A Time-Asymmetric Process in Central Force Scatterings

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We show that in attractive force fields a high speed particle with a small mass speeding through space, statistically loses energy by colliding softly and transversely with large masses that move slowly and randomly. Furthermore, we show that the opposite holds in repulsive force fields: the light particle statistically gains energy. We consider the large masses to be distant from one another such that a series of two-body scatterings would suffice for the statistical analysis. The difference of this process from equilibration is discussed.

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Dynamical Description- In addition to the well known gravitational and Coulomb, nearly all other interactions in nature such as intermolecular forces and interaction of vortices in superconductors are central. In conservative fields, the central force on each particle can be derived from a potential function by $F = -\nabla V(r)$ where $r$ is the distance between the two bodies and $V(r) = \frac{\alpha}{r}$; here $\alpha$ is the strength of the interaction depending on the parameters of the problem, $k$ defines the range of the interaction [1, 2]. Lastly, $\alpha < 0$ and $\alpha > 0$ correspond to attractive and repulsive force fields respectively [3].

In this paper we study a fast small mass passing through a dilute system of randomly moving central forces, where changes in the state of the small mass can be well approximated by a series of two-body scatterings. For the purposes of this section, it is best to think of the scatterings in the lab frame. Consider a scattering between two interacting particles $m_1$ and $m_2$ where $m_1$ is much lighter ($m_1 \ll m_2$) yet much faster than the other ($v_1 \gg v_2$) but nevertheless $m_1 v_1^2 \ll m_2 v_2^2$. For example one can visualize a small comet ($m_1$) undergoing a small angle scattering in the gravitational field of the planet Jupiter ($m_2$). For the sake of concreteness take the potential to be attractive for now. In a typical scattering both masses are initially moving. The questions we are interested in investigating are: What statistically invariant features are shared by such scatterings in central force fields? Would many such small angle scatterings have a net effect on the energy of $m_1$?

While there is an active frontier of numerical work on many-body simulations [4], there are still interesting statistical inferences that can be derived from close analysis of two-body collisions. Further, such inferences in the cases where the collisions are head on have been extensively studied as in the Fermi acceleration mechanism [5]. In contrast the main contribution to the effect presented in this paper is from collisions that are transverse; i.e., the trajectory of the massive particle, roughly speaking, is perpendicular to that of the small particle.

The organization is as following: 1) We give a heuristic description of the effect, 2) We provide a numerical verification 3) We prove the result by giving a mathematical derivation, 4) Lastly we discuss this effect in a larger statistical context and discuss its difference from the equilibration process.

Let us consider two extreme cases that would convey the gist of what underlies this work. In the first case the massive particle, $m_2$, slowly and transversely veers away from the trajectory of $m_1$ that is speeding by. In the second case, $m_2$ slowly and transversely approaches the trajectory of $m_1$ (see Figure 3a).

In the first case where $m_2$ is moving away, $m_1$ falls into the potential well of $m_2$ and so long as it is approaching the point of minimum distance it gains kinetic energy. After passing this point, $m_1$ starts climbing up the potential well and pays back the gained kinetic energy by restoring it into the potential energy of the two-body system. However, on the way out of the well it climbs a well that is effectively smaller than the one it fell into as $m_2$ is on average farther away from it. Therefore, in the case that the big mass is moving transversely away from the particle, the small particle emerges with a gain in the kinetic energy i.e. $\frac{1}{2} m_1 |v_{m_1}|_2^2 < \frac{1}{2} m_1 |v_{m_1}|_+^2$.

The exact opposite effect holds in the second case, where $m_2$ is moving towards $m_1$. In this case, $m_1$ enters the potential well set up by $m_2$ and, as in the previous case, gains kinetic energy so long as it is approaching the minimum distance between the two masses. However, on the way out of the well it faces a more demanding climb as $m_2$ is on average closer to it and the potential well is steeper and deeper than before. Therefore, in the case that the big mass is transversely approaching the particle (Figure 4), the small particle emerges with a loss in the kinetic energy i.e. $\frac{1}{2} m_1 |v_{m_1}|_2^2 > \frac{1}{2} m_1 |v_{m_1}|_+^2$.

