Abstract: We describe a few properties of the XXX spin chain with long range interaction. The plan of these notes is:
1 — The Hamiltonian.
2 — Symmetry of the model.
3 — The irreducible multiplets.
4 — The spectrum.
5 — Wave functions and statistics.
6 — The spinon description.
7 — The thermodynamics.

Introduction. The XXX spin chain with long range interaction is a variant of the spin half Heisenberg chain, with exchange inversely proportional to the square distance between the spins. It possesses the remarkable properties that its spectrum is additive and that the elementary excitations are spin half objects obeying a half-fractional statistics intermediate between bosons and fermions. In this sense, it gives a model for an ideal gas of particles with fractional statistics. The model is gapless; its low energy properties belong to the same universality class as the Heisenberg model, and are described by the level one $su(2)$ WZW conformal field theory.

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1 The Hamiltonian.

The Hamiltonian of the trigonometric isotropic spin chain with long range interaction is given by:

$$ H = \left( \frac{\pi}{N} \right)^2 \sum_{i \neq j} \frac{(P_{ij} - 1)}{\left(2 \sin \left( \frac{\pi(i-j)}{2N} \right) \right)^2} $$

where $P_{ij}$ is the operator which exchange the spins at the sites $i$ and $j$. We restrict ourselves to the $su(2)$ case, in which case the spin variables can only take two values: $\sigma_i = \pm$. The sum is over all the distinct pairs of sites labeled by integers $i,j, \cdots$ ranging from 1 to $N$.

The spectrum of (1), which has been conjectured by Haldane, possesses a remarkable additivity property as well as a rich degeneracy. It can be described as follow. To each eigenstate multiplet is associated a set of rapidities $\{m_p\}$ which are non-consecutive integers ranging from 1 to $(N-1)$. The energy of an eigenstate $|\{m_p\}\rangle$ with rapidities $\{m_p\}$ is:

$$ H|\{m_p\}\rangle = \left( \sum_p \epsilon(m_p) \right) |\{m_p\}\rangle \quad \text{with} \quad \epsilon(m) = \left( \frac{\pi}{N} \right)^2 m(m-N) $$

The degeneracy of the multiplet with rapidities $\{m_p\}$ is described by its $su(2)$ representation content as follows. Encode the rapidities in a sequence of $(N-1)$ labels 0 or 1 in which the 1's indicate the position of the rapidities; add two 0's at both extremities of the sequence which now has length $(N+1)$. Since the rapidities are never equal nor differ by a unit, two labels 1 cannot be adjacent. A sequence can be decomposed into the product of elementary motifs. A motif is a series of $Q$ consecutives 0’s, and it corresponds to a spin $Q-\frac{1}{2}$ representation of $su(2)$. The representation content of a sequence is then the tensor product of its motifs.

The degeneracy of the spectrum can also be described in terms of path, in a way surprisingly similar to the path description of the six-vertex corner transfer matrix [4].

2 Symmetry of the model.

The symmetry algebra responsible for the degeneracy of the model was identified as the $su(2)$ Yangian [5]. A Yangian is an infinite dimensional associative algebra generated by elements $T_n^{ab}$, with $n$ a positive integer, and $a, b = \pm$ in the $su(2)$ case. These generators satisfy quadratic relations which can be arranged into a Yang-Baxter equation by introducing the transfer matrix $T(x)$, with matrix elements, $T^{ab}(x) = \delta^{ab} + \sum_{n\geq0} x^{-n-1}T_n^{ab}$. The commutation relations then take the following form:

$$ R(x-y) \left( 1 \otimes T(x) \right) \left( T(y) \otimes 1 \right) = \left( T(y) \otimes 1 \right) \left( 1 \otimes T(x) \right) R(x-y) $$

(3)

The matrix $R(x)$ is the solution of the Yang-Baxter equation given by: $R(x) = x + P$, where $P$ is the permutation operator which exchanges the two auxiliary spaces. The transfer matrix was constructed in [8]. Its expression is:

$$ T^{ab}(x) = \delta^{ab} + \sum_{n=0}^{N} X^{ab}_n \left( \frac{1}{x} \right)^n $$

(4)
with \( L_{ij} = (1 - \delta_{ij}) \theta_{ij} P_{ij} \), \( \theta_{ij} = z_i / z_{ij} \) with \( z_{ij} = z_i - z_j \), and \( X_i^{ab} \) is the canonical matrix \(|a\rangle \langle b|\) acting on the \( i^{th} \) spin only. The transfer matrix (4) form a representation of the exchange algebra (3) for any values of the complex numbers \( z_j \). The center of the \( \text{su}(2) \) Yangian algebra (3) is generated by the so-called quantum determinant \( \text{Det}_q T(x) \) defined by (5):

\[
\text{Det}_q T(x) = T_{--}(x - 1)T_{++}(x) - T_{+-}(x - 1)T_{-+}(x)
\]

