Quench dynamics of topological maximally entangled states

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Abstract
We investigate the quench dynamics of the one-particle entanglement spectra (OPES) for systems with topologically nontrivial phases. By using dimerized chains as an example, it is demonstrated that the evolution of OPES for the quenched bipartite systems is governed by an effective Hamiltonian which is characterized by a pseudospin in a time-dependent pseudomagnetic field $S(k, t)$. The existence and evolution of the topological maximally entangled states (tMESs) are determined by the winding number of $S(k, t)$ in the $k$-space. In particular, the tMESs survive only if nontrivial Berry phases are induced by the winding of $S(k, t)$. In the infinite-time limit the equilibrium OPES can be determined by an effective time-independent pseudomagnetic field $S_{\text{eff}}(k)$. Furthermore, when tMESs are unstable, they are destroyed by quasiparticles within a characteristic timescale in proportion to the system size.

(Some figures may appear in colour only in the online journal)

1. Introduction

Topological phases that are not characterized by local order parameters have been subjects of key interest in condensed matter physics due to the anomalous properties associated with these phases. The recent discovery of time-reversal invariant topological insulators [1–6] has triggered further intense investigations on the characterization of topological phases. One of the important features of topological phases is the possibility of creating nonlocal properties associated with the topology, which is often realized as the entanglement between the system and its environment. In particular, the entanglement spectrum (ES) i.e. the eigenvalues of the reduced density matrix of the system, provides an ideal tool to characterize the topological phase [7–9]. In general, not only can the ES be used to distinguish different classifications of topological phases but it can also detect the existence of edge modes at zero energy [10, 11]. The existence of edge states at zero energy reflects the nontrivial topology of the underlying quantum state. From the point of view of manipulating quantum information, these edge modes represent the maximally entangled states (MESs) of the system and its environment and are called topological maximally entangled states (tMESs). Therefore, they are the natural candidate for qubits with the potential advantage of stability due to their topological nature.

In order for tMESs to be a viable candidate for qubits, it is necessary to examine if they could survive under quantum information processing. Since typical quantum manipulations involve the rapid change of the coupling to the environments, it is therefore important to examine the quench dynamics of the tMESs. Recent investigations indicate that the thermalization of integrable systems due to quench depends strongly on the initial conditions [12–21]. These studies, however, are confined to the bulk properties. There are few papers concerning the quench dynamics of topological edge states [22, 23]. In the presence of topological edge states, the system can be maximally entangled with the environment and the thermalization of quench dynamics could be entirely different. It is thus important to examine the quench dynamics.
of the tMESs. In this paper, by taking dimerized chains as an example, we investigate how edge states affect the thermalization and the quench dynamics of one-particle entanglement spectra (OPES) as defined below. In particular, we show that the existence and evolution of tMESs are determined by the winding number of a pseudomagnetic field $S(k, \ell)$. The tMESs survive only if nontrivial Berry phases are induced by the winding of $S(k, \ell)$.

2. Entanglement and topological maximally entangled states

Consider the ground state $|\Psi_{AB}\rangle$ of a bipartite total system $AB$ that consists of the system $A$ and the environment $B$. The reduced density matrix of the system $A$ is $\rho_A = \operatorname{Tr}_B|\Psi_{AB}\rangle\langle\Psi_{AB}|$. The entanglement entropy (EE), defined as $S_A \equiv -\operatorname{Tr}\rho_A \log_2 \rho_A$, has been widely used to measure the bipartite entanglement between the system $A$ and the environment $B$ [24].

Maximally entangled states play an important role in the quantum information theory. For example, the system of a Bell state $|0\rangle_A |1\rangle_B + |1\rangle_A |0\rangle_B)/\sqrt{2}$, where $|0\rangle_A (|1\rangle_A)$ is a state in the system and $|0\rangle_B (|1\rangle_B)$ in the environment, can be represented by the reduced density matrix $\rho_A = 1/2(0\rangle_A 0\langle_A + 1\rangle_B 1\langle_B)$. The entanglement entropy $S_A = 1$. The system–environment implies that the system represents the property that EE diverges at the critical points provides (the quantum information theory. For example, the system equation (1).

