The Lie–Poisson structure of the reduced \( n \)-body problem

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Abstract

The classical \( n \)-body problem in \( d \)-dimensional space is invariant under the Galilean symmetry group. We reduce by this symmetry group using the method of polynomial invariants. One novelty of our approach is that we do not fix the centre of mass but rather use a momentum shifting trick to change the kinetic part of the Hamiltonian to arrive at a new, dynamically equivalent Hamiltonian which is easier to reduce. As a result we obtain a reduced system with a Lie–Poisson structure which is isomorphic to \( \mathfrak{sp}(2n-2) \), independently of \( d \). The reduction preserves the natural form of the Hamiltonian as a sum of kinetic energy that depends on velocities only and a potential that depends on positions only. This splitting allows us to construct a Poisson integrator for the reduced \( n \)-body problem which is efficient away from collisions for \( n = 3 \). In particular, we could integrate the figure eight orbit in 18 time steps.

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(Some figures may appear in colour only in the online journal)

1. Introduction

The \( n \)-body Hamiltonian in \( \mathbb{R}^d \) is

\[
H = \sum_{i=1}^{n} \frac{||p_i||^2}{2m_i} + \sum_{1 \leq i < j \leq n} V_{ij}(||q_i - q_j||^2),
\]

(1)

with position vectors \( q_i \in \mathbb{R}^d \) and conjugate momentum vectors \( p_i \in \mathbb{R}^d, i = 1, \ldots, n \). All of the following is valid for a more general potential \( V \) that is a function of pair-wise distances \( ||q_i - q_j||^2 \) only. In the gravitational case the pair-wise potential function is
The equations of motion are
\[ \dot{q}_i = \frac{1}{m_i} p_i, \quad \dot{p}_i = -2 \sum_{j \neq i} (q_i - q_j) V_{ij}(q_i - q_j)^2. \] (2)

The Hamiltonian (1) is invariant under translations in space \( T(q_i, p_i) = (q_i + q_0, p_i) \), \( i = 1, \ldots, n \) with \( q_0 \in \mathbb{R}^d \) and rotations \( R(q_i, p_i) = (Rq_i, Rp_i), i = 1, \ldots, n \) with \( R \in O(d) \). In addition Galilean boosts \( B(q_i, p_i) = (q_i + v_0 t, p_i + m_i v_0) \), \( v_0 \in \mathbb{R}^d \) leave the equations of motion (2) invariant but not the Hamiltonian (1). We denote the total symmetry group generated by \( T, B \) and \( R \) by \( G(d) \), which is the Galilean group without the time translation generated by \( H \), acting diagonally on \((\mathbb{R}^d)^n\).

We are going to follow Lagrange’s footsteps [11], as was recently done by Albouy and Chenciner [1, 2, 6], by introducing scalar products of difference vectors as new coordinates. The fact that the Poisson structure we are going to derive exists is mentioned in [2], but no further analysis of it is done. Here we are going to derive explicit expressions for the Poisson structure in a particularly suitable basis and then use the Poisson structure to construct a geometric integrator for the reduced three-body problem.

Reduction using invariants in the case of two bodies is well known, see e.g. [8]. An approach similar to ours has also been taken in [3], but there mainly the case of vanishing angular momentum for three bodies is considered. The Poisson reduction of rotational \( O(d) \) symmetry has been discussed for arbitrary \( n \) in [13] from the point of view of singular reduction. In many ways our analysis is similar to [13]. The main difference is that they only reduced the \( O(d) \) symmetry, but not the larger \( G(d) \) symmetry. After this work was completed I became aware of two related papers. The paper [20] identifies the Lie–Poisson algebra as \( \mathfrak{sp}(2n - 2) \), but uses a different less symmetrical set of heliocentric invariants. The paper [5] deals with the description of collective motion of many body (quantum) Hamiltonians in nuclear physics.

The general idea of Poisson reduction using invariants is as follows. Say we have a symmetry given as a group action \( \phi^g \) of the group \( G \) on a Poisson manifold \( M \). Then by definition an invariant \( f : M \mapsto \mathbb{R} \) satisfies \( f \circ \phi^g = f \) for all \( g \in G \). If for every \( g \) the mapping \( \phi^g \) preserves the Poisson bracket, then in addition we have that for any two invariants \( f \) and \( g \) their Poisson bracket \( \{ f, g \} = \{ f \circ \phi^g, g \circ \phi^g \} = \{ f, g \} \circ \phi^g \) is also invariant under \( \phi^g \). Assume that the set of invariants is closed under the Poisson bracket. If in addition the bracket is linear in the invariants, the so called Lie–Poisson case, then the Jacobi identity is inherited from the Lie-group structure of the brackets of the invariants. Thus a reduced Poisson structure is obtained.

Poisson reduction using invariants differs significantly from symplectic reduction, as initiated by Marsden and Weinstein [16], specifically for the \( n \)-body problem see, e.g. [14]. To my knowledge, no matter what coordinates are used, the fully symplectically reduced Hamiltonians have a kinetic energy that depends on the positions. Using invariants for the reduction preserves this special structure of the Hamiltonian and thus enables us to construct a numerical integration schemes that preserve the Poisson structure using a splitting method [10, 12, 17].

The plan of the paper is as follows. In section 2 we do the centre of mass reduction to prepare the system for reduction. In section 3 we introduce the invariant polynomials and express the Hamiltonian in terms of the invariants. The main part of the paper is the description of the algebra of invariant quadratic forms in section 4. The main theorem 4 shows that the set of \( 2 \times 2 \) block-Laplacian matrices (i.e. matrices with row sum 0) corresponds to the quadratic invariants of \( G(d) \) and that the algebra is isomorphic so \( \mathfrak{sp}(2n - 2) \). Explicit details of the reduced Poisson structure are given in section 5 in particular for \( n = 3 \). In section 6 we use
the reduced Poisson structure to construct a Poisson structure preserving integrator, and in the final section we illustrate the properties of this integrator by computing an ultra-discretization of the figure eight choreography with three equal bodies in only 18 steps.

2. Centre of mass decomposition

Usually reduction by translation symmetry is achieved by introducing the centre of mass as a coordinate. Here, instead we proceed by decomposing the Hamiltonian and angular momentum so that they become invariant under boosts and translations. This preparatory step which shifts positions and momenta is crucial when using invariants for reduction.

The $G(d)$ group action is given after equation (2). Invariance of the Hamiltonian $H$ under translations $T$ leads to the conservation of linear momentum $P = \sum p_i$. Invariance of the Hamiltonian under rotations $R$ leads to the conservation of angular momentum with respect to the origin $L = \sum q_i \wedge p_i$. In general the two-form $L$ may be viewed as an antisymmetric matrix, but for the special case $d = 3$ it can also be identified with a vector in $\mathbb{R}^3$. The constant of motion $MC - tP$ generates the boosts $B$, where $M = \sum m_i$ is the total mass and $C = (\sum m_i q_i)/M$ is the centre of mass. The behaviour of all the integrals under the Galilean group is well known, see, e.g. [22].

