Operational calculus for holonomic distributions in the framework of $D$-module theory

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Abstract

Let $f$ be a real polynomial of $x = (x_1, \ldots, x_n)$ and $\varphi$ be a locally integrable function of $x$ which satisfies a holonomic system of linear differential equations. We study the distribution $f^\lambda \varphi$ with a meromorphic parameter $\lambda$, especially its Laurent expansion and integration, from an algorithmic viewpoint in the framework of $D$-module theory.

1 Introduction

Let $f$ be a non-constant real polynomial in $x = (x_1, \ldots, x_n)$ and $\varphi$ be a locally integrable function on an open subset $U$ of $\mathbb{R}^n$. Then $\varphi$ can be regarded as a distribution (generalized function in the sense of L. Schwartz) on $U$. We assume that there exists a left ideal $I$ of the ring $D_n$ of differential operators with polynomial coefficients in $x$ which annihilates $\varphi$ on $U_f := \{ x \in U \mid f(x) \neq 0 \}$, i.e., $P\varphi$ vanishes on $U_f$ for any $P \in I$. Moreover, we assume that $M := D_n/I$ is a holonomic $D_n$-module. In this situation, $\varphi$ is called a (locally integrable) holonomic function or a holonomic distribution.

Let us consider the distribution $f^\lambda \varphi$ on $U$ with a holomorphic parameter $\lambda$. This distribution can be analytically extended to a distribution-valued meromorphic function of $\lambda$ on the complex plane $\mathbb{C}$. Such a distribution was systematically studied by Kashiwara and Kawai in [2] with $f$ being, more generally, a real-valued real analytic function. Their investigation was focused on a special case where $M$ has regular singularities but most of the arguments work without this assumption.

The main purpose of this article is to give algorithms to compute

1. A holonomic system for the distribution $f^{\lambda_0} \varphi$ with $\lambda_0$ not being a pole of $f^\lambda \varphi$. 


2. A holonomic system for each coefficient of the Laurent series of $f^λ + \varphi$ about an arbitrary point.

3. Difference equations for the local zeta function $Z(\lambda) = \int_{\mathbb{R}^n} f^λ \varphi \, dx$.

As was pointed out in [2], an answer to the first problem provides us with an algorithm to compute a holonomic system for the product of two locally $L^2$ holonomic functions. Note that the product does not necessarily satisfies the tensor product of the two holonomic systems for both functions.

In Section 2, we review the theoretical properties of $f^λ \varphi$ mostly following Kashiwara [1] and Kashiwara and Kawai [2] in the analytic category; i.e., under a weaker assumption that $f$ is a real-valued real analytic function and that $\varphi$ satisfies a holonomic system of linear differential equations with analytic coefficients.

In Section 3, we give algorithms to compute holonomic systems considered in Section 2. As a byproduct, we obtain an algorithm to compute difference equations for the local zeta function, which was outlined in [4].

2 Theoretical background

Let $\mathcal{D}_{\mathbb{C}^n}$ be the sheaf on $\mathbb{C}^n$ of linear partial differential operators with holomorphic coefficients, which is generated by the derivations $\partial_j = \partial x_j = \partial / \partial x_j$ $(j = 1, \ldots, n)$ over the sheaf $\mathcal{O}_{\mathbb{C}^n}$ of rings of holomorphic functions on $\mathbb{C}^n$, with the coordinate system $x = (x_1, \ldots, x_n)$ of $\mathbb{C}^n$.

We denote by $\mathcal{D} b$ the sheaf on $\mathbb{R}^n$ of the Schwartz distributions. Assume that $f = f(x)$ is a nonzero real-valued real analytic function defined on an open connected set $U$ of $\mathbb{R}^n$. Let $\varphi$ be a locally integrable function on $U$. Then $f^λ \varphi$ is also locally integrable on $U$ for any $\lambda \in \mathbb{C}$ with $\text{Re} \, \lambda \geq 0$, where $f^λ(x) = \max\{f(x), 0\}$.

Let $\mathcal{M}$ be a holonomic $\mathcal{D}_{\mathbb{C}^n}$-module defined on an open set $\Omega$ of $\mathbb{C}^n$ such that $U \subset \Omega \cap \mathbb{R}^n$. We say that a distribution $\varphi$ is a solution of $\mathcal{M}$ on $U$ if there exist a section $u$ of $\mathcal{M}$ on $U$ and a $\mathcal{D}_{\mathbb{C}^n}$-linear homomorphism $\Phi : \mathcal{D}_{\mathbb{C}^n} u \to \mathcal{D} b$ defined on $U$ such that $\Phi(u) = \varphi$. As a matter of fact, we have only to assume that $\varphi$ is a solution of $\mathcal{M}$ on $U_f := \{x \in U \mid f(x) \neq 0\}$ and that $\mathcal{M}$ is holonomic on $\Omega_f := \{x \in \Omega \mid f(x) \neq 0\}$.

