Supermembranes and Super Matrix Models

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ABSTRACT

We review recent developments in the theory of supermembranes and their relation to matrix models.
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1 Introduction

Supermembranes [1] were proposed as a consistent quantum-mechanical extension of 11-dimensional supergravity [2], inspired by the way in which string theory defines a quantum-mechanical extension of 10-dimensional supergravity theories. Although there are similarities in the theoretical description of superstrings and supermembranes, there are also a number of features that are distinctly different. An elementary superstring can be formulated as a field theory on the 2-dimensional worldsheet swept out by the string in Minkowski space. This field theory is free and describes an infinite number of states with a discrete equidistant mass spectrum with steps measured by $1/\sqrt{\alpha'}$, the fundamental mass scale of string theory. Likewise the supermembrane theory can be formulated as a field theory on a 3-dimensional world volume. But unlike the previous case, this theory is not a free but an interacting theory of a complicated structure. Furthermore the mass spectrum of the supermembrane is continuous, rather than discrete [3]. This is not a generic feature of quantized extended objects, but crucially rests on the presence of supersymmetry. At an early stage the question was raised whether, in view of Haag’s theorem, the supermembrane should not be regarded as a second- rather than a first-quantized theory, with a unitarily non-implementable evolution matrix [4]. As it turns out, both issues are resolved in the context of a more recent perspective in which the continuity of the spectrum is seen as arising from multi-membrane states. The theory, set up initially to define a first-quantized supermembrane, captures also the presence of multi-membrane states as described in a second-quantized theory. Again this feature strongly hinges on supersymmetry: for the generic theory there is no reason why states of several interacting membranes should give rise to a continuous mass spectrum.

The continuity of the supermembrane spectrum is due to the fact that, quantum-mechanically, the supermembrane can develop stringlike zero-area ‘spikes’ which do not contribute to the mass. Consequently a membrane can be pinched into two or more membranes connected by these stringlike configurations of arbitrary length, which become indistinguishable from the multi-membrane state obtained by suppressing the connecting strings. In this way, not only are single- and multi-membrane states indistinguishable, but so are certain states of different topology and states with and without winding (so that topology changes will correspond to smooth transitions in the moduli space that parametrizes these states). Thus the first-quantized theory of spherical supermembranes ultimately describes also membranes of nontrivial topology, multi-membrane states and (if the target space has compact coordinates) supermembranes with winding.

In 11 spacetime dimensions the supermembrane can consistently couple to a superspace background that satisfies a number of constraints which are equivalent to the supergravity equations of motion. The supermembrane action can also exist in 4, 5 and 7 spacetime dimensions, in the same way as the Green-Schwarz superstring [5] is classically consistent in 3, 4, 6 and 10 dimensions. In the context
of string theory it was natural to expect that the massless states of the superme-
mbrane would correspond to those of 11-dimensional supergravity. However, in
the presence of a continuous mass spectrum \([3]\) the possible existence of massless
states is difficult to prove or disprove \([4, 6, 7]\). The inability to make sense of
the mass spectrum and the fact that 11-dimensional supergravity seemed to have
no place in string theory, formed an obstacle for further development of the the-
ory. More recently, however, interest in supermembranes was rekindled by the
realization that 11-dimensional supergravity does have its role to play as the long-
distance approximation to M-theory \([8, 9, 10, 11]\). M-theory is the conjectured
framework for unifying all five superstring theories and 11-dimensional super-
gravity. It turns out that supermembranes, M-theory and super-matrix-models
are all intricately related.

An important observation was that it is possible to regularize the superme-
mbrane in terms of a super matrix model based on some finite group, such as
\(U(N)\). In the limit of infinite \(N\) one then recovers the supermembrane \([4]\). These
supersymmetric matrix models were constructed long ago \([12]\) and can be ob-
tained from supersymmetric Yang-Mills theories in the zero-volume limit. More
recently it was realized that these models describe the short-distance behaviour
of \(N\) Dirichlet particles \([13]\). The continuity of the spectrum is then understood
directly in terms of the spectrum of \(N\)-particle states. A bold conjecture was
that these super matrix models capture the degrees of freedom of M-theory \([14]\).
In the large-\(N\) limit, where one considers the states with an infinite number
of particles, the supermembranes should then re-emerge. Furthermore there is ev-
idence meanwhile that the supermembrane has massless states \([13]\), which will
presumably correspond to the states of 11-dimensional supergravity, although
proper asymptotic states do not exist. The evidence is based on the matrix
model regularization of the supermembrane for low values of \(N\). For fixed value
of \(N\) the existence of such states was foreseen on the basis of identifying the
Kaluza-Klein states of M-theory compactified on \(S^4\) with the Dirichlet particles
and their bound states in type-IIA string theory.

From this viewpoint it is natural to consider the supermembrane in curved
backgrounds associated with 11-dimensional supergravity. Such backgrounds con-
sist of a nontrivial metric, a three-index gauge field and a gravitino field. This
provides us with an action that transforms as a scalar under the combined (local)
supersymmetry transformations of the background fields and the supermembrane
embedding coordinates. Here it is important to realize that the supersymmetry
transformations of the embedding coordinates will themselves depend on the
background. When the background is supersymmetric, then the action will be
supersymmetric as well. In the light-cone formulation this model will lead to
models invariant under area-preserving diffeomorphisms, which in certain situa-
tions can be approximated by matrix models in curved backgrounds. The area-
preserving diffeomorphisms are then replaced by a finite group, such as \(U(N)\), but
target-space diffeomorphisms are no longer manifestly realized. Matrix models
in curved space have already been studied in [16]. Recently toroidal compactifications of matrix theory were considered in which the three-form gauge field of 11-dimensional gravity plays a crucial role [17]. These compactifications exhibit interesting features in which the noncommutative torus appears as a new solution to compactified matrix theory. We should also point out that classical supermembrane solutions in nontrivial backgrounds have been discussed before, see, e.g. [18]. In view of the relation between near-horizon geometries and conformal field theories [19] interesting classes of backgrounds are the ones where the target space factorizes locally into the product of an $AdS$ space and some compact space.

In this lecture we review many of these topics starting from the supermembrane point of view. We should stress that there remain many open questions and problems, both for supermembranes and for super matrix models. For instance, the large-$N$ behaviour is still poorly understood as are features related to matrix models and membranes in nontrivial backgrounds. But the most intriguing questions concern the precise role that these theories play in M-theory, the theory that encompasses all known perturbative string theories. For other reviews, we refer to [20]; a number of related topics was also discussed in the workshop and can be found in this volume.

2 Supermembranes

Fundamental supermembranes can be described in terms of actions of the Green-Schwarz type, possibly in a nontrivial but restricted (super)spacetime background [1]. Such actions exist for supersymmetric $p$-branes, where $p = 0, 1, \ldots, d - 1$ defines the spatial dimension of the brane. Thus for $p = 0$ we have a superparticle, for $p = 1$ a superstring, for $p = 2$ a supermembrane, and so on. The dimension of spacetime in which the superbrane can live is very restricted. These restrictions arise from the fact that the action contains a Wess-Zumino-Witten term, whose supersymmetry depends sensitively on the spacetime dimension. If the coefficient of this term takes a particular value then the action possesses an additional fermionic gauge symmetry, the so-called $\kappa$-symmetry. This symmetry is necessary to ensure the matching of (physical) bosonic and fermionic degrees of freedom. In the following we restrict ourselves to supermembranes (i.e., $p = 2$) in 11 dimensions.

The 11-dimensional supermembrane [1] is written in terms of superspace embedding coordinates $Z^M(\zeta) = (X^\mu(\zeta), \theta^a(\zeta))$, which are functions of the three world-volume coordinates $\zeta^i$ ($i = 0, 1, 2$). It couples to the superspace geometry of 11-dimensional supergravity, encoded by the supervielbein $E_M^A$ and the
antisymmetric tensor gauge superfield $B_{MNP}$, through the action

$$S[Z(\zeta)] = \int d^3\zeta \left[ -\sqrt{-g(Z(\zeta))} - \frac{1}{6} \varepsilon^{ijk} \Pi_i^A \Pi_j^B \Pi_k^C B_{CBA}(Z(\zeta)) \right], \quad (1)$$

where $\Pi_i^A = \partial Z^M / \partial \zeta^i E^{MA}$ is the pull-back of the supervielbein to the membrane worldvolume. Here the induced metric equals $g_{ij} = \Pi_i^r \Pi_j^s \eta_{rs}$, with $\eta_{rs}$ being the constant Lorentz-invariant metric. This action is invariant under local fermionic $\kappa$ transformations alluded to above, given that certain constraints on the background fields hold, which are equivalent to the equations of motion of 11-dimensional supergravity [1].

Flat superspace is characterized by

$$E_{\mu r} = \delta_{\mu r}^r, \quad E_{\alpha a} = \delta_{\alpha a}^a, \quad B_{\mu \alpha} = (\bar{\theta} \Gamma_{\mu} \alpha), \quad B_{\mu \alpha \beta} = (\bar{\theta} \Gamma_{\mu})_{(\alpha} (\bar{\theta} \Gamma_{\mu})_{\beta)}, \quad B_{\mu \nu \rho} = 0. \quad (2)$$

The gamma matrices are denoted by $\Gamma^r$; gamma matrices with more than one index denote antisymmetrized products of gamma matrices with unit weight. In flat superspace the supermembrane Lagrangian, written in components, reads (in the notation and conventions of [4]),

$$L = -\sqrt{-g(X, \theta)} - \varepsilon^{ijk} \bar{\theta} \Gamma_{\mu \nu \rho} \delta \kappa \theta \left[ \frac{1}{2} \partial_i X^\mu \left( \partial_j X^\nu + \bar{\theta} \Gamma_{\nu} \partial_j \theta \right) + \frac{1}{6} \bar{\theta} \Gamma^\mu \partial i \theta \bar{\theta} \Gamma^\nu \partial j \theta \right], \quad (3)$$

The target space can have compact dimensions which permit winding membrane states [21]. In flat superspace the induced metric,

$$g_{ij} = (\partial_i X^\mu + \bar{\theta} \Gamma_{\mu} \partial_i \theta)(\partial_j X^\nu + \bar{\theta} \Gamma_{\nu} \partial_j \theta) \eta_{\mu \nu}, \quad (4)$$

is supersymmetric. Therefore the first term in (3) is trivially invariant under spacetime supersymmetry,

$$\delta X^\mu = -\varepsilon \Gamma^\mu \theta, \quad \delta \theta = \epsilon. \quad (5)$$

In 4, 5, 7, or 11 spacetime dimensions the second term in the action proportional to $\varepsilon^{ijk}$ is also supersymmetric (up to a total divergence) and the full action is invariant under $\kappa$-symmetry.

In the case of the open supermembrane, $\kappa$-symmetry imposes boundary conditions on the fields [22, 23]. They must ensure that the following integral over the boundary of the membrane world volume vanishes,

$$\int_{\partial M} \left[ \frac{1}{2} dX^\mu \wedge (dX^\nu + \bar{\theta} \Gamma_{\nu} d\theta) \bar{\theta} \Gamma_{\mu \nu \delta, \theta} \delta \kappa \theta + \frac{1}{6} \bar{\theta} \Gamma^\mu d\theta \wedge \bar{\theta} \Gamma^\nu d\theta \bar{\theta} \Gamma_{\mu \nu \delta, \theta} \right] + \frac{1}{2} (dX^\mu - \frac{1}{3} \bar{\theta} \Gamma_{\mu \nu} d\theta \wedge \bar{\theta} \Gamma^\nu \delta \kappa \theta) = 0. \quad (6)$$

\(^1\)Our notation and conventions are as follows. Tangent-space indices are $A = (r, a)$, whereas curved indices are denoted by $M = (\mu, \alpha)$. Here $r, \mu$ refer to commuting and $a, \alpha$ to anticommuting coordinates. Moreover we take $\epsilon_{012} = -\epsilon^{012} = 1.$

4
This can be achieved by having a “membrane D-\(p\)-brane” at the boundary with \(p = 1, 5, \) or 9, which is defined in terms of \((p + 1)\) Neumann and \((10 - p)\) Dirichlet boundary conditions for the \(X^\mu\), together with corresponding boundary conditions on the fermionic coordinates. More explicitly, we define projection operators

\[
P_\pm = \frac{1}{2} \left( 1 \pm \Gamma_{p+1} \Gamma_{p+2} \cdots \Gamma_{10} \right),
\]

and impose the Dirichlet boundary conditions

\[
\partial_\parallel X^M = 0, \quad M = p + 1, \ldots, 10, \\
P_- \theta = 0,
\]

where \(\partial_\perp\) and \(\partial_\parallel\) define the world-volume derivatives perpendicular or tangential to the surface swept out by the membrane boundary in the target space. Note that the fermionic boundary condition implies that \(P_- \partial_\parallel \theta = 0\). Furthermore, it implies that spacetime supersymmetry is reduced to only 16 supercharges associated with spinor parameters \(P_+ \epsilon\), which is chiral with respect to the \((p+1)\)-dimensional world volume of the D-\(p\)-brane at the boundary. With respect to this reduced supersymmetry, the superspace coordinates decompose into two parts, one corresponding to \((X^M, P_- \theta)\) and the other corresponding to \((X^m, P_+ \theta)\) where \(m = 0, 1, \ldots, p\). While for the five-brane these superspaces exhibit a somewhat balanced decomposition in terms of an equal number of bosonic and fermionic coordinates, the situation for \(p = 1, 9\) shows heterotic features in that one space has an excess of fermionic and the other an excess of bosonic coordinates. Moreover, we note that supersymmetry may be further broken, e.g. by choosing different Dirichlet conditions on nonconnected segments of the supermembrane boundary.

