ROBUST LOCALIZATION OF THE BEST ERROR WITH FINITE ELEMENTS IN THE REACTION-DIFFUSION NORM

FRANCESCA TANTARDINI, ANDREAS VEEGER, AND RÜDIGER VERFÜRTH

Abstract. We consider the approximation in the reaction-diffusion norm with continuous finite elements and prove that the best error is equivalent to a sum of the local best errors on pairs of elements. The equivalence constants do not depend on the ratio of diffusion to reaction. As application, we derive local error functionals that ensure robust performance of adaptive tree approximation in the reaction-diffusion norm.

1. Introduction

Finite element methods are well-established for the numerical solution of elliptic and parabolic problems. An important aspect in their mathematical understanding and foundation are the approximation properties of finite elements spaces. In view of adaptive mesh refinement, the local features of the latter under minimal regularity assumptions are of interest.

The most basic finite element approach to the homogeneous Dirichlet problem for Poisson’s equation leads to the following approximation problem: Approximate a function \( u \in H^1_0(\Omega) \) in the \( H^1 \)-seminorm with functions from a space \( S \) consisting of continuous piecewise polynomials of degree \( \leq \ell \) associated with a given simplicial mesh \( T \). In this context one of the authors [10] proved that

\[
\inf_{v \in S} \| \nabla (u - v) \|_\Omega \approx \left( \sum_{K \in T} \inf_{P \in \mathbb{P}_\ell(K)} \| \nabla (u - P) \|_K^2 \right)^{1/2}.
\]

i.e. the global best error is equivalent to the \( \ell_2 \)-norm of the local best errors on elements. Notice that the right-hand side does not involve any coupling between elements and that no additional regularity of \( u \) is invoked. If \( u \) disposes of additional piecewise regularity, this result and the Bramble-Hilbert Lemma readily imply error bounds. Moreover, it shows that adaptive tree approximation [3] by P. Binev and R. DeVore with the local best errors as error functionals yields near best meshes for the global best error on the left-hand side.

In view of problems with extreme parameters, it is important that approximation properties are robust. An important and basic example for such a problem is given by reaction-dominated diffusion, whose stationary variant is also of interest in the discretization of the heat equation. In this context the \( H^1 \)-seminorm in (1.1) is replaced by the so-called reaction-diffusion norm

\[
\| \cdot \| := \| \cdot \|^2 + \varepsilon \| \nabla \cdot \|^2,
\]

and one is interested in a variant of (1.1) where the hidden constants are independent of the parameter \( \varepsilon \). The exact counterpart of (1.1) for the reaction-diffusion norm cannot be robust; this arises from the fact that, for \( \varepsilon = 0 \), a discontinuous...
piecewise constant function yields 0 for the sum of the local best errors, but not in general for the global best error. We find the following robust replacements:

$$\inf_{v \in S} \|u - v\|_{\Omega} \approx \left( \sum_{E \in \bar{T}} \inf_{P \in S_{\omega_T(E)}} \|u - P\|_{\omega_T(E)}^2 \right)^{1/2}$$

$$(1.3)$$

$$\lesssim \left( \sum_{K \in T} \inf_{P \in S_{\partial(K)}} \left( \|u - P\|_K^2 + \frac{|K|}{|\partial K|} \|u - P\|_{\partial K}^2 \right) \right)^{1/2},$$

where the infima in the first sum are localized to continuous piecewise polynomials on pairs of elements $\omega_T(E)$ sharing an internal face $E \in \bar{E}$. Notice that the second sum is ready to apply the Bramble-Hilbert lemma, while the first one is not. The second sum however does not provide a robust lower bound of the global best error on the left-hand side. The reason lies in the fact that, for $\varepsilon = 0$, it requires an additional 1/2-derivative. Accordingly, adaptive tree approximation with the local contributions of the second sum cannot provide near best meshes for the reaction-diffusion in a robust manner. The local contributions of the first sum are also not suited for adaptive tree approximation, but for another reason: they do not allow to define local error functionals depending solely on the target function and a given element. Adopting however the idea of minimal rings in [3] to pairs, we provide a modification of the first sum that is suitable for tree approximation.

The article is organized as follows. In §2 we show that the hidden constant of the nontrivial inequality of (1.1) for the reaction-diffusion norm blows up for $\varepsilon \searrow 0$. In [3] we fix notations, while in §4 we show that localization results like the first part of (1.3) follow from a suitable property of a quasi-interpolation operator. This is exploited, in §6 and §7 respectively, to prove the first part of (1.3) and its counterpart for minimal pairs. Section §8 analyzes the non-robustness of §2 more precisely, thereby deriving an alternative way to compute the local best errors in the first part of (1.3) and showing its second part. Finally, we extend in §8 our results to conforming approximation of functions with vanishing boundary values.

2. Decoupling of Elements is not Robust

The purpose of this section is to show that, for the reaction-diffusion norm $\|\cdot\|$, the $\lesssim$-part in (1.1) cannot hold with a constant independent of $\varepsilon$. The counterexample provides functions $u_\varepsilon \in H^1(\Omega)$ converging to a discontinuous function $u_0 \notin H^1(\Omega)$ such that the global best error is bounded from below independently of $\varepsilon$, while the local best errors decrease with $\sqrt{\varepsilon}$.

We consider the domain $\Omega = (-2, 2) \times (-1, 1) \subset \mathbb{R}^2$, with the subdomains $\Omega_+ = (0, 2) \times (-1, 1)$ and $\Omega_- = (-2, 0) \times (-1, 1)$. Let $\mathcal{T}$ be any conforming simplicial triangulation of $\Omega$, that is subordinate to the decomposition $\Omega = \Omega_+ \cup \Omega_-$, and let $S$ be the space of continuous functions that are piecewise polynomial of degree at most $\ell$ with respect to $\mathcal{T}$. We denote by $\mathcal{P}_\mathcal{T}$ the $L^2$-projection onto $S$, by $\mathcal{R}_\mathcal{T}$ the Ritz projection onto $S$ with respect to the reaction-diffusion norm and by $\mathcal{R}_{\mathcal{T}}^+$ the local counterpart of $\mathcal{R}_\mathcal{T}$, by $\mathcal{R}_{\mathcal{T}}^-$ the local counterpart of $\mathcal{R}_{\mathcal{T}}^+$. The functions $u_\varepsilon \in H^1(\Omega)$ and $u_0 \in L^2(\Omega)$ are given by

$$u_\varepsilon(x) = \begin{cases} -1 & \text{for } x_1 < -\sqrt{\varepsilon}, \\ \frac{\varepsilon}{x_1} & \text{for } -\sqrt{\varepsilon} < x_1 < \sqrt{\varepsilon}, \\ 1 & \text{for } \sqrt{\varepsilon} < x_1, \end{cases}$$

$$u_0(x) = \begin{cases} -1 & \text{for } x_1 < 0, \\ 1 & \text{for } 0 < x_1. \end{cases}$$
On one hand, for the local best errors, \( u_0 \in \mathbb{P}_l(K) \) for every \( K \in \mathcal{T} \) implies
\[
\sum_{K \in \mathcal{T}} \| u_\varepsilon - R_T^\varepsilon u_\varepsilon \|_K^2 \leq \sum_{K \in \mathcal{T}} \| u_\varepsilon - u_0 \|_K^2
\]
\[
= \int_{-\sqrt{\varepsilon}}^{\sqrt{\varepsilon}} \int_{-\sqrt{\varepsilon}}^{\sqrt{\varepsilon}} | \partial u_\varepsilon / \partial x_1 |^2 + \int_{-\sqrt{\varepsilon}}^{\sqrt{\varepsilon}} \int_{-\sqrt{\varepsilon}}^{\sqrt{\varepsilon}} | u_\varepsilon - u_0 |^2 = \frac{16}{3} \varepsilon.
\]
On the other hand, for the global best error, there holds
\[
\| u_\varepsilon - R_T^\varepsilon u_\varepsilon \|_{L^2(\Omega)} \geq \| u_\varepsilon - P_T u_\varepsilon \|_{L^2(\Omega)} \rightarrow \| u_0 - P_T u_0 \|_{L^2(\Omega)} > 0 \quad \text{for} \ \varepsilon \rightarrow 0,
\]
since \( u_\varepsilon \) converges to \( u_0 \) in the \( L^2 \)-norm, the \( L^2 \)-projection onto \( S \) is continuous and \( u_0 \notin S \). Consequently, the constant in the nontrivial inequality of (1.1) for the reaction-diffusion norm has to grow at least with \( \varepsilon^{-1/4} \).

This simple example reflects a more general situation. Consider in fact any couple of adjacent elements \( K \) and \( \bar{K} \). Taking \( u_\varepsilon = \min\{1, \varepsilon^{-1/2} \text{dist}(\mathbb{R}^d \setminus K)\} \) and \( u_0 = \chi_K \) and reasoning as above shows that the constant blows up as \( \varepsilon \searrow 0 \). These observations suggest to modify (1.1) for the reaction-diffusion norm by invoking local best errors that incorporate the continuity constraint through a face.

3. Meshes and basis functions

We denote by \( \mathcal{T} \) a conforming simplicial mesh of a polyhedral domain \( \Omega \subset \mathbb{R}^d \), by \( \mathcal{E} \) the set of its faces, and by \( \mathcal{E}_i \) the subset of \( \mathcal{E} \) of those faces which are not contained in the boundary \( \partial \Omega \). If \( K \in \mathcal{T} \) is an element and \( E \in \mathcal{E} \) is a face, we write \( |K| \) and \( |E| \) for its \( d \)-dimensional Lebesgue and \((d-1)\)-dimensional Hausdorff measure, respectively. For every face \( E \in \mathcal{E}_i \), the set
\[
\omega_T(E) := \bigcup\{ K \in \mathcal{T} : \partial K \supseteq E \}
\]
is the union of elements sharing the face \( E \). It consists of two elements if \( E \in \mathcal{E}_i \) and of one element otherwise. We stress that \( E \) belongs to various meshes and that \( \omega_T(E) \) actually depends on \( \mathcal{T} \) too.

