The Liouville equation differs from the von Neumann equation ‘only’ by a characteristic superoperator. We demonstrate this for Hamiltonian dynamics, in general, and for the Jaynes-Cummings model, in particular. – Employing superspace (instead of Hilbert space), we describe time evolution of density matrices in terms of path integrals which are formally identical for quantum and classical mechanics. They only differ by the interaction contributing to the action. This allows to import tools developed for Feynman path integrals, in order to deal with superoperators instead of quantum mechanical commutators in real time evolution. Perturbation theory is derived. Besides applications in classical statistical physics, the “classical path integral” and the parallel study of classical and quantum evolution indicate new aspects of (dynamically assisted) entanglement (generation). Our findings suggest to distinguish intra- from inter-space entanglement.

Keywords: path integral; Liouville equation; von Neumann equation; superoperator; entanglement; Jaynes-Cummings model.

1. Introduction

The quantum-classical divide has been intensely studied in recent years with profound impact on various areas of research. In particular, this concerns the foundations of quantum mechanics, new quantum technologies (quantum information processing, precision measurements, designer materials, etc.), recent observations of quantum coherent processes in biology, and, last not least, unresolved issues surrounding “quantum gravity”.

Not surprisingly, these modern topics, which touch the foundations of quantum mechanics in one way or another, increase the impetus to try to reconstruct and

\[\text{a}^{\text{I.e.}, \text{the conflict between quantum mechanics necessitating an external time and diffeomorphism invariance in general relativity, for example, which defies its existence. Despite its great successes in describing the statistical aspects of experiments, quantum theory itself presents problems of interpretation, which are brought to the forefront in quantum cosmology. They arise from its indeterministic features and are clearly seen, for example, in the measurement problem.}\]
to better understand the emergence of quantum mechanics from simpler dynamical structures beneath or more profound theoretical principles.

Indeed, there is a growing number of deterministic models of quantum mechanical objects which are based on conjectured fundamental information loss, coarse graining, or dissipation mechanisms. We recall that 't Hooft’s existence theorem shows that the evolution of all quantum mechanical objects that are characterized by a finite dimensional Hilbert space can be captured by a dissipative process. This holds also for objects that are described by a set of mutually commuting Hermitian operators.

However, a theory is lacking that would generally explain the emergence of quantum features of common objects, at the scales where they are observed.

In order to make progress in these matters, it may be useful to further examine the quantum-classical divide. Presently, we look more carefully into the common as well as the distinctive features of classical and quantum dynamics, as described by the Liouville and the von Neumann equations, respectively.

We will derive a new path integral representation of the propagator for density matrices in the classical theory. It is identical with the usual one at the kinematic level, employing the Feynman propagator of quantum mechanics, this allows ‘external sources’ in the relevant action that are coupled to terms linear or quadratic in the generalized coordinates. Yet the interaction part differs in a characteristic way. The new formalism based on superoperators will be presented and illustrated here by perturbation theory applied to an anharmonic oscillator.

Similarly, as a case study, we will re-derive the Jaynes-Cummings model – the well-known benchmark model of quantum optics and cavity QED – based on classical dynamics described by a Liouville equation. Thus, when applied to the two-level dynamics of Rydberg atoms coupled to one mode of the photon field in a suitably tuned cavity, we find surprisingly that it is “almost classical”, with quantum and classical dynamics differing by a characteristic superoperator.

We conclude by pointing out some interesting problems, concerning the preparation of entangled states, in particular. Here, the quantum-classical divide shows new aspects, which may help to further unravel the hidden dynamics that must be involved when it is crossed – be it in the “classical limit” or following axiomatic “quantization rules”.

2. Hamiltonian dynamics and the Liouville superoperator

To begin with, we will consider an object with a single continuous degree of freedom. We will treat an atom interacting with the electromagnetic field in a later chapter, while a relativistic field theory has been studied elsewhere.
Let us assume that there are only conservative forces and that Hamilton’s equations are determined by the generic Hamiltonian function:

\[ H(x, p) := \frac{1}{2}p^2 + V(x) , \]

defined in terms of generalized coordinate \( x \) and momentum \( p \) (a mass parameter will be inserted in due time, but is omitted here for simplicity), and where \( V(x) \) denotes the potential. An ensemble of such objects, for example, following trajectories with different initial conditions, is described by a distribution function \( \rho \) in phase space, i.e., by the probability \( \rho(x, p; t) \) to find a member of the ensemble in an infinitesimal volume at point \((x, p)\). This distribution evolves according to the Liouville equation:

\[ -\partial_t \rho = \frac{\partial H}{\partial p} \cdot \frac{\partial \rho}{\partial x} - \frac{\partial H}{\partial x} \cdot \frac{\partial \rho}{\partial p} = \{ p\partial_x - V'(x)\partial_p \} \rho , \]

with \( V'(x) := \frac{dV(x)}{dx} \). - We recall that the relative minus sign in the Poisson bracket, or between terms here, reflects the symplectic phase space symmetry. This will translate into the familiar commutator structure in Eq. (5).

A Fourier transformation, \( \rho(x, p; t) = \int d\gamma e^{-ip\gamma} \rho(x, \gamma; t) \), replaces the Liouville equation by:

\[ i\partial_t \rho = \{-\partial_\gamma \partial_x + yV'(x)\} \rho , \]

without changing the symbol for the distribution function, whenever changing variables. Thus, momentum is eliminated in favour of doubling the number of coordinates. Finally, with the transformation:

\[ Q := x + y/2 , \quad q := x - y/2 \]

we obtain the Liouville equation in the form:

\[ i\partial_t \rho = \{ \hat{H}_Q - \hat{H}_q + \mathcal{E}(Q, q) \} \rho , \]

\[ \hat{H}_\chi := -\frac{1}{2}\partial_\chi^2 + V(\chi) , \quad \text{for} \quad \chi = Q, q \]

\[ \mathcal{E}(Q, q) := (Q - q)V'(\frac{Q + q}{2}) - V(Q) + V(q) = -\mathcal{E}(q, Q) . \]

We remark that the presented reformulation of classical dynamics is rather independent of the number of degrees of freedom. It applies to matrix valued as well as to Grassmann valued variables, representing the “pseudoclassical” fermion fields introduced by Casalbuoni and by Berezin and Marinov. Field theories require a classical functional formalism.

Furthermore, the Eq. (5) appears as the von Neumann equation for a density operator \( \hat{\rho}(t) \), considering \( \rho(Q, q; t) \) as its matrix elements. We automatically recover the Hamiltonian operator \( \hat{H} \) related to the Hamiltonian function, Eq. (1), as in quantum theory. However, an essential difference consists in the interaction \( \mathcal{E} \).
between bra- and ket-states. The Hilbert space and its dual here are coupled by a superoperator \(c\).

