Renormalons in the effective potential of the vectorial \((\vec{\phi}^2)^2\) model

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Abstract

We study the properties of ultraviolet renormalons in the vectorial \((\vec{\phi}^2)^2\) model. This is achieved by studying the effective potential of the theory at next to leading order of the \(1/N\) expansion, the appearance of the renormalons in the perturbative series and their relation to the imaginary part of the potential. We also consider the mechanism of renormalon cancellation by “irrelevant” higher dimensional operators.
I. INTRODUCTION

The $(\vec{\varphi}^2)^2$ theory and QED suffer from non-perturbative singularities at large energy scales \( [1] \). This is a common behaviour of renormalizable “infrared free” quantum field theories in four dimensions and it is a consequence of the presence of the so-called Landau ghost in the running coupling constant. These theories are considered “trivial”: renormalization group arguments tell us that there is no consistent way perturbatively to take the large cut-off limit of the regularized theory without driving to zero the renormalized coupling constant \( [2] \). Another problem is that the next to leading \( 1/N \) order effective potential of the vectorial \( (\vec{\varphi}^2)^2 \) model is complex \( [3] \) for the same reason as that for the Landau ghost, i.e. the ultraviolet behaviour of the running coupling constant. All these aspects are connected to the behaviour of high order terms in the perturbative expansion. The perturbative series may indeed diverge due to two different reasons:

1) proliferation in the number of Feynman graphs. The number of diagrams that contribute to a fixed perturbative order \( n \) grows as \( n! \) (instanton contributions);

2) pathological behaviour associated to single diagrams. The renormalized amplitude of single graphs that at the \( n \)-th perturbative order have a number of subdivergencies of order \( n \) goes as \( n! \) (renormalon contributions) \( [4] \).

The \( 1/N \) expansion allows us to resum the second class of diagrams which are those studied in this paper. It has been argued by \( [5] \) that in this case their contribution is the dominant one. The Borel trasform method provides us a general recipe to treat divergent series \( [4] \), when the Borel trasform exists. According to this method, given some physical quantity whose perturbative series in the coupling constant “\( g \)” is

\[
F[g] = \sum_{n=1}^{\infty} F_n \ g^n = F_1 \ g + F_2 \ g^2 + ... , \tag{1}
\]

one defines its Borel trasform by

\[
B[F[g], t] = \sum_{n=0}^{\infty} F_{n+1} \ \frac{t^n}{n!} = F_1 + F_2 \ t + F_3 \ \frac{t^2}{2} + ... . \tag{2}
\]

Then, formally
\[ F[g] = \int_0^{\infty} dt e^{-t/g} B[F[g], t], \quad (3) \]

in the sense that the expansion for \( B[F[g], t] \) reproduces the expansion for \( F[g] \) term by term. The inverse transform in eq. (3) is meaningful if the integral converges and \( B[F[g], t] \) has no singularities in the range of integration. Unfortunately the \((\phi^2)^2\) theory, QED and QCD are not even Borel summable \[4,8\]. In these theories the Borel transforms of physical quantities have singularities on the positive axis of Borel plane. In order to do the integral (3), one should then specify a prescription to go around the singularities. Unfortunately the result will depend on the prescription which has been chosen. The singularity closest to the origin in \( t \), which we denote as “first renormalon”, gives the leading contribution to the ambiguity of the perturbative series.

Parisi \[7\] conjectured that UV renormalon ambiguities in the \( n \)-point 1-PI Green functions are proportional to the insertion of local “irrelevant” composite operators in these Green functions. The first renormalon can be removed considering the full set of the operators of dimension six \((\phi^6, \phi^2 \partial_\mu \phi \partial^\mu \phi, \ldots)\), the second renormalon considering operators of dimension eight and so on.

In this paper we compute the first renormalon contribution to the effective potential at the next-to-leading order (NLO) in \( 1/N \) and check whether the conjecture of Parisi applies. Since at the order \( 1/N \) at which we work, the effective potential is built using all zero momentum 1-PI Green functions the renormalon contributions are completely removed with the help of higher dimensional operators without derivatives. We show indeed that in order to remove the first renormalon in the effective potential it is sufficient to add to the Lagrangian the operator \((\phi^2)^3\) in agreement with the conjecture of Parisi. Thus, the fact that the NLO effective potential develops an imaginary part does not mean at all that the theory is inconsistent. The Borel transform and the renormalon theory give us a method to isolate this imaginary contribution and cancelling it with the insertion of higher dimensional operators.

This paper is organized as follows: in section II we recall the basic ingredients for the
calculation of the effective potential and introduce the notation; in section III we compute the Borel transform of the potential; in section IV we discuss its renormalization. In section V then, we isolate the first renormalon contribution to the effective potential and we compare the result to Parisi’s hypothesis on 1-PI green functions. The conclusion can be found in section VI.

II. THE EFFECTIVE POTENTIAL IN THE LEADING AND NEXT-TO-LEADING $1/N$ EXPANSION

In this section we give the basic elements and fix the notation of the $1/N$ expansion of the effective potential. Our derivation follows closely those which can be found in the past literature on this subject \cite{3,9}. The starting point is a $$(\vec{\phi}^2)^2$$-vector theory, with $N$ scalar fields $\phi_a$.

