Non-Kähler String Backgrounds and their Five Torsion Classes

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ABSTRACT

We discuss the mathematical properties of six–dimensional non–Kähler manifolds which occur in the context of $\mathcal{N} = 1$ supersymmetric heterotic and type IIA string compactifications with non–vanishing background H–field. The intrinsic torsion of the associated $SU(3)$ structures falls into five different classes. For heterotic compactifications we present an explicit dictionary between the supersymmetry conditions and these five torsion classes. We show that the non–Ricci flat Iwasawa manifold solves the supersymmetry conditions with non–zero H–field, so that it is a consistent heterotic supersymmetric groundstate.
1 Introduction

Superstring compactifications on internal spaces with non–vanishing expectation values for additional ‘matter’ fields are interesting for various reasons. In particular, they offer new prospects for obtaining more realistic string models capable of describing (part of the) four–dimensional spacetime physics. For instance, in superstring compactifications with non–vanishing H–field fluxes (see eg. [1]–[26]) a potential gets generated that results in the lifting of some of the flat directions, i.e. to a fixing of the value of some of the geometric moduli of the underlying compact internal space. Moreover, in the context of intersecting brane world models with D6–branes wrapped around internal 3–cycles, it is possible to construct standard model like spectra in a systematic way (see eg. [27, 28] and other references therein). Warped compactification with fluxes and/or branes may also explain the origin of various hierarchies in string theory [29]–[35],[13].

Spacetime supersymmetry imposes strong restrictions on the allowed form of the string background fields. For instance, it is well known that in order to obtain purely geometric compactifications (without any further background expectation values) with either $\mathcal{N} = 1$ spacetime supersymmetry in the heterotic context or $\mathcal{N} = 2$ spacetime supersymmetry in the type II context, the internal space must be a complex Kähler manifold with vanishing Ricci tensor, i.e. a Calabi–Yau space [36]. However, when turning on additional background fields, which may correspond to the presence of branes, of fluxes, of a non–constant dilaton and/or of a non–trivial warp factor, the internal geometry will not remain Ricci–flat anylonger, and may not even remain complex. Since in this more generic situation the tools of complex geometry do not apply, the mathematical structure of these spaces is much less understood than in the case of Calabi–Yau spaces.

Heterotic string compactifications on an internal space with non–trivial warp factor, dilaton and H–field background were first discussed in [1]. Related compactifications in type I, type II and eleven–dimensional supergravity were also investigated in [2]. The requirement of $\mathcal{N} = 1$ spacetime supersymmetry forces the internal six–dimensional space to come equipped with a certain $SU(3)$ connection with torsion, provided by the H–field [1]. It turns out that certain properties of six–dimensional manifolds with torsion and $SU(3)$ structure have been systematically worked out in the mathematical literature [37]. It has been shown [38] that the allowed torsion tensors have to fall into five different classes. The resulting manifolds are quite restricted and may or may not be complex. One aim of the present paper is to provide an explicit dictionary between the mathematical classification of torsion of compact six–dimensional spaces with $SU(3)$ structure and the conditions leading to $\mathcal{N} = 1$ supersymmetric backgrounds in heterotic string theory.

The same mathematical classification of torsion may also be used in the context of type IIA compactifications on six–dimensional spaces $M$ with additional H–field backgrounds or wrapped D–branes. Again the gravitational backreaction of the fluxes or the branes is so strong that the internal type IIA geometry ceases to be of the Calabi–Yau type. One nice way to understand this feature is provided by the lift of type IIA theory to eleven–dimensional M–theory. Let us consider the case when, upon lifting to eleven dimensions, the type IIA

\[ \nabla J^m = H^p_{\, \, mn}J^p_n + H_{\, \, mn}^sJ^s_n. \]
H–fluxes become completely geometrized, that is to say that they completely originate from the eleven–dimensional metric. This is, for instance, the case for the background of the Ramond one–form gauge potential, as can be seen by a generalized Hopf procedure. Similarly, it is well known (see e.g. [39]) that the dilaton and H–field background of wrapped D6–branes become purely geometrical in eleven dimensions, namely the D6–branes correspond to the metric of the eleven–dimensional Eguchi–Hanson instanton. In all these purely geometric M–theory compactifications with \( \mathcal{N} = 1 \) supersymmetry in four spacetime dimensions it is known that the internal space must be a seven–dimensional manifold with \( G_2 \) holonomy. By constructing the \( G_2 \) space as a fibration over the six–dimensional base \( M \), one can deduce the corresponding \( SU(3) \) structure of \( M \) in type IIA from the associated \( G_2 \) structure in M–theory. Conversely, one can give conditions on the torsion of \( M \) which allow for an uplift of \( M \) to a seven–dimensional \( G_2 \) manifold [38].

The main results of this paper can be summarised as follows. We give an extensive dictionary between the known mathematical classification of torsion associated with compact six–dimensional manifolds with \( SU(3) \) structure and the necessary and sufficient conditions derived in [1] for four–dimensional \( \mathcal{N} = 1 \) spacetime supersymmetry in heterotic string compactifications. Since spacetime supersymmetry requires the six–dimensional manifolds to be complex, the associated torsion \( \tau \) has to lie in

\[
\tau \in \mathcal{W}_3 \oplus \mathcal{W}_4 \oplus \mathcal{W}_5. \quad (1.1)
\]

In addition, spacetime supersymmetry requires that

\[
2 \mathcal{W}_4 + \mathcal{W}_5 = 0 \quad (1.2)
\]

with both

\[
\mathcal{W}_4 \text{ and } \mathcal{W}_5 \text{ exact and real.} \quad (1.3)
\]

The presence of torsion results in the introduction of a generalized spin connection. We will denote the associated generalized Riemann tensor by \( \tilde{R}_{mnpq} \). Spacetime supersymmetry then also implies the vanishing of the generalized Ricci two–form

\[
\tilde{R}_{mnpq} J^{pq} = 0. \quad (1.4)
\]

The associated generalized Ricci tensor \( \tilde{R}_{m} \), however, need not vanish.

An example of a six–dimensional manifold satisfying (1.1), (1.2) and (1.3) is given by the Iwasawa manifold. This manifold was, to a certain extent, already considered in [1]. The moduli space of its complex structures consists of two disconnected components. For any choice of a complex structure one finds that \( \mathcal{W}_4 = \mathcal{W}_5 = 0 \). Explicit computation of the associated generalized Ricci two–form and of the generalized Ricci tensor shows that while the former vanishes, the latter doesn’t. In order for the Iwasawa manifold to be a consistent background for the heterotic string, certain additional conditions need to be met in the gauge sector. We show that the Bianchi identity for the three–form \( H \) is non–trivial and that it may be solved by turning on an abelian background gauge field. We note that this way of solving the Bianchi identity is very different from the usual procedure of embedding the spin in the gauge and of the associated construction of \( SU(3) \) instantons.
The Iwasawa manifold possesses $\mathcal{W}_4 = \mathcal{W}_5 = 0$. One may wonder whether it is possible to construct complex manifolds with $\mathcal{W}_4 \neq 0$, $\mathcal{W}_5 \neq 0$ satisfying (1.2) and (1.3). A possible technique for finding such spaces is given by the following. Suppose that one has constructed a complex manifold $P$ for which both $\mathcal{W}_4$ and $\mathcal{W}_5$ are exact and real, so that $\mathcal{W}_4 = d\Lambda$, $\mathcal{W}_5 = d\Sigma$, and hence $2\mathcal{W}_4 + \mathcal{W}_5 = d(2\Lambda + \Sigma) \equiv df$. Then, by performing the following rescaling of the metric, $g \rightarrow e^{2f}g$, it can be shown that $2\mathcal{W}_4 + \mathcal{W}_5 - df = 0 \rightarrow 2\mathcal{W}_4 + \mathcal{W}_5 = 0$. Therefore, the complex manifold $P$ has a chance of providing a consistent background for the heterotic string.

The paper is organized as follows. In section 2 we give a review of the mathematical classification of the intrinsic torsion of $SU(3)$ structures in terms of five classes as well as of the derivation of $G_2$ structures from $SU(3)$ structures [38]. In section 3 we present the dictionary between the mathematical classification of torsion and the necessary and sufficient conditions derived in [1] for four–dimensional $\mathcal{N} = 1$ spacetime supersymmetry in heterotic string compactifications. In section 4 we show that the Iwasawa manifold provides a consistent background for the propagation of the heterotic string. Section 5 contains an outlook. Appendix A contains a review on dynamical and fibered $G_2$ structures and their relation to type IIA compactifications. Appendix B contains a review of the derivation of the conditions for $\mathcal{N} = 1$ spacetime supersymmetry obtained in [1].

2 $SU(3)$ structures and the five classes of torsion

In this section we review [38] the classification of the intrinsic torsion of $SU(3)$ structures in terms of five classes, as well as the derivation of $G_2$ structures from $SU(3)$ structures.

We consider a six–dimensional manifold $M$ equipped with a Riemannian metric $g$ and an almost–complex structure $J$, with its associated two–form. At least locally one can choose a local orthonormal basis $(e^1, \ldots, e^6)$ such that the almost complex structure is

$$J = e^1 \wedge e^2 + e^3 \wedge e^4 + e^5 \wedge e^6. \tag{2.1}$$

It follows that a basis of $(1,0)$–forms is given by

$$e^i + i J \cdot e^i \in \Lambda^{(1,0)}, \tag{2.2}$$

where $J \cdot e^a = J^a_b e^b$ and consequently $J \cdot J = -1$. The above defines a $U(3)$ structure. An $SU(3)$ structure, on the other hand, is determined by the $(3,0)$–form

$$\Psi = (e^1 + i e^2) \wedge (e^3 + i e^4) \wedge (e^5 + i e^6). \tag{2.3}$$

This form has norm 1 and it is subject to the compatibility relations $J \wedge \psi_\pm = 0$, $\psi_+ \wedge \psi_- = \frac{2}{3} J \wedge J \wedge J$, where we used the further split into real and imaginary parts

$$\Psi = \psi_+ + i \psi_- \tag{2.4}$$

In the following, by an abuse of notation, we will use the same symbol $J$ for both the almost–complex structure and its associated two–form.
with $\psi_- = J \cdot \psi_+$.  

