Global large solutions to the two-dimensional compressible Navier–Stokes equations

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Abstract. We obtain the global large solutions to the compressible Navier–Stokes equations in $\mathbb{R}^2$. The solution is large in the sense that there is no smallness assumption applied to one component of the initial incompressible velocity.

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1. Introduction and the main result

The present paper is dedicated to the global well-posedness for the following compressible Navier–Stokes equations in $\mathbb{R}^2$:

\[
\begin{align*}
\partial_t \rho + \text{div}(\rho \mathbf{u}) &= 0, \\
\partial_t (\rho \mathbf{u}) + \text{div}(\rho \mathbf{u} \otimes \mathbf{u}) - \mu \Delta \mathbf{u} - (\lambda + \mu) \nabla \text{div} \mathbf{u} + \nabla P(\rho) &= 0, \\
\rho|_{t=0} &= \rho_0, \quad \mathbf{u}|_{t=0} = \mathbf{u}_0 \defeq (u_0^1, u_0^2),
\end{align*}
\]

(1.1)

where $\rho$ is the density, $\mathbf{u}$ is the velocity field, and $P(\rho) \in C^1$ is the pressure field, which satisfies that $P'(\rho) > 0$ and $P'(1) = 1$. The parameters $\mu$ and $\lambda$ are shear viscosity and volume viscosity coefficients, respectively.

As one of the most popular fluid motion models in the field of the analysis and applications, the compressible Navier–Stokes equations system has attracted much attention and there is a large literature important to mathematical analysis and fluid mechanics. One may mention in particular the works by Matsumura and Nishida [22], Xin [26], Lions [21], Danchin [7], Hoff [17], Feireisl [11], Feireisl et al. [12–14], Villani [24], Chen et al. [3–5], Charve and Danchin [2], Haspot [15], Huang et al. [19], Danchin [8,9], Kotschote [20], He et al. [16] and Zhai et al. [27]. To summarize, the most of the previous global well-posedness results were established by assuming that the density is close in some sense to a constant state and the initial velocity field is assumed to be small. Here, we only recall part of those results in the critical Besov spaces. By studying the behaviors of a hyperbolic–parabolic system, Danchin [7] constructed the global small solutions of (1.2) in $L^2$-type Besov space. With the aid of Green matrix, Chen et al. [3] and Charve and Danchin [2] obtained the global well-posedness result in the critical $L^p$ framework, respectively. Later, Haspot [15] gave a new proof via the so-called effective velocity. Wang et al. [25] proved the global well-posedness of three-dimensional compressible Navier–Stokes equations for some classes of large initial data, which may have large oscillation for the density and large energy for the velocity. Recently, based on the dispersion property of acoustic waves, Fang et al. [10] established the global strong solutions to (1.1) in $\mathbb{R}^3$ which allows the low-frequency part of the initial velocity field to be large. He et al. [16] also obtained the global solutions to (1.1) in $\mathbb{R}^3$ with the large vertical component of the incompressible part of the initial velocity. Very recently, Danchin and Mucha [9] proved
the global solutions to (1.1) in $\mathbb{R}^d (d \geq 2)$ with large initial velocity and almost constant density, if the volume viscosity is large enough. This result was further extended by Chen and Zhai [6] in a critical $L^p$ framework. It should be mentioned that there is no smallness restriction on the incompressible part of the initial velocity in [6,9] in $\mathbb{R}^2$. The proof of [6,9] is based on the global well-posedness for the two-dimensional classical Navier–Stokes equations and the large volume viscosity.

A natural question which arises is: When the volume viscosity is fixed, whether we can construct global solutions to the compressible Navier–Stokes equations in $\mathbb{R}^2$, without smallness restriction on the incompressible part of the initial velocity? As far as we know, it is still an interesting open problem. The main difficulty lies in that there has one bad term $Q(Pu \cdot \nabla Pu)$ [see the definition of $P, Q$ in (1.4)] in the equation of the compressible part [see the second equation of (3.3)]. We cannot use perturbation method to disappear this term. As an attempt, we construct global solutions to the compressible Navier–Stokes equations in $\mathbb{R}^2$ with the large component of the incompressible velocity with fixed volume viscosity. This is significantly different to [6,9] assuming the volume viscosity being sufficiently large. The method used here relies heavily on the algebraical structure of the equation for the incompressible part [see (3.2)], that is to say, the equation of each component for the incompressible velocity is a linear equation with coefficients depending on the density, the velocity of the compressible part and the other component of the incompressible velocity. Similar ideas have been used in [18,23] for inhomogeneous incompressible fluids. In the compressible case, it is much more complicated to deal with extra terms involved in $Qu$.

For simplicity of notation, we use the viscosity coefficient values $\mu = 1$ and $\lambda = 0$ throughout the paper.

Because we shall seemingly consider the density functions as perturbations of the reference density 1, it is natural to set $a = \rho - 1$, so that system (1.1) translates into

$$
\begin{align*}
\partial_t a + \text{div } u + \text{div } (a u) &= 0, \\
\partial_t u + u \cdot \nabla u - \Delta u - \nabla \text{div } u + \nabla a &= -L(a)\Delta u - L(a)\nabla \text{div } u + k(a)\nabla a, \\
(a, u)_{t=0} &= (a_0, u_0),
\end{align*}
$$

(1.2)

where

$$
L(a) \overset{\text{def}}{=} \frac{a}{1+a}, \quad k(a) \overset{\text{def}}{=} -\frac{P'(1+a)}{1+a} + P'(1).
$$

The present study lies on the homogeneous Littlewood–Paley decomposition

$$
z = \sum_{j \in \mathbb{Z}} \hat{\Delta}_j z \in S'(\mathbb{R}^2),
$$

which can be truncated into lower and higher oscillation parts expressed as

$$
z^l = \sum_{2^j \leq N_0} \hat{\Delta}_j z \quad \text{and} \quad z^h = \sum_{2^j > N_0} \hat{\Delta}_j z
$$

(1.3)

for a large integer $N_0 \geq 1$.

In the following, we use the Leray projection operators

$$
Q = \nabla \Delta^{-1} \text{div }, \quad P = \mathcal{I} - Q
$$

(1.4)

and vector components

$$
v = J_1(Pu), \quad w = J_2(Pu), \quad v_0 = J_1(Pu_0), \quad w_0 = J_2(Pu_0),
$$

where $J_1 \mathbf{M} = M^1$ is the first component of $\mathbf{M} = (M^1, M^2)$ and $J_2 \mathbf{M}$ is the second component of $\mathbf{M} = (M^1, M^2)$.

