Order Parameter Flow in the SK Spin-Glass I: Replica Symmetry

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Abstract

We present a theory to describe the dynamics of the Sherrington-Kirkpatrick spin-glass with (sequential) Glauber dynamics in terms of deterministic flow equations for macroscopic parameters. Two transparent assumptions allow us to close the macroscopic laws. Replica theory enters as a tool in the calculation of the time-dependent local field distribution. The theory produces in a natural way dynamical generalisations of the AT- and zero-entropy lines and of Parisi’s order parameter function $P(q)$. In equilibrium we recover the standard results from equilibrium statistical mechanics. In this paper we make the replica-symmetric ansatz, as a first step towards calculating the order parameter flow. Numerical simulations support our assumptions and suggest that our equations describe the shape of the local field distribution and the macroscopic dynamics reasonably well in the region where replica symmetry is stable.
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1 Introduction

The Sherrington-Kirkpatrick (SK) spin-glass model describes a collection of $N$ Ising spins, coupled by exchange interactions which are drawn at random from a Gaussian distribution. These interactions represent quenched (frozen) disorder. The equilibrium statistical mechanical description of the SK model seems to have reached a stable fixed-point, built on replica theory with, at least in the spin-glass phase, broken replica symmetry a la Parisi. A clear and extensive description of the formalism developed since 1975 and most of the relevant references can be found in textbooks like the ones by Mezard et al. and Fisher and Hertz.

With respect to the dynamical properties of the SK model, the situation seems different. The early dynamical studies, like, were more or less of a pilot character, employing mean field approximations (MFA) and linearisations of the exact dynamic ensemble averages. Analytical work beyond MFA published so far has mostly concentrated on Langevin dynamics for soft spins, as opposed to Ising spins. In the Langevin case the standard procedure (described in detail in e.g. and ) is to construct a generating functional from a path integral representation of the microscopic state probability, which can subsequently be averaged over the quenched disorder (i.e. the random exchange interactions). This leads to a saddle-point problem, the limit $N \to \infty$ can be taken and one obtains a complicated set of equations for correlation- and response functions. These can be interpreted in terms of a Langevin equation for a single spin with a retarded self-interaction and a noise term with non-trivial moments. In order to proceed from this stage, additional assumptions, restrictions or approximations are needed, like expansions near critical lines or near equilibrium. By construction, in these theories only timescales which do not diverge with $N$ are described. The approach followed by Sompolinsky in is different: here a hierarchy of time-scales is introduced, all of which diverge for $N \to \infty$, but in a strict order. The case of Glauber dynamics for Ising spins was studied by Sommers, who developed a path integral formalism by performing manipulations on the solution (in the form of a time-ordered product) of the master equation. His method, although subsequently applied by other authors to related models like the non-symmetric SK model, was later criticised by Lusakowski. As far as we are aware, the issue of the correctness (or otherwise) of the Sommers approach has not been settled.

At zero temperature the SK model shows strong remanence effects (see
e.g. Kinzel \cite{17}], with a non-exponential decay of the magnetisation. Only recently numerical evidence has been published \cite{18} which suggests that infinite-range models such as the SK model even exhibit ageing effects of the type observed in experiments on real spin-glasses \cite{4,19}, which until now were always assumed to be typical for finite-range models and therefore explained using scaling arguments for growing domains.

Motivated by the non-trivial dynamical phenomena exhibited by the SK model and by the restricted theoretical understanding of the Glauber dynamics (as opposed to the continuous Langevin approach), we develop in this paper a theory to describe the Glauber dynamics of the SK model in terms of deterministic flow equations for two macroscopic state variables: the magnetisation $m$ and the spin-glass contribution $r$ to the energy. Our reasons for choosing these two quantities as dynamic order parameters, in favour of a dynamical equivalent of the spin-glass order parameter $q$ or its distribution $P(q)$, are:

1. On finite time-scales both $m$ and $r$ evolve in time deterministically in the limit $N \to \infty$.

2. The Hamiltonian of the SK model can be expressed solely in terms of $m$ and $r$.

3. In thermal equilibrium $m$ and $r$ are self-averaging with respect to the quenched disorder in the limit $N \to \infty$ (since $m$ and the free energy are \cite{20}).

4. Both $m$ and $r$ are instantaneous functions of time for a single system, whereas $P(q)$ involves correlations between different times or systems.

The key to closing the deterministic laws is to calculate the distribution of time-dependent local alignment fields. Two transparent physical assumptions allow us to calculate this distribution analytically and find a \emph{closed} set of flow equations for our two order parameters. The theory produces in a natural way dynamical generalisations of the AT- and zero-entropy lines and of Parisi’s order parameter function $P(q)$. In equilibrium we recover the standard results from equilibrium statistical mechanics. The present formalism has previously been applied successfully to a related model: the Hopfield neural network model near saturation \cite{21}.

In our view the main appeal of our formalism is its transparency. The theory is formulated in terms of two directly observable macroscopic state variables and, apart from two simple assumptions, derived directly from the
microscopic stochastic equations. Secondly, an interesting difference with existing approaches is the way in which replica theory enters. In the standard Langevin approach (after having taken the limit $N \to \infty$) one ends up with quantities and equations very much like the ones encountered in equilibrium replica theory, with replica indices replaced by time arguments. In Sompolinsky’s theory replica indices are replaced by labels of the hierarchy of time-scales. In contrast, in the present formalism replica theory enters as a mathematical tool in calculating the time-dependent distribution of local alignment fields. The only uncertainty in the status of the theory originates from the two closure assumptions, since all subsequent calculations can in principle be performed exactly. Both are supported to a certain extent by evidence from numerical simulations. A recent study of an exactly solvable toy model \cite{22}, stimulated by the work reported here and in \cite{21}, suggests that the proposed closure procedure succeeds in capturing the main physics in a closed set of transparent deterministic equations and is exact for $t = 0$ and $t = \infty$, but does not reproduce all temporal characteristics for intermediate times. Since the closure procedure is based on the elimination of microscopic memory effects, the theory can contribute to a better understanding of the relation between the microscopic processes and correlations and the macroscopic measures of complexity, such as the order parameter $P(q)$.

In this paper we develop the general formalism. However, in calculating the order parameter flow explicitly we will make the replica-symmetric RS ansatz. We will show that in most of the flow diagram replica symmetry is stable. In the region where the RS solution is unstable the flow direction is still described correctly and the RS theory even predicts non-exponential relaxation for $T \to 0$, but the RS equations fail to describe a rigorous slowing down which, according to simulations, sets in near the de Almeida-Thouless line. In a subsequent paper we shall address the implications of replica symmetry breaking.

2 Dynamics of the Sherrington-Kirkpatrick Spin-Glass
2.1 Definitions and Macroscopic Laws

The Sherrington-Kirkpatrick (SK) spin-glass model [1] describes \( N \) Ising spins \( \sigma_i \in \{-1, 1\} \) with infinite-range exchange interactions \( J_{ij} \):

\[
J_{ij} = \frac{1}{N} J_0 + \frac{1}{\sqrt{N}} J_{zij} \quad (i < j)
\]

where the quantities \( z_{ij} \), which represent quenched disorder, are drawn independently at random from a Gaussian distribution with \( \langle z_{ij} \rangle = 0 \) and \( \langle z_{ij}^2 \rangle = 1 \).

The evolution in time of the microscopic state probability \( p_t(\sigma) \) is of the Glauber [13] form, described by a continuous-time master equation:

\[
\frac{d}{dt} p_t(\sigma) = \sum_{k=1}^{N} \left[ p_t(F_k \sigma) w_k(F_k \sigma) - p_t(\sigma) w_k(\sigma) \right]
\]

in which \( F_k \) is a spin-flip operator \( F_k \Phi(\sigma) \equiv \Phi(\sigma_1, \ldots, -\sigma_k, \ldots, \sigma_N) \) and the transition rates \( w_k(\sigma) \) are

\[
w_k(\sigma) \equiv \frac{1}{2} \left[ 1 - \sigma_k \tanh[\beta h_k(\sigma)] \right]
\]

which leads to the required standard equilibrium distribution

\[
p_{\text{eq}}(\sigma) \sim e^{-\beta H(\sigma)} \quad H(\sigma) \equiv -\sum_{i<j} \sigma_i J_{ij} \sigma_j - \sum_i \theta_i \sigma_i
\]

