CARLEMAN-VEKUA EQUATION WITH A SINGULAR POINT

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Abstract. In this article unconditional solvability of the Carleman-Vekua equation with a singular point is proved, the Riemann-Hilbert problem is solved, integral representations of solutions, the structures of their zeros and poles are received.

Keywords: Carleman - Vekua equation; complex plane; singular point; Riemann-Hilbert boundary value problem; holomorphic functions.

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Introduction.

Let $G$ be a bounded domain of the complex plane $\mathbb{C}$ with boundary $\Gamma \in C^{1,\alpha}, 0 < \alpha < 1$, with the inner point $z = a$.

Let $S(G)$ be the set of measurable, essentially bounded functions $f(z)$ in $G$ with the norm

$$||f||_{S(G)} = \text{ess sup} \, |f(z)| = \lim_{p \to \infty} ||f||_{L^p(G)}.$$

Now the spaces used below are defined:

$S_\nu(G, a)$ is the class of functions $f(z)$, for which $f(z)|z - a|^{\nu} \in S(G)$. The norm in $S_\nu(G, a)$ is defined by the formula

$$||f||_{S_\nu(G, a)} = \text{ess sup} \, (|z - a|^{\nu}|f(z)|),$$

where $\nu$ is a real number.

$C_\nu(\overline{G}, a)$ is the class of functions $f(z)$, for which $f(z)|z - a|^{\nu} \in C(\overline{G})$. The norm in $C_\nu(\overline{G}, a)$ is defined by the formula

$$||f||_{C_\nu(\overline{G}, a)} = \max_{z \in G} (|z - a|^{\nu}|f(z)|).$$

$U_0(G)$ is the class of holomorphic functions in $G$.

$W_p^1(G), p > 1$ is the Sobolev space, see [1].

Let us consider the equation

$$\partial_\zbar V + A(z)V + B(z)\overline{V} = F(z), \quad (1)$$

in $G$, where $\partial_\zbar = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right)$; $A(z), B(z) \in S_1(G, a); F(z) \in S_\beta(G, a), \text{max}\{0, 1 - 8\mu\} < \beta < \frac{2}{q}$. Here $\mu = ||A||_{S_1(G, a)} + ||B||_{S_1(G, a)}$; $2 < q < \frac{2}{1 - 8\mu}$ if $\mu < \frac{1}{8}$ and $q > 2$ if $\mu \geq \frac{1}{8}$.

For $a = 0$ we receive the equation

$$\partial_\zbar V + \frac{A_0(z)}{|z|}V + \frac{B_0(z)}{|z|}\overline{V} = F(z), \quad (2)$$

where $A_0(z), B_0(z) \in S(G); F(z) \in S_\beta(G, 0)$.

It is obvious, that $F(z) \in L_q(G), q > 2$. 

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The equation (2) with \( F(z) \equiv 0 \) arise in the theory of infinitesimal deformations of surfaces of positive curvature with a flat point, see [2, 3]. There it is required to prove the existence of continuous solutions of equation (2) in the neighborhood of a singular point \( z = 0 \). In this connection the equation (2) is studied in many works of L.G. Mikhailov, Z.D Usmanov, A. Tungatarov, see [4-7], etc.

In all results usually a sufficient smallness of the coefficients \( A_0(z) \) and \( B_0(z) \) or the smallness of the domain \( G \) is supposed. In the present article we prove existence of solutions \( V(z) \) of the equation (2), satisfying to the condition \( V(0) = 0 \), without any conditions on the smallness of the coefficients or on the smallness of the domain \( G \). Integral representations of solutions of equation (1) in the class \( W^1_{q}(G) \cap C_{\beta-1}(\overline{G},a) \), \( 0 < \beta < \frac{2}{q}, q > 2 \) (3) are also received. The Riemann – Hilbert problem for the equation (2) is solved in the class (3) without any smallness conditions on the coefficients assumed in [4-7]. The structures of the zeros and poles of the solutions of the equation (1) in the class (3) are investigated.

