ON SOBOLEV NORMS FOR LIE GROUP REPRESENTATIONS

HEIKO GIMPERLEIN

Maxwell Institute for Mathematical Sciences and Department of Mathematics, Heriot–Watt University, Edinburgh, EH14 4AS, United Kingdom

BERNHARD KRÖTZ

Institut für Mathematik, Universität Paderborn, Warburger Straße 100, 33098 Paderborn

Abstract. We define a continuous scale of Sobolev norms for a Banach representation \((\pi, E)\) of a Lie group, with regard to a single differential operator \(D = d\pi(R^2 + \Delta)\). Here, \(\Delta\) is a Laplace element in the universal enveloping algebra, and \(R > 0\) depends explicitly on the growth rate of the representation. In particular, we obtain a spectral gap for \(D\) on the space of smooth vectors of \(E\). The main tool is a novel factorization of the delta distribution on a Lie group.
1. Introduction

Let $G$ be a Lie group and $(\pi, E)$ be a Banach representation of $G$, that is, a morphism of groups $\pi : G \to \text{GL}(E)$ such that the orbit maps

$$\gamma_v : G \to E, \ g \mapsto \gamma(g)v,$$

are continuous for all $v \in E$.

We say that a vector $v$ is $k$-times differentiable if $\gamma_v \in C^k(G, E)$ and write $E^k \subset E$ for the corresponding subspace. The smooth vectors are then defined by $E^\infty = \bigcap_{k=0}^{\infty} E^k$.

The space $E^k$ carries a natural Banach structure: if $p$ is a defining norm for the Banach structure on $E$, then a $k$-th Sobolev norm of $p$ on $E^k$ is defined as follows:

$$(1.1) \quad p_k(v) := \left( \sum_{m_1 + \ldots + m_n \leq k} p(d\pi(X_1^{m_1} \ldots X_n^{m_n})v)^2 \right)^{\frac{1}{2}} \quad (v \in E^k).$$

Here $X_1, \ldots, X_n$ is a fixed basis for the Lie algebra $\mathfrak{g}$ of $G$, and $d\pi : U(\mathfrak{g}) \to \text{End}(E^\infty)$ is, as usual, the derived representation for the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$. Then $E^k$, endowed with the norm $p_k$, is a Banach space and defines a Banach representation of $G$. Furthermore, $E^\infty$ carries a natural Fréchet structure, induced from the Sobolev norms $(p_k)_{k \in \mathbb{N}_0}$. The corresponding $G$-action on $E^\infty$ is smooth and of moderate growth, i.e. an $SF$-representation in the terminology of [2].

In case $(\pi, H)$ is a unitary representation on a Hilbert space $H$, there is an efficient way to define the Fréchet structure on $H^\infty$ via a Laplace element

$$(1.2) \quad \Delta = -\sum_{j=1}^{n} X_j^2$$

in $U(\mathfrak{g})$. More specifically, one defines the $2k$-th Laplace Sobolev norm in this case by

$$(1.3) \quad \Delta p_{2k}(v) := p(d\pi((1 + \Delta)^k)v) \quad (v \in E^{2k}).$$

The unitarity of the action then implies that the standard Sobolev norm $p_{2k}$ is equivalent to $\Delta p_{2k}$.

For a general Banach representation $(\pi, E)$ we still have $E^\infty = \bigcap_{k=0}^{\infty} \text{dom}(d\pi(\Delta^k))$, but it is no longer true that $\Delta p_{2k}$, as defined in (1.3), is equivalent to $p_{2k}$: it typically fails that $p_{2k}$ is dominated by $\Delta p_{2k}$, for example if $-1 \in \text{spec}(d\pi(\Delta))$ or if elliptic regularity fails as in Remark 4.2 below.

In the following we use $\Delta$ for the expression (2.1), a first-order modification of $\Delta$ as defined in (1.2), in order to make $\Delta$ selfadjoint on $L^2(G)$. In case $G$ is unimodular, we remark that the two notions (2.1) and (1.2) coincide.

