Existence and non-existence of ground states of bi-harmonic equations involving constant and degenerate Rabinowitz potentials

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Abstract
Recently, the authors of the current paper established in Chen et al. (Calc Var Partial Differ Equ 59:1–38, 2020) the existence of a ground-state solution to the following bi-harmonic equation with the constant potential or Rabinowitz potential:

\[ (-\Delta)^2 u + V(x) u = f(u) \text{ in } \mathbb{R}^4, \tag{0.1} \]

when the nonlinearity has the special form \( f(t) = t(\exp(t^2) - 1) \) and \( V(x) \geq c > 0 \) is a constant or the Rabinowitz potential. One of the crucial elements used in Chen et al. (Calc Var Partial Differ Equ 59:1–38, 2020) is the Fourier rearrangement argument. However, this argument is not applicable if \( f(t) \) is not an odd function. Thus, it still remains open whether the Eq. (0.1) with the general critical exponential nonlinearity \( f(u) \) admits a ground-state solution even when \( V(x) \) is a positive constant. The first purpose of this paper is to develop a Fourier rearrangement-free approach to solve the above problem. More precisely, we will prove that there is a threshold \( \gamma^* \) such that for any \( \gamma \in (0, \gamma^*) \), the Eq. (0.1) with the constant potential \( V(x) = \gamma > 0 \) admits a ground-state solution, while does not admit any ground-state solution for any \( \gamma \in (\gamma^*, +\infty) \). The second purpose of this paper is to establish the existence of a ground-state solution to the Eq. (0.1) with any degenerate Rabinowitz potential \( V \) vanishing on some bounded open set. Among other techniques, the proof also relies on a critical Adams inequality involving the degenerate potential which is of its own interest.

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1 Introduction

We begin with considering the following nonlinear partial differential equation

\[(−Δ)^m u + V(x)u = f(u) \text{ in } \mathbb{R}^n,\]  

(1.1)

where \(m\) is a positive integer, \(V(x)\) is some nonnegative potential. Equations (1.1) with subcritical and critical growth have been extensively studied by many authors. In the case \(n > 2m\), the subcritical and critical growth means that the nonlinearity cannot exceed the polynomial growth of degree \(\frac{n+2m}{n-2m}\) by the Sobolev embedding. While in the case \(n = 2m\), we say that \(f(s)\) has critical exponential growth at infinity if there exists \(\alpha_0 > 0\) such that

\[\lim_{|t| \to +\infty} \frac{|f(t)|}{\exp(\alpha t^2)} = \begin{cases} 0, & \text{for } \alpha > \alpha_0 \\ +\infty, & \text{for } \alpha < \alpha_0. \end{cases} \]  

(1.2)

The critical exponential growth in the case \(m = 1, n = 2\) is consistent with the Trudinger—Moser inequality \([37, 46]\), while in the case \(m \geq 2, n = 2m\) is given by the Adams inequality \([1]\). The study of the existence of solutions for the Eq. (1.1) with the critical exponential growth involves a lack of compactness, i.e. the Palais-Smale compactness condition may fail at some special level. However, unlike the equations on bounded domain (see e.g., \([3, 7, 14, 19, 23, 25]\)), the loss of compactness for Eq. (1.1) may be produced not only by the concentration phenomena but also by the vanishing phenomena.

The earlier study of the existence of solutions for Eq. (1.1) with the critical exponential growth can date back to the work of Atkinson and Peletier \([4, 5]\). Indeed, the authors obtained the existence of ground state solutions for Eq. (1.1) by assuming that there exists some \(y_0 > 0\) such that \(g(t) = \log f(t)\) satisfies

\[g'(t) > 0, \quad g''(t) \geq 0,\]

for any \(t \geq y_0\). This kind of growth condition allows us to take the nonlinearity \(f(t) = (t^2 - t) \exp(t^2)\), which has critical exponential growth.

As far as we are concerned, having a positive lower bound has become a standard assumption on the potential \(V(x)\) in dealing with the existence of solutions to the Eq. (1.1) in the literature, we will briefly describe some of the relevant works below.

When \(V(x)\) is a coercive potential, that is,

\[V(x) \geq V_0 > 0, \text{ and additionally either } \lim_{x \to \infty} V(x) = +\infty \text{ or } \frac{1}{V} \in L^1(\mathbb{R}^n),\]

the existence results of Eq. (1.1) can be found in the papers e.g., \([15, 16, 20, 31, 47–49]\) and the references therein. Their proofs depend crucially on the compact embeddings given by the coercive potential, and the vanishing phenomena can be ruled out.

When \(V(x)\) is the constant potential, i.e. \(V(x) = \gamma > 0\), the natural space for a variational treatment of (1.1) is \(W^{m,2}(\mathbb{R}^n)\). It is well known that the embedding \(W^{m,2}(\mathbb{R}^n) \hookrightarrow L^2(\mathbb{R}^n)\) is continuous but not compact, even in the radial case.

In the case \(m = 1\) and \(n = 2\), the authors of \([2]\) obtained the existence of a ground solution to Eq. (1.1) under the assumptions that for any \(p > 2,\)

\[f(s) \geq \eta_p s^{p-1}, \forall s \geq 0,\]  

(1.3)
where $\eta_p$ is some constant depending on $p$. In [41], the authors also obtained the existence of a ground state solution to (1.1) under

$$
\lim_{|s| \to \infty} \frac{sf(s)}{\exp(4\pi s^2)} \geq \beta_0 > 0.
$$

(1.4)

In general, (1.3) and (1.4) are not comparable. In [17], the authors proved that there exists a positive $\gamma^*$ such that for any $0 < V = \gamma < \gamma^*$, the Eq. (1.1) has a ground state solution under a weaker assumptions than both (1.3) and (1.4).

In the case $m = 2$, the existence of a nontrivial solution to Eq. (1.1) was obtained in [9] under the assumption that (1.3) holds (see also [6]), and in [44] under the assumption that (1.4) holds. The existence of a nontrivial solution to (1.1) under the assumption weaker than both (1.3) and (1.4) was established in [11]. Furthermore, the existence of a ground state solution to (1.1) was only recently proved in [11] (see more detailed discussions below).

We recall that the following Trudinger–Moser inequality holds (see [30, 42]):

$$
\sup_{u \in W^{1,n}(\mathbb{R}^n), \int_{\mathbb{R}^n}(|\nabla u|^n + |u|^n)dx \leq 1} \int_{\mathbb{R}^n} \Phi_n(\alpha_n|u|^\frac{n}{n-1})dx < \infty.
$$

(1.5)

where $\omega_{n-1}$ denotes the area of the unit sphere in $\mathbb{R}^n$ and $\Phi_n(t) := e^t - \sum_{l=0}^{n-2} \frac{t^l}{l!}$. The proof of the Trudinger–Moser inequality in [42] and [30] relies on the Pólya-Szegő inequality and the symmetrization argument. Subsequently, the authors in [24] used a symmetrization-free approach to give a simple proof for the sharp Trudinger–Moser inequalities in $W^{1,n}(\mathbb{R}^n)$. This symmetrization-free method also applies to settings where the Pólya-Szegő inequality fails, such as on the sharp Trudinger–Moser inequalities on the Heisenberg group and Riemannian manifolds (see [21, 26, 29, 31, 32]) and also weighted Trudinger–Moser inequalities in $\mathbb{R}^n$ [10, 27]. Using a symmetrization-free argument, we have proved more recently in [13] and [12] the following Trudinger–Moser inequality in higher dimension $\mathbb{R}^n$ ($n \geq 2$) under the less restrictive constraint

$$
\int_{\mathbb{R}^n} (|\nabla u|^n + V(x)|u|^n)dx \leq 1,
$$

where $V(x) \geq 0$ satisfying:

(V1) $V(x) = 0$ at $B_\delta(0)$ and $V(x) \geq c_0$ in $\mathbb{R}^n \setminus B_{2\delta}(0)$ for some $c_0, \delta > 0$.

**Theorem A** Assume that the potential $V(x)$ satisfies the condition (V1). Then

$$
\sup_{u \in W^{1,n}(\mathbb{R}^n), \int_{\mathbb{R}^n}(|\nabla u|^n + V(x)|u|^n)dx \leq 1} \int_{\mathbb{R}^n} \Phi_n(\alpha_n|u|^\frac{n}{n-1})dx < \infty.
$$

(1.6)

We note that the loss of a positive lower bound of the potential $V(x)$ makes this inequality become fairly nontrivial. A sharp Trudinger–Moser inequality with degenerate potential on the Heisenberg group has also been proved recently in [8].

Sharp Adams inequalities on the entire space $\mathbb{R}^n$ were studied in [43] under the constraint

$$
\left\{ u \in W^m, \frac{m}{n} \left\| (I - \Delta)^{\frac{m}{2}} u \right\|_{\frac{m}{n}} \leq 1 \right\},
$$

when $m$ is an even integer. When the order $m$ of the derivatives is odd, a sharp Adams inequality was established in [22]. The same authors in [24] give a unified approach for all
orders $m$ of derivatives including fractional orders of derivatives through the rearrangement-free argument. Furthermore, Lam an Lu also proved in [24] the following sharp Adams inequality under the less restrictive Sobolev norm constraint: let $\tau > 0$,

$$
\sup_{u \in W^{2,2}(\mathbb{R}^4)} \int_{\mathbb{R}^4} \left( \exp(\beta |u(x)|^2) - 1 \right) dx \begin{cases} 
C & \text{if } \beta \leq 32\pi^2, \\
+\infty & \text{if } \beta > 32\pi^2.
\end{cases} \tag{1.7}
$$

As an application of critical Adams inequality (1.7) on the whole space $\mathbb{R}^4$, the authors of [11] obtained the existence of a non-trivial radial solution to the following bi-harmonic equation with the constant potential:

$$
(-\Delta)^2 u + \gamma u = f(u) \text{ in } \mathbb{R}^4, \tag{1.8}
$$

when the nonlinearity $f(t)$ has the critical exponential growth at infinity. However, the existence of a ground state solution was not proved in [11]. More precisely, in [11] the following was proved:

**Theorem B** [11] Assume that $f$ satisfies $f(0) = 0$ and the conditions (i), (ii), (iii) and (iv) in Sect. 2, then there exists $\gamma^* \in (0, +\infty]$ such that for any $\gamma \in (0, \gamma^*)$, the Eq. (1.8) admits a non-trivial radial solution. Moreover, $\gamma^*$ is equal to the radial Adams’ ratio:

$$
C_A^* = \sup \left\{ \frac{2}{\|u\|_2^2} \int_{\mathbb{R}^4} F(u) dx \mid u \in W^{2,2}_r(\mathbb{R}^4) \setminus \{0\}, \|\Delta u\|_2^2 \leq \frac{32\pi^2}{\alpha_0} \right\},
$$

where $W^{2,2}_r(\mathbb{R}^4)$ is the collection of all radial functions in $W^{2,2}(\mathbb{R}^4)$ and $F(t) = \int_0^t f(s) ds$. In particular, $\gamma^* = +\infty$ is equivalent to

$$
\lim_{|t| \to \infty} \frac{t^2 F(t)}{\exp(\alpha_0 t^2)} = +\infty.
$$

Furthermore, if the nonlinearity has the special form $f(t) = \lambda t \exp(2|t|^2)$, the authors can further prove that the solutions obtained are ground-state solutions:

**Theorem C** [11] For any $\gamma \in (0, +\infty)$, the equation

$$
(-\Delta)^2 u + \gamma u = \lambda u \exp(2|u|^2) \text{ in } \mathbb{R}^4
$$

admits a radial ground state solution if $\lambda \in (0, \gamma)$.

**Remark 1.1** We cannot use the Schwarz symmetrization principle directly in the proof of Theorem C due to the presence of the higher order derivatives. In order to overcome this difficulty, in [11] the authors applied the Fourier rearrangement proved by Lenzmann and So k in [28] to obtain a radially minimizing sequence for the infimum on the Pohozaev manifold. We stress that the Fourier rearrangement argument requires that $f(s)$ must be odd and all the coefficients of the Taylor series for the primitive function $F(s)$ must be positive.