We are now in the position to state the main result of the present paper:
Theorem 1.1. Let \( a_0 \in \dot{B}^0_{2,1} \cap \dot{B}^1_{2,1}(\mathbb{R}^2) \), \( u_0 \in \dot{B}^0_{2,1}(\mathbb{R}^2) \). Assume that there exist two positive constants \( c_0 \) and \( C_0 \) such that
\[
\|a_0\|_{\dot{B}^0_{2,1} \cap \dot{B}^1_{2,1}} + \|(Q_{u_0}, v_0)\|_{\dot{B}^0_{2,1}} \leq c_0 \exp \left(-C_0 \left(\|a_0\|_{\dot{B}^0_{2,1} \cap \dot{B}^1_{2,1}} + \|(Q_{u_0}, v_0, w_0)\|_{\dot{B}^0_{2,1}}\right)^2\right),
\]
then the system (1.2) admits a unique global solution \((a, u)\) satisfying, for \( t > 0 \),
\[
\|\left(a, Q_{u}, v\right)\|_{L^\infty_t(B^0_{2,1})} + \|a\|_{L^\infty_t(B^1_{2,1})} + \|\dot{a}\|_{L^1_t(B^1_{2,1})} + \|\dot{(Q_{u}, v)}\|_{L^1_t(B^0_{2,1})} \leq C \left(\|a_0, Q_{u_0}, v_0\|_{\dot{B}^0_{2,1}} + \|a_0\|_{\dot{B}^1_{2,1}}\right)\exp \left(\|a_0, Q_{u_0}, v_0, w_0\|_{\dot{B}^0_{2,1}} + \|a_0\|_{\dot{B}^1_{2,1}}\right)^2,
\]
and
\[
\|w\|_{L^\infty_t(B^0_{2,1})} + \|w\|_{L^1_t(B^1_{2,1})} \leq C \left(\|a_0, Q_{u_0}, v_0, w_0\|_{\dot{B}^0_{2,1}} + \|a_0\|_{\dot{B}^1_{2,1}}\right)
\]
for a constant \( C \).

Remark 1.2. In (1.5), there is no smallness restriction on \( w_0 \); thus, the above theorem improves the results of [3] \((p = 2)\) and [7] in \( \mathbb{R}^2 \).

2. Preliminaries

Denote by \( C \) a generic constant, which may vary from line to line. Let us first recall the theory of the Littlewood–Paley decomposition. The symbol \( \mathcal{F} \) represents the Fourier transform in \( S'(\mathbb{R}^2) \), the space of tempered distributions in \( \mathbb{R}^2 \). Let \( \varphi \) be a nonnegative smooth function supported in an annulus of \( \mathbb{R}^2 \) so that
\[
\sum_{j \in \mathbb{Z}} \varphi_j(\cdot) = 1 \quad \text{in} \quad \mathbb{R}^2 \setminus \{0\} \quad \text{for} \quad \varphi_j(\cdot) = \varphi(2^{-j} \cdot).
\]
Therefore, the homogeneous dyadic block is defined as
\[
\Delta_j u = \mathcal{F}^{-1}(\varphi_j \mathcal{F} u).
\]
Denote by \( S'_k(\mathbb{R}^2) \) the subspace of \( u \in S'(\mathbb{R}^2) \) such that the homogenous decomposition \( u = \sum_{j \in \mathbb{Z}} \Delta_j u \) holds true. Hence, we have the homogeneous Besov space
\[
\dot{B}^s_{2,1}(\mathbb{R}^2) = \left\{ u \in \mathcal{S}'(\mathbb{R}^2) \mid \|u\|_{\dot{B}^s_{2,1}} \overset{\text{def}}{=} \sum_{j \in \mathbb{Z}} 2^{js} \|\Delta_j u\|_{L^2} < \infty \right\} \quad \text{for} \quad s \in \mathbb{R}.
\]
For convenience, we use the symbols
\[
L^p_T(\dot{B}^s_{2,1}(\mathbb{R}^2)) \overset{\text{def}}{=} L^p(0, T; \dot{B}^s_{2,1}(\mathbb{R}^2)),
\]
\[
\widetilde{L}^p_T(\dot{B}^s_{2,1}(\mathbb{R}^2)) \overset{\text{def}}{=} \left\{ u : [0, T] \rightarrow S'(\mathbb{R}^2) \mid \|u\|_{\widetilde{L}^p_T(\dot{B}^s_{2,1})} \overset{\text{def}}{=} \sum_{j \in \mathbb{Z}} 2^{js} \|\Delta_j u\|_{L^p(0, T; L^2(\mathbb{R}^2))} < \infty \right\}.
\]
We will also repeatedly use the following Bernstein inequality:
\[
C^{-1} a^k \|u\|_{L^q} \leq \|\nabla^k u\|_{L^q} \leq C a^{k + \frac{d}{2} - \frac{d}{p}} \|u\|_{L^p} \quad \text{when} \quad \text{Supp} \mathcal{F} u \subset \sigma \mathcal{C}
\]
for \( 1 \leq p \leq q \leq \infty \), \( k \in \mathbb{Z} \), \( \mathcal{C} \) an annulus of \( \mathbb{R}^2 \) and a constant \( C \) independent of the scale parameter \( \sigma > 0 \).

Moreover, we will use the following pointwise product law [6, Lemma 2.7]
\[
\|uv\|_{\dot{B}^{s_1+s_2-1}_{2,1}} \lesssim \|u\|_{\dot{B}^{s_1}_{2,1}} \|v\|_{\dot{B}^{s_2}_{2,1}}, \quad s_1 \leq 1, \quad s_2 \leq 1, \quad s_1 + s_2 > 0
\]
and the estimate [1, Remark 2.102]
\[
\sum_{j \in \mathbb{Z}} 2^{js} \left\| (u \cdot \nabla, \hat{\Delta} j) v \right\|_{L^2} \lesssim \|\nabla u\|_{B^1_{2,1}} \|v\|_{B^2_{2,1}}, \quad s = 0, 1
\]
(2.3)
for the commutator \([u \cdot \nabla, \hat{\Delta} j] v = u \cdot \nabla \hat{\Delta} j v - \hat{\Delta} j (u \cdot \nabla v)\). Here and in what follows, \(a \lesssim b\) means the inequality \(a \leq C b\) for a generic constant \(C\).

Finally, we recall a composition estimate [1, Theorem 2.61]
\[
\|F(f)\|_{B^s_{2,1}} \lesssim \|f\|_{B^s_{2,1}}, \quad s > 0,
\]
(2.4)
where \(F\) with \(F(0) = 0\) is a smooth function defined on an open interval \(I\) containing 0 and \(f\) is valued in a bounded interval \(A \subset I\).

3. Proof of the main theorem

The global solutions are to be obtained by extending existing local solutions with respect to time.

