STRONGLY EXPOSED POINTS IN THE BALL OF THE BERGMAN SPACE

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Abstract. We investigate which boundary points in the closed unit ball of the Bergman space $A^1$ are strongly exposed. This requires study of the Bergman projection and its kernel, the annihilator of Bergman space. We show that all polynomials in the boundary of the unit ball are strongly exposed.

1. Introduction.

In Banach space theory one often seeks to determine the geometry of the unit ball of a given Banach space. A common way to distinguish “round” and “flat” parts of the boundary of the unit ball is through extreme and non-extreme points. Among the extreme points, or “round” parts of the boundary, further refinements can be made, for example exposed and strongly exposed points. In this paper we study these sets for the (unweighted) Bergman space $A^1$ of the unit disc $D$ of $\mathbb{C}$. These questions were inspired by and can be stated in the context of Hardy spaces of the unit ball in $\mathbb{C}^n$. However, we attempt to frame them within the theory of Bergman spaces. For an excellent survey of the theory of Bergman spaces, we refer the reader to [3]. Our main result identifies a large class of strongly exposed points, which includes all normalized polynomials. We also exhibit exposed points which are not strongly exposed. In the process we find opportunity to study the subspace $(A^1)^⊥ + \mathcal{C}(\partial D)$ of $L^∞(D)$, which is the analog of $H^∞ + \mathcal{C} \subset L^∞(\mathbb{T})$ ($\mathbb{T} = \partial D$). As is $H^∞ + \mathcal{C}$, the space $(A^1)^⊥ + \mathcal{C}(\partial D)$ is closed, but contrary to $H^∞ + \mathcal{C}$ it turns out not to be an algebra. However, it is a $C$-module.

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2. The Bergman Space $A^1$.

Let $D$ be the (open) unit disc in $\mathbb{C}$, with boundary $\mathbb{T}$, and let $dA = \frac{1}{2\pi} r \, dr \, d\theta$ be the normalized Lebesgue measure on $D$. The space of all holomorphic functions on $D$ will be denoted by $H(D)$. The (unweighted) Bergman space $A^1$ consists of all functions $f$ in $H(D)$ that are area-integrable on $D$. Equipped with the Bergman norm,

$$\|f\| := \int_D |f(z)| \, dA(z),$$

the Bergman space becomes a Banach space.
More generally, for $1 \leq p < \infty$, the space $A^p = H(D) \cap L^p(D, dA)$ with the $L^p$-norm is a Banach space. Under the $L^2$-inner product, $A^2$ is a Hilbert space. The orthogonal projection $P : L^2 \to A^2$, the so-called Bergman projection, will play an important role later on, so we mention its explicit representation:

\begin{align}
Pf(z) &= \int_D \frac{f(w)}{(1 - z\bar{w})^2} dA(w) \\
&= \sum_{n=0}^{\infty} ((n+1) \cdot \int_D f(w)\bar{w}^n dA(w)) z^n.
\end{align}

3. Extreme, exposed and strongly exposed points.

Let $X$ be a Banach space. We say that $x \in X$ is extreme if it is an extreme point of the unit ball of $X$ (in particular, $\|x\| = 1$). We say that $x \in \partial \text{Ball}(X)$ is exposed in $X$ if there exists a functional $L \in X^*$ that attains its norm at $x$ and at no other point of the closed unit ball. The functional $L$ is often assumed to be of norm 1 and is (then) called an exposing functional for $x$. Of course an exposed point is also extreme, but the converse need not hold in general. However, we have the following lemma, the simple proof of which we omit.

**Lemma 1.** Let $X$ be a Banach space in which every point of unit norm is extreme. Then all points of unit norm are also exposed.

The concept of exposedness can be refined in the following manner. We call $f \in \partial \text{Ball}(X)$ strongly exposed if there exists $L \in X^*$ with the properties: $L(f) = \|L\| = 1$ and for any sequence $(f_n)_{n=1}^\infty$ in the ball of $X$ such that $\lim_{n \to \infty} L(f_n) = 1$, it follows that $f_n$ converges to $f$ in norm. It is not difficult to see that a strongly exposed point is (indeed) exposed (by the same functional). By a theorem of Phelps [6], in a separable dual Banach space the closed unit ball is the closure of the convex hull of the strongly exposed points. In particular, because the Bergman space $A^1$ is a dual space, section 6, there exist many strongly exposed points.

4. A criterion for strongly exposed points in $A^1$.

Let us illustrate these definitions for the Bergman space $A^1$.

**Lemma 2.** Let $f \in A^1$ be of unit norm. Then the functional 

$$L : g \in A^1 \mapsto \int_D |g\bar{f}|/|f| dA$$

is exposing for $f$. In particular, all functions of unit norm are exposed in the unit ball of $A^1$.

It is not hard to show that all functions of unit norm in $A^1$ are extreme in the unit ball. Thus the claim follows from Lemma 1. However we opt for another proof: 

**Proof.** Let $f \in A^1$ be of unit norm. Suppose first that (some) $L \in (A^1)^*$ is such that $L(f) = \|L\| = 1$. Then, by the Hahn-Banach theorem, there exists $\psi \in L^\infty$ such that $\|\psi\| = 1$ and $L(g) = \int_D g\bar{\psi} dA$ for all $g \in A^1$. Because also

$$L(f) = \int_D |g\bar{f}|/|f| dA,$$

...
\[ \int_D |f| dA = 1 \] it follows that \( \psi = f/|f| \) (almost everywhere). In particular, an exposing functional for \( f \) is unique (if it exists). We finish the proof by showing that the functional \( L(g) = \int_D g\overline{f}/|f| dA \) is indeed exposing for \( f \). Clearly, \( L \) attains its norm at \( f \). Suppose that \( g \in \text{closed unit ball of } A^1 \) is such that \( L(g) = 1 \). Then by the above reasoning, \( f/|f| = g/|g| \) almost everywhere on \( D \). Hence \( g/f \) is a positive meromorphic function on \( D \), thus constant. Because \( \|g\| = \|f\| = 1 \), \( g \) must equal \( f \). This concludes the proof.

By contrast, not all functions of unit norm are strongly exposed in the unit ball of \( A^1 \). This is contained in the following proposition.

**Proposition 1.** The function

\[ f(z) = \frac{cz^2}{(1-z)^2 \log^2(1-z)} \]

where \( c \) is normalizing so that \( \|f\| = 1 \), is not strongly exposed in the unit ball of \( A^1 \).

