Global existence and optimal decay rate of weak solutions to the co-rotation Hooke model

Wenjie Deng\textsuperscript{1}*, Zhaonan Luo\textsuperscript{1†} and Zhaoyang Yin\textsuperscript{1,2‡}

\textsuperscript{1}Department of Mathematics, Sun Yat-sen University, Guangzhou 510275, China
\textsuperscript{2}Faculty of Information Technology, Macau University of Science and Technology, Macau, China

Abstract
In this paper, we mainly study global existence and optimal $L^2$ decay rate of weak solutions to the co-rotation Hooke model. This micro-macro model is a coupling of the Navier-Stokes equation with a nonlinear Fokker-Planck equation. Based on the defect measure propagation method, we prove that the co-rotation Hooke model admits a global weak solution provided the initial data under different integrability conditions. Moreover, we obtain optimal $L^2$ decay rate for the weak solutions by the Fourier splitting method.

2010 Mathematics Subject Classification: 35Q30, 76B03, 76D05, 76D99.

Keywords: The co-rotation Hooke model; global weak solutions; optimal $L^2$ decay rate.

Contents

1 Introduction 2
1.1 Reviews for the polymeric fluid models .......................... 2
1.2 Main results .................................................. 3

2 Preliminaries .... 5

3 Priori estimates .... 9

4 Compactness
4.1 Compactness on the velocity $u$ .................................. 12
4.2 Compactness on the polymeric distribution $g$ .................... 13

5 Optimal decay rate 18

*email: detective2028@qq.com
†E-mail: 1411919168@qq.com
‡E-mail: mcsyzy@mail.sysu.edu.cn
1 INTRODUCTION

In this paper we consider the Hooke model of polymeric fluids \[2, 6, 13\]:

\[
\begin{aligned}
\partial_t u &+ \text{div} (u \otimes u) - \nu \Delta u + \nabla P = \text{div} \tau, \\
\partial_t \psi &+ \text{div} (u \psi) - \alpha \text{div} q \left( \frac{\psi}{\psi_{\infty}} \right) \psi_{\infty} = \text{div} \sigma(u) + \mu \Delta \psi, \\
\tau(\psi) &= \int_{\mathbb{R}^d} q \otimes \nabla q \mu dq - Id, \\
\psi &\in L^1
\end{aligned}
\]

(1.1)

In (1.1), the parameters \(a, \mu\) and \(\nu\) are nonnegative. Here \(u(t, x)\) stands for the velocity of the polymeric liquid and \(\psi(t, x, q)\) denotes the distribution function for the internal configuration. The polymer elongation \(q \in \mathbb{R}^d\), which means that the extensibility of the polymers is infinite and \(x \in \mathbb{R}^d\). Stress tensor \(\tau\) is generated by the polymer particles effect. Note that \(\Omega = \frac{\nabla u - \nabla u^T}{2}\) and \(D(u) = \frac{\nabla u + (\nabla u)^T}{2}\).

In general, \(\sigma(u) = \nabla u\). For the co-rotation case, \(\sigma(u) = \Omega\). From the Hooke model (1.1), one can derive the following Oldroyd-B equation with \(\int_{\mathbb{R}^d} \psi dq = \int_{\mathbb{R}^d} \psi dq = 1\):

\[
\begin{aligned}
\partial_t u &+ u \cdot \nabla u + \nabla P = \text{div} \tau + \nu \Delta u, \\
\partial_t \tau &+ u \cdot \nabla \tau + Q(\nabla u, \tau) = bD(u) + \mu \Delta \tau, \\
u(t)|_{t=0} &= u_0, \quad \tau(t)|_{t=0} = \tau_0
\end{aligned}
\]

(1.2)

with \(Q(\nabla u, \tau) = \tau \Omega - \Omega \tau + b(D(u)\tau + \tau D(u))\). The co-rotation case means \(b = 0\).

1.1. Reviews for the polymeric fluid models

Let us review some mathematical results about the polymeric fluid models.

Take \(\nu > 0\) and \(\mu = 0\) in (1.1) and (1.2). M. Renardy \[27\] established the local well-posedness in Sobolev spaces with potential \(U(q) = (1 - |q|^2)^{\frac{1}{2}}\) for \(\sigma > 1\). Later, B. Jourdain, T. Lelièvre, and C. Le Bris \[14\] proved local existence of a stochastic differential equation with \(U(q) = -k \log (1 - |q|^2)\) and \(k > 3\) for a Couette flow. For the co-rotation case, P. L. Lions and N. Masmoudi \[19\] constructed global weak solutions for the Oldroyd-B model. By virtue of the defect measure propagation method, they obtained strong convergence of an approximating sequence. Global existence of weak solutions to the FENE model and the FENE-P model was proved in \[24, 23\]. J. Y. Chemin and N. Masmoudi \[9\] showed a sufficient condition of non-breakdown for viscoelastic fluids. N. Masmoudi, P. Zhang and Z. Zhang \[25\] obtained global well-posedness for the 2-D co-rotation Hooke model without any small conditions. In addition, Z. Lei, N. Masmoudi and Y. Zhou \[15\] improved the blow-up criteria for the Oldroyd-B model.

Take \(\nu = 0\) and \(\mu > 0\) in (1.1) and (1.2). T. M. Elgindi and F. Rousset \[8\] proved global regularity for the 2-D Oldroyd-B type model (1.2). Later on, T. M. Elgindi and J. Liu \[9\] obtained global strong solutions of (1.2) with \(d = 3\) and initial data small enough. For the 2-D co-rotation Oldroyd-B type model and the corresponding Hooke model, W. Deng, Z. Luo and Z. Yin \[15\] proved global existence with a class of large initial data.

The long time behavior for polymeric fluid models is noticed by N. Masmoudi \[10\]. Take \(\nu > 0\) and \(\mu = 0\) in (1.1). Under a low-frequency assumption, L. He and P. Zhang \[12\] obtained that the \(L^2\) decay rate of the velocity for the Hooke model is \((1 + t)^{-\frac{3}{4}}\) with \(d \geq 3\). Recently, M. Schonbek \[30\] studied the
$L^2$ decay of the velocity for the co-rotation FENE dumbbell model with $U(R) = -k \log(1 - (|R|/|R_0|)^2)$, and obtained the decay rate $(1 + t)^{-d + \frac{1}{2}}$, $d \geq 2$ with $u_0 \in L^1$. Moreover, she conjectured that the sharp decay rate should be $(1 + t)^{-d}$, however, she failed to get it because she could not use the bootstrap argument as in [28] due to the additional stress tensor. More recently, W. Luo and Z. Yin [21, 22] improved Schonbek’s result and showed that the decay rate is $(1 + t)^{-d}$ with $d \geq 2$. For the 2-D Oldroyd-B type model (1.2) with $\nu = 0$ and $\mu > 0$, W. Deng, Z. Luo and Z. Yin [5] shown $H^1$ decay rate for global solutions constructed by T. M. Elgindi and F. Rousset in [8].

1.2. Main results

Let $\sigma(u) = \Omega$. One can verify that $(0, \psi_\infty)$ with $\psi_\infty(q) = e^{-\frac{1}{2} |q|^2}$ is a trivial solution of (1.1). Take $\nu = a = 1$ and $\mu = 0$ in (1.1). Consider the perturbations near the global equilibrium $u = u$ and $g = \frac{\psi - \psi_\infty}{\psi_\infty}$, then we can rewrite (1.1) as the following system:

\[
\begin{align*}
\partial_t u + \text{div} (u \otimes u) - \Delta u + \nabla P &= \text{div}(g), \quad \text{div} u = 0, \\
\partial_t g + \text{div} (u g) - \mathcal{L}g &= \frac{1}{\psi_\infty} \text{div}_q [\Omega \cdot q g \psi_\infty], \\
\tau(g) &= \int_{\mathbb{R}^q} q \otimes \nabla q g \psi_\infty dq,
\end{align*}
\]

where $\mathcal{L}g = \frac{1}{\psi_\infty} \text{div}_q [\nabla q g \psi_\infty]$.

**Definition 1.1.** Set $\varphi \in \mathcal{D}([0, T) \times \Lambda)$, $\Phi \in \mathcal{D}([0, T) \times \Lambda \times \mathbb{R}^d)$ and $\Lambda = \mathbb{T}^d$ or $\mathbb{R}^d$, then we say $(u, g)$ is weak solution for system (1.3) if the following conditions hold:

(a) $\forall T > 0$, $u \in C([0, T); L^2) \cap \mathcal{L}_t L^2(0, T; H^1)$, $\tau \in C([0, T); L^2)$, $g \in C([0, T); L^2(\mathcal{L}^2))$ and $\nabla_q g \in L^2([0, T); L^2(\mathcal{L}^2))$. Moreover, $P \in L^1([0, T); W^{2,1}) \cap L^2([0, T); W^{1,1}) \cap L^2([0, T); L^r) + C([0, T); L^2)$ with $q \in [1, \infty)$, $r = \frac{d}{d - 2}$.

(b) For any $\varphi$ and $\Phi$, it holds that

\[
\int_{\mathbb{R}^d} u \cdot (\partial_t \varphi + u \cdot \nabla \varphi) - \nabla u \cdot \nabla \varphi + \text{Pdiv} \varphi dxdt + \int_{\Lambda} u_0 \varphi_0 dx = \int_{\mathbb{R}^d} \int_{\Lambda} \tau(g) : \nabla \varphi dxdt,
\]

and

\[
\int_{\mathbb{R}^d} \int_{\Lambda} \psi_\infty g (\partial_t \Phi + u \cdot \nabla \Phi) - \nabla_q g \psi_\infty \cdot \nabla_q \Phi + \Omega \cdot q g \psi_\infty g \cdot \nabla_q \Phi dq dxdt = - \int_{\mathbb{R}^d} \int_{\Lambda} \psi_\infty g_0 \Phi_0 dq dx.
\]

Global existence of weak solutions to (1.1) with $\mu = 0$ is still open problem [24]. In this paper, we first study the global weak solutions of the co-rotation case (1.3). The proof was based on additional weighted energy estimates and the defect measure propagation method. The main difficulty is to prove the weak compactness: $\nabla q \cdot (\Omega^q gq \psi_\infty) \rightharpoonup \nabla q \cdot (\Omega q g \psi_\infty)$. Our strategy is to prove strong convergence for $g$ in $L^2(\mathcal{L}^2)$. It is worthy mentioning that there exist two main differences between (1.3) and (1.2). Firstly, the nonlinear term $\text{div}_q [\sigma(u) \cdot q g \psi_\infty]$ contains infinite elongation $q \in \mathbb{R}^d$, which requires energy estimates with the weight $\langle q \rangle = \sqrt{1 + |q|^2}$. We prove the weak compactness with high integrability $\langle q \rangle g_0 \in L^\infty(\mathcal{L}^2).$ Secondly, the linear term $\text{div}_q [\nabla q g \psi_\infty]$ causes many difficulties in the process of constructing renormalization equation, which is useful for lowering integrability. N. Masmoudi [24] indicated that the $L^2$ bound of $\tau$ is sufficient to obtain global existence of weak solution
to polymeric fluid models. The main difficulty in renormalization process of (1.3) is to prove that the sum of measures produced by $|\nabla q g^n|^2$ is positive. Using a new energy estimate, we obtain the equi-integrability of $|\nabla q g^n|^2$. Then, we prove the weak compactness with lower integrability by virtue of the defect measure propagation method. Moreover, we prove exponential decay rate for $g$ and optimal $L^2$ decay rate for velocity $u$. Firstly, we obtain initial logarithmic decay rate for $u$ in $L^2$ by additional weighted energy estimates and the Fourier splitting method. Then, by virtue of the logarithmic decay rate and the time weighted energy estimate, we improve the time decay rate to $(1 + t)^{-\frac{3}{4}}$. Finally, we establish the lower bound of $L^2$ decay rate for velocity $u$, which implies that the $L^2$ decay rate we obtained is optimal.

Our main result can be stated as follows.

**Theorem 1.2.** Let $d = 2, 3$ and $\Lambda = \mathbb{T}^d$ or $\mathbb{R}^d$. Assume that a divergence-free field $u_0 \in L^2$, $\int_{\mathbb{R}^d} g_0 \psi_\infty \, dq = 0$ and $\langle q \rangle g_0 \in L^2 \cap L^\infty(L^2)$. Then system (1.3) admits a global weak solution $(u, g)$ in the sense of Definition 1.1. Furthermore, $u \in L^\infty(\mathbb{R}^+; L^2) \cap L^2(\mathbb{R}^+; \dot{H}^1)$, $\langle q \rangle g \in C([0, \infty); L^2 \cap L^\infty(L^2))$ and $\langle q \rangle \nabla q g \in L^2(\mathbb{R}^+; L^2(L^2))$.

