SIMILARITY REDUCTIONS OF PEAKON EQUATIONS: THE $b$-FAMILY

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The $b$-family is a one-parameter family of Hamiltonian partial differential equations of nonevolutionary type, which arises in shallow water wave theory. It admits a variety of solutions, including the celebrated peakons, which are weak solutions in the form of peaked solitons with a discontinuous first derivative at the peaks, as well as other interesting solutions that have been obtained in exact form and/or numerically. In each of the special cases $b = 2$ and $b = 3$ (the Camassa–Holm and Degasperis–Procesi equations, respectively), the equation is completely integrable, in the sense that it admits a Lax pair and an infinite hierarchy of commuting local symmetries, but for other values of the parameter $b$ it is nonintegrable. After a discussion of traveling waves via the use of a reciprocal transformation, which reduces to a hodograph transformation at the level of the ordinary differential equation satisfied by these solutions, we apply the same technique to the scaling similarity solutions of the $b$-family and show that when $b = 2$ or $b = 3$, this similarity reduction is related by a hodograph transformation to particular cases of the Painlevé III equation, while for all other choices of $b$ the resulting ordinary differential equation is not of Painlevé type.

Keywords: peakon, Painlevé equation, reciprocal transformation, hodograph transformation

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1. Introduction

The one-parameter family of partial differential equations (PDEs) given by

$$u_t - u_{xxt} + (b + 1)uu_x = bu_x u_{xx} + uu_{xxx},$$

(1.1)

where $b \in \mathbb{R}$ is a parameter, is known as the $b$-family. It was originally introduced in work by one of us with Degasperis and Holm [1], [2], in order to analyze the integrable case $b = 3$ which had been found a little earlier by Degasperis and Procesi [3], and facilitate comparison with the celebrated Camassa–Holm case $b = 2$, which was derived in the physical context of shallow water theory in [4], although its integrability could already be understood within the theoretical framework of hereditary symmetries and recursion operators described in [5]. It was subsequently shown in [6], [7] that all of the equations (1.1) apart from $b = -1$ are asymptotically equivalent by means of a suitable Kodama transformation, while

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in [8] (see also [9]) the equation for parameter values $b \geq 10/11$ or $b \leq -10$ was derived from a model of shallow water flowing over a flat bed, with $u$ being the horizontal component of the fluid velocity at the level line $\theta = \sqrt{\frac{11b-10}{12b}}$, $0 \leq \theta \leq 1$.

In the application to shallow water models, Eq. (1.1) appears with the inclusion of additional linear dispersion terms, namely, multiples of $u_x$ and $u_{xxx}$. However, such terms can always be removed by a combination of a Galilean transformation, going to a moving frame with independent variables $x' = x - vt$, $t' = t$, together with a shift to a new dependent variable $u' = u + h$, where the velocity $v$ and background $h$ are constants. Thus, for the purposes of what follows, we work with the dispersionless form of the $b$-family equation, bearing in mind that the addition of linear dispersion to (1.1) changes the boundary conditions of the solution, so that some of the solutions we consider with a nonzero (constant) background correspond to solutions that vanish at infinity when dispersion is introduced into (1.1). In the dispersionless case, it is convenient to rewrite (1.1) in terms of a momentum density $m$, in the more compact form

$$m_t + um_x + bu_x m = 0, \quad m = u - u_{xx}. \quad (1.2)$$

In the latter form, the $b$-family can be viewed as a nonlocal evolution equation for $m$, where the nonlocality arises from the fact that

$$u = g * m = \int_R g(x - y)m(y) \, dy, \quad (1.3)$$

where $g$ is the Green’s function for the Helmholtz operator $1 - D_x^2$ on the real line, that is,

$$g(x) = \frac{1}{2} e^{-|x|}, \quad (1.4)$$

so that $(1 - D_x^2)g(x) = \delta(x)$.

The dispersionless version of Eq. (1.1), or equivalently (1.2), is distinguished by the remarkable fact, first observed by Camassa and Holm in the case $b = 2$, that with vanishing boundary conditions at infinity it admits weak soliton solutions called peakons, which for any positive integer $N$ are given by a linear superposition of $N$ peaked solitons, that is,

$$u(x,t) = \sum_{j=1}^N p_j(t) \exp(-|x - q_j(t)|), \quad m(x,t) = 2 \sum_{j=1}^N p_j(t) \delta(x - q_j(t)), \quad (1.5)$$

subject to the requirement that the positions $q_j(t)$ and amplitudes $p_j(t)$ satisfy the system of ordinary differential equations (ODEs)

$$\frac{dq_j}{dt} = \frac{\partial \tilde{H}}{\partial p_j}, \quad \frac{dp_j}{dt} = -(b-1) \frac{\partial \tilde{H}}{\partial q_j}, \quad j = 1, \ldots, N, \quad (1.6)$$

with

$$\tilde{H} = \frac{1}{2} \sum_{j,k} p_j p_k e^{-|q_j - q_k|}.$$ 

When $b = 2$, the latter ODEs have the form of a Hamiltonian system with $q_j$ and $p_j$ being canonical positions and momenta, and $\tilde{H}$ being the Hamiltonian function, and in this particular case Hamilton’s equations are also completely integrable in the Liouville–Arnold sense. For other values of $b$, it turns out that ODEs (1.6) can still be considered as a finite-dimensional Hamiltonian system, but with respect to
a noncanonical Poisson bracket [10], and the two-body problem (N = 2) is integrable for any b. In fact, ODEs (1.6) always have two first integrals [11], given by

\[ H = \sum_{j=1}^{N} p_j, \quad P = \prod_{j=1}^{N} \prod_{k=1}^{N-1} (1 - e^{-|q_k - q_{k+1}|})^{b-1}, \]

but it seems almost certain that the equations of motion for the peakons with N > 2 can only be explicitly integrated in the special cases b = 2 and b = 3, since the exact solutions respectively obtained in [12], [13] and [14], [15] rely heavily on the use of an appropriate spectral problem derived from the underlying Lax pair for the corresponding integrable PDE in each of those cases.

For any b \neq 0, 1, Eq. (1.2) can be derived from the least action principle \( \delta S = 0 \) with \( S = \int\int L \, dx \, dt \), where the Lagrangian density is

\[ L = \frac{\varphi_t}{2\varphi_x} \left( (\log \varphi_x)_{xx} + 1 \right) - \frac{\varphi_x^b}{b-1}, \]  

which arises by rewriting Eq. (1.2) as the conservation law

\[ p_t + (up)_x = 0, \quad p = m^{1/b}, \]  

and then introducing \( \varphi \) as a potential such that

\[ p = \varphi_x, \quad u = -\frac{\varphi_t}{\varphi_x}. \]  

An appropriate Legendre transformation leads to the Hamiltonian form of Eq. (1.2), namely,

\[ m_t = \frac{1}{b-1} B \frac{\delta H}{\delta m}, \]  

with

\[ H = \int m \, dx, \quad B = (bmD_x + m_x)(D_x - D^2_x)^{-1}(-bD_xm + m_x), \]  

valid for any b \neq 1; for two different proofs of the Jacobi identity for the skew-symmetric operator B; see [10], [16]. (The same Hamiltonian operator B works for b = 1 in (1.10) with the replacement \((b - 1)^{-1}H \rightarrow \int m \log m \, dx\).) This operator has two independent Casimir functionals, namely,

\[ C_1 = \int m^{1/b} \, dx, \quad C_2 = \int m^{-1/b} \left( \frac{m_x^2}{b^2m^2} + 1 \right) \, dx, \]  

where the density of the first one corresponds to conservation law (1.8). The matter of determining appropriate classes of solutions for which these functionals are well-defined, or require appropriate regularization, and how this depends on the value of b, is a delicate one. (See, e.g., [17], where a Banach subspace of a weighted Sobolev space was considered in order to prove an orbital stability property of stationary solutions when \( b < -1 \).)

