Comparison of Splitting methods for Gross-Pitaevskii Equation

Jürgen Geiser\textsuperscript{1} and Amirbahador Nasari\textsuperscript{2}

\textsuperscript{1} Ruhr University of Bochum,
Department of Electrical Engineering and Information Technology,
Universitätstrasse 150, D-44801 Bochum, Germany
juergen.geiser@ruhr-uni-bochum.de

\textsuperscript{2} Ruhr University of Bochum,
Department of Civil and Environmental Engineering,
Universitätstrasse 150, D-44801 Bochum, Germany
amirbahador.nasari@ruhr-uni-bochum.de

Abstract. In this paper, we discuss the different splitting approaches to solve the Gross-Pitaevskii equation numerically. We consider conservative finite-difference schemes and spectral methods for the spatial discretisation. Further, we apply implicit or explicit time-integrators and combine such schemes with different splitting approaches. The numerical solutions are compared based on the conservation of the $L^2$-norm with the analytical solutions. The advantages of the splitting methods for large time-domains are presented in several numerical examples of different solitons applications.

Keywords: nonlinear Schrödinger equation, Gross-Pitaevskii equation, Bose-Einstein condensates, splitting methods, splitting spectral methods, convergence analysis, conservation methods

AMS subject classifications. 35K25, 35K20, 74S10, 70G65.

1 Introduction

Bose-Einstein condensate (BEC) nowadays is an actual modelling problem for theoretical and also experimental studies, see [6]. The evolution equation of the Bose-Einstein condensate (BEC) order parameter for weakly interacting bosons is done with the Gross-Pitaevskii equation, see [1], [7] and [3]. The weakly interacting bosons supports dark solitons for repulsive interactions and bright solitons for attractive interactions. A solitary wave or soliton solution is a localised travelling wave solution, that retain its size, shape and speed, when it moves. It does not spread or disperse, see [17]. The modelling equation has two parts, a defocusing effect, which is based on the dispersive term and a steeping effect, which is based on the nonlinear term. To obtain a equation balance of such a localised profile for the solution, we need a special nonlinearity, see [17]. Also after a collision of two solitons, each wave is unscathed with its size, shape and speed,
therefore, we have a special collision property, see [2]. Therefore, the numerical methods should also have conservational behaviours to solve such a specialised balance of nonlinearity (steepness) and diffusivity (smoothness) to obtain the sharp localised soliton solutions.

We are motivated to analyse such numerical methods, which allow to conserve such behaviours, see [17] and [12]. Additionally, we apply different splitting approaches in combination with finite difference schemes or spectral schemes to solve the Gross-Pitaevskii equation, see [1].

Numerically, two different ideas exist to solve the GPE:

– Conservative finite difference schemes, which are nonlinear schemes and need more computational amount, see [17]. Further, they can be constructed to conserve the solution, the momentum and the energy.
– Splitting schemes, which decompose the different parts of the GPE and are simple to implement. But they have energy conservation and stability problems, see [17]

Based on the different ideas, we propose a combination of the splitting approaches and the uses of the conservation properties based on the conservative finite-difference schemes, see [17]. Therefore, we could use the benefits of conservation approaches, see [17] and the splitting approaches, see [9] and [11] to stabilise and accelerate the solver processes. Such a combination allows to reduce the time-consuming procedures of the conservative FD schemes and stabilises the splitting approaches based on the conservative approaches.

The paper is outlined as following. The model is introduced in Section 2. In Section 3, we discuss the different numerical methods and present the convergence analysis. The numerical experiments are done in Section 4 and the conclusion is presented in Section 5.

2 Mathematical Model

The modelling is based on many-body Hamiltonian for a system of $N$ interacting particles (e.g., bosons) for the external field $V_{\text{ext}}$ and particle-particle interaction potential with $V(r-r')$:

$$
H = -\int u^\dagger \left(\frac{\hbar^2}{2m} \nabla^2 - V_{\text{ext}}(r) + \mu \right) u 
\, dr + 
\frac{1}{2} \int u^\dagger(r) u^\dagger(r') V(r-r') \, u(r) u(r') 
\, dr' \, dr,
$$

(1)

where $u$ is the particle (boson) field operator and we satisfy the commutation relation $[u(r), u^\dagger(r')] = \delta(r-r')$. Further, $V(r-r')$ is the two-body interaction and $\mu$ is the chemical potential, see [3]. Then the time-evolution of the field
operator $u$ is given as:

$$
\frac{i \hbar}{\partial t} u(r, t) = [u, H], \tag{2}
$$

$$
\frac{i \hbar}{\partial t} u(r, t) = \left( -\frac{\hbar^2}{2m} \nabla^2 + V_{\text{ext}}(r) - \mu + \int u^\dagger(r', t) V(r - r') u(r', t) \, dr' \right) u(r, t). \tag{3}
$$

Further, the BEC order parameter, or called as condensate wave function, is given as $u = \langle u \rangle$, where $\langle u \rangle$ is the expectation value of the Bose operator.

We have two possibilities:

- $\langle u \rangle = 0$, for $T > T_c$ and
- $\langle u \rangle \neq 0$, for $T < T_c$,

where $T_c$ is the Bose-Einstein condensation temperature.

In the following, we discuss the weakly interacting bosons.

### 2.1 Weakly interacting Bosons

We deal with the following Assumption 1.

**Assumption 1** — We consider dilute gas, while we assume, that the range $r_0$ of the interatomic forces is much more smaller, than the distance between the atoms, means $r_0 << d = n^{-1/3}$, where $n$ is the density of the atoms.

- For $T < T_c$, we obtain small momenta, such that the scattering amplitude is independent of the energy. Therefore, one could replace it by a low-energy-value, which is determined by the solitary wave with scattering length $a$.
- We replace the potential $V(r - r')$ with the effective soft potential $V_{\text{eff}}$, which has the same scattering properties, and it is defined as:

$$
g = \int V_{\text{eff}}(r) \, dr = \frac{4\pi \hbar a}{m} \tag{4}
$$

where $m$ is the atomic mass. Further we replace $V(r - r') = g \delta(r - r')$.

- We transform $u \rightarrow u \exp(i \mu t/\hbar)$.

- The expectation value is given as $u = \langle u \rangle$.

We apply the Assumption (1) to the evolution equation of the interacting particle system (3) and obtain the Gross-Pitaevskii equation with the condensate order parameter $u$ for weakly interacting bosons:

$$
\frac{i \hbar}{\partial t} u = \left( -\frac{\hbar^2}{2m} \nabla^2 + g |u(x, t)|^2 \right) u, \quad (x, t) \in \mathbb{R}^3 \times [0, T], \tag{5}
$$

where $g$ is the interaction term with the following characteristics:
– $g > 0$ implies a repulsive interaction, where $a > 0$,
– $g < 0$ implies an attractive interaction, where $a < 0$.