Based on the above result, by exploiting the relationship between the Nehari manifold and the corresponding limiting Nehari manifold, the authors can also obtain the existence of ground state solutions of the bi-harmonic equation with the non-radial Rabinowitz type potential introduced in [39]:

\[ \text{Springer} \]
Theorem D [11] Assume that $V(x)$ is a continuous function satisfying

$$0 < \lambda < V_0 = \inf_{x \in \mathbb{R}^4} V(x) < \sup_{x \in \mathbb{R}^4} V(x) = \gamma < +\infty,$$

the equation

$$(-\Delta)^2 u + V(x)u = \lambda u \exp(2|u|^2) \text{ in } \mathbb{R}^4$$

admits a ground state solution which is not necessarily radial.

The following remarks are in order. First, as we pointed out earlier, the method in [11] of using the Fourier rearrangement to establish the existence of a ground state solution is not applicable to more general nonlinearity $f$ than the special form $f(t) = \lambda t \exp(2|t|^2)$ (see Theorem C). Therefore, new methods without using the Fourier rearrangement need to be developed to deal with the more general nonlinearity $f$. Our method in this paper does not rely on the Pólya-Szegö inequality nor the Fourier rearrangement. Second, in our earlier work [11] we assume that the potential $V$ has a positive low bound in the entire space $\mathbb{R}^4$. Another novelty in this paper is that the potential $V$ can be degenerate on an open bounded set in $\mathbb{R}^4$.

Third, the Adams inequality under the Sobolev norm associated with the degenerate potential $V$ is established and is of its independent interest. This Adams embedding is necessary to establish the existence of the ground state solution. Fourth, our result is fairly sharp in the sense that we have identified a threshold $\gamma^*$ for the bi-harmonic equation with constant potential $(-\Delta)^2 u + \gamma u = f(u)$ in $\mathbb{R}^4$ such that the existence of a ground state solution is guaranteed for any $\gamma \in (0, \gamma^*)$ and the nonexistence of any ground state solution is assured for any $\gamma \in (\gamma^*, \infty)$.

Therefore, the main purpose of this paper is to answer the following two questions:

1. Can the solution in Theorem B be a ground-state solution and does Theorem C still hold when the nonlinearity $f$ is a more general function satisfying the critical exponential growth and the Ambrosetti–Rabinowitz condition rather than having a special form $f(t) = \lambda t \exp(2|t|^2)$?

2. Does Theorem D still hold when the potential $V$ is a degenerate Rabinowitz type potential and furthermore the nonlinearity $f$ is a more general function satisfying the critical exponential growth and the Ambrosetti–Rabinowitz condition rather than having a special form $f(t) = \lambda t \exp(2|t|^2)$?

2 The main results

Motivated by the results just described, in this paper, we first consider the following bi-harmonic equation with the constant potential:

$$(-\Delta)^2 u + \gamma u = f(u) \text{ in } \mathbb{R}^4,$$

where the nonlinearity $f(t)$ satisfies $f(0) = 0$ and the following properties:

(i) Has critical exponential growth (1.2).

(ii) (Ambrosetti–Rabinowitz condition) (A–R) [3, 38, 40] There exists $\mu > 2$ such that $0 < \mu F(t) = \mu \int_0^t f(s)ds \leq tf(t)$ for any $t \in \mathbb{R}$;

(iii) There exist $t_0$ and $M_0 > 0$ such that $F(t) \leq M_0|f(t)|$ for any $|t| \geq t_0$.

(iv) $f(t) = o(t)$ as $t \to 0$.

(v) $f(t) \in C^1$ and $\frac{f(t)}{t}$ is strictly increasing in $(0, +\infty)$ and decreasing in $(-\infty, 0)$.
Remark 2.1 The condition (ii) implies that $F(t) = o(t^2)$ as $t \to 0$. Indeed, the condition (ii) implies that $\left( \frac{F(t)}{t} \right)' > 0$, from which one can immediately get $F(t) = o(t^2)$ as $t \to 0$. From conditions (i), (ii) and (iv) above, one can obtain the following growth condition for $f(t)$: for any $\varepsilon > 0$ and $\beta_0 > \alpha_0$, there exists $C_\varepsilon$ such that

$$|f(t)| \leq \varepsilon |t| + C_\varepsilon |t|^\mu \left( e^{\beta_0 t^2} - 1 \right), \quad \forall \, t \in \mathbb{R}. \quad (2.2)$$

From (v), one can also easily check that the function $f(t) = t^2 F(t)$ is increasing in $(0, +\infty)$ and decreasing in $(-\infty, 0)$.

Our first result is the following

Theorem 2.2 Assume that $f$ satisfies $f(0) = 0$ and the conditions (i), (ii) and (iii), (iv) and (v), then there exists $\gamma^* \in (0, +\infty)$ such that the Eq. (2.1) admits a ground-state solution for any $\gamma \in (0, \gamma^*)$, and does not admit any ground-state solution for any $\gamma \in (\gamma^*, +\infty)$, where $\gamma^*$ is equal to the Adams’ ratio:

$$\gamma^* = \sup \left\{ \frac{2}{\|u\|_2^2} \int_{\mathbb{R}^4} F(u) \, dx \mid u \in W^{2,2}(\mathbb{R}^4) \setminus \{0\}, \|\Delta u\|_2^2 \leq \frac{32\pi^2}{\alpha_0} \right\}.$$

In particular, $\gamma^* = +\infty$ is equivalent to

$$\lim_{|t| \to \infty} \frac{t^2 F(t)}{\exp(\alpha_0 t^2)} = +\infty.$$

The above theorem reveals an interesting relation between an Adams type inequality and the nonexistence of a ground-state solution of bi-harmonic equation with the critical exponential growth. As an immediate consequence of Theorem 2.2, we can conclude the following

Corollary 2.3 Assume that $f$ satisfies $f(0) = 0$ and the conditions (i), (ii), (iii) and (iv) and (v), then the following Adams type inequality

$$\sup_{\|\Delta u\|_2 \leq \frac{32\pi^2}{\alpha_0}} \frac{\int_{\mathbb{R}^4} F(u) \, dx}{\int_{\mathbb{R}^4} |u(x)|^2 \, dx} \leq C$$

holds for some $0 < C < \infty$ if and only if there exists some $\beta^* \in \mathbb{R}$ such that for $\beta > \beta^*$, $(-\Delta)^2 u + \beta u = f(u)$ does not admit any ground-state solution.

As we mentioned before, the loss of compactness for the Eq. (1.1) may be produced not only by the concentration phenomena but also by the vanishing phenomena. In the literature, in order to exclude the vanishing phenomena, one can introduce the coercive potential (see [20, 47, 49]), or apply some symmetrization argument (see [2, 6, 9, 11, 17, 36, 41]). However, for our bi-harmonic Eq. (2.1), the symmetrization argument fails, since the nonlinearity $f(t)$ in Theorem 2.2 needn’t be an odd function. Hence, neither the Schwarz symmetrization principle nor the Fourier rearrangement used in [11] can be applied to prove Theorem 2.2. For this reason, we will explore the relationship between the Nehari manifold and Pohozaev manifold, and develop a rearrangement-free approach to exclude the vanishing phenomena (see Lemma 3.5 in Sect. 3). This rearrangement-free approach has its own interests and can be used in the settings where the symmetrization technique does not work. Indeed, Chen, Lu and Zhu have recently established in [8] the existence of a ground state solution to the
subelliptic quasilinear Schrödinger equation with constant or degenerate potentials on the Heisenberg group (or more general stratified groups).

In the recent work [13], the authors of this paper established the existence of ground-state solution for the following Schrödinger equation involving the degenerate Rabinowitz potential:

\[-\Delta u + V(x)u = f(u) \text{ in } \mathbb{R}^2,\]

where \( V(x) \geq 0 \) and may vanish on an open set of \( \mathbb{R}^2 \) and \( f \) has the critical exponential growth. This is the first existence result for elliptic equation involving critical exponential growth without using the standard assumption that the potential has positive lower bound. More recently, the authors established in [12] the existence of ground state solutions to the following quasilinear Schrödinger equation with the degenerate potential \( V \):

\[
\begin{align*}
\begin{cases}
-\text{div}(|\nabla u|^{n-2}\nabla u) + V(x)|u|^{n-2}u = f(u) \text{ in } \mathbb{R}^n, \\
u \in W^{1,n}(\mathbb{R}^n).
\end{cases}
\end{align*}
\] (2.3)

Motivated by the works in [13] and [12], we are interested to study the existence of ground state solutions to the following bi-harmonic equation:

\[-(\Delta)^2 u + V(x)u = f(u), \quad x \in \mathbb{R}^4,\] (2.4)

where \( f(t) \) satisfies \( f(0) = 0 \) and the conditions (i)–(v), and the potential \( V(x) \geq 0 \) satisfies

(V1) \( V(x) = 0 \) at \( B\delta(0) \) and \( V(x) \geq c_0 \) in \( \mathbb{R}^4 \setminus B_{\frac{3}{2}\delta}(0) \) for some \( c_0, \delta > 0 \),

(V2) \( \sup_{x \in \mathbb{R}^4} V(x) = \lim_{|x| \to \infty} V(x) = V_\infty > 0 \).

To this end, we first need to establish the following sharp critical Adams inequality involving the degenerate potential \( V(x) \).

**Theorem 2.4** Assume that the potential \( V(x) \) is non-negative, continuous and satisfies the condition (V1), then

\[
\sup_{u \in W^{2,2}(\mathbb{R}^4), f_{\mathbb{R}^d}(|\Delta u|^2 + V(x)|u|^2)d\mathbf{x} \leq 1} \int_{\mathbb{R}^4} (e^{32\pi^2 u^2} - 1)d\mathbf{x} < \infty.
\] (2.5)

**Remark 2.5** It should be noted that the loss of a positive lower bound of the potential \( V(x) \) makes this problem become fairly complicated and classical methods such as symmetrization argument and blow-up analysis fail in dealing with this problem. Furthermore, because \( \|\Delta(|u|)\|_{L^2(\mathbb{R}^d)} \leq \|\Delta(u)\|_{L^2(\mathbb{R}^d)} \) does not hold, it is not sufficient to only prove that this inequality (2.5) holds for all positive functions in Sobolev space \( W^{2,2}(\mathbb{R}^4) \). We will improve classical rearrangement-free argument developed by Lam and Lu in [21, 22] to overcome this difficulty.

Based on Theorems 2.2 and 2.4, by exploring the relationship between the Nehari manifold and the corresponding limiting Nehari manifold, we can obtain the following result.

**Theorem 2.6** Assume that \( V(x) \) is non-negative, continuous and satisfies the condition (V1) and (V2), then for any \( V_\infty \in (0, \gamma^*) \), the Eq. (2.4) admits a ground-state solution, where \( \gamma^* \) is defined as in Theorem 2.2.
This paper is organized as follows. Section 3 is devoted to the proofs of the existence and nonexistence of a ground-state solution to bi-harmonic Eq. (2.1) for general critical exponential growth. In Sect. 4, we will prove the critical Adams inequalities involving degenerate potential. In Sect. 5, we will prove the existence of ground state solutions for the bi-harmonic Eq. (2.4) with the degenerate Rabinowitz type potential.

Throughout this paper, the letter $c$ always denotes some positive constant which may vary from line to line.

3 Existence and nonexistence of a ground-state solution of bi-harmonic equations with constant potentials: Proof of Theorem 2.2

In this section, we are concerned with the ground states of the quasilinear bi-harmonic Eq. (2.1) with the constant potential and the nonlinearity $f(t)$ satisfying (i)-(iii). Namely, we will prove Theorem 2.2.

The associated functional and Nehari manifold are

$$I_\gamma(u) = \frac{1}{2} \int_{\mathbb{R}^4} (|\Delta u|^2 + \gamma |u|^2) \, dx - \int_{\mathbb{R}^4} F(u) \, dx,$$

$N_\gamma = \{ u \in W^{2,2}(\mathbb{R}^4) \mid u \neq 0, \, N_\gamma(u) = 0 \}$

respectively, where

$$N_\gamma(u) = \int_{\mathbb{R}^4} (|\Delta u|^2 + \gamma |u|^2) \, dx - \int_{\mathbb{R}^4} f(u)u \, dx.$$

One can easily verify that if $u \in N_\gamma$, then

$$I_\gamma(u) = \frac{1}{2} \int_{\mathbb{R}^4} (f(u)u - 2F(u)) \, dx.$$

In the following, we will denote the Sobolev norms by

$$\|u\|_{W^{2,2}_{\gamma}(\mathbb{R}^4)} = \left( \int_{\mathbb{R}^4} (|\Delta u|^2 + \gamma |u|^2) \, dx \right)^{1/2}$$

and

$$\|u\|_{W^{2,2}_{V}(\mathbb{R}^4)} = \left( \int_{\mathbb{R}^4} (|\Delta u|^2 + V(x)|u|^2) \, dx \right)^{1/2},$$

respectively.

