Some Serrin type blow-up criteria for the three-dimensional viscous compressible flows with large external potential force

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We provide a Serrin type blow-up criterion for the 3-D viscous compressible flows with large external potential force. For the Cauchy problem of the 3-D compressible Navier–Stokes system with potential force term, it can be proved that the strong solution exists globally if the velocity satisfies the Serrin's condition and the sup-norm of the density is bounded. Furthermore, in the case of isothermal flows with no vacuum, the Serrin's condition on the velocity can be removed from the claimed criterion.

KEYWORDS
blow-up criteria, compressible flow, Navier-Stokes equations, potential force

MSC CLASSIFICATION
35Q30

1 INTRODUCTION

In this present work, we are interested in the 3-D compressible Navier–Stokes equations with an external potential force in the whole space \( \mathbb{R}^3 \) (\( j = 1, 2, 3 \)):

\[
\begin{aligned}
\rho_t + \text{div}(\rho u) &= 0, \\
(\rho u_j)_t + \text{div}(\rho u u_j) + (P)_{x_j} &= \mu \Delta u_j + \lambda (\text{div } u)_{x_j} + \rho f^j.
\end{aligned}
\] (1.1)

Here, \( x \in \mathbb{R}^3 \) is the spatial coordinate, and \( t \in [0, \infty) \) stands for the time. The unknown functions \( \rho = \rho(x, t) \) and \( u = (u^1, u^2, u^3)(x, t) \) represent the density and velocity vector in a compressible fluid. The function \( P = P(\rho) \) denotes the pressure, \( f = (f^1(x), f^2(x), f^3(x)) \) is a prescribed external force, and \( \mu, \lambda \) are positive viscosity constants. The system (1.1) is equipped with initial condition

\[
(\rho(\cdot, 0) - \bar{\rho}, u(\cdot, 0)) = (\rho_0 - \bar{\rho}, u_0),
\] (1.2)

where the nonconstant time-independent function \( \bar{\rho} = \bar{\rho}(x) \) (known as the steady state solution to 1.1) can be obtained formally by taking \( u \equiv 0 \) in (1.1):

\[
\nabla P(\bar{\rho}(x)) = \bar{\rho}(x) f(x).
\] (1.3)

The Navier–Stokes system (1.1) expresses conservation of momentum and conservation of mass for Newtonian fluids, which has been studied by various teams of researchers. The local-in-time existence of classical solution to the full Navier–Stokes equations was proved by Nash\(^1\) and Tani.\(^2\) Later, Matsumura and Nishida\(^3\) obtained the global-in-time
existence of $H^3$-solutions when the initial data were taken to be small (with respect to $H^3$ norm), the results were then generalized by Danchin who showed the global existence of solutions in critical spaces. In the case of large initial data, Lions obtained the existence of global-in-time finite energy weak solutions, yet the problem of uniqueness for those weak solutions remains completely open. In between the two types of solutions as mentioned above, a type of “intermediate weak” solutions was first suggested by Hoff in previous studies and later generalized by Matsumura and Yamagata, Cheung and Suen, and other systems which include compressible magnetohydrodynamics (MHD), compressible Navier–Stokes–Poisson system, and chemotaxis systems. Solutions as obtained in this intermediate class are less regular than those small-smooth type solutions obtained by Matsumura and Nishida and Danchin in such a way that the density and velocity gradient may be discontinuous across some hypersurfaces in $\mathbb{R}^3$. On the other hand, those intermediate weak solutions would have more regularity than the large-weak type solutions developed by Lions so that the uniqueness and continuous dependence of solutions may be obtained; see Hoff and other compressible system.

Nevertheless, the global existence of smooth solution to the Navier–Stokes system (1.1) with arbitrary smooth data is still unknown. From the seminal work given by Xin, it was proved that smooth solution to (1.1) will blow up in finite time in the whole space when the initial density has compact support. Motivated by the well-known Serrin’s criterion on the Leray–Hopf weak solutions to the 3-D incompressible Navier–Stokes equations, Huang et al later proved that the strong solution exists globally if the velocity satisfies the Serrin’s condition and either the sup-norm of the density or the time integral of the $L^\infty$-norm of the divergence of the velocity is bounded. Under an extra assumption on $\lambda$ and $\mu$, Sun et al obtained a Beale–Kato–Majda blow-up criterion in terms of the upper bound of the density, which is analogous to the Beal–Kato–Majda criterion for the ideal incompressible flows. The results from Sun et al were later generalized to other compressible systems.

In this present work, we extend the results from Huang et al and Sun et al to the case of compressible Navier–Stokes equations with large potential force. The main novelties of this current work can be summarized as follows:

- We successfully extend the results from Huang et al and obtain a Serrin type blow-up criterion for (1.1) in which initial vacuum state is allowed;
- For the isothermal case, under the assumption that initial density is away from zero, we obtain a blow-up criterion in terms of density only. Such result is also consistent with the case studied in Suen when the magnetic field is removed.
- We introduce some new methods in controlling extra terms originated from the external force which is absent in Suen.

We give a brief description on the analysis applied in this work, and the main idea of the following discussion comes from Hoff. Due to the presence of the external force $f$, one cannot simply apply the same method given in Suen for obtaining the required blow-up criteria for the solutions. To understand the issue, we consider a decomposition on the velocity $u$ given by

$$u = u_p + u_s,$$

for which $u_p$ and $u_s$ satisfy

$$\begin{align*}
\mu \Delta (u_p) + \lambda \nabla \cdot \nabla (u_p) &= \nabla (P - P(\bar{\rho})), \\
\rho(u_s)_t - \mu \Delta u_s - \lambda \nabla \cdot (u_s) &= -\rho u \cdot \nabla u - \rho (u_p)_t + \bar{\rho}^{-1}(\rho - \bar{\rho}) \nabla P(\bar{\rho}).
\end{align*}$$

(1.4)

The decomposition of $u$ is important for obtaining some better estimates on the velocity $u$, which allows us to control $u$ in terms of $u_s$ and $u_p$ separately. Since we are addressing solutions around the steady state $(\bar{\rho}, 0)$, it is natural to consider the difference $P - P(\bar{\rho})$ as appeared in (1.4). Yet the term $P(\bar{\rho})$ will create extra terms in the following sense:

- On the one hand, since $P(\bar{\rho})$ is not necessary a constant, there is an extra term $\nabla P(\bar{\rho})$ arising from $\nabla (P - P(\bar{\rho}))$;
- On the other hand, using the identity (1.3), we can express $f$ in terms of $\bar{\rho}$ and $P$, so that the term $\rho f$ from (1.1) can be combined with $\nabla P(\bar{\rho})$ to give $\bar{\rho}^{-1}(\rho - \bar{\rho}) \nabla P(\bar{\rho})$ in (1.4).

