ONE LEVEL DENSITY OF LOW-LYING ZEROS OF QUADRATIC HECKE $L$-FUNCTIONS TO PRIME MODULI

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Abstract. In this paper, we study the one level density of low-lying zeros of a family of quadratic Hecke $L$-functions to prime moduli over the Gaussian field under the generalized Riemann hypothesis (GRH) and the ratios conjecture. As a corollary, we deduce that at least 75% of the members of this family do not vanish at the central point under GRH.

Mathematics Subject Classification (2010): 11M06, 11M26, 11M50

Keywords: low-lying zeros, one level density, quadratic Hecke $L$-function

1. Introduction

Understanding the behavior of the low-lying zeros of families of $L$-functions has significant applications in problems such as determining the size of the Mordell-Weil groups of elliptic curves and the size of class numbers of imaginary quadratic number fields. For this reason, much work (see [2, 3, 8, 9, 12, 19, 21–23, 25, 28, 32, 34, 37]) has been done towards the density conjecture of N. Katz and P. Sarnak [26, 27], which relates the distribution of zeros near the central point of a family of $L$-functions to the eigenvalues near 1 of a corresponding classical compact group.

In this paper, we focus on $L$-functions attached to quadratic characters. For the family of quadratic Dirichlet $L$-functions, this is initiated by A. E. Özlük and C. Snyder in [31], who studied the 1-level density of low-lying zeros of the family. Subsequent investigations were carried out in [13, 29, 35], in which the cardinalities of families considered all have positive densities in the set of all such $L$-functions. For these families, the density conjecture is verified when the Fourier transforms of the test functions are supported in $(-2, 2)$ if one assumes the Generalized Riemann Hypothesis (GRH) and the underlying symmetric of such families is unitary symplectic (USp).

Recently, J. C. Andrade and S. Baluyot [1] studied the 1-level densities of quadratic Dirichlet $L$-functions over prime modulus. This is a sparse family in the sense that its cardinality has density 0 in the set of all such $L$-functions. It is shown in [1] that the symmetric of this family is also USp and it also verifies the density conjecture when the Fourier transforms of the test functions are supported in $(-1, 1)$ under GRH.

It is then interesting to study the 1-level density of various families of $L$-functions of sparse sets. Motivated by the above result of Andrade and Baluyot, we investigate in this paper the 1-level density of quadratic Hecke $L$-functions in the Gaussian field $\mathbb{Q}(i)$ over prime modulus. Previous, we studied in [17] the same family but over a set of positive density in the set of all such $L$-functions.

Throughout the paper, we denote $K = \mathbb{Q}(i)$ for the Gaussian field and $\mathcal{O}_K = \mathbb{Z}[i]$ for the ring of integers in $K$. Note that in $\mathcal{O}_K$, every ideal co-prime to 2 has a unique generator congruent to 1 modulo $(1+i)^3$. Such a generator is called primary. We shall reserve the letters $\wp$ and $\varpi$ for primary primes in $K$.

Recall that the quadratic symbol $(\frac{\cdot}{\varpi})$ is defined in [17] Sect. 2.1 and we shall denote $\chi_n$ for $(\frac{n}{\varpi})$. It is also shown in [15] Section 2.1 that the symbol $\chi_{(1+i)^5c}$ defines a primitive quadratic Hecke character modulo $(1+i)^5$ of trivial infinite type when $c \in \mathcal{O}_K$ is odd and square-free. Here we recall that a Hecke character $\chi$ of $K$ is said to be of trivial infinite type if its component at infinite places of $K$ is trivial and we say that any $c \in \mathcal{O}_K$ is odd if $(c, 2) = 1$ and $c$ is square-free if the ideal $(c)$ is not divisible by the square of any prime ideal.

In what follows, we reserve the symbol $\varpi$ for primes, which means that $(\varpi)$ is a prime ideal in $\mathcal{O}_K$. We would like to consider the family of $L$-functions consisting of $L(s, \chi_{\varpi})$ for $\varpi$ being prime. Even though this is a natural choice, we consider instead in this paper the family

$F = \{L(s, \chi_{(1+i)^5\varpi}) : \varpi \text{ primary}\}$

(1.1)
since the modulus of $\chi_{(1+i)^5\varpi}$ is easier to describe. We remark here that our treatment for the above family certainly carries over to the family $\{L(s, \chi_{\varpi})\}$ as well.

