Symmetric coupling of angular momenta, quadratic algebras and discrete polynomials

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Abstract. Eigenvalues and eigenfunctions of the volume operator, associated with the symmetric coupling of three SU\((2)\) angular momentum operators, can be analyzed on the basis of a discrete Schrödinger–like equation which provides a semiclassical Hamiltonian picture of the evolution of a ‘quantum of space’, as shown by the authors in [1]. Emphasis is given here to the formalization in terms of a quadratic symmetry algebra and its automorphism group. This view is related to the Askey scheme, the hierarchical structure which includes all hypergeometric polynomials of one (discrete or continuous) variable. Key tool for this comparative analysis is the duality operation defined on the generators of the quadratic algebra and suitably extended to the various families of overlap functions (generalized recoupling coefficients). These families, recognized as lying at the top level of the Askey scheme, are classified and a few limiting cases are addressed.

1. Introduction and a brief review

In [1] a family of orthogonal polynomials has been introduced based on a three–term recursion relationship which plays the role of a discrete Schrödinger equation describing the action of a ‘volume’ operator. This operator occurs in the symmetric treatment of the quantum few–body problem as well as in spin–network modeling of a quantum of space, as pointed out originally in [2]. In this section a short introduction to the necessary mathematical background will be given, together with a summary of a few significant results found by the authors in [1]. Improved insights into algebraic and analytical aspects of the subject will be provided in the next sections.

The theory of (re)coupling of eigenstates of three SU\((2)\) angular momentum operators \(J_1, J_2, J_3\) to states of sharp total angular momentum \(J_4\) (with projection \(J_0\) along the quantization axis) is usually carried out in the setting of ‘binary couplings’ (see [3], Topic 12 and original references therein). Referring to the ordered triple as above, the admissible schemes are \(J_1 + J_2 = J_{12}; J_{12} + J_3 = J_4, J_2 + J_3 = J_{23}; J_1 + J_{23} = J_4\), respectively. The corresponding

\(^{\dagger}\) Talk delivered by Annalisa Marzuoli
eigenvectors are denoted
\[ |j_{12} > := |(j_1 j_2 j_{12} j_3 j_{12} m) > \text{ and } |j_{23} > := |j_1 (j_2 j_3) j_{23} j_{4} m' > , \]
where small js are labelings of the eigenvalues associated with the angular momentum operators (e.g. \( J_2^3 |j_{12} > = j_1 (j_1 + 1) |j_{12} > \), \( J_{12}^2 |j_{12} > = j_2 (j_2 + 1) |j_{12} > \)) running over \( \{0, 1/2, 1, 3/2, \ldots \} \), in \( h \) units, and \( m (m') \) is the eigenvalue of \( J_0 \) with \(-j_4 \leq m, m' \leq j_4 \) in integer steps. Thus the ket vectors above belong to Hilbert spaces representing simultaneous angular momentum (a configuration that can be associated with a not necessarily planar 'spin–network' models for 3–dimensional discretized quantum gravity and quantum computing Euclidean tetrahedron, a fact which is at the basis of the huge amount of literature about theorem). Recall in passing that the 6
\[ \Phi = j_1 + j_2 + j_3 \]
and the weights \((2j + 1)\) are the dimensions of the spin–\( j \) representations of \( SU(2) \) which provide the standard normalization of such a ‘recoupling coefficient’ as encoded in the shorthand notation in the left–hand side. Therefore a basis transform is simply written as \( |j_{23} > = \sum_{j_{12}'} |< j_{12} | j_{12} > |j'_{12} > \) while the inverse one is achieved by the transpose \( < j_{12} | j_{23} > \) (all non–null matrix elements obey the selection rule \( m = m' \) by Wigner–Eckart theorem). Recall in passing that the 6j symbol in (2) encodes naturally the symmetry of an Euclidean tetrahedron, a fact which is at the basis of the huge amount of literature about ‘spin–network’ models for 3–dimensional discretized quantum gravity and quantum computing flourished in the past two decades (see [4] and references therein).

