A LOCAL VERSION OF KATONA’S INTERSECTION THEOREM

MARCELO SALES AND BJARNE SCHÜLKE

ABSTRACT. Katona’s intersection theorem states that every intersecting family \( \mathcal{F} \subseteq [n]^{(k)} \) satisfies \( |\partial \mathcal{F}| \geq |\mathcal{F}| \), where \( \partial \mathcal{F} = \{ F \setminus x : x \in F \in \mathcal{F} \} \) is the shadow of \( \mathcal{F} \). Frankl conjectured that for \( n > 2k \) and every intersecting family \( \mathcal{F} \subseteq [n]^{(k)} \), there is some \( i \in [n] \) such that \( |\partial \mathcal{F}(i)| \geq |\mathcal{F}(i)| \), where \( \mathcal{F}(i) = \{ F \setminus i : i \in F \in \mathcal{F} \} \) is the link of \( \mathcal{F} \) at \( i \). Here, we prove this conjecture in a very strong form for \( n \geq k \geq 2 \). In particular, our result implies that for any \( j \in [k] \), there is a \( j \)-set \( \{a_1, \ldots, a_j\} \in [n]^{(j)} \) such that \( |\partial \mathcal{F}(a_1, \ldots, a_j)| \geq |\mathcal{F}(a_1, \ldots, a_j)| \). A similar statement is also obtained for cross-intersecting families.

§1. Introduction

Throughout the paper, let \( n, k, \ell \) be positive integers. Let \( [n] = \{1, \ldots, n\} \) and for a set \( X \) let \( X^{(k)} = \{ A \subseteq X : |A| = k \} \) be the set of \( k \)-subsets of \( X \). A family \( \mathcal{F} \subseteq [n]^{(k)} \) is called intersecting if \( F \cap F' \neq \emptyset \) for all \( F, F' \in \mathcal{F} \) and the shadow of \( \mathcal{F} \) is

\[
\partial \mathcal{F} = \{ F \setminus x : x \in F \in \mathcal{F} \}.
\]

Extremal properties of shadows and intersecting families are amongst the most prominent topics in extremal set theory. For instance, two cornerstones of the area are the Erdős–Ko–Rado theorem [7], which determines the maximum size of an intersecting family and the Kruskal–Katona theorem [8, 9], which provides a solution for the minimisation problem of the shadow.

The following celebrated theorem due to Katona [5] combines these two concepts by bounding the size of the shadow of an intersecting family.

**Theorem 1.1** ([5]). Suppose \( \mathcal{F} \subseteq [n]^{(k)} \) is intersecting. Then \( |\partial \mathcal{F}| \geq |\mathcal{F}| \).

Theorem 1.1 was proved in a more general setting in [5]. Recently, the result was improved by Frankl and Katona [4] and Liu and Mubayi [6] for intersecting families of larger size. It is also worth to note the cross-intersecting variant of Theorem 1.1 that Frankl proved in [1]. Given integers \( k, \ell \geq 1 \), a pair of families \( \mathcal{F} \subseteq [n]^{(k)}, \mathcal{G} \subseteq [n]^{(\ell)} \) is cross-intersecting if for every \( F \in \mathcal{F} \) and \( G \in \mathcal{G} \) we have \( F \cap G \neq \emptyset \).

**Key words and phrases.** Extremal set theory, intersecting families, shadow.
**Theorem 1.2** ([1]). Let \( 1 \leq k, \ell \leq n \) be positive integers and \( \mathcal{F} \subseteq [n]^{(k)} \) and \( \mathcal{G} \subseteq [n]^{(\ell)} \) be cross-intersecting families. Then either \(|\partial \mathcal{F}| \geq |\mathcal{F}| \) or \(|\partial \mathcal{G}| \geq |\mathcal{G}|\).