Note that the Hamiltonian $H$ is not invariant under boosts $B$. Clearly the potential energy $V$ is invariant under the full symmetry group, hence the kinetic energy $K$ is not invariant under boosts. To remedy this we split the kinetic energy into the kinetic energy of the motion $V$ is invariant under the full symmetry group, hence the kinetic energy $K_c$ of the centre of mass and the remainder denoted by $K_r$. To remedy this we split the kinetic energy into the kinetic energy of the motion $V$ is invariant under the full symmetry group, hence the kinetic energy $K_c$ of the centre of mass and the remainder denoted by $K_r$. In general the two-form $L$ may be viewed as an antisymmetric matrix, but for the special case $d = 3$ it can also be identified with a vector in $\mathbb{R}^3$. The constant of motion $MC - tP$ generates the boosts $B$, where $M = \sum m_i$ is the total mass and $C = (\sum m_i q_i)/M$ is the centre of mass. The behaviour of all the integrals under the Galilean group is well known, see, e.g. [22].

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$$K_c = \frac{1}{2} \sum_{i=1}^{n} \frac{1}{m_i} ||p_i - \frac{m_i}{M} P||^2 = \frac{1}{2} \sum_{i=1}^{n} \frac{1}{m_i} ||p_i||^2 - \frac{1}{2M} ||P||^2$$

so that $H_c = K_c + V = H - \frac{1}{2M} ||P||^2$ is invariant under translations and boosts.

Note that the angular momentum $L$ is not equivariant under all of $G(d)$. As in the construction of $K_c$ we replace $q_i$ by $q_i - C$ and $p_i$ by $p_i - \frac{m_i}{M} P$ which are invariant under $T$ and $B$. Thus define the $T$ and $B$ invariant angular momentum about the centre of mass

$$L_c = \sum (q_i - C) \wedge \left( p_i - \frac{m_i}{M} P \right) = \sum q_i \wedge p_i - C \wedge P$$

so that $L_c = L - C \wedge P$. Since $L_c$ is equivariant under $R$ and invariant under $T$ and $B$, its length squared $||L_c||^2$ is invariant under the full $G(d)$ action. What we have done here is the well-known derivation of the $G(d)$ invariants $H_c$ and $||L_c||^2$, see e.g. [22]. This prepares the system for reduction using quadratic polynomial invariants.

3. Reduction using polynomial invariants

The Hilbert–Weyl theorem (see, e.g. [9]) guarantees the existence of a so called Hilbert basis, i.e. a finite set of polynomials that generate the ring of invariant polynomials for a compact group acting linearly on a vector space. By a theorem of Schwarz [21] even every smooth invariant function can be expressed as a smooth function of the basic polynomial invariants. E.g. the invariant functions $K_c$, $V$ and $||L_c||$ can all be written in terms of the basic polynomial
invariants. The symmetry group \( G(d) \) is not compact, and the \( G(d) \) action is affine instead of linear, so the theorem does not apply in our case. Nevertheless, we will see that the invariants of the \( G(d) \) action are quadratic functions in the original variables and that they are sufficient for reduction because their Poisson bracket is closed, i.e. every bracket of invariants can again be expressed in terms of invariants.

The invariants are introduced in two simple steps. First form difference vectors

\[
q_{ij} = q_i - q_j \quad \text{and} \quad v_{ij} = \dot{q}_i - \dot{q}_j = p_i/m_i - p_j/m_j
\]

which are invariant under translations and boosts. Notice that momentum differences are not invariant under boosts unless all masses are equal. Second take scalar product of these difference vectors that are invariant under rotations as well. Hence scalar products of these difference vectors are invariant under \( G(d) \). Since difference vectors are linear in the original coordinates, these invariants are quadratic in the original coordinates.

There are \( n(n-1)/2 \) non-zero difference vectors between \( q_i \), similarly for \( v_j \), but only \( n-1 \) vectors of each group are independent. A possible choice of basis difference vectors are the \( 2n-2 \) vectors \( q_{ij}, v_{ij}, j = 2, \ldots, n \), similarly for any other fixed first (or second) index. The elements of this vector space are invariant under translations and boosts.

Now we have reduced the non-compact part of \( G(d) \) by introducing difference vectors. In the next step we introduce \( G(d) \) invariants based on these differences vectors that are \( O(d) \) invariant: any scalar product between two vectors from the space of difference vectors is invariant under \( O(d) \), and hence invariant under the full symmetry group. The basis of the space of difference vectors has dimension \( 2n-2 \), and forming all pairs there are \((2n-2)(2n-1)/2\) fully invariant scalar products.

All the scalar products between the basic difference vectors can be conveniently combined in a Gram matrix. Instead of taking the entries of the Gram matrix as invariants we will choose certain linear combinations that are more natural because they appear in the reduced Hamiltonian. In particular the potential depends on the \( n(n-1)/2 \) mutual distances \( \rho_{ij} = ||q_{ij}||^2, i < j \leq n \),

\[
V = \sum_{1 \leq i < j \leq n} V_{ij}(\rho_{ij}) = -G \sum_{1 \leq i < j \leq n} \frac{m_i m_j}{\sqrt{\rho_{ij}}},
\]

where the second equality holds for the gravitational \( n \)-body problem. The full kinetic energy of the Hamiltonian (1) cannot be written in terms of the invariants because it is not invariant under boosts. However, the kinetic energy relative to the centre of mass \( K_c \) is \( G(d) \) invariant and can thus be rewritten in terms of the \( n(n-1)/2 \) relative speeds \( v_{ij} = ||v_{ij}||^2, i < j \leq n \) as

\[
K_c = \frac{1}{2M} \left( M \sum_{1 \leq i < j \leq n} m_i ||q_{ij}||^2 - \left( \sum_{1 \leq i < j \leq n} m_i q_{ij} \right)^2 \right) = \frac{1}{2M} \sum_{1 \leq i < j \leq n} m_i m_j v_{ij}.
\]

The equivariant angular momentum relative to the centre of mass can be written in terms of difference vectors using the identity \( M(q_i - C) = \sum_j m_j q_{ij} \) and its time derivative, so that

\[
L_c = \frac{1}{M} \sum_{i,j,k} m_i m_j m_k q_{ij} \wedge v_{ik}.
\]

Using the Lagrange identity \((a \wedge b) \cdot (c \wedge d) = (a \cdot c)(b \cdot d) - (a \cdot d)(b \cdot c)\) the invariant \( ||L_c||^2 \) can then be written in terms of invariants\(^1\). The resulting expression depends not only on \( \rho_{ij} \) and \( v_{ij} \), but also on other scalar products of difference vectors. Exactly which scalar products

\(^1\) For \( d > 3 \) where \( L \) is an antisymmetric rank 2 matrix \( (a \wedge b) = ab' - ba' \) the scalar product between two such matrices \( A \) and \( B \) is given by \( A \cdot B = tr(AB')/2 \), and the Lagrange identity holds.
to choose as a basis of the vector space of quadratic invariants we leave open until section 5. Until then we work in a basis-independent formulation.

