$N = 2$ String-String Duality and Holomorphic Couplings

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We review aspects of $N=2$ duality between the heterotic and the type IIA string. After a description of string duality intended for the non-specialist the computation of the heterotic prepotential and the $F_1$ function for the ST, STU and STUV model (V a Wilson line) and the matching with the Calabi-Yau instanton expansions are given in detail. Relations with BPS spectral sums in various connections are pointed out.
## Contents

1. **Introduction** .................................................. 1

2. **$N = 2$ supergravity** ........................................... 7

3. **$N = 2$ Heterotic Vacua** ....................................... 10
   - 3.1 String Tree Level Prepotential ............................ 10
     - 3.1.1 STU model ............................................ 12
     - 3.1.2 ST model ............................................. 13
     - 3.1.3 STUV model .......................................... 13
   - 3.2 Perturbative Corrections to the Prepotential ............... 15
     - 3.2.1 STU model ............................................ 15
     - 3.2.2 ST model ............................................. 20
     - 3.2.3 STUV model .......................................... 21
   - 3.3 Higher gravitational couplings ............................. 25
     - 3.3.1 ST model ............................................. 26
     - 3.3.2 STU model ............................................ 27
     - 3.3.3 STUV model .......................................... 28

4. **The dual Calabi-Yau spaces** ................................ 31
   - 4.1 ST model .................................................. 33
   - 4.2 STU model ................................................ 38
   - 4.3 STUV model ............................................... 40

5. **BPS Spectral Sums** ........................................... 42
   - 5.1 BPS States ................................................ 42
     - 5.1.1 The $N = 4$ situation ................................ 42
     - 5.1.2 The $N = 2$ situation ................................ 44
   - 5.2 BPS spectral sums ......................................... 45
     - 5.2.1 $F_1$ function and the conifold ....................... 45
     - 5.2.2 The topological free energy .......................... 47
   - 5.3 threshold corrections and couplings ....................... 50
     - 5.3.1 ST model ............................................. 53
     - 5.3.2 STU model ............................................ 54
     - 5.3.3 STUV model .......................................... 54

6. **The duality** ................................................... 56
   - 6.1 ST model .................................................. 59
   - 6.2 STU model ................................................ 61
   - 6.3 STUV model ............................................... 66

7. **Summary and Outlook** ...................................... 68
### CONTENTS

**A Modular forms**  
A.1 Ordinary modular forms ....................................................... 69  
A.2 Siegel modular forms .......................................................... 71  
A.3 Jacobi forms ................................................................. 73  
A.4 Product expansions ............................................................ 74  
A.5 Theta functions and Jacobi forms .......................................... 75  
A.6 Lie algebra lattices and Jacobi forms ...................................... 77  
A.7 Tables ................................................................. 80

**B Bibliography**  

1 Introduction

In this section a description of string theory and duality intended for the non-specialist is given.

The starting point for the following considerations is represented by the experimentally well-tested standard model of elementary particle physics with gauge group $SU(3) \times SU(2) \times U(1)$. Now what is missing in this model? First, most of the theoretical insights and also many of the phenomenological predictions, which are tested in high precision experiments of the Standard Model, are limited to perturbation theory. However one would like to include into the picture, especially in the infrared regime of Quantum Chromodynamics, also solitonic states and nonperturbative effects at strong coupling. Furthermore there are 19 free parameters, and so the theory does not look quite fundamental. Also missing is an understanding of the occurrence of exactly these gauge groups; a problem which is not cured by embedding them in a grand unified theory (GUT) group like $SU(5)$, $SO(10)$ or even $E_6$. Similarly the number of generations has to be explained. Of course there is the most fundamental gap of not including gravity at the quantum level. Finally one is confronted with the hierarchy problem of explaining $m_{\text{weak}} \ll m_{\text{Pl}}$.

The last problem can be solved by the introduction of supersymmetry (SUSY), i.e. by postulating the existence of superpartners for all particles, which stabilize scalar field masses by avoiding their quadratic divergencies in loop orders. Of course, as these partners were not seen, supersymmetry has to be broken in nature. To avoid the Goldstino of broken global SUSY and to make a vanishing cosmological constant possible one is led to local SUSY and thus has automatically included gravity. But as the simplest version of local SUSY, namely $N=1$ supergravity (SUGRA), is still plagued by infinities one is led to introduce higher Supergravities. But – even those are plagued by infinities at higher loop-orders; furthermore the maximal group occurring is $SO(8)$ – by far to small to accomodate the data already known to occur in the atomic world as given by the Standard Model gauge group.

The superstring \cite{69,93} is an offer to get rid of all these problems. The ultraviolet infinities are (probably to all orders) avoided as point particles are replaced by extended objects, so there is no 'singular' interaction point in the Feynman diagrams, which themselves are organised in a series of two-dimensional Riemann surfaces (coming from the world-sheets) of ascending topological complexity as measured by the genus. Supergravity is included as effective theory in the point particle limit. The spacetime dimension, which is a free parameter in field theory, is restricted to be less or equal to 10 for superstring consistency. More precisely the requirement of cancellation of the conformal anomaly in the world-sheet approach fixes the possible space-time dimension $D_{\text{crit}}$ to be 10 (resp. 26 for the bosonic string). Similarly cancellation of the chiral anomalies fixes the gauge group. Note that in theories where spinors are coupled to gauge and gravitational fields one has to worry about the issue of anomalies \cite{4,113} and that gravitational anomalies are possible in $4k + 2$ dimensions like $D = 10$. That the theory is free of anomalies was the clue of the revival of string theory in the first string revolution of 1984 as the most promising candidate for a unified theory of all
elementary forces and particles. As said this also constrains the tendimensional (10 \( D \)) gauge group \([12]\): it has to be one of the two rank 16 groups \( SO(32) \) and \( E_8 \times E_8 \) of even self-dual Lie algebra lattice and thus explains the GUT gauge groups as we will see in a moment. Furthermore the construction had extremely little freedom – there were only five fundamental consistent theories:

- the type I string (including) open strings of gauge group \( SO(32) \) and \( N = 1 \) SUGRA as 10\( D \) field theory limit
- the heterotic string \([67]\) of gauge group \( SO(32) \) resp. \( E_8 \times E_8 \) and \( N = 1 \) SUGRA as limit (called heterotic as it consists of a bosonic string ’compactified’ on a rank 16 lattice for the left movers and a superstring for the right movers)
- the type II string \([68]\), which does not lead to non-abelian gauge groups but has \( N = 2 \) SUGRA as limit (having for both left- and right- movers a version of the superstring) and comes in two variants according to the two gravitini having the same (IIB) or different chirality (IIA)

So, to reduce from the critical dimension \( D = 10 \) to e.g. \( D = 4 \), one has like in the old Kaluza-Klein approaches to compactify some dimensions (resp. to take the approach of going down to \( D = 4 \) not on a space but on a conformal field theory \([1, 73, 87]\)). The dynamical evolution of this process requires certainly non-perturbative dynamics. Now the conformal invariance is maintained by choosing the six-dimensional compactification space Ricci-flat (and embedding its spin-connection into the gauge connection) and the perturbative stability is saved by choosing \( N = 1 \) SUSY. This in turn is equivalent to the existence of a covariantly constant spinor, whose existence furthermore affirms that the compactification space is complex Kähler. Such a Ricci-flat Kähler space (of complex dimension three) is called a Calabi-Yau space \([38]\).

Now Kähler manifolds of complex dimension \( n \) have holonomy group \( U(n) \) instead of the \( SO(2n) \) for the general compact manifold. Furthermore the condition of having a Ricci-flat metric is equivalent to have vanishing first Chern class \( c_1 \). Then the theorem of Calabi and Yau states that for a Kähler manifold the further reduction of the holonomy group to \( SU(n) \) is equivalent to \( c_1 = 0 \) (the hard part is that the existence of a metric of \( SU(n) \) holonomy is implied by \( c_1 = 0 \)).

Let us now return to our special case \( n = 3 \), i.e. compactification of the heterotic string on a Calabi-Yau (of strict \( SU(3) \) holonomy) leading to \( N = 1 \) in 4\( D \). The standard embedding of the holonomy group in the gauge group \( E_8 \times E_8 \) (constituting one of the favoured possibilities of anomaly cancellation) leaves besides an ‘hidden’ \( E_8 \) an \( E_6 \) GUT group; furthermore the geometry/physics dictionary leads to a generation number given by (the absolute value of) half the Euler number of the Calabi-Yau.

One is also interested in compactifications on spaces of smaller holonomy leading to higher SUSY in 4\( D \). Now the analogue of the CY for \( n = 3 \), which exists in quite diverse topological forms, is the elliptic curve \( T^2 \) for \( n = 1 \) and the \( K3 \) surface for \( n = 2 \). Compactification of the heterotic string on \( K3 \times T^2 \) of holonomy \( SU(2) \) leads then to \( N = 2 \) SUSY in 4\( D \) whereas compactification on the flat space \( T^6 \) leads to \( N = 4 \) SUSY in 4\( D \). If we had started instead with a type II string, of twice as much
SUSY in $D = 10$ as the heterotic string, the amount of SUSY in $D = 4$ would be equally twice as much. So note especially that the compactifications of the heterotic string on $T^4 \times T^2$ and the type II string on $K3 \times T^2$ both lead to $N = 4$ SUSY in $D = 4$ (obviously it is possible to formulate the observation already in $D = 6$) and similarly the heterotic string on $K3 \times T^2$ and the type II string on $CY$ both lead to $N = 2$ SUSY in $D = 4$.

Now, if one has direct phenomenological wishes, these models with higher SUSY are certainly not very convincing as extended SUSY has no chiral fermions since the representations of the Clifford algebra then split in $c + \bar{c}$ for the $D = 4$ Dirac algebra. Nevertheless the strategy to study first a version of the theory with higher symmetry has been here as in the past extremely successful.

To put these assertions into perspective we have to recall now the second string revolution going on since 1994, where a unification of all string theories as well as an inclusion of the non-perturbative regimes took place under the headline “duality”. As these developments were accompanied and partly proceeded by corresponding explorations in SUSY field theories let us discuss the latter first.

The first basic observation is the connection given by the fact that electro-magnetic duality of $N = 4$ supersymmetric 4D Yang-Mills theory corresponds to the exchange of elementary and solitonic states (magnetic monopoles) and so also to strong-weak coupling duality (the non-perturbative solitonic states have $m_{sol} \sim \frac{1}{g^2}$; in string theory there exist solitons with $m_{sol} \sim \frac{1}{g}$). At first a corresponding duality was considered impossible to hold also for $N = 2$ supersymmetric gauge theories as the gauge bosons and the magnetic monopoles are in different $N = 2$ multiplets (unifying in one multiplet states which differ in spin up to two steps of spin 1/2): the vector multiplet consisting of states of spin 1, 1/2 and 0 resp. the hypermultiplet consisting of states of spin 1/2, 0 and 1/2 of the other helicity. Note that because of the scalar $a$ in the vector multiplet one always has a Coulomb phase with $a$ as Higgs scalar and classical non-abelian gauge symmetry enhancement, say to $SU(2)$ in the simplest case, for $a = 0$. Now the breakthrough of Seiberg and Witten consisted in the determination of the exact non-perturbative low-energy effective action of $N = 2$ Yang-Mills theory in $D = 4$. If we call the perturbative expansion regime for a moment local in the coupling parameter, then they could gain the exact global – in the qualified sense – structure of the ‘characteristic function’, called prepotential (whose second derivatives give the gauge couplings), of the theory by combining its holomorphy with information about the singularities and the corresponding monodromies. Namely here the holomorphy corresponds to having $N = 2$ SUSY in the sense that holomorphy unifies analytic dependence on a real and an imaginary part like SUSY connects bosonic and fermionic states. Actually one has here - to get the proper conclusions - to consider the difference between physical and Wilsonian couplings (for a review cf. [77]), where the latter are good behaved with respect to the question of the connection between SUSY and holomorphy.

Now, in particular, one has the gauge symmetry enhancement in the Coulomb phase at $a = 0$ only classically, whereas quantum mechanically one finds a splitting of this one point of enhancement of the massless spectrum (namely by the non-abelian gauge bosons) in two points in the moduli space, where massless monopoles/dyons occur. The important technical tools to get and to express these results consist in the translation
of this setup: one starts with a variation of data over a complex one-dimensional base manifold, where the data degenerate at some points leading to monodromies; this is then reformulated in a geometrical picture involving the variation of the Seiberg-Witten torus, where one studies the periods around 1-cycles of a certain form. Near the points where the magnetic solitons occur one has then an effective description of the theory in variables, for which these states appear as elementary.

Now similarly as in supersymmetric field theories one has in the $N = 4$ heterotic string [60, 106] a strong-weak coupling duality (S-duality). Furthermore it was discovered during the last years that there is a duality of descriptions between perturbative effects in one string theory at weak coupling and non-perturbative effects in a dual theory at strong coupling, the so-called string-string duality (for a review cf. [59]). In this way all string theories are now believed to be unified just as different perturbative expansions of one basic underlying theory, often called $M$-theory. For instance [115] – to give an impression – one can think of the heterotic string, which is chiral with non-abelian gauge group, as a description of the real world at high energies (GUT’s) whereas the type II string, non-chiral and with abelian gauge group only, as a description of the low energy world. More precisely Hull and Townsend [73] and also Witten [114] found that the $N = 4$ string-string duality holds, if one includes in the comparison of heterotic string on $T^4$ and type IIA string on $K3$ also solitonic states, which are given in string theory by higher dimensional extended objects like membranes and p-branes. The occurrence of non-abelian gauge symmetry on the type II side is exactly explained by these membranes wrapping certain 2-cycles of $K3$, which become massless as the 2-cycles shrink leading to a $K3$ with singularities as classified by the Dynkin diagrams $ADE$ of simply laced Lie groups; one can already see the $E_8 \times E_8$ of the heterotic string occurring in the cohomology of $K3$ where $H^2(K3) = E_8 \oplus E_8 \oplus H \oplus H \oplus H$ ($H$ being the hyperbolic plane). This same observation lead also to the detection of the heterotic string (in 6D, on $T^4$) as a soliton of the type IIA string (on $K3$) [71].

Now in the next step, it was demonstrated [75] that there exists a string-string duality of $N = 2$ string compactifications to 4D given on the one hand by the heterotic string on $K3 \times T^2$ and on the other hand the type IIA string on $CY$. This duality becomes clear remembering the fact that $K3$ can be represented as fibration of $T^2$ over $P^1$ and so $K3 \times T^2$ as fibration of $T^2 \times T^2$ over $P^1$; in this way one gets by this adiabatic extension principle [112] that the 4D duality can be understood as spreading out the 6D duality over $P^1$. This then leads to the conclusion that one should expect a duality with type IIA on a $CY$ which is a $K3$ fibration over $P^1$. As we have just seen that it is possible to match the classical gauge symmetry enhancements it is of interest to study the mentioned ‘characteristic function’ (prepotential) and its higher relatives (involving higher gravitational couplings (so-called $F_1$ function)) in the possible dual models [56, 75].

More precisely we will study the prepotential in its vector moduli dependence, i.e. on the Coulomb branch. On the heterotic side, where the dilaton sits in a vector multiplet, one can at first study the prepotential in the perturbative regime only. Thanks to $N = 2$ SUSY, perturbative corrections occur only at 1-loop. Actually one uses then the perturbative quantum symmetries (T-duality generalising the exchange of momentum and winding modes related to the $R \rightarrow 1/R$ on $S^1$) and the singularity structure
corresponding to the gauge symmetry enhancement loci to determine the one-loop correction of the gauge couplings, which gives then direct information about the second derivatives of the prepotential. Here again the crucial role of singularities in the moduli space of vacua is demonstrated. So let us make more precise what is really meant with such a singularity. If the singularity is physical (and not an artefact of perturbation theory and smoothed out once quantum corrections are taken into account) it signals the breakdown of some approximation like the appearance of additional massless degrees of freedom on a subspace of the moduli space. Almost everywhere on the moduli space these degrees of freedom are heavy and thus have been integrated out of the effective theory. In 4D SUSY Yang-Mills below the threshold scale $M$ of the heavy states the gauge couplings of the light modes obey (a model dependent constant)

$$g_{\text{low}}^{-2} = g_{\text{high}}^{-2} + c \log M,$$

where the mass is a function of the moduli $M(\phi)$ with a zero somewhere in the moduli space, where the gauge coupling $g_{\text{low}}^{-2}$ becomes singular due to the inappropriate approximation of integrating out the heavy states. As we discussed, quantum mechanically the number and position of the singularities and the interpretation of the fields becoming massless at the singularities can be quite different to the classical picture [109].

Now by contrast on the type IIA side the vector moduli do not interfere with the dilaton, which for type II lies in a hypermultiplet, and so the space-time quantum field theory is given by the uncorrected string tree level result. The technical tool used on the type IIA side to evaluate the prepotential is mirror symmetry [64] to the type IIB string on the mirror CY, where the the vector moduli do not correspond to the $h^{1,1}$ cohomology like in type IIA but are now related to the $h^{2,1}$ cohomology which is no longer subjected to corrections by world-sheet instantons (i.e. rational curves giving 2-cycles).

So the subject of this review will be to provide several quantitative tests for the $N = 2$ string-string duality between the heterotic string on $K3 \times T^2$ and the type IIA string on a CY. We will first pick specific CY’s matching the spectrum, i.e. the number of vector- and hypermultiplets, of heterotic string models and will then go on to the heart of the investigation, which consists in making finer tests of the potentially dual pairs by comparing the couplings.

In more detail the content is as follows.

In section 2 we review the necessary formalism of $N = 2$ supergravity, especially the multiplets the splitting of vector- and hypermultiplet moduli spaces and the prepotential with its relation to all the other relevant quantities in the theory like the Kaehler potential, the gauge couplings and the BPS masses.

In section 3 we start with the perturbative analysis of the $N = 2$ heterotic vacua studying the spectrum, enhanced symmetry loci and the thereby gained information on the prepotential; then we go on and perform the corresponding analysis for the first of the higher gravitational couplings, the socalled $F_1$ function. Here and in the following we especially focus on three models of one resp. two, resp. three vector moduli besides the dilaton called ST resp. STU resp. STUV model.
In section 4 then the corresponding questions on the CY side are treated. Especially the instanton expansions for the prepotential and the $F_1$ function are given and the geometrical situation in the different potentially dual CY spaces are discussed.

In section 5 we discuss the notion of BPS states, including the situation in $N = 4$ in view of a later application of this to the question of the $S - T$ exchange symmetry. Then we go on and study the notion of BPS spectral sums, i.e. the function given by the sum over the BPS spectrum of the theory, in different approaches, especially in that of Harvey and Moore [70], where an intimate connection to the prepotential is made.

In section 6 at last we match the expressions for prepotential and $F_1$ function we have got on the heterotic and the type IIA side.

In the appendix we review the necessary background of the special functions used.
2 \( N = 2 \) supergravity

The \( N = 2 \) supergravity multiplets relevant for us to consider are the following:

- the gravitational multiplet containing the graviton \( g_{\mu\nu} \), two gravitini \( \psi_{I\mu} \) and an Abelian graviphoton \( \gamma_{\mu} \); in terms of \( N = 1 \) multiplets it is the sum of the \( N = 1 \) gravitational multiplet and a gravitino multiplet \( \Psi \) which contains a gravitino and an Abelian vector

- the vector multiplet \( V \) with a gauge field \( A_{\mu} \), two gauginos \( \lambda_{I}^{\alpha} \) and a complex scalar \( \phi \); it consists of an \( N = 1 \) vector multiplet \( V \) and a chiral multiplet \( \Phi \)

- furthermore matter fields arise from hypermultiplets \( H \) with two Weyl spinors \( \chi_{I}^{I} \) and four real scalars \( q^{IJ} \); it consists of two chiral multiplets (chiral plus antichiral)

Supersymmetry prohibits gauge neutral interactions between vector and hypermultiplets and therefore the moduli space locally has to be a direct product

\[
\mathcal{M} = \mathcal{M}_H \times \mathcal{M}_V, \tag{2.1}
\]

where \( \mathcal{M}_H \) is the (quaternionic) moduli space parameterized by the scalars of the hypermultiplets and \( \mathcal{M}_V \) is the (complex) moduli space spanned by the scalars in the vector multiplets.

\( \mathcal{M}_V \) has actually the structure of a special Kähler manifold, which is a Kähler manifold whose geometry obeys an additional constraint \([11, 13, 27]\), which constrains both the Kähler potential and the Wilsonian gauge couplings of the vector multiplets in an \( N = 2 \) effective Lagrangian to be determined by a single holomorphic function of the scalar fields \( \phi \) – the prepotential \( F(\phi) \).

One way to express this constraint is the statement that the Kähler potential \( K \) is not an arbitrary real function (as in \( N = 1 \) supergravity) but determined in terms of a holomorphic prepotential \( F \). Here \( F(X) \) is a homogeneous function of \( X^I \) of degree 2: \( X^IF_I = 2F \). The \( X^I, I = 0, \ldots, n_V + 1 \) are \( n_V + 2 \) holomorphic functions of the \( n_V + 1 \) complex scalar fields \( \phi^I, I = 1, \ldots, n_V + 1 \) which reside in the vector multiplets and the dilaton multiplet. \( F_I \) abbreviates the derivative, i.e. \( F_I \equiv \frac{\partial F(X)}{\partial X^I} \).

The mentioned connection with the Kähler potential is given by (cf. also \([2, 13, 19, 107]\))

\[
e^{-K} = i \left( X^I(\bar{\phi})F_I(X) - X^I(\phi)F_I(X) \right). \tag{2.2}
\]
The above description is slightly redundant. The variable $X^0$, which represents the graviphoton, can be eliminated by an appropriate choice of coordinates, the so-called special coordinates defined by:

$$ \phi^I = \frac{X^I}{X^0}, $$

i.e. the graviphoton is set to one.

In these special coordinates the Kähler potential (2.2) reads

$$ K = -\log \left( 2(\mathcal{F} + \bar{\mathcal{F}}) - (\phi^I - \bar{\phi}^I)(\mathcal{F}_I - \bar{\mathcal{F}}_I) \right), $$

where $\mathcal{F}(\phi)$ is an arbitrary holomorphic function of $\phi^I$ related to $F(X)$ via $F(X) = -i(X^0)^2 \mathcal{F}(\phi)$.

The metric $G_{IJ}$ on the Kaehler manifold is expressed in terms of the Kähler potential $K$ by

$$ G_{IJ} = \frac{\partial}{\partial \phi^I} \frac{\partial}{\partial \bar{\phi}^J} K(\phi, \bar{\phi}). $$

Furthermore of central importance in the formalism is the period vector

$$ \Omega = (X^I, F_I), $$

where $X^I$ constitutes the so called electric, $F_I$ the magnetic part. This terminology is motivated by considering the stringy realisation of the effective $N = 2$ supergravity as a compactification of the type IIB string on a Calabi-Yau. There the $X^I$ and the $F_I$ arise really as periods of the holomorphic 3-form $\Omega$

$$ X^I = \int_{\Gamma_{\alpha I}} \Omega, \quad F_I = \int_{\Gamma_{\beta I}} \Omega, $$

where the $\Gamma_{\alpha I}, \Gamma_{\beta J}, I, J = 1, \ldots, h^{2,1} + 1$ span a symplectic basis of $H_3(CY, \mathbb{Z})$ with the $\alpha$- and $\beta$- cycles dual under the intersection product (spoken in cohomology the "+1" corresponding to $h^{3,0}$ corresponds to the graviphoton). Furthermore we will discuss later that one has a BPS mass formula

$$ M_{BPS} = M_I X^I + N^J F_J, $$

where $M_I$ constitutes an electric and $N^J$ a magnetic charge vector. This by the way makes the easiest contact with the results of Seiberg/Witten [109] in the field theory limit by considering the analogy to the situation involving 1-cycles on the Seiberg/Witten torus [85].

This whole structure is acted on by duality transformations $\Gamma \in Sp(2(n_V + 2), \mathbb{Z})$

$$ \begin{pmatrix} X^I \\ F_I \end{pmatrix} \rightarrow \begin{pmatrix} U & Z \\ W & V \end{pmatrix} \begin{pmatrix} X^I \\ F_I \end{pmatrix}, $$

where $\begin{pmatrix} U & Z \\ W & V \end{pmatrix}$ constitutes a so called monodromy matrix (with $U^T V - W^T Z = 1$, etc.).
The $W_{IKM} = \partial_I \partial_K \partial_M F$ are referred to as the Yukawa couplings since for a particular class of heterotic $N = 1$ vacua (vacua which have a global $(2, 2)$ worldsheet supersymmetry) they correspond to space-time Yukawa couplings \cite{46}.

$N = 2$ supergravity not only constrains the Kähler manifold of the vector multiplets but also relates the gauge couplings $N_{IJ}$ to the holomorphic prepotential. They are essentially \cite{97} given by the second derivatives of the prepotential (which would constitute the exact expression in global $N = 2$ SUSY)

$$N_{IJ} = \partial_I \partial_J F + \cdots \quad (2.9)$$
3 $N = 2$ Heterotic Vacua

So far we reviewed the effective $N = 2$ supergravity without any reference to a particular string theory. The aim of this section is to determine the prepotential $\mathcal{F}$ for $N = 2$ heterotic vacua (cf. [11] for a review).

As we already discussed, for heterotic vacua the dilaton is part of a vector multiplet (actually of the equivalent dual vector-tensor multiplet) whereas in type IIA (IIB) the dilaton sits in a hypermultiplet (actually in the equivalent dual tensor (double tensor) multiplet).

From the fact that the dilaton organizes the string perturbation theory together with product structure of the moduli space one derives a non-renormalization theorem [15, 108] saying that the heterotic moduli space of the hypermultiplets is determined at the string tree level and receives no further perturbative or non-perturbative corrections, i.e. heterotically

$$\mathcal{M}_H = \mathcal{M}_H^{(0)}.$$  

(3.1)

On the other hand the PQ-symmetry can be used to derive a second non-renormalization theorem. The loop corrections of the prepotential $\mathcal{F}$ are organized in an appropriate power series expansion in the dilaton. The holomorphy of $\mathcal{F}$ and the PQ-symmetry only allow a very limited number of terms, so the prepotential $\mathcal{F}$ only receives contributions at the string tree level $\mathcal{F}^{(0)}$ (of order $S$), at one-loop $\mathcal{F}^{(1)}$ (dilaton independent) and non-perturbatively $\mathcal{F}^{(np)}$ (only constrained by the discrete PQ-symmetry), i.e.

$$\mathcal{F} = \mathcal{F}^{(0)}(S,M) + \mathcal{F}^{(1)}(M) + \mathcal{F}^{(np)}(e^{-8\pi^2 S}, M).$$  

(3.2)

3.1 String Tree Level Prepotential

The dilaton dependence of the tree level Kähler potential is constrained by the PQ-symmetry and the fact that the dilaton arises in the universal sector. Therefore it cannot mix with any other scalar field at the tree level and one necessarily has

$$e^{-K^{(0)}} = (S + \bar{S})e^{-K(M,\bar{M})}.$$  

(3.3)

This separation of the dilaton piece together with the constraint (2.4) uniquely fixes the tree level contribution to be [58]:

$$\mathcal{F}^{(0)} = -SM^i \eta_{ij} M^j = -S(TU - \phi^i \phi^j),$$  

(3.4)

where

$$\eta_{ij} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ \vdots & \ddots & \ddots \\ -1 & & & -1 \end{pmatrix}.$$  

(3.5)

3 Also often denoted in the literature by $h$ and similarly for its components to follow below; we will follow also this very common notation.
and the $\phi^i$ ($i = 1, \ldots, n_V - 3$; we count the dilaton in the $n_V$ for convinience in the later comparison with the type IIA spectrum) are the vector multiplet moduli of the factor $G'$. Inserted into (2.4) the tree level Kähler potential is found to be

$$K^{(0)} = -\log(S + \bar{S}) - \log(\text{Re}M^i \eta_{ij} \text{Re}M^j).$$

(3.6)

The metric derived from this Kähler potential is the metric of the coset space

$$\mathcal{M}_V^{(0)} = \frac{SU(1,1)}{U(1)} \times \frac{SO(2, n_V - 1)}{SO(2) \times SO(n_V - 1)},$$

(3.7)

(modulo discrete identifications related to the duality group) where the first factor of the moduli space is spanned by the dilaton and the second factor by the other vector multiplets. Thus one has generically a gauge group $U(1)^{n_V + 1}$ (the ”+1” coming from the graviphoton) which enhances at special loci.

There are two Abelian vector multiplets $T$ and $U$ with moduli scalars $T = \sqrt{g} + ib, U = g_{12}^{-1}(\sqrt{g} + ig_{12})$ of $T^2$ as well as further model-dependent Abelian or possibly non-Abelian vector multiplets. The total perturbative gauge group is

$$G = G' \times U(1)_S \times U(1)_T \times U(1)_U \times U(1)_\gamma,$$

(3.8)

where $G'$ refers to the additional Abelian or non-Abelian part of the gauge group and $U(1)_\gamma$ corresponds to the graviphoton.

Note that one has the $T$-duality group $Sl(2, \mathbb{Z})_T \times Sl(2, \mathbb{Z})_U \times \mathbb{Z}^{T+U}$ as a symmetry of string perturbation theory, so the effective action should be invariant under it.

The rank of $G$ is bounded by the central charge $\hat{c}_{int} = 22$. For $N = 2$ vacua one has with the two additional $U(1)$ gauge bosons in the universal sector corresponding to the graviphoton and the superpartner of the dilaton that

$$\text{rank}(G) \leq 22 + 2.$$ 

(3.9)

To get more information about the spectrum let us consider now the heterotic string in 6D, i.e. on $K3$. This supergravity is chiral and thus gauge and gravitational anomaly cancellation imposes constraints on the allowed spectrum. The anomaly can be characterized by the anomaly eight-form $I_8$, which is exact $I_8 = dI_7$, given by

$$I_8 = a \text{tr} R^4 + b \left(\text{tr} R^2\right)^2 + c \text{tr} R^2 \text{tr} F^2 + d \left(\text{tr} F^2\right)^2,$$

(3.10)

where $R$ is the curvature two-form, $F$ is the Yang-Mills two-form and $a, b, c, d$ are real coefficients which depend on the spectrum of the theory. The anomaly can only be cancelled if $a$ vanishes. One finds

$$n_H - n_V = 244.$$ 

(3.11)

The remaining anomaly eight form has to factorize in order to employ a Green–Schwarz mechanism. More precisely, one needs $I_8 \sim X_i \wedge \bar{X}_i$ ($i$ labels the factors $G_i$ of the
gauge group $G = \otimes_i G_i$ and $v_i, \tilde{v}_i$ are constants which depend on the massless spectrum \[ \mathbf{I}, \mathbf{I} \]

\[ N = 2 \]

Heterotic vacua

12

\[ g \]

\[ \otimes \]

\[ G \]

\[ i \]

\[ G_i \]

\[ v_i \]

\[ \tilde{v}_i \]

are constants which depend on the massless spectrum \[ \mathbf{63}, \mathbf{101} \]

\[ X_4 = \text{tr} R^2 - \sum_i v_i \left( \text{tr} F_i^2 \right) 
\]

\[ \tilde{X}_4 = \text{tr} R^2 - \sum_i \tilde{v}_i \left( \text{tr} F_i^2 \right). \]

(3.12)

In the Green–Schwarz mechanism one defines a modified field strength $H$ for the antisymmetric tensor $H = dB + \omega^L - \sum_i v_i \omega_i^{YM}$ such that $dH = X_4$ ($\omega^L$ is a Lorentz–Chern–Simons term and $\omega_i^{YM}$ is the Yang–Mills Chern–Simons term).