A collection \( W \) of subdomains of \( \Omega \) is a \( \beta \)-finite covering of \( \mathcal{T} \) if for every \( K \in \mathcal{T} \)
- there exists \( \omega \in W \) with \( \omega \supseteq K \) and
- there holds \( \sum_{\omega \in W} \chi_\omega \leq \beta \) on \( \bar{K} \),
where \( \chi_\omega \) stands for the characteristic function of \( \omega \). The collections \( \{ K \}_{K \in \mathcal{T}} \) and \( \mathcal{W}_T := \{ \omega_T(E) \}_{E \in \mathcal{E}_i} \) in (1.1) and (1.3) are 1-finite and \((d+1)\)-coverings, respectively. Notice that \( \beta \) arises in (1.1) and (1.3) as multiplicative constant in the straight-forward inequality. Another \( \beta \)-finite covering appears in (7).

The space
\[
S := \mathbb{S}_{\ell,0}(\mathcal{T}) = \{ v \in C^0(\Omega) : v \in \mathbb{P}_l(K), \ \forall K \in \mathcal{T} \}
\]
consists of all continuous functions that are piecewise polynomial over \( \mathcal{T} \). Given a set \( \omega \subset \Omega \), we indicate its restriction with
\[
S|_\omega := \{ v \in C^0(\omega) : \exists \tilde{v} \in S, \ \tilde{v}|_\omega = v \}.
\]
In particular we have \( S|_{\omega_T(E)} = \{ v \in C^0(\omega_T(E)) : v \in \mathbb{P}_l(K), \ \forall K \in \mathcal{T}, \ K \subseteq \omega_T(E) \} \). Furthermore, we denote by \( \mathcal{N} \) the set of nodes of \( S \). A subscript \( K, \Omega, \) etc. to \( \mathcal{N} \) indicates that only the nodes contained in the index-set are considered. We denote by \( \{ \phi_z \}_{z \in \mathcal{N}} \) the nodal basis, that is, for every \( z \in \mathcal{N} \),
\[
\phi_z \in S \quad \text{and} \quad \forall y \in \mathcal{N} \ \phi_z(y) = \delta_{yz}.
\]
Given an element $K \in \mathcal{T}$, the $L^2(K)$-dual basis functions $\{\psi^K_z\}_{z \in N_K}$ are such that
\[
\int_K \psi^K_z \phi_y = \delta_{zy},
\]
for every $y, z \in N_K$. We thus have, for every $p \in \mathbb{P}_l(K)$ and for every $z \in N_K$,
\[
p(z) = \int_K p \psi^K_z.
\]
We also recall some basic scaling properties of different norms of $\phi_z$ and $\psi^K_z$. We denote by $\hat{K} := \text{conv}\{0, e_1, \ldots, e_d\}$ the reference $d$-simplex, by $\hat{h} := \text{diam}(\hat{K})$ the diameter of $\hat{K}$, by $\{\phi_z\}, \{\psi_z\}$ respectively the basis and dual basis functions on $K$, and by $\|\cdot\|_\omega$ the $L^2$-norm on the set $\omega$. For every element $K \in \mathcal{T}$, there exists an affine transformation $F : \mathbb{R}^d \to \mathbb{R}^d$ with $F(\hat{K}) = K$, and $F(\hat{z}) = z$. There may be different choices for $\hat{z}$, which nevertheless lead to the same value of $\|\phi_z\|_K$. However, $\|\nabla \phi_z\|_K$ depends on the chosen node. For this reason, we take a $\hat{z}$ with minimal sum of the coordinates, so that $\|\nabla \phi_z\|_K$ is unique. Since $\psi_z = (\det B) \psi^K_z \circ F$, where $B$ is the non-singular matrix associated to $F$, the transformation rule and the proof of [4] Theorems 15.1 and 15.2 imply
\[
\|\phi_z\|_K = \frac{|K|^{1/2}}{|\hat{K}|^{1/2}} \|\phi_z\|_{\hat{K}},
\]
\[
\|\psi^K_z\|_K = \frac{|\hat{K}|^{1/2}}{|K|^{1/2}} \|\psi_z\|_{\hat{K}},
\]
\[
\|\nabla \phi_z\|_K \leq \frac{\hat{h}|K|^{1/2}}{\rho_K |\hat{K}|^{1/2}} \|\nabla \phi_z\|_{\hat{K}},
\]
where $\rho_K$ denotes the maximum diameter of a ball inscribed in $K$.

4. Localization and Interpolation

In this section we reduce the problem of localizing the global best error to the problem of defining a global quasi-interpolation operator that is locally near best. Roughly speaking, the latter means that the difference between the interpolant and a local best approximation is bounded, up to a constant, by a finite sum of local best errors. The results of this section are used in [6] and [7].

Let $\|\cdot\|_\omega$ be a norm on $H^1(\omega)$ such that its square is set-additive. Then there holds, e.g. $\|\cdot\|_\omega^2 = \sum_{K \in \mathcal{T}} \|\cdot\|_\omega^2$. Moreover let $W$ be a $\beta$-finite covering. For every subdomain $\omega \in W$, let $Q_\omega : H^1(\omega) \to S|_\omega$ be a local operator which maps a function to a corresponding best approximation in $S|_\omega$ with respect to $\|\cdot\|_\omega$. We thus have
\[
\|u - Q_\omega u\|_\omega = \inf_{v \in S|_\omega} \|u - v\|_\omega,
\]
for all $\omega \in W$ and every $u \in H^1(\omega)$.

Definition 4.1 (Local near best interpolation). An interpolation operator $\Pi$ is locally near best with respect to $\|\cdot\|_\omega$, $\omega \in W$, if there are constants $C_{\text{loc}} \geq 0$ and $\alpha_1, \alpha_2 \geq 1$ with the following property: for every element $K \in \mathcal{T}$, there exists a set $A_K \subset W$ and a subdomain $\omega \in A_K$ such that $\omega \supseteq K$ and
\[
\sum_{z \in N_K} |Q_\omega u(z) - \Pi u(z)| \|\phi_z\|_K \leq C_{\text{loc}} \sum_{\omega' \in A_K} \|u - Q_{\omega'} u\|_\omega,
\]
where the set $A_K$ consists of “neighbouring” subdomains, subject to the following two conditions:
- there holds $\#A_K \leq \alpha_1$ for every $K \in \mathcal{T}$,
- for every $\omega \in W$, there are at most $\alpha_2$ elements $K' \in \mathcal{T}$ with $\omega \in A_{K'}$. 
Using (4.2) and inserting back up to the first inequality, we get

\[
\inf_{v \in S} \|u - v\|_{\Omega} \leq C \left( \sum_{\omega \in \mathcal{W}, v \in S} \inf_{\omega \in \mathcal{W}, v \in S} \|u - v\|_{\omega}^{2} \right)^{1/2} \leq C\beta \inf_{v \in S} \|u - v\|_{\Omega}
\]

with 

\[
C = \sqrt{\alpha_1 \alpha_2 (1 + C_{\text{loc}})}
\]

for every \( u \in H^{1}(\Omega) \).

**Proof.** Let \( \Pi : H^{1}(\Omega) \rightarrow S \) be an interpolation operator that is locally near best with respect to \( \|\cdot\|_{\omega}, \omega \in \mathcal{W} \). Bounding the infimum on the left-hand side of (4.3) by \( \|u - \Pi u\|_{\Omega} \) and writing the norm as a sum over elements results in

\[
\inf_{v \in S} \|u - v\|_{\Omega} \leq \left( \sum_{K \in \mathcal{T}} \|u - \Pi u\|_{K}^{2} \right)^{1/2}.
\]

Fix an element \( K \in \mathcal{T} \) and choose a subdomain \( \omega \supseteq K \) as in Definition 4.1. The triangle inequality then yields

\[
\|u - \Pi u\|_{K} \leq \|u - Q_{\omega} u\|_{K} + \|\Pi u - Q_{\omega} u\|_{K}.
\]

Since both \( \Pi u|_{K} \) and \( Q_{\omega} u|_{K} \) are in \( \mathbb{P}(K) \), we can represent them in terms of the local nodal basis \( \{ \phi_{a} \}_{a \in \mathcal{N}_K} \) and obtain

\[
\|\Pi u - Q_{\omega} u\|_{K} \leq \sum_{z \in \mathcal{N}_K} \|Q_{\omega} u(z) - \Pi u(z)\| \|\phi_{z}\|_{K}.
\]

Using (4.2) and inserting back up to the first inequality, we get

\[
\inf_{v \in S} \|u - v\|_{\Omega} \leq \left( \sum_{K \in \mathcal{T}} \left(1 + C_{\text{loc}} \right) \sum_{\omega \in A_K} \|u - Q_{\omega} u\|_{\omega}^{2} \right)^{1/2}.
\]

As \( \# A_K \leq \alpha_1 \) we can use

\[
\left( \sum_{a=1}^{n} a_{a} \right)^2 \leq n \sum_{a=1}^{n} a_{a}^2 \text{ with } n \leq \alpha_1.
\]

Moreover every \( \omega \in \mathcal{W} \) belongs to at most \( \alpha_2 \) of the sets \( A_K \) and so, upon rearranging terms, we arrive at

\[
\inf_{v \in S} \|u - v\|_{\Omega} \leq \sqrt{\alpha_1 \alpha_2} (1 + C_{\text{loc}}) \left( \sum_{\omega \in \mathcal{W}} \|u - Q_{\omega} u\|_{\omega}^{2} \right)^{1/2}.
\]

In view of (4.1), this proves the first inequality in (4.3). To verify the second one, take \( v \in S \), observe \( \|u - Q_{\omega} u\|_{\omega} \leq \|u - v\|_{\omega} \) for any subdomain \( \omega \in \mathcal{W} \) and recall that every element \( K \in \mathcal{T} \) appears in at most \( \beta \) subdomains \( \omega \in \mathcal{W} \).

