FAST TRACK COMMUNICATION

Chirality of tensor perturbations for complex values of the Immirzi parameter

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Abstract

In this communication, we generalize previous work on tensor perturbations in a de Sitter background in terms of Ashtekar variables to cover all complex values of the Immirzi parameter $\gamma$ (previous work was restricted to imaginary $\gamma$). Particular attention is paid to the case of real $\gamma$. Following the same approach as in the imaginary case, we can obtain physical graviton states by invoking reality and torsion-free conditions. The Hamiltonian in terms of graviton states has the same form whether $\gamma$ has a real part or not; however, changes occur for the vacuum energy and fluctuations. Specifically, we observe a $\gamma$-dependent chiral asymmetry in the vacuum fluctuations only if $\gamma$ has an imaginary part. Ordering prescriptions also change this asymmetry. We thus present a measurable result for CMB polarization experiments that could shed light on the workings of quantum gravity.

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(Some figures may appear in colour only in the online journal)

1. Introduction

Although loop quantum gravity [1–3] is endowed with a rigorous mathematical structure, it is still difficult to obtain GR as a low-energy limit from it and make contact with experiments. However, progress has recently been made on the computation of the graviton propagator [4, 5], and in a previous publication [6] we have identified graviton states within the Hamiltonian framework for a self-dual (or anti-self-dual) connection (for which the Immirzi parameter is $\gamma = \pm i$). This work is built upon early work on graviton states for the self-dual connection [7]. The detailed calculation for general imaginary values of $\gamma$ was provided in [8]. To identify the graviton states that correspond to the dynamical, fluctuating part of spacetime we compared our approach to cosmological perturbation theory. After taking several subtleties into account (for more details see [8]) the Ashtekar Hamiltonian indeed reduces on-shell to the standard tensor perturbation Hamiltonian [10], but novelties come about. We found that only half of the graviton states are physical, retaining only the standard two polarizations.
for gravitons after reality conditions are imposed. For the physical states we discovered a \(\gamma\)-dependent chirality in the vacuum energy (VE) as well as the two-point function.

In this communication, these results will be generalized to a complex \(\gamma\) in a Lorentzian theory. This is a non-trivial algebraic exercise with significant modifications in the results for the intermediate steps, but the final result is remarkably simple. For details on how to derive the second-order Hamiltonian for gravitons \([8]\) should be consulted; here, we just summarize the framework and highlight the changes that occur for general \(\gamma\). These are most notably the reality conditions and commutation relations between the canonical variables. It turns out that, in spite of these modifications, the final result is very simple. The vacuum chirality derived in \([6, 8]\) is only present if \(\gamma\) has an imaginary part; for real \(\gamma\) the two graviton polarizations are symmetric.

The plan of the communication is as follows. In section 2, we introduce the perturbed metric and connection variables and their classical solution. Section 3 explains the reality conditions and commutation relations for general \(\gamma\). We present a representation of the Hamiltonian in terms of graviton states in section 4. In section 5, we explain how a complex \(\gamma\) leads to a chirality in the vacuum fluctuations, but only provided that \(\gamma\) has an imaginary part. The special case of real \(\gamma\) will be investigated in section 6. We finish with a concluding section summarizing our results.

2. Notation and classical solution

In this section, we lay down the notation, referring the reader to previous publications \([6, 8]\) for details. We consider tensor fluctuations around de Sitter spacetime described in the flat slicing, \(ds^2 = a^2[-d\eta^2 + (\delta_{ab} + h_{ab})dx^adx^b]\), where \(h_{ab}\) is a symmetric TT tensor, \(a = -1/H\eta\), \(H^2 = \Lambda/3\) and \(\eta < 0\). Using the convention \(\Gamma^i = -\frac{1}{2}\epsilon^{ijk}\Gamma^j\delta^{k}\) (where \(\Gamma^{ij}\) is the spin connection), the Ashtekar–Immirzi–Barbero connection is given by \(A^i = \Gamma^i + \gamma \Gamma^0\), with \(\gamma\) the Immirzi parameter. Making use of the Cartan equations for the zeroth-order solution, the canonical variables can be expressed as

\[
A^i_a = \gamma H a \delta^i_a + \frac{\delta^i_a}{a},
\]

\[
E^a_i = a^2 \delta^a_i - a \delta^a e^i,
\]

where \(E^a_i\) is the densitized inverse triad, canonically conjugate to \(A^i_a\). As in \([6, 8]\) we define \(\delta e^i_a\) via the triad, \(e^i_a = a \delta^i_a + \delta e^i_a\); we then raise and lower indices in all tensors with the Kronecker-\(\delta\), possibly mixing group and spatial indices. This simplifies the notation and is unambiguous if it is understood that \(\delta e\) is originally the perturbation in the triad. It turns out that \(\delta e^i_j\) is proportional to the ‘\(v\)’ variable used by cosmologists \([10, 11]\).

