Fourier Transforms of Positive Definite Kernels and the Riemann $\xi$-Function

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Abstract The purpose of this paper is to investigate the distribution of zeros of entire functions which can be represented as the Fourier transforms of certain admissible kernels. The principal results bring to light the intimate connection between the Bochner–Khinchin–Mathias theory of positive definite kernels and the generalized real Laguerre inequalities. The concavity and convexity properties of the Jacobi theta function play a prominent role throughout this work. The paper concludes with several questions and open problems.

Keywords Fourier transforms · Laguerre–Pólya class · Positive definite kernels · Log-concavity · Riemann $\xi$-function

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1 Introduction

To date, there are no known explicit necessary and sufficient conditions that even a “nice” kernel (cf. Definition 1.2), $K(t)$ could satisfy in order that its Fourier transform
have only real zeros (cf. [48, p. 17] and [49]). The questions investigated in this paper are motivated, in part, by several recent results [3,11,23,32,41–44] and our understanding that it is desirable to discover properties of the kernel, \( K \), which (hopefully) will lead to information about the distribution of zeros of the entire function \( F \). The main leitmotif of this note pertains to certain inequalities, known as the \textit{generalized Laguerre inequalities} (Sect. 2), which play a pivotal role in the study of functions in the Laguerre–Pólya class (cf. Definition 1.1). Notwithstanding the extensive research in this area and the impressive results dealing with the Riemann \( \xi \)-functions, it is curious that to date so little progress has been made in proving some of the simplest Laguerre inequalities that \( F \) must satisfy in order that it possess only real zeros (cf. Open Problem 4.7).

An outline of this work is as follows. In the remainder of this introduction, we recall some relevant definitions and nomenclature that will be used in the rest of this paper. In Sect. 2, we review several classical and new results involving the Laguerre and the generalized real Laguerre inequalities (Theorem 2.4) and prove two important, albeit elementary, propositions (Propositions 2.2 and 2.3) which anticipate some of the applications in Sect. 4. With the aid of the classical theorems of Bochner [1], A. Khinchin [33], and Mathias [41], we extend the work of Jensen [31] and Pólya [48] (Sect. 3). Our main results here (cf. Theorems 3.6 and 3.8) establish precise relationships between certain positive definite kernels and the generalized real Laguerre inequalities. In particular, Theorem 3.8 provides a new \textit{necessary and sufficient} condition for the Fourier transform of a strictly logarithmically concave kernel to possess only real zeros. Concavity plays a prominent role throughout this paper (see, for example, the technical result of Theorem 3.6) and it is the \textit{sine qua non} for analyzing the Jacobi theta function and related kernels. In Sect. 4, we apply the foregoing results and derive \textit{new} necessary and sufficient conditions for the Fourier transform of the Jacobi theta function, the Riemann \( \xi \)-function, to belong to the Laguerre–Pólya class. The paper ends with several (6) open problems (Sect. 4).

In the present investigation, we will adopt the following notation and nomenclature associated with real entire functions whose zeros lie in a strip. Let \( S(\tau) \) denote the closed strip of width 2\( \tau \), \( \tau \geq 0 \), in the complex plane, \( \mathbb{C} \), symmetric about the real axis:

\[
S(\tau) = \{ z \in \mathbb{C} : \|\text{Im}(z)\| \leq \tau \}. \tag{1.2}
\]

\textbf{Definition 1.1} We say that a real entire function \( f \) belongs to the class \( \mathcal{G}(\tau) \), if \( f \) can be expressed in the form

\[
f(z) = Ce^{-az^2+bz}z^m \prod_{k=1}^{\omega} (1 - z/z_k)e^{z/z_k}, \quad (0 \leq \omega \leq \infty), \tag{1.3}
\]

where \( a \geq 0, b \in \mathbb{R}, z_k \in S(\tau)\setminus\{0\}, \sum_{k=1}^{\infty} 1/|z_k|^2 < \infty \). We allow functions in \( \mathcal{G}(\tau) \) to have only finitely many zeros by letting, as usual, \( z_k = \infty \) and \( 0 = 1/|z_k|, k \geq k_0, \)
so that the canonical product in (1.3) is a finite product. By convention, the empty product is one. If \( f \in \mathcal{S}(\tau) \), for some \( \tau \geq 0 \), and if \( f \) has only real zeros (i.e., if \( \tau = 0 \)), then \( f \) is said to belong to the Laguerre–Pólya class, and we write \( f \in \mathcal{L} - \mathcal{P} \).

In addition, we write \( f \in \mathcal{L} - \mathcal{P}^* \), if \( f = pg \), where \( g \in \mathcal{L} - \mathcal{P} \) and \( p \) is a real polynomial. Thus, \( f \in \mathcal{L} - \mathcal{P}^* \) if and only if \( f \in \mathcal{S}(\tau) \), for some \( \tau \geq 0 \), and \( f \) has at most finitely many non-real zeros.

The significance of the class \( \mathcal{S}(\tau) \) in the theory of entire functions stems from the fact that \( f \in \mathcal{S}(\tau) \) if and only if \( f \) is the uniform limit on compact sets of a sequence of real polynomials having zeros only in the strip \( S(\tau) \) (cf. [25, p. 202] and [36, pp. 373–374]). It follows from the Gauss–Lucas Theorem [40, pp. 8–22], [52, p. 71] that this class of polynomials is closed under differentiation, and thus so is \( \mathcal{S}(\tau) \).

For various properties and algebraic and transcendental characterizations of functions in the Laguerre–Pólya class, we refer the reader to Pólya and Schur [51, p. 100], [50], [45, Kap. II] or [38, Ch. VIII].

In what follows, we will confine our attention to special kernels which we term admissible kernels and define as follows.

**Definition 1.2** A function \( K : \mathbb{R} \rightarrow \mathbb{R} \) is called an admissible kernel, if it satisfies the following properties: (i) \( K(t) \in C^\infty(\mathbb{R}) \), (ii) \( K(t) > 0 \) for \( t \in \mathbb{R} \), (iii) \( K(t) = K(-t) \) for \( t \in \mathbb{R} \), (iv) \( K'(t) < 0 \) for \( t > 0 \), and (v) for some \( \varepsilon > 0 \) and \( n = 0, 1, 2, \ldots \),

\[
K^{(n)}(t) = O \left( \exp \left( -|t|^{2+\varepsilon} \right) \right) \quad \text{as} \ t \rightarrow \infty. \tag{1.4}
\]

Thus, the assertions that \( F(x) \) (cf. 1.1) is a real entire function readily follows if we assume that \( K(t) \) is an admissible kernel. Moreover, a calculation shows [51, p. 269] that \( F(x) \) is an entire function of order \( \frac{2+\varepsilon}{1+\varepsilon} < 2 \). Also, by the Riemann-Lebesgue Lemma \( F(x) \rightarrow 0 \) as \( |x| \rightarrow \infty \). Observe that if we omit the requirement that \( K(t) \) is even [see, Definition 1.2 (iii)], then its transform, \( F \), cannot have only real zeros. This claim follows from integrating by parts and invoking the Riemann–Lebesgue Lemma (cf. [48]).

