Conformal Invariance and Particle Aspects in General Relativity

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Abstract

We study the breakdown of conformal symmetry in a conformally invariant gravitational model. The symmetry breaking is introduced by defining a preferred conformal frame in terms of the large scale characteristics of the universe. In this context we show that a local change of the preferred conformal frame results in a Hamilton-Jacobi equation describing a particle with adjustable mass. In this equation the dynamical characteristics of the particle substantially depends on the applied conformal factor and local geometry. Relevant interpretations of the results are also discussed.

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1 Introduction

Conformal invariance has been playing a particularly important role in the investigation of gravitational models ever since the emergence of such theories. In a system which includes matter, it is well known that conformal invariance requires the vanishing of the trace of the
stress tensor in the absence of dimensional parameters. In the presence of dimensional parameters, the conformal invariance can be established for a large class of theories [4] if the dimensional parameters are conformally transformed according to their dimensions. One general feature of conformally invariant theories is, therefore, the presence of varying dimensional coupling constants. In particular, one can say that the introduction of a constant dimensional parameter into a conformally invariant theory breaks the conformal invariance in the sense that a preferred conformal frame is singled out, namely that in which the dimensional parameter has the assumed constant configuration. Thus the breakdown of conformal invariance may be established by introducing a constant dimensional parameter into the theory. The determination of the corresponding preferred conformal frame depends on the nature of the problem at hand. In a conformally invariant gravitational model one usually considers the symmetry breaking as a cosmological effect. This would mean that one breaks the conformal symmetry by defining a preferred conformal frame in terms of the large scale properties of cosmic matter distributed in a finite universe. In this way, the breakdown of conformal symmetry was found to be a framework in which one can look for the origin of the gravitational coupling of matter, both classical [1] and quantum [2], at large cosmological scales.

The purpose of this paper is to show that the cosmological breakdown of conformal invariance in a conformally invariant gravitational model together with a local change of the corresponding preferred conformal frame may be used to model a particle concept in general relativity.

The organisation of the paper is as follows: We first study the breakdown of conformal symmetry in a conformally invariant gravitational model and define the resulting preferred conformal frame in terms of some cosmological characteristics in close correspondence to the work in [1]. We then show that, by a local change of the preferred conformal frame, it is possible to derive a Hamilton-Jacobi type equation with adjustable mass. In our presentation there is a substantial dynamical interplay between a particle in the ensemble and the applied conformal factor in a form which is similar to the dynamical effect of the quantum potential on a particle in the context of the causal interpretation of relativistic quantum mechanics [3]. This result seems to be interesting because it suggests that the emergence of quantal behaviour of matter may be a general feature of a conformal invariant gravitational model. We shall work with a metric having the signature (- + + +).

## 2 Breakdown of conformal invariance

In this section we briefly review the work in [4]. Consider the action functional

\[ S[\phi] = \frac{1}{2} \int d^4x \sqrt{-g} \left( g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \frac{1}{6} R \phi^2 \right) \]  \hspace{1cm} (1)

which describes a system consisting of a real scalar field \( \phi \) non-minimally coupled to gravity, \( R \) is the scalar curvature. Variations with respect to \( \phi \) and \( g_{\mu\nu} \) lead to the equations

\[ (\Box - \frac{1}{6} R) \phi = 0 \]  \hspace{1cm} (2)

\[ G_{\mu\nu} = 6 \phi^{-2} \tau_{\mu\nu}(\phi) \]  \hspace{1cm} (3)
where \( G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R \) is the Einstein tensor and

\[
\tau_{\mu\nu}(\phi) = -[\nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2}g_{\mu\nu} \nabla_\alpha \phi \nabla^\alpha \phi] - \frac{1}{6} (g_{\mu\nu} \Box - \nabla_\mu \nabla_\nu) \phi^2
\]

with \( \nabla_\mu \) denoting the covariant derivative. Taking the trace of (3) gives

\[
\phi(\Box - \frac{1}{6} R) \phi = 0
\]

which is consistent with equation (2). This is a consequence of the conformal symmetry of action (1) under the conformal transformations

