The sub-$k$-domination number of a graph with applications to $k$-domination

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Abstract

In this paper we introduce and study a new graph invariant derived from the degree sequence of a graph $G$, called the sub-$k$-domination number and denoted $\text{sub}_k(G)$. We show that $\text{sub}_k(G)$ is a computationally efficient sharp lower bound on the $k$-domination number of $G$, and improves on several known lower bounds. We also characterize the sub-$k$-domination numbers of several families of graphs, provide structural results on sub-$k$-domination, and explore properties of graphs which are sub$_k(G)$-critical with respect to addition and deletion of vertices and edges.

Keywords: sub-$k$-domination number, $k$-domination number, degree sequence index strategy

AMS subject classification: 05C69

1 Introduction

Domination is one of the most well-studied and widely applied concepts in graph theory. A set $S \subseteq V(G)$ is dominating for a graph $G$ if every vertex of $G$ is either in $S$, or is adjacent to a vertex in $S$. A related parameter of interest is the domination number, denoted $\gamma(G)$, which is the cardinality of the smallest dominating set of $G$. Much of the literature on domination is surveyed in the two monographs of Haynes, Hedetniemi, and Slater [11, 12]. For more recent results on domination, see [5, 6, 10, 24] and the references therein.

In 1984, Fink and Jacobson [9] generalized domination by introducing the notion of $k$-domination and its associated graph invariant, the $k$-domination number. Given a positive integer $k$, $S \subseteq V(G)$ is a $k$-dominating set for a graph $G$ if every vertex not in $S$ is adjacent to at least $k$ vertices in $S$. The minimum cardinality of a $k$-dominating set of $G$ is the $k$-domination number of $G$, denoted $\gamma_k(G)$. When $k = 1$, the 1-domination number is precisely the domination number; that is, $\gamma_1(G) = \gamma(G)$. Like domination, $k$-domination has also been extensively studied; for results on $k$-domination related to this paper, we refer the reader to [2, 4, 8, 13, 21, 22].

Computing the $k$-domination number is $NP$-hard [17], and as such, many researchers have sought computationally efficient upper and lower bounds for this parameter. In general, the degree sequence of a graph can be a useful tool for bounding $NP$-hard graph invariants. For example, the residue and annihilation number of a graph are derived from its degree sequence, and are

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respectively lower and upper bounds on the graph’s independence number (cf. [7,20]). Another example is a lower bound on the domination number due to Slater [23], which will be discussed in the sequel. Recently, Caro and Pepper [1] introduced the degree sequence index strategy, or DSI-strategy, which provides a unified framework for using the degree sequence of a graph to bound NP-hard invariants. In this paper we introduce a new degree sequence invariant called the sub-\(k\)-domination number, which is a sharp lower bound on the \(k\)-domination number; our investigation contributes to the known literature on both degree sequence invariants and domination.

Throughout this paper all graphs are simple and finite. Let \(G = (V(G), E(G))\) be graph. Two vertices \(v\) and \(w\) in \(G\) are adjacent, or neighbors, if there exists an edge \(vw \in E\). A vertex is an isolate if it has no neighbors. The complement of \(G\) is the graph \(\overline{G}\) with the same vertex set, in which two vertices are adjacent if and only if they are not adjacent in \(G\). A set \(S \subseteq V(G)\) is independent if no two vertices in \(S\) are adjacent; the cardinality of the largest independent set in \(G\) is denoted \(\alpha(G)\). For any edge \(e \in E(G)\), \(G - e\) denotes the graph \(G\) with the edge \(e\) removed; For any vertex \(v \in V(G)\), \(G - v\) denotes the graph \(G\) with the vertex \(v\) and all edges incident to \(v\) removed; for any edge \(e \in E(G)\), \(G + e\) denotes the graph \(G\) with the edge \(e\) added. The degree of a vertex \(v\), denoted \(d(v)\), is the number of vertices adjacent to \(v\). We will use the notation \(n(G) = |V(G)|\) to denote the order of \(G\), \(\Delta(G)\) to denote the maximum degree of \(G\), and \(\delta(G)\) to denote the minimum degree of \(G\); when there is no scope for confusion, the dependence on \(G\) will be omitted. We will also use \(d_i\) to denote the \(i\)th element in the degree sequence of \(G\), denoted \(D(G) = \{\Delta = d_1 \geq d_2 \geq \cdots \geq d_n = \delta\}\), which lists the vertex degrees in non-increasing order. We may abbreviate \(D(G)\) by only writing distinct degrees, with the number of vertices realizing each degree in superscript. For example, the star \(K_{n-1,1}\) may have its degree sequence written as \(D(K_{n-1,1}) = \{n-1,1^{n-1}\}\), and the complete graph \(K_n\) may have degree sequence written as \(D(K_n) = \{(n-1)^n\}\). For other graph terminology and notation, we will generally follow [15].

