HOMOTOPY PROPERTIES OF SPACES OF SMOOTH FUNCTIONS ON 2-TORUS

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Abstract. Let \( f : T^2 \to \mathbb{R} \) be a Morse function on a 2-torus, \( S(f) \) and \( O(f) \) be its stabilizer and orbit with respect to the right action of the group \( D(T^2) \) of diffeomorphisms of \( T^2 \), \( D_{id}(T^2) \) be the identity path component of \( D(T^2) \), and \( S'(f) = S(f) \cap D_{id}(T^2) \). We give sufficient conditions under which
\[
\pi_1 O(f) \cong \pi_1 D_{id}(T^2) \times \pi_0 S'(f) \equiv \mathbb{Z}^2 \times \pi_0 S'(f).
\]

In fact this result holds for a larger class of smooth functions \( f : T^2 \to \mathbb{R} \) having the following property: for every critical point \( z \) of \( f \) the germ of \( f \) at \( z \) is smoothly equivalent to a homogeneous polynomial \( \mathbb{R}^2 \to \mathbb{R} \) without multiple factors.

1. Introduction

Let \( M \) be a smooth closed oriented surface and \( D(M) \) be its groups diffeomorphisms acting from the right of the space \( C^\infty(M, \mathbb{R}) \) of smooth functions by the following rule:
\[
(f, h) \mapsto f \circ h : M \xrightarrow{h} M \xrightarrow{f} \mathbb{R},
\]
for \( f \in C^\infty(M, \mathbb{R}) \) and \( h \in D(M) \). Denote by
\[
S(f) = \{ f \in D(M) \mid f \circ h = f \}, \quad O(f) = \{ f \circ h \mid h \in D(M) \}
\]
respectively the stabilizer and the orbit of \( f \in C^\infty(M, \mathbb{R}) \) under the action (1). Endow the spaces \( D(M) \) and \( C^\infty(M, \mathbb{R}) \) with strong Whitney \( C^\infty \)-topologies. These topologies induce certain topologies on \( S(f) \) and \( O(f) \). Let also \( D_{id}(M) \) and \( S_{id}(f) \) be the path components of the identity map \( id_M \) of the groups \( D(M) \) and \( S(f) \), and let \( O_f(f) \) be the path component of \( f \) in its orbit \( O(f) \).

Denote by \( \text{Morse}(M, \mathbb{R}) \subset C^\infty(M, \mathbb{R}) \) the subset consisting of all Morse functions, that is the functions having only non-degenerate critical points. It is well known that \( \text{Morse}(M, \mathbb{R}) \) is open and everywhere dense in \( C^\infty(M, \mathbb{R}) \), e.g. [11]. Path components of \( \text{Morse}(M, \mathbb{R}) \) are computed in [14] [4] [7], and its homotopy type is described in [5].

Recall that the germs of smooth functions \( f, g : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0) \) are smoothly equivalent at point \( 0 \in \mathbb{R}^2 \) if there exist germs of diffeomorphisms \( h : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0) \) and \( \phi : (\mathbb{R}, 0) \to (\mathbb{R}^2, 0) \) such that \( \phi \circ g = f \circ h \).

Let \( \mathcal{F}(M, \mathbb{R}) \) be the subset of \( C^\infty(M, \mathbb{R}) \) consisting of functions \( f \) having the following property:

Property (L). For each critical point \( z \) of \( f \) its germ at \( z \) is smoothly equivalent to some homogeneous polynomial \( \mathbb{R}^2 \to \mathbb{R} \) without multiple factors.

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Notice that if \( z \) is a nondegenerate critical point of a smooth function \( f : M \to \mathbb{R} \) then the germ of \( f \) at \( z \) is equivalent to a homogeneous polynomial \( \pm x^2 \pm y^2 \) which obviously has no multiple factors. Hence we have an inclusion

\[
\text{Morse}(M, \mathbb{R}) \subset \mathcal{F}(M, \mathbb{R}).
\]

It is known, \([12, 13]\), see also \([8, \S 11]\), that for functions from \( \mathcal{F}(M, \mathbb{R}) \) the natural map

\[
p : D(M) \longrightarrow O(f), \quad p(h) = f \circ h,
\]

is a Serre fibration.

