SOME BIANCHI TYPE BULK VISCOUS STRING COSMOLOGICAL MODELS IN $f(R)$ GRAVITY

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Abstract. In this paper, we investigate bianchi type $II, VII, III$ and $IX$ Bulk viscous string cosmological models in the context of $f(R)$ gravity. Here, we obtained the solutions of the field equations in the presence of cosmic strings under some specific possible physical conditions. Some physical and geometrical features of the models are also discussed.

Keywords: Bianchi type-$II, VII, III$ and $IX$ Metrics, bulk viscosity, $f(R)$ gravity, modified gravity, Cosmic strings.

1. Introduction
The development of cosmology and gravitation can be seen as one of the scientific triumphs of the 20th century. Contemporary cosmological observational data [1, 4] shows that the expansion of the universe is accelerating. The cause of the far side of this observed acceleration is unknown, and is usually referred to as the "Dark energy problem". In order to explain this cosmic acceleration, two approaches have been suggested. The first approach is to propose and study various dynamical DE models such as scalar field model (Quintessence, Phantom, Quintom), Chaplygin gas [5], Holographic [6, 7], Pilgrim [8], DE models, etc. Second approach is to modify Einstein-Hilbert action has been generalized to various alternative theories of gravitation, $f(R)$, $f(R, T)$, $f(T)$, where $R$ is the Ricci scalar, $T$ is the trace of the energy momentum tensor, and $T$ denotes the torsion scalar. Among these theories, $f(R)$ theory of gravity is significant in which a general function of the Ricci scalar, $f(R)$, replaces $R$ in the standard Einstein-Hilbert Lagrangian. This provides an easy unification of early time inflation and late time acceleration [9]. The theory also gives a natural gravitational alternative to dark energy. Nojiri and Odinstov [10] and Capozziello and Laurent's [11] studied many aspects of $f(R)$ gravity.

Friedmann Robertson Walker (FRW) models are spatially homogeneous and isotropic in nature, best for the representation of the large scale structure of the present universe. However it is a conviction that the early universe may not have been exactly uniform [12]. It is also established that some large angle anomalies [14] seem in Cosmic Microwave Background Radiation [CMBR]. Bianchi type models, which are homogeneous but not necessarily isotropic, seem to be the friendliest explanation of these anomalies. Jaffe et al. [15] establish that removing a Bianchi component from the Wilkinson Microwave Anisotropy Problem (WMAP) data can account for several large angle anomalies go away from the universe to be isotropic. Thus, the models with the anisotropic background are the most acceptable to narrate the advanced stages of the universe. Bianchi type models occur among the simplest model with anisotropic
background. Newly, Bianchi type dark energy cosmological models in various theories of gravitation have been widely discussed in the literature [16, 24]. Sharif and Kausar [26] and Aditya et al. [27] discussed various non-vacuum Bianchi type models in $f(R)$ theory of gravity. Reddy et al. [28] discussed Birkhoff’s theorem in $f(R)$ theory of gravity. Shamir [29] studied exact vacuum solutions of Bianchi type-I, III and Kantowski-Sachs space-times in the metric version of $f(R)$ gravity. Shamir [30] explored plane symmetric vacuum solutions of Bianchi type-III cosmology in $f(R)$ gravity. Yilmaz et al. [31] studied quark and strange quark matter in $f(R)$ gravity for Bianchi type-I and V-spaces times. Reddy et al. [32], Amir and Sattar [33] discussed different exact vacuum solutions of Bianchi space-times in $f(R)$ theory of gravity. Shamir [34, 35] has studied static plane-symmetric solution in both $f(R)$ and $f(R, T)$ modified theories of gravity.

