TWO-LOOP CHIRAL PERTURBATION THEORY
AND THE PION-PION PHASE SHIFTS
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Abstract

We want to test the predictive power of Chiral Perturbation Theory (ChPT). In this work, we use the ChPT pion-pion scattering amplitude, including two-loop contributions, and we obtain S- and P-wave low-energy phase shifts. We show that, by varying just one free parameter, the resulting S- and P-wave phase shifts are in reasonable agreement with the experimental data.

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1 Introduction

The method of Chiral Perturbation Theory (ChPT) [1] aims to explore the low energy structure of Quantum Chromodynamics (QCD). Considering a simultaneous expansion of the generating functional in terms of the quark masses and momenta, this method gives the low energy expansion of QCD Green’s functions and in particular meson-meson scattering amplitudes.

It is known that tree level ChPT calculations are equivalent to the current algebra low-energy theorems and in particular that it reproduces the pion-pion Weinberg amplitude [2]. On the other hand, loop diagrams in ChPT give quite large corrections to leading current algebra results even at threshold.

The other method to describe low energy meson-meson scattering was invented in the early sixties - the hard-meson current algebra technique. Even ignoring the underlying theory, the chiral current algebra implies a set of Ward identities and this method consists in solving the system of Ward identities under suitable assumptions [3].

In this context we have introduced unitarity corrections to current algebra soft-meson result and we call this approach unitarization program of current algebra (UPCA). The application of UPCA allows one to go beyond threshold of meson-meson scattering and to access even the resonance region.

As ChPT and UPCA follow from chiral symmetric Ward identities we were interested in comparing their results. We have compared these two methods for the one-loop ChPT pion-pion and kaon-pion amplitudes [4] and for the two-loop ChPT pion-pion amplitude [5]. From this comparison we have concluded that UPCA quasi-unitarized amplitudes, published long time ago [6], have the same analytical structure as the corresponding ChPT ones: They have the same dependence on $s$, $t$ and $u$, they are crossing symmetric and they respect approximate unitarity for partial waves. The expressions for the amplitude differ only in their energy polynomial parts because the free parameters have different origins: In the UPCA approach they are related to subtraction constants, inherent to the dispersive technique, and in ChPT they are coupling constants of the Lagrangian.
From the very beginning of the application of ChPT to describe meson-meson scattering it was clear that one could not relate all its free parameters to the QCD scale and to the quark masses. Like in UPCA, some have to be obtained phenomenologically. Certain constraints like the experimental value of the D-wave pion-pion scattering length and the electromagnetic charge radius of the pion were used to fix these parameters, leaving however still an uncertainty range of 40\% to 60\%. We have been interested in the possibility of fixing these free parameters by fitting low-energy meson-meson phase shifts. Analyzing phase shifts based on ChPT amplitudes calculated at next-to-leading order, we have fixed $\bar{\ell}_1$ and $\bar{\ell}_2$ by fitting pion-pion experimental data and we have fixed $L_1^r, L_2^r$ and $L_5^r$ by fitting kaon-pion experimental data [8].

Recently the complete $O(p^6)$ ChPT pion-pion amplitude was determined [8]. The final expression contains six linear combinations of ten free parameters. Therefore the question concerning the predictive power of such a result arises. One possibility is to consider some estimates for relevant low-energy constants of $O(p^6)$ from another effective theory [9], to keep $O(p^4)$ constants in the same range of values previously used and to assume that ChPT is restricted to very low energies. The results in Ref. [8] for two different scales are consistent with data but the predictive power of the theory remains questionable. Here we present a different proposal. We would like to test the predictive power of ChPT by using the smallest possible number of parameters. Thereby, we deal only with analytical expressions for the partial waves and we try to describe pion-pion scattering by varying just one parameter. As a result, we obtain S- and P-wave phase shifts that are in reasonable agreement with experimental data. We interpret the ability of ChPT to describe these phase shifts, once given the $\rho$ mass, as an indication that its predictive power does not depend on the large number of free parameters. In the next section we present the ChPT pion-pion amplitude, the phase-shift definition, numerical results and an analysis of the near-threshold behavior. In the conclusions we include a short discussion concerning the D-wave amplitude.