Our task is now reduced to find an operator \( \Pi \) that is locally near best. Since point values are in general not defined for the norms of our interest, the definition of \( \Pi \) below in (4.2) typically entails that \( \Pi u|_{K} \) on some element \( K \in \mathcal{T} \) depends also on \( u|_{\tilde{K}} \) on certain other elements \( \tilde{K} \). To deal with this dislocation, we invoke suitable paths of overlapping subdomains. The following proposition applies also to the covering in \( \mathcal{T} \) whose subdomains are in general not unions of elements of the mesh \( \mathcal{T} \).

**Proposition 4.3** (Dislocation control). Let \( \mathcal{W} \) be a \( \beta \)-finite covering of \( \mathcal{T} \) and \( K, \tilde{K} \in \mathcal{T} \) two elements sharing a node \( z \in \mathcal{N} \). If there exist a finite sequence \( \{ \omega_{j} \}_{j=1}^{n} \subset \mathcal{W} \) and \( v \in (0, 1) \) such that

- \( \omega_1 \supseteq K \) and \( \omega_n \supseteq \tilde{K} \),
any intersection $\omega_j \cap \omega_{j+1}$ is a simplex $T_j$ containing $z$ and there is an element $K_j \in T$ which again contains $z$ and satisfies $|T_j| \geq \nu|K_j|$, then

$$|Q_{\omega_j}u(z) - \int_K u\tilde{\psi}_z^K| \leq 2\nu^{-1/2}\|\tilde{\psi}_z^K\|_K \sum_{j=1}^n |\tilde{\psi}_z^{K_j}|^{1/2} \|Q_{\omega_j}u - u\|_{L^2(\omega_j)},$$

with $K_n = \tilde{K}$.

Comparing with Proposition 4.2, we notice that the bound involves not best errors but $L^2$-errors of best approximations.

\begin{proof}
For every $j = 2, \ldots, n$, we add and subtract $Q_{\omega_j}u(z)$, which is well-defined thanks to $z \in \omega_j$, and use the triangle inequality to get

$$|Q_{\omega_j}u(z) - \int_K u\tilde{\psi}_z^K| \leq |Q_{\omega_j}u(z) - \int_K u\tilde{\psi}_z^K| + \sum_{j=1}^{n-1} |Q_{\omega_j}u(z) - Q_{\omega_{j+1}}u(z)|.$$

We bound the terms on the right-hand side separately. Exploiting property (3.1) and the Cauchy-Schwarz inequality, we obtain

$$|Q_{\omega_j}u(z) - \int_K u\tilde{\psi}_z^K| = \int_{K_j} (Q_{\omega_j}u - u)\tilde{\psi}_z^K \leq \|Q_{\omega_j}u - u\|_{L^2(K)} \|\tilde{\psi}_z^K\|_{L^2(K)},$$

and similarly, for every $j = 1, \ldots, n-1$,

$$|Q_{\omega_j}u(z) - Q_{\omega_{j+1}}u(z)| = \int_{T_j} (Q_{\omega_j}u - Q_{\omega_{j+1}}u)\tilde{\psi}_z^T \leq \|Q_{\omega_j}u - Q_{\omega_{j+1}}u\|_{L^2(T)} \|\tilde{\psi}_z^T\|_{L^2(T)}.$$

When summing the last inequality over $j$, the $L^2$-norm of $Q_{\omega_j}u - u$ appears on both $T_j$ and $T_{j-1}$. We bound both contributions by the $L^2$-norm on $\omega_j$ and combine this with the scaling property (5.3) of $\psi_z$ and $|T_j| \geq \nu|K_j|$. Finally, for simplification, we incorporate the term with $\tilde{K}$ into the sum and obtain the claimed inequality. \qed

For the covering $W_T = \{\omega_T(E)\}_{E \in E}$, the existence of the path in Proposition 4.3 follows from the following property of the mesh $T$; see [10] for a discussion of its implications on the regularity of the boundary $\partial \Omega$.

\begin{definition}
(\text{Face-connectedness}) A simplicial mesh $T$ is face-connected, if for every element pair $K, \tilde{K} \in T$ sharing a node $z \in \mathcal{N}$, there exists a pairwise disjoint finite sequence $\{K_j\}_{j=1}^n \subset T$ such that

- $K_1 = K$ and $K_n = \tilde{K}$,
- each intersection $K_j \cap K_{j+1} \in E$ is an interelement face containing $z$.

The length $n$ of the path is bounded in terms of

$$\tilde{n} := \max_{z \in \mathcal{N}} \#\{K \in T : K \ni z\}.$$

\end{definition}

Essentially, Propositions 4.1 and 4.3 cover both 1.1 and 1.3; in fact, [10] uses a variant of Proposition 4.3 where the intersection of subdomains is faces. The following definition specifies the property of the covering $W$ that is crucial for the robustness in the first part of (1.3).

\begin{definition}
(\text{Internal face covering}) A $\beta$-finite covering $W$ covers interelement faces internally if for every interelement face $E \in \mathcal{E}$, there exists $\omega \in W$ such that its interior almost contains $E$, i.e. $E \subset \omega$ and $|E \cap \omega| = |E|$.

\end{definition}
While \( W_\mathcal{T} \) covers interelement faces internally, \( \mathcal{T} \) does not. As a consequence, the local best errors associated with \( W_\mathcal{T} \) take into account the continuity constraint across interelement faces, a feature that is crucial for robustness by the observations in [2].

5. ROBUST LOCALIZATION TO PAIRS OF ELEMENTS

The purpose of this section is to prove the first part of (1.3). The reaction-diffusion norm (1.2) has the \( L^2 \)-norm and the \( H^1 \)-seminorm as limiting cases. We consider these cases first in a unified manner by applying Proposition 4.2 with the covering \( W_\mathcal{T} = \{ \omega_\mathcal{T}(E) \}_{E \in \mathcal{E}} \) and start with the \( L^2 \)-norm, which appears the more critical limiting case.

5.1. Pure reaction norm. We first introduce an interpolation operator and then show that it is locally near best with respect to \( \| \cdot \| = \| \cdot \|_{L^2(\omega)} \), \( \omega \in W_\mathcal{T} \). For simplicity, we write \( P_{EK}u \) for the best approximation to \( u \) in \( S|_{\omega_\mathcal{T}(E)} \) with respect to \( \| \cdot \|_{\omega_\mathcal{T}(E)} = \| \cdot \|_{L^2(\omega_\mathcal{T}(E))} \).

The definition of the interpolation operator relies on a classification of the nodes. For nodes that are interior to an element, we define the corresponding nodal value \( \Pi^0u \) with the help of a best approximation. For the other nodes that belong to several elements or are on the boundary \( \partial \Omega \), we use the averaging technique of L. R. Scott and S. Zhang [8]. More precisely: for every element \( K \in \mathcal{T} \), we fix a face \( E = E_K \in \mathcal{E}_i \) such that \( E_K \subset \partial K \) and, moreover, for every \( z \in \mathcal{N} \cap \Sigma \) with \( \Sigma := \bigcup_{K \in \mathcal{T}} \partial K \), we fix an element \( K_z \in \mathcal{T} \) such that \( z \in \partial K_z \). Given \( u \in H^1(\Omega) \), we then set

\[
\Pi^0u := \sum_{z \in \mathcal{N}} u_z \phi_z
\]

where

\[
u_z = \begin{cases} 
P_{E_k}u(z) & \text{if } z \in \mathcal{N}_{\hat{K}}^c \text{ for some } K \in \mathcal{T}, \\
\int_{K_z} u\psi_{K_z} & \text{if } z \in \mathcal{N} \cap \Sigma. \end{cases}
\]

Notice that in general \( \Pi^0|_{\partial K} \) depends not only on \( u|_K \) but also on \( u|_{\hat{K}} \) for neighbouring elements \( \hat{K} \).