(for numerical simulations we resort to a discrete-time sequential process, where the \( w_k(\sigma) \) are interpreted as transition probabilities and with iteration steps of duration \( 1/N \). For \( N \to \infty \) this must reproduce the physics of the continuous-time equation [24]). The energy per spin can be written in terms of two macroscopic quantities

\[
m(\sigma) = \frac{1}{N} \sum_i \sigma_i \quad r(\sigma) = \frac{1}{N \sqrt{N}} \sum_{i<j} \sigma_i z_{ij} \sigma_j
\]

\[
H(\sigma)/N = -\frac{1}{2} J_0 m^2(\sigma) - \theta m(\sigma) - J r(\sigma) + \frac{1}{2} J_0/N
\]

These two observables, the magnetisation and the energy contribution induced by the quenched variables \( \{z_{ij}\} \), will be used to define a macroscopic state. The corresponding macroscopic probability distribution is

\[
P_t(m, r) \equiv \sum_{\sigma} \delta (m - m(\sigma)) \delta (r - r(\sigma))
\]
By inserting the microscopic equation (2) and after defining the ‘discrete
derivatives’ $\Delta_i f(\sigma) \equiv f(F_i \sigma) - f(\sigma)$, we obtain
\[
\frac{d}{dt} P_t(m, r) = -\frac{\partial}{\partial m} \left\{ P_t(m, r) \langle \sum_{i=1}^{N} w_i(\sigma) \Delta_i m(\sigma) \rangle_{m,r,t} \right\}
\]
\[-\frac{\partial}{\partial r} \left\{ P_t(m, r) \langle \sum_{i=1}^{N} w_i(\sigma) \Delta_i r(\sigma) \rangle_{m,r,t} \right\} + O(N \Delta^2)
\]
with the sub-shell average
\[
\langle \Phi(\sigma) \rangle_{m,r,t} \equiv \frac{\sum_{\sigma} p_t(\sigma) \delta [m-m(\sigma)] \delta [r-r(\sigma)] \Phi(\sigma)}{\sum_{\sigma} p_t(\sigma) \delta [m-m(\sigma)] \delta [r-r(\sigma)]}
\]
The local alignment fields and the ‘discrete derivatives’ are given by
\[
h_i(\sigma) = J_0 m(\sigma) + J z_i(\sigma) + \theta + O\left(\frac{1}{N}\right)
\]
\[
z_i(\sigma) \equiv \frac{1}{\sqrt{N}} \sum_{j \neq i} z_{ij} \sigma_j
\]
\[
\Delta_i m(\sigma) = -\frac{2}{N} \sigma_i \quad \Delta_i r(\sigma) = -\frac{2}{N} \sigma_i z_i(\sigma)
\]
With these expressions and the transition rates (3) we can evaluate (4):
\[
\frac{d}{dt} P_t(m, r) = -\frac{\partial}{\partial m} \left\{ P_t(m, r) \left[ \langle \frac{1}{N} \sum_{i=1}^{N} \tanh \beta (J_0 m + J z_i(\sigma) + \theta) \rangle_{m,r,t} - m \right] \right\}
\]
\[-\frac{\partial}{\partial r} \left\{ P_t(m, r) \left[ \langle \frac{1}{N} \sum_{i=1}^{N} z_i(\sigma) \tanh \beta (J_0 m + J z_i(\sigma) + \theta) \rangle_{m,r,t} - 2r \right] \right\} + O\left(\frac{1}{N}\right)
\]
In the limit $N \to \infty$ equation (5) acquires the Liouville form and describes
deterministic flow at the macroscopic level $(m, r)$. The evolution of the
dynamic order parameters $(m, r)$ is governed by the flow equations
\[
\frac{d}{dt} m = \int dz \ D_{m,r,t}[z] \tanh \beta (J_0 m + J z + \theta) - m
\]
\[
\frac{d}{dt} r = \int dz \ D_{m,r,t}[z] z \tanh \beta (J_0 m + J z + \theta) - 2r
\]
All complicated terms are concentrated in the distribution of spin-glass con-
tributions $z_i(\sigma)$ to the local fields:
\[
D_{m,r,t}[z] \equiv \lim_{N \to \infty} \frac{\sum_{\sigma} p_t(\sigma) \delta [m-m(\sigma)] \delta [r-r(\sigma)] \frac{1}{N} \sum_i \delta [z-z_i(\sigma)]}{\sum_{\sigma} p_t(\sigma) \delta [m-m(\sigma)] \delta [r-r(\sigma)]}
\]
Thus far no approximations have been used; equations (4,5) are exact for
$N \to \infty$. 7
Figure 1: Trajectories in the \((m, r)\) plane obtained by performing sequential simulations of the SK model with \(T = 0.1\), \(J = 1\) and \(J_0 = 0\), for \(t \leq 10\) iterations/spin.

2.2 Closure of the Macroscopic Laws

The flow equations are not yet closed: they contain the distribution \(D_{m,r,t}[z]\) (10), which is defined in terms of the solution \(p_t(\sigma)\) of the microscopic equation (2). In order to close the set (8,9) we make two simple assumptions on the asymptotic \((N \to \infty)\) form of the local field distribution \(D_{m,r,t}[z]\):

(i) The deterministic laws describing the evolution in time of the order parameters \((m, r)\) are self-averaging with respect to the distribution of the quenched contributions \(z_{ij}\) to the exchange interactions. Therefore the local field distribution \(D_{m,r,t}[z]\) is self-averaging as well.
Figure 2: Trajectories in the \((m, r)\) plane obtained by performing sequential simulations of the SK model with \(T = 0.1\), \(J = 1\) and \(J_0 = 1\), for \(t \leq 10\) iterations/spin.
Figure 3: Trajectories in the \((m, r)\) plane obtained by performing sequential simulations of the SK model with \(T = 0.1\), \(J = 1\) and \(J_0 = 2\), for \(t \leq 10\) iterations/spin.
In view of (i) we assume that, as far as the calculation of $D_{m,r,t}[z]$ is concerned, we may assume equipartitioning of probability in the macroscopic $(m,r)$ subshells of the ensemble.

Assumption (i) allows us to simplify the problem by performing an average over the (quenched) random variables $\{z_{ij}\}$. As a consequence of assumption (ii) the explicit time-dependence in the flow equations (8,9) and the dependence on microscopic initial conditions are removed, since the distribution $D_{m,r,t}[z]$ will be replaced by:

$$D_{m,r}[z] \equiv \lim_{N \to \infty} \frac{\sum_{\sigma} \delta [m - m(\sigma)] \delta [r - r(\sigma)] \frac{1}{N} \sum_{i} \delta [z - z_{i}(\sigma)]}{\sum_{\sigma} \delta [m - m(\sigma)] \delta [r - r(\sigma)]} \langle \{z_{ij}\} \rangle \quad (11)$$

For sequential dynamics, the first of our two assumptions is clearly supported by experimental evidence (sequential simulations at $T = 0.1$), which we present in figures 1 (for $J_0 = 0$, where the system evolves towards a true spin-glass state), 2 (for $J_0 = 1$, which marks the onset of a non-zero equilibrium magnetisation) and 3 (for $J_0 = 2$, where the system evolves towards a ferro-magnetic state). Each of the flow graphs corresponds to one particular realisation of the quenched disorder $\{z_{ij}\}$. The initial states generating the different trajectories (labelled by $\ell = 0, \ldots, 10$) were drawn at random according to $p_0(\sigma) \equiv \prod_{i} \left[ \frac{1}{2} \left[ 1 + \frac{1}{10} \right] \delta_{\sigma_{i},1} + \frac{1}{2} \left[ 1 - \frac{1}{10} \right] \delta_{\sigma_{i},-1} \right]$, such that that $\langle m \rangle_{t=0} = 0.1 \ell$ and $\langle r \rangle_{t=0} = 0$. With increasing system size, fluctuations in individual trajectories eventually vanish and well-defined flow lines emerge, which no longer depend on the disorder realisation. The second closure assumption can only be tested in such a direct manner by comparing the actual local field distribution, measured during simulations, with the result of evaluating (11). This will be done in a subsequent section.