The Gross-Pitaevskii equation is a nonlinear partial differential equation with a cubic nonlinearity, means we deal with higher order nonlinearities, see also nonlinear Schrödinger equation [16].

In the following, we concentrate on the one-dimensional Gross-Pitaevskii equation, where we assume $\hbar = 1.0$ and the atomic mass $m = 1$. We also deal with a external potential $V(x,t) \equiv 0$ and we deal with the following form of the GPE:

$$i \frac{\partial u}{\partial t} = \left( -\frac{1}{2} \frac{\partial^2}{\partial x^2} + g |u(x,t)|^2 \right) u, \quad (x,t) \in [-L,L] \times [0,T],$$

$$u(x,t) = 0, \quad x \in \{-L,L\}, \quad \text{and} \ t \in [0,T],$$

$$u(x,0) = u_0(x), \quad x \in [-L,L],$$

where the Hamiltonian operator is given as $H = \left( -\frac{1}{2} \frac{\partial^2}{\partial x^2} + g |u(x,t)|^2 \right)$. Further, we assume $g = -1$, means we discuss attractive interactions.

### 3 Numerical Methods

For the numerical methods, we deal with the two standard ideas to approximate the GPE:

1. Splitting methods: The idea is to split the differential equations into some simpler parts and solve each simpler differential equation with fast PDE or ODE solvers. The results are summary approximated, e.g., via coupling the solution of the predecessor-solution as initial conditions of the successor-solution, or averaging the summarised results, see [15] and [8]. The benefits are the fast solver methods and a simple numerical construction with the simple implementation into a program-code, see [15] and [13]. The drawback is that the methods are not long-time stable and they preserve only on invariant of the solution, see [17].

2. Conservative Finite Element Schemes: The idea is to design a finite difference scheme, which preserve the square of $L_2$-norm of the solution, the impulse functional and the energy functional. Based on such a construction of finite-difference approaches, e.g., a well-known conservative FD scheme is the semi-implicit Crank-Nicolson method, see [14], we conserve all the three invariants, see [5], and we obtain stable and long-time behaviours of the solutions. The drawback of such schemes are the nonlinearity in the methods, e.g., we need additional nonlinear solvers, therefore the schemes are highly computational intensive comparing to fast splitting approaches, see [17].

We propose a mixture of the splitting approaches plus the application of the conservation finite-difference schemes, while we apply schemes for the GPE,
which is given as:
\[
\frac{\partial u}{\partial t} = -iHu, \quad x \in \Omega, \quad t \in [0, 1], \\
u(x, 0) = u_0(x), \quad x \in \Omega, \\
u(x, t) = 0.0, \quad x \in \partial\Omega, \quad t \in [0, 1],
\]
with \( Hu = \left( -\frac{1}{2} \frac{\partial^2}{\partial x^2} + g|u|^{2\sigma} \right) u \), \( \sigma = 1.0 \) and we have applied Dirichlet boundary conditions. Further, we apply \( g = -1 \), that means the attractive interaction case.

For an application of a single soliton, the exact solution is given as
\[
u(x, t) = A_0 \operatorname{sech} \left( \frac{|g|}{\sqrt{2}} (x - v_d t) A_0 \right) \exp \left( iv_p (x - v_p t) / 2 \right),
\]
where \( A_0 = \sqrt{(v_d^2 - 2v_p) / 2 |g|} \), \( v_d \) and \( v_p \) are the speeds of the density profile and phase profile, see the derivation of the exact solutions in [3].

**Assumption 2** We apply the absolute value as:
\[
|\nu(x, t)| = \sqrt{\eta(x, t)^2 + \xi(x, t)^2}.
\]
Further we have the following complex relations:
\[
u(x, t) = \eta(x, t) + i\xi(x, t), \\
\exp(\text{i}\theta) = \cos(\theta) + i\sin(\theta), \\
\operatorname{sech}(\theta) = \frac{1}{\cosh(\theta)}.
\]

### 3.1 Conservation Laws of the GPE
The GPE is given as in Equation (9)-(11) and we have the following invariants:
- Mass conservation, which is given as the square of \( L_2 \)-norm of the solution
\[
N(t) = \int_{\Omega} |\nu(x, t)|^2 \, dx,
\]
with \( N(t) = N(0) = \text{const} \).
- Impulse conservation, which is given as the impulse functional of the solution
\[
P(t) = \int_{\Omega} u^\dagger(x, t) (-i \frac{\partial}{\partial x}) u(x, t) \, dx,
\]
with \( P(t) = P(0) = \text{const} \), with \( u^\dagger \) is the conjugate of \( u \).
- Energy conservation, which is given as the energy functional of the solution
\[
E(t) = \frac{1}{2} \int_{\Omega} u^\dagger(x, t) Hu(x, t) \, dx,
\]
with \( E(t) = E(0) = \text{const} \), with \( u^\dagger \) is the conjugate of \( u \).

**Remark 1.** The conservation laws are proved for the GPE in the paper [17]. Further, the conservation laws are also proved for the general Schrödinger equations in the paper [5].

In the following, we present a conservative finite difference scheme.
3.2 Conservative finite difference schemes

We apply the discretisation of the GPE (9)-(11) with the following finite difference method, see also [17]:

\[
\frac{u_{j}^{n+1} - u_{j}^{n}}{\Delta t} = -\frac{1}{2} \left( \frac{u_{j-1}^{n+1} - 2u_{j}^{n+1} + u_{j+1}^{n+1}}{\Delta x^2} + \frac{u_{j-1}^{n} - 2u_{j}^{n} + u_{j+1}^{n}}{\Delta x^2} \right) + \frac{1}{2} g(|u_{j}^{n+1}|^2 + |u_{j}^{n}|^2) \frac{u_{j}^{n+1} + u_{j}^{n}}{2}, \quad j = 1, \ldots, M - 1, \tag{20}
\]

\[
u_{0}^{n} = u_{0}(x_{j}), \quad j = 0, \ldots, M, \tag{21}
\]

\[
u_{0}^{n} = u_{M}^{n} = 0, \quad n = 0, 1, \ldots, N, \tag{22}
\]

where \(M\) is the number of spatial grid points and \(N\) is the number of time grid points.

Here, we have a conservative spatial finite difference scheme, which has to be solved as a nonlinear equation system with fixpoint or Newton’s solvers, see [17].

Remark 2. The conservative behaviour of the semi-implicit Crank-Nicolson is proved in [17].

3.3 Asymptotic conservative finite difference schemes

Here, we apply the idea of the conservative finite difference scheme and reformulate the scheme into a splitting approach.

Therefore, we obtain asymptotic behaviours, while we have splitted the full equations. Based on such a splitting approach, see [8], we have to apply additional iterative steps to obtain the full coupled approximated conservative finite difference scheme, see [10].