It is proved in \([8, 9]\) that \( S_{id}(f) \) is contractible for every \( f \in \mathcal{F}(M, \mathbb{R}) \) except for the case when \( f : S^2 \to \mathbb{R} \) is a Morse function having exactly two critical point one of which is a maximum and another one is a minimum. In that case \( S_{id}(f) \) is homotopy equivalent to the circle \( S^1 \).

So assume that \( S_{id}(f) \) is contractible. Then it follows from the description of the homotopy type of groups \( D_{id}(M) \), see \([11, 2, 3]\), exact sequence of homotopy groups of fibration (2), and from results of \([9, 10]\) that \( \pi_i O(f) = \pi_i M \) for \( i \geq 3 \), \( \pi_2 O(f) = 0 \), and for \( \pi_1 O(f) \) we have a short exact sequence

\[
1 \longrightarrow \pi_1 D_{id}(M) \xrightarrow{p_1} \pi_1 O(f) \xrightarrow{\partial_1} \pi_0 S'(f) \longrightarrow 1
\]

where \( S'(f) = S(f) \cap D_{id}(M) \).

Notice that if \( M \) is distinct from the 2-sphere \( S^2 \) and 2-torus \( T^2 \), then the group \( D_{id}(M) \) is contractible, and so we get an isomorphism \( \pi_1 O(f) \cong \pi_0 S'(f) \).

However if \( M = S^2 \) or \( T^2 \) then the structure of the sequence (3) is not understood.

The aim of the present note is to give a sufficient conditions when the sequence (3) splits for the case \( M = T^2 \), see Theorem 2 below.

1.1. **Graph of a smooth function.** Let \( f \in \mathcal{F}(M, \mathbb{R}) \), \( t \in \mathbb{R} \), and \( \omega \) be the connected component of the level set \( f^{-1}(t) \). We will say that \( \omega \) is critical if it contains a critical point \( f \). Otherwise \( \omega \) will be called regular.

Consider the partition of \( M \) into the connected components of level sets of \( f \). Let also \( \Gamma(f) \) be the corresponding factor space. It is well known that \( \Gamma(f) \) has a structure of a one-dimensional CW-complex and often called the Kronrod-Reeb graph or simply the graph of \( f \). The vertices of \( \Gamma(f) \) are critical components of level sets of \( f \), while open edges of \( \Gamma(f) \) correspond to connected components of the complement of \( M \) to the union of all critical components of level sets of \( f \).

Notice that \( f \) can be represented as a composition

\[
f = \phi \circ p_f : M \xrightarrow{p_f} \Gamma(f) \xrightarrow{\phi} \mathbb{R},
\]

where \( p_f \) is the factor map and \( \phi \) is the function of \( \Gamma(f) \) induced by \( f \).

1.2. **Action of \( S(f) \) on \( \Gamma(f) \).** Let \( h \in S(f) \), that is \( f \circ h = f \), and so \( h(f^{-1}(t)) = f^{-1}(t) \) for all \( t \in \mathbb{R} \). Therefore \( h \) interchanges connected components of level sets of \( f \),
i.e. the points of $\Gamma(f)$. It is easy to check that $h$ induces a certain homeomorphism $\rho(h)$ of $\Gamma(f)$ such that the following diagram is commutative:

$$
\begin{array}{ccc}
M & \xrightarrow{p_f} & \Gamma(f) \\
\downarrow h & & \downarrow \rho(h) \\
M & \xrightarrow{p_f} & \Gamma(f)
\end{array}
\xrightarrow{\phi} \mathbb{R} \tag{4}
$$

and that the correspondence $h \mapsto \rho(h)$ is a homomorphism $\rho : S(f) \to \text{Aut}(\Gamma(f))$ into the group of all automorphisms of $\Gamma(f)$.

Consider the group $S'(f) = S(f) \cap D(T^2)$ from the right part of sequence (3), and let

$$G := \rho(S'(f))$$

be its image in $\text{Aut}(\Gamma(f))$. Thus $G$ is the group of automorphisms of $\Gamma(f)$ induced by isotopic to $\text{id}_M$ diffeomorphisms from $h \in S(f)$. Let us emphasize that a particular isotopy between $h$ and $\text{id}_{T^2}$ does not necessarily consist of diffeomorphisms belonging to $S(f)$.

