Computing the Bergsma Dassios sign-covariance

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Abstract

Bergsma and Dassios (2014) introduced an independence measure which is zero if and only if two random variables are independent. This measure can be naively calculated in $O(n^4)$. Weihs et al. (2015) showed that it can be calculated in $O(n^2 \log n)$. In this note we will show that using the methods described in Heller et al. (2016), the measure can easily be calculated in only $O(n^2)$.

1 Introduction

Testing whether two ordinal random variables are independent given a sample $(x_i, y_i)_{i=1}^n$ is a classic problem in statistics. Early efforts such as Pearson’s correlation and Kendall’s $\tau$ focused on testing against linear or monotone relationships. The first test for any type of independence was provided by Hoeffding (1948). This test is based on multiple partitions of the $(X,Y)$ plane into four quadrants where the number of points in a quadrant is compared to what it would be under independence. This score can be calculated in $O(n \log n)$ by counting the inversions of the permutation from the ranks of $(x_i)_{i=1}^n$ to $(y_i)_{i=1}^n$ as described in Heller et al. (2013). Additional rank based scores were suggested by others, typically based on finer partitions of the plane (see Heller et al. (2016) for review, algorithms and a powerful method that takes into account all possible partitions).

Bergsma and Dassios (2014) present another measure of independence which in the case of no ties is, also based on partitions of the plane into four quadrants. This
method is in some sense a generalization of Kendall’s $\tau$. Using the notation of [Weihs et al. (2015)](in their equation 1) the statistic is defined as:

$$t^* := \frac{(n - 4)!}{n!} \sum_{1 \leq i,j,k,l \leq n} a(x_i, x_j, x_k, x_l)a(y_i, y_j, y_k, y_l)$$

(1.1)

where

$$a(z_1, z_2, z_3, z_4) := \text{sign}(\mid z_1 - z_2 \mid + \mid z_3 - z_4 \mid - \mid z_1 - z_3 \mid - \mid z_2 - z_4 \mid).$$

This definition clearly shows that the statistic can be naively calculated in $O(n^4)$ since one can simply go over all quadruples of points. However, [Weihs et al. (2015)] show that the statistic can be calculated in $O(n^2 \log n)$ using red black trees. In this note we will show that the statistic can in fact be calculated in only $O(n^2)$ using methods described in [Heller et al. (2016)].

2 The algorithms

As [Bergsma and Dassios (2014)] show, the score is actually based on the number of concordant quadruples vs. the number of discordant quadruples. In a manner analogous to Kendall’s $\tau$ concordance and discordance are defined for a quadruple $(x_i, y_i)_{i=1}^4$ as follows:

**Definition 2.1.** A quadruple is called **concordant** if either $(\max(x_1, x_2) < \min(x_3, x_4)$ and $\max(y_1, y_2) < \min(y_3, y_4))$ or $(\max(x_1, x_2) < \min(x_3, x_4)$ and $\min(y_1, y_2) > \max(y_3, y_4))$

**Definition 2.2.** A quadruple is called **discordant** if $\max(x_1, x_2) < \min(x_3, x_4)$ and $\max(y_1, y_2) > \min(y_3, y_4)$ and $\min(y_1, y_2) < \max(y_3, y_4)$

We start with the simple case without ties.

2.1 The algorithm without ties

It is easy to see that in this case the statistic reduces to (equation 4 in [Weihs et al. (2015)])

$$t^* = \frac{(n - 4)!}{n!} (24 \cdot N_c) - \frac{1}{3},$$

where $N_c$ is the number of concordant quadruples. Therefore all we need to do is calculate $N_c$. Again as in [Weihs et al. (2015)] et. al clearly:
\[ N_c = \sum_{3 \leq k \leq n-1} \sum_{k < l \leq n} \left( \frac{M_<(k, l)}{2} \right) + \left( \frac{M_>(k, l)}{2} \right) \]

where they define
\[ M_<(k, l) := |\{i : x_i < \min(x_k, x_l), y_i < \min(y_k, y_l)\}|, \]
\[ M_>(k, l) := |\{i : x_i < \min(x_k, x_l), y_i > \max(y_k, y_l)\}|. \]

