The classical limit of minimal length uncertainty relation: revisit with the Hamilton-Jacobi method

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Abstract. The existence of a minimum measurable length could deform not only the standard quantum mechanics but also classical physics. The effects of the minimal length on classical orbits of particles in a gravitation field have been investigated before, using the deformed Poisson bracket or Schwarzschild metric. In this paper, we first use the Hamilton-Jacobi method to derive the deformed equations of motion in the context of Newtonian mechanics and general relativity. We then employ them to study the precession of planetary orbits, deflection of light, and time delay in radar propagation. We also set limits on the deformation parameter by comparing our results with the observational measurements. Finally, comparison with results from previous papers is given at the end of this paper.

Keywords: gravity, modified gravity

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1 Introduction

One of the predictions shared by various quantum theories of gravity is the existence of a minimal observable length. For example, this fundamental minimal length scale could arise in the framework of the string theory [1–3]. For a review of a minimal length in quantum gravity, see [4]. Some realizations of the minimal length from various scenarios have been proposed. Specifically, one of popular models is the generalized uncertainty principle (GUP) [5, 6] derived from the deformed fundamental commutation relation:

\[ [X, P] = i\hbar(1 + \beta P^2), \]  

(1.1)

where \( \beta \) is some deformation parameter, and the minimal measurable length is \( \Delta_{\text{min}} = \hbar \sqrt{\beta} \). For a review of the GUP, see [7]. The deformed fundamental commutator (1.1) have been widely discussed in the context of quantum mechanics, such as the harmonic oscillator [8], Coulomb potential [9, 10], and gravitational well [11, 12]. Since there is a UV-IR mixing embodied in the deformed commutation relation [13], it is also important to study effects of the minimal length in a classical context. For example, the effects of GUP on the classical quantum cosmology were discussed in [14], effects on classical harmonic oscillator in [15], and effects on equivalence principle in [16].

General relativity is the standard theory of gravity. The observational tests of gravity have been performed on Earth and in the solar system, such as the precession of the perihelia of orbit of Mercury, deflection of light by the Sun, and time delay of radar echoes passing the Sun. To set limits on new physics beyond General relativity, effects of the deformed commutation relation on these observational tests have been considered in [13, 17–20]. Specifically, by replacing the deformed quantum mechanical commutator by the deformed Poisson bracket via

\[ \frac{1}{i\hbar} [\hat{A}, \hat{B}] \Rightarrow \{ A, B \}, \]  

(1.2)
the authors of [13, 17, 18] found the deformed equations of motion and orbit of Mercury in the context of Newtonian dynamics. Since the procession of the perihelia of orbit of Mercury is predicted by general relativity not Newtonian mechanics, it is more appropriate to study them in the context of general relativity. Furthermore, it is impossible to calculate the deformed trajectory of a photo in Newtonian dynamics. Motivated by these considerations, the authors of [19] proposed a modification of the Schwarzschild metric to reproduce the modified Hawking temperature derived from the deformed fundamental commutation relation (1.1). Using this deformed metric, they computed corrections to the standard general relativistic predictions for the light deflection and perihelion precession. In [19], only the metric was deformed and the equation of motion of a test particle was still given by the standard geodesic equation. As pointed out in [19], a more profound way to obtain the deformed geodesic in general relativity would be from the deformed field equations of general relativity not just assuming a deformed solution (as in [19]), or a deformed kinematics (as in [13]). However, such deformed field equations are not available yet. Alternatively, the geodesics can be obtained using the Hamilton-Jacobi method. In [21], we derived the deformed Hamilton-Jacobi equation in 1D Newtonian mechanics and used it to study several examples. Moreover, the deformed Hamilton-Jacobi equations in curved spacetime have been derived when corrections, caused by the deformed fundamental commutator (1.1), to the Hawking temperature were studied using the Hamilton-Jacobi method [22–24].

In this paper, we use the Hamilton-Jacobi method to study effects of the minimal length on geodesic motions of particles in the context of Newtonian dynamics and general relativity. Concretely, in section 2 we calculate the deformed precession angle of planetary orbits in the context of Newtonian dynamics after the deformed Hamilton-Jacobi equation is derived. It turns out that our result (2.25) agrees with these obtained in [13] with $\beta' = 2\beta$ and [17], where the method of the deformed Poisson bracket was used. In section 3, we derive the deformed Hamilton-Jacobi equations in curved spacetime and the deformed precession angle of planetary orbits in the context of general relativity. Contrary to what was found in [19], our results show that the leading correction to the precession angle caused by deformations coincides with these obtained in [13, 17]. This discrepancy may come from an implicit assumption made in [19] about the energy of planets, which is discussed in detail in section 5. The deflection of light and time delay in radar propagation are also considered in section 3. We place constraints on the deformation parameter by comparing our results with the observational measurements in section 4. Section 5 is devoted to our discussion and conclusion. For simplicity, we set $c = k_B = 1$ in this paper.

2 Hamilton-Jacobi method in Newtonian dynamics

In this section, we first derive the deformed Hamilton-Jacobi equation for a nonrelativistic system and then apply it to the motion in a central potential.

2.1 Deformed Hamilton-Jacobi equation

In three dimensions, a generalization of the deformed algebra (1.1) reads [6]

\[
\begin{align*}
[X_i, P_j] &= i\hbar \left[(1 + \beta P^2)\delta_{ij} + \beta' P_i P_j\right], \\
[X_i, X_j] &= i\hbar \frac{(2\beta - \beta') + (2\beta + \beta') \beta P^2}{1 + \beta P^2} \left(P_i X_j - P_j X_i\right), \quad (2.1) \\
[P_i, P_j] &= 0,
\end{align*}
\]

(2.1)
where $\beta, \beta' > 0$ are two deformation parameters, and the minimal length becomes $\Delta X_{\text{min}} = \hbar \sqrt{\beta + \beta'}$. To study the Schrodinger equation incorporating the minimal length commutation relations (2.1), we need the representations of $X_i$ and $P_i$ in terms of some differential operators. In our paper, we consider the Brau reduction [10], where $\beta' = 2\beta$ and the commutators taken between different components of the position $X_i$ vanish to first order in $\beta$ and $\beta'$.