Since the interaction \(E\) is antisymmetric under \(Q \leftrightarrow q\), the complete (Liouville) operator on the right-hand side of Eq. (5) has a symmetric spectrum with respect to zero and, in general, will not be bounded below. Therefore, with this coupling of the Hilbert space and its dual by the superoperator, corresponding to the absence of a stable ground state, our reformulation of Hamiltonian dynamics does not qualify as a quantum theory. Related observations were discussed, for example, in Refs. 4, 5, 6, 9.

However, the following fact has been discussed in Refs. 12:

\[ E \equiv 0 \iff \text{potential } V(x) \text{ is constant, linear, or harmonic,} \]

with an eye on the possibility of having quantum phenomena emerge due to discrete spacetime structure. Analogously, the vanishing of \(E\) in a field theory is equivalent with having massive or massless free fields, with or without external sources, and with or without bilinear couplings. Generally, in these cases, anharmonic forces or interactions are absent.

In the following main parts of this work, we will study in more detail the classical Hamiltonian dynamics described by Eq. (5), or by its appropriate generalizations, and pay particular attention to the presence of the superoperator \(E\), when comparing with the von Neumann equation.

Concluding this introductory section, we recall relevant aspects of the interpretation of the density operator \(\hat{\rho}(t)\), which we invoked here.

### 2.1. Expectations, operators and the Born rule

We begin with the normalization of the classical probability distribution:

\[ 1 \equiv \int \frac{dx dp}{2\pi} \rho(x, p; t) = \int dQ dq \, \delta(Q - q) \rho(Q, q; t) =: \text{Tr } \hat{\rho}(t) , \]

incorporating the transformations of Section 2. Consider a complete set of orthonormal eigenfunctions of the operator \(\hat{H}_x\) of Eq. (6), defined by \(g_j(\chi; t) := \exp(-iE_j t)g_j(\chi)\) and \(\hat{H}_x g_j(\chi) = E_j g_j(\chi)\), respectively, with a discrete spectrum, for simplicity. Then, we may expand \(\rho\):

\[ \rho(Q, q; t) = \sum_{j,k} \rho_{jk}(t)g_j(Q; t)g_k^*(q; t) . \]

Employing this, the normalization condition (9) can be stated as:

\[ 1 \equiv \sum_{j,k} \rho_{jk}(t)e^{-i(E_j - E_k)t} \int dQ \, g_j(Q)g_k(Q) = \sum_j \rho_{jj}(t) . \]

\(c\)This superoperator is of a very specific form, which leads to the antisymmetry in Eq. (7). It differs from a Lindblad superoperator, often obtained as a symmetric double commutator structure, in the case of open quantum mechanical systems [19].
Since the classical phase space distribution is real, the expansion coefficients form a Hermitean matrix, \( \rho_{ij} = \rho_{ji}^* \), which we also denote by \( \hat{\rho} \).

The classical expectation values are calculated as follows:

\[
\langle x \rangle := \int \frac{dx dp}{2\pi} x \rho(x, p; t) = \int dQ dq \ \delta(Q - q) \left( \frac{Q + q}{2} \rho(Q, q; t) \right),
\]

\[
= \text{Tr} \left( \hat{X} \hat{\rho}(t) \right),
\]

\[
\langle p \rangle := \int \frac{dx dp}{2\pi} p \rho(x, p; t) = \int dQ dq \ \delta(Q - q) (-i) \frac{\partial Q - \partial q}{2} \rho(Q, q; t),
\]

\[
= \text{Tr} \left( \hat{P} \hat{\rho}(t) \right),
\]

introducing the operators \( \hat{X} \) and \( \hat{P} \), with matrix elements \( X(q, Q) = \delta(Q - q)(Q + q)/2 \) and \( P(q, Q) = -i(\delta(Q - q) \frac{\partial Q}{\partial q} - \frac{\partial q}{\partial Q} \delta(Q - q)) \) (derivatives act left or right, as indicated). Eliminating one of the two integrations in the above equations with the help of the \( \delta \)-functions and suitable partial integrations, these operators are recognized as the coordinate and momentum operators of quantum theory.

Similarly, we find:

\[
\int \frac{dx dp}{2\pi} xpp(x, p; t) = \frac{1}{2} \text{Tr} \left( (\hat{X}\hat{P} + \hat{P}\hat{X})\hat{\rho}(t) \right),
\]

which constitutes an example of the symmetric Weyl ordering, when replacing classical phase space quantities by quantum operators. – However, we remark that Hilbert space operators appear here by rewriting classical statistical formulae and not by following a quantization rule.

The Eqs. (9), (12)–(16) are in accordance with the interpretation of \( \rho(Q, q; t) \) as matrix elements of a density operator \( \hat{\rho}(t) \). – However, there is an important caveat: The eigenvalues of normalized quantum mechanical density operators are usually constrained to lie between zero and one, corresponding to the interpretation as standard probabilities. This is not necessarily the case with the operator \( \hat{\rho} \) obtained from a classical probability distribution. Similarly, the Wigner distribution – obtained from the matrix elements of a quantum mechanical density operator by applying the transformations leading from \( \rho(x, p) \) to \( \rho(Q, q) \) in reverse – generally, is not positive semi-definite on phase space, even though its marginal distributions are. Therefore, it does not qualify as a classical probability density.

As we have indicated before, there is clearly a dynamical feature missing, which governs the crossing of the quantum-classical divide, if not done ‘by hand’, as in any of the usual “quantization prescriptions”. Last not least, this must establish the Born rule by eliminating negative probabilities or by leading to their satisfactory interpretation.

3. From Hilbert space to superspace

In this section, we reformulate the notions relevant for the dynamics of density operators, at which we arrived in the previous section, in a more convenient way,
introducing the concept of *superspace*[^1], also called *Liouville space* – see Ref.^[20] for a concise presentation and numerous applications.

Considering a physical object characterized by the Hamiltonian \( \hat{H} \), as in quantum theory, we introduce a complete set of basis states, \( \{ |j\rangle \} \) \( (j = 1, \ldots, N) \), assuming that the relevant Hilbert space is \( N \)-dimensional. Then, taking matrix elements of the von Neumann equation, for example, we have:

\[
i \partial_t \rho_{jk} = [\hat{H}, \hat{\rho}]_{jk} , \quad j, k = 1, 2, \ldots, N ,
\]

with a density matrix \( \rho \) of \( N^2 \) elements. Which may be written as:

\[
i \partial_t \rho_{jk} = \sum_{l,m} \hat{L}_{jk,lm} \rho_{lm} ,
\]

in terms of the *Liouville superoperator* \( \hat{L} \) defined by:

\[
\hat{L}_{jk,lm} := H_{jl} \delta_{km} - H_{km}^* \delta_{jl} .
\]

This definition suggests to introduce a space where the density operator is a vector, which is the Liouville space (or superspace). The dynamics of the density operator can then be more conveniently described, completely in parallel for classical and quantum mechanics, as we shall see.