In this paper we establish a local version of Theorem 1.1. For a family \( \mathcal{F} \subseteq [n]^{(k)} \) and sets \( A, B \subseteq [n] \), we define

\[
\mathcal{F}(A, B) = \{ F \setminus A : F \in \mathcal{F} \text{ such that } A \subseteq F \text{ and } B \cap F = \emptyset \}
\]

as the link of \( \mathcal{F} \) at \( A \) induced on \([n] \setminus B\). Notice that for \( B = \emptyset \), the family \( \mathcal{F}(A) := \mathcal{F}(A, \emptyset) \) is just the usual link of \( \mathcal{F} \) at \( A \). If on the other hand \( A = \emptyset \), the family \( \mathcal{F}(B) := \mathcal{F}(\emptyset, B) \) is just the induced hypergraph on the set \([n] \setminus B\). Observe that \( \partial(\mathcal{F}(A)) = (\partial \mathcal{F})(A) \) and \( \partial(\mathcal{F}(A, B)) \subseteq (\partial \mathcal{F})(A, B) \). We write \( \partial \mathcal{F}(A, B) := \partial(\mathcal{F}(A, B)) \).

Frankl [3] conjectured the following local version of Theorem 1.1: Let \( n > 2k \) and let \( \mathcal{F} \subseteq [n]^{(k)} \) be an intersecting family. Then there exists a vertex \( i \in [n] \) such that the link of \( i \) satisfies \(|\partial \mathcal{F}(i)| \geq |\mathcal{F}(i)|\). We prove this conjecture for \( n > \binom{k+1}{2} \).

**Theorem 1.3.** Let \( n > \binom{k+1}{2} \) and let \( \mathcal{F} \subseteq [n]^{(k)} \) be intersecting. Then there exists an \( i \in [n] \) such that \(|\partial \mathcal{F}(i)| \geq |\mathcal{F}(i)|\).

Frankl [3] further conjectured that a local version of Theorem 1.2 should hold. We prove it for \( n > k\ell \).

**Theorem 1.4.** Suppose that for \( n > k\ell \), the families \( \mathcal{F} \subseteq [n]^{(k)} \) and \( \mathcal{G} \subseteq [n]^{(\ell)} \) are cross-intersecting. Then there is some \( i \in [n] \) such that

\[
|\partial \mathcal{F}(i)| \geq |\mathcal{F}(i)| \text{ or } |\partial \mathcal{G}(i)| \geq |\mathcal{G}(i)|.
\]

Theorems 1.3 and 1.4 will be deduced from a more general result. To formulate it compactly, let us further introduce the following definition.

**Definition 1.5.** We say that \( \mathcal{F} \subseteq [n]^{(k)} \) is pseudo-intersecting if \(|\partial \mathcal{F}(X)| \geq |\mathcal{F}(X)|\) for all \( X \subseteq [n] \).

Note that for \( A \subseteq [n] \) and \( \mathcal{F} \subseteq [n]^{(k)} \), the link \( \mathcal{F}(A) \) being pseudo-intersecting means that \(|\partial \mathcal{F}(A, X \setminus A)| \geq |\mathcal{F}(A, X \setminus A)|\) for every \( X \subseteq [n] \). The term pseudo-intersecting comes from the following observation: If \( \mathcal{F} \subseteq [n]^{(k)} \) is intersecting, then for every \( X \subseteq [n] \) we have \(|\partial \mathcal{F}(X)| \geq |\mathcal{F}(X)|\). This is a consequence of Theorem 1.1 and the fact that \( \mathcal{F}(X) \subseteq \mathcal{F} \) is intersecting. Note that if \( \mathcal{F}(A) \) is pseudo-intersecting for \( A \in [n]^{(j)} \) with \( j < k \), then \(|\partial \mathcal{F}(A)| \geq |\mathcal{F}(A)|\). That is, the pseudo-intersecting property of \( \mathcal{F}(i) \) implies the local version of Katona’s intersecting theorem.