The b-family has various interesting geometric properties, in addition to its Lagrangian and Hamiltonian structures. There is the conservation equation

\[ m(q, t)q_x^b = m(x, 0), \]  

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where \( x \mapsto q(x,t) \) is a diffeomorphism of \( \mathbb{R} \) defined by the initial value problem

\[
q_t = u(q,t), \quad q(x,0) = x.
\]

Equation (1.13) holds for all \( t \) in the domain of existence of the solution of (1.2), and for solutions with \( m(x,0) > 0 \) it implies that \( m \) remains positive as long as the solution exists. Moreover, if the periodic solutions of the equation are considered, taking \( S^1 \) instead of the real line \( \mathbb{R} \), then the \( b \)-family equation can be regarded as the geodesic equation for a suitable connection on the diffeomorphism group of the circle [18] (for generic \( b \), this is a nonmetric connection, but the case \( b = 2 \) gives an Euler–Poincaré equation for geodesics with respect to the \( H^1 \) metric [19]).

The case of solutions with positive \( m > 0 \) (or with fixed sign everywhere) is especially relevant in what follows, as it allows the definition of the reciprocal transformation

\[
dX = p\,dx - up\,dt, \quad dT = dt
\]

(1.14)

associated with conservation law (1.8), which transforms (1.2) to a third-order PDE for \( p = p(X,T) \) as a function of the independent variables \( X \) and \( T \):

\[
\frac{\partial}{\partial T} \frac{1}{p} + \frac{\partial}{\partial X} (p\log p)_X - p^b = 0
\]

(1.15)

(where, by an abuse of notation, we use the same letter \( p \) for the dependent variable, considered as a function of the new variables \( X \) and \( T \)). Equation (1.15) can naturally be regarded as an extension of the sine-Gordon equation when \( b = 2 \), or of the Tzitzeica equation when \( b = 3 \) (up to replacing \( m = p^b \to -p^b \); details can be found in [16]), but for other values of \( b \) it fails the Painlevé test [20], which is consistent with the results of other integrability tests applied to the \( b \)-family [3], [21].

Any reciprocal transformation sends a conservation law in the original independent variables to another conservation law in terms of the new variables. Hence there is another way to rewrite Eq. (1.15) in conservation form, which corresponds to applying the reciprocal transformation (1.14) to the conservation law associated with the conserved density for the Casimir \( C_2 \) in (1.12): for any \( b \neq 1 \), we have

\[
\frac{\partial V}{\partial T} + \frac{b}{2(b-1)} \frac{\partial p^{b-1}}{\partial X} = 0,
\]

(1.16)

where the quantity \( V \) is defined in terms of \( p \) by

\[
pp_{XX} - \frac{1}{2}p_X^2 + 2Vp^2 + \frac{1}{2} = 0.
\]

(1.17)

Regarded as an ODE for \( p \) as a function of \( X \), with \( V \) given, the latter equation is known as Ermakov’s equation [22], or the Ermakov–Pinney equation (see [16] for further references). Then \( V \) (up to a sign) can be interpreted as the potential for a Schrödinger operator, and the general solution of (1.17) can be written in the form of a product

\[
p = \psi_+\psi_-, \quad \text{for} \quad (D_X^2 + V)\psi_\pm = 0, \quad \text{with} \quad W(\psi_+\psi_-)^2 = 1,
\]

(1.18)

where

\[
W(\psi_+\psi_-) = \psi_+X\psi_- - \psi_-X\psi_+
\]
is the Wronskian of the two solutions of the Schrödinger equation. The logarithmic derivatives of these two wave functions can be written in terms of $p$, as

$$
\frac{d}{dX} \log \psi_\pm = \frac{pX \pm 1}{2p}.
$$

We use the representation (1.18) of $p$ when we consider similarity reductions of (1.15) in what follows.

Some time ago, Holm and Staley did a series of extensive numerical studies of the solutions of the $b$-family, and observed remarkable bifurcation phenomena controlled by the parameter $b$ [23], [24]: given initial data vanishing at infinity, they found a train of peakons was produced for $b > 1$; but for $-1 < b < 1$ the same initial value problem appears to form something that resembles the ramp/cliff profile seen in Burgers’ equation, given by the similarity solution (ramp)

$$
u(x, t) = \frac{x}{(b + 1)t} \quad (1.19)
$$

in a compact region, joined to a rapidly decaying cliff; while for $b < -1$ the initial profile develops into a train of leftons solutions, consisting of solitary waves that move to the left before becoming stationary (see (2.5) below for the explicit form of a lefton solution). As such, the $b$-family (at least for $|b| > 1$, the range where peakons/ leftons appear) seems to provide support for the soliton resolution conjecture (see, e.g., [25]), which says that for any suitable dispersive evolutionary PDE (not necessarily integrable), generic initial data should decompose into a train of solitary waves together with radiation which decays to zero as $t \to \infty$. Apart from intensive studies of the integrable cases $b = 2, 3$, further analytical and numerical support for the behavior reported by Holm and Staley has taken a while to materialize: an orbital stability result for a single lefton when $b < -1$ was proved in [17], while there are various well-posedness and rigidity results (see [26], [27], and the references therein); yet linear stability/instability results for peakons, ramp/cliff solutions and leftons across the full range of $b$ values have been found only very recently [28], [29].

In this paper, we are concerned with describing scaling similarity solutions of the $b$-family (1.1) which generalize the ramp (1.19). Our main result is that (for $b \neq 0$) such solutions satisfy an autonomous third-order ODE, which is related via a hodograph transformation to a nonautonomous second-order ODE that closely resembles the third Painlevé equation

$$
\frac{d^2w}{d\zeta^2} = \frac{1}{w} \left( \frac{dw}{d\zeta} \right)^2 - \frac{1}{\zeta} \left( \frac{dw}{d\zeta} \right) + \frac{1}{\zeta} (\alpha w^2 + \beta) + \gamma w^3 + \delta.
$$

(1.20)

Unlike the latter, for generic values of $b$ the second-order equation we find does not have the Painlevé property, with the exception of the special values $b = 2, 3$, which turn out to correspond to particular instances of (1.20). (Some details for the Camassa–Holm case $b = 2$ were already derived in [30].)