The treatment of the ‘symmetric’ coupling scheme for the addition of three \( SU(2) \) angular momenta \( J_1, J_2, J_3 \) to give \( J_4 \) (with projection \( J_0 \)) is characterized in terms of a ‘volume’ operator \( K = J_1 \cdot J_2 \times J_3 \). Unlike what happens with binary coupling schemes, the Js appear now to be all on the same footing, indicating that the volume operator can be thought of as acting democratically on either a composite system of four objects with vanishing total angular momentum (a configuration that can be associated with a not necessarily planar quadrilateral vector diagram \( J_1 + J_2 + J_3 + J_4 = 0 \)), or a system of three objects with total angular momentum \( J_4 \) (see again [3], Topic 12, last section, and original references therein). The present scheme is characterized by the six commuting Hermitian operators \( J_1^2, J_2^2, J_3^2, K, J_4^2 \) and \( J_0 \), so that eigenvectors and eigenvalues of \( K \) are given formally (consistently with the notation used in (1)) as
\[ |k > := |(j_1 j_2 j_3) j_4 m > \text{ with } K |k > = \lambda_k |k > . \]
Eigenvalues and matrix elements of \( K \) are naturally found within an imaginary antisymmetric representation based on a three–terms recursion relationship [2], which can be turned into a real, time–independent Schrödinger equation which governs the dynamics of a ‘quantum of space’ as a function of a discrete variable denoted \( \ell \) (see below). This has been achieved in [1] (which can be referred to also for a complete list of previous and related papers) where the introduction of discrete, potential–like functions highlights the surprising crucial role of ‘hidden’ symmetries, first discovered by Regge [5] for the 6j symbols. The Schrödinger equation is discretized with respect to a lattice variable given by the label of the operator \( J_{12} \) which characterizes the first of the binary schemes in (1) and reads
\[ \lambda_k \Psi^{(k)}_{\ell} + \alpha_{\ell+1} \Psi^{(k)}_{\ell+1} + \alpha_{\ell} \Psi^{(k)}_{\ell-1} = 0 \text{ with } \ell \equiv j_{12} \in \{j_{12}^{\text{min}}, j_{12}^{\text{min}} + 1, \ldots, j_{12}^{\text{max}}\} , \]
where the matrix elements \( \alpha_{\ell} \) are expressed in terms of geometric quantities, namely
\[ \alpha_{\ell} = F(\ell; j_1+1/2; j_2+1/2) F(\ell; j_3+1/2; j_4+1/2) / [(2\ell+1)(2\ell-1)]^{1/2} . \]
Here $F(A, B, C) = \frac{1}{4}[(A + B + C) (-A + B + C) (A - B + C) (A + B - C)]^{\frac{1}{2}}$ is the Heron’s formula for the area of a triangle with side lengths $A, B$ and $C$. Thus $\alpha_\ell$ is proportional to the product of the areas of the two triangles sharing the side of length $\ell$ and forming a quadrilateral of sides $j_1 + \frac{1}{2}, j_2 + \frac{1}{2}, j_3 + \frac{1}{2}$ and $j_4 + \frac{1}{2}$. Such a parameter quadrilateral, together with its Regge-conjugate (see below), is the guiding tool of the combinatorial and geometric analysis, in the asymptotic limit, of the Hamiltonian dynamics governing both tetrahedral and ‘fluttery’ quadrilateral configurations, see sections 2 and 3 of [1] for more details.

