Automatic realization of Hopf Galois structures

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Abstract

We consider Hopf Galois structures on a separable field extension $L/K$ of degree $p^4$, for $p$ an odd prime number. For $p > 3$, we prove that $L/K$ has at most one abelian type of Hopf Galois structures. For several nonabelian groups $N$ of order $p^4$ we prove that if $L/K$ has a Hopf Galois structure of type $N$, then it has a Hopf Galois structure of type $A$, an abelian group of the same exponent as $N$.

1 Introduction

A Hopf Galois structure on a finite extension of fields $L/K$ is a pair $(H,\mu)$, where $H$ is a finite cocommutative $K$-Hopf algebra and $\mu$ is a Hopf action of $H$ on $L$, i.e. a $K$-linear map $\mu : H \to \text{End}_K(L)$ giving $L$ a left $H$-module algebra structure and inducing a $K$-vector space isomorphism $L \otimes_K H \to \text{End}_K(L)$. Hopf Galois structures were introduced by Chase and Sweedler in [5]. For separable field extensions, Greither and Pareigis [10] give the following group-theoretic equivalent condition to the existence of a Hopf Galois structure.

Theorem 1. Let $L/K$ be a separable field extension of degree $g$, $\bar{L}$ its Galois closure, $G = \text{Gal}(\bar{L}/K), G' = \text{Gal}(\bar{L}/L)$. Then there is a bijective correspondence between the set of isomorphism classes of Hopf Galois structures on $L/K$ and the set of regular subgroups $N$ of the symmetric group $S_g$ normalized by $\lambda_G(G)$, where $\lambda_G : G \hookrightarrow S_g$ is the monomorphism given by the action of $G$ on the left cosets $G/G'$.

For a given Hopf Galois structure on a separable field extension $L/K$ of degree $g$, we will refer to the isomorphism class of the corresponding group $N$ as the type of the Hopf Galois structure. Given a regular subgroup $N$ of $S_g$, normalized by $\lambda_G(G)$, the corresponding Hopf Galois structure $(H,\mu)$ is obtained by Galois descent.

Childs [6] gives an equivalent condition to the existence of a Hopf Galois structure introducing the holomorph of the regular subgroup $N$ of $S_g$. Let $\lambda_N : N \to \text{Sym}(N)$ be the morphism given by the action of $N$ on itself by left translation. The holomorph $\text{Hol}(N)$ of $N$ is the normalizer of $\lambda_N(N)$ in $\text{Sym}(N)$. As abstract groups, we have $\text{Hol}(N) = N \rtimes \text{Aut}(N)$. We state the more precise formulation of Childs’ result due to Byott [3] (see also [7] Theorem 7.3).

Theorem 2. Let $G$ be a finite group, $G' \subset G$ a subgroup and $\lambda_G : G \to \text{Sym}(G/G')$ the morphism given by the action of $G$ on the left cosets $G/G'$. Let $N$ be a group of order $[G : G']$ with identity element $e_N$. Then there is a bijection between

$$\mathcal{N} = \{ \alpha : N \hookrightarrow \text{Sym}(G/G') \text{ such that } \alpha(N) \text{ is regular} \}$$

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and
\[ G = \{ \beta : G \hookrightarrow \text{Sym}(N) \text{ such that } \beta(G') \text{ is the stabilizer of } e_N \} \]

Under this bijection, if \( \alpha, \alpha' \in \mathcal{N} \) correspond to \( \beta, \beta' \in G \), respectively, then \( \alpha(N) = \alpha'(N) \) if and only if \( \beta(G) \) and \( \beta'(G) \) are conjugate by an element of \( \text{Aut}(N) \); and \( \alpha(N) \) is normalized by \( \lambda_G(G) \) if and only if \( \beta(G) \) is contained in the holomorphic \( \text{Hol}(N) \) of \( N \).

Recently a relationship has been found between Hopf Galois structures and an algebraic structure called brace. Classical braces were introduced by W. Rump \[12\], as a generalisation of radical rings, in order to study the non-degenerate involutive set-theoretic solutions of the quantum Yang-Baxter equation. Recently, skew braces were introduced by Guarnieri and Vendramin \[11\] in order to study the non-degenerate (not necessarily involutive) set-theoretic solutions. This connection is further exploited in \[13\], where the relation of braces with other algebraic structures is established.

**Definition 3.** A left brace is a set \( B \) endowed with two binary operations \( \cdot \) and \( \circ \) such that \((B, \cdot)\) and \((B, \circ)\) are groups and the two operations are related by the brace property

\[ a \circ (b \cdot c) = (a \circ b) \cdot a^{-1} \cdot (a \circ c), \text{ for all } a, b, c \in B, \]

where \( a^{-1} \) denotes the inverse of \( a \) in \((B, \cdot)\). The groups \((B, \cdot)\) and \((B, \circ)\) are called respectively the additive group and the multiplicative group of the brace \( B \). The brace is called classical when its additive group is abelian, skew otherwise.

A map between braces is a brace morphism if it is a group morphism both between the additive and the multiplicative groups.

The relation between braces and Hopf-Galois structures was first proved by Bachiller for classical braces (see \[11\] Proposition 2.3) and generalized by Guarnieri and Vendramin to skew braces.

**Proposition 4** (\[11\] Proposition 4.3). Let \((N, \cdot)\) be a group. There is a bijective correspondence between isomorphism classes of left braces with additive group isomorphic to \((N, \cdot)\) and classes of regular subgroups of \( \text{Hol}(N) \) under conjugation by elements of \( \text{Aut}(N) \).

For a finite separable field extension \( L/K \) we denote by \( \widetilde{L} \) a normal closure of \( L/K \), by \( G' \) the Galois group of \( \widetilde{L}/L \). We shall call a pair of groups \((G, N)\) realizable if a separable field extension \( L/K \) of degree \(|N|\) such that \( \text{Gal}(\widetilde{L}/K) \simeq G \) has a Hopf Galois structure of type \( N \). By Theorems \[1\] and \[2\] a pair of groups \((G, N)\), such that \( G \) has a subgroup \( G' \) with \(|G : G'| = |N|\), is realizable if and only if there exists a group monomorphism \( \beta : G \hookrightarrow \text{Hol}(N) \) such that \( \beta(G') \) is the stabilizer of \( e_N \). In particular a pair of groups \((G, N)\) with \(|G| = |N|\) is realizable if a Galois field extension \( L/K \) with Galois group isomorphic to \( G \) has a Hopf Galois structure of type \( N \). In this case, by Theorem \[2\] and Proposition \[1\] \((G, N)\) is realizable if and only if there exists a left brace \( B \) with additive group isomorphic to \( N \) and multiplicative group isomorphic to \( G \).

In this paper we obtain that if a pair of groups \((G, N)\) is realizable where \( N \) is an abelian group of order \( p^4 \), with \( p \) an odd prime number, then no pair \((G, N')\) is realizable, where \( N' \) is an abelian group of order \( p^4 \) nonisomorphic to \( N \). This generalizes a result in \[4\]. We also prove that for some nonabelian groups \( N \) of order \( p^4 \), if a pair of groups \((G, N)\) is realizable, then \((G, A)\) is realizable, where \( A \) is an abelian group of order \( p^4 \) with same exponent as \( N \).
2 Groups of order $p^4$

For $p$ an odd prime, there are fifteen groups of order $p^4$, up to isomorphism, which are described in [2] V §73. From these, five are abelian, which are

$$N_1 = C_{p^4}, \quad N_2 = C_{p^3} \times C_p, \quad N_3 = C_{p^2} \times C_{p^2}, \quad N_4 = C_{p^2} \times C_p \times C_p, \quad N_5 = C_p \times C_p \times C_p \times C_p.$$  