2.1 Fundamental lemmas

Under the assumptions above, $f^λ \varphi$ is a $\mathcal{D}b(U)$-valued holomorphic function of $\lambda$ on the right half-plane

$$\mathbb{C}_+ := \{\lambda \in \mathbb{C} \mid \text{Re} \, \lambda > 0\}.$$
In other words, let $\mathcal{OD}b$ be the sheaf on $\mathbb{C} \times \mathbb{R}^n \ni (\lambda, x)$ of distributions with a holomorphic parameter $\lambda$. Then $f_+^A \varphi$ belongs to

$$\mathcal{OD}b(\mathbb{C}_+ \times U) = \left\{ v(\lambda, x) \in \mathcal{D}b(\mathbb{C}_+ \times U) \mid \frac{\partial v}{\partial \lambda} = 0 \right\}.$$ 

Let $s$ be an indeterminate corresponding to $\lambda$. The following lemma (Lemma 2.9 of [2]) plays an essential role in the following arguments.

**Lemma 2.1 (Kashiwara-Kawai [2])** Let $\Omega$ be an open set of $\mathbb{C}^n$ such that $V := \mathbb{R}^n \cap \Omega$ is non-empty. Assume $P(s) \in \mathcal{D}[\mathbb{C}^n(\Omega)[s]$ and $P(\lambda)(f_+^A \varphi) = 0$ holds in $\mathcal{OD}b(\mathbb{C}_+ \times V_f)$ with $V_f := \{ x \in V \mid f(x) \neq 0 \}$. Then $P(\lambda)(f_+^A \varphi) = 0$ holds in $\mathcal{OD}b(\mathbb{C}_+ \times V)$.

Let us generalize this lemma slightly. For a positive integer $m$, let us define a section $f_+^A \varphi$ of the sheaf $\mathcal{OD}b$ on $\mathbb{C}_+ \times U$ by

$$\langle f_+^A (\log f_+)^m \varphi, \psi \rangle = \int_{\{ x \in U \mid f(x) > 0 \}} \varphi(x) f(x)^A (\log f(x))^m \varphi(x) \psi(x) \, dx \quad (\forall \psi \in \mathcal{C}_0^\infty(U)),$$

where $\mathcal{C}_0^\infty(U)$ denotes the space of $C^\infty$ functions on $U$ with compact supports.

In fact, $f_+^A (\log f_+)^m \varphi$ is the $m$-th derivative of the distribution $f_+^A \varphi$ with respect to $\lambda$.

**Lemma 2.2** Let $\Omega$ be an open set of $\mathbb{C}^n$ such that $V := \mathbb{R}^n \cap \Omega$ is non-empty. Let $\varphi_0, \ldots, \varphi_m$ be locally integrable functions on $V$. Assume $P_k(s) \in \mathcal{D}[\mathbb{C}^n(\Omega)[s]$ $(k = 0, 1, \ldots, m)$ and

$$\sum_{k=0}^m P_k(\lambda)(f_+^A (\log f_+)^k \varphi_k) = 0 \quad (1)$$

holds in $\mathcal{OD}b(\mathbb{C}_+ \times V_f)$. Then (1) holds in $\mathcal{OD}b(\mathbb{C}_+ \times V)$.

Proof: We follow the argument of the proof of Lemma 2.9 in [2]. Let $\phi$ belong to $\mathcal{C}_0^\infty(V)$ with $K := \text{supp} \, \phi$. Let $\chi(t)$ be a $C^\infty$ function of a variable $t$ such that $\chi(t) = 1$ for $|t| \leq 1/2$ and $\chi(t) = 0$ for $|t| \geq 1$. Then we have

$$\left\langle \sum_{k=0}^m P_k(\lambda)(f_+^A (\log f_+)^k \varphi_k), \phi \right\rangle = \left\langle \sum_{k=0}^m P_k(\lambda)(f_+^A (\log f_+)^k \varphi_k), \chi(\frac{t}{\tau}) \phi \right\rangle$$

$$= \sum_{k=0}^m \int_V f_+^A (\log f_+)^k \varphi_k P_k(\lambda) \left( \chi(\frac{t}{\tau}) \phi \right) \, dx$$

3
Lemma 2.3 Let \( P_k(\lambda) \) denote the adjoint operator of \( P_k(\lambda) \). Let \( m_k \) be the order of \( P_k(s) \) and \( d_k \) be the degree of \( P_k(s) \) in \( s \). Then there exist constants \( C_k \) such that

\[
\sup_{x \in K} |tP_k(\lambda)(\chi(\frac{f(x)}{\tau})\phi(x))| \leq C_k(1 + |\lambda|)^{d_k \tau^{-m_k}} \quad (0 < \forall \tau < 1).
\]

Assume \( \text{Re} \lambda > \max\{m_k + 1 \mid 0 \leq k \leq m\} \) and \( 0 < \tau < 1 \). Then we have

\[
\begin{align*}
\left| \int_{V} f^k_+ (\log f_+)^k \varphi_k^t P_k(\lambda)(\chi(\frac{f}{\tau})\phi) \, dx \right| & \leq C_k(1 + |\lambda|)^{d_k \tau^{-m_k}} \int_{\{x \in V \mid \log f_+ \leq \tau\}} |f^k_+ (\log f_+)^k \varphi_k(x)| \, dx \\
& \leq k!C_k(1 + |\lambda|)^{d_k \tau^{-m_k-1}} \int_{\{x \in V \mid \log f_+ \leq \tau\}} |\varphi_k(x)| \, dx
\end{align*}
\]

since \( |\log t|^k \leq k!t^{-1} \) holds for \( 0 < t < 1 \). This implies

\[
\left\langle \sum_{k=0}^{m} P_k(\lambda)(f^k_+ (\log f_+)^k \varphi_k), \phi \right\rangle = \lim_{\tau \to +0} \sum_{k=0}^{m} \int_{V} f^k_+ (\log f_+)^k \varphi_k^t P_k(\lambda)(\chi(\frac{f}{\tau})\phi) \, dx = 0.
\]

