moments of inertia along the three principal axes, settles to a constant value \( 1:0.7:0.4 \) after some oscillation in the first rotation period. One can see that the cross term \( I_{xy} \) measured in the rest frame follows closely a sinusoidal curve with a constant amplitude and constant period, which is half of the bar’s period \( 2\pi/\Omega = 0.1 \) Gyr. A more quantitative inspection of the pattern speed finds that it is a slowly decreasing function of time with a relatively large oscillation in the first rotation period.

It is interesting to compare our N-body experiment for the Galactic bar with the experiments for Schwarzschild’s triaxial galaxy model by Smith & Miller (1982). Their simulations were carried out for 100 000 particles for about 6 dynamical time-scales in both non-self-gravitating and self-gravitating conditions. The major semi-axis typically increased by 20 per cent in the inner region and dropped by 5 per cent in the outer region in the initial quarter of a dynamical time, followed then by a lasting gentle contraction at all radii until the end of the run (see their fig. 5). Overall the major semi-axis shortened by 20 per cent while the intermediate and the minor semi-axis appeared to have little evolution (see their fig. 4). Based on these results, they claimed that the model of Schwarzschild, which was in rigorous equilibrium by design, was robust without growing disturbance of more than 0.5 per crossing time. Comparing with our experiments, although our bar is not designed as consistently as the Schwarzschild model, it has similar amplitudes of the initial oscillations and seems to have settled more quickly to quasi-equilibrium (in one rotation). The strongest evolution is in the pattern speed, a unique property of bars, which declines with less than 10 per cent per Gyr. While the long term stability and interactions with a live disc and halo remain to be investigated, the bar made from our orbit model is in a stable quasi-equilibrium.

Contrary to the expectations (e.g. Hasan, Pfenniger & Norman 1993), our bar is not destroyed by the nucleus. While how this preliminary result depends on N-body codes and softening parameters also remains to be investigated, here we only comment that \( x \) orbits of a bar do not pass very close to the centre as box orbits do. About 5–10 per cent of our bar’s mass is enclosed inside 0.5 kpc (about the radius of the inner Lindblad resonance), which may not be sufficient to destroy the bar. Also previous models used a rigid nucleus while our nucleus is initially populated from orbits.

In a way the above findings are expected since the phase space of the model is constrained by matching a system made by nature. The thickness of the bar (Fig. 8) and the absence of counter rotation, strong anisotropy and sharp variations in the velocity distribution (Figs 5 and 6), all
argue against the bending instability of a thin and cold bar (Merritt & Sellwood 1994).

7 SUMMARY

We have built a 3D steady-state dynamical model for the Galactic bar using a generalized Schwarzschild technique. 325 regular direct boxy orbits are integrated in a rapidly rotating bar potential with corotation at 3.3 kpc, which includes an MN disc, the Dwek et al. bar and the $r^{-1.85}$ nucleus. These direct regular orbits make up the dominant fraction (55 per cent) of the mass of the bar, including 10 per cent banana (2:2:1) orbits. The remaining 45 per cent of the mass is distributed in 160 composite orbits, the distribution functions of which are delta functions of the Jacobi integral $E_J$. These composite orbits imply about 15 per cent of the mass of the bar in retrograde orbits and 30 per cent in irregular orbits. The mass of each orbit is determined with the NNLS method by fitting the observations with a smooth distribution in the phase space.

The model fits Dwek et al.'s luminosity model for the COBE bulge (see Figs 2, 4 and 5) and existing kinematic data. In particular, the orbit model fits the velocity dispersion at Baade's window and the observed solid-body rotation curve of the bulge tracers. The observed fall-off of radial dispersion along the minor axis, the vertex deviation and the proper motion anisotropy at Baade's window follow naturally from the model (see Fig. 6). The model is also in agreement with the flat gas rotation curve of the inner Galaxy and the asymmetry of light seen in the COBE infrared maps of the bulge.

The bar is also evolved with an N-body code and is found to be stable.

8 OTHER APPLICATIONS

Steady-state bar models have many other potential astronomical applications. Up to now the MACHO team and the OGLE team have together obtained more than 100 microlensing events in several fields of the bulge. If most of the lenses are at the front end of the bar (Zhao et al. 1995), then these experiments can in principle yield the lower end of the mass function of stars 6–8 kpc from us. However, a reliable prediction on mass function depends critically on a correct statistical distribution for the proper-motion velocity and distance of both the lens and the source. In this aspect it is the kind of steady-state bar models presented here that are most suited.

As we have shown in Section 6, steady-state models are robust tools to set up nearly equilibrium initial conditions for N-body studies of stability and secular evolution. They are most useful for general systems without analytical distri-
bution functions. One can in principle set up the full range of equilibria, including theoretically interesting triaxial or axisymmetric models with three integrals of motion and models that fit particular observations. These initial states are well suited to address the stability, the life span and the range of bars, and the interactions among the bar, the disc and the halo.

