Construction of geodesics on Teichmüller spaces of Riemann surfaces with \( \mathbb{Z} \) action

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Abstract

Teichmüller space \( \text{Teich}(R) \) of a Riemann surface \( R \) is a deformation space of \( R \). In this paper, we prove a sufficient condition for extremality of the Beltrami coefficients when \( R \) has the \( \mathbb{Z} \) action. As an application, we discuss the construction of geodesics. Earle-Kra-Krushkal proved that the necessary and sufficient conditions for the geodesics connecting \([0]\) and \([\mu]\) to be unique are \( \|\mu_0\|_\infty = |\mu_0|(z) \) (a.e.\( z \)) and “unique extremality”. As a byproduct of our results, we show that we cannot exclude “unique extremality”. To show the above claim, we construct a point \([\mu_0]\) in \( \text{Teich}(\mathbb{C} \setminus \mathbb{Z}) \), satisfying \( \|\mu_0\|_\infty = |\mu_0|(z) \) (a.e.\( z \)) and there exists a family of geodesics \( \{\gamma_\lambda\}_{\lambda \in D} \) connecting \([0]\) and \([\mu_0]\) with complex analytic parameter, where \( D \) is an open set in \( l^\infty \).

1 Introduction

Teichmüller space of a Riemann surface \( R \) is the deformation space of complex structure of \( R \). It is a high-dimensional complex manifold and complete metric space with respect to Teichmüller distance, and whether its dimension is finite or infinite depends on whether \( R \) is of finite or infinite type. Since the 1950s, these basic properties have been actively studied, as Teichmüller space theory with quasiconformal maps was developed by Ahlfors, Bers, and others. The detailed history and applications of Teichmüller spaces are described in many articles, for example, see Ahlfors [A], Gardiner [G], and Hubbard [H]. Riemann surfaces of infinite type appear quite naturally, but they are often transcendental. As for the geodesics in Teichmüller spaces treated in this paper, the geodesic connecting two points is unique in the finite-dimensional case, but this is generally not the case in the infinite-dimensional case, see section 4 for a detailed explanation. Therefore, when dealing with infinite types, it is important to impose a topological or analytical restriction on \( R \). Hence, in this paper, we deal with the case where \( R \) has a \( \mathbb{Z} \)-action. Imposing conditions on Automorphism of \( R \) is, for example, Strebel’s problem. Ohtake [O] showed that in the case of finitely generated Abe groups, the lifting by covering the extremal Beltrami coefficients is extremal, see Remark 1 for detail.

First, we recall the deformation theory of Riemann surfaces. Let \( R \) be a Riemann surface whose universal covering surface is the upper half plane \( \mathbb{H} \) with a covering map \( \pi : \mathbb{H} \to R \), and let be represented by a Fuchsian group \( \Gamma \) acting on \( \mathbb{H} \) as \( R = \mathbb{H}/\Gamma \). Denote by \( L^\infty(\Gamma) \) the complex Banach space of the bounded measurable \textit{Beltrami differentials} for \( \Gamma \) supported on \( \mathbb{H} \). Each element \( \mu \in L^\infty(\Gamma) \) satisfies that \( \mu \circ \gamma \cdot \gamma' = \mu \cdot \gamma' \) for every \( \gamma \in \Gamma \). Its open unit ball \( \text{Bel}(\Gamma) \) is the space of Beltrami coefficients for \( \Gamma \). Each element \( \mu \in \text{Bel}(\Gamma) \), the unique quasiconformal homeomorphism \( w^\mu : \mathbb{H} \to \mathbb{H} \) with Beltrami coefficient \( \mu \) that leaves 0, 1 and \( \infty \) fixed, is compatible for \( \Gamma \). Two elements \( \mu \) and \( \nu \) in \( \text{Bel}(\Gamma) \) are said to be \textit{Teichmüller equivalent} if \( w^\mu|_R \equiv w^\nu|_R \) holds. An element \( \mu \in \text{Bel}(\Gamma) \) is said to be \textit{trivial} if it is Teichmüller equivalent to 0. The \textit{Teichmüller space} \( \text{Teich}(\Gamma) \) of \( \Gamma \) is the quotient space of \( \text{Bel}(\Gamma) \) by the Teichmüller equivalence relation. For each \( \mu \in \text{Bel}(\Gamma) \), let \([\mu]\) denote the point of \( \text{Teich}(\Gamma) \) determined by \( \mu \). It is known that \( \text{Teich}(\Gamma) \) is a Banach manifold equipped with a complex structure and the Teichmüller distance. The \textit{Teichmüller distance} \( d_{T(\Gamma)}([\mu],[\nu]) \) of two points \([\mu]\) and \([\nu]\) in \( \text{Teich}(\Gamma) \) is defined by

\[
d_{T(\Gamma)}([\mu],[\nu]) := \inf \tanh^{-1}(K(w^\mu \circ (w^\nu)^{-1})),
\]
where \( K(w^\phi \circ (w^\mu)^{-1}) \) represent the maximal dilatation, defined in Section 2.1, of \( w^\phi \circ (w^\mu)^{-1} \). Let \( \text{Bel}(\Gamma) \) and \( \text{Teich}(\Gamma) \) be written as \( \text{Bel}(R) \) and \( \text{Teich}(R) \), respectively. See section 2.1 for their detailed relations.

From the above and Section 2.1, studying the geometry of Teichmüller spaces means clarifying the complex structure of Riemann surfaces. In this paper, we work on the former. More specifically, we consider geodesics with respect to the Teichmüller distance. First of all, the extremal Beltrami coefficients, closely related to geodesics, are discussed. A Beltrami differential \( \mu_0 \) in \( \text{Bel}(\Gamma) \) is said to be extremal if \( \|\mu_0\|_\infty = \inf\{|\mu|_\infty | \mu \in [\mu_0]\} \). From the Hamilton–Krushkal condition, if \( \mu_0 \) is an extremal Beltrami coefficients, then the map \([0,1] \ni t \rightarrow [t\mu_0] \) is a geodesic from \([0]\) to \([\mu_0]\), see Section 2.1. Hence, we consider extremality under the condition that \( R \) has \( Z \) action. The result presented in Theorem 1 is a generalization of Ohtake’s results on covering maps and extremality [O Theorem 1].

**Theorem A** (Theorem 1).

Let \( R \) be an analytic infinitely Riemann surface that is covering \( S \) and the covering group is an infinite cyclic group \( \langle \gamma \rangle \). Let \( \mu \) be in \( \text{Bel}(R) \), which satisfies that there exist an integrable quadratic holomorphic differentials \( \varphi \) with \( \int_S |\varphi| = 1 \) and \( k \in [||\mu||_\infty,1) \) such that

\[
\mu \circ \gamma_n \cdot \frac{\gamma_n'}{\gamma_n'} \xrightarrow{n \to \infty} k |\tilde{\varphi}| \quad (a.e. \ z \in \omega_0)
\]

where \( \tilde{\varphi} \) is the lift of the element by the covering, \( \gamma_n := \gamma^\omega_n \), and \( \omega_0 \) is fundamental domain for \( \langle \gamma \rangle \). Then \( \mu \) is extremal on \( R \).

**Remark 1**

Theorem 1 includes the known fact that the lifts of Teichmüller Beltrami coefficients on \( S \) are extremal in \( \text{Bel}(R) \). There was an important issue involved in these discussions, called Strebel’s problem. Let \( \pi: R \to S \) be a covering of a hyperbolic Riemann surface \( S \) with a covering transformation group \( \Gamma \). In this situation, there is a natural problem; Under what conditions does \( \pi: R \to S \) (or \( \Gamma \)) satisfy, does the induced map \( \pi^*: \text{Bel}(S) \ni \mu \mapsto \mu \circ \pi^* \in \text{Bel}(R) \) preserve extremality? This problem has already been solved. Ohtake [O Theorem 1] showed that it is sufficient that \( \Gamma \) is a finitely generated Abelian. Finally, McMullen [Mc Theorem 1.1] showed that it is equivalent to \( \Gamma \) is amenable, when \( S \) is of finite type. It should be noted, that Ohtake’s result can be applied even if \( R \) and \( S \) are of infinite type, while McMullen’s result requires that \( S \) be of finite type. Theorem 1 can be regarded as a generalization of Ohtake’s result by weighting.