This paper is organized as follows. In the next section, we introduce the sub-\(k\)-domination number of a graph and show that it is a lower bound on the \(k\)-domination number. In Section 3, we characterize the sub-\(k\)-domination numbers of several families of graphs and provide other structural results on sub-\(k\)-domination. In Section 4, we compare the sub-\(k\)-domination number to other known lower bounds on the \(k\)-domination number. In Section 5, we explore the properties of sub\(_k\)(\(G\))-critical graphs. We conclude with some final remarks and open questions in Section 6.

## 2 Sub-\(k\)-domination

In this section we introduce the sub-\(k\)-domination number of a graph and prove that it is a lower bound on the \(k\)-domination number. We first recall a definition and result due to Slater [23], which is a special case of our result. For consistency in terminology, we will refer to Slater’s definition as the sub-domination number of a graph; this invariant was originally denoted \(\text{sl}(G)\), and for our purposes will be denoted \(\text{sub}(G)\).

**Definition 1** ([23]). The sub-domination number of a graph \(G\) is defined as

\[
\text{sub}(G) = \min \left\{ t : t + \sum_{i=1}^{t} d_i \geq n \right\}.
\]

**Theorem 1** ([23]). For any graph \(G\), \(\gamma(G) \geq \text{sub}(G)\), and this bound is sharp.
For any \( k \geq 1 \), the \( k \)-domination number is monotonically increasing with respect to \( k \); that is, \( \gamma_k(G) \leq \gamma_{k+1}(G) \). Keeping monotonicity in mind, it is natural that a parameter generalizing \( \text{sub}(G) \) will need to increase with respect to increasing \( k \). This idea motivates the following definition.

**Definition 2.** Let \( k \geq 1 \) be an integer, and \( G \) be a graph. The \( sub-k \)-domination number of \( G \) is defined as

\[
\text{sub}_k(G) = \min \left\{ t : t + \frac{1}{k} \sum_{i=1}^{t} d_i \geq n \right\}
\]

Since the vertex degrees of \( G \) are integers between 0 and \( n-1 \), the sorted degree sequence of \( G \) can be obtained in \( O(n) \) time by counting sort (assuming vertex degrees can be accessed in \( O(1) \) time). By maintaining the sum of the first \( t \) elements in \( D(G) \) and incrementing \( t \), \( \text{sub}_k(G) \) can be computed in linear time; we state this formally below.

**Observation 2.** For any graph \( G \) and positive integer \( k \), \( \text{sub}_k(G) \) can be computed in \( O(n) \) time.

Taking \( k = 1 \) in Definition 2, we observe \( \text{sub}_1(G) = \text{sub}(G) \), and hence \( \text{sub}_1(G) \leq \gamma_1(G) \) by Theorem 1. More generally, we will now show that the \( k \)-domination number of a graph is bounded below by its \( sub-k \)-domination number.

**Theorem 3.** For any graph \( G \) and positive integer \( k \), \( \gamma_k(G) \geq \text{sub}_k(G) \), and this bound is sharp.

**Proof.** Let \( S = \{v_1, \ldots, v_t\} \) be a minimum \( k \)-dominating set of \( G \). By definition, each of the \( n - t \) vertices in \( V(G) \setminus S \) is adjacent to at least \( k \) vertices in \( S \). Thus, the sum of the degrees of the vertices in \( S \), i.e. \( \sum_{i=1}^{t} d(v_i) \), is at least \( k(n - t) \). Dividing by \( k \) and rearranging, we obtain

\[
t + \frac{1}{k} \sum_{i=1}^{t} d(v_i) \geq n.
\]

Since the degree sequence of \( G \) is non-increasing, it follows that \( \sum_{i=1}^{t} d_i \geq \sum_{i=1}^{t} d(v_i) \). Thus,

\[
t + \frac{1}{k} \sum_{i=1}^{t} d_i \geq n. \tag{1}
\]

Since \( \text{sub}_k(G) \) is the smallest index for which (1) holds, we must have \( \text{sub}_k(G) \leq t = \gamma_k(G) \).