2. Quadratic symmetry algebras
Following [6, 7], the quantum version of a classical dynamical algebra associated with a pair of ‘mutually integrable’ dynamical variables calls into play a triple $K_1, K_2, K_3$ of linear operators acting on a (suitably defined) Hilbert space with $K_{1,2}$ Hermitian and algebraically independent and $K_3 := [K_1, K_2]$ anti-Hermitian. The request that these generators do fulfill the Jacobi identity constrains the fundamental commutation relations to be of the form ($\{\,\,\,\,\,\,\,\,\}$ is the anticommutator)

$$[K_1, K_2] = K_3$$
$$[K_2, K_3] = 2R K_2 K_1 K_2 + A_1 \{K_1, K_2\} + A_2 K_2^2 + C_1 K_1 + D K_2 + G_1$$
$$[K_3, K_1] = 2R K_1 K_2 K_1 + A_1 K_1^2 + A_2 \{K_1, K_2\} + C_2 K_2 + D K_1 + G_2,$$

where $R, A_{1,2,3}, C_{1,2,3}, D, G_{1,2}$ are real parameters (the structure constants) and the right-hand sides of the last two relations contain only Hermitian terms. Such a kind of algebraic structures was actually introduced by Sklyanin [8] and they are called ‘quadratic’ algebras for the obvious reason that the commutators (Poisson brackets in the classical cases) are combinations of quadratic (and linear) terms in each of the generators. Mutual integrability is a sharper requirement with respect to the original formulation, and amounts to look at the symmetry algebra as a dynamical one—where $K_1$ is a constant of the motion for $K_2$ taken as the Hamiltonian operator, as well as the other way around. Further improvements in the study of classical, quantum and $q$-deformed symmetries along these lines have been provided over the past few decades by a number of authors. Often the admissible structures associated with (2) and listed in the table below [6] are referred to as ‘Zhedanov’s algebras’ in the literature. Note that for completeness the last line includes the two ‘standard’ Lie algebras on three generators (whose commutation relations are by definition linear).

| Classification of quadratic algebras | $R$ | $A_1$ | $A_2$ | $C & D$ |
|--------------------------------------|-----|-------|-------|---------|
| **AW(3)** (Askey–Wilson)             | *   | *     | *     | *       |
| **R(3)** (Racah)                     | 0   | *     | *     | *       |
| **H(3)** (Hahn)                      | 0   | 0     | *     | *       |
| **J(3)** (Jacobi)                    | 0   | 0     | *     | *       |
| Lie algebras:                        |     |       |       |         |
| $su(2)$, $su(1, 1)$                  | 0   | 0     | 0     | *       |

The denominations of the algebras, Askey–Wilson, Racah, ..., are strongly reminiscent of the Askey–Wilson scheme of hypergeometric orthogonal polynomials of one (continuous or discrete) variable [9]. This is not accidental: rather, this remark turns out to be crucial in order to recognize the deep connection between algebraic symmetries of (quantum) systems and special function theory in a quite straightforward way. Indeed the ‘overlap functions’ stemming from the
analysis of the eigenvalue problems for the operators \( K_1, K_2, K_3 \) which generate the quadratic algebras are, under mild conditions, orthogonal families of Wilson, Racah, Hanh, Jacobi, ..., Hermite polynomials. In what follows an account of a few technical details is given for the case of the Racah algebra \( \mathbf{R}(3) \) which corresponds to set \( R = 0 \) in (2).

Suppose that the Hermitian operators \( K_1 \) and \( K_2 \) –defined on a separable Hilbert space and possibly depending on a same (finite) set of real parameters– are both ladder operators, namely possess discrete, evenly–spaced spectra, and start considering the eigenvalue problem for \( K_1 \)

\[
K_1 \psi_p = \chi_p \psi_p, \quad p = 0, 1, 2, \ldots \quad \text{with} \quad \chi_{p+1} = \chi_p + 1.
\]

Then it can be easily shown that the operator \( K_2 \) is tridiagonal in this basis

\[
K_2 \psi_p = a_{p+1} \psi_{p+1} + a_p \psi_{p-1} + b_p \psi_p
\]

and, similarly, by exchanging the role of \( K_1 \) and \( K_2 \), one would get

\[
K_2 \phi_s = \mu_s \phi_s, \quad s = 0, 1, 2, \ldots \quad \text{with} \quad \mu_{s+1} = \mu_s + 1
\]

and

\[
K_1 \phi_s = c_{s+1} \phi_{s+1} + c_s \phi_{s-1} + d_s \phi_s.
\]

