Properties of an affine transport equation and its generalized holonomy

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Abstract. We investigate properties of a transport equation that was recently used to study the observer dependence of angular momentum in general relativity. The associated map between the tangent spaces at two points on a curve is affine, and for this reason, the operation was called “affine transport”. The map consists of a homogeneous (linear) part given by the parallel transport map along the curve, plus an inhomogeneous part which is related to the development of a curve in a manifold into an affine tangent space (also described as the rolling of a manifold along a tangent space without slipping or twisting). For closed curves, the affine transport equation defines a “generalized holonomy”. We use covariant bitensor calculus to compute the generalized holonomy around geodesic polygon loops, specifically for triangles and “parallelogramoids” with sides formed from geodesic segments. For small loops, we recover the well-known result for the leading-order holonomy of parallel transport ($\sim \text{Riemann} \times \text{area}$), and we derive the leading-order inhomogeneous part of the generalized holonomy ($\sim \text{Riemann} \times \text{area}^{3/2}$), as well as corrections to both results through order area squared.

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1. Introduction

Recently, in [1], one of the authors and a collaborator investigated what they called the observer dependence of angular momentum in general relativity: the fact that, in asymptotically flat spacetimes, two observers near future null infinity will disagree on changes in values of locally measured angular momenta when, to compare their measured values, they use a covariant procedure designed to recover the agreed-upon method in flat spacetime. The covariant prescription used by [1] was based on a method for transporting a vector $\xi^a$ along a curve $x(\lambda)$, according to the equation of "affine transport",

$$\dot{x}^b \nabla_b \xi^a = \dot{x}^a, \quad (1)$$

where $\dot{x}^a = dx^a/d\lambda$ is the tangent to the curve. The solution to this differential equation, $\xi^a$ at $x = x(\lambda)$, given the vector $\xi^a'$ at an initial point $x' = x(0)$, can be written as a sum of homogeneous and inhomogeneous parts:

$$\xi^a = \Lambda^a_{a'} \xi'^a + \Delta \xi^a. \quad (2)$$

Here $\Lambda^a_{a'}(\lambda)$ is the parallel transport map along the curve from $x'$ to $x$, which satisfies

$$\dot{x}^b \nabla_b \Lambda^a_{a'} = 0, \quad \Lambda^a_{a'}(0) = \delta^a_{a'}, \quad (3)$$

and $\Delta \xi^a(\lambda)$ is the inhomogeneous part of the solution, satisfying

$$\dot{x}^b \nabla_b \Delta \xi^a = \dot{x}^a, \quad \Delta \xi^a(0) = 0. \quad (4)$$

The vector $\Delta \xi^a$ is related to certain aspects of classical differential geometry: the development of a curve on a manifold into the affine tangent space at the curve’s starting point, which is equivalent to rolling the manifold along the initial tangent space without slipping or twisting [2, 3, 4]. More specifically, a vector $\Delta \xi^a$ at $x'$ is equivalent to the displacement vector in the initial affine tangent space that points between the initial and final values of the rolling (or developing) curve, and $\Delta \xi^a = \Lambda^a_{a'} \Delta \xi'^a$ is its parallel transport along the curve in the manifold from $x'$ to $x$. In flat spacetime, $\Delta \xi^a$ is the net displacement vector from $x'$ to $x$, and in curved spacetime, $\Delta \xi^a$ provides a curve-dependent notion of a displacement vector between the two points.

The two parts of the solution ($\Lambda^a_{a'}, \Delta \xi^a$) are an affine map from the tangent space at $x'$ to that at $x$. In [1], this map was the basis of a prescription for transporting a pair of tensors ($P^a', J^{a'b'}$) at $x'$, to ($P^a, J^{ab}$) at $x$, through the relations

$$P^a = \Lambda^a_{a'} P'^a, \quad J^{ab} = \Lambda^a_{a'} \Lambda^b_{b'} J^{a'b'} + 2 \epsilon^{[a} \Delta \xi^{b]}. \quad (5)$$

The pair ($P^a, J^{ab}$) represent locally measured linear and angular momenta of a spacetime, respectively. In flat spacetime, in Minkowski coordinates $x^\mu$, the matrix $\Lambda^a_{a'}$ is the identity matrix, and $\Delta \xi^\mu = (x - x')^\mu$, so that [5] reproduces the correct observer dependence (namely, dependence on a choice of origin) of the conserved momentum and angular momentum of an isolated source in special relativity; in addition, the transport is independent of the curve.

In this paper, we investigate some of the geometrical properties of the affine transport equation [1] and its “generalized holonomy” (defined by solving the affine transport equation around a closed curve). While [1] focused on the generalized holonomies near future null infinity in asymptotically flat, gravitational-wave spacetimes and their implications for transporting angular momentum, here, we compute the generalized holonomy around small (contractible) loops in a generic
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(sMOOTH) pseudo-Riemannian manifold. Our aim is to elucidate the local properties of the affine transport equation and its generalized holonomy by using covariant bitensors.

