On The Multichannel Kondo Model

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A detailed and comprehensive study of the one-impurity multichannel Kondo model is presented. In the limit of a large number of conduction electron channels $k \gg 1$, the low energy fixed point is accessible to a renormalization group improved perturbative expansion in $1/k$. This straightforward approach enables us to examine the scaling, thermodynamics and dynamical response functions in great detail and make clear the following features: i) the criticality of the fixed point; ii) the universal non-integer degeneracy; iii) that the compensating spin cloud has the spatial extent of the order of one lattice spacing.

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I. INTRODUCTION

Recently, the non-Fermi liquid infrared fixed point of the multichannel Kondo model, which describes a system of \( k \) identical conduction bands interacting with an impurity spin \( S \), has received considerable attention \([?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?\]. The interest is aroused by its potential application in the metallic glasses \([?,?,?,?\], and heavy fermion Uranium alloys \([?,?,?]\) where more examples have been found with their low temperature behavior falling out of the usual Fermi liquid expectation \([?,?]\). It is enhanced by the more interesting observation of non-Fermi liquid behavior in the normal state of the high Tc cuprates \([?,?]\) and its resemblance to the low energy behavior of the multichannel Kondo model \([?,?]\).

Although this model is more than ten years old \([?\] and has been attacked by various methods \([?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?]\), there is still room left for a simple interpretation(or reinterpretation) of the physics of the low energy fixed point which is often said to be nontrivial. This task is easily achieved in the limit \( k \gg 1 \) where the low energy fixed point has a value of order \( 1/k \) for the coupling constant and is accessible to a renormalization group(RG) improved perturbative expansion in \( 1/k \). Some physical quantities difficult to calculate by the other methods are readily obtained by this perturbative approach and the results provide new insight into the fixed point. That the nature of the fixed point remains qualitatively the same when continuing \( k \) to small values as long as \( k > 2S \) has been demonstrated by the results of Bethe Ansatz or conformal field methods \([?,?,?]\). In this paper, we calculate a long list of physical quantities to the leading or sub-leading order in \( 1/k \). A simple physical picture can be sketched based on these results and the previous understanding.

For an antiferromagnetic Kondo interaction between the the impurity spin and the conduction electrons, the impurity spin pulls in conduction electrons with opposite spin. This increases the spin-flip exchange which in turn enhances the attraction of the conduction electrons with opposite spin to the impurity spin. The result of this cooperative enhancement of the Kondo interaction is that one conduction electron from each of the \( k \) channels is pulled...
in to screen the impurity spin. Since $k > 2S$, the conduction electrons overscreen the impurity spin. Additional conduction electrons must come in to remedy the over-performance. The process keeps going on resulting in a critical system. It is important to realize that the system is only critical along the time axis. As short time details are averaged out, the system approaches a universal limiting behavior given by the infrared fixed point. The impurity spin is asymptotically screened in the ground state which is a spin singlet. However, the infrared fixed point can be reached only in an infinite system in which the energy levels form a continuum. In finite systems, the typical distance between discrete energy levels, of order $1/L$ for a system of linear dimension $L$, cuts off the scaling toward the fixed point. The ground state of a finite system has a residual spin and is doubly degenerated. A common definition of the residual spin is $\sqrt{T \chi(T)}$ as the temperature $T \to 0$, where $\chi(T)$ is the magnetic susceptibility. As we shall see, it is nonzero for a finite system. We shall also show that the entropy as given by the coefficient of the linear $T$ term in the free energy is indeed $\ln(2S + 1)$ for a finite system.

An intriguing feature of the multichannel Kondo model is that the entropy of an infinite system reduces to a smaller universal value although the ground state is a spin singlet. If the entropy is still defined as the logarithm of the ground state degeneracy, the ground state could have a universal non-integer degeneracy. Since the entropy is deduced by calculating the free energy at small but finite temperature $T$ which cuts off the scaling toward the fixed point, it is natural to relate the entropy to the effective residual spin at $T$. A necessary implication of this relation is that a tiny magnetic field will lift the degeneracy at $T \to 0$ as we shall see later.

Another interesting issue pertaining to all kind of Kondo problems, either overscreened or exactly screened, is the spatial size of the conduction electron screening cloud. This is a point underlying the resonant level nature of the Kondo problems. Considering the one-channel Kondo problem, it is well known that below the Kondo temperature $T_K$ the effective Kondo interaction enters the strong coupling regime. The energy gain from screening the impurity spin is of order $T_K$. If the screening cloud formed a localized bound state with the impurity
spin, it would have a spatial spreading of $v_F/T_K \gg 1/k_F$ [?]. NMR experiment ruled out any conduction electron screening cloud bigger than one lattice spacing [?]. Although the physical argument for the screening cloud to have a size of $1/k_F$ has been given [?,?], here we calculate the Knight shift for the multichannel Kondo model and explicitly show that the only length scale is $1/k_F$. We also explain why the same conclusion can be extended to the exactly screened case (k=1).

The paper is organized as follows. In section II, the multichannel Hamiltonian and the Popov method are briefly recaptured. In section III, the general integral expression for the conduction electron self-energy to the order $O(k^{-4})$ is derived which can serve as a future reference. In section IV, the integrals in the self-energy are evaluated at $T = 0$ and the results are used to derive the RG equation and running coupling constant. In section V, the scaling solutions for the conduction electron scattering rate and resistivity are obtained. The free energy is calculated in section VI from which the specific heat and entropy are deduced. The magnetic susceptibility and field dependent magnetization are calculated in section VII for an equal spin gyromagnetic ratio for the conduction electron and impurity spin. The dynamical susceptibility for the impurity spin is calculated in section VIII and the relaxation rate is deduced. The general case with different gyromagnetic ratios is considered in section IX. The Knight shift in the space surrounding the impurity spin is calculated in section X. The last section is devoted to a discussion of related issues. Part of the results has been briefly reported in [?]. Some details for the dynamical spin correlation functions are included in the appendix B.

II. POPOV TECHNIQUE

Without losing generality, we consider the local impurity to be a spin $S = 1/2$. The multichannel Kondo Hamiltonian in the magnetic field is

$$H = \sum_{\vec{k}, \mu, \lambda} (\epsilon_{\vec{k}} + \mu \hbar) c_{\vec{k}\mu\lambda}^+ c_{\vec{k}\mu\lambda} + 2 \mu S \vec{\hbar} \cdot \vec{S} + \frac{J}{N} \sum_{\vec{k}, \vec{k}'} \sum_{\mu, \nu, \lambda} \sum_{\lambda=1..k} c_{\vec{k}\mu\lambda}^+ \sigma_{\mu\nu} \frac{1}{2} c_{\vec{k}'\nu\lambda} \cdot \vec{S}, \quad (1)$$
where $\vec{S}$ is a spin-1/2 operator, $\vec{\sigma}_{\mu\nu}$ are the Pauli matrices, $J$ is the Kondo interaction strength and $N$ is the number of lattice sites. We have set the conduction electron gyromagnetic ratio and Bohr magneton equal to one so that the magnetic field $h$ has the dimension of energy. We have introduced a parameter $\mu_S$ to account for possible difference between the conduction electron and impurity spin: $\mu_S$ is equal to the gyromagnetic ratio of the impurity spin divided by that of the conduction electron. We shall adopt the usual cutoff scheme for the conduction electron band: $-D < \epsilon_{\vec{k}} < D$, with a constant density of states $\rho$ per spin per channel.

