Compatible metrics of constant Riemannian curvature: local geometry, nonlinear equations and integrability

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1 Introduction

In the present paper, the nonlinear equations describing all the nonsingular pencils of metrics of constant Riemannian curvature are derived and the integrability of these nonlinear equations by the method of inverse scattering problem is proved. These results were announced in our previous paper [1]. For the flat pencils of metrics the corresponding statements and proofs were presented in the present author’s work [2], [3], where the method of integrating the nonlinear equations for the nonsingular flat pencils of metrics was proposed. In [4] the Lax pair for the nonsingular flat pencils of metrics was demonstrated. This Lax pair is generalized to the case of arbitrary nonsingular pencils of metrics of constant Riemannian curvature (see examples and interesting applications in [4]).

In this paper, it is proved that all the nonsingular pairs of compatible metrics of constant Riemannian curvature are described by special integrable reductions of nonlinear equations defining orthogonal curvilinear coordinate systems in the spaces of constant curvature. Note that the problem of description for the pencils of metrics of constant Riemannian curvature is equivalent to the problem of description for compatible nonlocal Poisson brackets of hydrodynamic type generated by metrics of constant Riemannian curvature (compatible Mokhov–Ferapontov brackets [5]) playing an important role in the theory of systems of hydrodynamic type.

Recall that two pseudo-Riemannian contravariant metrics \( g_{ij}^1(u) \) and \( g_{ij}^2(u) \) are called compatible if for any linear combination of these metrics \( g_{ij}^j(u) = \lambda_1 g_{ij}^1(u) + \lambda_2 g_{ij}^2(u) \), where \( \lambda_1 \) and \( \lambda_2 \) are arbitrary constants for which \( \det(g_{ij}^j(u)) \neq 0 \), the coefficients of the corresponding Levi-Civita connections and the components of the corresponding tensors of Riemannian curvature are related by the same linear formula: \( \Gamma^i_{jk}(u) = \lambda_1 \Gamma^i_{jk}^1(u) + \lambda_2 \Gamma^i_{jk}^2(u) \) and \( R^{ij}_{kl}(u) = \lambda_1 R^{ij}_{kl}^1(u) + \lambda_2 R^{ij}_{kl}^2(u) \) (in this case, we shall say also that the metrics \( g_{ij}^1(u) \) and \( g_{ij}^2(u) \) form a pencil of metrics [1]). Flat pencils of metrics, that is nothing but compatible nondegenerate local Poisson brackets of hydrodynamic type (compatible Dubrovin–Novikov brackets [6]), were introduced in [7]. Two pseudo-Riemannian contravariant metrics \( g_{ij}^1(u) \) and \( g_{ij}^2(u) \) of constant Riemannian curvature \( K_1 \) and \( K_2 \) respectively are called compatible if any linear combination of these metrics \( g_{ij}^j(u) = \lambda_1 g_{ij}^1(u) + \lambda_2 g_{ij}^2(u) \), where \( \lambda_1 \) and \( \lambda_2 \) are arbitrary constants for which \( \det(g_{ij}^j(u)) \neq 0 \), is a metric of constant Riemannian curvature.

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curvature $\lambda_1K_1 + \lambda_2K_2$ and the coefficients of the corresponding Levi-Civita connections are related by the same linear formula: $\Gamma^0_i_k(u) = \lambda_1\Gamma^{ij}_{1,k}(u) + \lambda_2\Gamma^{ij}_{2,k}(u)$. In this case, we shall also say that the metrics $g_1^{ij}(u)$ and $g_2^{ij}(u)$ form a pencil of metrics of constant Riemannian curvature \cite{1}. It is obvious that all these definitions are mutually consistent, so that if compatible metrics are metrics of constant Riemannian curvature, then they form a pencil of metrics of constant Riemannian curvature, and if compatible metrics are flat, then they form a flat pencil of metrics. A pair of pseudo-Riemannian metrics $g_1^{ij}(u)$ and $g_2^{ij}(u)$ is called nonsingular if the eigenvalues of this pair of metrics, that is, the roots of the equation $\det(g_1^{ij}(u) - \lambda g_2^{ij}(u)) = 0$, are distinct (a pencil of metrics which is formed by a nonsingular pair of metrics is also called nonsingular). In \cite{1}, it is proved that an arbitrary nonsingular pair of metrics is compatible if and only if there exist local coordinates $u = (u^1, \ldots, u^N)$ such that both the metrics are diagonal in these coordinates and have the following special form (one can consider that one of the metrics, here $g_2^{ij}(u)$, is an arbitrary diagonal metric): $g_2^{ij}(u) = g^i(u)\delta^j$ and $g_1^{ij}(u) = f^i(u')g^j(u)\delta^j$, where $f^i(u')$, $i = 1, \ldots, N$, are functions of single variable (generally speaking, complex). It is obvious that the eigenvalues of the considered pair of metrics are given by the functions $f^i(u')$, $i = 1, \ldots, N$.