The failure of the holonomy group of the Riemann connection of $g$ to reduce to $SU(3)$ can be measured by the intrinsic torsion $\tau$, which is identified with $\nabla J$, if $\nabla$ is the covariant derivative with respect to the Riemannian connection. The space to which the torsion belongs can be decomposed into five classes \[ W_1 \oplus W_2 \oplus W_3 \oplus W_4 \oplus W_5, \] described by the decomposition of $\tau$ into $SU(3)$ irreducible representations:

\[ (1 + 1) + (8 + 8) + (6 + \bar{6}) + (3 + \bar{3}) + (3 + \bar{3}). \]  

We note that $W_1, \ldots, W_4$ is the space determining the torsion of the $U(3)$ structure \[ [38]. \]  

A closer analysis of these classes and of how they are determined will let us determine many important properties of the manifold. The five $W_i$ are expressed in terms of $dJ$ and $d\Psi$ in the following manner \[ [38]: \]

\[ W_1 \leftrightarrow [dJ]^{(3,0)}, \quad W_2 \leftrightarrow [d\Psi]^{(2,2)}, \]
\[ W_3 \leftrightarrow [dJ]^{(2,1)}, \quad W_4 \leftrightarrow J \wedge dJ, \]
\[ W_5 \leftrightarrow [d\Psi]^{(3,1)}. \] \[ (2.7) \]

The subscript 0 is used to remove all the forms which are not primitives, i.e. $\beta \in \Lambda_0^{(2,2)}$ if $J \wedge \beta = 0$, and $\gamma \in \Lambda_0^{(2,1)}$ if $J \wedge \gamma = 0$. A more precise definition \[ [38]\] comes as follows. The first class $W_1$ is given by two real numbers $W_1 = W_1^+ + W_1^-$. These components are selected by taking the exterior product with $\Psi$ and, using (2.4), one can define

\[ d\psi_+ \wedge J = \psi_+ \wedge dJ = W_1^+ J \wedge J \wedge J, \]
\[ d\psi_- \wedge J = \psi_- \wedge dJ = W_1^- J \wedge J \wedge J, \] \[ (2.8) \]

where $J^3$ is essentially the volume form. Similarly, one can see that the $\Lambda_0^{(2,2)}$ part of $d\Psi$ is determined by two $(1,1)$–forms, since one removes all the non–primitive forms. The two components of $W_2$ ($W_2 = W_2^+ + W_2^-$) are defined as follows,

\[ (d\psi_+)^{2,2} = W_1^+ J \wedge J + W_2^+ \wedge J, \]
\[ (d\psi_-)^{2,2} = W_1^- J \wedge J + W_2^- \wedge J. \] \[ (2.9) \]

To define the components of $W_4$ and $W_5$, one further needs the introduction of the contraction operator

\[ \downarrow : \bigwedge^k T^* \otimes \bigwedge^n T^* \to \bigwedge^{n-k} T^* \]

\[ (L(k), M(n)) \leftrightarrow \frac{1}{n!} \binom{n}{k} L^{a_1 \ldots a_k} M_{a_1 \ldots a_n} e^{a_{k+1}} \ldots e^{a_n} \] \[ (2.10) \]

with the convention that $e^1 \wedge e^2 \downarrow e^1 \wedge e^2 \downarrow e^3 \wedge e^4 = e^3 \wedge e^4$. Using this operator, we can now define $W_4$ and $W_5$, which are the following one–forms:

\[ W_4 = \frac{1}{2} J \downarrow dJ, \]
\[ W_5 = \frac{1}{2} \psi_+ \downarrow d\psi_+. \] \[ (2.11) \]
$W_3$ can be read off from

$$dJ^{(2,1)} = [J \wedge W_4]^{(2,1)} + W_3.$$  \hspace{1cm} (2.13)

Examples of manifolds which fall into this classification are given by Calabi–Yau spaces. In this case one has strict $SU(3)$ holonomy, namely the torsion is zero and therefore the r.h.s of these equations vanishes. This is equivalent to the fact that we have a Kähler structure satisfying $dJ = 0$ and a closed holomorphic 3–form of constant norm $d\Psi = 0$. However, we will see that there are several other cases of physical interest.

Depending on the classes of torsion one can obtain many different types of manifolds (complex or not), whose properties have often been studied in the mathematical literature. Here, we will list those manifolds which we will be using in the following. First of all it is important to distinguish between complex and non–complex manifolds.

i) **Complex manifolds**

The components of the Nijenhuis tensor completely determine $W_1 \oplus W_2$, therefore:

$$W_1 = W_2 = 0 \iff \text{Hermitian manifold.}$$  \hspace{1cm} (2.14)

These equations tell us that these manifolds have a complex structure $J$ such that the Hodge type is preserved upon differentiation,

$$dJ \in \Lambda^{(2,1)} \oplus \Lambda^{(1,2)} \quad \text{and} \quad d\Psi \in \Lambda^{(3,1)} \oplus \Lambda^{(1,3)}.$$  \hspace{1cm} (2.15)

The condition (2.14) is also equivalent to

$$d\alpha^{(0,2)} = 0, \quad \forall \alpha \in \Lambda^{(1,0)}.$$  \hspace{1cm} (2.16)

Inside the class of **complex manifolds** we find

\begin{align*}
W_1 = W_2 = W_4 = 0, & \quad \tau \in W_3 \oplus W_5 \iff \text{balanced manifolds,} \\
W_1 = W_2 = W_4 = W_5 = 0, & \quad \tau \in W_3 \iff \text{special–hermitian manifolds,} \\
W_1 = W_2 = W_3 = W_4 = 0, & \quad \tau \in W_5 \iff \text{Kähler manifolds,} \\
W_1 = W_2 = W_3 = W_4 = W_5 = 0, & \quad \tau = 0 \iff \text{Calabi–Yau manifolds.}
\end{align*}  \hspace{1cm} (2.17)

Another class of complex manifolds which does not lie in a definite sector of the above classification is given by the **Strong Kähler Torsion** (SKT) manifolds. These manifolds are such that

$$\partial \bar{\partial} J = 0, \quad \text{with } dJ \neq 0.$$  \hspace{1cm} (2.18)

We will see that these manifolds are related to solutions of the Bianchi identity $dH = 0$. A result of [14] tells us that if a manifold of dimension $d \geq 6$ is special–hermitian (i.e. $\tau \in W_3$) then it cannot be SKT. A short alternative proof is the following: a complex manifold with $W_4 = 0$ satisfies $J \wedge dJ = 0$. If we now take a $\bar{\partial}$ derivative on it we get

$$\bar{\partial}(J \wedge dJ) = \bar{\partial}J \wedge \bar{\partial}J + J \wedge \bar{\partial} \partial J = 0,$$  \hspace{1cm} (2.19)
where the last term vanishes because of the SKT condition. What remains can also be written as $\partial J \wedge \partial J = 0$ from which it follows that the manifold has also to be Kähler $\partial J = \bar{\partial} J = 0$.

ii) Non–complex manifolds

Some non–complex manifolds which will occur later on are

$$\tau \in W_1 \iff \text{nearly–Kähler manifolds,} \quad (2.20)$$

$$\tau \in W_2 \iff \text{almost–Kähler manifolds.} \quad (2.21)$$

Again, in terms of forms, nearly–Kähler manifolds are defined by a closed $(3,0)$–form and an almost–complex structure satisfying

$$dJ \in \Lambda^{(3,0)}, \quad d\Psi = W_1 J \wedge J. \quad (2.22)$$

Almost–Kähler manifolds are instead defined by a closed almost–complex structure, but a $(3,0)$ form satisfying

$$dJ = 0, \quad d\Psi \in \Lambda^{(2,2)}_0. \quad (2.23)$$

The last definition we want to introduce here and that will be very important in our follow–up is given by the so–called half–flat manifolds $[38]$. These manifolds can be either complex or not complex and satisfy

$$\tau \in W_1^- \oplus W_2^- \oplus W_3, \quad (2.24)$$

which implies

$$d\psi_+ = 0 \quad \text{and} \quad J \wedge dJ = 0. \quad (2.25)$$

We will see that such manifolds will be useful in constructing seven–dimensional manifolds of $G_2$ holonomy.

Finally, we want to mention a result of $[40]$ about conformally rescaled metrics. Conformally rescaling the metric adds components to the torsion in $W_4 \oplus W_5$. In particular, the following combination remains fixed $[38]$,

$$3W_4 + 2W_5. \quad (2.26)$$

For instance, conformally rescaled Calabi–Yau manifolds will be the proper subset of the manifolds in the class

$$\tau \in W_4 \oplus W_5 \quad (2.27)$$

for which $2W_4 = -3W_5$.

Let us now briefly review a construction by Hitchin $[42]$ whereby a six–dimensional half–flat manifold is lifted to a seven–dimensional manifold with $G_2$ holonomy. This is achieved as follows. Starting with a six–dimensional manifold $M$ equipped with an $SU(3)$ structure $(M, J, \Psi)$, a seven–dimensional manifold is then constructed as a warped product $X_7 = M \times I$, where $I \subset \mathbb{R}$, and the $G_2$ structure is defined by

$$\phi = J \wedge dr + \psi_+.$$

(2.28)
where $dr$ is the line element on $I$ and $(J, \psi_+, \psi_-)$ are now $r$–dependent. The $G_2$ structure of $X_7$ satisfies

$$
d\phi = \left( \hat{d}J - \frac{\partial \psi_+}{\partial r} \right) \wedge dr + \hat{d} \psi_+, \\
d \star \phi = \left( \hat{d} \psi_- + J \wedge \frac{\partial J}{\partial r} \right) \wedge dr + J \wedge \hat{d} J, \tag{2.29}
$$

where the forms have now been promoted to seven–dimensional forms and $\hat{d}$ denotes the six–dimensional differential operator. Since $M$ is half–flat, its $SU(3)$ structure is such that $\hat{d} \psi_+ = J \wedge \hat{d} J = 0$. Therefore, demanding that the seven–dimensional manifold $X_7$ has $G_2$ holonomy, i.e. $d\phi = d \star \phi = 0$, yields

$$
\left\{ \begin{array}{l}
\hat{d} J = \frac{\partial \psi_+}{\partial r}, \\
\hat{d} \psi_- = -J \wedge \frac{\partial J}{\partial r}.
\end{array} \right. \tag{2.30}
$$

These are the flow equations of Hitchin [42].