One of the main results of this note is that every Banach representation $(\pi, E)$ admits a constant $R = R(E) > 0$ such that the operator $d\pi(R^2 + \Delta) : E^\infty \to E^\infty$ is invertible, see Corollary 3.3. The constant $R$ is closely related to the growth rate of the representation, i.e. the growth of the weight $w_\pi(g) = \|\pi(g)\|$. More precisely, for the Laplace Sobolev norms defined as

$$(1.4) \quad \Delta p_{2k}(v) := p(d\pi((R^2 + \Delta)^k)v) \quad (v \in E^{2k})$$
we show that the families \((p_{2k})_k\) and \((\Delta p_{2k})_k\) are equivalent in the following sense: Let \(m\) be the smallest even integer greater equal to \(1 + \text{dim} G\). Then there exist constants \(c_k, C_k > 0\) such that
\[
c_k \cdot p_{2k}(v) \leq p_{2k+m}(v) \leq C_k \cdot \Delta p_{2k+m}(v) \quad (v \in E^\infty).
\]

The above mentioned results follow from a novel factorization of the delta distribution \(\delta_1\) on \(G\), see Proposition 2.3 in the main text for the more technical statement. This in turn is a consequence of the functional calculus for \(\sqrt{\Delta}\), developed in [3], and previously applied to representation theory in [7] to derive factorization results for analytic vectors. The functional calculus allows us to define Laplace Sobolev norms for any order \(s \in \mathbb{R}\) by
\[
\Delta p_s(v) := p(d\pi((R^2 + \Delta)\hat{v})v) \quad (v \in E^\infty).
\]

On the other hand [2] provided another definition of Sobolev norms for any order \(s \in \mathbb{R}\); they were denoted \(S_p\) and termed induced Sobolev norms there. The norms \(S_p\) were based on a noncanonical localization to a neighborhood of \(1 \in G\), identified with the unit ball in \(\mathbb{R}^n\), and used the \(s\)-Sobolev norm on \(\mathbb{R}^n\). We show that the two notions \(\Delta p_s\) and \(S_p\) are equivalent up to constant shift in the parameter \(s\), see Proposition 4.3. The more geometrically defined norms \(\Delta p_s\) may therefore replace the norms \(S_p\) in [2].

Our motivation for this note stems from harmonic analysis on homogeneous spaces, see for example [1] and [11]. Here one encounters naturally the dual representation of some \(E^k\) and in this context it is often quite cumbersome to estimate the dual norm of \(p_k\), caused by the many terms in the definition (1.1). On the other hand the dual norm of \(\Delta p_s\), as defined by one operator \(d\pi((\Delta + R^2)\hat{v})\), is easy to control and simplifies a variety of technical issues.

2. SOME GEOMETRIC ANALYSIS ON LIE GROUPS

Let \(G\) be a Lie group of dimension \(n\) and \(g\) a left invariant Riemannian metric on \(G\). The Riemannian measure \(dg\) is a left invariant Haar measure on \(G\). We denote by \(d(g, h)\) the distance function associated to \(g\) (i.e. the infimum of the lengths of all paths connecting group elements \(g\) and \(h\)), by \(B_r(g) = \{x \in G \mid d(x, g) < r\}\) the ball of radius \(r\) centred at \(g\), and we set
\[
d(g) := d(g, 1) \quad (g \in G).
\]

Here are two key properties of \(d(g)\), which will be relevant later, see [3]:

**Lemma 2.1.** If \(w : G \to \mathbb{R}_+\) is locally bounded and submultiplicative (i.e. \(w(gh) \leq w(g)w(h)\)), then there exist \(c_1, C_1 > 0\) such that
\[
w(g) \leq C_1 e^{c_1 d(g)} \quad (g \in G).
\]

