GENUINELY RAMIFIED MAPS AND STABILITY OF PULLED-BACK PARABOLIC BUNDLES

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ABSTRACT. Let $f : X \rightarrow Y$ be a genuinely ramified map between irreducible smooth projective curves defined over an algebraically closed field. Let $P$ be a branch data on $Y$ such that $P(y)$ and $B_f(y)$, where $B_f$ is branch data for $f$, are linearly disjoint for every $y \in Y$. Further assume that either $P$ is tame or $f$ is Galois. Then the pullback, by $f$, of any stable parabolic bundle on $Y$ with respect to $P$ is actually a stable parabolic bundle on $X$ with respect to $f^*P$.

1. Introduction

Let $f : X \rightarrow Y$ be a separable surjective map between irreducible smooth projective curves defined over an algebraically closed field. In [BP] the following was proved:

If the homomorphism of étale fundamental groups $f_* : \pi_1^{et}(X) \rightarrow \pi_1^{et}(Y)$ induced by is surjective, and $E$ is a stable vector bundle on $Y$, then the pullback $f^*E$ is also stable.

Any map $f$ as above such that the corresponding $f_* : \pi_1^{et}(X) \rightarrow \pi_1^{et}(Y)$ is surjective is called a genuinely ramified map; see [BP] Proposition 2.6 and [BP] Lemma 3.1, [BHS] p. 574, Proposition 3.3] for equivalent reformulation of it.

Given a smooth projective curve $Y$ defined over an algebraically closed field of characteristic zero, and some marked points $S$ of $Y$, parabolic bundles on $Y$ with parabolic structure over $S$ were introduced in [MS] by Mehta and Seshadri. They carried out a detailed investigation of the parabolic bundles, and proved that the stable parabolic bundles of parabolic degree zero over a complex curve are in bijection with the irreducible unitary representations of the complement of the parabolic points of the curve. In [Bo] a correspondence between the parabolic bundles and the orbifold bundles was established (see also [Bo1] and [Bo2]).

In [KP] and [KM] the definition of parabolic bundles was extended to smooth projective curves defined over an algebraically closed field of arbitrary characteristic. This was carried out using the formal orbifolds. A formal orbifold is a curve $\hat{Y}$ together with a branch data $P$ on a finite subset $S \subseteq Y$, which comprises of a nontrivial Galois extension $P(y)$ of $K_{Y,y}$, the fraction field of $\hat{O}_{Y,y}$ for each marked point $y_i \in S$. A parabolic structure on a vector bundle $V$ over $Y$ with branch data $P$, roughly speaking, consists of local equivariant structure on pullback of $V$ over the formal neighbourhood of $y_i$ which become trivial on the “punctured formal disc”, for all $y_i$ in $S$ (see Section 2 for the precise definition).

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A natural question to ask is whether an analogue of the above mentioned theorem of \cite{BP} continues to hold in the more general set-up of parabolic bundles.

Let \( f : X \to Y \) be a genuinely ramified map between irreducible smooth projective curves. Then \( f \) produces a branch data \( B_f \) on \( Y \).

We prove the following theorem (see Theorem 5.2):

**Theorem 1.1.** Suppose \( B_f(y) \) and \( P(y) \) are linearly disjoint over \( K_{Y,y} \) for all \( y \in Y \). Also assume that either \( P \) is tame or the map \( f \) is Galois. Then the pullback, by \( f \), of every stable parabolic bundle on \( Y \) with respect to \( P \) is a stable parabolic bundle on \( X \) with respect to \( f^*P \).

## 2. Parabolic bundles

The base field \( k \) is algebraically closed. Let \( Y \) be an irreducible smooth projective curve. We recall some definitions from \cite{KP}, Section 2]. For each closed point \( y \in Y \), let \( K_{Y,y} \) be the fraction field of the completion \( \hat{O}_{Y,y} \). A branch data \( P \) on \( Y \) assigns a finite Galois extension \( P(y) \) of \( K_{Y,y} \) for every closed point \( y \in Y \) such that \( P(y) \) is a trivial extension of \( K_{Y,y} \) for all \( y \) outside some finitely many closed points \( y_1, \cdots, y_n \). The collection \( \{y_1, \cdots, y_n\} \) is called the support of \( P \). If \( n = 0 \), i.e., the support is empty, then the branch data will be called trivial. The branch data \( P \) is called tame if each of these extension of local fields is a tame extension.