The assertion of the lemma follows from the uniqueness of analytic continuation. \( \square \)

2.2 Generalized \( b \)-function and analytic continuation

We assume that there exists on \( \Omega \) a sheaf \( \mathcal{I} \) of coherent left ideals of \( \mathcal{D}_{\mathbb{C}^n} \) which annihilates \( \varphi \) on \( U_f = \{ x \in U \mid f(x) \neq 0 \} \), namely, \( P\varphi = 0 \) holds on \( W \cap U_f \) for any section \( P \) of \( \mathcal{I} \) on an open set \( W \) of \( \mathbb{C}^n \). We set \( \mathcal{M} = \mathcal{D}_{\mathbb{C}^n}/\mathcal{I} \) and denote by \( u \) the residue class of \( 1 \in \mathcal{D}_X \) modulo \( \mathcal{I} \). In the sequel, we assume that \( \mathcal{M} \) is holonomic on \( \Omega_f = \{ z \in \Omega \mid f(z) \neq 0 \} \), i.e., that \( \text{Char}(\mathcal{M}) \cap \pi^{-1}(\Omega_f) \) is of dimension \( n \), where \( \text{Char}(\mathcal{M}) \) denotes the characteristic variety of \( \mathcal{M} \) and \( \pi : T^*\mathbb{C}^n \to \mathbb{C}^n \) is the canonical projection.

Let \( \mathcal{L} = \mathcal{O}_{\mathbb{C}^n}[f^{-1}, s]f^s \) be the free \( \mathcal{O}_{\mathbb{C}^n}[f^{-1}, s] \)-module generated by the symbol \( f^s \). Then \( \mathcal{L} \) has a natural structure of left \( \mathcal{D}_{\mathbb{C}^n}[s] \)-module induced by the derivation \( \partial_s f^s = s(\partial f/\partial x_i)f^{-1}f^s \). Let us consider the tensor product \( \mathcal{L} \otimes_{\mathcal{O}_{\mathbb{C}^n}} \mathcal{M} \) of \( \mathcal{O}_{\mathbb{C}^n} \)-modules, which has a natural structure of left \( \mathcal{D}_{\mathbb{C}^n}[s] \)-module.

**Lemma 2.3** Let \( v \) and \( P(s) \) be sections of \( \mathcal{M} \) and \( \mathcal{D}_{\mathbb{C}^n}[s] \) respectively on an open subset of \( \Omega \). Then \( P(s)(f^s \otimes v) = 0 \) holds in \( \mathcal{L} \otimes_{\mathcal{O}_{\mathbb{C}^n}} \mathcal{M} \) if and only if \( (f^{m-s} P(s)f^s)(1 \otimes v) = 0 \) holds in \( \mathbb{C}[s] \otimes_{\mathbb{C}} \mathcal{M} \) for a sufficiently large \( m \in \mathbb{N} \).
Proof: Set \( \mathcal{M}[s] = \mathbb{C}[s] \otimes_{\mathcal{O}} \mathcal{M} \), which has a natural structure of left module over \( \mathbb{C}[s] \otimes_{\mathcal{O}} \mathcal{D}_{\mathbb{C}} = \mathcal{D}_{\mathbb{C}}[s] \). Then we have \( \mathcal{L} \otimes_{\mathcal{O}[s]} \mathcal{M} = \mathcal{L} \otimes_{\mathcal{O}[s]} \mathcal{M}[s] \) as left \( \mathcal{D}_{\mathbb{C}}[s] \)-module. Let \( v \) be a section of \( \mathcal{M}[s] \). Since \( \mathcal{L} \) is isomorphic to \( \mathcal{O}[f^{-1}, s] \) as \( \mathcal{O}[s] \)-module, \( f^s \otimes v \) vanishes in \( \mathcal{L} \otimes_{\mathcal{O}[s]} \mathcal{M}[s] \) if and only if \( 1 \otimes v \) vanishes in \( \mathcal{O}[f^{-1}, s] \otimes_{\mathcal{O}[s]} \mathcal{M}[s] \). First, let us show that this happens if and only if \( f^m v = 0 \) in \( \mathcal{M}[s] \) with some \( m \in \mathbb{N} \).