Our method for deriving the hodograph transformation for scaling similarity solutions is based on reduction of the reciprocal transformation (1.14), so as a warm-up exercise, we show in Sec. 2 how the same method works in the slightly more straightforward context of the smooth traveling wave solutions of (1.1), which can be reduced to a quadrature. After applying the hodograph transformation in the cases $b = 2, 3$, the traveling waves are given explicitly in parametric form in terms of Weierstrass functions, for which we give full details (omitting a discussion of the degenerate case of a vanishing discriminant, $g_2^3 - 27g_3^2 = 0$, which produces the smooth 1-soliton solution of Camassa–Holm/Degasperis–Procesi, described elsewhere [30]–[32]). We also present the periodic traveling waves for $b = -1$, which are given parametrically in terms of trigonometric functions. Section 3 is devoted to the scaling similarity solutions of (1.1), and the corresponding parametric formulas obtained via a hodograph transformation. Once again, after describing the general case, we focus on the special parameter values $b = 2, 3$ and clarify the connection with particular cases of the third Painlevé equation, before ending with a brief section of conclusions.
2. Traveling waves and hodograph transformation

We start by considering traveling waves of (1.1), setting

\[ u(x,t) = U(z), \quad m(x,t) = M(z), \quad z = x - ct, \]

where \( c \) is the wave velocity, and we also write \( P(z) \) for the quantity \( M^{1/b} \). The conservation law (1.8) becomes a total \( z \)-derivative, so integrating this we obtain

\[ (U - c)P + d = 0, \tag{2.1} \]

where \( d \) is an integration constant. We assume henceforth that \( d \neq 0 \), as the case \( d = 0 \) implies that either \( U = \text{const} \) or \( P = 0 \), if we are considering smooth solutions; but the 1-peakon solution with \( U = ce^{-|z-z_0|}, M = 2c\delta(z - z_0) \) (\( z_0 \) arbitrary) can be viewed as a weak limit of strong (analytic) solutions with \( d = 0 \) (see the discussion, e.g., in [31] or [33]).

From the formula relating \( m \) and \( u \), as in (1.2), we then find

\[ U_{zz} - U + P^b = 0. \tag{2.2} \]

If we substitute \( U = c - d/P \) from (2.1) in the last formula, then we obtain a second-order equation for \( P \),

\[ P_{zz} - 2 \frac{P^2}{P} - cd^{-1}P^2 + P + d^{-1}P^{b+2} = 0. \tag{2.3} \]

This equation can be integrated to yield a first-order equation, that is,

\[ P_z^2 = F(P), \quad \text{where} \quad F(P) = P^2 - 2cd^{-1}P^3 + eP^4 - \frac{2P^{b+3}}{d(b - 1)}, \tag{2.4} \]

with \( e \) being another integration constant (and it is necessary to assume \( b \neq 1 \), otherwise a term with \( \log P \) appears). Thus the determination of traveling waves reduces to the quadrature \( \int dP/\sqrt{F(P)} = z + \text{const.} \)

We observe that the assumption \( d \neq 0 \) puts constraints on the boundary conditions of the traveling waves, depending on the sign of \( b \). When \( b > 0 \), the combination of relation (2.2) with \( U = c - d/P \) implies that there is no smooth solution \( U(z) \) that vanishes at infinity, and hence only periodic or unbounded waves are possible in that case. (This can also be seen by considering the phase portrait of (2.3) in the \((P, P_z)\) plane.) However, when \( b < 0 \), it is possible to have solutions with \( U \to 0 \) and \( P \to \infty \) as \( |z| \to \infty \): for instance, when \( b < -1 \) and \( c = 0 \), there are the stationary lepton solutions, given explicitly by

\[ U(z) = A(\cosh \gamma(z - z_0))^{-1/\gamma}, \]
\[ M(z) = \frac{A(1 - b)}{2} (\cosh \gamma(z - z_0))^{b/\gamma}, \quad \gamma = -\frac{b + 1}{2}, \tag{2.5} \]

where \( A \) is an arbitrary constant that can be written in terms of \( d \) and \( b \), corresponding to setting \( c = 0 \) and \( e = 0 \) in (2.4).

It is instructive to see how the same solutions arise via reduction of the Lagrangian density (1.7) for the PDE. If we replace \( p = \varphi_x \to \phi_z = P \) and then note that we require

\[ u = -\frac{\varphi_t}{\varphi_x} \to U = c - \frac{d}{P} = c - \frac{d}{\phi_z}, \]

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then we obtain the Lagrangian

\[ L = -\frac{1}{2} \left( c - \frac{d}{\phi_z} \right) ((\log \phi_z)_{zz} + 1) - \frac{\phi^b_z}{b-1}, \]

but we can subtract the terms involving \( c \) (a constant plus a total \( z \)-derivative) since these do not affect the Euler–Lagrange equation, to obtain \( L \to \bar{L} \), where

\[ \bar{L} = \frac{d}{2\phi_z} ((\log \phi_z)_{zz} + 1) - \frac{\phi^b_z}{b-1}. \]  

(2.6)

Because \( \phi \) does not appear, the Euler–Lagrange equation for (2.6) is a total derivative:

\[ \frac{d}{dz} \left( -\frac{d^2}{dz^2} \frac{\partial \bar{L}}{\partial \phi_{zzz}} + \frac{d}{dz} \frac{\partial \bar{L}}{\partial \phi_{zz}} - \frac{\partial \bar{L}}{\partial \phi_z} \right) = 0. \]

The above equation can be integrated, with an integration constant \( C \), or we can replace \( \phi_z \) and its derivatives in terms of \( P \) and add a term with \( C \) as a Lagrange multiplier: \( \bar{L}(\phi_z, \phi_{zz}, \phi_{zzz}) \to \bar{L}(P, P_z, P_{zz}) + C(P - \phi_z) \); then the leading terms involving derivatives are

\[ \frac{d}{dz} \left( P_{zz}P^{-2} - P_z^2P^{-3} \right) = \frac{d}{dz} \left( (P_zP^{-2})_z + P_z^2P^{-3} \right), \]

and hence removing the total derivative leads to an equivalent Lagrangian in terms of \( P \):

\[ \hat{L} = \frac{d}{2} \left( \frac{P_z^2}{P^3} + \frac{1}{P} \right) - \frac{P^b}{b-1} + CP. \]  

(2.7)

The Euler–Lagrange equation for \( P \) obtained from \( \hat{L} \) is different from (2.3), but applying a Legendre transformation to (2.7) yields the conjugate momentum to \( P \) and a conserved Hamiltonian, namely,

\[ \pi = \frac{\partial \hat{L}}{\partial P_z} = \frac{dP_z}{P^3} \implies h = \pi P_z - \hat{L} \]

whence on a fixed level set \( h = \text{const} \) we have

\[ h = \frac{d}{2} \left( \frac{P_z^2}{P^3} - \frac{1}{P} \right) + \frac{P^b}{b-1} - CP, \]

which agrees with (2.4) when we identify \( c = -h, \ e = 2C/d, \)