Let \( f : X \to Y \) be a separable (same as generically smooth) nonconstant morphism of smooth projective curves. We will call such a morphism a (ramified) covering. It will be called a Galois covering if the induced field extension \( k(X)/k(Y) \) is Galois. Using \( f \), we will construct a branch data \( B_f \) on \( Y \). For \( y \in Y \), define \( B_f(y) \) to be the compositum of the Galois closure of \( K_{X,x}/K_{Y,y} \) for all \( x \in X \) with \( f(x) = y \). Therefore, the support of \( B_f \) is the branch locus of \( f \). We note that if \( f \) is tamely ramified, then \( B_f(y) \) is the cyclic field extension of \( K_{Y,y} \) whose degree is the least common multiple of the multiplicities of \( f \) at the points of \( f^{-1}(y) \).

A formal orbifold curve is a pair \((Y, P)\) where \( Y \) is a smooth projective curve and \( P \) is a branch data on \( Y \). A morphism \( f : (X, Q) \to (Y, P) \) of formal orbifolds is a finite separable morphism \( f : X \to Y \) such that for all \( x \in X \) the field \( Q(x) \) contains \( P(f(x)) \). This morphism is called étale if \( Q(x) = P(f(x)) \) for all \( x \in X \). We say that \((Y, P)\) is geometric, or \( P \) is a geometric branch data, if there exists an étale covering \( f : (X, 0) \to (Y, P) \) with 0 being the trivial branch data. Note that in this case \( P = B_f \).

Fix a finite subset \( S := \{y_1, \cdots, y_n\} \subset Y \) of a smooth projective curve \( Y \). Fix a branch data \( P \) with support \( S \). For any \( y \in S \), let \( I_{y} = \text{Gal}(P(y)|K_{Y,y_i}) \) be the Galois group. The integral closure of \( O_{Y,y_i} \) in \( P(y_i) \) will be denoted by \( R_i \).

Let \( V \) be a vector bundle over \( Y \); the stalk of \( V \) (viewed as a locally free sheaf) at \( y_i \) is denoted by \( V_{y_i} \). We recall from \cite{KM} that a parabolic structure on \( V \) over \( S \) with branch data \( P \) consists of an \( I_{y_i} \)-equivariant structure on \( V_{y_i} \otimes O_{Y,y_i} \) \( R_i \) together with an isomorphism between the induced \( I_{y_i} \)-equivariant structure on \( V_{y_i} \otimes O_{Y,y_i} \) \( P(y_i) \) and the
trivial $I_y$–equivariant structure on $V_{y_i} \otimes_{\mathcal{O}_{Y,y_i}} P(y_i)$, for every $y_i \in S$. See [KM, Section 4] for more details.

Let $f : (X, 0) \to (Y, P)$ be an étale Galois covering of formal orbifold curves; the Galois group of $f$ will be denoted by $\Gamma$. We know that the category $\mathrm{PVect}(Y,P)$ of parabolic bundles on $Y$ with the branch data $P$ is equivalent to the category of $\Gamma$–equivariant vector bundles on $X$ [KM, Theorem 4.11]. In fact in [KP] the category of vector bundles $\mathrm{Vect}(Y,P)$ on $(Y,P)$ was defined as $\Gamma$–equivariant bundles on $X$, and it was shown to be independent of the choice of the étale Galois covering $f$ [KP, Proposition 3.6]. Note that if $P' \geq P$, then there is an embedding of categories

$$\mathrm{PVect}(Y,P) \to \mathrm{PVect}(Y,P')$$

[KM Corollary 5.10]. Furthermore, if $P$ and $P'$ are geometric, then there is an embedding $\mathrm{Vect}(Y,P) \to \mathrm{Vect}(Y,P')$ [KP, Theorem 3.7]. An orbifold bundle on $Y$ is an object of $\mathrm{PVect}(Y,P)$, and a parabolic bundle on $Y$ is an object of $\mathrm{PVect}(Y,P)$. The functors between $\mathrm{Vect}(Y,P)$ and $\mathrm{PVect}(Y,P)$ induce an equivalence between the category of orbifold bundles and the category of parabolic bundles $\mathrm{PVect}(Y)$ on $Y$.