### 3 $\mathcal{N} = 1$ supersymmetric compactifications of heterotic string theory

In the following, we will be interested in $\mathcal{N} = 1$ supersymmetric compactifications of heterotic string theory on spaces with metric given by

$$
ds^2 = g^{0}_{MN} dx^M \otimes dx^N = e^{2\Delta(y)} (dx^\mu \otimes dx^\nu \hat{g}_{\mu\nu}(x) + dy^m \otimes dy^n \hat{g}_{mn}(y)). \tag{3.1}
$$

Here $\hat{g}_{\mu\nu}(x)$ denotes the metric of a four–dimensional maximally symmetric spacetime and $\Delta$ denotes a warp factor which we take to only depend on the internal coordinates $y^m$. It can be shown [1],[2] that the ten–dimensional supersymmetry equations (in the absence of gaugino condensates) can be cast into the following form

$$
\delta \psi_M = \mathcal{D}_M \epsilon \equiv \nabla_M \epsilon - \frac{1}{4} H_M \epsilon, \tag{3.2a}
$$

$$
\delta \chi = -\frac{1}{4} \Gamma^{MN} \epsilon F_{MN}, \tag{3.2b}
$$

$$
\delta \lambda = \nabla \phi + \frac{1}{24} H \epsilon, \tag{3.2c}
$$

where $H \equiv \Gamma^{MNP} H_{MNP}, H_M \equiv H_{MNP} \Gamma^{NP}$, and where the covariant derivative $\nabla$ is constructed from the rescaled metric $g_{MN} = e^{-2\phi} \hat{g}_{MN}$. Necessary and sufficient conditions for $\mathcal{N} = 1$ spacetime supersymmetry in four dimensions were derived in [1] and are given by:

1. the four–dimensional spacetime has to be Minkowski, i.e. $\hat{g}_{\mu\nu} = \eta_{\mu\nu}$;
2. the internal six–dimensional manifold has to be complex, i.e. the Nijenhuis tensor $N_{mnp}$ has to vanish;

3. up to a constant factor, there is exactly one holomorphic $(3,0)$–form $\omega$, whose norm is related to the complex structure $J$ by

$$ \star d \star J = i (\bar{\partial} - \partial) \log ||\omega||; \quad (3.3) $$

4. the Yang–Mills background field strength must be a $(1, 1)$–form and must satisfy

$$ \text{tr} F \wedge F = \text{tr} \tilde{R} \wedge \tilde{R} - i \partial \bar{\partial} J \quad (3.4) $$

as well as

$$ F_{mn} J^{mn} = 0; \quad (3.5) $$

5. the warp factor $\Delta$ and the dilaton $\phi$ are determined by

$$ \Delta(y) = \phi(y) + \text{constant}, $$

$$ \phi(y) = \frac{1}{8} \log ||\omega|| + \text{constant}; \quad (3.6) $$

6. the background three–form $H$ is determined in terms of $J$ by

$$ H = \frac{i}{2} (\bar{\partial} - \partial) J, \quad (3.7) $$

where $i(\partial - \bar{\partial}) = dx^n J_n^m \partial_m$.

Inspection of $(3.4)$ shows that if $\text{tr} \tilde{R} \wedge \tilde{R}$ is non–vanishing, then it has to be a $(2,2)$–form for consistency.

We note that the integrability associated with the vanishing of the internal gravitino equation $(3.2a)$ (cf. appendix B),

$$ [\mathcal{D}_m, \mathcal{D}_n] \eta_+ = \frac{1}{4} \tilde{R}_{mn}^{pq} \Gamma_{pq} \eta_+ = 0, \quad (3.8) $$

implies the vanishing of the Ricci two–form

$$ \tilde{R}_{mn}^{pq} J_{pq} = 0, \quad (3.9) $$

where $\tilde{R}_{mn}^{pq}$ denotes the generalized Riemann tensor constructed out of

$$ \mathcal{D}_m \equiv \partial_m + \frac{1}{4} (\omega_m^{np} - H_m^{np}) \Gamma_{np}. \quad (3.10) $$

Here $H_m^{np}$ denotes a torsion term. We also note that in general it does not follow that the generalized Ricci tensor $\tilde{R}_{mn} = \tilde{R}^{q}{}_{mqn}$ vanishes. Usually, the vanishing of the Ricci tensor


is derived by multiplying (3.8) with $\Gamma^n$ and using that $R_{m[npq]} = 0$. In general, however, $\tilde{R}_{m[npq]} \neq 0$ because of the torsion terms, and hence one cannot conclude that $\tilde{R}_{mn} = 0$.

Next we will reformulate the conditions just mentioned in terms of torsional constraints, using the language of section 2. Since the internal manifold is taken to be complex, it immediately follows from (2.14) that

$$\mathcal{W}_1 = \mathcal{W}_2 = 0.$$  \hspace{1cm} (3.11)

The torsion is therefore left in

$$\tau \in \mathcal{W}_3 \oplus \mathcal{W}_4 \oplus \mathcal{W}_5,$$  \hspace{1cm} (3.12)

but it cannot be completely generic, because there is one further geometric constraint to be satisfied, namely (3.3). This equation relates the dual of the complex structure to the holomorphic $(3,0)$–form and therefore can be interpreted as a relation among the $\mathcal{W}_4$ and $\mathcal{W}_5$ classes. The $\mathcal{W}_4$ class is determined by $J \wedge dJ$ which, using the duality relation $*J = \frac{1}{2} J \wedge J$, can be interpreted as $d * J$. This implies that information about this class is encoded in the left–hand side of equation (3.3), as this is given by the one–form $*d * J$. Moreover, from the definition of $\mathcal{W}_4$ (2.11), it follows that it must be described by a one–form, so it is interesting to establish the precise relation among the two quantities. Following the definition of the contraction operator (2.10), we can rephrase the Hodge star dual and show that

$$\mathcal{W}_4 = \frac{1}{2} J \lrcorner dJ = \frac{1}{2} J \cdot (d * J).$$  \hspace{1cm} (3.13)

The proof follows directly from the definition of the contraction operator

$$J \lrcorner dJ = \frac{3}{2} J^{sn} dx^p \nabla_{[s} J_{np]} = dx^n J^p_s \nabla_p J_n^s,$$  \hspace{1cm} (3.14)

and of the Hodge dual:

$$J \cdot (d * J) = -dx^n J^s_n \nabla_p J^p_s = dx^n J^p_s \nabla_p J_n^s.$$  \hspace{1cm} (3.15)

Going back to (3.3), and in order to determine the precise relation between $\mathcal{W}_4$ and $\mathcal{W}_5$, we better consider multiplying (3.3) with $J$. In this way the equation gets simplified to

$$\mathcal{W}_4 = -\frac{1}{2} d \log ||\omega||,$$  \hspace{1cm} (3.16)

which gives a further constraint on $\mathcal{W}_4$, namely that it is an exact real 1–form. On the right hand side of this equation we find the norm of the holomorphic form, which is related to $\mathcal{W}_5$. Our classification of the torsion relies on the definition of a unit norm $(3,0)$–form, which in this case is simply

$$\Psi = \frac{\omega}{||\omega||}.$$  \hspace{1cm} (3.17)

This form is not holomorphic anymore for a generic dilaton profile (which is then related to the $\omega$ norm) and that implies $\mathcal{W}_5 \neq 0$. From the definition of the unit–norm $(3,0)$–form (3.17) it follows that

$$d \psi_+ = \frac{1}{2} (d \Psi + d \bar{\Psi}) = -d \log ||\omega|| \wedge \psi_+.$$  \hspace{1cm} (3.18)
The contraction with $\psi_+$ will therefore lead to
\[
\mathcal{W}_5 = \frac{1}{2} \psi_+ \wedge d\psi_+ = d\log ||\omega|| ,
\] (3.19)
and this finally translates into
\[
2\mathcal{W}_4 + \mathcal{W}_5 = 0 .
\] (3.20)
Since the remaining equations to obtain supersymmetric solutions do not give rise to further geometrical constraints, but rather to equations in the gauge sector, we can conclude that the torsion of the complex manifolds giving rise to supersymmetric compactifications of heterotic string theory has to satisfy
\[
\tau \in \mathcal{W}_3 \oplus \mathcal{W}_4 \oplus \mathcal{W}_5 ,
\](3.21)
with
\[
2\mathcal{W}_4 + \mathcal{W}_5 = 0
\] and $\mathcal{W}_4, \mathcal{W}_5$ exact and real.

Note that the condition of $\mathcal{W}_4$ and of $\mathcal{W}_5$ being exact is a global condition. Conversely, (3.21) implies that the six–dimensional manifold is complex, that (3.3) holds and that there exists a holomorphic $(3,0)$–form without zeroes or poles, as we will now show. Let $\mathcal{W}_5 = d\Sigma$ with $\Sigma$ real and define $\omega = e^\Sigma \Psi$. One has $d\psi_+ = \psi_+ \wedge d\Sigma$ and hence $d\psi_- = \psi_+ \wedge J \cdot d\Sigma$. Since $\psi_+ \wedge J \cdot d\Sigma = \psi_- \wedge d\Sigma$, it follows that $d\Psi = \Psi \wedge d\Sigma$, and hence $d\omega = 0$.

We can now try to discuss which classes of the manifolds discussed in section 2 can be used as consistent solutions of the torsion constraints (3.21).

The easiest way to solve (3.21) non–trivially is to consider manifolds which have both $\mathcal{W}_4$ and $\mathcal{W}_5$ vanishing independently. This implies that the torsion is left only in the $\mathcal{W}_3$ sector, which means that a particular class of solutions is given by special–hermitian manifolds. In this case the solutions will have a constant dilaton profile and therefore a closed (holomorphic) $(3,0)$–form of constant norm. The only difference between this new class of solutions and the known Calabi–Yau manifolds is given by a non–trivial three–form $H$, which drives the change of the geometry and makes the manifold non–K"ahler. Examples of special–hermitian manifolds are the nilmanifolds [43] and the Moishezon manifolds [44].

