Expanding space-time and variable vacuum energy

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Abstract. The paper describes a cosmological model which contemplates the presence of a vacuum energy varying, very slightly (now), with time. The constant part of the vacuum energy generated, some 6 Gyr ago, a deceleration/acceleration transition of the metric expansion; so now, in an aged Universe, the expansion is inexorably accelerating. The vacuum energy varying part is instead assumed to be eventually responsible of an acceleration/deceleration transition, which occurred about 14 Gyr ago; this transition has a dynamic origin: it is a consequence of the general relativistic Einstein-Friedmann equations. Moreover, the vacuum energy (constant and variable) is here related to the zero-point energy of some quantum fields (scalar, vector, or spinor); these fields are necessarily described in a general relativistic way: their structure depends on the space-time metric, typically non-flat. More precisely, the commutators of the (quantum field) creation/annihilation operators are here assumed to depend on the local value of the space-time metric tensor (and eventually of its curvature); furthermore, these commutators rapidly decrease for high momentum values and they reduce to the standard ones for a flat metric. In this way, the theory is “gravitationally” regularized; in particular, the zero-point (vacuum) energy density has a well defined value and, for a non static metric, depends on the (cosmic) time. Note that this varying vacuum energy can be negative (Fermi fields) and that a change of its sign typically leads to a minimum for the metric expansion factor (a “bounce”).

1. Introduction. Energy-momentum tensors
On the space-time four-dimensional manifold are assigned a metric/gravitational tensor $g$ and an energy-momentum (macroscopic) tensor $T$; their local expressions, in a Friedmann synchronous chart $(t,r_1,r_2,r_3; U) = (t,r; U)$ are $(i,j = 1, 2, 3; U$ is an open subset of the space-time manifold):

$$g_{00}(t,r) = c^2(t), \quad g_{ij}(t,r) = -a^2(t)\gamma_{ij}(r) \quad (1a)$$

$$T_{00}(t,r) = \rho(t), \quad T_{ij}(t,r) = -\sigma(t)a^2(t)\gamma_{ij}(r) \quad (1b)$$

while $g_{0j}(t,r) = T_{0j}(t,r) = 0$; $a$ (the expansion factor), $g_{00}$ and $T_{00}$ are always $> 0$. The curvature of the 3-dimensional metric expressed by $a^2(t)\gamma_{ij}$ is assumed (isotropy) to depend only on $t$; this “spatial” curvature will be denoted by $K(t) (= K_0/a^2(t), -\infty < K_0 < +\infty)$. The purely spatial metric can be diagonalized: $\gamma_{ij}(r) = \delta_{ij}/b^2(r)$, where $b(r) = 1 + K_0 r^2$. Usually (and here) the chart is chosen in such a way that $c = 1$, but other options are available (for example $\rho = 1$); in any case, when $c = 1$, $\rho$ can be identified to the (proper) energy density. See Friedmann [4] (Annex 1) or, for example, Einstein [2], Hawking and Ellis [6], Tolman [9], Weinberg [10].
The energy conservation law, \( a(t)\rho'(t) + 3a'(t) (\rho(t) - \sigma(t)) = 0 \) leads, assuming (in a first approximation) that \( \sigma \) is constant, to:

\[
\rho(t) = \sigma + \frac{\mu}{a^3(t)}; 
\]

\( \mu = (\rho - \sigma) a^3 \) is an (universal) constant. If \( \sigma > 0 \), \( \rho \) is always \( > \sigma \) or \( < \sigma \) according to the sign of \( \mu \), but when \( \mu < 0 \) the (weak) energy condition \( \rho \geq 0 \), \( \rho \geq \sigma \) ([6], Chapter 4) is badly violated. For a \( \sigma < 0 \), \( \mu \) has to be \( > 0 \) and \( a \) is bounded \( (a^3 < \mu/|\sigma|) \). Note that, because \( \sigma' = -\sigma(a' \rho - \sigma) \), \( \rho \) increase (decrease) when \( a \) decrease (increase), if \( \rho > \sigma \).

