Unlocking Starlight Subtraction in Full-data-rate Exoplanet Imaging by Efficiently Updating Karhunen–Loève Eigenimages

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Abstract

Starlight subtraction algorithms based on the method of Karhunen–Loève eigenimages have proved invaluable to exoplanet direct imaging. However, they scale poorly in runtime when paired with differential imaging techniques. In such observations, reference frames and frames from which starlight is to be subtracted are drawn from the same set of data, requiring a new subset of references (and eigenimages) for each frame processed to avoid self-subtraction of the signal of interest. The data rates of extreme adaptive optics instruments are such that the only way to make this computationally feasible has been to downsample the data. We develop a technique that updates a precomputed singular value decomposition of the full data set to remove frames (i.e., a “downdate”) without a full recomputation, yielding the modified eigenimages. This not only enables analysis of much larger data volumes in the same amount of time, but also exhibits near-linear scaling in runtime as the number of observations increases.

We apply this technique to archival data and investigate its scaling behavior for very large numbers of frames \(N\). The resulting algorithm provides speed improvements of \(2.6 \times\) (for 200 eigenimages at \(N = 300\)) to \(140 \times\) (at \(N = 10^4\)) with the advantage only increasing as \(N\) grows. This algorithm has allowed us to substantially accelerate Karhunen–Loève image projection (KLIP) even for modest \(N\), and will let us quickly explore how KLIP parameters affect exoplanet characterization in large-\(N\) data sets.

Unified Astronomy Thesaurus concepts: Exoplanet detection methods (489); Exoplanets (498); Direct imaging (387); Astronomy data reduction (1861); Astronomy data analysis (1858)

1. Introduction

Direct imaging of exoplanets and disks depends on our ability to remove the effects of starlight on the signal of interest, whether before the focal plane with a coronagraph or after with data postprocessing techniques. The locally optimal combination of images (LOCI) technique of Lafrenière et al. (2007) achieved gains in contrast through postprocessing by combining a reference library of images to approximate the frame under analysis. The techniques of Karhunen–Loève image projections (KLIP) and principal component analysis (PCA) for starlight subtraction were introduced to the field of high-contrast imaging by Soummer et al. (2012) and Amara & Quanz (2012), bringing improved performance relative to LOCI. The former described a scheme for computing a Karhunen–Loève (KL) basis for modeling of the point-spread function (PSF) from reference image covariance matrices, and the latter the principal components of the references via the singular value decomposition (SVD). (These operations are equivalent, as shown in Section 3.)

To decorrelate the target signal from the star signal, differential imaging techniques are used. These exploit differing behavior of the two signals with angle in angular differential imaging (ADI, Marois et al. 2006), with wavelength in spectral differential imaging (Sparks & Ford 2002; Biller et al. 2004), or with polarization in polarization differential imaging. When combined with KLIP, the recorded realizations of the PSF are both the data from which starlight is to be subtracted and the reference library from which to construct a model. By computing a new KL basis for each frame while drawing references from the other frames, subtraction of any signal of interest is minimized.

Unfortunately, this recomputation of the KL basis at each frame increases the computation time faster than quadratically with the number of frames used (Section 6.2). As data volumes from extreme adaptive optics (ExAO) instruments increase, making the same analyses possible in comparable amounts of time will require either scaling up computing resources massively (e.g., Haug-Baltzell et al. 2016) or identifying algorithmic improvements.

The usual solution is to combine frames in a sequence until the computation time is acceptably short. In the case of ADI, image combination will necessarily change either the amount of sky rotation in a single frame or the statistical properties of the images, presenting challenges. Furthermore, Macintosh et al. (2005) report that most atmospheric speckles have a lifetime of less than one second in simulation. Short integrations capture information about short-lifetime speckles, but they will be combined in the coadded frames, potentially causing the KL basis to capture spurious correlations.