In the representation (4), the quantum determinant is a pure number for any values of the \( z_j \)'s given by:

\[
\text{Det}_q T(x) = 1 + \sum_{i,j=1}^{N} \left( \frac{1}{x - \Theta} \right)_{ij} = \frac{\Delta_N(x + 1)}{\Delta_N(x)}
\]

with \( \Delta_N(x) \) the characteristic polynomial of the \( N \times N \) matrix \( \Theta \) with entries \( \theta_{ij} \):

\[
\Delta_N(x) = \det(x - \Theta).
\]

The trigonometric spin chain corresponds to \( z_j = \omega^j \) with \( \omega \) a primitive \( N^{th} \) root of the unity. For these values of \( z_j \), the transfer matrix (4) commutes with the Hamiltonian (1). For \( z_j = \omega^j \), the matrix \( \Theta \) can be diagonalized giving the following expression for \( \Delta_N \):

\[
\Delta_N(x) = \prod_{j=1}^{N} \left( x + \frac{N + 1}{2} - j \right)
\]

3 The irreducible multiplets.

Solving the model consists in finding all the irreducible components of the Yangian symmetry algebra and computing the energy in each of these blocks. For the values of the \( z_j \)'s induced by the spin chain, \( z_j = \omega^j \), the representation (4) is reducible. It is completely reducible since the transfer matrix is hermitic: \( t_n^{ab} \dagger = t_n^{ba} \). Each irreducible sub-representation possesses a unique highest weight (h.w.) vector \(|\Lambda\rangle\) which is annihilated by \( T_{+-}(x) \) and which is an eigenvector of the diagonal components \( T_{\pm\pm}(x) \) of the transfer matrix:

\[
T(x)|\Lambda\rangle = \begin{pmatrix} t_{++}(x) & 0 \\ \ast & t_{--}(x) \end{pmatrix} |\Lambda\rangle
\]

Here, \( t_{\pm\pm}(x) \) are rational functions in \( x \), but not operators. Since the quantum determinant (4) take the same value in any of the irreducible block, these two functions are related by:

\[
\frac{\Delta_N(x + 1)}{\Delta_N(x)} = t_{--}(x - 1)t_{++}(x)
\]

Hence, only one of them, say \( t_{--}(x) \), is independent. It uniquely characterizes the \( \text{su}(2) \) Yangian representation. We therefore have to compute all the functions \( t_{--}(x) \) arising from the decomposition of the Yangian representation induced by the spin chain, but
Obviously, the ferromagnetic vacuum \(|\Omega\rangle = |++\cdots++\rangle\) is a h.w. vector: the corresponding \(t_{--}(x)\) is one, and the energy is zero. The h.w. vectors in the one-magnon sector are \(|m\rangle = \sum_j \omega^{m \sigma_j} |\Omega\rangle\), with \(1 \leq m \leq (N-1)\): the corresponding eigenvalue is \(t_{--}(x) = \frac{P_1(x+1)}{P_1(x)}\), with \(P_1(x) = (x + \frac{N+1}{2} - m)\), and the one-magnon energy is \(\epsilon(m) = \left(\frac{\pi}{N}\right)^2 m(m - N)\).

In order to determine all the highest weight vectors, we decompose the Hilbert space into subspaces of fixed magnon number. A \(M\)-magnon state \(|\Psi\rangle\) has \(M\) spin reversed:

\[
|\Psi\rangle = \sum_{n_1, \ldots, n_M} \psi_{n_1, \ldots, n_M} \sigma_{n_1}^{-} \cdots \sigma_{n_M}^{-} |\Omega\rangle
\]  

where \(\sigma_n^{-}\) denote the Pauli matrices acting on the spin located on the site \(n\). By construction, the coefficients \(\psi_{n_1, \ldots, n_M}\) of the \(M\)-magnon wave functions are symmetric in their indices. The wave function coefficients are unspecified for two coincident indices \(\psi_{n, \ldots, n, \ldots}\). By convention, we choose these coefficients to be zero.