**2 The Laguerre Inequalities**

One important property, shared by all functions in \( \mathcal{L} - \mathcal{P} \), is logarithmic concavity; that is, if \( f(x) \in \mathcal{L} - \mathcal{P} \), then \( f(x)^2 \frac{\log f(x)}{f''} \leq 0 \) for all \( x \in \mathbb{R} \). To verify this claim, one need only to consider the derivative of the logarithmic derivative of \( f(x) \in \mathcal{L} - \mathcal{P} \) using the (Hadamard) factorization (1.3), (see, for example, [5–7]). The logarithmic concavity, in conjunction with the closure property of \( \mathcal{L} - \mathcal{P} \) under differentiation, implies that if \( f \in \mathcal{L} - \mathcal{P} \), then \( f \) satisfies the following inequalities, known as the Laguerre inequalities, [10–12,14,17,21,24,46]

\[
L_{1,p}(x; f) := (f^{(p)}(x))^2 - f^{(p-1)}(x)f^{(p+1)}(x) \geq 0, \quad p = 1, 2, 3, \ldots, \quad \text{for all} \ x \in \mathbb{R}. \tag{2.1}
\]
For the sake of simplicity of notation, we set \( L_{1,1}(x; f) := L_1(x; f) := L_1(x) \). In what follows, we will be primarily concerned with the case when \( p = 1 \) in (2.1); that is, \( L_1(x) \). The reason for the subscript “1” will become clear when we consider the generalized real Laguerre inequalities (see Theorem 2.4). We remark that one of the simplest manifestations of the existence of a non-real zero of a real entire function \( f \in \mathcal{S}(\tau) \), occurs when \( f \) possesses a positive local minimum or a negative local maximum. It is this observation that motivates us to consider the Laguerre inequalities. We emphasize here that the Laguerre inequalities are only necessary conditions and, in general, are not sufficient for an entire function to have only real zeros. Indeed, \( f(x) := e^{-x^2}(1 + x^2) \notin \mathcal{L} - \mathcal{P} \), while a calculation shows that \( L_1(x) = 2e^{-2x^2}x^2(3 + x^2) \geq 0 \) for all \( x \in \mathbb{R} \).

**Remark 2.1** To illustrate by an example a relationship between logarithmic concavity and the Turán inequalities, consider again an admissible kernel \( K(t) \) and its Fourier cosine transform \( F_c(x) \). Then via the change of variables, \( u = -x^2 \), we obtain the entire function

\[
F_c(u) := \sum_{k=0}^{\infty} \frac{k!\beta_k}{(2k)!} u^k := \int_0^{\infty} K(t) \cosh(t\sqrt{u}) \, dt,
\]

where \( \beta_k := \int_0^{\infty} K(t)t^{2k} \, dt, \quad k = 0, 1, 2, \ldots \).

Now set \( \gamma_k := \frac{k!\beta_k}{(2k)!} \) for \( k = 0, 1, 2, \ldots \). If \( \log(K(\sqrt{t})) \) is strictly concave for all \( t > 0 \), then we can infer that the Taylor coefficients of \( F_c(x) \) satisfy the Turán inequalities; that is, \( L_{1,p}(0; F_c) := (F_c^{(p)}(0))^2 - F_c^{(p-1)}(0)F_c^{(p+1)}(0) = \gamma_p^2 - \gamma_{p-1}\gamma_{p+1} \geq 0 \), for \( p = 1, 2, 3, \ldots \) (see, for example, [7,13,15,20]). Once again, the Turán inequalities are only necessary conditions for \( F_c \) (and whence for \( F \)) to belong to the Laguerre–Pólya class.

Our next proposition asserts that if a real entire function \( f \in \mathcal{S}(\tau), \tau = 1 \), has only real zeros in a vertical strip \( A \leq \text{Re}z \leq B, B - A > 2 \), then \( L_1(x) \geq 0 \) for \( x \in [A + 1, B - 1] := I \). Thus, on the interval \( I, f \) cannot have a positive local minimum or a negative local maximum.

**Proposition 2.2** [21] Let \( f \in \mathcal{S}(\tau) \), where \( \tau = 1 \) and suppose that \( f(0) \neq 0 \). Let \( \{x_k\}_{k=1}^{\infty} \) denote the real zeros and let \( z_j = \alpha_j + i\beta_j, \ j = 1, 2, \ldots, \omega, 1 \leq \omega \leq \infty, \) denote the non-real zeros of \( f \). If there is an interval \( [A, B] \), with \( B - A > 2 \), such that \( \alpha_j \notin [A, B] \) for all \( j \geq 1 \), then

\[
L_1(x) \geq 0 \quad \text{for all} \quad x \in [A + 1, B - 1]. \tag{2.2}
\]

**Proof** Using the product representation (1.3), logarithmic differentiation yields

\[
L_1(x) = (f(x))^2 \left\{ 2\alpha + \sum_{k=1}^{\omega} \frac{1}{(x-x_k)^2} + 2 \sum_{j=1}^{\omega} \frac{(x-\alpha_j)^2 - \beta_j^2}{[(x-\alpha_j)^2 + \beta_j^2]^2} \right\}. \tag{2.3}
\]
Since \((x - \alpha_j)^2 - \beta_j^2 > 0\) for any \(x \in [A + 1, B - 1]\), \((2.3)\) gives the desired result \((2.2)\).

\[\text{Proposition 2.3} \ [21] \text{ Let } g(x) \text{ be a real entire function and define} \]

\[f(x) := \left( (x - \alpha)^2 + \beta^2 \right)^m g(x) \quad (\alpha \in \mathbb{R}, \beta > 0, m \in \mathbb{N}), \tag{2.4}\]

so that \(\alpha \pm i\beta\) are two non-real zeros of order \(m\) of \(f\). If \(g(\alpha) \neq 0\), then

\[L_1(\alpha; f) = -2m\beta^{4m-2}(g(\alpha))^2 + \beta^{4m}L_1(\alpha; g). \tag{2.5}\]

Thus, there exists \(M > 0\) sufficiently small such that

\[L_1(\alpha; f) < 0 \quad \text{for all } \ 0 < \beta < M. \tag{2.6}\]

\[\text{Proof} \quad \text{Since } f(\alpha) = \beta^{2m}g(\alpha), \text{ a straightforward calculation, using logarithmic differentiation, yields (2.5) and whence the desired result (2.6) follows.} \]

A heuristic description of Proposition 2.2 is as follows. A conjugate pair of non-real zeros \(\alpha \pm i\beta\) of \(f(x)\), when \(\beta > 0\) is sufficiently small, forces \(L_1(\alpha; f)\) to be negative.

We consider next the so-called \textit{generalized real Laguerre inequalities} (see, for example, \[19, 24\]) that are both necessary and sufficient for membership in the Laguerre–Pólya class.

\[\text{Theorem 2.4 (The Generalized Real Laguerre Inequalities [19, Thm. 2.9]) Let } f \text{ denote a real entire function, } f \neq 0. \text{ For } n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\} \text{ and } x \in \mathbb{R}, \text{ set} \]

\[L_n(x) := L_{n,1}(x; f) := \sum_{j=0}^{2n} \left( -1 \right)^{j+n} \left( \frac{2n}{j} \right) f^{(j)}(x) f^{(2n-j)}(x). \tag{2.7}\]

If \(f(x) \in \mathcal{L} - \mathcal{P}\), then \(L_n(x) \geq 0 \quad \text{for all } n \in \mathbb{N}_0 \text{ and for all } x \in \mathbb{R}. \tag{2.8}\)

Conversely, suppose that

\[f(x) = e^{-ax^2}g(x), \quad a \geq 0, \text{ where the genus of } g(x) \text{ is } 0 \text{ or } 1. \tag{2.9}\]

If \(L_n(x) \geq 0 \quad \text{for all } n \in \mathbb{N}_0 \text{ and for all } x \in \mathbb{R}, \text{ then } f(x) \in \mathcal{L} - \mathcal{P}. \tag{2.10}\)

\[\text{Remark 2.5} \text{ Observe that } L_0(x) = f(x)^2 \text{ and to justify the appellation “generalized Laguerre expression”, note that } L_1(x) = f'(x)^2 - f(x)f''(x). \text{ In addition, we remark that if the real entire function } f(x) \text{ satisfies the generalized real Laguerre inequalities,} \]
Let \( L_n(x) \geq 0 (n \in \mathbb{N}_0, x \in \mathbb{R}) \), then \( f(x) \) has only real zeros cf. [19, p. 343]. For the sake of completeness, we mention here the following representation of \(|f(x + iy)|^2\) which can be derived by a direct calculation (see, for example, [19, 46, 48] or using a recursion relation [5]):

\[
|f(x + iy)|^2 = f(x + iy)f(x - iy) = \sum_{n=0}^{\infty} L_n(x)y^{2n}, \quad (x, y \in \mathbb{R}), \quad (2.11)
\]

where \( L_n(x) \) is defined in (2.7).