In the second place, we discuss the construction of a family of geodesics through some two points. When \( R \) is analytically finite type, a geodesic connecting any two points in \( \text{Teich}(R) \) exists uniquely from Teichmüller theorem. However, when \( R \) is of analytically infinite type, it is known to be incorrect. Li [L Theorem 1] constructed two points whose geodesics are not unique in the universal Teichmüller space \( \text{Teich}(\mathbb{D}) \). In general infinite type, Tanigawa [T Theorem 3.5] and Li [L Theorem 4.1] independently derive sufficient conditions for the non-uniqueness of geodesics. In their settings, there exists a family of geodesics through some fixed two points with a complex analytic parameter in \( \mathbb{D} \). On the other hand, necessary and sufficient conditions for geodesics to be unique are also given in [EKK Theorem 6] and [L Theorem 3]:

1. The Beltrami coefficient \( \mu \) is uniquely extremal in \([\mu]\),
2. \(|\mu(z)| = ||\mu||_\infty \quad \text{a.e.} \quad z \in R\).

Condition (1) is called unique extremality, and condition (2) is called absolutely constant. Bozin-Lakic-Markovic-Mateljevic [BLMM Theorem 10] proved that condition (1) does not imply condition (2). Here, we prove that condition (2) does not imply condition (1), constructing a family of geodesics while maintaining condition (2).
**Theorem B (Theorem 1).**

If $R$ has $\mathbb{Z}$ action, then there exists a Beltrami coefficient $\mu$ satisfies the following properties:

1. $\mu$ is extremal.
2. $|\mu|$ is constant.
3. There exists a family of geodesics $\{g_{\lambda} \mid \lambda \in \Lambda\}$, where $\Lambda$ is an open set in $\mathbb{C}$, through $[0]$ and $[\mu]$ with a complex analytic parameter. In particular, $\lambda_1 \neq \lambda_2$ implies that the geodesic $[0, 1] \ni t \mapsto [t\mu_{\lambda_1}]$ is not equal to $[0, 1] \ni t \mapsto [t\mu_{\lambda_2}]$.

This means that we cannot exclude the condition (1) “unique extremality”, which is one of the necessary and sufficient conditions for the unique existence of a geodesic given by Earle-Kra-Krushkal $\text{[EKK]}$ Theorem 6. In addition, if $R = \mathbb{C} \setminus \mathbb{Z}$, we construct an infinite-dimensional family of geodesics through some fixed two points with a complex analytic parameter in an open set of $l^\infty$.

**Theorem C (Theorem 11).**

Suppose $R = \mathbb{C} \setminus \mathbb{Z}$, then there exists a Beltrami coefficient $\mu$ satisfies following properties:

1. $\mu$ is extremal.
2. $|\mu|$ is constant.
3. There exists a family of geodesics $\{\lambda_{(\lambda_n)} \mid (\lambda_n) \in \Omega\}$, where $\Omega$ is an open set in $l^\infty(\mathbb{C})$, through $[0]$ and $[\mu]$ with a complex analytic parameter. In particular, $\lambda_1 \neq \lambda_2$ implies that the geodesic $[0, 1] \ni t \mapsto [t\mu_{\lambda_1}]$ is not equal to $[0, 1] \ni t \mapsto [t\mu_{\lambda_2}]$.

**Remark 2**

Historically, the weak version of Theorem B and Theorem C was proved in $\text{[L1]}$ Theorem 1, $\text{[L2]}$ Theorem 4.1 and $\text{[L3]}$ Theorem 3.5. However, their results do not consider the condition absolutely constant. Moreover, they constructed only one complex parameter family of geodesics, which is an important extension of the family of geodesics in Theorem C which has complex analytic parameters in an infinite-dimensional open set.

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# 2 Preliminaries

In this section, we recall some basic facts about quasiconformal maps in the plane, integrable holomorphic quadratic differentials, extremal Beltrami coefficients, and the general Poincaré series.

## 2.1 Quasiconformal maps and Teichmüller Theory

First, we explain the quasiconformal maps that play an important role in constructing Teichmüller spaces. Let $K > 1$ be a real number. A homeomorphic map $f$ that preserves the orientation defined by a planar domain $\Omega \subset \mathbb{C}$ is called a $K$-quasiconformal if $f$ has locally integrable distributional derivatives which satisfy $|f_z| \leq K|f_z|$ a.e. $\Omega$, where $k := (K - 1)/(K + 1)$. And $K$ is called maximal dilatation of the homeomorphism $f$. If $K$ (or $k$) is not specified, the function $f$ is simply called quasiconformal. It is well known that $f$ satisfies $f_z \neq 0$ a.e. $\Omega$. The ratio $\text{bel}(f) := f_z/f_z$ is a well-defined measurable function on $\Omega$ with $||\text{bel}(f)||_\infty \leq k$, and called complex dilatation of $f$ or Beltrami coefficient of $f$. Conversely, from the measurable Riemann mapping theorem, for every bounded measurable function $\mu$ on $\mathbb{C}$ with $||\mu||_\infty < 1$, there exists a unique quasiconformal map $f^\mu$ with Beltrami coefficient $\mu$ that leaves 0, 1 and $\infty$ fixed, see $\text{[A]}$. 


2.2 Quadratic differentials and Beltrami differentials

The space \( \Gamma \) and \( Q \)

ideal boundary of \( R \)

is a quasiconformal map and the dilatation of the map \( \mu \) satisfies \( \| \mu \|_\infty < 1 \).

Next, we briefly explain that Teichmüller space \( \text{Teich}(\Gamma) \) can also be regarded as a deformation space of the complex structure of Riemann surface \( R \), using quasiconformal maps in the plane. Let \( L^\infty(R) \) denote the complex Banach space of all bounded measurable \((-1,1)\)-forms on \( R \), where \( \mu \) is a bounded measurable \((-1,1)\)-form means \( \mu = \mu dz/d\bar{z} \). The elements contained in its open unit ball \( \text{Bel}(\Gamma) \) are called the Beltrami coefficients on \( R \). Note that an element \( \mu \) of \( L^\infty(R) \) lifts to an element \( \hat{\mu} \) of \( L^\infty(\Gamma) \) by the formula \( \hat{\mu} = \mu \circ \pi \). The map \( \mu \mapsto \hat{\mu} \) is an isometry between \( L^\infty(R) \) to \( L^\infty(\Gamma) \). Let \( \mu \) be an element in \( \text{Bel}(\Gamma) \), then \( \hat{\mu} \) is an element in \( \text{Bel}(\Gamma) \).

Since \( w^\mu \) is a homeomorphism and compatible with \( \Gamma \), \( \Gamma^\mu \) is a discrete group acting on \( \mathbb{H} \), hence \( R^\mu := \mathbb{H}/\Gamma^\mu \) becomes a Riemann surface again. Moreover \( w^\mu \) induces a quasiconformal map \( f^\mu \) between \( R \to R^\mu \). From classical results, for two elements \( \mu \) and \( \nu \) in \( \text{Bel}(R) \), \( \hat{\mu} \) is Teichmüller equivalent to \( \nu \) if and only if there exists a conformal map \( c : R^\mu \to R^\nu \) such that \((f^\nu)^{-1} \circ c \circ f^\mu \) is homotopic to \( \text{id}_R \) relative ideal boundary of \( R \). The Teichmüller space of \( R \) is \( \text{Bel}(R) \) factored by Teichmüller equivalence relation, and denoted by \( \text{Teich}(R) \). \( \text{Teich}(R) \) and \( \text{Teich}(\Gamma) \) are naturally isomorphic, see [G] Proposition 1 in §5.

2.2 Quadratic differentials and Beltrami differentials

Let \( Q(R) \) denote the complex Banach space of all integrable holomorphic quadratic differentials for \( \Gamma \). The space \( Q(\Gamma) \) is identified with the cotangent space of \( \text{Teich}(\Gamma) \) at \( [0] \). The complex dimension of \( \text{Teich}(\Gamma) \) is \( 3 \). Extremality under the action of \( \mathbb{Z} \).