Also notice that it follows from (4) and the observation that the function $\phi : \Gamma(f) \to \mathbb{R}$ is monotone of edges of $\Gamma(f)$ that the group $G$ is finite.

Let $v$ be a vertex of $\Gamma(f)$, and

$$G_v = \{ g \in G \mid g(v) = v \}$$

be the stabilizer of $v$ with respect to $G$. By a star $\text{star}(v)$ of $v$ we will mean an arbitrary connected closed $G_v$-invariant neighbourhood of $v$ in $\Gamma(f)$ containing no other vertices of $\Gamma(f)$.

Let us fix any star $\text{star}(v)$ of $v$ and denote by

$$G_v^{\text{loc}} = \{ g|_{\text{star}(v)} \mid g \in G_v \}$$

the subgroups of $\text{Aut}(\text{star}(v))$ consisting of restrictions of elements from $G_v$ onto $\text{star}(v)$. We will call $G_v^{\text{loc}}$ the local stabilizer of the vertex $v$ with respect the group $G$. Evidently $G_v^{\text{loc}}$ does not depend on a particular choice of a star $\text{star}(v)$.

The aim of this note is to prove the following two statements.

**Proposition 1.** Let $f \in \mathcal{F}(T^2, \mathbb{R})$ be such that its graph $\Gamma(f)$ is a tree. Then there exists a unique vertex $v$ of $\Gamma(f)$ such that the complement $T^2 \setminus p_f^{-1}(v)$ is a disjoint union of open 2-disks.

**Theorem 2.** Let $f \in \mathcal{F}(T^2, \mathbb{R})$ be such that its graph $\Gamma(f)$ is a tree, and $v$ be the vertex of $\Gamma(f)$ described in Proposition 1. Suppose that the local stabilizer $G_v^{\text{loc}}$ of $v$ is a trivial group. Then the sequence (3) splits, and so

$$\pi_1 \mathcal{O}_f(f) \cong \pi_1 D_{\text{id}}(T^2) \times \pi_0 S'(f) \cong \mathbb{Z}^2 \times \pi_0 S'(f).$$

2. Proof of Proposition 1

Let $f \in \mathcal{F}(T^2, \mathbb{R})$ such that $\Gamma(f)$ is a tree. The following lemma is evident.
Lemma 3. Let $e$ be an open edge of the tree $\Gamma(f)$, $z \in e$ be a point, and $C = p_f^{-1}(z)$ be the corresponding regular component of some level set of $f$, so $C$ is a simple closed curve in $T^2$. Then

1. $z$ divides $\Gamma(f)$;
2. $C$ divides $T^2$ and therefore only one of two connected components of $T^2 \setminus C$ is a 2-disk. \hfill \square

Let $e = (u_0u_1)$ be an open edge of the tree $\Gamma(f)$, $z \in e$, and $C = p_f^{-1}(z)$ be the same as in Lemma 3. For $i = 0, 1$ denote by $T_{zu_i}$ the closure of those connected component of $\Gamma(f) \setminus z$ which contains the point $u_i$. Put

$$X_i = p_f^{-1}(T_{zu_i}).$$

Then by Lemma 3 exactly one of two subsurfaces either $X_0$ or $X_1$ is a 2-disk. Let us orient the edge $e$ from $u_0$ to $u_1$ whenever $X_0$ is a 2-disk and from $u_1$ to $u_0$ otherwise.

Then each edge of $\Gamma(f)$ obtains a canonical orientation and so $\Gamma(f)$ is a directed tree.

Lemma 4. For each vertex $u$ of the directed tree $\Gamma(f)$ there exists at most one edge going from $u$.

Proof. Suppose that there are two edges going from $u$ and finishing at vertices $v$ and $v'$ respectively. Choose arbitrary points $z_0 \in (uv)$ and $z_1 \in (uv')$ and denote

$$A = p_f^{-1}(T_{zu_0}), \quad A' = p_f^{-1}(T_{zu_1}), \quad B = p_f^{-1}(T_{zu_1}), \quad B' = p_f^{-1}(T_{zu_1}),$$

see Figure 1. By definition of orientation of edges, $A$ and $B$ are 2-disks. Moreover, since $T^2 = A \cup A' = B \cup B'$,

$$A' \subset B, \quad B' \subset A,$$

and the intersections $A \cap A' = p_f^{-1}(z_0)$ and $B \cap B' = p_f^{-1}(z_1)$ are simple closed curves, it follows that each of subsurfaces $A'$ and $B'$ is a torus with one hole. But then neither $A'$ nor $B'$ can be embedded into a 2-disk which contradicts to the inclusions (5). Hence for each vertex $u$ of the directed tree $\Gamma(f)$ there exists at most one edge going from $u$. \hfill \square

Let $v$ be a vertex of $\Gamma(f)$. Notice that the complement $T^2 \setminus p_f^{-1}(v)$ is a union of 2-disks if and only if all edges incident to $v$ go into $v$, that is $v$ is a sink.

