ZIMMER’S CONJECTURE: SUBEXPONENTIAL GROWTH, MEASURE RIGIDITY, AND STRONG PROPERTY (T)

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ABSTRACT. We prove several cases of Zimmer’s conjecture for actions of higher-rank, cocompact lattices on low-dimensional manifolds. For example, if \( \Gamma \) is a cocompact lattice in \( \text{SL}(n, \mathbb{R}) \), \( M \) is a compact manifold, and \( \omega \) a volume form on \( M \) we show that any homomorphism \( \alpha : \Gamma \to \text{Diff}(M) \) has finite image if the dimension of \( M \) is less than \( n-1 \) and that any homomorphism \( \alpha : \Gamma \to \text{Diff}(M, \omega) \) has finite image if the dimension of \( M \) is less than \( n \). The key step in the proof is to show that any such action has uniform subexponential growth of derivatives. This is established using ideas from the smooth ergodic theory of higher-rank abelian groups, structure theory of semisimple groups, and results from homogeneous dynamics. Having established uniform subexponential growth of derivatives, we apply Lafforgue’s strong property (T) to establish the existence of an invariant Riemannian metric.

1. INTRODUCTION

1.1. Results, history, and motivation. As a special case of our main result, Theorem 2.7 below, we confirm of Zimmer’s conjecture for actions of cocompact lattices in \( \text{SL}(n, \mathbb{R}) \).

Theorem 1.1. For \( n \geq 3 \), let \( \Gamma < \text{SL}(n, \mathbb{R}) \) be a cocompact lattice. Let \( M \) be a compact manifold. If \( \dim(M) < n-1 \) then any homomorphism \( \Gamma \to \text{Diff}^2(M) \) has finite image. In addition, if \( \omega \) is a volume form on \( M \) and \( \dim(M) = n-1 \) then any homomorphism \( \Gamma \to \text{Diff}^2(M, \omega) \) has finite image.

The key step in the proof is to establish that the derivatives of group elements for such an action grow subexponentially relative to the word length. This is inspired by the third author’s paper on the Burnside problem for diffeomorphism groups [Hur]. To prove subexponential growth of derivatives in this context, we study the induced \( G \)-action on a suspension space and apply a number of measure rigidity results including Ratner’s theorem and recent work of the first author with Rodriguez Hertz and Wang. Having established subexponential growth of derivatives, the main theorem is established by using the strong Banach property (T) of Lafforgue to find an invariant Riemannian metric. The proof has many surprising features, including its use of hyperbolic dynamics to prove an essentially elliptic result and its use of results from homogeneous dynamics to prove results about non-linear actions. We include a detailed sketch of the proof at the end of the introduction.

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The result stated Theorem 1.1 lies in the context of the Zimmer Program. In [Zim2] Zimmer made a number of conjectures concerning smooth volume-preserving actions of lattices in higher-rank semisimple groups on low-dimensional manifolds. These conjectures were clarified in [Zim4, Zim5] and extended to the case of smooth non-volume-preserving actions by Farb and Shalen in [FS].

The Zimmer program is motivated by earlier results on rigidity of linear representations of lattices in higher-rank Lie groups. The history of the subject begins in the early 1960s with results of Selberg and Weil which established that cocompact lattices in simple Lie groups other than $\text{PSL}(2, \mathbb{R})$ were locally rigid: any perturbation of a lattice is given by conjugation by a small group element [Sel, Wei1]. In the late 60s and early 70s, this was improved by Mostow to a global rigidity theorem showing that any isomorphism between cocompact lattices in the same class of groups extended to an isomorphism of the ambient Lie group [Mos]. The global rigidity result was extended by Margulis and Prasad to include non-uniform lattices [Mar1, Pra]. These developments led to Margulis’ work on superrigidity and arithmeticity in which Margulis classified all linear representations of lattices in Lie groups of higher real-rank [Mar2] and established that all such lattices are arithmetic.

Inspired by Margulis’ superrigidity theorem, in the early 1980s Zimmer proved a superrigidity theorem for cocycles from which he to proved results about orbit equivalence of higher-rank group actions [Zim1]. Motivated by earlier results in the rigidity of linear representations and the cocycle superrigidity theorem, Zimmer proposed studying non-linear representations of lattices in higher-rank simple Lie groups. That is, given a lattice $\Gamma \subset G$, rather than studying linear representations $\rho: \Gamma \to \text{GL}(d, \mathbb{R})$, Zimmer proposed studying representations $\alpha: \Gamma \to \text{Diff}(M)$ where $M$ is a compact manifold. The main objective of the Zimmer program is to show that all such non-linear representations $\alpha$ are of an “algebraic origin.” In particular, the Zimmer conjecture states that if the dimension of $M$ is sufficiently small (relative to data associated to $G$) then any action $\alpha: \Gamma \to \text{Diff}(M)$ should preserve a smooth Riemannian metric or factor through the action of a finite group. See Conjecture 1.2 for a precise formulation.

In this paper we establish the non-volume-preserving case of Zimmer’s conjecture for actions of cocompact lattices in higher-rank split simple Lie groups as well as certain volume-preserving cases. While there have been a number of sharp results for actions on extremely low-dimensional manifolds (for manifolds of dimension 1 or 2) or under strong regularity conditions on the action or algebraic conditions on the lattice, prior to this paper the exact result conjectured by Zimmer was only known for non-uniform lattices in $\text{SL}(3, \mathbb{R})$. Our results provide a class of higher-rank Lie groups and a large collection of lattices such that the critical dimension is as conjectured in the non-volume-preserving and either as conjectured or almost as conjectured in the volume-preserving case. In addition to establishing the conjecture for cocompact lattices in split simple Lie groups, we also give strong partial results for actions of cocompact lattices in non-split simple Lie groups.

In the case of volume-preserving actions, the conjecture is motivated by the following corollary of Zimmer’s cocycle superrigidity theorem: all volume-preserving actions in sufficiently low dimensions preserve a measurable Riemannian metric [Zim1]. From this point of view, the main step in proving the conjecture is to promote a measurable metric to a smooth metric. Conditional and partial results verifying the existence of a smooth invariant metric in the volume-preserving case are contained in many papers of Zimmer of which [Zim5] provides an excellent overview.
Perhaps the best evidence for the conjecture in the case of volume-preserving actions is
Zimmer’s result that all actions satisfying the conjecture have discrete spectrum [Zim6]. In
the non-volume-preserving case, evidence for the conjecture follows from the work of Farb
and Shalen on analytic actions and work of Nevo and Zimmer that produces measurable
projective quotients for actions which do not preserve a measure [FS, NZ].

Other strong evidence for the conjectures is provided by a plethora of results concerning
actions on compact manifolds of dimension 1 or 2. The earliest results were those of Witte
Morris proving that all $C^0$ actions on $S^1$ of $\text{SL}(n, \mathbb{Z})$ and $\text{Sp}(2n, \mathbb{Z})$ and their finite-index
subgroups factor through finite groups [Wit]. Later results of Burger and Monod and of
Ghys show similar results for $C^1$ actions of all lattices in higher-rank simple Lie groups
[BM, Ghy]. Ghys’ result also includes results for irreducible lattices in products of rank-1
groups, which admit infinite actions on the circle. In dimension 2, results of Polterovich
and of Franks and Handel show that all volume-preserving actions of non-uniform lattices
on surfaces are also all finite [FH, Pol]. Moreover, Franks and Handel showed that for
any surface of genus at least 1, any action by a non-uniform lattice in a higher-rank simple
Lie group which preserves a Borel probability measure is finite. Some earlier results on
actions on surfaces, such as those of Farb and Shalen in the analytic category, do not
require an invariant measure but instead make stronger assumptions on the acting group
and the regularity of the action. Combined with results of [FH] and [BRHW3], we resolve
the conjecture almost completely for $C^2$-actions on surfaces of genus at least 1 in Theorem
1.5. Above dimension 2, very little is known. See the second author’s survey of the Zimmer
program [Fis] for a detailed history as well as earlier surveys by Feres and Katok, Labourie,
and Witte Morris and Zimmer [FK, Lab, ZM].

We recall the full conjecture of Zimmer as extended by Farb and Shalen. Given a
semisimple Lie group $G$, let $n(G)$ denote the minimal dimension of a non-trivial real rep-
presentation of the Lie algebra $\mathfrak{g}$ of $G$ and let $v(G)$ denote the minimal codimension of a
maximal (proper) parabolic subgroup $Q$ of $G$. Let $d(G)$ denote the minimal dimension of
all non-trivial homogenous spaces $K/C$ as $K$ varies over all compact real-forms of all
simple factors of the complexification of $G$. Take $\bar{d}(G)$ to be the smallest complex dimen-
sion of a simple factor of the complexification of $G$, or alternately the smallest dimension of a compact factor of a compact form of $G$ and let $d'(G)$ be the smallest integer satisfying
\[ \frac{d'(G)(d'(G)+1)}{2} \geq \bar{d}(G). \]

We have $d(G) \geq d'(G)$.

**Conjecture 1.2** (Zimmer’s Conjecture). Let $G$ be a connected, semisimple Lie group with
finite center, all of whose almost-simple factors have real-rank at least 2. Let $\Gamma \leq G$ be a
lattice. Let $M$ be a compact manifold and $\omega$ a volume form on $M$. Then

1. if $\dim(M) < \min(n(G), d(G), v(G))$ then any homomorphism $\alpha : \Gamma \to \text{Diff}(M)$ has finite image;
2. if $\dim(M) < \min(n(G), d(G))$ then any homomorphism $\alpha : \Gamma \to \text{Diff}(M, \omega)$ has finite image;
3. if $\dim(M) < n(G)$ then for any homomorphism $\alpha : \Gamma \to \text{Diff}(M, \omega)$, the image
   $\alpha(\Gamma)$ preserves a Riemannian metric;
4. if $\dim(M) < v(G)$ then for any homomorphism $\alpha : \Gamma \to \text{Diff}(M)$, the image
   $\alpha(\Gamma)$ preserves a Riemannian metric.

The conjecture is almost sharp in several senses. In dimension $v(G)$, any subgroup of $G$
adopts an infinite image, non-isometric, non-volume-preserving action in dimension $v(G)$
namely, the left action on $G/Q$ where $Q$ is a parabolic subgroup of codimension $v(G)$.
These actions are the natural analogue of the action of $\text{SL}(n, \mathbb{R})$ and its lattices on $\mathbb{R}P^{n-1}$.
In dimension \( n(G) \), there is always a semisimple Lie group with finite center \( \hat{G} \) with the same Lie algebra as \( G \), a lattice \( \Gamma \subset G \), and a volume-preserving, non-isometric action in dimension \( n(G) \). However, in these examples the lattice \( \Gamma \) is, in fact, the integer points of \( \hat{G} \) with respect to the rational structure for which the representation in dimension \( n(G) \) is rational; in particular, in such examples \( \Gamma \) is necessarily non-uniform. This construction is the natural analogue of the action of \( \text{SL}(n, \mathbb{Z}) \) on \( \mathbb{T}^n \). In particular, \( n(G) \) is a sharp bound for results about actions of all lattices in a Lie group \( G \) but may not be sharp for results about actions of a particular lattice; given our results it is natural to ask if sharper bounds can be established for cocompact lattices. Lastly, the number \( d(G) \) bounds the dimension in which infinite isometric actions can occur. The existence of an invariant Riemannian metric \( g \) for the action \( \alpha \) implies that the action is given by a homomorphism \( \alpha : \Gamma \to K \) where \( K = \text{Isom}(M, g) \) is a compact Lie group. Margulis’ superrigidity theorem implies that \( \alpha(\Gamma) \) cannot be infinite below dimension \( d(G) \). In fact, in the presence of an invariant metric for low dimensional actions, Margulis’ superrigidity theorem classifies the possible isometry groups and elementary geometry gives sharper results on manifolds admitting infinite, isometric actions.

**Historical Remarks.** Items (2) and (3) are due to Zimmer. Zimmer stated (2) in terms of \( d' \) instead of \( d \). Item (1) is a natural extension by Farb-Shalen. The conjecture as stated in both [FS, Fis] assumed erroneously that one always has \( v(G) = n(G) - 1 \) so the conjecture is slightly misstated in those references. Item (4) is new here, but is a natural extension of the other conjectures. We are intentionally vague concerning regularity of the diffeomorphisms in the conjecture. Zimmer originally considered mostly \( C^\infty \) actions. Most evidence for the conjecture including existing results requires the action to be at least \( C^1 \) but the conjecture might be true for actions by homeomorphisms, see particularly [Wei2, BGV] for discussion and evidence in this regularity.

The group \( \text{SL}(n, \mathbb{R}) \) is the standard split simple Lie group with restricted root system of type \( A_n \). We denote by \( \text{Sp}(2n, \mathbb{R}) \) the group of real symplectic \( 2n \times 2n \) matrices, the standard split simple Lie group of rank \( n \) with restricted root system of type \( C_n \).

**Theorem 1.3.** Conjecture 1.2 holds for cocompact lattices in \( \text{Sp}(2n, \mathbb{R}) \) for \( n \geq 2 \). In particular if \( M \) is a compact manifold with \( \dim(M) < 2n - 1 \) and \( \Gamma < \text{Sp}(2n, \mathbb{R}) \) is a cocompact lattice then any homomorphism \( \alpha : \Gamma \to \text{Diff}^2(M) \) has finite image. In addition, if \( \dim(M) = 2n - 1 \) and \( \omega \) is a volume form on \( M \) then any homomorphism \( \alpha : \Gamma \to \text{Diff}(M, \omega) \) has finite image.

The fact that all actions in Theorem 1.1 and 1.3 factor through finite quotients follows from the existence of an invariant Riemannian metric and the fact that, for these cases, \( v(G) + 1 = n(G) \leq d'(G) \) where \( v(\text{SL}(n, \mathbb{R})) = n - 1 \) and \( v(\text{Sp}(n, \mathbb{R})) = 2n - 1 \). See Section 7 for full discussion.

The remaining split classical Lie groups are \( \text{SO}(n, n) \) and \( \text{SO}(n, n + 1) \). Note that \( \text{SO}(2, 2) \) is not simple and we omit below the higher-rank simple groups \( \text{SO}(2, 3) \) and \( \text{SO}(3, 3) \) as their identity components are double covered by \( \text{Sp}(4, \mathbb{R}) \) and \( \text{SL}(4, \mathbb{R}) \), respectively. For \( G = \text{SO}(n, n) \) with \( n \geq 4 \), we have

\[
n(G) = 2n, \quad d(G) = d'(G) = 2n - 1, \quad \text{and} \quad v(G) = 2n - 2
\]

and similarly for \( G = \text{SO}(n, n + 1) \) with \( n \geq 3 \), we have

\[
n(G) = 2n + 1, \quad d(G) = d'(G) = 2n, \quad \text{and} \quad v(G) = 2n - 1.
\]
Theorem 1.4. The non-volume-preserving case of Conjecture 1.2 holds for cocompact lattices $\Gamma$ in $\text{SO}(n, n)$ with $n \geq 4$ and for $\text{SO}(n, n+1)$ with $n \geq 3$; the volume-preserving case holds up to dimension 1 less than conjectured.

More precisely, let $M$ be a compact connected manifold and $\omega$ a volume form on $M$.

1. If $\Gamma < \text{SO}(n, n)$ is a cocompact lattice and $\dim(M) < 2n - 2$ then any homomorphism $\alpha : \Gamma \to \text{Diff}^2(M)$ has finite image. If $\dim(M) = 2n - 2$ then any homomorphism $\alpha : \Gamma \to \text{Diff}^2(M, \omega)$ has finite image.

2. If $\Gamma < \text{SO}(n, n+1)$ is a cocompact lattice and $\dim(M) < 2n - 1$ then any homomorphism $\alpha : \Gamma \to \text{Diff}^2(M)$ has finite image. If $\dim(M) = 2n - 1$ then any homomorphism $\alpha : \Gamma \to \text{Diff}^2(M, \omega)$ has finite image.

Again, the finiteness of the action follows from Theorem 2.7 below and a computation of the value of $d'(G) = d(G)$.