More general classes of solutions have a non–constant dilaton profile, as the torsion in the $\mathcal{W}_4$ and $\mathcal{W}_5$ classes is not vanishing anylonger. However, as shown in (3.21), it is not sufficient to find a generic manifold lying in these classes, because the one–forms defining $\mathcal{W}_4$ and $\mathcal{W}_5$ have to be exact and linearly dependent. It is interesting to point out that the definition of $\mathcal{W}_4$ in terms of $\theta \equiv J \cdot (\ast d \ast J)$, as follows from (3.13), can be translated as a condition on the so–called Lee–form, which is precisely given by $\theta$ [13]. This form defines the balanced manifolds, which are those with vanishing $\theta$, and the conformally balanced manifolds, which have $\theta$ exact, i.e. $\theta = d\Lambda$. The conformally balanced manifolds are also known to admit a holomorphic $(3,0)$–form [14], and therefore could be used as proper solutions of the torsional constraints. Of course, in this class one has to select those manifolds whose $\mathcal{W}_4$ and $\mathcal{W}_5$ classes are linearly dependent through the relation (3.20).

An interesting subcase is given by the manifolds which lead to a closed three–form flux $dH = 0$, which are the so–called SKT manifolds (2.18). In this case one may solve the gauge
sector by the usual “spin in the gauge” procedure. However it has been proved [17, 16, 18, 19] that a compact, conformally balanced, SKT manifolds admits holonomy in $SU(3)$ only if it is also Kähler (and then CY). An example of a SKT manifold is the $S^3 \times S^3$ manifold, which for a proper choice of the almost–complex structure may also be shown to be complex [14]. However it does not admit a holomorphic $(3,0)$–form and therefore it is not conformally balanced.

We finish this section with some comments on the use of manifolds derived by rescalings of the metric. In section 2 we mentioned that a class of manifolds with torsion in $\mathcal{W}_4$ and $\mathcal{W}_5$ is given by conformally rescaled metrics, among which are the conformally rescaled Calabi–Yau manifolds. These latter manifolds cannot be used as solutions because the torsion has to satisfy (3.20), whereas the conformally rescaled CY metrics have to fulfill $3\mathcal{W}_4 + 2\mathcal{W}_5 = 0$.

Nevertheless, the trick of conformally rescaling the metric may yield more general solutions to the torsional constraints. Indeed, if one finds a complex manifold such that the $\mathcal{W}_4$ and $\mathcal{W}_5$ classes are defined by exact real forms, $\mathcal{W}_4 = d\Lambda, \quad \mathcal{W}_5 = d\Sigma$, (3.22)

then their linear combination is also an exact form, and in particular

$$2\mathcal{W}_4 + \mathcal{W}_5 = d(2\Lambda + \Sigma) \equiv df.$$  (3.23)

From this equation it follows that one can now satisfy the torsional constraints (3.21) upon rescaling the metric by

$$g \to e^{2f}g,$$  (3.24)

as follows. Under the rescaling (3.24) it can be shown [38] that

$$\mathcal{W}_4 \to \mathcal{W}_4 + 2df,$$
$$\mathcal{W}_5 \to \mathcal{W}_5 - 3df,$$  (3.25)

and hence

$$2\mathcal{W}_4 + \mathcal{W}_5 - df = 0 \to 2\mathcal{W}_4 + \mathcal{W}_5 = 0.$$  (3.26)

The holomorphic $(3,0)$–form associated with (3.23) is given by $\omega = e^\Sigma \Psi$. Upon rescaling of the metric, $\omega$ will rescale as $\omega \to e^{3f}\omega \equiv \hat{\omega}$. Note, however, that $\hat{\omega}$ is not anymore holomorphic. A $(3,0)$–form, which is also holomorphic after the rescaling, is instead given by $\omega$, whose norm in the rescaled metric is given by $||\omega|| = e^{\Sigma - 3f}$.

4 The heterotic string on the Iwasawa manifold

In this section we will construct an explicit example of a manifold in the class described by (3.21) which solves the supersymmetry equations (3.2). We will, for simplicity, consider special–hermitian manifolds, which are manifolds whose torsion is contained only in $\mathcal{W}_3$. These manifolds are the only complex manifolds which are also half–flat, the latter being
upliftable to $G_2$ spaces. Since for such manifolds $\mathcal{W}_4 = \mathcal{W}_5 = 0$, the dilaton stays constant and equation (3.3) is identically solved, because both terms are zero independently.

To further simplify the analysis we will consider the so-called six-dimensional nilmanifolds. These manifolds are constructed from simply-connected nilpotent Lie groups $G$ by quotinging with a discrete subgroup $\Gamma$ of $G$ for which $G/\Gamma$ is compact. There are 34 classes of such manifolds and they do not admit a Kähler metric, implying that any solution involving these manifolds will yield a non-trivial departure from the well known Calabi–Yau examples.

Out of these 34 classes, 18 admit a complex structure \[52, 43\]. An interesting example coming from a two–step algebra inside this classification is given by the Iwasawa manifold, which is going to be the one we will analyze in the following. It is interesting to point out that for any of the 18 classes one can choose a complex structure which is compatible with the metric and which has $\mathcal{W}_4 = 0$ \[11\]. Moreover, since on a nilmanifold which admits a complex structure one can choose a basis of one–forms such that $d\alpha \in \Lambda^{(2,0)}$ for any given $\alpha \in \Lambda^{(1,0)}$ \[43\], also $\mathcal{W}_5 = 0$. One can check that one can choose a complex structure such that both conditions can be satisfied simultaneously and that therefore the resulting torsion lies entirely in $\mathcal{W}_3$.

The Iwasawa manifold is a nilmanifold obtained as the compact quotient space $M = \Gamma \setminus G$, where $G$ is the complex Heisenberg group and $\Gamma$ is the subgroup of the Gaussian integers. The complex Heisenberg group is given by the set of matrices

\[
G = \left\{ \begin{pmatrix} 1 & z & u \\ 0 & 1 & v \\ 0 & 0 & 1 \end{pmatrix} : u, v, z \in \mathbb{C} \right\}
\]

under multiplication and the discrete subgroup $\Gamma$ is defined by restricting $u, v, z$ to Gaussian integers:

\[
\begin{align*}
v & \to v + m, \\
z & \to z + n, \\
u & \to u + p + nv,
\end{align*}
\]

\[m, n, p \in \mathbb{Z} \oplus i \mathbb{Z}. \quad (4.2)
\]

Left invariant one–forms on $G$ are given by

\[
\begin{align*}
\alpha &= dz, \\
\beta &= dv, \\
\gamma &= -du + z dv.
\end{align*}
\]

These are also holomorphic $(1,0)$–forms with respect to the standard complex structure as we will see later. The Lie bracket is determined by the equations

\[
\begin{align*}
d\alpha &= 0, \\
d\beta &= 0, \\
d\gamma &= \alpha \wedge \beta.
\end{align*}
\]

3Such $\Gamma$ subgroup exists if the structure equations of its Lie algebra are rational \[50, 51\].

4An algebra $g$ is $n$–step nilpotent if $g^n = 0$. 

13
Setting
$$\alpha = e^1 + i e^2, \quad \beta = e^3 + i e^4, \quad \gamma = e^5 + i e^6,$$
one can introduce a real basis for the dual of the Lie algebra corresponding to $G$, which is also the cotangent space. In this way the real form of equation (4.4) is given by
$$\begin{align*}
de^i &= 0, \quad i = 1, \ldots, 4, \\
de^5 &= e^1 \wedge e^3 + e^4 \wedge e^2, \\
de^6 &= e^1 \wedge e^4 + e^2 \wedge e^3.
\end{align*}$$
(4.6)
Thus we see that, since there is a four–dimensional kernel of the differential map, one can find a geometrical description of the manifold $M$ as a principal $T^2$–fibration over $T^2 \times T^2$.

An important and established mathematical result [52, 53] is that the set of almost complex structures on $M$ which are compatible with the Riemannian metric $ds^2 = \sum_{i=1}^6 (e^i)^2$ are isomorphic to $\mathbb{C}P^3$. In particular one can define the fundamental 2–form at an arbitrary point of the moduli space of almost complex structures as
$$J = e^5 \wedge (\cos \theta e^6 + \sin \theta f^1) - f^2 \wedge (\cos \theta f^1 - \sin \theta e^6) + f^3 \wedge f^4,$$
(4.7)
where $f^i$ are an $SO(4)$ rotated basis for $e^1, \ldots, e^4$, with the definition $f^i = P_{ij} e^j$, $P \in SO(4)$. Note that inequivalent complex structures arise only for $P \in SO(4)/U(2)$. The appropriate $(1,0)$–form basis relative to the almost complex structure (4.7) is given by
$$\begin{align*}
\alpha &= \cos \theta f^1 - \sin \theta e^6 + i f^2, \\
\beta &= f^3 + i f^4, \\
\gamma &= e^5 + i (\cos \theta e^6 + \sin \theta f^1).
\end{align*}$$
(4.8)
In terms of these $(1,0)$–forms the almost complex structure is given by
$$J = \frac{i}{2} (\alpha \wedge \bar{\alpha} + \beta \wedge \bar{\beta} + \gamma \wedge \bar{\gamma}),$$
(4.9)
and the norm one $(3,0)$–form by
$$\Psi = \alpha \wedge \beta \wedge \gamma.$$
(4.10)

If we want the Iwasawa manifold to be a solution of the torsional constraints, we must choose the parameters in (4.7) such that the torsion lies in $\mathcal{W}_3 \oplus \mathcal{W}_4 \oplus \mathcal{W}_5$, satisfying (3.21). Demanding the torsion to be in these classes, it has been shown in [52] that $\mathcal{W}_4 = 0$.