The next result shows that one can always find a pseudo-intersecting link in an intersecting family. In particular, it implies Theorem 1.3 for \( n > \binom{k+1}{2} \).
Theorem 1.6. Let $\mathcal{F} \subseteq [n]^{(k)}$ be intersecting. Then there are sets $M_1 \subseteq \cdots \subseteq M_k \subseteq [n]$ with $|M_i| \geq n - \sum_{i \leq j \leq k} j$ such that for all $A \subseteq M_i^{(i)}$, the family $\mathcal{F}(A)$ is pseudo-intersecting.

Note that the link of every subset of $M_1$ is pseudo-intersecting. In particular, for any intersecting family $\mathcal{F} \subseteq [n]^{(k)}$, the inequality $|\partial \mathcal{F}(i)| \geq |\mathcal{F}(i)|$ holds for all but at most $\binom{k+1}{2}$ vertices $i$.

Our general result for cross-intersecting families reads as follows. For $n > k\ell$ it implies Theorem 1.4.

Theorem 1.7. Let $\mathcal{F} \subseteq [n]^{(k)}$ and $\mathcal{G} \subseteq [n]^{(\ell)}$ be cross-intersecting. Then there are sets $M_1 \subseteq \cdots \subseteq M_k \subseteq [n]$ with $|M_i| \geq n - (k+1-i)\ell$ such that one of the following holds.

- The family $\mathcal{F}(A)$ is pseudo-intersecting for all $i \in [k]$ and $A \subseteq M_i^{(i)}$,
- or $\mathcal{G}(B)$ is pseudo-intersecting for all $B \subseteq M_2$.

A family $\mathcal{F} \subseteq \mathcal{P}([n])$ is called $t$-union if $|F \cup F'| \leq t$ for every $F, F' \in \mathcal{F}$. A family $\mathcal{F}$ is an antichain if $F \nsubseteq F'$ for every $F, F' \in \mathcal{F}$. Kiselev, Kupavskii and Patkós made the following conjecture on the minimum degree of $(2\ell+1)$-union antichains.1

Conjecture 1.8 ([3]). Suppose that $1 \leq 2\ell + 1 < n$, and that $\mathcal{F} \subseteq \mathcal{P}([n])$ is a $(2\ell+1)$-union antichain. Then $\delta(\mathcal{F}) \leq \binom{n-1}{\ell-1}$.

In [3], Frankl solved Conjecture 1.8 for $n \geq \ell^3 + \ell^2 + \frac{3}{2}\ell$. He also noted that one could obtain better bounds by proving a local version of Theorem 1.1 (see Proposition 3.4(i), [3]). In particular, by using his reduction, Theorem 1.3 implies Conjecture 1.8 for $n > \binom{\ell+2}{2}$.

§2. Tools

In this section we introduce the main technical lemma in the proof. Roughly speaking, in our proof we will inductively construct sets $M_i \subseteq M_{i+1}$ such that $\mathcal{F}(A)$ is pseudo-intersecting for all $A \subseteq M_i^{(i)}$. The following lemma will help with the induction step. We directly formulate it in the setup in which it will be used.

Lemma 2.1. Let $\mathcal{F} \subseteq [n]^{(k)}$ and let $A, M \subseteq [n]$. If $\mathcal{F}(A, \overline{M \setminus A})$ is pseudo-intersecting and $\mathcal{F}(A \cup x)$ is pseudo-intersecting for all $x \in M \setminus A$, then $\mathcal{F}(A)$ is pseudo-intersecting.

Proof. Let $X \subseteq [n]$ be given. We need to show that $|\partial \mathcal{F}(A, \overline{X \setminus A})| \geq |\mathcal{F}(A, \overline{X \setminus A})|$. Note that $|\partial \mathcal{F}(A, (\overline{X \cup M} \setminus A))| \geq |\mathcal{F}(A, (\overline{X \cup M} \setminus A))|$ holds by assumption. Hence, it is enough to show that if

$$|\partial \mathcal{F}(A, \overline{X' \setminus A})| \geq |\mathcal{F}(A, \overline{X' \setminus A})|$$

\[ 2.1 \]