Vanishing of $\int_{K3} dH$ gives with $n_i = \int_{K3} \left( \text{tr} F_i^2 \right)$

\[ \sum_i n_i = \int_{K3} \text{tr} R^2 = 24 \]

(3.13)

so that one gets for instanton numbers $n_i$ ($i = 1, 2$) in the two $E_8$

\[ n_1 + n_2 = 24. \]

(3.14)

The dimension (counted quaternionically, i.e. giving the number of hypermultiplets) of the moduli space of instantons on $K3$ in $SU(2)$ of instanton number $n$ is given by

\[ \dim M_n = 2n - 3. \]

(3.15)

For $n_i \geq 4$ one has the breaking $E_8 \times E_8 \to E_7 \times E_7$ with $\frac{1}{2} n_i - 2 \mathbf{56}$’s and 62 singlets which consist of $h_{1,1} = 20$ universal hypermultiplets of K3 and the sector $M_{n_1} \times M_{n_2}$. Complete Higgsing of the $E_7 \times E_7$ is possible in the cases $(n_1, n_2) = (12, 12), (11, 13), (10, 14)$; one can show that the cases $(12, 12)$ and $(10, 14)$ are equivalent \[ \mathbf{35}, \mathbf{94} \]. This gives $(\frac{1}{2} 24 - 4) \cdot 56 - 2 \cdot 133 = 182$ further hypermultiplets giving in total 244 which corresponds in view of the anomaly condition for the spectrum to the fact that the gauge group is then completely broken. Alternatively one could count instead the moduli of instantons in the group $E_8$ (of dual Coxeter number $h = 30$) from the beginning giving directly the $30 \cdot 24 - 2 \cdot 248 = 224$ moduli.

We will study the following three cases

- **ST-model**: at $T = U$ with $(10, 10; 4)$ instanton embedding
- **STU-model**: generic $T^2$ with $(10, 14)$ resp. $(12, 12)$ embedding
- **STUV-model**: generic $T^2$ with $(10, 14)$ embedding and breaking only $E_7^{(2)} \to SU(2)_V$

### 3.1.1 STU model

To consider this in greater detail start with a compactification of the heterotic $E_8^{(1)} \times E_8^{(2)}$ string on $K3$ with $SU(2)$ bundles with instanton numbers $(d_1, d_2) = (12 - n, 12 + n)$ ($n \geq 0$). For $0 \leq n \leq 8$, the gauge group is $E_7^{(1)} \times E_7^{(2)}$, and the spectrum of massless hypermultiplets follows from the index theorem \[ \mathbf{83}, \mathbf{85} \] as

\[ \frac{1}{2}(8 - n)(\mathbf{56}, \mathbf{1}) + \frac{1}{2}(8 + n)(\mathbf{1}, \mathbf{56}) + 62(\mathbf{1}, \mathbf{1}). \]

(3.16)
For the standard embedding, \( n = 12 \), the gauge group is \( E_8^{(1)} \times E_7^{(2)} \) with massless hypermultiplets

\[
10(1, 56) + 65(1, 1).
\]  

(3.17)

These gauge groups can be further broken by giving vev’s to the charged hypermultiplets. Specifically, \( E_7^{(2)} \) can be completely broken through the chain

\[
E_7 \rightarrow E_6 \rightarrow SO(10) \rightarrow SU(5) \rightarrow SU(4) \rightarrow SU(3) \rightarrow SU(2) \rightarrow SU(1),
\]  

(3.18)

where \( SU(1) \) denotes the trivial group consisting of the identity only. In the following, we will concentrate on the cases where we break \( E_7^{(2)} \) either completely or down to \( SU(2) \). On the other hand, \( E_8^{(1)} \) can be perturbatively broken only to some terminal group \( G_0^{(1)} \) that depends on \( n \) (see [24] for details); e.g. for \( n = 4 \) this group is given by \( G_0^{(1)} = SO(8) \). For \( n = 8 \) it is \( G_0^{(1)} = E_8 \). It is only for \( n = 0, 1, 2 \) that \( E_7^{(1)} \) can be completely broken. Finally, when compactifying to four dimensions on \( T^2 \), three additional vector fields arise, namely the fields \( S, T \) and \( U \).

The enhancements in the \( T, U \) moduli space arise along the critical line \( T = U \) (mod \( SL(2, \mathbb{Z}) \)) where two additional massless gauge fields appear and the \( U(1)_T \times U(1)_U \) is enhanced to \( SU(2) \times U(1) \). Further enhancement appears at \( T = U = 1 \), which is the intersection of the two critical lines \( T = U \) and \( T = 1/U \). In this case one has 4 extra gauge bosons and an enhanced gauge group \( SU(2) \times SU(2) \). The intersection at the critical point \( T = U = \rho = e^{2\pi i/12} \) gives rise to 6 massless gauge bosons corresponding to the gauge group \( SU(3) \). Altogether we have:

\[
\begin{align*}
T = U: & \quad U(1)_T \times U(1)_U \rightarrow SU(2) \times U(1) \\
T = U = 1: & \quad U(1)_T \times U(1)_U \rightarrow SU(2) \times SU(2) \\
T = U = \rho: & \quad U(1)_T \times U(1)_U \rightarrow SU(3)
\end{align*}
\]

3.1.2 ST model

A variation of the theme is going in the moduli space of the \( T^2 \) (on which the six-dimensional theory on \( K3 \) is further compactified) to the locus \( T = U \) leading to an extra \( SU(2) \) gauge symmetry which opens the possibility of a threefold distribution of the required 24 among the gauge group factors in the form \( (10, 10; 4) \) say, leading to \( n_H = 2[(3 \cdot 56 - 133) + (2 \cdot 10 - 3)] + (2 \cdot 4 - 3) + 20 = 129 \). Note that this is not a simple truncation of the \( STU \) model as the \( SU(3) \) enhancement at \( T = U = \rho \) is lost and here only the possible further \( SU(2) \) enhancement at \( T = U = i \) remains.

3.1.3 STUV model

We will be also interested in the case where the first \( E_7 \) is not completely broken but a \( SU(2)_V \) survives; here we let \( V \) denote the fourth vector field besides the three fields \( S, T, U \), i.e. the Wilson line in the Cartan subalgebra of \( SU(2)^{(2)} \), so this model is connected to the pure \( STU \)-model by a Higgs transition. The commutant of \( SU(2)^{(2)} \) in \( E_7^{(2)} \) is \( SO(12)^{(2)} \). Then, it follows from the index theorem that the charged spectrum
consists of $\frac{1}{2}(8 - n)$ 56 of $E^{(1)}_7$, as well as of $\frac{1}{2}(8 + n)$ 32 of $SO(12)^{(2)}$ plus 62 gauge neutral moduli.

As for the $STU$ models, it is only possible to perturbatively higgs the $E^{(1)}_7 \times SO(12)^{(2)}$ completely for $n = 0, 1, 2$. Thus, these heterotic models will have a massless spectrum comprising $n_V = 4$ vector multiplets, $S, T, U, V$ (plus the graviphoton), as well as neutral hypermultiplets. Note that, unlike in the $STU$ models with $n_H = 244$, the number of hypermultiplets now depends on $n$. Furthermore, as we will discuss, for the four-parameter models also the vector multiplet couplings are sensitive to $n$ already at the perturbative level.

Now consider the mentioned Higgs transition. At the transition point $V = 0$, the $U(1)$ associated with the Wilson line modulus $V$ becomes enhanced to an $SU(2)$. Let $n'_V = 2$ and $n'_H$ denote the number of additional vector- and hypermultiplets becoming massless at this transition point. Then

$$\frac{1}{2} (n'_H - n'_V) = 6n + 15. \quad (3.20)$$

This will prove to be a useful relation later on. It follows from the fact that the Euler number of the Calabi–Yau space $\chi(X_n)$ and of the $STU$ models ($\chi = -480$) differ by $2(n'_H - n'_V) = \chi(X_n) + 480$.

Let us consider the question of gauge symmetry enhancement loci for the $STUV$ model in more detail. In addition to the $V = 0$ locus of gauge symmetry enhancement, there are also the enhancement loci (such as $T = U$), associated with the toroidal moduli $T$ and $U$, already known from the $STU$ model. All these loci correspond to surfaces/lines of gauge symmetry enhancement in the heterotic perturbative moduli space $\mathcal{H}_2 = \frac{SO(3,2)}{SO(3) \times SO(2)}$ and have a common description as follows.

Consider the Narain lattice $\Gamma = \Lambda \oplus U(-1)$ of signature (3, 2), where $U(-1)$ denotes the hyperbolic plane $\left( \begin{array}{cc} 0 & 1 \\ -1 & -1 \end{array} \right)$, and where $\Lambda = U(-1) \oplus <2> = \left( \begin{array}{cc} 0 & -1 \\ -1 & 0 \end{array} \right)$ is a basis that we will denote by $(f_2, f_{-2}, f_3)$; we will use the coordinate $z = iT f_{-2} + iU f_2 - iV f_3$ in $\Lambda \otimes \mathbb{C}$. Note here that the perturbative moduli space $\mathcal{H}_2 = \frac{SO(3,2)}{SO(3) \times SO(2)}$, which is a Hermitian symmetric space, has a representation as a bounded domain of type IV, that is, as a connected component of $\mathcal{D} = \{ [\omega] \in \mathbb{P}(\Gamma \otimes \mathbb{C}) | \omega^2 = 0, \omega \cdot \bar{\omega} > 0 \} = \Lambda \otimes \mathbb{R} + iC(\Lambda) \subset \Lambda \otimes \mathbb{C}$, where $C(\Lambda) = \{ x \in \Lambda \otimes \mathbb{R} | x^2 < 0 \}$ (this last condition ensures again that $2ReT ReU - 2(ReV)^2 > 0$; the connected component can then be realized as $\mathcal{D}^+ = \Lambda \otimes \mathbb{R} + iC^+(\Lambda)$, where $C^+(\Lambda)$ denotes the future light cone component of $C(\Lambda)$).

Now in the basis $\varepsilon_1 = f_{-2} - f_2, \varepsilon_2 = f_3, \varepsilon_3 = f_2 - f_3$, $\Lambda$ is equivalent to the intersection matrix $A_{1,0} = \left( \begin{array}{ccc} 2 & 0 & -1 \\ -1 & -2 & -2 \\ 0 & 2 & 2 \end{array} \right)$ associated to the Siegel modular form $C_{35}$ of $[55]$. To each element $\varepsilon_i$, which squares to 2, is associated the Weyl reflection $s_i : x \rightarrow x - (x \cdot \varepsilon_i) \varepsilon_i$. The fixed loci of these Weyl reflections give the enhancement loci $[11]$. As these reflection planes are given by planes orthogonal to the elements $\varepsilon_i$, this gives rise to the following loci: the orthogonality conditions $(a \varepsilon_1 + b \varepsilon_2 + c \varepsilon_3) \varepsilon_i = 0$
yield \( c = 2a, b = c \) and \( a = 2(c - b) \). Since \( a, b \) and \( c \) are related to \( T, U \) and \( V \) by \( a = iT, b = iT + iU + iV \) and \( c = iT + iU \), as can be seen by comparing \( a\varepsilon_1 + b\varepsilon_2 + c\varepsilon_3 = a(f_2 - f_2) + b(f_3 + c) + c(f_2 - f_3) = af_2 + (c - a)f_2 + (b - c)f_3 \) with \( z = iTf_2 + iUf_2 - iVf_3 \), the above orthogonality conditions result in the enhancement loci \( T = U, V = 0 \) and \( T - 2V = 0 \). Note that these are the conditions for enhancement loci related to \( C_{35} = C_{30} \cdot C_5 \) (cf. Appendix A2). Also note that the locus \( T - 2V = 0 \) goes over into the locus \( T - U = 0 \) under the target space duality transformation \[ T \rightarrow T + U + 2V, U \rightarrow U, V \rightarrow V + U. \] Thus, the enhancement lines of the STU model have become the Humbert surfaces \( H_4 \) and \( H_1 \) (see the discussion about rational quadratic divisors given in ch. 5 of [70] \((s = 1)\) as well as in [65]).

### 3.2 Perturbative Corrections to the Prepotential

We use the perturbative quantum symmetries and the singularity structure corresponding to the gauge symmetry enhancement loci to determine the one-loop correction of the gauge couplings.

#### 3.2.1 STU model

Now \( h^{(1)}(T,U) \) does have singularities inside the fundamental domain. As mentioned there are additional (gauge neutral) massless abelian vector multiplets on the subspace \( T = U \), which induce a logarithmic singularity in the \( U(1) \) gauge couplings, which are related to \( h^{(1)} \) via its second derivatives. So the 1-loop prepotential \( h^{(1)} \) exhibits logarithmic singularities exactly at the lines (points) of the classically enhanced gauge symmetries and is therefore not a single valued function when transporting the moduli fields around the singular lines \( \mathbb{Z} \). The ratio of the number of additional massless states is \( 1 : 2 : 3 \), which leads in connection with the modular properties and the corresponding vanishing properties of the \( j \)-function at the special points to the occurrence of \( (r \in \mathbb{Z} \neq 0) \) a term \( \log(j(T) - j(U))^r \) (cf. [30]).

\( h^{(1)} \) had to be a modular form of weight \(-2\) if it were nowhere singular. In the presence of singularities one has to allow for integer ambiguities of the \( \theta \)-angles which results in

\[
h^{(1)}(T,U) \rightarrow \frac{h^{(1)}(T,U) + \Xi(T,U)}{(iT + d)^2}
\]

for an \( SL(2, \mathbb{Z})_T \) transformation. For \( SL(2, \mathbb{Z})_U T \) and \( U \) are interchanged in eq. \( (3.21) \). \( \Xi \), which is related to the monodromy around the enhancement loci, is an arbitrary quadratic polynomial in the variables \( (1, iT, iU, TU) \) and parameterizes the most general allowed ambiguities in the \( \theta \)-angles; so \( \Xi \) obeys \( \partial_T^2 \Xi = \partial_U^2 \Xi = 0 \). As said, for \( \Xi = 0 \) \( h^{(1)} \) is a modular form of weight \((-2, -2)\) with respect to \( SL(2, \mathbb{Z})_T \times SL(2, \mathbb{Z})_U \) but for non-zero \( \Xi \) it has no good modular properties; instead \( \partial_T^2 h^{(1)} \) is a single valued modular form of weight \((4, -2)\) and similarly \( \partial_U^2 h^{(1)} \) has weight \((-2, 4)\).
The singularity of the prepotential near $T = U$ can be determined by purely field theoretic considerations [102]. For the case at hand one finds [45, 102, 109]

$$h^{(1)}(T \sim U) = \frac{1}{16\pi^2}(T - U)^2 \log(T - U)^2 + \text{finite} \quad (3.22)$$

where the coefficient $\frac{1}{16\pi^2}$ is set by the $SU(2)$ $\beta$-function. More precisely one finds from the number of additional massless states the following behaviour of $2\pi F^1$ at the various enhancement loci

$$\begin{align*}
(T - U)^2 \log(T - U)^2 & \quad T = U \quad (3.23) \\
(T - 1)^2 \log(T - 1)^4 & \quad T = U = 1 \quad (3.24) \\
(T - \rho)^2 \log(T - \rho)^6 & \quad T = U = \rho. \quad (3.25)
\end{align*}$$

This can be summarised in a covariant way as

$$F^1_{TU} = -\frac{1}{4\pi^2} \log(j(iT) - j(iU)) + \text{finite} \quad (3.26)$$

which furthermore can be thought of as the sum over massive states in the internal line for the gauge coupling $F^1_{TU} = \log \det M = \sum \log \mathcal{M}_{TV}$ (cf. section 5).

Since the moduli dependence of the gauge coupling is related to the second derivative of $h$ and in the decompactification limit $(T, U \to \infty)$ the gauge coupling cannot grow faster than a single power of $T$ or $U$ one sees that $\partial_T \partial_U h^{(1)}$, $\partial_T^2 h^{(1)}$ and $\partial_U^2 h^{(1)}$ cannot grow faster than $T$ or $U$ in the decompactification limit. Hence one has the asymptotic behaviour $\partial^2 h^{(1)} \to \text{const.}$ for $T \to \infty$ (analogously with $T$ and $U$ interchanged). The properties of $h^{(1)}$ which we have assembled so far can be combined in the Ansatz

$$\partial^3_T h^{(1)} = \frac{X_{4,-2}(T, U)}{j(iT) - j(iU)}, \quad (3.27)$$

If one furthermore uses a factorisation ansatz for the numerator then its factors cannot have any pole inside the modular domain while for large $T, U$ they have - divided by the $j$ function - to go to a constant, which properties uniquely determine the factors to be $E_4(iT)$ and $\frac{E_4 E_6}{\eta^{24}}(iU)$, so

$$\partial^3_T h^{(1)} = \frac{1}{2\pi} \frac{E_4(iT) E_4(iU) E_6(iU)}{[j(iT) - j(iU)] \eta^{24}(iU)}, \quad (3.28)$$

where the coefficient is determined by eq. (3.22) or rather the $SU(2)$ $\beta$-function. The same analysis holds for $T$ and $U$ interchanged.

In addition to the transformation law of $h$ (eq. (3.21)) also the $N = 2$ dilaton is no longer invariant at the quantum level [45]. Instead, under an $SL(2, \mathbb{Z})_T$ transformation one finds

$$S \to S + \frac{1}{2} \partial_T \partial_U \Xi - \frac{i c \partial_U (h^{(1)} + \Xi)}{2(icT + d)} + \text{const.} \quad (3.29)$$
This result can be understood from the fact that in perturbative string theory the relation between the dilaton and the vector-tensor multiplet is fixed. However, the duality relation between the vector-tensor multiplet and its dual vector multiplet containing $S$ is not fixed but suffers from perturbative corrections in both field theory and string theory. Nevertheless, it is possible to define an invariant dilaton

$$S^{\text{inv}} = S - \frac{1}{2} \partial_T \partial_U h^{(1)} - \frac{1}{8\pi^2} \log[j(iT) - j(iU)].$$

The last term is added such that $S^{\text{inv}}$ is finite so that altogether $S^{\text{inv}}$ is modular invariant and finite. However, $S^{\text{inv}}$ it is no longer an $N = 2$ special coordinate.

The analysis just performed only determines the third derivatives of $h^{(1)}$ because these are modular forms. $h^{(1)}$ itself has been calculated in [70] by explicitly calculating the appropriate string loop diagram.

It was shown in [8, 46] that threshold corrections in $N = 2$ heterotic string compactifications can be written in terms of the supersymmetric index

$$\frac{1}{\eta^2} \text{Tr}_R F(-1)^F q^{L_0 - c/24} \bar{q}^{\bar{L}_0 - \bar{c}/24}. $$

This quantity is, as shown in [70], also related to the computation of the perturbative heterotic $N = 2$ prepotential. For the model with instanton number embedding $(d_1, d_2) = (0, 24)$, the supersymmetric index (3.31) was calculated in [70] and found to be equal to

$$\frac{1}{\eta^2} \text{Tr}_R F(-1)^F q^{L_0 - c/24} \bar{q}^{\bar{L}_0 - \bar{c}/24} = -2iZ_{2,2} \frac{E_4E_6}{\Delta},$$

where $Z_{2,2}$ denotes the sum over the Narain lattice $\Gamma^{2,2}$, $Z_{2,2} = \sum_{p \in \Gamma^{2,2}} q^{p^2} \bar{q}^{\bar{p}^2}$, and where $\frac{E_4E_6}{\Delta} = \sum_{n \geq 1} \tilde{c}_{STU}(n)q^n$. Here the subscript on the trace indicates the Ramond sector as right-moving boundary condition; $F$ denotes the right-moving fermion number, $F = F_R$.

Let us recall how this expression came about. First, one can reduce (3.31) to

$$\frac{1}{\eta^2} \text{Tr}_R (-1)^F q^{L_0 - c/24} \bar{q}^{\bar{L}_0 - \bar{c}/24},$$

where the contributions are weighted with $\pm 2\pi i$ depending on whether a BPS hyper- or vector multiplet contributes. The expression resulting from the evaluation of the trace consists of the product of three terms, namely of $Z_{2,2}/\eta^4$, of the partition function for the first $E_8^{(1)}$ in the bosonic formulation (leading to the contribution $E_4/E_8^2$), and of the elliptic genus for the second $E_8^{(2)}$ containing the gauge connection on $K3$. This last quantity now decomposes additively (taking into account the appropriate weightings) into contributions from the following sectors, namely: 1) the $(NS, R)$ sector, which we will also denote by $(NS^+, R)$, 2) the “twisted” sector $(NS^-, R)$, where a factor $(-1)^F_L$ is inserted in the trace (this contribution is weighted with $(-1)$) and 3) the $(R, R)$ sector, which we will also denote by $(R^+, R)$. Since we are using the fermionic representation for $E_8^{(2)}$, we decompose the fermionic $D_8^{(2)} \subset E_8^{(2)}$, so that each of these contributions splits again multiplicatively into a free $D_6^{(2)}$ part and a $D_2$ part, to be
called $D_2^{(2)}K3$, containing the gauge connection $A_1$ which describes the corresponding gauge bundle on $K3$. The corresponding contributions are summarized in the following table, where we also indicate the connection to the generic elliptic genus

$$Z(\tau, z) = \text{Tr}_{R, R} y^{F_L} (-1)^{F_L + F_R} q^{L_o - c/24} q^{L_o - c/24} = \exp \frac{\eta^2 \theta_1^2 \theta_2^2 (\tau, z)}{\eta^4} - \exp \frac{\eta^3 \theta_1^3 \theta_2^3 (\tau, z)}{\eta^4}, \text{ where } y = \exp(2\pi i z) \text{ (see [54, 80]).}$$

|             | $D_6$ | $K3D_2$         |
|-------------|-------|-----------------|
| $(NS^+, R)$ | $\frac{\theta_1^2}{\eta^2}$ | $-2\frac{\theta_1^2 \theta_2^2}{\eta^4} = q^{\frac{1}{2}} Z(\tau, \frac{\tau + 1}{2})$ |
| $(NS^-, R)$ | $\frac{\theta_2^2}{\eta^2}$ | $2\frac{\theta_1^2 \theta_2^2}{\eta^4} = q^{\frac{1}{2}} Z(\tau, \frac{\tau}{2})$ |
| $(R^+, R)$  | $\frac{\theta_1^3}{\eta^3}$ | $2\frac{\theta_1^3 \theta_2^3}{\eta^6} = Z(\tau, \frac{\tau}{2})$ |
| $(R^-, R)$  | $\frac{\theta_2^3}{\eta^3}$ | $6\frac{\theta_1^3 \theta_2^3}{\eta^6} = 24 = Z(\tau, 0)$ |

Now recall that $E_4$ and $E_6$ have the following \(\theta\)-function decomposition:

\[
2E_4 = \theta_1^2 \cdot \theta_2^2 + \theta_2^3 \cdot \theta_2^2 + \theta_3 \cdot \theta_1^2 + \theta_1^3 \cdot \theta_2^3
\]

\[
2E_6 = -\theta_1^2 (\theta_2^3 + \theta_2^3) \cdot \theta_2^2 + \theta_2^3 (\theta_2^3 - \theta_2^3) \cdot \theta_2^3 + \theta_3 (\theta_2^3 + \theta_3) \cdot \theta_2^3
\]  \hspace{1cm} (3.33)

the \(\theta_i^2\) contributions \((i = 2, 3, 4)\) are due to the $SO(4)$ piece in the fermionic decomposition of $E_8 \supset SO(12) \times SO(4)$. Hence the sum of the three non-vanishing terms in the table precisely leads to (3.32).

On the other hand, in the case of a general \((d_1, d_2)\) embedding (using now a fermionic representation for both $E_8$’s), one first has to decompose the $D_2^{(1)}K3D_2^{(2)}$ part into $D_2^{(1)}K3 \times D_2^{(2),\text{free}} + D_2^{(1),\text{free}} \times K3D_2^{(2)}$, where the factors in each summand are now in different, and hence commuting, $E_8$’s. Furthermore, since the rudimentary $K3$ gauge bundles are structurally completely the same as before, the amount of contribution realized by them can — by comparison with the “complete” $K3$ bundle considered above — be read off from the $R^-$ sector. Note that $Z(\tau, 0)$ is the Witten index, which gives the Euler number of $K3$ resp. the second Chern class of the relevant vector bundle.

This results in a contribution proportional to

\[
\frac{1}{\Delta} (d_4 E_6 \cdot E_4 + E_4 \cdot \frac{d_2}{24} E_6) = \frac{1}{\Delta} E_4 E_6 = \sum_{n \geq -1} c_{STU}(n) q^n =: f_{STU}(q) \hspace{1cm} (3.34)
\]

which is independent of the particular instanton embedding.

Via an explicit string 1-loop computation involving the quantity

\[
I \sim \int \frac{d^2 \tau}{\tau_2} \left[ \sum_{p \in \Gamma^{2,2}} q^\frac{k_1^2}{2} q^\frac{k_2^2}{2} \right] f_{STU}(q)
\]  \hspace{1cm} (3.35)

the semiclassical prepotential can then be written in the following explicit form \cite{70} \((y = (T, U), \text{ cf. Appendix A1})\)
\[ F = -STU + \frac{1}{384\pi^2} \gamma_{ABC}^2 y^A y^B y^C \]
\[ -\frac{1}{(2\pi)^4} \sum_{k,l \geq 0} c_1(kl) Li_3(e^{-2\pi(kT+\text{iu}T)}) - \frac{1}{(2\pi)^4} Li_3(e^{-2\pi(T-U)}), \quad (3.36) \]

\( F \) has a branch locus at \( T = U \). \( F \) given in (3.36) is defined in the fundamental Weyl chamber \( T > U \) (meaning that the real part of \( T \) is larger than the real part of \( U \)). The cubic coefficients \( \gamma_{ABC}^2 \) will be determined below. We have ignored a possible constant term as well as a possible additional quadratic polynomial in \( T \) and \( U \). The cubic terms cannot be uniquely fixed, since the prepotential contains an ambiguity \([6, 45]\) which is a quadratic polynomial in the period vector \((1, T, U, TU)\). Hence, the ambiguity is at most quartic in the moduli and at most quadratic in \( T \) and \( U \). It follows that the third derivative in \( T \) or in \( U \) is unique; \( \frac{\partial^2 h}{\partial T \partial U} \), however, is still ambiguous. Specifically, in the chamber \( T > U \), the cubic terms have the following general form \([7, 15]\)

\[ d_{ABC}^2 y^A y^B y^C = -32\pi \left( 3(1 + \beta)T^2U + 3\alpha TU^2 + U^3 \right). \quad (3.37) \]

The cubic term in \( U \) is unique, whereas the parameters \( \alpha \) and \( \beta \) correspond to the change induced by adding a quadratic polynomial in \((1, T, U, TU)\). So with our later favoured choice \( \alpha = 0, \beta = -1 \) the cubic term will come out as \(-\frac{1}{12\pi}U^3\). As discussed it is convenient to introduce a dilaton field \( S^{inv} \), which is invariant under the perturbative \( T \)-duality transformations at the one-loop level. It is defined as follows

\[ S^{inv} = S - \frac{1}{2} \frac{\partial h^{(1)}}{\partial T \partial U} - \frac{1}{8\pi^2} \log(j(T) - j(U)) \]
\[ = S + \frac{1}{4\pi} (1 + \beta)T + \frac{\alpha}{4\pi} U + \frac{1}{8\pi^2} \sum_{k,l \geq 0} kl c_1(kl) Li_1(e^{-2\pi(kT+\text{iu}T)}) \]
\[ -\frac{1}{8\pi^2} Li_1(e^{-2\pi(T-U)}) - \frac{1}{8\pi^2} \log(j(iT) - j(iU)). \quad (3.38) \]

In the decompactification limit to \( D = 5 \) \([7]\), obtained by sending \( T, U \to \infty \) \((T > U)\), the invariant dilaton \( S^{inv} \) has a particularly simple dependence on \( T \) and \( U \). Namely, by using \( \log j(T) \to 2\pi T \), one obtains that

\[ S^{inv} \to S^{inv}_\infty = S + \frac{\beta}{4\pi} T + \frac{\alpha}{4\pi} U, \]
\[ (3.39) \]

which gives for the mentioned later favoured choice \( \alpha = 0, \beta = -1 \)

\[ S^{inv}_\infty = S - \frac{1}{4\pi} T. \]
\[ (3.40) \]

Substituting \( S^{inv}_\infty \) back into the heterotic prepotential (3.36) yields that

\[ F = -S^{inv}_\infty TU - \frac{1}{12\pi}U^3 - \frac{1}{4\pi} T^2U - \frac{1}{(2\pi)^4} \sum_{k,l \geq 0} c_1(kl) Li_3(e^{-2\pi(kT+\text{iu}T)}) \]
\[ -\frac{1}{(2\pi)^4} Li_3(e^{-2\pi(T-U)}). \quad (3.41) \]
Note that the ambiguity in $\alpha$ and $\beta$ is hidden away in $S^{inv}_\infty$. Let us make some remarks on a specific symmetry occurring here. The string–string duality\[^5\] in $d = 6$ which is closely related to the duality of the heterotic string on $T^4$ and the type IIA on $K3$ manifests itself also in $K3$ compactifications of the heterotic string.

For $n_1 = n_2 = 12$ there is no strong coupling singularity so that this class of heterotic vacua is well defined for all values of the dilaton. It has been shown that there is a self-duality among the $n_1 = n_2 = 12$ heterotic vacua \[^{15, 52, 94}\]. More precisely, the theory is invariant under

\[
D \rightarrow -D, \quad g_{\mu\nu} \rightarrow e^{-D} g_{\mu\nu},
H \rightarrow e^{-D} * H, \quad X_4 \leftrightarrow \tilde{X}_4
\]

(3.42)

if in addition perturbative gauge fields are replaced by non-perturbative gauge fields and the hypermultiplet moduli spaces are mapped non-trivially onto each other; note that for $n_1 = n_2 = 12$ there is no strong coupling singularity so that this class of heterotic vacua is well defined for all values of the dilaton\[^4\].