Compared with the previous work Suen, the term $\bar{\rho}^{-1}(\rho - \bar{\rho}) \nabla P(\bar{\rho})$ is distinctive for the present system (1.1), and we have to develop new method for dealing with it. By examining the regularity, one can see that $\bar{\rho}^{-1}(\rho - \bar{\rho}) \nabla P(\bar{\rho})$ is more regular than $\nabla (P - P(\bar{\rho}))$; hence, it can be used for obtaining $H^1$ estimates on $u_s$ provided that $\|\bar{\rho}^{-1}(\rho - \bar{\rho}) \nabla P(\bar{\rho})\|_{L^2}$ is under control. Thanks to the $L^2$-energy balance law given by (3.1), we can control $\|\bar{\rho}^{-1}(\rho - \bar{\rho}) \nabla P(\bar{\rho})\|_{L^2}$ if $\rho$ is bounded. This is a crucial step for obtaining estimates for $u$ in some higher regularity classes, and the details will be carried out in Section 4.
Another key of the proof is to extract some “hidden regularity” from the velocity \( u \) and density \( \rho \), which is crucial for decoupling \( u \) and \( \rho \). In order to achieve our goal, we introduce an important canonical variable \( F \) associated with the system (1.1), which is known as the effective viscous flux. To see how it works, by the Helmholtz decomposition of the mechanical forces, we can rewrite the momentum Equation (1.1) as follows (summation over \( k \) is understood):

\[
\tilde{\rho} \tilde{u}^i = (\tilde{\rho} F)_{x_i} + \mu \omega^{j,k}_{x_i} + \rho f^j - P(\tilde{\rho})_{x_i},
\]

where \( \tilde{u}^i = u^i_t + u \cdot u^i \) is the material derivative on \( u^i \), \( \omega^{j,k} = u^j_{x_k} - u^k_{x_j} \) is the vorticity, and the effective viscous flux \( F \) is defined by

\[
\tilde{\rho} F = (\mu + \lambda) \text{div} u - (P(\rho) - P(\tilde{\rho})).
\]

By differentiating (1.5) with respect to \( x_j \) and using the anti-symmetry from \( \omega \), we obtain the following Poisson equation for \( F \):

\[
\Delta (\tilde{\rho} F) = \text{div}(\tilde{\rho} \tilde{u} - \rho f + \nabla P(\tilde{\rho})).
\]

The Poisson Equation (1.6) can be viewed as the analog for compressible Navier–Stokes system of the well-known elliptic equation for pressure in incompressible flow. For sufficiently regular steady state \( \tilde{\rho} \), by exploiting the Rankine–Hugoniot condition (see Suen and Hoff\(^{14} \) for example), one can deduce that the effective viscous flux \( F \) is relatively regular than \( \text{div}(u) \) or \( P(\rho) \), which turns out to be crucial for the overall analysis in the following ways:

(i) Equation (1.5) allows us to decompose the acceleration density \( \tilde{\rho} \tilde{u} \) as the sum of the gradient of the scalar \( F \) and the divergence-free vector field \( \omega^{k,j} \). The skew symmetry of \( \omega \) insures that these two vector fields are orthogonal in \( L^2 \), so that \( L^2 \) bounds for the terms on the left side of (1.5) immediately give \( L^2 \) bounds for the gradients of both \( F \) and \( \omega \). These, in turn, will be used for controlling \( \nabla u \) in \( L^2 \) when the estimates of \( u(\cdot, t) \) in \( H^2 \) are unknown, which are crucial for estimating different functionals in \( u \) and \( \rho \); also refer to Lemma 3.2 and Remark 3.3. The details will be carried out in Section 3.

(ii) As we have seen before, we aim at applying a decomposition of \( u \) given by \( u = u_p + u_s \) with \( u_p \) satisfying (1.4). To estimate the term \( \partial_t u_p \), if we apply time derivative on the above identity, then there will be the term \( \nabla (\partial_t P(\rho)) \) appearing in the analysis. In view of the strongly elliptic system (1.4), we can obtain estimates on \( \|\partial_t u_p\|_{L^2} \) in terms of the lower order term \( \|P(\rho) u\|_{L^2} \) if we have

\[
\nabla (\partial_t P(\rho)) = \nabla \text{div}(-P(\rho) u),
\]

which is valid when the system is isothermal, that is, for the case when \( \gamma = 1 \) in (1.7); also refer to Lemma 4.4 and the estimate (4.14).

We now give a precise formulation of our results. For \( r \in [1, \infty] \) and \( k \in [1, \infty) \), we let \( L'^r := L'^r(\mathbb{R}^3) \), \( W^{k,r} := W^{k,r}(\mathbb{R}^3) \) and \( H^k := H^k(\mathbb{R}^3) \) be the standard Sobolev spaces, and we define the following function spaces for later use (also refer to other works\(^{20,21,25} \) for similar definitions):

\[
\begin{align*}
D^{k,r} & := \{ u \in L^1_{loc}(\mathbb{R}^3) : \| \nabla^k u \|_{L^r} < \infty \}, \| u \|_{D^{k,r}} := \| \nabla^k u \|_{L^r}, \\
D^k & := D^{k,2}, D^1_0 := \{ u \in L^2 : \| \nabla u \|_{L^2} < \infty \}.
\end{align*}
\]

We define the system parameters \( P, f, \mu, \lambda \) as follows. For the pressure function \( P = P(\rho) \) and the external force \( f \), we assume that

\[
P(\rho) = a \rho^\gamma \quad \text{with constants} \quad a > 0 \quad \text{and} \quad \gamma \geq 1;
\]

\[
\text{there exists} \quad \psi \in H^2 \quad \text{such that} \quad f = \nabla \psi \quad \text{and} \quad \psi(x) \to 0 \quad \text{as} \quad |x| \to \infty.
\]

The viscosity coefficients \( \mu \) and \( \lambda \) are assumed to satisfy

\[
7 \mu > \lambda > 0.
\]
Next, we define $\bar{\rho}$ as mentioned at the beginning of this section. Given a constant density $\rho_\infty > 0$, we say that $(\bar{\rho}, 0)$ is a steady state solution to (1.1) if $\bar{\rho} \in C^2(\mathbb{R}^3)$, and the following holds:

$$\begin{align*}
\nabla P(\bar{\rho}(x)) &= \bar{\rho}(x) \nabla \psi(x), \\
\lim_{|x| \to \infty} \bar{\rho}(x) &= \rho_\infty.
\end{align*}$$

(1.10)

Given $\rho_\infty > 0$, we further assume that

$$- \int_0^{\rho_\infty} \frac{P'(\rho)}{\rho} d\rho < \inf_{x \in \mathbb{R}^3} \psi(x) \leq \sup_{x \in \mathbb{R}^3} \psi(x) < \int_{\rho_\infty}^{\infty} \frac{P'(\rho)}{\rho} d\rho,$$

(1.11)

and by solving (1.10), $\bar{\rho}$ can be expressed explicitly as follows:

$$\bar{\rho}(x) = \begin{cases} 
\rho_\infty \exp \left( \frac{1}{a} \psi(x) \right), & \text{for } \gamma = 1 \\
\left( \rho_\infty^{-1} + \frac{1}{a} \psi(x) \right)^{-1}, & \text{for } \gamma > 1.
\end{cases}$$

We recall a useful lemma from Li and Matsumura26 about the existence of steady state solution $(\bar{\rho}, 0)$ to (1.1) which can be stated as follows:

**Lemma 1.1.** Given $\rho_\infty > 0$, if we assume that $P, f, \psi$ satisfy (1.7)–(1.9) and (1.11), then there exists positive constants $\rho_1, \rho_2, \delta$ and a unique solution $\bar{\rho}$ of (1.10) satisfying $\bar{\rho} - \rho_\infty \in H^2 \cap W^{2, 6}$ and

$$\rho_1 < \rho_1 + \delta \leq \bar{\rho} \leq \rho_2 - \delta < \rho_2.$$  

(1.12)

From now on, we fix $\rho_\infty > 0$ and choose $\bar{\rho}$ satisfying Lemma 1.1. And for the sake of simplicity, we also write $P = P(\rho)$ and $P = P(\bar{\rho})$ unless otherwise specified.