For this particular case, there is a very simple reduction of the form to first order in $\beta$:

$$X_i = x_i,$$
$$P_i = p_i \left(1 + \beta p^2\right),$$

where $x_i$ and $p_i$ are the conventional momentum and position operators satisfying

$$[x_i, p_j] = i\hbar \delta_{ij}, [x_i, x_j] = [p_i, p_j] = 0,$$

and $p^2 = \sum_i p_i p_i$. In the pseudo-position representation, one then has

$$x_i = x_i \text{ and } p_i = \frac{\hbar}{i} \frac{\partial}{\partial x_i}. \quad (2.4)$$

For simplicity, we could put eqs. (2.2) in a form:

$$X_i = x_i,$$
$$P_i = p_i f \left(\beta p^2\right),$$

where $f (x) = 1 + x$ for the Brau reduction.

We now consider a nonrelativistic quantum system with a Hamiltonian of the general form:

$$H = \frac{\vec{P} \cdot \vec{P}}{2m} + V (X).$$

(2.6)

The deformed time dependent Schrodinger equation is

$$\frac{\vec{P} \cdot \vec{P}}{2m} \psi (\vec{x}, t) + V (x) \psi (\vec{x}, t) = i\hbar \frac{\partial \psi (\vec{x}, t)}{\partial t},$$

(2.7)

where $\vec{P} = \vec{p} f \left(\beta p^2\right)$. Substituting the ansatz $\psi (x, t) = \exp \left[\frac{iS (x, t)}{\hbar}\right]$ into eq. (2.7) and taking the limit $\hbar \rightarrow 0$, one finds that the leading order of eq. (2.7) gives the classical Hamilton-Jacobi equation in deformed spaces:

$$\frac{1}{2m} \left(\vec{\nabla} S \cdot \vec{\nabla} S\right) f^2 \left(\beta \vec{\nabla} S \cdot \vec{\nabla} S\right) + V (\vec{x}) + \frac{\partial S}{\partial t} = 0,$$

(2.8)

where $S (\vec{x}, t)$ is the classical action, and $\vec{p} \psi = \vec{\nabla} S \psi$. If the potential $V (\vec{x})$ does not depend explicitly on time, we can separate the variables as

$$S = W (\vec{x}) - Et,$$

(2.9)

where $E$ can be identified with the total energy.
2.2 Motion in a central potential

When a particle is moving in a central potential $V(r)$, the Hamilton-Jacobi equation can be solved in the spherical coordinates. In the spherical coordinates, one has for $\vec{\nabla}$

$$\vec{\nabla} = \hat{r} \frac{\partial}{\partial r} + \hat{\theta} \frac{\partial}{r \partial \theta} + \hat{\phi} \frac{\partial}{r \sin \theta \partial \phi}. \quad (2.10)$$

Thus, we find

$$\vec{\nabla} S \cdot \vec{\nabla} S = \left( \frac{\partial S}{\partial r} \right)^2 + \frac{1}{r^2} \left( \frac{\partial S}{\partial \theta} \right)^2 + \frac{1}{r^2 \sin^2 \theta} \left( \frac{\partial S}{\partial \phi} \right)^2. \quad (2.11)$$

Since there are no explicit $t$- and $\phi$-dependence in the Hamilton-Jacobi equation, we assume that

$$S = S_1 (r) + S_2 (\theta) - Et + L_z \phi, \quad (2.12)$$

where $E$ and $L_z$ have the meaning of the energy and $z$-component of the orbital angular momentum, respectively. To separate the variable $\theta$ from $r$, one can introduce a constant $L$ and has the equation for $S_2 (\theta)$

$$\left( \frac{dS_2}{d\theta} \right)^2 + \frac{L_z^2}{\sin^2 \theta} = L^2, \quad (2.13)$$

where $L$ represents the orbital angular momentum. The equation for $S_1 (r)$ then becomes

$$\frac{1}{2m} \left( \left( \frac{dS_1}{dr} \right)^2 + \frac{L_z^2}{r^2} \right) f^2 \left( \beta \left( \left( \frac{dS_1}{dr} \right)^2 + \frac{L_z^2}{r^2} \right) \right) + V(r) = E. \quad (2.14)$$

For the function $f(x)$, one can solve the equation $y = xf^2(x)$ for $x$ in terms of $y$ and express the solution as $x = yg(y)$. For the Brau reduction, we have $g(x) = 1 - 2x + O(x^2)$. Solving eq. (2.14) gives

$$S_1 = \int dr \sqrt{2m \left[ E - V(r) \right] g \left( 2m \beta \left[ E - V(r) \right] \right)} - \frac{L_z^2}{r^2}. \quad (2.15)$$

We can choose the $z$-axis such that the motion of the particle is in the $x$-$y$ plane. Then $\sin \theta = 1$ and $L = L_z$. In this case, we use the inverse Legendre transform to find [25]

$$\phi = -\frac{\partial S_1}{\partial L} = \frac{1}{r} \int \frac{dr}{r \sqrt{2m \left[ E - V(r) \right] g \left( 2m \beta \left[ E - V(r) \right] \right)} - \frac{L_z^2}{r^2}}. \quad (2.16)$$

Considering the Kepler motion with $E < 0$, we have a specific form of the potential:

$$V(r) = -\frac{k}{r}, \quad (2.17)$$

where $k = GMm$. In this case, eq. (2.16) takes on the form:

$$\phi = \int \frac{d\tilde{r}}{\tilde{r}} \frac{1 - \varepsilon}{(\tilde{r} - \tilde{r}_{\text{min}})(\tilde{r}_{\text{max}} - \tilde{r})} + O(\varepsilon^2), \quad (2.18)$$

