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John–Nirenberg inequalities for parabolic BMO

Juha Kinnunen¹ · Kim Myyryläinen¹ · Dachun Yang²

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Abstract
We discuss a parabolic version of the space of functions of bounded mean oscillation related to a doubly nonlinear parabolic partial differential equation. Parabolic John–Nirenberg inequalities, which give exponential decay estimates for the oscillation of a function, are shown in the natural geometry of the partial differential equation. Chaining arguments are applied to change the time lag in the parabolic John–Nirenberg inequality. We also show that the quasihyperbolic boundary condition is a necessary and sufficient condition for a global parabolic John–Nirenberg inequality. Moreover, we consider John–Nirenberg inequalities with medians instead of integral averages and show that this approach gives the same class of functions as the original definition.

Mathematics Subject Classification 42B35 · 42B37

1 Introduction

Functions of bounded mean oscillation (BMO) are essential in harmonic analysis and partial differential equations. A particularly useful result is the John–Nirenberg lemma which gives an exponential decay estimate for the mean oscillation of a function in BMO. Functions of bounded mean oscillation and the John–Nirenberg lemma were first discussed in [11] and the corresponding time-dependent theory was initiated by Moser in [18, 19]. The proof of the parabolic John–Nirenberg lemma requires

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genuinely new ideas compared to the time independent case. The main challenge is that the definition of parabolic BMO consists of two conditions on the mean oscillation of a function, one in the past and the other one in the future with a time lag between the estimates, see Definition 2.4 below. Moreover, Euclidean cubes are replaced by rectangles that respect the natural geometry of the related parabolic partial differential equation. Fabes and Garofalo [6] gave a simpler proof for the parabolic John–Nirenberg lemma and the general approach of Aimar [1] applies in spaces of homogeneous type. Martín-Reyes and de la Torre [17] studied one-sided BMO in the one-dimensional case. Berkovits [2, 3] discussed this approach in the higher dimensional case, but the geometry is not related to nonlinear parabolic partial differential equations. In contrast with the extensive literature on the classical BMO, there are few references to the corresponding parabolic theory.

We discuss a parabolic BMO space tailored to a doubly nonlinear equation

$$\frac{\partial}{\partial t}(|u|^{p-2}u) - \text{div}(|Du|^{p-2}Du) = 0, \quad 1 < p < \infty.$$  \hspace{1cm} (1.1)

For $p = 2$ we have the standard heat equation. Gradient and divergence are taken with respect to the spatial variable only. Observe that a solution to (1.1) can be scaled, but constants cannot be added to a solution. The equation is nonlinear in the sense that the sum of two solutions is not a solution in general. Our discussion applies to more general equations of the type

$$\frac{\partial}{\partial t}(|u|^{p-2}u) - \text{div} A(x, t, u, Du) = 0,$$

where $A$ is a Caratheodory function that satisfies the structural conditions

$$A(x, t, u, Du) \cdot Du \geq C_0|Du|^p \quad \text{and} \quad |A(x, t, u, Du)| \leq C_1|Du|^{p-1}$$

for some positive constants $C_0$ and $C_1$. In the natural geometry of (1.1), we consider space-time rectangles where the time variable scales to the power $p$. This is in accordance with the following scaling property: if $u(x, t)$ is a solution, so does $u(\alpha x, \alpha^p t)$ with $\alpha > 0$. Different values of $p$ lead to different $p$-geometries and different parabolic BMO spaces. For recent regularity results for the doubly nonlinear equation, we refer to Bögelein, Duzaar, Kinnunen and Scheven [5], Bögelein, Duzaar and Liao [4], Kuusi, Siljander and Urbano [15] and Kuusi, Laleoglu, Siljander and Urbano [16].

The following scale and location invariant Harnack inequality for positive solutions to (1.1) has been obtained by Moser [18] for $p = 2$ and by Trudinger [26] for $1 < p < \infty$. See also Gianazza and Vespri [8] and Kinnunen and Kuusi [12]. Assume that $u > 0$ is a weak solution to (1.1) in $\Omega_T = \Omega \times (0, T)$ and let $0 < \gamma < 1$ be a time lag. There exist an exponent $\delta = \delta(n, p, \gamma) > 0$ and constants $c_i = c_i(n, p, \gamma) > 0$, $i = 1, 2, 3$, such that
\[
\sup_{R^{-}(\gamma)} u \leq c_1 \left( \frac{\int_{2R^-} u^\delta}{\int_{2R^-} u} \right)^{\frac{1}{\delta}} \leq c_2 \left( \frac{\int_{2R^+} u^{-\delta}}{\int_{2R^+} u^{-\delta}} \right)^{-\frac{1}{\delta}} \leq c_3 \inf_{R^+(\gamma)} u \quad (1.2)
\]

for every parabolic rectangle with \(2R \subset \Omega_T\). See Definition 2.1 for the parabolic rectangles. Harnack’s inequality gives a scale and location invariant pointwise bound for a positive solution at a given time in terms of its values at later times. This indicates that the parabolic rectangles in the \(p\)-geometry respect the natural geometry of the doubly nonlinear equation. The time lag \(\gamma > 0\) is an unavoidable feature of the theory rather than a mere technicality. The fact that the result is not true with \(\gamma = 0\) can be seen from the heat kernel already when \(p = 2\), since Harnack’s inequality does not hold on a given time slice. The first and the last inequalities in (1.2) are based on a successive application of Sobolev’s inequality and energy estimates. The remaining inequality follows from the fact that a logarithm of a positive weak solution belongs to the parabolic BMO with a uniform estimate and a parabolic John–Nirenberg lemma as in \([18, 26]\). The proof in \([12]\) applies an abstract lemma of Bombieri instead of the parabolic John–Nirenberg lemma. See also Kinnunen and Saari \([13]\) and Saari \([22]\). The parabolic John–Nirenberg lemma with a general parameter \(p\) has applications in the theory of parabolic Muckenhoupt weights in Kinnunen and Saari \([13, 14]\).

We discuss several versions of the parabolic John–Nirenberg inequality in the \(p\)-geometry with \(1 < p < \infty\). Theorem 3.1 gives an exponential bound for the mean oscillation in terms of integral averages. To our knowledge this result is new already for \(p = 2\) and generalizes the corresponding result for the standard BMO in \([11]\) to the parabolic case. A more common version of the parabolic John–Nirenberg inequality is stated in Corollary 3.2. This generalizes the results of Moser \([18, 19]\) and Fabes and Garofalo \([6]\) to the \(p\)-geometry with \(1 < p < \infty\). A general approach of Aimar \([1]\) applies in metric measure spaces and also covers the \(p\)-geometry with \(1 < p < \infty\) by considering the parabolic metric

\[
d((x, t), (y, s)) = \max\{\|x - y\|_\infty, |t - s|^{\frac{1}{p}}\}.
\]

We prefer giving a direct and transparent proof in the \(p\)-geometry that also allows further investigation of the theory of parabolic BMO. The argument is based on a Calderón–Zygmund decomposition in the \(p\)-geometry. The John–Nirenberg inequality implies that a parabolic BMO function is locally integrable to any positive power with reverse Hölder-type bounds, see Corollary 3.3. Corollary 5.4 gives a stronger result which states that a parabolic BMO function is locally exponentially integrable with uniform estimates on parabolic rectangles. A chaining argument in the proof of Theorem 4.1 shows that the size of the time lag can be changed in parabolic BMO. Parabolic chaining arguments have been previously studied by Saari \([22, 23]\) and our approach complements these techniques. We also discuss the John–Nirenberg inequality up to the spatial boundary of a space-time cylinder by applying results of Saari \([22]\) and Smith and Stegenga \([24]\). In particular, we show that the quasihyperbolic boundary condition is a necessary and sufficient condition for a global parabolic John–Nirenberg inequality. This extends some of the results in \([24]\) to the parabolic setting.
John [10] observed that it is possible to relax the a priori local integrability assumption in the definition of BMO and to create a theory that applies to measurable functions. We extend this theory to the parabolic context. This approach is based on the notion of median, for example, see Jawerth and Torchinsky [9], Poelhuis and Torchinsky [21] and Strömberg [25]. It is remarkable that the John–Nirenberg inequality can be proved starting from this condition and as a consequence, these functions are locally integrable to any positive power, see Theorem 6.6. Corollary 6.8 shows that the parabolic BMO with medians coincides with the original definition of parabolic BMO.

2 Definition and properties of parabolic BMO

The underlying space throughout is $\mathbb{R}^{n+1} = \{(x, t) : x = (x_1, \ldots, x_n) \in \mathbb{R}^n, t \in \mathbb{R}\}$. Unless otherwise stated, constants are positive and the dependencies on parameters are indicated in the brackets. The Lebesgue measure of a measurable subset $A$ of $\mathbb{R}^{n+1}$ is denoted by $|A|$. A cube $Q$ is a bounded interval in $\mathbb{R}^n$, with sides parallel to the coordinate axes and equally long, that is, $Q = Q(x, L) = \{y \in \mathbb{R}^n : |y_i - x_i| \leq L, i = 1, \ldots, n\}$ with $x \in \mathbb{R}^n$ and $L > 0$. The point $x$ is the center of the cube and $L$ is the side length of the cube. Instead of Euclidean cubes, we work with the following collection of parabolic rectangles in $\mathbb{R}^{n+1}$.

\textbf{Definition 2.1} Let $1 < p < \infty$, $x \in \mathbb{R}^n$, $L > 0$ and $t \in \mathbb{R}$. A parabolic rectangle centered at $(x, t)$ with side length $L$ is

$$R = R(x, t, L) = Q(x, L) \times (t - L^p, t + L^p)$$

and its upper and lower parts are

$$R^+(\gamma) = Q(x, L) \times (t + \gamma L^p, t + L^p) \quad \text{and} \quad R^- (\gamma) = Q(x, L) \times (t - L^p, t - \gamma L^p),$$

where $-1 < \gamma < 1$ is called the time lag.

Note that $R^- (\gamma)$ is the reflection of $R^+(\gamma)$ with respect to the time slice $\mathbb{R}^n \times \{t\}$. The spatial side length of a parabolic rectangle $R$ is denoted by $l_s(R) = L$ and the time length by $l_t(R) = 2L^p$. For short, we write $R^\pm$ for $R^\pm (0)$. The top of a rectangle $R = R(x, t, L)$ is $Q(x, L) \times \{t + L^p\}$ and the bottom is $Q(x, L) \times \{t - L^p\}$. The $\lambda$-dilate of $R$ with $\lambda > 0$ is denoted by $\lambda R = R(x, t, \lambda L)$. We observe that the Lebesgue differentiation theorem holds on the collection of parabolic rectangles.

\textbf{Definition 2.2} A sequence $(A_i)_{i \in \mathbb{N}}$ of measurable sets $A_i \subset \mathbb{R}^{n+1}$, $i \in \mathbb{N}$, converges regularly to a point $(x, t) \in \mathbb{R}^{n+1}$, if there exist a constant $c > 0$ and a sequence $(R_i)_{i \in \mathbb{N}}$ of parabolic rectangles $R_i$, $i \in \mathbb{N}$, such that $|R_i| \to 0$ as $i \to \infty$, $A_i \subset R_i$, $(x, t) \in R_i$ and $|A_i| \leq |R_i| \leq c|A_i|$ for every $i \in \mathbb{N}$.

The integral average of $f \in L^1(A)$ in measurable set $A \subset \mathbb{R}^{n+1}$, with $0 < |A| < \infty$, is denoted by

$$f_A = \int_A f \, dx \, dt = \frac{1}{|A|} \int_A f(x, t) \, dx \, dt.$$
The Lebesgue differentiation theorem below can be proven in a similar way as in the classical case using a covering argument for parabolic rectangles.

**Lemma 2.3** Let $f \in L^1_{\text{loc}}(\mathbb{R}^{n+1})$. Then

\[
\lim_{i \to \infty} \int_{A_i} |f(y, s) - f(x, t)| \, dy \, ds = 0
\]

for almost every $(x, t) \in \mathbb{R}^{n+1}$, whenever $(A_i)_{i \in \mathbb{N}}$ is a sequence of measurable sets converging regularly to $(x, t)$.

The positive and the negative parts of a function $f$ are denoted by

\[
f_+ = \max\{f, 0\} \quad \text{and} \quad f_- = -\min\{f, 0\}.
\]

Let $\Omega \subset \mathbb{R}^n$ be an open set and $T > 0$. A space-time cylinder is denoted by $\Omega_T = \Omega \times (0, T)$. It is possible to consider space-time cylinders $\Omega \times (t_1, t_2)$ with $t_1 < t_2$, but we focus on $\Omega_T$.

This section discusses basic properties of parabolic BMO. We begin with the definition. The differentials $dx \, dt$ in integrals are omitted in the sequel.

**Definition 2.4** Let $\Omega \subset \mathbb{R}^n$ be a domain, $T > 0$, $-1 < \gamma < 1$, $-\gamma \leq \delta < 1$ and $0 < q < \infty$. A function $f \in L^q_{\text{loc}}(\Omega_T)$ belongs to $\text{PBMO}^{+}_{\gamma,\delta,q}(\Omega_T)$ if

\[
\|f\|_{\text{PBMO}^{+}_{\gamma,\delta,q}(\Omega_T)} = \sup_{R \subset \Omega_T} \inf_{c \in \mathbb{R}} \left( \int_{R^+(\gamma)} (f - c)^q_+ + \int_{R^{-}(\delta)} (f - c)^q_- \right)^{\frac{1}{q}} < \infty.
\]

If the condition above holds with the time axis reversed, then $f \in \text{PBMO}^{-}_{\gamma,\delta,q}(\Omega_T)$.