The lagrangian density, after the introduction of a constraint-field (lagrangian multiplier) $\hat{\chi}$, can be written as

$$L = \frac{1}{2} \partial_\mu \phi_a \partial_\mu \phi_a + \frac{m_0^2}{2} \phi_a \phi_a + \frac{g_0}{8} (\phi_a \phi_a)^2 - \frac{1}{2g_0} \left( \frac{\hat{\chi}}{\sqrt{N}} - \frac{g_0}{2} \phi_a \phi_a - m_0^2 \right)^2 =$$

$$= \frac{1}{2} \partial_\mu \phi_a \partial_\mu \phi_a - \frac{\hat{\chi}^2}{2g_0N} + \frac{\hat{\chi}}{\sqrt{N}} \phi_a \phi_a + \frac{m_0^2}{g_0} \frac{\hat{\chi}}{\sqrt{N}}.$$

(4)

The fields $\phi_a$ and the constraint $\hat{\chi}$ can be expressed as their vacuum expectation value ($\phi$ and $\chi$ respectively) plus quantum fluctuations. By taking only the first component of the fields $\phi_a$ to have a non-zero vacuum expectation value (VEV), we have

$$\phi_1(x) = \sigma(x) + \varphi \sqrt{N}, \quad \hat{\chi}(x) = \tilde{\chi}(x) + \chi \sqrt{N},$$

$$\phi_a(x) = \varphi_a(x), \quad a = 2, 3, ..., N.$$  

(5)

Up to an irrelevant constant and linear terms, the lagrangian is then given by

$$L = \frac{\infty}{\in} \partial_\mu \varphi_{\pm} \partial_\mu \varphi_{\mp} + \frac{\chi}{\in} \varphi_{\pm} \varphi_{\mp} + \frac{\in}{\in} \partial_\mu \sigma \partial_\mu \sigma + \frac{\chi}{\in} \sigma^\varepsilon + \varphi \sigma \varphi_{\varepsilon} - \frac{\varphi_{\varepsilon}}{\in \varepsilon} \frac{\varphi_{\varepsilon}}{\varepsilon \varepsilon} + \frac{\varphi_{\varepsilon}}{\varepsilon \varepsilon} (\sigma^\varepsilon + \varphi_{\varepsilon} \varphi_{\varepsilon}),$$

where the sum over repeated Latin indices goes now from 2 to N.
A. Basic equations and their tree level solution

The effective potential $V(\varphi, \chi)$ of the modified theory eq. (6) reduces to the effective potential of the original theory when $\chi$ satisfies the requirement

$$\frac{\partial V(\varphi, \chi)}{\partial \chi} \bigg|_{\varphi} = 0.$$  \hfill (7)

We use (7) to define $\chi$ as a function of $\varphi$.

In order to renormalize the potential, we impose the following renormalization conditions

$$\chi(\varphi)\big|_{\varphi=0} = \mu^2,$$
$$\frac{d V(\varphi, \chi(\varphi))}{d \varphi^2} \bigg|_{\varphi=0, \chi=\mu^2} = \frac{\mu^2}{2},$$
$$\frac{d^2 V(\varphi, \chi(\varphi))}{d^2 \varphi^2} \bigg|_{\varphi=0, \chi=\mu^2} = \frac{g N}{4},$$  \hfill (8)

where $\mu^2$ and $g$ are the renormalized mass and coupling constant.

Finally, we introduce the expansion of the potential in powers of $N$

$$V = N V_L + V_{NL} + ..., \quad (9)$$

where $N V_L$ is of order $N$, $V_{NL}$ of order 1, etc.

A similar expansion holds also for $\varphi$ and $\chi$. Thus, for example we may write the solution of eq. (7) as

$$\chi = \chi_0 + \frac{1}{N} \chi_1 + ...$$  \hfill (10)

At lowest order in $1/N$, $\chi_0$ is the solution of

$$\frac{\partial V_L}{\partial \chi} \bigg|_{\chi=\chi_0} = 0.$$  \hfill (11)

It is straightforward to derive the tree level vacuum potential, expressed in terms of the fields $\varphi$ and $\chi$

$$V_{tree} = -\frac{\chi^2}{2g_0} + \frac{N}{2} \varphi^2 + \frac{m_0^2}{g_0} \chi.$$  \hfill (12)
At tree level the inverse propagators of the relevant fields, as derived from the lagrangian (13), are

\[ D^{-1}_{ab} = \delta_{ab}(k^2 + \chi) , \quad D^{-1}_{\sigma\sigma} = k^2 + \chi , \quad D^{-1}_{\sigma\tilde{\chi}} = \varphi , \quad D^{-1}_{\tilde{\chi}\tilde{\chi}} = -\frac{1}{g_0 N} , \]

and they are drown in fig. 1.

Using eqs. (7), (8) and expanding the potential as in eq. (9), we are now ready to compute the effective potential at higher orders in the $1/N$ expansion. Now we show the result of the calculation of the first two terms of the potential expanded as in eq. (13).

**B. Leading order effective potential**

In order to compute the effective potential beyond the tree level, we add to $V_{\text{tree}}$ the contribution of the bubble diagrams shown in fig. 2.a. These diagrams give

\[ \frac{N - 1}{2} \int \ln(k^2 + \chi) \sim \frac{N}{2} \int \ln(k^2 + \chi) , \]

where we have taken only the term proportional to $N$. The subleading term $-1/2 \int \ln(k^2 + \chi)$ contributes to NLO effective potential $V_{NL}$, see below.

Using eq. (14), the unrenormalized LO potential at one loop is given by

\[ N V_L = -\frac{\chi^2}{2g_0} + \frac{N}{2} \chi \varphi^2 + \frac{m_0^2}{g_0} \chi + \frac{N}{2} \int \ln(k^2 + \chi) . \]

In order to renormalize $V_L$ we impose the renormalization conditions eq. (8). At this order these conditions are given by

\[ \chi_0(\varphi^2)_{|\varphi=0} = \mu^2 , \]

\[ \frac{d V_L^{\text{Ren}}(\varphi, \chi(\varphi))}{d \varphi^2} \bigg|_{\varphi=0, \chi_0=\mu^2} = \frac{\mu^2}{2} , \]

\[ \frac{d^2 V_L^{\text{Ren}}(\varphi, \chi(\varphi))}{d^2 \varphi^2} \bigg|_{\varphi=0, \chi_0=\mu^2} = \frac{g N}{4} . \]

The final result for the renormalized LO potential is readily found
\[ V_{L}^{\text{Ren}} = -\frac{\chi_0^2}{2g} + \frac{N}{2} \chi_0 \varphi^2 + \frac{\mu^2}{g} \chi_0 \left( 1 + \frac{gN}{2(4\pi)^2} \right) + \frac{N}{4(4\pi)^2} \chi_0^2 \left( \ln(\frac{\chi_0}{\mu^2}) - \frac{3}{2} \right), \quad (17) \]

where \( \chi_0 \) is the solution of eq. (11) which written in terms of the renormalized potential and fields becomes

\[ -\frac{\chi_0}{g} + \frac{N}{2} \varphi^2 + \frac{\mu^2}{g} \left( 1 + \frac{gN}{2(4\pi)^2} \right) + \frac{N}{2(4\pi)^2} \chi_0 \left( \ln(\frac{\chi_0}{\mu^2}) - 1 \right) = 0. \quad (18) \]