Let $\chi = \chi_n$ for some $n \in \mathcal{O}_K$, we denote $L(s, \chi)$ for the corresponding Hecke $L$-function and the non-trivial zeroes of $L(s, \chi)$ by $\frac{1}{2} + i\gamma_{\chi,j}$. Without assuming GRH (so that $\gamma_{\chi,j}$ is not necessarily real), we order them as

$$\ldots \leq \Re\gamma_{\chi,-2} \leq \Re\gamma_{\chi,-1} < 0 \leq \Re\gamma_{\chi,1} \leq \Re\gamma_{\chi,2} \leq \ldots .$$

We further normalize the zeros by letting

$$\hat{\gamma}_{\chi,j} = \frac{2\gamma_{\chi,j}}{2\pi} \log X$$

and define, for an even Schwartz class function $\phi$, the 1-level density for the single $L$-function $L(s, \chi)$ as the sum

$$S(\chi, \phi) = \sum_j \phi(\hat{\gamma}_{\chi,j}).$$

We remark here that the kernel of the integral

$$\lim_{X \to +\infty} D(\phi; w, X) = \int_{\mathbb{R}} \phi(x) W_{USp}(x) dx,$$

where

$$W_{USp}(x) = \sum_{\varpi \equiv 1 \mod (1+i)^3} w \left( \frac{N(\varpi)}{X} \right).$$

We point out here that our formulation of the 1-level density is the more commonly used one in the literature, while in \[1\], the 1-level density is formed using a form factor, as initially used by "Ozluk and Snyder in \[31\]. Now we state our result on the one level density as follows.

**Theorem 1.1.** Assuming GRH. Let $w(t)$ be an even, non-zero and non-negative Schwartz function and $\phi(x)$ an even Schwartz function whose Fourier transform $\hat{\phi}(u)$ has compact support in $(-1, 1)$, then

$$\lim_{X \to +\infty} D(\phi; w, X) = \int_{\mathbb{R}} \phi(x) W_{USp}(x) dx, \quad \text{where } W_{USp}(x) = 1 - \frac{\sin(2\pi x)}{2\pi x}.$$
Lemma 2.2. Let $L$ sum over zeros of an $\mathcal{L}$-function to a sum over primes. Moreover, as $x \to \infty$,

$$(1.6) \quad D(\phi; w, X) = \frac{1}{W(X)} \sum_{\varpi \equiv 1 \mod (1+i)^3} w \left( \frac{N(\varpi)}{X} \right) \mathcal{I}(\varpi) + O_x \left( X^{-1/2+\epsilon} \right) ,$$

where

$$\mathcal{I}(\varpi) = \frac{1}{2\pi} \int_{\mathbb{R}} \left( 2 \beta c_{1}(1+2it) \zeta_{K}(1+2it) + 2A_{\alpha}(it, it) + \log \left( \frac{32N(\varpi)}{\pi^2} \right) + \frac{\Gamma'}{\Gamma} \left( \frac{1}{2} - it \right) + \frac{\Gamma'}{\Gamma} \left( \frac{1}{2} + it \right) \right) \left( \frac{1}{2} + it \right) \zeta_{K}(1-2it)A(-it, it) \phi \left( \frac{t \log X}{2\pi} \right) dt.$$

One certainly expects that the result given in Theorem 1.2 implies the assertion of Theorem 1.1. We show this indeed is the case in Section 3 by adapting an approach in [1].

One important application of the density conjecture is to address the non-vanishing issues of families of $L$-functions at the central point. It is a conjecture that goes back to S. Chowla [5] that $L(1/2, \chi) \neq 0$ for all primitive Dirichlet character $\chi$. It is shown in [11] Theorem 3] that at least 75% of the family of quadratic Dirichlet $L$-functions to prime moduli do not vanish at the central point. analogue to this, the following corollary shows that exactly the same percentage of the family of quadratic Hecke $L$-functions to prime moduli do not vanish at the central point.

Corollary 1.3. Assuming GRH and that $1/2$ is a zero of $L(s, \chi(1+i)^s)$ of order $m_\varpi \geq 0$. As $X \to \infty$,

$$\sum_{\varpi \equiv 1 \mod (1+i)^3} m_\varpi \phi \left( \frac{N(\varpi)}{X} \right) \leq \left( \frac{1}{4} + o(1) \right) W(X).$$

Moreover, as $X \to \infty$

$$\# \{ N(\varpi) \leq X : L\left(1/2, \chi(1+i)^s \varpi \right) \neq 0 \} \geq \left( \frac{3}{4} + o(1) \right) \frac{X}{\log X} .$$

As the proof is standard (see that of [4 Corollary 2.1]), we omit it here by only pointing out here that it follows from Lemma 2.5 below that $W(X) \sim \tilde{w}(1) \frac{X}{\log X}$ when $X \to \infty$ with $\tilde{w}(s)$ being the Mellin transform of $w$ defined in (2.3).

2. Preliminaries

2.1. The Explicit Formula. Our proof of Theorem 1.1 starts with the following explicit formula, which converts a sum over zeros of an $L$-function to a sum over primes.

Lemma 2.2. Let $\phi(x)$ be an even Schwartz function whose Fourier transform $\hat{\phi}(u)$ is compactly supported. For any square-free $c \in \mathcal{O}_K$, $N(c) \leq X$, we have

$$S(\chi_c, \phi) = \hat{\phi}(0) \frac{\log N(c)}{\log X} - \frac{1}{2} \int_{-\infty}^{\infty} \hat{\phi}(u) du - 2S(\chi_c, X; \hat{\phi}) + O \left( \frac{\log \log 3X}{\log X} \right) ,$$

with the implicit constant depending on $\phi$. Here

$$S(\chi_c, X; \hat{\phi}) = \sum_{\varpi \equiv 1 \mod (1+i)^3 \atop N(\varpi) \leq X} \chi_c(\varpi) \log N(\varpi) \frac{\hat{\phi} \left( \frac{\log N(\varpi)}{\log X} \right)}{\sqrt{N(\varpi)}} .$$

We omit the proof of Lemma 2.2 here since it is standard and follows by combining the proof of [20 Lemma 4.1] and [17 Lemma 2.4].

