LIMIT THEOREMS FOR JACOBI ENSEMBLES WITH LARGE PARAMETERS

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Abstract. Consider $\beta$-Jacobi ensembles with the distributions
\[ c_{k_1,k_2,k_3} \prod_{1 \leq i < j \leq N} (x_j - x_i)^{k_3} \prod_{i=1}^{N} (1 - x_i)^{\frac{k_1+k_2}{2} - \frac{1}{2}} (1 + x_i)^{\frac{k_2}{2} - \frac{1}{2}} dx \]
of the eigenvalues on the alcoves $A := \{ x \in \mathbb{R}^N | -1 \leq x_1 \leq \ldots \leq x_N \leq 1 \}$. For $(k_1,k_2,k_3) = \kappa \cdot (a,b,1)$ with $a, b > 0$ fixed, we derive a central limit theorem for these distributions for $\kappa \to \infty$. The drift and the covariance matrix of the limit are expressed in terms of the zeros of classical Jacobi polynomials. We also determine the eigenvalues and eigenvectors of the covariance matrices.

These results are related to corresponding limits for $\beta$-Hermite and $\beta$-Laguerre ensembles for $\beta \to \infty$ by Dumitriu and Edelman and by Voit.

1. Introduction

We derive a central limit theorem (CLT) for $\beta$-Jacobi random matrix ensembles for fixed dimension $N$ where all parameters of the models tend to infinity. These ensembles are usually described (see e.g. [F, K, KN, M]) via their joint eigenvalue distributions $\mu_k$ on the alcoves
\[ A := \{ x \in \mathbb{R}^N | -1 \leq x_1 \leq \ldots \leq x_N \leq 1 \} \]
with the Lebesgue densities
\[ c_{k_1,k_2,k_3} \prod_{1 \leq i < j \leq N} (x_j - x_i)^{k_3} \prod_{i=1}^{N} (1 - x_i)^{\frac{k_1+k_2}{2} - \frac{1}{2}} (1 + x_i)^{\frac{k_2}{2} - \frac{1}{2}} \]
with parameters $k := (k_1,k_2,k_3) \in [0,\infty[^3$ and a normalization $c_k$ which can be determined via a Selberg integral; see [FW] for the background.

It is known from Kilip and Nenciu [KN] that all measures $\mu_k$ appear as joint distributions of the ordered eigenvalues of some tridiagonal random matrix models similar to the tridiagonal models for $\beta$-Hermite and $\beta$-Laguerre models of Dumitriu and Edelman [DE1]. Another matrix model in the Jacobi case is given in [L].

The tridiagonal models for $\beta$-Hermite and $\beta$-Laguerre models of [DE1] are used in [DE2] to derive limit theorems for $\beta \to \infty$. In particular, [DE2] contains an CLT where the covariance matrices $\Sigma$ of the limits are described in terms of the zeros of the $N$-th Hermite or Laguerre polynomial respectively. Moreover, these CLTs were derived in [V] directly where there formulas appear for the inverses $\Sigma^{-1}$.

In the present paper we transfer the approach of [V] to $\beta$-Jacobi ensembles. For

\[ c_{k_1,k_2,k_3} \prod_{1 \leq i < j \leq N} (x_j - x_i)^{k_3} \prod_{i=1}^{N} (1 - x_i)^{\frac{k_1+k_2}{2} - \frac{1}{2}} (1 + x_i)^{\frac{k_2}{2} - \frac{1}{2}} \]

\[ \prod_{i=1}^{N} (1 - x_i)^{\frac{k_1+k_2}{2} - \frac{1}{2}} (1 + x_i)^{\frac{k_2}{2} - \frac{1}{2}} \]

\[ \prod_{i=1}^{N} (1 - x_i)^{\frac{k_1+k_2}{2} - \frac{1}{2}} (1 + x_i)^{\frac{k_2}{2} - \frac{1}{2}} \]

\[ \prod_{i=1}^{N} (1 - x_i)^{\frac{k_1+k_2}{2} - \frac{1}{2}} (1 + x_i)^{\frac{k_2}{2} - \frac{1}{2}} \]

\[ \prod_{i=1}^{N} (1 - x_i)^{\frac{k_1+k_2}{2} - \frac{1}{2}} (1 + x_i)^{\frac{k_2}{2} - \frac{1}{2}} \]

\[ \prod_{i=1}^{N} (1 - x_i)^{\frac{k_1+k_2}{2} - \frac{1}{2}} (1 + x_i)^{\frac{k_2}{2} - \frac{1}{2}} \]
Consider multiplicity parameters \( k = (k_1, k_2, k_3) = \kappa \cdot (a, b, 1), N \in \mathbb{N}, a \geq 0, b > 0 \) fixed, \( \kappa \to \infty \), we prove an CLT where the drift and the inverse \( \Sigma^{-1} \) of the covariance matrices are described in terms of the zeros of some Jacobi polynomial \( P_{N}^{(a, \beta)} \); see Theorem 2.4. Our CLT is closely related to the CLT A.1 in the appendix A of [BG]. Moreover, our CLT with its inverse covariance matrices is used in [AHV] to compute the covariance matrices themselves. We expect that the results of the present paper can be used to derive limit results for \( \kappa \to \infty \) and then \( N \to \infty \) as in [AHV] [GK] for Hermite ensembles. Further related CLTs can be found in Proposition 2.3 of [N] and in [J, KN].

We mention that for all \( k := (k_1, k_2, k_3) \in [0, \infty]^3 \), the measures \( \mu_k \) on \( A \) are the stationary distributions of so-called \( \beta \)-Jacobi processes \( (X_t^k)_{t \geq 0} \); see [Dem]. These processes are diffusions on \( A \) with reflecting boundaries where the generators of the associated Feller semigroups are second order differential operators \( D_k \) which appear in the Heckman-Opdam theory of hypergeometric functions associated with root systems; see [HS]. The Heckman-Opdam Jacobi polynomials form multivariate systems of orthogonal polynomials with the \( \mu_k \) as orthogonality measures, and they are eigenfunctions of the \( D_k \). In the case of Hermite and Laguerre ensembles, the associated diffusions are multivariate Bessel processes which appear in the study of Calogero-Moser-Sutherland particle models [DV, F]. Limit theorems for the Bessel processes for large parameters were studied in this context in [AKM1, AKM2, AV1, VW]. We expect that similar results are available for \( \beta \)-Jacobi processes.

A comment about our parameters \( (k_1, k_2, k_3) \) which come from the special functions associated with the root system \( BC_N \); see [HS] [AV1] [AV2] [V] [VW]. In the random matrix community usually our \( \kappa = k_3 \) is denoted by \( \beta \).

This paper is organized as follows: In Section 2 we show that the measures \( \mu_{\kappa \cdot (a, b, 1)} \) tend to some point measure \( \delta_z \) for \( \kappa \to \infty \) where the coordinates of \( z \in A \) consist of the ordered zeros of the classical Jacobi polynomials \( P_{N}^{(a, \beta)} \) with \( \alpha := a + b - 1 > -1 \) and \( \beta = b - 1 > -1 \). This result is in principle known (see Section 6.7 of [S], Section 3.5 of [I], or Appendix A of [BG]) and is needed for our CLT, the main result of this paper. We shall state this CLT in algebraic and trigonometric coordinates. Moreover we discuss how our CLT is related to the corresponding CLTs for Hermite and Laguerre ensembles in [DE2, V]. Section 3 is then devoted to the proof of the CLT and the eigenvalues and eigenvectors of the covariance matrices in trigonometric coordinates.

