Multi-objective Contextual Multi-armed Bandit Problem with a Dominant Objective

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Abstract—In this paper, we propose a new multi-objective contextual multi-armed bandit problem with two objectives, where one of the objectives dominates the other objective. Unlike single-objective bandit problems in which the learner obtains a random scalar reward for each arm it selects, in the proposed problem, the learner obtains a random reward vector, where each component of the reward vector corresponds to one of the objectives and the distribution of the reward depends on the context that is provided to the learner at the beginning of each round. We call this problem contextual multi-armed bandit with a dominant objective (CMAB-DO). In CMAB-DO, the goal of the learner is to maximize its total reward in the non-dominant objective while ensuring that it maximizes its total reward in the dominant objective. In this case, the optimal arm given a context is the one that maximizes the expected reward in the non-dominant objective among all arms that maximize the expected reward in the dominant objective. First, we show that the optimal arm lies in the Pareto front. Then, we propose the multi-objective contextual multi-armed bandit algorithm (MOC-MAB), and define two performance measures: the 2-dimensional (2D) regret and the Pareto regret. We show that both the 2D regret and the Pareto regret of MOC-MAB are sublinear in the number of rounds. We also compare the performance of the proposed algorithm with other state-of-the-art methods in synthetic and real-world datasets. The proposed model and the algorithm have a wide range of real-world applications that involve multiple and possibly conflicting objectives ranging from wireless communication to medical diagnosis and recommender systems.

Index Terms—Online learning, contextual bandits, multi-objective bandits, dominant objective, multi-dimensional regret, Pareto regret.

I. INTRODUCTION

With the rapid increase in the generation speed of the streaming data, online learning methods are becoming increasingly valuable for sequential decision making problems. Many of these problems, including recommender systems [1], [2], medical screening [3], cognitive radio networks [4], [5] and wireless network monitoring [6] may involve multiple and possibly conflicting objectives. For this problem, we propose a multi-objective contextual bandit problem with dominant and non-dominant objectives. For this problem, we construct a multi-objective contextual bandit algorithm named MOC-MAB, which maximizes long-term reward of the non-dominant objective conditioned on the fact that it maximizes the long-term reward of the dominant objective.

In this problem, the learner observes a multi-dimensional context in the beginning of each round. Then, it selects one of the available arms and receives a random reward vector, which is drawn from a fixed distribution that depends on the context and the selected arm. No statistical assumptions are made on the way the contexts arrive, and the learner does not have any a priori information on the reward distributions. The optimal arm is defined as the one that maximizes the expected reward of the non-dominant objective among all arms that maximize the expected reward of the dominant objective given the context.

The learner’s performance is measured in terms of its regret, which measures the loss that the learner accumulates due to not knowing the reward distributions beforehand. We introduce two new notions of regret: the 2D regret and the Pareto regret. The 2D regret is a vector whose $i$th component corresponds to the difference between the expected total reward of an oracle in objective $i$ that selects optimal arm for each context and that of the learner by time $T$. On the other hand, the Pareto regret measures sum of the distances of the arms selected by the learner to the Pareto front. For this, we extend the Pareto regret proposed in [7] to take into account the dependence of the Pareto front on the context.

We prove that MOC-MAB achieves $\tilde{O}(T^{(2\alpha+d)/(3\alpha+d)})$ 2D regret, where $d$ is the dimension of the context and $\alpha$ is a constant that depends on the similarity information that relates the distances between contexts to the distances between expected rewards of an arm. This shows that MOC-MAB is average-reward optimal in the limit $T \to \infty$ in both objectives. We also show that the optimal arm lies in the Pareto front, and MOC-MAB also achieves $\tilde{O}(T^{(2\alpha+d)/(3\alpha+d)})$ Pareto regret. Then, we argue that it is possible to make the Pareto regret of MOC-MAB $\tilde{O}(T^{(\alpha+d)/(2\alpha+d)})$ by adjusting its parameters, such that the Pareto regret becomes order optimal up to a logarithmic factor [8], but this comes at an expense of making the regret in the non-dominant objective of MOC-MAB linear in the number of rounds.

To the best of our knowledge, our work is the first to formulate a contextual multi-objective bandit problem and prove sublinear bounds on the 2D regret and the Pareto regret. In addition, we also evaluate the performance of MOC-MAB through simulations and compare it with other single and multi-objective bandit algorithms and various offline methods. Our results show that MOC-MAB outperforms its competitors, which are not specifically designed to deal with problems involving dominant and non-dominant objectives.

The rest of the paper is organized as follows. Related work is given in Section II. Problem formulation, definitions of the 2D regret and the Pareto regret, and possible applications of
CMAB-DO are given in Section [IV] MOC-MAB is introduced in Section [V] and its regrets are analyzed in Section [V]. An extension of MOC-MAB that deals with dynamically changing reward distributions is proposed in Section [VI]. Numerical results are presented in Section [VII] and concluding remarks are provided in Section [VIII].

II. RELATED WORK

In the past decade, many variants of the classical multi-armed problem have been introduced (see [11] for a comprehensive discussion). Two notable examples are contextual bandits [9], [12], [13] and multi-objective bandits [7]. While these examples have been studied separately in prior works, in this paper we aim to fuse contextual bandits and multi-objective bandits together. Below, we discuss the related work on the classical multi-armed bandit problem, contextual bandits and multi-objective bandits. The differences between our work and related works are summarized in Table I.

A. The Classical Multi-armed Bandit Problem

The classical multi-armed bandit problem involves $K$ arms with unknown reward distributions. The learner sequentially selects arms and observes noisy reward samples from the selected arms. The goal of the learner is to use the knowledge it obtains through these observations to maximize its long-term reward. For this, the learner needs to identify arms with high rewards without wasting too much time on arms with low rewards. In conclusion, it needs to strike the balance between exploration and exploitation.

A thorough technical analysis of the classical multi-armed bandit problem is given in [14], where it is shown that $O(\log T)$ regret is achieved asymptotically by index policies that use upper confidence bounds (UCBs) for the rewards. This result is tight in the sense that there is a matching asymptotic lower bound. Later on, it is shown in [15] that it is possible to achieve $O(\log T)$ regret by using index policies constructed using the sample means of the arm rewards. The first finite-time logarithmic regret bound is given in [15]. Strikingly, the algorithm that achieves this bound computes the arm indices using only the information about the current round, the sample mean arm rewards and the number of times each arm is selected. This line of research has been followed by many others, and new algorithms with tighter regret bounds have been proposed [17].

B. The Contextual Bandit Problem

In the contextual bandit problem, different from the classical multi-armed bandit problem, the learner observes a context (side information) at the beginning of each round, which gives a hint about the expected arm rewards in that round. The context naturally arises in many practical applications such as social recommender systems [18], medical diagnosis [19] and big data stream mining [20]. Existing work on contextual bandits can be categorized into three based on how the contexts arrive and how they are related to the arm rewards.

The first category assumes the existence of similarity information (usually provided in terms of a metric) that relates the variation in the expected reward of an arm as a function of the context to the distance between the contexts. For this category, no statistical assumptions are made on how the contexts arrive. However, given a particular context, the arm rewards come from a fixed distribution parameterized by the context.

This problem is considered in [8], and the Query-Ad-Clustering algorithm that achieves $O(T^{1-1/(2+d_c)+\epsilon})$ regret for any $\epsilon > 0$ is proposed, where $d_c$ is the covering dimension of the similarity space. In addition, $\Omega(T^{1-1/(2+d_p)-\epsilon})$ lower bound on the regret, where $d_p$ is the packing dimension of the similarity space, is also proposed in this work. The main idea behind Query-Ad-Clustering is to partition the context space into disjoint sets and to estimate the expected arm rewards for each set in the partition separately. A parallel work [9] proposes the contextual zooming algorithm which partitions the similarity space non-uniformly, according to both sampling frequency and rewards obtained from different regions of the similarity space. It is shown that contextual zooming achieves $\tilde{O}(T^{1-1/(2+d_z)})$ regret, where $d_z$ is the zooming dimension of the similarity space, which is an optimistic version of the covering dimension that depends on the size of the set of near-optimal arms.

In this category of contextual bandits, reward estimates are accurate as long as the contexts that lie in the same set of the context space partition are similar to each other. However, when dimension of the context is high, the regret bound becomes almost linear. This issue is addressed in [21], where it is assumed that the arm rewards depend on an unknown subset of the contexts, and it is shown that the regret in this case only depends on the number of relevant context dimensions.