**Block form of the total Poisson structure.** The canonical variables \( q_i, p_i \) satisfy the Poisson bracket \([ (q_i)_k, (p_j)_l ] = \delta_{ij} \delta_{kl}, i, j = 1, \ldots, n, \) and \( k, l = 1, \ldots, d. \) A second index outside parenthesis denotes the component of a vector, not to be confused with the double index without parenthesis \( q_j \) for the difference between \( q_i \) and \( q_j.\)

Since the velocities are more important than the momenta in our construction it is useful to pass to a new non-standard symplectic structure in which the momenta are replaced by velocities as variables. This is a non-canonical transformation that changes the symplectic structure. It introduces scalar factors originating from \( k_i \) to \( \nu_i, \) and \( v_k \) to \( \dot{v}_k.\) Thus the usual identity block-matrices in the standard symplectic structure are replaced by diagonal matrices with entries \( 1/m_j.\)

To verify that the equations of motion separate into the centre of mass motion and the non-trivial part described by invariants we need to show that the Poisson bracket between any quadratic invariant and the centre of mass and its derivative (the linear momentum) vanishes. Clearly \([ q_{ij} \cdot v_{ki}, C ] = 0 \) and \([ v_{ij} \cdot v_{kl}, P ] = 0.\) Now verify that \([ q_{ij} \cdot v_{ki}, C ] = \delta_{ik} q_{ij} \cdot v_{kl}/M + \delta_{kl} q_{ij} \cdot v_{ki}/M = 0, \) similarly \([ q_{ij} \cdot v_{ki}, P ] = 0.\) Finally \([ q_{ij} \cdot \nu_{kl}, P ] = \sum_{m=i, j, k, l} \delta_{mn} q_{ij} \cdot \nu_{kl} = 0, \) and similarly \([ v_{ij} \cdot \nu_{kl}, C ] = 0.\)

As a result the equations of motion generated by \( H = H_c + \frac{1}{2\mu} ||P||^2 \) reproduce the equations \( C = [ H, C ] = P/M \) and \( P = [ H, P ] = 0.\) More importantly, the equations of motion for any quadratic invariant \( I \) is determined by \( H_c, \) only, since \( I = [ H, I ] = [ H_c, I ].\) Thus from now on \( H_c \) is our Hamiltonian.

**Two bodies.** Before we proceed to the general case we briefly treat \( n = 2 \) where everything can be done by direct computation. As a basis for the three invariants we choose \( \rho_{12} = ||q_{12}||^2, \) \( v_{12} = ||v_{12}||^2, \) and \( \sigma_{12} = q_{12} \cdot v_{12}.\) The relative kinetic energy is \( K_r = \mu v_{12}^2, \) where the reduced mass \( \mu \) is given by \( 1/\mu = 1/m_1 + 1/m_2.\) The relative momentum is \( ||L||^2 = \mu^2 (\rho_{12} v_{12} - \sigma_{12}^2).\) The reduced relative Hamiltonian is \( H_r = \frac{1}{2} \mu v_{12}^2 = \frac{G M}{\sqrt{\rho_{12}}} \) By direct computation one finds the closed bracket

\[ \{ \rho_{12}, v_{12} \} = \frac{4}{\mu} \sigma_{12}, \quad \{ \rho_{12}, \sigma_{12} \} = \frac{2}{\mu} \rho_{12}, \quad \{ v_{12}, \sigma_{12} \} = -\frac{2}{\mu} v_{12}. \]

Notice that the bracket is independent of the spatial dimension \( d.\) Accordingly the structure matrix \( B \) of the new Poisson bracket is

\[ B = \frac{2}{\mu} \begin{pmatrix} 0 & 2\sigma_{12} & \rho_{12} \\ -2\sigma_{12} & 0 & -v_{12} \\ -\rho_{12} & v_{12} & 0 \end{pmatrix}. \]

This is the Lie–Poisson bracket of \( \mathfrak{sp}(2), \) see e.g. [15]. As expected \( ||L||^2 \) is a Casimir function of this Poisson bracket, i.e. it has vanishing bracket with any function.

**4. Algebra of invariant quadratic forms**

We now derive the reduced Poisson bracket between the invariants for general \( n \) without choosing a particular basis in the vector space of quadratic invariants. Denote by \( Z \) the vector of \( 2m \) variables corresponding to (possibly non-standard) symplectic \( 2m \times 2m \) matrix \( J.\) The Poisson bracket is defined by \( \{ f, g \} = (\nabla_Z f, J \nabla_Z g) \) where \( (, ) \) is the Euclidean standard scalar product. Each invariant is a quadratic form. Denote an arbitrary quadratic form by
The Poisson bracket of quadratic forms induces an algebra of symmetric 2m × 2m matrices defined by
\[ \{Q_A, Q_B\} = (\nabla Z Q_A, J \nabla Z Q_B) = (AZ, J BZ) = (Z, A J B Z) = \frac{1}{2} (Z, (A J B - B J A) Z) \]
where in the last step we symmetrized the matrix A J B . Thus the multiplication in the induced algebra of symmetric matrices of even dimension is defined by
\[ A \ast B = A J B - B J A = 2 [A J B]_{\text{sym}}. \tag{5} \]
where \([U]_{\text{sym}} = \frac{1}{2} (U + U^t)\) gives the symmetric part of a matrix. Obviously we have
\[ A \ast B = - B \ast A, \]
so it is in fact a Lie algebra. This leads to the well-known lemma, see, e.g. [13]:

**Lemma 1.** The symplectic Poisson bracket on \( \mathbb{R}^{2m} \) induces an algebra on quadratic forms that is isomorphic to \( \mathfrak{sp}(2m) \).

The simple observation is that by defining \( A = \hat{A} J^{-1} \) the algebra multiplication \( A \ast B \) is turned into the commutator \([\hat{A}, \hat{B}]\) of Hamiltonian matrices. We are going to show that the set of quadratic invariants of the \( G(d) \) symmetry of the \( n \)-body problem is a sub-algebra of the algebra of quadratic forms.

Denote the phase space variables in \( \mathbb{R}^{2nd} \) by \( Z = (q_1, \ldots, q_n, v_1, \ldots, v_n) \) (where \( q_i \) represents the \( d \) components of the vector \( q_i \), similarly for \( v_j \)) with the modified Poisson structure \( \{(q_i)_l, (v_j)_l\} = \delta_{ij} \delta_{kl} / m_j \). The corresponding symplectic matrix is denoted by
\[ J_{nd} = \begin{pmatrix} 0 & M_{nd} \\ -M_{nd} & 0 \end{pmatrix} \tag{6} \]
with diagonal matrix \( M_{nd} \) containing the inverse masses. Now write two symmetric matrices \( A \) and \( B \) in block form as
\[ A = \begin{pmatrix} R_v & W_v \\ W_v & P_v \end{pmatrix}, \quad B = \begin{pmatrix} R_b & W_b \\ W_b & P_b \end{pmatrix}, \]
with symmetric blocks \( R \) and \( P \), and off-diagonal block \( W \) each of size \( nd \times nd \). Then the algebra multiplication of \( A \) and \( B \) induced by the Poisson bracket as given in (5) with symplectic matrix (6) can be explicitly written as
\[ A \ast B = \begin{pmatrix} 2[W_v M_{nd} R_b - W_b M_{nd} R_v]_{\text{sym}} & [W_v, W_b]_M + R_b M_{nd} P_a - R_a M_{nd} P_b \\ [W_v, W_b]_M + P_a M_{nd} R_b - P_b M_{nd} R_a & 2[P_a M_{nd} W_b - P_b M_{nd} W_a]_{\text{sym}} \end{pmatrix} \]
\[ \begin{pmatrix} 2 [W_v M_{nd} R_b - W_b M_{nd} R_v]_{\text{sym}} & [W_v, W_b]_M + R_b M_{nd} P_a - R_a M_{nd} P_b \\ [W_v, W_b]_M + P_a M_{nd} R_b - P_b M_{nd} R_a & 2 [P_a M_{nd} W_b - P_b M_{nd} W_a]_{\text{sym}} \end{pmatrix} \tag{7} \]
where \([U, V]_M = U M_{nd} V - V M_{nd} U\) is a ‘twisted’ commutator of matrices.