Given the Hilbert space, as above, the density operator can be expanded as:

\[
\hat{\rho} = \sum_{j,k} \rho_{jk} |j\rangle \langle k| .
\]

We may think of the family of \( N^2 \) operators \( |j\rangle \langle k| \), with \( j, k = 1, \ldots, N \), as a complete set of matrices, or vectors, and express the density operator as:

\[
|\rho\rangle = \sum_{j,k} \rho_{jk} |jk\rangle \rangle ,
\]

where the “ket” \( |jk\rangle \rangle \) denotes the Liouville space *vector* representing the Hilbert space *operator* \( |j\rangle \langle k| \). Similarly, we introduce a “bra” vector \( \langle\langle k| \) as the Hermitean conjugate of \( |jk\rangle \rangle \).

In Liouville space, any operator \( \hat{A} \) is represented by a vector and denoted by \( |A\rangle \rangle \). It can be expanded as:

\[
|A\rangle \rangle = \sum_{j,k} A_{jk} |jk\rangle \rangle ,
\]

where \( A_{jk} \) are the usual matrix elements \( \langle j| \hat{A} |k\rangle \). Furthermore, we can define a “bra” vector \( \langle\langle B| \), representing \( \hat{B}^\dagger \), and the scalar product of two operators:

\[
\langle\langle B|A\rangle \rangle := \text{Tr}(\hat{B}^\dagger \hat{A}) .
\]

[^1]: The notion of superspace here, at first sight, has little in common and should not be confused with the corresponding term relating to supersymmetry.
Then, the following orthonormality condition holds:
\[
\langle\langle jk | mn \rangle\rangle = \text{Tr}(|k\rangle\langle j|mn\rangle) = \delta_{kn}\delta_{jm},
\]
which is analogous to \( \langle j|k \rangle = \delta_{jk} \). Finally, consider the scalar product:
\[
\langle\langle jk | A \rangle\rangle = \text{Tr}(|k\rangle\langle j|\hat{A}) = \sum_l \langle l|k\rangle\langle j|\hat{A}|l\rangle = \langle j|\hat{A}|k \rangle \equiv A_{jk}.
\]
Upon substitution in Eq. (22), this yields:
\[
|A\rangle\rangle = \sum_{j,k} |jk\rangle\rangle\langle\langle jk | A \rangle\rangle.
\]
This is consistent with the following completeness relation in Liouville space:
\[
\sum_{j,k} |jk\rangle\rangle\langle\langle jk | = 1.
\]
Following these considerations, it can be verified that Liouville space is a linear space, in which the density operator \( \hat{\rho} \) is a vector. In this space, a linear operator can be defined by:
\[
\hat{\mathcal{F}} := \sum_{j,k,m,n} |jk\rangle\rangle\langle\langle jk |\hat{\mathcal{F}}|mn\rangle\rangle\langle\langle mn | \equiv \sum_{j,k,m,n} \mathcal{F}_{jk,mn}|jk\rangle\rangle\langle\langle mn |,
\]
i.e., in terms of appropriate matrix elements.

The importance of Liouville space for classical and quantum dynamics is that the Liouville and von Neumann equations, both, can be written in the form:
\[
i\partial_t \hat{\rho} = \hat{L} \hat{\rho},
\]
with an appropriate superoperator \( \hat{L} \), cf. Eqs. (5)–(7) and (17)–(19), respectively. Thus, there is a formal analogy (even isomorphism) between the structure of these equations and the Schrödinger equation, \( i\partial_t \Psi = \hat{H} \Psi \).

These observations suggest that techniques or formal results concerning the solution of the Schrödinger equation can be transferred to the case of the Liouville or von Neumann equations with the help of Liouville space notions. This concerns perturbation theory (and nonperturbative methods) as much as a path integral formulation, which we shall discuss in turn.

First of all, we introduce the Liouville space evolution operator \( \hat{U} \) satisfying:
\[
i\partial_t \hat{U}(t,t_0) = \hat{L}(t)\hat{U}(t,t_0) \quad ,
\]
with the initial condition \( \hat{U}(t,t_0) = 1 \). This implies:
\[
\hat{\rho}(t) = \hat{U}(t,t_0)\hat{\rho}(t_0) \quad ,
\]
as the solution of the density operator equation of motion. For a time independent Liouville superoperator this yields:
\[
\hat{U}(t,t_0) = \exp(-i\hat{L}(t-t_0)) \quad .
\]
Thus, time evolution of the density matrix is implemented by a superoperator in Liouville space, while in Hilbert space the evolution is described by:

\[ \hat{\rho}(t) = U(t, t_0)\hat{\rho}(t_0)U(t, t_0)^\dagger, \tag{33} \]

with \( U(t, t_0) := \exp(-i\hat{H}t) \). The Eqs. (31) and (33) represent the evolution of the same object, although in different spaces. For a time dependent Hamiltonian, we have instead:

\[ \hat{U}(t, t_0) = T \exp\left( -i \int_{t_0}^{t} d\tau \hat{L}(\tau) \right) \tag{34} \]

\[ := 1 + \sum_{n=1}^{\infty} (-i)^n \int_{t_0}^{t} d\tau_n \int_{t_0}^{\tau_n} d\tau_{n-1} \ldots \int_{t_0}^{\tau_1} d\tau \hat{L}(\tau_n)\hat{L}(\tau_{n-1})\ldots\hat{L}(\tau_1), \tag{35} \]

in terms of the time-ordered exponential.

For later purposes, we also introduce the “interaction picture” in Liouville space. Considering a Liouville operator which consists of two parts:

\[ \hat{L} \equiv \hat{L}_0(t) + \hat{L}'(t), \tag{36} \]

we obtain the evolution operator in the following form:

\[ \hat{U}(t, t_0) = \hat{U}_0(t, t_0)\hat{U}_I(t, t_0), \tag{37} \]

with:

\[ \hat{U}_0(t, t_0) = T \exp\left( -i \int_{t_0}^{t} d\tau \hat{L}_0(\tau) \right), \tag{38} \]

and:

\[ \hat{U}_I(t, t_0) = T \exp\left( -i \int_{t_0}^{t} d\tau \hat{L}'_I(\tau) \right), \tag{39} \]

with \( \hat{L}'_I(\tau) := \hat{U}_0^\dagger(\tau, t_0)\hat{L}'(\tau)\hat{U}_0(\tau, t_0) \). For an operator \( \hat{U}_0 \) that can be treated exactly, study of time evolution essentially concerns the operator \( \hat{U}_I \) – for this, Eq. (39) presents the starting point of perturbation theory (expanding the exponential).

4. The quantum/classical path integral for the propagator of density matrices

The technical ingredients needed for the Feynman path integral approach, for the derivation of quantum mechanical propagators in particular, are very well known. We import these here, especially from Ref. [10] in order to derive a path integral for the propagator of density matrices based on the Liouville space formulation of the preceding Section 3.

Our derivation relies on the close formal similarity between the classical Liouville equation and the von Neumann equation on one hand side and the Schrödinger equation on the other, in an appropriate representation, as we have discussed.