Since these indices range from 1 to \(N\), to any \(M\)-magnon state is associated a symmetric polynomial \(\Psi(z_1, \ldots, z_M)\) in \(M\) variables of degree less than \((N-1)\) such that \(\Psi(\omega^{n_1}, \ldots, \omega^{n_M}) = \psi_{n_1, \ldots, n_M}\). In the following, we restrict ourselves to the class of magnon states deriving from polynomials of the following form:

\[
\Psi(z_1, \ldots, z_M) = \prod_{p<q} (z_p - z_q)^2 R(z_1, \ldots, z_M)
\]  

with \(R(z_1, \ldots, z_M)\) a symmetric polynomial of degree less than \((N-2M+1)\). This class of states does not include all the states of the spin chain but, as we will see, all the highest weight vectors are in this class.

As explained in the Appendix, the operators \(T_{--}(x)\) and \(T_{+-}(x)\) act on this class of states. Therefore, the action of these operators on these magnon states induces an action on the polynomials. As shown in the Appendix, we find:

\[
T_{--}(x) \Psi(z) = \left(1 + \sum_{p=1}^{M} \frac{1}{x + \frac{N+1}{2} - D_p}\right) \Psi(z)
\]  

and

\[
T_{+-}(x) \Psi(z) = \left(1 + \sum_{p=1}^{M} \frac{1}{x + \frac{N+1}{2} - \hat{D}_p}\right) \Psi(z_1 = 0, z)
\]

Here we have introduced differential operators \(D_p\) which have recently been proved useful in the Calogero-Sutherland model [4, 13, 12, 8]. For the following, we also need another set of differential operators \(\hat{D}_p\). Both are defined by:

\[
D_p = z_p \partial_{z_p} + \sum_{p \neq q} \theta_{pq} K_{pq}
\]

\[
\hat{D}_p = z_p \partial_{z_p} + \sum_{q > p} \theta_{pq} K_{pq} - \sum_{q < p} \theta_{qp} K_{pq}
\]  

Here, \(\theta_{pq} = z_p / z_{pq}\) and, the operator \(K_{pq}\) exchanges the positions: \(K_{pq} z_p = z_q K_{pq}\).
whereas the $\hat{D}_p$’s are not. On the other hand, the differentials $\hat{D}_p$ commute, $[\hat{D}_p, \hat{D}_q] = 0$, but the differentials $D_p$’s does not: $[D_p, D_q] = (D_p - D_q)K_{pq}$. The sum of the $m^{th}$ powers of both differentials form two sets of commuting operators. However, they are not independent thanks to the following relation:

$$1 + \sum_{p=1}^{M} \frac{1}{x - D_p} = \left(1 + \frac{1}{x - \hat{D}_1}\right) \cdots \left(1 + \frac{1}{x - \hat{D}_M}\right)$$  \hspace{1cm} (13)$$

In Eq. (13) it is understood that the operators are acting on functions symmetric in their arguments.

From eq. (11), we learn that the highest weight vectors correspond to polynomials $\Psi(z)$ with $R(z)$ given by:

$$R(z_1, \cdots, z_M) = \left(\prod_{p=1}^{M} z_p\right)\phi(z_1, \cdots, z_M)$$  \hspace{1cm} (14)$$

with $\phi(z_1, \cdots, z_M)$ a symmetric polynomial of degree less than $(N - 2M)$. Thus, in the $M$-magnon sector, there are $(\frac{(N-M)!}{M!(N-2M)!})$ independent highest weight vectors.

From eq. (10), we learn that the eigenfunctions of $T_{-\cdot}(x)$ are the eigenvectors of the commuting hamiltonians of the Calogero-Sutherland model, or equivalently of the commuting operators $\hat{D}_p$. The symmetric eigenfunctions $\Psi^{(m_p)}(z)$ with

$$\sum_p (\hat{D}_p)^n \Psi^{(m_p)}(z) = \sum_p m_p^n \Psi^{(m_p)}(z),$$

are polynomials with degree between 1 and $(N - 1)$ if $1 \leq m_p \leq (N - 1)$. Hence, the M-magnon highest weight vectors $\{|m_p\}$, with wave function given by $\Psi^{(m_p)}(z)$, are labeled by sets of M integers $\{m_1, \cdots, m_M\}$. Due to the Vandermond prefactor in eq.(9), these integers never coincide nor differ by a unit. Using the factorisation relation (13), one find that the value of $t_{-\cdot}(x)$ for these highest weight vectors are:

$$t_{-\cdot}(x) = \frac{P_1(x+1)}{P_1(x)} \quad \text{with} \quad P_1(x) = \prod_{p=1}^{M} (x + \frac{N+1}{2} - m_p)$$  \hspace{1cm} (15)$$