The canonical variables have a symplectic structure

\[
\{A^i_a(x), E^b_j(y)\} = \gamma l_2^2 \delta^i_a \delta^b_j \delta(x - y),
\]

where \(l_2^2 = 8\pi G\) parametrizes the strength of gravity, in units in which \(c = 1\) (without reference to \(\hbar\) at this stage). Equation (3) implies \([8]\)

\[
\{\delta e^i_a(x), \delta e^j_b(y)\} = -\gamma l_2^2 \delta^i_a \delta^b_j \delta(x - y).
\]
To make contact with cosmological perturbation theory and standard perturbative quantum field theory we use mode expansions (see [8] for a full explanation)

\[ \delta e_{ij} = \int \frac{d^3 k}{(2\pi)^3} \sum_r e_{ij}^r(k) \tilde{e}_{r+}(k, \eta) e^{ik \cdot x} + e_{ij}^{\ast r}(k) \tilde{e}_{r-}(k, \eta) e^{-ik \cdot x} \]

\[ a_{ij} = \int \frac{d^3 k}{(2\pi)^3} \sum_r e_{ij}^r(k) \tilde{a}_{r+}(k, \eta) e^{ik \cdot x} + e_{ij}^{\ast r}(k) \tilde{a}_{r-}(k, \eta) e^{-ik \cdot x}, \tag{5} \]

where \( \tilde{e}_{r+}(k, \eta) = e_{r+}(k) \Psi_c(k, \eta) \) and \( \tilde{a}_{r+}(k, \eta) = a_{r+}(k) \Psi_c(k, \eta) \), and \( e_{ij}^r \) are the polarization tensors. Amplitudes \( \tilde{a}_{r+}(k) \) and \( \tilde{e}_{r+}(k) \) have two indices (contrasting with previous literature, e.g. [12, 13]): \( r = \pm 1 \) for right and left helicities, and \( p \) for graviton \((p = 1)\) and anti-graviton \((p = -1)\) modes. The \( a_{r+} \) and \( e_{r+} \) can be chosen so as not to carry any time dependence, and for simplicity we will assume that they are equal. After imposing on-shell conditions we will find that functions \( \Psi_c(k, \eta) \) must then carry an \( r \) and \( p \) dependence.

The classical solution in terms of these variables can be read off from cosmological perturbation theory [8]. Since \( \Psi_c \) is proportional to the ‘\( v \)’ variable used in cosmology [10, 11], it must satisfy the well-known equation

\[ \Psi_c'' + (k^2 - \frac{2}{\Omega}) \Psi_c = 0 \]

where \( \Omega \) is proportional to the ‘\( v \)’ variable used in cosmology. This has solution

\[ \Psi_c = e^{-ik \eta} \frac{1}{2\sqrt{k}} \left( 1 - \frac{i}{k \eta} \right), \tag{6} \]

where the normalization ensures that the amplitudes \( e_{r+} \) become annihilation operators upon quantization. The connection can then be inferred from Cartan’s torsion-free condition \[ T^I = de^I + \Gamma_i^I \wedge e^i = 0. \] To first order, this is solved by

\[ \delta \Gamma_i^I = \frac{1}{a} \delta e_{ij}^I dx^j, \tag{7} \]

\[ \delta \Gamma_{ij} = -\frac{2}{a} \partial_i \partial_j e_{ij} dx^i. \tag{8} \]

These imply \[ \delta \Gamma_i = \frac{1}{a} \epsilon^{ikj} \partial_k \partial e_{ij} dx^i, \] so that

\[ a_{ij} = \epsilon_{ij} \partial_k \partial e_{ik} + \gamma \delta e_{ij}^I, \tag{9} \]

