THE YAMABE EQUATION ON COMPLETE MANIFOLDS WITH FINITE VOLUME

NADINE GROSSE

Abstract. We prove the existence of a solution of the Yamabe equation on complete manifolds with finite volume and positive Yamabe invariant. In order to circumvent the standard methods on closed manifolds which heavily rely on global (compact) Sobolev embeddings we approximate the solution by eigenfunctions of certain conformal complete metrics.

This also gives rise to a new proof of the well-known result for closed manifolds and positive Yamabe invariant.

1. Introduction

Yamabe examined whether a closed $n$-dimensional Riemannian manifold $(M, g)$ ($n \geq 3$) possesses a metric $\tilde{g}$ conformal to $g$ with constant scalar curvature. His striking idea was the consideration of the so-called Yamabe invariant, see Definition 1 which gave the possibility to view the question as a variational problem. Works from Aubin [1], Schoen [10] and Trudinger [14] answered the question of Yamabe affirmatively.

There are several possibilities to generalize the Yamabe problem to open (i.e. noncompact and without boundary) manifolds. One possibility is simply to pose the same question as Yamabe did. On open manifolds, this gives much more freedom. We want to make this more precise by comparing to the closed case:

On closed manifolds, if the Yamabe invariant $Q$ is nonpositive, then every conformal metric having constant scalar curvature $c$ and unit volume fulfills $c = Q$. In case that $Q < 0$ this conformal metric is even unique. If $Q > 0$ and there is a conformal metric with constant scalar curvature $c$, one see immediately that $c \geq Q$. But nevertheless, on closed manifolds in all cases a conformal metric with constant scalar curvature has the same sign as the corresponding Yamabe invariant.

On open manifolds, this is no longer true, an easy example is given by an open ball in the Euclidean space. Its Yamabe invariant is the one of the standard sphere, but it carries conformal metrics of constant scalar curvature of all signs: The original Euclidean metric has zero scalar curvature, the spherical metric has constant positive and the hyperbolic metric has constant negative scalar curvature. But all those metrics are conformally equivalent.

That’s why the question is often posed more restrictively. A first possibility is to fix the sign of the constant scalar curvature and/or ask additionally for completeness. This was done by many authors and many results with positive and negative answers were obtained, see for example [2], [7], [13].
Theorem 2. The role of these subcritical solutions and we obtain as test functions for the critical problem. In our approach here the eigenfunctions will play equations. This allows to show converges of a sequence of those solutions which then serve solutions of differential equations that are somehow 'near' to the desired Euler-Lagrange.

In the standard proof on closed manifolds one uses the subcritical Yamabe problem to get do not even exist continuous Sobolev embeddings. On complete open manifolds of finite volume there 

The standard proof for the Yamabe problem on closed manifolds heavily relies on the existence of compact Sobolev embeddings. On complete open manifolds of finite volume there do not even exist continuous Sobolev embeddings $H^2 \to L^p$. [6, Lem. 3.2]. That’s why we will use a different approach by approximating the desired solution by certain eigenfunctions.

The aim of this paper is to study the existence of a smooth positive solution of (1) for complete manifolds of finite volume.

Let $(M^n, g)$ be an $n$-dimensional complete connected Riemannian manifold of finite volume and $n \geq 3$. Let $L_g = a_n \Delta_g + \text{scal}_g$ be the conformal Laplacian where $\text{scal}_g$ is the scalar curvature of the metric $g$ and $a_n = 4^{n-2}n(n-2)$. 

Definition 1. The Yamabe invariant of $(M, g)$ is given by

$$Q(M, g) = \inf \left\{ Q_g(v) := \frac{\int_M v L_g v \, \text{vol}_g}{\|v\|_{L^p(g)}^p} \mid v \in C_c^\infty(M), v \neq 0 \right\}$$

where $p = \frac{2n}{n-2}$ and $C_c^\infty(M)$ denotes the set of compactly supported real valued functions on $M$.

$Q$ is conformally invariant which is seen from the conformal transformation formula of the conformal Laplacian: For $\overline{g} = f^2 g$ where $f \in C_0^\infty(M)$ is a smooth positive real function on $M$ we have

$$L_{\overline{g}}v = f L_g v \text{ where } \overline{\tau} = f^{-\frac{n+2}{2}} \tau.$$ 

The Yamabe invariant is given as a variational problem. Its Euler-Lagrange equation is

$$L_g v = Q v^{p-1} \quad v \in H^1_\text{loc}, \|v\|_{L^p(g)} = 1. \quad (1)$$

The aim of this paper is to study the existence of a smooth positive solution of (1) for complete manifolds of finite volume.