The Dirichlet boundary conditions can be supplemented by the following Neumann boundary conditions,

\[
\partial_\perp X^m = 0, \quad m = 0, 1, \ldots, p, \\
P_+ \partial_\perp \theta = 0.
\]

These do not lead to a further breakdown of the rigid spacetime symmetries.

We now continue and follow the light-cone quantization described in [4] for a closed membrane without winding. In the light-cone gauge the light-cone coordinate \(X^+ = (X^1 + X^0)/\sqrt{2}\) is linearly identified with to the world-volume time denoted by \(\tau\) and the fermionic coordinates are subject to the gauge condition \(\gamma_+ \theta = 0\). The momentum \(P_-\) is time independent and proportional to the center-of-mass (CM) value \(P_0^+ = (P_-)_{0} \) times some density \(\sqrt{\omega(\sigma)}\) of the spacesheet, whose spacesheet integral is normalized to unity. Hence we have

\[
P_0^+ = \int d^2\sigma \, P^+(\sigma).
\]
The CM momentum $P_0^-$ is equal to minus the Hamiltonian and takes the form

$$H = \frac{1}{P_0^+} \int d^2\sigma \sqrt{w} \left[ \frac{P_0^+ P_a}{2w} + \frac{1}{4} \{ X^a, X^b \}^2 - P_0^+ \bar{\theta} \gamma_a \{ X^a, \theta \} \right]. \quad (11)$$

Here the integral runs over the spatial components of the world volume denoted by $\sigma^1$ and $\sigma^2$, while $P_a(\sigma) (a = 2, \ldots, 9)$ are the momenta conjugate to the transverse coordinates $X^a$. Furthermore we made use of the Poisson bracket $\{ A, B \}$ defined by

$$\{ A(\sigma), B(\sigma) \} = \frac{1}{\sqrt{w(\sigma)}} \varepsilon^{rs} \partial_r A(\sigma) \partial_s B(\sigma). \quad (12)$$

Note that the coordinate $X^- = (X^1 - X^0)/\sqrt{2}$ itself does not appear in the Hamiltonian (11). It is defined via

$$P_0^+ \partial_r X^- = -P_0^+ \bar{\theta} \gamma_r \partial_r \theta, \quad (13)$$

and implies that the right-hand side of (13) must be closed; without winding in $X^-$, it must be exact. This constraint is important later on.

The other CM coordinates and momenta are

$$P_0 = \int d^2\sigma P, \quad X_0 = \int d^2\sigma \sqrt{w(\sigma)} X(\sigma), \quad \theta_0 = \int d^2\sigma \sqrt{w(\sigma)} \theta(\sigma). \quad (14)$$

In the light-cone gauge we are left with the transverse coordinates $X$ and corresponding momenta $P$, which transform as vectors under the SO(9) group of transverse rotations. Only sixteen fermionic components $\theta$ remain, which transform as SO(9) spinors. Furthermore we have the CM momentum $P_0^+$ and the center-of-mass coordinate $X_0^-$ (the remaining modes in $X^-$ are dependent).

The supermembrane Hamiltonian (11) can be decomposed as follows,

$$H = \frac{P_0^2}{2P_0^+} + \frac{\mathcal{M}^2}{2P_0^+}. \quad (15)$$

Because the Hamiltonian is equal to $-P_0^-$, $\mathcal{M}$ is the supermembrane mass operator, which does not depend on any of the CM coordinates or momenta. The explicit expression for $\mathcal{M}^2$ is

$$\mathcal{M}^2 = \int d^2\sigma \sqrt{w(\sigma)} \left[ \frac{P_2(\sigma)}{w(\sigma)} \right] + \frac{1}{2} \left( \{ X^a, X^b \} \right)^2 - 2P_0^+ \bar{\theta} \gamma_a \{ X^a, \theta \}, \quad (16)$$

where $[P^2]'$ indicates that the contribution of the CM momentum $P_0$ is suppressed.

The structure of the Hamiltonian (15) shows that the wave functions for the supermembrane now factorize into a wave function of the CM modes and a wave
function of the supersymmetric quantum-mechanical system that describes the other modes. For the latter the mass operator plays the role of the Hamiltonian of a supersymmetric model in quantum mechanics. The aspects related to supersymmetry will be discussed in the next section.

In the light-cone gauge there is still a residual invariance associated with area-preserving diffeomorphisms of the membrane spacesheet. These are defined by transformations

$$\sigma^r \to \sigma^r + \xi^r(\sigma),$$

with

$$\partial_r(\sqrt{w(\sigma)}\xi^r(\sigma)) = 0.$$  \hfill (18)

It is convenient to rewrite this condition in terms of dual spacesheet vectors by

$$\sqrt{w(\sigma)}\xi^r(\sigma) = \varepsilon^{rs}\xi_s(\sigma).$$  \hfill (19)

In the language of differential forms the condition (18) then implies that $\xi_r$ corresponds to a closed one-form. The trivial solutions are the exact forms, in components,

$$\xi_r = \partial_r\xi(\sigma),$$

for any globally defined function $\xi(\sigma)$. The nontrivial solutions are the closed forms which are not exact. On a Riemann surface of genus $g$ there are precisely $2g$ linearly independent non-exact closed forms, whose integrals along the homology cycles are normalized to unity. In components we write

$$\xi_r = \phi(\lambda)_r, \quad \lambda = 1, \ldots, 2g.$$  \hfill (21)

The presence of the closed but non-exact forms is crucial for describing the winding of the embedding coordinates. More precisely, while the momenta $P(\sigma)$ and the fermionic coordinates $\theta(\sigma)$ remain single valued on the spacesheet, the embedding coordinates, written as one-forms with components $\partial_r X(\sigma)$ and $\partial_r X^-(\sigma)$, are decomposed into closed one-forms. Their non-exact contributions are multiplied by an integer times the length of the compact direction.

### 3 Gauge theory of area-preserving diffeomorphisms

It turns out that the light-cone formulation of the supermembrane can be described as a gauge theory of area-preserving diffeomorphisms. Under these diffeomorphisms the fields $X^a$, $X^-$ and $\theta$ transform according to

$$\delta X^a = \frac{\varepsilon^{rs}}{\sqrt{w}} \xi_r \partial_s X^a, \quad \delta X^- = \frac{\varepsilon^{rs}}{\sqrt{w}} \xi_r \partial_s X^-, \quad \delta \theta^a = \frac{\varepsilon^{rs}}{\sqrt{w}} \xi_r \partial_s \theta.$$  \hfill (22)
where the time-dependent reparametrization $\xi^r$ consists of closed exact and non-exact parts. The commutator of two infinitesimal area-preserving diffeomorphisms is determined by the product rule

$$\xi^{(3)}_r = \partial_r \left( \frac{\varepsilon^{st}}{\sqrt{w}} \xi^{(2)}_s \xi^{(1)}_t \right),$$

(23)

where both $\xi^{(1,2)}_r$ are closed vectors. Because $\xi^{(3)}_r$ is exact, the exact vectors thus generate an invariant subgroup of the area-preserving diffeomorphisms. As we shall discuss in the next section this subgroup can be approximated by SU($N$) in the large-$N$ limit, at least for closed membranes. For open membranes the boundary conditions on the fields (8) lead to a smaller group, such as SO($N$). Accordingly there is a gauge field $\omega_r$, which is therefore closed as well and transforming as

$$\delta \omega_r = \partial_0 \xi_r + \partial_r \left( \frac{\varepsilon^{st}}{\sqrt{w}} \xi_s \omega_t \right).$$

(24)

Corresponding covariant derivatives are

$$D_0 X^a = \partial_0 X^a - \frac{\varepsilon^{rs}}{\sqrt{w}} \omega_r \partial_s X^a, \quad D_0 \theta = \partial_0 \theta - \frac{\varepsilon^{rs}}{\sqrt{w}} \omega_r \partial_s \theta,$$

(25)

and likewise for $D_0 X^-$. The action corresponding to the following Lagrangian density is then gauge invariant under the transformations (22) and (24),

$$\mathcal{L} = P_0^+ \sqrt{w} \left[ \frac{1}{2} (D_0 X)^2 + \bar{\theta} \gamma_- D_0 \theta - \frac{1}{4} (P_0^+)^{-2} \{ X^a, X^b \}^2 \right. + (P_0^+)^{-1} \bar{\theta} \gamma_- \gamma_a \{ X^a, \theta \} + D_0 X^- \left. \right],$$

(26)

where we draw attention to the last term proportional to $X^-$, which can be dropped in the absence of winding. Moreover, we note that for open supermembranes, (26) is invariant under the transformations (22) and (24) only if $\xi_\parallel = 0$ holds on the boundary. This condition defines a subgroup of the group of area-preserving transformations, which is consistent with the Dirichlet conditions (8). Observe that here $\partial_\parallel$ and $\partial_\perp$ refer to the $\text{spacesheet}$ derivatives tangential and perpendicular to the membrane boundary $\parallel$.

The action corresponding to (26) is also invariant under the supersymmetry transformations

$$\delta X^a = -2 \bar{\epsilon} \gamma^a \theta, \quad \delta \theta = \frac{1}{2} \gamma_+ (D_0 X^a \gamma_a + \gamma_-) \epsilon + \frac{1}{4} (P_0^+)^{-1} \{ X^a, X^b \} \gamma_+ \gamma_a \epsilon,$$

$$\delta \omega_r = -2 (P_0^+)^{-1} \bar{\epsilon} \partial_r \theta.$$  

(27)

\(^2\text{Consistency of the Neumann boundary conditions (9) with the area-preserving diffeomorphisms (22) further imposes } \partial_\perp \xi_\parallel = 0 \text{ on the boundary, where indices are raised according to (19).} \)
The supersymmetry variation of $X^-$ is not relevant and may be set to zero. For open membranes one finds that the boundary conditions $\omega_\parallel = 0$ and $\epsilon = P_+ \epsilon$ must be fulfilled in order for (27) to be a symmetry of the action. In that case the theory takes the form of a gauge theory coupled to matter. The pure gauge theory is associated with the Dirichlet and the matter with the Neumann (bosonic and fermionic) coordinates.

In the case of a ‘membrane D-9-brane’ one now sees that the degrees of freedom on the ‘end-of-the world’ 9-brane precisely match those of 10-dimensional heterotic strings. On the boundary we are left with eight propagating bosons $X^m$ (with $m = 2, \ldots, 9$), as $X^{10}$ is constant on the boundary due to (8), paired with the 8-dimensional chiral spinors $\theta$ (subject to $\gamma_+ \theta = P_- \theta = 0$), i.e., the scenario of Hořava-Witten [11].

The full equivalence with the membrane Hamiltonian is now established by choosing the $\omega_r = 0$ gauge and passing to the Hamiltonian formalism. The field equations for $\omega_r$ then lead to the membrane constraint (13) (up to exact contributions), partially defining $X^-$. Moreover the Hamiltonian corresponding to the gauge theory Lagrangian of (26) is nothing but the light-cone supermembrane Hamiltonian (11). Observe that in the above gauge theoretical construction the space-sheet metric $w_{rs}$ enters only through its density $\sqrt{w}$ and hence vanishing or singular metric components do not pose problems.

We are now in a position to study the full 11-dimensional supersymmetry algebra of the winding supermembrane. For this we decompose the supersymmetry charge $Q$ associated with the transformations (27), into two 16-component spinors,

$$Q = Q^+ + Q^-, \quad \text{where} \quad Q^\pm = \frac{1}{2} \gamma_\pm \gamma_\mp Q,$$

(28)
to obtain

$$Q^+ = \int d^2\sigma \left( 2 P_0^a \gamma_a + \sqrt{w} \{ X^a, X^b \} \gamma_{ab} \right) \theta,$$

$$Q^- = 2 P_0^+ \int d^2\sigma \sqrt{w} \gamma_- \theta.$$

(29)

In the presence of winding the supersymmetry algebra takes the form [21]

$$( Q^+_a, \bar{Q}^+_\beta )_{DB} = 2 (\gamma_+)_{a\beta} H - 2 (\gamma_a \gamma_+)_{a\beta} \int d^2\sigma \sqrt{w} \{ X^a, X^- \},$$

$$( Q^-_a, \bar{Q}^-_\beta )_{DB} = -(\gamma_a \gamma_+ \gamma_-)_{a\beta} P_0^a - \frac{1}{2} (\gamma_{ab} \gamma_+ \gamma_-)_{a\beta} \int d^2\sigma \sqrt{w} \{ X^a, X^b \},$$

$$( Q^\alpha_+, \bar{Q}^-_\beta )_{DB} = -2 (\gamma_-)_{a\beta} P_0^+, \quad (30)$$

where use has been made of the Dirac brackets of the phase-space variables and the defining equation (13) for $\partial_r X^-$. The new feature of this supersymmetry algebra is the emergence of the central charges in the first two anticommutators, which are generated through the winding contributions. They represent topological quantities obtained by integrating
the winding densities
\[ z^a(\sigma) = \varepsilon^{rs} \partial_r X^a \partial_s X^- \] (31)
and
\[ z^{ab}(\sigma) = \varepsilon^{rs} \partial_r X^a \partial_s X^b \] (32)
over the space-sheet. It is gratifying to observe the manifest Lorentz invariance of (30). Here we should point out that, in adopting the light-cone gauge, we assumed that there was no winding for the coordinate \( X^+ \). In [24] the corresponding algebra for the matrix regularization was studied. The result coincides with ours in the large-\( N \) limit, in which an additional longitudinal five-brane charge vanishes, provided that one identifies the longitudinal two-brane charge with the central charge in the first line of (30). This identification requires the definition of \( X^- \) in the matrix regularization, a topic that we return to in the next section. The form of the algebra is another indication of the consistency of the supermembrane-supergravity system.