Given \(a_0 \in \dot{B}^0_{2,1} \cap \dot{B}^1_{2,1}(\mathbb{R}^2), u_0 \in \dot{B}^0_{2,1}(\mathbb{R}^2)\) with \([a_0]\|_{\dot{B}^1_{2,1} \cap \dot{B}^2_{2,1}}\) being sufficiently small, it follows from [8] that there exists a positive time \(T\) so that (1.2) has a unique local solution \((a, u)\) with
\[
a \in C((0, T]; \dot{B}^0_{2,1} \cap \dot{B}^1_{2,1}), \quad u \in C((0, T]; \dot{B}^0_{2,1} \cap L^1(0, T; \dot{B}^2_{2,1})).
\]
(3.1)
Denote \(T^*\) to be the largest time \(T\) in (3.1). Hence, to prove Theorem 1.1, we only need to prove that \(T^* = \infty\). To do so, we need to produce a priori estimates to the solutions.

By using operators \(P\) and \(Q\), we can decompose the system (1.2) into two subsystems:
\[
\begin{cases}
\partial_t v + u \cdot \nabla v + J_1([\mathcal{P}, u \cdot \nabla] u) - \Delta v = -J_1(\mathcal{P}(L(a) \Delta u + L(a) \nabla \div u)), \\
\partial_t w + u \cdot \nabla w + J_2([\mathcal{P}, u \cdot \nabla] u) - \Delta w = -J_2(\mathcal{P}(L(a) \Delta u + L(a) \nabla \div u)),
\end{cases}
\]
(3.2)
and
\[
\begin{cases}
\partial_t a + u \cdot \nabla a + \div Q u = -\div u, \\
\partial_t Q u + u \cdot \nabla Q u - \Delta Q u - \div \nabla Q u + \nabla a = Q G,
\end{cases}
\]
(3.3)
with
\[
Q G = -[Q, u \cdot \nabla] u - Q(L(a) \Delta u) - Q(L(a) \nabla \div u) + Q (k(a) \nabla a).
\]

Lemma 3.1. Let \((a, u)\) be the global smooth solution to (1.2), then there holds the following inequality:
\[
\|Q G\|_{\dot{B}^0_{2,1}} \lesssim \|v\|_{\dot{B}^1_{2,1}} \|w\|_{\dot{B}^2_{2,1}} \\
+ \left(\|(a, v, Q u)\|_{\dot{B}^0_{2,1}} + \|a\|_{\dot{B}^1_{2,1}}\right) \left(\|a^h\|_{\dot{B}^1_{2,1}} + \|a^\ell, v, w, Q u\|_{\dot{B}^2_{2,1}}\right).
\]

Proof. It follows from \(u = P u + Q u\) that
\[
[Q, u \cdot \nabla] u = [Q, (Q u + P u) \cdot \nabla] (Q u + P u) \\
= [Q, (Q u) \cdot \nabla] P u + [Q, (Q u) \cdot \nabla] Q u \\
+ [Q, (P u) \cdot \nabla] Q u + [Q, (P u) \cdot \nabla] P u.
\]
(3.4)
Applying (2.2), we have
\[
\begin{align*}
\|Q, (Q u) \cdot \nabla] P u\|_{\dot{B}^2_{2,1}} & \lesssim \|Q((Q u) \cdot \nabla P u)\|_{\dot{B}^0_{2,1}} \lesssim \|(v, w)\|_{\dot{B}^2_{2,1}} \|Q u\|_{\dot{B}^2_{2,1}}, \\
\|Q, (Q u) \cdot \nabla] Q u\|_{\dot{B}^2_{2,1}} & \lesssim \|(Q u) \cdot \nabla Q u\|_{\dot{B}^2_{2,1}} \lesssim \|Q u\|_{\dot{B}^2_{2,1}} \|Q u\|_{\dot{B}^2_{2,1}}.
\end{align*}
\]
(3.5)
It is not hard to check
\[ \hat{\Delta}_j([Q, (Pu) \cdot \nabla]Q u) = [\hat{\Delta}_j Q, (Pu) \cdot \nabla]Q u - [\hat{\Delta}_j, (Pu) \cdot \nabla]Q u. \]

Thus, from (2.3), one infer that
\[ \|([Q, (Pu) \cdot \nabla]Q u)\|_{B^2_{2,1}} \lesssim \|(v, w)\|_{B^2_{2,1}} \|Q u\|_{B^2_{2,1}}. \tag{3.6} \]

Thanks to the divergence free property \( \partial_2 w = -\partial_1 v \), we deduce
\[ \|Q, (Pu) \cdot \nabla Pu\| \lesssim \|Pu\| \cdot \nabla Pu \]
\[ \lesssim \|Pu\| \cdot \nabla v + \|Pu\| \cdot \nabla w \]
\[ \lesssim |w\partial_1 v - v\partial_2 w| + |v\partial_1 w - w\partial_1 v| \]
\[ \lesssim |v\partial_2 w| + |w\partial_2 v| + |v\partial_1 w| + |w\partial_1 v|. \tag{3.7} \]

By using (2.2) again, we have
\[ \|v\partial_2 w\|_{B^2_{2,1}} + \|v\partial_1 w\|_{B^2_{2,1}} \lesssim \|v\|_{B^2_{2,1}} \|w\|_{B^2_{2,1}}, \]
\[ \|w\partial_2 v\|_{B^2_{2,1}} + \|w\partial_1 v\|_{B^2_{2,1}} \lesssim \|v\|_{B^2_{2,1}} \|w\|_{B^2_{2,1}}. \tag{3.8} \]

The combination of (3.5)–(3.8) yields
\[ \|([Q, u \cdot \nabla]u)\|_{B^2_{2,1}} \lesssim \|v\|_{B^2_{2,1}} \|w\|_{B^2_{2,1}} \]
\[ + \left( \|Q u\|_{B^2_{2,1}} + \|v\|_{B^2_{2,1}} \right) \left( \|Q u\|_{B^2_{2,1}} + \|(v, w)\|_{B^2_{2,1}} \right). \tag{3.9} \]

Throughout we make the assumption that
\[ \sup_{t \in \mathbb{R}_+, x \in \mathbb{R}^2} |a(t, x)| \leq \frac{1}{2} \tag{3.10} \]
which will enable us to use freely the composition estimate stated in (2.4). Note that as \( \dot{B}^1_{2,1}(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2) \), condition (3.10) will be ensured by the fact that the constructed solution about \( a \) has small norm.