We reformulate the finite difference scheme (20)-(22) in the operator notation:

\[
U_{n+1} = U_{n} + i \frac{\Delta t}{2} (A_{1}U_{n+1} + A_{1}U_{n}) + i \frac{\Delta t}{2} (A_{2}(U_{n+1}) + A_{2}(U_{n})) \frac{(U_{n+1} + U_{n})}{2}, \tag{23}
\]

where the matrices are given as:

\[
A_{1} = \frac{1}{2} \frac{1}{\Delta x^2} \begin{bmatrix}
-2 & 1 & 0 & \cdots & 0 \\
1 & -2 & 1 & \cdots & 0 \\
0 & 1 & -2 & 1 & \cdots \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \cdots & -2
\end{bmatrix} \in \mathbb{R}^{M-1 \times M-1}, \tag{24}
\]

\[
A_{2}(U_{n}) = -g |u_{n}|^2 \begin{bmatrix}
-2 & 1 & 0 & \cdots & 0 \\
1 & -2 & 1 & \cdots & 0 \\
0 & 1 & -2 & 1 & \cdots \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \cdots & -2
\end{bmatrix} \in \mathbb{R}^{M-1 \times M-1}, \tag{25}
\]
where with $U^n = (u^n_1, \ldots, u^n_{M-1})^t$ is the vector at the grid points $u^n_j = u^n(x_j)$ for $j = 1, \ldots, M - 1$. Further $I \in \mathbb{R}^{M-1 \times M-1}$ is the identity matrix and $\text{abs}(U^n)^2 = (|u^n_1|^2, \ldots, |u^n_{M-1}|^2)^t \in \mathbb{R}^{M-1}$ is a vector.

Further, the time-steps are given as $\Delta t = t^{n+1} - t^n$, with $n = 0, \ldots, N - 1$ and $\ell^0 = 0$ and $i$ is the imaginary number.

We apply the following asymptotic approximation, based on the Picard-fixpoint scheme, we reformulate the operator scheme (23)-(25) as following:

$$U_k^{n+1} = U^n + i \frac{\Delta t}{2} \left(A_1 U_k^{n+1} + A_1 U^n\right) +$$
$$+ i \frac{\Delta t}{2} \left(A_2(U_{k-1}^{n+1}) + A_2(U^n)\right) \frac{(U_{k-1}^{n+1} + U^n)}{2}, \quad (26)$$

where $k = 1, \ldots, K$ is the iteration index and we have $U_0^{n+1} = U^n$ as the initialisation of the iteration, while we have the stopping criterion $||U_k^{n+1} - U_{k-1}^{n+1}|| \leq \text{err}$ and $\text{err}$ is an error-bound, e.g., $\text{err} = 10^{-5}$, or we stop at $k = K$, while $K$ is a fixed integer, e.g., $K = 5$.

We reformulate in a scaled $\frac{1}{2}AB$ and $\frac{1}{2}BA$ splitting approach. Here, we obtain a first order splitting approach for both splitting approaches, see [8], see the Algorithm 3.

**Algorithm 3** We apply the time-steps $n = 1, \ldots, N - 1$, where $N$ are the number of the time-steps. The initialisation is $U^0 = U(0)$ and we start with $n = 1$.

1. $\frac{1}{2}AB$

$$\hat{U}_k^{n+1} = U^n + i \frac{\Delta t}{2} A_1 \hat{U}_k^{n+1} +$$
$$+ i \frac{\Delta t}{2} \left(A_2(U_{k-1}^{n+1}) + A_2(U^n)\right) \frac{U_{k-1}^{n+1}}{2}, \quad (27)$$

where the starting condition at $k = 1$ is $U_0^{n+1} = U^n$.

2. $\frac{1}{2}BA$

$$\hat{U}_k^{n+1} = \hat{U}^n + i \frac{\Delta t}{2} A_1 \hat{U}^n +$$
$$+ i \frac{\Delta t}{2} \left(A_2(U_{k-1}^{n+1}) + A_2(\hat{U}^n)\right) \frac{\hat{U}^n}{2}, \quad (28)$$

where the starting condition at $k = 1$ is $U_0^{n+1} = U^n$, further we have $\hat{U}^n = \hat{U}_k^{n+1}$. The solution is given as $U_k^{n+1} = \hat{U}_k^{n+1}$.

If $k = K$ or $||U_k^{n+1} - U_{k-1}^{n+1}|| \leq \text{err}$, we are done and goto step 3.,
else we go to the next iterative-step and we apply $k = k + 1$ and goto step 1.

3. If $n + 1 = N$, we are done,
else go to the next time-step and we apply $n = n + 1$ and goto step 1.
We solve the two $B$-steps exactly and reformulate the asymptotic conservative finite difference scheme (26) with respect to the splitting approach, we call it the A-B-A(semiCN) splitting approach, see the Algorithm 4.

Here the $A$ operator is the linear term with the FD scheme discretised, while the $B$ operator is the nonlinear term and is exactly solved. We apply an additional iterative procedure to approach the semi-implicit CN method.

**Algorithm 4**

\[
U^{n+1}_1 = (I - i \Delta t/2 \ A_1)^{-1} \ U^n, \text{with timestep } \Delta t/2 \ (\text{implicit Euler}), \tag{29}
\]
\[
U^{n+1}_2 = \exp(-i \ g \ A_2 \Delta t) \ U^{n+1}_1, \text{with timestep } \Delta t \ (\text{spectral method}), \tag{30}
\]
\[
U^{n+1}_i = (I + i \Delta t/2 \ A_1) \ U^{n+1}_i, \text{with timestep } \Delta t/2 \ (\text{explicit Euler}), \tag{31}
\]

where

\[
A_1(t, x) = \frac{1}{2} \ \frac{1}{\Delta x^2} \begin{bmatrix} -2 & 1 & 0 & \ldots & 0 \\ 1 & -2 & 1 & 0 & \ldots \\ 0 & 1 & -2 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \\ 0 & 0 & 0 & 0 & \ldots & -2 \end{bmatrix} \in \mathbb{R}^{M-1 \times M-1}, \tag{32}
\]

\[
A_2(t, x, U^n, U^{n+1}_{i-1}) = I \ \frac{1}{2} \ (\text{abs}(U^n)^2 + \text{abs}(U^{n+1}_{i-1})^2) \in \mathbb{R}^{M-1 \times M-1}, \tag{33}
\]

where with spatial vector $x = (x_1, \ldots, x_{M-1})^t$ and $M$ are the number of spatial points. Further $U^n = (u^n_1, \ldots, u^n_{M-1})^t$ is the vector at the grid points $u^n_j = u^n(x_j)$ for $j = 1, \ldots, M - 1$.