2 Equations for nonsingular pairs of compatible metrics of constant Riemannian curvature

Consider the problem of description for nonsingular pairs of compatible metrics of constant Riemannian curvature. As is proved in \cite{1} (see also Introduction above), it is sufficient to classify pairs of diagonal metrics of constant Riemannian curvature of the special type $g_2^{ij}(u) = g^i(u)\delta^j$ and $g_1^{ij}(u) = f^i(u')g^j(u)\delta^j$, where $f^i(u')$, $i = 1, \ldots, N$, are arbitrary (generally speaking, complex) functions of single variable, $g^i(u)$, $i = 1, \ldots, N$, are arbitrary functions.

The problem of description for $N \times N$ diagonal metrics of constant Riemannian curvature $K$, that is, metrics $g_2^{ij}(u) = g^i(u)\delta^j$ such that their tensor of Riemannian curvature has the form $R_2^{ij} = K(\delta_i^k\delta^j - \delta_i^j\delta^k)$, where $K = \text{const}$, is the classical problem of differential geometry. This problem is equivalent to the problem of description of orthogonal curvilinear coordinate systems in $N$-dimensional spaces of constant curvature $K$. Recently Zakharov showed that the Lamé equations describing orthogonal curvilinear coordinate systems in Euclidean or pseudo-Euclidean $N$-dimensional spaces are integrated by the method of inverse scattering problem \cite{3} (see also an algebraic-geometric approach in \cite{4}). Similarly, the equations describing orthogonal curvilinear coordinate systems in $N$-dimensional spaces of constant curvature are integrated by the method of inverse scattering problem.

Introduce the standard classical notation

$$g^i(u) = \frac{e^i}{(H_i(u))^2}, \quad ds^2 = \sum_{i=1}^{N} e^i(H_i(u))^2(du^i)^2, \quad (2.1)$$

$$\beta_{ik}(u) = \frac{1}{H_i(u)} \frac{\partial H_k}{\partial u^i}, \quad i \neq k, \quad (2.2)$$
where $H_i(u)$ are the Lamé coefficients and $\beta_k(u)$ are the rotation coefficients, $\varepsilon^i = \pm 1$, $i = 1, ..., N$. Although, in our case, all the functions are, generally speaking, complex, we shall use formulae, which are convenient for using also in the purely real case.

**Theorem 2.1** Nonsingular pairs of compatible metrics of constant Riemannian curvature $K_1$ and $K_2$ are described by the following consistent integrable nonlinear systems:

\[
\frac{\partial \beta_{ij}}{\partial u^k} = \beta_{ik} \beta_{kj}, \quad i \neq j, \quad i \neq k, \quad j \neq k,
\]

\[
\varepsilon^i \frac{\partial \beta_{ij}}{\partial u^i} + \varepsilon^j \frac{\partial \beta_{ji}}{\partial u^j} + \sum_{s \neq i, s \neq j} \varepsilon^s \beta_{si} \beta_{sj} = -K_2 H_i H_j, \quad i \neq j,
\]

\[
\varepsilon^i f^i(u^i) \frac{\partial \beta_{ij}}{\partial u^i} + \frac{1}{2} \varepsilon^i (f^i)' \beta_{ij} + \varepsilon^j f^j(u^j) \frac{\partial \beta_{ji}}{\partial u^j} + \frac{1}{2} \varepsilon^j (f^j)' \beta_{ji} + \sum_{s \neq i, s \neq j} \varepsilon^s f^s(u^s) \beta_{si} \beta_{sj} = -K_1 H_i H_j, \quad i \neq j,
\]

where $f^i(u^i), i = 1, ..., N$, are arbitrary given functions of one variable.

**Remark 2.1** Equations (2.3) are the well-known n-wave equations (the Darboux equations [10]), and equations (2.3), (2.4) describe orthogonal curvilinear coordinate systems in $N$-dimensional spaces of constant curvature $K_2$, in particular, for $K_2 = 0$ we get the famous Lamé equations. Equations (2.3), (2.4) define a nontrivial nonlinear differential reduction of equations (2.3), (2.4).

**Remark 2.2** If there are coinciding nonzero constants among the functions $f^i(u^i)$, then equations (2.3)–(2.5) also describe a pencil of metrics of constant Riemannian curvature, but, obviously, this pencil is not nonsingular.