We give the definitions for strong solution and maximal time of existence as follows.

**Definition 1.2.** We say that $(\rho, u)$ is a (local) strong solution of (1.1) if for some $T > 0$ and $q' \in (3, 6]$, we have

$$\begin{align*}
0 &\leq \rho \in C([0, T], W^{1, q'}), \quad \rho_1 \in C([0, T], L^{q'}), \\
u &\in C([0, T], D^1 \cap D^2) \cap L^2(0, T; D^{2, q'}), \\
\frac{1}{\rho^2}u_t &\in L^\infty(0, T; L^2), \quad u_t \in L^2(0, T; D^1).
\end{align*}$$

(1.13)

Furthermore, $(\rho, u)$ satisfy the following conditions:

- For all $0 \leq t_1 \leq t_2 \leq T$ and $C^1$ test functions $\varphi \in D(\mathbb{R}^3 \times (-\infty, \infty))$ which are Lipschitz on $\mathbb{R}^3 \times [t_1, t_2]$ with $\text{supp} \varphi(\cdot, \tau) \subset K$, $\tau \in [t_1, t_2]$, where $K$ is compact, it holds

$$\int_{\mathbb{R}^3} \rho(x, \cdot) \varphi(x, \cdot) dx \bigg|_{t_1}^{t_2} = \int_{t_1}^{t_2} \int_{\mathbb{R}^3} (\rho \varphi_t + \rho u \cdot \nabla \varphi) dx d\tau.$$

(1.14)

- For test functions $\varphi$ which are locally Lipschitz on $\mathbb{R}^3 \times [0, T]$ and for which $\varphi, \varphi_t, \nabla \varphi \in L^2(\mathbb{R}^3 \times (0, T))$, $\nabla \varphi \in L^\infty(\mathbb{R}^3 \times (0, T))$ and $\varphi(\cdot, T) = 0$, it holds

$$\begin{align*}
\int_{\mathbb{R}^3} (\rho u^i)(x, \cdot) \varphi(x, \cdot) dx \bigg|_{t_1}^{t_2} &= \int_{t_1}^{t_2} \int_{\mathbb{R}^3} \left[ \rho u^i \varphi_t + \rho u^i u \cdot \nabla \varphi + (P - \bar{P}) \varphi \varphi_x \right] dx d\tau \\
&- \int_{t_1}^{t_2} \int_{\mathbb{R}^3} \left[ \mu \nabla u^i \cdot \nabla \varphi + \lambda (\text{div}(u)) \varphi \varphi_x \right] dx d\tau \\
&+ \int_{t_1}^{t_2} \int_{\mathbb{R}^3} (\rho f - \nabla \varphi) \cdot \varphi dx d\tau.
\end{align*}$$

(1.15)
Definition 1.3. We define $T^* \in (0, \infty)$ to be the maximal time of existence of a strong solution $(\rho, u)$ to (1.1) if for any $0 < T < T^*$, $(\rho, u)$ solves (1.1) in $[0, T) \times \mathbb{R}^3$ and satisfies (1.13)–(1.15). Moreover, the conditions (1.13)–(1.15) fail to hold when $T = T^*$.

We are ready to state the following main results of this paper which are summarized in Theorems 1.4 and 1.5:

**Theorem 1.4.** Given $\rho_\infty > 0$, let $\bar{\rho}$ be the steady state solution to (1.10). Let $(\rho, u)$ be a strong solution to the Cauchy problem (1.1) satisfying (1.7)–(1.9) with $\gamma > 1$. Assume that the initial data $(\rho_0, u_0)$ satisfy

$$\rho_0 \geq 0, \quad \rho_0 - \bar{\rho} \in L^1 \cap H^1 \cap W^{1,q}, \quad u_0 \in D^1 \cap D^2,$$

for some $q > 3$ and the compatibility condition

$$-\mu \Delta u_0 - \lambda \nabla \text{div}(u_0) + \nabla P(\rho_0) - \rho_0 f = \rho_0^\frac{1}{2} g,$$

for some $g \in L^2$. If $T^* < \infty$ is the maximal time of existence, then we have

$$\lim_{T \to T^*} \left( \sup_{[\mathbb{R}^3 \times [0, T]} |\rho| + \left\| \rho^\frac{1}{2} u \right\|_{L^2(0, T; \mathbb{R}^3)} \right) = \infty,$$

for some $r, s$ that satisfy

$$\frac{2}{s} + \frac{3}{r} \leq 1, \quad r \in (3, \infty], \quad s > \frac{3}{2}.$$

**Theorem 1.5.** Let $(\rho, u)$ be a strong solution to the Cauchy problem (1.1) satisfying (1.7)–(1.9) with $\gamma = 1$. Assume that the initial data $(\rho_0, u_0)$ satisfy (1.16) and (1.17). Suppose that the initial density $\rho_0$ further satisfies

$$\inf_{x \in \mathbb{R}^3} \rho_0(x) > 0.$$

If $T^* < \infty$ is the maximal time of existence, then we have

$$\lim_{T \to T^*} \sup_{[\mathbb{R}^3 \times [0, T]} |\rho| = \infty.$$

The rest of the paper is organized as follows. In Section 2, we recall some known facts and useful inequalities which will be used in later analysis. In Section 3, we give the proof of Theorem 1.4 by obtaining some necessary bounds on the strong solutions. In Section 4, we give the proof of Theorem 1.5 by introducing a different approach for the isothermal case $\gamma = 1$.

## 2 | PRELIMINARIES

In this section, we give some known facts and useful inequalities. We first state the following local-in-time existence and uniqueness of strong solutions to (1.1) with nonnegative initial density (references can be found in Nash\(^1\) and Tani\(^2\)):

**Proposition 2.1.** Let $\bar{\rho}$ be a steady state solution and $(\rho_0 - \bar{\rho}, u_0)$ be given initial data satisfying (1.16) and (1.17), then there exists a positive time $T > 0$ and a unique strong solution $(\rho, u)$ to (1.1) defined on $\mathbb{R}^3 \times (0, T]$.