A Beltrami differential \( \mu_0 \) in \( \text{Bel}(\Gamma) \) is said to be extremal if \( \| \mu_0 \|_\infty = \inf\{\| \mu \|_\infty | \mu \in [\mu_0] \} \). It is known that a Beltrami differential \( \mu \) is extremal if and only if there exists a sequence \((\varphi_j) \in \mathbb{N} \) of elements in \( Q(\Gamma) \) with unit integrable norm such that \( \| \mu \|_\infty = \lim_{n \to \infty} \text{Re} \int_{\mathbb{H}/\Gamma} \mu \varphi_n \), it is called the Hamilton–Krushkal condition, see [G] Theorem 1 and Theorem 6 in §6.7. Such a sequence is called a Hamilton sequence for \( \mu \). From the Hamilton–Krushkal condition, if \( \mu \) is extremal, we see that the following map is geodesic from \([0]\) to \([\mu]\) with respect to the Teichmüller distance:

\[
[0,1] \ni t \mapsto [t\mu].
\]

A Beltrami differential of the form \( z|\varphi| / \varphi \) with some \( z \in \mathbb{D} \) and \( \varphi \in Q(\Gamma) \) is called Teichmüller Beltrami differential, where \( \mathbb{D} \) is the unit disk in the complex plane. It is obvious that Teichmüller Beltrami differentials are extremal. It is known that a point \([\mu]\) in \( \text{Teich}(\Gamma) \) which contains a Teichmüller Beltrami differential is the unique extremal differential in equivalence class \([\mu]\), see [G] Theorem 2 in §6.2, Theorem 3 in §6.3.

Let \( S \) be a Reimann surface with the universal covering transformation group \( \Gamma' \) acting \( \mathbb{H} \). Suppose that \( R \) is a covering surface of \( S \), then \( \Gamma' \) is a subgroup of \( \Gamma' \). For any element \( \varphi \in Q(R) \), we set

\[
\Theta(\varphi)(z) := \sum_{g \in \Gamma'/\Gamma} \varphi \circ g \cdot (g')^2(z),
\]

where \( \Gamma'/\Gamma \) is the coset. It is said to be a (general) Poincaré series. The series is absolutely and uniformly convergence on \( \omega_0 \), where \( \omega_0 \) is a fundamental domain in \( \mathbb{H} \) for \( \Gamma' \). Moreover \( \Theta(\varphi) \) is an element in \( Q(S) \). The linear operator \( \Theta : Q(R) \to Q(S) \) is bounded and surjective.

3 Extremality under \( \mathbb{Z} \) action

We discuss the extremality of the Beltrami coefficient, which is closely related to the geodesics of Teichmüller space. The Hamilton-Krushkal condition is a well-known necessary and sufficient condition: A Beltrami coefficient \( \mu \) on a Riemann surface \( R \) is extremal if and only if

\[
\| \mu \|_\infty = \sup \left\{ \int_R |\mu| \varphi |Q(R), \| \varphi \|_{Q(R)} = 1 \right\},
\]
where $\omega$ is a fundamental domain. The proof is founded in many articles, for example, see Theorem 1 and Theorem 6 in §6.7. Now we prove the following which plays a critical role in the discussion below.

**Theorem 1.**

Let $R$ be an analytic infinitely Riemann surface that is covering $S$ and the covering group is an infinite cyclic group $\langle \gamma \rangle$. Let $\mu$ be in $\text{Bel}(R)$, which satisfies that there exist $\varphi \in Q(S)$ with $\|\varphi\|_{Q(S)} = 1$ and $k \in [\|\mu\|_{\infty}, 1)$ such that

$$
\mu \circ \gamma_n \cdot \frac{\gamma_n}{\gamma_n^*} \rightarrow k |\varphi| \text{ (a.e. } z \in \omega_0)
$$

where $\varphi$ is the lift of the element $\varphi$ in $Q(S)$, $\gamma_n := \gamma^n$, and $\omega_0$ is fundamental domain for $\langle \gamma \rangle$. Then $\mu$ is extremal.

**Proof:**

Let $\omega_0$ is a fundamental domain for $\langle \gamma \rangle$ in $R$, $\omega_n := \gamma_n(\omega_0)$, and $D_n := \bigcup_{0 \leq j \leq n} \omega_j$. Since the general Poincaré series $\Theta : Q(R) \rightarrow Q(S)$ is surjective, there is an element $f$ in $Q(R)$ such that $\tilde{\varphi} = \Theta(f)$ on $\omega_0$. Set

$$
F_n := \frac{1}{\|\Theta(f)\|_{Q(S)}} \sum_{-n \leq j \leq 0} f \circ \gamma_j \cdot (\gamma_j^\prime)^2, \quad \varphi_n := \frac{1}{\|F_n\|_{Q(R)}} F_n.
$$

We will prove that $(\varphi_n)$ is a Hamilton sequence for $\mu$. Namely, we prove that

$$
\left| \int_R \mu \varphi_n \right| \frac{n \rightarrow \infty}{k}.
$$

Since the following inequality:

$$
k \geq \left| \int_R \mu \varphi_n \right|
= \frac{1}{\|F_n\|_{Q(R)}} \left( \int_{D_n} \mu \tilde{\varphi} + \int_{D_n} \mu (F - \tilde{\varphi}) + \int_{R \setminus D_n} \mu F_n \right)
\geq \frac{1}{\|F_n\|_{Q(R)}} \left( \left| \int_{D_n} \mu \tilde{\varphi} \right| - \left| \int_{D_n} \mu (F - \tilde{\varphi}) \right| - \left| \int_{R \setminus D_n} \mu F_n \right| \right)
= \frac{n + 1}{\|F_n\|_{Q(R)}} \left( \frac{1}{n + 1} \left| \int_{D_n} \mu \tilde{\varphi} \right| - \frac{1}{n + 1} \left( \left| \int_{D_n} \mu (F - \tilde{\varphi}) \right| + \left| \int_{R \setminus D_n} \mu F_n \right| \right) \right),
$$

it is sufficient to show that the following four sequences converge as shown below:

$$
\lim_{n \rightarrow \infty} \frac{1}{n + 1} \int_{D_n} \mu \tilde{\varphi} = k; \quad \lim_{n \rightarrow \infty} \frac{1}{n + 1} \int_{D_n} |\tilde{\varphi} - F_n| = 0,
$$

$$
\lim_{n \rightarrow \infty} \frac{1}{n + 1} \int_{R \setminus D_n} \mu F_n = 0; \quad \lim_{n \rightarrow \infty} \frac{\|F_n\|_{Q(R)}}{n + 1} = 1.
$$

First, let us $x_n := \int_{\omega_0} \mu \tilde{\varphi}$, then we get

$$
x_n = \int_{\omega_0} \mu \circ \gamma_n \cdot \tilde{\varphi} \cdot \gamma_n \cdot |\gamma_n^\prime|^2 = \int_{\omega_0} \mu \circ \gamma_n \cdot \tilde{\varphi} \cdot \frac{1}{(|\gamma_n^\prime|^2)} \cdot |\gamma_n^\prime|^2 = \int_{\omega_0} \mu \circ \gamma_n \cdot \gamma_n^*: \tilde{\varphi}
= \int_{\omega_0} k |\tilde{\varphi}| = k \|\tilde{\varphi}\|_{Q(S)} = k,
$$
because $\mu$ is Beltrami differential and $\tilde{\varphi}$ is holomorphic quadratic differential. Hence, from Cesàro mean, it follows that
\[
\frac{1}{n+1} \int_{D_n} \mu \tilde{\varphi} = \frac{1}{n+1} \sum_{0 \leq j \leq n} x_j \xrightarrow{n \to \infty} k.
\]

Second, by the definition of $\tilde{\varphi}$ and $F_n$, we calculate
\[
\frac{1}{n+1} \int_{D_n} |\tilde{\varphi} - F_n| \leq \frac{1}{n+1} \sum_{0 \leq l \leq n} \int_{0^+} \omega_i \sum_{j > 0} |f \circ \gamma_j \cdot |(\gamma_j')^2|
\]
\[
= \frac{1}{n+1} \sum_{0 \leq l \leq n} \int_{0^+} \omega_i \sum_{j > 0} |f \circ \gamma_j \circ \gamma_l | \cdot |(\gamma_j')^2||(|\gamma_l')^2|
\]
\[
= \frac{1}{n+1} \sum_{0 \leq l \leq n} \int_{0^+} \omega_i \sum_{j > 0} |f \circ \gamma_m | \cdot |(\gamma_m')^2| \xrightarrow{n \to \infty} 0.
\]

Third, in the same way as above, the following calculation holds:
\[
\frac{1}{n+1} \int_{R \setminus D_n} \sum_{-n \leq j \leq 0} |f \circ \gamma_j \cdot (\gamma_j')^2| = \frac{1}{n+1} \sum_{0 > l \text{ or } l > 0} \int_{\omega_i} \sum_{-n \leq j \leq 0} |f \circ \gamma_j \cdot (\gamma_j')^2|
\]
\[
= \frac{1}{n+1} \sum_{0 > l \text{ or } l > 0} \int_{\omega_i} \sum_{-n \leq j \leq -l} |f \circ \gamma_j \cdot (\gamma_j')^2| \xrightarrow{n \to \infty} 0.
\]