**Remark 1.3.** By Lemma 2.4, we have $\|\langle q \rangle g\|_{L^2(L^2)} \leq C \|\nabla_q g\|_{L^2(L^2)}$. Taking a divergence-free field $u_0 \in L^2$ and $\nabla_q g_0 \in L^2 \cap L^p(L^2)$ for some $p > 4$, global existence of system (1.3) can be proved by the same way in the proof of Theorem 1.2. The more critical cases $p \in (2, 4)$ will be studied in Theorem 1.4.

**Theorem 1.4.** Let $d = 2, 3$ and $\Lambda = \mathbb{T}^d$ or $\mathbb{R}^d$. Assume that a divergence-free field $u_0 \in L^2$, $\int_{\mathbb{R}^d} g_0 \psi_\infty \, dq = 0$ and $\nabla_q g_0 \in L^2 \cap L^p(L^2)$ for some $p > 2$. Then system (1.3) admits a global weak solution $(u, g)$ in the sense of Definition 1.1. Furthermore, $u \in L^\infty(\mathbb{R}^+; L^2) \cap L^2(\mathbb{R}^+; \dot{H}^1)$, $\langle q \rangle g \in C([0, \infty); L^2 \cap L^p(L^2))$ and $\nabla_q g \in C([0, \infty); L^2 \cap L^p(L^2))$.

**Remark 1.5.** Take $q = 0$, then the Hooke model (1.3) is reduced to the Navier-Stokes equation. Theorems 1.2 and 1.4 cover the J. Leray celebrated results about global existence of weak solutions to the Navier-Stokes equation [16, 17, 18]. When $d = 2$, the high integrability of $g$ yield results about uniqueness and further regularity [22].

**Theorem 1.6.** Let $d = 2, 3$ and $\Lambda = \mathbb{R}^d$. Let $(u, g)$ be global weak solution constructed in Theorem 1.2 or Theorem 1.4. Suppose $u_0 \in L^1(\mathbb{R}^d)$ and $g_0 \in L^1(L^2)$, then there exist constants $c$ and $C$ such that

\begin{equation}
\|u\|_{L^2} \leq C(1 + t)^{-\frac{3}{4}} \quad \text{and} \quad \|g\|_{L^2(L^2)} \leq Ce^{-2ct}.
\end{equation}

Moreover, if $\int_{\mathbb{R}^d} u_0 \, dx \neq 0$ and $\|u_0\|_{L^2(L^2)} \leq \delta$ for some constant $\delta$ small enough, then there exists a constant $c$ such that

\begin{equation}
\|u\|_{L^2} \geq c(1 + t)^{-\frac{3}{4}}.
\end{equation}

**Remark 1.7.** In [22], M. Schonbek proved that $(1 + t)^{-\frac{3}{4}}$, $d = 2, 3$ is the optimal $L^2$ decay rate for the Navier-Stokes equations with the additional low frequency condition $u_0 \in L^1$. Theorem 1.6 covers the M. Schonbek results of optimal $L^2$ decay rate of weak solutions to the Navier-Stokes equation.

The paper is organized as follows. In Section 2 we introduce some notations and give some preliminaries which will be used in the sequel. In Section 3 we derive some priori estimates for (1.3). In Section 4 we present the compactness on velocity $u$ and distribution $g$. In Section 5 we present optimal $L^2$ decay rate for global weak solutions of (1.3) by virtue of the Fourier splitting method.
2 Preliminaries

In this section we will introduce some notations and useful lemmas which will be used in the sequel.

We are only concerned with the case $\Lambda = \mathbb{R}^d$, since the periodic case is more easier. For $p \geq 1$, we denote by $L^p$ the space

$$L^p = \{ f \mid \| f \|_{L^p} = \left( \int_{\mathbb{R}^d} |\psi_{\infty} f|^p dq \right)^{\frac{1}{p}} < \infty \}. $$

We will use the notation $L^p_x(L^q)$ to denote $L^p[\mathbb{R}^d; L^q]$

$$L^p_x(L^q) = \{ f \mid \| f \|_{L^p_x(L^q)} = \left( \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |\psi_{\infty} f|^q dq \right)^{\frac{1}{p}} dx \right)^{\frac{1}{q}} < \infty \}. $$

We now introduce the Littlewood-Paley decomposition theory and and Triebel-Lizorkin spaces.

**Proposition 2.1.** [3, 4, 7] Let $C$ be the annulus $\{ \xi \in \mathbb{R}^d : \frac{3}{8} \leq |\xi| \leq \frac{3}{4} \}$. There exist radial functions $\chi$ and $\varphi$, valued in the interval $[0,1]$, belonging respectively to $D(B(0, \frac{3}{4}))$ and $D(C)$, and such that

$$\forall \xi \in \mathbb{R}^d, \chi(\xi) + \sum_{j \geq 0} \varphi(2^{-j} \xi) = 1, $$

$$\forall \xi \in \mathbb{R}^d \setminus \{0\}, \sum_{j \in \mathbb{Z}} \varphi(2^{-j} \xi) = 1, $$

$$|j - j'| \geq 2 \Rightarrow \text{Supp} \varphi(2^{-j} \cdot) \cap \text{Supp} \varphi(2^{-j'} \cdot) = \emptyset, $$

$$j \geq 1 \Rightarrow \text{Supp} \chi(\cdot) \cap \text{Supp} \varphi(2^{-j} \cdot) = \emptyset. $$

The set $\bar{C} = B(0, \frac{3}{4}) + C$ is an annulus, and we have

$$|j - j'| \geq 5 \Rightarrow 2^j C \cap 2^j \bar{C} = \emptyset. $$

Further, we have

$$\forall \xi \in \mathbb{R}^d, \frac{1}{2} \leq \chi^2(\xi) + \sum_{j \geq 0} \varphi^2(2^{-j} \xi) \leq 1, $$

$$\forall \xi \in \mathbb{R}^d \setminus \{0\}, \frac{1}{2} \leq \sum_{j \in \mathbb{Z}} \varphi^2(2^{-j} \xi) \leq 1. $$

$\mathcal{F}$ represents the Fourier transform and its inverse is denoted by $\mathcal{F}^{-1}$. Let $u$ be a tempered distribution in $\mathcal{S}'(\mathbb{R}^d)$. For all $j \in \mathbb{Z}$, define

$$\Delta_j u = 0 \text{ if } j \leq -2, \quad \Delta_{-1} u = \mathcal{F}^{-1}(\chi \mathcal{F} u), \quad \Delta_j u = \mathcal{F}^{-1}(\varphi(2^{-j} \cdot) \mathcal{F} u) \text{ if } j \geq 0, \quad S_j u = \sum_{j' < j} \Delta_{j'} u. $$

Then the Littlewood-Paley decomposition is given as follows:

$$u = \sum_{j \in \mathbb{Z}} \Delta_j u \text{ in } \mathcal{S}'(\mathbb{R}^d). $$

Let $s \in \mathbb{R}, 1 \leq p, r \leq \infty$. The nonhomogeneous Triebel-Lizorkin Space $F^s_{p,r}$ is defined by

$$F^s_{p,r} = \{ u \in \mathcal{S}' : \| u \|_{F^s_{p,r}} = \left\| \left( 2^n \Delta_j u \right)_{j \in \mathbb{Z}} \right\|_{L^p} < \infty \}. $$
In particular, \[ \mathcal{H}^1 = F_{1,2}^0 = \{ u \in S' : \| u \|_{\mathcal{H}^1} = \left\| (\Delta_j u)_j \right\|_{L^1} < \infty \} \]
and \[ BMO = F_{0,2}^0 = \{ u \in S' : \| u \|_{BMO} = \left\| (\Delta_j u)_j \right\|_{L^\infty} < \infty \} , \]
with duality \( \mathcal{H}^{1*} = BMO \). The following embedding hold \[ \mathcal{H}^1 \hookrightarrow L^1 \text{ and } L^\infty \hookrightarrow BMO. \]
If \( \{ u_i \}_{i=1}^2 \) satisfies \( \text{div} \; u_1 = 0 \) and \( \nabla \times u_2 = 0 \), then there exists positive \( C \) such that
\[ \| u_1 \cdot u_2 \|_{\mathcal{H}^1} \leq C \| u_1 \|_{L^2} \| u_2 \|_{L^2}. \]
Moreover, denote \( R_k \) be Riesz operator such that \( \hat{R}_k = \frac{\hat{u}_k}{\| \hat{u}_k \|} \). Then for \( n \geq 2 \), there exists a positive constant \( C_n \) such that for \( f \in \mathcal{H}^1(\mathbb{R}^n) \), we have
\[ C_n^{-1} \| R_k f \|_{\mathcal{H}^1} \leq \| f \|_{\mathcal{H}^1} \leq C_n (\| f \|_{L^1} + \sum_{m=1}^n \| R_m f \|_{L^1}). \]

We introduce some notations about Lorentz spaces.

**Proposition 2.2.** \([1]\) Let \( f \) be a measurable function on a measure space \((X, \mu)\) and \( 0 < p, q \leq \infty \). The distribution function of \( f \) is defined as \( d_f(\alpha) = \mu(\{ x \in X : |f(x)| > \alpha \}) \). The decreasing rearrangement of \( f \) is defined as \( f^* = \inf \{ s > 0 : d_f(s) \leq t \} \). Set
\[ \| f \|_{L^{p,q}} = \left\{ \begin{array}{ll}
( \int_0^\infty (t^p f^*(t))^{q \frac{dt}{t}} )^{\frac{1}{q}} & \text{if } q < \infty, \\
\sup_{t>0} t^p f^*(t) & \text{if } q = \infty,
\end{array} \right. \]
and Lorentz space \( L_{p,q}(X, \mu) = \{ f : \| f \|_{L^{p,q}} < \infty \} \). Note that \( L^{p,p} = L^p \). Suppose \( 0 < q < r \leq \infty \), then there exists a constant \( C_{p,q,r} \) such that
\[ \| f \|_{L^{p,r}} \leq C_{p,q,r} \| f \|_{L^{p,q}}. \]

The interpolation lemma is as follows.

**Lemma 2.3.** \([26]\) Let \( d \geq 2 \), \( p \in \left[ 2, +\infty \right) \) and \( 0 \leq s, s_1 \leq s_2 \), then there exists a constant \( C \) such that
\[ \| \Lambda^s f \|_{L^p} \leq C \| \Lambda^{s_1} f \|_{L^2}^{1-\theta} \| \Lambda^{s_2} f \|_{L^2}^\theta, \]
where \( 0 \leq \theta \leq 1 \) and \( \theta \) satisfy
\[ s + d(\frac{1}{2} - \frac{1}{p}) = s_1 (1 - \theta) + \theta s_2. \]
Note that we require that \( 0 < \theta < 1 \), \( 0 \leq s_1 \leq s \), when \( p = \infty \).

The following lemma allows us to estimate \( g \).

**Lemma 2.4.** \([33]\) Assume \( g \in L^2(\mathbb{L}^2) \) with \( \int_{\mathbb{R}^d} g \psi \infty dq = 0 \), then there exists positive \( C \) such that
\[ \| \langle q \rangle g \|_{L^2(\mathbb{L}^2)} \leq C \| \nabla q g \|_{L^2(\mathbb{L}^2)}, \] \[ \| |q|^2 g \|_{L^2(\mathbb{L}^2)} \leq C \| \langle q \rangle \nabla q g \|_{L^2(\mathbb{L}^2)}, \]
and
\[ \| |q|^2 \nabla q g \|_{L^2(\mathbb{L}^2)} \leq C \| \langle q \rangle \nabla q \nabla q g \|_{L^2(\mathbb{L}^2)} + C \| \nabla q g \|_{L^2(\mathbb{L}^2)}. \]
The following lemma is useful for showing optimal decay rate.

**Lemma 2.5.** [7] Let \( r_1 > 1, r_2 \in [0, r_1] \). Then there exists positive constant \( C \) such that

\[
\begin{align*}
\int_0^1 (1 + s)^{-r_2} e^{-(1 + t - s)} \, ds & \leq C(r_2)(1 + t)^{-r_2}, \\
\int_0^1 (1 + t - s)^{-r_2} (1 + s)^{-r_2} \, ds & \leq C(r_1, r_2)(1 + t)^{-r_2}.
\end{align*}
\]

We give a commutator lemma, which is useful in renormalization process [17].