Thus, one can obtain
\[ \|Q G\|_{B^2_{2,1}} \lesssim \|Q, u \cdot \nabla |u\|_{B^2_{2,1}} + \|L(a)\Delta u + L(a)\nabla \text{div } u\|_{B^2_{2,1}} + \|k(a)\nabla a\|_{B^2_{2,1}} \]
\[ \lesssim \|v\|_{B^2_{2,1}} \|w\|_{B^2_{2,1}} + \|L(a)\|_{B^2_{2,1}} \|u\|_{B^2_{2,1}} + \|k(a)\|_{B^2_{2,1}} \|a\|_{B^2_{2,1}} \]
\[ + \left( \|(v, Q u)\|_{B^2_{2,1}} \right) \left( \|Q u\|_{B^2_{2,1}} + \|(v, w)\|_{B^2_{2,1}} \right) \]
\[ \lesssim \|v\|_{B^2_{2,1}} \|w\|_{B^2_{2,1}} \]
\[ + \left( \|a\|_{B^2_{2,1}} + \|(v, Q u)\|_{B^2_{2,1}} \right) \left( \|a^h\|_{B^2_{2,1}} + \|(a', v, w, Q u)\|_{B^2_{2,1}} \right), \tag{3.11} \]
in which we have used the fact:
\[ \|a\|_{B^2_{2,1}} \lesssim \|a'\|_{B^2_{2,1}} + \|a^h\|_{B^2_{2,1}} \lesssim \|a'\|_{B^2_{2,1}} \|a\|_{B^2_{2,1}} + \|a^h\|_{B^2_{2,1}} \]
\[ \lesssim \left( \|a\|_{B^2_{2,1}} + \|a\|_{B^2_{2,1}} \right) \left( \|a\|_{B^2_{2,1}} + \|a\|_{B^2_{2,1}} \right). \]

This completes the proof of Lemma 3.1. \( \square \)

Now, we present the energy estimates for \( a, Q u, v \) and \( w \), respectively, in the framework of Besov spaces.

Applying \( \hat{\Delta}_j \) to the first equation in (3.3) gives
\[ \partial_t \hat{\Delta}_j a + u \cdot \nabla \hat{\Delta}_j a + \text{div } Q \hat{\Delta}_j u + [\hat{\Delta}_j, u \cdot \nabla]a = -\hat{\Delta}_j (a \text{ div } u). \tag{3.12} \]
Taking $L^2$ inner product of $\dot{\Delta}_a$ with (3.12) and using integrating by parts, we have
\[
\frac{1}{2} \frac{d}{dt} \|\dot{\Delta}_a\|_{L^2}^2 + \int_{\mathbb{R}^2} \dot{\Delta}_a \cdot \text{div} \dot{\Delta}_a \, dx
= \frac{1}{2} \int_{\mathbb{R}^2} \text{div} u |\dot{\Delta}_a|^2 \, dx - \int_{\mathbb{R}^2} [\dot{\Delta}_a, u] \cdot \dot{\Delta}_a \, dx - \int_{\mathbb{R}^2} \dot{\Delta}_a (\text{div} u) \cdot \dot{\Delta}_a \, dx. \tag{3.13}
\]
Applying $\dot{\Delta}_a$ to the second equation in (3.3) gives
\[
\partial_t \dot{\Delta}_a u + u \cdot \nabla \dot{\Delta}_a u - 2\Delta \dot{\Delta}_a u + \nabla \dot{\Delta}_a = \dot{\Delta}_a QG - [\dot{\Delta}_a, u] \cdot \nabla \dot{\Delta}_a. \tag{3.14}
\]
We can get by using a similar derivation of (3.13) that
\[
\frac{1}{2} \frac{d}{dt} \|\dot{\Delta}_a u\|_{L^2}^2 + 2\|\nabla \dot{\Delta}_a u\|_{L^2}^2 - \int_{\mathbb{R}^2} \dot{\Delta}_a \cdot \text{div} \dot{\Delta}_a u \, dx
= \frac{1}{2} \int_{\mathbb{R}^2} \text{div} u |\dot{\Delta}_a|^2 \, dx - \int_{\mathbb{R}^2} [\dot{\Delta}_a, u] \cdot \dot{\Delta}_a \, dx + \int_{\mathbb{R}^2} \dot{\Delta}_a QG \cdot \dot{\Delta}_a u \, dx. \tag{3.15}
\]
Applying the gradient $\nabla$ on (3.12), we have
\[
\partial_t \nabla \dot{\Delta}_a + u \cdot \nabla \dot{\Delta}_a + \nabla \text{div} \dot{\Delta}_a u = F_j(a, u) \tag{3.16}
\]
with $F_j(a, u) \overset{\text{def}}{=} -\nabla ([\dot{\Delta}_a, u] \cdot \nabla a) - \nabla \dot{\Delta}_a (\text{div} u) - \nabla u \cdot \nabla \dot{\Delta}_a$.

Taking $L^2$ inner product of $\nabla \dot{\Delta}_a$ with the previous equation gives
\[
\frac{d}{dt} \|\nabla \dot{\Delta}_a\|_{L^2}^2 + 2\|\nabla \dot{\Delta}_a \cdot \text{div} \dot{\Delta}_a u\|_{L^2}^2
= \int_{\mathbb{R}^2} \text{div} u |\nabla \dot{\Delta}_a|^2 \, dx + 2\int_{\mathbb{R}^2} F_j(a, u) \cdot \nabla \dot{\Delta}_a \, dx. \tag{3.17}
\]
Testing (3.14) by $\nabla \dot{\Delta}_a$ and (3.16) by $\dot{\Delta}_a u$, we get
\[
\frac{d}{dt} \int_{\mathbb{R}^2} \dot{\Delta}_a u \cdot \nabla \dot{\Delta}_a \, dx + \int_{\mathbb{R}^2} |\nabla \dot{\Delta}_a|^2 \, dx
+ \int_{\mathbb{R}^2} \nabla \text{div} \dot{\Delta}_a u \cdot \dot{\Delta}_a u \, dx - 2\int_{\mathbb{R}^2} \Delta \dot{\Delta}_a u \cdot \nabla \dot{\Delta}_a \, dx
= \int_{\mathbb{R}^2} F_j(a, u) \cdot \dot{\Delta}_a u \, dx + \int_{\mathbb{R}^2} \dot{\Delta}_a u \cdot \nabla \dot{\Delta}_a \, dx
+ \int_{\mathbb{R}^2} (\dot{\Delta}_a QG - [\dot{\Delta}_a, u] \cdot \nabla \dot{\Delta}_a) \, dx \tag{3.18}
\]
in which we have used the following equality:
\[
\int_{\mathbb{R}^2} u \cdot \nabla (\dot{\Delta}_a u \cdot \nabla \dot{\Delta}_a) \, dx = -\int_{\mathbb{R}^2} \dot{\Delta}_a u \cdot \nabla \dot{\Delta}_a \, dx. \]
Denote
\[
L_j^2 \overset{\text{def}}{=} \int_{\mathbb{R}^2} (|\hat{\Delta}_j a|^2 + |\hat{\Delta}_j Q u|^2 + 2 \hat{\Delta}_j Q u \cdot \nabla \hat{\Delta}_j a + 2 |\nabla \hat{\Delta}_j a|^2) \, dx.
\]