For $0 \leq \delta = \gamma < 1$, we abbreviate $\text{PBMO}^{+}_{\gamma,q}(\Omega_T) = \text{PBMO}^{+}_{\gamma,\delta,q}(\Omega_T)$. In addition, we shall write $\text{PBMO}^{+}$ and $\|f\|$ whenever parameters are clear from the context or are not of importance. Observe that $f \in L^q_{\text{loc}}(\Omega_T)$ belongs to $\text{PBMO}^{+}_{\gamma,\delta,q}(\Omega_T)$ if and only if for every parabolic rectangle $R$ there exists a constant $c \in \mathbb{R}$, that may depend on $R$, with

\[
\int_{R^+(\gamma)} (f - c)^q_+ \leq M \quad \text{and} \quad \int_{R^{-}(\delta)} (f - c)^q_- \leq M,
\]

where $M \in \mathbb{R}$ is a constant that is independent of $R$.

**Remark 2.5** Assume that $u > 0$ is a weak solution to the doubly nonlinear equation in $\Omega_T$ and let $0 < \gamma < 1$. By Kinnunen and Saari [13] and Saari [22], we have

\[
f = -\log u \in \text{PBMO}^{+}_{\gamma,q}(\Omega_T),
\]
with \( q = (p-1)/2 \) and

\[
\| f \| = \| f \|_{\text{PBMO}^+_{\gamma, q}(\Omega_T)} \leq c(n, p, \gamma) < \infty.
\]

Observe that \( 0 < q < 1 \) for \( p < 3 \). By Hölder’s inequality, we may take \( q = 1 \) for \( p \geq 3 \). Observe that the bound for the PBMO norm is independent of the solution.

**Example 2.6** Let \( 1 < p < \infty \) and \( 0 < \gamma < 1 \). The function

\[
u(x, t) = t^{\frac{-n}{p(p-1)}} e^{-\frac{p-1}{p} \left( \frac{|x|^p}{pt} \right)^{\frac{1}{p-1}}}, \quad x \in \mathbb{R}^n, \ t > 0,
\]
is a solution of the doubly nonlinear equation in the upper half space \( \mathbb{R}^n \times (0, \infty) \). By Remark 2.5, we conclude that the function

\[
f(x, t) = -\log \nu(x, t) = -\log t^{\frac{-n}{p(p-1)}} e^{-\frac{p-1}{p} \left( \frac{|x|^p}{pt} \right)^{\frac{1}{p-1}}}, \quad x \in \mathbb{R}^n, \ t > 0,
\]

belongs to \( \text{PBMO}^+_{\gamma, q}(\mathbb{R}^n \times (0, \infty)) \) with \( q = (p-1)/2 \). Corollary 4.2 below implies that \( f \) belongs to \( \text{PBMO}^+_{\gamma, q}(\mathbb{R}^n \times (0, \infty)) \) for every \( 0 < q < \infty \).

The next lemma shows that for every parabolic rectangle \( R \), there exists a constant \( c_R \), depending on \( R \), for which the infimum in the definition above is attained. In the sequel, this minimal constant is denoted by \( c_R \).

**Lemma 2.7** Let \( \Omega_T \subset \mathbb{R}^{n+1} \) be a space-time cylinder, \( -1 < \gamma < 1 \), \( -\gamma \leq \delta < 1 \) and \( 0 < q < \infty \). Assume that \( f \in \text{PBMO}^+_{\gamma, \delta, q}(\Omega_T) \). Then for every parabolic rectangle \( R \subset \Omega_T \), there exists a constant \( c_R \in \mathbb{R} \), that may depend on \( R \), such that

\[
\int_{R^+(\gamma)} (f - c_R)^q_+ + \int_{R^-(\delta)} (f - c_R)^q_- = \inf_{c \in \mathbb{R}} \left( \int_{R^+(\gamma)} (f - c)^q_+ + \int_{R^-(\delta)} (f - c)^q_- \right).
\]

In particular, it holds that

\[
\sup_{R \subset \Omega_T} \left( \int_{R^+(\gamma)} (f - c_R)^q_+ + \int_{R^-(\delta)} (f - c_R)^q_- \right)^{\frac{1}{q}} = \| f \|_{\text{PBMO}^+_{\gamma, \delta, q}(\Omega_T)}.
\]

**Proof** Let \( R \subset \Omega_T \) be a parabolic rectangle. Consider a sequence \((c_i)_{i \in \mathbb{N}}\) of real numbers such that

\[
\int_{R^+(\gamma)} (f - c_i)^q_+ + \int_{R^-(\delta)} (f - c_i)^q_- < \inf_{c \in \mathbb{R}} \left( \int_{R^+(\gamma)} (f - c)^q_+ + \int_{R^-(\delta)} (f - c)^q_- \right) + \frac{1}{2^i}
\]
for every \( i \in \mathbb{N} \). Note that
\[
\int_{R^+ (\gamma)} (f - c_i)_+^q + \int_{R^- (\delta)} (f - c_i)_-^q \leq \int_{R^+ (\gamma)} f_+^q + \int_{R^- (\delta)} f_-^q + 1 < \infty
\]
for every \( i \in \mathbb{N} \), since \( f \in L^q_{\text{loc}} (\Omega_1 T) \). If \( 0 < q \leq 1 \), then it holds that
\[
(c_i)_-^q - \int_{R^+ (\gamma)} f_-^q + (c_i)_+^q - \int_{R^- (\delta)} f_+^q \leq \int_{R^+ (\gamma)} (c_i - f)_-^q + \int_{R^- (\delta)} (c_i - f)_+^q
= \int_{R^+ (\gamma)} (f - c_i)_+^q + \int_{R^- (\delta)} (f - c_i)_-^q
\leq \int_{R^+ (\gamma)} f_+^q + \int_{R^- (\delta)} f_-^q + 1.
\]
This implies
\[
|c_i|^q = (c_i)_-^q + (c_i)_+^q \leq \int_{R^+ (\gamma)} (f_+^q + f_-^q) + \int_{R^- (\delta)} (f_-^q + f_+^q) + 1
= \int_{R^+ (\gamma)} |f|^q + \int_{R^- (\delta)} |f|^q + 1 < \infty
\]
for every \( i \in \mathbb{N} \). On the other hand, if \( 1 < q < \infty \), we have
\[
2^{1-q} (c_i)_-^q - \int_{R^+ (\gamma)} f_-^q + 2^{1-q} (c_i)_+^q - \int_{R^- (\delta)} f_+^q \leq \int_{R^+ (\gamma)} (c_i - f)_-^q + \int_{R^- (\delta)} (c_i - f)_+^q
= \int_{R^+ (\gamma)} (f - c_i)_+^q + \int_{R^- (\delta)} (f - c_i)_-^q
\leq \int_{R^+ (\gamma)} f_+^q + \int_{R^- (\delta)} f_-^q + 1,
\]
and thus
\[
2^{1-q} |c_i|^q = 2^{1-q} (c_i)_-^q + (c_i)_+^q \leq \int_{R^+ (\gamma)} (f_+^q + f_-^q) + \int_{R^- (\delta)} (f_-^q + f_+^q) + 1
= \int_{R^+ (\gamma)} |f|^q + \int_{R^- (\delta)} |f|^q + 1 < \infty
\]
for every \( i \in \mathbb{N} \). This shows that the sequence \((c_i)_{i \in \mathbb{N}}\) is bounded. Therefore, there exists a subsequence \((c_{i_k})_{k \in \mathbb{N}}\) that converges to \( c_R \in \mathbb{R} \). Then \((f - c_{i_k})_\pm\) converges uniformly to \((f - c_R)_\pm\) in \( R \) as \( k \to \infty \). This implies the convergence in \( L^q (R) \), and thus
\[ \inf_{c \in \mathbb{R}} \left( \int_{R^+} (f - c)^q_+ + \int_{R^-} (f - c)^q_- \right) = \lim_{k \to \infty} \left( \int_{R^+} (f - c_i)^q_+ + \int_{R^-} (f - c_i)^q_- \right) = \int_{R^+} (f - c_R)^q_+ + \int_{R^-} (f - c_R)^q_- . \]

This completes the proof. \( \square \)

We list some properties of parabolic BMO below. In particular, parabolic BMO is closed under addition and scaling by a positive constant. On the other hand, multiplication by negative constants reverses the time direction.

**Lemma 2.8** Let \(-1 < \gamma < 1, -\gamma \leq \delta < 1\) and \(0 < q < \infty\). Assume that \(f\) and \(g\) belong to \(\text{PBMO}^+_{\gamma, \delta, q}(\Omega_T)\) and let \(\text{PBMO}^\pm = \text{PBMO}^\pm_{\gamma, \delta, q}(\Omega_T)\). Then the following properties hold true.

(i) \(\| f + a \|_{\text{PBMO}^+} = \| f \|_{\text{PBMO}^+}, a \in \mathbb{R}.\)

(ii) \(\| f + g \|_{\text{PBMO}^+} \leq \max\{2^{1-q-1}, 2^{1-\frac{1}{q}}\} (\| f \|_{\text{PBMO}^+} + \| g \|_{\text{PBMO}^+}).\)

(iii) \(\| af \|_{\text{PBMO}^+} = \begin{cases} a \| f \|_{\text{PBMO}^+}, & a \geq 0, \\ -a \| f \|_{\text{PBMO}^+}, & a < 0. \end{cases}\)

(iv) \(\| \max\{f, g\} \|_{\text{PBMO}^+} \leq \max\{1, 2^{1-q-1}\} (\| f \|_{\text{PBMO}^+} + \| g \|_{\text{PBMO}^+}),\)

\(\| \min\{f, g\} \|_{\text{PBMO}^+} \leq \max\{1, 2^{1-q-1}\} (\| f \|_{\text{PBMO}^+} + \| g \|_{\text{PBMO}^+}).\)

**Remark 2.9** The constants in (ii) and (iv) can be avoided by considering the norm

\[ \sup_{R \subset \Omega_T} \inf_{c \in \mathbb{R}} \left[ \left( \int_{R^+} (f - c)^q_+ \right)^{\frac{1}{q}} + \left( \int_{R^-} (f - c)^q_- \right)^{\frac{1}{q}} \right], \]

which is an equivalent norm with our definition. However, the current definition will be convenient in the proof of the John–Nirenberg lemma below.

**Proof of Lemma 2.8** (i) We observe that

\[ \inf_{c \in \mathbb{R}} \left( \int_{R^+} (f + a - c)^q_+ + \int_{R^-} (f + a - c)^q_- \right)^{\frac{1}{q}} = \inf_{c' \in \mathbb{R}} \left( \int_{R^+} (f - c')^q_+ + \int_{R^-} (f - c')^q_- \right)^{\frac{1}{q}} \]

with \(c' = c - a\). Taking supremum over all parabolic rectangles \(R \subset \Omega_T\) completes the proof.

(ii) We note that

\[ \int_{R^\pm} (f + g - c_R^f - c_R^g)^q_{\pm} \leq \max\{1, 2^{q-1}\} \left( \int_{R^+} (f - c_R^f)^q_+ + \int_{R^-} (g - c_R^g)^q_- \right) \]

\(\square\) Springer
for $c^f_R, c^g_R \in \mathbb{R}$. This implies

$$\inf_{c \in \mathbb{R}} \left( \frac{\int_{R^+ (\gamma)} (f + g - c)^q_s + \int_{R^- (\delta)} (f + g - c)^q_s}{q} \right)$$

$$\leq \left( \frac{\int_{R^+ (\gamma)} (f + g - c^f_R - c^g_R)^q_s + \int_{R^- (\delta)} (f + g - c^f_R - c^g_R)^q_s}{q} \right)$$

$$\leq \max \{1, 2^\frac{1}{\gamma} \} \left( \frac{\int_{R^+ (\gamma)} (f - c^f_R)^q_s + \int_{R^- (\delta)} (g - c^g_R)^q_s}{q} \right) + \max \{2^\frac{1}{\gamma} - 1, 2^\frac{1}{\gamma} \} \left( \frac{\int_{R^+ (\gamma)} (f - c^f_R)^q_s + \int_{R^- (\delta)} (g - c^g_R)^q_s}{q} \right)$$

By taking supremum over all parabolic rectangles $R \subset \Omega_T$, we obtain

$$\| f + g \|_{\text{PBMO}^+} \leq \max \{2^\frac{1}{\gamma} - 1, 2^\frac{1}{\gamma} \} (\| f \|_{\text{PBMO}^+} + \| g \|_{\text{PBMO}^+}).$$

(ii) Observe that

$$\inf_{c \in \mathbb{R}} \left( \frac{\int_{R^+ (\gamma)} (af - c)^q_s + \int_{R^- (\delta)} (af - c)^q_s}{q} \right)$$

$$= \inf_{c' \in \mathbb{R}} \left( \frac{\int_{R^+ (\gamma)} (af - ac')^q_s + \int_{R^- (\delta)} (af - ac')^q_s}{q} \right)$$

$$= \begin{cases} a \inf_{c' \in \mathbb{R}} \left( \frac{\int_{R^+ (\gamma)} (f - c')^q_s + \int_{R^- (\delta)} (f - c')^q_s}{q} \right), & a \geq 0, \\ -a \inf_{c' \in \mathbb{R}} \left( \frac{\int_{R^+ (\gamma)} (f - c')^q_s + \int_{R^- (\delta)} (f - c')^q_s}{q} \right), & a < 0, \end{cases}$$

with $c' = c/a$. The claim follows from this observation.