2. Results and summary

Our primary results are given in (8)–(11), which are the parts of the generalized holonomy when computed along the curves in figure 1. Our primary methods are those of covariant bitensor calculus [5, 6, 7]. The review paper [7] is a useful resource for understanding the bitensor formalism, and we adopt most of the notation of [7], except that we use Latin rather than Greek tensor indices, and the reader should be warned that many of our formulae have the roles of primed and unprimed indices switched relative to [7].

We consider a closed curve beginning and ending at a point \( x \), and we note that the solution to the affine transport equation (1) around the loop defines an affine map on the tangent space at \( x \). It takes an initial vector \( \xi^a_0 \) at \( x \) and returns a final vector \( \xi^a \) at \( x \):

\[
\dot{x}^b \nabla_b \xi^a = \dot{x}^a \Rightarrow \xi^a = \Lambda^a_{\ b} \xi^b_0 + \Delta \xi^a.
\]

The linear map \( \Lambda^a_{\ b} \) is the holonomy of parallel transport, and the vector \( \Delta \xi^a \) is the inhomogeneous contribution to generalized holonomy, \( (\Lambda^a_{\ b}, \Delta \xi^a) \).

We first compute the generalized holonomy around a small geodesic triangle, illustrated in figure 1 defined by three points \( x, x', \) and \( x'' \) which are assumed to be sufficient close to one another that there exist unique geodesic segments connecting them. The solution can be expressed in terms of the two vectors \( u^a \) and \( v^a \) at \( x \) that yield the points \( x' \) and \( x'' \) under the exponential map. Given in terms of derivatives of Synge’s world function \( \sigma \) [5, 7], they are

\[
u^a = -\sigma^a(x, x') \quad \text{and} \quad v^a = -\sigma^a(x, x'').
\]

Stated differently, \( x' \) (or \( x'' \)) is the point reached by following the affinely parametrized geodesic beginning at \( x \) with initial tangent \( u^a \) (or \( v^a \)) for a unit-parameter interval. A unique geodesic segment then links \( x' \) to \( x'' \). For the loop followed counterclockwise \( (x \rightarrow x' \rightarrow x'' \rightarrow x) \), the "u,v triangle", \( \triangle_{u,v} \), for short, we show that the holonomy of parallel transport is given by

\[
\Lambda^a_{\ b}(\triangle_{u,v}) = \delta^a_{\ b} + \frac{1}{2} R^a_{\ bcd} u^c u^d + \frac{1}{6} R^a_{\ bcd;e} v^c u^d (v^e + u^e) + O(4),
\]

and the inhomogeneous solution is given by

\[
\Delta \xi^a(\triangle_{u,v}) = \frac{1}{6} R^a_{\ bcd} (v^b + u^b) v^c u^d + O(4).
\]

Here \( O(n) \) stands for terms with \( n \) or more factors of the vectors \( u^a \) and \( v^a \) (i.e., \( n \) powers of distance). Our primary new result (9) shows that the leading-order inhomogeneous part of the generalized holonomy scales as the area of the triangle to the three-halves power (three powers of distance). We also give an exact series solution to all orders in distance (written in terms of usual two-point coincidence limits of the parallel propagator), and we present explicit results through fourth-order in distance in [Appendix A].

We then treat small “Levi-Civita parallelogramoids” [8], which are quadrilaterals formed from geodesic segments that are the closest approximation in curved space to
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Figure 1. Left: What we call the "u,v triangle", denoted by $\triangle_{u,v}$, which is traversed counterclockwise (the $x \to x' \to x'' \to x$ direction). The generalized holonomy of this loop is given by (8) and (9). The points $x'$ and $x''$ are the images of the exponential maps of $u^a$ and $v^a$ at $x$. Right: There are two possible parallelogramoid loops that can be defined by a pair of vectors $\chi^a$ and $\tau^a$ at $x$, which we label by $x \to y' \to z_1 \to y'' \to x$ and $x \to y' \to z_2 \to y'' \to x$. The points $y'$ and $y''$ are obtained through the exponential maps of $\chi^a$ and $\tau^a$ at $x$, respectively. The vector $\tau^a$ at $y'$ is the parallel transport (along the $x \to y'$ geodesic) of $\tau^a$ at $x$, and the point $z_1$ comes from the exponential map of $\tau^a$ at $y'$; there is then a unique geodesic linking $z_1$ to $y''$ and completing the first parallelogramoid. The vector $\chi^a$ at $y''$ is the parallel transport (along the $x \to y''$ geodesic) of $\chi^a$ at $x$, and the point $z_2$ is the result of the exponential map of $\chi^a$ at $y''$; there is then a unique geodesic that links $y'$ to $z_2$, thereby closing the second parallelogramoid. Through third order in distance, the generalized holonomy is the same around either loop, and (10) and (11) give the holonomy of what we call the $\chi,\tau$ parallelogramoid loop, $\phi_{\chi,\tau}$.

