SINGULARITIES OF SCHRÖDER MAPS AND
UNHYPERBOLICITY OF RATIONAL FUNCTIONS

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Dedicated to Professor Walter K. Hayman on his eightieth birthday

Abstract. We study transcendental singularities of a Schröder
map arising from a rational function $f$, using results from complex
dynamics and Nevanlinna theory. These maps are transcendental
meromorphic functions of finite order in the complex plane. We
show that their transcendental singularities lie over the set where
$f$ is not semihyperbolic (unhyperbolic). In addition, if they are
direct, then they lie over only attracting periodic points of $f$, and
moreover, if $f$ is a polynomial, then both direct and indirect sin-
gularities lie over attracting, parabolic and Cremer periodic points
of $f$. We also obtain concrete examples of both kinds of transcen-
dental singularities of Schröder maps as well as a new proof of
the Pommerenke-Levin-Yoccoz inequality and a new formulat-
on of the Fatou conjecture.

1. Introduction

Let $f$ be a rational function on $\hat{\mathbb{C}}$ of degree $d = \deg f \geq 2$, i.e.,
the critical set $C(f) := \{f'(c) = 0\} \neq \emptyset$. Denote its $k$-th iterate
($k \in \mathbb{N} \cup \{0\}$) by $f^k$. For details of complex dynamics, see, for example,
[14], [15], [20]. For every repelling periodic point $z_0$ of $f$ of period $p$,
there exists a unique meromorphic map $h$ on $\mathbb{C}$, which is called the
Schröder map of $f$ at $z_0$, such that $h(0) = z_0$, $h'(0) = 1$ and
\begin{equation}
  f^p \circ h = h \circ \lambda
\end{equation}
on $\mathbb{C}$. Here the multiplier $\lambda := (f^p)'(z_0) (|\lambda| > 1)$ also denotes multipli-
cation by $\lambda$ on $\mathbb{C}$. Using complex dynamics and Nevanlinna theory, we
study the relationship between singularities of Schröder maps $h$ and the
unhyperbolicity of $f$. Following Carleson-Jones-Yoccoz [4], we say that
$f$ is not semihyperbolic or, more conveniently, unhyperbolic at $a \in \hat{\mathbb{C}}$ if

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ity, complex dynamics, Nevanlinna theory, Pommerenke-Levin-Yoccoz inequality,
Fatou conjecture.
for every open neighborhood $U$ of $a$,
\begin{equation}
\sup_{k \in \mathbb{N}} \max_{V^{-k}} \deg(f^k : V^{-k} \to U) = \infty,
\end{equation}
where $V^{-k}$ ranges over all components of $f^{-k}(U)$. We denote by $UH(f)$ (unhyperbolic) the set of all such $a \in \hat{C}$.

**Notation 1.3.** $U_r(a)$ is the spherical open disk centered at $a \in \hat{C}$ and of radius $r > 0$. Let $F(f)$ and $J(f)$ be the Fatou and Julia sets of $f$, respectively, and let $AT(f), PB(f)$ and $CM(f)$ be the attracting, parabolic and Cremer periodic points of $f$, respectively.

If $g$ is transcendental meromorphic on $\mathbb{C}$, we can consider more general singularities than its critical set $C(g)$: let $\mathcal{N}$ be the set of decreasing families $\mathcal{A} = \{A_r\}_{r > 0} \subset 2^C$, so that $A_r \subset A_r$ if $s < r$. Let $TS(g) \subset \mathcal{N}$ be the set of $\mathcal{A} \in \mathcal{N}$ such that there exists (the unique) $\gamma = a_\mathcal{A} \in \hat{C}$ such that for every $r > 0$, $A_r$ is a component of $g^{-1}(U_r(a))$ and in addition that $\bigcap_{r > 0} A_r = \emptyset$ (cf. [3]). Each $\mathcal{A} \in TS(g)$ is called a transcendental singularity of $g$, and we extend $g$ to the map from $\mathbb{C} \cup TS(g)$ to $\hat{C}$ by setting $g(\mathcal{A}) := a$ for $\mathcal{A} \in TS(g)$. Following terminology due to Iversen [3], $\mathcal{A}$ is said to be direct if the point $g(\mathcal{A})$ is not contained in $g(A_r)$ for some $r > 0$, and indirect otherwise.

For a sequence $(z_k) \subset \mathbb{C}$, we say that $z_k \to A$ as $k \to \infty$ if for each $r > 0$, $z_k \in A_r$ for all large $k \in \mathbb{N}$. Similarly, for an arc $\gamma : (-\infty, \infty) \to \mathbb{C}$, we say that $\gamma(t) \to A$ as $t \to \infty$ if for each $r > 0$, $\gamma(t) \in A_r$ for all large $t > 0$. We call $\gamma$ an asymptotic arc of $g$ if $\lim_{t \to \infty} \gamma(t) = \infty$ and $\lim_{t \to \infty} g(\gamma(t)) \in \hat{C}$ exists. For every asymptotic arc $\gamma$ of $g$, there is the (unique) $\mathcal{A} \in TS(g)$ associated with $\gamma$, so that $\gamma(t) \to \mathcal{A}$ as $t \to \infty$. Conversely, for every $\mathcal{A} \in TS(g)$, there is an asymptotic arc $\gamma$ of $g$ to which $\mathcal{A}$ is associated; indeed many such arcs.

**Remarks.** This definition of $\mathcal{A}$ slightly modifies the classical one (eg. in [16], §XI], where $g(\mathcal{A})$ is called a transcendental singularity of $g^{-1}$. In the study of entire-meromorphic maps, the term asymptotic curve is more common than asymptotic arc, but we prefer the latter since it seems more in keeping with usage in dynamics.

For $\mathcal{A} = \{A_r\}, \mathcal{B} = \{B_r\} \in \mathcal{N}$, we say $\mathcal{A} \sim \mathcal{B}$ if
- for every $r > 0$, there exists $s > 0$ such that $A_s \subset B_r$,
- for every $r > 0$, there exists $s > 0$ such that $B_s \subset A_r$.
This defines an equivalence relation on $\mathcal{N}$.

When $g$ is a Schröder map $h$ as (1.1), we call the map $\Lambda = \Lambda_h$ below the natural extension of the multiplication action of $\lambda$ on $\mathbb{C}$ since from (1.5), we have $h \circ \Lambda = f^p \circ h$ on $TS(h)$.

**Theorem 1.4.** Let $f$ and $h$ be as above. Then there exists a map $\Lambda = \Lambda_h : TS(h) \to TS(h)$ such that for each $\mathcal{A} = \{A_r\} \in TS(h)$,
\begin{equation}
\Lambda \mathcal{A} \sim \{\lambda A_r\}_{r > 0}.
\end{equation}
The map $\Lambda$ is bijective and preserves the direct or indirect character of $A \in TS(h)$, i.e., $A$ is direct if and only if $\Lambda A$ is direct.

**Definition 1.6.** An $A \in TS(h)$ is periodic if it is periodic under $\Lambda_h$.

Theorem 1.4 is shown by a careful chase of the functional equation (1.1). For reader’s convenience, we include the proof in [3].

