Soft gluon emission at large angles

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Abstract

I review recent results on soft gluon radiation at large angles.

1 Gribov’s legacy on soft radiation

Although the major interest of Vladimir Gribov [1] in QCD was on non-perturbative aspects (notably colour confinement), his work on soft photons or soft pions emission in the pre-QCD era [2] is the basis for understanding general features of perturbative QCD dynamics. In principle soft emission is “trivial”, it reduces essentially to classical physics. That is, it depends only on external charges which are treated as classical currents (asymptotic, large-time states), thus it does not reveal the internal short distance structure of emitting systems.

In spite of this “trivial” aspects, universality of soft emission allows one to compute multi-soft gluon emission distributions and thus to obtain important information on the QCD properties in general. The key point is that in computing multi-soft gluon amplitudes one deals with multi-soft scales. The softest gluon does not “see” any internal structure. However the next-to-soft gluon can “see” the softest one. In this way one can recursively reconstruct the detailed deep structure of the emitting system up to the primary hard emitting system. Therefore, while the result is “trivial” in QED since only the primary emitting charges are involved in soft photon emission, in QCD instead, due to direct gluon interaction, also the successive emitted gluons are involved and then successively resolved.

In this talk I discuss various results obtained in the course of years by exploiting soft gluon emission properties obtained in this way.

2 Multi-soft gluon amplitudes

I start by recalling how one obtains [3] (see also [4]) the amplitude for the emission of \( n \) soft gluons \( q_1 \cdots q_n \), off a colour singlet dipole of primary hard quark antiquark \( p_b, p_b \) as emitted for instance in \( e^+e^- \) annihilation at high c.m. energy \( Q \) (generalization to other
dipoles is straightforward). The general colour structure is given as sum of Chan-Paton factors
\[ \mathcal{M}_n(p_a, p_b; q_1 \cdots q_n) = \sum_{\text{perm.}} \{ \lambda^{b_1} \cdots \lambda^{b_n} \}_{\beta \bar{\beta}} \mathcal{M}_n(p_a q_i \cdots q_n p_b), \quad (1) \]
with \( \beta, \bar{\beta} \) and \( b_i \) the quark, antiquark and gluon colour indices (in \( SU(N) \) in general). In \( \mathcal{M}_n \) (colour ordered amplitudes) the soft gluon momenta \( q_i \) are ordered as in the Chan-Paton factor.

In the soft limit \( \mathcal{M}_n \) can be obtained by recurrence relations. Take \( q_j \) as the softest among the soft gluons. One uses eikonal emission (including three-gluon vertex) and colour conservation and obtains
\[ \mathcal{M}_n(\cdots q_{\ell} q_j q_{\ell}' \cdots) = g_s \mathcal{M}_n-1(\cdots q_{\ell} q_{\ell}' \cdots) \cdot \left( \frac{q_{\ell}'^{\mu_j}}{(q_{\ell}' q_j)} - \frac{q_{\ell}^{\mu_j}}{(q_{\ell} q_j)} \right), \quad q_j \text{ softest gluon}. \quad (2) \]

Here \( \mu_j \) is the Lorentz index of softest gluon and \( q_{\ell}, q_{\ell}' \) are the momenta "near in colour" to \( q_j \) in the permutation considered (they are soft gluons or primary quark or antiquark). At this stage no information is needed for the deeper internal structure of \( \mathcal{M}_{n-1} \).

The factorization structure (2) is completely general and can be iterated. At each stage \( m < n \) one factorizes the dipole contribution of the softest gluon in \( \mathcal{M}_m \) and recovers the entire colour amplitudes \( \mathcal{M}_{m-1} \). The multi-soft gluon amplitude \( \mathcal{M}_n \) is finally obtained and expressed in terms of successive (energy ordered) dipole factors for all soft gluons and the Born amplitude \( \mathcal{M}_0 \) for the primary emitting system.

A direct consequence of the fact that at each stage one recovers the entire colour amplitudes is that one does not need to specify the gauge frame since, at each stage, one is dealing with gauge invariant amplitudes and dipole emission factors.

