Abstract

The visual complexity of a plane graph drawing is defined to be the number of basic geometric objects needed to represent all its edges. In particular, one object may represent multiple edges (e.g., one needs only one line segment to draw a path with an arbitrary number of edges). A crossing-free straight-line drawing of a graph is called monotone if there is a monotone path between any pair of vertices with respect to some direction. We study drawings of trees, outerplanar graphs, and general planar graphs with few segments on a polynomial size grid. For trees, the grid size is $n \times n$. For 3-connected planar graphs and biconnected outerplanar graphs we compute such drawings that are also monotone.

1 Introduction

The quality of a graph drawing can be assessed in a variety of ways: area, crossing number, bends, angular resolution, etc. All these measures have their justification, but in general it is challenging to optimize all of them in a single drawing. Recently, the visual complexity was suggested as another quality measure for drawings [34]. The visual complexity denotes the number of simple geometric entities used in the drawing.

We consider segment drawings, that is, all entities are straight-line segments. The idea is that we can use, for example, a single segment to draw a path of collinear edges. The hope is that a drawing that consists of only a few geometric entities is easy to perceive. A recent user study [25] suggests that visual complexity may positively influence aesthetics, depending on the background of the observer, as long as it does not introduce unnecessarily sharp corners. It is natural to ask for a drawing of a graph with the smallest visual complexity. Unfortunately, it is an NP-hard problem to determine the smallest number of segments necessary in a segment drawing [11]. However, we can still expect to prove bounds for certain graph classes.

A path $P$ in a straight-line drawing of a graph is monotone if there exists a line $l$ such that the orthogonal projections of the vertices of $P$ on $l$ appear along $l$ in the order induced by $P$. A drawing is monotone if there is a monotone path between every pair of vertices.

In this paper, we study drawings of planar graph classes with a small number of segments. For some of the results, we use techniques that have been used for monotone drawings; as a result, these drawings are also monotone. To improve the readability of the graph, we further require the drawings to lie on a polynomial size grid. Monotone drawings are motivated by the geodesic tendency in path-finding tasks, that is, eyes follow edges that go in the direction of the target vertex [21].
Related work. Dujmović et al. [9] were the first to study drawings with few segments and provided upper and lower bounds for several planar graph classes. Since then, some of the bounds were improved and some different graph classes were studied; the results are summarized in Table 1. However, most of these bounds do not require the drawings to be on the grid. In his thesis, Mondal [10] gives an algorithm for triangulations with \( n \) vertices using \( 8n/3 - O(1) \) segments on a grid of size \( 2^{O(n \log n)} \) in general and of size \( 2^O(n) \) for triangulations of bounded degree. Even with this large grid, the algorithm uses substantially more segments than the best-known algorithm for triangulations without the grid requirement by Durocher and Mondal [10], which uses \( 7n/3 - O(1) \) segments. Recently, Hültenschmidt et al. [22] presented algorithms to draw trees, planar 3-trees, and maximal outerplanar graphs with few segments on a polynomial size grid.

Known results for drawings with few segments are summarized in Table 1. There are three trivial lower bounds for the number of segments required to draw any graph \( G \). Let \( v \) be a triangulation. Assume that we selected a face as the outer face with vertices \( v, T_1, T_2, \) and \( T_3 \). We decompose the interior edges into three trees: \( T_1, T_2, \) and \( T_3 \) rooted at \( v, v_2, \) and \( v_3, \) respectively. The edges of the trees are oriented to their roots. For \( k \in \{1, 2, 3\} \), we call each edge in \( T_k \) a \( k \)-edge and the

Our Contribution. Our contribution is summarized in Table 1. We first consider drawings of trees with few segments in Section 2. We show that every tree has a drawing with at most \( 3n/4 - 1 \) segments on an \( n \times n \) grid, improving the area bound by Hültenschmidt et al. [22]. We then focus on drawing 3-connected planar graphs in Section 3. Using a combination of Schnyder realizers and orderly spanning trees, we show that every 3-connected planar graph can be drawn monotone with \( m - (n + 2)/3 \leq (8n - 14)/3 \) segments on an \( O(n) \times O(n^2) \) grid. Finally, in Section 4, we use this result to draw 4-connected triangulations monotone with \( 5n/2 - 4 \) segments, biconnected outerplanar graphs monotone with \( 3n/2 - 3 \) segments, connected outerplanar graphs with \( 7n - 9)/4 \) segments, and connected planar graphs with \( (17n - 38)/6 \) segments on an \( O(n) \times O(n^2) \) grid each. All our proofs are constructive and yield algorithms to obtain such drawings in \( O(n \log n) \) time for trees and in \( O(n) \) time for the other graph classes. As a side result, we also prove that the total number of leaves in every Schnyder realizer of a 3-connected planar graph is at most \( 2n + 1 \), which was only known for triangulated graphs [2, 28].

Preliminaries. For a graph \( G = (V, E) \), we denote \( n = |V| \) and \( m = |E| \). Let \( G \) be a triangulation. Assume that we selected a face as the outer face with vertices \( v_1, v_2, \) and \( v_3 \). We decompose the interior edges into three trees: \( T_1, T_2, \) and \( T_3 \) rooted at \( v_1, v_2, \) and \( v_3, \) respectively. The edges of the trees are oriented to their roots. For \( k \in \{1, 2, 3\} \), we call each edge in \( T_k \) a \( k \)-edge and the
Drawing planar graphs with few segments on a polynomial grid

Table 1: Upper and lower bounds on the visual complexity of segment drawings. Here, \( n \) is the number of vertices, \( m \) is the number of edges, \( \bar{v} \) is the number of odd-degree vertices, and \( b \) is the number of maximal biconnected components. Constant additions or subtractions have been omitted. Entries marked by a * are monotone drawings.