Fortunately, low-rank representations of many-dimensional data are broadly useful outside astronomy, and their efficient computation is the subject of much work in machine learning and image processing. For instance, “online” algorithms such as incremental PCA (Ross et al. 2008) enable incremental computation of outputs as data arrives without ever storing the entire data set in memory at once. Within astronomy, Savransky (2015) explored the sequential calculation of covariance matrices and their inverses used in LOCI. In this work we apply a different mathematical identity to the SVD with the same goal of enabling analyses of larger data sets with only a linear increase in runtime.

Low-rank SVD modifications have applications not only to data postprocessing but also to control of the adaptive optics system that acquires the data. At the core of the empirical orthogonal functions formalism for predictive wavefront control described in Guyon & Males (2021) lies an SVD that is iteratively updated to include new information on changing observatory conditions, for example.
Removing data from the low-rank representation after the fact, however, has not been explored in as much detail (Hall et al. 2000; Brand 2006). By describing the Karhunen–Loève transform in terms of an SVD of the image data, we develop an algorithm that uses the SVD of the full data set to efficiently obtain the truncated decomposition of that data set modified by the addition or removal of observations. This enables \( \sim2–100\times \) speedups that grow with the size of the data set (when compared with existing algorithms), enabling fast analyses of the full data volume from an ExAO system without compromises.

2. The Method of Karhunen–Loève Image Projections

To define our terms and describe KLIP in terms of matrix operations, we restate the core algorithm of Soummer et al. (2012). We begin with a target image \( T \) and \( N \) reference images \( \{R_1, \ldots, R_N\} \) that have had a mean image subtracted (such that the mean value of each pixel taken separately through the stack of images is zero). Each observation is rearranged into an \( \mathbb{R}^p \) vector of pixel values through an arbitrary mapping of \( p \) pixel locations to vector elements, giving us the column vector \( t \) for our target image \( T \) and a matrix \( R \) whose columns are \([n_1, \ldots, n_N]\) for our reference images.

For our reference set \( R \), the discrete KL transform takes a series of \( p \) dimensional vectors that represent observations of \( p \) variables and transforms them into an orthogonal \( p \)-dimensional basis such that truncating the basis at the first \( k \) vectors minimizes the mean squared error in reconstruction of the original data. To subtract starlight without subtracting planet light, we exclude our target vector \( t \) (and, optionally, frames adjacent in angle or wavelength) from the reference set and we truncate the KL basis at \( k \) vectors, or modes.

To obtain the KL transform, we first compute the image-to-image covariance matrix \( C = RR^T \). We then solve for its eigenvectors \( V = [v_1, \ldots, v_N] \) and corresponding eigenvalues \( \lambda_1 > \ldots > \lambda_N \) such that \( V^T CV = \lambda v \) = \( \lambda e_1 \), \ldots, \( \lambda e_N \).

Let \( \text{diag}(\lambda) = \text{diag}(\lambda_1^{-1/2}, \ldots, \lambda_N^{-1/2}) \). The optimal KL basis is then obtained from

\[
Z^{\text{KL}} = RV \text{diag}(\lambda).
\]

The columns of \( Z^{\text{KL}} \), when rearranged into images, are called the eigenimages of \( R \). To subtract starlight, we first define \( Z_{\text{kl}} \) as the first \( k \) columns of \( Z^{\text{KL}} \), truncating the basis set to its \( k \) most significant components. The estimate of the starlight in \( t \), \( \tilde{t} \), is then

\[
\tilde{t} = Z_k^{\text{KL}}(Z_k^{\text{KL}})^T t
\]

and the residual signal \( a \) is

\[
a = t - \tilde{t}.
\]

By mapping the entries of \( a \) back to pixel indices, the final image is obtained. These can then be combined, e.g., by median or rotate-and-stack ADI, to further enhance the signal-to-noise ratio of any astrophysical sources of interest.

