NOTES ON FINITELY GENERATED FLAT MODULES

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Abstract. In this article, the projectivity of a finitely generated flat module of a commutative ring is studied through its exterior powers and invariant factors. Consequently, the related results of Endo, Vasconcelos, Wiegand, Cox-Rush and Puninski-Rothmaler on the projectivity of f.g. flat modules are generalized.

1. Introduction

The main purpose of the present article is to investigate the projectivity of finitely generated flat modules of a commutative ring. It is worthy to mention that this has been the main topic of many articles in the literature over the years and it is still of current interest, see e.g. [1], [2], [4], [6], [7], [8]. Note that in general there are f.g. flat modules which are not necessarily projective, see Example 3.17, also see [3, Tag 00NY] for another example. We use f.g. in place of “finitely generated”.

In this paper, the projectivity of a finitely generated flat module of a commutative ring is studied through its exterior powers and invariant factors. The important outcome of this study is that some major results in the literature on the projectivity of f.g. flat modules are re-proved directly (e.g. without using the homological methods) and at the same time most of them are vastly generalized. In particular, Theorem 3.10 vastly generalizes [2, Theorem 1], Theorem 3.12 generalizes [7, Theorem 2.1] and Theorem 3.13 generalizes [8, Theorem 2], [1, Proposition 2.3], it also generalizes [6, Proposition 5.5 and Corollary 5.6] in the commutative case. In fact, Theorem 3.13 can be viewed as a generalization of all of the above mentioned results.

The main motivation to investigate the projectivity of f.g. flat modules essentially originates from the fact that “every f.g. flat module over a local ring is free”. In this article we also prove a more general result, Theorem 3.13. This result, in particular, implies the above fact.
see Corollary 3.5

For reading the present article having a reasonable knowledge from the “exterior powers of a module” is necessary. In this article, all of the rings are commutative.

2. Preliminaries

Lemma 2.1. Let \( M \) be a f.g. \( R \)-module, let \( I = \text{Ann}_R(M) \) and let \( S \) be a multiplicative subset of \( R \). Then \( S^{-1}I = \text{Ann}_{S^{-1}R}(S^{-1}M) \).

Proof. Easy. □

It is well-known that if \( R \to S \) is a ring map and \( M \) is an \( R \)-module then \( \Lambda^n(M) \otimes_R S \) as \( S \)-module is canonically isomorphic to \( \Lambda^n_S(M \otimes_R S) \). It is also well-known that if \( M \) is a projective (resp. flat) \( R \)-module then for each natural number \( n \), \( \Lambda^n(M) \) is a projective (resp. flat) \( R \)-module. Finally, if \( M \) is a f.g. \( R \)-module then \( \Lambda^n(M) \) is a f.g. \( R \)-module. We shall use these facts freely throughout this article.

3. Main results

Lemma 3.1. The annihilator of a f.g. projective module is generated by an idempotent element.