Combining with estimates (3.13), (3.15), (3.17) and (3.18), we get
\[
\frac{1}{2} \frac{d}{dt} L_j^2 + \int_{\mathbb{R}^2} (|\nabla \hat{\Delta}_j Q u|^2 + |\nabla \hat{\Delta}_j a|^2) \, dx
= \frac{1}{2} \int_{\mathbb{R}^2} \text{div} \, u |\hat{\Delta}_j a|^2 \, dx - \int_{\mathbb{R}^2} |\hat{\Delta}_j, u \cdot \nabla| a \cdot \hat{\Delta}_j a \, dx + \int_{\mathbb{R}^2} \hat{\Delta}_j (\text{div} \, u) \cdot \hat{\Delta}_j a \, dx
+ \frac{1}{2} \int_{\mathbb{R}^2} \text{div} \, u |\hat{\Delta}_j Q u|^2 \, dx + \int_{\mathbb{R}^2} \hat{\Delta}_j Q u \cdot \hat{\Delta}_j a \, dx + \int_{\mathbb{R}^2} \hat{\Delta}_j G \cdot \hat{\Delta}_j Q u \, dx
+ \int_{\mathbb{R}^2} \text{div} \, u |\hat{\Delta}_j| \text{div} \, u \, dx + \int_{\mathbb{R}^2} (\hat{\Delta}_j Q G - |\hat{\Delta}_j, u \cdot \nabla| Q u) \cdot \nabla \hat{\Delta}_j a \, dx. \tag{3.19}
\]

It is readily seen that
\[
L_j \approx \| (\hat{\Delta}_j Q u, \hat{\Delta}_j a, 2 \nabla \hat{\Delta}_j a) \|_{L^2} \quad \text{for all} \quad j \in \mathbb{Z},
\]
\[
\int_{\mathbb{R}^2} (|\nabla \hat{\Delta}_j Q u|^2 + |\nabla \hat{\Delta}_j a|^2) \, dx \geq c \min(2^{2j}, 2^{-2}) L_j^2
\]
for a constant c. Therefore, we deduce from (3.19) that
\[
\frac{1}{2} \frac{d}{dt} L_j^2 + c \min(2^{2j}, 2^{-2}) L_j^2
\lesssim \| \nabla u \|_{L^\infty} L_j^2 + \left( \| (\hat{\Delta}_j (\text{div} \, u), \hat{\Delta}_j Q G) \|_{L^2} + \| F_j(a, u) \|_{L^2} + \| \hat{\Delta}_j, u \cdot \nabla \| a \|_{L^2} + \| \hat{\Delta}_j, u \cdot \nabla \| Q u \|_{L^2} \right) L_j.
\]

Hence, with the aid of Hölder’s inequality, integrating in time, we can finally get
\[
L_j(t) + c \min(2^{2j}, 2^{-2}) \int_0^t L_j \, d\tau
\lesssim L_j(0) + \int_0^t \| \nabla u \|_{L^\infty} L_j \, d\tau + \int_0^t \left( \| (\hat{\Delta}_j (\text{div} \, u), \hat{\Delta}_j \nabla (\text{div} \, u), \hat{\Delta}_j Q G) \|_{L^2} + \| \nabla (\hat{\Delta}_j, u \cdot \nabla \| a \|_{L^2} + \| \hat{\Delta}_j, u \cdot \nabla \| Q u \|_{L^2} \right) \, d\tau. \tag{3.20}
\]

From (3.15), we get
\[
\frac{1}{2} \frac{d}{dt} \| \hat{\Delta}_j Q u \|_{L^2}^2 + 2^{2j} \| \hat{\Delta}_j Q u \|_{L^2}^2
\lesssim \| \nabla u \|_{L^\infty} \| \hat{\Delta}_j Q u \|_{L^2}^2 + (\| \nabla \hat{\Delta}_j a \|_{L^2} + \| \hat{\Delta}_j, u \cdot \nabla \| Q u \|_{L^2} + \| \hat{\Delta}_j Q G \|_{L^2}) \| \hat{\Delta}_j Q u \|_{L^2},
\]
which further implies

$$\|\Delta_j \mathcal{Q} u(t)\|_{L^2} + \int_0^t 2^{2j} \|\Delta_j \mathcal{Q} u\|_{L^2} \, d\tau$$

$$\lesssim \|\Delta_j \mathcal{Q} u(0)\|_{L^2} + \int_0^t \|\nabla u\|_{L^2} \|\Delta_j \mathcal{Q} u\|_{L^2} \, d\tau$$

$$+ \int_0^t \|\nabla \Delta_j a\|_{L^2} + \|\Delta_j \cdot \nabla \mathcal{Q} u\|_{L^2} + \|\Delta_j \mathcal{Q} G\|_{L^2}) \, d\tau. \quad (3.21)$$

This together with (3.20) yields

$$\|(a, \mathcal{Q} u)(t)\|_{\dot{B}_{2,1}^0} + \|a(t)\|_{\dot{B}_{2,1}^1} + \int_0^t \left(\|(a^f, \mathcal{Q} u)\|_{\dot{B}_{2,1}^2} + \|a^h\|_{\dot{B}_{2,1}^1}\right) \, d\tau$$

$$\lesssim \|(a_0, \mathcal{Q} u_0)\|_{\dot{B}_{2,1}^0} + \|a_0\|_{\dot{B}_{2,1}^1} + \int_0^t \|\nabla u\|_{L^2} \|(a, \nabla a, \mathcal{Q} u)\|_{\dot{B}_{2,1}^0} \, d\tau$$

$$+ \int_0^t \left(\|\text{adiv} \mathcal{Q} u\|_{\dot{B}_{2,1}^2} + \|\nabla (\text{adiv} \mathcal{Q} u)\|_{\dot{B}_{2,1}^0} + \|\mathcal{Q} G\|_{\dot{B}_{2,1}^0}\right) \, d\tau$$

$$+ \int_0^t \sum_{j \in \mathbb{Z}} \left(\|\Delta_j \cdot \nabla a\|_{L^2} + \|\nabla (\Delta_j \cdot \nabla a)\|_{L^2} + \|\Delta_j \cdot \nabla \mathcal{Q} u\|_{L^2}\right) \, d\tau. \quad (3.22)$$