The study of string theory has received considerable attention in cosmology. Cosmic strings are important during structure formation in the early stages of evolution of the universe before the particle creation. Cosmic strings are known as spontaneous symmetry results in a random network of stable one-dimensional topological defects. It is generally known that massive strings serve as a starting point for the large structures like clusters of galaxies in the universe. So these string models have gained considerable attention from research works. Mahanta and Mukherjee [36] and Bhattacharjee and Baruah [37], Stachel [38], Latelier [39], Vilenkin et al. [40], Banerjee et al. [41], Reddy [42], Rao et al. [43, 44] and Tripathy et al. [45] are some of the authors who have investigated strings cosmological models in various theories of gravitation. S.D.Katore [25] examined Bianchi type-II, VIII, and IX string cosmological models in $f(R)$ gravity. In order to study the evolution of the universe, the neutrino coupling in the early stages of the universe, the matter behaved as a viscous fluid. The coefficient of viscosity decreases as the universe and the strong dissipation, due to the neutrino viscosity, may considerably reduce the anisotropy of black body radiation, which has been discussed by several authors [46, 48]. Mohanty and Pattanaik [49], Singh and Shriram [50] are some of the authors who have investigated cosmological models with bulk viscosity in general relativistic FRW model. The effective total negative pressure, which leads to a repulsive gravity in bulk viscosity, overcomes the attractive expansion of the universe. Many authors like Wang [51], Bali and Pradhan [52], Bali and Dave [53], Rao and Sireesha [54], Tripathy et al. [55] have explored various Bianchi type cosmological models in the presence of cosmic strings and bulk viscosity in different theories of gravitation. Rao et al. [56] obtained anisotropic universe with cosmic strings and bulk viscosity in a scalar-tensor theory of gravitation. Sagar et al. [57] explored Bianchi type III bulk viscous cosmic string models in Brans Dicke theory of gravitation. Recently Santhi et al. [12] explored the Bianchi type III bulk viscous cosmic strings models in $f(R)$ theory of gravitation.

Motivated by these investigations, in this paper, we focus our attention on the Bianchi type II, VIII and IX bulk viscous string models in metric $f(R)$ theory of gravity. The plan of the paper is as follows: In sect.2, we derive $f(R)$ gravity field equations for Bianchi type-II, VIII and IX metrics in the presence of bulk viscous fluid with one dimensional strings. In sect. 3, bulk viscous string models are obtained by using Hybrid expansion law proposed by (Akarsu et al. [13]) for average scale factor. Some physical parameters are also evaluated for our models in section 4. In the last section, we summarize the results.

2. Metric and Field Equations

We consider the spatially homogeneous Bianchi type II, VIII and IX metrics of the form

$$ds^2 = -dt^2 + R^2 \left[ d\theta^2 + f^2(\theta) d\phi^2 \right] + S^2 \left[ d\psi + h(\theta) d\phi \right]^2 \quad (1)$$

where $(\theta, \phi, \psi)$ are the Eulerian angles, $R$ and $S$ are function of $t$ only. It represents
Bianchi type-$II$ if \( f (\theta) = 1 \) and \( h (\theta) = \theta \)
Bianchi type-$VIII$ if \( f (\theta) = \cosh \theta \) and \( h (\theta) = \sinh \theta \)
Bianchi type-$IX$ if \( f (\theta) = \sin \theta \) and \( h (\theta) = \cos \theta \)

The field equations of \( f(R) \) gravity are obtained from the action

\[
S = \int \sqrt{-g} \left( \frac{1}{16\pi G} f(R) + L_m \right) d^4x,
\]

where \( f(R) \) is a general function of the Ricci scalar and \( L_m \) is the matter Lagrangian. Variation of action (2) with respect to metric gives the following field equations:

\[
F(R)R_{ij} - \frac{1}{2} f(R) g_{ij} - \nabla_i \nabla_j F(R) + g_{ij} \Box F(R) = \kappa T_{ij},
\]

where \( F(R) = \frac{df}{dR} \) and \( \Box = \nabla^i \nabla_i \), \( \nabla_i \) is the covariant derivative. Contracting field equations (3), we get

\[
F(R)R - 2 F(R) + 3 \Box F(R) = \kappa T
\]