The (real) matrix coefficients \( a, b, c, d \) can be evaluated explicitly in terms of the commutation relations (2) and contain also the parameters which the operators may depend on (such parameters are dropped in the present simplified treatment aimed to point out the overall structural properties). Once chosen suitable normalizations for the two sets of eigenbases (7) and (9), it is possible to introduce two families of overlap functions by resorting to the Dirac braket convention (in which for instance \( \langle x | \psi > \) stands for the eigenfunction \( |\psi > \) of a system in the position representation)

\[
\langle \phi_s | \psi_p \rangle \equiv \langle s | \psi_p \rangle \equiv \langle s | p \rangle \quad \text{and} \quad \langle \psi_p | \phi_s \rangle \equiv \langle p | \phi_s \rangle \equiv \langle p | s \rangle
\]

which are both hypergeometric orthogonal polynomials of one discrete variable (the spectral parameter \( \mu_s \) and \( \chi_p \) respectively) to be identified, up to suitable rearrangements of the hidden parameters, with the Racah polynomial on the top of the Askey scheme [9].

In the \( K_1 \)-eigenbasis the operator \( K_3 \) satisfies

\[
K_3 \psi_p = (\chi_{p+1} - \chi_p) a_{p+1} \psi_{p+1} - (\chi_p - \chi_{p-1}) a_p \psi_{p-1},
\]

where eigenvalues \( \chi \) and matrix elements \( a \) are iteratively evaluated from (7) and (8). \( K_3 \) has a discrete, in general not evenly–spaced spectrum found as a solution of

\[
K_3 \varphi_n = \nu_n \varphi_n, \quad n = 0, 1, 2, \ldots
\]

It is worth noting that in general the diagonalization of \( K_3 \) cannot be carried out analytically, except in a few cases in which at least the lowest eigenvalues turn out to be representable in closed algebraic forms. The associated families of (normalized) overlap functions are denoted

\[
\langle \varphi_n | \psi_p \rangle \equiv \langle n | p \rangle \quad \text{and} \quad \langle \psi_p | \varphi_n \rangle \equiv \langle p | n \rangle \quad (n = 0, 1, 2, \ldots; p = 0, 1, 2, \ldots)
\]

and can be shown to be orthogonal (on different suitably defined lattices), each depending on one discrete variable, but in principle they might not be included into the Askey scheme.
Similarly, other two families of (normalized) overlap functions associated with the pair \( K_2, K_3 \) can be defined by notation consistency as

\[
< \varphi_n | \phi_s > \equiv < n | s > \quad \text{and} \quad < \phi_s | \varphi_n > \equiv < s | n > \quad (n = 0, 1, 2, \ldots; s = 0, 1, 2, \ldots). \quad (15)
\]

A crucial feature of the Racah algebra \( R(3) \) and associated overlap functions is the duality property. It relies on the following transformation of the generators

\[
K_1 \mapsto K_2; \quad K_3 \rightarrow -K_3
\]

which can be easily shown to represent an automorphism of the Racah algebra \( R(3) \). The notion of duality is extended to (all of) the sets of overlap functions introduced above. More precisely

i) Under the automorphism (16) the discrete, evenly–spaced variables of the two hypergeometric families of overlap functions associated with \( K_1, K_2 \) given in (11) and their degrees as polynomials are interchanged. Since in the present case the operator \( K_3 \) is not called into play, the stronger property of ‘self–duality’ of these families holds true: both of them are recognized as Racah polynomials, as already mentioned above.

ii) Referring to the families in (14), under the automorphism (16) the discrete, not evenly–spaced spectral variable \( \nu_n \) of the first family, which is orthogonal on the evenly–spaced lattice \( p = 0, 1, 2, \ldots \), is turned into the second family, where the variable is \( p \) and the polynomial degree is given in terms of the labels \( n = 0, 1, 2, \ldots \) of the eigenvalues of \( K_3 \).

