TORSION IN TENSOR PRODUCTS OVER ONE-DIMENSIONAL DOMAINS

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Abstract. Over a one-dimensional Gorenstein local domain $R$, let $E$ be the endomorphism ring of the maximal of $R$, viewed as a subring of the integral closure $\overline{R}$. If there exist finitely generated $R$-modules $M$ and $N$, neither of them free, whose tensor product is torsion-free, we show that $E$ must be local with the same residue field as $R$.

1. Introduction

Finding interesting examples of non-zero, finitely generated modules $M, N$ over a commutative Noetherian ring $R$, with $M \otimes_R N$ torsion-free (meaning that no non-zero element of $M \otimes_R N$ is killed by a regular element of $R$) is a non-trivial task. Of course there are boring examples: take one of the modules to be torsion-free and the other to be projective. Or, if $R$ is not local, take $M = R/\mathfrak{m}$ and $N = R/\mathfrak{n}$, where $\mathfrak{m}$ and $\mathfrak{n}$ are distinct maximal ideals. A slightly less boring example is obtained by taking $R = \mathbb{Q}[x,y]/(xy)$ and $M = N = R/(x)$.

Question 1.1. Let $R$ be a local domain, and let $M$ and $N$ be finitely generated modules, neither one of them free. Must $M \otimes_R N$ always have non-zero torsion?

Again, the answer is “no”, and here is the connection with numerical semigroups:

Example 1.2. Let $R = k[[t^4, t^5, t^6]]$, $M = (t^4, t^5)$, and $N = (t^4, t^6)$. Then $M \otimes_R N$ is torsion-free [5, 4.3].

In fact, the only known examples where Question 1.1 has a negative answer are numerical semigroup rings. This leads to a (somewhat halfhearted, since it is probably false) conjecture:

Conjecture 1.3. Suppose $R$ is a one-dimensional local domain whose integral closure $\overline{R}$ is finitely generated as an $R$-module. If there exist finitely generated modules $M$ and $N$, neither of them free, with $M \otimes_R N$ torsion-free, then $\overline{R}$ is local, and the inclusion $R \subseteq \overline{R}$ induces an isomorphism on residue fields.

2. Some evidence

In this section we will prove the result stated in the abstract, which gives some support (admittedly rather sketchy) for Conjecture 1.3.

Throughout, $(R, \mathfrak{m}, k)$ is a one-dimensional Gorenstein local domain, with maximal ideal $\mathfrak{m}$ and residue field $k = R/\mathfrak{m}$. We let $K$ denote the quotient field of $R$. 

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But Ext of End applications), and thus the last short exact sequence shows that $E/R$ generating set for $k$ and using the fact that $m$ would split, giving a decomposition $m = \{\alpha \in K \mid \alpha m \subseteq m\}$. Then $R \subseteq E \subseteq \overline{R}$, where $\overline{R}$ is the integral closure of $R$ in $K$. The next lemma is due to Bass [2].

**Lemma 2.1.** Assume $m$ is not a principal ideal. Then $E/R$ is a simple $R$-module, and $E$ is minimally generated, as an $R$-module, by $\{1, y\}$, where $y$ is an arbitrary element of $E \setminus R$.

**Proof.** Since $m$ is indecomposable, there is no surjection $m \to R$. (Such a surjection would split, giving a decomposition $m \cong R \oplus H$, with $H \neq 0$, as $m$ is not principal; but clearly $m$ is indecomposable, since $R$ is a domain.) This gives the second equality in the display

\begin{equation}
(2.1.1) \quad m^* = \text{Hom}_R(m, R) = \text{Hom}_R(m, m) = E.
\end{equation}

Dualizing the short exact sequence

$$0 \to m \to R \to k \to 0,$$

and using the fact that $k^* = 0$, we get an exact sequence

$$0 \to R^* \to m^* \to \text{Ext}_R^1(k, R) \to 0.$$

But $\text{Ext}_R^1(k, R) \cong k$, as $R$ is one-dimensional and Gorenstein. The identification of $\text{End}_R(m)$ with $E$ is compatible with the identification of $R^*$ with $R$ (via multiplications), and thus the last short exact sequence shows that $E/R \cong k$. The next assertion is clear from simplicity of $E/R$ and the fact that 1 is part of a minimal generating set for $E$, as $1 \notin m = mE$. \hfill \Box

**Lemma 2.2.** Let $S$ be a subring of $K$ containing $R$ and finitely generated as an $R$-module. Let $M$ and $N$ be finitely generated $S$-modules such that $M \otimes_R N$ is torsion over $R$. Then the natural surjection $M \otimes_R N \to M \otimes_S N$ is an isomorphism.

