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2022
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Wigner’s jellium is a model for a gas of electrons. The model consists of \( N \) particles with negative unit charge in a sea of neutralizing homogeneous positive charge spread out according to Lebesgue measure, and interactions are governed by the Coulomb potential. We consider the quantum jellium on quasi-one-dimensional spaces with Maxwell–Boltzmann statistics. Using the Feynman–Kac representation, we replace particle locations with Brownian bridges. We then adapt the approach of Leblé and Serfaty (2017) to prove a process-level large deviation principle for the empirical fields of the Brownian bridges.

1. Introduction

The jellium, introduced by Wigner [64], is a model for a gas of electrons moving in a (locally) uniformly smeared neutralizing background of positive charge. Wigner predicted that when the potential energy of the system overwhelms the kinetic energy, the electrons would form a “close-packed lattice configuration”.

Figure 1. Illustration of the Brownian-bridge interpretation of the quasi-one-dimensional quantum jellium.

Christian Hirsch acknowledges the financial support of the CogniGron research center and the Ubbo Emmius Funds (University of Groningen). Paul Jung was funded in part by the National Research Foundation of Korea grants NRF-2017R1A2B2001952 and NRF-2019R1A5A1028324.

MSC2020: primary 60F10, 60K35, 82B21; secondary 82B10.

Keywords: Coulomb systems, jellium, quasi-one-dimensional systems, large deviations principle, Feynman–Kac representation, screening, marked point process.
We are interested in the quasi one-dimensional quantum jellium, which is a model of electrons inside an insulated conducting wire with some thickness (quasi one-dimensional systems are infinite in one direction and bounded in all other directions, e.g., an infinite cylinder).

In the physics literature, Deshpande and Bockrath [17] observed Wigner-crystal type behavior in experiments on carbon nanotubes (see also [18; 45]), and thermal effects on crystallization were studied in [37]. In the mathematics literature, the classical quasi-one-dimensional jellium, locally two-dimensional, was considered in [11; 24], where the system was seen to be exactly solvable when the inverse temperature satisfies $\beta = 2$ (with electrons having unit charge). An expansion of the free energy of the system, in terms of the period of a hypercube in the bounded directions, is found in [23]. Translation symmetry breaking was shown (in support of Wigner’s prediction) when $\beta$ is an even integer in [63; 34], where the latter article focused on a connection to Laughlin states in cylindrical geometry. This symmetry breaking was extended to all values of $\beta$ in [2].

Wigner’s original model was of course in the quantum setting. For the quantum one-dimensional jellium, crystallization (or translation symmetry breaking) was proved in [7], by introducing the well-known Brascamp–Lieb inequality, for the one-dimensional quantum system at sufficiently low temperatures. This was extended to all temperatures in [33].

In this work we investigate the free energy, in the thermodynamic limit, of the quantum quasi one-dimensional jellium at low density through the lens of large deviations. The large deviations approach to investigating Coulomb gases has recently been used in [3; 10; 20; 27; 44], among other works. At the level of fluctuations or level three (process level) large deviations, results have been obtained in [41; 59] for $d \geq 2$ (in one dimension, they employ the log-potential, which is not the Coulomb potential for that dimension, but allows for connections to random matrix theory). In order to extend this approach to the quantum setting, we use the Feynman–Kac representation and replace particle locations with Brownian bridges or Brownian loops, which are sometimes referred to as filaments. One may equivalently take the view that the particle locations are “marked” with Brownian bridges. In the context of Coulomb gases this representation goes back to [31] and was used in [7; 33] (see also [9]). The introduction of Brownian bridge filaments into the picture of the Gibbs measures complicates the mathematics considerably, and to our knowledge, no such rigorous study for the thermodynamic limit of the quantum jellium has yet been established beyond the one-dimensional setting.

While the model in this work is geared towards the free energy of the jellium in the quantum setting, most of the analysis below can be adapted to other random marked point configurations. Such point configurations, with marks in path spaces, come up naturally in the study of infinite-dimensional interacting diffusions [25; 39; 60]; see [15; 19]. A Gibbs variational principle for interacting diffusions with infinite time-horizon, seen as space-time Gibbs random fields, was proven by Dai Pra, Roelly, and Zessin [12]. The aforementioned references focus on diffusions interacting via superstable pair potentials, but singular interactions—including interactions of the Coulomb type—have been studied as well, in part motivated by random matrices and Dyson’s model [50; 61; 62]. The large deviation techniques explored here could be of relevance to diffusions with singular interactions.

One complication that arises in the Feynman–Kac representation using Brownian bridge filaments is how
to deal with the Fermi–Dirac statistics inherent in the representation. For one-dimensional systems, this is easy since the Brownian reflection principle allows such statistics to be equated with the consideration of nonintersecting Brownian bridges (see [33, Section 3.1] for details). In higher dimensions, assuming the absence of hard cores, particles do not collide, and thus no such simplification is possible. Since the quasi-one-dimensional jellium is locally of higher dimension than one, we will simplify things by considering Boltzmann statistics. Such a simplification is reasonable at low density, when the Debye length is much smaller than the interparticle distance, as noted in [8; 9]. In this work, we will take as a starting point, the so-called “magic formula” (see Section V.1 of [9]) which in our case amounts to replacing electrons with Brownian bridges of time-length $\beta$.

In Sections 2A–2G, we introduce the model in detail. In Section 2H, we state our main result establishing a (level three) large deviation principle (LDP) for Brownian bridge configurations in the quasi-one-dimensional jellium, and in Section 2I, we summarize our main contributions. In Section 3, we give a top level proof of our LDP. Sections 4–6 concern the technically most challenging part of our result, establishing the so-called quasicontinuity of the specific energy, which leads to the lower bound of our LDP.

2. Model definition and main results

Let $D := [0, 1]^k$ be the unit cube in dimension $k \geq 1$. The jellium describes $N \geq 1$ negatively charged particles in the finite quasi one-dimensional domain

$\Lambda_N := [0, N] \times D$ (1)

of positive background charge, where particles are subject to pairwise interaction by the $(1+k)$-dimensional Coulomb potential. Since $|D| = 1$, the system is charge neutral upon equipping $\Lambda_N$ with a positive smeared background charge with density one [64]. The potential energy of a collection of $N \geq 1$ particles

$\vec{z}_N = (z_1, \ldots, z_N) = ((x_1, y_1), \ldots, (x_N, y_N)) \in \Lambda_N^N$

is given by

$\mathcal{H}_{\Lambda_N}(\vec{z}_N) = \sum_{i<j} g(z_i - z_j) - \sum_{i \leq N} \int_{\Lambda_N} g(z - z_i) \, dz + \frac{1}{2} \int \int_{\Lambda_N^2} g(z - z') \, dz \, dz'$,

where $g(z)$ is the Green’s function of the Laplacian on the domain

$S := \mathbb{R} \times D$.

That is, writing $o \in S$ for the origin, $-\Delta g = \delta_o$, where we use free boundary conditions in the $x$-direction and periodic boundary conditions in the $y$-directions. The Green’s function admits the eigenfunction expansion (see [2, Section 2])

$g(z - z') = -\frac{1}{2} |x - x'| + \sum_{n \geq 1} (2\sqrt{\lambda_n})^{-1} e^{-|x - x'|\sqrt{\lambda_n}} \varphi_n(y) \varphi_n(y')$

in terms of the eigenfunctions $-\Delta_D \varphi_n(y) = \lambda_n \varphi_n(y)$. 

2A. Short-distance regularization. The mathematical interest in Coulomb systems stems from their long-range interactions. Besides decaying slowly over long distances, the Coulomb potential also exhibits a singularity at the origin, thereby creating additional technical difficulties. We circumvent the issues arising from the singularity by following [41] and using Onsager’s approach of smearing out point charges uniformly in a sphere $\partial B_\eta(z)$ of a fixed small radius $\eta < \frac{1}{4}$ around a particle located at $z \in \Lambda_N$ [49]. With regards to the smearing, we view smeared particles to be in the larger space $S$ so that there is no ambiguity for how to smear $z$’s with $x$-coordinates near the boundaries of $[0, N]$. We write

$$\delta^\eta_z := \frac{1}{|\partial B_\eta(z)|} \int_{\partial B_\eta(z)} \delta_{z'} dz'$$

for the measure of the smeared point charge. This operation replaces the potential energy $H_{\Lambda_N}^{\eta}(\vec{z}_N)$ by

$$H_{\Lambda_N}^{\eta}(\vec{z}_N) = \sum_{i<j} g^{\eta,\eta}(z_i - z_j) - \sum_{i \leq N} \int_S \iint_{\Lambda_N} g(z - z') dz \delta^\eta_{z_i}(dz') + \frac{1}{2} \iint_{\Lambda_N^2} g(z - z') dz dz', \quad (4)$$

where

$$g^{\eta,\eta}(z) := \iint_{S^2} g(z' - z'') \delta^\eta_z (dz') \delta_{\eta}(dz'') .$$

The function $H_{\Lambda_N}^{\eta}(\vec{z}_N)$ is invariant with respect to permutation of the arguments $z_1, \ldots, z_N$; by some abuse of notation we use the same letter $H_{\Lambda_N}^{\eta}$ for the function of the set $\{z_1, \ldots, z_N\}$, with the $z_i$’s pairwise distinct.

We note that in [41], the $\eta$-smearing was eventually removed by taking $\eta \to 0$ in the thermodynamic limit. However, the techniques from [41] do not extend smoothly to the quantum setting. Indeed, their central step [41, Lemma 5.9] consists of a comparison to a regularized configuration, where particles that are too close to each other are carefully separated. In the setting of Brownian bridges, such regularizations would differ at different times, and it is not clear whether this can be achieved at low entropic cost while simultaneously satisfying the continuity restrictions induced by the Brownian bridge paths. For instance, a significant technical difficulty arises when one Brownian particle enters the $2\eta$-sphere of another particle, and exits on the opposite side, nowhere close to its entry point.\(^1\)

2B. Configuration space. When thinking of particles as electrons, the quantum setting employs Fermi–Dirac statistics as in [33]. However, the standard one-dimensional method of employing the Brownian reflection principle, as in [33], does not extend to the quasi-1D setting. We therefore work with the simpler Boltzmann statistics, which, as mentioned already, becomes a reasonable approximation at low densities [9].

\[^1\]The naive approach of replacing exceptional bridges by near-straight line paths (approximately constant in time, with small fluctuations around their starting points) seems difficult. Indeed, such a replacement would lead to major changes of the interaction potential with other bridges, which seem challenging to control. The sort of changes of the interaction potential resulting from any short-range regularization involving near-straight line paths can be seen in Proposition 5.8. However, unlike the situation there, changes to configurations due to any short-range regularization would not be restricted to bridges near the boundaries of the region.
Invoking the Feynman–Kac representation [9; 32], the Boltzmann statistics for the quantum jellium can be deduced by analyzing a system of standard Brownian bridges subject to Coulomb interactions in each time slice.

More precisely, let \( C([0, \beta]) \) be the space of continuous paths from \([0, \beta]\) to \( S\), with times 0 and \( \beta \) identified, i.e., \( \omega(0) = \omega(\beta) \), and let Conf denote the space of all configurations of the form

\[
\omega = \bigcup_{j \in J} \{(b_j(0), b_j)\}
\]

with \( J \) a countable set (possibly finite) and \((b_j)_{j \in J}\) a locally finite collection of bridges, i.e., elements in \( C([0, \beta]) \), such that for every compact set \( K \subseteq S \), the set \( K \cap \{b_j(0) : j \in J\} \) is finite. The space \( C([0, \beta]) \) is equipped with the supremum norm and associated Borel \( \sigma \)-algebra, and \( S \times C([0, \beta]) \) with the product \( \sigma \)-algebra. The \( \sigma \)-algebra of Conf is the smallest one such that the maps

\[
\omega \mapsto \#\{j \in J : (b_j(0), b_j) \in T\}
\]

are measurable for all measurable \( T \subseteq S \times C([0, \beta]) \) (\# denotes cardinality). In particular, by shifting the marks in the sense of \( \{(b_j(0), b_j - b_j(0))\}_{j \in J} \), our framework corresponds to that of marked point processes as presented in [30]. Let \( \mathcal{P} \) be the space of probability measures on Conf.

The projection \( \pi_\Lambda : \text{Conf} \to \text{Conf} \) to a measurable set \( \Lambda \subseteq S \) maps the configuration \( \omega \) given by (5) to

\[
\pi_\Lambda(\omega) = \omega_\Lambda := \{(b_j(0), b_j) \in \omega : b_j(0) \in \Lambda\}. \tag{6}
\]

Further, let

\[
P_\Lambda = P \circ \pi_\Lambda^{-1} \tag{7}
\]

be the image of \( P \in \mathcal{P} \) under \( \pi_\Lambda \) and let

\[
\text{Conf}(\Lambda) := \pi_\Lambda(\text{Conf}) \tag{8}
\]

denote the space of configurations with starting points in \( \Lambda \).

We will typically use the shorthand \( b \in \omega \), or similarly, suppress the first coordinate of the ordered pair \((b(0), b)\) since this can be easily found by evaluating \( b \) at time 0. We also use the notation

\[
\omega(t) := \{b(t) : b \in \omega\} \subseteq S, \quad t \in [0, \beta]. \tag{9}
\]

2C. Brownian bridge measure. In the following \( \mathbb{P}_{BB,N}(db) \) is the distribution of a standard Brownian bridge with uniformly distributed starting point in the background \( \Lambda_N \). Note, however, that the bridge is allowed to leave the region \( \Lambda_N \). Denote by \( \mathbb{P}_N \) the image of the product measure

\[
\mathbb{P}_{BB,N}(db_1) \otimes \cdots \otimes \mathbb{P}_{BB,N}(db_N) \tag{10}
\]

under the mapping \( C([0, \beta])^N \to \text{Conf}(\Lambda_N) \) given by \((b_1, \ldots, b_N) \mapsto \{b_i : i = 1, \ldots, N\} \). Equivalently, \( \mathbb{P}_N \) is the distribution of a binomial point process with independent Brownian bridge marks.
Henceforth, we analyze the Gibbs measure

\[ \mu_{N,\beta}(d\omega) := \frac{1}{Z_{N,\beta}} \exp\left( -\int_0^\beta \mathcal{H}_{\Lambda N}^\eta(\omega(t)) \, dt \right) \mathbb{P}_N(d\omega), \]  

with partition function

\[ Z_{N,\beta} = \int_{\text{Conf}(\Lambda N)} \exp\left( -\int_0^\beta \mathcal{H}_{\Lambda N}^\eta(\omega(t)) \, dt \right) \mathbb{P}_N(d\omega). \]

In other words, \( \mu_{N,\beta}(d\omega) \) is absolutely continuous with respect to \( \mathbb{P}_N(d\omega) \) with Radon–Nikodym density

\[ \frac{1}{Z_{N,\beta}} \exp\left( -\int_0^\beta \mathcal{H}_{\Lambda N}^\eta(\omega(t)) \, dt \right). \]

2D. Empirical field. Discrete shifts by \( k \in \mathbb{Z} \) act on \( (x, y) \in S \) and \( b \in C([0, \beta]) \) as follows:

\[ \theta_k(x, y) = (x - k, y), \quad (\theta_k b)(t) = \theta_k(b(t)). \]

The main result of this paper is an LDP on the level of empirical fields, sometimes referred to as a level-three or process-level LDP. To put this into a general context, note that the paradigm of statistical physics is to consider large systems of particles and describe the behavior of observables that are realized as an average over such a system \([52, \text{Chapter 6}]\). More precisely, for a collection of bridges \( \omega \), we consider the empirical field

\[ \text{Emp}_N(\omega) := \frac{1}{N} \sum_{0 \leq i \leq N-1} \delta_{\theta_i \omega}, \]

as an element of the space \( P \) of probability measures on Conf. Our choice of averaging only over integer shifts reflects the occurrence of crystallization and the appearance of fractional charges as striking characteristics of quasi one-dimensional Coulomb systems \([1, 2]\). We remark that since \( \omega \) is a configuration consisting only of bridges, i.e., not including any background charge, the \( \theta_k \)-shifts do not affect the background charge.

We would like to consider the limiting behavior of observables on \( N \)-particle systems as \( N \) grows, so we next describe the relevant topology for these limits.

2E. Topology and \( \sigma \)-algebra on \( P \). We now specify the topology on \( P \) entering the large deviation principle for the distribution of the empirical fields. A bounded measurable function \( f : \text{Conf} \rightarrow \mathbb{R} \) is local — in symbols \( f \in \mathcal{L}_b \) — if it depends only on bridges with starting points in some bounded region, i.e., if there exists a bounded set \( \Lambda \subseteq S \) such that \( f(\omega) = f(\omega_{\Lambda}) \) for all \( \omega \). The \( \tau_{\mathcal{L}_b} \)-topology is the smallest topology such that all evaluation maps \( P \mapsto \mathbb{E}_P[f(\omega)] \) for \( f \in \mathcal{L}_b \) are continuous. We endow \( P \) with the smallest \( \sigma \)-algebra such that for every bounded or nonnegative measurable \( f : \text{Conf} \rightarrow \mathbb{R} \), the map \( P \mapsto \mathbb{E}_P[f(\omega)] \) is measurable as well \([30]\).

Since the \( \tau_{\mathcal{L}_b} \)-topology is neither metrizable nor separable, many arguments become more technical than their analogs in the weak topology of \([41]\). The latter is weaker than the \( \tau_{\mathcal{L}_b} \)-topology since it relies on bounded local test functions that are continuous on Conf where point configurations in Conf
are identified with counting measures and equipped with the vague topology. However, we found the restriction to continuous test functions to be not versatile enough when dealing with configurations of bridges instead of particles. For example, the upper-semicontinuity argument in [41, Lemma 5.7] uses that in a certain infimum over vector fields a minimum is attained. This is not clear if the fields vary over time.

In particular, the $\sigma$-algebra and topology on $\mathcal{P}$ are the analogs of the cylinder $\sigma$-algebra $\mathcal{B}_{cy}$ and the $\tau$-topology described in the context of Sanov’s theorem in [14, Chapter 6.2]; see the comments after [28, Definition 0.2]. One should keep in mind that the $\sigma$-algebra on $\mathcal{P}$ is not the Borel $\sigma$-algebra of the $\tau_{L_b}$-topology and so, in particular, not every open set is measurable. As a consequence, large deviation principles must be formulated carefully, and compactness of level sets of rate functions no longer implies exponential tightness; see [22] and the remarks preceding [14, Lemma 1.2.18].

Remark 2.1. For point processes with finite intensity, another choice is the class $\mathcal{L}$ of tame local functions and the associated $\tau_{L}$-topology [30]. For point processes without marks, a local function is tame if $|f(\omega)| \leq C(1 + \#\omega_\Lambda)$ for some $C > 0$, some bounded $\Lambda$, and all $\omega$. The class of tame local functions is strictly larger than the class of bounded local functions, therefore the topology $\tau_{L}$ is finer than the topology $\tau_{L_b}$; however, as noted in the proof of Proposition 2.6 in [30], on sublevel sets $\{\text{ent}(P) \leq c\}$ for $c > 0$, the topologies coincide.

In view of the above discussion, it is helpful to single out the measurable open neighborhoods around some $P \in \mathcal{P}$. We use the notation

$$U_{\text{meas}}(P) \text{ is the family of measurable neighborhoods of } P.$$  \hfill (14)

The set $U_{\text{meas}}(P)$ contains, in particular, a basis of the $\tau_{L_b}$-topology of cylinder sets

$$U_{F, P} := \bigcap_{f \leq n} \{ Q \in \mathcal{P} : |E_Q[f_j] - E_P[f_j]| < \delta \}$$  \hfill (15)

with $\delta > 0$, $n \geq 1$, and $F = \{f_1, \ldots, f_n\}$ a finite set of bounded local functions.

2F. Entropy. A bridge configuration refers to a deterministic locally finite collection of $C([0, \beta])$-marked points in $S$, whereas the space $\mathcal{P}_s$ consists of shift-invariant probability measures ($P \circ \theta_k = P$ for all $k \in \mathbb{Z}$) describing stationary bridge processes.

