COMPUTING TROPICAL VARIETIES IN MACAULAY2

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Abstract. This paper introduces a package for doing tropical computations in Macaulay2. The package draws on the functionality of Gfan and Polymake while making the process as simple as possible for the end user. This provides a powerful and user friendly tool for computing tropical varieties requiring little prerequisite knowledge.

The package can be downloaded from:

http://homepages.warwick.ac.uk/staff/D.Maclagan/papers/TropicalPackage.html

1. Introduction

The purpose of this package is to facilitate computations in tropical geometry using Macaulay2 [3]. At the moment the main computational tool for tropical geometry is the program Gfan [6] by Jensen. This computes the Gröbner fan of an ideal $I$ and includes functions to compute only the subfan of the Gröbner fan given by the tropicalization of the variety $V(I)$. The polyhedral geometry program Polymake [2] also has some tropical functionality that is not implemented in Gfan.

The package gfanInterface2 [5], implemented in Macaulay2, allows the user to interface with Gfan while retaining the computational speed provided by Macaulay2 for Gröbner basis computations. A drawback of this package is that it requires good knowledge of the functions and conventions of Gfan. The goal of the Tropical package is to provide a user friendly tool to do these computations in Macaulay2 without requiring any knowledge of these conventions. The package includes different strategies for the same function depending on the input, and calls functions from Gfan, via gfanInterface2, and Polymake, as appropriate. Moreover the package implements some extra functionality not yet available in Gfan, such as computing multiplicities for tropical varieties of non-prime ideals and allowing the user to swap between the min and max conventions.

2. Mathematical Background

We follow the conventions of chapters two and three of [8]. Let $K$ be a field with valuation $v$.

Definition 2.1. Let $f = \sum_{u \in \mathbb{Z}^n} a_u x^u$ be a polynomial in $S = K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. The tropicalization of $f$ is the function $\text{trop}(f) : \mathbb{R}^n \to \mathbb{R}$ given by

$$\text{trop}(f)(w) = \min\{v(a_u) + w \cdot u : a_u \neq 0\}.$$ 

The tropical hypersurface defined by $f$ is

$$\text{trop}(V(f)) = \{w \in \mathbb{R}^n : \text{the minimum in trop}(f)(w) is achieved at least twice}\.$$
Let $I$ be an ideal in $S$. The tropicalization of the variety $V(I)$ is

$$\text{trop}(V(I)) = \bigcap_{f \in I} \text{trop}(V(f)).$$

The same definitions can be formulated using max instead of min. The Tropical package allows the user to choose their convention when loading the package.

If the ideal $I$ is generated by $f_1, ..., f_s$ then it is not true in general that trop $V(I)$ is the intersection of the tropical hypersurfaces associated to the $f_i$s. The intersection trop $V(f_1) \cap ... \cap \text{trop} V(f_s)$ is a tropical prevariety.

**Definition 2.2.** Let $I = \langle f_1, ..., f_s \rangle$ be an ideal in $S = K[x_1^{\pm1}, ..., x_n^{\pm1}]$. Then $f_1, ..., f_s$ are a tropical basis of $I$ if trop $V(I) = \bigcap_{i=1}^{s} \text{trop} V(f_i)$.

The tropical variety trop $V(I)$ is a polyhedral complex ([8, Proposition 3.2.8]) contained in the Gröbner complex of the ideal $I$. If the valuation is trivial, the tropical variety is a rational polyhedral fan and is a subfan of the Gröbner fan of the ideal. Moreover we can associate to each maximal cell an integer number called a multiplicity such that a balancing condition holds (see [8, Definition 3.3.1]).

The Tropical package takes as input an ideal $I$ in a usual (non-Laurent) polynomial ring $K[x_1, ..., x_n]$. The tropical variety computed is the variety $V(J) \subset (K^*)^n$ of the ideal $J = IK[x_1^{\pm1}, ..., x_n^{\pm1}]$.

### 3. Examples

In this section we give explicit examples in order to give a short overview of the package. The computations are all over the field $\mathbb{Q}$ of rational numbers with trivial valuation, hence all tropical varieties are polyhedral fans.

**Example 3.1.** Consider the algebraic variety $X = V(I) \subset (\mathbb{C}^*)^2$, where $I = \langle x + y + 1 \rangle$. The tropicalization of this variety can be computed using the function tropicalVariety(I). The package outputs this as a tropical cycle: a fan with a list of multiplicities corresponding to integer weights on the maximal cones. We extract information about the tropical cycle using associated functions. For example rays gives the generators of the rays as the columns of a matrix.
The tropical variety $\text{trop} V(I)$ is the standard tropical line in the plane: a 1-dimensional fan in $\mathbb{R}^2$ whose rays are $(-1, -1), (1, 0)$, and $(0, 1)$.

