Bragg resonances for tunneling between edges of a 2D Quantum Hall system

M. B. Hastings and L. S. Levitov

Physics Department, Massachusetts Institute of Technology

Abstract

A theory is presented for tunneling between compressible regions on the sides of a narrow incompressible Quantum Hall strip. Assuming that electron interactions lead to formation of a Wigner crystal on the edges of the compressible regions, we consider the situation when the non-conservation of electron momentum required for transport is provided, in the absence of disorder, by umklapp scattering on the crystal. The momentum given to the crystal is quantized due to the Bragg condition, which leads to resonances in tunneling conductivity as a function of the incompressible strip width, similar to those reported recently by N. Zhitenev, M. Brodsky, R. Ashoori, and M. Melloch.
The structure of the edge of a Quantum Hall system is central for understanding the QH physics \[1\]. Theories of the QH edge predict that the edge states form an interesting strongly interacting one-dimensional system. Different aspects of the edge dynamics are described by the chiral Luttinger liquid theories \[2\], and by theories taking into account the long-range character of electron interaction \[3\], exchange \[1\] and correlation effects \[5\]. One expects that the QH edge has a rich structure. To probe it, one can study electric transport across the edge, e.g., by tunneling into the edge \[6\], or by measuring an ac response. The latter was realized in a recent ac capacitance measurement \[7\], in which a capacitance probe attached above the 2D gas was used to accumulate charge under the probe and to form a sharp gradient of electron density near the probe edge. When the densities in the 2D gas and under the probe are greater and less than an integer, correspondingly, an incompressible strip is formed near the probe edge. Since electric field near the edge is strong, it was possible to create a very narrow incompressible strip of width estimated as 8–10 magnetic lengths.

Measuring the frequency dependent ac capacitance gives tunneling conductivity across the strip. As a function of the potential \(V_g\) on the probe, the conductivity displays resonances, uniformly spaced in \(V_g\) \[7\]. The resonances were studied at different magnetic fields and temperatures. The resonances shift as the field changes, however, the shift is much weaker than corresponding change of filling fraction, i.e., the resonances are more sensitive to \(V_g\) than to the field. It was found that the resonances broaden with increasing temperature, and that the peak values of conductivity increase roughly as \(T^\alpha\), with \(\alpha \geq 1\).

It was demonstrated that the peaks are not caused by resonance tunneling through impurities: the number of peaks is about the same for the probes of different sizes, and the conductivity at the peaks scales as the probe size. The high degree of the peaks ordering, as well as the temperature dependence, inconsistent with resonance tunneling, suggests looking for a different mechanism. Since the strip is narrow, it may be possible to tunnel across it without scattering off of impurities. Momentum conservation then requires that the tunneling electron give up momentum to other electrons. If the other electrons (or holes in the region of less than integer filling factor) form a Wigner crystal \[3,10\], momentum can
only be given up at multiples of the inverse lattice spacing. This effect would lead to Bragg resonances in tunneling conductivity when these multiples coincide with the momentum required to tunnel across the strip.

In all theories of the QH edge predicting structure of a few magnetic lengths wide \cite{3,4}, there is room for a 1D Wigner crystal. For example, the theory of edge reconstruction \cite{3} leads to an edge composed of Luttinger liquids, one chiral, and several non-chiral, each of the latter representing nothing but a 1D crystal. One expects no long-range order in a 1D system, so it is equally correct to view a 1D crystal as a correlated fluid. The correlation length, however, varying as the inverse of temperature, becomes large at low temperatures. This leads to finite widths of the peaks in the structure factor, scaling with temperature.

We will discuss a Wigner crystal with varying density, formed near the incompressible edge, and study tunneling into such a crystal. For the edge structure we use the long-range screening picture \cite{3}, however, since only one row of the crystal will be important, our predictions will hold for any reconstructed edge. We consider Bragg resonances in tunneling conductivity, and calculate the dynamical Debye-Waller factor due to the crystal zero-point and thermal motion, giving the temperature dependence of the peaks. The peak intensities, thermal broadening, and shifts caused by varying temperature, are found to be consistent with the experimental observations \cite{7}.

