Optimization and variational principles for the shear strength reduction method

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Funding information
Czech Science Foundation, Grant/Award Number: 19-11441S

Abstract
In this paper, a modified shear strength reduction method (MSSR) and its optimization variant (OPT-MSSR) are suggested. The idea of MSSR is to approximate the standard shear strength reduction to be more stable and rigorous from the numerical point of view. The MSSR method consists of a simplified associated elasto-plastic model completed by the strength reduction depending on the dilatancy angle. Three Davis’ modifications suggested by Tschuchnigg et al. (2015) are interpreted as special cases of MSSR and their factors of safety are compared. The OPT-MSSR method is derived from MSSR on the basis of rigid plastic assumption, similarly as in limit analysis. Using the variational approach, the duality between the static and kinematic principles of OPT-MSSR is shown. The numerical solution of OPT-MSRR is obtained by performing a regularization method in combination with the finite element method, mesh adaptivity and a damped Newton method. In-house codes (Matlab) are used for the implementation of this solution concept. Finally, two slope stability problems are considered, one of which follows from analysis of a real slope. The softwares packages Plaxis and Comsol Multiphysics are used for comparison of the results.

KEYWORDS
convex optimization, finite elements and mesh adaptivity, regularization, shear strength reduction method, slope stability, static and kinematic principles

Nomenclature: SSR, shear strength reduction; MSSR, modified shear strength reduction; OPT-MSSR, optimization variant of MSSR; LA, limit analysis; FoS, factor of safety; c′, effective cohesion; φ′, effective friction angle; ψ′, dilatancy angle; λ, control parameter for strength reduction, λ ≥ 0; cλ, reduced cohesion depending on λ; φλ, reduced friction angle depending on λ; ψλ, reduced dilatancy angle depending on λ; q(λ), function defining the strength reduction. We distinguish the following particular choices of q: qa, qA, qB, and qC for the associated model and for Davis A-C approaches; ̃cλ, reduced cohesion depending on λ; ̃φλ, reduced friction angle depending on λ; ̃ψλ, reduced dilatancy angle depending on λ; σ, Cauchy stress tensor; σ1, σ3, maximal and minimal principal stresses of σ; Φ, Mohr-Coulomb yield function; Ω, bounded domain represented an investigated body; ∂Ω, a part of the boundary of Ω; n, the outward unit normal to the boundary of Ω; F, f, volume and surface forces; ω∗, factor os safety for the OPT-SSR method. We distinguish the following particular choices of ω∗: λ∗a, λ∗A, λ∗B, and λ∗C for the associated model and for Davis A-C approaches.; ζ∗, load factor for the limit analysis method; Λ, factor of safety for limit analysis depending on the strength parameter λ; V, space of velocity field; Σ, space of stress fields; v, velocity field; Σ, strain-rate tensor depending on v; L, functional of external forces; P̃q(λ), set of plastically admissible stress fields; Λ, set of statically admissible stress fields; D, local dissipation function; α, regularization parameter; λ∗a, approximations of ω given by the regularization; Da, regularized local dissipation function, Da = Da for α = 1; T, derivative of Da with respect to the stress variable, T = T for α = 1; γ, unit weight for a homogeneous slope; E, Young’s modulus; μ, Poisson’s ratio; γunsat, specific weight for unsaturated material; γsat, specific weight for saturated material.
1 | INTRODUCTION

This paper deals with slope stability assessment, which includes the determination of the factor of safety (FoS) and the estimation of failure zones (slip surfaces) for a critical state of the slope. Limit equilibrium (LE), shear strength reduction (SSR) or limit analysis (LA) can be used for the determination of FoS. The methods arise from elastic-perfectly plastic models containing mainly the Mohr-Coulomb yield criterion.

The LE method is based on predefined failure zones, see, for example, Refs. 1, 2. It does not necessarily require numerical computation, and neither stress equilibrium at every point in the domain around the slope. Due to these facts, this method is simple, time-saving and widely used in geotechnical practice. On the other hand, accuracy of the solution cannot be easily verified, especially if anisotropic or inhomogeneous materials are considered or if a complex geometry is defined.

The SSR method3–6 has been suggested mainly for the slope stability assessment. It is a conventional method based on a displacement variant of the finite element method (FEM) and on reduction of strength parameters defining the Mohr-Coulomb model. It has also been implemented within some commercial softwares like Plaxis 7 or Midas GTS NX. 8 Strong dependence on the finite element mesh density or even a nonunique determination of FoS can occur in SSR with a non-associated plastic flow rule. 9,10

LA is a universal method that can be used for various stability problems, not only for the ones in geotechnical practice. FoS is derived from a critical (limit) value of the load factor. Originally, this method was purely analytical, see, for example, Refs. 11, 12. Now, it is rather a numerical method based on optimization, duality between kinematic and static principles and performed within the framework of FEM. 23–15 LA is supported by mathematical and numerical analyzes 13,15–18 and has been implemented, for example, within the software OPTUMG2.19 In slope stability; however, the FoS is defined according to strength parameters. In addition, there is also no rigorous solution in the case of non-associated plasticity. To overcome these drawbacks, various modifications of LA were suggested. 10,14,20

The usage of the non-associated plastic flow rule is supported by laboratory experiments and enables to control the inelastic volume changes of soils subjected to shearing, see, for example, Refs. 21, 22. On the other hand, mathematical theory of non-associated elastic-plastic problems is missing or at least incomplete. Especially, standard implicit discretizations of time (pseudo-time) variables lead to problematic numerical behavior. The drawbacks of non-associated models can be suppressed by variational approaches based on theory of bipotentials23,24 or semi-implicit time schemes. 25 Within the LA method, Davis 26 suggested to modify the strength parameters and consequently approximate the non-associated model by the associated one. In Refs. 9, 10, the Davis approach has been modified for purposes of the SSR method. The modification leads to an iterative solution scheme based on LA that was originally suggested by Sloan. 14 We also refer to recent papers 27,28 for comparison of the standard and the modified SSR method.

The drawback of the modified SSR method developed in Tschuchnigg et al. 10 is related to the fact that the FoS is defined by an iterative process requiring to solve the LA analysis problem in each iteration. In this paper, we propose a much straightforward modification of the SSR method (MSSR) which is independent of LA. The idea of MSSR is to solve the associated elasto-plastic problem and reduce strength parameters using a scalar function $q$ which depends on the effective friction and dilatancy angles and on a scalar factor driving the reduction. It is shown that Davis modifications suggested in Tschuchnigg et al. 10 correspond to appropriate choices of the function $q$. In case of associated plasticity, the modified and the standard SSR methods coincide.

Next, we propose an optimization variant of the MSSR method (OPT-MSSR) based on rigid plasticity, similarly to Krabbenhoff and Lyamin. 20 The OPT-MSSR problem can be analyzed regardless its space discretization. We use this fact for analytical comparison of safety factors of various Davis’ modifications suggested in Ref. 10. Next, we derive the duality between static and kinematic settings of the OPT-MSSR problem using variational principles.

For numerical solution of MSSR, one can use either standard incremental elasto-plasticity and finite elements or more advanced optimization methods arising from the OPT-MSSR problem. For example, one can consider the iterative LA approach mentioned above. 10,14,20 As an alternative, we propose a regularization method for the solution of OPT-MSSR which does not require to solve the LA problem in each iteration. This method is universal and has been successfully used in recent papers. 15,17,18,29–33 We analyze convergence with respect to the regularization parameter in order to relate the OPT-MSSR problem with its regularized counterpart. The regularization enables to solve the problem with standard finite element methods and with the damped Newton method as suggested in Ref. 34. Mesh adaptivity is also used to compute more accurate results.

The suggested numerical solution is implemented within in-house Matlab codes. These codes (based on elastic-plastic solvers) have been systematically developed and described in Refs. 35–37. Some of the codes are available for download. 38
For comparison of the results with the standard SSR method, the software Plaxis is used. We also use the software Comsol Multiphysics which is not specialized on the SSR method, but the studies show that this software is capable to solve the suggested MSSR method.