The system of equations (2.2)–(2.5) is consistent for any functions $f^i(u^i)$ what can be easily checked by direct calculations. Moreover, the general solution of the nonlinear system (2.2)–(2.5) depend on $N^2$ arbitrary functions of one variable (the functions $\beta_{ij}(u)$ and $H_j(u)$ can be arbitrarily defined on the $j$th coordinate line).

Let us consider the conditions that the diagonal metrics $g^{ij}_2(u) = g^i(u)\delta^{ij}$ and $g^{ij}_1(u) = f^i(u^i)g^i(u)\delta^{ij}$, where $f^i(u^i), i = 1, ..., N$, are arbitrary functions of one indicated variable (these functions must not be only equal to zero identically), are metrics of constant Riemannian curvature $K_2$ and $K_1$ respectively.

Recall that for any diagonal metric $\Gamma_{jk}^i(u) = 0$ if all the indices $i, j, k$ are distinct. It is also obvious that $R^{ij}_{kl}(u) = 0$ if all the indices $i, j, k, l$ are distinct. In addition, by virtue of the well-known symmetries of the tensor of Riemannian curvature, we have:

\[
R^{ij}_{kl}(u) = R^{ij}_{lk}(u) = 0, \quad R^{ij}_{il}(u) = -R^{ij}_{li}(u) = R^{ij}_{li}(u) = -R^{ij}_{il}(u).
\]
Thus, it is sufficient to consider the condition $R_{kl}^{ij}(u) = K_2(\delta^i_k \delta^j_l - \delta^i_l \delta^j_k)$ (the necessary and sufficient condition that the corresponding metric is a metric of constant Riemannian curvature $K_2$) only for the following components of the tensor of Riemannian curvature: $R_{kl}^{ij}(u)$, where $i \neq j$, $i \neq l$:

$$R_{kl}^{ij} = -K_2 \delta^i_l, \quad i \neq j, \; i \neq l.$$ 

For an arbitrary diagonal metric $g^{ij}(u) = g^i(u)\delta^{ij}$, we get

$$\Gamma_{2,ik}^{ij}(u) = -1 \frac{\partial g^i}{\partial u^k}, \quad i \neq j, \; i \neq l.$$ 

Equations (2.7) are equivalent to the equations

$$\frac{\partial^2 H_i}{\partial u^j \partial u^k} = 1 \frac{\partial H_i}{\partial u^j} \frac{\partial H_j}{\partial u^k} + 1 \frac{\partial H_j}{\partial u^i} \frac{\partial H_i}{\partial u^k}, \quad i \neq j, \; i \neq k, \; j \neq k,$$ 

which are equivalent, in turn, to equations (2.3).

2) $j = l$.

$$R_{ij}^{ij} = -K_2, \quad i \neq j.$$
\[ R_{2,ij}^{ij}(u) = g^i(u) \left( \frac{\partial \Gamma_{2,ii}^{ij}}{\partial u^i} - \frac{\partial \Gamma_{2,ii}^{ji}}{\partial u^i} + \Gamma_{2,ii}^{ij}(u) \Gamma_{2,ij}^{ii}(u) + \Gamma_{2,ij}^{ii}(u) \right) \]

\[ \Gamma_{2,ij}^{ii}(u) \Gamma_{2,ij}^{ii}(u) - \sum_{s=1}^{N} \Gamma_{2,si}^{ij}(u) \Gamma_{2,ji}^{ii}(u) = \]

\[ -\frac{1}{2} g^i(u) \frac{\partial}{\partial u^i} \left( \frac{1}{g^j(u)} \frac{\partial g^j(u)}{\partial u^i} \right) - \frac{1}{2} g^i(u) \frac{\partial}{\partial u^i} \left( \frac{g^i(u)}{(g^j(u))^2} \frac{\partial g^j(u)}{\partial u^i} \right) - \frac{1}{4} \left( g^i(u) \cdot \frac{\partial g^i}{\partial u^i} \right)^2 \frac{\partial H_j}{\partial u^i} \frac{\partial H_j}{\partial u^i} + \frac{1}{4} \left( g^i(u) \cdot \frac{\partial g^i}{\partial u^i} \right)^2 \frac{\partial u^s}{\partial u^i} \frac{\partial u^s}{\partial u^i} = -K_2. \quad (2.9) \]

Equations (2.9) are equivalent to the equations

\[ \varepsilon^i \frac{\partial}{\partial u^i} \left( \frac{1}{H_i(u)} \frac{\partial H_j}{\partial u^i} \right) + \varepsilon^j \frac{\partial}{\partial u^j} \left( \frac{1}{H_j(u)} \frac{\partial H_i}{\partial u^j} \right) + \]

\[ \sum_{s \neq i, s \neq j} \frac{\varepsilon^s}{(H_s(u))^2} \frac{\partial H_i}{\partial u^s} \frac{\partial H_j}{\partial u^s} = -K_2 H_i H_j, \quad i \neq j, \quad (2.10) \]

which are equivalent, in turn, to equations (2.4).