§1. Representations of solutions and solvability of the equation (1)

The solutions of the equation (1) are looked for in the class (3). As in [1] it is possible to prove, that the equation (1) is equivalent to the equation

\[
V(z) + (P_GV)(z) = (T_GF)(z) + \Phi(z), \quad z \in G,
\]

where

\[
(T_Gf)(z) = -\frac{1}{\pi} \int_G \frac{f(\zeta)}{\zeta - z} dG_{\zeta}, \quad (dG_{\zeta} = d\xi d\eta, \quad \zeta = \xi + i\eta),
\]

\[
\Phi(z) \in U_0(G), \quad (P_Gf)(z) = (T_Gf^*)(z), \quad f^* = A(z)f + B(z)\bar{f}.
\]

As the function \( V(z) \) belongs to the class (3), then \( V(a) = 0 \). Therefore from (4) we receive

\[
(P_GV)(a) = (T_GF)(a) + \Phi(a).
\]

From (4) and (5) it follows

\[
V(z) + (P_G,aV)(z) = (T_G,aF)(z) + (z-a)\Phi(z), \quad z \in G,
\]

where

\[
(P_G,a)f)(z) = (P_Gf)(z) - (P_Gf)(a),
\]

\[
(T_G,aF)(z) = (T_Gf)(z) - (T_Gf)(a), \quad \Phi(z) \in U_0(G).
\]

Thus, any solution of equation (1) from the class (3) satisfies the equation (6). A solution of the equation (6) is looked for in the class \( C_{\beta-1}(\overline{G},a) \). Let us show, that any solution of the equation
(6) from the class \( C_{\beta-1}(G, a) \) belongs to the class (3) and almost everywhere in \( G \) satisfies the equation (1). Hereafter \( M \) denotes positive constants, not depending on the factor involved.

**Lemma 1.** The operator \((T_{G,a}f)(z)\) maps the class \( S_\beta(G, a) \), \( 0 < \beta < \frac{2}{q} \), \( q > 2 \) into the class \( C_{\beta-1}(G, a) \cap C^\alpha(G) \), \( \alpha = 1 - \frac{2}{q} \), \( 0 < \beta < \frac{2}{q} \), \( q > 2 \) and moreover, the estimates

\[
\|(T_{G,a}f)(z)\|_{G_{\beta-1}(G,a)} \leq M_\beta(G)\|f\|_{S_\beta(G,a)},
\]

(7)

\[
\|(T_{G,a}f)(z)\|_{C^\alpha(G)} \leq M\|f\|_{L_q(G)}
\]

(8)

are hold, where

\[
M_\beta(G) = \sup_{z \in G} \frac{|z - a|^{\beta}}{\pi} \int_G \frac{dG_\zeta}{|\zeta - a|^{1+\beta}|\zeta - z|}.
\]

**Proof.** Let \( f(z) \in S_\beta(G, a) \), \( 0 < \beta < \frac{2}{q} \), \( q > 2 \). Using the Hadamard inequality, we have

\[
|(T_{G,a}f)(z)| \leq M_\beta(G)\|f\|_{S_\beta(G,a)}|z - a|^{1-\beta}.
\]

From here the inequality (7) follows. As \( f(z) \in L_q(G) \), \( q > 2 \), the inequality (8) follows from the estimation (6.8) of the work [1, ch.1, §6]. The lemma is proved.

**Lemma 2.** The operator \((P_{G,a}f)(z)\) maps the space \( C_{\beta-1}(G, a) \) into the space \( C_{\beta-1}(G, a) \cap C^\alpha(G) \), \( \alpha = 1 - \frac{2}{q} \), \( 0 < \beta < \frac{2}{q} \), \( q > 2 \), and moreover, the estimates

\[
\|(P_{G,a}f)(z)\|_{G_{\beta-1}(G,a)} \leq \mu M_\beta(G)\|f\|_{C_{\beta-1}(G,a)},
\]

(9)

\[
\|(P_{G,a}f)(z)\|_{C^\alpha(G)} \leq \mu M\|f\|_{C_{\beta-1}(G,a)},
\]

(10)

\[
\|(P_{G,a}f_1)(z) - (P_{G,a}f_2)(z)\|_{G_{\beta-1}(G,a)} \leq \mu M_\beta(G)\|f_1 - f_2\|_{C_{\beta-1}(G,a)},
\]

(11)

are true, where \( \mu = \|A\|_{S_1(G,a)} + \|B\|_{S_1(G,a)} \); \( f_1, f_2 \) are arbitrary functions from the class \( C_{\beta-1}(G, a) \).