**Remark 2.6** The action of the non-linear operators \( \{L_n\}_{n=0}^{\infty} \) taking a real entire function \( f(x) \) to \( L_n(x; f) := L_n(x) \) is given implicitly by Eq. (2.11). We mention here, parenthetically, a couple facts about these operators. It is known that the operators \( L_n \) satisfy a simple recursive relation [5, Thm. 2.1] and that \( L_n(x) \) is also a real entire function [5, Rem. 2.4]. Interesting generalizations of these operators are given by Dilcher and Stolarsky [26] and Cardon [2] (see also Sect. 3). Recently, Vishnyakova and the author [24] have shown that the various sufficient conditions for a real entire function, \( f(x) \), to belong to the Laguerre–Pólya class, expressed in terms of Laguerre-type inequalities, do not require the a priori assumptions about the order and type of \( f(x) \). Thus, for instance, implication (2.10) remains valid if we omit assumption (2.9). In light of the results in [24], we can state the complex Laguerre inequalities as follows. Suppose \( f, f \neq 0 \), is a real entire function. Once again we do not stipulate conditions on the order and type of \( f \) [24]. Then \( f \in \mathcal{L} - \mathcal{P} \) if and only if

\[
|f'(z)|^2 \geq \Re \left( f(z)f''(z) \right) \quad \text{for all } z \in \mathbb{C}. \quad (2.12)
\]

It may be of interest to note that the complex Laguerre expression can be also formulated in terms of two real Laguerre-type expressions [24]. Indeed, if \( f(x + iy) = U(x, y) + iV(x, y) \) is a real entire function, then a calculation shows that for all \( z = x + iy \in \mathbb{C}, \)

\[
\frac{1}{2} \frac{\partial^2}{\partial y^2} |f(x + iy)|^2 = |f'(z)|^2 - \Re \left( f(z)f''(z) \right) = U_x^2 - U U_{xx} + V_x^2 - V V_{xx}. \quad (2.13)
\]

### 3 Positive Definite Functions and the Laguerre Inequalities

We mention at the outset that it was Mathias [41] who in 1923, motivated by the results of Carathéodory and Toeplitz (cf. [54, p. 412]), first defined and studied the properties of positive definite functions. In this section, after reviewing some definitions, we will summarize a couple of classical results due to Mathias [41], Bochner [1], Khinchin [33] and Pólya [47]. We note that there are many excellent treatises in the literature dealing with positive definiteness and here we merely cite Lukacs [39], Kawata [34], Mathias [41] and Stewart [54], together with the original works of Bochner [1], Khinchin [33] and Pólya [47]. The interested reader will find 125 additional references in J.
Stewart’s outstanding survey article [54]. In the second part of this short section, our goal is to bring to light the connection between positive definiteness and the Laguerre inequalities.

**Definition 3.1** [34, p. 377] A continuous function \( \varphi : \mathbb{R} \to \mathbb{R} \) is said to be **positive definite** (or more precisely **non-negative definite**), if

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi(t-s) \rho(t) \overline{\rho(s)} \, dt \, ds \geq 0,
\]

where \( \rho : \mathbb{R} \to \mathbb{C} \) is any measurable function with compact support.

An equivalent definition of positive definiteness [34, p. 377] is the following discrete formulation. A continuous function \( \varphi \) is positive definite if the Hermitian form

\[
\sum_{j=1}^{n} \sum_{k=1}^{n} \varphi(x_j - x_k) \rho_j \overline{\rho_k} \geq 0 \quad \text{for every } x_1, \ldots, x_n \in \mathbb{R} \text{ and } \rho_1, \ldots, \rho_n \in \mathbb{C}.
\]

By way illustration, we note that \( \varphi(x) = \cos x \) is positive definite, since

\[
\sum_{j=1}^{n} \sum_{k=1}^{n} \cos(x_j - x_k) \rho_j \overline{\rho_k} = \left| \sum_{j=1}^{n} \rho_j \cos x_j \right|^2 + \left| \sum_{k=1}^{n} \rho_k \sin x_k \right|^2 \geq 0
\]

for every \( x_1, \ldots, x_n \in \mathbb{R} \) and \( \rho_1, \ldots, \rho_n \in \mathbb{C} \).

Similarly, it is easy to check that \( e^{itx}, (t \in \mathbb{R}) \), is positive definite; while it is not so straightforward to verify that the functions \( e^{-|x|}, e^{-x^2} \) and \( \frac{1}{1+x^2} \) are positive definite.

For the sake of clarity, we define one more term. By a **distribution function** we shall mean a non-decreasing function \( V(x) \) such that \( V(-\infty) = 0 \) and \( V(+\infty) = 1 \). The Fourier–Stieltjes transform of \( V \),

\[
f(t) = \int_{-\infty}^{\infty} e^{itx} dV(x) \quad (-\infty < t < \infty),
\]

is called the **characteristic function** corresponding to the given distribution function \( V \).

In 1932, S. Bochner proved the following celebrated theorem that bears his name.

**Theorem 3.2** [1], [38, p. 71] A continuous function, \( f(t) \), with \( f(0) = 1 \), is a characteristic function if and only if \( f(t) \) is positive definite.

We remark that since \( e^{itx}, (t \in \mathbb{R}) \) is positive definite, it is easy to show that a characteristic function is positive definite. The converse implication is the difficult part of Theorem 3.2 (see, for example, Kawata [34, p. 377] or Lukacs [39, p. 71]). For our purposes the following version of the Khinchin’s criterion [33] for a characteristic function will suffice (see also Lukacs [39, Thm. 4.2.4, Thm. 4.2.5]).
Theorem 3.3 [34, p. 387] A function of the form

\[ f(t) = \frac{1}{c} \int_{-\infty}^{\infty} \varphi(x + t)\overline{\varphi(x)} \, dx, \]  

(3.4)

where \( \varphi(x) \) is any function in \( L^2(\mathbb{R}) \) with \( ||\varphi||_2 = c > 0 \), or the local uniform limit of such functions, is a characteristic function. The converse is also true.

Theorem 3.3 implies Mathias’s result [40, Satz 15] which may be stated as follows. If \( \varphi \in L^2(\mathbb{R}) \), then the function

\[ f(t) = \int_{-\infty}^{\infty} \varphi(s + t)\overline{\varphi(s - t)} \, ds, \]  

(3.5)

is positive definite. We remark that if \( \varphi : \mathbb{R} \to \mathbb{R} \) is an admissible kernel, then \( \varphi \) is a bounded integrable function. Moreover, it is not difficult to demonstrate that \( \varphi \) satisfies the conditions of Fourier’s inversion theorem (cf. [55, Pringsheim’s Theorem, p. 16]). Thus, with the terminology adopted here we can express Mathias’s main theorem (cf. [41, Hauptsatz, p. 108] or [54, p. 412]) in the following form.

**Theorem 3.4** Let \( \varphi \) be an admissible kernel and let

\[ f(t) := \int_{-\infty}^{\infty} \varphi(x) \cos(xt) \, dx. \]  

(3.6)

Then \( \varphi \) is positive definite if and only if \( f(t) \geq 0 \) for all \( t \in \mathbb{R} \).

The above necessary and sufficient conditions for a characteristic function are, in general, not readily applicable in order to determine whether a given function is a characteristic function. There is, however, a beautiful and simple criterion due to Pólya [47] (see also Lukacs [39, p. 85]).

**Theorem 3.5** (Pólya’s criterion) Suppose that \( f : \mathbb{R} \to \mathbb{R} \) is continuous and satisfies the following conditions: (i) \( f(0) = 1 \), (ii) \( f(-t) = f(t) \), (iii) \( f \) is convex for \( t > 0 \) and (iv) \( \lim_{t \to \infty} f(t) = 0 \). Then \( f(t) \) is the characteristic function of an absolutely continuous distribution function \( V(x) \).