Next, we recall the following Gagliardo–Nirenberg inequalities:

**Proposition 2.2.** For $p \in [2, 6]$, $q \in (1, \infty)$ and $r \in (3, \infty)$, there exists some generic constant $C > 0$ such that for any $h_1 \in H^1$ and $h_2 \in L^q \cap D^{1,r}$, we have

$$\|h_1\|_{L^p} \leq C \|h_1\|_{L^2}^{\frac{6-p}{2}} \|\nabla h_1\|_{L^2}^{\frac{3p-6}{2}},$$

(2.1)

$$\|h_2\|_{L^q} \leq C \|h_2\|_{L^3}^{\frac{1}{3}} \|\nabla h_2\|_{L^3}^{\frac{q}{3}}.$$  

(2.2)
We also recall the following two canonical functions, namely, the effective viscous flux $F$ and vorticity $\omega$, which are defined by

$$\hat{\rho}F = (\mu + \lambda) \text{div } u - (P(\rho) - P(\hat{\rho})), \quad \omega = \omega^{jk} = u^j_{,k} - u^k_{,j}. \quad (2.3)$$

The following lemma gives some useful estimates on $u$ in terms of $F$ and $\omega$.

**Lemma 2.3.** For $r_1, r_2 \in (1, \infty)$ and $t > 0$, there exists a universal constant $C$ which depends on $r_1, r_2, \mu, \lambda, a, \gamma$, and $\hat{\rho}$ such that the following estimates hold:

$$\|\nabla F\|_{L^{r_1}} + \|\nabla \omega\|_{L^{r_1}} \leq C \left( \|\hat{\rho}^{\frac{1}{2}} u\|_{L^{r_1}} + \|\rho - \hat{\rho}\|_{L^{r_1}} \right)$$

$$\leq C \left( \|\hat{\rho}^{\frac{1}{2}} u\|_{L^{r_1}} + \|\rho^{\frac{1}{2}} u \cdot \nabla u\|_{L^{r_1}} + \|\rho - \hat{\rho}\|_{L^{r_1}} \right) \quad (2.4)$$

$$\|\nabla u(\cdot, t)\|_{L^{r_2}} \leq C \left( \|F(\cdot, t)\|_{L^{r_2}} + \|\omega(\cdot, t)\|_{L^{r_2}} + \|\rho - \hat{\rho}(\cdot, t)\|_{L^{r_2}} \right). \quad (2.5)$$

**Proof.** By the definitions of $F$ and $\omega$, and together with (1.1), $F$ and $\omega$ satisfy the elliptic equations

$$\Delta(\hat{\rho}F) = \text{div}(\rho \dot{h} - \rho f + \nabla P(\hat{\rho})) = \text{div}(\rho u_t + \rho u \cdot \nabla u - \rho f + \nabla P(\hat{\rho})), \quad (2.4)$$

where $\dot{h} := \partial h + u \cdot \nabla h$ is the material derivative on $h$. Hence by applying standard $L^p$-estimate, the estimates (2.4) and (2.5) follow. □

Finally, we recall the following inequality which was first proved in Beale et al. for the case $\text{div}(u) \equiv 0$ and was proved in Huang et al. for compressible flows.

**Proposition 2.4.** For $q \in (3, \infty)$, there is a positive constant $C$ which depends on $q$ such that the following estimate holds for all $u \in L^2 \cap D^{1,q}$:

$$\|\nabla u\|_{L^q} \leq C(\|\text{div}(u)\|_{L^q} + \|\nabla \times u\|_{L^q} \ln(e + \|\nabla^2 u\|_{L^q})) + C\|\nabla u\|_{L^q} + C, \quad (2.6)$$

where $e$ is the base of the natural logarithm.

### 3 PROOF OF THEOREM 1.4

In this section, we give the proof of Theorem 1.4. Let $(\rho, u)$ be a strong solution to the system (1.1) as described in Theorem 1.4. By performing standard $L^2$-energy estimate (see Suen for example), we readily have

$$\sup_{0 \leq t \leq T} \left( \left\| \rho^{\frac{1}{2}} u(\cdot, \tau) \right\|_{L^2}^2 + \int_0^T G(\rho(x, s)) ds \right) + \int_0^T \|\nabla u(\cdot, \tau)\|_{L^2}^2 d\tau \leq C_0, \quad (3.1)$$

for all $t \in [0, T^*)$, where $C_0$ depends on the initial data but is independent of both $t$ and $T^*$. Here, $G$ is a functional given by

$$\rho \int_0^\rho \frac{P(s) - P(\hat{\rho})}{s^2} ds = \rho \int_0^\rho \frac{as^2 - a\hat{\rho}^2}{s^2} ds.$$

In order to prove Theorem 1.4, for the sake of contradiction, suppose that (1.18) does not hold. Then there exists some constant $M_0 > 0$ such that

$$\lim_{T \rightarrow T^*} \left( \sup_{\mathbb{R}^3 \times [0, T]} \|\rho\| + \left\| \rho^{\frac{1}{2}} u \right\|_{L^1(0,T,L^r)} \right) \leq M_0. \quad (3.2)$$

We first obtain the estimates on $\nabla u$ and $u_t$ under (3.2):
Lemma 3.1. Assume that (3.2) holds, then for \( t \in [0, T^*] \), we have

\[
\sup_{0 \leq t \leq T} \| \nabla u(\cdot, t) \|_{L^2}^2 + \int_0^t \int_{\mathbb{R}^3} \rho \| u \|_{L^2}^2 \, dx \, dt \leq C,
\]

where and in what follows, \( C \) denotes a generic constant which depends on \( \mu, \lambda, a, f, \gamma, \beta, M_0, T^*, \) and the initial data.

Proof. We multiply the momentum Equation (1.1) by \( u_t \) and integrate to obtain

\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} \left( \mu \| \nabla u \|_{L^2}^2 + \lambda (\text{div}(u))^2 \right) dx + \int_{\mathbb{R}^3} \rho \| u_t \|_{L^2}^2 \, dx
= \int_{\mathbb{R}^3} P \text{div}(u) dx - \int_{\mathbb{R}^3} \rho u \cdot \nabla u \cdot u_t dx + \int_{\mathbb{R}^3} \rho f \cdot u_t dx.
\]

Using Hölder's inequality and Young's inequality, the term involving \( f \) can be bounded by

\[
\left| \int_{\mathbb{R}^3} \rho f \cdot u_t dx \right| \leq \left( \int_{\mathbb{R}^3} \rho \| u_t \|_{L^2}^2 \, dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^3} \rho \| f \|_{L^2}^2 \, dx \right)^{\frac{1}{2}}
\leq \frac{1}{2} \left( \int_{\mathbb{R}^3} \rho \| u_t \|_{L^2}^2 \, dx \right) + C \left( \int_{\mathbb{R}^3} \rho \| f \|_{L^2}^2 \, dx \right).
\]

Hence by following the steps given in Huang et al., we arrive at

\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} \left( \| \nabla u \|_{L^2}^2 + (\text{div}(u))^2 - P \text{div}(u) \right) dx + \frac{1}{2} \int_{\mathbb{R}^3} \rho \| u_t \|_{L^2}^2 \, dx \leq C \| \nabla u \|_{L^2}^2 + C \int_{\mathbb{R}^3} \rho \| u \cdot \nabla u \|_{L^2}^2 \, dx + C.
\]