**Proof.** Let \( f(z) \in C_{\beta-1}(G, a) \). Then \( f^*(z) \in S_\beta(G,a) \subset \subset L_q(G) \), \( 2 < q < \frac{2}{\beta} \) and the estimation

\[
\|f^*\|_{S_\beta(G,a)} \leq \mu \|f\|_{C_{\beta-1}(G,a)}
\]

is hold. Therefore, from (7) the inequality (9) follows. The inequality (11) can be proved similarly. As

\[
\|f^*\|_{L_q(G)} \leq M\mu\|f\|_{C_{\beta-1}(G,a)}
\]

inequality (10) follows from the estimation (6.8) of the work [1, ch.1, §6]. The lemma is proved.
From the form of the equation (6) and lemmas 1 and 2 by virtue of the results in [1], it follows, that any solution of this equation from the class $C_{\beta-1}(G, a)$ belongs to the class $W^{1}_q(G)$, $0 < \beta < \frac{2}{q}$, $q > 2$, and satisfies the equation (1) almost everywhere in $G$. Thus, the next result is true.

**Theorem 1.** Any solution of equation (1) from the class (3) satisfies the equation (6). And vice versa, if $\Phi(z) \in U_0(G)$, then any solution of the equation (6) from the class $C_{\beta-1}(G, a)$ belongs to the class $W^{1}_q(G)$, $0 < \beta < \frac{2}{q}$, $q > 2$, and satisfies the equation (1) almost everywhere in $G$.

Let $$(K_G f)(z) = \frac{1}{2\pi i} \int \frac{f(t)}{t-z} dt, \quad (K_{\Gamma,a} f)(z) = (K_G f)(z) - (K_{\Gamma} f)(a).$$

Applying the operator $(K_G f)(z)$ for $z \in G$ to both parts of equality (6), we receive

$$(K_G V)(z) + (P_G V)(a) = (T_G F)(a) + (z - a)\Phi(z). \quad (12)$$

From here for $z = a$ we have

$$(K_G V)(a) + (P_G V)(a) = (T_G F)(a). \quad (13)$$

From (12) and (13) it follows

$$(z - a)\Phi(z) = (K_{\Gamma,a} V)(z).$$

Hence, the integral representation of the first type for the solutions of equation (1) from the class (3) has the form

$$V(z) = -(P_{G,a} V)(z) + (K_{\Gamma,a} V)(z) + (T_{G,a} F)(z), \quad z \in G.$$

**Theorem 2.** The equation (6) is solvable in the class $C_{\beta-1}(G, a)$ for any right-hand side from the same class.

**Proof.** As $0 < \beta < 1$, then from $V(z) \in C_{\beta-1}(G, a)$ it follows, that $V(z) \in C(G)$ and $V(a) = 0$. Therefore, from lemma 2 and Arzela’s theorem it follows, that $P_{G,a}$ is a completely continuous operator from $C_{\beta-1}(G, a)$ into $C_{\beta-1}(G, a) \cap C^\alpha(G)$. Hence, for the proof of the solvability of equation (6) in class $C_{\beta-1}(G, a)$ it is enough to show, that the corresponding homogeneous equation

$$V(z) + (P_{G,a} V)(z) = 0, \quad z \in G \quad (14)$$

has only the trivial solution in $C_{\beta-1}(G, a)$.

Let us prove now, that the homogeneous equation (14) has only the trivial solution in class $C_{\beta-1}(G, a)$. The proof is performed by contradiction.

Assume that equation (14) has a not-trivial solution $V(z)$ in the class $C_{\beta-1}(G, a)$.
Applying the operator $\partial z$ to both sides of the equation (14) we have
\[
\partial z V + A(z) V + B(z) \overline{V} = 0.
\]

Hence, the representation for solution
\[
V(z) = \Phi(z) \exp(-\omega(z)) \quad (15)
\]
holds, see [1,4], where $\Phi(z) \in U_0(G)$, $\omega(z) = (T_G V^\wedge)(z)$, $V^\wedge = \frac{V}{\overline{V}}$.