Among the nonabelian ones, there is one group of exponent $p^3$ with presentation

$$N_6 = \langle a, b \mid a^{p^3} = 1, b^p = 1, b^{-1}ab = a^{1+p^2} \rangle \cong C_{p^3} \times C_p,$$

seven groups of exponent $p^2$ with presentations

$$N_7 = \langle a, b, c \mid a^{p^2} = 1, b^p = 1, c^p = 1, c^{-1}bc = ba^p, c^{-1}ac = a, b^{-1}ab = a \rangle,$$

$$N_8 = \langle a, b \mid a^{p^2} = 1, b^p = 1, b^{-1}ab = a^{1+p} \rangle,$$

$$N_9 = \langle a, b, c \mid a^{p^2} = 1, b^p = 1, c^p = 1, c^{-1}bc = b, c^{-1}ac = a^{1+p}, b^{-1}ab = a \rangle,$$

$$N_{10} = \langle a, b, c \mid a^{p^2} = 1, b^p = 1, c^p = 1, c^{-1}bc = b, c^{-1}ac = ab, b^{-1}ab = a \rangle,$$

$$N_{11} = \langle a, b, c \mid a^{p^2} = 1, b^p = 1, c^p = 1, c^{-1}bc = b, c^{-1}ac = ab, b^{-1}ab = a^{1+p} \rangle,$$

$$N_{12} = \langle a, b, c \mid a^{p^2} = 1, b^p = 1, c^p = 1, c^{-1}bc = a^p b, c^{-1}ac = ab, b^{-1}ab = a^{1+p} \rangle,$$

$$N_{13} = \langle a, b, c \mid a^{p^2} = 1, b^p = 1, c^p = 1, c^{-1}bc = a^p b, c^{-1}ac = ab, b^{-1}ab = a^{1+p} \rangle,$$

where $\alpha$ is any nonresidue mod $p$.

and two groups of exponent $p$ with presentations

$$N_{14} = \langle a, b, c, d \mid a^p = 1, b^p = 1, c^p = 1, d^p = 1, d^{-1}cd = ac, d^{-1}bd = b, d^{-1}ad = a,$$

$$c^{-1}bc = b, c^{-1}ac = a, b^{-1}ab = a \rangle,$$

$$N_{15} = \langle a, b, c, d \mid a^p = 1, b^p = 1, c^p = 1, d^p = 1, d^{-1}cd = cb, d^{-1}bd = ba, d^{-1}ad = a,$$

$$c^{-1}bc = b, c^{-1}ac = a, b^{-1}ab = a \rangle.$$

We note that, for $p = 3$, the groups $N_{11}, N_{12}, N_{13}$ and $N_{15}$ must be defined differently, namely,

$$N_{11} = \langle a, b, c \mid a^9 = 1, b^3 = 1, c^3 = 1, c^{-1}bc = b, c^{-1}ac = ab, b^{-1}ab = a^4 \rangle,$$

$$N_{12} = \langle a, b, c \mid a^9 = 1, b^3 = 1, c^3 = a^3, c^{-1}bc = b, c^{-1}ac = ab^{-1}, b^{-1}ab = a^4 \rangle,$$

$$N_{13} = \langle a, b, c \mid a^9 = 1, b^3 = 1, c^3 = a^{-3}, c^{-1}bc = b, c^{-1}ac = ab^{-1}, b^{-1}ab = a^4 \rangle,$$

$$N_{15} = \langle a, b, c, d \mid a^9 = 1, b^3 = 1, c^3 = 1, c^{-1}bc = a^{-3}b, c^{-1}ac = ab, b^{-1}ab = a \rangle.$$

In particular, there is just one nonabelian group of order 81 and exponent 3.

3 Hopf Galois structures of abelian type

Let $p$ denote an odd prime number. We proved in [8] Proposition 4 that if a separable extension of degree $p^n$ has a Hopf Galois structure of cyclic type, then it has no structure of noncyclic type. In the case of separable extensions of degree $p^3$, we obtained in [8] Theorem 9 that, for $p > 3$, the two abelian noncyclic types of Hopf Galois structures do not occur on the same
extension. In this section we prove that two different abelian types of Hopf Galois structures do not occur on a separable extension of degree \( p^4 \), for \( p > 3 \). For a Galois extension of degree \( p^4 \), with \( p > 3 \), this result is obtained by applying [4], Theorem 1, where the authors prove that if \((N, +)\) is a finite abelian \( p \)-group of \( p \)-rank \( m \) where \( m + 1 < p \), then every regular abelian subgroup of the holomorphic of \( N \) is isomorphic to \( N \). As a consequence we obtain that two classical braces of order \( p^4 \) with isomorphic multiplicative group must have isomorphic additive groups.

We determine first the automorphism groups of \( N_2, N_3, N_4 \) and \( N_5 \) and their \( p \)-Sylow subgroups.

1) Let \( N_2 = C_{p^3} \times C_p = \langle a \rangle \times \langle b \rangle \). An automorphism \( \varphi \) of \( N_2 \) is given by

\[
\varphi : \begin{align*}
a &\mapsto a^mb^n, \quad p \nmid m \\
b &\mapsto a^{p^2}b^j, \quad p \nmid j
\end{align*}
\]

Hence \( |\text{Aut}(N_2)| = p^4(p - 1)^2 \). Let us consider the automorphisms

\[
\varphi_1 : a \mapsto a^{1+p}, \quad \varphi_2 : a \mapsto ab \quad \varphi_3 : a \mapsto a
\]

which have orders \( p^2, p, p \), respectively. Moreover, they satisfy the relations \( \varphi_1 \varphi_2 = \varphi_2 \varphi_1 \), \( \varphi_1 \varphi_3 = \varphi_3 \varphi_1 \varphi_3^{-1} = \varphi_2 \varphi_2 \), hence generate a subgroup of \( \text{Aut}(N_2) \) of order \( p^4 \) and exponent \( p^2 \) isomorphic to \( N_7 \). We may check that \( \langle \varphi_1, \varphi_2, \varphi_3 \rangle \) is normal in \( \text{Aut}(N_2) \) hence is the only \( p \)-Sylow subgroup of \( \text{Aut}(N_2) \).

2) Let \( N_3 = C_{p^2} \times C_{p^2} = \langle a \rangle \times \langle b \rangle \). An automorphism \( \varphi \) of \( N_3 \) is given by

\[
\varphi : \begin{align*}
a &\mapsto a^ib^j, \\
b &\mapsto a^{p^2}b^j, \quad p \nmid il - jk
\end{align*}
\]

Hence \( |\text{Aut}(N_3)| = p^5(p + 1)(p - 1)^2 \). Let us consider the automorphisms

\[
\varphi_1 : a \mapsto ab \quad \varphi_2 : a \mapsto a \quad \varphi_3 : a \mapsto a^{1+p}
\]

which have orders \( p^2, p, p \), respectively. Let \( \psi := \varphi_1^{-1} \varphi_2^{-1} \varphi_1 \varphi_2 \). We may check that \( \psi \) has order \( p \), and that the following relations are fulfilled: \( \psi \psi_1 \psi = \varphi_1^{1-2p}, \psi \varphi_2 = \varphi_2 \psi \). Hence the subgroup \( \langle \varphi_1, \varphi_2 \rangle \) is isomorphic to \( N_{11} \). Now \( \varphi_3 \) satisfies \( \varphi_3 \varphi_1 \varphi_3^{-1} = \varphi_1^{1-p}, \varphi_2 \varphi_3 = \varphi_3 \varphi_2 \), hence the subgroup \( \langle \varphi_1, \varphi_2, \varphi_3 \rangle = \langle \varphi_1, \varphi_2 \rangle \times \langle \varphi_3 \rangle \) has order \( p^5 \) and is then a \( p \)-Sylow subgroup of \( \text{Aut}(N_3) \).

