Noise-resistant entanglement of strongly interacting spin systems

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We propose and analyze a scheme that makes use of interactions between spins to protect certain correlated many-body states from decoherence. The method exploits the finite energy gap of properly designed Hamiltonians to generate a manifold insensitive to local noise fluctuations. We apply the scheme to achieve decoherence-resistant generation of many particle GHZ states and show that it can improve the sensitivity in precision spectroscopy with trapped ions. Finally we also show that cold atoms in optical lattices interacting via short range interactions can be utilized to engineer the required long range interactions for a robust generation of entangled states.

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Quantum entanglement has recently emerged as an important resource in quantum information science and metrology\(^1\),\(^2\). For example, entangled many-particle states can be used to perform long-distance quantum communication and scalable quantum computation and to enhance the spectroscopic sensitivity in quantum-limited measurements. However, many-particle entangled states are difficult to prepare and to maintain since they are extremely fragile: in practice, noise and decoherence rapidly collapse them into classical statistical mixtures.

In this Letter we propose and analyze a new method for the robust generation of entangled states and its protection against decoherence. Our approach is based on creating a degenerate many-body subspace, which can be isolated from the rest of the Hilbert space by properly designed interactions. We focus specifically on interacting two-level systems where the two states are associated with spin sublevels, e.g trapped ions or neutral atoms\(^3\),\(^4\),\(^5\),\(^6\),\(^7\). Such systems are of particular importance for applications in precision measurements. Few-particle entangled states have already been prepared with trapped ions, and proof-of-principle experiments demonstrating the improvement of spectroscopic sensitivity have been carried out\(^3\),\(^4\). Our aim is to devise a technique to increase the robustness of entangled collective states. We discuss potential applications of this protected evolution to perform precision measurements with trapped ions and to generate many-particle-GHZ-type states in optical lattices.

Our approach can be best understood by considering a simple example a multi-spin system with isotropic ferromagnetic interactions. These interactions will naturally align the spins. While all of the spins can be rotated together around an arbitrary axis without cost of energy, local spin flips are energetically forbidden. Consequently ferromagnetic interactions enable the generation of superpositions (suitable for example for precision spectroscopy) with a substantial suppression of decoherence (See Fig.1). We note that in Ref.\(^8\) gapped quantum systems have been proposed for decoherence-free entanglement generation by adiabatic ground-state transitions. Specifically, Ref.\(^8\) exploited the finite energy gap to protect arbitrary axis without cost of energy, local spin flips are energetically suppressed. Panel (d) shows that the MPM lies on the Bloch surface with radius \(J = J_{\text{max}} = N/2\).

non-degenerate ground states. The present mechanism is capable of protecting degenerate multi-state manifolds with a large number of quantum degrees of freedom and use nonequilibrium dynamics as the mechanism for entanglement generation.

We consider a collection of \(N\) spin 1/2 atoms with isotropic infinite range ferromagnetic interactions described by the Hamiltonian \(\hat{H} = \hat{H}_\text{prot} + \hat{H}_m\), where

\[
\hat{H}_\text{prot} = -\lambda \hat{\sigma}_z^{(0)} \quad \text{and} \quad \hat{H}_m = \omega_0 \hat{\sigma}_z^{(0)}.
\]

We used \(\hat{\sigma}_\alpha^{(0)}\) to denote collective spin operators of the \(N\) atoms: \(\hat{\sigma}_\alpha^{(0)} = \frac{1}{2} \sum_i \hat{\sigma}_\alpha^{(i)}\), where \(\alpha = x, y, z\) and \(\hat{\sigma}_\alpha^{(i)}\) is a Pauli operator acting on the \(i\)th atom. Moreover we have identified the two relevant internal states of the atoms with the effective spin index \(\sigma = \uparrow, \downarrow\). These states have energy splitting \(\omega_0\) in units such that \(\hbar = 1\).

An appropriate basis to describe the dynamics of the system is the basis spanned by collective pseudo-spin states denoted as \(|J, M, \beta\rangle_z\). These states satisfy the eigenvalue relations \(\hat{j}_z^{(0)}|J, M, \beta\rangle_z = J(J + 1)|J, M, \beta\rangle_z\) and \(\hat{j}_z^{(0)}|J, M, \beta\rangle_z = M|J, M, \beta\rangle_z\), with \(J = N/2, \ldots, 0\) and \(-J \leq M \leq J\). \(\beta\) is an additional quantum number associated with the permutation group which is required to form a complete set of labels.