By considering how the traveling-wave solutions of (1.2) behave under the action of the reciprocal transformation (1.14), it is not hard to see that the roles of the parameters \( c \) and \( d \) are reversed: for traveling waves of (1.2) with velocity \( c \), the parameter \( d \) appears as an integration constant from (1.8), while traveling waves of (1.15) with velocity \( d \) satisfy the same relation \( U = c - d/P \), but now \( c \) appears as an integration constant when rewriting (1.15) as \( (p^{-1})_T - u_X = 0 \) and setting

\[ p(X, T) = P(Z), \quad u(X, T) = U(Z), \quad Z = X - dT, \]

and then integrating the reduced equation

\[ -\frac{d}{dZ}(U + dP^{-1}) = 0. \]
Hence the reciprocal transformation reduces to a hodograph transformation, that is,
\[
dZ = d(X - dT) = p\,dx - (up + d)\,dt = P(z)\,d(x - ct) = P(z)\,dz, \tag{2.8}
\]
where we note that once again we abuse notation by using the same letter \( P \) for the dependent variable viewed as a function of either argument \( (z \) or \( Z \)), i.e., \( P(Z) \equiv P(z(Z)) \). Hence, under this hodograph transformation, the derivatives transform as \( \frac{d}{dz} = P\frac{d}{dZ} \), and it follows from (2.4) that the first-order ODE for \( P(Z) \) is just
\[
\left( \frac{dP}{dZ} \right)^2 = F^*(P), \quad \text{where} \quad F^*(P) = 1 - 2cd^{-1}P + eP^2 - \frac{2P^{b+1}}{d(b-1)}. \tag{2.9}
\]

One can also obtain a second-order ODE for \( P(Z) \) by starting from the action \( S = \int L\,dz \), after replacing \( L \rightarrow L + \text{const} \) and \( dz \rightarrow P^{-1}dZ \) from (2.8), to obtain a new action \( S^* = \int L^*\,dZ \), and then (2.8) is satisfied on each level set of an appropriate Hamiltonian, obtained via a Legendre transformation applied to \( L^* \). Also, the \( X \)-derivatives in Ermakov’s equation (1.17) all become \( Z \)-derivatives under this reduction, and we can write the potential \( V = V(Z) \) as
\[
V = -\frac{1}{2P} \left( \frac{d^2P}{dZ^2} \right) + \frac{1}{4P^2} \left( \left( \frac{dP}{dZ} \right)^2 - 1 \right), \tag{2.10}
\]
but then for the traveling-wave solutions, the conservation law (1.16) becomes a total \( Z \)-derivative, and this can be integrated to yield
\[
V = \frac{b}{2d(b-1)}P^{b-1} - \frac{e}{4}, \tag{2.11}
\]
where the value of the integration constant is found by comparing (2.10) with (2.9). Hence we arrive at the main result in this section.

**Theorem 2.1.** The traveling-wave solutions of the \( b \)-family equation (1.2) for \( b \neq 1 \), with constant \( d \neq 0 \), are given in parametric form by \( P = P(Z), \, z = z(Z) \), where
\[
\int^{P(Z)} \frac{ds}{\sqrt{F^*(s)}} = Z + \text{const}, \quad z(Z) = \ln \frac{\psi_+(Z)}{\psi_-(Z)} + \text{const}, \tag{2.12}
\]
with \( F^* \) defined by (2.9), where \( \psi_\pm \) are two independent solutions of the same Schrödinger equation
\[
\left( \frac{d^2}{dZ^2} + V(Z) \right)\psi_\pm = 0 \tag{2.13}
\]
having the Wronskian \( W(\psi_+, \psi_-) = 1 \), subject to the requirement that \( P(Z) = \psi_+(Z)\psi_-(Z) \), and \( V(Z) \) is given in terms of \( P = P(Z) \) by (2.11).

**Proof.** The quadrature for \( P(Z) \) in (2.12) follows immediately from (2.9). To obtain the formula for \( z(Z) \), we first of all note that from the theory of Ermakov’s equation (1.17), if \( V(Z) \) is fixed by (2.11), then relation (2.10) implies that there is a pair of independent solutions \( \psi_\pm \) of (2.13) with the Wronskian 1 such that \( P = \psi_+\psi_- \). Then we have
\[
dz = \frac{1}{P(Z)}\,dZ = \frac{W(\psi_+, \psi_-)}{\psi_+\psi_-}\,dZ = d\log \frac{\psi_+(Z)}{\psi_-(Z)},
\]
and the result follows.
Example 2.1: The Camassa–Holm equation. In the case $b = 2$, for the analytic traveling-wave solutions of the Camassa–Holm equation, the hodograph-transformed ODE is

$$\left(\frac{dP}{dZ}\right)^2 = 1 - 2cd^{-1}P + eP^2 - 2d^{-1}P^3$$

which is solved in terms of elliptic functions. Up to the freedom to replace $Z \to Z + \text{const}$ (which is useful to exploit, shifting by a suitable half-period in order to obtain nonsingular solutions that are periodic and bounded for real $Z$), the solution can be written in terms of the Weierstrass function $\wp(Z) = \wp(Z; g_2, g_3)$ with arbitrary invariants $g_2$ and $g_3$ and another arbitrary parameter $W$ as

$$P(Z) = \frac{\wp(Z) - \wp(W)}{\wp'(W)},$$

(2.14)

with the coefficients in the ODE for $P(Z)$ given by

$$c = \frac{\wp''(W)}{2\wp'(W)^2}, \quad d = -\frac{1}{2\wp'(W)}, \quad e = 12\wp(W),$$

and Eq. (2.11) gives

$$V = d^{-1}P - \frac{e}{4} = -2\wp(Z) - \wp(W),$$

so that Schrödinger equation (2.13) corresponds to the simplest case of Lamé’s equation, and the two independent solutions with Wronskian 1 are given in terms of the Weierstrass sigma function by

$$\psi_{\pm}(Z) = \frac{1}{\sqrt{\wp'(W)}} \frac{\sigma(W \mp Z)}{\sigma(W)\sigma(Z)} e^{\pm \zeta(W)Z},$$

and these satisfy $P = \psi_+ \psi_-$. Thus, up to shifting by an arbitrary constant, the traveling-wave variable $z$ for the original equation has the form

$$z(Z) = \log \frac{\sigma(W - Z)}{\sigma(W + Z)} + 2\zeta(W)Z.$$

This explicit parametric form for the periodic traveling waves of Camassa–Holm was given in [30]. (For higher-genus analogues corresponding to finite-gap solutions of the Camassa–Holm equation, see, e.g., [34].)

