Invariants for systems of two linear hyperbolic-type equations by complex methods

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Abstract. The invariants of a general linear system of two hyperbolic equations have been derived under transformations of the dependent and independent variables by the real infinitesimal method. Here a subclass of the general system of linear hyperbolic partial differential equations (PDEs) is investigated for the associated invariants by complex as well as real methods. The complex procedure relies on the correspondence of the system and associated invariants with the base complex equation and related complex invariants, respectively. A comparison of all the invariant quantities obtained by complex and real methods is presented which shows that the complex procedure provides a few invariants different from those extracted by real symmetry analysis.

1. Introduction

Invariants of a system of two linear hyperbolic equations

\begin{align*}
  u_{tx} + a_1(t, x)u_t + a_2(t, x)v_t + b_1(t, x)u_x + b_2(t, x)v_x + c_1(t, x)u + c_2(t, x)v &= 0, \\
  v_{tx} + a_3(t, x)u_t + a_4(t, x)v_t + b_3(t, x)u_x + b_4(t, x)v_x + c_3(t, x)u + c_4(t, x)v &= 0,
\end{align*}

where \(t\) and \(x\) in the subscripts denote the derivatives with respect to these independent variables, under an invertible change of the dependent variables, have been derived in \cite{1} by the real infinitesimal method. Moreover, the joint invariants for this general class of systems of hyperbolic equations under the transformations of both the dependent and independent variables, are also deduced in \cite{1}. The most general group of equivalence transformations \cite{2, 3}, i.e. invertible change of the variables that maps a system of hyperbolic equations to itself with, in general different coefficients, is obtained first and then employed to find the invariants \cite{1}. Complex symmetry analysis is invoked in order to investigate the semi-invariants associated with a subclass of the system of hyperbolic equations \cite{1} under a change
of only the dependent variables [4]. This subclass of systems is represented by the following two hyperbolic-type equations

\[
\begin{align*}
u_{tx} + \alpha_1(t, x)u_t - \alpha_2(t, x)v_t + \beta_1(t, x)u_x - \beta_2(t, x)v_x + \gamma_1(t, x)u - \gamma_2(t, x)v &= 0, \\
v_{tx} + \alpha_2(t, x)u_t + \alpha_1(t, x)v_t + \beta_2(t, x)u_x + \beta_1(t, x)v_x + \gamma_2(t, x)u + \gamma_1(t, x)v &= 0.
\end{align*}
\]

(2)

This system of hyperbolic-type equations corresponds to a scalar complex hyperbolic equation

\[
w_{tx} + \alpha(t, x)w_t + \beta(t, x)w_x + \gamma(t, x)w = 0,
\]

(3)

if \(\alpha_1 + i\alpha_2 = \alpha, \beta_1 + i\beta_2 = \beta, \gamma_1 + i\gamma_2 = \gamma\) and \(u_1 + iv_2 = w\) [4]. The set of equivalence transformations used to obtain the semi-invariants of system (2) had a specific Cauchy-Riemann (CR) structure. Therefore the deduced invariants also satisfy the CR-equations. These quantities were also found to correspond to their complex analogues given in [5, 6], associated with the base complex equation by means of a change of the complex dependent variable. All the invariants of the scalar linear hyperbolic equation are determined in [8, 7].

Semi-invariants of the linear parabolic equation via a transformation of only the dependent variable have been derived [5]. Semi-invariants for the independent variables are given in [9, 11] and joint invariants for both the dependent and independent variables are found as well [10, 9, 11]. Laplace-type invariants of the linear parabolic equation have been extended to Ibragimov-type semi-invariants for a system of two parabolic-type PDEs by complex symmetry analysis [12]. These invariants also satisfy the CR-equations as the complex procedure relies on the CR-structure. Though complex and real symmetry analysis have been employed to find the semi-invariants of systems of parabolic and hyperbolic-type PDEs, these results have not been compared earlier. This comparison of both the real and complex procedures brings out the significance of the complex symmetry analysis (CSA). By comparing the invariants associated with the system of two hyperbolic-type PDEs to the real procedure, it is shown that CSA goes beyond the real procedure and yields a few new invariant quantities different from those obtained by real symmetry method developed for systems.

The plan of the paper is as follows. The second section is on the preliminaries where the infinitesimal method as used in earlier works is presented for the hyperbolic and parabolic PDEs. The subsequent section contains the derivation of the invariants of a system of hyperbolic-type equations by real symmetry analysis. The fourth section is on obtaining the invariants for the same system by the complex procedure and the comparison of these with the invariants obtained in the third section. Concluding remarks are presented in the last section.
2. Preliminaries

Semi-invariants of the linear hyperbolic equation

\[ w_{z_1 z_2} + \alpha(z_1, z_2) w_{z_1} + \beta(z_1, z_2) w_{z_2} + \gamma(z_1, z_2) w = 0, \]  

under a transformation of (only) the dependent variables

\[ w(z_1, z_2) = \sigma(z_1, z_2) u(z_1, z_2), \]  

are derived in [5, 6]. The transformations (5) correspond to the following infinitesimal change of the dependent variables

\[ w = u + \epsilon \eta(z_1, z_2) u, \]  

which leads to a generator of the form

\[ Z = \eta_{z_2} \partial_\alpha + \eta_{z_1} \partial_\beta + (\eta_{z_1 z_2} + \alpha \eta_{z_1} + \beta \eta_{z_2}) \partial_\gamma, \]  

that is readable from the transformed hyperbolic equation after implementing the infinitesimal change of the dependent variable (6) in equation (4). The following first order semi-invariants, viz. the Laplace invariants first reported in [6]

\[ h = \alpha_{z_1} + \alpha \beta - \gamma, \quad k = \beta_{z_2} + \alpha \beta - \gamma, \]  

are obtained by applying the once extended generator (7) on \( J(\alpha, \beta, \gamma, \alpha_{z_1}, \beta_{z_2}, \gamma_{z_1}, \alpha_{z_2}, \beta_{z_2}, \gamma_{z_2}) \).

Similarly, a change of the independent variables

\[ z_1 = \phi(t), \quad z_2 = \psi(x), \]  

which can be written in the infinitesimal form as

\[ z_1 = t + \epsilon \xi_1(t), \quad z_2 = x + \epsilon \xi_2(x), \]  

maps the linear hyperbolic equation (4) to an equation of the same family, i.e. those independent variables transformations that preserve the form, linearity and homogeneity of the hyperbolic PDE (4). This change of (only) the independent variables leads to a generator, i.e. the operator obtained after transforming the linear hyperbolic equation corresponding to the infinitesimal change (10) and reading it from the transformed new coefficients as

\[ Z = \xi_1 \partial_t + \xi_2 \partial_x - \alpha \xi_{2,x} \partial_\alpha - \beta \xi_{1,t} \partial_\beta - \gamma (\xi_{1,t} + \xi_{2,x}) \partial_\gamma. \]
Applying this generator on \( J(\alpha, \beta, \gamma) \) yields a zeroth order semi-invariant

\[
I_1 = \frac{\gamma}{\alpha \beta}.
\]  

(12)

Further, in order to find the first order semi-invariants one needs to invoke the first extension of (11) on \( J(\alpha, \beta, \gamma, \alpha_t, \beta_t, \gamma_t, \alpha_x, \beta_x, \gamma_x) \) which results in a system of linear PDEs that results in the following invariant quantities

\[
I_2 = \frac{\alpha \beta}{\alpha_t}, \quad I_3 = \frac{\beta_x}{\alpha_t}, \quad I_4 = \frac{\gamma}{\alpha_t}, \quad I_5 = \frac{\alpha (\beta \gamma_t - \gamma \beta_t)}{\beta \alpha_t^2}, \quad I_6 = \frac{\alpha \gamma_x - \gamma \alpha_x}{\alpha^2 \alpha_t}.
\]  

(13)