We first claim

**Lemma 3.1** For any $u \in W^{2,2}(\mathbb{R}^4) \setminus \{0\}$, there exists a $t_u > 0$ such that $t_u u \in N_\gamma$.

**Proof** For any $u \in W^{2,2}(\mathbb{R}^4)$, we have

$$N_\gamma(tu) = t^2 \int_{\mathbb{R}^4} (|\Delta u|^2 + \gamma |u|^2) \, dx - \int_{\mathbb{R}^4} f(tu)(tu) \, dx. \quad (3.1)$$

Obviously, from the expression of $N_\gamma(tu)$ and the conditions (i) and (iv), it is not hard to find that $N_\gamma(tu) < 0$ for large $t$ and $N_\gamma(tu) > 0$ for small $t$. Hence, there exists a $t_u > 0$ such that $t_u u \in N_\gamma$. $\square$

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We recall that a solution \( u \) of (2.1) is called a ground state if
\[
I_\gamma(u) = \inf \{ I_\gamma(w) \mid w \neq 0, w \text{ is a weak solution of (2.1)} \}.
\]
Set \( m_\gamma = \inf \{ I_\gamma(u) \mid N_\gamma(u) = 0 \} \). Then the existence of ground-state solution of Eq. (2.1) is equivalent to the attainability of \( m_\gamma \). We claim

**Lemma 3.2** There holds
\[
0 < m_\gamma \leq \frac{16\pi^2}{\alpha_0}. \tag{3.2}
\]

**Proof** We first show that \( m_\gamma > 0 \). We prove this by contradiction. Assume that there exists some sequence \( u_k \in N_\gamma \) such that \( I_\gamma(u_k) \rightarrow 0 \), that is,
\[
\lim_{k \rightarrow +\infty} \int_{\mathbb{R}^4} (f(u_k)u_k - 2F(u_k)) \, dx = 0,
\]
which together with the (A–R) condition and \( u_k \in N_\gamma \) yields that
\[
\lim_{k \rightarrow +\infty} \int_{\mathbb{R}^4} (|\Delta u_k|^2 + \gamma |u_k|^2) \, dx = 0. \tag{3.3}
\]

On one hand, it follows from (2.2) and \( u_k \in N_\gamma \) that
\[
1 = \int_{\mathbb{R}^4} f(u_k) \frac{u_k}{\|u_k\|^2_{W^{2,2}(\mathbb{R}^4)}} \, dx
\leq \int_{\mathbb{R}^4} \left( \varepsilon \frac{|u_k|^2_{W^{2,2}(\mathbb{R}^4)}}{ \|u_k\|^2_{W^{2,2}(\mathbb{R}^4)}} + C_\varepsilon \frac{|u_k|^\mu+1_{W^{2,2}(\mathbb{R}^4)}}{ \|u_k\|^2_{W^{2,2}(\mathbb{R}^4)}} (e^{\beta_0 u_k^2} - 1) \right) \, dx \tag{3.4}
\]
On the other hand, by (3.3), Adams inequality (1.7) and the fact that \( \mu > 2 \), we get for any \( p > 1 \),
\[
\frac{1}{\|u_k\|^2_{W^{2,2}(\mathbb{R}^4)}} \int_{\mathbb{R}^4} |u_k|^\mu+1_{W^{2,2}(\mathbb{R}^4)} (e^{\beta_0 u_k^2} - 1) \, dx
\leq \frac{1}{\|u_k\|^2_{W^{2,2}(\mathbb{R}^4)}} \left( \int_{\mathbb{R}^4} |u_k|^\mu+1 \, dx \right)^{1/p} \left( \int_{\mathbb{R}^4} (e^{p^\prime \beta_0 u_k^2} - 1) \, dx \right)^{1/p^\prime}
\leq c \|u_k\|^{\mu-1}_{W^{2,2}(\mathbb{R}^4)} \rightarrow 0, \text{ as } k \rightarrow \infty
\]
which is a contradiction with (3.4).

Next, we prove that \( m_\gamma \leq \frac{16\pi^2}{\alpha_0} \). Let \( w \in W^{2,2}(\mathbb{R}^4) \) such that \( \int_{\mathbb{R}^4} (|\Delta w|^2 + \gamma |w|^2) \, dx = 1 \). Then there exists \( t_w > 0 \) such that
\[
\int_{\mathbb{R}^4} (|\Delta (t_w w)|^2 + \gamma (t_w w)^2 - f(t_w w)(t_w w)) \, dx = 0,
\]
which implies that
\[
m_\gamma \leq \frac{1}{2} \int_{\mathbb{R}^4} (|\Delta (t_w w)|^2 + \gamma (t_w w)^2) \, dx - \int_{\mathbb{R}^4} F(t_w w)
\leq \frac{t_w^2}{2} \int_{\mathbb{R}^4} (|\Delta w|^2 + \gamma |w|^2) \, dx = \frac{t_w^2}{2}. \tag{3.5}
\]
On the other hand, \( f\left(\frac{t w}{t}\right) w \) is monotone increasing about the variable \( t \). Set \( m_\gamma = \frac{t_\gamma^2}{2} \). Then we derive that

\[
\int_{\mathbb{R}^4} \frac{f(t_\gamma w)}{t_\gamma} w \, dx \leq \int_{\mathbb{R}^4} \frac{f(t_w w)}{t_w} w \, dx
\]

\[
= \int_{\mathbb{R}^4} (|\Delta w|^2 + \gamma |w|^2) \, dx = 1,
\]

which implies that

\[
\sup_{f_{\mathbb{R}^4}(|\Delta w|^2 + \gamma |w|^2) \leq 1} \int_{\mathbb{R}^4} \frac{f(t_\gamma w)}{t_\gamma} w \, dx < \infty.
\]

Next, we claim \( m_\gamma = \frac{t_\gamma^2}{2} \leq \frac{16\pi^2}{\alpha_0} \). Indeed, the classical Adams inequality on \( W^{2,2}(\mathbb{R}^4) \) under the less restrictive constraint proved by Lam and Lu in [24] states that

\[
\sup_{u \in W^{2,2}(\mathbb{R}^4), \int_{\mathbb{R}^4} (|\Delta u|^2 + \gamma |u|^2) \, dx \leq L} \int_{\mathbb{R}^4} (e^{\alpha u^2} - 1) \, dx < \infty
\]

holds if and only if \( \alpha \leq 32\pi^2 \). Since \( f \) has the critical exponential growth, if \( m_\gamma > \frac{16\pi^2}{\alpha_0} \), then \( \lim_{s \to +\infty} \frac{f(t_\gamma s)}{e^{32\pi^2 s^2}} = +\infty \). Then one can construct the well-known Moser sequence (that is a concentration sequence which blows up at some point) to deduce that

\[
\sup_{u \in W^{2,2}(\mathbb{R}^4), \int_{\mathbb{R}^4} (|\Delta u|^2 + \gamma |u|^2) \, dx \leq L} \int_{\mathbb{R}^4} \frac{f(t_\gamma w)}{t_\gamma} w \, dx = +\infty,
\]

which is a contradiction. This proves that \( m_\gamma \leq \frac{16\pi^2}{\alpha_0} \).

Now, we introduce the Adams ratio:

\[
C_A^L = \sup_{\|u\|_2^2} \frac{2}{\|u\|_2^2} \int_{\mathbb{R}^4} F(u) \, dx \mid u \in W^{2,2}(\mathbb{R}^4) \setminus \{0\}, \|\Delta u\|_2^2 \leq L.
\]

The Adams threshold \( R(F) \) is given by

\[
R(F) = \sup\{L > 0 \mid C_A^L < +\infty\}.
\]

We denote by \( \gamma^* = C_A^{R(F)} \) the ratio at the threshold \( R(F) \). It follows from the critical exponential growth of nonlinearity \( f \) and the Adams inequality with the exact growth condition in \( W^{2,2}(\mathbb{R}^4) \) (see [35] and also [34]) that \( R(F) = 32\pi^2/\alpha_0 \).

Next, we claim that

**Lemma 3.3** If \( \gamma < \gamma^* \), then \( m_\gamma < \frac{16\pi^2}{\alpha_0} \).

**Proof** The proof is divided into two steps:

**Step 1:** Define the Pohozaev manifold \( \mathcal{P}_\gamma \) by

\[
\mathcal{P}_\gamma = \{u \in W^{2,2}(\mathbb{R}^4) \mid \int_{\mathbb{R}^4} \gamma |u|^2 \, dx = 2 \int_{\mathbb{R}^4} F(u) \, dx\},
\]

and \( M_p = \inf\{I_\gamma(u), u \in \mathcal{P}_\gamma\} \), we claim \( m_\gamma \leq M_p \).
Assume that \( \{u_k\}_k \) is a minimizing sequence for \( M_p \), that is \( u_k \in \mathcal{P}_\gamma \) and
\[
\lim_{k \to +\infty} \frac{1}{2} \| \Delta u_k \|_2^2 = M_p.
\]
Choosing \( \lambda_k \) such that
\[
\int_{\mathbb{R}^4} |\Delta (u_k(\lambda_k x))|^2 \, dx + \gamma \int_{\mathbb{R}^4} |u_k(\lambda_k x)|^2 \, dx = \int_{\mathbb{R}^4} f(u_k(\lambda_k x)) u(\lambda_k x) \, dx.
\]
Direct computations yields
\[
\lambda_k^4 = \frac{\int_{\mathbb{R}^4} (f(u_k)u_k - 2F(u_k)) \, dx}{\| \Delta u_k \|_2^2},
\]
which together with the (A–R) condition gives that \( \lambda_k > 0 \). Obviously \( u_k(\lambda_k x) \in \mathcal{P}_\gamma \cap \mathcal{N}_\gamma \) and
\[
m_\gamma \leq \lim_{k \to +\infty} I(u_k(\lambda_k x)) = \lim_{k \to +\infty} \frac{1}{2} \int_{\mathbb{R}^4} |\Delta (u_k(\lambda_k x))|^2 \, dx = \lim_{k \to +\infty} \frac{1}{2} \int_{\mathbb{R}^4} |\Delta u_k|^2 \, dx = M_p.
\]

(3.7)

**Step 2:** We claim that if \( \gamma < \gamma^* \), then \( M_p < \frac{16\pi^2}{\alpha_0} \).

We distinguish between the case \( \gamma^* < +\infty \) and \( \gamma^* = +\infty \).

In the case \( \gamma^* < +\infty \), since \( \gamma < \gamma^* \), then \( \gamma < \gamma^* - \epsilon_0 \) for some \( \epsilon_0 > 0 \). It follows from the definition of \( \gamma^* \) that there exists some \( u_0 \in W^{2,2}(\mathbb{R}^4) \) with \( \| \Delta u_0 \|_2^2 \leq R(F) \) satisfying
\[
\gamma^* - \epsilon_0 < \frac{2}{\| u_0 \|_2^2} \int_{\mathbb{R}^4} F(u_0) \, dx.
\]
Consequently,
\[
\gamma \| u_0 \|_2^2 < 2 \int_{\mathbb{R}^4} F(u_0) \, dx.
\]
Let \( h(s) = \gamma \int_{\mathbb{R}^4} |su_0|^2 \, dx - 2 \int_{\mathbb{R}^4} F(su_0) \, dx \) for \( s > 0 \). Since \( h(1) < 0 \) and \( h(s) > 0 \) for \( s > 0 \) small enough, then there exists \( s_0 \in (0, 1) \) satisfying \( h(s_0u_0) = 0 \). Therefore, we have \( s_0u_0 \in \mathcal{P}_\gamma \) and
\[
M_p \leq \frac{1}{2} \| \Delta (s_0u_0) \|_2^2 = \frac{1}{2} s_0^2 \| \Delta u_0 \|_2^2 < \frac{1}{2} \frac{16\pi^2}{\alpha_0} R(F) = \frac{16\pi^2}{\alpha_0}.
\]

In the case \( \gamma^* = +\infty \), for any \( \gamma > 0 \), there exists \( u_0 \in W^{2,2}(\mathbb{R}^4) \) with \( \| \Delta u_0 \|_2^2 \leq R(F) \) satisfying
\[
\gamma \| u_0 \|_2^2 < 2 \int_{\mathbb{R}^4} F(u_0) \, dx.
\]
Hence we can repeat the same arguments as case \( \gamma^* < +\infty \) to get the conclusion. Combining Step 1 and Step 2, we conclude that if \( \gamma < \gamma^* \), then \( m_\gamma < \frac{16\pi^2}{\alpha_0}. \)

We now consider a minimizing sequence \( \{u_k\}_k \subset \mathcal{N}_\gamma \) for \( m_\gamma \). According to the (A–R) condition (ii) and \( I(u_k) \to m_\gamma > 0 \) and \( u_k \in \mathcal{N}_\gamma \), we can obtain
\[
\left( 1 - \frac{2}{\mu} \right) \lim_{k \to +\infty} \int_{\mathbb{R}^4} (|\Delta u_k|^2 + |u_k|^2) \, dx \leq m_\gamma,
\]
which implies that \( u_k \) is bounded in \( W^{2,2}(\mathbb{R}^4) \). Then up to a subsequence, there exists \( u \in W^{2,2}(\mathbb{R}^4) \) such that
\( u_k \rightarrow u \) weakly in \( W^{2,2}(\mathbb{R}^4) \) and in \( L^p(\mathbb{R}^4) \), for any \( p > 1 \),
\( u_k \rightarrow u \) in \( L^p_{\text{loc}}(\mathbb{R}^4) \),
\( u_k \rightarrow u \), a.e.