3) Let \( N_4 = C_{p^2} \times C_{p} \times C_p = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \). An automorphism \( \varphi \) of \( N_4 \) is given by

\[
\varphi : \begin{align*}
a &\mapsto a^ib^jc^k, \quad p \nmid i \\
b &\mapsto a^{p^m}b^p c^l, \\
c &\mapsto a^{p^r}b^s c^t, \quad p \nmid nt - qs
\end{align*}
\]

Hence \( |\text{Aut}(N_4)| = p^6(p + 1)(p - 1)^3 \). Let us consider the automorphisms of order \( p \).
\[
\begin{aligned}
\varphi_1 &: a \mapsto a^{1+p} & \varphi_2 &: a \mapsto ab & \varphi_3 &: a \mapsto ac \\
b &\mapsto b & b &\mapsto b & b &\mapsto b \\
c &\mapsto c & c &\mapsto c & c &\mapsto c
\end{aligned}
\]

\[
\varphi_4 &: a \mapsto a & \varphi_5 &: a \mapsto a & \varphi_6 &: a \mapsto a
\] 
\[
b &\mapsto a^pb & b &\mapsto bc & b &\mapsto b \\
c &\mapsto c & c &\mapsto c & c &\mapsto a^pc
\]

\[\text{(3)}\]

We may check that \(\varphi_1, \varphi_2, \varphi_3, \varphi_4\) generate a group isomorphic to \(N_{14}\), that this group is normalized by \(\varphi_5\) and that \(\varphi_6\) normalizes \(\langle \varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5 \rangle\). Hence the six automorphisms together generate a \(p\)-Sylow subgroup of \(\text{Aut}(N_4)\).

4) Let \(N_5 = C_p^4\). Then \(\text{Aut}(N_5) \cong \text{GL}(4, p)\) and \(|\text{Aut}(N_5)| = p^6(p^4 - 1)(p^3 - 1)(p^2 - 1)(p - 1)\). We note that the subgroup \(U\) of upper unitriangular matrices is a \(p\)-Sylow subgroup of \(\text{GL}(4, p)\) and has exponent \(p\).

In the following lemmas, we study the \(p\)-Sylow subgroups of the holomorphs of \(N_3, N_4\) and \(N_5\).

**Lemma 5.** Let \(N_3 = C_p \times C_p^2 = \langle a, b \rangle\), with \(p\) an odd prime number, and \(H\) denote its holomorph. Let \(\text{Syl}_p(H)\) denote a \(p\)-Sylow subgroup of \(H\). Then \(\text{Syl}_p(H)\) has exponent \(p^2\). If \(p > 3\), the elements of order \(\leq p\) in \(\text{Syl}_p(H)\) form a subgroup of order \(p^6\).

**Proof.** We may take \(\text{Syl}_p(H) = \langle a, b, \varphi_1, \varphi_2, \varphi_3 \rangle\), with \(\varphi_1, \varphi_2, \varphi_3\) as in (2). Let \(\psi = \varphi_1^{-1} \varphi_2^{-1} \varphi_1 \varphi_2\).

We may write any element \(\varphi \in \text{Syl}_p(H)\) as \(\varphi = \varphi_1^i \varphi_2^j \varphi_3^k, 1 \leq i \leq p^2, 1 \leq j, k, l \leq p\). By computation, we obtain

\[
\varphi(a) = a^{1+(l-j)p} b^{i+p(l-i+j)} \\
\varphi(b) = a^{pk} b^{1+p(l+j-ik)}
\]

and, for \(n \geq 1\),

\[
\varphi^n(a) = a^{1+n(l-j)p+n(n-1)ipk/2} b^{n+2(n-1)id/2-ni+j+nj+n(n-1)n(n-2)k/6} \\
\varphi^n(b) = a^{npk} b^{n+2(n-1)ik/2}.
\]

Now we have \((p - 1)p(p + 1)/6 = 4\), for \(p = 3\), and \(p \mid (p - 1)p(p + 1)/6\), for \(p > 3\). Hence, we have

\[
\varphi^p(a) = ab^{p^i}, \quad \varphi^p(b) = b, \quad \text{if } p > 3, \\
\varphi^3(a) = a^{3^{i+3}j^{k}}, \quad \varphi^3(b) = b, \quad \text{if } p = 3,
\]

hence \(\varphi^{p^2} = \text{Id}\). Now, if \((z, \varphi) \in \text{Syl}_p(H)\), we have \((z, \varphi)^p = (z\varphi(z) \ldots \varphi^{p-1}(z), \varphi^p)\). By computation, we obtain

\[
a \varphi(a) \ldots \varphi^{-1}(a) = a^{p(1+p(p-1)(l-j)/2+(p-2)(p-1)pk/6)} b^{p(1+p(p-1)(l-j)/2+(p-2)(p-1)pk/6)}
\]

\[
= \begin{cases} 
  a^{pb^{(p-1)i/2}} & \text{if } p > 3 \\
  a^{3^{i+3}j^{k^{i+3l+3i}}} & \text{if } p = 3
\end{cases}.
\]

\[
b \varphi(b) \ldots \varphi^{-1}(b) = a^{2^p(p-1)k/2} b^{2^p(p-1)k/2}.
\]

\[
= \begin{cases} 
  b^p & \text{if } p > 3 \\
  b^{3^{i+3}k} & \text{if } p = 3
\end{cases}.
\]
For $z = a^* b^*$, we have then for any odd prime $p$, $(z, \varphi)^p \in \langle a^p, b^p \rangle \times \langle \varphi_1^p \rangle$ and $(z, \varphi)^{p^2} = (1, \text{Id})$, since $\varphi_1^p(a^p) = a^p$ and $\varphi_1(b) = b$. We have then proved that $\text{Syl}_p(H)$ has exponent $p^2$.

Moreover, for $p > 3$, we have $(z, \varphi)^p = (a^{p(p^p+rp(p-1)/2)}, \varphi_1^p) = (1, \text{Id})$ if and only if $p \mid i$ and $z$ has order $\leq p$. Since $\varphi_1^p, \psi, \varphi_2, \varphi_3$ generate a subgroup of $\text{Aut}(N_5)$ isomorphic to $C_4^4$, we obtain that the elements of order $\leq p$ in $H$ form a subgroup of order $p^6$.

**Lemma 6.** Let $N_4 = C_p^2 \times C_p \times C_p = \langle a, b, c \rangle$, with $p$ an odd prime number, and $H$ denote its holomorph. Let $\text{Syl}_p(H)$ denote a $p$-Sylow subgroup of $H$. Then $\text{Syl}_p(H)$ has exponent $p^2$. If $p > 3$, the elements of order $\leq p$ in $\text{Syl}_p(H)$ form a subgroup of order $p^6$.

**Proof.** We may take $\text{Syl}_p(H) = \langle a, b, c, \varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5, \varphi_6 \rangle$, with $\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5, \varphi_6$ as in [3]. For $\varphi := \varphi_1^i \varphi_2^j \varphi_3^k \varphi_4^l \varphi_5^m \varphi_6^n, n \geq 1$, we obtain

\[
\varphi^n : a \mapsto a^{1+ni+np+n(n-1)i2i4p/2+n(n-1)i3i6p/2p^2ni2c(n(n-1)i2i5/2+n3i5}
\]

\[
b \mapsto a^{ni+np+n(n-1)i3i6p/2p^2ni5}
\]

\[
c \mapsto a^{ni+pc}
\]

Hence $\varphi^p = \text{Id}$. Now, we obtain

\[
a\varphi(a) \ldots \varphi^{p-1}(a) = a^p, \quad b\varphi(b) \ldots \varphi^{p-1}(b) = 1, \quad c\varphi(c) \ldots \varphi^{p-1}(c) = 1, \quad \text{if } p > 3,
\]

\[
a\varphi(a)\varphi^2(a) = a^{3(1+i2i4+i3i6)c^2i2i5}, \quad b\varphi(b)\varphi^2(b) = a^{3i6i5}b, \quad c\varphi(c)\varphi^2(c) = c, \quad \text{if } p = 3,
\]

hence, for $z = a^{m_1} b^{m_2} c^{m_3}$, we have $(z, \varphi)^{p^2} = (1, \text{Id})$. We obtain then that $\text{Syl}_p(H)$ has exponent $p^2$.

If $p > 3$, $(z, \varphi)^p = (a^{pm_1}, \text{Id})$, hence $(z, \varphi)$ has order $p$ if and only if $p \mid m_1$. Then the elements in $H$ of order $p$ form a subgroup of order $p^9$.

**Lemma 7.** Let $N_5 = C_p^4$, for $p$ an odd prime number, and $H$ denote its holomorph. Let $\text{Syl}_p(H)$ denote a $p$-Sylow subgroup of $H$. Then $\text{Syl}_p(H)$ has exponent $p$.