In order to verify that \( \Pi^0 \) is locally near best, we fix an element \( K \in \mathcal{T} \), write \( E := E_K \) for short and choose \( \omega = \omega_\mathcal{T}(E) \) in (1.2). This is an admissible choice for \( \omega \) since \( \omega_\mathcal{T}(E) \supseteq K \). It is also a natural choice because in this way, for every \( z \in \mathcal{N}^c_K \), we have

\[
|P_Eu(z) - \Pi^0u(z)| = 0.
\]

Otherwise, if \( z \in \mathcal{N}_K \cap \Sigma \), we exploit Proposition 4.2 with \( \nu = 1 \) and get

\[
|P_Eu(z) - \Pi^0u(z)| \leq 2\|\hat{\psi}_z\|_K \sum_{j=1}^n \frac{|K_j|^{1/2}}{|K_{j+1}|^{1/2}} \|P_{E_j}u - u\|_{\omega_j}.
\]
with $K_n := K_2$. Combining (5.2) and (5.3) with the scaling property (3.2), we obtain
\[
\sum_{z \in N_K} |\mathcal{P}_E u(z) - \Pi^0 u(z)| \|\phi_z\|_K \leq 2 \sum_{z \in N_{K_n}} \|\tilde{\phi}_z\|_K \sum_{j=1}^{n} \frac{|K|^{1/2}}{|K_j|^{1/2}} \|\mathcal{P}_E u - u\|_{\omega_j} \leq M_0 \max_{K \in \tilde{\omega}_\gamma(K)} \frac{|K|^{1/2}}{|K|^{1/2}} \sum_{E \in \gamma_\omega(K)} \|\mathcal{P}_E u - u\|_{\omega_\gamma(E)},
\]
where
\[
(5.4) \quad M_0 = M_0(\ell, d) := 2 \sum_{z \in \partial K} \|\tilde{\phi}_z\|_K \|\hat{\phi}_z\|_K
\]
and $\tilde{\omega}_\gamma(K)$ is the union of the elements $\tilde{K} \in \mathcal{T}$ touching $K$ and $\gamma_\omega(K) := \cup\{E \in \mathcal{E}_i : E \cap \partial K \neq \emptyset\}$ is the skeleton of $\tilde{\omega}_\gamma(K)$. In order to achieve a bound that is independent of $K$, we introduce
\[
(5.5) \quad \mu_\mathcal{E} := \max_{K \in \mathcal{T}} \max_{K \in \tilde{\omega}_\gamma(K)} \frac{|K|^{1/2}}{|K|^{1/2}}
\]
and arrive at
\[
(5.6) \quad \sum_{z \in N_K} |\mathcal{P}_E u(z) - \Pi^0 u(z)| \|\phi_z\|_K \leq \mu_\mathcal{E} M_0 \sum_{E \in \gamma_\omega(K)} \|\mathcal{P}_E u - u\|_{\omega_\gamma(E)}.
\]

We observe that $\#\gamma_\omega(K)$ is bounded in terms of $d$ and $\tilde{n}$ from (4.4). Given $\tilde{E} \in \mathcal{E}_i$, the same holds for $\#\{K \in \mathcal{T} : \gamma_\omega(K) \ni \tilde{E}\}$. We can therefore apply Proposition 1.2 and get the following theorem.

**Theorem 5.1** (Localization of best $L^2$-error). For every $u \in L^2(\Omega)$ it holds
\[
\inf_{v \in S} \|u - v\|_\Omega \leq C \left( \sum_{E \in \mathcal{E}_i} \inf_{P \in S_{\omega_\gamma(E)}} \left\|u - P\right\|^2_{\omega_\gamma(E)} \right)^{\frac{1}{2}} \leq C(d + 1) \inf_{v \in S} \|u - v\|_\Omega
\]
where the constant $C$ depends on the polynomial degree $\ell$, the dimension $d$, $\tilde{n}$ from (4.4), and $\mu_\mathcal{E}$ from (5.5).

Notice that there is no explicit dependence on the shape regularity of $\mathcal{T}$ but a dependence on the local quasi-uniformity of $\mathcal{T}$ through $\tilde{n}$ and $\mu_\mathcal{E}$.

5.2. **Pure diffusion seminorm.** The counterpart of Theorem 5.1 for the $H^1$-seminorm follows from (1.1). In this subsection we present an alternative approach relying on an interpolation operator that is very close to the one in (5.1). The obtained inequalities will turn out useful for dealing with the reaction-diffusion dependence on the local quasi-uniformity of $\mathcal{T}$ through $\tilde{n}$ and $\mu_\mathcal{E}$.

Since $\|\nabla\|_{\omega_\gamma} = \|\nabla\|_{L^2(\omega)}$ is only a seminorm, best approximations in $S_{\omega_\gamma}$ are only unique up to a constant. This freedom allows to bound the $L^2$-errors appearing in Proposition 4.3 by local best errors in the $H^1$-seminorm with the help of the Poincaré inequality. We are thus led to the following local best approximation operators: given $u \in H^1(\Omega)$ and any $E \in \mathcal{E}_i$, let $\mathcal{R}_E$ be the best approximation to $u$ in $S_{\omega_\gamma(E)}$ with respect to $\|\nabla\|_{\omega_\gamma(E)}$ such that
\[
\int_{\omega_\gamma(E)} \mathcal{R}_E u = \int_{\omega_\gamma(E)} u.
\]
The interpolation operator $\Pi^\infty$ is then given by (5.1) where $\mathcal{P}_{E_k}$ is replaced by $\mathcal{R}_{E_k}$. Consequently, $\Pi^0$ and $\Pi^\infty$ differ only in the involved local best approximations. Before embarking on the proof that $\Pi^\infty$ is locally near best, we provide the following tailor-made Poincaré inequality.
Lemma 5.2 (Poincaré inequality for element pairs). Let \( \omega \) be the union of two adjacent elements \( K_1, K_2 \) sharing a face \( E = K_1 \cap K_2 \). For every \( v \in H^1(\omega) \) it holds

\[
\left\| v - \frac{1}{|\omega|} \int_\omega v \right\|_\omega \leq C_P h_\omega \| \nabla v \|_\omega,
\]

where \( C_P \leq \left( \frac{1}{\pi} + \frac{1}{d} \right)^{1/2} \) and \( h_\omega := \max\{\text{diam}(K_1), \text{diam}(K_2)\} \).

Proof. Since the mean value on \( \omega \) is the best approximating constant with respect to the \( L^2 \)-norm, we can substitute it with the mean value on the common face \( E \):

\[
\left\| v - \frac{1}{|\omega|} \int_\omega v \right\|^2_\omega \leq \left\| v - \frac{1}{|E|} \int_E v \right\|^2_\omega = \sum_{i=1}^2 \left\| v - \frac{1}{|E|} \int_E v \right\|^2_{K_i}.
\]

Thanks to the trace identity from [7] Proposition 4.2, we may write

\[
(5.8) \quad \frac{1}{|E|} \int_E v = \frac{1}{|K_i|} \int_{K_i} v + \frac{1}{d|K_i|} \int_{K_i} \hat{q}_{i,E} \cdot \nabla v,
\]

where \( \hat{q}_{i,E}(x) := x - z_i \) and \( z_i \) is the vertex of \( K_i \) opposite to \( E \). Moreover the classical Poincaré inequality on convex domains, see [1, 7], implies

\[
\| v \|_{K_i}^2 = \frac{1}{|K_i|} \int_{K_i} v^2 + \frac{1}{d|K_i|} \left( \int_{K_i} \hat{q}_{i,E} \cdot \nabla v \right)^2 \leq \text{diam}(K_i)^2 \left( \frac{1}{\pi^2} + \frac{1}{d^2} \right) \| \nabla v \|_{K_i}^2.
\]

Combining (5.8) and the two inequalities yields the claim. \( \square \)

In order to show that \( \Pi^\infty \) is locally near best with respect to \( \| \nabla \|_{\omega_T(E)} \), \( E \in \mathcal{E}_l \), we fix an element \( K \in \mathcal{T} \), write \( E := E_K \) for short and, as in (3.1), we choose \( \omega = \omega_T(E) \) in (4.2). If \( z \in \mathcal{N}_K \), we have again

\[
(5.9) \quad |R_E u(z) - \Pi^\infty u(z)| = 0.
\]

Otherwise, if \( z \in \mathcal{N}_K \cap \Sigma \), we apply also Proposition 4.3 with \( \nu = 1 \) and obtain

\[
|R_E u(z) - \Pi^\infty u(z)| \leq 2 \| \hat{\psi}_2 \|_K \sum_{j=1}^n \frac{|\hat{K}|^{1/2}}{|K_j|^{1/2}} \left\| R_{E_j} u - u \right\|_{\omega_j}
\]

with \( K_n = K_2 \). Here we invoke the Poincaré inequality Lemma 5.2 which yields

\[
(5.10) \quad |R_E u(z) - \Pi^\infty u(z)| \leq 2 C_P \| \hat{\psi}_2 \|_K \sum_{j=1}^n h_{\omega_T(E_j)} \frac{|\hat{K}|^{1/2}}{|K_j|^{1/2}} \left\| \nabla (R_{E_j} u - u) \right\|_{\omega_j},
\]

with \( h_{\omega_T(E)} := \max_{K \in \mathcal{T}, K \subseteq \omega_T(E)} \text{diam}(K) \). Combining (5.9) and (5.10) with (3.4) leads to

\[
\sum_{z \in \mathcal{N}_K} |R_E u(z) - \Pi^\infty u(z)| \| \nabla \hat{\psi}_2 \|_K
\]

\[
\leq 2 h \sum_{z \in \mathcal{N}_K} \| \hat{\psi}_2 \|_K \| \nabla \hat{\psi}_2 \|_K C_P \sum_{j=1}^n h_{\omega_T(E_j)} \frac{|\hat{K}|^{1/2}}{|K_j|^{1/2}} \left\| \nabla (R_{E_j} u - u) \right\|_{\omega_T(E_j)}
\]

\[
\leq M_\infty \max_{\hat{K} \in \mathcal{E}_{\gamma}(K)} \frac{|\hat{K}|^{1/2}}{|K|^{1/2}} \max_{E \in \mathcal{E}_{\gamma}(K)} \frac{h_{\hat{K}}}{\rho K} \sum_{E \in \mathcal{E}_{\gamma}(K)} \left\| \nabla (R_E u - u) \right\|_{\omega_T(E)},
\]
where $h_K := \text{diam}(K)$ and
\[
M_\infty = M_\infty(\ell, d) := 2\sqrt{2}C_P \sum_{z \in \partial K} \|\hat{\varphi}_z\|_K \|\nabla \hat{\varphi}_z\|_K.
\]
We thus see that the use of the Poincaré inequality compensates the scaling of $\|\nabla \hat{\varphi}_z\|_K$, if the shape parameter
\[
(5.11) \quad \sigma_T := \max_{K \in \mathcal{T}} \max_{K \in \mathcal{W}_T(K)} \frac{h_K}{\rho_K}
\]
of $T$ is moderate. We therefore have
\[
(5.12) \quad \sum_{z \in N_K} |R_E u(z) - \Pi^\infty u(z)| \|\nabla \phi_z\|_K
\]
and can apply Proposition 1.2 Apart from the dependencies listed in Theorem 5.1, the involved constant depends in addition on the shape coefficient of $T$ in (5.11).

