Direct Calculations of the Odderon Intercept in the Perturbative QCD

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Abstract. The odderon intercept is calculated directly, from its expression via an average energy of the odderon Hamiltonian, using both trial wave functions in the variational approach and the wave function recently constructed by R.A.Janik and J.Wosiek. The results confirm their reported value for the energy. Variational calculations give energies some 30% higher. However they also predict the odderon intercept to be quite close to unity. In fact, for realistic values of $\alpha_s$, the intercept calculated variationally is at most 2% lower than the exact one: 0.94 instead of 0.96. It is also found that the solution for $q_3 = 0$ does not belong to the odderon spectrum. The diffusion parameter is found to be of the order 0.6.
1 Introduction

There has recently been much interest in the odderon, both from experimental and theoretical points of view. On the experimental side, various processes mediated by a $C$-odd object with an intercept around unity are planned or already under investigation at HERA: the pseudoscalar meson production in ep collisions offers a direct probe for the odderon [1]. On the theoretical side, estimates of the relevant cross-sections have been made, using simplest model for the odderon, just a $C$-odd state of three non-interacting gluons [2] or a non-perturbative Regge pole at $j = 1$ [3]. Parallel to this there has been much activity in studying the gluon interaction effects, which presumably change the odderon intercept from exactly unity. In the perturbative QCD approach the odderon is a $C$-odd state of three reggeized gluons, which interact pairwise with a well-defined potential. The relevant equation is known since long ago [4]. It was shown that it is conformal invariant and splits into a pair of equations for the holomorphic and antiholomorphic parts of the wave function in exactly the same manner as for the pomeron [5]. An operator, later called $\hat{q}_3$, was also found, which, on the one hand, commutes with the odderon Hamiltonian $\hat{H}$ and, on the other hand, has a much simpler form than $\hat{H}$, thus opening a way to simplify the solution of the odderon problem considerably [6]. Finally, much effort has been put into directly relating the operators $\hat{q}_3$ and $\hat{H}$ to avoid using explicit wave functions in search of the odderon intercept [7-10]. Following this latter line of approach R.Janik and J.Wosiek (JW) calculated the odderon energy for arbitrary complex eigenvalue of $\hat{q}_3$ in [8]. In a later publication [9] they diagonalized the $\hat{q}_3$ operator and found both the odderon intercept and its wave function. If one relates the odderon intercept to the odderon ”energy” $\epsilon$ as [1]

$$\alpha_O(0) = 1 - (9\alpha_s/2\pi)\epsilon,$$

the result of JW for the ground state is

$$\epsilon = 0.16478...$$

Thus they confirm our old conclusion that the perturbative odderon intercept lies slightly below unity [11].

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1Different authors use different definitions of the odderon energy and therefore of its relation to the intercept. This has to be kept in mind comparing our numerical values with other references.
However a discussion about the validity of their procedure relating $q_3$ and $H$ is still going on [12]. In particular, for the eigenvalue $q_3 = 0$ the result which follows from their procedure is contested [12] (see also Sec. 5).

In view of this dispute a direct calculation of the odderon intercept, which starts from its explicit expression in terms of the wave function, acquires certain importance. It resolves in a unique manner any ambiguities involved in formal relations between ill-defined operators $q_3$ and $H$ and associated with boundary conditions to be imposed on their respective eigenfunctions. Reporting on these direct calculations in this paper, we also present two independent variational results, which can be obtained starting from the direct expression of the odderon energy in terms of its wave function. They present some interest, since the functions determined in a variational procedure may be a good approximation to the exact one, which is quite complicated and ill-suited for practical calculations.

Our direct calculations confirm the value (2) found by JW to be the ground state odderon energy. Our variational calculations with two different forms of the trial function give

$$\epsilon = 0.223 \text{ and } \epsilon = 0.226$$

(3)

that is, some 30% higher value for the energy. This change in $\epsilon$ corresponds to a 2% change in the odderon intercept, for realistic values of $\alpha_s$. From our calculations it also follows that the value $q_3 = 0$ does not lead to any physical odderon state.

The paper is organized as follows. In Sections 2 and 3 we review the basic formulas which serve as a starting point for our calculations. Sections 4 and 5 present the results of direct calculations of the odderon energy with a variational and exact wave functions respectively. Some conclusions are drawn in Section 6.

## 2 The odderon energy

The odderon energy can be sought as a ratio

$$\epsilon = E/D$$

(4)

where $E$ and $D$ are energy and normalization functionals, quadratic in the odderon wave function $Z(r,\phi)$ [13]. Explicitly

$$E = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} d\nu |\alpha_n(\nu)|^2$$

(5)
Here
\[ \epsilon_n(\nu) = 2 \Re \left( \psi \left( \frac{1 + |n|}{2} + i\nu \right) - \psi(1) \right), \]  
(6)
\[ \alpha_n(\nu) \] is a double Fourier transform of \( Z(r, \phi) \):
\[ \alpha_n(\nu) = \int_0^{\infty} dr r^{2-2i\nu} \int_0^{2\pi} d\phi e^{-in\phi} \]
\[ \left( i\nu + \frac{n+1}{2} + re^{i\phi}(h - i\nu - \frac{n-1}{2}) \right) \left( i\nu - \frac{n-1}{2} \right)(-\tilde{h} + i\nu - \frac{n-1}{2}) \]
(7)
where \( h \) and \( \tilde{h} \) are the two conformal weights for the holomorphic and antiholomorphic parts of the wave function. They have a general form
\[ h = \frac{1}{2} + n - i\nu, \quad \tilde{h} = \frac{1}{2} - n + i\nu, \quad -\infty < \nu < \infty, \quad n = \ldots -1, 0, 1, \ldots \]  
(8)
For the lowest branch of the odderon spectrum, which leads to the highest intercept, supposedly \( n = 0 \). This will always be assumed in the following. For the ground state also \( \nu = 0 \). \( D \) is obtained substituting \( \epsilon_n(\nu) \) by unity in (5).