2.3. Conditional Estimations on GRH. In this section, we include two lemmas that are obtained by assuming the truth of GRH. The first is about sums over primes.

Lemma 2.4. Assuming the truth of GRH. For any Hecke character $\chi$ (mod $m$) of trivial infinite type, we have for $X \geq 1$, $N(\varpi) \leq X$,

$$(2.1) \quad S(X, \chi) = \sum_{\varpi \equiv 1 \mod (1+i)^3 \atop N(\varpi) \leq X} \chi(\varpi) \log N(\varpi) = \delta_X X + O \left( X^{1/2} \log^2(2X) \log N(m) \right) ,$$

where $\delta_X$ is the number of real primes $\leq X$.
where \( \delta_{\chi} = 1 \) if and only if \( \chi \) is principal and \( \delta_{\chi} = 0 \) otherwise. Moreover, we have

\[
(2.2) \quad \sum_{\omega \equiv 1 \mod (1+i)^3} \frac{\log N(\omega)}{N(\omega)} = \log X + O(1).
\]

Proof. The formula in (2.2) follows directly from [24, Theorem 5.15] and (2.5) follows from (2.2) by taking \( \chi \) to be the principal character and using partial summation. \( \square \)

We recall that the Mellin transform of \( w \) is given by

\[
(2.3) \quad \widetilde{w}(s) = \int_0^\infty w(t)t^s \frac{dt}{t}.
\]

Integrating by parts implies that for \( \Re(s) < 1 \) and any integer \( \nu \geq 1 \),

\[
(2.4) \quad \widetilde{w}(s) \ll \frac{1}{|s||s-1|^\nu}.
\]

Let \( \zeta_K(s) \) denote the Dedekind zeta function of \( K \) and \( \Lambda_{[i]}(n) \) for the von Mangoldt function on \( K \), which is the coefficient of \( N(n)^{-s} \) in the Dirichlet series of \( \zeta_K(n)/\zeta_K(s) \). Our next result gives estimations on various sums needed in this paper.

**Lemma 2.5.** Assuming GRH. For any even, non-zero and non-negative Schwartz function \( w \), let \( W(X) \) be given as in (1.3) for \( X \geq 1 \). Let \( z \in \mathbb{C} \) such that \( |z| \leq 1 \) and that \( 0 \leq \Re(z) \leq \frac{1}{2} \). Then we have for any \( \varepsilon > 0 \),

\[
(2.5) \quad \sum_{\omega \equiv 1 \mod (1+i)^3} w\left(\frac{N(\omega)}{X}\right) \log N(\omega) = \widetilde{w}(1)X + O\left(X^{1/2+\varepsilon}\right),
\]

\[
(2.6) \quad W(x) = \widetilde{w}(1)X \log X + O\left(\frac{X}{(\log X)^2}\right)
\]

and

\[
(2.7) \quad \frac{1}{W(X)} \sum_{\omega \equiv 1 \mod (1+i)^3} w\left(\frac{N(\omega)}{X}\right) N(\omega)^{-z} = X^{-z} + O\left(|z|^2 \log X + \frac{1}{\log X}\right).
\]

Proof. We prove (2.5) first. Due to rapid decay of \( w \) given in (2.4), we have that

\[
\sum_{\omega \equiv 1 \mod (1+i)^3} w\left(\frac{N(\omega)}{X}\right) \log N(\omega) = \sum_{(n)} w\left(\frac{N(n)}{X}\right) \Lambda_{[i]}(n) + O\left(\sum_{(\omega)} w\left(\frac{N(\omega)}{X}\right) \log N(\omega)\right),
\]

where we write \( \sum_{(n)} \) and \( \sum_{(\omega)} \) for the sum over non-zero integral and prime ideals of \( \mathcal{O}_K \), respectively.

Note that

\[
(2.8) \quad \sum_{(\omega)} w\left(\frac{N(\omega)}{X}\right) \log N(\omega) \ll X^\varepsilon \quad \sum_{(\omega)} \log N(\omega) \ll X^{1/2+\varepsilon}.
\]

Now we apply Mellin inversion to get

\[
\sum_{(n)} w\left(\frac{N(n)}{X}\right) \Lambda_{[i]}(n) = -\frac{1}{2\pi i} \int_{(2)} \frac{\zeta_K'(s)}{\zeta_K(s)} \widetilde{w}(s)X^s ds.
\]