5.3. Reaction-diffusion Norm. We now turn to the main result of this section: the robust localization of the best error in the reaction-diffusion norm. To this end, we follow the lines of norms on a finite dimensional space, there exist constants $c_\ell$ and $\sigma_\ell$, write (5.6) and (5.12) for $\Pi^\ell$ and $\Pi^\infty$, respectively. Inequality (5.14) and the counterpart of (5.6) imply
\[
\sum_{z \in N_K} |R_E u(z) - \Pi^\ell u(z)| \|\phi_z\|_K \leq \mu_T M_0 \sqrt{1 + \varepsilon \frac{C_\ell^2 h^2}{\rho K^2}} \sum_{E \in \mathcal{W}_T(K)} \|R_E u - u\|_{\omega_T(E)}.
\]
while \((5.15)\) and the counterpart of \((5.12)\) give
\[
\sum_{z \in \mathcal{N}_K} |R^z_E u(z) - \Pi^z u(z)| \|\phi_z\| \\
\leq \mu_T \sigma T M_\infty \sqrt{1 + \frac{1}{\varepsilon} C_1^2 \frac{h^2}{\rho_K^2}} \sum_{E \in \gamma_T(K)} \varepsilon^{1/2} \left\| \nabla (R^z_E u - u) \right\|_{\omega_T(E)}.
\]
Combining the last two inequalities, we arrive at
\[
(5.16) \quad \sum_{z \in \mathcal{N}_K} |R^z_E u(z) - \Pi^z u(z)| \|\phi_z\| \leq \mu_T M_\varepsilon \sum_{E \in \gamma_T(K)} \|R^z_E u - u\|_{\omega_T(E)}
\]
where
\[M_\varepsilon := \min \left\{ M_0 \sqrt{1 + \frac{C^2 \frac{h^2}{\rho_K^2}}{\varepsilon}} \mu_T \sqrt{1 + \frac{1}{\varepsilon} C_1^2 \frac{h^2}{\rho_K^2}} \right\}\]
satisfies \(\lim_{\varepsilon \to 0} M_\varepsilon = M_0\) and \(\lim_{\varepsilon \to \infty} M_\varepsilon = M_\infty\) as well as
\[M_\varepsilon \leq \max \left\{ M_0, M_\infty \sigma T \right\} \sqrt{1 + \frac{C_\infty \frac{h^2}{\rho_K^2}}{\varepsilon}}\]
Using this in Proposition \(4.2\) provides the main result of this section:

**Theorem 5.3** (Robust localization for reaction-diffusion norm). There is a constant \(C\) such that for any \(\varepsilon > 0\) and every \(u \in H^1(\Omega)\) it holds
\[
\inf_{v \in S} \|u - v\|_\Omega \leq C \left( \sum_{E \in \mathcal{E}_i} \inf_{P \in S_{\omega_T(E)}} \|u - P\|_{\omega_T(E)}^2 \right)^{1/2} \leq C(d + 1) \inf_{v \in S} \|u - v\|_\Omega
\]
where \(\|\cdot\|\) stands for \((\|\cdot\|^2 + \varepsilon \|\nabla \cdot \|^2)^{1/2}\). The constant \(C\) is bounded in terms of the polynomial degree \(d\), the dimension \(\bar{n}\) from \(1.3\), \(\mu_T\) from \(5.4\), and the shape parameter \(\sigma_T\) from \(5.1\). If \(\varepsilon\) is small with respect to \(\min_{K \in \mathcal{E}} \rho_K^2\), the dependence on the shape parameter \(\sigma_T\) disappears.

### 6. Local best errors: single elements versus pairs

According to \(2\) and \(6.3\) the best errors in the reaction-diffusion norm \(1.2\) on single elements do not provide a robust localization, while those on pairs of elements do. This section analyzes their difference. As side-products, we derive the second part of \(1.3\) and an alternative way to compute best errors on element pairs.

Throughout this section \(\omega\) is the union of two simplices \(K_1 \text{ and } K_2\) with common face \(E = K_1 \cap K_2\). For example, \(\omega = \omega_T(E)\) with \(E \in \mathcal{E}_i\). We write \(h_i := h_{K_i}\) and \(\rho_i := \rho_{K_i}\) for short, \(i = 1, 2\), and set
\[\sigma_E := \frac{\max\{h_1, h_2\}}{\min\{\rho_1, \rho_2\}}\] and \[h_E := \frac{\min\{|K_1|, |K_2|\}}{|E|}\]
Moreover denote by \(P_E\) the best approximation in \(S|_\omega := \mathcal{S}\) with respect to the reaction-diffusion norm \(\|\cdot\|_\omega\), whence
\[
\|u - P_E\|_\omega = \inf_{P \in S|_\omega} \|u - P\|_\omega.
\]
Section \(2\) shows that the best errors on the single elements are missing something for a robust upper bound. We first determine a quantity that provides a remedy.
Lemma 6.1 (Jump augmentation). Let $C_P > 0$ be a constant and $u \in H^1(\omega)$. If $P_i \in \mathbb{P}_\ell(K_i)$ satisfy

$$
\|u - P_i\|_{K_i} \leq C_P h_i \|\nabla(u - P_i)\|_{K_i}, \quad i = 1, 2, 
$$

then there holds

$$
\|u - P_E\|_{\omega} \leq C \left( h_E^{1/2} \|P_1 - P_2\|_E + \sum_{i=1}^2 \|u - P_i\|_{K_i} \right),
$$

where $C$ depends on $C_P$, $\ell$, $d$, $\sigma_E$, but is independent of $\varepsilon$.

Proof. We may assume $|K_2| \geq |K_1|$ without loss of generality. Define $\tilde{P} \in S_{\omega}$ by

$$
\tilde{P}(z) := \begin{cases} 
P_i(z) & z \in N_{K_1 \setminus E} \text{ with } i \in \{1, 2\}, \\
P_2(z) & z \in N_E. 
\end{cases}
$$

Thanks to (6.1) we have $\|u - P_E\|_{\omega} \leq \|u - \tilde{P}\|_{\omega}$. Observing

$$
\|u - \tilde{P}\|_{K_i} = \|u - P_i\|_{K_2} \quad \text{and} \quad \|u - \tilde{P}\|_{K_1} \leq \|u - P_i\|_{K_1} + \|P_i - \tilde{P}\|_{K_1},
$$

we are left with establishing a suitable bound for $\|P_i - \tilde{P}\|_{K_1}$. To this end, we proceed similarly to (5.3). We expand $P_1 - \tilde{P}$ with respect to the nodal basis functions on $K_1$ and obtain

$$
\|P_1 - \tilde{P}\|_{K_1} \leq \sum_{z \in N_E} |P_1(z) - P_2(z)| \|\phi_z\|_{K_1},
$$

because $|P_1(z) - \tilde{P}(z)| = 0$ for every $z \in N_{K_1 \setminus E}$. For the shared nodes $z \in N_E$, we have

$$
|P_1(z) - P_2(z)| \leq \int_E |(P_1 - P_2)|^2 |\psi_z|^2 \leq \frac{|E|^{1/2}}{|E|^{1/2}} \|P_1 - P_2\|_E \|\psi_z\|_E
$$

by (5.3). In view of (5.14) and (5.2), the last two inequalities imply

$$
\|P_1 - \tilde{P}\|_{K_1} \leq \sum_{z \in N_E} |P_1(z) - P_2(z)| \sqrt{1 + \varepsilon \frac{C_N^2 h^2}{\rho_i^2}} \|\phi_z\|_{K_1},
$$

$$
\leq m_0 \sqrt{1 + \varepsilon \frac{C_N^2 h^2}{\rho_i^2}} \|P_1 - P_2\|_E
$$

with

$$
m_0 = \frac{|E|^{1/2}}{|K|^{1/2}} \sum_{z \in E} \|\psi_z\|_E \|\phi_z\|_K.
$$

To derive an alternative bound when $\varepsilon$ is big, we first observe that the trace identity (5.8) and the Poincaré inequalities (5.2) yield

$$
\|u - P_i\|_{\omega}^2 \leq C_P \left( C_P + \frac{2}{d} \right) \frac{h_i^2 |E|}{|K_i|} \|\nabla(u - P_i)\|_{K_i}^2, \quad i = 1, 2.
$$