2. CENTRAL LIMIT THEOREMS IN THE FREEZING REGIME

Consider multiplicity parameters \( k = (k_1, k_2, k_3) = \kappa \cdot (a, b, 1) \) where we fix \( a \geq 0, b > 0 \). We study the limit \( \kappa \to \infty \) of the probability measures \( \mu_k \) with densities (1.2) on the alcove \( A \) defined in (1.1). For this let \( X_\kappa \) be \( \mathbb{R}^N \)-valued random variables with the distributions

\[
\mu_\kappa := \mu_{\kappa \cdot (a, b, 1)}.
\]

As the \( \mu_\kappa \) have Lebesgue-densities \( f_\kappa \) of the form

\[
f_\kappa(x) = c_\kappa g(x) \phi(x)^\kappa \quad \text{with} \quad c_\kappa := c_{\kappa \cdot (a, b, 1)} \quad (2.1)
\]
on \( A \) with suitable continuous functions \( g, \phi \) and suitable constants \( c_\kappa \), we use the following well-known Laplace method to obtain a first limit law:

**Lemma 2.1.** Let \( g, \phi : \mathbb{R}^N \to \mathbb{R}_+ \) be continuous functions such that \( \phi \) has a unique global maximum at \( x_0 \in \mathbb{R}^N \). If \( g(x_0) > 0 \), and if \( g \cdot \phi^\kappa \in L^1(\mathbb{R}^N, \lambda^N) \) for \( \kappa \geq 1 \),
Theorem 2.3. Let \( b > 0 \) and \( \alpha, \beta \) be random variables as above. Let \( z = (z_1, \ldots, z_N) \) be the vector in the interior of \( A \) which consists of the ordered zeros of \( P_N^{(\alpha,\beta)} \) with \( \alpha, \beta \) as in Lemma 2.2. Then, for \( \kappa \to \infty \) the \( X_\kappa \) converge to \( z \) in probability.

Proof. Lemmas 2.1 and 2.2 imply that the distributions \( \mu_\kappa \) of the \( X_\kappa \) tend weakly to \( \delta_z \). This fact is equivalent to the statement of the theorem.
We now study the Jacobi ensemble in trigonometric coordinates which fits to the theory of special functions associated with the root systems. For this we define the probability measures \( \tilde{\mu}_k \) on the trigonometric alcoves

\[
\tilde{\mathcal{A}} := \{ t \in \mathbb{R}^N \mid \frac{\pi}{2} \geq t_1 \geq \ldots \geq t_N \geq 0 \}
\]

with the Lebesgue densities

\[
\tilde{c}_k \cdot \prod_{1 \leq i < j \leq N} (\cos(2t_j) - \cos(2t_i))^{k_3} \prod_{i=1}^N (\sin(t_i)^{k_1} \sin(2t_i)^{k_2})
\]

(2.6)

with a suitable normalization \( \tilde{c}_k > 0 \). A short computation shows that the measures \( \mu_k \) on \( A \) with the densities (2.6) for \( \kappa > 0 \) are the pushforward measures of the \( \tilde{\mu}_k \) under the transformation

\[
T : \tilde{\mathcal{A}} \rightarrow A, \quad T(t_1, \ldots, t_N) := (\cos(2t_1), \ldots, \cos(2t_N)). \quad (2.7)
\]

Using this transformation, Theorem 2.3 reads as follows:

**Theorem 2.4.** Let \( a \geq 0 \) and \( b > 0 \). Let \( \tilde{X}_\kappa \) be \( \tilde{\mathcal{A}} \)-valued random variables with the distributions \( \tilde{\mu}_{\kappa(a,b,1)} \) with the densities (2.5) for \( \kappa > 0 \). Then, for \( \kappa \rightarrow \infty \) the \( \tilde{X}_\kappa \) converge to \( T^{-1}(z) = (\frac{1}{2} \arccos z_1, \ldots, \frac{1}{2} \arccos z_N) \) in probability.

We now turn to a CLT for the random variables \( \tilde{X}_\kappa \) in trigonometric form which is the main result of this paper. It will be proved in Section 3.

**Theorem 2.5.** Let \( a \geq 0 \) and \( b > 0 \). Let \( \tilde{X}_\kappa \) be random variables with the distributions \( \tilde{\mu}_\kappa \). Then

\[
\sqrt{\kappa}(\tilde{X}_\kappa - T^{-1}(z))
\]

converges in distribution for \( \kappa \rightarrow \infty \) to the centered \( N \)-dimensional normal distribution \( N(0, \Sigma) \) with covariance matrix \( \Sigma \) whose inverse \( \Sigma^{-1} := \tilde{S} = (\tilde{s}_{i,j})_{i,j=1,\ldots,N} \) satisfies

\[
\tilde{s}_{i,j} = \begin{cases}
4 \sum_{\substack{\not=j \not=0 \ldots N \mid z_i z_j}} \frac{1-z^4}{(z_i-z_j)^3} + 2(a + b) \frac{1+z_i}{1-z_i} + 2b \frac{1-z_i}{1+z_i} & \text{for } i = j \\
a \sqrt{\frac{(1-z_i)(1-z_j)}{(z_i-z_j)^3}} & \text{for } i \not=j
\end{cases}
\]

Furthermore the eigenvalues of \( \tilde{\Sigma}^{-1} \) are simple and given by

\[
\lambda_k = 2k(2N + \alpha + \beta + 1 - k) > 0 \quad (k = 1, \ldots, N).
\]

Each \( \lambda_k \) has a eigenvector of the form

\[
v_k := \left( q_{k-1}(z_1) \sqrt{1-z_1^2}, \ldots, q_{k-1}(z_N) \sqrt{1-z_N^2} \right)^T
\]

for polynomials \( q_{k-1} \) of order \( k-1 \) which are orthonormal w.r.t the discrete measure

\[
\mu_{N,\alpha,\beta} := (1-z_1^2)\delta_{z_1} + \ldots + (1-z_N^2)\delta_{z_N}
\]

This CLT can be transfered clearly into a CLT in algebraic coordinates. However, in these coordinates, the eigenvalues and eigenvectors are more complicated:

**Theorem 2.6.** Let \( a \geq 0 \) and \( b > 0 \). Let \( X_\kappa \) be random variables with the distributions \( \mu_\kappa \) as described in the beginning of this section. Then

\[
\sqrt{\kappa}(X_\kappa - z)
\]
converges for $κ \to \infty$ to the centered $N$-dimensional normal distribution $N(0, Σ)$ with covariance matrix $Σ$ whose inverse $Σ^{-1} =: S = (s_{i,j})_{i,j=1,...,N}$ is given by

$$s_{i,j} = \begin{cases} \frac{1}{z_i - z_j} & \text{for } i = j \\ \frac{a+b+1}{2} \frac{1}{(1-z_j)^2} & \text{for } i = j \\ \frac{b}{2} \frac{1}{(1+z_j)^2} & \text{for } i \neq j \end{cases}$$

Our CLTs 2.5 and 2.6 are closely related with the following determinantal formula for the zeros of the Jacobi polynomials. It will be also proved in the next section.