$$- 4 -$$
where
\[ \varepsilon = 2m\beta |E|, \quad A = \frac{k}{L} \sqrt{\frac{2m}{|E|}}, \quad \tilde{r} = \frac{2m|E|r}{L^2}, \]
and
\[ \tilde{r}_{\text{max}}/\text{min} = \frac{A + 4\varepsilon A \pm \sqrt{A^2 - 4 - 8\varepsilon}}{2 + 4\varepsilon}. \]
It follows from eq. (2.18) that \( \tilde{r} \) reached its minimum and maximum values \( \tilde{r}_{\text{min}} \) and \( \tilde{r}_{\text{max}} \) at perihelion and aphelion. Integrating eq. (2.18) leads to
\[ \phi (\tilde{r}) = \frac{(1 - \varepsilon)}{\sqrt{\tilde{r}_{\text{min}}\tilde{r}_{\text{max}}}} \arccos \left( \frac{2\tilde{r}_{\text{min}}\tilde{r}_{\text{max}} - (\tilde{r}_{\text{min}} + \tilde{r}_{\text{max}}) \tilde{r}}{\tilde{r}_{\text{max}} - \tilde{r}_{\text{min}}} \right) - \phi_0 + \mathcal{O} (\varepsilon^2), \]
In particular, one finds from eq. (2.21) that
\[ \phi (\tilde{r}_{\text{max}}) - \phi (\tilde{r}_{\text{min}}) = (1 - A^2\varepsilon) \pi + \mathcal{O} (\varepsilon^2). \]
It precesses by an angle of \(-2\pi A^2\varepsilon\) per revolution. For \( \varepsilon \ll 1 \), the precession angle is
\[ \Delta \omega_\beta = -2\pi A^2\varepsilon + \mathcal{O} (\varepsilon^2). \]
In the leading order, one has that
\[ e = \sqrt{1 - \frac{2|E|L^2}{mk^2}} \quad \text{and} \quad a = \frac{k}{2|E|}, \]
where \( a \) is the semi-major axis of the planet’s orbit, and \( e \) is its eccentricity. Thus, eq. (2.23) becomes
\[ \Delta \omega_\beta \approx -2\pi A^2\varepsilon + \mathcal{O} (\varepsilon^2), \]
which perfectly coincides with the result of [13] (eq. (66)) with \( \beta' = 2\beta \).

3 Hamilton-Jacobi method in relativity theory

In section 2, we consider the nonrelativistic case and obtain the deformed Hamilton-Jacobi equation (2.8) by using WKB approximation to find the leading order in \( \hbar \) of the deformed Schrodinger equation (2.7). In this section, we derive the deformed Hamilton-Jacobi equations in the relativistic case. To do so, we first find the deformed Klein-Gordon, Dirac, and Maxwell’s equations incorporating the minimal length commutation relations (2.1). After the deformed Hamilton-Jacobi equations are given, motions of massive and massless particles through the Schwarzschild metric are investigated. We also discuss effects of the minimal length on the experimental tests of general relativity, e.g. precession of planetary orbits, the bending of light, and time-delay in radar propagation.

3.1 Deformed Hamilton-Jacobi equation

When the deformed commutation relations (2.1) are considered, the deformed Klein-Gordon equation for a scalar particle of mass \( m \) has been suggested in [26, 27]:
\[ \left( \frac{\partial_i^2}{\hbar^2} + \frac{P_i^2}{\hbar^2} + \frac{m^2}{\hbar^2} \right) \Phi = 0, \]

\[ \]
where
\[ P_i = p_i (1 + \beta p^2) \equiv p_i f(\beta p^2), \quad p_i \Phi = \hbar \partial_i \Phi, \] (3.2)
and the index \( i \) runs over spatial coordinates. Expressing \( \Phi \) in terms of \( S(\vec{x}, t) \):
\[ \Phi(x, t) = \exp \left[ \frac{iS(\vec{x}, t)}{\hbar} \right], \] (3.3)
one finds the lowest order in \( \hbar \) gives the deformed Hamilton-Jacobi equation for a scalar particle:
\[ \left( \frac{\partial S}{\partial t} \right)^2 - \left( \vec{\nabla} S \cdot \vec{\nabla} S \right) f^2 \left( \beta \vec{\nabla} S \cdot \vec{\nabla} S \right) = m^2. \] (3.4)

Similarly, the deformed Dirac equation for a spin-1/2 fermion of mass \( m \) takes the form:
\[ \left( \gamma_0 \partial_t + \frac{\vec{\gamma} \cdot \vec{P}}{\hbar} - \frac{m}{\hbar} \right) \Psi = 0, \] (3.5)
where \( \gamma_0 \) and \( \vec{\gamma} \) are Gamma matrices. Multiplying \( \left( \gamma_0 \partial_t + \frac{\vec{\gamma} \cdot \vec{P}}{\hbar} + \frac{m}{\hbar} \right) \) by eq. (3.5) and using the gamma matrices anticommutation relations, the deformed Dirac equation can be written as
\[ \partial_t^2 \Psi = - \left( \frac{P^2}{\hbar^2} + \frac{m^2}{\hbar^2} \right) \Psi + \frac{[\gamma_i, \gamma_j]}{2} P_i P_j \Psi. \] (3.6)

To obtain the Hamilton-Jacobi equation for the fermion, the ansatz for \( \Psi \) takes the form:
\[ \Psi = \exp \left[ \frac{iS(\vec{x}, t)}{\hbar} \right] v, \] (3.7)
where \( v \) is a vector function of the spacetime. Substituting eq. (3.7) into eq. (3.6) and noting that the second term on r.h.s. of eq. (3.6) does not contribute to the lowest order in \( \hbar \), we thus find that the deformed Hamilton-Jacobi equation for a fermion is the same as that for a scalar, which is eq. (3.4).

The deformed Maxwell’s equations for a massless vector field \( A_\mu \) is
\[ \tilde{\partial}^\mu \tilde{F}_{\mu \nu} = 0, \] (3.8)
where \( \tilde{\partial}_t = \partial_t, \tilde{\partial}_i = f(-\beta \hbar^2 \partial_\mu) \partial_i, \) and \( \tilde{F}_{\mu \nu} = \tilde{\nabla}_\mu A_\nu - \tilde{\nabla}_\nu A_\mu. \) We then make the WKB ansatz:
\[ A_\mu = a_\mu \exp \left( \frac{iS(\vec{x}, t)}{\hbar} \right), \] (3.9)
where \( a_\mu \) is the polarization vector, and \( S(\vec{x}, t) \) is the action. Plugging the WKB ansatz into eq. (3.8), we find that leading order in \( \hbar \) gives
\[ a_\nu S^\nu S_\mu - a_\mu S^\nu S_\nu = 0, \] (3.10)
where
\[ S_t \equiv \frac{\partial S}{\partial t} \] and \( S_i \equiv f(\beta \tilde{\nabla} S \cdot \tilde{\nabla} S) \partial_i S. \) (3.11)

To simplify eq. (3.10), one could impose the Lorentz gauge:
\[ \tilde{\partial}^\mu A_\mu = 0, \] (3.12)
whose leading order is
\[ a_\nu S^\nu = 0. \] (3.13)

By plugging eq. (3.13) into eq. (3.10), it shows that the Hamilton-Jacobi equation for a massless vector boson is also given by eq. (3.4) with \( m = 0 \).