Further, the joint invariants of the hyperbolic equation have been deduced in [7, 8]. These invariants are derived by applying an operator of the form (11) in the space of the Laplace semi-invariants \( h \) and \( k \) given in (8). Therefore, the first task is to transform the operator (11) in the variables \( h, k \) and then apply it on the functions of these variables \( J(h, k) \) and their derivatives \( J(h, k, h_t, k_t, h_x, k_x) \) and so on in order to get the zeroth, first and higher order joint invariants of the linear hyperbolic equation, respectively. After writing the generator (11) in the space of the Laplace invariants \( h, k \), i.e.

\[
Z = Z(h) \partial_h + Z(k) \partial_k,
\]  

(14)

its first extension reads as

\[
Z^{[1]} = \xi_1(t) \partial_t + \xi_2(x) \partial_x - (\xi_{1,t} + \xi_{2,x}) h \partial_h - (\xi_{1,t} + \xi_{2,x}) k \partial_k - (\xi_{1,t} h + 2 \xi_{1,t} h_t + \xi_{2,x} h_t) \partial_{h_t} \\
- (\xi_{1,t} h_x + \xi_{2,x} h_x + 2 \xi_{2,x} h_x) \partial_{h_x} - (\xi_{1,t} h + 2 \xi_{1,t} k_t + \xi_{2,x} k_t) \partial_{k_t} - (\xi_{1,t} k_x + \xi_{2,x} k_x) \partial_{k_x} + 2 \xi_{2,x} k_x) \partial_{k_x}.
\]  

(15)

It yields the following joint invariants [7, 8]

\[
J_1 = \frac{k}{h},
\]

\[
J_2 = \frac{(hk_t - kh_t)(hk_x - kh_x)}{h^5},
\]

\[
J_3 = \frac{kh_{tx} + hk_{tx} - h_t k_x - h_x k_t}{h^3},
\]

\[
J_4 = \frac{(hk_x - kh_x)^2 (hk_{xx} - h^2 k_{xx} - 3kh_x^2 + 3hh_x k_x)}{h^9},
\]

\[
J_5 = \frac{(hk_t - kh_t)^2 (hk_{tx} - h^2 k_{tx} - 3kh_x^2 + 3hh_x k_x)}{h^9},
\]

\[
J_6 = \frac{k(hh_{tx} - h_t h_x)}{h^4},
\]

(16)

of the scalar linear hyperbolic equation (4).

We utilize the operators and invariants reviewed here in the next sections by employing complex methods.
3. Invariants of a system of two hyperbolic-type equations by real procedure

Invariants of a system of two linear hyperbolic PDEs (1) have been determined by using the infinitesimal method [1]. The derivation of these invariants starts with the determination of the most general group of the equivalence transformations that maps the system of two linear hyperbolic equations to itself with different coefficients. It requires the application of the generator

\[ Z = \xi^1 \partial_t + \xi^2 \partial_x + \eta^1 \partial_u + \eta^2 \partial_v + \eta^1_t \partial_{u_t} + \eta^2_x \partial_{u_x} + \eta^1_x \partial_{u_x} + \eta^2_v \partial_{v_x} + \eta^1_{tx} \partial_{u_{tx}} + \eta^2_{tx} \partial_{v_{tx}} \\
+ \mu^{11} \partial_{\alpha_1} + \mu^{12} \partial_{\alpha_2} + \mu^{13} \partial_{\alpha_3} + \mu^{14} \partial_{\alpha_4} + \mu^{21} \partial_{\beta_1} + \mu^{22} \partial_{\beta_2} + \mu^{23} \partial_{\beta_3} + \mu^{24} \partial_{\beta_4} \\
+ \mu^{31} \partial_{\gamma_1} + \mu^{32} \partial_{\gamma_2} + \mu^{33} \partial_{\gamma_3} + \mu^{34} \partial_{\gamma_4}, \]

(17)

on both the equations of the system (1), here \( \xi^\kappa, \eta^\kappa \) are functions of \((t, x, u, v)\), where \( \kappa = 1, 2, \) and \( \mu^{1\lambda}, \mu^{2\lambda}, \mu^{3\lambda} \) for \( \lambda = 1, 2, 3, 4 \), are functions of \((t, x, u, v, a_\lambda, b_\lambda, c_\lambda)\). This procedure yields the most general group of equivalence transformations of the dependent and independent variables. Using the generators associated with these infinitesimal transformations the invariants of (1) have been derived [1]. The invariants associated with the subclass (2) of the system of hyperbolic equations (1) are determined in the remaining part of this section.

The system of two hyperbolic-type PDEs (2) is obtainable from a hyperbolic PDE with two independent variables, when the dependent variable of equation (3) is considered complex. Such systems have a CR-structure due to this correspondence. Thus they are said to be CR-structured systems. The group of equivalence transformations associated with (2) is obtained when the following generator

\[ Z = \xi^1 \partial_t + \xi^2 \partial_x + \eta^1 \partial_u + \eta^2 \partial_v + \eta^1_t \partial_{u_t} + \eta^2_x \partial_{u_x} + \eta^1_x \partial_{u_x} + \eta^2_v \partial_{v_x} + \eta^1_{tx} \partial_{u_{tx}} + \eta^2_{tx} \partial_{v_{tx}} \\
+ \mu^{11} \partial_{\alpha_1} + \mu^{12} \partial_{\alpha_2} + \mu^{21} \partial_{\beta_1} + \mu^{22} \partial_{\beta_2} + \mu^{31} \partial_{\gamma_1} + \mu^{32} \partial_{\gamma_2}, \]

(18)

where \( \xi^\kappa, \eta^\kappa \) are functions of \((t, x, u, v)\) and \( \mu^{1\kappa}, \mu^{2\kappa}, \mu^{3\kappa} \) are functions of \((t, x, u, v, \alpha_\kappa, \beta_\kappa, \gamma_\kappa)\), acts on both the equations of the system (2). The solution of the system of linear PDEs obtained after action of (18) is

\[
\begin{align*}
\xi_1 &= F_1(t), \quad \xi_2 = F_2(x), \\
\mu^{11} &= -F_{3,x} - \alpha_1 F_{2,x}, \quad \mu^{12} = F_{4,x} - \alpha_2 F_{2,x}, \\
\mu^{21} &= -F_{3,t} - \beta_1 F_{1,t}, \quad \mu^{22} = F_{4,t} - \beta_2 F_{1,t}, \\
\mu^{31} &= -F_{3,tx} - \alpha_1 F_{3,t} - \alpha_2 F_{4,t} - \beta_1 F_{3,x} - \beta_2 F_{4,x} - \gamma_1 (F_{3,t} + F_{2,x}), \\
\mu^{32} &= F_{4,tx} + \alpha_1 F_{4,t} - \alpha_2 F_{3,t} + \beta_1 F_{4,x} - \beta_2 F_{3,x} - \gamma_2 (F_{3,t} + F_{2,x}),
\end{align*}
\]

(19)

where \( F_3 \) and \( F_4 \) depends on \((t, x)\). Inserting the above in (18) leads to a generator that corresponds to changes of both the dependent and independent variables in the system (2).
Hence, the resulting transformations of the coefficients are characterized by (18) with the above insertions.