Thus, Pólya’s criterion provides a sufficient condition for a continuous function \( f : \mathbb{R} \to \mathbb{R} \) to be a characteristic function. There is however a caveat in order: our admissible kernels do not satisfy the convexity hypothesis of Theorem 3.5. Having set up our tool-box, we shall now relate positive definiteness to the various Laguerre-type inequalities presented in Sect. 2. Our first result in this direction shows that if \( \varphi(t) \) is an admissible kernel such that \( \log \varphi(t) \) is strictly concave (i.e., \( d^2/dt^2 \log \varphi(t) < 0 \) for \( t > 0 \)), then for each \( n \in \mathbb{N} \cup \{0\} \) we can associate with \( \varphi(t) \) a (canonical) kernel \( K_n \) which is also an admissible kernel.

\[ \text{Springer} \]
Theorem 3.6 Let $\varphi(t)$ be an admissible kernel. If $\log\varphi(t)$ is strictly concave for $t > 0$, then for each non-negative integer $n$, the associated kernel

$$K_n(t) := \int_{-\infty}^{\infty} \varphi(s + t)\varphi(s - t)s^{2n} \, ds \quad (n = 0, 1, 2, \ldots), \quad (3.7)$$

is also an admissible kernel.

Proof Fix a non-negative integer $n$. We readily deduce that $K_n(t)$ satisfies the properties (i), (ii), (iii) and (v) of Definition 1.2. Thus, it remains to show that $K_n'(t) < 0$ for $t > 0$. Invoking Leibniz’s rule to justify the differentiation under the integral, we have

$$K_n'(t) = 2\int_{0}^{\infty} \left[ \varphi'(t + s)\varphi(t - s) + \varphi(t + s)\varphi'(t - s) \right] s^{2n} \, ds. \quad (3.8)$$

Next, we fix $t > 0$ and consider the intervals of integration $I_1 := (0, t)$ and $I_2 := (t, \infty)$. Since $\log\varphi(t)$ is strictly concave for $t > 0$, $\varphi'(t)\varphi(t)$ is strictly decreasing for $t > 0$. Hence, for $s > 0$, we claim that

$$\frac{\varphi'(t + s)}{\varphi(t + s)} < \frac{-\varphi'(t - s)}{\varphi(t - s)}. \quad (3.9)$$

If $s \in I_1$, then $0 < s < t$ and $-\varphi'(t - s) > 0$. Since $\varphi'(t + s) < 0$, we see that (3.9) holds. On the other hand, if $s \in I_2$, then $t - s < 0$. Since $\varphi(t)$ is an even function, $\varphi(t - s) = \varphi(s - t)$. Also, $0 < s - t < s + t$, and thus, (3.9) holds, since

$$\frac{\varphi'(t + s)}{\varphi(t + s)} < \frac{\varphi'(s - t)}{\varphi(s - t)} = \frac{-\varphi'(t - s)}{\varphi(t - s)}. \quad \Box$$

Following Pólya’s work involving Jensen’s Nachlass [48, pp. 278–308], we next establish an important relationship between a given strictly logarithmically concave admissible kernel and the associated admissible kernel $K_n(t)$ defined in (3.7).

Lemma 3.7 If $\varphi(t)$ is a strictly logarithmically concave admissible kernel for $t > 0$, then

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi(t)\varphi(s)e^{ix(s+t)}(s - t)^{2n} \, dt \, ds = 2 \cdot 2^{2n} \int_{-\infty}^{\infty} K_n(v) \cos(2xv) \, dv,$$

$$(n = 0, 1, 2, \ldots), \quad (3.10)$$

where $K_n$ is the associated admissible kernel defined by (3.7).
\textbf{Proof} (A sketch) Using (i) Euler’s formula \(e^{ix+it} = \cos(x(s+t)) + i \sin(x(s+t))\), (ii) the fact that the odd functions integrate to zero and (iii) the absolute value of the Jacobian of the transformation, \(s \to u + v\) and \(t \to u - v\), we obtain

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi(t)\varphi(s)e^{ix(s+t)}(s-t)^{2n} \, dt \, ds
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi(t)\varphi(s) \cos(x(s+t))(s-t)^{2n} \, dt \, ds
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi(t)\varphi(s) \cos(x(s-t))(s+t)^{2n} \, dt \, ds
= 2 \cdot 2^n \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi(u + v)\varphi(u - v) \cos(2xv)(u)^{2n} \, du \, dv
= 2 \cdot 2^n \int_{-\infty}^{\infty} K_n(v) \cos(2xv) \, dv. \quad (3.11)
\]

\(\square\)

**Theorem 3.8** Let \(\varphi(t)\) be a strictly logarithmically concave (for \(t > 0\)) admissible kernel and let \(K_n\) \((n = 0, 1, 2, \ldots)\) denote the associated admissible kernel defined by (3.7). Let \(F(x) := \int_{-\infty}^{\infty} e^{ix} \varphi(t) \, dt\). Then,

\[
L_n(x) := L_n(x; F) := \frac{2 \cdot 2^n}{(2n)!} \int_{-\infty}^{\infty} K_n(t) \cos(2xt) \, dt, \quad (n = 0, 1, 2, \ldots),
\]

where \(L_n(x)\) is the generalized real Laguerre expression \([\text{cf. (2.7) of Theorem 2.4}]\) for the entire function \(F\). Moreover, \(F \in \mathcal{L} - \mathcal{P}\) if and only if \(K_n\) is a positive definite kernel for all \(n = 0, 1, 2, \ldots\).

**Proof** We recall from Sect. 2 \([\text{see (2.11) of Remark 2.5}]\) that the Taylor coefficient of \(y^{2n}\) in the expansion \(|F(x + iy)|^2\) is precisely \(L_n(x)\). Moreover, since both \(\varphi(t)\) and \(K_n(t)\) are admissible kernels (Theorem 3.6), the following calculations are valid:

\[
|F(x + iy)|^2 = F(x + iy)F(x - iy)
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi(t)\varphi(s)e^{ix(s+t)} \cosh((s-t)y) \, dt \, ds
= \sum_{n=0}^{\infty} \frac{y^{2n}}{(2n)!} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi(t)\varphi(s)e^{ix(s+t)}(s-t)^{2n} \, dt \, ds. \quad (3.13)
\]

Hence, (3.12) follows from Lemma 3.7 and (3.13). Next, by Theorem 2.4, \(F \in \mathcal{L} - \mathcal{P}\) if and only if \(L_n(x) \geq 0\) for \(n = 0, 1, 2, \ldots\) and for all \(x \in \mathbb{R}\). By Theorem 3.4 and (3.12), \(L_n(x) \geq 0\) for \(n = 0, 1, 2, \ldots\), and for all \(x \in \mathbb{R}\), if and only if the kernel \(K_n\) \((n = 0, 1, 2, \ldots)\) is positive definite. \(\square\)
Remark 3.9 (a) In [40, Satz 15], Mathias has proved that the kernel \( K_0(t) = \int_{-\infty}^{\infty} \varphi(s+t)\varphi(s-t) \, ds \) is positive definite. This is clear in our setting, since \( L_0(t) = |F(t)|^2 \). The case when \( n = 1 \); that is,

\[
K_1(t) = \int_{-\infty}^{\infty} \varphi(s+t)\varphi(s-t) \, s^2 \, ds \quad \text{and} \quad L_1(x) = (F'(x))^2 - F(x)F''(x)
\]

appears to be much more difficult. (b) The desideratum to characterize Fourier transforms in the Laguerre–Pólya class, in terms of the indicated kernels, is achieved by Theorem 3.8. However, the elusive nature of positive definiteness certainly remains as an issue. (c) Note that in conjunction with the Bochner and Khinchin results (cf. Theorems 3.2 and 3.3), our Theorem 3.8 gives rise to new families of characteristic functions when the kernels \( K_n \) associated with functions \( F \in \mathcal{L}-\mathcal{P} \) are appropriately normalized.

Open Problem 3.10 Characterize the logarithmically concave admissible kernels \( \varphi(t) \) such that the associated admissible kernel \( K_1(t) \) is positive definite.

Striving for simplicity, we propose here another, direct, approach for showing that \( \int_{0}^{\infty} K_1(t) \cos(xt) \, dt \geq 0 \) for all \( x \in \mathbb{R} \).