a parallelogram in flat space. As described in figure 1 and section 7, one can use a pair of vectors $\chi^a$ and $\tau^a$ at a point $x$ to define two distinct parallelogramoid loops starting and ending at $x$. However, both loops in figure 1 when traversed in the counterclockwise direction from $x$, have the same generalized holonomy through third order in distance. It is given by

$$\Lambda^a_b(\phi_{\chi,\tau}) = \delta^a_b + R^a_{bcd} \tau^c \chi^d + \frac{1}{2} R^a_{bcd,e} \tau^c \chi^d (\tau^e + \chi^e) + O(4), \quad (10)$$

and

$$\Delta \xi^a(\phi_{\chi,\tau}) = \frac{1}{2} R^a_{bcd}(\tau^b + \chi^b) \tau^c \chi^d + O(4). \quad (11)$$

We show how the parallelogramoid solution can be obtained from a composition of the solutions for two triangles (which, through this order, is particularly simple in that the solution is additive).

The remainder of the paper is organized as follows. In section 3 we give the solution to the affine transport equation along a geodesic segment in terms of fundamental bitensors, specifically, the parallel propagator and derivatives of Sygne’s world function. Section 4 describes the mathematical framework for computing holonomies around geodesic triangles. The framework is similar to that used in [9, 5] to derive the curvature corrections to the law of cosines (a result which we reproduce below). Section 5 derives the holonomy of parallel transport around the triangle, and...
section\textsuperscript{[6]} derives the inhomogeneous contribution to the generalized holonomy. We treat the generalized holonomy of the parallelogramoid in section\textsuperscript{[7]} and we conclude in section\textsuperscript{[5]}. The appendix contains fourth-order terms for the generalized holonomy of the geodesic triangle.

3. Affine transport along a geodesic segment

Consider an affinely parametrized geodesic $x'(\lambda)$ with tangent $u^a'(\lambda)$,

\begin{equation}
\frac{dx^a'}{d\lambda}, \quad \frac{Du^a'}{D\lambda} = u^b \nabla_b u^a' = 0, \tag{12}
\end{equation}

and let $x = x'(0)$ be a fixed initial point on the geodesic, where the tangent is $u^a$. The affine transport equation,

\begin{equation}
\eta^b b^a' = u^a', \tag{13}
\end{equation}

has a formal solution along the (assumed unique) geodesic connecting $x$ to $x'(\lambda)$, which in the language of bitensor calculus \textsuperscript{[5, 6, 7]} is given by

\begin{equation}
\xi^a = g_{ab}(x,x') \xi^b + \sigma^a = -\lambda u^a, \tag{14}
\end{equation}

Here $\xi^a$ is the solution at $x'$, $\xi^a$ is the initial value at $x$, $g_{ab}(x,x')$ is the parallel propagator, and $\sigma^a (x,x')$ is the covariant derivative at $x'$ of Synge’s world function $\sigma(x,x')$.

\begin{figure}[h]
\centering
\includegraphics{figure2.png}
\caption{The affinely parametrized geodesic $x'(\lambda)$ with initial point $x = x'(0)$. The tangent is parallel transported from $u^a$ at $x$ to $u^a'$ at $x'$. The derivatives of the world function both point outward from the geodesic segment: $\sigma^a = -\lambda u^a$ and $\sigma^a' = \lambda u^a'$.}
\end{figure}

That (14) satisfies (13) follows from the following properties. The world function is related to the tangents and the affine parameter interval by

\begin{equation}
\sigma(x,x') = \frac{1}{2} \lambda^2 u^2, \quad \sigma^a = -\lambda u^a, \quad \sigma^a' = \lambda u^a', \tag{15}
\end{equation}

and it satisfies

\begin{equation}
\sigma^b \sigma^a = 0, \quad \sigma^b' \sigma^a' = 0. \tag{16}
\end{equation}

Dividing the second equation of (16) by $\lambda$ and using the last equation of (15) shows that the second term in (14) is the inhomogeneous (particular) solution to (13). That the first term of (14) is the homogeneous solution follows from the second of the identities

\begin{equation}
\sigma^b g^a}_{a;b} = 0, \quad \sigma^b g^a}_{a;b} = 0, \tag{17}
\end{equation}

and the condition $g_{a'} = \delta^a_{a'}$ as $x' \rightarrow x$, which define the parallel propagator. Also note that while the tangent is parallel transported, the world function derivatives are minus the parallel transports of each other:

\begin{equation}
\sigma^a = -g^a}_{a} \sigma^a. \tag{18}
\end{equation}

The above properties will be used often throughout the remainder of the paper.
4. Geodesic triangles