**Proof.** For \(-2 < \beta < 0\), let \( f_\beta(z) = c_\beta(1-z)^\beta \), where the constant \( c_\beta > 0 \) is normalizing, i.e. \( \|f_\beta\| = 1 \). Let \( \varphi_\beta = f_\beta/|f_\beta| \) and \( \varphi_{-2} = \frac{1-z}{z \log(1-z)} \). By construction, \( \int_D f_\beta \varphi_{-2} dA = 1 \) for all \( \beta \). Let

\[ \varphi = f/|f| = \varphi_{-2} \cdot \frac{z \log(1-z)}{z \log(1-z)} \]

Then the \( f \)-exposing functional \( L \) is given by

\[ L : g \in A^1 \mapsto \int_D g\overline{\varphi} dA. \]

As \( \|\varphi_\beta - \varphi_{-2}\|_\infty \to 0 \) for \( \beta \downarrow -2 \), it follows that \( \lim_{\beta \downarrow -2} \int_D f_\beta \overline{\varphi_{-2}} dA = 1 \). Next, because \( \frac{z \log(1-z)}{z \log(1-z)} \to 1 \) as \( D \ni z \to 1 \), the bounded function \( \varphi - \varphi_{-2} \) is continuous on \( \overline{D} \setminus \{0\} \) and vanishes at \( z = 1 \). Because \( \|f_\beta\| \to \infty \) as \( \beta \downarrow -2 \), the normalizing constants \( c_\beta \) tend to 0 as \( \beta \downarrow -2 \), thus the functions \( f_\beta \) tend to 0 uniformly on \( \overline{D} \setminus B(1, \varepsilon) \) for every \( \varepsilon > 0 \), as \( \beta \downarrow -2 \). Hence, \( \int_D f_\beta(\overline{\varphi} - \overline{\varphi_{-2}}) dA \to 0 \) as \( \beta \downarrow -2 \). Consequently,

\[ L(f_\beta) = \int_D f_\beta \overline{\varphi} dA \to L(f) = 1. \]

Because the functions \( f_\beta \) tend to zero pointwise, they do not converge to \( f \) in norm. This demonstrates that \( f \) is not strongly exposed.

Now let

\[ (A^1)^\perp = \{ \psi \in L^\infty : \int_D f \overline{\psi} dA = 0 \text{ for all } f \in A^1 \} \]

denote the annihilator of \( A^1 \) contained in \( L^\infty \). The space \( (A^1)^\perp \) is quite large but we do not know of a structural description of its elements. Finally let \( C \) denote the continuous functions on \( \overline{D} \).

We are now ready to give an abstract characterization of the strongly exposed points of \( A^1 \).
Theorem 1. Let \( f \in A^1 \) be of unit norm. Then \( f \) is strongly exposed if and only if the \( L^\infty \)-distance of \( f/|f| \) to the space \((A^1)^{1} + C\) is less than one.

Proof. We first show that the distance condition is necessary. We argue by contradiction. Thus let \( f \in A^1 \) be strongly exposed and suppose the \( L^\infty \)-distance of \( f/|f| \) to \((A^1)^{1} + C\) is one. Pick a point \( z_0 \) such that \( f(z_0) \neq 0 \). Let \( A^1_{z_0} \) denote the subspace of all Bergman functions vanishing at \( z_0 \). We let \( L' \) denote the restriction of the \( f \)-exposing functional \( L : g \mapsto \int gT/|f| dA \) to \( A^1_{z_0} \). By the Hahn-Banach theorem the operator norm of \( L' \) equals the \( L^\infty \)-distance of \( f/|f| \) to \((A^1_{z_0})^\perp \). Now if \( \psi \in (A^1_{z_0})^\perp \), then with the choice \( c = \int_D \psi dA \), the function \( \psi(w) - \frac{c}{1 - \overline{z}_0 w} \) annihilates both \( A^1_{z_0} \)-functions and constants. This shows that \((A^1_{z_0})^\perp \subset (A^1)^{1} + C\). By the assumption on \( f/|f| \), we conclude that \( L' \) has operator norm 1. Hence we find a sequence of functions \( f_n \) in the unit ball of \( A^1_{z_0} \) for which \( L(f_n) = L'(f_n) \to 1 \). Yet contrary to the assumption of strong exposedness of \( f \), the functions \( f_n \) do not converge to \( f \) in norm. Indeed, norm convergence implies pointwise convergence, which fails at the point \( z_0 \).

Next we show that the distance condition is sufficient. This distance condition strongly resembles one in a theorem of the second author on the strongly exposed points in the Hardy space \( H^1 \) of the unit ball \( B_2 \) of \( \mathbb{C}^n \) [10]. There it is proven that an exposed point \( F \) is strongly exposed in \( H^1 \) if and only if the \( L^\infty \)-distance of the function \( F/|F| \) on the sphere \( S \) of \( \mathbb{C}^2 \) to the space \((H^1)^{1} + C(S)\) is less than one. By Theorem 7.2.4 in [8], the Bergman space \( A^1 \) is isometrically contained in the Hardy space \( H^1 \) of the unit ball \( B_2 \) in \( \mathbb{C}^2 \); namely, look at all holomorphic functions \( F(z, w) \) on \( B_2 \) which depend only on \( z \): \( F(z, w) = F(z, 0) \). Then \( F \) is in \( H^1(B_2) \) if and only if \( f(z) := F(z, 0) \) is in \( A^1 \) and the corresponding norms are then the same. Similarly, \((A^1)^{1} \) can be interpreted as a subspace of \((H^1)^{1} \) and \( C(TD) \) as a subspace of \( C(S) \).

Let our function \( f \in A^1 \) correspond with \( F \in H^1 \). Because \( F \) is continuous on an open subset of \( S \) (in fact, all of \( S \) except possibly \( T \times \{0\} \)), \( F \) is exposed in \( H^1 \). Because of the inclusion \((A^1)^{1} + C \subset (H^1)^{1} + C(S)\), \( F/|F| \) has \( L^\infty \)-distance less than one to \((H^1)^{1} + C(S)\). Hence \( F \) is strongly exposed in \( H^1 \) by [10]. Thus \( F \), or rather \( f \), is strongly exposed in \( A^1 \subset H^1 \). This finishes the proof. \( \square \)

The question now is: how can we estimate the distance in \( L^\infty \) of \( \varphi = f/|f| \) to \((A^1)^{1} + C\), where \( f \) is a given function in \( A^1 \)? Clearly the distance cannot exceed one. Throughout the remainder we will use various techniques to estimate said distances.

5. The functions \((z - \alpha)^\beta\), Part I.

In order to simplify the necessary calculations we will test strong exposedness on functions of a particularly simple form, i.e. polynomials. It will later be shown (section 8) that we may then even restrict to simple polynomials of the form \( f(z) = c(z - \alpha)^n \), where \( c \) is normalizing. Having then obtained our results for these functions it is easy to generalize to functions of the form \( f(z) = c(z - \alpha)^\beta \) for non-integer \( \beta \) (in which case \( |\alpha| \geq 1 \), obviously).
So let us first look at polynomials: \( f(z) = c(z - \alpha)^n \). We assume \( n \geq 1 \) because unimodular constants are clearly strongly exposed. The case where \( |\alpha| > 1 \) is the easiest: \( f/|f| \) is continuous on \( \overline{D} \), so \( f \) will be strongly exposed. When \( |\alpha| < 1 \) the proof that \( f \) is again strongly exposed is a little more involved. Let us write \( f = \varphi/|\varphi| \). If we can show that the Bergman projection \( P\varphi \) is continuous on \( \overline{D} \) (thus bounded), we will be done because it will then follow that \( \varphi = (\varphi - P\varphi) + P\varphi \) is contained in \( (A^1)^\perp + C \) rather than \( (A^2)^\perp + C \). Write \( \varphi = \psi_1 + \psi_2 \), where \( \psi_1 \) is compactly supported in \( D \) and \( \varphi \equiv \psi_1 \) on a neighborhood of \( \alpha \). From (2.1) we see that \( P\psi_1 \) is holomorphic across the unit circle because of the support of \( \psi_1 \). For the other function, using the series expansion (2.2) for the Bergman projection, we see that the smoothness of \( \psi_2 \) implies continuity (smoothness) of \( P\psi_2 \) on \( \overline{D} \). This proves that \( P\varphi \) is continuous on \( \overline{D} \) and we conclude that \( f \) is strongly exposed.