The starting condition for $U^{n+1}_0 = U^n$.

**Remark 3.** We reformulated the semi-CN scheme into an ABA-splitting approach, while the reformulation has also second order terms, we have at least for such an approximation, only a first order scheme, see [8].

### 3.4 Standard Finite Difference Methods and Standard Splitting Approaches

In the following, we discuss the different standard finite difference method and standard Splitting approaches, which are related to the finite difference schemes for the Gross-Pitaevskii equation.

#### 3.4.1 Splitting methods with finite difference schemes

We apply the semi-discretisation of the diffusion operator with a finite difference scheme (second order), where we deal with $M$ discrete spatial points.
Further, we employ the following transformation and change of variables with \( u = \eta + i\xi \in (\mathbb{R}^M + i\mathbb{R}^M) \) and obtain:

\[
U^{n+1} = U^n + i\,\Delta t\,A(t, x, U^n)U^n \tag{34}
\]

\[
A(t, x, U^n) = A_1(t, x) + A_2(t, x, U^n), \tag{35}
\]

\[
A_1(t, x) = \frac{1}{2\,\Delta x^2} \begin{bmatrix}
-2 & 1 & 0 & \ldots & 0 \\
1 & -2 & 1 & \ldots & 0 \\
0 & 1 & -2 & 1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & -2
\end{bmatrix} \in \mathbb{R}^{M-1\times M-1}, \tag{36}
\]

\[
A_2(t, x, U^n) = \epsilon \, I \, \text{abs}(U^n)^2 \in \mathbb{R}^{M-1\times M-1}, \tag{37}
\]

where with \( U^n = (u^n_1, \ldots, u^n_{M-1})^t \) is the vector at the grid points \( u^n_j = u^n(x_j) \) for \( j = 1, \ldots, M-1 \).

Further, the time-steps are given as \( \Delta t = t^{n+1} - t^n \), with \( n = 0, \ldots, N-1 \) and \( t^0 = 0 \) and \( i \) is the imaginary number.

- Implicit Euler method:

\[
U^{n+1} = (I - i\,\Delta t\,A(t, x, U^n))^{-1}U^n, \tag{38}
\]

where, we start with \( U^0 \).

- CN-method:

\[
U^{n+1} = (I - i\,\Delta t/2\,A(t, x, U^n))^{-1}(I - i\,\Delta t/2\,A(t, x, U^n))^{-1}U^n, \tag{39}
\]

where, we start with \( U^0 \).

- A–B splitting, where we deal with implicit for the diffusion and explicit time discretisation for the nonlinear term:

\[
U^{n+1} = U^n + i\,\Delta t\,(A_1(t, x)U^{n+1} + A_2(t, x, U^n))U^n, \tag{40}
\]

\[
U^{n+1} = (I - i\Delta t\,A_1(t, x))^{-1}(I + i\Delta t\,A_2(t, x, U^n))U^n, \tag{41}
\]

where we start with \( U^0 \).

- A–B splitting, where we deal with explicit for the diffusion and explicit time discretisation for the nonlinear term:

\[
U^{n+1} = U^n + i\,\Delta t\,(A_1(t, x)U^n + A_2(t, x, U^n))U^n, \tag{42}
\]

\[
U^{n+1} = (I + i\Delta t\,A_1(t, x) + i\Delta t\,A_2(t, x, U^n))U^n, \tag{43}
\]

where we start with \( U^0 \).

### 3.5 Standard Spectral Methods and Combinations with Splitting and Finite Difference schemes

In the following, we present spectral and mixed schemes, combining spectral and finite difference schemes with splitting approaches.
The spectral methods applied the Fourier transformation or Fourier spectral method, see [4]. The spectral methods can be applied to the linear part (spatial derivation) and nonlinear part (interaction or potential) of the GPE, see [17].

In the following, we apply the different splitting approaches with respect to the spectral methods.

3.5.1 Time-splitting spectral method We apply the spectral method in $t \in [t^n, t^{n+1}]$

We have two parts of the equation:

- Linear part:

  \[
  \frac{\partial u}{\partial t} = i \frac{1}{2} \frac{\partial^2 u}{\partial x^2}, \quad (x,t) \in [-L, L] \times [0, T],
  \]

  \[
  u(x,t) = 0, \quad x \in \{-L, L\}, \quad t \in [0, T],
  \]

  where we start to apply the Fourier transform for the input $u^n$ and obtain:

  \[
  \hat{u}^n = \sum_{j=\frac{M-1}{2}}^{\frac{M+1}{2}} u^n_j \exp(-i \mu_l (x_j - L)), \quad l = -\frac{M}{2}, \ldots, \frac{M}{2} - 1,
  \]

  \[
  \mu_l = \frac{\pi l}{L}, \quad l = -\frac{M}{2}, \ldots, \frac{M}{2} - 1.
  \]

  We apply the Fourier transform to the linear term and obtain the result in the Fourier transformed space and the inverse Fourier transform and obtain the result:

  \[
  u^{n+1} = \frac{1}{M} \sum_{l=-\frac{M}{2}}^{\frac{M}{2}-1} \exp(-i \mu_l^2 \Delta t/2) \hat{u}_l^n \exp(i \mu_l (x_j - L))
  \]

  \[
  \text{with timestep } \Delta t/2,
  \]

  \[
  \hat{u}^{n+1/2} = \frac{1}{M} \sum_{l=-\frac{M}{2}}^{\frac{M}{2}-1} \exp(-i \mu_l^2 \Delta t/2) \hat{u}^n_l \exp(i \mu_l (x_j - L)),
  \]

  \[
  \text{with timestep } \Delta t,
  \]

  \[
  u^{n+1} = \exp(-i g |u^n|^2 \Delta t/2) U_2^{n+1}, \quad \text{with timestep } \Delta t/2,
  \]

  \[
  U_1^{n+1/2} = \exp(-i g |u^n|^2 \Delta t/2) U^n, \quad \text{with timestep } \Delta t/2,
  \]

The algorithm for the splitting approach is given as:

\textbf{Algorithm 5} We apply the Time-splitting spectral method as following:

\[
U_1^{n+1/2} = \exp(-i g |u^n|^2 \Delta t/2) U^n, \quad \text{with timestep } \Delta t/2,
\]

\[
U_2^{n+1} = \frac{1}{M} \sum_{l=-\frac{M}{2}}^{\frac{M}{2}-1} \exp(-i \mu_l^2 \Delta t/2) \hat{u}_l^{n+1/2} \exp(i \mu_l (x_j - L)),
\]

\[
\text{with timestep } \Delta t,
\]

\[
U^{n+1} = \exp(-i g |u^n|^2 \Delta t/2) U_2^{n+1}, \quad \text{with timestep } \Delta t/2,
\]
where \( \hat{U}_{n+1}^{1/2} = \sum_{j=-M+1}^{M-1} U_{1,j}^{n+1/2} \exp(-i \mu_j (x_j - L)), l = -\frac{M}{2}, \ldots, \frac{M}{2} - 1 \) and \( \mu_j = \frac{\pi}{L}, l = -\frac{M}{2}, \ldots, \frac{M}{2} - 1 \).