From Conjecture 1.2, one expects that in dimension $n(G) - 1 = d(g) = v(g) + 1$ all volume-preserving actions necessarily preserve a Riemannian metric. In this case, Margulis’ superrigidity theorem would imply the action is finite unless the manifold is the $(n(G) - 1)$-dimensional sphere or projective space. While the techniques of this paper impose certain restrictions on volume-preserving actions in dimension $n(G) - 1$, it seems additional ideas are needed to obtain the conjectured result in dimension $n(G) - 1$.

We remark that the conclusions of Theorems 1.1, 1.3, and 1.4 continue to hold for actions of cocompact lattices in Lie groups isogenous to the groups in the theorems. That is, if $G$ is a Lie group with finite center whose Lie algebra is isomorphic to the Lie algebra of a group in Theorems 1.1, 1.3, or 1.4, then the conclusion of the corresponding theorem continues to hold for cocompact lattices in $G$.

Combined with the main results of [FH] and [BRHW3] we obtain the following theorem for actions of lattices on surfaces.

Theorem 1.5 ([FH, Corollary 1.7] + [BRHW3, Theorem 1.6] + Theorem 2.7). Let $S$ be a closed, oriented surface of genus at least 1. Let $G$ be a simple Lie group with finite center and real-rank at least 2 and assume the restricted root system of the Lie algebra of $G$ is not of type $A_2$. Let $\Gamma \subset G$ be a lattice. Then any homomorphism $\alpha : \Gamma \to \text{Diff}^2(S)$ has finite image.

Note that the hypothesis that the restricted root system of $G$ is not of type $A_2$ ensures the number $r(G)$ defined in Section 2.2 below is at least 3. Up to isogeny, the three simple Lie groups of type $A_2$ are $\text{SL}(3, k)$ where $k = \mathbb{R}, \mathbb{C}$, or $\mathbb{H}$. We remark that the conclusion of Theorem 1.5 is expected to hold for lattices in $\text{SL}(3, \mathbb{C})$ and $\text{SL}(3, \mathbb{H})$, and for lattices in $\text{SL}(3, \mathbb{R})$ assuming that $S$ is not the 2-sphere.

We defer the statement of our main theorem, Theorem 2.7, which includes partial results for non-split and exceptional Lie groups, until we have made some requisite definitions. For non-split groups, our main theorem does not recover the full conjecture but does imply finiteness of actions in a dimension that grows linearly with the rank.

1.2. Outline of the proof. We will illustrate the main ideas of the proof of Theorem 1.1 by considering the case where $\Gamma \subset G = \text{SL}(n, \mathbb{R})$ is a cocompact lattice acting on a closed manifold $M$ and $\dim(M) < n - 1$. In this case, if the action preserves a measure $\mu$, Zimmer’s cocycle superrigidity theorem implies that the derivative cocycle is measurably cohomologous to a cocycle taking values in a compact subgroup or, equivalently, that the action preserves a measurable Riemannian metric [Zim1]. This implies, in particular, that all Lyapunov exponents for all elements of $\Gamma$ are zero. As remarked above, the conjecture would follow from promoting the invariant measurable metric to a smooth invariant metric.
It was originally observed by Zimmer that much lower regularity could be used to complete a proof of the conjecture, namely that one only needed the metric to be bounded above and below. (An argument that shows this is contained in Section 7.) Very early on, Zimmer also observed that one might get better regularity by noting that the metric was invariant, so its growth along orbits was controlled by the derivative cocycle. Using this he could show that the metric was, in a sense, in $L^\infty$, for very small values of $\varepsilon > 0$ [Zim3]. A more sophisticated, non-linear, attempt to average metrics in order to produce invariant smooth metrics was proposed by the second author in [Fis]. Both of these attempts fail to produce good results because even with a measurable (or even slightly more regular) invariant metric, the only a priori bound on growth of derivatives along orbits is exponential.

The first step in the proof of Theorem 1.1 is to show that any action $\alpha : \Gamma \to \text{Diff}^2(M)$ for $\Gamma$ and $M$ as in Theorem 1.1 has uniform subexponential growth of derivatives: for every $\varepsilon > 0$, there is $C_\varepsilon$ such that

$$\|D\alpha(\gamma)\| \leq C_\varepsilon e^{\varepsilon l(\gamma)}$$

where $\|D\alpha(\gamma)\| = \max_{x \in M} \|D_x\alpha(\gamma)\|$ denotes the norm of the derivative and $l(\cdot)$ denotes the word-length with respect to some choice of finite generating set for $\Gamma$.

To illustrate how we establish uniform subexponential growth of derivatives, consider a more elementary fact from classical smooth dynamics: a diffeomorphism $f : M \to M$ of a compact manifold $M$ has uniform subexponential growth of derivatives if and only if all Lyapunov exponents of $f$ are zero with respect to any $f$-invariant probability measure. Clearly, uniform subexponential growth of derivatives implies that all Lyapunov exponents vanish for any measure. To prove the converse, observe that if $\|D_x g^n\| \geq e^{\varepsilon n}$ for some $x_n \in M$, an appropriate accumulation point of the sequence of measures $\mu_n := \frac{1}{n} \sum_{i=1}^n g^i \ast \delta_{x_n}$ will be a measure with non-zero Lyapunov exponents.

To implement the above, in Section 3.1 we induced from the $\Gamma$-action on $M$ a $G$-action on an auxiliary manifold $M^\alpha$. This space has the structure of an $M$ bundle over $G/\Gamma$. With $A \subset \text{SL}(n, \mathbb{R})$ the subgroup of positive diagonal matrices (that is, a maximal split Cartan subgroup), the failure of the action $\alpha$ to have uniform subexponential growth of derivatives implies the existence of an element $s \in A$ and an $s$-invariant probability measure $\mu$ on $M^\alpha$ with a positive Lyapunov exponent for the fiberwise derivative cocycle. The key new idea is to construct from $\mu$ a $G$-invariant measure $\mu'$ on $M^\alpha$ such that the fiberwise derivative cocycle continues to have a positive Lyapunov exponent for some $s' \in A$. This yields a contradiction with Zimmer’s cocycle superrigidity theorem as there are no non-trivial linear representations in dimension less then $n$. We thus obtain the uniform subexponential growth of derivatives for the action $\alpha$.

To construct a $G$-invariant measure $\mu'$, starting with our $s$-invariant measure $\mu$ we build a sequence of measures by averaging: given a measure $\mu$ that has a positive fiberwise Lyapunov exponent for some $s \in A$, by averaging $\mu$ along $A$ or a unipotent subgroup commuting with $s$ we obtain a new measure $\mu'$ which is invariant by a larger subgroup and has positive fiberwise exponent for some $s' \in A$. There is some similarity here to Margulis’ original proof of the superrigidity theorem using Oseledec’s theorem where it is used (see [Mar2]) that higher-rank semisimple Lie groups can be generated by centralizers of certain elements of the diagonal.

While we cannot average directly to obtain a $G$-invariant measure on $M^\alpha$, we may average so as to obtain an $A$-invariant measure on $M^\alpha$ whose projection to $G/\Gamma$ is the Haar measure and has positive fiberwise exponent for some $s' \in A$. This step requires a careful choice of subgroups over which to average and employs Ratner’s theorem on measures.
invariant under unipotent subgroups and an improvement due to Shah concerning averages of measures along unipotent subgroups. As the general averaging argument requires understanding the combinatorics of irreducible root systems, we explain this step for the special case of $\text{SL}(n, \mathbb{R})$ in Section 5.2.

To show such a measure is, in fact, $G$-invariant we use a result (Proposition 5.8 below) from the work of the first author with Rodriguez Hertz and Wang where its shown that—under the same dimension bounds as in Theorem 1.1—any $P$-invariant measure on $M^\alpha$ is, in fact, $G$-invariant [BRHW3]. Here $P$ denotes the group of upper triangular matrices. As $P$ contains $A$ and as any $P$-invariant measure on $G/\Gamma$ is necessarily Haar, we are in a more general setting than is studied in [BRHW3]. The key idea in the proof of Proposition 5.8 from [BRHW3] is to relate the Haar-entropy of elements of the $A$-action in $G/\Gamma$ with the $\mu$-entropy of elements of the $A$-action in $M^\alpha$. For the Haar measure on $G/\Gamma$, the entropy of elements of $A$ is computed in terms of the roots of $G$. Moreover, the contribution from the fiber to the $\mu$-entropy of elements of the $A$-action is constrained by the dimension assumption. Many key ergodic theoretic notions for these argument are developed in [BRH, Bro, BRHW1].

Both the main result in [BRHW3] and our use of their techniques here employ the philosophy that “non-resonance implies invariance.” This philosophy was introduced by the same authors in their study of global rigidity of Anosov actions of higher-rank lattices in [BRHW2]. Given a $G$-action and an $A$-invariant (or equivariant) object $O$, such as a measure or a semiconjugacy to a linear action, one may try to associate to $O$ a class of linear functionals $\mathcal{O}$. In the case of an $A$-invariant measure, the functionals are the Lyapunov exponents; in the case of a conjugacy to a linear action, the functionals are the weights of the representation corresponding to the linear action. The philosophy, implemented in both [BRHW2] and [BRHW3], is that, given any root $\beta$ of $G$ that is not positively proportional to an element of $\mathcal{O}$, the object $O$ will automatically be invariant (or equivariant) under the unipotent root group $G^\beta$. If one can find enough such non-resonant roots, the object $O$ is automatically $G$-invariant (or $G$-equivariant).

The second step in the proof of Theorem 1.1, is to use the strong property (T) of V. Lafforgue to produce an invariant metric from uniform subexponential growth of derivatives. Strong property (T) was introduced by Lafforgue who proved that all simple Lie groups containing $\text{SL}(3, \mathbb{R})$ and their cocompact lattices have strong property (T) with respect to Hilbert spaces. The precise results we use here are an extension of Lafforgue’s due to de Laat and de la Salle [Laf, dLdLS]. We state a special case of their theorems.

**Theorem 1.6** ([dLdLS]). Let $\mathcal{H}$ be a Hilbert space and $\Gamma$ as in Theorem 1.1. There exists $\varepsilon > 0$, such that for any representation $\pi: \Gamma \to \mathcal{B}(\mathcal{H})$, if there exists $C_\varepsilon > 0$ such that
\[
\|\pi(g)\| \leq C_\varepsilon e^{\varepsilon l(\gamma)},
\]
then there exists a sequence of averaging operators $p_n = \sum w_i \pi(\gamma_i)$ in $\mathcal{B}(\mathcal{H})$—where $w_i > 0$, $\sum w_i = 1$, and $w_i = 0$ for every $\gamma_i \in \Gamma$ of word length larger than $n$—such that for any vector $v \in \mathcal{H}$ the sequence $v_n = p_n(v) \in \mathcal{H}$ converges to a $\Gamma$-invariant vector $v^\ast$. Moreover the convergence is exponential: there exists $0 < \lambda < 1$ (independent of $\pi$) and a $C$ so that $\|v_n - v^\ast\| \leq C\lambda^n\|v\|$.

In the case of $C^\infty$ actions, we may apply this theorem to a Sobolev space of sections of the bundle of symmetric 2-tensors on $M$ (where Riemannian metrics are a subset). As the uniform subexponential growth of derivatives implies subexponential growth of derivatives of higher order, we verify the slow norm growth required in Theorem 1.6 holds and obtain a $\Gamma$-invariant symmetric 2-tensor. To verify that the tensor is in fact a metric (that is, that
the 2-tensor is non-degenerate) we use that the norms decay at a subexponential rate under the averaging operator while the convergence to the limit is exponentially fast.

We remark that a somewhat similar use of subexponential growth of derivatives along a central foliations also occurs in the work of the first author with Kalinin and Spatzier on rigidity for Anosov actions of abelian groups [FKS]. In that work, subexponential growth is verified from the existence of a Hölder conjugacy and is used in conjunction with exponential decay of matrix coefficients for abelian groups. These ideas are also applied in the work of Rodriguez-Hertz and Wang [RHW].

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2. Background, main result, and first reductions

2.1. Facts from the structure of Lie groups. To state our main results we recall some facts and definitions from the structure theory of real Lie groups. A standard reference is [Kna]. Let $G$ be a connected, semisimple Lie group with finite center. As usual, write $\mathfrak{g}$ for the Lie algebra of $G$. Fix once and for all a Cartan involution $\theta$ of $\mathfrak{g}$ and write $\mathfrak{t}$ and $\mathfrak{p}$, respectively, for the $+1$ and $-1$ eigenspaces of $\theta$. Denote by $\mathfrak{a}$ the maximal abelian subalgebra of $\mathfrak{p}$ and by $\mathfrak{m}$ the centralizer of $\mathfrak{a}$ in $\mathfrak{t}$. We let $\Sigma$ denote the set of restricted roots of $\mathfrak{g}$ with respect to $\mathfrak{a}$. Note that the elements of $\Sigma$ are real linear functionals on $\mathfrak{a}$. Recall that $\dim_{\mathbb{R}}(\mathfrak{a})$ is the $\mathbb{R}$-rank of $G$. We fix $\mathfrak{a}$ for the remainder.

Recall that a base (or a collection of simple roots) for $\Sigma$ is a subset $\Pi \subset \Sigma$ that is a basis for the vector space $\mathfrak{a}^*$ and such that every non-zero root $\beta \in \Sigma$ is either a positive or a negative integer combination of elements of $\Pi$. For a choice of $\Pi$, elements $\beta \in \Pi$ are called simple (positive) roots. Relative to a choice of base $\Pi$, let $\Sigma_+ \subset \Sigma$ be the collection of positive roots and let $\Sigma_-$ be the corresponding set of negative roots. For $\beta \in \Sigma$ write $\mathfrak{g}^\beta$ for the associated root space. Then $\mathfrak{n} = \bigoplus_{\beta \in \Sigma_+} \mathfrak{g}^\beta$ is a nilpotent subalgebra. A subalgebra $\mathfrak{q}$ of $\mathfrak{g}$ is said to be a standard parabolic subalgebra or simply parabolic (relative to the choice of $\theta$ and $\Pi$) if $\mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n} \subset \mathfrak{q}$ where $\mathfrak{n}$ is defined relative to $\Pi$. We have that the standard parabolic subalgebras of $\mathfrak{g}$ are parametrized by exclusion of simple (negative) roots: for any sub-collection $\Pi' \subset \Pi$ let

$$\mathfrak{q}_{\Pi'} = \mathfrak{m} \oplus \mathfrak{a} \oplus \bigoplus_{\beta \in \Sigma_+ \cup \text{Span}(\Pi')} \mathfrak{g}^\beta.$$  

Then $\mathfrak{q}_{\Pi'}$ is a Lie subalgebra of $\mathfrak{g}$ and all standard parabolic subalgebras of $\mathfrak{g}$ are of the form $\mathfrak{q}_{\Pi'}$ for some $\Pi' \subset \Pi$ [Kna, Proposition 7.76].

Let $A, N$, and $K$ be the analytic subgroups of $G$ corresponding to $\mathfrak{a}, \mathfrak{n}$ and $\mathfrak{t}$. Then $G = KAN$ is the corresponding Iwasawa decomposition of $G$. As $G$ has finite center, $K$ is compact. Note that the Lie exponential $\exp: \mathfrak{g} \to G$ restricts to diffeomorphisms between $\mathfrak{a}$ and $A$ and $\mathfrak{n}$ and $N$. Fixing a basis for $\mathfrak{a}$, we often identify $A = \exp(\mathfrak{a}) = \mathbb{R}^d$. Via this identification we extend linear functionals on $\mathfrak{a}$ (in particular, the restricted roots of $\mathfrak{g}$) to functionals on $A$. Write $\mathcal{M} = C_K(\mathfrak{a})$ for the centralizer of $\mathfrak{a}$ in $K$. Then $P = MAN$ is the standard minimal parabolic subgroup. Since $M$ is compact, it follows that $P$ is amenable. A standard parabolic subgroup (relative to the choice of $\theta$ and $\Pi$ above) is any closed subgroup $Q \subset G$ containing $P$. The Lie algebra of any standard parabolic subgroup $Q$ is a standard parabolic subalgebra and the correspondence between standard parabolic subgroups and subalgebras is 1-1.
We say two restricted roots $\beta, \tilde{\beta} \in \Sigma$ are coarsely equivalent if there is some $c > 0$ with $\tilde{\beta} = c\beta$.