FIG. 1: Many-body protected manifold (MPM) generated by the Hamiltonian \(\hat{H} = \hat{H}_\text{prot} + \hat{H}_m\), where

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The isotropic Hamiltonian \( \hat{H}_{prot} \) has a ground state manifold spanned by a set of \( N + 1 \) degenerate states. They lie on the surface of the Bloch sphere with maximal radius \( J = N/2 \) (see Fig.1) and are totally symmetric, i.e. invariant with respect to particle permutations. This set of states are fully characterized by \( J \) and \( M \), and we denote them as \( | M/2, M \rangle \) with the additional label \( \beta \) omitted. There is a finite energy gap \( E_g = \lambda N \) that isolates the ground state manifold from the rest of the Hilbert space. This gap is the key for the many-body protection against decoherence. We will refer to the ground state manifold as the many-body protected manifold (MPM).

To understand the protection within the MPM, we first assume that the dominant type of decoherence is single-particle dephasing. Such dephasing comes from processes that, while preserving the populations in the atomic levels, randomly change the phases leading to a decay of the off-diagonal density matrix elements. We model the phase decoherence by adding to Eq. (1) the following Hamiltonian [14]

\[
\hat{H}_{env} = \frac{1}{2} \sum_i h_i(t) \hat{\sigma}_i^z, \tag{2}
\]

where the \( h_i(t) \) are assumed to be independent stochastic Gaussian processes with zero mean and with autocorrelation function \( \langle h_i(t) h_j(t') \rangle = \delta_{ij} \Gamma(t-t') \). Here the bar denotes averaging over the different random outcomes. In what follows we will use the property that zero mean Gaussian variables satisfy the property \( \langle \exp[-i \int_0^t d \tau \Gamma(\tau)] \rangle = \exp[-\Gamma(t)] \), with \( \Gamma(t) = \frac{1}{2} \int_0^t dt_1 \int_0^t dt_2 f(t_1-t_2) \).

Let us first study the case \( \lambda = 0 \) dynamics, with the system in the ground manifold at \( t = 0 \). Phase decoherence modifies the free precession of the Bloch vector generated by \( \hat{H}_m \). While \( \langle \hat{J}_+^{(0)}(t) \rangle \) remains a conserved quantity, the \( x \) and \( y \) projections decohere exponentially with rate \( \Gamma(t) \), i.e. \( \langle \hat{J}_x^{(0)}(t) \rangle = e^{-\Gamma(t)} \langle \hat{J}_x^{(0)}(0) \rangle \big|_{t=0} \) and \( \langle \hat{J}_y^{(0)}(t) \rangle \big|_{t=0} \). In addition, due to local phase fluctuation which drive the system out of the ground state manifold and deplete the \( J = N/2 \) levels also \( \langle \hat{J}^{(0)2} \rangle \) decays exponentially.

The effect of the environment on the free evolution is significantly reduced by \( \hat{H}_{prot} \). The protection can be best understood by using the basis of collective states. In terms of collective spin operators \( \hat{H}_{env} \) can be written as:

\[
\hat{H}_{env} = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \sqrt{N} g^k(t) \hat{J}_z^{(k)}, \tag{3}
\]

where \( g^k(t) = \frac{1}{\sqrt{N}} \sum_{j=1}^N \hat{h}_j(t) e^{-i2\pi k} \) and \( \hat{J}_z^{(k)} = \frac{1}{2} \sum_{j=1}^N \hat{\sigma}_j e^{i2\pi k j} \). In the presence of a large energy gap \( E_g \), one can distinguish two different types of processes: (i) Decoherence effects that take place within the MPM due to the collective dynamics induced by the \( k = 0 \) component of \( \hat{H}_{env} \), and (ii) transitions across the gap induced by the inhomogeneous terms with \( k \neq 0 \). The latter couple the MPM with the rest of the system. All allowed transitions must conserve \( M \) since both the system and noise Hamiltonian commute with \( \hat{J}_z^{(0)} \).