We let $\text{Pois}$ denote the law of a unit-intensity homogeneous Poisson point process on $S$, marked with independent Brownian bridges of diffusion parameter $1$ and time-length $\beta$. See [40, Chapter 5.3] for an abstract definition of independent markings. For $P \in \mathcal{P}_s$, let

$$\text{ent}(P) := \lim_{N \to \infty} \frac{1}{N} \int \log \left( \frac{dP_{\Lambda_N}}{d\text{Pois}_{\Lambda_N}}(\omega) \right) P_{\Lambda_N}(d\omega)$$  \hfill (16)

denote the specific relative entropy of $P$ with respect to $\text{Pois}$ [30], applying the convention that $\text{ent}(P) = \infty$ if, for some $N$, $P_{\Lambda_N}$ is not absolutely continuous with respect to $\text{Pois}_{\Lambda_N}$. It can be seen from subadditivity arguments in [28; 30; 52] that this limit exists. In order to state the LDP formally in the space $\mathcal{P}$ instead of $\mathcal{P}_s$, we set $\text{ent}(P) = \infty$ if $P \in \mathcal{P} \setminus \mathcal{P}_s$. 

2G. Energy. A crucial property of Coulomb systems is an intimate relation between the energy (2), and the energy of an associated electric field (for instance, this appears in the splitting formula in [59, Proposition 3.3]). More precisely, consider a bounded region \( \Lambda \) and a configuration \( \omega \in \text{Conf}(\Lambda) \). For each time \( t \leq \beta \), the points \( b(t) \) of the Brownian bridges \( b \in \omega \) together with the homogeneous opposing background charge in \( \Lambda \) create the electrostatic potential

\[
V_t(z, \omega, \Lambda) := \int_S g(z - z') \left( \sum_{b \in \omega, \Lambda} \delta_{b(t)} - I_\Lambda \right) (dz').
\]

(17)
The integral is over \( S \) rather than \( \Lambda \) because Brownian bridges \( b(t) \) may leave \( \Lambda \). Let

\[
H_{\beta, \Lambda}(\omega) := \frac{1}{2\beta} \int_{S \times [0, \beta]} |\nabla V_t(z, \omega, \Lambda)|^2 d(z, t).
\]

(18)
Denote by

\[
\text{Neut}(\Lambda) := \{ \omega \in \text{Conf}(\Lambda) : \#\omega = |\Lambda| \}
\]

(19)
the space of all charge-neutral configurations. In the Appendix, we integrate by parts to prove the following variation of a standard identity.

**Lemma 2.2** (energy in terms of electric field). Let \( \Lambda = [L_-, L_+] \times D \) with \( L_-, L_+ \in \mathbb{Z} \) and let \( \omega \in \text{Neut}(\Lambda) \). Then

\[
\frac{1}{\beta} \int_0^\beta \mathcal{H}_\Lambda(\omega(t)) \, dt = H_{\beta, \Lambda}(\omega) - Ng^{\eta, \eta}(\omega).
\]

(20)
Relying on Lemma 2.2, the expected energy per unit length along the \( x \)-direction for a stationary bridge process \( P \in \mathcal{P}_x \) is now defined in two steps. Proceeding naively, one could define the energy content of a finite window \( \Lambda_N \) by minimizing over all possible boundary conditions and then taking the expectation and the limit over an increasing window size. Minimizing over boundary conditions is not necessarily natural from a physics point of view but it turns out to be technically convenient. In statistical mechanics it is customary to impose conditions like temperedness on boundary conditions [55] that give a handle on interactions between a bounded window \( \Lambda_N \) and the outside \( \Lambda_N^c \). In our context, the interactions we need to control are caused by bridges starting outside \( \Lambda_N \) but invading \( \Lambda_N^c \) deep inside. We impose the following conditions. For an interval \( I = [L_-, L_+] \subset \mathbb{R} \), we first define the eroded interval

\[
I^- := [L_+ + \lfloor |I|^{7/8} \rfloor, L_+ - \lceil |I|^{7/8} \rceil],
\]

and for \( K = I \times D \), we put \( K^- = I^- \times D \). Then

\[
\Theta(K) := \{ \omega \in \text{Conf} : b \in \omega \text{ and } b(0) \not\in K \text{ implies } b \cap K^- = \emptyset \},
\]

(21)
with \( b \cap K^- = \emptyset \) a shorthand for \( \{ b(t) : t \in [0, \beta] \} \cap K^- = \emptyset \), consists of all configurations \( \omega \in \text{Conf} \) for which any bridge in \( \omega \) with starting point outside of \( K \), stays outside of \( K^- \) for all times.

**Remark 2.3** (choice of exponents). The exact choice of the exponent \( \frac{7}{8} \) above is not important. We have chosen this value to allow for the proof of Proposition 3.2 (see (32)), as well as for reasonable choices of parameters later (see Remark 5.2). The exponents in Lemma 3.4 below are also chosen with (32) in mind.
In the following, \( \mathbb{E}_P \) denotes the expectation with respect to \( P \in \mathcal{P} \).

**Definition 2.4 (expected specific energy).** The expected specific energy of \( P \in \mathcal{P}_s \) with \( \text{ent}(P) < \infty \) is defined, whenever this limit exists, as

\[
\mathbb{W}_\beta(P) := \lim_{M \to \infty} \mathbb{W}_\beta^M(P),
\]

(22)

where

\[
\mathbb{W}_\beta^M(P) := \lim_{N \to \infty} \mathbb{E}_P[(N^{-1} \tilde{H}_{\beta, N}(\omega) \wedge M)],
\]

(23)

and for \( \omega \in \text{Conf} \) and \( K \subseteq \mathcal{S} \),

\[
\tilde{H}_{\beta, K}(\omega) := \inf_{\substack{\Lambda \supset K, \tilde{\omega} \in \text{Neut}(\Lambda) \\wedge \tilde{\omega} \in \Theta(K), \tilde{\omega}_K = \omega_K}} \left\{ \frac{1}{2\beta} \int_{K \times [0, \beta]} |\nabla V(t, \tilde{\omega}, \Lambda)|^2 \, d(z, t) \right\}.
\]

(24)

The infimum considers only domains \( \Lambda \) of the form \( \Lambda = I \times D \) for an interval \( I \subseteq \mathbb{R} \) with integer endpoints.

**Remark 2.5.** By approximating arbitrary Brownian bridges by piecewise linear functions with rational interpolation points, we may replace the infimum in (24) by the infimum over a countable set. In particular, there are no issues with the measurability of \( \omega \mapsto \tilde{H}_{\beta, K}(\omega) \) when forming expectations.

Let us relate the definition of \( \mathbb{W}_\beta(P) \) to the analogous definition in [41]. Firstly, as mentioned above, we found it necessary due to the presence of highly fluctuating Brownian bridges to impose the boundary condition \( \Theta(K) \). Moreover, in (22) the order of forming the expectation and taking the limit is reversed. When working with bridges instead of just their endpoints, we found it difficult to control the asymptotic behavior of the specific energy of configurations in large volumes on the level of realizations. Loosely speaking, after forming expectations the effects of the fluctuation are less severe and a large-volume analysis becomes tractable.

**2H. Main result: large deviation principle.** Defining the free energy as

\[
\mathcal{F}_\beta(P) := \mathbb{W}_\beta(P) + \beta^{-1} \text{ent}(P)
\]

if \( P \in \mathcal{P}_s \) and \( \text{ent}(P) < \infty \), and \( \mathcal{F}_\beta(P) = \infty \) otherwise, we now state the large deviation principle, where \( \bar{A} \) and \( A^o \) refer to the closure and interior with respect to the \( \tau_{\mathcal{L}_b} \)-topology of a measurable set \( A \subseteq \mathcal{P} \).

**Theorem 2.6 (LDP for the empirical fields).** The empirical fields \( \{\text{Emp}_N(\omega)\}_{N \geq 1} \) under \( \{\mu_{N, \beta}\}_{N \geq 1} \) satisfy the LDP at speed \( N \) in the \( \tau_{\mathcal{L}_b} \)-topology on \( \mathcal{P} \) with good rate function

\[
P \mapsto \beta \mathcal{F}_\beta(P) - \inf_{P' \in \mathcal{P}} \beta \mathcal{F}_\beta(P').
\]

That is, for every measurable set \( A \subseteq \mathcal{P} \),

\[
\limsup_{N \to \infty} \frac{1}{N} \log \mu_{N, \beta}(\{\omega_{\Lambda_N} : \text{Emp}_N(\omega_{\Lambda_N}) \in A\}) \leq - \sup_{P \in \bar{A}} \left( \beta \mathcal{F}_\beta(P) - \inf_{P' \in \mathcal{P}} \beta \mathcal{F}_\beta(P') \right),
\]

and

\[
\liminf_{N \to \infty} \frac{1}{N} \log \mu_{N, \beta}(\{\omega_{\Lambda_N} : \text{Emp}_N(\omega_{\Lambda_N}) \in A\}) \geq - \sup_{P \in A^o} \left( \beta \mathcal{F}_\beta(P) - \inf_{P' \in \mathcal{P}} \beta \mathcal{F}_\beta(P') \right).
\]
An important byproduct of our proof in Section 3, is that we are also able to describe the asymptotics of the free energy log \( E_N[\exp(-\beta H_{\Lambda_N}(\omega))] \) for large \( N \), where the expectation is with respect to \( \mathbb{P}_N \) defined above (10).

Having formulated the LDP in terms of Brownian bridges, we now translate it back to the quantum setting under Maxwell–Boltzmann statistics. Essentially, this means specializing Theorem 2.6 to the starting points of the bridges. After stating the result, we expound on possible implications of this result from the perspective of mathematical physics.

In the operator-theoretic setting, the quantum-mechanical Hamiltonian for the quasi-1D jellium becomes

\[
H_{\Lambda_N}^\eta = -\frac{1}{2} \sum_{i \leq N} \Delta z_i + \sum_{i < j} g^{\eta}(z_i - z_j) - \sum_{i \leq N} \int_{\Lambda_N} \int_{\Lambda_N} g(z - z') \, dz \delta_{z_i}^\eta (d z') + \frac{1}{2} \int_{\Lambda_N^2} g(z - z') \, dz \, dz',
\]

which is an operator on the Hilbert space \( L^2(S^N) \). (Recall that we use free boundary conditions in the \( x \)-direction and periodic boundary conditions in the \( y \)-directions.) The expected value of an observable \( a \) (self-adjoint operator in \( L^2(S^N) \)) in the finite-volume Gibbs state is

\[
\langle a \rangle_{N,\beta} = \frac{\text{Tr} \, a \exp(-\beta H_{\Lambda_N}^\eta)}{\text{Tr} \, \exp(-\beta H_{\Lambda_N}^\eta)}. \tag{26}
\]

The Feynman–Kac formula implies that expected values of observables that depend on particle positions \( z_1, \ldots, z_N \) (but not on momenta) are completely determined by the distribution of the initial points of the Brownian bridges: if \( a \) is a multiplication operator with some function \( G_N(z_1, \ldots, z_N) \), then

\[
\langle G_N(z_1, \ldots, z_N) \rangle_{N,\beta} = \int G_N(b_1(0), \ldots, b_N(0)) \mu_{N,\beta}(d b).
\]

Among the position-dependent observables of interest are extensive two-body quantities

\[
G_N(z_1, \ldots, z_N) = \sum_{1 \leq i < j \leq N} h(z_j - z_i)
\]

with bounded and compactly supported \( h \). To make the connection with the stationary empirical field, notice that

\[
G_N(z_1, \ldots, z_N) = \sum_{k=0}^{N-1} u([z_i - k]_{i=1,\ldots,N}) \tag{27}
\]

with

\[
u([z_i]_{i=1,\ldots,N}) = \frac{1}{2} \sum_{i=1}^N 1_{[0,1) \times D}(z_i) \sum_{j \neq i} h(z_j - z_i).
\]

The function \( u \) is bounded by a constant times the number of particles in \([0, 1) \times D\) and it only depends on the configuration in some neighborhood of that set because \( h \) is compactly supported. Thus, \( G_N / N \) is the integral of the local and tame function \( u \) against the stationary empirical field of the point configuration \( \{z_1, \ldots, z_N\} \). Theorem 2.6, (26) and the contraction principle yield a large deviation principle for all macroscopic observables of that type.
Corollary 2.7. Let \( u \) be a local and tame function of point configurations on the strip, and \( G_N \) as in (27). Define \( I_u : \mathbb{R} \to \mathbb{R} \cup \{\infty\} \) by
\[
I_u(x) := \inf \left\{ \beta F_\beta(P) : P \in \mathcal{P}_s, \ F_\beta(P) < \infty, \ \int_{\text{Conf}} u(\{b(0) : b \in \omega\}) P(d\omega) = x \right\}.
\]
Then \( I_u \) is convex with compact sublevel sets and for all measurable subsets \( A \subseteq \mathbb{R} \),
\[
\limsup_{N \to \infty} \frac{1}{N} \log \langle 1_{\{G_N/N \in A\}} \rangle_{N, \beta} \leq - \inf_{x \in A} (I_u(x) - \min I_u),
\]
\[
\liminf_{N \to \infty} \frac{1}{N} \log \langle 1_{\{G_N/N \in A\}} \rangle_{N, \beta} \geq - \inf_{x \in A^c} (I_u(x) - \min I_u).
\]

Very loosely speaking, one consequence of Corollary 2.7 is that with a high probability, the \( G_N/N \)-observable is asymptotically close to one of the minimizers of \( I_u \). The constraint appearing in the definition of \( I_u(x) \) means that the expected value of the real-valued observable \( u \) under the probability measure \( P \) should be given by \( x \).

LDPs have a long history and deep connection with statistical physics; recent examples of this connection include [27; 41; 44] and in particular, in the quantum setting, [36]. LDPs of the type in this work are fairly standard in classical statistical mechanics with short-range interactions. Let us briefly review a few relevant aspects. LDPs play an important role in clarifying foundational questions in statistical mechanics. Firstly, as explained already by Lanford [38] for Gibbs measures with finite-range interactions, when the rate function \( I_g \) has a unique minimizer, the LDP implies that the equilibrium distribution of the intensive observable \( G_N/N \) is sharply peaked around its mean value, as it should be in light of the statistical mechanics approach for thermodynamics. Of course the hard work then consists in actually proving that \( I_g \) has a unique minimizer, a question intimately tied to uniqueness of Gibbs measures and absence of phase transitions. Secondly, the LDP for the stationary empirical field is closely related to the Gibbs variational principle [52] and it can be used to investigate the equivalence of ensembles on the level of states [29].

On a related note, the proof of Corollary 2.7 provides a variational representation of the limiting free energy. A consequence is the existence of the thermodynamic limit for our quasi-1D quantum systems. While such existence should follow from standard methods, to the best of our knowledge a proof for the systems considered here has not yet appeared in the literature. Moreover, while subadditivity arguments only yield the mere existence of the limit, the variational representation adds the conceptual insight that it results from balancing well-defined contributions from specific energy and specific entropy of an infinite system. The standard subadditivity arguments are also no longer available for inhomogeneous systems, and the work of [41] illustrates that the LDP approach is flexible enough to extend to such a setting.

In the quantum setting, large deviations are less well understood. Large deviation estimates for particle numbers in the quantum ideal weakly interacting gas, and quantum gas with Boltzmann statistics and repulsive interaction at all densities and temperatures were studied by Lebowitz, Lenci and Spohn [43] and Gallavotti, Lebowitz and Mastropietro [26]. For quantum lattice systems, large deviations for broader classes of observables are available; see Ogata and Rey-Bellet [48]. Subtleties of quantum large
deviations are also discussed by Netočný and Redig [47] and some remarks in relation with the asymptotic equipartition theorem are made by de Roeck, Maes and Netočný [13]. Quantum large deviations have also attracted interest in the context of quantum statistics and quantum information theory; see, for example, the quantum versions of the Shannon–McMillan and Sanov theorems [4; 5].

None of the aforementioned references on quantum large deviations deal with long-range interactions or Coulomb systems. The long-range nature of Coulomb interactions creates technical difficulties even for classical systems, as demonstrated in [41]. Our results constitute a first step towards quantum large deviations for Coulomb systems.

Finally, from a more probabilistic point of view, while a law of large numbers and a central limit theorem capture the typical behavior and the typical fluctuations of a physical system, the LDP allows to describe the configurations under rare events. In particular, the distribution under a rare event becomes accessible via the Gibbs conditioning principle [14, Theorem 7.3.3] and importance sampling is also made possible; for an application see [53].

One last comment is that, although Theorem 2.6 is stated for the Coulomb potential, many of the arguments remain valid more generally. The essential ingredients in our arguments are bounds on the number of bridges crossing various boundaries and bounds on the change in energy due to bridges deviating far from their starting points. For instance, the recent results obtained in [16] pave the way towards extending Theorem 2.6 to the 1D log-potential.

2I. Summary of contributions. The goal of the present work is to provide a first example for which the fine-scale large deviation analysis achieved in the classical setting in [41] may be extended to the quantum setting via Feynman–Kac formalism. However, as demonstrated by our work here, passing from particles to bridges is far from just a formal step, but instead requires substantial novel arguments to address technical challenges. We now highlight the main novel contributions briefly, and provide a more detailed discussion in the corresponding subsections of the manuscript.

• When using configurations of bridges instead of particles, one needs to use a stronger topology as discussed in Section 2E. The natural topology in our setting, taken from [30], is neither metrizable nor separable, and thus the analog of various arguments in [41] become more technical.

• Related to the above, in the classical setting, a crucial ingredient in the proof of the LDP is a fine coercivity property from [54, Lemma 4.2]. Loosely speaking, starting from point configurations and electric fields in bounded windows, the lemma allows to pass to a subsequence that converges to a deterministic limiting configuration in the entire domain. In the setting of bridges such a subsequence would need to be produced through the Arzelà–Ascoli theorem, which is difficult to apply due to the lack of equicontinuity of the bridges. Although we are still using coercivity properties of the electric energy to derive energy bounds, these arguments do not give full compactness because of the extra bridge degree of freedom. Therefore, we develop a different route and rely on compactness results in the space $\mathcal{P}$ itself.

• The energy of configurations in the classical setting was defined by taking the infimum of all possible electric fields that are compatible with a given particle configuration [41, Lemma 2.3]. One difficulty in our setting, is that in any given region of space, Brownian bridges that start in that region, may leave
The Brownian bridges could potentially be subject to extreme fluctuations, particularly in the setting of large deviations, and thereby prevent a screening construction that can be carried out consistently over the entire time horizon. To deal with this issue, we introduce concepts of \( \varepsilon \)-dense abscissas (Definition 4.2) and \((M, \varepsilon)\)-regular configurations (Definition 5.1) and show that irregular configurations are so rare that they do not interfere with the screening construction. In particular, both Section 4 and Section 5A are devoted to dealing with issues that arise due to the fluctuations of bridges, and they have no analog in previous works.

- In Section 6B, we show that relying on the probabilistic concept of a coupling allows to shorten many of the volume computations from [41].

3. Top-level proof of the main result

To prove Theorem 2.6, we proceed in three steps. First, in Section 3A, we show that the specific energy \( \mathbb{W}_\beta \) is lower semicontinuous. Due to the quantum fluctuations introduced by the presence of Brownian bridges, the absence of superadditivity makes this step more difficult than one might expect. Second, in Section 3B, we establish the LDP upper bound. Third, in Section 3C, we outline how the LDP lower bound reduces to finding configurations of low energy. The construction of such configurations relies on the screening idea from [41; 51; 54; 57; 58] and comprises the bulk of this paper, Sections 4–6. In the Appendix, we prove a result expressing the energy of a configuration in terms of the electric field. A common motif throughout the manuscript is that the additional fluctuations coming from the Brownian bridges should not be seen as a simple model extension, since some of the arguments that were immediate in [41] require more attention and new ideas.

In the sequel, to simplify notation, we drop the \( \beta \)-dependence and write

\[
H_\Lambda(\omega) \equiv H_{\beta,\Lambda}(\omega), \quad \tilde{H}_\Lambda(\omega) \equiv \tilde{H}_{\beta,\Lambda}(\omega), \\
\mathbb{W}(P) \equiv \mathbb{W}_\beta(P), \quad \mathbb{W}^M(P) \equiv \mathbb{W}^M_\beta(P), \quad \mathcal{F} \equiv \mathcal{F}_\beta.
\]

3A. Lower semicontinuity. Our main results of this subsection are

**Proposition 3.1** (existence of the energy density). If \( \text{ent}(P) < \infty \), then the limit (23) exists.

**Proposition 3.2** (lower semicontinuity of free energy). The free energy \( \mathcal{F} \) in (25) is lower semicontinuous.