A similar property is shared by the families associated with the pair \( K_2, K_3 \) given in (15).

More details on the nature of the automorphism group and on the statements about the overlap functions will be reported in the next section when dealing with a specific ‘realization’ of the Racah algebra.

3. Generalized recoupling theory, Regge symmetry and duality

The realization of the Racah algebra \( R(3) \) within the setting of generalized \( SU(2) \) recoupling theory was actually the issue addressed originally in [7] which has inspired further work on quadratic algebras. Combining the definitions and notation of section 2 with those of section 1 it is straightforward to recognize the following correspondence

\[
K_1 = J^2_{12}; \quad K_2 = J^2_{23}; \\
K_3 = [J^2_{12}, J^2_{23}] = -4i J^1 \cdot (J_2 \times J_3) \equiv -4i K
\]

between the abstract ordered set of operators \( K_1, K_2, K_3 \) and its realization as \( J^2_{12}, J^2_{23}, K \).

The next step would consist in associating eigenvalue equations and three–term recursion relations of the abstract approach with their realizations in generalized quantum (re)coupling theory. Here we do not enter into much details about this matter since the translation of (8) based on the pair \( K_1, K_2 \) represents the three–term recursion relation for the 6j coefficient in disguise (see e.g. [10]). The analysis for the pair \( K_1, K_3 \) which gives the abstract three–term relation as written in (12) is examined in details in [2] (and references therein) while its symmetrized counterpart is nothing but the discretized Schrödinger–like equation displayed already in (4) [1].

Focusing on the specific issue regarding the families of solutions of such relationships, one would directly be lead to establish the correspondence

\[
\text{overlap functions} \quad \mapsto \quad \text{binary and symmetric recoupling coefficients}, \quad (18)
\]

where the arrow stands for the specific realization (17) of \( R(3) \). To achieve this goal in a transparent and consistent way a few more steps are needed, the first one of which consists in
establishing suitable notations for all of the recoupling coefficients. The $6j$ symbol in (2) and
the functions $\Psi^{(k)}_\ell$ in (4) are thus denoted and defined respectively as

$$
<j_{23} | j_{12} > = < \ell | \ell > \quad \text{and} \quad \Psi^{(k)}_\ell := < \ell | k > .
$$

Actually this is not a mere question of notation, since in this way the objects $< \bullet | \circ >$ may reveal their ‘double’ meaning as i) quantum mechanical transition amplitudes, namely the square modulus $| < \bullet | \circ > |^2$ is the probability that a system, prepared in the state $| \circ >$, be measured to be in the state $| \bullet >$; ii) eigenfunctions of the operator whose quantum number is in $| \circ >$ in the representation labeled by the eigenvalue of the other operator, namely through the projection onto $< \bullet |$. The latter interpretation will be under focus in what follows and more details about the correspondence (18) can be worked out by introducing explicitly the (so far ignored) parameters of the problem. Upon replacement of the original (ordered) set of labeling of the four angular momenta forming a quadrilateral according to

$$
(j_1, j_2, j_3, j_4) \mapsto (a, b, c, d),
$$

the functionals are rewritten as

$$
< \ell | \ell > (a, b, c, d) \propto \begin{bmatrix} a & b & \ell \\ c & d & \bar{\ell} \end{bmatrix} \quad \text{and} \quad \Psi^{(k)}_\ell (a, b, c, d) = < \ell | k > (a, b, c, d).
$$