In some ways, this paper may be viewed as a continuation of [5], which studies the growth with $k$ of the proximity function $m(a, f^k)$ as $a$ varies in $\hat{C}$. Thus as in [5], we consider the omega-limit set

$$\omega_f(c) := \{z \in \hat{C}; \exists k_j \to \infty \text{ such that } \lim_{j \to \infty} f^{k_j}(c) = z\}$$

for each $c \in \hat{C}$, and define the Mañé set of $f$ as

$$M(f) := \bigcup_{c \in C(f) \cap J(f)} \omega_f(c).$$

We will recall in [2] that $CM(f) \subset M(f)$ and that $AT(f) \cup PB(f) \cup M(f)$ coincides with $UH(f)$.

One of our principal results is:

**Theorem 1.** Let $h$ be a Schröder map of the rational function $f$. Then

(1.7) \hspace{1cm} h(TS(h)) \subset AT(f) \cup PB(f) \cup M(f),

(1.8) \hspace{1cm} h(\{A \in TS(h); \text{periodic}\}) \subset AT(f) \cup PB(f) \cup CM(f),

(1.9) \hspace{1cm} h(\{A \in TS(h); \text{direct}\}) \subset AT(f).

In general, the inclusion (1.9) is proper. As an example, we have:

**Theorem 2.** Let $h$ be a Schröder map of the rational function $f$ at a repelling fixed point $z_0$ of $f$ of multiplier $\lambda$, let $D$ an immediate basin of $a \in AT(f) \cup PB(f)$, and suppose that a component $W$ of $h^{-1}(D)$ is periodic (under $\lambda$) in that $\lambda^N W = W$ for some $N \in \mathbb{N}$ (so $f^N(D) = D$). Then for every $w_0 \in W$, there is an asymptotic arc $\gamma : (-\infty, \infty) \to W$ of $h$ with $\gamma(0) = w_0$ and $\lim_{t \to -\infty} h(\gamma(t)) = a$ such that for every $t \in (-\infty, \infty)$,

(1.10) \hspace{1cm} \gamma(t + 1) = \lambda^N \gamma(t).

Suppose that $a \in AT(f)$. If $f^{-N}(a) \cap D \neq \{a\}$, then $W \cap h^{-1}(a) \neq \emptyset$. If $\gamma(0) = w_0 \in W \cap h^{-1}(a)$, then $A \in TS(h)$ associated with this $\gamma$ is indirect. If there exists an indirect $A = \{A_r\} \in TS(h)$ with $h(A) = a$ and $A_r \subset W$ for some $r > 0$, then $D \cap C(f^N) \neq \{a\}$.

Replacing $f$ by $f^n$ for an appropriate $n \in \mathbb{N}$, we may apply Theorem 2 to every immediate basin $D$ of $a \in AT(f) \cup PB(f)$, by a theorem of Przytycki-Zdunik [19, Theorem A] (see also Pommerenke [18, §2] when $D$ is simply connected): $\partial D$ contains a dense subset of repelling periodic points $z_0$ of $f$ accessible from $D$ along an arc $\Gamma = \Gamma_{z_0} : [0, 1] \to \partial D$ such
that $\Gamma(0) = z_0$, $\Gamma((0, 1]) \subset D$ and $\Gamma \subset f^n(\Gamma)$ for some $n \in \mathbb{N}$. We note that $z_0$ is fixed by $f^n$, and put $\lambda := (f^n)'(z_0)$. For all small $s > 0$, the Schröder map $h$ of $f^n$ at $z_0$ is univalent on $\{|w| < s\}$, and we can assume that $\Gamma \subset (f^n)(\Gamma) \subset h(\{|w| < s\})$. Choose $W$ as $(h(\{|w| < s\})^{-1}(\Gamma) \subset W$. Then we have $W = \lambda W$.

**Corollary 1.** For every $a \in AT(f) \cup PB(f)$ of the rational function $f$, there exists a Schröder map $h$ of $f$ with $a \in h(TS(h))$.

We apply these results to two concrete dynamical issues.

**Pommerenke-Levin-Yoccoz inequality.** Suppose that the rational function $f$ is a polynomial and that $p = 1$, i.e., $f(z_0) = z_0$. Then $h$ is an entire function of order $\rho = (\log d)/\log |\lambda|$ (see [2]). Let $D_\infty$ be the immediate basin of $\infty$, and let $q_\infty = q_\infty(h)$ be the number of components of $h^{-1}(D_\infty)$. Eremenko and Levin [6] proved that $q_\infty \leq 2\rho \vee 1$, that every component of $h^{-1}(D_\infty)$ is periodic under $\lambda$ and ([6, p. 1260]) that if $J_0$ is the component of $J(f)$ with $z_0 \in J_0$ and

\[(EL) \quad J_0 \neq \{z_0\}\]

(eg., if there are at least two components of $h^{-1}(D_\infty)$), then

\[q_\infty \leq 2\rho. \tag{1.11}\]

A spiral version of Denjoy's conjecture (see Theorem 6.2 below), which was considered by Ahlfors [1] and proved unambiguously by Hayman/Jenkins (cf. [7, Theorem 8.21], [9]) establishes a refinement of (EL): condition (EL) implies that for every asymptotic arc $\gamma : (-\infty, \infty) \to \mathbb{C}$ of $h$ with $\lim_{t \to \infty} h(\gamma(t)) = \infty$,

\[q_\infty \cdot \left(1 + \limsup_{t \to \infty} \frac{\arg \gamma(t)}{\log |\gamma(t)|}\right)^2 \leq 2\rho. \tag{1.12}\]

As a special case, (1.12) has a dynamical implication: for each component $W$ of $h^{-1}(D_\infty)$, let $q_W$ be the least $N \in \mathbb{N}$ such that $\lambda^N W = W$, and let $\gamma_W : (-\infty, \infty) \to W$ be an asymptotic arc of $h$ obtained by Theorem 2. Then by (1.10), for every $k \in \mathbb{N}$, we have

\[\gamma_W(k) = \lambda^{k-q_W} \gamma_W(0). \tag{1.13}\]

If (EL) holds, then we can define a single-valued branch $\arg_W(\cdot)$ of $\arg(\cdot)$ on $W$, and there exists a (unique) $p_W \in \{0, 1, \ldots, q_W - 1\}$ such that for some branch of $\arg \lambda$,

\[\arg_W(\lambda^{q_W} w) - \arg_W(w) = q_W \arg \lambda - 2\pi p_W. \tag{1.14}\]

for every $w \in W$. We can also show that both $q = q_W$ and $p = p_W$ are independent of $W$ (see the discussion of (6.3) below). Therefore,

**Corollary 2.** Let $h$ be a Schröder map of a polynomial $f$ of degree $d \geq 2$ at a repelling fixed point $z_0$ of $f$ having multiplier $\lambda$. Assume
and let \( q_\infty, p, q \) be as above. Then there is a branch of \( \arg \lambda \) so that
\[
q_\infty \cdot \left( 1 + \left( \frac{\arg \lambda - 2\pi p/q}{\log |\lambda|} \right)^2 \right) \leq 2\rho = 2 \log \frac{d}{\log |\lambda|}.
\]

Inequality (1.15) was shown by Pommerenke [18] and Levin [11] in somewhat weaker form, by Yoccoz (unpublished) in an equivalent form to (1.15) under the assumption \( J_0 = J(f) \), and by Jin [10] under (EL).