The point however is that the two cases are not symmetric. The decreasing of the magnitude of the force with distance breaks the symmetry between the two cases. This is shown in Figure 4 In an attractive force field, $m_1$ has a greater loss (in magnitude) of energy when $m_2$ approaches it than a gain (in magnitude) when $m_2$ recedes away from it. This asymmetry, deduced from dynamical principles, has consequences for the statistical mechanics of $m_1$ that we will explore.

In repulsive force fields, $\alpha > 0$, the potential in Fig-
is moving along or opposite to the direction of its asymptotic motion, a satellite can be carefully steered so that it has higher energies is not new. In the gravity assist or slingshot effect, a satellite can be carefully steered so that it can lose or gain energy by scattering from larger masses (e.g., stars in a galaxy) that randomly either approach it or move away from it in the manner described above, the small mass statistically loses (gains) energy to (from) the big masses when the force fields are attractive (repulsive).

The fact that a high speed particle with low energy can lose or gain energy by scattering from larger masses with higher energies is not new. In the gravity assist or the slingshot effect, a satellite can be carefully steered so that, by elastically scattering from a planet or a star that is moving along or opposite to the direction of its asymptotic motion ($t \to \infty$), it acquires a boost in energy or brakes to save fuel. The intuition in gravity assist, as in this work, is that to boost the energy of the satellite we have to throw it behind the approaching planet and vice versa. Slingshot effect, however, is not a net statistical effect resulting from an underlying random process.

We proceed to examine a scenario where the large masses move in random directions.

A Numerical Experiment- We simulate a process where $m_2 / m_3 = 5 \times 10^{-8}$, $\alpha = -0.13346$, and $k = 1$. The initial conditions for $m_1$ and initial position of $m_2$ are taken to be the same in all the simulations (see Appendix for the numerical details). To single out the effect of the direction of the motion of the large mass on the statistical energetics of $m_1$, we take the velocity of $m_2$ to be: $v_{m_2}^x = v_2 \cos \theta$ and $v_{m_2}^y = v_2 \sin \theta$, where $v_2$ is kept constant, $v_1 / v_2 \approx 10$ and $\theta$ is uniformly distributed on a circle $\theta = [0, \pi/2, \ldots, 2\pi]$. We find that $m_1$ on average loses energy to $m_2$ (Figure 2). It is instructive to look at the points corresponding to the two extreme cases discussed in the previous section. When $m_2$ moves towards the trajectory of $m_1$ along the path of minimum approach ($\theta \sim \pi/2$), maximum loss is seen; furthermore, when $m_2$ veers away from the path of minimum distance ($\theta \sim 0$), the maximum gain is seen. Both are congruent with our expectations. In Figure 2 the $\theta$-average of the change in the kinetic energy of $m_1$ is shown in red, which is seen to be less than the kinetic energy of $m_1$ when $m_2$ is pinned down (i.e. $v_2 = 0$ for all times). In the latter case, used as a calibration here, the asymptotic kinetic energy of $m_1$ is conserved (shown with a black line in Figure 2).

Analytical Derivation- We are considering an standard elastic collision between two particles. Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be the velocities of the two bodies in the lab frame; the velocities in the center of mass ($C$ system denoted by the suffix 0) are related to the velocities in the lab by $\mathbf{v}_{10} = m_2 \mathbf{v} / (m_1 + m_2)$, $\mathbf{v}_{20} = -m_1 \mathbf{v} / (m_1 + m_2)$, where $\mathbf{v} = \mathbf{v}_1 - \mathbf{v}_2$. In the $C$ system the collision simply rotates the velocities, which remain opposite in direction and unchanged in magnitude. If we denote by $\mathbf{n}_0$ the unit vector in the direction of the velocity of $m_1$ after the collision, then the velocities of the two particles after the collision (distinguished by primes) are $\mathbf{v}_{10}' = m_2 \mathbf{n}_0 / (m_1 + m_2)$ and $\mathbf{v}_{20}' = -m_1 \mathbf{n}_0 / (m_1 + m_2)$. In order to obtain the final velocities in the lab frame, denoted by $\mathbf{v}_1$ and $\mathbf{v}_2$, we add the velocity of the center of mass $\mathbf{V} = (m_1 \mathbf{v}_1 + m_2 \mathbf{v}_2) / (m_1 + m_2)$ to $\mathbf{v}_{10}$ and $\mathbf{v}_{20}$.