Using this in (6.4) gives

$$
|P_1(z) - P_2(z)| \leq \sqrt{C_P \left( C_P + \frac{2}{d} \right) |E|} \|\psi_z\|_E \sum_{j=1}^2 \frac{h_j}{|K_j|^{1/2}} \|\nabla(u - P_j)\|_{K_j},
$$

and

$$
\|u - P_E\|_{\omega} \leq \sqrt{C_P \left( C_P + \frac{2}{d} \right) |E|} \|\psi_z\|_E \sum_{j=1}^2 \frac{h_j}{|K_j|^{1/2}} \|\nabla(u - P_j)\|_{K_j},
$$

for every $z \in N_E$. The proof is complete.
which together with (6.3), (6.15) and (3.3) implies
\[
\|P_i - \tilde{P}\|_{K_i} \leq \sum_{z \in N_E} |P_i(z) - P_2(z)| \left( 1 + \frac{1}{\varepsilon} \frac{h_i^2}{c} \right)^{1/2} \|\nabla \phi_i\|_{K_i},
\]
(6.7)
\[
\leq m_\infty \sqrt{1 + \frac{1}{\varepsilon} \frac{h_i^2}{c} \rho_i^2} \sum_{j=1}^2 \frac{h_j \varepsilon^{1/2}}{\rho_i} \|\nabla (u - P_j)\|_{K_i},
\]
where
\[
m_\infty = m_\infty (C_p, \ell, d) := \sqrt{C_p \left( \frac{C_p + 2}{d} \right) \frac{\hat{E}}{|K|} \hat{h} \sum_{z \in E} \|\hat{\psi}_i\|_{E} \|\nabla \hat{\phi}_i\|_{\hat{K}}.}
\]
Combining (6.5) and (6.7) yields
\[
\|P_i - \tilde{P}\|_{K_i} \leq C \left( h_E^{1/2} \|P_1 - P_2\|_E + \sum_{i=1}^2 \|u - P_i\|_{K_i} \right)
\]
with
\[
C = \max\{m_0, m_\infty \sigma_E\} \left( 1 + \frac{C_N \hat{h} h_1}{c N \hat{\rho} \rho_1} \right)
\]
and so the claimed inequality is established. \(\square\)

Next, we check that the jump term in the upper bound in Lemma 6.1 does not overestimate, if we choose \(P_i \in \mathbb{P}_\ell(K_i)\) as the best approximations to \(u\) with respect to the reaction-diffusion norm \(\|\cdot\|\) that is
\[
\|u - P_i\|_{K_i} = \inf_{P \in \mathbb{P}_\ell(K_i)} \|u - P\|_{K_i}, \quad i = 1, 2.
\]
(6.8)
This leads to the following characterization of the best error on a pair by the best approximation on its single elements.

**Theorem 6.2** (Sharp jump augmentation). For any \(u \in H^1(\omega)\), it holds
\[
\|u - P_E\|_{\omega} \approx \left( h_E \|P_1 - P_2\|_E^2 + \sum_{i=1}^2 \|u - P_i\|_{K_i}^2 \right)^{1/2}
\]
whenever \(P_i\) are given by (6.3). The hidden constants depend on \(\ell, d, \sigma_E\), but are independent of \(\varepsilon\).

**Proof.** We start by bounding \(\|u - P_E\|_{\omega}\) from above. The choice (6.3) implies \(\int_{K_i} P_1 = \int_{K_i} u\); see (5.2). Therefore the classical Poincaré inequality on convex domains, see [1, 7], ensures that (6.2) holds with \(C_p \leq \frac{1}{\varepsilon}\) and Lemma 6.1 yields the desired bound.

In order to bound \(\|u - P_E\|_{\omega}\) from below, we first observe
\[
\sum_{i=1}^2 \|u - P_i\|_{K_i}^2 \leq \sum_{i=1}^2 \|u - P_E\|_{K_i}^2 = \|u - P_E\|_{\omega}^2.
\]
using (6.8). Therefore the critical term is the jump term. To bound it, we first add and subtract \(P_E\) and use the triangle inequality:
\[
\|P_1 - P_2\|_E \leq \|P_1 - P_E\|_E + \|P_2 - P_E\|_E.
\]
(6.10)
Since \(P_1\) and \(P_E\) are both polynomials on \(E\), we write their expansion with respect to the nodal basis functions on \(E\). Every \(z \in N_E\) is also a node of \(K_i\) and so we can
use (3.1) on $K_i$. Using also the Cauchy-Schwarz inequality, the scaling properties (3.2) and (3.3), we derive the following explicit inverse inequality:

\[ h_E^{1/2} \| P_1 - P_E \|_E \leq h_E^{1/2} \sum_{z \in N_E} \int_{K_i} |(P_1 - P_E)\psi_z^K| \|\psi_z\|_E \]

\[ \leq \frac{|K_1|^{1/2}}{|E|^{1/2}} \sum_{z \in N_E} \| P_1 - P_E \|_{K_i} \| \hat{\psi}_z \|_E \| \hat{\phi}_z \|_E \frac{|K_1|^{1/2}}{|E|^{1/2}} \| E \|^{1/2} \| K_1 \|^{1/2} \]

\[ \leq \tilde{m}_0 \| P_1 - P_E \|_{K_i}, \]

where we have assumed $|K_2| \geq |K_1|$ without loss of generality and \[ \tilde{m}_0 = \tilde{m}_0(\ell, d) := \frac{|K_1|^{1/2}}{|E|^{1/2}} \sum_{z \in E} \| \hat{\psi}_z \|_E \| \hat{\phi}_z \|_E. \]

Inserting $u$ and recalling (6.8), we get

\[ h_E^{1/2} \| P_1 - P_E \|_E \leq 2\tilde{m}_0 \| u - P_E \|_{K_i}. \]

Combining (6.9), (6.10) and (6.11), we conclude

\[ h_E \| P_1 - P_2 \|_E^2 + \sum_{i=1}^2 \| u - P_i \|_{K_i}^2 \leq (1 + 8\tilde{m}_0^2) \| u - P_E \|_E^2, \]

where the constant depends only on $\ell$ and $d$. \[ \square \]

Theorem 6.2 shows that best approximations on elements augmented with interelement jumps provide also a robust localization. As a consequence, the non-robustness in (2) is caused by the absence of these jump terms. This fact is illustrated by the following corollary for the limiting case $\varepsilon = 0$.

**Corollary 6.3** (Sharp jump augmentation for $L^2$). Let $u \in L^2(\omega)$ and denote by $P_F, P_i$ the best approximations to $u$ in $S|_\omega$ and $P_i(K_i)$, respectively with respect to the $L^2$-norm. Then

\[ \| u - P_E \|_\omega \approx \left( h_E \| P_1 - P_2 \|_E^2 + \sum_{i=1}^2 \| u - P_i \|_{K_i}^2 \right)^{1/2}, \]

where the hidden constants depend on $\ell$ and $d$, but are independent of $\sigma_E$.

**Proof.** Consider only the $L^2$-part of the reaction-diffusion norm in the proof of Theorem 6.2. \[ \square \]

The localization associated with the right-hand side in Theorem 6.2 is less costly to compute than the one in Theorem 6.3. The jumps between the best approximations however require communication between elements. At the price of a slight overestimation, the following proposition, which establishes the second part of (1.3), uses only best errors on elements and in particular avoids this communication.

**Proposition 6.4** (Trace augmentation). For any $u \in H^1(\omega)$, it holds

\[ \| u - P_E \|_\omega \leq C \sum_{i=1}^2 \inf_{P_i \in P_i(K_i)} \left( \| u - P \|_{K_i}^2 + \frac{|K_i|}{|\partial K_i|} \| u - P \|_{\partial K_i}^2 \right)^{1/2}, \]

where the constant $C$ depends on $\ell$, $d$, $\sigma_E$, but is independent of $\varepsilon$.

**Proof.** Let $P_i, i = 1, 2$, be the best approximations associated with the two infima in the claimed bound. It suffices to verify that they satisfy (6.2). In fact, inserting
If $E$ in the jump term of Lemma 5.1 yields the claim. Fix $i = 1, 2$ and write $K := K_i$ and $v := u - P_i$ for short. Since

$$
\int_K v + \frac{|K|}{|\partial K|} \int_{\partial K} v = 0,
$$

we can write

$$
\|v\|^2_K = \left\| v - \frac{1}{|K|} \int_K v \right\|^2_K + \frac{1}{4|K|} \left( \frac{|K|}{|\partial K|} \int_{\partial K} v - \int_K v \right)^2.
$$

Adding the trace identity, cf. (5.8), for every face of $K$ in a suitable weighted manner, we obtain a vector field $q_K$ such that

$$
\left\| \frac{|K|}{|\partial K|} \int_{\partial K} v - \int_K v \right\| = \int_K q_K \cdot \nabla v \leq \frac{h_K}{d} |K|^{1/2} \|\nabla v\|_K.
$$

Consequently the classical Poincaré inequality on convex domains, see [1, 7], shows that (6.2) holds with $C_P \leq \left( \frac{1}{\pi} + \frac{4\pi}{3d^2} \right)^{1/2}$. □

7. Robust localization for tree approximation

Adaptive tree approximation of P. Binev and R. DeVore [3] constructs near best meshes within a hierarchy by means of so-called local error functionals. In this section we propose local error functionals that are suitable for the reaction-diffusion norm and, by modifying Theorem 5.3 we show that they ensure a robust performance of tree approximation.

We start by fixing our setting for tree approximation. As refinement procedure, we adopt bisection of (tagged) simplices as described, e.g., [6, §4.1] or R. Stevenson [9]. Recall that the refinement edge of a simplex is the edge that will be halved when the simplex is bisected and that the refinement edges of the two children simplices are assigned in a unique manner.

Let $T_0$ be a conforming mesh of $\Omega$ into simplices such that the so-called matching condition, see e.g. [6, §4.2], is satisfied. Denote by $T_c$ the set of all conforming meshes that can be generated from $T_0$ by successive bisections. Thanks to the matching condition it contains in particular all uniform refinements of $T_0$. Moreover we have a graph $B^*$ where the nodes correspond to simplices and each simplex is connected with its two children. This graph is a forest of infinite binary trees, whose roots are the elements of $T_0$, and it is often called master tree. Every conforming mesh $T \in T_c$ is represented by a subtree $B$ of the master tree $B^*$. The set of leaves and interior nodes of a tree $B$ are denoted by $L(B)$ and $N(B)$, respectively.

A local error functional is a map $e$ that associates a positive real $e(K) \geq 0$ to any simplex $K \in B^*$. Roughly speaking, tree approximation uses $e$ to construct trees $B$ that almost minimize the global error functional

$$
E(B) := \sum_{K \in L(B)} e(K)
$$

within trees of similar cardinality.