**Proposition 2.7.** For $N ∈ \mathbb{N}$ consider the ordered zeros $z_1 \leq ... \leq z_N$ of $P_N^{(α, β)}$ with $α, β > -1$. Then the determinant of the matrix $S := (s_{i,j})_{i,j=1,...,N}$ with

$$s_{i,j} = \begin{cases} \frac{1}{z_i - z_j} & \text{for } i = j \\ \frac{a+1}{2} \frac{1}{(1-z_j)^2} & \text{for } i = j \\ \frac{b}{2} \frac{1}{(1+z_j)^2} & \text{for } i \neq j \end{cases}$$

satisfies

$$\det(S) = \frac{N! ((N + α + β + 1)N_3)}{2^{3N} (α + 1)_N (β + 1)_N}.$$  

**Remark 2.8.** Theorem 2.6 and Proposition 2.7 are closely related to corresponding results for Hermite and Laguerre ensembles in [V]. Moreover, the distributions of Hermite and Laguerre ensembles may be seen as limits of Theorem 2.6 and Proposition 2.7. For this we fix $α, β > 0$ and consider $S := (s_{i,j})_{i,j=1,...,N}$ with

$$s_{i,j} = \begin{cases} \frac{1}{z_i - z_j} & \text{for } i = j \\ \frac{a+1}{2} \frac{1}{(1-z_j)^2} & \text{for } i = j \\ \frac{b}{2} \frac{1}{(1+z_j)^2} & \text{for } i \neq j \end{cases}$$

which appears in the CLT for Hermite ensembles in [V]. Proposition 2.7 and (2.8) now show that

$$\det(S^H) = \lim_{α \to \infty} \frac{1}{α^N} \det(S^{(α)}) = N!.$$  

In summary, these limit results agree perfectly with the results in Section 2 of [V].

**Remark 2.9.** In a similar way, the results in Section 3 of [V] for Laguerre ensembles can be seen as limits of Theorem 2.6 and Proposition 2.7. For this we fix $β > 0$, i.e. $β > -1$, and consider $α \to \infty$, i.e. $α \to \infty$. We recapitulate from (4.1.3) and (5.3.4) of [S] that

$$\lim_{α \to \infty} P_N^{(α, β)}(2x/α - 1) = (-1)^N \lim_{α \to \infty} P_N^{(β, α)}(1 - 2x/α) = (-1)^N L_N^{(β)}(x).$$
We now denote the ordered zeros of $P_N^{(\alpha,\beta)}$ by $z_1^{(\alpha)}, \ldots, z_N^{(\alpha)}$, and the ordered zeros of $L_N^{(\beta)}$ by $z_1^{L}, \ldots, z_N^{L}$. We then have
\[
\lim_{\alpha \to \infty} \alpha \left(1 + z_j^{(\alpha)}\right) = z_j^{L} \quad (j = 1, \ldots, N).
\]
(2.12)

We now insert these limits into the matrices $S^{(\alpha)}$ of Theorem 2.6 and obtain
\[
\lim_{\alpha \to \infty} \frac{8}{\alpha^2} S^{(\alpha)} = S^L
\]
with the matrix $S^L = (s_{i,j}^L)_{i,j=1,\ldots,N}$ with entries
\[
s_{i,j}^L := \begin{cases} \frac{\beta+1}{(z_i^L)^2} + 2 \sum_{l \neq i} (z_i^L - z_l^L)^{-2} & \text{for } i = j \\ -2(z_i^L - z_j^L)^{-2} & \text{for } i \neq j \end{cases}
\]
(2.14)

Proposition 2.7 and (2.13) now imply readily that
\[
\det(S^L) = \lim_{\alpha \to \infty} \frac{8^N}{\alpha^{2N}} \det(S^{(\alpha)}) = \frac{N!}{(\beta + 1)_N}.
\]
(2.15)

The inverse limit covariance matrix $S^L$ from (2.14) and its determinant in (2.15) fits with the inverse limit covariance matrix in the CLT 3.3 of [V] and its determinant in Corollary 3.4 in [V] (for the starting point 0 and time $t = 1$ there). This connection is not obvious as the Laguerre ensembles in Section 3 of [V] are transformed, which is motivated by the theory of multivariate Bessel processes. To explain this connection, we recapitulate that in Section 3 of [V], in the notation of the present paper, random vectors $\tilde{Y}_{\beta+1,\alpha}$ are studied with the Lebesgue densities
\[
\tilde{c}_{\beta+1,\alpha}^B e^{-\|x\|^2/2} \prod_{i < j} (x_i^2 - x_j^2)^{2\alpha} \cdot \prod_{i=1}^{N} x_i^{2(\beta+1)\alpha}
\]
(2.16)
on the Weyl chambers
\[
C_N^B := \{ x \in \mathbb{R}^N : x_1 \geq x_2 \geq \ldots \geq x_N \geq 0 \}
\]
with suitable normalizations $\tilde{c}_{\beta+1,\alpha}^B > 0$ for fixed $\beta > -1$ and $\alpha \to \infty$. We now use the zeros $z_i^L \geq \ldots \geq z_N^L$ of $L_N^{(\beta)}$ as well as the vector
\[
r = (r_1, \ldots, r_N) \in C_N^B \quad \text{with} \quad 2(z_1^L, \ldots, z_N^L) = (r_1^2, \ldots, r_N^2).
\]
(2.17)

The CLT 3.3 and its Corollary 3.4 in [V] now state that
\[
\tilde{Y}_{\beta+1,\alpha} - \sqrt{\alpha} \cdot r
\]
converges for $\alpha \to \infty$ to the centered $N$-dimensional distribution $N(0, (\tilde{S}^L)^{-1})$ with the covariance matrix $(\tilde{S}^L)$ where the matrix $\tilde{S}^L = (\tilde{s}_{i,j}^L)_{i,j=1,\ldots,N}$ satisfies
\[
\tilde{s}_{i,j}^L := \begin{cases} 1 + \frac{2(\beta+1)}{r_i^2} + 2 \sum_{l \neq i} (r_i - r_l)^{-2} + 2 \sum_{l \neq i} (r_i + r_l)^{-2} & \text{for } i = j \\ 2(r_i + r_j)^{-2} - 2(r_i - r_j)^{-2} & \text{for } i \neq j \end{cases}
\]
(2.18)
and
\[
\det(\tilde{S}^L) = N! \cdot 2^N.
\]
(2.19)
It is clear that the random vectors \( Y_{β+1,α} := \tilde{Y}_{β+1,α}^2 / 2 \) (where the squares are taken in each component) have the Lebesgue densities
\[
c_{β+1,α}^2 e^{-\left(\sum_{i=1}^{N} x_i \right) / 2} \prod_{i<j} x_i x_j^{(β+1)α-1/2} \tag{2.20}
\]
on \( C_N^β \) with suitable normalizations \( c_{β+1,α}^2 > 0 \). The Delta-method for the central limit theorem of random variables, which are transformed under some smooth transform (see Section 3.1 of [vV]) now implies that
\[
\frac{1}{\sqrt{α}} \left( Y_{β+1,α} - \frac{α}{2} \right) = \frac{1}{2} \left( \tilde{Y}_{β+1,α} - \sqrt{α} \cdot r \right) \cdot \tilde{Y}_{β+1,α} + \sqrt{α} \cdot r
\]
converges for \( α \to ∞ \) to the centered \( N \)-dimensional distribution \( N(0, S^{-1}_L) \) with transformed covariance matrix \( S^{-1}_L = D(\tilde{S}^L)^{-1}D \) with the diagonal matrix \( D = diag(r_1, \ldots, r_N) \). If we use the equation in Lemma 3.1(2) of [V] for the \( r_i \), we obtain easily that the matrix \( S_L = D^{-1}\tilde{S}_L D^{-1} \) is equal to the matrix \( S^L \) in (2.14). Moreover, (2.14) and (2.17) yield that
\[
\prod_{i=1}^{N} z_i^L = (β + 1)^N; \tag{2.21}
\]
see also (5.1.7) and (5.1.8) in [S]. (2.21) and (2.17) now lead to
\[
\det S_L = \frac{1}{2} \frac{N}{(β + 1)_N} \det \tilde{S}_L = \frac{N!}{(β + 1)_N} = \det S^L.
\]
These results fit to (2.14) and (2.15) as claimed.

