Spectrum of Soft Compact Linear Operator with Properties

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Abstract. Soft operators defined on soft normed spaces are relatively modern concepts. Many properties of these operators have not been thoroughly studied. We have introduced in this paper some new concepts related to the soft compact operators such as spectrum of soft compact operator, null space of soft compact operator and rang of soft compact operator. Many important theorems related to these concepts were introduced.

Keywords: soft compact operator, null space of soft compact operator, range of soft compact operator.

1. Introduction
Molodtsov [1] in 1999 started the concept of soft sets as a new mathematical instrument for dealing with uncertainties. He introduces some presentations of this theory for solving several real-world problems in engineering, economy, medical science, community science, etc. Few years later Maji et al. [2] introduced a number of operations on soft sets and used soft sets to solve decision making problems. Feng et al.2010 in [3] describe some new operations on soft sets in 2009. In the reduction line and addition of parameters of the soft sets, some work was done by Chen [4]. Aktas and Cagman [5] introduced the notion of soft group and discussed various properties. Feng et al.2008 [6] worked on soft ideals, soft semiring and idealistic soft semiring. Shabir and Naz, [7] were given the idea of soft topological spaces. Mappings among soft sets were explained by Majumdar and Samanta [8]. Feng et al. [3] worked on soft sets together with fuzzy sets and rough sets. Das and Samanta [8] presented a concept of soft real (complex) sets, soft real (complex) numbers respectively, and some of their basic effect have been check out. They present some submissions of soft real sets (number) in Virtual reality problems. And later they presented the ideas of soft metric In addition to soft norm, soft inner product. Many properties of soft metric spaces, soft normed spaces and soft inner product spaces have been check out and many examples and counter examples were introduced.

2. Preliminaries
The basic definitions and theorems were introduced in this section that may found in earlier studies.
Definition 2.1 [1] Suppose X is a universe set; T is a set of parameters. Consider ϕ(X) is the set of all subsets of X and B ≠ ∅ is a subset of T. An ordered pair (H, B) is named a soft set over X, where H is a mapping given by H: B → ϕ(X). We can say that a soft set over X is parameterized kindred of subsets of the universe X. H (ω) can consider like a set of ω - approximate elements of (H, B) for all ω ∈ B.

Definition 2.2 [3] Suppose (H, B) and (J, D) are two soft sets over a shared set X, Then (H, B) is a soft subset of (J, D) if:
(1) B ⊆ D.
(2) For all ω ∈ B, H (ω) ⊆ J (ω). We write (H, B) ⊆ (J, D).

Definition 2.3 [3] Two soft sets (H, B) and (J, D) over a shared set X are called identical, if (H, B) and (J, D) are soft subset of each other.

Definition 2.4 [2] Let (H, B), (J, D) be two soft sets over the shared set X. The union of (H, B) and (J, D) is the soft set (L, M); Assuming M = B ∪ D and for all ω ∈ M,

\[ H(\omega) \quad \text{if} \quad \omega \in B - D \]
\[ J(\omega) \quad \text{if} \quad \omega \in D - B \]
\[ H(\omega) \cup G(\omega) \quad \text{if} \quad \omega \in B \cap D \]

In Mathematical expression (H, B) ∪ (J, D) = (L, M).

Definition 2.5 [6] Let (H, B), (J, D) be two soft sets over the shared set X. The intersection of (H, B) and (J, D) is the soft set (K, M); Assuming M = B ∩ D and for all ω ∈ M, K (ω) = H(ω) ∩ J(ω). In Mathematical expression (H, B) ∩ (J, D) = (K, M).

Suppose X be an initial universal set and B is a non-flatulent set of parameters. In the upstairs definitions the set of parameters may differ from soft set to another, but in our considerations, through this paper all soft sets have the same set of parameters B. The upstairs definitions are also useable for these types of soft sets as a particular case of those definitions.

Definition 2.6 [8] For a soft set (F, B), the complement of (F, B) is symbolized by (F, B)^c = (F^c, B), assuming F^c: B → ϕ(X) defined by F^c(λ) = X - F (λ), with any ω ∈ B.

Definition 2.7 [2] A soft set (F, A) over X is called an absolute soft set symbolized via X if F (ω) = X with every ω ∈ B.

Definition 2.8 [2] A soft set (F, A) over X is called a null soft set symbolized via ∅ if F (ω) = ∅ with every ω ∈ B.

Definition 2.9 [7] Let (H, B), (J, D) be two soft sets over the shared set X. The difference (H, B) of (F, B) and (G, B), symbolized by (F, B) \ (J, B), is defined via H (ω) = F(ω) \ G(ω) with any ω ∈ B.

Proposition 2.10 [7] for two soft sets (F, B) and (J, D) we have:
(i) \[ (F, B) \cup (J, B))^c = (F, B)^c \cap (J, B)^c. \]
(ii) \[ (F, B) \cap (J, B))^c = (F, B)^c \cup (J, B)^c. \]

Definition 2.11 [9] Let X is a non-flatulent set of elements and B ≠ ∅ is a set of parameter. The function ε: B → X is called a soft element of X. A soft element ε of X is belongs to a soft set R of X, it is symbolized with ε ∊ R, if ε(ω) ∈ R \ (ω) for every ω ∈ A. consequently, for a soft set R of X we obtained that R (ω) = {ε (ω), ε ∈ R}, ω ∈ B.

We can recognized each singleton soft set (a soft set (H, B) for which H (ω) is a singleton set, for every ω ∈ B) with a soft element by just recognizing the one element set with the element that it contains, wit any ω ∈ B.

Definition 2.12 [10] Suppose ℝ (ℝ) be the collection of all non-empty bounded subsets of ℝ (ℝ is real number) and B booked as a parameters set. Then, a mapping H: B → ℝ (ℝ) is named a soft real set. and symbolized with (H, B). If specifically (H, B) is a singleton soft set, then when recognizing (H, B) with the agreeing soft element, it will be called a soft real number.

The collection of each soft real numbers is symbolized with ℝ (B) while the collection of non-negative only is symbolized with ℝ(B)^+.
Definition 2.13 [11] suppose \( \mathcal{P} (\mathbb{C}) \) be the collection of each non-flatulent bounded subsets of the set of complex numbers \( \mathbb{C} \). \( B \) is a set of parameters. Therefore, a mapping \( H : B \rightarrow \mathcal{P} (\mathbb{C}) \) is named a soft complex set symbolized with \((H, B)\). If in specific \((H, B)\) is a singleton soft set, then recognizing \((H, B)\) with the agreeing soft element, it will be named a soft complex number.

The collection of each soft complex numbers is symbolized with \( \mathbb{C}(B) \).

Definition 2.14 [11] supposes \((H, B)\) be a soft complex set. The complex conjugate of \((H, B)\) is symbolized with \( \overline{H} \), and \( (H, B) \) is defined by \( \overline{H} = \{ \overline{z} : \overline{z} \in H(\omega) \} \), for every \( \omega \in B \), assuming \( \overline{z} \) is complex conjugate of the ordinary complex number \( z \). The complex conjugate of a soft complex number \((H, B)\) is \( \overline{H}(\omega) = \{ \overline{z} : z = H(\lambda) \} \), for every \( \omega \in B \).