The condition that the metric \( g^i_j(u) = f^i(u) g^i(u) \delta^j \) is also a metric of constant Riemannian curvature gives \( N(N - 1)/2 \) additional equations (2.5). Note that in this case components (2.7) of the tensor of Riemannian curvature vanish automatically, and the condition (2.9) gives the corresponding \( N(N - 1)/2 \) equations.

Actually, for the metric \( g^i_j(u) = f^i(u) g^i(u) \delta^j \), the Lamé coefficients and the rotation coefficients have the form

\[ \bar{H}_i(u) = \frac{H_i(u)}{\sqrt{e^i f^i(u)}}, \quad f^i(u) g^i(u) = \frac{e^i e^j}{(H_i(u))^2}, \quad e^i = \pm 1, \quad (2.11) \]

\[ \bar{\beta}_{ik}(u) = \frac{1}{H_i(u)} \frac{\partial H_k}{\partial u^i} = \]

\[ \frac{\sqrt{e^i f^i(u)}}{\sqrt{e^k f^k(u)}} \left( \frac{1}{H_i(u)} \frac{\partial H_k}{\partial u^i} \right) = \frac{\sqrt{e^i f^i(u)}}{\sqrt{e^k f^k(u)}} \beta_{ik}(u), \quad i \neq k. \quad (2.12) \]

Accordingly, equations (2.3) are automatically satisfied also for the rotation coefficients \( \bar{\beta}_{ik}(u) \), and equations (2.4) for \( \bar{\beta}_{ik}(u) \) give equations (2.5).

Note that, for our further purposes, it is more convenient to write the system (2.3)–(2.7) namely in this form in order to emphasize the reduction (2.11), (2.12) playing an important role in our method of integrating this system (see the next section), although it is easy to show that equations (2.4), (2.3) for nonsingular pairs of metrics (that is, all the functions \( f^i(u) \) must be distinct also in the case if they are constants) are equivalent to the following equations (in particular, it is more convenient to use these equations for checking
the consistency of system (2.2)–(2.5)

\[
\frac{\partial \beta_{ij}}{\partial u^t} = \frac{1}{2} \frac{(f^i(u^i))'}{(f^i(u^i) - f^j(u^j))} \beta_{ij} + \frac{\varepsilon^i \varepsilon^j}{2} \frac{(f^j(u^j))'}{(f^j(u^j) - f^i(u^i))} \beta_{ji} - \\
- \sum_{s \neq i, s \neq j} \varepsilon^i \varepsilon^s \left( \frac{f^j(u^j) - f^s(u^s)}{(f^j(u^j) - f^i(u^i))} \beta_{sj} + \varepsilon^i \frac{K_1 - K_2 f^j(u^j)}{(f^j(u^j) - f^i(u^i))} H_i H_j, \ i \neq j. \right) \quad (2.13)
\]

Example 2.1 If all the functions \(f^i(u^i)\) are arbitrary distinct nonzero constants: \(f^i(u^i) = c^i, c^i = \text{const} \neq 0, i = 1, \ldots, N, c^i \neq c^j\) for \(i \neq j\), then the integrable system (2.2)–(2.5) takes the following form:

\[
\begin{align*}
\frac{\partial \beta_{ij}}{\partial u^k} &= \beta_{ik} \beta_{kj}, \quad i \neq j, \ i \neq k, \ j \neq k, \quad (2.14) \\
\frac{\partial \beta_{ij}}{\partial u^i} + \frac{\partial \beta_{ji}}{\partial u^j} + \sum_{s \neq i, s \neq j} \beta_{si} \beta_{sj} &= 0, \quad i \neq j, \quad (2.15) \\
\frac{\partial H_j}{\partial u^i} &= \beta_{ij} H_i, \quad i \neq j. \quad (2.16)
\end{align*}
\]

In the flat case, for \(K_1 = K_2 = 0\), system (2.14), (2.15) was considered in [4] (it is related to a curious triple of pairwise commuting Monge–Ampère equations). So the system of equations (2.14)–(2.16) is also of special interest for applications.