In equilibrium studies the above two assumptions are in fact the basic building blocks of analysis as well, where (i) is assumed and (ii) is a consequence of the Boltzmann form of the microscopic equilibrium distribution. Our aim is to calculate analytically the $N \to \infty$ flow illustrated in figures 1 to 3, by combining equations (8,9) with (11). The distribution (11) will be calculated using the replica method.
2.3 The Local Field Distribution

We use the following replica expression for writing expectation values of a given state variable $\Phi$ over a given measure $W$:

$$
\langle \Phi(\sigma) \rangle_W \equiv \frac{\langle \Phi(\sigma) W(\sigma) \rangle_{\sigma}}{\langle W(\sigma) \rangle_{\sigma}} = \lim_{n \to 0} \langle \Phi(\sigma^1) \prod_{\alpha=1}^n W(\sigma^\alpha) \rangle_{\sigma^v}
$$

which allows us to write (13) in the replica form. By writing the delta-functions in integral representation we obtain

$$
D_{m,r}[z] = \int \frac{dx}{2\pi} e^{i x z} \lim_{n \to 0} \left[ \frac{N}{2\pi} \right]^{2n} \times
$$

$$
\int d\hat{m} d\hat{r} \; e^{i N \sum_{\alpha} [x \hat{\sigma}_{\alpha} + m \hat{\sigma}_{\alpha}]} \langle e^{-i \sum_{\alpha} \hat{\sigma}_{\alpha}^2 \sum_{k} \hat{\sigma}_{\alpha}^k M(\sigma^\alpha)} \rangle_{\sigma^vl;ij}
$$

with

$$
M(\sigma^\alpha) \equiv \langle e^{-\frac{i}{2N} \sum_{k>1} \hat{\sigma}_{\alpha} \hat{\sigma}_{\alpha}^k - \frac{i}{\sqrt{N}} \sum_{\alpha} \hat{\sigma}_{\alpha} \sum_{k>1} \hat{\sigma}_{\alpha} \hat{\sigma}_{\alpha}^k} \rangle_{\sigma^vl;ij}
$$

We now perform the average over the quenched variables $\{z_{ij}\}$ in $M(\sigma^\alpha)$, with the result:

$$
M(\sigma^\alpha) = e^{-\frac{i}{4} x^2 - \frac{N}{4} \sum_{\alpha,\beta} \hat{r}_{\alpha} q_{\alpha,\beta}(\sigma) \hat{r}_{\beta} + \frac{i}{4} [\sum_{\alpha} \hat{r}_{\alpha}]^2 - x \sum_{\alpha} \hat{r}_{\alpha} q_{\alpha,\beta}(\sigma) \hat{\sigma}_{\alpha}^2 + O(1/N)}
$$

in which we have introduced the familiar order parameters $q_{\alpha,\beta}(\sigma) \equiv \frac{1}{N} \sum_{\alpha} \hat{\sigma}_{\alpha}^\alpha \hat{\sigma}_{\alpha}^\beta$. If we again introduce appropriate delta-functions,

$$
1 = \int dq \; \delta[q - q(\sigma)] = \left[ \frac{N}{2\pi} \right]^n \int dq dq \; e^{i N \sum_{\alpha,\beta} q_{\alpha,\beta} [q_{\alpha,\beta} - q_{\alpha,\beta}(\sigma)]}
$$

we can reduce the spin-averages to single-site ones. The result can then be written in terms of an $n$-replicated Ising spin $(\sigma_1, \ldots, \sigma_n)$:

$$
D_{m,r}[z] = \int \frac{dx}{2\pi} e^{-\frac{i}{4} x^2 + i x z} \lim_{n \to 0} \left[ \frac{N}{2\pi} \right]^{n^2+2n} \times \int d\hat{m} d\hat{r} d\hat{q} dq \; e^{i \sum_{\alpha} \hat{r}_{\alpha}^2 + O(1/N)}
$$

$$
\times e^{N \Psi(\hat{m}, \hat{r}, \hat{q}, q)} \langle e^{-\sum_{\alpha} \sigma_{\alpha} [x \hat{r}_{\alpha} q_{\alpha,\beta} + i \hat{m}_{\alpha}]} - i \sum_{\alpha,\beta} \hat{q}_{\alpha,\beta} \sigma_{\alpha} \sigma_{\beta}] \rangle_{\sigma}
$$

$$
\Psi(\hat{m}, \hat{r}, \hat{q}, q) = i \sum_{\alpha} [x \hat{r}_{\alpha} + m \hat{m}_{\alpha}] + i \sum_{\alpha,\beta} q_{\alpha,\beta} q_{\alpha,\beta} - \frac{1}{4} \sum_{\alpha,\beta} \hat{r}_{\alpha} q_{\alpha,\beta}^2 \hat{r}_{\beta}
$$

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\[ log(e^{-i\sum_{\alpha} \hat{m}_\alpha \sigma_{-i} \sum_{\alpha \beta} \hat{q}_{\alpha \beta} \sigma_{\alpha} \sigma_{\beta}}) \sigma \] (12)

For large \( N \) the integral is evaluated by steepest descent and we obtain

\[ D_{m,r}[z] = \int dx e^{-\frac{1}{2}x^2 + i x z} \lim_{n \to 0} \frac{\langle e^{-\sum_{\alpha} \sigma_{[x\hat{r}_\alpha q_{1\alpha} + i \hat{m}_\alpha] - i \sum_{\alpha \beta} \hat{q}_{\alpha \beta} \sigma_{\alpha} \sigma_{\beta}} \rangle \sigma}{\langle e^{-i \sum_{\alpha} \hat{m}_\alpha \sigma_{-i} \sum_{\alpha \beta} \hat{q}_{\alpha \beta} \sigma_{\alpha} \sigma_{\beta}} \rangle \sigma} \] (13)

in which the order parameters \( \{ \hat{m}, \hat{r}, \hat{q}, q \} \) are found by selecting the saddle-point of \( \Psi \) (12), which gives a minimum with respect to variation of the order parameters \( q_{\alpha \beta} \). Variation of \( q_{\alpha \beta} \) allows us to eliminate already one set of conjugate parameters: \( \hat{q}_{\alpha \beta} = -\frac{1}{2} i q_{\alpha \beta} \hat{r}_{\alpha} \hat{r}_{\beta} \). The remaining conjugate parameters, uniquely determined by the saddle-point requirement, turn out to be purely imaginary: \( \hat{r}_{\alpha} \equiv i \rho_{\alpha} \) and \( \hat{m}_{\alpha} \equiv i \mu_{\alpha} \), with which we obtain the following saddle-point equations:

\[ m = \frac{\langle \sigma_{\alpha} \sigma_{\beta} e^{\sum_{\gamma} \mu_{\gamma} \rho_{\gamma} + \frac{1}{2} \sum_{\gamma \delta} q_{\gamma \delta} \rho_{\gamma} \rho_{\delta} \sigma_{\gamma} \sigma_{\delta}} \rangle \sigma}{\langle e^{\sum_{\gamma} \mu_{\gamma} \rho_{\gamma} + \frac{1}{2} \sum_{\gamma \delta} q_{\gamma \delta} \rho_{\gamma} \rho_{\delta} \sigma_{\gamma} \sigma_{\delta}} \rangle \sigma} \] (14)

\[ q_{\alpha \beta} = \frac{\langle \sigma_{\alpha} \sigma_{\beta} e^{\sum_{\gamma} \mu_{\gamma} \rho_{\gamma} + \frac{1}{2} \sum_{\gamma \delta} q_{\gamma \delta} \rho_{\gamma} \rho_{\delta} \sigma_{\gamma} \sigma_{\delta}} \rangle \sigma}{\langle e^{\sum_{\gamma} \mu_{\gamma} \rho_{\gamma} + \frac{1}{2} \sum_{\gamma \delta} q_{\gamma \delta} \rho_{\gamma} \rho_{\delta} \sigma_{\gamma} \sigma_{\delta}} \rangle \sigma} \] (15)

\[ \sum_{\beta} q_{\alpha \beta}^2 \rho_{\beta} = 2r \] (16)

The exponent \( \Psi \) can be simplified to:

\[ \Psi = -r \sum_{\alpha} \rho_{\alpha} - m \sum_{\alpha} \mu_{\alpha} - \frac{1}{4} \sum_{\alpha \gamma} \rho_{\alpha} q_{\alpha \gamma}^2 \rho_{\gamma} + \log(\langle e^{\sum_{\gamma} \mu_{\gamma} \rho_{\gamma} + \frac{1}{2} \sum_{\gamma \delta} q_{\gamma \delta} \rho_{\gamma} \rho_{\delta} \sigma_{\gamma} \sigma_{\delta}} \rangle \sigma) \] (17)

and the distribution \( D_{m,r}[z] \) becomes

\[ D_{m,r}[z] = \int dx e^{-\frac{1}{2}x^2 + i x z} \lim_{n \to 0} \frac{\langle e^{-ix \sum_{\gamma} \sigma_{\gamma} \rho_{\gamma} + \sum_{\gamma} \mu_{\gamma} \rho_{\gamma} + \frac{1}{2} \sum_{\gamma \delta} q_{\gamma \delta} \rho_{\gamma} \rho_{\delta} \sigma_{\gamma} \sigma_{\delta}} \rangle \sigma}{\langle e^{\sum_{\gamma} \mu_{\gamma} \rho_{\gamma} + \frac{1}{2} \sum_{\gamma \delta} q_{\gamma \delta} \rho_{\gamma} \rho_{\delta} \sigma_{\gamma} \sigma_{\delta}} \rangle \sigma} \] (18)