### 3.2 Motion in the Schwarzschild metric

We now generalize the deformed Hamilton-Jacobi equation (3.4) in flat spacetime to the Schwarzschild metric:

\[
ds^2 = h(r) \, dt^2 - \frac{dr^2}{h(r)} - r^2 \left( d\theta^2 + \sin^2\theta \, d\phi^2 \right),
\] (3.14)

where we have
\[ h(r) = 1 - \frac{2GM}{r}. \] (3.15)

In [24], the deformed Hamilton-Jacobi equations in the Schwarzschild metric for scalars and spin 1/2 fermions have been derived from the deformed Klein-Gordon and Dirac equations in the Schwarzschild metric. Here, we use an easier but less rigorous way to obtain the deformed Hamilton-Jacobi equations in curved spacetime. First consider the Hamilton-Jacobi equation without GUP modifications. In [28], we showed that the unmodified Hamilton-Jacobi equation in curved spacetime with the metric \( ds^2 = g_{\mu\nu} dx^\mu dx^\nu \) was
\[
g_{\mu\nu} \partial_\mu S \partial_\nu S - m^2 = 0.
\] (3.16)

Therefore, the unmodified Hamilton-Jacobi equation in the Schwarzschild metric becomes
\[
\frac{(\partial_t S)^2}{h(r)} - h(r) (\partial_r S)^2 - \frac{(\partial_\theta S)^2}{r^2} - \frac{(\partial_\phi S)^2}{r^2 \sin^2\theta} = m^2.
\] (3.17)

On the other hand, the unmodified Hamilton-Jacobi equation in flat spacetime can be obtained from eq. (3.4) by taking \( \beta = 0 \):
\[
(\partial_t S)^2 - (\partial_r S)^2 - \frac{(\partial_\theta S)^2}{r^2} - \frac{(\partial_\phi S)^2}{r^2 \sin^2\theta} = m^2.
\] (3.18)

Comparing eq. (3.17) to eq. (3.18), one finds that the Hamilton-Jacobi equation in the Schwarzschild metric can be obtained from that in flat spacetime by making replacements \( \partial_r S \to \sqrt{h(r)} \partial_r S \) and \( \partial_t S \to \frac{\partial_t S}{\sqrt{h(r)}} \). Likewise, by making replacements \( \partial_t S \to \sqrt{h(r)} \partial_t S \) and \( \partial_r S \to \frac{\partial_r S}{\sqrt{h(r)}} \), the deformed Hamilton-Jacobi equation in flat spacetime (3.4) leads to that in the Schwarzschild metric
\[
\frac{1}{h(r)} \left( \frac{\partial S}{\partial t} \right)^2 - X f^2 (\beta X) = m^2,
\] (3.19)

where we have
\[ X = h(r) (\partial_r S)^2 + \frac{(\partial_\theta S)^2}{r^2} + \frac{(\partial_\phi S)^2}{r^2 \sin^2\theta}. \] (3.20)

Since there are no explicit \( t \)- and \( \phi \)-dependence in the Schwarzschild metric, we assume that
\[ S = S_1(r) + S_2(\theta) - Et + L_z \phi, \] (3.21)
where $E$ and $L_z$ can be identified as the energy and $z$-component of the orbital angular momentum, respectively. Introducing a constant $L$ representing the orbital angular momentum, one finds that $S_2(\theta)$ satisfies
\[
\left(\frac{dS_2}{d\theta}\right)^2 + \frac{L_z^2}{\sin^2 \theta} = L^2.
\] (3.22)

The equation for $S_1(r)$ is
\[
\mathcal{X} f^2(\beta \mathcal{X}) = \frac{E^2}{h(r)} - m^2,
\] (3.23)
where we have
\[
\mathcal{X} = h(r) \left(\frac{dS_1}{dr}\right)^2 + \frac{L^2}{r^2}.
\] (3.24)

Solving eq. (3.23) gives
\[
S_1 = \int dr \sqrt{\frac{1}{h(r)} \left(\frac{E^2}{h(r)} - m^2\right) g\left(\beta \left(\frac{E^2}{h(r)} - m^2\right)\right) - \frac{L^2}{h(r) r^2}},
\] (3.25)
where $g(x) = 1 - 2x + O(x^2)$ for the Brau reduction. Because of the spherical symmetry of the Schwarzschild metric we can therefore, with no loss of generality, confine our attention to particles moving in the equatorial plane given by $\theta = \frac{\pi}{2}$. In this case, one has that $\sin \theta = 1$, $L = L_z$, and the trajectory is
\[
\phi = -\frac{\partial S_1}{\partial L} = L \int dr \frac{dr}{r^2 \sqrt{[E^2 - h(r) m^2] g\left(\beta \left(\frac{E^2}{h(r)} - m^2\right)\right) - \frac{h(r)L^2}{r^2}}}.
\] (3.26)

The time-dependence of the motion is then obtained by the inverse Legendre transformation:
\[
t = \frac{\partial S_1}{\partial E}.
\] (3.27)

### 3.2.1 Precession of planetary orbits

For massive particles, differentiate eq. (3.26) with respect to $r$ gives
\[
\left(\frac{Adu}{d\phi}\right)^2 = \left[\tilde{E}^2 - (1 + \varepsilon) (1 - 2uA^2)\right] - (1 - 2uA^2) A^2 u^2 - \varepsilon \frac{\tilde{E}^4}{(1 - 2uA^2)} + 2\varepsilon \tilde{E}^2 + O(\varepsilon^2),
\] (3.28)
where we have
\[
A = \frac{GMm}{L}, \quad u = \frac{L^2}{GMm^2 r}, \quad \tilde{E} = \frac{E}{m}, \quad \text{and} \quad \varepsilon = 2\beta m^2.
\] (3.29)

One then differentiates eq. (3.28) with respect to $\phi$ and obtains a second-order equation for $u(\phi)$:
\[
\frac{d^2u}{d\phi^2} = 1 + \varepsilon \left(1 - \tilde{E}^4\right) - u - 4\varepsilon \tilde{E}^4 A^2 u + 3A^2 u^2 + O\left(A^4, \varepsilon^2\right).
\] (3.30)