3 Compatible flat metrics and the Zakharov method of differential reductions

In this section, we demonstrate the method of integrating the system of equations (2.3)–(2.5) in the flat case (for \(K_1 = K_2 = 0\)) (see [4], [3]), that is, the following system (in this section \(\varepsilon^i = 1, i = 1, \ldots, N\) in all formulae):

\[
\begin{align*}
\frac{\partial \beta_{ij}}{\partial u^k} &= \beta_{ik} \beta_{kj}, \quad i \neq j, \ i \neq k, \ j \neq k, \quad (3.1) \\
\frac{\partial \beta_{ij}}{\partial u^i} + \frac{\partial \beta_{ji}}{\partial u^j} + \sum_{s \neq i, s \neq j} \beta_{si} \beta_{sj} &= 0, \quad i \neq j, \quad (3.2) \\
f^i(u^i) \frac{\partial \beta_{ij}}{\partial u^i} + \frac{1}{2} (f^i)' \beta_{ij} + f^j(u^j) \frac{\partial \beta_{ji}}{\partial u^j} + \frac{1}{2} (f^j)' \beta_{ji} + \\
\sum_{s \neq i, s \neq j} f^s(u^s) \beta_{si} \beta_{sj} &= 0, \quad i \neq j, \quad (3.3)
\end{align*}
\]

where \(f^i(u^i), i = 1, \ldots, N, \) are arbitrary given functions of one variable.

Recall the Zakharov method for integrating the Lamé equations (3.1) and (3.2) [8].
We must choose a matrix function $F_{ij}(s, s', u)$ with some special properties and solve the linear integral equation

$$K_{ij}(s, s', u) = F_{ij}(s, s', u) + \int_{s}^{\infty} \sum_{l} K_{il}(s, q, u) F_{lj}(q, s', u) dq. \quad (3.4)$$

Then we obtain a one-parameter family of solutions of the Lamé equations by the formula

$$\beta_{ij}(s, u) = K_{ji}(s, s, u). \quad (3.5)$$

In particular, if $F_{ij}(s, s', u) = f_{ij}(s - u^i, s' - u'^j)$, where $f_{ij}(x, y)$ is an arbitrary matrix function of two variables, then formula (3.5) produces solutions of equations (3.1). To construct solutions of equations (3.1), Zakharov proposed to impose on the “dressing matrix function” $F_{ij}(s - u^i, s' - u'^j)$ a certain additional linear differential relation. If the function $F_{ij}(s - u^i, s' - u'^j)$ satisfies the Zakharov differential relation (we shall present it below), then the rotation coefficients $\beta_{ij}(u)$ additionally satisfy equations (3.2).

Let us describe our scheme of integrating all the system of equations (3.1)–(3.3).

**Lemma 3.1** If both the function $F_{ij}(s - u^i, s' - u'^j)$ and the function

$$\tilde{F}_{ij}(s - u^i, s' - u'^j) = \sqrt{f_j(u^j - s^j)} f_{ij}(s - u^i, s' - u'^j) \quad (3.6)$$

satisfy the Zakharov differential relation, then the corresponding rotation coefficients $\beta_{ij}(u)$ (3.5) satisfy all the equations (3.1)–(3.3).

Actually, if $K_{ij}(s, s', u)$ is the corresponding to the function $F_{ij}(s - u^i, s' - u'^j)$ solution of the linear integral equation (3.4), then

$$\tilde{K}_{ij}(s, s', u) = \frac{\sqrt{f_j(u^j - s^j)}}{\sqrt{f_j(u^j - s^j)}} K_{ij}(s, s', u) \quad (3.7)$$

is the corresponding to function (3.6) solution of equation (3.4). It is easy to prove multiplying the integral equation (3.4) by $\sqrt{f_j(u^j - s^j)}/\sqrt{f_j(u^j - s^j)}$:

$$\tilde{K}_{ij}(s, s', u) = \tilde{F}_{ij}(s - u^i, s' - u'^j) + \int_{s}^{\infty} \sum_{l} \tilde{K}_{il}(s, q, u) \tilde{F}_{lj}(q - u^l, s' - u'^l) dq. \quad (3.8)$$

Then both $\tilde{\beta}_{ij}(s, u) = \tilde{K}_{ji}(s, s, u)$ and $\beta_{ij}(s, u) = K_{ji}(s, s, u)$ satisfy the Lamé equations (3.1) and (3.2). Besides, we have

$$\tilde{\beta}_{ij}(s, u) = \tilde{K}_{ji}(s, s, u) = \frac{\sqrt{f_j(u^j - s^j)}}{\sqrt{f_j(u^j - s^j)}} K_{ji}(s, s, u) = \frac{\sqrt{f_j(u^j - s^j)}}{\sqrt{f_j(u^j - s^j)}} \beta_{ij}(s, u). \quad (3.9)$$