Then, we start again with \( U^{n+1} \) in step A.

### 3.5.2 AB Splitting Methods with finite difference and spectral schemes

We deal with the different AB-splitting methods:

- 1.) TSSP Method: A and B are in the spectral version
- 2.) A-B splitting: A operator is the nonlinear term with the spectral method for the reaction
  B operator is the linear term and is in the FD scheme
- 3.) A-B splitting: A operator is the nonlinear term with the FD scheme
  B operator is the linear term in spectral method
- 4.) A-B splitting: A operator is the nonlinear term with the FD scheme
  B operator is the linear term is in FD scheme

- 1.) TSSP Method: A and B are in the spectral version

**Algorithm 6** We apply the Time-splitting spectral method as following:

\[
U_1^{n+1} = \exp(-i g |u^n|^2 \Delta t) U^n, \text{ with timestep } \Delta t, \tag{54}
\]

\[
U^{n+1} = \frac{1}{M} \sum_{l=-M/2}^{M/2-1} \exp(-i \mu_l^2 \Delta t) \hat{U}_1^{n+1/2} \exp(i \mu_l (x_j - L)), \tag{55}
\]

with timestep \( \Delta t \),

where \( \hat{U}_1^{n+1} = \sum_{j=-M+1}^{M-1} U_{1,j}^{n+1} \exp(-i \mu_j (x_j - L)), l = -\frac{M}{2}, \ldots, \frac{M}{2} - 1 \) and \( \mu_j = \frac{\pi}{L}, l = -\frac{M}{2}, \ldots, \frac{M}{2} - 1 \). Then, we start again with \( U^{n+1} \) in step A.

- 2.) A-B splitting: A operator is the nonlinear term with the spectral method for the reaction
  B operator is the linear term and is in the FD scheme.

**Algorithm 7** We apply the combined FD and spectral method as:

\[
U_1^{n+1} = \exp(-i g |u^n|^2 \Delta t) U^n, \text{ with timestep } \Delta t, \tag{56}
\]

\[
U^{n+1} = (I + A_1(t,x)) U_1^{n+1}, \text{ with timestep } \Delta t, \tag{57}
\]

where

\[
A_1(t,x) = i \frac{1}{2} \frac{\Delta t}{\Delta x^2} \begin{bmatrix}
-2 & 1 & 0 & 0 & \ldots & 0 \\
1 & -2 & 1 & 0 & \ldots & 0 \\
0 & 1 & -2 & 1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \ldots & -2
\end{bmatrix} \in \mathbb{R}^{M-1 \times M-1}. \tag{58}
\]

Then, we start again with \( U^{n+1} \) in step A.
3.) A-B splitting: A operator is the nonlinear term with the FD scheme B operator is the linear term in spectral method

Algorithm 8 We apply the Time-splitting spectral method as following:

\begin{align*}
U_{n+1}^1 &= U_n + (-igA_2\Delta t) U^n, \text{ with timestep } \Delta t, \\
U^{n+1} &= \frac{1}{M} \sum_{i=-M/2}^{M/2-1} \exp(-i\mu_l^2 \Delta t) \hat{U}_{1,i}^{n+1/2} \exp(i\mu_l(x_j-L)),
\end{align*}

with timestep \( \Delta t \),

where \( \hat{U}_{1,i}^{n+1} = \sum_{j=-M+1}^{M-1} U_j^{n+1} \exp(-i\mu_l(x_j-L)), l = -\frac{M}{2}, \ldots, \frac{M}{2}-1 \) and

\begin{equation}
A_2(t,x,U) = \begin{bmatrix}
f(\eta_1,\xi_1,t^n,x_1) & 0 & 0 & 0 & \cdots & 0 \\
0 & f(\eta_2,\xi_2,t^n,x_2) & 0 & 0 & \cdots & 0 \\
0 & 0 & f(\eta_3,\xi_3,t^n,x_3) & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \cdots & f(\eta_{M-1},\xi_{M-1},t^n,x_{M-1})
\end{bmatrix} \in \mathbb{R}^{M-1 \times M-1},
\end{equation}

where \( f(\eta_j,\xi_j,t^n,x_j) = (\eta(t^n,x_j))^2 + (\xi(t^n,x_j))^2 \) for \( j = 1, \ldots, M-1 \) with the spatial vector \( x = (x_1, \ldots, x_{M-1}) \) and \( M \) are the number of spatial points. Further \( U = (u_1, \ldots, u_{M-1})^t \) is the vector at the grid points \( u_j = u(x_j) \) for \( j = 1, \ldots, M-1 \).

Then, we start again with \( U_{n+1} \) in step A.

4.) A-B splitting: A operator is the nonlinear term with the FD scheme B operator is the linear term in FD scheme

Algorithm 9 We apply the splitting approach with the FD schemes as:

\begin{align*}
U_{1,n+1} = U^n + (-igA_2\Delta t) U^n, \text{ with timestep } \Delta t, \\
U^{n+1} = U^n + (iA_1\Delta t) U_{1,n+1}, \text{ with timestep } \Delta t,
\end{align*}

where

\begin{equation}
A_1(t,x) = \frac{1}{2} \frac{1}{\Delta t^2} \begin{bmatrix}
-2 & 1 & 0 & 0 & \cdots & 0 \\
1 & -2 & 1 & 0 & \cdots & 0 \\
0 & 1 & -2 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \cdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & -2
\end{bmatrix} \in \mathbb{R}^{M-1 \times M-1},
\end{equation}

\begin{equation}
A_2(t,x,U) = \text{I abs}((U^n)^2) \in \mathbb{R}^{M-1 \times M-1},
\end{equation}

where with spatial vector \( x = (x_1, \ldots, x_{M-1}) \) and \( M \) are the number of spatial points. Further \( U^n = (u_1^n, \ldots, u_{M-1}^n)^t \) is the vector at the grid points \( u_j^n = u^n(x_j) \) for \( j = 1, \ldots, M-1 \).
5.) A-B-A(CN) splitting: A operator is the linear term with the FD scheme, B operator is the nonlinear term in spectral method.