The moduli space of complex structures turns out to have two disconnected components which are given by
$$\theta = 0 \text{ and } P = I,$$
(4.11)
and
$$\theta = \pi \text{ and } P \in SO(4).$$
(4.12)
The case $\theta = 0$ and $P$ the identity gives the standard complex structure $J_0$, whereas (1.12) describes a complex projective line of inequivalent complex structures (an ‘edge’). We will show below that $\mathcal{W}_5 = 0$ on both components of the moduli space.
4.1 The standard complex structure $J_0$

Consider picking the standard complex structure $J_0 = e^1 \wedge e^2 + e^3 \wedge e^4 + e^5 \wedge e^6$. For this choice of $J$ the manifold is complex and therefore we already satisfy the first geometrical constraint. The $(1,0)$–forms are then given by

$$\alpha = e^1 + ie^2, \quad \beta = e^3 + ie^4, \quad \gamma = e^5 + ie^6.$$  \hspace{1cm} (4.13)

Complex coordinates $(z,v,u)$ can be introduced accordingly, in order to evaluate the various geometrical quantities explicitly:

$$\alpha = dz, \quad \beta = dv, \quad \gamma = -du + zdv.$$  \hspace{1cm} (4.14)

These are the natural coordinates inherited from the definition of the manifold as a coset (c.f. (4.3)). The forms $\alpha, \beta$ and $\gamma$ are holomorphic with respect to $J_0$. The differentials $dz$, $dv$ and $du$ are closed differentials, i.e. $d^2z = d^2v = d^2u = 0$. This is consistent with evaluating $d\gamma = e^1 \wedge e^3 + e^4 \wedge e^2 + i(e^1 \wedge e^4 + e^2 \wedge e^3) = dz \wedge dv$ (cf. (4.3)). The complex structure in these coordinates is given by

$$J_0 = i \frac{1}{2} [dz \wedge d\bar{z} + dv \wedge d\bar{v} + (du - zdv) \wedge (d\bar{u} - \bar{z}d\bar{v})],$$  \hspace{1cm} (4.15)

and the metric is

$$ds^2 = [dzd\bar{z} + dvd\bar{v} + (du - zdv)(d\bar{u} - \bar{z}d\bar{v})].$$  \hspace{1cm} (4.16)

This makes it evident that the Iwasawa manifold can be viewed as a $T^2$ fibration over a $T^2 \times T^2$ base.

We find the following closed forms modulo exact forms (note that the Euler number of the torus fibration is zero)

$$b_1 = 4: \quad \alpha, \beta,$$
$$b_2 = 8: \quad \alpha \gamma, \beta \gamma, \quad \alpha \bar{\alpha}, \alpha \bar{\beta}, \beta \bar{\alpha}, \beta \bar{\bar{\beta}}, \quad \bar{\alpha} \bar{\gamma}, \bar{\beta} \bar{\gamma},$$
$$b_3 = 10: \quad \alpha \beta \gamma, \alpha \gamma \bar{\alpha}, \alpha \gamma \bar{\beta}, \beta \gamma \bar{\alpha}, \beta \gamma \bar{\beta}, \quad \alpha \bar{\alpha} \bar{\gamma}, \beta \bar{\alpha} \bar{\gamma}, \alpha \bar{\beta} \bar{\gamma}, \beta \bar{\beta} \bar{\gamma}, \quad \bar{\alpha} \bar{\beta} \bar{\gamma}.$$  \hspace{1cm} (4.17)

where the wedges between forms are understood.

The norm one $(3,0)$–form is given by

$$\Psi = \alpha \wedge \beta \wedge \gamma = -dz \wedge dv \wedge du.$$  \hspace{1cm} (4.18)

It is obvious from (4.4) that this form is also closed and therefore holomorphic. Hence $\omega = \Psi$, and it follows from (3.19) that

$$\mathcal{N}_5 = 0.$$  \hspace{1cm} (4.19)

It can also be explicitly checked that

$$J_0 \wedge dJ_0 = 0.$$  \hspace{1cm} (4.20)
Therefore also
\[ \mathcal{W}_4 = 0 \] (4.21)
and hence (3.21) is satisfied.

Having checked that the conditions on the torsion imposed by supersymmetry are satisfied, we proceed to solve the supersymmetry conditions in the gauge sector. We therefore compute the three–form field \( H \), which we find to be non–vanishing and given by
\[
H \equiv \frac{i}{2} (\bar{\partial} - \partial) J_0 = -\frac{1}{4} (du - zdv) \wedge d\bar{z} \wedge d\bar{v} - \frac{1}{4} (d\bar{u} - zd\bar{v}) \wedge dz \wedge dv
\] (4.22)
which differs from (4.22) by a relative sign. It then follows from (2.13) that
\[
\mathcal{W}_3 = \frac{i}{2} \alpha \wedge \beta \wedge \bar{\gamma} . \tag{4.24}
\]

The \( H \)--field also defines the torsion of the generalized connection in the co–variant derivative \( \mathcal{D} \), see (3.10), from which the generalized Riemann two–form \( \tilde{R}^m_{\ n} \) can be constructed. The generalized connection is compatible with the metric and the complex structure is covariantly constant with respect to this connection:
\[
\mathcal{D}_m g_{np} = 0 , \quad \mathcal{D}_m J_{np} = 0 . \tag{4.25}
\]
The generalized Riemann two–form has the following non–zero components,
\[
\begin{align*}
\tilde{R}^z_v &= dv \wedge d\bar{v} , & \tilde{R}^z_u &= d\bar{v} \wedge dz , \\
\tilde{R}^v_z &= -dv \wedge d\bar{z} , & \tilde{R}^v_u &= dz \wedge d\bar{z} , \\
\tilde{R}^u_z &= -zdv \wedge d\bar{z} , & \tilde{R}^u_v &= zdv \wedge d\bar{v} + 2z dz \wedge d\bar{z} , \\
\tilde{R}^u_u &= -dv \wedge d\bar{v} - dz \wedge d\bar{z} ,
\end{align*}
\] (4.26)
as well as the complex conjugate entries. Note that there are no non–vanishing \( \tilde{R}^a_b \) two–form components which shows that the holonomy is contained in \( U(3) \). The generalized Ricci tensor is \emph{not} vanishing and is given by
\[
\tilde{R}_{z\bar{z}} = \tilde{R}_{\bar{z}z} = \tilde{R}_{v\bar{v}} = \tilde{R}_{\bar{v}v} = -1 . \tag{4.27}
\]
Furthermore, it can be explicitly checked that the generalized Ricci two–form is vanishing,
\[
\tilde{R}_{mn} \epsilon^{pq} J_{pq} = 0 , \tag{4.28}
\]

which implies a further reduction of the holonomy group of the Iwasa manifold to be contained in $SU(3)$ \cite{14, 11}. Therefore there exist two covariantly constant Killing spinors $\eta_{\pm}$ which are annihilated by the holonomy generators,

$$\tilde{R}_{mn}^{\ pq}\Gamma_{pq}\eta_{\pm} = 0 . \quad (4.29)$$

In particular, by explicit computations it can be checked that the Iwasa manifold endowed with the standard complex structure $J_0$ has $SU(2) \times U(1)$ holonomy.

In order to solve the Bianchi identity (3.4) for $H$ we need to evaluate the four–forms $i\partial\bar{\partial}J_0$ and $\text{tr}(\tilde{R} \wedge \tilde{R})$. The first one is given by

$$i\partial\bar{\partial}J_0 = \frac{1}{2}dz \wedge dv \wedge d\bar{z} \wedge d\bar{v} = \alpha \wedge \beta \wedge \bar{\alpha} \wedge \bar{\beta} , \quad (4.30)$$

and is therefore proportional to the volume of the $T^2 \times T^2$ base. The other is identically zero

$$\text{tr}(\tilde{R} \wedge \tilde{R}) = 0 . \quad (4.31)$$

This implies that a solution of the Bianchi identity will be now provided by a $(1, 1)$–form $F$ such that $Tr(F \wedge F)$ is proportional to the volume of the base space and that $F_{mn}J^{mn} = 0$. In the following we will use an Abelian field strength configuration, and not the more common $SU(n)$ configurations. Let us therefore recall the origin of the terms in the Bianchi identity. The first and second Chern class of an $U(2)$ vector bundle $V$ is defined from the following expansion of the full Chern class $c(V) = 1 + c_1(V) + c_2(V)$, where

$$c(V) = \det \left(1 + \frac{i}{2\pi}F^{ab}\right) , \quad (4.32)$$

and where $a, b$ are the group indices of the $U(2)$ bundle, i.e. they refer to the matrix indices 1, 2 of the element of $\text{Lie}U(2)$. Therefore one gets

$$c_1(V) = \frac{i}{2\pi}trF ,$$

$$c_2(V) = -\frac{1}{4\pi^2}(F^{11} \wedge F^{22} - F^{21} \wedge F^{12}) = \frac{1}{8\pi^2}(trF \wedge F - trF \wedge trF) . \quad (4.33)$$

Thus, for the actual anomaly term, we obtain

$$c_2(V) - \frac{1}{2}c_1^2(V) = \frac{1}{8\pi^2}trF \wedge F . \quad (4.34)$$

Note that in our case the left hand side of the Bianchi identity is proportional to $\alpha \wedge \bar{\alpha} \wedge \beta \wedge \bar{\beta}$ (cf. (4.30)). This entails two things. First, since on the one hand this form denotes the volume form of the base $T^2 \times T^2$, whereas on the other hand it is nothing but $dH$ and hence it is also exact, there appears to be an immediate contradiction. This apparent contradiction is, however, avoided by noticing that no section exists for a nontrivial $T^2$ fibration over the $T^2 \times T^2$ base, i.e. the base does not lie embedded in the total space. Secondly, it is precisely
this combination of four leg directions in $dH$ which makes it possible to solve $dH \sim tr F \wedge F$ with just an Abelian gauge field configuration. This is so because in the Abelian case the field strength two–form $F$ is closed, and $H^{1,1}(M)$ is precisely generated by $\alpha \wedge \bar{\alpha}, \alpha \wedge \beta, \beta \wedge \bar{\alpha}, \beta \wedge \beta$ (cf. (4.17)).