| Class       | Segments | Segments on the grid |
|-------------|----------|----------------------|
|             | Lower b. | Upper b.             |                     |
| tree        | \( \vartheta/2 \) | \( \vartheta/2 \) | \( 3n/4 \) \( O(n^2) \times O(n^{1.58}) \) [22] |
| max. outerpl. | \( n \) | \( n \) | \( 3n/2 \) \( O(n) \times O(n^2) \) [22] |
| 2-conn. outerpl. | \( n \) | * \( m - n/2 \) | \( 3n/2 \) \( O(n) \times O(n^2) \) Th. 4 |
| outerplanar | \( n \) | * \( 3n/2 \) | \( 3n/2 + b \) \( O(n) \times O(n^2) \) Th. 5 |
| 2-tree      | \( 3n/2 \) | \( 3n/2 \) | \( 8n/3 \) \( O(n) \times O(n^2) \) [22] |
| 3-tree      | \( 2n \) | \( 2n \) | \( m - n/3 \) \( O(n) \times O(n^2) \) Th. 2 |
| 2-conn.     | \( \vartheta/2 \) | \( 8n/3 \) | \( m - n/3 \) \( O(n) \times O(n^2) \) Th. 2 |
| cubic 3-conn. | \( n/2 \) | \( 5n/2 \) | \( 8n/3 \) \( O(n) \times O(n^2) \) Cor. 1 |
| triangulation | \( 2n \) | \( 7n/3 \) | \( n/2 \) \( O(n) \times O(n) \) [23] |
| 4-conn.     | \( 2n \) | \( 12n/8 \) | * \( 8n/3 \) \( O(n) \times O(n^2) \) Cor. 1 |
| 4-conn. triang. | \( 2n \) | \( 9n/4 \) | * \( 5n/2 \) \( O(n) \times O(n^2) \) Th. 3 |
| planar      | \( 2n \) | \( 17n/3 \) | \( 17n/6 \) \( O(n) \times O(n^2) \) Cor. 4 |

parent of a vertex in \( T_k \) is its \( k \)-parent. The decomposition is a Schnyder realizer \[^{[33]}\] if at every interior vertex the edges are counter-clockwise ordered as: outgoing 1-edge, incoming 3-edges, outgoing 2-edge, incoming 1-edges, outgoing 3-edge, incoming 2-edges; see Fig 1. \( T_1, T_2, \) and \( T_n \) are called Schnyder trees. The trees of a Schnyder realizer are also called canonical ordering trees, as each describes a canonical order on the vertices of \( G \) by a (counter-)clockwise pre-order traversal \[^{[7]}\]. There is a unique minimal realizer such that any interior cycle in the union of the three trees is oriented clockwise \[^{[3]}\]: this realizer can be computed in linear time \[^{[33]}\]. The number of such cycles is denoted by \( \Delta_0 \) and is upper bounded by \( \lfloor (n - 1)/2 \rfloor \) \[^{[39]}\]. Bonichon et al. \[^{[2]}\] prove that the total number of leaves in a minimal realizer is at most \( 2n + 1 - \Delta_0 \).

For 3-connected graphs, Schnyder realizers also exist \[^{[8]}\] \[^{[14]}\], but the edges can be bidirected: an edge \((u, v)\) is bidirected if it is an outgoing \( i \)-edge at \( u \) and an outgoing \( j \)-edge at \( v \) with \( i \neq j \). All other edges are unidirected, that is, they are an outgoing \( i \)-edge at \( u \) and an incoming \( i \)-edge at \( v \) (or vice-versa). The restriction on the cyclical ordering around each vertex remains the same, but now the Schnyder trees are not necessarily disjoint.

Chiang et al. \[^{[6]}\] have introduced the notion of orderly spanning trees. Recently, orderly spanning trees were redefined by Hossain and Rahman \[^{[20]}\] as good spanning trees. We will use the definition by Chiang et al., but note that they are equivalent. Two vertices in a rooted spanning tree are unrelated if neither of them is an ancestor of the other one.

Let \( G = (V, E) \) be a plane graph and let \( r \in V \) lie on the outer face. Let \( T \) be an ordered spanning tree of \( G \) rooted in \( r \) that respects the embedding of \( G \). Let \( v_1, \ldots, v_n \) be the vertices of \( T \) as encountered in a counter-clockwise pre-order traversal. For any vertex \( v_i \), let \( p(v_i) \) be its parent in \( T \), let \( C(v_i) \) be the children of \( v \) in \( T \), let \( N(v_i) \) be the neighbors of \( v_i \) that are unrelated to \( v_i \). Further, let \( N^-(v_i) = \{ v_j \in N(v_i) \mid j < i \} \) and \( N^+(v_i) = \{ v_j \in N(v_i) \mid j > i \} \). Then, \( T \) is called orderly if the neighbors around every vertex \( v_i \) are
counter-clockwise order $\rho(v_i), N^-(v_i), C(v_i), N^+(v_i)$. In particular, this means that there is no edge in $T$ between $v_i$ and an ancestor that is not its parent and there is no edge in $T$ between $v_i$ and a successor that is not its child.

2 Trees

Let $T = (V, E)$ be a tree of size $n$. In this section, we describe an algorithm to draw $T$ with at most $3n/4 - 1$ segments on an $n \times n$ grid in $O(n \log n)$ time.

If $T$ consist only of vertices of degree 1 and 2, then it is a path and we can draw it with 1 segment and $n \times 1$ area. So we will assume that there is at least one vertex with higher degree. We choose such a vertex as the root of $T$. Denote the number of degree-2 vertices by $\beta$ and the number of leaves by $\alpha$. In the first step, we create another tree $T'$ with $n - \beta$ vertices by contracting all degree-2 vertices of $T$. We say that a degree-2 vertex $u$ belongs to a vertex $v$ if $v$ is the first successor of $u$ in $T$ that has not degree 2. Note that $T'$ has the same number of leaves as $T$. In the next step, we remove all leaves from $T'$ and obtain a tree $T''$ with $n - \beta - \alpha$ vertices; see Fig. 2.

The main idea of our algorithm is as follows. Draw $T''$ in some way with $n - \beta - \alpha$ segments. Then, add the $\alpha$ leaves in such a way that they either extend the segment of an edge, or that two of them share a segment, adding at most $\alpha/2$ new segments to the drawing. Finally, place the $\beta$ degree-2 vertices onto the segments without increasing the number of segments. This way, we get a drawing with $n - \beta - \alpha/2$ segments. Since $T'$ has no degree-2 vertices, we have that more than half of its vertices are leaves, so $\alpha > (n - \beta)/2$. Hence, the drawing has at most $3(n - \beta)/4 < 3n/4$ segments. We will traverse the tree $T''$ bottom-up to create such a drawing of $T$ in $n \times n$ area.

Let $v$ be a vertex in $T''$, and let $T[v]$ be the subtree of $T$ rooted in $v$. We will draw $T[v]$ inside a box $B_v$ of width $w_v = \ell_v + r_v$ and height $h_v = t_v + b_v$ such that no vertex of $T[v]$ lies to the top-left of $v$; see Fig. 3a. We now describe how to draw $T[v]$ such that $v$ lies at coordinate $(0, 0)$. We first recursively place the children of $v$ in $T''$ and their subtrees, then add the degree-2 vertices that belong to these children, and finally add the vertices of $T[v]$ that are not in $T''$.

