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Gravitational waves from scattering of stellar-mass black holes in galactic nuclei

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ABSTRACT
Stellar-mass black holes (BHs) are expected to segregate and form a steep density cusp around supermassive black holes (SMBHs) in galactic nuclei. We follow the evolution of a multimass system of BHs and stars by numerically integrating the Fokker–Planck energy diffusion equations for a variety of BH mass distributions. We find that the BHs ‘self-segregate’, and that the rarest, most massive BHs dominate the scattering rate closest to the SMBH (≲10−1 pc). BH–BH binaries form out of gravitational wave emission during BH encounters. We find that the expected rate of BH coalescence events detectable by Advanced LIGO is ∼1–102 yr−1, depending on the initial mass function of stars in galactic nuclei and the mass of the most massive BHs. We find that the actual merger rate is likely ∼10 times larger than this due to the intrinsic scatter of stellar densities in many different galaxies. The BH binaries that form this way in galactic nuclei have significant eccentricities as they enter the LIGO band (90 per cent with e > 0.9), and are therefore distinguishable from other binaries, which circularize before becoming detectable. We also show that eccentric mergers can be detected to larger distances and greater BH masses than circular mergers, up to ∼700 M⊙. Future ground-based gravitational wave observatories will be able to constrain both the mass function of BHs and stars in galactic nuclei.

Key words: black hole physics – gravitational waves – galaxies: kinematics and dynamics – galaxies: nuclei.

1 INTRODUCTION

1.1 Motivation
The coalescence of stellar-mass black hole–black hole (BH–BH) binaries is one of the most anticipated sources of gravitational waves (GWs) for ground-based interferometers such as LIGO1 or VIRGO.2 Whether formed primordially (Belczynski, Kalogera & Bulik 2002; Belczynski, Sadowski & Rasio 2004; Sadowski et al. 2008) or in dense star clusters (Portegies Zwart & McMillan 2000; Gültekin, Miller & Hamilton 2004, 2006; O’Leary et al. 2006; O’Leary, O’Shaughnessy & Rasio 2007; Miller & Lauburg 2009), nearly all of these binaries are expected to be circularized by GW emission before they are detectable by ground-based observatories (Gültekin et al. 2006; O’Leary et al. 2006). In order to be detected with a high signal-to-noise ratio (S/N), matched filtering algorithms sift through the LIGO data stream looking for such circular inspirals (Abbott et al. 2008), but might miss many eccentric events (Martel & Poisson 1999; Tessmer & Gopakumar 2008; Mandel et al. 2008).

In principle, eccentric inspirals are well suited for detection as well, however, they have not been expected to be a significant source for terrestrial detectors (except see Wen 2003).

In this paper, we propose an important additional source of GW sources, which preferentially form binaries that are still eccentric as they merge. In galactic nuclei with supermassive black holes (SMBH) of mass M_{SMBH} < 10^7 M⊙, relaxation times are often less than a Hubble time, and can result in the formation of steep density cusp of stars and stellar-mass BHs. Indeed, as many as ∼20 000 BHs are expected to have segregated into the inner ≈1 pc of the Milky Way (Morris 1993; Miralda-Escudé & Gould 2000; Freitag, Amaro-Seoane & Kalogera 2006, hereafter FAK06; Hopman & Alexander 2006b, hereafter HA06). When two BHs have a close encounter, they can release sufficient amount of energy in GWs to form a tight binary, and merge in less than a few hours. These mergers are almost always eccentric, in contrast to circular inspirals expected in globular clusters. A similar process can occur in the runaway growth of BHs when no SMBH is present that will also result in eccentric mergers; however, such systems are inherently...
unstable and not long lived (Quinlan & Shapiro 1987, 1989, 1990; Lee 1993).

The GWs generated during the evolution of these orbits are endowed with a rich structure. For sufficiently small impact parameters, the GWs can be detected with second generation GW instruments already during first passage (Kocsis, Gáspár & Márka 2006). Subsequently, the eccentricity is close to unity for several thousand orbits after the formation of the binary, generating a train of short-duration distinct GW bursts with a continuous frequency spectrum. As the eccentricity gradually decreases, the separation between subsequent bursts decreases, and the waveform becomes continuous in the time-domain and decouples into discrete harmonics in frequency. Interestingly, as we show below, the eccentricity is non-negligible all the way to coalescence. The full waveform, consisting of a correlated set of distinct GW bursts, which evolves into a continuous eccentric inspiral signal, would not be discovered in the GW data by existing data analysis techniques such as GW burst search algorithms, nor by small eccentricity inspiral templates, since the contribution of individual bursts to the GW power is small and the contribution of higher harmonics is significant throughout the evolution (see Yunes et al. 2008, for a similar discussion for extreme mass ratio GW bursts during encounters with a SMBH).

Eccentric inspirals are more luminous in GWs than circular inspirals and extend to higher frequencies, and so may be detectable to higher redshifts. Furthermore, these events may be capable of detecting intermediate-mass BHs using terrestrial instruments (Brown et al. 2007; Mandel et al. 2008). The evolution of the signal during the initial phase of eccentric inspirals can be described using a small number of parameters, allowing one to construct very efficient and sensitive data analysis algorithms to search for these signals in the LIGO data. Additionally, these signals spend a longer physical time in the LIGO band, possibly reducing the false alarm rate for a fixed number of templates that cover the observational period. The modulation introduced by the Earth’s rotation during such an event can be utilized to further improve the measurement accuracy for the source position.

The GW waveforms have been studied for eccentric inspirals through several different approaches. Post-Newtonian (PN) waveforms exist up to 1.5PN order beyond the leading order gravitational radiation effects ($\alpha v^4$) for general mass ratios with spins (Vasuth & Majar 2007; Majar & Vasuth 2008), 2PN without spins ($\alpha v^4$, including the effects of eccentricity and radiation reaction; Damour, Gopakumar & Iyer 2004) and 3PN including eccentricity but without spins and also neglecting radiation reaction ($\alpha v^6$; Arun et al. 2008, and references therein), 5.5PN for extreme mass ratios without spins ($\alpha v^{11}$; Tanaka, Tagoshi & Sasaki 1996). The evolution of a binary is chaotic at (or beyond) the 2PN approximation if the component BHs have spins (Levin 2006; Wu & Xie 2007) and also possibly for non-spinning BHs approaching the unstable circular orbit (i.e. exhibiting zoom-whirl orbits; Pretorius & Khurana 2007; Washik et al. 2008). Numerical relativity has mostly focused on the circular inspiral for comparable mass BHs (Baker et al. 2007; Berti et al. 2007; Boyle et al. 2008, and references therein), or extreme mass ratios (Martel 2004; Babak et al. 2007; Pretorius & Khurana 2007). Recently, however, the waveforms have been evaluated for a handful of equal mass eccentric initial conditions (Sperhake et al. 2008; Hinder et al. 2008a,b; Washik et al. 2008).

In this paper, we perform two separate analyses to two different problems. First, in Section 2, we determine the multimass distribution of BHs in galactic nuclei for a variety of mass models. In Section 3, we determine the formation rate of binaries due to GW capture in a variety of galactic nuclei. In Section 4, we describe the general features of the generated GW waveform, calculate the expected S/N for its detection with second-generation terrestrial GW instruments and determine the expected detection rate of such mergers. We also show that the eccentricity distribution of such sources will distinguish binaries formed in the manner outlined in this paper from BH–BH binaries formed dynamically in massive star clusters.

2 MASS SEGREGATION

Bahcall & Wolf (1976, hereafter BW76) were the first to correctly analyse the relaxation of a stellar population around a central point mass, although their analysis was highly idealized. They first derived the Fokker–Planck equations for a spherically symmetric distribution of a single-mass population of low-mass stars around a massive BH ($M_\ast \ll M_{\text{SMBH}}$). They found that after about one half of a relaxation time-scale the mass density profile of stars reaches a steady state and forms a power-law cusp with respect to radius, $\alpha r^{-\alpha}$, around the massive central object, with $\alpha = 7/4$. In a second paper, they extended their analysis to look at a multimass system and included effects of the loss cone (Shapiro & Lightman 1976) that results from the disruption of stars by the central BH (Bahcall & Wolf 1977, hereafter BW77). They found that in a two-mass system the more massive objects segregate from the lower mass stars by forming a steeper power-law density profile than the stars. This is in stark contrast to the evolution of a two-mass cluster of stars without a SMBH, which can eventually lead to the so-called Spitzer instability (Spitzer 1969), in which the high-mass stars decouple from the low-mass objects entirely.

For a population of stars that are sufficiently old ($\gtrsim 100$Myr), stellar-mass BHs with mass $\sim 10M_\odot$ are expected to become the most massive objects in the system and will begin to segregate from the main-sequence stars due to dynamical friction (Morris 1993). Looking at our own Galactic Centre and using a typical stellar-mass function, Miralda-Escudé & Gould (2000) estimated that $\sim 2.5 \times 10^3$ BHs should have segregated into the inner pc. Indeed, HA06 followed the work of BW77 by solving the time-dependent Fokker–Planck equations and confirmed these results. They found that $\sim 18000$ BHs of $10M_\odot$ each should segregate to form a steep density cusp around the SMBH in our Galaxy, Sgr A*, with $\alpha_{BH} \approx 2$, and that the stars and lighter compact objects form a shallower cusp with $\alpha_s \approx 1.4$.

FAK06 have so far provided the most robust analysis of the segregation and distribution of a multimass system of BHs and stars in the nuclei of galaxies (for a review, see Amaro-Seoane et al. 2007). The authors combined large $N \approx 10^5$ Monte Carlo Fokker–Planck simulations with a population synthesis code (Belczynski et al. 2002) to simulate the segregation and subsequent formation of a density cusp for an old population of stars. Their simulations were in remarkably good agreement with previous analyses (e.g. Morris 1993; Miralda-Escudé & Gould 2000; HA06). In their simulations of a Milky-Way-like galactic nucleus they found a similar overall number of stellar-mass BHs ($\sim 20000$) within 1 pc of a $3.5 \times 10^6 M_\odot$ SMBH. However, they found the BHs follow a slightly shallower density profile than HA06, with $\alpha_{BH} \approx 1.8$. Despite the level of detail in the simulations of FAK06, the simulations still have their limitations. Because of the large range in scales needed to follow the mass segregation, and subsequently the large number of particles in that volume, each star cannot be modelled by a single particle. The densest, most interesting regions of their
simulations still suffer from small number statistics, and cannot
reveal the precise expectations for the mass distribution of stars
and BHs. In order to calculate the rate of GW capture events one
needs to apply a high-resolution method (like HA06) to multimass
distribution of BHs for calculating the rates and detectability of the
events.

Although the BHs are expected to dominate relaxation and dy-
namical encounters in the inner 0.1 pc around Sgr A*, their presence
has so far eluded detection. X-ray observations of our own Galaxy
and others may provide the first window to this interesting stellar
population. In our own Galaxy, Muno et al. (2005) have found an
overabundance of X-ray sources in the vicinity of Sgr A*, which
seems to follow the underlying distribution of stars (Muno et al.
2006). For large distances, the X-ray sources appear to follow the
BH distribution as well (FAK06). In other galaxies, many BHs
may interact with the dense accretion discs around SMBHs, and in
some circumstances be intrinsically more luminous than the SMBH
(Deeg & Nayakshin 2007; Nayakshin & Sunyaev 2007).

The existence of a nuclear population of BHs may also be re-
vealed through dynamical encounters with observable stars. Indeed,
Weinberg, Milosavljević & Ghez (2005) have shown that a decade
of observations with the next generation of 30-m class telescopes
may detect the scattering of the BHs with the so-called ‘S stars’
that are within ≈0.04 pc of Sgr A*. Microlensing by the BHs is
expected to produce a much weaker signature (Miralda-Escudé &
Gould 2000; Alexander & Loeb 2001; Chanamé, Gould & Miralda-
Escudé 2001). Strong encounters between the BHs and the same
population of S stars may dynamically eject the stars with high
enough velocity to escape the potential of the galaxy (O’Leary &
Loeb 2008) and populate the halo with the observed hypervelocity
stars (Brown et al. 2005, 2006). Strong encounters are also expected
to tidally spin-up long-lived stars, and so the BH population may be
inferred through spectroscopic measurements of the spin of lower
mass stars (Alexander & Kumar 2001). Alternatively, one may infer
the presence of the BHs by looking at density distribution of old
pulsars (Chanamé & Gould 2002).

Finally, future GW observations with the space-based GW in-
terferometer LISA 1 or the Earth-based second generation GW in-
struments are expected to detect the inspiral of such BHs into
SMBHs (Gair et al. 2004; Rubbo, Holley-Bockelmann & Finn 2006;
Hopman, Freitag & Larson 2007; Yunes et al. 2008) or IMBHs
(Brown et al. 2007; Mandel et al. 2008), respectively, several times
per year. Given the test particle limit of the BH, such encounters are
expected to be a stringent test of General Relativity in the strong-
field regime (Collins & Hughes 2004; Arun et al. 2006).