By using (2.2), we have

$$\|\text{adiv} \mathcal{Q} u\|_{\dot{B}_{2,1}^2} \lesssim \|a\|_{\dot{B}_{2,1}^2}, \|\text{div} \mathcal{Q} u\|_{\dot{B}_{2,1}^1} \lesssim \|a\|_{\dot{B}_{2,1}^1}, \|\mathcal{Q} u\|_{\dot{B}_{2,1}^0},$$

$$\|\nabla (\text{adiv} \mathcal{Q} u)\|_{\dot{B}_{2,1}^0} \lesssim \|\text{adiv} \mathcal{Q} u\|_{\dot{B}_{2,1}^2} \lesssim \|a\|_{\dot{B}_{2,1}^2}, \|\text{div} \mathcal{Q} u\|_{\dot{B}_{2,1}^1} \lesssim \|a\|_{\dot{B}_{2,1}^1}, \|\mathcal{Q} u\|_{\dot{B}_{2,1}^0}. \quad (3.23)$$

The terms involved in the commutator can be obtained from (2.3) that

$$\sum_{j \in \mathbb{Z}} \|\Delta_j \cdot \nabla a\|_{L^2} \lesssim \sum_{j \in \mathbb{Z}} \|\Delta_j \cdot \mathcal{P} u \cdot \nabla a\|_{L^2} + \sum_{j \in \mathbb{Z}} \|\Delta_j \cdot \mathcal{Q} u \cdot \nabla a\|_{L^2}$$

$$\lesssim \|(v, w, \mathcal{Q} u)\|_{\dot{B}_{2,1}^1} \|a\|_{\dot{B}_{2,1}^0}, \quad (3.24)$$

$$\sum_{j \in \mathbb{Z}} \|\nabla (\Delta_j \cdot \nabla a)\|_{L^2} \lesssim \sum_{j \in \mathbb{Z}} \|\nabla (\Delta_j \cdot \mathcal{P} u \cdot \nabla a)\|_{L^2} + \sum_{j \in \mathbb{Z}} \|\nabla (\Delta_j \cdot \mathcal{Q} u \cdot \nabla a)\|_{L^2}$$

$$\lesssim \|(v, w, \mathcal{Q} u)\|_{\dot{B}_{2,1}^1} \|a\|_{\dot{B}_{2,1}^0}, \quad (3.25)$$

$$\sum_{j \in \mathbb{Z}} \|\Delta_j \cdot \nabla \mathcal{Q} u\|_{L^2} \lesssim \sum_{j \in \mathbb{Z}} \|\Delta_j \cdot \mathcal{P} u \cdot \nabla \mathcal{Q} u\|_{L^2} + \sum_{j \in \mathbb{Z}} \|\Delta_j \cdot \mathcal{Q} u \cdot \nabla \mathcal{Q} u\|_{L^2}$$

$$\lesssim \|(v, w, \mathcal{Q} u)\|_{\dot{B}_{2,1}^1} \|\mathcal{Q} u\|_{\dot{B}_{2,1}^0}. \quad (3.26)$$
Inserting (3.23)–(3.26) into (3.22) and using Lemma 3.1, we have
\[
\| (a, Q u) (t) \|_{\dot{B}^1_{2,1}} + \| a (t) \|_{\dot{B}^1_{2,1}} + \int_0^t \left( \| (a^\ell, Q u) \|_{\dot{B}^1_{2,1}} + \| a^h \|_{\dot{B}^1_{2,1}} \right) \, d\tau \\
\lesssim \| (a_0, Q u_0) \|_{\dot{B}^1_{2,1}} + \| a_0 \|_{\dot{B}^1_{2,1}} \\
+ \int_0^t \left( \| a \|_{\dot{B}^1_{2,1}} + \| (a, v, Q u) \|_{\dot{B}^1_{2,1}} \right) \left( \| a^h \|_{\dot{B}^1_{2,1}} + \| (a^\ell, v, w, Q u) \|_{\dot{B}^1_{2,1}} \right) \, d\tau.
\]
(3.27)

Applying \( \dot{\Delta}_j \) to the first equation in (3.2) and using a standard energy argument, we have
\[
\| v \|_{\dot{L}^\infty (\dot{B}^1_{2,1})} + \| v \|_{L^1 (\dot{B}^1_{2,1})} \\
\lesssim \| v_0 \|_{\dot{B}^1_{2,1}} + \int_0^t \| \nabla u \|_{L^\infty} \| v \|_{\dot{B}^1_{2,1}} \, d\tau + \int_0^t \sum_{j \in \mathbb{Z}} \| \dot{\Delta}_j, u \cdot \nabla \| v \|_{L^2} \, d\tau \\
+ \int_0^t \| \mathcal{P} (L(a) \Delta u + L(a) \nabla \| \nabla u \|_{\dot{B}^1_{2,1}} \, d\tau + \int_0^t \| \mathcal{P}, u \cdot \nabla \| u \|_{\dot{B}^1_{2,1}} \, d\tau.
\]
(3.28)

Using the embedding relation \( \dot{B}^1_{2,1} (\mathbb{R}^2) \hookrightarrow L^\infty (\mathbb{R}^2) \), we get
\[
\| \nabla u \|_{L^\infty} \| v \|_{\dot{B}^1_{2,1}} \lesssim \| \nabla u \|_{\dot{B}^1_{2,1}} \| v \|_{\dot{B}^1_{2,1}} \lesssim \| Q u \|_{\dot{B}^1_{2,1}} \| v \|_{\dot{B}^1_{2,1}}.
\]
(3.29)

The term about commutator can be estimated similarly to (3.24) that
\[
\sum_{j \in \mathbb{Z}} \| [\dot{\Delta}_j, u \cdot \nabla] v \|_{L^2} \lesssim \| (v, w, Q u) \|_{\dot{B}^1_{2,1}} \| v \|_{\dot{B}^1_{2,1}}.
\]
(3.30)

Thanks to (2.2) and (2.4), we infer that
\[
\| \mathcal{P} (L(a) \Delta u + L(a) \nabla \|_{\dot{B}^1_{2,1}} \lesssim \| L(a) \|_{\dot{B}^1_{2,1}} \| \nabla^2 u \|_{\dot{B}^1_{2,1}} \\
\lesssim \| a \|_{\dot{B}^1_{2,1}} \| (v, w, Q u) \|_{\dot{B}^1_{2,1}}.
\]
(3.31)

In order to deal with the last term in (3.28), we deduce from \( u = \mathcal{P} u + Q u \) that
\[
[\mathcal{P}, u \cdot \nabla] u = [\mathcal{P}, (Q u + \mathcal{P} u) \cdot \nabla] (Q u + \mathcal{P} u) \\
= [\mathcal{P}, (Q u) \cdot \nabla] \mathcal{P} u + [\mathcal{P}, (Q u) \cdot \nabla] Q u \\
+ [\mathcal{P}, (\mathcal{P} u) \cdot \nabla] Q u + [\mathcal{P}, (\mathcal{P} u) \cdot \nabla] \mathcal{P} u.
\]
(3.32)