When \( k = 1 \), note that \( \text{sub}(K_{n-1,1}) = 1 = \gamma(K_{n-1,1}) \). When \( k > 1 \), let \( G \) be a complete bipartite graph with a perfect matching removed where each part of the vertex partition is of size \( k + 1 \). Then \( \text{sub}_k(G) = \min\{t : t + \frac{1}{k} \sum_{i=1}^{t} k \geq n\} = k + 1 = \gamma_k(G) \). Thus, the bound is sharp for all \( k \).

In the next section, we compute \( \text{sub}_k(G) \) for several families of graphs and investigate graphs for which \( \text{sub}_k(G) = \gamma_k(G) \).
Proposition 4. Let \( G \) be a graph with \( \Delta \geq n - 2 \). Then, \( \text{sub}(G) = \gamma(G) \).

Proof. If \( \Delta = n - 1 \) then \( \gamma(G) = 1 \) and thus \( \text{sub}(G) = \gamma(G) \), since by Theorem 3, \( 1 \leq \text{sub}(G) \leq \gamma(G) = 1 \). If \( \Delta = n - 2 \), then \( \gamma(G) = 2 \) since no single vertex can dominate the graph, but a maximum degree vertex and its non-neighbor is a dominating set. Moreover, \( \text{sub}(G) \neq 1 \) since \( 1 + (n - 2) < n \); thus, \( 2 \leq \text{sub}(G) \leq \gamma(G) = 2 \).

If \( G \) is a graph with \( \Delta \leq n - 3 \), then \( \text{sub}(G) \) may not be equal to \( \gamma(G) \). For example, let \( G \) be the graph obtained by appending a degree one vertex to two leaves of \( K_{1,3} \); it can be verified that \( \gamma(G) = 3 \) and \( \text{sub}(G) = 2 \).

Proposition 5. Let \( G \) be a graph with \( \gamma(G) \leq 2 \). Then \( \text{sub}(G) = \gamma(G) \).

Proof. From Theorem 3 if \( \gamma(G) = 1 \) then \( \text{sub}(G) = 1 \). Conversely, if \( \text{sub}(G) = 1 \), then \( 1 + d_1 \geq n \) and hence from Proposition 4, \( \gamma(G) = 1 \). Similarly, if \( \gamma(G) = 2 \) then \( \text{sub}(G) \leq 2 \); however, since \( \text{sub}(G) = 1 \) if and only if \( \gamma(G) = 1 \), it follows that \( \text{sub}(G) = 2 \).

If \( G \) is a graph with \( \gamma(G) \geq 3 \), then \( \text{sub}(G) \) may not be equal to \( \gamma(G) \). For example, let \( G \) be the graph obtained by appending two pendants to each vertex of \( K_3 \); it can be verified that \( \gamma(G) = 3 \) and \( \text{sub}(G) = 2 \).

We next characterize the sub-\( k \)-domination number of regular graphs. This will reveal some families of graphs for which \( \text{sub}_k(G) = \gamma_k(G) \) for \( k \geq 2 \).

Theorem 6. If \( G \) is an \( r \)-regular graph, then \( \text{sub}_k(G) = \lceil \frac{kn}{r+k} \rceil \).

Proof. Since \( G \) is \( r \)-regular, \( d_i = r \) for \( 1 \leq i \leq n \). Then, from the definition of sub-\( k \)-domination, we have

\[
\text{sub}_k(G) + \frac{\text{sub}_k(G)r}{k} = \text{sub}_k(G) + \frac{1}{k} \sum_{i=1}^{\text{sub}_k(G)} d_i \geq n.
\]

Rearranging (2), we obtain

\[
\frac{kn}{r+k} \leq \text{sub}_k(G).
\]

Since \( \text{sub}_k(G) \) is the smallest integer that satisfies (3), it follows that \( \text{sub}_k(G) = \lceil \frac{kn}{r+k} \rceil \). 

Note that \( \gamma_k(G) = n \) whenever \( k > \Delta(G) \). We therefore restrict ourselves to the more interesting case of \( k \leq \Delta \). The next example shows an infinite family of graphs for which the sub-\( k \)-domination number equals the \( k \)-domination number for all \( k \leq \Delta \).

Observation 7. Let \( C_n \) be a cycle. For all \( k \leq \Delta \), \( \text{sub}_k(C_n) = \gamma_k(C_n) \).