The dimension of the irreducible multiplets are encoded in the transfer matrix eigenvalues. The eigenvalues $t_{-\cdot}(x)$ are given by eq. (13). The remaining eigenvalues $t_{++}(x)$ are computed from the relation (4). They can also be written in a product form. The result is:

$$T(x)|\{m_p\}⟩ = \frac{P_1(x+1)}{P_1(x)} \begin{pmatrix} P_0(x+1) \\ P_0(x) \\ 0 \\ * \\ 1 \end{pmatrix} |\{m_p\}⟩$$  \hspace{1cm} (16)$$

with $P_1(x)$ given in eq.(13), and $P_0(x)$ and $P_1(x)$ factorize $\Delta_N(x)$:

$$\Delta_N(x) = P_0(x) \ P_1(x) P_1(x-1).$$  \hspace{1cm} (17)$$

One can check that the factorization equation (17) admits solutions only if the roots of
Let us decompose the sequence of rapidities \( \{m_p\} \) in elementary motifs as explained in Section 1. To each motif of length \( Q \), we associate a canonical transfer matrix defined by:

\[
T_{\text{motif}}^{ab}(x) = \delta^{ab} + \frac{S^{ab}}{x - x_0}
\]

(18)

where \( S^{ab} \) are the matrices forming the spin \( \frac{Q-1}{2} \) representation of \( su(2) \) and \( x_0 \) is the position of the most left label 0 of the motif. It is easy to check that the matrix (18) satisfy the commutation relations (3). The representation induced by the transfer matrix (16) is then seen to be equivalent to the irreducible tensor product of the transfer matrices associated to each motifs:

\[
T(x) \cong \bigotimes_{\text{motifs}} T_{\text{motif}}(x + \frac{N + 1}{2})
\]

(19)

This is proved by comparing the eigenvalues of the diagonal transfer matrix elements on the h.w. vector. Therefore, we find that the multiplet of a rapidity sequence is the tensor product of each of its motifs.

We have found one (and only one) irreducible representation for each rapidity sequence. Their direct sum is a vector space of dimension \( 2^N \). Therefore, it fills the Hilbert space of the spin chain, and there is no other irreducible multiplet.

4 The spectrum.

Since all the irreducible multiplets are now identified, finding the spectrum consists in computing the action of the Hamiltonian on the highest weight vectors. The hamiltonian (11) is \( su(2) \) invariant, hence it acts on the \( M \)-magnon subspace. In the magnon basis, this action is:

\[
(H\psi)_{n_1,\ldots,n_M} = -2 \left( \frac{\pi}{N} \right)^2 \sum_p \sum_{k_p \neq n_p} \frac{\omega^{k_p}\omega^{n_p}}{(\omega^{k_p} - \omega^{n_p})^2} (\psi_{n_1,\ldots,k_p,\ldots,n_M} - \psi_{n_1,\ldots,n_p,\ldots,n_M})
\]

- \( \left( \frac{\pi}{N} \right)^2 \sum_{pq} \frac{\omega^{n_p}\omega^{n_q}}{(\omega^{n_p} - \omega^{n_q})^2} \psi_{n_1,\ldots,n_M}
\]

The Hamiltonian act on the state of the form (9). Using eq. (43) given in the Appendix, one realizes that the action induced on the polynomials \( \Psi(z) \) is:

\[
(H_M \Psi)(z) = \left( \frac{\pi}{N} \right)^2 \sum_{p=1}^{M} \left( \sum_{p=1}^{M} z_p \partial_{z_p} (z_p \partial_{z_p} - N) + 4 \sum_{p<q} \frac{z_p z_q}{z_p z_q} \right) \Psi(z)
\]

(20)

In Eq. (20) one recognizes the Calogero-Sutherland Hamiltonian at a special value of the coupling constant. In other words, the spin chain in the \( M \)-magnon sector has been mapped on the \( M \)-body Calogero problem. The last equality in (20), gives the energy of a multiplet \( \{m_p\} \):

\[
E(\{m_p\}) = \sum_p \left( \frac{\pi}{N} \right)^2 m_p(m_p - N)
\]
5 Wave functions and statistics.