The second category assumes that the expected reward of an arm is a linear combination of the elements of the context. For this model, Li et al. [1] proposed the LinUCB algorithm. A modified version of this algorithm, named SupLinUCB, is considered by Chu et al. [10], and is shown to achieve $\tilde{O}(\sqrt{dT})$ regret, where $d$ is the dimension of the context. Valko et al. [22] mixed LinUCB and SupLinUCB with kernel functions and proposed an algorithm whose regret is $\tilde{O}(\sqrt{dT})$ where $d$ is the effective dimension of the kernel feature space.

The third category assumes that the contexts and arm rewards are jointly drawn from a fixed but unknown distribution. For this case, Langford et al. [12] proposed the epoch greedy algorithm with $O(T^{2/3})$ regret and later works [13], [23] proposed more efficient learning algorithms with $O(T^{1/2})$ regret.

Our problem is similar to the problems in the first category in terms of the context arrivals and existence of the similarity information.

C. The Multi-objective Bandit Problem

In the multi-objective bandit problem, the learner receives a multi-dimensional reward in each round. Since the rewards are no longer scalar, the definition of a benchmark to compare the learner against becomes obscure. Existing work on multi-objective bandits can be categorized into two: Pareto approach and scalarized approach.
In the Pareto approach, the main idea is to estimate the Pareto front set which consists of the arms that are not dominated by any other arm. Dominance relationship is defined such that if the expected reward of an arm \( a^* \) is greater than the expected reward of another arm \( a \) at least at one objective, and the expected reward of the arm \( a \) is not greater than the expected reward of the arm \( a^* \) in any objective, then the arm \( a^* \) dominates the arm \( a \). This approach is proposed in [7], and a learning algorithm called Pareto-UCB1 that achieves \( O(\log(T)) \) Pareto regret is proposed. Essentially, this algorithm computes UCB indices for each objective-arm pair, and then, uses these indices to estimate the Pareto front arm set, after which it selects an arm randomly from the Pareto front set. A modified version of this algorithm where the indices depend both on the estimated mean and the estimated standard deviation is proposed in [24]. Numerous other variants are also considered in prior works, including the Pareto Thompson sampling algorithm in [25] and the Annealing Pareto algorithm in [26].

On the other hand, in the scalarized approach [7], [27], a random weight is assigned to each objective at each round, from which for each arm a weighted sum of the indices of the objectives are calculated. In short, this method turns the multi-objective bandit problem into a single-objective problem. For instance, Scalarized UCB1 in [7] achieves \( O(S' \log(T/S')) \) scalarized regret where \( S' \) is the number of scalarization functions used by the algorithm.

The regret notion used in the Pareto and scalarized approaches are very different from our 2D regret notion. In the Pareto approach, the regret at round \( t \) is defined as the minimum distance that should be added to expected reward vector of the chosen arm at round \( t \) to move the chosen arm to the Pareto front. On the other hand, scalarized regret is the difference between scalarized expected rewards of the optimal arm and the chosen arm. Different from these definitions, which define the regret as a scalar quantity, we define the 2D regret as a two-dimensional vector. Hence, our goal is to minimize a multi-dimensional regret measure conditioned on the fact that we minimize the regret in the dominant objective. We show that by achieving this, we also minimize the Pareto regret.

III. PROBLEM DESCRIPTION

A. System Model

The system operates in a sequence of rounds indexed by \( t \in \{1, 2, \ldots\} \). At the beginning of round \( t \), the learner observes a \( d \)-dimensional context denoted by \( x_t \). Without loss of generality, we assume that \( x_t \) lies in the context space \( \mathcal{X} := [0, 1]^d \). After observing \( x_t \) the learner selects an arm \( a_t \) from a finite set \( \mathcal{A} \), and then, observes a two dimensional random reward \( r_t = (r_1^t, r_2^t) \) that depends both on \( x_t \) and \( a_t \). Here, \( r_1^t \) and \( r_2^t \) denote the rewards in the dominant and the non-dominant objectives, respectively, and are given by \( r_1^t = \mu_{a_t^1}(x_t) + \kappa_{a_t^1}^1 \) and \( r_2^t = \mu_{a_t^2}(x_t) + \kappa_{a_t^2}^2 \), where \( \mu_{a_t}(x_t), i \in \{1,2\} \) denotes the expected reward of arm \( a_t \) in objective \( i \) given context \( x_t \), and the noise process \( \{\kappa_{a_t^1}^1, \kappa_{a_t^2}^2\} \) is such that the marginal distribution of \( \kappa_{a_t^i}^i, i \in \{1,2\} \) is conditionally 1-sub-Gaussian, i.e.,

\[
\forall \lambda \in \mathbb{R} \quad \mathbb{E}[e^{\lambda \kappa_{a_t^i}^i}] \leq (1 + \lambda^2)^{(\kappa_{a_t^i}^i)} \leq \exp(\lambda^2/2)
\]

where \( b_{1:t} := (b_1, \ldots, b_t) \). The expected reward vector for context-arm pair \((x, a)\) is denoted by \( \mu_a(x) := (\mu_{a^1}(x), \mu_{a^2}(x)) \).

The set of arms that maximize the expected reward for the dominant objective for context \( x \) is given as \( \mathcal{A}^*(x) := \arg \max_{a \in \mathcal{A}} \mu_{a^1}(x) \). The set of optimal arms is given as the set of arms in \( \mathcal{A}^*(x) \) with the highest expected rewards for the non-dominant objective. Without loss of generality, we assume that there is a single optimal arm, and denote it by \( a^*(x) \). Hence, we have \( a^*(x) = \arg \max_{a \in \mathcal{A}^*(x)} \mu_{a^2}(x) \). Let \( \mu_a^1(x) \) and \( \mu_a^2(x) \) denote the expected rewards of arm \( a^*(x) \) in the dominant and the non-dominant objectives, respectively, given context \( x \). We assume that the expected rewards are Hölder continuous in the context, which is a common assumption in the contextual bandit literature [8], [19], [20].

Assumption 1. There exists \( L > 0, \alpha > 0 \) such that for all \( i \in \{1,2\}, \alpha \in \mathcal{A} \) and \( x', x'' \in \mathcal{X} \), we have

\[
|\mu_{a^1}(x) - \mu_{a^1}(x')| \leq L \|x - x'\|^\alpha.
\]

Another common way to compare arms when the rewards are multi-dimensional is to use the notion of Pareto optimality, which is described below.

\footnote{Examples of 1-sub-Gaussian distributions include the Gaussian distribution with zero mean and unit variance, and any distribution defined over an interval of length 2 with zero mean [23]. Moreover, our results generalize to the case when \( \kappa_{a_t}^i \) is conditionally \( R \)-sub-Gaussian for \( R \geq 1 \). This only changes the constant terms that appear in our regret bounds.}
Definition 1 (Pareto Optimality). (i) An arm \(a\) is weakly dominated by arm \(a'\) given context \(x\), denoted by \(\mu_a(x) \leq \mu_{a'}(x)\) or \(\mu_{a'}(x) \geq \mu_a(x)\), if \(\mu_{a'}^i(x) \leq \mu_a^i(x), \forall i \in \{1, 2\}\). (ii) An arm \(a\) is dominated by arm \(a'\) given context \(x\), denoted by \(\mu_a(x) < \mu_{a'}(x)\) or \(\mu_{a'}(x) > \mu_a(x)\), if it is weakly dominated and \(\exists i \in \{1, 2\}\) such that \(\mu_{a'}^i(x) < \mu_a^i(x)\).

Remark 1. Note that \(a^*(x) \in \mathcal{O}(x)\) for all \(x \in \mathcal{X}\) since \(a^*(x)\) is not dominated by any arm. For all \(a \in \mathcal{A}\), we have \(\mu_a^1(x) \geq \mu_{a^*}^1(x)\). By definition of \(a^*(x)\) if there exists an arm \(a\) for which \(\mu_a^1(x) > \mu_{a^*}^1(x)\), then we must have \(\mu_a^1(x) < \mu_{a^*}^1(x)\). Such an arm \(a\) will be incomparable with \(a^*(x)\).