In this section we can solve the case when \( |\alpha| = 1 \) only partially, that is, depending on the degree \( n \) of the polynomial. We may then of course assume that \( \alpha = 1 \). Let us write \( f_n(z) = c_n(1 - z)^n \) and \( \varphi_n = f_n/|f_n| \). The corresponding exposing functional \( L \) for \( f_2 \) is given by

\[
L(g) = \int_D g(z) \frac{1 - \pi}{1 - z} dA(z) = \int_D \frac{g(z)}{1 - z} (1 - \pi) dA(z).
\]

Integrating first over circles we see that there exist constants \( C_0 \) and \( C_1 \) (independent of \( g \)) such that \( L(g) = C_0 g(0) + C_1 g'(0) \). Thus there exists a polynomial \( \psi \) such that \( L(g) = \int_D g \psi dA \). (Alternatively, verify that \( \psi = P\varphi_2 \) is a polynomial.) But this means that \( \varphi_2 - \psi \) is contained in the annhilator of \( A^1 \), hence that \( \varphi_2 \in (A^1)^\perp + C \) and subsequently \( f_2 \) is strongly exposed.

Quite similarly one shows that for all even \( n \), \( \varphi_n \) is contained in \( (A^1)^\perp + C \) and that \( f_n \) is strongly exposed in \( A^1 \).

We come to the following "odd" proposition on real powers.

**Proposition 2.** Let \( f_\beta(z) = c_\beta(1 - z)^\beta \). Then for all \( \beta > -1 \), the \( L^\infty \)-distance of \( \varphi_\beta \) to \( (A^1)^\perp + C \) is at most \( |\sin(\frac{\pi \beta}{2})| \). In particular, for all \( \beta > -1 \), \( \beta \neq 1, 3, 5, \ldots \), the function \( f_\beta \) is strongly exposed in the unit ball of \( A^1 \).

**Proof.** Of course, there is nothing to prove for odd \( \beta \), so we take \( \beta > -1 \) not odd. We will exploit the fact that the functions \( \varphi_0, \varphi_2, \varphi_4, \ldots \) are contained in the space \( (A^1)^\perp + C \). We find an integer \( n \geq 0 \) such that \( \beta \in (2n - 1, 2n + 1) \). Let \( \theta = |\beta - 2n| < 1 \). Because \( \varphi_{a+b} = \varphi_a \varphi_b \),

\[
\|\varphi_\beta - \cos(\frac{\pi \theta}{2})\varphi_{2n}\|_\infty = \|\varphi_\theta - \cos(\frac{\pi \theta}{2})\|_\infty = \sup_{|t| < \frac{\pi \theta}{2}} |e^{it} - \cos(\frac{\pi \theta}{2})| = \sin(\frac{\pi \theta}{2}) = |\sin(\frac{\pi \beta}{2})|.
\]

This gives the desired upper bound for the \( L^\infty \)-distance of \( \varphi_\beta \) to \( (A^1)^\perp + C \).

By Theorem 1, \( f_\beta \) is strongly exposed. \( \Box \)

In section 8 we investigate the odd powers in greater detail. Before doing so, we need to investigate the Bergman projection further.
6. The Bloch space.

Recall the Bergman projection $P : L^2 \to A^2$,

$$Pf(z) = \int_D \frac{f(w)}{(1-\overline{z}w)^2} dA(w).$$

We have already used the Bergman projection $P$ to prove strong exposedness, namely in those cases where $P$ projects the bounded function $\varphi = f/|f|$ to a continuous function on $\overline{D}$. However, a priori we cannot even expect $P$ to project bounded functions to bounded functions. Obviously we would like to understand better how $P$ acts on bounded functions. For this we need to discuss the Bloch space.

The Bloch space $B$ consists of all holomorphic functions $f$ on $D$ with the property that $(1-|z|^2)|f'(z)|$ is bounded on $D$. Equipped with the norm

$$(6.1) \quad \|f\|_B := |f(0)| + \sup_{z \in D} (1-|z|^2)|f'(z)|,$$

$B$ becomes a Banach space. The set of all functions $f$ in $B$ for which the expression $(1-|z|^2)|f'(z)| \to 0$ as $|z| \to 1$ is a closed subspace of $B$, called the little Bloch space $B_0$. Finally, let $C_0(D)$ denote the continuous functions on $\overline{D}$ that are zero on $T$.

**Theorem 2 ([1]).** The Bergman projection $P$ maps $L^\infty$ boundedly onto $B$. Furthermore, $P$ maps both $C$ and $C_0$ boundedly onto $B_0$.

**Proof.** Cf. [3], Theorem 1.12. □

For future reference we remark that the proof of Theorem 2 in [3] gives that the norm of $P$ is at most $\frac{7}{8}$ and that if $f \in B$ satisfies $f(0) = f'(0) = 0$, then the $L^\infty$-function $\psi = (1-|w|^2)f'(w)/|w|$ is mapped to $f$ under $P$.

It can be shown the Bloch space is the dual of the Bergman space $A^1$, while the Bergman space is the dual of the little Bloch space $B_0$. However, the resulting operator norms are equivalent with, but not equal to the standard norms that we defined previously. See [3], Chapter 1. The strongly exposed points in the Bergman space under the operator norm have been described by C. Nara [5].

7. The space $(A^1)^\perp + C$.

We recall that $(A^1)^\perp + C$ plays the same role in Theorem 1 with respect to the Bergman space as $(H^1)^\perp + C(\mathbb{T})$ does with respect to $H^1$ of the unit ball in $\mathbb{C}$. In $\mathbb{C}$, the space $(H^1)^\perp + C$ is nothing other than the space $H^\infty + C(\mathbb{T})$ which has been studied extensively. It is a famous result ([4],[9]) that $H^\infty + C(\mathbb{T})$ is a closed subspace of $L^\infty$. From this then it follows relatively easily that $H^\infty + C(\mathbb{T})$ is in fact an algebra. We will now discuss how these results extend to the space $(A^1)^\perp + C$.