Let $v_1, \ldots, v_k$ be the children of $v$ in $T''$. Assume that $T[v_1], \ldots, T[v_k]$ have already been drawn. (We will notate the boxes and their dimensions by $B_i, h_i, \ldots$ instead of $B_v, h_v, \ldots$ in the following.) We aim to place $v_1$ directly below $v$, and each box $B_i, i \geq 2$, to the right of box $B_{i-1}$, aligning $v_i$ with the top boundary of $B_{i-1}$; see Fig. 3b. We place $v_1$ at coordinate $(0, -\sum_{i=1}^k t_i - 1)$, and each $v_i$ at coordinate $(x(v_i-1), y(v_i-1)) + (r_{i-1} + \ell_i + 1, t_i - 1)$.

Let $\beta_i$ be the number of degree-2 vertices that belong to $v_i$. We move the box $B_i$ downwards by $\beta_i$, and place the degree-2 vertices directly above $v_i$; see Fig. 3c. This does not change any edge incident to $v$, the boxes are only moved downwards and are still disjoint, so the drawing remains plane.

It remains to add the children of $v$ that are not in $T''$, that is, the leaves in $T[v]$ and the degree-2 vertices that belong to these leaves. Let $u_1, \ldots, u_n$ be the leaf-children of $v$ in $T'$, that is, the leaves in $T[v]$ except $v$ itself. Let $b_i$ be the number of degree-2 vertices that belong to $u_i$. Assume w.l.o.g. that $b_1 \geq \ldots \geq b_n$. We
place the leaves alternately to the bottom-left and to the top-right of v; we draw each $u_{2i-1}$ and $u_{2i}$ on a segment through v with slope $1/i$. To this end, we place $u_{2i-1}$ at coordinate $(-(b_{2i-1}+1) \cdot i, -b_{2i-1} - 1)$ and $u_{2i}$ at coordinate $((b_{2i} + 1) \cdot i, b_{2i} + 1)$. This leaves enough room to place the degree-2 vertices that belong to the leaves; see Fig. 3d.

If $a$ is odd, then we sometimes treat $u_a$ differently. If v is a leaf, then we place $u_a$ below v at coordinate $(0, -b_a - 1)$. If v is not a leaf, no degree-2 vertex belongs to v, and v is not the first child of its parent in $T''$ (that is, there will be no edge that leaves v vertically above), then we place $u_a$ above v at coordinate $(0, b_a + 1)$ such that it shares a segment with $(v, v_{1})$.

By construction, these segments cannot intersect $B_2, \ldots, B_k$, but there might be an intersection between the segment from $u_1$ to v and $B_1$. In this case, we move $B_1$ downwards until the crossing disappears, which makes the drawing planar again. Thus, we created a drawing of $T[v]$ inside a box $B_v$, and the dimensions of $B_v$ can easily be calculated during the construction.

**Lemma 1.** For any vertex $v$ of $T''$, $T[v]$ is drawn on an $n \times n$ grid.

**Proof.** We perform the proof bottom-up in $T''$, so we can assume that the bound holds for all children of $v$ in $T''$.

Let $v_1, \ldots, v_k$ be the children of $v$ in $T''$, and let $\beta_1, \ldots, \beta_k$ be the number of degree-2 vertices that belong to $v_1, \ldots, v_k$, respectively. Assume that the area bound holds for the drawings of $T[v_1], \ldots, T[v_k]$. Before inserting the degree-2 vertices, we have drawn $T[v_1], \ldots, T[v_k]$ next to each other without a gap, and they overlap vertically (see Fig. 3b), so the total width and height of their drawings are both at most $\sum_{i=1}^{k} n_{v_i}$. Each degree-2 vertex that belongs to some $v_i$ moves the drawing of $T[v_i]$ downwards by one, so the height of the drawing increases by at most $\sum_{i=1}^{k} \beta_i$, while the width remains the same.

Let $u_1, \ldots, u_a$ be the leaf-children of $v$ in $T'$, and let $b_1, \ldots, b_a$ the number of degree-2 vertices that belong to $u_1, \ldots, u_a$, respectively, with $b = \sum_{i=1}^{a} b_i$. We will now analyze the width and height of the part of the drawing of $T[v]$ that consists of these vertices. Recall that we have drawn each $u_{2i-1}$ at coordinate $(-(b_{2i-1}+1) \cdot i, -b_{2i-1} - 1)$ and each $u_{2i}$ at coordinate $((b_{2i} + 1) \cdot i, b_{2i} + 1)$. Let further $b^L = \sum_{i=1}^{\lfloor a/2 \rfloor} b_{2i-1}$ and $b^R = \sum_{i=1}^{\lceil a/2 \rceil} b_{2i}$ be the number of degree-2 vertices drawn to the left/right of $v$, respectively.

Recall that $b_1 \geq \ldots \geq b_a$. Hence, the vertices with the highest and lowest $y$-coordinate are $u_2$ at $y(u_2) = b_2 + 1$ and $u_1$ at $y(u_1) = -b_1 - 1$, respectively. Thus, the height of this part of the drawing is $1 = 1 + a + b$ if $a = 0$; $b_1 + 2 = 1 + a + b$ if $a = 1$; and $3 + b_1 + b_2 \leq 1 + a + b$ if $a \geq 2$, so at most $1 + a + b$ in total.

For analyzing the width of this part of the drawing, we first consider those vertices that are drawn to the right of $v$. Let $r$ be such that $u_{2r}$ is the rightmost vertex at $x$-coordinate $(b_{2r} + 1) \cdot r$. Since $b_1 \geq \ldots \geq b_a$, we have that

$$b^R = \sum_{i=1}^{\lfloor a/2 \rfloor} b_{2i} \geq \sum_{i=1}^{r} b_{2r} = r \cdot b_{2r}.$$
Symmetrically, let \( \ell \) be such that \( u_{2\ell-1} \) is the leftmost vertex at \( x \)-coordinate \(- (b_{2\ell-1}+1) \cdot \ell \). We have that
\[
b^k = \sum_{i=1}^{\lceil a/2 \rceil} b_{2i-1} \geq \sum_{i=1}^\ell b_{2\ell-i} = \ell \cdot b_{2\ell-1}.
\]
Hence, the total width of this part of the drawing is at most
\[
1 + (b_{2r} + 1) \cdot r + (b_{2\ell-1} + 1) \cdot \ell \leq 1 + \ell + r + b^r + b^\ell \leq 1 + a + b.
\]
In total, we have \( n_v = 1 + \sum_{i=1}^k (n_v + \beta_i) + a + b \). Hence, the width and the height of \( T[v] \) is at most \( n_v \).

