Multidimensional Riemann Problems: Transonic Shock Waves and Free Boundary Problems

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Abstract

We are concerned with global solutions of multidimensional Riemann problems for nonlinear hyperbolic systems of conservation laws, focusing on their global configurations and structures. We present some recent developments in the rigorous analysis of two-dimensional Riemann problems involving transonic shock waves through several prototypes of hyperbolic systems of conservation laws and discuss some further multidimensional Riemann problems and related problems for nonlinear partial differential equations (PDEs). In particular, we present four different two-dimensional Riemann problems through these prototypes of hyperbolic systems and show how these Riemann problems can be reformulated/solved as free boundary problems with transonic shock waves as free boundaries for the corresponding nonlinear conservation laws of mixed elliptic-hyperbolic type and related nonlinear PDEs.

Keywords: Riemann problems, M-D, 2-D, transonic shocks, solution structure, free boundary problems, mixed elliptic-hyperbolic type, global configurations, large-time asymptotics, global attractors, shock capturing methods

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1 Introduction

We are concerned with global solutions of multidimensional (M-D) Riemann problems for nonlinear hyperbolic systems of conservation laws, focusing on their global configurations and structures. In this paper, we present some recent developments in the rigorous analysis of two-dimensional (2-D) Riemann problems involving transonic shock waves (shocks, for short) through several prototypes of hyperbolic systems of conservation laws and discuss some further M-D Riemann problems and related problems for nonlinear partial differential equations (PDEs). These Riemann problems can be reformulated as free boundary problems with transonic shocks as free boundaries for the corresponding nonlinear conservation laws of mixed elliptic-hyperbolic type and related nonlinear PDEs.

The study of Riemann problems has an extensive history, which dates back to the pioneering work of Riemann [74] in 1860. For the one-dimensional (1-D) Riemann problem, a theory has been established for the appropriate amplitude of the Riemann data for general strictly hyperbolic systems (cf. [55, 66]) and for general Riemann data for the compressible Euler equations (cf. [12, 70, 78, 87] and the references cited therein). The 1-D Riemann problem has been essential in the development of the 1-D mathematical theory of hyperbolic conservation laws and associated shock capturing methods for the construction and computation of global entropy solutions; see [35, 42, 44, 54, 55, 57, 66, 77] and the references cited therein. More importantly, general global entropy solutions can be locally approximated by the Riemann solutions that are regarded as fundamental building blocks of the entropy solutions (cf. [35, 42, 55, 78]). Moreover, the Riemann solutions usually determine the large-time asymptotic behaviors and global attractors of general entropy solutions of the Cauchy problem. On the other hand, it is the simplest Cauchy problem (initial value problem) whose solutions have fine explicit structures.

The M-D Riemann problems are more challenging mathematically, and the corresponding M-D Riemann solutions are of much richer global configurations and structures; see [9–12, 34, 35, 43, 44, 56, 75, 90] and the references cited therein. Thus, the Riemann solutions often serve as standard test models for analytical and numerical methods for solving nonlinear hyperbolic systems of conservation laws and related nonlinear PDEs. Theoretical results for first-order scalar conservation laws are available in [12, 27, 45, 65, 79, 86, 91] and the references cited therein. During recent decades, some significant developments for the 2-D Riemann problems for first-order hyperbolic systems and second-order hyperbolic equations of conservation laws have been made. In this paper, we present four different 2-D Riemann problems involving transonic shocks through the prototypes of nonlinear hyperbolic PDEs and demonstrate how these Riemann problems can be reformulated and then solved as free boundary problems for nonlinear conservation laws of mixed elliptic-hyperbolic type and related nonlinear PDEs. These are achieved by developing further the nonlinear method and related ideas/techniques introduced in Chen-Feldman [20–22] for solving free boundary problems with transonic shocks as free boundaries.
for nonlinear conservation laws of mixed elliptic-hyperbolic type and related nonlinear PDEs; also see [14, 23].

The organization of this paper is as follows: In Section 2, we first show how the solutions of M-D Riemann problems for hyperbolic conservation laws can be formulated as the self-similar solutions for nonlinear conservation laws of mixed elliptic-hyperbolic type and then we introduce the notion of Riemann solutions in the self-similar coordinates in the distributional sense. In Section 3, we present the first 2-D Riemann problem, Riemann Problem I, involving two shocks and two vortex sheets for the pressure gradient system and show how Riemann Problem I can be reformulated/solved as a free boundary problem with transonic shocks as free boundaries for a second-order nonlinear conservation law of mixed elliptic-hyperbolic type and related nonlinear PDEs.

In Section 4, we present the second 2-D Riemann problem, Riemann Problem II – the Lighthill problem for shock diffraction by convex cornered wedges through the nonlinear wave equations, and show how Riemann Problem II can be solved as another free boundary problem. In Section 5, we present the third 2-D Riemann problem, Riemann Problem III – the Prandtl-Meyer problem for unsteady supersonic flow onto solid wedges through the Euler equations for potential flow and show how Riemann Problem III can be reformulated/solved as a free boundary problem for a second-order nonlinear conservation law of mixed elliptic-hyperbolic type.

Then, in Section 6, we present the fourth 2-D Riemann problem, Riemann Problem IV – the von Neumann problem for shock reflection-diffraction by wedges for the Euler equations for potential flow, and show how Riemann Problem IV can be solved again as a free boundary problem. We give our concluding remarks in Section 7 and discuss several further M-D Riemann problems and related problems for nonlinear PDEs.

2 Multidimensional Riemann Problems and Nonlinear Conservation Laws of Mixed Elliptic-Hyperbolic Type

In this section, we first show how the solutions of the M-D Riemann problems for nonlinear hyperbolic conservation laws can be formulated as the self-similar solutions for nonlinear conservation laws of mixed elliptic-hyperbolic type, and then introduce the notion of Riemann solutions in the self-similar coordinates in the distributional sense.

Consider both the M-D first-order quasilinear hyperbolic systems of conservation laws of the form:

\[ \partial_t U + \nabla_x \cdot F = 0 \quad \text{for } t \in \mathbb{R}_+ = [0, \infty) \text{ and } x \in \mathbb{R}^n \]  

(2.1)

with \( U \in \mathbb{R}^m \) and nonlinear mapping \( F : \mathbb{R}^m \to \mathbb{R}^m \times \mathbb{R}^n \), and the M-D second-order quasilinear hyperbolic equations of conservation laws of the form:

\[ \partial_t G_0(\partial_t \Phi, \nabla \Phi) + \nabla_x \cdot G(\partial_t \Phi, \nabla_x \Phi) = 0 \quad \text{for } t \in \mathbb{R}_+ \text{ and } x \in \mathbb{R}^n \]  

(2.2)
with \( \Phi \in \mathbb{R} \) and nonlinear mapping \((G_0, G) : \mathbb{R}^{n+1} \to \mathbb{R} \times \mathbb{R}^n\).

A prototype of (2.1) is the full Euler equations in the conservation form (2.1) with

\[
\begin{align*}
U := (\rho, \rho u, \rho E)^T, \\
F := (\rho u, \rho u \otimes u + pI, (\rho E + p)u)^T,
\end{align*}
\]

where \( \rho > 0 \) is the density, \( u \in \mathbb{R}^n \) the velocity, \( p \) the pressure, and \( E = \frac{|u|^2}{2} + e \) the total energy per unit mass with the internal energy \( e \) given by \( e = \frac{p}{(\gamma - 1)\rho} \) for the adiabatic constant \( \gamma > 1 \) for polytropic gases.

A prototype of (2.2) can be derived from the Euler equations for potential flow, which is governed by the conservation law of mass and the Bernoulli law for the density function \( \rho \) and the velocity potential \( \Phi \) (i.e., \( u = \nabla \Phi \)):

\[
\begin{align*}
\partial_t \rho + \nabla \cdot (\rho \nabla \Phi) &= 0, \\
\partial_t \Phi + \frac{1}{2} |\nabla \Phi|^2 + h(\rho) &= B,
\end{align*}
\]

where \( B \) is the Bernoulli constant and \( h(\rho) \) is given by

\[
h(\rho) = \frac{\rho^{\gamma - 1} - 1}{\gamma - 1} \quad \text{for the adiabatic exponent } \gamma > 1. \tag{2.5}
\]

By (2.4)–(2.5), \( \rho \) can be expressed as

\[
\rho(\partial_t \Phi, \nabla \Phi) = h^{-1}(B - \partial_t \Phi - \frac{1}{2} |\nabla \Phi|^2).
\]

Then system (2.4) can be rewritten as the second-order nonlinear wave equation (2.2) with

\[
(G_0, G) = (\rho(\partial_t \Phi, \nabla \Phi), \rho(\partial_t \Phi, \nabla \Phi) \nabla \Phi) \tag{2.7}
\]

and \( \rho(\partial_t \Phi, \nabla \Phi) \) determined by (2.6).

A **standard Riemann problem** for (2.1) is a special Cauchy problem:

\[
\begin{align*}
U|_{t=0} &= U_0(x)
\end{align*}
\]

so that the initial data function \( U_0(x) \) is invariant under the self-similar scaling in \( x \):

\[
U_0(\alpha x) = U_0(x) \quad \text{for any } \alpha > 0
\]

that is, \( U_0(x) \) is constant along the ray originating from \( x = 0 \); in other words, \( U_0 \) depends only on the angular directions of the rays originating from \( x = 0 \) in \( \mathbb{R}^n \).

A **lateral Riemann problem** for (2.1) is a special initial-boundary problem in a unbounded domain \( \mathcal{D} \) that contains the origin and is invariant under the self-similar scaling (i.e., if \( x \in \mathcal{D} \), then \( \alpha x \in \mathcal{D} \) for any \( \alpha > 0 \)) so that the initial data and boundary data are also invariant under the self-similar scaling.
Since system (2.1) is invariant under the time-space self-similar scaling, the standard/lateral Riemann problems are also invariant under the time-space self-similar scaling:

\[(t, x) \rightarrow (\alpha t, \alpha x) \quad \text{for any } \alpha > 0.\]  

(2.9)

Thus, we seek self-similar solutions of the Riemann problems:

\[U(t, x) = \mathbf{V}(\frac{x}{t}).\]  

(2.10)

Denote \(\xi = \frac{x}{t}\) as the self-similar variables. Then \(\mathbf{V}(\xi)\) is determined by

\[D \cdot \mathbf{F}(\mathbf{V}) - \xi \cdot D\mathbf{V} = 0,\]

that is,

\[D \cdot (\mathbf{F}(\mathbf{V}) - \mathbf{V} \otimes \xi) + n \mathbf{V} = 0,\]  

(2.11)

where \(D = (\partial_{\xi_1}, \cdots, \partial_{\xi_n})\) is the gradient with respect to the self-similar variables \(\xi = (\xi_1, \cdots, \xi_n) \in \mathbb{R}^n\), and \(\mathbf{V} \otimes \xi = (V_i \xi_j)_{1 \leq i, j \leq n}\). Even though system (2.1) is hyperbolic, system (2.11) generally is of mixed elliptic-hyperbolic type, even composite-mixed elliptic-hyperbolic type. In particular, for a bounded solution \(\mathbf{V}(\xi)\), system (2.11) may be purely hyperbolic in the far field, i.e., outside a large ball in the \(\xi\)-coordinates, but generally is of mixed type or composite-mixed type in a bounded domain containing origin \(\xi = 0\).

For the full Euler system (2.1) with (2.3), the self-similar solutions are governed by the following system:

\[
\begin{cases}
\text{div}(\rho \mathbf{v}) + n \rho = 0, \\
\text{div}(\rho \mathbf{v} \otimes \mathbf{v}) + Dp + (n + 1)\rho \mathbf{v} = 0, \\
\text{div}(\frac{1}{2} \rho |\mathbf{v}|^2 + \frac{\gamma p}{\gamma - 1})\mathbf{v}) + n\left(\frac{1}{2} \rho |\mathbf{v}|^2 + \frac{\gamma p}{\gamma - 1}\right) = 0,
\end{cases}
\]

(2.12)

where \(\mathbf{v} = \mathbf{u} - \xi\) is the pseudo-velocity with \(\mathbf{V} = (\rho, \rho \mathbf{v}, \frac{1}{2} \rho |\mathbf{v}|^2 + \rho e)^\top\).

The weak solutions of system (2.11) can be defined as follows:

\textbf{Definition 2.1} (Weak Solutions). A function \(\mathbf{V} \in L^\infty_{\text{loc}}(\Omega)\) in a domain \(\Omega \subset \mathbb{R}^n\) is a weak solution of system (2.11) in \(\Omega\), provided that

\[
\int_{\Omega} \left\{ (\mathbf{F}(\mathbf{V}) - \mathbf{V} \otimes \xi) \cdot D\zeta(\xi) - n \mathbf{V} \zeta(\xi) \right\} \, d\xi = 0
\]

(2.13)

for any \(\zeta \in C^1(\Omega)\).

It can be shown that any weak solution of system (2.11) in the \(\xi\)-coordinates in the sense of Definition 2.1 is a weak solution of system (2.1) in
the \((t, x)\)-coordinates. Then any co-dimension-one \(C^1\)-discontinuity \(S\) satisfies the Rankine-Hugoniot conditions along \(S\) in the \(\xi\)-coordinates:

\[
([F(V)] - [V] \otimes \xi) \cdot \nu_s = 0,
\]
or equivalently,

\[
[(F(V) - V \otimes \xi) \cdot \nu_s] = 0,
\]  
(2.14)

where \(\nu_s\) is a unit normal to \(S\), and \([\cdot]\) denotes the difference between the traces of the corresponding quantities on the two sides of the co-dimension-one surface \(S\).

Similarly, the Riemann problems for Eq. (2.2) are invariant under the time-space self-similar scaling:

\[
(t, x, \Phi(t, x)) \to (\alpha t, \alpha^2 x, \Phi(\alpha t, \alpha^2 x)) \quad \text{for any } \alpha > 0. \quad (2.15)
\]

Thus, we seek self-similar solutions of the Riemann problem:

\[
\Phi(t, x) = t\phi\left(\frac{x}{t}\right). \quad (2.16)
\]

Then \(\phi(\xi)\) is determined by

\[
\text{div}\left(G(\phi - \xi \cdot D\phi, D\phi) - \xi \cdot DG_0(\phi - \xi \cdot D\phi, D\phi)\right) = 0,
\]
that is,

\[
\text{div}(G(\phi - \xi \cdot D\phi, D\phi) - G_0(\phi - \xi \cdot D\phi, D\phi)\xi) + nG_0(\phi - \xi \cdot D\phi, D\phi) = 0. \quad (2.17)
\]

Again, even though Eq. (2.2) is hyperbolic, Eq. (2.17) generally is of mixed elliptic-hyperbolic type. In particular, for a gradient bounded solution \(\phi(\xi)\), Eq. (2.17) may be purely hyperbolic in the far field, i.e., outside a large ball in the \(\xi\)-coordinates, but generally is of mixed type in a bounded domain containing the origin.