Proof. Let \( M \) be a \( R \)-module, let \( I = \text{Ann}_R(M) \) and let \( J \) be the ideal of \( R \) which is generated by the elements \( f(m) \) where \( f : M \to R \) is a \( R \)-linear map and \( m \in M \). Clearly \( IJ = 0 \). Consider a free \( R \)-module \( F \) with basis \( \{e_i\} \) and an onto \( R \)-linear map \( \psi : F \to M \). For each \( i \) there is a \( R \)-linear map \( h_i : F \to R \) such that \( h_i(e_j) = \delta_{ij} \). If \( M \) is \( R \)-projective then there exists a \( R \)-linear map \( \varphi : M \to F \) such that \( \psi \circ \varphi \) is the identity map. Put \( f_i = h_i \circ \varphi \) for all \( i \). Then for each \( m \in M \) we may write \( m = \sum f_i(m)\psi(e_i) \) where \( f_i(m) = 0 \) for all but a finite number of indices \( i \). This implies that \( JM = M \). If moreover \( M \) is a f.g. \( R \)-module then, by [3, Tag 00DW], we may find an element \( b \in J \) such that \( 1 + b \in I \). Let \( a = 1 + b \). Then clearly \( a = a^2 \) and \( I = Ra \). □
Remark 3.2. In a flat module both of the scalars and vectors involved
in a linear relation have very peculiar properties. More precisely, let
\( M \) be a \( R \)-module and consider a linear relation \( \sum_{i=1}^{n} a_i x_i = 0 \) in \( M \)
where \( a_i \in R \) and \( x_i \in M \) for all \( i \). Let \( I = (a_1, \ldots, a_n) \) and consider
the map \( \psi : R^n \to I \) which maps each \( n \)-tuple \( (r_1, \ldots, r_n) \) of \( R^n \)
into \( \sum_{i=1}^{n} r_i a_i \). Then we get the following exact sequence of \( R \)-modules
\[
K \otimes_R M \xrightarrow{i \otimes 1} R^n \otimes_R M \xrightarrow{\psi \otimes 1} I \otimes_R M
\]
where \( K = \ker \psi \). If moreover \( M \) is \( R \)-flat then \( \sum_{i=1}^{n} \epsilon_i \otimes x_i \in \ker(\psi \otimes 1) \) where \( \epsilon_i = (\delta_{i,k})_{1 \leq k \leq n} \)
for all \( i \). Because by the flatness of \( M \), \( I \otimes_R M \) is canonically isomorphic
to \( IM \). Therefore there exist a natural number \( m \geq 1 \) and also
\( \epsilon_j \in M \) with \( \sum_{i=1}^{n} \epsilon_i \otimes x_i = 0 \) for all \( i \).
Moreover for each \( j \), \( \sum_{i=1}^{n} r_{i,j} a_i = 0 \) since \( \psi(s_j) = 0 \).

Under the light of Remark 3.2, the following result is obtained.

Theorem 3.3. Let \( (R, m) \) be a local ring and let \( M \) be a flat \( R \)-module.
Let \( S \) be a subset of \( M \) such that its image under the canonical map
\( M \to M/mM \) is linearly independent over \( k = R/m \). Then \( S \) is linearly independent over \( R \).

Proof. Suppose \( \sum_{i=1}^{n} a_i x_i = 0 \) where \( a_i \in R \) and \( \{x_1, \ldots, x_n\} \subseteq S \).
To prove the assertion we shall use an induction argument on \( n \). If \( n = 1 \) then by Remark 3.2
there are elements \( z_1, \ldots, z_d \in M \) and \( r_1, \ldots, r_d \in R \) such that \( x_1 = \sum_{j=1}^{d} r_j z_j \) and \( r_j a_1 = 0 \) for all \( j \). By the
hypotheses, \( x_1 \notin mM \). Therefore \( r_j \notin m \) for some \( j \). It follows that \( a_1 = 0 \). Now let \( n > 1 \). Again by Remark 3.2
there are elements \( y_1, \ldots, y_m \in M \) and \( r_{i,j} \in R \) such that \( x_i = \sum_{j=1}^{m} r_{i,j} y_j \) and \( \sum_{i=1}^{n} r_{i,j} a_i = 0 \)
for all \( i,j \). There is some \( j \) such that \( r_{n,j} \notin m \) since \( x_n \notin mM \). It
follows that \( a_n = \sum_{i=1}^{n-1} r_{n,i}^{-1} r_{i,j} a_i \). Then we get \( \sum_{i=1}^{n-1} a_i (x_i + r_{n,j}^{-1} r_{i,j} x_n) = 0 \).

Let \( c_i = r_{n,j}^{-1} r_{i,j} \). Note that the image of \( \{ x_i + c_i x_n : 1 \leq i \leq n - 1 \} \) under the canonical map \( M \to M/mM \) is linearly independent (because if \( \{ x_1, ..., x_n \} \) is a linearly independent subset of a module then \( \{ x_i + r_i x_n : 1 \leq i \leq n - 1 \} \) is also a linearly independent subset where the \( r_i \) are arbitrary scalars). Therefore, by the induction hypothesis, \( a_i = 0 \) for all \( 1 \leq i \leq n - 1 \). This also implies that \( a_n = 0 \). □

**Corollary 3.4.** Let \((R, m)\) be a local ring and let \( M \) be a flat \( R \)-module. Then there is a free \( R \)-submodule \( F \) of \( M \) such that \( M = F + mM \). In particular, if either \( M/F \) is finitely generated or the maximal ideal is nilpotent then \( M \) is a free \( R \)-module.