Representing the impurity spin in terms of pseudofermions

$$\vec{S} = \frac{1}{2} \sum_{\mu,\nu=\pm} f^\dagger_\mu \vec{\sigma}_{\mu\nu} f_\nu,$$

Popov made an observation that the Kondo Hamiltonian without imposing constraint on the pseudofermions is disconnected in the pseudofermion charge sectors. That is

$$Z = \text{Tr} e^{-\beta H} = Z_0 + Z_1 + Z_2 = \text{Tr} \delta(\hat{n}_f)e^{-\beta H} + \text{Tr} \delta(\hat{n}_f - 1)e^{-\beta H} + \text{Tr} \delta(\hat{n}_f - 2)e^{-\beta H}.$$

Among these three separate contributions, only $Z_1$ from the subspace

$$\hat{n}_f = \sum_{\sigma=\pm} f^\dagger_\sigma f_\sigma = 1$$

is physical. Popov’s technique \cite{popov} is to add an imaginary chemical potential, $i\omega_0 = i\pi/(2\beta)$, to the pseudofermions so that $Z_0 + Z_2 = 0$. Using this trick, the partition function in the path integral formalism can be represented as

$$Z = \int D[\bar{c}, c, \bar{f}, f] \exp \left[ -\int_0^\beta d\tau (L_0 + H) \right], \quad \text{(2)}$$

$$L_0 = \sum_{\vec{k},\mu,\lambda} \bar{c}_{\vec{k}\mu\lambda} \partial_\tau c_{\vec{k}\mu\lambda} + \sum_\mu \bar{f}_\mu(\partial_\tau + i\omega_0)f_\mu. \quad \text{(3)}$$

There is no additional constraint. The standard perturbation method then follows from this path integral representation. The impurities are assumed to be randomly distributed in space. The averaging over the impurity distribution is done according to the standard recipe as in the case of the spinless impurities \cite{spinless,spinless2} and we only keep contributions linear in the impurity density $n_i$. The Feynman rules for constructing diagrams are listed in the appendix A.
III. CONDUCTION ELECTRON SELF-ENERGY

The first thing we shall calculate is the conduction electron self-energy in the absence of the external magnetic field. Up to the order \(J^4, J^5k\) and \(J^6k^2\), the relevant diagrams are given in Figure [I]. The two subscripts in each term of the self-energy expansion indicate the powers of \(J\) and \(k\) respectively. After lengthy algebra, the final results at finite temperature are

\[
\Sigma(i\omega_n, T) = \Sigma(2,0) + \Sigma(3,0) + \Sigma(4,1) + \Sigma(4,0) + \Sigma(5,1) + \Sigma(6,2),
\]

\[
\Sigma(2,0)(i\omega_n) = -\frac{3n_i}{16} J^2 \rho \int \frac{d\epsilon}{i\omega_n - \epsilon},
\]

\[
\Sigma(3,0)(i\omega_n) = \frac{3n_i}{16} J^3 \rho^2 \int d\epsilon_1 d\epsilon_2 \tanh\left(\frac{\beta\epsilon_1}{2}\right) \frac{1}{(i\omega_n - \epsilon)(\epsilon - \epsilon_1)}
\]

\[
\Sigma(4,1)(i\omega_n) = \frac{3n_i}{16} J^4 \rho^3 k \int d\epsilon_1 d\epsilon_2 d\epsilon_3 d\epsilon_4 \frac{\varphi_{\epsilon_1\epsilon_2\epsilon_3\epsilon_4}}{(\epsilon - \epsilon_1)(\epsilon_2 - \epsilon_1)},
\]

\[
\Sigma(4,0)(i\omega_n) = \frac{9n_i}{32} J^4 \rho^3 \int d\epsilon_1 d\epsilon_2 \frac{3}{(i\omega_n - \epsilon)(\epsilon - \epsilon_2)} \left[\frac{\varphi_{\epsilon_1}}{\varphi_{\epsilon_2}} - \frac{3}{8}\right],
\]

\[
\Sigma(5,1)(i\omega_n) = -\frac{3n_i}{32} J^5 \rho^4 k \int \frac{d\epsilon_1 d\epsilon_2 d\epsilon_3 d\epsilon_4 d\epsilon_5}{(\epsilon_1 - \epsilon_2)(\epsilon_4 - \epsilon_3)(\epsilon_1 - \epsilon_2 + \epsilon_4 - \epsilon_3)(i\omega_n - \epsilon_4)}
\]

\[
- \frac{3n_i}{32} J^5 \rho^4 k \int \frac{d\epsilon_1 d\epsilon_2 d\epsilon_3 d\epsilon_4}{(\epsilon_1 - \epsilon_2)(\epsilon_4 - \epsilon_3)(\epsilon_1 - \epsilon_2 + \epsilon_4 - \epsilon_3)(i\omega_n - \epsilon_4 + \epsilon_2 - \epsilon_1)}
\]

\[
+ \frac{3n_i}{16} J^5 \rho^4 k \int \frac{d\epsilon_1 d\epsilon_2 d\epsilon_3 d\epsilon_4}{(\epsilon_1 - \epsilon_2)(i\omega_n - \epsilon_4)(\epsilon_1 - \epsilon_2 + \epsilon_4 - \epsilon_1)} \frac{1}{\epsilon_4 - \epsilon_3} + \frac{1}{\epsilon_2 - \epsilon_3}
\]

\[
- \frac{21n_i}{128} J^5 \rho^4 k \int \frac{d\epsilon_1 d\epsilon_2 d\epsilon_3 d\epsilon_4}{(\epsilon_1 - \epsilon_2)(\epsilon_1 - \epsilon_3)(\epsilon_2 - \epsilon_3)(i\omega_n - \epsilon_4 + \epsilon_2 - \epsilon_1)},
\]

\[
\Sigma(6,2)(i\omega_n) = -\frac{3n_i}{32} J^6 \rho^5 k^2 \int \frac{d\epsilon_1 d\epsilon_2 d\epsilon_3 d\epsilon_4 d\epsilon_5}{(i\omega_n - \epsilon_5)(\epsilon_1 - \epsilon_2)(\epsilon_3 - \epsilon_4)}
\]

\[
\times \left[\frac{\varphi_{\epsilon_1\epsilon_2\epsilon_3\epsilon_4\epsilon_5}}{(i\omega_n - \epsilon_5 + \epsilon_3 - \epsilon_4)(\epsilon_1 - \epsilon_2 + \epsilon_3 - \epsilon_4)} + \frac{\varphi_{\epsilon_1\epsilon_2\epsilon_3\epsilon_4\epsilon_5}}{i\omega_n - \epsilon_5 + \epsilon_1 - \epsilon_2 + \epsilon_3 - \epsilon_4}
\]

\[
\times \frac{2}{i\omega_n - \epsilon_5 + \epsilon_3 - \epsilon_4 + \epsilon_1 - \epsilon_2 + \epsilon_3 - \epsilon_4}\right],
\]

where the integration range is always \([-D, D]\). Whenever necessary, the above integrals (and integrals throughout this paper) take principal value. We have introduced following shorthand notations,
\[ \varphi_{\varepsilon_1} = f(\varepsilon_1) f(-\varepsilon_2) + f(-\varepsilon_1) f(\varepsilon_2), \]
\[ \varphi_{\varepsilon_4 \varepsilon_5} = f(\varepsilon_3) f(-\varepsilon_4) f(-\varepsilon_5) + f(-\varepsilon_3) f(\varepsilon_4) f(\varepsilon_5), \]
\[ \xi_{\varepsilon_2 \varepsilon_3} = f(\varepsilon_1) f(-\varepsilon_2) f(-\varepsilon_3) - f(-\varepsilon_1) f(\varepsilon_2) f(\varepsilon_3), \]
\[ \varphi_{\varepsilon_2 \varepsilon_4 \varepsilon_5} = f(\varepsilon_1) f(-\varepsilon_2) f(\varepsilon_3) f(-\varepsilon_4) f(-\varepsilon_5) + f(-\varepsilon_1) f(\varepsilon_2) f(-\varepsilon_3) f(\varepsilon_4) f(\varepsilon_5), \]
where \( f(\varepsilon) = 1/(e^{\beta\varepsilon} + 1) \), is the Fermi-Dirac function.

**IV. RENORMALIZATION GROUP EQUATION AND SOLUTION**

Under the analytic continuation \( i\omega_n \to \omega + i0^+ \), \( \Sigma(\omega + i0^+) = \Sigma'(\omega) + i\Sigma''(\omega) \). The imaginary part of the self-energy \( \Sigma''(\omega) \) is proportional to the total scattering rate of the conduction electron of energy \( \omega \). At \( T = 0 \), the integrations in the imaginary part of the self-energy (8)-(10) can be carried out and we obtain

\[
\Sigma''(\omega, D, g) = \frac{3\pi n_i}{16\rho} g^2 \left( P_0 + P_1 g \ln \bar{\omega} + P_2 g^2 k \ln \bar{\omega} + P_3 g^2 k + P_4 g^2 \ln^2 \bar{\omega} + P_5 g^3 k \ln^2 \bar{\omega} + P_6 g^3 k \ln \bar{\omega} + P_7 g^3 k + P_8 g^4 k^2 \ln^2 \bar{\omega} + P_9 g^4 k^2 \ln \bar{\omega} + P_{10} g^4 k^2 \right),
\]

where \( g = J\rho \) and \( \bar{\omega} = \omega/D \). All coefficients \( P_0 \) to \( P_{10} \) are known: \( P_0 = 1, P_1 = -2, P_2 = 1, P_3 = \ln 2 - 1, P_4 = 3, P_5 = -7/2, P_6 = 5 - 3 \ln 2, P_8 = 1, P_9 = 2 \ln 2 - 5/2 \). The coefficients \( P_7 \) and \( P_{10} \) are not needed for deriving the RG equation to the sub-leading order. But they can be found from (9) and (10). Note that the \( g^4 \) order contributions from (8) contain only a \( \ln^2(\bar{\omega}) \) term. Since we expect the fixed point \( g^* \) to be of order \( 1/k \) [?], our result (11) includes all contributions up to the order \( O(k^{-4}) \).