(4)

Using (4) in (3), the field equations take the form

\[
F(R)R_{ij} - \nabla_i \nabla_j F(R) - \kappa T_{ij} = g_{ij} \left( \frac{F(R)R - \Box F(R) - \kappa T}{4} \right)
\]

(5)

Equation (4) is an important relationship between \( f(R) \) and \( F(R) \), which will be used to simplify the field equations and to evaluate \( f(R) \). The energy momentum tensor for a bulk viscous fluid containing one dimensional cosmic string is given by

\[
T_{ij} = (\rho + p) u_i u_j - pg_{ij} - \lambda x_i x_j,
\]

\[
p = p - 3 \xi H (= \omega \rho),
\]

where \( p = \omega_0 \rho (0 \leq \omega_0 \leq 1) \). Here \( p \) is the total pressure, which includes the proper pressure \( p \); \( \lambda \) is the string tension density; \( \rho \) is the rest energy density of the system; \( \xi(t) \) is the coefficient of bulk viscosity. \( 3 \xi H \) is generally known as bulk viscous pressure; \( H \) is the Hubble parameter of the model; and \( \omega = \omega_0 - \zeta \) (where \( \omega_0 \) and \( \zeta \) are constants). We consider \( \rho, p \) and \( \lambda \) as functions of time \( t \) only.

Also, \( u_i \) is the four velocity vector, \( x_i \) is a space like vector, which represents the anisotropic directions of the string and they satisfy

\[
g^{ij} u_i u_j = -x^i x_j = 1, \quad u^i x_i = 0
\]

(8)

We assume the string to be lying along the \( z \)-axis. The one dimensional strings are assumed to be loaded with particle and energy density as \( \rho_p = \rho - \lambda \). Using co-moving coordinates, field equations (5) for metric (1) yield the following equations:

\[
F \left( \frac{\dot{R}}{R} + \frac{\dot{R}^2 + \delta}{R^2} + \frac{\dot{R} S}{RS} - \frac{S^2}{2R^4} \right) + \frac{1}{2} f(R) - \left( \frac{\dot{R}}{R} + \frac{\dot{S}}{S} \right) \dot{F} - \ddot{F} = \kappa \ddot{p}.
\]

(9)

\[
F \left( \frac{\ddot{S}}{S} + \frac{2R \dot{R} S}{RS} + \frac{S^2}{2R^4} \right) + \frac{1}{2} f(R) - \frac{2 \dot{R}}{R} \dot{F} - \ddot{F} = \kappa (\ddot{p} - \lambda)
\]

(10)

\[
F \left( \frac{2 \dot{R}}{R} + \frac{\ddot{S}}{S} \right) + \frac{1}{2} f(R) - \left( \frac{2 \dot{R}}{R} + \frac{\dot{S}}{S} \right) \dot{F} = -\kappa \rho
\]

(11)

Here the over head ‘dot’ denotes differentiation with respect to ‘\( t' \). When \( \delta = 0, -1, +1 \), the field equations (9) - (11) correspond to the Bianchi type-$II$, $VIII$ and $IX$ universes respectively.
3. Solution of the field equations

In this case, the set of equations (9) - (11) forms a system of three independent equations with six unknowns: \(R\), \(S\), \(F(R)\), \(\rho\), \(\lambda\) and \(\rho\). Hence, to find a determinate solutions of these highly nonlinear differential equations, we use the following physically viable conditions:

(i) The shear scalar \(\sigma\) is proportional to scalar expansion \(\theta\), which leads to a relationship between the metric potentials, so that we can take, (Collins et al. [58])

\[
R = S^n, \tag{12}
\]

where \(n \neq 0\) is a constant and preserves the anisotropic character of the space-time.