An interesting method for detecting a dense population of BHs
is through the GWs they produce as they scatter on each other and
get captured into a tight binary. Kocsis et al. (2006) estimated the
expected detection rates of the first hyperbolic passage between
such stellar-mass BHs in globular clusters, and found rates to be
typically less than 0.1 yr 1/2 for second generation GW instruments.
Here, we consider the detectability of the much stronger GWs that
follow once the eccentric binary has formed in galactic nuclei in this
way and evolves toward coalescence. We explore this new method
in Sections 3 and 4. However, in order to completely determine this
rate we need to know how a multimass distribution of BHs relaxes
around a SMBH in galactic nuclei. In the remainder of this section
we calculate the steady-state distribution of BHs for a variety of
models.

2.1 Fokker–Planck equations

In our calculations, we follow the analysis of BW77, who derived
the multimass Fokker–Planck equations for a spherically symmetric
and isotropic distribution of stars around a SMBH. \( f_{\text{d}}(r, v) =
\frac{d}{dt}f_{\text{d}}(E), \) where \( E = (1/2)Mv^2 - GM_{\text{SMBH}}/r \) is the mechanical
energy of bound orbits in the galactic cusp that are outside the
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loss cone (below). Much of our analysis follows that of HA06,
After the integration is complete, we calculate the number density of stars at radius $r$:

$$n_M(r) = 2\pi^{-3/2} n_* \int_{-\infty}^{\rho_0(r)} \frac{g_M(x)}{\phi(r) - x}^{1/2} dx,$$

(6)

where $\phi(r)$ is the dimensionless specific potential. Until now, we have assumed that the stars were on entirely Keplerian orbits around the SMBH [i.e. that $\phi(r) = r_i/r$]. This assumption was necessary in order to simplify the Fokker–Planck equations to equation (2) (BW76). However, for our calculations in Section 3, it is important that the BH density goes to zero as $r$ approaches infinity. We therefore calculate $\phi(r)$ and $n_M(r)$ iteratively after calculating the steady-state distribution functions $g_M(x)$. We do this by sequentially solving for $n_M(r)$ (equation 6) and

$$\phi(r) = r_i/r - \int_0^{M(<r)/M_{\text{BH}}} \frac{M(<r)}{M_{\text{BH}}} \frac{dM}{d\Phi} d\Phi,$$

(7)

where $M(<r) = \sum_M \int_0^{r_i} 4\pi r^2 n_M(r) dr$ is the total mass of stars interior to $r$, thus accounting for the stars’ contribution to the potential as well as the SMBH’s contribution. We find that we converge to a single solution for $\phi(r)$ and $n_M(r)$ before about six iterations. This solution to the number density and potential is fully consistent with the distribution functions $g_M(x)$. However, equations (2) and (3) implicitly assume that, as $r \to \infty$, the potential approaches zero and the number density approaches a constant. This assumption is approximately followed as long as the core radius of the isothermal sphere $r_c = \left[ 9\pi^2 / (4\pi G \rho_0) \right]^{1/3}$ (Binney & Tremaine 1987) is larger than the radius of influence of the SMBH. We also note that the number density is the statistical average. In cases of the small numbers of BHs, we assume that values we determine will match the average over many galaxies. However, we caution that the dynamics of such systems will likely behave differently than assumed here.

### 2.2 Initial conditions and BH mass distributions

The initial conditions of our models are described entirely by the distribution functions of unbound stars $g_M(x < 0)$. Sufficiently far from the radius of influence of the SMBH the population of visible stars appears to be similar to an isothermal sphere, with a constant velocity dispersion and a number density distribution that scales as $\sigma \propto r^{-1.5}$ (Genzel et al. 2003; Schödel et al. 2007). We therefore set $g_M(x) = \exp(x)$ for $x < 0$, which corresponds to an isothermal sphere of stars. We look at only one population of stars of mass $M_\star = 1 M_\odot$. This is justified by previous analyses of mass segregation that show that the underlying stellar population, being considerably less massive than BHs, tends to follow the same distribution function independent of mass where the BHs dominate the relaxation (H0A06, FAK06). We confirm this conclusion in Section 2.3. Although there is a significant fraction of more massive stars in nuclei, their lifetimes are expected to be less than the relaxation time-scale at most $r$ and are in such small number that they should not contribute to the dynamics significantly.

We assume that the BHs initially follow the distribution of the stars, and amount to a total fraction, $C_{\text{BH}} = \sum_M C_M$, of the number density. BH natal kick velocities are expected to be much less than the velocity dispersions of the systems we are interested in (White & van Paradijs 1996; Willems et al. 2005), and so we expect the BHs’ velocity dispersion to be similar to the stars’. Therefore, we set $g_M(x) = C_M \exp(x)$, yielding a BH number density of $C_M n_\star$ near $r_i$. This is in contrast to the work of BW77 and HA06a, who both assume that the population of stars was already in thermal equipartition (i.e. $M_\beta_M = M_\beta_\alpha$). BW77 addressed the distribution of stars in globular clusters, which have half-mass relaxation times much shorter than the Hubble time and so the entire system should reach equipartition. In contrast, in galactic nuclei the relaxation time-scale for large radii is usually longer than the age of the universe, and so the source population could not reach a complete equilibrium. HA06 made a numerical error in how they normalized the BH distribution and so had reasonable agreement between their work and previous works. However, with our normalization, they would have found the number of BHs to be $(M/M_\odot)^{3/2} = 10^{12} \times$ times larger than their results indicated. This is because in the case of a velocity dispersion independent of mass, the ratio of number densities of objects is described by $\sum_M n_\star = C_M / C_{\text{BH}}$, whereas for a population in thermal equilibrium we get $n_M/M = C_M M^{1/2}/(C_M M_\odot^{1/2})$. Because the steady-state distribution function is most sensitive to $g(0)$ and not the functional form $g(x < 0)$, their results are similar to ours when one uses the incorrect normalization.

One goal of our analysis is to see if the mass distribution of BHs may be determined either through dynamical interactions with their environment, or through the release of GWs. We therefore look at a variety of time-independent BH mass functions parametrized as power laws. The models are determined by four basic parameters: the total number fraction of BHs, $C_{\text{BH}} = \sum_M C_M$; the slope of the BH mass function, $\beta$, where $dM_{\text{BH}}/dM \propto M^{-\beta}$; the minimum BH mass, $M_{\text{min}}$, and, finally, the maximum BH mass in the nucleus, $M_{\text{max}}$. In our calculations, we approximate the distribution of the BHs with $N = 9$ discrete distribution functions with

$$C_i = \int_{M_{\text{min}} + \Delta M}^{M_{\text{max}} + \Delta M} \gamma M^{-\beta} dM,$$

(8)

where $\Delta M = (M_{\text{max}} - M_{\text{min}})/N$, and $\gamma$ is normalized such that $\sum_i C_i = C_{\text{BH}}$. We set the mass for each distribution function to be the average mass for that bin,

$$M_i = \int_{M_{\text{min}} + \Delta M}^{M_{\text{max}} + \Delta M} (\gamma/C_i) M^{-\beta + 1} dM.$$

(9)

We describe the numbers used above in detail below, and outline them in Table 1.

The number fraction of BHs in galactic nuclei is sensitive to the initial mass function (IMF) of high-mass stars. For a Kroupa IMF (Kroupa & Weidner 2003) we estimate $C_{\text{BH}} \approx 0.001$ by assuming that all stars with mass $M > 20 M_\odot$ become BHs. A similar distribution of stars formed uniformly throughout time was found to be consistent with the $K$-band luminosity distribution of stars in the galactic centre (Alexander & Sternberg 1999). However, more recent observations of our Galactic Centre suggest the IMF slope of high-mass stars may be considerably shallower than those in regular clusters (see e.g. Nayakshin & Sunyaev 2005; Nayakshin et al. 2006; Paumard et al. 2006; Maness et al. 2007). Paumard et al. (2006) found that the stars in an apparent disc around Sgr A* are best fit with a flattened IMF $\propto m^{-\beta_{\text{imf}}}$ with $\beta_{\text{imf}} \approx 0.85$–1.35 and a depletion of low-mass stars. This is consistent with the work of Nayakshin & Sunyaev (2005), who constrained the number of young low-mass objects with X-ray observations, and determined that BHs are more efficient than stars of mass $\lesssim 3 M_\odot$. Most recently, Maness et al. (2007) looked at late-type giants and concluded that the MF of high-mass stars may be as shallow as $\beta_{\text{imf}} \approx 0.5$. A shallow MF is also expected theoretically (Morris 1993) and is consistent with the most recent hydrodynamic simulations of star formation in the Galactic Centre (Alexander et al. 2008). Considering the abundant evidence for an alternative IMF in galactic nuclei, we therefore look at a wide range of number fractions, setting...
C_{BH} = 0.001, 0.01 and 0.1, which roughly corresponds to $\beta_{init} = 2.3, 1.5$ and 0.8, respectively.

The mass distribution of BHs is relatively unconstrained. Currently, the best constraints come from the couple of tens of X-ray binaries with dynamically determined BH masses (Remillard & McClintock 2006). Each measurement often has considerable uncertainty, but the masses seem to span a range of $\sim 3–18\,M_{\odot}$. There is now strong dynamical evidence for BHs with even greater masses, up to $23–34\,M_{\odot}$, in low-metallicity environments (Orosz et al. 2007; Prestwich et al. 2007; Silverman & Filippenko 2008). Nevertheless, these observations suffer from a severe observational bias; the BHs must be in a close binary to be observed. Theoretical estimates for the mass distribution are also highly uncertain.

Belczynski et al. (2004) have used their sophisticated population synthesis models to determine the expected distribution of BH masses in a variety of environments. Typically, they expect most BHs in high-metallicity environments to have a uniform distribution of masses between $\sim 5$ and $10\,M_{\odot}$, but do find that significantly more massive BHs may form from the merger of the BH with a high-mass companion star. Unfortunately, these massive BHs would not be found in X-ray binaries, unless they were introduced into one dynamically. Given these uncertainties, we choose to parametrize the mass distribution of BHs as a power law $dN/d\log m \propto m^{-\beta}$, where we consider $\beta = 2$ and 3. We look at the importance of the highest mass BHs, by modelling different upper limits on BH mass, $M_{\text{max}}$. We also have a model based on the work of Belczynski et al. (2004). In Model BSR, we use a flat $\beta = 0$ model of BHs with masses between $5$ and $10\,M_{\odot}$ with a fraction $\sim 0.01$ of BHs with mass $25\,M_{\odot}$. This is consistent with their Model C2 of solar metallicity stars and based on their fig. 7.

For the remainder of Section 2, we focus our results on Milky-Way-like nuclei, and assume that $M_{\text{SMBH}} = 3.5 \times 10^6\,M_{\odot}$ (Eisenhauer et al. 2005; Ghez et al. 2005) and that the velocity dispersion of the stars and stellar BHs is $\sigma_{*} = 75\,\text{km}\,\text{s}^{-1}$ (HA06). In Section 3 we also consider other galaxies that have relaxation times short enough to reach steady state in a Hubble time with $M_{\text{SMBH}} \sim 10^3–10^4\,M_{\odot}$, in order to calculate the overall rate of GWs sources in the universe.

2.3 Results and implications

For all of our models, independent of the BH mass function (namely, $\beta$ and the mass range of BHs), the most massive BHs always become the dominant BH species in the inner $\sim 0.1\,\text{pc}$ of the galactic nucleus. In Fig. 1, we plot the number density of the stars and BHs as a function of radius. As has been found previously (BW77; FAK06; HA06), deep in the Keplerian potential of the SMBH, the distribution functions of the stars and BHs become power laws of the negative specific energy of the objects $g_{\epsilon}(x) \propto x^{-p_{w}}$, and hence have a power-law density profile $\rho \propto r^{-3/2-p_{w}}$. We find that throughout all of our simulations, the exponent $p_w$ is best fitted by a linear relationship between the mass ratio of the object, $m$, and the most massive BH, $M_{\text{max}}$.

\[ p_{w} = p_{0} + \frac{m}{M_{\text{max}}}, \tag{10} \]

where $p_{0}$ is $0.5–0.6$. We have found that $p_{0}$ usually has a small scatter ($\sim 20$ per cent), depending on $M_{\text{max}}$ and $\beta$; however, given $p_{0}$, equation (10) is often accurate to $\lesssim 1$ per cent. This relationship is similar to that found in BW77, who found that $p_{0} \approx 0.25–0.3$, when they looked at two different components with comparable number densities. Alexander & Hopman (2008) attribute the steeper density profiles to ‘strong’ mass segregation, where the relaxation of the system is determined by the many low-mass objects. In their calculations, $p_{w}$ increases monotonically with $M_{\text{max}}$, however, the rate at which it increases is rather slow after $p_{w} \approx 0.5$. In contrast, in our simulations with a mass spectrum of massive objects and accounting for the loss cone, we do not find that the maximum power-law index $p_{w}$ to depend sensitively on the number or mass of the highest mass objects. However, we have not looked at the large range of parameters of Alexander & Hopman (2008).

Mass segregation in galactic nuclei ceases when the BHs begin to dominate the relaxation process in the inner cusp. Typically, this is expected when $n_{\text{BH}}/n_{*} > n_{*} M_{\text{BH}}^{2}$. One therefore may try to detect the presence of a cluster of BHs through their interactions with the luminous stars in its vicinity. We can estimate the relaxation time-scale at a radius $r$ from equations (1) and (5), in terms of the
Figure 1. The number density of stars and BHs for Model B (left) and E-2 (right). The top dash–dotted line is the number density of the stars as a function of radius. The alternating dotted and dash–dotted lines show the number density of the separate mass bins used in our calculations with the most massive BHs having the largest number density interior to \( r \approx 0.3–0.6 \text{ pc} \). The solid blue line is the total BH number density \( = \sum M n_M(r) \). Near \( r \approx 1 \text{ pc} \) the potential of the stars and BHs is equal to the Keplerian potential of the SMBH. The rapid drop in the number of BHs drops rapidly here because of the rigid boundary condition at \( \phi = 0 \). In all of our simulations, the bin of the most massive BHs dominates the number density of BHs in the inner \( \sim 0.1 \text{ pc} \).