Definition 2.15 [11] Let \((L, B), (J, B) \in \mathbb{C}(B)\). Then, the Addition, subtraction, multiplication and division are interpreted with:

1. \((L + J) (\omega) = z + p, z \in L (\omega), p \in J (\omega)\), for all \( \omega \in B \).
2. \((L - J) (\omega) = z - p, z \in L (\omega), p \in J (\omega)\), for all \( \omega \in B \).
3. \((LJ) (\omega) = zp, z \in L (\omega), p \in J (\omega)\), for all \( \omega \in B \).
4. \((L/J) (\omega) = z/p, z \in L (\omega), p \in J (\omega)\), on condition that \( J (\omega) \neq 0 \), for all \( \omega \in B \).

Definition 2.16 [11] Let \((L, B)\) be a soft complex number. The modulus of \((L, B)\) is symbolized by \(|L|\) and is defined by \(|L| (\omega) = |z|, z \in L (\omega)\), for all \( \omega \in B \), assuming \( z \) is an usual complex number.

Because the modulus of all usual complex number (or real number) are a non-negative real number together with definition of soft real numbers, We can conclude that \(|L|\) is a non-negative soft real number with any soft complex number \((L, B)\).

Consider \( X \) is a non-flatulent set and \( \bar{X} \) be the absolute soft set i.e., \( V(\omega) = X \), for each \( \omega \in B \), where \( V(B) = \bar{X} \). Suppose \( S(\bar{X}) \) be the collection of all soft sets \((H, B)\) over \( X \) with condition \( H(\omega) \neq \emptyset \), for all \( \omega \in B \) collected with the null soft set \( \bar{N} \). Let \((H, B) \neq \emptyset \in S(\bar{X})\), then the gathering of all soft elements of \((H, B)\) will be symbolized by \( SE(H, B) \). For a gathering \( \mathfrak{Y} \) of soft elements of \( \bar{X} \), the soft set created by \( \mathfrak{Y} \) is symbolized with \( SS(\mathfrak{Y}) \).

Definition 2.17 [12] A mapping \( M : SE(\bar{X}) \times SE(\bar{X}) \rightarrow \mathbb{R}(B)^* \), is called a soft metric on the soft set \( \bar{X} \) if it fulfills the cases listed below:

1. \( M(\bar{x}; \bar{y}) \geq 0 \), with any \( \bar{x}, \bar{y} \in \bar{X} \).
2. \( M(\bar{x}; \bar{y}) = 0 \), if and only if \( \bar{x} = \bar{y} \).
3. \( M(\bar{x} + \bar{y}, \bar{z}) = M(\bar{y}, \bar{z}) \) with any \( \bar{x}, \bar{y}, \bar{z} \in \bar{X} \).
4. With any \( \bar{x}, \bar{y}, \bar{z} \in \bar{X} \), \( M(\bar{x}; \bar{y}) + M(\bar{y}; \bar{z}) + M(\bar{z}; \bar{x}) \).

The soft set \( \bar{X} \) together with above mapping \( M \) on \( \bar{X} \) is called a soft metric space and is symbolized with \((\bar{X}, M, B)\) or \((\bar{X}, \mathfrak{Y}, B)\).

Definition 2.18 [13] Let \( Q \) be a vector space over a field \( K \) and \( B \) is a set of parameters. Let \( L \) be a soft set over \((Q, B)\). If for all \( \omega \in B, L(\omega) \) is a vector subspace of \( Q \), then \( L \) is called a soft vector space of \( Q \) over \( K \).

Definition 2.19 [14] Suppose \( L \) is a soft vector space of \( Q \) over \( K \). Let \( H : B \rightarrow \mathcal{P}(Q) \) be a soft set over \((Q, B)\). If for each \( \omega \in B, H(\omega) \) is a vector subspace of \( Q \) over \( K \) and \( L(\omega) \supseteq H(\omega) \), then \( H \) is called a soft vector subspace of \( L \).

Definition 2.20 [13] Suppose LL is a soft vector space of \( Q \) over a field \( K \), then, a soft element of \( L \) is called a soft vector of \( L \). In the same sense a soft element of the soft set \((K, B)\) is called a soft scalar.

Definition 2.21 [13] Let \( \bar{x}, \bar{y} \) be soft vectors of \( L \) and \( \bar{k} \) be a soft scalar. The addition \( \bar{x} + \bar{y} \) of \( \bar{x} \), \( \bar{y} \) and scalar multiplication \( \bar{k}\bar{x} \) of \( \bar{k} \) and \( \bar{x} \) are defined by \( \bar{k}\bar{x} = \bar{x} + \bar{y} \) and \( \bar{k}\bar{x} = \bar{k}(\omega)\bar{x}(\omega) \), \( \bar{k}\bar{x}(\omega) = \bar{k}(\omega)\bar{x}(\omega) \) for all \( \omega \in B \). Obviously, \( \bar{x} + \bar{y}, \bar{k}\bar{x} \) are soft vectors of \( L \).

Definition 2.22 [15] Suppose \( \bar{X} \) be the absolute soft vector space i.e., \( \bar{X}(\omega) = X \), for all \( \omega \in B \). Then a mapping \( \| . \| : SE(\bar{X}) \rightarrow R(B)^{\ast} \) is called a soft norm on the soft vector space \( \bar{X} \) if \( \| . \| \) fulfills the succeeding situations:

1. \( \| \bar{x} \| \geq 0 \) for every \( \bar{x} \in \bar{X} \).
(2). \(||x|| = 0\) if and only if \(x = \Theta\).
(3). \(||ax|| = |a||x||\) for each \(a \in \mathbb{K}\) as well as for each scalar \(\bar{a}\).
(4). With any \(\bar{a}, \bar{y} \in \bar{X}\), \(||\bar{a}|\bar{y}||| \leq ||\bar{a}||||\bar{y}||\).

The soft vector space \(\bar{X}\) with a soft norm \(||.||\) on \(\bar{X}\) is called a soft normed linear space and is symbolized with \((\bar{X}, ||.||, B)\) or \((\bar{X}, ||.||, \mathbb{K})\). The exceeding conditions are called soft norm axioms.

**Theorem 2.23** [13] Consider a soft norm \(||.||\) on \(\bar{X}\) and any \(\bar{w}' \in \bar{B}\) the set \((||\bar{x}||(\bar{w}': \bar{x}(\bar{w}) = \xi')\) is a one element set. Then with any \(\bar{w} \in \bar{B}\), the function \(||\bar{w}|| : X \to R^+\) defined with \(||\bar{x}|| := ||\bar{x}||(\bar{w})\), with any \(\xi \in X\) and \(\bar{x} \in \bar{X}\) such that \(\bar{x}(\bar{w}) = \xi\), can be considered as a norm on \(X\).

**Definition 2.24** [14] Consider \((\bar{X}, ||.||, B)\) a soft normed linear space, \(\bar{F} \supseteq \bar{0}\) is a soft real number.

We realize the following concepts:
\(\mathbb{B}(\bar{x}, \bar{r}) = \{ \bar{y} \in \bar{X} : ||\bar{x} - \bar{y}|| < \bar{r} \} \subset SE(\bar{x}),\)
\(\mathbb{B}(\bar{x}, \bar{r}) = \{ \bar{y} \in \bar{X} : ||\bar{x} - \bar{y}|| \leq \bar{r} \} \subset SE(\bar{x}),\)
\(S(\bar{x}, \bar{r}) = \{ \bar{y} \in \bar{X} : ||\bar{x} - \bar{y}|| = \bar{r} \} \subset SE(\bar{x}),\)
\(\mathbb{B}(\bar{x}, \bar{r}), \mathbb{B}(\bar{x}, \bar{r}), S(\bar{x}, \bar{r})\) respectively called an open ball, a closed ball and a sphere with center at \(\bar{x}\) and radius \(\bar{r}\).