By construction \( \mathcal{M}_n \) is the leading soft part of the amplitude, proportional to the inverse of all soft gluon energies \( \mathcal{M}_n \sim (\omega_1 \cdots \omega_n)^{-1} \). It is valid in any angular configuration even away from collinear configurations. Therefore the factorization form (2) is adequate for our discussion on physics of large angle soft radiation.

Next-to-leading corrections in the soft limit are known [5], see also [6]. They are expressed in terms of a factorized 2-soft gluon emission current. These corrections are needed in order to reconstruct the argument of the running coupling \( \alpha_s \) entering the distribution as the soft gluon transverse momentum in the center of mass of the emitting dipole.

For the square of the colour ordered amplitude \( \mathcal{M}_n \) the recurrence (2) gives [3]
\[ |\mathcal{M}_n(p_a q_1 \cdots q_n p_b)|^2 = \left( \frac{\alpha_s}{2\pi} \right)^n W_{ab}(q_1 \cdots q_n) \cdot |\mathcal{M}_0|^2, \quad W_{ab}(q_1 \cdots q_n) = \frac{(p_a p_b)}{(p_a q_1) \cdots (q_n p_b)}. \quad (3) \]
The distribution \( W_{ab}(1 \cdots n) \) contains collinear singularities when any two partons which are near in colour become parallel. However, this expression is valid in any angular configuration, even at large angles (hedgehog configurations). There are three remarkable properties for this colour distribution.

The first is that in any energy-ordered configuration for the soft gluons one obtains the same expression for the distribution [3]. This in spite of the fact that the expression of the colour ordered amplitude \( \mathcal{M}_n \), obtained from the recurrence relation (2), depends on the soft gluon energy-ordering.
The second property is the factorization structure
\[ W_{ab}(1 \cdots n) = W_{ab}(\ell) \cdot W_{\ell a}(1 \cdots \ell - 1) W_{\ell b}(\ell + 1 \cdots n), \]
with \( \ell \) an arbitrary soft gluon. This branching structure allows the use of parton language. The colour-ordered distribution factorizes into the emission of \( \ell \) off the primary \( ab \)-dipole with the remaining soft gluons emitted by two multi-dipole with “primary” partons \( a\ell \) and \( \ell b \).

The third property refers to the multi-soft-gluon distribution obtained in the same way [3] in the pure Yang-Mills case (primary dipole partons \( p, \bar{p} \) are hard gluons and the Chan-Paton factor in (1) is a trace). One finds that the square colour amplitude obtained in the soft limit coincides with the square of the exact MHV amplitude obtained by Parke-Taylor [7].

The full distribution \( |M_n|^2 \) is obtained by squaring (1). To leading order in \( N \) it is given by the sum of permutations of colour ordered distributions [3] (colour indices organized in the planar way). The non-planar contributions (subleading in \( N \)) are given by the product of two colour amplitudes for different permutations. They are easily computed from (2) for any fixed \( n \), however there is not a close expression valid for arbitrary values of \( n \). As a general feature, non-planar corrections depend in general on the energy ordering and are less collinear singular than the planar contribution [3], one of the collinear singularities cancels [3]. For these reasons, in studies in which collinear singularities are relevant, it is justified to neglect non-planar contributions. For studies of soft radiation at large angle in which collinear enhancements are not relevant neglecting non-planar contribution could be less justified.

In the following I schematically report results on physics of large angle soft emission. First, using the multi-soft gluon distributions in the planar approximation, I deduce the generating functional which provides a “complete” description of soft radiation emission. It can be formulated as a Markov process and then I describe the corresponding Monte Carlo simulation procedure. I describe the use of the generating functional to study the inclusive distribution of energy deposited in a region away from jets. This distribution takes contributions, to leading order, from successive soft gluon branching thus involves the full structure of the multi-soft gluon distribution. Finally I discuss distributions for which the planar approximation is not necessary. In particular I discuss the fifth form factor in hadron collision and the non-relativistic heavy quark multiplicity.