**Algorithm 10** We apply the ABA-splitting approach with FD schemes and spectral schemes as:

\[ U_{n+1}^{1} = (I - i \Delta t/2 \ A_1)^{-1} U_n, \text{ with timestep } \Delta t/2 \text{ (implicit Euler)}, \]

\[ U_{n+1}^{2} = \exp(-i g A_2 \Delta t) U_{n+1}^{1}, \text{ with timestep } \Delta t \text{ (spectral method)}, \]

\[ U_{n+1} = (I + i \Delta t/2 \ A_1) U_{n+1}^{2}, \text{ with timestep } \Delta t/2 \text{ (explicit Euler)}, \]

where

\[ A_1(t,x) = \frac{1}{2} \begin{bmatrix} -2 & 1 & 0 & \ldots & 0 \\ 1 & -2 & 1 & \ldots & 0 \\ 0 & 1 & -2 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & -2 \end{bmatrix} \in \mathbb{R}^{M-1 \times M-1}, \]

\[ A_2(t,x,U^n) = I \ \text{abs}((U^n))^2 \in \mathbb{R}^{M-1 \times M-1}, \]

where with spatial vector \( x = (x_1, \ldots, x_{M-1}) \) and \( M \) are the number of spatial points. Further \( U^n = (u_1^n, \ldots, u_{M-1}^n)^t \) is the vector at the grid points \( u_j^n = u^n(x_j) \) for \( j = 1, \ldots, M-1 \).

### 4 Numerical experiments

For the numerical experiments, we test two models:

- Single soliton with exact solution as corresponding solution.
- Collision of two solitons with numerically fine solution as corresponding solution.

For the errors, we apply the \( L_2 \)-norm and use:

\[ \text{err} \left( \frac{L_2}{\text{num.}}, \Delta t \right) = \left( \int_{[0,T]} \int_{\Omega} \left| u_{\text{exact}}(x,t) - u_{\text{num}}(x,t) \right|^2 dx \ dt \right)^{1/2} = \left( \Delta t \Delta x \sum_{n=1}^{N} \sum_{i=1}^{M} \left| u_{\text{exact}}(x_i,t^n) - u_{\text{num}}(x_i,t^n) \right|^2 \right)^{1/2}, \]

where \( \left| u_{\text{exact}}(x_i,t^n) - u_{\text{num}}(x_i,t^n) \right| = \text{abs}(u_{\text{exact}}(x_i,t^n) - u_{\text{num}}(x_i,t^n)). \)

We apply a convergence-tableau based on the different spatial- and time-steps, means we apply \( 16\Delta t, \ldots, \Delta t/8 \) and \( \Delta x, \ldots, \Delta x/8 \) with the underlying errors.

In the following, we apply different numerical experiments to validate our numerical method.
4.1 First example: GPE with one soliton

We consider the GPE in order to apply for the numerical schemes in a suitable rewriting:

\[
\frac{\partial u}{\partial t} = -i H u, \quad x \in \Omega, \ t \in [0, 1],
\]

\[
u(x, 0) = \text{sech}(\frac{1}{\sqrt{2}}(x - 25)) \exp(i \frac{x}{20}), \ x \in \Omega,
\]

\[
u(x, t) = 0.0, \ x \in \partial \Omega, \ t \in [0, 1],
\]

with \( H u = \left( -\frac{1}{2} \frac{\partial^2}{\partial x^2} + g|u|^2\sigma \right) u, \ \sigma = 1.0 \) and we have applied Dirichlet boundary conditions.

We applied for the analytical solution \( g = -1, \ v_d = \frac{1}{10} \) and \( v_p = -\frac{199}{200} \) and the analytical solution is given as:

\[
u(x, t) = \text{sech}(\frac{1}{\sqrt{2}}(x - \frac{t}{10} - 25)) \exp(i(\frac{x}{20} - \frac{199}{400})), \ (x, t) \in [-L, L] \times [0, T].
\]

We deal with the following methods:

- implicit Euler method (all operators are done with the implicit method),
- Crank-Nicolson scheme (all operators are done with the CN method),
- AB-splitting:
  - linear operator is done with the Spectral method and nonlinear operator is done with the spectral method,
  - linear operator is done with the FD method and nonlinear operator is done with the spectral method,
  - linear operator is done with the Spectral method and nonlinear operator is done with the FD method,
  - linear operator is done with the FD method and nonlinear operator is done with the FD method.
- ABA-splitting:
  - linear operator is done with the Spectral method and nonlinear operator is done with the spectral method.
- ABA-CN and ABA-iCN:
  - linear operator is done with the finite difference method, while the nonlinear operator is done with the spectral method.
  - for the iterative scheme, we apply different iterative steps.

The convergence-tableaus of the different numerical methods are given in the Tables 1-10.

The computational times and the errors of the different methods for the single soliton solutions are given in Table 11 and 12.

The Figure 1 present the solutions of the one soliton results and the convergence tableau.

The Figure 2 present the solutions with the approximated conservation finite difference scheme.
| $\Delta x/4$ | $\Delta x/8$ | $\Delta x/16$ |
|---------------|---------------|---------------|
| 4$\Delta t$   | 1.5474e-06    | 3.9236e-11    |
| 8$\Delta t$   | 3.667e-05     | 9.2733e-10    |
| 16$\Delta t$  | 7.334e-05     | 1.8547e-09    |

**Table 1.** Convergence tableau for the method implicit Euler.

| $\Delta x/4$ | $\Delta x/8$ | $\Delta x/16$ |
|---------------|---------------|---------------|
| 4$\Delta t$   | 1.312e-05     | 3.3337e-10    |
| 8$\Delta t$   | 3.667e-05     | 9.2733e-10    |
| 16$\Delta t$  | 7.334e-05     | 1.8547e-09    |

**Table 2.** Convergence tableau for the method Crank-Nicolson.

| $\Delta x/4$ | $\Delta x/8$ | $\Delta x/16$ |
|---------------|---------------|---------------|
| 4$\Delta t$   | 1.583e-05     | 4.2155e-10    |
| 8$\Delta t$   | 3.667e-05     | 9.2733e-10    |
| 16$\Delta t$  | 7.334e-05     | 1.8547e-09    |

**Table 3.** Convergence tableau for the method AB-splitting: A and B operators are spectral.

| $\Delta x/4$ | $\Delta x/8$ | $\Delta x/16$ |
|---------------|---------------|---------------|
| 4$\Delta t$   | 1.312e-05     | 3.3337e-10    |
| 8$\Delta t$   | 3.667e-05     | 9.2733e-10    |
| 16$\Delta t$  | 7.334e-05     | 1.8547e-09    |

**Table 4.** Convergence tableau for the method AB-splitting: A spectral, B FD.

| $\Delta x/4$ | $\Delta x/8$ | $\Delta x/16$ |
|---------------|---------------|---------------|
| 4$\Delta t$   | 1.583e-05     | 4.2155e-10    |
| 8$\Delta t$   | 3.667e-05     | 9.2733e-10    |
| 16$\Delta t$  | 7.334e-05     | 1.8547e-09    |