The first order semi-invariants

\[ h_1^r = \alpha_{1,t} + \alpha_1 \beta_1 - \alpha_2 \beta_2 - \gamma_1, \]
\[ h_2^r = \alpha_{2,t} + \alpha_1 \beta_2 + \alpha_2 \beta_1 - \gamma_2, \]
\[ k_1^r = \beta_{1,t} + \alpha_1 \beta_1 - \alpha_2 \beta_2 - \gamma_1, \]
\[ k_2^r = \beta_{2,t} + \alpha_1 \beta_2 + \alpha_2 \beta_1 - \gamma_2, \]

(20)

associated with the system (2) due to a change of (only) dependent variables

\[ \eta^1 = F_3 u + F_4 v, \quad \eta^2 = F_3 v - F_4 u, \]

(21)

are obtained by employing the generator that is associated with the change of the dependent variable, viz.

\[ X = -F_{3,x} \partial_{\alpha_1} + F_{4,x} \partial_{\alpha_2} - F_{3,t} \partial_{\beta_1} + F_{4,t} \partial_{\beta_2} - (F_{3,tx} + \alpha_1 F_{3,t} + \alpha_2 F_{4,t} + \beta_1 F_{3,x} + \beta_2 F_{4,x}) \partial_{\gamma_1} + (F_{4,tx} + \alpha_1 F_{4,t} - \alpha_2 F_{3,t} + \beta_1 F_{4,x} - \beta_2 F_{3,x}) \partial_{\gamma_2}, \]

(22)

on \( J(\alpha_1, \beta_1, \gamma_1, \alpha_{1,t}, \beta_{1,t}, \gamma_{1,t}, \alpha_{1,x}, \beta_{1,x}, \gamma_{1,x}) \) and solving the resulting linear system of PDEs.

Considering only a change of the independent variables leads to an infinitesimal generator

\[ Z_I = F_1(t) \partial_t + F_2(x) \partial_x - \alpha_1 F_{2,x} \partial_{\alpha_1} - \alpha_2 F_{2,x} \partial_{\alpha_2} - \beta_1 F_{1,t} \partial_{\beta_1} - \beta_2 F_{1,t} \partial_{\beta_2} - \gamma_1 (F_{1,t} + F_{2,x}) \partial_{\gamma_1} - \gamma_2 (F_{1,t} + F_{2,x}) \partial_{\gamma_2}. \]

(23)

Applying it on \( J(\alpha_1, \beta_1, \gamma_1) \) yields the following zeroth order invariants

\[ I_1^r = \frac{\alpha_2}{\alpha_1}, \quad I_2^r = \frac{\beta_2}{\beta_1}, \quad I_3^r = \frac{\gamma_1}{\alpha_1 \beta_1}, \quad I_4^r = \frac{\gamma_2}{\alpha_1 \beta_1}. \]

(24)

Further, the first order invariants are obtained when the once extended generator (23) acts on \( J(\alpha_1, \beta_1, \gamma_1, \alpha_{1,t}, \beta_{1,t}, \gamma_{1,t}, \alpha_{1,x}, \beta_{1,x}, \gamma_{1,x}) \), and this leads to a system of PDEs which gives the following quantities

\[ I_5^r = \frac{\alpha_{1,t}}{\alpha_1 \beta_1}, \quad I_6^r = \frac{\alpha_{2,t}}{\alpha_1 \beta_1}, \quad I_7^r = \frac{\beta_{1,t}}{\alpha_1 \beta_1}, \quad I_8^r = \frac{\beta_{2,t}}{\alpha_1 \beta_1}, \]
\[ I_9^r = \frac{\beta_{1,2,t} - \beta_2 \beta_{1,t}}{b^2}, \quad I_{10}^r = \frac{\beta_1 \gamma_{1,t} - \gamma_1 \beta_{1,t}}{\alpha_1 \beta_1^3}, \quad I_{11}^r = \frac{\beta_1 \gamma_{2,t} - \gamma_2 \beta_{1,t}}{\alpha_1 \beta_1^3}, \]
\[ I_{12}^r = \frac{\alpha_1 \alpha_{2,x} - \alpha_2 \alpha_{1,x}}{\alpha_1^2}, \quad I_{13}^r = \frac{\alpha_1 \gamma_{1,x} - \gamma_1 \alpha_{1,x}}{\alpha_1^2 \beta_1}, \quad I_{14}^r = \frac{\alpha_1 \gamma_{2,x} - \gamma_2 \alpha_{1,x}}{\alpha_1^2 \beta_1}. \]

(25)
including the four zeroth order invariants (24).

The joint invariants of the system (2)

\[ J^r_1 = \frac{h_2^r}{h_1^r}, \quad J^r_2 = \frac{k_1^r}{h_1^r}, \quad J^r_3 = \frac{k_2^r}{h_1^r}, \]

are found when the following PDE

\[ h_1^r \frac{\partial h_1^r}{\partial h_1} + h_2^r \frac{\partial h_2^r}{\partial h_2} + k_1^r \frac{\partial k_1^r}{\partial k_1} + k_2^r \frac{\partial k_2^r}{\partial k_2} = 0, \]

is solved. This equation appears due to action of the infinitesimal generator (23) that is associated with the change of the independent variables to the space of invariants \( h_1^r, k_1^r \).

4. Invariants of a system of two hyperbolic-type equations by complex procedure

Semi-invariants associated with a system of two hyperbolic-type equations (2) that is obtained from a scalar linear hyperbolic equation (3), are derived in this section by complex methods. The generator of the form (7) associated with the equation (4) becomes complex due to the presence of the complex dependent variable and the complex coefficients split (7) into two operators

\[
X_1 = \eta_{1,z_2} \partial_{\alpha_1} + \eta_{2,z_2} \partial_{\alpha_2} + \eta_{1,z_1} \partial_{\beta_1} + \eta_{2,z_1} \partial_{\beta_2} + (\eta_{1,z_1 z_2} + \alpha_1 \eta_{1,z_1} - \alpha_2 \eta_{2,z_1} + \beta_1 \eta_{1,z_2} - \beta_2 \eta_{2,z_2}) \partial_{\gamma_1} + (\eta_{2,z_1 z_2} + \alpha_2 \eta_{1,z_1} + \alpha_1 \eta_{2,z_1} + \beta_2 \eta_{1,z_2} + \beta_1 \eta_{2,z_2}) \partial_{\gamma_2}, \]

\[
X_2 = \eta_{2,z_2} \partial_{\alpha_1} - \eta_{1,z_2} \partial_{\alpha_2} + \eta_{2,z_1} \partial_{\beta_1} - \eta_{1,z_1} \partial_{\beta_2} + (\eta_{2,z_1 z_2} + \alpha_2 \eta_{1,z_1} + \alpha_1 \eta_{2,z_1} + \beta_2 \eta_{1,z_2} + \beta_1 \eta_{2,z_2}) \partial_{\gamma_1} - (\eta_{1,z_1 z_2} + \alpha_1 \eta_{1,z_1} - \alpha_2 \eta_{2,z_1} - \beta_1 \eta_{1,z_2} + \beta_2 \eta_{2,z_2}) \partial_{\gamma_2}. \]

There are four first order semi-invariants

\[
h_1 = \alpha_{1,z_1} + \alpha_1 \beta_1 - \alpha_2 \beta_2 - \gamma_1, \]
\[
h_2 = \alpha_{2,z_1} + \alpha_2 \beta_1 + \alpha_1 \beta_2 - \gamma_2, \]
\[
k_1 = \beta_{1,z_2} + \alpha_1 \beta_1 - \alpha_2 \beta_2 - \gamma_1, \]
\[
k_2 = \beta_{2,z_2} + \alpha_2 \beta_1 + \alpha_1 \beta_2 - \gamma_2, \]

that are found to be associated with the system (2) on employing the pair of operators (28) and (29). These are exactly the same as represented by \( h_1^r, k_1^r \) in (20). Therefore, in this case
the real and complex procedures lead to the same semi-invariants of the system (2). Notice that all the four semi-invariants (30) are readable from the first order semi-invariants associated with the complex hyperbolic linear equation (4) and satisfy

\[ X_1^1 h_1 \big|_{h_1=0} = X_2^1 h_2 \big|_{k_2=0} = X_1^1 k_1 \big|_{k_1=0} = X_2^1 k_2 \big|_{k_2=0} = 0. \]  

(31)

The linear combination \( X_3 \) of both the operators \( X_1 \) and \( X_2 \) results in the following relations

\[ X_3^1 h_1 \big|_{h_1=0} = X_3^1 h_2 \big|_{h_2=0} = X_3^1 k_1 \big|_{k_1=0} = X_3^1 k_2 \big|_{k_2=0} = 0. \]  

(32)