**Definition 2.25** [13] A sequence of soft elements \(\{\bar{x}_n\}\) in a soft normed space \((\bar{X}, ||.||, B)\) called convergent sequence, if \(||\bar{x}_n - \bar{x}|| \to 0\) as \(n \to \infty\), \(A\) is called a soft element \(\bar{x}\). In other words for each \(\bar{e} \geq 0\), there exists \(N \in \mathbb{N}\) such that \(\bar{e} \leq ||\bar{x}_n - \bar{x}|| \leq \bar{e}\) every time \(n > N\).

**Definition 2.26** [13] A sequence \(\{\bar{x}_n\}\) of soft elements in a soft normed space \((\bar{X}, ||.||, B)\) is called a soft Cauchy sequence in \(\bar{X}\), if matching to each \(\bar{e} \geq 0\), there exists \(m \in \mathbb{N}\) satisfy:
\(||\bar{x}_j - \bar{x}_i|| \leq \bar{e}\), for all \(i,j \geq m\) i.e., \(||\bar{x}_i - \bar{x}_j|| \to 0\) as \(i, j \to \infty\).

**Definition 2.27** [13] Suppose \((\bar{X}, ||.||, B)\) is a soft normed space. Then, \(\bar{X}\) is called soft complete if each soft Cauchy sequence in \(\bar{X}\) converges to a soft element of \(\bar{X}\). In this case the space is called a soft Banach Space.

**Theorem 2.28** [13] Each soft Cauchy sequence in \(\mathbb{R}(\bar{X})\) is convergent provided that the set of parameters is finite, i.e., \(\mathbb{R}(\bar{X})\) together with its usual modulus soft norm is a soft Banach space, provided that the set of parameters is finite.

**Definition 2.29** [14] A series \(\sum_{k=1}^{\infty} \bar{x}_{k}\) of soft elements called soft convergent, if the partial sum of the series \(\bar{S}_n = \sum_{k=1}^{n} \bar{x}_k\) is soft convergent.

Let \(\bar{X}, Y\) be the agreeing absolute soft normed spaces, i.e., \(\bar{X}(\bar{w}) = X, Y(\bar{w}) = Y\), for all \(\bar{w} \in \bar{B}\). We usage the symbolization \(\bar{x}, y, \bar{z}\) to represent soft vectors in \(\bar{X}\).

**Definition 2.30** [13] Suppose \(T : SE(\bar{X}) \to SE(\bar{Y})\) is an operator. \(T\) is called soft linear, if
\((L1)\). \(T(\bar{x}_1 + \bar{x}_2) = T(\bar{x}_1) + T(\bar{x}_2)\) with any soft elements \(\bar{x}_1, \bar{x}_2 \in \bar{X}\).
\((L2)\). \(T(k \bar{x}) = k T(\bar{x})\), with any soft scalar \(k\) and any soft elements \(\bar{x} \in \bar{X}\).
The condition \((L1)\) and \((L2)\) can be combined in one condition \(T(k \bar{x}_1 + \bar{x}_2) = k T(\bar{x}_1) + T(\bar{x}_2)\) for every soft elements \(\bar{x}_1, \bar{x}_2 \in \bar{X}\) and every soft scalars \(k_1, k_2\).

**Definition 2.31** [13] \(T : SE(\bar{X}) \to SE(\bar{Y})\) is called soft continuous operator at \(\bar{x}_0 \in \bar{X}\), if with any soft sequence \(\{\bar{x}_n\}\) of soft elements of \(\bar{X}\) with \(\bar{x}_n \to \bar{x}_0\) as \(n \to \infty\), the image \(T(\bar{x}_n) \to T(\bar{x}_0)\) as \(n \to \infty\).
to \infty. i.e., \|x_n - x_0\| \to 0 as n goes to \infty implies \|T(x_n) - T(x_0)\| \to 0 as n goes to \infty. If T is soft continuous at every soft element of \(\tilde{X}\), then T is called a soft continuous operator.

**Theorem 2.32**[13] Suppose \(X, Y\) are two soft normed spaces and T: SE(\(\tilde{X}\)) \to SE(\(\tilde{Y}\)) be a soft linear operator. If T is soft continuous at some soft element \(x_0\in \tilde{X}\), then T is soft continuous at every soft element of \(\tilde{X}\).

**Definition 2.33**[13] Let \(X, Y\) are two soft normed spaces and T: SE(\(\tilde{X}\)) \to SE(\(\tilde{Y}\)) be a soft linear operator. The operator T is said to be soft bounded if there exists some positive soft real number \(M\) such that for each \(x\in \tilde{X}\), \(\|T(x)\| \leq M\|x\|\).

**Theorem 2.34**[13] Let \(X, Y\) are two soft normed spaces and T: SE(\(\tilde{X}\)) \to SE(\(\tilde{Y}\)) be a soft linear operator. T is soft bounded implies that T is soft continuous.

**Theorem 2.35**[13] (Decomposition Theorem) Suppose \(X, Y\) are two soft normed spaces and T: SE(\(\tilde{X}\)) \to SE(\(\tilde{Y}\)) be a soft linear operator fulfills the situation (L3). That is, for \(x\in X\), and \(\omega\in B\) the set \(\{T(x)\omega): x\in \tilde{X}\}\) such that \(x\in X\) and \(\omega\in B\) fulfills (L3) and T \((\omega) = \xi\), for all \(\omega\in B\) which fulfills (L3) and T \((\omega) = \xi\), for all \(\omega\in B\).

**Theorem 2.36**[13] Suppose \(X, Y\) are two soft normed spaces and T \(\omega: X\to Y\) be a family of crisp linear operators for all \(\omega\in B\). Let \(\tilde{X}, \tilde{Y}\) be the consistent absolute soft vector spaces. Then, there exists a soft linear operator T: SE(\(\tilde{X}\)) \to SE(\(\tilde{Y}\)) defined by T \((\tilde{x})\omega = T_{\omega}(\xi)\) if \(\tilde{x}\in \tilde{X}\) and \(\tilde{\omega}\in \tilde{Y}\) such that \(\tilde{x}\omega = \tilde{\xi}\), \(\omega\in B\). Which fulfills (L3) and T \((\omega) = \xi\), for all \(\omega\in B\).

**Theorem 2.37**[13] Let \(\tilde{X}\) and \(\tilde{Y}\) be two soft normed spaces fulfills (N5) and T: SE(\(\tilde{X}\)) \to SE(\(\tilde{Y}\)) a soft linear operator Achieve (L3). T is soft continuous implies that T is soft bounded.

**Theorem 2.38**[13] Suppose \(\tilde{X}\) and \(\tilde{Y}\) be two soft normed spaces fulfills (N5) and T: SE(\(\tilde{X}\)) \to SE(\(\tilde{Y}\)) a soft linear operator Achieve (L3). Under the condition \(\tilde{X}\) is of finite dimension, T will be soft bounded. Consequently, T will be soft continuous.

**Definition 2.39**[13] Consider T is a soft bounded linear operator from SE(\(\tilde{X}\)) into SE(\(\tilde{Y}\)). Then, the norm of the operator T symbolized by \(\|T\|\), is a soft real number and it can be defined as:

With any \(\omega\in B\), \(\|T\|\omega = \inf\{t\in R: \|T(x)\omega\| \leq t.\|x\|\omega\}\) for each \(x\in \tilde{X}\).