Finally, we will prove that
\[
\|\Theta(f)\|_{Q(S)} - \frac{1}{n+1} \int_{D_n} |F_n| \xrightarrow{n \to \infty} 0,
\]
it implies that
\[
\frac{1}{n+1} \int_{D_n} |F_n| \xrightarrow{n \to \infty} \|\Theta(f)\|_{Q(S)}.
\]

From the definition of Poincaré series, it is seen that
\[
\left|\|\Theta(f)\|_{Q(S)} - \frac{1}{n+1} \int_{D_n} |F_n|\right| = \frac{1}{n+1} \left|(n+1)\|\Theta(f)\|_{Q(S)} - \int_{D_n} \sum_{-n \leq j \leq 0} f \circ \gamma_j \cdot (\gamma_j')^2\right|
\]
\[
\leq \frac{1}{n+1} \left|\sum_{0 \leq l \leq n} \int_{\omega_0} |\Theta(f)| - \sum_{0 \leq l \leq n} \int_{\omega_0} \sum_{-n \leq j \leq -l} f \circ \gamma_j \cdot (\gamma_j')^2\right|
\]
\[
\leq \frac{1}{n+1} \left|\sum_{0 \leq l \leq n} \int_{\omega_0} \Theta(f) - \sum_{-n \leq j \leq -l} f \circ \gamma_j \cdot (\gamma_j')^2\right|
\]
\[
\leq \frac{1}{n+1} \left|\sum_{0 \leq l \leq n} \int_{\omega_0} \sum_{-n \leq j \leq -l} f \circ \gamma_j \cdot (\gamma_j')^2\right| \xrightarrow{n \to \infty} 0
\]

Thus, we get
\[
\frac{\|\Theta(f)\|_{Q(S)} \|F_n\|_{Q(R)}}{n+1} = \frac{1}{n+1} \left(\int_{D_n} + \int_{R \setminus D_n}\right) |F_n| \xrightarrow{n \to \infty} \|\Theta(f)\|_{Q(S)} + 0.
\]

Summing up the above calculations, we can show that the sequence $(\varphi_n)$ in $Q(S)$ satisfies
\[
\|\varphi_n\|_{Q(S)} = 1 \quad (\forall n), \quad \lim_{n \to \infty} \int_R \mu \varphi_n = k = \|\mu\|_\infty.
\]
Hence, It can be seen that $\mu$ is extremal from the Hamilton-Kruskal condition.
We compose an example that satisfies the assumption of Theorem 1

**Example 1**
Let us look at the infinitely analytic Riemann Surface $R := \mathbb{C} \setminus \mathbb{Z}$, then $R$ has action by infinite cyclic group generated by $z \mapsto z + 1$. Set $S := R/\langle \gamma : z \mapsto z + 3 \rangle$, and $\omega_0 := \{ z \in R | 0 < \text{Re} z \leq 3 \}$. Since $S$ is conformal equivalent to five punctured sphere, it follows that dim$_c Q(S)$ is equal to 2 from Riemann-Roch Theorem. Let $\varphi^r$ and $\varphi^l$ from $Q(S)$ be linear independent, and let $(a_n)$ and $(b_n)$ be sequences in $[0, 1)$ which satisfies that $a_n + b_n \neq 0$ and
\[
\lim_{n \to \infty} a_n = 1, \quad \lim_{n \to \infty} a_n = 0, \quad \lim_{n \to \infty} b_n = 0, \quad \lim_{n \to \infty} b_n = 1.
\]
We define Beltrami coefficient $\mu$ on $R$ as follows:
\[
\mu(= \mu(a_n, b_n)) := k \frac{|a_n \varphi^r + b_n \varphi^l|}{a_n \varphi^r + b_n \varphi^l} (z \in \omega_n),
\]
where $k$ is a constant in $[0, 1)$, $\omega_n := \gamma^n(\omega_0)$, and we consider $\varphi^r$ and $\varphi^l$ as elements in $Q(S)$ identifying $\omega_j$ with $S$. A simple computation shows that $\mu$ satisfies the assumption of Theorem 1. Indeed,
\[
\mu \circ \gamma_n \cdot \frac{\gamma_n}{\gamma_n}(z) = k \frac{|a_n \varphi^r + b_n \varphi^l|}{a_n \varphi^r + b_n \varphi^l} (z) \xrightarrow{n \to \infty} k \frac{|\varphi^r|}{\varphi^r}(z) \quad (\text{for all } z \in \omega_0).
\]
For instance, if we take $a_n = 1, b_n = 0$, then $\mu(a_n, b_n)$ can be regarded as the lift of $\varphi^r$. Therefore Theorem 1 is a generalization of Ohtake’s result ([O] Theorem 1).

**4 Construction of geodesics**

The uniqueness of geodesics plays an important role in considering the geometry of Teichmüller space. In the finite-dimensional case, the geodesic connecting any two points is unique. By contrast, in the infinite-dimensional case, the geodesic connecting any two points is unique. By contrast, in the infinite-dimensional case, there exists a point $\mu$ which satisfies the geodesic connecting $[0]$ and $[\mu]$ is not unique. Of course, there are two points whose geodesics are unique.

The first result on this issue is that Li [L1] Theorem 1 constructed an example in Teich($D$). In general infinite-dimensional Teichmüller spaces, Li [L2] Theorem 4.1 and Tanigawa [Thi] Theorem 3.5 independently constructed examples. On the other hand, Li [L1] Theorem 3 and Earle-Kra-Krushkal [EKK] Theorem 6 proved that the necessary and sufficient condition for a geodesic connecting $[0]$ and $[\mu]$ to be unique is that the following two conditions hold:

1. The Beltrami coefficient $\mu$ is uniquely extremal in $[\mu]$,
2. $|\mu(z)| = \|\mu\|_\infty \quad \text{a.e. } z \in R$.

In [L1], Li asked if condition (1) implies condition (2). However, for this problem, Bozin-Lakic-Markovic-Mateljevic gave a counterexample ([BLMM] Theorem 10). That is to say, they constructed that $\mu$ is uniquely extremal and $\mu$ is not absolutely constant in a certain Riemann Surface. From the above, we consider how many geodesics can be constructed while preserving conditions (2). Namely, we configure a family of Beltrami coefficients $\{ \mu_\lambda \}_{\lambda \in \Lambda}$ which satisfies the following properties:

a). For all $\lambda \in \Lambda$, $\mu_\lambda$ is extremal.
b). For all $\lambda_1, \lambda_2 \in \Lambda$, $\mu_{\lambda_1}$ is Teichmüller equivalent to $\mu_{\lambda_2}$.
c). $\lambda_1 \neq \lambda_2$ implies $[0, 1] \ni t \mapsto [t \mu_{\lambda_1}]$ and $[0, 1] \ni t \mapsto [t \mu_{\lambda_2}]$ are distinct.
d). There exists $\lambda_0 \in \Lambda$ such that $|\mu_{\lambda_0}(z)| = \|\mu_{\lambda_0}\|_\infty \quad (\text{a.e. } z \in R)$.

The fact that the tangent vectors at the origin of each geodesic are different is a sufficient condition for condition c). Such a characterization is given by Li [L2].
4.1 Construction of the family \( \{\tau_\lambda\} \)

First, to define \( \{\mu_\lambda\} \), we construct a family of Beltrami coefficients on \( U \).

**Lemma 3.**

Let \( T_R := \{\zeta \in \mathbb{C} \mid 0 < \text{Im}\zeta < R\} \), and \( \Lambda := \{\lambda \in \mathbb{C} \mid |\text{Im}\lambda| < 1, |\text{Re}\lambda| > 0\} \) \( (R \in (0, \infty)) \). There exists a family of Beltrami coefficients \( \{\hat{\tau}_\lambda\}_{\lambda \in \Lambda} \) on \( T_R \) which satisfies the following properties:

- For all \( \lambda \in \Lambda \), \( \hat{\tau}_\lambda \) is Teichmüller equivalent to 0 in \( \text{Bel}(T_R) \).
- \( \lambda \in \mathbb{R} \cap \Lambda \) implies that \( |\hat{\tau}_\lambda| \) is constant.
- \( \lambda_1 \neq \lambda_2 \) implies that \( \int_{T_R/\langle \zeta \mapsto \zeta + 2\pi\rangle} (\hat{\tau}_{\lambda_1} - \hat{\tau}_{\lambda_2}) \partial_{\zeta} \partial_{\zeta} \leq 0 \).
- \( \hat{\tau}_\lambda : \Lambda \ni \lambda \mapsto \hat{\tau}_\lambda \in L_\infty(T_R) \) is holomorphic map.