As noticed by [9] the effective potential is afflicted by pathologies for large values of \( \varphi^2 \). The solution of (18) increases monotonically in \( \varphi^2 \) from the value \( \chi_0 = \mu^2 \) at \( \varphi^2 = 0 \) to \( \chi_0 = \mu^2 \exp(32\pi^2/gN) \) at \( \varphi^2 = \varphi_{\text{max}}^2 \), where

\[ \varphi_{\text{max}}^2 = \mu^2 \left[ -\frac{2}{gN} \left( 1 + \frac{gN}{32\pi^2} \right) + \exp\left( \frac{32\pi^2}{gN} \right) \right]. \quad (19) \]

For \( \varphi^2 > \varphi_{\text{max}}^2 \) the solution of equation (18) is found at complex values of \( \chi_0 \). Consequently also \( V_L \) becomes complex.

C. NLO potential

In order to compute the NLO effective potential we proceed as follows.

a) First we calculate the propagator of \( \bar{\chi} \) by resumming all \( \varphi_\alpha \)-bubble diagrams, as shown in fig. 2.b. The result is

\[ D_{\bar{\chi}\chi}^{-1} = -\frac{1}{gN} (1 + gNI(k^2, \chi)), \quad (20) \]

where \( I(k^2, \chi) \) is the renormalized bubble diagram. In dimensional regularization we obtain

\[ I(k^2, \chi) = \frac{1}{2} \left[ \mu^{2\epsilon} \int d\bar{p} \left( \frac{1}{(p^2 + \chi)((k + p)^2 + \chi)} - \frac{1}{4(4\pi)^2} \frac{1}{\epsilon - \gamma_E} \right) \right] = \frac{1}{2} \frac{4\chi}{k^2} \ln \left( \frac{k^2 + 2\chi + \sqrt{k^2 + 4\chi}}{2\chi} \right) - \ln \left( \frac{\chi}{\mu^2} \right), \quad (21) \]

where \( d\bar{p} = d^{4-2\epsilon}p/(2\pi)^{4-2\epsilon} \) is the \( d \)-dimensional phase space and \( \epsilon = 2 - d/2 \).

b) We construct with \( D_{\bar{\chi}\chi}^{-1} \) the NLO effective potential from the diagrams shown in fig. 2.c. We have
\[
\frac{1}{2} \ln \left\{ \det \begin{pmatrix}
\varphi & -1 + gNI(k^2, \chi) \\
1 + gN & \varphi
\end{pmatrix} \right\}.
\]

This term must be added to the subleading part of equation (14), as we mentioned before.

The final formula for the unrenormalized NLO potential is given by
\[
V_{NL} = \frac{1}{2} \int \left[ -\ln \left( \frac{gN}{1 + gNI} \right) + \ln \left( 1 + \frac{gN}{k^2 + \chi} \right) \right].
\]

1. Renormalization of $V_{NL}$

In order to renormalize the potential $V = V_L + V_{NL}$ we have to impose the renormalization conditions of eq. (8). Since at the lowest order we have imposed the conditions in eq. (16), the renormalization conditions for $V_{NL}$ are given by

\[
\chi_1(\varphi^2)|_{\varphi=0} = 0, \quad \frac{dV_{NL}^{\text{Ren}}(\varphi, \chi(\varphi))}{d\varphi^2}|_{\varphi=0, \chi_0=\mu^2} = 0, \quad \frac{d^2V_{NL}^{\text{Ren}}(\varphi, \chi(\varphi))}{d^2\varphi^2}|_{\varphi=0, \chi_0=\mu^2} = 0,
\]

The renormalization procedure is complicated by the presence of quadratic divergences in some of the integrals. The details of the calculations are given in appendix A. Here we give the final result:

\[
V_{NL}^{\text{Ren}} = V_1 + V_2,
\]

\[
V_1 = \frac{1}{2} \int \ln \left( 1 + \frac{\varphi^2}{k^2 + \chi} \right) - \frac{\varphi^2}{2} \int \frac{\Delta}{k^2 + \mu^2} + \frac{\varphi^4}{4} \int \frac{\Delta^2}{(k^2 + \mu^2)^2} + \frac{\varphi^2}{2} (\chi - \mu^2) \int \frac{\Delta}{k^2 + \mu^2} \left( \frac{1}{k^2 + \mu^2} + I'(k^2, \mu^2) \Delta \right) + \frac{\varphi^2}{2} f(\chi) \frac{1}{(4\pi)^2} \int \frac{\Delta^2}{(k^2 + \mu^2)^2}
\]

\[
V_2 = -\frac{1}{2} \int \ln (-\Delta) + \frac{(\chi - \mu^2)^2}{2} \int \frac{\Delta}{k^2 + \mu^2} \left( \frac{1}{k^2 + \mu^2} + I'(k^2, \mu^2) \right) - \frac{(\chi - \mu^2)^2}{4} \int \Delta \left( I''(k^2, \mu^2) - \Delta I'(k^2, \mu^2) \right) + \frac{3}{2(4\pi)^2} g(\chi) \int \frac{\Delta^2}{k^2 + \mu^2} \left( I'(k^2, \mu^2) \right) + \frac{3}{2(4\pi)^2} h(\chi) \int \frac{\Delta^2}{k^2 + \mu^2} \left( I'(k^2, \mu^2) \right) - \frac{j(\chi)}{4(4\pi)^4} \int \frac{\Delta^2}{k^2 + \mu^2}.
\]
where \( \Delta = gN/[1 + gNI(k^2, \mu^2)] \) and the exact form of \( f(\chi), g(\chi), h(\chi), l(\chi), j(\chi) \) is given in the appendix A.