It is known that the scaling law of EE provides a way to distinguish different quantum phases [26]. Furthermore, the property that EE diverges at the critical points provides a useful tool to examine the quantum criticality [27, 28]. In addition to global properties associated with EE, it is useful to explore detailed microscopic quantum phenomena using OPES, defined as the set of $\lambda_m$s with $\rho_A = \bigotimes_m \begin{pmatrix} \lambda_m & 0 \\ 0 & 1-\lambda_m \end{pmatrix}$. EE and OPES are related through the relation $S_A = \sum_m S_m$ where $S_m = -\lambda_m \log_2 \lambda_m - (1-\lambda_m) \log_2 (1-\lambda_m)$. The OPES have been used to investigate disorder lines [29], Berry phase [11, 30] and zero-energy edge states [11, 30]. It is clear that the eigenvalue $\lambda_m = 1/2$ corresponds to the situation when the system A and the environment B are maximally entangled so that $S_m = 1$.

To investigate the quench dynamics of the tMESs, we consider a one-dimensional (1D) dimerized chain characterized by the Hamiltonian

$$H_b = -\sum_i (1 + \delta (-1)^i) (c_i^\dagger c_{i+1} + \text{h.c.}),$$

(1)

where $i$ is the site index and $\delta \in [-1, 1]$. This model is known as the SSH model [31] with the ground state being a topological insulator in the BDI class [32–34]. For two infinite chains with $\delta = \pm |\delta|$, their ground states are degenerate in energy and are related by shifting a lattice constant. The degeneracy of these two ground states is lifted in the presence of an edge, exhibiting the following nontrivial topology: for a semi-infinite chain occupying $i \geq 1$, the ground state undergoes a phase transition from a topologically trivial phase ($\delta < 0$) to a topologically nontrivial phase ($\delta > 0$) as $\delta$ is varied across the phase boundary $\delta_c = 0$. Therefore, if the region $i \geq 1$ represents the system A and $i < 0$ represents the environment B, the topological edge mode appears if $\delta > 0$. Otherwise the edge mode does not exist [35–37]. In this paper we study the quench dynamics of the tMESs for a dimerized chain by suddenly quenching the parameter $\delta$.

The nontrivial topology of (1) has its origin in chiral symmetry. Consider an infinite chain, by defining a spinor $c_{x_1}$, $c_{x_2})$ where $x_1 = 2i - 1$ and $x_2 = 2i$ and performing a Fourier transformation of the corresponding bulk Hamiltonian (1), we find

$$H_b = -\sum_{k \in \text{BZ}} c_k^\dagger [\mathbf{R}_b(k) \cdot \mathbf{a}] c_k, $$(2)

where $\mathbf{a} = (\sigma_x, \sigma_y, \sigma_z)$ are Pauli matrices and

$$\mathbf{R}_b(k) = ((1-\delta) + (1+\delta) \cos k, (1+\delta) \sin k, 0)$$

(3)

is a pseudomagnetic field with the magnitude $R_b(k) = 2(\cos^2 k/2 + \delta^2 \sin^2 k/2)^{1/2}$. The system fulfills chiral symmetry because $\mathbf{R}_b(k)$ lies on a plane. For $\delta \in (0, 1]$ the closed loop $\ell$ of $\mathbf{R}_b(k)$ encloses the origin $O$ as $k$ runs through the Brillouin zone. Consequently, the Berry phase (or Zak’s phase) $\chi$, defined as a line integral of the curvature of the filled band, is $\pi$. Due to the fact that $\ell$ can be continuously deformed into a unit circle without crossing the origin, topological argument ensures that the original Hamiltonian corresponding to $\ell$ contains at least one pair of one zero-energy edge states as a consequence of chiral symmetry [10]. On the other hand, if $\delta \in [-1, 0)$, one obtains trivial Berry phases and no topological edge state occurs (see the upper panels of figure 1). The occurrence of tMESs has the same topological origin. Consider a total system of infinite size, and partition it into the system A with $x_1 = 1, \ldots, L/2$ and the environment B with $x_2$ to be the remaining part. The reduced density matrix of the system $\rho_A$ for the ground state can be determined
Figure 1. Upper two panels: representative loops $\ell$ of $R_\delta(k)$ for phase I and II with $\delta = 0.5$ and $\delta = -0.5$ respectively. In phase I ($\delta \in (0, 1]$) the Berry phase is $\pi$ due to circling of $R_\delta(k)$ around the origin, while in phase II ($\delta \in [-1, 0)$) the Berry phase is trivial. Lower four panels: evolutions of two $\lambda_m(t)$ that are closest to $1/2$ for different quench processes: (a) phase I ($\delta = 0.5$) to I ($\delta = 0.2$), (b) phase I ($\delta = 0.5$) to II ($\delta = -0.5$), (c) phase II ($\delta = -0.5$) to I ($\delta = 0.5$) and (d) phase II ($\delta = -0.5$) to II ($\delta = -0.2$). Insets: loop $\ell$ of $S_{\text{eff}}$.