The semi-invariants of the system of two hyperbolic-type PDEs under a transformation of the independent variables are

\[ I_1^c = \frac{(\alpha_1 \gamma_1 + \alpha_2 \gamma_2) \beta_1 + (\alpha_1 \gamma_2 - \alpha_2 \gamma_1) \beta_2}{(\alpha_1^2 + \alpha_2^2)(\beta_1^2 + \beta_2^2)}, \]
\[ I_2^c = \frac{(\alpha_1 \gamma_2 - \alpha_2 \gamma_1) \beta_1 - (\alpha_1 \gamma_1 + \alpha_2 \gamma_2) \beta_2}{(\alpha_1^2 + \alpha_2^2)(\beta_1^2 + \beta_2^2)}, \]
\[ I_3^c = \frac{(\alpha_1 \beta_1 - \alpha_2 \beta_2) \alpha_1 + (\alpha_2 \beta_1 + \alpha_1 \beta_2) \alpha_2}{\alpha_1^2 + \alpha_2^2}, \]
\[ I_4^c = \frac{(\alpha_2 \beta_1 + \alpha_1 \beta_2) \alpha_1 - (\alpha_1 \beta_1 - \alpha_2 \beta_2) \alpha_2}{\alpha_1^2 + \alpha_2^2}, \]
\[ I_5^c = \frac{\alpha_1 \beta_1 x + \alpha_2 \beta_2 x}{\alpha_1^2 + \alpha_2^2}, \quad I_6^c = \frac{\alpha_1 \beta_2 x - \alpha_2 \beta_1 x}{\alpha_1^2 + \alpha_2^2}, \]
\[ I_7^c = \frac{\alpha_1 \gamma_1 + \alpha_2 \gamma_2}{\alpha_1^2 + \alpha_2^2}, \quad I_8^c = \frac{\alpha_1 \gamma_2 - \alpha_2 \gamma_1}{\alpha_1^2 + \alpha_2^2}, \]
\[ I_9^c = \frac{(\alpha_1^2 - \alpha_2^2 x^2)(\beta_1^2 + \beta_2^2)}{(\alpha_1^2 + \alpha_2^2)(\beta_1^2 + \beta_2^2)} [\alpha_1 \beta_1(\beta_1 \gamma_1 t - \beta_2 \gamma_2 t - \gamma_1 \beta_1 t + \gamma_2 \beta_2 t) - \alpha_2 \beta_1(\beta_2 \gamma_1 t + \beta_1 \gamma_2 t - \gamma_2 \beta_2 t - \gamma_1 \beta_2 t)] + \alpha_1 \beta_1(\beta_2 \gamma_1 t + \beta_1 \gamma_2 t - \gamma_2 \beta_2 t - \gamma_1 \beta_2 t) - \alpha_2 \beta_1(\beta_1 \gamma_1 t - \beta_2 \gamma_2 t - \gamma_1 \beta_1 t + \gamma_2 \beta_2 t) \]  

(33)

\[ I_{10}^c = \frac{(\alpha_1^2 - \alpha_2^2 x^2)(\beta_1^2 + \beta_2^2)}{(\alpha_1^2 + \alpha_2^2)(\beta_1^2 + \beta_2^2)} [\alpha_2 \beta_1(\beta_1 \gamma_1 t - \beta_2 \gamma_2 t - \gamma_1 \beta_1 t + \gamma_2 \beta_2 t) + \alpha_1 \beta_1(\beta_2 \gamma_1 t + \beta_1 \gamma_2 t - \gamma_2 \beta_2 t - \gamma_1 \beta_2 t)] - \alpha_2 \beta_1(\beta_2 \gamma_1 t + \beta_1 \gamma_2 t - \gamma_2 \beta_2 t - \gamma_1 \beta_2 t) + \alpha_1 \beta_1(\beta_1 \gamma_1 t - \beta_2 \gamma_2 t - \gamma_1 \beta_1 t + \gamma_2 \beta_2 t) \]  

(34)
The correspondence of these semi-invariants of independent variables with the system of the hyperbolic-type equations is established due to the following operators

\[ I'_{11} = \frac{(a_2^2 - a_2^2)}{(a_1^2 + a_2^2)(a_1^2 + a_2^2)} \left[ (a_1 \gamma_{1,x} - a_2 \gamma_{2,x} - \gamma_1 \alpha_{1,x} + \gamma_2 \alpha_{2,x}) \alpha_{1,t} + (a_2 \gamma_{1,x} + a_1 \gamma_{2,x} - \gamma_2 \alpha_{2,x} - \gamma_1 \alpha_{2,x}) \alpha_{2,t} \right] \]

(35)

\[ I'_{12} = \frac{(a_2^2 - a_2^2)}{(a_1^2 + a_2^2)(a_1^2 + a_2^2)} \left[ (a_2 \gamma_{1,x} + a_1 \gamma_{2,x} + \gamma_2 \alpha_{2,x} - \gamma_1 \alpha_{1,x}) \alpha_{1,t} - (a_1 \gamma_{1,x} - a_2 \gamma_{2,x} - \gamma_1 \alpha_{2,x} + \gamma_2 \alpha_{1,x}) \alpha_{2,t} \right] \]

(36)

\[
I_{11} = \frac{(a_2^2 - a_2^2)}{(a_1^2 + a_2^2)(a_1^2 + a_2^2)} \left[ (a_1 \gamma_{1,x} - a_2 \gamma_{2,x} - \gamma_1 \alpha_{1,x} + \gamma_2 \alpha_{2,x}) \alpha_{1,t} + (a_2 \gamma_{1,x} + a_1 \gamma_{2,x} - \gamma_2 \alpha_{2,x} - \gamma_1 \alpha_{2,x}) \alpha_{2,t} \right]
\]

(37)

\[
I_{12} = \frac{(a_2^2 - a_2^2)}{(a_1^2 + a_2^2)(a_1^2 + a_2^2)} \left[ (a_2 \gamma_{1,x} + a_1 \gamma_{2,x} + \gamma_2 \alpha_{2,x} - \gamma_1 \alpha_{1,x}) \alpha_{1,t} - (a_1 \gamma_{1,x} - a_2 \gamma_{2,x} - \gamma_1 \alpha_{2,x} + \gamma_2 \alpha_{1,x}) \alpha_{2,t} \right]
\]

(38)

that are the real and imaginary parts of the complex generator \((\mathbf{11})\). Using these operators it is observed that

\[
\begin{align*}
X_1 & = 2 \xi_1 \partial_x + 2 \xi_2 \partial_x - \alpha_1 \xi_2 \partial_{\alpha_1} - \alpha_2 \xi_2 \partial_{\alpha_2} - \beta_1 \xi_{1,t} \partial_{\beta_1} - \beta_2 \xi_{1,t} \partial_{\beta_2} - \gamma_1 (\xi_{1,t} + \xi_{2,x}) \partial_{\gamma_1} \\
X_2 & = -\alpha_2 \xi_{2,x} \partial_{\alpha_1} + \alpha_1 \xi_{2,x} \partial_{\alpha_2} - \beta_2 \xi_{2,x} \partial_{\beta_1} + \beta_1 \xi_{2,x} \partial_{\beta_2} - \gamma_2 (\xi_{1,t} + \xi_{2,x}) \partial_{\gamma_1}
\end{align*}
\]

(39)

It is seen that the above invariants are complex splits of their real analogues \((13)\). Similarly, the linear combination of both \(X_1\) and \(X_2\), if denoted by \(X_3\), satisfy the relations

\[
\begin{align*}
\mathbf{X}_1^{[1]} I_1^c & = \mathbf{X}_2^{[1]} I_2^c = 0, \\
\mathbf{X}_1^{[1]} I_5^c & = \mathbf{X}_2^{[1]} I_6^c = 0, \\
\mathbf{X}_1^{[1]} I_9^c & = \mathbf{X}_2^{[1]} I_{10}^c = 0
\end{align*}
\]

(40)