\begin{align}
\mathbf{v}_1' &= m_2 \mathbf{n}_0 / (m_1 + m_2) + \mathbf{V}, \\
\mathbf{v}_2' &= -m_1 \mathbf{n}_0 / (m_1 + m_2) + \mathbf{V}.
\end{align}
No further information about the collision can be obtained from the laws of conservation of momentum and energy. The direction of the unit vector $\hat{n}_0$ depends on the particular law of interaction and positions during the collision. We assume the massive particles are far enough from one another that a sequence of two-body scatterings would be adequate to describe the process \cite{9}. The change in the kinetic energy of $m_1$ before and after a collision is $\Delta K_{E_{m_1}} = \frac{\mu v}{n^2} \left( v^2_1 - v^2_1 \right)$ which reads

$$\Delta K_{E_{m_1}} = \mu v \cdot \left[ n^2_0 - n^2_0 \right] \equiv \mu v V \cdot n$$ \hspace{1cm} (3)

where, $\mu = \frac{m_1 m_2}{m_1 + m_2}$ is the reduced mass, $n_0^2$ and $n_0^2$ denote the unit vectors pointing in the direction of motion of $m_1$ before and after the collision in the $C$ frame and $n = n_0^2 - n_0^2$ is their difference. Before calculating Eq. 3 as a function of $\theta$, once again, we look at the two special cases discussed above. First consider an attractive force field. Suppose $V$ lies on the line of the minimum distance as shown in Figure 3. Clearly if $m_2$ recedes away from $m_1$ then $V$ points in the same direction as $n$ and the dot product on the right hand side of 3 evaluates to be positive, whereas if $V$ and $n$ point in opposite directions the right hand side is negative. In the case of repulsion the signs would be the opposite (see Figure 3i).

What we now show is that the two cases are not symmetric. That is the kinetic energy loss (gain) in the approaching case is larger than the gain (loss) in the receding case for an(a) attractive (repulsive) potential. The reason is that if we keep the setting the same but only change the direction of $V$, both $v$ and $n$ will change. In the attractive case (Figure 3i), if $m_2$ moves towards $m_1$ the angle between the asymptotes, $2\psi$, is smaller than it would be if $m_2$ moved away. Therefore, $|n| = 2\cos\psi$ is larger in the approaching case. For very high speed encounters, $|v_1| \gg |v_2|$, we can approximate $v$ to be the same in the two cases, therefore Eq. 3 becomes:

$$\Delta K_{E_{m_1}} = \mu v V \cdot n \approx \left\{ \begin{array}{ll} -\frac{2\mu v \cos\psi_a}{2\mu v \cos\psi_r}, & \cos\psi_a > \cos\psi_r, \\
\end{array} \right.$$ \hspace{1cm} (4)

We next consider the case where $m_2$ moves in random directions. We do so by applying the above to the cases of gravitational and Coulomb interactions, where the equations of motion are particularly simple to integrate \cite{10} and analytical results can be compared to the numerical experiments of the previous section. We first parametrize: $v_2 = v_2\cos\hat{a} + v_2\sin\hat{a}$ which is equal to $V$ in the limit where $m_2 \gg m_1$. The orbits of scatterings in $V(r) = -\alpha/r$ (here $\alpha$ is taken to be positive) are hyperbolas parametrized by \cite{8}: $r = a(\cos\hat{\xi} - 1), \quad x = a(e - \cos\hat{\xi}), \quad y = a\sqrt{e^2 - 1}\sin\hat{\xi}$

and $t = \sqrt{\frac{a}{\alpha}}(\sinh\hat{\xi} - \xi)$, where $\xi$ (hence time) varies from $-\infty$ to $\infty$ and $v = \frac{\partial}{\partial t} \frac{\partial}{\partial \hat{\xi}} + \frac{\partial}{\partial \hat{p}} \frac{\partial}{\partial \hat{p}}$. We are interested in the change in the kinetic energy of the smaller mass in the lab frame \cite{11}. Since, $m_2 \gg m_1 \Rightarrow \mu = m_1$; we have for $\Delta K_{E_{m_1}} \equiv \lim_{t \to \infty}(K_{E_{m_1}}) - \lim_{t \to \infty}(K_{E_{m_1}})$, which reads