We now describe our framework for analyzing geodesic triangles, illustrated in figure 3. We start at a fixed base point $x$ with two vectors $u^a$ and $v^a$ at $x$, and we then follow the geodesics with initial tangents $u^a$ and $v^a$, for affine parameter intervals $\lambda$ and $\varepsilon$, to reach the points $x'$ and $x''$, respectively. As in (15), the tangents are related to world-function derivatives by

$$\lambda u^a = -\sigma^a(x, x'), \quad \varepsilon v^a = -\sigma^a(x, x'').$$  \hspace{1cm} (19)

This defines affinely parametrized geodesics $x'(\lambda)$ and $x''(\varepsilon)$ emanating from $x$. The tangents to these geodesics at $x'$ and $x''$ are given by

$$\lambda u^a' = \sigma^a'(x', x''), \quad \varepsilon v^a'' = \sigma^a''(x, x''),$$  \hspace{1cm} (20)

which are parallel transports of (19) [cf. (18)]. We assume there is then a unique geodesic segment connecting $x'$ to $x''$. Its tangents are denoted by $w^a'$ at $x'$ and $w^a''$ at $x''$, and are assumed to be normalized so that the affine parameter interval from $x'$ to $x''$ is 1. In terms of derivatives of the world function, they are given by

$$w^a' = -\sigma^a'(x', x''), \quad w^a'' = \sigma^a''(x, x''),$$  \hspace{1cm} (21)

and they are related to each other by parallel transport along the geodesic connecting $x'$ and $x''$.

![Figure 3. The geodesic triangle associated with the three points $x$, $x'$, and $x''$. We find it convenient to treat the triangle as a function of a fixed point $x$ and fixed initial tangents $u^a$ and $v^a$ at $x$ with two affine parameters $\lambda$ and $\varepsilon$ that parameterize two of the geodesic legs. The tangents are parallel transported along the legs: $u^a$ at $x$ to $u^a'$ at $x'$, $v^a$ at $x$ to $v^a''$ at $x''$, and $w^a$ at $x'$ to $w^a''$ at $x''$.](image)

In our calculations below, we will fix the base point $x$ and the vectors $u^a$ and $v^a$ at $x$, and we will vary the affine parameters $\lambda$ and $\varepsilon$. From this perspective, the points $x'$ and $x''$ vary along the fixed geodesics determined by $u^a$ and $v^a$ at $x$. Quantities expressible as functions of $x'$ and $x''$, such as $w^a = -\sigma^a(x', x'')$ or $g^a'_{a'}(x', x'')$, can then be expressed as functions of $\lambda$ and $\varepsilon$, and can be differentiated according to

$$\frac{D}{D\lambda} = u^a \nabla_a', \quad \frac{D}{D\varepsilon} = v^a'' \nabla_a''.$$  \hspace{1cm} (22)
Note that these two derivatives commute, because $\nabla_{a'}$ and $\nabla_{a''}$ commute, and because $u^{a'}$ is independent of $\varepsilon$ and $u^{a''}$ is independent of $\lambda$. Also note that the quantities $u^{a'}$ and $g^{a''}(x, x'')$ depend only on $\lambda$; $v^{a''}$ and $g^{a''}(x, x'')$ depend only on $\varepsilon$; and $u^{a''}$, $w^{a''}$, and $g^{a''}(x, x'')$ depend on both $\lambda$ and $\varepsilon$. These are all of the quantities necessary to define the generalized holonomy around the triangle.

5. The holonomy of parallel transport around a geodesic triangle

We now turn to the calculation of the holonomy of parallel transport around the geodesic triangle loop of the previous section. When following the loop counterclockwise $(x \to x' \to x'' \to x)$, the holonomy of parallel transport is given by

$$\Lambda^a_b(\Delta_{u,v}) = g^{a''}_{a''b} g^{a''}_{a''b} g^{b'}_b = \sum_{m,n=0}^{\infty} \lambda^m \varepsilon^n \frac{\Lambda^a_{b(m,n)}}{m!n!}, \quad (23)$$

We leave out the arguments of the parallel propagators, because they can be understood from their indices. Assuming $x$, $u^{a'}$, and $v^{a''}$ fixed, while $\lambda$ and $\varepsilon$ vary to change the locations of $x'$ and $x''$, we write the holonomy tensor as a covariant Taylor series in $\lambda$ and $\varepsilon$ (the second equality). The coefficients $\Lambda^a_{b(m,n)}$ are constant tensors at $x$ that will depend on $u^{a'}$, $v^{a''}$, and the local geometry at $x$.