We evaluate the above integral by shifting the line of integration to \( \Re(s) = 1/2 + \varepsilon \). The only pole we encounter is at \( s = 1 \) with residue \( \widetilde{w}(1)X \). The integration on \( \Re(s) = 1/2 + \varepsilon \) can be estimated to be \( O(X^{1/2+\varepsilon}) \) using (2.2) and the following estimation (see [24, Theorem 5.17]) for \( \zeta_K'(s)/\zeta_K(s) \) when \( \Re(s) \geq 1/2 + \varepsilon \):

\[
\frac{\zeta_K'(s)}{\zeta_K(s)} \ll \log(1 + |s|).
\]

The expression given in (2.5) now follows.
We note that (2.4) is obtained from (2.3) via partial summation. It therefore remains to establish (2.7). For this, we set

\[ f(z) = \sum_{\varpi \equiv 1 \mod (1+i)^3} w \left( \frac{N(\varpi)}{X} \right) N(\varpi)^{-z}. \]

Then we have

\[ f'(z) = - \sum_{\varpi \equiv 1 \mod (1+i)^3} w \left( \frac{N(\varpi)}{X} \right) (\log N(\varpi))N(\varpi)^{-z} = - \sum_{(n)} w \left( \frac{N(n)}{X} \right) \Lambda[i_1](n)N(n)^{-z} + O(X^{1/2+\varepsilon}), \]

where the last estimation above follows from (2.8).

Now we apply Mellin inversion to get

\[ \sum_{(n)} w \left( \frac{N(n)}{X} \right) \Lambda[i_1](n)N(n)^{-z} = - \frac{1}{2\pi i} \int_{(2)} \frac{\zeta'(s+z)}{\zeta(s+z)} \overline{w}(s)X^s \, ds. \]

We evaluate the above integral by shifting the line of integration to \( \Re(s) = 1/2 - \Re(z) + \varepsilon \). The only pole we encounter is at \( s = 1 - z \) with residue \( \overline{w}(1-z)X^{1-z} \). Thus we obtain that

\[ f'(z) = \overline{w}(1-z)X^{1-z} + O \left( X^{1/2+\varepsilon} \right) = \overline{w}(1)X^{1-z} + O \left( |z|X + X^{1/2+\varepsilon} \right). \]

It follows that

\[ f(z) - f(0) = \int_0^z f'(v) \, dv = \overline{w}(1)X^{1-z} - \overline{w}(1) \frac{X}{\log X} + O \left( |z|^2 X + X^{1/2+\varepsilon} \right). \]

Note that we have \( f(0) = W(X) \) so that we deduce from the above and (2.6) that

\[ f(z) = \overline{w}(1)X^{1-z} + O \left( |z|^2 X + \frac{X}{(\log X)^2} + X^{1/2+\varepsilon} \right). \]

Combining this with (2.6) again for the evaluation of \( W(X) \), we readily deduce (2.7) and this completes the proof of the lemma. \( \square \)

2.6. The approximate functional equation. Let \( \chi \) be a primitive quadratic Hecke character modulo \( m \) of trivial infinite type defined on \( \mathcal{O}_K \). As shown by E. Hecke, \( L(s, \chi) \) admits analytic continuation to an entire function and satisfies the functional equation (24, Theorem 3.8)

\[ \Lambda(s, \chi) = W(\chi)(N(m))^{-1/2} \Lambda(1-s, \chi), \]

where \( |W(\chi)| = (N(m))^{1/2} \) and

\[ \Lambda(s, \chi) = (|D_K|N(m))^{s/2}(2\pi)^{-s}\Gamma(s)L(s, \chi). \]

For \( s \in \mathbb{C} \), we evaluate the integral

\[ \frac{1}{2\pi i} \int_{(2)} \Lambda(u+s, \chi)G(u)x^u \frac{du}{u} \]

by shifting the line of integration to \(-2\) and proceed in a similar manner as [14, Section 2.5], we get that if

\[ W(\chi) = N(m)^{1/2}, \]

then for any \( x > 1 \),

\[ L(s, \chi) = \sum_{0 \neq A \subset \mathcal{O}_K} \frac{\chi(A)}{N(A)^s} V_s \left( \frac{2\pi N(A)}{x} \right) \]

(2.11)

\[ + \left( \frac{(2\pi)^2}{|D_K|N(m)} \right)^{s-1/2} \frac{\Gamma(1-s)}{\Gamma(s)} \sum_{0 \neq A \subset \mathcal{O}_K} \frac{\chi(A)}{N(A)^{1-s}} V_{1-s} \left( \frac{2\pi N(A)}{|D_K|N(m)} \right), \]

where

\[ V_s(x) = \frac{1}{2\pi i} \int_{(2)} \frac{\Gamma(s+u)}{\Gamma(s)} x^{-u} \frac{du}{u}. \]

(2.12)
We note that it is shown in [16] Lemma 2.2] that \( \frac{\log N(x)}{\log X} \) holds if \( \chi = \chi_{(1+i)^c} \) for any odd, square-free \( c \in \mathcal{O}_K \). Thus the approximate functional equation \( (2.11) \) is valid for \( L(s, \chi_{(1+i)^c}) \).