2 ChPT amplitudes and numerical results

The amplitude for elastic pion scattering obtained from ChPT Lagrangian [8], including two loop contributions with $m_\pi = 1$ and $F_\pi = 93.2/140$ is:

$$A^{(1)}(s, t, u) = \frac{(s - 1)/F_\pi^2 + \left[ b_1 + b_2 s + b_3 s^2 + b_4 (t - u)^2 \right] / F_\pi^4 + \left[ F^{(1)}(s) + G^{(1)}(s, t) + G^{(1)}(s, u) \right] / F_\pi^4 + \left[ b_5 s^3 + b_6 s (t - u)^2 \right] / F_\pi^6 + \left[ F^{(2)}(s) + G^{(2)}(s, t) + G^{(2)}(s, u) \right] / F_\pi^6}{(s - 1)/F_\pi^2 + \left[ b_1 + b_2 s + b_3 s^2 + b_4 (t - u)^2 \right] / F_\pi^4 + \left[ F^{(1)}(s) + G^{(1)}(s, t) + G^{(1)}(s, u) \right] / F_\pi^4 + \left[ b_5 s^3 + b_6 s (t - u)^2 \right] / F_\pi^6 + \left[ F^{(2)}(s) + G^{(2)}(s, t) + G^{(2)}(s, u) \right] / F_\pi^6}$$

(1)

where:

$$F^{(1)}(s) = \frac{1}{2} J(s) \left( s^2 - 1 \right),$$

$$F^{(2)}(s) = J(s) \left\{ \frac{1}{16\pi^2} \left[ \frac{503}{108} s^3 - \frac{929}{54} s^2 + \frac{887}{27} s - \frac{140}{9} \right] + b_1 (4s - 3) + b_2 (s^2 + 4s - 4) + \frac{1}{3} b_3 (8s^3 - 21s^2 + 48s - 32) + \frac{1}{3} b_4 (16s^3 - 71s^2 + 112s - 48) \right\}$$

$$+ \frac{1}{18} K_1(s) \left[ 20s^3 - 19s^2 + 210s - 135 - \frac{9}{16} \pi^2 (s - 4) \right] + \frac{1}{32} K_2(s) (s\pi^2 - 24) + \frac{1}{9} K_3(s) (3s^2 - 17s + 9).$$
\[ G^{(1)}(s, t) = \frac{1}{6} J(t) \left( 14 - 4s - 10t + st + 2t^2 \right), \]
\[ G^{(2)}(s, t) = J(t) \left\{ \frac{1}{16\pi^2} \left[ \frac{412}{27} - \frac{s}{54} (t^2 + 5t + 159) - t \left( \frac{267}{216} t^2 - \frac{727}{108} t + \frac{1571}{108} \right) \right] \\
+ b_1(2 - t) + \frac{1}{3} b_2(t - 4)(t^2 + s - 5) - \frac{1}{6} b_3(t - 4)^2(3t + 2s - 8) \\
+ \frac{1}{6} b_4(2s(3t - 4)(t - 4) - 32t + 40t^2 - 11t^3) \right\} \\
+ \frac{1}{36} K_1(t) \left[ 174 + 8s - 10t^3 + 72t^2 - 185t - \frac{1}{16} \pi^2(t - 4)(3s - 8) \right] \\
+ \frac{1}{9} K_2(t) \left[ 1 + 4s + \frac{1}{64} \pi^2 t(3s - 8) \right] \\
+ \frac{1}{9} K_3(t) \left( 1 + 3st - s + 3t^2 - 9t \right) + \frac{5}{3} K_4(t) (4 - 2s - t). \]