**Theorem 3.** The space $(A^1)^\perp + C$ is a proper, closed subspace of $L^\infty$.

**Proof.** The kernel of the map $P : L^\infty \to B$ is $(A^1)^\perp$. Because $B_0$ is closed in $B$, $P^{-1}(B_0)$ is closed in $L^\infty$ by the continuity of $P$. By Theorem 2, $L^\infty \neq P^{-1}(B_0) = (A^1)^\perp + C$ and we are done. □

**Theorem 4.** The space $(A^1)^\perp + C$ is a $C$-module.
Before we give the proof we need a lemma.
Let \( L^\infty_0 \) be the subspace of \( L^\infty \) consisting of all \( L^\infty \)-functions that satisfy
\[
\lim_{r \to 1} \operatorname{ess sup}_{r < |z| < 1} |f(z)| = 0.
\]

**Lemma 3.** The space \( L^\infty_0 \) is a closed subspace of \( (A^1)^\perp + C \) and \( (A^1)^\perp + C \) is closed under multiplication by functions in \( L^\infty_0 \).

**Proof.** Clearly, \( L^\infty_0 \) is closed. Also, the product of a function in \( L^\infty_0 \) and a bounded function will again be in \( L^\infty_0 \) so what remains is to show that \( L^\infty_0 \) is contained in \( (A^1)^\perp + C \). Take \( \psi \in L^\infty_0 \). We write \( \psi = \psi_1 + \psi_2 \), where \( \psi_1 \) is the restriction of \( \psi \) to the disc around zero with radius \( r \). If \( r \) is close enough to 1, then \( \|\psi_2\|_\infty \) will be arbitrarily small by the assumption on \( \psi \).

Hence the \( B \)-norm of \( P\psi_2 \) will be arbitrarily small by the continuity of \( P \).

On the other hand, \( P\psi_1 \) is holomorphic across \( \mathbb{T} \), so \( P\psi_1 \in \mathcal{B}_0 \). It follows that the \( B \)-distance of \( P\psi \) to \( \mathcal{B}_0 \) will be at most the \( B \)-norm of \( P\psi_2 \), i.e. arbitrarily small. Because \( \mathcal{B}_0 \) is closed in \( \mathcal{B} \), we conclude that \( P\psi \in \mathcal{B}_0 \).

By the proof of Theorem 3, \( \psi \in (A^1)^\perp + C \) and we are done. \( \square \)

We proceed with the proof of Theorem 4:

**Proof.** The space \( (A^1)^\perp \) is closed under multiplication by \( \overline{\mathbb{Z}} \). Because \( (A^1)^\perp + C \) is closed and by the Stone-Weierstrass theorem, we need then only show that \( zg(z) \in (A^1)^\perp + C \) when \( g \in (A^1)^\perp \). Take \( f \in A^1_0 = zA^1 \), say \( f(z) = zF(z), F \in A^1 \). Then, with the \( L^2 \)-inner product \( \langle ., . \rangle \), \( \langle f, zg \rangle = \langle F, |z|^2 g \rangle \). Observe that the function \( |z|^2 g \) is in \( (A^1)^\perp + C \) because \((1 - |z|^2)g(z) \in L^\infty_0 \) and by Lemma 3. Let’s say, \( |z|^2 g = g_1 + \varphi_1 \), where \( g_1 \in (A^1)^\perp \) and \( \varphi_1 \in C \). Then \( \langle f, zg \rangle = \langle F, \varphi_1 \rangle \). Next we approximate \( \varphi_1 \) uniformly with a trigonometric polynomial \( p_1 = p_1(\varphi_1) \), i.e. \( \|\varphi_1 - p_1\|_\infty < \varepsilon \).

The integral \( \langle F, p_1 \rangle \) depends on the Taylor coefficients of \( F \) in a finite fixed set of places. Because \( f = zF \) has the same coefficients, albeit shifted, we can find a polynomial \( p_2 \) such that \( \langle F, p_1 \rangle = \langle f, p_2 \rangle \), for all \( f = zF \in A^1_0 \).

Now, \( \langle f, zg \rangle = \langle F, \varphi_1 \rangle = \langle F, p_1 \rangle + \langle F, \varphi_1 - p_1 \rangle = \langle f, p_2 \rangle + \langle F, \varphi_1 - p_1 \rangle \), so that \( |\langle f, zg - p_2 \rangle| \leq \varepsilon \|F\|_{A^1} \). We remark that the \( A^1 \)-norms of \( f \) and \( F \) are equivalent in the sense that for all \( F \in A^1 \):
\[
\|zF\|_{A^1} \leq \|F\|_{A^1} \leq 4\|zF\|_{A^1}.
\]

Hence, \( |\langle f, zg - p_2 \rangle| \leq 4\varepsilon \|f\|_{A^1} \). By the Hahn-Banach theorem, the \( L^\infty \)-distance of \( zg - p_2 \) to the annihilator of \( A^1_0 \) is at most \( 4\varepsilon \). And since \( p_2 \) is continuous, and \( (A^1)^\perp + C = (A^1)^\perp + C \), the \( L^\infty \)-distance of \( zg \) to \( (A^1)^\perp + C \) is at most \( 4\varepsilon \), thus zero. By Theorem 3, \( zg \in (A^1)^\perp + C \) and the proof is complete. \( \square \)

It is well-known that the space \( H^\infty + C(\overline{\mathbb{D}}) \) is closed in \( L^\infty(D) \) (Theorem 6.5.5. in [8], [9]). Let us write \( \mathcal{A} := L^\infty_0 + \overline{H^\infty} + C(\overline{\mathbb{D}}) \), where the bar denotes complex conjugation. By the preceding remarks, the space \( \mathcal{A} \) is a non-trivial closed algebra contained in \( (A^1)^\perp + C \), and the space \( (A^1)^\perp + C \) is an \( \mathcal{A} \)-module. It should be stressed however that \( (A^1)^\perp + C \) is not an algebra.

**Lemma 4.** Let \( f_\beta = (1 - z)^\beta \) and let \( \varphi_\beta = f_\beta / |f_\beta| \) for \( \beta \in \mathbb{R} \). Then \( \varphi_{-4} \in (A^1)^\perp \), but \( \varphi_{-2} \) is not contained in the space \( (A^1)^\perp + C \).
Proof. Using the Stokes theorem one obtains that, at least formally, for every polynomial $F$:

$$
\int_D F \overline{\varphi} \, dA = \int_D \overline{\partial} [F(z) \frac{(1-z)^2}{(1-\overline{z})}] \, dA
= \int_S F(z) \frac{(1-z)^2 \, dz}{2\pi (1-\overline{z})} = \int_S -F(z)(1-z) \frac{dz}{2\pi} = 0.
$$

(In fact, by the same argument, $\int_D F(z) \overline{\varphi} \, dA = 0$ for all $k = 2, 3, 4, \ldots$.)

Here we have used the identity $\frac{z^k}{\overline{z}} = 1$ on $S$ to simplify the integrals over the circle. We conclude that (formally) $P\varphi_{-4} = 0$, that is, $\varphi_{-4} \in (A^1)\perp$.