The physical meaning of the order parameters \( q_{\alpha \beta} \), which in the present theory are functions of the two macroscopic state variables \( m \) and \( r \), can be inferred in the usual manner by considering two spin systems, \( \sigma \) and \( \sigma' \), with the same microscopic realisations of the quenched disorder. For such
systems we define the disorder-averaged probability distribution \( P_{mr}(q) \) for the mutual overlap between microscopic configurations if both systems are constrained on the same macroscopic \((m, r)\) subshell:

\[
P_{mr}(q) \equiv \frac{\sum_{\sigma, \sigma'} \delta \left[ q - \frac{1}{N} \sum_k \sigma_k \sigma'_k \right] \delta \left[ m - m(\sigma) \right] \delta \left[ r - r(\sigma) \right] \delta \left[ m - m(\sigma') \right] \delta \left[ r - r(\sigma') \right]}{\sum_{\sigma, \sigma'} \delta \left[ m - m(\sigma) \right] \delta \left[ r - r(\sigma) \right] \delta \left[ m - m(\sigma') \right] \delta \left[ r - r(\sigma') \right]} \langle \{z_{ij}\} \rangle
\]

\[
= \lim_{n \to 0} \frac{1}{n(n-1)} \sum_{\alpha \neq \beta} \langle \delta \left[ q - \frac{1}{N} \sum_k \sigma_k^\alpha \sigma_k^\beta \right] \prod_{\gamma=1}^n \delta \left[ m - m(\sigma^\gamma) \right] \delta \left[ r - r(\sigma^\gamma) \right] \rangle \langle \{\mathbf{s}_\alpha\} , \{z_{ij}\} \rangle - 1
\]

\[
= \lim_{n \to 0} \frac{1}{n(n-1)} \sum_{\alpha \neq \beta} \delta \left[ q - q_{\alpha\beta} \right] (19)
\]

This dynamical equivalent of Parisi’s [2] equilibrium order parameter function will, in the present theory, depend on time through the values of the two macroscopic parameters \((m, r)\).

The saddle-point exponent \( \Psi \) that is extremised in the replica calculation of the local field distribution has an entropic physical interpretation. We define the entropy per spin \( \tilde{S} \) for the instantaneous macroscopic state \((m, r)\) as

\[
\tilde{S} \equiv \lim_{N \to \infty} \frac{1}{N} \log \sum_{\sigma} \delta \left[ m - m(\sigma) \right] \delta \left[ r - r(\sigma) \right] (20)
\]

Using the replica trick \( \log Z = \lim_{n \to 0} \frac{1}{n} \left[ Z^n - 1 \right] \) and averaging over the quenched disorder allows us to express \( \tilde{S} \) in terms of the saddle-point problem encountered in calculating \( D_{mr}[z] \):

\[
\tilde{S} = \log 2 + \lim_{N \to \infty} \lim_{n \to 0} \frac{1}{Nn} \left[ \langle \prod_{\alpha=1}^n \delta \left[ m - m(\sigma^\alpha) \right] \delta \left[ r - r(\sigma^\alpha) \right] \rangle \langle \{\mathbf{s}^\alpha\} , \{z_{ij}\} \rangle - 1 \right]
\]

\[
= \log 2 + \lim_{n \to 0} \frac{1}{n} \Psi (21)
\]

in which \( \Psi \) is the saddle-point exponent (17). The entropy \( \tilde{S} \) again depends on time through the values of the macroscopic state variables \((m, r)\).
2.4 Equilibrium

For large times the microscopic probability distribution $p_t(\sigma)$ converges to the static Boltzmann expression $Z^{-1}e^{-\beta H(\sigma)}$ (with the partition function $Z \equiv \sum_{\sigma} e^{-\beta H(\sigma)}$). Since $H(\sigma)$ (4) can be written in terms of the macroscopic state variables $m(\sigma)$ and $r(\sigma)$, at equilibrium we automatically obtain equipartitioning of probability in the $(m, r)$ sub-shells of the ensemble (equipartitioning in the energy shells is an even stronger statement). This removes the need for the second of our closure assumptions, leaving need only for our assumption that the evolution of $m$ and $r$ is self-averaging. We will now demonstrate that in equilibrium we do recover the full standard results from equilibrium statistical mechanics, including the replica symmetry breaking (RSB) equations.

The standard replica formalism as applied to the SK model (see e.g. [3] or [4]) leads in the thermodynamic limit $N \to \infty$ to the following expressions for the disorder-averaged free energy per spin $\mathcal{F}$

$$\mathcal{F} = -\frac{1}{\beta} \log 2 + \lim_{n \to 0} \min \left\{ F(m, q) \right\}$$

$$F(m, q) = \frac{J_0}{2n} \sum_{\alpha} m_\alpha^2 + \frac{\beta J^2}{4n} \sum_{\alpha \gamma} q_{\alpha \gamma} - \frac{1}{\beta n} \log \left\langle e^{\beta \sum_{\alpha} \sigma_\alpha (J_0 m_\alpha + \theta) + \frac{1}{2} \beta^2 J^2 \sum_{\alpha \gamma} \sigma_\alpha \sigma_\gamma q_{\alpha \gamma}} \right\rangle$$

The corresponding saddle point equations are

$$m_\gamma = \frac{\left\langle \sigma_\gamma e^{\beta \sum_{\alpha} \sigma_\alpha (J_0 m_\alpha + \theta) + \frac{1}{2} \beta^2 J^2 \sum_{\alpha \gamma} \sigma_\alpha \sigma_\gamma q_{\alpha \gamma}} \right\rangle}{\left\langle e^{\beta \sum_{\alpha} \sigma_\alpha (J_0 m_\alpha + \theta) + \frac{1}{2} \beta^2 J^2 \sum_{\alpha \gamma} \sigma_\alpha \sigma_\gamma q_{\alpha \gamma}} \right\rangle}$$

$$q_{\gamma \delta} = \frac{\left\langle \sigma_\gamma \sigma_\delta e^{\beta \sum_{\alpha} \sigma_\alpha (J_0 m_\alpha + \theta) + \frac{1}{2} \beta^2 J^2 \sum_{\alpha \gamma} \sigma_\alpha \sigma_\gamma q_{\alpha \gamma}} \right\rangle}{\left\langle e^{\beta \sum_{\alpha} \sigma_\alpha (J_0 m_\alpha + \theta) + \frac{1}{2} \beta^2 J^2 \sum_{\alpha \gamma} \sigma_\alpha \sigma_\gamma q_{\alpha \gamma}} \right\rangle}$$

and the physical interpretation in terms of the two (disorder-averaged) functions $P(q)$ and $P(m)$ is:

$$P(q) \equiv \left\langle Z^{-2} \sum_{\sigma \sigma'} \delta \left[ q - \frac{1}{N} \sum_k \sigma_k \sigma'_k \right] e^{-\beta H(\sigma) - \beta H(\sigma')} \right\rangle_{\{z_{ij}\}}$$

$$= \lim_{n \to 0} \frac{1}{n(n - 1)} \sum_{\alpha \neq \gamma} \delta [q - q_{\alpha \gamma}]$$

$$= \lim_{n \to 0} \frac{1}{n(n - 1)} \sum_{\alpha \neq \gamma} \delta [q - q_{\alpha \gamma}]$$

15
\[ P(m) \equiv \langle Z^{-1} \sum_{\sigma} \delta \left[ m - \frac{1}{N} \sum_k \sigma_k \right] e^{-\beta H(\sigma)} \rangle_{\{z_{ij}\}} \]

\[ = \lim_{n \to 0} \frac{1}{n} \sum_{\alpha} \delta [m - m_{\alpha}] \]  

(26)

According to Parisi’s [2] theory the magnetisation is self-averaging, even in the regime where replica symmetry is broken [20], so \( P(m) \) is a delta-function and \( m_{\alpha} = m \) for all \( \alpha \). From the internal energy in thermal equilibrium \( E/N = [1 + \beta \partial_{\beta}] f \), which is also self-averaging [20], we obtain the equilibrium expression for our dynamic order parameter \( r \):

\[ r_{eq} = \frac{1}{2} \beta J \left[ 1 - \int dq \ P(q)q^2 \right] \]

For \( \beta J_0 < 1 \), where \( m_{eq} = 0 \), the continuous transition at \( \beta J = 1 \) from the paramagnetic phase with \( P(q) = \delta(q) \) to the spin-glass phase, is therefore marked by \( r_{eq} = \frac{1}{2} \).