Proposition 3.11 Let \( F(x) := \int_{0}^{\infty} \varphi(t) \cos xt \, dt \) and set \( K_1(t) := \int_{t}^{\infty} \varphi(s+t)\varphi(s-t) \, s^2 \, ds \). Let \( \overline{G}(t) := \int_{t}^{\infty} K_1(u) \, du \) and \( A := \overline{G}(0) \). Then \( K_1(t) \) is positive definite if and only if

\[
\int_{0}^{\infty} \overline{G}(t) \sin xt \, dt \leq \frac{A}{x} \quad \text{for all} \quad x \neq 0.
\]

Remark 3.12 Before we prove Proposition 3.11, we recall that Pólya’s argument [47] shows that in general, the non-negativity of the Fourier sine transform is easier to demonstrate than that of the Fourier cosine transform. Indeed, consider the function \( \overline{G}(t) \) defined in Proposition 3.11. Then for each fixed \( x > 0 \),

\[
I(x) := \int_{0}^{\infty} \overline{G}(t) \sin xt \, dt = \sum_{k=0}^{\infty} \int_{\pi k/x}^{\pi (k+1)/x} \overline{G}(t) \sin xt \, dt \quad \left( t = s + \frac{\pi k}{x} \right)
\]

\[
= \sum_{k=0}^{\infty} \int_{0}^{\pi/x} \overline{G} \left( s + \frac{\pi k}{x} \right) \sin(xs + \pi k) \, ds
\]

\[
= \sum_{k=0}^{\infty} (-1)^k \int_{0}^{\pi/x} \overline{G} \left( s + \frac{\pi k}{x} \right) \sin(xs) \, ds.
\]

Since \( \overline{G}(s) > 0 \), \( \overline{G}'(s) < 0 \) (\( s > 0 \)), and \( \overline{G}(s) \to 0 \) as \( s \to \infty \), it follows from the alternating series test that \( I(x) \to 0 \) for \( x > 0 \).
Proof of Proposition 3.11 Integration by parts yields,
\[
\int_0^\infty K_1(t) \cos(\pi t) \, dt = \int_0^\infty K_1(u) \, du - \pi \int_0^\infty \left( \int_t^\infty K_1(u) \, du \right) \sin \pi t \, dt
\]
and hence inequality (3.14) follows if and only if \( K_1(t) \) is positive definite.

We conclude this section with a concrete example which demonstrates that, if \( K_n \) is positive definite, then in general, \( K_{n+1} \) need not be positive definite. There are several ways we can illustrate this fact. The kernel we will use is a Gaussian, \( e^{-t^2} \), times a polynomial and therefore, it will not satisfy condition (v) of Definition 1.2. Nevertheless, our choice facilitates the exact evaluation of the required integrals. The calculations are sufficiently involved to warrant the use of a computer.

Example 3.13 Let \( \varphi(t) := e^{-t^2}(15 + t^2 + t^4) \). Then it is easy to confirm that \( \varphi(t) \) satisfies conditions (i)–(iv) [but not (v)] of Definition 1.2. In addition, \( \log(\varphi(t)) \) is strictly concave for \( t > 0 \). In the subsequent calculations, we will denote by \( c_j, j \geq 1 \), a positive constant whose exact value is irrelevant. Then
\[
F(x) = \int_{-\infty}^\infty \varphi(t) \cos \pi t \, dt = c_1 e^{-x^2/4}(260 - 16x^2 + x^4).
\]
Since \( F(x) > 0 \), \( F \) has 4 non-real zeros (i.e., \( F \notin \mathcal{L} - \mathcal{P} \)) and whence by Theorem 3.8 at least one of the kernels \( K_n \) [cf. (3.7)] fails to be positive definite. Since \( \int_{-\infty}^\infty K_1(t) \cos 2\pi t \, dt = c_2 e^{-x^2/2}(84,240 - 13,536x^2 + 712x^4 - 24x^6 + x^8) > 0 \) for all \( x \in \mathbb{R} \), \( K_1 \) is (strictly) positive definite. On the other hand, \( \int_{-\infty}^\infty K_2(t) \cos 2\pi t \, dt = c_3 e^{-x^2/2}(107,088 - 18,496x^2 + 696x^4 - 16x^6 + x^8) \) has 4 simple real zeros and consequently \( K_2 \) is not positive definite.

4 Scholia: the Jacobi Theta Function and the Riemann \( \xi \)-Function

The purpose of this section is three-fold: (i) to investigate the properties of the Jacobi theta function [cf. (4.2)] and related kernels, (ii) apply the results of Sect. 3 (Theorems 3.6 and 3.8) and provide new necessary and sufficient conditions for \( H(x) := \xi(x/2)/8 \in \mathcal{L} - \mathcal{P} \) [cf. (4.1)], and (iii) formulate some open problems involving kernels associated with the Jacobi theta function.

By way of background information, we commence with Riemann’s definition of his \( \xi \)-function [48, p. 10]; that is,
\[
\xi(iz) := \frac{1}{2} \left( z^2 - \frac{1}{4} \right) \pi^{-z/2-1/4} \Gamma\left( \frac{z}{2} + \frac{1}{4} \right) \xi\left( z + \frac{1}{2} \right).
\]

Then it is known [48, p. 11], [56, p. 255] or [51, p. 286] that \( \xi(x) \) admits the integral representation of the form
\[
H(x) := \frac{1}{8} \xi\left( \frac{x}{2} \right) := \int_0^\infty \Phi(t) \cos(\pi t) \, dt,
\]
where the Jacobi theta function, (without the usual factor 4) is defined as

\[
\Phi(t) := \sum_{n=1}^{\infty} \pi n^2 (2\pi n^2 e^{4t} - 3) \exp \left( 5t - \pi n^2 e^{4t} \right).
\]  

(4.2)

The Riemann Hypothesis is equivalent to the statement that all the zeros of \( H(x) \) are real (cf. [56, p. 255]). We also recall that \( H(x) \) is an entire function of order one [56, p. 16] of maximal type (cf. [18, App. A]). Thus, with the above nomenclature (cf. Sect. 1) the Riemann Hypothesis is true if and only if \( H \in \mathcal{L} - \mathcal{P} \). It is also known [30, p. 7] that all the zeros of \( H \) lie in the interior of the strip \( S(1) \), so that \( H(x) \in \mathcal{G}(\tau) \), with \( \tau = 1 \) and that \( H(x) \) has an infinite number of real zeros [56, p. 256]. Before we begin with a synopsis of results, we emphasize that the raison d’être for investigating the kernel \( \Phi_1 \) is that there is an intimate connection (the precise meaning of which is yet unknown) between the properties of \( \Phi_1 \) and the distribution of the zeros of its Fourier transform \( H(x) \) [cf. (4.1)].

**Theorem 4.1** [15, Thm. A] Consider the function \( \Phi_1 \) of (4.2) and set

\[
\Phi(t) = \sum_{n=1}^{\infty} a_n(t), \quad \text{where} \quad a_n(t) := \pi n^2 (2\pi n^2 e^{4t} - 3) \exp \left( 5t - \pi n^2 e^{4t} \right) (n = 1, 2, \ldots).
\]

Then, the following are valid:

(i) for each \( n \geq 1, a_n(t) > 0 \) for all \( t \geq 0 \), so that \( \Phi(t) > 0 \) for all \( t \geq 0 \);
(ii) \( \Phi(z) \) is analytic in the strip \(-\pi/8 < \text{Im } z < \pi/8\);
(iii) \( \Phi \) is an even function, so that \( \Phi^{(2m+1)}(0) = 0 \) \( (m = 0, 1, \ldots) \);
(iv) for any \( \varepsilon > 0 \), \( \lim_{t \to \infty} \Phi^{(n)}(t) \exp \left[ (\pi - \varepsilon)e^{4t} \right] = 0 \);
(v) \( \Phi'(t) < 0 \) for all \( t > 0 \).

The proofs of statements (i)–(iv) can be found in Pólya [48], whereas the proof of (v) is in Wintner [58] (see also Spira [53]).