**Proof.** If $A$ is an upper unitriangular matrix in $\text{GL}(4, p)$, we have $\text{Id} + A + \cdots + A^{p-1} = 0$, hence if $\varphi \in \text{Aut}(N_5)$ corresponds to $A$ by the isomorphism from $\text{GL}(4, p)$ to $\text{Aut}(N_5)$, we have $z\varphi(z) \ldots \varphi^{p-1}(z) = 1$ for any $z \in N_5$. We have then that $H$ has exponent $p$.

We shall also need to consider the $p$-Sylow subgroup of a transitive subgroup of the holomorph of a group of order $p^4$.

**Lemma 8.** Let $G$ be a subgroup of $\text{Hol}(N)$, for $N$ a group of order $p^4$, where $p$ is an odd prime number. Then $G$ is transitive if and only if $\text{Syl}_p(G)$ is transitive.

**Proof.** Clearly, if $G$ is a subgroup of $\text{Hol}(N)$, then $\text{Syl}_p(G)$ is a subgroup of $\text{Syl}_p(\text{Hol}(N))$. Now, $G$ is transitive if and only if $[G : G \cap \text{Stab}_{\text{Hol}(N)}(0)] = p^4$. We have the following equalities between indices.

\[
[G : \text{Syl}_p(G) \cap \text{Stab}_{\text{Hol}(N)}(0)] = [G : \text{Syl}_p(G)] [\text{Syl}_p(G) : \text{Syl}_p(G) \cap \text{Stab}_{\text{Hol}(N)}(0)]
\]

\[
= [G : G \cap \text{Stab}_{\text{Hol}(N)}(0)] [G \cap \text{Stab}_{\text{Hol}(N)}(0) : \text{Syl}_p(G) \cap \text{Stab}_{\text{Hol}(N)}(0)].
\]

Since $[G : \text{Syl}_p(G)]$ and $[G \cap \text{Stab}_{\text{Hol}(N)}(0) : \text{Syl}_p(G) \cap \text{Stab}_{\text{Hol}(N)}(0)]$ are prime to $p$, we obtain that, $G$ is transitive if and only if $\text{Syl}_p(G)$ is transitive.
Proposition 9. Let $G$ be a group, $p$ an odd prime number.

1) Let $A$ be an abelian group of order $p^4$. If the pairs $(G, N_2)$ and $(G, A)$ are realizable, then $A \simeq N_2$.

2) Let $p > 3$ and $A_1$ and $A_2$ be abelian groups of order $p^4$. If the pairs $(G, A_1)$ and $(G, A_2)$ are realizable, then $A_1 \simeq A_2$.

Proof. 1) By [8] Proposition 4, we know that if $(G, C_{p^n})$ is realizable, then $(G, N)$ is not realizable for any noncyclic group $N$ of order $p^n$.

Let us assume that $(G, N_2)$ is realizable and $(G, N_i)$ is realizable, where $i = 3, 4$ or 5. By Theorem 2 $G$ is isomorphic to a transitive subgroup of Hol($N_i$), hence by Lemmas 5 and 7 Syl$_p(G)$ has exponent $< p^2$. If $(G, N_2)$ is realizable, then $G$ is isomorphic to a transitive subgroup of Hol($N_2$). We shall see that a subgroup $G$ of Hol($N_2$) such that Syl$_p(G)$ has exponent $< p^2$ is not transitive and in this way reach a contradiction. We write $N_2 = \langle a \rangle \times \langle b \rangle \simeq C_{p^3} \times C_p$ and consider the automorphisms $\varphi_i, 1 \leq i \leq 3$, as in (11). The subgroup $\langle a, b, \varphi_1, \varphi_2, \varphi_3 \rangle$ is a p-Sylow subgroup of Hol($N_2$) and we may check that it is normal, hence the unique p-Sylow subgroup of Hol($N_2$). Now, for $(z, \varphi) \in$ Syl$_p($Hol($N_2$))$, we write $z = a^rb^s, 1 \leq r \leq p^3, 1 \leq s \leq p, \varphi = \varphi_1^i\varphi_2^j\varphi_3^k, 1 \leq i \leq p^2, 1 \leq j \leq p, 1 \leq k \leq p$, and obtain

$$\varphi(a) = a^{1+ipj}, \quad \varphi(b) = a^{2+kp^2}. $$

For $n \geq 1$, we have

$$\varphi^n : a \mapsto a^{1+np+n(n-1)jp^2/2+n(n-1)j^2p^2/2p^3}, \quad b \mapsto a^{np}b.$$ 

We further obtain

$$(z, \varphi)^p = \begin{cases} (a^{rp+ri(p-1)p^2/2}, \varphi^p), & \text{if } p > 3, \\ (a^{3+9i+9jk}, \varphi^2), & \text{if } p = 3. \end{cases} \quad (z, \varphi)^p = (a^{rp^2}, \text{Id}).$$

We conclude then that $(z, \varphi)^p = (1, \text{Id})$ if and only if $p \mid r$. Hence if Syl$_p(G)$ has exponent $\leq p^2$, it is not transitive and by Lemma 8 neither is $G$. We have then obtained the searched contradiction.

2) For the rest of the proof, we take $p > 3$. Let us assume $(G, N_5)$ is realizable and $(G, N_i)$ is realizable, where $i = 3$ or 4. By Theorem 2 $G$ is isomorphic to a transitive subgroup of Hol($N_i$), hence by Lemma 7 Syl$_p(G)$ has exponent $p$. If $L/K$ has a Hopf Galois structure of type $N_3$ (resp. type $N_4$), then $G$ is isomorphic to a transitive subgroup of Hol($N_3$) (resp. Hol($N_4$)). We shall see that a subgroup $G$ of Hol($N_3$) (resp. Hol($N_4$)) such that Syl$_p(G)$ has exponent $p$ is not transitive and in this way reach a contradiction. In the proof of Lemma 5 we have proved that if an element $(z, \varphi) \in$ Syl$_p($Hol($N_3$)) has order $p$, then $z^p = 1$. We have then that if Syl$_p(G)$ has exponent $p$, it cannot be isomorphic to a transitive subgroup of Hol($N_3$), and the same apply to $G$ by Lemma 8. In the proof of Lemma 5 we have proved that if an element $(z, \varphi) \in$ Syl$_p($Hol($N_4$)) has order $p$, then $z \in \langle a^p, b, c \rangle$, where $N_4 = \langle a, b, c \rangle$. We have then that if Syl$_p(G)$ has exponent $p$, it cannot be isomorphic to a transitive subgroup of Hol($N_4$), and the same apply to $G$ by Lemma 8.

It remains to prove that if $(G, N_3)$ is realizable, then $(G, N_4)$ is not realizable. Let us assume that $(G, N_3)$ is realizable. Then $G$ is a transitive subgroup of Hol($N_3$). We have seen
in Lemma 5 that the set of elements of \( \text{Hol}(N_3) \) of order \( \leq p \) form a subgroup of order \( p^6 \). We denote it by \( F_3 \). Clearly, \( F_3 \) is a normal subgroup of \( \text{Hol}(N_3) \). Writing \( N_3 = \langle a, b \rangle \) and with \( \varphi_1 \) as in (2), we have that \( \text{Syl}_p(\text{Hol}(N_3)) \) is isomorphic to the Heisenberg group of order \( p^3 \). Indeed \( \bar{a}, \bar{b}, \bar{\varphi_1} \) have order \( p \), \( \bar{b} \) commutes with \( \bar{a} \) and \( \bar{\varphi_1}a\bar{\varphi_1}^{-1} = \bar{a} \). If \( G \) is a transitive subgroup of \( \text{Hol}(N_3) \), by Lemma 8 so is \( \text{Syl}_p(G) \). Then \( \text{Syl}_p(G) \) contains some elements of the form \((a, *)\) and \((b, *)\). Hence \( \text{Syl}_p(G)/(F_3 \cap \text{Syl}_p(G)) \) has order \( \geq p^2 \). We have seen in Lemma 6 that the set of elements of \( \text{Hol}(N_4) \) of order \( \leq p \) form a subgroup \( F_4 \) of order \( p^9 \). We have then \([\text{Syl}_p(\text{Hol}(N_4)) : F_4] = p\). Now if \( G \) is a subgroup of \( \text{Hol}(N_4) \), we have a monomorphism from \( \text{Syl}_p(G)/(F_4 \cap \text{Syl}_p(G)) \) to \( \text{Syl}_p(H_4)/F_4 \). Hence \( \text{Syl}_p(G)/(F_4 \cap \text{Syl}_p(G)) \) has order \( \leq p \) in contradiction with the above. Then \( G, N_4 \) is not realizable.