The standard proof for the Yamabe problem on closed manifolds heavily relies on the existence of compact Sobolev embeddings. On complete open manifolds of finite volume there do not even exist continuous Sobolev embeddings $H^2 \to L^p$. [6, Lem. 3.2]. That’s why we will use a different approach by approximating the desired solution by certain eigenfunctions of conformal metrics, cp. Section 4.

In the standard proof on closed manifolds one uses the subcritical Yamabe problem to get solutions of differential equations that are somehow 'near' to the desired Euler-Lagrange equations. This allows to show converges of a sequence of those solutions which then serve as test functions for the critical problem. In our approach here the eigenfunctions will play the role of these subcritical solutions and we obtain

Theorem 2. Let $(M, g)$ be an open complete manifold of finite volume with $0 < Q(M, g) < \overline{Q}(M, g)$ and $\|\text{scal}_g\|_\infty < \infty$ where $(\text{scal}_g)_- := -\min\{\text{scal}, 0\}$.

Then, there exists a smooth positive solution $v \in H^1_\text{loc}$ of $L_g v = Q v^{p-1}$ with $\|v\|_{L^p(g)} = 1$.

$\overline{Q}(M, g)$ is the Yamabe invariant at infinity, see Definition 4 and replaces $Q(S^n)$ that appears at this point in the closed case, cf. Remark 5.

The non-existence of a continuous Sobolev embedding $H^2 \to L^p$ has the following straightforward implications: If $Q > 0$, $\text{scal}_g$ cannot be bounded from above. Moreover, if $v$ is a solution as in Theorem 2, $\overline{g} = v^\frac{-2}{n+2} g$ is a metric with finite volume and constant scalar curvature and for all $v \in C_c^\infty(M) \|v\|_{L^p(\overline{g})} \leq (\max\{a_n, Q\})^\frac{2}{p} \|v\|_{H^2(\overline{g})}$. Thus, $\overline{g}$ cannot be complete.
The method used to prove Theorem\[2\] also gives rise to a different proof for the closed case with positive Yamabe invariant, see Theorem\[14\]. Moreover, the method can be adapted to similar contexts, e.g. one can obtain similar results for the spinorial Yamabe invariant, cf. [4].

2. Preliminaries

In this section we collect some facts on the Yamabe invariant.

**Remark 3.** On complete manifolds instead of taking the infimum over $C_c^\infty$ in the definition of the Yamabe invariant \[1\] one could as well take the infimum over $v \in L^2 \cap H^1_{loc}$ with $\int_M v L_g v \, dvol_g < \infty$. This is seen when considering $Q_g(\eta, v)$ for suitable cut-off functions $\eta$ with $\eta \to 1$.

**Definition 4.** \[2\] Let $(M, g)$ be an open $n$-dimensional manifold with a compact exhaustion $K_i$ fulfilling $K_i \subset K_{i+1} \subset M$ and $\cup_i K_i = M$. Then the Yamabe invariant at infinity is defined as

$$\overline{Q}(M, g) := \lim_{i \to \infty} Q(M \setminus K_i, g).$$

Note that $Q(M \setminus K_i, g) \leq Q(M \setminus K_{i+1}, g)$ since when considering only a subset less test functions can be used in Definition \[1\]. Together with $Q(M, g) \leq Q(S^n)$ \[11\] where $Q(S^n) = n(n-1) \vol(S^n) \frac{2}{n}$ is the Yamabe invariant of the sphere with the standard metric the sequence $Q(M \setminus K_i, g)$ is monotonically increasing and bounded. Thus, $\overline{Q}$ always exists and it holds $\overline{Q}(M, g) \leq Q(S^n)$. Furthermore, $\overline{Q}$ does not depend on the choice of the sequence $K_i$.

**Remark 5.** We note that the condition $\overline{Q}(M, g) \leq Q(S^n)$ in Theorem\[2\] replaces $Q(M, g) \leq Q(S^n)$ that appears in the closed case. This can be seen since for $p \in M$ we have $Q(M, g) = Q(M \setminus \{p\}, g)$ \[11\] Lem. 2.1 and $\overline{Q}(M \setminus \{p\}, g) = \lim_{i \to \infty} Q(B_i(p), g) = Q(S^n)$ where $B_i(p)$ is a ball around $p$ with radius $\epsilon$.

The blow-up argument in the standard proof of the Yamabe problem \[12\] which rules out concentration phenomena at a fixed point shows that for fixed $x \in M$ $Q(B_i(x), g) \to Q(\mathbb{R}^n, g_\text{st}) = Q(S^n, g_\text{st})$ as $\epsilon \to 0$. We will need the following slight generalization:

**Lemma 6.** For all compact subsets $U \subset M$ and $\delta > 0$ there is an $\epsilon = \epsilon(U, \delta) > 0$ such that for all $x \in U$: $Q(B_i(x), g) \geq Q(S^n) - \delta$.