The pathologies of \( V_{NL} \) look even more serious than in the LO case. As noticed in [3], \( V_{NL} \) is complex at every value of \( \varphi \). The reason is that \( D_{\chi\chi}(k^2, \chi) \) crosses the (Landau) pole, \( \Lambda_{Landau}^2 = \mu^2 \exp(2 + 32\pi^2/Ng) \), for large values of \( k^2 \), so that the integration over \( k^2 \) always leads to a complex result. The analytic properties of the potential are better understood in the framework of the renormalon theory. In order to isolate the contribution of the different renormalons it is better to work in the space of the Borel transform.

### III. BOREL TRAFOFORM OF \( V_{NL} \)

In this section we perform the Borel transform of the effective potential. The essential ingredients of this calculations are the Borel trasform of \(-gN/(1 + gNI)\) and its powers:

\[
B\left[ -\frac{gN}{1 + gNI} \right] = N \exp(-NtI), \quad B\left[ -\frac{gN}{1 + gNI} \right]^n = N \frac{(Nt)^{(n-1)}}{(n-1)!} \exp(-NtI), \quad (28)
\]

where \( t \) is the Borel parameter conjugate of the coupling constant \( g \) of eq. (2). It is important to note that even though \( I = I(k^2, \chi) \) is a function of the constraint \( \chi \), and depends on the coupling constant \( g \) via eq. (18), we can perform the Borel trasform before imposing the constraint eq. (18). In this way we can treat \( \chi \) as a variable independent of \( g \).

The Borel trasform of the different terms appearing in \( V_{NL}^{Ren} \) can be obtained in the following way.

i) For the first term in eq. (27) we have:

\[
B\left[ \ln\left( -\frac{gN}{1 + gNI} \right) \right] = \frac{d}{dh} \bigg|_{h=0} B\left[ \left( -\frac{gN}{1 + gNI} \right)^h \right] = \\
= \frac{d}{dh} \bigg|_{h=0} \frac{(Nt)^{h-1}}{\Gamma(h)} B\left[ \left( -\frac{gN}{1 + gNI} \right)^h \right] = \frac{1}{Nt} B\left[ \left( -\frac{gN}{1 + gNI} \right)^h \right]. \quad (29)
\]

ii) The first term in eq. (28) is given by
\[
B \left[ \ln \left( 1 + \frac{\varphi^2}{k^2 + \chi} \frac{gN}{1 + gN} \right) \right] = B \left[ \sum_{n=1}^{\infty} (-1)^{n+1} \left( \frac{\varphi^2}{k^2 + \chi} \right)^n \left( \frac{gN}{1 + gN} \right)^n \right] = \\
= -\frac{1}{Nt} B \left[ \frac{gN}{1 + gNI} \right] \sum_{n=1}^{\infty} \frac{1}{n!} \left( \frac{Nt\varphi^2}{k^2 + \chi} \right)^n = \frac{1}{Nt} \exp\left( \frac{Nt\varphi^2}{k^2 + \chi} \right) B \left[ -\frac{gN}{1 + gNI} \right]. \tag{30}
\]

iii) The Borel transform of all the other terms can be done straightforwardly substituting \(\Delta, \Delta^2\) respectively with \(-N \exp(-NtI)\) and \(N^2t \exp(-NtI)\), according to eq. (28).

From i) and ii) we get the final expression of the Borel transform of unrenormalized \(V_{NL}\)

\[
-\frac{1}{2t} \int \exp \left[ Nt \left( \frac{\varphi^2}{k^2 + \chi} - I(k^2, \chi) \right) \right], \tag{31}
\]

and iii) is used in the whole renormalized potential, that is \(V_{NL}^{\text{Ren}}\). Notice that, if in eq. (31) we consider only the asymptotic behaviour of \(I(k^2, \chi)\),

\[
\exp(-NtI) \sim e^{-2u} \left( \frac{k^2}{\mu^2} \right)^{u} \quad \quad u = \frac{Nt}{32\pi^2}, \tag{32}
\]

then some renormalon contributions are missed in \(V_{NL}\).

On the other hand, since the exact calculation of the Borel transform is very hard, some approximation is necessary. This is the argument of the next section.

**IV. FIRST RENORMALON IN THE NLO EFFECTIVE POTENTIAL**

Using the Borel trasform of eq. (25), it is possible to isolate the contribution of the first renormalon to the potential and to write the corresponding renormalon contribution to the 1-PI green functions. Our strategy is the following:

a) we expand the Borel trasform of \(V_{NL}^{\text{Ren}}(\varphi, \chi)\) in powers of \(\chi\) and \(\varphi\) around the point \(\varphi = 0, \chi = \mu^2;\)

b) in this expansion we encounter integrals of the form

\[
\int \frac{[I^{(a1)}(k^2, \mu^2)]^{b_1} [I^{(a2)}(k^2, \mu^2)]^{b_2} \ldots}{(k^2 + \mu^2)^c} \exp[-NtI(k^2, \mu^2)], \tag{33}
\]

where \(I^{(a)}(k^2, \mu^2) = \left[ \partial^n I(k^2, \chi) / \partial \chi^n \right]_{\chi = \mu^2}\). Renormalon contributions come from the UV behaviour of such integrals, the integrands of which at large momenta typically
behave as \( \ln^m(k^2)/(k^2)^{u-n} \), where \( u = Nt/(32\pi^2) \) is the Borel variable. In dimensional regularization, \( u \) acts as a regulator and the above UV behaviour yields poles for \( u = n - 2 \). For the unrenormalized NLO potential, \( n \geq 1 \), so that poles develop at \( u = -1, 0, 1, \ldots \). The singularities in \( u = -1, 0 \) correspond to the quadratic and the logarithmic divergencies of the potential and are removed by renormalization. Thus in the renormalized NLO potential, the pole closest to the origin is located at \( u = 1; \)

iii) we calculate the first renormalon contribution by considering all integrands whose ultraviolet behaviour is \( \ln^m(k^2) k^{u-3} \). Let us note that the counterterms added in order to renormalize \( V_{\text{NL}} \) not only remove the poles for \( u = -1, 0 \), but give also contributions to the renormalons at \( u = 1, 2, 3, \ldots \). Thus the counterterms are fundamental to obtain the final result.