### 3.2.2 ST model

Now one has (let $\tilde{S} = 4\pi S$, so $q_i \tilde{S} = e^{-8\pi^2 S}$)

\[
\mathcal{F} = \frac{1}{2} S T^2 + h(T) + h^{np}(e^{-8\pi^2 S}, T).
\]

(3.43)

The T duality group $Sl_2(\mathbb{Z})_T$ acts as $iT \rightarrow \frac{aiT + b}{ciT + d}$ with

\[
h \rightarrow \frac{h(T) + \Xi(T)}{(ciT + d)^4},
\]

(3.44)

where the quartic polynomial $\Xi$ arises from the multivaluedness of $h(T)$, i.e. in the absence of the logarithmic singularities $h$ would be a modular form of weight $-4$. $\partial^2_T h$ is then a true modular form of weight $6$ and the $U(1)_T$ factor of the gauge group $U(1)_T \times U(1)_S \times U(1)_\gamma$ is at $iT = i$ enhanced: $U(1)_T \rightarrow SU(2)_T$, so

\[
\partial^2_T h = -\frac{b_{SU(2)}}{8\pi^2} \log(iT - i) + \text{finite} = \frac{2}{4\pi^2} \log(iT - i) + \text{finite},
\]

(3.45)

\[^4\]This duality requires the existence of non-perturbative gauge fields with properties reminding one of $SO(32)$ heterotic strings. This posed a slight puzzle since for $E_8 \times E_8$ vacua the singularity of small instantons is caused by a non-critical tensionless string rather than additional massless gauge bosons. However, by mapping the $n_1 = n_2 = 12$ heterotic vacuum to a particular type I vacuum \[^{11}\] which indeed does have non-perturbative gauge fields this issue has been resolved and the gauge fields are shown to exist also for this class of heterotic vacua \[^{19}\].
which gives\(^5\) by modularity \((j(iT) - j(i))\) vanishes to second order)

\[
\partial_T^3 h = \frac{1}{4\pi^2} \log(j(iT) - j(i)) + \text{finite.} \tag{3.46}
\]

Also one defines again an invariant dilaton

\[
S^{\text{inv}} = S = \frac{1}{3} (\partial_T^2 h - \frac{1}{4\pi^2} \log(j(iT) - j(i)). \tag{3.47}
\]

### 3.2.3 STUV model

In the presence of a Wilson line, which we will take to lay in the second \(E_8^{(2)}\), the symmetry between the two \(E_8\)'s is broken and thus, contrary to the three-parameter case, the prepotential will already depend perturbatively on the type \((d_1, d_2)\) of the instanton embedding (we take \(d_2 \geq d_1\)).

The supersymmetric index \([3.31]\) will now have the form

\[
\frac{1}{\eta^2} \text{Tr}_R F(-1)^F q^{L_0-c/24} q^{L_0-\epsilon/24} = -2i Z_{3.2}(\tau, \bar{\tau}) \ast F(\tau), \tag{3.48}
\]

where \([3.31, 3.33]\)

\[
Z_{3.2}(\tau, \bar{\tau}) \ast F(\tau) = \left( \sum_{b \text{ even}} q^{\frac{p_1^2}{2}} q^{\frac{p_2^2}{2}} \right) F_0(\tau) + \left( \sum_{b \text{ odd}} q^{\frac{p_1^2}{2}} q^{\frac{p_2^2}{2}} \right) F_1(\tau). \tag{3.49}
\]

Here, \(F(\tau) = F_0(\tau) + F_1(\tau)\), and \(Z_{3.2}\) denotes the sum over the Narain lattice \(\Gamma_{3.2}\),

\[
Z_{3.2} = \sum_{p \in \Gamma_{3.2}} q^{\frac{p_1^2}{2}} q^{\frac{p_2^2}{2}} = (\sum_{b \text{ even}} + \sum_{b \text{ odd}}) q^{\frac{p_1^2}{2}} q^{\frac{p_2^2}{2}}.
\]

The presence of the Wilson line in \(E_8^{(2)}\) has the following effect on the \(\theta_1^2\) pieces appearing in the decomposition \([3.33]\) of \(E_4\) and \(E_6\)

\[
2 E_{4,1}(\tau, z) = \theta_2^1 \cdot \theta_2^2(\tau, z) + \theta_3^6 \cdot \theta_3^2(\tau, z) + \theta_4^4 \cdot \theta_4^2(\tau, z), \tag{3.50}
\]

\[
2 E_{6,1}(\tau, z) = -\theta_2^6(\theta_4^3 + \theta_4^1) \cdot \theta_2^2(\tau, z) + \theta_3^3(\theta_4^4 - \theta_2^2) \cdot \theta_3^2(\tau, z) + \theta_4^6(\theta_4^1 + \theta_3^4) \cdot \theta_4^2(\tau, z), \tag{3.51}
\]

where

\[
\begin{align*}
\theta_2^1(\tau, z) &= \theta_2(2\tau) - \theta_3(2\tau), \\
\theta_2^2(\tau, z) &= \theta_2(2\tau) + \theta_3(2\tau), \\
\theta_3^2(\tau, z) &= \theta_3(2\tau) + \theta_2(2\tau), \\
\theta_4^2(\tau, z) &= \theta_3(2\tau) - \theta_2(2\tau).
\end{align*}
\]

\(^5\)One can also go on in an analogous way as for the \(STU\) model now \([3.2]\). But as, in contrast to the \(STUV\) model with its very special function theory, the \(ST\) and \(STU\) models have a lot of the formalism in common we will take in discussing both of these models two different attitudes: whereas in the \(ST\) model we will read of the mirror map from the instanton expansion in closed form - as \(1/i\) - and then go on with closed expressions in the business of comparing couplings between the heterotic and the type II side, by contrast - to change methods - in the case of the \(STU\) model we will go on the heterotic side to the fully (and not only up to integrations to be made) solved prepotential with its coefficients (in front of the trilogarithm terms) of the specifying modular form - the supersymmetric index - being then later compared to the explicit instanton numbers on the type IIA CY.
are the two $SU(2)$ characters of the surviving $A_1$ when written in the boundary condition picture instead of the usual conjugacy class picture. We refer to appendices A.5 and A.6 for a description and interpretation of the hatting procedure.

The replacement $E_4 \to \hat{E}_{4,1}$, in particular, amounts to replacing the $E_8$ partition function $P_{E_8} = P_{E_7}^{(0)} \cdot P_{A_1}^{(0)} + P_{E_7}^{(1)} \cdot P_{A_1}^{(1)}$ with $P_{E_7}^{(0)} + P_{E_7}^{(1)}$. This precisely describes the breaking of the $E_8^{(2)}$ to $E_7^{(2)} \times U(1)$ when turning on a Wilson line.

Thus, the effect of turning on a Wilson line can be described as follows. Introducing $E$ the breaking of the tations to the heterotic prepotential $P$

\[ A_{\tau} = \sum_{n \in \mathbb{Z}, \tau + \frac{3}{4}} c_n(4N)q^n \]

as well as

\[ F_0(\tau) = \sum_{n \in \mathbb{Z}} c_n(4N)q^n \]

\[ F_1(\tau) = \sum_{n \in \mathbb{Z} + \frac{1}{4}} c_n(4N)q^n \]

it follows that turning on a Wilson line results in the replacement

\[ Z_{2,2} \frac{1}{\Delta} E_4 E_6 \to Z_{3,2} \ast F = \left( \sum_{b \text{ even}} q^{b_2} \hat{q}^{b_2} \right) F_0 + \left( \sum_{b \text{ odd}} q^{b_2} \hat{q}^{b_2} \right) F_1 \]

The product $\tau_2 Z_{3,2} \ast F$ is invariant under modular transformations $[31, 35]$. As discussed in the previous section, the supersymmetric index is given in terms of

\[ F(\tau) = A_n = \sum_{n \in \mathbb{Z}, \tau + \frac{3}{4}} c_n(4N)q^n \]

As explained in appendix A3, the modular function $A_n(\tau)$ is in one-to-one correspondence with the index-one Jacobi form with the same expansion coefficients $c_n(k, b) = c_n(4k - b^2)$: $A_n(\tau) = A_n(\tau, z)$, $A_n(\tau, z) = \frac{1}{\Delta(\tau)} \left( \frac{d_{11}}{24} E_6(\tau) \cdot E_{4,1}(\tau, z) + E_4(\tau) \cdot \frac{d_{11}}{24} E_{6,1}(\tau, z) \right) = \sum_{k, b} c_n(4k - b^2)q^k \hat{q}^b$. (The first few expansion coefficients of $A_0, A_1, A_2$ and $A_12$ are listed in the second table in appendix A7)

The expansion coefficients $c_n(4N)$ of $F(\tau)$ govern the perturbative, i.e. one-loop, corrections to the heterotic prepotential $F_0^{\text{het}}[74]$. For the class of $STUV$ models considered here, the perturbative heterotic prepotential is given by

\[ F_0^{\text{het}} = -S(TU - V^2) + p_n(T, U, V) - \frac{1}{4\pi^2} \sum_{k, l, b \in \mathbb{Z}, \langle k, l, b \rangle > 0} c_n(4kl - b^2)Li_3(e^{[kiT + liU + biV]}) \]

\[ F_0^{\text{het}} = -S(TU - V^2) + p_n(T, U, V) - \frac{1}{4\pi^2} \sum_{k, l, b \in \mathbb{Z}, \langle k, l, b \rangle > 0} c_n(4kl - b^2)Li_3(e^{[kiT + liU + biV]}) \]

\[ F_0^{\text{het}} = -S(TU - V^2) + p_n(T, U, V) - \frac{1}{4\pi^2} \sum_{k, l, b \in \mathbb{Z}, \langle k, l, b \rangle > 0} c_n(4kl - b^2)Li_3(e^{[kiT + liU + biV]}) \]

\[ F_0^{\text{het}} = -S(TU - V^2) + p_n(T, U, V) - \frac{1}{4\pi^2} \sum_{k, l, b \in \mathbb{Z}, \langle k, l, b \rangle > 0} c_n(4kl - b^2)Li_3(e^{[kiT + liU + biV]}) \]

\[ F_0^{\text{het}} = -S(TU - V^2) + p_n(T, U, V) - \frac{1}{4\pi^2} \sum_{k, l, b \in \mathbb{Z}, \langle k, l, b \rangle > 0} c_n(4kl - b^2)Li_3(e^{[kiT + liU + biV]}) \]
where $e[x] = \exp 2\pi ix$. The first term $-S(TU - V^2)$ is the tree-level prepotential of the special Kähler space $SO(3,2)/SO(3) \times SO(2)$; $p_n(T, U, V)$ denotes the one-loop cubic polynomial that depends on the particular instanton embedding $n$

$$p_n(T, U, V) = -\frac{1}{12\pi} U^3 + \frac{1}{4\pi} \left[ (\frac{n_1}{2} - 6)TV^2 + (\frac{n_1}{2} - 5)UV^2 - (\frac{4}{3} + n_1 - 12)V^3 \right] \quad (3.57)$$

The condition $(k, l, b) > 0$ means that: either $k > 0, l, b \in \mathbb{Z}$ or $k = 0, l > 0, b \in \mathbb{Z}$ or $k = l = 0, b < 0$ (cf. [70]). Next, consider truncating an $STUV$ model to the $STU$ model by setting $V = 0$. Then, the sum over $b$ in (3.56) yields independently from $n$ the coefficients of the three-parameter model,

$$c_{STU}(kl) = \sum_b c_n(4kl - b^2) \quad (3.58)$$

as is obvious by the dehatting procedure and can be checked by explicit comparison. Therefore the prepotential (3.56) truncates correctly to the prepotential for the $STU$ model.

The (Wilsonian) Abelian gauge threshold functions are related (see [45] for details) to the second derivatives of the one-loop prepotential $h(T, U, V) = p_n(T, U, V) - \frac{1}{4\pi} \sum_{(k, l, b) > 0} c_n(4kl - b^2) Li_3(e^{kiT + liU + biV})$. At the loci of enhanced non-Abelian gauge symmetries, some of the Abelian gauge couplings will exhibit logarithmic singularities due to the additional massless states. First, consider $\partial_T \partial_U h$. At the line $T = U$ one $U(1)$ is extended to $SU(2)$ without additional massless hypermultiplets. It can be easily checked that, as $T \to U$,

$$\partial_T \partial_U h = -\frac{1}{\pi} \log(T - U) \quad (3.59)$$

as it should. The Siegel modular form, which vanishes on the $T = U$ locus and has modular weight 0, is given by $\frac{C^2_{30}}{C^2_{12}}$. It can be shown that, as $V \to 0$,

$$\frac{C^2_{30}}{C^2_{12}} \to (j(T) - j(U))^2 \quad (3.60)$$

up to a normalization constant. Hence one deduces that

$$\partial_T \partial_U h = -\frac{1}{2\pi} \log \frac{C^2_{30}}{C^2_{12}} + \text{regular.} \quad (3.61)$$

On the other hand, at the locus $V = 0$, a different $U(1)$ gets enhanced to $SU(2)^{(2)}$, and at the same time $n'_H$ hypermultiplets, being doublets of $SU(2)^{(2)}$, become massless. Using eq. (3.20), $n'_V = 2$ and that $2c_n(-1) = -n'_H, 2c_n(-4) = n'_V$, it can be checked that, as $V \to 0$,

$$-\frac{1}{4} \partial_V^2 h = \frac{3}{2\pi} (2 + n) \log V = -\frac{1}{\pi} \left( 1 - \frac{1}{8} n'_H \right) \log V \quad (3.62)$$
Observe that the factor \((1 - \frac{1}{8} n'_{H})\) is precisely given by the \(N = 2\) \(SU(2)\) gauge \(\beta\)-function coefficient with \(n'_{H}/2\) hypermultiplets in the fundamental representation of \(SU(2)\). The Siegel modular form, which vanishes on the \(V = 0\) locus and has modular weight 0, is given by \(\frac{C_5}{C_{12}^{5/12}}\). It can be shown that, as \(V \to 0\),

\[
C_5 \to V (\Delta(T)\Delta(U))^{\frac{1}{2}}, \tag{3.63}
\]

So we now conclude that

\[
- \frac{1}{4} \partial_{V}^{2} h = \frac{3}{4\pi} (2 + n) \log \left( \frac{C_5}{C_{12}^{5/12}} \right)^{2} + \text{regular}. \tag{3.64}
\]
3.3 Higher gravitational couplings

Up to now we concentrated on the gauge couplings, which are lowest order (two derivatives) couplings, because of their special analytic properties. In addition, there is a class of higher derivative curvature terms in the effective action \[3, 21, 22\] whose couplings \(g_n\) are also determined by holomorphic functions \(F_n(S, M^i)\) of the vector moduli. The prepotential \(F\) as well as these higher derivative couplings \(F_n\) arise from chiral integrals (F-terms) in \(N = 2\) superspace. They occur at 1-loop on the heterotic side; on the type II side they arise at \(n\)-loop level, each representing a topological partition function of the twisted Calabi-Yau sigma model. The higher derivative couplings of vector multiplets \(X\) to the Weyl multiplet \(W\) of conformal \(N = 2\) supergravity can be expressed as a power series \[11\]

\[
F(X, W^2) = \sum_{g=0}^{\infty} F_g(X)(W^2)^g.
\]

(3.65)

Specifically for \(n = 1\) we will consider a term (\(R\) is the Riemann tensor)

\[
\mathcal{L} \sim g_1^{-2} R^2 + \ldots,
\]

(3.66)

of almost harmonic coupling \(g_1\) with

\[
g_1^{-2} = \text{Re} F_1(S, M^i) + A_1.
\]

(3.67)

At tree level \(A_1 = 0\) thus \(g_1^{-2}\) is a harmonic function. However exactly as for the gauge coupling the \(g_1^{-2}\) cease to be harmonic as soon as quantum corrections are included, i.e. a holomorphic anomaly \(A_1 \neq 0\) is induced \[20\].

Thus, one has analogously

\[
F_1 = F_1^{(0)}(S, M^i) + F_1^{(1)}(M^i) + F_1^{(np)}(e^{-8\pi^2 S}, M^i).
\]

(3.68)

Furthermore in a convenient normalization

\[
F_1^{(0)} = 24S.
\]

(3.69)

At 1-loop one has

\[
g_1^{-2} = \text{Re} F_1 + \frac{b_{\text{grav}}}{16\pi^2}(\log \frac{M_{Pl}^2}{p^2} + K),
\]

which leads to the anomaly

\[
\partial_i \bar{\partial}_j g_1^{-2} = \frac{b_{\text{grav}}}{16\pi^2} \partial_i \bar{\partial}_j K,
\]

(3.70)

where \(b_{\text{grav}} = 2(n_H - (n_V - 1) + 22) = 2((n_H - 1) - n_V + 24) = 48 - \chi\) is the 1-loop coefficient of the 'gravitational' beta-function.

\(\chi\) the Euler characteristic of the dual Calabi-Yau.
3.3.1 ST model

One has

\[ F_1 = 24S + h_1(T) + \text{non-pert.} \] (3.71)

and at \( T \sim 1 \) the singular contribution to \( h_1(T) \) coincides with the one for \( \partial_T^2 h \) as no additional gauge singlets become massless

\[ h_1 = \frac{1}{4\pi^2} \log(j(iT) - j(i)) + \text{finite.} \] (3.72)

On the other hand the modular transformation properties of \( h_1 \) follow from the holomorphic/modular anomaly of this coupling, namely

\[ g_{1}^{-2} = \text{Re}F_1 + \frac{b_{\text{grav}}}{16\pi^2}(\log \frac{M_{Pl}^2}{p^2} + K(S, T)) \] (3.73)

has to be modular invariant. Here one has contrary to the tree level Kaehler potential \( e^{-K} = (S + \bar{S})(T + \bar{T})^2 \) that at one loop

\[ e^{-K} = (S + \bar{S})(T + \bar{T})^2 - (4(h + \bar{h}) - 2(T + \bar{T})(\partial_T h + \partial_{\bar{T}} \bar{h})) \] (3.74)

i.e.

\[ K = -\log(S + \bar{S} - V_{\text{GS}}) - \log(T + \bar{T})^2 \] (3.75)

with the Green-Schwarz term \( V_{\text{GS}} = \frac{4(h + \bar{h}) - 2(T + \bar{T})(\partial_T h + \partial_{\bar{T}} \bar{h})}{(T + \bar{T})^2} \).

Now with the compensation ansatz

\[ F_1 = 24S^{\text{inv}} + \frac{1}{4\pi^2} \log(j(iT) - j(i)) + \frac{x}{4\pi^2} \log \frac{1}{\eta^2(iT)} \] (3.76)

one gets from the invariance requirement for \( \text{Re}F_1 + \frac{b_{\text{grav}}}{4\pi^2} \frac{1}{4} K \) in view of the facts that \( \frac{1}{4} K = \log(2\text{Im}(iT))^{-1/2} + \text{stuff} \) and \( \text{Im}(iT) \) transforms as absolute value of a modular form of weight \(-2\) that \( x = b_{\text{grav}} \), i.e.

\[ F_1 = 24S^{\text{inv}} + \frac{1}{4\pi^2} \log(j(iT) - j(i)) + \frac{b_{\text{grav}}}{4\pi^2} \log \frac{1}{\eta^2(iT)} \] (3.77)
3.3.2 STU model

At the one-loop level $F_1$ is given by \[30, 41, 78\]
\[
F_1 = 24 S^{inv} + \frac{1}{4\pi^2} \log(j(iT) - j(iU))^2 + \frac{1}{4\pi^2} b_{grav} \log \frac{1}{\eta^2(iT)\eta^2(iU)}. \tag{3.78}
\]

Note that the square in $(j(iT) - j(iU))^2$ comes from the fact that now the vanishing of $j(iT) - j(iU)$ in $T - U$ is of first order in contrast to the situation in the $ST$ model where $j(iT) - j(i)$ vanished in $iT - i$ to the second order. Furthermore the relative $\frac{1}{2}$ factor in front of $b_{grav}$ comes from the fact that it would on the diagonal exponentiate the term $\eta^2\eta^2$.

For the model we are discussing one has that $b_{grav} = 48 - \chi = 528$. Later we will see that the one-loop $F_1$ can be written as a sum over the $N = 2$ BPS states.

For later use let us also rewrite the result in the variable $\widetilde{y} = e^{-8\pi^2 S^{inv}}$

\[
F_1 = \frac{6}{4\pi^2} \log[\widetilde{y}^{-2} (j(iT) - j(iU))^{2/6} (\eta^2(iT)\eta^2(iU))^{1/6} b_{grav}]. \tag{3.79}
\]

Inserting $S^{inv}$ given in \((3.38)\) into $F_1$ yields \[27\]
\[
F_1 = 24 \left( S - \frac{1}{768\pi^2} \partial_T \partial_U \left( \rho^a_{abc} y^b y^c \right) - \frac{1}{8\pi^2} \log(j(T) - j(U)) \right.
\]
\[
+ \frac{1}{8\pi^2} \sum_{k,l \geq 0} klc_1(kl) Li_1(e^{-2\pi(kT+U)}) - \frac{1}{8\pi^2} Li_1(e^{-2\pi(T-U)}) \bigg)
\]
\[
+ \frac{1}{4\pi^2} \log(j(T) - j(U))^2 + \frac{b_{grav}}{8\pi^2} \log \frac{1}{\eta^2(T)\eta^2(U)} \bigg)
\]
\[
= 24S + 12 + 6\beta \frac{T}{\pi} + 11 + 6\alpha \frac{U}{\pi}
\]
\[
- \sum_{k,l} \left( \frac{3}{\pi^2} klc_1(kl) - \frac{5}{2\pi^2} c(kl) \right) \log(1 - e^{-2\pi(kT+U)})
\]
\[
+ \frac{1}{4\pi^2} \log(j(T) - j(U))^2 + \frac{b_{grav}}{8\pi^2} \log \frac{1}{\eta^2(T)\eta^2(U)}. \tag{3.80}
\]

Thus, $F_1$ is essentially determined by two types of coefficients of modular functions, namely $c_1(n)$ and $c(n)$.

At last let us for convenience of extending the preceding results to the $STUV$ model

First the perturbative Wilsonian gravitational coupling for the $STU$ model is given by\[8\] (in the chamber $T > U$)
\[
F_1 = 24 S^{inv} - \frac{b_{grav}}{\pi} \log \eta(T)\eta(U) + 2 \frac{1}{\pi} \log(j(T) - j(U)) \tag{3.81}
\]

\[8\] For convenience of notation involving the quantity $\tilde{S}$ we will denote here and in the $STUV$ subsections (3.3.3) and (3.2.3) $\tilde{S}$ already by $S$, i.e. the dilaton is defined to be $S = 4\pi/g^2 - i\theta/2\pi$. 


Using that \[ S^{inv} = \tilde{S} + \frac{1}{8} L , \]
\[ \tilde{S} = S - \frac{1}{2} \partial_T \partial_U h , \quad L = -\frac{4}{\pi} \log(j(T) - j(U)) , \]
it follows that \( F_1 \) can be rewritten as
\[ F_1 = 24 \tilde{S} - \frac{1}{\pi} \left[ 10 \log(j(T) - j(U)) + b_{grav} \log \eta(T) \eta(U) \right] . \] (3.83)

The perturbative gravitational coupling is related to the perturbative Wilsonian coupling by
\[ \frac{1}{g_{grav}^2} = \text{Re} F_1 + \frac{b_{grav}}{4\pi} K = 12(S + \tilde{S} + V_{GS}) + \Delta_{grav} . \] (3.84)

This relates the Wilsonian gravitational coupling \( F_1 \) to the supersymmetric index, that is to \( \Delta_{grav} = -\frac{2}{4\pi} \tilde{I}_{2,2} [27] \), where \( [70] \) (note that \( b_{grav} = 48 - \chi = -2(c_1(0) - 24) = -2\tilde{c}_1(0) \))
\[ \tilde{I}_{2,2} = \frac{1}{d} \int \frac{d^2\tau}{\tau_2} \left[ Z_{2,2} E_4 E_6 \left( E_2 - \frac{3}{\pi \tau} \right) - \tilde{c}_1(0) \right] . \] (3.85)

It follows from (3.84) that
\[ F_1 = 24S - \frac{2}{\pi} \sum_{r > 0} \tilde{c}_1 \left( -\frac{r^2}{2} \right) L_{t_1} \]
\[ = 24 \tilde{S} - \frac{1}{\pi} \left[ 10 \log(j(T) - j(U)) + b_{grav} \log \eta(T) \eta(U) \right] . \] (3.86)

Here, we have ignored the issue of ambiguities in (3.86) linear in \( T \) and in \( U \).

### 3.3.3 \textit{STUV} model

The classical moduli space of a heterotic \textit{STUV} model is locally given by the Siegel upper half plane \( \mathcal{H}_2 = \frac{SO(3,2)}{SO(3) \times SO(2)} \). Because of target space duality invariance, one has to consider modular forms on \( \mathcal{H}_2 \), i.e. Siegel modular forms (cf. appendix A).

The Siegel modular form, which vanishes on the \( T = U \) locus and has modular weight 0, is given by \( \frac{C_{30}^2}{C_{12}^5} \). It can be shown that, as \( V \to 0 \),
\[ \frac{C_{30}^2}{C_{12}^5} \to (j(T) - j(U))^2 , \] (3.87)

up to a normalization constant. On the other hand, the Siegel modular form, which vanishes on the \( V = 0 \) locus and has modular weight 0, is given by \( \frac{C_5}{C_{12}^{7/2}} \). It can be shown that, as \( V \to 0 \),
\[ C_5 \to V (\Delta(T)\Delta(U))^{\frac{1}{2}} , \] (3.88)
3 \ N = 2 \ HETEROTIC \ VACUA

up to a proportionality constant. Finally, the Siegel form \( C_{12} \) generalizes \( \Delta(T)\Delta(U) \), that is

\[
C_{12} \rightarrow \Delta(T)\Delta(U)
\]  

(3.89)
as \( V \rightarrow 0 \).

Then, in analogy to (3.81), the perturbative Wilsonian gravitational coupling for an \( STUV \) model is now given by (in the chamber \( T > U \))

\[
F_1 = 24S^{inv} - \frac{b_{grav}}{24\pi} \log C_{12} + \frac{1}{\pi} \log \frac{C_{30}^2}{C_{12}^5} - \frac{1}{2\pi} \left( n_H' - n_V' \right) \log \left( \frac{C_5}{C_{12}^{5/12}} \right)^2 .
\]  

(3.90)

Here, \( n_V' \) and \( n_H' \) denote the vector and the hypermultiplets, which become massless at the \( V = 0 \) locus. Since at \( V = 0 \) there is a gauge symmetry restoration \( U(1) \rightarrow SU(2) \), we have \( n_V' = 2 \).

The invariant dilaton \( S^{inv} \) is given by \[15\]

\[
S^{inv} = \bar{S} + \frac{1}{10} L ,
\]

\[
\bar{S} = S - \frac{4}{10} \left( \partial_T \partial_U - \frac{1}{4} \partial_V^2 \right) h ,
\]  

(3.91)

where the role of the quantity \( L \) is to render \( S^{inv} \) free of singularities. Using eqs. (3.61) and (3.64), it follows that

\[
\bar{S} = S + \frac{1}{5\pi} \log \frac{C_{30}^2}{C_{12}^5} - \frac{3}{10\pi} (2 + n) \log \left( \frac{C_5}{C_{12}^{5/12}} \right)^2 + \text{regular}
\]  

(3.92)

and, hence,

\[
L = -\frac{2}{\pi} \log \frac{C_{30}^2}{C_{12}^5} + \frac{3}{\pi} (2 + n) \log \left( \frac{C_5}{C_{12}^{5/12}} \right)^2 .
\]  

(3.93)

It follows that the Wilsonian gravitational coupling (3.90) can be rewritten into

\[
F_1 = 24\bar{S} - \frac{1}{\pi} \left[ \frac{19}{5} \log \frac{C_{30}^2}{C_{12}^5} + \frac{b_{grav}}{24} \log C_{12} + \frac{72}{10} (2 + n) + \frac{1}{2} \log \left( n_H' - n_V' \right) \log \left( \frac{C_5}{C_{12}^{5/12}} \right)^2 \right] .
\]  

(3.94)

Now recall from (3.20) that \( n_H' - n_V' = 12n + 30 \). Inserting this into (3.94) yields

\[
F_1 = 24\bar{S} - \frac{1}{\pi} \left( \frac{19}{5} \log \frac{C_{30}^2}{C_{12}^5} + \frac{3}{5} (1 - 2n) \log C_5^2 \right) .
\]  

(3.95)

Note that the \( \log C_{12} \) terms have completely cancelled out!

Now consider the perturbative gravitational coupling, which is again related to the perturbative Wilsonian coupling by

\[
\frac{1}{g_{grav}^2} = \text{Re}F_1 + \frac{b_{grav}}{4\pi} K = 12(S + \bar{S} + V_{GS}) + \Delta_{grav} ,
\]  

(3.96)
where this time $\Delta_{\text{grav}} = -\frac{2}{4\pi} \tilde{I}_{3,2}$ with

$$\tilde{I}_{3,2} = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} [Z_{3,2} \ast A_n(E_2 - \frac{3}{\pi\tau_2}) - d_n(0)] . \quad (3.97)$$

Here, we have introduced

$$B_n(\tau) = E_2 A_n = \frac{12 - n}{24} \frac{E_2 E_6 E_{4,1}}{\Delta} + \frac{12 + n}{24} \frac{E_2 E_4 E_{6,1}}{\Delta} = \sum_{N \in \mathbb{Z} \text{ or } \mathbb{Z} + \frac{3}{4}} d_n(4N)q^N . \quad (3.98)$$

The worldsheet integral (3.97) can be evaluated (cf. [26], App. B) using the techniques of [46, 70, 81, 82, 95]. Then we find, from (3.96), that

$$F_1 = 24S - \frac{2}{\pi} \sum_{(k,l,b) > 0} d_n(4kl - b^2) Li_1$$

$$= 24S - \frac{1}{\pi} \left( \frac{19}{5} \log C_{30}^2 + \frac{3}{5}(1 - 2n) \log C_5^2 \right) . \quad (3.99)$$
4 The dual Calabi-Yau spaces

The massless spectrum of a type II vacuum compactified on a Calabi–Yau threefold \( Y \) is characterized by the Hodge numbers \( h_{1,1} \) and \( h_{1,2} \). For type IIA one finds \([34, 103]\)

\[ n_V = h_{1,1}, \quad n_H = h_{2,1} + 1 \]

where the extra hypermultiplet counts the dilaton multiplet of the universal sector. For type IIB vacua one has \( n_V = h_{2,1} \) and \( n_H = h_{1,1} + 1 \). So the role of \( h_{1,1} \) and \( h_{1,2} \) is interchanged between type IIA and type IIB. Note that type IIA on a Calabi–Yau threefold \( Y \) is by mirror symmetry equivalent to type IIB on the mirror Calabi–Yau \( \tilde{Y} \).