Let \( \rho : \mathcal{O}[s, t] \to \mathcal{O}[s, f^{-1}] \) be the homomorphism defined by \( \rho(h(s, t)) = h(s, f^{-1}) \) for \( h(s, t) \in \mathcal{O}[s, t] \). Let \( \mathcal{K} \) be the kernel of \( \rho \). Then we have an exact sequence

\[
\mathcal{K} \otimes_{\mathcal{O}[s]} \mathcal{M}[s] \to \mathcal{O}[s, t] \otimes_{\mathcal{O}[s]} \mathcal{M}[s] \xrightarrow{\rho \otimes \text{id}} \mathcal{O}[s, f^{-1}] \otimes_{\mathcal{O}[s]} \mathcal{M}[s] \to 0.
\]

Hence \( 1 \otimes v \) vanishes in \( \mathcal{O}[s, f^{-1}] \otimes_{\mathcal{O}[s]} \mathcal{M}[s] \) if and only if there exists \( h(s, t) = \sum_{k=0}^{m} h_k(s) t^k \in \mathcal{K} \) such that \( 1 \otimes v = h(s, t) \otimes v \) holds in \( \mathcal{O}[s, t] \otimes_{\mathcal{O}[s]} \mathcal{M}[s] \), which is equivalent to \( h_k(s)v = \delta_{0k}v \) \( (k = 0, 1, \ldots, m) \) since \( \mathcal{O}[s, t] \) is free over \( \mathcal{O}[s] \). On the other hand, \( \sum_{k=0}^{m} h_k(s)f^{-k} = \rho(h(s, t)) = 0 \) implies

\[
0 = f^mh_0(s)v + f^{m-1}h_1(s)v + \cdots + fh_{m-1}(s)v + h_m(s)v = f^mv.
\]

Conversely, if \( f^m v = 0 \) for some \( m \in \mathbb{N} \), then we have \( 1 \otimes v = f^{-m} \otimes f^m v = 0 \) in \( \mathcal{O}[s, f^{-1}] \otimes_{\mathcal{O}[s]} \mathcal{M} \).

Let \( P(s) \) be a section of \( \mathcal{D}_{\mathbb{C}}[s] \) of order \( m \). For \( i = 1, \ldots, n \),

\[
\partial_i(f^s \otimes v) = f^{s-1} \otimes (sf_i + f\partial_i)v = f^{s-1} \otimes (f^{1-s}\partial_i f^s)v
\]

holds in \( \mathcal{L} \otimes_{\mathcal{O}[s]} \mathcal{M}[s] \) with \( f_i = \partial f/\partial x_i \). This allows us to show that

\[
P(s)(f^s \otimes v) = f^{s-m} \otimes (f^{m-s}P(s)f^s)v
\]

holds in \( \mathcal{L} \otimes_{\mathcal{O}[s]} \mathcal{M}[s] \). (Note that \( f^{m-s}P(s)f^s \) belongs to \( \mathcal{D}_{\mathbb{C}}[s] \).) Summing up, we have shown that \( P(s)(f^s \otimes v) \) vanishes in \( \mathcal{L} \otimes_{\mathcal{O}[s]} \mathcal{M}[s] \) if and only if \( (f^{l-s}P(s)f^s)v \) vanishes in \( \mathcal{M}[s] \) for some \( l \geq m \). \( \square \)

Lemma 2.3 with \( P(s) = 1 \) immediately implies

**Proposition 2.4** Let \( \mathcal{M}[f^{-1}] := \mathcal{O}[f^{-1}] \otimes_{\mathcal{O}[s]} \mathcal{M} \) be the localization of \( \mathcal{M} \) with respect to \( f \), which has a natural structure of left \( \mathcal{D}_{\mathbb{C}} \)-module. Then the natural homomorphism \( \mathcal{L} \otimes_{\mathcal{O}[s]} \mathcal{M} \to \mathcal{L} \otimes_{\mathcal{O}[s]} \mathcal{M}[f^{-1}] \) is injective.

**Proposition 2.5** Let \( P(s) \) be a section of \( \mathcal{D}_{\mathbb{C}}[s] \) on an open set \( \Omega \) of \( \mathbb{C}^n \) and suppose \( P(s)(f^s \otimes u) = 0 \) in \( \mathcal{L} \otimes_{\mathcal{O}[s]} \mathcal{M} \). Set \( V = U \cap \Omega \). Then \( P(\lambda)(f^s \varphi) = 0 \) holds in \( \mathcal{ODb}(\mathbb{C}^+ \times V) \).
Proof: Let $\mathcal{O}_{+\infty}Db$ be the sheaf on $\mathbb{R}^n$ associated with the presheaf

$$\mathcal{O}_D b\{\{\lambda \in \mathbb{C} \mid \text{Re } \lambda > a\} \times W\}$$

for every open set $W$ of $\mathbb{R}^n$, where the inductive limit is taken as $a \to \infty$. The $\mathbb{C}$-bilinear sheaf homomorphism

$$\mathcal{L} \times \mathcal{M} \ni (a(s)f^{s-m}, Pu) \mapsto (a(\lambda)f_{++}^{\lambda-m})P\varphi \in \mathcal{O}_{+\infty}Db$$

with $a(s) \in \mathcal{O}_X[s]$, $m \in \mathbb{N}$, $P \in \mathcal{D}_X$, which is well-defined and $\mathcal{O}_{\mathbb{C}^n}$-balanced on $V_f$ such that $f_{++}^{\lambda-m}$ is real analytic there, induces a $\mathcal{D}_{\mathbb{C}^n}$-linear homomorphism

$$\Psi : \mathcal{L} \otimes_{\mathcal{O}_{\mathbb{C}^n}} \mathcal{M} \to \mathcal{O}_{+\infty}Db$$
on $V_f$ such that $\Psi(a(s)f^{s-m} \otimes Pu) = a(\lambda)f_{++}^{\lambda-m}P\varphi$. In particular, if $P(s) \in \mathcal{D}_{\mathbb{C}^n}[s]$ satisfies $P(s)(f^s \otimes u) = 0$ in $\mathcal{L} \otimes_{\mathcal{O}_{\mathbb{C}^n}} \mathcal{M}$, then $P(\lambda)(f_{++}^{\lambda}\varphi) = 0$ holds in $\mathcal{O}_{+\infty}Db(V_f)$, hence also in $\mathcal{O}_{+\infty}Db(V)$ by Lemma 2.1. Since $f_{++}^{\lambda}\varphi$ belongs to $\mathcal{O}Db(\mathbb{C}_+ \times V)$, it follows that $P(f_{++}^{\lambda}\varphi) = 0$ holds in $\mathcal{O}Db(\mathbb{C}_+ \times V)$. This completes the proof. □