Recall that geometrically the first functional is associated with a tetrahedron ($\ell$ and $\bar{\ell}$ being a pair of opposite edges) and the second one to a quadrilateral (actually two triangles hinged by one of its diagonal, $\ell$ or $\bar{\ell}$) bounding, so to speak, a portion of volume of amount $\lambda_k$, the eigenvalue of the volume operator given in (3). In order to select in a convenient way the Hilbert space on which the volume operator acts and all the functionals above can be defined consistently, the role of Regge symmetries, originally introduced for the $6j$ [5], is crucial. Such symmetries in their original formulation are recognized as functional relations on the arguments (namely they cannot be derived by interchanging the $6j$ arguments as happens for the so-called ‘classical’ or tetrahedral symmetries) and read

$$
\begin{bmatrix} a & b & \ell \\ c & d & \bar{\ell} \end{bmatrix} = \begin{bmatrix} s - a & s - b & \ell \\ s - c & s - d & \bar{\ell} \end{bmatrix} := \begin{bmatrix} a' & b' & \ell \\ c' & d' & \bar{\ell} \end{bmatrix},
$$

where $s = (a + b + c + d)/2$ is the semi–perimeter of the parameter quadrilateral and in the last equality the new set $(a', b', c', d')$ is defined. It can be checked that the total number of classical and Regge symmetries is 144, which equals the order of the product permutation group $S_4 \times S_3$.

Denoting by $a$ the smallest value among the eight parameters $(a, b, c, d, a', b', c', d')$, it can be shown that a consistent ordering of the other parameters compatible with all the due inequalities is given by $\{ a \leq b \leq c \leq d : d - (b - a) \leq c \leq d + (b - a) \}$. This sort of gauge fixing implies that the whole problem becomes finite–dimensional and workable out for each fixed values of the parameters $(a, b, c, d) \in \mathbb{R}^4$. Moreover: i) the tetrahedron $< \ell | \ell > (a, b, c, d)$ can be chosen as the reference one, calling $< \ell | \ell > (a', b', c', d')$ its Regge–conjugate; ii) the same thing holds for the quadrilateral denoted $< \ell | k > (a, b, c, d)$ and its conjugate $< \ell | k > (a', b', c', d')$. More technical details about this specific parametrization and the denomination Regge–‘conjugate’ (as well as the proof that the volume operators and all quantities in its three–term recursion relation (4) are Regge–invariant) can be found in [11] and [1] respectively.

Coming back to the statement regarding the correspondence (18), the remarks above should have made clear that Regge symmetry is strictly related to the duality property of the Racah algebra discussed at the end of section 2. Note that in [7] it had been already recognized that (classical + Regge) symmetries do have the group structure given by $S_4 \times S_3$, to be identified with the automorphism group of the Racah algebra.
4. Classification of discrete polynomial families

In this section the focus will be on interconnections among the families of discrete orthogonal polynomials in view of the formalization presented in section 2 and summarized there in items i) and ii). This analysis—not addressed elsewhere to our knowledge—is just sketched here, leaving aside a number of technical details that can be found in [12]. The various cases, together with the most significant properties of each family, are summarized in the following table.

**Finite families of discrete orthogonal polynomials [(a, b, c, d) fixed]**

| #  | family | orthogonality on lattice | eigenvalue (related to the variable) | degree related to |
|----|--------|--------------------------|--------------------------------------|------------------|
| I.A | < ĝ | ℓ | Σ_ℓ < ℓ | ℓ' > < ĝ | ℓ > = δ_ℓℓ | ℓ(ℓ + 1) | ĝ |
| I.B | < ℓ | ĝ | Σ_ℓ < ℓ | ℓ' > < ℓ | ĝ > = δ_ℓĝ | ĝ(ℓ + 1) | ℓ |
| II.A | < ℓ | k | Σ_ℓ < ℓ | k' > < ℓ | k > = δ_ℓk | λ_k | ℓ |
| II.B | < k | ℓ | Σ_k < k | ℓ' > < k | ℓ > = δ_kℓ | ℓ(ℓ + 1) | k |
| III.A | < ĝ | k | Σ_ℓ < ĝ | k' > < ĝ | k > = δ_ĝk | λ_k | ĝ |
| III.B | < k | ĝ | Σ_k < k | ĝ' > < k | ĝ > = δ_kĝ | ĝ(ℓ + 1) | k |

Comparing the notations adopted here—the bar stands for complex conjugation or simply transposition in the real cases—with those of section 2, it is straightforward to recognize that the classes I, II and III are in correspondence with the overlap functions in (11), (14) and (15) (restricted to finite sets by suitable choices of the omitted parameters), respectively.