**Proof.** Let $U$ and $\delta$ be fixed. Then for each $x \in U$ let $\epsilon(x)$ be the maximal radius such that $Q(B_i(x), g) \geq Q(S^n) - \delta$ is fulfilled. Set $\epsilon = \inf_{x \in U} \epsilon(x)$. Suppose $\epsilon = 0$. Then there is a sequence $x_i \in U$ with $\epsilon(x_i) \to 0$. Since $U$ is compact, $x_i \to x \in U$. Note that on closed manifolds $Q$ depends smoothly on $g$ in the $C^2$-topology \[3\] Proof of Prop. 7.2]. Thus, $\epsilon(x_i) \to \epsilon(x) > 0$ which is a contradiction. Thus, $\epsilon > 0$.\qed

3. Nonnegative Yamabe invariants and the $L^2$-spectrum

On closed manifolds and if $Q \geq 0$,

$$Q(M, g) = \inf \{ \mu(L_g) \mid \mathfrak{g} \in [g] \}$$

where $\mu(L_g)$ is the lowest eigenvalue of the conformal Laplacian $L_g$ and $[g] := \{ \mathfrak{g} = f^2 g \mid f \in C_c^\infty(M) \}$ denotes the conformal class of $g$.

On general manifolds the spectrum of $L_g$ does not only contain eigenvalues but there can be residual and continuous spectrum. Moreover, in general $L_g$ is even not essentially self-adjoint.
We consider
\[ \mu(L_g) = \inf \left\{ \frac{\int_M vL_g v \text{dvol}_g}{\|v\|_{L^2(g)}^2} : v \in C_c^\infty(M) \right\}. \]

If \( L_g \) is essentially self-adjoint, \( \mu(L_g) \) is the minimum of the spectrum of \( L_g \).

**Remark 7.** If \( Q \geq 0 \) and \( \text{vol}(M, g) = 1 \), then \( \int_M vL_g v \text{dvol}_g \geq Q\|v\|_2^2 \geq Q\|v\|_2^2 \), i.e. \( \mu(L_g) \geq Q \) and \( L_g \) is bounded from below. Then, \( L_g \) is essentially self-adjoint on \( C_c^\infty(M) \), [13] Thm. 1.1 and possesses only eigenvalues and essential spectrum. Moreover, the spectrum is real.

If it is clear from the context to which Riemannian manifold \((M, g)\) we refer, we abbreviate \( \|\cdot\|_s := \|\cdot\|_{L^s(g)} \).

**Lemma 8.** Let \((M, g)\) be a Riemannian manifold with \( Q \geq 0 \). Then
\[ Q(M, g) = \inf \{ \mu(L_\mathcal{F}) : \mathcal{F} \in [g], \text{vol}(M, \mathcal{F}) = 1 \}. \]

If \((M, g)\) has additionally unit volume,
\[ Q(M, g) = \inf \{ \mu(L_\mathcal{F}) : \mathcal{F} = f^2 g, \text{vol}(M, \mathcal{F}) = 1, \exists \text{ a compact subset } K_f \subset M : f|_{M \setminus K_f} = 1 \}. \]

If for a function \( f \in C_c^\infty(M) \) such a compact subset \( K_f \) exists, we shortly say that \( f \equiv 1 \) near infinity. The proof of the first part is the same as in the closed case. But since we are not aware of a reference we shortly give the proof.

**Proof.** Without loss of generality we can assume that \( g \) already has unit volume. Since \( Q \geq 0 \), \( \int_M vL_g v \text{dvol}_g \geq 0 \) for all \( v \in C_c^\infty(M) \).

From Remark 7 we have \( \mu(L_\mathcal{F}) \geq Q(M, g) \) for all conformal metrics \( \mathcal{F} \in [g] \) with unit volume. On the other hand, let \( v_i \in C_c^\infty(M) \) be a minimizing sequence for \( Q \) with \( \|v_i\|_p = 1 \) and \( \int_M v_i L_g v_i \text{dvol}_g \to Q \). Set \( g_i = (\|v_i + i^{-1}\|_p^{-1}(v_i + i^{-1}))^{\frac{1}{p-2}} g \). Then, \( \text{vol}(M, g_i) = \int_M (\|v_i + i^{-1}\|_p^{-1}(v_i + i^{-1}))^p \text{dvol}_g = 1 \) (Note that \( \|v_i + i^{-1}\|_p \leq \|v_i\|_p + i^{-1} = 1 + i^{-1} \) is finite.). Moreover,
\[
\|\mathcal{F}_i\|_{L^2(g_i)}^2 = \int_M (\|v_i + i^{-1}\|_p^{-1}(v_i + i^{-1}))^{\frac{1}{p-2}} v_i^2 \text{dvol}_g \\
\geq \int_M (1 + i^{-1})^{-\frac{1}{p-2}} (v_i + i^{-1})^{\frac{1}{p-2}} v_i^2 \text{dvol}_g \\
\geq (1 + i^{-1})^{-\frac{1}{p-2}} \int_M v_i^2 \text{dvol}_g = (1 + i^{-1})^{-\frac{1}{p-2}}.
\]