In this expression

\[ (16\pi^2)^2 J(s) = \sigma(s) h(s) + 2, \quad (16\pi^2)^2 K_1(s) = h^2(s), \quad (16\pi^2)^2 K_2(s) = \sigma^2(s) h^2(s) - 4, \]
\[ (16\pi^2)^2 K_3 = \frac{1}{s^2} h^3(s) + \frac{\pi^2}{s^2} h(s) - \frac{\pi^2}{2}, \quad \text{and} \quad K_4 = \frac{1}{s^2} \left( \frac{4}{3} K_1 + \frac{1}{3} K_3 + \frac{1}{16\pi^2} J + s \frac{\pi^2 - 6}{192\pi^2} \right), \]

where

\[ h(x) = \ln \frac{\sigma(x) - 1}{\sigma(x) + 1}, \quad \text{with} \quad \sigma(x) = \sqrt{1 - \frac{4}{x}}. \quad (2) \]

The total amplitudes \( T_l \) for the isospin \( I = 0, 1 \) and 2 channels are:

\[ T_1(s, t) = A(t, s) - A(u, t), \quad T_2(s, t) = A(t, s) + A(u, t) \quad \text{and} \quad T_0(s, t) = 3A(s, t) + T_2(s, t). \]

We work with S-, P- and D-waves obtained from the total amplitudes by:

\[ t_{I\ell}(s) = \frac{1}{64\pi} \int_{-1}^{1} T_l(s, t, u) P_\ell(x) dx, \]

where \( 2t = (s - 4)(x - 1) \), \( 2u = -(s - 4)(x + 1) \) and \( P_\ell \) are Legendre polynomials.

We take this opportunity to introduce some of the usual phase shift definitions in order to establish the difference between the one employed in the present paper and that used in other approaches.

It is well known that, in the elastic region \( 4 < s < 16 \), unitary implies that the partial wave amplitudes \( t_{I\ell} \) fulfil the relation:

\[ I_m t_{I\ell}(s) = \frac{\sigma}{32\pi} | t_{I\ell}(s) |^2. \]

This relation allows one to define 'exact' phase shifts \( \delta_{I\ell} \) so that

\[ t_{I\ell}(s) = \frac{32\pi}{\sigma} \exp \left( i \delta_{I\ell}(s) \right) \sin \delta_{I\ell}(s), \]

However, ChPT partial waves do not satisfy exact elastic unitarity and, for this reason, some authors introduced another definition, namely

\[ \delta_{I\ell}^I(s) = \frac{\sigma}{32\pi} R_e \ t_{I\ell}(s), \]

which is a good approximation to the 'exact' definition for small values of \( \delta_{I\ell} \).
In order to exploit the right-hand-side discontinuity of the amplitude generated by the loop calculation, leading to an imaginary part, we have adopted another definition for \(\delta_I\ell\) in applications of the UPCA \[3\] which we also use in the present analysis, namely:

\[
\delta_I\ell(s) = \tan^{-1} \frac{I_m t_I\ell(s)}{R_e t_I\ell(s)}.
\]  

(3)

In this paper we fix almost all parameters to be zero and we vary just \(b_4\). We show the S- and P-phase shifts in Fig. 1 and Fig. 2 respectively for the one-loop amplitude with \(b_4 = -.005\) and in Fig. 3 and Fig. 4 respectively for the two-loop amplitude with \(b_4 = .025\).

To analyze our results it is convenient to divide the total amplitude (Eq.(1)) in three parts: A first part which contains the function \(h(s)\) up to power three; a second part which contains \(h(t)\) (or \(h(u)\)) up to the third power and a rest which then contains neither \(h(s)\) nor \(h(t)\). This last part includes the lowest order term in the chiral expansion as obtained by Weinberg \[2\].