The \( G(d) \) invariant quadratic forms have special structure for two reasons. First, there is some redundancy because all the components of the vectors \( q_i \) and \( v_j \) are treated in the same way (lemma 2). Second, there is special structure because the quadratic forms are invariant under translations and boosts (lemma 3).

The ordering of variables used is such that the components of vectors are consecutive entries in \( Z \). Since all our invariant quadratic forms come from forming scalar products of differences of vectors all the \( d \) components of vectors are treated in the same way.

**Lemma 2.** The sub-algebra of \( G(d) \) invariant quadratic forms is independent of the spatial dimension \( d \).

**Proof.** The matrix \( A \) of an invariant quadratic form \( Q_A \) has the form \( A = \hat{A} \otimes 1_d \) where \( 1_d \) is the \( d \)-dimensional identity matrix and \( \otimes \) denotes the Kronecker product; similarly \( J_{nd} = J \otimes 1_d \). Using the general identity \((\hat{A} \otimes 1)(\hat{B} \otimes 1) = (\hat{A} \hat{B} \otimes 1)\) the dimension \( d \) drops out:
\[ (\hat{A} \otimes 1) \ast (\hat{B} \otimes 1) = 2[\hat{A} \hat{B} \otimes 1]_{\text{sym}} = 2[\hat{A} \hat{B}]_{\text{sym}} \otimes 1. \]
Thus we can define an induced algebra of $2n \times 2n$ matrices by
\[
\tilde{A} \ast \tilde{B} = \tilde{A} \tilde{J} \tilde{B} - \tilde{B} \tilde{J} \tilde{A}, \quad \tilde{J} = \begin{pmatrix} 0 & \tilde{M} \\ -\tilde{M} & 0 \end{pmatrix}, \quad \tilde{M} = \text{diag} \left( \frac{1}{m_1}, \ldots, \frac{1}{m_n} \right).
\]
(8)

The dimension reduced $2n \times 2n$ matrices have the same composition law in block form as stated in (7), except that $M_{nd}$ is replaced by $\tilde{M}$.

The second interesting structure of the sub-algebra comes from the fact that $G(d)$ invariant quadratic forms are invariant under the symmetry group operations of translations and boosts. A quadratic form $Q_A(Z)$ is called shift invariant if it is invariant under translations and boosts. After passing from $nd$ to $n$ dimensions this simply means that adding the same multiple of $(1, 0)$ to all pairs $(q_i, v_i)$ (translations), and adding the same multiple of $(t_1, 1)$ to all pairs $(q_i, v_i)$ (boosts) does not change the value of the invariant form. Define the vector $t_1$ with the first $n$ components equal to 1, and the remaining $n$ components equal to 0, and the vector $t_2$ with the first $n$ components equal to 0, and the remaining $n$ components equal to 1. Shift invariant quadratic forms satisfy $Q_A(Z + t_1) = Q_A(Z)$ and $Q_A(Z + t_2) = Q_A(Z)$. Accordingly the matrix $A$ of a shift invariant quadratic form has the vectors $t_1$ and $t_2$ in its kernel. The specific form of the two vectors now implies that $A$ is $2 \times 2$ block Laplacian. A Laplacian has a vector with all 1’s in its kernel, or, equivalently, a matrix that has all row sums equal to zero. Clearly the $G(d)$ invariant quadratic forms are shift invariant.

The main observation is that the symmetric block-Laplacian matrices form a sub-algebra of all symmetric matrices under compositions $\ast$. Note that the diagonal blocks $\tilde{R}$ and $\tilde{P}$ are symmetric, while the off diagonal block $\tilde{W}$ in general is not. It can be decomposed into symmetric Laplacian part $\tilde{S}$ and antisymmetric Laplacian part $\tilde{D}$. Invariants of the form $q_{ij}, q_{kl}, v_{ij} \cdot v_{kl}$ and $q_{ij} \cdot v_{kl}$ correspond to the blocks $\tilde{R}, \tilde{P}$, and $\tilde{W}$, respectively.

**Lemma 3.** The symmetric $2n \times 2n$ matrices of the form
\[
\begin{pmatrix} \tilde{R} & \tilde{S} + \tilde{D} \\ \tilde{S} - \tilde{D} & \tilde{P} \end{pmatrix}
\]
where $\tilde{R}, \tilde{P},$ and $\tilde{S}$ are $n \times n$ symmetric Laplacian matrices, and $\tilde{D}$ is antisymmetric Laplacian form a sub-algebra of the symmetric $2n \times 2n$ matrices under composition (8).

**Proof.** By definition, a (symmetric or antisymmetric) matrix is Laplacian if the vector $(1, \ldots, 1)^t$ is in the kernel of the matrix. From this definition it is clear that the Laplacian property is preserved under matrix multiplication. The algebra multiplication only involves matrix multiplication and addition of blocks, and thus $\tilde{A} \ast \tilde{B}$ is block Laplacian if $\tilde{A}$ and $\tilde{B}$ are block Laplacian. In particular the symmetric and antisymmetric part of the off-diagonal block can again be written explicitly in terms of matrix multiplications and addition of blocks. Therefore the symmetric and antisymmetric part of the off-diagonal block are both again Laplacian.

We already recalled in lemma 1 that the algebra of quadratic forms is isomorphic to the symplectic algebra. We just showed in lemma 3 that the block-Laplacian matrices form a sub-algebra. This brings us to our central result.

**Theorem 4.** The Algebra of $G(d)$ invariant quadratic forms of the $n$-body problem in $\mathbb{R}^d$ is isomorphic to $\text{sp}(2n - 2)$ and the invariant quadratic forms are given by $2 \times 2$ block-Laplacian matrices.
Proof. The subspace $U = \text{span}\{t_1, t_2\}$ is a two-dimensional symplectic subspace of $\mathbb{R}^{2n}$, since the symplectic form restricted to $U$ is non-degenerate, namely $(t_1, Jt_2) = \sum 1/m_i \neq 0$. Consequently the symplectic orthogonal complement $U''$ is a symplectic subspace of $\mathbb{R}^{2n}$ of dimension $2n - 2$, see, e.g., [4]. We already recalled in lemma 1 that $\mathfrak{sp}(2n)$ is isomorphic to the algebra of quadratic forms of $\mathbb{R}^{2nd}$ with composition $\ast$ induced by the Poisson bracket, and by lemma 2 we set $d = 1$. By lemma 3 block-Laplacian matrices whose kernel is the subspace $U$ form a sub-algebra of this algebra. Hence by restriction to the symplectic subspace $U''$ this defines an algebra of quadratic forms isomorphic to $\mathfrak{sp}(2n - 2)$.