Let $\mathrm{PVect}^t(Y)$ be the full subcategory of $\mathrm{PVect}(Y)$ whose objects come from the objects in $\mathrm{PVect}(Y,P)$ for some tame branch data $P$ on $Y$. Similarly, we can define the full subcategory of tame orbifold bundles whose objects come from objects in $\mathrm{Vect}(X,P)$ for some tame branch data $P$. The restriction of the above functor to these full subcategories is an equivalence as well.

When the base field has characteristic zero, parabolic bundles were defined by Mehta and Seshadri ([MS] as follows. Let $E$ be a vector bundle on $Y$ of rank $r$. Let $E^*$ be a parabolic structure on $E$ with parabolic divisor

$$S := \{y_1, \ldots, y_n\} \subset Y.$$ 

For every $y_i \in S$, let

$$E^*_{p_i} = E_i,1 \supset E_i,2 \supset \cdots \supset E_i,l_i \supset E_i,l_i+1 = 0$$

be the quasiparabolic filtration, and let

$$0 \leq \alpha_{i,1} < \alpha_{i,2} < \cdots < \alpha_{i,l_i} < 1$$

be the corresponding parabolic weights (see [MS], [Bi]). Then the parabolic degree of $E^*$ is defined to be

$$\text{par-deg}(E^*) := \text{degree}(E) + \sum_{i=1}^n \sum_{j=1}^{l_i} \dim(E_{i,j}/E_{i,j+1}) \cdot \alpha_{i,j},$$

and the parabolic slope of $E^*$ is defined to be

$$\text{par-}\mu(E^*) := \frac{\text{par-deg}(E^*)}{r}.$$ 

The parabolic bundle $E^*$ is called stable (respectively, semistable) if

$$\text{par-}\mu(F^*) < \text{par-}\mu(E^*)$$

(respectively, $\text{par-}\mu(F^*) \leq \text{par-}\mu(E^*)$)

for every subbundle $0 \neq F \subsetneq E$ with the parabolic structure induced by that of $E^*$ (see [MS], [Bi]). The parabolic bundle $E^*$ is called polystable if
• $E_*$ is semistable, and
• $E_*$ is a direct sum of stable parabolic bundles.

When the base field has positive characteristic, the parabolic degree, slope, stability and semistability were defined in terms of equivariant bundles (see Remark 3.9 of [KP] and the paragraph following the remark). In characteristic zero, the definition of (semi)stability in terms of equivariant bundles and the above definition agree (see Proposition 5.1).

In [KM, Proposition 5.15] it was shown that if the base field $k$ has characteristic zero then the category $\text{PVect}(Y)$ of parabolic bundles on $Y$ is equivalent to the category of parabolic bundles with rational weights on $Y$ in the sense of [MS], [MY]. When characteristic $p$ of $k$ is positive, the same proof verbatim gives us the following proposition.

**Proposition 2.1.** The category of $\text{PVect}(Y)$ is equivalent to the category of parabolic bundles in the sense of Mehta-Seshadri, [MS], with rational weights satisfying the condition that the denominators in weights are coprime to the characteristic $p$.

Given a branch data $P$ on $Y$, and a covering $f : X \rightarrow Y$, for a closed point $x \in X$, let $(f^*P)(x)$ be the compositum $P(f(x))K_{X,x}$. Then $f^*P$ is a branch data on $X$; it is defined by $x \mapsto (f^*P)(x)$.

Given a covering $f : X \rightarrow Y$, in [KP] the pullback functor from the category Vect$(Y,P)$ to the category Vect$(X,f^*P)$ was defined. This was used in [KM] to define the functor

$$\text{PVect}(Y,P) \rightarrow \text{PVect}(X,f^*P).$$

Note that if $P$ is a tame branch data on $Y$ then $f^*P$ is also tame. Hence tame orbifold (respectively, parabolic) bundles on $Y$ pull back to tame orbifold (respectively, parabolic) bundles on $X$.