Thus, in this case, the rotation coefficients $\beta_{ij}(u)$ satisfy exactly all equations (3.1), (3.3), that is, they generate the corresponding compatible flat metrics.
The Zakharov reduction is given by the following differential relations \[8\]:
\[
\frac{\partial F_{ij}(s, s', u)}{\partial s'} + \frac{\partial F_{ji}(s', s, u)}{\partial s} = 0.
\] (3.10)

To resolve these relations for the matrix function \(F_{ij}(s - u^i, s' - u^j)\), we can introduce \(N(N - 1)/2\) arbitrary functions of two variables \(\Phi_{ij}(x, y)\), \(i < j\), and put for \(i < j\)
\[
F_{ij}(s - u^i, s' - u^j) = \frac{\partial \Phi_{ij}(s - u^i, s' - u^j)}{\partial s},
\]
and, besides, put for any \(i\)
\[
F_{ii}(s - u^i, s' - u^i) = \frac{\partial \Phi_{ii}(s - u^i, s' - u^i)}{\partial s},
\] (3.11)
where \(\Phi_{ii}(x, y)\), \(i = 1, \ldots, N\), are arbitrary skew-symmetric functions:
\[
\Phi_{ii}(x, y) = -\Phi_{ii}(y, x),
\] (3.13)
see \[8\].

For the function
\[
\tilde{F}_{ij}(s - u^i, s' - u^j) = \frac{\sqrt{f^j(u^j - s)}}{\sqrt{f^i(u^i - s)}} F_{ij}(s - u^i, s' - u^j)
\] (3.14)
the Zakharov differential relations (3.10) give exactly \(N(N - 1)/2\) linear partial differential equations of the second order for \(N(N - 1)/2\) functions of two variables \(\Phi_{ij}(s - u^i, s' - u^j)\), \(i < j\):
\[
\frac{\partial}{\partial s'} \left( \frac{\sqrt{f^j(u^j - s)}}{\sqrt{f^i(u^i - s)}} \frac{\partial \Phi_{ij}(s - u^i, s' - u^j)}{\partial s} \right) - \frac{\partial}{\partial s} \left( \frac{\sqrt{f^i(u^i - s)}}{\sqrt{f^j(u^j - s)}} \frac{\partial \Phi_{ij}(s - u^i, s' - u^j)}{\partial s'} \right) = 0, \quad i < j,
\] (3.15)
which are equivalent to the equations
\[
2 \frac{\partial^2 \Phi_{ij}(s - u^i, s' - u^j)}{\partial u^i \partial u^j} \left( f^j(u^j - s) - f^i(u^i - s') \right) + \frac{\partial \Phi_{ij}(s - u^i, s' - u^j)}{\partial u^j} \frac{df^i(u^i - s)}{du^i} - \frac{\partial \Phi_{ij}(s - u^i, s' - u^j)}{\partial u^i} \frac{df^j(u^j - s)}{du^j} = 0, \quad i < j.
\] (3.16)

It is interesting that all these equations (3.16) for the functions \(\Phi_{ij}(s - u^i, s' - u^j)\) are of the same type and coincide with the single equation in the two-component case \((N = 2)\),
which was derived directly from the conditions of vanishing the corresponding tensors of Riemannian curvature (see [1]).

Besides, for \( N \) functions \( \Phi_{ii}(s - u^i, s' - u^i) \), we have also \( N \) linear partial differential equations of the second order, which are derived from the Zakharov differential relations (3.10):

\[
\frac{\partial}{\partial s'} \left( \frac{\sqrt{f^i(u^i - s)}}{\sqrt{f^i(u^i - s')}} \frac{\partial \Phi_{ii}(s - u^i, s' - u^i)}{\partial s} \right) + \frac{\partial}{\partial s} \left( \frac{\sqrt{f^i(u^i - s')}}{\sqrt{f^i(u^i - s)}} \frac{\partial \Phi_{ii}(s - u^i, s' - u^i)}{\partial s'} \right) = 0
\]

(3.17)

or equivalently

\[
2 \frac{\partial^2 \Phi_{ii}(s - u^i, s' - u^i)}{\partial s \partial s'} \left( f^i(u^i - s) - f^i(u^i - s') \right) - \frac{\partial \Phi_{ii}(s - u^i, s' - u^i)}{\partial s} \frac{df^i(u^i - s)}{ds} + \frac{\partial \Phi_{ii}(s - u^i, s' - u^i)}{\partial s'} \frac{df^i(u^i - s)}{ds} = 0.
\]

(3.18)

Any solution of the linear partial differential equations of the second order (3.16) and (3.18) generates a one-parameter family of solutions of system (3.1)–(3.3) by linear relations and explicit formulae (3.11), (3.12), (3.4), (3.5). Thus system (3.1)–(3.3) is linearized.