Kashiwara proved in [1] (Theorem 2.7) that on a neighborhood of each point $p$ of $\Omega$, there exist nonzero $b(s) \in \mathbb{C}[s]$ and $P(s) \in \mathcal{D}_{\mathbb{C}^n}[s]$ such that

$$P(s)(f^{s+1} \otimes u) = b(s)f^s \otimes u \quad \text{in } \mathcal{L} \otimes_{\mathcal{O}_{\mathbb{C}^n}} \mathcal{M}.$$ 

Such $b(s)$ of the smallest degree $b(s) = b_p(s)$ is called the (generalized) $b$-function for $f$ and $u$ at $p$.

Assume $p \in U$. Then by the proposition above,

$$P(\lambda)(f_{++}^{\lambda+1}\varphi) = b(\lambda)f_{++}^{\lambda}\varphi$$

holds in $\mathcal{O}Db(\mathbb{C}_+ \times V)$ with an open neighborhood $V$ of $p$. It follows that $f_{++}^{\lambda}\varphi$ is a $\mathcal{D}b(V)$-valued meromorphic function of $\lambda$ on $\mathbb{C}$. It is easy to see that we can replace $V$ by an arbitrary relatively compact subset of $U$. The poles of $f_{++}^{\lambda}\varphi$ are contained in

$$\{\lambda - k \mid b_p(\lambda) = 0 \ (\exists p \in V), \ k \in \mathbb{N}\}.$$

**Proposition 2.6 (Lemma 2.10 of [2])** There exists a positive real number $\varepsilon$ such that $f_{++}^{\lambda}\varphi$ belongs to $\mathcal{O}Db\{\{\lambda \in \mathbb{C} \mid \text{Re } \lambda > -\varepsilon\} \times U\}$.

Proof: Let $\lambda_0$ be an arbitrary pole of $f_{++}^{\lambda}\varphi$. There exists $\psi \in C_0^\infty(U)$ such that $\lambda_0$ is a pole of $Z(\lambda) := \langle f_{++}^{\lambda}\varphi, \psi \rangle$. In particular, $|Z(\lambda_0 + t)|$ tends to infinity as $t \to +0$. On the other hand, $Z(\lambda)$ is continuous on $\{\lambda \in \mathbb{C} \mid \text{Re } \lambda \geq 0\}$. This implies $\text{Re } \lambda_0 < 0$. The conclusion follows since there are at most a finite number of poles of $f_{++}^{\lambda}\varphi$ in the set $\{\lambda \in \mathbb{C} | \text{Re } \lambda > -1\}$. □

In conclusion, $f_{++}^{\lambda}\varphi$ is a $\mathcal{D}b(U)$-valued meromorphic function on $\mathbb{C}$ whose poles are contained in $\{\lambda \in \mathbb{C} \mid \text{Re } \lambda < 0\}$. 6
2.3 Holonomicity of \( f_+^\lambda \varphi \) and its applications

Let \( f, \varphi, \mathcal{M} = \mathcal{D}_{\mathbb{C}^n}/\mathcal{I} \) be as in the preceding subsection. Let \( \mathcal{N} = \mathcal{D}_{\mathbb{C}^n}[s](f^s \otimes u) \) be the left \( \mathcal{D}_{\mathbb{C}^n}[s] \)-submodule of \( \mathcal{L} \otimes_{\mathcal{O}_{\mathbb{C}^n}} \mathcal{M} \) generated by \( f^s \otimes u \). Theorem 2.5 of Kashiwara [1] guarantees that \( \mathcal{N}_{\lambda_0} := \mathcal{N}/(s - \lambda_0)\mathcal{N} \) is a holonomic \( \mathcal{D}_{\mathbb{C}^n} \)-module on \( \Omega \) for any \( \lambda_0 \in \mathbb{C} \).

**Proposition 2.7** Let \( \lambda_0 \) be an arbitrary complex number and \( f^{\lambda_0} \otimes \varphi \) the residue class of \( f^s \otimes u \in \mathcal{N} \) modulo \( (s - \lambda_0)\mathcal{N} \).

1. \( \mathcal{N}_0 \) is isomorphic to \( \mathcal{M} \) as \( \mathcal{D}_{\mathbb{C}^n} \)-module on \( \Omega_f \).

2. If \( \mathcal{M} \) is \( f \)-saturated, i.e., if \( f v = 0 \) with \( v \in \mathcal{M} \) implies \( v = 0 \), then there is a surjective \( \mathcal{D}_{\mathbb{C}^n} \)-homomorphism \( \Phi : \mathcal{N}_0 \to \mathcal{M} \) on \( \Omega \) such that \( \Phi(f^0 \otimes u) = u \). Moreover, \( \Phi \) is an isomorphism on \( \Omega_f \).