Note that $c$ takes values only in $\{\frac{1}{2}, 1, 2\}$ and this occurs only if the root system $\Sigma$ has a factor of type $BC_{\ell}$. Let $\hat{\Sigma}$ denote the set of coarse restricted roots; that is, the set of coarse equivalence classes $[\beta]$ in $\Sigma$. Note that for $[\beta] \in \hat{\Sigma}$, $\mathfrak{g}^{[\beta]} := \bigoplus_{\beta' \in [\beta]} \mathfrak{g}^{\beta'}$ is a nilpotent subalgebra and the Lie exponential restricts to a diffeomorphism between $\mathfrak{g}^{[\beta]}$ and the corresponding analytic subgroup which we denote by $G^{[\beta]}$.

Let $\mathfrak{q}$ denote a standard parabolic subalgebra of $\mathfrak{g}$. Observe that if $\mathfrak{g}^{\beta} \cap \mathfrak{q} \neq 0$ for some $\beta \in \Sigma$ then, from the structure of parabolic subalgebras, $\mathfrak{g}^{[\beta]} \subset \mathfrak{q}$ where $[\beta] \in \hat{\Sigma}$ is the coarse restricted root containing $\beta$. A proper subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is maximal if there is no subalgebra $\mathfrak{h}'$ with $\mathfrak{h} \subset \mathfrak{h}' \subset \mathfrak{g}$. Note that maximal standard parabolic subalgebras are of the form $\mathfrak{q}_{\Pi, \langle \beta \rangle}$ for some $\beta \in \Pi$.

2.2. Resonant codimension and related lemmas. Consider a Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ that is saturated by coarse root spaces. For such a subalgebra define the resonant codimension, $\bar{r}(\mathfrak{h})$, of $\mathfrak{h}$ to be the cardinality of the set

$$\{[\beta] \in \hat{\Sigma} \mid \mathfrak{g}^{[\beta]} \not\subset \mathfrak{h}\}.$$ 

For a subgroup $H \subset G$ whose Lie algebra is saturated by coarse root spaces, we will also refer to the resonant codimension of the group $H$.

Note that standard parabolic subalgebras $\mathfrak{q}$ are automatically saturated by coarse root spaces whence the resonant codimension is defined for all standard parabolic subalgebras. As in [BRHW3], given a (semi)simple Lie algebra $\mathfrak{g}$ as above we define a combinatorial number $r(\mathfrak{g})$. As the number depends only on the Lie algebra $\mathfrak{g}$, we use both the notation $r(G)$ and $r(\mathfrak{g})$ interchangeably.

Definition 2.1. The minimal resonant codimension of $\mathfrak{g}$ or $G$, denoted by $r(\mathfrak{g})$ or $r(G)$, is defined to be the minimal value of the resonant codimension $\bar{r}(\mathfrak{q})$ of $\mathfrak{q}$ as $\mathfrak{q}$ varies over all (maximal) proper parabolic subalgebras of $\mathfrak{g}$.

Remark 2.2. In the case that the Lie algebra $\mathfrak{g}$ of $G$ is a split real form, the minimal resonant codimension $r(\mathfrak{g})$ coincides with minimal codimension of all maximal parabolic subalgebras. In general, we have $r(\mathfrak{g}) \leq v(G)$.

In the case that $\mathfrak{g}$ is semisimple then $r(\mathfrak{g})$ is the minimal value of $r(\mathfrak{g}')$ as $\mathfrak{g}'$ varies over all non-compact simple ideals of $\mathfrak{g}$. In particular, if $\mathfrak{g}$ has rank-1 factors then $r(\mathfrak{g}) = 1$.

Example 2.3. We compute $r(\mathfrak{g})$ for a number of classical real simple Lie algebras. Note that it follows from definition that the minimal resonant codimension depends only on the restricted root system of $\mathfrak{g}$ and not on the Lie algebra $\mathfrak{g}$.

Type $A_n$: $r(\mathfrak{g}) = n$. This includes $\mathfrak{sl}(n + 1, \mathbb{R})$, $\mathfrak{sl}(n + 1, \mathbb{C})$, $\mathfrak{sl}(n + 1, \mathbb{H})$.

Type $B_n$, $C_n$, and $(BC)_n$: $r(\mathfrak{g}) = 2n - 1$. This includes $\mathfrak{sp}(2n, \mathbb{R})$, $\mathfrak{so}(n, m)$ for $n < m$, and $\mathfrak{su}(n, m)$ and $\mathfrak{sp}(n, m)$ for $n \leq m$.

Type $D_n$, $n \geq 4$: $r(\mathfrak{g}) = 2n - 2$. This includes $\mathfrak{so}(n, n)$ for $n \geq 4$.

Type $E_6$: $r(\mathfrak{g}) = 16$.

Type $E_7$: $r(\mathfrak{g}) = 27$.

Type $E_8$: $r(\mathfrak{g}) = 57$.

Type $F_4$: $r(\mathfrak{g}) = 15$.

Type $G_2$: $r(\mathfrak{g}) = 5$. 

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We note that for all root systems above, the minimal resonant codimension \( r(g) \) corresponds to the codimension of the maximal parabolic subalgebra \( \mathfrak{p}_{\Pi \setminus \{\alpha_i\}} \) where the simple roots are as enumerated as in the Dynkin diagrams in Table 1.

For the remainder of this subsection, we show that certain subgroups of \( G \) with resonant codimension at most \( r(G) \) are parabolic. With \( g \) the Lie algebra of \( G \), let \( \Sigma = \Sigma(g) \) be the restricted root system of \( g \), and let
\[
g = m \oplus a \oplus \bigoplus_{\beta \in \Sigma} g^\beta
\]
be the restricted root space decomposition (relative to the choice of Cartan involution \( \theta \)). Note that \( g^\beta \) is not a Lie subalgebra if \( 2\beta \) is a root; in this case let \( \langle g^\beta \rangle \) denote the Lie-subalgebra generated by \( g^\beta \).

**Lemma 2.4.** Let \( h \subset g \) be a Lie subalgebra with \( m \oplus a \subset h \). Then for every \( \beta \in \Sigma \) with \( h \cap g^\beta \neq 0 \), we have
\[
\langle g^\beta \rangle \subset h.
\]

**Proof.** The proof follows closely the proof of [Kna, Lemma 7.73]. Let \( 0 \neq X \in g^\beta \cap h \). Then \( Y = \theta X \in g^{-\beta} \) [Kna, Proposition 6.40] and the Lie subalgebra spanned by \( X \) and \( Y \) is isomorphic to \( \mathfrak{sl}(2, \mathbb{R}) \). Normalizing \( X \), we may assume \( X \) corresponds to \( e \), \( Y \) corresponds to \( -f \) and \( [X, Y] \) corresponds to \( h \) in the standard basis for \( \mathfrak{sl}(2, \mathbb{R}) \) [Kna, page 29]. Denote by \( \mathfrak{s}l_X \subset g \) this copy of \( \mathfrak{sl}(2, \mathbb{R}) \).

\( \mathfrak{s}l_X \) acts via the adjoint representation on the vector space \( V = \bigoplus_{c \in \mathbb{Z}} g^{c\beta} \). We have either
\[
V = g^{-2\beta} \oplus g^{-\beta} \oplus (m \oplus a) \oplus g^{-\beta} \oplus g^{-2\beta}
\]
or
\[
V = g^{-\beta} \oplus (m \oplus a) \oplus g^{-\beta}.
\]

Decomposing \( V \) into \( \mathfrak{s}l_X \)-irreducible components and using the structure theory of irreducible \( \mathfrak{s}l_X \)-modules it follows that
\begin{enumerate}
\item \( (\text{ad}X)(g^{-\beta}) \subset (m \oplus a) \);
\item \( (\text{ad}X)^2(g^{-2\beta}) \subset (m \oplus a) \);
\item \( (\text{ad}X)^2 \) is a bijection between \( (g^{-\beta}) \) and \( \langle g^\beta \rangle \);
\item \( (\text{ad}X)^4 \) is a bijection between \( (g^{-2\beta}) \) and \( \langle g^{2\beta} \rangle \).
\end{enumerate}

In particular, it necessarily follows that
\[
(\text{ad}X) : (\text{ad}X)(g^{-\beta}) \to g^\beta
\]
is a bijection and since \( (\text{ad}X)(g^{-\beta}) \subset m \oplus a \subset h \) it follows that \( g^\beta \subset h \) whence \( \langle g^\beta \rangle \subset h \). \( \square \)

**Lemma 2.5.** Let \( h \subset g \) be a subalgebra whose codimension is at most the minimal codimension of all proper parabolic subalgebras of \( g \). Then \( h \) is parabolic.

**Proof.** It suffices to prove the lemma for \( G \) simple, as the general case follows from that one. As the Lemma is only about Lie algebras, replacing \( G \) with its image \( \text{Ad}(G) \) in \( \text{Aut}(g) \), we assume for the rest of the proof that \( G \) is a (real) linear algebraic group.

Let \( H \subset G \) be the Zariski closed subgroup with Lie algebra \( \mathfrak{h} \). \( G \) acts via the adjoint representation on exterior powers \( \Lambda^n g \). For \( n = \dim \mathfrak{h} \), \( H \) stabilizes the vector \( \Lambda^n \mathfrak{h} \). The full stabilizer of \( \Lambda^n \mathfrak{h} \) is the normalizer \( N_G(H) \) which is a connected group whose Lie algebra \( n \) is the normalizer of \( h \). Let \( V^G \subset \Lambda^n g \) be the set of \( G \)-invariant vectors and identify \( \mathbb{R}^k = \Lambda^n g / V^G \). Let \( v \in \mathbb{R}^k \) be the image of \( \Lambda^n \mathfrak{h} \) which is stabilized by \( H \).
Note that if \( \bigwedge^n h \in V^G \) then \( N_G(H) = G \) in which case \( H \) is normal in \( G \) and so equal to \( G \) by simplicity, completing the proof. We may thus assume \( \nu \) is non-zero.

\( G \) acts on the projective space \( \mathbb{R}P^{k-1} \). The \( G \)-orbit of \( \nu \) coincides with the homogeneous space \( G/N_G(H) \). We claim the \( G \)-orbit of \( \nu \) is closed in \( \mathbb{R}P^{k-1} \). Indeed, if not, the closure of the orbit \( G \cdot \nu \) contains a closed orbit \( G \cdot u \) of strictly smaller dimension. The orbit \( G \cdot u \) coincides with a homogeneous space \( G/H' \) where \( H' \) has dimension strictly larger than \( N_G(H) \). \( G/H' \) is a projective variety whence it follows that the Lie algebra \( h' \) of \( H' \) is a parabolic subalgebra. As the action of \( G \) has no fixed points in \( \mathbb{R}P^{k-1} \), it follows that \( \dim H \leq \dim N_G(H) < \dim H' < \dim G \) contradicting our choice of \( h \).

It follows that \( G/N_G(H) \) is a projective variety whence \( N_G(H) \neq G \) is parabolic. By hypothesis on \( h \), it follows that \( \dim H = \dim N_G(H) \) whence \( h \) is parabolic.

**Proposition 2.6.** Let \( h \subset g \) be a Lie subalgebra with \( m \oplus a \subset h \). If the cardinality of the set \( \{ [\beta] \in \hat{\Sigma}(g) : g^{[\beta]} \not\subset h \} \) is at most \( r(g) \), then \( h \) is parabolic.

**Proof.** We may assume \( g \) is simple. Let \( \ell \) be the real-rank of \( g \) and let \( \Sigma = \Sigma(g) \) be the restricted root system of \( g \). Let \( \Sigma' = \Sigma \) if \( g \) is not of type \( BC \ell \) and let \( \Sigma' = B_\ell \) otherwise.

First consider \( g' \) a split real form with restricted root system \( \Sigma' \). Let \( \Pi = \{ \alpha_1, \ldots, \alpha_k \} \) and \( \Pi' = \{ \alpha'_1, \ldots, \alpha'_k \} \) be bases for \( \Sigma \) and \( \Sigma' \) respectively. Given \( \beta \in \Sigma \) such that \( 1/2 \beta \notin \Sigma \) write \( \beta = \sum c_i \alpha_i \) and let \( \beta' = \sum c_i \alpha_i' \); if \( 1/2 \beta \in \Sigma \) let \( \beta' = \frac{1}{2} \sum c_i \alpha_i' \).

Consider \( h \subset g \) as in the proposition. Let \( \hat{\Delta} \subset \hat{\Sigma}(g) \) be

\[
\hat{\Delta} = \{ [\beta] \in \hat{\Sigma} : g^{[\beta]} \subset h \}
\]

and let \( \Delta' \subset \Sigma' \) be the corresponding collection of roots under the correspondence described above. Let \( h' \) be the Lie subalgebra containing

\[
m' \oplus a' \oplus \bigoplus_{\beta' \in \Delta'} g^{[\beta']}
\]

Note that the subspace \( m' \oplus a' \oplus \bigoplus_{\beta' \in \Delta'} g^{[\beta']} \) has codimension \( r(g) \) in \( g' \). We have that \( h' \) has codimension at least \( r(g) \) in \( g' \). From Lemma 2.5 we have that either \( h' = g \) or \( h' \) is parabolic and moreover has codimension \( r(g) \). In particular, if \( h' \neq g' \) then

\[
h' = m' \oplus a' \oplus \bigoplus_{\beta' \in \Delta'} g^{[\beta']}
\]

As bases for \( \Sigma' \) are in one-to-one correspondence with bases for \( \Sigma \), it follows from the definition of \( \hat{\Delta} \) that if \( h' \neq g' \) then \( h \) is parabolic.

On the other hand, if \( h' = g' \) it follows that \( h \cap g^{[\beta]} \neq \emptyset \) for every \( \beta \in \Sigma(g) \). If \( \Sigma(g) \) is not of type \( (BC)_\ell \), it follows from Lemma 2.4 that \( h = g \). If \( \Sigma(g) \) is of type \( (BC)_\ell \) then Lemma 2.4 implies that \( h \) contains (as a vector space) all root spaces corresponding to long roots (roots \( \beta \) with \( 1/2 \beta \in \Sigma \)), all root spaces corresponding to middle roots (roots \( \beta \) with \( c \beta \notin \Sigma \) for \( c \neq \pm 1 \)), and at least \( 2\ell - n(g) = 2\ell - (2\ell - 1) = 1 \) root spaces corresponding to short roots (roots \( \beta \) with \( 2\beta \in \Sigma \)). As there is at least one short root and all middle roots, \( h \) is saturated by all root spaces of \( \Sigma(g) \). It follow that \( h = g \) whence \( h \) is parabolic.

**2.3. Main theorem.** Our main theorem gives a partial solution to Zimmer’s conjecture for actions of cocompact lattices in any semisimple Lie group all of whose non-compact, almost-simple factors are of higher rank. Results stated in the introduction follow from Theorem 2.7 and Margulis’ superrigidity theorem as explained in Section 7.
Theorem 2.7. Let $G$ be a connected, semisimple real Lie group with finite center, all of whose non-compact, almost-simple factors have real-rank at least 2. Let $\Gamma \subset G$ be a cocompact lattice and for $k \geq 2$, let $\alpha : \Gamma \to \text{Diff}^k(M)$ be an action. Suppose that either
\begin{enumerate}
  \item $\dim(M) < r(G)$, or
  \item $\dim(M) = r(G)$ and $\alpha$ preserves a smooth volume.
\end{enumerate}
Then $\alpha(\Gamma)$ preserves a Riemannian metric which is $C^{k-1-\delta}$ for all $\delta > 0$.

Theorem 2.7 gives a partial solution to Zimmer’s conjecture for cocompact lattices in any higher-rank simple Lie group $G$. In particular, the number $r(G)$ provides a critical dimension—which grows linearly in the rank of $G$—for which the conclusion of Zimmer’s conjecture holds. Moreover, the number $r(G)$ gives the optimal result for non-volume-preserving actions when $G$ is a split real form.