**Remark 10.** In Table 1 we show the number of Hopf Galois structures of type \( N \) on a Galois field extension of degree 81 with Galois group \( G \). We observe in particular that Hopf Galois structures of types \( N_3, N_4 \) and \( N_5 \) do occur on the same extension.

## 4 Hopf Galois structures of nonabelian type

We proved in 8 that if a separable field extension of degree \( p^3 \) has a nonabelian Hopf Galois structure of type \( N \), then it has an abelian structure whose type has the same exponent as \( N \). Here we prove similar results for separable field extensions of degree \( p^4 \). More precisely, given a group \( G \) and some groups \( A \) and \( N \) such that \( A \) is abelian, \( N \) is nonabelian and \( A \) and \( N \) have both order \( p^4 \) (or \( p^n, n \geq 3 \)) and the same exponent, we shall prove that if \( (G, N) \) is realizable, then \( (G, A) \) is realizable. To this end, by Theorem 2 it suffices to prove that \( \text{Hol}(A) \) contains a regular subgroup isomorphic to \( N \) such that its normalizer in \( \text{Hol}(A) \) is equal to its normalizer in \( \text{Sym}(A) \), that is, has order equal to \( \text{Hol}(N) \).

**Proposition 11.** Let \( p \) be an odd prime number, \( n \geq 3 \), \( G \) a group. If \( (G, C_{p^{n-1}} \rtimes C_p) \) is realizable then \( (G, C_{p^{n-1}} \rtimes C_p) \) is realizable.

**Proof.** We write \( A := C_{p^{n-1}} \times C_p, N := C_{p^{n-1}} \rtimes C_p \). Let us note that there are exactly two morphisms from \( C_p \) to \( \text{Aut}(C_{p^{n-1}}) \cong C_{p^{n-2}(p-1)} \), the trivial one, corresponding to \( A \), and the injective one, corresponding to \( N \). We determine first the automorphism groups of \( A \) and \( N \). The groups \( A \) and \( N \) have presentations

\[
A := \langle a, b \mid a^{p^{n-1}} = 1, b^p = 1, bab^{-1} = a \rangle, \quad N := \langle c, d \mid c^{p^{n-1}} = 1, d^p = 1, dcd^{-1} = c^{1+p^{n-2}} \rangle.
\]

An automorphism of \( A \) is given by

\[
a \mapsto a^k b^l, \quad b \mapsto a^{p^{n-2}i} b^j, \quad \text{with} \quad 1 \leq k \leq p^{n-1}, 1 \leq i, j, l \leq p, p \nmid k, p \nmid j.
\]

An automorphism of \( N \) is given by

\[
c \mapsto c^k d^l, \quad d \mapsto c^{p^{n-2}i} d^l, \quad \text{with} \quad 1 \leq k \leq p^{n-1}, 1 \leq i, l \leq p, p \nmid k.
\]

Hence \( |\text{Aut}(A)| = p^n(p-1)^2 \), \( |\text{Aut}(N)| = p^n(p-1) \). We consider the following elements in \( \text{Hol}(A) \).

- \( C := (a, \varphi) \), with \( \varphi \) defined by \( \varphi(a) = a, \varphi(b) = a^{p^{n-2}}b \),
\[ D := (b, \psi), \text{ with } \psi \text{ defined by } \psi(a) = a^{1-p^n-2}, \psi(b) = b. \]

Clearly \( \varphi \) has order \( p \), \( C \) has order \( p^{n-1} \), \( \psi \) has order \( p \) and \( D \) has order \( p \). Moreover we have 
\[ DC(D^{-1} = (b, \psi)(a, \varphi)(b^{-1}, \psi^{-1}) = (ba^{1-p^n-2}, \psi\varphi)(b^{-1}, \psi^{-1}) = (a^{1-2p^n-2}, \varphi) = (a, \varphi)^{1-2p^n-2}, \]

hence \( \langle C, D \rangle \) is isomorphic to \( N \). For any integers \( r, s \) with \( 1 \leq r \leq p^{n-1}, 1 \leq s \leq p \), we have 
\[ C^{r(1-\psi p^n-2)}D^s = (a^{r(1-\psi p^n-2)}, \varphi^r)(b^s, \psi^s) = (a^{r(1-\psi p^n-2)}\varphi^r(b^s), \varphi^r\psi^s) = (a^rb^s, \varphi^r\psi^s), \]

hence \( \langle C, D \rangle \) is a regular subgroup of \( \text{Hol}(A) \).

We determine now the normalizer \( \text{Nor} \) of the subgroup \( \langle C, D \rangle \) in \( \text{Hol}(A) \). We have
\[
\begin{align*}
    a(a, \varphi)a^{-1} &= (a, \varphi), & a(b, \psi)a^{-1} &= (a, \varphi)^{p^n-2}(b, \psi), \\
    b(a, \varphi)b^{-1} &= (a, \varphi)^{1-p^n-2}, & b(b, \psi)b^{-1} &= (b, \psi).
\end{align*}
\]

Hence \( A \subset \text{Nor} \). We consider now the automorphisms \( \chi \) of \( A \) defined by
\[
\chi(a) = a^kb^l, \quad \chi(b) = a^{p^n-2}b, \quad \text{with } 1 \leq k \leq p^{n-1}, 1 \leq l, k \leq p, p \nmid k.
\]

They form a subgroup \( B \) of \( \text{Aut}(A) \) of order equal to \( |\text{Aut}(N)| \) and satisfy
\[
\begin{align*}
    \chi(a, \varphi)^{-1} &= (\chi(a), \chi\varphi^{-1}) = (a^kb^l, \varphi^{k(1-\psi p^n-2)}b^l), & \chi(b, \psi)^{-1} &= (\psi^{p^n-2}b, \psi) = (a, \varphi)^{p^n-2}(b, \psi).
\end{align*}
\]

We have obtained then \( A \ltimes B \subset \text{Nor} \), hence \( |\text{Nor}| = |\text{Hol}(N)| \) as wanted.

**Proposition 12.** Let \( p \) be an odd prime number, \( G \) a group. If \( (G, N_8) \) is realizable then \( (G, C_{p^2} \times C_{p^2}) \) is realizable.

**Proof.** Let us write
\[
\begin{align*}
    N_3 &= C_{p^2} \times C_{p^2} = \langle a, b \mid a^{p^2} = 1, b^{p^2} = 1, b^{-1}ab = a \rangle, \\
    N_8 &= \langle c, d \mid c^{p^2} = 1, d^{p^2} = 1, d^{-1}cd = c^{1+p} \rangle.
\end{align*}
\]

The automorphisms of \( N_8 \) are of the form
\[
\begin{align*}
    c \mapsto c^i d^j, & \quad d \mapsto c^k d^l, & \text{with } p \nmid i, p \mid j, l \equiv 1 \pmod{p},
\end{align*}
\]

hence \( |\text{Aut}(N_8)| = p^5(p - 1) \). We consider the following elements in \( \text{Hol}(N_3) \)

- \( C := a \)
- \( D := (b, \psi), \) with \( \psi \) defined by \( \psi(a) = a^{1-p}, \psi(b) = b^{1-p} \).

We have that \( C \) has order \( p^2 \), \( D \) satisfies \( D^p = (b^p, \text{Id}) \), so \( D \) has order \( p^2 \) and \( D^{-1}CD = (b^{-1-p}, \psi^{-1})a(b, \psi) = a^{1+p} = C^{1+p} \), hence \( \langle C, D \rangle \) is a subgroup of \( \text{Hol}(N_3) \) isomorphic to \( N_8 \). Now for integers \( r, s \) with \( 1 \leq r \leq p^2 \), we have \( C^rD^s = a^r(b, \psi)s = (a^rb^s)^{ps(s-1)/2}, \psi^s \). Since \( s(s - 1)/2 \pmod{p} \) depends only on \( s \pmod{p} \), we obtain that \( \langle C, D \rangle \) is a regular subgroup of \( \text{Hol}(N_3) \).