Example 2.2: The Degasperis–Procesi equation. In the case $b = 3$, for the analytic traveling-wave solutions of the Degasperis–Procesi equation, the hodograph-transformed ODE is

$$\left(\frac{dP}{dZ}\right)^2 = 1 - 2cd^{-1}P + eP^2 - 2d^{-1}P^4$$

(2.15)

which defines a curve of genus one in the $(P, P^2)$ plane, and is solved in terms of elliptic functions. By making a birational transformation from the quartic curve defined by (2.15) to a Weierstrass cubic, we find that the solution is given explicitly by

$$P(Z) = \frac{1}{\alpha \wp'(W_1)} \left( \frac{\wp(Z) - \wp(W_1)}{\wp(Z) - \wp(W_2)} \right)^\alpha,$$

(2.16)
being specified by the three parameters $g_2, g_3$, and $W_2$, where $\alpha$ and the quantity $W_1$ that fixes the zeros of $P$ are determined by

$$\alpha = -\frac{1}{2} \frac{\psi''(W_2)}{\psi'(W_2)^2} = (\varphi(W_1) - \varphi(W_2))^{-1}, \quad (2.17)$$

while the coefficients in (2.15) are fixed by

$$e = 12 \varphi(W_2) - \frac{3}{2} \left( \frac{\psi''(W_2)}{\psi'(W_2)} \right)^2, \quad d = -\frac{16 \psi'(W_2)^6}{\psi'(W_1)^2 \psi''(W_2)^2},$$

$$\frac{e}{d} = \frac{8}{\psi'(W_1)} \left( \frac{\psi'(W_2)^4}{\psi''(W_2)^2} - 3 \frac{\varphi(W_2) \psi'(W_2)^2}{\psi''(W_2)} + \frac{1}{4} \psi''(W_2) \right). \quad (2.18)$$

To obtain the original traveling-wave variable $z$ for the Degasperis–Procesi equation parametrically in terms of $Z$, we note that we can write

$$P^{-1} = \alpha \varphi'(W_1) + \frac{\varphi'(W_1)}{\varphi(Z) - \varphi(W_1)},$$

and then by standard elliptic function identities this can be integrated with respect to $Z$ to yield

$$z(Z) = (\alpha \varphi'(W_1) + 2 \zeta(W_1))Z + \log \left( \frac{\sigma(W_1 - Z)}{\sigma(W_1 + Z)} \right) \quad (2.19)$$

(up to a constant). Having obtained the explicit form of $z(Z)$, we can then apply Theorem 2.1 in reverse, writing $z(Z) = \log(\psi_+ / \psi_-)$, $P = \psi_+ \psi_-$ to find that

$$\psi_{\pm}(Z) = \frac{1}{\sqrt{\alpha \varphi'(W_1) \sigma(W_1) \sigma(Z)}} \left( \varphi(Z) - \varphi(W_2) \right)^{-1/2} \exp \left( \pm \left( \zeta(W_1) + \frac{1}{2} \alpha \varphi'(W_1) \right) Z \right). \quad (2.20)$$

have Wronskian 1 and satisfy the same linear equation

$$\left( \frac{d^2}{dZ^2} + V(Z) \right) \psi_{\pm} = 0, \quad V = \frac{3}{4d}P^2 - \frac{e}{4},$$

from (2.11), with the potential given explicitly by

$$V(Z) = \frac{3}{4d} \frac{1}{\alpha \varphi'(W_1)^2} \left( \frac{\varphi(Z) - \varphi(W_1)}{\varphi(Z) - \varphi(W_2)} \right)^2 - \frac{e}{4}. \quad (2.21)$$

The linear equation for $\psi_{\pm}$ as given by (2.20) can be verified directly by rewriting it as

$$V = -\frac{d^2}{dZ^2} \log \psi_{\pm} - \left( \frac{d}{dZ} \log \psi_{\pm} \right)^2,$$

and then noting that both the left- and right-hand sides above are elliptic function of $Z$ with double poles at points congruent to $\pm W_2$ modulo the period lattice of the Weierstrass curve, and nowhere else, with the same leading-order Laurent expansions

$$\frac{3}{4}(Z \mp W_2)^{-2} + O(1) \quad \text{as} \quad Z \to \pm W_2.$$

Comparing the value of the function on each side of (2.21) at $Z = 0$, we find the identity

$$\frac{3}{4d} \frac{1}{\alpha \varphi'(W_1)^2} - \frac{e}{4} = \frac{1}{\alpha} - \frac{1}{4} \alpha^2 \varphi'(W_1)^2,$$

which is a consequence of (2.18) and the given expression (2.17) for $\alpha$ in terms of elliptic functions with argument $W_2$. 

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Example 2.3: Genus-zero solutions for \( b = -1 \). In the case \( b = -1 \), Eq. (2.9) becomes
\[
\left( \frac{dP}{dZ} \right)^2 = 1 + d^{-1} - 2cd^{-1}P + eP^2, \tag{2.22}
\]
which defines a genus-zero curve (a conic) in the \((P, P_Z)\) phase plane. If we rule out the case of parabolas \((e = 0)\), then there are two types of solution for \( P(Z) \): unbounded solutions given in terms of hyperbolic functions, when the curve is a hyperbola, and bounded periodic solutions when the curve is an ellipse. We focus on the latter, and consider solutions of the form
\[
P(Z) = A + B \sin(\gamma Z), \tag{2.23}
\]
with parameters \( A \geq |B| > 0, \gamma > 0 \), whence \( P \geq 0 \). This corresponds to taking the parameters
\[
e = -\gamma^2, \quad d = -\frac{1}{1 + \gamma^2(A^2 - B^2)}, \quad c = Ade
\]
in (2.22). Upon integrating \( P^{-1} \) with respect to \( Z \), the traveling-wave variable for (1.2) is found to be
\[
z(Z) = \frac{2}{\gamma \sqrt{A^2 - B^2}} \arctan \left( \frac{A \tan(\gamma Z/2) + B}{\sqrt{A^2 - B^2}} \right),
\]
up to a constant. By Theorem 2.1, the latter can be rewritten as a logarithm of the ratio of two independent solutions of (2.13) with the potential
\[
V(Z) = -\frac{1 + \gamma^2(A^2 - B^2)}{4(A + B \sin(\gamma Z))^2} + \frac{\gamma^2}{4},
\]
but we omit further details.

3. Scaling similarity reductions and Painlevé equations

Each member of the \( b \)-family of equations (1.1), apart from the case \( b = 0 \), admits a scaling similarity reduction, which is obtained by taking
\[
u(x, t) = t^{-1}U(z), \quad z = x + ab^{-1} \log t, \tag{3.1}
\]
where \( a \) is an arbitrary parameter. For the variable (momentum density) \( m \) in (1.2), this means that we can write
\[
m(x, t) = t^{-1}M(z), \quad M = U - U_{zz} \quad \implies \quad p = m^{1/b} = t^{-1/b}P(z). \tag{3.2}
\]
Under this reduction, the equation in its original form (1.1) reduces to an autonomous third-order ODE for \( U(z) \):
\[
(U + ab^{-1})(U_{zzz} - U_z) + (bU_z - 1)(U_{zz} - U) = 0. \tag{3.3}
\]
The ramp profile (1.19) for \( b \neq -1 \) corresponds to the solution
\[
U(z) = \frac{z - z_0}{b + 1} \tag{3.4}
\]
when \( a = 0 \) (with \( z_0 \) being an arbitrary choice of origin for the ramp). As it stands, in general there appears to be no way to integrate Eq. (3.3) further. However, by exploiting the reciprocal transformation (1.14), it is
possible to obtain the solutions of this equation in parametric form from the solutions of a nonautonomous second-order ODE that is related to (3.3) via a hodograph transformation.