B. Definitions of the 2D Regret and the Pareto Regret

Initially, the learner does not know the expected rewards; it learns them over time. The goal of the learner is to compete with an oracle, which knows the expected rewards of the arms for every context and chooses the optimal arm given the current context. Hence, the 2D regret of the learner by round \(T\) is defined as the tuple \((\text{Reg}^1(T), \text{Reg}^2(T))\), where

\[
\text{Reg}^i(T) := \sum_{t=1}^{T} \mu_a^i(x_t) - \sum_{t=1}^{T} \mu_{a^*}^i(x_t), \quad i \in \{1, 2\}
\]

for an arbitrary sequence of contexts \(x_1, \ldots, x_T\).

Another interesting performance measure is the Pareto regret [7], which measures the loss of the learner with respect to arms in the Pareto front. To define the Pareto regret, we first define the Pareto suboptimality gap (PSG).

Definition 2 (PSG of an arm). The PSG of an arm \(a \in \mathcal{A}\) given context \(x\), denoted by \(\Delta_a(x)\), is defined as the minimum scalar \(\epsilon \geq 0\) that needs to be added to all entries of \(\mu_a(x)\) such that \(a\) becomes a member of the Pareto front. Formally,

\[
\Delta_a(x) := \inf_{\epsilon \geq 0} \text{ s.t. } (\mu_a(x) + \epsilon) \parallel \mu_a(x), \forall a' \in \mathcal{O}(x)
\]

where \(\epsilon\) is a 2-dimensional vector, whose entries are \(\epsilon\).

Based on the above definition, the Pareto regret of the learner by round \(T\) is given by

\[
\text{PR}(T) := \sum_{t=1}^{T} \Delta_a(x_t).
\]

C. Applications of CMAB-DO

In this subsection we describe two possible applications of CMAB-DO.

1) Multichannel Communication: Consider a multi-channel communication scenario in which a user chooses a channel \(Q \in \mathcal{Q}\) and a transmission rate \(R \in \mathcal{R}\) in each round after receiving context \(x_t := \{\text{SNR}_Q, t\}_{Q \in \mathcal{Q}}\), where \(\text{SNR}_Q, t\) is the signal to noise ratio of channel \(Q\) in round \(t\).

In this setup, each arm corresponds to a transmission rate-channel pair denoted by \(a_{R,Q}\). Hence, the set of arms is \(\mathcal{A} = \mathcal{R} \times \mathcal{Q}\). When the user completes its transmission at the end of round \(t\), it receives a two dimensional reward where the dominant one is related to throughput and the non-dominant one is related to reliability. Here, \(r^2_t \in \{0, 1\}\) where 0 and 1 correspond to failed and successful transmission, respectively. Moreover, the success rate of \(a_{R,Q}\) is equal to \(\mu_{a_{R,Q}}^1(x_t) = 1 - \rho_{\text{out}}(R, Q, x_t)\), where \(\rho_{\text{out}}(\cdot)\) denotes the outage probability. Here, \(\rho_{\text{out}}(R, Q, x_t)\) also depends on the gain on channel \(Q\) whose distribution is unknown to the user. On the other hand, for \(a_{R,Q}, r^2_t \in \{0, R\}\) and \(\mu_{a_{R,Q}}^1(x_t) = R(1 - \rho_{\text{out}}(R, Q, x_t))\).

2) Online Binary Classification: Consider a medical diagnosis problem where a patient with context \(x_t\) (including features such as age, gender, medical test results etc.) arrives in round \(t\). Then, this patient is assigned to one of the experts in \(\mathcal{A}\) who will diagnose the patient. In reality, these experts can either be clinical decision support systems or humans, but the classification performance of these experts are context dependent and unknown a priori. In this problem, the dominant objective can correspond to accuracy while the non-dominant objective can correspond to false negative rate. For this case, the rewards in both objectives are binary, and depend on whether the classification is correct and a positive case is correctly identified.

IV. THE LEARNING ALGORITHM

We introduce MOC-MAB in this section. Its pseudocode is given in Algorithm 1.

MOC-MAB uniformly partitions \(\mathcal{X}\) into \(m^d\) hypercubes with edge lengths \(1/m\). This partition is denoted by \(\mathcal{P}\). For each \(p \in \mathcal{P}\) and \(a \in \mathcal{A}\) it keeps: (i) a counter \(N_{a,p}\) that counts the number of times the context was in \(p\) and arm \(a\) was selected before the current round, (ii) the sample mean of the rewards obtained from rounds prior to the current round in which the context was in \(p\) and arm \(a\) was selected, i.e., \(\hat{\mu}_{a,p}^1\) and \(\hat{\mu}_{a,p}^2\) for the dominant and non-dominant objectives, respectively. The idea behind partitioning is to utilize the similarity of arm rewards given in Assumption 1 to learn together for groups of similar contexts. Basically, when the number of sets in the partition is small, the number past samples that fall into a specific set is large; however, the similarity of the past samples that fall into the same set is small. The optimal partitioning should balance the inaccuracy in arm reward estimates that results form these two conflicting facts.

At round \(t\), MOC-MAB first identifies the hypercube in \(\mathcal{P}\) that contains \(x_t\), which is denoted by \(p^*\). Then, it calculates the following indices for the rewards in dominant and non-dominant objectives:

\[
\hat{g}^i_{a,p^*} := \hat{\mu}_{a,p^*}^i + u_{a,p^*}, \quad i \in \{1, 2\}
\]

Note that in this example, given that arm \(a_{R,Q}\) is selected, we have \(\kappa_1^1 = r^1_t - \mu_{a_{R,Q}}^1(x_t)\) and \(\kappa_2^1 = r^2_t - \mu_{a_{R,Q}}^2(x_t)\). Clearly, both \(\kappa_1^1\) and \(\kappa_2^1\) are zero mean. Moreover, \(\kappa_1^1 \in [-R, R]\) and \(\kappa_2^1 \in [-1, 1]\). Hence, \(\kappa_1^1\) is \(R\)-sub-Gaussian and \(\kappa_2^1\) is 1-sub-Gaussian.
the randomness in the rewards and the partitioning of $X$ by creating a safety margin below the maximal index $g_{a_1^*,p^*}$ for the dominant objective, when its confidence for $a_1^*$ is high, i.e., when $u_{a_1^*,p^*} > \beta v$, where $\beta > 0$ is a constant. For this, it calculates the set of candidate optimal arms given as

$$\hat{A}_*: \{a \in A : g_{a,p^*} \geq \hat{\mu}_{a,p^*} - u_{a_1^*,p^*} - 2v\}$$

Here, the term $-u_{a_1^*,p^*} - 2v$ accounts for the joint uncertainty over the sample mean rewards of arms $a$ and $a_1^*$. Then, MOC-MAB selects $a_t = \arg \max_{a \in A} g_{a,p^*}$.

On the other hand, when its confidence for $a_1^*$ is low, i.e., when $u_{a_1^*,p^*} > \beta v$, it has a little hope even in selecting an optimal arm for the dominant objective. In this case it just selects $a_t = a_1^*$ to improve its confidence for $a_1^*$. After its arm selection, it receives the random reward vector $r_t$, which is then used to update the counters and the sample mean rewards for $p^*$.

**Remark 2.** At each round, finding the set in $P$ that $x_t$ belongs to requires $d$ computations. Moreover, each of the following processes requires $|A|$ computations: (i) finding maximum value among the indices of the dominant objective, (ii) creating a candidate set and finding maximum value among the indices of the non-dominant objective. Hence, MOC-MAB requires $dT + 3|A|T$ computations in $T$ rounds. In addition, the memory complexity of MOC-MAB is $O(m^d)$.

V. Regret Analysis

In this section we prove that both the 2D regret and the Pareto regret of MOC-MAB are sublinear functions of $T$. Hence, MOC-MAB is average reward optimal in both regrets. First, we introduce the following as preliminaries.

For an event $\mathcal{F}$, let $\mathcal{F}^c$ denote the complement of that event. For all the parameters defined in Section V, we explicitly use the random index $t$, when referring to the value of that parameter at the beginning of round $t$. For instance, $N_{a,p}(t)$ denotes the value of $N_{a,p}$ at the beginning of round $t$. Let $N_p(t)$ denotes the number of context arrivals to $p \in \mathcal{P}$ by round $t$. $\tau_p(t)$ denote the round in which a context arrives to $p \in \mathcal{P}$ for the $t$th time, and $R_p^i(t)$ denote the random reward of arm $a$ in objective $i$ at round $t$. Let $\tilde{x}_p(t) := x_{\tau_p(t)}$, $\tilde{R}_p^i(t) := R_p^i(\tau_p(t))$, $\tilde{N}_{a,p}(t) := N_{a,p}(\tau_p(t))$, $\tilde{\mu}_{a,p}(t) := \tilde{\mu}_{a,p}(\tau_p(t))$, $\tilde{\mu}_{a,p}(t) := \tilde{\mu}_{a,p}(\tau_p(t))$, $\tilde{\mu}_{a,p}(t) := \tilde{\mu}_{a,p}(\tau_p(t))$, $\tilde{\mu}_{a,p}(t) := \tilde{\mu}_{a,p}(\tau_p(t))$.

