Reconstruction of omega-categorical structures from their endomorphism monoids

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Joint with Manuel Bodirsky, Michael Kompatscher and Michael Pinsker.
Non-reconstructibility

Fact
There exist separable profinite groups $G_1, G_2$ which are isomorphic as groups, but not as topological groups.

Theorem (DE + P. Hewitt, 1990)
There exist two countable, $\omega$-categorical structures $\mathcal{M}_1, \mathcal{M}_2$ whose automorphism groups are isomorphic as groups, but not as topological groups.

Theorem (M. Bodirsky + DE + M. Kompatscher + M. Pinsker, ’14)
There exist two countable, $\omega$-categorical structures $\mathcal{M}_1, \mathcal{M}_2$ whose endomorphism monoids are isomorphic as monoids, but not as topological monoids.

- Can use the same $\mathcal{M}_1, \mathcal{M}_2$.
- Question asked by Lascar (’87); Bodirsky, Pinsker, Pongrácz (’14).
Endomorphisms

Relational structure with domain $A$: $\mathcal{A} = (A; (R_i : i \in I))$, where $R_i \subseteq A^{n_i}$, $n_i \in \mathbb{N}$.

Endomorphism of $\mathcal{A}$: $\alpha : A \rightarrow A$, $\alpha(R_i) \subseteq R_i$ for all $i \in I$.

End$(\mathcal{A})$: monoid of endomorphisms of $\mathcal{A}$.

Caveat: Sensitive to the language (ie. choice of the atomic relations $R_i$).

Aut$(\mathcal{A})$: group of units in End$(\mathcal{A})$.

Topological monoid: End$(\mathcal{A}) \subseteq A^A$. 
Translations

Closed subgroups of $\text{Sym}(A) \leftrightarrow \text{Aut}(A)$, $A$ relational structure with domain $A$.

Closed submonoids of $A^A \leftrightarrow \text{End}(A)$, $A$ relational structure with domain $A$.

Suppose $A$ is countable:

Closed oligomorphic subgps of $\text{Sym}(A) \leftrightarrow \text{Aut}(A)$, $A$ $\omega$-categorical.

Oligomorphic: finitely many orbits on $A^n$, for all $n \in \mathbb{N}$.

Closed submonoids of $A^A \leftrightarrow \text{End}(A)$, $A$ $\omega$-categorical.

with oligomorphic unit group

If $A$ is $\omega$-categorical the closure of $\text{Aut}(A)$ in $\text{End}(A)$ is the monoid $E\text{Emb}(A)$ of elementary embeddings $A \to A$. 

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Suppose $\mathcal{A}_1, \mathcal{A}_2$ are countable, $\omega$-categorical structures.
Suppose $X$ denotes $\text{Aut}$, $\text{End}$ or $\text{EEmb}$.

Suppose $X(\mathcal{A}_1)$ and $X(\mathcal{A}_2)$ are isomorphic as algebraic objects. How are $\mathcal{A}_1$ and $\mathcal{A}_2$ related?

REMARK: If $\text{Aut}(\mathcal{A}_1)$ and $\text{Aut}(\mathcal{A}_2)$ are isomorphic as topological groups, then $\mathcal{A}_1, \mathcal{A}_2$ are biinterpretable.
Failure of automatic continuity

Theorem (Bodirsky, Pinsker, Pongrácz, 2014)

Let $\mathcal{A}$ be countable $\omega$-categorical. Then there is a monoid homomorphism $\xi : EEmb(\mathcal{A}) \to A^A$ which is not continuous.
Lascar’s Theorem

**Definition:** (1) If $S$ is a topological group, denote by $S^\circ$ the intersection of the closed subgroups of finite index in $S$.
(2) A countable, $\omega$-categorical structure $\mathcal{A}$ is *G-finite* if for every open subgroup $U \leq \text{Aut}(\mathcal{A})$ the subgroup $U^\circ$ is of finite index in $U$.