Since the dimensionless scattering rate (11) must be invariant under RG transformation, we obtain the following equation,

\[
\left( \frac{\partial}{\partial \ln D} + \beta(g) \frac{\partial}{\partial g} \right) \rho \Sigma''(\omega, D, g) = 0.
\]

The beta-function, \( \beta(g) \), of the Kondo interaction \( g \) can be deduced from (11) using the standard RG technique [?]. The equation (12) can be regarded as an equation which generates logarithmic series with fixed \( g \).
\[ \Sigma''(\omega, D, g) = -\beta(g) \frac{\partial}{\partial g} \int d(\ln D) \Sigma''(\omega, D, g) + \text{const.} \]  

(13)

Substituting (11) into (13), the integration over \( \ln D \) can be carried out easily. Expressing \( \beta(g) \) to the sub-leading order as

\[ \beta(g) = g^2(\kappa_1 + \kappa_2 g + \kappa_3 g^2 + \kappa_4 g^3 + \kappa_5 g^4) \]  

(14)

and substituting it into (13), we obtain following equations for the coefficients \( \kappa_1 \) to \( \kappa_5 \) from (13) by equating order by order the coefficients of the polynomials on the left and right sides,

\[ \kappa_1 = \frac{P_1}{2P_0} = -1, \]  

(15)

\[ \kappa_2 = \frac{P_2}{2P_0} = \frac{1}{2}, \]  

(16)

\[ \kappa_3 = 0, \]  

(17)

\[ \kappa_4 = \frac{P_6 - 4P_3 \kappa_1}{2P_0} = \frac{1 + \ln 2}{2}, \]  

(18)

\[ \kappa_5 = \frac{P_9 - 4P_3 \kappa_2}{2P_0} = \frac{1}{4}. \]  

(19)

There are three consistency equations

\[ P_4 = \frac{3}{2} P_1 \kappa_1, \]

\[ P_5 = 2P_2 \kappa_1 + \frac{3}{2} P_1 \kappa_2, \]

\[ P_8 = 2P_2 \kappa_2, \]

which are all satisfied.

The obtained beta-function can be written explicitly as

\[ \beta(g) = -g^2 + \frac{k}{2} g^3 + \frac{k}{2}(1 + \ln 2) g^4 - \frac{k^2}{4} g^5. \]  

(20)

The intermediate fixed point \( g^* \), determined by \( \beta(g^*) = 0 \), and the slope of the \( \beta(g) \) at the fixed point are then easily found to be

\[ g^* = \frac{2}{k} \left(1 - \frac{2 \ln 2}{k}\right), \]

(21)

\[ \Delta = \beta'(g^*) = \frac{2}{k} \left(1 - \frac{2}{k}\right). \]  

(22)
We find indeed \( g^* \sim 1/k \) thus allowing a reliable expansion in the \( k \to \infty \) limit. The slope which determines the critical exponent is universal. The perturbative result \(^{(22)}\) agrees with the conformal field result \( 2/(k+2) \) \(^{(2)}\) up to the sub-leading order in the \( 1/k \) expansion.

The running coupling constant \( g_R(\omega) \) is determined by the differential equation

\[
\frac{dg_R}{d\ln \omega} = \beta(g_R),
\]

(23)

with the initial condition \( g_R(\omega = D) = g \). With our perturbative \( \beta(g) \) of \(^{(20)}\), the solution of \(^{(23)}\) covering the full range of energy scale from \( \omega \ll T_K \) to \( \omega \gg T_K \) can be obtained. We rewrite \(^{(23)}\) in the integral form,

\[
\int_D^\omega d\ln \omega' = \int_g^{g_R(\omega)} \frac{dg'}{\beta(g')} \simeq g^* \int_g^{g_R(\omega)} \frac{dg'}{(g')^2(g' - g^*)} \left[ 1 + g' \ln 2 - \frac{k}{2}(g')^2 \right]^{-1}.
\]

This leads to the full solution

\[
|g_R(\omega) - g^*| = |g - g^*| \left( \frac{\omega}{T_K} \right)^\Delta [g_R(\omega)]^{k\Delta/2} e^{-\Delta/g_R(\omega)},
\]

(24)

where \( T_K = D g^{k/2} \exp(-1/g) \). At \( \omega < T_K \), it has an asymptotic form of,

\[
g_R(\omega) = g^* - \zeta \left( \frac{\omega}{T_K} \right)^\Delta, \quad \zeta = (g^* - g) (g^*)^{k\Delta/2} e^{-\Delta/g^*}.
\]

(25)

For an initial condition of weak coupling, \( g \to 0 \) and \( D \to \infty \), \( \zeta = (g^*)^{1+k\Delta/2} e^{-\Delta/g^*} \). From \(^{(24)}\), it is interesting to note that the running coupling constant has a power law behavior not only at low energy \( \omega < T_K \) but also at high energy \( \omega > T_K \), underlying the critical nature of the system.

V. SCATTERING RATE AND RESISTIVITY

To obtain the scaling solution for the conduction electron scattering rate \( \rho \Sigma'' \), we can use the RG invariance,

\[
\rho \Sigma''(\omega, D, g_R(D) = g) = \rho \Sigma''(\omega, D', g_R(D')).
\]

(26)

Choosing \( D' = \omega \), the logarithmic terms in \( \rho \Sigma''(\omega, g_R(\omega)) \) of \(^{(11)}\) drop out and the scaling form for \(^{(11)}\) is
\[ \rho \Sigma''(\omega, D, g) = \frac{3\pi n_i}{16} \left[ g_R^2(\omega) - (1 - \ln 2)kg_R^4(\omega) \right] \simeq \frac{3\pi n_i}{4(k + 2)^2} \left[ 1 - A_0 \left( \frac{\omega}{T_K} \right)^\Delta \right], \quad (27) \]

where \( A_0 = k\zeta[1 - (4 - 6 \ln 2)/k] \). A similar RG procedure will be performed repeatedly later. Note that instead of a Lorentzian frequency dependence in the exactly screened Kondo problem, \( \rho \Sigma'' \sim T_K^2/(\omega^2 + T_K^2) \), the scattering rate \( (27) \) has a cusp at \( \omega = 0 \). At nonzero temperature, the cusp is expected to be smoothed out.

In order to calculate the resistivity, we need the temperature dependent imaginary part of the conduction electron self-energy \( \Sigma''(\omega, T) \). Identifying the transport relaxation time due to the Kondo exchange as \( \tau_{ex}(\omega, T) = 1/[2\Sigma''(\omega, T)] \) since there is only \( s \)-wave scattering \([?]\), the total scattering rate is \( 1/\tau(\omega, T) = 1/\tau_0 + 1/\tau_{ex}(\omega, T) \), where \( \tau_0 \) is the ordinary relaxation time in the absence of the Kondo interaction. The ordinary scattering \( 1/\tau_0 \) could arise from spinless impurities or defects which are assumed to be located at different lattice sites and uncorrelated to the impurity spins. A low impurity spin density is assumed, \( n_i \ll 1 \), such that \( \tau_0 \ll \tau_{ex} \).

The total relaxation time is substituted into the conductivity expression \([?]\),

\[ \sigma(T) = \frac{n_e e^2}{m_e} \int_0^\infty d\omega \frac{\tau(\omega)}{2T \cosh^2 \left( \frac{\omega}{2T} \right)} \simeq \frac{n_e e^2}{m_e} \int_0^\infty d\omega \frac{\tau_0}{2T \cosh^2 \left( \frac{\omega}{2T} \right)} \left( 1 - \frac{\tau_0}{\tau_{ex}(\omega)} \right), \quad (28) \]

with \( e, m_e \) and \( n_e \) denoting the conduction electron charge, mass and density respectively. With our perturbative expression for \( \Sigma''(\omega, T) \) to the order \( k^{-3} \), we find for the resistivity due to Kondo scattering,

\[ \delta \rho_e(T) = \frac{3\pi}{16} \frac{m_e}{n_e e^2 \rho} n_i g^2 \int_0^\infty \frac{dx}{\cosh^2 \frac{x}{T}} \left[ 1 - g \int_{-D/T}^{D/T} dy \tan \frac{y}{2} - g^2 k \Psi \left( \frac{x}{T}, \frac{D}{T} \right) \right] \]

\[ = \frac{3\pi}{8} \frac{m_e}{n_e e^2 \rho} n_i g^2 \left[ 1 + O(g, g^2 k) + O(g, g^2 k) \times \ln \frac{D}{T} \right], \quad (29) \]

where

\[ \Psi \left( \frac{x}{T}, \frac{D}{T} \right) = \int_{-D/T}^{D/T} dy \int_{-D/T}^{D/T} dy_1 \frac{1}{\cosh \frac{x}{T}} \frac{1}{e^{y+1} e^{y_1+1} - e^{y+y_1+1}} \left( 1 - \frac{1}{e^{y-y-x} + 1} - \frac{1}{e^{y-y+x} + 1} \right). \quad (30) \]