(ii) \(f(R)\) theory of gravity has been shown equivalent to scalar tensor theory of gravity, which is incompatible with solar system tests of general relativity, as long as the scalar field propagates over solar system scales (Chiba et al. [59]). The power law relation between scalar field and average scale factor has already been used by Johri and Sudharsan [60] in the context of FRW models with bulk viscosity in Brans - Dicke theory. However, Uddin et al. [61] have established a result in the context of \(f(R)\) theory of gravity that shows that

\[
F(R) \propto (a(t))^m \tag{13}
\]

where \(m\) is an arbitrary constant. Thus using power law relation between \(F\) and the average scale factor given by

\[
F(R) = F_0[a(t)]^m \tag{14}
\]

Where \(F_0\) is proportionality constant.

(iii) We take the average scale factor \(a(t)\) combination of power law and exponential law (Akarsu et al. [13])

\[
a(t) = t^\alpha e^{t\beta} \tag{15}
\]

where \(\alpha\) and \(\beta\) are non-negative constants. Here, when \(\alpha = 0\) we get the exponential law and when \(\beta = 0\) we obtain power law. Thus, Eq.(15) gives the combination of exponential and power law which is usually known as Hybrid expansion law. This choice of average scale factor leads to a time dependent deceleration parameter. The solution gives inflation and radiation dominance era with subsequent transition from decelerating to accelerating phase of the universe. The average scale factor \(a(t)\) is defined as

\[
a(t) = V^{1/3} = (R^2 S f(\theta))^{1/3}. \tag{16}
\]

Now using Eqs.(16) (12) and (15) we obtained the metric potentials as

\[
S = \left( \frac{(t^\alpha e^{t\beta})^3}{f(\theta)} \right)^{1/(2n+1)} \tag{17}
\]

\[
R = \left( \frac{(t^\alpha e^{t\beta})^3}{f(\theta)} \right)^{n/(2n+1)}, \tag{18}
\]

where \(f(\theta) = 1\), \(\cosh\theta\) and \(\sin\theta\) for Bianchi type-II, VIII and IX spaces times respectively. From Eqs.(14) and (15), the function \(F(R)\) becomes

\[
F = F_0 \left( t^\alpha e^{t\beta} \right)^m \tag{19}
\]
From Eqs. (9) - (11), we get the string energy density and tension density
\[
\rho = \frac{F_0(t^\alpha t^\beta)^m}{\kappa(\omega + 1)} \left[ \left( \frac{\alpha}{t} + \beta \right)^2 \left( \frac{9(n-1)}{2n+1} + \frac{3n}{2n+1} - m^2 \right) + \frac{3\alpha(n+1)}{(2n+1)t^2} \right. \\
+ \frac{m\alpha}{t^2} + \frac{6n}{2n+1} \left( f(\theta) \right)^{\frac{2n}{2n+1}} - \frac{1}{2} \left( t^\alpha t^\beta \right)^{\frac{6-12n}{2n+1}} \left( f(\theta) \right)^{\frac{4n-2}{2n+1}} \right] \\
\lambda = \frac{F_0(t^\alpha t^\beta)^m}{\kappa} \left[ \left( \frac{\alpha}{t} + \beta \right)^2 \left( \frac{9(n-1)}{2n+1} + \frac{3n}{2n+1} - m^2 \right) + \frac{3\alpha(1-n)}{(2n+1)t^2} \right. \\
+ \frac{m\alpha}{t^2} + \frac{6n}{2n+1} \left( f(\theta) \right)^{\frac{2n}{2n+1}} - \frac{1}{2} \left( t^\alpha t^\beta \right)^{\frac{6-12n}{2n+1}} \left( f(\theta) \right)^{\frac{4n-2}{2n+1}} \right] 
\]