The paper is organized as follows. In Section 2, we recall the standard SSR method and introduce the MSSR method. In Section 3, various Davis’ modifications suggested in Ref. 10 are interpreted as special cases of the MSSR method. In Section 4, the OPT-MSSR method is introduced, analyzed and used for analytical comparison of the Davis modifications. In Section 5, the LA approach for the solution of OPT-SSR is introduced. It enables to relate the OPT-SSR method to the approaches from Refs. 10, 14. In Section 6, variational principles, duality and the kinematic approaches to the OPT-SSR method are presented. The regularization method, which is built on the variational principles, is introduced and analyzed in Section 7. Numerical examples illustrating the efficiency of the suggested numerical methods are presented in Section 8. Concluding remarks are given in Section 9. The appendix contains a closed form of a regularized dissipative function for the Mohr-Coulomb yield criterion.

2 | THE STANDARD SSR METHOD AND ITS MODIFICATION

The standard SSR method is based on the elastic-perfectly plastic problem including the Mohr-Coulomb yield criterion. For the complete definition of this problem, see Refs. 36, 39. Such a model includes the elastic deformational material parameters (Young’s modulus and Poisson’s ratio) and the following strength parameters: the effective cohesion \( c' \), the effective friction angle \( \phi' \), and the dilatancy angle \( \psi' \). It is assumed that \( 0 \leq \psi' \leq \phi' \). In case of \( \psi' = \phi' \), we arrive at an associated flow rule.

The SSR method is based on the reduction of the parameters \( c' \), \( \phi' \) and \( \psi' \):

\[
c_\lambda := \frac{c'}{\lambda}, \quad \phi_\lambda := \arctan \frac{\phi'}{\lambda}, \quad \psi_\lambda := \arctan \frac{\psi'}{\lambda},
\]

where \( \lambda > 0 \) is the reduction parameter. Alternatively, one can use the following formula for \( \psi_\lambda \) (see also Ref. 10):

\[
\psi_\lambda := \psi' \text{ as long as } \psi' < \phi_\lambda; \text{ otherwise } \psi_\lambda := \phi_\lambda.
\]

FoS of the SSR method is defined as a maximum of \( \lambda \) for which the elastic-perfectly plastic problem has a solution with respect to the parameters \( c_\lambda, \phi_\lambda, \) and \( \psi_\lambda \). This definition is from the mathematical point of view rather formal, because the solvability of the elasto-plastic problem requires to introduce convenient functional spaces and a weak form of the problem (see, e.g., Ref. 40). Such an analysis is problematic for the case of the non-associated plasticity.

Inspired by the paper, 10 we propose the modified variant of the SSR method (MSSR), which is based on the associated model and the following reduction of the parameters \( c' \) and \( \phi' \):

\[
c_\lambda := \frac{c'}{q(\lambda; \phi', \psi')}, \quad \phi_\lambda := \frac{\phi'}{q(\lambda; \phi', \psi')}, \quad \psi_\lambda := \phi_\lambda.
\]

where \( q \) is a function with the following general properties:

(A1) \( q \) is positive and continuous for any \( \lambda > 0 \) and any \( \phi', \psi' \) such that \( 0 \leq \psi' \leq \phi' \);
(A2) \( q \) is increasing with respect to the variable \( \lambda \geq 0 \);
(A3) \( q \) is non-increasing with respect to the variable \( \psi' \in [0, \phi'] \);
(A4) \( q(\lambda; \phi', \psi') \geq \lambda \) for any \( \lambda \geq 0 \) and any \( \phi', \psi' \) such that \( 0 \leq \psi' \leq \phi' \);
(A5) if \( \psi' = \phi' \) then \( q(\lambda; \phi', \psi') = \lambda \).

FoS for the MSSR method is then defined as a maximum of \( \lambda \) for which the associated elastic-perfectly plastic problem has a solution with respect to the parameters \( c_\lambda, \phi_\lambda, \) and \( \psi_\lambda = \phi_\lambda \).

The assumptions (A1) and (A2) ensure that the strength parameters are reduced, that is, the values \( c_\lambda \) and \( \phi_\lambda \) decrease with increasing \( \lambda \). The assumption (A3) enables to include the influence of the dilatancy angle \( \psi' \) on FoS. The reduction of the strength parameters is larger for smaller values of \( \psi' \) than for larger ones. Therefore, the smaller the values of \( \psi' \)
are, the lower the values of FoS are expected. Due to (A4) and (A5) one can expect that FoS for the associated elasto-plastic model is larger than FoS for the non-associated one. If $\psi' = \phi'$, associated plasticity is considered and in this case, the assumption (A5) means that the standard SSR method and the MSSR method are identical.

The examples of $q$ related to the paper\textsuperscript{10} are introduced in the next section. Nevertheless, the MSSR method is not limited to these choices. The choice of the function $q$ can also be optimized, for example, by inverse analysis.

# 3 DAVID'S MODIFICATIONS OF THE SSR METHOD AND THEIR COMPARISON

The aim of this section is to introduce three different functions $q$ representing Davis’ modifications of the SSR method suggested in Ref. 10 and compare these functions. Since the Davis modifications are denoted as Davis A, Davis B and Davis C, we use the notation $q_A$, $q_B$, and $q_C$ instead of $q$ to distinguish these particular cases. Their definitions are following:

$$q_A(\lambda; \phi', \psi') = \lambda \frac{1 - \sin \psi' \sin \phi'}{\cos \psi' \cos \phi'},$$  \hspace{1cm} (3.1)

$$q_B(\lambda; \phi', \psi') = \lambda \frac{1 - \sin \phi' \sin \phi}{\cos \phi \cos \phi'},$$  \hspace{1cm} (3.2)

$$q_C(\lambda; \phi', \psi') = \begin{cases} \lambda \frac{1 - \sin \psi' \sin \phi'}{\cos \psi \cos \phi'}, & \text{if } \phi \geq \psi', \\ \lambda, & \text{if } \phi \leq \psi', \end{cases}$$  \hspace{1cm} (3.3)

where the functions $\lambda \mapsto \phi$ and $\lambda \mapsto \psi$ are defined by (2.1), that is,

$$\tan \phi := \frac{\tan \phi'}{\lambda}, \quad \tan \psi := \frac{\tan \psi'}{\lambda}. \hspace{1cm} (3.4)$$

Remark 3.1. The function $q$ (or its particular cases $q_A$, $q_B$, and $q_C$) was not directly introduced in Ref. 10. Instead of this, the following function was introduced there:

$$\beta(\lambda; \phi', \psi') := \frac{\lambda}{q(\lambda; \phi', \psi')}.$$  \hspace{1cm} (3.5)

Next, advantages and disadvantages of the DAVIS A–C approaches were discussed in Ref. 10. For example, due to the fact that the difference between $\phi$ and $\psi$ defines the amount of non-associativity, the methods Davis B is considered to be more appropriate compared to Davis A and Davis C.

In order to easily analyze the properties of $q_A$, $q_B$, $q_C$ and compare these functions, it is convenient to replace the sine and the cosine with the tangent using the following well-known formulas:

$$\cos \phi' = \frac{1}{\sqrt{1 + \tan^2 \phi'}}, \quad \sin \phi' = \frac{\tan \phi'}{\sqrt{1 + \tan^2 \phi'}}.$$  \hspace{1cm} (3.6)

These formulas can be easily derived from $\tan \phi' = \sin \phi'/\cos \phi'$ and $\sin^2 \phi' + \cos^2 \phi' = 1$. Using (3.6) and (3.4), one can write the functions $q_A$, $q_B$, $q_C$ as follows:

$$q_A(\lambda; \phi', \psi') = \lambda \left[\sqrt{(1 + \tan^2 \psi')(1 + \tan^2 \phi') - \tan \psi' \tan \phi'}\right].$$  \hspace{1cm} (3.7)
\[ q_B(\lambda; \phi', \psi') = \frac{1}{\lambda} \left[ \sqrt{\left( \lambda^2 + \tan^2 \psi' \right) \left( \lambda^2 + \tan^2 \phi' \right)} - \tan \psi' \tan \phi' \right] \tag{3.8} \]

\[
q_C(\lambda; \phi', \psi') = \begin{cases} 
\sqrt{\left( \lambda^2 + \tan^2 \phi' \right) \left( 1 + \tan^2 \psi' \right)} - \tan \phi' \tan \psi', & \text{if } \tan \phi' \geq \lambda \tan \psi', \\
\lambda, & \text{if } \tan \phi' \leq \lambda \tan \psi'.
\end{cases} \tag{3.9}
\]

**Lemma 3.2.** The functions \( q_A \) and \( q_B \) satisfy the assumptions (A1)–(A5) introduced in Section 2. The function \( q_C \) satisfies (A1)–(A4) for any \( \lambda \geq 0 \) and (A5) for any \( \lambda \geq 1 \).