The eigenvalues and eigenvectors of the \( S^H \) and \( \tilde{S}_L \) were determined in [AV2] explicitly. On the other hand, it is more complicated to determine the eigenvectors and eigenvalues of the matrix \( S^L \) for the Laguerre ensembles (2.20). Therefore, the difficulty of finding the eigenvectors and eigenvalues depends heavily on the parametrization of the random matrix ensembles.

3. Proof of the Main Results

In this section we prove the CLTs 2.5 and 2.6 and Proposition 2.7. The proofs are divided into several parts. In the first step we derive a restricted version of Theorem 2.6, where we shall only get vague instead of weak convergence.

Step 1. The representation (1.2) of the densities \( f_κ \) of the variables \( X_κ \) implies that the random variables \( \sqrt{κ}(X_κ - z) \) have the Lebesgue densities
\[
\hat{f}_κ(x) := \frac{1}{κ} f_κ \left( \frac{x}{\sqrt{κ}} + z \right) \tag{3.1}
\]
\[
= \frac{c_κ}{κ^2} \prod_{j=1}^{N} \left( 1 - \left( \frac{x}{\sqrt{κ}} + z_j \right)^2 \right)^{κ/2} \times
\]
\[
\times \left( \prod_{i<j} \left( \frac{x_i - x_j}{\sqrt{κ}} + z_j - z_i \right) \prod_{j=1}^{N} \left( 1 - \frac{x_j}{\sqrt{κ}} - z_j \right)^{\frac{κ+1}{2}} \right)^κ
\]
on $\sqrt{\kappa}(A - z)$ and zero elsewhere. We split this formula into two parts

$$f_\kappa(x) = C_\kappa h_\kappa(x)$$

(3.2)

where $h_\kappa$ depends on $x$ and $C_\kappa$ is constant w.r.t. $x$. More precisely, we put

$$C_\kappa := \frac{c_\kappa}{\kappa^{-\frac{N}{2}}} \prod_{j=1}^{N} \frac{1}{(1 - z_j^2)^{\frac{b}{2}}} \left( \prod_{i<j} (z_j - z_i) \prod_{j=1}^{N} ((1 - z_j) - \frac{a+b}{2}(1 + z_j)^{\frac{b}{2}}) \right)^\kappa$$

(3.3)

and

$$h_\kappa(x) := \prod_{j=1}^{N} \left( \frac{1 - z_j^2)^{\frac{b}{2}}}{(1 - \frac{x}{\sqrt{\kappa}} + z_j^2)^{\frac{b}{2}}} \times \left( \prod_{i<j} \left(1 + \frac{x_j - x_i}{\sqrt{\kappa}(z_j - z_i)} \right) \prod_{j=1}^{N} \left(1 - \frac{x}{\sqrt{\kappa}(1 - z_j)} \right) \right) \right)^\kappa.$$  

(3.4)

We first investigate $C_\kappa$. We here first focus on the constants $c_\kappa$ in (1.2) and recapitulate from [FW] the Selberg integral

$$\int_{[0,1]^N} \prod_{i<j} |x_i - x_j|^\kappa \prod_{i=1}^{N} (1 - x_i)^{\mu - 1} x_i^{\nu - 1} dx = \prod_{j=1}^{N} \frac{\Gamma(1 + j \rho) \Gamma(\mu + (j - 1) \rho) \Gamma(\nu + (j - 1) \rho)}{\Gamma(1 + \rho) \Gamma(\mu + \nu + (N + j - 2) \rho)}$$

(3.5)

for $\mu, \nu, \rho > 0$. The substitution $x_i = 2y_i - 1$ ($i = 1, \ldots, N$) then yields

$$\frac{1}{c_\kappa} = \int_{A_{1 \leq i < j \leq N}} (x_j - x_i)^{\kappa} \prod_{i=1}^{N} (1 - x_i)^{\frac{c(a+b)}{2}} (1 + x_i)^{\frac{a+b}{2}} dx$$

$$= \frac{1}{N!} \int_{[-1,1]^N} \prod_{1 \leq i < j \leq N} |x_j - x_i|^\kappa \prod_{i=1}^{N} (1 - x_i)^{\frac{c(a+b)}{2}} (1 + x_i)^{\frac{a+b}{2}} dx$$

$$= \frac{1}{N!} \int_{[0,1]^N} \prod_{1 \leq i < j \leq N} (2 |y_j - y_i|^\kappa \prod_{i=1}^{N} \left(2 (1 - y_i)^{\frac{c(a+b)}{2}} (2 y_i)^{\frac{a+b}{2}} \right) dy$$

$$= \frac{2^{2\kappa(N-1+a+2b)}}{N!} \int_{[0,1]^N} \prod_{1 \leq i < j \leq N} |y_j - y_i|^\kappa \prod_{i=1}^{N} (1 - y_i)^{\frac{c(a+b)}{2}} (y_i)^{\frac{a+b}{2}} dy$$

$$= \frac{2^{2\kappa(N-1+a+2b)}}{N!} \prod_{j=1}^{N} \Gamma(1 + j \frac{\beta}{2}) \Gamma(\frac{a}{2} + \frac{\beta}{2} + (j - 1) \frac{\alpha}{2}) \Gamma(\frac{\beta}{2} + \frac{1}{2} + \frac{a+b}{2} \frac{\alpha}{2} + \frac{1}{2} + (N + j - 2) \frac{\alpha}{2})$$

(3.6)

$$= \frac{2^{2\kappa(N+a+b+1)}}{N!} \prod_{j=1}^{N} \Gamma(1 + j \frac{\beta}{2}) \Gamma(\frac{a+b}{2} + \frac{1}{2}) \Gamma(\frac{a}{2} + \frac{1}{2})$$

where the notation $\alpha = a + b - 1$ and $\beta = b - 1$ from Lemma (2.2) was used. In order to study the limit behavior of (3.6) for $\kappa \to \infty$, we use the notation

$$f(x) \sim g(x) : \iff \lim_{x \to \infty} \frac{f(x)}{g(x)} = 1,$$
We also recapitulate Stirling’s formula and two of its well-known consequences:

\[
\Gamma(1 + x) = x \Gamma(x) \sim \sqrt{2\pi x} \left( \frac{x}{e} \right)^x, \quad (3.7)
\]

\[
\frac{\Gamma(\frac{1}{2} + x)}{\Gamma(1 + x)} \sim x^{\frac{1}{2} - 1} = \frac{1}{\sqrt{x}}, \quad \Gamma(\frac{1}{2} + x) \sim \sqrt{2\pi} \left( \frac{x}{e} \right)^x. \quad (3.8)
\]