Hence,
\[
0 \leq \mu(L_{g_i}) \leq \frac{\int_M \mathcal{F}_i L_g \mathcal{F}_i \text{dvol}_{g_i}}{\|\mathcal{F}_i\|_{L^2(g_i)}^2} \leq \frac{\int_M v_i L_g v_i \text{dvol}_g}{(1 + i^{-1})^{-\frac{1}{p-2}}} \to Q(M, g)
\]
as \( i \to \infty \) which finishes the proof of the first claim.

Let now \((M, g)\) be complete and of finite volume and \( v_i \) be the test sequence of above. Let \( K_i \) be a sequence of compact subsets with \( v_i \subset K_i \). The value \( \|\mathcal{F}_i\|_{L^2(g_i)}^2 \int_M \mathcal{F}_i L_g \mathcal{F}_i \text{dvol}_{g_i} \) only depends on the metric \( g_i \) on \( \text{supp} \ v_i \). Thus, we can deform \( g_i \) such that the conformal factor \( f_i = 1 \) outside a compact subset \( K_{f_i} \), with \( K_i \subset K_{f_i} \subset M \), \( f_i^2 g \equiv g_i \) on \( K_i \), and \( \text{vol}(f_i^2 g) = 1 \).

In particular, if \( g \) was complete, all those \( \mathcal{F} \) are also complete. \( \square \)

Next we study the Yamabe invariant if essential spectrum is present.
Lemma 9. Let \((M, g)\) be a complete Riemannian manifold of unit volume. Let \(L_g\) be essentially self-adjoint on \(C_c^\infty(M)\) and let the essential spectrum of \(L_g\) be non-empty. Then \(Q(M, g) = Q(M, g) \leq 0\).

Proof. Let \(\mu\) be in the essential spectrum of \(L_g\). Then there is a sequence \(v_i \in C_c^\infty(M \setminus B_i)\) where \(B_i\) a ball with radius \(i\) around a fixed point \(z \in M\) such that \(|(L_g - \mu)v_i|_2 \to 0\), \(|v_i|_2 = 1\) and \(v_i \to 0\) weakly in \(L^2\). Then, using \(1 = |v_i|_2 \leq |v_i|_p \text{vol}(M \setminus B_i)^{\frac{1}{p}}\) and, thus, \(|v_i|_p \geq 1\) we estimate

\[
Q(M \setminus B_i) \leq \frac{\int_M v_i L_g v_i \text{dvol}_g}{|v_i|_p^2} \leq \frac{|L_g v_i|_2^2 |v_i|_2}{|v_i|_p^2} \leq \frac{(|(L_g - \mu)v_i|_2^2 + |\mu| |v_i|_p \text{vol}(M \setminus B_i)^{\frac{1}{p}})}{|v_i|_p^2} \leq \frac{(|(L_g - \mu)v_i|_2^2 + |\mu| \text{vol}(M \setminus B_i)^{\frac{1}{p}})}{|v_i|_p^2}
\]

where the right hand-side goes to zero as \(i \to \infty\).

From that and Remark 7 it follows directly

Corollary 10. If \((M, g)\) is a complete Riemannian manifold of unit volume with \(Q(M, g) > 0\) or \(\overline{Q}(M, g) > Q(M, g) = 0\), there exists a sequence of \(g_i = f_i^2 g\) with \(f_i \in C_c^\infty(M)\), \(\int_M f_i^p \text{dvol}_g = 1\) and eigenvalues \(\mu_i := \mu(L_{g_i}) \to Q\) as \(i \to \infty\).