Recall that we moved the drawing of \( T[v_1] \) downwards if it crossed the segment between \( u_1 \) and \( v \). However, observe that this crossing can only happen if \( y(u_1) < y(v_1) \), and that there are \(-y(u_1)\) vertices between \( u_1 \) and \( v \), so the height of the drawing remains at most \( n_v \).

We will now move to the number of segments. Let \( v \) be a vertex in \( T'' \). Let \( T^+[v] \) be \( T[v] \) together with all the degree-2 vertices that belong to \( v \). Let \( n_v \) be the number of vertices in \( T^+[v] \). Let \( e_v \) be the last edge of \( T \) on the path from \( v \) to its parent in \( T'' \). Further, let \( s_v \) be the number of segments used in the drawing of \( T^+[v] \) plus the edge \( e_v \). We show the following.

**Lemma 2.** For any vertex \( v \neq v \) of \( T'' \), if \( e_v \) is drawn vertical, then \( s_v \leq (3n_e - 1)/4 \), otherwise \( s_v \leq 3n_e / 4 \).

**Proof.** We perform the proof bottom-up in \( T'' \), so we can assume that the bound holds for all children of \( v \) in \( T'' \).

Recall that \( u_1, \ldots, u_a \) are the leaf-children of \( v \) in \( T' \), \( v_1, \ldots, v_k \) are the children of \( v \) in \( T'' \), and \( v_1 \) is connected to \( v \) by a vertical segment. Let \( b \) be the number of degree-2 vertices that belong to \( v \). Let \( n' = \sum_{i=1}^b n_{v_i} \); then, \( n_v \geq n' + a + b + 1 \). (There might be degree-2 vertices between \( v \) and its leaf-children in \( T' \) which we do not count.) By induction, \( s_{v_1} \leq 3(n_{v_1} - 1)/4 \) and \( s_{v_i} \leq 3n_{v_i}/4 \) for \( 2 \leq i \leq k \), so \( \sum_{i=1}^k s_{v_i} = (3n' - 1)/4 \).

It remains to analyze the number of segments for the leaves in \( T[v] \), the degree-2 vertices that belong to them, for the degree-2 vertices that belong to \( v \), and for \( e_v \).

**Case 1.** \( v \) is not a leaf in \( T'' \) and \( b = 0 \). We have \( n_v \geq n' + a + 1 \).

**Case 1.1.** \( a = 0 \) and \( e_v \) is vertical. Then, \( n_v = n' + 1 \) and \( e_v \) lies on a vertical segment with the edge \( e_{v_1} \). Hence, \( s_v \leq (3n' - 1)/4 + 1 = 3n_v/4 \).

**Case 1.2.** \( a = 0 \) and \( e_v \) is not vertical. Again, \( n_v = n' + 1 \). We use one segment for \( e_v \), so we have \( s_v \leq (3n' - 1)/4 + a + 1 = 3n_v/4 \).

**Case 1.3.** \( a \geq 2 \) is even. We use \( a/2 \) segments for \( u_1, \ldots, u_a \), and one more for \( e_v \). Hence, \( s_v \leq (3n' - 1)/4 + a/2 + 1 \leq 3n_v/4 - a/4 \leq (3n_v - 2)/4 \).

**Case 1.4.** \( a \) is odd and \( e_v \) is vertical. We use \( (a + 1)/2 \) segments for \( u_1, \ldots, u_a \), but \( e_v \) shares its vertical segment with \( e_{v_1} \). Hence, \( s_v \leq (3n' - 1)/4 + (a + 1)/2 \leq 3n_v/4 + (a - 1)/2 \leq (3n_v - 2)/4 \).

**Case 1.5.** \( a \) is odd and \( e_v \) is not vertical. In this case, we place \( u_a \) above \( v \) such that it lies on a segment with \( e_{v_1} \). We use \( (a - 1)/2 \) segments for \( u_1, \ldots, u_a - 1 \) and one segment for \( e_v \), so we have \( s_v \leq (3n' - 1)/4 + (a - 1)/2 + 1 \leq 3n_v/4 - a - 1/2 \leq (3n_v - 2)/4 \).

**Case 2.** \( v \) is not a leaf in \( T'' \) and \( b > 0 \). We have \( n_v \geq n' + a + b + 1 \geq n' + a + 2 \).

**Case 2.1.** \( a \) is even. We use \( a/2 \) segments for \( u_1, \ldots, u_a \). The degree-2 vertices that belong to \( v \) share a vertical segment with \( e_{v_1} \), and we need one more segment for \( e_v \), so \( s_v \leq (3n' - 1)/4 + a/2 + 1 \leq 3(n_v - a - 3)/4 \leq (3n_v - 3)/4 \).

**Case 2.2.** \( a \) is odd and \( e_v \) is vertical. We use the exact same number of segments as in Case 1.4, so \( s_v \leq (3n_v - 2)/4 \).

**Case 2.3.** \( a \) is odd and \( e_v \) is not vertical. We use \( (a + 1)/2 \) segments for \( u_1, \ldots, u_a \). The degree-2 vertices that belong to \( v \) share a vertical segment with \( e_{v_1} \), and we need one more segment for \( e_v \). Hence, \( s_v \leq (3n' - 1)/4 + (a + 1)/2 + 1 \leq 3n_v/4 - a - 1/2 \leq (3n_v - 2)/4 \).

**Case 3.** \( v \) is a leaf in \( T'' \) and \( b = 0 \). Then, \( n_v \geq a + 1 \). Since \( v \) is a leaf in \( T'' \), it has at least two children in \( T \), so \( a \geq 2 \).