3. Equivalence of Image-to-image Covariance and Direct SVD

It is often more computationally efficient to compute \( Z^{\text{KL}} \) by finding the eigenvectors of the image-to-image covariance matrix (see Section 6.2). However, the SVD is mathematically equivalent. Suppose \( A \in \mathbb{R}^{m \times n} \). Its SVD is then given by the orthogonal matrices \( U \in \mathbb{R}^{m \times m} \) and \( V \in \mathbb{R}^{n \times n} \) such that

\[
U^T AV = \Sigma = \text{diag}(\sigma_1, \ldots, \sigma_q) \quad \text{where} \quad q = \min(m, n) \quad \text{and} \quad \sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_q.
\]

By this definition, \( A = V \Sigma^2 U^T \), \( UU^T = I \), and \( VV^T = I \), so

\[
A^T A = (V \Sigma U^T)(U \Sigma^T V^T),
\]

\[
A^T A = V \Sigma \Sigma^T V^T, \quad \text{and} \quad A^T A = V \text{diag}(\sigma_1^2, \ldots, \sigma_q^2) V^T.
\]

If we set \( A = R \), we see that this \( V \) and the \( V \) above that diagonalizes the covariance matrix \( C \) must be equivalent and that the singular values are related to the eigenvalues of the covariance by \( \sigma_i^2 = \lambda_i \). To compute \( Z^{\text{KL}} \) from this representation, we can write

\[
\Sigma^{-1} = \text{diag}(1/\sigma_1^2, \ldots, 1/\sigma_N^2) = \text{diag}(\sigma_1^{-1}, \ldots, \sigma_N^{-1}).
\]

Equation (1) becomes

\[
Z^{\text{KL}} = RV \Sigma^{-1}.
\]

Replacing \( R \) with its SVD gives

\[
Z^{\text{KL}} = (U \Sigma V^T) V \Sigma^{-1} = U \Sigma V V^T.
\]

Thus, the left singular vectors \( U \) of the reference matrix \( R \) are the basis \( Z^{\text{KL}} \) we computed in Section 2.

For \( N \) reference images and \( p \) pixels where \( p \gg N \), it is advantageous to compute \( Z^{\text{KL}} \) from the \( N \times p \) image-to-image covariance matrix rather than the \( N \times p \) matrix of vectorized images. Eigenvectorization of the image-to-image covariance is an \( O(N^3) \) problem, while the SVD requires \( O(pN^2) \) operations. For data where \( N \) exceeds \( p \), the SVD is preferred because the matrix can always be transposed such that the cost is \( O(pN^2) \).

The case where \( N \) exceeds \( p \) may arise not only when an exceptionally long sequence of images is taken, but also when the number of pixels \( p \) is reduced through masking or dividing the image into search regions that each contain far fewer pixels than the entire image does.

4. Modification of the SVD to Obtain Updated Eigenimages

When processing differential imaging data from ground-based telescopes, \( t \) and \( R \) are just submatrices of a larger observations matrix \( X \in \mathbb{R}^{p \times (N+1)} \) whose columns we will call

\[
X = [x_1, \ldots, x_{N+1}].
\]

While processing all \( N+1 \) frames, each column of \( X \) will take its turn as the target \( t \) and all other columns will form the reference. Other exclusion schemes are possible, e.g., based on field rotation or wavelength difference to prevent overlap of an astrophysical signal. What follows removes one column of \( X \), but will readily generalize to removal of multiple observations.

To process frame \( i \),

\[
t = x_i \quad \text{and} \quad R = [x_{i-1}, x_{i+1}, \ldots, x_{N+1}].
\]

To compute \( \tilde{t} \), we would find the SVD of \( R = U \Sigma V^T \) and truncate to the first \( k \) singular values and vectors (or, equivalently, compute a rank-\( k \) thin SVD) to obtain \( R \approx U_k \Sigma_k V_k^T \). Then, we would calculate Equation (2) with \( Z_k^{\text{KL}} = U_k \).