From (15) it follows
\[
\Phi(z) = V(z) \exp \omega(z) \quad (16)
\]
Substituted (15) into (14) gives
\[
\Phi(z) = -\exp(-\omega(z)) \cdot (P_{G,a} V)(z), \quad \text{where} \quad V = \Phi(z) \exp(-\omega(z)).
\]

As $\exp(-\omega(z)) \cdot (P_{G,a} V)(z) \in U_0(E \setminus \overline{G})$, then from the last equality it follows that the function $\Phi(z)$ analytically single extends to the entire complex plane $E$.

Let $0 < \varepsilon < 1$ and $G_0 = \{z : |z - a| < \varepsilon\} \subset G$. Using the Hadamard inequality, we have
\[
|\omega(z)| \leq \frac{\mu}{\pi} \left( \iint_{G \setminus G_0} \frac{dG_\zeta}{|\zeta - a||\zeta - z|} - \iint_{G_0} \frac{dG_\zeta}{|\zeta - a||\zeta - z|} \right) \leq \mu \left( M(G \setminus G_0) + 8 \ln \frac{1}{|z - a|} \right),
\]
where $M(G \setminus G_0) > 0$ is a constant, depending only on $G \setminus G_0$. From here it follows
\[
M|z - a|^{8\mu} \leq |e^{\omega(z)}| \leq \frac{M}{|z - a|^{8\mu}}, \quad z \in G
\]
Hence by virtue of (16) it follows that $\Phi(z) \in C_{8\mu + \beta - 1} (\overline{G}, a)$.

As $\beta > 1 - 8\mu$ if $1 - 8\mu > 0$ and $\beta > 0$ if $1 - 8\mu \leq 0$ then from here it follows that $8\mu + \beta - 1 > 0$ and $\Phi(\infty) = 0$.

Therefore, the function $\Phi(z)$ analytically single extends to the entire complex plane and is equal to zero at point $z = \infty$. Then by Liouville theorem $\Phi(z) \equiv 0$. Therefore, $V(z) \equiv 0$ in $G$.

The theorem is proved.

From this theorem by virtue of theorem 1 the next result it follows.

**Theorem 3.** The equation (1) is solvable in the class (3).

§2. Zeros and poles of solutions of equation (1)

Let $k$ be an integer number. Let us consider the equation (1) in $G$, where $F(z) \in S_{\beta-k}(G, a)$, $0 < \beta < 1$; $A(z), B(z) \in S_1(G, a)$.

The solution of equation (1) from the class
\[
W_1^1(G) \cap S_{\beta-k-1}(G, a) \quad (17)
\]
is looked in the form

\[ V(z) = (z - a)^k W(z), \]  

where \( W(z) \) is a new unknown function from the class (3).

Substituted (18) into (1) we get

\[ \partial_z W + A(z)W + B_k(z)\overline{W} = F_k(z), \]  

where

\[ B_k(z) = B(z) \exp(-2ik\varphi), F_k(z) = (z - a)^{-k} F(z), \varphi = \arg(z - a). \]

It is obvious, that \( B_k(z) \in S_1(G, a) \), \( F_k(z) \in S_{\beta}(G, a) \). Therefore, by virtue of the results of §1 the equation (19) has solutions from the class (3). They can be found from the equation

\[ W(z) + (P_{G,0}^\Delta W)(z) = (T_{G,0} F_k)(z) + (z - a)\Phi(z), \]  

where

\[ (P_{G,0}^\Delta f)(z) = (T_{G,0} f_k^*)(z), \quad f_k^*(z) = A(z)f + B_k(z)\bar{f}. \]

Thus the next theorem is proved.

**Theorem 4.** The equation (1), where \( F(z) \in S_{\beta-k}(G, a), 0 < \beta < \frac{2}{q}, q > 2, k \) is an integer number, \( A(z), B(z) \in S_1(G, a) \), has solutions from the class (17), which can be found by formulas (18), (20).