**Case 3.1.** \( a \) is even. Leaves are always placed on common segments with the degree-2 vertices belonging to
Drawing planar graphs with few segments on a polynomial grid

them, and by construction the leaves are paired up to share segments. Hence, the number of segments is \( a/2 \) for the leaves and their degree-2 vertices, plus one for the edge \( e_v \). Hence, \( s_v \leq a/2 + 1 \leq (n_v - 1)/2 + 1 \leq (3n_v - 1)/4 \) since \( n_v \geq 3 \\

Case 3.2. \( a \) is odd, so \( a \geq 3 \) and \( n_v \geq 4 \). We have drawn \( a - 1 \) of the leaves paired up, and the last leaf was placed below \( v \). If \( e_v \) is not vertical, then it needs another segment. Hence, we have \( s_v \leq (a - 1)/2 + 2 \leq n_v/2 + 1 \leq 3n_v/4 \\

Case 4. \( v \) is a leaf in \( T'' \) and \( b > 0 \). Then, \( a \geq 2 \) and \( n_v \geq a + b + 1 \geq a + 2 \geq 4 \\

Case 4.1. \( a \) is even and \( e_v \) is vertical. We use \( a/2 \) segments for \( u_1, \ldots, u_2 \), and the degree-2 vertices that belong to \( v \) lie on a vertical segment with \( e_v \). Hence, we have \( s_v \leq a/2 + 1 \leq n_v/2 \leq 3n_v/4 - 1 \\

Case 4.2. \( a \) is even and \( e_v \) is not vertical. We again have \( n_v \geq 4 \). The degree-2 vertices that belong to \( v \) now lie on a different segment than \( e_v \), so we have one more segment than in Case 4.1, so \( s_v \leq 3n_v/4 \\

Case 4.3. \( a \) is odd. We have \( a \geq 3 \) and thus \( n_v \geq 5 \). We have drawn \( u_1, \ldots, u_{a-1} \) paired up. We have drawn \( u_a \) on a vertical segment with the degree-2 vertices that belong to \( v \), and we have possibly one more segment for \( e_v \). Hence, we have \( s_v \leq (a - 1)/2 + 2 \leq (n_v + 1)/2 + 1 \leq (3n_v - 3)/4 \\

Now, we can bound the total number of segments in the drawing of \( T \\

Lemma 3. Our algorithm draws \( T \) with at most \( 3n_v/4 - 1 \) segments if \( n_v \geq 3 \\

Proof. If \( T \) is a path with \( n \geq 3 \), then the bound trivially holds. If \( T \) is a subdivision of a star, then the bound also clearly holds. Otherwise, \( T'' \) consists of more than one vertex. Let \( v_1, \ldots, v_k \) be the children of \( r \) in \( T'' \) such that \( v_k \) is connected by a vertical edge. Let \( n' = \sum_{i=1}^{k} n_{v_i} \). By Lemma 2, they contribute at most \( (3n' - 1)/4 \) segments to the drawing of \( T \). Let \( a \) be the number of leaf children of \( r \) in \( T'' \). If \( a \) is even, then we use \( a/2 \) segments to draw them. If \( a \) is odd, then we align one of them with the vertical segment of \( v_1 \), and draw the remaining with \( (a - 1)/2 \) segments. Since \( n \geq n' + a + 1 \), the total number of segments is at most \( (3n' - 1)/4 + a/2 \leq 3n/4 - a/4 - 1 \leq 3n/4 - 1 \\

All steps of the algorithm work in linear time, except sorting the leaf children by the number of degree-2 vertices that belong to them. Thus, Theorem 1 follows. Fig. 2d shows the result of our algorithm for the tree of Fig. 2a.

Theorem 1. Any tree with \( n \geq 3 \) can be drawn planar on a \( n \times n \) grid with \( 3n/4 - 1 \) segments in \( O(n \log n) \) time.

3 3-connected planar graphs

In this section, we present an algorithm to compute drawings with at most \( 8n/3 - 14/3 \) segments for 3-connected planar graphs.

Angelini et al. [1] have introduced the notion of a slope-disjoint drawing of a rooted tree, which is defined as follows; see Fig. 4.

(S1) For every vertex \( u \) in \( T \), there exist two angles \( \alpha_1 \) and \( \alpha_2 \) with \( 0 < \alpha_1(u) < \alpha_2(u) < \pi \), such that, for every edge \( e \) that is either \( (p(u), u) \) or lies in \( T(u) \), it holds that \( \alpha_1(u) < \text{slope}(e) < \alpha_2(u) \\

(S2) for every directed edge \( (u, v) \) in \( T \), it holds that \( \alpha_1(u) < \alpha_1(v) < \alpha_2(v) < \alpha_2(u) \); and

(S3) for every two vertices \( u, v \) in \( T \) with \( p(u) = p(v) \), it holds that either \( \alpha_1(u) < \alpha_2(u) < \alpha_1(v) < \alpha_2(v) \) or \( \alpha_1(v) < \alpha_2(v) < \alpha_1(u) < \alpha_2(u) \).

Further, Angelini et al. proved the following lemma:

Lemma 4 [1]. Every slope-disjoint drawing of a tree is planar and monotone.

We will now create a special slope-disjoint drawing for rooted ordered trees.

Lemma 5. Let \( T = (V, E) \) be an ordered tree rooted in a vertex \( r \) with \( \lambda \) leaves. Then, \( T \) admits a slope-disjoint drawing with \( \lambda \) segments on an \( O(n) \times O(n^2) \) grid such that all slopes are integer. Such a drawing can be found in \( O(n) \) time.
P. Kindermann, T. Mchedlidze, T. Schneck, and A. Symvonis

Proof. Let \( v_1, \ldots, v_n = r \) be the vertices of \( T \) as encountered in a counter-clockwise post-order traversal. Let \( e_i = (p(v_i), v_i), 1 \leq i < n \). We assign the slopes to the edges of \( T \) in the order \( e_1, \ldots, e_{n-1} \). We start with assigning slope \( s_1 = 1 \) to \( e_1 \). For any other edge \( e_i, 1 < i < n \), if \( v_i \) is a leaf, then we assign the slope \( s_i = s_{i-1} + 1 \) to \( e_i \). Otherwise, since we traverse the vertices in a post-order, \( p(v_{i-1}) = v_i \) and we assign the slope \( s_i = s_{i-1} \) to \( e_i \).

We create a drawing \( \Gamma \) of \( T \) as follows. We place \( r = v_n \) at coordinate \((0,0)\). For every other vertex \( v \) with parent \( p \) that is drawn at coordinate \((x,y)\), we place \( v \) at coordinate \((x+1, y+\text{slope}(v))\).