We would repeat this procedure for every frame \( i \in [1, N+1] \). This gives KLIP with differential imaging unfortunate \( O(pN^3) \) scaling. (For calculation of \( Z_k^{\text{KL}} \) from the covariance, it is \( O(N^3) \).) However, we can exploit the fact that the left singular vectors we recompute correspond to reference matrices \( R \) that differ from \( X \) only by the presence or absence of some columns.
Brand (2006) developed an identity for additive (and subtractive) modifications of an SVD, which we will apply to our matrices. Suppose that we have already computed the SVD of our full data set through a standard solver to obtain 

\[ X = U_0 \Sigma_0 V_0^T. \]

After truncation, this gives a low-rank approximation, \( X \approx U_{0,k} \Sigma_{0,k} V_{0,k}^T. \) We want to find the low-rank or truncated SVD of \( R \), that is, \( X \) without column \( i \).

To modify the data matrix to remove \( x_i \), we express \( R = X + AB^T \), the sum of \( X \) and the product of two low-rank matrices \( A \in \mathbb{R}^{p \times c}, B \in \mathbb{R}^{(N+1) \times c} \). The addition of \( AB^T \) will zero \( c \) columns of the resulting matrix by adding \(-x_i\) to column \( i \) for every \( i \) removed. It can be verified that the left singular vectors of an \( m \times (n+1) \) matrix with an excess column of zeros are identical to those for an \( m \times n \) matrix where the column is omitted entirely—up to a sign factor. (This sign factor is unimportant for starlight subtraction, because the product \( Z^{KL}_{\text{fit}}(Z^{KL}_{\text{fit}})^T \) in Equation (2) will cancel it out, but it should be accounted for when comparing singular vectors produced by different algorithms.)

By substituting this diagonalization for \( X \) into Equation (2), we obtain

\[ X + AB^T = U_{0,k} \Sigma_{0,k} V_{0,k}^T. \]

Thus we have developed an identity for additive and subtractive modifications of an SVD, which we will apply to our matrices. Suppose that we have already computed the SVD of our full data set through a standard solver to obtain 

\[ X = U_0 \Sigma_0 V_0^T. \]

After truncation, this gives a low-rank approximation, \( X \approx U_{0,k} \Sigma_{0,k} V_{0,k}^T. \) We want to find the low-rank or truncated SVD of \( R \), that is, \( X \) without column \( i \).

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In this case, \( c = 1 \), \( A = x_i \), and \( B \) is a vector of zeros where the \( i \)th component has been set to \(-1\). When removing multiple columns, each column of \( A \) corresponds to the contents of the column to remove, and each column of \( B \) has a \(-1 \) in the row corresponding to the index of the removed column in the original matrix.

The sum \( X + AB^T \) can be expressed in terms of \( X \)'s SVD as

\[
X + AB^T = [U_0 A] \begin{bmatrix} \Sigma_0 & 0 \\ 0 & I \end{bmatrix} [V_0 B]^T.
\]  

(4)

We want to modify the truncated SVD of rank \( k \), so Equation (4) becomes

\[
X + AB^T \approx [U_{0,k} A] \begin{bmatrix} \Sigma_{0,k} & 0 \\ 0 & I \end{bmatrix} [V_{0,k} B]^T.
\]  

(5)

The left and right block matrices may be rewritten as

\[
[U_{0,k} A] = [U_{0,k} P] \begin{bmatrix} I & U_{0,k}^T A \\ 0 & R_A \end{bmatrix}
\]  

(6)

and

\[
[V_{0,k} B] = [V_{0,k} Q] \begin{bmatrix} I & V_{0,k}^T B \\ 0 & R_B \end{bmatrix}
\]  

(7)

where \( P \) is an orthogonal basis of the column space of \((I - U_{0,k} U_{0,k}^T)A\) (which may be obtained by QR decomposition) and \( R_A = P^T (I - U_{0,k} U_{0,k}^T) A \). \( Q \) and \( R_B \) are defined similarly for \( V_{0,k} B \) and \( B \).