Only the wave functions of the h.w. vectors are relevant since those of their descendents are obtained by recursive action of the transfer matrix. We now show how recent results on the Calogero models can be used to find explicit expressions for these wave functions. The latter are based on the construction of operators, which we denote by \( \Lambda_M \) in the \( M \)-magnon sector, intertwining the Calogero Hamiltonian and the free Hamiltonian:

\[
H \Lambda_M = \Lambda_M \Delta \quad \text{with} \quad \Delta = \sum_{p=1}^{M} z_p \partial_{z_p} (z_p \partial_{z_p} - N) \quad (21)
\]

These intertwiners were defined in [10, 13]. One of their definitions is the following Vandermonde determinant of the operators \( D_p \):

\[
\Lambda_M = \sum_{\sigma \text{ perm.}} \text{sign}(\sigma) \prod_{p=1}^{M} D_{\sigma_p}^{M-1} \cdot \ldots \cdot D_{\sigma_2} D_{\sigma_1}
\]

In this formula, it is understood that \( \Lambda_M \) is acting on antisymmetric functions. For example, for two magnons: \( \Lambda_2 = z_1 \partial_{z_1} - z_2 \partial_{z_2} - \frac{z_1 z_2}{z_1 - z_2} \).

The operators \( \Lambda_M \) are antisymmetric. Therefore, the symmetric wave functions are obtained by acting first with the antisymmetrizer on the plane waves \( z_1^{m_1} \cdots z_M^{m_M} \), and then with \( \Lambda_M \):

\[
\Psi(z_1, \ldots, z_M) = \Lambda_M \left( \det(z_{\sigma_p}^{m_q}) \right)
\]

(22)

It is easy to check that the wave functions (22) are symmetric polynomials vanishing at coincident points. If the rapidities are such that \( 1 \leq m_p \leq (N - 1) \), these polynomials have degree less than \( (N - 1) \) and satisfy the condition (14). They thus are the wave functions of the h.w. vectors. In other words, since the plane waves \( z^m \) are the wave functions of the one-magnon h.w. vectors, the operator \( D = \Lambda_M \circ \det \) map tensor products of \( M \) one-magnon states into \( M \)-magnon states:

\[
|m_1\rangle \otimes \cdots \otimes |m_M\rangle \xrightarrow{D} \{|\{m_1, \ldots, m_M\}\rangle
\]

(23)

This map is a generalization of the Slater determinant in the sense that it implements the rapidity selection rules: if two of the rapidities \( m_p \) and \( m_q \) are either equal or differ by a unit, then the resulting wave function vanish identically. The fact that the result vanish if two of the rapidities coincide is obvious from the definition, while the fact that it vanishes if they differ by a unit results from an explicit check on the two-magnon case (which has all generality thanks to the symmetry of the wave functions and the recursive definition of \( \Lambda_M \) given in [13]).

6 The spinon description.

The magnons are the excitations over the ferromagnetic vacuum; the excitations over the antiferromagnetic vacuum are conveniently described in terms of spinons.
For $N$ even, the antiferromagnetic vacuum corresponds to the alternating sequence of symbols $010101\cdots010$. Its rapidities sequence is $\{m_j = 2j - 1\}_{j=1}^{N/2}$. The excitations are obtained by flipping and moving the symbols 0 and 1. We classify the sequence by their number $M$ of rapidities. The spinon number $N_{sp}$ of a sequence is then defined by $M = \frac{N - N_{sp}}{2}$. Since $M$ is an integer, $(N - N_{sp})$ is always even. The spin $S_z$ of the Yangian highest weight vector of the sequence is $S_z = \frac{1}{2}(N - 2M) = \frac{N_{sp}}{2}$.

A sequence of rapidities $\{m_j; j = 1, \cdots, M = \frac{N - N_{sp}}{2}\}$, in the $N_{sp}$ sector, can be decomposed into $(M + 1)$ elementary motifs. As we defined them in Sect.1, an elementary motif is a series of consecutive 0. We will think about the elementary motifs as the possible orbitals for spin half objects, which are called spinons. At fixed $N_{sp}$, there are $(1 + \frac{N - N_{sp}}{2})$ orbitals. To a spinon in the $j^{th}$ orbital, with $j = 0, \cdots, \frac{N - N_{sp}}{2}$, we assign a momentum $k = \frac{2\pi j}{N}$. Thus, the spinon momenta vary from zero to $k_0 = \frac{2\pi}{N} \left(\frac{N - N_{sp}}{2}\right)$.

By convention, a sequence of rapidities $\{m_j\}$ corresponds to the filling of the $(1 + \frac{N - N_{sp}}{2})$ orbitals with respective occupation numbers $n_j = n_+^j + n_-^j = (m_{j+1} - m_j - 2)$, with $m_0 = -1$ and $m_{M+1} = N+1$. The length of the $j^{th}$ elementary motif is then $Q_j = n_j + 1$.