**Theorem 2.40**[13] Let \(\tilde{X}, \tilde{Y}\) be two soft normed spaces Achieves (N5) and T: SE(\(\tilde{X}\)) \to SE(\(\tilde{Y}\)) Achieve (L3). Then for any \(\omega\in B\), \(\|T\|\omega = \|T_{\omega}\|_{\omega}\\), where \(\|T_{\omega}\|_{\omega}\\) is the norm of the linear operator \(T_{\omega}: X\to Y\).

**Theorem 2.41**[13] \(\|T\|\omega \leq \|T\|\|x\|\omega\\) for all \(x\in \tilde{X}\).

**Theorem 2.42**[13] Let \(\tilde{X}\) and \(\tilde{Y}\) be two soft normed linear spaces Achieves (N5) and T: SE(\(\tilde{X}\)) \to SE(\(\tilde{Y}\)) be a soft linear operator Achieve (L3). Then:

(i) \(\|T\|\omega = \sup\{\|T(x)\omega\|: \|x\|\leq 1\} = \|T_{\omega}\|_{\omega}\), for each \(\omega\in B\).

(ii) \(\|T\|\omega = \sup\{\|T(x)\omega\|: \|x\| = 1\} = \|T_{\omega}\|_{\omega}\), for each \(\omega\in B\).

(iii) \(\|T\|\omega = \sup\{\|F(x)\omega\|: \|F\|\mu \neq 0, for all \mu\in B\}\) = \(\|T_{\omega}\|_{\omega}\), for each \(\omega\in B\)
**Theorem 2.43** ([13]). Let $\tilde{X}$ and $\tilde{Y}$ be two soft normed linear spaces Achieve (N5) and $T: SE(\tilde{X}) \to SE(\tilde{Y})$ a soft continuous linear operator Achieve (L3). Then, $T_{\omega}$ is continuous on $X$ with any $\omega \in B$.

**Theorem 2.44** [14] Let $\tilde{X}$ and $\tilde{Y}$ be two soft normed linear spaces Achieves (N5). Let $\{ T_{\omega}, \omega \in B \}$ be a family of continuous linear operators such that $T_{\omega} : X \to Y$ with any $\omega$. Then, the soft linear operator $T: SE(\tilde{X}) \to SE(\tilde{Y})$, defined with $(T(x))(\omega) = T_{\omega}(\tilde{x}(\omega))$, with any $\omega \in B$ is a continuous soft linear operator Achieve (L3).

**Definition 2.45** [13] Suppose $\tilde{X}$, $\tilde{Y}$ are two soft normed space and $T: SE(\tilde{X}) \to SE(\tilde{Y})$ is a soft linear operator. Then, $T$ is named injective or one-to-one if $T(x) = T(y)$ implies $x = y$. It is named surjective or onto if Rang$(T) = SE(\tilde{Y})$. The operator $T$ is said to be bijective provided that $T$ is both one-to-one and onto.

**Theorem 2.46** [18] A soft eigenvectors $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \ldots$. Corresponding to different eigenvalues $\mu_1, \mu_2, \mu_3, \ldots$ of a soft linear operator $T$ on a soft normed space $\tilde{X}$ constitute a linearly independent set.

3. **Spectrum of soft compact linear operator**

In this section we introduce several theorems related to soft compact linear operator. In the beginning, we prove an important lemma that we need to prove these theorems.

**Theorem 3.1.** (F.Riesz’s lemma in soft normed space)

Let $Y = (G,P)$ and $Z = (H,P)$ be soft subspace of a soft normed space $\tilde{X}$ (of any dimension) and suppose that $Y$ is soft closed and is a proper soft subset of $Z$, we can find for any soft real number $\tilde{k}$ such that $0 < \tilde{k} \leq 1$ a soft element $\tilde{z} \in Z$ such that $\| \tilde{z} \| = 1$, $\| \tilde{z} - \tilde{y} \| \geq \tilde{k}$ for all $\tilde{y} \in Y$.

**Proof:** we consider any $\tilde{y} \in Z - Y$ and denote its distance from $Y$ by $\tilde{a}$ i.e., $\tilde{a} = \inf_{\tilde{y} \in Y} \| \tilde{y} - \tilde{y} \|$, since $Y$ is soft closed, clearly $\tilde{a} \geq 0$ for all $\tilde{y} \in P$. Let $\tilde{k} \in R(P)^*$ such that $0 < \tilde{k} \leq 1$, by definition of infimum there exist $\tilde{y}_0 \in Y$ such that $\tilde{a} \leq \| \tilde{y} - \tilde{y}_0 \| \leq \frac{\tilde{a}}{\tilde{k}} \leq \frac{\tilde{a}}{\tilde{k}}$ . note that $\frac{\tilde{a}}{\tilde{k}} \geq \tilde{a}$ for all $\tilde{y} \in Y$ since $0 < \tilde{k} \leq 1$. Let $\tilde{z} = \frac{\tilde{y} - \tilde{y}_0}{\| \tilde{y} - \tilde{y}_0 \|}$ then $\| \tilde{z} \| = 1$ and $\| \tilde{z} - \tilde{y} \| = \frac{\| \tilde{y} - \tilde{y}_0 \|}{\| \tilde{y} - \tilde{y}_0 \|} = \| \tilde{y} - \tilde{y}_0 - \tilde{y} - \tilde{y}_0 \| = \| \tilde{y} - \tilde{y}_0 \| \frac{\tilde{a}}{\| \tilde{y} - \tilde{y}_0 \|} \leq \tilde{k}$.

Where $\tilde{y}_1 = \tilde{y}_0 + \frac{\tilde{a}}{\| \tilde{y} - \tilde{y}_0 \|}$ , $\tilde{y}_1 \in Y$ since $Y$ is a soft subspace. Hence $\| \tilde{y} - \tilde{y}_1 \| \geq \tilde{a}$ by definition of $\tilde{a}$.

**Theorem 3.2:** the set of eigenvalues of soft compact linear operator $T: SE(\tilde{X}) \to SE(\tilde{X})$ on a soft normed space $\tilde{X}$ is countable (perhaps finite or even empty) and the only possible point $\mu$ of accumulation is $\mu = 0$.

**Proof:** Obviously, if we can prove that for every $K > 0$ the set of all eigenvalues $\mu \in \sigma_p(T)$ such that $|\mu| \geq K$ is finite, the result is the same. 

Suppose the contrary for some $K_0 > 0$. Then there is a sequence $\{ \mu_n \}$ of infinitely many distinct eigenvalues such that $|\mu_n| \geq K_0$. Also $T\tilde{x}_n = \mu_n \tilde{x}_n$ for some $\tilde{x}_n \neq \tilde{0}$, the set of all $\tilde{x}_n$'s is linearly independent by (theorem 2.49).

Let $M_n = \text{span} \{ \tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n \}$ then every $\tilde{x} \in M_n$ has a unique representation $\tilde{x} = \alpha_1\tilde{x}_1 + \alpha_2\tilde{x}_2 + \ldots + \alpha_n\tilde{x}_n$.

We apply $T - M_n I$ and use $T\tilde{x}_i = M_i\tilde{x}_i$.

$(T-M_n I)\tilde{x} = \alpha_1(M_1 - M_n)\tilde{x}_1 + \alpha_2(M_2 - M_n)\tilde{x}_2 + \ldots + \alpha_n(M_n-1-M_n)\tilde{x}_n-1$

Hence $(T-M_n I)\tilde{x} \in M_{n-1}$ for all $\tilde{x} \in M_n$. 

