Closed-Flux Solutions to the
Quantum Constraints for
Plane Gravity Waves

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Abstract

The metric for plane gravitational waves is quantized within the Hamiltonian framework, using a Dirac constraint quantization and the self-dual field variables proposed by Ashtekar. The z axis (direction of travel of the waves) is taken to be the entire real line rather than the torus (manifold coordinatized by (z,t) is RxR rather than S₁ x R). Solutions to the constraints proposed in a previous paper involve open-ended flux lines running along the entire z axis, rather than closed loops of flux; consequently, these solutions are annihilated by the Gauss constraint at interior points of the z axis, but not at the two boundary points. The solutions studied in the present paper are based on closed flux loops and satisfy the Gauss constraint for all z.

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1 Introduction

This is one of a series of papers discussing quantization of plane gravitational waves [1, 2]. The emphasis is on the Ashtekar approach

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a major goal is to understand how radiative phenomena are expressed using the Ashtekar language. The method of quantization is canonical. Due to the high degree of symmetry (full translational symmetry in the x,y directions), four of the seven scalar, vector, and Gauss constraints can be solved and eliminated from the problem before quantization. (The four are: the two vector constraints generating x,y spatial diffeomorphisms; and the two Gauss constraints generating “internal” rotations around the X and Y axes.) The remaining three constraints are imposed after quantization, and the key step is finding a wavefunctional which is annihilated by these constraints. A previous paper (II) wrote down the three surviving constraints in the Ashtekar formalism, quantized them by replacing one half of each (A, \( \tilde{E} \)) canonical pair by a functional derivative, and wrote down a class of wavefunctionals \( \psi \) annihilated by the the constraints [2].

The \( \psi \) constructed in II are unusual. It is usual to have the \( \psi \) depend on the connection fields \( A \), in order to exploit analogies with Yang-Mills theory, whereas these \( \psi \) depend on only one connection field, \( A^Z_z \). The remaining dependence is on densitized inverse triad fields \( \tilde{E} \). Each \( \psi \) is a string of \( \tilde{E} \) operators separated by holonomy matrices depending on \( A^Z_z \). If the holonomies are visualized in the usual way as lines of flux, then these solutions are flux lines extending along the z axis, with \( \tilde{E} \) fields inserted into the line.

These solutions satisfy the Gauss constraint at finite \( z \), but not at infinity. This can also be understood in visual terms. The flux line is an open, rather than closed curve, since the flux line does not loop back to close on itself. Since flux exits at infinity, the Gauss constraint is not satisfied there.

The present paper constructs solutions which are “closed flux” in nature and are annihilated by the Gauss constraint everywhere. Since the present paper is for the most part an extension of II, I have refrained from repeating many matters of philosophy and motivation already discussed in that paper. I have repeated here some of the more detailed results of II, however, so as to make the calculations of the present paper self-contained: see the first half of section 2 and the opening paragraphs of section 3.

The first half of section 2 reviews the quantization procedure of
paper II and writes out the three constraints using Ashtekar variables. The second half of section 2 constructs a point-splitting regularization for one of the terms in the scalar constraint. (In II it was not necessary to regulate this term, because it vanishes identically when acting on the solutions of II.) Section 3 presents the ansatz for $\psi$ and shows that the three constraints annihilate the ansatz. Section 4 presents conclusions and directions for further research.

My notation is typical of papers based upon the Hamiltonian approach with concomitant 3+1 splitup. Upper case indices $A, B, \ldots, I, J, K, \ldots$ denote local Lorentz indices ("internal" SU(2) indices) ranging over $X, Y, Z$ only. Lower case indices $a, b, \ldots, i, j, \ldots$ are also three-dimensional and denote global coordinates on the three-manifold. Occasionally the formula will contain a field with a superscript $(4)$, in which case the local Lorentz indices range over $X, Y, Z, T$ and the global indices are similarly four-dimensional; or a $(2)$, in which case the local indices range over $X, Y$ (and global indices over $x, y$) only. The $(2)$ and $(4)$ are also used in conjunction with determinants; e. g., $g$ is the usual 3x3 spatial determinant, while $(2)\tilde{E}$ denotes the determinant of the 2x2 $X, Y$ subblock of the densitized inverse triad matrix $\tilde{E}_{\alpha}^a$. I use Levi-Civita symbols of various dimensions: $\epsilon_{TXYZ} = \epsilon_{XYZ} = \epsilon_{XY} = +1$. The Minkowski metric convention is $\eta_{TT} = -1$. The basic variables of the Ashtekar approach are an inverse densitized triad $\tilde{E}_{\alpha}^a$ and a complex SU(2) connection $A_{\alpha}^A$:

$$\tilde{E}_{\alpha}^a = \sqrt{g} e_{\alpha}^a;$$  \hspace{1cm}  (1)

$$[\tilde{E}_{\alpha}^a, A_{\beta}^B] = -\hbar \delta(x-x')\delta_{\alpha\beta}^a. \hspace{1cm} (2)$$

e_{\alpha}^A$ is the usual 3x3 triad matrix and $e_{\alpha}^a$ is its inverse. The local Lorentz indices are vector rather than spinor, and strictly speaking the internal symmetry is O(3) rather than SU(2). After the gauge-fixing allowed by the planar symmetry, the internal symmetry is O(2) rather than U(1).