For non-split simple Lie groups, our critical dimension falls below the conjectured result. In particular, while we recover the complete conjecture as stated in Conjecture 1.2 for cocompact lattices in $\text{SL}(n, \mathbb{R})$ with $n > 2$, for lattices in $\text{SL}(n, \mathbb{C})$ and $\text{SL}(n, \mathbb{H})$ our critical dimension $r(G)$ is, respectively, one half and one quarter of the conjectured critical value. For lattices in $\text{SO}(n, m)$ we obtain the conjectured result in the split case where $m = n$ or $m = n+1$. However, for fixed $n$ our critical dimension $r(G)$ for $G = \text{SO}(n, m)$, $m > n$, is constant in $m$ and thus the defect between the critical dimension in Theorem 2.7 and the conjectured critical dimension becomes arbitrarily large as $m \to \infty$.

The obstruction to improving our results for non-split simple Lie groups is to improve the results of [BRHW3], particularly the result quoted below in Proposition 5.8. In particular, the method of proof of Proposition 3.7 below can not distinguish between actions of lattices in two groups with the same restricted root system.

2.4. Proof of Theorem 2.7. We prove Theorem 2.7 in two steps. Let $G$ be a finitely generated group. Let $l : \Gamma \to \mathbb{N}$ denote the word-length function relative to some fixed finite set of generators. Let $\alpha : \Gamma \to \text{Diff}^1(M)$ be an action of $\Gamma$ on a compact manifold $M$ by $C^1$ diffeomorphisms. We say the action $\alpha$ has uniform subexponential growth of derivatives if for all $\varepsilon > 0$ there is a $C_\varepsilon$ such that for all $\gamma \in \Gamma$ we have
$$
\|D\alpha(\gamma)\| \leq C_\varepsilon e^{\varepsilon l(\gamma)}
$$
where $\|D\alpha(\gamma)\| = \sup_{x \in M} \|D_x \alpha(\gamma)\|$.

To prove Theorem 2.7 we first establish uniform subexponential growth of derivatives for actions of cocompact lattices in the low-dimensional settings consider above.

Theorem 2.8. Let $G$ be a connected, semisimple Lie group with finite center. Let $\Gamma \subset G$ be a cocompact lattice and let $\alpha : \Gamma \to \text{Diff}^{1+\beta}(M)$ be an action for $\beta > 0$. Suppose that either
\begin{enumerate}
  \item $\dim(M) < r(G)$, or
  \item $\dim(M) = r(G)$ and $\alpha$ preserves a smooth volume.
\end{enumerate}
Then $\alpha$ has uniform subexponential growth of derivatives.

When $G$ is rank-1 or has rank-1 factors we have $r(G) = 1$. In this case, Theorem 2.8 is trivial if $\dim(M) < r(G)$ and is nearly as trivial if $\dim(M) = r(G)$ and $\alpha$ preserves a smooth volume since any group of diffeomorphisms preserving a smooth volume form on the circle is smoothly conjugate to a group of isometries.

Having established Theorem 2.8, the second step in the proof of Theorem 2.7 is to show that for a group with strong property $(T)$, any action with subexponential growth of derivatives preserves a smooth Riemannian metric.
Theorem 2.9. Let \( \Gamma \) be a finitely generated group, \( M \) a compact manifold, and \( \alpha: \Gamma \to \text{Diff}^k(M) \) an action on \( M \) by \( C^k \) diffeomorphisms for \( k \geq 2 \). If \( \Gamma \) has strong property (T) and if \( \alpha \) has uniform subexponential growth of derivatives then \( \alpha \) preserves a Riemannian metric which is \( C^{k-1+\delta} \) for all \( \delta > 0 \).

Theorem 2.7 is an immediate consequence of Theorems 2.9 and 2.8.

Note that Theorem 2.7 implies Conjecture 1.2 for non-volume-preserving actions of cocompact lattices in all split simple Lie groups. Moreover, as the minimal non-trivial linear representations of \( \mathfrak{sl}(n, \mathbb{R}) \) and \( \mathfrak{sp}(2n, \mathbb{R}) \) occur in dimensions \( n \) and \( 2n \), respectively, Theorem 2.7 implies the volume-preserving case of Conjecture 1.2 for lattices in (groups isogenous to) \( \text{SL}(n, \mathbb{R}) \) and \( \text{Sp}(2n, \mathbb{R}) \). For the split orthogonal groups, the minimal linear representations occur in dimensions \( 2n = r(g) + 2 \) for \( g = \mathfrak{so}(n, n) \) and \( 2n + 1 = r(g) + 2 \) for \( g = \mathfrak{so}(n, n+1) \) and thus we are unable to recover the full conjecture for volume-preserving actions from Theorem 2.7.

3. Suspension action and proof of Theorem 2.8

We begin by introducing the suspension action with which we work for the remainder of the proof of Theorem 2.8. We then give some general background on Lyapunov exponents and state the two key propositions used in the proof of Theorem 2.8.

3.1. Suspension space. Recall we fix \( G \) to be a semisimple Lie group with real-rank at least 2. Let \( \Gamma \subset G \) be a cocompact lattice and let \( \alpha: \Gamma \to \text{Diff}^{1+\beta}(M) \) be an action for \( \beta > 0 \).

We construct an auxiliary space on which the action \( \alpha \) of \( \Gamma \) on \( M \) embeds as a Poincaré section for an associated \( G \)-action. On the product \( G \times M \) consider the right \( \Gamma \)-action

\[
(g, x) \cdot \gamma = (g\gamma, \alpha(\gamma^{-1})(x))
\]

and the left \( G \)-action

\[
a \cdot (g, x) = (ag, x).
\]

Define the quotient manifold \( M^\alpha := G \times M / \Gamma \). As the \( G \)-action on \( G \times M \) commutes with the \( \Gamma \)-action, we have an induced left \( G \)-action on \( M^\alpha \). For \( g \in G \) and \( x \in M^\alpha \) we denote this action by \( g \cdot x \) and denote the derivative of the diffeomorphism \( x \mapsto g \cdot x \) by \( Dg \). We write \( \pi: M^\alpha \to G/\Gamma \) for the natural projection map. Note that \( M^\alpha \) has the structure of a fiber-bundle over \( G/\Gamma \) induced by the map \( \pi \) with fibers diffeomorphic to \( M \). Note that the \( G \)-action preserves the fibers of \( M^\alpha \). As the action of \( \alpha \) is by \( C^{1+\beta} \) diffeomorphism, \( M^\alpha \) is naturally a \( C^{1+\beta} \) manifold. Equip \( M^\alpha \) with a \( C^\infty \) structure compatible with the \( C^{1+\beta} \) structure.

Equip \( M^\alpha \) with a Riemannian metric whose restriction to \( G \) orbits coincides under push-forward by the projection \( M^\alpha \to G/\Gamma \) with the metric on \( G/\Gamma \) induced by the right-invariant metric on \( G \).

3.2. Lyapunov exponents and Oseledec’s theorem. Let \( X \) be a locally compact metric space equipped with a (left) \( G \)-action. Let \( \mathcal{A}: G \times X \to \text{GL}(d, \mathbb{R}) \) be a bounded measurable linear cocycle; that is

1. \( \sup_{(g, x) \in K \times X} \| \mathcal{A}(g, x) \| \) is bounded for every compact \( K \subset G \);
2. \( \mathcal{A}(e, x) = \text{Id} \);
3. \( \mathcal{A}(g', g \cdot x) \mathcal{A}(g, x) = \mathcal{A}(g'g, x) \).
Given $s \in G$ and an $s$-invariant Borel probability measure $\mu$ on $X$ we define the average top (or leading) Lyapunov exponent of $A$ to be

$$
\lambda_+(s, \mu, A) := \inf_{n \to \infty} \frac{1}{n} \int \log \|A(s^n, x)\| \, d\mu(x).
$$

(2)

By the Kingman subadditive ergodic theorem, if $\mu$ is ergodic, the function $\frac{1}{n} \log \|A(s^n, x)\|$ converges $\mu$-a.e. to $\lambda_+(s, \mu, A)$.

We have the following elementary fact.

**Claim 3.1.** If the function $A(s, \cdot) : X \to \text{GL}(d, \mathbb{R})$ is continuous then $\mu \mapsto \lambda_+(s, \mu, A)$ is upper-semicontinuous on the set of all $s$-invariant Borel probability measures equipped with the weak-$\ast$ topology.

We also note that for an $s$-invariant measure $\mu$, the sequence $\frac{1}{n} \int \log \|A(s^n, x)\| \, d\mu(x)$ is subadditive whence the infimum in (2) maybe replaced by a limit.

We recall the following standard fact which is crucial in our later averaging arguments.

**Lemma 3.2.** Let $A : G \times X \to \text{GL}(d, \mathbb{R})$ be a bounded, continuous linear cocycle. Let $s \in G$ and let $\mu$ be an $s$-invariant, Borel probability measure on $X$. Let $H \subset G$ be an amenable subgroup contained in the centralizer of $s$ in $G$. Let $F_m$ be a Følner sequence in $H$ and let $\mu'$ be a Borel probability measure that is weak-$\ast$ subsequential limit of the sequence of measures $\{F_m \ast \mu\}$. Then

(a) $\mu'$ is $s$-invariant;

(b) $\lambda_+(s, \mu', A) \geq \lambda_+(s, \mu, A)$.

**Proof.** (a) follows as each $\{F_m \ast \mu\}$ is $s$-invariant and $s$-invariance is closed under weak-$\ast$ convergence.

For (b), first note that as $A$ is assumed bounded, it follows from the cocycle relation that $\lambda_+(s, F_m \ast \mu, A) = \lambda_+(s, \mu, A)$ for every $m$. Indeed, for any $t \in H$ the number $C_t = \sup x \in X \log \|A(t \pm 1, x)\|$ and let $C_m = \sup_{t \in F_m} C_t$. For $x \in M$ and $t \in F_m$ the cocycle property and subadditivity of norms yields

$$
\begin{align*}
\log \|A(s^n, tx)\| &\leq C_t + \log \|A(s^n t, x)\| \\
&= C_t + \log \|A(ts^n, x)\| \\
&\leq 2C_t + \log \|A(s^n, x)\| \\
&\leq 2C_m + \log \|A(s^n, x)\|.
\end{align*}
$$
Similarly we can prove that $\log \|A(s^n,tx)\| \geq \log \|A(s^n,x)\| - 2C_m$. 
Thus,
\[
\int \log \|A(s^n, x)\| \, d(F_m * \mu)(x) = \frac{1}{|F_m|} \int_{F_m} \int \log \|A(s^n, x)\| \, dt \, \mu(x) \\
= \frac{1}{|F_m|} \int_{F_m} \int \log \|A(s^n, tx)\| \, d\mu(x) \\
\leq \frac{1}{|F_m|} \int_{F_m} \left( 2C_m + \int \log \|A(s^n, x)\| \, d\mu(x) \right) \, dt \\
\leq 2C_m + \int \log \|A(s^n, x)\| \, d\mu(x)
\]
Dividing by $n$ yields $\lambda_+(s, F_m * \mu, A) \leq \lambda_+(s, \mu, A)$. The reverse inequality is similar. Conclusion $(b)$ follows from the upper semicontinuity in Claim $3.1$. 

Consider $A \subset G$ any abelian subgroup isomorphic to $\mathbb{R}^k$. Equip $A \cong \mathbb{R}^k$ with any norm $|\cdot|$. Consider an $A$-invariant, $A$-ergodic probability measure $\mu$ on $X$. For a bounded measurable linear cocycle $A: A \times X \to GL(d, \mathbb{R})$ we have the following consequence of the higher-rank Oseledec’s multiplicative ergodic theorem (c.f. [BRH, Theorem 2.4]).

**Proposition 3.3.** There are

1. an $\alpha$-invariant subset $\Lambda_0 \subset X$ with $\mu(\Lambda_0) = 1$;
2. linear functionals $\lambda_i: \mathbb{R}^k \to \mathbb{R}$ for $1 \leq i \leq p$;
3. and splittings $\mathbb{R}^d = \bigoplus_{i=1}^p E_{\lambda_i}(x)$ into families of mutually transverse, $\mu$-measurable subbundles $E_{\lambda_i}(x) \subset \mathbb{R}^d$ defined for $x \in \Lambda_0$

such that

(a) $A(s, x)E_{\lambda_i}(x) = E_{\lambda_i}(s \cdot x)$ and 
(b) $\lim_{|s| \to \infty} \frac{1}{|s|} \log |A(s, x)(v)| - \lambda_i(s) = 0$

for all $x \in \Lambda_0$ and all $v \in E_{\lambda_i}(p) \smallsetminus \{0\}$.

Note that $(b)$ implies for $v \in E_{\lambda_i}(x)$ the weaker result that for $s \in A$,

$$
\lim_{k \to \pm \infty} \frac{1}{k} \log |A(s^k, x)(v)| = \lambda_i(s).
$$

Also note that for $s \in A$, and $\mu$ an $A$-invariant, $A$-ergodic measure that

$$
\lambda_+(s, \mu, A) = \max_i \lambda_i(s).
$$

In the case that $\mu$ is $A$-invariant but not $A$-ergodic, Proposition 3.3 holds on each $A$-ergodic component of $\mu$. In this case we have the following construction which will be used later to avoid passing to ergodic components.

**Lemma 3.4.** If $\mu$ is an $A$-invariant measure on $X$ then for any $s' \in A$, there is a linear functional $\lambda_{+, s'}: A \to \mathbb{R}$ so that

1. $\lambda_{+, s'}(ts') = \lambda_+(ts', \mu, A)$ for any $t \geq 0$;
2. $\lambda_+(s, \mu, A) \geq \lambda_{+, s'}(s)$ for all $s \in A$.

**Proof.** Let $\{\mu_{x}^r\}$ be the decomposition of $\mu$ into $A$-ergodic components and let $\mathcal{L}_x = \{\lambda_{i,x}, 1 \leq i \leq r(x)\}$ be the collection of Lyapunov exponents for the cocycle $A$ and the measure $\mu_{x}^r$. Let $x \mapsto \lambda_{+, s', x} \in \mathcal{L}_x$ be a measurable $A$-invariant assignment satisfying

$$
\lambda_{+, s', x}(s') = \max_i \lambda_{i,x}(s').
$$
Take $\lambda_{+,s'}: A \to \mathbb{R}$ to be

$$\lambda_{+,s'}(s) = \int \lambda_{+,s',x}(s) \, d\mu(x).$$

$\lambda_{+,s'}(s)$ satisfies the properties of the Lemma. \hfill \Box

### 3.3. Subexponential growth of fiberwise derivatives.

We return to the setting introduced in Section 3.1. With $\pi: M^\alpha \to G/\Gamma$ the projection, let $F = \ker(D\pi)$ denote the fiberwise tangent bundle of $M^\alpha$.

We say the induced action of $G$ on $M^\alpha$ has uniform subexponential growth of fiberwise derivatives if for all $\epsilon > 0$ there is a $C$ such that

$$\|Dg|_F\| \leq Ce^{d(e,g)}$$

where $\|Dg|_F\| = \sup_{x \in M^\alpha} \|Dg(x)|_{F_x}\|$. As $\Gamma$ is cocompact, there is a clear relation between the growth of the fiberwise derivatives for the $G$-action and the growth of derivatives of the $\Gamma$-action.

**Claim 3.5.** $\alpha$ has uniform subexponential growth of derivatives if and only if the induced action of $G$ on $M^\alpha$ has uniform subexponential growth of fiberwise derivatives.

### 3.4. Proof of Theorem 2.8.

We let $A$ denote the fiberwise derivative cocycle for the action of $G$ on $M^\alpha$; that is $A(g, x) = D_xg|_F$. Let $A = \exp a \subset G$ be a maximal split Cartan subgroup. Given $s \in A$ and an $s$-invariant Borel probability $\mu$ we write

$$\lambda^+_F(s, \mu) := \lambda^+_F(s, \mu, A) = \inf_{n \to \infty} \frac{1}{n} \int \log \|D_x(s^n)|_F\| \, d\mu(x)$$

for the average top fiberwise Lyapunov exponent of $s$ with respect to $\mu$.

The proof of Theorem 2.8 is by contradiction. Assuming Theorem 2.8 fails, from Claim 3.5 we first establish the following.

**Proposition 3.6.** Suppose the induced action of $G$ on $M^\alpha$ fails to have uniform subexponential growth of fiberwise derivatives. Then there is an $s \in A$ and an $A$-invariant measure $\mu$ with $\lambda^+_F(s, \mu) > 0$.