by eigenvalues $\lambda_m$ of the block correlation function matrix (CFM) $[38–40]$ $G^{\alpha\beta}_{\alpha\beta}(x_i, x_j) = \text{Tr}(\rho c_{\alpha i}^\dagger c_{\beta j})$, where $G^\delta$ is the correlation function of the infinite chain with $x_\alpha$ and $x_\beta$ being evaluated at sites of the system $\Lambda, \alpha, \beta = 1, 2$, and $\rho$ is the ground state density matrix of the total system. Hence the CFM can be considered as an effective Hamiltonian that determines the OPES. In the Fourier $k$-space, $G^{\alpha\beta}_\delta(k)$ is given by

$$G^{\alpha\beta}_\delta(k) = \frac{1}{2}[1 + \hat{R}_\delta(k) \cdot \sigma_{\alpha\beta}], \quad (4)$$

where $\hat{R}_\delta(k) = R_\delta(k)/R_\delta(k)$ and $k$ takes continuous values in the Brillouin zone for (1). It takes almost the same form as the Hamiltonian (2) except for a constant and a positive normalization factor $R_\delta$, which leads to the conclusion that they share the same topology. If $R_\delta(k)$ encloses the origin in the parameter space with Berry phase equal to $\pi$, a pair of zero-energy edge states appears for the Hamiltonian (2), while for the CFM, we obtain a pair of tMESs with $\lambda_m = 1/2$. Therefore the states with $\lambda_m = 1/2$ are mid-gap states which are topologically protected. Notice that the tMESs are not only the MESs but also the edge states, which can be seen in figure 3.

3. Sudden quench

Consider now a sudden quench at $t = 0$ by changing the parameter of the dimerized chain in (1) from $\delta$ at $t < 0$ to $\delta'$ at $t > 0$. Using cold atoms and counter-propagating laser beams a fermionic dimerized chain can be experimentally realized by a superlattice with two standing laser waves where one of the standing waves has twice the frequency of the other one [41]. One can load fermions into the superlattice [42] and use the Feshbach resonance to reduce the atom–atom...
interaction down to zero. The sudden quench can actually be fulfilled by a sudden change of a phase $\pi$ of a standing wave.

Denoting the phase whose $\delta \in (0, 1]$ with tMESs as phase I, and the phase whose $\delta \in [-1, 0)$ as phase II, we perform four possible quench processes, including I to I, I to II, II to I, and II to II. The OPES $\lambda_{\text{op}}(t)$ at time $t$ can be obtained by diagonalizing a time-dependent CFM which is defined as $G_{\text{eff}}(x_i, x_j, t) = \text{Tr}(\rho^{tH_{\text{I}}^{t}}c_{x_i}c_{x_j}e^{-iH_{\text{I}}t})$ where $x_i, x_j \in A$ [43]. In figures 1(a)–(d), we show the time evolution of OPES for states whose eigenvalues are closest to 1/2. In the cases of I to I and I to II, the tMESs persist for a while and then split into two different evolutions. Only in the case of I to I, however, the two splitting eigenstates evolve back to two tMESs at infinite time. In contrast, for the case of I to II, the two splitting eigenstates closest to 1/2 remain splitting forever. On the other hand, if one starts with initial states without tMESs such as the cases of II to I and II to II, the tMESs cannot be created at a later time.

It is instructive to define a time-dependent pseudomagnetic field $S(k, t)$ from the time-dependent CFM $G(k, t)$ in the Fourier space through the relation

$$G_{\text{eff}}(k, t) = 1/2[1 + S(k, t) \cdot \sigma]_{\text{I} \beta}. \tag{5}$$

$S(k, t)$ can be further written as a summation of three different contributions

$$S_1(k, t) = \cos(2R_{\text{I}}t)\hat{R}_{\text{I}}(k), \tag{6}$$

$$S_2(k, t) = \sin(2R_{\text{I}}t)\hat{R}_{\text{I}}(k) \times \hat{R}_{\text{I}}(k), \tag{7}$$

$$S_3(k, t) = (1 - \cos(2R_{\text{I}}t))\hat{R}_{\text{I}}(k) \cdot \hat{R}_{\text{I}}(k)\hat{R}_{\text{I}}(k). \tag{8}$$

We refer to the appendix for the detailed derivations of the above equations. The thermalization of EE observed in figure 1 can be understood by considering the infinite-time limit of the time-dependent pseudomagnetic field (6)–(8). Since the sinusoidal parts dephase out, the long-time behavior of the time-dependent CFM, $G_{\text{eff}}(k, t = \infty)$, is solely determined by the effective pseudomagnetic field

$$S_{\text{eff}}(k) = \langle \hat{R}_{\text{I}}(k) \cdot \hat{R}_{\text{I}}(k) \rangle\hat{R}_{\text{I}}(k). \tag{9}$$