The function $\text{tropicalVariety}$ uses one of two different algorithms depending on the input ideal. If the ideal is prime, the tropical variety is connected through codimension one ([1, Theorem 3.1]) and the $\text{Gfan}$ commands $\text{gfan\_tropicalstartingcone}$ and $\text{gfan\_tropicaltraverse}$, which implement the algorithm described in [1], are used. However if the ideal is not prime, this algorithm might fail. The package then calls the more computationally expensive command $\text{gfan\_tropicalbruteforce}$, which computes the entire Gröbner fan. The multiplicities are then computed separately. The package does not require that the user knows these intricacies, but simply requires that they flag when the ideal is not prime.

```
### i9 : elapsedTime(
    tropicalVariety I);
-- 0.088835 seconds elapsed
### i10 : elapsedTime(
    tropicalVariety(
        I,Prime=>false));
-- 0.103651 seconds elapsed
```

For most functions $\text{Gfan}$ requires the input to be homogeneous. The $\text{Tropical}$ package will accept non-homogeneous input, and do the pre- and post-processing to put it into a format acceptable for $\text{Gfan}$. Small additions such as this help decrease the prerequisite knowledge for the package.

**Example 3.2.** A tropical cycle is a fan with multiplicities attached to its maximal cones; it need not be the tropicalization of an algebraic variety. Therefore the package allows the user to create a tropical cycle manually by defining a fan via its maximal cones and attaching multiplicities to each of those cones. The following example shows how we can construct $\text{trop} V(I)$ manually.

```
### i11 : C1=posHull(matrix{{1},{0}});
### i12 : C2=posHull(matrix{{0},{1}});
### i13 : C3=posHull(matrix{{-1},{-1}});
### i14 : F=fan({C1,C2,C3})
### o14 = F
### i15 : mult={1,1,1}
### o15 = {1, 1, 1}
### i16 : S=tropicalCycle(F,mult)
### o16 : TropicalCycle
### i17 : isBalanced S
### o17 : true
```

The $\text{tropicalCycle}$ command does not check that the resulting weighted fan is balanced. To verify this we use the $\text{isBalanced}$ command.

**Example 3.3.** Consider the tropical hypersurfaces $\text{trop} V(f)$ and $\text{trop} V(g)$ cut out by the polynomials $f = x + y + z$ and $g = x^2 + y^2 + z^2$. Their intersection cuts out a tropical prevariety. We would like to compute whether this prevariety is equal to the tropical variety $\text{trop} V(I)$ where $I = \langle f, g \rangle$. 
The polynomials $f, g$ are not a tropical basis for $I$ and therefore the prevariety given by them is not equal to trop $V(I)$. We can see from our computation that the prevariety has a two-dimensional cone, while trop $V(I)$ is one-dimensional.

**Example 3.4.** For two curves $V(f)$ and $V(g)$ in $\mathbb{P}^2$, Bézout’s Theorem states that $|V(f) \cap V(g)|$ equals $\deg(f) \cdot \deg(g)$ counting multiplicities. The tropical analogue of Bézout’s Theorem states that the stable intersection of trop $V(f)$ and trop $V(g)$ is $\deg(f) \cdot \deg(g)$ points counting multiplicities. The following example shows how the package and the `stableIntersection` function can be used to verify examples of tropical Bézout’s Theorem.

The above code considers the stable intersection of a tropical line and a plane quadric. The resulting tropical cycle is a single point, the origin, with multiplicity two, verifying the claim of tropical Bézout’s theorem.

The function `stableIntersection` has two different strategies for computation depending on the software available to the user. If the user has a recent version of Polymake installed, the default strategy is to use `atint [4]`, a Polymake extension for tropical intersection theory by Simon Hampe. If this is not available, the package instead uses Gfan to compute the stable intersection.

### 4. Future Plans

We plan for the `Tropical` package to become the umbrella package for all tropical computations in Macaulay2. This will include implementing alternate strategies for some of the core commands as algorithms improve, before they are included into Gfan and Polymake.

In addition there are still functions available in Gfan and Polymake that are not yet available in the package. We particularly highlight the treatment of nontrivial
valuations, which is available in \textit{Gfan}, and the visualization of low-dimensional tropical varieties, which is available in \textit{Polymake}.

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