### 3 Generating functional (soft and planar limit)

Complete information on the emission of soft gluons \( \{q_1 \cdots q_n\} \) off the dipole \( p_a p_b \) is contained in the generating functional
\[ \Sigma_{ab}[Q, u] = \sum_n \frac{1}{n!} \int \frac{d\sigma_{ab}^{(n)}}{\sigma_{ab} T} \prod_{i=1}^{n} u(q_i), \]
with \( Q \) the hard scale, \( \sigma_{ab} \) the dipole distributions and \( u(q) \) the source which serves to specify the radiation observable considered. In the planar approximation the multi-soft gluon distribution \( d\sigma_{ab}^{(n)}(q_1 \cdots q_n) \) is proportional to \( W_{ab}(q_1 \cdots q_n) \). Taking the derivative
with respect to $Q$ (with energy ordered phase space) and using the branching factorization property [8] one obtains the evolution equation [8]

$$Q \partial_Q \Sigma_{ab}[Q, u] = \int \frac{d\Omega_q}{4\pi} \bar{\alpha}_s w_{ab}(q) \left\{ u(q) \Sigma_{aq}[Q, u] \cdot \Sigma_{qb}[Q, u] - \Sigma_{ab}[Q, u] \right\},$$

(6)

with

$$\bar{\alpha}_s = \frac{N \alpha_s}{\pi}, \quad w_{ab}(q) = \omega^2_{ab} W_{ab}(q) = \frac{\xi_{ab}}{\xi_{aq} \xi_{qb}}, \quad \xi_{ij} = 1 - \cos \theta_{ij}. \quad (7)$$

The initial condition is set at a scale $Q_0$ corresponding to no emission, $\Sigma_{ab}[Q_0, u] = 1$. Virtual corrections (the last term in (3)) are included to satisfy unitarity $\Sigma[Q, u] = 1$. This realizes the real-virtual cancellation of the collinear singularity of $w_{ab}(q)$ for $\xi_{aq} \to 0$ or $\xi_{qb} \to 0$. This evolution equation reflects the factorization branching structure (3) of the multi-dipole distribution.

The generating functional (5) fully describes soft emission even at large angles and thus it is valid beyond the collinear singularity approximation. However it is difficult to generalize the evolution equation (6) beyond soft approximation including non-soft emission and recoil.

In the next Sections I will report applications exploiting (6).

4 Large angle soft emission and Monte Carlo

The evolution equation (6) for $\Sigma_{ab}$ can be solved numerically by Monte Carlo simulation. In the simulation one generates events with any number of soft gluons which can be used to compute any final state observables. First one needs to resum virtual corrections. To split the real and virtual terms in (6) one introduces a cutoff $Q_0$ in the dipole transverse momentum. The resummation of virtual corrections gives rise to the Sudakov form factor

$$\ln S_{ab}(Q, Q_0) = -\int_{Q_0}^{Q} \frac{d\omega_q}{\omega_q} \frac{d\Omega_q}{4\pi} \bar{\alpha}_s(q_{abt}) w_{ab}(q) \cdot \theta(q_{abt} - Q_0), \quad q_{abt}^2 = \frac{2\omega_q^2}{w_{ab}(q)}. \quad (8)$$

The evolution equation (6) can be written as

$$\Sigma_{ab}[Q, u] = S_{ab}(Q, Q_0) + \int_{Q_0}^{Q} dP_{ab}(q) u(q) \Sigma_{aq}[\omega_q, u] \cdot \Sigma_{qb}[\omega_q, u],$$

(9)

with $dP_{ab}(q)$ the splitting probability which can be casted in the following factorized form

$$dP_{ab}(q) = d \left( \frac{S_{ab}(Q, Q_0)}{S_{ab}(\omega_q, Q_0)} \right) \cdot dR_{ab}(\Omega_q), \quad \omega_q < Q,$$

(10)

with

$$\frac{dR_{ab}(\Omega_q)}{d\Omega_q} = \frac{\bar{\alpha}_s(q_{abt}) w_{ab}(q)}{N_{ab}(\omega_q)} \cdot \theta(q_{abt} - Q_0), \quad \int dR_{ab}(\Omega) = 1. \quad (11)$$

From (11) one has that the Sudakov form factor $S_{ab}(Q, Q_0)$ can be interpreted as probability for no-emission of a soft gluon with dipole transverse momentum larger than $Q_0$.