**Lemma 2.2.** There exists \(c_G > 0\) such that for all \(C > c_G\), \(\int_G e^{-Cd(g)} \, dg < \infty\).

Denote by \(V(G)\) the space of left-invariant vector fields on \(G\). It is common to identify the Lie algebra \(\mathfrak{g}\) with \(V(G)\) where \(X \in \mathfrak{g}\) corresponds to the vector field \(\tilde{X}\) given by
\[
(\tilde{X}f)(g) = \left. \frac{d}{dt} \right|_{t=0} f(g \exp(tX)) \quad (g \in G, f \in C^\infty(G)).
\]
We note that the adjoint of $\tilde{X}$ on the Hilbert space $L^2(G)$ is given by
\[ \tilde{X}^* = -\tilde{X} - \text{tr}(\text{ad} \ X), \]
and $\tilde{X}^* = -\tilde{X}$ in case $\mathfrak{g}$ is unimodular. Let us fix an orthonormal basis $\mathcal{B} = \{X_1, \ldots, X_n\}$ of $\mathfrak{g}$ with respect to $\mathfrak{g}$. Then the Laplace–Beltrami operator $\Delta = d^*d$ associated to $\mathfrak{g}$ is given explicitly by
\[
(2.1) \quad \Delta = \sum_{j=1}^n (-\tilde{X}_j - \text{tr}(\text{ad} \ X_j)) \tilde{X}_j.
\]
As $(G, \mathfrak{g})$ is complete, $\Delta$ is essentially selfadjoint. We denote by
\[ \sqrt{\Delta} = \int \lambda \, dP(\lambda) \]
the corresponding spectral resolution. It provides us with a measurable functional calculus, which allows to define
\[ f(\sqrt{\Delta}) = \int f(\lambda) \, dP(\lambda) \]
as an unbounded operator $f(\sqrt{\Delta})$ on $L^2(G)$ with domain
\[ D(f(\sqrt{\Delta})) = \left\{ \varphi \in L^2(G) \mid |f(\lambda)|^2 \, d\langle P(\lambda)\varphi, \varphi \rangle < \infty \right\}. \]
We are going to apply the above calculus to a certain function space. To do so, for $R' > 0$ we define a region
\[ \mathcal{W}_{R',\theta} = \{ z \in \mathbb{C} \mid |\text{Im} \ z| < R' \} \cup \{ z \in \mathbb{C} \mid |\text{Im} \ z| < \theta |\text{Re} \ z| \}. \]
For $R > 0$, $s \in \mathbb{R}$, the relevant function space is then defined as
\[ \mathcal{F}_{R,s} = \{ f \in C^\infty(\mathbb{R}, \mathbb{C}) \mid f \text{ even, } \exists \theta > 0 \, \exists R' > R : f \in \mathcal{O}(\mathcal{W}_{R',\theta}), \quad \forall k \in \mathbb{N} : \sup_{z \in \mathcal{W}_{R',\theta}} |\partial_z^k f(z)|(1 + |z|)^{k-s} < \infty \}. \]
The resulting operators $f(\sqrt{\Delta})$ are given by a distributional kernel $K_f \in \mathcal{D}'(G \times G)$,
\[ \langle f(\sqrt{\Delta}) \varphi, \psi \rangle = \langle K_f, \varphi \otimes \psi \rangle \quad \text{for all } \varphi, \psi \in C^\infty_c(G). \]
$K_f$ has the following properties:
- smooth outside diagonal: For $\Delta(G) = \{ (g, g) | g \in G \}$, $K_f \in C^\infty(G \times G \setminus \Delta(G))$,
- left invariant: $K_f(gx, gy) = K_f(x, y)$,
- hermitian: $K_f(x, y) = \overline{K_f(y, x)}$.