We skip all the details of this complicated calculation and we report only the final result. Denoting \( \tilde{V}_1 \) and \( \tilde{V}_2 \) as the first renormalon contributions to \( V_1 \) and \( V_2 \), eqs. (26) and (27), we have

\[
\tilde{V}_1 = \frac{\varphi^6(Nt)^2}{12(1-u)} + \frac{\varphi^4}{2} \left[ \frac{Nt}{1-u} (\chi - \mu^2) + \frac{(Nt)^2}{2(4\pi)^2} \left( \frac{\mu^2 - \chi}{(1-u)^2} + \frac{f(\chi)}{1-u} \right) \right] + \\
+ \frac{\varphi^2(\chi - \mu^2)^2}{2} \left[ \frac{1}{1-u} - \frac{2(Nt)}{(4\pi)^2(1-u)^2} + \frac{(Nt)^2}{(4\pi)^4(1-u)^3} \right] + \\
+ \frac{3\varphi^2(Nt)^2}{2(4\pi)^2(1-u)} \left[ \frac{7\chi^2}{8} - \mu^2 \chi + \frac{\chi^2}{8} + \frac{\chi^2}{4} \left( \ln \frac{\chi}{\mu^2} - 3 \ln \frac{\chi}{\mu^2} \right) \right].
\]

(34)

For \( \tilde{V}_2 \) it is more convenient to present its third derivative \( \partial^3 \tilde{V}_2/\partial x^3 \) respect to \( \chi \). Thus, \( \tilde{V}_2 \) can easily be obtained integrating three times in \( \chi \) with the conditions \( \tilde{V}_2(\mu^2) = \tilde{V}_2'(\mu^2) = \tilde{V}_2''(\mu^2) = 0 \),

\[
\frac{\partial^3 \tilde{V}_2}{\partial \chi^3} = - \frac{3(Nt)^2}{(4\pi)^6(1-u)^4} + \frac{1}{(4\pi)^4(1-u)^3} \left[ 6(Nt) + \frac{(Nt)^2}{(4\pi)^2} \left( 1 - \frac{\chi}{\mu^2} + 7 \ln \frac{\chi}{\mu^2} \right) \right] + \\
+ \frac{1}{(4\pi)^2(1-u)^2} \left[ -6 + \frac{(Nt)}{(4\pi)^2} \left( -11 \ln \frac{\chi}{\mu^2} + 2 \frac{\mu^2}{\chi} + \frac{\chi}{\mu^2} - \frac{15}{2} \right) \right] + \\
+ \frac{(Nt)^2}{(4\pi)^4} \left[ -\frac{7}{2} \ln \frac{\chi}{\mu^2} + \frac{3}{4} \ln \frac{\chi}{\mu^2} + \frac{7}{8} - \frac{\chi}{\mu^2} + \frac{\chi^2}{8\mu^4} \right] + \frac{1}{(4\pi)^2(1-u)} \left[ 6 \ln \frac{\chi}{\mu^2} + 16 - \frac{2\mu^2}{\chi} + \\
+ \frac{1}{(4\pi)^2} \left( 1 - \frac{\chi}{\mu^2} + 7 \ln \frac{\chi}{\mu^2} \right) \right].
\]
\[-\frac{\mu^4}{2\chi^2} + \frac{(Nt)}{(4\pi)^2} \left( 4 \ln \frac{\chi}{\mu^2} + 14 \ln \frac{\chi}{\mu^2} - \frac{\mu^2}{2\chi} \ln \frac{\chi}{\mu^2} - \frac{3\chi}{2\mu^2} + \frac{3\mu^2}{\chi} + \frac{9}{2} \right) + \]
\[+ \frac{(Nt)^2}{(4\pi)^4} \left( \frac{7}{6} \ln^3 \frac{\chi}{\mu^2} - \frac{3}{8} \ln^2 \frac{\chi}{\mu^2} - \frac{7}{8} \ln \frac{\chi}{\mu^2} - \frac{\chi^2}{16\mu^4} + \frac{\chi}{\mu^2} - \frac{15}{16} \right) \] \tag{35}

In the previous formulae (34) and (35) we have omitted an overall factor $e^{-2u}/(4\pi)^2\mu^2$, which is, however, essential to the calculation of the residue at $u = 1$.

The formulae (34) and (35) represent the main result of this paper and some comments are necessary for a better understanding.

First of all we recall that by applying $n$ total derivatives
\[
\frac{d^n}{(d\varphi)^n} = \left( \frac{\partial}{\partial \varphi} + \frac{d\chi}{d\varphi} \frac{\partial}{\partial \chi} \right)^n, \tag{36}
\]
computed at $\varphi = 0$, $\chi = \mu^2$ to the effective potential we obtain all zero momentum 1-PI Green functions $\Gamma^{(n)}(p = 0, g)$. Thus, acting by eq. (36) on $\tilde{V} = \tilde{V}_1 + \tilde{V}_2$ we compute the first renormalon contribution to all 1-PI Green functions. We see that the first renormalon appears for the first time in the 6-points 1-PI Green function. Actually we have
\[
d\tilde{V}/d\varphi^2|_{\varphi=0} = d^2\tilde{V}/d^2\varphi^2|_{\varphi=0} = 0 \quad \text{and} \quad \frac{d^3\tilde{V}}{d(\varphi^2)^3}|_{\varphi=0} = (\chi_0')^3 \frac{\partial^3 \tilde{V}}{\partial \chi^3} + 3(\chi_0')^2 \frac{\partial^3 \tilde{V}}{\partial \chi^2 \partial \varphi^2} + 3\chi_0' \frac{\partial^3 \tilde{V}}{\partial \chi \partial \varphi^4} + \frac{\partial^3 \tilde{V}}{\partial \varphi^6}, \tag{37}
\]
where, from eq. (38)
\[
\chi_0' = \frac{d\chi}{d\varphi^2}|_{\varphi=0} = \frac{gN}{2}. \tag{38}
\]