**Proof.** We consult the following commutative diagram:

\begin{equation}
(2.2.1) \quad M \otimes_R N \xrightarrow{\delta} K \otimes_R (M \otimes_R N) \xrightarrow{\alpha} (K \otimes_R M) \otimes_K (K \otimes_R N) \xrightarrow{\beta} M \otimes_S N \xrightarrow{\gamma} K \otimes_S (M \otimes_S N) \xrightarrow{\gamma^*} (K \otimes_S M) \otimes_K (K \otimes_S N)
\end{equation}

The map $\delta$ is injective because $M \otimes_R N$ is torsion-free. One checks (by clearing denominators) that a subset of an $S$-module is linearly independent over $S$ if and only if it is linearly independent over $R$, and so its rank as an $S$-module equals its rank as an $R$-module. Thus $r := \dim_K(K \otimes_R M) = \dim_K(K \otimes_S M)$ and $s := \dim_K(K \otimes_R N) = \dim_K(K \otimes_S N)$. The surjective map $\gamma$ is therefore an isomorphism, since its domain and target both have the same $K$-dimension, namely $rs$. From the diagram, we see that $\beta$ must be an isomorphism too, and hence $\alpha$ is injective. \hfill \Box

**Theorem 2.3.** Let $(R, m, k)$ be a Gorenstein local domain of dimension one, and let $E = \text{End}_R(m)$, viewed as a ring between $R$ and its integral closure $\overline{R}$. Assume that there exist finitely generated modules $M$ and $N$, neither of them free, such
that \( M \otimes_R N \) is torsion-free. Then \( E \) is local, and the inclusion \( R \to E \) induces a bijection on residue fields.

**Proof.** If \( \mathfrak{m} \) is a principal ideal, then \( R \) is a discrete valuation ring, and \( R = E = \overline{R} \). Therefore we assume from now on that \( \mathfrak{m} \) is not principal.

We begin with some reductions. We first get rid of free summands, by writing \( M = M' \oplus R^m \) and \( N = N' \oplus R^n \), where both \( M' \) and \( N' \) are non-zero, and neither has a non-zero free direct summand. Notice that \( M' \otimes_R N' \), being a direct summand of \( M \otimes_R N \), is torsion-free. Replacing \( M \) by \( M' \) and \( N \) by \( N' \), we may assume that neither \( M \) nor \( N \) has a non-zero free direct summand.

Next, we have a reduction that goes back to Auslander’s 1961 paper [1]. Let \( TX \) denote the torsion submodule of a module \( X \), and put \( \perp X = X/(TX) \). By [3] Lemma 2.2, \( (\perp M) \otimes_R (\perp M) \) is torsion-free. Moreover, both \( \perp M \) and \( \perp N \) are non-zero, since otherwise \( M \otimes_R N \) would be a non-zero torsion module. We claim that \( \perp M \) has no non-zero free summand. For, suppose there is a surjection \( \perp M \to R \). Composing this with the natural surjection \( M \to \perp M \), we get a surjection \( M \to R \), and hence \( M \cong R \oplus L \), a contradiction. Similarly, \( \perp N \) has no non-zero free summand. Replacing \( M \) and \( N \) by their reductions modulo torsion, we may assume that both \( M \) and \( N \) are non-zero torsion-free \( R \)-modules, and that neither \( M \) nor \( N \) has a non-zero free direct summand.

As in [2], we note that every homomorphism \( M \to R \) has its image in \( \mathfrak{m} \), and so \( M^* = \text{Hom}_R(M, \mathfrak{m}) \), which has a natural \( E \)-module structure extending the \( R \)-module structure. Therefore \( M^* \) is also an \( E \)-module. Since \( R \) is Gorenstein and \( M \) is torsion-free (= maximal Cohen-Macaulay), the natural map \( M \to M^* \) is an isomorphism, and hence \( M \) itself has an \( E \)-module structure compatible with the original \( R \)-module structure. By symmetry, \( N \) too has a compatible \( E \)-module structure. Lemma [2.2] shows that the natural surjection \( M \otimes_R N \to M \otimes_E N \) is an isomorphism and, in particular, \( M \otimes_E N \) is torsion-free.

Suppose, by way of contradiction, that \( E \) is not local, and put \( A = E/\mathfrak{m}E \). This is a 2-dimensional \( k \)-algebra, and it is not local and hence must be isomorphic to \( k \times k \). Let \( e \) be the idempotent of \( A \) supported on first coordinate. Then neither \( 1 - e \) nor \( 1 - \mathfrak{e} \) is a unit of \( A \). Let \( \mathcal{M} = M/\mathfrak{m}M \) and \( \mathcal{N} = N/\mathfrak{m}N \). We claim that \( e\mathcal{M} \neq 0 \). For suppose \( e\mathcal{M} = 0 \). Lift \( e \) to an element \( \mathfrak{e} \in E \). Then \( e\mathcal{M} \subseteq \mathfrak{m}M \). Moreover, \( \mathfrak{e}M + (1 - \mathfrak{e})M + \mathfrak{m}M = M \), and hence \( (1 - \mathfrak{e})M = M \) by Nakayama’s Lemma. The Determinant Trick yields an element \( a \in (1 - \mathfrak{e})E \) such that \( (1 + a)M = 0 \). But \( M \) is faithful as an \( R \)-module and hence as an \( E \)-module (clear denominators). Therefore \( 1 + a = 0 \), and hence \( -1 \in (1 - \mathfrak{e})E \). But then \( -1 \in (1 - \mathfrak{e})A \), contradicting the fact that \( 1 - \mathfrak{e} \) is not a unit. This proves the claim and shows that \( e\mathcal{M} \neq 0 \). By symmetry, \( (1 - e)\mathcal{N} \neq 0 \), and hence \( e\mathcal{M} \otimes_k (1 - e)\mathcal{N} \neq 0 \). However, the isomorphism \( \alpha : M \otimes_R N \overset{\cong}{\to} M \otimes_E N \) induces an isomorphism \( \mathcal{M} \otimes_k \mathcal{N} \overset{\cong}{\to} \mathcal{M} \otimes_A \mathcal{N} \), carrying the non-zero module \( e\mathcal{M} \otimes_k (1 - e)\mathcal{N} \) onto \( e\mathcal{M} \otimes_A (1 - e)\mathcal{N} = 0 \), a contradiction. This completes the proof that \( E \) is local.

