A Sextic with 35 Cusps

Oliver Labs

Abstract. Recently, W. Barth and S. Rams discussed sextics with up to 30 $A_2$-singularities (also called cusps) and their connection to coding theory [4]. In the present paper, we find a sextic with 35 cusps within a four-parameter family of surfaces of degree 6 in projective three-space with dihedral symmetry $D_5$. This narrows the possibilities for the maximum number $\mu_{A_2}(6)$ of $A_2$-singularities on a sextic to $35 \leq \mu_{A_2}(6) \leq 37$. To construct this surface, we use a general algorithm in characteristic zero for finding hypersurfaces with many singularities within a family.

Introduction

Since the middle of the 19th century algebraic geometers are interested in the question: How many isolated singularities of a given topological type can a surface of given degree $d$ in $\mathbb{P}^3 := \mathbb{P}^3(\mathbb{C})$ contain? For surfaces of degree $d \leq 3$ the complete answer to this question is already known since Schlafli’s work [16] on cubic surfaces in 1863, see [10] for explicit equations and illustrating pictures. The treatment of the case $d = 4$ was completed in 1997 using computers (see [20] and [17]). For higher degree much less is known, even when restricting to surfaces with $A_k$-singularities with local equation $x^{k+1} + y^2 + z^2 = 0$. We denote by $\mu_{A_k}(d)$ the maximum number of $A_k$-singularities a surface of degree $d$ in $\mathbb{P}^3$ can have. The maximum number $\mu_{A_1}(d)$ of nodes on a surface of degree $d$ is only known for $d \leq 6$ (see [1] and [11] for the case $d = 6$). In [13] we improved the lower bound for the next open case by constructing a surface of degree 7 with 99 nodes which narrowed the possibilities for $\mu_{A_1}(7)$ to: $99 \leq \mu_{A_1}(7) \leq 104$. For most higher degrees, Chmutov [5] described the currently best known construction.

Even less is known for $A_2$-singularities. The currently best known upper bounds follow from Miyaoka’s result [15] and for $d = 4, 7$ from [19]. The lower bounds are usually achieved by constructions. The cases $d \leq 3$ are classical, $\mu_{A_2}(4) = 8$ follows from Yang’s article mentioned above in an abstract way, and Barth gave an explicit construction for an 8-cuspidal quartic in [3]. In [2], he showed $\mu_{A_2}(5) \geq 15$ by constructing a 15-cuspidal quintic which is nicely connected to Clebsch’s Cubic Diagonal Surface. For the construction of sextics with up to 30 cusps (this was the maximum known prior to the present article), based on an idea already used by Rohn in the 19th century, see [4]. In this article, Barth and Rams also study the codes connected to these cuspidal sextics.

We also base our construction on Rohn’s idea. This gives a 4-parameter family of sextics with dihedral symmetry $D_5$ and 30 cusps. Then we use the computer
algebra programm SINGULAR [9] to find a sextic with 35 cusps within this family which shows:

\[ \mu_{A_2}(6) \geq 35. \]

The following table lists the known restrictions to \( \mu_{A_2}(d) \). For \( d \geq 7 \), they follow from our generalization of Chmutov’s construction of nodal surfaces to higher \( A_j \)-singularities, see [13].

| degree | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | d |
|--------|---|---|---|---|---|---|---|---|----|----|---|
| \( \mu_{A_2} \leq \) | 0 | 3 | 8 | 20 | 37 | 62 | 98 | 144 | 202 | 275 | \( \frac{1}{3}d(d-1)^2 \) |
| \( \mu_{A_2} \geq \) | 0 | 3 | 8 | 15 | 35 | 52 | 70 | 126 | 159 | 225 | \( \approx \frac{2}{3}d^3 \) |

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1. The Sextics with 35 Cusps

In previous works (e.g., [15], [1], [6]), the authors used geometric arguments to reduce a problem depending on several parameters to polynomials each depending only on one parameter. The roots of these polynomials could then easily be found by hand or by computer algebra. But what can we do when there is no geometric argument available to reduce the problem to equations in one variable each? In this case, we can still use a similar approach by replacing root-finding of a polynomial in one variable by primary decomposition.