**Proof.** Every vector space has a basis. So let \( \{ x_i + mM : i \in I \} \) be a \( k \)-basis of \( M/mM \) where \( k = R/m \). By Theorem 3.3, \( F = \sum_{i \in I} Rx_i \) is a free \( R \)-module. Clearly \( M = F + mM \). If \( M/F \) is finitely generated then by the Nakayama lemma, \( M = F \). If \( m \) is nilpotent then there is a natural number \( n \geq 1 \) such that \( m^n = 0 \). It follows that \( M/F = m^n(M/F) = 0 \). □

As an immediate consequence of the above corollary we obtain the following result which plays a major role in this article:

**Corollary 3.5.** Every f.g. flat module over a local ring is free. □

As a first application of Corollary 3.5 we obtain:

**Lemma 3.6.** The annihilator of a f.g. flat module is an idempotent ideal.

**Proof.** Let \( M \) be a f.g. flat module over a ring \( R \). Let \( I = \text{Ann}_R(M) \). Let \( p \) be a prime ideal of \( R \). By Lemma 2.1, \( I_p = \text{Ann}_{R_p}(M_p) \). By Corollary 3.5, \( M_p \) is a free \( R_p \)-module. Therefore \( I_p \) is either the whole localization or the zero ideal. If \( I_p = 0 \) then \( (I^2)_p = 0 \) since \( I^2 \subseteq I \). But if \( I_p = R_p \) then \( I \) is not contained in \( p \). Thus we may choose some \( a \in I \setminus p \). Clearly \( a^2 \in I^2 \setminus p \) and so \( (I^2)_p = R_p \). Therefore \( I = I^2 \). □
If \( M \) is a \( R \)-module then the \( n \)-th invariant factor of \( M \), denoted by \( I_n(M) \), is defined as the annihilator of the \( n \)-th exterior power of \( M \). Therefore \( I_n(M) = \text{Ann}_R (\Lambda^n(M)) \). We have:

**Lemma 3.7.** The invariant factors of a f.g. flat module are idempotent ideals.

**Proof.** If \( M \) is a f.g. flat \( R \)-module then \( \Lambda^n(M) \) is as well. Thus, by Lemma 3.6, \( I_n(M) \) is an idempotent ideal. □

**Remark 3.8.** Let \( M \) be a f.g. flat \( R \)-module. Then Corollary 3.5 leads us to a function \( \psi : \text{Spec} \ R \to \mathbb{N} = \{0, 1, 2, \ldots\} \) which is defined as \( p \mapsto \text{rank}_R(M_p) \). It is called the rank map of \( M \). It is also easy to see that \( \text{Supp}(\Lambda^n(M)) = \{p \in \text{Spec}(R) : \text{rank}_R(M_p) \geq n\} \).

**Theorem 3.9.** Let \( M \) be a f.g. flat \( R \)-module. Then the following conditions are equivalent.

(i) \( M \) is \( R \)-projective.

(ii) The invariant factors of \( M \) are f.g. ideals.

(iii) The rank map of \( M \) is locally constant.

**Proof.** (i) \( \Rightarrow \) (ii): It is well-known that \( \Lambda^n(M) \) is a f.g. projective \( R \)-module and so by Lemma 3.1, \( I_n(M) \) is a principal ideal.

(ii) \( \Rightarrow \) (iii): It suffices to show that the rank map of \( M \) is Zariski continuous. By Lemma 3.7 and [3, Tag 00DW], there exists some \( a \in I_n(M) \) such that \((1-a)I_n(M) = 0\). Clearly \( a = a^2 \) and \( I_n(M) = Ra \). By Remark 3.8, \( \psi^{-1}(\{n\}) = \text{Supp}(N) \cap (\text{Spec}(R) \setminus \text{Supp} N') \) where \( N = \Lambda^n(M) \) and \( N' = \Lambda^{n+1}(M) \). But \( \text{Supp} N = D(1-a) \). Moreover, \( \text{Supp} N' = V(I_{n+1}(M)) \) since \( N' \) is a f.g. \( R \)-module. Therefore \( \psi^{-1}(\{n\}) \) is an open subset of \( \text{Spec} R \).