Until now we discussed the general case of a flat target space with possible winding states. To make the identification with the matrix models more explicit, let us again ignore the winding and split off the center-of-mass (CM) variables as in the previous section. As discussed there the structure of the Hamiltonian [34] shows that the wave functions for the supermembrane now factorize into a trivial wave function pertaining to the CM modes and a wave function of the supersymmetric quantum-mechanical system that describes the other (interacting) modes. For the latter the mass operator plays the role of the Hamiltonian. When the mass operator vanishes on the state, then the 32 supercharges act exclusively on the CM coordinates and generate a massless supermultiplet of eleven-dimensional supersymmetry. In case there is no other degeneracy beyond that caused by supersymmetry, the resulting supermultiplet is the one of supergravity, describing the graviton, the antisymmetric tensor and the gravitino. In terms of the \( SO(9) \) helicity representations, it consists of \( 44 \oplus 84 \) bosonic and \( 128 \) fermionic states. For an explicit construction of these states, see [25]. When the mass operator does not vanish on the states, we are dealing with huge supermultiplets consisting of multiples of \( 2^{15} + 2^{15} \) states.

4 The matrix approximation

One may expand the supermembrane coordinates and momenta on the spacesheet in a complete set of functions \( Y_A \) with \( A = 0, 1, 2, \ldots, \infty \). It is convenient to choose \( Y_0 = 1 \). Furthermore we choose a basis of the closed one-forms, consisting of the exact ones, \( \partial_r Y_A \), and a set of closed nonexact forms denoted by \( \phi(\lambda)_r \). Completeness of the \( Y_A \) implies the following decompositions,

\[ \{Y_A, Y_B\} = f_{AB}^C Y_C, \]
\[ \varepsilon^{rs} \phi(\lambda)_r \partial_s Y_A = f_{\lambda A}^B Y_B, \]
\[ \varepsilon^{rs} \phi(\lambda)_r \phi(\lambda')_s = f_{\lambda \lambda'}^A Y_A, \] (33)

so that the constants \( f_{ABC}, f_{\lambda A}^B \) and \( f_{\lambda \lambda'}^A \) represent the structure constants of the infinite-dimensional group of area-preserving diffeomorphisms. Lowering of indices can be done with the help of the invariant metric
\[ \eta_{AB} = \int d^2 \sigma \sqrt{w(\sigma)} Y_A(\sigma) Y_B(\sigma). \] (34)

There is no need to introduce a metric for the \( \lambda \) indices. Observe that we have \( \eta_{00} = 1 \). Furthermore it is convenient to choose the functions \( Y_A \) with \( A \geq 1 \) such that \( \eta_{0A} = 0 \). Completeness implies
\[ \eta^{AB} Y_A(\sigma) Y_B(\rho) = \frac{1}{\sqrt{w(\sigma)}} \delta^{(2)}(\sigma, \rho). \] (35)

After lowering of upper indices, the structure constants are defined as follows [26, 21],
\[ f_{ABC} = \int d^2 \sigma \varepsilon^{rs} \partial_r Y_A(\sigma) \partial_s Y_B(\sigma) Y_C(\sigma), \]
\[ f_{\lambda BC} = \int d^2 \sigma \varepsilon^{rs} \phi(\lambda)_r Y_A(\sigma) \partial_s Y_B(\sigma) Y_C(\sigma), \]
\[ f_{\lambda \lambda' C} = \int d^2 \sigma \varepsilon^{rs} \phi(\lambda)_r (\sigma) \phi(\lambda')_s (\sigma) Y_C(\sigma). \] (36)

Note that we have \( f_{AB0} = f_{AB0} = 0 \).

Using the above basis one may write down the following mode expansions for the phase-space variables of the supermembrane,
\[ \partial_\sigma X(\sigma) = \sum \lambda \, X^\lambda (\sigma) + \sum A \, X^A \partial_\sigma Y_A(\sigma), \]
\[ P(\sigma) = \sum A \, \sqrt{w(\sigma)} P^A Y_A(\sigma), \]
\[ \theta(\sigma) = \sum A \, \theta^A Y_A(\sigma), \] (37)

introducing winding modes for the transverse coordinates \( X \). A similar expansion exists for \( X^- \).

Other tensors are needed, for instance, to write down the Lorentz algebra generators [26]. An obvious tensor is given by
\[ d_{ABC} = \int d^2 \sigma \sqrt{w(\sigma)} Y_A(\sigma) Y_B(\sigma) Y_C(\sigma), \] (38)
which is symmetric in all three indices and satisfies $d_{AB0} = \eta_{AB}$. Another tensor, whose definition is more subtle, arises when expressing $X^-$ in terms of the other coordinates and momenta. We recall that $X^-$ is restricted by (13), which implies the following Gauss-type constraint,

$$\varphi^A = f_{BC}^A \left[ P^B \cdot X^C + P_0^+ \tilde{\theta}^B \gamma^{-C} \right] + f_{B\lambda}^A P^B \cdot X^\lambda \approx 0. \quad (39)$$

The coordinate $X^-$ receives contributions proportional to $Y^A(\sigma)$, which can be parametrized by ($A \neq 0$)

$$X^A_\lambda \approx \frac{1}{2P_0} c_{BC}^A \left[ P^B \cdot X^C + P_0^+ \tilde{\theta}^B \gamma^{-C} \right] + \frac{1}{2P_0} c_{B\lambda}^A P^B \cdot X^\lambda. \quad (40)$$

In addition $X^-$ has CM and winding modes. Observe that the tensors $c_{BC}^A$ and $c_{B\lambda}^A$ are ambiguous, as (40) is only defined up to the constraints (39). The symmetric component of $c_{BC}^A$ is, however, fixed and given by $c_{BC}^A + c_{CB}^A = -2d_{ABC}$. Note that $c_{B0}^A = 0$. There are many other identities between the various tensors which can be derived by using completeness. Some examples are

$$f_{[AB}^E f_{C]}^D E = d_{(AB}^E f_{C)}^D = d_{ABC} f_{[DE}^B f_{FG]}^C = c_{DE}^A f^{BC]E} = d_{E[A} d_{B]C]} D^E = 0. \quad (41)$$

The first identity is just the Jacobi identity for the structure constants of the group of area-preserving diffeomorphisms and the second expresses the fact that $d_{ABC}$ is a group-invariant tensor.

It is possible to replace the group of the area-preserving diffeomorphisms by a finite group, so that (16) defines the Hamiltonian of a supersymmetric quantum-mechanical system based on a finite number of degrees of freedom [27]. In a suitable limit to the infinite-dimensional group we thus recover the supermembrane. This observation enables one to regularize the supermembrane in a supersymmetric way by considering a limiting procedure based on a sequence of groups whose limit yields the area-preserving diffeomorphisms. For membranes of certain topology it is known how to approximate a (sub)group of the area-preserving diffeomorphisms as a particular $N \to \infty$ limit of $\text{SU}(N)$. To be precise, it can be shown that the structure constants of $\text{SU}(N)$ tend to those of the invariant subgroup of the diffeomorphisms generated by the exact vectors, up to corrections of order $1/N^2$. The structure of the corresponding truncations are shown in Fig. 1 for a spherical and a toroidal membrane. While some of the identities (11) remain valid at finite $N$, others receive corrections of order $1/N^2$. Furthermore, the tensors $c_{BC}^A$ and $c_{B\lambda}^A$ are intrinsically undefined at finite $N$. Therefore, the expression for $X^-$ is ambiguous for the matrix model and Lorentz invariance holds only in the large-$N$ limit [26, 28]. We should add that the matrix regularization works also for the case of open supermembranes. In that case one deals with certain subgroups of $\text{SU}(N)$. We refer to [23] for further details.
Figure 1: Truncation of spherical harmonics and Fourier modes corresponding
to an expansion on $S^2$ and $T^2$, respectively. The figure shows the case $N = 7$.
The constant modes associated with the origin correspond to the U(1) generator
while the other $N^2 - 1$ modes are associated with SU($N$).

The correspondence between the membrane expressions and the matrix ex-
pressions are summarized below:

\[
\begin{align*}
\int d^2\sigma \sqrt{w} Y_A &= 0 \quad \iff \quad \mathrm{Tr} (T_A) = 0 \\
\{Y_A, Y_B\} &= f_{AB} C Y_C \quad \iff \quad [T_A, T_B] = f_{AB} C T_C \\
f_{ABC} &= \int d^2\sigma \sqrt{w} Y_A \{Y_B, Y_C\} \quad \iff \quad f_{ABC} = \mathrm{Tr} (T_A [T_B, T_C]) \\
\eta_{AB} &= \int d^2\sigma \sqrt{w} Y_A Y_B \quad \iff \quad \eta_{AB} = \mathrm{Tr} (T_A T_B) \\
\int d^2\sigma \sqrt{w} \{Y_A, Y_B\} &= 0 \quad \iff \quad \mathrm{Tr} ([T_A, T_B]) = 0
\end{align*}
\]

We should stress that the nature of the large-$N$ limit itself is subtle and is
connected to the membrane topology. As long as $N$ is finite, no distinction can
be made with regard to the topology and clearly the generators of U($N$) as found
for different topologies are related by a simple similarity transformation. In this
way one may establish a mapping between functions on the sphere decomposed
into a finite number of spherical harmonics with $l < N$ and functions on the torus
decomposed into a finite number of Fourier modes belonging to some fundamental
lattice (see Fig. 1). But in fact there are inequivalent $N \to \infty$ limits. This is in
line with the fact that there exists no mapping between differentiable functions
on the sphere and the torus in general, in view of their different topological
structure (cf. the discussion in appendix B of [26]). But when taking the trace
the precise nature of the large-$N$ limit seems less relevant. However, at this point,
the diffeomorphisms associated with the harmonic vectors remain problematic;
as it turns out they cannot be incorporated for finite $N$, at least not at the
level of the Lie algebra. This was shown in [23], where it was established that
the finite-$N$ approximation to the structure constants $f_{ABC}$ violates the Jacobi
identities for a toroidal membrane. Therefore it seems impossible to present a
matrix model regularization of the supermembrane with winding contributions.
There exists a standard prescription for dealing with matrix models with winding
[29], however, which is therefore conceptually different. The consequences of this
difference are not well understood. The prescription amounts to adopting the
gauge group \([U(N)]^M\), for winding in one dimension, which in the limit \(M \to \infty\) leads to supersymmetric Yang-Mills theories in \(1 + 1\) dimensions \[29\]. Hence, in this way it is possible to extract extra dimensions from a suitably chosen infinite-dimensional gauge group. This approach can obviously be generalized to a hypertorus.

5 Supersymmetric matrix models and their energy spectrum

The models that one obtains by a truncation of the gauge group of area-preserving
diffeomorphisms to a finite group belong to a class of models proposed long ago as extended models of supersymmetric quantum mechanics with more than four supersymmetries \[12\]. These theories can also be obtained from a supersymmetric
gauge theory in the zero-volume limit. They are based on the Hamiltonian,

\[
H = \frac{1}{g} \text{Tr} \left[ \frac{1}{2} \mathbf{P}^2 - \frac{1}{4} [X^a, X^b]^2 + \frac{1}{2} g \theta^T \gamma_a [X^a, \theta] \right],
\]

and depend on a number of \(d\)-dimensional coordinates \(X = (X^1, \ldots, X^d)\), corresponding momenta \(\mathbf{P}\), as well as real spinorial anticommuting coordinates \(\theta_\alpha\), all taking values in the matrix representation of some Lie algebra. The phase space is restricted to the subspace invariant under the corresponding (compact) Lie group and is therefore subject to Gauss-type constraints. These constraints coincide with the ones discussed in the previous section. The spatial dimension \(d\) and the corresponding spinor dimension are restricted. The models exist for \(d = 2, 3, 5\), or 9 dimensions; the (real) spinor dimension equals 2, 4, 8, or 16, respectively. Naturally this is also the number of independent supercharges.

Just as for the supermembrane we restrict ourselves to the highest-dimensional case. In that case the model contains 16 supercharges, denoted by \(Q^+\). However, additional charges can be obtained when the gauge group has abelian factors by including the zero modes of the fermion field belonging to the abelian supermultiplet (the supercharge of the abelian supermultiplet is already contained in the 16 supercharges, in order that one obtains the total Hamiltonian from the anticommutator of these supercharges). The extra charges will be denoted by \(Q^-\).