## 2 Quantization

The first part of this section summarizes the main results and formulas on quantization of plane waves from II, for the convenience of
readers who do not have a copy of II at hand. The second part discusses a regularization needed to define unambiguously the action of one term in the scalar constraint.

I follow the quantization procedure used by Husain and Smolin \[6\]. These authors use the planar symmetry to solve and eliminate four constraints (the x and y vector constraint and the X and Y Gauss constraint) and correspondingly eliminate four pairs of \((\tilde{E}_A^a, A_a^A)\) components. The 3x3 \(\tilde{E}_A^a\) matrix then assumes a block diagonal form, with one 1x1 subblock occupied by \(\tilde{E}_Z^Z\) plus one 2x2 subblock which contains all \(\tilde{E}_A^a\) with \(a = x, y\) and \(A = X, Y\). The 3x3 matrix of connections \(A_a^A\) assumes a similar block diagonal form. None of the surviving fields depends on x or y.

After these simplifications, the total Hamiltonian reduces to a linear combination of the three surviving constraints,

\[
H_T = N'[\langle(2)\tilde{E}_C^ab\tilde{E}_C^ab\rangle - 1\tilde{E}_a^a A_b^b/2 + \epsilon_{MN}E^b_{MN} F^N_{ab}]
+ iN\tilde{E}_M^b F^M_{ab} - iN_G \partial_z \tilde{E}_Z^Z - \epsilon_{NQ} A_a^Q A_b^Q + S.T.
\]

\[\equiv N'H_S + N'H_z + N_G H_G + S.T., \tag{3}\]

where

\[
F^N_{ab} = \partial_z A_b^N - \epsilon_{NQ} A_z^Q A_b^Q. \tag{4}\]

\(H_S, H_z,\) and \(H_G\) are the surviving scalar, vector, and Gauss constraints. Strictly speaking these are Hamiltonian densities; for simplicity I have suppressed an integration over the z axis. S.T. denotes “surface” terms (terms evaluated at the two endpoints on the z axis). The detailed form of these terms is worked out in II but will not be needed here.

The theory may be quantized by replacing half of each conjugate momentum-coordinate pair by a functional derivative, in the usual way,

\[
A_a^A \rightarrow \hbar \delta/\delta \tilde{E}_A^a; A = X, Y; a = x, y; \tag{5}\]
\[
\tilde{E}_Z^Z \rightarrow -\hbar \delta/\delta A_z^Z, \tag{6}\]

in order to realize the canonical commutation relations, equation (2).

In II I modified the Husain-Smolin quantization procedure in certain respects. Those authors consider the closed topology (the
z axis is topologically a circle $S_1$) whereas I consider the open one (the $z$ axis is the real line $R$). Persons familiar with open topology calculations in three spatial dimensions might suppose that therefore the constraint equation (3) should be integrated over the entire real axis, $z = -\infty$ to $z = +\infty$. In the present case (effectively one space dimension rather than three, because of the planar symmetry) the space does not become flat at $z$ goes to infinity. Moving the boundary points to spatial infinity does not especially simplify matters, and in fact integrals are better behaved if all integrations are taken from $z_0$ to $z_{2n+1}$ (in the notation to be used in section 3), where the points $z_0$ and $z_{2n+1}$ are a finite distance from the origin. The radiation is assumed to be wave packets confined to the region $z_0 < z < z_{2n+1}$ inside the boundaries, so that the gravitational field is relatively simple (but not flat) at boundaries. Again, the details of the fields at the boundary will not be needed here, but perhaps it is helpful to point out that the result that the space does not become flat as $z$ goes to infinity does agree with one’s intuition from Newtonian gravity, where the potential in one spatial dimension due to a bounded source grows as $z$ at large $z$.