Using this branching structure and the factorized form of the splitting probability (10) one can construct a Monte Carlo event generator (see also [9]). To generate an event
(with resolution $Q_0$) one proceeds as follows. Starting from the $ab$-dipole with virtuality $Q$, one tries to emit a soft gluon $q$ using (10). To this end one first sorts a random number $0 < r < 1$:

- for $r < S(Q, Q_0)$ the dipole does not branch (with $Q_0$ cutoff);
- for $r > S(Q, Q_0)$ a soft gluon is emitted with energy $\omega_q$ obtained by solving

$$S_{ab}(\omega_q, Q_0) \cdot r = S_{ab}(Q, Q_0).$$

Then, using (11), one finds the direction $\Omega_q$. If the emission takes place, one has to deal with the two dipoles $aq$ and $qb$ which are further emitting soft gluons with energy smaller than $\omega_q$. The procedure is repeated for each dipole till no dipole is emitting (withing the cutoff $Q_0$). Such an event generator can be used to study radiation quantities in the leading soft limit. Due to strong ordering, no recoil is considered. To make a realistic simulation valid beyond soft limit one needs to include parton recoil, full collinear structure and quark branching. Finally hadronization had to be accounted for.

A way to account for non-soft emission consists in taking the collinear limit of (3) and one obtains [3] the evolution equation in angles (instead of in energy). Using angular evolution one can easily introduce recoil, the full splitting functions (including quarks and the full $N$ dependence) and initial state emission. In the angular evolution large angle soft emission are accounted for only in an average sense (coherence through angular ordering). The angular evolution is the basis for the Monte Carlo simulations [10] presently used. There is a continuous upgrading of these Monte Carlo simulations [11]. It would be important to match the angular evolution with the one which fully accounts for large angle soft emission as given in (3).

### 5 Large angle soft emission and jet characteristics

Measuring final state characteristics in hard interactions supplements the overall hardness scale $Q$ with the second scale $Q_0 \ll Q$ that quantifies small deviation of the final state system from the Born kinematics. The ratio of these two scales being a large parameter calls for analysis and resummation of double (DL) and single logarithmic (SL) radiative corrections in all orders. Logarithmically enhanced (both DL and SL) contributions of collinear origin take contributions from bremsstrahlung radiation off primary hard partons and are resummed into exponential Sudakov form factors. SL effects due to soft gluons radiated at large angles, when present, pose more problems. In the following I discuss two cases in which these SL contributions are actually present.

The first case refers to the so called non-global observables that acquire contributions from a restricted phase space window. Examples are particle energy flow $E = Q_0$ in a given inter-jet direction, Sterman-Weinberg distribution (energy in a cone), photon isolation, rapidity cuts in DIS, hadron-hadron inter jet string/drag effects, profile of a separate jet (current hemisphere).

The second case consists in measuring final state characteristics in hard hadron–hadron interactions.
5.1 Non-global logarithms

Non-global observables acquire SL contributions from ensembles of $n$-energy ordered gluons radiated at arbitrary (large) angles which are resummed [12], at the planar level, by the soft gluon evolution equation (6). The simplest case is the study in $e^+e^-$ of the radiation emitted in the region “out” away from the thrust axis as shown here

![Diagram showing thrust axis and radiation regions]

Most of emission is contributing to the two jets around the thrust axis. Therefore, to study the distribution in energy $E_{\text{out}}$ away from jet

$$\Sigma_{\text{out}}(Q, E_{\text{out}}) = \sum_n \int \frac{d\sigma_n}{\sigma_T} \Theta \left( E_{\text{out}} - \sum_{\text{out}} q_{ti} \right)$$

(13)

one needs a careful description of soft gluons emitted at large angle, that is the transverse momentum evolution equation (6). Generalizing to the emission off a general $ab$-dipole and specializing the soft gluon source $u(q)$ to this specific observable, one obtains [8]

$$\partial_\tau \Sigma_{ab}(\tau) = - (\partial_\tau R_{ab}) \cdot \Sigma_{ab}(\tau) + \int_{\text{in}} \frac{d\Omega}{4\pi} w_{ab}(q) \left[ \Sigma_{aq}(\tau) \cdot \Sigma_{qb}(\tau) - \Sigma_{ab}(\tau) \right],$$

(14)

where, at SL accuracy, the distribution $\Sigma_{ab}(\tau)$ is a function of the two scales $Q$ and $E_{\text{out}}$ via the logarithmic integral of the running coupling

$$\tau = \int_{Q_0}^{Q} dq_t \frac{\bar{\alpha}_s(q_t)}{q_t}, \quad Q_0 = E_{\text{out}},$$