Assuming one abelian component associated with the matrix trace, we thus have 32 supercharges defined by

\[
Q^+ = \text{Tr} \left[ (P^a \gamma_a + \frac{1}{2} i [X^a, X^b] \gamma_{ab}) \theta \right], \quad Q^- = g \text{Tr} [\theta].
\]

The \(Q^+\) generate the familiar supersymmetry algebra (in the group-invariant subspace),

\[
\{Q^+_\alpha, Q^+_\beta\} \approx H \delta_{\alpha\beta}.
\]
It is possible, though subtle, to also evaluate the central charges of the supersymmetry algebra for the matrix models [24], which at large $N$ tend to the winding charges exhibited in (30).

As explained in the previous section the supermembrane in the light-cone formulation is described by a quantum-mechanical model of the type above with an infinite-dimensional gauge group corresponding to the area-preserving diffeomorphisms of the membrane spacesheet [4] and a coupling constant $g$ given by the total light-cone momentum $(P_\perp)_0$. The fact that finite truncations of the gauge group are possible allows one to study supermembranes in a convenient regularization. The connection with the supermembrane shows that the manifest SO(9) symmetry, which from the viewpoint of the supermembrane is simply the exact transverse rotational invariance of the lightcone formulation, extends to the full 11-dimensional Lorentz group in the limit of an appropriate infinite-dimensional gauge group [26, 28].

Classical zero-energy configurations require all commutators to vanish,

$$[X^a, X^b] = 0.$$  \hspace{1cm} (46)

Dividing out the gauge group implies that zero-energy configurations are parametrized by $\mathbb{R}^{9N}/S_N$. The zero-energy valleys characterized by (46) extend all the way to infinity where they become increasingly narrow. Their existence raises questions about the nature of the spectrum of the Hamiltonian (13). In the bosonic versions of these models the wave function cannot freely extend to infinity, because at large distances it becomes more and more squeezed in the valley. By the uncertainty principle, this gives rise to kinetic-energy contributions which increase monotonically along the valley. Another way to understand this effect is by noting that oscillations perpendicular to the valleys give rise to a zero-point energy, which induces an effective potential barrier that confines the wave function. This confinement causes the spectrum to be discrete. However, for the supersymmetric models defined by (13) the situation is different. Supersymmetry can cause a cancellation of the transverse zero-point energy. Then the wave function is no longer confined, indicating that the supersymmetric models have a continuous spectrum. The latter was rigourously proven for the gauge group SU($N$) [3].

Whether or not the Hamiltonian (13) allows normalizable or localizable zero-energy states, superimposed on the continuous spectrum, is a subtle question. Early discussion on the existence of such zero-energy states can be found in [4, 6]; more recent discussions can be found in [7, 15]. According to [15] such states do indeed exist in $d = 9$. We should emphasize that there is an important difference between states whose energy is exactly equal to zero and states of positive energy. The supersymmetry algebra implies that zero-energy states are annihilated by the supercharges. Hence, they are supersinglets. The positive-energy states, on the other hand, must constitute full supermultiplets. So they are multiplets consisting
of multiples of $128 + 128$ bosonic + fermionic states (for $d = 9$). However, the presence of the extra supersymmetry charge causes a further degeneration by $128 + 128$ states, so that one obtains zero-energy multiplets of 256 states or positive-energy multiplets comprising (multiples of) 65536 states.

For the supermembrane, the classical zero-mass configurations correspond to zero-area stringlike configurations of arbitrary length, characterized by the condition that

$$\{X^a(\sigma), X^b(\sigma)\} = 0.$$  

(47)

As the supermembrane mass is described by a Hamiltonian of the type (43), the mass spectrum of the supermembrane is continuous for the same reasons as given above. For a supermembrane moving in a target space with compact dimensions, winding may raise the mass of the membrane state. This is so because winding in more than one direction gives rise to a nonzero central charge in the supersymmetry algebra, which sets a lower limit on the membrane mass. This fact should not be interpreted as an indication that the spectrum becomes discrete. The possible continuity of the spectrum hinges on the two features mentioned above. First the system should possess continuous valleys of classically degenerate states. Qualitatively one recognizes immediately that this feature is not directly affected by winding. A classical membrane with winding can still have stringlike configurations of arbitrary length, without increasing its area. Hence the classical instability persists. The second feature is supersymmetry. Without winding it is clear that the valley configurations are supersymmetric, so that one concludes that the spectrum is continuous. With winding the latter aspect is more subtle. However, we note that, when the winding density is concentrated in one part of the spacesheet, then valleys can emerge elsewhere corresponding to stringlike configurations with supersymmetry. Hence, as a space-sheet local field theory, supersymmetry can be broken in one region where the winding is concentrated and unbroken in another. In the latter region stringlike configurations can form, which, at least semiclassically, will not be suppressed by quantum corrections [21]. However, in this case we can only describe the generic features of the spectrum. These arguments do not preclude the existence of mass gaps. Because massless states exist for the $d = 9$ matrix models, we should expect them to exist for the supermembrane. In a flat target space these massless states will constitute massless supermultiplets in 11 spacetime dimensions and will presumably coincide with supermultiplet of states of 11-dimensional supergravity.

The continuous mass spectrum of the supermembrane forms an obstacle in interpreting the membrane states as elementary particles, in analogy to what is done in string theory. Instead the continuity of the spectrum should be viewed as a result of the fact that supermembrane states do not really exist as asymptotic states. As we discussed already in section 1 the membrane collapses into stringlike configurations and the resulting states are to be interpreted as multi-membrane states which possess a continuous mass spectrum. Qualitatively, the situation for
the matrix models [13] based on a finite number of degrees of freedom, is the same as for the supermembrane. Among the zero-energy states there are those where the matrices take a block-diagonal form, which can be regarded as a direct product of states belonging to lower-rank matrix models [14]. The fact that the moduli space of ground states, whose nature is protected by supersymmetry at the quantum-mechanical level, is isomorphic to $\mathbb{R}^{9N}/S_N$, is already indicative of a corresponding description in terms of an $N$-particle Fock space.

The finite-$N$ matrix models have an independent interpretation in string theory. Strings can end on certain defects by means of Dirichlet boundary conditions. These defects are called D-branes (for further references, see [30]). They can have a $p$-dimensional spatial extension and carry Ramond-Ramond charges [31]. D-Branes play an important role in the nonperturbative behaviour of string theory. The models of this section are relevant for D0-branes (Dirichlet particles). The effective short-distance description for D-branes can be derived from simple arguments [13]. As the strings must be attached to the $p$-dimensional branes, we are dealing with open strings whose endpoints are attached to a $p$-dimensional subspace. At short distances, the interactions caused by these open strings are determined by the massless states of the open string, which constitute the ten-dimensional Yang-Mills supermultiplet, propagating in a reduced $(p+1)$-dimensional spacetime. Because the endpoints of open strings carry Chan-Paton factors the effective short-distance behaviour of $N$ D-branes can be described in terms of a $U(N)$ ten-dimensional supersymmetric gauge theory reduced to the $(p+1)$-dimensional world volume of the D-brane. The $U(1)$ subgroup is associated with the center-of-mass motion of the $N$ D-branes.

In the type-IIA superstring one has Dirichlet particles moving in a 9-dimensional space. As the world volume of the particles is one-dimensional ($p = 0$), the short-distance interactions between these particle is thus described by the model of section 1 with gauge group $U(N)$ and $d = 9$. The continuous spectrum without gap is natural here, as it is known that, for static D-branes, the Ramond-Ramond repulsion cancels against the gravitational and dilaton attraction, a similar phenomenon as for BPS monopoles. With this gauge group the coordinates can be described in terms of $N \times N$ hermitean matrices. The valley configurations correspond to the situation where all these matrices can be diagonalized simultaneously. The eigenvalues then define the positions of $N$ D-particles in the 9-dimensional space. As soon as one or several of these particles approach each other then the $[U(1)]^N$ symmetry that is left invariant in the valley, will be enhanced to a non-abelian subgroup of $U(N)$. Clearly there are more degrees of freedom than those corresponding to the D-particles, which are associated with the strings stretching between the D-particles. As we alluded to above the model naturally incorporates configurations corresponding to widely separated clusters of D-particles, each of which can be described by a supersymmetric quantum-mechanics model based on the product of a number of $U(k)$ subgroups forming a maximal commuting subgroup of $U(N)$. When all the D-particles move further apart this corresponds
to configurations deeper and deeper into the potential valleys. These D-particles thus define an independent perspective on the models introduced in this section, which can be used to study their dynamics. We refer to [32] for work along these lines.

The study of D-branes was further motivated by a conjecture according to which the degrees of freedom of M-theory are fully captured by the U(N) supermatrix models in the $N \to \infty$ limit [14]. The elusive M-theory is defined as the strong-coupling limit of type-IIA string theory and is supposed to capture all the relevant degrees of freedom of all known string theories, both at the perturbative and the nonperturbative level [3, 11]. In this description the various string-string dualities are fully incorporated. At large distances M-theory is described by 11-dimensional supergravity. A direct relation between supermembranes and type-IIA string theory was emphasized in [9], based on the relation between extremal black holes in 10-dimensional supergravity [33] and the Kaluza-Klein states of 11-dimensional supergravity in an $S^1$ compactification. In this compactification the Kaluza-Klein photon coincides with the Ramond-Ramond vector field of type-IIA string theory. Therefore Kaluza-Klein states are BPS states whose Ramond-Ramond charge is proportional to their mass. Hence they have the same characteristics as the Dirichlet particles. From this correspondence with the Kaluza-Klein spectrum one may infer that the corresponding matrix models must possess zero-energy bound states, whose existence was indeed established in [15]. Furthermore, the effective interaction between infinitely many Dirichlet particles must lead to a theory that is identical to that of an elementary supermembrane. There are alternative compactifications of M-theory which make contact with other string theories. Supermembranes have been used to provide evidence for the duality of M-theory on $R^{10} \times S_1/Z_2$ and 10-dimensional $E_8 \times E_8$ heterotic strings [11]. Finally let us mention the so-called double-dimensional reduction of membranes, which is a truncation that leads to fundamental strings [34]. Whether this truncation remains relevant in the context of the full supermembrane theory is an open question.

6 Membranes and matrix models in curved space

So far we considered supermembranes moving in a flat target superspace. Their description follows from substituting the flat superspace expressions (2) into the supermembrane action (1). However, these expressions can also be evaluated for nontrivial backgrounds, such as those induced by a nontrivial target-space metric, a target-space tensor field and a target-space gravitino field, corresponding to the fields of (on-shell) 11-dimensional supergravity. This background can in principle be cast into superspace form by a procedure known as ‘gauge completion’ [35]. For 11-dimensional supergravity, the first steps of this procedure were carried out long ago [36] and recently the results were determined to second order in the
fermionic coordinates $\theta$.

To elucidate the generic effects of nontrivial backgrounds for membrane theories, let us confine ourselves for the moment to the purely bosonic theory and present the light-cone formulation of the membrane in a background consisting of the metric $G_{\mu\nu}$ and the tensor gauge field $C_{\mu\nu\rho}$. In the subsequent sections we will include the fermionic coordinates. The Lagrangian density for the bosonic membrane follows directly from (1),

$$L = -\sqrt{-g} - \frac{1}{6} \epsilon^{ijk} \partial_i X^\mu \partial_j X^\nu \partial_k X^\rho C_{\rho\mu,\nu},$$

where $g_{ij} = \partial_i X^\mu \partial_j X^\nu \eta_{\mu\nu}$. In the light-cone formulation, the coordinates are decomposed in the usual fashion as $(X^+, X^-, X^a)$ with $a = 1 \ldots 9$. Furthermore we use the diffeomorphisms in the target space to bring the metric in a convenient form,

$$G_{--} = G_{a-} = 0.$$  

Just as for a flat target space, we identify the time coordinate of the target space with the world-volume time, by imposing the condition $X^+ = \tau$. Moreover we denote the spacesheet coordinates of the membrane by $\sigma^r$, $r = 1, 2$. Following the same steps as for the membrane in flat space, one arrives at a Hamiltonian formulation of the theory in terms of coordinates and momenta. These phase-space variables are subject to a constraint, which takes the same form as for the membrane theory in flat space, namely,

$$\phi_r = P_a \partial_r X^a + P_- \partial_r X^- \approx 0.$$  

Of course, the definition of the momenta in terms of the coordinates and their derivatives does involve the background fields, but at the end all explicit dependence on the background cancels out in the phase-space constraints.