By left invariance $f(\sqrt{\Delta})$ is a convolution operator with kernel $\kappa_f(x^{-1}y) := K_f(1, x^{-1}y) = K_f(x, y)$:
\[ \langle f(\sqrt{\Delta}) \varphi, \psi \rangle = \langle K_f, \varphi \otimes \psi \rangle = \langle \kappa_f(x^{-1}y), (\varphi \otimes \psi)(x, y) \rangle = \langle \varphi \ast \kappa_f, \psi \rangle \]
for all $\varphi, \psi \in C^\infty_c(G)$. The distribution $\kappa_f \in \mathcal{D}'(G)$ is a smooth function on $G \setminus \{1\}$. Because $K_f$ is hermitian, $\kappa_f(x) = \kappa_f(x^{-1})$, so that $\kappa_f$ is left differentiable at $x \in G \setminus \{1\}$, if and only if it is right differentiable at $x$.

We define the weighted $L^1$-Schwartz space on $G$ by
\[ \mathcal{S}_R(G) := \{ f \in C^\infty(G) \mid \forall u, v \in U(\mathfrak{g}) : (\hat{u} \otimes \hat{v}_r)f \in L^1(G, e^{Rd(g)}dg) \}. \]
where \( \tilde{u}_l \), resp. \( \tilde{v}_r \), is the left, resp. right, invariant differential operator on \( G \) associated with \( u,v \in \mathcal{U}(g) \).

A theorem by Cheeger, Gromov and Taylor \([3]\) allows us to describe the global behavior of \( \kappa_f \):

**Theorem 2.3.** Let \( R, \varepsilon > 0, s \in \mathbb{R} \) and \( f \in \mathcal{F}_{R,s} \). Then \( \kappa_f = \kappa_1 + \kappa_2 \), where

1. \( \kappa_1 \in \mathcal{E}'(G) \) is supported in \( B_\varepsilon(1) \), and \( K_1(x,y) = \kappa_1(x^{-1}y) \) is the kernel of a pseudodifferential operator on \( G \) of order \( s \),
2. \( \kappa_2 \in \mathcal{S}_R(G) \).

From part (1) and the kernel estimates for pseudodifferential operators, we obtain \( \kappa_1 \in C^{-s-n-\varepsilon}(G) \) for all \( \varepsilon > 0 \), provided \( s < -n \).

Applying the theorem to the function \( f(z) = (R'^2 + z^2)^{-m} \) for \( m \in \mathbb{N} \), which lies in \( \mathcal{F}_{R,-2m} \) for any \( R < R' \), we conclude the following factorization of the Dirac distribution \( \delta_1 \):

**Proposition 2.4.** Let \( R' > R > 0, m \in \mathbb{N} \). Then

\[
\delta_1 = (R'^2 + \Delta)^m \delta_1 \ast \kappa ,
\]

where \( \kappa = \kappa_1 + \kappa_2 \) satisfies the estimates from Theorem 2.3 with \( s = -2m \).

### 3. Banach representation of Lie groups

In this section we briefly recall some basics on Banach representation of Lie groups and apply Proposition 2.4 to the factorization of vectors in \( E^k \).

For a Banach space \( E \) we denote by \( GL(E) \) the associated group of isomorphisms. By a **Banach representation \((\pi, E)\)** of a Lie group \( G \) we understand a group homomorphism \( \pi : G \to GL(E) \) such that the action

\[
G \times E \to E, \quad (g, v) \mapsto \pi(g)v,
\]

is continuous. For a vector \( v \in E \) we denote by

\[
\gamma_v : G \to E, \quad g \mapsto \pi(g)v,
\]

the corresponding continuous orbit map. Given \( k \in \mathbb{N}_0 \), the subspace \( E^k \subset E \) consists of all \( v \in E \) for which \( \gamma_v \in C^k \). We write \( E^\infty = \bigcap_k E^k \) and refer to \( E^\infty \) as the space of smooth vectors. Note that all \( E^k \) for \( k \in \mathbb{N}_0 \cup \{\infty\} \) are \( G \)-stable.