**Proof:** Since \( \mathcal{M}[f^{-1}] = \mathcal{M} \) on \( \Omega_f \), we may assume that \( \mathcal{M} \) is \( f \)-saturated. In view of Lemma 2.3 and the definition of \( \mathcal{N}_0 \), \( P \in \mathcal{D}_{\mathbb{C}^n} \) annihilates \( f^0 \otimes u \) if and only if there exist \( Q(s) \in \mathcal{D}_{\mathbb{C}^n}[s] \) and an integer \( m \geq \operatorname{ord} Q(s) \) such that \( (f^{m-s}Q(s)f^s)(1 \otimes u) = 0 \) in \( \mathcal{M}[s] \) and \( P = Q(0) \). If there exist such \( Q(s) \) and \( m \), set

\[
(f^{m-s}Q(s)f^s) = Q_0 + Q_1 s + \cdots + Q_m s^m \quad (Q_i \in \mathcal{D}_{\mathbb{C}^n}).
\]

Then \( Q_i u = 0 \) holds for any \( i \). In particular, \( Q_0 = f^m P \) annihilates \( u \). This implies \( Pu = 0 \) since \( \mathcal{M} \) is \( f \)-saturated. Hence the homomorphism \( \Phi \) is well-defined.

Now assume \( f \neq 0 \) and \( Pu = 0 \). Then \( Q(s) := f^s P f^{-s} \) belongs to \( \mathcal{D}_{\mathbb{C}^n}[s] \) and annihilates \( f^s \otimes u \) by Lemma 2.3. Hence \( P = Q(0) \) annihilates \( f^0 \otimes u \). This implies that \( \Phi \) is an isomorphism on \( \Omega_f \). \( \square \)

**Theorem 2.8** If \( \lambda_0 \) is not a pole of \( f_+^\lambda \varphi \), then \( f_+^{\lambda_0} \varphi \) is a solution of \( \mathcal{N}_{\lambda_0} \).

**Proof:** Assume that \( \lambda_0 \in \mathbb{C} \) is not a pole of \( f_+^\lambda \varphi \). Let \( P \) be a section of \( \mathcal{D}_{\mathbb{C}^n} \) which annihilates \( f_+^{\lambda_0} \otimes u \). Then there exist \( Q(s), R(s) \in \mathcal{D}_{\mathbb{C}^n}[s] \) such that

\[
P = Q(s) + (s - \lambda_0)R(s), \quad Q(s)(f^s \otimes u) = 0 \quad \text{in} \ \mathcal{N}.
\]

Proposition 2.5 implies that \( Q(\lambda)(f_+^\lambda \varphi) \) vanishes as section of the sheaf \( \mathcal{O} Db \). In particular, \( P(f_+^{\lambda_0} \varphi) = Q(\lambda_0)(f_+^{\lambda_0} \varphi) = 0 \) holds as distribution. Thus the homomorphism

\[
\mathcal{D}_{\mathbb{C}^n}(f_+^{\lambda_0} \otimes u) \ni P(f_+^{\lambda_0} \otimes u) \mapsto P(f_+^{\lambda_0} \varphi) \in \mathcal{D}b
\]

is well-defined and \( \mathcal{D}_{\mathbb{C}^n} \)-linear. Hence \( f_+^{\lambda_0} \varphi \) is a solution of \( \mathcal{N}_{\lambda_0} \). \( \square \)

The following two theorems are essentially due to Kashiwara and Kawai [2] although they are stated with additional assumptions and stronger results.
Theorem 2.9 \( \varphi \) is a solution of the holonomic \( \mathcal{D}_\mathbb{C}^n \)-module \( N_0 \).

Proof: First note that \( \mathcal{O}_{\mathbb{C}^n}[f^{-1},s](-f)^* \) is isomorphic to \( \mathcal{O}_{\mathbb{C}^n}[f^{-1},s]f^* \) as left \( \mathcal{D}_{\mathbb{C}^n}[s] \)-module since \( \partial_i(-f)^* = sf_i f^{-1}(-f)^* \) holds in \( \mathcal{O}_{\mathbb{C}^n}[f^{-1},s](-f)^* \) with \( f_i = \partial f / \partial x_i \). Assume that \( P(f^0 \otimes u) = 0 \) holds in \( N_0 = \mathcal{N} / s\mathcal{N} \). Then there exist \( Q(s), R(s) \in \mathcal{D}_{\mathbb{C}^n}[s] \) such that

\[
P = Q(s) + sR(s), \quad Q(s)(f^* \otimes u) = 0 \text{ in } \mathcal{N}.
\]

Let \( \theta(t) \) be the Heaviside function; i.e., \( \theta(t) = 1 \) for \( t > 0 \) and \( \theta(t) = 0 \) for \( t \leq 0 \). Then we have \( \theta(f) = f^*_+ \) and \( \theta(-f) = (-f)^*_+ \). Theorem 2.8 implies that \( P = Q(0) \) annihilates both \( \theta(f)\varphi \) and \( \theta(-f)\varphi \), and hence also \( \varphi = \theta(f)\varphi + \theta(-f)\varphi \). Thus \( \varphi \) is a solution of \( N_0 \). \( \square \)