**Lemma 2.6.** Let \( \Omega \subset \mathbb{R}^d \) be a domain. Let \( f \in \mathcal{L}^2 \) and \( \psi^\pm_\infty, \psi^{\pm}_q \in \mathcal{W}^{1,2}(\Omega; \mathbb{R}^d) \) be given functions with \( \psi^\pm_\infty \) bounded in any compact set \( K \subset \Omega \). Then

\[
\| [\nabla_q (f \psi^{\pm}_\infty)]_q^\varepsilon - \nabla_q ([f]_q^\varepsilon \psi^{\pm}_\infty) \|_{L^1(K)} \leq C(K)\|f\|\mathcal{L}_x\|\psi^\pm_\infty\|_{\mathcal{W}^{1,2}(\Omega; \mathbb{R}^d)},
\]

and

\[
\nabla_q (f \psi^{\pm}_\infty)]_q^\varepsilon - \nabla_q ([f]_q^\varepsilon \psi^{\pm}_\infty) \rightarrow 0 \text{ in } L^1(K) \text{ as } \varepsilon \rightarrow 0.
\]

Here \( \nu \mapsto [\nu]_q^\varepsilon = \theta^\varepsilon_x \ast \nu \) is smoothing operator with

\[
\theta^\varepsilon_x = \varepsilon^{-d} \theta\left(\frac{x}{\varepsilon}\right) \quad \text{and} \quad \theta = \frac{e^{\frac{1}{|x|^2-1}}}{\int_{\mathbb{R}^d} e^{\frac{1}{|x|^2-1}} \, dx} \cdot 1_{\{|x| \leq 1\}}.
\]

**Proof.** To begin with, we observe that the following quantity

\[
[\nabla_q (f \psi^{\pm}_\infty)]_q^\varepsilon - \nabla_q ([f]_q^\varepsilon \psi^{\pm}_\infty)
\]

is well defined on \( K \) whenever \( \varepsilon \) sufficiently small. Moreover, \((2.8)\) holds for any \( f \in \mathcal{C}^\infty \), which is dense in \( L^1(K) \). According to Banach-Steinhaus theorem, we ensure \((2.8)\) by showing the bound \((2.7)\).

To this end, we write

\[
[\nabla_q (f \psi^{\pm}_\infty)]_q^\varepsilon - \nabla_q ([f]_q^\varepsilon \psi^{\pm}_\infty) = [f]_q^\varepsilon \nabla_q \psi^{\pm}_\infty
\]

\[
+ \int_{\mathbb{R}^d} f(x - z) \frac{\psi^{\pm}_\infty(x) - \psi^{\pm}_\infty(x - z)}{|z|} \cdot \nabla_q \theta^\varepsilon(|z|)|z|dz.
\]

According to Minkowski’s inequality, we infer that

\[
\int_K \int_{\mathbb{R}^d} f(x - z) \frac{\psi^{\pm}_\infty(x) - \psi^{\pm}_\infty(x - z)}{|z|} \cdot \nabla_q \theta^\varepsilon(|z|)|z|dzdx \leq C(K)\|f\|\mathcal{L}_x\|\psi^\pm_\infty\|_{\mathcal{W}^{1,2}(\Omega; \mathbb{R}^d)}.
\]

We thus complete the proof of Lemma \ref{lem:commutator}.

The following proposition is a straightforward consequence of Lemma \ref{lem:commutator}.

**Proposition 2.7.** Let \( \Omega \subset \mathbb{R}^d \) with \( d \geq 2 \) be an arbitrary domain. Let

\[
g \in \mathcal{C}([0, T); L^2(\Omega, \mathcal{L}^2)) \text{ and } \nabla_q g \in L^2(0, T; L^2(\Omega, \mathcal{L}^2)),
\]

\[
u \in L^2(0, T; \mathcal{W}^{1,2}(\Omega)) \text{ and } h \in L^1(0, T; L^1(\Omega, \mathcal{L}^2)),
\]

such that

\[
\int_K \int_{\mathbb{R}^d} f(x - z) \frac{\psi^{\pm}_\infty(x) - \psi^{\pm}_\infty(x - z)}{|z|} \cdot \nabla_q \theta^\varepsilon(|z|)|z|dzdx \leq C(K)\|f\|\mathcal{L}_x\|\psi^\pm_\infty\|_{\mathcal{W}^{1,2}(\Omega; \mathbb{R}^d)}.
\]
satisfy
\begin{equation}
\partial_t g + \text{div}_x(gu) - \mathcal{L}g = h \text{ in } \mathcal{D}'((0,T) \times \Omega \times \Omega),
\end{equation}
with \( \mathcal{L}g = \frac{1}{\psi_\infty} \nabla_q \cdot (\nabla_q g \psi_\infty) \). Then we have
\begin{equation}
\partial_t B(g) + \text{div}_x(B(g)u) - \mathcal{L}B(g) + B''(g)|\nabla_q g|^2 = B'(g)h \text{ in } \mathcal{D}'((0,T) \times \Omega \times \Omega)
\end{equation}
for any \( B \in C^2[0,\infty) \).

**Proof.** Firstly, we prove (2.13) for any \( B \in \mathcal{D}(0,\infty) \). Applying the regularizing operators \( v \mapsto [v]_{x,q} \) to both sides of (2.12), we obtain
\begin{equation}
\partial_t [g]_{x,q} + \nabla_x [g]_{x,q} \cdot u - \mathcal{L}[g]_{x,q} = [h]_{x,q} + s^\varepsilon + r^\varepsilon \text{ a.e. on } O,
\end{equation}
for any bounded open set \( O \subset \bar{O} \subset (0,T) \times \mathbb{R}^d \times \mathbb{R}^d \) provided \( \varepsilon > 0 \) small enough with
\begin{equation}
\begin{align*}
s^\varepsilon &= |\mathcal{L}[g]_{x,q} - \mathcal{L}[g]_{x,q}|_{x,q} \in L^1(O), \\
r^\varepsilon &= \text{div}_x \{([gu]_{x,q} - ([g]_{x,q}u) \} \in L^1(O). 
\end{align*}
\end{equation}
It follows from the proof of Lemma 2.6 that
\begin{equation}
[f \frac{q}{\psi_\infty}]_{\partial_\varepsilon} - [f \frac{q}{\psi_\infty}]_{\partial_\varepsilon} \to 0 \text{ in } L^1(K) \text{ as } \varepsilon \to 0.
\end{equation}
Set \( F = \nabla_q g \psi_\infty \in L^2 \). Notice that
\begin{equation}
s^\varepsilon = [F \frac{q}{\psi_\infty}]_{\partial_\varepsilon} - [F \frac{q}{\psi_\infty}]_{\partial_\varepsilon} + \text{div}_q \{[F \psi_\infty^{-1}]_{\partial_\varepsilon} - [F \psi_\infty^{-1}]_{\partial_\varepsilon} + \psi_\infty^{-1} \text{div}_q \{[\nabla_q g \psi_\infty] - [\nabla_q g] \psi_\infty \} \).
\end{equation}
According to Lemma 2.6 we thus deduce that
\begin{equation}
s^\varepsilon + r^\varepsilon \to 0 \text{ in } L^1(O) \text{ as } \varepsilon \to 0.
\end{equation}
Multiplying \( B'(g) \) to both sides of (2.14), we obtain
\begin{equation}
\partial_t [g]_{x,q} + \nabla_x B([g]_x^z) \cdot u - \mathcal{L}B([g]_{x,q}) + B''([g]_{x,q})|\nabla_q g|_{x,q}^2 = ([h]_{x,q} + s^\varepsilon + r^\varepsilon)B'([g]_{x,q}),
\end{equation}
which yields (2.13) for \( B \in \mathcal{D}(0,\infty) \) by passing \( \varepsilon \to 0 \). For any given \( B \in C^2[0,\infty) \), take \( B_n \in \mathcal{D}(0,\infty) \) which is uniformly bounded and converges to \( B \) uniformly on compact sets in \( [0,\infty) \). We thus complete the proof of Proposition 2.7 by using Lebesgue’s convergence theorem.

**Lemma 2.8.** ([24])(Existence of Diperna-Lions flow) If \( u \in L^2(0,T;H^1(\mathbb{R}^d)) \) and \( \text{div } u = 0 \). Then there exists a unique flow \( X(t,t_0,x) \) such that for all \( t_0 \in (0,T) \) and for a.e.\( x \in \mathbb{R}^d, t \mapsto X(t,t_0,x) \) is absolutely continuous and satisfies
\begin{equation}
\begin{cases}
\frac{\partial X}{\partial t}(t,t_0,x) = u(t,X(t,t_0,x)), & t \in (0,T), \\
X(t_0,t_0,x) = x,
\end{cases}
\end{equation}
and for \( t,t_0 \in (0,T) \), the map \( x \mapsto X(t,t_0,x) \) is measure-preserving.
Lemma 2.9. (Mild formulation) Assume that \( u \in L^2(0,T;H^1_0(\Omega)) \) and that \( X(t,x) \) is its Diperna-Lions flow. Let \( f \in L^\infty((0,T) \times \Omega) \), \( f_0 \in L^\infty(\Omega) \) and \( h \in L^1((0,T) \times \Omega) \). The following three systems are equivalent:

\[
\begin{align*}
\partial_t f + u \cdot \nabla f & \geq h \text{ in } \mathcal{D}'((0,T) \times \Omega), \\
f(t = 0, x) & \geq f_0(x), \\
\end{align*}
\]

In this case, we also have that \( \int_\Omega u \cdot g \, dq \) [24](Mild formulation) Assume that

\[
\begin{align*}
\int_\Omega u \cdot g \, dq \\
\int_\Omega u \cdot g \, dq 
\end{align*}
\]

\( \in \mathcal{D}'((0,T) \times \Omega) \), \( f(t = 0, x) \geq f_0(x) \),

\[
\begin{align*}
\frac{d}{dt}[f(t,X(t,x))] & \geq h(t,X(t,x)) \text{ in } \mathcal{D}'((0,T)) \text{ for a.e. } x \in \Omega, \\
f(t = 0, x) & \geq f_0(x), \\
\end{align*}
\]

Moreover, let \( u \in H^2(\Omega) \) such that \( u \) is smooth solution of system [1.3] with \( u_0 \in L^2 \), \( \int_{\mathbb{R}^d} g_0 \psi_{\infty} \, dq = 0 \) and \( g_0 \in L^2(\mathbb{R}^2) \), then

\[
\begin{align*}
\|u\|_{L^\infty L^2} + \|u\|_{L^2_{\mathbb{R}^d}} & \leq \|u_0\|_{L^2} + C\|g_0\|_{L^2(H^1_0)} + \|g\|_{L^\infty L^2(L^2)} + \|\nabla g\|_{L^\infty L^2(L^2)} \leq \|g_0\|_{L^2}.
\end{align*}
\]

Moreover, if \( g_0 \in L^p(\mathbb{R}^2) \), then we obtain

\[
\|g\|_{L^\infty L^p(L^2)} \leq \|g_0\|_{L^p(L^2)},
\]

for any \( p \in [1, \infty] \).

We establish additional weighted energy estimates in the following proposition.