(15)

and $R_{ab}(\tau)$ the $ab$-radiator evaluated in the region away from the jets (region “out”)

$$R_{ab}(\tau) = \int_{E_{\text{out}}}^{Q} dq_t \frac{\bar{\alpha}_s(q_t)}{q_t} \int_{\text{out}} \frac{d\Omega}{4\pi} w_{ab}(q) \simeq 2\tau \ln \frac{2}{\theta_{\text{in}}}. $$

(16)

In (14) the radiator term corresponds to bremsstrahlung emission while the integral term corresponds to the soft gluon branching which takes place inside the jet region (region “in”) and gives rise to “non-global logs”. The solution of (14) has the form

$$\Sigma_{ab}(\tau) = e^{-R_{ab}(\tau)} \cdot C_{ab}(\tau).$$

(17)

The first is the Sudakov form factor. The second factor $C_{ab}$ comes from the branching inside the jet region$^1$. Collinear singularities in $w_{ab}(q)$ partially cancel between real and virtual contributions. At large $\tau$ virtual corrections are overwhelming real contributions and one finds the Gaussian behaviour

$$C_{ab}(\tau) \sim e^{-c^2 \tau^2}, \quad c = 4.883...$$

(18)

$^1$The integral term in (14), and then the correlation $C_{ab}$, is absent for global observables which involve the full phase space.
The asymptotic behaviour (18) is independent on the geometry of the recorded region away from the jet. The value of the constant $c$ is related to universal features of the evolution equation in the asymptotic regime and enters also in the small $x$ physics as we shall see. In (17) the Sudakov form factor is negligible with respect to the correlation function at large $\tau$.

Berger, Kúcs and Sterman [13] have formulated a programme of how to avoid non-global logarithms in a measurement of transverse energy flow away from jets. They suggested to squeeze the jets by taking the thrust $T$ close to one and thus suppress multi-gluon branching inside the jet region. They introduced the jet/shape correlation $\Sigma_{fs}(Q, V, E_{\text{out}})$ in the two variables, the energy-flow $E_{\text{out}}$ and the global jet-shape $V$, for instance $V = 1 - T$. They showed that for $V \sim E_{\text{out}}/Q \ll 1$ the flow/shape correlation reduces to the global jet-shape distribution $\Sigma(Q, V)$. This result has been extended [14] for general (small) values $V$ and $E_{\text{out}}/Q$. At SL accuracy one finds the factorized result
\[
\Sigma_{fs}(Q, V, E_{\text{out}}) \simeq \Sigma(Q, V) \cdot \Sigma_{\text{out}}(VQ, E_{\text{out}}),
\]
so that the associated measure is equivalent to the two independent ones, the global one $\Sigma(Q, V)$ at the scale at $Q$ and the non-global one at $VQ$.

Analysis of non-global observables can be generalized (in planar approximation) to hard hadron-hadron collisions. In planar approximation, the hard square matrix element could be expressed as a superposition of dipoles and each $ab$-dipole contributes with the distribution $\Sigma_{ab}(\tau)$ in (17). In hadron collision, contributions beyond planar approximations have been obtained only for global jet-shape observables as we shall discuss in the following.

5.2 Hadron interaction and the fifth form factor

We discuss now global jet characteristics in hard hadron–hadron interaction with the underlying parton scattering process $p_1, p_2 \rightarrow p_3, p_4$ at a scale $Q$. Examples are out-of-event-plane energy production, near-to-backward particle correlations, inter-jet energy flows, etc. They are characterized by a scale $Q_0 \ll Q$.

For global observables all enhanced contributions at DL and SL level comes from bremsstrahlung radiation off the four primary partons. This implies that the problem reduces to the analysis of virtual corrections due to multiple gluons with $q_\ell > Q_0$ attached to primary hard partons only. They can be treated iteratively and fully exponentiated.

In addition to the standard collinear contributions leading to Sudakov for factors associated to the four primary partons $p_i$, one has SL enhancements coming from large angle soft emission. This is due to the fact that soft gluon emission changes the colour state of the hard parton system which in turn affects successive radiation of a softer gluon. Resummation of these soft logarithms generates a fifth form factor in addition to the Sudakov form factors.