We will now analyze the number of segments used in \( \Gamma \). The root \( r \) is an endpoint of \( \text{deg}(r) \) segments and every leaf is an endpoint of exactly 1 segment. For every other vertex \( v \), its incoming edge and one of its outgoing edges lie on the same segment, so it is an endpoint of \( \text{deg}(v) - 2 \) segments. Since every segment has two endpoints the total number of segments is

\[
\frac{1}{2} \left( \text{deg}(r) + \sum_{v \text{ not leaf}, v \neq r} (\text{deg}(v) - 2) + \sum_{v \text{ leaf}} \text{deg}(v) \right)
\]

\[
= \frac{1}{2} \left( \sum_{v} \text{deg}(v) - 2(n - \lambda - 1) \right) = \frac{1}{2} (2n - 2 - 2n + 2\lambda + 2) = \lambda.
\]

We now prove that \( \Gamma \) is slope-disjoint. Let \( \varepsilon > 0 \) be arbitrary small, \( 1/n \) should suffice. We set \( \alpha_1(r) = 1 - \varepsilon \) and \( \alpha_2(r) = s_n + \varepsilon \). We assign the remaining values of \( \alpha_1 \) and \( \alpha_2 \) in pre-order. Let \( u \) be a vertex that has already been handled, that is, (S1) holds for \( u \), (S2) holds for \( (p(u), u) \), and (S3) holds for \( u \) and all of its siblings. Obviously, this holds for \( r \) in the beginning. Let \( u_1, \ldots, u_k \) be the children of \( u \) in counter-clockwise order. By construction, we have \( \text{slope}(u_k) = \text{slope}(u) < \alpha_2(u) \) and \( \text{slope}(u_k) > \alpha_1(u) \). Furthermore, by construction, all slopes in \( T(u_i), 1 \leq i \leq k \) are larger than \( \text{slope}(u_{i-1}) \) and at most \( \text{slope}(u_i) \). Hence, choosing \( \alpha_1(u_1) = \alpha_1(u) + \varepsilon, \alpha_1(u_i) = \text{slope}(u_{i-1}) + \varepsilon, 1 < i \leq k \), and \( \alpha_2(u_i) = \text{slope}(u_i) + \varepsilon, 1 \leq i \leq k \) satisfies (S1) for every \( u_i \), (S2) for every edge \((u, u_i)\), and (S3) for every pair \((u_i, u_j)\). Repeating this construction establishes the conditions for every vertex, every edge, and every pair of siblings, so \( \Gamma \) is slope-disjoint.

Finally, since every vertex is placed one \( x \)-coordinate to the right of its parent, we use at most \( n \) columns, and since the highest slope is \( n-1 \), the drawing lies on a grid of size \( O(n) \times O(n^2) \). Our algorithm consists of doing one post-order traversal and then placing the vertices, so it clearly takes \( O(n) \) time.

We will use this algorithm to draw planar graphs.

Lemma 6. Let \( G = (V,E) \) be a planar graph and let \( T \) be an orderly spanning tree of \( G \) with \( \lambda \) leaves. Then, \( G \) admits a planar monotone drawing with at most \( m - n + 1 + \lambda \) segments on an \( O(n) \times O(n^2) \) grid in \( O(n) \) time.

Proof. We first create a drawing of \( T \) according to Lemma 5. Now, we will plug this tree drawing into the algorithm by Hossain and Rahman [20].

The algorithm of Hossain and Rahman takes a slope-disjoint drawing of an orderly spanning tree of \( G \). Then, they stretch the edges of \( T \) such that the remaining edges of \( G \) can be inserted without crossings. In this stretching operation, the slopes of all edges of \( T \) remain unchanged. Furthermore, the total width of the drawing only increases by a constant factor. Since \( T \) is drawn slope-disjoint, this produces a planar monotone drawing of \( G \) on an \( O(n) \times O(n^2) \) grid. Their algorithm runs in \( O(n) \) time.
To count the number of segments, assume that every edge of $G$ that does not lie on $T$ is drawn with its own segment. We have drawn $T$ with $\lambda$ segments and the slopes of the edges of $T$. Hence, our algorithm draws $G$ with $\lambda$ segments for $T$ and with $m - n + 1$ segments for the remaining edges.

Both Chiang et al. [6] and Hossain and Rahman [24] have shown that every planar graph has an embedding that admits an orderly spanning tree. However, we do not know anything about the number of leaves in an orderly spanning tree. Miura et al. [28] have shown that Schnyder trees are orderly spanning trees, and it is known that every 3-connected planar graph has a Schnyder realizer.

**Lemma 7** ([25]). Let $G = (V,E)$ be a 3-connected planar graph and let $T_1$, $T_2$, and $T_3$ be the Schnyder trees of a Schnyder realizer of $G$. Then, $T_1$, $T_2$, and $T_3$ are orderly spanning trees of $G$.

Bonichon et al. [3] showed that there is a Schnyder realizer for every triangulated graph such that the total number of leaves in $T_1$, $T_2$, and $T_3$ is at most $2n + 1$, which already gives us a good bound on the number of segments for triangulations. We will now show that the same holds for every Schnyder realizer of a 3-connected graph.

Let $v$ be a leaf in one of the Schnyder trees $T_k$, $k \in \{1, 2, 3\}$, that is not the root of a Schnyder tree, so $v$ has no incoming $k$-edge. Hence, the outgoing $(k + 1)$-edge $(v, u)$ and the outgoing $(k - 1)$-edge $(v, w)$ are consecutive in the cyclical ordering around $v$, so they lie on a common face $f$. We will assign the pair $(v, k)$ to $f$ and write $F_k(v) = f$. We first show two lemmas.

**Lemma 8.** Let $u_1, u_2, \ldots, u_p$ be the vertices on an interior face $f$ in counter-clockwise order. If $F_i(u_1) = F_i(u_2) = F_i(u_3) = f$ for some $i, j, k \in \{1, 2, 3\}$, then $i = k = j$.

**Proof.** We first show that $i = k$: see Fig. 5a. Since $u_2$ is a leaf in $T_k$, $(u_1, u_2)$ cannot be an outgoing $k$-edge at $u_1$. Hence, $(u_1, u_2)$ is either an incoming $(k+1)$-edge at $u_1$ (if it is undirected), or an outgoing $(k-1)$-edge (if it is bidirected); it cannot be an outgoing $k$-edge since bidirected edges have to belong to two Schnyder trees. For $(u_1, i)$ to be assigned to $f$, $u_1$ must have two outgoing edges at $f$, so we are in the latter case. Hence, $(u_1, u_p)$ is outgoing at $u_1$, and by the cyclical ordering of the edges around $u_1$, it is an outgoing $(k-1)$-edge. Thus, $u_1$ has an outgoing $(k+1)$-edge and an outgoing $(k-1)$-edge at $f$, so $i = k$.