As discussed above, Theorem 2.8 holds trivially in the case where $G$ has rank-1 factors. To complete the proof of Theorem 2.8 we may thus assume that all non-compact, almost-simple factors of $G$ are of higher-rank. The following proof of the following proposition contains the major technical innovations in this paper.

**Proposition 3.7.** Let $G$ be a connected, semisimple Lie group with finite center, all of whose non-compact, almost-simple factors are of real-rank at least 2. Let $\Gamma \subset G$ be a cocompact lattice and let $\alpha: \Gamma \to \text{Diff}^{1+\beta}(M)$ be an action. Suppose that either

1. $\dim(M) < r(G)$, or
2. $\dim(M) = r(G)$ and $\alpha$ preserves a smooth volume

and that there is an $s \in A$ and an $A$-invariant measure $\mu$ on $M^\alpha$ with $\lambda^+_F(s, \mu) > 0$. Then there is a $G$-invariant measure $\mu'$ and $s' \in A$ with $\lambda^+_F(s', \mu') > 0$.

From Proposition 3.7 we immediately obtain a contradiction with Zimmer’s cocycle superrigidity theorem and the fact that there are no non-trivial linear representations of $G$ into $\text{GL}(r(G), \mathbb{R})$ [Zim5]. Theorem 2.8 follows immediately from Propositions 3.6, 3.7 and Claim 3.5.
4. Proof of Proposition 3.6

To establish Proposition 3.6, suppose the induced action of \( G \) on \( M^\alpha \) fails to have uniform subexponential growth of fiberwise derivatives. Then there exist \( \varepsilon > 0 \), a sequence of elements \( g_n \) in \( G \) with \( d(e, g_n) \to \infty \), a sequence of base points \( x_n \in M^\alpha \), and a sequence of unit vectors \( v_n \in T_{x_n} M^\alpha \cap F \) tangent to the fibers of \( M_n \), satisfying

\[
\|D_{x_n} g_n(v_n)\| \geq e^{3cd(e, g_n)}.
\]

Let \( G = KAK \) be the Cartan decomposition of \( G \) (c.f. [Kna, Theorem 7.39]). For each \( g_n \), write \( g_n = k_n a_n k_n^\prime \) where \( k_n, k_n^\prime \in K \) and \( a_n \in A \). Note that \( a_n \to \infty \) as \( n \to \infty \). As \( K \) is a compact, the fiberwise derivative \( \sup_{k \in K} \|D_k|_F\| \) is bounded above and thus

\[
\|D_{x_n} a_n(v_n)\| \geq e^{2cd(a_n, e)}
\]

for all sufficiently large \( n \).

Recall that the Lie exponential \( \exp: g \to G \) restricts to a diffeomorphism from \( a \) to \( A \); moreover, \( \exp: a \to A \) is an isometry. Write \( a_n = \exp(t_n u_n) \) where \( u_n \) is a unit vector in \( a \) and \( t_n = d(a_n, e) \). Given \( t \in \mathbb{R} \) let \( \lfloor t \rfloor \) denote the integer part of \( t \). Then for sufficiently large \( n \) we have

\[
\|D_{x_n} \exp([t_n]u_n)(v_n)\| \geq e^{\lfloor t_n \rfloor}.
\]

Passing to a subsequence, we assume \( u_n \) converges to a unit vector \( u \in a \). The element \( s = \exp(u) \in A \) will be the element satisfying the conclusion of the proposition.

Recall \( F = \ker(D\pi) \) denotes the fiberwise tangent bundle of \( M^\alpha \). Let \( UF \) denote the unit-sphere bundle; that is, the quotient of \( F \) under the equivalence relation of positive proportionality in each fiber \( F(x) \) of \( F \). We represent elements of \( UF \) by pairs \( (x, v) \) where \( x \in M^\alpha \) and \( v \) is a unit vector in the fiber \( F(x) \). The derivative of the \( G \)-action on \( M^\alpha \) induces a \( G \)-action on \( F \) by fiber-bundle automorphisms; the map intertwining fibers is denoted by \( D_x g: F(x) \to F(gx) \). The \( G \)-action of \( F \) induces a \( G \)-action on \( UF \); we denote the map intertwining fibers by \( UF \) by \( UD_x g: UF(x) \to UF(gx) \).

For each \( n \), we define a Borel probability measure \( \nu_n \) on \( UF \). Given a continuous \( \phi: UF \to \mathbb{R} \) let

\[
\int \phi \, d\nu_n := \frac{1}{\lfloor t_n \rfloor} \sum_{m=0}^{\lfloor t_n \rfloor - 1} \phi(\exp(mu_n) \cdot (x_n), UD_x \exp(mu_n)(v_n)).
\]

Given \( g \in G \) and a probability measure \( \nu \) on \( UF \) consider the expression

\[
\psi(g, \nu) = \int_{UF} \log \left( \frac{\|D_x g(v)\|_F}{\|v\|_x} \right) \, d\nu(x, v).
\]

From the definition of \( \nu_n \) we have for every \( n \) that

\[
\psi(\exp(u_n), \nu_n) \geq \varepsilon.
\]

Consider any weak-* accumulation point \( \nu \) of the sequence of probability measures \( \{\nu_n\} \) on \( UF \). We have that \( \nu \) is invariant under \( s := \exp(u) \). Indeed, let \( f: UF \to \mathbb{R} \) be a continuous function. Then

\[
\int_{UF} f \circ s - f \, d\nu_n = \int_{UF} f \circ \exp(u) - f \circ \exp(u_n) \, d\nu_n + \int_{UF} f \circ \exp(u_n) - f \, d\nu_n.
\]
The first integral converges to zero as the functions $f \circ \exp(u) - f \circ \exp(u_n)$ converges uniformly to zero in $n$. The second integral clearly converges to zero by compactness and the definition of $\nu_n$. Taking $n \to \infty$,

$$\int_{UM} f \circ s \, d\nu = \int_{UM} f \, d\nu.$$ 

From uniform convergence and (3) we have

$$\psi(s, \nu) = \lim_{n \to \infty} \psi(\exp(u_n), \nu_n) \geq \varepsilon \quad (4)$$

Replacing $\nu$ with an ergodic component of $\nu$ satisfying (4), we can suppose $\nu$ is $s$-ergodic.

Let $\mu$ denote the natural projection of $UF$ onto $M^\alpha$, and let $\mu' = p_*\nu$. Clearly $\mu'$ is $s$-invariant and ergodic. We show that $\mu'$ has at least one non-zero fiberwise Lyapunov exponent. Indeed for $\nu$-almost every $(x_0, v_0)$ in $UF$, it follows from the pointwise ergodic theorem and the chain rule that

$$\varepsilon \leq \int_{UF} \log \left( \frac{\|D_x s(v)\|}{\|v\|} \right) \, d\nu(x, v)$$

$$= \lim_{N \to \infty} \frac{1}{N} \sum_{k=0}^{N-1} \log \left( \frac{\|D_{k,s} x_0 s(U D_{x_0} s^{k} v_0)\|}{\|U D_{x_0} s^{k} v_0\|} \right)$$

$$= \lim_{N \to \infty} \frac{1}{N} \log \left( \|D_{x_0} s^N(v_0)\| \right).$$

As inf$_{N \to \infty} \frac{1}{N} \log \left( \|D_{x_0} s^N |_{U}\| \right) \geq \varepsilon$ for $\mu'$-a.e. $x_0$, it follows $\lambda^F_{+}(s, \mu') \geq \varepsilon$.

Finally, averaging $\mu'$ against a Følner sequence in $A$ and passing to a subsequential limit $\mu$, from Lemma 3.2 we have that $\mu$ is $A$-invariant and $\lambda^F_{+}(s, \mu) \geq \lambda^F_{+}(s, \mu') > 0$. This completes the proof of Proposition 3.6.

5. Proof of Proposition 3.7

To prove Proposition 3.7 we apply an averaging argument to improve the regularity of the $A$-invariant measure on $M^\alpha$ with positive exponents produced in Proposition 3.6. Using measure rigidity results from homogeneous dynamics, the projection of the averaged measure $\bar{\mu}$ to $G/\Gamma$ will be the Haar measure. Using the key technical proposition of [BRHW3] and the algebraic results in subsection 2.2, we deduce that $\bar{\mu}$ is in fact $G$-invariant. We first recall some facts from homogeneous dynamics, particularly a number of results related to Ratner’s measure classification theorem, and then describe the averaging arguments in the proof. To illustrate the general argument, the averaging argument is explained for the special case of $SL(n, \mathbb{R})$ in Section 5.2.

5.1. Facts from homogeneous dynamics. Let $G$ be a connected, semisimple Lie group and let $\Gamma \subset G$ be a lattice. Recall that a nilpotent subgroup $U \subset G$ is called unipotent if $\text{ad}(u) - \text{Id}$ is a nilpotent for every element $u \in U$. Let $U = \exp u \subset G$ be a unipotent subgroup. Let $\{b_1, \ldots, b_k\}$ be a regular basis for $u$ (see [Sha]) and for $m = (m_1, \ldots, m_k) \in [0, \infty)^k$ let

$$F_m = \{\exp(t_1 b_1) \cdot \cdots \cdot \exp(t_k b_k) : 0 \leq t_j \leq m_j\} \subset U.$$ 

Let $|F_m|$ denote the Haar measure of $F_m$ in $U$. Recall for $x \in G/\Gamma$ we write $\nu_{x}^{F_m} = F^{\circ x} F_{m} + b_x$.

**Theorem 5.1** (Ratner, Shah). Let $X = G/\Gamma$ and let $U$ be a unipotent subgroup. The following hold:
(a) Every ergodic, $U$-invariant measure is homogeneous [Rat, Theorem 1].
(b) The orbit closure $O_x := \{ u \cdot x : u \in U \}$ is homogenous for every $x \in G/\Gamma$ [Rat, Theorem 3].
(c) The orbit $F_m \cdot x$ equidistributes in $O_x$; that is $\nu_{x}^{F_m}$ converges to the Haar measure on $O_x$ as $m_1 \to \infty, \ldots, m_k \to \infty$ [Sha, Corollary 1.3].
(d) Let $A = \exp a$ be a maximal split Cartan, let $\beta$ be a restricted root of $g$ relative to $a$, and let $\mu$ be a $G^{[\beta]}$-invariant Borel probability measure. If $\mu$ is $A$-invariant then $\mu$ is $G^{[-\beta]}$-invariant.

Note that (d) follows from [Rat, Theorem 9] and the structure of $sl(2, \mathbb{R})$-triples.

Given $x \in G/\Gamma$, let $m^U_x$ denote the Haar measure on the homogeneous manifold $O_x$ in Theorem 5.1(b). Given a measure $\mu$ on $G/\Gamma$ let

$$U \ast \mu = \int m^U_x \, d\mu(x).$$

**Proposition 5.2.** Let $A = \exp a$ be a maximal split Cartan subgroup and let $U = \exp u$ be a unipotent subgroup normalized by $A$. Let $\mu$ be a Borel probability measure on $G/\Gamma$. Then

(a) $F_m \ast \mu \to U \ast \mu$ for any $m_1 \to \infty, \ldots, m_k \to \infty$;
(b) If $\mu$ is $A$-invariant, so is $U \ast \mu$;
(c) If $\mu$ is $A$-invariant, and $A$-ergodic then $U \ast \mu$ is $A$-ergodic.

**Proof.** For $x \in G/\Gamma$ we have that

$$\nu_{x}^{F_m} := F_m \ast \delta_x$$

converges to the Haar measure $m^U_x$ on the orbit closure $O_x$ of $U \cdot x$. By dominated convergence we have

$$F_m \ast \mu = \int \nu_{x}^{F_m} \, d\mu(x) \to \int m^U_x \, d\mu(x) = U \ast \mu$$

and (a) follows.

For (b), note that if $s \in A$ and if $\{ F_m \}$ is a Følner sequence as above, then $\{ sF_m s^{-1} \}$ is also a Følner sequence as above. From the $s$-invariance of $\mu$ and equidistribution in Theorem 5.1(c) we have that

$$s_*(U \ast \mu) = s_* \left( \lim \int \nu_{x}^{F_m} \, d\mu(x) \right)$$

$$= \lim s_* \left( \int \nu_{x}^{F_m} \, d\mu(x) \right)$$

$$= \lim \int s_* \nu_{s x}^{F_m s^{-1}} \, d\mu(x)$$

$$= \lim \int \nu_{x}^{F_m s^{-1}} \, d\mu(x)$$

$$= \int m^U_x \, d\mu(x).$$

For (c) we note that as $U = \exp u$ is unipotent and normalized by $A$, there is an $s \in A$ such that $U$ is contracted by $s$; that is

$$u \subset \bigoplus_{\beta : \beta(s) < 0} g^\beta.$$
For such an $s$, the pointwise ergodic theorem implies that the $s$-ergodic components of $U \ast \mu$ are refined by the measurable hull of the partition of $G/\Gamma$ into $U$-orbits. In particular, if $\phi$ is a bounded, $A$-invariant measurable function then

$$\phi(x) = \int \phi(x) \, dm^U_x$$

for $U \ast \mu$-a.e. $x$. But $x \mapsto \int \phi(x) \, dm^U_x$ is an $A$-invariant, $\mu$-measurable function, whence by the $A$-ergodicity of $\mu$, is constant $\mu$-a.e. Then $\phi$ is constant $(U \ast \mu)$-a.e. \hfill \Box

5.2. Averaging argument for $G = \text{SL}(n, \mathbb{R})$. We explain the first step of the proof of Proposition 3.7 in the case $G = \text{SL}(n, \mathbb{R})$, $n \geq 3$. Taking the Cartan involution $\theta: \mathfrak{sl}(n, \mathbb{R}) \to \mathfrak{sl}(n, \mathbb{R})$ to be $\theta(X) = -X^t$ we have

$$A = \text{diag}(e^{t_1}, e^{t_2}, \ldots, e^{t_n}) = \begin{pmatrix} e^{t_1} & & & \\ & e^{t_2} & & \\ & & \ddots & \\ & & & e^{t_n} \end{pmatrix}$$

where $t_1 + t_2 + \cdots + t_n = 0$. Also, $m = \{0\}$, $M$ is the finite group with $\pm 1$ along the diagonals, $K = \text{SO}(n)$ and (relative to the standard base)

$$N = \begin{pmatrix} 1 & * & * & \cdots & * \\ 1 & * & \cdots & * \\ & \ddots & \vdots & \vdots \\ & & 1 & * \\ & & & 1 \end{pmatrix}.$$ 

For $i \neq j \in \{1, \ldots, n-1\}$ let $\beta_{i,j}: A \to \mathbb{R}$ be the linear functional

$$\beta_{i,j}(\text{diag}(e^{t_1}, e^{t_2}, \ldots, e^{t_n})) = t_i - t_j.$$ 

These are the roots of $\mathfrak{sl}(n, \mathbb{R})$ and the standard base for $\Sigma(\mathfrak{sl}(n, \mathbb{R}))$ is

$$\Pi = \{\alpha_1 = \beta_{1,2}, \alpha_2 = \beta_{2,3}, \ldots, \alpha_{n-1} = \beta_{n-1,n}\}.$$ 

To prove Proposition 3.7 it is enough to find an $A$-invariant measure $\mu'$ on $M^\alpha$ with a non-zero fiberwise Lyapunov exponent projecting to the Haar measure on $G/\Gamma$. By Proposition 2.6 and Proposition 5.8 below, such a measure will automatically be $G$-invariant.

By the hypotheses of Proposition 3.7, we have an ergodic, $A$-invariant measure $\mu$ with a non-zero fiberwise Lyapunov exponent $\lambda_\mu^F: A \to \mathbb{R}$. Note that $\mu$ need not project to the Haar measure on $G/\Gamma$. Our goal will be to average $\mu$ over various subgroups of $G$ in order to obtain a new $A$-invariant measure $\mu'$ projecting to Haar. The subtlety of the argument is to choose the subgroups so that the fiberwise Lyapunov exponents do not vanish after averaging.