This claim can be made precise by a limit argument involving integration over the unit disc with a small disc around the point $z = 1$ punched out. Alternatively, one can directly calculate the Bergman projection of $\varphi_{-4}$. We omit the details. By Theorem 1 combined with the calculations in the proof of Proposition 1, we conclude that the $L^\infty$-distance of $\varphi_{-2} \mapsto (A^1)\perp + C$ is 1. In particular, the function $\varphi_{-2}$ is not contained in $(A^1)\perp + C$. \hfill \qed

**Corollary 1.** The space $(A^1)\perp + C$ is not an algebra.

Indeed, $\varphi_{-4}$ and $\varphi_2$ are both contained in $(A^1)\perp + C$, but their product $\varphi_{-4} \cdot \varphi_2 = \varphi_{-2}$ is not.

Next, let $u$ be an automorphism (Möbius map) of $D$. If $\psi$ is an element of $L^\infty(D)$ one can define the composition $\psi \circ u$ in $L^\infty$ of $\psi$ and $u$ as (represented by) the composition of $\Psi$ with $u$, where $\Psi$ is any representative of $\psi$. That this yields a well-defined element of $L^\infty$ follows from the fact that $u$ and its inverse map sets of Lebesgue measure zero to sets of Lebesgue measure zero. It is easily seen that the map $\psi \mapsto \psi \circ u$ is an isometric isomorphism of $L^\infty$.

**Proposition 3.** The space $(A^1)\perp + C$ is invariant under composition with automorphism of $D$.

Proof. Clearly, the space $C$ is invariant under composition with automorphisms of $D$. Take an element $g \in (A^1)\perp$, and let $u$ be an automorphism of $D$. We will show that $g \circ u$ is contained in $(A^1)\perp + C$. Let $f$ be an element of $A^1$. Then $\int_D \overline{g} \circ u \, dA = \int_D (f \circ u^{-1}) \overline{g} J_R(u^{-1}) \, dA$, where $J_R(u^{-1})$ is the real Jacobian of $u^{-1}$, an element of $C$. By Theorem 4 there exist $g^* \in (A^1)\perp$ and $h \in C$ such that $g J_R(u^{-1}) = g^* + h$. Thus, because $f \circ u^{-1}$ is contained in $A^1$, $\int_D \overline{g} \circ u \, dA = \int_D (f \circ u^{-1}) \overline{h} \, dA = \int_D f \overline{(h \circ u)} J_R(u) \, dA$. We conclude that $g \circ u - (h \circ u)J_R(u)$ annihilates the Bergman space, hence $g \circ u \in (A^1)\perp + C$. \hfill \qed

**Proposition 4.** Let $f$ be a strongly exposed point in $A^1$.

(a) If $u$ is an automorphism of $D$, then the normalized function $f_1 = C_1(f \circ u)$ is strongly exposed.

(b) If $v \in C$ is holomorphic on $D$ and zero-free on the circle, then the normalized function $f_2 = C_2 f v$ is strongly exposed.

Furthermore, the functions $\varphi = f/|f|$, $\varphi_1 = f_1/|f_1|$ and $\varphi_2 = f_2/|f_2|$ have the same $L^\infty$-distance to $(A^1)\perp + C$. 
Proof. (a) There exist \( g \in (A^1)\perp \), \( h \in C \) such that \( \|\varphi - g - h\|_\infty < 1 \). By Proposition 3, \( g \circ u \) is again contained in \((A^1)\perp + C\). Because \( \varphi_1 = \varphi \circ u \),

\[ \|\varphi_1 - g \circ u - h \circ u\|_\infty < 1, \]

and we conclude that \( f_1 \) is strongly exposed. Also, the \( L^\infty \)-distance of \( \varphi_1 \) to \((A^1)\perp + C\) does not exceed that of \( \varphi \). Replacing \( u \) by its inverse, the reverse inequality follows. (b) With \( g \) and \( h \) as above and \( \varphi_2 = \varphi \circ h \), \( \|\varphi_2 - g \circ h - h \circ h\|_\infty < 1 \). One finishes the proof as before, using Lemma 3 and the fact that \( \frac{\varphi}{|\varphi|} \) is invertible in \( L_0^\infty + C \).

\[ \square \]

8. The functions \((z - \alpha)^\beta\), Part II.

We saw in section 4 that the functions \( f_\beta = c_\beta (1-z)^\beta \) are strongly exposed in the unit ball of \( A^1 \) for all \( \beta > -1 \) except possibly when \( \beta = 1, 3, 5, \ldots \). This was deduced from rather straightforward estimates of the \( L^\infty \)-distances of the functions \( \varphi_\beta = f_\beta / |f_\beta| \) to the space \((A^1)\perp + C\) (Proposition 2). In this section we will sharpen these estimates and answer the question of strong exposedness for odd exponents.

**Theorem 5.** For all \( \beta \geq 0 \), the Bloch distance of the function \( P\varphi_\beta \) to \( B_0 \) equals \( \frac{4}{\pi} \frac{|\sin(\frac{\beta \pi}{2})|}{|\beta + 2|} \).

**Proof.** We showed in section 5 that the functions \( P\varphi_\beta \) are contained in \( B_0 \) so henceforth we will assume that \( \beta \) is not even. It is convenient to rewrite \( \varphi_\beta \) as \( \varphi_\beta(w) = (1 - w)^{\beta/2}/(1 - \overline{w})^{\beta/2} \). Using the series expansions for the Bergman kernel \( 1/(1 - z\overline{w})^2 \) (see (2.2)), as well as for \( (1 - w)^{\beta/2} \) and \( 1/(1 - \overline{w})^{\beta/2} \), we evaluate the Bergman projection \( P\varphi_\beta \). One obtains \( P\varphi_\beta = \sum_{n=0}^\infty c_\beta_n z^n \), where

\[ c_\beta_n = \frac{\Gamma(m + \frac{\beta}{2})}{\Gamma(-\frac{\beta}{2}) \Gamma(\frac{\beta}{2})} \sum_{m=0}^\infty \frac{\Gamma(m + n + \frac{\beta}{2}) \Gamma(m + n - \frac{\beta}{2})}{m!(m + n + 1)!} \cdot \]

we claim that for fixed \( \beta > 0 \):

\[ \sum_{m=0}^\infty \frac{\Gamma(m + \frac{\beta}{2}) \Gamma(m + n - \frac{\beta}{2})}{m!(m + n + 1)!} = \frac{4}{n^2 \beta(\beta + 2)} (1 + o(1)), \]

where the \( o(1) \)-term tends to zero as \( n \to \infty \). This implies that

\[ c_\beta_n = \frac{4}{\Gamma(-\frac{\beta}{2}) \Gamma(\frac{\beta}{2}) n\beta(\beta + 2)} (1 + o(1)) = \frac{-2 \sin(\frac{\beta \pi}{2})}{\pi(\beta + 2)} n (1 + o(1)), \]

where the \( o(1) \)-term vanishes as \( n \to \infty \). (Here we have used the functional equations \( \Gamma(z + 1) = z \Gamma(z) \) and \( \Gamma(z) \Gamma(1 - z) \sin(\pi z) = \pi \)). But then,