The odderon wave function \( Z \) can be considered as a function of \( z = r \exp(i\phi) \) and its conjugate. It has a form
\[ Z(z, z^*) = |z(1 - z)|^{2h/3} \Phi(z, z^*) \]  
(9)
where function \( \Phi \) has to be invariant under substitutions
\[ z \to 1 - z, \quad z \to 1/z, \]  
(10)
which follows from the requirement of Bose symmetry in the three gluons.

Eqs. (4), (5) present a direct way of calculating the odderon energy once its wave function is known. It also opens up a possibility of finding \( \epsilon \) by variational techniques minimizing the value of (4) in a given space of functions which should satisfy (9) and (10). Note that if one drops the restricting condition of Bose symmetry (10), then function \( Z \) and consequently \( \alpha_n(\nu) \) become arbitrary, with the only requirement that (5) exist. Then it becomes clear that the minimal value of \( E/D \) is realized by \( |\alpha_n(\nu)|^2 = \delta_{n0}\delta(\nu) \), which gives \( \epsilon = \epsilon_0(0) = -4 \ln 2 \), that is the pomeron energy. Since the space of functions obeying (10) is smaller, one gets an evident bound [14]
\[ \epsilon > -4 \ln 2 \]  
(11)
In fact this bound is very crude (cf. (2)).
3 The $\hat{q}_3$ operator

An operator $\hat{q}_3$ which commutes with the odderon Hamiltonian was found in [6]:

$$\hat{q}_3^2 = -r_{12}^2 r_{23}^2 r_{31} q_1 q_2 q_3^2, \quad [\hat{q}_3^2, H] = 0.$$

(12)

Evidently, the odderon ground state (nondegenerate) should also be an eigenstate for $\hat{q}_3^2$. But, in contrast to the odderon Hamiltonian $H$, the operator $\hat{q}_3^2$ is a finite order differential operator, which does not contain logarithms of neither momenta nor co-ordinates. It splits into a product of two differential operators $\hat{q}_3$ and its conjugate, each of the third order in complex variables $z$ and $z^*$ respectively. In this section we closely follow the derivation of the spectrum of $\hat{q}_3$ given in [9]. The eigenvalue equation for $\hat{q}_3$ and its conjugate can easily be obtained using the explicit form (12). For the holomorphic part of the function $\Phi$ (see Eq. (9)) it reads [6,9]

$$a(z) \frac{d^3}{dz^3} \Phi(z) + b(z) \frac{d^2}{dz^2} \Phi(z) + c(z) \frac{d}{dz} \Phi(z) + d(z) \Phi(z) = 0,$$

(13)

where

$$a(z) = z^3(1-z)^3, \quad b(z) = 2z^2(1-z)^2(1-2z),$$

$$c(z) = z(z-1) (z(z-1)(3\mu +2)(\mu -1) + 3\mu^2 -\mu),$$

$$d(z) = \mu^2(1-\mu)(z+1)(z-2)(2z-1) - iq_3 z(1-z),$$

and $\mu = h/3$.

This is a third order linear differential equation with three regular singular points at $z = 0,1$ and $\infty$. It has three linearly independent solutions $u_{i}^{(0)}$, $i = 1,2,3$ which can be chosen so as to possess a given behaviour in the vicinity of $z = 0$:

$$u_1^{(0)}(z) \sim z^{2h/3}, \quad u_2^{(0)}(z) \sim z^{1-h/3}, \quad u_3^{(0)}(z) \sim z^{1-h/3} \log z + az^{-h/3}, \quad z \to 0$$

(14)

obtained from the characteristic equation corresponding to (13) at small $z$.

The final eigenfunction $\Phi(z, z^*)$ should be constructed as a sum of products of these solutions with those for the antiholomorphic part.

$$\Phi(z, z^*) = \sum_{i,k=1}^{3} \tilde{u}_{i}^{(0)}(z^*) A_{ik}^{(0)} u_{k}^{(0)}(z) \equiv \tilde{u}^{(0)} A^{(0)} u^{(0)}$$

(15)

Apart from the Bose symmetry requirements (10) it has also to be a single-valued function in the complex $z, z^*$ plane. This latter condition puts evident restrictions
on the form of the coupling matrix $A^{(0)}$. From the behaviour (14) one finds that the function $\Phi$ will be uniquely defined in the vicinity of $z = 0$ if and only if

$$A^{(0)}_{12} = A^{(0)}_{21} = A^{(0)}_{13} = A^{(0)}_{31} = 0, \quad A^{(0)}_{23} = A^{(0)}_{32} = 0 \tag{16}$$