*In this chapter, we reinstate \( h \) explicitly.*
4.1. **Essentials of the Feynman path integral**

We recall that the (forward propagating) operator Green’s function $\hat{G}$,

$$
\hat{G}(t, t_0) \equiv \theta(t - t_0) \exp \left( -i\hat{H}(t - t_0)/\hbar \right),
$$

(40)

allows one to write the solution of the time dependent Schrödinger equation as:

$$
|\psi(t)\rangle = \hat{G}(t, t_0) |\psi(t_0)\rangle.
$$

Correspondingly, for $t > t_0$, we have the coordinate space matrix elements:

$$
G(x, t; y, t_0) = \langle x | e^{-i\hat{H}(t-t_0)/\hbar} | y \rangle,
$$

(41)

from which one derives the path integral representation of these amplitudes, for a generic Hamiltonian $\hat{H} = \hat{p}^2/2m + V(\hat{x})$, describing a particle of mass $m$ in an external potential $V$, cf. Eq. (1), through the following steps [16]:

- Cut the time interval from $t_0$ to $t$ into a large number $N$ of equal pieces.
- Write the exponential of the Hamiltonian operator $\times$ time as a product of identical factors, each factor representing the propagator for a small time interval $\propto 1/N$.
- Separate the kinetic and potential terms contributing to $\hat{H}$ in each one of these factors with the crucial help of the Trotter product formula.
- Alternatingly, insert complete sets of momentum and coordinate eigenstates, such as $\int dx |x\rangle\langle x| = 1$ (and correspondingly for momentum eigenstates) between the factors of exponentials involving either momentum or coordinate operators and evaluate the resulting Gaussian integrals over momentum variables.
- Realize that the obtained phases in the product of exponentials can be summed up to represent a discretized version of the classical action pertaining to the Hamiltonian function (corresponding to $\hat{H}$).

Taking the limit $N \to \infty$ in the end, one obtains the following Feynman path integral representation of the amplitudes in question:

$$
\lim_{N \to \infty} \left( \frac{m}{2\pi i\hbar} \right)^{N/2} \int dx_1 \ldots dx_{N-1} \exp \left( \frac{i\epsilon}{\hbar} \sum_{j=0}^{N-1} \left( \frac{m}{2} \left( \frac{x_{j+1} - x_j}{\epsilon} \right)^2 - V(x_j) \right) \right)
$$

$$
= \int \mathcal{D}x \exp \left( \frac{i}{\hbar} \int_{t_0}^{t} d\tau \left( \frac{m}{2} \dot{x}^2 - V(x) \right) \right)
$$

(43)

$$
\equiv \int \mathcal{D}x \exp \left( \frac{i}{\hbar} S[\dot{x}, x] \right),
$$

(44)

where $\epsilon := (t-t_0)/N$, $\dot{x} := dx/d\tau$ and where $S$ denotes the relevant classical action, which is to be evaluated for each one of the paths contributing to the integral, with the boundary conditions $x(t) = x$ and $x(t_0) = y$. 
4.2. The Liouville space propagator as a path integral

We are now ready to appreciate the economy of the Liouville space representation introduced in Section 3. In particular, the formal solution of the classical Liouville equation as well as of the quantum mechanical von Neumann equation, both, can be written in the form:

$$|\rho(t)\rangle\rangle = e^{-i\hat{H}(t-t_0)/\hbar} |\rho(t_0)\rangle\rangle ,$$  \hspace{1cm} (45)

where $\hat{H}$ is the appropriate super-Hamiltonian. Generally, we have:

$$\langle\langle Q,q|\hat{H}|Q',q'\rangle\rangle = \delta(Q-Q')\delta(q-q')(\hat{H}(Q) - \hat{H}(q) + E(Q,q)) ,$$  \hspace{1cm} (46)

where $\hat{H}$ denotes the appropriate Hamilton operator in coordinate representation, as indicated, which alone is relevant for the von Neumann equation, while $E$ represents the additional superoperator for classical dynamics, cf. Section 2.

In order to solve the problem of time evolution in the present case, we need to know the (super)matrix elements entering the propagation equation:

$$\langle\langle Q,q|\rho(t)\rangle\rangle = \int dQ' dq' \langle\langle Q,q|e^{-i\hat{H}(t-t_0)/\hbar}|Q',q'\rangle\rangle \langle\langle Q',q'|\rho(t_0)\rangle\rangle ,$$  \hspace{1cm} (47)

which appears formally analogous to evolution of a state vector according the Schrödinger equation. Thus, not surprisingly, we go through the steps indicated in the preceding Section 4.1, in order to construct the path integral representation of the propagator here.

However, in this derivation, we have to pay attention to the crucial role of the Trotter product formula. It turns out that it can be generalized for our purposes, where superoperators present the new feature, in a straightforward way; the relevant definitions and details of the proof will be given elsewhere [21].

Rewriting the Eq. (47) as:

$$\rho(Q,q;t) = \int dQ' dq' \mathcal{G}(Q,q;t|Q',q';t_0)\rho(Q',q';t_0) ,$$  \hspace{1cm} (48)

our interest is to know the superpropagator $\mathcal{G}$. Next, we will follow the recipe to arrive at a path integral representation, as summarized above, in Section 4.1.

In particular, here we make use of suitably inserted complete sets of superspace vectors, such as:

$$\int dQ dq |Q,q\rangle\rangle\langle\langle Q,q| = 1 ,$$  \hspace{1cm} (49)

and, correspondingly, for momentum space, cf. Eq. (27). Using the plane wave relation between coordinate and momentum eigenfunctions, we also employ:

$$\langle\langle P,p| = \frac{1}{2\pi\hbar} \int dQ dq \exp\left( -\frac{i}{\hbar} (PQ - pq) \right)\langle\langle Q,q| .$$  \hspace{1cm} (50)

Furthermore, the orthogonality relation $\langle\langle Q,q|Q',q'\rangle\rangle = \delta(Q-Q')\delta(q-q')$, cf. Eq. (24), implies:

$$\langle\langle P,p|Q,q\rangle\rangle = \frac{1}{2\pi\hbar} \exp\left( -\frac{i}{\hbar} (PQ - pq) \right) .$$  \hspace{1cm} (51)
Then, with all following steps of the derivation in parallel with the usual ones in quantum mechanics, it is straightforward to obtain in the present case:

\[ G(Q_f, q_f; t_f|Q_i, q_i; t_0) = \int DQDq \exp \left( \frac{i}{\hbar} S[\dot{Q}, Q; \dot{q}, q] \right), \tag{52} \]

with the boundary conditions \( Q(t_i) = Q_i, q(t_i) = q_i, Q(t_f) = Q_f, \) and \( q(t_f) = q_f, \)
and where the superaction \( S, \) corresponding to the super-Hamiltonian \( H' \) above, is defined as follows:

\[ S \equiv \int_{t_0}^{t_f} d\tau \left( T(\dot{Q}, \dot{q}) - V(Q, q) \right) \tag{53} \]

\[ := \int_{t_0}^{t_f} d\tau \left( \frac{m^2}{2} \dot{Q}^2 - V(Q) - \left( \frac{m^2}{2} \dot{q}^2 - V(q) \right) - \mathcal{E}(Q, q) \right). \tag{54} \]

We recall that \( \mathcal{E} \equiv 0 \) corresponds to evolution according to the von Neumann equation, whereas \( \mathcal{E} \neq 0 \) represents classical dynamics in accordance with the Liouville equation, cf. Eqs. (5)–(7).