$$\Delta K_{E_{m_1}} = -2\sqrt{\frac{m_1}{\alpha}}\sinh\hat{\xi} - \xi,$$ \hspace{1cm} (5)

with $D$ being the impact parameter, which also depends on the initial angle between the velocity of the two masses. The strength of interaction for gravitational and Coulomb attractions are $\alpha = Gm_1m_2$ and $\alpha = \frac{|q_1q_2|}{4\pi\varepsilon_0}$ respectively.

For the two cases discussed above we have for the approaching case: $\theta = 0$ and $v_0 \equiv v'$. And for the receding case $\theta = \pi$ and $v_0 \equiv v'$. It is easy to see that $v' < v'$; therefore we yet again arrive at the same conclusion.

In Figure 3 we plot Eq. 5 as a function of $\theta$ and see that our theoretical treatment and numerical experiment Figure 2 agree.

A formal statistical treatment depends on the particular laws of interaction as well as the statistical distribution of the large masses. For $V = \alpha/r$ it remains to show that under the assumption of microscopic chaos in the direction of motion of the large masses, the effect holds for any generic choice of initial conditions. Equation 5 can be written entirely in terms of initial conditions in the lab frame as (see Appendix H1C)

$$\Delta K_{E_{m_1}} \simeq -\frac{2\alpha V}{Dv_\infty} \cos \theta = -\frac{2\alpha V}{|p|} \cos \theta \left\{ \begin{array}{ll} \sin \theta - q/p, \\
\end{array} \right.$$ \hspace{1cm} (6)
where, $p \equiv (x_2 - x_1) v_2$ and $q \equiv -(y_2 - y_1) v_1$ encode the dependence on the initial conditions. Now let $z = q/p$ and $\langle \cdot \rangle_{\theta}$ denote the $\theta$- average, we have

$$\langle \Delta KE_{m_i} \rangle_{\theta} \propto f(z) \equiv \left( \frac{-\cos \theta}{|\sin \theta - z|} \right)_{\theta} \quad (7)$$

where $f$ encodes the dependence on the choice of initial conditions. Figure 5 establishes the existence of the effect irrespective of particular choices of initial conditions in the scattering process as long as the assumptions leading up to Eq. 5 are intact.

One can use mechanics of similarity to calculate $\Delta KE_{m_1}$ when the law of interaction is Coulombic [3]. If we fix the corresponding paths we can related the change in the kinetic energy of the two: $\Delta KE_{m_1} = \frac{d}{dz} \Delta KE_{m_1}$, where we have explicitly labeled the quantities corresponding to the gravitational and the Coulomb interactions by $g$ and $c$ respectively. Furthermore, the corresponding time of travel between two fixed points on the path are related by $t_g = \frac{t_c}{\alpha}$. Consequently, for repulsive Coulomb interaction our final result reads,

$$\Delta KE_{m_1} = \frac{2\alpha_c V}{Dv_\infty} \cos \theta \quad (8)$$

where $\alpha_c = \frac{\alpha_g}{4\pi e_0}$ as before. We see that the equation we get is the negative of Eq. 5 which, as expected, corresponds to a net gain in energy when averaged over $\theta$. A calculation from first principles yields the same result (see Eq. 13 in Appendix II).

**Statistical Context:** A phenomena worth considering is dynamical friction [14, 15]. Dynamical friction, however, is like Brownian motion [12] as a big mass enters a medium of many smaller particles and slows down as a result. But it is distinct from Brownian motion as the interactions are long ranged. It is found that in dynamical friction “only stars with velocities less than the one under consideration contribute to the effect” [16] and [8], p. 299.

The main requirement in our work is that the particle scatters from the time-dependent field of other massive particles, that are moving randomly and slowly in space, through a series of two-body scatterings. In addition, it is required that during the effective scattering the scatterer remains on one side of the scattering particle. This is equivalent to the assumption of the slow motion, in the dilute medium, of the large masses.