3. Proof of Theorem 1.1

Note that \( \hat{\phi}(u) \) is smooth with support contained in \((-1+\varepsilon, 1-\varepsilon)\) for some \( 0 < \varepsilon < 1 \). We set \( Y = X^{1-\varepsilon} \) so that \( \hat{\phi}(\log N(x)/\log X) \neq 0 \) only when \( N(x) \leq Y \). Now we apply Lemma \( (2.2) \) to sum \( S(\chi_{(1+i)^c} \chi x, X; \hat{\phi}) \) over the primary primes \( x \) against the weight function \( w \) to arrive at

\[
D(\phi; w, X) = \frac{\hat{\phi}(0)}{W(X) \log X} \sum_{\chi = 1 \mod (1+i)^3} w \left( \frac{N(x)}{X} \right) \log N(x)
\]

\[
\approx \frac{1}{2} \int_{-\infty}^{\infty} \hat{\phi}(u) du - \frac{2}{W(X)} S(X, Y; \phi, w) + O \left( \frac{\log \log 3}{\log X} \right).
\]

where the last estimation follows from \( (2.5), (2.6) \) and \( \phi \) against the weight function \( w \) to arrive at

\[
S(X, Y; \phi, w) = \sum_{\chi = 1 \mod (1+i)^3} \sum_{x = 1 \mod (1+i)^3} \frac{\chi_{(1+i)^c}(\chi x)}{\sqrt{N(\chi x)}} \hat{\phi} \left( \frac{\log N(\chi x)}{\log X} \right) \frac{N(x)}{X} \chi(\chi x) w \left( \frac{N(x)}{X} \right).
\]

We note that by the quadratic reciprocity \( (17, (2.1)) \) for Hecke characters, we have \( \chi(\chi x) = \chi(\chi x) \) when \( \chi x \) are both primary. As it is shown in \( [16, \text{Sect. 2.1}] \) that \( \chi(\chi x) \) is a Hecke character modulo \( 16\chi x \) of trivial infinite type, we can apply \( (2.1) \) and partial summation by noting that \( \log N(\chi x) \ll \log X \) to see that

\[
\sum_{\chi = 1 \mod (1+i)^3} \chi(\chi x) w \left( \frac{N(x)}{X} \right) \ll X^{1/2+\varepsilon}.
\]

It follows that

\[
S(X, Y; \phi, w) \ll \sum_{\chi = 1 \mod (1+i)^3} \frac{\log N(\chi x) \chi(\chi x)}{\sqrt{N(\chi x)}} \hat{\phi} \left( \frac{\log N(\chi x)}{\log X} \right) \sum_{\chi = 1 \mod (1+i)^3} \chi(\chi x) w \left( \frac{N(x)}{X} \right)
\]

\[
\ll X^{1/2+\varepsilon} \sum_{\chi = 1 \mod (1+i)^3} \frac{\log N(\chi x)}{\sqrt{N(\chi x)}} \ll X^{1/2+\varepsilon} X^{1/2+\varepsilon},
\]

where the last estimation follows from \( (2.2) \) and partial summation.

We then deduce that when \( Y = X^{1-\varepsilon} \), \( S(X, Y; \phi, w) = O(X^{1-\varepsilon}) = o(W(x)) \) by Lemma \( (2.5) \) By taking \( X \to \infty \) on both sides of \( (4.1) \), we obtain \( (4.4) \) and this completes the proof of Theorem \( 1.1 \).
where \( xy = N((1 + i)^5c) \) and

\[
X_\varepsilon(s) = \frac{\Gamma(1 - s)}{\Gamma(s)} \left( \frac{\pi^2}{32N(c)} \right)^{s-1/2}.
\]

(4.3)

On the other hand, by writing \( \mu_{[i]} \) for the M"obius function on \( K \), we have for \( \Re(s) > 1 \),

\[
\frac{1}{L(s, \chi_{(1+i)^5c})} = \sum_{m \neq 0} \mu_{[i]}(m) \chi_{(1+i)^5c}(m) N(m)^s,
\]

(4.4)

where the summation above is over all non-zero ideals \( m \) of \( O_K \).

Substituting both (4.2) and (4.4) into (4.1), we see that

\[
\left( 1 + O_\varepsilon \right) \sum_{n \equiv 1 \mod (1+i) \gamma} w \left( \frac{N(n)}{\chi_{(1+i)^5c}(nm)} \right) 1_{\gamma} = 1 \quad \text{equals} \quad \sum_{n \equiv 1 \mod (1+i) \gamma} w \left( \frac{N(n)}{\chi_{(1+i)^5c}(nm)} \right) 1_{\gamma} \sim 1. \]

We now summarize our discussions above in the following version of the ratios conjecture.