However simple this result may seem, the Eqs. (52)–(54) present a new approach to describe time evolution of the full density matrix, with the particular feature that classical and quantum mechanical motion are formally treated in parallel, differing only in the action entering the phase in the path integral.

We emphasize that our derivations are not confined to one-dimensional or single-particle physics, but can be extended as well all the way to relativistic field theories. Various applications come to mind here, some of which will be discussed in the following and in the concluding section.

4.3. Perturbation theory and superpropagator Dyson equation

Considering the splitting of the superaction as in Eq. (53), the perturbation theory naturally departs from organizing contributions to the full superpropagator, Eq. (52), according to powers of the “perturbation” \( V. \) Sometimes it may be advantageous to include parts of the perturbation into the “free” part \( T. \) This must be familiar from quantum mechanics, which presents a special case of our general considerations here.

To begin with, if \( V(Q, q) \equiv V(Q) - V(q), \) corresponding to the superoperator related to the von Neumann equation, then the path integrals in Eq. (52) factorize and we recover quantum mechanics.

In the absence of an external potential or other interactions \( (V \equiv 0), \) the zeroth

\( ^{\text{We remark that a complementary approach, based on the effective action generating equal-time correlation functions for nonequilibrium statistical systems, has been presented in Ref. 22, which results in evolution equations for a truncated set of correlation functions.}} \)
order or free superpropagator $G_0$ is obtained as:

$$G_0(Q_f, q_f; t(Q_1, q_1; t_0) = \int DQ Dq \exp \left( \frac{i}{\hbar} \int_{t_0}^{t} d\tau \, T(\dot{Q}, \dot{q}) \right)$$

$$= G_0(Q_f, t; Q_1, t_0) G_0^*(q_f, t; q_1, t_0),$$

in terms of the well known free quantum mechanical propagator $G_0$, cf. Eqs. (42–43), which is explicitly given by (16):

$$G_0(x, t; y, t_0) \equiv G_0(x, y; T) := \left( \frac{m}{2\pi i \hbar T} \right)^{1/2} \exp \left( \frac{i m}{2 \hbar T} (x - y)^2 \right),$$

for a free nonrelativistic particle of mass $m$. Remarkably, this zeroth order result is identical for classical and quantum mechanical propagation.

Despite the fact that the free propagator for the Schrödinger equation incorporates such phenomena as the quantum mechanical spreading of a wave packet, we learn here that it also describes the propagation of a free classical particle. It is straightforward to verify – following the transformations between Eqs. (1) and Eq. (7) – that a massive particle, initialized as $\rho(x, p, t_0) := 2\pi \delta(x - x_0) \delta(p - p_0)$ is propagated to $\rho(x, p, t) := 2\pi \delta(x - x_0 - T p/m) \delta(p - p_0)$, as expected.

For the perturbative expansion, we employ the standard formula:

$$\exp \left( -\frac{i}{\hbar} \int_0^t d\tau \, \mathcal{V}(Q, q) \right) = \sum_{n=0}^{\infty} \frac{1}{n!} \left( -\frac{i}{\hbar} \int_0^t d\tau \, \mathcal{V}(Q, q) \right)^n$$

$$= \sum_{n=0}^{\infty} \left( \frac{-i}{\hbar} \right)^n \int_0^t d\tau_1 \mathcal{V}(Q, q) \int_0^{\tau_1} d\tau_2 \mathcal{V}(Q, q) \cdots \int_0^{\tau_{n-1}} d\tau_n \mathcal{V}(Q, q) \tau_n,$$

with $\mathcal{V}(Q, q) \tau_n := \mathcal{V}(Q(\tau_n), q(\tau_n))$.

In order to analyze such terms at a given order, we make use of the important semigroup property of the (free) propagator and obtain to first order in the perturbation (21)

$$\mathcal{G}(Q, q; t|Q', q'; t_0) = \mathcal{G}_0(Q, q; t|Q', q'; t_0)$$

$$- \frac{i}{\hbar} \int_{t_0}^{t} d\tau \int dx dy \, \mathcal{G}_0(Q, q; t|x, y; \tau) \mathcal{V}(x, y) \mathcal{G}_0(x, y; \tau|Q', q'; t_0) + O(V^2),$$

(60)

to be illustrated explicitly by the result for an anharmonic potential shortly.

We remark that on the right-hand side of Eq. (60) the superpotential is preceded (and followed) by a zeroth order propagator. This observation, which similarly holds at every order of this expansion, leads to a recursion relation of the $k$-th order propagator in terms of the $(k - 1)$-th order one. This allows us to resum the perturbation series in the form of a Dyson integral equation for the full superpropagator:

$$\mathcal{G}(Q, q; t|Q', q'; t_0) = \mathcal{G}_0(Q, q; t|Q', q'; t_0)$$

$$- \frac{i}{\hbar} \int_{t_0}^{t} d\tau \int dx dy \, \mathcal{G}_0(Q, q; t|x, y; \tau) \mathcal{V}(x, y) \mathcal{G}(x, y; \tau|Q', q'; t_0),$$

(61)
The whole procedure follows the usual one in quantum mechanics, yet includes the case of classical mechanics, and possibly others, for a suitably chosen superaction.

4.4. Illustration: the case of an anharmonic potential

In order to make our general derivations more concrete and to extract some interesting general aspects, it may be useful to consider the example of a massive particle in an anharmonic potential, \( V(x) := \lambda x^4 \), where \( \lambda \) is the coupling constant. We recall that for constant, linear, or harmonic coupling terms there is no difference between classical and quantum dynamics, cf. (8), in the representation that we have developed in this article.

The calculations evaluating the superpropagator to first order, here with:

\[
V(x,y) \equiv \frac{x + y}{2} = \frac{\lambda}{2} (x^4 - y^4 + 2(x^3y - xy^3)) ,
\]

for classical dynamics \( V(x,y) \equiv V(x) - V(y) \) for quantum mechanics) consist in straightforward (if tedious) multiple Gaussian integrals, according to Eqs. (55)–(57) and Eq. (60). The final result is:

\[
G(Q,q; t | Q',q'; t_0) = G_0(Q,q; t | Q',q'; t_0) \cdot \left( 1 - \frac{i}{\hbar} \lambda \left[ C_1 \Gamma_{QM}(Q,q; Q',q'; T) + C_2 \Gamma_{CL}(Q,q; Q',q'; T) \right] \right) + O(\lambda^2) ,
\]

(63)

where \( T := t - t_0 \), the coefficients for classical dynamics, \( C_1 := 1/2, C_2 := 1/2 \) (\( C_1 := 1, C_2 := 0 \) for quantum mechanics), and with the function:

\[
\Gamma_{QM}(Q,q; Q',q'; T) := \frac{T}{5} \left[ 1 + \frac{i}{\hbar} \lambda \left[ C_1 \Gamma_{QM}(Q,q; Q',q'; T) + C_2 \Gamma_{CL}(Q,q; Q',q'; T) \right] \right] + O(\lambda^2) ,
\]

(64)

\[
\Gamma_{CL}(Q,q; Q',q'; T) := \frac{T}{5} \left( \frac{i}{\hbar} \lambda \left[ C_1 \Gamma_{QM}(Q,q; Q',q'; T) + C_2 \Gamma_{CL}(Q,q; Q',q'; T) \right] \right) + O(\lambda^2) ,
\]

(65)

This result shows several interesting features. First of all, the perturbative expansion turns out to be a short-time expansion, with the overall scale of the first order correction set by \( \lambda T \). Furthermore, different contributing terms differ by a scale set by \( T/m \), i.e., by \( T \times \) Compton wavelength of the particle. Numerical studies visualizing the outcome here are presently underway [21].