We determine now the normalizer \( \text{Nor} \) of the subgroup \( \langle C, D \rangle \) in \( \text{Hol}(N_3) \). We have
\[
\begin{align*}
    a(b, \psi)a^{-1} &= a^p(b, \psi), & b(b, \psi)b^{-1} &= (b^{1+p}, \psi) = (b, \psi)^{1+p},
\end{align*}
\]

hence \( N_3 \subset \text{Nor} \). We consider the automorphisms of \( N_3 \) of the form
\[
\chi(a) = a^ib^j, \quad \chi(b) = a^kb^l, \quad \text{with } p \nmid i, p \mid j, l \equiv 1 \pmod{p}.
\]

They form a subgroup \( B \) of \( \text{Aut}(N_3) \) of order \( p^5(p - 1) \) and we have

\[
\chi a \chi^{-1} = \chi(a) = a^ib^j = a^i(b, \psi)^j, \quad \text{since } p \mid j
\]
\[
\chi(b, \psi)a^{-1} = (\chi(b), \chi\psi\chi^{-1}) = (a^kb^l, \psi) = a^k(b, \psi)^l, \quad \text{since } l \equiv 1 \pmod{p}.
\]

Hence \( N_3 \rtimes B \subset \text{Nor} \) and we obtain \( |\text{Nor}| = |\text{Hol}(N_3)| \) as wanted. \( \square \)

**Proposition 13.** Let \( p \) be an odd prime number, \( G \) a group. If \( (G, N_7) \) is realizable, then \( (G, C_{p^2} \times C_p \times C_p) \) is realizable.

**Proof.** Let us write

\[
N_4 = C_{p^2} \times C_p \times C_p = \langle a, b, c \mid a^{p^2} = 1, b^p = 1, c^p = 1, b^{-1}ab = a, c^{-1}ac = a, c^{-1}bc = b \rangle,
\]
\[
N_7 = \langle d, e, f \mid df^2 = 1, e^p = 1, f^p = 1, f^{-1}ef = ed^p, f^{-1}df = d, e^{-1}de = d \rangle.
\]

The automorphisms of \( N_7 \) are of the form

\[
d \mapsto d^i, \quad e \mapsto d^i e^b f^l, \quad f \mapsto d^i e^c f^l \quad \text{with } p \nmid i, p \mid j, p \mid r, kt - ls \equiv 1 \pmod{p},
\]

hence \( |\text{Aut}(N_7)| = p^4(p - 1)^2(p + 1) \). We consider the following elements in \( \text{Hol}(N_4) \)

- \( D := a \)
- \( E := (b, \varphi) \), with \( \varphi \) defined by \( \varphi(a) = a, \varphi(b) = b, \varphi(c) = a^{(p + p^2)/2}c \).
- \( F := (c, \psi) \), with \( \psi \) defined by \( \psi(a) = a, \psi(b) = a^{(p^2 - p)/2}b, \psi(c) = c \).

We have that \( D \) has order \( p^2 \), \( E \) and \( F \) have order \( p \) and \( DE = (ab, \varphi) = ED, DF = (ac, \psi) = FD, F^{-1}EF = (c^{-1}, \psi^{-1}))(b, \varphi)(c, \psi) = (a^p b, \varphi) = a^p(b, \varphi) = ED^p \), hence \( \langle D, E, F \rangle \) is a subgroup of \( \text{Hol}(N_4) \) isomorphic to \( N_7 \). Now for integers \( r, s, t \) with \( 1 \leq r \leq p^2, 1 \leq s, t \leq p \), we have \( D^{r - (p + p^2)st/2}E^sF^t = (a^b b^s c^t, c^s \psi^t) \). Hence \( \langle D, E, F \rangle \) is a regular subgroup of \( \text{Hol}(N_4) \).

We determine now the normalizer \( \text{Nor} \) of the subgroup \( \langle D, E, F \rangle \) in \( \text{Hol}(N_4) \). We have

\[
a(b, \varphi)a^{-1} = (b, \varphi), \quad a(c, \psi)a^{-1} = (c, \psi), \quad b(b, \varphi)b^{-1} = (b, \varphi),
\]
\[
b(c, \psi)b^{-1} = a^{p^2 + p/2}(c, \psi), \quad c(b, \varphi)c^{-1} = a^{(p^2 - p)/2}(b, \varphi), \quad c(c, \psi)c^{-1} = (c, \psi),
\]

hence \( N_4 \subset \text{Nor} \). We consider the automorphisms of \( N_4 \) of the form

\[
\chi : \quad a \mapsto a^i, \quad p \nmid i
\]
\[
b \mapsto a^{pm b^n a^q}, \quad c \mapsto a^{p^2 b^s c^t}, \quad i(nt - qs) \equiv 1 \pmod{p}
\]

They form a subgroup \( B \) of \( \text{Aut}(N_4) \) of order \( p^4(p - 1)^2(p + 1) \) and we have

\[
\chi a \chi^{-1} = \chi(a) = a^i
\]
\[
\chi(b, \varphi) \chi^{-1} = \chi(b), \chi(\varphi \chi^{-1}) = (a^{pm b^n c^q}, \varphi^p \psi^q) = a^{pm - (p^2 + p)mq/2}(b, \varphi)^n(c, \psi)q
\]
\[
\chi(c, \psi) \chi^{-1} = (c, \chi \psi \chi^{-1}) = (a^{p^2 b^s c^t}, \varphi^s \psi^t) = a^{pm - (p^2 + p)st/2}(b, \varphi)^s(c, \psi)^t.
\]

Hence \( N_4 \rtimes B \subset \text{Nor} \) and we obtain \( |\text{Nor}| = |\text{Hol}(N_7)| \) as wanted. \( \square \)
Proposition 14. Let $p$ be an odd prime number, $G$ a group. If $(G,N_9)$ is realizable, then $(G,C_{p^2} \times C_p \times C_p)$ is realizable.

Proof. Let us write

\[ N_4 = C_{p^2} \times C_p \times C_p = \langle a, b, c \mid a^{p^2} = 1, b^p = 1, c^p = 1, b^{-1}ab = a, c^{-1}ac = a, c^{-1}bc = b \rangle, \]
\[ N_9 = \langle d, e, f \mid df^2 = 1, e^p = 1, f^p = 1, f^{-1}ef = e, f^{-1}df = d^{1+p}, e^{-1}de = d \rangle. \]

The automorphisms of $N_9$ are of the form

\[ d \mapsto d^i e^j f^k, \quad e \mapsto d^p e^m, \quad f \mapsto d^{pr} e^s f \quad \text{with} \ p \nmid i, m, \]

hence $|\text{Aut}(N_9)| = p^6(p - 1)^2$. We consider the following elements in $\text{Hol}(N_4)$

- $D := (a, \varphi)$, with $\varphi$ defined by $\varphi(a) = a, \varphi(b) = b, \varphi(c) = a^{(p^2+p)/2}c$.
- $E := b$.
- $F := (c, \psi)$, with $\psi$ defined by $\psi(a) = a^{1+(p^2-p)/2}, \psi(b) = b, \psi(c) = c$.

We have that $\varphi$ has order $p$, $(a, \varphi)^{p^6} = (a^p, \text{Id})$, hence $D$ has order $p^2$, $E$ has order $p$, $\psi$ has order $p$ and $F$ has order $p$. Moreover $DE = (ab, \varphi) = ED, F^{-1}DF = (c^{-1}, \psi^{-1}(a, \varphi)(c, \psi) = (a^{1+p}, \varphi) = (a, \varphi)^{1+p} = D^{1+p}, EF = (bc, \psi) = FE$, hence $\langle D, E, F \rangle$ is a subgroup of $\text{Hol}(N_4)$ isomorphic to $N_9$. Now for integers $t, u, v$ with $1 \leq t \leq p^2, 1 \leq u, v \leq p$, we have $D^{t-(p+p^2)tu/2}E^uF^v = (a^ib^u c^v, \varphi^i\psi^v)$. Hence $\langle D, E, F \rangle$ is a regular subgroup of $\text{Hol}(N_4)$.