The gauge group is always Abelian and given by \( n_V + 1 \) \( U(1) \) factors (the extra \( U(1) \) is the graviphoton in the universal sector). So

- (IIA) \( n_V = h_{1,1} \) and \( n_H - 1 = h_{2,1} \)
- (IIB) \( n_V = h_{2,1} \) and \( n_H - 1 = h_{1,1} \)

and in both cases one has an additional hypermultiplet corresponding to the type II dilaton. So for type II vacua \( \mathcal{M}_V \) is dilaton independent and therefore given by the uncorrected tree level result. This tree level prepotential can be computed exactly, i.e. including worldsheet instanton corrections, by using mirror symmetry and it shows logarithmic singularities (at tree level!) along the conifold locus; the latter behaviour can be understood as an 1-loop effect (thanks to the specific string coupling behaviour of RR-fields) due to the appearence (at the conifold locus) of massless hypermultiplets corresponding to charged black holes in the internal line.

More specifically let us consider the IIA string on a Calabi-Yau space \( X \) at the large radius limit. \( \mathcal{M}_V \) is described by the Kähler moduli, i.e. an element in moduli space is given by the (complexified) Kähler form \( B + iJ \in H^2(X, \mathbb{C}) \), which expands in a basis \((e_\alpha)\) of integral cohomology \( H^2(X, \mathbb{Z}) \) as

\[
B + iJ = \sum_\alpha t_\alpha e_\alpha,
\]

where in the complex parameters \( t_\alpha = B_\alpha + iJ_\alpha \) (they are \( N = 2 \) special coordinates) the \( B_\alpha \) are the moduli corresponding to the antisymmetric tensor and \( J_\alpha \) the moduli which are associated to the deformation of the metric. In the large radius limit the moduli \( B_\alpha \) are periodic variables and thus enjoy a discrete ‘PQ-like’ symmetry. More precisely, one finds that the low energy effective theory is invariant under \( B_\alpha \to B_\alpha + 1 \) or equivalently \( t_\alpha \to t_\alpha + 1 \).

For the holomorphic prepotential (at large radius) one has then

\[
\mathcal{F} = -\frac{i}{6} \sum (D_\alpha \cdot D_\beta \cdot D_\gamma) t_\alpha t_\beta t_\gamma + \frac{1}{(2\pi)^3} \sum_{(d_i)_{i=1,\ldots,n}} n_{d_1,\ldots,d_n} Li_3(\prod_{i=1}^n q_i^{d_i}) , \tag{4.1}
\]

with \( D_\alpha \) the associated divisor to \( e_\alpha \) and in the additioinal term involving the contribution of worldsheet instanton corrections, which vanish in the limit \( t_\alpha \to \infty \), the
$n_{d_1,\ldots,d_n}$ denote the rational instanton numbers, which are computed via mirror symmetry ($q_j = e^{2\pi i t_j}$). The coefficients of the cubic part are the classical intersection numbers of the $(1,1)$ forms defined by $f_Y e_\alpha \wedge e_\beta \wedge e_\gamma$.

The higher derivative couplings $F_n$ which were defined in eqs. (3.66),(3.67) and which similarly only depend on the vector multiplets, also obey the type II non-renormalization theorem. For these couplings one finds that they are proportional to the genus $n$ topological partition functions of a twisted Calabi–Yau $\sigma$–model [20]. They receive a contribution only at fixed string loop order $n$. In the large radius limit one has

$$F_{II}^I = -i \sum_{i=1}^h t_i c_2 \cdot J_i - \frac{1}{\pi} \sum_n \left[ 12 n_{d_1,\ldots,d_n}^e \log(\tilde{\eta}\left(\prod_{i=1}^h q_i^{d_i}\right)) + n_{d_1,\ldots,d_n}^r \log(1 - \prod_{i=1}^h q_i^{d_i}) \right].$$

(4.2)

Here $\tilde{\eta}(q) = \prod_{m=1}^\infty (1 - q^m)$, and the $n_{d_1,\ldots,d_n}^e$ denote the elliptic genus one instanton numbers; $c_2$ is the second Chern class of the Calabi–Yau manifold.

Now in tests of the proposed string/string duality [75] one compares the heterotic $\mathcal{M}_V$ at weak coupling with the exact $\mathcal{M}_V$ of the type II vacuum. In particular one identifies the heterotic dilaton with one of the vector moduli, say $t_S = (B + i J)_S$, of the type IIA string, namely $t_S = 4\pi i S$.

Now in the considered examples of dual pairs [75],[84] a certain preference of the dual Calabi-Yau space to have the structure of a K3 fibration (over $P^1(C)$) was observed - first "experimentally", then made plausible by noting the property of $D_S$ having zero selfintersection by [3,4], so being fibrelike [44], then at last stated under precise conditions [16] using in particular the additional property $D_S \cdot c_2(X) = 24 = \chi_{K3}$ coming from equ. (3.69).

By computing the spectrum for suitable choices of the gauge bundles one gets predictions for the hodge numbers of the dual Calabi-Yau leading to the proposal of the spaces (the upper postscript indicates the Euler number, the lower ones the $h^{1,1}$ and $h^{2,1}$)

- $ST$ model: $P_{1,1,2,2,6}(12)^{252}_{2,128}$
- $STU$ model: $P_{1,1,2,8,12}(24)^{-480}_{5,243}$
- $STUV$ model: $P_{1,1,2,6,10}(20)^{-372}_{4,190}$

\footnote{For the normalization compare the shift symmetries $B_S \rightarrow B_S + 1$ and $\theta \rightarrow \theta + 2\pi$}
4.1 ST model

The typical defining polynomial

\[ p = z_1^{12} + z_2^{12} + z_3^6 + z_4^6 + z_5^2 \]  

leads to the singular curve \( C \) (the singularity coming from the common factor of the weight indices) of genus 2 (as being isomorphic to \( P_{2,2,6}(12) = P_{1,1,3}(6) \), a 2-fold covering of \( P^1 = P_{1,1} \) with 6 branch points) on the Calabi-Yau

\[ z_3^6 + z_4^6 + z_5^2 = 0 \quad , \quad z_1 = z_2 = 0 \]  

which is resolved by pointwise insertion of \( P^1 \)'s leading to an exceptional divisor \( E \) representing a ruled surface over the curve \( C \). The projective ratio of the first two coordinates gives a base \( P^1 \) coordinate with K3 fibre (as being a double cover of \( P^2 \) branched along a sextic) \( P_{1,2,2,6}(12) = P_{1,1,1,3}(6) \) for our Calabi-Yau space. Let us for later use introduce the notation \( L \) for that fibre class (the cohomology class of the divisors of the corresponding linear system \( |L| \) defined by the degree 1 polynomials (generated by \( z_1 \) and \( z_2 \))) and similarly let us denote the linear system of degree 2 polynomials by \( |H| \). As quadratic polynomials in \( z_1 \) and \( z_2 \) are the subsystem of \( |H| \) vanishing on \( C \) one has \( H = 2L + E \). \( L \) and \( H \) are the relevant generators of \( h^{1,1} \), whose topological intersection numbers will give later important information concerning the matching of quantities in establishing tests of string duality.

Note that the mirror, which is a certain orbifold of the generic element in the subclass of spaces

\[ p = z_1^{12} + z_2^{12} + z_3^6 + z_4^6 + z_5^2 - 12\psi z_1 z_2 z_3 z_4 z_5 - 2\phi z_1^2 z_2^2 \]  

(having now only the 2 shown complex deformations but on the other hand 128 independent divisors from resolution of the generated orbifold singularities, i.e. the mirror exchange of hodge numbers has taken place), has a conifold singularity along \( \Delta_{\text{Can}} = (8\psi^4 + \phi)^2 - 1 = 0 \), a more complicated singularity along \( \phi^2 - 1 = 0 \), which we call in view of later considerations the strong coupling singularity; let us call (only slightly abusing the terminology of [23]) these loci \( C_{\text{con}} \) and \( C_1 \) (there are two further discriminant loci [23], of which we want only to mention the locus \( C_0 \) corresponding to \( \psi = 0 \)).

The mirror map for the two complex structure deformations of the mirror Calabi-Yau coming from

\[ p = z_1^{12} + z_2^{12} + z_3^6 + z_4^6 + z_5^2 + a_0 z_1 z_2 z_3 z_4 z_5 + a_1 z_1^6 z_2^6 \]

is given by \( q_j = e^{2\pi i t_j} \)
\[ x = j(q_1)^{-1} + O(q_2), y = g(q_1)q_2 + O(q_2^2) \]

(with uniformizing variables at large complex structure \( x = a_1a_0^{-6}; y = a_1^{-2} \); also used are the parameters \( \bar{x} = \alpha x, \bar{y} = 4y \) with \( \alpha = 12^3 = 1728 = j(i) \). The parameters of \([78]\) and \([25]\) are related by \( a_0 = -12\psi, a_1 = -2\phi \). Note that the discriminant is essentially given by

\[ \Delta = (1 - \bar{x})^2 - \bar{x}^2\bar{y}. \tag{4.6} \]

The investigation of the mirror map proceeds by investigating the Picard-Fuchs equations satisfied by the cohomology classes when transported along a way in moduli space surrounding the discriminant divisors; the corresponding monodromy considerations are relevant when identifying the large complex structure limit to find the correct variables, in which the mirror is to be expressed. According to the "regularity theorem" the relevant differential equation describing the transport has regular singular points, so that in a suitable basis the corresponding matrix has at worst first order poles, whose residue matrix has rational eigenvalues resp. in fact integer eigenvalues for unipotent monodromy. This leads to unique identification of large complex structure limit point in (blown up) moduli space \([25]\).

The modular property of the mirror map (at \( \bar{y} = 0 \) \( \bar{x} = \frac{\alpha}{j(q_1)} \)) was noted first experimentally \([25]\), then explained \([84]\) on the basis of the property of the Calabi-Yau being a K3-fibration together with the known reduction of the mirror map for certain 1-parameter families of K3 to the elliptic case.

Then with the expression for the mirror map at weak coupling \((S \approx \infty, \text{i.e. } y \approx 0)\)

\[ x = \frac{1}{j(q_1)} \]
\[ y = q_2g(q_1) \]

one gets \([72]\) that at weak coupling (so \( \Delta = (1 - \bar{x})^2 \))

\[ K_{\bar{x}\bar{x}} = \frac{1}{4\bar{x}^3 \Delta} = \frac{j^5}{4\alpha^3 (j - \alpha)^2} \]
\[ K_{\bar{x}\bar{y}} = \frac{1 - \bar{x}}{2\bar{x}^2 \bar{y} \Delta} = \frac{j^3}{2\alpha^2 j - \alpha \bar{y}} \]

respectively with \( \partial_{\tau} \bar{x} = -\frac{j}{j^3}, \partial_{\tau} \bar{y} = \bar{y} \partial_{\tau} \log g \) (listing only the terms contributing in the limit \( y \to 0 \) (cf. \([25]\]))(cf. appendix A1)

\[ K_{\tau\tau} = \left( \frac{d\bar{x}}{d\tau} \right)^3 K_{\bar{x}\bar{x}} + 3K_{\bar{x}\bar{y}} \left( \frac{d\bar{x}}{d\tau} \right) \left( \frac{d\bar{y}}{d\tau} \right) \]
\[
\frac{j_\tau^3}{4j(j - \alpha)^2} + \frac{3}{2j(j - \alpha)} \partial_\tau \log g \\
\sim -\frac{j_\tau^2}{j(j - \alpha)} \left[ \frac{j_\tau}{j - \alpha} - \frac{3}{2} \partial_\tau \log g \right] \\
= 4\pi^2 E_4 \partial_\tau [\log(j - \alpha) - \frac{3}{2} \log g]
\]

or (in \(\tau = t_1 = iT\))

\[
K_{TTT} \sim -4\pi^2 E_4 \partial_\tau [\log(j - \alpha) - \frac{3}{2} \log g]
\]

so that after going to the special gauge where \(F_0 = 1\), i.e. after dividing by \(X_0^2\) (which corresponds here to the square of the fundamental period: \(\omega_0^2\), which in turn equals at weak coupling \(E_4(iT)\) (cf. [90])) and taking a suitable overall normalization

\[
\partial_\tau^2 F = \frac{1}{4\pi^2} \partial_\tau [\log(j - \alpha) - \frac{3}{2} \log g] 
\]  

(4.7)

so

\[
\partial_\tau^2 F = S + \frac{1}{4\pi^2} \log(j - \alpha) - \frac{3}{2} \frac{1}{4\pi^2} \log g. 
\]  

(4.8)

Now concerning the \(F_1\) function by the holomorphic anomaly we will get that (up to an additive constant)

\[
F_1^{II} = \log[y^{-\alpha}(j - \alpha)^\beta].
\]  

(4.9)

Namely

\[
F_1^{II} = \log[(\frac{12\psi}{\omega_0})^{\frac{5 - \chi}{12}} \frac{\partial(\psi, \chi)}{\partial(t_1, t_2)} f]
\]

with \(f = \Delta_{\text{Can}}^\phi(\phi^2 - 1)^b\psi^c\), \(\Delta_{\text{Can}} = \frac{\Delta}{x^2 y} = (\frac{1 - x}{x^2 y})^2 = \bar{y} - 1\alpha^2(j - \alpha)^2\), \(\phi^2 - 1 = \bar{y} - 1\), \(\psi \sim y^{-1}\frac{E_{4^{1/2}}}{\eta^4} \frac{\partial(\psi, \phi)}{\partial(t_1, t_2)} \sim y^{-1}\frac{1}{\psi^5}\) and \(c = 1\) and \(\omega_0\) the fundamental period (note that \(\omega_0^2|_{y=0} = E_4\)).

One can obtain the values \(a = -\frac{1}{6}\), \(b = -\frac{3}{2}\) by comparing in the large radius limit with the topological intersection numbers [23]. This leads in total to the announced expression of \(F_1^{II}\) with

\[
\alpha = 1 + a + b + \frac{1}{12}(c - \frac{\chi}{12}) = 2 \\
\beta = 2a + \frac{1}{2} = \frac{1}{6} \\
\gamma = 2(c + 3 - \frac{\chi}{12}) = \frac{1}{6} b_{\text{grav}},
\]
i.e.

\[ F_1^{II} = \log[y^{-2} \frac{(j - \alpha)^{\frac{1}{2}}}{\eta^2(t_1)^{\frac{1}{2}b_{grav}}}]. \quad (4.10) \]

For later use let us come back to the mentioned asymptotic evaluations

\[ \alpha = \frac{1}{12} c_2 \cdot L = \frac{1}{12} c_2(L) = \frac{1}{12} \chi_{K3} \quad (4.11) \]

and

\[ \beta + \frac{\gamma}{12} = \frac{1}{12} c_2 \cdot H = \frac{1}{12} 2(\chi_{K3} - \chi_C) = \frac{1}{12} 2(4 - \frac{\chi}{12} + 1), \]

so \( 2(4 - \frac{\chi}{12})(12\frac{\beta}{\gamma} + 1) = \gamma(12\frac{\beta}{\gamma} + 1) = 2(4 - \frac{\chi}{12}) + 2 \)

\[ \frac{\gamma}{\beta} = 48 - \chi = b_{grav} \quad (4.12) \]

showing also \( \frac{\alpha}{\beta} = \frac{\chi_{K3}}{2}. \)

Furthermore the consideration of the monodromy around the conifold\footnote{in going to the second line we used the relation \( E^3 = -8 = 4\chi_C \) between the singular curve \( C \) of the Calabi-Yau and the ruled surface \( E \) of its pointwise resolution \cite{24} which gives \( c_2 \cdot H = \frac{1}{12}[2\chi_{K3} + c_3(E) - E^3] = \frac{1}{12}[2\chi_{K3} + 2\chi_C - 4\chi_C] \) and in going to the third line we used that because of the K3-fibration \( \chi = (\chi_{P1} - 12)\chi_{K3} + 12(\chi_C + 1) \) one has \( -\frac{\chi}{12} = -4 + \chi_{K3} - \chi_C - 1 \) (the +1 comes from the possibility of having \( z_3 = z_4 = z_5 = 0 \))} in its operation on the periods leads via mirror map to its operation on the coordinates \( t_1, t_2 \) in \( H^2 = H^{1,1} \) (remember \( B + iJ = t_1 H + t_2 L \)). This in turn leads then again to an corresponding operation on \( H_2 \), i.e. on classes of the embedded instanton worldsheets. This leads to the symmetry \( n_{jk} = n_{j,j-k} \) in the instanton expansions around the large complex structure limit point of the Yukawa couplings (which of course in the end are to be identified (after a certain ‘gauging’ by the fundamental period) with the corresponding third derivatives of the prepotential on the heterotic side \cite{25}), for example in \cite{25} (\( q_r = e^{2\pi i t_r}, F^{0}_{abc} \) the topological intersection numbers)

\[ F_{111} = F^{(0)}_{111} + \sum_{j,k} j^3 n_{jk} \frac{q_i^j q_k^k}{1 - q_i^j q_k^k} \quad (4.13) \]

These coefficients are similarly relevant in the later considered expression (with the modified Dedekind function \( \eta(q) = \prod_{n=1}^{\infty} (1 - q^n) \) and the genus one instanton contributions \( d_{jk} \) besides the usual genus zero contributions)\footnote{Actually one considers a slightly modified quantity; namely one has still to correct by the (simple structured) monodromy around \( C_0 \), which just corresponds to the operation \( (\psi, \phi) \rightarrow (\mu \psi, -\phi) \).}
\[ F_{1}^{\text{top}} = -\frac{2\pi i}{12} c_2 \cdot (B + iJ) - \sum_{j,k} [2d_{jk} \log \eta(q_{1}^{j}q_{2}^{k}) + \frac{n_{jk}}{6} \log (1 - q_{1}^{j}q_{2}^{k})] \] (4.14)

This transformation - in the end caused by the conifold monodromy! - has led to the proposal [84] that - as the symmetry in the instanton coefficients translates to the symmetry \( q_1 \to q_1 q_2, q_2 \to 1/q_2 \) in the expansion variables - one has actually to identify \( q_2 = q_s/q_T \) leading to the nonperturbative exchange symmetry S-T!

As for later use in tests of S-T exchange symmetry let us state a further property of the \( n_{jk} \) and at the same time let us look at the problem from a more explicit point of view, which makes the instanton properties - as they are the cornerstone of the nonperturbative exchange symmetry - more directly visible (cf. [25]).

To understand \( n_{jk} = n_{j,j-k} \) let \( f : X \to X' \) the resolution of the Calabi-Yau with \( f : E \to C \) the contained resolution of the singular curve by the surface \( E \) as \( P^1 \) bundle over \( C \), so the fibers of \( E \) over \( C \) are smooth rational curves \( l \) with \( l \cdot E = l \cdot (H - 2L) = -2 \) (as \( l \cdot H = 0, l \cdot L = 1 \)) and let \( \Gamma \subset X \) an irreducible rational embedded instanton curve.

If \( \Gamma \not\subset E \) (otherwise \( \Gamma \) is one of the fibers \( l \)), then \( \Gamma \) intersects \( E \) in \( n_{\Gamma} = \Gamma \cdot E \geq 0 \) points \( p_i \) and taking the union of \( \Gamma \) with the fibers \( l_i \) through these points one gets a connected curve \( \Gamma' \) homologous to \( \Gamma + nl,a \) (contributing) degenerate instanton. Now if you already started with a connected degenerated instanton \( \Gamma \), where the possibility to contain (at an intersection point with \( E \)) the corresponding fibre is realized for a subset of the intersection points (\( \Gamma \) containing, say, \( m \) fibers as components, so \( \tilde{\Gamma} = \Gamma - \sum_{i=1}^{m} l_i \) is still connected, \( m \leq n = n_{\Gamma} \geq 0 \)), throwing away these fibers and plugging in the fibers at the other intersection points will give you an involution \( \Gamma \to \Gamma' \) on the set of degenerate instantons (with the exception of the fibers), where \( \Gamma' \), which is homologous to \( \Gamma + (\Gamma \cdot E)l = \Gamma + ((\tilde{\Gamma} + ml) \cdot E)l = \Gamma + (n - 2m)l = \Gamma - ml + (n - m)l \) (showing here something like the mirror reflected Picard-Lefshetz monodromy), contributes now to \( n_{j,j-k} \) when \( \Gamma \) contributed to \( n_{jk} \):

\[
\Gamma' \cdot H = \Gamma \cdot H + 0 = j
\]
\[
\Gamma' \cdot L = \Gamma \cdot L + (\Gamma \cdot E)l \cdot L = k + \Gamma \cdot E
\]
\[
= k + \Gamma \cdot (H - 2L) = k + j - 2k = j - k
\] (4.15)

Similarly, to understand the later also used property \( n_{jk} = 2\delta_{j0}\delta_{k1} \) for \( j > k \), note that \( n_{01} = 2 = -\chi_C \) and that (let \( r := \Gamma \cdot L \)) for \( j > 0 \) one has \( k = \Gamma \cdot L = \tilde{\Gamma} \cdot L + ml \cdot L = r + m \leq j - r \leq j \) as \( 0 \leq m \leq \tilde{\Gamma} \cdot E = \tilde{\Gamma} \cdot (H - 2L) = j - 2r \).
4.2 STU model

The defining polynomial of $P_4^{1,1,2,8,12}(24)_{3,243}$ is

$$p = z_1^{24} + z_2^{24} + z_3^{12} + z_4^3 + z_5^2 + a_0 z_1 z_2 z_3 z_4 z_5 + a_1 (z_1 z_2 z_3)^4 + a_2 (z_1 z_2)^{12},$$

which shows already the deformations of the mirror Calabi-Yau with uniformizing variables at large complex structure (also used are the parameters $a_0 = -12 \psi_0, a_1 = -2 \psi_1, a_2 = -\psi_2$ and $x = \frac{\alpha}{4} x, \bar{y} = 4 y, \bar{z} = 4 z$, where $\alpha = j(i) = 1728$)

$$x = -\frac{2}{\alpha^2} \psi_0^{-6} \psi_1, \ y = \psi_2^{-2}, \ z = -\frac{1}{4} \psi_1^{-2} \psi_2.$$

For later use let us also record the inverse relations

$$\psi_0 \sim \bar{y}^{-\frac{1}{12}} (\bar{x}^2 \bar{z})^{-\frac{1}{12}}$$
$$\psi_1 \sim (\bar{y}^{-\frac{1}{2}} \bar{z}^{-1})^{\frac{1}{2}}$$
$$\psi_2 \sim \bar{y}^{-\frac{1}{2}}.$$

Like [25] I will use discriminant factors shifted compared to [72, 84] (we are in the limit $\bar{y} \to 0$)

$$\tilde{\Delta}_1 = \bar{y}^{-1} \left( 1 - \frac{\bar{z}}{\bar{x}} \right)^2$$
$$\tilde{\Delta}_2 = \bar{y}^{-1} (\bar{x}^2 \bar{z})^{-2} ((1 - \bar{x})^2 - \bar{x}^2 \bar{z})^2$$
$$\tilde{\Delta}_3 = \frac{1 - \bar{y}}{\bar{y}} \approx \bar{y}^{-1}.$$

In the limit $\bar{y} \to 0$ the mirror map is given by ($q_1 := q_T, q_3 := q_U/q_T; j := j(iT), k := j(iU)$) [84, 89]

$$\bar{x} = \frac{\alpha}{2 j k + \sqrt{j(j - \alpha) \sqrt{k(k - \alpha)}}} = q_1 + \sum_{m+n>1} a_{mn} q_1^m q_3^n$$
$$\bar{y} = q_3 f(q_1, q_3) + O(q_2^2) = q_2 \sum_{m+n\geq 1} c_{mn} q_1^m q_3^n + O(q_2^2)$$
$$\bar{z} = (\frac{\alpha}{2})^2 \frac{1}{j k x^2} = q_3 + \sum_{m+n>1} b_{mn} q_1^m q_3^n.$$

One has $t_1 = t_T = iT, t_2 = 4 \pi i S, t_3 = t_U - t_T, t_U = iU$; so $q_4 = e^{-8 \pi^2 S}$, where $S$ is the tree-level dilaton of the heterotic string, and analogously to the ST case again $\bar{y} = e^{-8 \pi^2 S_{\text{inv}}}$ with the modular invariant dilaton [45, 78].

Note that in the divisor picture one has here besides the hyperplane section $J$ first again a ruled surface $E$ corresponding to the necessary resolution caused by the common factor 2 of the last three weights, and then still a further divisor $F$ corresponding
to the higher common divisibility of the last two weights by 4 introducing the Hirzebruch surface $F_2$ over which the resolved CY is eliptically fibered (the trace of which projection map in the space $P_{1,1,2,8,12}(24)$ is seen by projection to the first three coordinates). One has then here the relation $J = 4L + 2E + F$ with the $K3$ fibre class $L$.

The Yukawa couplings are given by [72]

$$F_{klm}^{II} = F_{klm}^0 + \sum_{d_1, \ldots, d_h} n^{r}_{d_1, \ldots, d_h} d_k d_l d_m \frac{h}{1 - \prod_{i=1}^{h} q_i^{d_i}},$$  \tag{4.17}

where $q_i = e^{-2\pi t_i}$. The $F_{klm}^0$ denote the intersection numbers, whereas the $n^{r}_{d_1, \ldots, d_h}$ denote the rational instanton numbers of genus zero. These instanton numbers are expected to be integer numbers. We will, in the following, work inside the Kähler cone $\sigma(K) = \{ \sum_i t_i J_i | t_i > 0 \}$. For points inside the Kähler cone $\sigma(K)$, one has for the degrees $d_i$ that $d_i \geq 0$. Integrating back yields that

$$F_0^{II} = F^0 - \frac{1}{(2\pi)^3} \sum_{d_1, \ldots, d_h} n^{r}_{d_1, \ldots, d_h} L J_3(\prod_{i=1}^{h} q_i^{d_i})$$  \tag{4.18}

up to a quadratic polynomial in the $t_i$. $F^0$ is cubic in the $t_i$. Thus we have three Kähler moduli $t_1, t_2, t_3$ and instanton numbers $n^{r}_{d_1, d_2, d_3}$. The classical Yukawa couplings $F_{klm}^0$ on the type II side are given by [72]

$$F_{t_1 t_1 t_1}^0 = 8, \quad F_{t_1 t_1 t_2}^0 = 2, \quad F_{t_1 t_1 t_3}^0 = 4,$$
$$F_{t_1 t_2 t_3}^0 = 1, \quad F_{t_1 t_3 t_3}^0 = 2.$$  \tag{4.19}

It follows that

$$F^0 = \frac{4}{3} t_1^3 + t_1^2 t_2 + 2t_1^2 t_3 + t_1 t_2 t_3 + t_1 t_3^2.$$  \tag{4.20}

Some of the instanton numbers $n^{r}_{d_1, d_2, d_3}$ can be found in [72]. When investigating the prepotential $F_0^{II}$ [84], two symmetries become manifest, namely

$$t_1 \rightarrow t_1 + t_3, \quad t_3 \rightarrow -t_3 \quad \text{for} \quad t_2 = \infty,$$  \tag{4.21}

and

$$t_2 \rightarrow -t_2, \quad t_3 \rightarrow t_2 + t_3.$$  \tag{4.22}

These symmetries are true symmetries of $F_0^{II}$, since the world-sheet instanton numbers $n^{r}$ enjoy the remarkable properties [84]

$$n^{r}_{d_1, 0, d_3} = n^{r}_{d_1, 0, d_3 - d_3} \quad \text{and} \quad n^{r}_{d_1, d_2, d_3} = n^{r}_{d_1, d_3 - d_2, d_3}.$$  \tag{4.23}

Observe that $F^0$ is completely invariant under the symmetry (4.22).
Concerning $F_1$ one gets from the holomorphic anomaly of $F_1$ \[20\] that up to an additive constant (cf. \[25\], \[11\] \[78\])

$$F_1^{II} = \log\left[\frac{24\psi_0}{\omega_0}^{3+3} \chi \frac{\partial(\psi_0, \psi_1, \psi_2)}{\partial(T, S, U)} f\right]$$

with a holomorphic function $f = \tilde{\Delta}_0^2 \tilde{\Delta}_2^2 \tilde{\Delta}_3^4 \psi_0^c$ with $c = 3$ and the fundamental period at weak coupling (as in all what follows) $\omega_0^3 = (E_4(T)E_4(U))^{1/2}$. One again gets the values $a = b = -\frac{1}{6}, d = -\frac{1}{2}$ by comparing in the large radius limit with the topological intersection numbers. Furthermore one can eventually find the crucial relation (cf. \[11\]) $\tilde{\Delta}_1 \tilde{\Delta}_2 \sim \tilde{y}^{-2}(j - k)^4$, which shows that generically there are no more complicated enhancement loci than those corresponding to $T = U$. Then one gets with $f = \tilde{\psi}_0^3(\tilde{\Delta}_3)^{-1/2}(\tilde{\Delta}_1 \tilde{\Delta}_2)^{-1/6} \sim \tilde{y}^{\frac{\chi}{24^2}}(j k)^{\frac{1}{2}}(j - k)^{-\frac{5}{3}}, \psi_0 \sim \tilde{y}^{\frac{\chi}{24^2}}(j k)^{\frac{1}{2}}, \frac{\tilde{\psi}_0}{\omega_0} \sim \tilde{y}^{-\frac{1}{24^2}} \frac{1}{\eta^2(t_T) \eta^2(t_U)} \frac{\partial(\tilde{\psi}_0, \tilde{\psi}_1, \tilde{\psi}_2)}{\partial(T, S, U)} \sim \tilde{y}^{-\frac{1}{2}} \frac{(j - k)^{\frac{1}{2}}}{\sqrt{jk}} \omega_0^3 \psi_0$ that the factors besides $\tilde{y}$ powers and $(j - k)$ powers collect (with $\tilde{\psi}_0 \sim (jk)^{\frac{1}{2}}$) as follows: $(\frac{\tilde{\psi}_0}{\omega_0})^{6 - \frac{5}{2} \frac{\chi}{44^2}} = (\frac{\tilde{\psi}_0}{\omega_0})^{4 - \frac{5}{2} \frac{\chi}{44^2}} = (\frac{1}{\eta^2(t_T) \eta^2(t_U)})^{\frac{1}{24^2}}$.

so that one has finally

$$F_1^{II} = \log[\tilde{y}^{-2}(\eta^2(t_T) \eta^2(t_U))^{\frac{1}{24^2}}]. \quad (4.24)$$

### 4.3 STUV model

In the dual type IIA description, based on compactifications on four-parameter Calabi–Yau threefolds $X_n$, the Euler numbers are $\chi(X_n) = 2(h_{1,1} - h_{2,1}) = 24n - 420$ and the Hodge numbers are given by $h_{1,1} = n_T = 4, h_{2,1} = n_H = 214 - 12n$. For the Hodge numbers compare the second column of table A.1 in \[24\]. The $n = 2$ Calabi–Yau threefold $X_2$, for instance, is given by the space $P_{1,1,2,6,10}(20)$ of \[18\]. The Calabi–Yau spaces $X_0$ and $X_1$ are given in \[24\], \[25\].