In eq. (37) the residue to the pole in $u = 1$ give us
\[
Im\Gamma^{(6)}(p = 0, g) = -\frac{375}{16\pi\mu^2} e^{-2g^3 e^{-32\pi^2/(gN)}} + O(e^{-64\pi^2/(gN)}) = \]
\[= -\frac{375}{16\pi\Lambda_{\text{Landau}}^2} g^3 + O\left( \frac{1}{\Lambda_{\text{Landau}}^4} \right). \tag{39}
\]

The $Im\Gamma^{(6)}(p = 0, g)$ has been also calculated by [10] summing all Feynman graphs that contribute to the first renormalon of the $\Gamma^{(6)}$. We can easily obtain this diagrammatic expansion replacing $\tilde{V}$ with $V_{NL}^{\text{Ren}}$, eq. (25), in eq. (37) as shown in appendix B. From
eq. (26) and (27) it follows also that, in general \( \frac{d^n\tilde{V}}{d^n\varphi^2}\big|_{\varphi=0} \neq 0 \) for \( n \geq 3 \) due to the presence of terms like \( \ln \chi/\mu^2 \) and \( 1/\chi \), i.e. the first renormalon contribution is present in any \( 2n \)-point Green function with \( n \geq 3 \).

Parisi hypothesis tell us that the first renormalon in the Green function can be removed with the insertion of all possible 6-dimension operators. Considering only the zero momentum Green function the whole set of such operators reduces to \((\vec{\varphi}^2)^3\). By adding to the Lagrangian the operator \((\vec{\varphi}^2)^3\) with a suitable coefficient the first renormalon contribution on all zero momentum Green functions can be removed. We have [7]

\[
Im\Gamma^{(n)}(p = 0, g) = C_6(g)\Gamma^{(n)}_{\varphi^6}(p = 0, g) + O(e^{-64\pi^2/(gN)})
\] (40)

where \( \Gamma^{(n)}_{\varphi^6}(p = 0, g) \) is the \( n \)-point 1-PI Green function with the insertion of \((\vec{\varphi}^2)^3\) at zero momentum calculated at the leading order in \( 1/N \) expansion, whose diagram is shown in fig. 3. At leading order \( \Gamma^{(6)}_{(\vec{\varphi}^2)^3}(p = 0, g) = 1 \), and so \( C_6(g) \) is given by eq (39). We notice that renormalization group analysis [7] tells us only that \( C_6(g) \) is proportional to \( g^3e^{-32\pi^2/(gN)} \).

By performing derivatives on \( \tilde{V} \) we can verify that eq. (40) holds for every \( n \). For example we have explicitly checked that \( Im\Gamma^{(8)}(p = 0, g) \) satisfies eq. (40).

V. CONCLUSIONS

One of the main features that emerges from the Borel trasform of \( V_{NL} \) is the structure of the singularities in the Borel parameter \( u \), due to the UV behaviour of Green functions. We have checked that for the unrenormalized potential these singularities come for integer values of \( u \) starting from \( u = -1 \). The renormalization procedure removes the poles at \( u = -1, 0 \). More precisely the insertion of operators of dimension 2 remove the singularity in \( u = -1 \) (mass renormalization), the operators of dimension four the one in \( u = 0 \) (coupling costant and wave function renormalization). In the renormalized potential the singularities are then only found starting from \( u = 1 \). In our study we have shown that indeed the pole in \( u = 1 \) is canceled by adding a term proportional to the operator \((\vec{\varphi}^2)^3\) to the Lagrangian. The
poles in $u = 2, 3, \text{etc.}$ are removed by the insertion of even higher dimension operators. The resulting theory thus requires fixing the renormalization conditions for all possible $2n$–point Green functions (including those containing derivatives) and this would introduce an infinite number of arbitrary parameters. In a “complete” theory we would expect that renormalon singularities do not appear and that all Green functions are determined in terms of only a finite number of parameters. For “complete” we mean a theory which appears as an effective $(\phi^2)^2$-theory below same physical scale that we call $\Lambda_{\text{Landau}}$.

We have seen that renormalon effects are suppressed by powers of $\Lambda_{\text{Landau}}$. The first renormalon give a contribution proportional to $\Lambda_{\text{Landau}}^{-2}$, the second one a contribution proportional to $\Lambda_{\text{Landau}}^{-4}$ and so on. The ambiguities become then more and more important as we move closer to the region of the Landau pole. The renormalon analysis gives us a systematic procedure to order the effects of the presence of the Landau pole (scale of new physics) in inverse powers of $\Lambda_{\text{Landau}}$.

VI. ACKNOWLEDGMENTS

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VII. APPENDIX A

In this appendix we describe in detail the renormalization of the effective potential. Let us start with

$$V_1 = \frac{1}{2} \int \ln \left( 1 + \frac{\varphi^2}{k^2 + \chi} \frac{g N}{1 + gNI(k^2, \chi)} \right). \quad (41)$$

In order to renormalize $V_1$ we calculate the two and four point 1-PI functions at the point $\varphi = 0$ and $\chi = \mu^2$. In the following we put $\Delta(k^2, \chi) = gN/[1 + gNI(k^2, \chi)]$. The $V_1$ contribution to $\Gamma^{(2)}$ is
\[
\frac{\partial V_1}{\partial \varphi^2} = \frac{1}{2} \int \frac{1}{k^2 + \chi} \Delta(k^2, \chi). \tag{42}
\]