(iii) \( \Rightarrow \) (i): Apply Corollary 3.5 and [3, Tag 00NX], □

The following result vastly generalizes [2, Theorem 1].

**Theorem 3.10.** Let \( A \subseteq B \) be an extension of rings and let \( M \) be a f.g. flat \( A \)-module. If \( M \otimes_AB \) is \( B \)-projective then \( M \) is \( A \)-projective.

**Proof.** First we shall prove that \( I = \text{Ann}_A(M) \) is a principal ideal. Let \( L = \text{Ann}_B(N) \) where \( N = M \otimes_A B \). We claim that \( IB = L \). Let
$q$ be a prime ideal of $B$. Clearly $N$ is a f.g. $B$–module. Thus, by Corollary 3.5, $L_q$ is either the whole localization or the zero ideal. If $L_q = 0$ then $(IB)_q = 0$ since $IB \subseteq L$. But if $L_q = B_q$ then $L$ is not contained in $q$ and so $N_q = 0$. Again by Corollary 3.5, $I_p$ is either the whole localization or the zero ideal where $p = A \cap q$. If $I_p = 0$ then $M_p \neq 0$ and so, by Corollary 3.5, $M_p \otimes A_p B_q \neq 0$. But $M_p \otimes A_p B_q$ is isomorphic to $N_q$. This is a contradiction. Therefore $I_p = A_p$. It follows that $(IB)_q = B_q$. This establishes the claim. By Lemma 3.1 there is an idempotent $e \in B$ such that $IB = Be$. Let $J = B(1-e) \cap A$. Clearly $IJ = 0$. We have $I + J = A$. If not, then there exists a prime ideal $p$ of $A$ such that $I + J \subseteq p$. Thus, by Corollary 3.5 $I_p = 0$. Therefore the extension of $IB$ under the canonical map $B \to B \otimes_A A_p$ is zero. Thus there exists an element $s \in A \setminus p$ such that $se = 0$ and so $s = s(1-e)$. Hence $s \in J$. But this is a contradiction. Therefore $I + J = A$. It follows that there is an element $c \in J$ such that $c = c^2$ and $I = Ac$. Now, let $n \geq 1$. But $\Lambda^n(M)$ is a f.g. flat $A$–module. Moreover, $\Lambda^n(M) \otimes_A B$ is $B$–projective since it is canonically isomorphic to $\Lambda^n_B(M \otimes_A B)$. Thus, by what we have proved above, $I_n(M)$ is a principal ideal. Hence, by Theorem 3.9 $M$ is $A$–projective. □

Lemma 3.11. Let $M$ be a f.g. flat $R$–module and let $J$ be an ideal of $R$. Let $I = \operatorname{Ann}_R(M)$ and $L = \operatorname{Ann}_R(M/JM)$. Then $L = I + J$.

Proof. Clearly $I + J \subseteq L$. Let $p$ be a prime ideal of $R$. By Lemma 2.1 $I_p = \operatorname{Ann}_{R_p}(M_p)$. Thus, by Corollary 3.5, $I_p$ is either the whole localization or the null ideal for all primes $p$. If $I_p = R_p$ then $(I + J)_p = L_p = R_p$ since $I \subseteq I + J \subseteq L$. But if $I_p = 0$ then $M_p \neq 0$ and so $\operatorname{Ann}_{R_p}(M_p/J_pM_p) = J_p$ (recall that if $F$ is a non-zero free $R$–module then $\operatorname{Ann}_R(F/JF) = J$). On the other hand, by Lemma 2.1 $L_p = \operatorname{Ann}_{R_p}(M_p/J_pM_p)$. Thus $(I + J)_p = J_p = L_p$. Hence $L = I + J$. □

The following result generalizes [7, Theorem 2.1].