As our starting point, we take the 4-parameter family \( f_{s,t,u,v} \subset \mathbb{P}^3 \) with dihedral symmetry \( D_5 \) defined by:

\[
\begin{align*}
p &:= z \cdot \prod_{j=0}^{3} \left[ \cos \left( \frac{2\pi j}{5} \right) x + \sin \left( \frac{2\pi j}{5} \right) y - z \right] \\
&= \frac{z}{16} \left[ x \left( x^4 - 2 \cdot 5 \cdot x^2 y^2 + 5 y^4 \right) \\
&\quad - 5 \cdot z \cdot (x^2 + y^2)^2 + 4 \cdot 5 \cdot z^3 \cdot (x^2 + y^2) - 16 z^5 \right], \\
q_{s,t,u,v} &:= s \cdot (x^2 + y^2) + t \cdot z^2 + u \cdot z w + v \cdot w^2, \\
f_{s,t,u,v} &:= p - q_{s,t,u,v}^3.
\end{align*}
\]

\( p \) is the product of \( z \) and 5 planes in \( \mathbb{P}^3(\mathbb{C}) \) meeting in the point \((0 : 0 : 0 : 1)\) with the symmetry \( D_5 \) of the 5-gon with rotation axes \( \{ x = y = 0 \} \). \( q_{s,t,u,v} \) is also \( D_5 \)-symmetric, because \( x \) and \( y \) only appear as \( x^2 + y^2 \).

The generic surface \( f_{s,t,u,v} \) has \( 15 \cdot 2 = 30 \) singularities of type \( A_2 \) at the intersections of the tripled quadric \( q_{s,t,u,v} \) with the \( \binom{3}{2} \) pairwise intersection lines of the 6 planes \( p \). \( 2 \cdot 5 = 10 \) of the singularities lie in the \( \{ z = 0 \} \) plane, the other \( 4 \cdot 5 = 20 \) not. The coordinates of the latter 20 can be obtained from the 4 singularities in the \( \{ y = 0 \} \) plane using the symmetry of the family. To see that the \( \{ y = 0 \} \) plane contains 4 cusps, note that \( p|_{y=0} = z \cdot (z - x) \cdot (x^2 - 2zx - 4z^2)^2 \). For generic values of the parameters, this doubled quadric factor meets the tripled quadric \( q_{s,t,u,v} \) in \( 2 \cdot 2 \) points.

Note that

\[
f_{s,t,u,v}(x, y, z, \lambda w) = f_{s,t,\lambda u,\lambda z v}(x, y, z, w) \forall \lambda \in \mathbb{C}^*,
\]

s.t. we can choose \( v := 1 \) (it is easy to see that \( v = 0 \) corresponds to a degenerate case). Therefore, we write:

\[
f_{s,t,u} := f_{s,t,u,1} \text{ and } q_{s,t,u} := q_{s,t,u,1}.
\]
To find surfaces in this 3-parameter family with more singularities, we compute the discriminant $\text{Disc}_{f,s,t,u} \in \mathbb{C}[s,t,u]$ of the family $f_{s,t,u}$ by first dividing out the base locus (the intersections of the double lines of $p$ with the quadric $q$) from the singular locus (we use saturation, because we have to divide out the base locus six times):

$$sl := \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}, \frac{\partial f}{\partial w} \right),$$

$$bl := \left( \frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}, \frac{\partial p}{\partial z}, \frac{\partial p}{\partial w}, q \right),$$

$$I := sl : bl^\infty.$$

Then we eliminate the variables $x, y, z$ from this quotient. In fact, because of the symmetry we restrict our attention to the $\{y = 0\}$ plane, which speeds up the computations: Every singularity in the plane $\{y = 0\}$ which is not on the rotation axes $\{x = y = 0\}$ generates an orbit of length 5 of singularities of the same type. A short SINGULAR computation then gives the discriminant $\text{Disc}_{f,s,t,u} \in \mathbb{Q}[s,t,u]$, which factorizes into $\text{Disc}_{f,s,t,u} = D_{f,1} \cdot D_{f,2} \cdot D_{f,3}$, where:

$$D_{f,1} = 2^{20} \cdot 3^6 \cdot s^5 \cdot (2^4 \cdot s^2 + 2^2 \cdot 3 \cdot s + t^2) \cdot (s + t)^2$$
$$+ (-2^{19} \cdot 3^6) \cdot s^5 \cdot (2 \cdot 11 \cdot s^2 + 19 \cdot s + 2 \cdot t^2) \cdot (s + t) \cdot u$$
$$+ 2^{16} \cdot 3^6 \cdot s^5 \cdot (41 \cdot s^2 + 2 \cdot 3 \cdot 7 \cdot s + 2 \cdot 3 \cdot t^2) \cdot u^4$$
$$+ (-2^{14} \cdot 3^3) \cdot s^3$$
$$\cdot (2 \cdot 3 \cdot s \cdot s^2 \cdot s^2 \cdot s^2 + 2^2 \cdot 3^2 \cdot s^3 \cdot s^2 \cdot s^2 \cdot s^2 + 5 \cdot 5^2 \cdot s^4 \cdot s^2 \cdot s^2 \cdot s^2 - 5^2 \cdot s^4 \cdot s^2 \cdot s^2 - 5^3 \cdot s^4 \cdot s^2 \cdot s^2$$
$$+ 2^{12} \cdot 3^3 \cdot s^3 \cdot (3^3 \cdot s \cdot s^2 \cdot s^2 \cdot s^2 + 2^5 \cdot 5^2 \cdot s^2 \cdot s^2 \cdot s^2 + 2 \cdot 3 \cdot 5^2 \cdot s^2 \cdot s^2 \cdot s^2 + 5^2 \cdot s^2 \cdot s^2 \cdot s^2) \cdot u$$
$$+ 2 \cdot 10 \cdot 3^3 \cdot s^3 \cdot (61 \cdot s + 3 \cdot 5 \cdot t) \cdot u^4$$
$$+ (-2^5 \cdot 3^4) \cdot (2^2 \cdot 3^3 \cdot s^3 \cdot u + 2^3 \cdot 5 \cdot 23 \cdot s^3 + 2^5 \cdot 3 \cdot 5 \cdot s^3 \cdot s^2 \cdot t + 2^2 \cdot 3^5 \cdot s^2 \cdot s^2 \cdot s^2 + 5^2 \cdot t^3)$$
$$+ 2 \cdot 4 \cdot 3^4 \cdot (2^5 \cdot s^2 + 2^3 \cdot 5 \cdot s \cdot t + 5 \cdot t^2) \cdot u^2$$
$$+ (-2^4 \cdot 3^5) \cdot (2^2 \cdot s + t) \cdot u^4$$
$$+ 5^5 \cdot (u^4 - 2^2 \cdot u^2 + 2^4) \cdot (u^2 + 2^2),$$
$$D_{f,2} = (-2^4) \cdot t^2 + 2^2 \cdot t \cdot (u^2 + 2) + (2 \cdot u - (u^2 + 2^2)) \cdot (2 \cdot u + (u^2 + 2^2)),$$
$$D_{f,3} = 2^2 \cdot t + (2 - u) \cdot (2 + u).$$

We hope that some singularities of the discriminant correspond to examples of surfaces $f_{s,t,u}$ with more $A_2$-singularities. Note that only $D_{f,1}$ depends on the parameter $s$. Using computer algebra, it is easy to verify that the intersections of two of the 3 components $D_{f,1}, D_{f,2}, D_{f,3}$ of $\text{Disc}_{f,s,t,u}$ do not yield to surfaces with many additional singularities.
So, we use SINGULAR again to compute the primary decomposition of the singular locus of $D_{f,1}$ over $\mathbb{Q}$: $sl(D_{f,1}) = sl_{f,1} \cap sl_{f,2} \cap sl_{f,3} \cap sl_{f,4}$, where

\[
sl_{f,1} = \left(2^2, (2^2s - t) + u^2, \ 2^6 \cdot 3^3 \cdot s^3 - 5\right)
\]

\[
sl_{f,2} = \left(-2^2 \cdot (2^2s + 5t) + 5u^2, \ 2^4 \cdot 3^2 \cdot s^2 + 2^2 \cdot 3 \cdot 5 \cdot s + 5^2\right)
\]

\[
sl_{f,3} = \begin{pmatrix}
2^{15} \cdot 3^3 \cdot t^6 - 2^{14} \cdot 3^4 \cdot t^5 \cdot u^2 + 2^{11} \cdot 3^4 \cdot 5 \cdot t^4 \cdot u^4 - 2^6 \cdot 3^3 \cdot 5^3 \cdot t^3 \cdot (2^5 \cdot u^6 - 11.31) \\
+2^4 \cdot 3^2 \cdot 5^2 \cdot t^2 \cdot (2^3 \cdot u^6 - 11.31) \cdot u^2 - 2^2 \cdot 3^2 \cdot t \cdot (2^4 \cdot u^6 - 5 \cdot 11.31) \cdot u^4 \\
+ (2^3 \cdot 3^3 \cdot u^{12} - 3^3 \cdot 5 \cdot 11.31 \cdot u^6 + 2^6 \cdot 5^2 \cdot 19^3),
\end{pmatrix}
\]

\[
2^{11} \cdot 3^2 \cdot t^4 - 2^{11} \cdot 3^2 \cdot t^3 \cdot u^2 + 2^8 \cdot 3^3 \cdot t^2 \cdot u^4 \\
-2^2 \cdot (2^5 \cdot 3^2 \cdot t^6 - 2^2 \cdot 5 \cdot 7 \cdot 19 \cdot 211 \cdot s - 5 \cdot 73 \cdot 193 \cdot t) \\
+ u^2 \cdot (2^3 \cdot 3^2 \cdot u^6 - 5 \cdot 73 \cdot 193)
\]

\[
sl_{f,4} = \left(2^2 \cdot 3 \cdot s - 5, \ -4 \cdot (t + 1) + u^2\right).
\]