6
The $M_n$’s are closed, and by Riesz’s lemma there is a sequence $\{ y_n \}$ such that $y_n \in M_n$, $\| y_n \| = 1$ and $\| y_n - x \| \geq \frac{1}{2}$ for all $x \in M_{n-1}$, for $n, m \in \mathbb{N}$, $n > m$.

$$T y_n - T y_m = x_n - x_m + \sum_{k=n}^{m} T x_k.$$

Since $m < n \Rightarrow m \leq n - 1$, we see that $y_n \in M_{n-1} = \text{span}\{ x_1, x_2, \ldots, x_{n-1} \}$.

Hence $T y_n = T x_1 + T x_2 + \cdots + T x_{n-1}$.

Therefore, $\mu_n y_n - T y_m = (T - \mu_n I) y_n \in M_{n-1}$ since $(T - \mu_n I) x_n \in M_{n-1}$ for all $x \in X$.

So, together $\| y_n \| \leq M_{n-1}$ and also $x = \mu_n y_n - T y_m$.

Hence $\| T y_n - T y_m \| = \| \mu_n y_n - w \| = \| \mu_n \| \| y_n - x \| \geq \frac{1}{2} |\mu_n| \geq \frac{1}{4} K_0$.

We prove the following theorem which will be used in this paper.

**Theorem 3.3 (finite dimension)**

If a soft normed space $X$ satisfy the situation that the soft closed unit ball $M = \{ x : \| x \| = 1 \}$ is soft compact, then $\tilde{X}$ is finite dimension.

**Proof:** we assume that $M$ is compact but dim ($\tilde{X}$) = $\infty$, we choose any $\tilde{x}$ such that $\| \tilde{x} \| = 1$, this soft vector generate a one dimensional subspace $\tilde{x}$, which is soft closed and is proper subspace of $\tilde{X}$ since dim($\tilde{X}$) = $\infty$. By Riesz’s lemma, there is a $\tilde{x} \in \tilde{X}$ of norm $\| \tilde{x} \| = 1$ such that $\| \tilde{x} - 1 \| \geq k = \frac{1}{\sqrt{2}}$.

The two soft vectors $\tilde{x}, \tilde{x}$ generate a two dimension subspace of $\tilde{X}$, by Riesz’s lemma there is $\tilde{x} \in \tilde{X}$ of norm $\| \tilde{x} \| = 1$ such that for all $\tilde{x} \in \tilde{X}$ we have $\| \tilde{x} \| \geq \frac{1}{\sqrt{2}}$ and in particular $\| \tilde{x} - 1 \| \geq \frac{1}{\sqrt{2}}$.

Proceeding by induction, we obtain a sequence $\{ \tilde{x}_n \}$ of element $\tilde{x}_n \in \tilde{X}$ such that $\| \tilde{x}_n - \tilde{x} \| \geq \frac{1}{\sqrt{2}} (n \neq m)$.

Hence $\{ \tilde{x}_n \}$ cannot have a convergent subsequence, this contradicts the compact of $M$, hence our assumption dim ($\tilde{X}$) = $\infty$ is false, hence dim ($\tilde{X}$) < $\infty$.

**Theorem 3.4: (Null space)**

Consider $T : SE(\tilde{X}) \rightarrow SE(\tilde{X})$ to be a soft compact linear operator on a soft normed space $\tilde{X}$, then for every $\mu \neq 0$ the null space $\mathcal{N}(T_{\mu})$ of $T_{\mu} = T - \mu I$ is finite dimension.

**Proof:** we try to show that the soft closed unit ball $M$ in $\mathcal{N}(T_{\mu})$ is compact.

Let $M$ be the soft closed unit ball $M = \{ \tilde{x} : \| \tilde{x} \| \leq 1 \}$, let $\{ x_n \}$ be in $M$, then $\{ x_n \}$ is bounded since $\| x_n \| \leq 1$, $T(\tilde{x}_n)$ has a convergent subsequence $\{ T(\tilde{x}_n) \}$ since $T$ is soft compact.

$\tilde{x}_n \in \tilde{X}$ implies $T(\tilde{x}_n) = (T - \mu I) \tilde{x}_n = T(\tilde{x}_n) - \mu \tilde{x}_n = \tilde{x}_n$, hence $\tilde{x}_n \rightarrow T(\tilde{x}_n)$ because $\mu \neq 0$.

Consequently, $\tilde{x}_n = \mu^{-1} T(\tilde{x}_n)$ also converge, the limit is in $M$ since $M$ is soft closed. Hence $M$ is compact by "definition of compact set" because $\{ x_n \}$ was arbitrary in $M$, this proves that dim $\mathcal{N}(T_{\mu})$ < $\infty$ by (Theorem 3.3).

**Proposition 3.5** Consider $T : SE(\tilde{X}) \rightarrow SE(\tilde{X})$ to be a soft compact linear operator on a soft normed space $\tilde{X}$. Then for every $\mu \neq 0$, $\dim \mathcal{N}(T_{\mu}^n)$ < $\infty$ and $\{ 0 \} = \mathcal{N}(T_{\mu}) \subset \mathcal{N}(T_{\mu}^2) \subset \mathcal{N}(T_{\mu}^3) \subset \ldots$.

**Proof:** since $T_{\mu}(\tilde{x}_n + \mu \tilde{x}_n) = (T - \mu I)(\tilde{x}_n + \mu \tilde{x}_n)$.

Therefore, $T_{\mu}(\tilde{x}_n + \mu \tilde{x}_n) = (T - \mu I)(\tilde{x}_n + \mu \tilde{x}_n) = (T - \mu I)(\tilde{x} - \mu \tilde{x}_n) = (T - \mu I)(\tilde{x} - \mu \tilde{x} - \beta \tilde{x}_n) = (T - \mu I)(\tilde{x} - \mu \tilde{x} - \beta \tilde{x}_n) = (T - \mu I)(\tilde{x} - \mu \tilde{x} - \beta \tilde{x}_n) = (T - \mu I)(\tilde{x} - \mu \tilde{x} - \beta \tilde{x}_n)$.

Hence $T_{\mu}$ is linear, it maps $\theta$ to $\theta$ and therefor $T_{\mu}^{n+1} \tilde{x} = \theta$ implies $T_{\mu}^{n+1} \tilde{x} = \theta$.

Therefore, $\mathcal{N}(T_{\mu}) \subset \mathcal{N}(T_{\mu}^2) \subset \mathcal{N}(T_{\mu}^3) \subset \ldots$.

Now, $T_{\mu}^n(T - \mu I)^n = (T - \mu I)^n + \sum_{k=1}^{n} (\mu I)^k T^{k-1} (\mu I)^{n-k}$.
\( T^n = W - kl \) where \( k = -(-\mu)^n \) and \( W = TS \), \( S = \sum_{k=1}^{n} \binom{n}{k} T^{k-1} (-\mu)^{n-k} \)

Since \( T \) is soft compact and \( S \) is bounded, \( W \) is soft compact. Hence \( \dim \mathcal{N}(W - kl) < \infty \)
i.e., \( \dim \mathcal{N}(T^n) < \infty \).

**Theorem 3.6:** (Rang Let \( T : \text{SE}(\tilde{X}) \rightarrow \text{SE}(\tilde{X}) \) be a soft compact linear operator on a soft normed space \( \tilde{X} \), then for every \( \mu \neq 0 \) the range of \( T^n = T - \mu I \) will be soft closed.

**Proof:** Assume \( T^n(\tilde{X}) \) is not soft closed, then there is \( \tilde{y} \in T^n(\tilde{X}) \) and \( \tilde{y} \notin T^n(\tilde{X}) \) and a sequence \( \{\tilde{x}_n\} \) in \( \tilde{X} \) such that \( \tilde{y} = T^n(\tilde{x}_n) \rightarrow \tilde{y} \).