Proposition 3.2. Let \( d \geq 2 \) and \( p \in [2, \infty] \). Assume \( (u,g) \) is smooth solution of system [1.3] with \( \int_{\mathbb{R}^d} g_0 \psi_{\infty} \, dq = 0 \) and \( \langle q \rangle g_0 \in L^p(\mathbb{R}^2) \). There exists positive constant \( C \) such that

\[
\|\langle q \rangle g\|_{L^\infty L^\infty L^p(L^2)} \leq \|\langle q \rangle g_0\|_{L^p(L^2)} e^{Ct}.
\]

Moreover, if \( p = 2 \), then there exists positive constant \( C \) such that

\[
\|\langle q \rangle g\|_{L^2 L^2 L^2} + \int_0^t \|\langle q \rangle \nabla g\|_{L^2(L^2)} \, ds \leq C\|\langle q \rangle g_0\|_{L^2(L^2)}.
\]

Proof. Taking \( L^2 \) inner product with \( \langle q \rangle^2 g \psi_{\infty} \) to [1.3], we infer that

\[
\frac{1}{2}\frac{d}{dt}\|\langle q \rangle g\|_{L^2}^2 + \frac{1}{2} u \cdot \nabla \|\langle q \rangle g\|_{L^2}^2 + \int_{\mathbb{R}^d} L g \cdot \psi_{\infty} g \langle q \rangle^2 dq = \int_{\mathbb{R}^d} \text{div}_y (\Omega q \cdot g \psi_{\infty}) \cdot \langle q \rangle^2 g dq.
\]

Notice that

\[
\int_{\mathbb{R}^d} \text{div}_y (\Omega q \cdot g \psi_{\infty}) \cdot \langle q \rangle^2 g dq = 0.
\]
and
\[
(3.7) \quad \int_{\mathbb{R}^d} \mathcal{L}g \cdot \psi_\infty g \langle q \rangle^2 dq = \int_{\mathbb{R}^d} \langle q \rangle^2 |\nabla_q g|^2 \psi_\infty dq + \int_{\mathbb{R}^d} |qg|^2 \psi_\infty dq - d \int_{\mathbb{R}^d} g^2 \psi_\infty dq.
\]
According to (3.5)–(3.7), we obtain
\[
(3.8) \quad \frac{1}{2} \frac{d}{dt} \|\langle q \rangle g\|^2_{L^2} + \frac{1}{2} u \cdot \nabla \|\langle q \rangle g\|^2_{L^2} + \|q \nabla_q g\|^2_{L^2} + \|qg\|^2_{L^2} = d\|g\|^2_{L^2},
\]
which implies that
\[
(3.9) \quad \frac{1}{2} \frac{d}{dt} \|\langle q \rangle g\|^2_{L^2} + \frac{1}{2} u \cdot \nabla \|\langle q \rangle g\|^2_{L^2} \leq d\|g\|^2_{L^2}.
\]
Multiplying \(\|\langle q \rangle g\|_{L^2}^{p-2}\) to (3.9), we deduce that
\[
(3.10) \quad \frac{1}{p} \frac{d}{dt} \|\langle q \rangle g\|_{L^p(L^2)}^p + \frac{1}{p} u \cdot \nabla \|\langle q \rangle g\|_{L^p(L^2)}^p \leq d\|\langle q \rangle g\|_{L^p(L^2)}^p.
\]
Integrating over \(\mathbb{R}^d\) with respect to \(x\), we obtain
\[
(3.11) \quad \frac{1}{p} \frac{d}{dt} \|\langle q \rangle g\|_{L^p(L^2)}^p \leq d\|\langle q \rangle g\|_{L^p(L^2)}^p,
\]
Applying Gronwall’s inequality to (3.11), we deduce that
\[
(3.12) \quad \|\langle q \rangle g\|_{L^p(L^2)}^p \leq \|\langle q \rangle g_0\|_{L^p(L^2)}^p e^{Ct},
\]
for any \(p \in [2, \infty]\). Moreover, integrating (3.8) over \([0, t] \times \mathbb{R}^d\) and using (3.1), we obtain
\[
(3.13) \quad \|\langle q \rangle g\|^2_{L^p(L^2)} + \int_0^t \|\nabla_q g\|^2_{L^2(L^2)} ds \leq C\|\langle q \rangle g_0\|^2_{L^2(L^2)}.
\]
We thus complete the proof of Proposition 3.2.

The new energy estimates in the following proposition play a key role in the proof of the equi-integrability of \(\|\nabla_q g\|_{L^2(L^2)}^2\).

**Proposition 3.3.** Let \(d \geq 2\). Assume \((u, g)\) is smooth solution of system with initial data \(\nabla_q g_0 \in L^p(L^2)\) for some \(p \in [2, \infty]\). Let \(\int_{\mathbb{R}^2} \psi_\infty dq = 0\). There exists positive constant \(C\) such that
\[
(3.14) \quad \|\nabla_q g\|_{L^p(L^2)} \leq \|\nabla_q g_0\|_{L^p(L^2)}.
\]
Moreover, if \(p = 2\), then there exists positive constant \(C\) such that
\[
(3.15) \quad \|\nabla_q g\|^2_{L^2(L^2)} + \int_0^t \|\nabla_q g\|^2_{L^2(L^2)} ds \leq \|\nabla_q g_0\|^2_{L^2(L^2)}.
\]

**Proof.** According to the antisymmetry of \(\Omega\), we obtain
\[
(3.16) \quad \int_{\mathbb{R}^2} \nabla_q \left( \frac{1}{\psi_\infty} \nabla_q \cdot (\Omega qg \psi_\infty) \right) \nabla_q g \psi_\infty dq
= \int_{\mathbb{R}^2} \nabla_q (\Omega_{ik} q_k \nabla_q g - \Omega_{ik} q_k q_g) \nabla_q g \psi_\infty dq
\]

10
\[
= \int_{\mathbb{R}^2} \left( \Omega^{ik} \delta^j \nabla^i g + \Omega^{ik} q_k \nabla^i g - \Omega^{ik} (\delta^j q_i + \delta^j q_k) g - \Omega^{ik} q_k q_i \nabla^i g \right) \nabla^j g \psi_\infty dq = 0.
\]

By virtue of Lemma 2.4 we deduce that

\begin{equation}
(3.17)
\int_{\mathbb{R}^2} \nabla_q \left( \frac{1}{\psi_\infty} \nabla_q \cdot (\nabla_q g \psi_\infty) \right) \nabla_q g \psi_\infty dq
= \int_{\mathbb{R}^2} \nabla_q \cdot (\nabla_q \nabla_q g \psi_\infty) - \nabla_q g \nabla_q g \psi_\infty dq
= \int_{\mathbb{R}^2} \nabla_q \cdot (\nabla_q \nabla_q g \psi_\infty) dq - \|\nabla_q g\|_{L^2}^2
= -\|\nabla_q g\|_{L^2}^2 - \|\nabla_q g\|_{L^2}^2.
\end{equation}

From \(1.3\), \(3.16\) and \(3.17\), we infer that

\begin{equation}
(3.18)
\frac{1}{2} \frac{d}{dt} \|\nabla_q g\|_{L^2}^2 + \frac{1}{2} u \cdot \nabla\|\nabla_q g\|_{L^2}^2 + \|\nabla_q^2 g\|_{L^2}^2 + \|\nabla_q g\|_{L^2}^2 = 0.
\end{equation}

Similarly, we deduce that for any \(p \geq 2\),

\begin{equation}
(3.19)
\frac{1}{p} \frac{d}{dt} \|\nabla_q g\|_{L^p(L^2)}^p \leq 0,
\end{equation}

which implies that \(\|\nabla_q g\|_{L^p(L^2)} \leq \|\nabla_q g_0\|_{L^p(L^2)}\). Moreover, integrating \(3.18\) over \([0,t] \times \mathbb{R}^d\), we obtain

\begin{equation}
(3.20)
\|\nabla_q g\|_{L^2(L^2)}^2 + \int_0^t \|\nabla_q^2 g\|_{L^2(L^2)}^2 ds \leq \|\nabla_q g_0\|_{L^2(L^2)}^2.
\end{equation}

We thus complete the proof of Proposition 3.3. \(\square\)

**Lemma 3.4.** \(7\) Let \(d = 3\). Assume \(u \in L_\infty^\infty(L^2) \cap L_\infty^3(H^1)\), then

\begin{equation}
(3.21)
\|u\|_{L^2(L^\beta)} < \infty,
\end{equation}

with \(\alpha \in [2, \infty)\) and \(\beta = \frac{6\alpha}{3\alpha - 4}\). Then we obtain

\begin{equation}
(3.22)
\|u \otimes u\|_{L^q(L^r)}; \|u \nabla u\|_{L^\gamma L^\delta} < \infty,
\end{equation}

with \(q \in [1, \infty), r = \frac{3q}{3q - 2}\) and \(\gamma \in [1, 2], \delta = \frac{3\gamma}{4\gamma - 2}\). Moreover, we have

\begin{equation}
(3.23)
\|u \nabla u\|_{L^1_t L^\frac{5}{2}_x} < \infty.
\end{equation}

The following property is crucial to obtain the weak compactness of the velocity \(u\).

**Proposition 3.5.** \(7\) Let \(d = 3\). Assume \(\text{div}\ u = 0\) and \(u \in L_\infty^\infty(L^2) \cap L_\infty^3(H^1)\), then

\begin{equation}
(3.24)
\Delta^{-1} \text{div}(u \cdot \nabla u) \in L_1^1 W^{2,1} \cap L_1^2 W^{1,1} \cap L_1^3 L^r,
\end{equation}

with \(q \in [1, \infty)\) and \(r = \frac{3q}{3q - 2}\).
Proof. Notice that $\partial_j(\partial_i u^j) = 0$ and $\nabla \times \nabla u^i = 0$. According to Proposition 2.1, we infer that
\begin{equation}
\|\nabla^2 \Delta^{-1} \partial_i u^j \partial_j u^i \|_{L^2_T H^1} \leq C \|\partial_i u^j \partial_j u^i \|_{L^2_T H^1} \leq C \|u\|_{L^2_T W^{2,1}},
\end{equation}
which implies $\Delta^{-1} \text{div}(u \cdot \nabla u) \in L^1_T \dot{W}^{2,1}$. Similarly, we obtain
\begin{equation}
\|\Delta^{-1} \text{div} \nabla (u \tau) \|_{L^2_T H^1} \leq C \|u\|_{L^2_T L^2} \|\nabla u\|_{L^2_T L^2},
\end{equation}
which implies $\Delta^{-1} \text{div}(u \cdot \nabla u) \in L^2_T \dot{W}^{1,1}$. Moreover, applying Lemma 3.4 we deduce that
\begin{equation}
\|\Delta^{-1} \text{div} \text{div}(u \otimes u)\|_{L^2_T L^r} \leq C \|u \otimes u\|_{L^2_T L^r},
\end{equation}
which implies that $\Delta^{-1} \text{div}(u \cdot \nabla u) \in L^2_T L^r$.
\end{proof}

4 Compactness

4.1. Compactness on the velocity $u$

In this section, we only consider the case $\Lambda = \mathbb{R}^d$ and $d = 3$, since the other case is more easier [20].

We first split $u$ into $u_1 + u_2 + u_3$ where $u_1, u_2$ and $u_3$ solve respectively
\begin{equation}
\begin{cases}
\partial_t u_1 - \Delta u_1 + \nabla P_1 = -\text{div} (u \otimes u), \\
\text{div } u_1 = 0, \quad u_1|_{t=0} = 0,
\end{cases}
\end{equation}
and
\begin{equation}
\begin{cases}
\partial_t u_2 - \Delta u_2 + \nabla P_2 = 0, \\
\text{div } u_2 = 0, \quad u_2|_{t=0} = u_0,
\end{cases}
\end{equation}
where $P_2$ is a given harmonic function and
\begin{equation}
\begin{cases}
\partial_t u_3 - \Delta u_3 + \nabla P_3 = \text{div } \tau(g), \\
\text{div } u_3 = 0, \quad u_3|_{t=0} = 0.
\end{cases}
\end{equation}

Let’s recall the following compactness property.

Lemma 4.1. [17] [18] Assume $u^n \in C(L^2_T \cap H^1_T \cap L^q_T (\dot{H}^1))$ such that for any $K \subset \mathbb{R}^3$, $u^n \to u$ in $L^q_T L^p(K)$ with $q \in [2, \infty]$ and $p < \frac{6q}{3q-2}$. Then there exists $u \in L^2_T (L^2) \cap L^2_T (H^1)$ such that $u^n \to u$ in $L^p_T L^q(K)$ with $K \subset \subseteq \mathbb{R}^3$. For more details, one can refer to [17] [18].

Proposition 4.2. Assume $u^n_1, u^n_2$ and $u^n_3$ solve (4.1), (4.2) and (4.3) respectively with $u^n \in L^q_T (L^2) \cap L^q_T (H^1)$ and $g^n \in L^q_T (L^2)$. Then there exist $u_1, u_2$ and $u_3$ such that
\begin{equation}
u^n_i \to u_i \in L^q_T L^p(K), \quad i = 1, 3 \quad \text{and} \quad u^n_2 \to u_2 \in L^q_T (L^2) \cap L^q_T (\dot{H}^1),
\end{equation}
with $K \subset \subseteq \mathbb{R}^3, p \in [2, \infty)$ and $q < \frac{6q}{3q-2}$. Moreover,
\begin{equation}
\nabla u^n_i \to \nabla u_i \in L^{r_0}_T L^{q_0} \cap L^1_T L^r
\end{equation}
with $p_0 < 2$ and $r_0 < 3$. 