In order to prove these propositions, we will leverage that \( \tilde{H}_{\Lambda_N} \) over increasing regions \( \{\Lambda_N\}_N \) is “close” to being superadditive. The biggest obstacle to superadditivity is the possibility of configurations having many highly fluctuating Brownian bridges which cross the boundaries of the regions \( \{\Lambda_N\} \). For a bridge \( b = (b(t))_{t \leq \beta} \), let us denote by \( b_x(t) \) and \( b_y(t) \) the spatial coordinates \( x \in \mathbb{R} \) and \( y \in D \), respectively, of the Brownian bridge at time \( t \). We first present two lemmas which will make use of the following \( x \)-range
functional of a bridge $b$ seen from its starting point:

$$
\psi(b) := \max_{t \leq \beta} |b_X(t) - b_X(0)|.
$$

This quantity is one of the main players in the proof of our main result. In particular, we stress that the bound $-\log(P(\{\psi(b) > r\}) \approx cr^2$ will be the key to handling fast Brownian bridges. The following two lemmas provide bounds on the configurations which are the worst offenders in crossing boundaries.

**Lemma 3.3** ($\Theta$ contains most configurations). Let $k \geq 1$ and $I_1, \ldots, I_k \subseteq \mathbb{R}$ be distinct intervals with integer endpoints and of length $R \geq 1$. For every $\omega \in \text{Conf}(\Lambda_N)$,

$$
\#\{i \leq k : \omega \not\in \Theta(I_i \times D)\} \leq 2 \sum_{b \in \omega} (\psi(b) - R^{7/8} + 2)_+.
$$

**Lemma 3.4** (bound on Brownian bridge $x$-ranges). Let $P \in \mathcal{P}_x$ with $\text{ent}(P) < \infty$. There exists $c > 0$, not depending on $P$, such that for all $\zeta > 1$

$$
\sup_{\Lambda \in \mathcal{S}} \frac{1}{|\Lambda|} \mathbb{E}_P \left[ \sum_{b \in \omega} (\psi(b)^{7/6} - \zeta)_+ \right] \leq (c + \text{ent}(P))\zeta^{-5/8}.
$$

Our choice of the exponent $\frac{7}{6}$ on the left side of the above inequality is due to later use of this lemma (see Definition 5.1).

As usual the supremum on the left-hand side is over domains of the form $[L_-, L_+] \times D$ with $L_\pm \in \mathbb{Z}$. The lemma is similar to Lemma 5.2 in [30]; it is based on entropy bounds. Some context for this kind of bound is provided in Section 4A.

Regarding the exponent choices in the above lemma, see Remark 2.3. Before proving Lemmas 3.3 and 3.4, let us see how they yield Proposition 3.1.

**Proof of Proposition 3.1.** Put

$$
W^M_N(\omega) := (N^{-1} \tilde{H}_{\Lambda_N}(\omega)) \land M.
$$

Let $\{N_i^-\}_{i \geq 1}$ and $\{N_i^+\}_{i \geq 1}$ be sequences realizing the lim inf and lim sup under $\mathbb{E}_P$. That is,

$$
\lim_{i \to \infty} \mathbb{E}_P[W^M_{N_i^-}] = \liminf_{N \to \infty} \mathbb{E}_P[W^M_N] \quad \text{and} \quad \lim_{i \to \infty} \mathbb{E}_P[W^M_{N_i^+}] = \limsup_{N \to \infty} \mathbb{E}_P[W^M_N],
$$

where we may assume without loss of generality that $N_i^- \geq (N_i^+)^2$ for all $i \geq 1$. Now, we set $N'_i = \lfloor (N_i^- - 4(N_i^-)^{7/8}) \rfloor$ and claim the almost-superadditive bound

$$
N'_i \mathbb{E}_P[W^M_{N_i^+}(\omega)] \leq N_i^- \mathbb{E}_P[W^M_{N_i^-}(\omega)] + 2M \mathbb{E}_P \left[ \sum_{b \in \omega \land \Lambda_{N_i^-}} (\psi(b) - (N_i^+)^{7/8} + 2)_+ \right].
$$

First, note that for any $r > 0$ and for any $R > 0$ with $R^{7/8} - 2 \geq (R/2)^{7/8}$ we have

$$
(r - R^{7/8} + 2)_+ \leq r \mathbb{1}\{r \geq (R/2)^{7/8}\} \leq r^{7/6} \mathbb{1}\{r^{7/6} \geq (R/2)^{49/48}\} \leq 2(r^{7/6} - (R/2)^{49/48}/2)_+.
$$
Now, divide both sides of (31) by $N_i^-$ and send $i \to \infty$. Lemma 3.4 shows that
\[
\frac{1}{N_i^-} \mathbb{E}_P \left[ \sum_{b \in \omega \cap N_i^-} (\psi(b) - (N_i^+)^{7/8} + 2)_+ \right] \leq \frac{2}{N_i^-} \mathbb{E}_P \left[ \sum_{b \in \omega \cap N_i^-} (\psi(b)^{7/6} - (N_i^+)^2)^{49/48}_+ \right]
\leq 2(c + \text{ent}(P))(N_i^+)^{-5(49)/(8-48)} \to 0, \quad (32)
\]
which concludes the proof, modulo the claim (31).

To establish (31), we let $K_1, \ldots, K_{N_i'} \subseteq [0, N_i^-]$ be $N_i'$ intervals of length $N_i^+$, placed within the interval $[0, N_i^-]$, such that $K_{i+1}$ is obtained by shifting $K_i$ to the right by 1 (along the $x$-coordinate) and such that each $K_j$ is at least distance $[(N_i^-)^{7/8}]$ from 0 and $N_i^-$. Now, let $\omega \in \text{Conf}(K_j)$ be arbitrary, and $\Lambda \supset K_j$, $\omega' \in \text{Neut}(\Lambda)$ be such that $\omega' \in \Theta(K_j)$ and $\omega_K = \omega_K$. Then by definition (see (24))
\[
\tilde{H}_{K_j}(\omega) \cap N_i^+ M \leq \frac{1}{2\beta} \int_{K_j \times [0,\beta]} |\nabla V_i(z, \omega, \Lambda)|^2 \text{d}(z, t) + MN_i^+ \mathbb{1}{(\omega \notin \Theta(K_j))}. \quad (33)
\]
Summing over all $j \leq N_i'$ and dividing by $N_i^+$ yields that
\[
\sum_{j \leq N_i'} \left( \frac{1}{N_i^+} \tilde{H}_{K_j}(\omega') \right) \cap M \leq \frac{1}{2\beta} \int_{[0,N_i^-] \times [0,\beta]} |\nabla V_i(z, \omega, \Lambda)|^2 \text{d}(z, t) + M \sum_{j \leq N_i'} \mathbb{1}{(\omega \notin \Theta(K_j))}.
\]
Applying Lemma 3.3 and taking expectations concludes the proof of (31). \qed

Next, we show that the free energy is lower semicontinuous.

**Proof of Proposition 3.2.** The specific relative entropy is lower semicontinuous by arguments similar to [30, Proposition 2.6]. Hence, if $\text{ent}(P) = \infty$, then for any level $\ell$ there is a neighborhood $U$ of $P$ such that $P' \in U$ implies $\text{ent}(P') > \ell$ thus we have lower semicontinuity of $\mathcal{F}$ at such $P$. Therefore, we may assume that $\text{ent}(P) < \infty$ from now on.

Since the supremum of lower semicontinuous functions remains lower semicontinuous, it suffices to show lower semicontinuity of the truncated free energy
\[
\mathcal{F}^M = \mathcal{W}^M + \beta^{-1} \text{ent}.
\]
Let $P \in \mathcal{P}_s$ and $\epsilon > 0$ be arbitrary. We need to show that there exists a neighborhood $U$ of $P$ such that
\[
\mathcal{F}^M(P) \leq \inf_{P' \in U} \mathcal{F}^M(P') + \epsilon. \quad (34)
\]
Since the left-hand side of (34) is at most $\alpha := M + \text{ent}(P)/\beta$ (with $\text{ent}(P) < \infty$), it suffices to show
\[
\mathcal{F}^M(P) \leq \inf_{P' \in U} \mathcal{F}^M(P') + \epsilon,
\]
By lower semicontinuity of the specific relative entropy, we may focus on the energy part. Here, for every $M, N \geq 1$ the observable $W_N^M(\omega)$ defined in (30) is local and bounded, so that
\[
P \mapsto f_N(P) := \mathbb{E}_P[W_N^M(\omega)]
\]
is continuous in \( P \). Moreover, the proof of Proposition 3.1 revealed that the energies are almost superadditive. If they were precisely superadditive, then the limit in \( N \) could be replaced by a supremum, so that again we could leverage lower semicontinuity of a supremum of continuous functions. In the rest of the proof, we show that this line of argumentation extends to the present setting of almost superadditivity.

More precisely, we provide a neighborhood \( U \) of \( P \) such that

\[
f(P) \leq \inf_{P' \in U \atop \text{ent}(P') \leq a} f(P') + \varepsilon,
\]

where we set

\[
f := \lim_{N \to \infty} f_N = \mathbb{W}^M.
\]

To achieve this goal, we decompose \( f(P) - f(P') \) as

\[
f(P) - f(P') = (f(P) - f_N(P)) + (f_N(P) - f_N(P')) + (f_N(P') - f(P')).
\]

The first term on the right-hand side tends to 0 as \( N \to \infty \). For fixed \( N \), the second term becomes arbitrarily small for all \( P' \) from a sufficiently small neighborhood of \( P \). Hence, it suffices to show that for fixed \( a > 0 \)

\[
\limsup_{N \to \infty} \sup_{N' \geq N^2} \sup_{P' : \text{ent}(P') \leq a} (f_N(P') - f_N(P')) \leq 0.
\]

Now, copying the proof of the bound (31) gives that

\[
(1 - 4(N^{'-1}/8))f_N(P') \leq f_N(P') + M\mathbb{E}_{P_N'} \left[ \sum_{b \in \omega} (\psi(b) - N^7/8 + 2)_+ \right].
\]

Hence, applying Lemma 3.4 concludes the proof. \( \square \)

It remains to prove Lemmas 3.3 and 3.4.

**Proof of Lemma 3.3.** The \( \{I_j\}_{j \leq k} \) are each of the form \( \theta_i(\Lambda_R) \) for some \( i \in \mathbb{Z} \). We have

\[
\sum_{i \in \mathbb{Z}} 1\{\omega \not\subset \Theta(\theta_i(\Lambda_R))\}
\leq \sum_{i \in \mathbb{Z}} \sum_{b \in \omega} 1\{b(0) \not\subset \theta_i(\Lambda_R) \text{ and } b \cap \theta_i(\Lambda_R^-) \neq \emptyset\}
\leq \sum_{b \in \omega} \#\{i \leq -b_x(0) : b \cap \theta_i(\Lambda_R^-) \neq \emptyset\} + \sum_{b \in \omega} \#\{i \geq -b_x(0) - R : b \cap \theta_i(\Lambda_R^-) \neq \emptyset\}.
\]

Now, if \( i \leq -b_x(0) \), then \( b \) can intersect \( \Lambda_R^- \) only if \( i \geq -b_x(0) - \psi(b) + [R^7/8] \). Hence, the total number of such \( i \) is at most \( (\psi(b) - R^7/8 + 2)_+ \). A similar argument applies in the case \( i \geq -b_x(0) - R \), which gives the first bound in the lemma. \( \square \)

**Proof of Lemma 3.4.** We adapt an argument from the proof of [30, Lemma 5.2]. By [30, (2.11)], the relative entropy \( \text{ent}_A(P_A) \) of \( P_A \) with respect to Pois_\Lambda satisfies

\[
\text{ent}_A(P_A) = \sup_{f} \left\{ \mathbb{E}_{P_A} [f(\omega)] - \log \mathbb{E}_{\text{Pois}_\Lambda} [\exp(f(\omega))] \right\},
\]

(35)
where the supremum is over all bounded measurable \( f : \text{Conf}(\Lambda) \to \mathbb{R} \). The inequality extends to nonnegative measurable \( f \) that are not necessarily bounded \([30, (5.3)]\). Set \( a = \zeta^{5/8} \),

\[
\begin{aligned}
f(\omega) &= \sum_{b \in \omega} (\psi(b)^{7/6} - \zeta)_+,
\end{aligned}
\]

and apply (35) to \( af \). Noting that the specific relative entropy \( \text{ent} \) defined in (16) is equal to \( \sup_{\Lambda} |\Lambda|^{-1} \text{ent}_\Lambda \) (see Remark 2.5 in \([30]\)), we find that

\[
\begin{aligned}
\frac{1}{|\Lambda|} E_{P_\Lambda}[f(\omega)] &\leq \frac{1}{a} \text{ent}(P) + \frac{1}{a|\Lambda|} \log E_{\text{Pois}_\Lambda}[^{\exp(af(\omega))}].
\end{aligned}
\]

Then, inserting the Laplace transform of a marked Poisson point process \([40, \text{Proposition 5.4}]\),

\[
\begin{aligned}
\log E_{\text{Pois}_\Lambda}[\exp(af(\omega))] &= |\Lambda|(\exp(\varphi(\zeta)) - 1),
\end{aligned}
\]

where for a Brownian bridge \( B \) on \([0, \beta]\),

\[
\begin{aligned}
\varphi(\zeta) &:= \log E_{BB}[\exp(a(\psi(B)^{7/6} - \zeta)_+)].
\end{aligned}
\]

We next evaluate for \( \zeta > 0 \):

\[
\begin{aligned}
e^{\varphi(\zeta)} - 1 &\leq E_{BB}[\exp(a(\psi(B)^{7/6}) 1\{\psi(B) > \zeta^{6/7}\})] \\
&\leq \left(E_{BB}[\exp(2a(\psi(B)^{7/6})]P(\psi(B) > \zeta^{6/7})\right)^{1/2}.
\end{aligned}
\]

Tail estimates on the maximum of a Brownian motion \([21, \text{Theorem 1.2.6}; 46, \text{Section 2.2.1}]\) show that

\[
\begin{aligned}
\log P(|\psi(B)| > \zeta^{6/7})
\end{aligned}
\]

is of order \(-\zeta^{12/7}\) for large \( \zeta \). Next, applying the Laplace principle to the function \( f(x) = 2ax^{7/6} - cx^2 \), which is maximized when \( a^{12/5} \sim x^2 \), shows that the exponential moment \( E_{BB}[e^{2a(\psi(B))^{7/6}}] \) is of order \( O(a^{12/5}) \) as \( \zeta \to \infty \). In particular, \( e^{\varphi(\zeta)} - 1 \) remains bounded, which concludes the proof. \( \square \)

### 3B. Proof of Theorem 2.6: LDP upper bound.

Using (20), we rewrite \( \mu_{N,\beta}(\text{Emp}_N(\omega) \in A) \) for \( A \subseteq \mathcal{P} \) measurable as

\[
\begin{aligned}
\mu_{N,\beta}(\text{Emp}_N(\omega) \in A) &= \frac{1}{Z_{N,\beta}} e^{N \beta g^{w,q}(\omega)} E_N[\mathbbm{1}\{\text{Emp}_N(\omega) \in A\} \exp(-\beta H_{\Lambda_N}(\omega))].
\end{aligned}
\]

After dividing by \( \mu_{N,\beta}(\text{Emp}_N(\omega) \in \mathcal{P}) = 1 \), we arrive at

\[
\begin{aligned}
\mu_{N,\beta}(\text{Emp}_N(\omega) \in A) &= \frac{E_N[\mathbbm{1}\{\text{Emp}_N(\omega) \in A\} \exp(-\beta H_{\Lambda_N}(\omega))]}{E_N[\exp(-\beta H_{\Lambda_N}(\omega))]},
\end{aligned}
\]

so that proving Theorem 2.6 reduces to understanding the asymptotics of the numerator in (39) at an exponential scale.

To prove the LDP upper bound, we leverage a lower bound for the energy, large deviations for Poisson point processes, and standard arguments from the proof of the upper bound in Varadhan’s integral lemma \([14]\). As a critical ingredient for the proof of the upper bound we use that if the particles do not interact, then the rate function is given by the relative entropy.
Lemma 3.5 (LDP upper bound without interaction). For every measurable \( A \subseteq \mathcal{P} \),
\[
\limsup_{N \to \infty} \frac{1}{N} \log \mathbb{P}_N(\text{Emp}_N(\omega) \in A) \leq - \inf_{P \in \overline{A}} \text{ent}(P).
\]

Recall from (14) that \( U_{\text{meas}}(P) \) is the set of measurable neighborhoods around \( P \).

Lemma 3.6 (measurable neighborhoods of high-enough energy). Let \( M \geq 1 \) and \( \varepsilon > 0 \) be given. For each \( P \in \mathcal{P}_s \), there exists a measurable neighborhood \( U_{F,P} \in U_{\text{meas}}(P) \) such that one can choose \( R(M, \varepsilon, P) \) and then \( N(M, \varepsilon, P, R) \) to satisfy the following property. If \( \omega \in \text{Neut}(\Lambda_N) \) is such that \( \text{Emp}_N(\omega) \in U_{F,P} \), then for all \( R \geq R(M, \varepsilon, P) \) and \( N \geq N(M, \varepsilon, P, R) \),
\[
\frac{1}{N} H_{\Lambda_N}(\omega) \geq \mathbb{W}^M(P) - \varepsilon - \frac{2M}{N} \sum_{b \in \omega} (\psi(b) - R^{7/8} + 2)_+ . \tag{40}
\]

Lemma 3.6 and Proposition 3.8 below allow us to apply arguments similar to the proof of Varadhan’s lemma, based on a large deviation principle for the empirical field, even though the energy is not directly a function of the empirical field. In [30], Georgii and Zessin deal with this difficulty by introducing a notion of asymptotic empirical functional [30, (3.1)], but the latter asks for some uniformity condition which cannot be applied to Coulomb systems.

Before proving Lemmas 3.5 and 3.6, we explain how they enter the proof of Theorem 2.6. By establishing lower and upper bounds for the nonnormalized measures in (39), it is enough to control the numerator in (39).

Proof of Theorem 2.6: upper bound for the numerator in (39). Let \( A \subseteq \mathcal{P} \) be measurable. Because the rate function is lower semicontinuous and \( \mathbb{W}^M \) is increasing in \( M \), it is enough to show that for each \( M \geq 1 \),
\[
\limsup_{N \to \infty} \frac{1}{N} \log \mathbb{E}_N[\exp(-\beta H_{\Lambda_N}(\omega)) \mathbb{1}[\text{Emp}_N(\omega) \in A]] \leq - \inf_{P \in \overline{A}} (\beta \mathbb{W}^M(P) + \text{ent}(P)). \tag{41}
\]

Indeed, if there is a finite constant \( C \) such that for every \( M \geq 1 \) there exists \( P^M \in \overline{A} \) with
\[
\mathbb{W}^M(P^M) + \beta^{-1} \text{ent}(P^M) \leq C ,
\]
then also
\[
\mathbb{W}(P^*) + \beta^{-1} \text{ent}(P^*) = \lim_{M \to \infty} \mathbb{W}^M(P^*) + \beta^{-1} \text{ent}(P^*) \leq C ,
\]
where \( P^* \in \overline{A} \) is any limit point of the \( P^M \). Here, the compactness of the entropy-sublevel set
\[
K_\alpha := \{ P \in \mathcal{P} : \text{ent}(P) \leq \alpha \}
\]
guarantees that some limit point \( P^* \) exists [30, Proposition 2.6].

By compactness, there exists a finite set \( \{ P_i \}_{i \leq k} \) and measurable neighborhoods \( U_i = U_{F_i,P_i} \) which cover \( K_\alpha \cap \overline{A} \) and satisfy the inequality from Lemma 3.6, for \( N \geq \max_i N(P_i, \varepsilon) \).

For \( \varepsilon > 0 \), set
\[
E_{R,\varepsilon} := \left\{ P \in \mathcal{P} : \mathbb{E}_P \left[ \sum_{b \in b_x(0) \in [0,1]} (\psi(b) - R^{7/8} + 2)_+ \right] \leq \varepsilon \right\}.
\]

On the event that \( \{ \text{Emp}_N(\omega) \in E_{R,\varepsilon} \} \), the error in Lemma 3.6 is small.
The set $E_{R, \varepsilon}$ is measurable. Indeed, let $f : \text{Conf} \to \mathbb{R}_+$ be a measurable local function and define $A := \{ P \in \mathcal{P} : \mathbb{E}_P[f] \leq \varepsilon \}$. Then for every $n \in \mathbb{N}$, the set $A_n := \{ P \in \mathcal{P} : \mathbb{E}_P[f \wedge n] \leq \varepsilon \}$ is the preimage of $[0, \varepsilon]$ under the map $P \mapsto \mathbb{E}_P[f \wedge n]$, which is measurable because $f \wedge n$ is local and bounded. Therefore $A_n$ is measurable. Moreover $A_n \downarrow A$, so $A$ is measurable as well. The obvious choice of $f$ yields that $E_{R, \varepsilon}$ is indeed measurable.