4. Proof of Theorem 2

From Corollary 10 we have: If \(Q > 0\), there exists a sequence of eigenfunctions \(\overline{\nu}_i\) with \(L_g \overline{\nu}_i = \mu_i \overline{\nu}_i\) and \(\int_M |\overline{\nu}_i|^2 \text{dvol}_g = 1\). \(\overline{\nu}_i\) is eigenfunction to the lowest eigenvalue \(\mu_i\) of \(g_i = f_i^2 g\) \((f_i = 1\) near infinity\) and, hence, positive. Viewing these equation w.r.t. the reference metric \(g\) we obtain the following setting

\[
L_g v_i = \mu_i f_i^2 v_i \quad \int_M f_i^2 v_i^2 \text{dvol}_g = 1, \int_M f_i^p \text{dvol}_g = 1, \mu_i \setminus Q, v_i > 0.
\]

Firstly we note that \(\int_M |\overline{\nu}_i|^2 \text{dvol}_g = \int_M f_i^2 v_i^2 \text{dvol}_g = 1\) and \(f_i = 1\) outside a compact subset implies \(v_i \in L^2(g)\).

Moreover, due to Remark 3 \(v_i\) can serve as a test function for \(Q\) and, thus, \(Q |v_i|^2_p \leq \int_M v_i L_g v_i \text{dvol}_g = \mu_i\). Since \(Q > 0\) \(v_i \in L^p(g)\) and if then \(i \to \infty\) we obtain \(|v_i|_p \to 1\).

Thus, \(v_i\) is uniformly bounded in \(L^p(g)\) and, due to the finite volume, also in \(L^2(g)\).

From

\[
\mu_i = \int_M v_i L_g v_i \text{dvol}_g = a_n |dv_i|^2 + \int_M \text{scal}_g v_i^2 \text{dvol}_g
\]

\[
\geq a_n |dv_i|^2 - \int_M (\text{scal}_g - \frac{n}{2}) v_i^2 \text{dvol}_g \geq a_n |dv_i|^2 - \|\text{scal}_g - \frac{n}{2}\| |v_i|^2_p
\]

and the assumption that \(|\text{scal}_g - \frac{n}{2}| < \infty\) we see that \(|dv_i|^2\) is also uniformly bounded. Summarizing \(v_i\) is uniformly bounded in \(H^1_0\) and, hence, \(v_i \to v \geq 0\) weakly in \(H^1_0\) and in \(L^p\). Moreover, \(\int_M f_i^p \text{dvol}_g = 1\) implies that there is \(f \in L^\infty\) such that \(f_i^2 \to f^2\) weakly in \(L^2\).

Lemma 11. Let \(f_i^2 \to f^2\) weakly in \(L^2\) and \(v_i \to v\) weakly in \(H_0^1\).

i) Then \(f_i^2 v_i \to f^2 v\) in \(L^s(U)\) for all compact subsets \(U \subset M\) and \(1 < s < q = \frac{2n}{n+2}\)

ii) If additionally \(L_g v_i = \mu_i f_i^2 v_i\) and \(\mu_i \to Q, f\) and \(v\) weakly fulfill \(L_g v = Q f^2 v\).
Proof. i) We fix \( w \in L^{s^*}(U) \) with \( \frac{1}{s} + \frac{1}{s^*} = 1 \). Then

\[
\left| \int_U (f_i^2 v_i - f^2 v) w \, d\text{vol}_g \right| \leq \int_U |f_i^2 - f^2| |v| w \, d\text{vol}_g + \int_U f_i^2 |v_i - v| |w| \, d\text{vol}_g
\]

The weak convergence \( v_i \to v \) in \( H^1_1 \) implies strong convergence \( v_i \to v \) on \( L^p(U) \) for all \( 1 \leq p' < p \). We choose \( q' \) such that \( p > q' > \frac{n}{n-2q-n} \). Then Hölder inequality implies

\[
\|vw\|_{L^{\frac{nq'}{p'-q}}(U)} \leq \|v\|_{L^{p'}(U)} \|w\|_{L^{(n-2q-n)p'-n}(U)} < \infty
\]

if \( \frac{nq'}{(n-2q-n)p'-n} \leq s^* \). The choice of \( q' \) implies \( p < \frac{nq'}{(n-2q-n)p'-n} < \infty \) and, thus, \( p < s^* < \infty \) and \( 1 < s < q \).