For the Euler equations for potential flow (2.2) with (2.6)–(2.7), the self-similar solutions are governed by the following second-order nonlinear equation for the pseudo-velocity \(\varphi = \phi - \frac{1}{2}\|\xi\|^2\):

\[
\text{div}(\rho(|D\varphi|^2, \varphi)D\varphi) + n\rho(|D\varphi|^2, \varphi) = 0, \quad (2.18)
\]

where \(\rho(|D\varphi|^2, \varphi) = \left(B_0 - (\gamma - 1)(\frac{1}{2}|D\varphi|^2 + \varphi)\right)^{\frac{1}{\gamma-1}}\) with \(B_0 = (\gamma - 1)B + 1\).

The weak solutions of Eq. (2.17) can be defined as follows:
Definition 2.2. A function $\phi \in W^{1,\infty}_{\text{loc}}(\Omega)$ in a domain $\Omega \subset \mathbb{R}^n$ is a weak solution of system (2.17) in $\Omega$, provided that
\[
\int_{\Omega} \{(G(\phi - \xi \cdot D\phi, D\phi) - G_0(\phi - \xi \cdot D\phi, D\phi)\xi) \cdot D\zeta(\xi) \\
- nG_0(\phi - \xi \cdot D\phi, D\phi)\zeta(\xi)\} \, d\xi = 0
\]
(2.19)
for any $\zeta \in C^1_0(\Omega)$.

Similarly, it can shown that any weak solution of Eq. (2.17) in the $\xi$–coordinates in the sense of Definition 2.2 is a weak solution of Eq. (2.2) in the $(t, x)$–coordinates. Then any co-dimension-one $C^1$–discontinuity $S$ satisfies the Rankine-Hugoniot conditions along $S$ in the $\xi$–coordinates:

\[
[\phi] = 0,
\]
\[
(G(\phi - \xi \cdot D\phi, D\phi) - G_0(\phi - \xi \cdot D\phi, D\phi)\xi) \cdot \nu_s = 0,
\]
or equivalently,
\[
[\phi] = 0,
\]
\[
[(G(\phi - \xi \cdot D\phi, D\phi) - G_0(\phi - \xi \cdot D\phi, D\phi)\xi) \cdot \nu_s] = 0,
\]
where $\nu_s$ is a unit normal to $S$.

3 Two-Dimensional Riemann Problem I: Two Shocks and Two Vortex Sheets for the Pressure Gradient System

In this section, we present the first 2-D Riemann problem, Riemann Problem I, through the pressure gradient system that is a hyperbolic system of conservation laws.

The pressure gradient system takes the following form:

\[
\begin{aligned}
&u_t + p x_1 = 0, \\
v_t + p x_2 = 0, \\
&E_t + (p u) x_1 + (p v) x_2 = 0,
\end{aligned}
\]
(3.1)

where $E = \frac{|u|^2}{2} + p$ with $u = (u, v)$. System (3.1) can be written in the form (2.2) with

\[
U = (u, E)^\top, \quad F_1 = (E - \frac{|u|^2}{2})(1, 0, u)^\top, \quad F_2 = (E - \frac{|u|^2}{2})(0, 1, v)^\top.
\]
(3.2)

There are two mechanisms for the fluid motion: the inertia and the pressure difference. Corresponding to a separation of these two mechanisms, the full Euler equations (2.1) with (2.3) in gas dynamics can be split into two
subsystems of conservation laws: the pressure gradient system and the pressureless Euler system, respectively; also see [1, 29, 62] and the references cited therein for this and similar flux-splitting ideas which have been widely used in order to design the so-called flux-splitting schemes and their high-order accurate extensions. Furthermore, system (3.1) can also be deduced from system (2.1) with (2.3) under the physical regime whereby the velocity is small and the adiabatic gas constant $\gamma$ is large; see Zheng [92]. An asymptotic derivation of system (3.1) has also been presented by Hunter as described in [94]. We refer the reader to [58, 95] for further background on system (3.1).

3.1 2-D Riemann Problem I: Two Shocks and Two Vortex Sheets

We now consider the following Riemann problem:

**Problem 3.1 (2-D Riemann Problem I: Two Shocks and Two Vortex Sheets).** Seek a global solution of system (3.1) with Riemann initial data that consist of four constant states in four sectorial regions $\Omega_i$ with symmetric sectorial angles (see Fig. 3.1):

$$ (p, u)(0, x) = (p_i, u_i) \quad \text{for} \quad x \in \Omega_i, \ i = 1, 2, 3, 4, \quad (3.3) $$

such that the four initial constant states are required to satisfy the following conditions:

$$ \begin{cases} 
A \text{ forward shock } S_{41}^+ \text{ is formed between states (1) and (4),} \\
A \text{ backward shock } S_{12}^- \text{ is formed between states (1) and (2),} \\
A \text{ vortex sheet } J_{23}^+ \text{ is formed between states (2) and (3),} \\
A \text{ vortex sheet } J_{34}^- \text{ is formed between states (3) and (4).} 
\end{cases} \quad (3.4) $$

![Fig. 3.1 Riemann Problem I: Riemann initial data (cf. [31, 93])](image-url)
This Riemann problem initially with the assumption that angle $\alpha_1 = \alpha_2$ is close to zero was first analyzed rigorously in Zheng [93], for which the two shocks bend slightly and the diffracted shock $\Gamma_{\text{shock}}$ does not meet the inner sonic circle $C_2$. In the recent work [31], this Riemann problem has been solved globally for the general case; that is, the angle between the two shocks is not necessarily close to $\pi$.

### 3.2 Reformulation of Riemann Problem I

As we discussed earlier, we seek self-similar solutions in the self-similar coordinates with the form:

$$(p, u)(t, x) = (p, u)(\xi) \quad \text{with} \quad \xi = \frac{x}{t}, \ t > 0.$$ 

In the $\xi$-coordinates, system (3.1) can be rewritten in form (2.11) with (3.2). The four waves in Riemann Problem I can be obtained by solving four 1-D Riemann problems in the self-similar coordinates $\xi$, which form the following configuration as shown in Fig. 3.2:

![Fig. 3.2 Riemann Problem I: Riemann solution configuration (cf. [31])](image)

More precisely, let $\xi_2 = f(\xi_1)$ be a $C^1$–discontinuity curve of a bounded discontinuous solution of system (2.11) with (3.2). From the Rankine-Hugoniot relations on $\xi_2 = f(\xi_1)$:

$$\begin{align*}
\left(\xi_1 f'(\xi_1) - f(\xi_1)\right)[u] - f'(\xi_1)[p] &= 0, \\
\left(\xi_1 f'(\xi_1) - f(\xi_1)\right)[v] + [p] &= 0, \\
\left(\xi_1 f'(\xi_1) - f(\xi_1)\right)[E] - f'(\xi_1)[pu] + [pv] &= 0,
\end{align*}$$
we find that $\xi_2 = f(\xi_1)$ can be one of the two nonlinear discontinuities:

$$\begin{cases}
\frac{df(\xi_1)}{d\xi_1} = \sigma_\pm = -\frac{|u|}{v} = \frac{\xi_1 f(\xi_1) \pm \sqrt{p(\xi_1^2 + |f(\xi_1)|^2 - \overline{p})}}{\xi_1^2 - \overline{p}}, \\
[p]^2 = \overline{p}(|u|^2 + |v|^2),
\end{cases}$$

or a vortex sheet (linearly degenerate discontinuity):

$$\begin{cases}
\sigma_0 = \frac{f(\xi_1)}{\xi_1} = \frac{|v|}{|u|}, \\
[p] = 0,
\end{cases}$$

where $\overline{p}$ is the average of the pressure on the two sides of the discontinuity.

A nonlinear discontinuity is called a shock if it satisfies (3.5) and the entropy condition: pressure $p$ increases across it in the flow direction; that is, the pressure ahead of the wave-front is larger than that behind the wave-front. There are two types of shocks $S^\pm$:

- $S = S^+$ if $Dp$ and the flow direction form a right-hand system;
- $S = S^-$ if $Dp$ and the flow direction form a left-hand system.

A discontinuity is called a vortex sheet if it satisfies (3.6). There are two types of vortex sheets $J^\pm$ determined by the signs of the vorticity:

$$J^\pm : \text{curl } u = \pm \infty.$$ 

It can be shown that, for fixed $(p_1, u_1)$ and $p_2 = p_3 = p_4$ satisfying $p_1 > p_2$, there exist states $u_i, i = 2, 3, 4$, such that the conditions in (3.4) for the Riemann initial data hold, and $u_i, i = 2, 3, 4$, depend on angles $(\alpha_1, \alpha_2)$.

![Fig. 3.3](image-url) The Riemann data and the global solution when $\alpha_1 = 0$ (cf. [31])
There is a critical case when $\alpha_1 = 0$. Then the Riemann initial data satisfy

\[ p_1 > p_2 = p_3 = p_4, \quad u_1 = u_2 = u_3 = u_4, \quad v_1 > v_2 = v_3 = v_4. \]

The global Riemann solution is a piecewise constant solution with two planar shocks: $S_{12}^-$ for $\xi_1 < 0$ and $S_{41}^+$ for $\xi_1 > 0$ on line $\xi_2 = -\sqrt{p}$ with

\[ [v] = -\frac{[p]}{\sqrt{p}}, \quad [u] = 0 \quad \text{for} \quad \bar{p} = \frac{p_1 + p_2}{2}, \]

and two vortex sheets $J_{23}^+$ and $J_{34}^-$, as shown in Fig. 3.3. The two planar shocks $S_{12}^-$ and $S_{41}^+$ are both tangential to circle $|\xi| = \sqrt{p}$ with the tangent point on the circle as the end-point. It follows from the expression of $J_{23}^+$ given in (3.6) that $p_2 = p_3$ on both sides of $J_{23}^+$. At the point where $J_{23}^+$ intersects with $S_{12}^-$, we see that $J_{23}^+$ does not affect the shock owing to $p_2 = p_3$. The intersection between $J_{34}^+$ and $S_{41}^+$ can be handled in the same way.

We now consider the general case: $\alpha_1 \in (0, \frac{\pi}{2})$. From system (2.11) with (3.2), we can derive the following second-order nonlinear equation for $p$:

\[
(p - \xi_1^2)p_{\xi_1} - 2\xi_1\xi_2p_{\xi_1}\xi_2 + (p - \xi_2^2)p_{\xi_2}\xi_2 + \frac{(\xi_1p_{\xi_1} + \xi_2p_{\xi_2})^2}{p} - 2(\xi_1p_{\xi_1} + \xi_2p_{\xi_2}) = 0.
\]

Eq. (3.7) is of mixed hyperbolic-elliptic type, which is hyperbolic when $|\xi| > \sqrt{p}$ and elliptic when $|\xi| < \sqrt{p}$ with the transition boundary – the sonic circle $|\xi| = \sqrt{p}$. Furthermore, in the polar coordinates: $(r, \theta) = (|\xi|, \arctan(\frac{\xi_2}{\xi_1}))$, Eq. (3.7) becomes

\[
Q_p := (p - r^2)p_{rr} + \frac{p}{r^2}p_{\theta\theta} + \frac{p}{r}p_r + \frac{1}{p}(\frac{r}{p}p^2 - 2rpp_r) = 0,
\]

which is hyperbolic when $p < r^2$ and elliptic when $p > r^2$. The sonic circle is given by $r = r(\theta) = \sqrt{p(r(\theta), \theta)}$.

In the $\xi$–coordinates, the four waves come from the far-field (at infinity corresponding to $t = 0$) and keep planar waves before the two shocks meet the outer sonic circle $C_1$ of state (1):

\[
C_1 := \{\xi : |\xi| = \sqrt{p_1}\}.
\]

When the two shocks $S_{12}^-$ and $S_{41}^+$ meet the sonic circle $C_1$ at points $P_3$ and $P_1$ respectively, the key point is whether they bend and meet to form a diffracted shock, denoted by $\Gamma_{\text{shock}}$; see Fig. 3.2. Since the whole configuration is symmetric with respect to the $\xi_2$–axis, $\Gamma_{\text{shock}}$ must be perpendicular to $\xi_1 = 0$ at point $P_2$ where the two diffracted shocks meet. It is not known a priori whether the diffracted shock may degenerate partially into a portion of the inner sonic circle $C_2$ of state (2). Once this case occurs, $p = p_2$ on the sonic circle, which
satisfies the oblique derivative condition on the diffracted shock automatically. Observe that the two vortex sheets $J_{23}^+$ and $J_{34}^-$ and the diffracted shock $\Gamma_{\text{shock}}$ have no influence on each other during the intersection. Therefore, from now on, we first ignore the two vortex sheets and focus mainly on the diffracted shock.