The total Hamiltonian now follows straightforwardly and is equal to

$$H = \int d^2 \sigma \left\{ \frac{G_{++}}{P_- - C_-} \left[ \frac{1}{2} \left( P_a - C_a - \frac{P_- - C_-}{G_{++}} G_{a+} \right)^2 + \frac{1}{4} (\epsilon^{rs} \partial_r X^a \partial_s X^b)^2 \right] - \frac{P_- - C_-}{2 G_{++}} G_{++} - C_+ - C_{++} + c^r \phi_r \right\}.$$  

where we have included the Lagrange multipliers $c^r$ coupling to the constraints. Observe that transverse indices are contracted with the metric $G_{ab}$ or its inverse. Furthermore we have made use of the following definitions,

$$C_a = -\epsilon^{rs} \partial_r X^- \partial_s X^b C_{-ab} + \frac{1}{2} \epsilon^{rs} \partial_r X^b \partial_s X^c C_{abc},$$

$$C_{\pm} = \frac{1}{2} \epsilon^{rs} \partial_r X^a \partial_s X^b C_{\pm ab},$$

$$C_{+-} = \epsilon^{rs} \partial_r X^- \partial_s X^a C_{+-a}.$$  

The gauge choice $X^+ = \tau$ still allows for $\tau$-dependent reparametrizations of the world-space coordinates $\sigma^r$, which in turn induce transformations on the
Lagrange multiplier \( c^r \) through the Hamilton equations of motion. In addition there remains the freedom of performing tensor gauge transformations of the target-space three-form \( C_{\mu\nu\rho} \). In order to rewrite (51) in terms of a gauge theory of area-preserving diffeomorphisms it is desirable to obtain a Hamiltonian which is polynomial in momenta and coordinates. For this the dynamics of \( P_\tau - C_\tau \) needs to become trivial, i.e. \( \partial_\tau (P_\tau - C_\tau) = 0 \), allowing us to set it equal to some space-sheet density \( \sqrt{w(\sigma)} \). The residual invariance group is then constituted by the area-preserving diffeomorphisms that leave \( \sqrt{w} \) invariant. The \( \tau \)-independence of \( P_\tau - C_\tau \) can be achieved by firstly assuming that the background fields are \( X^{\pm} \)-independent. Secondly one uses the tensor gauge transformations to set \( C_{\tau ab} \) equal to a constant antisymmetric matrix. One then has

\[
\partial_\tau (P_\tau - C_\tau) \approx \partial_\tau \left[ -\varepsilon^{rs} \partial_s X^a C_{+a} + (P_\tau - C_\tau) c^r \right].
\] (53)

We now choose a gauge such that the right-hand side of this equation vanishes. In that case the total Hamiltonian takes the following form,

\[
H = \int d^2\sigma \left\{ \frac{G_+}{P_\tau - C_\tau} \left[ \frac{1}{2} \left( P_a - C_a - \frac{P_\tau - C_\tau}{G_+} G_{a+} \right)^2 + \frac{1}{4} (\varepsilon^{rs} \partial_r X^a \partial_s X^b)^2 \right] \\
- \frac{P_\tau - C_\tau}{2 G_+} G_{++} - C_+ \\
+ \frac{1}{P_\tau - C_\tau} \left[ \varepsilon^{rs} \partial_r X^a \partial_s X^b P_a C_{+b} + C_\tau C_{++} \right] \right\},
\] (54)

where \( P_\tau - C_\tau \propto \sqrt{w} \) and \( C_{-ab} \) constant.

At this point one can impose further gauge choices and set \( G_{++} = 1 \) and \( C_{+a} = 0 \). Taking also \( C_{-ab} = 0 \) the corresponding Hamiltonian can be cast in Lagrangian form in terms of a gauge theory of area-preserving diffeomorphisms \[10\],

\[
w^{-1/2} \mathcal{L} = \frac{1}{2} (D_0 X^a)^2 + D_0 X^a \left( \frac{1}{2} C_{abc} \{ X^b, X^c \} + G_{a+} \right) \\
- \frac{1}{4} \{ X^a, X^b \}^2 + \frac{1}{2} G_{++} + \frac{1}{2} C_{+ab} \{ X^a, X^b \},
\] (55)

where the covariant derivatives were introduced in section 3. For convenience we have set \( (P_\tau)_0 = 1 \). In the case of compact dimensions, it may not always be possible to set \( C_{+a} \) and \( C_{-ab} \) to zero, although they can be restricted to constants. One can then follow the same procedure as above. As alluded to in the first reference of \[17\], the Lagrangian then depends explicitly on \( X^- \), a feature that was already exhibited earlier for the winding membrane (cf. (26). However, in the case at hand, the constraint makes the resulting expression for \( X^- \) extremely nontrivial. This is clearly an issue that deserves more study. Recently the antisymmetric constant matrix \( C_{-ab} \) was conjectured to play a role for the matrix model compactification on a noncommutative torus \[17\]. It should be interesting to see what the role is of (54) in this context.
With a reformulation of the membrane in background fields as a gauge theory of area-preserving diffeomorphisms at one's disposal, one may consider its regularization through a matrix model by truncating the mode expansion for coordinates and momenta along the lines explained in section 4. This leads to a replacement of Poisson brackets by commutators, integrals by traces and products of commuting fields by symmetrized products of the corresponding matrices. At that point the original target-space covariance is affected, as the matrix reparametrizations in terms of symmetrized products of matrices do not possess a consistent multiplication structure; this is just one of the underlying difficulties in the construction of matrix models in curved space \[16\]. Finally, one may now study interactions between membranes by considering the behaviour of a test membrane in a background field induced by another membrane \[41\].

7 Supergravity in 11 dimensions

Before moving to the more general superspace backgrounds associated with supergravity in 11 spacetime dimensions, we give a brief summary of this theory in order to establish our conventions. The theory is based on an “elfbein” field $e_{\mu r}$, a Majorana gravitino field $\psi_\mu$ and a 3-rank antisymmetric gauge field $C_{\mu\nu\rho}$. Its Lagrangian\[4\] can be written as follows \[2\],

\[
\mathcal{L} = -\frac{1}{2} e R(e, \omega) - 2 e \bar{\psi}_{\mu} \Gamma^{\mu\nu\rho} D_\nu \left[ \frac{1}{2} (\omega + \hat{\omega}) \right] \psi_\rho - \frac{1}{96} e \left( F_{\mu\nu\rho\sigma} \right)^2 \\
- \frac{1}{216} \epsilon^{\mu_1 \cdots \mu_11} F_{\mu_1\mu_2\mu_3\mu_4} F_{\mu_5\mu_6\mu_7\mu_8} C_{\mu_9\mu_{10}\mu_{11}} \\
- \frac{1}{96} \epsilon \left( \bar{\psi}_{\lambda} \Gamma^{\mu\rho\sigma\lambda} \psi_\tau + 12 \bar{\psi}^\mu \Gamma^{\mu\rho\sigma} \psi^\sigma \right) \left( F + \hat{F} \right)_{\mu\nu\rho\sigma},
\]

where $e = \det e_{\mu r}$, $\omega_{\mu rs}$ denotes the spin connection and $F_{\mu\nu\rho\sigma}$ the field strength of the antisymmetric tensor. A caret denotes that these quantities have been made covariant with respect to local supersymmetry. The derivative $D_\mu (\omega)$ is covariant with respect to local Lorentz transformations,

\[
D_\mu (\omega) \epsilon = \left( \partial_\mu - \frac{1}{2} \omega_\mu^{rs} \Gamma_{rs} \right) \epsilon.
\]

The supersymmetry transformations are equal to

\[
\delta e_\mu^r = 2 \bar{\epsilon} \Gamma^r \psi_\mu, \\
\delta \psi_\mu = D_\mu (\hat{\omega}) \epsilon + T_{\mu\nu\rho\kappa} \epsilon \hat{F}_{\nu\rho\kappa}, \\
\delta C_{\mu\nu\rho} = -6 \bar{\epsilon} \Gamma_{[\mu\nu} \psi_{\rho]}.
\]

with

\[
T_{r s t u v} = \frac{1}{288} \left( \Gamma_{r s t u v} - 8 \delta_r^{[s} \Gamma^t_{u v]} \right).
\]

\[^3\]Gamma matrices satisfy $\{ \Gamma_{r}, \Gamma_{s} \} = 2 \eta_{rs}$, where $\eta_{rs}$ is the tangent-space metric $\eta_{rs} = \text{diag}(-, +, \cdots, +)$. Gamma matrices with multiple indices denote symmetrized products with unit strength. In particular $\Gamma_{r_{1} r_{2} \cdots r_{11}} = \Gamma_{1}^{r_{1} r_{2} \cdots r_{11}} = 1 \epsilon^{r_{1} r_{2} \cdots r_{11}}$. The Dirac conjugate is defined by $\bar{\psi} = i \psi^\dagger \Gamma^0$ for a generic spinor $\psi$. 

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and $\hat{F}_{\mu\nu\rho\sigma}$ is the supercovariant field strength,

$$\hat{F}_{\mu\nu\rho\sigma} = 4 \partial_{[\mu} C_{\nu\rho\sigma]} + 12 \bar{\psi}_{[\mu} \Gamma_{\nu\rho} \psi_{\sigma]} .$$

The supercovariant spin connection $\hat{\omega}^{rs}_{\mu}$ is the one that corresponds to a vanishing supercovariant torsion tensor.

The Lagrangian (56) is derived in the context of the so-called “1.5-order” formalism, in which the spin connection is defined as a dependent field determined by its (algebraic) equation of motion, whereas its supersymmetry variation in the action is treated as if it were an independent field [42]. Furthermore we note the presence of a Chern-Simons-like term $F \wedge F \wedge C$ in the Lagrangian. Under tensor gauge transformations,

$$\delta_C C_{\mu\nu\rho\sigma} = 3 \partial_{[\mu} \xi_{\nu\rho\sigma]} ,$$

the corresponding action is thus only invariant up to surface terms.

We have the following bosonic field equations and Bianchi identities,

$$R_{\mu\nu} = \frac{1}{144} g_{\mu\nu} F_{\rho\sigma \lambda\tau} F^{\rho\sigma \lambda\tau} - \frac{1}{12} F_{\mu\rho\sigma\lambda} F^{\rho\sigma\lambda} ,$$

$$\partial_{[\mu} (e F^{\rho\sigma\lambda\tau}] = \frac{1}{1152} \varepsilon^{\rho\sigma\lambda\mu_1...\mu_8} F_{\mu_1\mu_2\mu_3\mu_4} F_{\mu_5\mu_6\mu_7\mu_8} ,$$

where $H_{\mu_1...\mu_7}$ is the dual field strength,

$$H_{\mu_1...\mu_7} = \frac{1}{1152} \varepsilon_{\mu_1...\mu_11} F_{\mu_8\mu_9\mu_{10}\mu_{11}} - \frac{1}{12} F_{[\mu_1\mu_2\mu_3\mu_4} C_{\mu_5\mu_6\mu_7]} .$$

When the third equation of (62) and (63) receive contributions from certain source terms on the right-hand side, the corresponding charges can be associated with the ‘flux’-integral of $H_{\mu_1...\mu_7}$ and $F_{\mu_1\mu_2\mu_3\mu_4}$ over the boundary of an 8- or a 5-dimensional spatial volume, respectively. This volume is transverse to a $p = 2$ and $p = 5$ brane configuration, and the corresponding charges are 2- and 5-rank Lorentz tensors. For solutions of 11-dimensional supergravity that contribute to these charges, see e.g. [44, 45, 46, 9].

It is straightforward to evaluate the supersymmetry algebra on these fields. The commutator of two supersymmetry transformations yields a general-coordinate transformation, a supersymmetry transformation, a local Lorentz transformation, and a gauge transformation associated with the tensor gauge field,

$$[\delta(\epsilon_1), \delta(\epsilon_2)] = \delta_{gct}(\xi^\mu) + \delta(\epsilon_3) + \delta_L(\lambda^{rs}) + \delta_C(\xi_{\mu\nu}) .$$
The parameters of the transformations on the right-hand side are given by

\[ \xi^\mu = 2 \bar{\epsilon}_2 \Gamma^\mu \epsilon_1, \]
\[ \epsilon_3 = -\xi^\mu \psi_\mu, \]
\[ \lambda^{rs} = -\xi^\mu \tilde{\omega}_{rs}^\mu + \frac{1}{72} \bar{\epsilon}_2 \left[ \Gamma^{rs\mu
u\rho\sigma} \hat{F}_{\mu\nu\rho\sigma} + 24 \Gamma_{\mu
u} \hat{F}^{rs\mu\nu} \right] \epsilon_1, \]
\[ \xi_{\mu\nu} = -\xi^\rho C_{\rho\mu\nu} - 2 \bar{\epsilon}_2 \Gamma_{\mu\nu} \epsilon_1. \] (66)

8 Superspace in terms of component fields

After the definition of the component fields and transformation rules of supergravity in 11 spacetime dimensions, we briefly introduce the method for constructing superspace backgrounds in terms of these component fields. At the end of this section we present the superspace quantities of interest to second order in the anticommuting coordinates \( \theta \). The superspace geometry with coordinates \( Z^M = (x^\mu, \theta^\alpha) \) is encoded in the supervielbein \( E^A_M \) and a spin-connection field \( \Omega_M{}^{AB} \). In what follows we will not pay much attention to the spin-connection, which is not an independent field. Furthermore we have an antisymmetric tensor gauge field \( B_{MNP} \), subject to tensor gauge transformations,

\[ \delta B_{MNP} = 3 \partial_{[M} \Xi_{NP]} \]. (67)

Unless stated otherwise the derivatives with respect to \( \theta \) are always left derivatives.