The method of differential reductions is applicable also to integrating arbitrary nonsingular pencils of metrics of constant Riemannian curvature (2.2)–(2.5). In the next section, we present the Lax pair with a spectral parameter for system (2.2)–(2.5), what also gives the possibility to integrate the system.

Note that equations (3.16), (3.18) can be easily integrated explicitly in the special functions. If the functions \( f^i(u^i) \) and \( f^j(u^j) \) are not constants, then, as is proved in [1], for the description of all the corresponding pencils of metrics, it is sufficient to consider the case \( f^i(u^i) = u^i, f^j(u^j) = u^j \), since in this case the functions \( f^i(u^i) \) and \( f^j(u^j) \) can be chosen as the corresponding new local coordinates. Accordingly equations (3.16) belong to one of the following three types:

\[
\frac{\partial^2 F}{\partial u^1 \partial u^2} = \frac{1}{2(u^1 - u^2)} \frac{\partial F}{\partial u^1} - \frac{1}{2(u^1 - u^2)} \frac{\partial F}{\partial u^2},
\]

(3.19)

\[
\frac{\partial^2 F}{\partial u^1 \partial u^2} = -\frac{1}{2u^2} \frac{\partial F}{\partial u^1},
\]

(3.20)

\[
\frac{\partial^2 F}{\partial u^1 \partial u^2} = 0.
\]

(3.21)

The general solutions of equations (3.20) and (3.21) are the functions

\[
F(u^1, u^2) = g(u^1)/\sqrt{u^2} + h(u^2) \quad \text{and} \quad F(u^1, u^2) = g(u^1) + h(u^2)
\]

respectively, where \( g(u^1) \) and \( h(u^2) \) are arbitrary functions.

After the change of variables \( u^1 = t + r, u^2 = t - r \), equation (3.19) becomes the equation

\[
\frac{\partial^2 F}{\partial t^2} = \frac{\partial^2 F}{\partial r^2} + \frac{1}{r} \frac{\partial F}{\partial r},
\]

(3.22)
which is well known in the classical geometry (one more well-known Darboux equation) and in the classical mathematical physics (the equation of axially symmetric oscillations of gas \((r)\) is the radial coordinate) or the equation for solutions of the wave equation which correspond to processes of radiation on the plane \((x,y)\), that is, solutions depending only on the radial coordinate \(r = \sqrt{x^2 + y^2}\) and the time \(t\).

In particular, the functions
\[
v(t,r) = \text{const} \int_{-1}^{+1} \frac{\psi(t + xr)}{\sqrt{1 - x^2}} dx,
\]
where \(\psi(t)\) is an arbitrary function, satisfy to the Darboux equation (3.22) with the initial conditions
\[
F(t,0) = \psi(t), \quad \frac{\partial F}{\partial r}(t,0) = 0.
\]
For the function \(\psi(t,z) = \psi(t)\), given on the plane \((t,z)\), the function \(v(t,r)\) gives the mean value over the circle of radius \(r\) with the centre at the point \((t,z)\).

The general solutions of equations (3.22) can be constructed by the method of separation of variables: \(F(t,r) = S(t)R(r)\),
\[
\frac{S''}{S} = \frac{R''}{R} + \frac{1}{rR'},
\]
\[
S'' = aS, \quad rR'' + R' - arR = 0, \quad a = \text{const}.
\]
It is easy to write the general solutions in terms of the special Bessel and modified Bessel functions or degenerate hypergeometric functions.

4 Lax pair for nonsingular pencils of metrics of constant Riemannian curvature

The Lax pair for nonsingular flat pencils of metrics (3.1)–(3.3) was demonstrated by Ferapontov in [4]:
\[
\frac{\partial \varphi_i}{\partial u^j} = \sqrt{\frac{(\lambda + f^i)}{(\lambda + f^j)}} \beta_{ij} \varphi_j, \quad i \neq j,
\]
\[
\frac{\partial \varphi_i}{\partial u^i} = -\sum_{k \neq i} \sqrt{\frac{(\lambda + f^k)}{(\lambda + f^i)}} \beta_{ki} \varphi_k,
\]
where \(\lambda\) is the spectral parameter.