By construction, the total occupation number is the spinon number:

\[\sum_{j=0}^{N - N_{sp}} (n_+^j + n_-^j) = N_{sp}\]  

Since an elementary motif of length $Q$ corresponds to a spin $\frac{Q-1}{2}$ representation of $su(2)$, the full degeneracy of the sequences is then recovered by assuming that, at fixed spinon number, the spinon behaves as bosons. Notice that this property is specific to the $su(2)$ spin chain.

The spinons are not bosons but “semions” since the number of available orbitals varies with the total occupation number $[14]$. In particular, the spinons are always created by pairs.

The energy of a collection of $N_{sp} = N - 2M$ spinons is:

\[E - E_{vac} = E_0(M) + \sum_j \sum_\sigma 2(M - j)(M - \frac{N}{2} + j)n_\sigma^j + \sum_{jj'} \sum_\sigma \sigma' (M - \sup(j, j')) n_\sigma^j n_{\sigma'}^{j'}\]  

with $E_0(M) = \frac{1}{3}M(M-1)(4M+1) - (N-1)M^2$.

The low energy, low temperature, behavior is classified $[15]$ as the level one $su(2)$ WZW conformal field theory. In the spinon description the states consist of semi-infinite sequences of symbols 0 and 1. The two primary states, which correspond to the two integrable representations of the $su(2)$ current algebra at level one, are the vacuum, with sequence $010101\cdots$ and the spin half primary, with sequence $0010101\cdots$. The excited states are given by finite rearrangement of the primary sequence. The Virasoro generator acts as $L_0 = \sum_j (m_j^0 - m_j)$, where $\{m_j^0\}$ are the primary sequences. Its spinon representation is:

\[L_0 = \left(\frac{N_{sp}}{2}\right)^2 + \sum_j j(n_+^j + n_-^j)\]  

(26)
with \( N_{sp} = \sum_j (n_j^+ + n_j^-) \), the total spinon number. The excitations over the vacuum possess an even number of spinons, whereas those over the spin half primary contain an odd number of spinons.

### 7 The thermodynamics.

Following ref. [3], the thermodynamics can be derived from the spinon description using methods similar to the thermodynamic Bethe ansatz [14].

First, we consider the system with a fixed spinon density \( D_{sp} = \frac{N_{sp}}{N} \). In the \( N \to \infty \) limit, the spinon orbitals are labeled by momenta \( k \) continuously varying from zero to \( k_0 = \frac{2\pi}{N} \left( \frac{N-N_{sp}}{2} \right) = \pi (1 - D_{sp}) \). Let \( n_\sigma(k) \) be the spinon occupation numbers of the \( k^{th} \) orbital. By definition, they satisfy:

\[
\sum_\sigma \int_0^{k_0} \frac{dk}{2\pi} n_\sigma(k) = D_{sp} \quad (27)
\]

In the continuum limit, the energy per unit of length is:

\[
\frac{E}{N} \equiv \mathcal{E}(D_{sp}; n_\pm(k)) = \mathcal{E}_0(D_{sp}) + \sum_\sigma \int_0^{k_0} \frac{dk}{2\pi} \epsilon_0(k)n_\sigma(k) + \sum_{\sigma\sigma'} \int_0^{k_0} \frac{dkdk'}{(2\pi)^2} V(k,k')n_\sigma(k)n_{\sigma'}(k') \quad (28)
\]

with \( \mathcal{E}_0(D_{sp}) = \left( \frac{\pi}{2} \right)^2 (1 - D_{sp})^2 (\frac{1+2D_{sp}}{3}) \), \( \epsilon_0(k) = \frac{1}{2}(k_0 - k)(k_0 + k - \pi) \) and \( V(k,k') = \frac{\pi}{2}(k_0 - \text{sup}(k,k')) \). In each orbital the spinons behave as bosons, therefore the entropy of a configuration of \( n_\pm(k) \) spinons is:

\[
\frac{S}{N} \equiv S(D_{sp}; n_\pm(k)) = \sum_\sigma \int_0^{k_0} \frac{dk}{2\pi} ((n_\sigma(k) + 1) \log(n_\sigma(k) + 1) - n_\sigma(k) \log n_\sigma(k)) \quad (29)
\]

The free energy per unit of length is:

\[
\frac{F}{N} = \mathcal{F} = \mathcal{E} - T\mathcal{S} - h\mathcal{M} \quad (30)
\]

with \( h \) the exterior magnetic field and \( \mathcal{M} = \int_0^{k_0} \frac{dk}{2\pi} (n_+(k) - n_-(k)) \) the magnetization.