\[
\phi \rightarrow \tilde{\phi} = \Omega^{-1}(x)\phi, \quad g_{\mu\nu} \rightarrow \tilde{g}_{\mu\nu} = \Omega^2(x)g_{\mu\nu}
\]

where the conformal factor \( \Omega(x) \) is an arbitrary, positive and smooth function of space-time. Adding a matter source \( S_m \) independent of \( \phi \) to the action (1) in the form

\[
S = S[\phi] + S_m
\]

yields the field equations

\[
(\Box - \frac{1}{6} R) \phi = 0 \quad \text{(8)}
\]

\[
G_{\mu\nu} = 6\phi^{-2}[\tau_{\mu\nu}(\phi) + T_{\mu\nu}] \quad \text{(9)}
\]

where \( T_{\mu\nu} \) is the matter energy-momentum tensor. The following algebraic requirement

\[
T^\mu_\mu = 0 \quad \text{(10)}
\]

then emerges as a consequence of comparing the trace of (8) with (8). This implies that only traceless matter can couple consistently to such gravity models.

We may break the conformal symmetry by adding a dimensional mass term \( \frac{1}{2} \int d^4x \sqrt{-g}\mu^2 \phi^2 \), with \( \mu \) being a constant mass parameter, to the action (7). This leads to

\[
\mu^2 \phi^2 = T^\mu_\mu \quad \text{(11)}
\]

Consequently, the field equations become

\[
(\Box - \frac{1}{6} R - \mu^2) \phi = 0 \quad \text{(12)}
\]

\[
G_{\mu\nu} - 3\mu^2 g_{\mu\nu} = 6\phi^{-2}[\tau_{\mu\nu}(\phi) + T_{\mu\nu}] \quad \text{(13)}
\]

Now the basic input is to consider the invariance breaking as a cosmological effect. This would mean that one may take \( \mu^{-1} \) as the length scale characterizing the typical size of the universe \( R_0 \) and \( T^\mu_\mu \) as the average density of the large scale distribution of matter \( \sim MR_0^{-3} \), where \( M \) is the mass of the universe. This leads, as a consequence of (11) to the estimation of the constant background value of \( \phi \)

\[
\phi^{-2} \sim R_0^{-2}(M/R_0^3)^{-1} \sim R_0/M \sim G
\]
where the well-known empirical cosmological relation $GM/R_0 \sim 1$ has been used. Inserting this background value of $\phi$ into the field equations (12), (13) leads to the following set of Einstein equations

$$G_{\mu\nu} - 3\mu^2 g_{\mu\nu} = 6\phi^{-2} T_{\mu\nu} \sim GT_{\mu\nu}$$

(15)

with a correct coupling constant $8\pi G$, and a term $3\mu^2$ which appears as an effective cosmological constant $\Lambda$ of the order of $R_0^{-2}$. The field equation (12) for $\phi$ contains no new information.

We should emphasize that the invariance breaking explained above dealt with a broken conformal invariance in the presence of a dimensional matter source which effectively appeared as a cosmological constant $\Lambda$. By implication, we get a preferred conformal frame $(\phi, g_{\mu\nu}, \mu)$ for the gravitational variables, namely that in which $\phi^2 \sim G^{-1}$, $\mu^2 \sim \Lambda$ and $g_{\mu\nu}$ determined by the field equations (14). This preferred conformal frame has the remarkable property that it incorporates a correct coupling of the cosmic matter to gravity. We shall call it the cosmological frame.

### 3 Particle interpretation

A key feature of any fundamental theory consistent with a given symmetry is that its breakdown would lead to effects which can have various manifestations of physical importance. Therefore, in the case of conformal symmetry, one would expect that the corresponding cosmological invariance breaking, considered in the last section, would have an important effect on the local structure of the underlying theory. Here we would like to study one particular effect which illustrates the local particle aspects.