Up until this point the calculation is valid for all complex \( \gamma \). The first novelty in this communications appears upon inserting decomposition (5) into (9), to determine torsion-free conditions in Fourier space. Using relation \( \epsilon_{ij} \epsilon_{kl} \) we obtain

\[ \Psi^{+}_a = \gamma \Psi^I + r k \Psi_c, \tag{10} \]

\[ \Psi^{-}_a = \gamma^* \Psi^I + r k \Psi_c, \tag{11} \]

and clearly \( \gamma^* = -\gamma \), used in [6], is only valid if \( \gamma \) is imaginary. By writing a generally complex \( \gamma \) as

\[ \gamma = \gamma_R + i\gamma_I, \tag{12} \]

we find that inside the horizon (\( |k \eta| \gg 1 \))

\[ \Psi^{+}_a = \Psi_c k (r - i\gamma_R + p\gamma_I), \tag{13} \]

generalizing the expression derived in [8]. We note that the \( p \) dependence of these functions only occurs if \( \gamma \) has an imaginary part. For a real \( \gamma \), \( \Psi_a \) is the same for both gravitons and anti-gravitons, as expected (a real connection would be expanded in terms of a single particle \( \tilde{a}_r \), so an index \( p \) would be unnecessary; see section 6 for a longer discussion). This is a first hint that the chirality found in [6, 8] is specific to non-real \( \gamma \).
3. Reality conditions and commutation relations

To be able to relate graviton and anti-graviton states (and their respective Hermitian conjugates), we need to impose reality conditions. As in [8], this will be done via the choice of the inner product, rather than as operator conditions. Nonetheless, it is important to see what these conditions look like in terms of operators (or as classical identities). As the metric is real \((\delta e_{ij} = e^{*}_{ij})\), we have

\[ e_{r+i}(k) = e_{r-i}(k). \]  

(14)

The definition of the connection implies

\[ \Re A^i = \Gamma^i + \gamma^R \Gamma^0, \]  

(15)

\[ \Im A^i = \gamma^I \Gamma^0. \]  

(16)

Compared to the corresponding expressions for imaginary \(\gamma\) (see [8]), we note that the real part of the connection now has a contribution from \(\gamma^R \Gamma^0\), i.e. the extrinsic curvature. The reality conditions for the connection should embody the non-dynamical torsion-free conditions, i.e. those not involving the extrinsic curvature, which in the Hamiltonian formalism becomes the time derivative of the metric. The full torsion-free conditions representing (9) are now

\[ a_{ij} + \bar{a}_{ij} = 2a \left( \delta \Gamma_{ij} + \gamma^R \delta \Gamma^0_{ij} \right) \]
\[ = 2\epsilon_{kl} \delta_{ij} + 2\gamma^R \delta e_{ij}, \]  

(17)

\[ a_{ij} - \bar{a}_{ij} = 2i\gamma^I \delta \Gamma^0_{ij} = 2i\gamma \delta e_{ij}, \]  

(18)

or, in terms of Fourier components

\[ \tilde{a}_{r+i}(k, \eta) + \tilde{a}_{r-i}(k, \eta) = 2rk\tilde{e}_{r+i}(k, \eta) + 2\gamma^R \tilde{e}'_{r+i}(k, \eta), \]  

(19)

\[ \tilde{a}_{r+i}(k, \eta) - \tilde{a}_{r-i}(k, \eta) = 2i\gamma \tilde{e}'_{r+i}(k, \eta). \]  

(20)

Combining (19) and (20) so as to eliminate the time derivative in the metric leads to the condition

\[ i\gamma^* \tilde{a}_{r+i}(k, \eta) - i\gamma \tilde{a}_{r-i}(k, \eta) = 2rk\gamma \tilde{e}_{r+i}(k, \eta). \]  

(21)

Its Hermitian conjugate is

\[ -i\gamma^* \tilde{a}_{r-i}^\dagger(k, \eta) + i\gamma^* \tilde{a}_{r+i}^\dagger(k, \eta) = 2rk\gamma \tilde{e}_{r-i}^\dagger(k, \eta), \]  

(22)

which also invokes (14). These expressions represent the reality conditions that should be imposed quantum mechanically by the choice of the inner product. They are very different from their counterparts for a purely imaginary \(\gamma\) and represent novelty number two in our calculation. For each \(r\) and \(k\), there are two independent conditions upon the four operators \(a_{r+i}(k)\) and \(e_{r+i}(k)\). Besides them there is an independent dynamical torsion-free condition.