At fixed spinon density, the thermodynamic equilibrium is determined by minimizing (30) with respect to the variation of the spinon occupation numbers subject to the constraint (27). This gives:

\[
n_\pm(k) = \frac{1}{e^{\beta(\epsilon(k) \mp h - A)} - 1} \quad (31)
\]

where \( A \) is the Lagrange multiplier and \( \epsilon(k) \) is the dressed energy defined by:

\[
\epsilon(k) = 2\pi \frac{\delta \mathcal{E}}{\delta n_\sigma(k)} = \epsilon_0(k) + 2 \sum_{\sigma\sigma'} \int_0^{k_0} \frac{dk'}{2\pi} V(k,k')n_{\sigma'}(k') \quad (32)
\]

At fixed density, the Lagrange multiplier is determined from the constraint (27). This
The spin chain corresponds to a spinon gas of arbitrary density; i.e. the spinon chemical potential \( \mu = \frac{\partial F}{\partial D_{sp}} \) is zero. Minimizing the free energy with respect to the density fixes \( A \) to be \( \beta A = -\log(2 \cosh(\beta h)) \). The constraint (27) then gives the mean density \( D_{sp} \).

The coupled eqs. (31,32) can be solved. Deriving twice eq.(32) with respect to \( k \) gives:

\[
\frac{\partial^2 \epsilon(k)}{\partial k^2} = -\left( 1 + \frac{1}{2} \sum_{\sigma} n_{\sigma}(k) \right)
\]

with the boundary conditions: \( \epsilon(k_0) = \epsilon(0) = 0 \), and \( \epsilon'(0) = -\epsilon'(k_0) = \frac{\pi}{2} \). Eq.(33) is integrated by introducing the dressed momenta \( p = \frac{d\epsilon}{dk} \). It varies from \( -\frac{\pi}{2} \) to \( \frac{\pi}{2} \), and it satisfies \( \frac{dp}{dk} = \frac{d^2 \epsilon}{dk^2} = \frac{d(p^2/2)}{dc} \). As function of \( p \), the occupation number are then given by:

\[
n_{\pm} = \exp \left[ \beta (\eta \pm \epsilon_{dr} \pm h) \right]
\]

with

\[
\eta(p) = \frac{1}{2} \left[ p^2 - \left( \frac{\pi}{2} \right)^2 \right] \quad \text{and} \quad \exp(\beta \eta) = \frac{\sinh(\beta \epsilon_{dr})}{\sinh(\beta h)}
\]

In the limit \( h \to 0 \), the dressed energy is \( \epsilon_{dr} \sim h e^{\beta \eta} \).

The free energy is found by integrating the thermodynamical relations:

\[
\left( \frac{\partial F}{\partial T} \right)_h = -S \quad \text{and} \quad \left( \frac{\partial F}{\partial h} \right)_T = -M
\]

The result is the following simple answer:

\[
F = -T \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dp \frac{dp}{\pi} \log \left[ \frac{\sinh(\beta (\epsilon_{dr} + h))}{\sinh(\beta h)} \right]
\]

Notice that this is the free energy of a gas of non-interacting particles which, in the limit \( h \to 0 \), have energies given by \( \eta(p) \).

8 Appendix.

In this Appendix we prove the eqs.(10,11). First we compute the action of \( T_{--}(x) \) on these magnon states. We recall that \( T_{--}(x) \) can be written as \( T_{--}(x) = 1 + \sum_{i,j} \left( \frac{1}{L} \right)_{ij} P^-_j \), where \( P^-_j \) is the projector on the spin \( (\sigma_j = -) \) acting on the \( j \)th spin only. The projector \( P^-_j \) on the \( M \)-magnon states (8) gives states with one spin down marked of the form:

\[
P^-_j |\Psi\rangle \equiv |\Psi_j\rangle = M \sum_{n_2,\ldots,n_M} \psi_{j,n_2,\ldots,n_M} \sigma^-_j \sigma^-_{n_2} \cdots \sigma^-_{n_M} |\Omega\rangle
\]