3. **Genuinely ramified covers relative to a branch data**

A separable surjective map $f : X \rightarrow Y$ between irreducible smooth projective curves is called **genuinely ramified** if the homomorphism of étale fundamental groups

$$f_* : \pi_1^\text{ét}(X) \rightarrow \pi_1^\text{ét}(Y)$$

induced by $f$ is surjective; see [BP] Definition 2.5 and [BP] Proposition 2.6.

**Lemma 3.1.** Let $f : X \rightarrow Y$ be a genuinely ramified map. Let $g : Z \rightarrow Y$ be a Galois cover such that $B_f(y)$ and $B_g(y)$ are linearly disjoint over $K_{Y,y}$ for all $y \in Y$. Suppose $k(Z)$ and $k(X)$ are linearly disjoint over $k(Y)$. Let $W$ be the normalization of the fiber product $X \times_Y Z$. Then the natural projection

$$p_Z : W \rightarrow Z$$

is genuinely ramified.

**Proof.** We first observe that the condition that $k(X)$ and $k(Z)$ are linearly disjoint over $k(Y)$ implies that $W$ is connected.

Let

$$\tilde{f} : \tilde{X} \rightarrow Y$$
be the Galois closure of $f$ with Galois group $G$, and let $\widetilde{W}$ be the Galois closure of $p_Z$. We note that $\widetilde{W}$ is the normalization of $\tilde{X} \times_Y Z$, and the Galois group of $k(\widetilde{W})/k(Z)$ is also $G$. Suppose $p_Z$ is not genuinely ramified. Then there exists an étale covering $Z' \to Z$ dominated by $W$. Let $H = \text{Gal}(k(\widetilde{W})/k(Z'))$, and let $Y'$ be the normalization of $Y$ in $k(\tilde{X})^H$. Then $Y'$ is dominated by $X$. Moreover, for any point $y \in Y$ and a point $y' \in Y'$ lying above $y$, we know $K_{Y',y'}$ is contained in $B_f(y)$ and $B_g(y)$. Hence $Y' \to Y$ is étale, which contradicts the condition that $f$ is genuinely ramified (see [BP, Proposition 2.6]). □

**Lemma 3.2.** Let $f : X \to Y$ be a covering of smooth projective curves. Let $P$ be a tame branch data on $Y$ such that $P(y)$ and $B_f(y)$ are linearly disjoint over $K_{Y,y}$ for all $y \in Y$. Then there exists a Galois covering $g : Z \to Y$ such that

1. $k(Z)$ and $k(X)$ are linearly disjoint over $k(Y)$,
2. $B_f(y)$ and $B_g(y)$ are linearly disjoint over $K_{Y,y}$ for all $y \in Y$, and
3. $P(y) = B_g(y)$ for all $y$ in support of $P$.

**Proof.** Let $y_1$ be a point in $Y$ outside the support of $B_f$. Let $N$ be the least common multiple of $[P(y) : K_{Y,y}]$ for all $y \in Y$. Take a rational function $\alpha$ on $Y$ which has

- a simple zero at $y_1$,
- a zero of multiplicity $N/P(y)$ at every $y$ in support of $P$ and
- the remaining zeroes and poles are outside the support of $B_f$.

Let $g : Z \to Y$ be the normalization of $Y$ in $k(Y)[\alpha^{1/N}]$. Note that $Z \to Y$ is totally ramified at $y_1$, and $X \to Y$ is étale over $y_1$, and hence $k(Z)$ and $k(X)$ are linearly disjoint over $k(Y)$. The rest of the statements follow as $g$ is ramified only at the zeros and poles of $\alpha$ and the ramification index of $g$ at a zero $y$ of $\alpha$ is $N/\text{ord}_y(\alpha)$. □

**Lemma 3.3.** Let $P$ be a branch data on $Y$. Let $S$ be a finite subset of $Y \setminus \text{Supp}(P)$. Then there exists a geometric branch data $P'$ on $Y$ such that $P'(y) = P(y)$ for all $y$ in the support of $P$ while the support of $P'$ is disjoint from $S$.