Comparison with the dynamical eqns. (14,15,16), shows that the two approaches yield identical equations if we impose the following conditions:

\[ \mu_\alpha = \mu \equiv \beta (J_0 m + \theta) \quad \rho_\alpha = \rho \equiv \beta J \]  

(27)

Below we show that these conditions turn out to be precisely those which imply dynamical stability with respect to the macroscopic flow (8,9);

\[ \frac{d}{dt} m = 0, \quad \frac{d}{dt} r = 0 \]

and hence they also describe the same equilibrium physics.

First we consider the evolution of \( m \), using the noise distribution (18) and the conditions (27). If we perform a shift of the integration line for \( z \) and perform the integral over \( x \) we arrive at:

\[ \frac{d}{dt} m = -m + \lim_{n \to 0} \int Dz \ \frac{\langle \tanh[\rho z + \mu + \rho^2 \sum \sigma_k] e^\mu \sum \sigma_\gamma + \frac{1}{2} \rho^2 \sum \sigma_\gamma \sigma_\delta \rangle_{\sigma}}{\langle e^\mu \sum \sigma_\gamma + \frac{1}{2} \rho^2 \sum \sigma_\gamma \sigma_\delta \rangle_{\sigma}} \]

with the abbreviation \( Dz \equiv (2\pi)^{-\frac{1}{2}} e^{-\frac{1}{2}z^2} dz \). In the numerator of this expression we perform the average over \( \sigma_1 \) explicitly, and use the identity

\[ e^{-u} \int Dz \ \tanh[\rho z - \rho^2 + u] + e^u \int Dz \ \tanh[\rho z + \rho^2 + u] = 2 \sinh[u] \]  

(28)
to arrive at:

\[
\frac{d}{dt}m = -m + \lim_{n \to 0} \langle \sigma_1 \left( e^{\mu \sum_{\gamma} \sigma_{\gamma} + \frac{1}{2} \rho^2 \sum_{\gamma \delta} q_{\gamma \delta} \sigma_{\gamma} \sigma_{\delta}} \right) \sigma \rangle = 0
\]

(utilizing (14)).

In a similar way we obtain for the evolution of \( r \):

\[
\frac{d}{dt}r = -2r + \lim_{n \to 0} \int Dz \langle [z + \rho \sum_{\alpha} q_{1\alpha} \sigma_{\alpha}] \tanh[\rho z + \mu + \rho^2 \sum_{\alpha} q_{1\alpha} \sigma_{\alpha}] \rangle
\]

\[
\left( e^{\mu \sum_{\gamma} \sigma_{\gamma} + \frac{1}{2} \rho^2 \sum_{\gamma \delta} q_{\gamma \delta} \sigma_{\gamma} \sigma_{\delta}} \right) \sigma
\]

Again we perform the average over \( \sigma_1 \) in the numerator explicitly, simplify the result with the identity

\[
e^{-u} \int Dz \left[ \rho z - \rho^2 + u \right] \tanh[\rho z - \rho^2 + u] + e^u \int Dz \left[ \rho z + \rho^2 + u \right] \tanh[\rho z + \rho^2 + u]
\]

\[= 2u \sinh[u] + 2 \rho^2 \cosh[u]
\]

and arrive at:

\[
\frac{d}{dt}r = -2r + \lim_{n \to 0} \\rho \sum_{\alpha > 1} q_{1\alpha} \left( \langle \sigma_1 \sigma_{\alpha} \rangle \left( e^{\mu \sum_{\gamma} \sigma_{\gamma} + \frac{1}{2} \rho^2 \sum_{\gamma \delta} q_{\gamma \delta} \sigma_{\gamma} \sigma_{\delta}} \right) \sigma \right)
\]

\[= \lim_{n \to 0} \rho \sum_{\alpha} q_{1\alpha}^2 - 2r = 0
\]

(utilizing (15,16)).

Finally we use the equilibrium conditions (27) to show that the thermodynamic entropy per spin \( S = \beta^2 \partial_{\beta} \tilde{S} \) in equilibrium coincides with the dynamic entropy per spin \( \tilde{S} \) given by (20):

\[
S = \log 2 - \lim_{n \to 0} \left\{ m\mu + \frac{3\rho^2}{4n} \sum_{\alpha \gamma} q_{1\alpha}^2 \right\} - \log \left( e^{\mu \sum_{\alpha} \sigma_{\alpha} + \frac{1}{2} \rho^2 \sum_{\alpha \gamma} \sigma_{\alpha} \sigma_{\alpha \gamma}} \right)
\]

According to (17) and (21) this expression is identical to the one we obtained for \( \tilde{S} \).
3 Replica Symmetry

3.1 Replica-Symmetric Local Field Distribution

We first make the replica-symmetric ansatz (RS) and assume $P_{mr}(q)$ to be a delta-function, so $q_{\alpha\beta} = \delta_{\alpha\beta} + q(1 - \delta_{\alpha\beta})$. From this ansatz the saddle-point equations (14,15,16) allow us to deduce $\mu_\alpha = \mu$ and $\rho_\alpha = \rho$. For $n \to 0$ we obtain:

$$m = \int Du \tanh(\rho \sqrt{q} u + \mu)$$

(30)

$$q = \int Du \tanh^2(\rho \sqrt{q} u + \mu)$$

(31)

$$\rho = \frac{2r}{1 - q^2}$$

(32)

The corresponding local field distribution $D_{mr}^{RS}[z]$ becomes:

$$D_{mr}^{RS}[z] = \int \frac{dx}{2\pi} e^{-\frac{1}{2}x^2 + ixz} \lim_{n \to 0} \int Du \cosh[\rho \sqrt{q} u + \mu - ix\rho] \cosh^{n-1}[\rho \sqrt{q} u + \mu - ixq\rho]$$

We first perform the shift $u \to v + ix\sqrt{q}$, after which the limit $n \to 0$ can be safely taken. In the resulting expression we can perform the integral over $x$. After some final transformations of integration variables we arrive at

$$D_{mr}^{RS}[z] = \frac{e^{-\frac{1}{2}[z + \rho(1-q)]^2}}{2\sqrt{2\pi}} \left\{ 1 + \int Dy \tanh \left[ \rho y \sqrt{q(1-q)} - \rho q [z + \rho(1-q)] - \mu \right] \right\}$$

$$+ \frac{e^{-\frac{1}{2}[z - \rho(1-q)]^2}}{2\sqrt{2\pi}} \left\{ 1 + \int Dy \tanh \left[ \rho y \sqrt{q(1-q)} + \rho q [z - \rho(1-q)] + \mu \right] \right\}$$

(33)

This expression cannot be simplified further, except for three special cases which we will discuss below.

From expression (33) and the saddle-point equations it is clear that $D_{mr}^{RS}[z]$ is Gaussian only along the line $r = 0$:

$$r = 0 : \quad D_{m,0}^{RS}[z] = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$$

(34)

For $r = 0$ we obtain $q = m^2$. We can identify such macroscopic states as purely ferromagnetic (for $m \neq 0$) or paramagnetic (for $m = 0$). The result
(34) is indeed what one would obtain in thermal equilibrium for $\beta J = 0$
(where only the para-magnetic and purely ferromagnetic states are found).

A second simplification of (33) results for $q = 0$ (the paramagnetic state),
which can only occur along the line $m = 0$. For $m = 0$ the RS saddle-point
equations reduce to

$$q = F(q) \equiv \int D u \ \tanh^2 \left[ \frac{2 u r \sqrt{q}}{1 - q^2} \right]$$

with the properties

$$F(1) = 1 \quad F(q) = 4 r^2 q - 32 r^4 q^2 + O(q^3)$$

from which we conclude that along the $m = 0$ line we find a paramagnetic
$(q = 0)$ state for $r < \frac{1}{2}$:

$$m = 0, \ r < \frac{1}{2} : \ D_{0,r}^{RS}[z] = \frac{1}{2\sqrt{2\pi}} e^{-\frac{1}{2}[z+2r]^2} + \frac{1}{2\sqrt{2\pi}} e^{-\frac{1}{2}[z-2r]^2}$$

This result is indeed what one would obtain for the field distribution in
thermal equilibrium in the paramagnetic region of the phase diagram [25].
For $r > \frac{1}{2}$, $m = 0$ we obtain a spin-glass with $q \neq 0$, where again we know
from equilibrium studies [25] that the local field distribution indeed has a
non-trivial form like that in (33).