Under superspace diffeomorphisms corresponding to \( Z^M \to Z^M + \Xi^M(Z) \), the super-vielbein and tensor gauge field transform as

\[ \delta E^A_M = \Xi^N \partial_N E^A_M + \partial_M \Xi^N E^A_N, \]
\[ \delta B_{MNP} = \Xi^Q \partial_Q B_{MNP} + 3 \partial_{[M} \Xi^Q B_{NP]} \]. (68)

Tangent-frame rotations are \( Z \)-dependent Lorentz transformations that act on the vielbein according to

\[ \delta E^A_M = \frac{1}{2} \left( \Lambda^{rs} L_{rs} \right)^A_B E^B_M, \] (69)

where the Lorentz generators \( L_{rs} \) are defined by

\[ \frac{1}{2} (\Lambda^{rs} L_{rs})^u_v = \Lambda^u_v, \quad \frac{1}{2} (\Lambda^{rs} L_{rs})^a_b = \frac{1}{4} \Lambda^{rs} (\Gamma_{rs})^a_b. \] (70)

The superspace that we are dealing with is not unrestricted but is subject to certain constraints and gauge conditions. Furthermore, we will not describe an off-shell situation as all superfields will be expressed entirely in terms of the three component fields of on-shell 11-dimensional supergravity, the elfbein \( e_\mu^r \),
the antisymmetric tensor gauge field $C_{\mu\nu\rho}$ and the gravitino field $\psi_\mu$. As a result of these restrictions the residual symmetry transformations are confined to 11-dimensional diffeomorphisms with parameters $\xi^\mu(x)$, local Lorentz transformations with parameters $\lambda^{rs}(x)$, tensor-gauge transformations with parameters $\xi_{\mu\nu}(x)$ and local supersymmetry transformations with parameters $\epsilon(x)$. To derive how the superfields are parametrized in terms of the component fields it is necessary to also determine the form of the superspace transformation parameters, $\Xi^M$, $\Lambda^{rs}$ and $\Xi_{MN}$, that generate the supersymmetry transformations. Here it is important to realize that we are dealing with a gauge-fixed situation. For that reason the superspace parameters depend on both the $x$-dependent component parameters defined above as well as on the component fields. This has two consequences. First of all, local supersymmetry transformations reside in the superspace diffeomorphisms, the Lorentz transformations and the tensor gauge transformations, as $\Xi^M$, $\Lambda^{rs}$ and $\Xi_{MN}$ are all expected to contain $\epsilon$-dependent terms. Thus, when considering supersymmetry variations of the various fields, one must in principle include each of the three possible superspace transformations. Secondly, when considering the supersymmetry algebra, it is crucial to also take into account the variations of the component fields on which the parameters $\Xi^M$, $\Lambda^{rs}$ and $\Xi_{MN}$ will depend.

The method of casting component results into superspace has a long history and is sometimes called ‘gauge completion’. For results in 4 spacetime dimensions we refer the reader to [35], while results in 11 dimensions in low orders of $\theta$ were presented in [36, 37] (see also [38]). Here we will follow [37] where results were obtained to second order in $\theta$. There are two, somewhat complimentary, ways to obtain information on the embedding of component fields in superspace geometry. One is to consider the algebra of the supersymmetry transformations as generated by the superspace transformations and to adjust it to the supersymmetry algebra of the component fields. This determines the superspace transformation parameters. The other is to compare the transformation rules for the superfields with the known transformations of the component fields. This leads to a parametrization of both the superfields and the transformation parameters in terms of the component fields and parameters. The evaluation proceeds order-by-order in the $\theta$-coordinates, but at each level one encounters ambiguities which can be fixed by suitable higher-order coordinate redefinitions and gauge choices. The first step in this iterative procedure is the identification at zeroth-order in $\theta$ of some of the component fields and transformation parameters with corresponding components of the superfield quantities. The underlying assumption is that this identification can always be implemented by choosing an appropriate gauge. An obvious
identification is given by [17, 18, 35, 36],

\[ E^r_{\mu}(x, \theta = 0) = e^r_{\mu}(x), \quad \Xi^\mu(x, \theta = 0) = \xi^\mu(x), \]
\[ E^a_{\mu}(x, \theta = 0) = \psi^a_{\mu}(x), \quad \Xi^a(x, \theta = 0) = \epsilon^a(x), \]
\[ B_{\mu\nu}(x, \theta = 0) = C_{\mu\nu}(x), \quad \Xi_{\mu\nu}(x, \theta = 0) = \zeta_{\mu\nu}(x). \]

As was explained above, the component supersymmetry transformations with parameters \( \epsilon(x) \) are generated by a linear combination of a superspace diffeomorphism, a local Lorentz and a tensor gauge transformation; their corresponding parameters will be denoted by \( \Xi^{M}(\epsilon) \), \( \Lambda^{rs}(\epsilon) \) and \( \Xi_{MN}(\epsilon) \), respectively. Given the embedding of the component fields into the superfields, application of these specific superspace transformations should produce the very same transformation rules that were defined directly at the component level in the previous section. The structure of the commutator algebra of unrestricted infinitesimal superspace transformations is obvious. Two diffeomorphisms yield another diffeomorphism, two Lorentz transformations yield another Lorentz transformation, according to the Lorentz group structure, while two tensor transformations commute. On the other hand, a diffeomorphism and a local Lorentz transformation yield another Lorentz transformation, and a diffeomorphism and a tensor gauge transformation yield another gauge transformation. All other combinations commute.

For further details we refer to [37] and we proceed directly to the results. For the supervielbein \( E^A_M \) the following expressions were found,

\[
E^r_{\mu} = e^r_{\mu} + 2 \bar{\theta} \Gamma^r \psi_{\mu} \\
+ \bar{\theta} \Gamma^r \left[ - \frac{1}{4} \tilde{\omega}^{st}_{\mu} \Gamma_{st} + T_{\mu}^{\nu\rho\sigma\lambda} \tilde{F}_{\nu\rho\sigma\lambda} \right] \theta + O(\theta^3),
\]
\[
E^a_{\mu} = \psi^a_{\mu} - \frac{1}{4} \tilde{\omega}^{rs}_{\mu} (\Gamma_{rs} \theta)^a + (T_{\mu}^{\nu\rho\sigma\lambda} \theta)^a \tilde{F}_{\nu\rho\sigma\lambda} + O(\theta^2),
\]
\[
E^r_{\alpha} = - (\bar{\theta} \Gamma^r)_{\alpha} + O(\theta^3),
\]
\[
E^a_{\alpha} = \delta^a_{\alpha} + M^a_{\alpha} + O(\theta^3),
\]

where \( M^a_{\alpha} \) characterizes the \( \tilde{F} \theta^2 \)-contributions, which have not been evaluated. Observe that \( E^a_{\mu} \) was determined only up to terms of order \( \theta^2 \). The result for the tensor field \( B_{MNP} \) reads as follows,

\[
B_{\mu\nu\rho} = C_{\mu\nu\rho} - 6 \bar{\theta} \Gamma_{[\mu\nu}\psi_{\rho]} \\
- 3 \bar{\theta} \Gamma_{[\mu\nu} \left[ - \frac{1}{4} \tilde{\omega}^{rs}_{\rho] \Gamma_{rs} + T_{\rho]}^{\sigma\lambda\kappa\tau} \tilde{F}_{\sigma\lambda\kappa\tau} \right] \theta - 12 \bar{\theta} \Gamma_{[\mu\nu} \theta \Gamma^{\rho} \psi_{\rho]} + O(\theta^3),
\]
\[
B_{\mu\nu\alpha} = (\bar{\theta} \Gamma_{\mu
u})_{\alpha} - \frac{3}{2} \bar{\theta} \Gamma^{\rho} \psi_{[\mu} (\theta \Gamma_{\nu]\rho)_{\alpha} + \frac{1}{3} (\bar{\theta} \Gamma^{\rho})_{\alpha} \bar{\theta} \Gamma_{\rho}[\mu \psi_{\nu]} + O(\theta^3),
\]
\[
B_{\mu\alpha\beta} = (\bar{\theta} \Gamma_{\mu\nu})_{(\alpha} (\bar{\theta} \Gamma^{\nu})_{\beta)} + O(\theta^3),
\]
\[
B_{\alpha\beta\gamma} = (\bar{\theta} \Gamma_{\mu\nu})_{(\alpha} (\bar{\theta} \Gamma^{\nu})_{\beta)} (\bar{\theta} \Gamma^{\mu})_{\gamma)} + O(\theta^3). \tag{73}
\]

For completeness we included the \( \theta^3 \)-term in \( B_{\alpha\beta\gamma} \) which were already known from the flat-superspace results [2].
Then we turn to some of the transformation parameters. The supersymmetry transformations consistent with the fields specified above, are generated by superspace diffeomorphisms, local Lorentz transformations and tensor gauge transformations. The corresponding parameters are as follows. The superspace diffeomorphisms are expressed by

\[
\Xi_\mu(\epsilon) = \bar{\theta} \Gamma^\mu \epsilon - \bar{\theta} \Gamma^\nu \epsilon \bar{\theta} \Gamma^\mu \psi_\nu + O(\theta^3),
\]

\[
\Xi^\alpha(\epsilon) = \epsilon^\alpha - \bar{\theta} \Gamma^\nu \epsilon \psi^\alpha_\mu + \frac{1}{4} \bar{\theta} \Gamma^\nu \epsilon \hat{\omega}^r s (\Gamma_{rs})^\alpha + e^\beta N^\alpha_\beta + O(\theta^3),
\]

where \(N^\alpha_\beta\) encodes unknown terms proportional to \(\hat{F} \theta^2\). The Lorentz transformation is given by

\[
\Lambda^{rs}(\epsilon) = \epsilon \Gamma^\mu \theta \hat{\omega}_r^s + \frac{1}{144} \bar{\theta} (\Gamma^{rs} \Gamma^{\rho \sigma} \hat{F}_{\rho \sigma} + 24 \Gamma^\mu \hat{F}^{rs} \epsilon) + O(\theta^2).
\]

The tensor gauge transformations are parametrized by

\[
\Xi_{\mu \nu}(\epsilon) = \epsilon (C_{\mu \rho} \Gamma^\rho + \Gamma_{\mu \nu}) \theta + \bar{\theta} \Gamma^\rho \epsilon \bar{\theta} (C_{\mu \rho} \Gamma^\rho + \Gamma_{\mu \nu}) \psi_\rho + \frac{1}{4} \bar{\theta} \Gamma^\rho \psi_\rho \bar{\theta} \Gamma^\rho \epsilon + O(\theta^3),
\]

\[
\Xi_{\alpha \mu}(\epsilon) = \frac{1}{6} \bar{\theta} \Gamma^\nu \epsilon (\bar{\theta} \Gamma_{\mu \nu})_\alpha + \frac{1}{6} (\theta \Gamma^\nu)_{\alpha} \bar{\theta} \Gamma_{\mu \nu} \epsilon + O(\theta^3),
\]

\[
\Xi_{\alpha \beta}(\epsilon) = O(\theta^3).
\]

Finally local Lorentz transformations are generated by a superspace local Lorentz transformation combined with a diffeomorphism. The corresponding expressions read

\[
\Lambda^{rs}(\lambda) = \lambda^{rs},
\]

\[
\Xi^\alpha(\lambda) = -\frac{1}{4} \lambda^{rs} (\Gamma_{rs})^\alpha.
\]

9 The supermembrane in a nontrivial background

The initial supermembrane action (1) is manifestly covariant under independent superspace diffeomorphisms, tangent-space Lorentz transformations and tensor gauge transformations. For the specific superspace fields associated with 11-dimensional on-shell supergravity that we presented in the previous section, this is no longer true and one has to restrict oneself to the superspace transformations corresponding to the component supersymmetry, general-coordinate, local Lorentz and tensor gauge transformations. When writing (1) in components, utilizing the expressions found in the previous sections, one thus obtains an action that is covariant under the restricted superspace diffeomorphisms (74) acting on the superspace coordinates \(Z^M = (X^\mu, \theta^\alpha)\) (including the spacetime arguments of the background fields) combined with usual transformations on the component
fields (we return to this point shortly). Note that the result does not constitute an invariance. Rather it implies that actions corresponding to two different sets of background fields that are equivalent by a component gauge transformation, are the same modulo a reparametrization of the supermembrane embedding coordinates. In order to be precise let us briefly turn to an example and consider the action of a particle moving in a curved spacetime background with metric $g_{\mu \nu}$,

$$S[X^\mu, g_{\mu \nu}(X)] = -m \int dt \sqrt{-g_{\mu \nu}(X(t))} \dot{X}^\mu(t) \dot{X}^\nu(t). \quad (78)$$

This action, which is obviously invariant under world-line diffeomorphisms, satisfies $S[X'^\mu, g'_{\mu \nu}(X')] = S[X^\mu, g_{\mu \nu}(X)]$, where $X'^\mu$ and $X^\mu$ are related by a target-space general coordinate transformation which also governs the relation between $g'_{\mu \nu}$ and $g_{\mu \nu}$. Of course, when considering a background that is invariant under (a subset of the) general coordinate transformations (so that $g = g'$), then the action will be invariant under the corresponding change of the coordinates. This is the situation that we will address in the next section, where we take a specific background metric with certain isometries. In that context the relevant target space for (78) is an anti-de-Sitter ($AdS_d$) space, which has isometries that constitute the group $SO(d - 1, 2)$, where $d$ is the spacetime dimension. Then (78) describes a one-dimensional field theory with an $SO(d - 1, 2)$ invariance group. In the particular case of $d = 2$ this invariance can be re-interpreted as a conformal invariance for a supersymmetric quantum mechanical system.\(^4\)