**Proof.** It is readily seen that the assumption (A1) holds for all the functions and that (A5) is satisfied for \( q_A \) and \( q_B \). If \( \lambda \geq 1 \) then (A5) also holds for \( q_C \). The assumptions (A2) and (A3) follow from the partial derivatives of \( q_A, q_B, q_C \) with respect to \( \lambda \) and \( \psi' \). For example, we have:

\[
\frac{\partial q_B}{\partial \lambda} = \frac{1 + \frac{1}{\lambda^2} \left[ \sqrt{\left( \lambda^2 + \tan^2 \psi' \right) \left( \lambda^2 + \tan^2 \phi' \right)} - \tan \psi' \tan \phi' \right]}{\sqrt{\left( \lambda^2 + \tan^2 \psi' \right) \left( \lambda^2 + \tan^2 \phi' \right)}} > 0,
\]

\[
\frac{\partial q_B}{\partial \psi'} = \frac{\lambda (\tan^2 \psi' - \tan^2 \phi') \cos^2 \psi'}{\sqrt{\lambda^2 + \tan^2 \psi'} \left( \tan \psi' \sqrt{\lambda^2 + \tan^2 \phi'} + \tan \phi' \sqrt{\lambda^2 + \tan^2 \psi'} \right)} < 0.
\]

To verify (A4) for \( q_A \), we use the following equality:

\[
\sqrt{\left( 1 + \tan^2 \psi' \right) \left( 1 + \tan^2 \phi' \right)} - 1 - \tan \psi' \tan \phi' = \frac{(\tan \phi' - \tan \psi')^2}{\sqrt{\left( 1 + \tan^2 \psi' \right) \left( 1 + \tan^2 \phi' \right) + 1 + \tan \psi' \tan \phi'}}. \tag{3.10}
\]

Similarly, one can prove (A4) for \( q_B \) and \( q_C \), see also formulas (3.11)–(3.13) introduced below.

From now on, we shall mainly analyze the dependence of \( q \) (or \( q_A, q_B, q_C \)) on \( \lambda \). Therefore, we write \( q := q(\lambda) \), for the sake of simplicity. We also define the function \( q_{ass}(\lambda) := \lambda \) representing the associated model with \( \psi' = \phi' \). This function is in accordance with the assumption (A5). We have the following results comparing the functions \( q_A, q_B, q_C \), and \( q_{ass} \).

**Lemma 3.3.** The following statements hold:

1. \( q_A \geq q_{ass}, q_B \geq q_{ass}, q_C \geq q_{ass} \);
2. \( q_A(1) = q_B(1) = q_C(1) \);
3. \( q_A(\lambda) \geq q_B(\lambda) \geq q_C(\lambda) \) for any \( \lambda \geq 1 \);
4. \( q_C(\lambda) \geq q_B(\lambda) \geq q_A(\lambda) \) for any \( \lambda \leq 1 \).

**Proof.** We use the following simplifying notation: \( a := \tan \phi' \) and \( b := \tan \psi' \), that is \( 0 \leq b \leq a \). We insert \( a \) and \( b \) into (3.7), (3.8), and (3.9). Next, these formulas for \( q_A, q_B, q_C \) can be arranged to the following forms:

\[
q_A(\lambda) = \lambda + \frac{\lambda(a - b)^2}{\sqrt{(1 + a^2)(1 + b^2) + 1 + ab}}, \tag{3.11}
\]
\[ q_B(\lambda) = \lambda + \frac{\lambda(a - b)^2}{\sqrt{(\lambda^2 + a^2)(\lambda^2 + b^2) + \lambda^2 + ab}}, \quad (3.12) \]

\[ q_C(\lambda) = \begin{cases} 
\lambda + \frac{(a - \lambda b)^2}{\sqrt{\lambda^2 + a^2}(\lambda^2 + b^2) + \lambda^2 + ab}, & \text{if } \lambda \leq a/b \\
\lambda, & \text{if } \lambda \geq a/b.
\end{cases} \quad (3.13) \]

Hence, it is readily seen that the first two statements hold. The relations between \( q_A \) and \( q_B \) also hold. To relate \( q_B \) and \( q_C \), we use the following inequalities:

\[
(a - \lambda b)^2 \leq (a - b)^2, \quad \text{if } 1 \leq \lambda \leq a/b,
\]

\[
(a - \lambda b)^2 \geq (a - b)^2, \quad \text{if } 1 \geq \lambda,
\]

and

\[
\lambda \left[ \sqrt{(\lambda^2 + a^2)(\lambda^2 + b^2) + \lambda^2 + ab} \right] \geq \sqrt{(\lambda^2 + a^2)(\lambda^2 + b^2) + \lambda^2 + ab}, \quad \text{if } 1 \leq \lambda,
\]

\[
\lambda \left[ \sqrt{(\lambda^2 + a^2)(\lambda^2 + b^2) + \lambda^2 + ab} \right] \leq \sqrt{(\lambda^2 + a^2)(\lambda^2 + b^2) + \lambda^2 + ab}, \quad \text{if } 1 \geq \lambda.
\]

Lemma 3.3 is important for comparison of FoS for the Davis A-C modifications of the SSR method. Such a comparison will be discuss in the next section.

4 \quad THE OPT-MSSR METHOD AND ITS ANALYSIS

The MSSR method has been introduced in Section 2 and its examples related to various Davis’ approaches have been discussed in Section 3. One can solve the MSSR problem based on the formula (2.3) similarly as the standard SSR method. In the following, we focus on an optimization variant of the MSSR method (OPT-MSSR) which is built on rigid plasticity. Similar treatment was considered in Ref. 20 and is also known in limit analysis. Consequently, we show that OPT-MSSR enables a deeper analysis of the MSSR method (regardless space discretization).

To define the OPT-MSSR problem, we introduce the required notation. The Mohr-Coulomb yield criterion for the parameters \( \tilde{c}_\lambda \) and \( \tilde{\phi}_\lambda \) (see Equation 2.3) reads

\[ (\sigma_1 - \sigma_3) + (\sigma_1 + \sigma_3) \sin \tilde{\phi}_\lambda - 2\tilde{c}_\lambda \cos \tilde{\phi}_\lambda \leq 0, \quad (4.1) \]

where \( \sigma_1 \geq \sigma_2 \geq \sigma_3 \) denote the principle effective stresses of the Cauchy stress tensor \( \sigma \) in the standard mechanical sign convention (positive sign for tension). Using the formulas (3.6) and (2.3), the criterion can be arranged to the form

\[ \Phi(q(\lambda); \sigma) := (\sigma_1 - \sigma_3) \sqrt{q^2(\lambda) + \tan^2 \phi' + (\sigma_1 + \sigma_3) \tan \phi' - 2c'} \leq 0 \quad (4.2) \]

which is more convenient for consequent analysis.