On-shell, i.e. using (13) and invoking (14), the connection can be written in terms of the metric according to the weak identity:

\[ \tilde{a}_{r-i}(k, \eta) \approx rk\tilde{e}_{r} + \gamma^* \tilde{e}'_{r} \rightarrow \tilde{e}_{r}(r - i\gamma^*)k \]
\[ \tilde{a}_{r+i}(k, \eta) \approx rk\tilde{e}_{r} + \gamma \tilde{e}'_{r} \rightarrow \tilde{e}_{r}(r - i\gamma)k, \]  

(23)

where the latter expression is valid in the limit \(k|\eta| \gg 1\). These will be useful in deriving the graviton operators for this theory. They render one of the graviton modes unphysical, fully relating metric and connection.
Before we can set up a quantum theory in terms of graviton operators we need to define the commutation relations in terms of modes. These are obtained, as usual, from the Poisson brackets (3) and (4), leading to

$$[A_i^a(x), E^b_j(y)] = i\gamma \frac{\delta^b_i \delta^a_j}{2}\delta(x - y),$$  \hspace{1cm} (24)

and

$$[\alpha^a_i(x), \delta E^b_j(y)] = -i\gamma \frac{\delta^b_i \delta^a_j}{2}\delta(x - y).$$  \hspace{1cm} (25)

The commutators for the mode expansions can be derived as in [8] and are

$$[\tilde{a}_{ir}(k), \tilde{c}_{ir}^+(k')] = -i(\gamma_R + p_i \gamma_I) \frac{\delta^2}{2}\delta_{r't'}\delta(k - k'),$$ \hspace{1cm} (26)

where \(\tilde{q} = -q\). (From this moment onwards we are assuming units with \(\hbar = 1\) in addition to \(c = 1\)). Compared to [8], the factor \(\gamma p\) has been replaced by \(\gamma_R + p_i \gamma_I\). This is algebraic novelty number three, the last one in our calculation. For real \(\gamma\) the \(p\) dependence is erased from the commutation relations.

4. The Hamiltonian

We now have all the ingredients to find a Hamiltonian in terms of graviton creation and annihilation operators (which will be linear combinations of the perturbations in the metric and connection variables). A surprise is in store at this point: in spite of the three novelties in the ingredients, spelled out above, the final result for the graviton operators and Hamiltonian is formally the same.

The gravitational Hamiltonian in terms of Ashtekar variables is given by

$$H = \frac{1}{2l_p^2} \int d^3x N E^a_i \frac{\epsilon^{ijk}}{2} \left[ F_{ij}^a + H^2 \epsilon_{abc} E^c_k - 2(1 + \gamma^2)K^j_{[a}K^i_{b]}\right],$$ \hspace{1cm} (27)

where

$$K^j_{a} = \frac{A^1_a - \Gamma^1_a(E)}{\gamma}$$ \hspace{1cm} (28)

is the extrinsic curvature of the spatial surfaces. The total Hamiltonian includes two further constraints, the Gauss and vector constraint, but they are automatically satisfied by expansions (5) and do not contribute to the order in perturbation theory we will consider [8]. The dynamics of the theory is encoded by the second-order Hamiltonian quadratic in first-order perturbations. To derive this Hamiltonian, a number of subtleties need to be taken into account which are spelled out in detail in [8]. To write the Hamiltonian as a product of graviton creation and annihilation operators inside the horizon, we need to express the second-order Hamiltonian in terms of the mode expansion (5) (see appendix III of [8]).