**Fatou conjecture.** We consider the unicritical polynomial family
\[
\{ f_c(z) = z^d + c; c \in \mathbb{C} \} \sim \mathbb{C},
\]
and note that \( C(f_c) \cap \mathbb{C} = \{ 0 \} \) while \( \infty \in AT(f_c) \). The Mandelbrot set and its hyperbolicity locus are defined as
\[
\mathcal{C} := \{ c \in \mathbb{C}; \lim_{k \to \infty} |f_c^k(0)| \neq \infty \}, \quad \mathcal{H} := \{ c \in \mathbb{C}; AT(f_c) \cap \mathbb{C} \neq \emptyset \}
\]
respectively. It is known that \( \mathcal{H} \) is an open and closed subset of \( \text{int} \mathcal{C} \) ([13, Theorem 4.4]).

We say that a covering selfmap \( g \) of \( \hat{\mathbb{C}} \), which is possibly ramified and not surjective, covers a point \( a \in \mathbb{C} \) completely if there exists \( r > 0 \) such that \( g^{-1}(U_r(a)) \) has no unbounded component; \( g \) itself is complete if it covers all \( a \in \mathbb{C} \) completely (cf. [2, I. 21A]).

**Corollary 3.** Let \( c \in \text{int} \mathcal{C} \). Then \( c \notin \mathcal{H} \) if and only if every Schröder map of \( f_c \) is a complete covering selfmap of \( \mathbb{C} \).

We remark that it has been expected for a long time that
\[
\text{int} \mathcal{C} = \mathcal{H};
\]
this is known as a Fatou conjecture (cf. [13, p. 58]). Perhaps our characterization of \( \mathcal{H} \) might be helpful in understanding this conjecture.

### 2. Dynamical and Nevanlinna-theoretic results

Let \( f \) be a rational function on \( \hat{\mathbb{C}} \) of degree \( d \geq 2 \).

**Mañé’s theorem and Siegel compacta.** Consider the set \( UH(f) \) in (1.2). By a standard argument (cf. [5, §2]), we have \( UH(f) \cap F(f) = AT(f) \) and \( UH(f) \cap J(f) \supset PB(f) \cup M(f) \), and Mañé’s theorem below sharpens the second containment to equality, so that
\[
UH(f) = AT(f) \cup PB(f) \cup M(f).
\]

**Theorem 2.1** ([12, Theorem II]). For every \( a \in J(f) \setminus (PB(f) \cup M(f)) \) and every \( \epsilon > 0 \), there exists an open neighborhood \( U \) of \( a \) such that
\[
\sup_{k \in \mathbb{N}} \max_{V^{-k}} \text{diam} V^{-k} \leq \epsilon, \quad \sup_{k \in \mathbb{N}} \max_{V^{-k}} \text{deg}(f^k : V^{-k} \to U) \leq d^{2d-2},
\]
(2.2)
\[
\lim_{k \to \infty} \max_{V^{-k}} \text{diam} V^{-k} = 0,
\]
where \( V^{-k} \) ranges over all components of \( f^{-k}(U) \).
We also use Pérèz-Marco’s theorem on indifferent fixed points.

**Theorem 2.3** ([17, Theorem 1]). Let \( \phi \) be an analytic germ at an indifferent fixed point \( x \in \hat{\mathbb{C}} \), which is univalent on an open set compactly containing a Jordan neighborhood \( U \subset \hat{\mathbb{C}} \) of \( x \). Then there exists a continuum \( K \subset \overline{U} \), which is called a Siegel compactum associated to \((\phi, U, x)\), such that \( x \in K = \phi(K) \not\subset U \) and \( \hat{\mathbb{C}} \setminus K \) is connected.

**Meromorphic maps of finite order.** Let \( g \) be a meromorphic map on \( \mathbb{C} \). The order of \( g \) is \( \rho = \rho_g := \limsup_{r \to \infty} (\log T(r, g))/\log r \in [0, +\infty) \) (cf. [16, p. 215]). When \( \rho < \infty \), as occurs here (compare (2.9) below), \( g \) is subject to two fundamental controls.

**Theorem 2.4** (Denjoy-Carleman-Ahlfors, cf. [16, p. 303, p. 307]).

\[
\# \{ A \in TS(g); A \text{ is direct} \} \leq 2\rho + 1.
\]

Moreover, if \( g \) is entire, then

\[
\# \{ A \in TS(g); g(A) \in \mathbb{C} \} \leq 2\rho.
\]

**Theorem 2.7** (Bergweiler-Eremenko [3, Theorem 1’]). If \( A = \{ A_r \} \in TS(g) \) is indirect, then there exists \( (c_k) \subset C(g) \setminus g^{-1}(g(A)) \) such that \( c_k \to A \) as \( k \to \infty \). In particular, \( g(A) \) is a derived point of \( g(C(g)) \).

We record one consequence of Theorem 2.4.

**Lemma 2.8.** Suppose that \( \rho_g < \infty \). If \( A = \{ A_r \} \in TS(g) \) is direct and \( r > 0 \) is small enough, then for every \( t \in (0, r) \), \( A_t \) is the only component of \( g^{-1}(U_t(g(A))) \) contained in \( A_r \).

**Proof.** Otherwise, there exists \( (r_j)_{j \in \mathbb{N}} \subset \mathbb{R}_{>0} \) decreasing to 0 such that for each \( j \in \mathbb{N} \), there is a component \( B_{r_{j+1}} \) of \( g^{-1}(U_{r_{j+1}}(g(A))) \) other than \( A_{r_{j+1}} \) and contained in \( A_{r_j} \). Since \( A \) is direct, we may assume that \( h(A_{r_j}) \not\subset g(A) \). Then for every \( j \geq 2 \), there exists \( B^j = \{ B^j_r \} \in TS(g) \) such that \( g(B^j) = g(A) \) and \( B^j_r \subset A_{r_{j-1}} \setminus A_{r_j} \), so that all \( B^j \) are not only direct but also mutually distinct. This contradicts (2.5). \( \square \)

**Schröder maps.** When \( g \) is a Schröder map \( h \) of \( f \) as in [11] Valiron calculated that

\[
(2.9) \quad \rho_h = \frac{\log d^p}{\log |\lambda|} < \infty
\]

(cf. [21, p. 160]), so those results may be applied to \( h \).

The next theorem seems well known. We sketch a proof for completeness.

**Theorem 2.10.** For every Schröder map \( h \) of \( f \) with \( p = 1 \),

\[
(2.11) \quad E(f) := \{ a \in \hat{\mathbb{C}}; f^{-2}(a) = \{ a \} \} = \{ a \in \hat{\mathbb{C}}; h^{-1}(a) = \emptyset \},
\]

\[
(2.12) \quad h(C(h)) = \bigcup_{k \in \mathbb{N}} f^k(C(f)) \setminus E(f).
\]
Proof. We recall (cf. [14, Lemma 4.9] or [15, Theorem 2.3.3]) that
\[ E(f) \subset C(f) \] and that
\[ (2.13) \quad E(f) = \{ a \in \hat{C}; \# \bigcup_{j \in \mathbb{N}} f^{-j}(a) < \infty \}. \]

Let \( E_P(h)^* \) be the last term in (2.11). Suppose that \( a \in \hat{C} \setminus E(f) \). Then by (2.13), \( \# \bigcup_{j \in \mathbb{N}} f^{-j}(a) = \infty (\geq 3) \), so that by Picard’s theorem, \( h(C) \cap \bigcup_{j \in \mathbb{N}} f^{-j}(a) \neq \emptyset \). This with (2.1) implies that \( h^{-1}(a) \neq \emptyset \), i.e., \( a \in \hat{C} \setminus E_P(h)^* \). Conversely, suppose that \( a \in \hat{C} \setminus E_P(h)^* \). Note that for every \( j \in \mathbb{N} \), by repeated use of (2.11),
\[ (2.14) \quad h^{-1}(f^{-j}(a)) = \lambda^{-j}(h^{-1}(a)). \]

If \( a \in E(f) \), then by the first equality in (2.11), we would have \( h^{-1}(a) = h^{-1}(f^{-2j}(a)) = \lambda^{-2j}(h^{-1}(a)) \) for every \( j \in \mathbb{N} \). Hence since \( |\lambda| > 1 \) and \( h \) is continuous at 0, we would have \( 0 \in h^{-1}(a) \), and \( z_0 = h(0) = a \in (E(f) \subset C(f) \)). This contradicts that \( f'(z_0) = \lambda \neq 0 \).

