Optimal quantum phase estimation in an atomic gyroscope

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Abstract

We investigate the optimal states for an atomic gyroscope which is used to measure mechanical rotation by phase estimation. In previous studies, various states such as the BAT state and the NOON state are employed as the probe states to estimate the uncertainty of the phase. In this paper, we propose a general method to find the optimal state which can give the maximal quantum Fisher information on both lossless and lossy conditions. As a result, we find that the squeezed entangled state can give a significant enhancement of the precision for moderate loss rates.

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I. INTRODUCTION

In recent decades, the technology of quantum physics plays an important role in precision measurements and the accuracy has been improved significantly in comparison with classical systems. The advantages of quantum technology are reflected in various fields such as the biological sensing [1–4], the measurements of physical constant [5–7], and the gravitational wave detection [8–13]. In quantum metrology, the optical interferometer such as the Mach-Zehnder (MZ) interferometer is frequently used to estimate the relative phase of two modes, and the precision obtained by using classic states can reach the standard quantum limit (SQL), i.e., \(1/\sqrt{N}\), where \(N\) is the total mean number of photons of two modes. In 1981, a typical scheme proposed by Caves [14] is to take a coherent state \(|\alpha\rangle\) and a squeezed vacuum state \(|\xi\rangle\) as the input states of the Mach-Zehnder interferometer, which can beat the SQL. On the other hand, the unique characteristics of quantum states such as entanglement provide us with a way to reach the Heisenberg limit, i.e., \(1/N\), and the NOON state [15–17] is widely studied since its quantum property of maximal entanglement and superior performance in metrology. Based on the result of the NOON state, the entangled coherent state (ECS) [18–21] which is viewed as a similar probe state, has an outperformance on both lossless and lossy conditions than the NOON state.

Similar researches for phase estimation can be applied to the atomic gyroscope, which is designed to measure the phase caused by rotation and composed of a collection of ultracold atoms trapped in an optical lattice loop of several sites [17]. Unlike optical interferometers, the modes of the atomic gyroscope is able to increase as the number of sites increases and input states such as multi-mode entangled states are of research value. In Ref. [17], they investigated three different input states, and found that the NOON state produces the best precision with scaling \(1/N\) under lossless conditions. However, considering the presence of particle loss in practical systems, the BAT state is a better choice because of its robustness in high loss regime. In the above studies, quantum Fisher information (QFI) [22–26] is an important concept in quantum metrology, which gives the lower limit of the variance of parameter \(\theta\) due to the quantum Cramér-Rao bound, i.e.,

\[
\Delta \theta \geq \frac{1}{\sqrt{\mu F_Q}},
\]

where \(\mu\) is the number of independent repeats of the experiment and \(F_Q\) is the QFI. Hence, the core task in many studies is trying every means to obtain larger QFI. In fact, the changes of quantum states can be regarded as a unitary transformation \(U(\theta)\) whether they evolve in the atomic
gyroscopes or in the optical devices. Therefore, for a pure state, the QFI with respect to the parameter $\theta$ is defined as

$$F_Q = 4\left(\langle H^2 \rangle - \langle H \rangle^2\right),$$

(2)

where

$$H \equiv i \left(\partial_\theta U^\dagger\right) U$$

(3)

is a Hermitian operator which contains Fisher information of the whole process and the QFI is determined by the variance of $H$. Therefore, for a definite procedure with a fixed $H$, the optimal probe state could be gotten by comparing various input states.