To have uniqueness around the two other singular points, 1 and $\infty$, one has to know the behaviour of (15) in their vicinity. Due to the symmetry of the Eq. (13) under the substitutions

$$z \rightarrow 1 - 1/z, \quad z \rightarrow 1/(1 - z) \tag{17}$$

one can easily construct two other systems of solutions $u^{(1)}_i$ and $u^{(\infty)}_i$, $i = 1, 2, 3$ with the same behaviour around $z = 1$ and $z = \infty$ respectively. They can be expressed as superpositions of the initial solutions, say,

$$u^{(1)}_i(z) = \sum_{k=1}^{3} R^{(10)}_{ik} u^{(0)}_k(z) \tag{18}$$

and similarly for $u^{(\infty)}(z)$. The "transfer matrix" $R^{(10)}$ from the solutions $u^{(0)}$ to solutions $u^{(1)}$, is a constant matrix, which can technically be calculated once the solutions $u^{(1)}$ and $u^{(0)}$ are known. Evidently, in terms of $u^{(1)}$ the wave function $\Phi$ can be written as

$$\Phi(z, z^*) = \bar{u}^{(1)} A^{(1)} u^{(1)} \tag{19}$$

where

$$A^{(1)} = (R^{(01)})^T A^{(0)} R^{(01)}, \quad R^{(01)} = (R^{(10)})^{-1} \tag{20}$$

For $\Phi$ to be a single valued function of $\phi$ in the vicinity of $z = 1$ it is necessary that $A^{(1)}$ has the same properties (16) as the matrix $A^{(0)}$. Moreover the Bose symmetry requires that these matrices coincide. This gives an equation

$$A = (R^{(01)})^T A R^{(01)} \tag{21}$$

for a matrix $A$ satisfying (16) [9]. It determines both the eigenvalues $q_3$ and non-zero elements of the matrix $A$, that is, the eigenfunction $\Phi(z, z^*)$ according to (15). Note that there is no need to additionally require that $\Phi$ should be a single valued function of $\phi$ around $z = \infty$, since a contour encircling this point can be made of two contours around $z = 0$ and $z = 1$.

In [9] the transfer matrix $R$ was determined numerically from the solutions $u$ calculated as a series in powers of $z$. Solution of (21) then determined the eigenvalues of...
\( \hat{q}_3 \). They are all pure imaginary. The one corresponding to the odderon ground state turned out to be

\[
q_3 = -0.20526 \, i
\]  

(22)

After that the value (2) for the odderon energy was obtained by JW using their procedure to relate the eigenvalues of \( \hat{q}_3 \) and \( H \) constructed in [8].

Our calculations use the form of the wave function (15) with the matrix \( A \) determined by Eq. (21) as an input to be substituted into (5), which gives the value of the odderon energy directly.

4 Variational calculations

4.1 Starting point [13,14]

The first calculations of the odderon energy starting from Eqs. (4), (5) were done in the variational technique. In fact, taking some trial function which satisfies (10) and putting it into (4) and (5) one obtains an upper bound for the odderon energy and thus a lower bound for its intercept. Of course the problem consists in choosing a good trial function, on the one hand, close enough to the exact one and, on the other hand, sufficiently simple to allow for the numerical treatment. On top of that there is a problem of satisfying conditions (10).

This latter problem can be resolved by choosing the trial function as a function of arguments which are invariant under (10) by themselves. One of such arguments was proposed in [13]. Let \( r = |z| \) and \( r_1 = |1 - z| \). Then it is easy to see that

\[
a = \frac{r^2 r_1^2}{(1 + r^2)(1 + r_1^2)(r^2 + r_1^2)}
\]  

(23)

is invariant under (10). This is not the only argument with this property. Let us take

\[
b = \frac{(1 - |z|^2)(1 - |z_1|^2)(|z|^2 - |z_1|^2)}{(1 + r^2)(1 + r_1^2)(r^2 + r_1^2)}
\]  

(24)

It is invariant under \( z \to 1 - z \) and changes sign under \( z \to 1/z \), so that \( b^2 \) is invariant. Thus any function \( \Phi(a, b^2) \) will satisfy (10). To further narrow a possible choice of the trial functions, one can impose condition that at \( z \to 0 \) it has the correct behaviour following from (14) and (15). Due to invariance under (10) it would mean that it would also have the correct behaviour near the other two singular points \( z = 1 \) and \( z = \infty \).
For the ground state $h = 1/2$ it means that at $z \to 0$

$$\Phi(z, z^*) \sim c_1 r^{2/3} + c_2 r^{5/3}(1 + c_3 \ln r + c_4 \cos 2\phi)$$

This suggests taking the trial function in the form

$$\Phi(a, b^2) = c_1 a^{1/3} + c_2 a^{5/6}(1 + c_3 \ln a + c_4 b^2/a)$$

In our calculations the trial functions were chosen in a more general form

$$\Phi = N - N_1 \sum_{k=1}^{N-N_1} c_k a^{k/2-1/6} + \sum_{k=1}^{N_1} d_k a^{k-1/6} \ln a + f b^2/a^{-1/6}$$

with $N + 1$ variational parameters $c_k, d_k$ and $f$ (one of them is determined by the normalization condition). Note that the last term in $\Phi$ contains an azimuthal dependence, which, due to the properties of $\epsilon_n(\nu)$, can only raise the energy. Therefore one cannot expect any improvement of the variational bound coming from it. Indeed, our calculations including this term always lead to $f$ close to zero for the minimizing function. Accordingly in the following we shall not discuss its influence and use (27) with $f = 0$.