For $A, \varepsilon \ll 1$, we put $u$ in a form of a Taylor series in terms of $\varepsilon$ and $A^2$:
\[
u = u_0 + A^2 x + \varepsilon y + A^2 \varepsilon z + O\left(A^4, \varepsilon^2\right),
\] (3.31)
where \( u_0 \) is a Newtonian solution while \( A^2 x \) is a small deviation due to general relativity, and \( \varepsilon y \) and \( A^2 \varepsilon z \) due to quantum gravity. Plugging eq. (3.31) into eq. (3.30), we obtain

\[
\begin{align*}
\frac{d^2 u_0}{d\phi^2} + u_0 &= 1, \\
\frac{d^2 x}{d\phi^2} + x &= 3u_0^2, \\
\frac{d^2 y}{d\phi^2} + y &= (1 - \tilde{E}^4), \\
\frac{d^2 z}{d\phi^2} + z &= -4\tilde{E}^4u_0 + 6\varepsilon u_0.
\end{align*}
\] (3.32)

For a bound orbit of a planet, the first equation in eqs. (3.32) has the solution:

\[
u_0 = 1 + e \cos \phi,
\] (3.33)

which describes an ellipse with the eccentricity \( e \). It follows from eq. (3.33) that the rest equations of eqs. (3.32) give

\[
x = 3 \left( 1 + \frac{1}{2} e^2 \right) + 3e\phi \sin \phi - \frac{1}{2} e^2 \cos 2\phi, \\
y = \left( 1 - \tilde{E}^4 \right), \\
z = \left( 6 - 10\tilde{E}^4 \right) + \left( 3 - 5\tilde{E}^4 \right) e\phi \sin \phi.
\] (3.34)

The first terms in expressions of \( x \), \( y \), and \( z \) in eqs. (3.34) are constant displacement, while the last ones in expressions of \( x \) and \( z \) oscillate around zero. However, the terms with \( \phi \sin \phi \) describe effects which accumulate over successive orbits. Combing these terms with \( u_0 \), we have

\[
u = 1 + e \cos \left[ (1 - \alpha) \phi \right],
\] (3.35)

where

\[
\alpha = 3A^2 \left[ 1 + \varepsilon \left( 1 - \frac{5\tilde{E}^4}{3} \right) \right] + \mathcal{O} \left( A^4, \varepsilon^2 \right).
\] (3.36)

We find, during each orbit of the planet, perihelion advances by an angle:

\[
\Delta \omega = 2\pi \alpha,
\] (3.37)

and the contribution from the minimal length is

\[
\Delta \omega_{\beta} = 2\pi \left( 3A^2 \right) \varepsilon \left( 1 - \frac{5\tilde{E}^4}{3} \right) + \mathcal{O} \left( \varepsilon^2 \right).
\] (3.38)

Since \( E \) is the energy of a planet including its rest energy, one has \( \tilde{E}^2 = 1 + \mathcal{O} \left( A^2 \right) \) and hence

\[
\Delta \omega_{\beta} \approx -2\pi \frac{4\beta GMm^2}{a \left( 1 - e^2 \right)},
\] (3.39)

where we use \( L^2 = GMm^2 \left( 1 - e^2 \right) a \). It follows from eq. (3.39) that \( \Delta \omega_{\beta} \) obtained in the context of general relativity is the same as \( \Delta \omega_{\beta} \) in eq. (2.25) and [13] with \( \beta' = 2\beta \), which have been computed in the context of Newtonian dynamics. This coincidence is not surprising, since perturbative methods have been used to expand solutions \( u \) around the Newtonian solution \( u_0 \) in both [13] and our paper.
3.2.2 Deflection of light

Since a massive object can have a significant effect on the propagation of photons, we can test the predictions of general relativity by investigating the slight deflection of light by, for example, the Sun.

For massless particles, eq. (3.26) leads to a second-order equation for $u(\phi)$:

$$\frac{d^2u}{d\phi^2} = -\varepsilon \left(1 + 4A^2u\right) - u + 3A^2u^2,$$

(3.40)

where

$$u = \frac{L^2}{GM E^2 r}, \quad \varepsilon = 2\beta E^2, \quad \text{and} \quad A = \frac{G M E}{L}.$$  

(3.41)

Plugging the ansatz (3.31) into eq. (3.40), we obtain

$$\frac{d^2u_0}{d\phi^2} + u_0 = 0,$$

$$\frac{d^2x}{d\phi^2} + x = 3u_0^2,$$

$$\frac{d^2y}{d\phi^2} + y = -1,$$

$$\frac{d^2z}{d\phi^2} + z = -4u_0 + 6yu_0.$$  

(3.42)

In the absence of matter, we may write the solution for $u_0$ as

$$u_0 = B \sin \phi,$$  

(3.43)

which represents a straight-line path with impact parameter $b = \frac{L^2}{G M E^2 r}$. Solving eqs. (3.32) for $x$, $y$, and $z$ gives

$$x = \frac{3B^2}{2} \left(1 + \frac{1}{3} \cos 2\phi \right),$$

$$y = -1,$$

$$z = 5B \phi \cos \phi.$$  

(3.44)

Combining eq. (3.43) with eqs. (3.44), we find

$$u = B \sin [(1 - \alpha) \phi] + \frac{3A^2B^2}{2} \left(1 + \frac{1}{3} \cos [2(1 - \alpha) \phi] \right) - \varepsilon + \mathcal{O} \left(A^4, \varepsilon^2 \right),$$

(3.45)

where

$$\alpha = -5\varepsilon A^2.$$  

(3.46)

Now consider the limit $r \to \infty$, i.e. $u \to 0$. Clearly, for a slight deflection we can take $\sin \phi \approx \phi$ and $\cos \phi \approx 1$ at infinity, to obtain

$$\phi = -2A^2B \left(1 - \frac{\varepsilon}{2A^2B^2} - 5\varepsilon A^2 \right) + \mathcal{O} \left(A^4, \varepsilon^2 \right).$$  

(3.47)

Thus the total deflection is

$$\Delta \phi = \frac{4GM}{b} \left(1 - \frac{\varepsilon}{2} - 5\varepsilon A^2 \right) + \mathcal{O} \left(A^4, \varepsilon^2 \right),$$

(3.48)

where we use $L = Eb$ for the leading order.
3.2.3 Time-delay in radar propagation