**Proof.** We first assume that $Y = \mathbb{P}^1$. Consider $\text{Supp}(P) = \{y_1, \ldots, y_r\}$. Let $n_i$ be the tame degree of $P(y_i)$, and $n_0 := \text{l.c.m.}\{n_i \mid 1 \leq i \leq r\}$. Take any $y_0 \in \mathbb{P}^1$ outside $S \cup \text{Supp}(P)$. Set $P'(y_0)$ to be the cyclic extension of $K_{Y,y_0}$ of degree $n_0$ and $P'(y) = P(y)$ for all $y \in Y$ different from $y_0$. Then $P'$ is geometric by [KP, Corollary 2.32].

For the general case of $Y$, take a point $y_0 \in Y$ outside $\text{Supp}(P)$, and also take a regular function $a$ on $Y \setminus y_0$ which takes different values at the points of $\text{Supp}(P) \cup S$. Then we get a covering $a : Y \to \mathbb{P}^1$ totally ramified at $y_0$. Let $Q(a(y)) = P(y)$ for $y \in \text{Supp}(P)$, and take $\alpha(\mathbb{P}^1 \setminus a(\text{Supp}(P) \cup S \setminus \{y_0\}))$. Then by the $Y = \mathbb{P}^1$ case there is a Galois covering $b : Z \to \mathbb{P}^1$ such that $B_0(x) = Q(x)$ for $x \in \mathbb{P}^1$ except at one point $x_0$. Now the normalized pullback $c : Z' \to Y$ of $b$ is a connected Galois cover of $Y$ as $b$ is étale over $a(y_0)$ and $a$ is totally ramified at $y_0$. Set $P' = B_c$. Then $P'$ satisfies all the statements in the lemma. □

**Lemma 3.4.** Let $f : X \to Y$ be a genuinely ramified Galois covering of smooth projective curves. Let $P$ be a branch data on $Y$ such that $P(y)$ and $B_f(y)$ are linearly disjoint over $K_{Y,y}$ for all $y \in Y$. Then there exists a tame branch data $P'$ on $Y$ such that $P'(y) = P(y)$ for all $y$ in the support of $P$, while the support of $P'$ is disjoint from $S$. □

Theorem 3.5. Let $f : X \to Y$ be a generically étale and generically unramified Galois covering of smooth projective curves. Then the Galois closure $\widetilde{W}$ is the normalization of $\tilde{X} \times_Y Z$, and the Galois group of $k(\widetilde{W})/k(Z)$ is also $G$. Suppose $p_Z$ is not genuinely ramified. Then there exists an étale covering $Z' \to Z$ dominated by $W$. Let $H = \text{Gal}(k(\widetilde{W})/k(Z'))$, and let $Y'$ be the normalization of $Y$ in $k(\tilde{X})^H$. Then $Y'$ is dominated by $X$. Moreover, for any point $y \in Y$ and a point $y' \in Y'$ lying above $y$, we know $K_{Y',y'}$ is contained in $B_f(y)$ and $B_g(y)$. Hence $Y' \to Y$ is étale, which contradicts the condition that $f$ is genuinely ramified (see [BP, Proposition 2.6]). □
disjoint over \( K_{Y,y} \) for all \( y \in Y \). Then there exists a Galois covering \( g : Z \to Y \) such that

1. \( k(Z) \) and \( k(X) \) are linearly disjoint over \( k(Y) \),
2. \( B_f(y) \) and \( B_g(y) \) are linearly disjoint over \( K_{Y,y} \) for all \( y \in Y \), and
3. \( P(y) = B_g(y) \) for all \( y \) in the support of \( P \).