Function $\Phi$ with $N = 3, N_1 = 1$ was used in the first variational calculations in [13].

The basic quantity $\alpha$ given by Eq. (7) can be presented in the form

$$\alpha_n(\nu) = (n/2 - i\nu)(n - 1/2 - i\nu)[(n + 1/2 + i\nu)f_n^{(1)}(\nu) + (1 - n/2 - i\nu)f_{n-1}^{(0)}(\nu)]$$

where

$$f_n^{(k)}(\nu) = \int_0^\infty dr r^{-1-k-2i\nu} \int_0^{2\pi} d\phi e^{-i\nu}\Phi(r, \phi)$$

Function $f_n^{(k)}(\nu)$ has the properties: $f_n^{(1)}(\nu) = f_n^{(0)}(-\nu) = (f_n^{(0)})^*$. Using them one can restrict the summation over $n$ and integration over $\nu$ to nonnegative values. The value of the $|\alpha_n(\nu)|^2$ can evidently be expressed via a single function $f_n^{(0)}(\nu)$, which will simply be denoted as $f_n(\nu)$ in the following. In this manner one obtains

$$E = \sum_{n=0}^{\infty} \int_0^\infty d\nu \epsilon_n(\nu)p_n(\nu)$$

where for $n > 0$

$$p_n(\nu) = (n^2 + \nu^2)((n - 1)^2 + \nu^2)[(n + 1)^2 + \nu^2]|f_n(\nu)|^2 + ((n - 2)^2 + \nu^2)|f_{n-1}(\nu)|^2 + 2 \Re (n + 1/2 - i\nu)(2 - n/2 - i\nu)f_n(\nu)f_{n-1}(\nu)$$

and

\[ p_0(\nu) = \nu^2 \left( \frac{1}{4} + \nu^2 \right) \left( \frac{1}{4} + \nu^2 \right) |f_0(\nu)|^2 + (1 + \nu^2) |f_1(\nu)|^2 + 2 \text{Re} \left( \frac{1}{2} - i\nu \right) (1 - i\nu) f_0(\nu) f_1(\nu) \]

(32)

The normalization functional \( D \) has the same form (30) with \( \epsilon_n(\nu) \to 1 \). Thus calculation of the odderon energy requires calculation of functions \( f_n(\nu) \) and \( \epsilon_n(\nu) \).

The main technical difficulty is the double Fourier transform (29). The energy \( \epsilon_n(\nu) \) in \( E \), Eq. (6), monotonously grows both with \( n \) and \( \nu \). It is negative only for \( n = 0 \) and small enough values of \( \nu \). So the problem with this formalism is that cutting in (5) summation over \( n \) and integration over \( \nu \) by some maximal values \( n_m \) and \( \nu_m \), one always gets \( E \) smaller than the exact value, corresponding to \( n_m \) and \( \nu_m \to \infty \).

Therefore in the course of the calculation one always approaches the variational value of \( \epsilon \) from below. As we shall see in the following, in fact, rather high values of \( n_m \) and \( \nu_m \) are necessary to obtain \( \epsilon \) with a good degree of accuracy. This is the reason why in [13] a negative value was obtained for \( \epsilon \) (corresponding to \( \alpha_O(0) \simeq 1.07 \) ): too small values of \( n_m \) and \( \nu_m \) were chosen there. On the other hand, with high \( n \) and \( \nu \), the double Fourier transform (29) becomes very difficult, especially having in mind that, due to the factors in (31), two first terms in the asymptotic expansion of \( \alpha \) at high \( n \) and \( \nu \) cancel. As a result, a trustworthy calculation of \( E \) and \( D \) turns out to be very complicated, in spite of its superficial transparency.

The important point in obtaining reliable results has been using analytic asymptotic expansions for \( f \) at high \( n \) and \( \nu \), which are briefly discussed in Appendix.

### 4.2 Numerical procedure

As mentioned numerical integration in the double Fourier transform (29) presents a formidable calculational task. As mentioned, our results were obtained by two groups working independently and using different choices of the number \( N_1 \) of logarithmic terms in (27) for a given total number of terms \( N \). We present here in some detail the calculational procedure adopted in the computation with \( N_1 = 1 \).