Looking beyond the COBE bar, steady-state models can have important applications in ellipticals, extragalactic bulges and nuclei, which share their basic dynamics with the Galactic bar. There is plenty of evidence from stellar light and kinematics which suggests that massive central black holes exist in nearby galactic nuclei. The main uncertainty is due to a well-known ambiguity: the unknown amount of radial anisotropy can masquerade as the gravity of a dark component. Previous models such as models for the M32 nucleus cannot lift this degeneracy because they rely on simplifications such as (i) the potential is spherical (Dressler & Richstone 1988), or at least axisymmetric, and (ii) the intrinsic velocity has an isotropic Gaussian distribution, or at least a distribution with only two integrals of motion \( f(E, L_z) \) (Qian et al. 1995; Dehnen 1995). Since there is no compelling reason to believe that real galactic nuclei satisfy the above assumptions, a water-tight case for a central black hole can only be established by searching the range of the solution space in triaxial models or at least axisymmetric models with three integrals of motion. The available high-quality data and the new techniques of deriving velocity profiles (for example van der Marel & Franx 1993) should give theorists additional incentives to explore beyond a few relatively easy-to-compute models.

Our orbit-construction program is well suited to these systems, as it works with the minimal assumptions of theoretical consistency, positivity and smoothness. It is a more proper technique than solving truncated equations, as one can actually fit the kinematic data by adjusting the weights of the orbits. This unique property allows one to make complete use of the data, which now includes skewness and the kurtosis of the line profile as well as the streaming velocity, dispersion and surface brightness distribution at subarcsec seeing at many positions of the nucleus. As a technical point, the Gaussian–Hermit expansion coefficients \( h_n \) with \( n = 1, 2, 3, 4, \ldots \) (e.g. Gerhard 1993), all can be programmed as a linear equation of orbit weights similar to equation 2.

The uniqueness of the Galactic bar model still needs to be understood: how many different combinations of orbits fit the observations for a given bar potential? How much can the parameters of the bar potential vary and still be consistent with the observations? The involved computation is huge. Based on some preliminary results given by Zhao (1994), where models with or without a disc or nucleus and models with or without direct or retrograde orbits are investigated, one can say quite certainly that the COBE map and the minor-axis radial velocity dispersions together are still not enough to constrain the bar model uniquely. But one can hopefully limit the parameters of the bar to a narrow range with the combined constraints from microlensing and gas kinematics (Zhao et al. 1996), and with detailed stellar radial-velocity and proper-motion data as well as theoretical consistency and stability.

Several ongoing stellar proper-motion studies of the bulge will yield interesting dynamical constraints in the next few years. Already several thousands of bulge giants at the \((-1°, -8°)\) field (Mendez et al. 1996) and dozens of infrared bright sources in the central pc (Eckart & Genzel 1996) have been measured for proper motions. The more informative 3D velocity ellipsoid and even orbits of real stars can be constructed if radial velocities and distances are also measured for the same proper-motion sample (Zhao et al. 1994). Such a data set now exists for a few hundred bright stars at Baade’s window (Spannhauer et al. 1992; Terndrup et al. 1995). The steady-state model here shows a first step towards linking a variety of observations of the Galaxy in a consistent and unique model.

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Similarly when the observable is a high-velocity moment \( \eta_n \) with \( n = 2, 3, 4 \) except that \( \gamma_{j,k} \) changes to the higher-velocity moments of the \( k \)th orbit.

A special case is when the observable \( \mu_i \) is the fraction of orbit mass inside the 3D cell, which corresponds to the volume density. In this case, \( \gamma_{j,k} \) is the fraction of time that the \( k \)th orbit spends in the 3D cell, and \( \beta_{j,k} = 1 \) for all \( j \) and \( k \). The equation is simplified to

\[
\mu_i = \frac{\sum_{k=1}^{N} \gamma_{j,k} w_k}{\sum_{k=1}^{N} w_k}, \tag{A1}
\]

which is almost identical to equation (1) of Schwarzschild (1979) except that the total mass of the orbits \( \left( \sum_{k=1}^{N} w_k \right) \) is left as a free parameter here.

**APPENDIX B: DETAILS ON LAUNCHING AND INTEGRATING ORBITS**

We control the initial orbit distribution in the following way. Most orbits are launched, similar to Schwarzschild (1979, 1982) and Pfenniger (1984), in close pairs perpendicularly from the \( xx-, yy- \) or \( xy \)-symmetry plane or the \( x- \) or \( y \)-symmetry axis. To guarantee completeness, a small fraction of orbit pairs are also launched tangentially either from the minor axis or from no particular positions. In all cases the initial radius is sampled uniformly between zero to 4 kpc, and the initial velocity is tangential and less than the local circular velocity so that the starting point is always a local apogalacticon. To reduce redundancy of orbits the specified angles of position and velocity vectors are assigned so that different orbits maximally avoid similar initial coordinates.