Applying (2.2), we have
\[
\| [\mathcal{P}, (Q u) \cdot \nabla] \mathcal{P} u \|_{\dot{B}^1_{2,1}} \lesssim \| (Q u) \cdot \nabla \mathcal{P} u \|_{\dot{B}^1_{2,1}} \lesssim \| (v, w) \|_{\dot{B}^1_{2,1}} \| Q u \|_{\dot{B}^1_{2,1}},
\]
\[
\| [\mathcal{P}, (Q u) \cdot \nabla] Q u \|_{\dot{B}^1_{2,1}} \lesssim \| (Q u) \cdot \nabla Q u \|_{\dot{B}^1_{2,1}} \lesssim \| Q u \|_{\dot{B}^1_{2,1}} \| Q u \|_{\dot{B}^1_{2,1}}.
\]
(3.33)

By virtue of
\[
\dot{\Delta}_j ([\mathcal{P}, (\mathcal{P} u) \cdot \nabla] Q u) = [\dot{\Delta}_j \mathcal{P}, (\mathcal{P} u) \cdot \nabla] Q u,
\]
(3.34)

and an argument similar to the derivation of (3.6), we have
\[
\| [\mathcal{P}, (\mathcal{P} u) \cdot \nabla] Q u \|_{\dot{B}^1_{2,1}} \lesssim \| (v, w) \|_{\dot{B}^1_{2,1}} \| Q u \|_{\dot{B}^1_{2,1}}.
\]
(3.35)

On the other hand, since
\[
\| [\mathcal{P}, (\mathcal{P} u) \cdot \nabla] \mathcal{P} u \| \lesssim \| (\mathcal{P} u) \cdot \nabla \mathcal{P} u |,
\]
we get from (3.7) that
\[
\|\mathcal{P} \cdot (\mathcal{P} \mathbf{u}) \cdot \nabla \mathcal{P} \mathbf{u}\|_{B_{2,1}^q} \lesssim \|v\|_{B_{2,1}^q} \|w\|_{B_{2,1}^q} + \|v\|_{B_{2,1}^q} \|w\|_{B_{2,1}^q}.
\]
(3.36)

The combination of the estimates (3.32)–(3.36) yields
\[
\|\mathcal{P} \cdot \nabla \mathbf{u}\|_{B_{2,1}^q} \lesssim \|v\|_{B_{2,1}^q} \|w\|_{B_{2,1}^q} + \left(\|\mathcal{Q} \mathbf{u}\|_{B_{2,1}^q} + \|v\|_{B_{2,1}^q} \right) \left(\|\mathcal{Q} \mathbf{u}\|_{B_{2,1}^q} + \|(v, w)\|_{B_{2,1}^q}\right).
\]
(3.37)

Inserting (3.29)–(3.31) and (3.37) into (3.28) gives
\[
\|v\|_{L^\infty_t(B_{2,1}^s)} + \|v\|_{L^1_t(B_{2,1}^s)} \
\lesssim \|v_0\|_{B_{2,1}^s} + \int_0^t \|v\|_{B_{2,1}^s} \|w\|_{B_{2,1}^s} \, d\tau \
+ \int_0^t \|(v, w, \mathbf{Q} \mathbf{u})\|_{B_{2,1}^s} \left(\|a\|_{B_{2,1}^s} + \|(v, \mathbf{Q} \mathbf{u})\|_{B_{2,1}^s}\right) \, d\tau.
\]
(3.38)

The combination of (3.27) and (3.38) produces the desired estimate:
\[
\|(a, \mathbf{Q} \mathbf{u}, v)(t)\|_{B_{2,1}^s} + \|a(t)\|_{B_{2,1}^s} + \int_0^t \left(\|(a^\ell, \mathbf{Q} \mathbf{u}, v)\|_{B_{2,1}^s} + \|a^h\|_{B_{2,1}^s}\right) \, d\tau \
\lesssim \|(a_0, \mathbf{Q} \mathbf{u}_0, v_0)\|_{B_{2,1}^s} + \|a_0\|_{B_{2,1}^s} + \int_0^t \|v\|_{B_{2,1}^s} \|w\|_{B_{2,1}^s} \, d\tau \
+ \int_0^t \left(\|a\|_{B_{2,1}^s} + \|(a, \mathbf{Q} \mathbf{u}, v)\|_{B_{2,1}^s}\right) \left(\|a^h\|_{B_{2,1}^s} + \|(a^\ell, v, w, \mathbf{Q} \mathbf{u})\|_{B_{2,1}^s}\right) \, d\tau.
\]
(3.39)

The global solution is to be deduced by a continuity argument based on the estimates obtained. Indeed, let \(0 < c_1 \ll 1\) be a sufficient small constant, which will be determined later on. Define
\[
T^{**} = \sup \left\{ t \in [0, T^*): \|(a, \mathbf{Q} \mathbf{u}, v)\|_{L^\infty_t(B_{2,1}^s)} + \|a\|_{L^1_t(B_{2,1}^s)} + \|(a^\ell, v, \mathbf{Q} \mathbf{u})\|_{L^1_t(B_{2,1}^s)} + \|a^h\|_{L^1_t(B_{2,1}^s)} \leq c_1 \right\}.
\]
(3.40)

According to the known local well-posedness, it is obvious that \(T^{**} > 0\). We shall prove \(T^{**} = \infty\) under the assumption (1.5).

By using the interpolation inequality, we can get
\[
\int_0^t \|v\|_{B_{2,1}^s} \|w\|_{B_{2,1}^s} \, d\tau \leq C \int_0^t \|v\|_{B_{2,1}^s}^{3/2} \|v\|_{B_{2,1}^s}^{3/2} \|w\|_{B_{2,1}^s} \, d\tau \
\leq \frac{1}{8} \int_0^t \|v\|_{B_{2,1}^s} \, d\tau + C \int_0^t \|v\|_{B_{2,1}^s} \|w\|_{B_{2,1}^s}^2 \, d\tau.
\]
(3.41)
Inserting (3.41) into (3.39) and using (3.40), we get for \( t \in [0, T^*] \) that

\[
\| (a, Qu, v)(t) \|_{\dot{B}^0_{2,1}} + \| a(t) \|_{\dot{B}^1_{2,1}} + \frac{1}{2} \int_0^t \left( \| (a^t, Qu, v) \|_{\dot{B}^0_{2,1}} + \| a^h \|_{\dot{B}^1_{2,1}} \right) \, d\tau
\]

\[
\lesssim \| (a_0, Qu_0, v_0) \|_{\dot{B}^0_{2,1}} + \| a_0 \|_{\dot{B}^1_{2,1}}
\]

\[
+ \int_0^t \left( \| w \|_{\dot{B}^0_{2,1}} + \| w \|_{\dot{B}^1_{2,1}}^2 \right) \left( \| a \|_{\dot{B}^1_{2,1}} + \| (a, Qu, v) \|_{\dot{B}^2_{2,1}} \right) \, d\tau
\]