To estimate the advection term on the right side of (3.4), for \( r, s \) satisfying (1.19), we use (2.1) and (2.5) to obtain

\[
\| \rho \frac{1}{2} u \cdot \nabla u \|_{L^r} \leq C \| \rho u \|_{L^r} \| \nabla u \|_{L^2}^\frac{r}{r-2}
\leq C \| \rho \frac{1}{2} u \|_{L^r} \left( \| F \|_{L^{2r}}^{\frac{1}{2}} \| \nabla F \|_{L^2}^\frac{1}{2} + \| \nabla \omega \|_{L^2}^{\frac{1}{2}} \| \nabla \omega \|_{L^2}^\frac{1}{2} + 1 \right).
\]

Using Young's inequality, for any \( \epsilon > 0 \) being small, there exists \( C_\epsilon > 0 \) such that

\[
\| \rho \frac{1}{2} u \|_{L^r} \left( \| F \|_{L^{2r}}^{\frac{1}{2}} \| \nabla F \|_{L^2}^\frac{1}{2} + \| \nabla \omega \|_{L^2}^{\frac{1}{2}} \| \nabla \omega \|_{L^2}^\frac{1}{2} + 1 \right)
\leq \epsilon (\| \nabla F \|_{L^2} + \| \nabla \omega \|_{L^2}) + C_\epsilon \| \rho \frac{1}{2} u \|_{L^r}^\frac{1}{2} (\| F \|_{L^2} + \| \nabla \omega \|_{L^2} + 1) + C_\epsilon,
\]

and hence, by applying (2.4), we obtain

\[
\| \rho \frac{1}{2} u \cdot \nabla u \|_{L^2} \leq C \| \rho u_t \|_{L^2} + C_\epsilon \| \rho \frac{1}{2} u \|_{L^r}^\frac{1}{2} (\| \nabla u \|_{L^2} + 1) + C_\epsilon.
\]

Applying (3.5) on (3.4) and choosing \( \epsilon > 0 \) small enough,

\[
\frac{d}{dt} \int_{\mathbb{R}^3} \left( \| \nabla u \|_{L^2}^2 + (\text{div}(u))^2 - P \text{div}(u) \right) dx + \frac{1}{2} \int_{\mathbb{R}^3} \rho \| u_t \|_{L^2}^2 \, dx
\leq C \left( \| \rho \frac{1}{2} u \|_{L^r}^\frac{1}{2} + 1 \right) (\| \nabla u \|_{L^2} + 1) + C \leq C (\| \nabla u \|_{L^2} + 1),
\]

where the last inequality follows by (3.2). Hence, the estimate (3.3) follows by using Grönwall's inequality on (3.6).
Next, we make use of Lemma 3.1 to obtain some higher order estimates on $u$ which can be stated in the following lemma:

**Lemma 3.2.** Assume that (3.2) holds, then for all $t \in [0, T^*)$, we have

\[
\sup_{0 \leq \tau \leq t} \rho \|\dot{\rho}(. \tau)\|^2_{L^2} + \int_0^t \int_{\mathbb{R}^3} \rho |\nabla \dot{\rho}|^2 d\tau \leq C. \tag{3.7}
\]

**Proof.** Following the steps given in Huang et al., we readily have

\[
\sup_{0 \leq \tau \leq t} \rho \|\dot{\rho}(. \tau)\|^2_{L^2} + \int_0^t \int_{\mathbb{R}^3} \rho |\nabla \dot{\rho}|^2 d\tau \leq C \int_0^t \|\nabla u(., \tau)\|^4_{L^4} d\tau + C \left( \int_{\mathbb{R}^3} |f| d\tau \right) + C.
\]

To estimate the term $\int_0^t \|\nabla u\|^4_{L^4} d\tau$, we apply (2.4) and (2.5) to get

\[
\int_0^t \|\nabla u\|^4_{L^4} d\tau \leq C \int_0^t (||F||^4_{L^4} + ||\omega||^4_{L^4}) d\tau + C \\
\leq C \int_0^t \left( ||F||^3_{L^4} ||\nabla F||^1_{L^2} + ||\omega||^3_{L^4} ||\nabla \omega||^1_{L^2} \right) d\tau + C \\
\leq C \int_0^t ||\nabla \dot{u}||^3_{L^2} d\tau + C;
\]

hence, by using Young’s inequality and Grönwall’s inequality, the estimate (3.7) holds for all $T \in [0, T^*)$. \hfill \Box

**Remark 3.3.** As pointed out in Section 1, it is important to use the effective viscous flux $F$ and the vorticity $\omega$ for estimating the term $\int_0^t \|\nabla u\|^4_{L^4} d\tau$, since there is no available a priori bound on $\|\nabla u\|_{H^1}$, and hence, we cannot merely apply the Sobolev embedding $H^1 \hookrightarrow L^4$ in Lemma 3.2.

We give the following estimate on the density gradient $\nabla \rho$ and the $H^1$ norm of $\nabla u$:

**Lemma 3.4.** Assume that (3.2) holds, then for all $t \in [0, T^*)$, we have

\[
\sup_{0 \leq \tau \leq t} (||\rho||_{H^1 \cap W^{1,q'}} + \|\nabla u\|_{H^1})(., \tau) \leq C, \tag{3.8}
\]

for all $q' \in (3, 6]$.

**Proof.** For any $p \in [2, 6]$, we have

\[
\frac{d}{dt} (|\nabla \rho|^p) + \operatorname{div}(|\nabla (\rho - \bar{\rho})|^p u) + (p - 1) |\nabla (\rho - \bar{\rho})|^p \operatorname{div}(u) \\
+ p |\nabla (\rho - \bar{\rho})|^{p-2} \nabla (\rho - \bar{\rho}) \nabla u |\nabla (\rho - \bar{\rho})| + p(\rho - \bar{\rho}) |\nabla (\rho - \bar{\rho})|^{p-2} \nabla (\rho - \bar{\rho}) \nabla \operatorname{div}(u) \\
= -p \nabla \operatorname{div}(\dot{\rho} u) \cdot (\nabla (\rho - \bar{\rho})) |\nabla (\rho - \bar{\rho})|^{p-2}.
\]

We integrate the above equation over $\mathbb{R}^3$ and use (2.1), (2.4), and (3.7) to obtain

\[
\frac{d}{dt} ||\nabla (\rho - \bar{\rho})||_{L^p} \leq C (1 + \|\nabla u\|_{L^\infty} + \|\nabla (\dot{\rho})\|_{L^p} + \|u\|_{L^p} + \|\nabla u\|_{L^p} ||\nabla (\rho - \bar{\rho})||_{L^p} \\
\leq C (1 + \|\nabla u\|_{L^\infty} + \|\nabla \dot{u}\|_{L^2}) ||\nabla (\rho - \bar{\rho})||_{L^p}. \tag{3.9}
\]

On the other hand, upon rearranging terms from the momentum Equation (1.1)$_2$, we have

\[
\mu \Delta u + \lambda \nabla \operatorname{div}(u) = \rho \dot{u} + \nabla (P(\rho) - P(\bar{\rho})) + \nabla \bar{P}(\bar{\rho} - \rho) \bar{\rho}^{-1}. \tag{3.10}
\]
Hence for each $q' \in (3, 6)$, by applying $L^{q'}$-estimate on $u$ in (3.10), we have

$$\|\nabla u\|_{L^{q'}} \leq C(1 + \|\nabla \tilde{u}\|_{L^2} + \|\nabla (\rho - \tilde{\rho})\|_{L^{q'}}).$$  (3.11)