First, we recall from the proof of Proposition 3.1 that for large enough $R > 0$ and all $r > 0$ we have $(r - R^{7/8} + 2)_{+} \leq 2(r^{7/6} - \frac{1}{2}(R/2)^{49/48})$. Hence, we conclude from the reasoning below (36) in the proof of Lemma 3.4 that

$$
\frac{1}{N} \log \mathbb{E}_N \left[ \exp \left( \frac{1}{2} R \right) \mathbb{E}_{\text{Emp}_N(\omega)} \left[ \sum_{b, b_i(0) \in [0,1]} (\psi(b) - R^{7/8} + 2)_{+} \right] \right]
$$

remains bounded as $R \to \infty$. Thus, by the Markov inequality,

$$
\lim_{R \to \infty} \limsup_{N \to \infty} N^{-1} \log \mathbb{P}_N(\text{Emp}_N(\omega) \in E_{R, \varepsilon}^c) = -\infty.
$$

In particular, we may restrict to the set $A \cap \overline{E}_{R, \varepsilon}$ for $R$ large enough.

The remaining part adapts the arguments of [14, Lemma 4.3.6]. By compactness of $\overline{A} \cap K_\alpha$, there exists a finite covering $\bigcup_{i \leq k} U_i \supset \overline{A} \cap K_\alpha$ by measurable neighborhoods $U_i = U_{F_i, P_i}$ satisfying the inequality from Lemma 3.6 for $R_i$ chosen appropriately and $N \geq \max_i N(M, \varepsilon, P_i, R_i)$. We now bound the numerator in (39). For all $N \geq \max_i N(R_i, M, P_i, \varepsilon)$, we have

$$
\mathbb{E}_N \left[ \exp(-\beta H_{\text{Emp}_N(\omega)}(\omega)) \mathbb{1}[\text{Emp}_N(\omega) \in A \cap \overline{E}_{R, \varepsilon}] \right] 
\leq \sum_{i \leq k} e^{-\beta N M^M(P_i) - 3 \varepsilon M} \mathbb{P}_N(\text{Emp}_N(\omega) \in U_i) + \mathbb{P}_N \left( \text{Emp}_N(\omega) \in \left( \bigcup_i U_i \right)^c \cap A \right). \tag{42}
$$

Since $(\bigcup_i U_i)^c \cap \overline{A} \subseteq K_\alpha^c$, we deduce from Lemma 3.5 that

$$
\limsup_{N \to \infty} \frac{1}{N} \log \mathbb{P}_N \left( \text{Emp}_N(\omega) \in \left( \bigcup_i U_i \right)^c \cap A \right) \leq -\inf_{P \in K_\alpha} \text{ent}(P) < -\alpha,
$$

which becomes arbitrarily negative for large $\alpha$. In light of Lemma 3.5, taking the limit $N \to \infty$ first and then $\varepsilon \to 0$ in (42) proves (41).

Next, we prove Lemma 3.5.

**Proof of Lemma 3.5.** Let $\text{Pois}$ be the reference measure introduced above (16). Then

$$
\mathbb{P}_N(\text{Emp}_N(\omega) \in A) = \text{Pois}_{\Lambda_N}(\text{Emp}_N(\omega) \in A \mid \# \omega = N), \tag{43}
$$

and hence

$$
\frac{1}{N} \log \mathbb{P}_N(\text{Emp}_N(\omega) \in A) \leq \frac{1}{N} \log \text{Pois}_{\Lambda_N}(\text{Emp}_N(\omega) \in A) - \frac{1}{N} \log \left( \frac{NN!}{N!} \exp(-N) \right). \tag{44}
$$

The last term vanishes by Stirling’s formula. For the probability under the Poisson process, putting $Z_i := \omega_{[i,i+1) \times D}$ and thinking of $\text{Emp}_N(\omega)$ as the empirical field of the independent spin system $\{Z_i\}_{i \in \mathbb{Z}}$ on $\mathbb{Z}$, we conclude with [28, Theorem 1.2].

□
Finally, to prove Lemma 3.6, we need one further auxiliary result. Set
\[ W^M_R(P) := \mathbb{E}_P[(R^{-1} \tilde{H}_{\Lambda_R}(\omega)) \wedge M] \]  
(45)
to be the expected truncated specific energies in \( \Lambda_R \). By Definition 2.4, \( \lim_{R \to \infty} W^M_R(P) = W^M(P) \).

**Lemma 3.7** (lower bound for the energy). Let \( 1 \leq R \leq N - 1 \) and \( \omega \in \text{Neut}(\Lambda_N) \) be a charge-neutral bridge configuration. Then
\[ H_{\Lambda_N}(\omega) \geq (N - R)W^M_R(\text{Emp}_{N-R}(\omega)) - 2M \sum_{b \in \omega} (\psi(b) - R^{7/8} + 2)_+. \]

**Proof.** The definition of Emp gives that
\[ (N - R)W^M_R(\text{Emp}_{N-R}(\omega)) = \sum_{i \leq N - R - 1} (R^{-1} \tilde{H}_{\Lambda_R}(\theta_i \omega)) \wedge M. \]
Since \( \Lambda_R \subseteq \theta_i(\Lambda_N) \) for every \( i \leq N - R - 1 \), the definition of \( \tilde{H}_{\Lambda_R} \) and the fact that \( \omega \in \text{Neut}(\Lambda_N) \) yields
\[ \tilde{H}_{\Lambda_R}(\theta_i \omega) \wedge RM \leq \frac{1}{2R^2} \int_{\theta_i(\Lambda_R) \times [0, \beta]} |\nabla V(z, \omega, \Lambda_N)|^2 \, d(z, t) + 1(\omega \notin \Theta(\theta_i(\Lambda_R))) RM. \]  
(46)
In particular, dividing by \( R \) and summing over \( i \leq N - R - 1 \) yields that
\[ (N - R)W^M_R(\text{Emp}_{N-R}(\omega)) \leq \frac{1}{2R^2} \sum_{i \leq N - R - 1} \int_{\theta_i(\Lambda_R) \times [0, \beta]} |\nabla V(z, \omega, \Lambda_N)|^2 \, d(z, t) + M \sum_{i \leq N - R - 1} 1(\omega \notin \Theta(\theta_i(\Lambda_R))), \]
where the first sum on the right-hand side is bounded by
\[ \frac{1}{2R^2} \int_{S \times [0, \beta]} |\nabla V(z, \omega, \Lambda_N)|^2 \, d(z, t) = H_{\Lambda_N}(\omega). \]
Applying Lemma 3.3 concludes the proof. \( \square \)

**Proof of Lemma 3.6.** There exists \( R = R(M, \varepsilon, P) > 0 \) such that
\[ \sup_{R \geq R(M, \varepsilon, P)} |W^M_R(P) - W^M(P)| < \frac{1}{3} \varepsilon. \]
Note that \( \tilde{H}_{\Lambda_R} \wedge M \) is a bounded local observable. Thus, \( W^M_R(Q) \) are arbitrarily close to \( W^M_R(P) \) for all \( Q \) belonging to a sufficiently small open neighborhood \( U_{F,P} \) of the form (15).

Also note that \( W^M_R(\text{Emp}_{N-R}(\omega)) \) gets arbitrarily close to \( W^M_R(\text{Emp}_N(\omega)) \) if \( N \) is sufficiently large. Hence,
\[ |W^M_R(\text{Emp}_{N-R}(\omega)) - W^M(P)| < \frac{2}{3} \varepsilon \]
if \( \text{Emp}_N(\omega) \in U_{F,P} \). In particular, by Lemma 3.7,
\[ H_{\Lambda_N}(\omega) \geq (N - R)(W^M_R(P) - \frac{2}{3} \varepsilon) - 2M \sum_{b \in \omega} (\psi(b) - R^{7/8} + 2)_+ \]  
(47)
for all \( \omega \in U_{F,P} \). Dividing both sides by \( N \) and choosing \( N \) large enough that \( (R/N)W^M(P) < \frac{1}{3} \varepsilon \) proves (40). \( \square \)
3C. Proof of Theorem 2.6: LDP lower bound. In order to prove the lower bound in Theorem 2.6, we establish a quasicontinuity property of the specific energy. To make this precise, we show the following analog of [41, Proposition 4.2]. In essence, Proposition 3.8 states that it is possible to construct a family of bridge configurations, with substantial probability mass, such that (i) the associated empirical fields are close to a given $P \in \mathcal{P}_s$, and (ii) the energy of the bridge configurations does not exceed $\mathbb{W}(P)$ substantially.

**Proposition 3.8** (quasicontinuity of the specific energy). Let $P \in \mathcal{P}_s$ be such that $\mathcal{F}_\beta(P) < \infty$. Then, for every measurable neighborhood $U_P \in \mathcal{U}_{\text{meas}}(P)$ and sufficiently small $\delta > 0$,

$$\liminf_{N \to \infty} \frac{1}{N} \log \mathbb{P}_N \left( \text{Emp}_N(\omega) \in U_P, N^{-1}H_{\Lambda_N}(\omega) \leq \mathbb{W}(P) + \delta \right) \geq -\text{ent}(P) - O(\delta). \quad (48)$$

The proof of Proposition 3.8 is the heart of the present paper. Before establishing Proposition 3.8, we show how it completes the proof of Theorem 2.6.

**Proof of Theorem 2.6, lower bound for the numerator in (39).** Let $A \subseteq \mathcal{P}$ be a measurable set, $P \in A^\circ \cap \mathcal{P}_s$ with $\mathcal{F}_\beta(P) < \infty$, and $\delta > 0$. Then there exists a measurable neighborhood $U_P$ of $P$ such that $U_P \subseteq A$; see (15). Note

$$\frac{1}{N} \log \mathbb{E}_N \left[ \mathbb{1}\{\text{Emp}_N(\omega) \in A\} \exp(-\beta H_{\Lambda_N}(\omega)) \right] \geq \frac{1}{N} \log \mathbb{E}_N \left[ \mathbb{1}\{\text{Emp}_N(\omega) \in A, N^{-1}H_{\Lambda_N}(\omega) \leq \mathbb{W}(P) + \delta\} \exp(-\beta H_{\Lambda_N}(\omega)) \right] \geq -\beta O(\delta) - \beta \mathbb{W}(P) + \frac{1}{N} \log \mathbb{P}_N \left( \text{Emp}_N(\omega) \in U_P, N^{-1}H_{\Lambda_N}(\omega) \leq \mathbb{W}(P) + \delta \right).$$

By Proposition 3.8,

$$\liminf_{N \to \infty} \frac{1}{N} \log \mathbb{E}_N \left[ \mathbb{1}\{\text{Emp}_N(\omega) \in A\} \exp(-\beta H_{\Lambda_N}(\omega)) \right] \geq -(\beta \mathbb{W}(P) + \text{ent}(P)) - O(\delta). \quad (49)$$

This holds true for every $\delta > 0$ and $P \in A^\circ \cap \mathcal{P}_s$, and hence

$$\liminf_{N \to \infty} \frac{1}{N} \log \mathbb{E}_N \left[ \mathbb{1}\{\text{Emp}_N(\omega) \in A\} \exp(-\beta H_{\Lambda_N}(\omega)) \right] \geq - \inf_{P \in A^\circ \cap \mathcal{P}_s} (\beta \mathbb{W}(P) + \text{ent}(P)). \quad (50)$$

On $A^\circ \setminus \mathcal{P}_s$ the free energy $\mathcal{F}_\beta$ is infinite, therefore the infimum over $A^\circ \cap \mathcal{P}_s$ is equal to the infimum over $A^\circ$ and the proof is complete.

4. Entropy and energy estimates

In order to implement the screening construction, we require a variety of subtle entropy and energy bounds. The present section is devoted to this topic. First, recall that a key ingredient in the theory of Gibbs measures is a control on interactions between distinct regions of space. In our setup, if we split the strip $S$ into two halves separated by an interface at $x = x_0$, we would like to control interactions between the left and right half-strips $x < x_0$ and $x > x_0$. Interactions can be large in particular if there are too many bridges crossing the interface, or too many bridges that have their initial point $b(0)$ close to $x_0$.\footnote{This terminology is taken from [41, Section 4.1.2] and [6].}
This section gathers the required estimates and explains how the estimates connect to classical problems in statistical mechanics.

The first type of bound is an entropy bound and it has actually already been stated and proven in Lemma 3.4. Section 4A provides some additional physical context.

Some events come with large energy penalties and can be excluded if we impose bounds on the energy. This second type of bound is exploited in Section 4B.

4A. Entropy bounds: Brownian bridges as unbounded spins. To get a feel for the difficulties caused by Brownian bridges, we may split the energy into a contribution from time-constant bridges (which amounts to classical mechanics) plus quantum correction terms of Boltzmann-type coming from the bridges not being constant:

\[
H_\Lambda(\omega) = \frac{1}{2} \int_\Lambda |\nabla V_0(z, \omega, \Lambda)|^2 \, dz + \frac{1}{2\beta} \int_{\Lambda \times [0, \beta]} (|\nabla V_t(z, \omega, \Lambda)|^2 - |\nabla V_0(z, \omega, \Lambda)|^2) \, d(z, t). \tag{51}
\]

The first term depends only on the initial positions of the bridges, the second term depends on the actual bridges. It is instructive to look at the bridge-dependent part of a pair interaction: letting \( b \) and \( \gamma \) be two bridges,

\[
\int_0^\beta g(b(t) - \gamma(t)) \, dt = \beta g(b(0) - \gamma(0)) + \int_0^\beta \bigl( g(b(t) - \gamma(t)) - g(b(0) - \gamma(0)) \bigr) \, dt. \tag{52}
\]

Consider for simplicity the purely one-dimensional system with Green’s function \( g(x) = -|x|/2 \). Then the additional contribution to the pair interaction from bridge degrees of freedom, i.e., the second contribution in (52), is equal to

\[
-\frac{1}{2} \int_0^\beta (|b(t) - \gamma(t)| - |b(0) - \gamma(0)|) \, dt, \tag{53}
\]

which can in principle be arbitrarily large for paths with large values for \( b(t) - b(0) \) or \( \gamma(t) - \gamma(0) \), regardless of the distance \( |b(0) - \gamma(0)| \). However, this should be unlikely because Brownian bridges in the bounded time interval \([0, \beta]\) should typically not stray away too much from their initial point (this unlikeness is at the level of entropy — without even having to take energy into account). Hence, a good control for the additional interaction induced by the bridge degrees of freedom requires probabilistic entropy estimates.

This situation is similar to the classical statistical mechanics for systems of unbounded spins on a lattice. The space \( C([0, \beta]) \) is a noncompact single-spin space, the path \((b(t) - b(0))_{t \in [0, \beta]}\) with initial point 0 plays the role of a spin. Ensuring the existence of Gibbs measures for unbounded spin systems requires additional conditions and estimates. Several approaches are possible. Superstability estimates build on a long series of technical estimates that culminate in bounds on the probability of large spin values with respect to the Gibbs measure \([42; 55; 56]\). For interacting diffusions seen as Gibbs measures on path space, a different condition is used \([15; 25]\). In our setting, we employ the set \( \Theta(K) \) in (21) together with the entropy bound in Lemma 3.4 for stationary measures, which is a variant of Lemma 5.2 in \([30]\). In particular, this bound plays a role in Lemma 6.5 where we show that, with high probability,
configurations behave in a reasonable way. We will additionally introduce notions of \((M, \varepsilon)\)-regular configurations and tame abscissas (Definitions 5.1 and 5.3).

4B. Energy bounds via charge imbalance. Proceeding in the vein of [2], we now rely on energy estimates to exclude configurations that have too many bridges close to a given abscissa \(x = x_0\). Intuitively, if there are too many bridges nearby \(x = x_0\), then there is large charge imbalance, hence a large electric field and large electrostatic energy. Thus, conversely an energy bound constrains the charge accumulating in any finite region.

As we put periodic boundary conditions in the \(y\)-direction, we henceforth write \(\partial(I \times D) = (\partial I) \times D\) for the \(x\)-boundary of a domain inside \(S\). Henceforth, set \(\bar{e}_x = (1, 0, \ldots, 0) \in \mathbb{R}^{k+1}\) and

\[
\mu_t(\omega, dz) := -\sum_{b \in \omega} \delta_{\bar{e}_b(t)}(dz) + 1_{\Lambda} dz,
\]  

(54)

so that \(\mu_t(\omega, A) = \mu_t(\omega)(A)\) computes the charge imbalance or net charge in a region \(A \subseteq S\). (We use here the terms “charge imbalance” and “net charge” synonymously but note that in previous work [2] charge imbalance instead refers to the difference of the total charge to left of some abscissa minus total charge to the right.)

Lemma 4.1 below is closely related to the discrepancy bounds [51, Lemma 2.2]. The essence of both results is that the electric energy in a domain grows at least in the order of the square of the charge imbalance. Whereas [51, Lemma 2.2] contains refined estimates that also apply to Riesz energies, the focus of Lemma 4.1 concerns time-varying bridges.

Lemma 4.1 (charge imbalance bound). Let \(L_- \leq x_- \leq x_+ \leq L_+\) and \(\omega \in \text{Neut}(\Lambda)\) with \(\Lambda = [L_-, L_+] \times D\).

1. For \(\Lambda^- := [L_-, x_-] \times D\),

\[
|\mu_0(\omega, \Lambda^-)| \leq \left( \frac{1}{\beta} \int_{D \times [0, \beta]} |\nabla V_t((x_-, y), \omega, \Lambda)|^2 d(y, t) \right)^{1/2} + \# \{ b \in \omega : \inf_{t \leq \beta} |b_x(t) - x_-| \leq 1 \}.
\]  

2. For \(\Lambda^' := [x_-, x_+] \times D\),

\[
|\mu_0(\omega, \Lambda^')| \leq \left( \frac{4}{\beta} \sum_{s \in \{\pm\}} \int_{D \times [0, \beta]} |\nabla V_t((x_s, y), \omega, \Lambda)|^2 d(y, t) \right)^{1/2} + \sum_{s \in \{\pm\}} \# \{ b \in \omega : \inf_{t \leq \beta} |b_x(t) - x_s| \leq 1 \}.
\]  

Proof: We start with part (1). Gauss’s theorem yields that for all \(t \leq \beta\)

\[
\lim_{u \to -\infty} \int_D \nabla V_t((u, y), \omega, \Lambda) \cdot (-\bar{e}_x) dy + \int_D \nabla V_t((x_-, y), \omega, \Lambda) \cdot \bar{e}_x dy = \mu_t(\omega, (-\infty, x_-] \times D). \]  

(55)

The first term on the left-hand side vanishes since the electric field at \(-\infty\) vanishes, which is easily proven with (3), charge neutrality, and properties of purely one-dimensional systems [1]. Hence, by Cauchy–Schwarz,

\[
\mu_t(\omega, (-\infty, x_-] \times D))^2 \leq \left( \int_D |\nabla V_t((x_-, y), \omega, \Lambda)| dy \right)^2 \leq \int_D |\nabla V_t((x_-, y), \omega, \Lambda)|^2 dy. \]  

(56)
Now note that for every \( t \leq \beta \), the charge imbalance is at least
\[
\mu_t(\omega, (\infty, x_-) \times D) \geq \mu_0(\omega, (\infty, x_-) \times D) - \#\{b \in \omega : \inf_{t \leq \beta} |b_x(t) - x_-| \leq 1\}.
\]
Hence, inserting this bound into (56) and integrating over time concludes the proof.

The arguments for part (2) are very similar. Gauss’s theorem yields that for all \( t \leq \beta \)
\[
\int_D \nabla V_t((x_+, y), \omega, \Lambda) \cdot \vec{e}_x \, dy + \int_D \nabla V_t((x_-, y), \omega, \Lambda) \cdot (-\vec{e}_x) \, dy = \mu_t(\omega, \Lambda').
\]
Hence, by Cauchy–Schwarz and the fact that \( |\partial^3 \mu^\prime| = 2 \),
\[
\mu_t(\omega, \Lambda')^2 \leq \left( \sum_{s \in \{+, -\}} \int_D |\nabla V_t((x_s, y), \omega, \Lambda)| \, dy \right)^2 \leq 4 \sum_{s \in \{+, -\}} \int_D |\nabla V_t((x_s, y), \omega, \Lambda)|^2 \, dy,
\] (57)
and we now conclude as above. \( \square \)

**Definition 4.2** (\( \varepsilon \)-dense abscissas). Let \( \varepsilon \in (0, \frac{1}{2}) \) and set \( \xi = \varepsilon^{-3} > 0 \). For \( \omega \in \text{Conf}(\Lambda_R) \), the set Dense\( _\varepsilon(\omega) \) of \( \varepsilon \)-dense abscissas consists of all
\[
x_0 \in [2R^{7/8}, R - 2R^{7/8}]
\]
such that
\[
\#\{b \in \omega_{\Lambda_R} : |b_x(0) - x_0| \leq \xi \} \geq \frac{4\xi}{\varepsilon}.
\] (58)

We bound the Lebesgue measure of dense abscissas in terms of the energy.