Theorem 2.10 Let \( \varphi_1 \) and \( \varphi_2 \) be locally \( L^p \) and \( L^q \) functions respectively on an open set \( U \subset \mathbb{R}^n \) with \( 1 \leq p, q \leq \infty \) and \( 1/p + 1/q = 1 \). Assume that \( \varphi_1 \) and \( \varphi_2 \) are solutions of holonomic \( \mathcal{D}_{\mathbb{C}^n} \)-modules \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) respectively on \( U \). Then for any point \( x_0 \) of \( U \), there exists a holonomic \( \mathcal{D}_{\mathbb{C}^n} \)-module \( \mathcal{M} \) on a neighborhood of \( x_0 \) of which the product \( \varphi_1 \varphi_2 \) is a solution.

Proof: There exist analytic functions \( f_1 \) and \( f_2 \) on a neighborhood \( V \) of \( x_0 \) such that the singular support (the projection of the characteristic variety minus the zero section) of \( \mathcal{M}_k \) is contained in \( f_k = 0 \) for \( k = 1, 2 \). Set \( f(z) = f_1(z)f_1(\overline{z})f_2(z)f_2(\overline{z}) \). Then \( f(x) \) is a real-valued real analytic function and \( \varphi_1 \) and \( \varphi_2 \) are real analytic on \( V_f \). Then it is easy to see, in the same way as in the proof of Theorem 2.8, that \( \varphi_1 \varphi_2 \) is a solution of \( \mathcal{M}_1 \otimes_{\mathcal{O}_{\mathbb{C}^n}} \mathcal{M}_2 \) on \( V_f \). To complete the proof, we have only to apply Theorem 2.9 to \( \mathcal{M}_1 \otimes_{\mathcal{O}_{\mathbb{C}^n}} \mathcal{M}_2 \) and \( f \). \( \square \)

2.4 Laurent coefficients of \( f^*_+ \varphi \)

Let \( f, \varphi, \mathcal{M} \) be as in preceding subsections.

Theorem 2.11 Let \( p \) be a point of \( U \). Then each coefficient of the Laurent expansion of \( f^*_+ \varphi \) about an arbitrary \( \lambda_0 \in \mathbb{C} \) is a solution of a holonomic \( \mathcal{D}_{\mathbb{C}^n} \)-module on a common neighborhood of \( p \).

Proof: Fix \( m \in \mathbb{N} \) such that \( \Re \lambda_0 + m > 0 \). By using the functional equation involving the generalized \( b \)-function, we can find a nonzero \( b(s) \in \mathbb{C}[s] \) and a germ \( P(s) \) of \( \mathcal{D}_{\mathbb{C}^n}[s] \) at \( p \) such that

\[
b(\lambda)f^*_+ \varphi = P(\lambda)(f^*_+ + m \varphi).
\]

8
Factor $b(s)$ as $b(s) = (s - \lambda_0)^l c(s)$ with $c(s) \in \mathbb{C}[s]$ such that $c(\lambda_0) \neq 0$ and an integer $l \geq 0$. Then we have

$$(\lambda - \lambda_0)^l f^\lambda_+ \varphi = \frac{1}{c(\lambda)} P(\lambda)(f^\lambda_+ m \varphi).$$

The right-hand side is holomorphic in $\lambda$ on an neighborhood of $\lambda = \lambda_0$. Let

$$f^\lambda_+ \varphi = \sum_{k=-l}^\infty (\lambda - \lambda_0)^k \varphi_k$$

be the Laurent expansion with $\varphi_k \in \mathcal{D}(U)$, which is given by

$$\varphi_k = \frac{1}{(l+k)!} \lim_{\lambda \to \lambda_0} \frac{\partial^{l+k}}{\partial \lambda^{l+k}} ((\lambda - \lambda_0)^l f^\lambda_+ \varphi) = \frac{1}{(l+k)!} \lim_{\lambda \to \lambda_0} \frac{\partial^{l+k}}{\partial \lambda^{l+k}} \left( \frac{1}{c(\lambda)} P(\lambda)(f^\lambda_+ m \varphi) \right).$$

Hence there exist $Q_{kj} \in \mathcal{D}_{\mathbb{C}^n}$ such that

$$\varphi_k = \sum_{j=0}^{l+k} Q_{kj} (f^\lambda_+ m (\log f_+)^j \varphi). \quad (2)$$

First let us show that $f^\lambda_+ m (\log f_+)^j \varphi$ with $0 \leq j \leq k$ satisfy a holonomic system. Consider the free $\mathcal{O}_{\mathbb{C}^n}[s, f^{-1}]$-module

$$\mathcal{L} := \mathcal{O}_{\mathbb{C}^n}[s, f^{-1}] f^s \otimes \mathcal{O}_{\mathbb{C}^n}[s, f^{-1}] f^s \log f \otimes \mathcal{O}_{\mathbb{C}^n}[s, f^{-1}] f^s (\log f)^2 \oplus \cdots,$$

which has a natural structure of left $\mathcal{D}_{\mathbb{C}^n}[s]$-module. Let