This also proves that the space of invariant quadratic forms has dimension $(2n - 1)(n - 1)$, which is the dimension of $\mathfrak{sp}(2n - 2)$.

As discussed in [13] the momentum map of the $O(d)$ action on $\mathbb{R}^{2nd}$ and the $\mathfrak{sp}(2n)$ action with Lie algebra $\mathfrak{sp}(2n)$ identified with invariant quadratic forms are a dual pair. If we replace $\mathbb{R}^{2nd}$ by the the vector space of difference vectors of dimension $2(n - 1)d$ the same construction gives a dual pair for the action of $O(d)$ and $\mathfrak{sp}(2n - 2)$ on difference vectors, or, similarly for the action of $G(d)$ and $\mathfrak{sp}(2n - 2)$ on the original space $\mathbb{R}^{2nd}$.

5. Structure of the reduced Poisson bracket

Now we describe a good choice of basis. As part of the basis for the quadratic $G(d)$ invariants we choose the mutual distances squared $\rho_{ij} = ||q_{ij}||^2$ and the mutually relative speeds squared $v_{ij} = ||v_{ij}||^2, i < j \leq n$, since these are needed in order to write the Hamiltonian $H$ in a simple way. Further choose the scalar products $\sigma_{ij} = q_{ij} \cdot v_{ij}, i < j \leq n$. We write $\rho$, $v$, $\sigma$ for the column vectors with entries $\rho_{ij}$, $v_{ij}$, $\sigma_{ij}$, $i < j \leq n$, respectively, together giving $3n(n - 1)/2$ basis elements. The invariants $\rho$, $v$, and $\sigma$ have Laplacian matrices (see lemma 3) with the only non-zero block being $\hat{R}$, $\hat{P}$, and $\hat{S}$, respectively.

Now we are still missing $(n - 1)(n - 2)/2$ invariants to complete the basis of dimension $(2n - 1)(n - 1)$. A natural choice is to have the remaining basis invariants be quadratic forms for which only the antisymmetric Laplacian block $\hat{D}$ is non-zero. The quadratic forms $C_{ij,kl} = q_{ij} \cdot v_{kl} - v_{ij} \cdot q_{kl}$ span this space. Fixing $j = k = n$ gives a basis for $i < l < n$, but except for $n = 3$ this does not give a nice symmetry in the resulting Poisson bracket. For now denote any choice that completes the basis in this way by $\delta_{ij}, i < j < n$.

There is a natural block-structure in the Poisson structure matrix $B$ that arrises by pairing variables of the four ‘types’ $(\rho, v, \sigma, \delta)$. The structure of these blocks can be computed from the algebra operation in block form (7). Symbolically we will denote these blocks by $\{\rho, \rho\}$, $\{\rho, v\}$, etc.

**Theorem 5.** The Poisson structure matrix $B$ for the $G(d)$ invariant variables $(\rho, v, \sigma, \delta)$ has the block form

\[ B = \begin{pmatrix}
0 & 2L(\sigma) - \Delta & L(\rho) & v(\rho) \\
. & 0 & -L(v) & v(v) \\
. & . & \Delta & v(\sigma) \\
. & . & . & \Sigma
\end{pmatrix}, \]

where $\Delta = \Delta(\delta)$ and $\Sigma = \Sigma(\sigma)$ and all four matrix-valued functions $L$, $v$, $\Delta$, $\Sigma$ are linear in their arguments and have coefficients that are of degree $-1$ in the masses $m_i$. In addition $L$ is symmetric, while $\Delta$ and $\Sigma$ are antisymmetric.

Note that for $n = 2$ the blocks $v$ and $\Sigma$ are absent and the block $\Delta = 0$. For $n = 3$ all blocks are present but still $\Sigma = 0$ since it is a $1 \times 1$ block and sits in the diagonal.
Proof. The following argument can be done with the original blocks $R$, $P$, $S$, $D$ or the reduced blocks $\hat{R}$, $\hat{P}$, $\hat{S}$, $\hat{D}$. It is easy to see that $\{\rho, \rho\} = 0$ since $P_a = P_b = W_a = W_b = 0$ in (7) implies $A \ast B = 0$, similarly $\{v, v\} = 0$. Entries of the type $\{\rho, \sigma\}$ are found from
\[
\begin{pmatrix}
R_a & 0 \\
0 & S_b
\end{pmatrix} \ast 
\begin{pmatrix}
0 & S_b \\
S_b & 0
\end{pmatrix} = 
\begin{pmatrix}
-2[S_b M R_a]_{\text{sym}} & 0 \\
0 & 0
\end{pmatrix}.
\]
(9)
The entries are located in the upper left block only, so that $\{\rho, \sigma\}$ is a linear function of $\rho$ only. We define this matrix-valued function to be $L(\rho)$. Similarly entries of the type $\{v, \sigma\}$ are found from
\[
\begin{pmatrix}
0 & 0 \\
P_a & 0
\end{pmatrix} \ast 
\begin{pmatrix}
0 & S_b \\
S_b & 0
\end{pmatrix} = 
\begin{pmatrix}
0 & 0 \\
2[P_b M S_b]_{\text{sym}} & 0
\end{pmatrix}.
\]
Now $[P_a M S_b]_{\text{sym}} = [S_b M P_a]_{\text{sym}}$ since all three matrices in the product are symmetric. By our choice of basis the matrix $R$ that corresponds to $\rho_{ij}$ is the same as the matrix $P$ that corresponds to $v_{ij}$. Thus for $\{v, \sigma\}$ up to a minus sign we find the same linear function $L$ as in $\{\rho, \sigma\}$, but since the block is located in the lower right here $L$ is a function of $v$. Entries of the type $\{\rho, v\}$ are found from
\[
\begin{pmatrix}
R_a & 0 \\
0 & 0
\end{pmatrix} \ast 
\begin{pmatrix}
0 & 0 \\
0 & P_a
\end{pmatrix} = 
\begin{pmatrix}
0 & -R_a M P_b \\
0 & 0
\end{pmatrix}.
\]
The off-diagonal block is now decomposed into symmetric and antisymmetric part and thus gives a linear function of $\sigma$ (the symmetric part of the off-diagonal block) and of $\delta$ (the antisymmetric part of the off-diagonal block). Since in our choice of basis $\sigma_{ij}$ is represented by a matrix $S$ that equals the matrix $R$ that represents $\rho_{ij}$ we see that up to a factor of two again we find the linear function $L$ now evaluated at $\sigma$. The antisymmetric part is linear in $\delta$ and called $\Delta$. It is given by $-R_a M P_b + P_b M R_a$. Up to a factor the same function $\Delta$ appears in $\{\sigma, \sigma\}$ which is computed from
\[
\begin{pmatrix}
0 & W_a \\
W_a & 0
\end{pmatrix} \ast 
\begin{pmatrix}
0 & W_b \\
W_b & 0
\end{pmatrix} = 
\begin{pmatrix}
0 & [W_a, W_b]_M \\
0 & 0
\end{pmatrix}
\]
where $W_a = S_a$ and $W_b = S_b$; hence $S_a M S_b = S_b M S_a$ which is antisymmetric, which again gives $\Delta(\delta)$. The same matrix operation is used to compute $[\delta, \delta]$ except that now antisymmetric Laplacian matrices $D_a$, $D_b$ are used so that $\Sigma(\delta)$ is obtained from $D_a M D_b - D_b M D_a$. The remaining blocks are $\{\rho, \delta\}$, $\{v, \delta\}$ and $\{\sigma, \delta\}$ which all lead to the same linear matrix-valued function $v$ evaluated at $\rho$, $v$, $\sigma$, respectively. The corresponding matrix products are $-2[D_a M R_a]_{\text{sym}}$, $2[P_b M D_b]_{\text{sym}}$ and $[S_a, D_b]_M$ which all define the same function $v$. □