We have shown (2.11). We can show (2.12) by repeated use of (1.1), the chain rule and the fact that \( h'(w) \neq 0 \) if \( |w| \) is small enough. \( \square \)

Remark 2.15. We note another description of \( E(f) \):
\[ E(f) = E_P(h) := \{ a \in \hat{C}; \# h^{-1}(a) < \infty \}. \]

By (2.11), \( E(f) = E_P(h)^* \subset E_P(h) \). Conversely, suppose that \( a \in E_P(h) \). Then by (2.14), we have for every \( j \in \mathbb{N} \), \( \# h^{-1}(f^{-j}(a)) = \# h^{-1}(a) < \infty \). Hence \( \bigcup_{j \in \mathbb{N}} f^{-j}(a) \subset E_P(h) \), and \( \# E_P(h) \leq 2 \) by Picard’s theorem. Thus by (2.13), we have \( a \in E(f) \).

Since \( E(f^k) = E(f) \) for every \( k \in \mathbb{N} \), we have \( E(f) = E_P(h)^* = E_P(h) \) even when \( p \geq 2 \).

Corollary 2.16. Let \( h \) be a Schröder map of \( f \). Then

(i) if \( f \) is a polynomial, then \( h \) is entire;

(ii) every direct \( A \in TS(h) \) is periodic. If \( h \) is also entire, then every \( A \in TS(h) \) is periodic;

(iii) for every indirect \( A \in TS(h) \), \( h(A) \) is a derived point of the critical orbit \( \bigcup_{k \in \mathbb{N}} f^k(C(f)) \) of \( f \).

Proof. The assertion (i) follows from (2.11) with \( a = \infty \in E(f) \), (ii) from Theorems 1.4 and 2.4, and (iii) from (2.12) and Theorem 2.7. \( \square \)

3. Proof of Theorem 1.4

For a meromorphic map \( g \) on \( \mathbb{C} \), the following is straightforward: for \( \mathcal{A} = \{ A_r \}, \mathcal{B} = \{ B_r \} \in TS(g) \subset \mathfrak{M} \),
\[ (3.1) \quad \mathcal{A} = \mathcal{B} \iff \mathcal{A} \sim \mathcal{B}. \]

Replacing \( f^p \) by \( f \) if necessary, we assume that \( p = 1 \). Put \( \tilde{h} := h \circ \lambda^{-1} \). For every \( \mathcal{A} = \{ A_r \} \in TS(h) \), we have \( \{ \lambda A_r \} =: \tilde{\mathcal{A}} = \{ \tilde{h}^{-1}(a) \} \in TS(\tilde{h}) \).
\{\tilde{A}_r\} \in TS(\tilde{h}). We show that there exists the (unique) \(B = \{B_r\} \in TS(f \circ \tilde{h}) = TS(h)\) since \(f \circ \tilde{h} = h\) with \(B \sim \tilde{A}\): put \(a := h(\tilde{A}) (= h(A))\) and \(b := f(a)\). For every \(r > 0\), there exists \(s > 0\) such that \(U_s(a) \subset f^{-1}(U_r(b))\), and let \(B_r\) be the component of \(\tilde{h}^{-1}(f^{-1}(U_r(b)))\) with \(\tilde{A}_s \subset B_r\). Then we have \(B := \{B_r\} \in \mathcal{G}\). Conversely, for every \(r > 0\), if \(s > 0\) is small enough, then \((f(\tilde{h}(B_s)) \subset U_s(b) \subset f(U_r(a))\), so that \(\tilde{h}(B_s) \subset f^{-1}(f(U_r(a)))\). Since \(\tilde{A}_t \subset B_s \cap A_r\) for some \(t > 0\), we have \(\emptyset \neq \tilde{h}(B_s) \cap U_r(a)\). Hence if \(r > 0\) is so small that \(f : U_r(a) \to f(U_r(a))\) is proper, then \(\tilde{h}(B_s) \subset U_r(a)\), and so \((\emptyset \neq \tilde{A}_t \subset B_s \subset \tilde{A}_r\). Hence we have \(B \sim \tilde{A}\), and \(B \in TS(f \circ \tilde{h})\).

We define \(\Lambda : TS(h) \to TS(h)\) by \(\Lambda A := B\), which is injective by (3.1). Moreover, if \(r > 0\) is small enough, then \(U_r(a) \cap f^{-1}(b) = \{a\}\), which implies that \(B = \Lambda A\) is direct if and only if so is \(A\).

We show the surjectivity of \(\Lambda : TS(h) \to TS(f \circ \tilde{h}) = TS(h)\): let \(B = \{B_r\} \in TS(f \circ \tilde{h})\) and put \(b := (f \circ \tilde{h})(B)\). For every \(t > 0\), the inclusion \(f(\tilde{h}(B_t)) \subset U_t(b)\) gives the component \(V_t^{-1}\) of \(f^{-1}(U_t(b))\) with \(\tilde{h}(B_t) \subset V_t^{-1}\). Then \(V_t^{-1} \subset V_r^{-1}\) if \(s < t\), and \(\bigcap_{t>0} V_t^{-1}\) is a singleton \(\{a\} \subset f^{-1}(b)\). Moreover, for every \(r > 0\), there is \(s > 0\) with \((\tilde{h}(B_s) \subset V_s^{-1} \subset U_r(a)\), and hence there exists the component \(\tilde{A}_r\) of \(\tilde{h}^{-1}(U_r(a))\) such that \(B_s \subset \tilde{A}_r\). Then \(\tilde{A} := \{\tilde{A}_r\} \in \mathcal{G}\). Conversely, for every \(r > 0\), if \(s > 0\) is small enough, then \((f \circ \tilde{h})(\tilde{A}_s) \subset f(U_s(a)) \subset U_r(b)\), so that \((B_t \subset \tilde{A}_s \subset B_r\) (for some \(t > 0\)). Hence \(\tilde{A} \sim B\) and \(\tilde{A} \in TS(\tilde{h})\). Putting \(A := \{\lambda^{-1}\tilde{A}_r\} \in TS(\tilde{h}) = TS(h)\), we have \(\Lambda A \sim \{\lambda(\lambda^{-1}\tilde{A}_r)\} = \tilde{A} \sim B\), so that \(\Lambda A = B\) from (3.1).