We are primarily interested to an expansion factor strictly increasing on some open, real line interval \([t_M, t_N]\) (and \( t_M < t_0 < t_N \), if \( t_0 \) is the present time); typically \( t_N = +\infty \) and \( \lim_{t \to +\infty} a(t) = +\infty \) (so \( K( +\infty) = 0 \)). Instead \( t_M \) can be \( > -\infty \); in this case or \( a_M = a(t_M) \) is a minimum of \( a \) and \( a(t) \) decrease for \( t < t_M \) or \( a(t) \) is defined only for \( t > t_M \) and \( \lim_{t \to t_M^+} a(t) = 0 \). In any case, for \( t_M < t < t_N \), \( \rho \) can be expressed as a function of \( a \), for example:

\[
\rho = F \left( \frac{1}{a} \right) = \sum_{n \geq 0} \frac{\rho(n)}{a^n}, \quad (3)
\]

that is \( \rho = F \circ 1/a; \) \( F \) is some differentiable (eventually analytical) function. Consequently \( a\rho' = -(a'/a) F'(\frac{1}{a}) = -3a' (\rho - \sigma) \) and

\[
\sigma = F \left( \frac{1}{a} \right) - \frac{1}{3a} F'(\frac{1}{a}) = \sum_{n \geq 0} \left( 1 - \frac{n}{3} \right) \frac{\rho(n)}{a^n}, \quad (4)
\]

Note that, for \( \sigma \), the coefficient of \( a^{-3} \) is always \( 0 \) and the one of \( a^{-4} \) is \( = -\frac{1}{3} \rho(4) \); this fact is, in our context, simply a consequence of the tridimensionality of the (ordinary) space. The energy condition \( \rho \geq 0 \), \( \rho \geq \sigma \) will hold when \( \sum_{n \geq 0} \frac{\rho(n)}{a^{n}} \) and \( \sum_{n \geq 0} n \frac{\rho(n)}{a^{n}} \) are both \( \geq 0 \).

If \( \rho^{(r e f)} \) is some constant, “reference” energy density and \( \Omega^{(n)} := \rho^{(n)} / \rho^{(r e f)} \),

\[
\rho = \rho^{(r e f)} \sum_{n \geq 0} \frac{\Omega^{(n)}}{a^n}; \quad (5)
\]

the traditional choice for \( \rho^{(r e f)} \) is \( \frac{1}{3} (H_0)^2 \), the critical density, \( \rho^{(c r i t)} \); \( \kappa = \frac{\pi}{2} G_N e w t o n \). \( \kappa \rho^{(n)} = (H_0)^2 \Omega^{(n)}, H := a' / a \) is the Hubble-Lemaître function ([4], Annex 1), \( H_0 = H(t_0) \) and \( t_0 \) is always the present time. A better choice for \( \rho^{(r e f)} \), not related to some particular value of \( t \), would be simply \( \rho(0) \) if \( \rho(0) \) is \( > 0 \). Observe that here \( \Omega^{(3)} \) and \( \Omega^{(4)} \) are only the coefficient of \( a^{-3} \) and \( a^{-4} \) in the energy density expansion: they do not necessary correspond to \( \Omega^{(m a t)} \) (non-relativistic matter) and to \( \Omega^{(r a d i a t i o n)} \) (relativistic matter, radiation): \( \rho = \rho^{(m a t)} + \rho^{(r a d i a t i o n)} + \rho^{(v a c)} \) is the total energy density, from matter, radiation and some other sources that we call collectively vacuum.

2. Dynamics of expansion

The Einstein-Friedmann equation \( a'' / a = -\frac{2}{3} (\rho - 3\sigma) \) can now be written, taking into account the energy conservation law, as

\[
a'' = \kappa \left( a F \left( \frac{1}{a} \right) - \frac{1}{2} F'(\frac{1}{a}) \right) = H_0^2 \sum_{n \neq 2} \left( 1 - \frac{n}{2} \right) \Omega^{(n)} a^{1-n}; \quad (6)
\]

this ordinary, semi-linear differential equation, supplemented by some initial conditions (for example \( a(t_0) = 1, a'(t_0) = H_0 \), if \( t_0 \) is the present time) has an unique solution, defined on
some open (eventually unbounded, Section 1) interval of the real line. See, for some numerical example, the Section 5.

The “other” Einstein-Friedmann equation, \((a'/a)^2 + K_0/a^2 = \kappa \rho\), can be written as

\[
(a')^2 = \kappa a^2 F \left( \frac{1}{a} \right) - K_0 = H_0^2 \sum_{n \neq 2} \Omega^{(n)} a^{2-n} + \left( H_0^2 \Omega^{(2)} - K_0 \right);
\]

it is, evidently, a first integral of the above, second order differential equation; the integration constant is related to the value of the spatial curvature \(K = K_0/a^2\) at time \(t_0\), if \(a(t_0) = 1\). As expected, \(\sum_{n \geq 0} \Omega^{(n)} = 1 + K_0/H_0^2\).