They corresponds to polynomials \( \Psi(z_1|z_2,\ldots,z_M) \) symmetric in \( z_2,\ldots,z_M \) with the point \( z_1 \) distinguished. To evaluate the action of \( \sum_{i,j} (L^n)_{ij} P^-_j \), we remark that on this class of states, \( L_{ij} \) acts as follows:

\[
\sum L_{ik} |\Psi_k\rangle = |(L\Psi)_i\rangle
\]
with,

\[(L\Psi)_{j;n_2,\ldots,n_M} = \sum_k \hat{\theta}_{jk}\Psi_{k;n_2,\ldots,n_M} + \sum_{q=1}^M \hat{\theta}_{jn_q}(K_{1q}\Psi)_{j;n_2,\ldots,n_M} \tag{40}\]

where \(\hat{\theta}_{jk} = \omega^j/(\omega^j - \omega^k)\), and \(K_{1q}\) permutes the indices in position 1 and q. Therefore, \(|(L\Psi)_i\rangle = \sum_j(L^n)_{ij}P_j^\dagger|\Psi\rangle\) can be recursively computed. Then,

\[\sum_{i,j}(L^n)_{ij}P_j|\Psi\rangle = M \sum_i |(L^n\Psi)_i\rangle \tag{41}\]

is obtained by symmetrizing the wave function coefficients of \(|(L^n\Psi)_i\rangle\) in all its indices. I.e. its wave function coefficients, denoted \((L^nP\Psi)_{n_1,\ldots,n_M}\) are:

\[(L^nP\Psi)_{n_1,\ldots,n_M} = \left(1 + \sum_{p\neq 1} K_{1p}\right) (L^n\Psi)_{n_1;n_2,\ldots,n_M} \tag{42}\]

We now translate this action on the wave function coefficients into an action on the polynomials. We recall that in the basis of polynomials \(Q^k\) in one variable of degree \((N - 1)\) specified by \(Q^k(\omega^n) = \delta^n_k\), the matrix elements of the derivatives are:

\[(z\partial_z - \frac{N+1}{2})Q_j(z) = -\sum_{k\neq j} \frac{\omega^j}{\omega^j - \omega^k} Q_k(z) \tag{43}\]

\[z\partial_z(z\partial_z - N)Q_j(z) = -2 \sum_{k\neq j} \frac{\omega^j\omega^k}{(\omega^j - \omega^k)^2} (Q_k(z) - Q_j(z))\]

The differentials \(D_p\), introduced in eq.(12), acts on polynomials of the form (4); i.e. they preserve the form of these polynomials. Moreover, by comparing the formula, we recognize in eq.(40) the operator \(\left(D_1 - \frac{N+1}{2}\right)\). Since the operators \(D_p\) are covariant by permutation of the coordinates, the polynomial of \((L^nP\Psi)\) is given by acting with \(\sum_p \left(D_p - \frac{N+1}{2}\right)^n\) on \(\Psi\). Resuming all the contributions, we obtain:

\[T_{-+}(x) \Psi(z.) = \left(1 + \sum_{p=1}^M \frac{1}{x + \frac{N+1}{2} - D_p}\right) \Psi(z.)\]

The action of \(T_{-+}(x)\) on these subclass of magnon states can be computed using the same methods. Its action is given by eq.(11). It is remarkable that the differentials \(D_p\) appear naturally in the study of the spin transfer matrix.

References

[1] F.D. Haldane, Phys.Rev.Lett. 60(1988)635.

[2] B.S. Shastry, Phys.Rev.Lett. 60(1988)639.

[3] F.D. Haldane, Phys.Rev.Lett. 66 (1991) 1529.

[4] E. Date, M. Jimbo, A. Kuniba, T. Miwa and M. Okado, Lett. Math. Phys.
[5] F.D. Haldane, Z.N. Ha, J.C. Talstra, D. Bernard and V. Pasquier, Phys.Rev.Lett. 69 (1992) 2021.

[6] V.G. Drinfel’d, “Quantum Groups.”, Proc. of the ICM, Berkeley (1987) 798.

[7] E.K. Sklyanin, Funct. Anal. Appl. 16 (1983) 263.

[8] D. Bernard, M. Gaudin, F.D. Haldane and V. Pasquier, J. Phys. A. , to be published.

[9] A.G. Izergin and V. Korepin, Sov.Phys.Dokl. 26 (1981) 653.

[10] E.M. Opdam, Invent. Math. 98 (1989) 1; 
G.J. Heckman, Invent. Math. 103 (1991) 341.

[11] A.P. Polychronakos, Phys.Rev.Lett. 69 (1992) 703.

[12] I. Cherednik, Invent. Math. 106 (1991) 411; and Publ. RIMS 27 (1991) 727; and reference therin.

[13] O. Chalykh and A. Veselov, Comm. Math. Phys. 152 (1993) 29.

[14] F.D. Haldane, Phys.Rev.Lett. 67(1991)937.

[15] H.W. Blöte, J.L. Cardy, and Nightingale, Phys.Rev.Lett. 56 (1986) 742; 
I. Affleck, Phys.Rev.Lett. 56 (1986) 746.

[16] C.N. Yang and C.P. Yang, Phys.Rev. 10(1969)1115.