Let $T_p := \{t \in \{1, \ldots, T\} : x_t \in p\}$ denote the set of rounds for which the context is in $p \in \mathcal{P}$.

Next, we define the following lower and upper bounds: $L_{a,p}(t) := \tilde{\mu}_{a,p}(t) - \tilde{\mu}_{a,p}(t)$ and $U_{a,p}(t) := \tilde{\mu}_{a,p}(t) + \tilde{\mu}_{a,p}(t)$ for $t \in \{1, 2\}$. Let

$$\text{UC}_{a,p} := \frac{N_p(T)}{t=1} \left\{ \mu_{a,p}(\tilde{x}_p(t)) \notin [L_{a,p}(t) - v, U_{a,p}(t) + v] \right\}$$

denote the event that the learner is not confident about its reward estimate in objective $i$ for at least once in rounds in which the context is in $p$ by time $T$. Here $L_{a,p}(t) - v$ and $U_{a,p}(t) + v$ are the lower confidence bound (LCB) and UCB.
for $\mu_a^i(\tilde{x}_p(t))$, respectively. Also, let $UC_a^i := \cup_{a \in A} UC_{a,p}$, $UC_p := \cup_{i \in I(1,2)} UC_a^i$ and $UC := \cup_{p \in P} UC_p$, and for each $i \in \{1, 2\}$, $p \in P$ and $a \in A$, let

$$p^i_{a,p} := \sup_{x \in p} \mu_a^i(x) \text{ and } \mu^i_{a,p} := \inf_{x \in p} \mu_a^i(x).$$

Let

$$Reg_i^p(T) := \sum_{t=1}^{N_p(T)} \mu^i_a(\tilde{x}_p(t)) - \sum_{t=1}^{N_p(T)} \mu^i_{a,p}(\tilde{x}_p(t))$$

denote the regret incurred in objective $i$ for rounds in $T_p$ (regret incurred in $p \in P$). Then, the total regret in objective $i$ can be written as

$$Reg_i^p(T) = \sum_{p \in P} Reg_i^p(T).$$

Thus, the expected regret in objective $i$ becomes

$$E[Reg_i^p(T)] = \sum_{p \in P} E[Reg_i^p(T)].$$

Next, we bound $E[Reg_i^p(T)]$. We have

$$E[Reg_i^p(T)] = E[Reg_i^p(T) \mid UC] \Pr(UC) + E[Reg_i^p(T) \mid UC^c] \Pr(UC^c) \leq C_i^{\max} N_p(T) \Pr(UC) + E[Reg_i^p(T) \mid UC^c]$$

where $C_i^{\max}$ is the maximum difference in the expected reward of the optimal arm and any other arm for objective $i$.

Having obtained the decomposition in (7), we proceed by bounding the terms in (7). For this, we first bound $\Pr(UC_p)$ in the next lemma.

**Lemma 1.** For any $p \in P$, we have $\Pr(UC_p) \leq 1/(m^d T)$.

**Proof.** The proof is given in Appendix A. \qed

Using the result of Lemma 1, we obtain

$$\Pr(UC) \leq 1/T \text{ and } \Pr(UC^c) \geq 1 - 1/T.$$  

To prove the lemma above, we use the concentration inequality given in [28] to bound the probability of $UC_{a,p}$. However, a direct application of this inequality is not possible to our problem, due to the fact that the context sequence $\tilde{x}_p(1), \ldots, \tilde{x}_p(N_p(t))$ does not have identical elements, which makes the mean values of $\bar{R}_{a,p}(1), \ldots, \bar{R}_{a,p}(N_p(t))$ different. To overcome this problem, we use the sandwich technique proposed in [19] in order to bound the rewards sampled from actual context arrivals between the rewards sampled from two specific processes that are related to the original process, where each process has a fixed mean value.

After bounding the probability of the event $\Pr(UC_p)$, we bound the instantaneous (single round) regret on event $\Pr(UC^c)$. For simplicity of notation, in the following lemmas we use $\mu_a^i(\tilde{x}_p(t))$ to denote the optimal arm, $\tilde{a}(t) := \tilde{a}_p(t)$ to denote the arm selected at round $\tau_p(t)$ and $\tilde{a}_1^*(t)$ to denote the arm whose first index is highest at round $\tau_p(t)$, when the set $p \in P$ that the context belongs to is obvious.

The following lemma shows that on event $UC^c_p$ the regret incurred in a round $\tau_p(t)$ for the dominant objective can be bounded as function of the difference between the upper and lower confidence bounds plus the margin of tolerance.

**Lemma 2.** When MOC-MAB is run, on event $UC^c_p$, we have

$$\mu^1_{a^*(t)}(\tilde{x}_p(t)) - \mu^1_{a_1^*(t)}(\tilde{x}_p(t)) \leq U_{\tilde{a}_1^*(t),p}^1(t) - L_{\tilde{a}_1^*(t),p}^1(t) + 2(\beta + 2)v$$

for all $t \in \{1, \ldots, N_p(T)\}$.

**Proof.** We consider two cases. When $\tilde{u}_{\tilde{a}_1^*(t),p}(t) \leq \beta v$, we have

$$U_{\tilde{a}_1^*(t),p}^1(t) \geq L_{\tilde{a}_1^*(t),p}^1(t) - 2v \geq U_{\tilde{a}_1^*(t),p}^1(t) - 2(\beta + 1)v.$$  

On the other hand, when $\tilde{u}_{\tilde{a}_1^*(t),p}(t) > \beta v$, the selected arm is $\tilde{a}(t) = \tilde{a}_1^*(t)$. Hence, we obtain

$$U_{\tilde{a}_1^*(t),p}^1(t) = U_{\tilde{a}_1^*(t),p}^1(t) \geq U_{\tilde{a}_1^*(t),p}^1(t) - 2(\beta + 1)v.$$  

Thus, for both cases, we have

$$U_{\tilde{a}_1^*(t),p}^1(t) \geq U_{\tilde{a}_1^*(t),p}^1(t) - 2(\beta + 1)v$$

and

$$U_{\tilde{a}_1^*(t),p}^1(t) \geq U_{a^*(t),p}^1(t).$$

On event $UC^c_p$, we also have

$$\mu^1_{a^*(t)}(\tilde{x}_p(t)) \leq U_{a^*(t),p}^1(t) + v$$

and

$$\mu^1_{\tilde{a}_1^*(t)}(\tilde{x}_p(t)) \geq L_{\tilde{a}_1^*(t),p}^1(t) - v.$$  

By combining (9)-(12), we obtain

$$\mu^1_{a^*(t)}(\tilde{x}_p(t)) - \mu^1_{a_1^*(t)}(\tilde{x}_p(t)) \leq U_{\tilde{a}_1^*(t),p}^1(t) - L_{\tilde{a}_1^*(t),p}^1(t) + 2(\beta + 2)v.$$

The lemma below bounds the regret incurred in a round $\tau_p(t)$ for the non-dominant objective on event $UC^c_p$ when the uncertainty level of the arm with the highest index in the dominant objective is low.

**Lemma 3.** When MOC-MAB is run, on event $UC^c_p$, for $t \in \{1, \ldots, N_p(T)\}$ if

$$\tilde{u}_{\tilde{a}_1^*(t),p}(t) \leq \beta v$$

holds, then we have

$$\mu^2_{a^*(t)}(\tilde{x}_p(t)) - \mu^2_{a_1^*(t)}(\tilde{x}_p(t)) \leq U_{a_1^*(t),p}^2(t) - L_{a_1^*(t),p}^2(t) + 2v.$$  

**Proof.** When $\tilde{u}_{\tilde{a}_1^*(t),p}(t) \leq \beta v$ holds all arms that are selected as candidate optimal arms have their index for objective 1 in the interval $[L_{\tilde{a}_1^*(t),p}^1(t) - 2v, U_{\tilde{a}_1^*(t),p}^1(t)]$. Next, we show that $U_{a^*_p(t),p}^1(t)$ is also in this interval.