(iv) For the positive part, we have

$$\int_{R^+ (\gamma)} (\max \{f, g\} - \max \{c^f_R, c^g_R\})^q_s$$

$$\leq \int_{R^+ (\gamma) \cap \{f \geq g\}} (\max \{f, g\} - c^f_R)^q_s + \int_{R^+ (\gamma) \cap \{f < g\}} (\max \{f, g\} - c^g_R)^q_s$$

$$= \int_{R^+ (\gamma) \cap \{f \geq g\}} (f - c^f_R)^q_s + \int_{R^+ (\gamma) \cap \{f < g\}} (g - c^g_R)^q_s$$
\[
\int_{R^+(\gamma)} (f - c_{f}^q)^+ + \int_{R^+(\gamma)} (g - c_{g}^q)^+.
\]

For the negative part, we have

\[
\int_{R^{-}(\delta)} (\max\{f, g\} - \max\{c_{f}^q, c_{g}^q\})_+^q \leq \int_{R^{-}(\delta)} (f - c_{f}^q)^- + \int_{R^{-}(\delta)} (g - c_{g}^q)^-.
\]

Hence, we obtain

\[
\inf_{c \in \mathbb{R}} \left( \int_{R^+(\gamma)} (\max\{f, g\} - c)^+ + \int_{R^{-}(\delta)} (\max\{f, g\} - c)^- \right)^{\frac{1}{q}}
\]
\[
\leq \left( \int_{R^+(\gamma)} (\max\{f, g\} - \max\{c_{f}^q, c_{g}^q\})_+^q + \int_{R^{-}(\delta)} (\max\{f, g\} - \max\{c_{f}^q, c_{g}^q\})^-_+^q + \int_{R^+(\gamma)} (g - c_{g}^q)^- + \int_{R^{-}(\delta)} (g - c_{g}^q)^- \right)^{\frac{1}{q}}
\]
\[
\leq \max\{1, 2^{\frac{1}{q} - 1}\} \left( \int_{R^+(\gamma)} (f - c_{f}^q)^+ + \int_{R^{-}(\delta)} (f - c_{f}^q)^- \right)^{\frac{1}{q}}
\]
\[
+ \max\{1, 2^{\frac{1}{q} - 1}\} \left( \int_{R^+(\gamma)} (g - c_{g}^q)^+ + \int_{R^{-}(\delta)} (g - c_{g}^q)^- \right)^{\frac{1}{q}}.
\]

By taking supremum over all parabolic rectangles \( R \subset \Omega_T \), we conclude that

\[
\|\max\{f, g\}\|_{\text{PBMO}^+} \leq \max\{1, 2^{\frac{1}{q} - 1}\} \left( \|f\|_{\text{PBMO}^+} + \|g\|_{\text{PBMO}^+} \right).
\]

The claim for \( \min\{f, g\} \) follows similarly. \( \Box \)

**Remark 2.10** Every function \( f \in \text{PBMO}^+_{\gamma, \delta, q}(\Omega_T) \) can be approximated pointwise by a sequence of bounded \( \text{PBMO}^+_{\gamma, \delta, q}(\Omega_T) \) functions, since the truncations

\[
f_k(x) = \min\{\max\{f(x), -k\}, k\}, \quad k \in \mathbb{N},
\]

belong to \( \text{PBMO}^+_{\gamma, \delta, q}(\Omega_T) \) with

\[
\|f_k\|_{\text{PBMO}^+_{\gamma, \delta, q}(\Omega_T)} \leq \max\{1, 2^{\frac{2}{q} - 2}\} \|f\|_{\text{PBMO}^+_{\gamma, \delta, q}(\Omega_T)}
\]

for every \( k \in \mathbb{N} \), see (i) and (iv) of Lemma 2.8, and it holds that \( f_k \rightarrow f \) pointwise and in \( L_{\text{loc}}^q(\Omega_T) \) as \( k \rightarrow \infty \).
3 Parabolic John–Nirenberg inequalities

This section discusses several versions of the John–Nirenberg inequality for parabolic BMO. We begin with a version where the exponential bound is given in terms of integral averages. For short, we suppress the variables \((x, t)\) in the notation and, for example, write

\[
R^+(\alpha) \cap \{(f - c_R)_+ > \lambda\} = \{(x, t) \in R^+(\alpha) : (f(x, t) - c_R)_+ > \lambda\}
\]

in the sequel.

**Theorem 3.1** Let \(R \subset \mathbb{R}^{n+1}\) be a parabolic rectangle, \(0 \leq \gamma < 1\), \(\gamma < \alpha < 1\) and \(0 < q \leq 1\). Assume that \(f \in \text{PBMO}^+_{\gamma, q}(R)\) and let \(\|f\| = \|f\|_{\text{PBMO}^+_{\gamma, q}(R)}\).

Then there exist constants \(c_R \in \mathbb{R}, A = A(n, p, q, \gamma, \alpha), B = B(n, p, q, \gamma, \alpha)\) and \(C = C(n, p, q, \gamma, \alpha)\) such that

\[
| R^+(\alpha) \cap \{(f - c_R)_+ > \lambda\}| \leq e^{-B(\lambda/\|f\|)^q} A \left( \frac{\|f\|^q}{c_R} \right)_{\gamma}^q
\]

and

\[
| R^-(\alpha) \cap \{(f - c_R)_- > \lambda\}| \leq e^{-B(\lambda/\|f\|)^q} A \left( \frac{\|f\|^q}{c_R} \right)_{\gamma}^q
\]

for every \(\lambda \geq C \|f\|\).

**Proof** Let \(R_0 = R = R(x_0, t_0, L) = Q(x_0, L) \times (t_0 - L^p, t_0 + L^p)\). By considering the function \(f(x + x_0, t + t_0)\), we may assume that the center of \(R_0\) is the origin, that is, \(R_0 = Q(0, L) \times (-L^p, L^p)\). By (i) and (iii) of Lemma 2.8, we may assume that \(c_{R_0} = 0\) and \(\|f\|^q = \frac{1}{q}(1 - \alpha)/(1 - \gamma)\). We note that it is sufficient to prove the first inequality of the theorem since the second inequality follows from the first one by applying it to the function \(-f(x, -t)\).

We claim that

\[
| R_0^+(\alpha) \cap \{f_+ > \lambda\}| \leq e^{-B(\lambda/\|f\|)^q} A \left( \frac{\|f\|^q}{c_R} \right)_{\gamma}^q
\]

for every \(\lambda \geq C \|f\|\). Let \(m\) be the smallest integer with \(3 + \alpha \leq 2^{pm+1}(\alpha - \gamma)\), that is,

\[
\frac{1}{p} \log_2 \left( \frac{3 + \alpha}{2(\alpha - \gamma)} \right) \leq m < \frac{1}{p} \log_2 \left( \frac{3 + \alpha}{2(\alpha - \gamma)} \right) + 1.
\]

Let \(S_0^+ = R_0^+(\alpha) = Q(0, L) \times (\alpha L^p, L^p)\). The time length of \(S_0^+\) is \(l_t(S_0^+) = (1 - \alpha)L^p\). We partition \(S_0^+\) by dividing each spatial edge \([-L, L]\) into \(2^m\) equally long intervals. If
we divide the time interval of $S_0^+$ into $\lfloor \frac{2^{pm}}{m} \rfloor$ equally long intervals. Otherwise, we divide the time interval of $S_0^+$ into $\lceil \frac{2^{pm}}{m} \rceil$ equally long intervals. We obtain subrectangles $S_1^+$ of $S_0^+$ with spatial side length $l_x(S_1^+) = l_x(S_0^+)/2^m = L/2^m$ and time length either

$$l_t(S_1^+) = \frac{l_t(S_0^+)}{\lfloor \frac{2^{pm}}{m} \rfloor} = \frac{(1 - \alpha)L^p}{\lfloor \frac{2^{pm}}{m} \rfloor} \quad \text{or} \quad l_t(S_1^+) = \frac{(1 - \alpha)L^p}{\lceil \frac{2^{pm}}{m} \rceil}.$$  

For every $S_1^+$, there exists a unique rectangle $R_1$ with spatial side length $l_x = L/2^m$ and time length $l_t = 2L^p/2^{mp}$ such that $R_1$ has the same top as $S_1^+$. We select those rectangles $S_1^+$ for which $\lambda < c_{R_1}$ and denote the obtained collection by $\{S_1^+, j\}$. If $\lambda \geq c_{R_1}$, we subdivide $S_1^+$ in the same manner as above and select all those subrectangles $S_2^+$ for which $\lambda < c_{R_2}$ to obtain family $\{S_2^+, j\}$. We continue this selection process recursively. At the $i$th step, we partition unselected rectangles $S_i^+$ by dividing each spatial side into $2^m$ equally long intervals. If

$$l_t(S_i^+) \leq \frac{(1 - \alpha)L^p}{\lfloor \frac{2^{pm}}{m} \rfloor}, \quad (3.1)$$

we divide the time interval of $S_i^+$ into $\lfloor \frac{2^{pm}}{m} \rfloor$ equally long intervals. If

$$l_t(S_i^+) \geq \frac{(1 - \alpha)L^p}{\lfloor \frac{2^{pm}}{m} \rfloor}, \quad (3.2)$$

we divide the time interval of $S_i^+$ into $\lceil \frac{2^{pm}}{m} \rceil$ equally long intervals. We obtain subrectangles $S_i^+$. For every $S_i^+$, there exists a unique rectangle $R_i$ with spatial side length $l_x = L/2^{mi}$ and time length $l_t = 2L^p/2^{pm}$ such that $R_i$ has the same top as $S_i^+$. Select those $S_i^+$ for which $\lambda < c_{R_i}$ and denote the obtained collection by $\{S_i^+, j\}$. If $\lambda \geq c_{R_i}$, we continue the selection process in $S_i^+$. In this manner we obtain a collection $\{S_i^+, j\} \cup_j$ of pairwise disjoint rectangles.

Observe that if (3.1) holds, then we have

$$l_t(S_i^+) = \frac{l_t(S_i^+)}{\lfloor \frac{2^{pm}}{m} \rfloor} \leq \frac{(1 - \alpha)L^p}{\lfloor \frac{2^{pm}}{m} \rfloor}.$$  

On the other hand, if (3.2) holds, then

$$l_t(S_i^+) = \frac{l_t(S_i^+)}{\lfloor \frac{2^{pm}}{m} \rfloor} \leq \frac{l_t(S_i^+)}{\lfloor \frac{2^{pm}}{m} \rfloor} \leq \cdots \leq \frac{(1 - \alpha)L^p}{\lfloor \frac{2^{pm}}{m} \rfloor}.$$
This gives an upper bound

\[ l_t(S_i^+) \leq \frac{(1 - \alpha)L^p}{2^{pmi}} \]

for every \( S_i^+ \).

Suppose that (3.2) is satisfied at the \( i \)th step. Then we have a lower bound for the time length of \( S_i^+ \), since

\[ l_t(S_i^+) = \frac{l_t(S_{i-1}^+)}{2^{pmi}} \geq \frac{[2^{pmi}] (1 - \alpha)L^p}{2^{pmi}} \geq 1 \frac{(1 - \alpha)L^p}{2}. \]

On the other hand, if (3.1) is satisfied, then

\[ l_t(S_i^+) = \frac{l_t(S_{i-1}^+)}{2^{pmi}} \geq \frac{l_t(S_{i-1}^+)}{2^{pmi}}. \]

In this case, (3.2) has been satisfied at an earlier step \( i' \) with \( i' < i \). We obtain

\[ l_t(S_i^+) \geq \frac{l_t(S_{i-1}^+)}{2^{pmi}} \geq \cdots \geq \frac{l_t(S_{i-1}^+)}{2^{pm(i-i')}} \geq \frac{(1 - \alpha)L^p}{2}. \]

by using the lower bound for \( S_{i'}^+ \). Thus, we have

\[ \frac{1}{2} \frac{1 - \alpha)L^p}{2^{pmi}} \leq l_t(S_i^+) \leq \frac{1 - \alpha)L^p}{2^{pmi}} \]

for every \( S_i^+ \). By using the lower bound for the time length of \( S_i^+ \) and the choice of \( m \), we observe that

\[ l_t(R_i) - l_t(S_i^+) \leq \frac{2L^p}{2^{pmi}} - \frac{1}{2} \frac{1 - \alpha)L^p}{2^{pmi}} = \frac{L^p}{2^{pmi}} (3 + \alpha) \leq \frac{(\alpha - \gamma)L^p}{2^{pm(i-1)}} - \frac{(1 - \gamma)L^p}{2^{pm(i-1)}} \leq l_t(R_{i-1}^+(\gamma)) - l_t(S_{i-1}^+). \]

This implies

\[ R_i \subset R_{i-1}^+(\gamma) \quad (3.3) \]

for a fixed rectangle \( S_{i-1}^+ \) and for every subrectangle \( S_i^+ \subset S_{i-1}^+ \) (see Fig. 1). By the construction of the subrectangles \( S_i^+ \), we have

\[ 2^{nm} [2^{pm}] |S_i^+| \leq |S_i^+| \leq 2^{nm} [2^{pm}] |S_i^+| \quad (3.4) \]
We have a collection \( \{ S^+_{1,j} \}_{i,j} \) of pairwise disjoint rectangles. However, the rectangles in the corresponding collection \( \{ S^-_{1,j} \}_{i,j} \) may overlap. Thus, we replace it by a subfamily \( \{ \tilde{S}^-_{i,j} \}_{i,j} \) of pairwise disjoint rectangles, which is constructed in the following way. At the first step, choose \( \{ S^-_{1,j} \}_{j} \) and denote it by \( \{ \tilde{S}^-_{1,j} \}_{j} \). Then consider the collection \( \{ S^-_{2,j} \}_{j} \) where each \( S^-_{2,j} \) either intersects some \( \tilde{S}^-_{1,j} \) or does not intersect any \( \tilde{S}^-_{1,j} \). Select the rectangles \( S^-_{2,j} \) that do not intersect any \( \tilde{S}^-_{1,j} \), and denote the obtained collection by \( \{ \tilde{S}^-_{2,j} \}_{j} \). At the \( i \)th step, choose those \( S^-_{i,j} \) that do not intersect any previously selected \( \tilde{S}^-_{i',j} \), \( i' < i \). Hence, we obtain a collection \( \{ \tilde{S}^-_{i,j} \}_{i,j} \) of pairwise disjoint rectangles. Observe that for every \( S^-_{i,j} \) there exists \( \tilde{S}^-_{i',j} \) with \( i' < i \) such that

\[
\text{pr}_x(S^-_{i,j}) \subset \text{pr}_x(\tilde{S}^-_{i',j}) \quad \text{and} \quad \text{pr}_t(S^-_{i,j}) \subset 3\text{pr}_t(\tilde{S}^-_{i',j}).
\] (3.6)