**Remark 3.1.** Let \((\pi, E)\) be a Banach representation. The uniform boundedness principle implies that the function

\[
w_\pi : G \to \mathbb{R}_+, \quad g \mapsto \|\pi(g)\|,
\]

satisfies the assumptions of Lemma 2.1.

Let

\[
c_\pi := \inf\{c > 0 \mid \exists C > 0 : w_\pi(g) \leq Ce_c\|(g)\}\).
\]

For \( R > 0 \) we introduce the exponentially weighted spaces

\[
\mathcal{R}_R(G) := L^1(G, w_Rdg), \quad w_R(g) = e^{Rd(g)}.
\]

It is clear that \( \mathcal{R}_R(G) \) defines a scale of Banach spaces. A different choice of the metric \( g \) leads to a scale which is equivalent to \( \mathcal{R}_R(G) \) up to affine reparametrization \( R \mapsto CR + C \).
Denote by $\pi_l$ the left regular representation of $G$ on $\mathcal{R}_R(G)$, and by $\pi_r$ the right regular representation. A simple computation shows that $\mathcal{R}_R(G)$ becomes a Banach algebra under left convolution
\[ \varphi \ast \psi(g) = \int_G \varphi(x) [\pi_l(x)\psi](g) \, dx \quad (\varphi, \psi \in \mathcal{R}_R(G), g \in G) \]
for $R > c_G$.

More generally, whenever $(\pi, E)$ is a Banach representation, Lemma 2.1 and Remark 3.1 imply that
\[ \Pi(\varphi)v := \int_G \varphi(g) \pi(g)v \, dg \quad (\varphi \in \mathcal{R}_R(G), v \in E) \]
defines an absolutely convergent Banach space valued integral for $R > R_E := c_\pi + c_G$. Hence the prescription
\[ \mathcal{R}_R(G) \times E \to E, \quad (\varphi, v) \mapsto \Pi(\varphi)v, \]
defines a continuous algebra action of $\mathcal{R}_R(G)$ (here continuous refers to the continuity of the bilinear map $\mathcal{R}_R(G) \times E \to E$).

As an example, the left-right representation $\pi_l \otimes \pi_r$ of $G \times G$ also induces a Banach representation on $\mathcal{R}_R(G)$.

Our concern is now with the $k$-times differentiable vectors $\mathcal{R}_R(G)^k$ of $(\pi_l \otimes \pi_r, \mathcal{R}_R(G))$. It is clear that $\mathcal{R}_R(G)^k$ is a subalgebra of $\mathcal{R}_R(G)$ and that
\[ \Pi(\mathcal{R}_R(G)^k) E \subset E^k, \]
whenever $(\pi, E)$ is a Banach representation and $R > R_E$.

**Theorem 3.2.** Let $R > 0$ and $k = 2m$ for $m \in \mathbb{N}$. Set $k' := k - \dim G - 1 \geq 0$. Then there exists a $\kappa \in \mathcal{R}_R(G)^{k'}$ such that: For all Banach representations $(\pi, E)$ with $R > R_E$ one has the following factorization of $k$-times differentiable vectors
\[ v = \Pi(\kappa)d\pi((R^2 + \Delta)^m)v \quad (v \in E^k). \]

**Proof.** Recall the factorization (2.2) of $\delta_1$,
\[ \delta_1 = [(R^2 + \Delta)^m \delta_1] \ast \kappa, \]
where $\kappa \in \mathcal{R}_R(G)^{k'}$. Given $v \in E^k$, we convolve this identity with the function $\varphi(g) = \gamma_v(g)$, to obtain
\[ \gamma_v = \Pi(\kappa)[(R^2 + \Delta)^m \gamma_v]. \]
The assertion follows by evaluating at $g = 1$. \qed