The additional column(s) in \( P \) capture the component(s) of \( A \) orthogonal to \( U_{0,k} \), and \( R_A \) represents that component in terms of \( P \). By substituting Equations (6) and (7) into Equation (4), we obtain

\[
X + AB^T \approx [U_{0,k} P] K [V_{0,k} Q]^T
\]  

(8)

where we have defined \( K \) as

\[
K = \begin{bmatrix} I & U_{0,k}^T A & \Sigma_{0,k} & 0 \\ 0 & R_A & 0 & I \\ 0 & 0 & 0 & R_B \end{bmatrix} [V_{0,k}^T B]^T.
\]  

(9)

Brand (2006) notes that the number of columns of \( P \) and number of rows of \( R_A \) are equal to the rank of this orthogonal component, and may be zero. Since we are removing columns of \( X \), our \( A \) matrix contains data already incorporated in \( U_{0,k} \) and our \( B \) matrix contains data already incorporated in \( V_{0,k} \).

Evaluating \( P, R_A, Q, \) and \( R_B \) when removing columns confirms that all their entries are very nearly zero. In fact, the component captured by \( P \) is exactly the residual starlight after subtracting the reconstruction, and incorporating it would produce an eigenbasis that almost perfectly subtracts \( t \). As we are interested in the residuals \( a = t - \hat{t} \), this is an undesirable effect. Therefore, we should ignore \( P, R_A, Q, \) and \( R_B \), treating them as if they had zero rows and columns, which simplifies Equations (8) and (9) to

\[
X + AB^T \approx U_{0,k} KV_{0,k}^T
\]  

(10)

and

\[
K = [I U_{0,k}^T A \begin{bmatrix} \Sigma \ 0 \\ 0 \ 1 \end{bmatrix} I V_{0,k}^T B].
\]

(11)

To obtain the updated SVD \( X + AB^T = U_k \Sigma_k V_k^T \) we must re-diagonalize \( K \) as \( K = W \Sigma_k Y^T \) by computing an SVD of reduced cost \( O(k^2) \).

By substituting this diagonalization for \( K \) into Equation (10), we obtain

\[
X + AB^T = R \approx (U_{0,k} W) \Sigma_k (V_{0,k} Y)^T
\]

\[
\approx U_k \Sigma_k V_k^T.
\]  

(12)

By the definition of the SVD, \( W \) and \( Y \) are orthogonal matrices, and so are the products \( U_{0,k} W \) and \( V_{0,k} Y \). Therefore \( U_k = U_{0,k} W \) are the left singular vectors, \( V_k = V_{0,k} Y \) are the right singular vectors, and \( \Sigma_k \) is the diagonal matrix of the updated singular values.

Thus we have updated the SVD by removing data from, or “downampling,” the lower-rank decomposed representation of \( X \) directly.

5. Practical Considerations for Implementation

This algorithm is straightforward to implement with basic matrix operations and an SVD solver from a library (e.g., LAPACK, Intel MKL). Table 1 summarizes the symbols used above and their dimensions for the benefit of the implementer.

Superfluous operations can be eliminated by noting that the right singular vectors \( V_k \) are not necessary for KLIP, and the product \( V_{0,k} Y \) need not be calculated. Furthermore, we have assumed that the rank-\( k \) representation of the data captures all the information necessary for starlight subtraction, so the product \( U_{0,k}^T A \) is just \(-\Sigma_{0,k} v^T \) where \( v \) is the \( i \)th row vector of \( V_{0,k} \). Similarly, \( V_{0,k}^T B \) is just \( v^T \). (In effect, we are removing the vector \( U_{0,k}^T U_{0,k}^T t \) rather than \( t \).)