Remark 4.1. If we consider that \( \sigma \) is an arbitrary stress tensor then the Mohr-Coulomb yield surface is the pyramid aligned with the hydrostatic axis, see, for example, Ref. 39. From (4.2), it follows that the apex of this pyramid is independent of \( \lambda \). By reducing the strength parameters (i.e., by enlarging \( \lambda \)) the slope of the Mohr-Coulomb pyramid is reduced.
Next, $\Omega$ denotes a bounded domain in 2D and 3D representing an investigated body, $F$ is a volume force (e.g., the weight of the body), $f$ is a prescribed surface force acting on the part $\partial \Omega_f$ of the boundary $\partial \Omega$ and $n$ denotes the outward unit normal to the boundary $\partial \Omega$.

The OPT-MSSR problem defines the factor of safety $\omega^*$ as follows:

$$\omega^* = \supremum \ of \ \lambda \geq 0 \ subject \ to$$

$$\begin{align*}
-\text{div } \sigma &= F \ in \ \Omega, \quad \sigma n = f \ on \ \partial \Omega_f, \\
\Phi(q(\lambda); \sigma) &\leq 0 \ in \ \Omega.
\end{align*}$$

(4.3)

Notice that the constraints on the first and second lines of (4.3) represent statically and plastically admissible stress fields, respectively. According to the literature on convex analysis, we rather use the supremum than the maximum in this definition, because for the critical value $\omega^*$, the admissible stress $\sigma$ satisfying (4.3) does not need to exist on functional spaces. Although the definition admits the case $\omega^* = +\infty$, one can expect that $\omega^*$ is finite in geotechnical boundary value problems.

It is worth noticing that the function $\Phi$ is convex with respect to the stress variable $\sigma$ but not with respect to both the variables $\lambda$ and $\sigma$. Due to this fact, the analysis of the OPT-MSSR method is more challenging than the limit analysis method. The following statement implies that (4.3) holds for any $\lambda \geq 0$ such that $\lambda < \omega^*$. Without this basic property, it would be very difficult to find $\omega^*$.

**Lemma 4.2.** Let the assumptions (A1)–(A2) hold. If (4.3) is satisfied for some $\lambda : = \overline{\lambda} > 0$ then (4.3) holds for any $\lambda < \overline{\lambda}$.

*Proof.* For any $\sigma$ fixed, $\sigma_1 - \sigma_3 \geq 0$ and thus the function $\lambda \mapsto \Phi(q(\lambda); \sigma)$ is non-decreasing as follows from the assumptions (A1)–(A2). Hence, if there exists $\sigma$ such that (4.3) holds for some $\lambda : = \overline{\lambda} > 0$ then for any $\lambda < \overline{\lambda}$, we have

$$\Phi(q(\lambda); \sigma) \leq \Phi(q(\overline{\lambda}); \sigma) \leq 0 \ in \ \Omega.$$  

(4.4)

The statement of the lemma easily follows from this inequality. $\square$

For the choices $q : = q_{\text{ass}}, \ q : = q_A, \ q : = q_B,$ and $q : = q_C$, we denote the corresponding FoS by $\omega^* : = \lambda_{\text{ass}}^*, \omega^*_A : = \lambda_A^*, \omega^*_B : = \lambda_B^*$, and $\omega^*_C : = \lambda_C^*$, respectively. Let us note that if $\lambda_A^* \geq \tan \phi' / \tan \phi''$ then $\omega^*_C = \lambda_{\text{ass}}^*$. If $\phi' = 0^\circ$ then $\lambda_B^* = \lambda_C^*$. In this case, the Davis B and Davis C approaches coincide.

Using Lemma 3.3, one can compare analytically the values $\lambda_{\text{ass}}^*, \lambda_A^*, \lambda_B^*, \lambda_C^*$.  

**Theorem 4.3.** The following statements hold:

1. $\lambda_A^* \leq \lambda_{\text{ass}}^*, \lambda_B^* \leq \lambda_{\text{ass}}^*, \lambda_C^* \leq \lambda_{\text{ass}}^*$.
2. Either $1 \leq \lambda_A^* \leq \lambda_B^* \leq \lambda_C^*$ or $1 \geq \lambda_A^* \geq \lambda_B^* \geq \lambda_C^*$.
3. If one of the values $\lambda_A^*, \lambda_B^*, \lambda_C^*$ is equal to one then the same holds for the remaining values.

*Proof.* Let $\lambda < \lambda_A^*$. Then the constraints in (4.3) are satisfied for $\lambda$ and $q : = q_A$. Since $q_A \geq q_{\text{ass}}$, we have

$$0 \geq \Phi(q_A(\lambda), \sigma) \geq \Phi(q_{\text{ass}}(\lambda), \sigma).$$

It means that the constraints in (4.3) are also satisfied for $\lambda$ and $q : = q_{\text{ass}}$. This implies $\lambda_{\text{ass}}^* \leq \lambda_A^*$. Analogously, one can prove $\lambda_B^* \leq \lambda_{\text{ass}}^*$ and $\lambda_C^* \leq \lambda_{\text{ass}}^*$, and thus the first statement holds.

Let one of the values $\lambda_A^*, \lambda_B^*, \lambda_C^*$ be greater than one. Then, using the equalities $q_A(1) = q_B(1) = q_C(1)$, the constraints in (4.3) are satisfied for $\lambda : = 1$ and $q : = q_A$ or $q : = q_B$ or $q : = q_C$. This implies $\lambda^*_A \geq 1$, $\lambda_B^* \geq 1$, $\lambda_C^* \geq 1$. From the inequalities $q_A(\lambda) \geq q_B(\lambda) \geq q_C(\lambda)$ which hold for any $\lambda \geq 1$, we consequently derive $1 \leq \lambda_A^* \leq \lambda_B^* \leq \lambda_C^*$ similarly as in the first part of the proof.

Let $\lambda_C^* \leq 1$. Then, for any $\lambda < 1$, we have $q_C(\lambda) \geq q_B(\lambda) \geq q_A(\lambda)$ implying $\lambda_A^* \geq \lambda_B^* \geq \lambda_C^*$. In addition, the inequality $\lambda_A^* \leq 1$ must hold as a consequence of the previous part of the proof. Therefore, the second statement holds.

Third statement is a direct consequence of the second statement. $\square$
The derived comparison of FoS is in accordance with numerical observations presented in Refs. [10, 28]

5 | ITERATIVE LIMIT ANALYSIS FOR THE SOLUTION OF OPT-MSSR

The aim of this section is to relate the iterative LA solution scheme from Ref. 14 to the OPT-MSSR problem. Consider a fixed value of \( \lambda \geq 0 \) and the corresponding reduction parameter \( q(\lambda) \) for a given function \( q \). With respect to this parameter, we define the LA problem as follows:

\[
\ell(\lambda) = \sup_{\zeta \geq 0} \Phi(\zeta; \sigma) \leq 0 \text{ in } \Omega,
\]

\[
\Phi(q(\lambda), \sigma) \leq 0 \text{ in } \Omega.
\]

The value \( \ell(\lambda) \) defines the safety factor of the LA problem depending on \( \lambda \). We shall discuss properties of the corresponding function \( \ell \).

**Lemma 5.1.** Let the function \( q \) satisfy the assumptions (A1)-(A2). Then the function \( \ell \) is non-increasing.

**Proof.** Let \( \lambda_1 \leq \lambda_2 \) be two arbitrary values. To prove \( \ell(\lambda_1) \geq \ell(\lambda_2) \) it suffices to show that the following implication holds for any \( \zeta \geq 0 \): if \( \zeta < \ell(\lambda_2) \) then \( \zeta \leq \ell(\lambda_1) \). Let us suppose that \( \zeta < \ell(\lambda_2) \). Then there exists \( \sigma \) satisfying (5.1) for \( \lambda := \lambda_2 \). Since the function \( q \) is non-decreasing by the assumption, we have

\[
\Phi(q(\lambda_1), \sigma) \leq \Phi(q(\lambda_2), \sigma) \leq 0.
\]

Hence, \( \sigma \) satisfies (5.1) also for \( \lambda := \lambda_1 \). Therefore, \( \zeta \leq \ell(\lambda_1) \). \( \square \)

If the function \( q \) is continuous, then one can also expect that \( \ell \) is continuous. By comparison of the constraints (4.3) and (5.1), we derive that the safety factor \( \omega^* \) of the OPT-SSR problem introduced in Section 4 is a solution of the following equation:

\[
\ell(\omega^*) = 1.
\]

If we solve this equation iteratively, we arrive, for example, at the algorithm introduced in Ref. 14. Consequently, the safety factors \( \lambda_A^*, \lambda_B^* \) and \( \lambda_C^* \) presented above should be very close to the safety factors computed with finite element limit analysis (FELA) Davis A-C as presented in Ref. 10.