We can determine the graviton operators inside the horizon (\(|k\eta| \gg 1\)) following the same procedure as in [8]. Before reality conditions are imposed there should be unphysical modes that vanish on-shell (and that will turn out to have negative energy and norm). The physical modes should commute with the non-physical modes and reduce, on-shell, to the correct expressions in terms of metric variables. Using these rules, and recalling (23) and (26), we define

$$G_{r+} = \frac{-r}{i\gamma} (\tilde{a}_{r+} - k(r + i\gamma)\tilde{c}_{r+})$$ \hspace{1cm} (29)

$$G_{r-} = \frac{-r}{i\gamma} (\tilde{a}_{r-} - k(r - i\gamma)\tilde{c}_{r+})$$ \hspace{1cm} (30)
\begin{align}
G_{r;\gamma}^+ &= \frac{r}{i\gamma} (a_{\gamma}^+ - k(r - iy)e_{\gamma}^+) \\
G_{r;\gamma}^- &= \frac{r}{i\gamma} (a_{\gamma}^- - k(r + iy)e_{\gamma}^+).
\end{align}

The index \( \mathcal{P} = \mathcal{P}_+, \mathcal{P}_- \) denotes physical and unphysical modes, respectively. The normalization ensures the right behaviour on-shell, i.e. \( G_{r;\mathcal{P}_-} \approx 0 \) and \( G_{r;\mathcal{P}_+} \approx 2rke_r \). Once the reality conditions (21)–(22)–(14) are enforced one can check that the \( G^\dagger \) are indeed the Hermitian conjugate operators of the \( G \). The commutation relations are, as required,

\begin{align}
\{ G_{r;\mathcal{P}_+}(k), G_{s;\mathcal{P}_-}^\dagger(k') \} &= \delta_{rs} \delta(k - k'),
\end{align}

These expressions are precisely the same as found in [8] for a purely imaginary \( \gamma \), in spite of the three algebraic novelties spelled out above. Somehow the modifications conspire to give the same graviton operators and commutators between them. This means that the Hamiltonian in terms of graviton states can be written in the same way as equation (105) of [8]. Just like before an inner product, enforcing the reality conditions, may be found in the representation diagonalizing the \( G^\dagger \) operators. The state \( \mathcal{P}_+ = 1 \) has positive energy and norm, and \( \mathcal{P}_- = -1 \) has negative energy and norm. On-shell, the Hamiltonian becomes

\begin{align}
H_{\text{ph eff}} &\approx \frac{1}{2l_p^2} \int \frac{dk}{2} \sum_r \left[ G_{r}^\dagger G_{r}^\dagger (1 + i\gamma) + G_{r}^\dagger G_{r} (1 - i\gamma) \right],
\end{align}

where \( G_{r}^\dagger = G_{r;\mathcal{P}_-} \).

The first term in the Hamiltonian we have just derived (which follows from a EEF ordering) needs to be normal ordered, leading to a chiral (i.e. \( r \)-dependent) VE \( V_r \propto 1 + i\gamma \). The chiral asymmetry is given by

\begin{align}
\frac{V_R - V_L}{V_R + V_L} &= i\gamma.
\end{align}

In [8] it was found that for imaginary \( \gamma \) the VE is chiral and that for \( |\gamma| > 1 \) one of the modes has negative VE. This flags a point of interest, since a negative VE is usually associated with fermionic degrees of freedom. We now find that for \( \gamma \) with a real part the VE for each mode is complex. The imaginary part, however, is maximally chiral and so cancels out, when right and left modes are added together. The real part never sees such a cancellation, except in the limit when \( |\Im(\gamma)| \to \infty \), when the total VE is indeed zero.

What is the origin of this result? We already pointed out in [8] that non-perturbatively the Hamiltonian is generally complex, a matter behind many of the novelties we have exposed. On-shell the Hamiltonian is zero and therefore real. The complexity of the Hamiltonian is not to be confused with its Hermiticity after quantization, and the inner product should enforce the Hermiticity of the quantum Hamiltonian. Perturbatively, however, the situation is more complicated. As explained in [8], even though the second-order Hamiltonian must still be zero on-shell, the portion dependent on first-order variables (to be seen as the perturbative Hamiltonian \( \mathcal{H}^{\text{pert}} \)) evades the Hamiltonian constraint. A number of other novelties of this sort appear when going from the full theory to perturbation theory, and to the fact that we now have dynamics instead of a pure constraint. It turns out that the classical perturbed Hamiltonian is always real on-shell, even if it is no longer zero. This is still true for a generally complex \( \gamma \). However, quantum mechanically the perturbative Hamiltonian is only Hermitian, on- and off-shell, if \( \gamma \) is imaginary. If \( \gamma \) has a real part the normal ordered Hamiltonian is still Hermitian,
but the VE is not. This can easily be seen from (35): obviously $G_{\text{ph}}^D G_{\text{ph}}^D \neq G_{\text{ph}}^D G_{\text{ph}}^D$. and $G_{\text{ph}}^D G_{\text{ph}}^D$ are still Hermitian under the chosen inner product, but their coefficients spoil Hermiticity before, but not after ordering.