Hence, \( \int_U |f_i^2 - f^2| |vw| \, d\text{vol}_g \to 0 \) as \( i \to \infty \). For the second summand of the above inequality we have

\[
\int_U f_i^2 |v_i - v| |w| \, d\text{vol}_g \leq \|v_i - v\|_{L^{p'}(U)} f_i^2 \|w\|_{L^{\frac{nq'}{p'-q}}(U)} \leq \|v_i - v\|_{L^{p'}(U)} f_i^2 \|w\|_{L^{(n-2q-n)p'-n}(U)} \to 0 \quad \text{as} \quad i \to \infty.
\]

ii) Let \( w \in C_c^\infty(M) \).

\[
\left| \int_M (L_g v - Q f^2 v) w \, d\text{vol}_g \right| = \left| \int_M (L_g v - L_g v_i + \mu_i f_i^2 v_i - Q f^2 v) w \, d\text{vol}_g \right| \\
\leq a_n \left| \int_M (dv - dv_i) w \, d\text{vol}_g \right| + \left| \int_M (v - v_i)(\text{scal}_g w) \, d\text{vol}_g \right| \\
+ \mu_i \left| \int_M f_i^2 v_i - f^2 v \, d\text{vol}_g \right| + |Q - \mu_i| \left| \int_M f^2 |v| \, d\text{vol}_g \right|
\]

All summands on the right-hand side tend to zero as \( i \to \infty \) since \( v_i \to v \) weakly in \( H^1_2 \) (note that \( \text{scal}_g w \in C_c^\infty \subset L^2 \)), part i) and \( \mu_i \to Q \).

\[ \square \]

In order to finish the proof of Theorem 2 it remains to show that \( f^2 = v^{p-2} \) and \( \|v\|_p = 1 \). We start with a non-vanishing result.

**Lemma 12.** In the setting of Lemma 11 and assuming \( 0 < Q(M,g) < Q(S^n) \), \( v \) does not vanish identically.

**Proof.** We prove by contradiction and assume \( v \equiv 0 \) and, hence, \( \int_U v_i^2 \, d\text{vol}_g \to 0 \) for all compact subsets \( U \subset M \) and \( 1 \leq s < p \).

Firstly, we want to show that then also \( \int_U v_i^2 \, d\text{vol}_g \to 0 \) for all compact subsets \( U \subset M \). For that, we assume the contrary, i.e. \( \|v_i\|_{L^p(U)} > C(U) > 0 \) and consider small balls \( B_{2\epsilon}(x) \) with \( x \in M \). We choose \( \epsilon \) small enough such that for all \( x \in U \) \( Q(B_{2\epsilon}(x)) > Q(M,g) \). Due to \( Q(S^n) > Q(M,g) \) and Lemma 6 this is always possible. Then we cover \( U \) by finitely many of those balls \( B_{2\epsilon}(x) \) and define smooth cut-off functions \( \eta_{\epsilon,x} \) compactly supported in \( B_{2\epsilon}(x) \) that are \( 1 \) on \( B_{\epsilon}(x) \) and \( |d\eta_{\epsilon,x}| \leq 2\epsilon^{-1} \). Then we estimate
Now we can estimate $$Q(B_2(x), g) \leq \int_{B_{2x}} \eta_\epsilon \chi v L_g (v \chi)_i dv_\epsilon = \int_{B_{2x}} \eta_\epsilon^2 \chi v L_g v_\epsilon^2 dv_\epsilon + a_n \int_{B_{2x}} |\eta_\epsilon^2 \chi v|_2^2 dv_\epsilon$$

$$\leq \frac{\mu_i}{\epsilon} \int_{B_{2x}} \eta_\epsilon^2 \chi v_\epsilon^2 dv_\epsilon + a_n \frac{4}{\epsilon^2} \int_{B_{2x}} v_\epsilon^2 dv_\epsilon \leq \frac{\mu_i}{\epsilon} + a_n \frac{4}{\epsilon^2} \int_{B_{2x}} v_\epsilon^2 dv_\epsilon$$

where in the second last step we used the Hölder inequality to estimate the summand including $$\mu_i$$. If $$i$$ tends to $$\infty$$, we obtain $$Q(B_2(x), g) \leq Q$$ which is a contradiction to $$Q(B_2(x)) > Q$$. Thus, $$\|v_i\|_{L^p(U)} \to 0$$ as $$i \to \infty$$.

Next, let $$\chi_R$$ be a smooth cut-off function with $$\chi_R = 0$$ on $$B_R := B_R(z)$$ for a fixed $$z \in M$$, $$\chi_R = 1$$ on $$M \setminus B_{2R}$$ and $$|d\chi_R| \leq 2R^{-1}$$. Then

$$Q = \lim_{i \to \infty} \int_M v_i L_g v_i dv_\epsilon = \lim_{i \to \infty} \left( \int_M \chi_R^2 v_i L_g v_i dv_\epsilon + \mu_i \int_M (1 - \chi_R^2) f_i^2 v_i^2 dv_\epsilon \right)$$

$$\geq \lim_{i \to \infty} \left( \int_M \chi_R^2 v_i L_g v_i dv_\epsilon + \mu_i \int_{B_R} f_i^2 v_i^2 dv_\epsilon \right)$$