Let \( \mathfrak{n} \) be the maximal ideal of \( E \), and put \( \ell = E/\mathfrak{n} \). Suppose \( \dim_k \ell > 1 \). The inclusion \( \mathfrak{m}E \to \mathfrak{n} \) induces a surjection \( E/\mathfrak{m}E \to E/\mathfrak{n} = \ell \). Since, by Lemma 2.1, \( \dim_k(E/\mathfrak{m}E) = 2 \), this surjection must be an isomorphism, and hence \( \mathfrak{n} = \mathfrak{m}E = \mathfrak{m} \).

Observe that the isomorphism \( \alpha : M \otimes_R N \to M \otimes_E N \) induces an isomorphism

\[
(2.3.1) \quad \mathcal{M} \otimes_k \mathcal{N} \overset{\cong}{\to} \mathcal{M} \otimes_\ell \mathcal{N}.
\]
Put \( u = \dim_k \mathcal{M} \) and \( v = \dim_k \mathcal{N} \). Then \( \dim_k (\mathcal{M} \otimes_k \mathcal{N}) = uv \), and hence \( \dim_k (\mathcal{M} \otimes_k \mathcal{N}) = 2uv \). On the other hand, \( \dim_k (\mathcal{M} \otimes_k \mathcal{N}) = (\dim_k \mathcal{M})(\dim_k \mathcal{N}) = (2u)(2v) = 4uv \). The isomorphism in (2.3.1) forces \( 4uv = 2uv \), and hence either \( u = 0 \) or \( v = 0 \), contradicting Nakayama’s Lemma. This shows that \( \dim_k \ell = 1 \), and the proof is complete. \( \square \)

One might hope, at least for a Gorenstein ring \( (R, \mathfrak{m}, k) \) with finite integral closure \( \overline{R} \), that \( E \) being local with residue field \( k \) would force \( \overline{R} \) to be local with residue field \( k \). Of course, Theorem 2.3 would then answer Conjecture 1.3 affirmatively. The next example dashes this hope.

**Example 2.4.** Let \( k \) be a field and \( D = k[X]/(X^2(X - 1)) \). Then \( D \) is a principal ideal domain with 2 maximal ideals. Let \( A = k[T]/(T^2), B = k[X]/(X^2) \times k[X]/(X - 1)^2 \), and define \( i : A \to B \) by \( i(a + bt) = (a + bx, a + b(x - 1)) \) where \( a, b \in k \), and decapitalization of the indeterminates indicates passage to cosets. Let \( \pi : D \to B \) be the composition of the natural projection of \( D \to D/(X^2(X - 1)^2) \) and the isomorphism \( D/(X^2(X - 1)^2) \xrightarrow{\cong} B \) provided by the Chinese Remainder Theorem. Define \( R \) to be the pullback of \( i \) and \( \pi \):

\[
\begin{array}{ccc}
R & \xrightarrow{\pi} & D \\
\downarrow & & \downarrow \\
A & \xrightarrow{i} & B
\end{array}
\]

By [3] Proposition 3.1], \((R, \mathfrak{m}, k)\) is a local one-dimensional domain, \( \overline{R} = D \), and \( \overline{R} \) is finitely generated as an \( R \)-module. Furthermore, letting \( f \) be the conductor, we have \( A \cong R/f \) and \( B \cong D/f \). Since the length of \( R/f \), namely 4, is twice the length of \( R/\mathfrak{m} \), [3 Corollary 6.5] guarantees that \( R \) is Gorenstein. One checks that \( E := \text{End}_R(\mathfrak{m}) \) is local, with residue field \( k \), but \( \overline{R} \) is not local.

This example cannot be promoted to a counterexample to Conjecture 1.3. To see this, first observe that \( B \) is generated by two elements as an \( A \)-module. It follows that \( \overline{R} = D \) is two-generated as an \( R \)-module. Therefore \( R \) has multiplicity two [4 Theorem 2.1], and hence every ideal of the completion \( \widehat{R} \) is two-generated. It follows that \( \widehat{R} \) is a hypersurface and therefore, by the main theorem of [5], the tensor product of any two non-free finitely generated \( R \)-modules has non-zero torsion.

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