We now apply these formulas to (3.6). For this we first observe that (3.7) leads to

\[
\prod_{j=1}^N \frac{\Gamma(1 + j\frac{\kappa}{2})}{\Gamma(1 + \frac{\kappa}{2})} \sim \prod_{j=1}^N \sqrt{\pi j \kappa} \left( \frac{2\pi}{\kappa} \right)^{\frac{j}{2}} = \prod_{j=1}^N j^{1/2} \left( \frac{K}{2\pi} \right)^{\frac{j}{2}}(3.9)
\]

For the second part of (3.6) we use (3.7) and (3.8) and get

\[
\Gamma \left( \frac{\kappa(\rho + j)}{2} + \frac{1}{2} \right) \sim \sqrt{2\pi} \left( \frac{\kappa(\rho + j)}{2e} \right)^{\frac{\kappa(\rho + j)}{2} \frac{1}{2}} \quad (\rho = \alpha, \beta), \quad \text{and}
\]

\[
\Gamma \left( \frac{\kappa(N + \alpha + \beta + j)}{2} + 1 \right) \sim \sqrt{\pi \kappa(N + \alpha + \beta + j)} \left( \frac{\kappa(N + \alpha + \beta + j)}{2e} \right)^{\frac{\kappa(N + \alpha + \beta + j)}{2}}.
\]

These results lead to

\[
\prod_{j=1}^N \frac{\Gamma(\frac{\kappa(\beta + j)}{2} + \frac{1}{2})}{\Gamma(\frac{\kappa(\alpha + j)}{2} + \frac{1}{2})} \sim \prod_{j=1}^N \frac{2\sqrt{\pi}}{\sqrt{\kappa(N + \alpha + \beta + j)}} \left( \frac{2e}{\kappa} \right)^{\frac{N}{2}} \left( \frac{(\alpha + j) \frac{\alpha + \beta + 1}{N} (\beta + j) \frac{\beta + 1}{N}}{(N + \alpha + \beta + j) \frac{N + \alpha + \beta + 1}{N}} \right)^{\kappa} \quad (3.10)
\]

\[
= \frac{2^N \pi \frac{N}{2}}{\kappa \frac{N}{2} \sqrt{(N + \alpha + \beta + 1)N}} \left( \frac{2e}{\kappa} \right)^{\frac{N}{2}} \frac{N}{2} \prod_{j=1}^N \frac{1}{\sqrt{(N + \alpha + \beta + j)}} \left( \frac{(\alpha + j) \frac{\alpha + \beta + 1}{N} (\beta + j) \frac{\beta + 1}{N}}{(N + \alpha + \beta + j) \frac{N + \alpha + \beta + 1}{N}} \right)^{\kappa}.
\]

Combining (3.9) and (3.10), we obtain

\[
\prod_{j=1}^N \frac{\Gamma(1 + j\frac{\kappa}{2})}{\Gamma(1 + \frac{\kappa}{2})} \sim \frac{\sqrt{N} 2^N \pi \frac{N}{2}}{\kappa \frac{N}{2} \sqrt{(N + \alpha + \beta + 1)N}} \left( \prod_{j=1}^N \frac{1}{(N + \alpha + \beta + j) \frac{N + \alpha + \beta + 1}{N}} \right)^{\kappa}.
\]
Finally, if we apply this to (3.6), we see that $1/c_\kappa$ behaves like

$$
\frac{2^{2N} (N+\alpha+\beta+1) 2^N \pi^{\frac{N}{2}}}{\sqrt{\kappa \pi^{\frac{N}{2}}} \sqrt{(N+\alpha+\beta+1)^N}} \left( \prod_{j=1}^{N} j^{\frac{1}{2}} \frac{(\alpha+j)^{\frac{\alpha+1}{2}} (\beta+j)^{\frac{\beta+1}{2}}}{(N+\alpha+j)^{\frac{\alpha+\beta+1}{2}}} \right)^{\kappa}.
$$

Having this limit of $c_\kappa$ in mind, we now determine the asymptotics of $C_\kappa$ in (3.3). For this we use (2.6) with the function $\phi$ there as well as (2.3), (2.4), and (3.11). Using the Pochhammer symbol $(x)_N := x(x+1) \cdots (x+N-1)$, we get

$$
C_\kappa = \frac{c_\kappa}{\kappa^{\frac{N}{2}}} (\phi(z))^\kappa \prod_{j=1}^{N} \frac{1}{(1-z_j^2)^{\frac{1}{2}}} = \frac{c_\kappa}{\kappa^{\frac{N}{2}}} \prod_{j=1}^{N} \frac{1}{(1-z_j)^{\frac{1}{2}} (1+z_j)^{\frac{1}{2}}} (\phi(z))^\kappa
$$

$$
= \frac{c_\kappa 2^{-N}}{\kappa^{\frac{N}{2}}} 2^{2N} (N+\alpha+\beta+1) \prod_{j=1}^{N} \left( j^{\frac{1}{2}} \frac{(j+\alpha)^{\frac{\alpha+1}{2}} (j+\beta)^{\frac{\beta+1}{2}}}{(N+\alpha+j)^{\frac{\alpha+\beta+1}{2}}} \right)^{\kappa} \frac{N+\alpha+\beta+j}{\sqrt{(\alpha+j)(\beta+j)}}
$$

$$
\sim \frac{\sqrt{N!}}{\sqrt{2^N \pi^{\frac{N}{2}}} \sqrt{(N+1)_N (\beta+1)_N}}.
$$

In summary,

$$
\lim_{\kappa \to \infty} C_\kappa = \frac{\sqrt{N!}}{\sqrt{2^N \pi^{\frac{N}{2}}} \sqrt{(N+\alpha+\beta+1)_N}}.
$$

We next turn to an asymptotics of $h_\kappa(x)$ in (3.4). We first observe that

$$
\prod_{j=1}^{N} \frac{(1-z_j^2)^{\frac{1}{2}}}{(1-(x_j \sqrt{\kappa} + z_j)^2)^{\frac{1}{2}}} \to 1 \quad (\kappa \to \infty).
$$

Hence, this factor can be ignored. It will be convenient to write the further factor $h_\kappa(x)$ of $h_\kappa(x)$ in the second line of (3.4) as $h_\kappa(x) = \exp(\log(h_\kappa(x)))$. We now have to investigate the term

$$
\exp \left( \kappa \left( \sum_{i<j} \log \left( 1 + \frac{x_j - x_i}{\sqrt{\kappa}} \right) \right) + \frac{a+b}{2} \sum_{j=1}^{N} \log \left( 1 - \frac{x_j}{\sqrt{\kappa}} \right) + \frac{b}{2} \sum_{j=1}^{N} \log \left( 1 + \frac{x_j}{\sqrt{\kappa}} \right) \right).
$$