The coefficient of bulk viscosity is given by
\[
\xi = \frac{(\omega_0 - \omega)}{3(\frac{\alpha}{t} + \beta)} \left[ \frac{F_0(t^\alpha t^\beta)^m}{\kappa(\omega + 1)} \left[ \left( \frac{\alpha}{t} + \beta \right)^2 \left( \frac{9(n-1)}{2n+1} + \frac{3n}{2n+1} - m^2 \right) + \frac{3\alpha(n+1)}{(2n+1)t^2} \right. \\
+ \frac{m\alpha}{t^2} + \frac{6n}{2n+1} \left( f(\theta) \right)^{\frac{2n}{2n+1}} - \frac{1}{2} \left( t^\alpha t^\beta \right)^{\frac{6-12n}{2n+1}} \left( f(\theta) \right)^{\frac{4n-2}{2n+1}} \right] 
\]

we obtain the total pressure as
\[
\bar{p} = \frac{F_0(\omega t^\alpha t^\beta)^m}{\kappa(\omega + 1)} \left[ \left( \frac{\alpha}{t} + \beta \right)^2 \left( \frac{9(n-1)}{2n+1} + \frac{3n}{2n+1} - m^2 \right) + \frac{3\alpha(n+1)}{(2n+1)t^2} \right. \\
+ \frac{m\alpha}{t^2} + \frac{6n}{2n+1} \left( f(\theta) \right)^{\frac{2n}{2n+1}} - \frac{1}{2} \left( t^\alpha t^\beta \right)^{\frac{6-12n}{2n+1}} \left( f(\theta) \right)^{\frac{4n-2}{2n+1}} \right] 
\]

The scalar curvature is
\[
R = \frac{2f''}{fR^2} - \frac{4\dot{R}}{R} - \frac{2\dot{S}}{S} - \frac{2\dot{R}^2}{R} - \frac{4\dot{R}\dot{S}}{RS} + \frac{S^2h^2}{2R^4f^2} \\
= \frac{\frac{2}{f(\theta)}}{\left( f(\theta) \right)^{\frac{6n}{2n+1}}} \left( t^\alpha t^\beta \right)^{\frac{6}{2n+1}} \left( f(\theta) \right)^{\frac{4n-2}{2n+1}}, 
\]

where overhead prime stands for ordinary differentiation with respect to \( \theta \)

Now from (9) (10) and (11), we get
\[
f(R) = \frac{2F_0(\omega t^\alpha t^\beta)^m}{3} \left[ \left( \frac{\alpha}{t} + \beta \right)^2 \left( \frac{2m^2 + 15mn + 6m}{2n+1} + \frac{2\omega - 1}{\omega + 1} \left( \frac{9n - 9}{(2n+1)^2} + \frac{3mn}{2n+1} - m^2 \right) \right. \right. \\
- \frac{1}{2n+1} \left( 9n - 9 + 3mn - 3m \right) - \frac{9}{(2n+1)^2} \left( 4n^2 + 3n + 2 \right) \right) \\
+ \frac{\alpha}{(2n+1)(\omega + 1)t^2} \left( 9(2n\omega + \omega + n) - 3m(2n+1) \right) - \frac{3\delta}{\omega + 1} \left( t^\alpha t^\beta \right)^{\frac{6n}{2n+1}} \left( f(\theta) \right)^{\frac{2n}{2n+1}} \\
\left. + \frac{3\delta}{2\omega + 2} \left( t^\alpha t^\beta \right)^{\frac{6-12n}{2n+1}} \left( f(\theta) \right)^{\frac{4n-2}{2n+1}} \right] 
\]

In all the above expressions i.e (20) - (25), \( f(\theta) = 1 \), \( cosh \theta \) and \( sin \theta \) for Bianchi type- \( II, VIII \) and \( IX \) spaces times respectively.

Now the metric (1) can be written as
\[
ds^2 = -dt^2 + \left( \frac{t^\alpha t^\beta}{f(\theta)} \right)^{\frac{2n}{2n+1}} [d\theta^2 + f^2(\theta)d\phi^2] + \left( \frac{t^\alpha t^\beta}{f(\theta)} \right)^{\frac{2n}{2n+1}} [d\psi + h(\theta)d\phi]^2 
\]
Thus, metric (26) together with (20) - (25) constitutes Bianchi type -II, VIII and IX anisotropic string cosmological model with bulk viscosity in $f(R)$ theory of gravity.