On $\Gamma_{\text{shock}}$, the Rankine-Hugoniot conditions in the polar coordinates must be satisfied:

\[
\begin{cases}
\begin{aligned}
 r[u] - (\cos \theta + \frac{1}{r} \frac{dr}{d\theta} \sin \theta) [p] &= 0, \\
 r[v] - (\sin \theta - \frac{1}{r} \frac{dr}{d\theta} \cos \theta) [p] &= 0, \\
 r[E] - (\cos \theta + \frac{1}{r} \frac{dr}{d\theta} \sin \theta) [pu] - (\sin \theta - \frac{1}{r} \frac{dr}{d\theta} \cos \theta) [pv] &= 0.
\end{aligned}
\end{cases}
\] (3.9)

Owing to $[pu] = p[u] + \pi [p]$, with $p$ as the average of the two neighboring states of $p$, we eliminate $[u]$ and $[v]$ in the third equation in (3.9) to obtain

\[
\left(\frac{dr}{d\theta}\right)^2 = \frac{r^2(r^2 - p)}{\bar{p}}.
\] (3.10)

The shock diffraction can also be regarded to be generated from point $P_2$ in two directions, which implies that $r'(\theta) > 0$ for $\theta \in [\frac{3\pi}{2}, \theta_1]$ and $r'(\theta) < 0$ for $\theta \in [\theta_3, \frac{3\pi}{2}]$, where $\theta_i$ are denoted as the $\theta$–coordinates of points $P_i$, $i = 1, 3$, respectively. Thus, we choose

\[
\frac{dr}{d\theta} = g(p(r(\theta), \theta), r(\theta)) := \begin{cases}
\begin{aligned}
 r \sqrt{\frac{r^2 - \bar{p}}{\bar{p}}} &\quad \text{for } \theta \in [\frac{3\pi}{2}, \theta_1], \\
 -r \sqrt{\frac{r^2 - \bar{p}}{\bar{p}}} &\quad \text{for } \theta \in [\theta_3, \frac{3\pi}{2}].
\end{aligned}
\end{cases}
\] (3.11)

It follows from (3.5), or (3.9), that $[p]^2 = \bar{p} ([u]^2 + [v]^2)$. Then taking the derivative $r'(\theta) \partial_r + \partial_\theta$ on both sides of this equation along the shock yields the derivative boundary condition on $\Gamma_{\text{shock}} = \{(r(\theta), \theta) : \theta_3 \leq \theta \leq \theta_1\}$:

\[
\beta_1 p_r + \beta_2 p_\theta = 0,
\] (3.12)

where $\beta = (\beta_1, \beta_2)$ is a function of $(p, p_2, r(\theta), r'(\theta))$ with

\[
\beta_1 = 2r'(\theta) \left(\frac{r^2 - \bar{p}}{r^2} - [p] + \bar{p}(r^2 - \bar{p})\right), \quad \beta_2 = \frac{4(r^2 - \bar{p})}{r^2} - \frac{[p]}{2 \bar{p}}.
\] (3.13)

The obliqueness becomes

\[
\mu := (\beta_1, \beta_2) \cdot (1, -r'(\theta)) = -2r'(\theta)(1 - \frac{p}{\bar{p}}).
\]
Note that $\mu$ vanishes at point $P_2$ where $r'(\frac{3\pi}{2}) = 0$ and

$$\beta_1 = 0, \quad \beta_2 = -\frac{|p|}{2p} < 0,$$

owing to $p > p_2$.

Let $\Gamma_{\text{sonic}}$ be the larger portion $\widehat{P_1P_3}$ of the sonic circle $C_1$ of state (1). On $\Gamma_{\text{sonic}}$, $p$ satisfies the Dirichlet boundary condition:

$$p = p_1.$$  \hspace{1cm} (3.13)

Let $\Omega$ be the bounded domain enclosed by $\Gamma_{\text{sonic}}$ and $\Gamma_{\text{shock}}$. Then Riemann Problem I (Problem 3.1) can be reformulated into the following free boundary problem:

**Problem 3.2 (Free Boundary Problem).** Seek a solution $(p(r,\theta), r(\theta))$ such that $p(r,\theta)$ and $r(\theta)$ are determined by Eq. (3.8) in $\Omega$ and the free boundary conditions (3.10)–(3.12) on $\Gamma_{\text{shock}}$ (the derivative boundary condition), in addition to the Dirichlet boundary condition (3.13) on $\Gamma_{\text{sonic}}$.

### 3.3 Global Solutions of Riemann Problem I: Free Boundary Problem, Problem 3.2

To solve Riemann Problem I, it suffices to deal with the free boundary problem, Problem 3.2, which has been solved as stated in the following theorem.

**Theorem 3.1** (Chen-Wang-Zhu [31]). There exists a global solution $(p(r,\theta), r(\theta))$ of Problem 3.2 in domain $\Omega$ with the free boundary

$$\Gamma_{\text{shock}} := \{(r(\theta), \theta) : \theta_3 \leq \theta \leq \theta_1\}$$

such that

$$p \in C^{2,\alpha}(\Omega) \cap C^\alpha(\overline{\Omega}), \quad r \in C^{2,\alpha}((\theta_3, \theta_1)) \cap C^{1,1}([\theta_3, \theta_1]),$$

where $\alpha \in (0,1)$ depends only on the Riemann initial data. Moreover, the global solution $(p(r,\theta), r(\theta))$ satisfies the following properties:

(i) $p > p_2$ on the free boundary $\Gamma_{\text{shock}}$; that is, $\Gamma_{\text{shock}}$ does not meet the sonic circle $C_2$ of state (2).

(ii) The free boundary $\Gamma_{\text{shock}}$ is convex in the self-similar coordinates.

(iii) The global solution $p(r,\theta)$ is $C^\alpha$ up to the sonic boundary $\Gamma_{\text{sonic}}$ and Lipschitz continuous across $\Gamma_{\text{sonic}}$.

(iv) The Lipschitz regularity of the solution across $\Gamma_{\text{sonic}}$ from the inside of the subsonic domain is optimal.

There are three main difficulties for the proof of Theorem 3.1:
(i) The diffracted shock $\Gamma_{\text{shock}}$ is a free boundary, which is not known \textit{a priori} whether it coincides with the inner sonic circle $C_2$ of state (2).

(ii) On the sonic boundary $\Gamma_{\text{sonic}}$, owing to $p_1 = r^2$, the ellipticity of Eq. (3.8) degenerates.

(iii) At point $P_2$ where the diffracted shock $\Gamma_{\text{shock}}$ meets the $\xi_2$-axis: $\xi_1 = 0$, the obliqueness of derivative boundary condition fails, since

$$(\beta_1, \beta_2) \cdot (1, -r'(\theta)) = 0.$$

In the proof of Theorem 3.1, we first assume that $p \geq p_2 + \delta$ holds on $\Gamma_{\text{shock}}$ for some $\delta > 0$; that is, $\Gamma_{\text{shock}}$ cannot coincide with the sonic circle $C_2$ of state (2), which is eventually proved. For the third difficulty, we may express this as a one-point Dirichlet condition $p(P_2) = \hat{p}$ by solving

$$2r(\theta_2) = p(r(\theta_2), \theta_2) + p_2.$$

More precisely, the existence proof is divided into four steps:

1. Since Eq. (3.8) degenerates on the sonic boundary, the differential operator $Q$ in Eq. (3.8) is replaced by the regularized operator:

$$Q^\varepsilon = Q + \varepsilon \Delta \xi.$$

The free boundary $\Gamma_{\text{shock}}$ is first fixed, then the equation and the derivative boundary condition are linearized, and the existence of solutions of the linear fixed mixed-type boundary problem is established for the regularized equation in the polar coordinates.

2. Based on the estimates of solutions to the linear fixed boundary problem obtained in Step 1, the existence of a solution of the nonlinear fixed boundary problem is proved via the Schauder fixed point theorem.

3. The existence of a solution of the free boundary problem with the oblique derivative boundary condition for the regularized elliptic equation is established by using the Schauder fixed point argument again. It follows that the free boundary never meets the sonic circle $C_2$ of state $p_2$.

4. Finally, the limiting solution as the elliptic regularization parameter $\varepsilon$ tends to 0 is proved to be a solution of Problem 3.2.

In Theorem 3.1, a global solution $p$ of the second-order equation (3.7) in $\Omega$ is constructed, which is piecewise constant in the supersonic domain. Moreover, $p$ is proved to be Lipschitz continuous across the degenerate sonic boundary $\Gamma_{\text{sonic}}$ from $\Omega$ to the supersonic domain. To recover the velocity $u = (u, v)$, we consider the first two equations in system (2.11) with (3.2). We can rewrite these equations in the radial variable $r$ as

$$\frac{\partial u}{\partial r} = \frac{1}{r} Dp,$$
and integrate from the boundary of the subsonic domain toward the origin. It is
direct to see that $u$ is at least Lipschitz continuous across $\Gamma_{\text{sonic}}$. Furthermore,$u$ has the same regularity as $p$ inside $\Omega$ except origin $r = 0$. However, $u$ may
be multi-valued at the origin (i.e., $r = 0$). Therefore, we have

**Theorem 3.2** (Chen-Wang-Zhu [31]). Let the Riemann initial data satisfy
(3.4). Then there exists a global solution $(p, u)(r, \theta)$ with the 2-D shock

$$\Gamma_{\text{shock}} = \{(r(\theta), \theta) : \theta_3 \leq \theta \leq \theta_1\}$$

such that

$$(p, u) \in C^{2, \alpha}(\Omega), \quad p \in C^\alpha(\bar{\Omega}), \quad r \in C^{2, \alpha}((\theta_3, \theta_1)) \cap C^{1,1}([\theta_3, \theta_1]),$$

and $(p, u)$ are piecewise constant in the supersonic domain. Moreover, the
global solution $(p, u)$ with shock $\Gamma_{\text{shock}}$ satisfies properties (i)--(ii) in Theorem
3.1 and

(a) The solution $(p, u)$ is $C^\alpha$ up to the sonic boundary $\Gamma_{\text{sonic}}$ and Lipschitz
continuous across $\Gamma_{\text{sonic}}$.

(b) The Lipschitz regularity of both solution $(p, u)$ across $\Gamma_{\text{sonic}}$ from the
subsonic domain $\Omega$ and shock $\Gamma_{\text{shock}}$ across points $\{P_1, P_3\}$ is optimal.

More details can be found in Chen-Wang-Zhu [31]. Similar results can
be obtained for the nonlinear wave system introduced in Section 4 below by
using the same approach and related techniques/methods. Furthermore, Rie-
mann Problem I for the Euler equations for potential flow has also been solved
recently in [16].

4 Two-Dimensional Riemann Problem II:
The Lighthill Problem for Shock Diffraction
for the Nonlinear Wave System

In this section, we present the second Riemann problem, Riemann problem II
– the Lighthill problem for shock diffraction by 2-D convex cornered wedges in
compressible fluid flow (Lighthill [63, 64]), through the nonlinear wave system;
also see [4, 17, 38, 39].

The nonlinear wave system consists of three conservation laws, which takes
the form:

$$
\begin{align*}
\rho_t + m_{x_1} + n_{x_2} &= 0, \\
m_t + p_{x_1} &= 0, \\
n_t + p_{x_2} &= 0,
\end{align*}
$$

for $(t, x) \in [0, \infty) \times \mathbb{R}^2$, where $\rho$ stands for the density, $p$ for the pressure, and
$(m, n)$ for the momenta in the $x$-coordinates. The pressure-density constitutive
relation is
\[ p(\rho) = \frac{\rho^\gamma}{\gamma} \quad \text{for } \gamma > 1, \quad (4.2) \]
by scaling without loss of generality. Then the sonic speed \( c = c(\rho) \) is determined by
\[ c(\rho) := \sqrt{p'(\rho)} = \rho^{(\gamma-1)/2}, \]
which is a positive, increasing function for all \( \rho > 0 \). System (4.1) can be written in form (2.1) with
\[ U = (\rho, m, n)^\top, \quad F_1 = (m, p, 0)^\top, \quad F_2 = (n, 0, p)^\top. \quad (4.3) \]

The 2-D nonlinear wave system (4.1) is derived from the compressible isentropic gas dynamics by neglecting the inertial terms, \textit{i.e.}, the quadratic terms in the velocity; see Canic-Keyfitz-Kim [7].

4.1 Riemann Problem II: The Lighthill Problem for Shock Diffraction by Convex Cornered Wedges

Let \( S_0 \) be the vertical planar shock in the \((t, x)\)–coordinates, with the left constant state \( U_1 = (\rho_1, m_1, 0) \) and the right state \( U_0 = (\rho_0, 0, 0) \), satisfying
\[ m_1 = \sqrt{(p(\rho_1) - p(\rho_0))(\rho_1 - \rho_0)} > 0, \quad \rho_1 > \rho_0. \]

When \( S_0 \) passes through a convex cornered wedge:
\[ W := \{ x = (x_1, x_2) : x_2 < 0, x_1 \leq x_2 \cot \theta_w \}, \]
shock diffraction occurs, where the wedge angle \( \theta_w \) is between \( -\pi \) and 0; see Fig. 4.1. Then the shock diffraction problem can be formulated as follows:

![Fig. 4.1 Riemann Problem II: The Lighthill problem (cf. [17])](image-url)
Problem 4.1 (Riemann Problem II: The Lighthill Problem for Shock Diffraction). Seek a solution of system (4.1)–(4.2) with the initial condition at \( t = 0 \):

\[
U|_{t=0} = \begin{cases} 
(\rho_0, 0, 0) & \text{in } \{ -\pi + \theta_w \leq \arctan(x_2x_1) \leq \frac{\pi}{2} \}, \\
(\rho_1, m_1, 0) & \text{in } \{ x_1 < 0, x_2 > 0 \},
\end{cases}
\]

and the slip boundary condition along the wedge boundary \( \partial W \):

\[
(m, n) \cdot \nu_w |_{\partial W} = 0,
\]

where \( \nu_w \) is the exterior unit normal to \( \partial W \) (see Fig. 4.1).