**Wigner crystal at the edge:** In magnetic field, electrons form a crystal at sufficiently low densities and temperatures, where potential energy overcomes kinetic energy \cite{11}. The critical density $\nu$ was estimated to be slightly below $1/5$ \cite{12}. At higher densities a QH fluid will form.

On the sides of an incompressible strip, formed by one or several fully filled Landau levels with integer density, the electron density is non-integer and varies smoothly on a scale, set by long-range electric forces, of many magnetic lengths \cite{3}. Thus the density of excess electrons and holes right near the strip must be quite small. Within the two small density regions there are several crystalline rows of electrons and holes, respectively (see Fig. 1.b). The number of the rows is determined by the external electric field $E$. In the continuum limit
the electron density measured from the integer within the strip goes to zero at some point with a square-root singularity. The discrete crystal will have its front row located near that point. The rows separation can be determined by balancing forces, $eE$ from confining potential against the force $e^2/\epsilon r^2$ from neighboring particles, which, up to a factor $\sim 1$, gives the spacing in the front row: $a = \sqrt{e^2/\epsilon E}$. In the experiment [7], the e- and h-crystals both have small density at the strip edge, and increase away from it. We estimate that $a$ is of the order of 2-3 magnetic lengths for the front row of each crystal. In the continuum limit [8], the particle spacing goes as the $-\frac{1}{4}$ power of the distance from the edge. Similarly, in numerical simulations [8], the particle spacing varies very slowly with the distance from the edge. However, the estimated density of the front row is close to the Wigner crystal critical density [12], and so we expect that beyond a few crystal rows a QH fluid is formed (Fig. 1.b shows e- and h-rows obtained by a simulation similar to [7]).

At constant density, a Wigner crystal forms a perfect triangular lattice. In the presence of the field forming the strip, the density of the crystal will increase along the field, however, because of the long-range interaction, the crystal forms well-ordered rows, even at the edge [8]. The largest simulations of a crystal in an external field that we know of were done by Pieranski using $1/r^3$-interacting dipoles [13] (see Fig. 1.a for a picture of the resulting crystal). These experiments also revealed the remarkable idea of the conformal crystal [13]. Since the electron interaction $1/r$ is longer-range than $1/r^3$, the front rows in the $1/r$ crystal are ordered even better than in the $1/r^3$ crystal (see Fig. 2 in [8]).

We expect that both electrons and holes will be surrounded by spin textures, topological excitations of the local spin [14]. We will ignore any effect this has on the crystal, as well as any question of how the spin is transferred across the strip. It may be that spin is transferred independently of charge, by spin waves.

**Bragg reflection** Since the tunneling amplitude falls exponentially with distance, the main contribution to the tunneling rate arises from transitions between the front e- and h-rows (Fig. 1.b). Therefore, we will focus attention on only one row on each side of the strip. We consider a strip of width $w$ going in the $x-$direction, with e- and h-crystal row spacing $a$. 
In Landau gauge, electron states can be written as

\[ \psi(x, y) = e^{ipx} f_n(y + pch/eB) \] (1)

where \( f_n \) is the harmonic oscillator \( n \)-th wave function, \( p \) labels the \( x \)-momentum.

The tunneling amplitude is given by the overlap of states (1) on opposite sides of the strip. In the absence of momentum non-conserving scattering, the amplitude will vanish because shifting \( y \) by \( w \) across the strip is equivalent to a change of \( p \), but the states with different \( p \)'s are orthogonal. In the presence of a crystal, electron momentum may change by a multiple of the Bragg vector, \( \delta p = 2\pi n/a \). (For simplicity, let us assume that the crystal period is the same on both sides of the strip.) On the other hand, since according to (1) the change in momentum is coupled to a \( y \)-shift, the overlap of the states will be maximal at \( \delta pch/eB = w \). The two conditions put together give that tunneling must be maximal at

\[ w_n = \frac{\Phi_0}{aB} n , \] (2)

where \( n \) corresponds to the Bragg resonance order, and \( \Phi_0 = hc/e \) is the flux quantum.