Specifically, let us pick a $U(1)$ subgroup of $U(2)$ in the following way,

$$F = \frac{1}{2} \begin{pmatrix} \mathcal{F} & 0 \\ 0 & \mathcal{F} \end{pmatrix}. \quad (4.35)$$

The $(1,1)$–form $\mathcal{F}$ can be determined as follows. Setting

$$\mathcal{F} = \mathcal{F}_{zz} \, dz \wedge d\bar{z} + \mathcal{F}_{v\bar{v}} \, dv \wedge d\bar{v} + \mathcal{F}_{z\bar{v}} \, dz \wedge d\bar{v} + \mathcal{F}_{vz} \, dv \wedge d\bar{z} \quad (4.36)$$

and requiring $\mathcal{F}$ to be real yields

$$\mathcal{F}_{zz} = -\mathcal{F}_{z\bar{z}}^*, \quad \mathcal{F}_{v\bar{v}} = -\mathcal{F}_{v\bar{v}}^*, \quad \mathcal{F}_{z\bar{v}} = -\mathcal{F}_{vz}^*. \quad (4.37)$$

Demanding that $\mathcal{F}_{mn}J^{\cdot mn} = 0$ yields $\mathcal{F}_{zz} = -\mathcal{F}_{v\bar{v}}$ in addition. It follows that

$$tr F \wedge F = \mathcal{F} \wedge \mathcal{F} = -2 (f^2 + |\mathcal{F}_{z\bar{v}}|^2) \, dz \wedge dv \wedge d\bar{z} \wedge d\bar{v}, \quad (4.38)$$

where we set $\mathcal{F}_{z\bar{z}} \equiv if$ with $f$ real. Setting $tr F \wedge F = -dH = -\frac{1}{2} dz \wedge dv \wedge d\bar{z} \wedge d\bar{v}$ yields

$$|\mathcal{F}_{z\bar{v}}|^2 = \frac{1}{4} - f^2. \quad (4.39)$$

Therefore

$$\mathcal{F}_{z\bar{v}} = e^{i\alpha} \sqrt{\frac{1}{4} - f^2}, \quad (4.40)$$

and hence

$$\mathcal{F} = if \, dz \wedge d\bar{z} - if \, dv \wedge d\bar{v} + e^{i\alpha} \sqrt{\frac{1}{4} - f^2} \, dz \wedge d\bar{v} - e^{-i\alpha} \sqrt{\frac{1}{4} - f^2} \, dv \wedge d\bar{z}. \quad (4.41)$$

Next, we evaluate the Bianchi identity $d\mathcal{F} = 0$. Equating the coefficients of the various $(2,1)$–forms yields

$$i\partial_v f + \partial_z \left( e^{-i\alpha} \sqrt{\frac{1}{4} - f^2} \right) = 0, \quad (4.42)$$

$$i\partial_z f + \partial_v \left( e^{i\alpha} \sqrt{\frac{1}{4} - f^2} \right) = 0. \quad (4.42)$$

One solution to (4.42) is given by

$$\mathcal{F}_{z\bar{v}} = 0, \quad f = \pm \frac{1}{2}. \quad (4.43)$$

Another solution is given by

$$\mathcal{F}_{z\bar{v}} = \frac{e^{i\alpha}}{2}, \quad f = 0, \quad \alpha = \text{constant}. \quad (4.44)$$
4.2 The edge

On the edge (4.12) the complex structure is given by \[ J = - f^1 \wedge f^2 + f^3 \wedge f^4 - e^5 \wedge e^6, \] (4.45)
and the associated (1, 0)–forms are
\[ \kappa = f^1 - if^2, \quad \lambda = f^3 + if^4, \quad \mu = e^5 - ie^6. \] (4.46)
These forms are related to the forms \( \alpha, \beta \) and \( \gamma \) introduced in (4.5) by
\[ \kappa = P \bar{\alpha}, \quad \lambda = P \beta, \quad \mu = \bar{\gamma}, \] (4.47)
where \( P \in SO(4) \). The norm one (3, 0)–form \( \Psi \) reads
\[ \Psi = \kappa \wedge \lambda \wedge \mu. \] (4.48)

On the edge \( W_4 = 0 \) [52]. We now show that also \( W_5 = 0 \), which implies that \( \Psi \) is holomorphic. Since \( P \) is linear, it follows that \( d\Psi = P(\bar{\alpha} \wedge \beta) \wedge (\bar{\alpha} \wedge \bar{\beta}) \). Following [52], we decompose the two–forms into self– and anti–selfdual parts with respect to the four–dimensional Hodge star operator \( *_4 \), i.e. \( \Lambda^2 D = \Lambda^2_+ D \oplus \Lambda^2_- D \) (where \( D = \oplus_{i=1}^4 e_i \mathbb{R} \)). Now \( \alpha \wedge \beta \) is in \( \Lambda^2_- D \) and \( \bar{\alpha} \wedge \bar{\beta} \) is in \( \Lambda^2_+ D \). Since \( P = P_+ + P_- \) respects the decomposition, and wedging a form in \( \Lambda^2_+ D \) with a form in \( \Lambda^2_- D \) gives zero, \( d\Psi = 0 \).

Computing \( dJ \) and \( H \) gives
\[ dJ = \frac{i}{2} \left( \alpha \wedge \beta \wedge \bar{\gamma} - \gamma \wedge \bar{\alpha} \wedge \bar{\beta} \right), \]
\[ H = \frac{1}{4} \left( \alpha \wedge \beta \wedge \bar{\gamma} + \gamma \wedge \bar{\alpha} \wedge \bar{\beta} \right). \] (4.49)
Therefore we see that on the edge, the values for \( dJ \) and for \( H \) precisely equal the values obtained from \( J_0 \). This implies that the rest of the analysis given in the previous subsection goes through also for the edge. The holonomy with respect to the generalized connection is again \( SU(2) \times U(1) \), and the Bianchi identity (3.4) can again be solved by an Abelian field strength.

Thus we have established that for any point in the moduli space of complex structures the Iwasawa manifold is a consistent \( \mathcal{N} = 1 \) supersymmetric heterotic background.

5 Outlook

Let us briefly describe a few directions for future research. First of all, it would be important to construct examples of complex, conformally balanced six–dimensional spaces satisfying \( 2W_4 + W_5 = 0 \). Finding such examples would enlarge the list of non–Kähler manifolds representing consistent backgrounds for the heterotic string provided that the conditions in
the gauge sector can also be met. Second, it would be instructive to directly obtain the super-symmetry conditions (3.3)–(3.7) from the ten–dimensional heterotic action by rewriting the latter in terms of squares of expressions which vanish for supersymmetric compactifications. This would also provide a check of the equations of motion, which we haven’t performed in this paper. It would be equally interesting to derive the $\mathcal{N} = 1$ supersymmetry conditions (3.3)–(3.7) from the minimization of a four–dimensional superpotential $W$, i.e. $W = \partial_\Phi W = 0$, where $\Phi$ denotes one of the background fields.

The Iwasawa manifold which, as we showed, provides a consistent background for the heterotic string, is an example of a half–flat manifold and can therefore be lifted to a $G_2$ manifold by using the flow equations (2.30) of Hitchin. It would be interesting to make this explicit.

In [54] it was shown that in the presence of both an $H$–field and a suitably tuned gaugino condensate it is possible to obtain supersymmetric backgrounds where the four–dimensional spacetime part is an $AdS_4$ space. It would be very interesting to redo the analysis of [1] for the case that both an $H$–field and a gaugino condensate are turned on and to reformulate the resulting geometrical conditions on the internal manifolds in terms of the mathematical classification of torsion described in section 2.

Another direction would be to discuss type II and M–theory compactifications with fluxes in the mathematical framework of section 2 (cf. appendix A).

Finally let us comment on the possibility of M–theory compactifications on $G_2$ manifolds with boundaries and their relation to six–dimensional heterotic string backgrounds (cf. [58]). As shown in [59] an M–theory background allows for a ten–dimensional heterotic string interpretation if it possesses two boundaries, namely the two ten–dimensional end of the world planes on which the heterotic gauge degrees of freedom are located. Hence, a seven–dimensional $G_2$ manifold with two boundaries is expected to be associated with a particular $\mathcal{N} = 1$ (non–perturbative) heterotic string background. As a candidate for this scenario consider a $G_2$ manifold constructed as a warped product $X_7 = M \times I$, where $I \subset \mathbb{R}$ is a compact interval. According to the Hořava–Witten picture, the radial $r$–direction would then correspond to the eleventh M–theory dimension. The manifold $M$ would be the geometric heterotic background, and the torsion could be due to a non–perturbative gaugino condensate. Whether the associated Bianchi identity can be solved and whether such a picture really survives remains to be seen. One may also study its possible relation to heterotic coset space compactifications [60, 61, 62].

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$^5$M–theory compactifications on orbifolds to four dimensions and their heterotic string interpretation were discussed in [54, 56, 57].
Note added
As we were about to publish the paper, we received [78] which has some overlap with the issues discussed in appendix A.

A  M–theory compactifications

In this appendix we will give some examples of the occurrence of the five classes of torsion in string theory and in M–theory compactifications.

Some interesting subclasses of six–dimensional manifolds with $SU(3)$ structure arise in an indirect way as bases of seven–dimensional manifolds with $G_2$ holonomy (see also e.g. the discussion in [63]). A $G_2$ structure on a seven–dimensional manifold $X_7$ is determined by a non–vanishing 3–form $\phi$ on $X_7$ given by

$$\phi = e^1 \wedge e^3 \wedge e^5 + e^1 \wedge e^2 \wedge e^7 + e^3 \wedge e^4 \wedge e^7 + e^5 \wedge e^6 \wedge e^7 - e^1 \wedge e^4 \wedge e^6 - e^2 \wedge e^3 \wedge e^6 - e^2 \wedge e^4 \wedge e^5. \quad (A.1)$$

It is well known that a necessary and sufficient condition for $G_2$ holonomy is that $\phi$ is closed as well as co–closed:

$$d\phi = 0, \quad d \star \phi = 0. \quad (A.2)$$

This condition is equivalent to that $X_7$ admits a covariantly constant Killing spinor $\eta$:

$$D_i \eta = 0, \quad i = 1, \ldots, 7. \quad (A.3)$$

Integrability of this equation leads to

$$R_{a(ij} \gamma^{a} \eta = 0.. \quad (A.4)$$

Since the torsion–free curvature satisfies $R_{a[ibcd]} = 0$ one can conclude (by multiplying eq. (A.4) by $\gamma^j$) that

$$R_{ij} \gamma^j \eta = 0, \quad (A.5)$$

which implies that $R_{ij} = 0$, i.e. $X_7$ is Ricci–flat.