Now consider the trajectory of a photon from the observer to the test object. Obviously, the trajectory will be deflected when the photon passes through the gravitational field of a massive object of mass $M$. The time taken to travel between the observer and test object can be given by eq. (3.27) with $m = 0$. Let $r_0$ be the coordinate distance of closest approach of the photon to the massive object. Thus, we have

$$\left( \frac{dr}{dt} \right)_{r_0} = 0.$$  

It then follows from eq. (3.27) that

$$\frac{E^2}{L^2} = \frac{h(r_0)}{r_0^2 g \left( \frac{\beta E^2}{\hbar} \right)}.$$  

Using eqs. (3.27) and (3.50), we find that the time taken to travel between points $r_0$ and $r$ is

$$t(r, r_0) = \int_{r_0}^r dr \frac{g \left( \frac{\varepsilon}{\hbar(r)} \right) + g' \left( \frac{\varepsilon}{\hbar(r)} \right)}{r \sqrt{h(r_0) g \left( \frac{\varepsilon}{\hbar(r_0)} \right) - h(r) r_0^2 g \left( \frac{\varepsilon}{\hbar(r_0)} \right)}} r \sqrt{h(r_0)}.$$  

where $\varepsilon = 2 \beta E^2$. Integrating this leads to

$$t(r, r_0) = \left( 1 - \frac{3 \varepsilon}{2} \right) \sqrt{r^2 - r_0^2} + GM \sqrt{\frac{r - r_0}{r + r_0}} \left( 1 - \frac{5 \varepsilon}{2} \right)$$

$$+ 2 \left( 1 - 3 \varepsilon \right) GM \ln \left( \frac{r + \sqrt{r^2 - r_0^2}}{r_0} \right) + O \left( \varepsilon^2, G^2 M^2 \right).$$  

Note that eq. (3.4) gives the energy-momentum dispersion relation of a photon in flat spacetime:

$$E \approx p \left( 1 + \beta p^2 \right),$$  

where we use $E = \frac{\partial S}{\partial t}$, $p_i = \frac{\partial S}{\partial x_i}$, and $p^2 = p_i p_i$. Thus, after effects of the minimal length are considered, the light speed in flat spacetime becomes

$$\frac{\partial E}{\partial p} \approx 1 + 3 \beta p^2 \approx 1 + \frac{3 \varepsilon}{2},$$  

which gives that the first term in eq. (3.52) is just what we would have expected if the light had been travelling in flat spacetime along a straight line. The second and third terms give us the extra time taken for the photon to travel along the curved path. If a radar beam is sent to the test object and bounces back to the observer, the excess time delay over a straight-line path is

$$\Delta t = 2 \left[ t(r_1, r_0) + t(r_2, r_0) - t(r_1, r_0) \big| G = 0 - t(r_2, r_0) \big| G = 0 \right],$$  

where $r_1$ and $r_2$ (both assumed $\gg r_0$) are the distances of the observer and test object from the massive object, respectively. Thus one obtains

$$t(r_1, r_0) - t(r_1, r_0) \big| G = 0 = GM \left( 1 - \frac{5 \varepsilon}{2} \right) + 2 \left( 1 - 3 \varepsilon \right) GM \ln \left( \frac{r_1}{r_0} \right),$$  

and likewise for $t(r_2, r_0)$. In this case, the time delay becomes

$$\Delta t \approx 4GM \left[ \left( 1 - \frac{5 \varepsilon}{2} \right) + (1 - 3 \varepsilon) \ln \left( \frac{r_1 r_2}{r_0^2} \right) \right].$$  

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4 Comparison with experiments

To make comparison with experiments, we often express the parameter $\beta$ in terms of a dimensionless parameter $\beta_0$:

$$\beta = \beta_0 \ell_p^2 / \hbar^2 = \beta_0 / m_{\text{pl}}^2, \quad (4.1)$$

where $m_{\text{pl}}$ is the Planck mass, and $\ell_p$ is the Planck length. For the Brau reduction, the minimal length associated with $\beta$ is $\Delta X_{\text{min}} = \hbar \sqrt{3 \beta} = \sqrt{3} \beta_0 \ell_p$. Naturally, if the minimal length is assumed to be order of the Planck length $\ell_p$, one has $\beta_0 \approx 1$. In [29], based on discussions of effects of the minimal length on the tunnelling current in a Scanning Tunnelling Microscope, an upper bound of $\beta_0$ was given by $\beta_0 < 10^{21}$.

We calculate the precession angle of a planet caused by deformations in context of Newtonian dynamics in section 2 and general relativity in section 3. Both of our results (2.25) and (3.39) perfectly coincide with the result of [13] (eq. (66)) with $\beta' = 2\beta$. In [13], the authors compared their result to the observed precession of the perihelion of Mercury and estimated an upper bound on $\beta$:

$$\hbar \sqrt{\beta_M} < 2.3 \times 10^{-68} \text{ m} \sim 10^{-33} \ell_p, \quad (4.2)$$

where the subscript $M$ of $\beta$ means that $\beta$ is for Mercury. It is quite surprising to note that this minimal length is 33 orders of magnitude below the Planck length. However, as pointed out in [30], this strangely small result stemmed from the assumption made in [13] that the deformation parameter $\beta$ for Mercury was the same as for elementary particles. It also was shown in [30] that if Mercury consists of $N$ quarks, the deformation parameter $\beta_M$ was substantially reduced by a factor $N^{-2}$:

$$\beta_M = \frac{\beta_q}{N^2}, \quad (4.3)$$

where $\beta_q$ is $\beta$ for quarks. Since $N \sim 10^{50}$, the upper bounds on the deformation parameter $\beta$ for quarks was given by

$$\hbar \sqrt{\beta_q} < 1.4 \times 10^{-17} \text{ m} \sim 10^{18} \ell_p \text{ and } \beta_q^2 < 10^{36}. \quad (4.4)$$