I have modified the Husain-Smolin quantization in another respect. In equation (3) the Lagrange multiplier $N'$ and the scalar constraint $H_S$ are rescaled versions of the usual Lagrange multiplier $\tilde{N}$ and Ashtekar scalar constraint $H$:

$$\tilde{N}H = [(\tilde{N}\tilde{E}_z)][H/\tilde{E}_z]$$

$$\equiv [(N')][H_S]$$  \hspace{1cm} (7)

A rescaling of this type, for a one space-dimensional theory, was first proposed by Teitelboim in the context of geometrodynamics [7]. This innocuous-looking rescaling has profound effects on the closure of the algebra of constraints. As shown in II, when the factors are ordered as in equation (3), with all functional derivatives to the right, then the constraint algebra is consistent, and simultaneously $H_z$ can be interpreted as the generator of diffeomorphisms.

However, since $H_S = H/\tilde{E}_z$, and $H$ is polynomial, the new scalar constraint $H_S$ is rational. I view this complication as a price which must be paid; but it is a small price, considering what one gets in return. The constraints close, which they
should as a matter of principle; and the quantum theory possesses
the diffeomorphism invariance characteristic of the classical theory.

Nevertheless, at some point a price must be paid: how does one
define the inverse operator \((\tilde{E}_Z)^{-1}\)? Fortunately, for the wavefunc-
tionals considered in this paper and in II there is a natural defini-
tion of the inverse. The variable \(A^Z\) conjugate to \(\tilde{E}_Z\) occurs only in
holonomies

\[
M(z_{i+1}, z_i) = \exp[i \int_{z_i}^{z_{i+1}} S_z A^Z(z')dz'].
\] (8)

\(S_z\) is the usual \(2j+1\) dimensional matrix representation of the gener-
ator for SU(2) rotations around the Z axis, and there is an explicit
factor \(i\) because \(S_z\) is Hermitean. From equation (6) the action of \(\tilde{E}_Z\)
on this function is especially simple, since it merely brings down a
factor of \(-iS_z = -im_z\) where \(m_z\) is the eigenvalue of \(S_z\). The natural
definition of the inverse \((\tilde{E}_Z)^{-1}\) is then

\[
(\tilde{E}_Z)^{-1}M = (-im_z)^{-1}M.
\] (9)

This works provided \(m_z\) is never allowed to assume the value zero.

So far I have been merely reviewing matters already discussed
in II. I now consider a new topic, regularization of the constraints. As
with the definition of the \((\tilde{E}_Z)^{-1}\) operator just given, the reg-
ularization will make the constraints finite and unambiguous when
the constraints act on the wavefunctionals considered in this paper;
no claim is made that the proposed regularization will work in all
circumstances. I follow Husain and Smolin [6], who suggest a sim-
ple point splitting regularization. This simple technique works be-
cause the planar case is simpler than the full three-dimensional case.
In the full case, the wavefunctionals typically involve integrals over
loops (Wilson loops) which are one-dimensional, while the functional
derivatives analogous to equation (5) give rise to three-dimensional
delta functions [8, 9, 10]. There are more delta functions than in-
tegrals, which is a recipe for divergences. In the planar case, the
wavefunctionals still involve one-dimensional integrals ( always over
the z axis), but the delta functions from the functional derivatives
are also one-dimensional.
Even when the dimensionality of the integrals matches the dimensionality of the delta functions, it is possible to get a divergence if the wavefunctional contains a product of two or more fields evaluated at the same point. (For example a wavefunctional of the form \( \int \! dz' |\tilde{E}(z')|^2 \) acted upon by a constraint operator containing a product \( [\delta/\delta \tilde{E}(z)]^2 \) gives rise to an undefined \( [\delta(z - z')]^2 \).) But this does not happen in the present case: the wavefunctionals studied here involve at most one field evaluated at each point.

In fact only the first term in equation (3) needs regularization, and the regularization is needed to remove an ambiguity rather than a divergence. This term is the only one containing three, rather than one functional derivative. The wavefunctional is path-ordered, so that one can talk meaningfully about “adjacent” fields. When the first term acts on three adjacent fields, the point splitting is needed to define unambiguously which three fields will be acted upon in which order.

I call the first term \( H_E \) \((E \) for the \( (2)\tilde{E} \) which it contains) and point split it as follows:

\[
H_E = N^{(2)}\tilde{E}(z)\epsilon_{AB}\epsilon_{ab}A^A(z + \epsilon)[\tilde{E}^B_Z(z)]^{-1}A^B_b(z - \epsilon)/2. \tag{10}
\]

One can now see how the point splitting removes an ambiguity. It defines unambiguously which holonomy \( M \) the \( [\tilde{E}^Z_Z]^{-1} \) operator acts on, when the wavefunctional contains a chain of holonomies separated by \( \tilde{E} \) fields.

\[
\int \! dz_1 dz_2 \cdots M(z_{i+1}, z_i)\tilde{E}(z_i)M(z_i, z_{i-1})\tilde{E}(z_{i-1})M \cdots. \tag{11}
\]
The integrals are path ordered,

\[
\cdots z_{i+1} \geq z_i \geq z_{i-1} \cdots \tag{12}
\]

When the two operators \( A = \delta/\delta \tilde{E} \) in \( H_E \) act on the two \( \tilde{E} \) in equation (11), the \( [\tilde{E}^Z_Z]^{-1} \) must act on the holonomy between the \( \tilde{E} \), because of the pattern of epsilons in \( H_E \). Another pattern of epsilons would have forced the \( [\tilde{E}^Z_Z]^{-1} \) to act on the holonomy to the right or to the left of the two \( \tilde{E} \).