4. Some important properties of the model
We define the following parameters for the Bianchi type-II, VIII and IX models. Which are important in discussion of cosmology of obtained anisotropic bulk viscous string model:

- Spatial volume and average scale factor of the models
  \[ V = \sqrt{-g} = R^2sf(\theta); \quad a(t) = V^{\frac{1}{3}} = (R^2sf(\theta))^{\frac{1}{3}} \]
  \[ V = (t^\alpha e^{t\beta})^{\frac{3}{2}}; \quad a(t) = t^\alpha e^{t\beta}. \]  
  \[ \text{(27)} \]

- The mean Hubble’s parameter $H$ is given by
  \[ H = \frac{H_1 + H_2 + H_3}{3} = \left( \frac{\alpha}{t} + \beta \right), \]
  where $H_1 = H_2 = \frac{\dot{R}}{R} = \frac{3n}{2n+1} \left( \frac{\alpha}{t} + \beta \right)$, $H_3 = \frac{\dot{S}}{S} = \frac{3}{2n+1} \left( \frac{\alpha}{t} + \beta \right)$ are directional Hubble’s parameters, which express the expansion rates of the universe in the directions of $x$, $y$ and $z$ respectively.

- Anisotropic parameter is
  \[ A_h = \frac{1}{3} \sum_{i=1}^{3} \left( \frac{H_i - H}{H} \right)^2 = \frac{2(n-1)^2}{(2n+1)^2}. \]
  \[ \text{(29)} \]

$A_h$ is the deviation from isotropic expansion and the universe expands isotropically if $A_h = 0$.

- Expansion scalar and shear scalar are given by
  \[ \theta = u^i_i = \frac{2\dot{R}}{R} + \frac{\ddot{S}}{S} = 3 \left( \frac{\alpha}{t} + \beta \right) \]
  \[ \sigma^2 = \frac{1}{2} \sigma^{ij}_i \sigma^{ij} = \frac{1}{2} \left( \sum_{i=1}^{3} H_i^2 - \frac{\theta^2}{3} \right) = \frac{3(n-1)^2}{(2n+1)^2} \left( \frac{\alpha}{t} + \beta \right)^2 \]
  \[ \text{(30)} \]

- Deceleration parameter is given by
  \[ q = \frac{d}{dt} \left( \frac{1}{H} \right) - 1 = \frac{\alpha}{(\alpha + \beta t)^2} - 1 \]
  \[ \text{(32)} \]

The universe exhibits accelerating volumetric expansion if $-1 \leq q < 0$, decelerating volumetric expansion if $q > 0$, and exhibits constant rate volumetric expansion if $q = 0$.

5. Summary and Conclusions
Here, we have investigated Bianchi type II, VIII and IX string cosmological models with bulk viscosity in metric $f(R)$ theory of gravity. We have obtained cosmological models corresponding to bulk viscous fluid and bulk viscous cosmic strings using some physically viable conditions.

We observed that the spatial volume of the models tend to zero at $t = 0$. At this epoch, all the physical and kinematical parameters diverge. As $t \to \infty$ spatial volume become infinite. As
t → ∞, Hubble’s parameter \( H \) is constant hence the universe expands forever with constant rate. The mean anisotropy parameter \( A_t \neq 0 \), the models are anisotropic throughout the evolution of the universe. Recent observations of SNe I \( \equiv \) expose that the present universe is accelerating and the value of the deceleration parameter lies in the range of \( -1 \leq q < 0 \). It is observed that in our model the deceleration parameter is time depended and exhibits a transition from early decelerated phase to current accelerated phase of the universe. Finally, the models (26) obtained by using hybrid expansion law provide a very nice description of the transition of the universe from the early deceleration to present cosmic acceleration, which is an essential feature for evolution of the universe.

6. References

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