Using the Sobolev inequality (2.2), together with the estimates (2.6) and (3.11), we have

$$\|\nabla u\|_{L^{\infty}} \leq C + C(\|\text{div}(u)\|_{L^{\infty}} + \|\omega\|_{L^{\infty}}) \ln(e + \|\nabla \tilde{u}\|_{L^2})$$
$$+ C(\|\text{div}(u)\|_{L^{\infty}} + \|\omega\|_{L^{\infty}}) \ln(e + \|\nabla (\rho - \tilde{\rho})\|_{L^{q'}}).$$  (3.12)

To estimate the time integral of $(\|\text{div}(u)\|_{L^{\infty}}^2 + \|\omega\|_{L^{\infty}}^2)$, using (2.2), (2.4), and (3.7), we readily have

$$\int_0^t (\|\text{div}(u)\|_{L^{\infty}}^2 + \|\omega\|_{L^{\infty}}^2)(\cdot, \tau)d\tau \leq C \int_0^t (\|F\|_{L^{\infty}}^2 + \|\omega\|_{L^{\infty}}^2)(\cdot, \tau)d\tau + C$$
$$\leq C \int_0^t \|\nabla \tilde{u}(\cdot, \tau)\|_{L^{2}}^2 d\tau + C \leq C.$$  (3.13)

Hence by applying (3.12) on (3.9) with $p = q'$, using Grönwall's inequality with the bounds (3.7) and (3.13), we obtain

$$\sup_{0 \leq \tau \leq t} \|\nabla (\rho - \tilde{\rho})(\cdot, \tau)\|_{L^{q'}} \leq C.$$  (3.14)

By combining (3.12) with (3.14) and (3.13), it further gives

$$\int_0^t \|\nabla u(\cdot, \tau)\|_{L^{\infty}} d\tau \leq C.$$  (3.15)

Integrating (3.9) with $p = 2$ over $t$ and together with (3.7) and (3.15), it follows that

$$\sup_{0 \leq \tau \leq t} \|\nabla (\rho - \tilde{\rho})(\cdot, \tau)\|_{L^{2}} \leq C.$$  (3.16)

which gives the bound on $\rho - \tilde{\rho}$ as claimed in (4). The bound on $u$ as appeared in (4) then follows from $L^2$-estimate on (3.10) with the bounds (3.3) and (3.16), and we finish the proof for (4). □

**Proof of Theorem 1.4.** The proof then follows from the same argument given in Huang et al.²⁰; namely, by choosing the function $(\rho, u)(x, T^*)$ to be the limit of $(\rho, u)(x, t)$ as $t \to T^*$, one can show that $(\rho, u)(x, T^*)$ satisfies the compatibility condition (1.17) as well. Therefore, if we take $(\rho, u)(x, T^*)$ to be the new initial data for the system $(1.1)$, then Proposition 2.1 applies and shows that the local strong solution can be extended beyond the maximal time $T^*$. □

## 4 | PROOF OF THEOREM 1.5

In this section, we prove Theorem 1.5 using a different approach compared with the proof of Theorem 1.4. We let $(\rho, u)$ be a strong solution to the system $(1.1)$ for the isothermal case as described in Theorem 1.5, and for the sake of contradiction, suppose that (1.21) does not hold. Then there exists a constant $\tilde{M}_0 > 0$ such that

$$\lim_{T \to T^*} \sup_{[0, T]} |\rho| \leq \tilde{M}_0.$$  (4.1)

Furthermore, together with the bound (3.15) on $\|\nabla u\|_{L^{\infty}}$ and the assumption (1.20) on $\rho_0$, we have

$$\inf_{\mathbb{R}^3 \times (0, T^*)} \rho \geq \tilde{M}_1,$$  (4.2)

where $\tilde{M}_1 > 0$ is a constant which depends on $\mu, \lambda, a, f, \tilde{\rho}, \tilde{M}_0, T^*$, and the initial data.
To facilitate our discussion, we introduce the following auxiliary functionals:

\[
\Phi_1(t) = \sup_{0 \leq \tau \leq t} \| \nabla u(\cdot, \tau) \|_{L^2}^2 + \int_0^t \| \frac{\partial}{\partial \tau} \hat{u}(\cdot, \tau) \|_{L^2}^2 \, d\tau, \tag{4.3}
\]

\[
\Phi_2(t) = \sup_{0 \leq \tau \leq t} \| \rho \frac{\partial}{\partial \tau} \hat{u}(\cdot, \tau) \|_{L^2}^2 + \int_0^t \| \nabla \hat{u}(\cdot, \tau) \|_{L^2}^2 \, d\tau, \tag{4.4}
\]

\[
\Phi_3(t) = \int_0^t \int_{\mathbb{R}^3} |\nabla u|^4 \, dx \, d\tau. \tag{4.5}
\]

We recall the following lemma which gives estimates on the solutions of the Lamé operator $\mu \Delta + \lambda \nabla \text{div}$. Details can be found in Sun et al.\textsuperscript{21}, pp. 39

**Lemma 4.1.** Consider the following equation:

\[
\mu \Delta v + \lambda \nabla \text{div}(v) = J, \tag{4.6}
\]

where $v = (v^1, v^2, v^3)(x)$, $J = (J^1, J^2, J^3)(x)$ with $x \in \mathbb{R}^3$ and $\mu$, $\lambda > 0$. Then for $r \in (1, \infty)$, we have:

- if $J \in W^{2,r}(\mathbb{R}^3)$, then $||\Delta v||_{L^r} \leq C ||J||_{L^r}$;
- if $J = \nabla \varphi$ with $\varphi \in W^{2,r}(\mathbb{R}^3)$, then $||\nabla v||_{L^r} \leq C ||\varphi||_{L^r}$;
- if $J = \nabla \text{div}(\varphi)$ with $\varphi \in W^{2,r}(\mathbb{R}^3)$, then $||v||_{L^r} \leq C ||\varphi||_{L^r}$.

Here, $C$ is a positive constant which depends on $\mu$, $\lambda$, and $r$.

One of the key for the proof of Theorem 1.5 is to estimate the $L^4$ norm of $\rho \frac{\partial}{\partial \tau} u$, and the results can be summarized in the following lemma:

**Lemma 4.2.** Assume that (4.1) holds, then for $t \in [0, T^*)$, we have

\[
\sup_{0 \leq \tau \leq t} \int_{\mathbb{R}^3} \rho |u|^4 \, dx \leq \tilde{C}, \tag{4.7}
\]

where, and in what follows, $\tilde{C}$ denotes a generic constant which depends on $\mu$, $\lambda$, $\alpha$, $\beta$, $\tilde{M}_0$, $T^*$, $\tilde{M}_1$, and the initial data.