To work out the joint invariants of the coupled system of two hyperbolic-type equations \((2)\), the operators \((37)\) and \((38)\) need to be transformed to the space of invariants \(h_\kappa, k_\kappa\). The same procedure was adopted in \((7)\) before using the generator \((11)\) in determining the joint invariants of the scalar linear hyperbolic equation. The complex generator was transformed to \(h\) and \(k\), i.e. to the space of the semi-invariants associated with the hyperbolic equation under a change of the dependent variables. The procedure to transform \((37)\) and \((38)\) to
$(h_\kappa, k_\kappa)$–space starts with splitting (14) when $Z(h)$ and $Z(k)$ are taken as complex, i.e. $Z(h) = Z(h)_1 + iZ(h)_2$ and $Z(k) = Z(k)_1 + iZ(k)_2$. The real and imaginary parts of (14) are

\[ X_1 = \frac{1}{2}[Z(h)_1\partial_{h_1} + Z(h)_2\partial_{h_2} + Z(k)_1\partial_{k_1} + Z(k)_2\partial_{k_2}], \]

\[ X_2 = \frac{1}{2}[Z(h)_2\partial_{h_1} - Z(h)_1\partial_{h_2} + Z(k)_2\partial_{k_1} - Z(k)_1\partial_{k_2}], \]

(41)

where

\[ Z(h)_1 = X_1 h_1 - X_2 h_2 = -(\xi_{1,t} + \xi_{2,x})h_1, \]

\[ Z(h)_2 = X_2 h_1 + X_1 h_2 = -(\xi_{1,t} + \xi_{2,x})h_2, \]

\[ Z(k)_1 = X_1 k_1 - X_2 k_2 = -(\xi_{1,t} + \xi_{2,x})k_1, \]

\[ Z(k)_2 = X_2 k_1 + X_1 k_2 = -(\xi_{1,t} + \xi_{2,x})k_2. \]

(42)

Using (42) in (41) yields the following two operators

\[ X_1 = -\frac{(\xi_{1,t} + \xi_{2,x})}{2}[h_1\partial_{h_1} + h_2\partial_{h_2} + k_1\partial_{k_1} + k_2\partial_{k_2}], \]

\[ X_2 = -\frac{(\xi_{1,t} + \xi_{2,x})}{2}[h_2\partial_{h_1} - h_1\partial_{h_2} + k_2\partial_{k_1} - k_1\partial_{k_2}], \]

(43)

that are the real and imaginary parts of the complex generator (15). These operators are used to arrive at the joint invariants for the system of two linear hyperbolic-type equations (2). We have the following joint invariants

\[ J_{11} = \frac{h_1 k_1 + h_2 k_2}{k_1^2 + k_2^2}, \]

\[ J_{12} = \frac{h_2 k_1 - h_1 k_2}{k_1^2 + k_2^2}, \]

\[ J_{13} = \frac{(h_1^5 - 10h_1^3 h_2^2 + 5h_1 h_2^4)(h_1 k_{1,t} - h_2 k_{2,t} - k_1 h_{1,t} + k_2 h_{2,t})}{(h_1^5 - 10h_1^3 h_2^2 + 5h_1 h_2^4)^2 + (5h_1^4 h_2 - 10h_1^2 h_2^3 + h_2^5)^2}(h_1 k_{1,x} - h_2 k_{2,x} - k_1 h_{1,x} + k_2 h_{2,x}) \]

\[ + \frac{(5h_1^4 h_2 - 10h_1^2 h_2^3 + h_2^5)(h_2 k_{1,t} + h_1 k_{2,t} - k_2 h_{1,t} - k_1 h_{2,t})}{(h_1^5 - 10h_1^3 h_2^2 + 5h_1 h_2^4)^2 + (5h_1^4 h_2 - 10h_1^2 h_2^3 + h_2^5)^2}(h_1 k_{1,x} - h_2 k_{2,x} - k_1 h_{1,x} + k_2 h_{2,x}) \]

\[ + \frac{(5h_1^4 h_2 - 10h_1^2 h_2^3 + h_2^5)(h_1 k_{1,t} - h_2 k_{2,t} - k_1 h_{1,t} + k_2 h_{2,t})}{(h_1^5 - 10h_1^3 h_2^2 + 5h_1 h_2^4)^2 + (5h_1^4 h_2 - 10h_1^2 h_2^3 + h_2^5)^2}(h_2 k_{1,x} + h_1 k_{2,x} - k_2 h_{1,x} - k_1 h_{2,x}) \]

\[ + \frac{(5h_1^4 h_2 - 10h_1^2 h_2^3 + h_2^5)(h_2 k_{1,t} + h_1 k_{2,t} - k_1 h_{1,t} - k_2 h_{2,t})}{(h_1^5 - 10h_1^3 h_2^2 + 5h_1 h_2^4)^2 + (5h_1^4 h_2 - 10h_1^2 h_2^3 + h_2^5)^2}(h_2 k_{1,x} + h_1 k_{2,x} - k_2 h_{1,x} - k_1 h_{2,x}), \]

\[ J_{14} = \frac{-(5h_1^4 h_2 - 10h_1^2 h_2^3 + h_2^5)(h_1 k_{1,t} - h_2 k_{2,t} - k_1 h_{1,t} + k_2 h_{2,t})}{(h_1^5 - 10h_1^3 h_2^2 + 5h_1 h_2^4)^2 + (5h_1^4 h_2 - 10h_1^2 h_2^3 + h_2^5)^2}(h_1 k_{1,x} - h_2 k_{2,x} - k_1 h_{1,x} + k_2 h_{2,x}) \]
\[
\begin{align*}
J_{15} &= \frac{(h_1^3 - 3h_1^2)(k_1k_{1,t} + k_2k_{2,t} - k_1k_{1,x} - h_1k_{2,x} - h_1k_{1,t} + h_2k_{2,t})}{(h_1^3 - 3h_1^2h_2^2 + 3h_1^2h_2^2 - h_2^3)k_2^2} + (3h_1^2h_2^2 - h_2^3), \\
J_{16} &= \frac{(h_2^3 - 3h_1^2h_2^2)(k_1k_{1,t} - k_2k_{2,t} + h_1k_{1,x} - h_2k_{2,x} - h_1k_{1,t} + h_2k_{2,t})}{(h_1^3 - 3h_1^2h_2^2 + 3h_1^2h_2^2 - h_2^3)k_2^2} + (3h_1^2h_2^2 - h_2^3), \\
J_{17} &= \frac{k_1(-6h_1^2h_2^2 + h_1^4 + 4h_2^4) + k_2(4h_1^3h_2 - 4h_1h_2^3)}{(-6h_1^2h_2^2 + h_1^4 + h_2^4) + (4h_1^3h_2 - 4h_1h_2^3)k_1(-6h_1^2h_2^2 + h_1^4 + h_2^4) + (4h_1^3h_2 - 4h_1h_2^3)}(h_1k_{1,x} - h_2k_{2,x} - h_1k_{1,t} + h_2k_{2,t}), \\
J_{18} &= \frac{k_2(-6h_1^2h_2^2 + h_1^4 + h_2^4) - k_1(4h_1^3h_2 - 4h_1h_2^3)}{(-6h_1^2h_2^2 + h_1^4 + h_2^4) + (4h_1^3h_2 - 4h_1h_2^3)k_2(-6h_1^2h_2^2 + h_1^4 + h_2^4) + (4h_1^3h_2 - 4h_1h_2^3)}(h_1k_{1,x} - h_2k_{2,x} - h_1k_{1,t} + h_2k_{2,t}), \\
J_{19} &= \frac{\mu_1v_1 + \mu_2v_2}{\mu_1^2 + \mu_2^2} + \frac{\mu_2v_1 - \mu_1v_2}{\mu_1^2 + \mu_2^2}, \\
J_{20} &= \frac{\mu_1v_2 - \mu_2v_1}{\mu_1^2 + \mu_2^2} + \frac{\mu_1v_1 + \mu_2v_2}{\mu_1^2 + \mu_2^2}, \\
J_{21} &= \frac{\mu_1v_3 + \mu_2v_4}{\mu_1^2 + \mu_2^2} + \frac{\mu_1v_4 - \mu_2v_3}{\mu_1^2 + \mu_2^2}, \\
J_{22} &= \frac{\mu_1v_4 - \mu_2v_3}{\mu_1^2 + \mu_2^2} + \frac{\mu_1v_3 + \mu_2v_4}{\mu_1^2 + \mu_2^2},
\end{align*}
\]