Using the previous results one may now write down the complete action of the supermembrane coupled to background fields up to order $\theta^2$. Direct substitution leads to the following result for the supervielbein pull-back,

$$\Pi'_i = \partial_i X^\mu \left( \epsilon^r_\mu + 2 \bar{\theta} \Gamma^r \psi_\mu - \frac{1}{4} \bar{\theta} \Gamma^{rs} \theta \bar{\omega}^{rst} + \bar{\theta} \Gamma^r T_{\mu}^{\nu \rho \sigma \lambda} \theta \hat{F}_{\nu \rho \sigma \lambda} \right) + \frac{3}{4} \bar{\theta} \Gamma^r \partial_i \theta + O(\theta^3),$$

$$\Pi^i = \partial_i X^\mu \left( \bar{\psi}_\mu^a - \frac{1}{4} \bar{\omega}^{rs} \Gamma_{rs} \theta \bar{\psi}_\mu^a + (T_{\mu}^{\nu \rho \sigma \lambda})^a \hat{F}_{\nu \rho \sigma \lambda} \right) + \partial_i \theta^a + O(\theta^2). \quad (79)$$

Consequently the induced metric is known up to terms of order $\theta^3$. Furthermore the pull-back of the tensor field equals

$$-\frac{1}{6} \varepsilon^{ijk} \Pi^A_i \Pi^B_j \Pi^C_k B_{CBA} = -\frac{1}{6} \varepsilon^{ijk} \partial_i Z^M \partial_j Z^N \partial_k Z^P B_{PMN} = \frac{1}{6} dX^{\mu \rho} \left[ C_{\mu \nu \rho} - 6 \bar{\theta} \Gamma_{\mu \nu} \psi_\rho + \frac{2}{4} \bar{\theta} \Gamma_{\mu \nu} \psi_\rho \right]$$

$$-3 \bar{\theta} \Gamma_{\mu \nu} \psi_\rho \hat{F}_{\sigma \lambda \kappa \tau} - 12 \bar{\theta} \Gamma_{\sigma \mu} \psi_\nu \bar{\theta} \Gamma_{\lambda \kappa \tau} \psi_\rho$$

$$-\varepsilon^{ijk} \bar{\theta} \Gamma_{\mu \nu} \partial_k \theta \left[ \frac{1}{2} \partial_i X^\mu (\partial_j X^\nu + \bar{\theta} \Gamma^\nu \partial_j \theta) + \frac{1}{6} \bar{\theta} \Gamma^\mu \partial_i \theta \bar{\theta} \Gamma^\nu \partial_j \theta \right]$$

\(^4\)This situation arises generically for any $p$-brane moving in a target space that is locally the product of $AdS_{p+2}$ and some compact space. The conformal interpretation was emphasized in...
\[+ \frac{1}{4} \varepsilon^{ijk} \partial_i X^\mu \partial_j X^\nu \left[ 4 \bar{\theta} \Gamma_{\rho \mu} \partial_k \theta \bar{\theta} \Gamma^\rho \psi_\nu - 2 \bar{\theta} \Gamma^\rho \partial_k \theta \bar{\theta} \Gamma_{\rho \mu} \psi_\nu \right] + \mathcal{O}(\theta^3), \quad (80)\]

where we have introduced the abbreviation \( dX^{\mu \nu \rho} = \varepsilon^{ijk} \partial_i X^\mu \partial_j X^\nu \partial_k X^\rho \) for the world-volume form. Observe that we included also the terms of higher-order \( \theta \)-terms that were determined in previous sections and listed in (73). The first formula of (79) and (80) now determine the supermembrane action (1) up to order \( \theta^3 \).

As an illustration of what we stated at the beginning of this section, one may consider the effect of the superspace diffeomorphisms (74) on \( \Pi^a_i \). We only need the variations to first order in \( \theta \), so that we substitute \( X^\mu \rightarrow X^\mu + \bar{\theta} \Gamma^\mu \epsilon \) and \( \theta \rightarrow \theta + \epsilon - \bar{\theta} \Gamma^\mu \epsilon \psi_\mu \) into (79). For \( \Pi^r_i \) this induces a variation which can be rewritten as

\[
\delta \Pi^r_i = \partial_i X^\mu \left[ \delta \epsilon^r_\mu + 2 \bar{\theta} \Gamma^r \delta \psi_\mu \right] - \Lambda^{rs}(\epsilon) \Pi^s_i + \mathcal{O}(\theta^2). \quad (81)
\]

The first term on the right-hand side represents the change of \( \Pi^r_i \) under the supersymmetry variations (58) of the background fields. The second term represents a Lorentz transformation whose parameter is given by (75). For the induced metric, given by \( g_{ij} = \Pi^r_i \Pi^s_j \eta_{rs} \), the Lorentz transformation drops out, so that the effect of the coordinate change of \((X^\mu, \theta^a)\) is the same as when performing a supersymmetry transformation of the background fields. This implies that the first term in the supermembrane action (1) has indeed the required transformation behaviour.

A similar result holds for the variation of \( \Pi^a_i \) under the coordinate change as well as for the pull-back of the tensor field. Again we refrain from giving further details, but refer instead to [37].

While the above results were guaranteed to hold on the basis of the procedure followed in the previous section, the \( \kappa \)-invariance of the action is an independent issue. The \( \kappa \)-symmetry transformations are defined in the unrestricted superspace and will be given below. In principle, it should be possible to derive the transformation rules in the gauge-fixed superspace situation that we are working with. However, it is not necessary to do so, because we are only interested in establishing the invariance of the action. Both the original and the gauge-fixed action should be \( \kappa \)-symmetric, so that we can just use the original superspace diffeomorphisms corresponding to \( \kappa \)-symmetry and substitute them in the gauge-fixed action. These \( \kappa \)-transformations take the form of superspace coordinate changes defined by

\[
\delta Z^M E_M^r = 0, \quad \delta Z^M E_M^a = (1 - \Gamma)^a b \kappa^b, \quad (82)
\]

where \( \kappa^a(\zeta) \) is a local fermionic parameter and the matrix \( \Gamma \) is defined by

\[
\Gamma = \frac{\varepsilon^{ijk} \Pi^r_i \Pi^s_j \Pi^t_k \Gamma_{rst}}{6 \sqrt{-g}}, \quad (83)
\]
with \( g = \det g_{ij} \). It satisfies the following properties,

\[
\Gamma^2 = 1, \quad \Gamma^r \Gamma^i = \Pi^r_i \Gamma^s \Gamma^r_s = \frac{1}{2} g_{ij} \varepsilon^{jkl} \Pi^r_k \Pi^s_l \Gamma^r_{rs}.
\] (84)

Therefore the matrix \((1 - \Gamma)\) in (82) is a projection operator. As a consequence, this allows one to gauge away half of the \( \theta \) degrees of freedom. With these definitions one can prove that the action is invariant under local \( \kappa \)-symmetry in the appropriate order in \( \theta \), up to a world-volume surface term which is a generalization of (3). At this level in \( \theta \) there are as yet no constraints on the background. These constraints will be required in higher orders of \( \theta \) and will take the form of the supergravity field equations. Again we refer to [37] for details.

10 Near-horizon geometries

In the previous section we discussed the determination of superspace quantities, i.e. the superspace vielbein and the tensor gauge field, in terms of the fields of 11-dimensional on-shell supergravity. The corresponding expressions are obtained by iteration order-by-order in \( \theta \) coordinates, but except for the leading terms it is hard to proceed with this program. Nevertheless these results enable one to write down the 11-dimensional supermembrane action coupled to a nontrivial supergravity component-field background to second order in \( \theta \), so that one can start a study of the supermembrane degrees of freedom in the corresponding background geometries. In analogy to the bosonic case discussed in section 6, the light-cone supermembrane turns out to be equivalent to a gauge theory of area-preserving diffeomorphisms coupled to background fields, modulo corresponding assumptions on the background geometry. This \( \text{U}(\infty) \) gauge theory may then in turn be regularized by a supersymmetric \( \text{U}(N) \) quantum-mechanical model in curved backgrounds with a certain degree of supersymmetry. Whether or not this will shed some light on the problem of formulating matrix models in curved spacetime is at present still an open question, as we have already been alluding to in section 6.

However, in specific backgrounds with a certain amount of symmetry, it is possible to obtain results to all orders in \( \theta \). Interesting candidates for such backgrounds are the membrane [14] and the five-brane solution [15] of 11-dimensional supergravity, as well as solutions corresponding to the product of anti-de-Sitter spacetimes with compact manifolds [18]. Coupling to the latter solutions, which appear near the horizon of black D-branes [33], seem especially appealing in view of the recent results on a connection between large-\( N \) superconformal field theories and supergravity on a product of AdS space with a compact manifold [19]. The target-space geometry induced by the \( p \)-branes interpolates between \( \text{AdS}_{p+2} \times B \) near the horizon, where \( B \) denotes some compact manifold (usually
a sphere), and flat \((p + 1)\)-dimensional Minkowski space times a cone with base \(B\).

This program has been carried out recently for the type-IIB superstring and the D3-brane in an IIB-supergravity background of this type \([50, 57, 59]\). In the context of 11-dimensional supergravity the \(AdS_4 \times S^7\) and \(AdS_7 \times S^4\) backgrounds stand out as they leave 32 supersymmetries invariant \([51, 51]\). These backgrounds are associated with the near-horizon geometries corresponding to two- and five-brane configurations and thus to possible conformal field theories in 3 and 6 spacetime dimensions with 16 supersymmetries, whose exact nature is not yet completely known. In this section we consider the supermembrane in these two backgrounds \([52]\). As the corresponding spaces are local products of homogeneous spaces, their geometric information can be extracted from appropriate coset representatives leading to standard invariant one-forms corresponding to the vielbeine and spin-connections. The approach of \([52]\) differs from that of \([53]\), which is also discussed at this conference; in the latter one constructs the geometric information exploiting simultaneously the kappa symmetry of the supermembrane action, while in \([52]\) the geometric information is determined independently from the supermembrane action. The results for the geometry coincide with those of \([54]\).

As is well known, the compactifications of the theory to \(AdS_4 \times S^7\) and \(AdS_7 \times S^4\) are induced by the antisymmetric 4-rank field strength of M-theory. These two compactifications are thus governed by the Freund-Rubin field \(f\), defined by (in Pauli-Källén convention, so that we can leave the precise signature of the spacetime open),

\[
F_{\mu\nu\rho\sigma} = 6f e \varepsilon_{\mu\nu\rho\sigma},
\]

with \(e\) the vierbein determinant. When \(f\) is purely imaginary we are dealing with an \(AdS_4 \times S^7\) background while for real \(f\) we have an \(AdS_7 \times S^4\) background. The nonvanishing curvature components corresponding to the 4- and 7-dimensional subspaces are equal to

\[
R_{\mu\nu\rho\sigma} = -4f^2(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}),
\]

\[
R_{\mu'\nu'\rho'\sigma'} = f^2(g_{\mu'\rho'}g_{\nu'\sigma'} - g_{\mu'\sigma'}g_{\nu'\rho'}).
\]

Here \(\mu, \nu, \rho, \sigma\) and \(\mu', \nu', \rho', \sigma'\) are 4- and 7-dimensional world indices, respectively. We also use \(m_{4,7}\) for the inverse radii of the two subspaces, defined by \(|f|^2 = m_4^2 = \frac{1}{4}m_7^2\). The Killing-spinor equations associated with the 32 supersymmetries in this background take the form

\[
(D_\mu - f\gamma_\mu\gamma_5 \otimes 1)\epsilon = (D_{\mu'} + \frac{f}{2}1 \otimes \gamma_{\mu'})\epsilon = 0,
\]

where we make use of the familiar decomposition of the (hermitean) gamma matrices \(\gamma_\mu\) and \(\gamma_{\mu'}\), appropriate to the product space of a 4- and a 7-dimensional
subspace (see [52]). Here $D_\mu$ and $D_{\mu'}$ denote the covariant derivatives containing the spin-connection fields corresponding to SO(3,1) or SO(4) and SO(7) or SO(6,1), respectively.

The algebra of isometries of the $AdS_4 \times S^7$ and $AdS_7 \times S^4$ backgrounds is given by $osp(8|4; \mathbb{R})$ and $osp(6,2|4)$. Their bosonic subalgebra consists of $so(8) \oplus sp(4) \simeq so(9) \oplus so(3,2)$ and $so(6,2) \oplus usp(4) \simeq so(6,2) \oplus so(5)$, respectively. The spinors transform in the $(8,4)$ of this algebra. Observe that the spinors transform in a chiral representation of $so(8)$ or $so(5)$.

One may decompose the generators of $osp(8|4)$ or $osp(6,2|4)$ in terms of irreducible representations of the bosonic $so(7) \oplus so(3,1)$ and $so(6,1) \oplus so(4)$ subalgebras. In that way one obtains the bosonic (even) generators, the fermionic (odd) generators, and the fermionic tangent-space indices by $a, b, \ldots$ and $a', b', \ldots$ for 4- or 7-dimensional indices.