From (2), the peak positions are dependent upon the strip width \( w \), and thus are very sensitive to the gate voltage \( V_g \). Increasing \( V_g \) leads to squeezing of the strip: roughly, \( w \) is inversely proportional to \( V_g \). For this reason, we expect that the dependence on \( V_g \) of the \( B \)-field needed to observe a given resonance is stronger than the dependence on \( V_g \) of the \( B \)-field needed to maintain a given filling fraction under the gate. This is seen in the experiment.

To estimate the order of a resonance one takes the product \( wa/l_H^2 \), where \( l_H \) is magnetic length. In the experiment, \( l_H \approx 10\text{nm} \), \( w \) is about \( 5 - 8 \ l_H \), and \( a \) is estimated above to be \( 2 - 3 \ l_H \). This gives the order of resonance between 10 and 25, consistent with the experiment.

Going from a 2D crystal to single row: To estimate the response of the crystal to injecting an electron, we assume that all relaxation occurs within one row, and treat it as a one-dimensional system. The other electrons will be taken into account only to the extent that
they produce a confining potential $V(y)$ for that row. This approximation underestimates the crystal stiffness, enhanced by the presence of other rows, and thus overestimates the peak widths. However, we believe the discrepancy is not large, because for a crystal with long-range interaction the shear modulus is much smaller than the compressibility, and also the particles in the front row, being more widely spaced, are more loose than in other rows. Thus, charge spreads most easily within the first row.

With this, one writes the Hamiltonian for one row as

$$\hat{H} = \sum_n \frac{(p_{n,x} - \frac{e}{c} By_n)^2}{2m} + \frac{p_{n,y}^2}{2m} + V(y_n) + \frac{1}{2} \sum_{m,n} U(r_n - r_m),$$

where $r_n = x_n \hat{i} + y_n \hat{j}$ are electron positions, and the interaction $U(r) = e^2/\varepsilon |r| - e^2/\varepsilon (r^2 + d^2)^{1/2}$ accounts for the screening by the gate.

In a magnetic field, kinetic energy is very small compared to potential energy, and so the fluctuations about mean positions in the crystal will be small. We assume that different Landau levels are not mixed by the interaction $U$, and then treat the displacements $x_n$ and $y_n$ as conjugate variables with the commutation relation $[x_n, y_m] = i\Phi_0/B$. Expanding the Hamiltonian, and keeping only quadratic terms, one obtains a harmonic chain problem. Diagonalizing the Hamiltonian, one gets elementary excitations. Their spectrum at small $qa \ll 1$ is acoustic: $\omega(q) = c|q|$, where $c = a\sqrt{\kappa/m_*}$. (Here $\kappa = \sum_n n^2 U''_{r=an}$, $m_* = m(1 + m\omega_c^2/V''_{r=0})$.) At small $q$, the displacements are along the row.

**Evaluating the tunneling rate:** The tunneling is described by transferring an electron across the strip, from the e-row to the h-row. Injecting an electron into the row is accompanied by motion of other electrons that give space to the new electron. This makes the tunneling a collective process, and requires considering relaxation in the row. Since at high magnetic field the effective mass of electrons in the row is large, $m_* \gg m$, the relaxation will be slow. This simplifies analytic treatment, making it possible to consider only the effect of acoustic modes, $qa \ll 1$. In order to evaluate the tunneling rate we assume that the transfer of an electron across the strip is a much faster process than charge relaxation within the row, and that the characteristic time scales are much larger than $a/c$. Such an assumption is
self-consistent at low $T < \hbar \omega_c$, which is within experimental range (see similar discussion in [16]).