We determine now the normalizer $\text{Nor}$ of the subgroup $\langle D, E, F \rangle$ in $\text{Hol}(N_4)$. We have

\[ a(a, \varphi)a^{-1} = (a, \varphi), \quad a(c, \psi)a^{-1} = a^{(p+p^2)/2}(c, \psi), \quad b(a, \varphi)b^{-1} = (a, \varphi), \]
\[ b(c, \psi)b^{-1} = (c, \psi), \quad c(a, \varphi)c^{-1} = (a, \varphi)^{(p^2-p)/2}, \quad c(c, \psi)c^{-1} = (c, \psi), \]

hence $N_4 \subseteq \text{Nor}$. We consider the automorphisms of $N_4$ of the form

\[ \chi: \begin{array}{c} a \mapsto a^{ib^j}c^k, \quad p \nmid i \\ b \mapsto a^{pm}b^p, \quad p \nmid m \\ c \mapsto a^{pr}b^s c \end{array}. \]

They form a subgroup $B$ of $\text{Aut}(N_4)$ of order $p^6(p - 1)^2$ and we have

\[ \chi(a, \varphi) \chi^{-1} = (\chi(a), \chi \varphi \chi^{-1}) = (a^{ib^j}c^k, \varphi^{(1-(p^2-p)/2)}c^k) = (a, \varphi)^{(1-(p^2-p)/2)}b^j(c, \psi)^k \]
\[ \chi(b) \chi^{-1} = \chi(b) = a^{pm}b^p = (a, \varphi)^{pm}b^p \]
\[ \chi(c, \psi) \chi^{-1} = (\chi(c), \chi \psi \chi^{-1}) = (a^{pr}b^s c, \psi) = (a, \varphi)^{pr}b^s(c, \psi). \]

Hence $N_4 \rtimes B \subseteq \text{Nor}$ and we obtain $|\text{Nor}| = |\text{Hol}(N_9)|$ as wanted.

\[ \square \]

Proposition 15. Let $p$ be an odd prime number, $G$ a group. If $(G,N_{10})$ is realizable, then $(G,C_{p^2} \times C_p \times C_p)$ is realizable.
Proof. Let us write
\[ N_4 = C_p^2 \times C_p \times C_p = \langle a, b, c \mid a^{p^2} = 1, b^p = 1, c^p = 1, b^{-1}ab = a, c^{-1}ac = a, c^{-1}bc = b \rangle, \]
\[ N_{10} = \langle d, e, f \mid db^2 = 1, e^p = 1, f^p = 1, f^{-1}ef = e, f^{-1}df = de, e^{-1}de = d \rangle. \]

The automorphisms of \( N_{10} \) are of the form
\[ d \mapsto d^{e^if^k}, \quad e \mapsto e^{it}, \quad f \mapsto f^{ip^r} \quad \text{with} \quad p \nmid i, \quad p \nmid t, \]

hence \( |\text{Aut}(N_{10})| = p^5(p-1)^2 \). We consider the following elements in \( \text{Hol}(N_4) \)

- \( D := (a, \varphi) \), with \( \varphi \) defined by \( \varphi(a) = a, \varphi(b) = b, \varphi(c) = b^{(p+1)/2}c \).
- \( E := (a, \psi) = (a, b) \).
- \( F := (c, \psi) \), with \( \psi \) defined by \( \psi(a) = ab^{(p-1)/2}, \psi(b) = b, \psi(c) = c \).

We have that \( \varphi \) has order \( p \), \( (a, \varphi)^p = (a^p, \text{Id}) \), hence \( D \) has order \( p^2 \), \( E \) has order \( p \), \( \psi \) has order \( p \) and \( F \) has order \( p \). Moreover \( DE = (ab, \varphi) = ED, F^{-1}DF = (c^{-1}, \psi^{-1})(a, \varphi)(c, \psi) = (ab, \varphi) = (a, \varphi)b = DE, EF = (bc, \varphi) = FE \), hence \( \langle D, E, F \rangle \) is a subgroup of \( \text{Hol}(N_4) \) isomorphic to \( N_{10} \). Now for integers \( t, u, v \) with \( 1 \leq t \leq p^2, 1 \leq u, v \leq p \), we have \( D^t E^{u-(p+1)tv/2}F^v = (a^{t}b^{up}c^{v}, \varphi^{t} \psi^{v}) \). Hence \( \langle D, E, F \rangle \) is a regular subgroup of \( \text{Hol}(N_4) \).

We determine now the normalizer \( \text{Nor} \) of the subgroup \( \langle D, E, F \rangle \) in \( \text{Hol}(N_4) \). We have
\[ a(a, \varphi)a^{-1} = (a, \varphi), \quad a(c, \psi)a^{-1} = b^{-(p+1)/2}(c, \psi), \quad b(a, \varphi)b^{-1} = (a, \varphi), \]
\[ b(c, \psi)b^{-1} = (c, \psi), \quad c(a, \varphi)c^{-1} = (a, \varphi)b^{-(p+1)/2}, \quad c(c, \psi)c^{-1} = (c, \psi), \]

hence \( N_4 \subset \text{Nor} \). We consider the automorphisms of \( N_4 \) of the form
\[ \chi : \quad a \mapsto a^{ib^jck}, \quad p \nmid i, \]
\[ \chi^i \]
\[ b \mapsto b^{it}, \quad \chi^j \]
\[ c \mapsto c^{ipb^rb^s}, \quad p \nmid t, \]

They form a subgroup \( B \) of \( \text{Aut}(N_4) \) of order \( p^5(p-1)^2 \) and we have
\[ \chi(a, \varphi)a^{-1} = (\chi(a), \chi \varphi \chi^{-1}) = (a^{i}b^{j}c^{k}, \varphi^{i} \psi^{k}) = (a, \varphi)^{i}b^{j+ik(p-1)/2}(c, \psi)^{k} \]
\[ \chi(b) = b^{it} \]
\[ \chi(c) = c^{ipb^rb^s}, \quad \psi(c) = c \]

Hence \( N_4 \cong B \subset \text{Nor} \) and we obtain \( |\text{Nor}| = |\text{Hol}(N_{10})| \) as wanted.

\[ \square \]

Proposition 16. Let \( p \) be an odd prime number, \( G \) a group. If \( (G, N_{14}) \) is realizably, then
\( (G, C_p^4) \) is realizably.

Proof. Let us write
\[ N_5 = C_p^4 \times C_p \times C_p = \langle a, b, c, d \mid a^{p^2} = 1, b^p = 1, c^p = 1, d^p = 1, b^{-1}ab = a, c^{-1}ac = a, d^{-1}ad = a, \]
\[ c^{-1}bc = b, d^{-1}bd = b, d^{-1}cd = c \rangle, \]
\[ N_{14} = \langle e, f, g, h \mid e^p = 1, f^p = 1, g^p = 1, h^p = 1, h^{-1}gh = eg, h^{-1}fh = f, h^{-1}eh = e, \]
\[ g^{-1}fg = f, g^{-1}eg = e, f^{-1}ef = e \rangle. \]

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The automorphisms of $N_{14}$ are of the form

$$
e \mapsto e^{m_{11}}, \quad f \mapsto e^{m_{12}}f^{m_{22}}, \quad g \mapsto e^{m_{13}}f^{m_{23}}g^{m_{33}}h^{m_{43}}, \quad h \mapsto e^{m_{14}}f^{m_{24}}g^{m_{34}}h^{m_{44}}$$

with $p \nmid m_{11}, p \nmid m_{22}, m_{33}m_{44} - m_{34}m_{43} \equiv m_{11} \pmod{p}$.

hence $|\text{Aut}(N_{14})| = p^6(p - 1)^2(p + 1)$. We consider the following elements in $\text{Hol}(N_5)$

- $E := a$,
- $F := b$,
- $G := (c, \varphi)$, with $\varphi$ defined by $\varphi(a) = a, \varphi(b) = b, \varphi(c) = c, \varphi(d) = a^{(p+1)/2}d$,
- $H := (d, \psi)$, with $\psi$ defined by $\psi(a) = a, \psi(b) = b, \psi(c) = a^{(p-1)/2}c, \psi(d) = d$.