In this article, we follow the work of the Ref. [17], and propose an equivalent procedure to calculate the highest precision we can reach with the atomic gyroscope introduced above. Although three input states, i.e., the uncorrelated state, the BAT state, and the NOON state, have been discussed before, the Hermitian operator $H$ obtained by utilizing unitary transformations shows that there is still plenty of room to improve the precision. We also investigate the effects of particle loss in the atomic gyroscope. In principle, the maximally entangled states such as the NOON state are vulnerable, and the phase information will be rapidly lost when they decoherence [17, 18, 27]. Therefore, it is important to find a state that can reach the Heisenberg limit and also has good robustness against decoherence. Apart from the ECS involved in Ref. [18], the squeezed entangled state (SES) [28] which is constructed by the superposition of NOON states with different particle numbers like the ECS shows great potential, but the difference is the number states used in the superposition are even number states. The comparison between the ECS and the SES is similar to the relationship between the uncorrelated state and the BAT state discussed in Ref. [17], which shows the superposition of even number states produces better precision and robustness than the superposition of the general number states. Furthermore, Refs. [28, 29] show that the Mandel Q-parameter of the single-mode in a path-symmetric state determines the upper limit of precision. Therefore, these advantages of the SES can be also used to improve the phase estimation of the atomic gyroscope under both lossless conditions and lossy conditions.

II. GENERAL EXPRESSION OF QFI

First of all, it is necessary to review the procedure of phase measurement in Ref. [17]. The atomic gyroscope is composed of $N$ ultracold atoms of mass $m$ trapped in an optical lattice loop
of three sites where the circumference of the loop is $L$, and a schematic diagram is shown in Fig. 1. This scheme can be described by Bose-Hubbard Hamiltonian

$$\frac{H}{\hbar} = \sum_{i=0}^{2} \varepsilon_i \hat{a}_i \hat{a}_i - \frac{2}{i=0} J_i (\hat{a}_i \hat{a}_{i+1} + \hat{a}_{i+1} \hat{a}_i) + \sum_{i=0}^{2} V_i \hat{a}_i \hat{a}_i^2,$$

(4)

where $\hat{a}_i^\dagger$ and $\hat{a}_i$ are creation and annihilation operators in site $i$. The first term accounts for energy offset, and it can be ignored by taking a fixed zero-energy for each site and, we set $\varepsilon_i = 0$. The second term is the coupling between site $i$ and site $i+1$, and the last term represents the interaction between atoms on each site. Moreover, the strengths of coupling energy and interaction energy are described by the parameters $J_i$ and $V_i$ respectively, and the latter is able to be ignored compared with the former when the potential barrier between two sites is reduced, i.e., $V_i \approx 0$. This system is described in detail in Ref. [30]. Therefore, the unitary transformation $U = e^{-iHt/\hbar}$ only depends on the coupling energy in the high coupling regime. It is convenient to make the Hamiltonian diagonalized by using the bases, i.e.,

$$\begin{pmatrix}
\alpha_{-1} \\
\alpha_0 \\
\alpha_1
\end{pmatrix} = \frac{1}{\sqrt{3}} \begin{pmatrix}
1 & e^{-i2\pi/3} & e^{i2\pi/3} \\
1 & 1 & 1 \\
1 & e^{i2\pi/3} & e^{-i2\pi/3}
\end{pmatrix} \begin{pmatrix}
\alpha_0 \\
\alpha_1 \\
\alpha_2
\end{pmatrix},$$

(5)

and the Hamiltonian in high coupling regime is given by

$$\frac{H}{\hbar} = -2J \sum_{j=-1}^{1} \cos (2\pi j / 3) \alpha_j^\dagger \alpha_j.$$

(6)

In this way, the equivalent two-port 50:50 beam splitter usually used in the optical interferometer and the three-port beam splitter (tritter) described in Ref. [30] are able to be realized. Specifically, the unitary transformation of 50:50 two-port beam splitter is expressed as

$$U_{2p} = e^{i\pi/4} (a_0^\dagger a_1 + a_1^\dagger a_0),$$

(7)

and the tritter is expressed as

$$U_{3p} = e^{i2\pi/9} (a_0^\dagger a_1 + a_1^\dagger a_2 + a_2^\dagger a_0 + h.c.).$$

(8)

The inverse operations are realized by changing the phase from $\pi/4$ to $3\pi/4$ for the two-port beam splitter and $2\pi/9$ to $4\pi/9$ for the tritter. The procedure of phase estimation can be briefly summarized as follows: (i) Prepare an initial state. (ii) Perform a tritter operation. (iii) Apply
FIG. 1. Schematic diagram of the phase estimation in the atomic gyroscope. A quantum state is prepared as an input state of the atomic gyroscope which is a ring configuration of three sites. Then the mechanical rotation of the atomic gyroscope causes a phase $\theta$, and after performing several operations the phase $\theta$ is able to read out.