### 4.2 Reformulation of Riemann Problem II

Notice that Problem 4.1 is invariant under the self-similar scaling: \((t, x) \rightarrow (\alpha t, \alpha x)\) for \( \alpha \neq 0 \). In the self-similar \( \xi \)–coordinates, system (4.1)–(4.2) can be rewritten in form (2.11) with (4.3). In the polar coordinates \((r, \theta)\), \( r = |\xi| \), the system can be further written as

\[
\partial_r \left( r \rho - m \cos \theta - n \sin \theta \right) + \partial_\theta \left( m \sin \theta - n \cos \theta \right) = \left( \rho + \frac{\cos \theta}{r} m + \frac{\sin \theta}{r} n \right) - \left( m + \frac{\cos \theta}{r} p(\rho) \right) n + \left( n + \frac{\sin \theta}{r} p(\rho) \right) m.
\]

The location of the incident shock \( S_0 \) for large \( r \gg 1 \) is:

\[
\xi_1 = \xi_1^0 := \sqrt{\frac{p(\rho_1) - p(\rho_0)}{\rho_1 - \rho_0}} > 0.
\]

Then Problem 4.1 can be reformulated as a boundary value problem in an unbounded domain (see Fig. 4.2): Seek a solution of system (2.11) with (4.3), or equivalently (4.6), with the asymptotic boundary condition when \( r \to \infty \):

\[
(\rho, m, n) \rightarrow \begin{cases} 
(\rho_0, 0, 0) & \text{in } \{ \xi_1 > \xi_1^0, \xi_2 > 0 \} \cup \{ -\pi + \theta_w \leq \arctan(x_2x_1) \leq 0 \}, \\
(\rho_1, m_1, 0) & \text{in } \{ \xi_1 < \xi_1^0, \xi_2 > 0 \},
\end{cases}
\]

and the slip boundary condition along the wedge boundary \( \partial W \):

\[
(m, n) \cdot \nu_w |_{\partial W} = 0.
\]

For a smooth solution \( U = (\rho, m, n) \) of system (2.11) with (4.3), we may eliminate \( m \) and \( n \) in (4.1) to obtain a second-order nonlinear equation for \( \rho \):

\[
((e^2 - \xi_1^2) \rho_{\xi_1} - \xi_1 \xi_2 \rho_{\xi_2} + \xi_1 \rho)_{\xi_1} + ((e^2 - \xi_2^2) \rho_{\xi_2} - \xi_1 \xi_2 \rho_{\xi_1} + \xi_2 \rho)_{\xi_2} - 2\rho = 0.
\]
Correspondingly, Eq. (4.10) in the polar coordinates \((r, \theta), r = |\xi|\), takes the form
\[
\left(\left(c^2 - r^2\right)\rho_r\right)_r + \frac{c^2}{r}\rho_r + \left(\frac{c^2}{r^2}\rho_\theta\right)_\theta = 0. \tag{4.11}
\]

In the self-similar \(\xi\)-coordinates, as the incident shock \(S_0\) passes through the wedge corner, \(S_0\) interacts with the sonic circle \(\Gamma_{\text{sonic}}\) of state (1): \(r = r_1\), and becomes a transonic diffracted shock \(\Gamma_{\text{shock}}\), and the flow in domain \(\Omega\) behind the shock and inside \(\Gamma_{\text{sonic}}\) becomes subsonic.

Consider system (4.6) in the polar coordinates. Then the Rankine-Hugoniot relations, i.e., the jump conditions, are
\[
[p][\rho] = [m]^2 + [n]^2, \quad \frac{dr}{d\theta} = r \frac{\sqrt{r^2 - \bar{c}^2(\rho, \rho_0)}}{\bar{c}(\rho, \rho_0)},
\]
with \(\bar{c}(\rho, \rho_0) = \sqrt{\frac{p(\rho)-p(\rho_0)}{\rho-\rho_0}}\), where the plus branch has been chosen so that \(\frac{dr}{d\theta} > 0\). Differentiating the first equation above along \(\Gamma_{\text{shock}}\) and using the equations obtained above, we have
\[
\beta_1 \rho_r + \beta_2 \rho_\theta = 0 \quad \text{on } \Gamma_{\text{shock}} := \{(r(\theta), \theta) : \theta \in [\theta_w, \theta_1]\}. \tag{4.12}
\]
where \(\beta = (\beta_1, \beta_2)\) is a function of \((\rho_0, \rho, r(\theta), r'(\theta))\) with
\[
\beta_1 = r'(\theta)\left(c^2(r^2 - \bar{c}^2) - 3\bar{c}^2(c^2 - r^2)\right), \quad \beta_2 = 3c^2(r^2 - \bar{c}^2) - \bar{c}^2(c^2 - r^2).
\]
Then the obliqueness becomes
\[
\mu := \beta \cdot (1, -r'(\theta)) = -2r^2(c^2 - \bar{c}^2)r'(\theta) \neq 0,
\]
where \((1, -r'(\theta))\) is the outward normal to \(\Omega\) on \(\Gamma_{\text{shock}}\). Note that \(\mu\) becomes zero when \(r'(\theta) = 0\), i.e., \(r = \bar{c}(\rho, \rho_0)\), where
\[
\beta_1 = 0, \quad \beta_2 = -\bar{c}^2(c^2 - r^2) < 0,
\]
since \(c^2(\rho) > \bar{c}^2(\rho, \rho_0) = r^2\) if \(\rho > \rho_0\).
The second condition on $\Gamma_{\text{shock}}$ is the shock evolution equation:

$$
\frac{dr}{d\theta} = r \sqrt{r^2 - \bar{c}^2(r, \rho_0)} := g(r, \theta, \rho(r, \theta)), \quad r(\theta_1) = r_1,
$$

(4.13)

where $(r_1, \theta_1)$ are the polar coordinates of $P_1 = (\xi_0^1, \xi_0^2)$.

At point $P_2$, $r'(\theta_w) = 0$, (4.12) does not satisfy the oblique derivative boundary condition. We may alternatively express this as a one-point Dirichlet condition by solving $r(\theta_w) = \bar{c}(\rho(r_w), \theta_w, \rho_0)$. In order to deal with this equation, we use the notation:

$$
a = (\bar{c}_b)^{-1}(r) \quad \text{when} \quad \bar{c}_b := \bar{c}(a, b) = r \quad \text{for fixed} \quad b,
$$

(4.14)

so that

$$
\rho(P_1) = \bar{\rho} = (\bar{c}_{\rho_0})^{-1}(r(\theta_w)).
$$

(4.15)

The boundary condition on the wedge is the slip boundary condition, i.e., $(m, n) \cdot \nu = 0$. Differentiating it along the wedge and combining this with the second and third equations in (4.1), we conclude that $\rho$ satisfies

$$
\rho \nu = 0 \quad \text{on} \quad \Gamma_0 := \partial \Omega \cap \{\{\theta = \pi\} \cup \{\theta = \theta_w\}\}.
$$

(4.16)

The Dirichlet boundary condition on $\Gamma_{\text{sonic}}$ is:

$$
\rho = \rho_1 \quad \text{on} \quad \Gamma_{\text{sonic}} := \partial \Omega \cap \partial B_{c_1}(0).
$$

(4.17)

On the Dirichlet boundary $\Gamma_{\text{sonic}}$, Eq. (4.11) becomes degenerate elliptic from the inside of $\Omega$.

With the derivation of the free boundary conditions on $\Gamma_{\text{shock}}$ and the fixed boundary conditions on $\Gamma_{\text{sonic}} \cup \Gamma_0$, Problem 4.1 is further reduced to the following free boundary problem for Eq. (4.11) in domain $\Omega$, with $(m, n)$ correspondingly determined by (4.6).

**Problem 4.2** (Free Boundary Problem). *Seek a solution $(\rho(r, \theta), r(\theta))$ such that $\rho(r, \theta)$ and $r(\theta)$ are determined by Eq. (4.11) in domain $\Omega$ and the free boundary conditions (4.12)–(4.15) on $\Gamma_{\text{shock}} = \{(r(\theta), \theta) : \theta_w \leq \theta \leq \theta_1\}$, in addition to the Neumann boundary condition (4.16) on wedge $\Gamma_0$ and the Dirichlet boundary condition (4.17) on the degenerate boundary $\Gamma_{\text{sonic}}$, the sonic circle of state (1) (cf. Fig. 4.2).*

### 4.3 Global Solutions of Riemann Problem II: Free Boundary Problem, Problem 4.2

To solve Riemann Problem II, it suffices to deal with the free boundary problem, Problem 4.2, which has been solved as stated in the following theorem.
Theorem 4.1 (Chen-Deng-Xiang [17]). Let the wedge angle \( \theta_w \) be between \(-\pi\) and 0. Then there exists a global solution, a density function \( \rho(r,\theta) \) in domain \( \Omega \) and a free boundary \( \Gamma_{\text{shock}} = \{(r(\theta), \theta) : \theta_w \leq \theta \leq \theta_1\} \), of Problem 4.2 such that
\[
\rho \in C^{2+\alpha}(\Omega) \cap C^\alpha(\overline{\Omega}), \quad r \in C^{2+\alpha}([\theta_w, \theta_1]) \cap C^{1,1}([\theta_w, \theta_1]).
\]
Moreover, solution \( (\rho(r,\theta), r(\theta)) \) satisfies the following properties:

(i) \( \rho > \rho_0 \) on the free boundary \( \Gamma_{\text{shock}} \); that is, \( \Gamma_{\text{shock}} \) is separated from the sonic circle \( C_0 \) of state \( (0) \).

(ii) The free boundary \( \Gamma_{\text{shock}} \) is strictly convex up to point \( P_1 \), except point \( P_2 \), in the self-similar \( \xi \)-coordinates.

(iii) The density function \( \rho(r,\theta) \) is \( C^{1,\alpha} \) up to \( \Gamma_{\text{sonic}} \) and Lipschitz continuous across \( \Gamma_{\text{sonic}} \).

(iv) The Lipschitz regularity of \( \rho(r,\theta) \) across \( \Gamma_{\text{sonic}} \) and at \( P_1 \) from the inside is optimal.

Similar to the proof of Theorem (3.1), Theorem 4.1 is established in two steps. First, the regularized approximate free boundary problem for (4.11) involving two small parameters \( \varepsilon \) and \( \delta \) is solved. Then the limits: \( \varepsilon \to 0 \) and \( \delta \to 0 \) are proved to yield a solution of Problem 4.2, i.e., (4.11)–(4.17), with the optimal regularity.

In Theorem 4.1, a global solution \( \rho \) of Eq. (4.11) in \( \Omega \) is constructed, by combining this function with \( \rho = \rho_1 \) in state \( (1) \) and \( \rho = \rho_0 \) in state \( (0) \). That is, the global density function \( \rho \) that is piecewise constant in the supersonic domain is Lipschitz continuous across the degenerate sonic boundary \( \Gamma_{\text{sonic}} \) from \( \Omega \) to state \( (1) \). To recover the momentum vector function \( (m,n) \), we can integrate the second and third equation in (4.6). These can also be written in the radial variable \( r \),
\[
\frac{\partial(m,n)}{\partial r} = \frac{1}{r} D_p(\rho)
\]
and integrated from the boundary of the subsonic domain toward the origin.

It has been proved that the limit of \( D_p(\rho) \) does not exist at \( P_1 \) as \( \xi \) in \( \Omega \) tends to \( \xi^0 \), but \( |Dc(\rho)| \) has a upper bound. Thus, \( p(\rho) \) is Lipschitz, which implies that \( (m,n) \) are at least Lipschitz across the sonic circle \( \Gamma_{\text{sonic}} \). Furthermore, \( (m,n) \) have the same regularity as \( \rho \) inside \( \Omega \) except for the origin \( r = 0 \). However, \( (m,n) \) may be multi-valued at origin \( r = 0 \). Therefore, we have

Theorem 4.2 (Chen-Deng-Xiang [17]). Let the wedge angle \( \theta_w \) be between \(-\pi\) and 0. Then there exists a global solution \( (\rho,m,n)(r,\theta) \) with the diffracted shock \( \Gamma_{\text{shock}} = \{(r(\theta), \theta) : \theta_w \leq \theta \leq \theta_1\} \) of Problem 4.2 such that
\[
(\rho,m,n) \in C^{2+\alpha}(\Omega), \quad \rho \in C^\alpha(\overline{\Omega}), \quad r \in C^{2+\alpha}([\theta_w, \theta_1]) \cap C^{1,1}([\theta_w, \theta_1]),
\]
and \( (\rho,m,n) = (\rho_1,m_1,0) \) in the domain \( \{\xi_1 < \xi^0_1, r > r_1\} \) and \( (\rho_0,0,0) \) in domain \( \{\xi_1 > \xi^0_1, \xi_2 > \xi^0_2\} \cup \{r > r(\theta), \theta_w \leq \theta \leq \theta_1\} \). Moreover, solution \( (\rho,m,n)(r,\theta) \) with the diffracted shock \( \Gamma_{\text{shock}} \) satisfies properties (i)–(ii) in Theorem 4.1 and
(a) $(\rho, m, n)$ is $C^{1,\alpha}$ up to $\Gamma_{\text{sonic}}$ and Lipschitz continuous across $\Gamma_{\text{sonic}}$.
(b) The Lipschitz regularity of solution $(\rho, m, n)$ across $\Gamma_{\text{sonic}}$ and at $P_1$ from the inside is optimal.
(c) The momentum vector function $(m, n)$ may be multi-valued at the origin.

In particular, Theorem 4.2 implies the following facts:
(a) The optimal regularity of $(\rho, m, n)(r, \theta)$ across $\Gamma_{\text{sonic}}$ and at $P_1$ from the inside is $C^{0,1}$, i.e., Lipschitz continuity.
(b) The diffracted shock $\Gamma_{\text{shock}}$ is definitely not degenerate at point $P_2$. This had been an open question even when the wedge angle is $\frac{\pi}{2}$ as in [50], though it is physically plausible.
(c) The curvature of the diffracted shock $\Gamma_{\text{shock}}$ away from point $P_2$ is strictly convex and has a jump at point $P_1$ from a positive value to zero, while the strict convexity of the curvature fails at $P_2$.

More details can be found in Chen-Deng-Xiang [17]. Similar results can be obtained for the pressure gradient equation introduced in Section 3 above. In Chen-Feldman-Hu-Wang [24], the loss of regularity of solutions of Problem 4.1 for the potential flow equation (2.4)–(2.5), or (2.2) with (2.6), has been shown, which implies that the solution configuration for this case is much more complicated.

5 Two-Dimensional Riemann Problem III: The Prandtl-Meyer Problem for Unsteady Supersonic Flow onto Solid Wedges for the Euler Equations for Potential Flow

Now we present the third Riemann problem, Riemann Problem III, for the Prandtl-Meyer problem for unsteady supersonic flow onto solid wedges for the Euler equations for potential flow in form (2.2) with (2.6)–(2.7), or (2.4)–(2.5); see also [3, 37, 71, 73].