Thus for the proof of Proposition 1 it suffices to prove that the oriented tree $\Gamma(f)$ has a unique sink. This statement is a direct consequence of the following lemma.
Lemma 5. Let \( \Gamma \) be an oriented tree.

(a) If \( \Gamma \) is finite, then it has maximal vertices.
(b) Suppose that for each vertex of \( \Gamma \) there exists at most one edge going from \( u \). Then \( \Gamma \) has at most one sink.

Proof. (a) Suppose that \( \Gamma \) has no sinks, so for each vertex \( v \) there exist at least one edge going from \( u \). Let \( v_0, \ldots, v_{n-1}, v_n \) be an arbitrary oriented path in \( \Gamma \) consisting of mutually distinct vertices. Since the edge \((v_{n-1}v_n)\) goes into \( v_n \), it follows from Proposition 1 that there exists an edge \((v_nv_{n+1})\) going from \( v_n \). Notice that \( v_{n+1} \neq v_i \), \( i = 0, \ldots, n \), otherwise \( v_0, \ldots, v_n, v_{n+1} \) would be a cycle in the tree \( \Gamma \) which is impossible. Therefore every oriented path in \( \Gamma \) can be extended to a longer one which contradicts to a finiteness of \( \Gamma \). Hence \( \Gamma \) must have sinks.

(b) Suppose that \( \Gamma \) has two sinks \( v_1 \) and \( v_2 \) and let \( \gamma : e_0, \ldots, e_k \) be a unique path connecting \( v_1 \) and \( v_2 \). Since the edges \( e_0 \) and \( e_k \) go into \( v_1 \) and \( v_2 \) respectively, it easily follows that for at least one vertex \( u \) of \( \gamma \) the edges incident to it goes from \( u \), which is impossible due to the assumption, see Figure 2. Hence \( \Gamma \) has at most one sink. \( \square \)

Figure 2. Path between \( v_1 \) and \( v_2 \)

Now existence of a sink in \( \Gamma(f) \) follows from (a) of Lemma 5 and its uniqueness from (b). Proposition 1 is completed.

3. Proof of Theorem 2

Let \( f \in \mathcal{F}(T^2, \mathbb{R}) \) be such that \( \Gamma(f) \) is a tree, and \( v \) be a unique maximal vertex of \( \Gamma(f) \) described in Proposition 1. Suppose that \( G^\text{loc}_v = 1 \). We should prove that the sequence

\[
1 \longrightarrow \pi_1\mathcal{D}_\text{id}(T^2) \xrightarrow{p_1} \pi_1\mathcal{O}_f(f) \xrightarrow{\partial_1} \pi_0\mathcal{S}'(f) \longrightarrow 1
\]

splits.

Notice that due to \( \mathcal{S} \) Lemma 2.2] the image \( p_1(\pi_1\mathcal{D}_\text{id}(T^2)) \) is contained in the center of the group \( \pi_1\mathcal{O}_f(f) \). Therefore for splitting of (5) it suffices to construct a section \( s : \pi_0\mathcal{S}'(f) \rightarrow \pi_1\mathcal{O}_f(f) \), that is a homomorphism such that \( \partial_1 \circ s = \text{id} \).