To indicate the significance of the next theorem, we consider the Taylor series of \( H(x) \) about the origin

\[
H(z) = \sum_{k=0}^{\infty} \frac{(-1)^k b_k}{(2k)!} z^{2k}, \quad \text{where} \quad b_k := \int_0^{\infty} t^{2k} \Phi(t) \, dt \quad (k = 0, 1, 2, \ldots).
\]

(4.3)

The change of variable, \( z^2 = -x \) in (4.3), yields the entire function

\[
F(x) := \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k, \quad \text{where} \quad \gamma_k := \frac{k! b_k}{(2k)!} > 0 \quad (k = 0, 1, 2, \ldots).
\]

(4.4)
Then it is easy to see that $F(x)$ is an entire function of order $\frac{1}{2}$ and that the Riemann Hypothesis is equivalent to the statement that all the zeros of $F(x)$ are real and negative. Now it is known (Pólya and Schur [50]) that a necessary condition for $F(x)$ to have only real zeros is that the moments $b_k$ in (4.3) satisfy the Turán inequalities; that is,

$$b_k^2 - \frac{2k-1}{2k+1}b_{k-1}b_{k+1} \geq 0 \quad \text{or equivalently} \quad (4.5)$$

$$T_k := \gamma_k^2 - \gamma_{k-1}\gamma_{k+1} \geq 0 \quad (k = 1, 2, 3, \ldots). \quad (4.6)$$

These inequalities have been established (cf. [15], for $m \geq 2$, and [20]) as a consequence of either one of the two concavity properties [(a) or (b)] of $\Phi_1$ stated in the following theorem. (For related interesting results see also Dimitrov and Lucas [29] and Dimitrov [27, 28]).

**Theorem 4.2** Let $\Phi$ be defined by (4.1). Then $\Phi$ satisfies the following concavity properties.

(a) [15, Prop. 2.1] If

$$K_\Phi(t) := \int_t^\infty \Phi(\sqrt{u}) \, du \quad (t \geq 0),$$

then $\log K_\Phi(t)$ is strictly concave for $t > 0$; that is, $\frac{d^2}{dt^2} \log K_\Phi(t) < 0$ for $t > 0$.

(b) [20, Thm. 2.1] The function $\log \Phi(\sqrt{t})$ is strictly concave for $t > 0$. □

**Remark 4.3** (a) A calculation shows that $\log \Phi(\sqrt{t})$ is strictly concave for $t > 0$ if and only if $g(t) := t \left[ (\Phi'(t))^2 - \Phi(t)\Phi''(t) \right] + \Phi(t)\Phi'(t) > 0$ for $t > 0$. Since $\Phi(t) > 0$ and $\Phi'(t) < 0$ for $t > 0$, it is easy to check that the inequality $g(t) > 0$ is stronger than the assertion that $\log(\Phi(t))$ is strictly concave for $t > 0$. Indeed, the inequality $\Phi'(t)^2 - \Phi(t)\Phi''(t) > 0$ does not imply, in general, the Turán inequalities (4.5) (see, for example, [4, Ex. 3.4]).

(b) Since $\Phi(t) > 0$ and $\Phi'(t) < 0$ for $t > 0$, we can also demonstrate that that the “average value” of $H(x)$, the Fourier cosine transform of $\Phi$ [cf. (4.1)], is positive. Indeed, for $t > 0$,

$$\int_0^t H(u) \, du = \int_0^\infty \Phi(x) \left( \int_0^t \cos xu \, du \right) \, dx = \int_0^\infty \Phi(x) \frac{\sin xt}{x} \, dx > 0,$$

where the last inequality can be established using the method of proof presented in Remark 3.12.

We append here yet another convexity result involving $\Phi$.

**Theorem 4.4** [10, pp. 43–44] The function $\Phi(\sqrt{t})$ is strictly convex for $t > 0$; (that is, $\frac{d^2}{dt^2} \Phi(\sqrt{t}) > 0$ for $t > 0$) and hence

$$\int_0^\infty \Phi(\sqrt{t}) \cos xt \, dt > 0 \quad \text{for all } x \in \mathbb{R}. \quad \Box$$
Having reviewed some of the salient properties of the Jacobi theta function, we are now in position to apply the results of Sect. 3.

**Theorem 4.5** The Jacobi theta function, \( \Phi(t) \), is a strictly logarithmically concave admissible kernel. Moreover, the associated kernel

\[
K_n(t) := K_n(t; \Phi) := \int_{-\infty}^{\infty} \Phi(s + t)\Phi(s - t)s^{2n} ds \quad (n = 0, 1, 2, \ldots),
\]

is also an admissible kernel.

**Proof** By Theorem 4.1, \( \Phi \) is an admissible kernel. Now, it follows from Theorem 4.2 and Remarks 4.3 (a) that \( \log \Phi(t) \) is strictly concave for \( t > 0 \). Thus, by Theorem 3.6, for each non-negative integer \( n \), the associated kernel \( K_n(t) := K_n(t; \Phi) \) is also an admissible kernel. \( \square \)

Finally, with the aid of Lemma 3.7, Theorem 3.6 and Theorem 3.8, we obtain the following equivalent formulation of the Riemann Hypothesis.

**Theorem 4.6** Let \( K_n := K_n(t; \Phi) \) \((n = 0, 1, 2, \ldots)\) denote the associated admissible kernel defined by (4.7). Let \( H(x) := \int_0^\infty \Phi(t) \cos xt dt \). Then, for \( n = 0, 1, 2, \ldots \),

\[
L_n(x) := L_n(x; H) := \frac{2 \cdot 2^n}{(2n)!} \int_{-\infty}^{\infty} K_n(t) \cos(2xt) dt,
\]

where \( L_n(x) \) is the generalized real Laguerre expression [cf. (2.7) of Theorem 2.4] for the entire function \( H \). Moreover, \( H \in L - P \) if and only if \( K_n \) is a positive definite admissible kernel for all \( n = 0, 1, 2, \ldots \).

We next state the following tantalizing open problem.

**Open Problem 4.7** (One of the simplest Laguerre inequalities for the Riemann \( \xi \)-function) Let \( \Phi \) denote the Jacobi theta function and let \( H(x) := \xi(x/2)/8 = \int_0^\infty \Phi(t) \cos xt dt \). Then, is it true that

\[
L_1(x) = (H'(x))^2 - H(x)H''(x) \geq 0 \quad \text{for all} \quad x \in \mathbb{R}.
\]

**Remark 4.8** The verification of the special Laguerre inequality (4.9) itself would be significant. If we could prove that \( L_1(x) > 0 \) for all real \( x \), then it would follow that all the real zeros of \( H \) are simple. Of course, should inequality (4.9) fail to hold for some \( x_0 \), then the Riemann Hypothesis would be false. Now it follows from the numerical results of van de Lune, te Riele, and Winter [57] that the zeros of \( H(x) \) are real and simple for \( |x| < 1.09 \cdots \times 10^9 \) and whence, by Proposition 2.2, \( L_1(x) > 0 \) for \( |x| < 1.09 \cdots \times 10^9 \).

Open Problem 4.7 need not be construed as an insurmountable barrier for further research. Indeed, in the interest of new investigations, we propose here a variant of the \( \text{Pólyaesque} \) approach: namely, if you cannot solve a problem change it (for example,
generalize it). In this spirit, we mention that in the study of the distribution of zeros of entire functions $f(x) \in \mathcal{S}(\tau)$ (of order $< 2$) under the action of the operator $e^{-tD^2}$, $(D := d/dx)$ there is a simple heuristic principle formulated by Pólya. If $t > 0$, then under the action of $e^{-tD^2}$ the zeros of $f(x)$ tend to be “attracted” to the real axis, while under the action of $e^{tD^2}$ the zeros of $f$ tend to be repelled by the real axis. Guided by this principle, we apply $e^{-tD^2}$ to the Riemann $\xi$-function [see (4.1)]. For convenience and to adhere to the notation employed in the papers cited below, we set $H(x) := H_0(x) := \xi(x/2)/8$. Let

$$H_t(x) = e^{-tD^2}H(x) = \int_0^\infty e^{ts^2} \Phi(s) \cos(xs) \, ds \quad \left( t \in \mathbb{R}; \ x \in \mathbb{C}, \ D := \frac{d}{dx} \right).$$

(4.10)

In 1950, de Bruijn [25] established that (i) $H_t(x)$ has only real zeros for $t \geq 1/2$ (this is a consequence of the fact that $H \in \mathcal{S}(\tau)$, with $\tau = 1$, and that $\cos(tD)H \in L - P$ for all $t \geq 1$) and (ii) if $H_t(x)$ has only real zeros for some real $t$, then $H_t(x)$ also has only real zeros for any $t' \geq t$. Subsequently, Newman [42] showed, in 1976, that there is a real constant $\Lambda$, which satisfies $-\infty \leq \Lambda \leq 1/2$, such that $H_t$ has only real zeros if and only if $t \geq \Lambda$. This constant $\Lambda$ is now called the de Bruijn–Newman constant in the literature, and the Riemann Hypothesis is equivalent to the statement that $\Lambda \leq 0$. We remark that Odlyzko, Smith, Varga and the author [16] has shown that $-5.895 \times 10^{-9} < \Lambda$ (see also [22]).