Here \( \text{pr}_x \) denotes the projection to \( \mathbb{R}^n \) and \( \text{pr}_t \) denotes the projection to the time axis. Rename \( \{ S^+_{i,j} \}_{i,j} \) and \( \{ \tilde{S}^-_{i,j} \}_{i,j} \) as \( \{ S^+_{i,j} \}_{i,j} \) and \( \{ \tilde{S}^-_{i,j} \}_{j} \), respectively. Let \( S(\lambda) = \bigcup_i S^+_{i,j} \). Note that \( S^+_{i,j} \) is spatially contained in \( S^-_{i,j} \), that is, \( \text{pr}_x S^+_{i,j} \subset \text{pr}_x S^-_{i,j} \). In the time direction, we have

\[
\text{pr}_t(S^+_{i,j}) \subset \text{pr}_t(R_i) \subset \frac{7 + \alpha}{1 - \alpha} \text{pr}_t(S^-_{i,j}).
\] (3.7)
since
\[ \left( \frac{7 + \alpha}{1 - \alpha} + 1 \right) \frac{l_t(S_i^-)}{2} \geq \frac{8}{1 - \alpha} \frac{(1 - \alpha)L^p_{2^pmi+2}}{2^pmi} = \frac{2L^p_{2^pmi}}{2^pmi} = l_t(R_i). \]

Therefore, by (3.6) and (3.7), it holds that
\[ |S(\lambda)| = \left| \bigcup_i S_i^+ \right| \leq c_1 \sum_j |\tilde{S}_j^-| \quad \text{with} \quad c_1 = \frac{7 + \alpha}{1 - \alpha}. \tag{3.8} \]

Let \( \lambda > \delta > 0 \) and consider the collection \( \{S_k^+\}_k \) for \( \delta \). Then every \( S_i^+ \) is contained in a unique \( S_k^+ \). Let \( J_k = \{ j \in \mathbb{N} : \tilde{S}_j^- \subset S_k^+ \} \). Using (3.5), we have
\[ \lambda^q < c_R^q \leq \int_{\tilde{S}_j^-} (c_{R_j} - f)_+^q + \int_{\tilde{S}_j^-} f_+^q \]
\[ \leq \frac{1 - \gamma}{1 - \alpha} \int_{R_j^-} (c_{R_j} - f)_+^q + \int_{\tilde{S}_j^-} f_+^q \]
\[ \leq \frac{1 - \gamma}{1 - \alpha} \|f\|^q + \int_{\tilde{S}_j^-} f_+^q \]
\[ = 1 + \int_{\tilde{S}_j^-} f_+^q \]
for every \( \tilde{S}_j^- \). By summing over \( j \), we obtain
\[ (\lambda^q - 1) \sum_j |\tilde{S}_j^-| \leq \sum_j \int_{\tilde{S}_j^-} f_+^q = \sum_k \sum_{j \in J_k} \int_{\tilde{S}_j^-} f_+^q. \tag{3.9} \]

Let \( k \in \mathbb{N} \). We have \( \tilde{S}_j^- \subset S_k^+ \) for every \( j \in J_k \), where \( S_k^+ \) was obtained by subdividing a previous \( S_k^- \) for which \( a_{R_k^-} \leq \delta \). Hence, it holds that \( \tilde{S}_j^- \subset R_k \) for every \( j \in J_k \). By (3.3), it follows that \( \tilde{S}_j^- \subset R_k^-(\gamma) \) for every \( j \in J_k \). Since \( \tilde{S}_j^- \) are pairwise disjoint, by applying (3.4) together with (3.5), we arrive at
\[ \sum_{j \in J_k} \int_{\tilde{S}_j^-} f_+^q \leq \sum_{j \in J_k} \int_{\tilde{S}_j^-} (f - a_{R_k^-} + \delta)_+^q \]
\[ \leq \sum_{j \in J_k} \int_{\tilde{S}_j^-} (f - a_{R_k^-})_+^q + \sum_{j \in J_k} \int_{\tilde{S}_j^-} \delta^q \]
\[ \leq \int_{R_k^-(\gamma)} (f - a_{R_k^-})_+^q + \delta^q \sum_{j \in J_k} |\tilde{S}_j^-| \]
\[ \leq \frac{1}{2} \frac{1 - \alpha}{1 - \gamma} |R_k^-(\gamma)| + \delta^q \sum_{j \in J_k} |\tilde{S}_j^-| \]
\[ \leq |S_k^-| + \delta^q \sum_{j \in J_k} |\tilde{S}_j^-| \]
\[ \leq 2^{nm} |2^{pm}| |S_k^+| + \delta^q \sum_{j \in J_k} |\tilde{S}_j^-| \]
\[ \leq c_2 |S_k^+| + \delta^q \sum_{j \in J_k} |\tilde{S}_j^-|, \]

where
\[ 2^{nm} |2^{pm}| \leq 2^{n+m+1} \leq 2^{1+(n+p)\left( \frac{1}{p} \log_2 \left( \frac{3+\alpha}{2(\alpha-\gamma)} \right) + 1 \right)} \]
\[ = 2^{1+n+p} \left( \frac{3 + \alpha}{2(\alpha - \gamma)} \right)^{1+\frac{n}{p}} = c_2. \]

By summing over \( k \) and applying (3.9), we obtain
\[ (\lambda^q - 1) \sum_j |\tilde{S}_j^-| \leq c_2 \sum_k |S_k^+| + \delta^q \sum_j |\tilde{S}_j^-|. \]

Thus, we have
\[ \sum_j |\tilde{S}_j^-| \leq \frac{c_2}{\lambda^q - \delta^q - 1} \sum_k |S_k^+| \]
for every \( \lambda^q > \delta^q + 1 \). By (3.8), we obtain
\[ |S(\lambda)| \leq \frac{c_1 c_2}{\lambda^q - \delta^q - 1} |S(\delta)| \]
for every \( \lambda^q > \delta^q + 1 \). By setting \( a = 2c_1 c_2 + 1 \) and replacing \( \lambda^q \) and \( \delta^q \) by \( \lambda + a \) and \( \lambda \), respectively, we have
\[ |S((\lambda + a)^{1/q})| \leq \frac{1}{2} |S(\lambda^{1/q})|. \tag{3.10} \]

Assume that \( \lambda \geq a \). Then there exists an integer \( N \in \mathbb{Z}_+ \) such that \( Na \leq \lambda < (N + 1)a \). By iterating (3.10), we arrive at
\[ |S(\lambda^{1/q})| \leq |S((Na)^{1/q})| \leq \frac{1}{2^{N-1}} |S(a^{1/q})| \]
\[ \leq 2^{-\frac{2}{\alpha+2}} |S(a^{1/q})| = 4e^{-\frac{\lambda}{\alpha+2}\gamma} \log^2 |S(a^{1/q})|. \]

Applying (3.8), (3.9) and (3.3), we get
\[ |S(a^{1/q})| \leq \frac{c_1}{a - 1} \int_{R^+_a(\gamma)} f^q = \frac{1}{4c_2} \frac{1}{1 - \gamma} \|f\|_q \int_{R^+_a(\gamma)} f^q. \]
This implies

\[ |S(\lambda)| \leq e^{-2B(\lambda/\|f\|^q)} \frac{A}{\|f\|^q} \int_{R^+_0(\gamma)} f^q_+ \]

for every \( \lambda \geq \frac{1}{a^{\frac{1}{q}}} \) with

\[ A = \frac{1}{c_2} \frac{1 - \alpha}{1 - \gamma} \quad \text{and} \quad B = \frac{1}{4} \frac{1 - \alpha}{1 - \gamma} \frac{\log 2}{c_1 c_2 + 1}. \]

If \((x, t) \in S_0^+ \setminus S(\lambda)\), then there exists a sequence \{S^+_l\}_{l \in \mathbb{N}} of subrectangles containing \((x, t)\) such that \(c^{R_l} \leq \lambda\) and \(|S^+_l| \to 0\) as \(l \to \infty\). This implies

\[ \int_{S^+_l} f^q_+ \leq \int_{S^+_l} (f - c^{R_l})^q_+ + \lambda^q \leq 1 + \lambda^q. \]

The Lebesgue differentiation theorem (Lemma 2.3) implies that \(f(x, t)^q_+ \leq 1 + \lambda^q\) for almost every \((x, t) \in S_0^+ \setminus S(\lambda)\). It follows that

\[ \{ (x, t) \in S_0^+ : f(x, t)^q_+ > 1 + \lambda^q \} \subset S(\lambda) \]

up to a set of measure zero. Given \( \lambda \geq 2^{\frac{1}{q}} \), we have \( \lambda^q \geq 1 + \frac{\lambda^q}{2} \). We conclude that

\[ |S_0^+ \cap \{ f^q_+ > \lambda \}| \leq |S_0^+ \cap \{ f^q_+ > \lambda^q \}| \leq |S_0^+ \cap \{ \lambda^q > 1 + \frac{\lambda^q}{2} \}| \leq |S(\lambda/2^{\frac{1}{q}})| \]

for every \( \lambda > (2a)^{\frac{1}{q}} = (4a(1 - \gamma)/(1 - \alpha))^{\frac{1}{q}} \|f\| = C \|f\|. \) This completes the proof. \( \Box \)

As a corollary of Theorem 3.1, we obtain a more standard version of the parabolic John–Nirenberg inequality.

**Corollary 3.2** Let \( R \subset \mathbb{R}^{n+1} \) be a parabolic rectangle, \( 0 \leq \gamma < 1, \gamma < \alpha < 1 \) and \( 0 < q \leq 1 \). Assume that \( f \in \text{PBMO}^{+}_{\gamma, q}(R) \) and let \( \|f\| = \|f\|_{\text{PBMO}^{+}_{\gamma, q}(R)} \). Then there exist constants \( c_R \in \mathbb{R}, A = A(n, p, q, \gamma, \alpha) \) and \( B = B(n, p, q, \gamma, \alpha) \) such that

\[ |R^+(\alpha) \cap \{ f - c_R \}^+ \geq \lambda | \leq A e^{-B(\lambda/\|f\|^q)} |R^+(\alpha)| \]

and

\[ |R^-(-\alpha) \cap \{ f - c_R \}^- \geq \lambda | \leq A e^{-B(\lambda/\|f\|^q)} |R^-(-\alpha)| \]

for every \( \lambda > 0 \).
Proof By Theorem 3.1, there exists a constant $C = C(n, p, q, \gamma, \alpha)$ such that

$$|R^+(\alpha) \cap \{(f - c_R)_+ > \lambda\}| \leq e^{-B(\lambda/\|f\|)^q} \frac{A}{\|f\|^q} \int_{R^+(\gamma)} (f - c_R)_+^q$$

$$\leq e^{-B(\lambda/\|f\|)^q} \frac{1}{c_2} \frac{1 - \alpha}{1 - \gamma} |R^+(\gamma)|$$

$$= \frac{1}{c_2} e^{-B(\lambda/\|f\|)^q} |R^+(\alpha)|$$

for every $\lambda \geq C \|f\|$. On the other hand, if $0 < \lambda < C \|f\|$, then

$$|R^+(\alpha) \cap \{(f - c_R)_+ > \lambda\}| \leq e^{1} e^{-1} |R^+(\alpha)| \leq e^{1} e^{-\frac{1}{c_2} (\lambda/\|f\|)^q} |R^+(\alpha)|.$$  

This proves the first inequality in the theorem. The second inequality follows similarly. $\square$

Another consequence of Theorem 3.1 is a weak reverse Hölder inequality for parabolic BMO. In particular, this implies that a function in parabolic BMO is locally integrable to any positive power.

Corollary 3.3 Let $R \subset \mathbb{R}^{n+1}$ be a parabolic rectangle, $0 \leq \gamma < 1$, $\gamma < \alpha < 1$, $0 < q \leq 1$ and $q \leq r < \infty$. Assume that $f \in \text{PBMO}^+_\gamma,q(R)$ and let $\|f\| = \|f\|_{\text{PBMO}^+_\gamma,q(R)}$. Then there exist constants $c_R \in \mathbb{R}$ and $c = c(n, p, q, r, \gamma, \alpha)$ such that

$$\int_{R^+(\alpha)} (f - c_R)_+^r \leq c \|f\|^{r-q} \int_{R^+(\gamma)} (f - c_R)_+^q$$

and

$$\int_{R^-(\alpha)} (f - c_R)_-^r \leq c \|f\|^{r-q} \int_{R^-(\gamma)} (f - c_R)_-^q.$$  

Proof Let

$$E(\lambda) = R^+(\alpha) \cap \{(f - c_R)_+ > \lambda\}.$$  

By using Cavalieri’s principle, we get

$$\int_{R^+(\alpha)} (f - c_R)_+^r = r \int_0^{\infty} \lambda^{r-1} |E(\lambda)| d\lambda$$

$$= r \int_0^{\|f\|} \lambda^{r-1} |E(\lambda)| d\lambda + r \int_{\|f\|}^{\infty} \lambda^{r-1} |E(\lambda)| d\lambda.$$
where \( C = C(n, p, q, \gamma, \alpha) \) is the constant in Theorem 3.1. We estimate the obtained integrals separately. For \( 0 \leq \lambda \leq C \| f \| \), we have \( \lambda^{r-1} \leq (C \| f \|)^{r-q} \lambda^{q-1} \), and thus

\[
\begin{align*}
 r \int_0^{C \| f \|} \lambda^{r-1} |E(\lambda)| \, d\lambda & \leq r (C \| f \|)^{r-q} \int_0^{\infty} \lambda^{q-1} |E(\lambda)| \, d\lambda \\
 & = \frac{r}{q} C^{r-q} \| f \|^{r-q} \int_{R^+(\alpha)} (f - c R)^q_+.
\end{align*}
\]

For the second integral, we apply Theorem 3.1 to obtain

\[
\begin{align*}
 r \int_{C \| f \|}^{\infty} \lambda^{r-1} |E(\lambda)| \, d\lambda & \leq \frac{Ar}{\| f \| q} \int_{R^+(\gamma)} (f - c R)^q_+ \left( \frac{\| f \|}{B^1} \right)^{r-1-\gamma} \int_{0}^{\infty} e^{-s \frac{q}{q}} ds \\
 & = \frac{Ar}{B^q q} \Gamma\left( \frac{r}{q} \right) \| f \|^{r-q} \int_{R^+(\gamma)} (f - c R)^q_+,
\end{align*}
\]

where we applied a change of variables \( s = B \lambda^q / \| f \|^q \). This implies

\[
\int_{R^+(\alpha)} (f - c R)^r_+ \leq c \| f \|^{r-q} \int_{R^+(\gamma)} (f - c R)^q_+ , \quad \text{with} \quad c = \frac{r}{q} \left( C^{r-q} + \frac{A}{B^\frac{r}{q}} \Gamma\left( \frac{r}{q} \right) \right).
\]

The second inequality of the theorem follows similarly.

\[\square\]

### 4 Chaining arguments and the time lag

Applying Corollary 3.2 with chaining arguments, we obtain a parabolic John–Nirenberg inequality, which allows us to change the time lag.