**Table 5.** Convergence tableau for the method AB-splitting: A FD, B spectral.

| $\Delta x/4$ | $\Delta x/8$ | $\Delta x/16$ |
|---------------|---------------|---------------|
| 4$\Delta t$   | 1.312e-05     | 3.3337e-10    |
| 8$\Delta t$   | 3.667e-05     | 9.2733e-10    |
| 16$\Delta t$  | 7.334e-05     | 1.8547e-09    |

**Table 6.** Convergence tableau for the method AB-splitting: A FD, B FD.

| $\Delta x/4$ | $\Delta x/8$ | $\Delta x/16$ |
|---------------|---------------|---------------|
| 4$\Delta t$   | 1.583e-05     | 4.2155e-10    |
| 8$\Delta t$   | 3.667e-05     | 9.2733e-10    |
| 16$\Delta t$  | 7.334e-05     | 1.8547e-09    |

**Table 7.** Convergence tableau for the ABA-Splitting method.
| Δx/4 | Δx/8 | Δx/16 |
|------|------|------|
| 4Δt | 1.583e-05 | 4.2155e-10 | 2.2106e-12 |
| 8Δt | 3.667e-05 | 9.2733e-10 | 4.6629e-12 |
| 16Δt| 7.334e-05 | 1.8547e-09 | 9.3258e-12 |

**Table 8.** Convergence tableau for the BAB-Splitting method.

| Δx/4 | Δx/8 | Δx/16 |
|------|------|------|
| 4Δt | 1.312e-05 | 3.3337e-10 | 1.6673e-12 |
| 8Δt | 3.667e-05 | 9.2733e-10 | 4.6629e-12 |
| 16Δt| 7.334e-05 | 1.8547e-09 | 9.3258e-12 |

**Table 9.** Convergence tableau for the ABA(CN)Splitting method.

| Δx/4 | Δx/8 | Δx/16 |
|------|------|------|
| 4Δt | 1.312e-05 | 3.3337e-10 | 1.6673e-12 |
| 8Δt | 3.667e-05 | 9.2733e-10 | 4.6629e-12 |
| 16Δt| 7.334e-05 | 1.8547e-09 | 9.3258e-12 |

**Table 10.** Convergence tableau for the ABA(semiCN) Splitting method.

**Table 11.** Computational times of one soliton with the different methods.

| | T=2.5 | T=5 | T=7.5 | T=10 |
|---|---|---|---|---|
| Implicit Euler method | 0.8313 | 1.6785 | 2.1124 | 2.9281 |
| Crank-Nicolson scheme | 2.0496 | 3.8930 | 5.7764 | 7.1148 |
| AB-splitting: A and B operators are spectral | 0.0271 | 0.0486 | 0.0785 | 0.1007 |
| AB-splitting: A Spectral , B FD | 1.8159 | 3.2140 | 4.6932 | 5.7207 |
| AB-splitting: A FD , B Spectral | 0.0466 | 0.0551 | 0.0668 | 0.0962 |
| AB-splitting: A FD , B FD | 2.4798 | 3.8211 | 5.7146 | 7.0136 |
| ABA-Splitting | 0.0352 | 0.0632 | 0.0940 | 0.1264 |
| BAB-Splitting | 0.0330 | 0.0396 | 0.0420 | 0.0488 |
| ABA(CN)-Splitting | 0.0343 | 0.0624 | 0.1003 | 0.1281 |
| ABA(semiCN)-Splitting | 0.9762 | 1.9774 | 2.6190 | 3.2304 |

**Table 12.** Numerical errors of one soliton with the different methods.

| | T=2.5 | T=5 | T=7.5 | T=10 |
|---|---|---|---|---|
| Implicit Euler method | 0.8977 | 1.9084 | 2.4616 | 2.6552 |
| Crank-Nicolson scheme | 0.9165 | 2.0208 | 2.7069 | 2.9775 |
| AB-splitting: A and B operators are spectral | 0.0330 | 0.0396 | 0.0420 | 0.0488 |
| AB-splitting: A Spectral , B FD | 0.9443 | 2.0468 | 2.7054 | 2.9648 |
| AB-splitting: A FD , B Spectral | 0.1333 | 0.3893 | 0.7650 | 1.2144 |
| AB-splitting: A FD , B FD | 0.9165 | 2.0208 | 2.7069 | 2.9775 |
| ABA-splitting | 0.0057 | 0.0080 | 0.0097 | 0.0111 |
| BAB-splitting | 0.0057 | 0.0080 | 0.0097 | 0.0111 |
| ABA(CN)-Splitting | 0.9178 | 2.0201 | 2.6952 | 2.9630 |
| ABA(semiCN)-Splitting | 0.9174 | 2.0208 | 2.7003 | 2.9740 |
Fig. 1. Results of the GPE with one soliton equation, here we have applied the ABA-splitting approach (left figure: numerical results, right figure: convergence results).

Fig. 2. Numerical solution with the ABA-CN method of the single solitons.

Remark 4. We see the benefits of the conservation schemes in the long time behaviour. But the drawbacks are the time-consuming computations. The balance based on the splitting approach including the conservative schemes are an alternative to reduce the time-consuming approaches and allow to obtain asymptotic conservative results with sufficient enough iterative steps.

4.2 Second Example: Collision of two solitons

We apply a collision of two solitons with the GPE. The evolution equation is given as:

\[ \frac{\partial u}{\partial t} = -iHu, \ x \in \Omega, \ t \in [0, 10], \]  
\[ u(x, 0) = \text{sech}\left(\frac{1}{\sqrt{2}}(x - 20)\right) \exp(-i\frac{x}{20}) + \]  
\[ + \text{sech}(x + 20) \exp(i\frac{x}{20}), \ x \in \Omega, \]  
\[ u(x, t) = 0.0, \ x \in \partial\Omega, \ t \in [0, T], \]
with \( Hu = \left( -\frac{1}{2} \frac{\partial^2}{\partial x^2} + g |u|^{2\sigma} \right) u, \sigma = 1.0. \)

We have two solitons starting in \( x = -20 \) and \( x = 20 \) and they collide at \( x = 0 \) at the time-point \( t = 5.0. \)

For the reference solution, we apply a fine spatial- and time-discretised solution with an ABA method.