Furthermore, using \( u_k \in \mathcal{N}_\gamma \) and the (A–R) condition (ii) again, we conclude that
\[
\mu \int_{\mathbb{R}^4} F(u_k) dx \leq \int_{\mathbb{R}^4} f(u_k) u_k dx = \int_{\mathbb{R}^4} (|\Delta u_k|^2 + |u_k|^2) dx < +\infty.
\]

**Lemma 3.4** If \( \gamma \neq \gamma^* \), then up to some translation, we can assume that the minimizing sequence \( u_k \) satisfies \( \lim \lim_{k \to +\infty} \int_{B_L} f(u_k) u_k dx \neq 0 \).

**Proof** Define \( M(L) = \lim_{k \to +\infty} \sup_{y \in \mathbb{R}^4} \int_{B_L(y)} f(u_k) u_k dx \), we will show that \( \lim_{L \to +\infty} M(L) \neq 0 \). We first show that there exists some \( R > 0 \) such that
\[
\lim_{k \to +\infty} \int_{\{|u_k| < R\}} f(u_k) u_k dx > 0.
\]

Suppose not, that is
\[
\lim_{k \to +\infty} \lim_{R \to +\infty} \int_{\{|u_k| < R\}} f(u_k) u_k dx = 0,
\]

and sequence \( \{u_k\} \) blows up. By the condition (iii), we can deduce that \( \lim_{|t| \to +\infty} \frac{f(t)}{F(t)} = +\infty \). In fact, suppose not, then there exist some \( R > 0 \) s.t. \( \|f(t_k)\|_2 \leq M \) for any \( |t_k| \rightarrow +\infty \) such that \( |t_k f(t_k)| \leq M |F(t_k)| \). This together with the condition (iii) yields
\[
|t_k f(t_k)| \leq M |F(t_k)| \leq M M_0 |f(t_k)|,
\]
which is a contradiction with the fact \( |t_k| \rightarrow +\infty \). According to \( \lim_{|t| \to +\infty} \frac{f(t)}{F(t)} = +\infty \), we derive that \( F(t) = f(t) t \cdot o(t)(1) \) as \( |t| \to +\infty \). Gathering this with the boundedness of \( \int_{\mathbb{R}^4} f(u_k) u_k dx \) and \( \lim_{k \to +\infty} \int_{\{|u_k| < R\}} f(u_k) u_k dx = 0 \), we derive that
\[
\lim_{k \to +\infty} \int_{\mathbb{R}^4} F(u_k) dx = \lim_{k \to +\infty} \int_{\{|u_k| < R\}} F(u_k) dx + \lim_{k \to +\infty} \int_{\{|u_k| \geq R\}} F(u_k) dx
\]
\[
= \lim_{k \to +\infty} \int_{\{|u_k| < R\}} F(u_k) dx + \lim_{k \to +\infty} \int_{\{|u_k| \geq R\}} F(u_k) dx = 0.
\]

(3.9)

From this claim and Lemma 3.3, we immediately get \( \lim_{k \to +\infty} \|\Delta u_k\|_{L^2(\mathbb{R}^4)}^2 + \|u_k\|_{L^2(\mathbb{R}^4)}^2 = 2\gamma < \frac{32\pi^2}{\alpha_0} \). Through Remark 2.1, we see that for any \( \beta_0 > \alpha_0 \) and \( \varepsilon > 0 \),
\[
\int_{\mathbb{R}^4} f(u_k) u_k dx \leq \varepsilon \|u_k\|_2 + C_\varepsilon \int_{\mathbb{R}^4} |u_k|^{\mu+1} \left( e^{\beta_0 u_k^2} - 1 \right) dx.
\]

(3.10)

Since \( \lim_{k \to +\infty} \|\Delta u_k\|_{L^2(\mathbb{R}^4)}^2 + \|u_k\|_{L^2(\mathbb{R}^4)}^2 = \frac{32\pi^2}{\alpha_0} \), through Adams inequality in \( \mathbb{R}^4 \) (1.7), we derive that there exists \( p > 1 \) sufficiently close to 1 such that \( \int_{\mathbb{R}^4} \left( e^{\beta_0 u_k^2} - 1 \right)^p dx \leq 1 \) since \( \beta_0 \) can be chosen close to \( \alpha_0 \). Hence by Hölder’s inequality, we obtain
\[
\int_{\mathbb{R}^4} |u_k|^{\mu+1} \left( e^{\beta_0 u_k^2} - 1 \right) dx \leq \left( \int_{\mathbb{R}^4} \left( e^{\beta_0 u_k^2} - 1 \right)^p dx \right)^{\frac{1}{p}} \left( \int_{\mathbb{R}^4} |u_k|^{p(\mu+1)} \right)^{\frac{1}{p}} \leq \left( \int_{\mathbb{R}^4} |u_k|^{p(\mu+1)} \right)^{\frac{1}{p}}.
\]
This together with \( \lim_{k \to +\infty} \int_{\mathbb{R}^4} F(u_k) \, dx = 0 \) gives that \( \lim_{k \to +\infty} C_\varepsilon \int_{\mathbb{R}^4} |u_k|^{\mu+1} (e^{\beta_0 R^2} - 1) \, dx = 0 \).

This proves that \( \lim_{k \to +\infty} \int_{\mathbb{R}^4} f(u_k) u_k \, dx \leq \varepsilon \lim_{k \to +\infty} \| u_k \|^2 \) holds for any \( \varepsilon > 0 \), which is a contradiction. Hence we conclude that there exists some \( R > 0 \) such that (3.8) holds.

Now, we are in position to prove that \( \lim_{L \to +\infty} M(L) \neq 0 \). In fact, if \( \lim_{L \to +\infty} M(L) = 0 \), then for any \( L > 0 \), \( \lim_{k \to +\infty} \sup_{y \in \mathbb{R}^4} \int_{B_L(y)} f(u_k) u_k \, dx = 0 \). It follows from the Lions lemma (see Lemma 1.1 in [33]) that \( \lim_{k \to +\infty} \| u \|_{L^q(\mathbb{R}^4)} = 0 \) for any \( q > 2 \). Hence by Remark 2.1, we can derive that for any \( R > 0 \),

\[
\lim_{k \to +\infty} \int_{|u_k| < R} |f(u_k) u_k| \, dx \leq \lim_{k \to +\infty} \varepsilon \| u_k \|_2 + C_\varepsilon \lim_{k \to +\infty} \int_{\mathbb{R}^4} |u_k|^{\mu+1} (e^{\beta_0 R^2} - 1) \, dx \leq C_\varepsilon,
\]

which is an contradiction with \( \lim_{k \to +\infty} \int_{|u_k| < R} f(u_k) u_k \, dx > 0 \). This accomplishes the proof of \( \lim_{L \to +\infty} M(L) \neq 0 \).

Hence there exists \( x_k \in \mathbb{R}^4 \) such that \( \lim_{L \to +\infty} \lim_{k \to +\infty} \int_{B_L(x_k)} f(u_k) u_k \, dx \neq 0 \). Denote \( \{ w_k \}_k \) by \( w_k(x) = u_k(x + x_k) \), then \( w_k \) is still a minimizing sequence for \( M_{\gamma} \) and

\[
\lim_{L \to +\infty} \lim_{k \to +\infty} \int_{B_L} f(w_k) w_k \, dx \neq 0.
\]

For convenience, we still denoted this new minimizing sequence by \( \{ u_k \}_k \).

Next, we claim that

**Lemma 3.5** If \( \gamma < \gamma^* \), it holds

\[
\lim_{k \to +\infty} \int_{\mathbb{R}^4} f(u_k) u_k \, dx = \lim_{L \to +\infty} \lim_{k \to +\infty} \int_{B_{L}(0)} f(u_k) u_k \, dx
\]

and

\[
\lim_{L \to +\infty} \lim_{k \to +\infty} \int_{\mathbb{R}^4 \setminus B_{L}(0)} f(u_k) u_k \, dx = 0.
\]

**Proof** Set

\[
M = \lim_{k \to +\infty} \int_{\mathbb{R}^4} f(u_k) u_k \, dx,
\]

\[
M^0 = \lim_{L \to +\infty} \lim_{k \to +\infty} \int_{B_{L}(0)} f(u_k) u_k \, dx
\]

and

\[
M^\infty = \lim_{L \to +\infty} \lim_{k \to +\infty} \int_{\mathbb{R}^4 \setminus B_{L}(0)} f(u_k) u_k \, dx.
\]

Obviously, we have \( M^0 + M^\infty = M \). We now show that \( (M^0, M^\infty) = (M, 0) \) or \( (M^0, M^\infty) = (0, M) \). Since \( u_k \in \mathcal{N}_\gamma \), then

\[
\int_{\mathbb{R}^4} (|\Delta u_k|^2 + \gamma |u_k|^2) \, dx = \lim_{L \to +\infty} \lim_{k \to +\infty} \int_{B_{L}(0)} f(u_k) u_k \, dx + \lim_{L \to +\infty} \lim_{k \to +\infty} \int_{\mathbb{R}^4 \setminus B_{L}(0)} f(u_k) u_k \, dx
\]

\[
= M^0 + M^\infty.
\]

(3.12)
Noticing that we can also write \( \int_{\mathbb{R}^4} (|\Delta u_k|^2 + \gamma |u_k|^2) \) as
\[
\int_{\mathbb{R}^4} (|\Delta u_k|^2 + \gamma |u_k|^2) \, dx = \lim_{L \to +\infty} \lim_{k \to +\infty} \int_{B_L(0)} (|\Delta u_k|^2 + \gamma |u_k|^2) \, dx \]

Hence we can assume that
\[
\lim_{L \to +\infty} \lim_{k \to +\infty} \int_{B_L(0)} (|\Delta u_k|^2 + \gamma |u_k|^2) \, dx = \lim_{L \to +\infty} \lim_{k \to +\infty} \int_{\mathbb{R}^4 \setminus B_L(0)} f(u_k) u_k \, dx \tag{3.13}
\]

or
\[
\lim_{L \to +\infty} \lim_{k \to +\infty} \int_{\mathbb{R}^4 \setminus B_L(0)} (|\Delta u_k|^2 + \gamma |u_k|^2) \, dx \leq \lim_{L \to +\infty} \lim_{k \to +\infty} \int_{\mathbb{R}^4 \setminus B_L(0)} f(u_k) u_k \, dx \tag{3.14}
\]

For a sufficiently large number \( R \), take a radial function \( \phi^0(r) \in C_0^\infty(\mathbb{R}^4) \) satisfying \( 0 \leq \phi^0(r) \leq 1, \phi^0(r) = 1 \) for \( r \leq L \), \( \phi^0(r) = 0 \) for \( r \geq L + 1 \), \( \phi'(r) \leq c \) and \( \phi''(r) \leq c \). Set \( \phi^\infty(r) = 1 - \phi^0(r) \) and \( u^\infty_{k,L} = u_k \phi^\infty_{k,L} \) (\( * = 0 \) or \( \infty \)), we can easily derive the following fact by an argument similar to that in Lemma 2.1 of [18].