First we recall the construction of the boundary homomorphism \( \partial_1 \). Let \( \omega_t \) be a loop in \( \mathcal{O}_f(f) \), that is a continuous map \( \omega : [0, 1] \rightarrow \mathcal{O}_f(f) \) such that \( \omega_0 = \omega_1 \). As \( p : \mathcal{D}(T^2) \rightarrow \mathcal{O}(f) \) is a Serre fibration, \( \omega \) can be lifted to a path in \( \mathcal{D}(T^2) \). In other words, there exists a continuous map \( h : [0, 1] \rightarrow \mathcal{D}(T^2) \) such that \( \omega = p \circ h \), that is \( \omega_t = p(h_t) = f \circ h_t \) for all \( t \in [0, 1] \). Then by definition \( \partial_1(\omega) = [h_1] \), where \([h_1]\) is the class of \( h_1 \) in \( \pi_0\mathcal{S}'(f) \).
Thus if $h \in S'(f)$ and $h : [0, 1] \to D(T^2)$ is a path such that $h_0 = \text{id}$ and $h_1 = h$, then $\omega_t = f \circ h_t$ is a loop in $O_f(f)$ such that $\partial_1(\omega) = h$.

Now Theorem 2 is a consequence of the following lemma.

Lemma 6. Let $v$ be a unique sink of $\Gamma(f)$, $(vu)$ be any open edge of $\Gamma(f)$ incident to $v$, $z \in (vu)$ be a point, and $C = p_f^{-1}(z)$ be the corresponding simple closed curve on $T^2$. If the group $C^\text{loc}$ is trivial, then the following statements hold true.

(i) Let $h \in S'(f)$. Then $h(C) = C$ and there exists an isotopy $h_t : T^2 \to T^2$, $t \in [0, 1]$, such that

\[ h_0 = \text{id}_{T^2}, \quad h_1 = h, \quad h_t(C) = C, \forall t \in [0, 1]. \tag{7} \]

(ii) If $\{h'_t\}$ is any other isotopy satisfying (7), then the paths $\{h_t\}$ and $\{h'_t\}$ are homotopic in $D(T^2)$ relatively their ends. In particular, the loops $\{f \circ h_t\}$ and $\{f \circ h'_t\}$ represent the same element of $\pi_1O_f(f)$. Denote that element by $s(h)$.

(iii) The map $s : h \mapsto s(h)$ is a homomorphism $s : \pi_0S'(f) \to \pi_1O_f(f)$ such that $\partial_1 \circ s = \text{id}$, i.e. a section of $\partial_1$. Hence the sequence (6) splits.

Proof. (i) We need the following lemma, see also [6].

Lemma 7. Let $M$ be a smooth compact surface, $f \in F(M, \mathbb{R})$, $\Gamma(f)$ be the graph of $f$, $\rho : S(f) \to \text{Aut}(\Gamma(f))$ be the action homomorphism of $S(f)$ on $\Gamma(f)$, $v$ be any vertex of $\Gamma(f)$, $\text{star}(v)$ be any star of $v$ in $\Gamma(f)$, and $N = p_f^{-1}(\text{star}(v))$. Let also $h \in S'(f)$ and $\rho(h) : \Gamma(f) \to \Gamma(f)$ be the corresponding automorphism of $\Gamma(f)$ induced by $h$. Suppose that $\rho(h)(v) = v$ and $\rho(h)|_{\text{star}(v)} = \text{id}$. Then there exists an isotopy $g_t : T^2 \to T^2$, $t \in [0, 1]$, such that

1. $g_0 = h$;
2. $g_t \in S'(f)$;
3. $g_t$ is fixed on $N$;
4. $\rho(h) = \rho(g_t) = \text{id}$ for each $t \in [0, 1]$.

In particular, $[h] = [g_t] \in \pi_0S'(f)$.

Proof. Let $V = p_f^{-1}(v)$ be the critical component of the critical level set of $f$ corresponding to $v$. Then $V$ is a finite graph embedded into $M$ and it follows from $\rho(h)(v) = v$ that $h(V) = V$. Since $h$ is isotopic to $\text{id}_{T^2}$ and trivially acts on $\text{star}(v)$, it follows from [8] Theorem 7.1 that $h$ preserves each edge $e$ of $V$ and keeps its orientation. Now existence of an isotopy satisfying (1)-(4) follows from [8] Lemmas 6.4 and 4.14. \qed

Let us prove (i). Not loosing generality one can assume that there are two stars $\text{star}_1(v)$ and $\text{star}_2(v)$ of $v$ such that $z \in \text{star}_1(v) \subset \text{Int}(\text{star}(v))$, where $\text{Int}(\text{star}(v))$ is the interior of $\text{star}(v)$ in $\Gamma(f)$. Hence if we put $N_1 = p_f^{-1}(\text{star}_1(v))$ and $N = p_f^{-1}(\text{star}(v))$, then $N_1 \subset \text{Int}(N)$.