Notice that the local best errors in Theorem 5.3 cannot be combined to define an error functional due to the dependence of $\omega_T(E)$ on $T$. To remedy, we mimic the idea of ‘minimal ring’ in P. Binev et al. [2] and introduce the following variant of $\omega_T(E)$: given a face $E$ of any simplex $K \in B^*$, we define

$$
\omega_*(E) := \bigcap_{T \in T_c : E \text{ is a face of } T} \omega_T(E).
$$

If $E$ is an interelement face of some mesh $T \in T_c$, then $\omega_*(E)$ is the union of two elements $K'_1$ and $K'_2$ that belong to some virtual refinement of $T$ and are such that $K'_1 \cap K'_2 = E$. See also Figure 1.
Remark 7.1 (Properties of $\omega_\ast$). Let $T \in \mathcal{T}_c$ be a conforming mesh and denote by $E$ the set of faces, and by $E_i$ its subset of the interelement faces. There hold:

(i) Let $K \in T$ and $E \in E$ such that $E \subseteq \partial K$. Then $K \subseteq \omega_\ast(E)$ if and only if the refinement edge $S$ of $K$ is contained in the face $E$.

(ii) Let $E \in E$. For every $K \in T$ with $K \subseteq \omega_\ast(E)$, we have that $\omega_\ast(E) \cap K$ is an element of $T$ or of some virtual refinement of $T$ so that $|\omega_\ast(E) \cap K| \geq |K|/2$.

(iii) For any $K \in T$, there exists $E \in E$ such that $\omega_\ast(E) \supseteq K$. Moreover it is possible to take $E \in E_i$, if there exists a face $E \subseteq \partial K$ such that $E \not\subseteq \partial \Omega$ and $E$ contains the refinement edge of $K$.

(iv) The collection $\{\omega_\ast(E)\}_{E \in E}$ is a $d$-finite covering of $T$.

Proof. We start with (i). Since $E \subseteq \partial K$, we have $K \subseteq \omega_\ast(E)$. We consider the two cases $S \subseteq E$ and $S \not\subseteq E$ separately. In the first case, if $K$ is bisected, $E$ is also bisected, and there does not exist any conforming refinement $T' \in \mathcal{T}_c$ of $T$ and any descendant $K'$ of $K$ such that $K' \subset \omega_\ast(E)$. Hence $K \subseteq \omega_\ast(E)$. On the other hand, consider the second case $S \not\subseteq E$. If $K$ is bisected, one of its children $K'$ still contains $E$ and its refinement edge is contained in $E$. Therefore $K' \subseteq \omega_\ast(E)$ and $K' \subseteq K$ entails $K \not\subseteq \omega_\ast(E)$.

In order to verify (ii), we note from the proof of (i) that either $\omega_\ast(E) \cap K = K$ or $\omega_\ast(E) \cap K = K'$, where $K'$ is a child of $K$. Since bisection yields $|K'| = |K|/2$, we have also $|\omega_\ast(E) \cap K| \geq |K|/2$.

Finally, taking a face which contains the refinement edge of $K$ and applying (i) shows (iii), which then together with (i) implies (iv). \qed
Motivated by the form (7.1) of the global error functional and the fact that the covering in Remark 7.1(iv) covers faces internally (see Definition 4.3), we introduce the following local error functional for the reaction-diffusion norm:
\begin{equation}
(7.2) 
    e(K) := \sum_{E \text{ face of } K} \inf_{P \in \mathcal{S}|\omega_{s}(E)} \|u - P\|_{\omega_{s}(E)}^{2}, \quad K \in B^{*}.
\end{equation}
This indeed depends only on $K$ because each $\omega_{s}(E)$ depends only on $E$ and the refinement edges in the mesh $\mathcal{T}_{0}$.

The finiteness of the covering in Remark 7.1(iv) is related also the following useful property of the local error functional, which \cite{[3]} calls ‘modified subadditivity’.

**Proposition 7.2** (Weak subadditivity). The functional $e$ in (7.2) has the following property: if $B \subset B^{*}$ is a tree with a single root $K$ and such that $\mathcal{L}(B)$ is a conforming mesh, then
\begin{equation}
(7.3) 
    \sum_{K' \in \mathcal{L}(B)} e(K') \leq 2d e(K).
\end{equation}

**Proof.** For every $E$ face of $K$, denote by $v_{E}$ the best approximation in $S|\omega_{s}(E)$ to $u$ with respect to the $\|\cdot\|_{\omega_{s}(E)}$-norm. For every $E' \subset K' \in \mathcal{L}(B)$, there exists $E \subset K$ such that $\omega_{s}(E') \subset \omega_{s}(E)$ and so
\[ \|u - v_{E'}\|_{\omega_{s}(E')} \leq \|u - v_{E}\|_{\omega_{s}(E)}. \]
Since a given point in $\omega_{s}(E)$ is contained in at most $d$ pairs $\omega_{s}(E')$, we thus obtain
\[ \sum_{K' \in \mathcal{L}(B)} e(K') \leq \sum_{K' \in \mathcal{L}(B)} \sum_{E' \text{ face of } K'} \|u - v_{E'}\|_{\omega_{s}(E')}^{2} \]
\[ \leq 2d \sum_{E' \text{ face of } \mathcal{L}(B)} \|u - v_{E'}\|_{\omega_{s}(E')}^{2} \]
\[ = 2d e(K). \]

Modifying the proof of Theorem 5.3, we can relate its corresponding global error functional with the best error in $S$ with respect to the reaction-diffusion norm (1.2).

**Theorem 7.3** (Localization for tree approximation). Let $E$ be given by (1.1) with (1.2). For any $u \in H^{1}(\Omega)$ and any conforming mesh $\mathcal{T} \in \mathcal{T}_{\epsilon}$, it holds
\[ \inf_{v \in S|\omega^{\epsilon}(\mathcal{T})} \|u - v\|_{\Omega} \approx E(B)^{1/2}, \]
where $B \subset B^{*}$ is the tree corresponding to $\mathcal{T}$ and the hidden constants depend only on the polynomial degree $\ell$, the dimension $d$, the mesh $\mathcal{T}_{0}$, but not on $\epsilon$.

**Proof.** Writing $S := S^{\ell,0}(\mathcal{T})$ for short, the results follows from the following inequalities:
\begin{equation}
(7.4) 
    \inf_{v \in S} \|u - v\|_{\Omega} \leq C \left( \sum_{E \in \mathcal{E}} \inf_{P \in \mathcal{S}|\omega_{s}(E)} \|u - P\|_{\omega_{s}(E)} \right)^{1/2} \leq C d \inf_{v \in S} \|u - v\|_{\Omega},
\end{equation}
where $C$ depends on $\ell$, $d$, and the shape parameter $\sigma_{T}$. In fact, $E(B)$ regroups only the terms of the sum inside the square root of (7.4) and the shape parameter $\sigma_{T}$ is bounded in terms of the $\sigma_{T}$; see, e.g., \cite{[3]} Corollary 4.1.

The proof of (7.4) resembles the one of Theorem 5.3 and we restrict ourselves to emphasize the differences.

In order to define the interpolation operator, we fix $K_{z}$ for every $z \in \mathcal{N} \cap \Sigma$ as before but, for every $K \in \mathcal{T}$, we fix $E_{K}$ such that $\omega_{s}(E_{K}) \supset K$. The latter allows to choose $\omega = \omega_{s}(E_{K})$ in the verification of (1.2) but requires to incorporate the faces on the domain boundary $\partial \Omega$ in the localization (7.3). The interpolation operator

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If \( \Pi_*^e \) is then given by (5.1) where \( P_{E_K} \) is replaced by \( R_{E_K}^\varepsilon \), the best approximation operator associated with \( S|_{\omega_*(E_K)} \) and the reaction-diffusion norm.

Next, we show that \( \Pi_*^e \) is locally near best with respect to the covering \( W^* := \{ \omega_*(E) \}_{E \in \mathcal{E}} \) in Remark 4.3 (iv). To this end, we fix \( K \in \mathcal{T} \), write \( E := E_K \) and choose \( \omega = \omega_*(E) \) in (7.4). Again, we have \( |R_{E_K}^\varepsilon u(z) - \Pi^e u(z)| = 0 \) for \( z \in \mathcal{N}_K \) and exploit Proposition 4.3 for \( z \in \mathcal{N}_K \cap \Sigma \). For the covering \( W_* \) however, the construction of the path of subdomains is more involved.