5.1 2-D Riemann Problem III: The Prandtl-Meyer Problem for Unsteady Supersonic Flow onto Solid Wedges for Potential Flow

Consider a supersonic flow with a constant density $\rho_0 > 0$ and a velocity $u_0 = (u_0, 0)$, $u_0 > c_0 := c(\rho_0)$, which impinges toward a symmetric wedge:

$$W := \{(x_1, x_2) : |x_2| < x_1 \tan \theta_w, x_1 > 0\}$$

(5.1)

at time $t = 0$. If $\theta_w$ is less than the detachment angle $\theta_w^d$, then the well-known shock polar analysis demonstrates that there are two different steady weak solutions: the steady weak shock solution $\Phi$ and the steady strong shock solution, both of which satisfy the entropy condition and the slip boundary
condition (see Fig. 5.1); see also [3, 14, 34]. Then the dynamic stability of the weak transonic shock solution for potential flow can be formulated as the following problem:

Problem 5.1 (Riemann Problem III: The Prandtl-Meyer Problem for Unsteady Supersonic Flow onto Solid Wedges). Given \(\gamma > 1\), fix \((\rho_0, u_0)\) with \(u_0 > c_0\). For a fixed \(\theta_w \in (0, \theta_w^d)\), seek a global entropy solution \(\Phi \in W^{1,\infty}_{loc}(\mathbb{R}_+ \times (\mathbb{R}^2 \setminus W))\) of Eq. (2.2) with (2.6)–(2.7) and \(B = \frac{u_0^2}{2} + h(\rho_0)\) so that \(\Phi\) satisfies the initial condition at \(t = 0:\)

\[
(\rho, \Phi)|_{t=0} = (\rho_0, u_0 x_1) \quad \text{for } x \in \mathbb{R}^2 \setminus W, \tag{5.2}
\]

and the slip boundary condition along the wedge boundary \(\partial W:\)

\[
\nabla_x \Phi \cdot \nu_w |_{\partial W} = 0, \tag{5.3}
\]

where \(\nu_w\) is the exterior unit normal to \(\partial W\). In particular, we seek a solution \(\Phi \in W^{1,\infty}_{loc}(\mathbb{R}_+ \times (\mathbb{R}^2 \setminus W))\) that converges to the steady weak oblique shock solution \(\bar{\Phi}\) corresponding to the fixed parameters \((\rho_0, u_0, \gamma, \theta_w)\) with \(\bar{\rho} = h^{-1}(B - \frac{1}{2} |\nabla \bar{\Phi}|^2)\), when \(t \to \infty\), in the following sense: For any \(R > 0\), \(\Phi\) satisfies

\[
\lim_{t \to \infty} \left\| (\nabla_x \Phi(t, \cdot) - \nabla_x \bar{\Phi}, \rho(t, \cdot) - \bar{\rho}) \right\|_{L^1(B_R(0) \setminus W)} = 0 \tag{5.4}
\]

for \(\rho(t, x)\) given by (2.6).
Definition 5.1 (Weak Solutions of Problem 5.1: Riemann Problem III). A function $\Phi \in W^{1,1}_{\text{loc}}(\mathbb{R}_+ \times (\mathbb{R}^2 \setminus W))$ is called a weak solution of Problem 5.1 if $\Phi$ satisfies the following properties:

(i) $B - (\partial_t \Phi + \frac{1}{2} |\nabla_x \Phi|^2) \geq h(0+) \ a.e. \ in \ \mathbb{R}_+ \times (\mathbb{R}^2 \setminus W)$,

(ii) For $\rho(\partial_t \Phi, |\nabla_x \Phi|^2) \ |\nabla_x \Phi| \in (L^1_{\text{loc}}(\mathbb{R}_+ \times (\mathbb{R}^2 \setminus W))^2$,

(iii) For every $\zeta \in C^\infty_c(\mathbb{R}_+ \times \mathbb{R}^2)$,

\[
\int_0^\infty \int_{\mathbb{R}^2 \setminus W} \left( \rho(\partial_t \Phi, |\nabla_x \Phi|^2) \partial_t \zeta + \rho(\partial_t \Phi, |\nabla_x \Phi|^2) \nabla \Phi : \nabla \zeta \right) \, dx \, dt \\
+ \int_{\mathbb{R}^2 \setminus W} \rho_0 \zeta(0,x) \, dx = 0.
\]

Since $\zeta$ does not need to be zero on $\partial \Lambda$, the integral identity in Definition 5.1 is a weak form of equation (2.2) with (2.6)–(2.7) and the boundary condition $\rho \nabla_x \Phi \cdot \nu_w = 0$ on $\partial W$. A weak solution is called an entropy solution if it satisfies the entropy condition that is consistent with the second law of thermodynamics (cf. [22, 34, 35, 55]). In particular, a piecewise smooth solution is an entropy solution if the discontinuities are all shocks.

5.2 Reformulation of Riemann Problem III

Notice that Eq. (2.2) with (2.6)–(2.7) is invariant under the self-similar scaling (2.15), so that it admits self-similar solutions in form (2.16). Then the pseudo-potential function $\varphi = \phi - \frac{1}{2} |\xi|^2$ satisfies the following equation:

\[
\text{div}(\rho(|D\varphi|^2, \varphi) D\varphi) + 2\rho(|D\varphi|^2, \varphi) = 0 \quad (5.5)
\]

for

\[
\rho(|D\varphi|^2, \varphi) = (B_0 - (\gamma - 1)(\frac{1}{2} |D\varphi|^2 + \varphi))^\frac{1}{\gamma - 1}, \quad (5.6)
\]

where $B_0 = (\gamma - 1)B + 1$. Eq. (5.5) written in the non-divergence form is

\[
(c^2 - \varphi_{\xi_1}^2)\varphi_{\xi_1, \xi_1} - 2\varphi_{\xi_1} \varphi_{\xi_2} \varphi_{\xi_1, \xi_2} + (c^2 - \varphi_{\xi_2}^2)\varphi_{\xi_2, \xi_2} + 2c^2 - |D\varphi|^2 = 0, \quad (5.7)
\]

where the sonic speed $c = c(|D\varphi|^2, \varphi)$ is determined by

\[
c^2(|D\varphi|^2, \varphi) = \rho^{\gamma - 1}(|D\varphi|^2, \varphi) = B_0 - (\gamma - 1)(\frac{1}{2} |D\varphi|^2 + \varphi). \quad (5.8)
\]

Eq. (5.5) is a nonlinear PDE of mixed elliptic-hyperbolic type. It is elliptic at $\xi$ if and only if

\[
|D\varphi| < c(|D\varphi|^2, \varphi) \quad \text{at} \ \xi, \quad (5.9)
\]

and is hyperbolic if the opposite inequality holds.
One class of solutions of (5.5) is that of \textit{constant states} which are the solutions with constant velocity \( v \in \mathbb{R}^2 \). Then the pseudo-potential of the constant state \( v \) satisfies \( D\varphi = v - \xi \) so that

\[
\varphi(\xi) = -\frac{1}{2}|\xi|^2 + v \cdot \xi + C,
\]

(5.10)

where \( C \) is a constant. For such \( \varphi \), the expressions in (5.6) and (5.8) imply that the density and sonic speed are positive constants \( \rho \) and \( c \), \textit{i.e.}, independent of \( \xi \). Then, from (5.9)–(5.10), the ellipticity condition for the constant state \( v \) is

\[ |\xi - v| < c. \]

Thus, Eq. (5.5) is elliptic inside the \textit{sonic circle} with center \( v \) and radius \( c \), and hyperbolic outside this circle.

Moreover, if density \( \rho \) is a constant, then the solution is also a constant state; that is, the corresponding pseudo-potential \( \varphi \) is of form (5.10).

Since the problem involves transonic shocks, we have to consider weak solutions of Eq. (5.5), which admit shocks. A shock is a curve across which \( D\varphi \) is discontinuous. If \( \Lambda^+ \) and \( \Lambda^- := \Lambda \setminus \Lambda^+ \) are two nonempty open subsets of a domain \( \Lambda \subset \mathbb{R}^2 \), and \( S := \partial \Lambda^+ \cap \Lambda \) is a \( C^1 \)-curve where \( D\varphi \) has a jump, then \( \varphi \in W^{1,1}_{\text{loc}} \cap C^1(\Lambda^+ \cup S) \cap C^2(\Lambda^-) \) is a global weak solution of (5.5) in \( \Lambda \) if and only if \( \varphi \) is in \( W^{1,\infty}_{\text{loc}}(\Lambda) \) and satisfies Eq. (5.5) and the Rankine-Hugoniot conditions on \( S \):

\[
\varphi|_{\Lambda^+ \cap S} = \varphi|_{\Lambda^- \cap S}, \tag{5.11}
\]

\[
\rho(|D\varphi|^2, \varphi)D\varphi \cdot \nu_s|_{\Lambda^+ \cap S} = \rho(|D\varphi|^2, \varphi)D\varphi \cdot \nu_s|_{\Lambda^- \cap S}. \tag{5.12}
\]

A piecewise smooth solution with the discontinuities is called an \textit{entropy solution} of (5.5) if it satisfies the entropy condition: \textit{density \( \rho \) increases in the pseudo-flow direction of \( D\varphi|_{\Lambda^+ \cap S} \) across any discontinuity.} Then such a discontinuity is called a shock.

As the upstream flow has the constant velocity \( u_0 = (u_0,0) \), the corresponding pseudo-potential \( \varphi_0 \) has the expression of

\[ \varphi_0 = -\frac{1}{2}|\xi|^2 + u_0\xi_1 \]

(5.13)

directly from (5.10) with the choice of \( B \) in Problem 5.1. Since the symmetry of the domain and the upstream flow in Problem 5.1 with respect to the \( x_1 \)-axis, Problem 5.1 can then be reformulated as the following boundary value problem in the domain:

\[ \Lambda := \mathbb{R}^2_+ \setminus \{\xi : \xi_2 \leq \xi_1 \tan \theta_w, \xi_1 \geq 0\} \]
Fig. 5.2 Self-similar solutions for $\theta_w \in (0, \theta^s_w)$ in the self-similar coordinates $\xi$ (cf. [3])

Fig. 5.3 Self-similar solutions for $\theta_w \in [\theta^s_w, \theta^d_w)$ in the self-similar coordinates $\xi$ (cf. [3])

in the self-similar coordinates $\xi$, which corresponds to domain $\{(t, x) : x \in \mathbb{R}^2_+ \setminus W, t > 0\}$ in the $(t, x)$–coordinates, where $\mathbb{R}^2_+ = \{\xi : \xi_2 > 0\}$: Seek a solution $\varphi$ of Eq. (5.5) in the self-similar domain $\Lambda$ with the slip boundary condition:

$$D\varphi \cdot \nu_w |_{\partial \Lambda} = 0$$

and the asymptotic boundary condition:

$$\varphi - \varphi_0 \longrightarrow 0$$

along each ray $R_\theta := \{\xi_1 = \xi_2 \cot \theta, \xi_2 > 0\}$ with $\theta \in (\theta_w, \pi)$ as $\xi_2 \to \infty$ in the sense that

$$\lim_{r \to \infty} \|\varphi - \varphi_0\|_{C(R_\theta \setminus B_r(0))} = 0.$$  

Given $M_0 > 1$, $\rho_1$ and $u_1$ are determined via the shock polar as shown in Fig. 5.1 for steady potential flow. For any wedge angle $\theta_w \in (0, \theta^s_w)$, line $v = u \tan \theta_w$ and the shock polar intersect at a point $\mathbf{u}_1 = (u_1, v_1)$ with $|u_1| > c_1$ and $u_1 < u_0$; while, for any $\theta_w \in [\theta^s_w, \theta^d_w)$, they intersect at a point $\mathbf{u}_1$ with $u_1 > u_d$ and $|u_1| < c_1$ where $u_d$ is the $u$–component of the unique detachment state $\mathbf{u}_d$ when $\theta_w = \theta^d_w$. The intersection state $\mathbf{u}_1$ is the velocity for steady potential flow behind an oblique shock $S_0$ attached to the wedge vertex with angle $\theta_w$. The strength of shock $S_0$ is relatively weak compared to the shock given by the other intersection point on the shock polar, hence $S_0$ is called a weak oblique shock, and the corresponding state $\mathbf{u}_1$ is a weak state. Moreover, such state $\mathbf{u}_1$ depends smoothly on $(u_0, \theta_w)$ and is supersonic when $\theta_w \in (0, \theta^s_w)$ and subsonic when $\theta_w \in [\theta^s_w, \theta^d_w)$.
Once \( u_1 \) is determined, by (5.11)–(5.13), the pseudo-potential \( \varphi_1 \) below the weak oblique shock \( S_0 \) is

\[
\varphi_1 = -\frac{1}{2} |\xi|^2 + u_1 \cdot \xi.
\]  

(5.17)

We seek a global entropy solution with two types of Prandtl-Meyer reflection configurations whose occurrence is determined by the wedge angle \( \theta_w \) for the two different cases: One contains a straight weak oblique shock \( S_0 \) attached to the wedge vertex \( O \) and connected to a normal shock \( S_1 \) through a curved shock \( \Gamma_{\text{shock}} \) when \( \theta_w < \theta_w^* \), as shown in Fig. 5.2; the other contains a curved shock \( \Gamma_{\text{shock}} \) attached to the wedge vertex and connected to a normal shock \( S_1 \) when \( \theta_w^* \leq \theta_w < \theta_w^d \), as shown in Fig. 5.3, in which the curved shock \( \Gamma_{\text{shock}} \) is tangential to the straight weak oblique shock \( S_0 \) at the wedge vertex. To achieve these, we need to compute the pseudo-potential function \( \varphi \) below \( S_0 \).