\[ \lim_{x \to 1^-} |(1 - x^2)(P\varphi_\beta)'(x)| = \frac{4|\sin(\frac{\beta \pi}{2})|}{\pi(\beta + 2)}, \]

so the Bloch distance of \( P\varphi_\beta \) to \( B_0 \) is at least \( \frac{4|\sin(\frac{\beta \pi}{2})|}{\pi(\beta + 2)} \). On the other hand, for large \( N \),

\[ |(\sum_{n=N}^\infty c_\beta_n z^n)'| \leq \sum_{n=N}^\infty |c_\beta_n| |z|^{n-1} \leq \frac{2 |\sin(\frac{\beta \pi}{2})|}{\pi(\beta + 2)} \cdot \frac{1 + o(1)}{1 - |z|}, \]
where the $o(1)$-term tends to zero as $N$ increases. Using the fact that the polynomials are contained in $B_0$ it follows that the Bloch distance of $P_{\varphi\beta}$ to $B_0$ is at most $\frac{|4\sin(\frac{\pi \beta}{2})|}{\pi(\beta+2)}$. This then proves the theorem.

We turn to the claim (8.1). Let us first assume $\beta > 2$. Given any large $n \in \mathbb{N}$, let $M = M_n$ be the integer nearest to $\sqrt{n}$. We write

$$
\sum_{m=0}^{\infty} \frac{\Gamma(m+\frac{\beta}{2})\Gamma(m+n-\frac{\beta}{2})}{m!(m+n+1)!} = \sum_{m=0}^{M-1} \sum_{m=M}^{\infty}.
$$

Because $\beta > 2$, $\frac{\Gamma(m+\frac{\beta}{2})}{m!}$ is increasing in $m$. On the other hand, $\frac{\Gamma(n+m-\frac{\beta}{2})}{(n+m+1)!}$ is decreasing in $m + n$. The first sum can thus be estimated by

$$
\sum_{m=0}^{M-1} \leq M \frac{\Gamma(M+\frac{\beta}{2})\Gamma(n-\frac{\beta}{2})}{(M)!} \frac{(n-\beta/2)}{(n+1)!}.
$$

Recall Stirling’s formula:

$$
\lim_{x \to \infty} \frac{\Gamma(x+1)}{\sqrt{2\pi x} x^x} = 1.
$$

By this result, there exists a constant $A = A_\beta$, independent of $n$, such that

$$
\sum_{m=0}^{M-1} \leq A \frac{M \cdot M^{\frac{\beta}{2}-1}}{n^{2+\frac{\beta}{2}}} = A \frac{\left(\frac{M}{n}\right)^{\frac{\beta}{2}}}{n^2}.
$$

Hence

$$
\sum_{m=0}^{M-1} = o(1),
$$

as $n \to \infty$. In the remaining sum, $\sum_{m=M}^{\infty}$, all the arguments in the Gamma functions and factorials tend to infinity as $n \to \infty$. Another application of Stirling’s formula seems in place. One obtains that, given a $\varepsilon > 0$, for all sufficiently large $n$ and all $m \geq M$,

$$
\left| \frac{\Gamma(m+\frac{\beta}{2})\Gamma(m+n-\frac{\beta}{2})}{m!(m+n+1)!} - 1 \right| < \varepsilon.
$$

In particular, for $\alpha > 0$,

$$
\left| \sum_{m=M}^{\infty} \frac{\Gamma(m+\frac{\beta}{2})\Gamma(m+n-\frac{\beta}{2})}{m!(m+n+1)!} - 1 \right| < \varepsilon,
$$

as $n \to \infty$. Therefore, by (8.3), the claim (8.1) follows once we show that

$$
\sum_{m=M}^{\infty} \frac{m^{\frac{\beta}{2}-1}}{(n+m)^{\frac{\beta}{2}+2}} \sim \frac{4}{n^{2\beta(\beta+2)}} \to 1,
$$

as $n \to \infty$. Let us investigate the functions $g_n(x) = \frac{x^{\frac{\beta}{2}-1}}{(n+x)^{\frac{\beta}{2}+2}}$. For all $x \geq 1$, $g_n(x) \leq \frac{1}{x(n+x)^2}$, so $g_n(x) \leq \frac{1}{x^{\beta/2}}$ when $x \geq M$. There is a number $x_{\beta,n} > 0$ such that $g_n(x)$ is increasing on the interval $(0,x_{\beta,n}]$ and decreasing
on the interval \([x_{\beta,n}, \infty)\). Hence, the sum \(\sum_{m=M}^{\infty} \frac{m^{\beta-1}}{(n+m)^{2+\beta}}\) and the integral
\[
\int_{M}^{\infty} \frac{\frac{x^{\beta-1}}{(n+x)^{2+\beta}}} \, dx \text{ differ at most } \frac{4}{n^{\frac{3}{2}}} = o(1). \]
By a change of variables,
\[
\int_{M}^{\infty} \frac{x^{\beta-1}}{(n+x)^{2+\beta}} \, dx = \frac{1}{n^2} \int_{\frac{M}{n}}^{\infty} \frac{x^{\beta-1}}{(1+x)^{2+\beta}} \, dx.
\]
Now, with \(B(\cdot, \cdot)\) the standard Beta-function,
\[
\int_{0}^{\infty} \frac{x^{\beta-1}}{(1+x)^{2+\beta}} \, dx = B\left(\frac{\beta}{2}, 2\right) = \frac{4}{\beta(\beta+2)}.
\]
On the other hand, as \(n \to \infty\),
\[
\int_{0}^{\frac{M}{n}} \frac{x^{\beta-1}}{(1+x)^{2+\beta}} \, dx = o(1).
\]
By the preceding estimates, the claim (8.1) now follows for all \(\beta > 2\).

When \(0 < \beta < 2\) we proceed as follows. Given a large \(n \in \mathbb{N}\), we let \(M = M_n\) be the integer nearest to \(n^2\). Now the terms in the sum
\[
\sum_{m=0}^{\infty} \frac{\Gamma(m+\frac{\beta}{2})\Gamma(m+n-\frac{\beta}{2})}{m!(m+n+1)!} = \sum_{m=0}^{M} \sum_{m=M}^{\infty}.
\]
are decreasing. The first sum can be estimated by
\[
\sum_{m=0}^{M-1} \leq M \frac{\Gamma\left(\frac{\beta}{2}\right)\Gamma\left(n-\frac{\beta}{2}\right)}{(n+1)!} \leq \frac{A_\beta}{n^{2+\frac{\beta}{2}}} = o(1).
\]
The second sum can be dealt with as before. (Now the functions \(g_n(x)\) are decreasing on \((0, \infty)\), which makes the analysis even simpler.) We omit the details. This finishes the proof of equation (8.1) for all \(\beta > 0\).