The commutation relations between even generators are

\[
[P_r, P_s] = -4f^2 M_{rs}, \quad [P_{r'}, P_{s'}] = f^2 M_{r's'},
\]

\[
[P_r, M_{st}] = \delta_{rs} P_t - \delta_{rt} P_s, \quad [P_{r'}, M_{s't'}] = \delta_{r's'} P_{t'} - \delta_{r't'} P_{s'},
\]

\[
[M_{rs}, M_{tu}] = \delta_{ru} M_{st} + \delta_{st} M_{ru} \quad [M_{r's'}, M_{t'u'}] = \delta_{r't'} M_{s'u'} + \delta_{s't'} M_{r'u'}
\]

\[-\delta_{rt} M_{su} - \delta_{su} M_{rt}, \quad -\delta_{r't'} M_{s'u'} - \delta_{s't'} M_{r'u'}.
\]

The odd-even commutators are given by

\[
[P_r, Q_{a'a'}] = -f(\gamma_r' \gamma_5)_{a'}^b Q_{b'a'}, \quad [P_{r'}, Q_{a'a'}] = -\frac{1}{2}f(\gamma_r')^a_{b'} Q_{ab'},
\]

\[
[M_{rs}, Q_{a'a'}] = -\frac{1}{2}(\gamma_{rs})_{a}^b Q_{ba'}, \quad [M_{r's'}, Q_{a'a'}] = -\frac{1}{2}(\gamma_{r's'})^a_{b'} Q_{ab'}.
\]

Finally, we have the odd-odd anti-commutators,

\[
\{Q_{a'a'}, Q_{b'b'}\} = -(\gamma_5 C)_{ab} \left(2(\gamma_{r'} C')_{a'b'} P^{r'} - f(\gamma_{r's'} C')_{a'b'} M^{r's'}\right)
\]

\[-C_{a'b'} \left(2(\gamma_r C)_{ab} P^r + 2f(\gamma_{rs} \gamma_5 C)_{ab} M^{rs}\right).
\]

All other (anti)commutators vanish. The normalizations of the above algebra were determined by comparison with the supersymmetry algebra in the conventions of [37] in the appropriate backgrounds.

However, one can return to 11-dimensional notation and drop the distinction between 4- and 7-dimensional indices so that the equations obtain a more compact form. In that case the above (anti)commutation relations that involve the supercharges can be concisely written as,

\[
[P_r, \bar{Q}] = \bar{Q} T_r^{stu} F_{stu}, \quad [M_{rs}, \bar{Q}] = \frac{1}{2} \bar{Q} \Gamma_{rs},
\]

\[
\{Q, \bar{Q}\} = -2\Gamma_r P^r + \frac{1}{144}\left[\Gamma^{rstuvw} F_{stu} + 24 \Gamma_{tu} F^{rstu}\right] M_{rs},
\]

31
where the tensor $T$ is was defined in (59). Note, however, that the above formulae are only applicable in the background where the field strength takes the form given in (87). In what follows, we will only make use of (91).

11 Coset-space representatives of $AdS_4 \times S^7$ and $AdS_7 \times S^4$

Both backgrounds that we consider correspond to homogenous spaces and can thus be formulated as coset spaces [55]. In the case at hand these (reductive) coset spaces $G/H$ are $OSp(8|4; \mathbb{R})/SO(7) \times SO(3, 1)$ and $OSp(6, 2|4)/SO(6, 1) \times SO(4)$.

To each element of the coset $G/H$ one associates an element of $G$, which we denote by $L(Z)$. Here $Z_A$ stands for the coset-space coordinates $x^r, \theta^a$ (or, alternatively, $x^r, y^r$ and $\theta^{a\alpha}$). The coset representative $L$ transforms from the left under constant $G$-transformations corresponding to the isometry group of the coset space and from the right under local $H$-transformations: $L \rightarrow L' = gLh^{-1}$.

The vielbein and the torsion-free $H$-connection one-forms, $E$ and $\Omega$, are defined through\footnote{A one-form $V$ stands for $V \equiv dZ^A V_A$ and an exterior derivative acts according to $dV \equiv -dZ^B \wedge dZ^A \partial_A V_B$. Fermionic derivatives are thus always left-derivatives.}

$$dL + L \Omega = LE,$$ (92)

where

$$E = E^r P_r + \bar{E}Q, \quad \Omega = \frac{1}{2} \Omega^{rs} M_{rs}.$$ (93)

The integrability of (92) leads to the Maurer-Cartan equations,

$$d\Omega - \Omega \wedge \Omega \frac{1}{2} E^r \wedge E^s \left[ P_r, P_s \right] - \frac{1}{256} \bar{E} \left[ \Gamma^{rstuvw} F_{tuvw} + 24 \Gamma_{tu} F^{rstu} \right] E M_{rs} = 0,$$

$$dE^r - \Omega^{rs} \wedge E_s - \bar{E} \Gamma^r \wedge E = 0,$$

$$dE + E^r \wedge T_{r}^{tuvw} E_{tuvw} - \frac{1}{4} \Omega^{rs} \wedge \Gamma_{rs} E = 0,$$ (94)

where we suppressed the spinor indices on the anticommuting component $E^a$.

The first equation in a fermion-free background reproduces (86) upon using the commutation relations (88).

Now the question is how to determine the vielbeine and connections to all orders in $\theta$ for the spaces of interest. First, observe that the choice of the coset representative amounts to a gauge choice that fixes the parametrization of the coset space. We will not insist on an explicit parametrization of the bosonic part of the space. It turns out to be advantageous to factorize $L(Z)$ into a group element of the bosonic part of $G$ corresponding to the bosonic coset space, whose parametrization we leave unspecified, and a fermion factor. Hence one may write

$$L(Z) = \ell(x) \hat{L}(\theta), \quad \text{with } \hat{L}(\theta) = \exp[\bar{\theta}Q].$$ (95)
There exists a convenient trick \cite{56, 57, 58} according to which one first rescales the odd coordinates according to \( \theta \rightarrow t\theta \), where \( t \) is an auxiliary parameter that we will put to unity at the end. Taking the derivative with respect to \( t \) of (92) then leads to a first-order differential equation for \( E \) and \( \Omega \) (in 11-dimensional notation),

\[
\dot{E} - \dot{\Omega} = d\bar{Q} + (E - \Omega) \dot{\theta}Q - \dot{\theta}Q (E - \Omega) \tag{96}
\]

After expanding \( E \) and \( \Omega \) on the right-hand side in terms of the generators and using the (anti)commutation relations \cite{91} we find the coupled first-order linear differential equations,

\[
\begin{align*}
\dot{E}^a &= (d\theta + E^r T_r^{stuv}\theta F_{stuv} - \frac{1}{4} \Omega^s \Gamma_{rs}\theta)^a, \\
\dot{E}^r &= 2 \bar{\theta} \Gamma^r E, \\
\dot{\Omega}^r &= \frac{1}{72} \bar{\theta} \left[ \Gamma^{rstuv} F_{tuvw} + 24 \Gamma_{tu} F^{rstu} \right] E.
\end{align*}
\tag{97}
\]

These equations can be solved straightforwardly \cite{57} and one finds

\[
\begin{align*}
E(x, \theta) &= \sum_{n=0}^{16} \frac{1}{(2n+1)!} \mathcal{M}^{2n} D\theta, \\
E^r(x, \theta) &= dx^\mu e^r_{\mu} + 2 \sum_{n=0}^{15} \frac{1}{(2n+2)!} \bar{\theta} \Gamma^r \mathcal{M}^{2n} D\theta, \\
\Omega^{rs}(x, \theta) &= dx^\mu \omega^{rs}_{\mu} + \frac{1}{72} \sum_{n=0}^{15} \frac{1}{(2n+2)!} \bar{\theta} \left[ \Gamma^{rstuv} F_{tuvw} + 24 \Gamma_{tu} F^{rstu} \right] \mathcal{M}^{2n} D\theta,
\end{align*}
\tag{98}
\]

where the matrix \( \mathcal{M}^2 \) equals,

\[
(M^2)_{ab} = 2 (T_r^{stuv}\theta)^a F_{stuv} (\bar{\theta} \Gamma^r)_b - \frac{1}{288} (\Gamma_{rs}\theta)^a (\bar{\theta} \left[ \Gamma^{rstuv} F_{tuvw} + 24 \Gamma_{tu} F^{rstu} \right])_b.
\tag{99}
\]

and

\[
D\theta^a \equiv \dot{E}^a|_{t=0} = \left( d\theta + e^r T_r^{stuv}\theta F_{stuv} - \frac{1}{4} \omega^{rs} \Gamma_{rs}\theta \right)^a.
\tag{100}
\]

It is straightforward to write down the lowest-order terms in these expansions,

\[
\begin{align*}
E^r &= e^r + \bar{\theta} \Gamma^r d\theta + \bar{\theta} \Gamma^r (e^m T_m^{stuv} F_{stuv} - \frac{1}{4} \omega^{st} \Gamma_{st}) \theta + \mathcal{O}(\theta^4), \\
E &= d\theta + (e^r T_r^{stuv} F_{stuv} - \frac{1}{4} \omega^{rs} \Gamma_{rs}) \theta + \mathcal{O}(\theta^3), \\
\Omega^{rs} &= \omega^{rs} + \frac{1}{144} \bar{\theta} \left[ \Gamma^{rstuv} F_{tuvw} + 24 \Gamma_{tu} F^{rstu} \right] d\theta + \mathcal{O}(\theta^4),
\end{align*}
\tag{101}
\]

which agree completely with those given in section 8 (and, for the spin-connection field, in \cite{36}).
This information can now be substituted into the first part of the supermembrane action (1). By similar techniques one can also determine the Wess-Zumino-Witten part of the action by first considering the most general ansatz for a four-form invariant under tangent-space transformations. Using the lowest-order expansions of the vielbeine (101) and comparing with (37) shows that only two terms can be present. Their relative coefficient is fixed by requiring that the four-form is closed, something that can be verified by making use of the Maurer-Cartan equations (94). The result takes the form

$$F^{(4)} = \frac{1}{4!} \left[ E^r \wedge E^s \wedge E^t \wedge E^u F_{rstu} - 12 E \wedge \Gamma_{rs} E \wedge E^r \wedge E^s \right].$$

(102)

To establish this result we also needed the well-known quartic-spinor identity in 11 dimensions. The overall factor in (102) is fixed by comparing to the normalization of the results given in [37].

Because $F^{(4)}$ is closed, it can be written locally as $F^{(4)} = dB$. The general solution for $B$ can be found by again exploiting the one-forms with rescaled $\theta$ coordinates according to $\theta \to t \theta$ and deriving a differential equation. Using the lowest order result for $B$ this equation can be solved and yields

$$B = \frac{1}{6} e^r \wedge e^s \wedge e^t C_{rst} - \int_0^1 dt \bar{\theta} \Gamma_{rs} E \wedge E^r \wedge E^s,$$

(103)

where the vielbein components contain the rescaled $\theta$'s. This answer immediately reproduces the flat-space result upon substitution of $F_{rstu} = \omega^{rst} = 0$.

In order to obtain the supermembrane action one substitutes the above expressions in the action (1). The resulting action is then invariant under local fermionic $\kappa$-transformations (1) as well as under the superspace isometries corresponding to $osp(8|4)$ or $osp(6,2|4)$.

We have already emphasized that the choice of the coset representative amounts to adopting a certain gauge choice in superspace. The choice that was made in [52] connects directly to the generic 11-dimensional superspace results, written in a Wess-Zumino-type gauge, in which there is no distinction between spinorial world and tangent-space indices. In specific backgrounds, such as the ones discussed here, gauge choices are possible which allow further simplifications. For this we refer to [57] and other contributions to this volume.

The results of this section provide a strong independent check of the low-order $\theta$ results obtained by gauge completion for general backgrounds [37, 36]. A great amount of clarity was gained by expressing our results in 11-dimensional language, so that both the $AdS_4 \times S^7$ and the $AdS_7 \times S^4$ solution could be covered in one go. Note that in both these backgrounds the gravitino vanishes.

We have no reasons to expect that the 11-dimensional form of our results will coincide with the expressions for a generic 11-dimensional superspace (with the gravitino set to zero) at arbitrary orders in $\theta$. 

34
12 Concluding remarks

In this lecture I discussed supermembranes in a variety of situations. Closed supermembranes can live in flat spaces, or in superspaces corresponding to supergravity in 11 spacetime dimensions. When the target space has compact dimensions there is the possibility of winding. Furthermore open supermembranes exist, though with rather restrictive boundary conditions. In many cases the supermembrane theory can be regularized, resulting in a super matrix model based on a finite number of degrees of freedom. These are the very same models that describe the short-distance dynamics of D0-branes. A fascinating feature that these models share is that their Hilbert space contains both single-particle and multi-particle states. For the supermembrane the same feature is present with respect to states with and without winding.

Yet many questions are still open, as was already stressed in the introduction. For instance, the nature of the supermembrane spectrum is hard to understand. One could be tempted and conjecture that the supermembrane mass spectrum (in flat space) corresponds simply to the single- and multiple-particle states of supergravity! At this moment I have no idea how to test the correctness of such a conjecture. Another open issue concerns the large-\(N\) limit of the super matrix models.

On the more technical side it is gratifying that explicit constructions of supermembranes in certain nontrivial backgrounds are now possible. The complete M-theory two-brane action in \(AdS_4 \times S^7\) and \(AdS_7 \times S^4\) to all orders in \(\theta\) represents a further step in the program of finding the complete anti-de-Sitter background actions for the superstring [56, 57] and the M2-, D3- [58] and M5-branes initiated for the bosonic part in [19]. Furthermore, by studying the interaction between a test membrane in the background of an M2- or an M5-brane, one may hope to learn more about the interactions between branes. Some of these issues have already been considered recently [41].

The material of this lecture is by no means complete. For instance, we did not discuss matrix strings, nor did we review the matrix-model calculations pertaining to supergraviton scattering. Some of these issues are discussed by other speakers at this workshop.

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