By (5.11)–(5.14), the pseudo-potential \( \varphi_2 \) below the normal shock \( S_1 \) is of the form:

\[
\varphi_2 = -\frac{1}{2} |\xi|^2 + u_2 \cdot \xi + k_2
\]  

(5.18)

for constant state \( u_2 \) and constant \( k_2 \); see (5.10). Then it follows from (5.6) and (5.17)–(5.18) that the corresponding densities \( \rho_1 \) and \( \rho_2 \) are constants in the form:

\[
\rho_k^{\gamma-1} = \rho_0^{\gamma-1} + \frac{\gamma - 1}{2} (u_0^2 - |u_k|^2) \quad \text{for } k = 1, 2.
\]  

(5.19)

Denote \( \Gamma_{\text{wedge}} := \partial W \cap \partial \Lambda \), and the sonic arcs \( \Gamma_{\text{sonic}}^1 := P_1 P_4 \) on Fig. 5.2 and \( \Gamma_{\text{sonic}}^2 := P_2 P_3 \) on Figs. 5.2–5.3. The sonic circle \( \partial B_{c_1}(u_1) \) of the uniform state \( \varphi_1 \) intersects line \( S_0 \), where \( c_1 = \frac{\gamma + 1}{\gamma - 1} \) by (5.8). For the supersonic case \( \theta_w \in (0, \theta_w^d) \), there are two arcs of this sonic circle between \( S_0 \) and \( \Gamma_{\text{wedge}} \) in \( \Lambda \). Note that \( \Gamma_{\text{sonic}}^1 \) tends to point \( O \) as \( \theta_w \rightarrow \theta_w^* \) and is outside of \( \Lambda \) for the subsonic case \( \theta_w \in [\theta_w^s, \theta_w^d) \). Similarly, the sonic circle \( \partial B_{c_2}(u_2) \) of the uniform state \( \varphi_2 \) intersects line \( S_1 \), where \( c_2 = \rho_2^{\gamma+1} \). There are two arcs of this circle between \( S_1 \) and the line containing \( \Gamma_{\text{wedge}} \). Notice that \( \varphi_1 > \varphi_2 \) on \( \Gamma_{\text{sonic}}^1 \) and \( \varphi_1 < \varphi_2 \) on \( \Gamma_{\text{sonic}}^2 \). Then Problem 5.1 can be further reformulated into the following free boundary problem:

**Problem 5.2** (Free Boundary Problem). For \( \theta_w \in (0, \theta_w^d) \), find a free boundary (curved shock) \( \Gamma_{\text{shock}} \) and a function \( \varphi \) defined in domain \( \Omega \), as shown in Figs. 5.2–5.3, such that \( \varphi \) satisfies

(i) Eq. (5.5) in \( \Omega \),

(ii) \( \varphi = \varphi_0 \) and \( \rho D \varphi \cdot \nu_s = \rho_0 D \varphi_0 \cdot \nu_s \) on \( \Gamma_{\text{shock}} \),

(iii) \( \varphi = \hat{\varphi} \) and \( D \varphi = D \hat{\varphi} \) on \( \Gamma_{\text{sonic}}^1 \cup \Gamma_{\text{sonic}}^2 \) when \( \theta_w \in (0, \theta_w^s) \) and on \( \Gamma_{\text{sonic}}^2 \cup \{O\} \) when \( \theta_w \in [\theta_w^s, \theta_w^d) \) for \( \hat{\varphi} := \max(\varphi_1, \varphi_2) \),

(iv) \( D \varphi \cdot \nu_w = 0 \) on \( \Gamma_{\text{wedge}} \).
where \( \nu_s \) and \( \nu_w \) are unit normals to \( \Gamma_{\text{shock}} \) and \( \Gamma_{\text{wedge}} \) pointing to the interior of \( \Omega \), respectively.

It can be shown that \( \varphi_1 > \varphi_2 \) on \( \Gamma_{\text{sonic}}^1 \) and the opposite inequality holds on \( \Gamma_{\text{sonic}}^2 \). This justifies the requirements in Problem 5.2(iii) above. The conditions in Problem 5.2(ii)–(iii) are the Rankine-Hugoniot conditions \((5.12)–(5.11)\) on \( \Gamma_{\text{shock}} \) and \( \Gamma_{\text{sonic}}^1 \cup \Gamma_{\text{sonic}}^2 \) or \( \Gamma_{\text{sonic}}^2 \cup \{O\} \), respectively.

5.3 Global Solutions of Riemann Problem III: Free Boundary Problem, Problem 5.2

To solve Riemann Problem III, it suffices to solve the free boundary problem, Problem 5.2, for all the wedge angles \( \theta_w \in (0, \theta_{\text{w}}^d) \). To obtain a global solution from \( \varphi \) that is a solution of Problem 5.2 such that \( \Gamma_{\text{shock}} \) is a \( C^1 \)–curve up to its endpoints and \( \varphi \in C^1(\overline{\Omega}) \), we consider two cases:

For the supersonic case \( \theta_w \in (0, \theta_{\text{s}}^w) \), we divide domain \( \Lambda \) into four separate domains; see Fig. 5.2. Denote by \( S_{0,\text{seg}} \) the line segment \( OP_1 \subset S_0 \), and by \( S_{1,\text{seg}} \) the portion (half-line) of \( S_1 \) with left endpoint \( P_2 \) so that \( S_{1,\text{seg}} \subset \Lambda \). Let \( \Omega_S \) be the unbounded domain below curve \( S_{0,\text{seg}} \cup \Gamma_{\text{shock}} \cup S_{1,\text{seg}} \) and above \( \Gamma_{\text{wedge}} \) (see Fig. 5.2). In \( \Omega_S \), let \( \Omega_1 \) be the bounded domain enclosed by \( S_0, \Gamma_{\text{sonic}}^1 \), and \( \Gamma_{\text{wedge}} \). Set \( \Omega_2 := \Omega_S \setminus \overline{\Omega_1} \). Define a function \( \varphi_* \) in \( \Lambda \) by

\[
\varphi_* = \begin{cases} 
\varphi_0 & \text{in } \Lambda \setminus \Omega_S, \\
\varphi_1 & \text{in } \Omega_1, \\
\varphi & \text{in } \Gamma_{\text{sonic}}^1 \cup \Omega \cup \Gamma_{\text{sonic}}^2, \\
\varphi_2 & \text{in } \Omega_2.
\end{cases}
\]  

By Problem 5.2(ii)–(iii), \( \varphi_* \) is continuous in \( \Lambda \setminus \Omega_S \) and \( C^1 \) in \( \overline{\Omega_S} \). In particular, \( \varphi_* \) is \( C^1 \) across \( \Gamma_{\text{sonic}}^1 \cup \Gamma_{\text{sonic}}^2 \). Moreover, using Problem 5.2(i)–(iii), we obtain that \( \varphi_* \) is a global entropy solution of Eq. (5.5) in \( \Lambda \).

For the subsonic case \( \theta_w \in [\theta_{\text{s}}^w, \theta_{\text{w}}^d) \), domain \( \Omega_1 \cup \Gamma_{\text{sonic}}^1 \) in \( \varphi_* \) reduces to one point \( \{O\} \); see Fig. 5.3. The corresponding function \( \varphi_* \) is a global entropy solution of Eq. (5.5) in \( \Lambda \).

**Definition 5.2** (Admissible Solutions). Let \( \theta_w \in (0, \theta_{\text{w}}^d) \). A function \( \varphi \in C^{0,1}(\overline{\Lambda}) \) is an admissible solution of Problem 5.2 if \( \varphi \) is a solution of Problem 5.2 extended to \( \Lambda \) by (5.20) and satisfies the following properties:

(i) The structure of solution is of the form:

- If \( \theta_w \in (0, \theta_{\text{s}}^w) \), then \( \varphi \) has the configuration shown on Fig. 5.2 such that \( \Gamma_{\text{shock}} \) is \( C^2 \) in its relative interior and

\[
\varphi \in C^{0,1}(\Lambda) \cap C^1(\Lambda \setminus (S_{0,\text{seg}} \cup \Gamma_{\text{shock}} \cup S_{1,\text{seg}})),
\]

\[
\varphi \in C^1(\overline{\Omega}) \cap C^2(\overline{\Omega} \setminus (S_{0,\text{seg}} \cup S_{1,\text{seg}})) \cap C^3(\Omega).
\]
• If $\theta_\ell \in [\theta^e_\ell, \theta^d_\ell)$, then $\varphi$ has the configuration shown on Fig. 5.3 such that $\Gamma_{\text{shock}}$ is $C^2$ in its relative interior and

$$\varphi \in C^{0,1}(\Lambda) \cap C^1(\Lambda \setminus (\Gamma_{\text{shock}} \cup \overline{S}_{1,\text{seg}})),$$

$$\varphi \in C^1(\Omega) \cap C^2(\overline{\Omega} \setminus (\{O\} \cup \overline{S}_{1,\text{seg}})) \cap C^3(\Omega).$$

(ii) Eq. (5.5) is strictly elliptic in $\overline{\Omega} \setminus \Gamma_{\text{sonic}}$: $|D\varphi| < c(|D\varphi|^2, \varphi)$ in $\overline{\Omega} \setminus \Gamma_{\text{sonic}}$.

(iii) $0 < \partial_{\nu_\theta} \varphi \leq \partial_{\nu_\theta} \varphi_0$ on $\Gamma_{\text{shock}}$, where $\nu_\theta$ is the unit normal to $\Gamma_{\text{shock}}$ pointing to the interior of $\Omega$.

(iv) The inequalities hold:

$$\max\{\varphi_1, \varphi_2\} \leq \varphi \leq \varphi_0 \quad \text{in } \Omega.$$  \hspace{1cm} (5.21)

(v) The monotonicity properties hold:

$$D(\varphi_0 - \varphi) \cdot e_{S_1} \geq 0, \quad D(\varphi_0 - \varphi) \cdot e_{S_0} \leq 0 \quad \text{in } \Omega,$$ \hspace{1cm} (5.22)

where $e_{S_0}$ and $e_{S_1}$ are the unit vectors along lines $S_0$ and $S_1$ pointing to the positive $\xi_1$-direction, respectively.

The monotonicity properties in (5.22) imply that

$$D(\varphi_1 - \varphi) \cdot e \leq 0 \quad \text{in } \overline{\Omega} \quad \text{for all } e \in \text{Cone}(-e_{S_1}, e_{S_0}),$$ \hspace{1cm} (5.23)

where $\text{Cone}(-e_{S_1}, e_{S_0}) = \{-a e_{S_1} + b e_{S_0} : a, b > 0\}$. Notice that $e_{S_0}$ and $e_{S_1}$ are not parallel if $\theta_\ell \neq 0$. Then we have the following theorem:

**Theorem 5.1** (Bae-Chen-Feldman [3]). Let $\gamma > 1$ and $u_0 > c_0$. For any $\theta_\ell \in (0, \theta^d_\ell)$, there exists a global entropy solution $\varphi$ of Problem 5.2 such that the following regularity properties are satisfied for some $\alpha \in (0, 1)$:

(i) If $\theta_\ell \in (0, \theta^e_\ell)$, the reflected shock $\overline{S}_{0,\text{seg}} \cup \Gamma_{\text{shock}} \cup \overline{S}_{1,\text{seg}}$ is $C^{2,\alpha}$-smooth, and $\varphi \in C^{1,\alpha}(\overline{\Omega}) \cap C^\infty(\overline{\Omega} \setminus (\overline{\Gamma}_{\text{sonic}} \cup \overline{\Gamma}_{\text{shock}}^{2,\alpha}))$.

(ii) If $\theta_\ell \in [\theta^e_\ell, \theta^d_\ell)$, the reflected shock $\overline{\Gamma}_{\text{shock}} \cup \overline{S}_{1,\text{seg}}$ is $C^{1,\alpha}$ near $O$ and $C^{2,\alpha}$ away from $O$, and $\varphi \in C^{1,\alpha}(\overline{\Omega}) \cap C^\infty(\overline{\Omega} \setminus (\{O\} \cup \overline{\Gamma}_{\text{shock}}^{2,\alpha}))$.

Moreover, in both cases, $\varphi$ is $C^{1,1}$ across the sonic arcs, and $\Gamma_{\text{shock}}$ is $C^\infty$ in its relative interior. Furthermore, $\varphi$ is an admissible solution in the sense of Definition 5.2, so $\varphi$ satisfies the additional properties listed in Definition 5.2.

To achieve this, for any small $\delta > 0$, the required uniform estimates of admissible solutions with wedge angles $\theta_\ell \in [0, \theta^d_\ell - \delta]$ are first obtained. Using these estimates, the Leray-Schauder degree theory can be applied to obtain the existence for each $\theta_\ell \in [0, \theta^d_\ell - \delta]$ in the class of admissible solutions, starting from the unique normal solution for $\theta_\ell = 0$. Since $\delta > 0$ is arbitrary, the existence of a global entropy solution for any $\theta_\ell \in (0, \theta^d_\ell)$ can be established.
More details can be found in Bae-Chen-Feldman [3]; see also Chen-Feldman [22] and related references cited therein.

Recently, we have also established the convexity of transonic shocks for the Prandtl-Meyer reflection configurations.

**Theorem 5.2** (Chen-Feldman-Xiang [25]). *If a solution of the Prandtl-Meyer problem is admissible in the sense of Definition 5.2, then its domain \( \Omega \) is convex, and the shock curve \( \Gamma_{\text{shock}} \) is a strictly convex graph. That is, \( \Gamma_{\text{shock}} \) is uniformly convex on any closed subset of its relative interior. Moreover, for the solution of Problem 5.2 extended to \( \Lambda \) by (5.20) (with the appropriate modification for the subsonic/sonic case) with pseudo-potential \( \varphi \in C^{0,1}(\Lambda) \) satisfying Definition 5.2(i)–(iv), the shock is strictly convex if and only if Definition 5.2(v) holds.*

With the convexity of reflected-diffracted transonic shocks, the uniqueness and stability of global regular shock reflection-diffraction configurations have also been established in the class of *admissible solutions*; see Chen-Feldman-Xiang [26] for the details.

The existence results in Bae-Chen-Feldman [3] indicate that the steady weak supersonic/transonic shock solutions are the asymptotic limits of the dynamic self-similar solutions, the Prandtl-Meyer reflection configurations, in the sense of (5.16) in Problem 5.1 for all \( \theta_w \in (0, \theta^d_w) \) and all \( \gamma > 1 \).

On the other hand, it is shown in Elling [36] and Bae-Chen-Feldman [3] that, for each \( \gamma > 1 \), there is no self-similar *strong* Prandtl-Meyer reflection configuration for the unsteady potential flow in the class of admissible solutions. This means that the situation for the dynamic stability of the strong steady oblique shocks is more sensitive.

### 6 Two-Dimensional Riemann Problem IV: the von Neumann Problem for Shock Reflection-Diffraction for the Euler Equations for Potential Flow

In this section, we present some recent developments in the analysis of the fourth Riemann problem, Riemann Problem IV – the von Neumann problem for shock reflection-diffraction by wedges for the Euler equations for potential flow in form (2.4)–(2.5), or (2.2) with (2.6)–(2.7).

#### 6.1 2-D Riemann Problem IV: The von Neumann Problem for Shock Reflection-Diffraction by Wedges

When a vertical planar shock perpendicular to the flow direction and separating two uniform states (0) and (1), with constant velocities \( u_0 = (0, 0) \) and \( u_1 = (u_1, 0), u_1 > 0 \), and constant densities \( \rho_0 < \rho_1 \) (state (0) is ahead or to
the right of the shock, and state (1) is behind the shock), hits a symmetric wedge W in (5.1) head-on at time \( t = 0 \), a reflection-diffraction process takes place when \( t > 0 \). Mathematically, the shock reflection-diffraction problem is a 2-D lateral Riemann problem in domain \( \mathbb{R}^2 \setminus W \).

**Problem 6.1** (Riemann Problem IV – the von Neumann Problem for Shock Reflection-Diffraction by Wedges). *Piecewise constant initial data, consisting of state (0) on \( \{ x_1 > 0 \} \setminus W \) and state (1) on \( \{ x_1 < 0 \} \) connected by a shock at \( x_1 = 0 \), are prescribed at \( t = 0 \). Seek a solution of \( \text{Eq. (2.2)} \) with \( (2.6)-(2.7) \) for \( t \geq 0 \) subject to the initial data and the boundary condition \( \nabla \Phi \cdot \nu_w = 0 \) on \( \partial W \).