\section{Proof of Theorem 1}

Let \(f\) be a rational function on \(\tilde{C}\) of degree \(d \geq 2\), and \(h\) a Schröder map of \(f\) as in 1.1. Replacing \(f^p\) by \(f\) if necessary, we assume that period \(p = 1\), and extend \(\lambda : \mathbb{C} \to \mathbb{C}\) to \(\Lambda : TS(h) \to TS(h)\) as in Theorem 1.4. Fix \(A = \{A_r\} \in TS(h)\), and put \(a := h(A)\) and \(U_r := U_r(a)\) \((r > 0)\). From (1.2), \(\Lambda^{-k} \sim \{\lambda^{-k} A_r\}_{r>0}\) for every \(k \in \mathbb{N}\).

We introduce some notation. The inclusion \(f^k(h(\lambda^{-k} A_r)) = h(A_r) \subset U_r\) gives the component \(V_r^{-k}\) of \(f^{-k}(U_r)\) such that \(h(\lambda^{-k} A_r) \subset V_r^{-k}\). From the proof of the surjectivity of \(\Lambda\) in 3.3, we have for every \(k \in \mathbb{N}\),

\[ \bigcap_{r>0} V_r^{-k} = \{h(\lambda^{-k} A)\}. \]

We first prove (1.8) and (1.9), leaving (1.7) to the end.

**Periodic \(A\):** in the case that \(A\) is periodic under \(\Lambda\), without loss of generality, we assume that \(\Lambda^{-1} A = A\), so that for every \(k \in \mathbb{N}\), \(A \sim \{\lambda^{-k} A_r\}_{r>0}\) and \(\bigcap_{r>0} V_r^{-k} = \{a\}\).

When \(f'(a) = 0\), we immediately have \(a \in AT(f)\). Suppose that \(f'(a) \neq 0\). If \(r > 0\) is small enough, then for every \(t \in (0, r]\), \(f : V_t^{-1} \to \mathbb{C}\)
$U_t$ is univalent and fixes $a$, and we denote its inverse by

$$f_t^{-1} : U_t \to V_t^{-1}.$$  

Since $\bigcap_{r>0} A_r = \emptyset$, diminishing $r > 0$ if necessary, we may suppose that

$$A_r \subset \{|w| \geq 1\}.$$  

If $a$ is a repelling or Siegel fixed point of $f$, then there exists $t \in (0, r)$ such that for every $k \in \mathbb{N}$, $(f_t^{-1})^k$ is well defined on $U_t$ and

$$h(\lambda^{-k}A_t) \subset V_t^{-k} = (f_t^{-1})^k(U_t) \subset U_r,$$  

so that $(A_k \subset \lambda^{-k}A_t \subset A_r$ (for some $s > 0$ since $A \sim \{\lambda^{-k}A_r\}$), which contradicts (4.1) since always $|\lambda|^{-1} < 1$. Now we have proved (1.8).

If $a \in PB(f) \cup CM(f)$, then for every $t \in (0, r)$, Theorem 2.3 yields a Siegel compactum $K_t$ associated to $(f_{t/2}^{-1}, U_{t/2}, a)$, so that $f_t^{-1}(K_t) = K_t \subset U_t$. For every component $L$ of $h^{-1}(K_t)$ with $L \subset A_t$, the inclusion $f(h(\lambda^{-1}L)) = h(L) \subset K_t$ implies that

$$h(\lambda^{-1}L) \subset h(\lambda^{-1}A_t) \cap f^{-1}(K_t) \subset V_t^{-1} \cap f^{-1}(K_t) = f_t^{-1}(K_t) = K_t.$$  

Let $\tilde{L}$ be the component of $h^{-1}(K_t)$ such that $\lambda^{-1}L \subset \tilde{L}$. Then $h(\lambda\tilde{L}) = f(\tilde{L}) \subset f(K_t) = f(\tilde{L}) = K_t$, so that $(L \subset \lambda\tilde{L} \subset L$. From $A = \Lambda^{-1}A \sim \{\lambda^{-1}A_r\}$, by decreasing $t \in (0, r)$ if necessary, we have

$$h(\lambda^{-1}L) \subset h(\lambda^{-1}A_t) \cap h^{-1}(K_t) \subset A_t \cap h^{-1}(U_t).$$  

If $A$ were also direct, then diminishing $r > 0$ if necessary, we even have $A_t \cap h^{-1}(U_t) = A_t$ from Lemma 2.8. Hence by induction, for every $k \in \mathbb{N}$, $\lambda^{-k}L$ must be a component of $h^{-1}(K_t)$ such that $\lambda^{-k}L \subset A_t(\subset A_r)$. This contradicts (4.1) as before.

Thus we have proved (1.9) since every direct transcendental singularity of $h$ is periodic under $\Lambda$ from Corollary 2.16 (iii).

**Indirect $A$:** we now assume that $A$ is non-periodic and indirect, and show (1.7) by eliminating any other possibility for $a = h(A)$.

Suppose that $a \in F(f)$. For every $k \in \mathbb{N} \cup \{0\}$, let $D^{-k}$ be the Fatou component of $f$ with $h(\Lambda^{-k}A) \in D^{-k}$, so that $f(D^{-k+1)} = D^{-k}$.

For every $k \in \mathbb{N} \cup \{0\}$, $D^{-k}$ is cyclic: otherwise, all $D^{-k}$ for $k$ large enough are not cyclic, and are mutually disjoint. Then since $\#C(f) < \infty$, we have for all large $k$, $D^{-k} \cap C(f) = \emptyset$, i.e., $D^{-k} \cap (\bigcup_{i \in \mathbb{N}} f^i(C(f))) = \emptyset$. Hence by Corollary 2.16 (iii), all $\Lambda^{-k}A$ for $k$ large enough are not only distinct but direct, which contradicts (2.5).

From this fact and Corollary 2.16 (iii), we must have one of two alternatives: $a = h(A) \in AT(f)$ (desirable) or all $D^{-k}$ are rotation domains of $f$. However, the second situation cannot occur: fix $r > 0$ with $U_r \subset D^0$. Then for every $k \in \mathbb{N}$, $h(\Lambda^{-k}A) \in V_r^{-k} \subset D^{-k}$, so that

$$h(\lambda^{-k}A_r) \subset D^{-k}.$$
If all $D^{-k}$ were rotation domains, then $f^{k} : D^{-k} \to D^0$ is univalent for every $k \in \mathbb{N}$. Since $h'(0) \neq 0$, $h\{|w| < t\}$ is univalent for $t > 0$ small enough, and then by repeated use of (1.1),

$$(h : \{w \in A_r; |w| < |\lambda|^k t\} \to D_0)$$

$$= (f^{k} : D^{-k} \to D^0) \circ (h : \{w \in \lambda^{-k} A_r; |w| < t\} \to D^{-k}) \circ \lambda^{-k}$$

is univalent for all $k \in \mathbb{N}$. Hence $A_r \cap C(h) = \emptyset$, which contradicts Theorem 2.7 since $A$ is indirect.