However, repetitive solution of the LA problem is expensive, especially, if the LA is combined with mesh adaptivity which improves the quality of the computed results significantly. In the next sections, more straightforward methods for solution of OPT-MSSR are discussed.

6 | VARIATIONAL PRINCIPLES, DUALITY AND KINEMATIC APPROACHES

We consider the abstract OPT-MSSR problem introduced in Section 4:

\[
\omega^* = \sup_{\lambda \geq 0} \Phi(\lambda) \leq 0 \text{ subject to}
\]

\[
\Phi(q(\lambda); \sigma) \leq 0 \text{ in } \Omega,
\]

where \( q \) satisfies the assumptions (A1)-(A5). This problem can be interpreted as the static principle of the OPT-MSSR method. The aim of this section is to derive the corresponding kinematic principle, which will be used for the numerical solution of the problem. Since a similar derivation is known in the limit analysis problem, 13,15,16 some technical details are skipped, for the sake of brevity.
We introduce the following functionals spaces:

\[ V = \{ v \in [H^1(\Omega)]^3 \mid v = 0 \text{ on } \partial \Omega \setminus \partial \Omega_f \}, \tag{6.2} \]

\[ \Sigma = \{ \sigma \in [L^2(\Omega)]^{3\times3} \mid \sigma_{ij} = \sigma_{ji} \text{ in } \Omega, \ i, j = 1, 2, 3 \}. \tag{6.3} \]

Similarly as in LA, the space \( V \) represents velocity fields and \( \Sigma \) is used for symmetric stress fields. \( L^2(\Omega) \) and \( H^1(\Omega) \) denote the Lebesgue and Sobolev spaces, respectively. More advanced functional spaces are considered in Ref. 13.

Using the space \( V \) we arrive at the weak form of (6.1):

\[ \int_{\Omega} \sigma : \varepsilon(v) \, dx = L(v) \quad \forall v \in V, \tag{6.4} \]

where \( \varepsilon \) denotes the strain-rate tensor field,

\[ \varepsilon(v) = \frac{1}{2}(\nabla v + (\nabla v)^T), \tag{6.5} \]

and \( L \) is the load functional defined by

\[ L(v) = \int_{\Omega} F \cdot v \, dx + \int_{\partial \Omega_f} f \cdot v \, ds. \tag{6.6} \]

Let \( \Lambda \) denote the set of stresses \( \sigma \in \Sigma \) satisfying (6.4) and let

\[ P_{q(\lambda)} := \{ \sigma \in \Sigma \mid \Phi(q(\lambda); \sigma) \leq 0 \text{ in } \Omega \}. \tag{6.7} \]

We see that the set \( P_{q(\lambda)} \) represents the constraint (6.1) and thus we can write

\[ \omega^* = \sup \{ \lambda \geq 0 \mid P_{q(\lambda)} \cap \Lambda \neq \emptyset \} \]

\[ = \sup_{\lambda \geq 0} \sup_{\sigma \in P_{q(\lambda)}} \{ \lambda \}. \tag{6.8} \]

From (6.4), we have

\[ \inf_{v \in V} \left[ \int_{\Omega} \sigma : \varepsilon(v) \, dx - L(v) \right] = \begin{cases} 0, & \text{if } \sigma \in \Lambda, \\ -\infty, & \text{otherwise.} \end{cases} \]

Hence, one can rewrite (6.8) as follows:

\[ \omega^* = \sup_{\lambda \geq 0} \sup_{\sigma \in P_{q(\lambda)}} \inf_{v \in V} \left[ \lambda + \int_{\Omega} \sigma : \varepsilon(v) \, dx - L(v) \right] \]

\[ = \sup_{\lambda \geq 0} \inf_{v \in V} \sup_{\sigma \in P_{q(\lambda)}} \left[ \lambda + \int_{\Omega} \sigma : \varepsilon(v) \, dx - L(v) \right] \]

\[ = \sup_{\lambda \geq 0} \inf_{v \in V} \left[ \lambda + \int_{\Omega} D(q(\lambda); \varepsilon(v)) \, dx - L(v) \right]. \tag{6.9} \]
where

\[ D(q(\lambda); \varepsilon) = \sup_{\sigma \in \mathbb{R}^{3 \times 3}_{\text{sym}}} \sup_{\Phi(q(\lambda); \sigma) \leq 0} \sigma : \varepsilon \]  

(6.10)

denotes the local dissipation function depending on \( \lambda \geq 0 \). The function \( D(q(\lambda); \cdot) \) is finite-valued only on a convex cone belonging to \( \mathbb{R}^{3 \times 3}_{\text{sym}} \). Therefore, the inner problem in (6.9) can be classified as cone programming. (6.9) can be interpreted as the kinematic principle of the OPT-MSSR method.

Let us note that the ordering of inf and sup has been interchanged during the derivation of (6.9). The corresponding equality is expected and partially justified by the results presented in Refs. 13, 15. Unlike limit analysis, we cannot eliminate the supremum over \( \lambda \) within the kinematic principle, because the function \( \Phi \) is not convex (with respect to both the variables \( \lambda \) and \( \sigma \)).

Discrete forms of the kinematic and static OPT-MSSR problems could be solved by extending the algorithm which was sketched in Ref. 20. This algorithm is based on verification of feasible sets depending on \( \lambda \) by the interior point method. As an alternative, we suggest to use a regularization method (introduced and analyzed in the next section).

7 | REGULARIZATION METHOD

In Refs. 15, 17, 18, 29–31 a regularization method has been systematically developed for the solution of the limit analysis (LA) problem. This method has also been used in strain-gradient plasticity and other applications.\(^{32,33}\) The aim of this section is to use the regularization for the solution of the OPT-MSSR problem and to study the relation between the original and the regularized problem.

We arise from (6.8) and regularize this problem with respect to a parameter \( \alpha > 0 \) as follows:

\[ \omega^*_\alpha = \sup_{\lambda \geq 0} \sup_{\sigma \in P_{q(\lambda)} \cap \Lambda} \left[ \lambda - \frac{1}{2\alpha} \int_{\Omega} C^{-1} \sigma : \sigma \ dx \right] \]  

(7.1)

where \( C \) is a positive definite fourth order tensor, for example, the elastic tensor. We have the following result.

**Lemma 7.1.** The sequence \( \{\omega^*_\alpha \}_{\alpha > 0} \) defined by (7.1) is nondecreasing and satisfying

\[ \omega^*_\alpha \leq \omega^*, \quad \lim_{\alpha \rightarrow +\infty} \omega^*_\alpha = \omega^*, \]  

(7.2)

where \( \omega^* \) is defined by (6.8).

**Proof.** It is readily seen that the inequalities \( \omega^*_{\alpha_1} \leq \omega^*_{\alpha_2} \leq \omega^* \) hold for any \( 0 < \alpha_1 \leq \alpha_2 \). Next, for any \( \lambda < \omega^* \) the intersection \( P_{q(\lambda)} \cap \Lambda \) is nonempty as follows from (6.8). Then the inner sup-problem in (7.1) has a unique solution \( \sigma_{\lambda} \in P_{q(\lambda)} \cap \Lambda \), because it contains the quadratic functional. Consequently,

\[ \lim_{\alpha \rightarrow +\infty} \omega^*_\alpha \geq \lim_{\alpha \rightarrow +\infty} \left[ \lambda - \frac{1}{2\alpha} \int_{\Omega} C^{-1} \sigma_{\lambda} : \sigma_{\lambda} \ dx \right] = \lambda \]

and thus

\[ \omega^* \geq \lim_{\alpha \rightarrow +\infty} \omega^*_\alpha \geq \sup\{\lambda \geq 0 \mid P_{q(\lambda)} \cap \Lambda \neq \emptyset\} = \omega^*. \]

This implies the limit in (7.2).