**§3. Generalized Riemann-Hilbert problem for equation (1)**

Let \( R > 0, G = \{z : |z| < R\}, \Gamma = \{t : |t| = R\} \). Let us consider equation (1) in \( G \), where \( A(z), B(z) \in S_1(G,0), F(z) \in S_{\beta}(G,0), 0 < \beta < \frac{2}{q}, q > 2. \)

We look for solutions of equation (1) in the class

\[ W^1_q(G) \bigcap S_{\beta-1}(G,0), \quad 0 < \beta < \frac{2}{q}, \quad q > 2. \]  

Let us consider in \( G \) the generalized Riemann-Hilbert problem in the canonical form, see [1]. The more general cases can be reduced to this form, see [1].

**Problem R–H.** It is necessary to find a solution of equation (1) in the class (21), satisfying the boundary condition

\[ \text{Re}[t^{-m}V(t)] = g(t), \quad t \in \Gamma, \]  

where \( m \) is an integer number, \( g(t) \in C^\alpha(\Gamma), 0 < \beta < \frac{2}{q}, q > 2. \)

1. Let \( m \geq 1. \) To solve the R–H problem the equation (6) is used with \( a = 0: \)

\[ V + (P_{G,0}V)(z) = (T_{G,0} F)(z) + z\Phi(z), \quad z \in G, \]  

where \( \Phi(z) \in U_0(G) \bigcap C^\alpha(\overline{G}), \alpha = 1 - \frac{2}{q}. \)
Following [1, ch.4, §7], the function \( \Phi(z) \) in equation (23) is represented in the form

\[
\Phi(z) = \Phi_0(z) - (P_m V)(z) + (Q_m F)(z),
\]

where

\[
\Phi_0(z) \in U_0(G) \cap C^\alpha(\overline{G}), \quad (P_m V)(z) = (Q_m V^*)(z), \quad V^* = A(z)V + B(z)\overrightarrow{V},
\]

\[
(Q_m f) = -\frac{z^{2m}}{\pi R^{2m}} \int_G \frac{f(\zeta)dG_\zeta}{R^2 - \zeta z} - \frac{z^{2m-1}}{\pi R^{2m}} \int_G \frac{f(\zeta)}{\zeta}dG_\zeta.
\]

Substituting the representation (23) for \( V(z) \) in the boundary condition (22), we have

\[
Re[t^{-m+1}\Phi_0(t)] = g(t).
\]

The general solution of this problem is given by the formula, see [1, ch. 4, §7],

\[
\Phi_0(z) = (D_{m-1}g)(z) + \Phi_{0m}(z),
\]

where

\[
(D_m g)(z) = \frac{z^m}{2\pi i} \int_G g(t) \frac{t + z}{t - z} dt,
\]

\[
\Phi_{0m}(z) = \sum_{k=0}^{m-2} (\alpha_k(z^k - R^{2(k-m-1)}z^{2m-k-2}) + i\beta_k(z^k - R^{2(k-m+1)}z^{2m-k-2})) + i\beta_m z^{m-1}, \tag{26}
\]

if \( m \geq 2 \) and \( \Phi_{0m}(z) = i\beta_m, \ if \ m = 1. \)

Here \( \alpha_k, \beta_k, k = 0, ..., m - 2; \beta_m \) are arbitrary real numbers.

From formulas (23) - (25) it follows that

\[
V(z) + (P_m^\wedge V)(z) = (D_m g)(z) + (Q_m^\wedge F)(z) + z\Phi_{0m}(z), \quad z \in G, \tag{27}
\]

where

\[
(P_m^\wedge V)(z) = (P_{G,0} V)(z) + z(P_m V)(z), \quad (Q_m^\wedge V)(z) = (T_{G,0} F)(z) + z(Q_m F)(z).
\]

Thus, with \( m \geq 1 \) the R-H problem is reduced to the equivalent equation (27). For any real numbers \( a_k \) and \( \beta_k, k = 0, ..., m - 2; \beta_m \) the solution of equation (27) is the solution of the R-H problem. Let us prove, that equation (27) has a solution in the class \( C_{\beta-1}(\overline{G}, 0) \). As in the case of the operator \( (P_{G,a} V)(z) \) it is proved, that the operator \( (P_m^\wedge V)(z) \) is completely continuous in the class \( C_{\beta-1}(\overline{G}, 0) \).