The third simplification occurs for $q \approx 1$. Expanding the saddle-point
equations in powers of $\epsilon \equiv 1 - q$ gives the leading orders

$$\rho = r \epsilon^{-1} + \ldots \quad \mu = r \epsilon^{-1} \sqrt{2} \ \text{erf}^{-1}(m) + \ldots \quad (36)$$

$$r = \sqrt{\frac{2}{\pi}} e^{-[\text{erf}^{-1}(m)]^2} \quad (37)$$

Equation (37) defines the line in the $(m, r)$ plane where the situation $q = 1$
actually occurs. Near this line we can use the scaling relations (36) to show
that expression (33) reduces to the Schwalter-Klein [26] form, which in
equilibrium would be obtained in the limit of zero temperature [25] (in RS
approximation):

$$D_{m,r(m)}^{RS}[z] = \frac{e^{-\frac{1}{2}[z+r(m)]^2}}{\sqrt{2\pi}} \theta \left[ -z - r(m) - \sqrt{2} \ \text{erf}^{-1}(m) \right] \quad (38)$$

$$+ \frac{e^{-\frac{1}{2}[z-r(m)]^2}}{\sqrt{2\pi}} \theta \left[ z - r(m) + \sqrt{2} \ \text{erf}^{-1}(m) \right]$$

in which $r(m)$ denotes the $q = 1$ line (37).
3.2 Special Lines in the Flow Diagram

In order to check the applicability of the RS ansatz we calculate the equivalent of the RS zero-entropy (‘freezing’) line in the \((m, r)\) plane (where the number of microscopic configurations contributing to our averages vanishes), and the de Almeida-Thouless (AT) line [23], where a replica-symmetry breaking (RSB) solution of the saddle-point equations bifurcates from the RS saddle-point.

In RS theory the dynamic entropy (20) is, according to (21), given by

\[
\tilde{S}_{RS} = \log 2 + \int Du \log \cosh \left[ \rho u \sqrt{q + \mu} - m \mu + \frac{1}{4} \rho^2 (1-q)^2 - pr \right] \tag{39}
\]

For \(r = 0\) (where there is no spin-glass alignment) the entropy reduces to

\[
\tilde{S}_{RS, r=0} = \log 2 - \int_{\tanh^{-1}(|m|)}^{0} ds \left[ 1 - \tanh^2(s) \right] \in [0, \log 2]
\]

with \(\tilde{S}_{RS, r=0} = \log 2\) at \(m = 0\) (the para-magnetic state) down to \(\tilde{S}_{RS, r=0} = 0\) at \(m = \pm 1\) (the fully ordered ferro-magnetic state). Along the line \(m = 0\), below \(r = \frac{1}{2}\), we find \(q = 0\) and \(\tilde{S}_{RS} = \log 2 - r^2 > 0\). Using the scaling relations (36) one can finally show that near the \(q = 1\) line (37) the RS entropy is negative, except for \(|m| = 1, r = 0\), where the \(q = 1\) line and the line \(\tilde{S}_{RS} = 0\) meet. Since the physical dynamical entropy cannot be negative this already signals an inadequacy in the RS ansatz, analogous to that found in the equilibrium RS theory of SK [1]. The full curve \(\tilde{S}_{RS} = 0\) signals this inadequacy in \((m, r)\) space.

An AT-line [23] signals the first continuous bifurcation of a saddle-point solution without replica symmetry from the replica-symmetric one. We follow the usual convention and assume that the first such bifurcation is the replicon mode:

\[
q_{\alpha\beta} \rightarrow q + \delta q_{\alpha\beta}, \quad \rho_\alpha = \rho, \quad \mu_\alpha = \mu
\]

Inserting this ansatz into the full saddle point equations shows that the RSB bifurcations are of the form \(\sum_{\alpha \neq \beta} \delta q_{\alpha\beta} = 0\). After some bookkeeping and after taking the limit \(n \rightarrow 0\) one then obtains the bifurcation condition which defines the dynamic AT line:

\[
1 - \rho^2 \int Du \cosh^{-4} [\rho \sqrt{qy} + \mu] = 0 \tag{40}
\]
Figure 4: AT line (large dashes), RS freezing line (dashes/dots) and $q = 1$ line (small dashes) in the $(m,r)$ plane.
The RS solution is stable as long as the left-hand side of (40) is positive. For \( r = 0 \) (with \(|m| < 1\)) the RS solution is indeed stable. The AT line intersects the line \( m = 0 \) at \( r = \frac{1}{2} \). Using the scaling relations (36) one can also show that near the \( q = 1 \) line (37) the RS solution is unstable, except for \(|m| = 1, r = 0\), where the \( q = 1 \) line and the AT line meet.

In figure 4 we show the freezing line (where \( S_{RS} = 0 \)) (39), the AT line (40) and the \( q = 1 \) line (37) in the \((m, r)\) plane. We note that the \( q = 1 \) line always lies above the \( S_{RS} = 0 \) line, which in turn lies above the AT line, except at \(|m| = 1, r = 0\). Thus the AT line is the critical one for replica symmetry. The separation between the AT line and the \( q = 1 \) line, which provides an effective boundary for the \((m, r)\) dynamics, is greatest for small \( m \) where the ferromagnetic order is small and occurs for large \( r \), when spin-glass alignment is greatest.

Below the AT line the RS solution is stable against RSB fluctuations. The RS solution breaks down in the region where ferromagnetic order is small and spin-glass type field-alignment dominates.

### 3.3 Replica-Symmetric Flow Equations

By combining the equations (34) with expression (33) we arrive at a closed set of autonomous differential equations describing the deterministic evolution of the macroscopic state \((m, r)\):

\[
\frac{d}{dt}m = \int \int DxDy \; M(m,r;x,y) - m
\]

\[
\frac{d}{dt}r = \int \int DxDy \; R(m,r;x,y) - 2r
\]

in which

\[
M(m,r;x,y) = \\
\frac{1}{2} \left[ 1 - \tanh \left[ x\rho \sqrt{q(1-q) + \rho qy + \mu} \right] \tanh \beta [J_0 m + J y + \theta - J \rho (1-q)] \right] \\
+ \frac{1}{2} \left[ 1 + \tanh \left[ x\rho \sqrt{q(1-q) + \rho qy + \mu} \right] \tanh \beta [J_0 m + J y + \theta + J \rho (1-q)] \right]
\]

\[
R(m, r; x, y) = \\
\frac{1}{2} \left[ y - \rho (1-q) \right] \left[ 1 - \tanh \left[ x\rho \sqrt{q(1-q) + \rho qy + \mu} \right] \tanh \beta [J_0 m + J y + \theta - J \rho (1-q)] \right]
\]
\[ \frac{1}{2} \left[ y + \rho (1-q) \right] \left[ 1 + \tanh \left( x \rho \sqrt{q(1-q)} + \rho q y + \mu \right) \right] \tanh \beta \left[ J_0 m + J y + \theta + J \rho (1-q) \right] \]

with \( \{q, \rho, \mu\} \) being functions of the macroscopic state \( (m,r) \), to be solved from the saddle-point equations (30,31,32).

In figure 3.3 we compare the flow defined by (41,42) with numerical simulations for \( N = 3000, \theta = 0, J = 1, J_0 \in \{0,1,2\} \) and four choices of the temperature \( T \). The parameters \( J_0 \) and \( T \) have been chosen in such a way that the corresponding equilibrium situations (according to standard equilibrium theory [4]) include spin-glass states \( (J_0 < 1, T < 1) \), states with ferro-magnetic order \( (J_0 > 1, T < J_0) \) and para-magnetic states \( (J_0 < T, T > 1) \). At intervals of \( \Delta t = 1 \) iteration/spin we measure the macroscopic order parameters \( (m,r) \) in the simulated system and calculate the derivatives \( \left( \frac{d}{dt} m, \frac{d}{dt} r \right) \) as predicted by (41,42). The initial states generating the trajectories (labelled by \( \ell = 0, \ldots, 10 \)) were drawn at random according to \( p_0(\vec{s}) \equiv \prod_i \left[ \frac{1}{2} \left[ 1 + \frac{1}{2} \delta_{s_i, \xi_1} + \frac{1}{2} \left[ 1 - \frac{1}{2} \delta_{s_i, -\xi_1} \right] \right] \right] \), such that that \( \langle m \rangle_{t=0} = 0.1 \ell \) and \( \langle r \rangle_{t=0} = 1 \). The figure indicates that the flow is described quite well by (41,42), except for those regions in the \( (m,r) \) plane where the RS solution is unstable (above the AT line). More detailed comparisons between theory and simulations will be made in a subsequent section.