On event $UC^c_p$, we have

$$\mu^1_{a^*_p(t)}(\tilde{x}_p(t)) \leq U_{a^*_p(t),p}^1(t) - v, U_{a^*_p(t),p}^1(t) + v$$

and

$$\mu^1_{\tilde{a}_1^*(t)}(\tilde{x}_p(t)) \leq L_{\tilde{a}_1^*(t),p}^1(t) - v, U_{\tilde{a}_1^*(t),p}^1(t) + v.$$
We also know that

\[ \mu_{a(t)}(\tilde{x}_p(t)) \geq \mu^1_{a(t)}(\tilde{x}_p(t)). \]

Using the inequalities above, we obtain

\[ U^1_{a(t),p}(t) \geq \mu^1_{a(t)}(\tilde{x}_p(t)) - v \geq \mu^1_{a(t)}(\tilde{x}_p(t)) - v \]

\[ \geq L^1_{a(t),p}(t) - 2v. \]

Since the selected arm has the maximum index for the non-dominant objective among all arms whose indices for the dominant objective are in \([L^1_{a(t),p}(t) - 2v, U^1_{a(t),p}(t)],\) we have \(U^2_{a(t),p}(t) \geq U^2_{a(t),p}(t).\) Combining this with the fact that \(UC^{c}_p\) holds, we get

\[ \mu^2_{a(t)}(\tilde{x}_p(t)) \geq L^2_{a(t),p}(t) - v \]

and

\[ \mu^2_{a(t)}(\tilde{x}_p(t)) \leq U^2_{a(t),p}(t) + v \leq U^2_{a(t),p}(t) + v. \]

Finally, by combining (13) and (14), we obtain

\[ \mu^2_{a(t)}(\tilde{x}_p(t)) - \mu^2_{a(t)}(\tilde{x}_p(t)) \leq U^2_{a(t),p}(t) - L^2_{a(t),p}(t) + 2v. \]

For any \(p \in P,\) we also need to bound the regret of the non-dominant objective for rounds in which \(\tilde{u}_{a^*_i}(t,p) > \beta v,\)

\[ \text{Lemma 4. When MOC-MAB is run, the number of rounds in } T_p \text{ for which } \tilde{u}_{a^*_i}(t,p) > \beta v \text{ happens is bounded above by} \]

\[ |A| \left( \frac{2A_mT}{\beta^2v^2} + 1 \right). \]

**Proof.** This event happens when \(\tilde{N}_{a^*_i}(t,p) < 2A_mT/(\beta^2v^2).\) Every such event will result in an increase in the value of \(N_{a^*_i}(t,p)\) by one. Hence, for \(p \in P\) and \(a \in A,\) the number of times \(\tilde{u}_{a,p}(t) > \beta v\) can happen is bounded above by \(2A_mT/(\beta^2v^2) + 1.\) The final result is obtained by summing over all arms. □

In the next lemmas, we bound \(Reg^1_p(t)\) and \(Reg^2_p(t)\) given that \(UC^{c}_p\) holds.

**Lemma 5. When MOC-MAB is run, on event \(UC^{c}_p,\) we have for all \(p \in P\)

\[ \text{Reg}^1_p(t) \leq |A|C^1_{\max} + 2B_mT \left( |A|N_p(t) + 2(\beta + 2)vN_p(t) \right). \]

where \(B_mT := 2\sqrt{2A_mT}.\)

**Proof.** The proof is given in Appendix B. □

**Lemma 6. When MOC-MAB is run, on event \(UC^{c}_p\) we have for all \(p \in P\)

\[ \text{Reg}^2_p(t) \leq C^2_{\max} |A| \left( \frac{2A_mT}{\beta^2v^2} + 1 \right) + 2B_mT \sqrt{|A|N_p(t)}. \]

**Proof.** The proof is given in Appendix C. □

Next, we use the result of Lemmas 5 and 6 to find a bound on \(Reg^i(t)\) that holds for all \(t \leq T\) with probability at least \(1 - 1/T.\)

**Theorem 1. When MOC-MAB is run, we have for any \(i \in \{1, 2\}\)

\[ \Pr(Reg^i(t) < \epsilon_i(t) \forall t \in \{1, \ldots, T\}) \geq 1 - 1/T \]

where

\[ \epsilon_1(t) = m^d|A|C^1_{\max} + 2B_mT \sqrt{|A|m^d} + 2(\beta + 2)vt \]

and

\[ \epsilon_2(t) = m^d|A|C^2_{\max} + 2B_mT \sqrt{|A|m^d} + 2(\beta + 2)vt. \]

**Proof.** By 5 and Lemmas 5 and 6 we have on event \(UC^c\):

\[ \text{Reg}^1(t) \leq m^d|A|C^1_{\max} + 2B_mT \sum_{p \in P} \sqrt{|A|N_p(t)} + 2(\beta + 2)vt \]

\[ \leq m^d|A|C^1_{\max} + 2B_mT \sqrt{|A|m^d} + 2(\beta + 2)vt. \]

and

\[ \text{Reg}^2(t) \leq m^d|A|C^2_{\max} + m^dC^2_{\max} |A| \left( \frac{2A_mT}{\beta^2v^2} \right) + 2B_mT \sum_{p \in P} \sqrt{|A|N_p(t) + 2vt} \]

\[ \leq m^d|A|C^2_{\max} + m^dC^2_{\max} |A| \left( \frac{2A_mT}{\beta^2v^2} \right) + 2B_mT \sqrt{|A|m^d} + 2vt \]

for all \(t \leq T.\) The result follows from the fact that \(UC^c\) holds with probability at least \(1 - 1/T.\) □

The following theorem shows that the expected 2D regret of MOC-MAB by time \(T\) is \(O(T^{1/3\alpha + \delta}).\)

**Theorem 2. When MOC-MAB is run with inputs \(m = \lceil T^{1/(3\alpha + \delta)} \rceil\) and \(\beta > 0,\) we have

\[ E[Reg^1(T)] \leq C^1_{\max} + m^d|A|C^1_{\max} T^{\frac{\delta}{1+\alpha}} \]

\[ + 2(\beta + 2)\alpha \gamma^{\alpha/2}T^{\frac{\delta}{1+\alpha}} + 2^d\gamma^{d+1} \beta T^{\frac{\delta}{1+\alpha}} \]

**Proof.** \(E[Reg^1(T)]\) is bounded by using the result of Theorem 1 and 7:

\[ E[Reg^1(T)] \leq E[Reg^1(T) | UC^c] + \sum_{p \in P} C^i_{\max}N_p(T) \Pr(UC) \]


\[ \leq \mathbb{E}[\text{Reg}_1(T) \mid UC^c] + \sum_{p \in P} C_{\text{max}}^i N_p(T)/T \]
\[ = \mathbb{E}[\text{Reg}_1(T) \mid UC^c] + C_{\text{max}}^i. \]

Therefore, we have
\[ \mathbb{E}[\text{Reg}_1(T)] \leq \epsilon_1(T) + C_{\text{max}}^1 \]
\[ \mathbb{E}[\text{Reg}_2(T)] \leq \epsilon_2(T) + C_{\text{max}}^2. \]

It can be shown that when we set \( m = \lceil T^{1/(2\alpha + 3/2)} \rceil \) regret bound of the dominant objective becomes \( O(T^{(\alpha + d)/(2\alpha + d)}) \) and regret bound of the non-dominant objective becomes \( O(T) \). The optimal value for \( m \) that makes both regrets sublinear is \( m = \lceil T^{1/(3\alpha + d)} \rceil \). With this value of \( m \), we obtain
\[ \mathbb{E}[\text{Reg}_1(T)] \leq 2^d |A| C_{\text{max}}^{d+1} T^{\frac{d+1}{2\alpha + 3}} + 2(\beta + 2) L d^{2/2} T^{\frac{2\alpha + d}{2\alpha + d}} \]
\[ + 2^{d/2+1} B_{m,T} \mathbb{E}[|A|^1 T^{\frac{1}{2\alpha + d}}] + C_{\text{max}}^1 \]

and
\[ \mathbb{E}[\text{Reg}_2(T)] \leq \left( 2 L d^{1/2} + \frac{C_{\text{max}}^2 |A|}{2^d} \right) T^{\frac{2\alpha + d}{2\alpha + d}} \]
\[ + \frac{C_{\text{max}}^2 + 2 d C_{\text{max}}^2 |A| T^{\frac{1}{2\alpha + d}}}{\beta^2 L d^{1/2}} \]
\[ + 2^{d/2+1} B_{m,T} \mathbb{E}[|A|^1 T^{\frac{1}{2\alpha + d}}]. \]

From the results above we conclude that both regrets are \( \tilde{O}(T^{(2\alpha + d)/(3\alpha + d)}) \), where for the first regret bound the constant that multiplies the highest order of the regret does not depend on \( A \), while the dependence on this term is linear for the second regret bound.