A $2\pi/3$ phase to site two with the unitary transformation $U_{2s} = \exp(i2\pi a_2^\dagger a_2/3)$. (iv) Rotate the systems with velocity $\omega$. (v) Apply a $-2\pi/3$ phase to site two with the unitary transformation $U_{2s}^\dagger = \exp(-i2\pi a_2^\dagger a_2/3)$. (vi) Perform an inverse tritter operation. (vii) Read out the phase $\theta$ caused by the rotation. In step (iv), the Hamiltonian in Eq. (5) becomes

$$H_r = -2J \sum_{j=-1}^{1} \cos \left( \frac{\theta}{3} - 2\pi j/3 \right) \alpha_j^\dagger \alpha_j.$$  (9)

To obtain the velocity of rotation more accurately, we need to reduce the variance of $\theta$ since $\Delta\omega = \Delta\theta \cdot (h/L^2m)$. It is worthy to note that the QFI for various initial states only depends on step (i) to (iv), hence the ultimate precision of the atomic gyroscope is able to be calculated by a general Hermitian operator $H_r$, which can be expressed as

$$H_r = U_{3p}^\dagger U_{2s}^\dagger H_r U_{2s} U_{3p}$$
$$= \frac{-2Jt_\omega}{3} \left[ \sin \left( \frac{\theta + 2\pi}{3} \right) n_0 + \sin \left( \frac{\theta - 2\pi}{3} \right) n_1 + \sin \left( \frac{\theta}{3} \right) n_2 \right],$$  (10)

where $H_r = i \left( \partial_\theta U_r^\dagger \right) U_r = H_r t_\omega / h$ according to Eq. (2) with $U_r = e^{-iH_r t_\omega / h}$ which is the unitary transformation of rotation, and $t_\omega$ is the time that the gyroscope takes to rotate. Utilizing Eqs. (1) and (9), a general expression of QFI for the atomic gyroscope is obtained, which is helpful to find the optimal state that corresponds to this scheme. In addition, the overall process of phase estimation can be simplified as an equivalent unitary transformation with Eqs. (3) and (9),

$$U_{eq} = \exp(i\phi_0 n) \exp(i\phi_1 n_1) \exp(i\phi_2 n_2)$$
$$= \exp(i\phi_0 n) \exp(i\phi_+ (n_1 + n_2)/2) \exp(i\phi_- (n_1 - n_2)/2),$$  (11)
where \( n_i = a_i^+ a_i \), \( n = n_0 + n_1 + n_2 \), and \( \phi_\pm = (\phi_1 \pm \phi_2) \), \( \phi_0 = 2Jt_\omega \cos(\theta/3 + 2\pi/3) \), \( \phi_1 = 2\sqrt{3}Jt_\omega \sin(\theta/3) \), and \( \phi_2 = 2\sqrt{3}Jt_\omega \sin(\theta/3 + \pi/3) \). We see that the operations to gyroscope can be regarded as corresponding linear phase shifts to different modes and it may be helpful to obtain the output state directly. Note that if the input state is a path-symmetric pure state \([28, 31]\), a QFI formula for relative phase \( \phi_- \) will be given by

\[
F_Q = \Delta^2 (n_1 - n_2) .
\]

And the phase \( \Delta \theta \) is able to be got via the error propagation formula \( \Delta \theta = \sqrt{3} \Delta \phi_-/2Jt_\omega \cos(\theta/3 - \pi/3) \).