Differentiation under the integral sign in Eq. (4.10) (which can be readily justified by Leibniz’s rule, see also [8]) shows that $H_t(x)$ satisfies the backward heat equation:

$$\frac{\partial(H_t(x))}{\partial t} = - \frac{\partial^2 H_t(x)}{\partial x^2}.$$  

(4.11)

This observation is the key ingredient in the proof of the following proposition.

**Proposition 4.9** [17, Prop. 1] Suppose that $H_0$ has a multiple real zero. Then $t_0 \leq \Lambda$. In particular, if $t > \Lambda$, then the zeros of $H_t$ are real and simple.

We next consider two open problems involving $H_\lambda(x)$ and the “new” kernels $\Phi_\lambda(t) := e^{\lambda t^2} \Phi(t)$ when (i) $\lambda < 0$ and when (ii) $\lambda > 0$.

**Open Problem 4.10** Fix $\lambda < 0$. Using the theory of positive definite kernels (see Sect. 3) show that for some non-negative integer $n$, the kernel

$$K_n(t) := K_n(t; \Phi_\lambda) := \int_{-\infty}^\infty \Phi_\lambda(s + t)\Phi_\lambda(s - t)s^{2n} \, ds,$$

is not positive definite.

(4.12)

Second, assume that $\lambda > 0$. In this case, the factor $e^{\lambda s^2}$ under the integral sign [cf. (4.10)] is an example of a function that Pólya termed an universal factor (see the beautiful papers by Pólya [49] and de Bruijn [25]). Universal factors preserve the

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Laguerre–Pólya class. In 2009, Ki et al. [37] proved that for every fixed \( \lambda > 0 \) but a finite number of the zeros of \( H_\lambda \) are real and simple. Thus, in particular, if \( \lambda > 0 \), then \( H_\lambda \in \mathcal{L} - \mathcal{P}^* \) (see Definition 1.1). Now, in 1987, T. Craven, W. Smith and the author proved the Pólya–Wiman Conjecture [9] (for a more elegant proof see Ki and Kim [35]); namely, if \( f(x) \in \mathcal{L} - \mathcal{P}^* \), then there is a positive integer \( m_0 \) such that \( f^{(m)}(x) \in \mathcal{L} - \mathcal{P} \) for all \( m \geq m_0 \). Therefore, it follows from the aforementioned results that for each fixed \( \lambda > 0 \), there is a positive integer \( m_0 = m_0(\lambda) \) such that for \( 2m \geq m_0 \) (we work with an even integer so that the new kernel is also even)

\[
H^{(2m)}(x) = \frac{d^{2m}}{dx^{2m}} e^{-\lambda x^2} f(x) = \int_0^\infty s^{2m} e^{\lambda x^2} \Phi(s) \cos(sx) \, ds \in \mathcal{L} - \mathcal{P}. \tag{4.13}
\]

Observe that the new kernel \( s^{2m} \Phi_\lambda(s) = s^{2m} e^{\lambda s^2} \Phi(s), (s > 0) \), is not monotone decreasing, it is not logarithmically concave and it tends to 0 (as \( s \to \infty \)) a “little” slower than \( \Phi \).

**Open Problem 4.11** With the above notation and assumptions, is the kernel

\[
K_1(t; \Phi_\lambda, m) := \int_{-\infty}^{\infty} \Phi_\lambda(s + t) \Phi_\lambda(s - t)(s^2 - t^2)^m s^2 \, ds, \quad \text{positive definite?}
\]

We conclude this paper with three additional open problems.

**Open Problem 4.12** Characterize the admissible kernels whose Fourier transforms have all their zeros located in the strip \( S(1) \).

**Open Problem 4.13** [4, Conj. 2.5] Show that the derivatives of the Jacobi theta function, \( \Phi(t) \), are (strictly) log-concave on \( \mathbb{R} \); that is, for each \( n \in \mathbb{N} \),

\[
J_n(t) := (\Phi^{(n)}(t))^2 - \Phi^{(n-1)}(t) \Phi^{(n+1)}(t) > 0 \quad \text{for} \quad t \in \mathbb{R}. \tag{4.14}
\]

Since \( \Phi(t) \) is an even function (cf. Theorem 4.1), \( J_n(t) \) is even and whence it suffices to establish (4.14) for \( t \geq 0 \).

Consider again the entire function \( F \) (cf. (4.4)) related to the Riemann \( \xi \)-function:

\[
F(x) := \sum_{k=0}^{\infty} \frac{k! y_k x^k}{(2k)!}, \quad \text{where} \quad y_k := \frac{k! b_k}{(2k)!}, \quad (k = 0, 1, 2, \ldots). \quad \text{Let} \quad T_k := y_k - y_{k-1} y_{k+1} \geq 0, \quad (k = 1, 2, 3, \ldots), \quad \text{and} \quad E_k := T_k^2 - T_{k-1} T_{k+1} \quad \text{for} \quad k = 2, 3, 4, \ldots.
\]

Then a necessary condition for the Riemann Hypothesis to hold is that the double Turán inequalities should hold; i.e., \( E_k \geq 0 \) for \( k = 2, 3, 4, \ldots \). In [13, Thm. 2.4], we derived a concavity condition (for an admissible kernel) which implies the double Turán inequalities (see also [27, 28]). Thus, an affirmative answer to the following conjecture, will establish yet another necessary condition for the validity of the Riemann Hypothesis.

**Open Problem 4.14** [13, Prob. 3.3] (*A new concavity condition of \( \Phi(t) \).*) Let \( s(t) := \Phi(\sqrt{t}) \) and set \( f(t) := s'(t)^2 - s(t)s''(t) \). By Theorem 4.2 (b), \( f(t) > 0 \) for \( t > 0 \). Then we conjecture that
\[
\frac{d^2}{dt^2} \log f(t) < 0 \quad \text{for} \quad t > 0.
\]