where

\[
\begin{align*}
\mu_1 &= h_1^9 - 36h_1^7h_2^2 + 126h_1^5h_2^4 - 84h_1^3h_2^6 + 9h_1h_2^8, \\
\mu_2 &= 9h_1^8h_2 - 84h_1^6h_2^4 + 126h_1^4h_2^6 - 36h_1^2h_2^8 + h_2^9,
\end{align*}
\]
\[ \nu_1 = k_2^2 h_{2,x}^2 + 2h_1 k_{2,x} k_{1,x} + 2h_2 k_{2,x} k_{2,x} + 2h_2 k_{1,x} k_{2,x} - 4k_1 h_{1,x} k_{2,x} - k_1^2 h_{2,x}^2 + h_2^2 k_{2,x} \\
+ 2h_2 k_{1,x} k_{2,x} - 2h_1 k_{1,x} k_{1,x} + 2h_2 k_{2,x} k_{1,x} + k_1^2 h_{1,x}^2 + h_1^2 k_{1,x}^2 + 2h_1 k_{2,x} k_{2,x} - h_2^2 k_{1,x}^2 \\
- 4h_1 k_{1,x} h_{2,x} - h_1^2 k_{2,x}^2 - h_2^2 h_{1,x}^2 + 2h_1 k_{1,x} k_{2,x}, \]
\[ \nu_2 = -2k_2^2 h_{2,x} k_1 - 2h_1 k_{1,x} k_{1,x} - 2k_1 h_{1,x} k_{2,x} + 2h_2 k_{2,x} k_{1,x} - 2k_2 h_{2,x} h_{1,x} + 2h_2 k_{2,x} k_{2,x} + 2k_2 h_{2,x} h_{2,x} + 2h_1 k_{2,x} k_{2,x} - 2h_2 k_{2,x} h_1 \\
+ 2h_1^2 k_{1,x} k_{2,x} + 2k_1 h_{1,x} h_{2,x} + 2h_1 k_{1,x} k_{2,x} + 2h_1^2 k_{1,x}^2 h_1 + h_1^3 k_{1,x}^2 + 2k_1 h_{1,x} h_{2,x} + 2h_1 k_{1,x} k_{2,x} + 2k_1^2 h_{1,x}^2 h_1, \]
\[ \nu_3 = -2h_2 k_{2,x} k_2 h_{2,t} - k_1^2 h_{2,x}^2 + 2h_1 k_{1,x} k_{2,x} - h_1^2 k_{1,x}^2 + 2h_2 k_{2,t} k_{1,x} + h_1^2 k_{1,x}^2 + 2h_1 k_{1,x} k_{1,x} - 2h_1 k_{1,x} k_{1,x} \\
+ k_1^2 h_{1,x}^2 + k_1^2 h_{1,t}^2 - 4h_1 k_{2,t} h_{1,t} - k_1^2 h_{1,t}^2 + 2h_1 k_{1,t} k_{2,t} + 2h_1 k_{2,t} k_{1,t} - h_1^2 k_{1,t}^2 \\
- 4k_1 h_{1,t} k_{2,t} - 2h_2 k_{1,t} k_{2,t} + 2h_1 k_{1,t} k_{2,t} + 2h_1 k_{1,t} k_{2,t} + h_1^2 k_{2,t}^2, \]
\[ \nu_4 = 2h_2 k_{2,x} k_2 h_{1,t} + 2h_1 k_{1,x} k_2 + 2h_1 k_{1,x} k_2 + 2h_2 k_{1,x} k_2 + 2k_1 h_{1,x} k_2 - 2h_1 k_{2,t} k_{1,x} + 2h_1^2 k_{1,x}^2 \\
- 2k_1 h_{1,x} h_{2,t} - 2h_2 k_{1,x} k_{2,t} + 2h_2 k_{1,x} k_{2,t} + 2h_2 k_{1,x} k_{2,t} + k_1^2 h_{1,t} h_{2,t} - k_1^2 h_{1,t} h_{2,t} \\
- 2h_2 k_{1,x} k_{1,t} - 2h_1 k_{1,t} k_{2,t} - 2h_1 k_{1,t} k_{2,t} - 2h_1^2 k_{1,t} h_{2,t}, \]
and
\[ \omega_1 = (h_1 k_1 - h_2 k_2) h_{1,t} + (h_2 k_1 - h_2 k_2) h_{2,t} + (h_1 k_1 + h_2 k_2) h_{2,t} + (h_1 k_1 + h_2 k_2) h_{2,t} + 2h_2 h_{2,t} - 3k_1 (h_{1,t} - h_{2,t}) \\
+ 6k_2 h_{1,t} h_{2,t} - (3h_1 h_{1,t} - 3h_2 h_{1,t}) k_{1,t} - (3h_2 h_{1,t} + 3h_1 h_{2,t}) k_{2,t}, \]
\[ \omega_2 = (h_2 k_1 + h_2 k_2) h_{1,t} + (h_1 k_1 - h_2 k_2) h_{2,t} + (h_1 k_1 - h_2 k_2) h_{2,t} + 2h_1 h_{1,t} + 3k_2 (h_{1,t} - h_{2,t}) \\
- 6k_1 h_{1,t} h_{2,t} - (3h_2 h_{1,t} + 3h_1 h_{2,t}) k_{1,t} + (3h_1 h_{1,t} - 3h_2 h_{2,t}) k_{2,t}, \]
\[ \omega_3 = (h_1 k_1 - h_2 k_2) h_{1,x} + (h_2 k_1 - h_2 k_2) h_{2,x} + (h_1 k_1 - h_2 k_2) h_{2,x} + 2h_1 h_{2,x} - 3k_1 (h_{1,x} - h_{2,x}) \\
+ 6k_2 h_{1,x} h_{2,x} + (3h_1 h_{1,x} - 3h_2 h_{1,x}) k_{1,x} - (3h_2 h_{1,x} + 3h_1 h_{2,x}) k_{2,x}, \]
\[ \omega_4 = (h_2 k_1 + h_2 k_2) h_{1,x} + (h_1 k_1 - h_2 k_2) h_{2,x} + (h_1 k_1 - h_2 k_2) h_{2,x} + 2h_1 h_{2,x} - 3k_2 (h_{1,x} - h_{2,x}) \\
- 6k_1 h_{1,x} h_{2,x} + (3h_2 h_{1,x} + 3h_1 h_{2,x}) k_{1,x} + (3h_1 h_{1,x} - 3h_2 h_{2,x}) k_{2,x}, \]
which are found to be associated with the system of two linear hyperbolic-type PDEs (2). These also can be observed to be the complex split of the joint invariants (16).

5. Applications

In this section a few examples of systems of hyperbolic-type equations are provided to illustrate the invariance criteria developed.