### III. OPTIMAL INPUT STATE

In this section, we investigate the variance of Hermitian operator \( \mathcal{H} \) with various input states under the condition of no particle loss. To compare different resources equivalently, we take into account the same average particle number \( \bar{n} = N \) for each state.

#### A. Particle number state

First of all, we consider the particle number state which is very common as an input state in quantum metrology. Eq. (9) shows that the Hermitian operator \( \mathcal{H} \) is the function of \( a_i^+ a_i \), and the precision of the phase will be more accurate with a larger variance of number operators. According to Refs. [15, 32], a general input state for three-mode is expressed as \( |\psi\rangle = \sum_{m+n=0}^{m+n=N} c_{m,n} |m, n, N - m - n\rangle \) where \( \sum_{m,n=0}^{m+n=N} |c_{m,n}|^2 = 1 \). From Eq. (11), one can see that the first term \( e^{i\phi_0 n} \) only provides a global phase and can be ignored. Therefore, it is easy to find that the QFI only depends on the relative number of particles from two modes, which means concentrating particles in two modes to increase the variance of number operators is the optimal way to improve the QFI.

In this article, we assume that the phase of site zero is invariant, and only the phase changes of the other two modes are considered here. For the uncorrelated state, the BAT state involved in Ref. [17], \( F_Q \) for parameter \( \phi_- \) are \( N \) and \( N(N/2 + N) \), respectively. And for the NOON state, which is the optimal state, has the maximal QFI of \( F_Q = N^2 \) in the lossless case in Ref. [17]. Moreover, we find that \( \Delta \theta \) can be improved slightly by utilizing \( \Delta \phi_+ \) and the maximally entangled state \( |\psi\rangle_M = 1/\sqrt{2} (|N, N\rangle + |0, 0\rangle) \) that also has the maximal QFI. Note that the error propagation
formula of $\Delta \phi_+$ is $\Delta \theta = \Delta \phi_+/2J_{t\omega} \cos(\theta/3 + \pi/6)$, and the minimal uncertainty of the phase $\theta$ is given by

$$\Delta \theta_{\text{min}} = \frac{1}{2NJ_{t\omega}}.$$  \hspace{1cm} (13)

Compared with the precision given by the NOON state, the parameter $\Delta \theta$ can be improved by $\sqrt{3}$ times with $\phi_+$ and $|\psi\rangle_M$.

B. Entangled coherent state

Entangled coherent state (ECS) shows the superiority for phase estimation in optical interferometers [18], and its advantage still exists in the atomic gyroscope. In general, the ECS is given by

$$|\psi\rangle_E = \mathcal{N}_\alpha (|\alpha, 0\rangle + |0, \alpha\rangle)$$ \hspace{1cm} (14)

where $|\alpha\rangle$ is a coherent state and the normalization factor is $\mathcal{N}_\alpha = 1/\sqrt{2 \left(1 + e^{-|\alpha|^2}\right)}$. The normalization factor must satisfy that $2\mathcal{N}_\alpha^2 |\alpha|^2 = N$ for the same average particle number $N$. On the situation of no particle loss, we make the approximation that $|\alpha|^2 \approx N$ and $\mathcal{N}_\alpha \approx 1/\sqrt{2}$. Note that if we use the parameter $\phi_-$ or Eq. (10) to estimate the phase $\theta$, then we will obtain the QFI of $F_Q \approx N(N+1)$. This is indeed correct for two-mode input states in the absence of a phase reference [28, 31], but for the gyroscope we consider here, there is an extra mode (i.e., site zero) which can be seen as a phase reference. It allows us to estimate $\phi_1$ or $\phi_2$ directly instead of the relative phase between different modes. Utilizing Eqs. (2), (3) and (11), the maximal QFI is given by $F_Q (\phi_{1(2)}) = \Delta^2 n_{1(2)} \approx N(N+2)$. Hence, the minimal uncertainty of the phase $\theta$ is given by,

$$\Delta \theta_{\text{min}} \approx \frac{\sqrt{3}}{2\sqrt{N(N+2)}J_{t\omega}}.$$ \hspace{1cm} (15)