We now give the explicit form of the block $L(\tau)$ where $\tau = \rho$, $v$, or $\sigma$. It is convenient to keep using two indices $ij$ where $i < j \leq n$ for the components of the vector $\tau$. Accordingly we denote the entries of $L$ by $L_{ij,kl}$.

Lemma 6. The entries in the matrix-valued function $L(\tau)$ are given by $L_{ij,ij} = 2\tau_{ij}/\mu_{ij}$ in the diagonal where $1/\mu_{ij} = 1/m_i + 1/m_j$, $L_{ij,kl} = 0$ if no two indices in $ij$ and $kl$ coincide, and the remaining non-zero entries are $L_{ij,ji} = (\tau_{ij} + \tau_{ji} - \tau_{ii})/m_j$.

Proof. The symmetric Laplacian matrix block corresponding to $\rho_{ij}$ (or $v_{ij}$ or $\sigma_{ij}$) is denoted by $E_{ij}$ for $i < j \leq n$ where the $ii$ and the $jj$ entry are $+1$, the $ij$ and $ji$ entry are $-1$ and all other entries are zero. We already showed that $L$ is found three times in $B$, we choose to compute its components from the $\{\rho, \sigma\}$ block, using (9).
First consider the diagonal of $L$ with $ij = kl$, so that from (9) we find $2[E_{ij}ME_{ij}]_{\text{sym}} = 2E_{ij}ME_{ij} = 2\delta_{ij}E_{ii}$, so that the diagonal entries of $L(\sigma)$ are $2\sigma_{ii}/m_{ii}$. Next consider the case that the pairs of indices $ij$ and $kl$ have no index in common. Then it is easy to see that $E_{ij}ME_{ij} = 0$, so the corresponding entry of $L$ vanishes. Finally consider the case in which there is exactly one index in common between $ij$ and $kl$, say $j = k$. By symmetry $E_{ij} = E_{ji}$, the other cases with one index in common can be reduced to this case. Then $2[E_{ij}ME_{ij}]_{\text{sym}} = (E_{ij} + E_{ji} - E_{ii})/m_{jj}$ as claimed.

Note that the $L(\tau)$ block is independent of the choice of basis for $\delta$. The explicit form of the further blocks $\Delta$, $v$ and $\Sigma$ depends on the choice of basis for $\delta$ and we give explicit formulæ for $n = 3$, in the following subsections.

Once a basis for the antisymmetric Laplacian matrices is chosen the computation of the components of $\Delta$, $v$, $\Sigma$ proceeds in a way similar to lemma 6. For example the entry $\Delta_{ij,kl}$ is given by the antisymmetric part of the product $E_{ij}ME_{kl}$. As before the results is zero if no indices are in common. In addition from anti-symmetry the result is also zero if both double-indices are the same. Hence the only non-zero entries are $E_{ij}ME_{kl} - E_{kl}ME_{ij}$ which now need to be expressed in terms of the basis of antisymmetric Laplacian matrices. A good choice of basis $\delta$ for $n = 3$ is $C_{23,31}$ and for $n = 4$ it is $\delta = (C_{12,43}, C_{23,41}, C_{24,31})$.

Three bodies. For $n = 3$ the 10 invariants are $\rho = (||q_{23}||^2, ||q_{13}||^2, ||q_{12}||^2)^t$, $\nu = (||v_{23}||^2, ||v_{13}||^2, ||v_{12}||^2)^t$, $\sigma = (q_{23} \cdot v_{23}, q_{13} \cdot v_{13}, q_{12} \cdot v_{12})^t$, and the single entry $\delta = q_{23} \cdot v_{31} - v_{23} \cdot q_{31}$. The corresponding antisymmetric Laplacian matrix $D$ of the quadratic form $\delta$ is the unique (up to scaling) antisymmetric Laplacian matrix in dimension three. The Poisson structure matrix is of the general form described in theorem 5 where $\Sigma = 0$. The building blocks of the Poisson structure matrix are

$$
\Delta(\delta) = \delta \begin{pmatrix}
0 & 1/m_3 & -1/m_2 \\
0 & 0 & 1/m_1 \\
. & . & 0
\end{pmatrix}.
$$

where $\Delta' = -\Delta$ and

$$
L(\tau_{23}, \tau_{13}, \tau_{12}) = \begin{pmatrix}
2\tau_{23}/\mu_{23} & (\tau_{23} + \tau_{13} - \tau_{12})/m_3 & (\tau_{23} - \tau_{13} + \tau_{12})/m_2 \\
. & 2\tau_{13}/\mu_{13} & (\tau_{23} + \tau_{13} + \tau_{12})/m_1 \\
. & . & 2\tau_{12}/\mu_{12}
\end{pmatrix},
$$

where $1/\mu_{ij} = 1/m_i + 1/m_j$ and $L' = L$. The column vector $v$ is

$$
v(\tau_{23}, \tau_{13}, \tau_{12}) = \begin{pmatrix}
(\tau_{23} + \tau_{13} - \tau_{12})/m_2 & (\tau_{23} - \tau_{13} + \tau_{12})/m_3 \\
(\tau_{23} + \tau_{13} + \tau_{12})/m_1 & (\tau_{23} - \tau_{13} + \tau_{12})/m_1
\end{pmatrix}.
$$

The reduced Poisson bracket has two Casimirs. One Casimir is the determinant of the Gram matrix of the vectors $(q_{23}, q_{13}, v_{23}, v_{13})$. In terms of invariants the symmetric Gram matrix is

$$
2G = \begin{pmatrix}
2\rho_{23} & \rho_{12} - \rho_{13} - \rho_{23} & 2\sigma_{23} & \delta + \sigma_{12} - \sigma_{13} - \sigma_{23} \\
. & 2\rho_{13} & -\delta + \sigma_{12} - \sigma_{13} - \sigma_{23} & 2\sigma_{13} \\
. & . & 2\nu_{23} & \nu_{12} - \nu_{13} - \nu_{23} \\
. & . & . & 2\nu_{13}
\end{pmatrix}.
$$

The Gram determinant is invariant under a change of basis of difference vectors, even though the Gram matrix itself is not invariant. The Gram determinant is homogeneous of degree 4
in the invariants. For \( d = 3 \) we necessarily have \( \det G = 0 \) since the 4-volume spanned by 3-vectors vanishes.