Theorem 3.12. Let $M$ be a f.g. flat $R$–module and let $J$ be an ideal of $R$ which is contained in the radical Jacobson of $R$. If $M/JM$ is $R/J$–projective then $M$ is $R$–projective.

Proof. First we shall prove that $I = \operatorname{Ann}_R(M)$ is a principal ideal. By Lemma 3.11 $L = I + J$. Also, by Lemma 3.11 $\operatorname{Ann}_{R/J}(M/JM) =$
L/J is a principal ideal. This implies that I = Rx + I ∩ J for some x ∈ R since L/J = I + J/J is canonically isomorphic to I/(I ∩ J). But I = Rx. Because let m be a maximal ideal of R. By Corollary 3.5, I_m is either the whole localization or the zero ideal. If I_m = 0 then (Rx)_m = 0 since Rx ⊆ I. But if I_m = R_m then I is not contained in m. Thus Rx is also not contained in m since I ∩ J ⊆ J ⊆ m. Hence (Rx)_m = R_m. Therefore I = Rx. Now let n ≥ 1 and let N = \Lambda^n(M). Then N/JN is R/J-projective. Because, N/JN as R/J-module is isomorphic to \Lambda^n(R/J)(M/JM) and \Lambda^n(R/J)(M/JM) is R/J-projective. But N is a f.g. flat R-module. Therefore, by what we have proved above, I_n(M) = Ann_R(N) is a principal ideal. Thus the invariant factors of M are f.g. ideals and so by Theorem 3.9, M is R-projective. □

A ring R is called an S-ring ("S" refers to Sakhajev) if every f.g. flat R-module is R-projective.

**Theorem 3.13.** Let ϕ : A → B be a ring map whose kernel is contained in the radical Jacobson of A. If M is a f.g. flat A-module such that M ⊗_A B is B-projective then M is A-projective. In particular, if B is an S-ring then A is as well.

**Proof.** Clearly M/JM is a f.g. flat A/J-module and M/JM ⊗_A/J B ≃ M ⊗_A B is B-projective where J = Ker ϕ. Moreover A/J can be viewed as a subring of B via ϕ. Therefore, by Theorem 3.10, M/JM is A/J-projective. Then apply Theorem 3.12. Finally, assume that B is an S-ring. If M is a f.g. flat A-module then M ⊗_A B is a f.g. flat B-module and so, by the hypothesis, it is B-projective. Therefore M is A-projective. □

**Remark 3.14.** Let S be a subset of a ring R. The polynomial ring R[x_s : s ∈ S] modulo I is denoted by S^{(-1)}R where the ideal I is generated by elements of the form sx_s^2 - x_s and s^2x_s - s with s ∈ S. We call S^{(-1)}R the pointwise localization of R with respect to S. Amongst them, the pointwise localization of R with respect to itself, namely R^{(-1)}R, has more interesting properties; for further information please consult with [5]. Note that Wiegand [3] utilizes the notation \(\hat{R}\) instead of R^{(-1)}R. Clearly η(s) = η(s)^2(x_s + I) and x_s + I = η(s)(x_s + I)^2 where η : R → S^{(-1)}R is the canonical map and the pair (S^{(-1)}R, η) satisfies in the following universal property: “for each such pair (A, ϕ) where ϕ : R → A is a ring map and for each s ∈ S there is some c ∈ A
such that \( \varphi(s) = \varphi(s)^2 c \) and \( c = \varphi(s)c^2 \) then there exists a unique ring map \( \psi : S^{(-1)}R \to A \) such that \( \varphi = \psi \circ \eta \).” Now let \( p \) be a prime ideal of \( R \) and consider the canonical map \( \pi : R \to \kappa(p) \) where \( \kappa(p) \) is the residue field of \( R \) at \( p \). By the above universal property, there is a (unique) ring map \( \psi : S^{(-1)}R \to \kappa(p) \) such that \( \pi = \psi \circ \eta \). Thus \( \eta \) induces a surjection between the corresponding spectra. This, in particular, implies that the kernel of \( \eta \) is contained in the nil-radical of \( R \). Using this, then the following result vastly generalizes [8, Theorem 2].