Finally, suppose that $a \in J(f) \setminus (PB(f) \cup M(f))$. We apply Theorem 2.1 to $U = U_r(a)$ for $r > 0$ small enough. Fix $t > 0$ such that $h\{|w| < t\}$ is univalent, and for this $t > 0$, put $\phi_t := (h\{|w| < t\})^{-1} : h\{|w| < t\} \to \{|w| < t\}$. Also fix $s > 0$ such that $U_{2s}(z_0) \in h\{|w| < t\}$ ($z_0 = h(0)$). For all large $k \in \mathbb{N}$, (2.2) shows that $\text{diam} V_\gamma^{-k} < s$, and since $0 \in \phi_t(U_{2s}(z_0))$,

$$\lambda^{-k} A_r \cap \phi_t(U_{2s}(z_0)) \neq \emptyset.$$  

Hence from $h(\lambda^{-k} A_r) \subset V_\gamma^{-k}$, we have $V_\gamma^{-k} \cap U_s(z_0) \neq \emptyset$, and since $\text{diam} V_\gamma^{-k} < s$, $V_\gamma^{-k} \subset U_{2s}(z_0)$. We recall again (1.2) and deduce that $\lambda^{-k} A_r \subset \phi_t(U_{2s}(z_0))$. This cannot be true since $A_r$ is unbounded. \qed

5. PROOF OF THEOREM 2

Replacing $f$ by an appropriate iterate if necessary, we assume that $N = 1$, so that $\lambda W = W$, $f(D) = D$ and $f(a) = a$.

Fix $w_0 \in W$, consider an arc $\gamma : [0, 1] \to W$ with $\gamma(0) = w_0$ and $\gamma(1) = \lambda w_0$, and extend the domain of $\gamma$ to $(-\infty, \infty)$ via the functional equation (1.10) (with $N = 1$). Then for every $k \in \mathbb{N}$ and every $s \in [0, 1]$, $\gamma(k+s) = \lambda^k \gamma(s) \in \lambda^k (\gamma([0,1]))$ and $h(\gamma(k+s)) \in f^k(h(\gamma([0,1])))$. Since $\gamma([0,1]) \subset (W \subset \mathbb{C}^\ast$ and $h(\gamma([0,1])) \subset D$, we have $\gamma(t) \to \infty$ and $h(\gamma(t)) \to a$ as $t \to \infty$. Thus $\gamma$ is as described in Theorem 2, to which $A^\gamma \subset TS(h)$ may be associated.

From now on, suppose that $a \in AT(f)$.

If $\gamma(0) = w_0 \in W \cap h^{-1}(a)$, then by $f(a) = a$ and (1.1), $\gamma(k) = \lambda^k w_0 \in h^{-1}(a)$ for every $k \in \mathbb{N}$. Hence, since $\gamma(k) \to A^h$ as $k \to \infty$, $A^\gamma$ is indirect.

Next, suppose that $f^{-1}(a) \cap D \neq \{a\}$. First, putting the backward orbits $BO := \bigcup_{n \in \mathbb{N}} (f : D \to D)^{-n}(a)$ of $a$ in $D$, we have $\#BO = \infty$: otherwise, since $BO$ is $(f : D \to D)^{-1}$-invariant, every $a' \in BO$ must be periodic, and then $a'$ must equal $a$, so that $f^{-1}(a) \cap D = \{a\}$, which is a contradiction. Second, we also have $\#(D \setminus h(W)) < \infty$: for each $b \in D \setminus h(W)$, there exists $B = \{B_r\} \subset \mathcal{U}$ such that for all small $r > 0$, $B_r$ is a component of $U_r(b)$ with $B_r \subset W$. We claim that $\bigcap_{r>0} B_r = \emptyset$: otherwise, $\bigcap_{r>0} B_r$ must be a singleton in $h^{-1}(b)$, and hence $h(W) \supset h(\bigcap_{r>0} B_r) = \{b\}$, which is a contradiction. Hence $B \in TS(h)$ with $h(B) = b$, and $B$ must be direct since $b \notin h(W)$. This
with \((2.3)\) yields \(#(D \setminus h(W)) \leq #(\text{direct singularities of } h) < \infty\). Consequently, \(#(BO \cap h(W)) = \infty\), which provides \(w_1 \in W\) such that \(f^n(h(w_1)) = a\) for some \(n \in \mathbb{N}\). Thus \(h(\lambda^n w_1) = a\) from \((1.1)\), so that \(\lambda^n w_1 \in \lambda^n W \cap h^{-1}(a) = W \cap h^{-1}(a)\). Hence \(W \cap h^{-1}(a) \neq \emptyset\).

Finally, suppose that some \(A = \{A_r\} \subset TS(h)\) with \(h(A) = a\) is indirect and that \(A_r \subset W\) for some \(r > 0\). Then by Theorem\(2.7\), there is \(c \in A_r \cap C(h) \setminus h^{-1}(a)(\subset W)\). Fix \(t > 0\) such that \(h|\{\{w| < t\}\) is univalent (using \(h'(0) \neq 0\)), and fix \(\ell \in \mathbb{N}\) such that \(\lambda^{-\ell} c \in \{\{w| < t\}\) (using \(|\ell| > 1\)). By \((1.1)\) and the chain rule,

\[
0 = h'(c) = (f^\ell \circ h \circ \lambda^{-\ell})'(c) = \prod_{i=0}^{\ell-1} f^i(h(\lambda^{-\ell} c)) \cdot h'(\lambda^{-\ell} c) \cdot \lambda^{-\ell},
\]

which implies that \(f^i(h(\lambda^{-\ell} c)) \in C(f)\) for some \(i \in \{0, \ldots, \ell-1\}\). For this \(i\), from \((1.1)\) and \(f(a) = a \neq h(c)\), we also have

\[
a \neq f^i(h(\lambda^{-\ell} c)) = h(\lambda^{-i} \lambda^{-\ell} c) \in h(\lambda^{-i} W) = h(W) \subset D.
\]

Hence \(D \cap C(f) \neq \{a\}\).

\[\square\]

6. Proofs of Corollaries 2 and 3

Proof of Corollary 2. For a transcendental entire function \(g\) and \(A = \{A_r\} \subset TS(g)\), we denote by \(A_r^\ast(0)\) the component of \(\mathbb{C} \setminus A_r\) containing \(0\) (if exists), and say that \(A\) is non-annular if \(A_r^\ast(0)\) is unbounded for all \(r > 0\). Note that if \(A = \{A_j\} \subset TS(g)\) with \(g(A) = \infty\) \((j = 1, \ldots, q')\) are mutually distinct, then \(\{A_j^*\}\) must be totally separated in that

\[
(6.1) \quad \bigcup_{i \neq j} A_i^j \subset (A_j^*)^*(0) \quad (j = 1, \ldots, q')
\]

for all small \(r > 0\). \((6.1)\) implies that when \(q' \geq 2\), all \(A_j^j (j = 1, \ldots, q')\) are non-annular.

To prove \((1.1.2)\), we need a spiral version of Ahlfors’s theorem (see our discussion of \((1.12)\)), which we formulate here as Theorem 6.2; the argument which seems useful to us is from Jenkins [9, §3] and Hayman [7, Theorem 8.21].

**Theorem 6.2.** Let \(g\) be an entire function of finite order \(\rho\), consider mutually distinct \(A_j^i \subset TS(h)\) with \(g(A_j^i) = \infty\) \((j = 1, \ldots, q')\), and suppose that \(A_1^j\) is non-annular when \(q' = 1\). Then for every asymptotic arc \(\gamma\) of \(h\) to which \(A_1^1\) is associated,

\[
q' \cdot \left(1 + \limsup_{t \to \infty} \left(\frac{\arg \gamma(t)}{\log |\gamma(t)|}\right)^2\right) \leq 2\rho.
\]

**Remark 6.4.** The assumption that \(A^j_i\) is non-annular is required because we work with the image of \(A^j_i\) under a branch of logarithm in the
proof. An entire function of order $< 1/2$ shows that such a condition is essential.