Hence \( \tilde{y} \notin \mathcal{N}(T^n) \) and \( \tilde{x}_n \notin \mathcal{N}(T^n) \) for all sufficiently large \( n \). Without loss of generality we may assume that this hold for every \( n \). Since \( \mathcal{N}(T^n) \) is closed, the distance \( \delta_n \) from \( \tilde{x}_n \) to \( \mathcal{N}(T^n) \) is soft positive.

That is \( \delta_n = \inf_{\tilde{z} \in \mathcal{N}(T^n)} \|\tilde{x}_n - \tilde{z}\| \geq 0 \). Hence by the definition of infimum there is a sequence \( \{\tilde{z}_n\} \) in \( \mathcal{N}(T^n) \) such that \( \tilde{a} = \|\tilde{x}_n - \tilde{z}_n\| < 2\delta_n \). We wish to show that \( \tilde{a} = \|\tilde{x}_n - \tilde{z}_n\| \rightarrow \infty \) as \( n \rightarrow \infty \). Assume not, then \( \{\tilde{x}_n - \tilde{z}_n\} \) has bounded sequence. Since \( T \) is compact, it follows that \( \{T(\tilde{x}_n - \tilde{z}_n)\} \) has a convergent subsequence. Now, \( T^n(\tilde{x}_n) \rightarrow \tilde{y} \) and \( T^n(\tilde{z}_n) \rightarrow \tilde{y} \).

Hence \( \tilde{y} \notin \mathcal{N}(T^n) \) and \( \{\tilde{x}_n - \tilde{z}_n\} \) has a convergent subsequence say \( \tilde{x}_n - \tilde{z}_n \rightarrow \tilde{v} \). Since \( T \) is compact, \( T \) is continuous and so \( T^n \).

Hence \( T^n(\tilde{x}_n - \tilde{z}_n) = T^n(\tilde{x}_n) - T^n(\tilde{z}_n) \rightarrow \tilde{v} \) i.e., \( T^n \tilde{v} = \tilde{y} \). Thus \( \tilde{y} \notin T^n(\tilde{X}) \) which contradicts \( \tilde{y} \notin T^n(\tilde{X}) \).

Hence \( \overline{\mathcal{N}}(n) = \|\tilde{x}_n - \tilde{z}\| \rightarrow \infty \) as \( n \rightarrow \infty \).

Now, let \( \tilde{w}_n = \frac{1}{\delta_n} (\tilde{x}_n - \tilde{z}_n) \). Thenfor \( \|\tilde{w}_n\| = 1 \). Since \( \tilde{a} \rightarrow \infty \) whereas \( T^n(\tilde{w}_n) = \tilde{a} \) and \( \{T^n(\tilde{w}_n)\} \) converge, it follows that:

\[
\frac{1}{\delta_n} T^n(\tilde{w}_n) = \tilde{y} \bigwedge
\]

Using again \( I = \mu^{-1}(T - T^\mu) \) we have \( \tilde{w}_n = \frac{1}{\mu} (T\tilde{w}_n - T^\mu \tilde{w}_n) \).

Since \( T \) is compact and \( \{\tilde{w}_n\} \) is bounded, \( \{T\tilde{w}_n\} \) has a convergent subsequence. Also \( \{T^n\tilde{w}_n\} \) converge ((\( T^n\tilde{w}_n = \frac{1}{\delta_n} T^n(\tilde{w}_n) \)) hence \( \{\tilde{w}_n\} \) has a convergent subsequence say \( \tilde{w}_n \rightarrow \tilde{w} \)

Since \( T^n\tilde{w}_n = \frac{1}{\delta_n} T^n(\tilde{w}_n) \rightarrow \tilde{y} \), \( T^n \tilde{w} = \tilde{y} \) and therefore, \( \tilde{w} \in \mathcal{N}(T^n) \). Since \( \tilde{w}_n \notin \mathcal{N}(T^n) \), \( \tilde{u}_n = \tilde{w}_n + \tilde{a} \tilde{w} \in \mathcal{N}(T^n) \).

Hence for the distance from \( \tilde{x}_n \) to \( \tilde{u}_n \) we must have \( \|\tilde{x}_n - \tilde{u}_n\| \leq \delta_n \).

\[
\delta_n \leq \||\tilde{x}_n - \tilde{u}_n|| = ||\tilde{u}_n| \tilde{w}|| = \|\tilde{a} \tilde{w} - \tilde{w} || < 2\delta_n ||\tilde{w} - \tilde{w}||
\]

Dividing by \( 2\delta_n \geq 0 \), we have \( \frac{1}{2} \leq \|\tilde{w} - \tilde{w}|| \) and this contradicts that \( \tilde{w}_n \rightarrow \tilde{w} \) and proves the theorem.

**Proposition 3.7:** Consider \( T : \text{SE}(\tilde{X}) \rightarrow \text{SE}(\tilde{X}) \) to be a soft compact linear operator on a soft normed space \( \tilde{X} \), then for every \( \mu \neq 0 \) the range of \( T^n \) is closed for every \( n = 1, 2, 3, \ldots \) Also
\[
\bar{x} = T^0_\mu(x) \supseteq T^1_\mu(x) \supseteq T^2_\mu(x) \supseteq \ldots.
\]

**Proof:** \(T^m_\mu = W - kI\) where \(k = (-\mu)^n\), \(W = TS\), \(S = \sum_{k=1}^{n} (n)_k T^{k-1} (-\mu)^{n-k}\) as in (proposition 3.5).

\(W\) is compact. Hence the range of \(W - kI\) is closed, i.e., range of \(T^m_\mu\) is closed for every \(n = 1, 2, \ldots\).

Now, we try to prove that \(T^m_\mu(x) = I(x) = \bar{x} \supseteq T^m_\mu(x)\).

Suppose that \(T^{m-1}_\mu(x) \supseteq T^m_\mu(x)\) and prove that the statement true when \(n = n+1\)

\[
T^{n+1}_\mu(x) = T_\mu[T^n_\mu(x)] \subset T_\mu[T^{n-1}_\mu(x)] = T^n_\mu(x)\]. Hence \(T^m_\mu(x) \supseteq T^{n+1}_\mu(x)\)

**Theorem 3.8:** Consider \(T: SE(\bar{x}) \rightarrow SE(\bar{x})\) to be a soft compact linear operator on a soft normed space \(\bar{x}\), and let \(\mu \neq 0\). Then there exist a smallest integer \(r\) (depending on \(\mu\)) such that from \(n = r\) on, the null space \(\mathcal{N}(T^m_\mu)\) are all equal, and if \(r > 0\) the inclusion \(\mathcal{N}(T^{m-1}_\mu) \subset \mathcal{N}(T^m_\mu)\) are all proper.