**Lemma 4.3** (a bound on \( \varepsilon \)-dense sets). Let \( M > 2, \varepsilon \in (0, 1/M^2), \) and \( R > (2/\varepsilon)^8 \) and let \( \omega \in \text{Conf}(\Lambda_R) \) be such that
\[
\frac{1}{2\beta} \int_{\Lambda_R \times D} |\nabla V_t(z, \omega, \Lambda_R)|^2 \, dz < MR
\]
and every bridge in \( \omega \) has a range less than \( R^{7/8} \). Then
\[
|\text{Dense}_\varepsilon(\omega)| \leq \varepsilon R.
\] (59)

Let us remark that the choices of parameters \( M, \varepsilon, \) and \( R \) are not optimal, but rather, chosen to be consistent with assumptions that we will use later on.

**Proof:** We proceed by contradiction. We first claim that if (59) does not hold, then
\[
\#\omega_{\Lambda} \geq 2R,
\] (60)
where \( \Lambda = [\frac{3}{2} R^{7/8}, R - \frac{3}{2} R^{7/8}] \times D \). If we have (60), then
\[
\sup_{t \leq \beta} \sup_{x \leq \frac{1}{2} L^{7/8}} \mu_t(\omega, [x, R - x] \times D) \leq -R.
\]
On the other hand, by (57),
\[
\mu_t(\omega, [x, R - x] \times D)^2 \leq 4 \int_D (|\nabla V_t(x, y)|^2 + |\nabla V_t(R - x, y)|^2) \, dy.
\]
After integrating over \([0, \frac{1}{2} R^{7/8}] \times [0, \beta]\), we arrive at
\[
\frac{1}{2\beta} \int_{[0, \frac{1}{2} R^{7/8}] \times [0, \beta]} \mu_i(\omega, [x, R - x] \times D)^2 \, dx \, dt < 4MR.
\]
But \(R > 8M > 16\), so we obtain the contradiction
\[
\frac{1}{4} R^{7/8} \cdot R^2 < 4MR \leq \frac{1}{2} R^2.
\]
It remains to show that if (59) does not hold, then \(\#\omega_\Lambda \geq 2R\). To prove this assertion, we integrate (58) over all \(x_0 \in \text{Dense}_e(\omega)\) to obtain that
\[
4\varepsilon^{-4} |\text{Dense}_e(\omega)| \leq \int_{[2R^{7/8}, R - 2R^{7/8}]} \# \{b \in \omega_\Lambda_R : |b \cdot x_0| \leq \varepsilon^{-3} \} \, dx_0
\]
\[
= \sum_{b \in \omega_\Lambda_R} \int_{[2R^{7/8}, R - 2R^{7/8}]} \mathbb{1} \{ |b \cdot x_0| \leq \varepsilon^{-3} \} \, dx_0
\]
\[
\leq 2\varepsilon^{-3} \#\omega_\Lambda.
\]
Hence, dividing both sides by \(2\varepsilon^{-3}\) concludes the proof. \(\square\)

5. Screening

Screening starts with the following: with every configuration \(\omega \in \text{Conf}(\Lambda_R)\) we will associate a new configuration \(\omega^{\text{scr}} \in \text{Neut}(\Lambda_R)\) and an electric field
\[
E^{\text{scr}} : \Lambda_R \times [0, \beta] \to \mathbb{R}^{k+1}, \quad (z, t) \mapsto E^{\text{scr}}_t(z)
\]
such that:

(i) For most \(\omega\), the configuration \(\omega^{\text{scr}}\) differs from \(\omega\) only near the boundaries of \(\Lambda_R\). A precise description of \(\omega^{\text{scr}}\) is in Section 5B.

(ii) The field \(E^{\text{scr}}\) is compatible with \(\omega^{\text{scr}}\) and a background charge distribution of Lebesgue measure in \(\Lambda_R\); in other words, in the region \(\Lambda_R\) it satisfies
\[
\nabla \cdot E^{\text{scr}}_t = -\sum_{b \in \omega^{\text{scr}}} \delta^\eta_{b(t)} + \mathbb{1}_\Lambda_R
\]
for all \(t \leq \beta\).

(iii) The field \(E^{\text{scr}}\) is screened; i.e.,
\[
E^{\text{scr}}_t(0, y) \cdot \vec{c}_x = E^{\text{scr}}_t(R, y) \cdot \vec{c}_x = 0
\]
for all \(y \in D\) and \(t \leq \beta\).

(iv) The field satisfies an energy bound of the type
\[
\frac{1}{2\beta} \int_{\Lambda_R \times [0, \beta]} |E^{\text{scr}}_t(z)|^2 \, d(z, t) \leq \tilde{H}_\Lambda(\omega) + \text{error term}\.
\]
The error term will be made precise in Proposition 5.8.
Even when restricting to configurations close to a well-behaved target process $P$, some of the bridges in the configurations may fluctuate too wildly for the envisioned screening construction. Thus, we need a preprocessing step. Recall that the $x$-range of $b$ is $\psi(b) = \sup_{t \leq \beta} |b_x(t) - b_x(0)|$ and remember Lemma 3.4.

**Definition 5.1** ($(M, \varepsilon)$-regularity). Let $M > 2$, $\varepsilon \in (0, 1/M^2)$ and $R > (2/\varepsilon)^8$. A bridge configuration $\omega \in \text{Conf}(\Lambda_R)$ is $(M, \varepsilon)$-regular if $\tilde{H}_{\Lambda_R}(\omega) + \varepsilon < MR$ and

$$\sum_{b \in \omega \Lambda_R} \left( \psi(b)^{7/6} - \varepsilon^{-7/3} \right)_+ < \frac{1}{2} \varepsilon R. \tag{64}$$

**Remark 5.2** (choice of parameters). As already mentioned, we have chosen $\varepsilon$ and $R$ according to later purposes. We however note that $R > (2/\varepsilon)^8$ is large enough that condition (64) forbids any single bridge with $x$-range bigger or equal to $R^{7/8}$ since then we would have the contradiction

$$R^{49/48} - \varepsilon^{-7/3} < \frac{1}{2} \varepsilon R.$$ 

Thus, Lemma 4.3 holds for all $(M, \varepsilon)$-regular configurations.

For $(M, \varepsilon)$-regular $\omega \in \text{Conf}(\Lambda_R)$, by definition of the energy and the regularity, there exists a domain $\Lambda \supset \Lambda_R$ and a configuration $\tilde{\omega} \in \text{Neut}(\Lambda)$ with $\tilde{\omega}_{\Lambda_R} = \omega$ such that

$$\frac{1}{2 \beta} \int_{\Lambda_R \times [0, \beta]} |\nabla V_t(z, \tilde{\omega}, \Lambda)|^2 d(z, t) \leq \tilde{H}_{\Lambda_R}(\omega) + \varepsilon < MR. \tag{65}$$

We choose such $\Lambda$ and $\tilde{\omega}$ and set

$$\tilde{E}_t(z) := \nabla V_t(z, \tilde{\omega}, \Lambda). \tag{66}$$

**5A. Choice of a good boundary abscissa $x_0$.** The construction of $\omega^{scr}$ depends on a choice of boundary abscissa which we denote as $x_0$. In the vein of the classical case from [41, Section 5], the screening construction modifies the configuration in boundary layers corresponding roughly to $[0, x_0] \times D$ and $[R - x_0, R] \times D$.

The boundary $x_0$ is chosen such that some electric field average is not too high. In addition, we ask that only few bridges cross the boundaries of $I_0 := [x_0, R - x_0]$. 

---

**Figure 2.** One-dimensional projection of a nonregular configuration.
Definition 5.3 ($\varepsilon$-tame abscissas). Let $\varepsilon \in (0, 1/M^2)$ and $R > (2/\varepsilon)^8$. Then, setting

$$\xi := \varepsilon^{-3},$$

we say that $x_0 \in [\varepsilon R, R - \varepsilon R]$ is $\varepsilon$-tame if

1. $x_0$ is not $\varepsilon$-dense, and
2. if $b \in \omega$ is such that $\inf_{t \leq \beta} |b_x(t) - x_0| \leq 1$, then $|b_x(0) - x_0| \leq \xi$.

Notice that $R > (2/\varepsilon)^8$ implies $\varepsilon R > R^{7/8}$, therefore item (1) of the definition says that the number of bridges that have their initial point $b_x(0)$ within distance $\xi$ of $x_0$ is smaller or equal to $4\xi / \varepsilon$.

The next lemma will be needed later to prove Lemma 5.6 and Proposition 5.8.

Lemma 5.4 (existence of good boundaries). Let $\varepsilon \in (0, 1/M^2)$, and $R > (2/\varepsilon)^8$. Furthermore, let $I \times D \subseteq \Lambda_{R/2}$ be such that $I \subseteq [\varepsilon R, R/2]$ is an interval of length at least $4\varepsilon R$. If $\omega$ is $(M, \varepsilon)$-regular, then there exists a point $x_0(\omega) \in I$ such that $x_0$ and $R - x_0$ are $\varepsilon$-tame, and the following bounds hold:

$$\int_{D \times [0, \beta]} (|\tilde{E}_t(x, 0, y)|^2 + |\tilde{E}_t(R - x, y)|^2) \, d(y, t) \leq \frac{1}{\varepsilon R} \int_{\Lambda_{R} \times [0, \beta]} |\tilde{E}_t(z)|^2 \, d(z, t),$$

(67)

and

$$\int_{J_{x_0, R} \times D \times [0, \beta]} |\tilde{E}_t(z)|^2 \, d(z, t) \leq \frac{1}{\varepsilon R |\sqrt{\varepsilon}|} \int_{\Lambda_{R} \times [0, \beta]} |\tilde{E}_t(z)|^2 \, d(z, t),$$

(68)

where

$$J_{x_0, R} = [x_0, x_0 + \sqrt{\varepsilon} R] \cup [R - x_0 - \sqrt{\varepsilon} R, R - x_0].$$

Proof: We bound the Lebesgue measure of the set of $x_0$’s violating the required conditions. First, the set $A_1 \subseteq I$ of abscissas violating condition (67) has Lebesgue measure at most $\varepsilon R$. Indeed, abbreviate

$$C = \frac{1}{2\beta R} \int_{\Lambda_{R} \times [0, \beta]} |\tilde{E}_t(z)|^2 \, d(z, t).$$

For $x \in [0, R/2)$, let

$$q(x) = \frac{1}{2\beta} \int_{D \times [0, \beta]} (|\tilde{E}_t(x, y)|^2 + |\tilde{E}_t(R - x, y)|^2) \, d(y, t).$$

Since $I \subseteq [0, R/2]$,

$$|\{x \in I : q(x) > C/\varepsilon\}| \leq \frac{\varepsilon}{C} \int_0^{R/2} q(x) \, dx = \frac{\varepsilon}{2\beta C} \int_{\Lambda_{R} \times [0, \beta]} |\tilde{E}_t(z)|^2 \, d(z, t) = \varepsilon R.$$

Second, the set $A_2 \subseteq I$ of $x_0$’s violating condition (68) has Lebesgue measure at most $\varepsilon R$. Indeed, let $q_2(x_0)$ be the left-hand side of (68). By Fubini’s theorem,

$$\int_0^{R/2} q_2(x_0) \, dx_0 \leq CR,$$

and hence

$$|\{x_0 \in I : q_2(x_0) > C/\varepsilon\}| \leq \varepsilon R.$$
Finally, we assert that for the set \( A_3 \) of \( x_0 \)'s that are not tame, \(|A_3| < 2\varepsilon R\). Then we conclude the proof of the lemma by combining the above bounds, as the measure of \( A_1 \cup A_2 \cup A_3 \) is strictly less than \( 4\varepsilon R \) and since \( I \) is of length \( 4\varepsilon R \) it is nonempty. To prove the assertion, note that since \( \omega \) is \((M, \varepsilon)\)-regular, we may use Lemma 4.3 to bound the Lebesgue measure of the set of \( \varepsilon \)-dense abscissas by \( \varepsilon R \). Next, let

\[
T_r(x_0) = [x_0 - r, x_0 + r] \times D
\]

be the product of an interval of length \( 2r \) centered at \( x_0 \) with the domain \( D \). We bound the measure of abscissas satisfying condition (64) but violating the second tameness condition by

\[
\sum_{b \in \omega} \left| \left[ b_x(0) - \psi(b), b_x(0) + \psi(b) \right] \setminus T_{2\varepsilon-1}(b_x(0)) \right| \leq 2 \sum_{b \in \omega} (\psi(b) - \varepsilon^{-2})_+.
\]

Hence, invoking the bound (64) shows \(|A_3| < 2\varepsilon R\).

5B. Construction of screened configurations \( \omega^{scr} \). Lemma 5.4 applies to the interval \([6\varepsilon R, 10\varepsilon R]\), which we will use throughout this section. In particular, we will assume \( x_0 \) designates a choice of the interface with this choice of interval. (A different application with the choice \([2\varepsilon R, 6\varepsilon R]\) is used following (117), and it will be convenient for our choice of the interval in this section to be to the right of \([2\varepsilon R, 6\varepsilon R]\); this is the reason for setting the left endpoint to \(6\varepsilon R\).) Having chosen a good boundary \( x_0 \), we proceed with the definition of \( \omega^{scr} \).

If \( \omega \in \text{Conf}(\Lambda_R) \) is not \((M, \varepsilon)\)-regular, then \( \omega^{scr} \) consists of time-constant bridges (straight lines) with initial points

\[
(i + \frac{1}{2}, 0), \quad i = 0, \ldots, R - 1.
\]

For \((M, \varepsilon)\)-regular \( \omega \in \text{Conf}(\Lambda_R) \), in a preliminary regularization step, we rearrange all bridges crossing the boundaries \( \partial I_0 \times D \), with \( I_0 = [x_0, R - x_0] \) with \( x_0 \) chosen above. By the choice of \( x_0 \), all these bridges have their starting point within \( T_{\xi}(x_0) \). We remove all bridges in \( \omega \) with starting point in \( T_{\xi}(x_0) \) and replace them with the same number of bridges inside \( [x_0 + \frac{1}{2}, x_0 + 1] \times D \subseteq T_{\xi}(x_0) \). The new bridges are time-constant straight lines with regularly spaced starting points \((x_i, 0)\) for \( i = 1, \ldots, n \), where \( n = \#\omega_{T_{\xi}(x_0)} \) and \( x_i = x_0 + \frac{1}{2} + \frac{1}{2n} + (i - 1) \frac{1}{2n} \). We proceed similarly in \( T_{\xi}(R - x_0) \).

The newly obtained

\[
\text{regularized configuration } \omega' \in \text{Conf}(\Lambda_R)
\]

coincides with \( \omega \) except possibly in \( T_{\xi}(x_0) \cup T_{\xi}(R - x_0) \). Moreover, the regularized configuration \( \omega' \) has no crossings of the interface at \( x_0 \) and \( R - x_0 \).

Changing \( \tilde{\omega} \) from (65) similarly results in a configuration \( \tilde{\omega}' \in \text{Neut}(\Lambda) \) with projection \( \tilde{\omega}'_{\Lambda_R} = \omega' \) and electric field

\[
\tilde{E}'(z) := \nabla V_t(z, \tilde{\omega}', \Lambda).
\]

We split the domain \([0, R/2] \times D\) into subdomains at the interfaces

\[
0 < x_- < x_0 < x_+ < R/2,
\]
where

\[ x_+ := x_0 + \xi \]  
(74)

and \( x_- \) is specified as follows. Recall from (65)–(66) our choice of \( \Lambda \), which we now write as \( \Lambda = [L_-, L_+] \times D \) and let

\[ d_0 := \#\tilde{\omega}'_{[L_-, x_0] \times D} - (x_0 - L_-) \]  
(75)

be the net negative charge (or deficit of charge) to the left of \( x_0 \) for the extended regularized configuration \( \tilde{\omega}' \in \text{Neut}(\Lambda) \). Note that \( d_0 + x_0 \in \mathbb{Z} \) since \( L_- \in \mathbb{Z} \) (it should be noted that if this is not an integer, then one runs into an issue with fractional charges — see the remark following (13)). In Lemma 5.6 below, we check that \( x_0 - 2|d_0| \geq 0 \) for sufficiently large \( R \), which we use later in (86). Set

\[ x_- := |x_0 - 2|d_0| \in \mathbb{N}_0. \]  
(76)

Here, we emphasize that \( x_0 \), and thus \( x_+ \) and \( x_- \), depend on the original configuration \( \omega \).

**Definition 5.5** \((\omega^{\text{scr}} \text{ for } (M, \varepsilon)\text{-regular configurations})\). The new configuration \( \omega^{\text{scr}} \) is defined separately in each subdomain. Then we proceed in the domain \([R/2, R] \times D\) similarly.

1. In the central subdomain

\[ \Lambda_+ := [x_+, R - x_+] \times D, \]  
(77)

the configuration \( \omega^{\text{scr}} \) coincides with \( \omega \), i.e., \( \omega^{\text{scr}}_{\Lambda_+} := \omega_{\Lambda_+} \).

2. In \([x_0, x_+] \times D\), the configuration \( \omega^{\text{scr}} \) coincides with the time-constant bridges as rearranged in the regularization step above. That is, \( \omega^{\text{scr}} \) and \( \omega' \) have the same restrictions to \([x_0, x_+] \times D\).

3. In \([x_-, x_0] \times D\), we place

\[ a_+ := x_0 - x_- + d_0 \]

time-constant bridges with regularly spaced starting points \((x_i, 0)\), where \( x_i = x_- + i(x_+ - x_-)/a_+ \) for \( i = 1, \ldots, a_+ \). Notice that \( x_0 + d_0 - x_- \) is an integer because \( x_0 + d_0 \) and \( x_- \) are, and \( x_0 + d_0 - x_- \geq 0 \) because \( x_- \leq x_0 - 2|d_0| \leq x_0 + d_0 \).

4. In \([0, x_-] \times D\), we place \( x_- \) time-constant bridges with regularly spaced starting points \((i - \frac{1}{2}, 0)\) for \( i = 1, \ldots, x_- \), so that in this region, \( \omega^{\text{scr}} \) is charge neutral.

We note that the construction of \( \omega^{\text{scr}} \) is robust in the sense that we could replace \( x_0 \) by a rational value sufficiently close to a good boundary, thereby making \( \omega^{\text{scr}} \) measurable in the input data \( \omega \).

**Lemma 5.6.** Let \( M > 2 \), \( \varepsilon \in (0, 1/M^2) \), and \( R > (2/\varepsilon)^8 \). Assume that \( \omega \in \text{Conf}(\Lambda_R) \) is \((M, \varepsilon)\)-regular. Then for any tame abscissa \( x_0 \in [6\varepsilon R, 10\varepsilon R] \), with \( d_0 \in \mathbb{R} \) as in (75),

\[ x_0 \geq 2|d_0|. \]

**Proof.** Let

\[ d := \#\tilde{\omega}'_{[L_-, x_0] \times D} - (x_0 - L_-) \]  
(78)
be defined in the same way as \( d_0 \) in (75) but with \( \tilde{\omega}' \) replaced by the nonregularized \( \tilde{\omega} \) and note that by tameness of \( x_0 \),

\[
|d - d_0| \leq 4\varepsilon^{-4}.
\]  

(79)

Now, by Lemma 4.1,

\[
|d| \leq \left( \frac{1}{\beta} \int_{D \times [0, \beta]} |\tilde{E}_t(x_0, y)|^2 \, d(y, t) \right)^{1/2} + \#\{b \in \tilde{\omega} : \inf_{t \leq \beta} |b_x(t) - x_0| \leq 1\}.
\]  

(80)

The inequality (67) from Lemma 5.4 yields

\[
\frac{1}{\beta} \int_{D \times [0, \beta]} |\tilde{E}_t(x_0, y)|^2 \, d(y, t) \leq \frac{2}{2\varepsilon \beta R} \int_{\Lambda R \times [0, \beta]} |\tilde{E}_t(z)|^2 \, d(z, t) \leq \frac{2M}{\varepsilon} \leq 2\varepsilon^{-3/2}.
\]  

(81)

Since \( x_0 \) is \( \varepsilon \)-tame the cardinality in (80) is bounded by \( 4\varepsilon^{-4} \), so that we arrive at

\[
|d| \leq 5\varepsilon^{-4}.
\]  

(82)

As \( R > (2/\varepsilon)^8 \) and \( x_0 \in [6\varepsilon R, 10\varepsilon R] \), it follows that

\[
|d_0| \leq 9\varepsilon^{-4} < \frac{1}{2}\varepsilon R \leq \frac{1}{2}x_0,
\]  

(83)

concluding the proof. \( \square \)

5C. Construction of screened electric fields \( E^{\text{scr}}_t \). We now construct the screened electric field \( E^{\text{scr}}_t \), along the lines of [41, Proposition 5.1], which is compatible with \( \omega^{\text{scr}} \) in the sense of (61). We define \( E^{\text{scr}}_t \) separately in each of the domains \([0, x_-] \times D, [x_-, x_0] \times D, \) and \([x_0, R/2] \times D\).