We have already hinted at the proof of Corollary [2] in [11].

By Eremenko and Levin, each component $W$ of $h^{-1}(D_\infty)$ is periodic under $\lambda$. Thus $W \subset h^{-1}(F(f))$, $0 \in \overline{W} \cap h^{-1}(J(f))$ (since $|\lambda| > 1$) and

$$0 \in \partial W.$$  

(6.5)

For a given $W$, Theorem [2] yields an asymptotic arc $\gamma_W : (-\infty, \infty) \to W$ with $\lim_{t \to \infty} h(\gamma_W(t)) = \infty$, to which $A^W = \{A^W_r\} \in TS(h)$ may be associated. We check that $A^W$ implies that all $A^W$ are non-annular. Then Theorem 6.2 may be applied to $A^W$, and all $\gamma_W$ satisfy (1.12) (for $q' = q_\infty$).

For all small $r > 0$, $U_r(\infty) \subset D_\infty$, so that

$$A^W_r \subset W.$$  

(6.6)

Hence (6.5) and (6.6) show that $\{A^W_r\}$ is totally separated.

Let $J_0$ be the component of $h^{-1}(J(f))$ containing 0. Since $W \subset h^{-1}(F(f))$, $J_0 \cap W = \emptyset$. Hence from (6.5) and (6.6), we have

$$J_0 \subset (A^W_k)^*(0),$$

(6.7)

using the notation introduced at the beginning of this section. By the $f$-invariance of $J(f)$ and (1.1), we also have

$$\lambda J_0 \subset J_0.$$  

From now on, we assume (EL), which is equivalent to

$$J_0 \neq \{0\}$$

since $h$ is analytic near 0 (and $h(0) = z_0$). Then $J_0$ is unbounded (since $|\lambda| > 1$) and $A^W$ is non-annular (by (6.7)).

The fact that $J_0$ is unbounded (when (EL) holds) also implies that $W$ can contain no closed curve winding around 0, so that there is a single-valued branch $arg_W(\cdot)$ of $arg(\cdot)$ on $W$.

Using Theorem [2] we may assume in addition that $\gamma_W$ satisfies (1.10), and hence (1.13). For each $W$ and each $R > 0$, let $t_W(R)$ be the least $t > 0$ such that $|\gamma_W(t)| = R$, and put $P_W(R) := \gamma_W(t_W(R))$. The set $\{P_W(R)\}_W$ has a natural cyclic order on the (counterclockwise-oriented) circle $\{|w| = R\}$, which induces a cyclic order of components $W$ of $h^{-1}(D_\infty)$ such that $W$ is the j-th component of $h^{-1}(D_\infty)$ if and only if $P_W(R)$ is the j-th point in $\{P_W(R)\}_W$. Let us denote by $W_j$ the j-th component of $h^{-1}(D_\infty)$ ($j = 0, \ldots, q_\infty - 1$). Replacing $\gamma_W$ if necessary, we may further assume that for every $W$, $\gamma_W = \lambda \cdot \gamma_W$, so that $P_W(|\lambda| R) = \lambda \cdot P_W(R)$. This yields the unique $p_\infty = p_\infty(h) \in \{0, \ldots, q_\infty - 1\}$ such that for each $j = 0, \ldots, q_\infty - 1$,

$$\lambda W_j = W_{j+p_\infty(\mod q_\infty)}.$$  

(6.8)
Hence, recalling that \( q_W = \min \{ N \in \mathbb{N}; \lambda^N W = W \} \), we observe that \( q := q_W \) is independent of \( W \) and that \( q_\infty / q =: m_\infty = m_\infty(h) \in \mathbb{N} \) is the number of cycles of components of \( h^{-1}(D_\infty) \) under \( \lambda \). We also recall \( p_W \) from (1.14), and observe that \( p_W \cdot q_\infty = q_W \cdot p_\infty \). Hence \( p_W = p_\infty / m_\infty =: p \) is also independent of \( W \).

Now the proof of Corollary 2 is completed by the spiral inequality (1.12) for \( \gamma_W \). The following, which uses standard ideas but appears to be new, may have independent interest. The hypothesis of finite order is essential. For completeness, we include the proof in §7.

**Theorem 3.** Let \( g \) be an entire function of finite order \( \rho \). If \( g \) does not cover \( a \in \mathbb{C} \) completely (in the same sense as discussed before Corollary 3), then \( a \in g(TS(g)) \).

By Corollary 2.16, every Schröder map \( h \) of \( f_c \) must be entire, and hence by Corollary 2.16, every \( A \in TS(h) \) is periodic.

Let \( c \in \text{int} \mathcal{C} \) (recall \( \mathcal{C} \) from the end of §1). Then \( f_c \) has no indifferent periodic point (cf. [13, Theorem 4.8]), and hence from (1.8) of Theorem 1, \( h(TS(h)) \subset AT(f_c) \) for every Schröder map \( h \) of \( f_c \).

If a Schröder map \( h \) of \( f_c \) does not cover \( a \in \mathbb{C} \) completely, then \( a \in h(TS(h)) \) by Theorem 3 so that as we just observed, we have even \( \Delta \nabla \subset AT(f_c) \), which implies that \( c \in \mathcal{H} \). Conversely, if \( c \in \mathcal{H} \), then there is \( a \in AT(f_c) \cap \mathbb{C} \), and by Corollary 1 there is a Schröder map \( h \) of \( f_c \) such that \( a \in h(TS(h)) \). Clearly, \( h \) cannot cover \( a \) completely. \( \square \)

### 7. Proof of Theorem 3

The hypotheses guarantee that \( g \) is transcendental. A consequence of this (with the identity theorem) is that for every \( r > 0 \), the cardinality of a subset of \( S(r) := \{ |z| = r \} \) on which either \( |g| \) or \( \arg g \) (mod 2\( \pi \)) is constant must be finite.

We may suppose that \( a = 0 \), and consider a sequence \( (r_n) \subset \mathbb{R}_{>0} \) with \( r_n \searrow 0 \) as \( n \to \infty \) and a sequence of unbounded components \( \Delta_n \) of preimages of \( \mathbb{D}_n = \{ |w| < r_n \} \) under \( g \). It is enough to show that if \( p > 2\rho \), then \( \Delta_n (n = 1, \ldots , p) \) cannot be mutually disjoint.

Suppose that \( \Delta_n (n = 1, \ldots , p) \) were mutually disjoint. Increasing each \( r_n \) slightly if necessary, we can assume that no critical value of
$g$ lies on $\bigcup_{n=1}^{p} \{ |w| = r_n \}$. From the observation above, the boundary $\partial \Delta_n$ $(n = 1, \ldots, p)$ consists of finitely many (analytic) Jordan arcs whose endpoints are both $z = \infty$.

The next lemma contains the main idea.

**Lemma 7.1.** Let $F$ be a simply-connected unbounded component of $\{|z| > r\} \setminus (\Delta_m \cup \Delta_n)$, where $m \neq n$, whose boundary consists of a subarc of $S(r)$ for some $r > 0$ and unbounded Jordan subarcs $\gamma$ of $\partial \Delta_m$ and $\gamma'$ of $\partial \Delta_n$, each of which has an endpoint on $S(r)$. Then $g$ is unbounded in $F$.