I have chosen the pattern in equation (10) because of its symmetry and simplicity. The calculations of the next section are not
unduly sensitive to this choice: one can still find solutions when the $[\tilde{E}_z^{-1}]$ acts on the holonomy to the left or the right, although they differ slightly from the solutions when the $[\tilde{E}_z^{-1}]$ acts in the middle. Note that another symmetric choice yields nothing new: reversing the sign of the epsilons in equation (10) does not change anything, since $H_E$ is even under interchange of the $A$’s. However, I have been unable to find a solution using an average over left, right, and middle holonomies, rather than the middle holonomy alone.

I have verified that the pattern of epsilons given in equation (10) is preserved by the constraint algebra, in the following sense. The $H_E$ term is contained in the scalar constraint $H_S$. Consider a constraint commutator where $H_S$ occurs on both sides of the equation.

$$[H_S, H_z] \propto H_S.$$  (13)

If the $H_E$ on the left is given the pattern of epsilons shown in equation (10) then the $H_E$ on the right will also follow the same pattern. (Also, there will be additional terms of order epsilon on the right, because the point splitting violates diffeomorphism invariance; but these terms disappear in the limit $\epsilon \to 0$.) If there is no $H_S$ on the right,

$$[H_S, H_S] \propto H_z;$$

$$[H_S, H_G] = 0,$$

then the right hand sides will contain additional terms of order epsilon, which again disappear in the limit $\epsilon \to 0$.

3 Solutions

This section proposes a solution, then verifies that it is annihilated by the constraints. In II the constraints were expressed in terms of fields which are eigenstates of the surviving gauge invariance $O(2)$ or $U(1)$ generated by $H_G$. That is, I used basis fields

$$\tilde{E}_a^\pm = [\tilde{E}_X^a \pm i\tilde{E}_Y^a]/\sqrt{2}, a=x,y,$$  (14)

rather than the usual $\tilde{E}_X^a$ and $\tilde{E}_Y^a$; similarly, I used $A_a^\pm$. The solutions of both II and the present paper are much simpler when expressed in
terms of these basis fields. Accordingly, I return to the Hamiltonian, equation (3) and break it up into eigenstates of O(2) by writing out the components of the Levi-Civita tensor,

$$\epsilon_{-+} = -\epsilon_{+-} = i,$$

while being careful to contract every + index with a - index, for example \(\epsilon^{MN}_{AB} A^M B^N = \epsilon_{-+} A^+ B^- + \cdots\).

$$H_T = \frac{1}{2} (N' - N^2) \left[ -\epsilon_{mn} \tilde{E}_m^a \tilde{E}_n^a (\tilde{E}_z^a)^{-1} \epsilon_{cd} A^+_c A^-_d - 2i\tilde{E}_+^b (\partial_z + iA_z^b) A^+_b \right]$$

$$+ \frac{1}{2} (N' + N^2) \left[ -\epsilon_{mn} \tilde{E}_m^a \tilde{E}_n^a (\tilde{E}_z^a)^{-1} \epsilon_{cd} A^+_c A^-_d + 2i\tilde{E}_-^b (\partial_z - iA_z^b) A^-_b \right]$$

$$- iN_G [\partial_z \tilde{E}_z^a - i(\tilde{E}_+^a A^-_a - \tilde{E}_-^a A^+_a)] + S.T. \quad (16)$$

It is perhaps also worth repeating the quantization equations using this O(2) eigenbasis, since the pattern of \(\pm\) signs may look unfamiliar.

$$A^\pm_a = \hbar \delta/\delta \tilde{E}_-^a;$$

$$\tilde{E}_z^a = -\hbar \delta/\delta A_z^a. \quad (17)$$

To understand the pattern of signs, note that the two dimensional Kronecker delta in equation (2) has only off-diagonal elements when expressed in terms of O(2) eigenstates: \(\delta_{\pm\pm} = +1.\)

The ansatz for the solution contains n factors of \(\tilde{E}^a S^+\), followed by an equal number of factors of \(\tilde{E}_-^a S_-\), where S\(\pm\) are SU(2) raising and lowering operators for the \(2j + 1\) dimensional representation of SU(2). The ES factors are separated by the z-axis holonomies M introduced at equation (8).