Assuming that only \(\rho^{(0)}, \rho^{(3)}\) and \(\rho^{(4)}\) are \(\neq 0\), we arrive to the “classical” cosmological models, old (when \(\rho^{(0)} = 0\) or \(\rho < 0\), see [10]) and new (when \(\rho^{(0)} > 0\), see [12]); \(\rho^{(3)}\) and \(\rho^{(4)}\) are always \(> 0\). In the following we will consider a more general model where, eventually, all the \(\rho^{(n)}\) are \(\neq 0\), that is when we are in presence of a variable vacuum energy density. Note that if \(\rho^{(\text{vac})}\) diverge negatively faster than \(\rho^{(\text{mat})} + \rho^{(\text{rad})}\) when \(a \to a_L\) (for some \(a_L \geq 0\)) then \(a(t)\) is typically defined for \(-\infty < t < +\infty\) and has here a global minimum (and eventually two flex, between the minimum and the present time, see the Section 5).

3. Digression. Quantum fields in a connection space

Now the quantum fields structure depends on the space-time connection, typically non-flat.

3.1. Affine connection spaces

\((X, \nabla)\) is an affine connection space: \(X\) is a \(n\)-dimensional, orientable manifold, \(\nabla\) is its absolute derivative (here assumed to be torsion-less); \(\tau_x\) and \(\tau'_x\) (the dual of \(\tau_x\)) are respectively the tangent and the cotangent vector spaces of \(X\), at \(x \in X\).

\(U\) is an open, geodetically convex, subset of \(X\): for every couple of points of \(U\), \(x\) and \(y\), there is one and only one (fixed a parametrization) \(\nabla\)-geodesic joining \(x\) to \(y\) and lying entirely in \(U\); \(v(x, y)\) is the vector of \(\tau_x\) tangent to the geodesic at \(x\) and such that \(v(x, x) = 0_x, \exp_x(0_x) = x\) and \(\exp_x(v(x, y)) = y\) (\(\exp_x\) is the canonical geodesic map from \(\tau_x\) to \(X\), generally defined only in a neighborhood of \(0_x\), \(V_x\); see [6], Section 2.5 or any differential geometry textbook, for example Chern, Chen and Lam [1], Section 5.2). If \((e_1(x), \ldots, e_n(x))\) is a basis of \(\tau_x\) and \(v(x, y) = \sum_i y^i e_i(x)\), the \(y^1, \ldots, y^n\) are the so called (geodetic) normal coordinates of \(y\), based on \(x\). The image by \(\exp_x^{-1}\) of a geodetic path crossing \(x\) is a (portion of a) straight line of \(\tau_x\), crossing \(0_x\).

In an affine space \((E, \mathcal{V})\) (for example in an Euclidean space or in a Galilean space-time) the vectors \(v(x, y)\) joining points of \(E\) (a generic set) belong to a fixed, \(n\)-dimensional vector space, \(\mathcal{V}\); they are usually denoted by \(y - x\) (or \(\overrightarrow{xy}\)) and \(\exp_x(v)\) is denoted by \(x + v\); then \(x + (v + w) = (x + v) + w\) while, fixed an \(x \in E\), \(v \mapsto x + v\) is a bijection from \(E\) to \(V\). The canonical connection induced by \(D\) (the “ordinary” derivative) is now globally flat, but, on \(E\), it is obviously possible to assign other, non-flat, derivatives. The more common space-time models are based on affine connection spaces, but sometimes are also considered affine spaces, with two (one flat, the other curved) physically recognizable connections. See Gupta [5] or Feynman [3]; it is possible, also in this case, to build quantum fields, perhaps more easily.