The programme of resumming soft SL effects due to large angle gluon emission in hadron–hadron collisions was pioneered by Botts and Sterman [15] and discussed in general also in [16]. The treatment for finite $N$ of large angle gluon radiation in global final state characteristics is based on the observation that the square of the eikonal current for emission of a soft gluon $q$ off the ensemble of four hard partons $p_i$ can be represented as
\[
-j^2(q) = T^2_1 W_{34}^{(1)}(q) + T^2_2 W_{34}^{(2)}(q) + T^2_3 W_{12}^{(3)}(q) + T^2_4 W_{12}^{(4)}(q) + T^2_t \cdot A_t(q) + T^2_u \cdot A_u(q).
\]
Here $T^2$ is the $SU(N)$ “colour charge” of parton $p_i$ and the two operators $T^2_i = (T_1 - T_3)^2$ and $T^2_u = (T_1 - T_3)^2$ are the charges exchanged in the $t$- and $u$- channels. In the first line one has the combination $W^{(c)}_{ab} = w_{ac} + w_{bc} - w_{ab}$ with the singular integral

$$
\int \frac{d\Omega}{4\pi} W^{(c)}_{ab} = \ln \frac{Q^2}{2m^2}, \quad Q^2 = \frac{tu}{s} = s \sin^2 \Theta_s ,
$$

with $s = (p_1 + p_2)^2$, $t = (p_1 - p_3)^2$ and $u = (p_1 - p_1)^2$. The distribution $W^{(c)}_{ab}$ is collinear singular only when $q$ is parallel to $p_c$. Here $m^2$ is a collinear cutoff which is replaced by the proper observable dependent scale $Q_0$ when real and virtual contributions are taken together. Its exponentiation leads to the Sudakov form factors $F_c(Q_0, Q)$ with colour charge of parton $c$ and common hard scale $Q$ in (21).

The last two terms in (20) are collinear regular

$$A_t = w_{12} + w_{34} - w_{13} - w_{24}, \quad \int \frac{d\Omega}{4\pi} A_t(q) = 2 \ln \frac{s}{-t},
$$

and similarly for $A_u$. Non-Abelian Coulomb corrections due to virtual gluons exchanges between the two incoming or two outgoing partons can be simply incorporated by replacing $-t, -u$ with $t, u$ in (22). Contrary to the first four DL contributions, this additional contribution originates from coherent gluon radiation at angles larger than the cms scattering angle $\Theta_s$. It gives rise to the fifth form factor which is a single logarithmic function of $\tau$ given in (15) with $E_{out}$ replaced by $Q_0$.

In summary, the two scale distributions are given by

$$\Sigma(Q_0, Q) = \Sigma^{\text{coll}}(Q_0, Q) \cdot S_X(\tau),
$$

Here $\Sigma^{\text{coll}}$ embodies the first four terms in (20) as well as collinear logarithms from parton distribution functions. $S_X(\tau)$ is the soft factor coming from the fifth form factor.

In the following I illustrate the solution in some relevant special cases for gluon–gluon scattering in $SU(N)$ that was first treated in [17]. The answer for the soft factor $S_X$ is expressed in terms of the following suppression factors

$$\chi_t(\tau) = \exp \left\{-2N\tau \cdot \ln \frac{s}{-t}\right\}, \quad \chi_u(\tau) = \exp \left\{-2N\tau \cdot \ln \frac{s}{-u}\right\}.
$$

**Scattering at $90^\circ$.** In this kinematical configuration ($t=u=-s/2$) one has

$$S_X(\tau) = \frac{\chi^2}{3} \left[ \frac{4 \chi^2}{N^2 - 1} + \chi + \frac{N - 3}{N - 1} \chi^{N+3} + \frac{N + 3}{N + 1} \chi^{-\frac{N+3}{N+1}} \right], \quad \chi(\tau) = \exp \{-2N\tau \ln 2\}.
$$

$N \to \infty$ limit. The soft factor becomes

$$S_X(\tau) = \chi_t \chi_u \frac{(m_t + m_u)^2 + (m_s - m_u)^2 \chi_t + (m_s + m_t)^2 \chi_u}{(m_t + m_u)^2 + (m_s - m_u)^2 + (m_s + m_t)^2}.
$$