In the following, we will discuss two types of $G_2$ structures, namely dynamical and fibered $G_2$ structures [38].

A.1  Manifolds with dynamical $G_2$ structures

As mentioned in section 2, a six–dimensional half–flat manifold $M$ can be lifted as a warped product $X_7 = M \times I$ to a seven–dimensional manifold with $G_2$ holonomy. We now specialize to the case where $I$ is the half–interval $I = \mathbb{R}^+$, and the metric $g$ of $X_7$ is the conical metric $g = r^2 \hat{g} + dr^2$. The physical relevance of this metric comes from the fact that chiral fermions in M–theory on a $G_2$ manifold are located at those points where $X_7$ exhibits a conical, codimension 7 singularity. In brane language, these are precisely the points where D6–branes intersect each other in a supersymmetric way.
The uplift of the six-dimensional (hatted) $SU(3)$–structure to seven dimensions follows from the form of the conical metric and is given by

\[ J = r^2 \hat{J}, \quad \psi_+ = r^3 \hat{\psi}_+, \quad \psi_- = r^3 \hat{\psi}_-. \]

(A.6)

Following section 2, the $G_2$ structure is defined by $\phi = J \wedge dr + \psi_+$. The Hitchin flow equations (2.30) reduce to

\[ \hat{d} \hat{\psi}_- = -2 \hat{J} \wedge \hat{J}, \quad \hat{d} \hat{J} = 3 \hat{\psi}_+. \]

(A.7)

This implies a further restriction of the allowed torsion to

\[ \tau \in W_1^- . \]

(A.8)

Therefore $M$ is nearly–Kähler (cf. (2.20)). $M$ is also referred to as having weak $SU(3)$ holonomy.

Examples of nearly–Kähler manifolds are the homogeneous spaces $M = Sp(4)/(SU(2) \times U(1))$, $M = SU(3)/(U(1) \times U(1))$ and $M = SU(2)^2/SU(2)$. The metrics of the three associated $G_2$ manifolds are well known. For $M = SU(2)^3/SU(2)$ the seven–space $X_7$ is a $\mathbb{R}^4$ bundle over $S^3$, whereas for $M = Sp(4)/(SU(2) \times U(1))$ or $M = SU(3)/U(1)^2$ the corresponding non–compact $G_2$ manifolds are a $\mathbb{R}^3$ bundle over a quaternionic space $Q$, with $Q = S^4$ or $Q = \mathbb{C}P^2$ respectively. Note that there also exist more general quaternionic spaces with less isometries which can be used to construct $G_2$ manifolds. In these cases $M$ will not be anymore a homogeneous coset space, but nevertheless one can still construct its weak $SU(3)$ structure [64].

For the homogeneous coset spaces the relevant torsion tensor can be explicitly constructed. Namely $\tau$ and also $\psi_-$ are directly proportional to the structure constants $f_{abc}$ of the coset space. These structure constants appear in the connection which possesses the $SU(3)$ structure discussed above. This connection is non–Ricci flat. However, there also exists another connection, with torsion again given by the structure constants $f_{abc}$, which is Ricci flat [61]. A priori it is not clear whether the manifolds with weak $SU(3)$ holonomy like the coset spaces listed above can be used as geometric background spaces in string theory. These are certainly not to be used as type IIA backgrounds, since they do not follow from a circle reduction.

### A.2 Type IIA compactifications: manifolds with fibered $G_2$ structures

The well–known case of Calabi–Yau compactifications of type IIA leads to an effective four–dimensional $\mathcal{N} = 2$ supergravity theory. However one can turn on additional background fields in order to achieve a partial supersymmetry breaking from $\mathcal{N} = 2$ to $\mathcal{N} = 1$ supersymmetry. One possibility is given by turning on appropriate Ramond 2–form H–fluxes through 2–cycles of the Calabi–Yau space. Another way is given by wrapping D6–branes around supersymmetric 3–cycles of the Calabi–Yau space. Examples are given by the resolved conifold
with \( N \) units of 2–form \( \mathrm{H} \)–flux through the \( S^2 \), resp. the deformed conifold with \( N \) wrapped D6–branes around the \( S^3 \) [32]. The resulting effective field theory possesses precisely \( \mathcal{N} = 1 \) supersymmetry. Both cases have in common that the internal background fields are not given only by the 6–dimensional metric but also by some additional ‘matter’ fields, the Ramond 1–form potential and the dilaton field. In fact, turning on these additional background fields has a strong backreaction on the internal geometry, such that the modified metric of the six–dimensional space \( M \) does not belong any more to a Calabi–Yau manifold, but instead to a non–Ricci flat space with a certain \( SU(3) \) structure.

Let us consider the case where the additional type IIA matter background fields become purely geometric when uplifting type IIA to M–theory. Asking for \( \mathcal{N} = 1 \) supersymmetry in four space–time dimensions, the 7–dimensional M–theory background space \( X_7 \) must have \( G_2 \) holonomy. E.g., for the 6–dimensional conifold with additional 2–form flux or with wrapped D6–branes this leads to a particular non–compact \( G_2 \) manifold, which topologically is an \( R^4 \) bundle over \( S^3 \) [66, 67, 68, 69]. For the IIA reduction one identifies a particular compact circle \( S^1 \) inside \( X_7 \). This circle is usually non–trivially fibered over a six–dimensional base \( M \) which then serves as the non–Ricci flat geometric background of the corresponding IIA superstring theory. In M–theory language the D6–branes correspond to co–dimension 4 A-D-E singularities of \( X_7 \). Moreover the brane intersection points, where the chiral fermions are localized, are given in terms of co–dimension 7 singularities of \( X_7 \).

In order to determine the \( SU(3) \) structure of the type IIA background \( M \), we consider the circle fibration [38]

\[
\pi : \ X_7 \to M , \quad (A.9)
\]

endowed with a metric of the form

\[
g = \alpha \otimes \alpha + \pi^* \hat{g} , \quad (A.10)
\]

where \( \hat{g} \) is the metric of \( M \), and \( d\alpha = \pi^* \rho \) with \( \rho \) some 2–form on \( M \). \( \rho \) corresponds to the Ramond 2–form field strength background and hence \( \alpha \) is the Ramond 1–form gauge potential. A \( G_2 \) structure of \( X_7 \) can be introduced as

\[
\phi = J \wedge \alpha + \psi_+ \quad (A.11)
\]

and it satisfies the following two relations (and omitting the pullback operator):

\[
\begin{align*}
\text{d} \phi &= \text{d} J \wedge \alpha + \text{d} \psi_+ + J \wedge \rho , \\
\text{d} \star \phi &= \text{d} \psi_- \wedge \alpha + J \wedge \text{d} J - \psi_- \wedge \rho .
\end{align*}
\quad (A.12)
\]

Now, requiring that \( X_7 \) is Ricci flat and \( g \) has \( G_2 \) holonomy implies that \( \text{d} \phi = \text{d} \star \phi = 0 \). This may be achieved by demanding that the almost complex structure is closed,

\[
\text{d} J = 0 , \quad (A.13)
\]

and that the \((3,0)\)–form \( \Psi \) satisfies

\[
\begin{align*}
\text{d} \psi_- &= 0 , \\
\text{d} \psi_+ &= - J \wedge \rho .
\end{align*}
\quad (A.14)
\]

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Moreover the remaining condition \( \rho \wedge \psi_- = 0 \) implies that

\[
\rho \in \Lambda^{(1,1)}_0,
\]

which means that \( \rho \) is a \((1,1)\)-form which cannot contain any piece proportional to \( J \). Therefore the torsion is in

\[
\tau \in \mathcal{W}^{-}_{2}.
\]

So the type IIA space \( M \) is an almost-Kähler manifold (cf. (2.21)).

Note that for case of Ramond 2–form flux through the 2–cycle of the resolved conifold, the non–closure of the 3–form \( \psi_+ \) of \( M \) corresponds to a non–vanishing NS 4–form flux as discussed in [63] (see also the remarks in section 5 of [70]). So turning on the non–geometrical Ramond 2–form flux induces as a backreaction on the six–dimensional internal space a geometrical 4–form flux, which is indeed needed to preserve \( \mathcal{N} = 1 \) supersymmetry. In the effective \( \mathcal{N} = 1 \) superpotential the non–geometrical 2–form flux as well as the geometrical 4–form flux are present, as they balance each other in the supersymmetric ground state of the scalar potential. In the type IIB mirror description both fluxes are non–geometrical, they correspond to a Ramond plus NS 3–form flux on the mirror geometry, which is still a Calabi–Yau space in type IIB. In the type IIB superpotential the contribution of the Ramond 3–form flux is balanced by the NS 3–form flux in order to get a supersymmetric groundstate. Therefore this means that type IIA on the non–Calabi Yau space \( M \) plus 2–form and 4–form background fluxes is mirror to IIB on a Calabi–Yau with appropriate 3–form H–fluxes.

### B Supersymmetric compactifications of the heterotic theory

In this appendix we review the derivation of the conditions leading to \( \mathcal{N} = 1 \) spacetime supersymmetry when compactifying heterotic string theory on a six–dimensional manifold [38, 4].