**Corollary 3.15.** Let \( M \) be a f.g. flat \( R \)-module. If there exists a subset \( S \) of \( R \) such that \( M \otimes_R S^{(-1)}R \) is \( S^{(-1)}R \)-projective then \( M \) is \( R \)-projective.

**Proof.** It is an immediate consequence of Theorem 3.13. \( \square \)

**Proposition 3.16.** Let \( I \) be an ideal of a ring \( R \). Then \( R/I \) is \( R \)-flat if and only if \( \text{Ann}_R(f) + I = R \) for all \( f \in I \).

**Proof.** First assume that \( R/I \) is \( R \)-flat. Suppose there is some \( f \in I \) such that \( \text{Ann}_R(f) + I \neq R \). Thus there exists a prime \( p \) of \( R \) such that \( \text{Ann}_R(f) + I \subseteq p \). Therefore, by Corollary 3.5 \( I_p = 0 \) and so there is an element \( s \in R \setminus p \) such that \( sf = 0 \). But this is a contradiction and we win. Conversely, let \( \varphi : M \to N \) be an injective \( R \)-linear map. To prove the assertion it suffices to show that the induced map \( M/IM \to N/IN \) given by \( m + IM \mapsto \varphi(m) + IN \) is injective. If \( \varphi(m) \in IN \) then we may write \( \varphi(m) = \sum_{i=1}^{n} f_i x_i \) where \( f_i \in I \) and \( x_i \in N \) for all \( i \). By the hypothesis, there are elements \( b_i \in \text{Ann}_R(f_i) \) and \( c_i \in I \) such that \( 1 = b_i + c_i \). It follows that \( 1 = (b_1 + c_1)(b_2 + c_2)...(b_n + c_n) = b + c \) where \( b = b_1b_2...b_n \) and \( c \in I \). Thus \( \varphi(m) = b\varphi(m) + c\varphi(m) = \varphi(cm) \). Therefore \( m = cm \in IM \). \( \square \)

As a final result in the following we give an example of a f.g. flat module which is not projective. Note that finding explicit examples of f.g. flat modules but not projective is not as easy as one may think at first.

**Example 3.17.** Let \( R = \prod_{i \geq 1} A \) be an infinite direct product of copies of a non-zero ring \( A \) and let \( I = \bigoplus_{i \geq 1} A \) which is an ideal of \( R \). Let \( f = (f_i) \)
be an element of \( I \). There exists a finite subset \( D \) of \( \{1, 2, 3, \ldots\} \) such that \( f_i = 0 \) for all \( i \in \{1, 2, 3, \ldots\} \setminus D \). Now consider the sequences \( g = (g_i) \) and \( h = (h_i) \) of elements of \( R \) with \( g_i = 0 \) and \( h_i = 1 \) for all \( i \in D \) and \( g_i = 1 \) and \( h_i = 0 \) for all \( i \in \{1, 2, 3, \ldots\} \setminus D \). Clearly \( g \in \text{Ann}_R(f) \), \( h \in I \) and \( 1_R = g + h \). Thus, by Proposition 3.16, \( R/I \) is \( R \)–flat. Suppose \( R/I \) is \( R \)–projective. Then, by Lemma 3.1, there is a sequence \( e = (e_i) \in R \) such that \( I = Re \). Thus there exists a finite subset \( E \) of \( \{1, 2, 3, \ldots\} \) such that \( e_i = 0 \) for all \( i \in \{1, 2, 3, \ldots\} \setminus E \). Clearly \( \{1, 2, 3, \ldots\} \setminus E \neq \emptyset \). Pick some \( k \in \{1, 2, 3, \ldots\} \setminus E \). There is some \( r = (r_i) \in R \) such that \( (\delta_{i,k})_{i \geq 1} = re \) where \( \delta_{i,k} \) is the Kronecker delta. In particular, \( 1_A = r_k e_k = r_k 0_A = 0_A \). This is a contradiction. Therefore \( R/I \) is not \( R \)–projective.

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