Here $m_s, m_t, m_u$ are pieces of the Born $gg$ scattering matrix element each containing the gluon exchange diagram in the corresponding channel (together with the piece of the four-gluon vertex with the same colour structure). One has

$$(m_t + m_u)^2 = 1 - \frac{st}{u^2} - \frac{us}{t^2} + \frac{s^2}{tu},$$

with the other two obtained by crossing symmetry.
**Regge limit**  In the case of close to forward scattering, \(|t| \ll s \simeq |u|\). Since for \(t \to 0\) scattering in the Born approximation is dominated by \(t\)-channel one gluon exchange, we are left with

\[
S(\tau) = \chi_t(\tau) = \left(\frac{s}{\tau}\right)^{-2N\tau},
\]

which exponent coincides with the (twice) Regge trajectory of the gluon.

The eigenvalues of the anomalous dimension matrix possesses a weird symmetry which interchanges of internal (colour group) and external (scattering angle) degrees of freedom:

\[
\frac{\ln s^2/tu}{\ln t/u} \iff N.
\]

In particular, this symmetry relates 90-degree scattering, \(t = u\), with the large-\(N\) limit of the theory. Giving the complexity of the expressions involved, such a symmetry being accidental looks highly improbable. Its origin remains mysterious and may points at existence of an enveloping theoretical context that correlates internal and external variables (string theory?).

### 6 Is there a BFKL Pomeron in jet emission?

It is surprising that large angle soft emission in jet physics has some relation with high-energy scattering and BFKL dynamics. This in spite of the difference in the relevant multi-soft gluon kinematical configurations. In the following I recall two examples in which the two dynamics are (formally) related.

**Kovchegov equation.** The evolution equation (14) for the jet observable \(\Sigma_{ab}(\tau)\) has some resemblance with the Kovchegov equation. To make the resemblance more striking, take a \(ab\)-dipole with small angle \(\theta_{ab} \ll 1\) and introduce the two-dimensional “angular” variable \(\vec{\theta}_{ab}\) and \(\Sigma_{ab}(\tau) = \Sigma(\tau, \vec{\theta}_{ab})\). The evolution equation (6) becomes

\[
\partial_\tau \Sigma(\tau, \vec{\theta}) = \int \frac{d^2\theta'}{2\pi} \frac{\theta'^2}{\theta'^2(\theta'-\vec{\theta})^2} \left[ \Sigma(\tau, \vec{\theta}')\Sigma(\tau, \vec{\theta}' - \vec{\theta}) - \Sigma(\tau, \vec{\theta}) \right].
\]

Here also the angular integration has been replaced by the two-dimensional integration over \(\vec{\theta}'\). One recognizes that (29) formally coincides with the Kovchegov equation for elastic S-matrix at high energy, \(S(\tau, \vec{x})\) where \(\vec{x}\) is the impact parameter and \(\tau = N \alpha_s/\pi Y\) with \(Y\) the rapidity and \(\alpha_s\) the fixed coupling. For large \(\tau\) one has (see the asymptotic behaviour of the correlation in (18))

\[
\Sigma(\tau, \vec{\theta}) \sim e^{-\frac{c}{2} \tau^2}, \quad \text{for } \theta > \theta_{\text{crit}}(\tau) \sim e^{-\frac{c}{2} \tau}, \quad c = 4.883...
\]

so that virtual corrections are overwhelming above \(\theta_{\text{crit}}\) (buffer region for the jet distribution [12]). The asymptotic behaviour (30) corresponds, in the case of the Kovchegov equation, to the saturation for the S-matrix with \(1/\theta_{\text{crit}}(\tau) \to Q_s(\tau)\), the saturation scale. The parameter \(c\) is determined [18] by the characteristic function of the BFKL equation which is the linear limit of (29), as I shall discuss next.
**BFKL equation.** An other case of a formal connection between high energy and jet physics is encountered [19] considering the multiplicity of non-relativistic heavy $Q\bar{Q}$ pair of total mass $M$ emitted by a $ab$-dipole. This observable $N(\rho_{ab}, \tau)$, with $\rho_{ab} = (1 - \cos \theta_{ab})/2$ and $\tau$ given in (15) for $Q_0$ replaced by $M$, satisfies the linearized version of (6) which, after azimuthal integration, is given by

$$\partial_\tau N(\rho, \tau) = \int_{1}^{\rho} \frac{d\eta}{1-\eta} \left( \eta^{-1} N(\eta\rho, \tau) - N(\rho, \tau) \right) + \int_{\rho}^{1} \frac{d\eta}{1-\eta} \left( N(\eta^{-1}\rho, \tau) - N(\rho, \tau) \right).$$ (31)