**Theorem 4.1** Let \( R \subset \mathbb{R}^{n+1} \) be a parabolic rectangle, \( 0 < \gamma < 1, -1 < \rho \leq \gamma, -\rho < \sigma \leq \gamma \) and \( 0 < q \leq 1 \). Assume that \( f \in \text{PBMO}^{+, \gamma, q}_R(R) \) and let \( \| f \| = \| f \|_{\text{PBMO}^{+, \gamma, q}_R(R)} \). Then there exist constants \( c \in \mathbb{R}, A = A(n, p, q, \gamma, \rho, \sigma) \) and \( B = B(n, p, q, \gamma, \rho, \sigma) \) such that

\[
|R^+(\rho) \cap \{(f - c)_+ > \lambda\}| \leq Ae^{-B(\lambda/\| f \|^q)|R^+(\rho)|}
\]

and

\[
|R^-(-\sigma) \cap \{(f - c)_- > \lambda\}| \leq Ae^{-B(\lambda/\| f \|^q)|R^-(-\sigma)|}
\]

for every \( \lambda > 0 \).
Proof Let $R_0 = R$. Without loss of generality, we may assume that the center of $R_0$ is the origin. Since $f \in \text{PBMO}_{p,q}^\gamma(R_0)$, Corollary 3.2 holds for any parabolic subrectangle of $R_0$ and for any $\gamma < \alpha < 1$. Let $m$ be the smallest integer with

$$m \geq \log_2 \left( \frac{1 + \alpha}{1 - \alpha} \right) + \frac{1}{p - 1} \left( 2 + \log_2 \frac{1 + \alpha}{\rho + \sigma} \right) + 2.$$ 

Then there exists $0 \leq \varepsilon < 1$ such that

$$m = \log_2 \left( \frac{1 + \alpha}{1 - \alpha} \right) + \frac{1}{p - 1} \left( 2 + \log_2 \frac{1 + \alpha}{\rho + \sigma} \right) + 2 + \varepsilon.$$ 

We partition $R_0^+ (\rho) = Q(0, L) \times (\rho L^p, L^p)$ by dividing each of its spatial edges into $2^m$ equally long intervals and the time interval into $\lfloor (1 - \rho)2^{mp}/(1 - \alpha) \rfloor$ equally long intervals. Denote the obtained rectangles by $U_{i,j}^+$ with $i \in \{1, \ldots, 2^m\}$ and $j \in \{1, \ldots, \lfloor (1 - \rho)2^{mp}/(1 - \alpha) \rfloor\}$. The spatial side length of $U_{i,j}^+$ is $l = l(U_{i,j}^+) = L/2^m$ and the time length is

$$l_t(U_{i,j}^+) = \frac{(1 - \rho)L^p}{\lfloor (1 - \rho)2^{mp}/(1 - \alpha) \rfloor}.$$ 

For every $U_{i,j}^+$, there exists a unique rectangle $R_{i,j}$ that has the same top as $U_{i,j}^+$. Our aim is to construct a chain from each $U_{i,j}^+$ to a central rectangle which is of the same form as $R_{i,j}$ and is contained in $R_0$. This central rectangle will be specified later. First, we construct a chain with respect to the spatial variable. Fix $U_{i,j}$. Let $P_0 = R_{i,j}$ and

$$P_0^+ = R_{i,j}^+(\alpha) = Q_i \times (t_j - (1 - \alpha)L^p, t_j).$$

We construct a chain of cubes from $Q_i$ to the central cube $Q(0, l)$. Let $Q_i' = Q_i = Q(x_i, l)$ and set

$$Q_k' = Q_k' - \frac{x_i}{|x_i|} \frac{\theta l}{2}, \quad k \in \{0, \ldots, N_i\},$$

where $1 \leq \theta \leq \sqrt{n}$ depends on the angle between $x_i$ and the spatial axes and is chosen such that the center of $Q_k$ is on the boundary of $Q_{k-1}$ (see Fig. 2). We have

$$\frac{1}{2^n} \leq \frac{|Q_k \cap Q_{k-1}|}{|Q_k|} \leq \frac{1}{2}, \quad k \in \{0, \ldots, N_i\},$$

and $|x_i| = \frac{\theta}{2} (L - bl)$, where $b \in \{1, \ldots, 2^m\}$ depends on the distance of $Q_i$ to the center of $Q_0 = Q(0, L)$. The number of cubes in the spatial chain $\{Q_k'\}_{k=0}^{N_i}$ is

$$N_i + 1 = \frac{|x_i|}{\frac{\theta l}{2}} + 1 = \frac{L}{l} - b + 1.$$
Next, we also take the time variable into consideration in the construction of the chain. Let

\[ P_k^+ = Q_k' \times (t_j - (1 - \alpha)l^p - k(1 + \alpha)l^p, t_j - k(1 + \alpha)l^p) \]

and \( P_k^- = P_k^+ - (0, (1 + \alpha)l^p) \), for \( k \in \{0, \ldots, N_i\} \), be the upper and the lower parts of a parabolic rectangle respectively. These will form a chain of parabolic rectangles from \( U_{i,j}^+ \) to the eventual central rectangle. Observe that every rectangle \( P_{N_i} \) coincides spatially for all pairs \((i, j)\). Consider \( j = 1 \) and such \( i \) that the boundary of \( Q_i \) intersects the boundary of \( Q_0 \). For such a cube \( Q_i \), we have \( b = 1 \), and thus \( N = N_i = \frac{L}{l} - 1 \). In the time variable, we travel from \( t_1 \) the distance

\[
(N + 1)(1 + \alpha)l^p + (1 - \alpha)l^p = (1 + \alpha)Ll^{p-1} + (1 - \alpha)l^p.
\]

We show that the lower part of the final rectangle \( P_N^- \) is contained in \( R_0 \). To this end, we subtract the time length of \( U_{i,1}^+ \) from the distance above and observe that it is less than half of the time length of \( R_0 \setminus (R_0^+(\rho) \cup R_0^-(\sigma)) \). This follows from the computation

\[
(1 + \alpha)Ll^{p-1} + (1 - \alpha)l^p - \frac{(1 - \rho)L^p}{[(1 - \rho)2^{mp}/(1 - \alpha)]}
= \left( \frac{1 + \alpha}{2^{m(p-1)}} + \frac{1 - \alpha}{2^{mp}} - \frac{1 - \rho}{[(1 - \rho)2^{mp}/(1 - \alpha)]} \right) L^p
\]
This implies that $P_0 \leq P_k$ than the previous rectangle was shifted, that is, we move each time direction a distance of $k \in \{1, 2, \ldots, N_i\}$ rectangles into the negative time direction such that the final rectangle coincides with $R_{j}$. For every $j \in \{2, \ldots, \lceil (1 - \rho)2^{mp}/(1 - \alpha) \rceil \}$, we consider a similar extension of the chain. The final rectangles of the chains coincide for fixed $j$ and for every $i$. Moreover, every chain is of the same length $N + 1$, and it holds that

$$m \geq \frac{1}{p - 1}\left(2 + \log_2 \frac{1 + \alpha}{\rho + \sigma}\right).$$

This implies that $P_N^- \subset R_0^+ (\rho - (\rho + \sigma)/2)$. Denote this rectangle $P_N$ by $R = R_R$. This is the central rectangle where all chains will eventually end.

Let $j = 1$ and assume that $i$ is arbitrary. We extend the chain $\{P_k\}_{k=0}^{N_i}$ by $N - N_i$ rectangles into the negative time direction such that the final rectangle coincides with the central rectangle $R$ (see Fig. 3a). More precisely, we consider $Q'_{k+1} = Q'_{N_i}$,

$$P_{k+1}^+ = P_k^+ - (0, (1 + \alpha)l^p) \text{ and } P_{k+1}^- = P_k^+ - (0, (1 + \alpha)l^p)$$

for $k \in \{N_i, \ldots, N - 1\}$. For every $j \in \{2, \ldots, \lceil (1 - \rho)2^{mp}/(1 - \alpha) \rceil \}$, we consider the time distance between the current ends of the chains for pairs $(i, j)$ and $(i, 1)$ is

$$\left\lfloor \frac{(j - 1)}{2^n} \frac{(1 - \rho) L^p}{(1 - \rho)2^{mp}/(1 - \alpha)} \right\rfloor.$$

Then we consider an index $j \in \{2, \ldots, \lceil (1 - \rho)2^{mp}/(1 - \alpha) \rceil \}$ related to the time variable. The time distance between the current ends of the chains for pairs $(i, j)$ and $(i, 1)$ is

$$\left\lfloor \frac{(j - 1)}{2^n} \frac{(1 - \rho) L^p}{(1 - \rho)2^{mp}/(1 - \alpha)} \right\rfloor.$$

Our objective is to have the final rectangle of the continued chain for $(i, j)$ to coincide with the end of the chain for $(i, 1)$, that is, with the central rectangle $R$. To achieve this, we modify $2^{m-1}$ intersections of $P_k^+$ and $P_{k+1}^-$ by shifting $P_k$ and also add a chain of $M_j$ rectangles traveling to the negative time direction into the chain $\{P_k\}_{k=0}^{N_i}$. We shift every $P_k, k \in \{1, \ldots, 2^{m-1}\}$, by a $\beta_j$-portion of their temporal length more than the previous rectangle was shifted, that is, we move each $P_k$ into the negative time direction a distance of $k \beta_j (1 - \alpha) l^p$ (see Fig. 3b). The values of $M_j \in \mathbb{N}$ and $0 \leq \beta_j < 1$ will be chosen later. In other words, modify the definitions of $P_k^+$ for $k \in \{1, \ldots, 2^{m-1}\}$ by

$$P_k^+ = Q_k' \times (t_j - (1 - \alpha)l^p - k(1 + \alpha + \beta_j(1 - \alpha)))l^p, t_j - k(1 + \alpha + \beta_j(1 - \alpha)) l^p,$$
and then add $M_j$ rectangles defined by
\[
P_{k+1}^+ = P_k^+ - (0, (1 + \alpha)l^p)
\quad \text{and} \quad
P_{k+1}^- = P_{k+1}^+ - (0, (1 + \alpha)l^p)
\]
for $k \in \{N, \ldots, N + M_j - 1\}$. Note that there exists $1 \leq \tau < 2$ such that
\[
\tau \frac{(1 - \rho)2^{mp}}{1 - \alpha} = \left\lceil \frac{(1 - \rho)2^{mp}}{1 - \alpha} \right\rceil.
\]
We would like to find such $0 \leq \beta_j < 1$ and $M_j \in \mathbb{N}$ that
\[
(j - 1) \frac{(1 - \rho)L^p}{[(1 - \rho)2^{mp}/(1 - \alpha)]} - M_j \frac{(1 + \alpha)L^p}{2^{mp}} = 2^{m-1} \beta_j \frac{(1 - \alpha)L^p}{2^{mp}},
\]
which is equivalent with
\[
(j - 1)\tau^{-1}(1 - \alpha) - M_j(1 + \alpha) = 2^{m-1} \beta_j(1 - \alpha).
\]
With this choice all final rectangles coincide. Choose $M_j \in \mathbb{N}$ such that
\[
M_j(1 + \alpha) \leq (j - 1)\tau^{-1}(1 - \alpha) < (M_j + 1)(1 + \alpha),
\]
that is,

\[ 0 \leq \xi = (j - 1)\tau^{-1}(1 - \alpha) - M_j(1 + \alpha) < 1 + \alpha \]

and

\[ \frac{(j - 1)(1 - \alpha)}{2(1 + \alpha)} - 1 \leq \frac{(j - 1)(1 - \alpha)}{\tau(1 + \alpha)} - 1 < M_j \leq \frac{(j - 1)(1 - \alpha)}{\tau(1 + \alpha)} \leq \frac{(j - 1)(1 - \alpha)}{1 + \alpha}. \]

By choosing \( 0 \leq \beta_j < 1 \) such that

\[ \xi = 2^{m-1}\beta_j(1 - \alpha) = 2^{\frac{2^m-1+\varepsilon}{p^{\gamma}}} \beta_j(1 + \alpha), \]

we have

\[ \beta_j = 2^{\frac{2}{p^{\gamma+1}}} \left( \frac{\rho + \sigma}{1 + \alpha} \right)^{\frac{1}{p^{\gamma}}} \frac{\xi}{1 + \alpha}. \]