\(h(s)\) is analytic in the complex s-plane cut along the positive real axis in the physical region \((s \geq 4)\), therefore s-channel contributions corresponding to powers of \(h(s)\) will give rise to a real \((t^{\text{right}})\) and an imaginary part \((t^{\text{imag}})\) for the partial wave amplitude. Crossed channel contributions, corresponding to the integration of \(h(t)\) and its powers, give rise to functions that are discontinuous on the left hand side \((s \leq 0)\) and we will call this part \(t^{\text{left}}\). The part of the amplitudes that contains neither \(h(s)\) nor \(h(t)\) gives rise to a polynomial contribution. This part includes the Weinberg amplitude, which we denote by \(t^{\text{ca}}\), and a remainder which we call \(t^{\text{free}}\). Needless to say that the parameters \(b_i\) appear in several parts of the amplitudes.

It is very instructive to know how each part of the amplitudes behaves near threshold when compared with the Weinberg amplitude. We show in Fig. 5 (for P-wave) and Fig. 6 (for S-wave) the ratio of each contribution to the soft-pion result \(t^{\text{ca}}\) (straight lines), for energies from 280 to 500 MeV. Curve (a) is the ratio \(t^{\text{free}}/t^{\text{ca}}\), curve (b) is the ratio \(t^{\text{right}}/t^{\text{ca}}\) and curve (c) is the ratio \(t^{\text{left}}/t^{\text{ca}}\). These figures help us to see that the leading contribution is \(t^{\text{ca}}\), but the corrections start to be important for energies bigger than 500 MeV.

At this point we would like to discuss another consequence of \(O(p^6)\) ChPT calculation. The UPCA second order corrected partial waves exhibit a discontinuity as follows:

\[
\text{Im} \ t^{(2)}_{\ell\ell}(s) = \frac{1}{16\pi} \sigma(s) \ t^{\text{ca}}_{\ell\ell}(s) \text{Re} \ t^{(1)}_{\ell\ell}(s) \quad \text{for} \quad s \geq 4.
\]

In this expression the superscripts indicate the order of the approximation. Since the Weinberg amplitude is linear in \(s\), there is no soft pion contribution \((t^{\text{ca}})\) for the D-wave and we conclude that the resulting D-wave amplitude is real for \(s \geq 4\). Now, since we have proven that the second order unitarity correction to current algebra by UPCA is equivalent to \(O(p^6)\) ChPT, this means that a two-loop calculation from ChPT does not provide any D-wave phase shifts using our definition. In principle, ChPT \(O(p^8)\) calculation allows a complex amplitude for the D-wave but we guess that the task of constructing this amplitude can be easier done by the UPCA method \[4\].

3 Conclusions

We have tested the predictive power of the ChPT amplitude for pion-pion scattering. We conclude that at \(O(p^4)\) and at \(O(p^6)\), with just one free parameter, ChPT provides a qualitative description of this process. On the hand we are calling attention to the fact that, up to the two loop approximation, ChPT still does not provide an imaginary part for the D-wave amplitude.
FIG. 1. $O(p^4)$ ChPT P-wave phase shifts with $b_4 = -0.005$ and experimental [10] data.

FIG. 2. $O(p^4)$ ChPT S-wave phase shifts with $b_4 = -0.005$ and experimental [10, 11] data.

FIG. 3. $O(p^6)$ ChPT P-wave phase shifts with $b_4 = 0.025$ and experimental [10] data.

FIG. 4. $O(p^6)$ ChPT S-wave phase shifts with $b_4 = 0.025$ and experimental [10, 11] data.

FIG. 5. Near the threshold behavior of P-wave amplitude components: Curve (a) is the ratio $t_{\text{free}}/t_{\text{ca}}$, curve (b) is the ratio $t_{\text{right}}/t_{\text{ca}}$ and curve (c) is the ratio $t_{\text{left}}/t_{\text{ca}}$.

FIG. 6. Near the threshold behavior of S-wave amplitude components: Curve (a) is the ratio $t_{\text{free}}/t_{\text{ca}}$, curve (b) is the ratio $t_{\text{right}}/t_{\text{ca}}$ and curve (c) is the ratio $t_{\text{left}}/t_{\text{ca}}$. 
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