The existence of the tMESs at infinite time is thus determined by the topology of $S_{\text{eff}}$. If $S_{\text{eff}}$ encircles origin, the Berry phase is $\pi$, the tMESs appear at $t = \infty$; otherwise there are no tMESs at a later time. It is clear that both the final $R_{\text{I}}$ and the initial $R_{\text{I}}$ determine $S_{\text{eff}}$. This implies that the long-time behavior carries the memory of the initial state, which is due to nonergodicity of integrable systems [19, 20]. The existence of the tMESs at infinite time requires that both the initial and final Hamiltonians possess nontrivial Berry phase. In the inset of figures 1(a)–(d), $S_{\text{eff}}$ is plotted for different quench processes with different initial and final Hamiltonians. It is clear that only for the quench from I to I, $S_{\text{eff}}$ encircles the origin, while in the other quenches $S_{\text{eff}}$ either does not encircle the origin (II to II) or passes through the origin (I to II or II to I).

The reason is that if the initial state is in the same phase as the final Hamiltonian (I to I or II to II), $R_{\text{I}}(k) \cdot R_{\text{I}}(k)$ is always positive, therefore the topology of $S_{\text{eff}}$ is the same as $R_{\text{I}}(k)$. On the other hand, if the system is quenched into a phase that is distinctly from the initial state, $R_{\text{I}}(k) = 0 \cdot R_{\text{I}}(k) = 0$ is positive while $R_{\text{I}}(k) \cdot R_{\text{I}}(k) = \pi$ is negative, then there must exist one point $k_0$ that $S_{\text{eff}}(k_0) \propto R_{\text{I}}(k_0) \cdot R_{\text{I}}(k_0) = 0$. Hence, $k_0$ destroys the topology of the infinite-time correlation function matrix. There exist no tMESs whenever the system is quenched across the topological phase boundary. This explains why only the quench from phase I to phase I creates the tMESs at infinite time.

4. Discussion and summary

For all cases shown in figure 1, the OPES $\lambda_{\text{op}}(t)$ of the dimerized chain fluctuate before it reaches the equilibrium. The intermediate regions can be explained by the appearance of $S_2$ in the time-dependent CFM. Since $S_2$ is proportional to $R_{\text{I}}(k) \times R_{\text{I}}(k)$ and is perpendicular to $S_1$ and $S_3$, the system is agitated by quasiparticles induced by $S_2$ and $S_3$ until the time-dependent sinusoidal functions dephase out and then the system reaches its equilibrate state with the effective $S_{\text{eff}}$ (9). Clearly, the thermalization depends on sizes of the system A. To check the size dependence, figure 2 shows several quench processes from phase I to phase II with different system sizes with different colors. At the first glance, one finds that the eigenvalues of maximally entangled states split out after a time $T^*$ and then after a long time the eigenvalues reach different asymptotic constants. In the lower panel, we compare eigenvalues at $t = 1000$ (red dots) evolved from the tMESs with eigenvalues obtained by diagonalization of $G_{\text{eff}}(k, \infty)$ (blue circles). The dependence on the system
size $L$ shows excellent agreement, indicating the validity of $S_{eff}$. It is interesting to notice that the two processes $I \rightarrow II$ and $II \rightarrow I$ are not invertible. This phenomenon is reflected in (9), which shows that the effective pseudomagnetic fields at $t = \infty$ for two processes are different. The reason can be traced back to the non-commutability of $H_\delta$ and $H_{\delta'}$. If one uses the Boltzmann factor to project out the ground state at $t = 0$, the final state of infinite time can be expressed as $\lim_{t \rightarrow \infty} \lim_{\beta \rightarrow \infty} e^{-i H_\delta t} e^{-H_{\delta'} / \beta} |\phi\rangle$, where $\phi$ is an arbitrary state. Clearly, to exchange $\delta$ and $\delta'$ does not lead to the same state. Therefore, the two processes are not invertible.