In order to isolate the divergences we develop \( \partial V_1/\partial \varphi^2 \) around \( \chi = \mu^2 \),
\[
\begin{align*}
\frac{\partial V_1}{\partial \varphi^2} &= \frac{1}{2} \int \frac{1}{k^2 + \mu^2} \Delta(k^2, \mu^2) + \\
&\quad -\frac{1}{2}(\chi - \mu^2) \int \frac{1}{k^2 + \mu^2} \Delta(k^2, \mu^2) \left( \frac{1}{k^2 + \mu^2} + I'(k^2, \mu^2) \Delta(k^2, \mu^2) \right) + \\
&\quad -\frac{1}{4}(\chi - \mu^2)^2 \int \frac{I''(k^2, \mu^2)}{k^2 + \mu^2} \Delta^2(k^2, \mu^2) + \\
&\quad +\frac{1}{2}(\chi - \mu^2)^2 \int \frac{1}{(k^2 + \mu^2)^2} \Delta(k^2, \mu^2) \left( \frac{1}{k^2 + \mu^2} + I'(k^2, \mu^2) \Delta(k^2, \mu^2) \right) + \\
&\quad +\frac{1}{2}(\chi - \mu^2)^2 \int \Delta^3(k^2, \mu^2)(I')^3(k^2, \mu^2) \frac{1}{k^2 + \mu^2} + \ldots . \tag{43}
\end{align*}
\]

At large momenta \( I(k^2) \sim \ln(k^2) \) and \( I', I'', \ldots \sim 1/k^2 \). Thus the divergent parts of the previous equation are all contained by the terms in the first three lines of eq. (43). The divergencies present in the first two terms can be removed by adding as counterterm
\[
-\frac{\varphi^2}{2} \int \left[ \frac{1}{k^2 + \mu^2} \Delta(k^2, \mu^2) + (\chi - \mu^2) \frac{1}{k^2 + \mu^2} \Delta(k^2, \mu^2) \left( \frac{1}{k^2 + \mu^2} + I'(k^2, \mu^2) \Delta(k^2, \mu^2) \right) \right]. \tag{44}
\]

The leading, large momentum behaviour of \( I'' \) is
\[
I''(k^2, \chi) = \frac{1}{(4\pi)^2} \left[ \frac{1}{\chi(k^2 + 4\chi)} + O(\frac{1}{k^4}) \right]. \tag{45}
\]

The fact that \( I''(k^2, \chi) - 1/[(4\pi)^2 \chi (k^2 + \mu^2)] = O(1/k^4) \) suggests that the missing counterterm has the following form
\[
\frac{1}{2} \varphi^2 f(\chi) \int \frac{1}{(4\pi)^2 (k^2 + \mu^2)^2} \Delta^2(k^2, \mu^2), \tag{46}
\]
where \( f(\chi) \) satisfies \( f(\mu^2) = f'(\mu^2) = 0 \) and \( f''(\chi) = 1/\chi \),
\[
f(\chi) = \mu^2 - \chi + \chi \ln \frac{\chi}{\mu^2}. \tag{47}
\]

The only non-zero contribution of \( V_1 \) to \( \Gamma^{(4)} \) for \( \varphi = 0 \) and \( \chi = \mu^2 \), is given by
\[
\frac{\partial^2 V_1}{\partial \varphi^4} = -\frac{1}{2} \int \left( \Delta(k^2, \chi) \frac{1}{k^2 + \chi} \right)^2. \tag{48}
\]
This can be cancelled by with the following counterterm
\[
\frac{\varphi^4}{4} \int \left( \Delta(k^2, \chi) \frac{1}{k^2 + \mu^2} \right)^2 .
\] (49)

We can apply a similar procedure to
\[
V_2 = -\frac{1}{2} \int \ln(-\Delta(k^2, \chi)) .
\] (50)

In this case there is no explicit dependence on \(\varphi^2\), and the expansion of \(V_2\) around \(\chi = \mu^2\) gives
\[
V_2 = \frac{1}{2} \int \left[ -\ln(-\Delta(k^2, \mu^2)) + (\chi - \mu^2)\Delta(k^2, \mu^2)I'(k^2, \mu^2) \right] + \\
+ \frac{(\chi - \mu^2)^2}{2} \Delta(k^2, \mu^2) \left[ I''(k^2, \mu^2) - \Delta(k^2, \mu^2)I'(k^2, \mu^2) \right] + \\
+ \frac{(\chi - \mu^2)^3}{6} \Delta(k^2, \mu^2) \left[ I'''(k^2, \mu^2) - 3\Delta(k^2, \mu^2)I'(k^2, \mu^2)I''(k^2, \mu^2) \right] + \\
+ \frac{(\chi - \mu^2)^3}{3} \Delta(k^2, \mu^2)^3 I^3(k^2, \mu^2) + ... .
\]

The divergences of the first two lines can be simply subtracted. The term
\[
-3\Delta^2(k^2, \mu^2)I'(k^2, \mu^2)I''(k^2, \mu^2),
\] (51)

is logarithmically divergent. It can be kept finite adding to \(V_2\)
\[
\frac{1}{2} \int \frac{3}{(4\pi)^2}\Delta^2(k^2, \mu^2) \frac{I'(k^2, \mu^2)}{k^2 + \mu^2} g(\chi) ,
\] (52)

with \(g(\chi)\) such that \(g(\mu^2) = \mu^2 = g''(\mu^2) = 0, g'''(\chi) = 1/\chi\). Thus
\[
g(\chi) = \frac{\chi^2}{2} \ln(\frac{\chi}{\mu^2}) - \frac{3}{4} \chi^2 + \mu^2 \chi - \frac{\mu^4}{4} .
\] (53)

A further derivative applied to eq. (51) shows that there is a further divergence due to
\[
-3\Delta^2(k^2, \mu^2)I''(k^2, \mu^2) .
\] (54)

This can be eliminated by
\[
\frac{1}{2} \int \frac{3}{(4\pi)^2}\Delta(k^2, \mu^2)^2 \frac{I''(k^2, \mu^2)}{k^2 + \mu^2} h(\chi) ,
\] (55)

15
with \( h(\mu^2) \) and its first three derivatives calculated in \( \chi = \mu^2 \) equal to zero and \( h^{iv}(\chi) = 1/\chi^2 \),

\[
h(\chi) = \frac{\chi^3}{6\mu^2} + \frac{\chi^2}{4} - \frac{\mu^2\chi}{2} + \frac{\mu^4}{12} - \chi^2\frac{\chi^2}{2}\ln\frac{\chi}{\mu^2}.
\] (56)