Therefore, our statement will be proved, if we show, that the homogeneous equation

\[
V + (P_m^\wedge V)(z) = 0, \quad z \in G \tag{28}
\]

has no non-trivial solution in the class \( C_{\beta-1}(G, 0) \). From (28) by virtue of the Cauchy integral formula, see [8], we get

\[
(K_F V)(z) - (P_G V)(0) = -z(P_m V)(z), \quad z \in G.
\]
If we compare the coefficients of the series expansions with respect to $z$ of the left-and right-hand sides of the last equality, then we obtain
\[ \int_{\Gamma} V(t)e^{-ik\theta}d\theta = 0, \quad k = 0, ..., 2m - 1; \quad t = Re^{i\theta}. \] (29)

Moreover, any solution of equation (29), can be represented as
\[ V(z) = \Phi(z) \exp \Omega(z), \] (30)
where
\[ \Phi(z) \in U_0(G) \bigcap C^\alpha(G), \quad \Phi(0) = 0, \]
\[ \Omega(z) = -\frac{1}{\pi} \int_{G} \left( \frac{V^\wedge(\zeta)}{\zeta - z} \right) dG\zeta, \quad V^\wedge(z) = \frac{V^*(z)}{V(z)}. \]

But $V(z)$ satisfies also the homogeneous boundary condition
\[ Re[t^{-m}V(t)] = 0, \quad t \in \Gamma. \] (31)

As $Re[i\Omega(t)] = 0, \quad t \in \Gamma$, then the boundary condition (31) by (30) is represented as
\[ Re[t^{-m}\Phi(t)] = 0. \]

The general solution of this problem with $\Phi(0) = 0$ has the form
\[ \Phi(z) = \sum_{k=1}^{2m-1} c_k z^k, \]
where $c_k, \quad k = 1, ..., 2m - 1$; are complex constants, satisfying the conditions $c_{2m-k} = \bar{c}_k$, $k = 1, ..., m$. Therefore, by virtue of (30) the solution of equation (28) has the form
\[ V(z) = \sum_{k=1}^{2m-1} c_k z^k \exp \Omega(z). \]

Inserting this into the equality (29), we get
\[ \sum_{k=1}^{2m-1} c_k \int_{\Gamma} t^k t^{-n} \exp \Omega(t)dt = 0, \quad n = 1, ..., 2m - 1. \] (32)

From here follows, that $c_k = 0, \quad k = 1, ..., 2m - 1$; because the determinant of the system (32) is different from zero as the Gram determinant for the system of the linearly independent functions
\[ t^k \exp \left( \frac{\Omega(t)}{2} \right), \quad k = 1, ..., 2m - 1; \quad \Omega(t) = \overline{\Omega(t)}, \quad t \in \Gamma. \]

This proves, that the homogeneous equation (28) has only the trivial solution. Hence, the integral equation (27) has a solution in $C_{\beta-1}(\overline{G}, 0)$ for any right-hand side belonging to $C_{\beta-1}(\overline{G}, 0)$. 

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Thus for \( m > 0 \) the non-homogeneous R-H problem is always solvable a solution and the homogeneous R-H problem \((F \equiv 0, \ g \equiv 0)\) has exactly \(2m - 1\) linearly independent solutions over the field of real numbers. The latter follows from the fact that the homogeneous problem is equivalent to the integral equation

\[
V + (P^m_0 V)(z) = z\Phi_0m(z),
\]

where \( \Phi_0m(z) \) is defined by formula (26), the right-hand side of which is a linear combination of \(2m - 1\) linearly independent functions. Thus the next theorem follows.

**Theorem 5.** For \( m > 0 \) the non-homogeneous R-H problem is always solvable and the homogeneous R-H problem has \(2m - 1\) linearly independent solutions over the field of real numbers.

2\(^0\). Let \( m = 0 \). Following to 1\(^0\), the function \( z\Phi(z) \) in the equation (23) is represented in the form

\[
z\Phi(z) = (D_0g)(z) + ic_0 - z(P_0V)(z) + z(Q_0F)(z),
\]

where \( c_0 \) is an arbitrary real number.

The function \( \Phi(z) \), given by formula (33), belongs to the class \( U_0(G) \), if the equalities

\[ (D_0g)(0) = 0, \]

\[ c_0 = 0 \]

are hold.