$$\psi(2n; j) = \prod_{i=1}^{2n} \int_{z_i}^{z_{i+1}} dz_i \theta(z_{i+1} - z_i) \theta(z_1, z_0) \times$$

$$\times M(z_{2n+1}, z_{2n}) \tilde{E}_+^{a_{2n}}(z_{2n}) S_- M(z_{2n}, z_{2n-1}) \cdots \tilde{E}_+^{a_{n+1}}(z_{n+1}) S_- M(z_{n+1}, z_n) \times$$

$$\times \tilde{E}_-^{a_n}(z_n) S_+ M(z_n, z_{n-1}) \tilde{E}_-^{a_{n-1}}(z_{n-1}) S_+ \cdots \tilde{E}_+^{a_1}(z_1) S_+ M(z_1, z_0) \times$$

$$\times M(z_0, z_{2n+1}) \epsilon_{a_{n+1}a_n}. \quad (18)$$

If the holonomy \(M(z_{i+1}, z_i)\) is pictured as a flux line extending from \(z_i\) to \(z_{i+1}\), then \(\psi\) corresponds to an open-ended flux line stretching.
from \( z_0 \) to \( z_{2n+1} \), with \( 2n \) \( \tilde{E} \) operators inserted along the line. The final \( M(z_0, z_{2n+1}) \) holonomy represents a flux line which loops back from \( z_{2n+1} \) to \( z_0 \) and turns the open-ended flux line into a closed loop. The SU(2) structure of equation (18) supports this interpretation. The \( S_\pm \) and \( M \) are \((2j+1) \times (2j+1)\) dimensional matrices with rows and columns labeled by the eigenfunctions \( m_z \) of \( S_z \). The \( m_z \) subscripts on these matrices have been suppressed for simplicity, but clearly, since there are an equal number of raising and lowering operators in the chain, the first and last holonomies in the chain have the same value of \( m_z \). This means that the return segment \( M(z_0, z_{2n+1}) \) has the right \( m_z \) subscripts to match the \( m_z \) of the flux exiting from the beginning and end of the open chain. Note that it is not necessary to sum over the \( m_z \) (it is not necessary to take a trace) because the internal gauge group SU(2) has been fixed to O(2), and the irreducible representations of O(2) are all one-dimensional, labeled by their \( m_z \) value. Later I use identities such as

\[
[S_z, S_\pm] = \pm S_\pm, \tag{19}
\]

but even here there is no need to sum over the intermediate values of \( m_z \) on the left, since only one intermediate value occurs anyway. If I wished, I could replace every \( S_i \) by the corresponding matrix element; e.g. replace \( S_z \) by \( m_z \); but that would be awkward, especially for the \( S_\pm \).

Since \( m_z \) is not summed over, the \( m_z \) value at (say) the start of the chain remains a free parameter for the moment. Later in this section we shall see that the scalar constraint is not satisfied unless the holonomy \( M(z_{n+1}, z_n) \) has \( m_z = -1/2 \). This particular holonomy occurs at the changeover point, where the chain shifts from \( S_+ \) to \( S_- \). Note also the \( a_i \) indices on either side of the changeover point must be antisymmetrized by the final epsilon tensor in equation (18). Again, this is required by the scalar constraint. The free parameters are therefore \( n, j \), and the \( a_i \) away from the changeover point.

Since every \( m_z \) occurs contracted with another \( m_z \), and every \( S_\pm \) is contracted with an \( \tilde{E}_\pm^a \), it should be clear that \( \psi \) is annihilated by the Gauss constraint. To verify this explicitly, note from equation (16) that this constraint is the sum of three terms,

\[
\partial_z \tilde{E}_z^a + i(\tilde{E}_-^a A_+^a - \tilde{E}_+^a A_-^a). \tag{20}
\]
The first term acts only on the holonomies:

$$
\partial_z \tilde{E}_z(z)M(z_{i+1}, z_i) = iS_z[\delta(z_{i+1} - z) - \delta(z_i - z)].
$$
(21)

The remaining two terms act on the $\tilde{E}_a \pm$ fields. These two terms, like the first term, also generate factors of $\pm i\delta(z_i - z)$, but no factors of $S_z$.