For the observational tests of general relativity involving null geodesics, we calculate the spatial deflection of star light by the Sun in section 3. From eq. (3.48), it follows that the deflection angle of a photon’s trajectory caused by deformations is

$$\Delta \phi_\beta \equiv \Delta \phi - \Delta \phi_0 \approx -\Delta \phi_0 \beta_p E^2, \quad (4.5)$$

where the third term in the bracket of eq. (3.48) is neglected since $A \ll 1$, $\Delta \phi_0 \equiv \frac{4GM}{b}$ is the deflection angle calculated in context of general relativity without deformation, $E$ is the energy of photons, and $\beta_p$ is the deformation parameter $\beta$ for photons. For light grazing the Sun it yields

$$\Delta \phi_0 = 1.75'' \quad \text{(4.6)}$$

Since the 1919 eclipse expedition led by Eddington to measure the deflection angle, several similar experiments were conducted. The Texas expedition to Chinguetti Oasis, Mauritania, at the eclipse of 30 June 1973 gave [31]

$$\Delta \phi_{\text{obs}} = (0.95 \pm 0.11) \Delta \phi_0, \quad (4.7)$$
where the error was 1σ. Comparison of eqs. (4.6) and (4.7) places a lower bound on $\Delta \phi_\beta$:

$$\Delta \phi_{\text{obs}} - \Delta \phi_0 = (-0.05 \pm 0.11) \Delta \phi_0 < \Delta \phi_\beta. \tag{4.8}$$

At 3σ eq. (4.8) gives

$$\beta_0^p < 10^{55}, \tag{4.9}$$

where $E = \frac{2\pi \hbar c}{\lambda}$, and we assume that $\lambda \sim 500$ nm for visible light. On the other hand, the tightest observational constraint to date on $\Delta \phi$ comes from observations of 87 VLBI sites and 541 radio sources over a period of 20 years. The typical frequencies of radio sources are around 10 GHz [32]. The result of this is [33]

$$\Delta \phi_{\text{obs}} = (0.99992 \pm 0.00023) \Delta \phi_0, \tag{4.10}$$

which is around 3 orders of magnitude better than the observations of eclipse expeditions. Similarly, one has at 3σ that

$$\beta_0^p < 10^{62}, \tag{4.11}$$

which is much less stringent than eq. (4.9) since the typical energies of radio sources [33] are much less than these of visible light observed in eclipse expeditions.

A currently more constraining test of general relativity using null trajectories involves the Shapiro time-delay effect which has been studied in section 3. From eq. (3.57), it follows that time delay over a straight-line path caused by deformations is

$$\Delta t_\beta \equiv \Delta t - \Delta t_0 = -\frac{\varepsilon \Delta t_0}{2} \frac{5 + 6 \ln \left(\frac{r_1 r_2}{r_0^2}\right)}{1 + \ln \left(\frac{r_1 r_2}{r_0^2}\right)} \approx -3 \varepsilon \Delta t_0, \tag{4.12}$$

where we use $r_1, r_2 \gg r_0$, and $\Delta t_0 = 4GM \left[1 + \ln \left(\frac{r_1 r_2}{r_0^2}\right)\right]$ is what we expect in the context of general relativity without deformation. In the gravitational field of the sun, the best constraint on this time-delay effect is obtained by using radio links with the Cassini spacecraft between the 6th of June and the 7th of July 2002 [34]. These observations result in the constraint

$$\Delta t_{\text{obs}} = (1.00001 \pm 0.00001) \Delta t_0. \tag{4.13}$$

The typical frequencies of radio photons transmitted from the ground to the Cassini spacecraft are around 10 MHz. At 3σ one has an upper bound on $\beta_0^p$:

$$\beta_0^p < 10^{66}. \tag{4.14}$$

5 Discussion and conclusion

In this paper, we have used the Hamilton-Jacobi method to investigate effects of the minimal length on the classical orbits of particles in a gravitation field. Specifically, we derived the deformed Hamilton-Jacobi equation and used it to study the deformed precession of planetary orbits in the context of Newtonian dynamics. In the context of general relativity, the deformed Hamilton-Jacobi equation in the Schwarzschild metric has also been obtained to calculate the precession angle of planetary orbits, deflection angle of light, and time delay in radar propagation. Comparison with the observational results places constraints on the deformation parameter $\beta_0$. 

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In [13], the precession of planetary orbits has also been studied in the classical limit of deformed spaces using the deformed Poisson bracket. The precession angle caused by deformations was calculated in the context of Newtonian dynamics. Our calculations confirm their results not only in the context of Newtonian dynamics but also in the context of general relativity, at least to the leading order in $\beta$.

Although incorporating the GUP into quantum mechanics is rather straightforward, incorporating the GUP into QFT is ambiguous. For example, it is not clear how to define the position operator in QFT. On the other hand, we found the Hamilton-Jacobi equations for a scalar, a spin-1/2 fermion, and a massless vector boson could be reduced to the same equation (3.4) in flat space-time. Moreover, eq. (3.4) seems to be the relativistic version of the Newtonian Hamilton-Jacobi equation (2.8), which supports our choice of the assumed deformed field equations (3.1), (3.5), and (3.8). The fact that the “general relativistic” result (3.39) reproduces the Newtonian correction obtained in [13] also lends support to these assumed deformed field equations.

The equivalence principle is one of the main cornerstones of general relativity as well as other metric theories of gravity [36]. It states that trajectory of any freely falling uncharged test body is independent of its internal composition or structure. In the metric theory of gravity, it is postulated that the trajectories of freely falling test bodies are geodesics of that metric. In our paper, we considered the GUP deformed geodesics of planets and photons. Specifically, we calculated the precession angle $\Delta \omega_{\beta}$ of a planet and deflection angle $\Delta \phi_{\beta}$ of a photon’s trajectory caused by deformations. According to eqs. (3.39) and (4.5), it appears that $\Delta \omega_{\beta}$ and $\Delta \phi_{\beta}$ depend on $\beta m^2$ and $\beta E^2$, respectively, which could lead to violations of the equivalence principle. These violations follow from the GUP deformed Hamilton-Jacobi equations. In fact, the GUP effects on the equivalence principle have been considered in [16, 21, 37, 38]. In [21], we used the deformed Hamilton-Jacobi equation to discuss the GUP effects on the Eotvos ratio [36]:

$$\eta(A, B) = \frac{2|a_A - a_B|}{|a_A + a_B|},$$

which quantifies the normalized difference in the gravitational accelerations between two different bodies $A$ and $B$. A non-zero Eotvos ratio $\eta(A, B)$ indicates violations of the equivalence principle. For the deformed fundamental commutation relation (1.1), we found

$$\eta(A, B) \sim |\beta_A m_A v_A^2 - \beta_B m_B v_B^2|,$$

which was usually non-zero. For the modern torsion-balance experiment which measures the gravitational accelerations of Beryllium and Titanium towards the Earth [39], we obtained

$$\eta(A, B) \sim 10^{-62} \beta_0^2 \lesssim (0.3 \pm 1.8) \times 10^{-13},$$

which gave $\beta_0^2 \lesssim 10^{49}$.