Keplerian potential of the SMBH:

\[
t_t(r) = \frac{3(2\pi v^2(r))^{3/2}}{32\pi^2 G^2 \ln \Lambda \sum M^2 n_M(r)},
\]

where \( v_c(r) = \sqrt{GM_{\text{SMBH}}/r} \) is the circular velocity around the SMBH at radius \( r \). In Fig. 2, we have plotted the relaxation time-scale as a function of radius for Models B and E-2. The BHs begin to dominate the local relaxation processes at \( r \sim 0.5 \text{ pc} \), consistent with the results of FAK06 and HA06. We find that the relaxation time-scale in the galactic centre is highly sensitive to the assumed mass distribution of the BHs, since the lower density of the BHs is easily outweighed by their greater mass. By finding a tracer of the relaxation time in the inner 0.1 pc of the galactic centre, it may be possible to determine if such heavy BHs even exist in our Galaxy (Chanamé & Gould 2002).

As discussed above, a population of massive BHs in the galactic centre may be revealed through strong encounters with their neighbouring stars. Because the BHs are more massive than the typical star, they tend to drive stars out of the nucleus, creating a shallow stellar density profile (FAK06; HA06; for observations see Genzel et al. 2003 and more recently Schödel et al. 2007). Miralda-Escudé & Gould (2000) proposed that a population of such BHs may be seen through stars on very radial orbits that are ejected by close encounters by the BHs. In some extreme encounters, O’Leary & Loeb (2008) showed that strong interactions between BHs and stars can lead to the stars being ejected from the Milky Way altogether as the observed hypervelocity stars (Brown et al. 2005, 2006). The

Figure 2. The relaxation time-scale as a function of radius for Model B (left) and E-2 (right) plotted in Gyr. The solid line shows the relaxation time-scale for the entire system. The long dashed and dotted lines show the contribution of the BHs and stars, respectively. Although the stars contribute to the formation of the cusp and are the most common objects in the system, their low mass precludes them from dominating the relaxation in the inner \( \sim 0.6 \text{ pc} \) of the cusp. The break in the relaxation time at \( \sim 0.0002 \text{ pc} \) is due to our boundary condition that no stars have \( x > 10^4 \).
results of O’Leary & Loeb (2008) were sensitive to the assumed BH mass distribution. Self-segregation of the BHs is an important consideration when evaluating the likelihood of the scenario, and the velocities of ejected HVs (Sesana, Haardt & Madau 2007).

As a more extreme example of a strong encounter, BHs and stars may physically collide into each other (Morris 1993; FAK06) producing strong X-ray and ultraviolet (UV) flares in galactic nuclei. Interestingly, we find that the total rate of physical collisions between stars and BHs can be comparable to the rate stars is tidally disrupted by the SMBH, and may be visible in variability surveys of galactic nuclei. Such encounters may have smaller bolometric luminosities than stars disrupted by the SMBH due to their lower Eddington luminosity.

3 BINARY FORMATION, INSPIRAL AND EVENT RATES

When two compact objects have a close encounter, they can emit sufficient energy through GWs that they become bound and form a binary. To see the importance of such a process in galactic nuclei, we first estimate the rate of binary formation for two typical 10 M⊙ BHs. We present more detailed calculations later in this section. To first order, two BHs with an initial relative velocity \( w \) will form a bound binary due to the emission of GWs if they come within a distance \( \approx 7.4 \times 10^3 \) R⊙ (w/100 km s\(^{-1}\))\(^{-4/7}\) at closest approach (see equation 18). This corresponds to an impact parameter (the distance of closest approach if the BHs moved in straight lines) of \((\text{equation } 17)\) \( b \lesssim 2.4 \) R⊙ (w/100 km s\(^{-1}\))\(^{-4/7}\).

We expect that a single BH will undergo such an encounter approximately every \( (\pi b^2 n_{\text{BH}} w)^{-1} \approx 3.4 \times 10^7 \) (nBH/100 pc\(^{-3}\))\(^{-3}\) \( (w/100 \text{ km s}^{-1})^{11/7} \) yr. The total rate of binary formation is then approximately (see e.g. equation 31) \( \int n_{\text{BH}}^2 \pi b^2 w^4 4\pi r^2 dr \), where \( w(r) \sim v_c(r) = \sqrt{GM_{\text{SMBH}}/r} \) is the Keplerian circular velocity at radius \( r \). Integrating from \( r = 10^{-2} \) to 1 pc, yields a rate of \( 3.4 \times 10^{-10} \) yr\(^{-1}\) \( [n_{\text{BH}}/(2 \times 10^3)]^2 \) for \( n_{\text{BH}} \) BHs in a density cusp around the SMBH with \( n_{\text{BH}} \propto r^{-2} \). This is in good agreement with our more detailed calculations presented in Section 3.2.

In this section, we calculate the formation rate of such GW capture binaries in greater detail. We first calculate the conditions necessary to form GW capture binaries and the properties of their eventual inspiral in different environments. We then calculate the rate of binary formation for the multimass, segregated clusters of BHs that we modelled in Section 2. Finally, we extrapolate our results for galactic models, including the distribution of SMBHs in the universe and accounting for the statistical variance in the number density of stars surrounding the SMBHs, in order to estimate the comoving rate density of binary formation and inspiral.

In our calculations, we express our equations with the symmetric mass ratio of the encounter \( \eta = (m M)/ (m + M)^2 \), and total mass of the BHs \( M_{\text{tot}} \). With these variables, the reduced mass is \( \mu = M_{\text{tot}} \eta \).

3.1 Orbital evolution

In the following sections, we shall work with dimensionless parameters in units \( G = c = 1 \), where the total mass \( M_{\text{tot}} \) has length and time dimensions. We define the dimensionless pericentre distance as \( \rho_p = r_p/M_{\text{tot}} \), dimensionless semimajor axis for an eccentric orbit with eccentricity \( e \) as \( a = a/M_{\text{tot}} = \rho_p/(1 - e) \). In these units, the mean orbital angular frequency from Kepler’s law is simply \( \omega_{\text{orb}} = M_{\text{tot}} \omega^{-3/2} = \rho_p^{-3/2} (1 - e)^{3/2} \), the angular frequency at pericentre passage is \( \omega_p = M_{\text{tot}}^{-1} \rho_p^{-3/2} (1 + e)^{3/2} \), the orbital time is \( \Delta t_{\text{orb}} = f_{\text{orb}}^{-1} = 2\pi \rho_p^{-1} \). We define the characteristic time of pericentre passage as \( \Delta t_p = f_p^{-1} = 2\pi \rho_p^{-1} \), which satisfies \( \Delta t_p = \Delta t_{\text{orb}} \) for circular encounters, but is much smaller for eccentric encounters.

3.1.1 First passage, formation of binaries

Two BHs of mass \( m \) and \( M \) will form a binary if they undergo a close encounter and release enough energy to become bound, \( \delta E > \eta M_{\text{tot}} w^2/2 \), where \( w = |v_\text{in} - v_\text{df}| \) is the magnitude of the relative velocity of the BHs at infinity, and we ignore the presence of the SMBH during the encounter and assume that the interaction is local. Given the relativistic nature of such events, and the comparatively low velocity dispersions in nuclei, the encounters are always nearly parabolic (Quinlan & Shapiro 1987; Lee 1993). In this limit, the amount of energy released during the encounter is (Peters & Mathews 1963; Turner 1977)

\[
\delta E \approx -\frac{85\pi}{12\sqrt{2}} \frac{\eta^2 M_{\text{tot}}^2}{r_p^3},
\]

where \( r_p \) is the distance of closest approach,

\[
r_p = \left( \frac{1}{b^2} + \frac{M_{\text{tot}}}{b^2 w^4} + \frac{M_{\text{tot}}}{b^2 w^2} \right)^{-1},
\]

and \( b \) is the impact parameter of the encounter, and in the second line we have expanded to first order in \( w/c \).

The final properties of the system are determined by the system’s final energy, \( E_{\text{final}} = M_{\text{tot}} \eta w^2/2 + \delta E \), and angular momentum, \( L_{\text{final}} = M_{\text{tot}} \eta b w + \delta L \), where

\[
\delta L \approx -6\pi M_{\text{tot}}^2 \eta^2 \frac{b^2 w^2}{r_p^3},
\]

is the amount of angular momentum lost in GWs (Peters 1964). For nearly all of the encounters we find \( |\delta L| \ll M_{\text{tot}} \eta b w \) and so we set \( \delta L = 0 \). If the total energy of the system is negative, \( E_{\text{final}} < 0 \), then the system will remain bound with a semimajor axis,

\[
a_0 = \frac{M_{\text{tot}}}{2E_{\text{final}}},
\]

eccentricity,

\[
e_0 = \sqrt{1 + \frac{2E_{\text{final}} b^2 w^2}{M_{\text{tot}} \eta}},
\]

and pericentre, \( r_{\text{p,0}} = a_0(1 - e_0) \).

With the criterion \( E_{\text{final}} < 0 \) for binary formation, the maximum impact parameter for two BHs with relative velocity \( w \) to become bound is

\[
b_{\text{max}} = \left( \frac{340\pi}{3} \right)^{1/7} \frac{M_{\text{tot}} \eta^{1/7} w^{-1/7}}{E_{\text{final}}^{1/7}}.
\]

Substituting in equation (13), this corresponds to a maximum pericentre distance,

\[
r_{\text{p, max}} = \left( \frac{85\pi}{6\sqrt{2}} \right)^{2/7} \frac{M_{\text{tot}} \eta^{2/7}}{w^{3/7}} \times \left[ 1 - \frac{1}{4} \left( \frac{85\pi}{3} \right)^{2/7} \frac{w^{10/7}}{(4\eta)^{2/7}} \right].
\]

We shall demonstrate below that event rates are dominated by the central regions of the galactic cusp. For \( m = M \) (\( \eta = 1/4 \) and
\[ w = 1000 \text{ km s}^{-1}, \quad b_{\text{max}} \approx 2900 \text{ M}_{\odot}, \quad \text{and} \quad r_{p,\text{max}} = 47 \text{ M}_{\odot}. \] The correction in equation (18) is clearly negligible for non-relativistic initial conditions.

### 3.1.2 Eccentric inspiral

After the binary forms with dimensionless pericentre distance \( r_{p0} = \frac{r_{p0}}{M_{\odot}} \) and eccentricity \( e_0 \), its orbit will decay through the emission of GWs (Peters 1964):

\[
\frac{dr_p}{dt} = -\frac{64}{5} \frac{M_{\odot}^2 \eta^2}{\rho_p^6} \frac{(1 - e)^{1/2}}{(1 + e)^{7/2}} \left(1 + \frac{73}{24} e^2 + \frac{37}{36} e^4\right),
\]

\[
\frac{de}{dt} = -\frac{304}{15} \frac{M_{\odot}^2 \eta}{\rho_p^6} \frac{(1 - e)^{3/2}}{(1 + e)^{7/2}} \left(1 + \frac{121}{304} e^4\right),
\]

where we assume \( \rho_p = \frac{r_p}{M_{\odot}} = a/[M_{\odot}(1 - e)] \) until the last stable orbit.

We evolve the orbits following Peters (1964) for the evolution of the dimensionless periapsis, \( \rho_p \), the time-to-merger, \( t \), as a function of the instantaneous eccentricity, \( e \). Note that if starting from a parabolic orbit, the initial condition for the orbit is determined from exactly one parameter, \( \rho_{0} \). Dividing equation (19) with equation (20) yields a separable differential equation for \( \rho_p(e) \).

The solution to the parabolic encounter initial condition \( \rho_p = \rho_{p0} \) at \( e = 1 \) is

\[
\rho_p(\rho_{p0}, e) = \rho_{p0} \kappa_p(e),
\]

where

\[
\kappa_p(e) = 2 \left(\frac{304}{425}\right)^{870/2299} e^{12/19} \left(1 + \frac{121}{304} e^4\right)^{870/2299} (1 + e)^{-1},
\]

for which \( \kappa_p = 1 \) at \( e = 1 \). The orbital evolution equations (21) and (22) are valid far from the horizon \( \rho_p \gg 1 \). Once the last stable orbit (LSO) is reached the evolution is no longer quasi-periodic and the binary quickly coalesces. In the leading order ratio approximation for infinite mass ratio and zero spins,

\[
\rho_p(\text{LSO}) = \frac{6}{1 + \epsilon_{\text{LSO}}^2} \rho_{p0},
\]

which we solve numerically for \( \epsilon_{\text{LSO}}(\rho_{p0}) \) using equations (21) and (22).