In the integral (29) the integration over \( r > 1 \) was transformed to \( r < 1 \) by a substitution \( r \to 1/r \). The integration over \( \phi \) was reduced to the interval \( 0 < \phi < \pi \) and \( \exp(-n\phi) \) was substituted by \( 2 \cos(n\phi) \). To soften the behaviour of the integrand at small values of \( r \), three first terms of its asymptotics at \( r \to 0 \) were subtracted and treated in an exact manner. The final formula for \( f_n(\nu) \) used in the calculations was
thus
\[
f_n(\nu) = 2 \int_0^\pi d\phi \cos \phi \int_0^1 dr (r^{-2i\nu}(\zeta(r, \phi) - \zeta_0) + r^{1+2i\nu}(\zeta(r, \phi) - \zeta_0 + \zeta_1 r \ln r - \zeta_2 r)) + \\
d_{n0}2\pi \left( \frac{\zeta_0}{2i\nu(1-2i\nu)} + \frac{\zeta_1}{(1+2i\nu)^2} + \frac{\zeta_2}{1+2i\nu} \right) \]
\tag{33}
\]
where \(\zeta = Z/r\) and the subtraction constants are
\[
\zeta_0 = c_1(1/2)^{1/3}, \quad \zeta_1 = -2c_N(1/2)^{5/6}, \quad \zeta_2 = (1/2)^{5/6}(c_2 - c_N \ln 2) \tag{34}
\]
Eq. (33) was used for numerical calculation of \(f_n(\nu)\) in the interval of \(0 \leq n < 30\) and \(0 < \nu < 15\). Integrations were performed by dividing the rectangle \(0 < r < 1, \ 0 < \phi < \pi\) into an \(M \times M\) grid, interpolating \(\zeta\) quadratically on the grid and then doing the integrals explicitly. The maximal value of \(M\) was 640. The achieved accuracy was about \(10^{-5}\). Thus calculated values of \(f_n(\nu)\) were summed over \(n\) and integrated over \(\nu\) as indicated in (30)-(32) to obtain \(E\) and \(D\). Stable results were obtained with the quite high maximal values \(n_m = 300\) and \(\nu_m = 150\). In the part of \((n, \nu)\) space outside the rectangle \(0 \leq n < 30, \ 0 < \nu < 15\) the asymptotic expressions were used for \(f_n(\nu)\) (see Appendix). As a result we calculated \(E\) and \(D\) as a quadratic form in the variational parameters \(c_k, d_k\). Afterwards the minimal value \(\epsilon\) of \(E\), subject to condition \(D = 1\), was found by standard methods.

Other calculations used (27) with \(N_1 = 1\) for \(N = 3, 4\) and \(N_1 = 2\) for \(N = 6\). The adopted numerical procedure was different, however the final results, as we shall presently see, are quite similar.

Our results for different number of parameters \(N\) are presented in Table 1 for the both choices: \(N_1 = 1\) always and \(N_1 = 1\) for \(N = 3, 4, \ N_1 = 2\) for \(N = 6\). The corresponding energies are denoted \(\epsilon_1\) and \(\epsilon_2\) respectively. Adjacent columns present values of the variational parameters for both cases (\(c_1 = 1\)). The standard precision corresponds to the \((r, \phi)\) grid 320\(\times\)320. To clarify the accuracy achieved we also present the results for \(\epsilon_1\) with a double precision (the grid 640\(\times\)640) for \(N = 5\).

Inspecting these results we see that the final accuracy in energy is of the order \(5.10^{-3}\). Also taking \(N > 6\) with \(N_1 = 1\) leads to no improvement within the precision achieved, since the corresponding change in energy is of the same order or less. So our conclusion is that the variational odderon energy with a trial function (27) is given by (3) and that with the accuracy achieved in the course of numerical integration, as described above, the maximal number of terms to be taken in the trial function is \(N = 6\), although already with \(N = 3\) the energy is obtained up to \(1\%\).
5 Exact wave functions

5.1 Problems of precision

In this section we report on the calculations of the odderon energy using Eqs. (4) and (5) with the odderon wave function (15) obtained after solving Eq. (21). These functions can be obtained in the form of power series in $z$ and $z^*$, different in different parts of the complex $z, z^*$ plane. The exact solution is, of course, analytic in each of the variables, except at the three mentioned singular points $z = 0, 1, \infty$. So it is absolutely smooth at the boundaries of the regions in which it is represented by different power series. However, in practice one knows it to some finite precision. This causes certain discontinuities in the function itself and its derivatives at the mentioned boundaries. At high values of $n$ and $\nu$ the double Fourier transform (29) is very sensitive to such local irregularities of the wave function. They result in a very poor precision for the calculated odderon energy and even in a completely wrong order of magnitude for it. Because of this, our first step was to obtain the odderon wave function with a higher precision than reported in [9]. To this end we set up a program which essentially repeats the procedure employed in [9] and allows to reduce discontinuities of $\Phi$ calculated in different variables to values of the order $10^{-9}$. The only difference is that we used the basic functions $u_i, i = 1, 2, 3$, multiplied by appropriate factors to make them real at points where the transfer matrices are calculated. This substantially facilitates achieving the desired accuracy.

For the eigenvalue $q_3$ and wave function parameters $\alpha, \beta$ and $\gamma$ we obtained the following values (precision $10^{-9}$)

$$iq_3 = 0.205257506, \quad \alpha = 0.709605410, \quad \beta = -0.689380668, \quad \gamma = 0.145651837$$

(35)

However even with these high precision values direct calculation of $\Phi$ in one of the three sets of variables $z, 1 - 1/z$ or $1/(1 - z)$ fails in the vicinity of the point $z_0 = \exp(i\pi/3)$ where none of the series converges absolutely. In spite of the fact that $z_0$ is only a single point in the $(r, \phi)$ plane, this makes it practically impossible to calculate the double Fourier transform for $|n| > 10$ and/or $|\nu| > 5$. To overcome this difficulty we had to redevelop the function $\Phi$ around the intermediate point $(1/2)z_0$. With this redevelopment 100 terms in the series proved to be sufficient to obtain reliable results.