1. A system of two hyperbolic-type PDEs
\[ u_{tx} + \left( a_1 - \frac{1}{x} \right) u_t - a_2 v_t + \left( b_1 + \frac{2}{t} \right) u_x - b_2 v_x + \left( c_1 - \frac{b_1}{x} + 2 \frac{a_1}{t} - \frac{2}{t,x} \right) u \]
corresponds to a complex hyperbolic equation in two independent variables

\[ w_{tx} + (a - \frac{1}{x}) w_t + (b + \frac{2}{t}) w_x + \left( c - \frac{b}{x} + 2\frac{a}{t} - \frac{2}{tx} \right) w = 0, \]  

where \( a \) is a complex constant \( a = a_1 + ia_2 \). The following complex transformation of the dependent variable \( w = (x/t^2)\overline{w} \) maps the complex equation (45) to

\[ \overline{w}_{tx} + a\overline{w}_t + b\overline{w}_x + c\overline{w} = 0. \]  

Both the complex hyperbolic equations (45) and (46) are transformable to each other because they have the same semi-invariants

\[ h = ab - c = k. \]  

The system of hyperbolic-type equations (44) is transformable to

\[ \overline{u}_{tx} + a_1\overline{u}_t - a_2\overline{u}_x + b_1\overline{u}_x - b_2\overline{u}_x + c_1\overline{u} - c_2\overline{u} = 0, \]
\[ \overline{v}_{tx} + a_2\overline{v}_t + a_1\overline{v}_x + b_2\overline{v}_x + b_1\overline{v}_x + c_2\overline{v} + c_1\overline{v} = 0, \]  

The real transformations of the dependent variables

\[ u = (x/t^2)\overline{w}, \quad v = (x/t^2)\overline{v}, \]  

are obtained by splitting the complex dependent transformation used to map the complex equations (45) and (46) into each other. Semi-invariants associated with both the systems (48) are

\[ h_1 = a_1 b_1 - a_2 b_2 - c_1 = k_1, \]
\[ h_2 = a_1 b_2 + a_2 b_1 - c_2 = k_2, \]  

therefore, both being transformable to each other.

2. An uncoupled system of PDEs

\[ u_{x_1 x_2} + 2a_1^2 u_{x_1} + 2b_1 u_{x_2} + 4c_1 u = 0, \]
\[ v_{x_1 x_2} + 2a_2^2 v_{x_1} + 2b_2 v_{x_2} + 4c_2 v = 0, \]  

13
is transformable to
\begin{align*}
    u_{tx} + a tu_t + b u_x + cu &= 0, \\
    v_{tx} + a tv_t + b v_x + cv &= 0,
\end{align*}
via invertible transformations of the independent variables
\[ z_1 = \sqrt{t}, \quad z_2 = \frac{1}{2}(x - 1). \] (53)
These are the invertible maps that also reduce a complex hyperbolic equation of the form
\[ w z_1 z_2 + 2a z_1^2 w z_1 + 2b z_1 w z_2 + 4c z_1 w = 0, \] (54)
with the semi-invariants
\[ I_1 = \frac{c}{abz_1^2}, \quad I_2 = b z_1^2, \quad I_3 = 0, \quad I_4 = \frac{c}{a}, \quad I_5 = 0 = I_6, \] (55)
to a simple linear form
\[ w_{tx} + atw_t + bw_x + cw = 0, \] (56)
with the following semi-invariants
\[ I_1 = \frac{c}{abt}, \quad I_2 = bt, \quad I_3 = 0, \quad I_4 = \frac{c}{a}, \quad I_5 = 0 = I_6. \] (57)
Notice that the semi-invariants (55) and (57) are the same by means of the transformations of the independent variables (53). The complex hyperbolic equation (54) does not only yield an uncoupled system of the hyperbolic-type equations (51). In fact it gives a coupled system
\begin{align*}
    u_{z_1 z_2} + 2a z_1^2 u_{z_1} - 2a_2 z_1^2 v_{z_1} + 2b z_1 u_{z_2} - 2b_2 z_1 v_{z_2} + 4c z_1 u - 4c z_1 v &= 0, \\
    v_{z_1 z_2} + 2a_2 z_1^2 u_{z_1} + 2a_1 z_1^2 v_{z_1} + 2b_2 z_1 u_{z_2} + 2b_1 z_1 v_{z_2} + 4c_2 z_1 u + 4c_1 z_1 v &= 0.
\end{align*}
(58)
This system of two hyperbolic-type equations can be mapped to
\begin{align*}
    u_{tx} + a_1 tu_t - a_2 tv_t + b_1 u_x - b_2 v_x + c_1 u - c_2 v &= 0, \\
    v_{tx} + a_2 tu_t + a_1 tv_t + b_2 u_x + b_1 v_x + c_2 u + c_1 v &= 0,
\end{align*}
(59)
under the transformations (53) that are already used to map the base complex equation to its canonical form.

3. Invoking the following change of the independent variables
\[ z_1 = e^t, \quad z_2 = \sqrt{x}, \] (60)
in a coupled system of two hyperbolic-type equations of the form
\[
\begin{align*}
  u_{z_1,z_2} + 2a_1 z_2 \ln z_1 u_{z_1} - 2a_2 z_2 \ln z_1 v_{z_1} + \frac{b_1}{z_1} u_{z_2} - \frac{b_2}{z_1} v_{z_2} + \frac{2c_1 z_2}{z_1} u - \frac{2c_2 z_2}{z_1} v &= 0, \\
  v_{z_1,z_2} + 2a_2 z_2 \ln z_1 u_{z_1} + 2a_1 z_2 \ln z_1 v_{z_1} + \frac{b_2}{z_1} u_{z_2} + \frac{b_1}{z_1} v_{z_2} + \frac{2c_2 z_2}{z_1} u + \frac{2c_1 z_2}{z_1} v &= 0,
\end{align*}
\]
transforms it to
\[
\begin{align*}
  u_t x + a_1 u_t - a_2 v_t + b_1 u_x - b_2 v_x + c_1 u - c_2 v &= 0, \\
  v_t x + a_2 u_t + a_1 v_t + b_2 u_x + b_1 v_x + c_2 u + c_1 v &= 0.
\end{align*}
\]
The transformation of these systems under the invertible change of the independent variables follows from the base complex hyperbolic equation
\[
\begin{align*}
  w_{z_1,z_2} + 2a \ln z_1 w_{z_1} + \frac{b}{z_1} w_{z_2} + \frac{2c z_2}{z_1} w &= 0.
\end{align*}
\]
It can be transformed to another linear form
\[
\begin{align*}
  w_t x + a w_t + b w_x + c w &= 0,
\end{align*}
\]
under the invertible transformations \((60)\). Similarly, the invertible transformations of the independent variables \((60)\) map the following system of PDEs
\[
\begin{align*}
  u_{z_1,z_2} + 2a_1 z_2 u_{z_1} - 2a_2 z_2 v_{z_1} + \frac{b_1}{z_1} u_{z_2} - \frac{b_2}{z_1} v_{z_2} + \frac{2c_1 z_2}{z_1} u - \frac{2c_2 z_2}{z_1} v &= 0, \\
  v_{z_1,z_2} + 2a_2 z_2 u_{z_1} + 2a_1 z_2 v_{z_1} + \frac{b_2}{z_1} u_{z_2} + \frac{b_1}{z_1} v_{z_2} + \frac{2c_2 z_2}{z_1} u + \frac{2c_1 z_2}{z_1} v &= 0,
\end{align*}
\]
to
\[
\begin{align*}
  u_t x + a_1 u_t - a_2 v_t + b_1 u_x - b_2 v_x + c_1 u - c_2 v &= 0, \\
  v_t x + a_2 u_t + a_1 v_t + b_2 u_x + b_1 v_x + c_2 u + c_1 v &= 0.
\end{align*}
\]

\textbf{4. Consider an uncoupled system of two hyperbolic type PDEs}
\[
\begin{align*}
  g_{1,t} x + \frac{\lambda}{2} (g_{1,t} + g_{1,x}) &= 0, \\
  g_{2,t} x + \frac{\lambda}{2} (g_{2,t} + g_{2,x}) &= 0,
\end{align*}
\]
for which \(h_1 = k_1 = \frac{\lambda^2}{4}\), and \(h_2 = k_2 = 0\). This implies that
\[
\begin{align*}
  J_1 = 1, \
  J_2 = \cdots = J_{12} = 0.
\end{align*}
\]
The system (67) is transformable to another system with the same invariants as given in (68) where \( h_1 = k_1 = -1, \ h_2 = k_2 = 0 \). The transformed system reads as
\[
\begin{align*}
    f_{1, z_1 z_2} + f_1 &= 0, \\
    f_{2, z_1 z_2} + f_2 &= 0.
\end{align*}
\] (69)