Using (33)–(34), from (23) we obtain

\[
V(z) + (P^\wedge V)(z) = (D_0g)(z) + (Q^\wedge F)(z), \ z \in G,
\]

where

\[
(P^\wedge V)(z) = (P_{G,0}V)(z) + z(P_0V)(z), \ (Q^\wedge V)(z) = (T_{G,0}F)(z) + z(Q_0F)(z).
\]

If equality (34) holds, then any solution of equation (36) belonging to \( C_{\beta-1} (\overline{G}, 0) \) satisfies the boundary condition (22) when \( m = 0 \). Solvability of equation (36) in the class \( C_{\beta-1} (\overline{G}, 0) \) can be proved similarly to the proof of the solvability of the equation (27). Hence, under the condition (34) the R-H problem is solvable. Thus, we have the next result.

**Theorem 6.** For the solvability of the R-H problem in the case \( m = 0 \) it is necessary and sufficient that the single condition (34) is satisfied. If the condition (34) holds, then the solution of the problem can be found from equation (36).

3\(^0\). Let \( m < 0 \). As the solutions of the R-H problem are sought from the class (21), in this case formula (27) is not suitable. Therefore, let us introduce the function \( W(z) = z^{k+1}V(z) \) into consideration, where \( k = -m \). The function \( W(z) \) satisfies the equation

\[
\partial_z W + A(z)W + B_{-k-1}(z)\overline{W} = F_{-k-1}W(z),
\]
where
\[ B_{-k-1}(z) = \exp(2(k+1)i\varphi)B(z), \quad F_{-k-1}(z) = z^{k+1}F(z), \quad \varphi = \arg z \]
and the boundary condition
\[ \text{Re}[t^{-1}W(t)] = g(t), \quad t \in \Gamma. \]
It is obvious, that \( B_{-k-1}(z) \in S_1(G, 0), \quad F_{-k-1}(z) \in S_\beta(G, 0). \)

Therefore this problem corresponds to the one considered in \( m = 1 \). Hence, the function \( W(z) \) satisfies the equation
\[ W(z) + (P_1^\lambda W)(z) = (D_1g)(z) + (Q_1^\lambda F_{-k-1})(z) + ic_0z, \quad z \in G, \]
where \( c_0 \) is an arbitrary real number,
\[ (P_1^\lambda W)(z) = (T_{G,0}W_{-k-1}^*)(z) + z(Q_1W_{-k-1}^*)(z), \quad W_{-k-1}^* = A(z)W + B_{-k-1}(z)\overline{W}. \]

Thus, when \( m < 0 \) the solution of the R-H problem can be found from the equation
\[ V(z) + (QV)(z) = \frac{z}{\pi i} \int_{\Gamma} \frac{g(t)dt}{t^{k+1}(t-z)} + (T_{G,0}F_{-k-1})(z) + \sum_{j=1}^{k} a_j z^{-j}, \quad z \in G, \quad (37) \]
where
\[ (QV)(z) = (T_{G,0}(z^{k+1}V^*(z)))(z) - \frac{z}{\pi i} \int_{\Gamma} \frac{\zeta^{2k}V^*(\zeta)}{1 - \zeta z} dG_\zeta, \]
\[ a_j = \frac{1}{\pi} \int_G \zeta^{-j}V^*(\zeta)dG_\zeta + \frac{1}{\pi} \int_G \zeta^{2k-j-1}V^*(\zeta)dG_\zeta + \frac{1}{\pi i} \int_{\Gamma} t^{j-k-1}g(t)dt, \quad g = 0, \ldots, k-1; \]
\[ a_k = \frac{1}{\pi} \int_G \zeta^{-k}V^*(\zeta)dG_\zeta + \frac{1}{2\pi i} \int_{\Gamma} \frac{g(t)}{t}dt + ic_0, \quad V^* = A(z)V + B(z)\overline{V}. \]