**Theorem (Lascar, 1980’s)**

Suppose $\mathcal{A}_1, \mathcal{A}_2$ are countable, G-finite, $\omega$-categorical structures and $\alpha : \text{EEmb}(\mathcal{A}_1) \to \text{EEmb}(\mathcal{A}_2)$ is an isomorphism of monoids. Then the restriction of $\alpha$ to $\text{Aut}(\mathcal{A}_1)$ is a topological isomorphism between $\text{Aut}(\mathcal{A}_1)$ and $\text{Aut}(\mathcal{A}_2)$. In particular, $\mathcal{A}_1$ and $\mathcal{A}_2$ are biinterpretable.

Start of proof: For $e, f \in \text{EEmb}(\mathcal{A}_1)$, write $e \leq f$ iff there is $k \in \text{EEmb}(\mathcal{A}_1)$ with $e = fk$. Note that this is preserved by $\alpha$ and $e \leq f$ iff $\text{im}(e) \subseteq \text{im}(f)$. So we can recover the poset of elementary submodels of $\mathcal{A}_1$ from the algebraic structure of $\text{EEmb}(\mathcal{A}_1)$. . .

**Question:** Can we recover $\text{EEmb}(\mathcal{A})$ from the algebraic structure of $\text{End}(\mathcal{A})$ (for $\omega$-categorical $\mathcal{A}$)?
Profinite quotients

Any separable profinite group $K$ embeds as a closed subgroup of $\Pi_{n \in \mathbb{N}} \text{Sym}(n)$.

**Fact (Cherlin - Hrushovski)**

There is a countable, $\omega$-categorical structure $\mathcal{A}$ and a continuous surjection $\theta : \text{Aut}(\mathcal{A}) \to \Pi_{n \in \mathbb{N}} \text{Sym}(n)$ with kernel $\Phi = (\text{Aut}(\mathcal{A}))^\circ$.

So if $K \leq \Pi_{n \in \mathbb{N}} \text{Sym}(n)$ is closed, then $\Sigma_K = \theta^{-1}(K)$ is a closed, oligomorphic group, $\Sigma_K^\circ = \Phi$, and $\Sigma_K / \Phi \cong K$.

**Remark:** If $K_1, K_2 \leq \Pi_{n \in \mathbb{N}} \text{Sym}(n)$ are closed and algebraically isomorphic, there does not seem to be any reason to expect that $\Sigma_{K_1}$ and $\Sigma_{K_2}$ should be algebraically isomorphic.
Examples for non-reconstructibility

Fact

There is a separable profinite group $G$ with the following properties:

- $G$ has a finite, central subgroup $F \neq 1$ such that $F$ has a complement in $G$ and any such complement is dense in $G$.
- $G$ is nilpotent of class 2 and the derived subgroup $G^{(1)}$ is a proper, dense subgroup of the centre $Z(G)$.

From the first point, there is a subgroup $E \leq G$ with $G = F \times E$, and any such $E$ is dense in $G$.

If $H = G/F$, then $H$ is algebraically isomorphic to $E$, but not topologically.

Thus $K = F \times H$ and $G$ are profinite groups which are isomorphic as groups.

Note that $Z(K) = F \times Z(H)$ and $K^{(1)} = 1 \times H^{(1)}$, so the derived group of $K$ is not dense in its centre. So $G, K$ are not topologically isomorphic.
Consider $G \xrightarrow{\pi} H = G/F$ and $\eta : H \rightarrow E$ given by $(\pi|E)^{-1}$ (discontinuous).

$G$ has a base $(G_i : i \leq \omega)$ of open neighbourhoods of 1 where $G_i \trianglelefteq G$ and $\bigcap_{i<\omega} G_i = F$.

Let $H_i = \pi(G_i)$ for $i < \omega$ and $H_\omega = \pi(G_\omega \cap E)$.

Let $X = \bigsqcup_{i<\omega} H/H_i$ and $C = H/H_\omega$.

The action of $H$ on $X$ gives a continuous embedding $H \rightarrow \text{Sym}(X)$.

The action of $H$ on $X \cup C$ gives an embedding $H \rightarrow \text{Sym}(X \cup C)$ which is not continuous. The closure of the image is topologically isomorphic to $G$.