Recall that $\lambda_\mu^F: A \to \mathbb{R}$ and each $\beta_{i,j}: A \to \mathbb{R}$ are non-zero linear functionals. Consider the linear span $V$ of $\{\alpha_2, \cdots, \alpha_{n-1}\}$ in $\mathfrak{a}^\ast$. It may be that $\lambda_\mu^F \in V$. However, given a permutation matrix (that is an element of the Weyl group) $P \in \text{SL}(n, \mathbb{R})$, let

$$P(\lambda_\mu^F)(s) = \lambda_\mu^F(P^{-1}sp).$$

One may check (as the Weyl group acts irreducibly on $\mathfrak{a}^\ast$) that for some $P$, $P(\lambda_\mu^F) \notin V$. Thus, up to conjugating $G$ by a permutation matrix, without loss of generality we may assume $\lambda_\mu^F: A \to \mathbb{R}$ is not in the linear span of $\{\alpha_2, \cdots, \alpha_{n-1}\}$.
Let $U$ be the unipotent subgroup
\[
U = \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 \\
1 & * & * & \cdots & * \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & * & & & *
\end{pmatrix}
\]
and let
\[
s_1 = \text{diag} \left( \frac{1}{6^{n-1}}, 6, \cdots, 6 \right) \in A
\]
Note that $s_1$ commutes with every element of $U$ and since $\lambda_\mu^F(s_1)$ is not in the linear span of $\{\alpha_2, \cdots, \alpha_{n-1}\}$, 
\[
\lambda_\mu^F(s_1) \neq 0.
\]
Replacing $s_1$ with $s_1^{-1}$, we may assume $\lambda_\mu^F(s_1) > 0$.

Take a Følner sequence along $U$, average the measure $\mu$, and pass to a subsequential limit $\mu_1$. From Theorem 5.2, we have that $\mu_1$ projects to an $AU$-invariant measure $\hat{\mu}_1$ in $G/\Gamma$. Note however that $\mu_1$ may not be $AU$-invariant. From Lemma 3.2 however, $\mu_1$ is $s_1$-invariant and $\lambda_\mu^F(s_1, \mu_1) > 0$. Averaging $\mu_1$ along a Følner sequence in $A$ and taking a subsequential limit $\mu_2$, we have $\mu_2$ is $A$-invariant, $\lambda_\mu^F(s_1, \mu_2) > 0$. Moreover, as the projection $\hat{\mu}_1$ of $\mu_1$ is an $AU$-invariant measure, $\mu_2$ and $\mu_1$ project to the same $AU$-invariant measure $\hat{\mu}_1 = \hat{\mu}_2$ in $G/\Gamma$. From Theorem 5.1(d), it follows $\hat{\mu}_1 = \hat{\mu}_2$ is $G'$-invariant where
\[
G' = \begin{pmatrix}
* & 0 & 0 & \cdots & 0 \\
0 & * & * & \cdots & * \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & * & * & & *
\end{pmatrix}
\]
Let $\lambda_{+, s_1, \mu_2} : A \to \mathbb{R}$ be the linear functional as in Lemma 3.4. Consider the two roots 
\[
\alpha_1 = \beta_{1,2} : A \to \mathbb{R}, \quad \delta = \beta_{1,n} : A \to \mathbb{R}
\]
(the simple root $\alpha_1$ and the highest root $\delta$.) Note that $\lambda_{+, s_1, \mu_2}$ is proportional to at most one of $\alpha_1$ or $\delta$.

Assume that $\lambda_{+, s_1, \mu_2}$ not proportional to $\alpha_1$. Let
\[
U' = \begin{pmatrix}
1 & * & 0 & \cdots & 0 \\
1 & 0 & & & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
1 & & & & 1 \\
1 & & & & 1
\end{pmatrix}
\]
and select any $s_2 \in \ker \alpha_1 \setminus \ker \lambda_{+, s_1, \mu_2}$.

Replacing $s_2$ with $s_2^{-1}$ if necessary, we have $\lambda_\mu^F(s_2, \mu_2) \geq \lambda_{+, s_1, \mu_2}(s_2) > 0$. Average $\mu_2$ along the one-parameter subgroup $U'$ and pass to a subsequential limit $\mu_3$. $\mu_3$ projects to an $AU'$-invariant measure $\hat{\mu}_3$ in $G/\Gamma$. Average $\mu_3$ along $A$ and pass again to a subsequential limit $\mu_4$. We then have
\begin{enumerate}
\item $\mu_4$ is $A$-invariant;
\item $\lambda_\mu^F(s_2, \mu_4) > 0$;
\item $\mu_4$ projects to an $AU'$-invariant measure $\hat{\mu}_4 = \hat{\mu}_3$ on $G/\Gamma$.
\end{enumerate}
We note that $U'$ commutes with the subgroup $H \subset G'$

$$H = \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
0 & * & * & \cdots & * \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & * & * & \cdots & *
\end{pmatrix}$$

whence $\hat{\mu}_3 = \hat{\mu}_4$ is also invariant under $H$ and $A$. From Theorem 5.1(d), it follows that the projection $\hat{\mu}_4 = \hat{\mu}_3$ in $G/\Gamma$ is invariant under the groups

$$\begin{pmatrix}
* & * & 0 & \cdots & 0 \\
* & * & 0 & \cdots & 0 \\
0 & 0 & 1 & 0 & \cdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1
\end{pmatrix}, \quad \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & * & * & \cdots & * \\
0 & * & * & \cdots & * \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & * & * & \cdots & *
\end{pmatrix}.$$

Since these generate $G$, the projection $\hat{\mu}_4$ is the Haar measure on $G/\Gamma$. Taking an appropriate $A$-ergodic component $\mu'$ of $\mu_4$ we have

1. $\mu'$ is $A$-invariant and $A$-ergodic;
2. $\mu'$ projects to the Haar measure on $G/\Gamma$;
3. $\lambda_{+}(s_2, \mu') > 0$ whence $\mu'$ has a non-zero fiberwise Lyapunov exponent.

Above we assumed $\lambda_{+}(s_1, \mu_2)$ was not proportional to $\alpha_1$. If $\lambda_{+}(s_1, \mu_2)$ is proportional to $\alpha_1$ then it is not proportional to $\delta$ and we may take

$$U' = \begin{pmatrix}
1 & 0 & 0 & \cdots & * \\
1 & 0 & 0 & \cdots \\
\ddots & \ddots & \ddots & \ddots & \ddots \\
1 & 0 \\
1
\end{pmatrix}$$

and select any $s_2 \in \ker \delta \setminus \ker \lambda_{+}(s_1, \mu_2)$. We may repeat the above arguments (which are now slightly simpler as $U'$ and $U$ commute) to obtain $\mu_4$ and $\mu'$ with the same properties as before.

### 5.3. Averaging algorithm on $G/\Gamma$

We present in this and the next subsection the generalization of the averaging procedure described in Section 5.2 for general Lie groups. Here, we describe what happens to the projection of the measure to $G/\Gamma$ as we average the measure on $M^\alpha$ over various subgroups of $G$.

Let $\mathfrak{g}$ be a semisimple Lie algebra. Let $\mathfrak{g}'$ be an ideal of $\mathfrak{g}$ with rank $\ell \geq 2$ and $G' \subset G$ be the corresponding analytic subgroup. Let $\Sigma$ be the set of restricted roots of $\mathfrak{g}'$ and let $\Pi$ be a choice of base generating a system of positive roots $\Sigma_+$. Let $\Pi = \{\alpha_1, \alpha_2, \ldots, \alpha_\ell\}$ be enumerated such that $\alpha_1$ is the left-most element in the corresponding Dynkin diagram as drawn in Appendix A.

**Proposition 5.3.** With respect to $\Pi$, let $\hat{\beta}$ be either

1. $\hat{\beta} = \delta$, the highest root, if $\mathfrak{g}'$ is of type $A_\ell$, $B_\ell$, $D_\ell$, $E_6$, or $E_7$;
2. $\hat{\beta} = \delta'$, the 2nd highest root, if $\mathfrak{g}'$ is of type $C_\ell$, $(BC)_\ell$, $E_8$, $F_4$, or $G_2$.

Let $u$ be the Lie subalgebra generated by $\{\mathfrak{g}^{\alpha_1}, \ldots, \mathfrak{g}^{\alpha_\ell}\}$ and let $U = \exp u$. Let $u'$ denote either the Lie subalgebra $\mathfrak{g}^{\alpha_1}$ or the Lie subalgebra $\mathfrak{g}^{\hat{\beta}}$ and let $U' = \exp u'$. 

Let $\Gamma \subset G$ be a lattice and let $\mu$ be an $A$-invariant measure on $G/\Gamma$. Then
\[ U' * (U * \mu) \]
is $G'$-invariant.

**Remark 5.4.** The choice of $\hat{\beta}$ as the highest root $\delta$ or second highest root $\delta'$ in Proposition 5.3 ensures the following two properties hold:

1. $U^\hat{\beta}$ and $U^{\alpha_j}$ commute for each $2 \leq j \leq \ell$;
2. there is a string of roots
   \[ \beta_0 = \alpha_1, \beta_2, \beta_3, \ldots, \beta_p = \hat{\beta} \]
   such that $\beta_{k_i} = \beta_{k_{i-1}} + \alpha_{j_i}$ for some $2 \leq j_i \leq \ell$ for each $1 \leq i \leq m$.

If $g'$ is of type $C_{\ell_1}, (BC)_{\ell_2}, E_8, F_4$, or $G_2$ the first property holds for the highest root $\hat{\beta} = \delta$ but the second property fails as $\alpha_1$ has a coefficient of 2 in $\delta$. (See Table 1 in Appendix A.)

The second property is used below to obtain $G$-invariance after two steps of averaging by obtaining invariance under subgroups which generate $G$.

Note also in the case that $\Sigma(g')$ is of type $(BC)_{\ell_1}$, neither $\hat{\beta} = \delta'$ nor $\hat{\beta} = \alpha_1$ is positively proportional to any other root. In particular $u' = g^\hat{\beta}$ is, in fact, a Lie subalgebra.

**Proof.** Note that $U * \mu$ is $U$-invariant. Let $\nu$ denote $U' * (U * \mu)$.

Consider first the case that $u' = g^\hat{\beta}$. From the choice of $\hat{\beta}$, $g^\hat{\beta}$ commutes with each of $g^{a_j}$ for every $2 \leq j \leq \ell$. From Lemma 3.2(a), $\nu$ is $U$-invariant. From Theorem 5.2(b) the measure $\nu$ is also $A$-invariant. It follows from Theorem 5.1(d) that $\nu$ is $\exp(g^{-a_j})$-invariant for $2 \leq j \leq \ell$. From the choice of $\hat{\beta}$ and examining tables of positive roots, there is a sequence of roots $a_1 = \beta_0, \beta_1, \ldots, \beta_p = \hat{\beta}$ where $\beta_{k_i} = \beta_{k_{i-1}} + (\alpha_{j_i})$ for some $2 \leq j_i \leq \ell$ and every $1 \leq k \leq p$. It follows that $\nu$ is $\exp(g^{a_1})$-invariant. It follows that $\nu$ is $G'$-invariant.

In the case that $u' = g^{a_1}$ we first observe that, as $U * \mu$ is $U$-invariant, $U * \mu$ is $\exp(g^{-a_j})$-invariant for every $2 \leq j \leq \ell$. Since $g^{a_1}$ commutes with $g^{-a_j}$ for every $2 \leq j \leq \ell$ it follows that $\nu$ is $\exp(g^{-a_1})$-invariant for every $2 \leq j \leq \ell$. As $\nu$ is $A$-invariant, it follows that $\nu$ is $U$-invariant and, as above, $\nu$ is $G'$-invariant. \square

### 5.4. Averaging algorithm on $M^\alpha$

Recall that $G$ is a connected, semisimple Lie group with finite center and all non-compact, almost-simple factors of real-rank at least 2. Recall the $G$-action on $X = M^\alpha$ preserves the fiberwise tangent bundle $F = \ker D\pi$. Let $A = \exp a \subset G$ be our fixed maximal split Cartan subgroup.

We assume as in Proposition 3.7 and that there is an $s \in A$ and an $A$-invariant Borel probability measure $\mu$ on $M^\alpha$ with $\lambda^F_k(s, \mu) > 0$. Let $g = \bigoplus_{k=1}^p g^k_k$ be the decomposition of $g$ into ideals. For each $g^k_k$ let $G^k_k \subset G$ be the corresponding analytic subgroup.

To complete the proof of Proposition 3.7, we show the following.

**Lemma 5.5.** For $1 \leq j \leq p$, if the projection of $\mu$ to $G/\Gamma$ is $G^k_k$-invariant for all $1 \leq k \leq j - 1 < p$ then there is an $s \in A$ and an $A$-invariant Borel probability measure $\mu'$ on $M^\alpha$ with $\lambda^F_k(s, \mu') > 0$ such that the projection of $\mu'$ to $G/\Gamma$ is $G^k_k$-invariant for all $1 \leq k \leq j \leq p$.

**Proof.** Fix such $G^j_j$ with Lie algebra $g^j_j$. Note that from Lemma 3.2, if $G^j_j$ is compact then, as $G^j_j$ commutes with $A$ and $G^k_k$ for all $1 \leq k \leq j - 1$, by averaging $\mu$ over the action of $G^j_j$ we obtain a measure $\mu'$ with the desired properties.

We may thus assume $G^j_j$ is non-compact and hence of rank at least 2 for the remainder. Let $U, U'$ be as in Proposition 5.3 where the choice of base II and $\hat{\beta}$ determining $U$ and $U'$
Claim 5.6. There is a choice of base $\Pi \subset \Sigma(g)$ and a choice of $\hat{\beta}$ in Proposition 5.3 such that for $U$ and $U'$ as in Proposition 5.3, Følner sequences $F_j$, $F_j'$, $F_j''$ as above, and $\mu'$ as above

(a) $\mu'$ projects to a measure on $G/\Gamma$ that is $G_k^\ell$-invariant for all $1 \leq k \leq j$;
(b) $\lambda^F_k(s', \mu') > 0$ for some $s' \in A$.

Lemma 5.5 follows immediately from the above claim. \hfill \Box

We finish the proof of Lemma 5.5 with the proof of Claim 5.6.

Proof of Claim 5.6. For any choice of $\Pi$ and choice of $\hat{\beta}$, let $\hat{\mu}_i$ denote the image of $\mu_i$ in $G/\Gamma$. We have that $\hat{\mu}_0$ is $A$-invariant. We have that $\hat{\mu}_1 = U \ast \hat{\mu}_0$ is $A U$-invariant whence $\hat{\mu}_2 = \hat{\mu}_1$. From Proposition 5.3 we have that $\hat{\mu}_3 = U' \ast (U \ast \hat{\mu}_0)$ is $G'_j \ast$-invariant. As $U \subset G'_j$ and $U' \subset G'_j$, and as $G'_k$ and $G'_k'$ commute for $k \neq k'$, it follows from Lemma 3.2(a) that $\hat{\mu}_3$ is $G'_j \ast$-invariant for all $1 \leq k \leq j - 1$. Then clearly $\hat{\mu}_4$ is $G'_j \ast$-invariant for all $1 \leq k \leq j$. (a) follows.

For (b) recall that we assume $\lambda^F_k(s, \mu_0) > 0$ for some $s \in A$. Recall the linear functional $\lambda_{+s, \mu_0}: A \rightarrow \mathbb{R}$ constructed in Lemma 3.4 with $\lambda_{+s, \mu_0}(s) = \lambda^F_k(s, \mu_0)$. Also, recall that restricted roots $\beta: A \rightarrow \mathbb{R}$ are linear functionals on $A$.

We claim there is a choice base $\Pi = \{\alpha_1, \ldots, \alpha_\ell\}$ so that $\lambda_{+s, \mu_0}$ not in the linear span of $\{\alpha_2, \ldots, \alpha_\ell\}$. Indeed, the Weyl group of $\Sigma(g'_j)$ acts irreducibly on $\ast'$ and simply transitively on bases $\Pi$ of $\Sigma(g'_j)$. Moreover the Weyl group preserves angles and lengths so if $\Pi = \{\alpha_1, \ldots, \alpha_\ell\}$ is a base of $\Sigma(g'_j)$ and $\Pi' = \{\alpha'_1, \ldots, \alpha'_\ell\} = \{w(\alpha_1), \ldots, w(\alpha_\ell)\}$ is the image of $\Pi$ under an element $w$ in the Weyl group, then the vertices $\{\alpha'_1, \ldots, \alpha'_\ell\}$ and $\{\alpha_1, \ldots, \alpha_\ell\}$ generate the same Dynkin diagram with the same ordering on the vertices. For a fixed $\Pi'' = \{\alpha'_1, \ldots, \alpha'_\ell\}$, there is an element $w$ of the Weyl group such that $w(\lambda_{+s, \mu_0})$ is not in the linear span of $\{\alpha_2, \ldots, \alpha_\ell\}$. Then, letting $\Pi = \{\alpha_1, \ldots, \alpha_\ell\}$ map to $\Pi'$ under $w$, we have that $\lambda_{+s, \mu_0}$ is not in the linear span of $\{\alpha_2, \ldots, \alpha_\ell\}$.