Some difficult integrals have to be evaluated to obtain the sub-leading terms in \( (29) \). So we limit ourselves to the leading order. Performing the same RG procedure as \( (27) \), we obtain

\[ \delta \rho_e = \frac{3\pi^2 n_i}{4k^2 n_e \rho e^{(0)}} \left[ 1 - k\zeta \left( \frac{T}{T_K} \right)^\Delta \right], \quad (31) \]
where \( \rho_e^{(0)} = 4\pi/(e^2k_F) \), is the resistivity in the unitary limit and \( k_F \) denotes the Fermi wave vector which is related to the conduction electron density of states through \( \rho = k_F m_e/(2\pi^2) \). The \( T = 0 \) value of the resistivity and the exponent \( \Delta \) have been reported [?] and an exact expression for the resistivity for \( k = 2 \) up to a constant factor \( \zeta \) has been derived recently [?]. As in the exactly screened Kondo problem, the corresponding resistivity decreases upon increasing temperature.

An important point is that even at \( T = 0 \), there is still inelastic scattering. In other words, the impurity spin together with its asymptotical screening cloud (see magnetization section) cannot be viewed as inert. Let’s see how the assumption of elastic scattering only at \( T = 0 \) would run into trouble. Following [?], the total resistivity can be divided into elastic and inelastic contributions,

\[
\delta \rho_e = \delta \rho_e^{\text{el}} + \cos(2\phi_0) \delta \rho_e^{\text{in}},
\]

where \( \phi_0 \) is the scattering phase shift at the Fermi energy. This separation is valid at least for weak inelastic scattering which would indeed be the case at low \( T \) if only elastic scattering were present at \( T = 0 \). For elastic scattering, we could identify

\[
\rho \Sigma''(\omega = 0, T = 0) \sim n_i \frac{S(S+1)}{\pi} \sin^2 \phi_0,
\]

which would lead to

\[
\phi_0 \sim \frac{\pi}{k}. \quad \text{(33)}
\]

The resistivity due to the elastic scattering can be calculated by using (27) as the scattering rate. To the leading order,

\[
\delta \rho_e^{\text{el}}(T) = \int_0^{\infty} \frac{d\omega}{2T \cosh^2 \frac{\omega}{2T}} \frac{3\pi^2 n_i}{4k^2 n_e} \rho_e^{(0)} \left[ 1 - A_0 \left( \frac{\omega}{T_K} \right)^\Delta \right]. \quad \text{(34)}
\]

Substituting (31), (33) and (34) into (32), we would obtain the inelastic contribution to the resistivity \( \delta \rho_e^{\text{in}} < 0 \) which certainly is impossible. This contradiction indicates that there are both elastic and inelastic scattering at \( T = 0 \). For the two channel case, \( k = 2 \), it has been suggested that there is actually only inelastic scattering [?].
VI. SPECIFIC HEAT AND ENTROPY

To calculate the free energy, we can use the linked cluster theorem,

\[ F = F_0 - \sum_{n=2}^{\infty} U_n, \tag{35} \]

where \( F_0 \) is the free energy of the non-interacting Fermi sea. The diagrams for the first three \( U_n \) are drawn in Figure 2. The results after completing the Matsubara frequency summations are

\[ U_2 = -\frac{3n_i}{16} g^2 k \int d\epsilon d\epsilon_1 \frac{f(\epsilon) - f(\epsilon_1)}{\epsilon - \epsilon_1} \tag{36} \]

\[ U_3 = \frac{n_i}{8} g^3 k \int d\epsilon_1 d\epsilon_2 d\epsilon_3 \left[ \frac{f(\epsilon_1) f(\epsilon_2) f(-\epsilon_3)}{(\epsilon_3 - \epsilon_1)(\epsilon_3 - \epsilon_2)} + 2 \text{ Permutations} \right] \tag{37} \]

\[ U_4 = \frac{3n_i}{64} g^4 k^2 \int d\epsilon_1 d\epsilon_2 d\epsilon_3 d\epsilon_4 f(-\epsilon_1) f(\epsilon_2) f(\epsilon_3) f(-\epsilon_4) \times \left[ \frac{1}{(\epsilon_1 - \epsilon_2)(\epsilon_3 - \epsilon_4)} \left( \frac{1 - e^{\beta(\epsilon_2 - \epsilon_1 + \epsilon_3 - \epsilon_4)}}{\epsilon_1 - \epsilon_2 - \epsilon_3 + \epsilon_4} + \frac{e^{\beta(\epsilon_2 - \epsilon_1) - e^{\beta(\epsilon_3 - \epsilon_4)}}}{\epsilon_1 - \epsilon_2 + \epsilon_3 - \epsilon_4} \right) \right]. \tag{38} \]

As noted by Kondo [?], it is quite delicate to extract the linear \( T \) terms in (37) and (38). Here we follow the method used by Kondo and present the calculations only for the integrals not calculated in [?].

The calculation of \( U_2 \) is simple and straightforward. The result is a constant plus \( T^2 \) corrections. To calculate \( U_3 \), we define a new integral \( I_3 \),

\[ I_3 = \int_{-D}^{D} d\epsilon_1 d\epsilon_2 d\epsilon_3 \frac{f(\epsilon_1) f(\epsilon_2) f(-\epsilon_3)}{(\epsilon_3 - \epsilon_1)(\epsilon_3 - \epsilon_2)}, \]

where we have introduced the notation

\[ \frac{1}{(x)_\delta} = \frac{x}{x^2 + \delta^2}, \quad \delta \to 0. \]

The \( \delta \to 0 \) limit is taken after the integrations are completed. Kondo showed [?],

\[ U_3 = \frac{3n_i}{8} g^3 k \left( I_3 - \frac{\pi^2}{6} T \right), \tag{39} \]

so we only sketch the calculation for \( I_3 \).
$$I_3 = \int d\epsilon_1 d\epsilon_2 f(\epsilon_1) f(\epsilon_2) \int_{-D}^{D} \frac{d\epsilon_3}{(\epsilon_3 - \epsilon_1)\delta (\epsilon_3 - \epsilon_2)\delta} - \int d\epsilon_1 d\epsilon_2 d\epsilon_3 \frac{f(\epsilon_1) f(\epsilon_2) f(\epsilon_3)}{(\epsilon_3 - \epsilon_1)\delta (\epsilon_3 - \epsilon_2)\delta}.$$  

The first integral has no linear $T$ contribution. Thus, up to a constant term, we have

$$I_3 = -\frac{1}{3} \int_{-D}^{D} d\epsilon_1 d\epsilon_2 d\epsilon_3 f(\epsilon_1) f(\epsilon_2) f(\epsilon_3) \left[ \frac{1}{(\epsilon_3 - \epsilon_1)\delta (\epsilon_3 - \epsilon_2)\delta} + \frac{1}{(\epsilon_3 - \epsilon_2)\delta (\epsilon_3 - \epsilon_1)\delta} + \frac{1}{(\epsilon_2 - \epsilon_1)\delta (\epsilon_2 - \epsilon_3)\delta} \right]$$

$$= -\frac{1}{3} \int_{-D}^{D} d\epsilon_1 d\epsilon_2 d\epsilon_3 f(\epsilon_1) f(\epsilon_2) f(\epsilon_3) \frac{((\epsilon_1 - \epsilon_2)^2 + (\epsilon_1 - \epsilon_3)(\epsilon_2 - \epsilon_3)) \delta^2}{[(\epsilon_1 - \epsilon_2)^2 + \delta^2] [(\epsilon_1 - \epsilon_3)^2 + \delta^2] [(\epsilon_2 - \epsilon_3)^2 + \delta^2]}$$

$$= -\frac{\pi^2}{3} \int_{-D}^{D} d\epsilon_1 d\epsilon_2 d\epsilon_3 f(\epsilon_1) f(\epsilon_2) f(\epsilon_3) \delta(\epsilon_1 - \epsilon_3) \delta(\epsilon_2 - \epsilon_3)$$

$$= -\frac{\pi^2}{3} \int_{-D}^{D} d\epsilon_1 [f(\epsilon_1)]^3 = -\frac{\pi^2}{2} T.$$  