What attitude should we take towards this result? One possibility is that there is nothing wrong with it. Obviously the VE couples to the Einstein equations, but the total (right and left helicities combined) is always real. Should we decide, however, that this feature is pathological then there are two possible implications. One is that a purely imaginary $\gamma$ should be favoured. Another is that a symmetric ordering of the Hamiltonian constraint is to be preferred. For more detail on the different ordering prescriptions see [8]; however it is obvious that $EFE$ or $1/2 (EEF + FEE)$ ordering would satisfy $\mathcal{H} = \mathcal{H}^\dagger$ on and off-shell, before and after ordering. In this case there would be no chirality in the VE; however, as the graviton modes are still the same, the vacuum fluctuations, or the two-point function, would still exhibit a chiral signature, as investigated in the next section.

5. Vacuum fluctuations

As in [8], we now want to compute the two-point function in terms of connection variables as it determines the vacuum fluctuation power spectrum. This is given by

$$\langle 0| A_r^\dagger (k) A_r (k') |0 \rangle = P_r (k) \delta (k - k'), \tag{37}$$

where $A_r (k)$ represents Fourier space connection variables with handedness $r$, i.e.

$$A_r (k) = a_r (k) e^{i k \cdot x} + a_r^\dagger (k) e^{i k \cdot x}. \tag{38}$$

Note that (37) depends on a specific ordering of the two-point function, and in general we have to consider

$$A \rightarrow \alpha A^\dagger + \beta A A^\dagger, \tag{39}$$

with $\alpha + \beta = 1$ and $\alpha, \beta > 0$. As (37) is a variance, it must always be real and positive (as opposed to the VE). Any chiral effects will then leave a measurable imprint on this quantity.

We need to relate the power spectrum to the physical graviton modes found in section 4. This can be done by substituting the on-shell conditions (23) into (29) and (31):

$$a_{r+}^{\text{ph}} = \frac{r + i \gamma}{2r} G_{r, p_+} \tag{40}$$

$$a_{r+}^{\text{ph}^\dagger} = \frac{r - i \gamma^*}{2r} G_{r, p_+}^{\dagger} \tag{41}$$

$$a_{r-}^{\text{ph}} = \frac{r - i \gamma}{2r} G_{r, p_-} \tag{42}$$

$$a_{r-}^{\text{ph}^\dagger} = \frac{r + i \gamma^*}{2r} G_{r, p_-}^{\dagger}. \tag{43}$$

We can see that these depend just on the graviton operators, so they will be the same for any ordering of the Hamiltonian. Plugging these expressions into (38), we obtain

$$A_r^{\text{ph}} (k) = \frac{r + i \gamma}{2r} G_{r, p_+} (k) e^{-i k \cdot x} + \frac{r - i \gamma}{2r} G_{r, p_-}^\dagger (k) e^{i k \cdot x},$$

$$A_r^{\text{ph}^\dagger} (k) = \frac{r - i \gamma^*}{2r} G_{r, p_+} (k) e^{-i k \cdot x} + \frac{r + i \gamma^*}{2r} G_{r, p_-}^\dagger (k) e^{i k \cdot x}. \tag{43}$$
so that
\[
\langle 0 | A_r^{ph}(k) A_r^{ph}(k') | 0 \rangle = P_r(\gamma)\langle 0 | G_{r_{PC}}(k) G_{r_{PC}}(k') | 0 \rangle,
\]
where
\[
P_r(\gamma) = \frac{(r + i\gamma)(r - i\gamma^*)}{4} = \frac{1 - 2\gamma r + |\gamma|^2}{4}.
\]
(45)
If \(\gamma r < 0\), \(P_r(\gamma)\) is obviously positive. Otherwise,
\[
P_r(\gamma) \propto 1 - 2|\gamma| + \gamma_I^2 + \gamma_R^2 = (1 - |\gamma|)^2 + \gamma_R^2
\]
so this is also positive for any complex \(\gamma\). Therefore, the two-point function is indeed always real and positive, as required. The chiral asymmetry in the power spectrum can be expressed as
\[
\frac{P_R - P_L}{P_R + P_L} = -\frac{2\gamma_I}{1 + |\gamma|^2},
\]
or, for a general ordering,
\[
\frac{P_R - P_L}{P_R + P_L} = \frac{2(\beta - \alpha)\gamma_I}{1 + |\gamma|^2}.
\]
This implies that for a real \(\gamma\) there is no asymmetry in the vacuum fluctuations for right and left gravitons. The chirality clearly traces to the fact that for an imaginary \(\gamma\) there must exist graviton and anti-graviton modes, i.e. the connection is a complex field. Note, however that the presence of a real part of the Immirzi parameter does affect the absolute value of the asymmetry due to the factor \(|\gamma|\) in the denominator of (47).