**Proof.** Apply Lemma 3.3 with \( S = \text{Supp}(B_f) \setminus \text{Supp}(P) \) to obtain a geometric branch data \( P' \) on \( Y \) such that \( P'(y) = P(y) \) for \( y \in \text{Supp}(P) \) and \( S \subseteq Y \setminus \text{Supp}(P') \). Let \( g : (Z, 0) \to (Y, P') \) be a Galois étale covering of formal orbifolds. Then \( g \) satisfy conclusion (2) and (3) of the lemma. Let \( K = k(Z) \cap k(X) \), and let \( a : W \to Y \) be the normalization of \( Y \) in \( K \). Then \( a : W \to Y \) is étale. This is because \( a \) is dominated by the coverings \( f \) and \( g \), the covering \( g : Z \to Y \) is étale over points in \( S \) and \( f \) is étale outside \( B_f \), hence the branch locus of \( a \) is contained in \( \text{Supp}(B_f) \cap \text{Supp}(P) \). Now for a point \( y \) in \( \text{Supp}(B_f) \cap \text{Supp}(P) \) and \( w \in W \) lying above \( y \), the given conditions that \( P(y) \) and \( B_f(y) \) are linearly disjoint and \( P'(y) = P(y) \) imply that the local field extension \( K_{W,w}/K_{Y,y} \) is trivial. Hence points in \( \text{Supp}(B_f) \cap \text{Supp}(P) \) are also not in the branch locus of \( a \). Since \( f \) is genuinely ramified \( a \) is the identity map, i.e., \( K = k(Y) \). Since \( k(X)/k(Y) \) and \( k(Z)/k(Y) \) are Galois, this implies \( k(Z) \) and \( k(X) \) are linearly disjoint over \( k(Y) \). \( \square \)

## 4. Pullback of equivariant stable bundles

Let \( X \) be a smooth projective curve equipped with an action of a finite group \( \Gamma \). A \( \Gamma \)-equivariant vector bundle \( E \) on \( X \) is called \( \Gamma \)-stable (respectively, \( \Gamma \)-semistable) if for all \( \Gamma \)-equivariant subbundle \( 0 \neq F \subseteq E \), the inequality

\[
\mu(F) := \frac{\text{degree}(F)}{\text{rank}(F)} < \mu(E) := \frac{\text{degree}(E)}{\text{rank}(E)} \quad (\text{respectively, } \mu(F) \leq \mu(E))
\]

holds.

When \( \Gamma = \{ e \} \), a \( \Gamma \)-stable (respectively, \( \Gamma \)-semistable) vector bundle is called stable (respectively, semistable). A semistable vector bundle is called polystable if it is a direct sum of stable vector bundles.

**Lemma 4.1.** Let \( E \) be a \( \Gamma \)-stable vector bundle on \( X \). Then \( E \) is polystable in the usual sense.

**Proof.** Note that \( E \) a is semistable bundle on \( X \) by [KP Lemma 3.10]. Let \( W \subseteq E \) be the socle (see [HL p. 23, Lemma 1.5.5]). From the uniqueness of the socle bundle we know that the action of \( \Gamma \) on \( E \) preserves \( W \). Since \( E \) is \( \Gamma \)-stable, this implies that \( W = E \). Hence \( E \) is polystable. \( \square \)

Let \( X \) and \( Y \) be irreducible smooth projective curves such that each of them is equipped with an action of \( \Gamma \), and let

\[
f : X \to Y
\]

be a nonconstant separable \( \Gamma \)-equivariant morphism. Given a \( \Gamma \)-equivariant bundle \( E \to Y \), the action of \( \Gamma \) on \( E \) produces an action of \( \Gamma \) on \( f^*E \), because the map \( f \) is \( \Gamma \)-equivariant. So \( f^*E \) is a \( \Gamma \)-equivariant bundle on \( X \).
Proposition 4.2. Assume that the map $f$ in (4.1) is genuinely ramified. Let $E$ be a $\Gamma$–stable vector bundle on $Y$. Then the pullback $f^*E$ is also $\Gamma$–stable.

Proof. From Lemma 4.1 we know that $E$ is polystable. Let

$$E = \bigoplus_{i=1}^{\ell} F_i$$

be a decomposition of $E$ into a direct sum of stable vector bundles with $\mu(F_i) = \mu(E)$ for all $1 \leq i \leq \ell$. Therefore, we have

$$f^*E = \bigoplus_{i=1}^{\ell} f^*F_i. \quad (4.2)$$

Now from [BP, Theorem 1] we conclude that each $f^*F_i$ is stable. Since

$$\mu(f^*F_i) = \text{degree}(f) \cdot \mu(F_i),$$

this implies that $f^*E$ is polystable.