**Proof.** Suppose on the contrary that $g$ is bounded in $F$. Then by Lindelöf’s theorem [16, p. 75], if $g(z)$ converges as $z \to \infty$ along two asymptotic arcs in $\overline{F}$ and the limits are both in $\mathbb{C}$, then they must coincide.

At once from the Cauchy-Riemann equations, (any fixed branch of) $\arg g$ (not mod $2\pi$) varies monotonically along each of the arcs $\gamma, \gamma' \subset \overline{F}$. If the variation of $\arg g$ were bounded on each of $\gamma$ and $\gamma'$, then $g(z)$ converges as $z \to \infty$ along each of them, and by Lindelöf’s theorem, the limits must be same, which cannot be since $r_m \neq r_n$.

Thus, with no loss of generality, we may choose any fixed branch of $\arg g$ on $\gamma$, and it will be unbounded on $\gamma$. On the other hand, there are (in fact uncountably many) distinct $\varphi_1, \varphi_2 \in [0, 2\pi)$ such that no critical value $w$ satisfies $\arg w = \varphi_1$ or $\varphi_2$ (mod $2\pi$).

For each $j = 1, 2$, we may choose infinitely many mutually distinct $z_{jk} \in \gamma$ $(k \in \mathbb{N})$ with $\arg g(z_{jk}) = \varphi_j$ (mod $2\pi$). For each $z_{jk}$, there exists the (maximal) lift $\Gamma_{jk} \subset \mathbb{C}_z$ by $g$ of the radial ray $(r_m, \infty) \ni t \rightarrow te^{i\varphi_j} \in \mathbb{C}_w$ with an endpoint $z_{jk}$. Since the ray $\{ w \in \mathbb{C}; \arg w = \varphi_j \}$ is free of critical values, $\Gamma_{jk} \cap \Gamma_{jk'} = \emptyset$ if $k \neq k'$. In addition, as we noted in the beginning of this section, $\Gamma_{jk}$ can intersect $S(r)$ for at most finitely many $k$. On the other hand, for every $k$, $\Gamma_{jk}$ cannot intersect $\gamma' \cup \gamma$ since $|(g|F)| > r_m$ near $\gamma$ and $|(g|F)| > r_n$ near $\gamma'$. Consequently, for all large $k$, the maximal lift $\Gamma_{jk}$ is a Jordan arc in $F$ tending to $\infty$. Then since $g$ is bounded in $F$, there must exist $t = t_{jk} \in (r_m, \infty)$ such that $g(z) \to te^{i\varphi_j}$ as $\Gamma_{jk} \ni z \to \infty$.

However, by Lindelöf’s theorem, all the limits coincide for $j = 1, 2$ and all large $k$, so that $\varphi_1 = \varphi_2$, which is a contradiction. \hfill $\Box$

Now we complete the proof of the theorem. Once we fix $r > 0$ large enough, we obtain at least $p$ of mutually disjoint domains $F$ satisfying the hypotheses of Lemma 7.1. However,

**Lemma 7.2.** If $p > 2\rho$, then $g$ must be bounded on at least one of these $F$.

**Proof.** We need a standard estimate of harmonic measure $\omega$ (cf. [16, XI. §4]). In the following, $C_1, C_2(> 2\log 2), C_3, C_4$ denote some positive constants independent of $R > 0$.
Let $F \subset \mathbb{C}$ be an arbitrary domain intersecting $S(t)$ so that $S(t) \setminus F \neq \emptyset$ for every $t \geq r$, and let $\theta_F(t)$ the angular measure of $F \cap S(t)$. Then

$$\omega(\cdot, F, F \cap S(R)) < C_1 \exp \left( -\pi \int_{2r}^{R/2} \frac{dt}{t \theta_F(t)} \right)$$
onumber

on $F \cap A(2r, R/2)$ as soon as $R/2 > 2r$. Let $F_1, \ldots, F_\ell \subset \mathbb{C}$ ($\ell \in \mathbb{N}$) be arbitrary mutually disjoint domains intersecting $S(t)$ for every $t \geq r$, with angular measure $\theta_j(t) := \theta_{F_j}(t)$ as above. Note that $\sum_{j=1}^\ell \theta_j(t) \leq 2\pi$ for all such $t$. Then by an argument involving the Cauchy-Schwarz inequality, we find some $j \in \{1, \ldots, \ell\}$ and a sequence $(R_k) \subset \mathbb{R}$ tending to $\infty$ as $k \to \infty$ such that for every $R = R_k$,

$$\pi \int_{2r}^{R/2} \frac{dt}{t \theta_j(t)} \geq \frac{\ell}{2} \left( \log \frac{R}{r} - C_2 \right).$$

Applying these estimates to $p$ of our domains $F$, we conclude that for one of these $F$, there is a sequence $(R_k) \subset \mathbb{R}$ with $R_k \not\to \infty$ as $k \to \infty$ such that for each $R = R_k$,

$$\omega(\cdot, F, F \cap S(R)) < C_3 (R/r)^{-p/2}$$

on $F \cap A(2r, R/2)$,

while (since $(r_m)$ decreases,) we have $\log |g| < \log r_1 + (\max_{S(r)} \log |g|) < C_4$ on $\partial F$. Since $g$ is entire, we also have

$$\rho = \limsup_{r \to \infty} \frac{\log \max_{|z|=r} \log^+ |g(z)|}{\log r},$$

from which, for all small $\epsilon \in (0, (p - 2\rho)/2)$, we have for all large $R$, $\log |g| < R^{p+\epsilon}$ on $S(R)$. Thus by the two-constants theorem, for all large $R = R_k$,

$$\log |g| \leq C_4 + R^{p+\epsilon} \cdot C_5 (R/r)^{-p/2} < 2C_4$$

on $F \cap A(2r, R/2)$.

Hence $g$ is bounded on $F$. \hfill \Box

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**Supplement by D. Drasin.** Since this issue is dedicated to Walter Hayman, I want to indicate the significant influence he has had in my career. I am sure that my experiences are not unusual, but usually there are no opportunities to make such comments.
His book *Meromorphic Functions* went through two editions, and even today may be the most efficient introduction to classical Nevanlinna theory in one complex variable (it was also the first text in English!). The first two chapters develop the fundamental theorems established by Rolf Nevanlinna in the 1920s, but every other chapter was centered on a topic which quickly led the reader to open questions, whose resolution has been a major activity for 30-plus years (Chapter 4 was my focus). His series of problem compilations, an activity beginning a few years before MF, efficiently covered a large variety of fields related to one complex variable (including some higher-dimensional questions). In those pre-internet days, assembling the material from colleagues all over the world required special effort. Until recent times (circa 1990) international contact was difficult in many countries, and his visits, letters and collections played an important role in supporting colleagues and students from these countries.

In his dealings with colleagues, he was always encouraging, and raising interesting questions for study. We met in 1967, Montreal, where I asked him a question about minimum modulus. Several weeks later, I received his example, which was the kernel of several of my later works (some joint with Dan Shea). At conferences he always had time to talk to participants, in a way that was always enthusiastic, supportive; he treated everyone with the same courtesy and spirit.

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