In the limit when the noise is sufficiently slow (i.e. when the spectral density of the noise has a cutoff frequency \( \omega_c \ll E_g \)) the effect of phase decoherence is dramatically reduced as type (ii) processes are energetically forbidden and only type (i) processes are effective. In this limit the noise acts as a uniform random magnetic field. If at \( t = 0 \) the density matrix, \( \hat{\rho} = \sum_{M, \hat{M}} \rho_{M, \hat{M}} | \frac{M}{2}, M \rangle \langle \frac{M}{2}, \hat{M} | \), then after time \( t \) each component \( \rho_{M, \hat{M}} \) acquires an additional phase \( e^{i(\theta_M(t) - \theta_{\hat{M}}(t))} \) with \( \theta_M(t) \equiv \frac{1}{\sqrt{N}} \sum_{j=1}^N g^0(\tau) \), such that

\[
\rho_{M, \hat{M}}(t) = \rho_{M, \hat{M}}(0) e^{i\omega_0(M-\hat{M})t} e^{-\Gamma(t)(M-\hat{M})^2/8}. \tag{4}
\]

Note the factor of \( \sqrt{N} \) in the denominator of \( \theta_M \).

It is fundamental for the reduction of the effect of decoherence within the MPM. For example, \( \langle \hat{J}_x^{(0)} \rangle \) decays \( N \) times slower than in the unprotected system: i.e. \( \langle \hat{J}_x^{(0)}(t) \rangle = e^{-\Gamma(t)/N} \langle \hat{J}_x^{(0)}(0) \rangle \big|_{t=0} \).

Let us now elaborate more on the necessary conditions required to suppress type (ii) processes. For this, we will assume the power spectrum of the noise, \( f(\omega) = \frac{f d\tau}{2\pi} e^{-i\omega \tau} \), to have a cut-off frequency \( \omega_c \) (e.g. \( f(\omega) = f \) for \( \omega \leq \omega_c \) and 0 otherwise with \( f < \lambda \)). A time dependent perturbation analysis predicts that as long as \( \omega_c < E_g \), the decay rate \( \Gamma_M(t) \) of the diagonal elements \( \rho_{M, M} \) of the density matrix due to type (ii) process is always bounded by

\[
\frac{\Gamma_M(t)}{N^2 - M^2} < \frac{(N^2 - M^2)^2}{N^2 - M^2}. \tag{5}
\]

The generation process can be understood by writing \( \langle \hat{J}_x^{(0)}(t) \rangle \) as \( \sum_{M} C_M \langle N/2, M \rangle \). During the evolution the Hamiltonian imprints an \( M^2 \) dependent phase to the different components. As the system evolves, at first the winding of the phases leads to a collapse of \( \langle \hat{J}_x^{(0)} \rangle \). However, at time \( t = t_{rev} \), the system becomes an \( N \)-particle GHZ state.

Recent experiments [12, 13] have used this type of scheme to generate GHZ states in trapped ions with the aim to perform precision measurements of \( \omega_0 \) (ideally the use of GHZ states should enhance the phase sensitivity to the fundamental Heisenberg limit [14]). However, decoherence significantly limited the applicability of the method and in practice even
for only six ions, the achieved phase sensitivity was significantly below the fundamental limit. The effect of decoherence can be quantified by calculating the fidelity of creating a GHZ state defined as \( \mathcal{F}(t_0) = \frac{\langle \psi^{GHZ}_0 | \rho(t_0) | \psi^{GHZ}_0 \rangle}{\langle \psi^{GHZ}_0 | \psi^{GHZ}_0 \rangle} \). It is possible to show that, after the evolution with the total Hamiltonian, \( H_{env} + H_z \), the fidelity is degraded to:

\[
\mathcal{F}(t_0) = \left( 1 + e^{-\Gamma(t_0)} \right)^N.
\]

The \( N \)-dependent decay of \( \mathcal{F}(t_0) \) and the exponential decay of \( \langle \hat{J}_z(0) \rangle \) reflect the rapid collapse of the entanglement as \( N \) increases (See Fig. 2). Consider now the same generation process within the MPM. The latter can be realized by replacing \( H_z \rightarrow H_{prot} + H_z \). In the slow noise limit the reduced density matrix evolves as Eq. (4) with the phase \( \omega_0(M - \bar{M}) \) replaced by \( \chi(M^2 - \bar{M}^2) \). For the initially polarized state the fidelity is given by:

\[
\mathcal{F}(t_0) = \frac{1}{\sqrt{1 + \Gamma(t_0)}}.
\]

The independence of \( \mathcal{F}(t_0) \) on \( N \), and the \( N \)-times reduced decay rate of \( \langle \hat{J}_x(0) \rangle \) demonstrate the usefulness of MPM to generate a large number of entangled particles.