We evaluate the tunneling rate semiclassically, by taking the saddle-point of the action for the dynamics in imaginary time corresponding to injection and removal of an electron in the rows. The path in imaginary time must be closed ("a bounce"), evolving the system to the initial state [15]. The tunneling rate is thus given by the sum over all injection and removal events, $(r_1, t_1), (r_2, t_2)$, with appropriate phase factors:

$$G(w) = \sum_{1,2} \int \mathcal{D}[u^{(e)}, u^{(h)}] e^{i \tilde{w}(r_1 - r_2)} e^{-S_{12}^{(e)} - S_{21}^{(h)}},$$

(3)

where 1 and 2 label different positions in the chain, $\tilde{w} = 2\pi B w / \Phi_0$, the labels (e) and (h) mark the electron and hole rows, and $u^{(e,h)}$ is the displacement field in the rows. It is convenient to rewrite the tunneling conductivity $G(w)$ as a convolution,

$$G(w) = \int F^{(e)}(\bar{w} - s) F^{(h)}(s) ds,$$

(4)

where $F^{(e)}$ and $F^{(h)}$ are the structure factors of the rows:

$$F(w) = \sum_{1,2} \int \mathcal{D}[u] e^{i \tilde{w}(r_1 - r_2)} e^{-S_{12}}.$$

(5)

The usefulness of using $F(w)$ is that the e- and h-rows may have different densities or elastic moduli, which will make spacings and widths of peaks of the conductivity $G(w)$ irregular, while the structure factor of each row still will be a simple function.

Let us calculate the structure factor of one row. Because of what has been said above we consider only the long-wavelength displacements $u(x, t)$. Choosing the units so that $a = 1$ and $c = 1$, the action for one row is

$$S = \int \int \left( \frac{\rho}{2} \dot{u}^2 + \frac{\rho}{2} u_x^2 + \lambda_x \dot{u} - \dot{\lambda} u_x \right) dx \, dt,$$

(6)

where $\rho = m^*/a$, and the Lagrange multiplier $\lambda(x, t)$ is a multivalued function changing by $\pm 2\pi i$ going around the points $(x_1, t_1), (x_2, t_2)$, respectively. The term with $\lambda$ is introduced to describe injecting an electron at $(x_1, t_1)$, and removing it at $(x_2, t_2)$. Qualitatively, the
particles are moving as shown in Fig. 1.c. The equation of motion then is \( \nabla^2 u = 0 \), with boundary conditions:

a) The displacement \( u(x, t) \) changes by \( \pm 1 \) going around the injection and removal points;

b) As a function of time, \( u(x, t) \) is periodic, with period \( \beta = 1/T \).

The solution is given by

\[
u_0(z) = \text{Im} \frac{1}{2\pi} \ln \frac{\sinh \pi T (z - z_1)}{\sinh \pi T (z - z_2)},
\]

where \( z = x + it \). Evaluating the action \( S[u_0(x, t)] \) one gets

\[
S_0 = \frac{\rho}{2\pi} \ln \left| \frac{\beta}{\pi} \sinh \left( \frac{\pi}{\beta} (z_1 - z_2) \right) \right|,
\]

which is equal to

\[
S_0 = \frac{\rho}{4\pi} \ln \frac{\beta^2}{\pi^2} \left( \sinh^2 \frac{\pi x_{12}}{\beta} + \sin^2 \frac{\pi t_{12}}{\beta} \right),
\]

where \( x_{12} = x_1 - x_2 \), \( t_{12} = t_1 - t_2 \).