The only formal difference between (31) and the BFKL equation for the $T$-matrix at high energy is the presence of the lower limit $\eta > \rho$ in the second integral which ensures that the argument of $N(\rho/\eta, \tau)$ remains within the physical region $\rho/\eta < 1$. We have

$$N(\rho, \tau) \sim e^{\tau^{-1} \ln 2} \cdot \frac{e^{-\ln^2 \rho/2D\tau}}{\tau \sqrt{2\pi D\tau \ln \rho}}.$$ (32)

Here $4 \ln 2$ and $D = 28 \zeta(3)$ are related to the BFKL characteristic function. The difference of this asymptotic behaviour with the one for the $T$-matrix in BFKL equation is given by $1/\tau$ in the prefactor which is the result of the fact that $\rho$ is an angular variable with constraint $\rho < 1$. For details on the jet physics solution and explanation on the origin of the additional $1/\tau$ suppression pre-asymptotic factor in $N(\rho, \tau)$, see [19].

The basis for the relation between the equations in jet-physics ((29), (31)) and the Kovchegov and BFKL ones in high-energy scattering is of course the (same) multi-soft gluon-distribution. However the dominant contributions in the jet and high energy case come from very different kinematical configurations as we shall discuss now.

**Jet-physics case.** Here all angles of emitted gluons are of same order. This is due to the fact that the observables considered in this Section do not contain collinear singularities. Moreover, in the (leading) infrared limit soft gluon energies are ordered so that also the emitted transverse momenta are ordered. The ordered variable $q_t$ enters the argument of the running coupling so that $\tau$ is never too large.

**High-energy scattering case.** Here all intermediate soft gluon transverse momenta are of same order (no singularities for vanishing transverse momentum differences). On the other hand, energy ordering implies in this case that intermediate gluon angles are ordered. Contrary to the previous case, the running coupling depends on transverse momenta which are all of same order. Therefore, in first approximation, one can take $\alpha_s$ fixed and so that $\tau$ becomes large at high energy.

### 7 Concluding remarks

I recalled recent results in the studies of soft gluon emission at large angles (hedgehog configurations). Here I present some of the possible developments of these studies.

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The well known result for the multiplicity in the collinear limit [3,4] is obtained by taking the collinear piece in (31) giving (for fixed $\alpha_s$)

$$\partial_\tau N_{\text{coll}}(\rho, \tau) = \int_{0}^{\rho} \frac{d\rho'}{\rho} N_{\text{coll}}(\rho', \tau), \quad N_{\text{coll}}(Q) = N_0 \exp \sqrt{2\alpha_s \ln Q/Q_0},$$

with $Q^2 = E^2 \rho$ and $Q_0$ a collinear cutoff.
Previous studies on non-perturbative corrections (power corrections) to perturbative results have been analysed for quantities involving emission in the collinear regions. It would be important to study power corrections for observables involving hedgehog soft gluon configurations (without collinear enhancements) in which takes place neutralization of colour between jets takes place.

Present Monte Carlo simulations involves contributions of large angle soft emission only in an approximate sense (angular ordering). It would be important to include exact contributions from hedgehog configuration. This requires to merge the Monte Carlo simulation here described with the one presently used.

Multi-soft gluon distributions |$\mathcal{M}_n$| are known for arbitrary $n$ only in the planar approximation. It would be important to have a close expression of the distribution for any $n$ and arbitrary $N$ or at least the subleading $1/N$ corrections. They are specially important for a correct description of hedgehog configurations.

Similarities and differences in the dynamics of high energy scattering and jet-physics (with non-global logs) should be further exploited in order to have new insights in both fields.

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