Proof. When \( k = 1 \), it is known that \( \gamma(C_n) = \lceil \frac{n}{2} \rceil \). Since cycles are 2-regular, Theorem 6 gives \( \text{sub}(C_n) = \lceil \frac{n}{2} \rceil \). Hence, \( \gamma(C_n) = \text{sub}(C_n) \) for all \( n \). When \( k = 2 \), Theorem 6 gives \( \lceil \frac{n}{2} \rceil \leq \text{sub}_2(C_n) \). Since we can produce a 2-dominating set for \( C_n \) by first picking any vertex \( v \) and adding all vertices whose distance from \( v \) is even, it follows that \( \gamma_2(C_n) \leq \lceil \frac{n}{2} \rceil \). Thus \( \text{sub}_2(C_n) = \gamma_2(C_n) \).

As another example, from Proposition 4 and Theorem 6, we see that \( \gamma(K_n) = \text{sub}(K_n) = 1 \) and \( \gamma_2(K_n) = \text{sub}_2(K_n) = 2 \) for all \( n \). When \( k \geq 3 \), \( \gamma_k(K_n) \) does not equal \( \text{sub}_k(K_n) \) for all \( n \). For example, \( \text{sub}_3(K_4) = 2 \) but \( \gamma_3(K_4) = 3 \); however, our next result shows that equality does hold when \( n \) is large enough.
Proposition 8. Let \( K_n \) be a complete graph and let \( k \leq n - 1 \) be a positive integer. Then \( \text{sub}_k(K_n) = \gamma_k(K_n) = k \) if and only if \( n > (k-1)^2 \).

Proof. First, note that \( \gamma_k(K_n) = k \) for \( k \leq n - 1 \), since any set of \( k \) vertices of \( K_n \) is \( k \)-dominating, while any set with at most \( k - 1 \) vertices is at most \( (k-1) \)-dominating. Next, since \( K_n \) is regular of degree \( n - 1 \) it follows from Theorem 6 that

\[
\text{sub}_k(K_n) = \left\lceil \frac{kn}{n-1+k} \right\rceil \leq k = \gamma_k(K_n).
\]

If \( \text{sub}_k(K_n) = k \), we must have

\[
\frac{kn}{n-1+k} > k - 1.
\]

Rearranging, we obtain that \( n > (k-1)^2 \).

Our last focus in this section is on the \( \text{sub}_k \)-domination number and \( k \)-domination number of 3-regular, or cubic, graphs. First, we recall an upper bound for the \( k \)-domination number due to Caro and Roditty [2].

Theorem 9 ([2]). Let \( G \) be a graph, and \( k \) and \( r \) be positive integers such that \( \delta \geq \frac{r+1}{r}k - 1 \). Then,

\[
\gamma_k(G) \leq \frac{r+1}{r+1} n.
\]

In particular, for cubic graphs, Theorem 8 and the Caro-Roditty bound (with \( r \) taken to be the smallest positive integer satisfying \( 3 \geq \frac{r+1}{r}k - 1 \)) imply the following intervals for the \( k \)-domination number.

Corollary 10. Let \( G \) be a cubic graph. Then,

1. \( \left\lceil \frac{n}{4} \right\rceil \leq \gamma(G) \leq \left\lfloor \frac{n}{2} \right\rfloor \),
2. \( \left\lceil \frac{2n}{5} \right\rceil \leq \gamma_2(G) \leq \left\lfloor \frac{n}{2} \right\rfloor \),
3. \( \left\lceil \frac{n}{2} \right\rceil \leq \gamma_3(G) \leq \left\lfloor \frac{3n}{4} \right\rfloor \).

We see from Corollary 10 that \( \text{sub}_k(G) = \gamma_k(G) \) for some cubic graphs with small values of \( n \); for example, \( \text{sub}(G) = \gamma(G) \) when \( n \leq 6 \) and \( \text{sub}_2(G) = \gamma_2(G) \) when \( n \leq 8 \).

4 Comparison to known bounds on \( \gamma_k(G) \)

A well-known lower bound on the domination number of a graph is \( \frac{n}{\Delta+1} \). This bound is not difficult to derive \( \text{a priori} \), but it immediately follows from the definition of \( \text{sub}\)(\( G \)) and Theorem 3. In [9], Fink and Jacobson generalized this bound by showing that \( \frac{kn}{\Delta+k} \leq \gamma_k(G) \); this also follows from a result of Hansberg and Pepper in [14]. In the following theorem, we show that \( \text{sub}_k(G) \) is an improvement on this bound.