One can also write

\[ \omega^*_\alpha = \max_{\lambda \geq 0} [\lambda - G_{\alpha}(\lambda)] = \lambda^*_\alpha - G_{\alpha}(\lambda^*_\alpha), \]  

(7.3)
where

\[
G_\alpha(\lambda) = \inf_{\sigma \in F_{q(\lambda)} \cap \Lambda} \frac{1}{2\alpha} \int_\Omega C^{-1} \sigma : \sigma \, dx
\]  

\[
= \begin{cases} 
\frac{1}{2\alpha} \int_\Omega C^{-1} \sigma : \sigma \, dx, & \text{if } P_{q(\lambda)} \cap \Lambda \neq \emptyset, \\
+\infty, & \text{otherwise}, 
\end{cases}
\]  

(7.5)

and \( \lambda^*_\alpha \) maximizes the middle term in (7.3). Since the value \( G_\alpha(\lambda^*_\alpha) \) is finite, we have \( P_{q(\lambda^*_\alpha)} \cap \Lambda \neq \emptyset \). This fact, (6.8) and (7.3) imply the following result.

**Lemma 7.2.** The sequence \( \{\lambda^*_\alpha\}_{\alpha > 0} \) defined by (7.4) satisfies

\[
\omega^*_\alpha \leq \lambda^*_\alpha \leq \omega^*, \quad \lim_{\alpha \to +\infty} \lambda^*_\alpha = \omega^*,
\]  

(7.6)

where \( \omega^* \) and \( \omega^*_\alpha \) are defined by (6.8) and (7.1), respectively.

From Lemmas 7.1 and 7.2, it follows that the values \( \omega^*_\alpha \) and \( \lambda^*_\alpha \) are close to \( \omega^* \) for sufficiently large \( \alpha \). The inequality \( \omega^*_\alpha \leq \lambda^*_\alpha \leq \omega^* \) implies that \( \lambda^*_\alpha \) is more accurate approximation of \( \omega^* \) than \( \omega^*_\alpha \). This is illustrated on a numerical example in Section 8.

Next, for the solution of (7.3), it is crucial to evaluate the function \( G_\alpha \). To this end, we use a similar duality approach as presented in Section 6. We arrive at the following kinematic definition of \( G_\alpha \):

\[
G_\alpha(\lambda) = -\inf_{\nu \in V} \left[ \int_\Omega D_\alpha(q(\lambda); \nu(\lambda)) : \nu(\lambda) \, dx - L(\nu) \right],
\]  

(7.7)

where

\[
D_\alpha(q(\lambda); \nu) = \sup_{\sigma \in \mathbb{R}^{3\times3}_{\text{sym}}} \left[ \sigma : \varepsilon - \frac{1}{2\alpha} C^{-1} \sigma : \sigma \right],
\]  

(7.8)

is the regularized dissipative function. In particular, \( D_\alpha \) is finite-valued and differentiable with respect to \( \varepsilon \) unlike the original dissipation \( D \), see, for example. Moreover, the second derivative of \( D_\alpha \) exists almost everywhere. Let \( T_\alpha(q(\lambda); \varepsilon) \in \mathbb{R}^{3\times3}_{\text{sym}} \) denote the derivative of \( D_\alpha(q(\lambda); \varepsilon) \) with respect to \( \varepsilon \). Then the problem (7.7) is equivalent to the following nonlinear variational equation:

\[
\text{find } \nu_{q(\lambda)} \in V : \quad \int_\Omega T_\alpha(q(\lambda); \varepsilon(\nu_{q(\lambda)})) : \nu(\lambda) \, dx = L(\nu) \quad \forall \nu \in V.
\]  

(7.9)

It is convenient to solve it by a non-smooth and damped version of the Newton method suggested in Ref. 34, because this method also finds descent directions of the functional in (7.7), which do not need to be bounded from below for some \( \lambda \).

Let \( D_1 := D_\alpha \) and \( T_1 := T_\alpha \) for \( \alpha = 1 \). Then the following formulas hold for any \( \alpha > 0, \lambda \geq 0 \) and \( \varepsilon \in \mathbb{R}^{3\times3}_{\text{sym}} \):

\[
D_\alpha(q(\lambda); \varepsilon) = \frac{1}{\alpha} D_1(q(\lambda); \alpha \varepsilon), \quad T_\alpha(q(\lambda); \varepsilon) = T_1(q(\lambda); \alpha \varepsilon).
\]  

(7.10)

These formulas simplify the construction of the operators \( D_\alpha \) and \( T_\alpha \) if the continuation over \( \alpha \) is used. In addition, \( T_1 \) is practically the same as the operator, which arises from the implicit Euler discretization of the elasto-plastic initial-value
FIGURE 1 Geometry of the considered slope with an inclination of 45°

constitutive problem. Its construction can be found for example, in Refs. 36,39 The closed form of $D_1$ is presented in the Appendix of this paper.

8 | NUMERICAL EXAMPLES

In this section, we present two different numerical examples on slope stability problems. The first example considers a homogeneous slope presented in Ref. 10. The aim is to illustrate our theoretical results and to verify that the computed FoS are in accordance with the published ones. The second example arises from an analysis of a real slope. Therefore, heterogeneous material conditions and the influence of the pore water pressure are considered in the analysis of this boundary value problem.

8.1 | Softwares and their numerical solution

We use and compare the results from three different softwares: in-house codes in Matlab, Plaxis, and Comsol Multiphysics. The in-house Matlab codes are based on elastic-plastic solvers, the finite element method and on mesh adaptivity. They have been systematically developed and described in Refs. 35–37. Some of the codes are available for download. 38 Within these codes, we have implemented the regularization method discussed in Section 7 to compute safety factors $\lambda^*_A$, $\lambda^*_B$, and $\lambda^*_C$ for associated plasticity and for Davis A-C approaches. In particular, six-noded triangular elements with the 7-point Gauss quadrature have been used and combined with the mesh adaptivity introduced in Refs. 15, 43. However, 15-noded triangular elements are also implemented in the code.

The software Plaxis enables to solve the shear strength reduction method for both associated and non-associated plasticity. The standard solver is based on the implicit Euler time discretization and the arc-length method. 7 15-noded triangular elements with a shape function of fourth order are used for the following studies. One can easily implement the Davis A approach in the existing SSR procedure. For the application of the Davis B-C approaches, an iterative procedure is used. Due to the utilization of this software we are able to compare the suggested OPT-MSSR method with current approaches of the shear strength reduction (SSR) method.

The software Comsol Multiphysics with its Geomechanical module neither includes the shear strength reduction method nor the arc-length method. However, the code allows to add a global equation to the elastic-plastic system of equations with respect to an unknown parameter and enables an optimization of this parameter. Therefore, the MSSR method including the Davis A-C modifications can be implemented in Comsol Multiphysics. Besides the regularization method, the standard incremental procedure for the solution of the elastic-plastic problem has been used. 15-noded triangular elements are considered.