The result also depends on the choice of the two-point function. For a completely symmetric ordering, \(\alpha = \beta\), (48) is zero. We want to stress that this is not a weakness of our prediction, but ordering ambiguities are an intrinsic problem of quantum mechanics. Ultimately, only experiment can tell us whether an adapted ordering is correct. We can see that the power spectrum for right and left helicities can be symmetric for two different reasons, a symmetric ordering or a purely real \(\gamma\). A negative result in measuring the asymmetry therefore does not really give us any new information. Measuring a non-zero value for the asymmetry, absent in the second-order formalism, would however provide a completely new result and, as we have shown, is very likely to be an intrinsic effect of quantum gravity.

The power spectrum asymmetry (47) is plotted against a range of values of \(\gamma\) figure 1. It is obviously anti-symmetric in \(\gamma_I\), the minimum and maximum being at \(\gamma = \pm i\) respectively which are the values that correspond to a SD/ASD connection. They display the maximum chirality because the Palatini action can naturally be split into a SD and ASD part [3]. The axis \(\gamma_I = 0\) corresponds to a real \(\gamma\) and therefore no asymmetry. The chirality also vanishes in the limit \(|\gamma| \rightarrow \infty\) which corresponds to the Palatini–Kibble theory.

6. A purely real \(\gamma\)

In everything we have derived so far we can take the limit \(3(\gamma) \rightarrow 0\) and regard the result as the real theory. The question remains as to whether this limit is the same as a purely real theory, in which all the variables are real from the start. In principle the two might be different, since some aspects of the construction are obviously discontinuous. For example, in a purely real theory, expansions (5) have modes \(a_r\) and \(e_r\) without a \(p\) index, so that for a fixed \(k\) and \(r\) we start off with two, rather than four modes. It is important to check that this discontinuity does not propagate into our results, leading to expressions different from those taking the limit \(3(\gamma) \rightarrow 0\) in the complex theory. In this section, we show that this is not the
case: at the very least it is possible to set up the real theory so that no discontinuities arise in any of the expressions in this communications, even though there is a jump in the number of independent degrees of freedom. Note that this is far from obvious since the statements \( \tilde{e}_{\gamma+} = \tilde{e}_{\gamma-} \) and \( \tilde{a}_{\gamma+} = \tilde{a}_{\gamma-} \) are second class constraints in the complex theory, and are not enforced as operator conditions, but as formal conditions on the inner product. The real theory results from imposing them as operator conditions.

Firstly, the commutation relations (26) continuously shrink to

\[
[\tilde{a}_r(k), \tilde{e}^s_{r'}(k')] = -i\gamma \frac{\hbar}{2} \delta_{rs} \delta(k - k'),
\]

for a real \( \gamma \). The reality conditions (21)–(22)–(14) are now trivial (stating \( 0 = 0 \)) and do not constrain the theory. However, for graviton operators we can still use definitions (29)–(32) for \( G_{r\gamma} \), simply dropping the \( p \) index from their right-hand side, for example,

\[
G_{r\gamma} = \frac{-r}{i\gamma} (\tilde{a}_r - r(1 + i\gamma)\tilde{e}_r).
\]