Assume that $f^*E$ is not $\Gamma$–stable. Let

$$0 \neq S \varsubsetneq f^*E \quad (4.3)$$

be a subbundle such that

- $\mu(S) = \mu(f^*E)$, and
- the action of $\Gamma$ on $f^*E$ preserves $S$

We will prove that $S$ is polystable. For that first note that if $V_1$ and $V_2$ are stable vector bundles with $\mu(V_1) = \mu(V_2)$, then any nonzero homomorphism $V_1 \to V_2$ is an isomorphism. From this it follows that any polystable vector bundle $\mathcal{V}$ can be uniquely expressed as

$$\mathcal{V} = \bigoplus_{i=1}^{\ell} V_i \otimes_k \text{Hom}(V_i, \mathcal{V}), \quad (4.4)$$

where $\{V_i\}_{i=1}^{n}$ are all stable vector bundles such that

- $\mu(\mathcal{V}) = \mu(V_i)$, and
- $\text{Hom}(V_i, \mathcal{V}) \neq 0$.

If $\mathcal{T} \subset \mathcal{V}$ is a subbundle of the polystable vector bundle $\mathcal{V}$ such that $\mu(\mathcal{V}) = \mu(\mathcal{T})$, then

$$\mathcal{T} = \bigoplus_{i=1}^{n} \mathcal{T} \cap (V_i \otimes_k \text{Hom}(V_i, \mathcal{V})) \quad (4.5)$$

with respect to the unique decomposition in (4.4). Now any subbundle $W$ of $V_i \otimes_k \text{Hom}(V_i, \mathcal{V})$ with $\mu(W) = \mu(V_i)$ must be of the form $V_i \otimes_k B$, where $B \subset \text{Hom}(V_i, \mathcal{V})$ is a subspace. This implies that $W$ is polystable. Since each nonzero $\mathcal{T} \cap (V_i \otimes_k \text{Hom}(V_i, \mathcal{V}))$ in (4.5) is polystable with $\mu(\mathcal{T} \cap (V_i \otimes_k \text{Hom}(V_i, \mathcal{V}))) = \mu(V_i) = \mu(\mathcal{V})$, we conclude that $\mathcal{T}$ is polystable.

Therefore, $S$ in (4.3) is polystable.
Since each $f^*F_i$ in (4.2) is stable, we know that the polystable bundle $S$ in (4.3) admits an isomorphism
\[
S \xrightarrow{\sim} \bigoplus_{j=1}^{\ell'} f^*F_{a_j},
\]
with $a_1 < \cdots < a_{\ell'}$ and $\ell' < \ell$. Denote
\[
V = \bigoplus_{j=1}^{\ell'} F_{a_j};
\]
from (4.6) it follows that
\[
S \xrightarrow{\sim} f^*V.
\]

Let $A$ and $B$ be two semistable vector bundles on $Y$ with $\mu(A) = \mu(B)$. From [BP, Lemma 4.3] we know that the natural homomorphism
\[
H^0(Y, \text{Hom}(A, B)) \rightarrow H^0(X, \text{Hom}(f^*A, f^*B))
\]
is an isomorphism. Therefore, there is a homomorphism
\[
\psi : V \rightarrow E
\]
such that $f^*\psi$ coincides with the inclusion map $S \hookrightarrow f^*E$ in (4.3) once the isomorphism in (4.7) is invoked. Note that since $V$ and $E$ are semistable with $\mu(V) = \mu(E)$, it follows that $\psi(V)$ is a subbundle of $E$. Since $f^*\psi$ coincides with the inclusion map $S \hookrightarrow f^*E$, it follows immediately that the subbundle $\psi(V) \subset E$ has the property that its pullback
\[
f^*\psi(V) \subset f^*E
\]
coincides with the subbundle $S \subset f^*E$ in (4.3).