Observe that \( 0 \leq \beta_j \leq \frac{1}{2} \) for every \( j \). For measures of the intersections of the modified rectangles, it holds that

\[ \frac{1}{2^{n+1}} \leq \frac{|P_k^+ \cap P_{k-1}^-|}{|P_k^+|} = \eta(1 - \beta_j) \leq 1 \]

for \( k \in \{1, \ldots, 2^{m-1}\} \), and thus

\[ \frac{1}{2^{n+1}} \leq \tilde{\eta}_j = \frac{|P_k^+ \cap P_{k-1}^-|}{|P_k^+|} \leq 1 \]

for every \( k \in \{1, \ldots, N + M_j\} \). Fix \( U_{i,j}^+ \). We conclude that

\[ (c_{R_{i,j}} - c_R)_+ = (c_{P_0} - c_{P_{N+M_j}})_+ \leq \sum_{k=1}^{N+M_j} (c_{P_{k-1}} - c_{P_k})_+ \]

\[ = \sum_{k=1}^{N+M_j} \int_{P_{k-1}^+ \cap P_k^+} (c_{P_{k-1}} - c_{P_k})_+ \]

\[ \leq \sum_{k=1}^{N+M_j} \left( \int_{P_{k-1}^+ \cap P_k^+} (c_{P_{k-1}} - f)_+^q + \int_{P_{k-1}^- \cap P_k^+} (f - c_{P_k})_+^q \right) \]

\[ \leq \sum_{k=1}^{N+M_j} \frac{1}{\tilde{\eta}_j} \left( \int_{P_{k-1}^+} (f - c_{P_{k-1}})_+^q + \int_{P_k^+} (f - c_{P_k})_+^q \right) \]
\[
\leq 2^{n+1} \sum_{k=0}^{N+M_j} \left( \int_{p_k^-} (f - c_{p_k})_+^r + \int_{p_k^+} (f - c_{p_k})_+^r \right) \\
\leq 2^{n+1}(N + 1 + M_j) \|f\|^q,
\]

where

\[
N + 1 + M_j = \frac{L}{l} + M_j \leq 2^m + (j - 1) \frac{1 - \alpha}{1 + \alpha} \\
\leq 2^m + \frac{(1 - \rho)2^{mp} 1 - \alpha}{1 - \alpha} \leq 2^m + 2^{mp+1} \leq 2^{mp+2} \\
\leq 2^{2p \tau + 3p + 2} \left( \frac{1 + \alpha}{\rho + \sigma} \right)^{\frac{p}{p - \tau}} \left( \frac{1 + \alpha}{1 - \alpha} \right)^p
\]

for every \( j \). Hence, we obtain

\[
(c_{R_i,j} - c_{\Omega})_+ \leq C \|f\| \quad \text{with} \quad C^q = 2^{2p \tau + 3p + n + 3} \left( \frac{1 + \alpha}{\rho + \sigma} \right)^{\frac{p}{p - \tau}} \left( \frac{1 + \alpha}{1 - \alpha} \right)^p.
\]

We observe that

\[
|R_i^+(\rho) \cap ((f - c_{\Omega})_+ > \lambda)| \leq \sum_{i,j} |R_i^+(\alpha) \cap ((f - c_{\Omega})_+ > \lambda)| \\
\leq \sum_{i,j} |R_i^+(\alpha) \cap ((f - c_{R_i,j})_+ > \frac{\lambda}{2})| + \sum_{i,j} |R_i^+(\alpha) \cap ((c_{R_i,j} - c_{\Omega})_+ > \frac{\lambda}{2})|.
\]

The first sum of the right-hand side can be estimated by Corollary 3.2 as follows

\[
\sum_{i,j} |R_i^+(\alpha) \cap ((f - c_{R_i,j})_+ > \frac{\lambda}{2})| \leq \sum_{i,j} A e^{-B(\lambda/\|f\|)^q} |R_i^+(\alpha)| \\
\leq A e^{-2^{-q}B(\lambda/\|f\|)^q} \sum_{i,j} \tau |U_{i,j}^+| \\
\leq 2A e^{-2^{-q}B(\lambda/\|f\|)^q} |R_0^+(\rho)|.
\]

To estimate the second sum above, assume that \( \lambda \geq 2C \|f\| \). This implies that

\[
|R_i^+(\alpha) \cap ((c_{R_i,j} - c_{\Omega})_+ > \frac{\lambda}{2})| \leq |R_i^+(\alpha) \cap (\|f\| > \frac{\lambda}{2})| = 0
\]

for every \( i, j \). Thus

\[
|R_i^+(\rho) \cap ((f - c_{\Omega})_+ > \lambda)| \leq 2A e^{-2^{-q}B(\lambda/\|f\|)^q} |R_0^+(\rho)|
\]
Proof

Let \( \mathcal{R} \subset \mathbb{R}^{n+1} \) be a space-time cylinder, \( 0 < \gamma < 1, -1 < \rho \leq \gamma, -\rho < \sigma \leq \gamma \) and \( 0 < q \leq r < \infty \). Then there exist constants \( c_1 = c_1(n, p, q, r, \gamma, \rho, \sigma) \) and \( c_2 = c_2(n, p, q, r, \gamma, \rho, \sigma) \) such that

\[
\| f \|_{\text{PBMO}_{\gamma, \delta, q}^+} \leq \| f \|_{\text{PBMO}_{\rho, \sigma, r}^+} \leq c_2 \| f \|_{\text{PBMO}_{\gamma, q}^+}.
\]

**Proof** Let \( R \) be a parabolic subrectangle of \( \Omega_T \). By Hölder’s inequality, we have

\[
\left( \int_{R^+} (f - c_R)^q + \int_{R^-} (f - c_R)^q \right)^{\frac{1}{q}} \leq \max\{1, 2^{\frac{1}{q} - 1}\} \left( \int_{R^+}^q (f - c_R)^\frac{q}{q} + \int_{R^-}^q (f - c_R)^\frac{q}{q} \right)^{\frac{1}{q}}
\]

\[
\leq \max\{1, 2^{\frac{1}{q} - 1}\} \left( \int_{R^+}^r (f - c_R)^\frac{r}{r} + \int_{R^-}^r (f - c_R)^\frac{r}{r} \right)^{\frac{1}{r}}
\]

\[
\leq c_0 \left( \int_{R^+}^\gamma (f - c_R)^\frac{r}{r} + \int_{R^-}^\gamma (f - c_R)^\frac{r}{r} \right)^{\frac{1}{r}},
\]

where \( c_0 = \max\{1, 2^{\frac{1}{q} - 1}\} \max\{1, 1^{\frac{1}{r}}\} \). We observe that

\[
\left( \int_{R^+} (f - c_R)^r + \int_{R^-} (f - c_R)^r \right)^{\frac{1}{r}} \leq \left( \frac{1 - \rho}{1 - \gamma} \int_{R^+} (f - c_R)^r + \frac{1 - \sigma}{1 - \gamma} \int_{R^-} (f - c_R)^r \right)^{\frac{1}{r}}.
\]
\[
\leq \left( \frac{1 - \min\{\rho, \sigma\}}{1 - \gamma} \right)^{\frac{1}{r}} \left( \int_{R^+(\rho)} (f - c_R)^r_+ + \int_{R^-(\sigma)} (f - c_R)^r_- \right)^{\frac{1}{r}}.
\]

By taking supremum over all parabolic subrectangles \( R \subset \Omega_T \), we arrive at

\[
\|f\|_{\text{PBMO}^+_{\rho,\sigma}(\Omega_T)} \leq c_0 \left( \frac{1 - \min\{\rho, \sigma\}}{1 - \gamma} \right)^{\frac{1}{r}} \|f\|_{\text{PBMO}^+_{\rho,\sigma,r}(\Omega_T)}.
\]

To show the second inequality, we make the restriction \( 0 < q \leq 1 \) so that we can apply Theorem 4.1. This is not an issue since after establishing the second inequality for \( 0 < q \leq 1 \) we can use the first inequality to obtain the whole range \( 0 < q \leq r \).

Cavalieri’s principle and Theorem 4.1 imply that

\[
\int_{R^+(\rho)} (f - c)^r_+ = \frac{r}{|R^+(\rho)|} \int_0^\infty \lambda^{r-1} |R^+(\rho) \cap \{(f - c)_+ > \lambda\}| \, d\lambda
\]

\[
\leq Ar \int_0^\infty \lambda^{r-1} e^{-B(\lambda/\|f\|_{\text{PBMO}^+_{\rho,\sigma,r}(R)})^q} \, d\lambda
\]

\[
= Ar \left( \frac{\|f\|_{\text{PBMO}^+_{\rho,\sigma,r}(R)}}{B^{\frac{1}{q}}} \right)^{r-1} \frac{\|f\|_{\text{PBMO}^+_{\rho,\sigma,r}(R)}}{B^{\frac{1}{q}}} \int_0^\infty s^{r-1} e^{-s} \, ds
\]

\[
= \frac{Ar}{B^{\frac{r}{q}}} \Gamma\left( \frac{r}{q} \right) \|f\|_{\text{PBMO}^+_{\rho,\sigma,r}(R)},
\]

where we applied a change of variables \( s = B^{\frac{q}{r}} \|f\|_{\text{PBMO}^+_{\rho,\sigma,r}(R)} \). Similarly, we obtain

\[
\int_{R^-(\sigma)} (f - c)^r_- \leq \frac{Ar}{B^{\frac{r}{q}}} \Gamma\left( \frac{r}{q} \right) \|f\|_{\text{PBMO}^+_{\rho,\sigma,r}(R)}.
\]

By adding up the two estimates above and taking supremum over all parabolic rectangles \( R \subset \Omega_T \), we conclude that

\[
\|f\|_{\text{PBMO}^+_{\rho,\sigma,r}(\Omega_T)} \leq c_2 \|f\|_{\text{PBMO}^+_{\rho,\sigma,r}(\Omega_T)} \quad \text{with} \quad c_2 = \left( \frac{2Ar}{B^{\frac{r}{q}}} \Gamma\left( \frac{r}{q} \right) \right)^{\frac{1}{r}}.
\]

\[\square\]

5 A global parabolic John–Nirenberg inequality

The results in the previous sections are local in the sense that they give estimates on a parabolic rectangle \( R \subset \Omega_T \). Next we discuss the corresponding estimates on the entire domain under the assumption that the domain \( \Omega \subset \mathbb{R}^n \) satisfies a quasihyperbolic boundary condition.
Definition 5.1 The quasihyperbolic metric in a domain $\Omega \subset \mathbb{R}^n$ is defined by setting, for any $x_1, x_2 \in \Omega$,

$$ k(x_1, x_2) = \inf_{\gamma_{x_1 x_2}} \int_{\gamma_{x_1 x_2}} \frac{1}{d(x, \partial\Omega)} \, ds(x), $$

where the infimum is taken over all rectifiable paths $\gamma_{x_1 x_2}$ in $\Omega$ connecting $x_1$ to $x_2$.

Definition 5.2 A domain $\Omega$ is said to satisfy the quasihyperbolic boundary condition if there exist $x_0 \in \Omega$ and constants $c_1$ and $c_2$ such that

$$ k(x_0, x) \leq c_1 \log \frac{1}{d(x, \partial\Omega)} + c_2 $$

for every $x \in \Omega$.

The class of the domains satisfying the quasihyperbolic boundary condition was first introduced in [7]. Note that a domain satisfying the quasihyperbolic boundary condition is bounded. In [22], a parabolic John–Nirenberg lemma was proven for domains satisfying the quasihyperbolic boundary condition. We state it here in its complete form.

Theorem 5.3 Assume that $\Omega \subset \mathbb{R}^n$ satisfies the quasihyperbolic boundary condition. Let $0 < \gamma < 1, 0 < q \leq 1$ and $0 < \tau_1 < \tau_2 < T$. Assume that $f \in \text{PBMO}^+_{\gamma,q}(\Omega_T)$ and let $\|f\| = \|f\|_{\text{PBMO}^+_{\gamma,q}(\Omega_T)}$. Then there exist constants $c \in \mathbb{R}, A = A(n, p, q, \gamma, \Omega, \tau_1, \tau_2)$ and $B = B(n, p, q, \gamma, \Omega, \tau_1, \tau_2)$ such that

$$ |\Omega \times (\tau_2, T) \cap \{(f - c)_+ > \lambda\}| \leq Ae^{-B(\lambda/\|f\|)^q} |\Omega \times (\tau_2, T)| $$

and

$$ |\Omega \times (0, \tau_1) \cap \{(f - c)_- > \lambda\}| \leq Ae^{-B(\lambda/\|f\|)^q} |\Omega \times (0, \tau_1)| $$

for every $\lambda > 0$.

As a corollary of Theorem 5.3, the positive part of a PBMO$^+$ function is exponentially integrable on upper parts of space-time cylinders and the negative part on lower parts.

Corollary 5.4 Assume that $\Omega \subset \mathbb{R}^n$ satisfies the quasihyperbolic boundary condition and let $0 < \tau_1 < \tau_2 < T$. Assume that $f \in \text{PBMO}^+(\Omega_T)$ and let $\|f\| = \|f\|_{\text{PBMO}^+(\Omega_T)}$. If $0 < \delta < B/\|f\|$, then

$$ \int_{\Omega \times (\tau_2, T)} e^{\delta(f - c)_+} < \infty \quad \text{and} \quad \int_{\Omega \times (0, \tau_1)} e^{\delta(f - c)_-} < \infty, $$

where $B$ and $c$ are the constants from Theorem 5.3.
Proof Let $E = \Omega \times (\tau_2, T)$. Cavalieri’s principle and Theorem 5.3 with $q = 1$ imply

\[
\int_E e^{\delta(f-c)_+} = \int_0^\infty |E \cap \{e^{\delta(f-c)_+} > \lambda\}| d\lambda \\
= \int_0^1 |E \cap \{e^{\delta(f-c)_+} > \lambda\}| d\lambda + \int_1^\infty |E \cap \{e^{\delta(f-c)_+} > \lambda\}| d\lambda.
\]

Here we applied a change of variables $\lambda = e^s$. The second inequality follows in a similar way.

The following theorem gives the characterization of the domains on which the parabolic John–Nirenberg lemma holds globally.

Theorem 5.5 Let $0 < \tau_1 < \tau_2 < T$. A domain $\Omega$ satisfies the quasihyperbolic boundary condition if and only if there exist $\delta > 0$ and $c \in \mathbb{R}$ such that

\[
\int_{\Omega \times (\tau_2, T)} e^{\delta(f-c)/\|f\|} \leq 2 \text{ and } \int_{\Omega \times (0, \tau_1)} e^{\delta(f-c)/\|f\|} \leq 2
\]

for every $f \in PBMO^+(\Omega_T)$ with $\|f\| = \|f\|_{PBMO^+(\Omega_T)}$.