1They formulated their conjecture as an upper bound on the diversity of an intersecting antichain, which is equivalent.
holds for some $X \subseteq X' \subseteq X \cup M$, then $|\partial F(A, (X' \setminus x) \setminus A)| \geq |F(A, (X' \setminus x) \setminus A)|$ for some $x \in X' \setminus (X \cup A)$. So let $X'$ with $X \subseteq X' \subseteq X \cup M$ satisfy (2.1) and let $x \in X' \setminus (X \cup A)$ be arbitrary. Given a family $\mathcal{H}$ and a vertex $x$, let $\mathcal{H}^{\setminus x} = \{H \cup \{x\} : H \in \mathcal{H}\}$. Observe that

$$F(A, (X' \setminus x) \setminus A) = F(A, X' \setminus A) \cup (F(A \cup x, X' \setminus (A \cup x)))^{\setminus x} \quad (2.2)$$

and

$$|\partial F(A, (X' \setminus x) \setminus A)| \geq |F(A, (X' \setminus x) \setminus A)| \quad (2.3)$$

Since we have (2.1) and since

$$|\partial F(A \cup x, X' \setminus (A \cup x))| \geq |F(A \cup x, X' \setminus (A \cup x))|$$

holds because $F(A \cup x)$ is pseudo-intersecting, (2.2) and (2.3) imply that

$$|\partial F(A, (X' \setminus x) \setminus A)| \geq |F(A, (X' \setminus x) \setminus A)|.$$

This is all we had to show. \hfill \Box

\section*{§3. Proof of Theorems 1.6 and 1.7}

\textbf{Proof of Theorem 1.6.} The proof proceeds inductively by constructing sets $M_i \subseteq M_{i+1}$ such that for all $A \in M_{i}^{(i)}$, the family $F(A)$ is pseudo-intersecting and $|M_i| \geq n - \sum_{i \leq j < k} j$.

We begin the backward induction with $i = k$. If $\mathcal{F} \neq \emptyset$, we would be done, so let $F \in \mathcal{F}$ and set $M_k = [n] \setminus F$ (in particular, $|M_k| \geq n - k$). Since $\mathcal{F}$ is intersecting, we have $\mathcal{F} \cap M_k^{(k)} = \emptyset$. Thus, for $A \in M_k^{(k)}$ and $X \subseteq [n]$ we have $F(A, X \setminus A) = \emptyset$. Hence, $|\partial F(A, X \setminus A)| \geq |F(A, X \setminus A)|$ for all $X \subseteq [n]$, meaning that $F(A)$ is pseudo-intersecting for $A \in M_k^{(k)}$.

Now assume that for some $i$ with $2 \leq i \leq k$, a set $M_i \subseteq [n]$ with $|M_i| \geq n - \sum_{i \leq j < k} j$ has been defined such that $F(A^+)$ is pseudo-intersecting for all $A^+ \in M_{i}^{(i)}$. Next, we will argue that we only need to delete at most one $(i - 1)$-set from $M_i$ to obtain a set $M_{i-1}$ as desired.

If for all $A \in M_{i}^{(i-1)}$ the family $F(A, M_i \setminus A)$ is pseudo-intersecting, then we set $M_{i-1} = M_i$. Since the induction hypothesis tells us that for every $A \in M_{i}^{(i-1)}$, and $x \in M_i \setminus A$, the family $F(A \cup x)$ is pseudo-intersecting, Lemma 2.1 implies that $F(A)$ is pseudo-intersecting. So let us assume that $F(B, M_i \setminus B)$ is not pseudo-intersecting for some $B \in M_{i}^{(i-1)}$. Then there is some $M \subseteq [n]$ with $M_i \subseteq M$ such that $|\partial F(B, M \setminus B)| < |F(B, M \setminus B)|$ and we set $M_{i-1} = M_i \setminus B$.

\textbf{Claim 3.1.} For every $A \in M_{i-1}^{(i-1)}$, the family $F(A, M_{i} \setminus A)$ is pseudo-intersecting.
Theorem

We set \( M \) yields that \( M \) is pseudo-intersecting.