Theorem 11. Let \( G \) be a graph; for every positive integer \( k \leq \Delta \),

\[
\frac{kn}{\Delta+k} \leq \text{sub}_k(G) \leq \gamma_k(G).
\]
Proof. The second inequality in (4) follows from Theorem 3. To prove the first inequality, fix \( k \) and let \( t = \text{sub}_k(G) \). By definition, \( t + \frac{1}{k} \sum_{i=1}^{t} d_i \geq n \). Since \( \Delta \geq d_i \) for \( 1 \leq i \leq n \), it follows that

\[
t + \frac{t \Delta}{k} = t + \frac{1}{k} \sum_{i=1}^{t} \Delta \geq t + \frac{1}{k} \sum_{i=1}^{t} d_i \geq n.
\]

Rearranging the above inequality gives

\[
\frac{kn}{\Delta + k} \leq t = \text{sub}_k(G).
\]

Recall from Theorem 6 that if \( G \) is regular of degree \( r \), then \( \text{sub}_k(G) = \lceil \frac{kn}{r+k} \rceil \). Thus, from Theorem 11, we see that regular graphs minimize the sub-\( k \)-domination number over all graphs with \( n \) vertices and maximum degree \( \Delta \). This suggests that in order to maximize the sub-\( k \)-domination number, we might consider graphs which are, in some sense, highly irregular with respect to vertex degrees. This motivates the following theorem and its corollary.

**Theorem 12.** Let \( G \) be a graph; for \( 1 \leq t \leq \Delta \) let \( n_t \) be the number of vertices of \( G \) with degree \( t \), let \( s_t = \sum_{i=1}^{s_t} n_{\Delta+1-i} \), and let \( \Delta_t = d_{s_t+1} \). If \( s_t + \sum_{i=1}^{s_t} d_i < n \) for some \( t \), then

\[
\frac{kn - \sum_{i=1}^{s_t} (\Delta + 1 - \Delta_t - i)n_{\Delta+1-i}}{k + \Delta_t} \leq \text{sub}_k(G).
\]

**Proof.** From the definition of \( \text{sub}_k(G) \), we have

\[
n \leq \text{sub}_k(G) + \frac{1}{k} \sum_{i=1}^{\text{sub}_k(G)} d_i.
\]

(5)

Since \( s_t + \sum_{i=1}^{s_t} d_i < n \), it follows that \( s_t < \text{sub}_1(G) \leq \text{sub}_k(G) \), and thus

\[
\sum_{i=1}^{\text{sub}_k(G)} d_i = \sum_{i=1}^{s_t} d_i + \sum_{i=s_t+1}^{\text{sub}_k(G)} d_i.
\]

(6)

Since \( s_t = n_\Delta + n_{\Delta-1} + \cdots + n_{\Delta-t+1} \) and since the degree sequence of \( G \) is non-increasing and has \( n_j \) elements with value \( j \), we have

\[
\sum_{i=1}^{s_t} d_i = \Delta n_\Delta + (\Delta - 1)n_{\Delta-1} + \cdots + (\Delta - t + 1)n_{\Delta-t+1}
\]

\[
= \sum_{i=1}^{t} (\Delta + 1 - i)n_{\Delta+1-i}.
\]

(7)

Again since \( D(G) \) is non-decreasing, we have that \( \Delta_t = d_{s_t+1} \geq d_{s_t+2} \geq \cdots \geq d_{\text{sub}_k(G)} \). Thus, it follows that

\[
\sum_{i=s_t+1}^{\text{sub}_k(G)} d_i \leq \sum_{i=s_t+1}^{\text{sub}_1(G)} \Delta_t = (\text{sub}_k(G) - s_t)\Delta_t.
\]

(8)
Substituting (6), (7), and (8) into the right-hand-side of (5) yields

\[ n \leq \text{sub}_k(G) + \frac{1}{k} \sum_{i=1}^{t} (\Delta + 1 - i)n_{\Delta+1-i} + \frac{1}{k}(\text{sub}_k(G) - s_t)\Delta_t. \]

By expanding \((\text{sub}_k(G) - s_t)\Delta_t\) and substituting \(s_t = \sum_{i=1}^{t} n_{\Delta+1-i}\), the above inequality can be rewritten as

\[ n \leq \text{sub}_k(G) \left(1 + \frac{\Delta_t}{k}\right) + \frac{1}{k} \sum_{i=1}^{t} (\Delta + 1 - \Delta_t - i)n_{\Delta+1-i}. \]

Rearranging the preceding inequality gives

\[ \frac{k n - \sum_{i=1}^{t}(\Delta + 1 - \Delta_t - i)n_{\Delta+1-i}}{k + \Delta_t} \leq \text{sub}_k(G). \]

We note that the bound in Theorem 12 is optimal when \(t\) is taken to be the maximum positive integer for which \(s_t + \sum_{i=1}^{s_t} d_i < n\). Theorem 12 can be used to give simple lower bounds for the \(k\)-domination number of a graph when certain restrictions on the order and maximum degree are met. These bounds also improve on the lower bound given in Theorem 11.