**Proof:**

Consider the following a self-affine map \( F_\lambda : T_R \rightarrow T(r) \),

\[
F_\lambda : T_R(\xi + i\eta \mapsto) := \begin{cases} 
\xi + \lambda\eta + i\eta & (0 < \eta \leq \frac{R}{2}), \\
\xi + \lambda(R - \eta) + i\eta & (\frac{R}{2} \leq \eta < R).
\end{cases}
\]

Let \( \hat{\tau}_\lambda \) be a beltrami coefficient of \( F_\lambda \). Since \( F_\lambda|_{\partial T_R} = \text{id}|_{\partial T_R} \), \( \hat{\tau}_\lambda \) is Teichmüller equivalent to 0 in \( \text{Bel}(T_R) \).

The explicit calculation of \( \tau \) is as follows:

\[
\hat{\tau}_\lambda = \begin{cases} 
\frac{i\lambda}{2 - i\lambda} & \left(0 < \eta \leq \frac{R}{2}\right), \\
\frac{-i\lambda}{2 + i\lambda} & \left(\frac{R}{2} \leq \eta < R\right).
\end{cases}
\] (1)

Thus, the remaining properties are obvious. \( \square \)

Since \( \pi : T_R \ni \zeta \mapsto e^{i\zeta} \in \mathbb{A}_{r_0} := \{z \in \mathbb{C} \mid r_0 < |z| < 1\} \), where \( r_0 := e^{-R} \), is a covering map, and \( F_c \) satisfies that \( F_c(\zeta + 2\pi) = F_c(\zeta) + 2\pi \), \( F_c \) induces a self quasiconformal map \( f_c : \mathbb{A}_{r_0} \). Let \( \nu_c \) be a Beltrami coefficient of \( f_c \), then the pullback of \( \nu_c \) by \( \pi \) coincide with \( \hat{\tau}_\lambda \). Note that the pullback of \( dz^2/\bar{z}^2 \) by \( \pi \) is \( d\zeta^2 \) and that the third complement is the coupling of the Beltrami coefficients and the holomorphic quadratic differentials the following proposition holds:
4.1 Construction of the family \( \{ \tau_\lambda \} \)

**Corollary 4.**

Set \( \mathcal{A}_r := \{ z \in \mathbb{C} \mid r < |z| < 1 \} \), where \( r \in [0, 1) \). There exists a family of Beltrami coefficients \( \{ \nu_\lambda \}_{\lambda \in \Lambda} \) on \( \mathcal{A}_r \) which satisfies the following properties:

- For all \( \lambda \in \Lambda \), \( \nu_\lambda \) is Teichmüller equivalent to 0 in Bel(\( \mathcal{A}_r \)).
- \( \lambda \in \mathbb{R} \cap \Lambda \) implies that \( |\nu_\lambda| \) is constant.
- \( \lambda_1 \neq \lambda_2 \) implies that there exists holomorphic function \( g \) on \( \mathcal{A}_r \) such that \( \int_{\mathcal{A}_r} (\nu_{\lambda_1} - \nu_{\lambda_2})g \, dx \, dy \neq 0 \).
- \( \nu_\star : \Lambda \ni \lambda \mapsto \nu_\lambda \in L_\infty(\mathcal{A}_r) \) is a holomorphic map.

**Proof:**

The first claim is more obvious from the properties of \( F_\lambda \). Explicitly calculating \( \nu_\lambda \) is as follows:

\[
\nu_\lambda = \begin{cases} 
-\frac{i\lambda}{2 + i\lambda} \frac{z}{z} & (r < |z| \leq \sqrt{r}), \\
\frac{i\lambda}{2 - i\lambda} \frac{z}{z} & (\sqrt{r} \leq |z| < 1).
\end{cases}
\]

A simple computation shows that \( \lambda \in \mathbb{R} \cap \Lambda \) implies \( |\tau_\lambda| \) is constant. The remaining properties follow from the previous discussion.

**Remark 3**

Since the lifts of the holomorphic quadratic differentials can distinguish \( \nu_\lambda \) in the infinitesimal sense is \( d\zeta^2 \), if the coefficient of \( z^{-2} \) of the holomorphic function \( h \) on \( \mathcal{A}_r \) is zero, then

\[
\int_{\mathcal{A}_r} \nu_\lambda h \, dx \, dy = 0.
\]

For later discussion, we consider the coupling between quadratic differentials and Beltrami coefficients on \( T_R \). Let \( \hat{\varphi}_\star \) be a holomorphic quadratic differential on \( T_R \), then \( \hat{\varphi}_\star \) satisfies

\[
\hat{\varphi}_\star(\zeta) = \hat{\varphi}_\star \circ \iota(\zeta) \cdot (\iota'(\zeta))^2 = \hat{\varphi}_\star(\zeta + 2\pi),
\]

where \( \iota(\zeta) = \zeta + 2\pi \). Namely, \( \hat{\varphi}_\star \) is regarded as a periodic function. In \( T_R := T_R/\langle \iota \rangle \), Fourier expansion yields:

\[
\int_{T_R} \nu_\lambda \hat{\varphi}_\star = \nu_\star(0) \text{Area}(T_R) \left( \frac{i\lambda}{2 - i\alpha} - \frac{i\lambda}{2 + i\alpha} \right) = \nu_\star(0) \text{Area}(T_R) \frac{\lambda^2}{4 + \lambda^2}.
\]

In the following, we will use the case \( h(z) = \frac{1}{z - \alpha} \), where \( |\alpha| \leq r_0 \), and we calculate it. Since the coefficient of \( z^{-2} \) of \( h \) is

\[
\frac{1}{2\pi i} \int_{|z|=r} \frac{z}{z - \alpha} \, dz = \frac{1}{2\pi i} \left( \int_{|z|=r} 1 + \frac{\alpha}{z - \alpha} \, dz \right) = \alpha,
\]

we get

\[
\int_{\mathcal{A}_r} \nu_\lambda \frac{1}{z - \alpha} = -\frac{2\pi \alpha}{i} \text{Area}(\mathcal{A}_r) \frac{\lambda^2}{4 + \lambda^2}.
\]

Taking the limit of \( r_0 \) to 0, we can construct a family of Beltrami coefficients on \( \mathbb{D} \) which satisfies the same properties. In the following, we describe different construction methods using elliptic functions.

**Theorem 5.**

Let \( \Lambda \subset \mathbb{C} \) take the same as above. There exists a family of Beltrami coefficients \( \{ \tau_\lambda \}_{\lambda \in \Lambda} \) on \( \mathbb{D} \) which satisfies the following properties:
a’). For all \( \lambda \in \Lambda \), \( \tau_\lambda \) is Teichmüller equivalent to 0 in \( \text{Bel}(D) \).

b’). \( \lambda \in \mathbb{R} \cap \Lambda \) implies that \( |\tau_\lambda| \) is constant.

c’). \( \lambda_1 \neq \lambda_2 \) implies that there exists holomorphic function \( g \) on \( D \) such that \( \int_D (\tau_{\lambda_1} - \tau_{\lambda_2}) g \, dx \, dy \neq 0 \).

d’). \( \tau_* : C \ni \lambda \mapsto \tau_\lambda \in L^\infty(D) \) is holomorphic map.

**Proof:**

Since Riemann mapping theorem, for all \( R \in (0, \infty) \), there uniquely exists \( t \in (0, 1) \) which satisfies that there exists a covering map \( \rho : T_R \to D_t \), where \( D_t \) is the set \( D \setminus [-t, t] \). In particular, we take \( \tilde{\rho} \) such that \( \rho(\{ \zeta | \text{Im} \zeta = R \}) = [-t, t] \), where \( \tilde{\rho} \) is extension of \( \rho \) to the closure of \( T_R \) to the closure of \( D_t \).

The \( \rho \) is the elliptic function theta satisfies the following differential equation:

\[
(r')^2 = (r - t)(r + t) \left( r - \frac{1}{t} \right) \left( r + \frac{1}{t} \right).
\]

We construct the family of Beltrami coefficients on \( D_t \) from the family of Beltrami coefficients \( \{F_c\} \) constructed by the Lemma 3 using \( \rho \). In fact, \( F_\lambda \) is compatible with \( \zeta \mapsto \zeta + 2\pi \), it induces a self quasiconformal map of \( D_t \):

\[
\begin{array}{ccc}
T_R & \overset{F_\lambda}{\longrightarrow} & T_R \\
\rho & \circ & \rho \\
D_t & \overset{\exists f_\lambda}{\ni} & D_t
\end{array}
\]

Let \( \tau_\lambda \) be a beltrami coefficient of \( \tilde{f}_\lambda \). It is easy to verify that the family of Beltrami coefficients \( \{\tau_\lambda\}_{\lambda \in \Lambda} \) satisfies a’), b’), and d’). We claim that the family satisfies c’).