Next, we show that the expected value of the Pareto regret of MOC-MAB given in (2) is also \( \tilde{O}(T^{(2\alpha + d)/(3\alpha + d)}) \).

**Theorem 3.** When MOC-MAB is run with inputs \( m = \lceil T^{1/(3\alpha + d)} \rceil \) and \( \beta > 0 \), we have
\[ \Pr(\text{PR}(t) < \epsilon_1(t) \quad \forall t \in \{1, \dotsc, T\}) \geq 1 - 1/T \]
where \( \epsilon_1(t) \) is given in Theorem 1 and
\[ \mathbb{E}[\text{PR}(T)] \leq C_{\text{max}}^{1} + 2^d |A| C_{\text{max}}^{d+1} T^{\frac{d+1}{2\alpha + 3}} \]
\[ + 2(\beta + 2) L d^{2/2} T^{\frac{2\alpha + d}{2\alpha + d}} \]
\[ + 2^{d/2+1} B_{m,T} \mathbb{E}[|A|^1 T^{\frac{1}{2\alpha + d}}]. \]

**Proof.** Consider any \( p \in P \) and \( t \in \{1, \dotsc, N_p(T)\} \). By definition \( \Delta_{\text{a}}(\tilde{x}_a(t)) \leq \mu_{\text{a}}^{-1}(\tilde{x}_a(t)) - \mu_{\text{a}}^{-1}(\tilde{x}_a(t)) \). This holds since for any \( \epsilon > 0 \), adding \( \mu_{\text{a}}^{-1}(\tilde{x}_a(t)) - \mu_{\text{a}}^{-1}(\tilde{x}_a(t)) + \epsilon \) to \( \mu_{\text{a}}^{-1}(\tilde{x}_a(t)) \) will either make it (i) dominate the arms in \( \mathcal{O}(\tilde{x}_a(t)) \) or (ii) incomparable with the arms in \( \mathcal{O}(\tilde{x}_a(t)) \). Hence, using the result in Lemma 2, we have on event \( UC^c \)
\[ \Delta_{\text{a}}(\tilde{x}_a(t)) \leq U_{\text{a}}(t) - L_{\text{a}}(t) + 2(\beta + 2) v. \]

Let \( \text{PR}_p(T) := \sum_{t=1}^{N_p(T)} \Delta_{\text{a}}(\tilde{x}_a(t)) \). Hence, \( \text{PR}(T) = \sum_{p \in P} \text{PR}_p(T) \). Due to this, the results derived for \( \text{Reg}_1(t) \) and \( \text{Reg}_2(t) \) in Theorems 1 and 2 also hold for \( \text{PR}_p(t) \) and \( \text{PR}_p(T) \).

**Corollary 1.** When MOC-MAB is run with inputs \( \hat{\xi}, \hat{\alpha}, m = \lceil T^{1/(3\alpha + d + 1)} \rceil \), and \( \beta > 0 \) using the extended context space \( \hat{\mathcal{X}} \) instead of the original context space \( \mathcal{X} \), we have
\[ \mathbb{E}[\text{Reg}_1(T)] = \hat{O}(T^{(2\alpha + d + 1)/(3\alpha + d + 1)}) \text{ for } i \in \{1, 2\}. \]

**Proof.** The proof simply follows from the proof of Theorem 2 by extending the dimension of the context space by one.

**VI. Learning Under Periodically Changing Reward Distributions**

In many practical cases, the reward distribution of an arm changes periodically over time even under the same context. For instance, in a recommender system the probability that a user clicks to an ad may change with the time of the day, but the pattern of change can be periodic on a daily basis and this can be known by the system. Moreover, this change is usually gradual over time. In this section, we extend MOC-MAB such that it can deal with such settings.

For this, let \( T_n \) denote the period. For the \( d \)-dimensional context \( x_t = (x_{1t}, x_{2t}, \dotsc, x_{dt}) \) received at round \( t \) let \( \hat{x}_t := (x_{1t}, x_{2t}, \dotsc, x_{dt+1}) \) denote the extended context where \( x_{dt+1} := (t \mod T_n)/T_n \) is the time context. Let \( \hat{\mathcal{X}} \) denote the \( d + 1 \) dimensional extended context space constructed by adding the time dimension to \( \mathcal{X} \). It is assumed that the following holds for the extended contexts.

**Assumption 2.** Given any \( \hat{x}, \hat{x}' \in \hat{\mathcal{X}} \), there exists \( \hat{\Delta} > 0 \) and \( \hat{\alpha} > 0 \) such that for all \( i \in \{1, 2\} \) and \( a \in \mathcal{A} \), we have
\[ |\mu_a^i(\hat{x}) - \mu_a^i(\hat{x}')| \leq \hat{\Delta} \||\hat{x} - \hat{x}'\|^\hat{\alpha}. \]

Note that Assumption 2 implies Assumption 1 with \( L = \hat{\Delta} \) and \( \alpha = \hat{\alpha} \) when \( \hat{x}_{dt+1} = \hat{x}_{dt+1} \). Moreover, for two contexts \( (x_1, \dotsc, x_d, x_{dt+1}) \) and \( (x_1, \dotsc, x_d, x'_{dt+1}) \), we have
\[ |\mu_a^i(\hat{x}) - \mu_a^i(\hat{x}')| \leq \hat{\Delta} \|x_{dt+1} - x'_{dt+1}\|^\hat{\alpha} \]
which implies that the change in the expected rewards is gradual. Under Assumption 2, the performance of MOC-MAB is bounded as follows.

**VII. Illustrative Results**

In order to evaluate the performance of MOC-MAB, we run three different experiments both with synthetic and real-world datasets. In the first two experiments, the task is classification. Hence, we compare MOC-MAB with well known classifiers. In the third experiment, we compare MOC-MAB with other bandit algorithms.
A. Experiment 1

In this experiment, we evaluate the performance of MOC-MAB on the breast cancer dataset from the UCI Machine Learning Repository \cite{29}. The dataset contains 569 instances and 30 features such as radius, texture, perimeter, area, smoothness, compactness etc. 357 instances are labeled as benign. The others are labeled as malignant.

First, we apply PCA to reduce the dimension of the feature vector to 3. We use these features in the rest of the experiment. The benchmarks we compare MOC-MAB against are support vector machines, multilayer perceptron and logistic regression. For each run the dataset is randomly split into 284 training and 285 test instances. All of the reported results are averaged over 50 runs. The test phase is carried out by randomly sampling data instances from the test instances. This allows us to increase the number of test rounds beyond the number of test instances.

The benchmarks are trained offline and are kept fixed during the test phase. Since MOC-MAB is an online method, it is not trained before testing. MOC-MAB has two arms: one arm always predicts benign and the other arm always predicts malignant. Note that false negative results miss the malignant tumor and have dangerous consequences for the patients. Hence, hyper-parameters of the methods are adjusted such that the false negative rate (FNR) is kept below 4%. This is achieved by shifting the hyperplanes in SVM, modifying the loss function in multilayer perceptron, changing the prior in logistic regression and adjusting $\beta$ in MOC-MAB. Also, the uncertainty level of MOC-MAB is multiplied by $1/10$ to reduce the number of explorations. The loss (negative of the reward) of MOC-MAB in the dominant and the non-dominant objectives are set as the misclassification and the false negative events, respectively.

Accuracy and false negative rate of the learning methods are presented in Fig. 1 as a function of the size of the test phase. This figure shows that the performance of MOC-MAB improves as it learns during the test phase, and it beats the other methods in both objectives as the size of the test phase gets larger.