The time evolution of the eccentricity can be written using equations (19) and (20) as

\[
t_{\text{merge}}(e) \approx \frac{15}{19} \left(\frac{304}{425}\right)^{3480/2299} M_{\odot} \eta^{-1} \rho_{p0}^4 \times \int_0^e \epsilon^{29/19} \left[1 + \frac{(121/304)e^2}{(1 - \epsilon^2)^{7/2}}\right]^{1181/2299} d\epsilon,
\]

where \( t_{\text{merge}}(e) \) denotes the time remaining until coalescence when the eccentricity is \( e \). Close to coalescence this is approximately

\[
t_{\text{merge}}(e) \approx \frac{5}{16} \left(\frac{304}{425}\right)^{3480/2299} M_{\odot} \eta^{-1} \rho_{p0}^4 48/19 \text{ if } e \ll 1,
\]

while the total merger time starting from a highly eccentric initial orbit (Peters 1964):

\[
t_{\text{merge}} \approx \frac{3}{85} \frac{a^4}{M_{\odot} \eta} (1 - e^2)^{7/2} \text{ if } e \approx 1.
\]

Substituting equations (15) and (16) for the initial separation and eccentricity \( a_0 \) and \( e_0 \) into equation (26), we get

\[
t_{\text{merge}} \approx \frac{3\sqrt{3}}{170\sqrt{85\pi}} \frac{b_{\text{FWHM}}^{21/2}}{M_{\odot}^2 \eta^{1/2}}.
\]

### 3.2 Event rates

Given the cross-section for binary formation \( \sigma_{\text{cm}} = \pi b_{\text{max}}^2 \), we can now calculate the expected rate of binary formation in a single galactic nucleus, \( \Gamma_{\text{IGN}} \). In order to evaluate the total contribution of different BH masses, \( (m, M) \), velocities, \( (v_u, v_M) \), and spatial position, \( r \), within a single galactic nucleus, we need to integrate

\[
\text{d}^3 \Gamma_{\text{IGN}} = \sigma_{\text{cm}} M_{\odot} \text{d}f_M(r, v_M) \text{d}f_M(r, v_M),
\]

for the final small eccentricity inspiral.
corresponding distributions. Here $w = |v_\nu - v_M|$ is still the magnitude of relative velocity between the BHs, and $f_\nu$ and $f_M$ are the six-dimensional distribution functions of the BHs with mass $m$ and $M$ derived in Section 2. Generally, this calculation requires us to evaluate an integral over nine variables:

$$\Gamma_{\text{IGN}} = \int_{r_{\text{min}}}^{r_{\text{max}}} dr \int_{m_{\text{min}}}^{m_{\text{max}}} dm \int_{M_{\text{min}}}^{M_{\text{max}}} dM \, d^3v_M \, f_M(r, m) \, f_\nu(r, v_\nu) \sigma_{\nu,w},$$

where the integration bounds are set consistent with the distribution domains (see Section 2). The multidimensional integration can be greatly simplified to only three variables using a few approximations. First in Section 2.3, we assumed that the the velocity distributions were isotropic. For fixed $m$ and $M$, we switch integration variables from $(v_\nu, m, M)$ to $(v_\nu - v_M, v_M + v_M)$, and adopt spherical coordinates. Since the integrand only depends on $w$, we can evaluate the integrals over the other five velocity components:

$$\int d^3v_\nu \int d^3v_M \, f_M(r, m) f_\nu(r, w) \sigma_{r,w} = n_m(r) n_M(r) \int dw \, \psi_{mM}(r, w) \sigma_{r,w},$$

where $\psi_{mM}(r, w)$ is the relative velocity distribution which can be expressed as a one-dimensional integral using the Dirac $\delta$ function for a power-law dimensionless energy distribution profile $g_m(x) \propto x^{p_m}$ (see Section 2 for a definition). Note that the phase-space distribution functions of the BHs and stars are well approximated by a power law $g_m \propto x^{p_m}$ for $x > 10$.

We have numerically integrated the remaining velocity integral over the possible range of $w$ for a variety of different slopes $p_m$ and $p_\nu$ and find that to within $\lesssim 10$ per cent the integrand is independent of the shape of the relative velocity distribution and only depends on the expected value of the relative velocities, and can be expressed with the expected value of the individual velocity magnitudes as

$$\int dw \, \psi_{mM}(r, w) \sigma_{r,w} \approx \pi b_{\text{max}}^2 v_\nu(r),$$

where $b_{\text{max}}^2$ is evaluated at $w = v_\nu(r)$ using equation (17), and $n_m(r) \propto r^{-p_\nu - 3.5}$. This is to be expected, since the encounters are practically parabolic such that the initial velocity profile is negligible compared to the velocity at periastron. Thus, equation (28) simplifies to

$$\frac{d\Gamma_{\text{IGN}}}{dr \, dm \, dM} = 4\pi^2 b_{\text{max}}^2 v_\nu(r) n_m(r) n_M(r)r^2,$$

so that the total rate in one galactic nucleus is

$$\Gamma_{\text{IGN}} = \int_{r_{\text{min}}}^{r_{\text{max}}} dr \int_{m_{\text{min}}}^{m_{\text{max}}} dm \int_{M_{\text{min}}}^{M_{\text{max}}} dM \frac{d\Gamma_{\text{IGN}}}{dr \, dm \, dM}.$$

In practice, we calculate $\Gamma$ for all of our simulations by calculating $n_m(r)$ and $n_M(r)$ for discrete masses and sum over $m < M$.

4 Some difficulties arise because of the integration bounds depend on the other parameters. However, the integrals can be evaluated under the approximation $r_{\text{min}} \leq r \leq r_{\text{max}}$ and $0 \leq v \leq v_\nu(r)$, where $r_{\text{min}}$ and $r_{\text{max}}$ are the minimum and maximum radii for a relaxed population of BHs in the nucleus and $v_\nu(r)$ is the escape velocity at radius $r$.

3.3 Results and discussion

The resulting event rates integrated over a single galactic nucleus are presented in the sixth column of Table 1 for all of our model nuclei. The estimated rates vary between $\sim 10^{-8}$ and $10^{-10}$ yr$^{-1}$ over the various models. We discuss the primary sources of uncertainties and other important aspects related to the event rates in detail below.

3.3.1 Number of black holes

Overall, we find that the merger rate is most sensitive to $C_{\text{BH}} = n_{\text{eq}}/n_s$, however, the accretion of BHs by the SMBH reduces the naive scaling relation $C_{\text{BH}}$. Thus, as the number fraction of BHs increases by a factor of 100, the rate of mergers only increases by a factor of 20. The merger rate depends far less so on $M_{\text{max}}$ and $M_{\text{min}}$, since for larger $M_{\text{max}}$ the total number of BHs tends to be reduced in the inner $\sim 0.1$ pc of the SMBH. The rate, however, remains relatively unchanged since the cross-section for binary capture increases with the BH mass.

3.3.2 Pericentre distance dependence

For parabolic encounters, the differential of pericentre distances that lead to binary formation, according to equation (13) is to leading order

$$dr_p \approx \frac{w^2 b \, db}{M_{\text{max}},}$$

where we have fixed $w$, the relative velocity at infinity. Since $d\Gamma/dw \propto b \propto dr_p/dw$, the pericentre distribution that lead to binary capture $(d\Gamma/dr_p)$ is uniform out to a maximum pericentre distance $r_{p,\text{max}}$. We can calculate the overall distribution of encounters that form binaries by changing the order of integration of equation (28) and leaving it as a function of $r_p$:

$$\frac{d\Gamma_{\text{IGN}}}{dr_p} = \frac{M_{\text{tot}}}{w^2} \psi_{mM}(r, w),$$

where $\psi_{mM}(r, w)$ is the distribution function of relative velocities given by equation (29). The limits of the integration determine the functional form in three main regimes. For $r_p < r_{p,\text{max}}[w_{\text{max}}(r_{\text{min}})]$ where $w_{\text{max}}(r) = 2v_{\text{esc}}(r)$ is the maximum relative velocity at radius $r$, $v_{\text{esc}}$ is the escape velocity and $r_{p,\text{max}}(w)$ is defined in equation (18), equation (34) is independent of $r_p$ and is integrated over the bounds

$$0 < w < w_{\text{max}}(r),$$

$$r_{\text{min}} < r < r_{\text{max}}.$$  

For $r_{p,\text{max}}[w_{\text{max}}(r_{\text{min}})] < r_p < r_{p,\text{max}}[w_{\text{max}}(r_{\text{max}})]$ the limits of integration are determined by $w_{\text{eq}}(r_p)$, the inverse of equation (18) and $r_{\text{eq}}(r_p) = 2GM_{\text{SMBH}}/w_{\text{eq}}(r_p)^2$. In this regime, the limits of integration are split into two regions

$$0 < w < w_{\text{eq}}(r_p),$$

$$r_{\text{min}} < r < r_{\text{eq}}(r_p),$$

and

$$w_{\text{eq}}(r_p) < w < w_{\text{max}}(r),$$

$$r_{\text{eq}}(r_p) < r < r_{\text{max}}.$$
Figure 4. Normalized probability distribution and integrated probability distribution of initial pericentre distances for binaries in galactic nuclei (a) and in Spitzer-unstable massive star clusters without a central massive BH (b; see Section 3.3.6 for details). To calculate the values in this figure, we have assumed that the merger rate is dominated by the 10 M⊙ BHs with a near Maxwellian velocity dispersion, and in galactic nuclei they have density profiles $\propto r^{-2}$, consistent with the density profile of the most massive objects in the nucleus (see Section 2.3). The solid line is the differential rate of binary formation $d\Gamma/d\rho_p$ and the dotted line represents the integrated probability $(\rho > \rho_p)$. For galactic nuclei, the probability distribution is perfectly uniform out to pericentre distance $\rho_p \approx 250 M_{\text{tot}}$, and it begins to drop off as a Gaussian profile for $\rho_p \gtrsim 540 M_{\text{tot}}$.

Finally, for $\rho_p > \rho_{p,\text{max}}[2\sqrt{2}v_c(\rho_{\text{max}})]$ the limits of integration are

$$0 < w < w_{\text{eq}}(\rho_p),$$

$$r_{\text{min}} < r < \rho_{\text{max}}.$$  \hspace{1cm} (38)

A normalized probability distribution of pericentre distances for encounters in galactic nuclei is plotted in Fig. 4(a), where we approximated $\psi_{\text{MM}}(r, w)$ as a Maxwellian distribution with variance $v_c(r)$. For comparison the distribution function for the Maxwellian core of a star cluster without a central massive BH is plotted in Fig. 4(b).

3.3.3 Eccentricity dependence

We can use equations (21), (22) and (24) to follow the secular evolution of the binary as it passes through the LIGO band and merges when it reaches its last stable orbit. In Fig. 5, we have plotted the secular evolution of formed binaries in 10 per cent probability intervals, and plotted the eccentricity after approximately every complete orbit with small circles. Nearly 90 per cent of all binaries actually form within the LIGO band, and $\approx94$ per cent have an eccentricity $e > 0.3$ as it enters the LIGO band (when $f_p = 10$ Hz). This signature is unique to an active cluster of BHs in a high velocity dispersion environment. For lower velocity dispersions as in star clusters, the merger distribution of pericentre distances $(d\Gamma/d\rho_p)$ stays uniform out to $\approx260 M_{\text{tot}}$. In contrast to galactic nuclei, we expect only $\approx10$ per cent of all binary GW sources in star clusters to have eccentricities higher than 0.3 when they enter the LIGO band, and $\approx8$ per cent to form within the LIGO band. Binaries which merge due to three- or four-body interactions are even less likely to have such a high eccentricity (Gültekin et al. 2006). O’Leary et al. (2006) calculated the expected distribution of eccentricities for binaries that form in dense star clusters (see their fig. 3). In their simulations which had over 1000 mergers from random encounters, they had no cases that had an eccentricity $e > 0.3$ when the binary’s GWs entered the LIGO band, and only...
three cases ($\approx 10^{-3}$) with eccentricity $>0.1$. Their calculations also included secular effects which could result in mergers with higher eccentricity (Wen 2003). However, in these cases, the binary must have merged after one Kozai cycle in a hierarchical triple. Since the Kozai cycle is a dynamical effect, it operates on a time-scale much shorter than the disruption time-scale of the cluster. Thus, any merger due to the Kozai effect would indicate an active cluster of BHs must still exist. This is in contrast to a delayed merger from an ejected binary, which takes of order a Hubble time to merge. We discuss other aspects of GW detection in Section 4.

In most instances, the first binary forming encounters occur within the LIGO band. However, the S/N of such encounters is too small to be detected for low masses (Kocsis et al. 2006). Therefore, encounters which remain eccentric throughout the inspiral, especially near plunge, may be the most readily detectable encounters. Therefore, we are interested in the eccentricity of the binary as it reaches the LSO (see equation 23). From equation (34), and the equation of evolution (Peters 1964, equation 24) we solve for the probability distribution of eccentricity at the LSO. We plot the eccentricity distribution at LSO in Fig. 6 for both galactic nuclei and globular clusters. For encounters that directly plunge, our calculation gives an eccentricity greater than 1. However, for normalization purposes, we include this in our calculations, as they interestingly comprise a significant fraction of merger events.

Until now, nearly all LIGO sources were expected to have a negligible eccentricity as they enter the LIGO band (but see Mandel et al. 2008, for intermediate-mass ratio inspirals in star clusters). The comparatively low eccentricity binary formed through few body encounters or standard binary evolution circularize before they enter the LIGO band and are detected. Therefore, the detection of eccentric inspirals is a strong test of the formation scenario of nuclear binaries, and can conclusively reveal the origin of the BHs.