The key observation is that, for \( b \neq 0 \), PDE (1.15) admits the similarity reduction

\[
p(X,T) = T^{-1/b}P(Z), \quad Z = XT^{1/b},
\]

(3.5)

and if the PDE is written in conservation form as \( (p^{-1})_X = u_X \), then under this reduction we find

\[
u(X,T) = p^b - p(ln p)_{XT} = T^{-1}U(Z), \quad \text{where} \quad U = P^b - b^{-1}P(Z(\ln P)_{ZZ} + (\ln P)_Z).
\]

Thus, after removing a factor of \( T^{-1} \), the reduced equation becomes a total \( Z \)-derivative, that is,

\[
\frac{d}{dZ}(b^{-1}ZP^{-1} - U) = 0,
\]

which can be integrated to yield

\[
U(Z) = \frac{1}{b}\left(\frac{Z}{P(Z)} - a\right),
\]

(3.6)

with \( a \) being an arbitrary integration constant. Upon replacing \( U \) in terms of \( P \) and its \( Z \)-derivatives, this gives the second-order equation

\[
P \frac{d}{dZ}(Z(\log P)_Z) - bP^b + \frac{Z}{P} - a = 0.
\]

(3.7)

At the level of the reciprocal transformation (1.14), this gives a relation between the similarity reductions of (1.1) and (1.15): we can identify the parameter \( a \), which plays different roles in these two reductions, to find that the similarity variable \( z \) in (3.1) satisfies

\[
dz = d(x + ab^{-1}\ln t) = p^{-1}dX + u\,dT + ab^{-1}T^{-1}dT =
\]

\[
= P^{-1}(Z)T^{1/b}dX + T^{-1}(U(Z) + ab^{-1})\,dT =
\]

\[
= P^{-1}(Z)(T^{1/b}dX + T^{-1}b^{-1}Z\,dT),
\]

using (3.5) and (3.6), hence

\[
dz = \frac{1}{P(Z)}dZ.
\]

(3.8)

This defines a hodograph transformation between the solutions of (3.3) and (3.7), where the latter can be rewritten as

\[
\frac{d^2P}{dZ^2} = \frac{1}{P} \left( \frac{dP}{dZ} \right)^2 - \frac{1}{Z} \left( \frac{dP}{dZ} \right) + \frac{1}{Z}(bP^b + a) - \frac{1}{P}.
\]

(3.9)

Hence we arrive at an analogue of Theorem 2.1 for these scaling similarity reductions.

**Theorem 3.1.** The scaling similarity solutions of the \( b \)-family equation (1.2) for \( b \neq 0 \), which satisfy Eq. (3.3), are given in parametric form by \( U = U(Z) \), \( z = z(Z) \), where \( U \) is given by (3.6) in terms of the solution \( P(Z) \) of nonautonomous second-order ODE (3.9), and \( z \) is determined from

\[
z(Z) = \log \frac{\psi_+(Z)}{\psi_-(Z)} + \text{const},
\]

(3.10)
where $\psi_{\pm}$ are two independent solutions of the same Schrödinger equation

$$\left(\frac{d^2}{dZ^2} + \nabla(Z)\right)\psi_{\pm} = 0 \tag{3.11}$$

with the Wronskian $W(\psi_{+}, \psi_{-}) = 1$, subject to the requirement that $P(Z) = \psi_{+}(Z)\psi_{-}(Z)$, with the potential $\nabla(Z)$ given in terms of $P = P(Z)$ by

$$\nabla = -\frac{1}{4P^2}\left(\left(\frac{dP}{dZ}\right)^2 - 1\right) + \frac{1}{2ZP}\left(\frac{dP}{dZ} - bP^b - a\right); \tag{3.12}$$

**Proof.** All the preceding statements follow from direct application of hodograph transformation (3.8), as described, apart from the form of Schrödinger equation (3.11). The form of the potential is obtained by replacing all the $X$-derivatives in (1.17) by $Z$-derivatives, to find that under the reduction we have $V(X,T) = T^{2/b}\nabla(Z)$, where $\nabla(Z)$ is given in terms of $P(Z)$ by the right-hand side of (2.10), and then Eq. (3.9) can be used to eliminate the second-derivative term, to yield formula (3.12).

The form of ODE (3.9) is very similar to the third Painlevé equation (1.20). However, by directly applying the Kowalewski–Painlevé analysis to the equation in the form (3.7), which is very similar to the corresponding analysis of PDE (1.15) carried out in [20], we see that $b = 2, 3$ are the only cases that have the Painlevé property. Indeed, movable singular points in (3.9) are obtained from the leading-order behavior of the form

$$P\kappa(Z - Z_0)^\mu,$$

and for generic $b$ the only possible balances have the leading exponent

$$\mu = \frac{2}{1 - b} \quad \text{or} \quad \mu = 1.$$

If $b \notin \mathbb{Z}$, then the $P^b$ term in the equation creates noninteger exponents in local series expansions with (at least one of) these leading order behaviors, and then the requirement that the leading exponent $2/(1 - b) \in \mathbb{Z}$ implies that $b = 2$ or $b = 3$ are the only possibilities.

As we see below, both these two special cases correspond to particular instances of (1.20). The nonautonomous Hamiltonian formulation of Painlevé equations was initially developed by Okamoto [35], but it was noted in [30] that a different type of Hamiltonian is required to cover the $b = 2$ case of (3.9). Interestingly, the same sort of Hamiltonian formulation extends to all values of $b \neq 1$: if we take the Hamiltonian to be

$$h = rZ^{-1}P^2\pi^2 + (s + (1 - as)Z^{-1}P)\pi - \frac{b}{2r(b - 1)}P^{b - 1}, \tag{3.13}$$

with $r \neq 0$ arbitrary, $s = \pm 1$ (and $\pi$ being the conjugate momentum to $P$), then Hamilton’s equations

$$\frac{dP}{dZ} = \frac{\partial h}{\partial \pi}, \quad \frac{d\pi}{dZ} = -\frac{\partial h}{\partial P}$$

are equivalent to (3.9). (For $b = 1$, the final term in (3.13) should be replaced with $\log P$.) Fixing the scale such that $r = 1$ when $b = 2$ corresponds to the choice made in [30], which gives $h(Z) = \frac{1}{dZ}\log \tau(Z)$, where the tau function $\tau$ has simple zeros at movable poles/zeros of $P(Z)$.

If we fix $z_0 = 0$, then for the ramp solution (3.4) we have $P(z) = (b + 1)^{-1/b}z^{1/b}$, and applying the hodograph transformation (3.8) in reverse we find that

$$P(Z) = b^{-1/(b+1)}Z^{1/(b+1)} \tag{3.14}$$

is a solution of (3.9) when $a = 0$, for any $b \neq -1$. 