Finally we have the term

\[
\Delta(k^2, \mu^2) I'''(k^2, \mu^2).
\] (57)

which at large momenta behaves as

\[
I'''(k^2, \chi) = -\frac{1}{(4\pi)^2} \left[ \frac{1}{\chi^2(k^2 + 4\chi)} + \frac{6}{\chi(k^2 + 4\chi)^2} + O\left(\frac{1}{k^6}\right) \right] \sim -\frac{1}{(4\pi)^2} \left( \frac{1}{\chi^2k^2} + \frac{2}{\chi^4} \right).
\] (58)

We can renormalize this contribution by adding as counterterm

\[
\frac{1}{2} \int \frac{1}{(4\pi)^2} \Delta(k^2, \mu^2) \left[ \frac{1}{k^2 + \mu^2} \left( 1 + \frac{\mu^2}{k^2 + \mu^2} \right) l(\chi) + \frac{2}{(k^2 + \mu^2)^2} g(\chi) \right]
\] (59)

where \( l(\chi) = -\chi \ln(\chi/\mu^2) + \chi^2/(2\mu^2) - \mu^2/2 \).

The residual divergencies are eliminated adding to the potential

\[
-\frac{h(\chi)}{2(4\pi)^2} \int \frac{\Delta^2(k^2, \mu^2)}{k^2 + \mu^2} I'(k^2, \mu^2) - \frac{j(\chi)}{4(4\pi)^2} \int \frac{\Delta^2(k^2, \mu^2)}{(k^2 + \mu^2)^2},
\] (60)

where \( j(\chi) \) is such that \( j(\mu^2) \) and its first four derivatives calculated in \( \chi = \mu^2 \) are zero and \( j^{(v)}(\chi) = 1/\chi^3 \),

\[
j(\chi) = h(\chi) + \frac{\chi^4}{24\mu^4} - \frac{\chi^3}{6\mu^2} + \frac{\chi^2}{4} - \frac{\mu^2\chi}{6} + \frac{\mu^4}{24}.
\] (61)

All these counterterm are included in eq. 26,27, section II.

VIII. APPENDIX B

In this appendix we give all the terms which contribute to first renormalon in the six points 1-PI Green function \( \Gamma^{(6)}(p = 0, g) \). We compare our result, obtained by using the effective potential, with the one of ref. 11 obtained in a diagrammatic way.
Replacing $\tilde{V}$ with $V_{NL}^{Ren}$ in eq. (37), $\Gamma^{(6)}(p = 0, g)$ is given by

$$\Gamma^{(6)}(p = 0, g) = 120 \left[ (\chi_0')^2 \frac{\partial^2 V_2}{\partial \chi^2} + 3 \chi_0' \frac{\partial^2 V_1}{\partial \chi \partial \varphi^4} + \frac{\partial^3 V_1}{\partial \varphi^6} + 3(\chi_0')^2 \frac{\partial^3 V_1}{\partial \chi^2 \partial \varphi^2} \right]_{\varphi=0, \chi=\mu^2}. \quad (62)$$

We can write the six points 1-PI function as the sum of the several terms

$$\Gamma^{(6)} = \Gamma^{(6)}_a + \Gamma^{(6)}_b + \Gamma^{(6)}_c + \Gamma^{(6)}_d, \quad (63)$$

where

$$\Gamma^{(6)}_a = 120 \frac{\partial^3 V_1}{\partial \varphi^6} = -120 \int \frac{\Delta^3(k^2, \mu^2)}{(k^2 + \mu^2)^3}, \quad (64)$$

$$\Gamma^{(6)}_b = 360 \chi_0' \frac{\partial^3 V_1}{\partial \chi \partial \varphi^4} = -180g \int \left[ \frac{\Delta^2(k^2, \mu^2)}{(k^2 + \mu^2)^3} + \frac{\Delta^3(k^2, \mu^2)}{(k^2 + \mu^2)^2} I'(k^2, \mu^2) \right], \quad (65)$$

$$\Gamma^{(6)}_c = 360(\chi_0')^2 \frac{\partial^3 V_1}{\partial \chi^2 \partial \varphi^2} = -45g^2 \int \left\{ \frac{2 \Delta^3(k^2, \mu^2)}{(k^2 + \mu^2)^3} I''(k^2, \mu^2) + \left[ \frac{\Delta^2(k^2, \mu^2)}{(k^2 + \mu^2)^2} I'(k^2, \mu^2) + \frac{1}{(4\pi)^2 \mu^2 (k^2 + \mu^2)^2} \right] \right\}, \quad (66)$$

$$\Gamma^{(6)}_d = 120(\chi_0')^3 \frac{\partial^3 V_2}{\partial \chi^3} = -15g^3 \int \left\{ \frac{3}{2} \frac{\Delta(k^2, \mu^2)}{(4\pi)^2 \mu^2 (k^2 + \mu^2)^2} + \Delta^2(k^2, \mu^2) I''(k^2, \mu^2) \right\} +$$

$$+ \left[ \frac{3}{2} \frac{\Delta(k^2, \mu^2)}{(4\pi)^2 \mu^2 (k^2 + \mu^2)^2} I'(k^2, \mu^2) + \frac{3}{2} \Delta^2(k^2, \mu^2) I'(k^2, \mu^2) I''(k^2, \mu^2) \right] +$$

$$+ \left[ \frac{1}{2} \Delta(k^2, \mu^2) I'''(k^2, \mu^2) - \frac{1}{2} \frac{1}{(4\pi)^2 \mu^4 (k^2 + \mu^2)^2} \Delta(k^2, \mu^2) \right]. \quad (67)$$

All these terms have a one-to-one correspondence to the Feynman diagrams listed in ref. [10].
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FIG. 1. Propagators for the theory involving the $\tilde{\chi}$ field.

FIG. 2. Contribution to $V(\varphi, \chi)$ at LO and NLO: (a) the LO contribution; (b) the propagator of the $\tilde{\chi}$-field at LO, i.e. with all $\varphi_0$-bubbles resummed; (c) the NLO contributions.
FIG. 3. $2(n + 2)$-point 1-PI Green functions with the insertion of $(\overline{\phi}^2)^3$ at the leading order.