Since \( \mathcal{T} \) is face-connected, there exists a sequence \( \{ K_i \}_{i=1}^r \) of elements of \( \mathcal{T} \) such that \( K_1 = K \), \( K_r = K_{\varepsilon} \), and each intersection \( K_i \cap K_{i+1} \in \mathcal{E}_i \) is an interelement face containing \( z \). We write \( \tilde{E}_i := K_i \cap K_{i+1} \) for the intersections and, for every element \( K_i \), we choose a face \( E_K^* \) such that \( \omega_*(E_K^*) \supseteq K_i \). Then we can construct the path \( \{ \omega_j \}_{j=1}^n := \{ \omega_*(E_j) \}_{j=1}^n \) of subdomains by means of the following algorithm:

\[
E_1 := E, j := 1 \\
\text{if } E_1 \neq \tilde{E}_1 \text{ then} \\
E_2 := \tilde{E}_1, j := j + 1 \\
\text{endif} \\
\text{for } i = 1, \ldots, r - 2 \text{ do} \\
\text{if } |\omega_*(\tilde{E}_i) \cap \omega_*(\tilde{E}_{i+1})| \geq |K_{i+1}|/2 \text{ then} \\
E_{j+1} := \tilde{E}_{i+1}, j := j + 1 \\
\text{else} \\
E_{j+1} := E_{K_{i+1}}, E_{j+2} := \tilde{E}_{i+1}, j := j + 2 \\
\text{endif} \\
\text{endfor} \\
E_{j+1} := E_{K_r}
\]

In view of Remark 7.1, this path is admissible for Proposition 4.3 with \( \nu = 1/2 \). We therefore can follow the lines in the proof of Theorem 5.3, replacing pairs by minimal pairs. Taking into account \( \nu = 1/2 \), \( h_{\omega_*(E')} \leq h_{\omega_*(E)} \) and the fact that more faces are involved, we derive

\[
(7.5) \quad \sum_{z \in \mathcal{N}_K} |R_{E_K}^\varepsilon u(z) - \Pi_{E_K}^e u(z)||\|_K \leq \sqrt{2\mu_T} M \sum_{\tilde{E} \in \gamma_T(K)} |||R_{E_{\tilde{E}}^*}^\varepsilon u - u|||_{\omega_*(E)};
\]

with \( \gamma_T(K) := \gamma_T(K) \cup \{ E \in \mathcal{E} : E \subset \partial \omega_*(K) \} \). Since \( \#\gamma_T(K) \) and \( \#\{ K \in \mathcal{T} : E \in \gamma_T(K) \} \) are still bounded in terms of \( n \) and \( d \), Proposition 4.2 ensures (7.4).

Let us illustrate the usefulness of the error functional proposed in (7.2). To this end, consider the Modified Second Algorithm in [3 §7] with (7.2): one could simplify the so-called 'new subdivision rule' therein in our context. A combination of Proposition 7.2 [3 Theorem 7.2] and Theorem 7.3 yields that any output mesh \( \mathcal{T} \) is near best in the following sense:

\[
\inf_{v \in S^{\ell_0}(\mathcal{T})} \| u - v \|_{\Omega} \leq CE_c \#\mathcal{T},
\]

where

\[
E_n := \min \left\{ \inf_{v \in S^{\ell_0}(\mathcal{T}')} \| u - v \|_{\Omega} : \mathcal{T}' \in \mathcal{T}_c, \#\mathcal{T}' \leq n \right\}
\]

and \( C \geq 1, c \in (0, 1) \) are constants depending on \( d, \ell, \mathcal{T}_0 \) but not on \( \varepsilon \). Moreover, counting an evaluation of \( e \) with 1 operation, the algorithm uses less than \( O(\#\mathcal{T} + \#\mathcal{T}_0) \) operations to create \( \mathcal{T} \).
8. ROBUST LOCALIZATION WITH DIRICHLET BOUNDARY CONDITIONS

In this section we briefly discuss the modifications of our results if the boundary values of the target function are imposed on the approximants. This is of interest, for example, when conforming finite element methods are applied to the homogeneous Dirichlet problem of the reaction-diffusion equation. For simplicity, we consider target functions in $H^1_0(\Omega)$ approximated by elements from $S_0 := S_0^{0,0}(T) := S^0(T) \cap H^1_0(\Omega)$. Considering $u_\varepsilon = \min\{1, \varepsilon^{-1/2}\text{dist}(\partial\Omega)\}$ as in [2] reveals the following: if any local best error on a subdomain $\omega$ with positive $(d-1)$-dimensional Hausdorff measure $|\partial\omega \cap \partial\Omega| > 0$ does not incorporate the boundary condition, robustness cannot hold. This suggests the following modification of the setting for, e.g., Theorem 5.3. We associate to every $E \in \mathcal{E}$ the local space

$$S_E := \begin{cases} S_0|_{\omega(E)} & \text{if } E \in \mathcal{E}_S, \\ S|_{\omega(E)} & \text{otherwise}, \end{cases}$$

where $\mathcal{E}_S := \{E' \in \mathcal{E} : \omega(E') \text{ has a face on } \partial\Omega\}$. Notice that we have $S_E \neq S_0|_{\omega(E)}$ if and only if $E \not\in \mathcal{E}_S$ but $\omega(E) \cap \partial\Omega \neq \emptyset$. This however, at least, does not create a problem for the second inequality in Theorem 5.3 since $S_E \supset S_0|_{\omega(E)}$.

The interpolation operator $\Pi_0^E$ now has to vanish on the domain boundary $\partial\Omega$. To this end, the elegant approach of averaging on boundary faces in [8], which has been adopted in [10], cannot be applied because the use of traces does not allow for [6,3] and so for robustness. We therefore use the original approach of suppressing the boundary nodes in $E$, Clément [5] and set

$$\Pi_0^E u := \sum_{z \in N_0} u_z \phi_z$$

with $u_z$ as in (5.11), where $\mathcal{P}_{E,k} u$ is replaced by the best approximation $R_{0,E,k}^\varepsilon u$ in $S_{\omega(E,k)}$ to $u$ with respect to the reaction-diffusion norm. Consequently, $\Pi^E$ and $\Pi_0^E$ differ only at boundary nodes and at nodes invoking a face in $\mathcal{E}_S$.

Nevertheless the counterpart of (5.6) holds. To see this, consider $K \in \mathcal{T}$ such that $K \cap \partial\Omega$ is non-empty and notice that only the boundary nodes $z \in N_K \cap \partial\Omega$ are critical. If $K$ has a face on the domain boundary $\partial\Omega$, then the same holds for $\omega(E,E_K)$ and so we have $u_z = 0 = R_{0,K}^\varepsilon u(z)$ for all boundary nodes $z \in N_K \cap \partial\Omega$. If the intersection $K \cap \partial\Omega$ is only a $k$-face with $k < d - 1$, we can find a path $\{\omega(E_1)\}_{i=1}^n$ of pairs such that $\omega(E_1) \supset K$ and $\omega(E_n)$ has a face on $\partial\Omega$. Hence we can bound $|R_{0,E_n}^\varepsilon u(z)| \leq |R_{0,K}^\varepsilon u(z) - R_{0,E_n}^\varepsilon u(z)|$ as in the proof of Proposition 5.3 for all boundary nodes $z \in N_K \cap \partial\Omega$.

Inequality (5.12) hinges on a Poincaré-type inequality on pairs. If $E' \in \mathcal{E}_S$, then (5.13) may not be correct and the counterpart of (5.12) is built on the following Friedrichs’ inequality, which is a tailor-made variant of Lemma 5.1 in [11].

**Lemma 8.1** (Friedrichs’ inequality on element pairs). Let $\omega$ be the union of two adjacent elements $K_1, K_2$ sharing a face $E = K_1 \cap K_2$. Moreover let $E_0$ be a face of $K_1$. For every $v \in H^1(\omega)$ with $\int_{E_0} v = 0$, it holds

$$\|v\|_\omega \leq C_{F,\omega} h_\omega \|\nabla v\|_\omega,$$

where $h_\omega := \max\{\text{diam}K_1, \text{diam}K_2\}$ and $C_F$ depends on $d$ and the shape parameter of $\{K_1, K_2\}$.

**Proof.** Adding and subtracting the mean value over the common face $E$, we obtain

$$\|v\|_\omega \leq \left\| v - \frac{1}{|E|} \int_E v \right\|_\omega + |\omega|^{1/2} \frac{1}{|E|} \int_E |v|.$$
We treat the first term on the right-hand side as in the proof of Lemma 5.2. For the second term, we use twice the trace identity (5.3) to get
\[
\frac{1}{|E|} \int_E v = \frac{1}{d|K_1|} \int_{K_1} (z_{E_0} - z_E) \cdot \nabla v,
\]
where \(z_{E_0}\) and \(z_E\) are the vertices opposite to \(E_0\) and \(E\), respectively. Hence
\[
|\omega|^{1/2} \frac{1}{|E|} \int_E v \leq \left( |K_1|^{1/2} + |K_2|^{1/2} \right) \frac{\|z_{E_0} - z_E\|_{\nabla v}}{d|K_1|^{1/2}} \leq \left( \frac{\text{diam}(K_1)}{d} + \frac{|K_2|^{1/2}|z_{E_0} - z_E|}{d|K_1|^{1/2}} \right) \|\nabla v\|_{K_1}.
\]
Since
\[
\frac{|K_2|^{1/2}|z_{E_0} - z_E|}{|K_1|^{1/2}} \leq \frac{|E|^{1/2}\text{diam}(K_2)|z_{E_0} - z_E|}{|E|^{1/2}\text{dist}(z_E, E)^{1/2}} \leq \frac{|z_{E_0} - z_E|^{1/2}}{\text{dist}(z_E, E)^{1/2}} h_{\omega}
\]
we conclude
\[
\|v\|_{H^\omega} \leq \left( C_P + \frac{1}{d} + \frac{|z_{E_0} - z_E|^{1/2}}{\text{dist}(z_E, E)^{1/2}} \right) h_{\omega} \|\nabla v\|_{\omega}.
\]

We thus obtain the following variant of Theorem 5.3.

**Theorem 8.2 (Robust localization with boundary condition).** Using (8.1), there holds
\[
\inf_{v \in S_{\omega}(T)} \|u - v\|_{\Omega} \approx \left( \sum_{E \in \mathcal{E}} \inf_{P \in S_{\omega}(E)} \|u - P\|_{\omega(E)}^2 \right)^{1/2}
\]
for every \(u \in H_0^1(\Omega)\). The hidden constants depend only on the polynomial degree \(d\), the dimension \(d\), the shape parameter \(\sigma\), but not on \(\omega\).

There holds a similar theorem where the covering \(\{\omega(E)\}_{E \in E}\) of ‘normal’ pairs is replaced by the one \(\{\omega_\epsilon(E)\}_{E \in E}\) of minimal pairs.

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