Similarly, we have

\[
\left( 1 + O_\varepsilon \right) \sum_{n \equiv 1 \mod (1+i) \gamma} w \left( \frac{N(n)}{\chi_{(1+i)^5c}(nm)} \right) 1_{\gamma} \sim 1.
\]

It follows that we have

\[
R_1(\alpha, \gamma) \sim \tilde{R}_1(\alpha, \gamma) = \sum_{nm = \text{odd } \square} \frac{\mu_{[i]}(m)}{N(m)^{1/2+\gamma}N(n)^{1/2+\alpha}},
\]

where we denote \( \square \) for a perfect square. We deduce by computing Euler products that

\[
\tilde{R}_1(\alpha, \gamma) = \frac{B_K(1 + 2\alpha)}{B_K(1 + \alpha + \gamma)} A(\alpha, \gamma),
\]

where \( A(\alpha, \gamma) \) is given in (1.5). Similarly, we have

\[
R_2(\alpha, \gamma) \sim \tilde{R}_2(\alpha, \gamma) = \frac{1}{W(X)} \sum_{n \equiv 1 \mod (1+i) \gamma} w \left( \frac{N(n)}{\chi_{(1+i)^5c}(nm)} \right) \tilde{1}_{\gamma}(\alpha, \gamma).
\]

We now summarize our discussions above in the following version of the ratios conjecture.

**Conjecture 4.2.** Let \( \varepsilon > 0 \) and let \( w \) be an even and nonnegative Schwartz test function on \( \Re \) which is not identically zero. Suppose that the complex numbers \( \alpha \) and \( \gamma \) satisfy \( |\Re(\alpha)| < 1/4, (\log X)^{-1} \ll \Re(\gamma) < 1/4 \) and \( S(\alpha), S(\gamma) \ll X^{1-\varepsilon} \). Then we have that

\[
\frac{1}{W(X)} \sum_{n \equiv 1 \mod (1+i) \gamma} w \left( \frac{N(n)}{\chi_{(1+i)^5c}(nm)} \right) L(1/2 + \alpha, \chi_{(1+i)^5c}(nm)) L(1/2 + \gamma, \chi_{(1+i)^5c}(nm)) = \frac{B_K(1 + 2\alpha)}{B_K(1 + \alpha + \gamma)} A(\alpha, \gamma)
\]

\[
+ \frac{1}{W(X)} \sum_{n \equiv 1 \mod (1+i) \gamma} w \left( \frac{N(n)}{\chi_{(1+i)^5c}(nm)} \right) X_{\varepsilon} \left( \frac{1}{2} + \alpha \right) \frac{B_K(1 - 2\alpha)}{B_K(1 - \alpha + \gamma)} A(-\alpha, \gamma) + O_\varepsilon(X^{-1/2+\varepsilon}),
\]

where \( A(\alpha, \gamma) \) is defined in (1.5) and \( X_{\varepsilon}(s) \) is defined in (1.3).

By taking derivatives with respect to \( \alpha \) on both sides of (4.5) and noting that the residue of the simple pole of \( \zeta K(s) \) at \( s = 1 \) equals \( \pi/4 \), we deduce from Conjecture 1.2 the following result.
Lemma 4.3. Assuming the truth of Conjecture [23], we have for any \( \varepsilon > 0 \), \((\log X)^{-1} \ll \Re(r) < 1/4\) and \( \Im(r) \ll X^{-1-\varepsilon}, \)

\[
\frac{1}{W(X)} \sum_{\omega \equiv 1 \text{ mod } (1+i)^3} w \left( \frac{N(\omega)}{X} \right) \frac{L'(1/2 + r, \chi(1+i)^{5}\omega)}{L(1/2 + r, \chi(1+i)^{5}\omega)} = \frac{\zeta'_K(1 + 2r)}{\zeta_K(1 + 2r)} + A_\alpha(r, r)
\]

(4.6)

\[-4 \frac{1}{\pi W(X)} \sum_{\omega \equiv 1 \text{ mod } (1+i)^3} w \left( \frac{N(\omega)}{X} \right) X \left( \frac{1}{2} + r \right) \zeta_K(1 - 2r)A(-r, r) + O_{\varepsilon}(X^{-1/2+\varepsilon}). \]

4.4. Derivation of the one level density from the ratios conjecture. In this section we prove Theorem 1.1 using the ratios conjecture and GRH. We recall the definition of \( D(\phi; w, X) \) from (1.2) and we note that GRH implies that the non-trivial zeros of the \( L \)-functions all have real parts \( 1/2 \). Thus by setting \( L = \log X \), we can recast \( D(\phi; w, X) \) as (4.7)

\[
D(\phi; w, X) = \frac{1}{W(X)} \sum_{\omega \equiv 1 \text{ mod } (1+i)^3} w \left( \frac{N(\omega)}{X} \right) \frac{1}{2\pi i} \int_{(a-1/2)} \left( \frac{L'(s, \chi(1+i)^{5}\omega)}{L(s, \chi(1+i)^{5}\omega)} - \frac{L'(1 - s, \chi(1+i)^{5}\omega)}{L(1 - s, \chi(1+i)^{5}\omega)} \right) \phi \left( \frac{iLr}{2\pi} \right) ds
\]

where \( 1/2 + 1/\log X < a < 3/4 \) and the fact that \( \phi \) is even is used in the derivation of the late equality above.