8.2 | Slope in homogeneous soil

Following Ref. 10 we consider a slope in homogeneous soil depicted in Figure 1. Its inclination is 45° and sizes (in meters) are given in Figure 1. The effective friction angle $\phi'$ is 45°, the effective cohesion $c'$ is 6.0 kPa and the unit weight $\gamma$
TABLE 1 Safety factors for the homogeneous slope and different approaches

|                     | MATLAB | COMSOL | Plaxis |
|---------------------|--------|--------|--------|
| $\psi = \phi = 45^\circ$, assoc. model | 1.52   | 1.52   | 1.51   |
| $\psi = 15^\circ$, Davis A       | 1.27   | 1.28   | 1.27   |
| $\psi = 15^\circ$, Davis B       | 1.36   | 1.37   | 1.35   |
| $\psi = 15^\circ$, Davis C       | 1.41   | 1.42   | 1.41   |
| $\psi = 15^\circ$, current approach | –      | –      | 1.46   |
| $\psi = 0^\circ$, Davis A       | 1.08   | 1.08   | 1.08   |
| $\psi = 0^\circ$, Davis B, C    | 1.15   | 1.16   | 1.16   |
| $\psi = 0^\circ$, current approach | –      | –      | 1.27–1.35 |

is 20.0 kN/m³. The dilatancy angle is either $\psi' = 0^\circ$, $\psi' = 15^\circ$ or $\psi' = 45^\circ$. The chosen values of $\phi'$ and $\psi'$ enable to highlight differences between the associated and non-associated material behavior and between the suggested and the currently used approaches of the SSR method.

Next, we set the following values for Young’s modulus and Poisson’s ratio: $E = 40$ MPa and $\nu = 0.3$. For the regularization method, the value $\alpha = 1000$ is used. This value is sufficiently large as will be discussed later.

Table 1 summarizes the computed factors of safety obtained with different softwares and different approaches. These values are practically the same as discussed in Ref. 10 where the software OPTUMG2 was used for Davis’ approaches A–C. The computed FoS are in accordance with Theorem 4.3, that is $1 \leq \lambda^*_{A} \leq \lambda^*_{B} \leq \lambda^*_{C} \leq \lambda^*_{\text{assoc}}$. We have used the current (standard) approach of the non-associated SSR method only in Plaxis. The corresponding value of FoS for $\psi' = 0^\circ$ cannot be uniquely determined due to oscillations of the method, which are a consequence of a varying failure mechanism during the strength reduction procedure (see Ref. 10 for more details). In Ref. 10 the current approach was also investigated for a very fine mesh leading to slightly lower factors of safety, namely 1.42 for $\psi' = 15^\circ$ and 1.21–1.27 for $\psi' = 0^\circ$. These results indicate again the distinct mesh dependency of FoS in the case of non-associated plasticity.

Unlike the current SSR approach, the results of OPT-MSSR method are practically insensitive if sufficiently fine meshes are used. It is illustrated in Figure 2 where the mesh adaptivity within the in-house Matlab codes is used. In this study, 20 levels of meshes are considered and the corresponding safety factors remain from about 1000 elements (level 10) onwards almost constant. The dependence of FoS on the mesh adaptivity has also been analyzed in Ref. 28. The finest mesh and the corresponding slip surface for the Davis B approach are depicted in Figure 3. The failure surface is visualized using the rate of the deviatoric strain.

The dependency of $\lambda^*_\alpha$ and $\omega^*_\alpha$ (see Section 7) on the regularization parameter $\alpha$ is depicted in Figure 4. One can see that these curves are increasing and approaches $\omega^*$. This confirms the theoretical results of the regularization method. We also see that $\lambda^*_\alpha$ approximates the safety factor $\omega^*$ even for relatively small values of $\alpha$. On the other hand, the bound $\omega^*_\alpha$
is very poor (too far from $\omega^*$) for small values of $\alpha$, see Figure 4(B). Hence, it is important to use sufficiently large values of the regularization parameter $\alpha$.

### 8.3 Case study – heterogeneous slope from locality Doubrava-Kozinec

The second example considers a real slope in Doubrava-Kozinec (near Karvina in the North-East part of the Czech Republic). This slope is located within a potentially unstable area with historical manifestations of landslide activity (within the quaternary clay layer). In Figure 5, the investigated slope including the soil conditions is illustrated. One can see that the slope is heterogeneous and consists of five soil layers. The particular materials and their parameters are specified in Table 2. It has to be mentioned that the sand and gravel layers in the investigated slope contain a small amount of fine
Table 2: Material parameters for the heterogeneous slope

|                  | Neogene Clay | Gravel | Quaternary Clay | Sand  | Clayed Sand |
|------------------|--------------|--------|-----------------|-------|-------------|
| $\phi'$ [°]      | 26           | 45     | 13              | 33    | 27          |
| $c'$ [kPa]       | 9            | 1      | 3               | 2     | 5           |
| $E$ [MPa]        | 16           | 140    | 10              | 14    | 27          |
| $\nu$            | 0.40         | 0.20   | 0.40            | 0.28  | 0.35        |
| $\gamma_{\text{unsat}}$ [kN/m$^3$] | 20.3       | 20.5   | 20.0            | 19.0  | 19.4        |
| $\gamma_{\text{sat}}$ [kN/m$^3$] | 20.7       | 20.6   | 20.5            | 20.5  | 21.4        |

During the evaluation of the slope, it turned out that the failure mechanism is located in the central part of the slope, or more precisely in quaternary clay and its interface with the clayed sand and the neogene clay layer. It is worth noticing that the effective friction angle of the quaternary clay is much lower than $\phi'$ of the other materials, thus it was expected that this layer is decisive for both, the obtained FoS and the computed failure mechanism. Numerical results presented below confirm the location of the failure mechanism.

The initial mesh for the computation in Comsol Multiphysics is depicted in Figure 6. This mesh reflects the heterogeneity of the soil conditions. This mesh has also been imported to Matlab. In Comsol Multiphysics, this mesh was then locally refined in the central part of the slope to obtain more accurate results in the region of interest (region of the expected failure surface). In Matlab, the original mesh was adaptively refined, where 15 mesh levels were considered.

A detail of the finest Matlab mesh for $\psi' = 0^\circ$ and the Davis B approach is depicted in Figure 7 (together with the corresponding failure surface). For the visualization of this zone, a norm of the rate of the deviatoric strain was used. The black curves in the figure depict the soil stratification. One can see that failure mechanism is not very deep and that a
large part of the slip surface lies on the transition of the quaternary clay layer to the neogene clay layer. This confirms again that this transition zone is decisive for the stability of the considered slope.

In Figure 8, we see the dependence of the safety factors on the mesh adaptivity computed in Matlab (for all three approaches). One can also see that these curves are practically constant after a sufficiently large number of mesh refinements (approximately 10). We also observe that the safety factor obtained with the Davis A approach is slightly lower than the one for Davis B. This is in accordance with Theorem 4.3.

The computed safety factors for the different approaches are summarized in Table 3. One can see that the safety factors are close to one for all investigated approaches indicating that the slope is close to its limit state, thus numerically close to failure. The observed failure of the slope can be explained by inhomogeneity of the quaternary clay layer. It can also be expected that in some parts of the slope the residual friction angle of quaternary clay is slightly lower than 13°, which in turn causes a reduction of the factor of safety. The values computed with the in-house Matlab codes are slightly lower than other FoS values, that is due to the usage of the local mesh adaptivity. We also observe that FoS for the Davis A and Davis B approaches practically coincide and are very close to one. This observation is in accordance with the third statement of Theorem 4.3.

9 | CONCLUSION

This work has been inspired by the recent paper,10 where the standard shear strength reduction (SSR) method was approximated by modified Davis approaches and a parametrized limit analysis (LA) method. These ideas have been generalized and extended in this paper and the so-called MSSR method has been introduced. In order to analyze the MSSR method in more details, its optimization variant called OPT-MSSR has been proposed. Using the OPT-MSSR method, factors of safety for various Davis modifications have been compared. Next, the OPT-MSSR method has been completed by variational principles and duality theory, similarly as in limit analysis. Hence, this approach can be interpreted as a rigorous method.
For the numerical solution, a regularization method has been used and combined with the finite element and the damped Newton method. This solution concept leads to similar solvers as standardly used in computational plasticity and thus can be easily implemented within existing elastic-plastic codes. In particular, in-house Matlab codes in combination with local mesh adaptivity have been used. Softwares Plaxis and Comsol Multiphysics have been utilized for comparison of the results and it has been shown that the MSSR can also be implemented within these softwares. One of the presented numerical examples can be classified as case history, since it deals with a real slope.