3.3.4 Radius dependence inside the galactic nucleus

In Fig. 7, we plot the cumulative binary formation rate for radii larger than $r$, $\Gamma(r) > r$, as well as $d\Gamma/d\ln r$. For most models, the total differential rate of binary formation per logarithmic bin is roughly flat. Thus, each logarithmic radius interval contributes equally to the rate. We therefore conclude that the rates determined are rather robust to the depletion of BHs very close to the SMBH as may be caused by resonant relaxation (Rauch & Tremaine 1996; Rauch & Ingalls 1998; Hopman & Alexander 2006a) or our choice of the innermost radius for BHs. In order for the rate to be dominated by mergers at large $r$, the number density of the BHs would have to decrease with an exponent $r^{-\alpha}$, where $\alpha = p + 3/2 < 3/2$ (see Section 2). This is precisely the reason we accounted for the stars in determining the potential in equation (6), and did not let the density profile of the BHs and stars go to a constant value as in previous analyses.

We do not expect these tight binaries in their subsequent inspiral phase to have any observable effect dynamically. Overall, we expect $\approx 10^3$ such binaries to merge over a Hubble time. This presents a much smaller source of energy than the SMBH, which accretes $\approx 10^5$ BHs over a Hubble time (FAK06). However, small the intrinsic rate in each galaxy, the cumulative merger rate of many galaxies is large enough to be detected by future ground-based GW observatories.

3.3.5 Cosmological merger rate density

Using the $M_{\text{SMBH}} - \sigma_*$ relationship found by Tremaine et al. (2002) for higher mass BHs,

$$M_{\text{SMBH}} \approx 1.3 \times 10^8 M_\odot (\sigma_*/200 \text{ km s}^{-1})^4,$$

we can extrapolate our results from Section 3.1.1 to a range of galactic nuclei and determine the overall rate of mergers in the Universe. Observations by Barth, Greene & Ho (2005) and Greene & Ho (2006) have demonstrated that equation (39) extends to active SMBHs with masses as low as $10^5 M_\odot$, well within the range of our interest. We expect the total mass of stars within the radius of influence to be comparable to the mass of the SMBH, $M(<r_i) \approx 2 M_{\text{SMBH}}$. If these stars follow a radial density profile of $n_r \propto r^{-3/2}$ as expected for a relaxed system (see Section 2), then their number density at the radius of influence is

$$n_r(M_{\text{SMBH}}) \approx 1.2 \times 10^9 \text{ pc}^{-3} \sqrt{10^8 M_\odot / M_{\text{SMBH}}}.$$  

(40)

This gives a number density about 45 per cent larger than assumed in Section 2 for the Milky Way and used in our calculations presented in Table 1, but is well within the expected scatter in number densities as we discuss later in this section. The actual number density at the radius of influence for a BH will depend on both the formation history of the galaxy as well as the merger history of the SMBH.

Because the merger rate is usually greatest at the smallest radii, we can determine the rate in any nucleus by scaling equation (32) and evaluating the integral at $r_{\text{min}} \propto \sqrt{M_{\text{SMBH}}}$, the radius where the merger time-scale of the BH into the SMBH is approximately a Hubble time-scale (approximately the inner radius of BHs). Coincidentally, this radius has the same scaling relation to the outer radius, $r_{\text{max}} = r_i \propto \sqrt{M_{\text{SMBH}}}$ where we use equation (39). The total merger rate, $\Gamma_*$, is simply proportional to $\propto n_{BH}(r)\sigma^2 \propto r^{-3/2}$. Substituting in $r_i$, we get $n_{BH}(r_i) \propto M_{\text{SMBH}}^{1/2}$, independent of the power-law distribution of BHs, $p$. The cross-section of binary capture times the relative velocity is

![Figure 6. The eccentricity distribution of events. Plotted is the eccentricity distribution of mergers at the last stable orbit ($d\Gamma/d\ln e_{\text{LSO}}$) for one example of a galactic nucleus (solid line) and a globular cluster (dotted line). Both lines are normalized so that they reach a maximum value of 1. $e_{\text{LSO}}$ corresponds to encounters that directly undergo a plunge. See the text and Fig. 4 for our assumptions and details of the calculation.](https://academic.oup.com/mnras/article-abstract/395/4/2127/971826/39542/12797168)
\[ \pi b^2 w \propto w^{-11/7} \propto M_{\text{SMBH}}^{-12/28}, \]

given \( w \propto (M_{\text{SMBH}}/r)^{1/2} \). Combining these dependencies, we find the merger rate has a relatively weak dependence on the mass of the SMBH, \( \Gamma \propto M_{\text{SMBH}}^{3/28} \). Over two orders of magnitude in mass, we expect the rate to change only by \( \approx 40-60 \) per cent. To test this scaling relationship we have run a simulation with \( M_{\text{SMBH}} = 10^9 \, M_\odot \) and \( \sigma_s = 30 \, \text{km s}^{-1} \), and found that the rate was in fact comparable to the relationship found here. Any discrepancy is likely due to the slight difference in the capture rate of the BHs and stars (see equation 4).

Although the merger rate of BHs is not very sensitive to the SMBH mass, it is sensitive to the intrinsic scatter of \( n_s \) for each galaxy, which is used to normalize the entire distribution of the BHs (see equation 6). The expected rate of mergers is determined by \( \langle n_s^2 \rangle \), the average of the number densities squared at the radius of influence for the distribution of galaxies in the universe. Given \( \langle n_s^2 \rangle = \langle n_s \rangle^2 + \sigma_s^2 \), where \( \sigma_s \) is the variance of the number density at the radius of influence of a population of SMBHs, we must rescale our results by a factor

\[ \xi = \frac{\langle n_s^2 \rangle}{\langle n_s \rangle^2} = 1 + \frac{\sigma_s^2}{\langle n_s \rangle^2}. \]  

(41)

Previous studies on GW event rate estimates neglected corrections due to cosmic variance, equation (41). However it is quite plausible that \( n_s \) all galaxies with the same SMBH mass have exactly the same number of stars in the central cusp. In fact, Merritt, Mikkola & Szell (2007) determined the relaxation times for galaxies in the ACS Virgo Survey (Côté et al. 2004), and found a relatively tight correlation, however, there was still significant scatter above the mean by about an order of magnitude. We have taken the results from Merritt et al. (2007) (specifically all nuclei with \( \sigma_s < 140 \, \text{km s}^{-1} \) in their fig. 1) and determined that \( \sigma_s^2/(\langle n_s \rangle^2) \sim 30 \). Although there is considerable uncertainty in the actual value of \( \xi \), both observationally and theoretically, we expect the merger rate to be larger than we have so far calculated by \( \xi \gtrsim 10-100 \), and scaled our results by \( \xi_{30} = \xi/30 \). In our calculations, however, we use the conservative slope of equation (40) to determine the mean value of \( n_s \), and not the shallower slope found by Merritt et al. (2007), which would give rate estimates to be an order of magnitude larger than found here.

We calculate the average cosmological merger rate density by convolving the rate per galaxy with the number density distribution of SMBHs in the universe. We extrapolate the results of Aller & Richstone (2002), who found the best-fitting number density distribution of massive SMBHs to be

\[ \frac{dn_{\text{SMBH}}}{dM_{\text{SMBH}}} = c_0 \left( \frac{M_{\text{SMBH}}}{M_\odot} \right)^{-1.25} e^{-M_{\text{SMBH}}/M_*} \]  

(42)

assuming this formula is valid all the way to \( M_{\text{SMBH}} = 10^4 \, M_\odot \), where \( c_0 = 3.2 \times 10^{-11} \, M_\odot \, \text{Mpc}^{-3} \) and \( M_* = 1.3 \times 10^8 \, M_\odot \). Finally, we get the cosmological merger rate by integration:

\[ \mathcal{R} = \int_{10^5 \, M_\odot}^{10^9 \, M_\odot} \Gamma_{\text{IGN}}(M_{\text{SMBH}}) \frac{dn_{\text{SMBH}}}{dM_{\text{SMBH}}} \, dM_{\text{SMBH}} \approx 3 \Gamma_{\text{IGN}} \xi_{30} \, \text{Mpc}^{-3}, \]  

(43)

where \( \Gamma_{\text{IGN}} \) is the expected rate of mergers for a single galactic nucleus of a specific model shown in Table 1. The normalization \( \xi_{30} \, \text{Mpc}^{-3} \) follows from the distribution given in equation (42), and also accounts for the intrinsic scatter of \( n_s \) for a population of galaxies, equation (41). For our fiducial model of the MW, Model B, we get a comoving rate density of \( 8.4 \times 10^{-10} \xi_{30} \, \text{yr}^{-1} \, \text{Mpc}^{-3} \). Because this integral is nearly flat in the log of \( M_{\text{SMBH}} \), it results in a similar rate density per logarithmic mass bin and is not very sensitive to the limits of the integration. Taking the lower limit of currently observed SMBH masses, of \( 10^2 \, M_\odot \), we get a rate \( \approx 60 \) per cent of what we calculated here. However, recent observations by Greene, Ho & Barth (2008) have shown that the \( M_{\text{SMBH}}-\sigma \) relation extends to the smallest SMBHs observed, some of which reside in galaxies without a classical bulge. Hence, there is still \( \approx 50 \) per cent uncertainty in the merger rate due to the true function of \( \Gamma_{\text{IGN}}(M_{\text{SMBH}}) \) discussed above, but the total rate density is relatively robust given the other uncertainties in our calculation, especially \( \xi_{30} \) and \( c_{\text{BH}} \).

For future calculations, we define \( d\mathcal{R}_{mM}/d\rho_{\text{BH}} \) as the rate density for fixed masses \( m \) and \( M \) analogous to equations (31) and (34).

3.3.6 Application to massive star clusters

BHs in massive star clusters without a central massive BH will also undergo an epoch of mass segregation (Kulkarni, Hut & McMillan 1993; Sigurdsson & Hernquist 1993), in which the BHs segregate to the cluster core, and effectively decouple from the stars forming...
their own subcluster (Portegies Zwart & McMillan 2000; Merritt et al. 2004; O’Leary et al. 2006; Miller & Lauburg 2009). This BH subcluster will continue to interact only with the BHs until a sufficient number of BHs are ejected dynamically, that they come back into equilibrium with the stars. This occurs approximately when \( N_{\text{BH}} \lesssim 100 \) (Watters, Joshi & Rasio 2000). For massive clusters, with \( \langle u_{\text{BH}} \rangle \sim 15 \text{ km s}^{-1} \) and \( n_{\text{BH}} \sim 10^6 \text{ pc}^{-3} \), the cluster evaporates before about \( \lesssim 10^8 - 10^9 \text{ yr} \). Scaling equation (32) to these parameters, the cluster will have a BH–BH merger rate of

\[
\Gamma_{\text{MSC}} \approx N_{\text{BH}}(N_{\text{BH}}u_{\text{BH}}c_s) \approx 1.8 \times 10^{-8} \text{ yr}^{-1}
\]

\[
\times \left( \frac{u_{\text{BH}}}{15 \text{ km s}^{-1}} \right)^{-1/7},
\]

(44)
during this period of evolution, if the number density of BHs is uniform in radius. This is comparable to, but slightly less than merger rate found by O’Leary et al. (2006) for the early evolution of a cluster of BHs due to three-body and four-body encounters alone. The detection rate of such early mergers depends on the number of young clusters (with \( t_{\text{age}} \lesssim 10^8 - 10^9 \text{ yr} \) within the detection limit of LIGO. Globular clusters, an important source of delayed mergers, are too old to still have a BH subcluster. However, the young clusters in starburst galaxies would be an excellent source if they survive sufficiently long in their hosts to undergo this process of mass segregation (O’Leary et al. 2007).

The eccentricity distribution of binary capture mergers in star clusters is plotted in Fig. 6. Overall, the rate of mergers in star clusters is dominated by small eccentricity events, which would be detected as circular inspirals by ground-based GW detectors. However, young star clusters may have as many, or even more, eccentric mergers than are expected in the nuclei of galaxies if there are a sufficient number of mergers in young star clusters. In massive star clusters, we expect \( \sim 10 \) per cent of all BH captures to merge with eccentricities similar to those in galactic nuclei. Therefore, the distribution of low-eccentricity events will be indicative of the source of BH–BH mergers, and may be useful in constraining the distribution and evolution of BHs in both galactic nuclei and massive star clusters.

4 DETECTION OF GRAVITATIONAL WAVES

To determine the expected detection rate of sources, we must now calculate the maximum luminosity distance to which these inspirals are detectable. In this section, we discuss the general properties of the waveform, calculate the maximum distance of detection and add up the total expected detection rate for second-generation terrestrial GW instruments.

4.1 General properties of the inspiral

The evolution of the binary and the GW signal can be separated into three phases.

[I] Highly eccentric encounters – train of distinct GW bursts in time, broad-band signal in frequency.

[II] Moderate–small eccentricity inspiral – continuous GW signal in time, dominated by distinct frequency harmonics.

[III] Merger and ringdown – short duration peak GW power and exponential decay.