We note that the functional equation [23] implies that

\[
\frac{L'(s, \chi(1+i)^{5}\omega)}{L(s, \chi(1+i)^{5}\omega)} = \frac{X^s}{X(1/2 + r)} - \frac{L'(1 - s, \chi(1+i)^{5}\omega)}{L(1 - s, \chi(1+i)^{5}\omega)}
\]

Inserting the above into (4.7), we obtain that

\[
D(\phi; w, X) = \frac{1}{W(X)} \sum_{\omega \equiv 1 \text{ mod } (1+i)^3} w \left( \frac{N(\omega)}{X} \right) \frac{1}{2\pi i} \int_{(a-1/2)} \left( \frac{2L'(1/2 + r, \chi(1+i)^{5}\omega)}{L(1/2 + r, \chi(1+i)^{5}\omega)} - \frac{X^s}{X(1/2 + r)} \right) \phi \left( \frac{iLr}{2\pi} \right) dr.
\]

Substituting (4.6) in (4.8), we deduce by noting the rapid decay of \( \phi \) that

\[
D(\phi; w, X) = \frac{1}{W(X)} \sum_{\omega \equiv 1 \text{ mod } (1+i)^3} w \left( \frac{N(\omega)}{X} \right) \frac{1}{2\pi i} \int_{(a-1/2)} \left( 2 \frac{\zeta'_K(1 + 2r)}{\zeta_K(1 + 2r)} + 2A_\alpha(r, r) - \frac{X^s}{X(1/2 + r)} \right) \phi \left( \frac{iLr}{2\pi} \right) dr + O_{\varepsilon}(X^{-1/2+\varepsilon}).
\]

(4.9)

As the function

\[
2 \frac{\zeta'_K(1 + 2r)}{\zeta_K(1 + 2r)} + 2A_\alpha(r, r) - \frac{X^s}{X(1/2 + r)} - \frac{8}{\pi X} \zeta_K(1 - 2r)A(-r, r)
\]

is analytic in the region \( \Re(r) \geq 0 \) (in particular it is analytical at \( r = 0 \)), we can now shift the line of integration in (4.9) to \( \Re(r) = 0 \) to deduce the reality of Theorem 1.2

4.5. Proof of Theorem 1.1 using the ratios conjecture. In this section, we give another proof of Theorem 1.1 by assuming the ratios conjecture. To achieve this, we recall that we set \( L = \log X \) and we first apply Lemma 2.5 to see that

\[
\frac{1}{W(X)} \sum_{\omega \equiv 1 \text{ mod } (1+i)^3} w \left( \frac{N(\omega)}{X} \right) \frac{1}{2\pi} \int_{\Re} \left( \log \left( \frac{32N(\omega)}{\pi^2} \right) \phi \left( \frac{tL}{2\pi} \right) dt = \frac{\phi(0)}{2\pi} \right) + O_{\varepsilon}(\frac{1}{L}).
\]

Moreover, similar to the treatment of \( T_1 \) in [IS (2.8)], we have that

\[
\frac{1}{W(X)} \sum_{\omega \equiv 1 \text{ mod } (1+i)^3} w \left( \frac{N(\omega)}{X} \right) \frac{1}{2\pi} \int_{\Re} \left( \frac{t'}{L} \frac{1}{2 - it} + \Gamma \left( \frac{1}{2} + it \right) \right) \phi \left( \frac{tL}{2\pi} \right) dt
\]

(4.10)

\[
= 2 \frac{\Gamma'}{\Gamma} \frac{\phi(0)}{L} + 2 \frac{L}{\Gamma} \int_{0}^{\infty} \frac{e^{-t/2}}{1 - e^{-t}} \left( \phi(0) - \phi \left( \frac{L}{t} \right) \right) dt = O_{\varepsilon}(\frac{1}{L}.
\]

(4.11)
We then deduce from (1.10), (4.10), (4.11) that
\[ D(\phi; w, X) = \hat{\phi}(0) + \frac{1}{W(X)} \sum_{\omega \equiv 1 \mod (1+i)^3} \frac{w(N(\omega))}{X} \frac{1}{2\pi i} \int_{\mathbb{R}} \left( 2 \frac{\zeta'_K(1 + 2it)}{\zeta_K(1 + 2it)} + 2A_\alpha(it, it) \right) \frac{i \phi(t \tau / 2\pi)}{2\pi} \frac{dt}{t} + O\left( \frac{1}{L} \right). \]

In view of (11.9) and the above expression for \( D(\phi; w, X) \), we see that we can recast it as
\[ D(\phi; w, X) = \hat{\phi}(0) + \frac{1}{W(X)} \sum_{\omega \equiv 1 \mod (1+i)^3} \frac{w(N(\omega))}{X} \frac{1}{2\pi i} \int_{(a')} \left( 2 \frac{\zeta'_K(1 + 2r)}{\zeta_K(1 + 2r)} + 2A_\alpha(r, r) \right) \frac{i \phi(t \tau / 2\pi)}{2\pi} \frac{dr}{r} + O\left( \frac{1}{L} \right). \]

where \( 1/\log X < a' < 1/4 \). By a straightforward computation we see that for \( 1/\log X < \Re(r) < 1/4 \), we have
\[ A_\alpha(r, r) + \frac{\zeta'_K(1 + 2r)}{\zeta_K(1 + 2r)} = -\sum_{(\omega)} \log N(\omega) N(\omega)^{1+2r} - 1. \]