We have that $E, F, G, H$ have order $p$ and satisfy $EF = ab = FE, EG = (ac, \varphi) = GE, EH = (ad, \psi) = HE, FG = (bc, \varphi) = GF, FH = (bd, \psi) = HF, H^{-1}GH = (d^{-1}, \psi^{-1})(c, \varphi)(d, \psi) = (ac, \varphi) = a(c, \varphi) = EG$, hence $\langle E, F, G, H \rangle$ is a subgroup of $\text{Hol}(N_5)$ isomorphic to $N_{14}$. Now for integers $r, s, t, u$ with $1 \leq r, s, t, u \leq p$, we have $E^{r-1}\langle p+1 \rangle/2 F^{s}G^{t}H = (a^rb^sd^u, c^r\psi^u)$. Hence $\langle E, F, G, H \rangle$ is a regular subgroup of $\text{Hol}(N_5)$.

We determine now the normalizer $\text{Nor}$ of the subgroup $\langle E, F, G, H \rangle$ in $\text{Hol}(N_5)$. We have

$$a(c, \varphi)a^{-1} = (c, \varphi), \quad b(c, \varphi)b^{-1} = (c, \varphi), \quad c(c, \varphi)c^{-1} = (c, \varphi), \quad d(c, \varphi)d^{-1} = a^{(p+1)/2}(c, \varphi)$$

$$a(d, \psi)a^{-1} = (d, \psi), \quad b(d, \psi)b^{-1} = (d, \psi), \quad c(d, \psi)c^{-1} = a^{(p+1)/2}(d, \psi), \quad d(d, \psi)d^{-1} = (d, \psi).$$

hence $N_5 \subset \text{Nor}$. We consider the automorphisms of $N_5$ of the form

$$\chi : \begin{align*}
a &\mapsto a^{n_{11}} \\
b &\mapsto a^{n_{12}}b^{n_{22}} \\
c &\mapsto a^{n_{13}}b^{n_{23}}c^{n_{33}}d^{n_{43}} \\
d &\mapsto a^{n_{14}}b^{n_{24}}c^{n_{34}}d^{n_{44}}
\end{align*}$$

with $p \nmid n_{11}, p \nmid n_{22}, n_{33}n_{44} - n_{34}n_{43} \equiv n_{11} \pmod{p}$. They form a subgroup $B$ of $\text{Aut}(N_5)$ of order $p^6(p - 1)^2(p + 1)$ and we have

$$\chi a \chi^{-1} = \chi(a) = a^{n_{11}}$$
$$\chi b \chi^{-1} = \chi(b) = a^{n_{12}}b^{n_{22}}$$
$$\chi(c, \varphi)\chi^{-1} = (\chi(c), \chi\varphi\chi^{-1}) = (a^{n_{13}}b^{n_{23}}c^{n_{33}}d^{n_{43}}, \varphi^{n_{33}}\psi^{n_{43}}) = a^{n_{13}+\frac{(p-1)n_{33}n_{43}}{2}}b^{n_{23}}(c, \varphi)^{n_{33}}(d, \psi)^{n_{43}}$$
$$\chi(d, \psi)\chi^{-1} = (\chi(d), \chi\psi\chi^{-1}) = (a^{n_{14}}b^{n_{24}}c^{n_{34}}d^{n_{44}}, \varphi^{n_{34}}\psi^{n_{44}}) = a^{n_{14}+\frac{(p-1)n_{34}n_{44}}{2}}b^{n_{24}}(c, \varphi)^{n_{34}}(d, \psi)^{n_{44}}$$

Hence $N_5 \rtimes B \subset \text{Nor}$ and we obtain $|\text{Nor}| = |\text{Hol}(N_{14})|$ as wanted.

\[ \square \]

**Remark 17.** For $p = 3$, in Table 1 one can see that a Galois field extension with Galois group isomorphic to $N_{11}$ has no Hopf Galois structures of type $N_3$ and that, for $i = 12, 13$ and 15, a Galois field extension with Galois group isomorphic to $N_i$ has Hopf Galois structures of type $N_3$ equivalently that $\text{Hol}(N_3)$ has regular subgroups isomorphic to $N_i$, but one may check with Magma that the order of the normalizer of these groups in $\text{Hol}(N_3)$ is not equal to the order
of Hol(N_i). Analogously, for i = 11, 12, 13 and 15, Hol(N_4) has regular subgroups isomorphic to N_i, but the order of the normalizer of these groups in Hol(N_4) is not equal to the order of Hol(N_i).

For p = 5, we have checked with Magma that Hol(N_4) has regular subgroups isomorphic to N_i, for i = 11, 12 and 13 but the order of the normalizer of these groups in Hol(N_4) is not equal to the order of Hol(N_i). Similarly Hol(N_5) has regular subgroups isomorphic to N_{15}, but the order of the normalizer of these groups in Hol(N_5) is not equal to the order of Hol(N_{15}).

For p = 2, the description of the Hopf Galois structures on a separable field extension of degree 16 can be found in [9].

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Table 1: Number of Hopf Galois structures of type $N$ on a Galois extension of degree 81 with Galois group $G$

| $G/N$ | $N_1$ | $N_2$ | $N_3$ | $N_4$ | $N_5$ | $N_6$ | $N_7$ | $N_8$ | $N_9$ | $N_{10}$ | $N_{11}$ | $N_{12}$ | $N_{13}$ | $N_{14}$ | $N_{15}$ |
|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|--------|--------|--------|--------|--------|--------|
| $N_1$ | 27    | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0      | 0      | 0      | 0      | 0      | 0      |
| $N_2$ | 0     | 135   | 0     | 0     | 0     | 270   | 0     | 0     | 0     | 0      | 0      | 0      | 0      | 0      | 0      |
| $N_3$ | 0     | 0     | 1161  | 0     | 0     | 0     | 0     | 6696  | 0     | 0      | 864    | 0      | 0      | 432    | 0      |
| $N_4$ | 0     | 0     | 1296  | 6345  | 648   | 0     | 24786 | 2592  | 19656 | 20952  | 19440  | 18144  | 23328  | 3024   | 5184   |
| $N_5$ | 0     | 0     | 0     | 224640| 110241| 0     | 0     | 0     | 1797120| 449280 | 4043520| 0      | 0      | 2162160| 4043520|
| $N_6$ | 0     | 135   | 0     | 0     | 0     | 270   | 0     | 0     | 0     | 0      | 0      | 0      | 0      | 0      | 0      |
| $N_7$ | 0     | 0     | 432   | 5049  | 216   | 0     | 24786 | 864   | 17064 | 17496  | 19440  | 16416  | 23328  | 1728   | 4320   |
| $N_8$ | 0     | 0     | 1161  | 0     | 0     | 0     | 0     | 6696  | 0     | 0      | 864    | 0      | 0      | 432    | 0      |
| $N_9$ | 0     | 0     | 324   | 5697  | 324   | 0     | 24786 | 648   | 18360 | 20952  | 19440  | 16200  | 23328  | 2376   | 4212   |
| $N_{10}$| 0    | 0     | 324   | 1755  | 216   | 0     | 1458  | 648   | 4644  | 5724   | 0      | 648    | 0      | 864    | 324    |
| $N_{11}$| 0   | 0     | 0     | 1080  | 324   | 0     | 972   | 0     | 4104  | 1512   | 2430   | 648    | 972    | 2052   | 324    |
| $N_{12}$| 0  | 0     | 108   | 648   | 0     | 0     | 1944  | 216   | 1296  | 0      | 1458   | 1026   | 1944   | 0      | 270    |
| $N_{13}$| 0  | 0     | 108   | 702   | 108   | 0     | 1458  | 216   | 2538  | 486    | 972    | 1512   | 1944   | 540    | 270    |
| $N_{14}$| 0 | 0     | 0     | 9504  | 10935 | 0     | 0     | 0     | 68256 | 13824  | 101088 | 0      | 0      | 145908 | 101088 |
| $N_{15}$| 0 | 0     | 108   | 1458  | 648   | 0     | 486   | 216   | 5346  | 486    | 0      | 1512   | 486    | 4212   | 2700   |