$$\geq \lim_{i \to \infty} \left( \int_M \chi_R v_i L_g (\chi_R v_i) dv_\epsilon - a_n \int_M |d\chi_R|^2 v_i^2 dv_\epsilon + \mu_i \int_{B_R} f_i^2 v_i^2 dv_\epsilon \right)$$

$$\geq \lim_{i \to \infty} \left( \int_M \chi_R v_i L_g (\chi_R v_i) dv_\epsilon - a_n \int_M |d\chi_R|^2 v_i^2 dv_\epsilon + \mu_i \int_{B_R} f_i^2 v_i^2 dv_\epsilon \right)$$

$$\geq \lim_{i \to \infty} \left( \int_M \chi_R v_i L_g (\chi_R v_i) dv_\epsilon - a_n \int_M |d\chi_R|^2 v_i^2 dv_\epsilon + \mu_i \int_{B_R} f_i^2 v_i^2 dv_\epsilon \right)$$

$$\geq \lim_{i \to \infty} \left( \int_M \chi_R v_i L_g (\chi_R v_i) dv_\epsilon - a_n \int_M |d\chi_R|^2 v_i^2 dv_\epsilon + \mu_i \int_{B_R} f_i^2 v_i^2 dv_\epsilon \right)$$

$$\geq \lim_{i \to \infty} \left( \int_M \chi_R v_i L_g (\chi_R v_i) dv_\epsilon - a_n \int_M |d\chi_R|^2 v_i^2 dv_\epsilon + \mu_i \int_{B_R} f_i^2 v_i^2 dv_\epsilon \right)$$

With $$\|v_i\|_{L^s(U)} \to 0$$ for $$1 \leq s \leq p$$ on compact subsets $$U \subset M$$, $$\|v_i\|_p \to 1$$ and

$$\int_{B_R} f_i^2 v_i^2 dv_\epsilon \leq \|f_i\|_n \|v_i\|_{L^p(B_R)}^2 \leq \|v_i\|_{L^p(B_R)}^2 \to 0$$

we obtain for all $$R$$ that

$$Q(M, g) \geq Q(M \setminus B_R, g).$$

That contradicts $$\overline{Q} > Q$$. Thus, $$v \neq 0$$. \qed

Now we can estimate

$$Q \leq \int_M v L_g v dv_\epsilon \leq Q \int_M f^2 v^2 dv_\epsilon \leq Q \frac{\|v\|^2_p}{\|v\|^2_p} \leq Q.$$

Hence, there is already equality. In particular, from the equality case in the used Hölder inequality we get $$f^2 = v^{p-2}$$ and $$1 = \|f\|_n = \|v\|_p$$. Smoothness of $$v$$ is obtained by standard local elliptic regularity theory. By the maximum principle one sees that $$v$$ is everywhere positive which concludes the proof of Theorem \[ \square \]

Standard local elliptic regularity also gives that $$v$$ is locally in $$C^{2,\alpha}$$. \[ \square \]
Remark 13 (On the assumption on the scalar curvature).
In Theorem [2] we assume that \( \| \text{scal}_g \|_{L_2}^2 (g) < \infty \). If the Yamabe invariant \( Q(g) = -\infty \), this could never be true. But in general it can happen that even though \( \| \text{scal}_g \|_{L_2}^2 (g) = \infty \), \( Q \) is finite and even positive. The easiest example is the standard hyperbolic space \( \mathbb{H}^n \) which has constant negative scalar curvature, infinite volume but the Yamabe invariant of the standard sphere. From this we can even easily construct an example with finite volume: Firstly, we note that \( \| \text{scal}_g \|_{L_2}^2 (g) \) is scale invariant. Let us take a ball \( B \) in the hyperbolic space with \( \| \text{scal}_g \|_{L_2}^2 (g) = 1 \) and then rescale it such that the rescaled ball \( B_i \) has volume \( i^{-2} \). If we consider the disjoint sum of the \( B_i \), we obtain an example for a (disconnected) Riemannian manifold of finite volume and \( \| \text{scal}_g \|_{L_2}^2 (g) = \infty \).
We assume that \( Q(g) = -\infty \) if and only if \( \| \text{scal}_g \|_{L_2}^2 (g) = \infty \) for all \( \gamma \in [g] \). But unfortunately we still cannot prove this. Even if this is true, this alone does not help in our context since we need a complete metric of finite volume with \( \| \text{scal}_g \|_{L_2}^2 (g) < \infty \) which probably cannot be achieved in general.

5. On closed manifolds

The method we used in Theorem [2] for complete manifolds of finite volume allows to reprove the result on closed manifolds with positive Yamabe invariant.