There are two characteristic scales of squared mass in the cosmological frame $(\phi, g_{\mu\nu}, \mu)$ which are subjected to the scale hierarchy

$$\mu^2 \ll \phi^2 \sim G^{-1}.$$  

(16)

In terms if this scale hierarchy, it is possible to single out a congruence of hypersurface orthogonal timelike curves, if we subject the corresponding generating vector field $\nabla_\mu S$ to the condition

$$\nabla_\mu S \nabla^\mu S = - (\phi^2 - \mu^2).$$  

(17)

We can observe that no known elementary particle can move along the congruence defined by (17) because of the large mass-scale defined by the right hand side of (17). Actually, we know from (16) that

$$\phi^2 - \mu^2 \sim \phi^2 \sim G^{-1} = m_p^2$$

where $m_p$ is the Plank mass. The situation, however, changes if we consider a local change of the cosmological frame. To illustrate this point, let us write equation (17) in a new frame $(\bar{\phi}, g_{\mu\nu}, \bar{\mu})$ which is locally connected to the frame $(\phi, g_{\mu\nu}, \mu)$ by a conformal transformation (8).

Taking into account that $S$ as a dimensionless quantity does not transform under conformal invariance, we find

$$\bar{\nabla}_\mu S \bar{\nabla}^\mu S = - (\bar{\phi}^2 - \bar{\mu}^2)$$  

(18)

\footnote{We use units in which $\hbar = c = 1$. In these units $\phi$ has the dimension of mass.}
with \( \bar{\phi} = \Omega^{-1} \phi, \bar{\mu} = \Omega^{-1} \mu \) and the bar quantities refering to the new frame. Now, from the field equation (12) we find

\[
\bar{\mu}^2 = \frac{\Box \bar{\phi}}{\bar{\phi}} - \frac{1}{6} \bar{R}
\]

which, if combined with equation (17) leads to

\[
\nabla_\mu S \nabla^\mu S = -\bar{\phi}^2 + \frac{\Box \bar{\phi}}{\bar{\phi}} - \frac{1}{6} \bar{R}.
\]  

(19)

Now, if the new conformal frame (\( \bar{\phi}, \bar{g}_{\mu\nu}, \bar{\mu} \)) is taken to be subjected to

\[
\nabla_\mu S \nabla^\mu \bar{\phi} = 0
\]

(20)

then equation (19) can be interpreted as a Hamilton-Jacobi equation for a particle with adjustable mass-scale \( \sim \bar{\phi}^2 \). The relation (20) ensures that this mass-scale will not change along the particle trajectory. In addition, we get a dynamical effect on the particle trajectories, reflected in the term \( \frac{\Box \bar{\phi}}{\bar{\phi}} - \frac{1}{6} \bar{R} \), which illustrates a modification of the particle mass due to the scalar curvature and the applied conformal transformation. In summary, we observe that starting from the given preferred cosmological frame (\( \phi, g_{\mu\nu}, \mu \)), we may derive a particle concept in terms of a corresponding local change of the conformal frame. In this approach no separate particle action needs to be introduced, that of a conformally invariant gravitational model suffices. The particle aspects emerge from an internal condition connecting the local properties of a time-like congruence of curves associated with a characteristic scale hierarchy in the cosmological frame with particle properties in a new conformal frame. This observation emphasizes that general relativity, if suitably formulated as a conformal invariant field theory, does not ascribe any special significance to a separate particle action. We should, however, note that there is a certain limitation on the applicability of the particle concept. Actually, in a universe in which \( \bar{\phi}^2 \) is smaller than \( \bar{\mu}^2 \), the above particle concept does not apply, for the mass scale of (17) becomes tachionic. This limitation, however, does not seem to be a weakness of our particle concept because the condition \( \bar{\phi}^2 < \bar{\mu}^2 \) describes a universe of trans-Plankian size.

### 4 Concluding remarks

In this paper we have shown that the cosmological breakdown of conformal symmetry in a conformally invariant gravitational model together with a local change of the corresponding preferred conformal frame leads to a picture consistent with a particle concept. This picture may be considered as a manifestation of Mach’s principle in that the particle concept emerges as a local effect emerging from large scale cosmological consideration.

We emphasize the similarity of the term \( \frac{\Box \bar{\phi}}{\bar{\phi}} \) on the right hand side of (19) with the quantum potential term in the context of the causal interpretation of relativistic quantum mechanics [3]. This similarity merits attention because it provides an indication for a possible geometrization of quantal behaviour of relativistic particles in the framework of a conformal invariant gravitational model.
References

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