Looking at the family I.A, observe that \(< ĝ | ℓ'| > := < ℓ' | ĝ > = < ĝ | ℓ' > > by the convention chosen for 6j symbols in (2) (and similarly for IB). Thus ‘self–duality’ relations for class I read either way

\[ \sum_ℓ < ℓ' | ĝ > < ĝ | ℓ > = δ_ℓℓ' \quad \text{and} \quad \sum_ℓ < ℓ' | ℓ > < ℓ | ĝ > = δ_ĝℓ, \]

once fulfilled the completeness relations Σ_ℓ < ĝ | ℓ > = I and Σ_ℓ < ℓ | ĝ > = I for the binary coupled eigenbases introduced in (1). Note that the operators associated with class I (J_{12} and J_{23}) represent a ‘Leonard pair’ so that the associated overlap functions (recoupling coefficients) are necessarily hypergeometric of Racah type [13]. More generally, in connection with the analysis of the other classes, a stringent result holds true: any finite system of orthogonal polynomials whose dual is a finite system of orthogonal polynomials must be (possibly q–deformed) Racah or one of its limiting cases which constitute finite systems (refer to [14] for a modern monograph on hypergeometric polynomials in the Askey–Wilson scheme). Indeed here
all of the families are consistently defined, for fixed parameters \( (a, b, c, d) \), as finite sets (recall the choice on the ordering discussed in connection with Regge symmetry) but the recognition of classes \( \text{II} \) and \( \text{III} \) as belonging to the Askey scheme is certainly not straightforward. (More precisely, the reduction process to specific hypergeometric functions of type \( 4F3 \) would require to find out a closed algebraic form for the sets of eigenvalues of the volume operator for given parameters, a task not yet accomplished.)

For what concerns duality within class \( \text{II} \), a first remark is about the bar operation: \( \langle \ell | k \rangle \) is \( \langle k | \ell \rangle \), but the latter, unlike what happens for the \( 6j \), is not necessarily equal to \( \langle \ell | k \rangle \) because this property actually depends on the volume operator \( K \) being Hermitian (imaginary antisymmetric) [2] or real symmetric (see [1] also for plots of the family of eigenfunctions \( \langle \ell | k \rangle \)). Anyway, both options can be included through a suitable notation into the duality relations

\[
\sum_{\ell} \langle k' | \ell \rangle \langle \ell | k \rangle = \delta_{k' k} \quad \text{and} \quad \sum_{k} \langle \ell' | k \rangle \langle k | \ell \rangle = \pm \delta_{\ell' \ell}
\]

(24)

according to the choice of the representation of \( K \). Duality relations in class \( \text{III} \) are similar to (24), with \( \ell \) taking the role of \( \ell' \).

To conclude this general overview on duality relationships, a further remarkable property –transversal with respect to the classes– has to be mentioned, namely

\[
\sum_{k} \langle \tilde{\ell} | k \rangle \langle k | \ell \rangle = \pm \langle \tilde{\ell} | \ell \rangle .
\]

(25)

Such a ‘triangular relation’ (and the other ones that can be derived by using the properties of the single classes given above) closely resembles the Racah identity satisfied by three \( 6j \) symbols and might be used also to explore a formalization of the whole subject within the general scheme of tensor categories.

### 5. Limiting cases

The issue of asymptotic (semiclassical) limits of angular momentum functions is of continuous interest in many fields, ranging from special function theory [14] to applied quantum mechanics [15]. Here just a few remarks concerning two limiting cases of families \( \text{II.A} \) and \( \text{III.B} \) are sketched.