It may appear that we are introducing too many modes. In the complex theory, for a fixed \( k \) and \( r \) we start with four modes, \( \tilde{a}_p \) and \( \tilde{e}_p \), from which we build four \( G_{r\gamma} \) and \( G^\dagger_{r\gamma} \). Three reality and torsion-free conditions then reduce them to a single physical operator, as explained after Equation (22). For the real theory we only have two modes, \( a_r \) and \( e_r \), from which we build four \( G_{r\gamma} \) and \( G^\dagger_{r\gamma} \) without having any reality conditions. However, upon closer inspection we see that for fixed \( k \) and \( r \) there are only two independent modes among \( G_{r\gamma} \) and \( G^\dagger_{r\gamma} \). In the complex theory we needed the reality conditions to ensure that \( G^\dagger_{r\gamma} \) were in fact the Hermitian conjugates of \( G_{r\gamma} \). If we drop the index \( p \) from their expressions, as in (50), then this fact follows trivially from their definitions and the linearity of the \( \dagger \) operation. Hence, by defining gravitons operators in the real theory we do preserve the number of independent degrees of freedom.
The issue persists on how to eliminate the non-physical mode. This is done by imposing the torsion-free condition, relating $a_r$ to $e_r$, which amounts to disqualifying the $G_{rP}$ mode. A possible implementation, even in the real theory, is to do this via the inner product. As in [8], we work in a holomorphic representation which diagonalizes $G_{rP}^\dagger \Phi_1(z)$, i.e. 
\[ G_{rP}^\dagger \Phi_1(z) = z_r \Phi_1(z). \]

Then (33) implies
\[ G_{rP} \Phi = \mathcal{P} k_l^2 \frac{\partial \Phi}{\partial z_r}. \]

Following the same procedure as in [8], we find
\[ \langle \Phi_1 | \Phi_2 \rangle = \int dz \, d\bar{z} \, e^{i \mu(z, \bar{z})} \Phi_1(\bar{z}) \Phi_2(z), \]
with
\[ \mu(z, \bar{z}) = \int dk \sum_{\mathcal{P}} \mathcal{P} z_r \mathcal{P} (k) \bar{z}_r (k), \]
rendering the states built from operators with $\mathcal{P} = \mathcal{P}_- = -1$ non-normalizable. As long as this procedure is adopted for the real theory the expressions found in this communications are continuous, and the limit $\Im(\gamma) \to 0$ does indeed represent the real theory.

7. Conclusion

In this communications, we have generalized the results of [8] to cover all values of the Immirzi parameter. Our analysis shows that an imaginary part of $\gamma$ is needed to produce a chiral effect in the vacuum fluctuations, whereas a purely real $\gamma$ would give the same physical Hamiltonian for right- and left-handed gravitons. The greatest asymmetry occurs for the values $\gamma = \pm i$, corresponding to a SD/ASD connection and the subject of [6]. Here, as in previous work, the chirality also depends on the ordering used for the two-point function. Although this implies that an observation of this asymmetry cannot be traced back to one single cause, it is still a striking prediction of quantum gravity in the Ashtekar formalism.

It was shown in [14] that even a small chiral effect in the gravitational wave background would greatly simplify its detection, making us hopeful that a test of our prediction could even be achieved by PLANCK. Note that other mechanisms exist that produce a similar chiral effect [15–17], but the one pointed out here is by far the simplest. It would be interesting to make contact with the work of [4], where a chiral contribution was found for the graviton propagator. However, in this publication a Euclidean signature and a real $\gamma$ were used, basically the opposite of our set-up, making the link between the predictions unclear.

Finally, we comment on the fact that our results depend on the ordering prescription employed and are not a general feature of the theory. This is not a weakness peculiar to our communications or indeed to quantum gravity: it is a general shortcoming of quantum mechanics. Many quantum ordering issues are ultimately decided by experiment. One should not be surprised by this: in playing the game of quantization one is inferring a deeper theory from one with less content. It is no surprise that the prescriptions used are often insufficient. There is therefore nothing shameful in appealing to experiment to remove ambiguities. It is in this spirit that we consider the dependence of our results upon ordering as an asset rather than a liability. Our results can assist us in resolving ordering issues via experiment. A related issue is the dependence of our results on the choice of vacuum (or base state). In a future publication we will make this state more explicit, by deriving it in the connection representation [9]. The ‘base state’ to be used in quantum cosmology might not be unique (Einstein’s equation, after all, has a large number of possible cosmological solutions). We just happen to live in the Universe we live in. It would be good to know which, though, and experiment is the only way to settle the matter.
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