To average the phase factor in (6), we write \( r_{1,2} \) in terms of the displacement \( u(x_{1,2}) \) from the mean position \( x_{1,2} \), explicitly showing the particle number in the row:

\[
F(w) = \sum_{x_{12} = n} (-)^n \langle e^{i\tilde{w}(n + u(x_1,t_1) - u(x_2,t_2))} \rangle e^{-S_0},
\]

where the sign is due to exchange of \( n + 1 \) fermions, and the Debye-Waller factor \( \langle ... \rangle \) describes dephasing due to the zero-point and thermal fluctuations. The gaussian average in (6) is performed in a standard fashion,

\[
\langle ... \rangle = e^{i\tilde{w}n} e^{-\frac{1}{4}\tilde{w}^2((u(x_1,-\tau/2) - u(x_2,\tau/2))^2)}
\]

where the second exponent equals \(-(\tilde{w}\hbar/\rho ac)^2S_0\). Taking the extremum of (8) in the time separation \( t_{12} \) gives optimal \( t_{12} = \beta/2 \). Finally, restoring \( c \) and \( a \), the resulting structure factor is

\[
F(w) = \sum_n e^{in(\tilde{w}a + \pi)} \left( \frac{\tilde{T}}{\cosh \tilde{T}n/2} \right)^{\alpha(\tilde{w})},
\]

where \( \tilde{T} = 2\pi aT/\hbar c \), \( \alpha(\tilde{w}) = \rho a^2 c/2\pi \hbar + \tilde{w}^2 \hbar/2\pi \rho c \). The structure factor (11) has peaks at \( \tilde{w}_m = (2\pi/a)m \), in agreement with (2). The width of the \( m \)–th peak scaled in \( a^{-1} \) is given by \( \alpha^{1/2}(\tilde{w}_m) \tilde{T} \), which equals
\[ [\gamma + m^2/\gamma]^{1/2} \tilde{T}, \quad \gamma = \rho a^2 c/2\pi \hbar. \quad (12) \]

The peak width increases with \( m \) and is proportional to temperature.

To compare with the experiment [7], we plot the ac capacitance
\[ C(w) = C/(1 + (C\omega/G(w))^2) \]
with \( G \) given by (4), (11), and \( C\omega \) being a parameter (Fig. 2).

The peaks appear at the transition between plateaus, and have temperature dependence consistent with [7].

**Qualitative discussion and conclusion** The distinction between the Bragg resonances and other explanations of the observed peaks can be made by studying thermal shifts of the peaks. In the Bragg condition (2) both the strip width \( w \) and the crystal period \( a \) depend on \( T \), which results in the shifts of the same sign for all the peaks, and of magnitude proportional to the order of the peak. The sign of the shifts is determined by the competition of two factors. The thermal expansion of the crystal will lead to \( a \) increasing, and \( w \) decreasing with temperature. According to (2), the first effect is more important, which leads to a shift consistent with experiment.

The effect of disorder is difficult to estimate, but in the setup [7] the sensitivity to disorder should be lower than in other existing devices: there is a stronger electric field, and thus a more narrow strip; both sides of the strip are free, and so the disorder potential will mainly result in the strip bending with lesser effect on the width.

We may conclude by listing several qualitative features of the experiment correctly described by the above theory. The number of peaks and their positions in the magnetic field–gate voltage plane are a result of the Bragg condition. The change in the height of peaks is a result of the temperature dependence of the action (3). The shifts in peak position are explained by a thermal expansion of the crystal.

**ACKNOWLEDGMENTS**

We greatly benefitted from discussions with Ray Ashoori and Nikolai Zhitenev.
REFERENCES

[1] B. I. Halperin, Phys. Rev. B25, 2185 (1982);
    A. H. MacDonald and P. Streda, Phys. Rev. 29, 1616 (1987);
    J. K. Jain and S. A. Kivelson, Phys. Rev. 37, 4276 (1988);
    C. W. J. Beenakker, Phys. Rev. Lett. 64, 216 (1990)

[2] X. G. Wen, Phys. Rev. Lett. 64, 2206 (1990); Phys. Rev. B43, 11025 (1991);
    C. L. Kane and M. P. Fisher, Phys. Rev. B46, 15233 (1992);
    K. Moon, et al., Phys. Rev. Lett. 71, 4381 (1993)