Furthermore \[12\], for the $K3$-fibre $P_{1,1,3,5}(10)$ of $X_n$, one finds that (cf. \[18\] for $n = 2$) in the basis $j_1, j_3, j_4$ (where we denote the intersections of the CY divisors with the $K3$ ($J_2$) by small letters) the intersection form is given by \[\begin{pmatrix} 2 & 1 & 0 & 0 \\ 4 & 4 & 2 & 6 \end{pmatrix}\], which is equivalent (over $\mathbb{Z}$) to $-A$ under the base change $f_2 = j_1 - j_3, f_2 = j_3$ and $f_3 = 2j_2 - j_3$. The enhancement loci become the conditions $t_3 = 0$ and $t_4 = 0$ for the Kähler moduli on the type II side.

The cubic parts of the type II prepotentials of $X_0$, $X_1$ and $X_2$ are given in \[18\], \[92\] and can be written in a universal, $n$-dependent function as follows:

$$F_{\text{cubic}}^{II} = t_2(t_1^2 + t_1 t_3 + 4 t_1 t_4 + 2 t_3 t_4 + 3 t_4^2)$$

\[12\] Note that, since the heterotic perturbative gauge group is reflected, on the dual type IIA side, in the (monodromy invariant part of the) Picard group of the generic $K3$ fibre of the Calabi–Yau \[11\], \[18\], the discussion presented here agrees precisely with the one of \[24\] concerning the zero divisor of the period map for the (mirror of the) $K3$. $D^+$ can be matched with the domain of the period map $\Phi(z)$. 
Note that for $t_4 = 0$, $F_{\text{cubic}}^{II}$ precisely reduces to the cubic prepotential of the three parameter models [72, 92].

Next, let us consider the contributions of the worldsheet instantons to the type II prepotential of a four-parameter model. Generically, they are given by

$$F_{\text{inst}}^{II} = -\frac{1}{(2\pi)^3} \sum_{d_1, d_2, d_3, d_4} n_{r_{d_1, d_2, d_3, d_4}}^{r} Li_3 \left( \prod_{i=1}^{4} q^{d_i} \right). \tag{4.26}$$

The $n_{r_{d_1, d_2, d_3, d_4}}^{r}$ denote the rational instanton numbers. The heterotic weak coupling limit $S \to \infty$ corresponds to the large Kähler class limit $t_2 \to \infty$. In this limit, only the instanton numbers with $d_2 = 0$ contribute in the above sum. Using the identification $kT + lU + bV = d_1 t_1 + d_3 t_3 + d_4 t_4$, it follows that (independently of $n$)

$$k = d_3, \quad l = d_1 - d_3, \quad b = d_4 - 2d_1. \tag{4.27}$$

Then, (4.26) turns into

$$F_{\text{inst}}^{II} = -\frac{1}{(2\pi)^3} \sum_{k, l, b} n_{r_{k, l, b}} Li_3 (e^{-2\pi (kT + lU + bV)}). \tag{4.28}$$

Later we will see by comparison with (3.56) that the rational instanton numbers have to satisfy the nontrivial constraint

$$n_{r_{k, l, b}} = n^r (4kl - b^2). \tag{4.29}$$
5 BPS Spectral Sums

5.1 BPS States

BPS states - the states saturating the Bogomolny bound $m \geq |Z|$ between mass and charge - play an important role in many difficult dynamical questions as they provide in some respects a rigid skeleton of a theory. This can be related to the issue of quantum uncorrectedness of their masses in certain situations or to their property of describing (via a certain index) something like a topological subsector of a theory. For example they are relevant for questions such as strong/weak coupling duality - say in tests of the nonperturbative $SL(2, \mathbb{Z})_S$ duality of the $N = 4$ theory in 4D given by the heterotic string on $T^6$ [105], where one can make checks of nonperturbative predictions as one expects the masses of these states to be uncorrected in the quantum theory. Another important framework is the conjectured 4D string/string/string triality [51]: namely in the intermediate step in six dimensions the heterotic string on $T^4$ is dual to a type IIA string [114]; in completing the compactification process in going to 4D on a further $T^2$ the heterotic string acquires beside the aforementioned $SL(2, \mathbb{Z})_S$ strong/weak coupling duality an duality relating to the moduli of $T^2$, namely the $SO(2, 2, \mathbb{Z})$ target space duality consisting of $SL(2, \mathbb{Z})_T \times SL(2, \mathbb{Z})_U$ and the exchange symmetry $T-U$. As the 6D duality leads to an exchange of $S$ and $T$ in 4D one gets the IIA string with fields $T,S,U$ (similar considerations lead to a 4D type IIB string with fields $U,T,S$). Under this S-T exchange the BPS spectra of the heterotic and the type IIA string get mapped into each other, whereas the BPS spectra of the individual strings don’t have this symmetry. The latter fact is due to the property of the BPS masses in the (4D) $N = 4$ situation as being given by the maximum of the 2 central charges $|Z_1|$ and $|Z_2|$ of the $N = 4$ algebra. So there exist two kinds of massive BPS multiplets called short and intermediate (see below). Now the point is that the states, which from the $N = 4$ point of view are intermediate, are short from the $N = 2$ point of view [27]. So one can expect an S-T symmetry of the BPS spectrum of certain $N = 2$ heterotic string compactifications.

A further area, in which BPS states play a central role, is the study of 1-loop threshold corrections to gauge and gravitational couplings in $N = 2$ heterotic string compactifications [70]. In cases, where the contributions are due to BPS states only, one can then also expect a S-T exchange symmetry of the concerned couplings.

Lastly a related theme is the resolution of the conifold singularity [108], in whose description the BPS states also play an essential role, as we will see.

5.1.1 The $N = 4$ situation

Before we come to the main issue of this report concerning $N = 2$ string compactifications let us point to some features of the $N = 4$ situation to put some questions into a broader perspective.

Consider the $N = 4$ heterotic string got by compactification on $T^6$. The two central charges of the $N = 4$ algebra are given by the 6-dimensinal electric/magnetic charge
vectors $\vec{Q}, \vec{P}$ as

$$|Z_{1,2}|^2 = \vec{Q}^2 + \vec{P}^2 \pm 2\sqrt{\vec{Q}^2 \vec{P}^2 - (\vec{Q} \cdot \vec{P})^2}$$

Now let us focus on the discussion of the moduli dependence of a $T^2$ subsector. One gets

$$|Z_{1,2}|^2 = \frac{1}{4(S + S)(T + T)(U + U)} |\mathcal{M}_{1,2}|^2$$

with $\mathcal{M}_1 = (\bar{M}_I + iS\bar{N}_I)\bar{P}^I$ and $\mathcal{M}_2 = (\bar{M}_I - iS\bar{N}_I)\bar{P}^I$ the integers $\bar{M}_I, \bar{N}_I$ being the electric (magnetic) charge quantum numbers ($I = 0,...,3$) of the gauge group $U(1)^4$. We will work in a rotated basis with $P = (1, -TU, iT, iU)^T$ and corresponding quantum numbers $M_I, N_I$.

Now as the BPS masses are being given by the maximum of the 2 central charges $|Z_1|$ and $|Z_2|$ of the $N = 4$ algebra there exist two kinds of massive BPS multiplets called

- short for

$$m_s^2 = |Z_1|^2 = |Z_2|^2$$

with the associated soliton background solution preserving 1/2 of the supersymmetries of $N = 4$, the multiplet containing maximal spin one

- intermediate for

$$m_I^2 = \max(|Z_1|^2, |Z_2|^2)$$

with the associated soliton background solution preserving 1/4 of the supersymmetries of $N = 4$, the multiplet containing maximal spin 3/2

The mass formula is invariant under the perturbative T-duality group $SL(2, \mathbb{Z})_T \times SL(2, \mathbb{Z})_U \times \mathbb{Z}_T^{2 \times U}$ as well as under the nonperturbative S-duality group $SL(2, \mathbb{Z})_S$.

Now the fundamental distinction for a $N = 4$ BPS state is whether it is short or intermediate, i.e. whether the (S-independent) expression $\Delta Z^2 = |Z_1|^2 - |Z_2|^2$ is generically zero or not. So the shortness condition means the parallelness $\vec{Q} \parallel \vec{P}$, i.e. $sM_I = pN_I$ with $s, p \in \mathbb{Z}$. This means for the holomorphic mass of a short multiplet that it factorizes into a S-dependent and a T,U-dependent term

$$\mathcal{M} = \mathcal{M}_S \mathcal{M}_{T,U}$$

with

$$\mathcal{M}_S = s + ipS$$
$$\mathcal{M}_{T,U} = m_2 - im_1 U + in_1 T - n_2 TU$$

$\mathcal{M}_{T,U} = T + U, \mathcal{P}^1 = i(1 + TU), \mathcal{P}^2 = T - U, \mathcal{P}^3 = -i(1 - TU)$

$\mathcal{M}_{T,U}$ via the identification $M = s\mathcal{V}, N = p\mathcal{V}$ ($\mathcal{V}$ the vector $\mathcal{V} = (m_2, n_2, n_1, -m_1)$ of the $T^2$ lattice)
So here the quantum numbers $m_i (n_i)$ are the momentum (winding) numbers associated to the $T^2$, $s (p)$ is the electric (magnetic) quantum number associated with the $U (1)_S$. Examples are the elementary electric heterotic string states with magnetic charge $p = 0$, where $M^2 \sim p_R^2$ with the right-moving $T^2$ lattice momentum $p_R$ and level matching

\[
\frac{p_I^2}{2} - \frac{p_R^2}{2} = m_1 n_1 + m_2 n_2 = N_R + h_R - N_L + 1/2 = 1 - N_L.
\]

For $S \to \infty$ an infinite number of these states becomes massless. Similarly for $S \to 0$ an infinite tower of magnetic monopoles with $s = 0$ ($p$ unrestricted) becomes arbitrary light.

One has two subclasses (again respected by the duality group)

- $m_1 n_1 + m_2 n_2 = 0$: KK excitations of elementary states and KK monopoles (no massless states for finite values of $T$ and $U$)

- $m_1 n_1 + m_2 n_2 = 1$: containing elementary states becoming massless along critical lines/points within $T, U$ moduli space (the cases $T = U, T = U = 1, T = U = \rho$ of classically enhanced gauge symmetries $SU (2), SU (2), SU (3)$); also containing H-monopoles

### 5.1.2 The $N = 2$ situation

The BPS masses are given by $m_{BPS}^2 = |Z|^2$ with $Z$ the (complex) central charge of the supersymmetry algebra. As a function of the $n_v + 1$ massless abelian vector multiplets $X^I$ of the abelian gauge group $U (1)^{n_v + 1}$ (the graviphoton $X^0$ will be set to one) one has with the holomorphic prepotential $F (X^I)$ and $\mathcal{M} = M_I X^I + i N^I F_I$ (with the electric resp. magnetic quantum numbers as coefficients) the mass formula

\[
m_{BPS}^2 = \epsilon^K |\mathcal{M}|^2 = \frac{|M_I X^I + i N^I F_I|^2}{X^I \overline{F}_I + \overline{X}^I F_I}
\]

with the expression $\epsilon^K = X^I \overline{F}_I + \overline{X}^I F_I$ involving the Kaehler potential. The relevant sector for us will then be given by $S = i X^1, T = -i X^2, U = -i X^3$. With the tree level prepotential $F^{(0)} = i X^1 X^2 X^3 X^6 = -STU$ one gets for the classical period vector

\[
\Omega = (1, TU, iT, iU|iSTU, iS, -SU, -ST)^T
\]

The electric period fields $P^I$ are only $T, U$-dependent, whereas the magnetic period fields $Q_I$ are proportional to $S$.

In this basis the holomorphic BPS mass becomes

\[
\mathcal{M} = M_0 + M_1 TU + i M_2 T + i M_3 U + i S (N_0 TU + N_1 + i N_2 U + i N_3 T)
\]

\footnote{In the standard heterotic normalization $S = \frac{1}{S} - i \frac{\theta}{\pi^2}$; the Peccei-Quinn symmetry associated with the axion is given by the shift $\theta \to \theta + 2\pi, S \to S - \frac{1}{S}$.}

\footnote{But note that the period vector $(X^I, i F_I)$ following from $F^{(0)}$ doesn’t lead to the behaviour that all classical gauge couplings become small as $S \to \infty$ \cite{STU, TUY}; whereas the couplings for $U (1)_T \times U (1)_U$ are proportional to Re$S$, the couplings involving $U (1)_S$ are constant or even growing; to get a period vector with all gauge couplings proportional to Re$S$ one has to make the symplectic transformation $(X^I, i F_I) \to (P^I, i Q_I)$ with $P^I = i F_I, Q_I = i X^I$ (the rest unchanged).}
5.2 BPS spectral sums

In this and the next subsection we review some contributions made over the years to the study and interpretation of quantities involving sums over BPS states \[ [57, 70, 110] \].

5.2.1 \( F_1 \) function and the conifold

Let us start from the phenomenon associated with the so-called conifold point in the Calabi-Yau moduli space. As one approaches such a point \( p \) a 3-cycle \( C_p \) together with its period \( f_{C_p} \Omega \) (which represents a coordinate value \( z(p) = 0 \) of the complex structure deformations) and thus together with the corresponding mass will vanish \( (m^2 = |z|^2) \) after convenient normalization of \( \Omega \). Here the associated classical state was at first constructed by Strominger \[ [108] \] by wrapping a 3-brane around the cycle, leading to a 4D extreme black hole carrying the relevant Ramond-Ramond \( U(1) \) gauge field (whereas the fundamental string excitations are neutral under this \( U(1) \)) saturating the Bogomolny bound and representing thus a stable BPS state (an N=2 hypermultiplet).

The assumption in \[ [108] \] (necessary for resolution of the conifold singularity) that there are no stable black holes corresponding to integer multiples of the vanishing cycle was established by Bershadsky, Sadov and Vafa \[ [22] \] using new insights \[ [21], [96] \] gained in the rapid development taking place immediately after the beginning \[ [98], [116] \] of the D-brane revolution.

It is now of interest to develop further the idea of resolution of the conifold singularity by the existence of the (becoming) massless black hole state in the full theory in that it gives an alternative explanation of divergent string amplitudes. To go ahead it is of course useful to consider a (perturbatively and nonperturbatively) uncorrected quantity. Moreover this will in the end bring us back to the free energy (represented as a BPS sum).

Concretely we will study a term for the type II string on the Calabi-Yau, which in turn is related to a topological amplitude. Namely let us consider again the moduli dependent topological 1-loop amplitude \( g_{1}^{-2} \), which gives terms for the vector multiplet related to gravitational couplings (\( R \) is the Riemann tensor)

\[
\int g_{1}^{-2} R^2.
\]

So our aim is now to understand the singular structure of \( g_{1}^{-2} \) near a conifold point in terms of the full theory including the nearly massless black hole. In the studied examples the singularity was found to be

\[
g_{1}^{-2} = -\frac{1}{12}\log |z|^2 + \cdots
\]

(the precise coefficient fixed by considering other limits), which gives \[ [11] \] for the correction of the effective action near the conifold point.

\[ \begin{align*}
17 & \text{After considerations of normalization \[ [110] \] using } \frac{1}{128 \pi^2} \int \epsilon^{abc} \epsilon^{cd} R_{abcd} R_{ijkl} = \frac{1}{2}.
\end{align*} \]
\[ \delta S = -\frac{\chi}{24} \log m^2 \]

But exactly such a term would come from a 1-loop computation [110] in the theory with a light black hole of mass \( m \) as the contribution of exactly one hypermultiplet, thus (again) confirming the consistency of Strominger’s proposal. Now in the effective theory near the conifold point a 1-loop term (proportional to \( \chi \)) is generated not only by the light hypermultiplet, but by all of them. So \( g_{-1}^2 \) conjecturally [110] contains (there could be further \( \chi \)-proportional terms in the effective theory near the conifold point) a term like (suitable regularization understood; also there is the jumping phenomenon which can destroy the (for \( F_1 \) needed) modular invariance of the BPS spectrum)

\[ -\frac{1}{12} \sum_{BPS} \log m^2 \]

(may be one has to restrict to a subspace of vanishing cycles and to discuss the jumping phenomenon). Note that in this setup the BPS mass formula \( m_{BPS}^2 = e^K |\mathcal{M}|^2 \) becomes

\[ m^2 = \frac{\int C \Omega \cdot \int C \bar{\Omega}}{\int \Omega \wedge \bar{\Omega}}. \]  

(5.3)

In the (complex) one dimensional situation (with the true winding states on \( T^2 \)) one finds (cf. the next subsection; appendix [57]) such a relation

\[ F = \log Z = \sum_{n,m} \log \left| n + \tau m \right|^2 \tau_2 = -\log(\det D^2) = g_{-1}^{-2} \]

giving \( g_{-1}^{-2} \) as a partition function for BPS states (\( D \) being the laplacian acting on the fields in the multiplet).

A more precise indication of what to expect comes from an anomaly comparison: the proposed BPS sum as an expression for \( g_{-1}^{-2} \) would give for \( \delta \bar{\delta} g_{-1}^{-2} \) something proportional to \( \delta \bar{\delta} K \) (as we had \( e^{-K} = \int \Omega \wedge \bar{\Omega} \)), but not the full anomaly as one has [20, 29]

\[ \partial_i \bar{\partial}_j g_{-1}^{-2} = \frac{1}{2} (3 + h^{1,1} - \frac{X}{12}) G_{ij} - \frac{1}{2} R_{ij} \]

leading to (say in the STU-model) the dependence in \( T,U \) (using the anomaly factor [20] in the first contribution to this sum and the moduli metric \( \partial \bar{\partial} K \) in the second; \( e^{-K} = (T + \bar{T})(U + \bar{U}) \))

\[ g_{-1}^{-2} = \log |\mathcal{F}_{hol}|^2 + (3 + h^{1,1} - \frac{X}{12}) \hat{K} - (h^{1,1} - 1) \log \frac{1}{(T + \bar{T})(U + \bar{U})} \]

\[ = 2 \text{Re} \log F_{hol} + \frac{b_{\text{grav}}}{12} \hat{K} \]
5.2.2 The topological free energy

This proposal was similar in spirit to the one made in [57], where a moduli dependent quantity (there called "topological free energy") $F = \log Z$ was defined as a certain nonholomorphic partition function for supersymmetric string compactifications. It corresponds to the generating functional of the effective action after integrating out the massive modes coming from the compactification (so in the toroidal case the Kaluza-Klein states and the winding states are concerned, but not the massive oscillator states, which are also present in the uncompactified theory; this is meant by the qualification as "topological"). It is given as (the log of) the determinant of the chiral mass matrix for the massive compactification modes and gives the effective description for amplitudes with external light states and possible massive (in the qualified sense) states in the loops. Its moduli dependence leads to an expression of $F$ in terms of automorphic functions (or forms) of an duality group: may it be the T-duality group for toroidal and/or orbifold compactifications or the symplectic duality group coming from the special Kaehler geometry of an $N = 2$ situation.

More concretely the process of integrating out the massive modes leads to

$$e^{F_{Bos}} = \int D\phi e^{\phi M^2 \phi^\dagger + \cdots}$$

where we have neglected higher orders (and derivative terms) in $\phi$ around the $\phi = 0$ vacuum. Because of the space time supersymmetry one has then

$$e^{F} = \det M^\dagger M$$

with the fermionic mass matrix, an expression, which can further be evaluated in the form (nonholomorphic part) $\times \text{holomorphic piece}^2$, namely (the nonholomorphic piece comes from an $e^K$ rescaling in the supergravity expression for $\det M^\dagger M$; $W$ is the superpotential, $K$ the Kaehler potential)

$$e^K |\det W_{ij}|^2$$

For example in the familiar case of compactification on (3 copies of) $T^2$, where one has $e^K = \frac{1}{S+S} (\frac{1}{T+T} (0+U)) = \frac{1}{S+S} e^\hat{K}$, one has explicitely (the prime on the sum means $(m_1, n_1, m_2, n_2) \neq (0, 0, 0, 0)$ and the constraint comes from $\vec{p}_L - \vec{p}_R = m_1 n_1 + m_2 n_2$ and the exclusion of oscillator states)

$$F = \sum_{m_1 n_1 + m_2 n_2 = 0} \log e^{\hat{K}} |\mathcal{M}_{T,U}|^2$$

(with a regularization of the infinite sum understood, which respects the T duality group $SL(2, \mathbb{Z})_T \times SL(2, \mathbb{Z})_U \times \mathbb{Z}_2$). Note that the constraint decomposes into two orbits: $n_2 = m_1 = 0$ and $n_2 = n_1 = 0$. So

$$F = \sum_{m_2, n_1} \log \frac{|m_2 + i m_1 T|}{T + T} + \sum_{m_2, m_1} \log \frac{|m_2 + i m_1 U|}{U + U}$$
or after the \( \zeta \)-function regularization

\[
F = \log(T + \bar{T})|\eta(T)|^4 + \log(U + \bar{U})|\eta(U)|^4
\]  

(5.4)

Note that exactly at the boundary points of moduli space, \( T,U = 0,\infty \), where additional KK- resp. winding-modes become massless (so that one expects an infinite contribution to the free energy), the expression diverges.

Or to argue in an other way (and writing in terms of the bosonic quantities now): the interpretation of the result

\[
Z = e^K |e^{\mathcal{F}}|^2 = \frac{|e^{\mathcal{F}}|^2}{(S + \bar{S})(T + \bar{T})(U + \bar{U})}
\]

coming from the "holomorphic free energy"

\[
\mathcal{F} = \sum_{BPS} \log \mathcal{M}
\]  

(5.5)

and \( m_{BPS}^2 = e^K |\mathcal{M}|^2 \) in the relation

\[
\log Z = \sum_{BPS} \log m_{BPS}^2
\]

shows that the appearence of the explicit expression above was to be expected because for \( Z \) to be invariant \( e^{\mathcal{F}} \) has to be a modular form of weight -1 with respect to the \( SL(2,\mathbb{Z}) \) groups in \( S,T \) and \( U \); so one gets up to an invariant function that

\[
e^{\mathcal{F}} = \frac{1}{\eta^4(s)\eta^4(t)\eta^4(U)}.
\]

This can be applied to the "toroidal sector" of the heterotic string on \( K3 \times T^2 \). In the setup of \( N = 2 \) supersymmetry the relevant expressions for the BPS masses and associated quantities in terms of the prepotential (not to be confused with the holomorphic free energy) \( \mathcal{F} \), and the special coordinates \( X_I \) are \( e^{-K} = X^I \bar{\mathcal{F}}_I + \bar{X}^I \mathcal{F}_I \) and

\[
\log Z = \sum_{M_I, N_I} \log \frac{|M_I X^I + iN_I \mathcal{F}_I|^2}{X^I \bar{\mathcal{F}}_I + \bar{X}^I \mathcal{F}_I}
\]

Note that the sum goes over the restricted subset of integers satisfying the topological level matching condition \( \vec{p}_L^2 - \vec{p}_R^2 = 0 \); and as the duality group \( \Gamma \subset Sp(2n + 2) \) leaves \( K \) invariant only up to a Kaehler transformation so then \( \log Z \) is invariant only if also a symplectic transformation on the integers \( M_I \) and \( N_I \) (accompanying the one on the \( X^I \) and \( i\mathcal{F}_I \)) is made , i.e. the sum runs over symplectic orbits.

To see the full analogy to the toroidal situation let us consider the Calabi-Yau example in greater detail. For definitiveness let us restrict ourselves first to the type IIA case. The chiral masses we want to describe come from the couplings \( X^I HH' \) between one N=2 vector multiplet and two N=2 hypermultiplets , i.e. the relevant massive string states are hypermultiplets with masses given by vev’s of the moduli scalars in the \( (1,1) \) vector multiplet ; so the chiral N=2 masses are just given by the holomorphic functions \( X^I \).

Obviously by considering these field theoretical masses \( X^I \) alone the spectrum can’t be
invariant as the $X^I$ are mixed under the symplectic duality transformations with the periods $i\mathcal{F}_I$. So the relevant ansatz for the hypermultiplet masses is $M = M_I X^I + i N^I \mathcal{F}_I$ with the integers satisfying symplectic constraints. Also the interpretation becomes analogous by considering first the field theory, i.e. large radius, limit. Let us describe the situation in the special gauge (special coordinate system) with $T^i = -i X^i / X^0$ and $f(T) = (X^0)^{-2} \mathcal{F}(X)$, where $K = -\log[\sum_i (T^i + \bar{T}^i)(f_i + \bar{f}_i) - 2(f + \bar{f})]$ and

$$
\log Z = \sum_{M_I, N^I} \log \frac{|M_0 + M_i T^i + i N^0 f + i N^i f_i|^2}{(T^i + \bar{T}^i)(f_i + \bar{f}_i) - 2(f + \bar{f})}
$$

With the harmonic $(1,1)$ forms $V_i$ and $J = \sum_i (T^i + \bar{T}^i)V_i$ one has $e^K = \int J \wedge J \wedge J = d_{ijk}(T^i + \bar{T}^i)(T^j + \bar{T}^j)(T^k + \bar{T}^k)$, where $f(T^i) = d_{ijk}T^i T^j T^k$ and the $d_{ijk}$ coincide in the field theory limit with the topological intersection numbers: $d_{ijk} = \int V_i \wedge V_j \wedge V_k$. With the expression

$$
m^2_{M_0} = \frac{M_0^2}{\int J \wedge J \wedge J} \quad (5.6)
$$

for those masses having only nonvanishing $M_0$ one recognizes (as now these masses are - in this field theory limit - inversely proportional to the volume of the Calabi-Yau space) these masses as the ordinary field theoretical Kaluza-Klein masses. So the states with only nonvanishing $M_0$ give here the momentum spectrum.

On the other hand for small radii nonperturbative instanton configurations giving non-trivial maps of the world-sheet into the Calabi-Yau space give quantum modifications to the aforementioned results. Also now the generalized winding states become light. As the duality invariance forces one to include states with nonvanishing $M_i$ and $N^I$, the masses given by $M_i T^i$ resp. $N^I \mathcal{F}_I$; i.e. we have now also identified the spectrum of generalized winding states.

This makes the toroidal analogy complete.

To include now also the type IIB string in the discussion note first that there the moduli in the vector multiplets are given by $(2,1)$ moduli representing the complex structure deformations of the underlying Calabi-Yau space. These are described now by the (true) periods $(\Omega)$, which are furthermore now quantum uncorrected by world-sheet instantons

$$
X^I = \int_{A^I} \Omega \quad , \quad i \mathcal{F}_J = \int_{B_J} \Omega
$$

of the holomorphic 3-form $\Omega$ integrated over a canonical symplectic homology basis $(A^I, B_I)$. So now these periods are the chiral masses of the massive hypermultiplets (here $e^{-K} = i \int \Omega \wedge \bar{\Omega}$).
5.3 threshold corrections and couplings

The occurrence of a sum over the BPS spectrum here is intimately related - as we will see - to the appearance of a similar expression in the study of threshold corrections [70]. Note first that the product structure $K3 \times T^2$ of the internal space of the heterotic string is reflected on the CFT level by the decomposition of the right-moving internal $c = 9$ superconformal algebra

$$A_{\text{int}} = A_{c=3}^{N=2} \oplus A_{c=6}^{N=4}$$

(5.8)

Let us denote the $U(1)$ current of the $c = 3$ theory by $J^{(1)}$; the representations of the $c = 6$ theory are labeled by the conformal weight and the representation of the (level one) $SU(2)$ Kac-Moody algebra as $(h, I)$. The total $U(1)$ current of the $c = 9$ theory is then $J = J^{(1)} + J^{(2)}$, where $J^{(2)} = 2J^3$ with the $SU(2)$ Cartan current. Furthermore the (local) factorization of the moduli space into vector multiplet and hypermultiplet contributions is in the $c = 6$ algebra reflected by the massless NS representations with $(h = 0, I = 0)$ and $(h = 1/2, I = 1/2)$ respectively (these are connected by the spectral flow of the $N = 4$ theory to massless Ramond representations). Namely as the (complex) central charge is determined by the right-moving momenta $p_R$ carried by the two free dimension $1/2$ superfields of the $c = 3$ theory and as the mass of a NS state is ($N_R$ the right-moving oscillator number coming from (spacetime-sector)$\oplus((c = 3) - \text{sector})$)

$$M^2 = (N_R - 1/2) + \frac{1}{2}p_R^2 + h$$

one has (corresponding to the massless NS representations of $A_{c=6}^{N=4}$) the description of the (bosonic) BPS states (with the slight abuse of language as in [70])

$$M^2 = \frac{1}{2}p_R^2 \Rightarrow \begin{cases} \text{vector} : N_R = 1/2, & (h = 0, I = 0) \\
\text{hyper} : N_R = 0, & (h = 1/2, I = 1/2) \end{cases}$$

Let us insert a remark here on the jumping phenomenon known in $N = 2$ theories [32, 36, 109] with their complex central charge, where in moving in moduli space along a path joining two given points one can’t avoid passing through configurations with collinear Bogomolny charges leading to soliton number jumping (BPS states suddenly (dis)appear under infinitesimal moduli perturbation) related [110] to the intersection numbers of the vanishing cycles, resp. (in view of the Picard-Lefschetz formula [89]) to the corresponding monodromy. Now with regard to this phenomenon, which seems at first to contradict the philosophy that the BPS spectrum gives some ‘rigid topological skeleton’ of a theory note [70] that as usual in such ‘topological’ connections one has to consider an index-like difference to get a deformation invariant quantity: namely these ‘chaotic’ BPS states always appear in vectormultiplet, hypermultiplet pairs; as one moves away from the special points they pair into long representations being no longer saturated.
This brings us to the connection with the threshold corrections. Note that such an index-like difference structure of BPS sums is reasonable to expect there: namely firstly there are no corrections in $N = 4$ and indeed $N = 4$ BPS states split up into a $N = 2$ hypermultiplet and a $N = 2$ vectormultiplet; and secondly one should expect at all only BPS states, i.e. short $N = 2$ representations as massive long $N = 2$ representations (of the extended supertranslation algebra) are short as $N = 4$ representations, so noncontributing.