**Proof:** for simplicity, we write \(\mathcal{N}_n = \mathcal{N}(T^m_\mu)\), the steps of proof are as follows:

1. We assume that \(\mathcal{N}_k = \mathcal{N}_{k+1}\) for no \(k\) and get a contradiction.

2. We show that \(\mathcal{N}_k = \mathcal{N}_{k+1}\) implies \(\mathcal{N}_r = \mathcal{N}_{r+1}\) for all \(r > k\).

We know that \(\mathcal{N}_r \subseteq \mathcal{N}_{k+1}\) by proposition 3.5. Suppose that \(\mathcal{N}_{k} = \mathcal{N}_{k+1}\) for no \(k\), then \(\mathcal{N}_n\) is a proper subspace of \(\mathcal{N}_{n+1}\) for every \(n\). Since these null spaces are closed, Riesz's lemma thus implies the existence of sequence \(\{\bar{y}_n\}\) such that \(\bar{y}_n \in \mathcal{N}_n\) with \(|\bar{y}_n| = 1\) and \(|\bar{y}_n - \bar{x}| \geq \frac{1}{2}\) for all \(\bar{x} \in \mathcal{N}_{n-1}\)

From \(T_\mu = T - \mu I\) we have \(T = T_\mu + \mu I\) and \(T\bar{y}_n - T\bar{y}_k = \mu \bar{y}_n - \bar{x}\) where \(\bar{x} = T_\mu \bar{y}_k + \mu \bar{y}_n - T_\mu \bar{y}_n\)

Let \(k < n\), since \(k \leq n - 1\) we clearly have \(\mu \bar{y}_k \in \mathcal{N}_k \subset \mathcal{N}_{n-1}\). Also \(\bar{y}_n \in \mathcal{N}_n\) implies

\[
\theta = T^{k}_\mu \bar{y}_k = T^{k-1}_\mu (T^{k}_\mu \bar{y}_k)\]. That is \(T^{k}_\mu \bar{y}_k \in \mathcal{N}_{k-1} \subset \mathcal{N}_{n-1}\). Similarly, \(\bar{y}_n \in \mathcal{N}_n\) implies \(T^{k}_\mu \bar{y}_k \in \mathcal{N}_{n-1}\). Together, \(\bar{z} \in \mathcal{N}_{n-1}\), also \(\bar{x} = \mu^{-1} \bar{z} \in \mathcal{N}_{n-1}\) so that \(|\mu \bar{y}_n - \bar{z}| = |\mu ||\bar{y}_n - \bar{z}|| \geq \frac{1}{2} |\mu|\)

\[
||T\bar{y}_n - T\bar{y}_k|| = |\mu ||\bar{y}_n - \bar{z}|| = |\mu ||\bar{y}_n - \bar{z}|| \geq \frac{1}{2} |\mu|
\]

So that \(T\bar{y}_n\) has no convergent subsequence because \(|\mu| > 0\), this contradicts the compactness of \(T\) since \(\{\bar{y}_n\}\) is bounded. Hence our assumption that \(\mathcal{N}_k = \mathcal{N}_{k+1}\) for no \(k\) is false and we must have \(\mathcal{N}_k = \mathcal{N}_{k+1}\) for some \(k\).

Now, we try to prove that \(\mathcal{N}_k = \mathcal{N}_{k+1}\) implies \(\mathcal{N}_n = \mathcal{N}_{n+1}\) for all \(n > k\). Suppose this not true. Then \(\mathcal{N}_n\) is a proper subspace of \(\mathcal{N}_{n+1}\) for some \(n > k\), let \(\bar{x} \in \mathcal{N}_{n+1} - \mathcal{N}_n\), by definition \(T^{n+1}_\mu \bar{x} = \theta\) but \(T^n_\mu \bar{x} \neq \theta\).
Since \( n > k \), we have \( n - k > 0 \), we set \( \bar{z} = T^k x \), then \( T^{k+1} \bar{z} = T^{n+1} x = \theta \) but \( T^{k+1} \bar{z} = T^n x \neq \theta \).

Hence \( \bar{z} \notin N_{k+1} \) but \( \bar{z} \notin N_k \), so that \( N_k \) is a proper subspace of \( N_{k+1} \). This contradicts \( N_k = N_{k+1} \).

i.e., there exist a smallest integer \( n \) (depending on \( \mu \)) such that from \( n = r \) on, the null spaces \( N(T^n \mu) \) are all equal.

Consequently, if \( r > 0 \) the inclusion stated in the theorem are proper.

**Theorem 3.9:** Consider \( T: SE(\bar{X}) \rightarrow SE(\bar{X}) \) to be a soft compact linear operator on a soft normed space \( \bar{X} \), and let \( \mu \neq 0 \). Then there exist a smallest integer \( q \) (depending on \( \mu \)) such that for \( q > 0 \) the inclusion \( T_q(\bar{x}) \supset T_q^2(\bar{x}) \supset \ldots \supset T_q^q(\bar{x}) \) are all proper.

**Proof:** for simplicity we write \( R_n = T^n(\bar{x}) \) , suppose that \( R_x = R_{s+1} \) for no \( s \), then \( R_{n+1} \) is a proper subspace of \( R_n \) for every \( n \) (by theorem 3.7) , since these ranges are closed (by theorem 3.6), Riesz's lemma thus implies the existence of a sequence \( \{\bar{x}_n\} \) such that \( \bar{x} \in R_n \) with \( ||\bar{x}_n|| = 1 \) and \( ||\bar{x}_n - \bar{x}|| \geq \frac{1}{2} \) for all \( \bar{x} \in R_{n+1} \).

Let \( k < n \). Since \( T = T_\mu + \mu \), we can write \( \bar{T} \bar{x}_k - T \bar{x}_n = \mu (\bar{x}_k - (-T_\mu \bar{x}_k + T_\mu \bar{x}_n + \mu \bar{x}_n) \)

On the right side, \( \mu \bar{x}_k \in R_k \) and \( \bar{x}_n \in R_k \), so that \( T_\mu \bar{x}_k \in R_{k+1} \).

Furthermore, since \( n > k \), \( T_\mu \bar{x}_n + \mu \bar{x}_n \in R_n \subset R_{k+1} \).

Hence \( \bar{T} \bar{x}_k - T \bar{x}_n = \mu (\bar{x}_k - (-T_\mu \bar{x}_k + T_\mu \bar{x}_n + \mu \bar{x}_n)) = \mu (\bar{x}_k - \bar{x}) \quad \bar{x} \in R_{k+1} \)

Consequently, \( ||\bar{T} \bar{x}_k - T \bar{x}_n|| = ||\mu ||\bar{x}_k - \bar{x}|| \geq \frac{1}{2} ||\mu || > 0 \). Since \( \{\bar{x}_n\} \) is soft bounded In addition to \( T \) is soft compact, then \( \{T \bar{x}_n\} \) has a convergent subsequence. This contradicts that \( ||T \bar{x}_k - T \bar{x}_n|| \geq \frac{1}{2} ||\mu || \) so that our assumption is false and \( R_x = R_{s+1} \) for some \( s \). Let \( q \) be the smallest such that \( R_x = R_{s+1} \), then if \( q > 0 \) the inclusion \( T_q x \supset T_q^2 x \supset \ldots \supset T_q^q x \) are all proper.

Furthermore, \( R_q = R_{q+1} \) means that \( T_q \) maps \( R_q \) onto itself. Hence repeated application of \( T_q \) gives \( R_n = R_{n+1} \) for every \( n > q \).

**Theorem 3.10:** Consider \( T: SE(\bar{X}) \rightarrow SE(\bar{X}) \) to be a soft compact linear operator on a soft normed space \( \bar{X} \), and let \( \mu \neq 0 \). Then there exist a smallest integer \( n = r \) (depending on \( \mu \)) such that:

\[ N(T^r \mu) \neq N(T^{r+1} \mu) \neq N(T^{r+2} \mu) = \ldots \ldots \text{ And } T^r x = T^{r+1} x = T^{r+2} x = \ldots \]

And if \( r > 0 \), the following inclusion is proper:

\[ N(T_0 x) \subseteq N(T_{r+1} x) \subseteq \ldots \ldots \subseteq N(T^r x) \]

**Proof:** (theorem 3.8) and (theorem 3.9) gives above statement. All we have to show is that \( q = r \). We write for simplicity \( N_n = N(T^n \mu) \) and \( R_n = T^n \mu(\bar{x}) \).
We have $R_q = R_{q+1}$ (by theorem 3.9) and that means $T_\mu(R_q) = R_q$.