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Example 3.1: The Camassa–Holm equation. When \( b = 2 \), Eq. (3.9) becomes

\[
\frac{d^2P}{dZ^2} = \frac{1}{P} \left( \frac{dP}{dZ} \right)^2 - \frac{1}{Z} \frac{dP}{dZ} + \frac{1}{Z} \left( 2P^2 + a \right) - \frac{1}{P} \tag{3.15}
\]

which is precisely the \( \gamma = 0 \) case of the third Painlevé equation; this reduction was first obtained in [30]. We can identify the variables and parameters in (1.20) and (3.15) as follows:

\[
w = P, \quad \zeta = Z, \quad \alpha = 2, \quad \beta = a, \quad \gamma = 0, \quad \delta = -1. \tag{3.16}
\]

It is known that Eq. (3.15) admits a Bäcklund transformation: for any solution \( P = P(Z) \), the quantities

\[
P_{\pm} = \frac{Z(\pm PZ + 1)}{2P^2} + \frac{(\mp 1 - a)}{2P} \tag{3.17}
\]

satisfy the same ODE but with the parameter replacement \( a \to a \pm 2 \). As pointed out in [30], this Bäcklund transformation can be derived from the Darboux–Crum transformation for Schrödinger equation (3.11). Moreover, if we take the Hamiltonian (3.13) for \( b = 2 \) with \( r = 1 \) and the two possible choices of sign for \( s \), then we find

\[
h_{\pm} = Z^{-1} P^2 P_{\pm}^2 + (a \pm 1) Z^{-1} P P_{\pm} - P_{\pm} - P.
\]

In other words, \( \pi = \pm P_{\pm} \) is the conjugate momentum for each of the respective choices of sign.

Adding the two equations in (3.17) implies that

\[
P_+ + P_- = \frac{Z}{P^2} - \frac{a}{P},
\]

but then using (3.15) this gives

\[
P_+ + P_- = 2P - \frac{d}{dZ} (Z (\log P)Z). \tag{3.18}
\]

Now if we introduce a tau function \( \sigma = \sigma(Z) \) such that

\[
P = -\frac{d}{dZ} (Z (\log \sigma)Z),
\]

and similarly introduce tau functions \( \sigma_{\pm} \) such that analogous relations hold for \( P_{\pm} \), then Eq. (3.18) implies that

\[
\frac{d}{dZ} \left( Z \left( \log \frac{\sigma_+ \sigma_-}{\sigma^2 P} \right)_Z \right) = 0,
\]

which can be integrated twice to yield

\[
\frac{\sigma_+ \sigma_-}{\sigma^2 P} = CZ^D, \tag{3.19}
\]

for some constants \( C \) and \( D \).

If we express \( P \) in terms of the tau function \( \sigma \), then (3.19) becomes a bilinear equation of Toda type,

\[
\sigma_+ \sigma_- + CZ^D \left( \frac{Z}{2} D_Z^2 \sigma \cdot \sigma + \sigma \sigma_Z \right) = 0,
\]

where \( D_Z \) denotes the Hirota derivative. Thus \( \sigma_- \), \( \sigma_+ \), and \( \sigma \) should be viewed as adjacent tau functions at points \( a - 2 \), \( a \), \( a + 2 \) on a lattice where each point is distance 2 away from the next.
As a particular example of a sequence of solutions generated in this way, we note that we can take $P = P_0(Z) = (Z/2)^{1/3}$ as a seed solution when $a = 0$, corresponding to the ramp solution of the Camassa–Holm equation, and then applying Bäcklund transformation (3.17) both forwards and backwards produces a sequence of algebraic solutions $P_n(Z)$ at parameter values $a = 2n$ for $n \in \mathbb{Z}$, which are rational functions of $Z^{1/3}$ (see [36] for a table with some of these solutions). In that case, we find an associated normalized sequence of tau functions $\sigma_{2n}$ such that $P_{2n} = -\frac{d}{dZ}(Z\log \sigma_{2n})_Z$, the corresponding potential $\mathbf{v} = \mathbf{v}_{2n}$, which also corresponds to an instance of the third Painlevé equation, namely, the case $\alpha = 2$, by which is apparent from the above table, as $|Z| \to \infty$ all of these algebraic solutions are asymptotic to the solution $P_0 = (Z/2)^{1/3}$, corresponding to the ramp profile of the Camassa–Holm equation.

**Example 3.2: The Degasperis–Procesi equation.** When $b = 3$, Eq. (3.9) becomes

$$\frac{d^2 P}{dZ^2} = \frac{1}{P} \left( \frac{dP}{dZ} \right)^2 - \frac{1}{Z} \frac{dP}{dZ} + \frac{1}{Z} \left( 3P^3 + a \right) - \frac{1}{P},$$

which also corresponds to an instance of the third Painlevé equation, namely, the case $\alpha = 0$, after making a slight change of dependent and independent variables (this was briefly mentioned in [1], but never elaborated on). We can identify the variables and parameters in (1.20) as follows:

$$w = \left( \frac{Z}{3} \right)^{-1/4}, \quad P = \left( Z \right)^{3/4}, \quad \alpha = 0, \quad \beta = \frac{4}{3} a, \quad \gamma = 1, \quad \delta = -1.$$

It is well known that, in the generic case $\gamma \delta \neq 0$, Eq. (1.20) can be rescaled such that it depends on only two essential parameters, which are associated with the root space $B_2$, and the corresponding affine Weyl group acts birationally on the parameter space and the dependent/independent variables via Bäcklund transformations. Here we have chosen the normalization $\gamma = -\delta = 1$ as in [36] (but see [38] for a different choice). In that case, given any seed solution with $\alpha = 0$ and $\beta$ arbitrary, we can use a composition of the Bäcklund transformation

$$w^{(1)} = \frac{1}{w} - \frac{\alpha + \beta + 2}{\zeta(w' + w^2 + 1) + (1 + \alpha)w}.$$
as in [36], with the prime denoting \( d/d\zeta \), which sends \( \alpha \to \alpha + 2, \beta \to \beta + 2 \), together with the transformation

\[
w^{(2)}(\zeta) = -\frac{1}{w} - \frac{\alpha - \beta - 2}{\zeta(w' - w^2 + 1) + (1 - \alpha)w'},
\]

which sends \( \alpha \to \alpha - 2, \beta \to \beta + 2 \). The overall effect is to send \( \alpha \to \alpha, \beta \to \beta + 4 \) (and there are corresponding inverse transformations that can be combined to yield \( \alpha \to \alpha, \beta \to \beta - 4 \)); equivalently, one can use the composition of the two Schlesinger transformations \( T_1 \) and \( T_2 \) for the Painlevé III equation, as described, e.g., in [38] (with a different choice of scaling for the parameters), which has the same overall effect: the main point is that one can keep the value \( \alpha = 0 \) fixed, and just shift \( \beta \) up or down. In terms of the original ODE (3.20) obtained by reduction from the Degasperis–Procesi equation, the effect is to shift the parameter \( a \to a + 3 \).