\[
\lim_{L \to +\infty} \lim_{k \to +\infty} \int_{\mathbb{R}^4} (|\Delta u_{k,L}^0|^2 + \gamma |u_{k,L}^0|^2) \, dx = \lim_{L \to +\infty} \lim_{k \to +\infty} \int_{B_L(0)} (|\Delta u_k|^2 + \gamma |u_k|^2) \, dx
\]

\[
\lim_{L \to +\infty} \lim_{k \to +\infty} \int_{\mathbb{R}^4} (|\Delta u_{k,L}^\infty|^2 + \gamma |u_{k,L}^\infty|^2) \, dx = \lim_{L \to +\infty} \lim_{k \to +\infty} \int_{\mathbb{R}^4 \setminus B_L(0)} (|\Delta u_k|^2 + \gamma |u_k|^2) \, dx
\]

\[
\lim_{L \to +\infty} \lim_{k \to +\infty} \int_{\mathbb{R}^4} f(u_{k,L}^0) \, dx = \lim_{L \to +\infty} \lim_{k \to +\infty} \int_{B_L(0)} f(u_k) u_k \, dx
\]

\[
\lim_{L \to +\infty} \lim_{k \to +\infty} \int_{\mathbb{R}^4} F(u_{k,L}^0) \, dx = \lim_{L \to +\infty} \lim_{k \to +\infty} \int_{\mathbb{R}^4 \setminus B_L(0)} F(u_k) \, dx
\]

If (3.13) holds, then there exists \( t^0_{k,L} \) such that \( t^0_{k,L} u_{k,L}^0 \in N_v \). We claim that
\[
\lim_{L \to +\infty} \lim_{k \to +\infty} t^0_{k,L} \leq 1. \]

Indeed, by the assumption (3.13) and the definition of \( u_{k,L}^0 \), we have
\[
\lim_{L \to +\infty} \lim_{k \to +\infty} \int_{\mathbb{R}^4} \left( |\Delta u_k|^2 + \gamma |u_{k,L}^0|^2 \right) \, dx \leq \lim_{L \to +\infty} \lim_{k \to +\infty} \int_{\mathbb{R}^4} f(u_{k,L}^0) u_{k,L}^0 \, dx. \tag{3.15}
\]

From the assumption (v), we know that \( f(t) = g(t)t \), where \( g(t) \) is increasing in \((0, +\infty)\) and decreasing in \((-\infty, 0)\). Hence if
\[
\lim_{L \to +\infty} \lim_{k \to +\infty} t^0_{k,L} = t_0 > 1,
\]
then
\[
\lim_{L \to +\infty} \lim_{k \to +\infty} \int_{\mathbb{R}^4} \left( |\Delta u_{k,L}^0 u_k|^2 + \gamma |u_{k,L}^0 u_{k,L}^0|^2 \right) \, dx
\]

\[
= t_0^2 \left( \lim_{L \to +\infty} \lim_{k \to +\infty} \int_{\mathbb{R}^4} \left( |\Delta u_k|^2 + \gamma |u_{k,L}^0|^2 \right) \, dx \right)
\]

\( \square \)
\[ t_0^2 \lim_{L \to +\infty} \lim_{k \to +\infty} \int_{\mathbb{R}^4} g(t_0 u_{k,L}^0)(u_{k,L}^0)^2 \, dx \]

\[
> t_0^2 \lim_{L \to +\infty} \lim_{k \to +\infty} \int_{\mathbb{R}^4} g(u_{k,L}^0)(u_{k,L}^0)^2 \, dx,
\]

that is

\[
\lim_{L \to +\infty} \lim_{k \to +\infty} \int_{\mathbb{R}^4} \left( |\Delta u_k|^2 + \gamma |u_{k,L}^0|^2 \right) dx > \lim_{L \to +\infty} \lim_{k \to +\infty} \int_{\mathbb{R}^4} f(u_{k,L}^0)u_{k,L}^0 \, dx,
\]

which is a contradiction with (3.15).

If \( t_{k,L}^0 \leq 1 \), then

\[
I_{\gamma}(t_{k,L}^0 u_{k,L}^0) = \frac{1}{2} \int_{\mathbb{R}^4} \left( f(t_{k,L}^0 u_{k,L}^0) t_{k,L}^0 u_{k,L}^0 - 2F(t_{k,L}^0 u_{k,L}^0) \right) dx 
\]

\[
\leq \frac{1}{2} \int_{\mathbb{R}^4} \left( f(u_{k,L}^0) u_{k,L}^0 - 2F(u_{k,L}^0) \right) dx, \tag{3.16}
\]

where we have used the monotonicity of function \( f(t) \) \( t - 2F(t) \), see Remark 2.1. If \( t_{k,L}^0 \geq 1 \), then

\[
I_{\gamma}(t_{k,L}^0 u_{k,L}^0) = \frac{1}{2} \int_{\mathbb{R}^4} \left( |\Delta u_{k,L}|^2 + \gamma |u_{k,L}^0|^2 \right) dx - \int_{\mathbb{R}^4} F(t_{k,L}^0 u_{k,L}^0) dx 
\]

\[
\leq \frac{1}{2} \int_{\mathbb{R}^4} \left( |\Delta u_{k,L}|^2 + \gamma |u_{k,L}^0|^2 \right) dx - \int_{\mathbb{R}^4} F(u_{k,L}^0) dx 
\]

\[
\leq \frac{1}{2} \int_{\mathbb{R}^4} \left( f(u_{k,L}^0) u_{k,L}^0 + o_{k,L}(1) \right) dx - \int_{\mathbb{R}^4} F(u_{k,L}^0) dx. \tag{3.17}
\]

Combining the above estimate, we derive that

\[
m_{\gamma} \leq \lim_{L \to +\infty} \lim_{k \to +\infty} I_{\gamma}(t_{k,L}^0 u_{k,L}^0) 
\]

\[
\leq \frac{1}{2} \lim_{L \to +\infty} \lim_{k \to +\infty} \int_{\mathbb{R}^4} (f(u_{k,L}^0) u_{k,L}^0 - 2F(u_{k,L}^0)) dx 
\]

\[
+ \frac{1}{2} \lim_{L \to +\infty} \lim_{k \to +\infty} \int_{\mathbb{R}^4} (f(u_{k,L}^\infty) u_{k,L}^\infty - 2F(u_{k,L}^\infty)) dx 
\]

\[
= \lim_{k \to +\infty} I(u_k) = m_{\gamma}. \tag{3.18}
\]

Thus, we can conclude that

\[
\lim_{L \to +\infty} \lim_{k \to +\infty} \int_{\mathbb{R}^4} (f(u_{k,L}^\infty) u_{k,L}^\infty - 2F(u_{k,L}^\infty)) dx = 0,
\]

that is

\[
\lim_{L \to +\infty} \lim_{k \to +\infty} \int_{\mathbb{R}^4 \setminus B_L(0)} (f(u_k) u_k - 2F(u_k)) dx = 0,
\]

which together with (A–R) condition implies that \( M^\infty = 0 \), that is \((M^0, M^\infty) = (M, 0)\). Similarly, we can prove that \((M^0, M^\infty) = (0, M)\) if we assume that (3.14) holds.

Now, it remains to show that \((M^0, M^\infty) = (0, M)\) is impossible to occur. In fact, according to Lemma 3.4, we get \( \lim_{L \to +\infty} \lim_{k \to +\infty} \int_{B_L(0)} f(u_k) u_k dx \neq 0 \), which implies that \( M^0 \neq 0 \). This accomplishes the proof of Lemma 3.5. \( \square \)
Lemma 3.6 There holds \( \lim_{k \to +\infty} \int_{\mathbb{R}^4} F(u_k) dx = \int_{\mathbb{R}^4} F(u) dx \).

**Proof** It follows from Lemma 3.5 that \( \lim_{L \to +\infty} \lim_{k \to +\infty} \int_{\mathbb{R}^4 \setminus B_L(0)} f(u_k)u_k dx = 0 \), which together with the (A–R) condition implies that \( \lim_{L \to +\infty} \lim_{k \to +\infty} \int_{\mathbb{R}^4 \setminus B_L(0)} F(u_k) dx = 0 \). In order to obtain the desired convergence, we only need to prove that

\[
\lim_{L \to +\infty} \lim_{k \to +\infty} \int_{B_L(0)} F(u_k) dx = \int_{\mathbb{R}^4} F(u) dx.
\]

Indeed, for any \( s > 0 \), we have

\[
\left| \int_{B_L(0)} F(u_k) dx - \int_{B_L(0)} F(u) dx \right| \\
\leq \left| \int_{B_L(0) \cap \{|u_k| < s\}} F(u_k) dx - \int_{B_L(0) \cap \{|u_k| < s\}} F(u) dx \right| \\
+ \left| \int_{B_L(0) \cap \{|u_k| \geq s\}} F(u_k) dx - \int_{B_L(0) \cap \{|u_k| \geq s\}} F(u) dx \right| \tag{3.19}
\]

A direct application of the dominated convergence theorem leads to \( I_{k,R,s} \to 0 \). For \( II_{k,R,s} \), from the condition (iii), we have

\[
\int_{B_L^c \cap \{|u_k| \geq s\}} F(u_k) dx \leq \frac{c}{s} \int_{\mathbb{R}^4 \cap \{|u_k| \geq s\}} f(u_k)u_k dx \\
= \frac{c}{s} \int_{\mathbb{R}^4} f(u_k)u_k dx \to 0, \quad \text{as} \quad s \to \infty,
\]

where we have used the fact that \( \int_{\mathbb{R}^4} f(u_k)u_k dx \) is bounded (See the beginning of Lemma 3.5). Consequently, \( II_{k,R,s} \to 0 \), and the lemma is finished. \( \square \)

Lemma 3.7 Let \( u_k \) be a bounded sequence in \( W^{2,2}(\mathbb{R}^4) \) converging weakly to non-zero \( u \). Furthermore, we also assume that \( \lim_{k \to +\infty} I_\gamma(u_k) < \frac{16\pi^2}{a_0} \) and \( \int_{\mathbb{R}^4} (|\Delta u|^2 + \gamma |u|^2) dx > \int_{\mathbb{R}^4} f(u)udx \), then

\[
\lim_{k \to +\infty} \int_{\mathbb{R}^4} f(u_k)u_k dx = \int_{\mathbb{R}^4} f(u)udx.
\]

**Proof** According to Lemma 3.5, we only need to prove that

\[
\lim_{L \to +\infty} \lim_{k \to +\infty} \int_{B_L(0)} f(u_k)u_k dx = \int_{\mathbb{R}^4} f(u)udx.
\]

It follows the lower semicontinuity of the norm in \( W^{2,2}(\mathbb{R}^4) \) that

\[
\lim_{k \to \infty} \int_{\mathbb{R}^4} (|\Delta u_k|^2 + \gamma |u_k|^2) dx \geq \int_{\mathbb{R}^4} (|\Delta u|^2 + \gamma |u|^2) dx.
\]

We divide the proof into the following two cases.