Recall the remark of Corollary 4 we prove that there exists a holomorphic function \( \varphi_* \) such that the pullback \( \tilde{\varphi}_* := \rho^*(\varphi_* \, dz^2) \) satisfies that \( \tilde{\varphi}_*(0) \neq 0 \). From simple calculations, we obtain

\[
\tilde{\varphi}_*(0) = \int_{-\pi}^{\pi} \tilde{\varphi}_*(\tilde{x}) \, d\tilde{x} = 2 \int_0^{\pi} \tilde{\varphi}_*(\tilde{x}) \, d\tilde{x} = 2 \int_0^{t} \varphi_*(x) \, (r')^2 \, dx = 2 \int_0^{t} \varphi_*(x) \, (r') \, dx = 2 \int_0^{t} \varphi_*(x) \cdot \sqrt{(x - t)(x + t)} \left( x - \frac{1}{t} \right) \left( x + \frac{1}{t} \right) \, dx.
\]

Note that integrand is positive on Integral interval, \( \tilde{\varphi}_*(0) \neq 0 \) under the condition which \( \varphi_*(0) \neq 0 \) and \( R \) is sufficiently large.

**Remark 4**

In [K], Reich constructed a family of Beltrami coefficients \( \{\tau_\lambda\}_{\lambda \in \Lambda} \) on \( D \) which satisfies the following properties, where \( C := \{\lambda \in \mathbb{C} \mid |\lambda| < 1/2\} \):

- For all \( \lambda \in C \), \( \tau_\lambda \) is Teichmüller equivalent to 0 in \( \text{Bel}(D) \).
- \( \lambda_1 \neq \lambda_2 \) implies that there exists holomorphic function \( g \) on \( D \) such that \( \int_D (\tau_{\lambda_1} - \tau_{\lambda_2}) g \, dx \, dy \neq 0 \).
- \( \tau_* : C \ni \lambda \mapsto \tau_\lambda \in L^\infty(D) \) is holomorphic map.

However, his example does not have elements whose absolute values are constant.
4.2 For sums of Beltrami coefficients whose supports are disjoint

Next, let us consider condition b) explained at the beginning of this section. Let \( R_1 \) be an infinitely analytic subsurface of \( R \) whose boundary is the union of relative compact analytic curves in \( R \), and \( R_2 := R \setminus R_1 \). In general, \( R_2 \) is probably not connected. Then, we consider the following function:

\[
\text{Bel}(R_1) \ni \tau \mapsto \tau + \mu \in \text{Bel}(R),
\]

where \( \mu \) is Beltrami coefficient on \( R_2 \) and \( \mu + \tau \) is extended to be identically \( \mu \) in \( R_2 \).

**Theorem 6.**
The above map induces the holomorphic map:

\[
\text{Teich}(R_1) \ni [\tau] \mapsto [\tau + \mu] \in \text{Teich}(R).
\]

In other words, if \( \tau_1 \) is Teichmüller equivalent to \( \tau_2 \) on \( R_1 \), \( \mu + \tau_1 \) is Teichmüller equivalent to \( \mu + \tau_2 \) on \( R \).

**Proof:**

First, we prove that \( \tau_1 + 0 \) is Teichmüller equivalent to \( \tau_2 + 0 \) on \( R \). Because \( \tau_1 \) is Teichmüller equivalent to \( \tau_2 \) on \( R_1 \), there exists a conformal map \( c : f^{\tau_1}(R_1) \to f^{\tau_2}(R_1) \) such that \((f^{\tau_2})^{-1} \circ c \circ f^{\tau_1}\) is homotopic to \( \text{id}_{R_1} \) in the following sense: There exists a homotopy \((g_t : R_1 \to R_1)_{t \in [0,1]}\) which extend continuously to the border of \( R_1 \) such that \( g_0 = \text{id}_{R_1}, g_1 = (f^{\tau_2})^{-1} \circ c \circ f^{\tau_1} \) and \( g_t|_{\partial R_1} = \text{id} \).

We consider the following maps:

\[
\tilde{c}(p) := \begin{cases} f^{\tau_2+0} \circ (f^{\tau_2})^{-1} \circ c \circ f^{\tau_1} \circ (f^{\tau_1+0})^{-1}(p) & p \in f^{\tau_1+0}(R_1) \\ f^{0+\tau_2}(f^{\tau_1+0})^{-1}(p) & p \in f^{\tau_1+0}(R_2) \end{cases}, \quad \hat{g}_t(p) := \begin{cases} g_t(p) & p \in R_1 \\ \text{id} & p \in R_2 \end{cases}.
\]

Note that \( g_t|_{\partial R_1} = \text{id} \) implies that \( \hat{g}_t \) is self-continuous map of \( R \).

\( \tilde{c} \) is conformal and \((f^{\tau_2+0})^{-1} \circ \tilde{c} \circ f^{\tau_1+0} \) is homotopic to \( \text{id}_R \). Indeed, bel\((f^{\tau_2} \circ (f^{\tau_1+0})^{-1}) = 0 \), since support of \( \tau_1 \) is contained in \( R_1 \). Moreover \( g_t \) is the homotopy which join \( \text{id}_{R_1} \) and \((f^{\tau_2})^{-1} \circ c \circ f^{\tau_1} \), thus \( \hat{g}_t \) join \( \text{id}_R \) and \((f^{\tau_2+0})^{-1} \circ \tilde{c} \circ f^{\tau_1+0} \).

Next we prove that \( \tau_1 + \mu \) is Teichmüller equivalent to \( \tau_2 + \mu \) on \( R \).

\[
\begin{aligned}
&f^{\tau_1+0}(R) & \overset{\hat{c}}{\longrightarrow} & f^{\mu+\tau_1}(R), \\
&f^{0+\tau_2}(R) & \overset{\hat{h}_{\tau_2}}{\longrightarrow} & f^{\mu+\tau_2}(R), \\
&f^{\mu+\tau_1}(R) & \overset{\hat{h}_{\tau_1}}{\longrightarrow} & f^{\mu+\tau_1}(R), \\
&f^{\mu+\tau_2}(R) & \overset{\hat{h}_{\tau_2}}{\longrightarrow} & f^{\mu+\tau_2}(R),
\end{aligned}
\]

where \( h_{\tau_j} := f^{\mu+\tau_j} \circ (f^{\mu+0})^{-1} \) (\( j = 1, 2 \)) and

\[
\hat{c} := f^{\mu+\tau_2} \circ (f^{0+\tau_2})^{-1} \circ \tilde{c} \circ f^{\mu+\tau_1} \circ (f^{\mu+\tau_1})^{-1}.
\]
Similarly to the above proof, we can show that $\mu + \tau_1$ and $\mu + \tau_2$ are Teichmüller equivalence in $f^{\mu+0}(R)$. In detail, $\hat{g}_1 := (f^{\mu+0})^{-1} \circ g_1 \circ f^{\mu+0}$ is the homotopy which joins $\text{id}_{f^{\mu+0}(R)}$ and $(h^{\tau_2})^{-1} \circ \hat{c} \circ h^{\tau_1}$. Therefore $\mu + \tau_1$ is Teichmüller equivalent to $\mu + \tau_2$ on $R$. \hfill \square

**Remark 5**
Taniguchi [Tm] and Maitani [Ma] considered a subsurface $R'$ of the Riemann surface $R$ such that if two Beltrami coefficients on the subsurface $R'$ which are Teichmüller equivalent, the homotopy appearing in the Teichmüller equivalence on $R'$ can be extended to $R \setminus R'$ by the identity map. Using their results, Tanigawa proved that if $R_1$ is simply connected, the induced map is injective, see [Th, Lemma 3.3, Lemma 3.4] for detail.

### 4.3 Main results

Before starting the main results, let us prepare a lemma on the construction of the integrable holomorphic quadratic differentials, which is necessary for the proof of them.

#### Lemma 7.
There exists an integrable holomorphic quadratic differential on $R$ which has a pole of order 1 at a puncture in $R$.