The supersymmetry transformations of the fermionic fields in the heterotic theory are given by [71, 72, 73]:

\[
\delta \psi_M = \nabla_M \epsilon + \frac{\sqrt{2}}{32} \varphi^{-3/4} \left( \Gamma_M \Gamma^{NPQ} - 12 \delta_M^N \Gamma^{PQ} \right) \epsilon H_{NPQ} + \frac{1}{256} \left( \Gamma_M \Gamma^{NPQ} - 8 \delta_M^N \Gamma^{PQ} \right) \epsilon \bar{\chi} \Gamma_{NPQ} \chi + \ldots ,
\]

\[
\delta \chi = - \frac{1}{4} \varphi^{-3/8} \Gamma^{MN} \epsilon F_{MN} + \ldots ,
\]

\[
\delta \lambda = - \frac{3\sqrt{2}}{8} \varphi^{-1} \nabla \varphi \epsilon + \frac{1}{8} \varphi^{-3/4} \Gamma^{MNP} \epsilon H_{MNP} + \frac{\sqrt{2}}{384} (\bar{\chi} \Gamma_{NPQ} \chi) \Gamma^{NPQ} \epsilon + \ldots .
\]

where the dots stand for other Fermi terms. The capital indices \( M, N \), label the ten–dimensional coordinates and \( \psi_M \) is the gravitino, \( \chi \) the gluino, \( \lambda \) the dilatino, \( \phi \) the dilaton,
$F$ a Yang–Mills field–strength and $H$ the three–form field that satisfies the Bianchi identity

$$dH = \text{tr} \tilde{R} \wedge \tilde{R} - \text{tr} F \wedge F,$$

where $\tilde{R}$ denotes the Riemann tensor computed with respect to a generalized connection that includes torsion terms. We have reported here only the relevant terms for both compactifications with $H$ fluxes and/or gaugino condensates $\langle \chi \Gamma^{MNP} \chi \rangle$.

We are interested here in compactifications where the metric reduces to:

$$ds^2 = g_{0}^{MN} dx^M \otimes dx^N = e^{2\Delta(y)} (dx^\mu \otimes dx^\nu \hat{g}_{\mu\nu}(x) + dy^m \otimes dy^n \hat{g}_{mn}(y)),$$

where $\Delta$ is the so–called warp factor, depending only on the internal coordinates, while the four–dimensional metric $\hat{g}_{\mu\nu}$ is that of a maximally symmetric spacetime. We will also require the dilaton to depend only on the internal coordinates $\phi = \phi(y)$, the $F$ and $H$ fields and the gaugino condensate to acquire expectation values only on the internal manifold. We also note that the Killing spinor $\epsilon$ will split into a four–dimensional part $\varepsilon(x)$ and a six–dimensional one $\eta(y)$.

In the following we will not consider gaugino condensates. A much simpler formulation of the supersymmetry equations (B.1) can then be obtained by rescaling them by appropriate powers of the dilaton field, such as to make them dilaton free. The transformations of the relevant fields are given by

$$\varphi = e^{-8/3\phi}, \quad g_{MN} = e^{-2\phi} g_{0}^{MN},$$

$$\lambda = \frac{1}{\sqrt{2}} e^{\phi/2} \lambda^0,$$

$$\epsilon = e^{-\phi/2} \epsilon^0,$$

$$\chi = e^{\phi/2} \chi^0,$$

$$H_{MNP} = \frac{3}{\sqrt{2}} H_{0}^{MNP},$$

$$F_{MN} = F_{0}^{MN},$$

where the quantities with a superscript 0 refer to the ones in (B.1). After these rescalings, the equations (B.1) can be rewritten as

$$\delta \psi_M = \nabla_M \epsilon - \frac{1}{4} H_M \epsilon,$$  \hspace{1cm} (B.5a)

$$\delta \chi = -\frac{1}{4} \Gamma^{MN} \epsilon F_{MN},$$  \hspace{1cm} (B.5b)

$$\delta \lambda = \nabla \phi + \frac{1}{24} H \epsilon,$$  \hspace{1cm} (B.5c)

where we further defined $H \equiv \Gamma^{MNP} H_{MNP}$, $H_M \equiv H_{MNP} \Gamma^{NP}$ and the covariant derivative $\nabla$ is constructed from the rescaled metric.

In the setup described above, once the variations of the Fermi fields are set to (B.5), the only way left to preserve some supersymmetry using a maximally symmetric four–dimensional space is that such space has vanishing cosmological constant [1] (in the case of both a gaugino condensate and an $H$–form flux, on the other hand, it is possible to have
a non–vanishing cosmological constant \[54\]). A short proof for this is given by the analysis of the integrability of the gravitino transformation rule on the four–dimensional part

\[ \delta \psi_\mu = \nabla_\mu \varepsilon = 0 . \]  

(B.6)

As said above, the covariant derivative is built in terms of the rescaled metric. The same formula in terms of the \( \hat{g}_{\mu\nu} \) connection reads (here the hat denotes quantities computed from \( \hat{g}_{\mu\nu} \))

\[ \hat{\nabla}_\mu \varepsilon + \frac{1}{2} \Gamma_\mu \Gamma^m \partial_m \log (\Delta - \phi) \varepsilon = 0 , \]  

(B.7)

whose consequences, for a compact six–dimensional manifold, are a condition relating the dilaton profile to the warp factor

\[ \Delta(y) = \phi(y) + \text{constant} , \]  

as well as the flatness of the spacetime

\[ \hat{R} = 0 . \]  

(B.9)

The space has been therefore simplified to the warped product of a Minkowski four–dimensional spacetime and an internal six–dimensional one, which we are now going to determine.

The same equation (B.5a) on the internal directions tells us that there must be a six–dimensional Killing spinor which is covariantly constant with respect to the covariant derivative

\[ D_m \equiv \partial_m + \frac{1}{4} (\omega_{m}^{np} - H_{m}^{np}) \Gamma_{np} . \]  

(B.10)

This shows that we have introduced a torsion term in the connection proportional to the three–form flux. If the holonomy of the space reduces to \( SU(3) \), there are two Weyl spinors which are covariantly constant with respect to this generalized connection

\[ D \eta_\pm = 0 , \]  

(B.11)

one with positive \( \eta_+ \) and one with negative chirality \( \eta_- \). They can be chosen to be normalized to one

\[ \eta_+^\dagger \eta_+ = 1 . \]  

(B.12)

Two important objects can be constructed from these spinors \[\text{I}]: the almost complex structure \( J \) and a \((3,0)\)–form \( \omega \) which will be shown to be holomorphic. It can be proved that the manifold is complex by defining

\[ J_m^n \equiv i \eta_+^\dagger \Gamma_m^n \eta_+ , \]  

(B.13)

which, of course, satisfies

\[ J_m^n J_n^p = - \delta_m^p , \]  

(B.14)

as can be seen by using the standard Fierz identity, plus some gamma algebra. This almost complex structure is covariantly constant with respect to the generalized connection

\[ D_m J_n^p = \nabla_m J_n^p - H_{sm}^p J_s^m - H_{mn}^s J_s^p = 0 . \]  

(B.15)
The fact that $J$ is really a complex structure follows from the analysis of the Nijenhuis tensor \( N_{mn}^p \equiv J_m^q J_{[n, q]}^p - J_n^q J_{[m, q]}^p \). Using the covariant constancy of $J$, it can be recast in the form

\[
N_{mnp} = H_{mnp} - 3 J_{[m}^q J_{n]}^r H_{pq}^r ,
\]

which can be proven to be zero using the equation for the dilatino (B.5c) and again some gamma algebra. This allows then to introduce complex coordinates and hence holomorphic $a, b, c$ and antiholomorphic $\bar{a}, \bar{b}, \bar{c}$ indices. It also follows from (B.13) by some more gamma algebra that the metric is hermitian

\[
g_{mn} = J_m^p J_n^q g_{pq} .
\]

Moreover, we can associate to $J$ the following two–form

\[
J = \frac{1}{2} J_m^p g_{pm} \, dx^n \wedge dx^m = i g_{ab} dz^a \wedge d\bar{z}^b ,
\]

which satisfies the duality relation

\[
\star J = \frac{1}{2} J \wedge J .
\]

This can be verified using once more the Fierz identity, some gamma algebra and the normalization of the Killing spinor. The definition for the Hodge dual used is

\[
\star \Phi_{(p)} = \frac{1}{p!(6-p)!} dx^{m_1} \wedge \ldots \wedge dx^{m_{6-p}} \epsilon_m \epsilon_{m_1} \ldots \epsilon_{m_{6-p}} \epsilon_{m_{6-p+1}} \ldots \epsilon_{m_6} \Phi_{m_{6-p+1} \ldots m_6}.
\]

Using the vanishing of the Nijenhuis tensor in (B.16), the covariant constancy of $J$ (B.15) and the properties of the complex structure, it can be shown that the three–form $H$ is expressed in terms of the complex structure by

\[
H = \frac{i}{2} (\bar{\partial} - \partial) J ,
\]

where the standard decomposition of the exterior derivative in holomorphic $\partial$ and antiholomorphic $\bar{\partial}$ components has been used. The Bianchi identity for $H$ follows then in terms of $J$ as

\[
dH = i \partial \bar{\partial} J = \text{tr} \bar{R} \wedge \bar{R} - \text{tr} F \wedge F .
\]

Not all the information of the supersymmetry equations (B.5) is contained in the previous relations. The dilatino equation can be recast into a relation between the dilaton and the complex structure

\[
8 i (\bar{\partial} - \partial) \phi = dx^n \nabla_p J_n^p = \star d \star J .
\]

A holomorphic (3,0)–form can be defined as follows,

\[
\omega \equiv e^{8 \phi} \eta^a \Gamma_{abc} \eta_{-} \, dz^a \wedge dz^b \wedge dz^c .
\]
The fact that this form is holomorphic follows from (B.5a), (B.23) and some more algebra. The key feature is that its norm is determined by the dilaton, up to a constant factor,

\[ ||\omega|| = \sqrt{\omega_{abc} \omega^{ab} g^{\bar{d \bar{e}}} g^{\bar{e} \bar{c}}} = e^{8(\phi - \phi_0)}. \]  

(B.25)

As a consequence, the relation (B.23) between the dilaton and the complex structure becomes a purely geometrical relation between the complex structure and the holomorphic (3,0)–form:

\[ i (\bar{\partial} - \partial) \log ||\omega|| = \star d \star J. \]  

(B.26)

The remaining gaugino equation (B.5b) becomes just a constraint for the two–form field–strength. In particular, from \( \eta^\dagger \Gamma_{mn} \delta \chi = 0 \), one obtains that \( F \) must be a (1, 1)–form further subjected to

\[ F_{mn} J^{mn} = 0, \]  

(B.27)

which is a consequence of the contraction of (B.5b) with \( \eta^\dagger \).

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