**Corollary 3.3.** Let $R > R_E$. Then
\[ d\pi(R^2 + \Delta) : E^\infty \to E^\infty \]
is invertible.

**Remark 3.4.** (Spectral gap for Banach representations) We can interpret Corollary 3.3 as a spectral gap theorem for Banach representations in terms of $R_E = c_\pi + c_G$. However, we note that the bound $R > R_E$ can be improved for special classes of representations. For example, if $(\pi, E)$ is a unitary representation, then
\[ \text{Re}(d\pi(\Delta)v, v) \geq 0 \]
for all \(v \in E^\infty\), and hence \(d\pi(\Delta) + R^2\) is injective for all \(R > 0\). Moreover, the Lax-Milgram theorem implies that \(d\pi(\Delta) + R^2\) is in fact invertible. On the other hand, our bound in Corollary 3.3 gives information about the convolution kernel of the inverse of \(d\pi(\Delta) + R^2\) for \(R > c_G\).

4. Sobolev norms for Banach representations

4.1. Standard and Laplace Sobolev norms. As before, we let \((\pi, E)\) be a Banach representation. On \(E^\infty\), the space of smooth vectors, one usually defines Sobolev norms as follows. Let \(p\) be the norm underlying \(E\). We fix a basis \(\mathcal{B} = \{X_1, \ldots, X_n\}\) of \(\mathfrak{g}\) and set

\[
p_k(v) := \left[ \sum_{m_1 + \ldots + m_n \leq k} p(d\pi(X_1^{m_1} \cdot \ldots \cdot X_n^{m_n})v)^2 \right]^\frac{1}{2} \quad (v \in E^\infty).
\]

Strictly speaking this notion depends on the choice of the basis \(\mathcal{B}\) and \(p_{k,\mathcal{B}}\) would be the more accurate notation. However, a different choice of basis, say \(\mathcal{C} = \{Y_1, \ldots, Y_n\}\) leads to an equivalent family of norms \(p_{k,\mathcal{C}}\), i.e. for all \(k\) there exist constants \(c_k, C_k > 0\) such that

\[
(4.1) \quad c_k \cdot p_{k,\mathcal{C}}(v) \leq p_{k,\mathcal{B}}(v) \leq C_k \cdot p_{k,\mathcal{C}}(v) \quad (v \in E^\infty).
\]

Having said this, we now drop the subscript \(\mathcal{B}\) in the definition of \(p_k\) and simply refer to \(p_k\) as the standard \(k\)-th Sobolev norm of \((\pi, E)\). Note that \(p_k\) is Hermitian (i.e. obtained from a Hermitian inner product) if \(p\) was Hermitian.

The completion of \((E^\infty, p_k)\) yields \(E^k\). In particular, \((E^k, p_k)\) is a Banach space for which the natural action \(G \times E^k \to E^k\) is continuous, i.e. defines a Banach representation.

The family \((p_k)_{k \in \mathbb{N}}\) induces a Fréchet structure on \(E^\infty\) (in view of \((1.1)\) of course independent of the choice of \(\mathcal{B}\)) such that the natural action \(G \times E^\infty \to E^\infty\) becomes continuous.

Now we introduce a family of Laplace Sobolev norms, first of even order \(k \in 2\mathbb{N}_0\), as follows. Let \(R > R_E\) and set

\[
\Delta p_k(v) := p \left( d\pi \left( (R^2 + \Delta)^{k/2} \right) v \right) \quad (v \in E^\infty).
\]

Of course, a more accurate notation would include \(R > 0\), i.e. write \(\Delta^R p_k\) instead of \(\Delta p_k\). In addition, \(\Delta\) also depends on the basis \(\mathcal{B}\). For purposes of readability we decided to suppress this data in the notation.