Case 1: \( \int_{\mathbb{R}^4} (|\Delta u_k|^2 + \gamma |u_k|^2) dx = \int_{\mathbb{R}^4} (|\Delta u|^2 + \gamma |u|^2) dx \), then according to convexity of the norm and the equivalence of norms, we see that \( u_k \to u \) in \( W^{2,2}(\mathbb{R}^4) \), hence \( u_k \to u \) in \( L^p(\mathbb{R}^4) \) for any \( p \geq 2 \).
We claim that
\[ \sup_k \int_{\mathbb{R}^4} (f(u_k)u_k)^{p_0} \, dx < \infty, \]
for any \( p_0 > 1 \). Indeed, from the assumption on \( f(t) \) (critical exponential growth, see (1.2)) and (iv), we see that \( f(u)u \leq c(\exp(\alpha u^2) - 1) \) for some \( \alpha > 0 \). Then we can obtain the claim by using the Adams inequality in \( W^{2,2}(\mathbb{R}^4) \). By (3.20) and the Vitali convergence theorem we have \( \lim_{k \to \infty} \sup_k \int_{B_L} f(u_k)u_k \, dx = \int_{B_L} f(u)u \, dx. \) Therefore, it follows that

\[ \lim_{L \to +\infty} \lim_{k \to \infty} \int_{B_L} f(u_k)u_k \, dx = \lim_{L \to +\infty} \int_{B_L} f(u)u \, dx = \int_{\mathbb{R}^4} f(u)u \, dx. \] (3.21)

Case 2: If \( \lim_{k \to \infty} \int_{\mathbb{R}^4} (|\Delta u_k|^2 + \gamma |u_k|^2) \, dx > \int_{\mathbb{R}^4} (|\Delta u|^2 + \gamma |u|^2) \, dx \), we set
\[ v_k := \frac{u_k}{\left(\|\Delta u_k\|^2 + \gamma \|u_k\|^2\right)^{\frac{1}{2}}} \quad \text{and} \quad v_0 := \frac{u}{\left(\|\Delta u_0\|^2 + \gamma \|u_0\|^2\right)^{\frac{1}{2}}}. \]
We claim there exists \( q_0 > 1 \) sufficiently 1 such that
\[ q_0(\|\Delta u_k\|^2 + \gamma \|u_k\|^2) < \frac{32\pi^2}{1 - (\|\Delta v_0\|^2 + \gamma \|v_0\|^2)}. \] (3.22)
Indeed, we can apply the Lemma 3.6 and (A–R) condition to obtain
\begin{align*}
\lim_{k \to \infty} \left(\|\Delta u_k\|^2 + \gamma \|u_k\|^2\right)(1 - (\|\Delta v_0\|^2 + \gamma \|v_0\|^2)) &= \lim_{k \to \infty} \left(\|\Delta u_k\|^2 + \gamma \|u_k\|^2\right) \left(1 - \frac{\|\Delta u\|^2 + \gamma \|u\|^2}{\|\Delta u_k\|^2 + \gamma \|u_k\|^2}\right) \\
&= 2 \lim_{k \to +\infty} I_\gamma(u_k) + 2 \int_{\mathbb{R}^4} F(u_k) \, dx - 2I(u) - 2 \int_{\mathbb{R}^4} F(u) \, dx \\
&< \frac{32\pi^2}{\alpha_0},
\end{align*}
where the last inequality holds because \( \int_{\mathbb{R}^4} (|\Delta u|^2 + \gamma |u|^2) \, dx > \int_{\mathbb{R}^4} f(u)u \, dx > 2 \int_{\mathbb{R}^4} F(u) \, dx \). Combining the above estimate with the concentration compactness principle (Theorem 3 of the reference [8]) associated with the Adams inequality in \( W^{2,2}(\mathbb{R}^4) \), one can derive that there exists \( p_0 > 1 \) such that
\[ \sup_k \int_{\mathbb{R}^4} (f(u_k)u_k)^{p_0} \, dx < \infty. \] (3.24)
Then it follows from the Vitali convergence theorem that
\[ \lim_{L \to +\infty} \lim_{k \to \infty} \int_{B_L} f(u_k)u_k \, dx = \lim_{L \to +\infty} \int_{B_L} f(u)u \, dx = \int_{\mathbb{R}^4} f(u)u \, dx. \]
Then the proof of Lemma 3.7 is accomplished. \( \square \)

Now we are in position to give the existence of ground-state solutions for the bi-harmonic equation with the constant potential \( \gamma < \gamma^* \), namely we will show that \( m_\gamma \) could be achieved.
Proof of the first part of Theorem 2.2  Recall that the minimizing sequence \( u_k \) for \( m_\gamma \) is bounded in \( W^{2,2}(\mathbb{R}^4) \) and weakly converges to \( u \) in \( W^{1,2}(\mathbb{R}^4) \). We will prove that \( u \) is the extremal for \( m_\gamma \). We first show that \( u \not= 0 \). We argue this by contradiction. If \( u \not= 0 \), then

\[
\lim_{k \to \infty} (\| \Delta u_k \|_2^2 + \gamma \| u_k \|_2^2) = 2 \lim_{k \to \infty} I_\gamma (u_k) + 2 \int_{\mathbb{R}^4} F(u_k) dx = 2 \lim_{k \to \infty} I_\gamma (u_k) = 2m_\gamma < \frac{32\pi^2}{\alpha_0}
\]

(3.25)

Then it follows from the Adams inequality in \( \mathbb{R}^4 \) and Lemma 3.5 that

\[
\lim_{k \to \infty} \int_{\mathbb{R}^4} f(u_k) u_k dx = 0,
\]

which implies that

\[
0 < m_\gamma = \lim_{k \to \infty} (\| \Delta u_k \|_2^2 + \gamma \| u_k \|_2^2) = \lim_{k \to \infty} \int_{\mathbb{R}^4} f(u_k) u_k dx = 0,
\]

which is a contradiction.

Next, we claim that

\[
\| \Delta u \|_2^2 + \gamma \| u \|_2^2 \leq \int_{\mathbb{R}^4} f(u) u dx.
\]

Suppose this is false, that is,

\[
\| \Delta u \|_2^2 + \gamma \| u \|_2^2 > \int_{\mathbb{R}^4} f(u) u dx.
\]

(3.26)

In view of Lemmas 3.3 and 3.7, we derive that

\[
\lim_{k \to \infty} \int_{\mathbb{R}^4} f(u_k) u_k dx = \int_{\mathbb{R}^4} f(u) u dx.
\]

This implies that

\[
\| \Delta u \|_2^2 + \gamma \| u \|_2^2 \leq \lim_{k \to \infty} \| \Delta u_k \|_2^2 + \gamma \| u_k \|_2^2
\]

\[
= \lim_{k \to \infty} \int_{\mathbb{R}^4} f(u_k) u_k dx
\]

\[
= \int_{\mathbb{R}^4} f(u) u dx < \| \Delta u \|_2^2 + \gamma \| u \|_2^2,
\]

(3.27)

which is a contradiction. This proves the claim.

Since

\[
\| \Delta u \|_2^2 + \gamma \| u \|_2^2 \leq \int_{\mathbb{R}^4} f(u) u dx,
\]

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there exists $\gamma_0 \in (0, 1]$ such that $\gamma_0 u \in \mathcal{N}_\gamma$. According to the definition of $m_\gamma$, we derive that
\[
m_\gamma \leq I_\gamma(\gamma_0 u) = \frac{1}{2} \int_{\mathbb{R}^4} (f(\gamma_0 u)(\gamma_0 u) - 2F(\gamma_0 u))dx \\
\leq \frac{1}{2} \int_{\mathbb{R}^4} (f(u)(u) - 2F(u))dx \\
\leq \lim_{k \to \infty} \frac{1}{2} \int_{\mathbb{R}^4} (f(u_k)(u_k) - 2F(u_k))dx \\
= \lim_{k \to \infty} I_\gamma(u_k) = m_\gamma.
\]
This implies that $\gamma_0 = 1$ and $u \in \mathcal{N}_\gamma$ and $I_\gamma(u) = m_\gamma$, which means that the Eq. (2.1) admits a ground-state solution for any $\gamma \in (0, \gamma^*)$.

In order to finish the proof of Theorem 2.2, we need the following result.

\textbf{Lemma 3.8} $m_\gamma < \frac{16\pi^2}{\alpha_0}$ if and only if $\gamma < \gamma^*$.

\textbf{Proof} Recalling Lemma 3.3, we have proved that if $\gamma < \gamma^*$, then $m_\gamma < \frac{16\pi^2}{\alpha_0}$. Hence we only need to prove that if $m_\gamma < \frac{16\pi^2}{\alpha_0}$, then $\gamma < \gamma^*$. Obviously, if $\gamma^* = +\infty$, then $\gamma < \gamma^*$ and the proof is finished. Therefore, without loss of generality, we may assume that $\gamma^* < +\infty$.

From the previously discussion, we know that if $m_\gamma < \frac{16\pi^2}{\alpha_0}$, then $m_\gamma$ could be achieved by some function $u \in W^{2,2}(\mathbb{R}^4) \setminus \{0\}$ which is a ground state solution to Eq. (2.1). Obviously, we have $u \in \mathcal{P}_\gamma$ (see the beginning of step 1 of Lemma 3.3), which implies that $M_p \leq m_\gamma$. Recalling Lemma 3.3, we have already proved that $M_p \geq m_\gamma$. Combining these facts, we conclude that $M_p$ is also achieved by $u$. Then according to the definition of the $M_p$, we have $\|\Delta u\|_2^2 < 32\pi^2/\alpha_0$ and $\gamma\|u\|_2^2 = 2 \int_{\mathbb{R}^4} F(u)dx$. Define
\[
g(s) = \frac{2}{s^2\|u\|_2^2} \int_{\mathbb{R}^4} F(su)dx,
\]
then $g(1) = \gamma$. From the (A–R) condition, then it is easy to see that $g(s)$ is monotone increasing. If we set $v = \frac{R(F)^{1/2}}{\|\Delta u\|_2} u$, then $\|\Delta v\|_2^2 = R(F)$ and
\[
\gamma^* \geq \frac{2}{\|v\|_2^2} \int_{\mathbb{R}^4} F(v)dx = g\left(\frac{R(F)^{1/2}}{\|\Delta u\|_2}\right) > g(1) = \gamma.
\]

Now, we give the proof for the non-existence of ground-state solutions for the bi-harmonic equation with the constant potential $\gamma > \gamma^*$.

\textbf{Proof of the second part of Theorem 2.2} We argue this by contradiction. We assume that there exists $\gamma_0 > \gamma^*$ such that the Eq. (2.1) admits a ground-state solution. From Lemma 3.2 and Lemma 3.8, we know that $m_{\gamma_0} = \frac{16\pi^2}{\alpha_0}$. Since $m_{\gamma_0}$ could be achieved by some function $u_0 \in W^{2,2}(\mathbb{R}^4)$, direct calculation gives that for any $\gamma \in (\gamma^*, \gamma_0)$, $m_\gamma < m_{\gamma_0} = \frac{16\pi^2}{\alpha_0}$. This is a contradiction with the fact: $m_\gamma < \frac{16\pi^2}{\alpha_0}$ if and only if $\gamma < \gamma^*$. This indicates that for any $\gamma > \gamma^*$, Eq. (2.1) does not admit a ground-state solution.
4 The Adams inequality with degenerate potentials in $\mathbb{R}^4$: Proof of Theorem 2.4

We note that the Trudinger–Moser inequalities with degenerate potentials have been established in Euclidean spaces (see [12, 13]) and on the Heisenberg group [8].

In this section, we will prove the critical Adams inequality involving degenerate potential, namely we will give the proof of Theorem 2.4. For this purpose, we need the following lemma.

**Lemma 4.1** Assume that $u \in W^{2,2}(\mathbb{R}^4)$ such that $\int_{\mathbb{R}^4} (|\Delta u|^2 + V(x)|u|^2) dx < +\infty$, where $V(x)$ is non-negative, continuous and satisfies the condition ($V1$). Then there exits some constant $c > 0$ depending on $\delta$ and $c_0$ such that

$$\int_{\mathbb{R}^4} u^2 dx \leq c \int_{\mathbb{R}^4} (|\Delta u|^2 + V(x)|u|^2) dx.$$

**Proof** Choose the cutoff function $\eta$ such that $\eta = 1$ in $B_{2\delta}$ and $\eta = 0$ in $\mathbb{R}^4 \setminus B_{4\delta}$. Obviously, $|\eta| \leq 1$ and $|\Delta \eta| \leq \frac{1}{\delta^2}$. By the Poincaré inequality and Young inequality, we derive that

$$\int_{B_{4\delta}} |u\eta|^2 dx \leq c\delta^4 \int_{B_{4\delta}} |\Delta(u\eta)|^2 dx \leq c\delta^4 \int_{B_{4\delta}} |\eta \Delta u + u \Delta \eta + \nabla u \nabla \eta|^2 dx \leq c\delta^4 \int_{B_{4\delta}} |\eta \Delta u|^2 dx + c\delta^4 \int_{B_{4\delta} \setminus B_{2\delta}} |u|^2 |\Delta \eta|^2 dx + c\delta^4 \int_{B_{4\delta} \setminus B_{2\delta}} |\nabla u \nabla \eta|^2 dx \leq c\delta^4 \int_{B_{4\delta}} |\Delta u|^2 dx + c\delta^4 \int_{B_{4\delta} \setminus B_{2\delta}} |u|^2 dx + c\delta \int_{B_{4\delta} \setminus B_{2\delta}} (|u|^2 + |\Delta u|^2) dx.$$

This gives that there exists $c_1$ depending on $\delta$ such that

$$\int_{B_{2\delta}} |u|^2 dx \leq c_1 \int_{B_{5\delta} \setminus B_{\frac{3}{2}\delta}} |u|^2 dx + c_1 \int_{B_{5\delta}} |\Delta u|^2 dx,$$

which together with $V(x) \geq c_0$ in $\mathbb{R}^4 \setminus B_{\frac{3}{2}\delta}$ implies that

$$\int_{\mathbb{R}^4} u^2 dx < c \int_{\mathbb{R}^4} (|\Delta u|^2 + V(x)|u|^2) dx,$$

where $c$ depends on $c_0$ and $\delta$. \hfill \Box

Now, we are in position to prove Theorem 2.4.