The other Casimir is the total angular momentum \( ||L_c||^2 \) with respect to the centre of mass which for \( n = 3 \) can be written in terms of the invariants as

\[
||L_c||^2 = \sum_{i<j} \frac{m_i^2 m_j^2}{M^2} (\rho_{ij} v_{ij} - \sigma_{ij}^2)
+ \frac{m_1 m_2 m_3}{2M} \left( \delta_{12}^2 + \sum_{i<j} \frac{m_i}{M} ((\rho_s - 2\rho_{ij})(v_s - 2v_{ij}) - (\sigma_s - 2\sigma_{ij})^2) \right),
\]

where \( \rho_s = \sum_{i<j} \rho_{ij}, v_s = \sum_{i<j} v_{ij}, \sigma_s = \sum_{i<j} \sigma_{ij} \). This Casimir is homogeneous quadratic in the invariants and hence defines a quadric, which is non-singular whenever the value of \( ||L_c||^2 \) is positive.

6. Poisson integrator

A Poisson map \( \phi : M \rightarrow N \) satisfies

\[
\{ f \circ \phi, g \circ \phi \}_M = \{ f, g \}_N \circ \phi.
\]

Let \( B \) be the structure matrix of the Poisson structure, and let \( \phi \) be a map from \( M \) to itself, as it arises when \( \phi \) is given by the flow of Poisson differential equations. Linearizing the definition in this case gives

\[
D\phi(x) B(x) D\phi(x)^\top = B(\phi(x)).
\]

An integrator with this property preserves the geometric structure of the flow. A symplectic map is a special case for which \( B(x) = J^{-1} \) is a constant even dimensional antisymmetric invertible matrix.

For a splitting integrator it is crucial that the parts of the Hamiltonian \( K_c(\nu) \) and \( V(\rho) \) produce integrable flows. Write \( Y = (\rho, \nu, \sigma, \delta) \). Recall that \( K_c(\nu) \) is linear in \( \nu \) and independent of the other variables. Hence \( K_c'(\nu) \) is a constant vector. The vector field generated by \( K_c(\nu) \) is given by

\[
\dot{Y} = X_K = B \cdot \nabla K_c = \begin{pmatrix} 2(L(\sigma) + \Delta)K_c' \\ 0 \\ -2v(\nu)' K_c' \\ -2v(\nu)' K_c' \end{pmatrix}.
\]

Therefore \( \nu \) is constant, and hence the derivative of \( \sigma \) and \( \delta \) is constant, so that they integrate to linear functions of \( t \). The remaining equations for \( \rho \) thus have a linear function of \( t \) on the right hand side, and integrate to quadratic functions of \( t \).

Using the particular form of \( K_c(\nu) \) one can show \( L(\tau) K_c' = \tau \). The simplest way to see that all other terms in the vector field vanish is to consider the flow of \( K_c \) in the original variables and the compute the invariants from that. The flow of \( K_c \) is \( q_i \rightarrow q_i + tv_i, v_i \rightarrow v_i \). Thus \( \rho_{ij} \rightarrow (q_{ij} + tv_{ij}) \cdot (q_{ij} + tv_{ij}) = \rho_{ij} + 2t\sigma_{ij} + t^2 v_{ij} \), similarly for \( \sigma_{ij} \). To see that \( \delta_{ij} \) is constant recall that \( C_{ijk} \) spans the set of basis invariants of type \( \delta \), but now \( (q_{ij} + tv_{ij}) \cdot v_{kl} = v_{ij} \cdot (q_{kl} + tv_{kl}) = q_{ij} \cdot v_{ij} - v_{ij} \cdot q_{kl} \) is constant. As a result the vector field \( X_K \) simply becomes

\[
\dot{\rho} = 2\sigma, \quad \dot{\nu} = 0, \quad \dot{\sigma} = \nu, \quad \dot{\delta} = 0,
\]
which is Newton’s first law in invariants. The explicit solution is the Poisson map
\[ \phi^t_k(\rho, v, \sigma, \delta) = (\rho + 2t\sigma + t^2v, v, \sigma + tv, \delta). \]
Notice that \( \phi^t_k \) is linear in the initial conditions.

The vector field generated by the potential \( V(\rho) \) is given by
\[
\dot{Y} = X_V = B \cdot \nabla V = \begin{pmatrix} 0 \\ -2(L(\sigma) + \Delta)V' \\ -L(\rho)V' \\ -2v(\rho)tV' \end{pmatrix}.
\]
Therefore \( \rho \) is constant, and thus the vector \( V'(\rho) \) is constant as well. Hence the derivative of \( \sigma \) and \( \delta \) is also constant, and they integrate to linear functions of time. The remaining equations for \( v \) thus have a linear function of \( t \) on the right hand side, and integrate to quadratic functions of \( t \). The explicit solution is the Poisson map
\[ \phi^t_v(\rho, v, \sigma, \delta) = (\rho, v - 2t(L(\sigma) + \Delta(\delta))V'(\rho) - t^2(L(a) + \Delta(b))V'(\rho, \sigma + ta, \delta + tb). \]
where \( a = -L(\rho)V'(\rho) \) and \( b = -v(\rho)tV'(\rho) \). Using linearity the solution for \( v \) can be rewritten as
\[ v(t) = v - (L(2t\sigma + t^2a) + \Delta(2t\delta + t^2b))V'(\rho). \]
The map \( \phi^t_v \) is linear in the initial conditions \( v, \sigma, \delta \), but non-linear in \( \rho \), unlike \( \phi^t_k \), which is linear in all initial conditions. As was done for \( X_K \), also for \( X_V \) one can solve the vector field before reduction and then project it onto the invariants. This gives an interesting elementary approach to computing \( X_V \) but no additional simplification is obtained.

Both combinations \( \phi^t_v \phi^t_k \) and \( \phi^t_k \phi^t_v \) are first order Poisson integrators for the reduced \( n \)-body problem. However, neither of them is reversible. Combining the two first order integrators so that the more expensive step \( \phi^t_v \) is only used once gives a second order integrator
\[ \Phi^t = \phi^{t/2}_k \circ \phi^t_v \circ \phi^{t/2}_k. \]
By construction the integrator is Poisson and exactly preserves the Casimirs. In addition the integrator \( \Phi^t \) is reversible, i.e. it satisfies \( \Phi^t \circ \Phi^{-t} = \text{id} \). This follows from the fact that each individual map \( \phi^t_k \) and \( \phi^t_v \) is a flow, and hence satisfies the flow property \( \phi^t_v \circ \phi^t_v = \phi^{2t}_v \).

A proof that \( \Phi \) is a second order integrator can be found in [10], this follows in general for splitting methods. Moreover, from the building blocks \( \phi_K \) and \( \phi_V \) also higher order integrators can be constructed, see [10, 23] and the references therein.