3.2. Quantum fields

Fixed a geodetically convex set \(U\) of \((X, \nabla)\) (the “observable” space-time), we want to build local quantum fields, defined in \(U\). Now \(\mathbf{A}\) is a \(\mathbb{C}\)-algebra, normed or semi-normed (and complete) with an involution (*) and an identity (I); \(a_x\) and \(a_x^*\), the local annihilation and creation “operators”, are, for every \(x \in U\), \(\mathbf{A}\)-valued (Radon) measures defined on \(\tau_x^2\): this means that \(f \mapsto a_x(f)\) and \(f \mapsto a_x^*(f) := (a_x(\overline{f}))^*\) are \(\mathbb{C}\)-linear and continuos maps, from a space of complex test
functions $\mathcal{D}_x$ (functions defined on $T_x$) to $A$: $a_x(f)$ is usually written as $\int f(p) \, da_x(p)$ (or also $\int f(p) \, a(x,p) \, dp$, with some abuse of notation). Moreover we have (Bose statistics):

$$[a_x(g), (a_x(f))^*]_− = B_x(f,g) \cdot I = \int \overline{f(p)}g(p) \, d\mu_x(p) \cdot I$$  

(8)

where $\mu_x$ is some real and positive measure, generally depending on the connection of $X$ (perhaps on its Ricci tensor). The sesquilinear form $B_x$ assigns a pre-Hilbert structure on $\mathcal{D}_x$, so, by completion, it is possible to build a Hilbert space $\mathcal{H}_x$ (and its Fock space).

The scalar and neutral quantum field $\varphi (\varphi^* = \varphi)$ is then defined by (here and in the following $\hbar_{\text{Planck}} = 2\pi$):

$$\varphi_x(y) := \int \left( e^{-ip.v(x,y)} \, da_x(p) + e^{+ip.v(x,y)} \, da_x^*(p) \right)$$  

(9)

or, setting $\phi(x,v) = \varphi_x(\exp_x(v))$,

$$\phi(x,v) := \int \left( e^{-ip.v} \, da_x(p) + e^{+ip.v} \, da_x^*(p) \right) = a_x(\chi_v) + a_x^*(\chi_v).$$  

(10)

In the above $x$ and $y \in U$, $v \in V_x \subset T_x$, $p \in T_x$; $\chi_v(p) := e^{-ip.v}$; $p.v(x,y) = \sum_i p_i \cdot v^i(x,y)$ ($= \sum_i p_i v^i$, if the $y^1, \ldots, y^n$ are the normal coordinates of $y$, based on $x$); the measures $a_x$ and $a_x^*$ (as $\mu_x$) are implicitly assumed to be prolongable to a space of continuous and bounded (in fact polynomially increasing) test functions. The $(y^1, \ldots, y^n) \mapsto \Phi_x(y^1, \ldots, y^n) (= \varphi_x(y))$ are differentiable, $A$-valued ordinary functions, defined on some open set of $\mathbb{R}^n$.

It is now possible to define a (curved) version of the Pauli-Jordan function (which is a true function):

$$\Delta(y,z) := \frac{1}{2} [\varphi_x(y), \varphi_x(z)]_− = \int \sin \left( p \cdot (v(x,y) - v(x,z)) \right) \, d\mu_x(p) = \int \sin \left( \sum_i p_i (y^i - z^i) \right) \, dm_x(p^1, \ldots, p^n).$$  

(11)

In an affine space-time, equipped with only the canonical connection (see the previous Subsection), $v(x,y) - v(x,z) = (y - x) - (z - x) = y - z$ and $\mu_x$ do not depend on $x$; therefore $\Delta(y,z) = \int_{\mathcal{V}} \sin \left( p \cdot (y - z) \right) \, d\mu(p)$ depends exclusively on the vector joining $y$ and $z$ (but it is generally “singular”; see, for example, Pauli [7] or Weinberg [11]).

3.3. Metric tensors

In all physically interesting cases the manifold $X$ is 4-dimensional and Riemannian, $g$ is its metric tensor (of signature $-2$, as in Section 1) and $(X,g)$ is time oriented by $g$. The measure $\mu_x$ is now concentrated on the positive (with respect to the local time orientation) $M$-mass hyperboloid of $g_x$:

$$d\mu_x(p) = C_x(p) \, d\mu_M(p) = (d\mu_M(p) = \delta^{(4)}_{\frac{p^2}{\mu^2} + M^2});$$  

(12)

here $p^2_2$ stands for $p_+^{-1}.p = (\sum_{ij} g^{ij}(x) p_i p_j)$, $M$ is some positive parameter characterizing the field and $C_x$ is a positive, rapidly decreasing (or also of compact support) continuous function. Chosen a $g_x$-orthogonal and positive basis on $T_x$ (and $T'_x$), if $(\epsilon, p_1, p_2, p_3) = (\epsilon, p)$ are the components of $p$, then $\epsilon > 0$, $\epsilon^2 - \mathbf{p}^2 = M^2$ and $d\mu_M(p)$ can be replaced by $\frac{1}{2} \left( \mathbf{p}^2 + M^2 \right)^{-1/2} \, dp$.