In the case of (3.20), it turns out that there are various interesting choices of the seed solution that can be used to generate explicit solutions for particular values of the parameter \( a \). The simplest choice is the one corresponding to the ramp solution, \( P = P_0(Z) = (Z/3)^{1/4} \) for \( a = 0 \). With the choice of normalization as in (3.21), this gives the constant seed solution \( w = 1 \) for the Painlevé III equation with the parameters \( \alpha = \beta = 0 \) and \( \gamma = -\delta = 1 \), and the action of Bäcklund transformations on this solution generates solutions that are rational in \( \zeta \), which can be expressed in terms of so-called Umemura polynomials (see [37] and the references therein for full details). If we apply the composition of (3.22) and (3.23), or the composition of their inverses, in order to maintain the requirement that \( \alpha = 0 \), then we obtain a particular sequence of these rational solutions for parameter values \( \beta = 4n, \ n \in \mathbb{Z} \), and under the change of variables (3.21) this produces a sequence of similarity solutions for the Degasperis–Procesi equation, which are given by functions \( P_{3n}(Z) \) that are rational in \( Z^{1/4} \) and satisfy (3.20) at parameter values \( a = 3n \) (see Table 2 below). Similarly to the case \( b = 2 \), as \( |Z| \to \infty \), all these algebraic solutions are asymptotic to \( P_0 = (Z/3)^{1/4} \), corresponding to the ramp profile for the Degasperis–Procesi equation.

### Table 2. Algebraic solutions for (3.20) in terms of \( \zeta = 4(Z/3)^{3/4} \)

| \( a = 3n \) | \( 3n = 0 \) | \( 3n = 3 \) | \( 3n = 6 \) | \( 3n = 9 \) |
|---|---|---|---|---|
| \( P_{3n} \) | \( \left( \frac{\zeta}{4} \right)^{1/3} \left( \frac{2\zeta - 3}{2\zeta - 1} \right) \) | \( \left( \frac{\zeta}{4} \right)^{1/3} \left( \frac{2\zeta - 3}{2\zeta - 1} \right) \) | \( \left( \frac{\zeta}{4} \right)^{1/3} \left( \frac{2\zeta - 3}{2\zeta - 1} \right) \) | \( \left( \frac{\zeta}{4} \right)^{1/3} \left( \frac{2\zeta - 3}{2\zeta - 1} \right) \) |

The Painlevé III equation also admits one-parameter families of classical solutions in terms of Bessel functions. With the choice of scaling in [38], the parameters in (1.20) are given by

\[
\alpha = -4v_2, \quad \beta = 4(v_1 + 1), \quad \gamma = -\delta = 4, \quad (3.24)
\]

where the pair \( (v_1, v_2) \) is associated with the \( B_2 \) root space. The classical solutions are obtained by starting from the line \( v_1 + v_2 = 0 \) in the parameter space. Along this line, there are special solutions such that the function \( w \) satisfies a Riccati equation, and linearizing the latter shows that such \( w \) are given in terms of the logarithmic derivative of the solution of a linear equation equivalent to Bessel’s equation with parameter \( v_1 \); hence, for \( v_1 \notin \mathbb{Z} \), this can be written using a linear combination of the modified Bessel functions \( I_{\pm v_1} \) with an argument proportional to \( \zeta \); we refer the reader Proposition 4.3 in [38] for the precise details. For our purposes, the main point is to see how this relates to particular solutions of (3.20). Upon comparing the choice of scale in (3.24) with (3.21), we see that the requirement \( \alpha = 0 \) together with \( v_1 + v_2 = 0 \) fixes
\( v_1 = v_2 = 0 \), while in general the parameter \( a \) is related to \( v_1 \) by \( a = \frac{3}{2}(v_1 + 1) \), so we obtain a Riccati equation for \( P \) at the parameter value \( a = 3/2 \), with a one-parameter family of solutions in terms of a combination of the Bessel functions \( J_0 \) and \( Y_0 \).

Then by applying the composition of two transformations (3.22) and (3.23), or their inverses, starting from a seed solution of this kind with \( a = 3/2 \), we obtain a sequence of related solutions of (3.20) at parameter values \( a = 3n + 3/2 \) for \( n \in \mathbb{Z} \). We note that it is in fact sufficient to just derive the solutions for nonnegative integers \( n \), since the ODE for \( P \) has the discrete symmetry \( P \to -P, a \to -a \); for negative \( n \), the solutions are therefore found immediately by applying this symmetry (and the same consideration applies to the algebraic solutions in Table 2).

**Remark.** Equation (3.9) with \( b = -1 \), that is,

\[
\frac{d^2 P}{dZ^2} = \frac{1}{P} \left( \frac{dP}{dZ} \right)^2 - \frac{1}{Z} \left( \frac{dP}{dZ} \right) + \frac{a}{Z} - \left( 1 + \frac{1}{Z} \right) \frac{1}{P},
\]

(3.25)

is extremely close to the special case \( \alpha = \gamma = 0 \) of the Painlevé III equation, which is one of the degenerate cases where (1.20) can be reduced to a quadrature and the general solution given in terms of elementary functions (see, e.g., [36]). However, the presence in (3.25) of the additional final term \( 1/(ZP) \) means that the reduction to a quadrature is no longer possible, and perhaps the best that can be done is to produce asymptotic series solutions for this \( b = -1 \) equation in the limit \( |Z| \to \infty \).

### 4. Conclusions

We are planning at least one article in the near future, in which we propose to describe the details of analogous scaling similarity reductions for other peakon equations. In particular, in [39] we have obtained related results for two integrable peakon equations with cubic nonlinearity, namely, the equation

\[
m_t + \left( m(u^2 - u_x^2) \right)_x = 0, \quad m = u - u_{xx},
\]

(4.1)

which was derived in [40] and [41], and considered more recently in [42], as well as Novikov’s equation

\[
m_t + u^2 m_x + 3wu_xm = 0, \quad m = u - u_{xx},
\]

(4.2)

which was obtained from a classification of such equations admitting infinitely many local symmetries in \( m \) [43]. It turns out that both of these equations admit similarity reductions that are connected via a hodograph transformation to certain equations of Painlevé type: for the reductions of (4.1), an equation of a second order and second degree arises, while for (4.2) we find a special case of the Painlevé V equation. Moreover, it happens that both of these reductions can be solved in terms of solutions of the Painlevé III equation, and hence the reduction of (4.1) is related to (3.15), while scaling similarity solutions of (4.2) are precisely the special cases of Painlevé V transcendents that are related to Painlevé III in the form (3.20). These connections are not entirely surprising in the light of the fact that, in a certain sense, (4.1) can be considered a modified Camassa–Holm equation, while (4.2) can be viewed as a modified version of the Degasperis–Procesi equation. However, the relevant connections with the cases \( b = 2, 3 \) of (1.1) are far from being straightforward, since reciprocal transformations are involved.

We also hope to obtain a more explicit description of the algebraic solutions in Table 1, using the Crum transformation for the corresponding Schrödinger equation (3.11), since it appears that a determinantal formula for the associated sequence of special polynomials in \( \zeta \) is currently lacking.

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