We can calculate the coefficients by repeatedly differentiating $\Lambda^a_{b(m,n)}$ using the operators $\frac{D}{D\lambda} = u^{a'} \nabla_{a'}$ and $\frac{D}{D\varepsilon} = v^{a''} \nabla_{a''}$ of (22). When doing so, we will frequently use the identities

$$u^{a'}_{;\beta} u^{b'}_{;\gamma} = 0 = v^{a''}_{;\beta} v^{b''}_{;\gamma}, \quad g^{a'}_{a'^b;\gamma} u^{b'} = 0 = g^{a''}_{a''b;\gamma} v^{b''}, \quad (24)$$

which are a restatement of the geodesic equations for $x' (\lambda)$ and $x'' (\varepsilon)$ and defining properties of the parallel propagators [cf. (17) and (20)]. The $(0,0)$ coefficient is given by the limit of (23) as $\lambda \to 0$ and $\varepsilon \to 0$, which is the identity map:

$$\Lambda^a_{b(0,0)} = \delta^a_b. \quad (25)$$

Acting on (23) with $m$ $\lambda$-derivatives and $n$ $\varepsilon$-derivatives and using (24), we find

$$\left(\frac{D}{D\lambda}\right)^m \left(\frac{D}{D\varepsilon}\right)^n \Lambda^a_{b} = g^{a''}_{a''b} g^{a''}_{a''b} g^{b'}_b = \Lambda^a_{b(m,n)} + O(\lambda) + O(\varepsilon). \quad (26)$$

Taking the $\varepsilon \to 0$ ($x'' \to x$) limit of this equation yields

$$\Lambda^a_{b(m,n)} + O(\lambda) = g^{a''}_{a''b} g^{a''}_{a''b} g^{b'}_b,$$

and then taking the $\lambda \to 0$ ($x' \to x$) limit yields the expansion coefficients in terms of usual two-point coincidence limits [7]:

$$\Lambda^a_{b(m,n)} = \left[ g^{a''}_{a''b} \cdots g^{a''}_{a''b} \right]_{x'' \to x} u^{c_1} \cdots u^{c_m} v^{d_1} \cdots v^{d_n}. \quad (27)$$

Recall that in this notation, the coincidence limit turns primed indices associated with the tangent space at $x'$ to unprimed indices associated with those at $x$. Note that, had we taken the limits in the opposite order, we would have obtained

$$\Lambda^a_{b(m,n)} = \left[ g^{a''}_{a''b} \cdots g^{a''}_{a''b} \right]_{x' \to x} u^{c_1} \cdots u^{c_m} v^{d_1} \cdots v^{d_n} = \left[ g^{a''}_{a''b} \cdots g^{a''}_{a''b} \right]_{x'' \to x} u^{c_1} \cdots u^{c_m} v^{d_1} \cdots v^{d_n}. \quad (28)$$
where the second line has inconsequentially renamed $x''$ to $x'$. We see that consistency requires the identity
\[
\left[ g^a b'_c(c'_1 \ldots c'_m)(d'_1 \ldots d'_n) \right]_{x' \rightarrow x} = \left[ g^{a'} b'_c(c_1 \ldots c_m)(d'_1 \ldots d'_n) \right]_{x' \rightarrow x}.
\]

(30)

This identity actually holds without the symmetrizations, and is a special case of the consequence of the independence of the path of approach to coincidence.‡

T holding for any bitensor $T$ with a well-defined coincidence limit, and being a consequence of the independence of the path of approach to coincidence. The two limits taken in (27) and (28), like the two derivatives (26), commute.

The coincidence limits necessary to compute $\Lambda^a_b$ through third order, via the expression (28), are given by
\[
\left[ g^{a}_{b';c} \right] = 0 = \left[ g^{a'}_{b';c'} \right],
\]
\[
\left[ g^{a}_{b';cd} \right] = -\frac{1}{2} R^a_{bcd}, \quad \left[ g^{a'}_{b';cde} \right] = \frac{1}{2} R^a_{bcd}, \quad \left[ g^{a'}_{b';c'd} \right] = \frac{1}{2} R^a_{bcd}
\]
\[
\left[ g^{a}_{b';cde} \right] = -\frac{2}{3} R^a_{bc(d;e)}, \quad \left[ g^{a'}_{b';c'de} \right] = -\frac{1}{3} R^a_{bc(c;d)},
\]
\[
\left[ g^{a}_{b';c'd} \right] = \frac{1}{3} R^a_{bc(d;c)}, \quad \left[ g^{a'}_{b';c'd'w} \right] = \frac{2}{3} R^a_{bc(d;c)}.
\]

The limits in the first two lines are well-known results [5, 7], and we discuss the computation of those in the last two lines in the appendix. Substituting these relations and (28) into (23) and setting $\lambda = \varepsilon = 1$ (or, equivalently, absorbing $\lambda$ into the definition of $u^a$ and $\varepsilon$ into that of $v^a$), we obtain the holonomy of parallel transport,
\[
\Lambda^a_b(\Delta_{u,v}) = \delta^a_b + \frac{1}{2} R^a_{bcd} v^c u^d + \frac{1}{6} R^a_{bcde} v^c u^d (v^e + u^e) + O(4).
\]

(32)

We list the fourth-order corrections to this result in Appendix A.

The term in (32) equal to $\frac{1}{2} R^a_{bcd} v^c u^d$ is half the value that appears in many textbook derivations of the holonomy around an “infinitesimal parallelogram” spanned by two vectors $v^a$ and $u^a$ (see, e.g., [10]). This is not surprising because the geodesic triangle has half its area. We were unable to find an equivalent calculation in the literature against which to check the third-order term $\frac{1}{2} R^a_{bcd} v^c u^d (v^e + u^e)$ in this series. It seems quite likely that it has been computed elsewhere, however. We now turn our attention to the affine part of the generalized holonomy.