All these prime ideals define smooth curves in the 3-dimensional parameter space. When projecting the curve $C_3$ defined by $sl_{f,3}$ to the $s,t$- or the $s,u$-plane, we get in both cases six straight lines defined by the equation

\[2^{15} \cdot 3^3 \cdot s^6 - 2^6 \cdot 3^3 \cdot 5 \cdot s^3 + 5^2 = 0.\]

This shows that $C_3$ consists in fact of the union of six plane curves. Over the algebraic extension $\mathbb{Q}(s)$, it is easy to compute the equation of these:

\[C_{3,s} = 5 \cdot u^2 - 2^2 \cdot 5 \cdot t - 2^{11} \cdot 3^2 \cdot s^4 - 2^4 \cdot 5 \cdot s \in \mathbb{Q}(s)[t,u].\]

To show that there is a surface with 35 $A_2$-singularities, we take the most simple point of this curve, the one with $u = 0$:

**Theorem 1.** Let $s_0 \in \mathbb{C}$ be one of the six roots of $[3]$. Let $(t_0, 0)$ be the point on $C_{3,s_0}$ with $u = 0$. Then the sextic $f_{s_0,t_0,0} \subset \mathbb{P}^3$ has exactly 35 singularities of type $A_2$ and no other singularities.

**Proof.** We use computer algebra. The SINGULAR script and its output can be downloaded from the webpage [12]. Here, we give the basic ideas:

With $u = 0$ in $C_{3,s_0}$, we find: $t_0 = -4 \cdot s_0 \left( \frac{2^7 \cdot 3^2}{5} \cdot s_0^3 + 1 \right)$. For the corresponding surface

\[S_{35} := f_{s_0, -4 \cdot s_0 \left( \frac{2^7 \cdot 3^2}{5} \cdot s_0^3 + 1 \right), 0}\]

we first check that the total milnor number is 70. Then we verify that the surface has 35 singularities of type $A_2$: For each orbit of singularities, we compute the ideal of one of the singularities and check explicitly that it is a cusp. To show this it suffices to verify that its milnor number is exactly two. E.g., for the orbit of the five non-generic singularities, we take the cusp $S_{yw}$ that lies in the $\{y = 0\}$ plane with coordinates:

\[S_{yw} = \left( -\frac{2^7 \cdot 3^2}{5} \cdot s_0^3 + 8 : 0 : 1 : 0 \right).\]
Corollary 2. The maximum number $\mu_{A_2}(6)$ of cusps a surface of degree 6 in $\mathbb{P}^3$ can have, satisfies:

$$35 \leq \mu_{A_2}(6) \leq 37.$$ 

Note that the coefficients of the surface $S_{35}$ are not real. In fact, the ideal $\mathcal{I}_{f,3}$ does not contain any real point, because equation (3) does not have any real root. In particular, it is not possible to use the software surf to draw an image of this sextic. This also holds for the more general family $f_{s,t,u,v}$ because of equation (2).

But in the case of the prime ideal $\mathcal{I}_{f,1}$, we get surfaces with 30 real $A_2$-singularities and 10 real $A_1$-singularities (see also fig. 1). Again, we choose a point in the parameter-space with $u = 0$:

Theorem 3. The sextic $f_{s_0,t_0,0} \subset \mathbb{P}^3$, where $s_0 := \frac{1}{\sqrt[5]{5}} \cdot \sqrt{5} \in \mathbb{R}$, $t_0 = 2^2 \cdot s_0 \in \mathbb{R}$, has exactly 30 singularities of type $A_2$, 10 singularities of type $A_1$, and no other singularities. Furthermore, all the singularities are real.

Proof. Similar to the preceding one. □

Figure 1. A sextic with 30 cusps and 10 nodes at infinity.

2. Concluding Remarks

The method we used to find the sextics with 35 cusps within a family of surfaces with many singularities is an algorithm that can be applied to many other families of hypersurfaces. The only limit is the time needed for the computations. The idea to such an algorithm is not new. In fact, our main observation was to notice that we can use features of the most recent versions of the computer algebra system SINGULAR to perform the algorithm on a computer in our particular case: Finding the equation of $S_{35}$ and verifying that it has exactly 35 cusps and no other singularities just takes a few seconds on our computer.

We could restrict our attention to a plane because of the symmetry of our family, so that the number of variables decreased. This speeded up the computations. But the case of septics with many nodes was too time-consuming to be treated in this way: Our construction of a 99-nodal surface of degree 7 involves computations in positive characteristics and then liftings to characteristic zero using the geometry of the examples, see [13].
In other applications, it might be easy to divide out the base locus and to compute the discriminant, e.g. by using the geometry of the family. Then it only remains to study the discriminant for finding examples which have more singularities than the generic member of the family.

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Johannes Gutenberg Universität Mainz, Germany

E-mail address: Labs@Mathematik.Uni-Mainz.de, mail@OliverLabs.net