Now it is quite reasonable to assume that the above $C_x(p)$ is built from $g$ and from the Ricci tensor of the connection, $\mathbf{R}$: this is apparently the simplest, non trivial choice. And it is usually assumed that $\nabla$ is the Levi-Civita connection of $g$ (but $\nabla$ and $g$ can be totally or partially independent: $\nabla$ can be, for example, the connection of $k^2 g$, where $k$ is some scalar
field). Anyway in the following we shall simply assume that $\nabla$ is the connection of $g$ and that $C_x$ depends only on its Ricci tensor, $\hat{R}$; for example

$$C_x (p) = N(x, p) \exp \left( -\frac{\lambda}{2M_p^2} \hat{R}_x (p, p) \right).$$

Here $\hat{R} := g^{-1}.R.g^{-1}$ (so $\hat{R} (p, p) = \sum_{ijkl} R_{kli} g^{ik} g^{lj} p_ip_j$), $M_P$ is a “very large” mass (the Plank mass, to say) and $\lambda$ is some universal parameter, eventually dependent on the statistics (Bose or Fermi) of the quantum field. Obviously $\lambda \hat{R}_x (p, p)$ has to be always $> 0$ (when $p^2 = M^2$); $N(x, p)$ is a normalization factor fixing the value of $C_x (p)$ on some chosen space-time point; or at a fixed time, in an “spatially homogeneous” space-time, as in Section 1. In this case we have $R_{00} = -3 (H^2 + \Lambda)$, $R_{ij} = (H^2 + 3H^2 + 2K) (a/b)^2 \delta_{ij}$ and $R_{0j} = 0$ ($i, j = 1, 2, 3$); therefore, on the $M$-mass hyperboloid

$$\frac{1}{2} \hat{R}_x (p, p) = -\frac{3\lambda}{2} (H^2 + \Lambda) M^2 + \lambda (K - H^2) p^2;$$

remember that $H(t) = a'(t)/a(t)$ and that $K(t) = K_0/a^2(t)$; $a''/a = H^2 + H^2$. Now $\lambda (K(t) - H^2(t))$ has to be $> 0$, for every $t > t_M$; accordingly the quantum fields are “gravitationally” regularized (nothing diverge).

4. Vacuum energy density

The zero-point (vacuum) energy density of some (scalar, vector or spinor) quantum field $\phi$, in a spatially homogeneous and non-static space-time, can now be written as (see, for example, Pauli [7], Section 9 or Zeldovich [13], Appendix 8, for the “flat” case):

$$\rho_{\phi}^{(\text{vac})} (t) = \int (p_0)^2 C_x (p).d\mu_{M_{\phi}} (p) \quad \text{(by definition)}$$

$$\approx M_C^2 \frac{N^{(0)}_{\phi}}{\theta(t)} + M_C^2 M_{\phi}^2 \frac{N^{(1)}_{\phi}}{\theta(t)} + \ldots \quad \text{(for $M_C \to +\infty$)}.$$  

In the above: the dimensionless numerical factors $N^{(0)}_{\phi}, N^{(1)}_{\phi}, \ldots$ depend on the field’s mass and type (spin and statistics, they are negative for a spinor field); the positive function $\theta := \Lambda/2 (M_C/M_P^2) (K - H^2)$ depends on the space-time metric and it is normalized, fixing the value of $M_C$, in such a way that $\theta(t) \to 1$ when $t \to +\infty$ (or, more cautiously, in such a way that $\theta_0 = \theta(t_0) = 1$); $M_C$ is again a cutoff mass and $M_\phi$ is the mass of the field $\phi$. The next term in the asymptotic expansion ($M_C \to +\infty$) is a logarithmic one, here irrelevant, but it is the main term in the self-mass or self-charge calculations.