This precision is basically consistent with the result obtained in Ref. [18], and it shows that the ECS produces a better precision than the NOON state involved in Ref. [17], especially when $N$ is modest. In addition, we can also use the parameter $\phi_+$ and a similar state with $|\psi\rangle_M$, that is, $|\psi\rangle_{EM} = \mathcal{N}_\alpha (|\alpha, \alpha\rangle + |0, 0\rangle)$, to improve $\Delta \theta$ slightly. Finally, the minimal $\Delta \theta$ is given by,

$$\Delta \theta_{\text{min}} \approx \frac{1}{2\sqrt{N(N+1)}J_{t\omega}}.$$ \hspace{1cm} (16)

The reason that the ECS has such a advantage is because the ECS can rewritten as [18]

$$|\psi\rangle_E = \mathcal{N}_\alpha e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} (|n, 0\rangle + |0, n\rangle),$$ \hspace{1cm} (17)
and it can be understood as a NOON-like state which is the superposition of the NOON state with different particle numbers, and the QFI is proportional to the square of the number of particles, so the NOON state with larger particle number would improve the precision significantly.

C. Squeezed entangled state

Inspired by the ECS, it is easy to obtain an idea that we can turn the coherent state in the ECS into the squeezed vacuum state to obtain the squeezed entangled state (SES), that is,

\[ |\psi\rangle_S = N_\xi (|\xi,0\rangle + |0,\xi\rangle) \tag{18} \]

where \(N_\xi = (2/\cosh r + 2)^{-1/2}, r = |\xi|\). And \(r\) is determined by \(\cosh r/(1 + \cosh r) \sinh^2 r = N\) in the case of the fixed average particle number. The SES has been discussed in Refs. [28, 29], and a generation scheme is proposed in Ref. [29]. Likewise, we can use \(\phi_1\) or \(\phi_2\) to obtain the variance of \(\theta\). With a phase reference, the maximal QFI is \(F_Q(\phi_1(2)) \approx 5N^2 + 4N\), which is almost 2 times better than the QFI of \(F_Q(\phi_-) \approx 3N^2 + 2N\) with respect to the relative phase \(\phi_-\) involved in Refs. [28, 29]. And the minimal \(\Delta \theta\) is given by

\[ \Delta \theta_{\text{min}} \approx \frac{\sqrt{3}}{2\sqrt{N(5N + 4)Jt_\omega}}. \tag{19} \]

Similarly, the uncertainty of the phase \(\theta\) obtained by utilizing \(\phi_+\) and \(|\psi\rangle_{SM} = N_\xi (|\xi,\xi\rangle + |0,0\rangle)\) is still smaller, that is

\[ \Delta \theta_{\text{min}} \approx \frac{1}{2\sqrt{N(3N + 2)Jt_\omega}}. \tag{20} \]

We see that the precision is improved \(\sqrt{3}\) times compared with the NOON state and the ECS. Fig. 2 shows the phase uncertainty \(\Delta \phi_1\) of various quantum states varies with respect to \(N\) in the lossless case. We find that when the average particle number is fixed, the ECS is superior to the NOON state only when \(N\) is small, however, if we choose the SES as the input state, the precision is improved significantly regardless of the number of particles. To appreciate this, Eq. (18) is rewritten as

\[ |\psi\rangle_S = N_\xi \sum_{n=0}^{\infty} C_{2n} (|2n,0\rangle + |0,2n\rangle), \tag{21} \]

where \(C_{2n} = 1/\sqrt{\cosh r (-e^{i\vartheta} \tanh r/2)^n \sqrt{(2n)!/n!}}, \vartheta = \text{arg}(\xi)\). It is easy to find that SES is a coherent superposition of the NOON states with different even particle numbers. It means larger particle number states are included in the SES under the condition of the same average particle
Phase uncertainty $\Delta \phi_1$ varies with the number of particles in a lossless case. $\Delta \phi_1$ of the NOON state (purple solid line) is approximately equal to that of the ECS (red dot-dashed line) for large $N$, whereas $\Delta \phi$ of the SES (blue dashed line) is better than the first two states in all ranges.

number, which is beneficial to improve the precision of the gyroscope. Furthermore, Refs. [28, 29] show that the QFI is proportional to Mandel Q-parameter of the single-mode in path-symmetric state, hence the squeezed vacuum state has an advantage over the coherent state.