It would be interesting to analyze the effect of microdynamics of the structure in the universe on the frequency shift of photons coming from distant sources. We expect a small loss of energy for photons that undergo weak lensing [9, 10] in the interstellar media of galaxies, clusters and super clusters [21]. Clearly the potential fluctuations, derived within this framework, in energy of the photons will be due to the dynamics of the lenses and disjoint from the essential time dependence of large scale space-time metric in the general theory of relativity as in Sachs-Wolfe and Rees-Sciama effects [17, 19].

At first sight the effect we are proposing seems to violate the equipartition of energy because a smaller and lighter particle “heats up” the medium of much larger particles that have higher average energies. But note that we are working with a non-equilibrium process in an open system [13]. In attractive potentials the small particle starts from non-equilibrium initial conditions and through a series of scatterings it statistically loses energy till a final scattering where the energy in the center of mass is negative. There on the small particle would have a bounded orbit about that final scatterer. This corresponds to the breakdown of small angle scattering assumption of this paper. In plasma physics this is known as shielding and in astrophysics it corresponds to capturing of a comet by a center of force. In repulsive potentials the energy of the small particle grows till the relativistic effects become significant and the transfer of energy between the small particle and the scatterers becomes of order unity with respect to the initial energy [22].
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1. APPENDIX: NUMERICAL DETAILS

The computer used is an IBM ThinkPad laptop. The experiments were done using an adaptive Bulirsch-Stoer algorithm \[24\] in C++. The simulation was done in three dimensions and I checked to make sure that the orbit stays in the plane and that the total energy is conserved; both to an accuracy of \(O(10^{-12})\). The data was then extracted and the plots were made in Matlab (Figure 2). The theoretical plots were all generated in Matlab. As stated before, we investigated the case where \(m_1/m_2 = 5 \times 10^{-8}\), \(\alpha = -0.13346\), and \(k = 1\).

For numerical purposes it is convenient to rewrite the equation

\[
\Delta KE = -2V \sqrt{m_1 \alpha \cos \psi \cos \theta}
\]

in terms of geometrical constants of the orbit, namely eccentricity \(e\) and \(a = \frac{\mu}{e^2}\). Hence, \(e \rightarrow e(\theta); a \rightarrow a(\theta)\) and are given by the energy and the angular momentum, which are in turn function of the initial conditions.

Therefore Eq. \[9\] reads \(\Delta KE_{m_1} = -2V \sqrt{\frac{m_1 \alpha}{a}} \cos \theta\), where \(a = \frac{\mu}{e^2}\) and \(e = \sqrt{1 + \frac{2KE}{m_1 \alpha}}\). Here \(l\) and \(E\) denote the magnitude of the angular momentum and the energy in the center of mass frame.

A gravitational model for this choice of parameters is one in which (in MKS units) \(m_1 = 1.0\), \(m_2 = 2 \times 10^9\), \(G = 6.67300 \times 10^{-11}\). Therefore we have \(\mu = \frac{m_1 m_2}{m_1 + m_2} = 0.9999999995\) and \(\alpha = 0.13346\). The big mass \(m_2\) was initially at the origin and \(m_1\) had the coordinates \(x_0 = -10000, y_0 = -2000\) with initial velocity \(v_{x0} = 0.008, v_{y0} = 0.005\). The initial velocity of \(m_2\) was \(v_2 = V \cos \theta \hat{i} + V \sin \theta \hat{j}\); therefore, \(v_{0}(\theta) = (v_{x0} - V \sin \theta) \hat{i} + (v_{y0} - V \cos \theta) \hat{j}\) from which we can calculate the constants of the orbit. The angular
momentum is conserved and is given by, \( \mathbf{I} = \mu r_0 \times \mathbf{v}_0 = \mu [x_0(v_{y0} - V \cos \theta) - y_0(v_{x0} - V \sin \theta)] \mathbf{\hat{k}} \), and the energy is given by, \( E = \frac{1}{2} \mu |\mathbf{v}_0(\theta)|^2 - \frac{\mu}{|r_0|} \). Therefore the orbit constants \( a \) and \( e \) are specified and so is \( \mu \) which is plotted in Figure 4 as a function of \( \theta \) and compared to the experimental case.