Let us now discuss physical implementations of the MPM. In Ref. [8] it has been shown that the Hamiltonian \( H_{prot} + H_z \) can be implemented by using the collective vibrational motion of the ions in a linear trap driven by illuminating them with a laser field. If the detuning of the laser, \( \delta \), from the internal transition frequency is large compared to the linewidth of the resonance but sufficiently different from the ion vibrational frequency, then the dominant processes are two-photon transitions which lead to simultaneous excitations of pairs of ions and thus to an effective Hamiltonian of the form:

\[
\hat{H}_{eff} = \chi(\hat{J}^{(0)2} - \hat{J}_z^{(0)2}) + \beta \hat{J}_z^{(0)}. \]

The cost of implementing \( H_{eff} \) instead of \( H_z \) is the additional echo techniques required for making the former insensitive to heating of the vibrational motion of the ions: while \( H_z \) is naturally insensitive to thermal vibrations due to the fact that it is generated by bichromatic beams [12] (which eliminate any dependence on vibrational quantum numbers due to the destructive interference between the transition paths) \( H_{eff} \) lacks this insensitivity as it is generated by monochromatic beams. On the other hand, the implementation of \( H_{eff} \) has the advantage in comparison with \( H_z \) that the MPM protects the system from local perturbations caused by non-ideal conditions, such as differences in the Rabi frequency experienced by the string of ions, or by external disturbances, such as magnetic field noise. It should be noted, however, that the system is not protected by the MPM against spontaneous emission and local perturbations.

**MPM with local interactions:** Up to now we have explored only the generation of an MPM via isotropic long-range interactions. In practice, however, it is desirable to have a similar kind of protection generated by systems with short range interactions such as those provided by cold atoms in optical lattices. We now show how an MPM can be created in lattice systems and can be used to robustly generate \( N \)-particle GHZ states in these systems. We consider ultracold bosonic atoms with two relevant internal states confined in an optical lattice with unity filling deep in the Mott insulator regime. Such systems is described by an effective spin XXZ Hamiltonian [13]:

\[
\hat{H}_{tot} = \hat{H}_H + \hat{H}_I = -\lambda \sum_{i,j}>\alpha \sigma_i^a \sigma_j^a - \chi \sum_{i,j>\alpha} \sigma_i^a \sigma_j^a.
\]

Here we have identified the two possible states at each site with the effective spin index \( \sigma = \uparrow, \downarrow \). The sum of \( (i,j) \) is over nearest neighbors. In Eq. (8) the coefficients are \( \lambda = \tau^2/4 \) and \( \chi = \tau^2 (U_{\uparrow \uparrow} + U_{\downarrow \downarrow} - 2U_{\uparrow \downarrow})/U^2 \), where \( U \) is the mean of the three different spin dependent on-site interactions energies \( U_{\uparrow \uparrow}, U_{\downarrow \downarrow}, U_{\uparrow \downarrow} \), which we assumed only slightly different between each other, and \( \tau \) is the tunneling energy between adjacent sites which we assume spin independent. Both \( U \) and \( \tau \) are functions of the lattice depth. For simplicity we restrict the analysis to one dimensional systems and assume periodic boundary conditions.

\[
\hat{H}_H \text{ is spherically symmetric and in terms of collective spin operators it can be written as}
\]

\[
\hat{H}_H = -\frac{4\lambda}{N} \hat{J}^{(0)2} - \frac{4\lambda}{N} \sum_{k,\sigma,\alpha} \hat{J}_\alpha^k \hat{J}_\alpha^{(k)\sigma} \cos \left( \frac{2\pi k}{N} \right).
\]

All the \( N + 1 \) states with \( J = N/2 \) are degenerate and span the ground state of \( \hat{H}_H \) thus form an MPM. \( \hat{H}_I \) is not spherically symmetric but we can also write it in terms of collective operators as

\[
\hat{H}_I = -\frac{4\chi}{N} \hat{J}_z^{(0)2} - \frac{4\chi}{N} \sum_{k=1}^{N-1} \hat{J}^{(k)} \hat{J}_z^{(k)} \cos \left( \frac{2\pi k}{N} \right).
\]
to fast oscillations in $\hat{\lambda}/\bar{\chi}$.