IV. EFFECTS OF PARTICLE LOSS

In practice, decoherence and particle loss are inevitable, so it is necessary to take into account the effects of particle loss of the atomic gyroscope in this section. In Refs. [17, 20, 32], the related issues have been involved. In this paper, we still use the model that inserting two fictitious "beam splitter" with the transmission rate of $\eta$ into two sites and, in general, the output state is described by a mixed state $\rho$. In this scheme, the particle loss is the loss from the momentum modes during $t_\omega$ [17]. Without loss of generality, we assume that both modes have the same loss rates $R = 1 - \eta$, and the model in Ref. [32] is used here to describe the effect of particle loss on the QFI. The situations of the particle number state and the ECS have been discussed in Refs. [18, 20, 32], so we focus on the calculation of the SES in this article.

First of all, we should calculate the reduced density matrix of the SES after particle loss. The
particle number states in Eq. (21) evolve into
\[ |2n,0\rangle \rightarrow \sum_{l_a=0}^{2n} \sqrt{B_{l_a}^{2n}} |2n-l_a,0\rangle \otimes |l_a,0\rangle , \]  
\[ |0,2n\rangle \rightarrow \sum_{l_b=0}^{2n} \sqrt{B_{l_b}^{2n}} |0,2n-l_b\rangle \otimes |0,l_b\rangle , \]

where
\[ B_{l_{a(b)}}^{2n} = \binom{2n}{l_{a(b)}} \eta^{2n-l_{a(b)}} (1-\eta)^{l_{a(b)}} , \]

and \(|l_a,0\rangle\) and \(|0,l_b\rangle\) are the states which represent \(l_a, l_b\) particles are lost from site one and site two, respectively. Then we obtain the reduced density matrix which is expressed as
\[ \rho = p_{0,0} \rho_{0,0} + \sum_{l_a=1}^{\infty} p_{l_a,0} \rho_{l_a,0} + \sum_{l_b=1}^{\infty} p_{0,l_b} \rho_{0,l_b} , \]

where \(p_{l,m} = \langle \psi_{l,m} | \psi_{l,m} \rangle\), and \(l, m\) are the number of particles lost. When \(l = m = 0\), we have
\[ |\psi_{0,0}\rangle = \frac{1}{\sqrt{P_{0,0}}} N_\xi \sum_{n=0}^{\infty} C_{2n} \eta^n (|2n,0\rangle + |0,2n\rangle) . \]

While \(l = l_a, m = 0\) and \(l = 0, m = l_b\), we also obtain
\[ |\psi_{l_a,0}\rangle = \frac{1}{\sqrt{P_{l_a,0}}} N_\xi \left( \sum_{n=1}^{\infty} C_{2n} \sqrt{B_{l_a}^{2n}} |2n-l_a,0\rangle \right) , \]
\[ |\psi_{0,l_b}\rangle = \frac{1}{\sqrt{P_{0,l_b}}} N_\xi \left( \sum_{n=1}^{\infty} C_{2n} \sqrt{B_{l_b}^{2n}} |0,2n-l_b\rangle \right) , \]

where
\[ \Gamma = \begin{cases} \frac{(l_{a(b)} + 1)}{2} & \text{for } l_{a(b)} \text{ is odd}, \\ \frac{l_{a(b)}}{2} & \text{for } l_{a(b)} \text{ is even}. \end{cases} \]