Similarly to Definition 5.1, we can define the notion of weak solutions of Problem 6.1, by noting that the boundary condition can be written as \( \rho \nabla \Phi \cdot \nu_w = 0 \) on \( \partial W \), which is the spatial conormal condition for Eq. (2.2) with \( (2.6)-(2.7) \).

The mathematical analysis of the shock reflection-diffraction by wedges was first proposed by John von Neumann in [82–84]. The complexity of reflection-diffraction configurations was first reported by Ernst Mach [68] in 1878, who observed two patterns of reflection-diffraction configurations: Regular reflection (two-shock configuration; see Figs. 6.1–6.2) and Mach reflection (three-shock/one-vortex-sheet configuration). It has been found later that the reflection-diffraction configurations can be much more complicated than what Mach originally observed; see also [5, 22, 34, 44, 46, 80] and the references cited therein.

### 6.2 Reformation of Riemann Problem IV

Problem 6.1 is invariant under self-similar (2.15), so it also admits self-similar solutions determined by Eq. (5.5)–(5.6), along with the appropriate boundary conditions. By the symmetry of the problem with respect to the \( \xi_1 \)-axis, we consider only the upper half-plane \( \{ \xi_2 > 0 \} \) and prescribe the boundary condition: \( \varphi \nu = 0 \) on the symmetry line \( \{ \xi_2 = 0 \} \). Then Problem 6.1 is reformulated as a boundary value problem in the unbounded domain

\[
\Lambda := \mathbb{R}^2_+ \setminus \{ \xi : |\xi_2| \leq \xi_1 \tan \theta_w, \xi_1 > 0 \}
\]

in the self-similar coordinates, where \( \mathbb{R}^2_+ := \mathbb{R}^2 \cap \{ \xi_2 > 0 \} \). The incident shock in the \( \xi \)-coordinates is the half-line: \( S_0 = \{ \xi = \xi_1^0 \} \cap \Lambda \), where

\[
\xi_1^0 := \rho_1 \sqrt{\frac{2(c_1^2 - c_0^2)}{(\gamma - 1)(\rho_1^2 - \rho_0^2)}} = \frac{\rho_1 u_1}{\rho_1 - \rho_0}, \tag{6.1}
\]

which is determined by the Rankine-Hugoniot conditions between states (0) and (1) on \( S_0 \). Then Problem 6.1 for self-similar solutions becomes the boundary value problem: *Seek a solution \( \varphi \) of Eq. (5.5)–(5.6) in the self-similar
domain $\Lambda$ with the slip boundary condition $D\varphi \cdot \nu|_{\partial \Lambda} = 0$ and the asymptotic boundary condition at infinity:

$$
\varphi \to \bar{\varphi} = \begin{cases} 
\varphi_0 & \text{for } \xi_1 > \xi_1^0, \xi_2 > \xi_1 \tan \theta_w, \\
\varphi_1 & \text{for } \xi_1 < \xi_1^0, \xi_2 > 0,
\end{cases} \quad \text{when } |\xi| \to \infty,
$$

where $\varphi_0 = -\frac{1}{2}|\xi|^2$ and $\varphi_1 = -\frac{1}{2}|\xi|^2 + u_1(\xi_1 - \xi_1^0)$.

Similarly, we can define the notion of weak solutions of the boundary value problem by observing that the boundary condition can be written as $\rho D\varphi \cdot \nu|_{\partial \Lambda} = 0$, which is the spatial conormal condition for Eq. (5.5)–(5.6). A weak solution is called an entropy solution if it satisfies the entropy condition: density $\rho$ increases in the pseudo-flow direction of $D\varphi|_{\Lambda^+ \cap S}$ across any discontinuity curve (i.e., shock).

If a solution has one of the regular shock reflection-diffraction configurations as shown in Figs. 6.1–6.2 (cf. [22]) and its pseudo-potential $\varphi$ is smooth in the subdomain $\Omega$ between the wedge and the reflected-diffracted shock, then it should satisfy the slip boundary condition on the wedge and the Rankine-Hugoniot conditions with state (1) across the flat shock $S_1 = \{\varphi_1 = \varphi_2\}$, which passes through point $P_0$ where the incident shock meets the wedge boundary. Define the uniform state (2) with pseudo-potential $\varphi_2(\xi)$ such that

$$
\varphi_2(P_0) = \varphi(P_0), \quad D\varphi_2(P_0) = \lim_{P \to P_0, P \in \Omega} D\varphi(P).
$$

Then the constant density $\rho_2$ of state (2) is equal to $\rho(|D\varphi|^2, \varphi)(P_0) = \rho(|D\varphi_2|^2, \varphi_2)(P_0)$ via (5.6). It follows that $\varphi_2$ satisfies the following three conditions at $P_0$:

$$
D\varphi_2 \cdot \nu_w = 0, \quad \varphi_2 = \varphi_1, \quad \rho(|D\varphi_2|^2, \varphi_2)D\varphi_2 \cdot \nu_{S_1} = \rho_1 D\varphi_1 \cdot \nu_{S_1} \quad (6.2)
$$

for $\nu_{S_1} = \frac{D(\varphi_1 - \varphi_2)}{|D(\varphi_1 - \varphi_2)|}$, where $\nu_w$ is the outward normal to the wedge boundary.
State (2) can be either supersonic or subsonic at $P_0$, which determines the supersonic or subsonic type of the configurations. The regular reflection solution in the supersonic domain is expected to consist of the constant states separated by straight shocks (cf. [76, Theorem 4.1]). Then, when state (2) is supersonic at $P_0$, it can be shown that the constant state (2), extended up to arc $\Gamma_{\text{sonic}} := P_1 P_4$ of the sonic circle of state (2), as shown in Fig. 6.1, satisfies Eq. (5.5) in the domain, the Rankine-Hugoniot conditions (5.12)–(5.11) on the straight shock $P_0 P_1$, and the slip boundary condition: $D\varphi_2 \cdot \nu_w = 0$ on the wedge $P_0 P_4$, and is expected to be a part of the configuration. Then the supersonic regular shock reflection-diffraction configuration on Fig. 6.1 consists of three uniform states (0), (1), (2), and a non-uniform state in domain $\Omega = P_1 P_2 P_3 P_4$, where Eq. (5.5) is elliptic. The elliptic domain $\Omega$ is separated from the hyperbolic domain $P_0 P_1 P_4$ of state (2) by the sonic arc $\Gamma_{\text{sonic}}$, on which the ellipticity in $\Omega$ degenerates. The subsonic regular shock reflection-diffraction configuration as shown in Fig. 6.2 consists of two uniform states (0) and (1), and a non-uniform state in domain $\Omega = P_0 P_2 P_3$, where Eq. (5.5) is elliptic, and $\varphi|_{\Omega}(P_0) = \varphi_2(P_0)$ and $D(\varphi|_{\Omega})(P_0) = D\varphi_2(P_0)$.

For the supersonic case in Fig. 6.1, we also use $\Gamma_{\text{shock}}$, $\Gamma_{\text{wedge}}$, and $\Gamma_{\text{sym}}$ for the curved part of $P_1 P_2$, the wedge boundary $P_3 P_4$, and the symmetry line segment $P_2 P_3$, respectively. For the subsonic case in Fig. 6.2, $\Gamma_{\text{shock}}$, $\Gamma_{\text{wedge}}$, and $\Gamma_{\text{sym}}$ denote $P_0 P_2$, $P_0 P_3$, and $P_2 P_3$, respectively. We unify the notations with the supersonic case by introducing points $P_1$ and $P_4$ for the subsonic case as

$$P_1 := P_0, \quad P_4 := P_0, \quad \Gamma_{\text{sonic}} := \{P_0\}. \quad (6.3)$$

The corresponding solution for $\theta_w = \frac{\pi}{2}$ is called the normal reflection. In this case, the incident shock normally reflects from the flat wall so that the reflected shock is also a plane $\{\xi_1 = \xi_1\}$, where $\xi_1 < 0$; see Fig. 6.3.

As indicated above, a necessary condition for the existence of a regular reflection solution is the existence of the uniform state (2) with pseudo-potential $\varphi_2$ determined by the system of algebraic equations (6.2) for constants $(u_2, v_2, \rho_2)$ of state (2) across the flat shock $S_1 = \{\varphi_1 = \varphi_2\}$ separating it from state (1) and satisfying the entropy conditions $\rho_2 > \rho_1$. For any
fixed densities $0 < \rho_0 < \rho_1$ of states (0) and (1), it can be shown that there exist a sonic angle $\theta^s_w$ and a detachment angle $\theta^d_w$ satisfying

$$0 < \theta^d_w < \theta^s_w < \frac{\pi}{2}$$

such that the algebraic system (6.2) has two solutions for each $\theta_w \in (\theta^d_w, \frac{\pi}{2})$ which become equal when $\theta_w = \theta^d_w$. Thus, for each $\theta_w \in (\theta^d_w, \frac{\pi}{2})$, there exist two states (2), weak versus strong, with densities $\rho^\text{weak}_2 < \rho^\text{strong}_2$. The weak state (2) is supersonic at the reflection point $P_0(\theta_w)$ for $\theta_w \in (\theta^d_w, \frac{\pi}{2})$, sonic for $\theta_w = \theta^s_w$, and subsonic for $\theta_w \in (\theta^d_w, \theta^s_w)$. The strong state (2) is subsonic at $P_0(\theta_w)$ for all $\theta_w \in (\theta^d_w, \frac{\pi}{2})$.

To determine which of the two states (2) for $\theta_w \in (\theta^d_w, \frac{\pi}{2})$, weak or strong, is physical for the local theory, it was conjectured that the strong shock reflection-diffraction configuration would be non-physical; indeed, it is shown in Chen-Feldman [21, 22] that the weak shock reflection-diffraction configuration tends to the unique normal reflection in Fig. 6.3, but the strong one does not, when the wedge angle $\theta_w$ tends to $\frac{\pi}{2}$. The entropy condition and the definition of weak state (2) imply that $0 < \rho_1 < \rho^\text{weak}_2$. With the weak state (2), the following conjectures were proposed (see von Neumann [82, 83]):

**The Sonic Conjecture:** There exists a supersonic regular shock reflection-diffraction configuration when $\theta_w \in (\theta^s_w, \frac{\pi}{2})$ for $\theta^s_w > \theta^d_w$. That is, the supersonicity of the weak state (2) implies the existence of a supersonic regular reflection solution, as shown in Fig. 6.1.

**The Detachment Conjecture:** There exists a regular shock reflection-diffraction configuration for any wedge angle $\theta_w \in (\theta^d_w, \frac{\pi}{2})$. That is, the existence of state (2) implies the existence of a regular reflection solution, as shown in Figs. 6.1–6.2.

In other words, the von Neumann detachment conjecture above is that the global regular shock reflection-diffraction configuration is possible whenever the local regular reflection at the reflection point is possible.

From now on, for the given wedge angle $\theta_w \in (\theta^d_w, \frac{\pi}{2})$, state (2) represents the unique weak state (2) and $\varphi_2$ is its pseudo-potential. State (2) is obtained from the algebraic conditions (6.2) which determines line $S_1$ and the sonic arc $\Gamma_{\text{sonic}}$ when state (2) is supersonic at $P_0$, and the slope of $\Gamma_{\text{shock}}$ at $P_0$ when state (2) is subsonic at $P_0$. Thus, the unknowns are both domain $\Omega$ and pseudo-potential $\varphi$ in $\Omega$, as shown in Figs. 6.1–6.2. Then, from (5.12)–(5.11), in order to construct a solution of Problem 6.1 for the supersonic or subsonic regular shock reflection-diffraction configuration, it suffices to solve the following problem:

**Problem 6.2** (Free Boundary Problem). For $\theta_w \in (\theta^d_w, \frac{\pi}{2})$, find a free boundary (curved reflected shock) $\Gamma_{\text{shock}} \subset \Lambda \cap \{\xi_1 < \xi_1 P_1\}$ and a function $\varphi$ defined in domain $\Omega$ as shown in Figs. 6.1–6.2 such that

(i) Eq. (5.5) is satisfied in $\Omega$ and is strictly elliptic for $\varphi$ in $\overline{\Omega} \setminus \Gamma_{\text{sonic}}$, 

(ii) $\varphi = \varphi_1$ and $D\varphi \cdot \nu_s = \rho D\varphi_1 \cdot \nu_s$ on the free boundary $\Gamma_{\text{shock}}$,
(iii) $\varphi = \varphi_2$ and $D\varphi = D\varphi_2$ on $P_1P_4$ in the supersonic case as shown in Fig. 6.1 and at $P_0$ in the subsonic case as shown in Fig. 6.1,
(iv) $D\varphi \cdot \nu_w = 0$ on $\Gamma_{\text{wedge}}$, and $D\varphi \cdot \nu_{\text{sym}} = 0$ on $\Gamma_{\text{sym}}$,

where $\nu_s$, $\nu_w$, and $\nu_{\text{sym}}$ are the interior unit normals to $\Omega$ on $\Gamma_{\text{shock}}$, $\Gamma_{\text{wedge}}$, and $\Gamma_{\text{sym}}$, respectively.

The conditions in Problem 6.2(ii) are the Rankine-Hugoniot conditions (5.12)–(5.11) on $\Gamma_{\text{shock}}$ between $\varphi|\Omega$ and $\varphi_1$. Since $\Gamma_{\text{shock}}$ is a free boundary and Eq. (5.5) is strictly elliptic for $\varphi$ in $\overline{\Omega \setminus \Gamma_{\text{sonic}}}$, then two conditions (the Dirichlet and oblique derivative conditions) on $\Gamma_{\text{shock}}$ are consistent with one-phase free boundary problems for nonlinear elliptic PDEs of second order.

A careful asymptotic analysis has been made for several reflection-diffraction configurations; see [44, 47–49, 72] and the references cited therein. Large or small scale numerical simulations have also been performed; cf. [5, 44, 88] and the references cited therein. However, most of the fundamental issues for the shock reflection-diffraction phenomena have not been understood, especially the global structures and the transition between the different patterns of shock reflection-diffraction configurations. This is partially because physical/numerical experiments are hampered by many difficulties and have not yielded clear transition criteria between the different patterns. In particular, some different patterns occur when the wedge angles are only fractions of a degree apart, a resolution even by sophisticated experiments and numerical simulations has been unable to reach (cf. [5, 67]). Therefore, the necessary approach to understand fully the shock reflection-diffraction phenomena, especially the transition criteria, is via rigorous mathematical analysis.