We now apply Taylor’s formula to all logarithms, i.e., for large $\kappa$,

$$
\log \left( 1 + \frac{x_j - x_i}{\sqrt{\kappa}} \right) = \frac{x_j - x_i}{\sqrt{\kappa}(z_j - z_i)} - \frac{(x_j - x_i)^2}{2 \kappa(z_j - z_i)^2} + O(\kappa^{-\frac{3}{2}})
$$

$$
\log \left( 1 - \frac{x_j}{\sqrt{\kappa}(1-z_j)} \right) = -\frac{x_j}{\sqrt{\kappa}(1-z_j)} - \frac{x_j^2}{2 \kappa(1-z_j)^2} + O(\kappa^{-\frac{3}{2}})
$$

$$
\log \left( 1 + \frac{x_j}{\sqrt{\kappa}(1+z_j)} \right) = \frac{x_j}{\sqrt{\kappa}(1+z_j)} - \frac{x_j^2}{2 \kappa(1+z_j)^2} + O(\kappa^{-\frac{3}{2}}).
$$
By (2.2),

\[
\sum_{i<j} \frac{\sqrt{\kappa}(x_j - x_i)}{(z_j - z_i)} = \frac{a + b}{2} \sum_{j=1}^{N} \frac{\sqrt{\kappa}x_j}{1 - z_j} + \frac{b}{2} \sum_{j=1}^{N} \frac{\sqrt{\kappa}x_j}{1 + z_j}
\]

(3.15)

\[
= \sqrt{\kappa} \sum_{j=1}^{N} x_j \left( \sum_{i=1, i \neq j}^{N} \frac{1}{z_j - z_i} - \frac{a + b}{2} \frac{1}{1 - z_j} + \frac{b}{2} \frac{1}{1 + z_j} \right) = 0
\]

and therefore (3.14) turns into

\[
\exp \left( -\frac{1}{2} \left( \sum_{i<j} \frac{(x_i - x_j)^2}{(z_j - z_i)^2} + \frac{a + b}{2} \sum_{j=1}^{N} \frac{x_j^2}{(1 - z_j)^2} + \frac{b}{2} \sum_{j=1}^{N} \frac{x_j^2}{(1 + z_j)^2} \right) \right)
\]

If we combine this with (3.13) we get

\[
\lim_{\kappa \to \infty} h_\kappa(x)
\]

\[
= \exp \left( -\frac{1}{2} \left( \sum_{i<j} \frac{(x_i - x_j)^2}{(z_j - z_i)^2} + \frac{a + b}{2} \sum_{j=1}^{N} \frac{x_j^2}{(1 - z_j)^2} + \frac{b}{2} \sum_{j=1}^{N} \frac{x_j^2}{(1 + z_j)^2} \right) \right)
\]

(3.16)

Now let \( f \in C_c(\mathbb{R}^N) \) be a continuous function with compact support. From (3.12), (3.11), and dominated convergence we get

\[
\lim_{\kappa \to \infty} \int_{\sqrt{\kappa}(A-z)} \frac{f(x)\tilde{f}_\kappa(x)dx}{\kappa} = C_\kappa \int_{\mathbb{R}^N} \frac{1}{\sqrt{\kappa}(A-z)}(x)f(x)h_\kappa(x)dx
\]

(3.17)

\[
= \frac{\sqrt{N!}}{2^{2N} \pi^N} \frac{1}{(N + \alpha + \beta + 1)^\frac{1}{2}} \frac{1}{(\alpha + 1)N(\beta + 1)N} \int_{\mathbb{R}^N} f(x)
\]

\[
\times \exp \left( -\frac{1}{2} \left( \sum_{i<j} \frac{(x_i - x_j)^2}{(z_j - z_i)^2} + \frac{a + b}{2} \sum_{j=1}^{N} \frac{x_j^2}{(1 - z_j)^2} + \frac{b}{2} \sum_{j=1}^{N} \frac{x_j^2}{(1 + z_j)^2} \right) \right) dx
\]

We briefly check that we can interchange the limit with integration in (3.17) by dominated convergence. For this we determine an integrable upper bound for \( C_\kappa \frac{1}{\sqrt{\kappa}(A-z)}(x)f(x)h_\kappa(x) \). We first observe that by (3.13), \( f \in C_c(\mathbb{R}^N) \) and a short calculation, we find constants \( C, \kappa_0 > 0 \) such that for all \( \kappa \geq \kappa_0 \) and \( x \in \mathbb{R}^N \),

\[
1 = \frac{1}{\sqrt{\kappa}(A-z)}(x)f(x)|\prod_{j=1}^{N} \frac{1}{1 - (\frac{x_j}{\sqrt{\kappa} + z_j})^2} \leq C
\]

(3.18)

holds. For the remaining factors we again use the Taylor expansion of \( \log(1 + x) \). Here the Lagrange remainder shows that

\[
\log \left( 1 + \frac{x_j - x_i}{\sqrt{\kappa}(z_j - z_i)} \right) = \frac{x_j - x_i}{\sqrt{\kappa}(z_j - z_i)} - \frac{(x_j - x_i)^2}{2\kappa(z_j - z_i)^2}w_{i,j},
\]

\[
\log \left( 1 - \frac{x_j}{\sqrt{\kappa}(1 - z_j)} \right) = \frac{-x_j}{\sqrt{\kappa}(1 - z_j)} - \frac{x_j^2}{2\kappa(1 - z_j)^2}w_j^-, \]

\[
\log \left( 1 + \frac{x_j}{\sqrt{\kappa}(1 + z_j)} \right) = \frac{x_j}{\sqrt{\kappa}(1 + z_j)} - \frac{x_j^2}{2\kappa(1 + z_j)^2}w_j^+
\]
with \( w_{i,j}, w_j^+ w_j^- \in (0, 1) \) for \( i, j = 1, ..., N \). If we set
\[
w := \min\{w_{i,j}, w_j^+ w_j^- | i, j = 1, ..., N\} \in (0, 1),
\]
we get
\[
1_{\sqrt{\pi(A - z)}(x)} \cdot |f(x)| \cdot h_n(x) \leq C \exp \left( -w \frac{1}{2} \left( \sum_{i<j} (x_i - x_j)^2 + a + b \sum_{j=1}^N \frac{x_j^2}{(1 - z_j)^2} + b \sum_{j=1}^N \frac{x_j^2}{(1 + z_j)^2} \right) \right).
\]
This and \( (3.18) \) show that dominated convergence in \( (3.17) \) is available. Eq. \( (3.17) \) means that \( \sqrt{n}(X_n - z) \) converges vaguely to the measure with the density
\[
\frac{\sqrt{N}}{2^{2N} \pi^{N/2}} \left( \frac{(N + \alpha + \beta + 1)N}{(\alpha + 1)N(\beta + 1)N} \right)^{N/2} \exp \left( -\frac{1}{2} x^T \Sigma^{-1} x \right),
\]
where \( \Sigma^{-1} = (s_{i,j})_{i,j=1,...,N} \) is given as in Theorem 2.6. As a vague limit of probability measures, this measure is a sub-probability measure. Moreover, this measure is the normal distribution in Theorem 2.5 for \( b > 0 \). We shall postpone the normalization to the end of this section.