However contrary to their algorithm we show that \( M_<(k, l) \) and \( M_>(k, l) \) can be calculated in \( O(1) \) and not in \( O(\log n) \) after a preprocessing step which takes \( O(n^2) \). This can be done by the methods described in [Heller et al. (2016)] as follows: We first note that the statistic is only based on ranks so we transform every pair \((x_i, y_i)\) to its respective ranks \((r_i, s_i)\) where \( r_i, s_i \in \{1, \ldots, n\} \), this can of course be done in \( O(n \log n) \).

We can now calculate the empirical cumulative distribution
\[ A(r, s) = \sum_{i=1}^{n} I(r_i \leq r \text{ and } s_i \leq s), \quad (r, s) \in \{0, 1, \ldots, n\}^2 \quad (2.1) \]

(where \( A(0, 0) = 0, A(r, 0) = 0 \) in \( O(n^2) \) time and space:

First, let \( B \) be the \((n + 1) \times (n + 1)\) zero matrix, and initialize to one \( B(r_i, s_i) \) for each observation \( i = 1, \ldots, n \). Next, go over the grid in \( s \)-major order, i.e., for every \( s \) go over all values of \( r \), and compute:

1. \( A(r, s) = B(r, s - 1) + B(r - 1, s) - B(r - 1, s - 1) + B(r, s) \)
2. \( B(r, s) = A(r, s) \).

It is easy to see that \( M_<(k, l) = A[\text{rank}(\min(x_k, x_l)) - 1, \text{rank}(\min(y_k, y_l)) - 1] \) and similarly that \( M_>(k, l) = \text{rank}(\min(x_k, x_l)) - A[\text{rank}(\min(x_k, x_l)), \text{rank}(\max(y_k, y_l))] \) and therefore for each \( k, l \) \( M_<(k, l) \) and \( M_>(k, l) \) can be calculated in \( O(1) \), resulting in a total of \( O(n^2) \) as desired.

### 2.2 The algorithm for data with ties

First, for ease of notation we order the samples such that \( x_1 \leq x_2 \ldots \leq x_n \). By Lemma 1 in [Weihs et al. (2015)] in this case the score reduces to
\[ t^* = \frac{(n - 4)!}{n!} (16 \cdot N_c - 8 \cdot N_d) \]
Therefore, greater care must be taken in this case as it requires calculating also \( N_d \) (where \( N_d \) is the number of discordant quadruples), which with ties is a little more subtle since a quadruple can be neither concordant nor discordant. We will use the following ranking scheme - \( n \) observations with \( m \) unique values will be transformed to \( n \) ranks in the range 1...\( m \) (so for example 2, 2, 3.5, 4, 4, 4 will be ranked as 1, 1, 2, 3, 3, 3). We first note that calculating the number of discordant pairs can be done in the same way as in the section above without ties, except that when we calculate the empirical cumulative distribution in the previous section. Our first goal will be to calculate in the same way as in the section above without ties, except that when we calculate the empirical cumulative distribution \( B(r, s) \) will be initialized to the number of observations with ranks \((r, s)\), which can be greater than one. We now turn to computing \( N_d \). Define \( N_d(k, l) = \{|i < j < k : i, j, k, l \text{ are discordant}\}| \). Clearly \( N_d = \sum_{3 \leq k \leq n-1} \sum_{k < l \leq n} N_d(k, l) \). Following [Weihls et al. (2015)] for any pair of samples \((x_k, y_k)\) and \((x_l, y_l)\) such that \( k < l \) we define:

\[
\begin{align*}
\text{top}(k, l) &= |\{i : x_i < x_k \text{ and } y_i > \max(y_k, y_l)\}|, \\
\text{mid}(k, l) &= |\{i : x_i < x_k \text{ and } \min(y_k, y_l) < y_i < \max(y_k, y_l)\}|, \\
\text{bot}(k, l) &= |\{i : x_i < x_k \text{ and } y_i < \min(y_k, y_l)\}|, \\
\text{eqMin}(k, l) &= |\{i : x_i < x_k \text{ and } y_i = \min(y_k, y_l)\}|, \\
\text{eqMax}(k, l) &= |\{i : x_i < x_k \text{ and } y_i = \max(y_k, y_l)\}|.
\end{align*}
\]