The argument for $k = j$ works symmetrically: see Fig. 5b. Both edges of $u_3$ at $f$ have to be outgoing, so $(u_2, u_3)$ has to be an outgoing $(k+1)$-edge and by the cyclical ordering the other edge $(u_3, u_4)$ (with $u_4 = u_1$ if $p = 3$) has to be an outgoing $(k-1)$-edge, so $j = k$.

**Lemma 9.** Let $u_1, u_2, \ldots, u_p$ be vertices on an interior face $f$ in counter-clockwise order. If $F_k(u_3) = \ldots = F_k(u_p)$ for some $k \in \{1, 2, 3\}$, then $F_i(u_1) \neq f$ and $F_j(u_2) \neq f$ for every $i, j \in \{1, 2, 3\}$.

**Proof.** Since $F_k(u_3) = F_k(u_p) = f$, we know that $(u_1, u_p)$ is an outgoing $(k+1)$-edge at $u_p$ and $(u_2, u_3)$ is an outgoing $(k-1)$-edge at $u_3$; hence, $(u_1, u_p)$ is either an incoming $(k+1)$-edge or an outgoing $(k-1)$-edge at $u_1$ and $(u_2, u_3)$ is either an incoming $(k-1)$-edge or an outgoing $(k+1)$-edge at $u_2$.

**Case 1:** $(u_1, u_p)$ is an incoming $(k+1)$-edge at $u_1$ and $(u_2, u_3)$ is an incoming $(k-1)$-edge at $u_2$. Then, neither $u_1$ nor $u_2$ has two outgoing edges at $f$, so there can be no pairs $(u, i)$ and $(v, j)$ assigned to $f$.

**Case 2:** $(u_1, u_p)$ is an outgoing $(k-1)$-edge at $u_1$ and $(u_2, u_3)$ is an incoming $(k-1)$-edge at $u_2$; see Fig. 6a. By Lemma 8, if $(u, i)$ is assigned to $f$, then $i = k$; hence, $(u_1, u_2)$ has to be an outgoing $(k+1)$-edge.
at $u_1$. By the cyclical ordering around $u_2$, $(u_1, u_2)$ has to be either an incoming $(k - 1)$-edge, order an outgoing $(k + 1)$-edge at $u_2$; both cases cannot be combined with an outgoing $(k + 1)$-edge at $u_1$. Hence, this case is not possible.

**Case 3:** $(u_1, u_p)$ is an incoming $(k + 1)$-edge at $u_1$ and $(u_2, u_3)$ is an outgoing $(k + 1)$-edge at $u_2$; see Fig. 6b. This case is symmetric to Case 2: $(u_1, u_2)$ has to be an outgoing $(k - 1)$-edge at $u_2$ and either an outgoing $(k - 1)$-edge or an incoming $(k + 1)$-edge at $u_1$, which is not possible.

**Case 4:** $(u_1, u_p)$ is an outgoing $(k - 1)$-edge at $u_1$ and $(u_2, u_3)$ is an outgoing $(k + 1)$-edge at $u_2$; see Fig. 6c. By the same arguments as in Cases 2 and 3, $(u_1, u_2)$ has to be an outgoing $(k + 1)$-edge at $u_1$ and an outgoing $(k - 1)$-edge at $u_2$. However, since $F_k(u_3) = \ldots = F_k(u_p)$, every edge $(u_q, u_{q+1}), 1 \leq q \leq p$ has to be an outgoing $(k + 1)$-edge at $u_q$, so there is a directed cycle in $T_{k+1}$; a contradiction to $T_{k+1}$ being a tree. Thus, this case is also not possible.

Combining Lemma 8 and Lemma 9 we can prove the desired bound on the number of leaves in a Schnyder realizer.

**Lemma 10.** Let $T_1, T_2, T_3$ be a Schnyder realizer of a 3-connected planar graph $G = (V, E)$. Then, there are at most $2n + 1$ leaves in total in $T_1$, $T_2$, and $T_3$.

**Proof 1.** Consider any interior face $f$ of $G$. By definition of the assignment, no vertex can be assigned to $f$ as a leaf twice. By Lemma 8, consecutive vertices on $f$ can only be assigned to $f$ as leaves from the same Schnyder tree, so by Lemma 9 at least two vertices on $f$ are not assigned to $f$ as a leaves, so we assign at most $\deg(f) - 2$ leaves to $f$. At the outer face $f^*$, every vertex that is not the root of a Schnyder trees can be assigned as a leaf at most once. However, the roots of the Schnyder trees have no outgoing edges, but they are a leaf in both other Schnyder trees. Hence, we assign at most $\deg(f^*) + 3$ leaves to the outer face. Let $F$ be the faces in $G$. Since every leaf gets assigned to exactly one face, the total number of leaves in $T_1$, $T_2$, and $T_3$ is at most

$$\sum_{f \in F} (\deg(f) - 2) + 5 \leq 2e - 2|F| + 5 = 2e + 2n - 2e - 4 + 5 = 2n + 1.$$  

Now we have the tools to proof the main result of this section.

**Theorem 2.** Any 3-connected planar graph can be drawn planar monotone on an $O(n) \times O(n^2)$ grid with $m - (n + 2)/3$ segments in $O(n)$ time.

**Proof.** Let $G = (V, E)$ be a 3-connected planar graph. We compute a Schnyder realizer of $G$, which is possible in $O(n)$ time. By Lemma 10 the Schnyder trees have at most $2n + 1$ leaves in total, so one of them, say $T_1$, has at most $(2n + 1)/3$ leaves. By Lemma 7, $T_1$ is an orderly spanning tree, so we can use Lemma 6 to obtain a planar monotone drawing of $G$ on an $O(n) \times O(n^2)$ grid with at most $m - n + 1 + (2n + 1)/3 = m - n/3 + 4/3$ segments in $O(n)$ time.

Since a planar graph has at most $m \leq 3n - 6$ edges, we have the following.

**Corollary 1.** Any 3-connected planar graph can be drawn planar monotone on an $O(n) \times O(n^2)$ grid with $(8n - 20)/3$ segments in $O(n)$ time.
4 Other planar graph classes

In this section, we use the results of Section 3 to obtain grid drawings with few segments for 4-connected triangulations, (biconnected) outerplanar graphs, and planar graphs.