In the next step we determine the eigenvectors and eigenvalues of the matrix \( \tilde{S} \) in Theorem 2.5 for \( a \geq 0, b > 0 \):

**Proposition 3.1.** The matrix \( \tilde{S} \) has the simple eigenvalues
\[
\lambda_k = 2k(2N + \alpha + \beta + 1 - k) > 0 \quad (k = 1, \ldots, N).
\]
Moreover, each \( \lambda_k \) has an eigenvector of the form
\[
v_k := \left( q_{k-1}(z_1) \sqrt{1 - z_1^2}, \ldots, q_{k-1}(z_N) \sqrt{1 - z_N^2} \right)^T
\]
for polynomials \( q_{k-1} \) of order \( k - 1 \) which are orthonormal w.r.t the discrete measure
\[
\mu_{N,\alpha,\beta} := (1 - z_1^2)\delta_{z_1} + \ldots + (1 - z_N^2)\delta_{z_N}
\]

The proof uses induction on \( k \). For \( k = 1 \) we have:

**Lemma 3.2.** The vector \( v_1 := (\sqrt{1 - z_1^2}, \ldots, \sqrt{1 - z_N^2})^T \) is an eigenvector of \( \tilde{S} \) associated with the eigenvalue \( \lambda_1 \).

**Proof.** By the definition of \( \tilde{S} \), the \( i \)-th component \( (i = 1, \ldots, N) \) of \( \tilde{S}v_1 \) is given by
\[
(\tilde{S}v_1)_i = 4 \sum_{l \neq i} \frac{1 - z_l^2}{(z_i - z_l)^2} \sqrt{1 - z_i^2} + 2(a + b) \frac{1 + z_i}{1 - z_i} \sqrt{1 - z_i^2}
\]
\[
+ 2b \frac{1 - z_i}{1 + z_i} \sqrt{1 - z_i^2} - 4 \sum_{l \neq i} \frac{(1 - z_l^2) \sqrt{1 - z_l^2}}{(z_i - z_l)^2}
\]
\[
= 4 \sqrt{1 - z_i^2} \left( \sum_{l \neq i} \frac{z_l^2 - z_i^2}{(z_i - z_l)^2} + \frac{a + b}{2} \frac{1 + z_i}{1 - z_i} + \frac{b}{2} \frac{1 - z_i}{2 + z_i} \right).
\]
Hence,

\[(\tilde{S}v_1)_i = 4 \sum_{l \neq i} \frac{1 - z_i^2}{(z_i - z_l)^2} \sqrt{1 - z_i^2} + 2(a + b) \frac{1 + z_i}{1 - z_i} \sqrt{1 - z_i^2} \]

\[+ 2b \frac{1 - z_i}{1 + z_i} \sqrt{1 - z_i^2} - 4 \sum_{l \neq i} \frac{(1 - z_i^2) \sqrt{1 - z_i^2}}{(z_i - z_l)^2} \]

\[= 4 \sqrt{1 - z_i^2} \left( \sum_{l \neq i} \frac{z_i^2 - z_l^2}{(z_i - z_l)^2} + \frac{a + b}{2} \frac{1 + z_i}{1 - z_i} + \frac{b}{2(1 + z_i)} \right) \]

\[= 4 \sqrt{1 - z_i^2} \left( \sum_{l \neq i} \frac{-2z_i + (z_i - z_l)}{z_i - z_l} + \frac{a + b}{2} \frac{1 + z_i}{1 - z_i} + \frac{b}{2(1 + z_i)} \right) \]

\[= 4 \sqrt{1 - z_i^2} \left( (N - 1) - 2z_i \sum_{l \neq i} \frac{1}{z_i - z_l} + \frac{a + b}{2} \frac{1 + z_i}{1 - z_i} + \frac{b}{2(1 + z_i)} \right). \]

(2.2) now leads to

\[\tilde{S}v_1 = 4 \sqrt{1 - z_i^2} \left( (N - 1) + \frac{a + b}{2} + \frac{b}{2} \right) \quad (i = 1, \ldots, N).\]

This proves readily that \(v_1\) is an eigenvector with eigenvalue \(\lambda_1\) as claimed. \(\Box\)

We next consider the \(\lambda_k\) for \(k > 1\). We here do not present the eigenvectors explicitly and prove a slightly weaker result:

**Lemma 3.3.** For \(k = 2, \ldots, N\) there exist polynomials \(p_k\) of order at most \(k - 2\), such that the vector \(v_k := \left( z_1^{k-1} \sqrt{1 - z_1^2}, \ldots, z_N^{k-1} \sqrt{1 - z_N^2} \right)^T \) satisfies

\[\tilde{S}v_k = \left( (\lambda_k z_1^{k-1} + p_k(z_1)) \sqrt{1 - z_1^2}, \ldots, (\lambda_k z_N^{k-1} + p_k(z_N)) \sqrt{1 - z_N^2} \right)^T.\]

**Proof.** We first consider the case \(k = 2\). We here have

\[\tilde{S}v_2 = 4 \sum_{l \neq i} \frac{1 - z_i^2}{(z_i - z_l)^2} z_i \sqrt{1 - z_i^2} + 2(a + b) \frac{1 + z_i}{1 - z_i} z_i \sqrt{1 - z_i^2} \]

\[+ 2b \frac{1 - z_i}{1 + z_i} z_i \sqrt{1 - z_i^2} - 4 \sum_{l \neq i} \frac{z_i(1 - z_i^2) \sqrt{1 - z_i^2}}{(z_i - z_l)^2}. \]

(3.21)
Hence,

\[
(\tilde{S}v_{2})_i = 4\sqrt{1 - z_i^2} \left( \sum_{l \neq i} \frac{(1 - z_l^2)z_i - (1 - z_i^2)z_l}{(z_i - z_l)^2} + \frac{a + b}{2} \frac{1 + z_i}{1 - z_i} + \frac{b}{2} \frac{1 - z_i}{1 + z_i} \right).
\]

\[
= 4\sqrt{1 - z_i^2} \left( \sum_{l \neq i} \frac{1 - z_l^2 - z_l z_i - z_i^2}{z_i - z_l} + \frac{a + b}{2} \frac{1 + z_i}{1 - z_i} + \frac{b}{2} \frac{1 - z_i}{1 + z_i} \right)
\]

\[
= 4\sqrt{1 - z_i^2} \left( \sum_{l \neq i} \frac{1 + z_l(z_i - z_l) + 2z_i(z_i - z_l) - 3z_i^2}{z_i - z_l}
\right.
\]

\[
\left. + \frac{a + b}{2} \frac{1 + z_i}{1 - z_i} + \frac{b}{2} \frac{1 - z_i}{1 + z_i} \right)
\]

\[
= 4\sqrt{1 - z_i^2} \left( (c - z_i) + 2z_i(N - 1) + (1 - 3z_i^2) \sum_{l \neq i} \frac{1}{z_i - z_l} + \frac{a + b}{2} \frac{1 + z_i}{1 - z_i} + \frac{b}{2} \frac{1 - z_i}{1 + z_i} \right)
\]

with \( c := \sum_{j=1}^{N} z_j \). A short computation and (2.2) now lead to

\[
(\tilde{S}v_{2})_i = 4\sqrt{1 - z_i^2} \left( (c - z_i) + 2z_i(N - 1) + \frac{\alpha + 1}{2} (2z_i + 1) + \frac{\beta + 1}{2} (2z_i - 1) \right)
\]

for \( i = 1, \ldots, N \). This proves readily that \( \tilde{S}v_{2} \) has the form as claimed in the lemma with some constant polynomial \( p_2 \).