Since $S$ is preserved by the action of $\Gamma$ on $f^*E$, it follows immediately that the subbundle $\psi(V) \subset E$ is preserved by the action of $\Gamma$ on $E$. But this contradicts the given condition that $E$ is a $\Gamma$–stable vector bundle; note that $\mu(\psi(V)) = \mu(E)$, because $\mu(S) = \mu(f^*E)$. Therefore, we conclude that the pullback $f^*E$ is a $\Gamma$–stable bundle. \qed

5. PARABOLIC STABILITY OF THE PULLBACK BUNDLE

Let $E_\ast$ be a parabolic bundle on $Y$ with respect to $P$. Let
\[
h : (X, 0) \rightarrow (Y, P)
\]
be an étale Galois covering of orbifolds with Galois group $\Gamma$. Let $W$ be the $\Gamma$–equivariant vector bundle on $X$ corresponding to $E_\ast$.

The following proposition is standard.

**Proposition 5.1.** The parabolic bundle $E_\ast$ is stable (respectively, semistable) if and only if $W$ is $\Gamma$–stable (respectively, $\Gamma$–semistable). Similarly, $E_\ast$ is polystable if and only if $W$ is $\Gamma$–polystable.

**Proof.** This is a tautology when the base field has positive characteristic. For the characteristic zero case we have $\text{degree}(h) \cdot \text{par-deg}(E_\ast) = \text{degree}(W)$. Also, the subbundles of $E_\ast$ with induced parabolic structure correspond to the $\Gamma$–equivariant subbundles of $W$ (see [Br]). The proposition follows immediately from these. \qed
Let \( f : X \to Y \) be a genuinely ramified map between smooth projective curves. Let \( P \) be a branch data on \( Y \). Let \( V \) be a parabolic bundle on \( Y \) with respect to \( P \). Then \( f^*V \) is a parabolic bundle on \( X \) with respect to \( f^*P \).

**Theorem 5.2.** Suppose \( B_f(y) \) and \( P(y) \) are linearly disjoint over \( K_{Y,y} \) for all \( y \in Y \). Also assume that either \( P \) is tame or \( f \) is Galois. Then the pullback, by \( f \), of every stable parabolic bundle on \( Y \) with respect to \( P \) is a stable parabolic bundle on \( X \) with respect to \( f^*P \).

**Proof.** By Lemma 3.2 and Lemma 3.4 there exists a \( \Gamma \)-Galois covering \( g : Z \to Y \) such that

1. \( B_g(y) = P(y) \) for all \( y \in \text{Supp}(P) \), and
2. \( B_f(y) \) and \( B_g(y) \) are linearly disjoint.

Moreover, the normalization \( W \) of \( Z \times_Y X \) is connected. By Lemma 3.1 the morphism \( f_Z : W \to Z \) is genuinely ramified. Also note that \( f_Z \) is \( \Gamma \)-equivariant. Now replacing \( P \) by \( B_g \) we may assume that \( P \) is geometric branch data; this is because \( P \leq B_g \).

Hence a parabolic bundle with respect to \( P \) is also a parabolic bundle with respect to \( B_g \). Consequently, \( V \) corresponds to a \( \Gamma \)-equivariant bundle \( \tilde{V} \) on \( Z \). Now by Proposition 4.2, the pullback \( f_Z^*\tilde{V} \) is a stable \( \Gamma \)-equivariant bundle on \( W \) (see Proposition 5.1). But \( (W, 0) \to (X, f^*B_g) \) is a \( \Gamma \)-Galois étale cover of formal orbifolds ([KP, Proposition 2.16]). So the \( \Gamma \)-bundle \( f_Z^*\tilde{V} \) on \( W \) corresponds to the parabolic bundle \( f^*V \) on \( X \) with respect to \( f^*P \). Hence \( f^*V \) is a stable parabolic bundle on \( Y \). \( \square \)

**Remark 5.3.** The additional conditions in Theorem 5.2 imply that the morphism \( f \) induce a surjection \( \pi_1(X, f^*P) \to \pi_1(Y, P) \). Though we expect the result to be true under a milder hypothesis, the proof in positive characteristic is not yet clear to us. The case of characteristic zero has been dealt with in [BKP].

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