**Proof.** It can be proved by the method given in Huang et al.\textsuperscript{20} (also refer to Hoff\textsuperscript{6} and Huang and Li\textsuperscript{27} for more details), and we omit here for the sake of brevity. We point out that the condition (1.9) is required for obtaining (4.7). \qed

We begin to estimate the functionals $\Phi_1$, $\Phi_2$, and $\Phi_3$. The following lemma gives an estimate on $\Phi_1$ in terms of $\Phi_3$:

**Lemma 4.3.** Assume that (4.1) holds. For any $0 \leq t < T^*$,

\[
\Phi_1(t) \leq \tilde{C}[1 + \Phi_3(t)]. \tag{4.8}
\]

**Proof.** Following the proof of Lemma 3.1, we have

\[
\sup_{0 \leq \tau \leq t} \int_{\mathbb{R}^3} |\nabla u|^2 \, dx + \int_0^T \int_{\mathbb{R}^3} \rho |\hat{u}|^2 \, dx \, d\tau \leq \tilde{C} + \tilde{C} \int_0^T \int_{\mathbb{R}^3} |\nabla u|^4 \, dx \, d\tau. \tag{4.9}
\]

The second term on the right side of (4.9) can be bounded by

\[
\int_0^T \int_{\mathbb{R}^3} |\nabla u|^4 \, dx \, d\tau \leq \left( \int_0^T \int_{\mathbb{R}^3} |\nabla u|^2 \, dx \, d\tau \right)^{\frac{1}{2}} \left( \int_0^T \int_{\mathbb{R}^3} |\nabla u|^4 \, dx \, d\tau \right)^{\frac{1}{2}} \leq \tilde{C}\Phi_3^{\frac{1}{2}}.
\]

Applying the above bounds on (4.9), the result follows. \qed
Before we estimate $\Phi_2$, we introduce the following decomposition on $u$ stated in Section 1. We write

$$u = u_p + u_s,$$  \hspace{1cm} (4.10)

where $u_p$ and $u_s$ satisfy (1.4), and we recall that $P := P(\bar{\rho})$. Then by using (4.1), for all $r > 1$, the term $u_p$ can be bounded by

$$\int_{\mathbb{R}^3} |\nabla u_p|^r \, dx \leq C \int_{\mathbb{R}^3} |P - P|^r \, dx \leq C \int_{\mathbb{R}^3} |\rho - \bar{\rho}|^r \, dx.$$  \hspace{1cm} (4.11)

On the other hand, the term $u_s$ can be estimated as follows.

**Lemma 4.4.** For any $0 \leq t < T^*$, we have

$$\sup_{0 \leq \tau \leq t} \int_{\mathbb{R}^3} |\nabla u_s|^2 \, dx \, d\tau + \int_0^T \int_{\mathbb{R}^3} \rho |\partial_t(u_s)|^2 \, dx \, d\tau + \int_0^T \int_{\mathbb{R}^3} |\Delta u|^2 \, dx \, d\tau \leq \tilde{C}. \hspace{1cm} (4.12)$$

**Proof.** We multiply (1.4) by $\partial_t(u_s)$ and integrate to obtain

$$\begin{align*}
&\int_{\mathbb{R}^3} \mu |\nabla u_s|^2 \, dx_0 + \int_0^T \int_{\mathbb{R}^3} (\mu + \lambda) \, \text{div} \, u_s |u|^2 \, dx \, d\tau + \int_0^T \int_{\mathbb{R}^3} \rho |\partial_t(u_s)|^2 \, dx \, d\tau \\
&= -\int_0^T \int_{\mathbb{R}^3} (\rho u - \nabla u) \cdot \partial_t(u_s) \, dx \, d\tau - \int_0^T \int_{\mathbb{R}^3} \rho \partial_t(u_p) \cdot \partial_t(u_s) \, dx \, d\tau \\
&\quad + \int_0^T \int_{\mathbb{R}^3} \bar{\rho}^{-1} (\rho - \bar{\rho}) \nabla P \cdot \partial_t(u_s) \, dx \, d\tau. \hspace{1cm} (4.13)
\end{align*}$$

We estimate the right side of (4.13) term by term. Using (4.7) and (4.11), the first integral can be bounded by

$$\begin{align*}
&\left( \int_0^T \int_{\mathbb{R}^3} \rho |u|^2 |\nabla u|^2 \, dx \, d\tau \right)^{1/4} \left( \int_0^T \int_{\mathbb{R}^3} \rho |\partial_t(u_s)|^2 \, dx \, d\tau \right)^{1/4} \\
&\leq \tilde{C} \left[ \int_0^T \left( \int_{\mathbb{R}^3} \rho |u|^4 \, dx \right)^{1/4} \left( \int_{\mathbb{R}^3} |\nabla u_s|^4 \, dx + \int_{\mathbb{R}^3} |\nabla u_p|^4 \, dx \right)^{1/4} \, d\tau \right]^{1/4} \\
&\quad \times \left( \int_0^T \int_{\mathbb{R}^3} \rho |\partial_t(u_s)|^2 \, dx \, d\tau \right)^{1/4} \\
&\leq \tilde{C} \left[ \int_0^T \left( \int_{\mathbb{R}^3} |\nabla u|^2 \, dx \right)^{1/4} \left( \int_{\mathbb{R}^3} |\Delta u|^2 \, dx \right)^{1/4} \, d\tau + \int_0^T \left( \int_{\mathbb{R}^3} |\rho - \bar{\rho}|^4 \, dx \right)^{1/4} \, d\tau \right]^{1/4} \\
&\quad \times \left( \int_0^T \int_{\mathbb{R}^3} \rho |\partial_t(u_s)|^2 \, dx \, d\tau \right)^{1/8} \\
&\leq \tilde{C} \left( \int_0^T \int_{\mathbb{R}^3} \rho |\partial_t(u_s)|^2 \, dx \, d\tau \right)^{1/8} \left( \int_0^T \int_{\mathbb{R}^3} |\Delta u|^2 \, dx \, d\tau \right)^{1/8} + 1.
\end{align*}$$

Next to estimate $-\int_0^T \int_{\mathbb{R}^3} (\rho \partial_t(u_p)) \cdot \partial_t(u_s) \, dx \, d\tau$, we differentiate (1.4) with respect to $t$ and use the assumption that $P(\rho) = a\rho$ to obtain

$$\mu \Delta \partial_t(u_p) + (\mu + \lambda) \text{div} \partial_t(u_p) = \nabla \text{div}(-P \cdot u).$$
Using Lemma 4.1 and the $L^2$-estimate (3.1) on $u$, we have

$$
\int_0^T \int_{\mathbb{R}^3} |\partial_t(u_p)|^2 \, dx \, dt \leq \tilde{C} \int_0^T \int_{\mathbb{R}^3} |P \cdot u|^2 \, dx \, dt \leq \tilde{C}.
$$

(4.14)

Therefore,

$$
- \int_0^T \int_{\mathbb{R}^3} (\rho \partial_t(u_p)) \cdot \partial_t(u_p) \, dx \, dt \leq \left( \int_0^T \int_{\mathbb{R}^3} |\rho \partial_t(u_p)|^2 \, dx \, dt \right)^{\frac{1}{2}} \left( \int_0^T \int_{\mathbb{R}^3} |\partial_t(u_p)|^2 \, dx \, dt \right)^{\frac{1}{2}} \leq \tilde{C} \left( \int_0^T \int_{\mathbb{R}^3} \rho |\partial_t(u_p)|^2 \, dx \, dt \right)^{\frac{1}{2}}.
$$