Construction in \([x_0, R/2] \times D\). We put \( E^{\text{scr}}_t(\cdot) = \tilde{E}'_t(\cdot) \) defined in (73), noting that in this region, this electric field is compatible with the configuration \( \omega^{\text{scr}} \) defined in Section 5B.

Construction in \([0, x_-] \times D\). Since the bridges are nicely separated, we define \( E^{\text{scr}}_t \) inside this domain by first constructing the electric field in the cell \([i - 1, i] \times D\) around each time-constant bridge \( b_i \) at \( x_i \) and then pasting the separate pieces together. In particular, for

\[
\text{div}(E^{\text{scr}}_t(x, y)) = -\delta^N_{x_i} + 1,
\]  

(84)
with $E_{t}^{\text{scr}}(x, y) \cdot \vec{e}_x = 0$ on the boundaries, a necessary and sufficient condition for the existence of a solution is charge neutrality, i.e., that the integral of the right-hand side over $[i - 1, i] \times D$ vanishes [35, Theorem 1.2].

Pasting together the fields defines the screened electric field $E_{t}^{\text{scr}}$ on $[0, x_-] \times D$. This pasting may lead to vector fields that are discontinuous across an interface so that just as in [41, Section 5.1], we need to work with distributional divergences. From [41, Section 5.1], we also recall that no additional divergences are created through the pasting operation since the normal derivatives along the interface coincide.

**Construction in** $[x_-, x_0] \times D$. Defining the electric field in $[x_-, x_0] \times D$ is the core of the screening construction. Indeed, here the task is to come up with a compatible electric field whose normal components agree with those of the field $E_{t}^{\text{scr}}(\cdot)$ from the previous step at the interface $[x_0] \times D$, and vanish at the interface $[x_-] \times D$.

For this purpose, we subdivide the domain $[x_-, x_0] \times D$ into $a_+ = x_0 - x_- + d_0$ congruent domains of the form $I^{(i)} \times D$,

$$I^{(i)} = [x_0 + (i + \frac{1}{2}) \ell, x_0 + (i + 1 + \frac{1}{2}) \ell], \quad \ell = \frac{x_0 - x_-}{a_+}, \quad i = 0, \ldots, a_+ - 1,$$  \hspace{1cm} (85)

each centered at a time-constant bridge $b_i$. Note that the choice of $x_-$ in (76) implies that

$$\frac{2}{3} \leq |I^{(i)}| = \ell \leq 2.$$

(86)

We think of the system in $[x_-, x_0] \times D$ as a superposition of two systems, a neutral system with a background of $y$-dependent charge density together with the time-constant bridges from $\omega^{\text{scr}}$, and an additional system with no particles and $y$-dependent charged background only. Accordingly the field is defined as a sum of two contributions.

Let us start with the latter contribution coming from a system with no particles. This has only components in the $x$-direction, and the $x$-component interpolates linearly between 0 and $E_{t}^{\text{scr}}(x_0, y)$: we define

$$E_{t}^{(1)}(x, y) := \frac{x - x_-}{x_0 - x_-} (E_{t}^{\text{scr}}(x_0, y) \cdot \vec{e}_x) \vec{e}_x.$$  \hspace{1cm} (87)

Notice

$$\text{div}(E_{t}^{(1)}(x, y)) = \frac{E_{t}^{\text{scr}}(x_0, y) \cdot \vec{e}_x}{x_0 - x_-} =: \rho_{t}^{(1)}(y),$$

(88)

and thus $E_{t}^{(1)}$ is compatible with a charged background of $y$-dependent charge density given by the right-hand side of (88).

For the second contribution, define $E_{t}^{(2,i)}$ as the solution in $I^{(i)} \times D$ of

$$\text{div}(E_{t}^{(2,i)}(x, y)) = -\delta_{b,(0)}^\eta + 1 - \rho_{t}^{(1)}(y)$$

(89)

with periodic boundary conditions in the $y$-direction and $E_{t}^{(2,i)}(x, y) \cdot \vec{e}_x = 0$ for $x \in \partial I^{(i)}$. As in (84), a
Figure 4. One-dimensional projection of the electric field $E^{\text{scr}}_t$, with positions of charges marked on the x-axis.

necessary and sufficient condition for the existence of a solution is that the integral of the right-hand side over the domain $I^{(i)}$ vanishes, i.e., that $I^{(i)}$ is charge-neutral. We compute

$$
\int_{I^{(i)} \times D} (1 - \rho_i^{(1)}(y)) \, d(x, y) = |I^{(i)}| - \frac{|I^{(i)}|}{x_0 - x_-} \int_D E^{\text{scr}}_t(x_0, y) \cdot \vec{e}_x \, dy
$$

$$
= |I^{(i)}| \left( 1 - \frac{-d_0}{x_0 - x_-} \right)
= \left( \frac{x_0 - x_-}{a_+} \right) \left( 1 - \frac{-d_0}{x_0 - x_-} \right) = 1.
$$

(90)

Since $|I^{(i)}| \geq \frac{2}{3} > 2\eta$ (see the remark after (85) and the assumption on $\eta$ in Section 2A), we see that indeed the cell $I^{(i)} \times D$ with background of charge density $1 - \rho_i^{(1)}(y)$ and smeared charge $\delta^\eta_{b_1(0)}$ is neutral.

Finally, we set

$$
E^{(2)}_t = \sum_i E^{(2,i)}_t \mathbb{1}_{I^{(i)} \times D}, \quad E^{\text{scr}}_t = E^{(1)}_t + E^{(2)}_t
$$

(91)

in $[x_-, x_0] \times D$ and note that the field $E^{\text{scr}}_t$ is compatible with $\omega^{\text{scr}}$ in that domain.

Altogether, we obtain a field $E^{\text{scr}}_t$ in $[0, R] \times D$ with the right properties. The combined construction is illustrated in Figure 4.

5D. Energy change due to screening electric fields. Using similar techniques to those employed in [2; 41], we will verify that the so-constructed screened electric field has an energy that does not exceed the original one by much. To this end we first need a lemma discussing the effect on the energy change due to moving a fixed number of charges. Let

$$
g^n(z) := \int_S g(z - z') \delta^n_0(dz')
$$

denote the Green’s function associated with a smeared charge.

Lemma 5.7 (moving charges). Let $z_1, \ldots, z_n \in \Lambda_R$ and $E : S \to \mathbb{R}^{k+1}$ be such that $-\text{div}(E) = \sum_{i \leq n} \delta^{(q)}_{z_i} - \mathbb{1}_{\Lambda_R}$. For $z_1' \in \Lambda_R$ and $z \in S$, define

$$
E'(z) := E(z) + \nabla g^n(z - z_1') - \nabla g^n(z - z_1).
$$
Then \(-\text{div}(E') = \delta_{z_1}^{(n)} + \sum_{i \geq 2} \delta_{z_i}^{(n)} - 1_{\Lambda_R}\). In addition there exists a constant \(c = c(D, \eta) > 0\) such that \(\sup_{z \in \Lambda_R} |E'(z) - E(z)| \leq c\). Moreover, for any \(\alpha > 0\) there exist \(c_{\text{mov}}, c'_{\text{mov}} > 0\) such that

\[
|E'(z) - E(z)| \leq c_{\text{mov}} \exp\left(-c'_{\text{mov}}(|x - x_1| \wedge |x - x'_1|)\right)
\]

holds whenever \(|x_1 - x'_1| \leq \alpha\).

\textbf{Proof}: The compatibility of \(E'\) is an immediate consequence of the definition. Moreover, since smearing removes the singular part of the Green’s function \(g\), the increment

\[
E'(z) - E(z) = \nabla g^n(z - z'_1) - \nabla g^n(z - z_1)
\]

remains bounded. For the final claim, we fix \(\alpha > 0\) and assume that \(|x_1 - x'_1| \leq \alpha\). The final claim then follows from (3). \hfill \Box

\textbf{Proposition 5.8} (energy change from screening electric fields). Let \(M > 2, \varepsilon \in (0, 1/M^2), R > (2/\varepsilon)^8\) and \(\omega \in \text{Conf}(\Lambda_R)\) be \((M, \varepsilon)\)-regular. Then the configuration \(\omega^{\text{scr}} \in \text{Neut}(\Lambda_R)\) satisfies:

(a) \(\omega^{\text{scr}}\) and \(\omega\) coincide on \([10\varepsilon R, R - 10\varepsilon R] \times D\); i.e.,

\[
\omega|_{[10\varepsilon R, R - 10\varepsilon R] \times D} = \omega^{\text{scr}}|_{[10\varepsilon R, R - 10\varepsilon R] \times D}.
\]

(b) The field \(E^{\text{scr}}\) is compatible with \(\omega^{\text{scr}}\) in \(\Lambda_R\); i.e., it satisfies (61).

(c) The field \(E^{\text{scr}}\) is screened; i.e., it satisfies (62).

(d) For every \(t \leq \beta\), the screened field has an electrostatic energy bounded as

\[
\int_{\Lambda_R} |E_t^{\text{scr}}(z)|^2 \, dz \leq \varphi_0(\tilde{E}_t, \varepsilon, R),
\]

where \(\varphi_0(\tilde{E}_t, \varepsilon, R) \geq 0\) is measurable in \(t\) and satisfies

\[
\frac{1}{2R\beta} \int_{[0, \beta]} \varphi_0(\tilde{E}_t, \varepsilon, R) \, dt \leq \frac{1}{2R\beta} \int_{\Lambda_R \times [0, \beta]} |\tilde{E}_t(z)|^2 \, d(z, t) + \varphi(\varepsilon, R)M,
\]

where \(\varphi(\varepsilon, R) \geq 0\) satisfies

\[
\lim_{\varepsilon \to 0} \lim_{R \to \infty} \varphi(\varepsilon, R) = 0.
\]

\textbf{Proof}: Parts (a), (b), and (c) hold true by definition of \(\omega^{\text{scr}}\) and \(E_t^{\text{scr}}\).

It remains to verify the energy condition (d).

\textbf{Energy in }\([0, x_-] \times D\).\textbf{\hspace{1em} Note that the energy each of the charge-neutral unit-volume constituent fields of }\tilde{E}_t'\textbf{\hspace{1em} corresponding to (4) in Definition 5.5 is of constant order (see for instance [54, Lemma 5.7]). Hence, the energy of }E_t^{\text{scr}}\textbf{\hspace{1em} in the domain }\([0, x_-] \times D\textbf{\hspace{1em} is of order }O(x_-) = O(\varepsilon R)\):

\[
\int_{[0, x_-] \times D} |E_t^{\text{scr}}(z)|^2 \, dz = O(\varepsilon R).
\]
Energy in \([x_-, x_0] \times D\). For the layer \([x_-, x_0] \times D\), remember \(E_t^{\text{scr}} = E_t^{(1)} + E_t^{(2)}\). Using the definition of \(E_t^{(1)}\) in (87), we compute
\[
\int_{[x_-, x_0] \times D} |E_t^{(1)}(z)|^2 \, dz = \frac{1}{2} (x_0 - x_-) \int_D (\mathcal{E}_t'(x_0, y) \cdot \vec{e}_x)^2 \, dy,
\]
and note by definition in (76) we have \(x_0 - x_- \leq 1 + 2|d_0|\). Next, the field \(E^{(2)}\) consists of \(a_+ \leq 3 |d_0| + 1\) constituents \(\{E^{(2,i)}\}_{i=1}^{a_+}\). Since the length of the interval \(I^{(i)}\) is of constant order, we conclude from [54, Lemma 5.7] that also the energy of \(E^{(2,i)}\) is of constant order. Hence,
\[
\int_{[x_-, x_0] \times D} |E_t^{(2)}(z)|^2 \, dz = O(a_+) = O(|d_0|).
\]
Using \(|E_t^{(1)} + E_t^{(2)}|^2 \leq 2|E_t^{(1)}|^2 + 2|E_t^{(2)}|^2\) we deduce
\[
\int_{[x_-, x_0] \times D} |E_t^{\text{scr}}(z)|^2 \, dz \leq O(|d_0|) \int_D (\mathcal{E}_t'(x_0, y) \cdot \vec{e}_x)^2 \, dy + O(|d_0|).
\]

Energy in \(I_0 \times D\). Remember the configuration \(\tilde{\omega} \in \text{Neut}(\Lambda)\) from (65) and the fields \(\tilde{E}_t\) and \(\tilde{E}_t'\) from (66) and (73). By definition of \(E_t^{\text{scr}}\),
\[
\int_{I_0 \times D} |E_t^{\text{scr}}(z)|^2 \, dz = \int_{I_0 \times D} |\tilde{E}_t'(z)|^2 \, dz.
\]
Altogether, we find that
\[
\int_{\Lambda_R} |E_t^{\text{scr}}(z)|^2 \, dz = \int_{I_0 \times D} |\tilde{E}_t'(z)|^2 \, dz + O(|d_0| + O(\varepsilon R) + O(|d_0|) \int_D (|\tilde{E}_t'(x_0, y)|^2 + |\tilde{E}_t'(R - x_0, y)|^2) \, dy. \tag{98}
\]

Next, we compare \(\tilde{E}_t\) and \(\tilde{E}_t'\) in \([x_0, R - x_0] \times D\). Remember \(\tilde{E}_t\) and \(\tilde{E}_t'\) are gradient fields created by configurations \(\tilde{\omega}, \tilde{\omega}' \in \text{Neut}(\Lambda)\), with \(\tilde{\omega}'\) obtained from \(\tilde{\omega}\) in the regularization step (72).

Lemma 5.7 bounds the change in electric field modulus caused by moving a single bridge in a given configuration. Since \(x_0\) is \(\varepsilon\)-tame, one may apply this lemma at most \(4\varepsilon^{-4}\) successive times to obtain, for \(x \in I_0\),
\[
|\tilde{E}_t'(x, y) - \tilde{E}_t(x, y)| \leq 4\varepsilon^{-4} c_{\text{mov}} \exp(-c_{\text{mov}}(x - x_0) \wedge (R - x_0 - x)). \tag{99}
\]
Combining \(a^2 = b^2 + (a - b)^2 + 2b(a - b)\) with the Cauchy–Schwarz inequality, we get
\[
\int_{D} |\tilde{E}_t'(x, y)|^2 \, dy \leq \int_{D} |\tilde{E}_t(x, y)|^2 \, dy + \int_{D} |\tilde{E}_t'(x, y) - \tilde{E}_t(x, y)|^2 \, dy
\]
\[
+ 2 \left( \int_{D} |\tilde{E}_t(x, y)|^2 \, dy \times \int_{D} |\tilde{E}_t'(x, y) - \tilde{E}_t(x, y)|^2 \, dy \right)^{1/2}.
\]
Applying (99) to the second and third terms on the right-hand side, we find that there exist \( c_0(\varepsilon) > 1 \) and \( c_0'(\varepsilon) > 0 \) such that for all \( x \in I_0 \),

\[
\int_D |\tilde{E}_t'(x, y)|^2 \, dy 
\leq \int_D |\tilde{E}_t(x, y)|^2 \, dy + c_0 \exp(-c_0'(x-x_0) \wedge (R-x_0-x)) \left( 1 + \int_D |\tilde{E}_t(x, y)|^2 \, dy \right). \tag{100}
\]

In particular,

\[
\int_D |\tilde{E}_t'(x, y)|^2 \, dy \leq 2c_0 \left( 1 + \int_D |\tilde{E}_t(x, y)|^2 \, dy \right). \tag{101}
\]

We return to (100). For the first term on the right-hand side of (98), we further subdivide \([x_0, R-x_0] \times D\). Integrating (101) over

\[
J_{x_0,R} = [x_0, x_0 + \sqrt{R}] \cup [R-x_0 - \sqrt{R}, R-x_0]
\]

shows that for all large \( R \),

\[
\int_{J_{x_0,R} \times D} |E_t^{scr}(z)|^2 \, dz \leq 2c_0 \left( \sqrt{R} + \int_{[x_0,x_0+\sqrt{R}] \times D} |\tilde{E}_t(z)|^2 \, dz \right). \tag{102}
\]

Integrating (100) over all \( x \in [x_0 + \sqrt{R}, R-x_0 - \sqrt{R}] \) shows that for all large \( R \),

\[
\int_{[x_0+\sqrt{R}, R-x_0-\sqrt{R}] \times D} |E_t^{scr}(z)|^2 \, dz \leq \int_{\Lambda_R} |\tilde{E}_t(z)|^2 \, dz + c_0 e^{-\sqrt{R}} \left( 1 + \int_{\Lambda_R} |\tilde{E}_t(z)|^2 \, dz \right). \tag{103}
\]

Combining the inequalities (98), (102) and (103), we get

\[
\int_{\Lambda_R} |E_t^{scr}(z)|^2 \, dz 
\leq \int_{\Lambda_R} |\tilde{E}_t(z)|^2 \, dz + c_0 e^{-\sqrt{R}} \left( 1 + \int_{\Lambda_R} |\tilde{E}_t(z)|^2 \, dz \right) + O(|d_0|) + O(\varepsilon R) 
+ 2c_0 \left( 2\sqrt{R} + \int_{J_{x_0,R}} |\tilde{E}_t(z)|^2 \, dz \right) + O(|d_0|)c_0 \int_{\partial I_0 \times D} |\tilde{E}_t(z)|^2 \, dz.
\]

To conclude, let \( \varphi_0 \) denote the right-hand side. Then we integrate \( t \) over \([0, \beta]\), divide by \( 2\beta \), and use the bounds from Lemma 5.4. This yields

\[
\frac{1}{2\beta} \int_0^\beta \varphi_0(\tilde{E}_t, \varepsilon, R) \, dt 
\leq \frac{1}{2\beta} \int_{\Lambda_R \times [0,\beta]} |\tilde{E}_t(z)|^2 \, dz(z, t) + c_0 e^{-\sqrt{R}} (1 + MR) + O(|d_0|) + O(\varepsilon R) 
+ 2c_0 \left( 2\sqrt{R} + \frac{M}{\varepsilon} \sqrt{R} \right) + O(|d_0|)c_0 \frac{M}{\varepsilon}.
\]
Remembering $|d_0| < \frac{1}{2} \epsilon R$ from (83), we obtain
\[
\frac{1}{2\beta} \int_{[0,\beta]} \varphi_0(\tilde{E}_t, \epsilon, R) \, dt \leq \frac{1}{2R\beta} \int_{\Lambda_R \times [0,\beta]} |\tilde{E}_t(z)|^2 \, d(z, t) + \varphi(\epsilon, R) \, M
\]  
with
\[
\lim_{R \to \infty} \varphi(\epsilon, R) = O(\epsilon).
\]

6. Quasicontinuity of the specific energy

In this section, we prove the quasicontinuity result, Proposition 3.8.