**Proof:**
Set $R' := R \cup \{a\}$. Let $R'$ be represented by a Fuchsian group $\Gamma'$ whose a covering map $\pi : D \to R'$ satisfies $\pi(0) = a$. Note that $\psi(z) := 1/z$ is an integrable meromorphic function on $D$. Hence, $\Psi := \Theta(\psi)$ has properties of the claim.

Through the above discussion, we obtain the main theorem in this paper.

#### Theorem 8.
If $R$ has action of infinite cyclic group $\langle \gamma \rangle$, then there exists a Beltrami coefficient $\mu$ satisfies the following properties:

1. $\mu$ is extremal.
2. $|\mu|$ is constant.
3. There exists a family of Beltrami coefficients $\{\mu_\lambda\}_{\lambda \in \Lambda}$ such that
   
   (a) $\mu \in \{\mu_\lambda\}_{\lambda \in \Lambda}$,
   
   (b) all included Beltrami coefficients are Teichmüller equivalence,
   
   (c) $\Lambda \ni \lambda \mapsto \mu_\lambda \in \text{Bel}(R)$ is holomorphic map,
   
   (d) $\lambda_1 \neq \lambda_2$ implies $\mu_{\lambda_1}$ is not infinitesimal equivalent to $\mu_{\lambda_2}$, and
   
   (e) there exists a domain $U \subset R$ such that $\mu_\psi|_{R \setminus U} = \mu|_{R \setminus U}$ and $\text{Cl}(U) \cap \{a_n\} \neq \emptyset$.

In particular, $U$ is conformal equivalent to $D$, $D \setminus \{0\}$, or $A$.

**Proof:**
Compose $\tilde{\mu}$ in the same way as in Eg 1. Namely, let $\varphi^r$ and $\varphi^l$ from $Q(R/\langle \gamma \rangle)$ be linear indipendent, and let $X_\alpha := (a_j, b_j)$ be sequences in $\mathbb{R}^2_0 \setminus \{(0, 0)\}$ with $\lim_{n \to \infty} X_\alpha = (0, 1)$. Using these, we define $\psi$ to be $a_n \varphi^r + b_n \varphi^l$, and

$$\tilde{\mu}(z) := t_0 \left| \psi_n \right| \psi_n(z) \quad (z \in \omega_n),$$

where $t_0$ is a constant in $[0, 1)$, $\omega_j := \gamma^{3j}(\omega_0)$, and we consider $\varphi^r$ and $\varphi^l$ as elements in $Q(R/\langle \gamma \rangle)$, identifying $\omega_j$ with $R/\langle \gamma \rangle)$. 

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4.3 Main results

Theorem 9.
Since the only nontrivial condition is condition c, we show this.
constructed in Corollary ??.

\( D \) (II). The above observation completes the proof.

\( \nu \) satisfies the conditions. Consider the family of Beltrami coefficients \( \{ \mu_\lambda \}_{\lambda \in \Lambda} \) and the Beltrami coefficient

\( \{ \mu_\lambda \}_{\lambda \in \Lambda} \) is Teichmüller equivalent to \( \chi_{R \setminus U} \bar{\mu} \). Moreover, suppose that \( \lambda_1 \neq \lambda_2 \). Let \( \varphi \) be in \( Q(R) \) satisfies \( \varphi(h^{-1}(0)) \neq 0 \). Then we get

\[ \int_R (\mu_{\lambda_1} - \mu_{\lambda_2})\varphi = \int_U (\bar{\tau}_{\lambda_1} - \bar{\tau}_{\lambda_2})\varphi \neq 0. \]

Hence, a) and c) are proved.

(II). Configuration where \( U \) is conformal equivalent to \( D^+ \)

Let \( U \) be a topological punctured disk in \( R \) with the analytic boundary. We denote by \( h \) a conformal map from \( U \) to \( D^+ \) and \( \alpha \) a puncture contained by \( U \). Replace \{\( \tau \)\} in the above configuration (I) with \{\( \nu \)\} constructed in Corollary ?? applying \( r = 0 \) to form a family. That is, we consider \( \{ \mu_\lambda := \nu_\lambda + \chi_{R \setminus U} \bar{\mu} \}_{\lambda \in \Lambda} \). Since the only nontrivial condition is condition c, we show this.

Let \( \lambda_1 \neq \lambda_2 \). Since Lemma ?? there exists \( \varphi \in Q(R) \) which has a poke of order 1 at \( \alpha \). Therefore

\[ \int_R (\mu_{\lambda_1} - \mu_{\lambda_2})\varphi = \int_U (\bar{\nu}_{\lambda_1} - \bar{\nu}_{\lambda_2})\varphi \neq 0. \]

(III). Configuration where \( U \) is conformal equivalent to \( \Lambda \)

Use the same domain \( U \), the Riemann map \( h \), and the puncture \( \alpha \) as the above configuration (II). Next, let \( D \) be an annulus contained in \( U \) such that \( \partial D \) contains the puncture \( \alpha \). Replace \{\( \nu \)\} in the above configuration with \{\( \tau \)\} constructed in Corollary ?? to form a family. That is, we consider \( \{ \mu_\lambda := \nu_\lambda + \chi_{R \setminus U} \bar{\mu} \}_{\lambda \in \Lambda} \). The only nontrivial condition c) is shown in the same way as the above configuration (II). The above observation completes the proof.

From this theorem, the following assertions about geodesics are derived.

Theorem 9.
If \( R \) has \( \mathbb{Z} \) action, then there exists a Beltrami coefficient \( \mu \) that satisfies following properties:
1. \( \mu \) is extremal.
2. \( |\mu| \) is constant.
3. There exists a family of geodesics \( \{ g_\lambda \mid \lambda \in \Lambda' \} \), where \( \Lambda' \) is an open set in \( \Lambda \), through \([0]\) and \([\mu]\) with a complex analytic parameter. In particular, \( \lambda_1 \neq \lambda_2 \) implies that the geodesic \([0, 1] \ni t \mapsto [t\mu_{\lambda_1}] \) is not equal to \([0, 1] \ni t \mapsto [t\mu_{\lambda_2}] \).

Proof:
To prove the theorem, we only need to construct a family consisting of extreme Beltrami coefficients that satisfies the conditions. Consider the family of Beltrami coefficients \( \{ \mu_\lambda \}_{\lambda \in \Lambda} \) and the Beltrami coefficient
4.3 Main results

$\mu$ constructed in the previous theorem. Put

$$
\Lambda' := \left\{ \lambda \in \Lambda \left| \frac{-i\lambda_0/2}{1 + i\lambda_0/2} < t_0(= ||\mu||) \right. \right\}.
$$

Then, for all $\lambda \in \Lambda'$, $||\mu_\lambda||$ is not greater than $t_0$. Hence, $\mu_\lambda$ is extremal from Theorem [1]. Finally, we set $g_\lambda : [0, 1] \ni t \mapsto [t\mu_\lambda]$.

As an above theorem, we can prove the following claim:

**Corollary 10.**

We cannot exclude “unique extremality”, which is one of the necessary and sufficient conditions for the geodesics connecting $[0]$ and $[\mu]$ to be unique in [EKK, Theorem 6].

Since the technical difficulty is the composition of integrable holomorphic quadratic differentials which distinguish geodesics, more geodesics can be constructed if $R$ is a subdomain of the plane, using what is known about integrable holomorphic quadratic differentials in detail. For example, more geodesics can be constructed when considered to $\mathbb{C} \setminus \mathbb{Z}$.

**Theorem 11.**

Suppose $R = \mathbb{C} \setminus \mathbb{Z}$, then there exists a Beltrami coefficient $\mu$ satisfies following properties:

1. $\mu$ is extremal.
2. $|\mu|$ is constant.
3. There exists a family of geodesics $\{\lambda_1(\lambda_n) | \lambda_n \in \Omega\}$, where $\Omega$ is an open set in $l^\infty(\mathbb{C})$, through $[0]$ and $[\mu]$ with a complex analytic parameter. In particular, $\lambda_1 \neq \lambda_2$ implies that the geodesic $[0, 1] \ni t \mapsto [t\mu_{\lambda_1}]$ is not equal to $[0, 1] \ni t \mapsto [t\mu_{\lambda_2}]$.

**Proof:**

Set $S := R/(z \mapsto z + 3)$, and make $\mu_0$ one of the components of $\text{Eg1}$. Let $K > 0$ satisfies

$$
\left| \frac{-iK}{1 - iK^2} \right| = ||\mu_0||.
$$

For this $K$, let denote by $\Omega := \{(\lambda_j) \in l^\infty | \lambda_j \in \Lambda, |\lambda_j| \in [0, K)\}$.