Now group together all terms in $H_G \psi$ containing a factor $\delta(z_i - z)$. If $z_i$ is not an endpoint $z_0$ or $z_{2n+1}$, there will be three such terms, one from $\tilde{E}_a \pm$, and two from the holonomies on either side of $\tilde{E}_a \pm$:

$$
i\delta(z_i - z)M(z_{i+1}, z_i)\{\pm S_z + [S_z, S_\mp]\}\tilde{E}_a \pm M(z_i, z_{i-1}) \cdots.
$$
(22)

From the commutation relations obeyed by the $S_i$, the curly bracket vanishes. If $z_i$ is an endpoint $z_0$ or $z_{2n+1}$, there will be no contributions from $\tilde{E}$ fields, only contributions from the three holonomies in $\psi$ which depend on the endpoints. But these holonomies can be combined so that all dependence on the endpoints disappears:

$$M(z_{2n+1}, z_{2n}) \cdots M(z_1, z_0)M(z_0, z_{2n+1}) = M(z_{2n}, z_1).
$$
(23)

Hence there is no contribution from the endpoints either, and the Gauss constraint annihilates $\psi$.

Now consider the diffeomorphism constraint $H_\gamma$. The proof that this annihilates $\psi$ proceeds along the same lines as the corresponding proof for the solutions presented in II, since again $\psi$ is a product of diffeomorphism-invariant integrals of the form $\int dz_i (\text{scalar density } \tilde{E})$. I mention only the one point of the proof which is not straightforward. Since the range of integration is $z_0$ to $z_{2n+1}$, rather than the whole real axis, the points $z_0$ and $z_{2n+1}$ are singled out. This seems to violate diffeomorphism invariance. However, the precise statement of this invariance requires that the wavefunctional be annihilated by the smeared constraint $\int dz \delta N_z H_\gamma dz$, where $\delta N_z$ is an infinitesimal change in the shift vector. Away from boundaries $\delta N_z$ is arbitrary, but at boundaries it must vanish, in order to preserve the boundary conditions $N_z = 0$. This vanishing is enough to guarantee the vanishing of any contributions to $\int dz \delta N_z H_\gamma dz \psi$ from boundary points, and the proof of diffeomorphism invariance is not affected by the finite range of integration.

Only the scalar constraint remains to be considered. From equation (16) this can be written as a sum of three terms.

$$H_S = H_+ + H_- + H_E.
$$
(24)
where
\[ H_\pm = \pm iN\tilde{E}_a^\pm (\partial_z \mp iA_z^\pm)A_a^\pm; \] 
(25)
\[ H_E = N'[\tilde{E}(z)\epsilon_{AB}\epsilon_{ab}A_a^A(z+\epsilon)(\tilde{E}_Z^a(z))^{-1}A_b^B(z-\epsilon)/2. \] 
(26)

It is understood that the \( A_a^A \) and \( \tilde{E}_a^\pm \) operators in these equations are to be replaced by the functional derivatives given at equations (17).

Consider the action of \( H_\pm \) first. This operator acts on the factors of \( \tilde{E}_a^\pm \) in \( \psi \) but does not remove the \( \tilde{E} \); instead, it removes one of the adjacent holonomies.

\[ H_\pm \psi = H_\pm[\cdots \theta(z_{i+1} - z_i)\theta(z_i - z_{i-1})M(z_{i+1}, z_i)\tilde{E}_a^\pm(z_i)S_\mp M(z_i, z_{i-1}) \cdots] \]
(27)
\[ = \sum_i[\cdots \theta(z_{i+1} - z_i)\theta(z_i - z_{i-1})M(z_{i+1}, z_i)(\pm iN\tilde{E}_a^\pm(z)) \times \]
\[ \times (\partial_z \mp iA_z^\pm)\delta(z - z_i)S_\mp M(z_i, z_{i-1}) \cdots]. \]
(28)

Now change the \( \partial_z \) on the delta function to a \( -\partial_z \) and integrate by parts on \( z_i \). The surface terms at \( z = z_0, z_{2n+1} \) again vanish because the smearing function \( N' \) is actually a \( \delta N' \) which vanishes at boundaries. The \( \partial_z \) acts on the \( M \) and the \( \theta \) factors in equation (27).

Since the action on the \( M \)’s brings down factors of \( A_z^\pm \), group these terms with the term in equation (27) which already contains an \( A_z^\pm \):

\[ (\partial_z \mp iA_z^\pm)[M(z_{i+1}, z_i)(\pm i)S_\mp M(z_i, z_{i-1})] \propto iA_z^\pm[-iS_zS_\mp + S_\mp iS_z \mp S_\mp] \]
\[ = 0. \] 
(29)

Thus the \( A_z^\pm \) term in equation (25) and equation (27) cancels terms where derivatives act on factors of \( M \), leaving the terms where derivatives act on the \( \theta \) functions.

\[ H_\pm \psi = \pm iN \sum_i[\cdots \delta(z - z_i)(\theta(z_{i+1} - z_i)\delta(z_i - z_{i-1}) - \delta(z_{i+1} - z_i)\theta(z_i - z_{i-1})) \times \]
\[ \times M(z_{i+1}, z_i)\tilde{E}_a^\pm(z_i)S_\mp M(z_i, z_{i-1}) \cdots]. \] 
(29)

The two delta functions in each term reduce one of the \( M \)’s to unity, in effect removing it completely. As stated previously, the \( H_\pm \) terms do not remove an \( \tilde{E} \), but do remove a neighboring holonomy.