We fix this choice of $\Pi = \{\alpha_2, \ldots, \alpha_\ell\}$ for the remainder.

Let $U$ be as in Proposition 5.3 for the above choice of $\Pi$. Fix $s_1 \in A \setminus \ker \lambda_{+s, \mu_0}$ such that $\alpha_j(s_1) = 0$ for all $2 \leq j \leq \ell$. Replacing $s_1$ with $s_1^{-1}$ if needed we have

1. $U$ commutes with $s_1$;
2. $\lambda^F_+(s_1, \mu_0) \geq \lambda_{+s, \mu_0}(s_1) > 0$.

In then follows from Lemma 3.2 that

1. $\mu_1$ is $s_1$-invariant;
2. $\lambda^F_+(s_1, \mu_1) \geq \lambda^F_+(s_1, \mu_0) > 0$;
3. $\lambda^F_+(s_1, \mu_2) \geq \lambda^F_+(s_1, \mu_1) > 0$.
We say a restricted root \( \lambda \) if no such \( \lambda \) exists.

We may select a \( \lambda^{\pm} \) in Proposition 5.3 (relative to the choice of \( \pi \) above).

Note that \( \beta \) and \( \alpha_1 \) are not proportional. In particular \( \lambda^{\pm} \) is proportional to at most one of \{\( \beta \), \( \alpha_1 \)\}. Let \( \beta' \in \{\beta, \alpha_1\} \) be such that \( \beta' \neq c\lambda^{\pm} \) for any \( c \in \mathbb{R} \) and take \( u' \) in Proposition 5.3 to be \( u' = g^{\beta'} \). Fix \( s_2 \in \mathbb{A} \) with \( \beta'(s_2) = 0 \) and \( \lambda^{\pm}(s_2) > 0 \).

From Lemma 3.2 we have that

1. \( \mu_3 \) is \( s_2 \)-invariant;
2. \( \lambda_1^F(s_2, \mu_3) \geq \lambda_1^F(s_2, \mu_2) \geq \lambda^{\pm}(s_2) > 0 \);
3. \( \lambda_1^F(s_2, \mu_3) > 0 \).

Taking \( s' = s_2 \) completes the proof of the claim.

5.5. Proof of Proposition 3.7. From Lemma 5.5 it follows that there exists an \( s \in A \) and an \( A \)-invariant Borel probability measure \( \mu' \) on \( M^\alpha \) with \( \lambda_1^F(s, \mu') > 0 \) such that the projection of \( \mu' \) to \( G/\Gamma \) is \( G \)-invariant. In particular, \( \mu' \) projects to the Haar measure on \( G/\Gamma \).

Let \( M \) denote the centralizer of \( A \) in \( K \), and \( \mu'' = M \star \mu' \). Since \( M \) commutes with \( A \) we have

1. \( \mu'' \) is \( MA \)-invariant;
2. \( \lambda_1^F(s, \mu'') \geq \lambda_1^F(s, \mu') > 0 \);
3. \( \mu'' \) projects to the Haar measure on \( G/\Gamma \).

Consider an \( MA \)-ergodic component \( \bar{\mu} \) of \( \mu'' \). As the Haar measure on \( G/\Gamma \) is \( MA \)-ergodic, it follows that any such \( \bar{\mu} \) projects to the Haar measure on \( G/\Gamma \). With \( s \) as above, we may select \( \bar{s} \) so that \( \lambda_1^F(s, \bar{\mu}) > 0 \).

**Definition 5.7.** Given an \( A \)-invariant, \( A \)-ergodic measure \( \mu \) on \( M^\alpha \), let \( \mathcal{L}^F = \{\lambda_1^F\} \) denote the Lyapunov exponent functionals for the fiberwise derivative cocycle for the measure \( \mu \).

We say a restricted root \( \beta \in \Sigma(g) \) is non-resonant with the fiberwise exponents of \( \mu \) if there is \( \lambda_i^F \in \mathcal{L}^F \) and \( c > 0 \) with

\[ \beta = c\lambda_i^F. \]

If no such \( \lambda_i^F \) and \( c \) we say \( \beta \) is non-resonant.

Note that resonance and non-resonance descend to coarse equivalence classes of restricted roots \( [\beta] \in \bar{\Sigma}(g) \).

We recall the following key observation from [BRHW3]. Note that if \( G \) has compact factors the Haar measure on \( G/\Gamma \) may fail to be \( A \)-ergodic.

**Proposition 5.8** ([BRHW3, Proposition 5.1]). Let \( \bar{\mu} \) be an \( A \)-invariant Borel probability measure on \( M^\alpha \) projecting to the Haar measure on \( G/\Gamma \). Let \( \mu \) be an \( A \)-invariant, \( A \)-ergodic component of \( \bar{\mu} \). Then, given a coarse restricted root \( [\beta] \in \bar{\Sigma} \) that is non-resonant with the fiberwise Lyapunov exponents of \( \mu \), the measure \( \mu \) is \( G^{[\beta]} \)-invariant.

Note that the group \( M \) acts transitively on the set of \( A \)-ergodic components of \( \bar{\mu} \). Moreover, as \( M \) commutes with \( A \), the group \( M \) preserves the Lyapunov exponents for the \( A \)-action with respect to distinct \( A \)-ergodic components of \( \bar{\mu} \). In particular, the set of roots of \( g \) that are non-resonant with the fiberwise exponents is constant for a.e. \( A \)-ergodic component of \( \bar{\mu} \). Let \( \Sigma_{NR, \mu} \) denote the a.s. constant collection of restricted roots of \( g \) that are non-resonant with the fiberwise exponents (of ergodic components of \( \bar{\mu} \)).
Let $\mathfrak{h} \subset \mathfrak{g}$ be the Lie subalgebra generated by
\[ \mathfrak{m} \oplus \mathfrak{a} \oplus \bigoplus_{\beta \in \Sigma_{NR, \mu}} \mathfrak{g}^{[\beta]} . \]

As there are at most $\dim(M)$ fiberwise Lyapunov exponents it follows that there are at most $\dim(M)$ resonant coarse restricted roots. It follows that $\mathfrak{h}$ has resonant codimension at most $\dim(M)$. As we assume $\dim(M) \leq r(\mathfrak{g})$, it follows from Proposition 2.6 that $\mathfrak{h}$ is parabolic.

Let $H \subset G$ be the analytic subgroup with Lie algebra $\mathfrak{h}$. Proposition 5.8 guarantees that $\bar{\mu}$ is $H$-invariant. We claim $H = G$. Indeed if $\dim(M) < r(G)$ then $\mathfrak{g} = \mathfrak{h}$ follows immediately from the minimality of $r(G)$. If $\dim(M) = r(g)$ and $H \neq G$ then, as $\mathfrak{h}$ is parabolic, we have
\[ \mathfrak{h} = \mathfrak{m} \oplus \mathfrak{a} \oplus \bigoplus_{\beta \in \Sigma_{NR, \mu}} \mathfrak{g}^{[\beta]} . \]

It follows that every fiberwise Lyapunov exponent is positively proportional with some restricted root $\beta$ with $\mathfrak{g}^{[\beta]} \cap \mathfrak{h} = 0$. In particular, there is an $s \in \mathfrak{a}$ such that $\lambda^F_s(s) < 0$ for every fiberwise Lyapunov exponent $\lambda^F_s \in \mathcal{L}^F$. However, the sum of all fiberwise exponents is zero in case that the $G$-action preserves a smooth volume in the fibers. It thus follows under the hypotheses of Proposition 3.7(2) that $\bar{\mu}$ is $G$-invariant. This completes the proof of the proposition.

6. Finding Smooth Metrics

In this section we prove Theorem 2.9. In particular, we establish the existence of an invariant Riemannian metric from uniform subexponential growth of derivatives in conjunction with the strong property (T) of Lafforgue.

6.1. Lafforgue’s strong property (T). We recall basic facts about strong property (T). The reader only interested in the case of $C^\infty$ actions may consider only representations into Hilbert spaces and ignore the class of Banach spaces $\mathcal{E}_{10}$ introduced in [dLdIS]. This in fact suffices to prove theorems for actions by $C^k$ diffeomorphisms on a manifold $M$ when $k = \frac{\dim(M)}{2} + 2$.

**Definition 6.1.** Let $\Gamma$ be a group with a length function $l$, $X$ a Banach space and $\pi: \Gamma \to B(X)$. Given $\varepsilon > 0$, we say $\pi$ has $\varepsilon$-subexponential norm growth if there exists a constant $L$ such that $\|\pi(\gamma)\| \leq Le^{\varepsilon l(\gamma)}$ for all $\gamma \in \Gamma$. We say $\pi$ has subexponential norm growth if it has $\varepsilon$-subexponential norm growth for all $\varepsilon > 0$.

Given a group $\Gamma$ and a generating set $S$ and let $l$ be the word length on $\Gamma$. Here we say a group $\Gamma$ has strong property (T) if it has strong property (T) on Banach spaces for Banach spaces of class $\mathcal{E}_{10}$ in the quantitative sense of [dLdIS, Section 6]. In what follows $X$ will denote a Banach space and $B(X)$ will denote the bounded operators on $X$. We will always be considering the operator norm topology on $B(X)$ and we will always mean the operator norm when we write $\|T\|$ for $T \in B(X)$.

**Definition 6.2.** A group $\Gamma$ has strong property (T) if there exists a constant $t > 0$ and a sequence of measures $\mu_n$ supported in the balls $B(n) = \{ \gamma \in \Gamma \mid l(\gamma) \leq n \}$ in $\Gamma$ such that for every Banach space $X \in \mathcal{E}_{10}$ the following holds: For any representation $\pi: \Gamma \to B(X)$ with $t$-subexponential norm growth the operators $\pi(\mu_n)$ converge exponentially quickly to a projection onto the space of invariant vectors. That is, there exists $0 < \lambda < 1$ (independent of $\pi$), a projection $P \in B(X)$ onto the space of $\Gamma$-invariant vectors, and an $n_0 \in \mathbb{N}$ such that $\|\pi(\mu_n) - P\| < \lambda^n$ for all $n \geq n_0$. 

...
We recall the following results obtained from combining results in [Laf, dLdLS]:

**Theorem 6.3.** Let $G$ be a connected semisimple Lie group with all simple factors of higher-rank and $\Gamma < G$ a cocompact lattice. Then $G$ and $\Gamma$ have strong property (T).

**Proof.** For the connected Lie group, this is proven explicitly in [dLdLS, Section 6]. For the cocompact lattices, this follows from that fact using the proof of [Laf, Proposition 4.3]. In particular the $\mu_n$ for $\Gamma$ are constructed there explicitly from $\mu'_n$ for $G$ and the properties we desire are all follow immediately from this definition since the function $f$ is chosen in $C_c(G)$. A priori, this produces a sequence of measures $\mu_n$ with support in $B(Dn)$ for some fixed number $D$, but by reindexing one can take measures $\mu_n$ supported in $B(n)$. This is not particularly relevant to applications. \qed

We summarize here some history of strong property (T) and some drift in the definitions of strong property (T). Lafforgue’s original definition only concluded the existence of a self-adjoint projection onto the invariant vectors [Laf]. In that paper, Lafforgue introduced strong property (T) and proved that the groups $\text{SL}(3, F')$ for $F'$ any local field, have strong property (T) for representations on Hilbert spaces. He also noted that this implied strong property (T) on Hilbert spaces for any Lie group containing $\text{SL}(3, \mathbb{R})$ and for cocompact lattice in all such groups. In subsequent papers, de la Salle and de Laat modified the definition to explicitly include that the projection was a limit of averaging operators defined by measures, but did not assume that the convergence to the limit was exponential [dIS2, dLdLS]. In [dIS2], de la Salle proved strong property (T) for a much wider class of Banach spaces for $\text{SL}(3, \mathbb{R})$ and in [dLdLS] de Laat and de la Salle proved strong property (T) for both $\text{SL}(3, \mathbb{R})$ and $\text{Sp}(4, \mathbb{R})$ and it’s universal cover for an even wider class of Banach spaces. These results combined with existing arguments imply strong property (T) for all higher rank simple Lie groups and for their cocompact lattices. More recently de la Salle has shown that the definition in [Laf] and the definition in [dIS2, dLdLS] are equivalent if one does not necessarily assume that the measures in question are positive [dIS1]. It does, however, follows from the proof of [dIS1, Theorem 3.9] that if one has positive measures converging to the projection then there are positive measures converging exponentially to the projection, namely the convolution powers of any measure close enough to the projection. All existing proofs of strong property (T) explicitly construct sequences of positive measures converging exponentially to a projection [Laf, dIS2, dLdLS]. While it is not explicitly relevant here, we remark that this is also true of the proof by Liao of strong Banach property (T) for higher rank simple algebraic groups over totally disconnected local fields [Lia]. We also remark that while many of these results extend the class to of Banach spaces satisfying strong property (T) to include some quite exotic Banach spaces, for our purposes it is enough to know the property holds for $\theta$-Hilbertian spaces.

6.2. **Sobolev spaces of inner products.** To prove Theorem 2.9 from Theorem 6.3, we need to realize various spaces of $k$-jets of metrics on $M$ as Banach spaces acted on by $\Gamma$. What follows is a special case of the discussion in [FM, Section 4] and we refer the reader there for more details and justifications. Any result stated in this subsection without a reference can be found there.

We will consider the bundle of symmetric two forms on $M$ written as $S^2(TM^*) \to M$. The $k$-jets of sections of $S^2(TM^*)$ are

$$J^k(S^2(TM^*)) \cong \bigoplus_{i=0}^{k} S^i(TM^*) \otimes S^2(TM^*).$$
A background Riemannian metric on $M$ defines Riemannian metrics on all associated
tensor bundles and hence on $J^k(S^2(TM^*))$. There is a natural inclusion
\[ C^k(M, S^2(TM^*)) \subset C^0(M, J^k(S^2(TM^*))) \]
as a closed subspace, but we note that not every section of $J^k(S^2(TM^*)) \to M$ is the $k$-jet
of a section of $S^2(TM^*)$. Given a fixed volume form $\omega$, we denote by $L^p(M, \omega, J^k(S^2(TM)))$
the space of $L^p$ sections of this bundle equipped with norm defined by
\[ \|\sigma\|_p^p = \int_M \|\sigma(m)\|^p d\omega(m). \]
Here the norm inside the integral is defined by the inner product on $S^2(TM^*_m)$ induced by
a fixed background Riemannian metric $g$ on $M$. Note that, as $M$ is compact, changing
the smooth volume $\omega$ or Riemannian metric $g$ gives an isomorphic $L^p$ space and the identity map between any pair of such spaces is bounded. The set of smooth
sections of $S^2(TM^*) \to M$ are naturally included in $L^p(M, \omega, J^k(S^2(TM^*)))$. Let $W^{p,k}(M, \omega, S^2(TM^*))$ be the completion of the set of smooth sections with respect to
the inner product. Thus
\[ W^{p,k}(M, \omega, S^2(TM^*)) \subset L^p(M, \omega, J^k(S^2(TM^*))) \]
is a closed subspace.

The following lemma verifies that all the Sobolev spaces discussed above are in the
class $\mathcal{E}_{10}$. The reader only interested in $C^\infty$ actions should consider the case $p = 2$ in
which all spaces discussed above are Hilbert.