Lemma

us that families (similarly as in the proof of Claim 2.1.2) for \( B \), \( A \) and \( F \) are cross-intersecting. Theorem 1.2 implies that \( |\partial H| \geq |H| \) has to hold for some \( H \in \{\mathcal{F}(A, M' \setminus A), \mathcal{F}(B, M \setminus B)\} \). By the choice of \( B \) and \( M \), this yields the statement of the claim. \( \Box \)

Together with the induction hypothesis, this claim allows us to apply Lemma 2.1 which yields that \( \mathcal{F}(A) \) is pseudo-intersecting for all \( A \in M_{i-1}^{(i-1)} \). Further note that in either case \( |M_{i-1}| \geq |M_i| - i + 1 \geq n - \sum_{i-1 \leq j \leq k} j \). Therefore, in either case \( M_{i-1} \) is as desired. \( \Box \)

Proof of Theorem 1.7. Again, we aim to inductively construct sets \( M_i \subseteq M_{i+1} \) such that for all \( A \in M_i^{(i)} \), the family \( \mathcal{F}(A) \) is pseudo-intersecting and \( |M_i| \geq n - (k - i + 1)\ell \). If at any point, we should not be able to proceed, i.e., we fail to construct the set \( M_{i-1} \), then \( \mathcal{G}(A) \) will be pseudo-intersecting for all \( A \subseteq M_i \).

We begin the backwards induction with \( i = k \). If \( \mathcal{G} = \emptyset \), we are done, so let \( G \in \mathcal{G} \) and set \( M_k = [n] \setminus G \). Since \( \mathcal{F} \) and \( \mathcal{G} \) are cross-intersecting, this means that \( \mathcal{F} \cap M_k^{(k)} = \emptyset \) and therefore, \( \mathcal{F}(A) \) is pseudo-intersecting for all \( A \in M_k^{(k)} \). Further, we have \( |M_k| \geq n - \ell \).

Now assume that for some \( i \) with \( 2 \leq i \leq k \) we have constructed sets \( M_j \) for all \( i \leq j \leq k \) as desired. First assume that \( \mathcal{G}(B, M_i \setminus B) \) is not pseudo-intersecting for some \( B \in M_i^{(\leq \ell)} \), i.e., there is some \( X \subseteq [n] \) with \( M_i \subseteq X \) such that \( |\partial \mathcal{G}(B, X \setminus B)| < |\mathcal{G}(B, X \setminus B)| \). Then we set \( M_{i-1} = M_i \setminus B \) and readily notice that \( |M_{i-1}| \geq n - (k - i + 2)\ell \). Further, observe that (similarly as in the proof of Claim 3.1) for all \( A \in M_{i-1}^{(i-1)} \) and \( M_i \subseteq M' \subseteq [n] \), the families \( \mathcal{F}(A, M' \setminus A) \) and \( \mathcal{G}(B, X \setminus B) \) are cross-intersecting since \( A \cap B = \emptyset \), \( A \cup B \subseteq M_i \subseteq M' \), \( X \), and since \( \mathcal{F} \) and \( \mathcal{G} \) are cross-intersecting. Thus, by the choice of \( B \) and \( X \), Theorem 1.2 yields that \( \mathcal{F}(A, M_i \setminus A) \) is pseudo-intersecting. Since the induction gives us that \( \mathcal{F}(A^+) \) is pseudo-intersecting for all \( A^+ \in M_i^{(i)} \), we are now in a position to apply Lemma 2.1 to conclude that \( \mathcal{F}(A) \) is pseudo-intersecting for all \( A \in M_{i-1}^{(i-1)} \).

Next assume that \( \mathcal{G}(B, M_i \setminus B) \) is pseudo-intersecting for all \( B \in M_i^{(\leq \ell)} \). In this case we set \( M_1 = M_2 = \cdots = M_i \).