The reference model of asymptotics is the well–know limit of the \( 6j \) symbol for three large entries (see [16, 10]), \( 6j \rightarrow 3j \), where the latter is the Wigner symbol, the symmetrized version of a Clebsch–Gordan coefficient. The counterpart of this operation in the Askey scheme is achieved by moving one step downwards from top, namely from \( 4F3 \) (Racah) to \( 3F2 \) (Hahn and dual Hahn) hypergeometric families.

A new change of notation is needed which consists in restoring the string \( (j_1, j_2, j_3, j_4) \) for the parameters (see (20)) and in writing down as an array the functions in (21) (equivalently, in family \( \text{II.A} \)) according to

\[
\Psi^{(k)}_{\ell} (j_1, j_2, j_3, j_4) \rightarrow \begin{cases} j_1 & j_2 \\ j_3 & j_4 \end{cases} | \ell > \lambda_k ,
\]

(26)

where the vertical bars in front of the last column of this symbol indicate that not all of the entries are constrained by standard triangular inequalities, as happens for the \( 6j \). To address any limit in which (some of) the arguments of the symbols become large –a fact that implies that all of the arguments can be ‘running’– a convenient notation is to substitute capital to small letters. Thus the formal limiting process for the symbol in (26) when the arguments of
the lower row become large can be displayed as a generalized $3j$ coefficient, denoted $3j$, related in turn to a generalized dual Hahn polynomial; schematically

$$
\frac{j_1}{J_3} | \ell, \Lambda_k, j_2 | \Lambda_k - J_3 \leftrightarrow 3j \text{ (dual Hahn family)}. \tag{27}
$$

On applying a similar procedure to family III.B, and denoting $\tilde{\ell}$ the previous generic argument $\ell$ (playing the role of $j_{23}$), the resulting correspondence would read

$$
\frac{j_1}{J_3} | \lambda_k, j_2 | \tilde{L}, J_4 \leftrightarrow \frac{j_1}{J_4 - \tilde{L}} | \lambda_k, \tilde{L} - J_3, J_3 - J_4 \leftrightarrow 3j \text{ (Hahn family)}. \tag{28}
$$

A few comments on these results are in order, leaving aside a more careful analysis and most technical details reported in [12]. As already noticed, the symbols in round brackets on the right–hand sides of (27) and (28) are generalized counterparts of $3j$ coefficients, the arguments in the lower row being interpreted as magnetic quantum numbers. They actually share with standard $3j$s a suitable formulation of Regge symmetry [17] and their properties as orthogonal families are inferred from three–term recursion relationships. The latter can in turn be derived as limits of the three–term recursions at the upper level (in particular, the relation for (27) can be quite easily worked out). The motivation for associating dual Hahn and Hahn families respectively is related with the specific lattices these three–term recursion relations are defined on. Thus it is found that the relation for (27) mimics the behavior of the relation of a $3j$ on a quadratic lattice ($\ell(\ell + 1)$), so that it is functionally similar to the standard dual Hahn polynomial family. Conversely, the relation for (28) mimics the behavior of the relation of a $3j$ on a linear lattice (given by scaling the quantum number $m \equiv J_3 - J_4$) and thus these functions represent counterparts of the Hahn polynomial family.

6. Outlook

Further developments can be addressed in parallel, from algebraic–analytical and geometric viewpoints. A schematic list of ongoing works (and still open questions) follows:

- improved interpretations of Regge symmetries on the geometric (scissor–congruent tetrahedra [18]) and algebraic (quaternionic reparametrization [12]) sides;
- convolution rules for overlap functions (specifically, symmetric recoupling coefficients) of Racah algebra;
- composition rules of collections of quadrilaterals able to provide new classes of integrable quantum systems to be associated with extended quantum geometries;
- $q$–deformed extensions and limiting cases of the dual sets of orthogonal polynomials also in view of applications in quantum chemistry.

In particular, a systematic study of limiting procedures –to be carried out on recurrence relations, on families of polynomials and possibly directly on the defining relations (2) of the underlying quadratic algebras– seems particularly promising also in view of recent analytical and numerical work on strictly related issues [19, 20, 21, 22].

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