The correspondence between the systems (67) and (69) is established by
\[
\begin{align*}
    z_1 &= \frac{\lambda}{2} t, \quad z_2 = -\frac{\lambda}{2} x, \\
    f_1 &= g_1 \exp\left(\frac{\lambda t + \lambda x}{2}\right), \quad f_2 = g_2 \exp\left(\frac{\lambda t + \lambda x}{2}\right).
\end{align*}
\] (70)

These transformations are obtainable from
\[
\begin{align*}
    z_1 &= \frac{\lambda}{2} t, \quad z_2 = -\frac{\lambda}{2} x, \\
    w &= u \exp\left(\frac{\lambda t + \lambda x}{2}\right),
\end{align*}
\] (71)

by \( w = f_1 + if_2 \) and \( u = g_1 + ig_2 \). The complex transformations map the complex scalar PDE
\[
w_{1, z_1 z_2} + \frac{\lambda}{2} (w_{z_1} + w_{z_2}) = 0,
\] (72)

with \( h = k = \frac{\lambda^2}{4} \) and \( p = 1 \), to an equation
\[
u_{,tx} + u = 0,
\] (73)

for which \( h = k = -1 \) and \( p = 1 \). Notice that the substitution \( \lambda = \lambda_1 + i\lambda_2 \), in the equation (72) results in a coupled system of two hyperbolic-type PDEs but it cannot be transformed by the complex method. The reason is the complex transformations (71) where the two independent variables split into four add extra dimensions. Therefore, the complex procedure fails for that case.

5. The complex transformations of the form
\[
\begin{align*}
    z_1 &= \frac{1}{t}, \quad z_2 = 2x, \\
    w &= \frac{u}{x},
\end{align*}
\] (74)

map the following Lie canonical form
\[
w_{1, z_1 z_2} + \alpha z_2^2 w_{z_2} + 2w = 0,
\] (75)

to
\[
u_{,tx} - \frac{1}{x} u_{,t} - \frac{\alpha x^2}{t^2} u_{,x} + \frac{1}{t^2} (\alpha x - 2) u = 0.
\] (76)

The invariant quantities associated with both the scalar Lie canonical form and the hyperbolic equations are \( h = -1, \ k = 2\alpha x - 1, \ p = 2(1 - \alpha x) \) and \( h = 2/t^2, \ k = \frac{2(1-\alpha x)}{t^2}, \ p = 1 - \alpha x \).
respectively. Inserting \( u = g_1 + ig_2 \) in the equation (76) while keeping \( \alpha \) a real constant yields an uncoupled system of two PDEs

\[
g_{1,t} - \frac{1}{x}g_{1,t} - \frac{\alpha x^2}{t^2} g_{1,x} + \frac{\alpha x - 2}{t^2} g_1 = 0, \\
g_{2,t} - \frac{1}{x}g_{2,t} - \frac{\alpha x^2}{t^2} g_{2,x} + \frac{\alpha x - 2}{t^2} g_2 = 0.
\] (77)

The system (77) is transformable to another system of the form

\[
f_{1,z_1z_2} + \alpha x^2 f_{1,z_2} + 2f_1 = 0, \\
f_{2,z_1z_2} + \alpha x^2 f_{2,z_2} + 2f_2 = 0,
\] (78)

under a change of the dependent and independent variables

\[
z_1 = \frac{1}{t}, \quad z_2 = 2x, \quad f_1 = \frac{g_1}{x}, \quad f_2 = \frac{g_2}{x}. \quad (79)
\]

These transformations are the real and imaginary parts of the complex transformations (74) and the transformed system is obtained by splitting the Lie canonical form (75) into the real and imaginary parts. The invariance criteria that ensure such a transformation of the system are satisfied. These quantities for both the systems (77) and (78) are

\[
h_1 = \frac{2}{t^2}, \quad k_1 = \frac{2(1 - \alpha x)}{t^2}, \quad h_2 = 0 = k_2, \quad p = \frac{-1}{\alpha x - 1}, \quad (80)
\]

and

\[
h_1 = -2, \quad k_1 = 2(\alpha x - 1), \quad h_2 = 0 = k_2, \quad p = \frac{1}{1 - \alpha x}, \quad (81)
\]

respectively.

A coupled system

\[
g_{1,t} - \frac{1}{x}g_{1,t} - \frac{\alpha_1 x^2}{t^2} g_{1,x} + \frac{\alpha x^2}{t^2} g_{2,x} + \frac{\alpha_1 x - 2}{t^2} g_1 - \frac{\alpha_2 x}{t^2} g_2 = 0, \\
g_{2,t} - \frac{1}{x}g_{2,t} - \frac{\alpha_2 x^2}{t^2} g_{1,x} - \frac{\alpha_1 x^2}{t^2} g_{2,x} + \frac{\alpha_2 x}{t^2} g_1 + \frac{\alpha_1 x - 2}{t^2} g_2 = 0.
\] (82)

with the invariants

\[
h_1 = \frac{2}{t^2}, \quad k_1 = \frac{2(1 - \alpha_1 x)}{t^2}, \quad h_2 = 0, \quad k_2 = \frac{-2\alpha_2 x}{t^2}, \\
J_1 = \frac{1 - \alpha_1 x}{(1 - \alpha_1 x)^2 + \alpha_2 x^2}, \quad J_2 = \frac{\alpha_2 x}{(1 - \alpha_1 x)^2 + \alpha_2 x^2}, \quad (83)
\]
is obtainable from the complex scalar PDE (76) when \( \alpha \) is also complex, i.e., \( \alpha = \alpha_1 + i\alpha_2 \). Employing the transformations (79) on (82) one arrives at a coupled system

\[
\begin{align*}
    f_{1, z_1 z_2} + \alpha_2 x^2 f_{1, z_2} - \alpha_2 x^2 f_{2, z_2} + 2f_1 &= 0, \\
    f_{2, z_1 z_2} + \alpha_2 x^2 f_{1, z_2} + \alpha_1 x^2 f_{2, z_2} + 2f_2 &= 0,
\end{align*}
\]

which is the real analogue of the complex transformed equation (73) and satisfies the invariance criteria, where

\[
\begin{align*}
    h_1 &= -2, \quad k_1 = 2(\alpha_1 x - 1), \quad h_2 = 0, \quad k_2 = 2\alpha_2 x, \\
    J_1 &= \frac{1 - \alpha_1 x}{(1 - \alpha_1 x)^2 + \alpha_2^2 x^2}, \quad J_2 = \frac{\alpha_2 x}{(1 - \alpha_1 x)^2 + \alpha_2^2 x^2}.
\end{align*}
\]

6. Conclusion

Semi-invariants of the hyperbolic and parabolic PDEs in two independent variables have been obtained by transforming the dependent or independent variables [2, 3, 5, 6, 7]. Further, the infinitesimal approach has been utilized to derive the joint invariants for the hyperbolic and parabolic equations [7, 8, 9, 10, 11]. The semi-invariants of the hyperbolic and parabolic PDEs have been extended to systems of such equations by complex symmetry analysis [4, 12]. The real and complex approaches were investigated in this work for the invariants of a system of two linear hyperbolic equations.

Semi-invariants of a special class of systems of two hyperbolic-type PDEs were derived here using real and complex methods developed for such systems of equations. Both the procedures are adopted to find the semi-invariants of the system of two hyperbolic-type equations that is obtainable from a complex hyperbolic PDE. Semi-invariants associated with the invertible change of the dependent as well as independent variables are deduced by both the real and complex methods. It is shown that same invariant quantities for the system of hyperbolic-type PDEs appear due to complex and real procedures, in the case of transformations of only the dependent variables. However, the semi-invariants of this system obtained by real symmetry analysis are different from those provided by the complex procedure. Furthermore, the joint invariants of this system of hyperbolic-type equations obtained by both the methods are also found to be different.

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