6A. Proof strategy. The proof is a variant of arguments by Leblé and Serfaty [41]. Let us sketch the heuristic argument before we embark on the details. The principal obstruction to Proposition 3.8 is the lack of a simple analog to Lemma 3.7 for an upper bound. In other words, it is difficult to show that $\text{Emp}_N(\omega) \approx P$ implies a bound of the type $(1/N)H_{\Lambda_N}(\omega) \lesssim W(P)$. Screening is the mechanism by which we will achieve this. In particular, we map a significant number of the configurations $\omega \in \text{Conf}(\Lambda_N)$ such that $\text{Emp}_N(\omega) \approx P$ to other configurations $\omega^{\text{scr}} \in \text{Neut}(\Lambda_N)$ that have a similar empirical field
\[
\text{Emp}_N(\omega^{\text{scr}}) \approx \text{Emp}_N(\omega) \approx P
\] and with energy
\[
\frac{1}{N} H_{\Lambda_N}(\omega^{\text{scr}}) \lesssim W(P).
\] The inequality holds true even though $(1/N)H_{\Lambda_N}(\omega)$ might be much larger than $W(P)$, so as far as the energy is concerned, the configuration $\omega^{\text{scr}}$ is better-behaved than $\omega$. We will find a probability measure $\mathbb{P}_N^{\text{Coup}}$ on $\text{Conf}(\Lambda_N) \times \text{Neut}(\Lambda_N)$ such that:

(i) The first marginal is the Poissonian reference measure projected to $\Lambda_N$, i.e., the distribution $\text{Pois}_{\Lambda_N}$ where Pois is a Poisson process with independent Brownian bridge marks defined above (16). The second marginal corresponds to a binomial process with independent Brownian bridge marks as defined around (10). We shall write elements of $\text{Conf}(\Lambda_N) \times \text{Neut}(\Lambda_N)$ as $(\omega^P, \omega^B)$.

(ii) With high probability, the coupled binomial process $\omega^B$ is close to the screened configuration $\omega^{\text{scr}} = (\omega^P)^{\text{scr}}$ in the sense that
\[
\frac{1}{N} H_{\Lambda_N}(\omega^B) \lesssim \frac{1}{N} H_{\Lambda_N}(\omega^{\text{scr}}), \quad \text{Emp}_N(\omega^B) \approx \text{Emp}_N(\omega^{\text{scr}}).
\]

Then we may estimate for some $c > 0$ and any $\delta > 0$,
\[
\mathbb{P}_N\left(\text{Emp}_N(\omega) \approx P, \frac{1}{N} H_{\Lambda_N}(\omega) \lesssim W(P)\right) = \mathbb{P}_N^{\text{Coup}}\left(\text{Emp}_N(\omega^B) \approx P, \frac{1}{N} H_{\Lambda_N}(\omega^B) \lesssim W(P)\right)
\geq \mathbb{P}_N^{\text{Coup}}\left(\text{Emp}_N(\omega^{\text{scr}}) \approx P, \frac{1}{N} H_{\Lambda_N}(\omega^{\text{scr}}) \lesssim W(P)\right)
\geq e^{-c\delta N} \mathbb{P}_N^{\text{Coup}}(\text{Emp}_N(\omega^P) \approx P)
\approx \exp\left(-N(\text{ent}(P) + c\delta)\right),
\]
which is the kind of statement that we are after.
For the rigorous proof, it is convenient to approximate empirical fields by block averages. Consider volumes
\[ \Lambda_N = \bigcup_{i \leq m} K_i, \quad K_i = [(i - 1)R, iR) \times D, \quad N = mR. \] (108)

The block average of a configuration \( \omega \in \text{Conf}(\Lambda_N) \) is
\[ \text{Avg}_R(\omega) := \frac{1}{m} \sum_{i \leq m} \delta_{\theta_{x_i}}(\omega_{K_i}), \quad x_i = (i - 1)R. \] (109)

The \( m \)- or \( N \)-dependence is suppressed from the notation. Notice that \( \text{Avg}_R(\omega) \) puts full mass on \( \text{Conf}(3R) \).

We shall see in Step 3 of the proof of Proposition 3.8 that if the block average is close to a given shift-invariant measure \( P \in \mathcal{P}_s \), then so is the empirical field. In our topology, the block average is not sensitive to small changes of the configuration in the vicinity of block boundaries \( \partial K_i \); this allows us to apply the screening construction from Section 5, block by block without changing too much the block and empirical averages.

The coupling measure is defined blockwise.

6B. Coupling within block \( K_1 \). Let \((\Omega, \mathcal{F}, \mathbb{P})\) be an enlarged probability space big enough to support all random variables introduced below. We now describe the coupling measure in the fixed block \( K_1 \), which will eventually be used to form \( \mathbb{P}^\text{Coup}_N \) on the full domain \( \Lambda_N \) with the correct marginal distributions as described in the beginning of the section. Throughout this subsection, we often suppress the dependence on \( K_1 \), by writing, for example, \((\omega^P, \omega^B) = (\omega^P_{K_1}, \omega^B_{K_1}) \). The joint distribution of \((\omega^P, \omega^B)\) under the coupling measure \( \mathbb{P}^\text{Coup}_K \) on \( K_1 \) will be defined using (regular) conditional probabilities. In particular, the first marginal \( \omega^P \) simply has the marginal distribution \( \text{Pois}_{K_1} \), and we proceed to define the conditional distribution of the second marginal given \( \omega^P \).

As a first step, we let the coupling measure be a mixture
\[ \mathbb{P}^\text{Coup} = \frac{1}{2} \mathbb{P}^I + \frac{1}{2} \mathbb{P}^D, \]
where under \( \mathbb{P}^I \), the marginals are independent, i.e., the configuration \( \omega^B \) is independent from \( \omega^P \) and has a distribution given by a binomial process with independent Brownian bridge marks as defined around (10). The independent portion \( \mathbb{P}^I \) is used later in Lemma 6.3 to handle configurations which are not \((M, \varepsilon)\)-regular.

Under \( \mathbb{P}^D \) the marginals are dependent, and we construct the coupling as follows. Fix \( \delta \in (0, \frac{1}{2}) \) (before any choice of \( M, \varepsilon, R \)) and put
\[ \Lambda_{R, \varepsilon} := [6\varepsilon R, R - 6\varepsilon R] \times D. \] (110)

It is convenient to label some of the bridges in the configuration \( \omega^P \). Suppose \( R^P, \{Z^P_i\}_{i \geq 1}, \{B^P_i\}_{i \geq 1} \) are independent random variables such that
- \( R^P \) is a Poisson random variable with parameter \( R - 12\varepsilon R \),
- \( Z^P_i \) is uniformly distributed in \( \Lambda_{R, \varepsilon} \),
- \( B^P_i \) is a Brownian bridge with \( B^P_i(0) = 0 \) and diffusivity 1.
and such that \( \omega_{\Lambda, \varepsilon}^P : \Omega \to \text{Conf} \) can be represented as
\[
\bigcup_{i \leq R^P} \{(Z_i^P, Z_i^P + B_i^P)\}.
\] (111)

We now specify the conditional distribution of \( \omega^B \) given \( \omega^P \) by fixing \( B_i^B := B_i^P \) for all \( i \leq R \) and by defining the initial points of the configuration \( (Z_1^B, \ldots, Z_R^B) \) as follows. Set
\[
\delta_1 := (\delta - 12\varepsilon R)/(1 - 12\varepsilon R).
\] (112)

Then, treating each \( i \leq R \) independently, with probability \( 1 - \delta \) the variable \( Z_i^B \) is set to be equal to \( Z_i^P \), with probability \( \delta_1 \) it is uniformly distributed in \( \Lambda_R \), and with probability \( \delta - \delta_1 \) it is uniformly distributed in \( \Lambda_R \setminus \Lambda_{R, \varepsilon} \).

If \( R^P \geq R \) this determines the vector \( (Z_1^B, \ldots, Z_R^B) \) — where in this case the points \( Z_{R+1}^P, \ldots, Z_{R^P}^P \) are simply discarded.

If \( R^P < R \), we complete the definition by letting \( Z_i^B \) be uniformly distributed in \( \Lambda_R \), for all \( R^P < i \leq R \) (we also produce extra independent random variables \( B_i^P \) for all \( R^P < i \leq R \)).

A technical description of the coupling under \( \mathbb{P}^D \), when \( \omega^P \) is \((M, \varepsilon)\)-regular, is via the probability kernel
\[
K_{\text{reg}}(z, dz') := (1 - \delta)\delta z(dz') + \delta_1 \frac{1}{R} \mathbb{1}_{\Lambda_{R, \varepsilon}}(z') dz' + (\delta - \delta_1) \frac{1}{12\varepsilon R} \mathbb{1}_{\Lambda_R \setminus \Lambda_{R, \varepsilon}}(z') dz'.
\] (113)

Then, \( \mathbb{P} \)-almost surely on the event \( R^P \leq R \), for all measurable \( A_1, \ldots, A_R \subseteq S \),
\[
\mathbb{P}
\bigg( Z_1^B \in A_1, \ldots, Z_R^B \in A_R \mid R^P, Z_1^P, \ldots, Z_{R^P}^P \bigg) = \left( \prod_{i \leq R^P} K_{\text{reg}}(Z_i^P, A_i) \right) \times \prod_{R^P + 1 \leq i \leq R} \frac{|A_i|}{R}. \] (114)

Finally, we put
\[
\omega^B = \bigcup_{i \leq R} \{(Z_i^B, Z_i^B + B_i^P)\},
\] (115)
and set \( \mathbb{P}^D \) on \( \text{Conf}(\Lambda_R) \cap \{(M, \varepsilon)\text{-regular}\} \times \text{Neut}(\Lambda_R) \) to be the joint law of \( (\omega^P, \omega^B) \). By construction, \( \mathbb{P}^{\text{Coup}} \) is indeed a coupling. That is, it has the correct marginals.

After construction of a measure with the correct marginals, the big space \( \Omega \) is no longer of any use. We change notation slightly and from now on use the letter \( \mathbb{P}^{\text{Coup}} \) for the distribution of \( (\omega^P, \omega^B) \).

**Lemma 6.1.** *The coupling measure \( \mathbb{P}^{\text{Coup}} \) on \( \text{Conf}(\Lambda_R) \times \text{Neut}(\Lambda_R) \) has as its first marginal distribution \( \text{Pois}_{\Lambda_R} \) as in (16), and has as its second marginal distribution a binomial bridge process as defined around (10).*

Next, we define the event of coupling to a screened configuration \( F^{\text{Scr}} \subseteq \text{Conf}(\Lambda_R) \times \text{Neut}(\Lambda_R) \) and bound \( \mathbb{P}^{\text{Coup}}(F^{\text{Scr}} | \omega^P) \) from below. Loosely speaking, \( F^{\text{Scr}} \) imposes that \( \omega^B \) is at most a small perturbation of the screened configuration \( \omega^{\text{scr}} \) defined in Section 5. As mentioned above, here the screening construction is applied to \( \omega^P \) and \( \omega^{\text{scr}} = (\omega^P)^{\text{scr}} \).
Lemma 4.1, this charge imbalance is at most $\frac{1}{16}$ for every $b \in \omega^{\text{scr}} \cap A_{\text{reg}}^c$ (116) and

$$F^{\text{Scr}} \cap A_{\text{reg}}^c := \{ \omega^{\text{Scr}}_{\Lambda_+} = \omega^{\text{scr}}_{\Lambda_+} \} \cap \{ \inf_{b' \in \omega^{\text{B}}} |b - b'|_\infty \leq \frac{1}{16} \text{ for every } b \in \omega^{\text{scr}} \setminus \omega^{\text{scr}}_{\Lambda_+} \} \cap A_{\text{reg}}.$$ (117)

The rest of this subsection is devoted to establishing a lower bound on the conditional probability of $F^{\text{Scr}}$.

Lemma 6.3 (probability of screening under the coupling). There exists $c = c(P)$ with the following property. For all $\delta \in (0, \frac{1}{2})$ and $M > 2$ there exist $\varepsilon_0(\delta) > 0$ and $R_0 = R_0(\varepsilon) > 0$ such that

$$\mathbb{P}^{\text{Coup}}[F^{\text{Scr}} \mid \omega^P] \geq \exp(-cR) + \exp(-c\delta R) 1_{\{\omega^P \text{ is } (M, \varepsilon)\text{-regular} \}}$$

holds for all $\varepsilon < \varepsilon_0(\delta)$ and $R \geq R_0(\varepsilon)$.

Proof: Case 1: $\omega^P$ is not $(M, \varepsilon)$-regular. Note that in this case $\omega^{\text{scr}}$ is just $R$ straight lines at locations (71). Under $\mathbb{P}^1$, the probability that the starting point $b(0)$ of a fixed bridge $b \in \omega^B$ is at distance at most $\frac{1}{32}$ from the starting point $b'(0)$ of a fixed bridge $b' \in \omega^{\text{scr}}$ equals $1/(16R)$. Moreover, let $q_0 > 0$ be the probability that a Brownian bridge does not deviate by more than $\frac{1}{32}$ from its starting point. Then the probability for that fixed bridge $b$ to satisfy $|b - b'|_\infty \leq \frac{1}{16}$ is at least $q_0/(16R)$. Since under $\mathbb{P}^1$, the coupled process $\omega^B$ consists of $R$ i.i.d. bridges independent of $\omega^P$, we conclude that for all large $R$, via Stirling’s approximation, that

$$\mathbb{P}^{\text{Coup}}(F^{\text{Scr}} \mid \omega^P) \geq \frac{1}{2} R! \left( \frac{q_0}{16R} \right)^R \geq \frac{1}{2} \left( \frac{q_0}{16\varepsilon} \right)^R. \quad (117)$$

Case 2: $\omega^P$ is $(M, \varepsilon)$-regular. To begin with, we assert that $R^P \leq R$. Indeed, choosing a good boundary $x_0 \in [2\varepsilon R, 6\varepsilon R]$ using Lemma 5.4, we derive a bound on the charge imbalance (= net charge) in the domain $I_0 \times D = [x_0, R - x_0] \times D$ at time 0 as in the proof of Lemma 5.6. More precisely, by part (2) of Lemma 4.1, this charge imbalance is at most

$$\left( \sum_{x' \in \partial I_0} \frac{4}{\beta} \int_{D \times [0, \beta]} \left| \mathcal{E}_t(x', y) \right|^2 d(y, t) \right)^{1/2} + \sum_{x' \in \partial I_0} \# \{ b \in \tilde{\omega} : \inf_{t \leq \beta} |b_\lambda(t) - x'| \leq 1 \}. \quad (118)$$

(Here $\tilde{\omega} = \tilde{\omega}^P$ is defined similarly to (65).) The cardinalities on the right are bounded by $4\varepsilon^{-4}$ because $x_0$ and $R - x_0$ are $\varepsilon$-tame. Hence, we conclude as in the proof of Lemma 5.6 that (118) is at most $9\varepsilon^{-4}$. On the other hand, if $R^P \geq R$, then the charge imbalance at time 0 is at least $R^P - (R - 4\varepsilon R) \geq 4\varepsilon R$. Thus, $4\varepsilon R \leq 9\varepsilon^{-4}$, which is absurd once $R > 3/\varepsilon^5$.

Next, we work with $\mathbb{P}^D$ and consider the domains $\Lambda_+$ and $\Lambda_R \setminus \Lambda_+$ separately. By the construction of the coupling, when considering only the first $R^P$ bridges of the coupling, i.e., those produced by $K(z, dz')$ in (114), we have that $\omega^{\text{B}}_{\Lambda_+} = \omega^{\text{scr}}_{\Lambda_+}$ has a probability of at least

$$\frac{1}{2} (1 - \delta) R^P \geq \frac{1}{2} (1 - \delta)^R. \quad (119)$$
Now, leveraging similar charge-imbalance arguments as in (118), we conclude that
\[ \#\omega^{\text{scr}}_{\Lambda_t} \geq R - 20\varepsilon R - 9\varepsilon^{-4}. \]  
(120)

At most \( a = 21\varepsilon R \) bridges remain to be coupled. Arguing as in the irregular case, the probability that the remaining bridges from \( \omega^B \) are close to the remaining bridges of \( \omega^{\text{scr}} \) has probability at least
\[ \frac{1}{2} \left( \frac{a\delta_1q_0}{16eR} \right)^a \geq \frac{1}{2} \exp \left( 21\varepsilon R \log \left( \frac{21\varepsilon\delta_1q_0}{(16e)} \right) \right), \]
which is of order \( \exp(-o(\varepsilon) R) \).

\[ \square \]

6C. **Screened configurations have compatible electric fields.** Continue to assume that the coupling measure is for the fixed block \( K_1 \). We have the following extension of *Proposition 5.8.*

**Lemma 6.4** (screened fields under the coupling). Let \( M > 2, \varepsilon \in (0, 1/M^2) \) and \( R > (2/\varepsilon)^8 \). There exists an \( M_0 > 1 \) not depending on \( M, R, \varepsilon \), and moreover, on the event \( E^{\text{Scr}} \), there exists a field \( \{E^B_t\}_{t \leq \beta} \) that is screened and compatible with \( \omega^B \) in the sense of (61) and (62) such that

1. if \( \omega^P \) is not \((M, \varepsilon)\)-regular, then for every \( t \leq \beta \)
   \[ \int_{\Lambda_R} |E^B_t(z)|^2 \, dz \leq \varphi_0(\tilde{E}_t, \varepsilon, R), \]
   and
   \[ \frac{1}{2\beta} \int_{[0,\beta]} \varphi_0(\tilde{E}_t, \varepsilon, R) \, dt \leq M_0 R, \]
   (121)

2. if \( \omega^P \) is \((M, \varepsilon)\)-regular, then for every \( t \leq \beta \)
   \[ \int_{\Lambda_R} |E^B_t(z)|^2 \, dz \leq \varphi_0(\tilde{E}_t, \varepsilon, R), \]
   and
   \[ \frac{1}{2\beta} \int_{[0,\beta]} \varphi_0(\tilde{E}_t, \varepsilon, R) \, dt \leq \tilde{H}_{\Lambda_R}(\omega^P) + \varphi(\varepsilon, R)MR, \]
   (122)
   where \( \varphi_0(\tilde{E}_t, \varepsilon, R) \geq 0 \) is measurable in \( t \) and \( \varphi(\varepsilon, R) \geq 0 \) satisfies
   \[ \lim_{\varepsilon \to 0} \lim_{R \to \infty} \varphi(\varepsilon, R) = 0. \]
   (123)

**Proof:** We give a sketch of proof. For the existence of \( E^B_t \) one revisits the proof of *Proposition 5.8* and checks the robustness of the construction with respect to the perturbations of \( \omega^B \) from \( \omega^{\text{scr}} \). We note that in the irregular case, the configuration \( \omega^{\text{scr}} \) consists simply of evenly spaced straight lines (see (71)). For the case where \( \omega \) is \((M, \varepsilon)\)-regular, the coupled bridges \( \{b_i\} \) of \( \omega^B \) stay away from the boundaries of the respective cells \( I^{(i)} \) or \( [i-1, i] \times D \) and also away from the boundary \( x_0 \) since \( \eta + \frac{1}{16} < \frac{1}{4} \); see *Definition 5.5* and (86). The energy bounds in [54, Lemma 5.7] are robust with respect to the perturbations of the smeared \( \delta \)-point charges as long as they stay inside the integration domain. Hence, the arguments for the energy of \( E^B_t \) in the domain \([0, x_+]\) and for the energy of \( E^{(2)} \) in the domain \([x_-, x_0]\) carry over to the perturbed setting. Moreover, the electric field \( E^{(1)} \) is independent of the perturbation. Finally, when
considering the energy in the bulk $\Lambda_+$, we note that the bounds from Lemma 5.7 also do not depend on the precise location of the bridges.

6D. Most configurations are regular. We now show that under bridge processes of finite energy and entropy, it is highly likely to see $(M, \varepsilon)$-regular configurations.

**Lemma 6.5** (most configurations are regular). Let $P \in \mathcal{P}_s$ be such that $\text{ent}(P) \vee \mathbb{W}(P) < \infty$. There exists $c = c(P) > 0$ with the following property. If $M > 2$ and $\varepsilon \in (0, 1/M^2)$, then

$$
\lim \inf_{R \to \infty} P(\om_{\Lambda_R} \text{ is } (M, \varepsilon)\text{-regular}) \geq 1 - \frac{\mathbb{W}(P)}{M} - c\varepsilon^{1/3}.
$$

(124)

It follows in particular that if $\mathbb{W}(P)$ and $\text{ent}(P)$ are both finite, then

$$
\lim_{M \to \infty} \lim \inf_{\varepsilon \to 0} \lim_{R \to \infty} P(\om_{\Lambda_R} \text{ is } (M, \varepsilon)\text{-regular}) = 1.
$$

(125)

**Proof.** By the Markov inequality and Proposition 3.1,

$$
\lim \sup_{R \to \infty} M P\left(\frac{1}{R} \tilde{H}_{\Lambda_R}(\om) \geq M\right) \leq \lim \sup_{R \to \infty} \mathbb{E}_P[(R^{-1} \tilde{H}_{\beta,\Lambda_R}(\om)) \wedge M] = \mathbb{W}^M(P).
$$

For the excursion condition of $(M, \varepsilon)$-regularity, we apply Lemma 3.4 with $\zeta = \varepsilon^{-7/3}$ to get

$$
P\left(\sum_{b \in \om}(\psi(b)^{7/6} - \varepsilon^{-7/3})_+ \geq \frac{\varepsilon R}{2}\right) \leq \frac{2}{\varepsilon R} \mathbb{E}_P\left[\sum_{b \in \om}(\psi(b)^{7/6} - \varepsilon^{-7/3})_+\right] \leq c\varepsilon^{11/24},
$$

where $c$ may depend on $\text{ent}(P)$.