The Fourier transform was again performed by interpolating $Z$ quadratically on the $(r, \phi)$ grid and doing the integrals in $r$ and $\phi$ analytically. Reasonable results are
obtained already with a 160 x 160 grid. We however also used 320 x 320 and 640 x 640 grids to analyse the precision achieved at this step.

Even with a 640 x 640 grid the numerical Fourier transform becomes unreliable for \(|n| > 30\) and/or \(|\nu| > 15\). For such high values of \(|n|\) and \(|\nu|\) we used asymptotic formulas for the Fourier transform, which can easily be obtained from the expansion of \(\Phi\) around \(z = 1\) in a similar manner as for variational wave functions in the preceding section (see Appendix).

Our final cutoffs were chosen to be \(|n| < 300\) and \(|\nu| < 150\), which proved to be quite sufficient for the determination of \(\epsilon\) with a precision 0.001.

5.2 Ground state energy

Results of our numerical calculation of \(D\) and \(E\) in the region \(|n| < 30\) and \(|\nu| < 15\) using an \(N \times N\) grid in the \((r, \phi)\) plane are presented in the Table 2 for \(N = 160, 320\) and 640.

To these values one has to add the contributions from the asymptotic region described in the preceding section. They are

\[
\Delta D = 0.002398, \quad \Delta E = 0.018451
\]

Taking the results at \(N = 640\) as the most accurate ones we finally have

\[
D = 1.644454, \quad E = 0.273083, \quad \epsilon = 0.1660
\]

Thus our result for \(\epsilon\) coincides with the value found in [9] up to \(10^{-3}\). With all the difficulties involved in the numerical calculations, we consider this agreement quite satisfactory. So direct calculation of the odderon energy confirms the result found by JW.

5.3 Higher eigenvalues of \(q_3\)

We have tried to check the result of [9] also for the excited state with the next higher value of \(iq_3\). Unfortunately in this case calculations proved to be still more difficult and we could not arrive at a result of a convincing accuracy.

Our precise calculations of the wave function gave for this state:

\[
\text{\(iq_3 = 2.343921063, \ \alpha = 0.391855163, \ \beta = -0.0533712012, \ \gamma = 0.918477570\)} \quad (38)
\]
Numerical calculation of $D$ and $E$ in the region $|n| < 15$, $|\nu| < 15$ gave results presented in Table 3. As one observes, the achieved accuracy does not exceed 15%. Analyzing these numbers one can see that all the error comes from the region of maximal $|n|$ and $|\nu|$ where the double Fourier transform is apparently performed inaccurately. From these numbers we can only conclude that for this excited state

$$\epsilon \simeq 2.0 \pm 0.3$$

(39)

The value found in [9] is 1.7231... Our result does not contradict this number.

5.4 The case $q_3 = 0$

We have also studied a degenerate solution for $q_3 = 0$, discussed in [9]. However in this case we were not able to construct a wave function unique in the ($r, \phi$) plane and satisfying the necessary boundary conditions and symmetry requirements. At $q_3 = 0$ the solution $u_3^{(0)}(z)$ has no logarithmic term. If, following [9], we seek $\Phi$ in the form (15) and require the same conditions (16) to be satisfied, then Eq. (21) has no solutions at all, which means that the Bose symmetry cannot be fulfilled. The problem is related to the fact, that, with the logarithmic term missing in $u_3$, a unique solution is obtained also with $A_{33} \neq 0$. As a result, starting from a solution with $A_{33} = 0$ one always obtains $A_{33} \neq 0$ after applying the transformation (21).

One may wonder if a solution containing a product $\bar{u}_3(z^*)u_3(z)$ is admissible as a physical odderon wave function. Such a solution does not vanish as the distance between any of the three gluons becomes small. This can be easily seen from the behaviour at $z \to 0$ of the complete wave function (9): the explicit factor $r^{2h/3}$ cancels against the leading term $r^{-2h/3}$ in the product $\bar{u}_3(z^*)u_3(z)$ and one gets a constant. In fact the solution with this property can be easily constructed explicitly [12]. It is just a sum of three pomeron wave functions

$$\Psi(r_1, r_2, r_3) = \Psi_{BFKL}(r_1, r_2) + \Psi_{BFKL}(r_2, r_3) + \Psi_{BFKL}(r_3, r_1)$$

(40)

This function is an evident eigenfunction of $\hat{q}_3$ for zero eigenvalue and it is conformal and Bose symmetric. However it cannot be considered as a physical odderon state, since the Hamiltonian $H$ cannot be applied to it due to its singularities at low gluon momenta. Take the first term, which does not depend on $r_3$. Then it is proportional to $\delta^2(k_3)$ in the momentum space $k_1, k_2, k_3$ of the three gluons. The part of the odderon
Hamiltonian depending on $k_3$ contains a term proportional to $\ln k_3^2$ and two interactions between the gluons 31 and 32. It can be easily seen that the latter give finite result, applied to a wave function containing $\delta^2(k_3)$. The $\ln k_3^2$, however, is infinite. So $H$ applied to (40) is not defined. This property can also be seen from the approximate relation $H \simeq \ln \hat{q}_3$ valid for small values of $z$ [5]. Evidently $H$ diverges as $\hat{q}_3 \to 0$.