Further, we also decouple the full equation after the spatial discretisation into a linear and nonlinear operator part, given as:

\[
HU^n = A(t, x, U^n) = A_1(t, x) + A_2(t, x, U^n),
\]

(80)

In the Table 13 and 14, we present the computational time and the numerical errors of the different methods for the two-solitons modelling problem.

| Method                          | \( T=2.5 \) | \( T=5 \) | \( T=7.5 \) | \( T=10 \) |
|---------------------------------|-------------|-----------|-------------|-----------|
| Implicit Euler method           | 2.4928      | 3.5601    | 4.9031      | 6.2648    |
| Crank-Nicolson scheme           | 4.5648      | 8.8923    | 13.8926     | 15.9353   |
| AB-splitting: A and B operators are spectral | 0.0342      | 0.0632    | 0.1004      | 0.1429    |
| AB-splitting: A Spectral, B FD  | 3.5292      | 6.7374    | 9.7638      | 12.9375   |
| AB-splitting: A FD, B Spectral  | 0.0349      | 0.0678    | 0.0965      | 0.1380    |
| AB-splitting: A FD, B FD        | 4.4182      | 8.5995    | 12.3866     | 16.4472   |
| ABA-splitting                   | 0.0445      | 0.0858    | 0.1408      | 0.1989    |
| BAB-splitting                   | 0.0425      | 0.0789    | 0.1524      | 0.1931    |
| ABA(CN)-Splitting               | 2.1821      | 4.4567    | 6.3876      | 7.7092    |
| ABA(semiCN)-Splitting           | 6.1543      | 10.5217   | 16.1007     | 19.6879   |

Table 13. Computational times of two solitons with the different methods.

| Method                          | \( T=2.5 \) | \( T=5 \) | \( T=7.5 \) | \( T=10 \) |
|---------------------------------|-------------|-----------|-------------|-----------|
| Implicit Euler method           | 1.0605      | 4.6478    | 5.0486      | 5.1546    |
| Crank-Nicolson scheme           | 1.0548      | 4.3745    | 9.1666      | 19.2207   |
| AB-splitting: A and B operators are spectral | 0.0866      | 0.1069    | 0.1501      | 0.1754    |
| AB-splitting: A Spectral, B FD  | 0.7579      | 1.8421    | 2.4654      | 2.8083    |
| AB-splitting: A FD, B Spectral  | 1.1001      | 6.0412    | 49.3173     | 114.0526  |
| AB-splitting: A FD, B FD        | 1.0548      | 4.3745    | 9.1666      | 19.2207   |
| ABA-splitting                   | 0.0296      | 0.0320    | 0.0410      | 0.0453    |
| BAB-splitting                   | 0.0295      | 0.0314    | 0.0405      | 0.0447    |
| ABA(CN)-Splitting               | 0.7024      | 1.8489    | 2.4949      | 2.7952    |
| ABA(semiCN)-Splitting           | 0.8599      | 2.6645    | 2.7771      | 2.7894    |

Table 14. Numerical errors of two solitons with the different methods.

The Figure 3 present the solutions and errors of the one soliton results. The solution of the two-solitons with the ABA-CN method in Figure 4.
Fig. 3. Results of the deterministic nonlinear Schrödinger equation with collisions of solitons (left figure: numerical results, right figure: exact results).

Fig. 4. Solution of the ABA-CN method for the two solitons.

Remark 5. We also obtain the same results as for the single soliton solutions. The alternative methods with the combination of the conservative schemes and the splitting approaches have small numerical errors and optimal computational times in the area of the fast splitting methods. With additional iterative steps, we could couple the ABA-iCN method more and achieve asymptotically the conservation schemes.

5 Conclusion

We propose an alternative ABA-iCN method, which combines the conservative finite difference scheme with a fast ABA splitting approaches. Such alternative methods allow to accelerate the solvers and stabilise the schemes to asymptotic conservative finite difference schemes. We apply different numerical test examples and verify our assumptions. In future, we have to analyse carefully the structure of the proposed methods with the underlying error analysis and present more real-life applications in the field of soliton collisions.
References

1. F.Kh. Abdullaev, A. Gammal, A.M. Kamechatnov and L. Tomio. *Dynamics of bright matter wave solitons in a Bose-Einstein condensate*. Int. J. Mod. Phys. B, 19(22):3415-3473, 2005.
2. R. Atre, P.K. Panigrahi and G.S. Agarwal. *Class of solitary wave solutions of the one-dimensional Gross-Pitaevskii equation*. Phys. Rev. E, 73(5):056611, 2006.
3. R. Balakrishnan and L.I. Satija. *Solitons in Bose–Einstein condensates*. Pramana, journal of physics, 77(5):929-947, 2011.
4. E.O. Brigham. *The Fast Fourier Transform: An Introduction to Its Theory and Application*. Prentice Hall, 1973.
5. Q. Chang, E. Jia and W. Sun. *Difference Schemes for Solving the Generalized Nonlinear Schrödinger Equation*. Journal of Computational Physics, 148:397-415, 1999.
6. F. Dalfovo, S. Giorgini, L.P. Pitaevskii and S. Stringari. *Theory of Bose-Einstein condensation in trapped gases*. Rev. Mod. Phys., 71(3):463-512, 1999.
7. T. Dauxois and M. Peyard. *Physics of Solitons*. Cambridge University Press, Cambridge, 2006.
8. J. Geiser. *Iterative Splitting Methods for Differential Equations*. Numerical Analysis and Scientific Computing Series, Taylor & Francis Group, Boca Raton, London, New York, 2011.
9. J. Geiser. *Multicomponent and Multiscale Systems: Theory, Methods, and Applications in Engineering*. Springer, Cham, Heidelberg, New York, Dordrecht, London, 2016.
10. J. Geiser. *Iterative splitting method as almost asymptotic symplectic integrator for stochastic nonlinear Schrödinger equation*. AIP Conference Proceedings 1863, 560005, 2017, https://doi.org/10.1063/1.4992688.
11. J. Geiser and A. Nasari. *Simulation of Multiscale Schroedinger Equation with Extrapolated Splitting Approaches*. AIP Conference Proceedings Paper, ICNAAM 2018 (13.-18. September, 2018), Rhodes, Greece, accepted August 2018.
12. S. Jiang, L. Wang and J. Hong. Stochastic Multi-Symplectic Integrator for Stochastic Nonlinear Schroedinger Equation. *Commun. Comput. Phys.*, 14(2):393–411, 2013.
13. R.I. McLachlan, G.R.W. Quispel. *Splitting methods*. Acta Numerica, 341-434, 2002.
14. J.M. Sanz-Serna and J.G. Verwer. *Conservative and non-conservative schemes for the solution of the nonlinear Schroedinger equation*. IMA J. Numer. Anal., 6(1):25-42, 1986.
15. G. Strang. *On the construction and comparison of differential schemes*. SIAM J. Numer. Anal., 5(3):506-517, 1968.
16. L.A. Takhtajan. *Quantum Mechanics for Mathematicians*. American Mathematical Society, Providence, Rhode Island, Graduate Series in Mathematics, vol. 95, 2008.
17. V.A. Trofimov and N.V. Peskov. *Comparison of finitedifference schemes for the GrossPitaevskii equation*. Mathematical Modelling and Analysis, 14(1):109-126, 2009.