12
Proof. Using standard $L^2$ energy estimate to (1.13) and $g^n \in L_7^\infty(L^2)$, we obtain

\begin{equation}
\tag{4.6}
u_n^3 \in \tilde{W}^{1,1}(L_7^2) \cap L_7^\infty(L^2) \cap L_7^2(H^1) \text{ and } P^n_3 = \Delta^{-1} \text{div} \, \tau^n \in L_7^\infty L^2.
\end{equation}

we infer from compact embedding theorem that there exists $u_3$ such that $u_n^3 \to u_3 \in C(0,T;W^{-r,2}(K))$ for any positive $\varepsilon$. By virtue of interpolation inequality, we deduce that $u_n^3 \to u_3 \in L^q L^p(K)$ with $K \subset \subset \mathbb{R}^d, p \in [2, \infty)$ and $q < \frac{6q}{3q-4}$. According to (4.2) and Duhamel’s principle, we infer that

\begin{equation}
\tag{4.7}
\int_0^t e^{(t-s)\Delta} \nabla P^n_2 \, ds.
\end{equation}

Taking harmonic functions $P^n_2 = [P_2]^{\frac{1}{2}}$ such that $P^n_2 \to P_2 \in H^2$, we deduce that there exists $u_2 \in L_7^\infty(L^2) \cap L_7^2(H^1)$ such that $u^n_2 \to u_2 \in L_7^\infty(L^2) \cap L_7^2(H^1) \to L_7^p L^p(K)$. Notice that $u^n_2 = u^n - (u^n_0 + u^n_3)$, there exists $u_1 \in L_7^\infty(L^2) \cap L_7^2(H^1)$ such that $u_1 \to u_1 \in L_7^p L^p(K)$. Moreover, according to (4.1) and Duhamel’s principle, we obtain

\begin{equation}
\tag{4.8}
u^n_1 = \int_0^t e^{(t-s)\Delta} \text{div} \, (u^n \otimes u^n) \, ds.
\end{equation}

According to Lemma 3.4 we have $\text{div} \, (u^n \otimes u^n) \in L_7^\infty H^1 \cap L_7^2 L^{\frac{3}{2}} \to L_7^k L^r$ with $k \in (1,2)$ and $r = \frac{\beta}{4k-\beta}$, we deduce that $u^n_1 \in L_7^k W^{2,r} \to L_7^k L^{\infty}$ with $r_0 < 6$. By virtue of compact embedding theorem and $u_1 \in L_7^2(H^1)$, we have $\nabla u^n_1 \to \nabla u_1 \in L_7^p L^{p_0} \cap L_7^2 L^{r_0}$ with $p_0 < 2$ and $r_0 < 3$. We thus complete the proof of Proposition 4.2.

Thus the compactness of $u_i, \ i = 1, 2, 3$ implies that $(u, g)$ satisfies system (1.13) in the sense of Definition 1.1.

4.2. Compactness on the polymeric distribution $g$

According to the compactness of $u$ we have discussed in Proposition 4.2 and the bound of $g$ in $L^2(L^2)$, we infer that

\[ \text{div} \, (u^n g^n) \to \text{div} \, (ug) \in \mathcal{D}'([0,T] \times \mathbb{R}^3 \times \mathbb{R}^3). \]

The main difficulty is to prove that the following weak compactness holds:

\begin{equation}
\tag{4.9}
\nabla_q \cdot (\Omega^n g^n \psi_\infty) \to \nabla_q \cdot (\Omega g \psi_\infty) \in \mathcal{D}'([0,T] \times \mathbb{R}^3 \times \mathbb{R}^3).
\end{equation}

According to Proposition 4.2 we infer that

\[ \nabla_q \cdot (\nabla(u^n_1 + u^n_2)g^n \psi_\infty) \to \nabla_q \cdot (\nabla(u_1 + u_2)g \psi_\infty) \in \mathcal{D}'([0,T] \times \mathbb{R}^3 \times \mathbb{R}^3). \]

Therefore, we are now concerned with $u_3$. We begin with the following equi-integrability for $\{g^n\}_{n \in N}$.

Proposition 4.3. Assume that $\{\nabla_q g^n\}_{n \in N}$ is bounded in $L_7^2 L^2(L^2)$ and $\{g^n\}_{n \in N}$ is bounded in $L_7^\infty L^2 \cap L^p(L^2)$ for some $p > 2$. Then $\{\|g^n\|^2\}_{n \in N}$ is equi-integrable in $L_7^1 L^1(\psi_{\infty} dq)$.

Proof. Applying Lemma 2.3 we have

\begin{equation}
\tag{4.10}\|g^n\|_{L_7^1 L^m(L^m)} \leq C\|g^n\|_{L_7^\infty L^p(L^2)}^\frac{1}{2-y} \|\nabla_q g^n\|_{L_7^2 L^2(L^2)}^\frac{1}{2-y},
\end{equation}

13
for $r = \frac{4m}{3(m - 2)}$ and $m = \frac{19p - 12}{p - 2} > 2$. Then we deduce that

$$\int_0^t \int_{\mathbb{R}^d} |g^n|^2 \psi_\infty 1(|g^n| \psi_\infty \geq M) dq dx ds \leq C \|g^n\|^2_{L^\infty L^m(\mathbb{L}^m)} \|g^n\|^2_{L^\infty L^2(\mathbb{L}^2)} M^{-(1 - \frac{2}{p})},$$

which implies that

$$\int_0^t \int_{\mathbb{R}^d} |g^n|^2 \psi_\infty 1(|g^n| \psi_\infty \geq M) dq dx ds \to 0 \text{ as } M \to \infty.$$

We thus complete the proof of Proposition 4.3.

**Proof of Theorem 1.2**: According to Propositions 3.1, 3.2 and 13, we introduce the following defect measures:

$$\begin{align*}
\begin{cases}
|\nabla (u_3^n - u_3)|^2 \to \mu \in \mathcal{M}, \\
|g^n - g|^2 \to \eta \in L^\infty L^1 (L^1 \cap L^\infty (\mathbb{L}^1)), \\
|\tau^n - \tau|^2 \to \alpha \in L^\infty (L^1 \cap L^\infty (\mathbb{L}^1)) \text{ with } \alpha \leq \int_{\mathbb{R}^d} \eta \psi_\infty dq, \\
\psi_\infty \nabla_q (g^n - g)^2 \to \kappa \in \mathcal{M}, \\
(\langle q \rangle)^{n \tau^n u_3^n} \to (\langle q \rangle)^{g \nabla u_3 + \beta} \in L^{\frac{2}{2}} (L^1 (\mathbb{L}^2)) \text{ with } |\beta| \leq \langle q \rangle \sqrt{\mu} \sqrt{\eta},
\end{cases}
\end{align*}$$

where $\mathcal{M}$ is space of bounded measure on $\mathbb{R}^3$. Note that all measure inequalities hold in the sense of almost everywhere. For simplify, we omit the notion a.e. here. Recalling that $u_3^n$ and $u_3$ solve the equation (4.3), we obtain

$$\begin{align*}
\begin{cases}
\frac{1}{2} \partial_t |u_3^n|^2 - \Delta |u_3^n|^2 + |\nabla u_3^n|^2 + \text{div} (u_3^n P^n) = \text{div} (u_3^n \tau^n) - Tr(\tau^n (\nabla u_3^n)^t), \\
\frac{1}{2} \partial_t |u_3|^2 - \Delta |u_3|^2 + |\nabla u_3|^2 + \text{div} (u_3 P_3) = \text{div} (u_3 \tau) - Tr(\tau (\nabla u_3)^t)
\end{cases}
\end{align*}$$

Passing to the limit in (4.14) and using the convergence properties already shown in (4.13), we obtain

$$\begin{align*}
\frac{1}{2} \partial_t |u_3|^2 - \Delta |u_3|^2 + |\nabla u_3|^2 + \mu + \text{div} (u_3 P_3) = \text{div} (u_3 \tau) - Tr(\tau (\nabla u_3)^t)
\end{align*}$$

Thus (4.15) minus (4.14) leads to

$$\mu = - \int_{\mathbb{R}^d} \beta_j \frac{q_i \nabla_j \mu}{\langle q \rangle} \psi_\infty dq,$$

which implies that $\mu \in L^\frac{2}{2} (L^1)$. According to (4.13), we obtain

$$\int_{\mathbb{R}^d} \beta_j \frac{q_i \nabla_j \mu}{\langle q \rangle} \psi_\infty dq \leq C \sqrt{\mu} \sqrt{\alpha},$$

which implies $\mu \leq C \alpha \leq C \int_{\mathbb{R}^d} \eta \psi_\infty dq$. Therefore, we infer that

$$\left( \int_{\mathbb{R}^d} \frac{|\beta|^2}{\langle q \rangle^2} \psi_\infty dq \right)^{\frac{1}{2}} \leq C \sqrt{\mu} \left( \int_{\mathbb{R}^d} \eta \psi_\infty dq \right)^{\frac{1}{2}} \leq C \int_{\mathbb{R}^d} \eta \psi_\infty dq.$$
Passing to the limit in (4.19) and using (4.13), we obtain

(4.20) \[ \partial_t \langle \|g\|_{L^2}^2 + \int \eta \psi_\infty dq \rangle + \text{div} \left( u \int \eta \psi_\infty dq \right) + 2\|\nabla_q g\|_{L^2}^2 + 2\|\kappa\|_{M(q)}^2 = 0. \]

According to (1.3), (4.13) and Proposition 2.7, we deduce that

(4.21) \[ \partial_t \|g\|_{L^2}^2 + \text{div} \left( u \|g\|_{L^2}^2 \right) + 2\|\nabla_q g\|_{L^2}^2 = \int \left( \beta_{ij} - \beta_{ji} \right) \frac{q_j}{q} \nabla_i g \psi_\infty dq. \]

Using (4.13), (4.20), (4.21) and Propositions 3.1 and 3.2, we obtain

(4.22) \[ \partial_t \int \eta \psi_\infty dq + \text{div}(u \int \eta \psi_\infty dq) + 2\|\kappa\|_{M(q)}^2 = \int \left( \beta_{ij} - \beta_{ji} \right) \frac{q_j}{q} \nabla_i g \psi_\infty dq \in L^1_T L^1_x. \]

According to Lemmas 2.8 and 2.9, we have

(4.23) \[ \partial_t \int \eta \psi_\infty dq + \text{div}(u \int \eta \psi_\infty dq) \leq C \|\langle q \rangle \nabla_q g\|_{L^2} \int \eta \psi_\infty dq \text{ a.e.}, \]

and thus

(4.24) \[ \int \eta \psi_\infty dq(X(t,x)) \leq \int \eta_0 \psi_\infty dq \cdot e^{C \int_0^t \|\langle q \rangle \nabla_q g\|_{L^2} ds} \text{ a.e.}, \]

where \( t \geq 0 \) and \( X \) is the unique a.e. flow such that

(4.25) \[ \dot{X} = u(t,X), \quad X(0,x) = x. \]

For each \( t \in [0,T] \), according to Proposition 3.2 and Minkowski’s inequality, we deduce that

(4.26) \[ \| \int_0^t \langle q \rangle \nabla_q g \|_{L^2 ds} \|_{L^2} \leq \int_0^t \|\langle q \rangle \nabla_q g\|_{L^2} ds < \infty. \]

Then \( e^{C \int_0^t \|\langle q \rangle \nabla_q g\|_{L^2} ds} \to 0 \) implies \( \int \eta \psi_\infty dq(X(t,x)) = 0 \) a.e. with \( \int \eta_0 \psi_\infty dq = 0 \). Using the invariance of Lebesgue measure, we deduce that \( \eta = 0 \) a.e. in \( x \) for all \( t \geq 0 \) and thus complete the proof of Theorem 1.2.

In the process of constructing renormalization equation, the main difficulty is to prove that the sum of measures produced by \( \|\nabla_q g^n\|^2 \) is positive with lower integrability of \( \nabla_q g^n \). Thanks to the new energy estimates in Proposition 4.3, we obtain the equi-integrability of \( |g^n|^2 \) and \( |\nabla_q g^n|^2 \).

**Proposition 4.4.** Assume that \( \{\nabla_q g^n\}_{n \in N} \) is bounded in \( L^p_T L^2 \cap L^p(L^2) \) for some \( p > 2 \) and \( \{\nabla_q g^n\}_{n \in N} \) is bounded in \( L^2_T L^2(L^2) \), then \( \{\nabla_q g^n\}_{n \in N} \) is equi-integrable in \( L^2_T L^2(L^2) \).