In [19], the authors introduced the deformed Schwarzschild metric to reproduce the Hawking temperature derived from the deformed fundamental commutation relation (1.1). Using this deformed metric, they computed corrections to the standard general Relativistic predictions for the light deflection and perihelion precession. Specifically, the deformed Schwarzschild metric takes the form:

$$ds^2 = F(r) dt^2 - F^{-1}(r) dr^2 - r^2 d\Omega^2,$$
where \( F(r) = 1 - \frac{R_H}{r} + \tilde{\varepsilon} \frac{R_H^n}{r^{n+1}} \), \( R_H = 2GM \), and the case with \( n = 2 \) was considered in [19]. Following calculations in [19], one finds that the horizons of the metric (5.4) is

\[
r_H = R_H \left( 1 - 2^{-n} \tilde{\varepsilon} \right) + \mathcal{O}(\tilde{\varepsilon}^2),
\]

and the Hawking temperature is

\[
T(\varepsilon) = \frac{\hbar}{4\pi R_H} \left[ 1 + 2^{-n} (2 - n) \tilde{\varepsilon} \right] + \mathcal{O}(\tilde{\varepsilon}^2).
\]  

(5.5)

Using eq. (2.31) in [19], one could relate \( \tilde{\varepsilon} \) to \( \beta_0 \) (which is \( \beta \) in [19]) in the cases with \( n \neq 2 \):

\[
\tilde{\varepsilon} \approx \frac{2^{n-4} \beta_0 \hbar}{\pi^2 GM^2 (2 - n)}.
\]  

(5.6)

For the case \( n = 2 \), it was shown in [19] that

\[
\tilde{\varepsilon}^2 \approx -\frac{\hbar \beta_0}{\pi^2 GM^2},
\]

(5.7)

which required that \( \beta_0 < 0 \).

By contrast, there are a number of differences between the methods used in our paper and in [19], which are as follows:

1. The authors of [19] calculated the precession angle of Mercury’s orbits in the deformed Schwarzschild metric (5.4) with \( n = 2 \) and found the correction was

\[
\frac{\Delta \phi - \Delta \phi_0}{\Delta \phi_0} \sim \tilde{\varepsilon},
\]  

(5.8)

which only depended on the mass of the Sun. It is naturally expected that eq. (5.8) is also true for the cases with \( n \neq 2 \). According to eqs. (5.6) and (5.7), the correction (5.8) is proportional to \( \sqrt{\beta_0} \) in the \( n = 2 \) case while it is proportional to \( \beta_0 \) in the \( n \neq 2 \) cases. On the other hand, we find that results in our paper and [13] are only proportional to \( \beta_0 \). It seems that results obtained using the method proposed in [19] may depend on the ansatz form of \( F(r) \). It also follows from eq. (5.6) that one does not need to require \( \beta_0 < 0 \) in the \( n \neq 2 \) cases.

2. Moreover, the results in our paper (see eq. (3.38)) and [13] show that

\[
\frac{\Delta \phi - \Delta \phi_0}{\Delta \phi_0} \sim \varepsilon = 2\beta m^2 \sim \beta E^2,
\]

(5.9)

where we use \( E \approx m \) for Mercury. The correction (5.9) depends on the energy \( E \) of Mercury while the correction (5.8) obtained in [19] does not. How can we reconcile this contradiction? One might note that the authors of [19] used eq. (2.20) from [19]: \( E = T \), to express the deformed Hawking temperature \( T \) in terms of the mass of the Schwarzschild black holes. If one substitutes

\[
E = T \sim \frac{\hbar}{8\pi GM},
\]  

(5.10)

into eq. (5.9), we find

\[
\varepsilon \sim \frac{\beta_0 \hbar}{16\pi^2 GM^2},
\]  

(5.11)
where we use $\frac{\hbar}{G} = 4m_{\text{pl}}^2$. It follows that eq. (5.11) is the same as eqs. (5.6) and (5.7) up to some numerical factors. In other words, there is implicit assumption made in [19] that the energy $E$ of Mercury was given by $E = T_s$, where $T_s$ was the Hawking temperature of the Schwarzschild black hole of 1 solar mass. Since $T_s$ is far less than the mass of Mercury, one could expect that the energy $E$ of Mercury was given by $E = T_s$, where $T_s$ was the Hawking temperature of the Schwarzschild black hole of 1 solar mass. Since $T_s$ is far less than the mass of Mercury, one could expect that the energy $E$ of Mercury in a more appropriate way using the method proposed in [19], one might need to resort to Gravity’s rainbow [35], where the minimal length deformations to the Schwarzschild black hole could depend on the energy of Mercury. This is expected since GUP is closely related to Doubly Special Relativity and Gravity’s rainbow [40]. Another way to understand this is to note that the deformed Hawking temperatures obtained using the Hamilton-Jacobi method [22–24] do depend on the energy of radiated particles. In this case, one possible way to find the deformations to the rainbow metric is using the deformed Hawking temperatures obtained in [22–24] instead.

Finally, we used the observational results to places constraints on $\beta_0$ in section 4. Comparing with constraints on $\beta_0$ from other papers, our results are much less stringent. In other words, it is difficult to observe quantum gravity effects on the deformed classical motions of particles. One of reasons for these difficulties is that the energy of the particles in classical motions is too small compared to the Planck mass $m_{\text{pl}}$. Typically, the corrections due to the minimal length is around $\beta_0 E^2/m_{\text{pl}}^2$. For photons, one has $E \sim 1$ eV for visible light and $E \sim 10^{-4}$ eV for radio of frequency 10 GHz. This also explains why the observations of eclipse expeditions put a stronger constraint on $\beta_0$ even though observations of VLBI have 3 orders of magnitude better precision. For nonrelativistic massive particles of mass $m$ in a weak gravitational field, one has $E \sim m$. Thus, the minimal length correction to the precession angle of planetary orbits is around $\frac{\beta_0 m_{\text{pl}}^2}{3^2 m_{\text{pl}}^4}$, where $m_{\text{pl}} \sim 1$ GeV is the mass of a nucleon. It follows that the observations of the precession of Mercury would place the strongest constraint on $\beta_0$ in our paper.

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