**Proposition 4.1.** (Comparison of the families \((p_{2k})_{k \in \mathbb{N}_0}\) and \((\Delta p_{2k})_{k \in \mathbb{N}_0}\))

For all \(k \in \mathbb{N}_0\) there exists \(C_k > 0\) such that for all \(v \in E^\infty\)

\[
\Delta p_{2k}(v) \leq C_k \cdot p_{2k}(v), \quad p_{2k-n-1}(v) \leq C_k \cdot \Delta p_{2k}(v).
\]

**Proof.** The first inequality follows directly from the definitions of \(p_{2k}\), \(\Delta p_{2k}\). The second is a consequence of the factorization \((3.1)\). \(\square\)

**Remark 4.2.** In general it is not true that \(p_{2k}\) is smaller than a multiple of \(\Delta p_{2k}\). In other words, an index shift as in Proposition \((4.1)\) is actually needed. As an example we consider \(E = C_0(\mathbb{R}^2)\) of continuous functions on \(\mathbb{R}^2\) which vanish at infinity, endowed with the sup-norm \(p(f) = \sup_{x \in \mathbb{R}^2} |f(x)|\). Then \(E\) becomes a Banach representation for the regular action of \(G = (\mathbb{R}^2, +)\) by translation in the
arguments. In this situation there exists a function \( u \in E \) such \( \Delta u \in E \) but \( \partial^2_y u \notin E \), see [6, Problem 4.9]. Hence \( p_2(u) = \infty \), while \( \Delta p_2(u) < \infty \).

4.2. Sobolev norms of continuous order \( s \in \mathbb{R} \).

4.2.1. Induced Sobolev norms. In [2] Sobolev norms for a Banach representation \((\pi, E)\) were defined for all parameters \( s \in \mathbb{R} \). We briefly recall their construction.

We endow the continuous dual \( E' \) of \( E \) with the dual norm
\[
p'(\lambda) := \sup_{p(v) \leq 1} |\lambda(v)| \quad (\lambda \in E').
\]
For \( \lambda \in E' \) and \( v \in E^\infty \) we define the matrix coefficient
\[
m_{\lambda,v}(g) = \lambda(\pi(g)v) \quad (g \in G),
\]
which is a smooth function on \( G \). Given an open relatively compact neighborhood \( B \subset G \) of \( 1 \), diffeomorphic to the open unit ball in \( \mathbb{R}^n \), we fix \( \phi \in C_c^\infty(G) \) such that \( \text{supp}(\phi) \subset B \) and \( \phi(1) = 1 \). The function \( \phi \cdot m_{\lambda,v} \) is then supported in \( B \) and upon identifying \( B \) with the open unit ball in \( \mathbb{R}^n \), say \( B_{\mathbb{R}^n} \), we denote by \( \|\phi \cdot m_{\lambda,v}\|_{H^s(\mathbb{R}^n)} \) the corresponding Sobolev norm. We then set
\[
Sp_s(v) := \sup_{\lambda \in E', p'(\lambda) \leq 1} \|\phi \cdot m_{\lambda,v}\|_{H^s(\mathbb{R}^n)} \quad (v \in E^\infty).
\]
In the terminology of [2] this defines a \( G \)-continuous norm on \( E^\infty \).

4.2.2. Laplace Sobolev norms. For \( R > R_E \) and \( s \in \mathbb{R} \), on the other hand the functional calculus for \( \sqrt{\Delta} \) also gives rise to a \( G \)-continuous norm on \( E^\infty \): We define
\[
\Delta p_s(v) := p((R^2 + \Delta)^{s/2} \gamma_v(g)|_{g=1}) \quad (v \in E^\infty).
\]

4.2.3. Comparison results.

Proposition 4.3. (Comparison of the families \((Sp_s)_{s \geq 0}\) and \((\Delta p_s)_{s \geq 0}\)) Let \( R > R_E \). Then for all \( s \geq 0, \varepsilon > 0 \), there exist \( c_s, C_s > 0 \) such that for all \( v \in E^\infty \)
\[
c_s \cdot Sp_s(v) \leq \Delta p_s(v) \leq C_s \cdot Sp_{s+\varepsilon/2+\varepsilon}(v).
\]

Proof. The first inequality was shown in [2] for \( k \in 2\mathbb{N} \). It follows for all \( s \geq 0 \) by interpolation.