We now turn to the case \( k \geq 3 \). We here have

\[
(\tilde{S}v_{k})_i = 4\sqrt{1 - z_i^2} \left( \sum_{l \neq i} \frac{1 - z_l^2}{(z_i - z_l)^2} z_i^{k-1} \sqrt{1 - z_i^2} + 2(a + b) \frac{1 + z_i}{1 - z_i} z_i^{k-1} \sqrt{1 - z_i^2} \right) + 2b \frac{1 - z_i}{1 + z_i} z_i^{k-1} \sqrt{1 - z_i^2} + 4 \sum_{l \neq i} \frac{z_i^{k-1}}{(z_i - z_l)^2} \frac{1 - z_i^2}{(z_i - z_l)^2}\right)
\]

(3.22)

thus

\[
(\tilde{S}v_{k})_i = 4\sqrt{1 - z_i^2} \left( \sum_{l \neq i} \frac{z_i^{k-1} - z_i^{k-1} - z_l^{k-1} + z_l^{k-1}}{(z_i - z_l)^2} + \right.
\]

\[
\left. + \frac{a + b}{2} \frac{1 + z_i}{1 - z_i} z_i^{k-1} + \frac{b}{2} \frac{1 - z_i}{1 + z_i} z_i^{k-1} \right)
\]
with
\[ z_i^{k-1} - z_i^{k+1} - z_i^{k-1} + z_i^{k+1} = \]
\[ = (z_i - z_i) \left( z_i^{k-2} + z_i^{k-3} z_i + \ldots + z_i^{k-2} - z_i^{k-1} z_i - \ldots - z_i^k \right) \]
\[ = (z_i - z_i) \left( (z_i - z_i) \left( z_i^{k-3} + 2z_i^{k-4} z_i + \ldots + (k - 2) z_i^{k-3} \right) + (k - 1) z_i^{k-2} \right) \]
\[ - (z_i - z_i) \left( z_i^{k-1} + 2z_i^{k-2} z_i + \ldots + k z_i^{k-1} \right) - (k + 1) z_i^k \]. (3.23)

We thus conclude that
\[ (\tilde{S}v_k)_i = 4 \sqrt{1 - z_i^2} \left( \sum_{l \neq i} \left( z_l^{k-1} + 2z_l^{k-2} z_i + \ldots + k z_l^{k-1} - z_l^{k-3} - 2z_l^{k-4} z_i - \ldots - (k - 2) z_l^{k-3} \right) \right. \]
\[ + \sum_{l \neq i} \frac{(k - 1) z_l^{k-2} - (k + 1) z_l^k}{z_l - z_i} + \frac{a + b}{2} \frac{1 + z_i}{1 - z_i} z_l^{k-1} + \frac{b}{2} \frac{1 - z_i}{1 + z_i} z_l^{k-1} \). \]

With (3.2), a suitable constant C, and with a suitable polynomials \( q_k, r_k^{(1)}, r_k^{(2)}, r_k^{(3)}, r_k^{(4)} \) of order at most \( k - 2 \) we thus obtain
\[ (\tilde{S}v_k)_i = 4 \sqrt{1 - z_i^2} \left( C - z_i^{k-1} - 2z_i^{k-1} - \ldots - (k - 1) z_i^{k-1} + k(N - 1) z_i^{k-1} + q_k(z_i) \right. \]
\[ + \frac{a + b}{2} \frac{z_i^k + z_i^{k-1} + (k - 1) z_i^{k-2} - (k + 1) z_i^{k-1}}{1 - z_i} \]
\[ + \frac{b}{2} \frac{z_i^{k-1} - z_i^k - (k - 1) z_i^{k-2} + (k + 1) z_i^k}{1 + z_i} \right) \]
\[ = 4 \sqrt{1 - z_i^2} \left( \left( k(N - 1) - \frac{(k - 1)k}{2} \right) z_i^{k-1} + r_k^{(1)}(z_i) \right. \]
\[ + \frac{a + b}{2} \left( k z_i^{k-1} + r_k^{(2)}(z_i) \right) + \frac{b}{2} \left( k z_i^{k-1} + r_k^{(3)}(z_i) \right) \right) \]
\[ = \sqrt{1 - z_i^2} \left( 2k(2N + \alpha + \beta + 1 - k) z_i^{k-1} + r_k^{(4)}(z_i) \right) \). (3.24) \]

This implies the lemma for \( k \geq 3 \).

**Proof of Proposition 3.1** Lemma 3.2, Lemma 3.3, induction on \( k = 1, \ldots, N \), and an obvious computation easily lead to the first statements of the Proposition for some polynomials \( q_k, r_k^{(1)}, r_k^{(2)}, r_k^{(3)}, r_k^{(4)} \) of order at most \( k \) for \( k = 0, \ldots, N - 1 \).

It remains to identify the polynomials \( q_k, r_k^{(1)}, r_k^{(2)}, r_k^{(3)}, r_k^{(4)} \) as finite sequence of orthogonal polynomials w.r.t. the discrete measure
\[ \mu_{N,\alpha,\beta} := (1 - z_1^2) \delta_{z_1} + \ldots + (1 - z_N^2) \delta_{z_N}. \] (3.25)
For this, consider a sequence of orthonormal polynomials \((q_{l}^{(\alpha,\beta)})_{l=0,\ldots,N-1}\) associated with \(\mu_{N,\alpha,\beta}\) as for instance in \([\mathbb{C}]\). We then have
\[
\sum_{i=1}^{N} q_{l}^{(\alpha,\beta)}(z_i) q_{k}^{(\alpha,\beta)}(z_i)(1-z_i^2) = \delta_{l,k} \quad (k, l = 0, \ldots, N - 1).
\] (3.26)

This orthogonality fits to the fact that we may write the symmetric matrix \(\tilde{S}\) as \(\tilde{S} = T^{-1} \cdot \text{diag}(\lambda_1, \ldots, \lambda_N) \cdot T\) with some orthogonal matrix \(T \in O(N)\). We thus obtain that the \(q_k\) in Corollary 3.1 are necessarily equal to the \(q_k^{(\alpha,\beta)}\) up to normalization constants as claimed.

We finally complete the proof of Theorems 2.5 and 2.6 and of Proposition 2.7.

**Proof of Proposition 2.7.** It can be easily checked that the matrices \(S\) and \(\tilde{S}\) from Theorems 2.5 and 2.6 are related by \(\tilde{S} = DSD\), where the matrix
\[
D := \text{diag} \left( \frac{1}{\sqrt{1-z_1^2}}, \ldots, \frac{1}{\sqrt{1-z_N^2}} \right)
\]
is the Jacobi matrix of the inverse Transformation \(T^{-1}\) of 2.7 at the position \(z = (z_1, \ldots, z_N)\). This and Proposition 3.1 ensure that
\[
\det(S) \cdot \det(D)^2 = \det(\tilde{S}) = 2^N \cdot N! \cdot (N + \alpha + \beta + 1)_N.
\]
Furthermore, \(\det(D)\) can be computed via (2.4) and (2.5) which finally leads to the proof of Proposition 2.7.

**Proof of Theorems 2.6 and 2.5.** Proposition 2.7 ensures that the measure with the density (3.19) is in fact a probability measure and hence the normal distribution \(N(0, \Sigma)\). As a consequence of the first step of the proof, we conclude that \(\sqrt{\pi}(X_k - z)\) converges in distribution to the normal distribution as claimed in Theorem 2.6.

The Delta-method for the central limit theorem of random variables (see Section 3.1 of [\mathbb{V}]) now immediately yields Theorem 2.5.

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