Now consider the \( H_E \) term in the scalar constraint. From equation (26), this removes two \( \tilde{E} \) and replaces them by an \( \tilde{E} \). The two \( \tilde{E} \) which are deleted must be adjacent along the chain; otherwise
there will be two delta functions $\delta(z - z_i)\delta(z - z_j)$ which will squeeze the range of integration of a $dz_k$ to zero, where $z_k$ lies between $z_i$ and $z_j$. Because of the $\epsilon_{AB}$ in equation (27), one of the adjacent $\tilde{E}$ must be an $\tilde{E}^a_+$, and the other must be an $\tilde{E}^a_-$. Therefore $H_E$ acts only at the changeover point, where the chain shifts from $S_+$ to $S_-$.

\[ H_E \psi = \cdots \cdot N'(\tilde{E}(z))\epsilon_{a_{n+1}}\delta(z + \epsilon - z_{n+1})\delta(z - \epsilon - z_n)/2 \times \]
\[ \times (\tilde{E}^a_+(z))^{-1}S_- M(z_{n+1}, z_n)S_+ \cdots \epsilon_{a_{n+1}} \]
\[ = \cdots \cdot N'(\tilde{E}(z))\epsilon_{a_{n+1}}^2\delta(z + \epsilon - z_{n+1})\delta(z - \epsilon - z_n)/2 \times \]
\[ \times S_-(-im_z)^{-1}M(z_{n+1}, z_n)S_+ \cdots \]

I have used equation (3) for the action of $\tilde{E}^a_+(z)^{-1}$ on $M$. Now take the limit $\epsilon \to 0$.

\[ H_E \psi \to \cdots \cdot N'(\tilde{E}(z))(-i)(+2)\delta(z - z_{n+1})\delta(z - z_n)/2 \times \]
\[ \times S_-(-im_z)^{-1}(1)S_+ \cdots \]
\[ = \cdots \cdot N'(\tilde{E}(z))\delta(z - z_{n+1})\delta(z - z_n)]S_- (m_z)^{-1}S_+ \cdots (30) \]

As predicted in the last section, the $z \pm \epsilon$ point splitting is not needed to regulate $(\delta(z - z_n))^2$ divergences; there are none. Rather, it is needed to define unambiguously which $M$ the $\tilde{E}^a_+(z)^{-1}$ is to act on: it acts on the $M$ between the two $\tilde{E}$ grasped by the A’s in $H_E$.

The action of $H_\pm$ and $H_E$ on $\psi$ has now been described; it is time to put the pieces together and describe the action of the entire scalar constraint on $\psi$. Each term in $H_S \psi$ contains a double delta function $\delta(z - z_i+1)\delta(z - z_i)$; cf. equation (24) and equation (30). It is therefore natural to group together terms having the same double delta function. Terms where $z_i+1$ or $z_i$ is a boundary point can be dropped, because $N'(z)$ (or more precisely, $\delta N'(z)$ ) vanishes at boundary points. For the moment ignore the $\delta(z - z_{n+1})\delta(z - z_n)$ terms at the crossover point, since these are the only terms to receive a contribution from the $H_E$ piece of $H_S$. $H_S \psi$ contains only two terms proportional to $\delta(z - z_{i+1})\delta(z - z_i)$, $i \neq n$. One of the two comes from the term shown explicitly in equation (24). This term is the $i$th term in a sum over $2n$ terms. The other $\delta(z - z_{i+1})\delta(z - z_i)$ term comes from the $(i+1)$st term in the same sum. The two terms have opposite sign and so cancel.
Now consider the $\delta(z-z_{n+1})\delta(z-z_n)$ terms, those at the crossover point. Since the crossover point is flanked by both an $\tilde{E}_{a_{n+1}}^+ (at z = z_{n+1})$ and an $\tilde{E}_{a_n}^- (at z = z_n)$, there will be contributions from both $H_+$ and $H_-$. For the $H_+$ contribution, set $i = n+1$ in equation (29); for the $H_-$ contribution, set $i = n$. These two contributions have the same sign and do not cancel.

$$(H_+ + H_-)\psi = \cdots N'\delta(z - z_{n+1})\delta(z - z_n)2i\tilde{E}_{a_{n+1}}^+(z)\tilde{E}_{a_n}^-(z)S_+ \cdots \epsilon_{a_{n+1}a_n}$$

This will be canceled by the $H_E$ contribution, equation (30), provided one chooses

$$m_z = -1/2.$$  \hfill (32)

This must be the value of $m_z$ at the changeover point. At the $z = z_0$ end of the chain, then, $m_z = -1/2 - n$. The $n S_+$ operators raise this to a maximum of $-1/2$ at the changeover point; then the $n S_-$ operators lower this back to $m_z = -1/2 - n$ at the $z = z_{2n+1}$ end of the chain. At no point does $m_z$ pass through the forbidden value of zero. This completes the proof that $\psi$, equation (18), is annihilated by all the constraints.