\(p_{0,0}, p_{l_a,0}\) and \(p_{0,l_b}\) are the normalization factors required for calculation of the mixed state \(\rho\). We set \(\vartheta = 0, \xi = r\), then Eq. (25) can be rewritten as
\[ |\psi_{0,0}\rangle = N_\xi (|\tilde{r},0\rangle + |0,\tilde{r}\rangle) , \]

where \(\tilde{r} = \text{arctanh}(\eta \tanh r)\), and \(N_\xi = (2/\cosh \tilde{r} + 2)^{-1/2}\). Utilizing Eqs. (25), (26) and (28), it is easy to obtain
\[ p_{0,0} = \frac{1 + \cosh \tilde{r}}{1 + \cosh r} , \]
\[ \sum_{l_a=0}^{\infty} p_{l_a,0} = \sum_{l_b=0}^{\infty} p_{0,l_b} = \frac{\cosh r - \cosh \tilde{r}}{2(1 + \cosh r)} . \]
FIG. 3. Phase uncertainty $\Delta \phi_1$ varies with transmission rate of $\eta$ for $N = 10$. The red dot-dashed line and the blue dashed line show the precision of $\phi_1$ for the ECS and the SES, respectively. The BAT state and the NOON state which are investigated in the previous studies are depicted in yellow dotted line and the purple solid line for comparison.

To obtain the QFI of the mixed state $\rho$, we have to diagonalize the mixed state as $\rho = \sum_m \lambda_m |\lambda_m\rangle \langle \lambda_m|$, where $\lambda_m$ are eigenvalues and $|\lambda_m\rangle$ are eigenvectors. Here we first estimate the parameter $\theta$ through $\phi_1$, then choose average particle number $\bar{n} = 10$, and then truncate the particle number state at $n = 40$ in Eq. (25). For a mixed state $\rho$, the QFI is expressed as

$$F_Q = 4 \sum_m \lambda_m \langle n_1 | \lambda_m \rangle^2 - \sum_{m,m'} \frac{8\lambda_m \lambda_{m'}}{\lambda_m + \lambda_{m'}} |\langle \lambda_m | n_1 | \lambda_{m'}\rangle|^2,$$

and the numerical simulation of QFI is shown in Fig. 3.

In this figure, we show the variation of $\Delta \phi_1$ with the change of $\eta$. In the small loss regime, the SES produces the best precision compared with the BAT state, the NOON state, and the ECS. However, the SES is inferior to the ECS when $\eta < 0.25$, which means the ECS has a better performance at the range of low transmission rate. In addition, we find that the NOON state and the BAT state have no advantage in all ranges of transmission rate, which means the states constructed by the superposition of the NOON states with different particle numbers have the potential to achieve higher precision and better robustness than general particle number states.
V. CONCLUSION

In this paper, we have investigated the optimal input state in the atomic gyroscope which is used to measure the small mechanical rotation, and the main approach is to find the state that produces the maximal QFI. We have provided a Hermitian operator $\mathcal{H}$ which contains Fisher information of the whole process, hence the measurement procedure is able to be simplified as an equivalent unitary transformation. To obtain the maximal QFI, we take the squeezed entangled state (SES) as a candidate to improve the precision of the atomic gyroscope.

Compared with the particle number state (especially the NOON state) and the entangled coherent state, the best precision is achieved by taking the SES as the input state in the ideal case. In addition, the existence of an extra mode allows us to set up a phase reference, which can make it possible to estimate the phase change of a mode directly. And on this basis, we found that using another form of entangled states, such as $|\psi\rangle_M$, $|\psi\rangle_{EM}$ and $|\psi\rangle_{SM}$ can obtain slightly better precision in the lossless case. Moreover, the QFI under practical conditions is also considered, and the SES shows its great robustness for moderate loss rates. Therefore, the SES is the preferred input state in this measurement scheme, and these results also show that the SES has a great application prospect in quantum metrology. Furthermore, we believe that the method we used in this article is helpful to the people who want to improve the precision of other measurement systems.

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