We stop the integration of an orbit when it has switched its radial velocity and turn back for more than \( 2 \times N = 200 \) times, which is roughly 50–100 rotation periods. During the integration of the orbits, we keep track of the effective integrals (time-averaged energy, angular momentum etc.) of an orbit, and trace the deviation between an orbit pair to classify it as a regular or irregular orbit. An orbit is taken as a regular orbit if (1) the effective integrals of the orbit do not differ by more than 1 per cent if time-averaged at \( T \) and at time \( T/2 \) (see Fig. 1), where \( T \) is the length of integration, (2) the pair orbit has similar effective integrals, and (3) the instantaneous deviation between the pair does not exceed 1 per cent. Orbits which do not satisfy all these are taken as irregular orbits, which appear as two kinds: in volume space some look like a featureless cocoon, others still have a clear structure like the regular orbits, typically a central hole at the end of integration. The effective integrals of the former cocoon orbits fluctuate at the 10 per cent level, and the latter at between 1 to 10 per cent level at the end of integration. Also based on the energy and angular momentum of an orbit at each time-step we can throw away orbits less interesting for our model. These include: (1) retrograde orbits, (2) orbits without fixed sense of rotation, and (3) orbits which have a tendency to escape from the bulge, that is, at one time they reached beyond 7 kpc or had a positive instantaneous energy \( E(i) \) (not the conserved \( E_i \)). In addition, we throw away (4) the irregular orbits.

**APPENDIX A: THE MEANINGS OF THE QUANTITIES IN EQUATION (1)**

When the observable \( \mu_i \) is the mean velocity \( \eta_i = V \) in a line of sight, the matrix element \( \beta_{j,k} \) specifies the fraction of time that the \( k \)th orbit spends in this line of sight (convolved with any observational cuts on the sample distance). The matrix element \( \gamma_{j,k} \) is the velocity when the \( k \)th orbit is in the line of sight or the average velocity if the orbit crosses several times. Summed over all orbits in the line of sight, the denominator or the right-hand side of equation (1) gives the surface density and the numerator gives the total flux. The ratio of the two gives the mean velocity.

\[ \gamma_{j,k} = \frac{w_k}{\sum_{k=1}^{N} w_k} \]

\[ \beta_{j,k} = \frac{\sum_{k=1}^{N} \gamma_{j,k} w_k}{\sum_{k=1}^{N} w_k} \]

\[ \eta_{j,k} = \mu_i \]

\[ \eta_{j,k} = V \]

\[ \eta_{j,k} = \frac{w_k}{\sum_{k=1}^{N} w_k} \]

\[ \eta_{j,k} = \frac{\sum_{k=1}^{N} \gamma_{j,k} w_k}{\sum_{k=1}^{N} w_k} \]

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To increase orbit integration efficiency and reduce storage, we use the Bulirsch–Stoer integrator (Press et al. 1992). We can integrate with large time-steps in a smooth potential, and store the position, velocity and acceleration to disc at fewer time-steps. After all the orbits are computed, the stored orbit points are interpolated with a fifth-order polynomial at finer time-steps piecewisely without propagating any small error from one big time-step to another. The interpolated orbit points are then binned according to the position, velocity, sky direction, or line-of-sight velocity to obtain the final information of the projected spatial and velocity distribution of the orbits.

For each composite orbit and direct regular orbit we need to compute and store its contribution to the bar’s: (a) volume density in a 3D, $10 \times 10 \times 10$ rectangular grid in the first octant with the cell size in the $x$, $y$ and $z$ directions being 200, 150 and 100 pc; (b) projected density with one square degree resolution in longitude $l$ and latitude $b$ within $0^\circ \leq b \leq 10^\circ$ and $-16^\circ \leq l \leq 16^\circ$, and (c) 2 non-zero observable flux moments ($\mu_\nu$, $\mu_\sigma$) and 4 non-zero pressure ellipsoid moments ($\mu_\nu^2$, $\mu_\nu \mu_\sigma$, $\mu_\sigma^2$, $\mu_\nu \mu_\pi$) at the same projected grid. Other moments, for example, $\mu_\nu_0$, $\mu_\nu_0 \nu_\pi$, $\mu_\nu_0 \pi_0$ are set to be zero.

Above we have exploited the symmetries of a bi-symmetric bar to save storage. Since the orbital equations in a bar potential preserve the reflection symmetries ($z \rightarrow -z$, $v_z \rightarrow -v_z$), ($y \rightarrow -y$, $v_y \rightarrow -v_y$), and ($x \rightarrow -x$, $v_x \rightarrow -v_x$), at most 16 orbits can be mirror images of each other. We always equally populate the 16 orbits in the model with the consideration that a steady-state model is unlikely to crucially depend on any minor-axis rotation or any $m=1$ or $m=2$ spiral arm modes.

It takes typically 1 m of CPU time on an IBM RS/6000 workstation (with computing power roughly equivalent to a Sparc 100) to select and integrate one orbit. One needs about 10 h of CPU and 0.1 Gbyte disc space to compute and store every 1000 orbits. The task of orbit integration can also be spread out among several processors computing different orbits at the same time as in Zhao (1994). The NNLS calculation takes a modest amount of CPU time (typically one or two h), but needs a large work space. To program the $n_c$ constraints and $n_p$ smoothness equations in double precision, one needs roughly $10(N + n_c/1000)(N/1000)$ Mbyte work space, where $N$ is the number of orbits. Typically with $N \sim 1000$–3000 the code can run on Sparc stations with medium swap space.