A fundamental property of the gravitational \( n \)-body problem is preserved by this integrator, which is the scaling invariance. The Hamiltonian and equations of motion are unchanged when time is scaled by \( \tau \) and space is scaled by \( \lambda \) such that \( \lambda^3 = \tau^2 \). The induced scaling of the invariants is \( S^g(\rho, v, \sigma, \delta) = (\lambda^2\rho, \lambda^{-1}v, \sqrt{\lambda}\sigma, \sqrt{\lambda}\delta) \). Now it is easy to check that for both, \( \phi_K \) and \( \phi_V \), and hence for \( \Phi \), we have
\[
\Phi^{\lambda^{1/3}} \circ S^g = S^g \circ \Phi^t.
\]
(10)
The Hamiltonian is scaled by \( \lambda^{-1} \) while the stepsize is scaled by \( \lambda^{3/2} \). Other homogeneous potentials have similar scaling laws.

The method presented here is useful for numerics probably only when \( n = 3 \) and \( d \gg 3 \), unless unusually high dimensions \( d > 3 \) are considered. The reason for that is that the number of \( G(d) \) invariants is \((n-1)(2n-1)\), see theorem 4, and grows quadratically with \( n \), while the symplectically fully reduced phase space has dimension \( 2(n-1)d - 2(d-1) \) which is linear in \( n \). The principal strengths of the current approach are that (1) Poisson reduction using invariants has no problem with singular reduction, (2) it is naturally independent of the dimension \( d \) and (3) it preserves the simple form of the Hamiltonian \( H_c \) which allows a splitting integrator to be constructed.
7. Numerical example: figure 8 in 18 steps

Since a Poisson integrator preserves the Casimirs and the geometric structure of the problem exactly, it may be sensible to consider unusually large time steps $h$, and still get qualitatively correct results. We use this kind of ultra-discretization to show that there is a periodic orbit of period 18 of the map $\Phi^h$ that has the same discrete symmetry and the same stability as the figure eight choreography with $n = 3$ bodies of Chenciner and Montgomery [7].

A numerical integrator is a map with the time step $h$ as a continuous parameter. Usually $h$ is chosen sufficiently small so that no essential change occurs when $h$ is changed. From the scaling relation (10) we see that in our particular problem orbits of the integrator appear in families. This reflects the well-known scaling symmetry of the $n$-body problem with homogeneous potential. Since the integrator in general does not conserve the Hamiltonian the only available parameter is the stepsize. Because of the global scaling property (10) orbits appear in families parametrized by $h$.

When using a Poincaré section to find periodic orbits the energy is fixed and the (continuous time) period of the periodic orbit is undetermined. When considering the integrator $\Phi^h$ as a discrete dynamical system there is no sense in fixing the energy since it is not conserved. Instead we look for (discrete) period $n$ orbits for fixed step size $h$ of the integrator (and hence fixed period $T = nh$ in the continuum limit). Because of the scaling symmetry (10) the step size can be arbitrarily fixed.

The figure eight choreography with $n = 3$ bodies can be discretized with 18 steps only, see figure 1. Period 18 amounts to only three iterates after factoring out the discrete $D_6/\mathbb{Z}_2$ symmetry (see [7]) in reduced space. With such a huge stepsize relative to the period $T = 6$ it is not obvious how to identify the discretized figure eight orbit. The precise claim is that
Montgomery and Moeckel [18] has elucidated the interplay between global regularization and reduction for \( n \)-body problem in \( d \) dimensions to a Lie–Poisson system with algebra \( \mathfrak{sp}(2n - 2) \). This enabled us to construct a splitting integrator for the fully symmetry reduced \( n \)-body problem, and the utility of the method was illustrated by integrating the figure eight choreography with \( n = 3 \) bodies in 18 steps only. The method is not restricted to computing the figure eight orbit; computation of some spatial choreographies with \( n = 3 \) bodies will be reported in a forthcoming paper.

We have shown how reduction using quadratic invariant polynomials can be used to reduce the \( n \)-body problem to a \( 3 \)-body problem for \( n = 3 \) and \( d = 2 \). A symplectic integration method for the globally regularized \( 3 \)-body problem in \( 2 \) dimensions to a Lie–Poisson system with algebra \( \mathfrak{sp}(3) \). This enabled us to construct a splitting integrator for the fully symmetry reduced \( n \)-body problem, and the utility of the method was illustrated by integrating the figure eight choreography with \( n = 3 \) bodies in 18 steps only. The method is not restricted to computing the figure eight orbit; computation of some spatial choreographies with \( n = 3 \) bodies will be reported in a forthcoming paper.

It must be noted that this integration method will fail at and near collision. A recent paper by Montgomery and Moeckel [18] has elucidated the interplay between global regularization and reduction for \( n = 3 \) and \( d = 2 \). A symplectic integration method for the globally regularized case with \( n = 3, d = 2 \) and angular momentum zero has recently been described in [19]. This leads to the interesting question whether reduction using polynomial invariants can also be used effectively for the \( n \)-body problem where binary collisions have been regularized.

8. Conclusion

There exists a period 18 orbit of \( \Phi^8 \) that has the same discrete symmetries as the figure eight, is linearly stable, and roughly follows the shape of the figure eight. For comparison a figure eight discretized with 150 steps is also shown in figure 1. Periodic orbits of \( \Phi^8 \) with period 6 or 12 with the correct symmetry do exist, but they are not linearly stable. Choosing initial conditions at the collinear configuration, the discrete symmetry forces the initial condition to be of the form \( \rho_{13} = \rho_{23}, v_{12} = 0, v_{13} = v_{23}, \sigma_{12} = 0, \sigma_{13} = -\sigma_{23} \), and imposing \( \det G = ||L_c||^2 = 0 \) in addition gives \( \rho_{12} = 4\rho_{13} \) and \( \delta_{12} = 2\sigma_{13} \) so that there are only three parameters for orbits with this symmetry. In order to find a symmetric period 6m solution the \( m \)th iterate is required to be at the isosceles configuration of the form \( \rho_{12} = \rho_{13}, v_{12} = v_{13}, \sigma_{12} = -\sigma_{13}, \sigma_{23} = 0 \). The numerical search uses a Newton method restricted to these symmetric subspaces.

The period 18 orbit of \( \Phi^8 \) is determined by \( \rho_{13} = 2.33107, v_{13} = 2.35105 \) and \( \sigma_{13} = 1.28227 \) for \( h = 1/3, T = 6 \). The period 6m orbit of \( \Phi^8 \) with stepsize \( h = 1/m \) converges to a phase space point on the figure eight orbit of the continuous system with period \( T = 6 \) and coordinates \( \rho_{13} = 2.34791, v_{13} = 2.3746, \sigma_{13} = 1.28904 \). The corresponding value of the Hamiltonian is \( H = -0.84 \). This orbit for \( m = 150/6 \) is shown in figure 1. Note that in the full space this corresponds to only half of the figure eight. Figure 2 shows the Casimirs and the Energy over 25 rounds of the figure eight.

Instead of scaling the stepsize with \( m \) we can also fix it at say \( h = 1 \) and scale the initial conditions with \( m \). Then we obtain the statement that the map \( \Phi^4 \) has a family of periodic orbits of period 6m for which \( \rho_{13} \rightarrow 2.34791m^{4/3}, v_{13} \rightarrow 2.3746m^{-2/3}, \sigma_{13} \rightarrow 1.28904m^{1/3} \) for large \( m \).
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