### 6.3 Global Solutions of Riemann Problem IV: Free Boundary Problem, Problem 6.2

If $\varphi$ is a solution of Problem 6.2, define its extension from $\Omega$ to $\Lambda$ by setting:

$$
\varphi = \begin{cases} 
\varphi_0 & \text{for } \xi_1 > \xi_1^0 \text{ and } \xi_2 > \xi_1 \tan \theta_w, \\
\varphi_1 & \text{for } \xi_1 < \xi_1^0 \text{ and above curve } P_0P_1P_2, \\
\varphi_2 & \text{in domain } P_0P_1P_4,
\end{cases}
$$

where we have used the notational convention (6.3) for the subsonic reflection case, in which domain $P_0P_1P_4$ is one point and curve $P_0P_1P_2$ is $P_0P_2$; see Figs. 6.1–6.2. Also, the extension by (6.4) is well-defined because of the requirement that $\Gamma_{\text{shock}} \subset \Lambda \cap \{\xi_1 < \xi_1 P_1\}$ in Problem 6.2.

In the supersonic case, the conditions in Problem 6.2(iii) are the Rankine-Hugoniot conditions on $\Gamma_{\text{sonic}}$ between $\varphi|\Omega$ and $\varphi_2$. Indeed, since state (2) is sonic on $\Gamma_{\text{sonic}}$, then it follows from (5.12)–(5.11) that no gradient jump occurs on $\Gamma_{\text{sonic}}$. Then, if $\varphi$ is a solution of Problem 6.2, its extension by (6.4) is a global entropy solution in the self-similar coordinates.
Since $\Gamma_{\text{sonic}}$ is not a free boundary, it is not possible in general to prescribe two conditions given in Problem 6.2(iii) on $\Gamma_{\text{sonic}}$ for a second-order elliptic PDE. In the iteration problem, we prescribe the condition: $\varphi = \varphi_2$ on $\Gamma_{\text{sonic}}$, and then prove that $D\varphi = D\varphi_2$ on $\Gamma_{\text{sonic}}$ by exploiting the elliptic degeneracy on $\Gamma_{\text{sonic}}$.

The key obstacle to establish the existence of regular shock reflection-diffraction configurations as conjectured by von Neumann [82, 83] is an additional possibility that, for some wedge angle $\theta_w^d \in (\theta_w^d, \frac{\pi}{2})$, shock $P_0P_2$ may attach to the wedge vertex $P_3$, as observed by experimental results (cf. [80, Fig. 238]). To describe the conditions of such a possible attachment, we note that

$$u_1 = (\rho_1 - \rho_0) \sqrt{\frac{2(\rho_1^{-1} - \rho_0^{-1})}{\rho_1^2 - \rho_0^2}} > 0, \quad \rho_1 > \rho_0, \quad c_1 = \rho_1^{\frac{\gamma - 1}{2}}.$$

Then it follows from the explicit expressions above that, for each $\rho_0$, there exists $\rho^c > \rho_0$ such that

$$u_1 \leq c_1 \quad \text{if} \quad \rho_1 \in (\rho_0, \rho^c]; \quad u_1 > c_1 \quad \text{if} \quad \rho_1 \in (\rho^c, \infty).$$

If $u_1 \leq c_1$, we can rule out the solution with a shock attached to the wedge vertex. This is based on the fact that, if $u_1 \leq c_1$, then the wedge vertex $P_3 = (0,0)$ lies within the sonic circle $B_{c_1}(u_1)$ of state (1), and $\Gamma_{\text{shock}}$ does not intersect $B_{c_1}(u_1)$, as we show below. If $u_1 > c_1$, there would be a possibility that the reflected shock could be attached to the wedge vertex, as the experiments show (e.g., [80, Fig. 238]).

To solve the free boundary problem (Problem 6.2) involving transonic shocks for all the wedge angles $\theta_w \in (\theta_w^d, \frac{\pi}{2})$, we define the following admissible solutions.

**Definition 6.1.** Let $\theta_w \in (\theta_w^d, \frac{\pi}{2})$. A function $\varphi \in C^{0,1}(\Lambda)$ is an admissible solution of the regular reflection problem if $\varphi$ is a solution of Problem 6.2 extended to $\Lambda$ by (6.4) and satisfies the following properties:

(i) The structure of solution:

- If $|D\varphi_2(P_0)| > c_2$, then $\varphi$ is of the supersonic regular shock reflection-diffraction configuration as shown on Fig. 6.1 and satisfies the conditions that the curved part of reflected-diffracted shock $\Gamma_{\text{shock}}$ is $C^2$ in its relative interior; curves $\Gamma_{\text{shock}}, \Gamma_{\text{sonic}}, \Gamma_{\text{wedge}},$ and $\Gamma_{\text{sym}}$ do not have common points except their endpoints; and

$$\varphi \in C^{0,1}(\Lambda) \cap C^1(\Lambda \ \setminus \ (S_0 \cup P_0P_1P_2)), \quad \varphi \in C^1(\Omega) \cap C^3(\Omega \ \setminus \ (\Gamma_{\text{sonic}} \cup \{P_2, P_3\})),$$
• If $|D\varphi_2(P_0)| \leq c_2$, then $\varphi$ is of the subsonic regular shock reflection-diffraction configuration shown on Fig. 6.2 and satisfies the conditions that the reflected-diffracted shock $\Gamma_{\text{shock}}$ is $C^2$ in its relative interior; curves $\Gamma_{\text{shock}}$, $\Gamma_{\text{wedge}}$, and $\Gamma_{\text{sym}}$ do not have common points except their endpoints; and

$$\varphi \in C^{0,1}(\Lambda) \cap C^1(\Lambda \setminus (S_0 \cup \Gamma_{\text{shock}})),$$

$$\varphi \in C^1(\Omega) \cap C^2(\Omega \setminus \{P_0, P_3\}).$$

Moreover, in both the supersonic and subsonic cases, the extended curve $\Gamma_{\text{shock}}^\text{ext} := \Gamma_{\text{shock}} \cup \{P_0\} \cup \Gamma_{\text{shock}}^{-}$ is $C^1$ in its relative interior, where $\Gamma_{\text{shock}}^{-}$ is the reflection of $\Gamma_{\text{shock}}$ with respect to the $\xi_1$-axis.

(ii) Eq. (5.5) is strictly elliptic in $\Omega \setminus \bar{\Gamma}_{\text{sonic}}$: $|D\varphi| < c(|D\varphi|^2, \varphi)$ in $\Omega \setminus \bar{\Gamma}_{\text{sonic}}$.

(iii) $\partial_{\nu_s} \varphi > \partial_{\nu_s} \varphi > 0$ on $\Gamma_{\text{shock}}$, where $\nu$ is the normal to $\Gamma_{\text{shock}}$ pointing to the interior of $\Omega$.

(iv) Inequalities hold:

$$\varphi_2 \leq \varphi \leq \varphi_1 \quad \text{in } \Omega. \tag{6.5}$$

(v) The monontonicity properties hold:

$$\partial_{\xi_2} (\varphi_1 - \varphi) \leq 0, \quad D(\varphi_1 - \varphi) \cdot e_{S_1} \leq 0 \quad \text{in } \Omega \text{ for } e_{S_1} = \frac{P_0 P_1}{|P_0 P_1|}. \tag{6.6}$$

Notice that (6.6) implies that

$$D(\varphi_1 - \varphi) \cdot e \leq 0 \quad \text{in } \overline{\Omega} \text{ for any } e \in \text{Cone}(e_{\xi_2}, e_{S_1}), \tag{6.7}$$

where $\text{Cone}(e_{\xi_2}, e_{S_1}) = \{ a e_{\xi_2} + b e_{S_1} : a, b > 0 \}$ with $e_{\xi_2} = (0, 1)$, and $e_{\xi_2}$ and $e_{S_1}$ are not parallel if $\theta_w \neq \frac{\pi}{2}$. Then we establish the following theorem:

**Theorem 6.1** (Chen-Feldman [21, 22]). There are two cases:

(i) If $\rho_0$ and $\rho_1$ are such that $u_1 \leq c_1$, then the supersonic/subsonic regular reflection solution exists for each wedge angle $\theta_w \in (\theta_{w}^d, \frac{\pi}{2})$. That is, for each $\theta_w \in (\theta_{w}^d, \frac{\pi}{2})$, there exists a solution $\varphi$ of Problem 6.2 such that

$$\Phi(t, x) = t \varphi\left(\frac{x}{t}\right) + \frac{|x|^2}{2t} \quad \text{for } \frac{x}{t} \in \Lambda, \ t > 0$$

with

$$\rho(t, x) = \left(\rho_0^{\gamma-1} - (\gamma - 1)(\Phi_t + \frac{1}{2} |\nabla_x \Phi|^2)\right)^{-\frac{1}{\gamma-1}}$$

is a global weak solution of Problem 6.1 in the sense of Definition 5.1 satisfying the entropy condition; that is, $\Phi(t, x)$ is an entropy solution.

(ii) If $\rho_0$ and $\rho_1$ are such that $u_1 > c_1$, then there exists $\theta_w^a \in [\theta_{w}^d, \frac{\pi}{2})$ so that the regular reflection solution exists for each wedge angle $\theta_w \in (\theta_{w}^a, \frac{\pi}{2})$, and the solution is of the self-similar structure described in (i) above.
Moreover, if $\theta^a_w > \theta^d_w$, then, for the wedge angle $\theta_w = \theta^a_w$, there exists an attached solution, i.e., $\varphi$ is a solution of Problem 6.2 with $P_2 = P_3$.

The type of regular shock reflection-diffraction configurations (supersonic as in Fig. 6.1 or subsonic as in Fig. 6.2) is determined by the type of state (2) at $P_0$:

(a) For the supersonic and sonic reflection case, the reflected-diffracted shock $P_0P_2$ is $C^{2,\alpha}$-smooth for some $\alpha \in (0,1)$ and its curved part $P_1P_2$ is $C^\infty$ away from $P_1$. The solution $\varphi$ is in $C^{1,\alpha}(\Omega) \cap C^\infty(\Omega)$, and is $C^{1,1}$ across the sonic arc which is optimal; that is, $\varphi$ is not $C^2$ across the sonic arc.

(b) For the subsonic reflection case (Fig. 6.2), the reflected-diffracted shock $P_0P_2$ and solution $\varphi$ in $\Omega$ is in $C^{1,\alpha}$ near $P_0$ and $P_3$ for some $\alpha \in (0,1)$, and $C^\infty$ away from $\{P_0, P_3\}$.

Moreover, the regular reflection solution tends to the unique normal reflection (as in Fig. 6.3) when the wedge angle $\theta_w$ tends to $\frac{\pi}{2}$. In addition, for both supersonic and subsonic reflection cases,

$$\varphi_2 < \varphi < \varphi_1 \quad \text{in } \Omega. \quad (6.8)$$

Furthermore, $\varphi$ is an admissible solution in the sense of Definition 6.1 below, so that $\varphi$ satisfies further properties listed in Definition 6.1.

Theorem 6.1 is proved by solving Problem 6.2. The first results on the existence of global solutions of the free boundary problem (Problem 6.2) were obtained for the wedge angles sufficiently close to $\frac{\pi}{2}$ in Chen-Feldman [21]. Later, in Chen-Feldman [22], these results were extended up to the detachment angle as stated in Theorem 6.1. For this extension, the techniques developed in [21], notably the estimates near the sonic arc, were the starting point. More details can be found in Chen-Feldman [22]; also see [21].

Furthermore, in Chen-Feldman-Xiang [25], we established the convexity of transonic shocks for the regular shock reflection-diffraction configurations.

Theorem 6.2 (Chen-Feldman-Xiang [25]). If a solution of the von Neumann problem for shock reflection-diffraction is admissible in the sense of Definition 6.1, then its domain $\Omega$ is convex, and the shock curve $\Gamma_{\text{shock}}$ is a strictly convex graph. That is, $\Gamma_{\text{shock}}$ is uniformly convex on any closed subset of its relative interior. Moreover, for the solution of Problem 6.2 extended to $\Lambda$ by (6.4), with pseudo-potential $\varphi \in C^{0,1}(\Lambda)$ satisfying Definition 6.1(i)–(iv), the shock is strictly convex if and only if Definition 6.1(v) holds.

Furthermore, with the convexity of reflected-diffracted transonic shocks, the uniqueness and stability of global regular shock reflection-diffraction configurations have also been established in the class of admissible solutions; see Chen-Feldman-Xiang [26] for details.
7 Concluding Remarks

In this paper, we have presented four different 2-D Riemann problems involving transonic shocks through several prototypes of hyperbolic systems of conservation laws and have showed how these Riemann problems can be formulated/solved as free boundary problems with transonic shocks as free boundaries for the corresponding nonlinear conservation laws of mixed elliptic-hyperbolic type and related nonlinear PDEs. In Li-Zheng [60, 61], another 2-D Riemann problem including the classical problem of the expansion of a wedge of gas into a vacuum for the isentropic Euler equations has also been solved; also see the recent work by Lai-Sheng [53] and the references cited therein on further related Riemann problems. The other types of 2-D Riemann problems are still wide open, even for the prototypes of hyperbolic systems of conservation laws as discussed in this paper.

For the full Euler equations (2.1) with (2.3), the 2-D Riemann problems involve vortex sheets and entropy waves, in addition to shocks and rarefaction waves; see [8–11, 22, 43, 52, 56, 58, 75, 95] and the references cited therein. Almost all of these Riemann problems for the full Euler equations (2.1) with (2.3) are still unsolved. In addition, all the 3-D or higher-D Riemann problems, including M-D wedge problems or M-D conic body problems, are completely open; see [15, 18, 19, 28] and the references cited therein for some recent developments for M-D steady problems. The nonlinear methods and related techniques/approaches originally developed in [20–22] as presented above for solving 2-D Riemann problems involving 2-D transonic shocks should be useful in the analysis of these longstanding Riemann problems and newly emerging problems for nonlinear PDEs; also see [14, 22, 23] and the references cited therein. Certainly, further new ideas, techniques, and methods still need to be developed in order to solve these mathematically challenging and fundamentally important problems.

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Declarations

There is no conflict of interest.
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