However, most interesting seem general similarities and differences between classical ("CL") and quantum mechanical ("QM") result in Eq. (63). The CL result has
the same zeroth order term as QM; at first order, CL has one term in common with QM which, however, is reduced by an overall factor $1/2$. This obviously stems from the varied expressions for $V$ between CL and QM, cf. Eq. (62). For the same reason, CL has additional terms, collected in $\Gamma_{\text{CL}}$, Eq. (65), which are absent in QM.

4.5. Intra- and inter-space entanglement

There is a qualitative difference between CL and QM, contained in $\Gamma_{\text{CL}}$ and based on the different superoperators that enter the full path integral, Eqs. (52)–(54). Equivalently, since the QM evolution is generated by a commutator of the Hamiltonian with the density operator $\hat{\rho}$, it superposes and, for multi-partite systems, generally, entangles underlying bra- and ket-states separately,

$$\propto H_{ij} \rho_{jk} - \rho_{ij} H_{jk}.$$  

For a bi-partite system, it is revealing to write such terms more clearly as:

$$[\hat{H}_{\text{int}}, \hat{\rho}] = \hat{H}_1 \hat{\rho}_1 \otimes \hat{H}_2 \hat{\rho}_2 - \hat{\rho}_1 \hat{H}_1 \otimes \hat{\rho}_2 \hat{H}_2 , \quad (66)$$

for an interaction $\propto \hat{H}_1 \otimes \hat{H}_2$, with the factors acting on subsystems “1” and “2”, respectively, and where $\hat{\rho} = \hat{\rho}_1 \otimes \hat{\rho}_2$, for a separable initial state. This has been called dynamically assisted entanglement generation, see, for example, Refs. 23, 24, 25.

It may come as a surprise that the CL evolution does this just as well, due to the contribution of $\Gamma_{\text{QM}}$ for the first two terms on the right-hand side of Eq. (62) or, generally, due to the superoperator $\mathcal{E}$ of our earlier considerations. For polynomial interactions, for example, this superoperator always contains a contribution proportional to the usual QM terms.

However, the CL evolution produces additional correlations in $\hat{\rho}$, due to the generator $\propto L_{ijkl} \rho_{kl}$, which possibly entangles bra- and ket-states. – In comparison with Eq. (66), for example, such terms can have the unfamiliar structure:

$$\hat{H}'_1 \hat{\rho}_1 \otimes \hat{\rho}_2 \hat{H}'_2 - \hat{\rho}_1 \hat{H}'_1 \otimes \hat{\rho}_2 \hat{H}'_2 , \quad (67)$$

which differs decidedly from a commutator. – This leads us to distinguish intra- (i.e., within given tensor product Hilbert space of subsystems “1” and “2”) and inter-space entanglement (i.e., between said Hilbert space and its dual).

For example, consider the anharmonic potential $V(x_1 - x_2) := \lambda (x_1 - x_2)^4$ for a bi-partite system consisting of particles “1” and “2”. Following and suitably generalizing our derivation in Section 2, this leads to the interaction:

$$V(Q_1, Q_2; q_1, q_2) = \frac{1}{2} \lambda (Q_1 - q_1 - (Q_2 - q_2))(Q_1 + q_1 - (Q_2 + q_2))^3 , \quad (68)$$

in terms of variables introduced previously, taking into account both subsystems; similarly as before, the $Q$ and $q$ variables refer to bra- and ket-states, respectively. Besides the separable terms, $\propto (Q_a - q_a)(Q_a + q_a)^3, \ a = 1, 2$, there are the terms which mix (and entangle) variables of both subsystems, as usual in QM. However, there are clearly additional terms that refer to Hilbert space and its dual simultaneously (and entangle corresponding states), for example, $\propto Q_a Q_b q_a^2, \ b \neq a.$
In retrospect, somehow, such difference between CL and QM evolution had to be expected: instead with superstates $|Q,q\rangle$, we could have worked with superstates $|x,p\rangle$, relating to coordinates and momenta of the classical theory. There, coordinates and momenta end up tightly correlated, due to Hamilton’s equations, and produce inter-space entanglement in an interacting bi-partite system.

Thus, we find that the confrontation of CL with QM, as in our side-by-side study, is quite revealing. In particular, we speculate that this opens new views on generating entanglement in multipartite systems, perhaps, by evolving through quasiclassical stages or by making use of decohered intermediary states. Concerning the quantum-classical divide, the present analysis shows that there is a deep formal similarity between CL and QM. However, this also demonstrates that what has been discussed in various ways as CL limit of QM – and which is similarly relevant for “emergent QM” – deserves more study.

While our work has been concerned mainly with the evolution of CL or QM objects, we recall that V.I. Man’ko and collaborators have pointed out that classical states may differ widely from what could be obtained as the “$\hbar \to 0$” limit of quantum mechanical ones. They show that all states can be classified by their ‘tomograms’ as either CL or QM, CL and QM, and neither CL nor CM.

The classical limit might be a “For All Practical Purposes” limit, gradually approached through decoherence or “$\hbar \to 0$”. However, in order to bridge (if at all) the qualitative difference between intra- and inter-space entanglement that we find, and explain the “Man’ko classes of states”, some unknown dynamics beneath still awaits to be uncovered.

5. The almost classical Jaynes-Cummings model

In this section, we apply our operator approach for the Liouville equation to a field theory, namely to a Rydberg atom interacting with the electromagnetic field. Following the approximations that lead to the quantum mechanical Jaynes-Cummings model, we will show that the dynamics of this celebrated model is almost of classical character. As we shall see, if it were not for the anharmonic Coulomb interaction between electron and atom, the dynamics would be entirely classical.

---

6Previous considerations of the semiclassical regime, such as in Refs. 23, 24, were motivated as suitable approximations of the quantum mechanical evolution, in particular, for studies of the different decoherence properties between classically regular and chaotic systems. Our results seem to show that crossing the quantum-classical divide may offer an additional resource for entanglement generation and related “truly quantum” phenomena. This might be related to the “underlying reality” of (CL and QM) physics, assumed in Refs. 26, consisting in statistical correlations.