In the inset we show $\langle J_z^{(0)}(t) \rangle$. The blue dot-dashed, dotted black, dashed green and solid red correspond to $\bar{\lambda} = 0, 5, 10, 20$ respectively. The plots are obtained by numerical evolution of Eq. (8) for GHZ state preparation in the lattice is degraded as $N$ grows because in the lattice the gap decreases and the generation time increases with $N$. This is observed in the inset where the fidelity $\hat{\lambda}$ decreases and the generation time increases with $N$. In this plot we used a system with $\omega_v = \bar{\chi}, \bar{\lambda} = 100\bar{\chi}$ and $\Gamma = 0.01\bar{\chi}$. At $N = 90$, $\Delta E_g = \omega_v$ and it explains the drop of $F$ for $N > 90$.

If the condition $\bar{\chi} \ll \bar{\lambda}$ is satisfied, which is naturally the case for spin independent lattices, the effect of the Ising term can be treated perturbatively. Assuming that $t = 0$ the initial state is prepared within the $J = N/2$ manifold, a perturbative analysis predicts that for times $t$ such that $\bar{\chi}t < \lambda/\bar{\chi}$, $\hat{H}_I$ confines the dynamics to the MPM and transitions outside can be neglected. As a consequence, only the projection of $H_I$ on $\hat{H}_I$ is effective. As $\mathcal{P} \hat{H}_I = \chi \hat{J}_z^{(0)2} + \text{const}$, with $\chi \equiv \frac{\bar{\lambda}}{N}[18]$, $H_I$ acts as a long range Hamiltonian.

In Fig. 3a we show the fidelity to create a GHZ state at $\lambda/\bar{\chi} = \pi/2$ vs $\bar{\lambda}/\bar{\chi}$ and contrast the dynamical evolution of $\langle J_z^{(0)}(t) \rangle$ for different $\bar{\lambda}/\bar{\chi}$ ratios assuming at time $t = 0$ all the spins are polarized in the $x$ direction. If only the Ising term is present, $\lambda = 0$, it induces local phase fluctuations that leads to fast oscillations in $\langle J_z^{(0)}(t) \rangle = N/2 \cos^2[2\bar{\chi}t]$. On the other hand, as the ratio $\bar{\lambda}/\bar{\chi}$ increases, the isotropic interaction inhibits the fast oscillatory dynamics and instead $\langle J_z^{(0)}(t) \rangle$ exhibits slow collapses and revivals. For $\bar{\lambda} > \bar{\chi}$ the dynamics exactly resembles the one induced by $\hat{H}_e$ and at $\chi e t = \pi/2$ the initial coherent state is squeezed into a GHZ state.

$\hat{H}_I$ also provides protection against phase decoherence. However, $\hat{H}_I$ is not as effective as $\hat{H}_\text{pro}$ because the energy gap between the MPM and the excited states of $\hat{H}_I$ (Eq. [11]) vanishes in the thermodynamic limit as $E_g \rightarrow \lambda/N^2$. This is a drawback of the short range Hamiltonian for the purpose of fully protecting the system from long wave length excitations [19]. Nevertheless, the many body interactions can still eliminate short-wavelength excitations since in the large $N$ limit they remain separated by a finite energy gap, $8\bar{\lambda}$. In Fig. 3b we quantify the protection provided by the lattice Hamiltonian against population decay by plotting $F$ as a function of $N$. As expected, an abrupt drop of the fidelity occurs at the value of $N$ at which $E_g = \omega_v$.

In summary we proposed and described a new method for the robust generation of entangled states and for their protection against decoherence and discussed its applicability for the generation of many-particle-GHZ-type states in ion traps and optical lattices. We emphasize that, even though we have limited the discussion to ensembles of spin $S = 1/2$ particles, the MPM ideas can be straightforwardly generalized to systems composed of higher spin atoms. Besides entanglement generation, the MPM might have also important applications for the implementation of good storage memories using for example nuclear spin ensembles in solid state [16] or photons [17].

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with $P$ the projection into the $J = N/2$ subspace.
[19] 1D systems are the worst scenario as the gap vanishes as $N^{-2/D}$ with $D$ the dimensionality.