Thus our conclusion is that the $q_3 = 0$ state does not correspond to any physical odderon state.

5.5 ”Moving” odderon

For conformal weights $h = \frac{1}{2} + i\sigma$ the odderon energy is supposed to behave at small $\sigma$ as

$$\epsilon(\sigma) = \epsilon_0 + a\sigma^2$$

(41)

where $\epsilon_0$ is the value (2) and $a$ is a parameter which determines the diffusion of the odderon wave function in the momentum space. This parameter has long been known for the pomeron to be $14\zeta(3) \sim 16.8$ (in units $3\alpha_s/\pi$). It is of certain interest to find $a$ for the odderon. To this aim we first found the parameters of the odderon wave function for various (small) $\sigma$ using the same method as employed for $h = 1/2$. Our results are presented in Table 4. The value of $iq_3$ turned out to be real for arbitrary $\sigma$, whereas, with $\alpha$ chosen to be real, both $\beta$ and $\gamma$ result complex. We chose $\alpha = 1$.

Inspecting these figures one immediately notes that $|\gamma| = iq_3$. This relation was predicted by L.N.Lipatov [15].

With the odderon parameters found, we calculated the odderon energies directly, using the same technique as for $\sigma = 0$. With $\sigma$ different from zero calculation becomes still more cumbersome and time and memory consuming due to lack of certain symmetries and overall complex arithmetics. For these reasons we had to limit ourselves with a maximal 160 x 160 grid in the $(r, \phi)$ plane and neglected the contribution from the asymptotical region $n > 30, |\nu| > 15$. Our results are shown in Table 5 together with the ones obtained via the solution of the Baxter equation [16].

Our energies lie a little below the ones obtained from the Baxter equation, which is natural since we have neglected the asymptotic part of the $n, \nu$. Having this in mind we find a complete agreement between our direct calculation results and the ones based
on the Baxter equation. From our energies we find for the parameter $a$ in (41)

$$a = 0.61.$$ 

More precise energies found in [14] lead to

$$a = 0.605.$$ 

Note however that already at $\sigma = 1$ the approximation (41) breaks down and more powers of $\sigma^2$ are needed to describe the energy behaviour. It is interesting that the parameter $a$ for the odderon is much smaller than the one for the pomeron. In fact their ratio is of the same order as the ratio of corresponding energies.

6 Conclusions

Our calculations confirm the results obtained by JW for the perturbative odderon intercept and thus seem to remove any doubts concerning the validity of their procedure to relate the $\hat{q}_3$ operator and the Hamiltonian.

The variational calculations give a result for $\epsilon$ which is $\sim$30% larger than the exact value. However they also convey the important message that the intercept of the odderon lies quite close to unity being slightly smaller. In fact, for realistic values of $\alpha_s$, the intercept $\alpha_O(0)$ calculated variationally is at most 2% lower than the exact one: 0.94 instead of 0.96.

The disputed eigenvalue $q_3 = 0$ does not seem to correspond to any physical odderon state.

In the course of the variational calculations a simple approximate form of the odderon wave function is found, which allows a realistic calculation of the odderon residues, important for the study of processes involving the odderon exchange.

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8 Appendix. Asymptotics at large $n$ and $\nu$ for variational calculations

Passing to the variable $\rho = -\ln r$ and introducing 2-dimensional vectors $x = (\rho, \phi)$ and $w = (z, n) = (2\nu, n)$ we rewrite (29) as

$$f_n(\nu) = \int_{-\infty}^{+\infty} d\rho \int_{-\pi}^{+\pi} d\phi e^{iwx} Z(x)$$  \hspace{1cm} (42)

The integration point $x = 0$ is obviously essential for the asymptotics at high $n$ and $\nu$. At $x \to 0$, keeping terms up to third order in small $\rho$ and $\phi$, we have ($x = \sqrt{\rho^2 + \phi^2}$):

$$Z_p = (rr_1)^{1/3} a^p = (1/2)^p x^{2p+1/3} (1 - (1/2)\rho + a_1\rho^2 + b_1\phi^2 + c_1\rho^3 + d_1\rho\phi^2)$$  \hspace{1cm} (43)

where

$$a_1 = 5/36 - (29/12)p \hspace{0.5cm} b_1 = -1/72 - (25/12)p$$

$$c_1 = -1/36 + (29/24)p \hspace{0.5cm} d_1 = 1/144 + (25/24)p$$

and

$$p = k/2 - 1/6 \hspace{0.5cm} k = 1, 2, \ldots$$

For the term with a logarithm, in the same manner we obtain

$$\tilde{Z}_p = (rr_1)^{1/3} a^p \ln a = (1/2)^p x^{2p+1/3} \ln(x^2) (1 - (1/2)\rho + a_1\rho^2 + b_1\phi^2 + c_1\rho^3 + d_1\rho\phi^2) +$$

$$\ln(1/2) - (1/2) \ln(1/2) (1 - (1/2)\rho + a_2\rho^2 + b_2\phi^2 + c_2\rho^3 + d_2\rho\phi^2)$$  \hspace{1cm} (44)

where

$$p = k - 1/6 \hspace{0.5cm} k = 1, 2, \ldots$$

and

$$a_2 = a_1 \ln(1/2) - 29/12 \hspace{0.5cm} b_2 = b_1 \ln(1/2) - 25/12 \hspace{0.5cm} c_2 = c_1 \ln(1/2) + 29/24 \hspace{0.5cm} d_2 = d_1 \ln(1/2) + 25/24$$