**Corollary 2.** Let \(d(\varphi_\beta, (A_1)^\perp + C)\) denote the \(L^\infty\)-distance of \(\varphi_\beta\) to \((A_1)^\perp + C\). Then for all \(\beta \geq 0\),
\[
1 \leq d(\varphi_\beta, (A_1)^\perp + C) \leq \frac{4}{\pi} \left| \sin\left(\frac{\beta\pi}{2}\right) \right| \leq \frac{2}{\pi},
\]
In particular, all \(f_\beta\) are strongly exposed for \(\beta \geq 0\).

**Proof.** Let \(q : \mathcal{B} \to \mathcal{B}/\mathcal{B}_0\) be the quotient map. By Theorem 2, the map \(q \circ P : L^\infty \to \mathcal{B}/\mathcal{B}_0\) is continuous and surjective. In the proof of Theorem 3 it was shown that the kernel of the map \(q \circ P\) is the space \((A_1)^\perp + C\). It follows that the derived map
\[
P^* : L^\infty / \left((A_1)^\perp + C\right) \to \mathcal{B}/\mathcal{B}_0
\]
is bijective and bounded by \(\frac{8}{\pi}\) (cf. the proof of Theorem 2). This gives the lower bound for \(d(\varphi_\beta, (A_1)^\perp + C)\), because \(\|P^*\varphi_\beta\| = \frac{4}{\pi} \left| \sin\left(\frac{\beta\pi}{2}\right) \right| / \beta + 2\).

By the closed graph theorem, the inverse \(P^{-1}\) of \(P^*\) is also bounded. Actually, we will show directly that \(\|P^{-1}\| \leq 1\), which in turn yields the upper bound for \(d(\varphi_\beta, (A_1)^\perp + C)\). Let us suppose that \(F \in \mathcal{B}/\mathcal{B}_0\) has norm
1. We need to show that $P^{*-1}(F)$ has norm at most 1 in $L^\infty/((A^1)^\perp + C)$. For any $\varepsilon > 0$, we can find a representative $f \in \mathcal{B}$ of the coset $F$ such that $\|f\|_\mathcal{B} < 1 + \varepsilon$. We recall from the proof of Theorem 2 that

$$\psi(w) = (1 - |w|^2) \cdot \frac{f'(w) - f'(0)}{w} \in L^\infty$$

satisfies $f(z) - P\psi(z) = f(0) + f'(0)z \in \mathcal{B}_0$. Thus $\psi$ is a representative of $P^{*-1}(F)$ in $L^\infty$. Hence, by Lemma 3,

$$\|P^{*-1}(F)\|_{L^\infty/((A^1)^\perp + C)} \leq \limsup_{r \to 1} \frac{\sup_{|w| < 1} |\psi(w)|}{1 + \varepsilon} \leq \frac{\|f\|_\mathcal{B} - 1}{\varepsilon}.$$ 

\textbf{Corollary 3.} Suppose that $g \in H(D) \cap C$ vanishes nowhere on $\mathbb{T}$. Let $z_1, z_2, \ldots, z_n \in \mathbb{T}$ be distinct and let $\beta_1, \beta_2, \ldots, \beta_n$ be real numbers greater than $-2$. Then the normalized function $f(z) = cg(z) \prod_{i=1}^n (1 - z\zbar)^{\beta_i}$ is strongly exposed in the unit ball of $A^1$ if and only if all functions $f_{\beta_i} = c\beta_i (1 - z)^{\beta_i}$ are strongly exposed. In particular, if all choices of $\beta_i > -1$ yield strongly exposed points and all normalized polynomials are strongly exposed in the unit ball of $A^1$.

\textbf{Proof.} By part (b) of Proposition 4, the factor $g(z)$ has no effect on strong exposedness of the function $f$. Let $d_i = d(\varphi_{\beta_i}, (A^1)^\perp + C)$ and let $\varphi = f/|f|$. We will show that $d(\varphi, (A^1)^\perp + C) = \max_i d_i$, which will give the desired result.

We find small pairwise disjoint neighborhoods $U_i$ of the $z_i$ and a partition $\chi_i$ of the unity relative to the $U_i$'s and $D$. That is to say, we find continuous functions $\chi_i \geq 0$ on $\overline{D}$ such that $\chi_i \equiv 1$ on $U_i$ and $\sum_i \chi_i \equiv 1$ on $D$. Then $\varphi = \sum_i \chi_i \varphi = \sum_i \varphi^{(i)}$. For every $i$, there exists a unimodular constant $\lambda = \lambda_i$ for which $\varphi^{(i)}(z) - \lambda \varphi_{\beta_i}(z\zbar) \in C$. Consequently, $d_i = d(\varphi^{(i)}, (A^1)^\perp + C)$. Using the $C$-module structure of $(A^1)^\perp + C$ and the fact that $\chi_i \leq 1$, it is easily seen that $d(\varphi^{(i)}, (A^1)^\perp + C) \leq d(\varphi, (A^1)^\perp + C)$, hence $\max_i d_i \leq d(\varphi, (A^1)^\perp + C)$.

Conversely, if the functions $g_i \in (A^1)^\perp + C$ are such that $\|\varphi^{(i)} - g_i\|_\infty < d_i + \varepsilon$, then

$$\| \sum_i \chi_i \varphi^{(i)} - \sum_i \chi_i g_i \|_\infty < \max_i d_i + \varepsilon.$$ 

Because $\varphi - \sum_i \chi_i \varphi^{(i)} = \sum_i \chi_i (1 - \chi_i) \varphi \in C$ and $\chi_i g_i \in (A^1)^\perp + C$, it follows that $d(\varphi, (A^1)^\perp + C) < \max_i d_i + \varepsilon$. \hfill \Box

We will now show that an estimate analogous to inequality (8.4) also holds for $\beta \in (-2, 0)$.

\textbf{Lemma 5.} Let $g_n = \frac{1}{n} (1 - z)^{-2 + \frac{1}{n}}$. Then $\lim_{n \to \infty} \|g_n\|_1 = 1$.

\textbf{Proof.} One can perform the calculation

$$\int_D |1 - z|^{-2 + \frac{1}{n}} dA(z) = \sum_{k=0}^{\infty} \frac{\Gamma^2(k + 1 - \frac{1}{2n})}{\Gamma^2(1 - \frac{1}{2n}) k!(k + 1)!}.$$ 

Using polar coordinates for the integral or hypergeometric functions for the sum, this expression can be evaluated to $\Gamma(1/n)/(\Gamma(1 + 1/2n))^2$. However
the following asymptotics are fairly simple and useful in the following Proposition.

The terms \( g_{n,k} = \frac{\Gamma^2(k+1-\frac{1}{2})}{k^2(k+1)!} \) are decreasing in \( k \). For large \( n \), let \( K = K_n = \sqrt{n} \). Then \( \sum_{k=1}^{K} g_{n,k} \leq \sqrt{n} \). In the remaining sum, we can approximate the terms using Stirling’s formula (8.2): \( g_{n,k} \sim \frac{1}{k^{1+\frac{1}{2}}} \). Therefore, \( \sum_{k=K+1}^{\infty} g_{n,k} \sim \int_{\sqrt{n}}^{\infty} \frac{1}{x^{1+\frac{1}{2}}} \sim n \). This proves the claim. \( \square \)

**Proposition 5.** For all \(-2 < \beta < 0\),

\[
(8.5) \quad \frac{2}{\pi} \frac{|\sin(\beta \pi)|}{\beta + 2} \leq d(\varphi_\beta, (A^1)_{\perp} + C),
\]

where again \( d(\varphi_\beta, (A^1)_{\perp} + C) \) denotes the \( L^\infty \)-distance of \( \varphi_\beta \) to \((A^1)_{\perp} + C\).