**Proof of Theorem 2.4:** Since $C_0^\infty(\mathbb{R}^4)$ is dense in $W^{2,2}(\mathbb{R}^4)$, we may assume that $u$ is a compactly supported smooth function. Furthermore, we assume that $\int_{\mathbb{R}^4} V(x)|u|^2 dx > 0$. In fact, if $\int_{\mathbb{R}^4} V(x)|u|^2 dx = 0$, then obviously $\text{supp} u \subseteq B_{2\delta}(0)$, through the classical Adams inequality on bounded domains, we have

$$\int_{\mathbb{R}^4} (e^{32\pi^2\cdot u^2} - 1) dx = \int_{B_{2\delta}} (e^{32\pi^2\cdot u^2} - 1) dx < c,$$

and the proof of Theorem 2.4 is completed. \hfill \Box
Hence it remains to consider the case when \( \int_{\mathbb{R}^4} V(x)u^2\,dx > 0 \). Set

\[
A(u) := \left( \int_{\mathbb{R}^4} V(x)u^2\,dx \right)^{\frac{1}{2}}
\]

and \( \Omega(u) := \{ x \in \mathbb{R}^4 \mid u > A(u) \} \) and \( \tilde{\Omega}(u) := \{ x \in \mathbb{R}^4 \mid -u > A(u) \} \). Then \( A(u) < 1 \) and

\[
\int_{\Omega(u) \cap B_{2\delta}^c} u^2\,dx \geq \int_{\Omega(u) \cap B_{2\delta}^c} A^2(u)\,dx \\
= \left( \int_{\mathbb{R}^4} V(x)u^2\,dx \right) |\Omega(u) \cap B_{2\delta}^c|,
\]

hence by (V1) we get

\[
\left| \Omega(u) \cap B_{2\delta} \right| \leq \frac{\int_{\Omega(u) \cap B_{2\delta}^c} u^2\,dx}{\int_{\mathbb{R}^4} V(x)u^2\,dx} \leq \frac{1}{c_0},
\]

and then \( |\Omega(u)| \leq |B_{2\delta}| + \frac{1}{c_0} \). Similarly, we can also obtain \( |\tilde{\Omega}(u)| \leq |B_{2\delta}| + \frac{1}{c_0} \).

Now, we rewrite

\[
\int_{\mathbb{R}^4} \left( e^{32\pi^2|u|^2} - 1 \right)\,dx = \int_{\Omega(u)} \left( e^{32\pi^2|u|^2} - 1 \right)\,dx + \int_{\tilde{\Omega}(u)} \left( e^{32\pi^2|u|^2} - 1 \right)\,dx \\
+ \int_{\mathbb{R}^4 \setminus (\Omega(u) \cup \tilde{\Omega}(u))} \left( e^{32\pi^2|u|^2} - 1 \right)\,dx \\
:= I_1 + I_2 + I_3,
\]

and we will prove that both \( I_1, I_2 \) and \( I_3 \) are bounded by a constant \( c \).

First, we estimate \( I_3 \). Since \( \int_{\mathbb{R}^4} (|\Delta u|^2 + V(x)u^2)\,dx \leq 1 \), through Lemma 4.1, we know that \( \int_{\mathbb{R}^4} u^2\,dx \) is also bounded by some constant, then

\[
I_3 \leq \int_{|u(x)| < 1} \sum_{k=1}^{\infty} \frac{(32\pi^2)^k}{k!} |u|^2k\,dx \leq \sum_{k=1}^{\infty} \frac{(32\pi^2)^k}{k!} \int_{\mathbb{R}^4} u^2\,dx \leq c.
\]

Since the estimate of \( I_1 \) and \( I_2 \) is similar, we only estimate \( I_1 \). Set

\[
v(x) = u(x) - A(u) \quad \text{in} \quad \Omega(u),
\]

then \( v \in W^{2,2}(\Omega(u)) \) with \( v = 0 \) on the \( \partial \Omega(u) \). Direct calculation gives that in \( \Omega(u) \),

\[
u^2(x) = (v(x) + A(u))^2 \\
= v^2(x) + A^2(u) + 2v(x)A(u) \\
\leq v^2(x) + A^2(u) + v^2(x)A^2(u) + 1 \\
= v^2(x)(1 + A^2(u)) + A^2(u) + 1.
\]

Let \( w(x) = v(x)(1 + A^2(u))^{\frac{1}{2}} \), then \( w(x) \in W^{2,2}(\Omega(u)) \cap W^{1,2}_0(\Omega(u)) \),

\[
u^2(x) \leq w^2(x) + 1 + A^2(u), \\
\Delta w(x) = (1 + A^2(u))^{\frac{1}{2}} \Delta v(x),
\]

\[
\xRightarrow{\text{Springer}}
\]
and

\[
\int_{\Omega(u)} |\Delta w(x)|^2 \, dx = (1 + A^2(u)) \int_{\Omega(u)} |\Delta v(x)|^2 \, dx \\
\leq (1 + A^2(u)) \left( 1 - \int_{\mathbb{R}^4} V(x) u^2 \, dx \right) \\
= \left( (1 + \int_{\mathbb{R}^4} V(x) u^2 \, dx) \right) \left( 1 - \int_{\mathbb{R}^4} V(x) u^2 \, dx \right) \\
\leq 1.
\]

Then using the Adams inequalities on bounded domain with the Navier boundary (see [45]), we get

\[
I_1 \leq \int_{\Omega(u)} \left( e^{32\pi^2 |u|^2} - 1 \right) \, dx \leq e^{32\pi^2 (1 + A^2(u))} \int_{\Omega(u)} e^{32\pi^2 |u|^2} \, dx \leq c,
\]

and the proof of Theorem 2.4 is finished.

5 Existence of the ground-state solution of bi-harmonic equations with degenerate potential: the proof of Theorem 2.6

In this section, we are concerned with the ground states of the quasilinear bi-harmonic Eq. (2.4), where \( f(t) \) has the critical exponential growth satisfying (i)-(v) and the potential \( V(x) \geq 0 \) satisfies (V1) and (V2).

The associated functional and Nehari Manifold are

\[
I_V(u) = \frac{1}{2} \int_{\mathbb{R}^4} (|\Delta u|^2 + V(x)|u|^2) \, dx - \int_{\mathbb{R}^4} F(u) \, dx
\]

and

\[
N_V = \{ u \in W^{2,2}(\mathbb{R}^4) | u \neq 0, N_V(u) = 0 \},
\]

respectively, where

\[
N_V(u) = \int_{\mathbb{R}^4} (|\Delta u|^2 + V(x)|u|^2) \, dx - \int_{\mathbb{R}^4} f(u) u \, dx.
\]

One can easily verify that if \( u \in N_V \), then

\[
I_V(u) = \frac{1}{2} \int_{\mathbb{R}^4} (f(u)u - 2F(u)) \, dx.
\]

Set \( m_V = \inf\{ I_V(u) \mid N_V(u) = 0 \} \), we will prove that if \( V_\infty < \gamma^* \), then \( m_V \) is achieved by some function \( u \in W^{2,2}(\mathbb{R}^4) \).

**Lemma 5.1** If \( V_\infty < \gamma^* \), then

\[
0 < m_V < \frac{16\pi^2}{\alpha_0}.
\]

**Proof** We first show that \( m_V > 0 \). We prove this by contradiction. Assume that there exists some sequence \( u_k \in N_V \) such that \( I_V(u_k) \to 0 \), that is,

\[
\lim_{k \to +\infty} \int_{\mathbb{R}^4} (f(u_k)u_k - 2F(u_k)) \, dx = 0,
\]
which together with the (A–R) condition and \( u_k \in \mathcal{N}_V \) yields that
\[
\lim_{k \to +\infty} \int_{\mathbb{R}^4} (|\Delta u_k|^2 + V(x)|u_k|^2) \, dx = 0. \tag{5.2}
\]

On one hand, it follows from (2.2) and \( u_k \in \mathcal{N}_V \) that
\[
1 = \int_{\mathbb{R}^4} f(u_k) \frac{u_k}{\|u_k\|_{W^{2,2}_V(\mathbb{R}^4)}^2} \, dx \\
\leq \int_{\mathbb{R}^4} \left( \epsilon \frac{|u_k|^2}{\|u_k\|^2_{W^{2,2}_V(\mathbb{R}^4)}} + C_\epsilon \frac{|u_k|^{\mu+1}}{\|u_k\|^2_{W^{2,2}_V(\mathbb{R}^4)}} \left( e^{\beta_0 u_k^2} - 1 \right) \right) \, dx \tag{5.3}
\]

On the other hand, by (5.2), Adams inequality involving the degenerate potential (Theorem 2.4) and the fact that \( \mu > 2 \), we get for any \( p > 1 \),
\[
\frac{1}{\|u_k\|^2_{W^{2,2}_V(\mathbb{R}^4)}} \int_{\mathbb{R}^4} |u_k|^{\mu+1} e^{\beta_0 u_k^2} \, dx \leq \frac{1}{\|u_k\|^2_{W^{2,2}_V(\mathbb{R}^4)}} \left( \int_{\mathbb{R}^4} |u_k|^{(\mu+1)p} \, dx \right)^{1/p} \left( \int_{\mathbb{R}^4} \left( e^{\beta_0 u_k^2} - 1 \right) \, dx \right)^{1/p} \leq c \|u_k\|^\mu_{W^{2,2}_V(\mathbb{R}^4)} \to 0, \text{ as } k \to \infty,
\]
which is a contradiction with (5.3).

Recalling the proof of Theorem 2.2, we have shown that \( m'_V < \frac{16\pi^2}{\alpha_0} \) if \( \gamma < \gamma^* \). From the definition of \( m_V \) and \( V(x) \leq V_\infty < \gamma^* \), we can easily obtain that \( m_V \leq m_\infty < \frac{16\pi^2}{\alpha_0} \). Then Lemma 5.1 is proved.

We now consider a minimizing sequence \( \{u_k\} \subset \mathcal{N}_V \) for \( m_V \). According to the (A–R) condition (ii) and \( I_V(u_k) \to m_V > 0 \), we derive that \( \{u_k\} \) is bounded in \( W^{2,2}(\mathbb{R}^4) \), then up to a subsequence, there exists some \( u \in W^{2,2}(\mathbb{R}^4) \) such that
\begin{itemize}
  \item \( u_k \rightharpoonup u \) weakly in \( W^{2,2}(\mathbb{R}^4) \) and in \( L^p(\mathbb{R}^4) \), for any \( p > 1 \),
  \item \( u_k \to u \) in \( L^p_{loc}(\mathbb{R}^4) \),
  \item \( u_k \to u \), a.e.
\end{itemize}

We claim
\begin{lemma}
\( u \neq 0 \).
\end{lemma}

\textbf{Proof} We prove this by contradiction. If \( u = 0 \), and \( u_k \to 0 \) in \( L^2_{loc}(\mathbb{R}^4) \). We first claim that:
\[
\lim_{k \to +\infty} \int_{\mathbb{R}^4} (V_\infty - V(x)) |u_k|^2 \, dx = 0. \tag{5.4}
\]

For any fixed \( \varepsilon > 0 \), we take \( R_\varepsilon > 0 \) such that
\[
|V_\infty - V(x)| \leq \varepsilon, \text{ for any } |x| > R_\varepsilon.
\]
Combining this and the boundedness of \( u_k \in W^{2,2}(\mathbb{R}^4) \), we derive that
\[
\int_{\mathbb{R}^4} (V_\infty - V(x)) |u_k|^2 \, dx = \int_{B_{R_\varepsilon}} (V_\infty - V(x)) |u_k|^2 \, dx + \int_{B_{R_\varepsilon}^c} (V_\infty - V(x)) |u_k|^2 \, dx \\
\leq c \int_{B_{R_\varepsilon}} |u_k|^2 \, dx + M \varepsilon,
\]

where \( M = \sup_{k} \int_{\mathbb{R}^4} |u_k|^2 \, dx \). This together with \( u_k \to 0 \) in \( L^2_{\text{loc}}(\mathbb{R}^4) \) as \( k \to \infty \) yields that

\[
\lim_{k \to +\infty} \int_{\mathbb{R}^4} (V - V(x))|u_k|^2 \, dx \leq M\varepsilon,
\]

which implies (5.4) holds.