The experiments were done by integrating Newton’s equations of motions in the lab frame subject to the same initial conditions. Namely, I integrated

\[
\begin{align*}
\ddot{x}_1 &= -\frac{Gm_2}{|r_1|^3} (x_1 - x_2) \\
\ddot{x}_2 &= -\frac{Gm_1}{|r_2|^3} (x_2 - x_1)
\end{align*}
\]

using Bulirsch-Stoer integrator. I then saved the state of both particles (position and velocity) as well as the change in the energy of the whole system to make sure it was constant. By conservation of angular momentum motion must take place in a plane, which I took to be the \( xy \) plane and monitored the \( z \) component of the position of both bodies. Accuracies were \( \mathcal{O}(10^{-12}) \). Furthermore I calculated, in the C++ code, the kinetic energies of both particles for values of \( \theta = p \mu \), where \( p = 0, \ldots, 19 \). These were written in .txt files that in turn were uploaded in a Matlab code which plotted the asymptotic value of the kinetic energy of \( m_1 \) vs. \( \theta \). The result is Figure 2.

II. APPENDIX: DERIVATIONS IN THE CASE OF COULOMB REPULSION FROM FIRST PRINCIPLES

The gain in the repulsive force field with \( k = 1 \) is done similarly. Let \( \xi \):

\[
\begin{align*}
r &= a (e \cos \xi + 1) , & t &= \sqrt{\frac{m_2}{m_1}} a (e \sin \xi + \xi) \\
x &= a (\cos \xi + e) , & y &= a \sqrt{(e^2 - 1)} \sin \xi.
\end{align*}
\]

Consequently, \( v = \sqrt{\frac{a}{m_2}} e \sin \xi + e \sqrt{\frac{m_2}{m_1}} \cos \xi \) and \( v \equiv \frac{\alpha}{m_2} \frac{e \sin \xi + e \sqrt{\frac{m_2}{m_1}} \cos \xi}{e \sin \xi + e \sqrt{\frac{m_2}{m_1}} \cos \xi + 1} \). As for the attractive case the motion of \( m_2 \) in the lab frame taken to be \( \mathbf{V} = V \cos \theta \mathbf{i} + V \sin \theta \mathbf{j} \). Eq. 3 becomes:

\[
\lim_{\xi \to \infty} \Delta \mathbf{K}_m = 2V \sqrt{\frac{\alpha}{m_2}} \cos \psi \cos \theta
\]

\[
\cos \psi = \frac{\alpha/m_1 v_0^2 D}{\sqrt{1 + (\alpha/m_1 v_0^2 D)^2}} \tag{12}
\]

for small angle scatterings \( \cos \psi \sim 0 \); therefore, \( \cos \psi \sim \alpha/m_1 v_0^2 D \ll 1 \). The final form becomes:

\[
\Delta \mathbf{K}_m \approx \frac{2\alpha V}{D} \cos \theta \tag{13}
\]

III. APPENDIX: TRANSFORMATION BETWEEN THE LAB AND CENTER OF MASS FRAME

A. Transformation from the Lab Frame to the Center of Mass Frame

Given the initial conditions in the lab frame we wish to obtain the parameters of the orbit which fully specify the problem in the center of mass frame.

| In the Lab at \( t_0 \) | In the COM |
|-------------------------|------------|
| \( r_1 = (x_1, y_1) \)  | \( E \)     |
| \( v_1 = (v_{x1}, v_{y1}) \) | 1          |
| \( r_2 = (x_2, y_2) \)  | \( D \)     |
| \( v_2 = (v_{x2}, v_{y2}) \) | \( v_\infty = |v_1 - v_2|_{t=f,\infty} \) |