6. The generalized holonomy around a geodesic triangle

Next, consider the inhomogeneous part $\Delta \xi^a$ of the generalized holonomy around the triangle. Under the same assumptions as in the calculation of the holonomy, we can write the exact solution by composing the solution (14) three times along each leg. Specifically, starting with $\xi^a = 0$ at $x$, we obtain $\sigma^a(x, x')$ at $x'$; then we parallel

‡ We thank Jordan Moxon for clarifying this point for us.
transport that to $x''$ and add $\sigma^{a''}(x'', x')$; finally, we parallel transport that to $x$ and add $\sigma^{a}(x, x'')$. The net result is

$$\Delta \xi^a = g^{a a''} \left( g^{a'' a'} \sigma^{a'} (x', x) + \sigma^{a'''} (x'', x') \right) + \sigma^{a} (x, x'')$$

$$= \Lambda^a b \lambda u^b + g^{a a''} w^{a''} - \varepsilon \sigma^a ,$$

where the second line has used (22) and the definitions of section 4. Having already found $\Lambda^a b$, we now need only to expand the quantity

$$\tilde{w}^a \equiv g^{a a''} w^{a''} = g^{a a''} \sigma^{a'''} (x'', x') = \sum_{m,n=0}^\infty \frac{\lambda^m \varepsilon^n}{m! n!} \tilde{w}^a (m,n) ,$$

the tangent at $x''$ to the $x' - x''$ leg, parallel transported to $x$. Its coincidence limit is

$$\tilde{w}^a (0,0) = 0 .$$

The coefficients $\tilde{w}^a (m,n)$ can be computed similarly to those of the holonomy in (26). After using the relations (24), derivatives of (34) have the simple form

$$\left( \frac{D}{D\lambda} \right)^m \left( \frac{D}{D\xi} \right)^n \tilde{w}^a = g^{a a''} g^{a'' a'''} c_1 c_2 \cdots c_m d_1 d_2 \cdots d_n u^{c_1} \cdots u^{c_m} v^{d_1} \cdots v^{d_n}$$

$$= \tilde{w}^a (m,n) + O(\lambda) + O(\varepsilon) .$$

Taking the $\varepsilon \to 0$ ($x'' \to x$) limit simplifies the expression to

$$\tilde{w}^a (m,n) + O(\lambda) = g^{a a''} g^{a'' a'''} c_1 c_2 \cdots c_m d_1 d_2 \cdots d_n u^{c_1} \cdots u^{c_m} v^{d_1} \cdots v^{d_n} ,$$

and taking the $\lambda \to 0$ ($x' \to x$) limit leaves

$$\tilde{w}^a (m,n) = \left[ \sigma^{a c_1} c_2 \cdots c_m d_1 d_2 \cdots d_n \right]_{x' \to x} u^{c_1} \cdots u^{c_m} v^{d_1} \cdots v^{d_n} .$$

As in the previous section, the two limits commute. To compute $\tilde{w}^a$ through fourth order, we first note that the coincidence limits of all third derivatives of the world function vanish. Then, the coincidence limits necessary to evaluate $\tilde{w}^a$ are given by (5 7)

$$\left[ \sigma_{ab} \right] = g_{ab} , \quad \left[ \sigma_{ab'} \right] = g_{ab} , \quad \left[ \sigma_{a'b'} \right] = g_{ab} ,$$

$$\left[ \sigma_{abcd} \right] = S_{abcd} , \quad \left[ \sigma_{abcd'} \right] = -S_{abcd} , \quad \left[ \sigma_{a'b'c'd'} \right] = S_{abcd} ,$$

$$\left[ \sigma_{ab'c'd'} \right] = -S_{bcd} , \quad \left[ \sigma_{a'b'c'd'} \right] = S_{abcd} ,$$

where $S_{abcd}$ is the symmetrized Riemann tensor:

$$S_{abcd} = S_{(ab)(cd)} = S_{(cd)(ab)} = -\frac{1}{3} \left( R_{abcd} + R_{adbc} \right) .$$

Substituting these results into (38) and the series (34), we obtain

$$\tilde{w}^a = g^{a a''} w^{a''} = v^a - u^a + \frac{1}{6} R^{a} b_{cd} (v^b - 2 u^b) v^c u^d + O(4) .$$

Putting this together with the results in (32) and (33) yields the inhomogeneous part of the generalized holonomy,

$$\Delta \xi^a (\Delta u,v) = \frac{1}{6} R^{a}_{bcd} (v^b + u^b) v^c u^d + O(4) .$$