To prevent the erosion of numerical accuracy, it is helpful to compute the decomposition of \( X \approx U_{0,k+1} \Sigma_{0,k+1} V_{0,k+1} \) and use \( k+1 \) singular vectors and values through the algorithm in Section 4, truncating to \( Z^{KL}_{\text{fit}} = U_k \) just before evaluating Equation (2). When removing more than one column per iteration, the rank of the initial decomposition should be increased by that number of singular values to mitigate errors at greater \( k \).

We have omitted discussion of the difference in mean for each set of reference images, assuming the difference between
the mean image of the entire set of references and the mean of a subset with some frames excluded is small. This approximation holds as long as the number of images removed, \( c \), is much smaller than \( N \).

Finally, note that the \( U_i \Sigma V_k \) computed in Equation (12) is the SVD of a modification not to \( X \) but rather to \( U_{0,k} \Sigma_{0,k} V_{0,k} \), and the two are not strictly equal unless \( \text{rank}(X) < k \). The numerical experiments in Section 6 show that this approximation has little impact on the recovered signal-to-noise ratio of a planet.

6. Performance

6.1. Impact on Recovered Signal-to-noise Ratio

To evaluate the impact of choice of algorithm on signal-to-noise ratio (S/N) with realistic data, we used observations of \( \beta \) Pictoris b from Abisil et al. (2013) made available as part of the VIP software package (Gonzalez et al. 2017). An example of the final KLIP+ADI output for the existing and modified SVD algorithms can be seen in Figure 1. The effect of choice of algorithm and number of modes \( k \) used is summarized by Figure 2. The SVD modification algorithm does not result in numerically identical values for S/N at various \( k \) modes when compared with direct computation of the singular vectors or eigenvectors, but we are satisfied that it performs comparably.

6.2. Theoretical

As we are most interested in the case where the number of observations exceeds the number of pixels, we will assume \( N > p \) in the following.

The cost of the image-to-image covariance calculation is \( O(N^3 + pN^2) \) per frame, where the cubed term is due to the eigenvectorization (Golub & Van Loan 2013, Section 8.3) and the other term is due to the matrix product to calculate the covariance matrix \( C \). The pixel-to-pixel covariance calculation, useful when \( N > p \), has complexity \( O(p^3 + Np^2) \) per frame. When only \( k \) eigenvectors are needed, the costs are \( O(N^2k + pN^2) \) and \( O(p^2k + Np^2) \), respectively.

The current state of the art in calculating intractably large SVDs is a randomized algorithm (Halko et al. 2011), which can provide a highly accurate approximation of the SVD in \( O(Np \log(k) + (N + p)k^2) \) time. Halko gives the scaling behavior for the conventional low-rank SVD as \( O(NpK) \).

So far, we have discussed the computational cost of computing \( Z_{KL} \) for each frame in isolation. When analyzing a sequence of frames, each of these estimates will be multiplied by an additional factor of \( N \) for the number of frames on which the procedure is repeated to obtain the total cost. The algorithm we have developed sidesteps this additional factor of \( N \) by...
amortizing the cost of the initial decomposition over the $N$ frames.

The SVD modification algorithm we have applied to KLIP costs $O((N+1)p^2)$ for an initial deterministic SVD and $O(k^3 + pk)$ for each iteration of the re-decomposition and computation of $U_k$ from $U_0$, $W$. Dividing the full initial decomposition cost by the number of frames gives a complexity of $O(p^2 + k^3 + pk)$ for finding the eigenimages of a single frame. In other words, the total cost will still increase linearly with $N$, but the cost per frame will remain constant. This is in contrast to the other algorithms mentioned above, in which the cost of processing a single frame depends on the total number of frames, with execution time growing quadratically (or faster) with $N$.

6.3. Empirical

To validate this algorithm we developed a Python implementation of KLIP+ADI into which we could substitute different schemes for recomputing the SVD, as well as different solver routines. After verifying that all algorithms produced equivalent signal-to-noise ratios on test data (Section 6.1), we compared execution times between KLIP implemented by:

1. finding eigenvectors of image-to-image covariance matrix $C$ for each frame,
2. computing the SVD of a new $R$ for each frame,
3. computing the randomized SVD of a new $R$ for each frame following Halko et al. (2011), using an implementation from scikit-learn (Pedregosa et al. 2011),
4. or modifying the rank-$k$ SVD of $X$ with our algorithm.