**Proof.** Fix any \( \beta \in (-2, 0) \) and let \( L = L_\beta \) be the functional \( L : g \in A^1 \mapsto \int_D g \varphi_\beta \, dA \). We will show that for the sequence \( g_n \) from Lemma 5,

\[
(8.6) \quad \lim_{n \to \infty} L(g_n) = \frac{2}{\pi} \frac{|\sin(\beta \pi)|}{\beta + 2}.
\]

From this we will get the desired lower bound as follows. The functions \( g_n \) and all their derivatives tend to zero uniformly on compact subsets of \( D \). Given any (fixed) integer \( N \) we define functions \( g^*_n(z) := g_n(z) - \sum_{k=0}^{N-1} \frac{g^{(k)}(0)}{k!} z^k \). It follows that \( \lim_{n \to \infty} \|g^*_n\|_1 = 1 \) and

\[
(8.7) \quad \lim_{n \to \infty} L(g^*_n) = \frac{2}{\pi} \frac{|\sin(\beta \pi)|}{\beta + 2}.
\]

Furthermore, by construction, the first \( N \) derivatives of the \( g^*_n \) vanish at the origin. If we let \( z^N A^1 \subset A^1 \) denote the closed subspace of all functions in \( A^1 \) whose first \( N \) derivatives vanish at the origin, then by equation (8.7), the norm of the functional \( L \) restricted to \( z^N A^1 \) is at least \( \frac{2}{\pi} \frac{|\sin(\beta \pi)|}{\beta + 2} \). By the Hahn-Banach theorem, the \( L^\infty \)-distance of \( \varphi_\beta \) to \( z^N A^1 \) is at least \( \frac{2}{\pi} \frac{|\sin(\beta \pi)|}{\beta + 2} \). Consequently, the \( L^\infty \)-distance of \( \varphi \) to \( (A^1)_{\perp} + C \) is at least \( \frac{2}{\pi} \frac{|\sin(\beta \pi)|}{\beta + 2} \). Observe that \( \mathcal{P} \) is uniformly dense in \((A^1)_{\perp} + C\), because it contains \((A^1)_{\perp}\) and all trigonometric polynomials. Therefore, the \( L^\infty \)-distance of \( \varphi \) to \((A^1)_{\perp} + C\) is at least \( \frac{2}{\pi} \frac{|\sin(\beta \pi)|}{\beta + 2} \).

Let us turn to formula (8.6). We calculate

\[
nL(g_n) = \int_D \frac{1}{(1-z)^{2+\frac{\beta}{2} - \frac{1}{\pi}}} \frac{1}{(1-\overline{z})^{-\frac{\beta}{2}}} \, dA(z)
\]

using the series expansions for \((1-z)^{\alpha}\) and \((1-\overline{z})^{\alpha}\). After a routine calculation, one obtains the following expression:

\[
nL(g_n) = \frac{1}{\Gamma(2 + \frac{\beta}{2} - \frac{1}{\pi}) \Gamma(-\frac{\beta}{2})} \sum_{k=0}^{\infty} \frac{\Gamma(2 + \frac{\beta}{2} - \frac{1}{\pi} + k)}{(k+1)!} \frac{\Gamma(k - \frac{\beta}{2})}{(k+1)!}.
\]

Now

\[
\frac{1}{\Gamma(2 + \frac{\beta}{2} - \frac{1}{\pi}) \Gamma(-\frac{\beta}{2})} \rightarrow \frac{1}{\Gamma(2 + \frac{\beta}{2}) \Gamma(-\frac{\beta}{2})} = \frac{2}{\pi} \frac{|\sin(\beta \pi)|}{\beta + 2}.
\]
as $n \to \infty$. So what’s left to do is to show that
\[
\frac{1}{n} \sum_{k=0}^{\infty} \frac{\Gamma(2 + \frac{\beta}{2} - \frac{1}{n} + k) \Gamma(k - \frac{\beta}{2})}{(k + 1)! \Gamma(k - \beta / 2)} \to 1,
\]
as $n \to \infty$. This can easily be done by following the proof of Lemma 5. □

We conclude with a conjecture on the functions $f_\beta$ for $-2 < \beta < 0$, for which strong exposedness is already implied by Proposition 2 when $-1 < \beta < 0$. Note that inequality (8.5) is “asymptotically sharp” for $\beta \downarrow -2$:
\[
\lim_{\beta \downarrow -2} \frac{2}{\pi} \frac{\sin(\frac{\beta \pi}{2})}{\beta + 2} = 1 = d(\varphi_{-2}, (A^1)_\perp + C).
\]

**Conjecture 1.** For all $-2 < \beta < 0$, $d(\varphi_{\beta}, (A^1)_\perp + C) = \frac{2}{\pi} \frac{\sin(\frac{\beta \pi}{2})}{\beta + 2}$. In particular, the functions $f_\beta$ are strongly exposed for all said $\beta$.

**References**

[1] R.R. Coifman, R. Rochberg, G. Weiss, *Factorization theorems for Hardy spaces in several variables*, Ann. of Math. (2), 103 (1976), 611–635.

[2] J. Garnett, *Bounded holomorphic functions*, Pure & Applied Mathematics, 96, Academic Press, Inc., New York-London, 1981.

[3] H. Hedenmalm, B. Korenblum, K. Zhu, *Theory of Bergman Spaces*, Springer Graduate Texts in Mathematics 199, 2000.

[4] H. Helson, D. Sarason, *Past and future*, Math. Scand. 21 (1967), 5–16.

[5] C. Nara, *Uniqueness of the predual of the Bloch space and its strongly exposed points*, Illinois J. Math, 34 (1990), no. 1, 98–107.

[6] R.R. Phelps, *Dentability and extreme points in Banach spaces*, J. Funct. Anal. 17 (1974), 78–90.

[7] E. Reich, *An extremum problem for analytic functions with area norm*, Annales Academiae Scientiarum Fennicae, Series A.I. Mathematica, Volumen 2, (1976), 429-445.

[8] W. Rudin, *Function Theory in the Unit Ball of $\mathbb{C}^n$*, Grundlehren der Mathematischen Wissenschaften, 241, Springer-Verlag, New York-Berlin, 1980.

[9] W. Rudin, *Spaces of type $H^\infty + C$*, Ann. Institut Fourier, 25, (1975), 99–125.

[10] J. Wiegerinck, *A characterization of strongly exposed points of the unit ball of $H^1$*, Indag. Mathem., N.S., 4 (4) (1993), 509–519.