First, for any $j \in \mathbb{N}$, let denote by

$$
A_j := \Delta \left(3j + \frac{3}{2} + \frac{1}{2j} + \frac{1}{2j} \right) \setminus \Delta \left(3j + \frac{3}{2} \right) = U := \bigcup_{j \in \mathbb{Z}} A_j.
$$

Moreover, denote $\alpha_j = 3j + 1, \beta_j := 3j + 2$, and $U \cup_{j \in \mathbb{Z}} A_j$.

Next, for each $(\lambda_n) \in \Omega$, we define

$$
\mu(\lambda_j) := \chi(\mathbb{C} \setminus \mathbb{Z}) \mu_0 + \sum_{j \in \mathbb{Z}} \nu_j \lambda_j,
$$

using $\nu_j$ is constructed in Corollary [4] applying $A_j$. Note that each satisfies the following equality:

$$
\int_{A_j} \nu_j \frac{1}{z - \alpha_j} = \frac{2\pi\alpha_j}{i} \text{Area} (A_j) \frac{A_j^2}{1 - A_j^2}, \quad \int_{A_j} \nu_j \frac{1}{z - \beta_j} = \frac{2\pi\beta_j}{i} \text{Area} (A_j) \frac{A_j^2}{1 - A_j^2},
$$

where $A_j := \frac{-i\lambda_j}{2}$. Moreover if $\varphi$ is a holomorphic on some simply connected domain containing the closure of $\Delta(3j + 3/2; 1)$, then $\int_{A_j} \nu_j \varphi = 0$.
We show that the constructed \( \{ \mu(\lambda_j) \} \) are all Teichmüller equivalences. Let \( \nu := \sum_{j \in \mathbb{Z}} \nu_{\lambda_j} \chi_{\lambda_j} \), then \( f^\nu \) is homotopic to \( \text{id}_{C \setminus \mathbb{Z}} \) relative \( \mathbb{Z} \), therefore \( f^{\chi(\mathbb{C} \setminus \mathbb{Z}) \cup \mu_0} \circ f^\nu \) is homotopic to \( f^{\chi(\mathbb{C} \setminus \mathbb{Z}) \cup \mu_0} \). Since the intersection of supports of \( \chi(\mathbb{C} \setminus \mathbb{Z}) \cup \mu_0 \) and \( \nu \) is empty, we get

\[
\text{bel}(f^{\chi(\mathbb{C} \setminus \mathbb{Z}) \cup \mu_0} \circ f^\nu) = \chi(\mathbb{C} \setminus \mathbb{Z}) \cup \mu_0 + \sum_{j \in \mathbb{Z}} \nu_{\lambda_j} \chi_{\lambda_j}.
\]

That is, \( \{ \mu(\lambda_j) \} \) is Teichmüller equivalent to \( \chi(\mathbb{C} \setminus \mathbb{Z}) \cup \mu_0 \).

Finally, we see that \( [0, 1] \ni t \mapsto [t \mu(\lambda_j)] \) gives different geodesics, using contra position. In detail, we prove that if \( [0, 1] \ni t \mapsto [t \mu(\lambda_j)] \) is equal to \( [0, 1] \ni t \mapsto [t \mu(\tilde{\lambda}_j)] \), then \( \lambda_j = \tilde{\lambda}_j \) for all \( j \in \mathbb{N} \). Given \( L > 0 \). Set

\[
\varphi_{\alpha_j} := \frac{1}{(z - \alpha_j)(z - 3L)(z - 6L)}, \quad \varphi_{\beta_j} := \frac{1}{(z - \beta_j)(z - 3L)(z - 6L)}.
\]

Let \((\gamma_j)_{|j| \leq L}\) be a finite string of length \( 2L + 1 \) consisting of \( \alpha \) and \( \beta \), and \( \varphi(\cdot) := \sum_{|j| \leq L} \varphi_j \). Since \( \mu(\lambda_j) \) and \( \mu(\tilde{\lambda}_j) \) give the same geodesic, applying the finite string \((\alpha, \alpha, \ldots, \alpha)\), we get

\[
0 = \int_{C \setminus \mathbb{Z}} (\mu(\gamma_j) - \mu(\tilde{\gamma}_j)) \varphi(\cdot)
= -\frac{2\pi}{L} \sum_{-L \leq j \leq L} \alpha_j \text{Area}(A_j) \left( \frac{A_j^2}{1 - A_j^2} - \frac{\tilde{A}_j^2}{1 - \tilde{A}_j^2} \right) + \sum_{|j| > L} \int_{A_j} (\mu(\gamma_j) - \mu(\tilde{\gamma}_j)) \varphi(\cdot).
\]

Besides, for any \(-L \leq J \leq L\), we replace the \( J \)-th of the string \((\alpha, \alpha, \ldots, \alpha)\) from \( \alpha \) to \( \beta \). By drawing the equation above against the equation obtained by applying the holomorphic function corresponding to the substituted string to the different geodesics, we obtain

\[
\frac{A_j^2}{1 - A_j^2} - \frac{\tilde{A}_j^2}{1 - \tilde{A}_j^2} = 0.
\]

Hence, \( \lambda_j = \tilde{\lambda}_j \).

References

[A] Lars. V. Ahlfors with appendix by C. J. Earle, I. Kra, M. Shishikura and J. H. Hubbard, Lectures on Quasiconformal Mappings, Second Edition American Mathematical Society, 2006.

[B] L. Bers, An extremal problem for quasiconformal mappings and a theorem by Thurston, Acta Mathematica, 141 (1978), 73–98.

[BLMM] V. Bozin, N. Lakic, V. Markovic, M. Mateljevic, Unique extremality, J. Anal. Math., 75 (1998), 299–338.

[EK] C. Earle, I. Kra and S. Krushkal, Holomorphic motions and Teichmüller spaces, Trans. Amer. Math. Soc., 343 (1994), 927–948.

[F] H. Fujino, The existence of quasiconformal homeomorphism between planes with countable marled points Kodai. Math., 38 (2015), no. 3, 732–746.

[FT] E. Fujikawa and M. Taniguchi, The Teichmüller space of a countable set of points on a Riemann surface, Conform. Geom. Dyn., 21 (2017), 64–77.

[G] Frederick. P. Gardiner, Teichmüller Theory and quadratic differentials, John Wiley and Sons, 1987.
REFERENCES

[GY] G. Yao, Y. Qi, On the modulus of extremal Beltrami coefficients, J. Math. Kyoto Univ., 46 (2006) 235–247.

[H] J. H. Hubbard, Teichmüller Theory and Applications to Geometry, Topology, and Dynamics, Vol. 1. Matrix Editions, Ithaca, NY, 2006.

[IT] Y. Imayoshi and M. Taniguchi, An introduction to Teichmüller spaces, Springer-Verlag, Tokyo, 1992.

[K] I. Kra, I, On Nielsen–Thurston–Bers type of self-maps of Riemann surfaces, Acta Math., 146 (1981), 231–270.

[L] Li Zhong, On the existence of extremal Teichmüller mappings, Comment. Math. Helv., 57 (1982), 511–517.

[L1] Li Zhong, Nonuniqueness of geodesics in infinite dimensional Teichmüller spaces(I), Complex Variables Theory Appl., 16 (1991), 261–272.

[L2] Li Zhong, Nonuniqueness of geodesics in infinite dimensional Teichmüller spaces(II), Ann. Acad. Sci. Fenn. Ser. Math., 18 (1993), 355–367.

[Mc] C. McMullen Amenability, Poincaré series and quasiconformal maps, Invent. Math., 97 (1989), 95–127.

[Ma] F. Maitani, On the rigidity of an end under conformal mappings preserving the infinite homology bases, Complex Variables Theory Appl., 24 (1994), no. 3–4, 281–287.

[O] H. Ohtake Lifts of extremal quasiconformal mappings of arbitrary Riemann surfaces, J. Math. Kyoto Univ., 22 (1982), 191–200.

[S] K. Strebel, On the existence of extremal Teichmüller mappings, J. Anal. Math., 30 (1976), 441–447.

[Th] H. Tanigawa, Holomorphic families of geodesic discs in infinite dimensional Teichmüller spaces, Nagoya Math. J., 127 (1992), 117–128.

[Tm] M. Taniguchi, On the rigidity of an infinite Riemann surface, Complex Variables Theory Appl., 14(1990), 161–167.

[YY] Yun Hu, Yuliang Shen, Some notes on geodesic segments in infinite dimensional Teichmüller spaces, J. Math. Anal. Appl., 424(2015) 237–247.

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