References

1. Bochner, S.: Vorlesungen über Fouriersche Integrale. Akademische Verlagsgesellschaft, Leipzig (1932)
2. Cardon, D.A.: Extended Laguerre Inequalities and a Criterion for Real Zeros, Progress in Analysis and its Applications. World Science Publishers, Hackensack (2010)
3. Cartwright, M.L.: The zeros of certain integral functions. Q. J. Math. 1, 38–59 (1930)
4. Coffey, M.W., Csordas, G.: On the log-concavity of a Jacobi theta function. Math. Comput. 82, 2265–2272 (2013)
5. Craven, T., Csordas, G.: Iterated Laguerre and Turán inequalities. JIPAM J. Inequal. Pure Appl. Math. 3, Article 39, pp. 14 (2002)
6. Craven, T., Csordas, G.: On a converse of Laguerre’s theorem. Electron. Trans. Numer. Anal. 5, 7–17 (1997)
7. Craven, T., Csordas, G.: Jensen polynomials and the Turán and Laguerre inequalities. Pacif. J. Math. 136, 241–260 (1989)
8. Craven, T., Csordas, G.: Differential operators of infinite order and the distribution of zeros of entire functions. J. Math. Anal. Appl. 186, 799–820 (1994)
9. Craven, T., Csordas, G., Smith, W.: The zeros of derivatives of entire functions and the Pólya–Wiman conjecture. Ann. Math. 125, 405–431 (1987)
10. Csordas, G.: Convexity and the Riemann \( \xi \)-function. Glas. Mat. Ser. III 33(53), 37–50 (1998)
11. Csordas, G.: Linear operators and the distribution of zeros of entire functions. Complex Var. Elliptic Equ. 51, 625–632 (2006)
12. Csordas, G.: The Laguerre inequalities and the zeros of the Riemann \( \xi \)-function. Complex Var. Elliptic Equ. 56, 49–58 (2011)
13. Csordas, G., Dimitrov, D.K.: Conjectures and theorems in the theory of entire functions. Numer. Algorithms 25, 109–122 (2000)
14. Csordas, G., Escassut, A.: The Laguerre inequality and the distribution of zeros of entire functions. Ann. Math. Blaise Pascal 12, 331–345 (2005)
15. Csordas, G., Norfolk, T.S., Varga, R.S.: The Riemann hypothesis and the Turán inequalities. Trans. Am. Math. Soc. 296, 521–541 (1986)
16. Csordas, G., Smith, W., Odlyzko, A.M., Varga, R.S.: A new Lehmer pair of zeros and a new lower bound for the de Bruijn–Newman constant \( \Lambda \). Electron. Trans. Numer. Anal. 1, 104–111 (1993)
17. Csordas, G., Smith, W., Varga, R.S.: Lehmer pairs of zeros and the Riemann \( \xi \)-function. Mathematics of Computation 1943–1993: a half-century of computational mathematics (Vancouver, BC, 1993). In: Proceedings of the Sympososium of Application Mathematics, 553–556. 48 American Mathematical Society, Providence (1994)
18. Csordas, G., Norfolk, T.S., Varga, R.S.: A lower bound for the de Bruijn-Newman constant \( \Lambda \). Math. Soc. 52, 483–497 (1988)
19. Csordas, G., Varga, R.S.: Necessary and sufficient conditions and the Riemann hypothesis. Adv. Appl. Math. 1, 328–357 (1990)
20. Csordas, G., Varga, R.S.: Moment inequalities and the Riemann hypothesis. Constr. Approx. 4, 175–198 (1988)
21. Csordas, G., Ruttan, A., Varga, R.S.: The Laguerre inequalities with applications to a problem associated with the Riemann hypothesis. Numer. Algorithms 1, 305–330 (1991)
22. Csordas, G., Smith, W., Varga, R.S.: Lehmer pairs of zeros, the de Bruijn-Newman constant \( \Lambda \), and the Riemann hypothesis. Constr. Approx. 10, 107–129 (1994)
23. Csordas, G., Yang, C.-C.: Finite Fourier transforms and the zeros of the Riemann \( \xi \)-function. II (2009). In: Proceedings of the 5th International ISAAC Congress, Catania, Italy, July 25–30, pp. 1295–1302. World Scientific, Hackensack (2005)
24. Csordas, G., Vishnyakova, A.: The generalized Laguerre inequalities and functions in the Laguerre–Pólya class. Cent. Eur. J. Math. 11, 1643–1650 (2013)
25. de Bruijn, N.G.: The roots of trigonometric integrals. Duke Math. J. 7, 197–226 (1950)
26. Dilcher, K., Stolarsky, K.B.: On a class of nonlinear operators acting on polynomials. J. Math. Anal. Appl. 170, 382–400 (1992)
27. Dimitrov, D.K.: Higher order Turán inequalities. Proc. Am. Math. Soc. **139**, 1013–1022 (2011)
28. Dimitrov, D.K.: Higher order Turán inequalities. Proc. Am. Math. Soc. **126**, 2033–2037 (1998)
29. Dimitrov, D.K., Lucas, F.R.: Higher order Turán inequalities for the Riemann $\xi$-function. Proc. Am. Math. Soc. **126**, 2033–2037 (1998)
30. Ivić, A.: The Riemann Zeta-Function. The Theory of the Riemann Zeta-Function with Applications. Wiley, New York (1985)
31. Jensen, J.L.W.V.: Reserches sur la théorie des équations. Acta Math. **36**, 181–195 (1913)
32. Katkova, O.M.: Multiple positivity and the Riemann zeta-function. Comput. Methods Funct. Theory **7**, 13–31 (2007)
33. Khinchin, A.: Zur Kennzeichnung der charakteristischen Funktionen. ull. Univ. Etat Moscou Ser. Int. Sect. A Math. et Mecan. Fasc. 5 **1**, 1–3 (1937)
34. Kawata, T.: Fourier Analysis in Probability Theory. Probability and Mathematical Statistics. No. 15 Academic Press, New York, London (1972)
35. Ki, H., Kim, Y.-O.: On the number of nonreal zeros of real entire functions and the Fourier–Pólya conjecture. Duke Math. J. **104**, 45–73 (2000)
36. Ki, H., Kim, Y.-O.: De Bruijn’s question on the zeros of Fourier transforms. J. Anal. Math. **91**, 369–387 (2000)
37. Ki, H., Kim, Y.-O., Lee, J.: On the de Bruijn–Newman constant. Adv. Math. **222**, 281–306 (2009)
38. Levin, B.J.: Distribution of Zeros of Entire Functions. Translation of Mathematical Monographs, vol. 5. American Mathematical Society, Providence (1964) (revised ed. 1980)
39. Lukacs, E.: Characteristic Functions, 2nd edn. Hafner Publishing Co, New York (1970)
40. Marden, M.: Geometry of Polynomials. Mathematical Surveys no. 3. American Mathematical Society, Providence (1966)
41. Mathias, M.: Über positive Fourier-Integrale. Math. Z. **16**, 103–125 (1923)
42. Newman, C.M.: Fourier transforms with only real zeros. Proc. Am. Math. Soc. **61**, 245–251 (1976)
43. Newman, C.M.: The GHS: inequality and the Riemann hypothesis. Constr. Approx. **7**, 389–399 (1991)
44. Nuttall, J.: Wronskians, cumulants, and the Riemann hypothesis. Constr. Approx. **38**, 193–212 (2013)
45. Obreschkoff, N.: Verteilung und Berechnung der Nullstellen reeller Polynome. VEB Deutscher Verlag der Wissenschaften, Berlin (1963)
46. Patrick, M.: Extensions of inequalities of the Laguerre and Turán type. Pac. J. Math. **44**, 675–682 (1973)
47. Pólya, G.: Remarks on characteristic functions. In: Proceedings of the Berkeley Symposium on Mathematical Statistics and Probability, pp. 115–123. University of California Press, Berkeley (1949)
48. Pólya, G.: Über die algebraisch-funktionentheoretischen Untersuchungen von J. L. W. V. Jensen. Kgl. Danske Vid. Sel. Math.-Fys. Medd. **7**, 3–33 (1927)
49. Pólya, G.: Über trigonometrische Integrale mit nur reellen Nullstellen. J. Reine Angew. Math. **158**, 6–18 (1927)
50. Pólya, G., Schur, J.: Über zwei Arten von Faktorenfolgen in der Theorie der algebraischen Gleichungen. J. Reine Angew. Math. **144**, 89–113 (1914)
51. Pólya, G.: Collected papers. In: Boas, R.P. (ed.) Location of Zeros, vol. II. MIT Press, Cambridge (1974)
52. Rahman, Q.I., Schmeisser, G.: Analytic Theory of Polynomials. London Mathematical Society Monographs. New Series, 26. The Clarendon Press, Oxford University Press, Oxford (2002)
53. Spira, R.: The integral representation for the Riemann $\Xi$-function. J. Number Theory **3**, 498–501 (1971)
54. Stewart, J.: Positive definite functions and generalizations, an historical survey. Rocky Mt. J. Math. **6**, 409–434 (1976)
55. Titchmarsh, E.C.: Introduction to the Theory of Fourier Integrals. Clarendon Press, Oxford (1937)
56. Titchmarsh, E.C.: The theory of the Riemann Zeta-function, 2nd edn. Oxford University Press, Oxford (1986) (revised by D. R. Heath-Brown)
57. van de Lune, J., te Riele, H.J.J., Winter, D.T.: On the zeros of the Riemann zeta function in the critical strip. IV. Math. Comp. **46**, 667–681 (1986)
58. Wintner, A.: A note on the Riemann $\xi$-function. J. Lond. Math. Soc. **10**, 82–83 (1935)