In figure 2, as discussed above, we also find that if initially the system is in phase I, the tMESs can persist for a long time $T^*$ before they are destroyed. This feature is striking because the tMESs are not destroyed in the beginning. Furthermore $T^*$ is independent of the initial conditions but depends on the final Hamiltonian and the system length $L$. This is because the tMESs reside on the edge [11] and they will disappear only when the quasiparticles created from the bulk reach the edges. Therefore, we expect $T^* \approx L / (2v_{\text{max}})$, where $v_{\text{max}}$ is the maximum velocity of the quasiparticles. For the energy dispersion one obtains $v_{\text{max}} = 2(1 - |\delta'|)$ hence

$$T^* \approx \frac{L}{4(1 - |\delta'|)}. \quad (10)$$

In the upper left inset of figure 2, we show the fitting of $T^*(L)$ to the function $a(L)/(1 - |\delta'|)$ using various system size $L$. We then fit $a(L) = b \times L$ as shown in the upper right inset of figure 2 to find $b \approx 0.232$. Combining these two fittings we find that $T^* \approx 0.232L/(1 - |\delta'|)$ which is very close to our approximation (10). Our results imply that edge modes remain to the infinite time if $L$ is infinite. Therefore, the edge modes serve as good candidates for qubits since they have the maximal entanglements and cannot be destroyed easily if the system is large enough.

The disappearance and recreation for different quench processes starting from phase I can be further explored by examining the probability sum of the two eigenstates with eigenvalues being closest to $1/2$. Figure 3(a) shows the case of I to I. It is seen that only small amounts of quasiparticles are excited. As a result, the edge modes remain from the beginning to the end. On the other hand, for the quench process I to II as shown in figure 3(b), quasiparticles strongly modify two edge modes until they disappear and the whole system is bulk-like without any edge mode.

In the above discussions, we assume that the dimerized chain is isolated. However, in reality, there must be interactions between the dimerized chain and its surroundings. Therefore, to examine the stability of qubits, one needs to consider interactions between qubits and the environment surrounding the dimerized chain. Furthermore, in practice, in order to construct a viable qubit, a weak perturbation between the dimerized chain and its surroundings is assumed. This is where the topology comes into play. The maximally entangled state discussed above has its topological origin, characterized by the winding number of the dimerized chain. The existence of such a state is protected by topology. It is known that as long as the interaction respects symmetries of the dimerized chain, the maximally entangled state exists and is protected. This is often true when the interaction with its surrounding is weak. Therefore, qubits are topologically protected and are stable under the weak interaction between qubits and environment.

![Figure 3](image-url)
In summary, using dimerized chains as an example, the quench dynamics of the tMESs is investigated by diagonalizing the time-dependent correlation function matrix. We find that the existence of the tMESs after long time quench is determined by an effective pseudomagnetic field \( \mathbf{S}_{\text{eff}} \), which depends on both the initial and final Hamiltonians. The topological properties at infinite time are thus determined by the initial states and the Hamiltonian after the quench. When the tMESs are unstable, they are destroyed by quasiparticles that move from the bulk to the edges with a characteristic timescale proportional to the system size.

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Appendix. The time-dependent correlation function matrix

In order to obtain the time-dependent correlation function matrix (5)–(8), one has to diagonalize \( H_k \) (2) in the Fourier space

\[
- \mathbf{R}_k \cdot \mathbf{\sigma} = R_k \sigma_z V_k \tag{A.1}
\]

where

\[
V_k = \frac{\mathbf{R}_k \cdot \mathbf{\sigma} - R_k \sigma_z}{\sqrt{2R_k(R_k - R_k^*})} \tag{A.2}
\]

Defining \( \alpha_\pm = (\alpha_+, \alpha_-) \) which diagonalizes \( H_k \), that means,

\[
H_k \sigma_\pm = \pm R_k \sigma_\pm \tag{A.3}
\]

The time-dependent CFM after a sudden quench in the Fourier space can be calculated as follows:

\[
\tilde{G}(k, t) = 1 - \tilde{G}(k, t) = \text{Tr}(\rho e^{iH_k t} \sigma_\pm e^{-iH_k t}). \tag{A.4}
\]

One should canonically transform \( \alpha \) twice to the proper operators using (A.2) and obtain

\[
\tilde{G}(k, t) = V_k e^{iR_k \sigma_i \sigma_j} V_k^\dagger \text{Tr}(\rho \sigma_\pm \sigma_j e^{-iR_k \sigma_i \sigma_j} V_k^\dagger V_k). \tag{A.5}
\]

For zero temperature, \( \rho \) is the density matrix of the ground state, therefore

\[
\text{Tr}(\rho \sigma_\pm \sigma_j) = \frac{1}{2} - \frac{R_k \cdot \mathbf{\sigma}}{2R_k}. \tag{A.6}
\]

Substituting (A.6) and (A.2) into (A.5) and using the properties of Pauli matrices, we obtain (5) with the relation (6)–(8).