Claim 3.2. The family \( \mathcal{G}(B) \) is pseudo-intersecting for all \( B \subseteq M_i = M_2 \).

Proof. For \( B \subseteq M_i \) with \( |B| > \ell \), the statement follows immediately. We proceed by backwards induction on \( j = |B| \) and begin with \( j = \ell \). The induction start follows because for \( B \) of size \( \ell \), we have \( \mathcal{G}(B, X \setminus B) = \mathcal{G}(B, M_i \setminus B) \) for all \( X \subseteq [n] \) and \( \mathcal{G}(B, M_i \setminus B) \) is pseudo-intersecting.
Given that for some \( j \leq \ell \), the family \( \mathcal{G}(B^+) \) is pseudo-intersecting for all \( B^+ \in M_i^{(j)} \), we can apply Lemma 2.1 (since we are in the case that \( \mathcal{G}(B, M_i \setminus B) \) is pseudo-intersecting for all \( B \in M_i^{(\leq \ell)} \)) to conclude that \( \mathcal{G}(B) \) is pseudo-intersecting for all \( B \in M_i^{(j-1)} \) which finishes the induction step.

Thus, we have shown that if we cannot construct all sets \( M_1, \ldots, M_k \) as desired by induction, then there is a set \( M_2 \subseteq [n] \) with \( |M_2| \geq n - (k - 1)\ell \) such that \( \mathcal{G}(B) \) is pseudo-intersecting for all \( B \subseteq M_2 \). In other words, we proved that indeed one of the statements in the theorem has to hold. \( \square \)

**Acknowledgments**

The authors thank Alexandre Perozim de Faveri for fruitful discussions and Peter Frankl and Andrey Kupavskii for reading earlier versions of this paper.

**References**

[1] P. Frankl, *Generalizations of theorems of Katona and Milner*, Acta Math. Acad. Sci. Hungar. **27** (1976), no. 3-4, 359–363, DOI 10.1007/BF01902114. MR414370 [1, 1.2]

[2] P. Frankl, *The shifting technique in extremal set theory*, Surveys in combinatorics 1987 (New Cross, 1987), London Math. Soc. Lecture Note Ser., vol. 123, Cambridge Univ. Press, Cambridge, 1987, pp. 81–110. MR905277 [1]

[3] P. Frankl, *Minimum degree and diversity in intersecting antichains*, Acta Math. Hungar. **163** (2021), no. 2, 652–662, DOI 10.1007/s10474-020-01100-y. MR4227804 [1, 1.8, 1]

[4] P. Frankl and G. O. H. Katona, *On strengthenings of the intersecting shadow theorem*, J. Combin. Theory Ser. A **184** (2021), Paper No. 105510, 21, DOI 10.1016/j.jcta.2021.105510. MR4297030 [1]

[5] G. Katona, *Intersection theorems for systems of finite sets*, Acta Math. Acad. Sci. Hungar. **15** (1964), 329–337, DOI 10.1007/BF01897141. MR168468 [1, 1.1, 1]

[6] X. Liu and D. Mubayi, *Tight bounds for Katona’s shadow intersection theorem*, European J. Combin. **97** (2021), Paper No. 103391, 17, DOI 10.1016/j.ejc.2021.103391. MR4282634 [1]

[7] P. Erdős, C. Ko, and R. Rado, *Intersection theorems for systems of finite sets*, Quart. J. Math. Oxford Ser. (2) **12** (1961), 313–320, DOI 10.1093/qmath/12.1.313. [1]

[8] J. B. Kruskal, *The number of simplices in a complex*, Mathematical optimization techniques, 1963, pp. 251–278. [1]

[9] G. Katona, *A theorem of finite sets*, Theory of graphs (Proc. Colloq., Tihany, 1966), 1968, pp. 187–207. [1]

Department of Mathematics, Emory University, Atlanta, USA
Email address: marcelo.tadeu.sales@emory.edu

Department of Mathematics, California Institute of Technology, Pasadena, USA
Email address: schuelke@caltech.edu