For the second inequality, we apply the standard Sobolev embedding theorem for \( \mathbb{R}^n \) and obtain that
\[
\|\phi \cdot m_{\lambda,v}\|_{H^{s+\varepsilon}(\mathbb{R}^n)} \geq \|\phi \cdot m_{\lambda,v}\|_{C^\infty(B_{\mathbb{R}^n})} \geq |\lambda((R^2 + \Delta)^{s/2} \pi(g)v)|_{g=1}.
\]
The assertion follows by taking the supremum over \( \lambda \in E' \) with \( p'(\lambda) \leq 1 \). \( \Box \)
4.3. **Sobolev norms of order** $s \leq 0$. The natural way to define negative Sobolev norms is by duality. For a Banach representation $(\pi, E)$ with defining norm $p$ and $k \in \mathbb{N}_0$ we let $p'_k$ be the norm of $(E')^k$ and define $p_{-k}$ as the dual norm of $p'_k$, i.e.

$$p_{-k} := (p'_k)' .$$

The norm $p_{-k}$ is naturally defined on $((E')^k)'$. Now observe that the natural inclusion $(E')^k \hookrightarrow E'$ is continuous with dense image and thus yields a continuous dual inclusion $E'' \hookrightarrow ((E')^k)'$. The double-dual $E''$ contains $E$ in an isometric fashion. Hence $p_{-k}$ gives rise to a natural norm on $E$, henceforth denoted by the same symbol, and the completion of $E$ with respect to $p_{-k}$ will be denoted by $E^{-k}$.

**Remark 4.4.** In case $E$ is reflexive, i.e. $E'' \simeq E$, the space $E^{-k}$ is isomorphic to the strong dual of $(E')^k$.

On the other hand we have seen that the families $(p_k)_k$ and $(\Delta p_k)_k$ are equivalent. In this regard we note that $\Delta p_{-k}$ as defined in (4.2) coincides with the dual norm of $\Delta p'_k$.

As a corollary of Proposition 4.1 (and interpolation to also non-even indices $k \in \mathbb{N}_0$) we have:

**Corollary 4.5.** For all $k \in \mathbb{N}_0$ there exist constants $c_k, C_k > 0$ such that

$$(4.3) \quad c_k \cdot p_{-k}(v) \leq \Delta p_{-k}(v) \leq C_k \cdot p_{-k+n+1}(v) \quad (v \in E^\infty) .$$

**References**

[1] J. Bernstein, *On the support of Plancherel measure*, Jour. of Geom. and Physics 5, No. 4 (1988), 663–710.

[2] J. Bernstein and B. Krötz, *Smooth Fréchet Globalizations of Harish-Chandra Modules*, Israel J. Math. 199 (2014), 45–111.

[3] J. Cheeger, M. Gromov and M. Taylor, *Finite Propagation Speed, Kernel Estimates for Functions of the Laplace Operator, and the Geometry of Complete Riemannian Manifolds*, J. Differential Geom. 17 (1982), 15–53.

[4] P. Delorme, F. Knop, B. Krötz and H. Schlichtkrull, *Plancherel theory for real spherical spaces: Construction of the Bernstein morphisms*, preprint (2018), arxiv: 1807.07541.

[5] L. Gårding, *Note on continuous representations of Lie groups*, Proc. Nat. Acad. Sci. USA 33 (1947), 331–332.

[6] D. Gilbarg and N.S. Trudinger, *Elliptic partial differential equations of second order*. Reprint of the 1998 edition. Classics in Mathematics. Springer-Verlag, Berlin, 2001.

[7] H. Gimperlein, B. Krötz and C. Liemau, *Analytic factorization of Lie group representations*, J. Funct. Anal. 262 (2012), 667–681.