Let $h \in S'(f)$ and $g_t : T^2 \to T^2$, $t \in [0, 1]$, be an isotopy satisfying (1)-(4) of Lemma 7. Then it follows from (3) that $\rho(g_t)(z) = z$, whence $g_t(C) = C$ for all $t \in [0, 1]$. Since $g_1$ is fixed on $N$ and the complement $T^2 \setminus N_1$ consists only of 2-disks, we see that $g_1$ is isotopic to $\text{id}_{T^2}$ via an isotopy fixed on $N_1$ and therefore on $C$.

Hence $h$ is isotopic to $\text{id}_{T^2}$ via an isotopy leaving $C$ invariant.
(ii) Let us recall the following well known statement.

**Lemma 8.** Let \( \omega : T^2 \times [0,1] \to T^2 \) be a loop in \( \mathcal{D}_{ld}(T^2) \), that is an isotopy \( \omega_0 = \omega_1 = \text{id}_{T^2} \). Let also \( q \in T^2 \) be a point and \( \omega_q : \{q\} \times [0,1] \to T^2 \) be a loop in \( T^2 \) given by the formula: \( \omega_q(t) = \omega(q,t) \). Then the loop \( \omega \) is null-homotopic in \( \mathcal{D}_{ld}(T^2) \) if and only if \( \omega_q \) is null-homotopic in \( T^2 \).

**Proof.** Since \( T^2 \) is a connected Lie group, it acts on itself by right shifts being diffeomorphisms. This action induces an embedding \( i : T^2 \hookrightarrow \mathcal{D}_{ld}(T^2) \). It is well known, [1, 3], that \( i \) is a homotopy equivalence. In particular, the induced homomorphism \( i^* : \pi_1 T^2 \to \pi_1 \mathcal{D}_{ld}(T^2) \) is an isomorphism. This easily implies that \( i^*([\omega_q]) = [\omega] \) and so \( \omega \) is null homotopic in \( \mathcal{D}(T^2) \) if and only if \( \omega_q \) is null-homotopic in \( T^2 \). \( \square \)

Let \( \alpha = \{h_t\} \) and \( \beta = \{h'_t\} \) be any two paths satisfying \( \{7\} \) and \( D \) be a 2-disk in \( T^2 \) bounded by \( C \). Consider the loop \( \omega = \alpha \beta^{-1} \) in \( \mathcal{D}_{ld}(T^2) \). Since \( \omega(C \times t) = C, t \in [0,1] \), we obtain that \( \omega(D \times t) = D \). Therefore for each \( q \in D \) the loop \( \omega_q : \{q\} \times [0,1] \to T^2 \) is null-homotopic in \( T^2 \). Hence by Lemma 8 the loop \( \omega \) is null-homotopic in \( \mathcal{D}_{ld}(T^2) \), that is \( \alpha \) and \( \beta \) are homotopic relatively their ends.

(iii) Let \( \{h_t\} \) and \( \{h'_t\} \) be paths in \( \mathcal{S}'(f) \) satisfying \( \{7\} \). Consider the following path

\[
g_t = \begin{cases} h_{2t}, & t \in [0, \frac{1}{2}], \\ h \circ h'_{2t-1}, & t \in [\frac{1}{2}, 1] \end{cases}
\]

in \( \mathcal{D}_{ld}(T^2) \) and the corresponding loop

\[
f \circ g_t = \begin{cases} f \circ h_{2t}, & t \in [0, \frac{1}{2}], \\ f \circ h \circ h'_{2t-1} = f \circ h'_{2t-1}, & t \in [\frac{1}{2}, 1] \end{cases}
\]

in \( \mathcal{O}_f(f) \). Then by definition of the multiplication in \( \pi_1 \mathcal{O}_f(f) \) we have that

\[
\{f \circ h_t\} \cdot \{f \circ h'_t\} = \{f \circ g_t\}.
\]

On the other hand, \( g_1 = h \circ h' \) and \( g_t(C) = C \) for all \( t \), that is \( \{f \circ g_t\} = s(h \circ h') \). Hence \( s(h) \circ s(h') = s(h \circ h') \). Lemma 8 is completed. \( \square \)

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