8A simple attractor model, motivated by assumptions about effects of fundamental spacetime discreteness, has been discussed in Ref. 20.
5.1. The classical model

We consider an electron (mass $m$) interacting electromagnetically with a positive charge (atom) fixed at the origin and with the radiation field. Thus, we depart from the classical Lagrangian:

$$L := \frac{m}{2} \dot{x}^2 + \int d^3r \left\{ \frac{1}{8\pi} (E^2 - B^2) - \rho \varphi + J \cdot A \right\}, \quad (69)$$

where the electric and magnetic fields, respectively, are given by:

$$E = -\dot{A} - \nabla \varphi, \quad B = \nabla \times A, \quad (70)$$
as usual, in terms of vector and scalar potential, $A$ and $\varphi$, respectively. The charge and current densities, $\rho$ and $J$, respectively, are given by:

$$\rho(r) = -e \delta^3(r - x) + \delta^3(r), \quad J(r) = -e \dot{x} \delta^3(r - x). \quad (71)$$

Next, we introduce Fourier modes of the fields, with $A(k) = A^*(-k)$ and $\varphi(k) = \phi^*(-k)$, since the fields are real. We choose the Coulomb gauge by imposing $A_{\parallel}(k) = 0$, which implies $\nabla \cdot A = 0$. Correspondingly rewriting the Lagrangian, we determine the canonical momenta, in order to obtain the Hamiltonian of the classical model:

$$H = \frac{1}{2m} \left( p + e \int d^3k \left\{ A(k)e^{ik\cdot x} + A^*(k)e^{-ik\cdot x} \right\} \right)^2 - \frac{e^2}{|x|}$$
$$+ \frac{1}{8\pi} \int d^3k \left\{ \Pi^*(k) \cdot \Pi(k) + k^2 A^*(k) \cdot A(k) \right\} \quad (72)$$
$$\equiv H(x, p; A, A^*; \Pi, \Pi), \quad (73)$$

where we indicate the canonically conjugated pairs of variables of the Hamiltonian; the momentum integrations have to take into account that not all Fourier modes are independent, for real fields.

We are now in the position, cf. Section 2, to describe this model in phase space. We proceed in four steps:

- First, we introduce the probability density (over phase space) $\rho(x, p; A, \Pi^*; A^*, \Pi)$, which will be interpreted, as before, as matrix element of a Hermitean density operator $\hat{\rho}$. We assume that the atom-electromagnetic-field system is confined to a cavity of finite volume $V$, thus replacing integrals by discrete mode sums, $\int d^3k g(k) \to V^{-1} \sum_k g_k$.
- Second, we obtain the Liouville equation, $-\partial_t \rho = \{ H, \rho \} = \ldots$, evaluating the relevant Poisson bracket.
- Third, we replace momenta by coordinates via Fourier transformation, $p \to y$, $\Pi_k^* \to B_k$, $\Pi_k \to B_k^*$. 
- Fourth, we perform the "Wigner rotations", $Q := x + y/2$, $q := x - y/2$, $Q_k := A_k + B_k/2$, and $q_k := A_k - B_k/2$.

Details and the following derivations will be reported elsewhere.\cite{30}
Introducing the following notation:

\[ V(\chi) := -\frac{e^2}{|\chi|}, \quad \text{for } \chi = Q, q ; \quad E(Q, q) := 4e^2\frac{Q^2 - q^2}{|Q + q|^3} - V(Q) + V(q), \quad (74) \]

where we suppress constant normalization factors etc., as before, we obtain the remarkable result that the classical evolution equation is:

\[ i\partial_t \rho = \left\{ \text{"von Neumann"} + E + \Gamma + \Sigma \right\} \rho \quad (75) \]

\[ \equiv \left\{ -\frac{1}{2m} \partial_Q^2 + V(Q) - \left( -\frac{1}{2m} \partial_q^2 + V(q) \right) \right. \\
+ \frac{1}{8\pi} \sum_k \left[ -\partial_{Q_k} \cdot \partial_{Q_k} + \omega_k^2 Q_k \cdot Q_k' - \left( -\partial_{q_k} \cdot \partial_{q_k} + \omega_k^2 q_k \cdot q_k' \right) \right] \\
- ie \sum_k \left[ e^{ikQ_k} \partial_Q + e^{iq_k} q_k \cdot \partial_q \right] \\
+ \frac{e^2}{2m} \sum_{k,k'} \left[ Q_k \cdot Q_k' e^{i(k-k') \cdot Q} - q_k \cdot q_k' e^{i(k-k') \cdot q} \right] \rho \\
\left. + \left\{ E(Q, q) + \Gamma + \Sigma \right\} \rho \right\} \quad (76) \]

with \( \rho \equiv \rho(Q, Q_k, Q_k', q, q_k, q_k'; t) \), where \( k \) runs over all modes, \( \omega_k := |k| \), and where \( \Gamma \) and \( \Sigma \) denote rather complicated terms that involve all phase space variables; they are given explicitly in Ref. 30. While the last line of Eq. (76) presents additional terms, in particular the superoperator \( E \), the previous terms represent exactly the terms of the quantum mechanical von Neumann equation for the atom-field system under consideration; besides further interaction terms, due to minimal coupling, we find the contribution of the free electromagnetic field in the second and those of the electron interacting with the Coulomb potential of the Rydberg atom in the first line, respectively.

We anticipate that in the dipole approximation we have \( \Gamma, \Sigma \to 0 \). Therefore, we do not study further the impact of those terms here 30.

Instead, we recall the well known additional approximations that turn the von Neumann terms of Eqs. (75)–(76) into those of the Jaynes-Cummings model 17:

- The dipole approximation, assuming that \( \tilde{k} \cdot l \ll 1 \), where \( \tilde{k} \) and \( l \) denote a typical photon momentum and linear size of a Rydberg electron orbit, respectively.
- The restriction to one cavity photon mode with energy \( \hbar \omega \). This yields the approximate Hamilton operator 30:

\[ \hat{H} = \sum_i \omega_i |i\rangle \langle i| + \omega (\hat{a}^\dagger \hat{a} + \frac{1}{2}) + i \sum_{i \neq j} d_{ij} (\hat{a} - \hat{a}^\dagger) |i\rangle \langle j| \]

where the sums run over the Rydberg levels, with energies \( \hbar \omega_i \), \( \hat{a}^{(1)} \) are photon annihilation (creation) operators, and where the last term involves...
the dipole transition amplitudes $d_{ij}$.

- The restriction to a two-level subspace, spanned by states $|g\rangle, |e\rangle$: the lower level energy is conveniently set to $\hbar \omega_g = 0$, while the physical realizations considered, usually, have $\omega_e \approx \omega$, i.e., approximately resonant photon and excited electron states.

- The rotating wave approximation, which yields the 'energy conserving' dipole interaction term $\hat{D} \propto \hat{a}|e\rangle \langle g| - |g\rangle \langle e| \hat{a}^\dagger$.