Combining the above result with (39), we obtain

$$U_3 = \text{const.} - \frac{n_i \pi^2}{4} g^3 k T. \quad (40)$$

Next, let’s turn to $U_4$ which can be calculated in a similar way. Defining a new function

$$I_4 = \int d\epsilon_1 d\epsilon_2 d\epsilon_3 d\epsilon_4 f(-\epsilon_1) f(\epsilon_2) f(\epsilon_3) f(-\epsilon_4) \left[ \frac{1}{(\epsilon_1 - \epsilon_2)\delta (\epsilon_3 - \epsilon_4)\delta} \left( \frac{1 - e^{\beta(\epsilon_2 - \epsilon_1 + \epsilon_3 - \epsilon_4)}}{(\epsilon_1 - \epsilon_2 - \epsilon_3 + \epsilon_4)\delta} + \frac{e^{\beta(\epsilon_2 - \epsilon_1)}}{(\epsilon_1 - \epsilon_2 + \epsilon_3 - \epsilon_4)\delta} \right) \right]$$

$$= 4 \int d\epsilon_1 d\epsilon_2 d\epsilon_3 d\epsilon_4 \frac{f(-\epsilon_1) f(\epsilon_2) f(\epsilon_3) f(-\epsilon_4)}{(\epsilon_1 - \epsilon_2)\delta (\epsilon_3 - \epsilon_4)\delta(\epsilon_1 - \epsilon_2 - \epsilon_3 + \epsilon_4)\delta},$$

then following the same steps Kondo employed to prove (39), it’s straightforward to show

$$U_4 - \frac{3n_i}{64} g^4 k^2 I_4 = \frac{3\pi^2 n_i}{32} g^4 k^2 T. \quad (41)$$

It can be further verified that $I_4$ has no contribution linear in $T$. Therefore,

$$U_4 = \text{const.} + \frac{3\pi^2 n_i}{32} g^4 k^2 T. \quad (42)$$

Substituting (40) and (42) into (35) and setting $n_i = 1$ for simplicity, we obtain the free energy shift due to the presence of the impurity spin,

$$F_{\text{imp}}(T) = -E_0 - T \ln 2 + \frac{\pi^2}{4} T \left( k g^3 - \frac{3}{8} k^2 g^4 \right) + O(g^4 k), \quad (43)$$
where $E_0$ is the ground state energy.

Since the free energy shift is RG invariant up to an additive constant, we obtain the scaling solution for the free energy shift at $T \to 0$ after carrying out standard RG procedure,

$$F_{\text{imp}}(T) = -E'_0 - T \ln 2 + \frac{\pi^2}{4} T \left[ k g_R^2(T) - \frac{3}{8} k^2 g_R^4(T) \right]$$

$$= -E'_0 - T(\ln 2 - \frac{\pi^2}{2k^2}) - \frac{3\pi^2}{4} \zeta^2 T \left( \frac{T}{T_K} \right)^{2\Delta} + \mathcal{O}(T(T/T_K)^{3\Delta}), \quad (44)$$

where $E'_0$ is in general different from $E_0$ of (43). The impurity specific heat is

$$C_{\text{imp}} = -T \frac{\partial^2 F_{\text{imp}}}{\partial T^2} = \frac{3\pi^2}{2} \zeta^2 \Delta \left( \frac{T}{T_K} \right)^{2\Delta}. \quad (45)$$

The critical exponent, $\alpha = 2\Delta$, agrees with the previous result [13,14]. The impurity entropy is reduced to a universal value [17]

$$S_{\text{imp}}(T = 0) = \ln 2 - \frac{\pi^2}{2k^2} \simeq \ln \left( 2 - \frac{\pi^2}{k^2} \right). \quad (46)$$

Actually, the correction to the entropy occurs only when one first takes the thermodynamic limit $N \to \infty$ and then the limit $T \to 0$. For a finite system, the integrals in $U_3$ and $U_4$ should be replaced by discrete momentum summations over the Brillouin zone which will not give rise to contributions linear in $T$ to the free energy, thus no correction to the bare $\ln 2$ term of the entropy. Therefore, the finite system always remains doubly degenerate.

VII. FIELD DEPENDENT MAGNETIZATION

A natural question following the finite entropy is whether or not there is a corresponding residual impurity spin. To answer this question we calculate the field dependent impurity magnetization defined as the total magnetization of the system subtracting the free Fermi sea contribution. In the presence of the external magnetic field, the leading order diagrams to the free energy are shown in Figure 3. This is an infinite series. The diagram with $l$ vertices has a contribution,

$$\delta F_l = -\frac{1}{\beta} (-1)^{l+1} \left[ -\frac{J}{N} \sum_{\vec{k},\omega_n} \sum_{\sigma=\pm} -\frac{\sigma/2}{i\omega_n - \epsilon_{\vec{k}} - \sigma h} \right]^{l} \sum_{\omega_m,\sigma' = \pm} \left( \frac{-\sigma'/2}{i\omega_m - i\omega_0 - \sigma' \mu h} \right)^{l}. \quad (47)$$
The momentum summation over the conduction band is readily carried out,

$$\frac{J}{N\beta} \sum_{\vec{k},\omega_n} \sum_{\sigma=\pm} \frac{\sigma}{\omega_n - \epsilon_{\vec{k}} - \sigma \hbar} = g \int_{-D}^{D} d\epsilon \left[ f(\epsilon + \hbar) - f(\epsilon - \hbar) \right] = -2g\hbar. \quad (48)$$

Summing up all the diagrams in Figure 3 and using (48), the free energy shift due to the impurity spin is

$$F_{\text{imp}}(T, \hbar) = \sum_{l} \delta F_{l} = \sum_{\sigma=\pm} \sum_{l=1}^{\infty} \left( -\frac{\sigma g k \hbar}{2} \right) \frac{1}{l!} \left[ \frac{d^{l-1} f(z)}{dz^{l-1}} \right]_{z=i\omega_0 + \sigma \mu \hbar}.$$

Introducing the primitive of $f(z)$: $du(z)/dz = f(z)$, the series can be summed up,

$$F_{\text{imp}}(T, \hbar) = \sum_{\sigma=\pm} \left[ u(i\omega_0 + \sigma \mu \hbar - \sigma k \hbar/2) - u(i\omega_0 + \sigma \mu \hbar) \right].$$

The corresponding magnetization is

$$\delta M(T, \hbar) = -\frac{\partial F_{\text{imp}}}{\partial \hbar} = \left( \mu_S - \frac{k g}{2} \right) \tanh \left[ \beta \hbar \left( \mu_S - \frac{k g}{2} \right) \right] - \tanh(\beta \mu \hbar).$$

Adding to the above result the bare magnetization of the impurity spin which just cancels the second term of the above expression, we finally obtain

$$M(T, \hbar) = \left( \mu_S - \frac{k g}{2} \right) \tanh \left[ \beta \hbar \left( \mu_S - \frac{k g}{2} \right) \right]. \quad (49)$$

This is the leading order temperature and field dependent magnetization.

In this section, we shall mainly consider the case $\mu_S = 1$, i.e. an equal gyromagnetic ratio. The expression (49) has the form characteristic of a reduced free spin,

$$M = S_{\text{eff}} \tanh(S_{\text{eff}} h/T).$$

The fact that $S_{\text{eff}} \to 0$ following a power law implies that the impurity spin is completely screened only for the infinite system. A finite system of size $L$ behaves as if there is a partially screened spin $S_{\text{eff}} \sim L^{-\Delta}$ since there is a minimum energy unit $1/L$ to cut off the scaling. It also suggests that the ground state degeneracy will be lifted by any small magnetic field at $T = 0$. Using the Maxwell relation $\partial S/\partial h = \partial M/\partial T$, the leading order magnetic field dependence of the entropy is
\[ S_{\text{imp}}(T, h) = \ln 2 + \int_{0}^{h} dh' \frac{\partial M(T, h')}{\partial T} \]

\[ = \ln 2 - \int_{0}^{\beta h[1-gk^2]} dx \frac{x}{\cosh^2 x} \to 0, \quad \text{for} \quad \begin{cases} T \to 0 \\ h \neq 0. \end{cases} \tag{50} \]

We expect that this is true to any order. In particular, the entropy change, \( \pi^2/(2k^2) \), in the next order \((49)\) will be removed by the magnetic field.