Proof One direction follows from the proof of Corollary 5.4 by choosing $0 < \delta \leq \frac{B}{1 + A \|f\|}$. For the other direction, we consider $f(x, t) = k(x_0, x)$, where $k(x_0, x)$ denotes the quasihyperbolic metric in $\Omega$. We note that

\[
\|f\|_{PBMO^+(\Omega_T)} \leq \|f\|_{BMO(\Omega)} \leq 2 \|f\|_{PBMO^+(\Omega_T)}.
\]

By [24, Theorem A], we conclude that $f \in BMO(\Omega)$. Thus, we have $f \in PBMO^+(\Omega_T)$ and the parabolic John–Nirenberg inequality of the claim applies for $f$. Set $\tilde{\delta} = \delta/4$. By applying Jensen’s inequality twice, Young’s inequality and the parabolic John–Nirenberg inequality, we obtain

\[
\int_{\Omega} e^{\tilde{\delta}|f - f_\Omega|/\|f\|} \leq \int_{\Omega} e^{\tilde{\delta}|f - f_\Omega|} \leq \int_{\Omega} e^{\tilde{\delta}|f - c|} \leq \left(\int_{\Omega} e^{\tilde{\delta}|f - c|} \right)^2 \leq \int_{\Omega} e^{2\tilde{\delta}|f - c|} \leq \int_{\Omega} e^{2\tilde{\delta}(f-c)_+} e^{2\tilde{\delta}(f-c)_-} \leq \int_{\Omega} e^{2\tilde{\delta}(f-c)_+} \leq \int_{\Omega} e^{2\tilde{\delta}(f-c)_-} \leq \int_{\Omega} e^{2\tilde{\delta}|f - f_\Omega|}.
\]
In general, the maximal s-median has the following properties.

Let $0 < s \leq 1$, then $m^s_f(E) = \inf \{a \in \mathbb{R} : |\{x \in E : f(x) > a\}| < s |E|\}$.

The maximal $s$-median of a function is an $s$-median [21]. In the next lemma, we list the basic properties of the maximal $s$-median. We refer to [20, 21] for the proofs of the properties.

**Definition 6.1** Let $0 < s \leq 1$. Assume that $E \subset \mathbb{R}^{n+1}$ is a measurable set with $0 < |E| < \infty$ and that $f : E \to [-\infty, \infty]$ is a measurable function. The maximal $s$-median of $f$ over $E$, if

$$|\{x \in E : f(x) > a\}| \leq s |E|$$

and

$$|\{x \in E : f(x) < a\}| \leq (1-s) |E|.$$  

In general, the $s$-median is not unique. To obtain a uniquely defined notion, we consider the maximal $s$-median as in [21].

**Lemma 6.2** Let $0 < s \leq 1$. Assume that $E \subset \mathbb{R}^{n+1}$ is a measurable set with $0 < |E| < \infty$ and that $f, g : E \to [-\infty, \infty]$ are measurable functions. The maximal $s$-median has the following properties.

(i) $m^s_f(E) \leq m^s_f(E)$ for any $0 < s \leq s' \leq 1$.

(ii) $m^s_f(E) \leq m^s_g(E)$ whenever $f \leq g$ almost everywhere in $E$.

(iii) If $E \subset E'$ and $|E'| \leq c |E|$ with some $c \geq 1$, then $m^s_f(E) \leq m^{s/c}_f(E')$.

(iv) $m^s_{\varphi f}(E) = \varphi(m^s_f(E))$ for an increasing continuous function $\varphi : f(E) \to [-\infty, \infty]$.

(v) $m^s_f(E) + c = m^s_{f+c}(E)$ for any $c \in \mathbb{R}$.

(vi) $m^s_{Ef}(E) = cm^s_{Ef}(E)$ for any $c > 0$.

(vii) $|m^s_f(E)| \leq m^{s}_{[f]}(E)$.

(viii) $m^s_{f+g}(E) \leq m^{t_1}_f(E) + m^{t_2}_g(E)$ whenever $t_1 + t_2 \leq s$.

By [24, Theorem A], the domain $\Omega$ satisfies the quasihyperbolic boundary condition.  

\[ \square \]

6 Parabolic BMO with medians

This section discusses John–Nirenberg inequalities for the median-type parabolic BMO. In many cases, it is preferable to consider medians instead of integral averages. Let $0 < s \leq 1$. Assume that $E \subset \mathbb{R}^{n+1}$ is a measurable set with $0 < |E| < \infty$ and that $f : E \to [-\infty, \infty]$ is a measurable function. A number $a \in \mathbb{R}$ is called an $s$-median of $f$ over $E$, if

$$|\{x \in E : f(x) > a\}| \leq s |E|$$

and

$$|\{x \in E : f(x) < a\}| \leq (1-s) |E|.$$  

In general, the $s$-median is not unique. To obtain a uniquely defined notion, we consider the maximal $s$-median as in [21].

**Definition 6.1** Let $0 < s \leq 1$. Assume that $E \subset \mathbb{R}^{n+1}$ is a measurable set with $0 < |E| < \infty$ and that $f : E \to [-\infty, \infty]$ is a measurable function. The maximal $s$-median of $f$ over $E$, if

$$|\{x \in E : f(x) > a\}| \leq s |E|$$

and

$$|\{x \in E : f(x) < a\}| \leq (1-s) |E|.$$  

In general, the $s$-median is not unique. To obtain a uniquely defined notion, we consider the maximal $s$-median as in [21].

**Lemma 6.2** Let $0 < s \leq 1$. Assume that $E \subset \mathbb{R}^{n+1}$ is a measurable set with $0 < |E| < \infty$ and that $f, g : E \to [-\infty, \infty]$ are measurable functions. The maximal $s$-median has the following properties.

(i) $m^s_f(E) \leq m^s_f(E)$ for any $0 < s \leq s' \leq 1$.

(ii) $m^s_f(E) \leq m^s_g(E)$ whenever $f \leq g$ almost everywhere in $E$.

(iii) If $E \subset E'$ and $|E'| \leq c |E|$ with some $c \geq 1$, then $m^s_f(E) \leq m^{s/c}_f(E')$.

(iv) $m^s_{\varphi f}(E) = \varphi(m^s_f(E))$ for an increasing continuous function $\varphi : f(E) \to [-\infty, \infty]$.

(v) $m^s_f(E) + c = m^s_{f+c}(E)$ for any $c \in \mathbb{R}$.

(vi) $m^s_{Ef}(E) = cm^s_{Ef}(E)$ for any $c > 0$.

(vii) $|m^s_f(E)| \leq m^{s}_{[f]}(E)$.

(viii) $m^s_{f+g}(E) \leq m^{t_1}_f(E) + m^{t_2}_g(E)$ whenever $t_1 + t_2 \leq s$.  

\[ \square \]
(ix) For any \( f \in L^p(E) \) with \( p > 0 \),

\[
m^s f(E) \leq \left( s^{-1} \int_E |f|^p \right)^{\frac{1}{p}}.
\]

(x) If \( E_i, i \in \mathbb{N} \), are pairwise disjoint measurable sets, then

\[
\inf_i m^s f(E_i) \leq m^s f(\bigcup_{i=1}^{\infty} E_i) \leq \sup_i m^s f(E_i).
\]

Lemma 2.3 can be combined with the proof of the Lebesgue differentiation theorem for medians in [21] to obtain the following lemma.

**Lemma 6.3** Let \( f : \mathbb{R}^{n+1} \to [-\infty, \infty] \) be a measurable function which is finite almost everywhere in \( \mathbb{R}^{n+1} \) and \( 0 < s \leq 1 \). Then

\[
\lim_{i \to \infty} m^s f(A_i) = f(x, t)
\]

for almost every \((x, t) \in \mathbb{R}^{n+1}\), whenever \((A_i)_{i \in \mathbb{N}}\) is a sequence of measurable sets converging regularly to \((x, t)\).

**Definition 6.4** Let \( \Omega \subset \mathbb{R}^n \) be a domain and \( T > 0 \). Given \( 0 \leq \gamma < 1 \) and \( 0 < s \leq 1 \), we say that a measurable function \( f : \Omega T \to [-\infty, \infty] \) belongs to the median-type parabolic BMO, denoted by \( \text{PBMO}^+_{\gamma, 0, s}(\Omega T) \), if

\[
\| f \|_{\text{PBMO}^+_{\gamma, 0, s}(\Omega T)} = \sup_{R \subset \Omega T} \inf_{c \in \mathbb{R}} \left( m^s_{(f-c)+}(R^+(\gamma)) + m^s_{(f-c)-}(R^-(\gamma)) \right) < \infty.
\]

If the condition above holds with the time axis reversed, then \( f \in \text{PBMO}^-_{\gamma, 0, s}(\Omega T) \).

The next lemma is a counterpart of Lemma 2.7. The proof is similar to that of Lemma 2.7 and thus is omitted here.

**Lemma 6.5** Let \( \Omega T \subset \mathbb{R}^{n+1} \) be a space-time cylinder, \( 0 \leq \gamma < 1 \) and \( 0 < s \leq 1 \). Assume that \( f : \Omega T \to [-\infty, \infty] \) is a measurable function. Then for every parabolic rectangle \( R \subset \Omega T \), there exists a constant \( c_R \in \mathbb{R} \), that may depend on \( R \), such that

\[
m^s_{(f-c_R)+}(R^+(\gamma)) + m^s_{(f-c_R)-}(R^-(\gamma)) = \inf_{c \in \mathbb{R}} \left( m^s_{(f-c)+}(R^+(\gamma)) + m^s_{(f-c)-}(R^-(\gamma)) \right).
\]

In particular,

\[
\sup_{R \subset \Omega T} \left( m^s_{(f-c_R)+}(R^+(\gamma)) + m^s_{(f-c_R)-}(R^-(\gamma)) \right) = \| f \|_{\text{PBMO}^+_{\gamma, 0, s}(\Omega T)}.
\]

The following John–Nirenberg lemma is a counterpart of Corollary 3.2. We apply the same decomposition argument as in the proof of Theorem 3.1.
Theorem 6.6 Let $R \subset \mathbb{R}_{n+1}$ be a parabolic rectangle, $0 \leq \gamma < 1$, $\gamma < \alpha < 1$ and $0 < s \leq s_0$, where $s_0$ is a small positive number. Assume that $f \in \text{PBMO}_{\gamma, 0, s}^+(R)$ and let $\|f\| = \|f\|_{\text{PBMO}_{\gamma, 0, s}^+(R)}$. Then there exist constants $c_R \in \mathbb{R}$, $A = A(n, p, \gamma, \alpha)$ and $B = B(n, p, \gamma, \alpha)$ such that

$$|R^+(\alpha) \cap \{(f - c_R)_+ > \lambda\}| \leq Ae^{-B\lambda/\|f\|} |R^+(\alpha)|$$

and

$$|R^-(\alpha) \cap \{(f - c_R)_- > \lambda\}| \leq Ae^{-B\lambda/\|f\|} |R^-(\alpha)|$$

for every $\lambda > 0$.

Proof We use the same notation as in the proof of Theorem 3.1 until Eq. (3.8) with the exception that we assume $\|f\| = 1$. We proceed from there. It holds that

$$|S(\lambda)| = \left| \bigcup_i S_i^+ \right| \leq c_1 \sum_j |\tilde{S}_j^-|,$$  

(6.1)

where $c_1 = 3(7 + \alpha)/(1 - \alpha)$. Take $\lambda > \delta > 0$ and form $\{S_k^+\}_k$ for $\delta$. Each $S_i^+$ is contained in a unique $S_k^+$. Set $J_k = \{j : \tilde{S}_j^+ \subset S_k^+\}$ and define

$$\mathcal{K} = \left\{ k \in \mathbb{N} : |S_k^+| \leq 2c_1 \sum_{j \in J_k} |\tilde{S}_j^-| \right\}.$$

By using properties (viii) and (iii) of Lemma 6.2 together with (3.5), we obtain

$$\lambda < c_{R_j} = m_{c_{R_j}}^1(\tilde{S}_j^-) \leq m_{(f - c_{R_j})_-}^{2s(1-\gamma)/(1-\alpha)}(\tilde{S}_j^-) + m_{f_+}^r(\tilde{S}_j^-) \leq m_{(f - c_{R_j})_-}^r(R_j^-(\gamma)) + m_{f_+}^r(\tilde{S}_j^-) \leq \|f\| + m_{f_+}^r(\tilde{S}_j^-) = 1 + m_{f_+}^r(\tilde{S}_j^-)$$

for every $\tilde{S}_j^-$, where $r = 1 - 2s(1-\gamma)/(1-\alpha)$. Since $\tilde{S}_j^-$ are pairwise disjoint for $j \in J_k$, property (x) of Lemma 6.2 implies that

$$\lambda - 1 \leq m_{f_+}^r\left( \bigcup_{j \in J_k} \tilde{S}_j^- \right)$$

for every $k \in \mathbb{N}$.