From this solution one can generate others. E.g., by interchanging plus and minus SU(2) subscripts in equation (18), one generates the "complex conjugate" wavefunctional, which is also a solution with all $m_z$ positive and equation (32) replaced by $m_z = +1/2$. Also, it is possible to relax the requirement that all $S\pm$ and $M$ have the same $j$; see the concluding section of II.

4 Discussion and Directions for Further Work

What is the physical interpretation of the solutions of section 3? In II I argued that, since $H_S$ generates time translations and $H_z$ generates space translations, it is reasonable to assume that the operators $H_S\pm H_z$ represent displacements along the light cone directions $t' \pm z'$ intrinsic to the plane wave metric. (In technical terms the metric considered here possesses two hypersurface orthogonal null vectors; and the corresponding hypersurfaces can be parameterized by $t' \pm z' = \text{constant}$, where the coordinates $(t',z')$ are related to $(t,z)$ by an
appropriate gauge transformation.) As a check, one can show that the commutator $[H_S + H_z, H_S - H_z]$ vanishes, as it should if it is the commutator of two independent translation generators. Note $(H_S \pm H_z)/2$ is the operator multiplying $N' \pm N'$ in equation (10), and this operator is linear in $A_\pm = \hbar \delta / \delta \tilde{E}_\pm$. The solutions in II depend on either $\tilde{E}_-^a$ or $\tilde{E}_+^a$ fields, but not both. This implies that the solutions obey either $(H_S + H_z)\psi = 0$ or $(H_S - H_z)\psi = 0$, which suggests that they depend on either $t' - z'$ or $t' + z'$, but not both. If the solutions of II represent radiation, then that radiation is unidirectional, traveling in a single direction along the $z$ axis. The solutions constructed in the present paper contain both $\tilde{E}_-^a$ and $\tilde{E}_+^a$ fields. By the same argument, if the solutions of this paper represent radiation, it is not unidirectional. These solutions correspond to scattering.

The argument just outlined is very limited. It does reveal the direction of motion of the field. By itself, however, it cannot establish that the field is radiative, much less give details of the polarization and amplitude of the radiation. One can ask to what extent it is possible to go farther and construct operators which explore these details. In the 1970’s, a great deal of work was done on characterization of *classical* radiative plane wave solutions, and most of this work is readily translated into the Ashtekar language. However, it is one thing to characterize classical radiation, and another to translate a classical criterion into a quantum criterion.

It is not hard to see what issues can arise. For simplicity, consider the corresponding problem in electrodynamics: given a radiation criterion valid for classical Maxwell electrodynamics, translate it into the corresponding criterion for QED. Consider, for example, the classical criterion that a plane wave traveling in the $z$ direction has only a single polarization:

$$(E_x + iB_y)_{cl} = 0.$$  \hfill (33)

The corresponding quantum criterion is not

$$(E_x + iB_y)\psi = 0,$$  \hfill (34)

where $\psi$ is now a state in the Hilbert space of the operators $E$ and $B$. To obtain the quantum analog of the classical equation equa-
tion (33), one must pass to large occupation numbers and use coherent states:
\[ < \text{coh} | (E_x + iB_y) | \text{coh} > = 0. \] (35)

Equation (34) is far too strong: the vacuum state with respect to a given polarization need not be annihilated by the corresponding operator. Equation (35), however, is too special, since it works for coherent states but not for eigenstates of the number operator.

If one has the full machinery of QED, of course the correct procedure is to break up E and B into creation and annihilation operators and demand that \( \hat{a} \psi = 0 \), \( \hat{a} \) an annihilation operator for the relevant polarization. In a generally covariant theory, creation and annihilation operators are not automatically available, since normally a preferred time coordinate is required for their definition, e.g. the time associated with the timelike Killing vector in a static spacetime. Here one has the preferred coordinates \( t' \pm z' \), but in these coordinates the metric is conformally flat rather than static. On the other hand, unidirectional plane waves are known to obey a superposition principle [11, 12], which suggests that the unidirectional case may possess particle-like modes. Even if no quantum criteria are forthcoming, the classical criteria by themselves are quite useful and illuminating, and I plan a paper which will (at a minimum) translate the classical criteria for plane radiation into the Ashtekar language.

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