Inserting these expressions into the integral (42) and extending the integration over $\phi$ to the whole real axis one obtains the asymptotical expansion of different terms in $f_n(\nu)$. In particular the asymptotical expansion of the term originating from $Z_p$ is found as

$$f_n^{(p)}(\nu) = c_p (1 + (1/2)i d/dz - a d^2/dz^2 - b d^2/dn^2 + ic b^3/dz^3 + id b^3/dz d n^2) w^{-\alpha - 1}$$
where $\alpha = 2p + 4/3$ and

$$c_p = 2^{\alpha+1-p}\Gamma^2(1/2 + \alpha/2)\cos(\pi\alpha/2)$$

Doing the derivatives, one obtains finally ($w = \sqrt{n^2 + 4\nu^2}$):

$$\text{Re} f_n^{(p)}(\nu) = c_p w^{-\alpha-1} \left( 1 + (\alpha + 1)w^{-4}[z^2(b - a(\alpha + 2) + n^2(a - b(\alpha + 2))] \right) \quad (45)$$

$$\text{Im} f_n^{(p)}(\nu) = -(1/2)c_p(\alpha + 1)zw^{-\alpha-3} \left( 1 + 2(\alpha + 3)w^{-4}[z^2(c(\alpha + 2) - d) + n^2(d(\alpha + 4) - 3c)] \right) \quad (46)$$

For the term with a logarithm only the part with $\ln x^2$ contributes. The result coincides with the formula above where the constant $c_p$ is substituted by $16\pi(1/2)^p$. If one puts these asymptotic expressions into (31) one finds that the two leading terms coming from $Z_{1/3}$ cancel. Numerically the asymptotic expansion begins to work at rather high values of $n$ and $\nu$: $\sqrt{n^2 + 4\nu^2} > 30$.

9 References

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Table 1. Odderon energy and parameters of the trial functions

| N  | $\epsilon_1 (N_1 = 1)$ | $c_2, c_3, \ldots; d_1$ | $\epsilon_2 (N_1 = 1, 2)$ | $c_2, c_3, \ldots; d_1, d_2$ |
|----|------------------------|--------------------------|---------------------------|--------------------------------|
| 3  | 0.22865                | -0.5036; 0.2895          | 0.23137                   | -0.50276; 0.28936              |
| 4  | 0.22632                | -0.2791,-0.3190;0.3609   | 0.22735                   | -0.21420,-0.41031;0.38136     |
| 5  | 0.22627                | -0.2009,-0.5052,0.1557; 0.3779 |                      |                               |
| 5* | 0.22634                | -0.2021,-0.5028,0.1543; 0.3775 |                      |                               |
| 6  | 0.22619                | -0.3735,0.08842,-0.9765, 1.003;0.3471 | 0.22269                   | 0.49231,-3.49272,-1.82821; 0.50316,-1.49528 |
| 7  | 0.22618                |                           |                           |                               |
| 8  | 0.22616                |                           |                           |                               |
| 9  | 0.22616                |                           |                           |                               |

*) double precision.

Table 2. $D$ and $E$ for the ground state

| N   | D       | E       |
|-----|---------|---------|
| 160 | 1.642162| 0.255480|
| 320 | 1.642085| 0.254852|
| 640 | 1.642056| 0.254632|
Table 3. $D$ and $E$ for the state with $iq_3 = 2.343921063$

| N  | D     | E     |
|----|-------|-------|
| 160| 2.92863| 6.08989|
| 320| 2.80693| 5.27381|
| 640| 3.05939| 6.96764|

Table 4. Odderon parameters for $h = \frac{1}{2} + i\sigma$

| $\sigma$ | $iq_3$     | $\beta$                                                          | $\gamma$                                                      |
|-----------|------------|-----------------------------------------------------------------|----------------------------------------------------------------|
| 0.01      | 0.205306079| -0.971740164-i0.014404102 | 0.205305637-i0.00425478                                       |
| 0.1       | 0.210089247| -0.995153863-i0.142974530 | 0.210052319-i0.003938872                                       |
| 0.3       | 0.247227544| -1.156524786-i0.415163678 | 0.247186043-i0.004529717                                       |
| 0.5       | 0.316528176| -1.395571390-i0.695891904 | 0.316214188+i0.014095104                                       |
| 1.0       | 0.619239545| -2.044631201-i1.672240784 | 0.591391973+i0.183611401                                       |

Table 5. Odderon energies for $h = \frac{1}{2} + i\sigma$

| $\sigma$ | $\epsilon$ | $\epsilon$ [14] |
|-----------|-------------|-----------------|
| 0.0       | 0.1534      | 0.16478         |
| 0.1       | 0.1597      |                 |
| 0.3       | 0.2085      | 0.21777         |
| 0.5       | 0.2980      | 0.30523         |
| 1.0       | 0.6269      | 0.63228         |