B. Experiment 2

In this experiment, we use the dataset and competitor algorithms from Experiment 1. The test phase size is fixed to be $T = 10^5$. The purpose of this experiment is to highlight the advantage of online learning over offline learning, when the distribution of the training and test sets are different. For this purpose offline benchmarks are trained with training sets that have different benign patient rates. Fig. 2 reports the error rate at $T = 10^5$ averaged over 10 runs. As expected, the error rate of the offline methods depend highly on the training set decomposition, and hence, they perform much worse than MOC-MAB.

C. Experiment 3

In this experiment, we compare MOC-MAB with other bandit algorithms on a synthetic multi-objective dataset. We take $\mathcal{X} = [0, 1]^2$ and assume that the context at each time
step is chosen uniformly at random from $X$. We consider 4 arms and the time horizon is set as $T = 10^5$. The expected arm rewards for 3 of the arms are generated as follows: We generate 3 multivariate Gaussian distributions for the dominant objective and 3 multivariate Gaussian distributions for the non-dominant objective. For the dominant objective, the mean vectors of the first two distributions are set as $[0.3, 0.5]$, and the mean vector of the third distribution is set as $[0.7, 0.5]$. Similarly, for the non-dominant objective, the mean vectors of the distributions are set as $[0.3, 0.7], [0.3, 0.3]$ and $[0.7, 0.5]$, respectively. For all the Gaussian distributions the covariance matrix is given by $0.3 + I$ where $I$ is the 2 by 2 identity matrix. Then, each Gaussian distribution is normalized by multiplying it with a constant, such that its maximum value becomes 1. These normalized distributions form the expected arm rewards. In addition, the expected reward of the fourth arm for the dominant objective is set as 0, and its expected reward for the non-dominant objective is set as the normalized multivariate Gaussian distribution with mean vector $[0.7, 0.5]$. We assume that the reward of an arm in an objective given a context $x$ is a Bernoulli random variable whose parameter is equal to the magnitude of the corresponding normalized distribution at context $x$.

We compare MOC-MAB with the following bandit algorithms:

**Pareto UCB1 (P-UCB1):** This is the Empirical Pareto UCB1 algorithm proposed in [7].

**Scalarized UCB1 (S-UCB1):** This is the Scalarized Multi-objective UCB1 algorithm proposed in [7].

**Contextual Pareto UCB1 (CP-UCB1):** This is the contextual version of P-UCB1 which partitions the context space in the same way as MOC-MAB does, and uses a different instance of P-UCB1 in each set of the partition.

**Contextual Scalarized UCB1 (CS-UCB1):** This is the contextual version of S-UCB1, which partitions the context space in the same way as MOC-MAB does, and uses a different instance of S-UCB1 in each set of the partition.

**Contextual Dominant UCB1 (CD-UCB1):** This is the contextual version of UCB1 [15], which partitions the context space in the same way as MOC-MAB does, and uses a different instance of UCB1 in each set of the partition. This algorithm only uses the rewards from the dominant objective to update the indices of the arms.

For S-UCB1 and CS-UCB1, the weights of the linear scalarization functions are chosen as $[1, 0], [0.5, 0.5]$ and $[0, 1]$. For all contextual algorithms, the partition of the context space is formed by choosing $m$ according to Theorem 2. For MOC-MAB, $\beta$ is chosen as 1. In addition, we scaled down the uncertainty level (also known as the confidence term or the inflation term) of all the algorithms by a constant chosen from $\{1, 1/5, 1/10, 1/15, 1/20, 1/25, 1/30\}$, since we observed that the regrets of all algorithms become smaller when the uncertainty level is scaled down. For MOC-MAB the optimal scale factor for the dominant objective is $1/15$, for CS-UCB1 and CD-UCB1, it is $1/5$, for CP-UCB1, it is $1/30$ and for the non-contextual algorithms, it is $1/20$. The regret results are obtained by using the optimal scale factor for each algorithm.

Every algorithm is run 100 times and the results are averaged over these runs. Simulation results given in Fig. 3 show the change in the regret of the algorithms in both objectives as a function of time (rounds). As observed from the results, MOC-MAB beats all other algorithms in both objectives except CD-UCB1. While the regret of CD-UCB1 in the dominant objective is slightly better than that of MOC-MAB, its regret is much worse than MOC-MAB in the non-dominant objective. This is expected since it only aims to maximize the reward in the dominant objective without considering the other objective. In the dominant objective, obtained total reward of MOC-MAB is 5.5% higher than that of CP-UCB1, 8.7% higher than that of CS-UCB1, 49.9% higher than that of P-UCB1 and 51.5% higher than that of S-UCB1 but 1% smaller than that of CD-UCB1. In the non-dominant objective, obtained total reward of MOC-MAB is 3.8% higher than that of CP-UCB1, 7% higher than that of CS-UCB1, 20.2% higher than that of CD-UCB1, and 68.6% higher than that of P-UCB1 and 69.4% higher than that of S-UCB1.
VIII. CONCLUSION

In this paper, we propose a new contextual bandit problem with two objectives in which one objective is dominant and the other is non-dominant. According to this definition, we propose two performance metrics: the 2D regret (which is multi-dimensional) and the Pareto regret (which is scalar). Then, we propose an online learning algorithm called MOC-MAB and show that it achieves sublinear 2D regret and Pareto regret. To the best of our knowledge, our work is the first to consider a multi-objective contextual bandit problem where the expected arm rewards and contexts are related through similarity information. We also evaluate the performance of MOC-MAB on both real-world and synthetic datasets and compare it with offline methods and other bandit algorithms. Our results demonstrate that MOC-MAB outperforms its competitors, which are not specifically designed to deal with problems involving dominant and non-dominant objectives.

APPENDIX A

OOF OF LEMMA II

Proof. From the definitions of $L_{a,p}^i(t)$, $U_{a,p}^i(t)$, and $U_{a,p}^{C_i}$, it can be observed that the event $U_{a,p}^{C_i}$ happens when $\mu_a^i(\hat{x}_p(t))$ does not fall into the confidence interval $[L_{a,p}^i(t) - v, U_{a,p}^i(t) + v]$ for some $t$. The probability of this event could be easily bounded by using the concentration inequality given in Appendix D if the expected reward from the same arm did not change over rounds. However, this is not the case in our model since the elements of $\{\hat{x}_p(t)\}_{t=1}^{N_p(T)}$ are not identical which makes the distributions of $R_{a,p}^i(t)$, $t \in \{1, \ldots, N_p(T)\}$ different.

In order to resolve this issue, we propose the following: Recall that

\[ \hat{R}_{a,p}^i(t) = \mu_a^i(\hat{x}_p(t)) + \tilde{\kappa}_p^i(t) \]

and

\[ \tilde{\mu}_{a,p}^i(t) = \sum_{l=1}^{i-1} \hat{R}_{a,p}^i(l)I(\bar{a}_p(l) = a) / \tilde{N}_{a,p}(t) \]

We define two new sequences of random variables, whose sample mean values will lower and upper bound $\tilde{\mu}_{a,p}^i(t)$. The best sequence is defined as $\{\tilde{R}_{a,p}^i(t)\}_{i=1}^{N_p(T)}$ where

\[ \tilde{R}_{a,p}^i(t) = \hat{R}_{a,p}^i(t) = \mu_a^i(\hat{x}_p(t)) + \tilde{\kappa}_p^i(t) \]

and the worst sequence is defined as $\{\bar{R}_{a,p}^i(t)\}_{i=1}^{N_p(T)}$ where

\[ \bar{R}_{a,p}^i(t) = \mu_a^i(\hat{x}_p(t)) + \bar{\kappa}_p^i(t) \]

Let

\[ \tilde{\mu}_{a,p}^i(t) := \sum_{l=1}^{i-1} \hat{R}_{a,p}^i(l)I(\bar{a}_p(l) = a) / \tilde{N}_{a,p}(t) \]

\[ \mu_a^i(\hat{x}_p(t)) \leq \tilde{\mu}_{a,p}^i(t) \leq \bar{\mu}_{a,p}^i(t) \forall t \in \{1, \ldots, N_p(T)\} \]

almost surely.