**Corollary 13.** Let \(G\) be a graph, let \(n_{\Delta}\) denote the number of maximum degree vertices of \(G\), and let \(\Delta'\) denote the second-largest degree of \(G\). If \(k\) is a positive integer and \(n_{\Delta} + \frac{\Delta n_{\Delta}}{k} < n\), then

\[ \frac{k n - n_{\Delta}(\Delta - \Delta')}{\Delta' + k} \leq \text{sub}_k(G) \leq \gamma_k(G). \] (9)

**Proof.** Take \(t = 1\) in the bound from Theorem 12 and note that \(s_1 = n_{\Delta}\) and \(\Delta_1 = d_{n_{\Delta}+1} = \Delta'\). Since \(n_{\Delta} + \frac{\Delta n_{\Delta}}{k} < n\), we have that \(s_1 + \frac{1}{k} \sum_{i=1}^{s_1} d_i = n_{\Delta} + \frac{1}{k} \sum_{i=1}^{n_{\Delta}} d_i = n_{\Delta} + \frac{\Delta n_{\Delta}}{k} < n\). Thus, the condition of Theorem 12 is satisfied, and we obtain the first inequality in (9); the second inequality in (9) follows from Theorem 3.

We see from Corollary 13 that if \(G\) has a unique maximum degree vertex, then

\[ \frac{k n - \Delta + \Delta'}{\Delta' + k} \leq \gamma_k(G). \]

Corollary 13 gives significant improvements on the lower bound in Theorem 11 whenever the difference between \(\Delta\) and \(\Delta'\) is large. For example, consider the corona of \(K_{1,n-1}\) \((n \geq 3)\) which is obtained by appending a vertex of degree 1 to each of the \(n - 1\) vertices of degree 1 in \(K_{1,n-1}\). The degree sequence of this graph is \(\{n - 1, 2^{n-1}, 1^{n-1}\}\) and its order is \(2n - 1\). This graph meets the conditions of Corollary 13 and the bound given in the corollary simplifies to \(\frac{(2k-1)n - (k-3)}{2+k}\), whereas the bound given by Theorem 11 is \(\frac{k(2n-1)}{n-1+k}\). To compare these two bounds, we first compute the difference between them:

\[ \frac{(2k-1)n - (k-2)}{2+k} - \frac{k(2n-1)}{n-1+k} = \frac{(2k-1)n^2 + (4-6k)n + 8k - k^2 - 3}{(2+k)(n-1+k)}. \]

When \(k\) is fixed, the difference between these two bounds approaches \(\infty\) as \(n \to \infty\).
\section{Critical graphs}

There are three natural ways to consider critical graphs in the context of sub-\(k\)-domination: graphs which are critical with respect to edge-deletion, edge-addition, and vertex-deletion.

\textbf{Definition 3.} Let \(G\) be a graph and \(k\) be a positive integer. We will say that

1. \(G\) is \textit{edge-deletion-sub}_{\(k\)}(\(G\))-\textit{critical} if for any \(e \in E(G)\), \(\text{sub}_{\(k\)}(G - e) > \text{sub}_{\(k\)}(G)\).

2. \(G\) is \textit{edge-addition-sub}_{\(k\)}(\(G\))-\textit{critical} if for any \(e \in E(\overline{G})\), \(\text{sub}_{\(k\)}(G + e) < \text{sub}_{\(k\)}(G)\).

3. \(G\) is \textit{vertex-deletion-sub}_{\(k\)}(\(G\))-\textit{critical} if for any \(v \in V(G)\), \(\text{sub}_{\(k\)}(G - v) > \text{sub}_{\(k\)}(G)\).

These properties will respectively be abbreviated as \(\text{sub}_{\(k\)}(G)\)-\textit{ED-critical}, \(\text{sub}_{\(k\)}(G)\)-\textit{EA-critical}, and \(\text{sub}_{\(k\)}(G)\)-\textit{VD-critical}.