From \((49)\), we obtain the low temperature magnetic susceptibility and its scaling form \([?,?]\),

\[ \chi_{\text{imp}}(T) = \beta \left( 1 - \frac{k g}{2} \right)^2 = \beta \left( 1 - \frac{k g_R(T)}{2} \right)^2 \left( \frac{k \zeta}{2} \right)^{2\Delta} \left( \frac{T}{T_K} \right)^{2\Delta}, \tag{51} \]

indicating that the fixed point is a spin singlet since \( S_{\text{eff}}^2 \sim T \chi_{\text{imp}}(T) \sim T^{2\Delta} \to 0 \) at \( T \to 0 \). The spin is quenched very slowly compared with the linear dependence in the exactly screened case. Also from \((49)\), we determine the field dependence of the zero temperature magnetization \([?]\),

\[ M(T = 0, h) = 1 - \frac{k g}{2} = 1 - \frac{k g_R(h)}{2} \left( \frac{h}{T_K} \right)^{\Delta}. \tag{52} \]

This shows that the spin operator has the scaling dimension \( \Delta \).

Using the well known results for the bulk specific heat and magnetic susceptibility:

\[ C_{\text{bulk}} = 2k \pi^2 T \rho/3 \]

and \( \chi_{\text{bulk}} = 2k \rho \) respectively \([?]\), the leading order Wilson ratio is determined from \((45)\) and \((51)\),

\[ W = \frac{\chi_{\text{imp}} C_{\text{bulk}}}{\chi_{\text{imp}} \chi_{\text{bulk}}} = \frac{\left( \frac{k \zeta}{2} \right)^{2\Delta} \left( \frac{T}{T_K} \right)_{2\Delta}^2}{3 \pi^2 \frac{k^2 T^2}{36}} = \frac{k^3}{36}, \tag{53} \]

in agreement with the conformal field result \([?]\). The Wilson ratio is universal because the only parameter having a possible dependence on the cut-off scheme is \( \zeta \) which is canceled out.

For \( \mu_S \neq 1 \), the external magnetic field couples to an unconserved spin operator. The magnetization will acquire an anomalous dimension and we shall discuss it in detail in section \( \text{IX} \).
VIII. DYNAMICAL SUSCEPTIBILITY

For the one-channel Kondo problem, it is well known that the impurity spin flips at a typical rate of Kondo temperature $T_K$. At $T < T_K$, the impurity spin is effectively inert leading to a fixed point of local Fermi liquid type described by a phase shift. To understand the multichannel case, we calculate the dynamical spin susceptibility. As we shall see, the typical spin flipping rate is given by the temperature.

We proceed as usual by first calculating Matsubara spin-spin correlation function.

$$\chi_f(\tau) = \langle \hat{T} \vec{S}(\tau) \cdot \vec{S}(0) \rangle = \frac{1}{\beta} \sum_n \chi_f(i\nu_n) e^{-i\nu_n\tau},$$  \hspace{1cm} (54)

where $\nu_n = \frac{2n\pi}{\beta}$. To the sub-leading order, the diagrams are shown in Figure 4. The result of calculating these diagrams is

$$\chi_f(i\nu_n) = \delta_{n,0} \frac{3\beta}{4} \left[1 - g^2 k (\ln \beta D - \ln 2 + I_0)\right] - \frac{3g^2 k}{4} K(i\nu_n),$$  \hspace{1cm} (55)

where $I_0 = \int_0^\infty dx [\tanh(x/2)/x - 1/(1 + x)] \simeq 0.125$ and

$$K(i\nu_n) = \int_{-D}^D \int_{-D}^D \frac{d\varepsilon_1 d\varepsilon_2}{(\varepsilon_1 - \varepsilon_2)^2 + \nu_n^2} \left[\frac{f(\varepsilon_1) - f(\varepsilon_2)}{\varepsilon_1 - \varepsilon_2} - f'(\varepsilon_1)\delta_{n,0}\right]$$

$$= -\frac{\pi}{|\nu_n|} (1 - \delta_{n,0}) + O(D^{-1}).$$  \hspace{1cm} (56)

To understand the result we just derived, let’s consider $\chi(z)$ in the complex frequency plane $i\nu_n \to z$ outside a disk of radius of order $T$ (Figure 5). Knowing its values at a set of discrete points on the imaginary axis, $z = i\nu_n$, is enough to give a unique analytic continuation,

$$\chi_f(i\nu_n) = \frac{3\pi}{4} g^2 k \frac{1}{|\nu_n|} \to \frac{3\pi}{4} g^2 k \frac{i \text{sgn}(\text{Im} z)}{z},$$  \hspace{1cm} (57)

where sgn(Im z) means taking the sigh of the imaginary part of $z$. When $z$ approaches the real axis, $z = \omega + i\delta$, $\chi_f(\omega + i\delta) = \chi_f'(\omega) + i \text{sgn}(\delta) \chi_f''(\omega)$, and we find

$$\chi_f''(\omega) = \frac{3\pi}{4} g^2 k \frac{1}{\omega}.\hspace{1cm} (58)$$

This is valid in the whole plane outside a disk of radius of order $T$. There is no energy scale to cut off $1/\omega$ dependence. Certainly for $\omega \ll T$, this dependence should be flattened.
out as seen from the $\nu_n = 0$ expression of (55). If we recall that in the exactly screened Kondo problem ($k = 1$), $\chi''_{f}(\omega)/\omega \sim 1/(\omega^2 + T_K^2)$, the system gives sizable magnetic response only when all the contributions up to the energy scale $T_K$ are included. In other words, the impurity spin is effectively quenched on time scale longer than $1/T_K$. In the multichannel case, the role of $T_K$ is played by the temperature $T$! The impurity is only marginally quenched since the time scale is $1/T$.

In the other limit $\omega \ll T$, carrying out the analytic continuation as in [2], we find

$$\frac{\chi''_{f}(\omega, T)}{\omega} \sim T^{2\Delta-1}.$$ (59)

Note that if $\mu_S \gg 1$, the coupling of the nuclear magnetic moment to the impurity spin dominates and the above result is proportional to the NMR relaxation rate $1/(T_1 T)$. In the limit $k \gg 1$, local probe basically sees a nearly free impurity spin. In the two-channel case, $k=2$, $2\Delta=1$, we see that (59) will have no power law divergence in $T$ except possible logarithmic dependence.

In the two-channel case, numerical work in the non-crossing approximation [2] and the solution at a special Toulouse point [2] have found $\chi''_{f}(\omega) \sim \tanh(\omega/2T)/(\omega^2 + T_K^2)$. The $\tanh(\omega/2T)$ term has also been produced by the conformal field theory method and is a property of the fixed point itself. It would be interesting to see if the remaining Lorentzian form with width $T_K$ can be reproduced from the perturbation of the leading irrelevant operator in the exact conformal field theory calculations. For $k > 2$, we do not expect a Lorentzian form with finite width $T_K$ for the following reason. The multiplicative renormalization factor for $\chi_{f}$ (see next section) will bring an anomalous dimension $\omega^{2\Delta}$ in (58). We see that $\chi''_{f}(\omega) \sim \omega^{2\Delta-1}$ for $T < \omega < T_K$. Since $2\Delta = 4/(k + 2) < 1$, this frequency dependence cannot be reconciled with a Lorentzian form with a finite width.

**IX. MAGNETIC SUSCEPTIBILITY FOR $\mu_S \neq 1$**

Besides the time ordered impurity spin-spin correlation function studied in the last section, we can also define
\[
\chi_{fe}(\vec{q}, \tau) = \langle \hat{T} \vec{S}_e(\vec{q}, \tau) \cdot \vec{S}(\tau=0) \rangle = \frac{1}{\beta} \sum_n \chi_{fe}(\vec{q}, i\nu_n) e^{-i\nu_n \tau},
\]
\[
\chi_e(\vec{q}, \tau) = \langle \hat{T} \vec{S}_e(\vec{q}, \tau) \cdot \vec{S}_e(0,0) \rangle = \frac{1}{\beta} \sum_n \chi_e(\vec{q}, i\nu_n) e^{-i\nu_n \tau},
\]
where the conduction electron spin operator is defined as
\[
\vec{S}_e(\vec{q}) = \frac{1}{2} \sum_{\vec{k}} \sum_{\mu,\nu=\pm} \sum_{\lambda=1..k} c_{\vec{k},\mu\lambda}^+ \vec{\sigma}_{\mu\nu} c_{\vec{k}+\vec{q},\nu\lambda}.
\]
The Feynman diagrams for these two correlation functions are shown in Figure 4 and calculated in the appendix B.