The resulting Jaynes-Cummings Hamiltonian is:

$$\hat{H}_{JC} = \omega_e |e\rangle \langle e| + \omega (\hat{a}^\dagger \hat{a} + \frac{1}{2}) + i d_{eg} (\hat{a}|e\rangle \langle g| - |g\rangle \langle e| \hat{a}^\dagger) .$$

(77)

Then, following the above derivation, the evolution equation becomes:

$$i \hat{D} = [\hat{H}_{JC}, \hat{\rho}] + \hat{E} \hat{\rho} ,$$

(78)

where we appropriately incorporated here the superoperator $\hat{E}$. This term presents the only difference between the classical dynamics described by Eqs. (75)–(76) and the usual quantum mechanical one.

Thus, we find in this 'standard model' of quantum optics a detailed example for the similarity between CL and QM evolution laws.

5.2. Dipole interaction and Coulomb superoperator as perturbations

In order to illustrate our findings, we will briefly study the influence of the classical superoperator on the evolution described by the Jaynes-Cummings model in perturbation theory, while a more complete analysis will be presented in Ref. [30].

Following Section 4.3, cf. Eq. (61), we presently treat the dipole interaction, $\hat{D} \propto \hat{a}|e\rangle \langle g| - |g\rangle \langle e| \hat{a}^\dagger$, together with the superoperator $\hat{E}$ of Eq. (78) as perturbation. Correspondingly, we choose the Rydberg atom states and one-mode photon number states for the tensor product basis of the relevant Hilbert space.

Then, the density matrix evolves according to:

$$\hat{\rho}(t) = \hat{G}(t) \hat{\rho}(0) = \hat{G}_0(t) \hat{\rho}(0) - i \int_0^t d\tau \hat{G}_0(t - \tau) (\hat{D} + \hat{E}) \hat{G}_0(\tau) \hat{\rho}(0) ,$$

(79)

to first order in $\hat{D} + \hat{E}$. While the dipole operator acts on atom and electromagnetic field states simultaneously, the superoperator acts only on the Rydberg states. Hence, the matrix elements of $\hat{E}$ are defined by:

$$E_{ab,cd} := \int d^3q \psi^*_a(Q)\psi_b(q)\mathcal{E}(Q, q)\psi_c(Q)\psi^*_d(q) ,$$

(80)

where $\psi \in (n, l, m)$ denote standard hydrogen-like wave functions. Since these are eigenstates of parity, $P_i = \pm 1$, we find $E_{ab,cd} = P_a P_b P_c P_d E_{ab,cd}$, which implies the selection rule: $E_{ab,cd} = 0$, if $P_a P_b P_c P_d = -1$. Furthermore, we have: $E_{ab,cd} = -E_{ba,dc}$. 

Consequently, the only nonzero matrix elements are $\mathcal{E}_{eg,eg}$, $\mathcal{E}_{ge,ge}$, $\mathcal{E}_{ee,gg}$, and $\mathcal{E}_{gg,ee}$. The latter two vanish for the specific ground and excited states used in cavity QED experiments.

Taking matrix elements of Eq. (79), we find that the superoperator only affects the evolution of the matrix elements $\rho_{eg|nn'}$ and $\rho_{ge|nn'} = \rho^*_{eg|nn'}$ (by hermiticity); here the Fock states are labelled by photon numbers $n, n'$. Then, we find:

$$\rho_{eg|nn'}(t) = e^{-it[\omega_e - \omega_g + \omega(n-n')]}
\left[ 1 + it\mathcal{E}_{eg,eg}\rho_{eg|nn'}(0) + d_{eg}t\left( |n \rangle \langle n| \rho_{gg}(0) \rho_{n+1 | n'}(0) - |n' \rangle \langle n'| \rho_{ee}(0) \rho_{n | n-1}(0) \right) \right],$$

(81)

where we assume that the initial state factorizes, for simplicity.

Thus, to first order in this perturbative expansion, we find that the superoperator competes with the dipole interaction, as far as the atom states are concerned; however, it does so without affecting the field states. Numerical estimates indicate that its matrix elements are not small compared to the ones of the dipole operator for cavity QED experiments (see Ref. 30 for further details).

To summarize, almost all dynamical (operator) features of the Jaynes-Cummings model can be derived in the classical framework, as we have shown.

Nevertheless, there is a noticeable difference between classical and quantum evolution in the version of the Jaynes-Cummings model that is related to cavity QED experiments. This is solely due to the classical superoperator, which stems from the Coulomb interaction between electron and Rydberg ion.

In distinction, had we considered a charged particle trapped by a linear or harmonic potential, then the superoperator would vanish identically, cf. Section 2, Eq. (8). The correspondingly modified Jaynes-Cummings model could be seen as of entirely classical origin, despite its quantum mechanical appearance.

6. Conclusions

Beginning with the Liouville equation of classical statistical mechanics, we have introduced a (super)operator formulation which brings it as close as possible to the von Neumann equation of quantum mechanics, provided suitable coordinates are chosen in superspace. Presently, we have concentrated on the similarities and differences between both evolution equations.

We have chosen the Jaynes-Cummings model, in particular, to illustrate both aspects and to show that this benchmark model of quantum optics and cavity QED can be interpreted to a large extent in terms of classical dynamics. Furthermore, this model serves as an example that our more general considerations apply not only to single- or few-particle systems, but to field theories as well.

While presently the relevant Hilbert space has been treated as tensor product space of the Rydberg electron single-particle and the photon Fock space, earlier also a functional approach combining fermion and boson fields has been discussed.

More generally, we discussed in parallel the formal solutions of the Liouville and
the von Neumann equations. Introducing suitable propagators, we derived a path
integral representation for both cases side by side.\textsuperscript{21}

The path integral for the propagator of the Liouville equation is new and may
have interesting applications in classical physics. We derived and illustrated the
related perturbation theory.

We discussed how these results and the action entering this path integral, in
particular, hint at the possibility that a form of entanglement is also generated
by classical dynamics, which has gone unnoticed before. It combines the quantum
mechanical dynamically assisted entanglement generation\textsuperscript{23,24,25} with a classical
counterpart. We call the former \textit{intra-space entanglement}, since it acts separately
within the Hilbert spaces of bra- and ket-states. In distinction, the classical dyna-
mics additionally produces \textit{inter-space entanglement}, i.e., it correlates the Hilbert
space and its adjoint in addition to what would, otherwise, be recognized as quan-
tum entanglement.

If the relative strength of intra- and inter-space entanglement can be manipu-
lated, for example, by driving a system dynamically between quantum and classical
behaviour, this may open additional ways to influence the quantum mechanical
entanglement, which is of central importance in research concerning quantum in-
formation and quantum foundations alike.

The close relation between classical and quantum mechanical dynamics that
we uncovered may help to address in new ways problems related to the nature of
classical or quantum states\textsuperscript{20,28,29}, to the pathways, if any, over the quantum-
classical divide\textsuperscript{1,12,4,5,9}, or the measurement problem\textsuperscript{3,7}.

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