Appendix A also contains the fourth-order corrections to this result.
Although we are unaware of another reference which has computed a quantity like the inhomogeneous part of the generalized holonomy, we can compute a closely related quantity that appears in classical differential geometry as a check of our result. We first note that an expansion similar to that above gives the tangent at \( x' \) to the \( x'-x'' \) leg, parallel transported back to \( x \), as

\[
g^{a}{}_{a'}w^{a'} = v^{a} - u^{a} + \frac{1}{6} R^{a}{}_{bcd} (u^{b} - 2v^{b}) v^{c}u^{d} + O(4) ,
\]

which is simply \([39]\) with \( u^{a} \leftrightarrow v^{a} \) and an over all minus sign. This result [or the result \([39]\) for \( g^{a}{}_{a'}w^{a'} \)] contracted with itself provides the leading-order correction to the law of cosines in curved space \([9, 5]\), expressing the squared geodesic interval along the \( x'-x'' \) leg in terms of \( u^{a} \) and \( v^{a} \):

\[
w^{2} = (v - u)^{2} - \frac{1}{3} R^{ab}{}_{cde}u^{a}v^{b}u^{c}v^{d} + O(5) .
\]

7. The parallelogramoid case

Finally, we turn to the calculation of the generalized holonomy around an infinitesimal parallelogramoid. There are two possible parallelogramoid loops which can be defined from two vectors \( \chi^{a} \) and \( \tau^{a} \) at a point \( x \), described and illustrated above in figure 1 and below in figure 2. The points \( y', y'', z_{1} \), and \( z_{2} \) are defined in terms of the vectors \( \chi^{a} \) and \( \tau^{a} \) at \( x \) by

\[
\chi^{a} = -\sigma^{a} (x, y'), \quad \tau^{a'} = g^{a'}{}_{a}(x, y') \tau^{a} = -\sigma^{a} (y', z_{1}) ,
\]

\[
\tau^{a} = -\sigma^{a} (x, y''), \quad \chi^{a''} = g^{a''}{}_{a}(x, y'') \chi^{a} = -\sigma^{a''} (y'', z_{2}) .
\]

We can define vectors \( \psi_{1}^{a} \) and \( \psi_{2}^{a} \) at \( x \) to be the tangents to the “diagonal” geodesics connecting \( x \) to \( z_{1} \) and \( z_{2} \), respectively, with unit affine parameter intervals along the segments. They have the form

\[
\psi_{1}^{a} = -\sigma^{a} (x, z_{1}) , \quad \psi_{2}^{a} = -\sigma^{a} (x, z_{2}) ,
\]

and we can show that they are related to \( \chi^{a} \) and \( \tau^{a} \) by

\[
\psi_{1}^{a} + O(3) = \psi_{2}^{a} + O(3) = \chi^{a} + \tau^{a} \equiv \psi^{a} ,
\]

with the following argument. First, let us identify the \( x-z_{2}-y'' \) triangle with the \( x-x'-x'' \) triangle of section 4; thus, \( \psi_{2}^{a} \) is identified with \( u^{a} \), \( \tau^{a} \) is identified with \( v^{a} \), and \( \chi^{a''} \) is identified with \( -w^{a''} \). The result \([39]\) then tells us that

\[
-\chi^{a} = -g^{a}{}_{a''} \chi^{a''} = \tau^{a} - \psi_{2}^{a} + O(3) .
\]

An analogous result holds for the \( x-z_{1}-y'' \) triangle, from which we obtain \([43]\).

The result of parallel or affine transport around the \( x \rightarrow y' \rightarrow z_{1} \rightarrow y'' \rightarrow x \) parallelogramoid loop (with \( z_{1} = z_{1} \) or \( z_{2} \)) will be the same as the result of transport around the \( x \rightarrow y' \rightarrow z_{1} \rightarrow x \) triangle loop followed by transport around the \( x \rightarrow z_{1} \rightarrow y'' \rightarrow x \) triangle loop; transport along the last leg of the first triangle is undone by transport along the first leg (the same leg, traversed in opposite directions). We find that, to this order, the same holonomies are obtained from either choice of the parallelogramoid loop. Using the relations \([32]\) and \([43]\), the holonomy of parallel transport is given by

\[
\Lambda^{a}{}_{b}(\Delta_{\chi, \tau}) = \Lambda^{a}{}_{c}(\Delta_{\chi, \psi}) \Lambda^{b}{}_{c}(\Delta_{\psi, \tau}) + O(4) \]

\[
= \delta^{a}{}_{b} + R^{a}{}_{bcd} \tau^{c} \chi^{d} + \frac{1}{2} R^{a}{}_{bcd} \tau^{c} \chi^{d} (\tau^{e} + \chi^{e}) + O(4) .
\]
Similarly, from (40) and (43), the inhomogeneous part of the generalized holonomy is given by

\[ \Delta \xi^a(\hat{\chi},\tau) = \Lambda^a_{\ b}(\Delta_{\psi},\tau) \Delta \xi^b(\Delta_{\chi},\psi) + \Delta \xi^a(\Delta_{\psi},\tau) + O(4) \]

\[ \hspace{1cm} = \Delta \xi^a(\Delta_{\chi},\psi) + \Delta \xi^a(\Delta_{\psi},\tau) + O(4) \]

\[ \hspace{1cm} = \frac{1}{2} R^a_{\ bcd}(\tau^b + \chi^b) \tau^c \chi^d + O(4). \]