This is reflected in the threshold corrections to the gauge coupling given by the 1-loop renormalization (for gauge group $G$, $b_G$ the 1-loop beta function coefficient, $Q$ a generator of $G$) \[ g^{-2} = \text{Re}S + \frac{b_G}{16\pi^2} \log \frac{M_{\text{string}}^2}{p^2} + \frac{1}{16\pi^2} \Delta^{\text{univ}} + \frac{1}{16\pi^2} \Delta^{\text{index}} \]

where we have separated the universal part of the threshold correction, which is related to the Green-Schwarz term and so can be expressed in terms of the 1-loop correction $h^1$ of the prepotential

\[
\frac{1}{16\pi^2} \Delta^{\text{univ}} = \frac{1}{2} V_{GS} = \frac{1}{(T + T)(U + U)} \text{Re}(2h^1 - (T + T)\partial_T h^1 - (U + U)\partial_U h^1)
\]

and the further piece $(J_0$ the mentioned total $U(1)$ charge)

\[
\Delta^{\text{index}} = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} \frac{-i}{\eta^2} Tr_{\mathcal{H}^{\text{int}}} \{J_0(-1)^{J_0}q^{L_0 - \frac{c}{24}}\bar{q}^{\bar{L}_0 - \frac{\bar{c}}{24}}[Q^2 - \frac{1}{8\pi\tau_2}]\} - b_G
\]

involving the supersymmetric index \[35, 36\] described by the trace over the Ramond sector of the internal $(c, \bar{c}) = (22,9)$ CFT

\[
Tr_{\text{Ramond}} J(-1)^J q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{\bar{c}}{24}}
\]

To put this definition into perspective recall that defines the usual partition function as $Tr q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{\bar{c}}{24}}$ and the elliptic genus’ \[88, 100, 115\] as $Tr_{\text{Ramond}} (-1)^J q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{\bar{c}}{24}}$. Now according to the decomposition (0.15) and as $J = J^{(1)} + J^{(2)}$ one is left with the index in the (left)×$(c = 3)$ part times the elliptic genus in the $N = 4$ part as the index for the $N = 4$ part vanishes since the eigenvalues of $J^3$ come in opposite pairs and $J^{(2)} = 2J^3$. Furthermore in the $(c = 3)$-part the BPS vectors come (in the Ramond sector) with $U(1)$-charge $\pm\frac{1}{2}$ leading to a $\frac{1}{2}e^{i\pi/2} - \frac{1}{2}e^{-i\pi/2} = i$ contribution to $J e^{i\pi J}$ while one has twofold this content for a BPS hypermultiplet leading to a $2i$ contribution; this is, including the elliptic genus factor from the $N = 4$ part, corrected to $-2i$ resp. $2i$ showing

\[
\frac{1}{\eta^2} Tr_{\text{Ramond}} J(-1)^J q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{\bar{c}}{24}} = -2i \left( \sum_{\text{vector}} BPS \ q^{\Delta} \bar{q}^{\bar{\Delta}} - \sum_{\text{hyper}} BPS \ q^{\Delta} \bar{q}^{\bar{\Delta}} \right)
\]
So one gets essentially

\[
g^{-2} \sim \frac{2}{16\pi^2} \left( \sum_{\text{vector}} \frac{BPS}{Q^2 \log m^2} - \sum_{\text{hyper}} \frac{BPS}{Q^2 \log m^2} \right)
\]

Now as we described the pay-off for the prepotential already in section 3 let us discuss here the connection with the gravitational \( F_1 \) coupling. The (nonperturbative) holomorphic free energy \( \mathcal{F} \), defined as a BPS sum, will diverge at those loci in the (nonperturbative) moduli space where additional BPS states become massless, i.e. along the loci of massless magnetic monopoles and dyons and along other singular strong coupling loci. By string/string duality this nonperturbative holomorphic free energy is given by the corresponding classical quantity of type II string involving a sum over classical BPS states, which is singular precisely along the discriminant locus \( \Delta \) of the (mirror) Calabi-Yau. Now the holomorphic free energy \( \mathcal{F} \) will be identified with \( \mathcal{F}^{\text{grav}} \) (our \( \mathcal{F}_1 \)), which is at tree level

\[
\mathcal{F}^{\text{grav}} = 24S, \quad g^{\text{grav}}_2 = \text{Re}\mathcal{F}^{\text{grav}} = 24\text{Re}S
\]

and at 1-loop (the last term reflects the classical symmetry enhancement)

\[
\mathcal{F}^{\text{grav}} = 24S^{\text{inv}} + \frac{1}{4\pi^2} \log(j(T) - j(U))^2 + \frac{1}{4\pi^2} \log \frac{1}{\eta^2(T)\eta^2(U)}
\]

which has the correct modular weight to render\footnote{an expression, which has in the general case to be supplemented by the quantity \( 12\frac{3-n}{16\pi^2} \log(S + \bar{S}) \), which vanishes for the 3 parameter model; \( K \) is again the tree level Kaehler potential with \( e^{-K} = (S + \bar{S})(T + \bar{T})(U + \bar{U}) \)}

\[
g^{\text{grav}}_2 = \text{Re}\mathcal{F}^{\text{grav}} + \frac{b^{\text{grav}}}{16\pi^2} \left( \log \frac{M^2_{11}}{p^2} + K \right) \tag{5.9}
\]

invariant under T-duality.

As we saw (by explicit evaluation in the case of the Dedekind-function resp. the symmetry enhancement argument in case of the j-function) the term involving the Dedekind resp. j-functions arises from the orbit \( m_1n_1 + m_2n_2 = 0 \) resp. 1. So it is natural to identify \( (\mathcal{M} = M_I P^I + iN^I Q_I) \)

\[
\mathcal{F}^{\text{grav}} \sim \mathcal{F} = \sum_{\text{vector}} \frac{BPS}{\log \mathcal{M}} - \sum_{\text{hyper}} \frac{BPS}{\log \mathcal{M}} \tag{5.10}
\]

where here in the sum over \( M_I \) and \( N^I \) the classical period vector \( \Omega = (P^I | iQ_I)^T = (1, TU, iT, iU | iSTU, iS, -SU, -ST)^T \) is concerned.
So even the tree level part $F_{grav} = 24S$ should arise from BPS states in (6.2) at $S \to \infty$, say from the term ($\hat{S} = 4\pi S$)

$$\log \frac{1}{\eta^2(S)} = \sum_{(s,p)\neq(0,0)} \log(s + ip\hat{S})$$ (5.11)

whose occurrence could be made plausible from the expression $M = (s + ipS)M_{T,U}$ for the holomorphic mass of the short $N = 4$ multiplet.

Now let us be more concrete about the 1-loop corrected gravitational coupling, which we rewrite (with the true heterotic loop counting parameter $2(\text{Re}S + \frac{1}{2}V_{GS})$) as

$$g_{grav}^{-2} = 24(\text{Re}S + \frac{1}{2}V_{GS}) + \frac{b_{grav}}{16\pi^2} \log \frac{M_{\text{string}}^2}{p^2} + \Delta_{grav}$$ (5.12)

with $(e^{-\hat{K}} = (T + \bar{T})(U + \bar{U}), M_{Pl}^2 \sim \frac{1}{g^2}M_{\text{string}}^2)$

$$\Delta_{grav} = 24(\text{Re}\sigma - \frac{1}{2}V_{GS}) + \frac{b_{grav}}{16\pi^2} \hat{K} + \text{Re}\left[\frac{1}{4\pi^2} b_{grav} \log \frac{1}{\eta^2(T)\eta^2(U)} \right. \\
+ \frac{1}{4\pi^2} \log(j(T) - j(U))^2]$$ (5.13)

In the $s = 0$ model of [70] with generic gauge group $E_8 \times E_7 \times U(1)^4$ one can explicitly compute [27] the 1-loop corrected gravitational coupling $F_{grav}$ obtaining in the limit $T \to \infty$ ($U$ finite) as result the tree level holomorphic gravitational coupling: $F_{grav} \to 24S$, showing a $N = 4$ like behaviour of this model in the decompactification limit $T \to \infty$ at weak coupling: $\text{Re}S > \text{Re}T \to \infty$.

Now as $N = 4$ compactifications of the heterotic and type II strings are related through exchange symmetries $S \leftrightarrow T, U$ by string/string/string triality [51] there could be such a nonperturbative exchange symmetry in the $s = 0$ model by the possible (in view of intermed$_{N=4} \rightarrow$ short$_{N=2}$) S-T symmetry of the BPS spectrum of certain $N = 2$ heterotic compactifications. Since contributions to the holomorphic gravitational coupling $F_{grav}$ arise from BPS states only (as shown at 1-loop [70]) let us discuss that symmetry for the object $F_{grav}$!

### 5.3.1 ST model

First, in the 2 parameter model (cf. also [7]), where we already observed the exchange symmetry S-T, the holomorphic gravitational coupling $F_{I}^{II} = \frac{2\pi^2}{3}F_{grav}$ is (with $t_1 = -iT, t_2 = -(i\hat{S} - iT)$; $c_2 \cdot (B + iJ) = 24t_2 + 52t_1 = -i(24\hat{S} + 28T)$)

$$F_{I}^{II} = -\frac{2\pi i}{12} c_2 \cdot (B + iJ) - \sum_{j,k} [2d_{jk} \log \eta(q_{1}^j q_{2}^k) + \frac{n_{jk}}{6} \log(1 - q_{1}^j q_{2}^k)]$$ (5.14)
So in the weak coupling limit $\text{Re} T < \text{Re} S \to \infty$ (so $q^j q^k \to 0$)

$$F^{II}_{1} \to -\frac{2\pi i}{12} c_2 \cdot (B + iJ) = \frac{2\pi}{12} (24\tilde{S} + 28T)$$

(5.15)

whereas in the strong coupling limit $\text{Re} S < \text{Re} T \to \infty$ (so $q^j q^k \not\to 0 \Rightarrow k > j$) where then $n_{jk} = 2\delta_{j0}\delta_{k1}$ and $d_{jk} = 0$, so $\sum_{j,k \geq 0} \frac{n_{jk}}{6} \log(1 - q^j q^k) = \frac{2}{6} \log(1 - q_2) = \frac{2}{6} \log(q_2) = \frac{4\pi i}{6} \log t^2$

$$F^{II}_{1} \to -\frac{2\pi i}{12} c_2 \cdot (B + iJ) + \frac{4\pi}{6}(\tilde{S} - T) = \frac{2\pi i}{12} (28\tilde{S} + 24T)$$

(5.16)

so that for $S, T \to \infty$

$$F^{II}_{1} \to \frac{2\pi i}{12} [(24\tilde{S} + 28T)\theta(\tilde{S} - T) + (28\tilde{S} + 24T)\theta(T - \tilde{S})]$$

(5.17)

showing $\tilde{S} \leftrightarrow T$ exchange symmetry and the corresponding chamber dependence (cf. [70]).

### 5.3.2 STU model

Second, in the $s = 0$ model with corresponding exchange symmetry assumed, one gets with the aid of the explicit expressions that analogously [27]

$$F_{\text{grav}} \to \frac{6}{\pi} [T\theta(T - \tilde{S}) + \tilde{S}\theta(\tilde{S} - T)]$$

(5.18)

### 5.3.3 STUV model

Lastly note again in the $STUV$ model the important role in the computation of the Wilsonian gravitational coupling $F_1$ is played by BPS states [27, 57, 70, 110],

$$F_1 \propto \log M$$

(5.19)

where $M$ denotes the moduli-dependent holomorphic mass of an $N = 2$ BPS state. For the $STUV$ models under consideration, the tree-level mass $M$ is given by [31, 81, 95]

$$M = m_2 - im_1 U + im_1 T + n_2(-UT + V^2) + ibV$$

(5.20)

Here, $l = (n_1, m_1, n_2, m_2, b)$ denotes the set of integral quantum numbers carried by the BPS state. The level matching condition for a BPS state reads

$$2(p^2_{L} - p^2_{R}) = 4n^T m + b^2$$

(5.21)

Of special relevance to the computation of perturbative corrections to $F_1$ are those BPS states, whose tree-level mass vanishes at certain surfaces/lines in the perturbative
moduli space $\mathcal{H}_2 = \frac{SO(3,2)}{SO(3) \times SO(2)}$. Note that $\mathcal{M} = 0$ is the condition (see appendix A.2) for a rational quadratic divisor

$$\mathcal{H}_1 = \left\{ \begin{pmatrix} iT & iV \\ iV & iU \end{pmatrix} \in \mathcal{H}_2 \middle| m_2 - im_1 U + in_1 T + n_2 (-UT + V^2) + ibV = 0 \right\} \quad (5.22)$$

of discriminant

$$D(l) = 2(p^2_L - p^2_R) = 4m_1 n_1 + 4n_2 m_2 + b^2. \quad (5.23)$$

Consider, for instance, BPS states becoming massless at the surface $V = 0$, the so-called Humbert surface $H_1$ (cf. appendix A.2). They lay on the orbit $D(l) = 1$, i.e. on the orbit $n^T m = 0, b^2 = 1$. On the other hand, BPS states becoming massless at $T = U$, the Humbert surface $H_4$, lay on the orbit $D(l) = 4$, which means that they carry quantum numbers satisfying $n^T m = 1, b^2 = 0$ [26].
6 The duality

The assertion of duality states that $N = 2$ heterotic vacua are quantum equivalent to $N = 2$ type IIA vacua \[75\]. Of course this can be combined - as is done in the actual procedure of establishing the duality checks - with the perturbative equivalence of IIA with IIB on the mirror CY.

Note first of all, for heterotic vacua the rank of the gauge group is bounded by the central charge to be less than 24 (eq. (3.9)) while in type II vacua the rank can certainly be much larger since both Hodge numbers easily exceed 22. Furthermore, we saw that the heterotic vacua have large non-Abelian gauge groups at special points in their moduli space while type II A vacua only have an Abelian gauge group. However, the analysis of Seiberg and Witten \[109\] taught us that asymptotically free non-Abelian gauge groups generically do not survive non-perturbatively but instead are broken to their Abelian subgroups. Conversely one has to show that in a particular limit corresponding to the heterotic weak coupling limit a type II vacuum can have a non-Abelian enhancement of its gauge group \[14, 115\].

Furthermore one has to keep in mind that the prepotential on the heterotic side $\mathcal{F}_{\text{het}}$ is only known perturbatively, that is in a weak coupling expansion. For a heterotic vacuum weak coupling corresponds to large $S$ and hence there has to be a type II modulus in a vector multiplet (so this cannot be the type II dilaton which sits in a hypermultiplet) which is identified with the heterotic dilaton; i.e. it has to be one of the $h_{1,1}$ Kähler deformations of the Calabi–Yau threefold. Thus one is interested in identifying this dual type II partner $t^s$ of the heterotic dilaton $S$. From the discrete PQ-symmetries one immediately infers that the relation must be

$$t^s \equiv 4\pi i S . \quad (6.1)$$

Once $t^s$ has been identified one can expand the type IIA prepotential $\mathcal{F}_{IIA}$ in a $t^s$ perturbation expansion around large $t^s$, i.e.

$$\mathcal{F}_{II} = \mathcal{F}_{II}(t^s, t^i) + \mathcal{F}_{II}(t^i) + \mathcal{F}_{II}(e^{-2\pi i t^s}, t^i) . \quad (6.2)$$

This expansion can be compared to the perturbative expansion of the heterotic prepotential. In particular one has to find19

$$\mathcal{F}_{II}(t^s, t^i) + \mathcal{F}_{II}(t^i) = \mathcal{F}_{\text{het}}^{(0)}(S, \phi^I) + \mathcal{F}_{\text{het}}^{(1)}(\phi^I) \quad (6.3)$$

(up to an appropriate overall normalization). Note that the left hand side is determined at tree level whereas the right hand side sums perturbative contributions at tree level and at one-loop.

Now generic properties should involve the heterotic dilaton since it couples universally for all heterotic vacua. Indeed, from eqs. (3.4),(6.3) one infers that the Calabi–Yau intersection numbers of a dual type IIA vacuum have to respect that $D_s \cdot D_s = 0$ and that $\text{sign}(d_{ij}) = (+, -, \ldots, -) = \text{sign}(\eta_{ij})$. In addition, eqs. (3.69), (4.2) imply $\int e_S \wedge c_2(Y) = 24$. These conditions imply that the Calabi–Yau manifold has to be

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19There can still be non-perturbative inequivalencies \[13, 18, 12, 24\].
a $K3$-fibration \[\text{[16]}\] \[\text{[24]},\] i.e. the Calabi–Yau is fibred over a $P^1$ base with $K3$ fibers. The size of the $P^1$ is parameterized by the modulus $t^s$ which is the type II dual of the heterotic dilaton. Over a finite number of points on the base, the $K3$ fibre can degenerate to a singular $K3$. The other Kähler moduli $t^i$ are either moduli of the $K3$ fibre or of the singular fibers. In general one finds $\text{sign}(d_{sij}) = (+, -, \ldots, -0, \ldots, 0)$, where the non-vanishing entries correspond to moduli from generic $K3$ fibers while the zeros arise from singular fibers. Since a $K3$ has at most 20 moduli the non-vanishing entries have to be less than 20.$^{20}$

Note that the $K3$ fibration is intuitively understood to arise by the argument of adiabatic extension of the 6D duality between the heterotic string on $T^4$ and type IIA on $K3$ \[\text{[112]}\]. Namely the heterotic compactification space $K3 \times T^2$ can be seen as fibration of $T^2_{K3} \times T^2$ over $P^1$ corresponding then to the type IIA picture.

Now the puzzle concerning non-abelian gauge symmetry enhancement in type IIA compactifications is easily understood. The non-abelian gauge symmetry enhancement arises for the heterotic string at weak coupling by the Frenkel-Kac-Segal mechanism. As the heterotic dilaton parameter corresponds to the size of the base $P^1$ of the Calabi-Yau on the type IIA side the weak coupling regime of the heterotic string is mapped to the 6D decompactification limit, i.e. effectively to type IIA on $K3$. There the enhanced gauge symmetry can be understood as arising from the $BPS$ soliton states given membranes wrapped around 2-cycles of $K3$. Namely the mass of theses $BPS$ states is proportional to the area of the 2-cycle, so as the 2-cycle shrinks to zero size the new state appears in the massless spectrum. Furthermore the local description of these degenerate $K3$ surfaces follows the pattern of $ADE$ singularities, so that the sought for heterotic gauge symmetry structure can arise from the intersection lattice $E_8 \oplus E_8 \oplus H \oplus H \oplus H$ of $K3$ (with the negative definite $E_8$ lattice and $H = \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right)$ the hyperbolic plane).

So the following picture about the matching of the vector moduli arises. $S_{\text{het}}$ corresponds to the cohomology class of the base $P^1$ of the $K3$ fibration of the Calabi-Yau; equivalently it corresponds to the 4-cycle represented by the generic $K3$ fibre. The perturbative heterotic gauge fields correspond to the 4-cycles which arise from 2-cycles of the generic $K3$ which are invariant under the monodromy group $\Gamma$ of the base $P^1$ (the fundamental group of the base $P^1$ with the finitely many points over which the $K3$ degenerates deleted) and so can be extended to 4-cycles of the Calabi-Yau. Lastly the nonperturbative heterotic gauge fields correspond to classes coming from the degenerate fibers.

Let us follow the connection of the classical heterotic gauge group and the classes in the Picard lattice $\text{Pic} := \text{Pic}(K3)^\Gamma$ more closely ($\text{Pic}(K3) = H^2(K3, \mathbb{Z}) \cap H^{1,1}(K3)$); we will denote the rank of Pic by $\rho$, the so-called Picard number. The moduli space of Kähler forms is then \[\text{[14]}\] given by the Grassmanian of space-like two-planes in $\Lambda \otimes_{\mathbb{Z}} \mathbb{R}$ where the lattice $\Lambda$ is defined by $\Lambda := \text{Pic} \oplus H \in H^*(K3, \mathbb{Z})$ with the $H$-part corresponding to $H^0 \oplus H^4$. Now as the signature of Pic is $(1, \rho - 1)$ one has (globally, i.e.

$^{20}$There is a possible subtlety here since this counts only geometrical $K3$ moduli. However, it is conceivable that quantum effects raise this number up to 22. \[\text{[13]}\]
up to division by a discrete group)

\[ \mathcal{M}_{\text{Käh}}(K3) = \frac{O(2, \rho)}{O(2) \times O(\rho)} \quad (6.4) \]

and the roots of the gauge group are giben by \( \{ \alpha \in \Lambda | \alpha^2 = -2 \text{ and } \alpha \in \mathcal{V} \} \), where \( \mathcal{V} \) is the space-like two-plane in \( \Lambda \otimes _\mathbb{Z} \mathbb{R} \).

For example in our three-parameter model \( P_{1,1,2,8,12}(24) \) with \( K3 \) given by \( P_{1,1,4,6}(12) \) one has \( \rho = 2 \) coming from the generic hyperplane section and the blow-up curve \( C \) of the \( \mathbb{Z}_2 \) singularity due to the common factor in the last two weights (correspondingly one has for the \( K3 \)'s of the two- and four-parameter model that \( \rho = 1 \) and \( \rho = 3 \)). One finds \( \Lambda = H \oplus H \) and \( \mathcal{M} = \frac{O(2,2)}{O(2) \times O(2)} \) (up to \( O(2,2; \mathbb{Z}) \) modding) and as the size parameter of \( C \) goes to zero a \( SU(2) \) arises which corresponds to the heterotic \( SU(2) \) arising for \( T - U \to 0 \).
6.1 ST model

Now the vector multiplet couplings are known for the type II vacuum [25, 72], whereas for the heterotic vacuum they have been studied pertubatively only [6, 45]. Recall that in the weak coupling expansion

$$F_{\text{het}} = \frac{1}{2}ST^2 + h(T) + h^{np}(e^{-8\pi^2 S}, T) \quad (6.5)$$

the dilaton independent 1-loop correction was found to transform (up to a quartic polynomial) under the $SL(2, \mathbb{Z})_T$ as a modular form of weight -4. The enhancement at $T = 1$ of the gauge group $U(1)^3$ (including the graviphoton) to $SU(2) \times U(1)^2$ is reflected in the singularity of $h$ at $T = 1$ (besides the one at $T = \infty$); because of the modular property one gets then from the logarithmic singularity $\partial^2_T h \sim \frac{1}{2\pi^2} \log(T - 1)$ of $\partial^2_T h$ that

$$\partial^2_T h = \frac{1}{4\pi^2} \log(j(iT) - j(i)) + \text{finite} \quad (6.6)$$

As the dilaton is in contrast to the tree level situation no longer invariant under T-duality one defines a corrected modular invariant coordinate with the correction term $\sigma = S^{\text{inv}} - S$ given by (up to a constant)

$$\sigma = \frac{1}{3} \left( \partial^2_T h - \frac{1}{4\pi^2} \log(j(iT) - j(i)) \right) \quad (6.7)$$

Now the fact that the discriminant factor (conifold part)

$$(1 - \bar{x})^2 - \bar{x}^2 \bar{y}$$

shows a complete square in $\bar{x}$ splitting as one moves $\bar{y}$ away from 0 was one of the identification properties used to match the variables in the string dual pair [75] as this is precisely the stringy realization of the expected Seiberg-Witten behaviour for a $SU(2) \ N = 2$ gauge theory, where the isolated singular point of classical $SU(2)$ restauration is split into two singular points of massless monopoles in the full quantum theory. So one was led at first [75] to identify $\bar{y} = e^{-S}$ as $\bar{y} = 0$ corresponds to large $S$ and (at least there) the classical enhancement point $T = 1$ with $\bar{x} = 1$. Now the 1-loop corrections of the prepotential can be matched [75] on the basis of the common occurence of the j-function on both sides of the proposed dual pair: coming on the heterotic side from the behaviour under T-duality, and on the Calabi-Yau side from the mentioned $K3$ fibre structure. We get the identification of the modular invariant dilaton (cf. also [45]) by matching the two prepotentials at weak coupling

$$\partial^2_T F_{\text{het}} = S + \partial^2_T h$$
and
\[ \partial_{F}^{2}F^{II} = S + \frac{1}{4\pi^{2}} \log(j - \alpha) - \frac{3}{2} \frac{1}{4\pi^{2}} \log g \]

so that we have in view of

\[ S^{\text{inv}} = S + \frac{1}{3} \left[ \partial_{T}^{2}h - \frac{1}{4\pi^{2}} \log(j - \alpha) \right] \]
\[ = S - \frac{1}{8\pi^{2}} \log g \quad (6.8) \]

or with \( \tilde{S}^{\text{inv}} = 4\pi S^{\text{inv}} \)

\[ t = t_{1} = iT \]
\[ y = q_{2}g(q_{1}) = e^{-8\pi^{2}S}g = e^{-8\pi^{2}S^{\text{inv}}} = q(i\tilde{S}^{\text{inv}}). \]

with \( t_{1} = iT \) and \( t_{2} = i\tilde{S}^{\text{inv}} = 4\pi i S^{\text{inv}} \).

Now after matching (at weak coupling) the prepotentials describing gauge couplings one can match the \( F_{1} \) functions describing gravitational couplings. One gets with \( b_{\text{grav}} = 48 - \chi = 300 \)

\[ F_{1} = 24S^{\text{inv}} + \frac{1}{4\pi^{2}} \log(j(iT) - j(i)) - \frac{300}{4\pi^{2}} \log \eta^{2}(iT) \]
\[ = \frac{6}{4\pi^{2}} \log[y^{-\alpha}(j - \alpha)^{\beta} \eta^{2}(t_{1})^{\gamma}] \]

with \( \alpha = 2, \beta = \frac{1}{6}, \gamma = \frac{1}{6}b_{\text{grav}}, \) i.e. \( \alpha/\beta = 12, \gamma/\beta = b_{\text{grav}} \) showing (after a suitable overall normalization) the coincidence with the type II side (the normalization constant 24 of \( S^{\text{inv}} \) corresponds to the Euler number of \( K3 \), the modularity compensation constant \( b_{\text{grav}} \) is matched).
6.2 STU model

Let us now compare the heterotic and the type II prepotentials and identify the \( t_i \) \((i = 1, 2, 3)\) with \( S, T \) and \( U \). In the following, we will actually match \(-4\pi F_0^{het}\) with \( F_0^{II} \).

First compare the cubic terms in (4.18) and (3.36). By assuming that the \( t_i \) and \( S, T \) and \( U \) are linearly related, the following identification between the Kähler class moduli and the heterotic moduli is enforced by the cubic terms

\[
\begin{align*}
  t_1 &= U \\
  t_3 &= T - U \\
  t_2 &= 4\pi S^{inv} = \tilde{S} + \beta T + \alpha U
\end{align*}
\]

where \( \tilde{S} = 4\pi S \). Recall that we are working inside the Kähler cone \( \sigma(K) = \{ \sum t_i J_i | t_i > 0 \} \). Now, in the heterotic weak coupling limit one has that indeed \( t_2 > 0 \). Demanding \( t_3 > 0 \) implies that one is choosing the chamber \( T > U \) on the heterotic side. The identification of \( t_1 \) and \( t_3 \) agrees, of course, with the one of [84]. The identification of \( 4\pi S^{inv} \) with the Kähler variable \( t_2 \) becomes very natural when performing the map to the mirror Calabi-Yau compactification with complex structure coordinates \( x, y, z \).

Here, since \( y \) is invariant under the CY monodromy group, \( y \) should be identified [84] with \( e^{-8\pi^2 S^{inv}} \). Thus, equation (3.38) provides the explicit mirror map; for large \( T, U \) the Kähler variable \( q_2 = e^{-2\pi t_2} \) and the complex structure field \( y \) completely agree.

Next, consider the exponential terms in the prepotential \( F_0^{II} \). In the heterotic weak coupling limit \( S \to \infty \), one has that \( t_2 \to \infty \) and, hence, \( q_2 = e^{-2\pi t_2} \to 0 \). Then, (4.18) becomes

\[
F_0^{II} = F^0 - \frac{1}{(2\pi)^3} \sum_{d_1,d_3} n^r_{d_1,0,d_3} \text{Li}_3(q_1^{d_1} q_3^{d_3}).
\]

Some of the instanton coefficients contained in (6.10) are as follows [72]

\[
\begin{align*}
  n^r_{d_1,0,0} &= n^r_{d_1,0,d_1} = 480 = -2(-240), \\
  n^r_{0,0,1} &= -2, \ n^r_{0,0,d_3} = 0, \ d_3 = 2, \ldots, 10; \\
  n^r_{2,0,1} &= 282888 = -2(-141444), \\
  n^r_{3,0,1} &= 17058560 = -2(-8529280), \\
  n^r_{4,0,1} &= 477516780 = -2(-238758390).
\end{align*}
\]

Note that the fact that \( n^r_{d_1,0,0} = n^r_{d_1,0,d_1} \) is a reflection of the \( T \leftrightarrow U \) exchange symmetry.

Now rewriting \( kT + lU = (l+k)U + k(T - U) = (l+k)t_1 + kt_3 \) and matching \( F_0^{II} = -4\pi F_0^{het} \) yields the following identifications

\[
d_1 = k + l, \ d_3 = k
\]

and so

\[
n^r_{d_1,0,d_3} = n^r_{k+l,0,k} = -2c_1(kl).
\]
Note that $d_3 = k \geq 0$ for points inside the Kähler cone. Also, if $d_3 = k = 0$, then $d_1 = l > 0$. On the other hand, if $d_3 = k > 0$, then $d_1 \geq 0$, that is $l \geq -k$.

Comparison of the instanton coefficients listed above with the $c_1$-coefficients occurring in the $q$-expansion of $F(q) = \frac{E_4E_6}{\eta^4}$ in equation (6.7), shows that the relation (6.13) is indeed satisfied.