Quoting equations 11 and 12 in [Weihls et al. (2015)] if \( y_k = y_l \) then

\[ N_d(k, l) = 0, \]

and if \( y_k \neq y_l \) then

\[
N_d(k, l) = \text{top}(k, l) \cdot (\text{mid}(k, l) + \text{eqMin}(k, l) + \text{bot}(k, l)) \\
+ \text{bot}(k, l) \cdot (\text{mid}(k, l) + \text{eqMax}(k, l)) \\
+ \text{eqMin}(k, l) \cdot (\text{mid}(k, l) + \text{eqMax}(k, l)) \\
+ \text{eqMax}(k, l) \cdot \text{mid}(k, l) \\
+ \binom{\text{mid}(k, l)}{2} - \sum_{y \in \text{unique}(k, l)} \left( \binom{|\{1 \leq i < k : x_k \neq x_i \text{ and } y_i = y\}|}{2} \right) \tag{2.2}
\]

where

\[
\text{unique}(k, l) := \{y_i : 1 \leq i < k \text{ and } x_i \neq x_k \text{ and } \min(y_k, y_l) < y_i < \max(y_k, y_l)\}.
\]

Clearly, \( \text{top}(k, l) \), \( \text{mid}(k, l) \), \( \text{bot}(k, l) \), \( \text{eqMin}(k, l) \), \( \text{eqMax}(k, l) \) can be calculated using the empirical cumulative distribution in \( O(1) \) as described in the previous section (e.g. \( \text{mid}(k, l) = A[\text{rank}(x_k) - 1, \text{rank}(\max(y_k, y_l)) - 1] - A[\text{rank}(x_k) - 1, \text{rank}(\min(y_k, y_l))] \) and \( \text{eqMin}(k, l) = A[\text{rank}(x_k) - 1, \text{rank}(\min(y_k, y_l))] - A[\text{rank}(x_k) - 1, \text{rank}(\min(y_k, y_l)) - 1] \).

We will now show how to calculate the last element in equation \[2.2\]. This will be done with a procedure similar to the one used to calculate the empirical cumulative distribution in the previous section. Our first goal will be to calculate in \( O(n^2) \).
\[ A(r, s) = \sum_{y \text{ s.t. } \text{rank}(y) \leq s} \left( |\{i : \text{rank}(x_i) < r \text{ and } y_i = y\}| \right) \]

We initialize \( A(0, s) = A(r, 0) = 0 \). We further set \( B(r, s) = |\{i : \text{rank}(x_i) = r \text{ and } \text{rank}(y_i) = s\}| \). We now compute the cumulative row sum \( R(r, s) = R(r - 1, s) + B(r, s) \) and then we compute row by row \( A(r, s) = A(r, s - 1) + \left( \frac{R(r, s)}{2} \right) \). Once we have \( A(r, s) \) we can easily calculate the last element in 2.2 in \( O(1) \).

\[
\sum_{y \in \text{unique}(k,l)} \left( \left( |\{1 \leq i < k : x_k \neq x_i \text{ and } y_i = y\}| \right) \right) = A[\text{rank}(x_k) - 1, \text{rank}(\max(y_k, y_l)) - 1] - A[\text{rank}(x_k) - 1, \text{rank}(\min(y_k, y_l))] 
\]

Thus completing the computation in \( O(n^2) \) as required.

3 Conclusion

We have shown how to calculate the Bergsma Dassios association measure in \( O(n^2) \). However, the question of the power of this method remains open. It would be interesting to compare its power to the power of methods based on finer partitions as in [Heller et al., 2016].

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