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Olivier Bernardi

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SHORT PROOF OF RAYLEIGH’S THEOREM AND EXTENSIONS

OLIVIER BERNARDI

Abstract. Consider a walk in the plane made of $n$ unit steps, with directions chosen independently and uniformly at random at each step. Rayleigh’s theorem asserts that the probability for such a walk to end at a distance less than 1 from its starting point is $1/(n+1)$. We give an elementary proof of this result. We also prove the following generalization valid for any probability distribution $\mu$ on the positive real numbers: if two walkers start at the same point and make respectively $m$ and $n$ independent steps with uniformly random directions and with lengths chosen according to $\mu$, then the probability that the first walker ends farther than the second is $m/(m+n)$.

We consider random walks in the Euclidean plane. Given some real positive random variables $X_1, X_2, \ldots, X_n$, we consider a random walk starting at the origin of the plane and made of $n$ steps of respective length $X_1, X_2, \ldots, X_n$, with the direction of each step chosen independently and uniformly at random. We denote by $X_1 \oplus X_2 \oplus \cdots \oplus X_n$ the random variable corresponding to the distance between the origin and the end of the walk. This definition is illustrated in Figure 1(a).

Figure 1. (a) The distance $X_1 \oplus X_2 \oplus X_3 \oplus X_4$ achieved after four steps. (b) Comparing the distances $D_m = m \odot X$ and $D_n = n \odot X$.

For a non-negative real random variable $X$, we denote by $n \odot X$ the random variable $X_1 \oplus \cdots \oplus X_n$, where $X_1, \ldots, X_n$ are independent copies of $X$. Hence $n \odot X$ represents the final distance from the origin after taking $n$ independent steps of lengths distributed like $X$ and directions chosen uniformly at random. Rayleigh’s theorem asserts that if $X = 1$, that is, each step has unit length, then for all $n > 1$,

$$\mathbb{P}(n \odot X < 1) = \frac{1}{n+1}.$$ 

This theorem was first derived from Rayleigh’s investigation of “random flights” in connection with Bessel functions (see [3]) and appears as an exercise in [2, p.104]. A simpler proof was given by Kenyon and Winkler as a corollary of their result on branched polymers [1]. The goal of this note is to give an elementary proof of the following generalization of Rayleigh’s theorem.

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1 The exercise calls for developing the requisite Fourier analysis for spherically symmetric functions in order to obtain an identity involving Bessel functions.
Theorem 1. Let $X$ be a real random variable taking positive values, and let $m, n$ be non-negative integers such that $m + n > 2$. If $D_m$ and $D_n$ are independent random variables distributed respectively like $m \circ X$ and $n \circ X$, then

$$
P(D_m > D_n) = \frac{m}{m + n}.
$$

In words, if two random walkers start at the origin and take respectively $m$ and $n$ independent steps with uniformly random directions and with lengths chosen according to the distribution of $X$, then the probability that the first walker ends farther from the origin than the second walker is $m/(m + n)$.

Theorem 1 is illustrated in Figure 1(b). Clearly, this extends Rayleigh’s theorem which corresponds to the case $m = 1$ and $X = 1$. Our proof of Theorem 1 starts with a lemma based on the fact that the angles of a triangle sum to $\pi$.

Lemma 2. For any random variables $A, B, C$ taking real positive values,

$$(1) \quad \mathbb{P}(A > B \oplus C) + \mathbb{P}(B > A \oplus C) + \mathbb{P}(C > A \oplus B) = 1.$$

Proof. By conditioning on the values of the random variables $A, B, C$, it is sufficient to prove (1) in the case where $A, B, C$ are non-random positive constants, and the randomness only resides in the directions of the steps. Now we consider two cases. First suppose that one of the lengths $A, B, C$ is greater than the sum of the two others. In this case, one of the probabilities appearing in (1) is 1 and the others are 0, hence the identity holds. Now suppose that none of the lengths $A, B, C$ is greater than the sum of the two others. In this case, there exists a triangle $T$ with side lengths $A, B, C$. The triangle $T$ is shown in Figure 2. The probability $\mathbb{P}(A > B \oplus C)$ is equal to $\alpha/\pi$, where $\alpha$ is the angle between the sides of length $B$ and $C$ in the triangle $T$ (because $A > B \oplus C$ if and only if the angle between the step of length $B$ and the step of length $C$ is less than $\alpha$ in absolute value). Summing this relation for the three probabilities appearing in (1) gives

$$
\mathbb{P}(A > B \oplus C) + \mathbb{P}(B > A \oplus C) + \mathbb{P}(C > A \oplus B) = \frac{\alpha + \beta + \gamma}{\pi} = 1.
$$

where $\alpha, \beta, \gamma$ are the angles appearing in Figure 2. \qed

![Figure 2. The triangle $T$ with side lengths $A, B, C$.](image)

We now complete the proof of Theorem 1. Let $s = m + n$ and let $D_0, D_1, \ldots, D_s$ be independent random variables distributed respectively like $0 \circ X, 1 \circ X, \ldots, s \circ X$. We denote $p_i = \mathbb{P}(D_i > D_{s-i})$ and want to prove $p_m = m/s$. Let $i, j, k$ be positive integers summing to $s$. Applying Lemma 2 to $A = D_i$, $B = D_j$, $C = D_k$ gives $p_i + p_j + p_k = 1$. Moreover, $p_k = 1 - p_{s-k}$ since $\mathbb{P}(D_k = D_{s-k}) = 0$ (recall that $s > 2$). Thus

$$
p_i + p_j = p_{i+j},
$$

for all $i, j > 0$ such that $i + j \leq n$. By induction, this implies $i p_1 = p_i$ for all $i \in \{1, \ldots, n\}$. In particular $p_1 = p_s/s = 1/s$, and $p_m = m p_1 = m/s$. This concludes the proof of Theorem 1.

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