Consequently for the total vacuum energy, considering all the fields and retaining only the first two terms in the asymptotic expansion, we have ($\rho^{(\text{vac})} := \sum_{\phi} \rho_{\phi}^{(\text{vac})}$, $N^{(2)} := \sum_{\phi} N^{(2)}_{\phi}$, $N^{(1)} := \sum_{\phi} (M_\phi/M_C)^2 N^{(1)}_{\phi}$, $M^{(\text{crit})} := \sqrt{\rho^{(\text{crit})}} \cong 2.46 \cdot 10^{-12}$ Gev, [8], Astrophysical Constants.):

$$\rho^{(\text{vac})} (t) \cong (M_C/M^{(\text{crit})})^4 \left( \frac{N^{(1)}_{\phi}}{\theta(t)} + \frac{N^{(2)}_{\phi}}{\theta^2(t)} \right) = \Omega^{(0)} \left( \frac{\eta^{(1)}}{\theta(t)} + \frac{\eta^{(2)}}{\theta^2(t)} \right).$$

When $t = +\infty$, $\rho^{(\text{vac})} = \rho^{(0)}$ and $\theta = 1$ so $1 \cong \eta^{(1)} + \eta^{(2)}$ (or, having set $\theta(t_0) = 1$, $\rho^{(\text{vac})}/\rho^{(0)} \cong \eta^{(1)} + \eta^{(2)}$). Note that the signs of $\eta^{(1)}$ and $\eta^{(2)}$ can be different: this is, in fact, the more interesting case.
For a spatially homogeneous and static space-time ([2], end of Chapter 3; [9], Chapter 10) the spatial curvature \( K (t) \) is a constant, as the vacuum energy density (which can be positive or negative, but reasonably \( \neq 0 \):) unfortunately, in this case, the space-time “singularity” \( \lim_{t \to t_M} a (t) = 0 \) seems unavoidable (see the Section 5). For a “generalized” de Sitter metric (the expansion parameter \( L_M \) is > 0 but \( \neq K_M = K (t_M) \), \( a (t) = a_M \cosh \sqrt{L_M} (t - t_M) \), \( H^2/L_M = 1 - (a_M/a)^2 \), \( H'/L_M = (a_M/a)^2 \); in this case \( \lambda (K_M - L_M) \) has to be > 0 and, setting \( \theta (t) = 1, \theta (t) = 1/a^2(t) \), physically quite unacceptable. In the Section 5 we shall consider the more general case, where

\[
\theta (t) \equiv 1 - \frac{\zeta}{a^2(t)},
\]

the parameter \( \zeta \) is now > 0 but \( \ll 1 \), \( a (t) \) will be > \( \sqrt{\zeta} \).

5. Numerical Examples

It is convenient to measure the time in term of \( 1/H_0 \) (= 14.5 Gyr): in this section, we shall set \( H_0 = a' (t_0) = 1 \) (\( t_0 \) is always the present time and \( a (t_0) = 1 \). Now the \( \Omega (n) \) are related to \( K_0 \) and to \( a_0'' = a'' (t_0) \), the acceleration parameter) by \( \sum_{n \geq 0} \Omega (n) = 1 + K_0 \) and by \( \sum_{n \geq 0} (1 - \frac{a_0'}{a_0}) \Omega (n) = a_0'' \) (Section 2) hence, if only \( \Omega (0) \) and \( \Omega (3) \) are significantly \( \neq 0 \) and \( K_0 \equiv 0 \),

\[
\Omega (0) = \frac{1}{3} \left( 1 + 2a_0' + K_0 \right) \approx \frac{1}{3} + \frac{2}{3} a_0'
\]

\[
\Omega (3) = \frac{2}{3} \left( 1 - a_0' + K_0 \right) \approx \frac{2}{3} - \frac{2}{3} a_0'
\]

The curvature parameter ([12], Section 1.5) is defined by \( \frac{|K(t)|}{H^2(t)} = \frac{|K_0|}{(a'(t))^2} \), therefore, if \( K_0 \neq 0 \),

\[
\frac{|K_0|}{(a'(-))^2} = \left| 1 - \frac{a^2}{K_0} \sum_{n \geq 0} \frac{\Omega (n)}{a_n} \right|^{-1} \; ; \text{then if } \Omega (0) > 0, a' (t) \to +\infty \text{ and } |K_0| / (a')^2 \to 0, \text{ for } t \to +\infty.
\]

When \( a'' = 0 \) \( (t = t_F) \) the expansion factor has a flex and the curvature parameter has a maximum.