Let us now determine the action of the symmetries (4.21) and (4.22) on the heterotic variables. Clearly the perturbative symmetry (4.21) corresponds to the exchange $T \leftrightarrow U$ for $S \to \infty$. The non-perturbative symmetry (4.22) corresponds to

$$S \to -(1 + \beta)S - \frac{\alpha(2 + \beta)}{4\pi} U - \frac{\beta(2 + \beta)}{4\pi} T,$$

$$T \to 4\pi S + (1 + \beta)T + \alpha U,$$

$$U \to U.$$  \hfill (6.14)

There is one very convenient choice for the parameters $\alpha$ and $\beta$, in which the non-perturbative symmetry (6.14) takes a very simple suggestive form. Namely, for $\alpha = 0$ and $\beta = -1$, this transformation becomes

$$4\pi S \leftrightarrow T,$$  \hfill (6.15)

that is, it just describes the exchange of the heterotic dilaton $S$ with the Kähler modulus $T$ of the two-dimensional torus. This choice for $\alpha$ and $\beta$ is very reasonable, since it is only in this case that the real parts of $S$ and $T$ remain positive after the exchange (6.14). At the end of this paper, by considering some six-dimensional one-loop gauge couplings, we will give some further arguments indicating that the choice $\beta = -1$ is the physically correct one. So, for the time being, we will set $\alpha = 0$ and $\beta = -1$ and discuss a few issues related to the exchange symmetry $4\pi S \leftrightarrow T$.

The non-perturbative exchange symmetry $4\pi S \leftrightarrow T$ is true for arbitrary $U$ in the chamber $S, T > U$. As already discussed in detail in [84], at the fixed point $t_2 = S^{\text{inv}}_{\infty} = 0$ of this transformation, one has that $S = T > U$, the complex structure field $y$ takes the value $y = 1$, and the discriminant of the Calabi-Yau model vanishes. The locus $S = T > U$ corresponds to a strong coupling singularity with additional massless states. In the model based on the Calabi-Yau space $P_{1,1,2,8,12}(24)$, a non-Abelian gauge symmetry enhancement with an equal number of massless vector and hypermultiplets takes place at $S = T > U$, such that the non-Abelian $\beta$-function vanishes [83, 86].

On the other hand, the non-perturbative exchange symmetry $4\pi S \leftrightarrow T$ implies that for $T \to \infty$ there is a ‘perturbative’ exchange symmetry $4\pi S \leftrightarrow U$ exchange symmetry. This symmetry is nothing but the $T - S$ transformed perturbative symmetry (4.21). Furthermore, for $T \to \infty$, there is a modular symmetry $SL(2, \mathbb{Z})_S \times SL(2, \mathbb{Z})_U$ and the corresponding ‘perturbative’ monodromy matrices of the prepotential can be computed in a straightforward way. Hence, for $T \to \infty$, there is a ‘perturbative’ gauge symmetry enhancement of either $U(1)^2$ to $SU(2) \times U(1)$ or to $SU(2)^2$ or to $SU(3)$ at the points $S = U$, $S = U = 1$ or $S = U = e^{i\pi/6}$, respectively, with no additional massless hyper multiplets [27].

Let us now investigate the gravitational coupling $F_1$. The non-exponential piece, which
dominates for large \( t_i \), reads \[86\]

\[-i \sum_{i=1}^{3} t_i c_2 \cdot J_i = 92t_1 + 24t_2 + 48t_3. \tag{6.16}\]

This expression is explicitly invariant under the non-perturbative symmetry \[1.22\]. Furthermore, by also explicitly checking some of the elliptic instanton numbers \( n_{d_1,d_2,d_3}^e \), one discovers that, just like in the case of \( n_r \),

\[ n_{d_1,0,d_3}^e = n_{d_1,0,d_1-d_3}^e \quad \text{and} \quad n_{d_1,d_2,d_3}^e = n_{d_1,d_1-d_2,d_3}^e. \tag{6.17}\]

It follows that \( F_1^{\text{III}} \) is symmetric under the two exchange symmetries \[1.21\] and \[1.22\].

Let us now compare the heterotic and the type II gravitational couplings \[28\]. We will match \( 4\pi F_1^{\text{het}} \) with \( F_1^{\text{III}} \). First take the decompactification limit to \( D=5 \), i.e. the limit \( T,U \to \infty \) (\( T > U \)). This eliminates all instanton contributions, i.e. all exponential terms. In the heterotic case we get in this limit

\[ F_1^{\text{het}} \to F_1^{\infty} = 24S_{\text{inv}}^\infty + \frac{12}{\pi} T + \frac{11}{\pi} U = 24S + \frac{12 + 6\beta}{\pi} T + \frac{11 + 6\alpha}{4\pi} U. \tag{6.18}\]

By comparing this expression with the type II large \( t_i \) limit given in \( \text{(6.16)} \), one finds that \( \text{(6.16)} \) and \( \text{(6.18)} \) match up precisely for the identification given in \( \text{(6.9)} \) between heterotic and type II moduli. When choosing \( \alpha = 0 \) and \( \beta = -1 \), it follows that \( F_1^{\infty} \) is symmetric under the exchange \( 4\pi S \leftrightarrow T \). This symmetry implies that in the limit \( T \to \infty \), \( F_1^{\text{het}} \) can be written \[27\] in terms of \( SL(2,\mathbb{Z})_S \) modular functions \( j(4\pi S) \) and \( \eta(4\pi S) \) by just replacing \( 4\pi S \) with \( T \) in equation \( \text{(3.80)} \).

Next, let us compare the exponential terms in \( F_1^{\text{III}} \) and \( F_1^{\text{het}} \). In the type II case we have to consider the weak coupling limit \( q_2 \to 0 \); hence only the terms with the instanton numbers \( n_{d_1,0,d_3}^{re} \) contribute to the sum. We will see that, when comparing with the heterotic expression, one gets a very interesting relation between the rational and elliptic instanton numbers for \( d_2 = 0 \). In order to do this comparison, we have to recall that \( \text{Li}_1(e^{-2\pi(kT+U)}) = -\log(1-e^{-2\pi(kT+U)}) \). The difference \( j(T) - j(U) \) can be written in the following useful form (in the chamber \( T > U \)) \[23, 70\]

\[ \log(j(T) - j(U)) = 2\pi T + \sum_{k,l} c(kl) \log(1-e^{-2\pi(kT+U)}), \tag{6.19}\]

where the integers \( k \) and \( l \) can take the following values \[70\]: either \( k = 1, l = -1 \) or \( k > 0, l = 0 \) or \( k = 0, l > 0 \) or \( k > 0, l > 0 \). First consider the terms with \( k = 1, l = -1 \) on the heterotic side. Matching the term \( \log(1-e^{-2\pi(T-U)}) \) contained in \( 4\pi F_1^{\text{het}} \) with \( F_1^{\text{III}} \) requires that

\[ 10c(-1) - 12c_1(-1) = 12n_{0,0,1}^e + n_{0,0,1}^r. \tag{6.20}\]

This is indeed satisfied, since \( c(-1) = c_1(-1) = 1 \) and \( n_{0,0,1}^e = 0, n_{0,0,1}^r = -2 \).

\[21\]In \[27\], a different choice was made for these two parameters, namely \( \alpha = -11/6 \) and \( \beta = -2 \). Hence it follows that \( F_1^{\infty} = 24S \).
Next, consider the terms in the sum with $k > 0, l = 0$ (and analogously $k = 0, l > 0$). Since $c(0) = 0$, only the term $\frac{b_{grav}}{8\pi^2} \log n^{-2}(T)$ contributes on the heterotic side ($b_{grav} = 528$). Matching $4\pi F^1_{het}$ with $F^I_{11}$ yields the following relation among the instanton numbers ($d_1 = d_3 = k$):

$$12 \sum_{i=1}^{s} n^e_{k,0,k} + n^r_{k,0,k} = b_{grav} = 528. \tag{6.21}$$

The $k_i$ ($i = 1, \ldots, s$) are the divisors of $k$ ($k_1 = k$, $k_s = 1$). Using Klemm’s list of explicit instanton numbers one checks that this relation is indeed true ($n^e_{1,0,1} = 4$, $n^r_{k,0,k} = 0$ for $k > 1$, $n^r_{k,0,k} = -\chi = 480$).

Finally, consider the case where $k > 0, l > 0$. By comparing the heterotic and type II expressions we derive the following interesting relation ($d_1 = k + l$, $d_3 = k$):

$$12 \sum_{i=1}^{s} n^e_{d_1,0,d_3} = -n^r_{k+l,0,k} + 10c(kl) + 12kc_1(kl) = 10c(kl) + (12kl + 2)c_1(kl). \tag{6.22}$$

Here $s$ is the number of common divisors $m_i$ ($i = 1, \ldots, s$) of $d_1 = k + l$ and $d_3 = k$ with $d_1^i = d_1/m_i$ and $d_3^i = d_3/m_i$ (where $m_1 = 1$). Again we can explicitly check the non-trivial relation (6.22) for the first few terms. For example, for $k = l = 1$ one has

$$n^e_{2,0,1} = -948 \quad \text{and} \quad n^r_{2,0,1} = 282888, \tag{6.23}$$

which, together with equations (A.7) and (A.10), confirms the above relation. For $k = 2$ and $l = 1$ one finds that $12n^e_{3,0,2} + n^r_{3,0,2} = 10c(2) + 24c_1(2)$ is indeed satisfied, since

$$n^e_{3,0,2} = -568640 \quad \text{and} \quad n^r_{3,0,2} = 17058560. \tag{6.24}$$

And finally, for $k = l = 2$ for instance, one finds that the relation $12 \left(n^e_{4,0,2} + n^e_{2,0,1}\right) + n^r_{4,0,2} = 10c(4) + 48c_1(4)$ indeed holds due to

$$n^e_{4,0,2} = -1059653772 \quad \text{and} \quad n^e_{2,0,1} = -948 \quad \text{and} \quad n^r_{4,0,2} = 8606976768. \tag{6.25}$$

One can (cf. appendix A1) rewrite equation (6.22) as

$$12 \sum_{i=1}^{s} n^e_{d_1,0,d_3} + n^r_{d_1,0,d_3} = 10c(kl) + 12kc_1(kl) = -2c_1(kl), \quad k > 0, l > 0. \tag{6.26}$$

At the end let us briefly comment on the relation to the heterotic/heterotic duality in six dimensions \cite{99,101} with the 6-dimensional heterotic string compactified on $K_3$. (The decompactification limit from $D = 4$ to $D = 6$ is obtained by sending $T \to \infty$ with $U$ finite; as discussed in \cite{119,122}, the $D = 6$ heterotic/heterotic duality becomes an exchange symmetry of $S$ with $T$ in $D = 4$.) We will concentrate on CY’s with Hodge numbers $(3,243)$, which are elliptic fibrations over $F_n$ with $n = 0, 2$. \cite{104}. Being elliptic fibrations, they can be used to compactify $F$-theory to six dimensions. In the $D = 6$ heterotic string, the integer $n$ is related to the number $s$ of $SU(2)$ instantons...
in one of the two $E_8$’s by $n = s - 12 \ [94, 104]$. First consider the case of an elliptic
fibration over $F_0$, corresponding to the symmetric embedding of the $SU(2)$ bundles
with equal instanton numbers $s = s' = 12$ into $E_8 \times E_8'$. This leads to a $D = 6$
heterotic model with gauge group $E_7 \times E_7'$ with $\tilde{v}_\alpha = \tilde{v}_\alpha' = 0 \ [72]$. There are 510
hypermultiplets transforming as $4(56, 1) + 4(1, 56) + 62(1, 1)$. The heterotic/heterotic
duality originates from the existence of small instanton configurations \[117\]. The model
is, however, not self-dual, since the non-perturbative gauge groups appear in different
points of the hyper multiplet moduli space than the original gauge groups \[52\]. For
generic vev’s of the hyper multiplets the gauge group is completely broken and one
is left with 244 hyper multiplets and no vector multiplets. Upon compactification to
$D = 4$ on $T_2$ one arrives at the heterotic string with gauge group $U(1)^4$, which is the
dual to the considered type II string on the CY $P_{1,1,2,8,12}(24)$. Semiclassically, at special
points in the hypermultiplet moduli space, this gauge group can be enhanced to a non-
Abelian gauge group, inherited from the $E_7 \times E_7'$ with $\tilde{v}_\alpha = 1/6$ and
$\tilde{v}_\alpha' = -1/6$ and hyper multiplets transforming as $5(56, 1) + 3(1, 56) + 62(1, 1)$. The
second $E_7$ can be completely Higgsed away, leading to a $D = 6$ heterotic model with
gauge group $E_7$ and hypermultiplets $5(56) + 97(1)$. As explained in \[3\], this model also
possesses a heterotic/heterotic duality, however without involving non-perturbative
small instanton configurations. Hence in this sense, this model is really self-dual. Just
like in the case of the symmetric embedding, the gauge group $E_7$ is spontaneously
broken for arbitrary vev’s of the gauge non-singlet hyper multiplets and one is again
left with 244 hyper multiplets and no vector multiplet. When compactifying on $T_2$ to
$D = 4$, one obtains the same heterotic string model with $U(1)^4$ gauge group as before.
For special values of the hyper multiplets a non-Abelian gauge group is obtained, now
however with $\beta$-function coefficient $b_\alpha = 12(1 + \tilde{w}_\alpha) = 24 \ [3]$. 
In summary, the symmetric $(12,12)$ model and the asymmetric $(14,10)$ model should
be considered as being the same \[3, 94\], since both are related by the Higgsing and
both lead to the same heterotic string in $D = 4$.
As already mentioned, we would like to provide a six-dimensional argument for why
$\beta = -1$ is the physically correct choice for one of the cubic parameters. We will
directly follow the discussion given in \[3\] and consider the one-loop gauge coupling for
the enhanced non-Abelian gauge groups that are inherited from the six-dimensional
gauge symmetries. Specifically, the gauge kinetic function is of the form \[3, 13\]
\[ f_\alpha = S^{inv} - \frac{b_\alpha}{8\pi^2} \log(\eta(T)\eta(U))^2. \] (6.27)
Using equation (3.39) this then in the decompactification limit $T \to \infty$ to $D = 6$
becomes $f_\alpha \to S + \frac{1+\beta+\tilde{w}_\alpha}{4\pi} T$. By comparing this expression with the six-dimensional
gauge coupling \[99\], it then follows that $\beta = -1$. 
6.3 STUV model

In order to match (4.25) with the cubic part of the heterotic prepotential given in (3.56), we will perform the following identification of type II and heterotic moduli (which differs from the one given in [18]):

\[ t_1 = U - 2V, \quad t_2 = S - \frac{n}{2}T - \left(1 - \frac{n}{2}\right)U, \]
\[ t_3 = T - U, \quad t_4 = V, \]  \(6.28\)

which is valid in the chamber \( T > U > 2V \). Then, (4.25) turns into

\[ F_{\text{cubic}}^{\text{II}} = - F_{\text{cubic}}^{\text{het}} = S(TU - V^2) + \frac{1}{3}U^3 + \left(\frac{4}{3} + n\right)V^3 - \left(1 + \frac{n}{2}\right)UV^2 - \frac{n}{2}TV^2 \]  \(6.29\)

Note that, using the heterotic moduli, the prepotential is independent of \( n \) in the limit \( V = 0 \).

The heterotic weak coupling limit \( S \to \infty \) corresponds to the large Kähler class limit \( t_2 \to \infty \). In this limit, only the instanton numbers with \( d_2 = 0 \) contribute in the above sum. Using the identification \( kT + lU + bV = d_1 t_1 + d_3 t_3 + d_4 t_4 \), it follows that (independently of \( n \))

\[ k = d_3, \quad l = d_1 - d_3, \quad b = d_4 - 2d_1. \]  \(6.30\)

Then, (1.20) turns into

\[ F_{\text{inst}}^{\text{II}} = - \frac{1}{(2\pi)^3} \sum_{k,l,b} n_{k,l,b} L_i e^{-2\pi(kT+lU+bV)} \]  \(6.31\)

The rational instanton numbers have now to satisfy

\[ n_{k,l,b}^r = -2c_n(4kl - b^2). \]  \(6.32\)

Note that the constraint (4.29) is non-trivial. Also note that \( 2c_n(0) = \chi(X_n) \) and recall that \( 2c_n(-1) = -n'_H, \ 2c_n(-4) = n'_V. \)

For concreteness, let us now check the above relations for the four-parameter model of [18], which has a dual type II description based on the Calabi–Yau space \( X_2 = P_{1,1,2,6,10}(20) \). Using the instanton numbers given in [18], it can be checked that both (1.23) and (6.32) for \( c_2 \) indeed hold, as can be seen from the second table in appendix A.6 and the table given below.
The truncation to the three-parameter Calabi–Yau model is made by setting $V = 0$. The instanton numbers $n^r_{k,l}$ of the $S$-$T$-$U$ model are then given by \[ n^r_{k,l} = \sum_b n^r (4kl - b^2) , \] \[ (6.33) \]
where the summation range over $b$ is finite. For example, $n^r_{0,1} = -2 + 56 + 372 + 56 - 2 = 480$ \[ [18] \]
7 Summary and Outlook

We have studied the duality between two different string theories, the heterotic string on $K3 \times T^2$ and type IIA string on CY, leading to $N=2$ SUSY in $D=4$. For this we picked specific CY’s to match the spectra of the heterotic theories. Actually one can extend such a study of different and related (by Higgs transitions) models on each side to the matching of whole webs [24] of compactifications for both string theories. Then we have presented quite explicit evidence that the proposed dual models are indeed dual even on the level of the couplings $F_0$ and $F_1$. This was possible by the use of the special functions occuring on the heterotic side, which are specific modular forms thanks to the operation of the T-duality group, and the instanton expansions on the type II side.

Actually one can also make a closer connection to the corresponding field theory results of Seiberg and Witten by taking the appropriate field theory limit [76, 85]. Also in string theory similar comparisons are possible between the type I string and the $SO(32)$ heterotic string [5].

Especially fruitfull is furthermore the reformulation in $D=6$ of the duality considered here in $D=4$, which is possible thanks to F-theory [111, 94]. Namely, as type IIA on a CY can be understood as F-theory on $T^2 \times CY$, one can cancel the $T^2$ part on both sides (if one has not used it in a special way like in the ST model). This leads to the duality between the heterotic string on $K3$ and F-theory on (an elliptically fibered) CY.

This makes it possible to directly ‘connect’ these $N=2$ studies to the more complicated models of $N=1$ in $D=4$, which is of course the next step, especially in view of coming closer to phenomenological relevance. There again one can study dual models: now heterotic on CY and on the type II side one has to use F-theory to make sense out of a non-CY base (a CY base would already give $N=2$). One possibility [12] is to orbifold known dual $N=2$ pairs by $\mathbb{Z}_2$. In this way one can even again match the relevant holomorphic coupling, now given by the superpotential, which is in the considered case again modular [47].

Another path one can follow already in the $N=2$ setup is a closer study of the higher gravitational couplings and the geometrical information about numbers of higher genus curves on the CY contained in them [5, 10, 20, 23].
A Modular forms

A.1 Ordinary modular forms

A modular form \( F_r(T) \) of weight \( r \) obeys the transformation law

\[
F_r(T) \rightarrow (icT + d)^r F_r(T) \quad . \tag{A.1}
\]

One can show that there are no modular forms of weight 0 and 2 while at weight 4 and 6 one has the Eisenstein functions

\[
E_4(q) = 1 + 240 \sum_{n=1}^{\infty} \frac{n^3 q^n}{1 - q^n} = 1 + 240q + 2160q^2 \ldots , \tag{A.2}
\]

\[
E_6(q) = 1 - 504 \sum_{n=1}^{\infty} \frac{n^5 q^n}{1 - q^n} = 1 - 504q - 16632q^2 \ldots ,
\]

where \( q = e^{2\pi i(T)} \). Both functions have no pole (including \( T = \infty \)) on the entire fundamental domain; \( E_4 \) has exactly one simple zero at \( T = \rho \) while \( E_6 \) has one simple zero at \( T = 1 \). One can construct modular forms of arbitrary even weight from products of these two Eisenstein functions.

The unique cusp form of weight 12 is \( \eta^{24} \), where

\[
\eta(q) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n) , \tag{A.3}
\]

the Dedekind \( \eta \)-function (\( \eta \) does not vanish at \( \rho \) or 1). The modular invariant \( j \) function defined by

\[
j(q) = \frac{E_4^3}{\eta^{24}} = q^{-1} + 744 + 196884q + \ldots \tag{A.4}
\]

has a simple pole at \( T = \infty \) and a triple zero at \( T = \rho \).

One also defines a function

\[
E_2 = 1 - 24 \sum_{n=1}^{\infty} \frac{nq^n}{1 - q^n} \tag{A.5}
\]

which is not quite automorphic.

Likewise the derivative of a modular form does again not quite transform like a modular form, instead a corresponding correction term arises.

This leads to the relations

\[
24 \frac{1}{2\pi i} \partial_T \log \eta = E_2 , \quad \frac{E_6}{E_4} = E_2 - \frac{3}{2\pi i} \partial_T \log E_4
\]

from which it follows that

\[
\frac{1}{2\pi i} \partial_T \log j = -\frac{E_6}{E_4} \Rightarrow j = \eta^{24} \Rightarrow j - \alpha = \frac{E_6^2}{\eta^{24}} \quad \text{one has} \quad \frac{1}{2\pi i} \partial_T \log j = -\frac{E_6 E_4^2}{\eta^{24}} \Rightarrow (\frac{1}{2\pi i})^2 j^2 = E_4 j(j - \alpha) , \quad \text{i.e.}
\]
\[ \frac{J^2}{j(j - \alpha)} = -4\pi^2 E_4. \]  

(A.6)

Note that one has despite the general difficulties with derivatives nevertheless the following: \( \partial^{k+1} f_{-k} \) is a modular form of weight \( k + 2 \) for \( f_{-k} \) a modular form of weight \( -k \).

The coefficients \( c_1(n) \) are defined by

\[
\begin{align*}
\frac{E_4 E_6}{\eta^{24}} = \sum_{n \geq -1} c_1(n) q^n & = \frac{1}{q} - 240 - 141444q - 8529280q^2 - 238758390q^3 \\
& \quad - 4303488384q^4 + \ldots
\end{align*}
\]

(A.7)

The coefficients \( \tilde{c}_1(n) \) are defined as follows [70]

\[
\frac{E_2 E_4 E_6}{\eta^{24}}(q) = \sum_{n = -1}^{\infty} \tilde{c}_1(n) q^n = \frac{1}{q} - 264 - 135756q - 5117440q^2 + \ldots
\]

(A.8)

We will also make use of the remarkable Borcherds identity [70]

\[
\log(j(T) - j(U)) = 2\pi T + \sum_{k,l} c(kl) \log(1 - e^{-2\pi(kT + lU)}),
\]

(A.9)

where the universal constants \( c(n) \) are defined as follows:

\[
\begin{align*}
\begin{array}{c}
j(q) - 744 = \sum_{n = -1}^{\infty} c(n) q^n = \frac{1}{q} + 196884q + 21493760q^2 + 864299970q^3 \\
& \quad + 20245856256q^4 + \ldots
\end{array}
\end{align*}
\]

(A.10)

Now consider the relation \( \left( \frac{E_4 E_6}{\eta^{24}} \right)' = -\frac{2\pi}{6} \left( \frac{E_2 E_6}{\eta^{12}} + 2 \frac{E_4^2}{\eta^{24}} + 3 \frac{E_6^3}{\eta^{36}} \right) \). From this we can, for \( n > 0 \), derive the useful equation \( 12nc_1(n) + 10c(n) = -2\tilde{c}_1(n) \).

Furthermore we will use the polylogarithmic functions

\[
\begin{align*}
Li_1(x) & = \sum_{j=1}^{\infty} \frac{x^j}{j} = -\log(1 - x) \\
Li_2(x) & = \sum_{j=1}^{\infty} \frac{x^j}{j^2} = \int_0^1 \frac{dt}{t} \frac{1}{1 - xt} \\
Li_3(x) & = \sum_{j=1}^{\infty} \frac{x^j}{j^3} = -\int_0^1 \frac{dt}{t} \int_0^1 \frac{ds}{s} \log(1 - xs) .
\end{align*}
\]

Note that \( \partial_k \partial_l Li_3 = (-2\pi)^2 d_k d_l Li_1 \), where \( Li_1(x) = -\log(1 - x) \).
A.2 Siegel modular forms

The classical moduli space of a heterotic STUV model is locally given by the Siegel upper half-plane \( \mathcal{H}_2 = \frac{SO(3,2)}{SO(3) \times SO(2)} \) (note the exceptional isomorphism \( SO(5) = B_2 = C_2 = Sp(4) \), here in a non compact formulation). The standard action of \( Sp(4, \mathbb{Z}) \) on an element \( \tau \) of the Siegel upper half-plane \( \mathcal{H}_2 \) is given by

\[
M \rightarrow M \cdot \tau = (a\tau + b)(c\tau + d)^{-1},
\]
where \( \tau = \begin{pmatrix} \tau_1 & \tau_3 \\ \tau_3 & \tau_2 \end{pmatrix} = \begin{pmatrix} i \nu & i \omega \\ i \omega & i \nu \end{pmatrix}, M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G = Sp(4, \mathbb{Z}) \) and where \( \det \text{Im}\tau = \text{Re}T\text{Re}U - (\text{Re}V)^2 > 0 \). Note that \( a, b, c \) and \( d \) denote \( 2 \times 2 \) matrices. A Siegel modular form \( F \) of even weight \( k \) transforms as

\[
F(M \cdot \tau) = (c\tau + d)^k F(\tau)
\]
for every \( M \in G = Sp(4, \mathbb{Z}) \), whereas a modular form of odd weight \( k \) transforms as

\[
F(M \cdot \tau) = \varepsilon(M) (c\tau + d)^k F(\tau).
\]
Here \( \varepsilon : G \rightarrow G/G(2) = S_6 \rightarrow \{\pm 1\} \) is the sign of the permutation in \( S_6 \); \( G(2) \) denotes the principal congruence subgroup of level 2.

The Eisenstein series are given by

\[
\mathcal{E}_k = \sum \det(c\tau + d)^{-k}.
\]

Now, recall that the usual modular forms of \( Sl(2, \mathbb{Z}) \) are generated by the (normalized) Eisenstein series \( E_4 \) and \( E_6 \). These are related to the two modular forms \( E_{12} \) and \( \Delta \) of weight 12 by

\[
aE_4^3 + bE_6^2 = (a + b)E_{12}, \\
E_4^3 - E_6^2 = \alpha \Delta,
\]
where \( \Delta = \eta^{24} \) is the cusp form, and where \( a = (3 \cdot 7)^2, b = 2 \cdot 5^3, c = a + b = 691, \alpha = 2^6 \cdot 3^3 = 1728 \).

Similarly, the ring of Siegel modular forms is generated by the (algebraic independent) Eisenstein series \( \mathcal{E}_4, \mathcal{E}_6, \mathcal{E}_{10}, \mathcal{E}_{12} \) and by one further cusp form of odd weight \( \mathcal{C}_{35} \), whose square can again be expressed in terms of the even generators. Alternatively, instead of using \( \mathcal{E}_{10} \) and \( \mathcal{E}_{12} \), one can also use the cusp forms \( \mathcal{C}_{10} \) and \( \mathcal{C}_{12} \).

A Siegel cusp form is defined as follows. Since a modular form \( f \) is invariant under the translation group \( U = \{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \in G \} \), where the integer-valued \( 2 \times 2 \) matrix \( b \) is symmetric, it has a Fourier expansion \( F = \sum_M a(M)e^{2\pi i trM\tau} \). Here, the summation extends over all symmetric half-integral \( 2 \times 2 \) matrices (that is over symmetric matrices that have integer-valued diagonal entries and half-integer-valued off-diagonal entries). The Fourier coefficient \( a(M) \) depends only on the class of \( M \) under conjugation by \( Sl(2, \mathbb{Z}) \), and it is zero unless \( M \) is positive semidefinite.

Now, consider the Siegel operator \( \Phi \), which associates, to every Siegel modular form \( F \) with Fourier coefficients \( a(M) \), the ordinary \( Sl(2, \mathbb{Z}) \) modular form \( \Phi F \) with Fourier coefficients \( a(n) = a(\begin{pmatrix} n & \cdot \\ \cdot & \cdot \end{pmatrix}) \). This yields a surjective homomorphism of graded rings.
of modular forms. The forms in the kernel are the cusp forms. Thus, identities between ordinary modular forms lead to Siegel cusp forms, as follows:

\[ E_4 E_6 = E_{10} \rightarrow \mathcal{E}_4 \mathcal{E}_6 - \mathcal{E}_{10} =: p \mathcal{C}_{10} , \]

\[ aE_4^3 + bE_6^2 = cE_{12} \rightarrow a\mathcal{E}_4^3 + b\mathcal{E}_6^2 - c\mathcal{E}_{12} =: \alpha^2 \frac{ab}{c} \mathcal{C}_{12} , \] (A.15)

where \( p \) denotes a normalization constant given by \( p = \frac{2^{10} \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 53}{43867} \). We will drop this normalization constant in the following, for notational simplicity.

Next, consider restricting the Siegel modular forms to the diagonal \( D = \{ \left( \begin{array}{cc} \tau_1 & 0 \\ 0 & \tau_2 \end{array} \right) \} \) (corresponding to the embedding \( \frac{SO(2,2)}{SO(2) \times SO(2)} \rightarrow \frac{SO(3,2)}{SO(3) \times SO(2)} \)). Then, interestingly,

\[ \mathcal{E}_k \left( \left( \begin{array}{cc} \tau_1 & 0 \\ 0 & \tau_2 \end{array} \right) \right) = E_k(\tau_1)E_k(\tau_2) . \] (A.16)

Specifically

\[ \mathcal{E}_4 \rightarrow E_4(\tau_1)E_4(\tau_2) , \]
\[ \mathcal{E}_6 \rightarrow E_6(\tau_1)E_6(\tau_2) , \]
\[ \mathcal{C}_{10} \rightarrow 0 , \]
\[ \mathcal{C}_{12} \rightarrow \Delta(\tau_1)\Delta(\tau_2) . \] (A.17)

More precisely, one finds that, up to a normalization constant, \( \mathcal{C}_{10} \rightarrow \tau_3^2 \Delta(\tau_1)\Delta(\tau_2) \) as \( \tau_3 \rightarrow 0 \).

Now, consider the behaviour on \( D \) of the odd cusp \( \mathcal{C}_{35} \). Since \( \mathcal{C}_{35} \) is a more complicated object, one first reexpresses its square in terms of the other, even generators. Namely, by using the results in [74], one finds that on the diagonal \( D, \mathcal{C}_{35} = 0 \) as well as

\[ \alpha^2 \frac{\mathcal{C}_{35}^2}{\mathcal{C}_{10}} = \mathcal{C}_{12}^2 j(\tau_1) - j(\tau_2))^2 = (\eta^2(\tau_1)\eta^2(\tau_2))^60(j(\tau_1) - j(\tau_2))^2 , \] (A.18)

where \( j(\tau) = E_4^3/\Delta \). Then, using \( \mathcal{C}_5 \) and \( \mathcal{C}_{30} \), which are related to the forms already defined by \( \mathcal{C}_{10} = \mathcal{C}_5^2 \) and \( \mathcal{C}_{35} = \mathcal{C}_{30}\mathcal{C}_5 \), respectively, it follows that

\[ \alpha^2 \mathcal{C}_{30}^2 \rightarrow \Delta^5(\tau_1)\Delta^5(\tau_2)(j(\tau_1) - j(\tau_2))^2 \] (A.19)

on the diagonal \( D \).