Using regular edge labelings, Zhang and He [36] proved that any 4-connected triangulation admits a Schnyder tree with at most \( \lceil n/4 \rceil \) leaves. Applying this to Lemma 6, we find that our algorithm uses at most \( m - n + 1 + \lceil (n + 1)/2 \rceil \leq 3n - 6 - n + 1 + n/2 + 1 = 5n/2 - 4 \) segments.

**Theorem 3.** Any 4-connected triangulation can be drawn planar monotone on an \( O(n) \times O(n^2) \) grid with \( 5n/2 - 4 \) segments in \( O(n) \) time.

We now consider outerplanar graphs.

**Theorem 4.** Any biconnected outerplanar graph can be drawn planar monotone on an \( O(n) \times O(n^2) \) grid with \( m - (n - 3)/2 \) segments in \( O(n) \) time.

**Proof.** Let \( G = (V, E) \) be a biconnected outerplanar graph. We construct a 3-connected planar graph \( G' = (V', E') \) by adding a vertex \( r \) and connecting it to all vertices of \( V \). Hence, we have \( n' = |V'| = n + 1 \) and \( m' = |E'| = m + n \). Then, we compute a Schnyder realizer of \( G' \) such that one of its Schnyder trees, say \( T_3 \), is rooted in \( r \). Since \( r \) is connected to all other vertices of \( V' \), \( T_3 \) consists of exactly those edges, so it has \( n' - 1 \) leaves. By Lemma 10, at least one of \( T_1 \) and \( T_2 \) has \( \lambda' \leq n'/2 + 1 \) leaves. We use Lemma 8 to obtain a planar monotone drawing of \( G' \) on an \( O(n) \times O(n^2) \) grid with at most

\[
m' - n' + 1 + \lambda' \leq m' - n' + 1 + \frac{n'}{2} + 1 = m' - \frac{n'}{2} + 2 = m + n - \frac{n + 1}{2} + 2 = m + \frac{n}{2} + \frac{3}{2}
\]

leaves in \( O(n) \) time. Since all edges of \( E' \setminus E \) lie in \( T_3 \), we can remove \( r \) and those edges without splitting any segment into two segments. Hence, we obtain a drawing of \( G \) with

\[
m + \frac{n}{2} + \frac{3}{2} - (m' - m) = m + \frac{n}{2} + \frac{3}{2} - n = m - \frac{n}{2} + \frac{3}{2}
\]

segments. Further, the monotonicity of \( G' \) depends only on the edges of its orderly spanning tree, so \( G \) is also drawn monotone.

Outerplanar graphs have at most \( m \leq 2n - 3 \) edges, so this Corollary follows.

**Corollary 2.** Any biconnected outerplanar graph can be drawn planar monotone on an \( O(n) \times O(n^2) \) grid with \( (3n - 3)/2 \) segments in \( O(n) \) time.

**Theorem 5.** Any connected outerplanar graph can be drawn planar on an \( O(n) \times O(n^2) \) grid with \( (3n - 5)/2 + b \) segments, where \( b \) is its number of maximal biconnected components, in \( O(n) \) time.

**Proof.** Let \( G = (V, E) \) be a connected outerplanar graph. We first augment \( G \) to a biconnected planar graph \( G^* = (V^*, E^*) \) by adding the minimum number of edges required. García et al. [15] gave an algorithm to do this in \( O(n) \) time, and Read [32] has shown that, if \( G \) consists of \( b \) maximal biconnected components, then the number of edges required is at most \( b - 1 \). Let \( d \leq b - 1 \) be the number of edges added by this algorithm. Hence, we have \( n^* = |V^*| = n \) and \( m^* = |E^*| = m + d \). We use Theorem 4 to obtain a planar drawing of \( G^* \) on an \( O(n) \times O(n^2) \) grid with at most \( m^* - n/2 + 3/2 \) segments. Removing the \( d \) added edges from \( G^* \) splits at most \( d \) segments into two, so we obtain a drawing of \( G \) with at most \( m^* + d - n/2 + 3/2 \) segments.

Since \( G^* \) is outerplanar, we have \( m^* \leq 2n - 3 \). Hence, the number of segments is at most \( 2n - 3 + d - n/2 + 3/2 \leq 3n/2 + b - 5/2 \).

By a simple case analysis, we can give a bound on the number of segments only in terms of \( n \). If \( m \leq 7n/4 - 9/4 \), then we draw \( G \) with \( m \) segments; otherwise, \( b \leq 2n - 3 - m + 1 \leq n/4 + 1/4 \) and we use Theorem 5 to draw \( G \) with at most \( 3n/2 + b - 5/2 \leq 3n/2 + n/4 + 1/4 - 5/2 = 7n/4 - 9/4 \) segments.
Corollary 3. Any connected outerplanar graph can be drawn planar on an $O(n) \times O(n^2)$ grid with $(7n - 9)/4$ segments in $O(n)$ time.

Finally, we consider drawings of planar graphs.

Theorem 6. Any planar graph can be drawn planar on an $O(n) \times O(n^2)$ grid with $(17n - 38)/3 - m$ segments in $O(n)$ time.

Proof. Let $G = (V, E)$ be a planar graph. We use a similar technique as for outerplanar graphs. We first augment $G$ to a 3-connected planar graph $G' = (V', E')$ with $n' = |V'| = n$ and $m' = |E'| = m + d$ by adding edges. Then, we use Theorem 2 to draw $G'$ on an $O(n) \times O(n^2)$ grid with at most $m' - n/3 - 2/3$ segments in $O(n)$ time. Finally, we remove the $d$ added edges to obtain a drawing with at most $m' - n/3 - 2/3 + d = m - n/3 - 2/3 + 2d$ segments.

Unfortunately, adding a minimum number of edges to a planar graph to make it 3-connected is NP-hard [24] and we are not aware of any good bounds on the number of edges required. In the worst case, $G'$ is a triangulation, so we have $m' \leq 3n - 6$ and $d \leq 3n - 6 - m$. Hence, our drawing uses at most $m - n/3 - 2/3 + 2d \leq m - n/3 - 2/3 + 6n - 12 - 2m = 17n/3 - m - 38/3$ segments.

We can again use a simple case distinction to give a bound on the number of segments purely in terms of $n$. If $m \leq 17n/6 - 38/6$, then we draw $G$ with $m$ segments. Otherwise, we use Theorem 6 to draw $G$ with at most $17n/3 - m - 38/3 \leq 17n/6 - 38/6$ segments.

Corollary 4. Any planar graph can be drawn planar on an $O(n) \times O(n^2)$ grid with $(17n - 38)6$ segments in $O(n)$ time.

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