From the RS saddle-point equations (30,31,32) we can directly recover all equilibrium results obtained by Sherrington and Kirkpatrick [1, 7]. Inserting the two relations \( \rho = \beta J \) and \( \mu = \beta (J_0 m + \theta) \) into our RS saddle-point equations gives

\[
\begin{align*}
m &= \int D u \, \tanh \beta (J_0 m + J \sqrt{q} u + \theta) \\
qu &= \int D u \, \tanh^2 \beta (J_0 m + J \sqrt{q} u + \theta) \\
r &= \frac{1}{2} \beta J \left[ 1 - q^2 \right]
\end{align*}
\]

We now use the identities (28,29) and perform a rotation in the space of the gaussian integrals in (41,42) to arrive for the RS thermal equilibrium state of [1, 7] at

\[
\begin{align*}
\frac{d}{dt} m &= \int D x \, \tanh \beta (J_0 m + J \sqrt{q} x + \theta) - m = 0 \\
\frac{d}{dt} r &= \beta J q \left[ q - \int D x \, \tanh^2 \beta (J_0 m + J \sqrt{q} x + \theta) \right] = 0
\end{align*}
\]
Figure 5: Trajectories in the \((m, r)\) plane obtained by performing sequential simulations of the SK model with \(N = 3000\) and zero external field, for \(t \leq 10\) iterations/spin (solid lines), together with the velocities as predicted by the theory (arrows, calculated at intervals of 1 iteration/spin for the instantaneous macroscopic state of the corresponding simulation, at the point of the base of the arrow). The first row of graphs corresponds to \(T = 1.5\), the second to \(T = 1.0\), the third to \(T = 0.5\) and the fourth to \(T = 0\). Dashed lines indicate the \(q = 1\) line (upper), the RS freezing line (middle) and the AT line (lower).

The RS order parameter equations in thermal equilibrium, as derived in \([1, 7]\), thus indeed define fixed-points of our flow equations, as also follows from our more general analysis of section 2.4.

If we insert the fixed-point relations into our expression (40) for the AT line, we obtain:

\[
1 = \beta^2 J^2 \int Dx \cosh^{-4} \beta (J_0 m + J \sqrt{q} x + \theta)
\]

which, again, corresponds exactly to the result obtained in thermal equilibrium \([23]\). This includes both the line segment separating paramagnetic from spin-glass phase, where \((m_{eq}, r_{eq}) = (0, \frac{1}{2})\), and the line segment separating the replica-symmetric and replica-symmetry broken ferromagnetic phases.

Our dynamical RS laws (41, 42) thus lead precisely to the thermal equilibrium described by Sherrington and Kirkpatrick \([1, 7]\) and de Almeida and Thouless \([23]\), including entropy and stability with respect to replica-symmetry breaking.

### 3.4 Relaxation Times

We will investigate the asymptotic behaviour of the RS flow equations. For simplicity and to suppress notation we restrict ourselves to the case \(J_0 = 0, J = 1\). We expand both RS flow equations (8, 9) around the equilibrium state \((m, r) = (0, r_{eq})\)

\[
m(t) = \epsilon \tilde{m}(t) + \mathcal{O}(\epsilon^2) \quad r(t) = r_{eq} - \epsilon \tilde{r}(t) + \mathcal{O}(\epsilon^2)
\]

\[
r_{eq} = \frac{1}{2} \beta (1 - q^2) \quad q = \int Dy \tanh^2 \left[ \frac{\beta \sqrt{q} y}{1 - q^2} \right]
\]

(43)
as well as the RS saddle-point equations (30,31,32):

\[
\left( \frac{\partial q}{\partial m} \right)_{0,\text{eq}} = 0, \quad \left( \frac{\partial \mu}{\partial m} \right)_{0,\text{eq}} = \frac{1}{1 - q}, \quad \left( \frac{\partial \mu}{\partial r} \right)_{0,\text{eq}} = 0 \tag{44}
\]

\[
\left( \frac{\partial q}{\partial r} \right)_{0,\text{eq}} = 4q \frac{\int D\upsilon u^2 \tanh^2(\beta \upsilon \sqrt{q}) - q}{\beta(1 + 3q^2)} - 2q(1 - q^2)(1 + 3q^2)^{-2} - \int D\upsilon u^2 \tanh^2(\beta \upsilon \sqrt{q}) + q \tag{45}
\]

The linearised flow equations decouple since \((\partial m)_{0,\text{RS}} \left[ z \right]\) and \((\partial r)_{0,\text{RS}} \left[ z \right]\) are respectively anti-symmetric and symmetric in \(z\) (this decoupling does not depend on replica symmetry). As a result we directly obtain the two relaxation times:

\[
\tau^{-1}_m = -\lim_{t \to \infty} \frac{1}{t} \log \left[ \frac{\hat{m}(t)}{m(0)} \right] = 1 - \int dz (\partial_m D)_{0,\text{eq}} \left[ z \right] \tanh [\beta z] \tag{46}
\]

\[
\tau^{-1}_r = -\lim_{t \to \infty} \frac{1}{t} \log \left[ \frac{\hat{r}(t)}{r(0)} \right] = 2 - \int dz (\partial_r D)_{0,\text{eq}} \left[ z \right] z \tanh [\beta z] \tag{47}
\]

In the paramagnetic temperature region \(T > 1\) the partial derivatives of the local field distribution \(\text{RS} \) can be calculated easily. We find

\[
\tau^{-1}_m = 1 - \int Dz \tanh(\beta z + \beta^2) \tag{48}
\]

\[
\tau^{-1}_r = 2 - 2 \int Dz z(\beta z + \beta^2) \tanh(\beta z + \beta^2) \tag{49}
\]

For \(T < 1\) more work is required to find the partial derivatives \((\partial D)\) and the relaxation times.

We first turn to the magnetisation. After some bookkeeping we can derive from (33):

\[
(\partial_m D)_{0,\text{eq}} \left[ z \right] = \frac{e^{-\frac{1}{2}(z-\Delta)^2}}{2(1-q)\sqrt{2\pi}} \left\{ 1 - \int Dy \tanh^2 \Delta \left[ y \left( \frac{q}{1-q} \right)^{\frac{1}{2}} + (z-\Delta) \frac{q}{1-q} \right] \right\} - \frac{e^{-\frac{1}{2}(z+\Delta)^2}}{2(1-q)\sqrt{2\pi}} \left\{ 1 - \int Dy \tanh^2 \Delta \left[ y \left( \frac{q}{1-q} \right)^{\frac{1}{2}} + (z+\Delta) \frac{q}{1-q} \right] \right\}
\]

where \(\Delta \equiv \beta(1-q)\). With this expression we obtain, after a rotation in the space of the integrals:

\[
\tau_m = \frac{1 - q}{\int D\upsilon \left[ 1 - \tanh^2(\beta \upsilon \sqrt{q}) \right] \int Dz \left[ 1 - \tanh \beta (y \sqrt{q} + z \sqrt{T - q + \beta(1-q)}) \right]}
\]

(50)
Figure 6: The two asymptotic RS relaxation times $\tau_m$ and $\tau_r$ for $J_0 = \theta = 0$ and $J = 1$ as a function of temperature.
For non-zero temperatures the asymptotic relaxation of $m$ described by the RS equations is indeed exponential. For $T \to 0$, however, we can use $q \sim 1 - \beta^{-1} \sqrt{\frac{T}{q}}$ to show that the relaxation becomes non-exponential:

$$\lim_{\beta \to \infty} \tau_m^{-1} = 1 - \frac{1}{2} \int dz \, e^{-\frac{1}{2} z^2} \text{sgn}[z + \sqrt{\frac{T}{q}}] \frac{d}{dz} \text{sgn}[z] = 0$$

Next we turn to the relaxation of $r$. Taking the appropriate derivatives results in

$$\tau_r^{-1} = 2 - \frac{2}{1+q} \int Dy \left[1 + \tanh(\beta y \sqrt{q})\right] \int Dz \left[\tanh(\beta Q) + \beta Q \left[1 - \tanh^2(\beta Q)\right]\right]$$

$$- \frac{2 \sqrt{q}}{1-q^2} \left[1 + \frac{\beta}{4q} (1+3q^2) \left(\frac{\partial q}{\partial r}\right)_{\text{eq}}\right] \int Dy \, y \left[1 - \tanh^2(\beta y \sqrt{q})\right] \int Dz \, Q \tanh(\beta Q)$$

$$- \left(\frac{\partial q}{\partial r}\right)_{\text{eq}} \int Dy \left[1 + \tanh(\beta y \sqrt{q})\right] \int Dz \left[\frac{y}{2 \sqrt{q}} - \frac{z - \beta (1-q^2)}{2 \sqrt{1-q} (1+q)^2}\right]$$

$$\times \left[\tanh(\beta Q) + \beta Q \left[1 - \tanh^2(\beta Q)\right]\right]$$

(51)

with the abbreviation $Q \equiv y \sqrt{q} + z \sqrt{1-q} + \beta (1-q)$ (to be used in combination with (45)). Numerical evaluation of the integrals in (48, 49, 50, 51) as a function of temperature results in figure 6. Note that in the absence of spin-glass interactions (i.e. for $J = 0$) one would simply find $\tau_m = 1$ for all $T$. Both relaxation times converge for $T \to 0$, with $\lim_{T \to 0} \tau_r/\tau_m = 0$.