6E. Proof of Proposition 3.8. Fix $P \in \mathcal{P}_s$ with $\text{ent}(P) \vee \mathbb{W}(P) < \infty$ and a neighborhood $U_{F,P}$ of the form (15) where we may assume that the local test functions in $F = \{f_1, \ldots, f_n\}$ satisfy $\max_k \|f_k\|_{\infty} \leq 1$. Since each of the test functions is local, we also write $d$ to be the maximum diameter of dependence over all $f_k$ in $F$. That is, $f_k(\om) = f_k(\om_\Delta)$, where

$$
\Delta := [-d/2, d/2] \times D \subseteq S.
$$

Step 1: Concatenate screened blocks. The coupling definition from Section 6B and Lemma 6.4 extend from the block $\overline{K}_1 = [0, R] \times D$ to other blocks $\overline{K}_i$, with the obvious modifications. This provides us with measures $\mathbb{P}^\text{Coup}_i$ on $\text{Conf}(K_i) \times \text{Neut}(K_i)$, events

$$
F^\text{Scr}_i \subseteq \text{Conf}(K_i) \times \text{Neut}(K_i),
$$

and fields $E^B_{i,t} : \overline{K}_i \to \mathbb{R}^{k+1}$ satisfying, among other conditions,

$$
E^B_{i,t}((i-1)R, y) \cdot \tilde{e}_x = E^B_{i,t}(iR, y) \cdot \tilde{e}_x
$$

(126)

for all $t \leq \beta$, $i \leq m$ and $y \in D$.

Writing $\{(\om^P_i, \om^B_i)\}_{i \leq m}$ for a collection of independent configurations distributed according to the product measure $\bigotimes_{i \leq m} \mathbb{P}^\text{Coup}_i$, the superposition $\bigcup_{i \leq m} \omega^B_i$ is distributed according to a binomial bridge process on $\Lambda_N$ conditioned on the event

$$
A_{N,R} := \bigcap_{i \leq m}\{\#\omega^B_{K_i} = R\}
$$
of having exactly \( R \) bridges in each block \( K_i \). Hence, we define the global coupling \( \mathbb{P}^{\text{Coup}} \) for \((\omega^P, \omega^B)\) on \( \Lambda_N \) as a mixture that has weight \( \mathbb{P}(A_{N,R}) \) on the product measure \( \bigotimes_{i \leq m} \mathbb{P}^{\text{Coup}} \) and weight \( 1 - \mathbb{P}(A_{N,R}) \) on an independent coupling between a Poisson bridge process on \( \Lambda_N \) and a binomial bridge process on \( \Lambda_N \) conditioned to have at least one block not containing exactly \( R \) bridges. We then set

\[
F_{\Lambda_N}^{\text{Scr}} := \bigcap_{i \leq m} \{(\omega^P_{K_i}, \omega^B_{K_i}) \in F_i^{\text{Scr}} \}.
\]

Since, on the event \( F_{\Lambda_N}^{\text{Scr}} \), the electric fields are screened for every \( i \leq m \) and \( t \leq \beta \), their superposition defines a globally screened electric field

\[
E^B_t(z) = \sum_{i \leq m} \mathbb{1}_{K_i}(z) E^B_{i,t}(z)
\]

without creating additional divergence. Hence, on \( F_{\Lambda_N}^{\text{Scr}} \),

\[
\text{div } E^B_t = -\sum_{b \in \omega^B} \delta^n_{b(t)} + \mathbb{1}_{\Lambda_N}.
\]

**Step 2: Compare energies.** Fix \( \delta < 1 \), which we may assume to satisfy \( 9(M_0 \mathbb{W}(P) + 3 + d)\delta < 1 \), and set \( M = 4/\delta^2 \).

Also let \( \varepsilon < 1/M^2 \) be such that \( \lim_{R \to \infty} \varphi(\varepsilon, R) < \delta^2/M \), such that Lemma 6.3 applies, and such that the error \( c \varepsilon^{1/3} \) from Lemma 6.5 is smaller than \( \delta^2 \).

Finally, choose \( R > (2/\varepsilon)^{\delta^2} \sqrt{18d/\delta} \) large enough that \( \mathbb{W}_R^M(P) \leq \mathbb{W}(P) + \delta^2 \), such that Lemmas 6.3 and 6.4 hold, and such that the error in Lemma 6.5 is smaller than \( \delta^2 \).

By [41, Lemma 3.10], replacing a screened field compatible with a charge-neutral configuration with the gradient of the potential associated to that configuration (which may possibly have different boundary conditions) can only decrease the energy. In other words, for every \( t \leq \beta \),

\[
\int_{\Lambda_N} |\nabla V_t(z, \omega^B, \Lambda_N)|^2 \, dz \leq \int_{\Lambda_N} |E^B_t(z)|^2 \, dz.
\]

Let

\[
\mathcal{I}(\omega) := \{i \leq m : \omega_{K_i} \text{ is } (M, \varepsilon)\text{-irregular}\}
\]

be the set of indices corresponding to blocks with \((M, \varepsilon)\)-irregular configurations. By (129) and Lemma 6.4, on the event \( F_{\Lambda_N}^{\text{Scr}} \),

\[
\frac{1}{N} H_{\Lambda_N}(\omega^B) \leq \frac{\# \mathcal{I}(\omega^P) M_0}{m} + \frac{1}{m} \sum_{i=1}^m \min\left( \frac{1}{R} \tilde{H}_{K_i}(\omega^P), M \right) + \varphi(\varepsilon, R) M
\]

\[
= \frac{\# \mathcal{I}(\omega^P) M_0}{m} + \mathbb{W}_R^M(\text{Avg}_R(\omega^P)) + \varphi(\varepsilon, R) M.
\]

Now, let

\[
U_1(P) := \{Q \in \mathcal{P} : |\mathbb{W}_R^M(Q) - \mathbb{W}_R^M(P)| < \delta^2 \},
\]

\[
U_2(P) := \{Q \in \mathcal{P} : |Q(\omega_{\Lambda_R} \text{ is } (M, \varepsilon)\text{-irregular}) - P(\omega_{\Lambda_R} \text{ is } (M, \varepsilon)\text{-irregular})| < \delta^2 \}.
\]
Then, if \( \text{Avg}_R(\omega^P) \in U_2 \), Lemma 6.5 and the choices of \( M, \varepsilon \) and \( R \) give that
\[
\frac{\# I(\omega^P)}{m} \leq P(\omega_{\Lambda_R} \text{ is } (M, \varepsilon)\text{-irregular}) + \delta^2 \leq (\mathcal{W}(P) + 3)\delta^2. \tag{133}
\]
Moreover, \( \mathcal{W}_R^M(P) \leq \mathcal{W}(P) + \delta^2 \) and \( \text{Avg}_R(\omega^P) \in U_1 \) imply that
\[
\mathcal{W}_R^M(\text{Avg}_R(\omega^P)) \leq \mathcal{W}_R^M(P) + \delta^2 \leq \mathcal{W}(P) + 2\delta^2. \tag{134}
\]
The assumption on \( \delta \) from the beginning of this case yields that on the event \( F_{\Lambda_N}^{\text{Scr}} \cap \{ \text{Avg}_R(\omega^P) \in U_1 \cap U_2 \}, \)
\[
\frac{1}{N} \mathcal{H}_{\Lambda_N}^X(\omega^B) \leq \mathcal{W}(P) + (M_0(\mathcal{W}(P) + 3) + 3)\delta^2 \leq \mathcal{W}(P) + \delta. \tag{135}
\]
Step 3: Compare block and empirical averages. We claim that, up to a null set of \( \mathbb{P}^{\text{Coup}}, \)
\[
F_{\Lambda_N}^{\text{Scr}} \cap \{ \text{Avg}_R(\omega^P) \in U_3 \} \subseteq \{ \text{Emp}_N(\omega^B) \in U_{F,P} \}, \tag{136}
\]
where
\[
U_3 := \{ Q \in \mathcal{P} : \max_{k \leq r} |E_Q[\bar{f}_k] - E_P[\bar{f}_k]| < \frac{1}{3}\delta \} \tag{137}
\]
with
\[
\bar{f}_k(\omega) := \frac{1}{R} \sum_{r \leq R - 1} f_k(\theta_r(\omega)). \tag{138}
\]
Since \( P \) is shift-invariant, we have \( E_P[\bar{f}_k] = E_P[f_k] \), so that
\[
|E_{\text{Emp}_N(\omega^B)}[f_k] - E_P[f_k]| \leq |E_{\text{Emp}_N(\omega^B)}[f_k] - E_{\text{Avg}_R(\omega^B)}[\bar{f}_k]| + |E_{\text{Avg}_R(\omega^B)}[\bar{f}_k] - E_{\text{Avg}_R(\omega^P)}[\bar{f}_k]| + |E_{\text{Avg}_R(\omega^P)}[\bar{f}_k] - E_P[\bar{f}_k]]. \tag{139}
\]
It suffices to show that each of the three summands is bounded by \( \frac{1}{3}\delta \).

If \( \text{Avg}_R(\omega^P) \in U_3 \), then the third term on the right-hand side is smaller than \( \frac{1}{3}\delta \).

For the first term, observe
\[
E_{\text{Emp}_N(\omega^B)}[f_k] = \frac{1}{m} \sum_{j \leq m} \bar{f}_k(\theta_{x_j}(\omega^B)), \tag{140}
\]
where \( x_j = (j - 1)R \) is the left endpoint of \( K_j \), and
\[
|E_{\text{Emp}_N(\omega^B)}[f_k] - E_{\text{Avg}_R(\omega^B)}[\bar{f}_k]| \leq \frac{1}{m} \sum_{j \leq m} |\bar{f}_k(\theta_{x_j}(\omega^B)) - \bar{f}_k(\theta_{x_j}(\omega^B_{K_j}))|. \tag{141}
\]
For \( j = 1 \) and \( \omega \in \text{Conf} \), we bound
\[
|\bar{f}_k(\omega) - \bar{f}_k(\omega_{K_1})| \leq \frac{1}{R} \sum_{r \leq R - 1} |f_k(\theta_r(\omega)) - f_k(\theta_r(\omega_{K_1}))|
\leq \frac{1}{R} \# \{ r \leq R - 1 : \pi_\Delta(\theta_r(\omega)) \neq \pi_\Delta(\theta_r(\omega_{K_1})) \}
\leq \frac{1}{R} \# \{ r \leq R - 1 : \theta_{r} \Delta \not\in K_1 \}, \tag{142}
\]
which is at most \( 2d/R \leq \frac{1}{3}\delta \).
Finally, for the second term on the right-hand side of (139), proceeding as in (142) yields
\[
\left| \mathbb{E}_{\text{Avg}_R(\omega^B)}[\tilde{f}_k] - \mathbb{E}_{\text{Avg}_R(\omega^P)}[\tilde{f}_k] \right| \\
\leq \frac{1}{m} \sum_{j \in I(\omega^P)} |\tilde{f}_k(\theta_{x_j}(\omega_{K_j}^B)) - \tilde{f}_k(\theta_{x_j}(\omega_{K_j}^P))| + \frac{1}{m} \sum_{j \notin I(\omega^P)} |\tilde{f}_k(\theta_{x_j}(\omega_{K_j}^B)) - \tilde{f}_k(\theta_{x_j}(\omega_{K_j}^P))| \\
\leq 2\#I(\omega^P) + 2\frac{d}{R},
\]
where we have used that on the event $F_{\Lambda_N}^{\text{Scr}} \cap \{\omega^P \text{ is } (M, \varepsilon)\text{-regular}\}$, the configurations $\omega_{K_j}^B$ and $\omega_{K_j}^P$ agree on $[6\varepsilon R, (1 - 6\varepsilon)R] \times D$. The right-hand side above is at most $\frac{1}{2} \delta$ by (133) and the choice of $R$.

Step 4: Conclusion. First, by Lemma 6.1, (135) and (136),
\[
\mathbb{P}(\text{Emp}_N(\omega) \in U_{F,P}, N^{-1}H_{\Lambda_N}(\omega) \leq \mathbb{W}(P) + \delta) = \mathbb{P}(\text{Coup}(\text{Emp}_N(\omega^B) \in U_{F,P}, N^{-1}H_{\Lambda_N}(\omega^B) \leq \mathbb{W}(P) + \delta) \\
\geq \mathbb{P}(\text{Coup}(F_{\Lambda_N}^{\text{Scr}} \cap \{\text{Avg}_R(\omega^P) \in U\})),
\]
where $U := U_1 \cap U_2 \cap U_3$ with $U_1$, $U_2$, and $U_3$ given in (132) and (137). Now, by Stirling’s approximation, the probability
\[
\mathbb{P}(A_{N,R}) = \mathbb{P}(\text{Coup}(\#\omega_{K_i}^B = R \text{ for every } i \leq m) = \frac{N!}{(R!)^m}
\]
of the binomial point process consisting of $R$ bridges in each block $K_i$ is bounded below by
\[
\exp \left( -\frac{cN \log R}{R} \right)
\]
for some $c > 0$.

Hence, by Lemma 6.3 and (133),
\[
\mathbb{P}(\text{Coup}(F_{\Lambda_N}^{\text{Scr}} | \omega^P) \geq \mathbb{P}(A_{N,R}) \prod_{i \leq m} \mathbb{P}(\text{Coup}(\omega_i^P, \omega_i^B) \in F_i^{\text{Scr}} | \omega^P) \geq \mathbb{P}(A_{N,R})e^{-c\delta N - cR\#I} \geq e^{-c_1\delta N}
\]
for some $c_1 > 0$. Therefore,
\[
\mathbb{E}(\text{Coup}[\text{Coup}(F_{\Lambda_N}^{\text{Scr}} | \omega^P) \mathbb{1}\{\text{Avg}_R(\omega^P) \in U\}]) \geq \mathbb{E}(\text{Coup}[\exp(-c_1\delta N) \mathbb{1}\{\text{Avg}_R(\omega^P) \in U\}])
\]
\[
= \exp(-c_1\delta N)\mathbb{P}(\text{Coup}(\text{Avg}_R(\omega^P) \in U)).
\]

Finally, by Sanov’s theorem [14, Theorem 6.2.10],
\[
\liminf_{N \to \infty} \frac{1}{m} \log \mathbb{P}(\text{Avg}_R(\omega^P) \in U) \geq -\inf_{Q \in U} \text{ent}(Q | \text{Pois}_\Lambda) \geq -\text{ent}(P_{A,R} | \text{Pois}_\Lambda).
\]

We conclude with ent$(P) = \sup_R \text{ent}_R(P_{A,R})$; see [30, Remark 2.5].

**Appendix: Energy in terms of electric field**

Here, we prove Lemma 2.2 using a standard integration by parts.

**Proof of Lemma 2.2.** In order to simplify notation, we fix $t$ and write
\[
V_N(z) = V(z, t, \omega_{\Lambda_N}, \Lambda_N).
\]
Also for such fixed \( t \), set \( \{ z_i \} = \{ b_i(t) : b_i \in \omega_{\Lambda_N}, i \leq N \} \). Partial integration gives

\[
\int_S |\nabla V_N(z)|^2 \, dz = \int_S V_N(z)(-\Delta V_N(z)) \, dz + \lim_{R \to \infty} \int_{\partial \Lambda_R} V_N(z) \nabla V_N(z) \cdot \vec{n} \, dz,
\]

where \( \vec{n} \) denotes the outer normal vector to the interface \( \partial \Lambda_R \). For each \( \omega_{\Lambda_N} \), there is an \( R \) such that \( \{ z_1, \ldots, z_N \} \subseteq \Lambda_R \), thus by charge neutrality, the second term on the right-hand side vanishes. On the other hand,

\[
\int_S V_N(z)(-\Delta V_N(z)) \, dz = \sum_{i \leq N} \frac{1}{|\partial B_\eta(o)|} \int_S V_N(z) \delta_{z_i}^\eta (dz) - \int_{\Lambda_N} V_N(z) \, dz.
\]

Now, insert the integral representation of the potential \( V_N \) via the Green’s function, i.e.,

\[
V_N(z) = \int_S g(z - z') \left( \sum_{i \leq N} \delta_{z_i}^\eta - \mathbb{1}_{\Lambda_N} \right)(dz').
\]

In particular, the sum on the right-hand side of (144) becomes

\[
\sum_{i, j \leq N} g_{\eta, \eta}(z_i - z_j) - \sum_{i \leq N} \int_{\partial B_\eta(z_i)} \int_{\Lambda_N} g(z - z') \, dz' \delta_{z_i}^\eta (dz),
\]

whereas the subtracted integral on the right-hand side of (144) is

\[
\sum_{i \leq N} \int_S \int_{\Lambda_N} g(z - z') \, dz \delta_{z_i}^\eta (dz') - \int_{\Lambda_N} \int_{\Lambda_N} g(z - z') \, dz \, dz'.
\]

Hence, splitting of the diagonal contributions in the double sum over \( i \) and \( j \) and rearranging terms concludes the proof. \( \square \)

**List of symbols**

| symbol | object | where defined |
|--------|--------|--------------|
| \( \Lambda_N \) | \([0, N] \times D \) where \( D = [0, 1]^k \) | (1) |
| \( \mathcal{H}_N^\eta \) | classical potential energy of a configuration with smeared pts. | (4) |
| \( \pi_\Lambda(\omega) \) | projection of configuration \( \omega \) onto \( \Lambda \) | (6) |
| \( P_\Lambda \) | image of \( P \) under \( \pi_\Lambda \) | (7) |
| \( \text{Conf}(\Lambda) \) | space of bridge configurations \( \omega \) in \( \Lambda \) | (8) |
| \( \mathbb{P}_N(d\omega), \mathbb{E}_N(\cdot) \) | bridge configuration measure on \( \text{Neut}(\Lambda_N) \) | (10) |
| \( \mu_{N, \beta}(d\omega) \) | Gibbs measure for bridge configurations | (11) |
| \( \text{Emp}_N(\omega) \) | empir. field constructed from \( \omega \); a prob. meas. on config. space | (13) |
| \( \mathcal{P} \) | space of probability measures on bridge configurations | below (13) |
| \( \mathcal{P}_s \) | space of stationary probability measures on bridge configurations | Section 2F |
| \( \mathcal{U}_{\text{meas}}(P) \) | family of neighborhoods of \( P \) | (14) |
| \( U_{F,P} \) | cylinder set | (15) |
| \( \text{ent}(P) \) | specific relative entropy of \( P \) | (16) |
\[ H_\Lambda(\omega) \quad \text{energy of } \omega \in \text{Conf}(\Lambda) \text{ in terms of electric field (18)} \]
\[ \text{Neut}(\Lambda) \quad \text{space of charge-neutral bridge configurations } \omega \text{ in } \Lambda (19) \]
\[ \mathbb{W}(P) \quad \text{expected specific energy (22)} \]
\[ \mathbb{W}^M(P) \quad \text{expected specific energy truncated at level } M (23) \]
\[ \mathbb{W}^M_R(P) \quad \text{truncated expected specific energy in a finite domain (45)} \]
\[ \tilde{H}_{\beta,K}(\omega) \quad \text{minimal energy of compatible configurations restricted to } K (24) \]
\[ b_x \quad x\text{-coordinate of a bridge } b \text{ above (28)} \]
\[ \psi(b) \quad x\text{-range functional of a bridge } b (28) \]
\[ \mu_t(\omega, A) \quad \text{net charge of } \omega \text{ in the region } A \text{ in the time slice } t (54) \]
\[ \omega' \quad \text{regularized configuration (72)} \]
\[ d_0 \quad \text{net negative charge to the left of } x_0 (75) \]
\[ \text{Avg}_R(\omega) \quad \text{block average of a configuration } \omega (109) \]

**Acknowledgements.** The authors thank the referees for providing us with high-quality feedback. Their constructive feedback helped us to improve the presentation in many places. The authors also thank D. Chafaï, A. van Enter and D. García-Zelada for useful remarks on earlier versions of the manuscript.

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Received 11 Mar 2021. Revised 29 Sep 2021. Accepted 8 Nov 2021.

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