**Proof.** Taking \( m = \frac{10p - 12}{3p - 2} \) with \( m \in (2, \frac{10}{3}) \), we deduce that

(4.27) \[ \|\nabla_q g^n\|_{L^p_T L^2(L^m(L^m))} \leq C \left( \int_0^T \left( \int \|\nabla_q g^n\|_{L^2}^{3 - \frac{m}{2}} \|\nabla_q g^n\|_{L^2}^{-3} dx \right) \frac{4m}{(m - 2)} dt \right) ^{\frac{1}{2(1 - m)}} \]

\[ \leq C \|\nabla_q g^n\|_{L^2_T L^2(L^2)} \|\nabla_q g^n\|_{L^2_T L^2(L^m)}, \]

which implies that \( \nabla_q g^n \in L^{m_1}_T L^m(L^m) \) with \( m_1 = \frac{4m}{3(1 - m)} > 2 \). Then there exist positive constant \( C_T \) such that

(4.28) \[ \int_{\{\nabla_q g^n \geq M\}} |\nabla_q g^n|^2 \psi_\infty dq dx \leq CM^{-1 + \frac{m}{2}} \|\nabla_q g^n\|_{L^2(L^2)}^2 \|\nabla q g^n\|_{L^m(L^m)}^2. \]
which implies that
\begin{equation}
(4.29) \quad \int_0^T \iint_{\{|\nabla \varphi g^n|^2 \psi \geq M\}} |\nabla \varphi g^n|^2 \psi \mathrm{d}q \mathrm{d}x \mathrm{d}t \to 0 \text{ as } M \to \infty.
\end{equation}
We thus complete the proof of Proposition \[4.4\]. \hfill \square

**Lemma 4.5.** Under the conditions in Proposition \[4.4\] and suppose that
\begin{equation}
(4.30) \quad \|\nabla \varphi g^n\|_{L^2}^2 \to \|\nabla \varphi g\|_{L^2}^2 + \tilde{\kappa},
\end{equation}
and
\begin{equation}
(4.31) \quad \frac{\|\nabla \varphi g^n\|_{L^2}^2}{(1 + \delta \|\varphi g^n\|_{L^2}^2)^2} \to \frac{\|\nabla \varphi g\|_{L^2}^2}{(1 + \delta \|\varphi g\|_{L^2}^2)^2} + \tilde{\kappa}_{\delta},
\end{equation}
with $\tilde{\kappa}, \tilde{\kappa}_{\delta} \in L^\infty \cap L^\infty_{\tilde{\sigma}}$ for some $p > 2$, then
\begin{equation}
(4.32) \quad \tilde{\kappa}_{\delta} \to \tilde{\kappa} \in L^\infty_{\tilde{\sigma}} \text{ as } \delta \to 0.
\end{equation}

**Proof.** It's sufficient to prove that $|\frac{\|\nabla \varphi g^n\|_{L^2}^2}{(1 + \delta \|\varphi g^n\|_{L^2}^2)^2} - \|\nabla \varphi g^n\|_{L^2}^2| \to 0 \in L^\infty \cap L^\infty_{\tilde{\sigma}}$ as $\delta \to 0$. Firstly, we deduce that
\begin{equation}
(4.33) \quad \frac{\|\nabla \varphi g^n\|_{L^2}^2}{(1 + \delta \|\varphi g^n\|_{L^2}^2)^2} - \frac{\|\nabla \varphi g^n\|_{L^2}^2}{(1 + \delta \|\varphi g^n\|_{L^2}^2)^2} = \frac{\|\nabla \varphi g^n\|_{L^2}^2}{(1 + \delta \|\varphi g^n\|_{L^2}^2)^2} (\delta^2 \|\varphi g^n\|_{L^2}^2 + 2\delta \|\varphi g^n\|_{L^2}^2)
\leq \|\nabla \varphi g^n\|_{L^2}^2 (\|\nabla \varphi g^n\|_{L^2}^2 > M) + 3M^2 \delta \|\varphi g^n\|_{L^2}^2.
\end{equation}
Applying Hölder inequality, then we have
\begin{equation}
(4.34) \quad \int_{R^3} \|\nabla \varphi g^n\|_{L^2}^2 \|\nabla \varphi g^n\|_{L^2}^2 \leq \|\nabla \varphi g^n\|_{L^p(L^2)} \|\nabla \varphi g^n\|_{L^2(L^2)}^{2-p} M^{2p-1}.
\end{equation}
This together with \[4.33\] implies that
\begin{equation}
(4.35) \quad \sup_{t \in [0, T]} \int_{R^3} \|\nabla \varphi g^n\|_{L^2}^2 \|\nabla \varphi g^n\|_{L^2}^2 \to 0 \text{ as } \delta \to 0.
\end{equation}
We thus complete the proof of Lemma \[4.5\]. \hfill \square

**Proof of Theorem 1.4:**
According to Propositions \[3.1, 3.2, 3.3, 3.8, 3.8\] and \[4.4\] we obtain the following defect measures:
\begin{equation}
(4.36) \quad \begin{cases}
|\nabla (u^n_3 - u_3)|^2 \to \mu \in \mathcal{M}, \\
|g^n - g|^2 \to \eta \in L^\infty_{\tilde{\sigma}} (L^1 \cap L^2_{\tilde{\sigma}} (L^1)), \\
|\tau^n - \tau|^2 \to \alpha \in L^\infty_{\tilde{\sigma}} (L^1 \cap L^2_{\tilde{\sigma}} (L^1)) \text{ with } \alpha \leq \int_{R^d} \eta \psi \mathrm{d}q, \\
|\nabla_q (g^n - g)|^2 \to \kappa \in L_{\tilde{T}^{m=25}}^\infty (L^1 \cap L^2_{\tilde{\sigma}} (L^2)) \text{ with } m = \frac{10p-12}{3p-2}, \\
(g^n)^T \nabla u^n_3 \to \langle g \rangle^n \nabla u_3 + \beta \in L^\infty_{\tilde{T}} (L^1 (L^2)) \text{ with } |\beta| \leq \sqrt{\mu \psi}.
\end{cases}
\end{equation}
Extracting subsequences if necessary, for each $\delta \in (0, 1)$, we assume that
\begin{equation}
(4.37) \quad \frac{\|g^n\|_{L^2}^2}{1 + \delta \|g^n\|_{L^2}^2} \to \frac{\|g\|_{L^2}^2}{1 + \delta \|g\|_{L^2}^2} + \tilde{\eta}_\delta, \quad 0 \leq \tilde{\eta}_\delta \leq \frac{1}{\delta},
\end{equation}
δ

Passing to the limit in (4.42) and using (4.37), (4.38), we deduce that

\[
\lim_{\delta \to 0} \frac{\|\nabla g^n\|^2}{\|g^n\|^2} + \frac{\|\nabla g^n\|^2}{\|g^n\|^2} = 0.
\]

Thus (4.43) minus (4.44) leads to

\[
\partial_t N + \frac{1}{N} \int (|\nabla g|^2 \psi) dq = 0.
\]

According to Proposition 2.3, we obtain

\[
\partial_t \|g^n\|^2 + \text{div} (u_3\|g^n\|^2) + 2\|\nabla g^n\|^2 = 0.
\]

Denote that \(\tilde{\eta} = \int \eta \psi dq\) and \(\tilde{\kappa} = \int \kappa \psi dq\). Passing to the limit in (4.40) and using the convergence properties in (4.39) and (4.30), we have

\[
\partial_t N + \frac{1}{N} \int (|\nabla g|^2 \psi dq) = 0.
\]

Multiplying \((1 + \delta\|g^n\|^2)^{-2}\) to (4.40), we infer that

\[
\partial_t \left[ \frac{\|g^n\|^2}{\|g^n\|^2} + \tilde{\eta} \right] + \text{div} \left[ u_3 \left( \frac{\|g^n\|^2}{\|g^n\|^2} + \tilde{\eta} \right) + \|\nabla g^n\|^2 \right] + 2\|\nabla g^n\|^2 = 0.
\]

Passing to the limit in (4.42) and using (4.37), (4.38), we deduce that

\[
\partial_t \left[ \frac{\|g^n\|^2}{\|g^n\|^2} + \tilde{\eta} \right] + \text{div} \left[ u_3 \left( \frac{\|g^n\|^2}{\|g^n\|^2} + \tilde{\eta} \right) + \|\nabla g^n\|^2 \right] + 2\tilde{\kappa} = 0.
\]

According to Proposition 2.3, we deduce that

\[
\partial_t \left[ \frac{\|g^n\|^2}{\|g^n\|^2} + \tilde{\eta} \right] + \text{div} \left[ u_3 \left( \frac{\|g^n\|^2}{\|g^n\|^2} + \tilde{\eta} \right) + \|\nabla g^n\|^2 \right] = \frac{1}{(1 + \delta\|g^n\|^2)} \int (\beta_{ji} - \beta_{ij}) \frac{q_j}{(q)} \nabla q g \psi dq.
\]

Thus (4.43) minus (4.44) leads to

\[
\partial_t \tilde{\eta} + \text{div} (u \tilde{\eta}) + 2\tilde{\kappa} = \frac{1}{(1 + \delta\|g^n\|^2)} \int (\beta_{ji} - \beta_{ij}) \frac{q_j}{(q)} \nabla q g \psi dq.
\]

Combining (4.41) and (4.45), we deduce that

\[
\partial_t \frac{\tilde{\eta}^2}{N^2} + \text{div} \left( \frac{\tilde{\eta}^2}{N^2} \right) = \frac{1}{N^2} \int (\beta_{ji} - \beta_{ij}) \frac{q_j}{(q)} \nabla q g \psi dq.
\]

According to Lemma 4.3, passing \(\delta\) to 0 leads to

\[
\partial_t \frac{\tilde{\eta}^2}{N^2} + \text{div} \left( \frac{\tilde{\eta}^2}{N^2} \right) = \frac{1}{N^2} \int (\beta_{ji} - \beta_{ij}) \frac{q_j}{(q)} \nabla q g \psi dq.
\]
Notice that \( \frac{1}{N^2} - \frac{\eta}{N^2} \geq 0 \) and \( 0 \leq \frac{1}{N} \leq 1 \). According to (4.18) and Proposition 3.2, we infer that
\[
\partial_t \frac{\eta}{N^2} + \text{div} \left( u \frac{\eta}{N^2} \right) \leq C \| \langle q \rangle \nabla q g \|_{L^2} \frac{\eta}{N^2} \in L^1_T L^1_x.
\]
According to Lemmas 2.8 and 2.9 with \( \frac{\eta}{N^2} \in L^\infty \), we infer that for any \( t \geq 0 \),
\[
\left( \frac{\eta}{N^2} \right) (X(t,x)) \leq \left( \frac{\eta_0}{N^2} \right) \parallel g_0 \parallel^2_{L^2} + \eta_0 + 1 e^{-C \int_0^t \| \langle q \rangle \nabla q g_0 \|_{L^2} ds} \text{ a.e. ,}
\]
where \( X \) is the unique a.e. flow such that
\[
\dot{X} = u(t, X), \quad X(0, x) = x.
\]
For each \( t \in [0, T] \), according to Proposition 3.2 and Minkowski’s inequality, we deduce that
\[
\parallel \int_0^t \| \langle q \rangle \nabla q g \|_{L^2} ds \|_{L^2} \leq \int_0^t \| \langle q \rangle \nabla q g \|_{L^2(L^2)} ds < \infty,
\]
which implies that \( e^{C \int_0^t \| \langle q \rangle \nabla q g \|_{L^2} ds} < \infty \) and thus \( \frac{\eta}{N^2} (X(t,x)) = 0 \) a.e. with \( \eta_0 = 0 \). Using the invariance of Lebesgue measure, we deduce that \( \eta = 0 \) a.e. in \( x \) for all \( t \geq 0 \) and thus complete the proof of Theorem 1.4.

\[\square\]

\textbf{Remark 4.6.} In Theorems 1.2 and 1.4, we obtain global existence of (1.3) with additional energy estimates. Global existence of (1.3) with standard energy estimates in 3.1 is an interesting problem. However, the technique in this paper fail to solve this problem. We are going to study about this problem in the future.

\section{Optimal decay rate}

\textbf{Proof of Theorem 1.6 :}

By the density argument, we assume that \( (u, g) \) is the smooth solution. We first prove the exponential decay rate of \( g \) in \( L^2(L^2) \). Taking \( L^2(L^2) \) inner product with \( g \) to (1.3)_2, we infer that
\[
\frac{d}{dt} \| g \|_{L^2(L^2)}^2 + 2 \| \nabla_q g \|_{L^2(L^2)}^2 = 0.
\]
According to Lemma 2.4, we have
\[
\| g \|_{L^2(L^2)}^2 \leq \| \nabla_q g \|_{L^2(L^2)}^2,
\]
which implies that
\[
\| g \|_{L^2(L^2)}^2 \leq \| g_0 \|_{L^2(L^2)}^2 e^{-2ct}.
\]
Applying Hölder inequality, we obtain
\[
\| \tau \|_{L^2}^2 \leq C \| g \|_{L^2(L^2)}^2 \leq C e^{-2ct}.
\]
Then we prove the optimal \( L^2 \) decay rate for velocity \( u \). The proof is divided into three steps. To start with, we get initial time decay rate \( \ln^{-l}(e + t) \) for \( u \) in \( L^2 \) for any \( t \in N^+ \) by the Fourier splitting
method. Then, by virtue of the time weighted energy estimate and the logarithmic decay rate, we improve the time decay rate to \((1 + t)^{-\frac{3}{2}}\). Finally, we establish the lower bound of \(L^2\) decay rate for velocity \(u\), which implies that \(L^2\) decay rate we obtain is optimal.