4 Further Comparisons with Numerical Simulations

In this section we present some more detailed simulation experiments, the outcome of which is compared to the predictions of our RS theory (the latter need not give sensible results above the AT line). The simulations can display two types of finite size effects: thermal fluctuations in the flow of the order parameters (i.e. finite size corrections to the Liouville equation (43)) and fluctuations in the local field distribution (i.e. finite size corrections to the steepest descent integration leading to (53)).
Figure 7: Trajectories in the \((m,r)\) plane obtained from sequential simulations of the SK model with \(N = 3200, J = 1\) and \(T = 0.1\), for three different choices of \(J_0\). Initial states: \((m, r) \sim (0.5, 0)\). Dots indicate times at which the spin-glass contributions to the local fields are measured in order to test the theory.
Figure 8: Comparison between RS theory (dashed lines) and the local field distribution as measured during the $J_0 = 0$ simulation.
Figure 9: Comparison between RS theory (dashed lines) and the local field distribution as measured during the $J_0 = 1$ simulation.
Figure 10: Comparison between RS theory (dashed lines) and the local field distribution as measured during the $J_0 = 2$ simulation.
4.1 The Local Field Distribution

First we compare our analytical result (33) directly with the outcome of measuring the spin-glass contributions to the local alignment fields during actual numerical simulations. In order to probe the different regions of the \((m, r)\) plane we performed simulations from the initial state \((m, r) \sim (0.5, 0)\) for \(J_0 = 0\), \(J_0 = 1\) and \(J_0 = 2\) and measured the instantaneous distribution of the spin-glass contributions to the local alignment fields at different times. In figure 7 we show the resulting trajectories in the \((m, r)\) plane (solid lines), together with the AT line (lower dashed line) and the \(q = 1\) line (upper dashed line). Dots indicate the instances were the relevant measurements were done: \(t = 0\), \(t = 1\), \(t = 5\) and \(t = 10\) (unit: iterations per spin). In figures 8, 9 and 10 the distributions as measured from the full microstate \(\sigma(t)\) (histograms) and calculated from (33) with only \(m(t)\) and \(r(t)\) as input (dashed lines) are shown. The RS theory leading to the distribution (33) turns out to give a good qualitative description of the simulation data; significant deviations are confined to the region above the AT line. Below the AT line these numerical results partially justify \textit{a posteriori} the ansätze of self-averaging and subshell equipartitioning, made to close the set of deterministic dynamical laws for the order parameters \(m\) and \(r\).

4.2 Cooling in a Small External Field

Next we study the evolution in time of the order parameters \(m\) and \(r\) that results after cooling the system instantaneously from \(T = \infty\), the paramagnetic state \((m, r) = (0, 0)\), to \(T = 0\). For simplicity we choose \(J_0 = 0\) and \(J = 1\). An external field \(\theta = 0.1\) is applied in order to obtain non-trivial evolution for the magnetisation (this field being small assures the macroscopic state vector eventually enters into the spin-glass region of the \((m, r)\) flow diagram, above the AT line).

In figures 12 and 11 we compare the result of performing numerical simulations (for an \(N = 3200\) system) with the result of solving numerically the RS flow equations (8,9). At least within the duration of the numerical experiments \((t \leq 10\ \text{iterations/spin})\), the direction of the flow in the \((m, r)\) plane is correctly described by the RS flow equations, even above the AT line. Within the limitations of our simulations the RS theory, however, breaks down even before the AT line is crossed, in that the RS flow equations fail to describe an overall slowing down of the macroscopic flow.
Figure 11: Flow in the \((m,r)\) plane of the order parameters \(m(t)\) and \(r(t)\), at \(T = 0.1\) with a small external field \(\theta = 0.1\). Initial state: \((m,r) = (0,0)\) (the paramagnetic state). Fluctuating lines: three independent \(N = 3200\) simulations. Smooth solid line: solution of RS flow equations. Dashed lines: the \(q = 1\) line (upper) and the AT line (lower).
Figure 12: Evolution in time of the order parameters $m(t)$ (left picture) and $r(t)$ (right picture), at $T = 0.1$ with a small external field $\theta = 0.1$. Initial state: $(m, r) = (0, 0)$ (the paramagnetic state). Fluctuating lines: three independent $N = 3200$ simulations. Smooth line: solution of RS flow equations below AT line. Dashed line: solution of RS flow equations above AT line.
4.3 Decay from a Fully Magnetized State

Finally we study the relaxation from the fully magnetized initial state \((m, r) = (1, 0)\) (a la Kinzel [17], albeit for short time-scales \(t \leq 10\) only). For simplicity we choose \(J_0 = \theta = 0\) and \(J = 1\). Figures 13 and 14 show the result of comparing numerical simulations for an \(N = 3200\) system with the result of solving numerically the RS flow equations (8,9). Again within the duration of the numerical experiments \((t \leq 10\) iterations/spin) the direction of the flow in the \((m, r)\) plane is correctly described by the RS flow equations, whereas the RS theory apparently fails to describe the overall slowing down that sets in even before the AT line is crossed (which gives rise to the familiar remanent magnetisation [17]). In order to describe the slow relaxation of this remanent magnetization above the AT line, measured rather in terms of a few thousand iterations per spin, we clearly need the RSB version of our theory.
Figure 14: Evolution in time of the order parameters $m(t)$ (left picture) and $r(t)$ (right picture), at $T = 0.0$ with $J_0 = \theta = 0$ and $J = 1$. Fluctuating lines: three independent $N = 3200$ simulations. Smooth line: solution of RS flow equations below AT line. Dashed line: solution of RS flow equations above AT line.
5 Discussion

In this paper we have developed a dynamical theory, valid on finite time-scales, to describe the Glauber dynamics of the SK model in terms of deterministic flow equations for two macroscopic state variables: the magnetisation and the spin-glass contribution to the energy. Two transparent physical assumptions, based on a systematic removal of microscopic memory effects, allow us to calculate the time-dependent distribution of local alignment fields in terms of the instantaneous order parameters only and thereby obtain a closed set of flow equations for our two order parameters. The theory produces in a natural way dynamical generalisations of the AT- and zero-entropy lines and of Parisi’s order parameter function $P(q)$. In equilibrium we recover the standard results from equilibrium statistical mechanics, including the full RSB equations.

In calculating the order parameter flow explicitly we have made the replica-symmetric (RS) ansatz, as a natural first step. A subsequent paper will be devoted to the implications of breaking the replica symmetry (RSB). We found that in most of the flow diagram replica symmetry is stable. Numerical simulations suggest that our equations describe the shape of the local field distribution and the macroscopic dynamics quite well in the region where replica symmetry is stable. In the region of the flow diagram where the RS solution is unstable the flow direction as given by the RS theory seems still correct and the RS theory even predicts non-exponential relaxation in the limit $T \to 0$. However, the RS theory fails to describe a rigorous slowing down which, according to simulations, sets in near the de Almeida-Thouless [23] line. Intuitively one expects the breaking up of phase space, as indicated by the breaking of replica symmetry, to have a slowing down effect on the macroscopic flow. Preliminary investigations of the effect of replica symmetry breaking, based on expansions just above the AT line and on a one-step symmetry breaking a la Parisi, show that this is indeed the case [27].

The main appeal of our formalism we consider to be its transparency. The theory is formulated in terms of two directly observable macroscopic state variables: the magnetisation and the spin-glass contribution to the energy per spin. Furthermore the macroscopic laws are derived directly from the underlying microscopic stochastic equations, given two key assumptions. An interesting difference with existing (mostly Langevin) approaches is that in the present formalism replica theory enters naturally as a mathematical tool in calculating the time-dependent distribution of local alignment fields.
One of the two assumptions on which our analysis is based (self-averaging of the macroscopic laws with respect to the frozen disorder) is quite standard. Both assumptions are supported by evidence from numerical simulations. Based on the agreement between theory and simulations in the RS region we believe that our two closure assumptions lead to a theory which captures the main physics of the order parameter flow of the SK model on finite time-scales, and that the impact of microscopic memory effects (which in the theory are explicitly removed) can be viewed, as in [21, 22], principally as an overall slowing down. Our next step will be to investigate in detail the RSB version of our dynamical laws, which will be the subject of a subsequent paper.

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