Fix $k \in \mathcal{K}$. We have $\tilde{S}_j^+ \subset S_k^+$ for all $j \in J_k$, where $S_k^+$ was obtained by subdividing a previous $S_k^-$ for which $a_{R_k^-} \leq \delta$. Hence, it holds that $\tilde{S}_j^- \subset R_k$ for all $j \in J_k$. By (3.3), it follows that $\tilde{S}_j^- \subset R_k^-(\gamma)$ for every $j \in J_k$ and thus also $\bigcup_{j \in J_k} \tilde{S}_j^- \subset R_k^-(\gamma)$. \(\square\) Springer
Using (3.4) and (3.5), we get
\[
|R_k^-(\gamma)| \leq 2 \frac{1 - \gamma}{1 - \alpha} |S_k^+| \leq 2 \frac{1 - \gamma}{1 - \alpha} 2^{nm} |2^{pm}| |S_k^+| \leq c_2 |S_k^+| \leq 2c_1 c_2 \left| \bigcup_{j \in \mathcal{J}_k} \tilde{S}_j^- \right| \tag{6.2}
\]
for every \(k \in \mathcal{K}\), where
\[
c_2 = \frac{1 - \gamma}{1 - \alpha} 2^{2n + p} \left( \frac{3 + \alpha}{2(\alpha - \gamma)} \right)^{1 + \frac{n}{p}}.
\]

By applying (iii), (ii) and (v) of Lemma 6.2, we have
\[
\lambda - 1 \leq m_{f+}^r \left( \bigcup_{j \in \mathcal{J}_k} \tilde{S}_j^- \right) \leq m_{f+}^{r/\tilde{c}} (R_k^+(\gamma)) \leq m_{(f - aR_k^-)_+}^{r/\tilde{c}} (R_k^+(\gamma)) + \delta \leq 1 + \delta \tag{6.3}
\]
for \(k \in \mathcal{K}\), where \(\tilde{c} = 2c_1 c_2\) and whenever \(s \leq r/\tilde{c}\). By combining estimates (6.2) and (6.3), we obtain
\[
(\lambda - \delta - 1) \sum_{j \in \mathcal{J}_k} |\tilde{S}_j^-| \leq |R_k^+(\gamma)| \leq c_2 |S_k^+|
\]
for \(k \in \mathcal{K}\). Thus, whenever \(\lambda > \delta + 1\), we have
\[
\sum_{k \in \mathcal{K}} \sum_{j \in \mathcal{J}_k} |\tilde{S}_j^-| \leq \frac{c_2}{\lambda - \delta - 1} \sum_{k \in \mathcal{K}} |S_k^+|.
\]

On the other hand, if \(k \notin \mathcal{K}\), we have
\[
\sum_{j \in \mathcal{J}_k} |\tilde{S}_j^-| \leq \frac{1}{2c_1} |S_k^+|,
\]
which implies
\[
\sum_{k \notin \mathcal{K}} \sum_{j \in \mathcal{J}_k} |\tilde{S}_j^-| \leq \frac{1}{2c_1} \sum_{k \notin \mathcal{K}} |S_k^+|.
\]

By combining the cases \(k \in \mathcal{K}\) and \(k \notin \mathcal{K}\), we obtain
\[
\sum_j |\tilde{S}_j^-| = \sum_k \sum_{j \in \mathcal{J}_k} |\tilde{S}_j^-| \leq \frac{c_2}{\lambda - \delta - 1} \sum_{k \in \mathcal{K}} |S_k^+| + \frac{1}{2c_1} \sum_{k \notin \mathcal{K}} |S_k^+|,
\]
whenever $\lambda > \delta + 1$. Applying (6.1), we arrive at

$$|S(\lambda)| \leq \frac{c_1 c_2}{\lambda - \delta - 1} \sum_{k \in K} |S_k^+| + \frac{1}{2} \sum_{k \notin K} |S_k^+|,$$

whenever $\lambda > \delta + 1$. Set $a = 2c_1 c_2 + 1$ and replace $\lambda$ and $\delta$ by $\lambda + a$ and $\lambda$, respectively. We have

$$|S(\lambda + a)| \leq \frac{1}{2} \sum_{k \in K} |S_k^+| + \frac{1}{2} \sum_{k \notin K} |S_k^+| = \frac{1}{2} \sum_k |S_k^+| \leq \frac{1}{2} |S(\lambda)|. \quad (6.4)$$

Assume that $\lambda \geq a$. Then there exists an integer $N \in \mathbb{Z}_+$ such that $Na \leq \lambda < (N + 1)a$. A recursive application of (6.4) gives

$$|S(\lambda)| \leq |S(Na)| \leq \frac{1}{2N-1} |S(a)| \leq 2^{-\frac{\lambda - a}{a} + 2} |S_0^+| = 4e^{-\frac{\lambda}{2c_1 c_2 + 1}} \log 2 |S_0^+|.$$ 

Hence, we have

$$|S(\lambda)| \leq A e^{-2B \lambda/\|f\|} |S_0^+|$$

for $\lambda \geq a$, where $A = 4$ and $B = \frac{1}{2} \log 2/(2c_1 c_2 + 1)$.

If $(x, t) \in S_0^+ \setminus S(\lambda)$, then there exists a sequence $\{S_l^+\}_{l \in \mathbb{N}}$ of subrectangles containing $(x, t)$ such that $c_{R_l} \leq \lambda$ and $|S_l^+| \to 0$ as $l \to \infty$. By (ii) and (v) of Lemma 6.2, we have

$$m_{f_+}^{2s(1-\gamma)/(1-\alpha)}(S_l^+) \leq m_{(f - c_{R_l})_+}^{2s(1-\gamma)/(1-\alpha)}(S_l^+) + \lambda \leq 1 + \lambda.$$ 

Lemma 6.3 then further implies that $f_+(x, t) \leq 1 + \lambda$ for almost every $(x, t) \in S_0^+ \setminus S(\lambda)$. It follows that

$$\{(x, t) \in S_0^+: f_+(x, t) > 1 + \lambda\} \subset S(\lambda)$$

up to a set of measure zero. For $\lambda \geq 2$, we have $\lambda \geq 1 + \frac{\lambda}{2}$. We conclude that

$$|S_0^+ \cap \{f_+ > \lambda\}| \leq |S_0^+ \cap \{f_+ > 1 + \frac{\lambda}{2}\}| \leq |S(\frac{\lambda}{2})| \leq A e^{-B \lambda/\|f\|} |S_0^+|$$

for every $\lambda \geq 2a$. If $0 < \lambda < 2a$, the claim follows from the estimate

$$|S_0^+ \cap \{f_+ > \lambda\}| \leq e^{1} e^{-1} |S_0^+| \leq e^{1} e^{-\frac{1}{2c_1 c_2 + 1}} \log 2 |S_0^+|.$$ 

Finally, we discuss the restriction on the median level parameter $s$, Springer
\[ s \leq \frac{r}{c} = \frac{1 - 2^{1-\gamma}s}{\tilde{c}} , \quad \text{that is,} \quad s \leq \frac{1}{\tilde{c} + 2^{1-\gamma}} = s_0. \]

The proof is complete. \qed

The following John–Nirenberg inequality is an analogy of Theorem 4.1 and its proof uses the same chaining argument.

**Theorem 6.7** Let \( R \subset \mathbb{R}^{n+1} \) be a parabolic rectangle, \( 0 < \gamma < 1, -1 < \rho \leq \gamma, -\sigma < \rho \leq \gamma \) and \( 0 < s \leq s_0 \). Assume that \( f \in \text{PBMO}_{\rho,0,s}^+(R) \) and let \( \|f\| = \|f\|_{\text{PBMO}_{\rho,0,s}^+(R)} \). Then there exist constants \( c \in \mathbb{R}, A = A(n, p, \gamma, \rho, \sigma) \) and \( B = B(n, p, \gamma, \rho, \sigma) \) such that

\[
|R^+(\rho) \cap \{(f - c)_+ > \lambda\}| \leq A e^{-B(\lambda/\|f\|^q)} |R^+(\rho)|
\]

and

\[
|R^-(\sigma) \cap \{(f - c)_- > \lambda\}| \leq A e^{-B(\lambda/\|f\|^q)} |R^-(\sigma)|
\]

for every \( \lambda > 0 \).

**Proof** We use the same notation as in the proof of Theorem 4.1 until the point where \((c_{R_i,j} - c_{\Omega})_+\) is estimated. By (v), (viii), (iii) and (i) of Lemma 6.2 in this order, we obtain

\[
(c_{R_i,j} - c_{\Omega})_+ = (c_{P_0} - c_{P_{N+M_j}})_+ \leq \sum_{k=1}^{N+M_j} (c_{P_{k-1}} - c_{P_k})_+
\]

\[
\leq \sum_{k=1}^{N+M_j} m^1_{(c_{P_{k-1}} - c_{P_k})_+} (P^-_{k-1} \cap P^+_k)
\]

\[
\leq \sum_{k=1}^{N+M_j} \left( m^1_{(c_{P_{k-1}} - f)_+} (P^-_{k-1} \cap P^+_k) + m^{1/2}_{(f - c_{P_k})_+} (P^-_{k-1} \cap P^+_k) \right)
\]

\[
\leq \sum_{k=1}^{N+M_j} \left( \tilde{m}^{1/2}_{(f - c_{P_{k-1}})_-} (P^-_{k-1}) + \tilde{m}^{1/2}_{(f - c_{P_k})_+} (P^+_k) \right)
\]

\[
\leq \sum_{k=0}^{N+M_j} \left( m^{1/2^{n+2}}_{(f - c_{P_k})_-} (P^-_k) + m^{1/2^{n+2}}_{(f - c_{P_k})_+} (P^+_k) \right)
\]

\[
\leq (N + 1 + M_j) \|f\|,
\]

whenever \( s \leq 1/2^{n+2} \). This is satisfied by the assumption \( s \leq s_0 \). We observe that

\[
N + 1 + M_j = \frac{L}{l} + M_j \leq 2^m + (j - 1) \frac{1 - \alpha}{1 + \alpha}
\]
\[
\leq 2^m + \frac{(1-\rho)2^{mp}}{1-\alpha} \leq 2^m + 2^{mp+1} \leq 2^{mp+2}
\]

\[
\leq 2^{\frac{2p}{p-1} + 3p + 2} \left( \frac{1+\alpha}{\rho+\sigma} \right)^{\frac{p}{p-1}} \left( \frac{1+\alpha}{1-\alpha} \right)^p = C
\]

for every \( j \). Hence, it holds that \( (c_{R_{i,j}} - c_{\Omega_j})_+ \leq C \| f \|. \) The proof now proceeds in the exactly same way as in Theorem 4.1 except applying Theorem 6.6 instead of Corollary 3.2. Thus, we can stop here. \( \square \)

As a corollary of Theorem 6.7, the median-type parabolic BMO coincides with the classical integral-type parabolic BMO, compare with Corollary 4.2. In particular, it follows that all results for the integral-type parabolic BMO also hold for the median-type parabolic BMO and vice versa.

**Corollary 6.8** Let \( \Omega_T \subset \mathbb{R}^{n+1} \) be a space-time cylinder, \( 0 < \gamma < 1 \), \( 0 < q < \infty \), \( 0 < \rho < 1 \) and \( 0 < s \leq s_0 \). Then there exist constants \( c_1 = c_1(n, p, q, \gamma, \rho, s) \) and \( c_2 = c_2(n, p, q, \gamma, \rho, s) \) such that

\[
c_1 \| f \|_{\text{PBMO}^{+}_{\gamma,0,s}(\Omega_T)} \leq \| f \|_{\text{PBMO}^{+}_{\rho,q}(\Omega_T)} \leq c_2 \| f \|_{\text{PBMO}^{+}_{\gamma,0,s}(\Omega_T)}.
\]

**Proof** Fix \( 0 < \gamma < 1 \) and assume that \( 0 < \rho \leq \gamma \). Let \( R \) be a parabolic subrectangle of \( \Omega_T \). By Lemma 6.2 (ix), we have

\[
m_{(f-c_R)_+}^{s}(R^+(\gamma)) + m_{(f-c_R)_-}^{s}(R^-(\gamma))
\]

\[
\leq \left( s^{-1} \int_{R^+(\gamma)} (f-c_R)^q_+ \right)^{\frac{1}{q}} + \left( s^{-1} \int_{R^-(\gamma)} (f-c_R)^q_- \right)^{\frac{1}{q}}
\]

\[
\leq c_0 \left( \int_{R^+(\gamma)} (f-c_R)^q_+ + \int_{R^-(\gamma)} (f-c_R)^q_- \right)^{\frac{1}{q}},
\]

where \( c_0 = s^{-\frac{1}{q}} \max \{ 1, 2^{1-\frac{1}{q}} \} \). We observe that

\[
\left( \int_{R^+(\gamma)} (f-c_R)^q_+ + \int_{R^-(\gamma)} (f-c_R)^q_- \right)^{\frac{1}{q}}
\]

\[
\leq \left( \frac{1-\rho}{1-\gamma} \right)^{\frac{1}{q}} \left( \int_{R^+(\gamma)} (f-c_R)^q_+ + \int_{R^-(\gamma)} (f-c_R)^q_- \right)^{\frac{1}{q}}.
\]

Taking supremum over all parabolic subrectangles \( R \subset \Omega_T \), we obtain

\[
\| f \|_{\text{PBMO}^{+}_{\gamma,0,s}(\Omega_T)} \leq c_0 \left( \frac{1-\rho}{1-\gamma} \right)^{\frac{1}{q}} \| f \|_{\text{PBMO}^{+}_{\rho,q}(\Omega_T)}
\]

for \( 0 < \rho \leq \gamma \).
For the second inequality, Cavalieri’s principle and Theorem 6.7 imply that

\[
\int_{R^+(\rho)} (f - c)^q_+ = \frac{q}{|R^+(\rho)|} \int_0^\infty \lambda^{q-1} |R^+(\rho) \cap \{(f - c)_+ > \lambda\}| d\lambda \\
\leq A q \int_0^\infty \lambda^{q-1} e^{-B\lambda/\|f\|_{\text{PBMO}^+_{\gamma,0,s}(R)}} d\lambda \\
= A q \left( \frac{\|f\|_{\text{PBMO}^+_{\gamma,0,s}(R)}}{B} \right)^{q-1} \frac{\|f\|_{\text{PBMO}^+_{\gamma,0,s}(R)}}{B} \int_0^\infty s^{q-1} e^{-s} d\lambda \\
= \frac{A q}{B^q} \Gamma(q) \|f\|^q_{\text{PBMO}^+_{\gamma,0,s}(R)},
\]

where we made a change of variables \( s = B\lambda/\|f\|_{\text{PBMO}^+_{\gamma,0,s}(R)} \). Similarly, we obtain

\[
\int_{R^-(\rho)} (f - c)^q_- \leq \frac{A q}{B^q} \Gamma(q) \|f\|^q_{\text{PBMO}^+_{\gamma,0,s}(R)}.
\]

By summing the two estimates above and taking supremum over all parabolic rectangles \( R \subset \Omega \), we get

\[
\|f\|_{\text{PBMO}^+_{\rho,q}(\Omega)} \leq c_2 \|f\|_{\text{PBMO}^+_{\gamma,0,s}(\Omega)} \quad \text{with} \quad c_2 = \left( \frac{2 A q}{B^q} \Gamma(q) \right)^{\frac{1}{q}}
\]

for \( 0 < \rho \leq \gamma \). Applying Corollary 4.2, we obtain the claim for the whole range \( 0 < \rho < 1 \). \( \Box \)

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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