Let

\[ L_{a,p}^i(t) := \bar{\mu}_{a,p}^i(t) - \tilde{\mu}_{a,p}^i(t) \]

\[ U_{a,p}^i(t) := \bar{\mu}_{a,p}^i(t) + \tilde{\mu}_{a,p}^i(t) \]

\[ L_{a,p}^i(t) := \mu_a^i(\hat{x}_p(t)) - \tilde{\mu}_{a,p}^i(t) \]

\[ U_{a,p}^i(t) := \mu_a^i(\hat{x}_p(t)) + \tilde{\mu}_{a,p}^i(t) \]

It can be shown that

\[ \{\mu_a^i(\hat{x}_p(t)) \notin [L_{a,p}^i(t) - v, U_{a,p}^i(t) + v]\} \]

\[ \subset \{\mu_a^i(\hat{x}_p(t)) \notin [L_{a,p}^i(t) - v, U_{a,p}^i(t) + v]\} \]

\[ \cup \{\mu_a^i(\hat{x}_p(t)) \notin [L_{a,p}^i(t) - v, U_{a,p}^i(t) + v]\}. \quad (15) \]

The following inequalities can be obtained from the Hölder continuity assumption:

\[ \mu_a^i(\hat{x}_p(t)) \leq \tilde{\mu}_{a,p}^i(t) \leq \mu_a^i(\hat{x}_p(t)) + L \left( \frac{\sqrt{d}}{m} \right)^\alpha \]

\[ \mu_a^i(\hat{x}_p(t)) - L \left( \frac{\sqrt{d}}{m} \right)^\alpha \leq \tilde{\mu}_{a,p}^i(t) \leq \mu_a^i(\hat{x}_p(t)). \quad (17) \]

Since $v = L \left( \sqrt{d/m} \right)^\alpha$, using (16) and (17) it can be shown that

\[ (i) \quad \{\mu_a^i(\hat{x}_p(t)) \notin [\tilde{R}_{a,p}^i(t) - v, U_{a,p}^i(t) + v]\} \]

\[ \subset \{\tilde{R}_{a,p}^i(t) \notin [\tilde{R}_{a,p}^i(t) - v, U_{a,p}^i(t) + v]\} \]

\[ \cup \{\mu_a^i(\hat{x}_p(t)) \notin [\tilde{R}_{a,p}^i(t) - v, U_{a,p}^i(t) + v]\}. \]

Plugging these into (15), we get

\[ \{\mu_a^i(\hat{x}_p(t)) \notin [L_{a,p}^i(t) - v, U_{a,p}^i(t) + v]\} \]

\[ \subset \{\mu_a^i(\hat{x}_p(t)) \notin [L_{a,p}^i(t) - v, U_{a,p}^i(t) + v]\} \]

Then, using the equation above and the union bound, we obtain

\[ \Pr(U_{a,p}^{C_i}) \leq \Pr(\bigcup_{i=1}^{N_p(T)} \{\tilde{R}_{a,p}^i(t) \notin [\tilde{R}_{a,p}^i(t) - v, U_{a,p}^i(t) + v]\}) \]

+ \Pr(\bigcup_{i=1}^{N_p(T)} \{R_{a,p}^i(t) \notin [R_{a,p}^i(t) - v, U_{a,p}^i(t) + v]\}) \]

Both terms on the right-hand side of the inequality above can be bounded using the concentration inequality in Appendix D. Using $\delta = 1/(4|A|m^dT)$ in Appendix D gives

\[ \Pr(U_{a,p}^{C_i}) \leq \frac{1}{2|A|m^dT} \]

since $1 + N_a(T) \leq T$. Then, using the union bound, we obtain

\[ \Pr(U_{a,p}^{C_i}) \leq \frac{1}{2m^dT} \]

and

\[ \Pr(U_{a,p}^i) \leq \frac{1}{m^dT}. \]

\[ \Box \]
APPENDIX B

PROOF OF LEMMA [5]

Let $T_{a,p} := \{1 \leq l \leq N_p(t) : \hat{a}_p(l) = a\}$ and $\bar{T}_{a,p} := \{l \in T_{a,p} : \bar{N}_a(l) \geq 1\}$. By Lemma [2] we have

\[
\Reg_p^1(t) = \sum_{a \in A} \sum_{l \in \bar{T}_{a,p}} \left( \mu_1^a(\bar{x}_p(l)) - \mu_1^{\hat{a}_p(l)}(\bar{x}_p(l)) \right)
\leq \sum_{a \in A} \sum_{l \in T_{a,p}} \left( U_{\hat{a}_p,l}(p)(l) - L_{\hat{a}_p,l}(p)(l) + 2(\beta + 2)v \right)
+ |A| C^1_{\max}
= \sum_{a \in A} \sum_{l \in T_{a,p}} \left( U_{\hat{a}_p,l}(p)(l) - L_{\hat{a}_p,l}(p)(l) \right)
+ 2(\beta + 2)v \bar{N}_p(t) + |A| C^1_{\max}.
\]

We also have

\[
\sum_{a \in A} \sum_{l \in \bar{T}_{a,p}} \left( U_{\hat{a}_p,l}(p)(l) - L_{\hat{a}_p,l}(p)(l) \right)
\leq \sum_{a \in A} \left( B_{m,T} \sum_{l \in \bar{T}_{a,p}} \sqrt{\frac{1}{N_{a,p}(l)}} \right)
\leq B_{m,T} \sum_{a \in A} \sqrt{N_{a,p}(t)} \sum_{k=0}^{N_{a,p}(t)-1} \frac{1}{1 + k}
\leq 2B_{m,T} \sum_{a \in A} \sqrt{N_{a,p}(t)}
\leq 2B_{m,T} \sqrt{|A| N_p(t)}
\]

where $B_{m,T} = 2\sqrt{2A_{m,T}}$, and (19) follows from the fact that

\[
\sum_{k=0}^{N_{a,p}(t)-1} \frac{1}{1 + k} \leq \int_{x=0}^{N_{a,p}(t)} \frac{1}{\sqrt{x}} dx = 2\sqrt{N_{a,p}(t)}.
\]

Combining (18) and (20), we obtain that on event $UC^c$

\[
\Reg_p^1(t) \leq |A| C^1_{\max} + 2B_{m,T} \sqrt{|A| N_p(t) + 2(\beta + 2)v \bar{N}_p(t)}.
\]

APPENDIX C

PROOF OF LEMMA [6]

Using the result of Lemma [4], the contribution to the regret of the non-dominant objective in rounds for which $\tilde{a}_{\hat{a}_p}(t),p(l) > \beta v$ is bounded by

\[
C^2_{\max}|A| \left( \frac{2A_{m,T}}{\beta^2 v^2} + 1 \right).
\]

We have on event $UC^c$

\[
\sum_{a \in A} \sum_{l \in T_{a,p}^2} \left( U_{\hat{a}_p,l}(p)(l) - L_{\hat{a}_p,l}(p)(l) \right)
\leq \sum_{a \in A} \sum_{l \in T_{a,p}^2} \left( B_{m,T} \sum_{l \in \bar{T}_{a,p}} \sqrt{\frac{1}{N_{a,p}(l)}} \right)
\leq B_{m,T} \sum_{a \in A} \sqrt{N_{a,p}(t)} \sum_{k=0}^{N_{a,p}(t)-1} \frac{1}{1 + k}
\leq 2B_{m,T} \sum_{a \in A} \sqrt{N_{a,p}(t)}
\leq 2B_{m,T} \sqrt{|A| N_p(t)}.
\]

where $B_{m,T} = 2\sqrt{2A_{m,T}}$. Combining (21), (22) and (23), we obtain

\[
\Reg_p^2(t) \leq C^2_{\max}|A| \left( \frac{2A_{m,T}}{\beta^2 v^2} + 1 \right) + 2v \bar{N}_p(t)
\]

\[
+ 2B_{m,T} \sqrt{|A| N_p(t)}.
\]

APPENDIX D

CONCENTRATION INEQUALITY [28], [31]

Consider an arm $a$ for which the rewards of objective $i$ are generated by a process $\{R^i_a(t)\}_{t=1}^T$ with $\mu^i_a = E[R^i_a(t)]$, where the noise $R^i_a(t) - \mu^i_a$ is conditionally 1-sub-Gaussian. Let $N_a(T) \geq 1$ denote the number of times $a$ is selected by the end of time $T$. Let $\hat{\mu}_a(T) = \sum_{t=1}^T \mathbb{I}(a(t) = a) R^i_a(t)/N_a(T)$. For any $\delta > 0$ with probability at least $1 - \delta$ we have

\[
|\hat{\mu}_a(T) - \mu_a| \leq \sqrt{\frac{2}{N_a(T)}} \left[ 1 + 2 \log \left( \frac{(1 + N_a(T))^{1/2}}{\delta} \right) \right] \forall T \in \mathbb{N}.
\]

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