Note that the linear problem for the Lamé equations (3.1), (3.2) was well known Darboux yet [10], see also, for example, [8], [9]:
\[
\frac{\partial \varphi_i}{\partial u^j} = \beta_{ij} \varphi_j, \quad i \neq j,
\]
\[
\frac{\partial \varphi_i}{\partial u^i} = -\sum_{k \neq i} \beta_{ki} \varphi_k.
\]
The condition of consistency for the linear system (4.3), (4.4) defines the Lamé equations (3.1), (3.2). The Lax pair (4.1), (4.2) can be easily derived from the classical linear problem (4.3), (4.4) for the Lamé equations. Actually, the system of equations (3.1)–(3.3) defining nonsingular flat pencils of metrics is derived from the condition that the diagonal metric $(\lambda + f^i(u^i))g^i(u)\delta_{ij}$ is flat for any $\lambda$. It is obvious that in this case (see (2.12) and [2], [3]) it is necessary to change $\beta_{ij}(u)$ to
\[ \hat{\beta}_{ij}(u) = \sqrt{\frac{\lambda + f^i(u^i)}{\lambda + f^j(u^j)}} \beta_{ij}(u) \]
in the Lamé equations (3.1), (3.2). Then the linear problem (4.3), (4.4) becomes the Lax pair (4.1), (4.2).

The Lax pair (4.1), (4.2) is generalized also to the case of arbitrary nonsingular pencils of metrics of constant Riemannian curvature (2.2)–(2.5) (see examples in [4]). The Lax pair for the system (2.2)–(2.5) can be also easily derived from the linear problem for the system (2.2)–(2.4) describing all the orthogonal curvilinear coordinate systems in $N$-dimensional spaces of constant curvature $K_2$:

\[
\frac{\partial \phi_i}{\partial u^j} = \sqrt{\varepsilon^i \varepsilon^j} \beta_{ij} \phi_j, \quad i \neq j, \quad \text{(4.5)}
\]

\[
\frac{\partial \phi_i}{\partial u^i} = -\sum_{k \neq i} \sqrt{\varepsilon^k \varepsilon^i} \beta_{ki} \phi_k + \sqrt{\varepsilon^i K_2 H_i \psi}, \quad \text{(4.6)}
\]

\[
\frac{\partial \psi}{\partial u^i} = -\sqrt{\varepsilon^i K_2 H_i \psi}, \quad \text{(4.7)}
\]

(the condition of consistency for the linear system (4.5)–(4.7) gives the equations (2.2)–(2.4)). Actually, the system of equations (2.2)–(2.5) defining the pencils of metrics of constant Riemannian curvature is equivalent to the condition that the diagonal metric $(\lambda + f^i(u^i))g^i(u)\delta_{ij}$ is a metric of constant Riemannian curvature $\lambda K_2 + K_1$ for any $\lambda$. In this case (see (2.11), (2.12) and [2], [3]), in the equations (2.3), (2.4), it is necessary replace $\varepsilon^i$ by $\varepsilon^i \epsilon^i$, $H_i(u)$ by
\[ \hat{H}_i(u) = \frac{H_i(u)}{\sqrt{\varepsilon^i (\lambda + f^i(u^i))}}, \]

$\beta_{ij}(u)$ by
\[ \hat{\beta}_{ij}(u) = \sqrt{\frac{\varepsilon^i (\lambda + f^i(u^i))}{\varepsilon^j (\lambda + f^j(u^j))}} \beta_{ij}(u), \]

$K_2$ by $\hat{K} = \lambda K_2 + K_1$, $\epsilon^i = \pm 1$. Then the linear problem (4.5)–(4.7) becomes the Lax pair with the spectral parameter for the system (2.2)–(2.3):

\[
\frac{\partial \phi_i}{\partial u^j} = \sqrt{\frac{\varepsilon^i (\lambda + f^i)}{\varepsilon^j (\lambda + f^j)}} \beta_{ij} \phi_j, \quad i \neq j, \quad \text{(4.8)}
\]

\[
\frac{\partial \phi_i}{\partial u^i} = -\sum_{k \neq i} \sqrt{\frac{\varepsilon^k (\lambda + f^k)}{\varepsilon^i (\lambda + f^i)}} \beta_{ki} \phi_k + \sqrt{\frac{\lambda K_2 + K_1}{\varepsilon^i (\lambda + f^i)}} H_i \psi, \quad \text{(4.9)}
\]
\[ \frac{\partial \psi}{\partial u^i} = - \sqrt{\frac{\lambda K_2 + K_1}{\epsilon^2 (\lambda + f^2)}} H_i \varphi_i, \]  \hspace{1cm} (4.10)

where \( \lambda \) is the spectral parameter. The condition of consistency for the linear system \((4.8)\)–\((4.10)\) is equivalent to the equations \((2.2)\)–\((2.5)\).

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