When the impurity spin has a different gyromagnetic ratio from the conduction electrons, a uniform external magnetic field couples to an unconserved spin operator \( \vec{S}_h = \vec{S}_e(\vec{q} = 0) + \mu_S \vec{S} \). The magnetic susceptibility is then only RG invariant up to a multiplicative renormalization factor,
\[
\chi_{imp}(T, g, D) = [\chi_{imp}(T, g, D')]^2 \chi_{imp}(T, gR(D'), D'),
\]
where \( Z_h \) is the multiplicative renormalization factor for the operator \( \vec{S}_h \) and we recall \( g_R(D) = g \). If \( \mu_S = 1 \), then \( Z_h \equiv 1 \). The RG equation for \( \chi \) is
\[
\left[ D \frac{\partial}{\partial D} + \beta(g) \frac{\partial}{\partial g} + 2\gamma_h(g) \right] \chi_{imp}(T, g, D) = 0,
\]
\[
\gamma_h(g) = D \frac{\partial \ln Z_h}{\partial D}.
\]
The perturbative result for \( \chi \) to the sub-leading order can be found from the results for three dynamical correlation functions calculated in the last section and appendix B,
\[
\chi_{imp}(T, g, D) = \frac{4}{3} \left[ \chi_e(\vec{q}=0, \nu_n=0) + 2\mu_S \chi_{fe}(\vec{q}=0, \nu_n=0) + \mu_S^2 \chi_{f}(\nu_n=0) \right]
\]
\[
= \beta \left\{ \left( \mu_S - \frac{gk}{2} \right)^2 \left[ 1 - g^2 k (\ln \beta D - \ln 2 + I_0) \right] - g^2 k \left( \mu_S - \frac{gk}{2} \right) \ln 2 \right\}.
\]
The leading term of \( \gamma_h(g) \) can be obtained from (66) using the same method of section IV,
\[
\gamma_h(g) = \frac{\mu_S - 1}{2\mu_S - gk} g^2 k.
\]
Its value at the fixed point, $\gamma_h(g^*)$, gives the anomalous dimension of the operator $\vec{S}_h$. The anomalous dimensions for the operators $\vec{S}$ and $\vec{S}_e(\vec{q}=0)$ are the limiting values at $\mu_S \to \infty$ and $\mu_S \to 0$ respectively.

To obtain the scaling solution, we first find the multiplicative renormalization factor,

$$Z_h(g, g_R(D')) = \int_g^{g_R(D')} dg' \frac{\gamma_h(g')}{\beta(g')} = \left( \frac{g - 2\mu_S}{g - 2} \right)^2 \left( \frac{g_R(D')k - 2}{g_R(D')k - 2\mu_S} \right)^2.$$  \hspace{1cm} (68)

Setting $D' = T$ in (63) and (68), we obtain the leading order susceptibility (since we only have leading order $Z_h$),

$$\chi_{\text{imp}}(T, g, D) = \left( \frac{g - 2\mu_S}{g - 2} \right)^2 \left[ \chi_{\text{imp}}(T, g, D) \right]_{\mu_S=1}.$$  \hspace{1cm} (69)

This is the main result of this section: the static magnetic response is the same, up to a constant factor, no matter whether or not the conduction electrons and the impurity spin have the same gyromagnetic ratio. I believe this conclusion remains true for all overscreened cases and even for the exactly screened case of $k = 1$. If the initial starting point belongs to weak coupling, $g \to 0$,

$$\chi_{\text{imp}}(T, g, D) = \mu_S^2 \left[ \chi_{\text{imp}}(T, g, D) \right]_{\mu_S=1}.$$  \hspace{1cm} (70)

This is the well known result for all kind of Kondo problems: the contribution to the magnetization from the conduction electrons is suppressed by a factor $T_K/D$ compared with that of the impurity spin [?]. Extending the treatment in this section to the field dependent magnetization and dynamical susceptibilities is straightforward.

**X. KNIGHT SHIFT**

Knight shift gives direct measurement of the spatial structure of the conduction electron screening cloud. Consider an impurity spin sitting at the origin and a uniform magnetic field $h$ being applied to the system, Knight shift measures the magnetization in the space surrounding the impurity spin,
\[ M(\vec{r}) = \sum_{\sigma=\pm} \sum_{\lambda=1}^{\cdots k} \sigma \langle \epsilon^\dagger_{\sigma,\lambda}(\vec{r}) \sigma_{\lambda}(\vec{r}) \rangle = \chi(\vec{r}) \ h, \quad \chi(\vec{r}) = \frac{1}{N} \sum_{\vec{q}} \chi(\vec{q}) \ e^{i\vec{q}\cdot\vec{r}}. \quad (71) \]

We implicitly assume that the free Fermi sea contribution to \( M(\vec{r}) \) has been subtracted. The result for the Knight shift is contained in the two general dynamical susceptibilities defined in the last section,

\[ \chi(\vec{q}) = \frac{4}{3} [\chi_e(\vec{q}, i\nu_n = 0) + \mu_s \chi_{fe}(\vec{q}, i\nu_n = 0)]. \quad (72) \]

Using the results from the appendix B, we find to the sub-leading order in \( 1/k \),

\[ \chi(\vec{q}) = -\frac{g^2}{2T} \Pi_0(\vec{q}) \left\{ \left[ \mu_s - \frac{g^2}{2} \right] \left[ 1 - g^2 k (\ln D/T - \ln 2 + I_0) \right] - \frac{\ln 2}{2} g^2 k \right\} \]
\[ -\ln \frac{2}{\ln 2} g^2 k \left[ \mu_s - \frac{g^2}{2} \right] \Pi_1(\vec{q}), \quad (73) \]
\[ \Pi_0(\vec{q}) = -\frac{1}{\rho N} \sum_{\vec{k}} \frac{f(\epsilon_\vec{k}) - f(\epsilon_{\vec{k}+\vec{q}})}{\epsilon_\vec{k} - \epsilon_{\vec{k}+\vec{q}}}, \quad (74) \]
\[ \Pi_1(\vec{q}) = \frac{1}{2 \ln 2} \frac{1}{\rho^2 N^2} \sum_{\vec{k},\vec{k}'} \tanh \frac{\epsilon_{\vec{k}}}{2T} \left[ \frac{f(\epsilon_\vec{k}) - f(\epsilon_{\vec{k}+\vec{q}})}{\epsilon_\vec{k} - \epsilon_{\vec{k}+\vec{q}}} - \frac{f(\epsilon_{\vec{k}}) - f(\epsilon_{\vec{k}+\vec{q}})}{\epsilon_{\vec{k}} - \epsilon_{\vec{k}+\vec{q}}} \right], \quad (75) \]

where \( I_0 \approx 0.125 \), given in the last section. We have included proper normalization factors in the definitions of \( \Pi_0(\vec{q}) \) and \( \Pi_1(\vec{q}) \) such that \( \Pi_0(\vec{q}=0) = 1 \) and \( \Pi_1(\vec{q}=0) = 1 \). Note that the Fourier transform of \( \Pi_0(\vec{q}) \) has the well known Friedel oscillation form in space [?]. The Fourier transform of \( \Pi_1(\vec{q}) \) should give similar spatial variation. The key point is that their Fourier transforms only depend on \( k_F r \), where \( r \) is the distance to the impurity spin.

Since \( \chi(\vec{q}) \) satisfies the RG equation,

\[ \left[ D \frac{\partial}{\partial D} + \beta(g) \frac{\partial}{\partial g} + \gamma_h(g) + [\gamma_h(g)]_{\mu_s=0} \right] \chi(\vec{q}) = 0. \quad (76) \]

The scaling solution can be found in a similar way to the last section,

\[ \chi(\vec{q}, T, g, D) = Z_h(g, g_R(T)) [Z_h(g, g_R(T))]_{\mu_s=0} \chi(\vec{q}, T, g_R(T), T). \quad (77) \]

Using the results for \( Z_h \) and \( g_R \), we obtain to the leading order,

\[ \chi(\vec{q}, T, g, D) = \frac{g^k (g^k - 2\mu_s)}{(g^k - 2)^2} \chi_{imp}(T) \Pi_0(\vec{q}). \quad (78) \]
where $\chi_{\text{imp}}(T)$ is the uniform susceptibility found in section VII. A knowledge of $Z_h$ to the
sub-leading order could extend (78) to the next order. When Fourier transforming (78) to
the real space, $M(\vec{r})$ is a function of $k_F r$. So the only length scale is $1/k_F$. Thinking over
why the length scale $v_F/T_K$ does not appear in $\chi(\vec{q})$, we observe that $\Pi_0$ and $\Pi_1$ are functions
of $q/k_F$. A new length scale $v_F/T_K$ would appear only if the logarithmic singularity were
cut off by $v_F q$ in the limit $v_F q > T$. This doesn’t happen! It means that the multichannel
system is critical only in the time direction, not in the space.