The distinction between phases I and II can be understood by studying the evolution of the relevant time-scales that determine the GW waveform: the orbital time (\( \Delta t_{\text{orb}} = \omega_{\text{orb}}^{-1} \)), and the time duration of pericentre passage (\( \Delta t_{\text{p}} \)). The units are minutes on both axes for \( (M, \eta, \rho_0, \Delta t_{\text{merger}}) = (20, 25, 40) \) and are different for other values as marked. Similar to Fig. 3, the orbits should not be extrapolated beyond the LSO for a particular \( \rho_0 \), shown with circles. The inspiral is quasi-periodic if \( \Delta t_{\text{orb}} \ll \Delta t_{\text{merger}} \) and \( \Delta t_{\text{p}} \ll \Delta t_{\text{merger}} \). The GW signal is burst-like if \( \Delta t_{\text{p}} \ll \Delta t_{\text{orb}} \) and is continuous if \( \Delta t_{\text{orb}} \sim \Delta t_{\text{orb}} \).

The GW waveform: the orbital time (\( \Delta t_{\text{orb}} = \omega_{\text{orb}}^{-1} \)), and the time duration of pericentre passage (\( \Delta t_{\text{p}} = \omega_{\text{p}}^{-1} \)). These are plotted in Fig. 8 as a function of time to merger \( t_{\text{merger}} \). For both phases I and II, \( |d\Delta t_{\text{orb}}/dt_{\text{merger}}| \ll 1 \), so the orbit is quasi-periodic and evolves gradually due to the emission of GWs.

The characteristic GW emission time-scale during each orbit is determined by \( \Delta t_{\text{p}} \). Fig. 8 shows that initially (phase I), \( \Delta t_{\text{p}} \ll \Delta t_{\text{orb}} \) implying that the waveform consists of a train of short \( \Delta t_{\text{p}} \) duration bursts arriving quasi-periodically with separation \( \Delta t_{\text{orb}} \). Later, when the burst duration time-scale \( \Delta t_{\text{p}} \) becomes comparable to \( \Delta t_{\text{orb}} \), the signal becomes continuous in time domain (phase II). Since the orbital evolution is quasi-periodic, the GW signal is approximately a sum of discrete frequency harmonics of the orbital frequency, \( \omega_{\text{orb}} \). When the eccentricity becomes relatively small \( e \lesssim 0.7 \), (phase II) the harmonic decomposition is quickly convergent, while during phase I, it is more convenient to work with the continuous limit of the frequency spectrum.

During phases I and II the equations of motion and the GW waveforms can be calculated accurately in various approximations (see Section 1.1). For phase II, the expected S/N of detection has been calculated by Barack & Cutler (2004), and the \( e = 1 \) parabolic case was examined in Kocsis et al. (2006). In the following, we generalize these studies to be applicable to phases I and II.

Once the orbital separation reaches the last stable orbit, the BHs fall in rapidly and form a common horizon. The last stable orbit, which marks the end of phase II, is determined by the initial pericentre distance \( \rho_{\text{p}} \) for an initially parabolic orbit shown by circles in Fig. 8. The GW waveform during phase III involves the calculation of the violently changing space–time and the eventual relaxation into a Kerr BH. This requires full numerical simulations of the Einstein equations. The detectability of the resulting waveforms have been examined for non-spinning binaries and quasi-circular initial conditions (Baker et al. 2007; Berti et al. 2007). These studies have shown that \( S/N = 10 \) can be reached up to a distance \( d_L = 1–6 \text{ Gpc} \) for total binary mass \( M_{\text{tot}} = 10–200 \text{ M}_\odot \) for AdLIGO. Eccentric mergers were considered very recently by Hinder et al. (2008a,b) and Washish et al. (2008) for the case of no initial spins.
who showed that the resulting GW power is comparable to (or sometimes larger than) the power released during quasi-circular mergers. Future studies should address spin effects during the coalescence, they might significantly modify the GW power and waveforms.

We note that the separation between phases I, II and III is valid only if the initial encounter has a minimum separation, \( r_p \), that is much larger than the unstable circular orbit, \( r_{UCC} \sim 2-4 M_{\odot} \), depending on BH spins. Direct captures, or orbits outside but repeatedly approaching the unstable circular orbit (the so-called ‘zoom-whirl’ orbits), are qualitatively different. Such encounters have been studied in the geodesic approximation appropriate for extreme mass ratios (Gair, Kennefick & Larson 2005, 2006; Pretorius & Khurana 2007; Levin & Perez-Giz 2008) and using full numerical simulation for equal masses (Hinder et al. 2008b). In this case the GW spectrum is considerably different and the power is considerably increased.

The purpose of this section is to derive the S/N for the quasi-periodic phases I and II and determine the maximum range of detection for second-generation terrestrial GW instruments. We leave the assessment of phase III and the zoom-whirl domain to future studies.

### 4.2 Signal-to-noise ratio

Here we briefly review the general calculation of the S/N for detecting the GW signal, which we can then utilize for the waveforms generated by GW capture events. We refer the reader to Flanagan & Hughes (1998) for more details.

In general, the S/N of a GW detection is defined as

\[
\frac{S^2}{N^2} = 4 \int_{f_{\text{min}}}^{f_{\text{max}}} \left| \frac{h(f, \theta)}{S_0(f)} \right|^2 \, df, \tag{45}
\]

where \( h(f, \theta) \) is the Fourier transform of the GW signal weighted by the antenna beam patterns, \( \theta = \{ \theta_i \} \) are the physical parameters describing the source and the detector orientation, \( S_0(f) \) is the one-sided noise spectral density in units of Hz\(^{-1}\) and \( f_{\text{min}} \leq f \leq f_{\text{max}} \) correspond to the frequency band of the instrument e.g. \( f_{\text{min}}, f_{\text{max}} \approx (10, 10^4) \) Hz for AdLIGO.\(^5\)

The sky position and binary orientation averaged root mean square S/N for a single orthogonal arm interferometric GW instrument is

\[
\left\langle \frac{S^2}{N^2} \rightangle = \int_{f_{\text{min}}}^{f_{\text{max}}} \frac{h_c(f)^2}{5 f S_0(f)} \, df, \tag{46}
\]

where \( h_c(f) \) is the characteristic isotropic GW amplitude defined as

\[
h_c = \frac{1}{3\pi} \sqrt{\frac{2E}{df}} = \frac{1}{3\pi} \sqrt{\frac{2E}{df}}, \tag{47}
\]

where \( d_L(z) \) is the luminosity distance to a source at a cosmological redshift \( z \), \( dE/df \) is the one-sided GW energy spectral density on a spherical shell at infinity. The second equality corresponds to the stationary phase approximation for a quasi-periodic signal sharply peaked at frequency \( f \). In this case, \( E \) is the total GW power at frequency \( f \), which evolves slowly in time according to its time derivative \( f' \).

Equations (46) and (47) can be generalized for a signal consisting of discrete harmonics \( f_n \), with negligible overlap as (Barack & Cutler 2004)

\[
\left\langle \frac{S^2}{N^2} \rightangle = \sum_{n=2}^{\infty} \int_{f_{\text{min}}}^{f_{\text{max}}} \frac{h_{c,n}(f_n)}{5 f_n S_0(f_n)} \, df_n \tag{48}
\]

and

\[
h_{c,n} = \frac{1}{\pi d_L} \sqrt{\frac{2E_n}{f_n}}, \tag{49}
\]

where \( E_n \) is the GW power radiated at frequency \( f_n \). For quasi-periodic orbits with an intrinsic orbital frequency \( f_{\text{orb}} \), the observed frequency harmonics at redshift \( z \) are given by

\[
f_n = n f_{\text{orb}, z} = n f_{\text{orb}} \frac{1}{1+z}. \tag{50}
\]

### 4.3 Application to the GW capture process

Let us now turn to the detectability of the GWs starting from the initial hyperbolic encounter and ending in the violent BH merger. Here we derive a computationally more efficient equivalent form of equation (48), which can be utilized for phases I and II.

Following Barack & Cutler (2004), we calculate the binary evolution and the GW signal waveform in the leading order approximation of Peters & Mathews (1963), where the interacting masses move on quasi-Newtonian trajectories and emit quadrupolar radiation. This approximation is adequate in terms of the angular-averaged S/N if the initial periapse is well outside the unstable circular orbit e.g. \( r_p \gg r_{UCC} \). At smaller initial \( r_p \), the Newtonian approximation underestimates the GW power and is therefore conservative.\(^6\) To lowest order,

\[
\bar{E}_n = \frac{32}{5} \frac{n^2}{n^3} M_{\odot}^{10/3} \frac{\alpha_{\odot}}{m_{\odot}} g(n, e), \tag{51}
\]

where \( g(n, e) \) determines the relative power of the \( n \)th harmonic for orbits with eccentricity \( e \), which is given by Peters & Mathews (1963) in the Newtonian approximation as

\[
g(n, e) = \frac{n^4}{32} \left[ \left( J_{n-2} - 2e J_{n-1} + \frac{3}{n} J_n + 2e J_{n+1} - J_{n+2} \right)^2 + (1-e^2) J_{n-2} - 2J_n + J_{n+2} \right]^2 + \frac{4}{3n^2} \right]. \tag{52}
\]

Here \( J_i(x) \) is the \( i \)th Bessel function, and we have suppressed the argument \( xe \), i.e. \( J_i(xe) \) for each \( i \) above. These waveforms generally have a maximum at frequency \( \omega_{\text{op}} \) for \( e \gtrsim 0.5 \) and have a steep cut-off such that the fractional GW power beyond \( 5 \omega_{\text{op}} \) is smaller than \( 10^{-3} \) (Turner 1977). For smaller \( e \), the harmonics beyond the first \( 2 \omega_{\text{op}} \) are greatly suppressed. The number of harmonics necessary to a precision \( 10^{-3} \) can be estimated as

\[
N_{\text{max}} = 5 \frac{\alpha_{\odot}}{m_{\odot}} = 5 \frac{(1+e)^{1/2}}{(1-e)^{1/2}} \tag{53}
\]

e.g. \( N_{\text{max}} = (10, 40, 10^3) \) for \( e = (0.3, 0.7, 0.99) \), respectively.

Note that equation (51) depends explicitly on \( e \). Therefore, in order to directly evaluate the S/N integral in equation (48) over \( f_n \) one needs to invert the frequency evolution equation \( f_n(t_{\text{ph}}, e_0, e) \).

---

\(^5\) http://www.ligo.caltech.edu/AdvLIGO/scripts/summary.shtml

\(^6\) Modulations due to general relativistic pericentre precession and relativistic beaming would be very important for the real data analysis, but not in terms of the calculation of event rates that depend on the angular averaged total S/N.
Computationally it is much more efficient to change the integration variable from $f_n$ to $e$,
\[
\frac{df_n}{f_n} = \frac{3}{2} \frac{\ln a}{a_0} = -18 \left( (73/24)e^2 + (37/96)e^4 \right) de,
\]
and reverse the order of the sum and the integral:
\[
\left\langle \frac{S^2}{N^2} \right\rangle = \frac{48 \eta M_{\text{tot}}^2 \rho_0^2}{95} \frac{dL}{d^3 \xi^{19}} \sum_{n=2}^{n_{\text{max}}} \frac{g(n, e)s(e, e_0)\, de}{n^2 S_h(f_n) / e},
\]
(54)

where $e_{\text{LSO}}$ corresponds to the particular $\rho_0$ (see equation 23), $e_0$ is given by equation (16), and
\[
s(e, e_0) = \left( \frac{e}{e_0} \right)^{24/19} \left[ \frac{1 + (121/304)e^2}{1 + (121/304)e_0^2} \right]^{1240/2299}
\times \frac{(1 + e^2)(1 - e^2)^{3/2}}{1 - (183/304)e^2 - (121/304)e^4}.
\]
(56)

Note, that $S_h(f) = \infty$ is assumed outside of $f_{\text{min}} \leq f \leq f_{\text{max}}$. The upper limit of the sum in this form can be adjusted to the required calculation precision using equation (53).

For phase II, we can rewrite equation (55) by changing the sum over $n$ to a continuous integral over $f_n$:
\[
\left\langle \frac{S^2}{N^2} \right\rangle = \frac{48 \eta M_{\text{tot}}^2 \rho_0^2}{95} \frac{dL}{d^3 \xi^{19}} \int_{e_{\text{LSO}}}^{e_{\text{max}}} \frac{g(n, e)s(e, e_0)\, df \, de}{f_{\text{orb}} \, S_h(f) \int e}.
\]
(57)

where $n = f/f_{\text{orb}}, f_{\text{orb}} = f_{\text{orb}}(r_{\text{ps}}, e, e_0)$ given above by equation (50), and $r_{\text{ps}} = M_{\text{tot}}, \rho_0 = (1 + e)M_{\text{tot}}\rho_0$.

In addition to their numerical advantages, equations (55) and (57) can be used to study the time-frequency evolution of the instantaneous S/N accumulation rate as the orbit evolves. Fig. 9 shows the contribution of the first $n \leq 100$ harmonics to the S/N for AdLIGO (i.e. before evaluating the sum or the integral in equation 55) for total masses $M_{\text{tot}} = (20, 60)M_\odot$ and initial periapsis $\rho_0 = (10, 20, 40)$. The figure illustrates the unique frequency evolution of the signal consistent with the expectations described above. Initially during phase I, it is broad-band in frequency, and decouples into discrete harmonics at smaller eccentricities during phase II. The contribution of upper harmonics is non-negligible even at LSO, especially if the initial periapsis satisfies $\rho_0 \lesssim 40$. Note that the maximum frequency of the S/N at large eccentricities $e \gtrsim 0.8$ in Fig. 9 is merely a consequence of not plotting harmonics beyond $n = 100$, leading to a large underestimate of S/N in phase II. In this case equation (57) becomes more useful than equation (55).