Theorem 14. Let \( (M, g) \) be a closed \( n \)-dimensional Riemannian manifold with \( 0 < Q < Q(S^n) \). Then, there is a smooth positive solution \( v \in H^2_\gamma \) of the Yamabe equation \( (\text{II}) \).

Proof. The proof in the closed case is essentially the same as the one presented in Section [4]. The only little difference occurs in the proof of Lemma [12] where the cut-off function \( \chi_R \) is introduced and \( Q \) is estimated. We make the following change – we take the smooth cut-off function \( \eta_i \) introduced before in Lemma [12]. Then with the same estimate as in Lemma [12] where \( M \setminus B \) substitutes \( B_2R \) and \( M \setminus B_{2R} \) replaces \( B_R \), we obtain

\[
Q \geq \lim_{i \to \infty} \left( Q(B_{2R}, g)(\| v_i \|_p - \| v_i \|_{L^2(M \setminus B_2R)})^2 - \frac{4a_n}{\epsilon^2} \| v_i \|_{L^2(M \setminus B_2R)} + \mu_i \int_{M \setminus B _{2R}} f_i^2 v_i^2 \text{dvol}_g \right)
\]

\[
= Q(B_{2R}, g)
\]

For \( \epsilon \) small enough this gives a contradiction to \( Q(M) < Q(S^n) \) due to Lemma [5]. Thus, following the rest of the proof in Section [4] we obtain that \( v \) is a smooth positive solution of \( L_g v = Qv^{p-1} \) with \( \| v \|_p = 1 \). Note that on closed manifolds the condition \( \| \text{scal}_g \|_{L_2}^2 (g) < \infty \) of Theorem [2] is trivially fulfilled. \( \square \)

References

[1] Aubin, T. Équations différentielles non linéaires et problème de Yamabe concernant la courbure scalaire. J. Math. Pures Appl. (9) 55, 3 (1976), 269–296.
[2] Aviles, P., and McOwen, R. C. Conformal deformation to constant negative scalar curvature on noncompact Riemannian manifolds. J. Differential Geom. 27, 2 (1988), 225–239.
[3] Berard-Bergery, L. Scalar curvature and isometry group. In Spectra of Riemannian Manifolds. Kagai Publications, Tokyo, 1983, pp. 9–28.
[4] Grosse, N. The spinorial Yamabe equation on complete manifolds of finite volume. in preparation.
[5] Grosse, N. The Yamabe equation on manifolds of bounded geometry. arXiv: 0912.4398v3.
[6] Hebey, E. Sobolev spaces on Riemannian manifolds, vol. 1635 of Lecture Notes in Mathematics. Springer-Verlag, Berlin, 1996.
[7] Jin, Z. R. A counterexample to the Yamabe problem for complete noncompact manifolds. In Partial differential equations (Tianjin, 1986), vol. 1306 of Lecture Notes in Math. Springer, Berlin, 1988, pp. 93–101.

[8] Kim, S. Scalar curvature on noncompact complete Riemannian manifolds. Nonlinear Anal. 26, 12 (1996), 1985–1993.

[9] Kim, S. An obstruction to the conformal compactification of Riemannian manifolds. Proc. Amer. Math. Soc. 128, 6 (2000), 1833–1838.

[10] Schoen, R. Conformal deformation of a Riemannian metric to constant scalar curvature. J. Differential Geom. 20, 2 (1984), 479–495.

[11] Schoen, R., and Yau, S.-T. Conformally flat manifolds, Kleinian groups and scalar curvature. Invent. Math. 92, 1 (1988), 47–71.

[12] Schoen, R., and Yau, S.-T. Lectures on differential geometry. Conference Proceedings and Lecture Notes in Geometry and Topology, I. International Press, Cambridge, MA, 1994. Lecture notes prepared by Wei Yue Ding, Kung Ching Chang [Gong Qing Zhang], Jia Qing Zhong and Yi Chao Xu. Translated from the Chinese by Ding and S. Y. Cheng, Preface translated from the Chinese by Kaising Tso.

[13] Shubin, M. Essential self-adjointness for semi-bounded magnetic Schrödinger operators on non-compact manifolds. J. Funct. Anal. 186, 1 (2001), 92–116.

[14] Trudinger, N. S. Remarks concerning the conformal deformation of Riemannian structures on compact manifolds. Ann. Scuola Norm. Sup. Pisa (3) 22 (1968).

[15] Zhang, Q. S. Nonlinear parabolic problems on manifolds, and a nonexistence result for the noncompact Yamabe problem. Electron. Res. Announc. Amer. Math. Soc. 3 (1997), 45–51 (electronic).