First at all we shall consider, as a reference, the case where the vacuum energy is constant and \( \Omega (0) = 0.685, \Omega (3) = 0.315 \) and \( \Omega (4) = 0.002 \). Therefore \( K_0 = 0.002, R_0 := 1/\sqrt{|K_0|}, \) if \( K_0 \neq 0 \) \( 22.4 (= 3.25 \cdot 10^{11} \text{ ly}), \) \( a'' = 0.525 \) and

\[
t_0 - t_M = 0.944 \text{ (13.70 Gyr)}, \quad t_F - t_M = 0.529 \text{ (7.67 Gyr)}.
\]

\( a_F = 0.617 \) while the curvature parameter at \( t_F \) is \( = 0.002 \) \( (0.002 \text{ at } t_0): a (t) \to 0 \text{ when } t \to t_M, \; t > t_M. \) A smaller value of \( \Omega (4) \) increases the value of \( t_0 - t_M: \) if, for example, \( \Omega (4) = 0.001, \) then \( t_0 - t_M = 0.947 \text{ (13.73 Gyr)}. \)

But if the vacuum energy varies with time, for example if (see the Section 4),

\[
\eta^{(1)} = 1.95, \quad \eta^{(2)} = -0.95, \quad \zeta = 0.001
\]

things change dramatically: the \( (t, a (t)) \) curve has a minimum at \( t_M \) and two flex (between \( t_M \) and \( t_0) \), at \( t_F \) (as in the “classical” case) and at \( t_F \) (I as inception), then

\[
t_0 - t_M = 0.939 \text{ (13.62 Gyr)}, \quad t_F - t_M = 0.523 \text{ (7.58 Gyr)}, \quad t_I - t_M = 3.9 \text{ M gyr}.
\]

Quite interesting is the behavior of the components of the energy density: the classical one \( (\rho^{(\text{mat})} + \rho^{(\text{rad})}) \) has a maximum at \( t_M \) then rapidly decrease, always respecting the energy conditions; the vacuum one \( (\rho^{(\text{vac})}) \) has a minimum at \( t_M \) (where it is < 0) then rapidly increase
and became $> 0$, never respecting the energy conditions; the total one ($\rho$) has a minimum at $t_M$ (where $a' = 0$ as $\rho'$ and $\rho_M \equiv 0$) then increase, reaches a maximum (near $t_M$) and finally regularly decrease. Obviously everything happens because $\eta^{(1)} > 0$ and $\eta^{(2)} < 0$; generally the behavior of $a$ and $\rho$ is strongly influenced by the values of the parameters: for example if $\eta^{(1)}$ is too large the flex are lost and $\rho_M$ is $< 0$; the flex are also lost if $\zeta$ is not enough small (and now the maximums and minimums are less sharp); for more, a realistic value of $\zeta$ (for our Universe) is perhaps much smaller than 0.001.

In any case, we can assume that different values of $\eta^{(1)}$, $\eta^{(2)}$ and $\zeta$ describe the contraction/expansion of some different massive object.

References

[1] Chern S S, Chen W H and Lam K S 2000 Lectures on Differential Geometry (Singapore: World Scientific)
[2] Einstein A 1955 The Meaning of Relativity (Princeton University Press)
[3] Feynman R P 1995 Feynman Lectures on Gravitation (Reading, Massachusetts: Addison-Wesley Publishing Company)
[4] Friedmann A and Lemaître G 1997 Essais de Cosmologie (Paris: Editions du Seuil)
[5] Gupta S N 1954 Gravitation and electromagnetism Physical Review 96 1683-1685
[6] Hawking S W and Ellis G F R 1973 The Large Scale Structure of Space-time (Cambridge University Press)
[7] Pauli W 1977 Pauli Lectures on Physics: Selected Topics on Field Quantization (Cambridge, Massachusetts: The MIT Press)
[8] Olive K A et al. (Particle Data Group) 2014 Chinese Physics C 38 090001
[9] Tolman R C 1987 Relativity, Thermodynamics and Cosmology (New York: Dover Publications)
[10] Weinberg S 1972 Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity (New York: John Wiley and Sons)
[11] Weinberg S 1995 The Quantum Theory of Fields I (Cambridge University Press)
[12] Weinberg S 2008 Cosmology (Oxford University Press)
[13] Zeldovich Ya B 1968 The cosmological constant and the theory of elementary particles Soviet Physics Uspekhi 11 381-393