A rational quadratic divisor of \( \mathcal{H}_2 \) is, by definition [12], the set

\[ \mathcal{H}_l = \{ \left( \begin{array}{cc} \tau & \nu \\ \nu & \mu \end{array} \right) \in \mathcal{H}_2 | im_1T + im_1U + ibV + n_2(-TU + V^2) + m_2 = 0 \} , \] (A.20)

where \( l = (n_1, m_1, b, n_2, m_2) \in \mathbb{Z}^5 \) is a primitive (i.e. with the greatest common divisor equals 1) integral vector. The number \( D(l) = b^2 - 4m_1n_1 + 4n_2m_2 \) is called the discriminant of \( \mathcal{H}_l \). This divisor determines the Humbert surface \( \mathcal{H}_D \) in the Siegel threefold \( Sp_4(\mathbb{Z}) \setminus \mathcal{H}_2 \). The Humbert surface \( \mathcal{H}_D \) is (the image in \( Sp_4(\mathbb{Z}) \setminus \mathcal{H}_2 \) of) the union of all \( \mathcal{H}_l \) of discriminant \( D(l) \). Each Humbert surface \( \mathcal{H}_D \) can be represented by a linear relation in \( T, U \) and \( V \). For instance, the divisor of \( \mathcal{C}_5 \) is the diagonal \( \mathcal{H}_1 = \{ Z = \left( \begin{array}{cc} \tau & \nu \\ \nu & \mu \end{array} \right) \in Sp_4(\mathbb{Z}) \setminus \mathcal{H}_2 \} \). Similarly, the divisor of the Siegel modular form \( \mathcal{C}_{30} \) is the surface \( \mathcal{H}_4 = \{ Z = \left( \begin{array}{cc} \tau & \nu \\ \nu & \mu \end{array} \right) \in Sp_4(\mathbb{Z}) \setminus \mathcal{H}_2 | T = U \} \). The divisor of the Siegel modular form \( \mathcal{C}_{35} \), on the other hand, is the sum (with multiplicity 1) of the surfaces \( \mathcal{H}_1 \) and \( \mathcal{H}_4 \).
A.3 Jacobi forms

A Siegel modular form \( F(T,U,V) \) of weight \( k \) has a Fourier expansion with respect to its variable \( iU \)

\[
F(T,U,V) = \sum_{m=0}^{\infty} \phi_{k,m}(T,V)s^m , \tag{A.21}
\]

where \( s = e[iU], e[x] = \exp 2\pi ix \). Each of the \( \phi_{k,m}(T,V) \) is a Jacobi form of weight \( k \) and index \( m \). That is, for each \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{Sl}(2,\mathbb{Z}) \) and \( \lambda,\mu \in \mathbb{Z} \)

\[
\phi_{k,m}(\begin{pmatrix} aT - ib \\ icT + d \end{pmatrix}) = (i\lambda T + \mu) e^{-2\pi i m (\lambda^2 iT + 2\lambda iV)} \phi_{k,m}(T,V) , \tag{A.22}
\]

A Jacobi form \( \phi_{k,m}(T,V) \) of index \( m \) has in turn an expansion

\[
\phi(T,V) = \sum_{n \geq 0} \sum_{l \in \mathbb{Z}} c(n,l)q^n r^l , \tag{A.23}
\]

where \( q = e[iT], r = e[iV] \). Of special relevance are the Jacobi forms \( \phi_{k,1} \) of index 1. The summation in \( l \) extends in the usual case, and for the generators introduced above, over \( 4n - l^2 \geq 0 \); for the forms divided by \( \Delta \), \( 4n - l^2 \geq -1 \) or \( -4 \), depending on whether the form is a cusp form or not. Furthermore

\[
c(n,l) = c(4n - l^2) . \tag{A.24}
\]

Consider, for instance, the Eisenstein series, which have the expansion

\[
E_k(T,U,V) = E_k(T) - \frac{2k}{B_k} E_{k,1}(T,V) s + \mathcal{O}(s^2) . \tag{A.25}
\]

Here, the \( B_k \) denote the Bernoulli numbers. Thus, for instance,

\[
E_4 = E_4 + 240E_{4,1}s + \cdots , \\
E_6 = E_6 - 504E_{6,1}s + \cdots . \tag{A.26}
\]

The Jacobi forms \( E_{4,1}(T,V) \) and \( E_{6,1}(T,V) \) of index 1 have the expansion (the expansion coefficients are listed in the first table of appendix A.6)

\[
E_{4,1} &= 1 + (r^2 + 56r + 126 + 56r^{-1} + r^{-2})q \\
&+ (126r^2 + 756 + 576r^{-1} + 126r^{-2})q^2 + \cdots , \\
E_{6,1} &= 1 + (r^2 - 88r - 330 - 88r^{-1} + r^{-2})q \\
&+ (-330r^2 - 4224r - 7524 - 4224r^{-1} - 330r^{-2})q^2 + \cdots . \tag{A.27}
\]

Note that \( E_{k,1} \rightarrow E_k \) as \( V \rightarrow 0 \).

Similarly, the cusp forms \( C_{10}(T,U,V) \) and \( C_{12}(T,U,V) \) have the expansion

\[
C_{10}(T,U,V) = \phi_{10,1}(T,V)s + \mathcal{O}(s^2) , \\
C_{12}(T,U,V) = \Delta(T) + \frac{1}{12} \phi_{12,1}(T,V)s + \mathcal{O}(s^2) , \tag{A.28}
\]
where
\[
\phi_{10,1} = \frac{1}{144} (E_6E_{4,1} - E_4E_{6,1}) \to 0 , \\
\phi_{12,1} = \frac{1}{144} (E_4^2E_{4,1} - E_6E_{6,1}) \to 12\Delta .
\] (A.29)

Here, we have indicated the behaviour under the truncation \( V \to 0 \). The Jacobi forms \( \phi_{10,1} \) and \( \phi_{12,1} \) of index 1 have the following expansion (the expansion coefficients are listed in the first table in appendix A.6):
\[
\phi_{10,1} = (r - 2 + r^{-1})q + (-2r^2 - 16r + 36 - 16r^{-1} - 2r^{-2})q^2 + \cdots , \\
\phi_{12,1} = (r + 10 + r^{-1})q + (10r^2 - 88r - 132 - 88r^{-1} + 10r^{-2})q^2 + \cdots .
\] (A.30)

### A.4 Product expansions

The Siegel modular forms \( C_5 \) and \( C_{30} = C_{35}/C_5 \) have the following product expansion
\[
C_5 = (qrs)^{1/2} \prod_{n,m,l \in \mathbb{Z}} (1 - q^n r^l s^m) f(4nm - l^2) , \\
C_{30} = (q^3rs^3)^{1/2} (q - s) \prod_{n,m,l \in \mathbb{Z}} (1 - q^n r^l s^m) f_2'(4nm - l^2) ,
\] (A.31)

where the condition \((n, m, l) > 0\) means that \( n \geq 0, m \geq 0 \) and either \( l \in \mathbb{Z} \) if \( n+m > 0 \), or \( l < 0 \) if \( n = m = 0 \). The coefficients \( f(4nm - l^2) \) and \( f_2'(4nm - l^2) \), which are listed in the first table in appendix A.6, are defined as follows. Consider the expansion of
\[
\phi_{0,1} := \frac{\phi_{12,1}}{\Delta(T)} = \sum_{n \geq 0} \sum_{l \in \mathbb{Z}} f(n, l) q^n r^l ,
\] (A.32)

where the sum over \( l \) is restricted to \( 4n - l^2 \geq -1 \). Then, \( f(N) = f(n, l) \) if \( N = 4n - l^2 \geq -1 \), and \( f(N) = 0 \) otherwise. The coefficients \( f_2'(N) \) are then given by \( f_2'(N) = 8f(4N) + 2(N - 3)f(N) + f(N^2) \). Here, \( (\frac{D}{2}) = 1, -1, 0 \) depending on whether \( D \equiv 1 \mod 8, 5 \mod 8, 0 \mod 2 \).

Using the product expansions \( (A.31) \), we can perform a check on the expansion \( (A.28) \) of \( C_{10} = qrs \prod (1 - q^n r^l s^m)^{2f} \). Namely, consider the term in \( C_{10} \) with \( n = m = 0, l = -1 \). It gives rise to \( qsr(1 - r^{-1})^2 = qs(r - 2 + r^{-1}) \), which indeed matches the \( q \)-term of \( \phi_{10,1} \).

Similarly, we can perform a check on \( (A.19) \). Setting \( r = 1 \) in \( (A.31) \), we see that the terms with \( m = 0 \) have \( f_2'(0) = 60 \); they thus match \( \Delta^{5/2}(T) = \eta^{60} \) occurring in \( C_{30} \propto \Delta^{5/2}(T) \Delta^{5/2}(j(T) - j(U)) \). The sum over \( l \) for the terms with \( m = n = 1 \), on the other hand, yields \( f_2'(4) + 2(f_2'(3) + f_2'(0)) = 196884 \), which matches the \( q \)-term in the expansion of \( j = 744 = q^{-1} + 196884q + \cdots \).
A.5 Theta functions and Jacobi forms

The standard Jacobi theta functions are defined as follows \( z = iV \)

\[
\begin{align*}
\theta_1(\tau, z) &= i \sum_{n \in \mathbb{Z}} (-1)^n q^{\frac{1}{2}(n-\frac{1}{2})^2} r^{n-\frac{1}{2}}, \\
\theta_2(\tau, z) &= \sum_{n \in \mathbb{Z}} q^{\frac{1}{2}(n-\frac{1}{2})^2} r^{n-\frac{1}{2}}, \\
\theta_3(\tau, z) &= \sum_{n \in \mathbb{Z}} q^\frac{1}{2} r^n, \\
\theta_4(\tau, z) &= \sum_{n \in \mathbb{Z}} (-1)^n q^{\frac{1}{2} n^2} r^n. 
\end{align*}
\]

(A.33)

It is useful to introduce

\[
\begin{align*}
\theta_{0,1}(\tau, z) &= \theta_3(2\tau, z) = \sum_{n \in \mathbb{Z}} q^{n^2} r^n, \\
\theta_{1,1}(\tau, z) &= \theta_2(2\tau, z) = \sum_{n \in \mathbb{Z}} q^{(n-\frac{1}{2})^2} r^{n-\frac{1}{2}}, 
\end{align*}
\]

(A.34)

as well as

\[
\begin{align*}
\theta_{ev}(\tau, z) &= \theta_{0,1}(\tau, 2z) = \sum_{n \equiv 0(2)} q^{n^2/4} r^n, \\
\theta_{odd}(\tau, z) &= \theta_{1,1}(\tau, 2z) = \sum_{n \equiv 1(2)} q^{(n-\frac{1}{2})^2} r^{n-\frac{1}{2}}. 
\end{align*}
\]

(A.35)

Next, consider setting \( z = 0 \). The \( \theta_i(\tau, 0) \) will be simply denoted by \( \theta_i \), whereas the \( \theta_i(2\tau, 0) \) will be denoted by \( \theta_i(2\cdot) \) \((i = 1, \ldots, 4)\). It is well known that \( \theta_1 = 0 \) and that \( \theta_3^4 = \theta_2^4 + \theta_4^4 \) as well as \( \theta_2 \theta_3 \theta_4 = 2\eta^3 \). Also

\[
\begin{align*}
E_4 &= \frac{1}{2} \left( \theta_2^8 + \theta_3^8 + \theta_4^8 \right), \\
E_6 &= \frac{1}{2} \left( \theta_2^4 \theta_3^4 + \theta_3^4 \theta_4^4 + \theta_4^4 \theta_2^4 \right) \\
&= \frac{1}{2} \left( -\theta_2^6 (\theta_3^4 + \theta_4^4) \theta_2^2 + \theta_3^6 (\theta_2^4 - \theta_4^4) \theta_3^2 + \theta_4^6 (\theta_2^4 + \theta_3^4) \theta_4^2 \right). 
\end{align*}
\]

(A.36)

Additional useful identities are given by

\[
\begin{align*}
2\theta_2(2\cdot) \theta_3(2\cdot) &= \theta_2^2, \\
\theta_2^2(2\cdot) + \theta_3^2(2\cdot) &= \theta_3^2, \\
\theta_3^2(2\cdot) - \theta_2^2(2\cdot) &= \theta_4^2, \\
2\theta_2^2(2\cdot) &= \theta_3^2 - \theta_4^2, \\
2\theta_3^2(2\cdot) &= \theta_2^2 + \theta_4^2, \\
\theta_4^2(2\cdot) &= \theta_3 \theta_4. 
\end{align*}
\]

(A.37)
Now consider Jacobi forms $f(\tau, z) = \sum_{n \geq 0} c(4n - l^2)q^n r^l$ of weight $k$ and index 1. The following examples provide useful identities between Jacobi forms of index 1 and Jacobi theta functions

$$
\phi_{10,1} = -\eta^{18}\theta_3^2(\tau, z) ,
\phi_{12,1} = 12\eta^{24}\frac{\theta_3^2(\tau, z)}{\theta_3^2} + (\theta_4^4 - \theta_2^4)[-\eta^{18}\theta_1^2(\tau, z)]
$$

as well as

$$
E_{4,1} = \frac{1}{2} \left( \theta_3^6\theta_2^2(\tau, z) + \theta_3^6\theta_3^2(\tau, z) + \theta_3^6\theta_4^2(\tau, z) \right) ,
E_{6,1} = \frac{1}{2} \left( -\theta_2^6(\theta_3^4 + \theta_4^4) \theta_2^2(\tau, z) + \theta_3^6(\theta_3^4 - \theta_2^4) \theta_3^2(\tau, z) + \theta_4^6(\theta_4^4 + \theta_3^4) \theta_4^2(\tau, z) \right) .
$$

A Jacobi form of index 1 has the following decomposition [53, 81, 82]

$$
f(\tau, z) = f_{ev}(\tau)\theta_{ev}(\tau, z) + f_{odd}(\tau)\theta_{odd}(\tau, z) ,
$$

where

$$
f_{ev} = \sum_{N \equiv 0(4)} c(N)q^{N/4} ,
f_{odd} = \sum_{N \equiv -1(4)} c(N)q^{N/4} .
$$

Consider, for instance, $E_{4,1}$. It has the decomposition [82]

$$
E_{4,1\, ev} = \theta_3^7(2\cdot) + 7\theta_3^3(2\cdot)\theta_3^3(2\cdot) ,
E_{4,1\, odd} = \theta_2^7(2\cdot) + 7\theta_2^3(2\cdot)\theta_3^3(2\cdot) .
$$

Furthermore one has (with $\theta_{ev} \equiv \theta_{ev}(\tau, z)$ and $\theta_{odd} \equiv \theta_{odd}(\tau, z)$)

$$
\theta_1^2(\tau, z) = \theta_2(2\cdot)\theta_{ev} - \theta_3(2\cdot)\theta_{odd} ,
\theta_2^2(\tau, z) = \theta_2(2\cdot)\theta_{ev} + \theta_3(2\cdot)\theta_{odd} ,
\theta_3^2(\tau, z) = \theta_3(2\cdot)\theta_{ev} + \theta_2(2\cdot)\theta_{odd} ,
\theta_4^2(\tau, z) = \theta_3(2\cdot)\theta_{ev} - \theta_2(2\cdot)\theta_{odd} .
$$

Next, consider the elliptic genus $Z(\tau, z)$ of $K3$, which is a Jacobi form of weight 0 and index 1, given by [80]

$$
Z(\tau, z) = 2\phi_{12,1}^2 = 24\frac{\theta_3^2(\tau, z)}{\theta_3^2} - 2\frac{\theta_4^4 - \theta_2^4}{\eta^4} \frac{\theta_4^2(\tau, z)}{\eta^2} .
$$

It has the decomposition

$$
Z_{ev} = 24\frac{\theta_3(2\cdot)}{\theta_3^2} - 2\frac{\theta_4^4 - \theta_2^4}{\eta^4} \frac{\theta_2(2\cdot)}{\eta^2} = 20 + 216q + 1616q^2 + \cdots ,
Z_{odd} = 24\frac{\theta_3(2\cdot)}{\theta_3^2} + 2\frac{\theta_4^4 - \theta_2^4}{\eta^4} \frac{\theta_3(2\cdot)}{\eta^2} = 2q^{-\frac{4}{3}} - 128q^{\frac{2}{3}} - 1026q^2 + \cdots .
$$
Now we introduce the hatted modular function \( \hat{f}(\tau, z) \) as
\[
\hat{f}(\tau, z) = f_{ev}(\tau) + f_{odd}(\tau) \quad .
\]

Hence the hatted modular function corresponds in a one-to-one way to the index 1 Jacobi form. In particular, the Jacobi form \( f(\tau, z) \) and its hatted relative \( \hat{f}(\tau, z) \) possess identical power series expansion coefficients \( c(N) \):
\[
f(\tau, z) = \sum_{n,l} c(4n - l^2) q^n r^l, \quad \hat{f}(\tau, z) = \sum_{N \in 4\mathbb{Z} \text{ or } 4\mathbb{Z}+3} c(N) q^{N/4} \quad .
\]

Note that an ordinary modular form (that is a form not having any \( z \)-dependence), if occurring as a multiplicative factor in front of a proper Jacobi form, is left untouched by the hatting procedure \((A.46)\). Thus, for instance,
\[
\hat{E}_{4,1} = \frac{1}{2} \left( \theta_2^6 \theta_3^2(\tau, z) + \theta_3^6 \theta_2^2(\tau, z) + \theta_4^6 \theta_4^2(\tau, z) \right) \quad \quad (A.48)
\]
\[
= \frac{1}{2} \left( \theta_2^6[\theta_2(2\cdot) + \theta_3(2\cdot)] + \theta_3^6[\theta_2(2\cdot) + \theta_3(2\cdot)] + \theta_4^6[\theta_3(2\cdot) - \theta_2(2\cdot)] \right) \quad ,
\]
\[
\hat{E}_{6,1} = \frac{1}{2} \left( -\theta_2^6(\theta_3^4 + \theta_4^4) \theta_2^2(\tau, z) + \theta_3^6(\theta_4^4 - \theta_2^4) \theta_3^2(\tau, z) + \theta_4^6(\theta_2^4 + \theta_3^4) \theta_4^2(\tau, z) \right) \quad ,
\]
and similarly
\[
\hat{Z} = Z_{ev} + Z_{odd} = 24 \frac{\theta_2(2\cdot) + \theta_3(2\cdot)}{\theta_2^3} - 2 \frac{(\theta_4^4 - \theta_2^4)(\theta_2(2\cdot) - \theta_3(2\cdot))}{\eta^4} \quad .
\]

Furthermore, consider introducing
\[
\tilde{f} = \tilde{f}(4\cdot) = \sum_{N \in 4\mathbb{Z} \text{ or } 4\mathbb{Z}+3} c(N) q^N \quad .
\]

Note that \( \tilde{f} \) is the \( \Gamma_0(4) \) modular form of half-integral weight \( k - 1/2 \) associated to a Jacobi form of weight \( k \) and index 1 \([B.3]\).

**A.6 Lie algebra lattices and Jacobi forms**

The relation between Lie algebra lattice sums and Jacobi forms will be established in three steps. We start by reviewing the well-known relationship between the Lie algebra lattice \( E_8 \) and the Eisenstein series \( E_4 \). Then we go on to show the relation between the Lie algebra lattice \( E_7 \) and the Jacobi Eisenstein series \( E_{4,1} \). Finally, we will relate the processes of splitting off an \( \gamma_1 \) and the hatting procedure. This will explain the relation between turning on a Wilson line and the hatting procedure.

First the relation between the Eisenstein series \( E_4 \) and the partition function of the \( E_8 \) lattice \( \Lambda = \{ x \in \mathbb{Z}^8 \cup \pi + \mathbb{Z}^8 \mid (x, \pi) \in \mathbb{Z} \} \) is well known \((\pi = (1/2, \cdots, 1/2) \in \mathbb{Z}^8)\) and reads
\[
E_4 = \sum_{x \in \Lambda} q^{\frac{1}{2} x^2} = \frac{1}{2} (\theta_2^8 + \theta_3^8 + \theta_4^8) \quad .
\]


Because of the lattice relation $\Lambda_{E_8} = \Lambda_{D_8(0)} + \Lambda_{D_8(s)}$, this also shows that the fermionically computed partition function $P_{D_8(0)} + P_{D_8(s)}$ of $E_8$ is identical to the bosonically computed one, if one recalls the relation between the bosonic conjugacy class picture and the fermionic boundary condition picture:

\[ P_{D_n}^{(0)} = \frac{\theta_3^n + \theta_4^n}{2} = \frac{N S^+ + N S^-}{2}, \]

\[ P_{D_n}^{(s)} = \frac{\theta_3^n - \theta_4^n}{2} = \frac{N S^+ - N S^-}{2}, \]

\[ P_{D_n}^{(s/c)} = \frac{\theta_3^n}{2} = \frac{R^+}{2}. \] (A.52)

Now consider the Jacobi form $E_{4,1}(\tau, z) = \sum c(4n - l^2)q^n r^l$. Since the expression $\sum_{x \in \Lambda} q^{\frac{1}{2} x^2} r^{(x, \pi)}$ has the correct weights (and truncation), and since the space in question is one-dimensional, this represents $E_{4,1}$. If one considers the $l = 0$ resp. $l = 1$ sector, one finds $\sum_{x \in \Lambda} q^{\frac{1}{2} x^2} = \sum_n c(4n) q^n = \sum_{N = 0(4)} c(N) q^{N/4}$ resp. $\sum_{x \in \Lambda} q^{\frac{1}{2} x^2} = \sum_n (4n - 1) q^n = \sum_{N = 1(4)} c(N) q^{N/4}$, i.e. $E_{4,1}^{(0)} = \sum_{x \in \Lambda} q^{\frac{1}{2} x^2}$ and $E_{4,1}^{(1)} = q^{-1/4} \sum_{x \in \Lambda} q^{\frac{1}{2} x^2} = \sum_{x \in \Lambda} q^{\frac{1}{2} x^2}$. Thus,

\[ E_{4,1}^{(0)} = \sum_{x \in \Lambda} q^{\frac{1}{2} x^2} = \sum_{x \in \Lambda} q^{\frac{1}{2} x^2} = P_{E_7}^{(0)}, \]

\[ E_{4,1}^{(1)} = \sum_{x \in \Lambda} q^{\frac{1}{2} x^2} = \sum_{x \in \Lambda} q^{\frac{1}{2} x^2} = P_{E_7}^{(1)}, \] (A.53)

where the lattice sums $P_{E_7}^{(i)} = \sum_{x \in \Lambda} q^{\frac{1}{2} x^2}$ run over vectors within the conjugacy class (i).

Besides this lattice theoretic argument, this can also be checked explicitly

\[
E_{4,1}^{(0)} = \theta_3(2 \cdot)(\theta_3^4(2 \cdot) + 7 \theta_2^2(2 \cdot)) = \theta_3(2 \cdot)[\theta_3^4(2 \cdot) + 8 \theta_2^4(2 \cdot)]
\]

\[
= \theta_3(2 \cdot)\frac{\theta_3^2 + \theta_4^2}{2}\left[\theta_3^2 \theta_4^2 + 2(\theta_3^2 - \theta_4^2)^2\right] = \theta_3(2 \cdot)[\theta_3^4 + \theta_4^4 - \frac{2\theta_3^2 \theta_4^2}{2}(\theta_3^2 + \theta_4^2)]
\]

\[
= \theta_3(2 \cdot)\frac{1}{2}[\theta_3^6 + \theta_4^6] + \theta_2(2 \cdot)\frac{1}{2}\theta_2^6 = P_{E_7}^{(0)} ;
\] (A.54)

similarly $E_{4,1}^{(1)} = P_{E_7}^{(1)}$.

The last relation in (A.54) follows by noting the lattice decomposition of $P_{E_7}^{(0)}$: $P_{E_7}^{(0)} = P_{D_6}^{(0)} \cdot P_{A_1}^{(0)} + P_{D_8}^{(s)} \cdot P_{A_1}^{(1)}$. Here one uses the following lattice sums for $A_1$, which has the root lattice $\Lambda_{A_1}^{(0)} = \sqrt{2}\mathbb{Z}$ and two conjugacy classes:

\[ P_{A_1}^{(0)} = \sum_{x \in \Lambda} q^{\frac{1}{2} x^2} = \sum_{n \in \mathbb{Z}} q^n = \theta_3(2 \cdot) , \]

\[ P_{A_1}^{(1)} = \sum_{x \in \Lambda} q^{\frac{1}{2} x^2} = \sum_{n \in \mathbb{Z}} q^{(n-1/2)^2} = \theta_2(2 \cdot) . \] (A.55)
Thus we obtain
\[ 2\hat{E}_{4,1} = \theta_2^3[\theta_2(2\cdot) + \theta_3(2\cdot)] + \theta_3^3[\theta_2(2\cdot) + \theta_3(2\cdot)] + \theta_4^3[\theta_3(2\cdot) - \theta_2(2\cdot)] \]
\[ = \theta_2^3 \cdot \theta_2^2(\tau, z) + \theta_3^3 \cdot \theta_2^2(\tau, z) + \theta_4^3 \cdot \theta_2^2(\tau, z) \]
\[ = 2(P_{E_8^{(0)}} + P_{E_7^{(1)}}), \quad (A.56) \]

which also holds, as is easily seen, in the dehatted version. Now we understand that the breaking of $E_8$ to $E_7$ by turning on a Wilson line, i.e. the splitting off of an $A_1^{\text{Wilson}}$, precisely corresponds to the replacement of $E_4$ by the hatted modular function $\hat{E}_{4,1}$.

On the other hand, note that the truncation
\[ V \to 0 \]
reflects the decomposition of $E_8 \supset E_7 \times A_1$
\[ P_{E_8} = P_{E_7^{(0)}} \cdot P_{A_1^{(0)}} + P_{E_7^{(1)}} \cdot P_{A_1^{(1)}}. \quad (A.58) \]

Let us again demonstrate the hatting procedure by considering the Wilson line breaking of $D_2 = A_1 \times A_1^{\text{Wilson}}$ to $A_1$. The lattice decomposition of $D_2$ under $A_1 \times A_1$ has the form
\[ P_{D_2^{(0)}} = \frac{\theta_3^2 + \theta_4^2}{2} = P_{A_1^{(0)}} \cdot P_{A_1^{(0)}} = \theta_3(2\cdot)^2, \]
\[ P_{D_2^{(V)}} = \frac{\theta_3^2 - \theta_4^2}{2} = P_{A_1^{(1)}} \cdot P_{A_1^{(1)}} = \theta_2(2\cdot)^2, \]
\[ P_{D_2^{(S,C)}} = \frac{\theta_3^2}{2} = P_{A_1^{(0)}} \cdot P_{A_1^{(1)}} = \theta_2(2\cdot)\theta_3(2\cdot). \quad (A.59) \]

Thus the corresponding hatted Jacobi forms become
\[ \frac{\theta_2^3(\tau, z) + \theta_4^3(\tau, z)}{2} = P_{A_1^{(0)}} = \theta_3(2\cdot), \]
\[ \frac{\theta_2^3(\tau, z) - \theta_4^3(\tau, z)}{2} = P_{A_1^{(1)}} = \theta_2(2\cdot), \]
\[ \frac{\theta_2^3(\tau, z)}{2} = \frac{1}{2}(P_{A_1^{(0)}} + P_{A_1^{(1)}}) = \frac{1}{2}(\theta_2(2\cdot) + \theta_3(2\cdot)). \quad (A.60) \]

Finally, going back from the conjugacy class picture to the boundary condition picture one has
\[ NS_{A_1}^\pm = P_{A_1^{(0)}} \pm P_{A_1^{(1)}} = \theta_3(2\cdot) \pm \theta_2(2\cdot) = \theta_{3/4}(\tau, z), \quad (A.61) \]
\[ R_{A_1}^+ = P_{A_1^{(0)}} + P_{A_1^{(1)}} = \theta_3(2\cdot) + \theta_2(2\cdot) = \theta_2^2(\tau, z). \quad (A.62) \]
### A.7 Tables

The first table displays some expansion coefficients of the Jacobi forms $E_{4,1}$, $E_{6,1}$, $\phi_{10,1}$, $\phi_{12,1}$ and of the Siegel forms $C_5$, $C_{30}$.

| $N$ | $e_{4,1}(N)$ | $e_{6,1}(N)$ | $c_{10,1}(N)$ | $c_{12,1}(N)$ | $f(N)$ | $f_2(N)$ |
|-----|--------------|--------------|--------------|--------------|--------|----------|
| −4  | −            | −            | −            | −            | −      | 1        |
| −1  | −            | −            | −            | −            | 1      | −1       |
| 0   | 1            | 1            | 0            | 0            | 10     | 60       |
| 3   | 56           | −88          | 1            | 1            | −64    | 32448    |
| 4   | 126          | −330         | −2           | 10           | 108    | 131868   |
| 7   | 576          | −4224        | −16          | −88          | −513   | ***      |
| 8   | 756          | −7524        | 36           | −132         | 808    | ***      |
| 11  | 1512         | −30600       | 99           | 1275         | −2752  | ***      |
| 12  | 2072         | −46552       | −272         | 736          | 4016   | ***      |
| 15  | 4032         | −130944      | −240         | −8040        | −11775 | ***      |
| 16  | 4158         | −169290      | 1056         | −2880        | 16524  | ***      |
| 19  | 5544         | −355080      | −253         | 24035        |        | ***      |
| 20  | 7560         | −464904      | −1800        | 13080        |        | ***      |

In the following table some expansion coefficients of $\frac{E_{4,1}E_6}{\Delta}$, $\frac{E_{4}E_{6,1}}{\Delta}$ and of $A_n$ (see eq. (3.52)) for $n = 0, 1, 2, 12$ are listed.

| $N$ | $E_{4,1}E_6/\Delta$ | $E_{4}E_{6,1}/\Delta$ | $2A_0$ | $2A_1$ | $2A_2$ | $2A_{12}$ |
|-----|---------------------|------------------------|--------|--------|--------|-----------|
| −4  | 1                   | 1                      | 2      | 2      | 2      | 2         |
| −1  | 56                  | −88                    | −32    | −44    | −56    | −176      |
| 0   | −354                | −66                    | −420   | −396   | −372   | −132      |
| 3   | −26304              | −27456                 | −52760 | −53356 | −53952 | −54912    |
| 4   | −88128              | −86400                 | −174528| −174384| −174240| −172800   |
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