As before we have \( \mu = \frac{m_1 m_2}{m_1 + m_2} \), and at \( t = t_0 : r_0 = (r_1 - r_2) \), \( v_0 = v_1 - v_2 \). The energy in the lab is given by \( E_L = \frac{m_1}{2} v_1^2 + \frac{m_2}{2} v_2^2 - U(r_0) \) and is related to the energy in the center of mass by:

\[
E = E_L - \frac{(m_1 + m_2)}{2} V_{COM}^2 \tag{14}
\]

where,

\[
V_{COM}^2 = \frac{(m_1 v_1 + m_2 v_2)^2}{m_1 + m_2} \tag{15}
\]

\[
= \frac{m_1^2 v_1^2 + m_2^2 v_2^2 + 2 m_1 m_2 (v_1 v_2)}{(m_1 + m_2)^2} \tag{16}
\]

\[
= V_0^2 + \frac{2 \mu_1 v_2 \cos \theta}{(m_1 + m_2)} \tag{17}
\]

\( V_{COM} \) is used to emphasize that it is obtained from the initial conditions; otherwise it is equivalent to \( V \).

The last equality is obtained by assuming \( v_1 \), with no loss of generality, to point along the \( x \)-axis; further \( V_0 \) is independent of \( \theta \). Therefore, the energy in the COM reads,

\[
E = E_0 - \mu_1 v_2 \cos \theta; \text{ where} \tag{18}
\]

\[
E_0 \equiv E_L - \frac{(m_1 + m_2)}{2} V_0^2 \tag{19}
\]

Angular Momentum: \( \mathbf{l} = \mu r_0 \times \mathbf{v}_0 \) denotes the angular momentum and relative velocity. At infinite separation, \( v_\infty \), is determined by the energy through, \( E = \frac{\mu v_\infty^2}{2} \).
Lastly, the impact parameter $D$ is determined by the angular momentum and $v_\infty$ via $D = \frac{1}{\mu v_\infty}$. With this, the problem is fully specified in the center of mass frame from the quantities given in the lab frame.

### B. From Center of Mass (COM) to The Lab Frame

This is more standard and is done in most textbooks. See for example [8, Section 14]. Though, what we said applies to all central forces, recall that $k = 1$ corresponds to gravitational and Coulomb attraction. For $k = 2$ the integrals can be integrated in terms of elementary functions [8, p. 40].

### C. Formulation of Our Results Entirely in The Lab Frame

Couple of remarks are in order before we proceed, first note that when $m_2 \gg m_1$, $\mu = m_1$ and therefore, $V_{COM} = \sqrt{V_0^2 + \frac{2m_1 m_2 v_1 v_2 \cos \theta}{(m_1 + m_2)}} \approx V_0$.

From section 8.3, we can obtain all the quantities in the lab frame, in particular we no loss of generality we take, $v_1 = v_1 \hat{i}$, $v_2 = v_2 \cos \theta \hat{i} + v_2 \sin \theta \hat{j}$. This specifies $v_0 \equiv v_1 - v_2$. In addition, $r_0 = (x_1 - x_2) \hat{i} + (y_1 - y_2) \hat{j}$. Hence, from Eqs. [9] and [8] we have,

$$\Delta KE_{m_1} = -\frac{2m_1 E_0 \cos \theta}{m_1 \alpha^2 + 2l^2 (E_0 - m_1 v_1 v_2 \cos \theta)} \frac{(2 E_0 - m_1 v_1 v_2 \cos \theta) \cos^2 \theta}{m_1 \alpha^2 + 2l^2 (E_0 - m_1 v_1 v_2 \cos \theta)}$$

Note that the expression depends on the square of the angular momentum. More importantly note that the pre-factor to the $\cos \theta$ depends on $\theta$. The angular momentum expressed in lab quantities reads,

$$\mathbf{l} = m_1 [(x_1 - x_2) (-v_2 \sin \theta) - (y_1 - y_2) (v_1 - v_2 \cos \theta)] \approx m_1 [v_2 (x_2 - x_1) \sin \theta - v_1 (y_1 - y_2)]$$

which when substituted into (21) gives

$$\Delta KE_{m_1} = -\frac{2V_0 \sqrt{2m_1 E_0 - m_1^2 v_1 v_2 \cos \theta \cos \theta}}{\sqrt{1 + \frac{2l^2 (E_0 - m_1 v_1 v_2 \cos \theta)}{m_1 \alpha^2}}}.$$