Where applicable, we compared between Intel MKL (CPU), MAGMA (CPU/GPU hybrid), and cuSolver (GPU) linear algebra solvers for the implementation of these algorithms.

Timings at a variety of $N$ for $k = 200$ modes are shown in Figure 3. In the interests of clarity, a subset of algorithm/solver combinations are shown (chosen to show each algorithm with the shortest runtime achieved, whichever device or library that may have used).

The characteristics of KLIP applied to differential imaging data, in which the references and target frames are drawn from the same set of observations, mean that our algorithm to perform $N$ updates to the SVD can outperform $N$ randomized SVD computations—which is the algorithm with the next-best asymptotic scaling exponent. The randomized SVD may nevertheless be useful for the initial computation of the rank-$k$ SVD that this algorithm modifies.

Classic rotate-and-stack ADI, in which a mean image is subtracted, the frames are rotated, and the images are combined by median or mean pixel values, scales linearly in $N$. Figure 3 shows that our SVD modification algorithm also exhibits (nearly) linear scaling over $300 < N < 10^4$.

Prior to developing this algorithm, we evaluated GPU computation as a possible path to speeding up KLIP. However, the general SVD solver for the GPU in cuSolver is notably slower than CPU and CPU/GPU hybrid algorithms. We note that cuSolver is slightly faster than MAGMA or MKL at solving symmetric eigenproblems at large $N$, but overall did not deliver a dramatic speedup to the existing algorithm. Fortunately, not only is CPU/GPU hybrid computation of the modified SVD algorithm faster, computers typically have far more RAM than GPU memory, allowing us to postpone investigating out-of-core algorithms.

Compared to the theoretical scaling for existing algorithms, execution time increases more slowly than expected with increasing $N$. This implies that our execution time is not dominated by the number of floating point operations. Nevertheless, the predicted linear scaling is accurate to a good approximation for the modified SVD algorithm.

7. Conclusions

We have described how to apply a mathematical identity to efficiently recompute the truncated SVD of a matrix after removing a subset its columns. We applied this to the computationally expensive problem of obtaining Karhunen–Loève eigenimages for starlight subtraction when the target and reference frames are drawn from the same set of observations. This technique computes $Z^{KL}$ for each frame in an ADI sequence in less than half the time at $N = 300$ and less than $1/140$ the time at $N = 10^4$ when compared with the most efficient algorithms. It exhibits near-linear scaling as $N \to \infty$, making it useful for large data sets. Our comparison of CPU and GPU implementations of the solver routines shows that hybrid techniques are advantageous.

Computing modified KL bases for each frame of large-$N$ data cubes was formerly intractable due to the scaling properties of KLIP with varying reference sets. We believe this technique will enable analysis of large-$N$, short-integration data cubes that capture short-lifetime speckles without compromises. This in turn accelerates KLIP hyperparameter searches, such as for search region geometry and exclusion amount. Future work will investigate the performance in starlight subtraction at large $N$ and $k$ for data cubes produced by MagAO-X and other ExAO systems.
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Figure 3. Actual runtimes for the algorithms tested. $N$ was varied by taking different slices from a 10,000 plane datacube. For algorithms that can take advantage of GPUs or alternative solvers, the combination that produced the best runtime at $N = 10^4$ frames was chosen to plot. After applying a mask to subset the data, the total number of pixels per plane used was $p = 6817$. Execution time was benchmarked on aUA HPC Ocelote cluster node with 250 GB RAM, an NVIDIA Tesla P100 GPU with 16 GB of GPU RAM, and a 28 core Intel Xeon E5-2680 v.4 CPU.