[3] D. B. Chklovskii, B. I. Shklovskii and L. I. Glazman, Phys. Rev. B46, 4026 (1992)

[4] J. D. Dempsey, B. Y. Gelfand, and B. I. Halperin, Phys. Rev. Lett. 70, 3639 (1993)

[5] C. Chamon and X.-G. Wen, Phys. Rev. B49, 8227 (1994);
    S.-R. E. M. Yang, A. H. MacDonald, and M. D. Johnson, Phys. Rev. Lett. 71, 3194 (1993);
    L. Brey, Phys. Rev. B50, 11861 (1994)

[6] F. P. Milliken, C. P. Umbach, and R. A. Webb, Solid State Comm. 97, 309 (1996);
    A. M. Chang, preprint (unpublished);

[7] N. B. Zhitenev, M. Brodsky and R. C. Ashoori, A New Class of Resonances at the Edge
    of a Two Dimensional Electron Gas, Phys. Rev. Lett., to appear, preprint cond/mat
    9601157

[8] H. A. Fertig, R. Côté, A. H. MacDonald, and S. Das Sarma, Phys. Rev. Lett. 69, 816
    (1992)

[9] H. A. Fertig, R. Côté, Phys. Rev. B48, 2391 (1993)

[10] Yu. V. Nazarov, Europhys. Lett. 32(5), 443 (1995)

[11] H. Fukuyama, P. M. Platzman and P. W. Anderson, Phys. Rev. B19, 5211 (1979)
[12] K. Maki and X. Zotos, Phys. Rev. B28, 4349 (1983);
    E. Y. Andrei, et al., Phys. Rev. Lett. 60, 2765 (1988);
    V. J. Goldman, et al., Phys. Rev. Lett. 65, 2189 (1990)

[13] P. Pieranski, in: Phase Transitions in Soft Condensed Matter, T. Riste, D. Sherrington
    eds., NATO ASI, Plenum, New York (1989);
    F. Rothen and P. Pieranski, Phys. Rev. E53, 2828 (1996)

[14] S. L. Sondhi, et al., Phys. Rev. B47, 16419 (1993);
    K. Moon, et al., Phys. Rev. B51, 5138 (1995);

[15] J. S. Langer, Ann. Phys. (N.Y.) 41, 108 (1967); S. Coleman, Phys. Rev. D 15, 2929
    (1977); I. Affleck, Phys. Rev. Lett. 46, 388 (1981)

[16] L. S. Levitov, A. V. Shytov, and A. Yu. Yakovets, Phys. Rev. Lett. 77, p. 370 (1995)
FIGURES

FIG. 1.  a) A large collection of magnetic dipoles which repelled each other was confined between two glass plates, and the glass was tilted; the in-page gravity component is down [13] (courtesy of P. Pieranski and F. Rothen).  b) Incompressible strip with few electron and hole rows on the sides is shown. Beyond the rows are regions of QH fluid. The coordinate $x$ is along the strip, and $y$ is normal to it.  c) The imaginary time dynamics picture. The world lines of the tunneling particle and of other particles are shown.

FIG. 2.  The ac capacitance $C(w) = C/(1 + (C\omega/G(w))^2)$, with $G$ given by (4) and (11), plotted as function of the strip width $\tilde{w} = 2\pi wB/\Phi_0$ at several temperatures, offset for clarity. Parameters used: $C\omega = 0.05$, $a^{(e)} = 1$, $a^{(h)} = 1.68$, $\gamma = 2$. Non-equal e- and h-periods are chosen to demonstrate the effect of possible period mismatch.
This figure "fig1.gif" is available in "gif" format from:

http://arxiv.org/ps/cond-mat/9607133v1
This figure "fig2.gif" is available in "gif" format from:

http://arxiv.org/ps/cond-mat/9607133v1