(3.42)

and the Gronwall inequality implies that

\[
\| (a, Qu, v)(t) \|_{\dot{B}^0_{2,1}} + \| a(t) \|_{\dot{B}^1_{2,1}} + \frac{1}{2} \int_0^t \left( \| (a^t, Qu, v) \|_{\dot{B}^0_{2,1}} + \| a^h \|_{\dot{B}^1_{2,1}} \right) \, d\tau
\]

\[
\lesssim \left( \| (a_0, Qu_0, v_0) \|_{\dot{B}^0_{2,1}} + \| a_0 \|_{\dot{B}^1_{2,1}} \right) \exp \left( \int_0^t \left( \| w \|_{\dot{B}^0_{2,1}} + \| w \|_{\dot{B}^1_{2,1}}^2 \right) \, d\tau \right)
\]

(3.43)

for \( t \in [0, T^*] \).

Next, we shall show that the integration \( \int_0^t (\| w \|_{\dot{B}^0_{2,1}} + \| w \|_{\dot{B}^1_{2,1}}) \, d\tau \) on the right-hand side of (3.43) can be controlled by the initial data if there holds (3.40). Indeed, along the same lines as the derivation of (3.28), we can infer from the second equation in (3.2) that

\[
\| w \|_{L^\infty_t(\dot{B}^0_{2,1})} + \| w \|_{L^1_t(\dot{B}^2_{2,1})} + \| w \|_{L^1_t(\dot{B}^1_{2,1})}
\]

\[
\lesssim \| w_0 \|_{\dot{B}^2_{2,1}} + \int_0^t \| u \cdot \nabla w \|_{\dot{B}^0_{2,1}} \, d\tau
\]

\[
+ \int_0^t \| \mathcal{P} (L(a) \Delta u + L(a) \nabla \text{div} u) \|_{\dot{B}^2_{2,1}} \, d\tau + \int_0^t \| [P, u \cdot \nabla] u \|_{\dot{B}^2_{2,1}} \, d\tau.
\]

(3.44)

The last two terms on the right-hand side of (3.44) have been bounded by (3.31) and (3.37). To control the second term, we have by using the fact \( \partial_1 v + \partial_2 w = 0 \) and product law in Besov spaces that

\[
\| u \cdot \nabla w \|_{\dot{B}^0_{2,1}} \lesssim \| v \partial_1 w \|_{\dot{B}^0_{2,1}} + \| w \partial_1 v \|_{\dot{B}^0_{2,1}} + \| Qu \cdot \nabla w \|_{\dot{B}^0_{2,1}}
\]

\[
\lesssim \| v \|_{\dot{B}^1_{2,1}} \| w \|_{\dot{B}^1_{2,1}} + \| Qu \|_{\dot{B}^0_{2,1}} \| w \|_{\dot{B}^1_{2,1}}.
\]

(3.45)

Plugging the estimates (3.31), (3.37) and (3.45) into (3.44) gives

\[
\| w \|_{L^\infty_t(\dot{B}^0_{2,1})} + \| w \|_{L^1_t(\dot{B}^2_{2,1})} + \| w \|_{L^1_t(\dot{B}^1_{2,1})}
\]

\[
\lesssim \| w_0 \|_{\dot{B}^2_{2,1}} + \int_0^t \| v \|_{\dot{B}^1_{2,1}} \| w \|_{\dot{B}^1_{2,1}} \, d\tau
\]

\[
+ \int_0^t \| (v, w, Qu) \|_{\dot{B}^2_{2,1}} \left( \| a \|_{\dot{B}^1_{2,1}} + \| (v, Qu) \|_{\dot{B}^2_{2,1}} \right) \, d\tau.
\]

(3.46)
Employing the interpolation inequality, we see that
\[
\int_0^t \|v\|_{\dot{B}^1_{2,1}} \|w\|_{\dot{B}^1_{2,1}} \, d\tau \lesssim \int_0^t \|v\|_{\dot{B}^0_{2,1}} \|v\|_{\dot{B}^1_{2,1}} \|w\|_{\dot{B}^1_{2,1}} \, d\tau
\lesssim \|v\|_{L^\infty_t(\dot{B}^0_{2,1})} \|v\|_{L^1_t(\dot{B}^1_{2,1})} \|w\|_{L^1_t(\dot{B}^1_{2,1})}. \tag{3.47}
\]
Hence, inserting (3.47) into (3.46) and using (3.40), we can get for \( t \in [0, T^{**}] \) that
\[
\|w\|_{L^\infty_t(\dot{B}^0_{2,1})} + \|w\|_{L^1_t(\dot{B}^1_{2,1})} \leq C \left( \|(a_0, Qu_0, v_0, w_0)\|_{\dot{B}^0_{2,1}} + \|a_0\|_{\dot{B}^1_{2,1}} \right). \tag{3.48}
\]
Taking the above inequality into (3.43), we can finally get
\[
\|(a, Qu, v)(t)\|_{\dot{B}^0_{2,1}} + \|a(t)\|_{\dot{B}^1_{2,1}} + \frac{1}{2} \int_0^t \left( \|(a^\ell, Qu, v)\|_{\dot{B}^1_{2,1}} + \|a^h\|_{\dot{B}^1_{2,1}} \right) \, d\tau
\leq C \left( \|(a_0, Qu_0, v_0, w_0)\|_{\dot{B}^0_{2,1}} + \|a_0\|_{\dot{B}^1_{2,1}} \right) \exp \left( C \left( \|(a_0, Qu_0, v_0, w_0)\|_{\dot{B}^0_{2,1}} + \|a_0\|_{\dot{B}^1_{2,1}} \right)^2 \right). \tag{3.49}
\]
Thus, if the smallness condition (1.5) is satisfied, then (3.49) implies that
\[
\|(a, Qu, v)\|_{L^\infty_t(\dot{B}^0_{2,1})} + \|a\|_{L^1_t(\dot{B}^1_{2,1})} + \|(a^\ell, v, Qu)\|_{L^1_t(\dot{B}^1_{2,1})} + \|a^h\|_{L^1_t(\dot{B}^1_{2,1})} \leq \frac{C_1}{2}
\]
for \( t \leq T^{**} \). This contradicts to (3.40). Hence, we conclude that \( T^{**} = \infty \) and the proof of Theorem 1.1 is complete.

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