The frequency-independent S/N accumulation rate can be obtained as a function of eccentricity (or time using the $e(t)$ dependence shown in Fig. 3), if evaluating the sum over $n$ in equation (55) or the integral over $d\omega/f$ in equation (57), but not the integral over eccentricity. Fig. 10 shows the result for the same masses and periapsis as Fig. 9, including an additional extreme case of $M_{\text{tot}} = 700M_\odot$. For BH masses below $M_{\text{tot}} \sim 100$, the S/N contribution of the high- and low-eccentricity phases are comparable. The S/N during highly eccentric encounters dominate for the typical case $M_{\text{tot}} \sim 20M_\odot$ for $\rho_0 \lesssim 20$, while small $e$ dominates for larger masses $M_{\text{tot}} \sim 60M_\odot$ for $\rho_0 \gtrsim 20$. It is very interesting that the eccentricity evolution of GWs as a function of eccentricity for various total mass and initial periapsis as labelled. The shading represents the expected S/N of the first $n = 100$ harmonics per logarithmic eccentricity bins for AdLIGO. The waveform is described by a broad-band spectrum at large $e$ (repeated burst phase) that later separates into distinct harmonics as eccentricity decreases until the LSO (eccentric inspiral phase). The S/N is substantial already at $e \gtrsim 0.7$.
more pronounced at much larger masses. A 100–100 M$_{\odot}$ eccentric inspiral is detectable up to three times farther than a standard 10–10 M$_{\odot}$ circular inspiral. For even larger masses, $d_{c}^{\text{max}}$ decreases, but remains non-negligible up to $M_{\text{tot}} = 700$ M$_{\odot}$. Currently less is known about the existence of such intermediate-mass BHs in galactic cusps, which makes it very difficult to make any theoretical estimates on the expected rates of such encounters. It is possible that AdLIGO will provide the first direct observational limit on the population of these objects. We explore this along with other estimates of the detection rate in Section 4.4.

4.4 Detection rate estimates

We set $(S^2/N^2)^{1/2} \geq 5$ as our detection threshold for a single GW instrument, and determine the volume averaged maximum luminosity distance for detection for each pair of masses, $d_{c}^{\text{max}} (m, M)$ from equation (58), assuming a uniform distribution of $r_p$ out to $r_{p,\text{max}}$. Thus, the total detection rate for Advanced LIGO is estimated to be

$$R = \int_{0}^{r_{p,\text{max}}} \int_{M_{\text{min}}}^{M_{\text{max}}} \int_{M_{\text{min}}}^{M_{\text{max}}} \int_{0}^{z_{\text{max}}} \int_{0}^{z_{\text{max}}} \frac{V_{\text{co}}}{dz} \frac{dV_{\text{co}}}{dz} \frac{\rho}{\rho_{0}} \frac{\partial^{3}R}{\partial \rho_{0} \partial dM \partial M} ,$$

where $\frac{\partial^{3}R}{\partial \rho_{0} \partial dM \partial M}$ is the comoving partial binary formation rate density between masses $(m, M)$ at initial periapse $r_{p,0}$, given by equations (31) and (43), averaged over the distribution of $M_{\text{SMBH}}$ and the number density normalization $n_{s}$ (see equations 42 and 43) at redshift $z$, $n_{\text{gal}}$ is the comoving number density of galaxies at redshift $z$, $z_{\text{max}}$ is the maximum detectable distance corresponding to $d_{c}^{\text{max}} (\rho_{0}, m, M)$ and $dV_{\text{co}}/dz$ is the comoving volume density corresponding to the given cosmology. In practice, however, we calculate $R$ discretely and ignore cosmological effects, which are not relevant for the next generation of GW instruments,

$$R \approx \sum_{M = m + M} \int_{0}^{r_{p,\text{max}}} \frac{dR_{\text{tot}}}{dr_{p,0}} \frac{d\rho_{0}}{\rho_{0}} \frac{4\pi}{3} (d_{L}^{\text{max}})^{3} ,$$

where $dR_{\text{tot}}/dr_{p,0}$ is the differential rate density of binary formation between mass bins $m$ and $M$ (see equations 31 and 43). The resulting total detection rates are shown in the last column of Table 1. Note that the dependence of $d_{c}^{\text{max}}$ on the binary parameters $(\rho_{0}, m, M)$ lead to an observational bias. For example, since $d_{c}^{\text{max}}$ is relatively larger for $\rho_{0} \sim 10$, the detection rate of these encounters is enhanced relative to larger and smaller $\rho_{0}$, even though the intrinsic rate of these encounters is independent of $\rho_{0}$. We have plotted the differential detection rate as a function of $\rho_{0}$ in Fig. 12 for all mergers as well as each mass bin.

Overall, the most massive BHs in galactic nuclei dominate the detection rate of mergers for AdLIGO. In Fig. 13, we have plotted the distribution of detectable mergers as a function of radius for the entire population of BHs, as well as for each mass bin. The clear domination of the high-mass BHs is caused by a combination of three important factors: (i) their number density is significantly enhanced by mass segregation (Section 2); (ii) the signal of the event is much stronger for larger masses (e.g. equations 55 and 57); and (iii) the cross-section for binary capture is greater for larger masses (e.g. equation 17).

As discussed in Section 4.3, we expect the actual inspiral of a BH with an IMBH may be revealed by AdLIGO if the event is sufficiently eccentric during plunge. Although we cannot properly account for this in our analysis, we can attempt to compare the rate to what we have done in this work. For a large mass ratio, $m \gg M$, the...
the overall cross-section for forming binaries increases roughly as $b^2 \propto m^{12/7}$. Compared to $10 \, M_\odot$ BHs, we expect a $1000 \, M_\odot$ BH to have a GW capture event $\sim 3000$ times as often as a single BH. However, the number of IMBHs in the region is very uncertain. If we take the optimistic number of $\sim 10$ IMBHs in a single galactic nucleus as a steady state distribution (Portegies Zwart et al. 2006), then we expect a comparable total number of events to $10-100 \, M_\odot$ BH–BH inspirals.

Our analysis in Section 2 suffers from inaccuracies for the most massive and rarest BHs. Equation (2) was derived assuming a constant density core (and constant relaxation time-scale) for large $r$ (BW76), which is clearly violated in most galactic nuclei. Typically, the total number of BHs in galactic nuclei decreased with $M_{\text{max}}$. Despite this decrease in number, we likely cannot extrapolate our calculations to higher mass BHs such as IMBHs, in order to see what effect they have on flattening the density profile of stellar-mass BHs.

5 SUMMARY AND DISCUSSION

In this paper, we have analysed two separate problems. First, we determined the multimass distribution of BHs in galactic nuclei, which we then used to analyse the detection rate and merger of GW capture binaries. We integrated the time-dependent Fokker–Planck equations for a variety BH mass distributions until they reached a steady state. We found, consistent with previous results, that within a relaxation time-scale at the radius of influence of the SMBH, the BHs and stars form approximately power-law density cusps ($\propto r^{-3/2-\rho_0/m/M_{\text{max}}}$) within the radius of influence of the SMBH. Because the BHs are more massive than the stars, they have steeper density profiles ($\propto r^{-2}$) and dominate the dynamics and relaxation processes in the inner $\sim 0.1$ pc near the SMBH.

In such dense population environments, the probability of close flybys between two BHs is non-negligible. Using our results for the steady-state distribution of BHs, we calculated the expected...
rate and distribution of GW capture binaries in galactic nuclei. We showed that after forming, these binaries rapidly inspiral and merge within hours. Unlike other sources of merging binaries, the BH binaries in galactic nuclei form with a characteristic GW frequency inside the AdLIGO frequency band, and the detectable GW signal duration is much longer compared to circular inspirals. In addition, for sufficiently small impact parameters, the waveforms for such eccentric inspirals are broad-band, and can be detected for much larger masses, up to $M_{\text{tot}} \gtrsim 700 M_{\odot}$. This exceeds the BH masses previously considered detectable (Brown et al. 2007; Mandel et al. 2008), and as such it opens a new avenue to probe for the existence of a population of intermediate-mass BHs in galactic nuclei.

Additionally, in Section 3.3.6, we also estimated the rate of GW captures in young, massive star clusters where BHs decouple from stars to form a subcluster in the centre. We found that the rate of GW captures in a single cluster may be intrinsically larger than in a galactic nucleus. Typically, most binaries circularize before becoming detectable, however a significant fraction ($\sim 10$ per cent) may still merge with residual eccentricity. Given the vastly different eccentricity distribution of mergers in galactic nuclei and massive star clusters, the dominant source of eccentric inspirals can be determined with only a few detections. The total rate of such events from massive clusters depends on the relatively unconstrained number of young clusters within the detection limits of AdLIGO.

Finally, we analysed the properties of the GW signal for the eccentric inspiral of comparable mass BHs, starting from the initial highly eccentric phase up to the final eccentric inspiral. We found that the maximum luminosity distance that such events may be detected using a single AdLIGO-type interferometer is 1.2 Gpc for component masses $m = M \sim 10 M_{\odot}$, and up to 3.3 Gpc for $m = M \sim 50 M_{\odot}$. We then used this to calculate the total detection rate of signals, by using a model merger rate convolved with a realistic population of SMBHs. We found that the most massive BHs dominate the detection rate of future ground-based GW detectors with $\sim (10^{-10}) \xi_{30}$ events expected per year, where $\xi_{30}$ is a measure of the expectation value for the square of number densities near the SMBH (see equation 41). The overall detection rate is sensitive to the number fraction of BHs as well as the maximum mass of BHs, and so future observations will be able to constrain both the average star formation properties and upper mass of BHs in galactic nuclei.

Overall, the expected AdLIGO detection rate for GW capture binaries in galactic nuclei is comparable to estimates for other sources of GWs. BH–BH binaries that form dynamically in massive star clusters may be detected by AdLIGO $\sim 10^{10}$ times per year, depending on the total fraction of star formation that occurs in massive, long lived clusters (Portegies Zwart & McMillan 2000; Gültekin et al. 2004, 2006; O’Leary et al. 2006, 2007; Miller & Lauburg 2009). Estimates for the population of merging binaries that formed in the early evolution of present day globular clusters are $\sim 10^{-10} \text{yr}^{-1}$ alone. Although globular clusters contain only a small fraction of the mass of stars in the universe, the merger rate of BHs is sufficiently enhanced by dynamical interactions that the rate is comparable to those expected from standard binary evolution for all other stars (Belczynski et al. 2002, 2004; Sadowski et al. 2008; however, see Belczynski et al. 2007). Merging neutron stars are also a promising source of GWs, whose progenitors have been directly observed. Using constraints from the modest population neutron star–neutron star binaries known in our Galaxy, Kalogera et al. (2004a,b) estimate that $\sim 20–600$ events per year may be detected by AdLIGO. We emphasize that the rates of GW captures calculated in this paper are independent of other sources of GWs, and will occur in addition to other sources. Given the estimates of sources from a variety of environments, GW detectors promise to open a new and interesting window into the physics, dynamics and evolution of compact binaries.

So far, we have not looked at the interaction between the BHs and other compact objects such as neutron stars (NSs). The NSs are not expected to segregate significantly in galactic nuclei, however, they are intrinsically more common than BHs. To test their importance we have performed an additional run of Model E-2 to look at the expected detection rate of NS–BH mergers. We found that the total detection rate of such events is about $\sim 1$ per cent of the total rate ($\sim 1 \text{yr}^{-1}$), and merits future study. GWs from BH–NS inspirals can provide interesting constraints on the equation of state for NSs (Faber et al. 2002), given model sources from numerical simulations (Faber et al. 2006; Rantsiou et al. 2008). To date, such analyses have focused on the circular inspiral of BHs with NSs, however, recent population synthesis studies suggest the number of BH–NS binaries that form may be as rare as those expected here, $\sim 1 \text{yr}^{-1}$ (Belczynski et al. 2007). Given that GW capture binaries are predominately eccentric throughout inspiral, we also expect the newly formed BH–NS binaries to be eccentric. However, with BH–NS binaries, the tidal effects of the BH on the NS can provide an additional source of energy dissipation in the binary, which can enrich the GW signal, and cause it to deviate from the point-mass approximations calculated here. Future numerical simulations with eccentric encounters must be done in order to test the importance of eccentricity on the GW signal for BH–NS binaries.

Another area we neglected is the influence of eccentricity on the merger rate of BHs in galactic nuclei. However, we can estimate their overall importance by looking at what fraction of binaries are hard enough to survive in galactic nuclei. Given the distribution of BH binaries in Belczynski et al. (2004), only a small fraction, $\lesssim 0.1$ per cent of the BH–BH binaries are sufficiently hard (with an orbital period $\lesssim 10 \text{d}$) not to be disrupted by repeated encounters with stars or BHs. Interestingly, this fraction is comparable to the fraction of BHs that merge due GW capture. The properties of the merger of such binaries have not been explored yet, however, given the evidence that such encounters produce few eccentric events in massive star clusters, we expect that they will not have many eccentric mergers in galactic nuclei. Unfortunately, the methods we have used here are not suitable for looking at higher order $N$-body interactions (such as exchanges, or binary–binary scattering); simulations similar to FAK06 are more adequate for this type of study. The rates we calculated might be enhanced if high-dispersion environments of protogalactic cores without a SMBH also exist, although such systems are old and not long lived (Quinlan & Shapiro 1987, 1989, 1990; Lee 1993).