ACKNOWLEDGMENTS
The authors acknowledge support for their work from the Czech Science Foundation through project No. 19-11441S. The authors also thank to Dr. Alexej Kolcun for fruitful discussions on mesh adaptivity for regular and irregular meshes.

AUTHOR CONTRIBUTIONS
This is an author contribution text.

FINANCIAL DISCLOSURE
None reported.

CONFLICT OF INTEREST
The authors declare no potential conflict of interests.

DATA AVAILABILITY STATEMENT
Data sharing not applicable - no new data generated, or the article describes entirely theoretical research.

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How to cite this article: Sysala S, Hrubešová E, Michalec Z, Tschuchnigg F. Optimization and variational principles for the shear strength reduction method. Int J Numer Anal Methods. 2021;45:2388–2407. https://doi.org/10.1002/nag.3270

APPENDIX: CLOSED FORM OF THE FUNCTION $D_1$

For the sake of completeness, we introduce the closed form of the function

$$D_1(q(\lambda); \varepsilon) = \sup_{\sigma \in \sigma_{\text{sym}} \Phi(q(\lambda), \sigma) \leq 0} \left[ \sigma : \varepsilon - \frac{1}{2} C^{-1} \sigma : \sigma \right].$$ (A.1)

from Section 7. In literature (see, e.g., Refs. 36, 39), one can find closed form of the derivative $T_1$ of $D_1$ representing the stress-strain relation but not $D_1$. Therefore, we try to fill this gap.
Besides \( \lambda \geq 0 \), this function also depends on the parameters \( c', \phi' \) and the elastic parameters \( K, G \). We construct the function \( D_1(q(\lambda); \varepsilon) \) only for such \( \lambda \) satisfying \( q(\lambda) = 1 \). For other choices of \( \lambda \), it suffices to replace \( c' \) and \( \phi' \) with

\[
e_{q(\lambda)}^c := \frac{c'}{q(\lambda)}, \quad \phi_{q(\lambda)} := \arctan \frac{\tan \phi'}{q(\lambda)}.
\]

see also (2.1). To be in accordance with the derivation presented in Ref. 36, we write the yield criterion \( \Phi(1; \sigma) \leq 0 \) in the form

\[
(1 + \sin \phi') \sigma_1 - (1 - \sin \phi') \sigma_3 - 2c' \cos \phi' \leq 0.
\]

Next, we assume that the strain tensor \( \varepsilon \) is given and its eigenvalues \( \varepsilon_1, \varepsilon_2, \varepsilon_3 \) satisfy \( \varepsilon_1 \geq \varepsilon_2 \geq \varepsilon_3 \). Let \( \text{tr} \varepsilon = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 \) denote the trace of \( \varepsilon \) and \( \Lambda = \frac{1}{2}(3K - 2G) \) denote the first Lamé coefficient. As in Ref. 39, we distinguish five possible cases:

- the elastic response,
- the return to the smooth portion of the Mohr-Coulomb pyramid (\( \sigma_1 > \sigma_2 > \sigma_3 \)),
- the return to the left edge (\( \sigma_1 = \sigma_2 > \sigma_3 \)),
- the return to the right edge (\( \sigma_1 > \sigma_2 = \sigma_3 \)),
- and the return to the apex of the pyramid (\( \sigma_1 = \sigma_2 = \sigma_3 \)).

**The elastic response.** This case happens if the elastic stress \( C \varepsilon \) satisfies \( \Phi(1; C \varepsilon) \leq 0 \), that is,

\[
2\Lambda(\text{tr} \varepsilon) \sin \phi' + 2G(1 + \sin \phi')\varepsilon_1 - 2G(1 - \sin \phi')\varepsilon_3 - 2c' \cos \phi' \leq 0.
\]

Then

\[
D_1(1; \varepsilon) = \frac{1}{2} C \varepsilon : \varepsilon = \frac{1}{2} \Lambda(\text{tr} \varepsilon)^2 + G(\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2).
\]

If the criterion (A4) does not hold then the plastic response occurs and we distinguish four possible cases of the return to the Mohr-Coulomb pyramid. We use the following auxiliary notation:

\[
\gamma_{s,l} = \frac{\varepsilon_1 - \varepsilon_2}{1 + \sin \phi'}, \quad \gamma_{s,r} = \frac{\varepsilon_2 - \varepsilon_3}{1 - \sin \phi'},
\]

\[
\gamma_{l,a} = \frac{\varepsilon_1 + \varepsilon_2 - 2\varepsilon_3}{3 - \sin \phi'}, \quad \gamma_{r,a} = \frac{2\varepsilon_2 - \varepsilon_2 - \varepsilon_3}{3 + \sin \phi'}.
\]

**The return to the smooth portion.** This case happens if (A4) is not satisfied and

\[
q_s(\varepsilon) < S \min\{\gamma_{s,l}, \gamma_{s,r}\}.
\]

where

\[
q_s(\varepsilon) = 2\Lambda(\text{tr} \varepsilon) \sin \phi' + 2G(1 + \sin \phi')\varepsilon_1 - 2G(1 - \sin \phi')\varepsilon_3 - 2c' \cos \phi',
\]

\[
S = 4\Lambda \sin^2 \phi' + 4G(1 + \sin^2 \phi').
\]

Then,

\[
D_1(1; \varepsilon) = \frac{1}{2} \Lambda(\text{tr} \varepsilon)^2 + G(\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2) - \frac{1}{2S}q_s^2(\varepsilon).
\]
The return to the left edge. This case happens if (A.4) is not satisfied and

\[ \gamma_{s,l} < \gamma_{l,a}, \quad L \gamma_{s,l} \leq q_l(\xi) < L \gamma_{l,a}, \]  

(A.8)

where

\[ q_l(\xi) = 2 \Lambda (\text{tr} \xi) \sin \phi' + G(1 + \sin \phi')(\varepsilon_1 + \varepsilon_2) - 2G(1 - \sin \phi')\varepsilon_3 - c' \cos \phi', \]

\[ L = 4 \Lambda \sin^2 \phi' + G(1 + \sin \phi')^2 + 2G(1 - \sin \phi')^2. \]

Then,

\[ D_1(1; \xi) = \frac{1}{2} \Lambda (\text{tr} \xi)^2 + G \left[ \frac{1}{2}(\varepsilon_1 + \varepsilon_2)^2 + \varepsilon_3^2 \right] - \frac{1}{2L} q_l^2(\xi). \]

(A.9)

The return to the right edge. This case happens if (A.4) is not satisfied and

\[ \gamma_{s,r} < \gamma_{r,a}, \quad R \gamma_{s,r} \leq q_r(\xi) < R \gamma_{r,a}, \]  

(A.10)

where

\[ q_r(\xi) = 2 \Lambda (\text{tr} \xi) \sin \phi' + 2G(1 + \sin \phi')\varepsilon_1 - G(1 - \sin \phi')(\varepsilon_2 + \varepsilon_3) - 2c' \cos \phi', \]

\[ R = 4 \Lambda \sin^2 \phi' + 2G(1 + \sin \phi')^2 + G(1 - \sin \phi')^2. \]

Then,

\[ D_1(1; \xi) = \frac{1}{2} \Lambda (\text{tr} \xi)^2 + G \left[ \varepsilon_1^2 + \frac{1}{2}(\varepsilon_2 + \varepsilon_3)^2 \right] - \frac{1}{2R} q_r^2(\xi). \]

(A.11)

The return to the apex. This case happens if (A.4) is not satisfied and

\[ q_a(\xi) \geq A \max\{\gamma_{l,a}, \gamma_{r,a}\}, \]  

(A.12)

where

\[ q_a(\xi) = 2K (\text{tr} \xi) \sin \phi' - 2c' \cos \phi', \quad A = 4K \sin^2 \phi'. \]

Then,

\[ D_1(1; \xi) = \frac{1}{2} K (\text{tr} \xi)^2 - \frac{1}{2A} q_a^2(\xi) = \frac{c'}{\tan \phi'}(\text{tr} \xi) - \frac{(c')^2}{2K \tan^2 \phi'}. \]

(A.13)