In this section, we present several structural results about sub-\(k\)-domination critical graphs, including connections to other graph parameters. Throughout the section, we will assume that given a graph \(G\) with \(V(G) = \{v_1, \ldots, v_n\}\) and \(D(G) = \{d_1, \ldots, d_n\}\) where \(d_1 \geq \cdots \geq d_n\), it holds that \(d_i = d(v_i)\) — in other words, the vertices of \(G\) are labeled according to a non-increasing ordering of their degrees.

We first present two results about \(\text{sub}_{\(k\)}(G)\)-\textit{ED-critical} graphs.

\textbf{Proposition 14.} Let \(G\) be a \(\text{sub}_{\(k\)}(G)\)-\textit{ED-critical} graph with \(\text{sub}_{\(k\)}(G) = t\). Then \(\{v_{t+1}, \ldots, v_n\}\) is an independent set of \(G\), and \(n - \text{sub}_{\(k\)}(G) \leq \alpha(G)\).

\textit{Proof.} Suppose for contradiction that \(\{v_{t+1}, \ldots, v_n\}\) is not an independent set and let \(e = v_xv_y\) be an edge with \(v_x, v_y \in \{v_{t+1}, \ldots, v_n\}\). Then, the degree sequence of \(G - e\) is \(d'_1 \geq \cdots \geq d'_n\), where \(d'_i = d_i\) for all \(1 \leq i \leq t\). Thus, \(t + \frac{1}{k} \sum_{i=1}^{t} d'_i = t + \frac{1}{k} \sum_{i=1}^{t} d_i \geq n\), which implies that \(\text{sub}_{\(k\)}(G - e) \leq t\); this contradicts the assumption that \(G\) is \(\text{sub}_{\(k\)}(G)\)-\textit{ED-critical}. Thus, \(\{v_{t+1}, \ldots, v_n\}\) is an independent set, so \(\alpha(G) \geq n - t\). \(\blacksquare\)

\textbf{Proposition 15.} Let \(G\) be a \(\text{sub}_{\(k\)}(G)\)-\textit{ED-critical} graph with no isolates and \(\text{sub}_{\(k\)}(G) = t\). Then \(\lfloor t + \frac{1}{k} \sum_{i=1}^{t} d_i \rfloor = n\), and for any \(e \in E(G)\), \(\text{sub}_{\(k\)}(G - e) = \text{sub}_{\(k\)}(G) + 1\).

\textit{Proof.} By definition of \(\text{sub}_{\(k\)}(G)\) and since \(n\) is an integer, we have that \(\lfloor t + \frac{1}{k} \sum_{i=1}^{t} d_i \rfloor \geq n\). Suppose for contradiction that \(\lfloor t + \frac{1}{k} \sum_{i=1}^{t} d_i \rfloor > n\). Since by Proposition 14 \(\{v_{t+1}, \ldots, v_n\}\) is an independent set of \(G\) and since \(G\) has no isolates, we can choose an edge \(e\) incident to exactly one vertex in \(\{v_1, \ldots, v_t\}\). The degree sequence of \(G - e\) is \(d'_1 \geq \cdots \geq d'_n\), where \(\sum_{i=1}^{t} d'_i = (\sum_{i=1}^{t} d_i) - 1\). Thus,

\[ t + \frac{1}{k} \sum_{i=1}^{t} d'_i = t + \frac{1}{k} \sum_{i=1}^{t} d_i - \frac{1}{k} \geq t + \frac{1}{k} \sum_{i=1}^{t} d_i - \frac{1}{k} \geq n + 1 - \frac{1}{k} \geq n, \]

meaning \(\text{sub}_{\(k\)}(G - e) = t\), which contradicts \(G\) being \(\text{sub}_{\(k\)}(G)\)-\textit{ED-critical}.

Now let \(e\) be any edge of \(G\) and \(d'_1 \geq \cdots \geq d'_n\) be the degree sequence of \(G - e\). The deletion of \(e\) decreases \(\sum_{i=1}^{t} d_i\) by at most 2, i.e., \(\sum_{i=1}^{t+1} d'_i \geq (\sum_{i=1}^{t} d_i) - 2\). Thus,

\[ (t + 1) + \frac{1}{k} \sum_{i=1}^{t+1} d'_i \geq (t + 1) + \frac{1}{k} \sum_{i=1}^{t} d_i - \frac{2}{k} = t + \frac{1}{k} \left(\sum_{i=1}^{t} d_i\right) + \frac{d_{t+1} - 2}{k} + 1 \geq n, \]
where in the last inequality $d_{t+1} \geq 1$ since $G$ has no isolates; this implies sub$_k(G - e) = t + 1 = \text{sub}_k(G) + 1$.