**Proof:** Identify $X$ with $\bigsqcup_{i<\omega} G/G_i$ and $C$ with $G/G_\omega$ via $\alpha : H/H_\omega \rightarrow G/G_\omega$ where $\alpha(hH_\omega) = \eta(h)G_\omega$. This is a bijection and $\eta(h)\alpha(kH_\omega) = \alpha(hkH_\omega)$. 
From $\Sigma_H$ to $\Gamma$

- Find $A$ countable, $\omega$-categorical, $\Sigma = \text{Aut}(A)$, with a continuous surjection $\nu : \Sigma \to H$ with kernel $\Phi = \Sigma^\circ$.
- Let $\Psi = \nu^{-1}(H_\omega)$; identify $C = H/H_\omega$ with $\Sigma/\Psi$.
- Let $B = A \cup C$ with $i : \Sigma \to \text{Sym}(B)$ the resulting action.
- Let $\Gamma$ be the closure of $i(\Sigma)$ in $\text{Sym}(B)$.

**Lemma**

1. $\Gamma$ is oligomorphic on $B$;
2. $\Gamma = i(\Sigma) \times \Gamma_A$ and $\Gamma_A \cong F$;
3. $\Gamma^\circ = i(\Phi)$;
4. $\Gamma/\Gamma^\circ$ is topologically isomorphic to $G$. 
Conclusion - for automorphism groups

- There is an $\omega$-categorical structure $\mathcal{M}_1$ with domain $B$ and automorphism group $\Gamma$.
- There is an $\omega$-categorical structure $\mathcal{M}_2$ with domain $B$ and automorphism group $\Delta = \Sigma \times F$ (topological product).

**Theorem**

$\text{Aut}(\mathcal{M}_1)$ and $\text{Aut}(\mathcal{M}_2)$ are isomorphic as groups, but not as topological groups.

**Proof:** The groups are both isomorphic to $\Sigma \times F$.
Suppose $\beta : \Gamma \to \Delta$ is an isomorphism of topological groups. Then $\beta(\Gamma^\circ) = \Delta^\circ$ and so we have a topological isomorphism $\Gamma / \Gamma^\circ \to \Delta / \Delta^\circ$.
But $\Gamma / \Gamma^\circ \cong G$ and $\Delta / \Delta^\circ \cong F \times H$ (topologically). Contradiction. $\square$
Endomorphism monoids

- Canonical language for $\mathcal{A}$: atomic relation for each $\text{Aut}(\mathcal{A})$-invariant subset of $A^n$ (all $n$).
- Let $\Lambda = \text{End}(\mathcal{A}) = \text{EEmb}(\mathcal{A}) = \tilde{\Sigma} \subseteq A^A$.
- $\nu : \Sigma \to H$ extends to a continuous monoid homomorphism $\mu : \Lambda \to H$.
- $\Lambda$ acts on $C = H/H_\omega = G/G_\omega$ by $f(hH_\omega) = \mu(f)hH_\omega$.
- Obtain embedding $j : \Lambda \to B^B$ (where $B = A \cup C$) extending $i$.
- Let $\Omega$ be the closure of $j(\Lambda)$ in $B^B$.

**Lemma**

1. $\Omega = j(\Lambda) \times \Omega_A$ and $\Omega_A = \Gamma_A$.
2. The group of units in $\Omega$ is $\Gamma$.
Conclusion - for endomorphism monoids

- Assume $\mathcal{M}_1, \mathcal{M}_2$ have their canonical languages.
- $\Gamma = \text{Aut}(\mathcal{M}_1)$ and $\Omega = \text{End}(\mathcal{M}_1)$.
- $\text{End}(\mathcal{M}_2)$ is isomorphic to the topological product $\Lambda \times F$.
- Both $\mathcal{M}_1, \mathcal{M}_2$ are countable, $\omega$-categorical.

**Theorem**

$\text{End}(\mathcal{M}_1)$ and $\text{End}(\mathcal{M}_2)$ are isomorphic as monoids, but not as topological monoids.

**Proof:** The monoids are isomorphic to $\Lambda \times F$. A topological isomorphism between them would induce a topological isomorphism between their groups of units, $\Gamma$ and $\Delta$, which is impossible. □