It follows from this and treatment similar to (11.11) Lemma 4.1 and (11.11) Lemma 3.7 that we have
\[ \frac{1}{2\pi i} \int_{(a')} \left( 2 \frac{\zeta'_K(1 + 2r)}{\zeta_K(1 + 2r)} + 2A_\alpha(r, r) \right) \frac{i \phi(t \tau / 2\pi)}{2\pi} \frac{dr}{r} = \frac{2}{L} \sum_{(\omega)} \log N(\omega) \frac{N(\omega)}{N(\omega)^{1+2r}} \phi \left( \frac{2j \log N(\omega)}{L} \right) = \frac{\phi(0)}{2} + O\left( \frac{1}{L} \right). \]

It now remains to treat the expression
\[ I = -\frac{8}{\pi W(X)} \sum_{\omega \equiv 1 \mod (1+i)^3} \frac{w(N(\omega))}{X} \frac{1}{2\pi i} \int_{(a')} \frac{X}{X} \left( \frac{1}{2} + r \right) \zeta_K(1 - 2r)A(-r, r) \phi \left( \frac{i \phi(t \tau / 2\pi)}{2\pi} \frac{dr}{r} \right). \]

Note that it follows from (15.5) that
\[ A(-\gamma, \gamma) = 2 - 2^r. \]

Combining this with the definition of \( X_\omega \) given in (15.6) and a change of variable \( r = 2\pi i \tau / L \), we deduce that
\[ I = \int_{\mathbb{C}} -\frac{8}{\pi} \frac{\Gamma(1/2 - 2\pi i \tau / L)}{\Gamma(1/2 + 2\pi i \tau / L)} \left( \frac{\tau}{32} \right)^{2\pi \tau / L} \left( 2 - 2^{4\pi i \tau / L} \right) \zeta_K \left( 1 - 4\pi i \tau / L \right) \frac{\phi(\tau)}{W(X)} \sum_{\omega \equiv 1 \mod (1+i)^3} \frac{w(N(\omega))}{X} N(\omega)^{2\pi i \tau / L}, \]

where \( C \) stands for the horizontal line \( \Im(\tau) = -L\alpha'/2\pi \).

We now deform \( C \) to the path \( C' = C_0 \cup C_1 \cup C_2 \), where
\[ C_0 = \{ \tau : \Im(\tau) = 0, |\Re(\tau)| \geq L^\varepsilon \}, \quad C_1 = \{ \tau : \Im(\tau) = 0, \eta \leq |\Re(\tau)| \leq L^\varepsilon \} \quad \text{and} \quad C_2 = \{ \tau : |\tau| = \eta, \Im(\tau) \leq 0 \}, \]

for some small \( \varepsilon, \eta > 0 \). The integration of \( I \) over \( C_0 \) can be estimated trivially by making use of the rapid decay of \( \phi \) to be of \( O(L^{-1}) \).

Using Taylor expansions, we see that on \( C_1 \cup C_2 \),
\[ -\frac{8}{\pi} \frac{\Gamma(1/2 - 2\pi i \tau / L)}{\Gamma(1/2 + 2\pi i \tau / L)} \left( \frac{\tau}{32} \right)^{2\pi \tau / L} \left( 2 - 2^{4\pi i \tau / L} \right) \zeta_K \left( 1 - 4\pi i \tau / L \right) = \frac{1}{2\pi i} + O\left( \frac{|\tau| + 1}{L} \right). \]
It follows that the integrand of $I$ on $C_1 \cup C_2$ equals
\[
\left( \frac{1}{2\pi i} + O \left( \frac{1}{|\tau|} \right) \right) \frac{\phi(\tau)}{W(X)} \sum_{\substack{w \equiv 1 \mod (1+i)^3}} w \left( \frac{N(w)}{X} \right) N(w)^{-2\pi i \tau / \mathcal{L}} \bigg|_{\mathcal{L}=1}^{\mathcal{L}} = \left( \frac{1}{2\pi i} + O \left( \frac{1}{|\tau|} \right) \right) \left( X^{-2\pi i \tau / \mathcal{L}} + O \left( \frac{|\tau|^2 + 1}{\mathcal{L}} \right) \right) \phi(\tau) 
\]
where the last equality above follows from the treatment of $I_1$ in the proof of [11 Lemma 4.6]. The assertion of Theorem 1.1 now follows by combining (4.12), (4.13) with the above expression.

Acknowledgments. P. G. is supported in part by NSFC grant 11871082 and L. Z. by the FRG grant PS43707 and the Goldstar Award PS53450 at the University of New South Wales (UNSW). Parts of this work were done when P. G. visited UNSW in September 2019. He wishes to thank UNSW for the invitation, financial support and warm hospitality during his pleasant stay.

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