**Lemma 6.4.** The Sobolev spaces $W^{p,k}(M, \omega, S^2(TM^*))$ are in the class $\mathcal{E}_{10}$.

**Proof.** We use only three facts about $\mathcal{E}_{10}$: that it contains Hilbert spaces, that the complex
interpolation of a space in $\mathcal{E}_{10}$ with any other space is in $\mathcal{E}_{10}$, and that $\mathcal{E}_{10}$ is closed under
taking subspaces. This is equivalent to saying that $\mathcal{E}_{10}$ contains all $\theta$-Hilbertian spaces.
Given any complex vector space $V$, the spaces $L^p(M, \omega, V)$ is an interpolation spaces of $L^2(M, \omega, V)$ with $L^{p'}(M, \omega, V)$ for any $p' > p$ and therefore in $\mathcal{E}_{10}$. Taking the complexification of $J^k(S^2(TM^*))$ and then passing back to the closed subspace of real
valued sections, we see that $L^p(M, \omega, J^k(S^2(TM^*)))$ is in $\mathcal{E}_{10}$. As the class $\mathcal{E}_{10}$ is closed
der under taking closed subspaces, $W^{p,k}(M, \omega, S^2(TM^*))$ is also in $\mathcal{E}_{10}$. \(\square\)

Denote by $C^k(M, S^2(TM^*))$ the space of $C^k$ sections of $S^2(TM^*)$. In the case
that $k$ is not integral, with $l = \lfloor k \rfloor$ and $\lambda = k - l$ elements of $C^k(M, S^2(TM^*)) = C^{l,\lambda}(M, S^2(TM^*))$ are sections of $S^2(TM^*)$ which are $l$-times differentiable and whose
order-$l$ derivatives are $\lambda$-Hölder. We will need the following special case of the Sobolev
embedding theorems.

**Theorem 6.5.** There is a bounded inclusion $W^{p,l}(M, \omega, S^2(TM^*)) \subset C^s(M, S^2(TM^*))$
where $s = l - \frac{n}{p}$.

As explained in [FM, Section 4], this is an easy consequence of the corresponding
embedding theorem for domains in $\mathbb{R}^n$ and the existence of partitions of unity. We remark
that the spaces $W^{p,l}(M, \omega, S^2(TM^*))$ are defined relative to a fixed volume form and metric. The background volume form and metric need not be preserved. In our arguments
below, the fact that the volume form and metric are not preserved is controlled by the uniform
subexponential growth of derivatives.

6.3. **Proof of Theorem 2.9.** To construct a $\Gamma$-invariant metric, we first check that the
induced action of $\Gamma$ on appropriate Sobolev spaces has subexponential norm growth. Note
that $C^k$ actions preserve the class of $C^{k-1}$ Riemannian metrics, since metrics are defined on the tangent bundle.

**Lemma 6.6.** Let $\alpha : \Gamma \to \text{Diff}^k(M)$ be an action with uniform subexponential growth of derivatives. Then the induced representation on $W^{p,k-1}(M, S^2(TM))$ has uniform subexponential norm growth.

To prove Lemma 6.6, the key is to see that subexponential growth of the first derivative implies subexponential growth of all derivatives. While this is already observed in [Hur], we include a proof for completeness. We recall a special case of [FM, Lemma 6.4]. Here given a diffeomorphism of $M$, we write $\|\phi\|_k$ for the norm of $\phi$ as an operator on $C^k$ vector fields or equivalently $\|\phi\|_k = \sup_{x \in M} \|J^k\phi(x)\|$ where $J^k\phi$ is the $k$-jet of $\phi$ or the induced map on $J^k(TM) \cong \oplus_{i=0}^k S^i(TM^*)$.

**Lemma 6.7.** Let $\phi_1, \ldots, \phi_n \in \text{Diff}^k(M)$. Let $N_k = \max_{1 \leq i \leq n} \|\phi_i\|_k$ and $N_1 = \max_{1 \leq i \leq n} \|\phi_i\|_1$. Then there exists a polynomial $Q$ depending only on the dimension of the leaves of the foliation and $k$ such that:

$$\|\phi_0 \circ \ldots \circ \phi_n\|_k \leq N_1^{kn} Q(nN_k)$$

for every $n \in \mathbb{N}$.

From this we deduce the following corollary on subexponential growth of higher derivatives.

**Corollary 6.8.** If $\Gamma$ is a finitely generated group, $M$ is a compact manifold and $\alpha : \Gamma \to \text{Diff}^k(M)$ has subexponential growth of derivatives then $\alpha$ also has subexponential growth of higher derivatives. More precisely, subexponential growth of derivatives for $\alpha$ implies that for all $\varepsilon > 0$ there exists $L_{\varepsilon,k}$ such that

$$\|\alpha(\gamma)\|_k \leq L_{\varepsilon,k} e^{\varepsilon l(\gamma)}$$

for all $\gamma \in \Gamma$.

**Proof of Lemma 6.6.** We first remark that exponential growth of derivatives is clearly equivalent to the fact that for all $\varepsilon > 0$ there exists an $n_0$ such that $\|\alpha(\gamma)\|_1 \leq e^{\varepsilon l(\gamma)}$ for all $\gamma$ with $l(\gamma) \geq n_0$. Applying Lemma 6.7 to words in $\Gamma$ of length $l(n_0)$ for $l \in \mathbb{N}$, we see that we have for such words that $\|\alpha(\gamma)\|_k \leq L e^{(k+1)\varepsilon l(\gamma)}$ where the $L$ and the $k+1$ instead of $k$ are to absorb the polynomial growth into the exponential. Letting $L' = \sup_{l(\gamma) < n_0} \|\alpha(\gamma)\|_k$, by writing all words as products of words of length $kn_0$ and words of length less than $n_0$, we see that $\|\alpha(\gamma)\|_k \leq L'L e^{(k+1)\varepsilon l(\gamma)}$ for all $\gamma \in \Gamma$. \hfill $\Box$

**Proof.** From Corollary 6.8, we have that for every $\varepsilon$ there is a an $L$ such that $\|\alpha(\gamma)\|_k < L e^{\varepsilon l(\gamma)}$. Up to relabelling $\varepsilon$ and $L$ to account for the action on $S^2(TM^*)$, this implies that for $\sigma \in J^k(M, S^2(TM^*))$, we have a pointwise bound $\|\alpha(\gamma) \ast \sigma(x)\| < \|\sigma(\alpha(\gamma)x)\| L e^{\varepsilon l(\gamma)}$. This yields the integral bound

$$\int_M \|\alpha(\gamma) \ast \sigma\|^p d\omega(x) \leq L^p e^{p\varepsilon l(\gamma)} \int_M \|\sigma(\alpha(\gamma)^{-1}m)\|^p d\omega(m).$$

Write $\Lambda \alpha(\gamma)$ for the Jacobian of derivative of $\alpha(\gamma)$. Uniform subexponential growth of derivatives implies that for every $\varepsilon > 0$ there is an $F > 1$ such that $\frac{1}{F} e^{-n\varepsilon l(\gamma)} \leq \Lambda \alpha(\gamma) \leq F e^{n\varepsilon l(\gamma)}$ where $n = \dim(M)$. By change of variable

$$\int_M \|\sigma(\alpha(\gamma)^{-1}m)\|^p d\omega(m) = \int_M \|\sigma(\alpha(\gamma)^{-1}m)\|^p (\Lambda \alpha(\gamma))^{-1} \Lambda(\alpha) d\omega(m)$$
so we have
\[ \int_M \| (\alpha(\gamma) \sigma)(m) \|^p \, d\omega(m) \leq F L e^{(p+n)\varepsilon l(\gamma)} \| \sigma \|^p_{p,k}. \]
As \( \varepsilon > 0 \) was arbitrary, this completes the proof. \( \square \)

**Proof of Theorem 2.9.** Fix an initial smooth metric \( g \). From Theorem 6.3 and Lemma 6.6, there exist measures \( \mu_n \) supported on \( B(n) \) in \( \Gamma \) such that \( g_n = \pi(\mu_n) g \) converge to an invariant metric \( g_{\text{fin}} \in W^{p,k-1}(M, S^2(TM)) \). Note that each \( g_n \) is a linear averages of \( g \) under the measure \( \mu_n \) on \( \Gamma \) and in particular does not depend on \( p \) or \( k \). Further note that \( \| g_n - g_{\text{fin}} \|_{p,k} \leq C_{p,k} \) for some \( O < C_{p,k} < 1 \) and all \( n \) sufficiently large. Applying Theorem 6.5, it follows that \( g \) is in \( C^{k-1-\frac{\delta}{2}} \) for all choices of \( p \) and is thus \( C^{k-1-\beta} \) for all \( \beta > 0 \). If the action is by \( C^\infty \) diffeomorphisms, this proves \( g_{\text{fin}} \) is \( C^\infty \). If the action is \( C^2 \), the metric \( g_{\text{fin}} \) is only Hölder.

It remains to check that \( g_{\text{fin}} \) is not degenerate. This follows as the averaged metrics \( g_n \) degenerate subexponentially while the convergence \( g_{\text{fin}} \) is exponentially fast. To see this explicitly, we check that \( g_{\text{fin}}(v,v) > 0 \) for any unit vector \( v \) in \( TM \). The Sobolev embedding theorems imply that \( \| g_n - g_{\text{fin}} \|_0 < KC^n \) for some \( 0 < C < 1, K > 0 \), and all sufficiently large \( n \). Choose \( \varepsilon > 0 \) with \( Ce^\varepsilon < 1 \). Uniform subexponential growth of derivatives implies that there is a constant \( L > 0 \) such that
\[ \| g(D\alpha(\gamma)(v), D\alpha(\gamma)(v)) \| \geq L e^{-\varepsilon l(\gamma)}. \]
This implies that
\[ g_n(v,v) = g(\pi(\mu_n)v, \pi(\mu_n)v) \geq L e^{-\varepsilon n} \| v \|^2. \]
If \( g_{\text{fin}}(v,v) = 0 \) then it would follow that \( g_n(v,v) \leq C^n \) whence \( Le^{-\varepsilon n} < KC^n \) for all sufficiently large \( n \). But then
\[ \frac{L}{K} \leq (Ce^\varepsilon)^n \]
for all sufficiently large \( n \), a contradiction. \( \square \)

**Remark:** It is likely Theorem 2.7 and the results stated in the introduction continue to hold for actions by \( C^{1+\delta} \) diffeomorphisms for any \( \delta > 0 \) by replacing the use of Sobolev spaces by a use of Besov or Triebel-Lizorkin spaces.

## 7. FROM METRICS TO COMPACT LIE GROUPS AND FINITE ACTIONS

To complete the proofs of our main theorem, we need to know that the isometry group of \( M \) is a compact Lie group of dimension at most \( \frac{1}{2} \dim(M)(\dim(M) + 1) \). For those cases where the metric constructed in Theorem 2.9 is at least \( C^1 \) this is an immediate consequence of the Myers-Steenrod Theorem and the fact that \( \text{Isom}(M) \) embeds in the bundle of orthogonal frames over \( M \) which is an \( O(\dim(M)) \) bundle [MS, Kob].

In the case where the metric is only Hölder, we use the solution of the Hilbert-Smith conjecture for bi-Lipschitz maps. We can define a distance on \( M \) as usual by infimizing lengths of smooth curves where we define lengths using our Hölder inner products on \( TM \). The group of isometries of the resulting compact metric space is a compact group \( K \). Using compactness of \( M \) and comparability of all inner products metrics on \( \mathbb{R}^{\dim(M)} \) it is easy to see that isometries of this distance function are bilipschitz on \( M \). By the result of Repovš and Ščepin, this implies \( \text{Isom}(M) \) has no small subgroups and so is a Lie group [RS].

To prove any of the theorems from the introduction, one now assumes that \( \alpha(\Gamma) \) is infinite. Using that compact groups are linear and applying Margulis’s superrigidity theorem, we see that the closure of \( \alpha(\Gamma) \) in \( \text{Isom}(M) \) is an almost direct product \( K = \)
$K_1 \times \cdots \times K_r$ of simple groups each of which is a compact form of $G$. As the action of $K$ on $M$ is faithful, there is at least one point $x \in M$ and at least one $K_i$ such that the $K_i$ orbit of $x$ is $K_i/C$ where $C$ is a closed proper subgroup of $K_i$. To complete the proofs of the results in the introduction, one computes the minimal dimension of $K_i/C_i$ for compact forms of the classical compact groups. In all cases, this implies the finiteness results in our theorems.

In the case where the $\Gamma$-orbits are dense it is possible to give a slightly softer argument due to Zimmer that does not use the solution of the bilipschitz Hilbert-Smith conjecture.
APPENDIX A. TABLES OF ROOT DATA

The following table includes Dynkin diagrams of all irreducible root systems and an enumeration of the simple roots relative to a choice of base II. We also include the highest and second highest roots $\delta$ and $\delta'$ relative to the base II and the resonant codimension of all maximal parabolic subalgebras $q_j := q_{II \setminus \{\alpha_j\}}$.

**Table 1. Roots systems, highest and 2nd highest roots, and resonant codimension of maximal parabolic subalgebras**

| Dynkin diagram and simple roots | Highest root $\delta$ and second-highest root $\delta'$; resonant codimension $\bar{r}(q_j)$ where $q_j = q_{II \setminus \{\alpha_j\}}$ |
|---------------------------------|-------------------------------------------------------------------------------------------------|
| $A_\ell$                        | $\delta = \alpha_1 + \cdots + \alpha_\ell$ $\bar{r}(q_j) = \frac{1}{2} ((\ell + 1)^2 - j^2 - (\ell + 1 - j)^2)$ |
| $B_\ell$                        | $\delta = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_\ell$ $\bar{r}(q_j) = \frac{1}{2} ((2\ell + 1)^2 - j^2 - (2\ell - j + 1))^2)$ |
| $C_\ell$                        | $\delta = 2\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{\ell-1} + \alpha_\ell$ $\delta' = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{\ell-1} + \alpha_\ell$ $\bar{r}(q_j) = \frac{1}{2} ((2\ell + 1)^2 - j^2 - (2\ell - j + 1))^2)$ |
| $BC_\ell$                       | $\delta = 2\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{\ell-1} + 2\alpha_\ell$ $\delta' = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{\ell-1} + 2\alpha_\ell$ |
| $D_\ell$                        | $\delta = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{\ell-2} + \alpha_{\ell-1} + \alpha_\ell$ $\bar{r}(q_j) = \frac{1}{2} ((2\ell - 1)^2 - j^2 - (\ell - j)^2)$ for $1 \leq j \leq \ell - 2$ $\bar{r}(q_j) = \frac{1}{2} ((2\ell - 1)^2 - \ell^2)$ for $\ell - 1 \leq j \leq \ell$ |
| $E_6$                           | $\delta = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + \alpha_5 + 2\alpha_6$ $\bar{r}(q_1) = 16$ $\bar{r}(q_2) = 25$ $\bar{r}(q_3) = 29$ $\bar{r}(q_4) = 26$ $\bar{r}(q_5) = 16$ $\bar{r}(q_6) = 21$ |
| $E_7$                           | $\delta = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + 2\alpha_7$ $\bar{r}(q_1) = 27$ $\bar{r}(q_2) = 42$ $\bar{r}(q_3) = 50$ $\bar{r}(q_4) = 53$ $\bar{r}(q_5) = 47$ $\bar{r}(q_6) = 33$ $\bar{r}(q_7) = 42$ |
| $E_8$                           | $\delta = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 5\alpha_4 + 6\alpha_5 + 4\alpha_6 + 2\alpha_7 + 3\alpha_8$ $\delta' = \alpha_1 + 3\alpha_2 + 4\alpha_3 + 5\alpha_4 + 6\alpha_5 + 4\alpha_6 + 2\alpha_7 + 3\alpha_8$ $\bar{r}(q_1) = 57$ $\bar{r}(q_2) = 83$ $\bar{r}(q_3) = 97$ $\bar{r}(q_4) = 105$ $\bar{r}(q_5) = 106$ $\bar{r}(q_6) = 98$ $\bar{r}(q_7) = 78$ $\bar{r}(q_8) = 92$ |
| $F_4$                           | $\delta = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4$ $\delta' = \alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4$ $\bar{r}(q_1) = 15$ $\bar{r}(q_2) = 20$ $\bar{r}(q_3) = 20$ $\bar{r}(q_4) = 15$ |
| $G_2$                           | $\delta = 2\alpha_1 + 3\alpha_2$ $\delta' = \alpha_1 + 3\alpha_2$ $\bar{r}(q_1) = 5$ $\bar{r}(q_2) = 5$ |
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