Hence $\bar{y} \in R_q$ implies $\bar{y} = T_\mu \bar{x}$ for some $\bar{x} \in R_q$. We want to prove that if $T_\mu \bar{x} = \theta$, $\bar{x} \in R_q$ implies $\bar{x} = \theta$.

Suppose that not, then $T_\mu \bar{x}_1 = \theta$ for some $\bar{x}_1 \in R_q$, $\bar{x}_1 \neq \theta$. Now, with $\bar{y} = \bar{x}_1$ gives $T_\mu \bar{x}_2$ for some $\bar{x}_2 \in R_q$. Similarly, $\bar{x}_2 = T_\mu \bar{x}_3$ for some $\bar{x}_3 \in R_q$. Hence for every $n$ we obtain by substitution:

\[
\theta \neq \bar{x}_1 = T_\mu \bar{x}_2 = \cdots = T_\mu^n(\bar{x}_n) \quad \text{but} \quad \theta = T_\mu^{n+1}(\bar{x}_n)
\]

Hence $\bar{x}_n = \bar{y} \in N_{n-1}$ but $\bar{x}_n \notin N_q$ (by theorem 3.8) and our present result shows that this inclusion is proper for every $n$ since $n$ is arbitrary. This contradicts (theorem 3.9) and the statement is proves, that is $\bar{y} = T_\mu \bar{x}$. Hence $\bar{y} \in R_q$. This proves that $T_\mu \bar{x} = \theta$, $\bar{x} \in R_q$ implies $\bar{x} = \theta$ (by theorem 3.5).

Now, we try to prove that $q = r$. If $q = 0$, this holds. Let $q \geq 1$, by definition of $q$ in (theorem 3.9) the inclusion $R_q \subseteq R_{q-1}$ is proper. Let $\bar{y} \in R_{q-1} - R_q$, then $\bar{y} = T_\mu^{q-1}(\bar{x}_0)$ for some $\bar{x}_0$. Also $T_\mu \bar{y} \in R_q = R_{q+1}$ implies that $T_\mu \bar{y} = T_\mu T_\mu^{q-1}(\bar{x}_0)$ for some $\bar{x}$. Since $T_\mu^{q-1}(\bar{x}_0) \in R_q$ but $\bar{y} \notin R_q$, we have $T_\mu^{q-1}(\bar{x}_0) = T_\mu^{q-1}(\bar{x}_0)$.

Hence $\bar{x} - T_\mu \bar{x} \notin N_{q-1}$. But $\bar{x} - T_\mu \bar{x} \in N_q$ because $T_\mu^{q-1}(\bar{x} - T_\mu \bar{x}) = T_\mu T_\mu^{q-1}(\bar{x}_0) = T_\mu \bar{y} - T_\mu \bar{y} = \theta$

This proves that $N_{q+1} = N_q$, so that $N_{q+1}$ is a proper subspace of $N_q$, this implies $q \leq r$ since $r$ is the smallest integer $n$ such that $N_n = N_{n+1}$ by (theorem 3.8). Hence $q = r$.

**Theorem 3.11:** Consider $T$: $SE(\bar{x}) \rightarrow SE(\bar{x})$ to be a soft compact linear operator on a soft Banach space $\bar{x}$, then every spectral value $\mu \neq 0$ of $T$ (if it exist) is a soft eigenvalue of $T$.

**Proof:** if $N(T_\mu) \neq \{\theta\}$ then $\mu$ is a soft eigenvalue of $T$ suppose that $N(T_\mu) = \{\theta\}$, where $\mu \neq 0$ then $T_\mu \bar{x} = \theta$ implies that $\bar{x} = \theta$ and $T_\mu^{-1}$: $T_\mu(\bar{x}) \rightarrow \bar{x}$ exist.

Since $\{\theta\} = N(T) = N(T_0)$, we have $r = 0$ by (theorem 3.10). It follows that $T_\mu$ is bijective, $T_\mu^{-1}$ is bounded since $\bar{x}$ is soft Banach space and $\mu \in \rho(T)$ by definition.

If $\mu = 0$ and $T$: $SE(\bar{x}) \rightarrow SE(\bar{x})$ be a soft compact linear operator on a soft normed space $\bar{x}$, then there are two cases if $\bar{x}$ is finite dimension then $\mu = 0$ or may not belong to $\sigma(T) = \sigma_0(T)$, that is, if $\dim(\bar{x}) < \infty$, we may have $0 \notin \sigma(T)$ and therefore $0 \in \rho(T)$, however if $\dim(\bar{x}) = \infty$, then we must have $0 \in \sigma(T)$ and all three cases $0 \in \sigma_0(T)$, 0 $\in \sigma_c(T)$, 0 $\in \sigma_e(T)$ are possible.

**References**
[1] Molodtsov D. 1999 Soft set theory first results Comput. Math. Appl. 37 19-31.

[2] Maji P K., Biswas R and Roy A R 2003 Soft set theory Comput. Math. Appl. 45 555-562.

[3] Feng F, Li C , Davvaz B and Ali M 2010. Soft sets combined with fuzzy sets and rough sets a tentative approach Soft Computing 14 899-911.

[4] Chen D 2005 The parametrization reduction of soft sets and its applications Comput. Math. Appl.49 757–763.

[5] Aktas H and Cagman N 2007 Soft sets and soft groups Inform. Sci. 177 2266–2735.

[6] Feng F, Jun Y and Zhao X 2008 Soft semirings Comput. Math. Appl. 56 2621-2628.

[7] Shabir M and Naz M 2011 On soft topological spaces Comput. Math. Appl. 61 1786 –1799.

[8] Das S, Majumdar B and Samanta S 2015 On Soft Linear Spaces and Soft Normed Linear Spaces Ann. Fuzzy Math. Inform. 9 (1) 91-109.

[9] Jafari S, Sadati S and Yaghobi A 2017. New Results on Soft Banach Algebra International Journal of Science and Engineering Investigations 6(68) September 17-25.

[10] Das S and Samanta S 2012 Soft real sets ,soft real numbers and their properties J. Fuzzy Math. 20 (3) 551-576.

[11] Das S and Samanta S 2013 On soft complex sets and soft complex numbers J. Fuzzy Math. 21 (1) 195-216.

[12] Das S and Samanta S 2013 Soft Metric Annals of Fuzzy Mathematics and Informatics Volume 6, No. 1(July) 77-94.

[13] Das S and Samanta S 2013 Soft linear operators in soft normed linear spaces Ann. Fuzzy Math. Inform. 6(2) 295-314.

[14] Das S and Samanta S 2014 Soft linear functionals in soft normed linear spaces Annals of Fuzzy Mathematics and Informatics Volume 7, No. 4, (April) 629 – 651.

[15] Thakur R, Samanta S 2015 Soft Banach Algebra Ann. Fuzzy Math. Inform. 10 (3) 397-412.

[16] Das S and Samanta S 2014 Operators on soft inner product spaces Fuzzy Inf. Eng. 6 435-450.

[17] Majumdar P and Samanta S 2010 On soft mappings Comput. Math. Appl. 60 2666–2672.

[18] Khaleefah Sabah and Ahmed Buthainah 2020 Invertible operators on soft Normed spaces Iraq journal of science 61(5) Currently in progress.