In order to estimate the detection rate of GW capture binaries in galactic nuclei we had to make assumptions that introduced uncertainty in our calculations. For the sake of clarity, we wish to summarize these uncertainties and the order of magnitude effect they may have on the actual detection rate for AdLIGO. The largest uncertainty in the rate calculation is the distribution of BHs in galactic nuclei, especially the number fraction of BHs ($\sim 10^2$) and their mass distribution ($\sim 10$) in the centres of galaxies. Observations of our own Galactic Centre suggest that the star formation may be top-heavy, and place the rate on the higher end of that reported here. One effect we neglect in our calculations is resonant relaxation, which may reduce the rate ($\sim 10$) by depleting the central cusp of BHs. Our final rate, however, is in practice only logarithmically sensitive to the inner edge of the cusp of BHs, and the densest cusps that dominate our rate will be least affected by resonant relaxation. The
final rate is also nearly logarithmically dependent on the population of SMBHs in the local universe (by a factor of at most $\sim 10$), but is much more sensitive to the distribution of densities at the radius of influence of the BHs (by a factor of $\sim 10^{10}$). Finally, there is some uncertainty in the detection rate given our calculation of the S/N of the mergers ($\sim 10$) since we do not consider corrections for zoom-whirl orbits and conservatively ignore the contribution of the merger and ringdown waveforms of the binary to the signal.

There are many aspects to the merger of GW capture binaries that require future work, which will be aided by advanced numerical simulations of the evolution of these binaries. In our calculations we used only the leading order (Newtonian) formula to calculate the evolution of the binaries and the GW waveforms, and do not account for the GW signal during the plunge and merger of the binary, nor do we consider the contribution of zoom-whirl orbits. The recent breakthrough of numerical simulations finally allows the full GW signal to be calculated directly. Eccentric mergers promise to have a richer signal than that for circular binaries. Future studies should incorporate the contribution of the final coalescence to the GW signal, which would increase the signal strength generated during the inspiral for the same capture event, and has a potential to further increase the maximum range of detection.

We also found that a fraction ($\sim 10$–20 per cent) of mergers will occur very close to, or within the estimated last stable orbit of BH binaries. These capture events may be zoom-whirl orbits, which dramatically increases the GW signal compared to a regular inspiral. These capture events may be zoom-whirl orbits, which will occur very close to, or within the estimated last stable orbit of BH binaries. These capture events may be zoom-whirl orbits, which will occur very close to, or within the estimated last stable orbit of BH binaries.

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REFERENCES

Abbott B. et al., 2008, Phys. Rev. D, 77, 062002
Alexander T., Hopman C., 2008, ApJ, submitted (arXiv:0808.3150)
Alexander T.,abin P., 2001, ApJ, 549, 948
Alexander T., Loeb A., 2001, ApJ, 551, 223
Alexander T., Sternberg A., 1999, ApJ, 520, 137
Alexander R. D., Armitage P. J., Cuadra J., Begelman M. C., 2008, ApJ, 674, 927
Aller M. C., Richstone D., 2002, AJ, 124, 3035
Amaro-Seoane P., Gair J. R., Freitag M., Miller M. C., Mandel I., Cutler C. J., Babak S., 2007, Class. Quantum Gravity, 24, 113
Arun K. G., Iyer B. R., Quilish M. S. S., Sathyaprakash B. S., 2006, Phys. Rev. D, 74, 024006
Arun K. G., Blanchet L., Iyer B. R., Quilish M. S. S., 2008, Phys. Rev. D, 77, 064035
Babak S., Fang H., Gair J. R., Glampedakis K., Hughes S. A., 2007, Phys. Rev. D, 75, 024005
Bahcall J. N., Wolf R. A., 1976, ApJ, 209, 214 (BW76)
Bahcall J. N., Wolf R. A., 1977, ApJ, 216, 883 (BW77)
Baker J. G., McWilliams S. T., van Meter J. R., Centrella J., Choi D.-I., Kelly B. J., Kopitz M., 2007, Phys. Rev. D, 75, 124024
Barack L., Cutler C., 2004, Phys. Rev. D, 69, 082005
Barth A. J., Greene J. E., Ho L. C., 2005, ApJ, 619, L151
Belczynski K., Kalogera V., Bulik T., 2002, ApJ, 572, 407
Belczynski K., Sadowski A., Rasio F. A., 2004, ApJ, 611, 1068
Belczynski K., Taam R. E., Kalogera V., Rasio F. A., Bulik T., 2007, ApJ, 662, 504
Berti E., Cardoso J., Cardoso V., Cavaglià M., 2007, Phys. Rev. D, 76, 104044
Binney J., Tremaine S., 1987, Galactic Dynamics. Princeton Univ. Press, Princeton, NJ
Boyle M., Buonanno A., Kidler L. E., Mroué A. H., Pan Y., Pfeiffer H. P., Schael M. A., 2008, Phys. Rev. D, 78, 104020
Brown W. R., Geller M. J., Kenyon S. J., Kurtz M. J., 2005, ApJ, 622, L33
Brown W. R., Geller M. J., Kenyon S. J., Kurtz M. J., 2006, ApJ, 640, L35
Brown D. A., Brink J., Fang H., Gair J. R., Li C., Lovelace G., Mandel I., Thorne K. S., 2007, Phys. Rev. Lett., 99, 211012
Chamamé J., Gould A., 2002, ApJ, 571, 320
Chamamé J., Gould A., Miralda-Escudé J., 2001, ApJ, 563, 793
Collins N. A., Hughes S. A., 2004, Phys. Rev. D, 69, 124022
Côte P. et al., 2004, ApJS, 153, 223
Damour T., Gopakumar A., Iyer B. R., 2004, Phys. Rev. D, 70, 064028
Deegan P., Nayakshin S., 2007, MNRAS, 377, 897
Eisenhauer F. et al., 2005, ApJ, 628, 246
Faber J. A., Grandclément P., Rasio F. A., Taniguchi K., 2002, Phys. Rev. Lett., 89, 231102
Faber J. A., Baumgarte T. W., Shapiro S. L., Taniguchi K., Rasio F. A., 2006, Phys. Rev. D, 73, 024012
Flanagan É. É., Hughes S. A., 1998, Phys. Rev. D, 57, 4535
Freitag M., Amaral-Seeane P., Kalogera V., 2006, ApJ, 649, 91 (FK06)
Gair J. R., Barack L., Creighton T., Cutler C., Larson S. L., Phinney E. S., Vallisineri M., 2004, Class. Quantum Gravity, 21, 1595
Gair J. R., Kennefick D. J., Larson S. L., 2005, Phys. Rev. D, 72, 084009
Gair J. R., Kennefick D. J., Larson S. L., 2006, Phys. Rev. D, 74, 100901
Genzel R. et al., 2003, ApJ, 594, 812
Ghez A. M., Salim S., Hornstein S. D., Tanner A., Lu J. R., Morris M., Becklin E. E., Duchêne G., 2005, ApJ, 620, 744
Greene J. E., Ho L. C., 2006, ApJ, 641, L21
Greene J. E., Ho L. C., Barth A. J., 2008, ApJ, 688, 159
Gütlekin K., Miller M. C., Hamilton D. P., 2004, ApJ, 616, 221
Gütlekin K., Miller M. C., Hamilton D. P., 2006, ApJ, 640, 156
Hinder I., Vaishnav B., Herrmann F., Shoemaker D. M., Laguna P., 2008a, Phys. Rev. D, 77, 081502
Hinder I., Herrmann F., Laguna P., Shoemaker D., 2008b, preprint (arXiv:0806.1037)
Hopman C., Alexander T., 2006a, ApJ, 645, 1152
Hopman C., Alexander T., 2006b, ApJ, 645, L133 (HA06)
Hopman C., Freitag M., Larson S. L., 2007, MNRAS, 378, 129
Kalogera V. et al., 2004a, ApJ, 601, L179
Kalogera V. et al., 2004b, ApJ, 614, L137
Kocsis B., Gaspár M. E., Märka S., 2006, ApJ, 648, 411
Kroupa P., Weidner C., 2003, ApJ, 598, 1076
Lee M. H., 1993, ApJ, 418, 147
Levin J., 2006, Phys. Rev. D, 74, 124027
Levin J., Perez-Giz G., 2008, Phys. Rev. D, 77, 103005
Majer J., Vasyut M., 2008, Phys. Rev. D, 77, 104005
Mandel I., Brown D. A., Gair J. R., Miller M. C., 2008, ApJ, 681, 1431
Maness H. et al., 2007, ApJ, 669, 1024
Martel K., Poisson E., 1999, Phys. Rev. D, 60, 124008
Merritt D., Piatak S., Portgez Szwart H., Hensendorf M., 2004, ApJ, 608, L25
Merritt D., Mikko S., Szel A., 2007, ApJ, 671, 53
Miller M. C., Lauburg V. M., 2009, ApJ, 692, 917

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Miralda-Escudé J., Gould A., 2000, ApJ, 545, 847
Morris M., 1993, ApJ, 408, 496
Muno M. P., Pfahl E., Baganoff F. K., Brandt W. N., Ghez A., Lu J., Morris M. R., 2005, ApJ, 622, L113
Muno M. P., Bauer F. E., Bandyopadhyay R. M., Wang Q. D., 2006, ApJS, 165, 173
Nayakshin S., Sunyaev R., 2005, MNRAS, 364, L23
Nayakshin S., Sunyaev R., 2007, MNRAS, 377, 1647
Nayakshin S., Dehnen W., Cuadra J., Genzel R., 2006, MNRAS, 366, 1410
O'Leary R. M., Loeb A., 2008, MNRAS, 383, 86
O'Leary R. M., Rasio F. A., Fregeau J. M., Ivanova N., O'Shaughnessy R., 2006, ApJ, 637, 937
O'Leary R. M., O'Shaughnessy R., Rasio F. A., 2007, Phys. Rev. D, 76, 061504
Orosz J. A. et al., 2007, Nat, 449, 872
Paumard T. et al., 2006, ApJ, 643, 1011
Peters P. C., 1964, Phys. Rev., 136, 1224
Peters P. C., Mathews J., 1963, Phys. Rev., 131, 435
Portegies Zwart S. F., McMillan S. L. W., 2000, ApJ, 528, L17
Portegies Zwart S. F., Baumgardt H., McMillan S. L. W., Makino J., Hut P., Ebisuzaki T., 2006, ApJ, 641, 319
Prestwich A. H. et al., 2007, ApJ, 669, L21
Pretorius F., Khurana D., 2007, Class. Quantum Gravity, 24, 83
Quinlan G. D., Shapiro S. L., 1987, ApJ, 321, 199
Quinlan G. D., Shapiro S. L., 1989, ApJ, 343, 725
Quinlan G. D., Shapiro S. L., 1990, ApJ, 356, 483
Rantsiou E., Kobayashi S., Laguna P., Rasio F. A., 2008, ApJ, 680, 1326
Rauch K. P., Ingalls B., 1998, MNRAS, 299, 1231
Rauch K. P., Tremaine S., 1996, New Astron., 1, 149
Remillard R. A., McClintock J. E., 2006, ARA&A, 44, 49
Rubbo L. J., Holley-Bockelmann K., Finn L. S., 2006, ApJ, 649, L25
Sadowski A., Belczynski K., Bulik T., Ivanova N., Rasio F. A., O'Shaughnessy R., 2008, ApJ, 676, 1162
Schédel R. et al., 2007, A&A, 469, 125
Sesana A., Haardt F., Madau P., 2007, MNRAS, 379, L45
Shapiro S. L., Lightman A. P., 1976, Nat, 262, 743
Sigurdsson S., Hernquist L., 1993, Nat, 364, 423
Silverman J. M., Filippenko A. V., 2008, ApJ, 678, L17
Sperhake U., Berti E., Cardoso V., González J. A., Brügmann B., Ansorg M., 2008, Phys. Rev. D, 78, 064069
Spitzer L. J., 1969, ApJ, 158, L139
Tanaka T., Tagoshi H., Sasaki M., 1996, Prog. Theor. Phys., 96, 1087
Tessmer M., Gopakumar A., 2008, Phys. Rev. D, 78, 084029
Tremaine S. et al., 2002, ApJ, 574, 740
Turner M., 1977, ApJ, 216, 610
Vasuth M., Majar J., 2007, Int. J. Mod. Phys. A, 22, 2405
Washik M. C., Healy J., Herrmann F., Hinder I., Shoemaker D. M., Laguna P., Matzner R. A., 2008, Phys. Rev. Lett., 101, 061102
Watters W. A., Joshi K. J., Rasio F. A., 2000, ApJ, 539, 331
Weinberg N. N., Milosavljević M., Ghez A. M., 2005, ApJ, 622, 878
Wen L., 2003, ApJ, 598, 419
White N. E., van Paradijs J., 1996, ApJ, 473, L25
Willems B., Henninger M., Levin T., Ivanova N., Kalogera V., McGhee K., Timmes F. X., Fryer C. L., 2005, ApJ, 625, 324
Wu X., Xie Y., 2007, Phys. Rev. D, 76, 124004
Yunes N., Sopuerta C. F., Rubbo L. J., Holley-Bockelmann K., 2008, ApJ, 675, 604

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