**Step 1:** Taking \(L^2\) energy estimate to \((5.8)\), we deduce that
\[
\frac{d}{dt} \|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 \leq C \|\tau\|_{L^2}^2.
\]
Define \(S_0(t) = \{\xi : |\xi|^2 \leq C_d \frac{|f'(t)|}{f(t)}\} \) with \(f(t) = \ln^3 (e + t)\) and the constant \(C_d\) large enough. According to Schonbek’s strategy, we have
\[
C_d \frac{f'(t)}{f(t)} \int_{S_0(t)^c} |\hat{u}(\xi)|^2 \, d\xi \leq \|\nabla u\|_{L^2}^2.
\]
According to \((5.4)\) and \((5.6)\), we infer
\[
\frac{d}{dt} \|u\|_{L^2}^2 + C_d \frac{f'(t)}{f(t)} \|u\|_{L^2}^2 \leq C_d \frac{f'(t)}{f(t)} \int_{S_0(t)} |\hat{u}(\xi)|^2 \, d\xi + C e^{-2ct}.
\]
Taking Fourier transform with respect to \(x\) in \((1.3)\), then we obtain
\[
\begin{cases}
\hat{u}_t + F(u \cdot \nabla u) + |\xi|^2 \hat{u} + i\xi \cdot \hat{P} = i\xi \cdot \hat{\tau}, \\
\hat{g}_t + F(u \cdot \nabla g) + \mathcal{L}(\hat{g}) = \text{div}_{\text{q}}(-\mathcal{F}(\Omega \cdot \text{g}_q\psi_{\infty})), \\
i\xi \cdot \hat{\tau}_t = -i\xi \cdot \hat{u} = 0.
\end{cases}
\]
Multiplying \(\hat{u}\) to \((5.8)\), we get
\[
\partial_t |\hat{u}|^2 \leq |\hat{\tau}|^2 + C |\mathcal{F}(u \otimes u)|^2.
\]
Taking \(L^2\) inner product to \((5.8)\) with \(\hat{g}\), we infer that
\[
\partial_t \|\hat{g}\|_{L^2}^2 + 2\|\nabla_q \hat{g}\|_{L^2}^2 \leq C \int_{\mathbb{R}^d} \psi_{\infty} |\mathcal{F}(u \cdot \nabla g)|^2 \, dq + C \int_{\mathbb{R}^d} \psi_{\infty} |\mathcal{F}(\nabla u \cdot \text{q}_g\psi_{\infty})|^2 \, dq.
\]
Applying Lemma \((2.4)\) we have
\[
|\hat{\tau}|^2 = (\int_{\mathbb{R}^d} q \otimes \nabla_q \mathcal{H}\hat{g}_q \psi_{\infty} \, dq) \leq C \|\nabla_q \hat{g}\|_{L^2}^2.
\]
Adding \((5.9)\) to \(\lambda \times (5.10)\) with \(\lambda\) large enough and integrating over \(S_0(t)\), we deduce that
\[
\int_{S_0(t)} |\hat{u}(t, \xi)|^2 + \lambda \|\hat{g}\|_{L^2}^2 \, d\xi \leq \int_{S_0(t)} |\hat{u}_0|^2 + \lambda \|\hat{g}_0\|_{L^2}^2 \, d\xi + C \int_{S_0(t)} \int_0^t |\mathcal{F}(u \otimes u)|^2 \, ds' \, d\xi
\]
\[
+ \lambda \int_{S_0(t)} \int_0^t \int_{\mathbb{R}^d} \psi_{\infty} |\mathcal{F}(u \cdot \nabla g)|^2 \, dq \, ds' \, d\xi + \lambda \int_{S_0(t)} \int_0^t \int_{\mathbb{R}^d} \psi_{\infty} |\mathcal{F}(\nabla u \cdot \text{q}_g\psi_{\infty})|^2 \, dq \, ds' \, d\xi.
\]
Under the assumption in Theorem \((1.6)\) we obtain
\[
\int_{S_0(t)} |\hat{u}_0|^2 + \|\hat{g}_0\|_{L^2}^2 \, d\xi \leq \int_{S_0(t)} d\xi \cdot \|\hat{u}_0\|^2 + \|\hat{g}\|_{L^2}^2 \|L^\infty(S(t))\|
\]
\[
\leq C \left( \frac{f'(t)}{f(t)} \right)^{\frac{1}{2}} (\|u_0\|_{L^1}^2 + \|\text{g}_0\|_{L^1(L^2)}^2).
\]
According to Minkowski’s inequality and (5.1), we get
\[
\int_{S_0(t)} \int_0^t |\mathcal{F}(u \otimes u)|^2 ds' d\xi = \int_0^t \int_{S_0(t)} |\mathcal{F}(u \otimes u)|^2 d\xi ds' \\
\leq C \int_{S_0(t)} d\xi \int_0^t \|\mathcal{F}(u \otimes u)\|_{L^\infty} ds' \\
\leq C \ln^{-1}(e + t).
\]

Using \( \text{div} \ u = 0 \) and (3.1), we have
\[
\int_{S_0(t)} \int_0^t \int_{\mathbb{R}^d} \psi_\infty |\mathcal{F}(u \cdot \nabla g)|^2 dq ds' d\xi \\
\leq C \int_{S_0(t)} |\xi|^2 d\xi \int_0^t \int_{\mathbb{R}^d} \|\psi_\infty \|_{L^\infty} |\mathcal{F}(ug)|^2 \|_{L^\infty} dq ds' \\
\leq C \left( \frac{f'(t)}{f(t)} \right)^{\frac{d}{2} + 1} \int_0^t \|u\|_{L^2} \|g\|_{L^2(\mathbb{R}^2)} ds' \\
\leq C \left( \frac{f'(t)}{f(t)} \right)^{\frac{d}{2}}.
\]

Applying Proposition 5.2 and Lemma 2.4, we deduce that
\[
\int_{S_0(t)} \int_0^t \int_q \psi_\infty |\mathcal{F}(u \cdot qg)|^2 dq ds' d\xi \\
\leq C \int_{S_0(t)} d\xi \int_0^t \|\psi_\infty \|_{L^\infty} \|\mathcal{F}(\nabla u \cdot qg)\|^2 \|_{L^\infty} dq ds' \\
\leq C \left( \frac{f'(t)}{f(t)} \right)^{\frac{d}{2}} \int_0^t \|\nabla u\|_{L^2} \|\langle qg\rangle\|_{L^2(\mathbb{R}^2)} ds' \\
\leq C \left( \frac{f'(t)}{f(t)} \right)^{\frac{d}{2}}.
\]

Combining the estimates for (5.12), we have
\[
\int_{S_0(t)} |\hat{u}(t, \xi)|^2 d\xi \leq C \ln^{-1}(1 + t).
\]

According to (5.17) and (5.17), we deduce that
\[
d \left[ \frac{d}{dt} \|u\|_{L^2}^2 \right] + C_d \frac{f'(t)}{f(t)} \|u\|_{L^2}^2 \leq C C_d \frac{f'(t)}{f(t)} \ln^{-1}(1 + t),
\]
from which we deduce that
\[
\|u\|_{L^2}^2 \leq C \ln^{-1}(e + t).
\]

Using the initial decay (5.19), we improve the \( L^2 \) decay rate by using the bootstrap argument.
\[
\int_{S(t)} \int_0^t |\mathcal{F}(u \otimes u)|^2 ds d\xi \leq C \left( \frac{f'(t)}{f(t)} \right)^{\frac{d}{2}} \int_0^t \|u\|_{L^2}^4 ds' \\
\leq C \left( \frac{f'(t)}{f(t)} \right)^{\frac{d}{2}} \int_0^t \ln^{-2}(e + s') ds' \\
\leq C \ln^{-3}(e + t).
\]

Then the proof of (5.18) implies that
\[
\|u\|_{L^2}^2 \leq C \ln^{-3}(e + t).
\]
Step 2 : Define \( S(t) = \{ \xi : |\xi|^2 \leq C_d(1 + t)^{-1} \} \), which will be useful to prove polynomial decay. By Schonbek’s strategy and (5.3), we have

\[
\frac{d}{dt} \| u \|_{L^2}^2 + \frac{C_d}{1 + t} \| u \|_{L^2}^2 \leq \frac{C_d}{1 + t} \int_{S(t)} |\dot{u}(\xi)|^2 d\xi + C e^{-2at}.
\] (5.22)

Adding (5.9) to \( \lambda \times (5.10) \) with \( \lambda \) large enough and integrating over \( S(t) \), we deduce that

\[
\int_{S(t)} |\dot{u}(t, \xi)|^2 + \lambda \| \dot{g} \|_{L^2}^2 d\xi \leq \int_{S(t)} \| \dot{u}_0 \|^2 + \lambda \| \dot{g}_0 \|_{L^2}^2 d\xi + C \int_{S(t)} \int_0^t |\mathcal{F}(u \otimes u)|^2 ds'd\xi
\]

\[+ \lambda \int_{S(t)} \int_0^t \int_{\mathbb{R}^d} \psi(\xi) |\mathcal{F}(u \cdot \nabla g)|^2 dqds'd\xi + \lambda \int_{S(t)} \int_0^t \psi(\xi) \mathcal{F}(\nabla u \cdot qg)|^2 dqds'd\xi.
\] (5.23)

Under the additional assumption in Theorem 1.6, we infer that

\[
\int_{S(t)} \| \dot{u}_0 \|^2 + \lambda \| \dot{g}_0 \|^2_{L^2} d\xi \leq \int_{S(t)} d\xi \cdot \| \dot{u}_0 \|^2 + \lambda \| \dot{g}_0 \|^2_{L^\infty(S(t))}
\]

\[\leq C(1 + t)^{-\frac{d}{2}} (\| u_0 \|^2_{L^1} + \lambda \| g_0 \|^2_{L^1(L^2)}) ,
\] (5.24)

and

\[
\int_{S(t)} \int_0^t |\mathcal{F}(u \otimes u)|^2 ds'd\xi \leq C(1 + t)^{-\frac{d}{2}} \int_0^t \| u \|^4_{L^2} ds.
\] (5.25)

Using \( \text{div} \ u = 0 \) and (5.11), we have

\[
\int_{S(t)} \int_0^t \int_{\mathbb{R}^d} \psi(\xi) \|\mathcal{F}(u \cdot \nabla g)\|^2 dqds'd\xi \leq C(1 + t)^{-\frac{d}{2} - 1} \int_0^t \| u \|^2_{L^2} \| g \|^2_{L^2(L^2)} ds'
\]

\[\leq C(1 + t)^{-\frac{d}{2} - 1}.
\] (5.26)

Applying Proposition 3.2 and Lemma 2.4, we obtain

\[
\int_{S(t)} \int_0^t \int_{\mathbb{R}^d} \psi(\xi) \|\nabla u \cdot qg\|^2 dqds'd\xi \leq C(1 + t)^{-\frac{d}{2}} \int_0^t \| \nabla u \|^2_{L^2} \|qg\|^2_{L^2(L^2)} ds'
\]

\[\leq C(1 + t)^{-\frac{d}{2}}.
\] (5.27)

Combining the estimates for (5.23), we obtain

\[
\int_{S(t)} |\dot{u}(t, \xi)|^2 d\xi \leq C \left[ (1 + t)^{-\frac{d}{2}} + (1 + t)^{-\frac{d}{2}} \int_0^t \| u \|^2_{L^2} ds' \right].
\] (5.28)

According to (5.22) and (5.28), we deduce that

\[
\frac{d}{dt} \| u \|^2_{L^2} + \frac{C_d}{1 + t} \| u \|^2_{L^2} \leq C C_d (1 + t)^{-\frac{d}{2} - 1} (1 + \int_0^t \| u \|^2_{L^2} ds').
\] (5.29)