From (37) it follows that for the continuity of the function \( V(z) \) inside of \( G \) it is necessary and sufficient that the equalities
\[ a_j = 0, \quad j = 0, \ldots, k; \quad (38) \]
are hold. The condition (38) contain \( 2k + 2 \) real equalities. One of them, namely \( \text{Im} \ a_k = 0 \), is possible to be satisfied by means of a suitable choice of the constant \( c_0 \). Hence, there remain \( 2k + 1 \) conditions. Thus, for \( m < 0 \) the R-H problem is reduced to the equation
\[ V(z) + (QV)(z) = \frac{z}{\pi i} \int_{\Gamma} \frac{g(t)dt}{t^{k+1}(t-z)} + (T_{G,0}F_{-k-1})(z), \quad z \in G. \quad (39) \]
The operator \( Q \) is completely continuous in \( C_{\beta-1}(G, 0) \) and mapping this space into \( C_{\beta-1}(G, 0) \cap C^\alpha(\overline{G}). \)

If in the equation
\[ V(z) + (QV)(z) = 0 \]
we replace \( V(z) \) by \( zW(z) \), then we obtain the equation (7.33) from [1, ch.4, §7], which has only the trivial solution. Therefore equation (39) is solvable in \( C_{\beta-1}(G,0) \) for any right-hand side from the same class. Thus, we have the next result.

**Theorem 7.** For the solvability of the R-H problem in the case \( m < 0 \) it is necessary and sufficient that the \( 2|m| + 1 \) real conditions (38) are satisfied.

§4. Riemann-Hilbert problem with an initial condition for equation (1)

Let \( G = \{ z : |z| < R \} \), \( \Gamma = \{ t : |t| = R \} \), \( \nu > 0 \), \( k = [\nu] \), \( k \neq \nu \), \( \beta = 1 - \nu + k \), \( R > 0 \). Let us consider the equation (1) in \( G \), where \( A(z), B(z) \in S_1(G,0) \), \( F(z) \in S_{1-\nu}(G,0) \).

Let us consider the Riemann-Hilbert problem with an initial condition in the following form.

**Problem \((R-H)_0\).** Find the solution of the equation (1) from the class

\[
C_{-\nu}(G,0) \cap W_q^{1}(G), \quad 2 < q < \frac{2}{\beta},
\]

satisfying the boundary condition

\[
Re[t^{-n}V(t)] = g(t), \quad t \in \Gamma,
\]

where \( g(t) \in C^\alpha(\Gamma) \), \( \alpha = 1 - \frac{2}{q} \), \( n \) is an integer number.

The solution of the \((R-H)_0\) problem is looked for in the form

\[
V(z) = z^kW(z),
\]

where \( W(z) \) is a new unknown function from the class

\[
W_q^{1}(G) \cap S_{\beta-1}(G,0), \quad 0 < \beta < \frac{2}{q}, \quad q > 2.
\]

**Remark 1.** If \( \nu < 1 \), the substitution (41) is not required.

**Remark 2.** If \( \nu > 0 \) is an integer number, then from \( V(z) = O(|z|^{\nu+\beta}) \), \( z \to 0 \), \( 0 < \beta < 1 \), it follows that \( V(z) = O(|z|^{\nu}) \), \( z \to 0 \). Therefore, the results which will be obtained for \([\nu] \neq \nu\), will also hold for \([\nu] = \nu\).

Substituting (41) into (1) and (40), respectively, we obtain

\[
\partial_z W + A(z)W + B_k(z)\overline{W} = F_k(z), \quad z \in G,
\]

where

\[
B_k(z) = B(z) \cdot \exp(-2ik\varphi), \quad F_k(z) = z^{-k}F(z)
\]

and

\[
Re[t^{-m}W(t)] = g(t), \quad t \in \Gamma, \quad m = n - k.
\]
It is obvious, that $B_k(z) \in S_1(G, 0)$, $F_k(z) \in S_\beta(G, 0)$, $0 < \beta < 1$. Hence, we obtain the Riemann-Hilbert problem solved in §3 for the equation (42). Therefore, from the results of §3 the next result follows.

**Theorem 8.** 1) For $n > [\nu]$ the problem $(R - H)_0$ is always solvable. The corresponding homogeneous problem has $2(n-[\nu])-1$ linearly independent solutions over the field of real numbers.

2) For $n = [\nu]$ for the solvability of the $(R - H)_0$ problem it is necessary and sufficient that the single condition (34) is satisfied.

3) For $n < [\nu]$ for the solvability of the $(R - H)_0$ problem it is necessary and sufficient that $2n-[\nu]+1$ conditions of the type (38) are satisfied, which are written with respect to the equation (42).

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