Next, we present two analogous results about sub$_k(G)$-EA-critical graphs.

**Proposition 16.** Let $G$ be a sub$_k(G)$-EA-critical graph with sub$_k(G) = t$. Then the vertices in \( \{v \in V(G) : d(v) < d_t^i\} \) form a clique.

*Proof.* Suppose on the contrary that there are two non-adjacent vertices $v_x$ and $v_y$ with $d_t^i > d_x^i \geq d_y^i$. Then, the degree sequence of $G + v_xv_y$ is $d_1^i \geq \cdots \geq d_n^i$, where $d_t^i = d_i$ for all $1 \leq i \leq t$. This implies that sub$_k(G + e) = \text{sub}_k(G)$, a contradiction. 

**Proposition 17.** Let $G$ be a sub$_k(G)$-EA-critical graph with no isolates and sub$_k(G) = t$. Then, for each $e \in E(G)$, sub$_k(G + e) = \text{sub}_k(G) - 1$.

*Proof.* Let $e$ be any edge in $G$ and $d_1^i \geq \cdots \geq d_n^i$ be the degree sequence of $G + e$. The addition of $e$ increases $\sum_{i=1}^t d_i$ by at most 2, i.e., $\sum_{i=1}^t d_i \leq (\sum_{i=1}^t d_i) + 2$. Thus,

\[
(t - 2) + \frac{1}{k} \sum_{i=1}^{t-2} d_i = (t - 2) + \frac{1}{k} \sum_{i=1}^t d_i^t - \frac{d_1^t + d_{t-1}^t - k}{k} \leq t + \frac{1}{k} \sum_{i=1}^t d_i - \frac{d_1^t + d_{t-1}^t - k}{k} \leq n - \frac{d_1^t + d_{t-1}^t}{k} < n,
\]

where in the last inequality $\frac{d_1^t + d_{t-1}^t}{k} > 0$ since $G$ has no isolates; this implies sub$_k(G + e) > t - 2$, so sub$_k(G + e) = \text{sub}_k(G) - 1$.

Graphs that are sub$_k(G)$-VD-critical differ from sub$_k(G)$-ED-critical graphs and sub$_k(G)$-EA-critical graphs, in the sense that it is possible for sub$_k(G - v)$ and sub$_k(G)$ to differ by much more than 1. For example, this is the case for the star $K_{n-1,1}$ when the center of the star is the vertex removed. We now show another result for sub$_k(G)$-VD-critical graphs.

**Proposition 18.** Let $G$ be a sub$_k(G)$-VD-critical graph with sub$_k(G) = t$. Then each vertex in \( \{v_{t+1}, \ldots, v_n\} \) is adjacent to at least $k + 1$ vertices in \( \{v_1, \ldots, v_t\} \).

*Proof.* Suppose that $v_x \in \{v_{t+1}, \ldots, v_n\}$ is adjacent to at most $k$ vertices in \( \{v_1, \ldots, v_t\} \). Then $G - v_x$ has degree sequence $d_1', \ldots, d_{t-1}'$ such that $\sum_{i=1}^t d_i' \geq (\sum_{i=1}^t d_i) - k$. Thus, $t + \frac{1}{k} \sum_{i=1}^t d_i' \geq t + \frac{1}{k} \sum_{i=1}^t d_i - 1 \geq n - 1$, which implies that sub$_k(G - v_x) \leq t$; this contradicts the assumption that $G$ is sub$_k(G)$-VD-critical.

6 Conclusion

In this paper, we introduced the sub-$k$-domination number and showed that it is a computationally efficient lower bound on the $k$-domination number of a graph. We also showed that the sub-$k$-domination number improves on several known bounds for the $k$-domination number, and gave some conditions which assure that sub$_k(G) = \gamma_k(G)$. This investigation was a step toward the following general question:

**Problem 1.** For each positive integer $k$, characterize all graphs for which $\gamma_k(G) = \text{sub}_k(G)$. 

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As another direction for future work, it would be interesting to define and study an analogue of sub-$k$-domination which is an upper bound to the $k$-domination number, or explore degree sequence based invariants which bound the connected domination number or the independent domination number of a graph.

**Acknowledgements**

This work is supported by the National Science Foundation, Grant No. 1450681 (B. Brimkov).

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