Multiplying \((1 + t)^\frac{d}{2} + 1\) to (5.29) and integrating on \([0, t]\), we have

\[
(1 + t)^{\frac{d}{2} + 1} \| u \|^2_{L^2} \leq C t + C \int_0^t \| u \|^2_{L^2} ds ds'.
\] (5.30)
Define \( M(t) = \sup_{0 \leq s' \leq t} (1 + s') \frac{\|u\|^2_{L^2(s')}}{\|u\|^2_{L^2}} \). Using (5.21) and (5.30), we deduce that

\[
(5.31) \quad M(t) \leq C + \int_0^t M(s')(1 + s')^{-\frac{3}{2}} \ln^{-\frac{3}{2}}(e + s') ds'
\]

Applying Gronwall’s inequality, then we get \( M(t) \leq C \) for any \( t > 0 \), which implies that

\[
(5.32) \quad \|u\|_{L^2} \leq C(1 + t)^{-\frac{3}{4}}.
\]

**Step 3**: We end up with establishing the lower bound of \( L^2 \) decay rate. Taking Leray projector \( \mathbb{P} \) and Fourier transform with respect to \( x \), we have

\[
(5.33) \quad \ddot{u} + |\xi|^2 \dot{u} = i\xi \hat{\mathbb{P}} \tau - \mathbb{P}(u \cdot \nabla u).
\]

Integrating over \([0, t]\) with respect to \( s \), we deduce that

\[
(5.34) \quad \dot{u} = e^{-|\xi|^2 t} \dot{u}_0 + \int_0^t e^{-|\xi|^2 (t-s)} i\xi [\hat{\mathbb{P}} \tau - \mathbb{P}(u \cdot \nabla u)] ds.
\]

Under conditions of Theorem 1.2, we can choose a ball \( B \) containing the origin such that \( \inf_{\xi \in B} \dot{u}_0 \geq c_0 \) for some positive constant \( c_0 \). Denote that \( d_0 = \alpha c_0^2 \) for some \( \alpha \) small enough. According to Minkowski inequality, we infer that

\[
(5.35) \quad \left( \int_B |\dot{u}|^2 d\xi \right)^{\frac{1}{2}} \geq \left( \int_B e^{-|\xi|^2 t} |\dot{u}_0|^2 d\xi \right)^{\frac{1}{2}} - \int_0^t \left( \int_B e^{-|\xi|^2 (t-s)} |\xi| [\hat{\mathbb{P}} \tau - \mathbb{P}(u \cdot \nabla u)] \right)_{L^2} ds
\]

\[
\geq \left( \int_B e^{-|\xi|^2 t} |\dot{u}_0|^2 d\xi \right)^{\frac{1}{2}} - \int_0^t \left( \int_B e^{-|\xi|^2 (t-s)} |\xi| \right)_{L^\infty} \|\hat{\mathbb{P}} \tau - \mathbb{P}(u \cdot \nabla u)\|_{L^2} ds
\]

\[
\geq d_0 (1 + t)^{-\frac{3}{4}} - \int_0^t (1 + t - s)^{-\frac{3}{4}} \left( \|\tau\|_{L^\frac{8}{3}} + \|u \otimes u\|_{L^\frac{4}{3}} \right) ds.
\]

Denote that

\[
E^{\alpha, \beta} = \|(u_0, ||g_0||_{L^2})\|_{L^1 \cap L^2}^{\alpha} \|(u_0, ||g_0||_{L^2})\|_{L^2}^{\beta},
\]

and

\[
B_d = \int_0^t (1 + t - s)^{-\frac{3}{4}} \left( \|\tau\|_{L^\frac{8}{3}} + \|u \otimes u\|_{L^\frac{4}{3}} \right) ds.
\]

Under conditions of Theorem 1.6, we can take \( \|(u_0, ||g||_{L^2})\|_{L^2} \) small enough that

\[
E^{0,1} + E^{1,1} + E^{1,1} + E^{1,1} \leq \frac{d_0}{2C}.
\]

For \( d = 3 \), applying (5.32), Lemma 2.8 and Proposition 8.1, we deduce that

\[
(5.36) \quad B_3 \leq \int_0^t (1 + t - s)^{-\frac{3}{4}} \left( \|\tau\|_{L^1}^{\frac{1}{2}} \|\tau\|_{L^2}^{\frac{1}{2}} + \|u\|_{L^2} \|\nabla u\|_{L^2} \right) ds
\]

\[
\leq CE^{0,1} (1 + t)^{-\frac{3}{2}} + CE^{1,1} \left[ \int_0^t (1 + t - s)^{-\frac{3}{4}} (1 + t)^{-\frac{3}{2}} ds \right]^\frac{1}{2}
\]

\[
\leq \frac{d_0}{2} (1 + t)^{-\frac{3}{2}}.
\]
Consider the critical case $d = 2$, we need more integrability in time for $\|\nabla u\|_{L^2}$. Let’s recalling $L^2$ energy estimate as follows

\begin{equation}
\frac{d}{dt}\|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 \leq C\|\tau\|_{L^2}^2.
\end{equation}

Multiplying $(1 + t)\frac{1}{2}$ to (5.37), we obtain

\begin{equation}
\frac{d}{dt}(1 + t)\frac{1}{2}\|u\|_{L^2}^2 + (1 + t)^{1/2}\|\nabla u\|_{L^2}^2 \leq C(1 + t)^{1/2}\|\tau\|_{L^2}^2 + \frac{1}{2}(1 + t)^{-\frac{1}{2}}\|u\|_{L^2}^2
\end{equation}

Integrating over $[0, t]$ with respect to $s$, we infer from (5.4) and (5.32) that

\begin{equation}
(1 + t)\frac{1}{2}\|u\|_{L^2}^2 + \int_0^t (1 + s)^{1/2}\|\nabla u\|_{L^2}^2 ds \leq CE^{2,0}.
\end{equation}

For $d = 2$, applying Lemma 2.3 (5.32) and (5.39), we deduce that

\begin{equation}
B_2 \leq \int_0^t (1 + t - s)^{-\frac{1}{2}}(\|\tau\|_{L^2} + \|u\|_{L^2}\|\nabla u\|_{L^2}) ds
\end{equation}

\begin{equation}
\leq CE^{0,1} \int_0^t (1 + t - s)^{-\frac{1}{2}}e^{-ct} ds + CE^{\frac{1}{2} + \frac{1}{4}} \left[ \int_0^t (1 + t - s)^{-1}(1 + t)^{-\frac{1}{2}} ds \right]^{\frac{1}{2}}
\end{equation}

\begin{equation}
\leq \frac{d_0}{2}(1 + t)^{-\frac{1}{2}}.
\end{equation}

According to (5.33), (5.36) and (5.40), we infer

\begin{equation}
\|u\|_{L^2} \geq \left( \int_B |\hat{u}|^2 d\xi \right)^{\frac{1}{2}} \geq \frac{d_0}{2}(1 + t)^{-\frac{1}{4}},
\end{equation}

which implies that the $L^2$ decay rate we obtain is optimal. □

**Acknowledgments** This work was partially supported by the National Natural Science Foundation of China (No.12171493), the Macao Science and Technology Development Fund (No. 098/2013/A3), and Guangdong Province of China Special Support Program (No. 8-2015).

**References**

[1] H. Bahouri, J.-Y. Chemin, and R. Danchin. *Fourier analysis and nonlinear partial differential equations*, volume 343 of Grundlehren der Mathematischen Wissenschaften. Springer, Heidelberg, 2011.

[2] R. B. Bird, R. C. Armstrong, and O. Hassager. *Dynamics of Polymeric Liquids*, volume 1. Wiley, New York, 1977.

[3] J.-Y. Chemin and N. Masmoudi. About lifespan of regular solutions of equations related to viscoelastic fluids. *SIAM J. Math. Anal.*, 33(1):84–112, 2001.

[4] R. Coifman, P.-L. Lions, Y. Meyer, and S. Semmes. Compensated compactness and Hardy spaces. *J. Math. Pures Appl.* (9), 72(3):247–286, 1993.

[5] W. Deng, Z. Luo, and Z. Yin. Global solutions and large time behavior for some Oldroyd-B type models in $\mathbb{R}^2$. https://doi.org/10.48550/arXiv.2107.12029.
[6] M. Doi and S. F. Edwards. *The Theory of Polymer Dynamics*. Oxford University Press, Oxford, 1988.

[7] R. Duan, S. Ukai, T. Yang, and H. Zhao. Optimal convergence rates for the compressible Navier-Stokes equations with potential forces. *Math. Models Methods Appl. Sci.*, 17(5):737–758, 2007.

[8] T. M. Elgindi and R. Frederic. Global regularity for some Oldroyd-B type models. *Comm. Pure Appl. Math.*, 68(11), 2015.

[9] T. M. Elgindi and J. Liu. Global wellposedness to the generalized Oldroyd type models in $\mathbb{R}^3$. *J. Differential Equations*, 259(5):1958–1966, 2015.

[10] Y. Giga and A. Novotný, editors. *Handbook of mathematical analysis in mechanics of viscous fluids*. Springer, Cham, 2018.

[11] L. Grafakos. *Modern Fourier analysis*, volume 250 of *Graduate Texts in Mathematics*. Springer, New York, third edition, 2014.

[12] L. He and P. Zhang. $L^2$ decay of solutions to a micro-macro model for polymeric fluids near equilibrium. *Siam J. Math. Anal.*, 40(5):1905–1922, 2009.

[13] N. Jiang, Y. Liu, and T.-F. Zhang. Global classical solutions to a compressible model for micro-macro polymeric fluids near equilibrium. *SIAM J. Math. Anal.*, 50(4):3149–3179, 2018.

[14] B. Jourdain, T. Lelièvre, and C. Le Bris. Existence of solution for a micro-macro model of polymeric fluid: the FENE model. *J. Funct. Anal.*, 209(1):162–193, 2004.

[15] Z. Lei, N. Masmoudi, and Y. Zhou. Remarks on the blowup criteria for Oldroyd models. *J. Differential Equations*, 248(2):328–341, 2010.

[16] J. Leray. Sur le mouvement d’un liquide visqueux emplissant l’espace. *Acta Math.*, 63(1):193–248, 1934.

[17] P.-L. Lions. *Mathematical topics in fluid mechanics. Vol. 1*, volume 3 of *Oxford Lecture Series in Mathematics and its Applications*. The Clarendon Press, Oxford University Press, New York, 1996. Incompressible models, Oxford Science Publications.

[18] P.-L. Lions. *Mathematical topics in fluid mechanics. Vol. 2*, volume 10 of *Oxford Lecture Series in Mathematics and its Applications*. The Clarendon Press, Oxford University Press, New York, 1998. Compressible models, Oxford Science Publications.

[19] P. L. Lions and N. Masmoudi. Global solutions for some Oldroyd models of non-Newtonian flows. *Chinese Ann. Math. Ser. B*, 21(2):131–146, 2000.

[20] P.-L. Lions and N. Masmoudi. Global existence of weak solutions to some micro-macro models. *C. R. Math. Acad. Sci. Paris*, 345(1):15–20, 2007.

[21] W. Luo and Z. Yin. The Liouville theorem and the $L^2$ decay for the FENE dumbbell model of polymeric flows. *Arch. Ration. Mech. Anal.*, 224(1):209–231, 2017.
[22] W. Luo and Z. Yin. The $L^2$ decay for the 2D co-rotation FENE dumbbell model of polymeric flows. *Adv. Math.*, 343:522–537, 2019.

[23] N. Masmoudi. Global existence of weak solutions to macroscopic models of polymeric flows. *J. Math. Pures Appl. (9)*, 96(5):502–520, 2011.

[24] N. Masmoudi. Global existence of weak solutions to the FENE dumbbell model of polymeric flows. *Invent. Math.*, 191(2):427–500, 2013.

[25] N. Masmoudi, P. Zhang, and Z. Zhang. Global well-posedness for 2D polymeric fluid models and growth estimate. *Phys. D*, 237(10-12):1663–1675, 2008.

[26] L. Nirenberg. On elliptic partial differential equations. *Ann. Scuola Norm. Sup. Pisa*, 13(2):115–162, 1959.

[27] M. Renardy. An existence theorem for model equations resulting from kinetic theories of polymer solutions. *SIAM J. Math. Anal.*, 22(2):313–327, 1991.

[28] M. E. Schonbek. $L^2$ decay for weak solutions of the Navier-Stokes equations. *Arch. Ration. Mech. Anal.*, 88(3):209–222, 1985.

[29] M. E. Schonbek. Lower bounds of rates of decay for solutions to the Navier-Stokes equations. *J. Amer. Math. Soc.*, 4(3):423–449, 1991.

[30] M. E. Schonbek. Existence and decay of polymeric flows. *SIAM J. Math. Anal.*, 41(2):564–587, 2009.