Since \( u_k \in N_V \), we know that there exists some sequence \( t_k \geq 1 \) such that \( t_k u_k \in N_{\infty} \), that is,

\[
\int_{\mathbb{R}^4} (|\nabla u_k|^2 + V_{\infty}|u_k|^2) \, dx - \int_{\mathbb{R}^4} f(t_k u_k) u_k \, dx = 0,
\]

(5.5)

On the other hand, since \( u_k \in N_V \), then

\[
\int_{\mathbb{R}^4} (|\nabla u_k|^2 + V(x)|u_k|^2) \, dx - \int_{\mathbb{R}^4} f(u_k) u_k \, dx = 0.
\]

(5.6)

Combining (5.5) and (5.6), we get

\[
\int_{\mathbb{R}^4} f(t_k u_k) t_k u_k \, dx - \int_{\mathbb{R}^4} f(u_k) u_k \, dx = 0.
\]

(5.7)

Next, we claim that \( t_k \to t_0 = 1 \) as \( k \to \infty \). We prove this by contradiction. Assume that \( t_0 > 1 \). Since

\[
IV(u_k) = \frac{1}{2} \int_{\mathbb{R}^4} (f(u_k) u_k - 2F(u_k)) \, dx \to m_V > 0, \text{ as } k \to \infty,
\]

we have \( \lim_{k \to \infty} \int_{\mathbb{R}^4} f(u_k) u_k \, dx > 0 \). Now, we claim that

\[
\lim_{k \to \infty} \int_{\mathbb{R}^4} \frac{f(t_0 u_k)}{t_0 u_k} u_k^2 \, dx > \lim_{k \to \infty} \int_{\mathbb{R}^4} f(u_k) u_k \, dx.
\]

(5.8)

Since \( f(t) = o(t) \) as \( t \to 0 \), and \( \|u_k\|_{L^2} \) is bounded, then

\[
\lim_{r \to 0} \lim_{k \to +\infty} \int_{\{|u_k| > r\}} f(u_k) u_k \, dx \leq \varepsilon \int_{\mathbb{R}^4} u_k^2 \, dx \leq C\varepsilon
\]

(5.9)

for any \( \varepsilon > 0 \). Hence we conclude that

\[
\lim_{r \to 0} \lim_{k \to +\infty} \int_{\{|u_k| < r\}} f(u_k) u_k \, dx = 0.
\]

(5.10)

Notice that \( \lim_{k \to \infty} \int_{\mathbb{R}^4} f(u_k) u_k \, dx > 0 \), hence one of the following two cases must occur:

Case 1: \( \lim_{R \to +\infty} \lim_{k \to +\infty} \int_{\{|u_k| > R\}} f(u_k) u_k \, dx = 0 \);

Case 2: There exist some \( r, R > 0 \) such that \( \lim_{k \to +\infty} \int_{\{r < |u_k| < R\}} f(u_k) u_k \, dx > 0 \).
If Case 1 occurs, since \( f(t) \) has critical exponential growth when \( t \) is large and \( \frac{f(t_k u_k)}{t_k u_k} \) is strictly increasing with the variable \( t \), then we have

\[
\lim_{R \to +\infty} \lim_{k \to \infty} \int_{\{|u_k| > R\}} \frac{f(t_k u_k)}{t_k u_k} u_k^2 \, dx > \lim_{R \to +\infty} \lim_{k \to \infty} \int_{\{|u_k| > R\}} f(u_k) u_k \, dx.
\]

If Case 2 occurs, since \( u_k \) is bounded on the set \( \{x | r \leq |u_k| \leq R\} \), then \( \{x | r \leq |u_k| \leq R\} > \beta \) for some \( \beta > 0 \). Hence

\[
\lim_{k \to \infty} \int_{\{r \leq |u_k| \leq R\}} \frac{f(t_k u_k)}{t_k u_k} u_k^2 \, dx > \lim_{k \to \infty} \int_{\{r \leq |u_k| \leq R\}} f(u_k) u_k \, dx.
\]

Combining the above estimates and the monotonicity of \( \frac{f(t)}{t} \), we derive that

\[
\lim_{k \to \infty} \int_{\mathbb{R}^4} \frac{f(t_k u_k)}{t_k u_k} u_k^2 \, dx \geq \lim_{k \to \infty} \int_{\mathbb{R}^4} \frac{f(t_k u_k)}{t_k u_k} u_k^2 \, dx > \lim_{k \to \infty} \int_{\mathbb{R}^4} f(u_k) u_k \, dx > 0
\]

which contradicts with (5.7).

Now, by (5.4), we can write

\[
m_\infty \leq \lim_{k \to +\infty} I_\infty(t_k u_k) = \lim_{k \to +\infty} \left( I_V(t_k u_k) + \frac{1}{2} t_k^2 \int_{\mathbb{R}^4} (V_\infty - V(x))|u_k|^2 \, dx \right)
= \lim_{k \to +\infty} I_V(t_k u_k) \leq \lim_{k \to +\infty} \frac{1}{2} t_k^2 \int_{\mathbb{R}^4} (|\nabla u_k|^2 + V(x)|u_k|^2) \, dx - \int_{\mathbb{R}^4} \frac{F(t_k u_k)}{t_k^2} u_k^2 \, dx.
\]

This together with the monotonicity of \( \frac{F(t)}{t^2} \) (see Remark 2.1) and \( \lim_{k \to \infty} t_k = 1 \) gives

\[
m_\infty \leq \lim_{k \to +\infty} I_V(u_k) = m_V
\]

which is a contradiction. This accomplishes the proof of Lemma 5.2. \( \square \)

The following Lemmas 5.3 and 5.4 is similar to the proof of Lemmas 3.5 and 3.6, we only state them without detailed proof.

**Lemma 5.3** It holds that

\[
\lim_{k \to +\infty} \int_{\mathbb{R}^4} f(u_k) u_k \, dx = \lim_{L \to +\infty} \lim_{k \to +\infty} \int_{B_L(0)} f(u_k) u_k \, dx, \quad \lim_{L \to +\infty} \lim_{k \to +\infty} \int_{\mathbb{R}^4 \setminus B_L(0)} f(u_k) u_k \, dx = 0.
\]

**Lemma 5.4** There holds \( \lim_{k \to +\infty} \int_{\mathbb{R}^4} F(u_k) \, dx = \int_{\mathbb{R}^4} F(u) \, dx. \)

**Lemma 5.5** Let \( u_k \) be a bounded sequence in \( W^{2,2}(\mathbb{R}^4) \) converging weakly to non-zero \( u \). Furthermore, we also assume that \( \lim_{k \to +\infty} I_V(u_k) < \frac{16\pi^2}{\alpha_0} \) and \( \int_{\mathbb{R}^4}(|\Delta u|^2 + V(x)|u|^2) \, dx > \int_{\mathbb{R}^4} f(u) u \, dx \), then

\[
\lim_{k \to +\infty} \int_{\mathbb{R}^4} f(u_k) u_k \, dx = \int_{\mathbb{R}^4} f(u) u \, dx.
\]

**Proof** According to Lemma 5.3, we only need to prove that

\[
\lim_{L \to +\infty} \lim_{k \to +\infty} \int_{B_L(0)} f(u_k) u_k \, dx = \int_{\mathbb{R}^4} f(u) u \, dx.
\]
Indeed, we can apply the condition (i) and (ii) to obtain that for any inequality which was established in [9], one can derive that there exists Combining the above estimate with the concentration compactness principle for the Adams which finishes the proof of Lemma 5.5.

We claim that there exists \( q > 1 \) sufficiently 1 such that

\[
q_0 \| u_k \|_{W^{2,2}_V(\mathbb{R}^4)}^2 < \frac{32\pi^2}{1 - \| u_0 \|_{W^{2,2}_V(\mathbb{R}^4)}^2}.
\] (5.12)

Indeed, we can apply the condition (i) and (ii) to obtain

\[
\lim_{k \to \infty} \| u_k \|_{W^{2,2}_V(\mathbb{R}^4)}^2 (1 - \| u_0 \|_{W^{2,2}_V(\mathbb{R}^4)}^2) = \lim_{k \to \infty} \| u_k \|_{W^{2,2}_V(\mathbb{R}^4)}^2 \left(1 - \frac{\| u \|_{W^{2,2}_V(\mathbb{R}^4)}^2}{\| u_k \|_{W^{2,2}_V(\mathbb{R}^4)}^2}\right) = 2 \int_{\mathbb{R}^4} F(u)dx - 2 V(\int_{\mathbb{R}^4} F(u)dx
\]

\[
< \frac{32\pi^2}{\alpha_0}.
\] (5.13)

Combining the above estimate with the concentration compactness principle for the Adams inequality which was established in [9], one can derive that there exists \( p_0 > 1 \) such that

\[
\sup_k \int_{\mathbb{R}^4} (f(u_k)u_k)^{p_0} dx < \infty.
\] (5.14)

Then it follows from the Vitali convergence theorem that

\[
\lim_{L \to +\infty} \lim_{k \to \infty} \int_{B_L} f(u_k)u_k dx = \int_{\mathbb{R}^4} f(u)udx,
\]

which finishes the proof of Lemma 5.5.

Now we are in the position to show the existence of ground-state solutions to Eq. (2.4) when \( V_\infty < \gamma^* \).
Proof of Theorem 2.6} We only need to prove that if $V_\infty < \gamma^*$, then $m_V$ is achieved by some $u$. We claim that

$$\int_{\mathbb{R}^4} (|\Delta u|^2 + V(x)|u|^2) \, dx \leq \int_{\mathbb{R}^4} f(u) \, dx.$$  

Suppose that this is false, then

$$\int_{\mathbb{R}^4} (|\Delta u|^2 + V(x)|u|^2) \, dx > \int_{\mathbb{R}^4} f(u) \, dx. \quad (5.15)$$

Since we have shown that $u \neq 0$ in Lemma 5.2, combining Lemmas 5.1 and 5.5, we derive

$$\lim_{k \to \infty} \int_{\mathbb{R}^4} f(u_k) u_k \, dx = \int_{\mathbb{R}^4} f(u) \, dx.$$  

This implies that

$$\int_{\mathbb{R}^4} (|\Delta u|^2 + V(x)|u|^2) \, dx \leq \lim_{k \to \infty} \int_{\mathbb{R}^4} (|\Delta u|^2 + V(x)|u_k|^2) \, dx$$

$$= \lim_{k \to \infty} \int_{\mathbb{R}^4} f(u_k) u_k \, dx = \int_{\mathbb{R}^4} f(u) \, dx \quad (5.16)$$

$$< \int_{\mathbb{R}^4} (|\Delta u|^2 + V(x)|u|^2) \, dx,$$

which is a contradiction. This proves the claim.

Since

$$\int_{\mathbb{R}^4} (|\Delta u|^2 + V(x)|u|^2) \, dx \leq \int_{\mathbb{R}^4} f(u) \, dx,$$

then there exists $\gamma_0 \in (0, 1]$ such that $\gamma_0 u \in N_V$. According to the definition of $m_V$, we have

$$m_V \leq I_V(\gamma_0 u) = \frac{1}{2} \int_{\mathbb{R}^4} (f(\gamma_0 u)(\gamma_0 u) - 2F(\gamma_0 u)) \, dx$$

$$\leq \frac{1}{2} \int_{\mathbb{R}^4} (f(u)(u) - 2F(u)) \, dx$$

$$\leq \lim_{k \to \infty} \frac{1}{2} \int_{\mathbb{R}^4} (f(u_k)(u_k) - 2F(u_k)) \, dx$$

$$= \lim_{k \to \infty} I_V(u_k) = m_V. \quad (5.17)$$

This implies that $\gamma_0 = 1$ and $u \in N_V$ and $I_V(u) = m_V$. We accomplish the proof of Theorem 2.6. □

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