We see that, through this order, the generalized holonomy for the parallelogramoid(s) can be found by simply adding the results (\(\Lambda^a_{\ b} - \delta^a_{\ b}\) or \(\Delta \xi^a\)) from the generalized holonomies of two appropriate triangles (which does not hold at higher orders). We note that, through this order, the same result would also be found by traversing the \(x \rightarrow y' \rightarrow z_1 \rightarrow y'' \rightarrow x\) pentagon.

8. Conclusion

In this paper, we focused on the local properties of an affine transport equation and its associated generalized holonomy. We used covariant bitensor methods to compute the generalized holonomy (including the usual holonomy of parallel transport) around a geodesic triangle to any order, in terms of coincidence limits of derivatives of the parallel propagator and Synge’s world function. We presented explicit results through fourth order in the vectors defining the geodesic triangle. Through third order in these vectors, the generalized holonomy (minus the identity map) around a parallelogramoid is just the sum of the generalized holonomies (minus the identity map) around the two triangles above and below its diagonal. The lowest-order part of the holonomy of parallel transport around the parallelogramoid reproduces the standard textbook treatments. The higher-order terms and the generalized holonomy are other quantities of potential physical interest.
Properties of an affine transport equation and its generalized holonomy

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Appendix A. The generalized holonomy for small geodesic triangles to fourth order in distance

The (symmetrized) coincidence limits needed to compute the holonomy of parallel transport around the triangle are given by

\[
\left[ g^a b^e; (cde) f \right] = 0, \\
\left[ g^a b^e; (cde) f^e \right] = -\frac{1}{4} R^a b_f (c; de) + \frac{1}{4} R^a b_g (R^d c; e) f, \\
\left[ g^a b^e; (cd) f^e f' \right] = \frac{1}{12} \left( R^a b (c; f) d - R^a b (c; d) f \right) + \frac{1}{2} R^a g (c; f) d b^g, \\
&\quad+ \frac{1}{4} \left( R^a b g (c; f) d b^g \right) - R^a b g (c; f) d b^g, \\
\left[ g^a b^e; (d' e') f' \right] = \frac{1}{4} R^a b c (d; e f) - \frac{1}{4} R^a b g (d; e f) c, \\
\left[ g^a b^e; (c' d' e') f' \right] = 0.
\]

These [and some of the coincidence limits above (32)] have been obtained by differentiating the coincidence expansions presented in Refs. [11, 12] and applying Synge’s rule [5, 7], while also employing the Bianchi identities and commuting derivatives of the Riemann tensor to simplify the resulting expressions. Following the calculation in section 3 the holonomy through fourth order is

\[
\Lambda^a_b = \delta^a_b + \frac{1}{2} R^a_{vcu} + \frac{1}{6} R^a_{vu; (v+u)} + \frac{1}{8} R^a_{cu; R^e_{vu}} + \frac{1}{48} \left\{ \left( R^a_{vu; (v+u) v} + R^a_{cuv; (2v-3u) v} \right) - \left( v \leftrightarrow u \right) \right\} + O(5),
\]

where \( u \) and \( v \) appearing in index slots denote contractions, of the Riemann tensor or its derivatives, with \( u^a \) and \( v^a \).

To compute the inhomogeneous part of the generalized holonomy at fourth order, we employ the following (symmetrized) coincidence limits [5]:

\[
\left[ \sigma^\alpha_{(b c d e)} \right] = 0 = \left[ \sigma^\alpha_{(b' c' d' e')} \right], \\
\left[ \sigma^\alpha_{(b c)} \right] = \frac{1}{6} R^a \left( (de)(bc) \right) + \frac{1}{2} R^a \left( (de)(bc) \right), \\
\left[ \sigma^\alpha_{(b c)} \right] = \frac{1}{2} R^a \left( (bc)(d e) \right), \\
\left[ \sigma^\alpha_{(bc)(d e)} \right] = -\frac{1}{2} R^a \left( (b c)(d e) \right) + \frac{1}{2} R^a \left( (b c)(d e) \right), \\
\left[ \sigma^\alpha_{b (c d e)} \right] = -\frac{1}{2} R^a \left( (b c)(d e) \right). 
\]

Using the results of section 3 the inhomogenous part has the form

\[
\Delta \xi^a (\Delta u, v) = \left\{ \left( \frac{1}{6} R^a_{vu; v} + \frac{1}{12} R^a_{v vu; v} + \frac{1}{24} R^a_{vu; v} \right) - \left( v \leftrightarrow u \right) \right\} + O(5).
\]

(A.2)
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