We can extend the statement that there is only one length scale $1/k_F$ to the one-channel
problem. In this case, the effective Kondo interaction flows to strong coupling at low energy.
From the renormalization group point of view, it means that summing up leading(or leading
plus sub-leading) logarithmic series is not enough. But reorganizing the perturbation expansion
in the coupling constant $g$ into successive logarithmic series(the first term of each series
has successive higher power in $g$) according to the renormalization group is still possible.
This is equivalent to write $\beta(g)$ as an expansion in $g$. The occurrence of a strong coupling
fixed point means that we need all terms in the expansion of $\beta(g)$ to describe the low energy
behavior. Suppose we could find all of them(infinite terms), we would have a correct low
energy theory by summing up all terms. The fact that the infrared singularity is not cut
off by $v_F q$ does not change merely because we have to include high order logarithmic series.
The necessary consequence is that $v_F/T_K$ will never have the chance to appear as a length
scale. Certainly, if the flowing of an effective interaction to strong coupling leads to a phase
transition like in higher dimensions, the above argument may not hold. But the impurity
problem has a dimension $0+1$, prohibiting a phase transition. Thus, our calculation pro-
vides an explicit analytical demonstration that the screening cloud has a spatial size of order
$1/k_F$, and agrees with the well known experimental result [?].
XI. DISCUSSION

We have carried out a comprehensive study on the non-Fermi liquid fixed point of the multichannel Kondo model emphasizing several intriguing aspects. As the actual realization of this model in heavy fermion and metallic glassy systems is still an open question, a thorough understanding of the model should be very helpful in devising new experimental tests and making comparison with experimental results. Part of the driving force behind the recent resurgence of interest is due to the resemblance of its low energy behavior to the normal state properties of high Tc cuprates. This similarity has been further exploited recently [?] arguing that a situation similar to the two-channel model is realized in the copper-oxide plane.

One interesting followup subject is to study the possibility of converting the local non-Fermi liquid fixed point into a coherent lattice one by forming an overscreened Kondo lattice, especially in two or three dimensions. The first step to study the two-channel-two-impurity problem has been carried out using the numerical renormalization group method [?]. The one-impurity fixed point is found unstable against developing correlations between the impurity spins. This is expected in the view of the asymptotical screening of the impurity spin. One artificial way to suppress magnetic correlations between impurity spins is to go to infinite dimension. Unfortunately, the presence of finite degeneracy on each site prevents development of true coherence. In any realistic situation, nontrivial magnetic correlations must intervene to lift the residual degeneracy of the impurity fixed point. If one wishes to follow the successful route of the heavy fermion theory from impurity to lattice once again, in the current intensive search of non-Fermi liquid models to describe the normal state of high Tc cuprates, only a non-degenerate fixed point of an impurity model has the chance to succeed. Development along this direction in search of a promising starting point of impurity model has been reported recently [?]. It is worthwhile mentioning that a quantum critical system with a Fermi surface has a true coherent non-Fermi liquid infrared fixed point in high dimensions. An example is an electron gas coupled to transverse gauge field where a
self-consistent solution of the infrared fixed point has been derived [?]. It is very interesting to note that this system is quantum critical at $T=0$ in $2D$ and $3D$ but without developing long range order.

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APPENDIX A: FEYNMAN RULES

To be self-contained, we list the rules for constructing Feynman diagrams in the presence of an external magnetic field. These rules also define our convention.

1. For the contributions of the $n$th order in the Kondo interaction, $\mathcal{O}(J^n)$, draw all topologically distinct diagrams with $n$ vertices.

2. For each conduction electron propagator, draw a solid line,

$$
\bar{k}, i\omega_n = G_{\mu\nu}^{(0)}(\bar{k}, i\omega_n) = \frac{-\delta_{\mu\nu}}{i\omega_n - \epsilon_{\bar{k}} - \mu h}.
$$

(A1)

There is a summation over each momentum $\bar{k}$ of the internal conduction electron propagator. For each pseudofermion propagator, draw a dashed line,

$$
\bar{k}, i\omega_n = G_{\mu\nu}^{(0)}(i\omega_n) = \frac{-\delta_{\mu\nu}}{i\omega_n - i\omega_0 - \mu sh}.
$$

(A2)

3. Each vertex is associated with a factor,

$$
\bar{k}', i\omega', \mu' \quad i\omega', \nu' \quad \bar{k}, i\omega, \mu \quad i\omega, \nu = -\frac{J}{4\sqrt{3}} \bar{\sigma}_{\mu'\mu} \cdot \bar{\sigma}_{\nu'\nu} \delta_{\omega_n + \omega_m, \omega'_n + \omega'_m}.
$$

(A3)
4. Each independent internal frequency is summed over.

5. Each conduction electron loop contributes a factor $-k$ and each pseudofermion loop contributes a factor $-1$.

6. For a diagram of order $J^n$, there is a numerical factor: $(\text{Number of different connections})/n!$. This combinatorial factor will be given explicitly in the figures of this paper for the diagrams we calculate.

APPENDIX B: DYNAMICAL SPIN-SPIN CORRELATION FUNCTIONS

Three dynamical correlation functions have been defined in the sections VII and IX. To the order $\mathcal{O}(1) + \mathcal{O}(1/k)$, their diagrams are shown in Figure 4. Each diagram contains two external vertices which are represented by two open ends at the left and right sides. The external vertex at each end is joined by two propagators. If the spin indices of these two propagators are $\sigma_1$ and $\sigma_2$, the corresponding vertex is associated with a vector of matrices $\vec{s}_{\sigma_2 \sigma_1}$. The two vectors of matrices from the two external vertices at the two ends form a scalar product $\vec{s}_{\sigma_2 \sigma_1} \cdot \vec{s}_{\sigma_4 \sigma_3}$. In $\chi_e$, only one of the two external vertices has the two joining conduction electron propagators differing in their momenta by $\vec{q}$.

The result for $\chi_f(i\nu_n)$ is given in (55). The results for the other two are

$$
\chi_f(q, i\nu_n) = -\delta_{n,0} \frac{3}{8} gk\beta \{\Pi_0(q) \left[ 1 - g^2k(\ln \beta D - \ln 2 + I_0) \right] + g \ln 2 \Pi_1(q) \}
$$

$$
+(1 - \delta_{n,0})\chi_f(q, i\nu_n),
$$

(B1)

$$
\chi_e(q, i\nu_n) = -\frac{g k}{2} \delta_{n,0} \chi_f(q, i\nu_n) + \frac{3}{16} \ln 2 g^3 k^2 \beta \Pi_0(q) - (1 - \delta_{n,0})\chi_f(q, i\nu_n),
$$

(B2)

where $\Pi_0$ and $\Pi_1$ are defined in section X and for $\nu_n \neq 0$,

$$
\chi_f(q, i\nu_n) = \frac{3}{8} g^2 k \left[ gk \Pi_0(q) K(i\nu_n) + 2K(q, i\nu_n) - 2L(q, i\nu_n) \right],
$$

(B3)

$$
K(q, i\nu_n) = \frac{1}{\rho^2 N^2} \sum_{\vec{k},\vec{k}'} \frac{1}{\epsilon_{\vec{k}} - \epsilon_{\vec{k}'} + \vec{q}} \frac{f(\epsilon_{\vec{k}'} - f(\epsilon_{\vec{k}})}{\epsilon_{\vec{k}} - \epsilon_{\vec{k}'} + \vec{q}^2 + \nu_n^2}
$$

(B4)

$$
L(q, i\nu_n) = \frac{1}{\rho^2 N^2} \sum_{\vec{k},\vec{k}'} \frac{1}{\epsilon_{\vec{k}} - \epsilon_{\vec{k}'} + \vec{q}} \frac{f(\epsilon_{\vec{k}'} - f(\epsilon_{\vec{k}})}{\epsilon_{\vec{k}} - \epsilon_{\vec{k}'} + \vec{q}^2 + \nu_n^2}.
$$

(B5)
Note that $K(\vec{q}=0, i\nu_n) = K(i\nu_n)$, is a generalization of $K(i\nu_n)$ to finite $\vec{q}$. In the expressions for $\chi_{fe}$ and $\chi_e$, we have dropped contributions of order $1/D$ to the $\nu_n = 0$ components with respect to $\beta$. 
FIGURES

FIG. 1. Feynman diagrams for the conduction electron self-energy.

FIG. 2. Diagrams for the free energy. The solid line represents the bare conduction electron propagator. The big circles stand for self-energies.

FIG. 3. Diagrams for the free energy in the magnetic field.

FIG. 4. Diagrams for three dynamical spin correlation functions.

FIG. 5. Analytic continuation in the complex frequency plane carried out in the region outside a disk of radius of order $T$. 