To estimate $\int_0^T \int_{\mathbb{R}^3} (\rho - \bar{\rho}) \nabla P \cdot \partial_t(u_p) \, dx \, dt$, using (3.1) and (4.2), we readily have

$$
\int_0^T \int_{\mathbb{R}^3} (\rho - \bar{\rho}) \nabla \bar{P} \cdot \partial_t(u_p) \, dx \, dt \leq \tilde{C} \left( \int_0^T \int_{\mathbb{R}^3} |\rho - \bar{\rho}|^2 \, dx \, dt \right)^{\frac{1}{2}} \left( \int_0^T \int_{\mathbb{R}^3} |\partial_t(u_p)|^2 \, dx \, dt \right)^{\frac{1}{2}} \leq \tilde{C} \left( \int_0^T \int_{\mathbb{R}^3} \rho |\partial_t(u_p)|^2 \, dx \, dt \right)^{\frac{1}{2}}.
$$

Combining the above, we have from (4.13) that

$$
\int_{\mathbb{R}^3} |\nabla u_p|^2(x, t) \, dx + \int_0^T \int_{\mathbb{R}^3} |\operatorname{div}(u_p)|^2 \, dx \, dt + \int_0^T \int_{\mathbb{R}^3} \rho |\partial_t(u_p)|^2 \, dx \, dt \leq \tilde{C} \left( \int_0^T \int_{\mathbb{R}^3} |\nabla u_p|^2 \, dx \, dt \right)^{\frac{1}{2}} \left( \int_0^T \int_{\mathbb{R}^3} |\Delta u_p|^2 \, dx \, dt \right)^{\frac{1}{2}} + \tilde{C}.
$$

(4.15)

It remains to estimate the term $\int_0^T \int_{\mathbb{R}^3} |\Delta u_p|^2$. Rearranging the terms in (1.4), we have that

$$
\mu \Delta u_p + (\mu + \lambda) \nabla \operatorname{div}(u_p) = \rho \partial_t(u_p) + \rho u \cdot \nabla u + \rho \partial_t(u_p) - \bar{\rho}^{-1}(\rho - \bar{\rho}) \nabla \bar{P}.
$$

Therefore, we can apply Lemma 4.1 and the bound (3.1) to get

$$
\int_0^T \int_{\mathbb{R}^3} |\Delta u_p|^2 \, dx \, dt \leq \tilde{C} \left[ \int_0^T \int_{\mathbb{R}^3} \left( |\rho \partial_t(u_p)|^2 + |\rho u \cdot \nabla u|^2 + |\rho \partial_t(u_p)|^2 + |\rho - \bar{\rho}|^2 \right) \, dx \, dt \right] \leq \tilde{C} \left( \int_0^T \int_{\mathbb{R}^3} \rho |\partial_t(u_p)|^2 \, dx \, dt + 1 \right).
$$

(4.16)

Applying the estimate (4.16) on (4.15) and using Gröwall’s inequality, we conclude that for $0 \leq t < T^*$,

$$
\int_{\mathbb{R}^3} |\nabla u_p|^2(x, t) \, dx \leq \tilde{C},
$$

and the result (4.12) follows.

We now give the estimate on $\Phi_2$ as defined in (4.4), which is given in the following lemma:

**Lemma 4.5.** Assume that (4.1) holds. For any $0 \leq t < T^*$,

$$
\Phi_2(t) \leq \tilde{C}[\Phi_1(t) + \Phi_3(t) + 1].
$$

(4.17)
Proof. Following the steps given in Hoff\(^6\), pp. 228-230 (by taking \(\sigma \equiv 1\)), we have

\[
\int_{\mathbb{R}^3} |\hat{u}(x, t)|^2 \, dx + \int_0^T \int_{\mathbb{R}^3} |\nabla \hat{u}|^2 \, dx \, \tau \leq \tilde{C} \left[ \sum_{1 \leq k, l \leq 3} \int_0^T \int_{\mathbb{R}^3} u_{x_k}^l u_{x_l}^l \, dx \, \tau \right]
\]

\[
+ \tilde{C} \left( \int_0^T \int_{\mathbb{R}^3} |u|^4 \, dx \, \tau + \Phi_1(t) + 1 \right).
\]

(4.18)

The summation term in (4.18) can be bounded by \(\tilde{C} \int_0^T \int_{\mathbb{R}^3} |\nabla u|^4\), and hence, it can be bounded by \(\tilde{C} \Phi_3^2\). The estimate (4.17) then follows by Cauchy's inequality.

Finally, we make use of \(u_s\) and \(u_p\) in (1.4) to estimate \(\Phi_3\):

**Lemma 4.6.** For any \(0 \leq t < T^*\),

\[
\Phi_3(t) \leq \tilde{C} \left[ \Phi_1(t)^2 + 1 \right].
\]

(4.19)

**Proof.** Using the decomposition (4.10) on \(u\) and the estimates (4.11) and (4.12), we have

\[
\Phi_3 \leq \int_0^T \int_{\mathbb{R}^3} |\nabla u_s|^4 \, dx \, \tau + \int_0^T \int_{\mathbb{R}^3} |\nabla u_p|^4 \, dx \, \tau
\]

\[
\leq \tilde{C} \int_0^T \left( \int_{\mathbb{R}^3} |\nabla u_s|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^3} |\Delta u_s|^2 \, dx \right)^{\frac{1}{2}} \, \tau + \int_0^T \int_{\mathbb{R}^3} |\rho - \tilde{\rho}|^4 \, dx \, \tau
\]

\[
\leq \tilde{C} \left[ \left( \sup_{0 \leq t \leq T} \int_{\mathbb{R}^3} |\Delta u_s(x, \tau)|^2 \, dx \right)^{\frac{1}{2}} + 1 \right].
\]

To estimate \(\int_{\mathbb{R}^3} |\Delta u_s|^2\), we rearrange the terms in (1.4) to obtain

\[
\mu \Delta u_s + (\mu + \lambda) \nabla \text{div}(u_s) = \rho u - \rho \nabla \phi.
\]

Therefore, Lemma 4.1 implies that

\[
\int_{\mathbb{R}^3} |\Delta u_s|^2 \, dx \leq \tilde{C} \left[ \int_{\mathbb{R}^3} (|\rho|^2 + |\rho \nabla \phi|^2) \, dx \right] \leq \tilde{C} (\Phi_2 + 1),
\]

and the result follows.

**Proof of Theorem.** In view of the bounds (4.8), (4.17), and (4.19), one can conclude that for \(0 \leq t < T^*\),

\[
\Phi_1(t) + \Phi_2(t) + \Phi_3(t) \leq \tilde{C}.
\]

(4.20)

Hence, using the bound (4.20) and applying the same argument given in the proof of Lemma 3.4, for \(T \in [0, T^*]\) and \(q \in (3, 6]\), we also have

\[
\sup_{0 \leq t \leq T} (\|\rho\|_{H^q W^{1,q}} + \|\nabla u\|_{H^q}) \leq \tilde{C}.
\]

Therefore, similar to the proof of Theorem 1.4, we can extend the strong solution \((\rho, u)\) beyond \(t = T^*\), which leads to a contradiction. This completes the proof of Theorem 1.5.

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