$$\mathcal{N}[k] := \mathcal{D}_{\mathbb{C}^n}[s](f^s \otimes u) + \mathcal{D}_{\mathbb{C}^n}[s](f^{s \log f} \otimes u) + \cdots + \mathcal{D}_{\mathbb{C}^n}[s](f^s (\log f)^k \otimes u)$$

be the left $\mathcal{D}_{\mathbb{C}^n}[s]$-submodule of $\mathcal{L} \otimes_{\mathcal{O}_{\mathbb{C}^n}} \mathcal{M}$ generated by $(f^s (\log f)^j) \otimes u$ with $j = 0, 1, \ldots, k$. It is easy to see that $\mathcal{N}[k]/\mathcal{N}[k-1]$ is isomorphic to $\mathcal{N} = \mathcal{N}[0]$ as left $\mathcal{D}_{\mathbb{C}^n}[s]$-module since

$$P(s)(f^s (\log f)^k \otimes u) \equiv (f^{s-m}(\log f)^k) \otimes (f^{m-s} P(s)f^s) u \mod \mathcal{N}[k-1]$$

holds for any $P(s) \in \mathcal{D}_{\mathbb{C}^n}[s]$ with $m = \text{ord } P(s)$. Moreover, $\mathcal{N}_{\lambda_0}[k] := \mathcal{N}[k]/(s - \lambda_0)\mathcal{N}[k]$ is a holonomic $\mathcal{D}_{\mathbb{C}^n}$-module since $\mathcal{N}_{\lambda_0}[k]/\mathcal{N}_{\lambda_0}[k-1]$ is isomorphic to $\mathcal{N}_{\lambda_0} = \mathcal{N}_{\lambda_0}[0]$, and hence is holonomic as left $\mathcal{D}_{\mathbb{C}^n}$-module.

Let $(f^{\lambda_0 + m}(\log f)^j) \otimes u \in \mathcal{N}_{\lambda_0 + m}[k]$ be the residue class of $(f^{s}(\log f)^j) \otimes u$ modulo $(s - \lambda_0 - m)\mathcal{N}[k]$. Suppose $\sum_{j=0}^k P_j((f^{\lambda_0 + m}(\log f)^j) \otimes u)$ vanishes in $\mathcal{N}_{\lambda_0 + m}[k]$ with $P_j$ being a section of $\mathcal{D}_{\mathbb{C}^n}$ on an open neighborhood of a point $p$ of $U$. Then there exist $Q_j(s) \in \mathcal{D}_{\mathbb{C}^n}[s]$ such that

$$\sum_{j=0}^k P_j((f^{s}(\log f)^j) \otimes u) = (s - \lambda_0 - m) \sum_{j=0}^k Q_j(s)((f^{s}(\log f)^j) \otimes u)$$
holds in $N[k]$. Then it is easy to see that
\[
\sum_{j=0}^{k} P_j(\lambda)(f^+_\lambda (\log f_+)^j \varphi) = (\lambda - \lambda_0 - m) \sum_{j=0}^{k} Q_j(\lambda)(f^+_\lambda (\log f_+)^j \varphi)
\] (3)
holds in $\mathcal{OD}b(\mathbb{C}_+ \times W)$ with an open neighborhood $W$ of $p$. Lemma 2.2 and analytic continuation imply that (3) holds in $\mathcal{OD}b(\mathbb{C}_+ \times W)$. Hence we have
\[
\sum_{j=0}^{k} P_j((f^+_{\lambda_0+m}(\log f_+)^j) \varphi) = 0.
\]
In conclusion, with $k$ replaced by $l+k$, there exists a $\mathcal{D}_{\mathbb{C}^n}$-homomorphism $\Phi : N_{\lambda_0+m}[l+k] \rightarrow \mathcal{D}b$ such that
\[
\Phi((f^+_{\lambda_0+m}(\log f_+)^j) \otimes u) = f^+_{\lambda_0+m}(\log f_+)^j \quad (0 \leq j \leq l+k).
\]
Set
\[
w := \sum_{j=0}^{l+k} Q_{kj}((f^+_{\lambda_0+m}(\log f_+)^j) \otimes u), \quad M_k := \mathcal{D}_{\mathbb{C}^n}w.
\]
Then $M_k$ is a $\mathcal{D}_{\mathbb{C}^n}$-submodule of $N_{\lambda_0+m}[l+k]$ and hence holonomic. Since $\Phi(w) = \varphi_k$ in view of (2), $\varphi_k$ is a solution of $M_k$. This completes the proof. □

3 Algorithms

We give algorithms for computing holonomic systems introduced in the previous section assuming that $f$ is a real polynomial and that $\mathcal{M}$ is algebraic, i.e., defined by differential operators with polynomial coefficients. Let $D_n := \mathbb{C}(x, \partial) = \mathbb{C}(x_1, \ldots, x_n, \partial_1, \ldots, \partial_n)$ be the ring of differential operators with polynomial coefficients with $\partial_j = \partial/\partial x_j$. The ring $D_n$ is also called the $n$-th Weyl algebra over $\mathbb{C}$.

In the sequel, let $f$ be a non-constant real polynomial of $x = (x_1, \ldots, x_n)$ and $\varphi$ be a locally integrable function on an open connected set $U$ of $\mathbb{R}^n$. We assume that there exists a left ideal $I$ of $D_n$ which annihilates $\varphi$ on $U_f$, i.e., $P \varphi = 0$ holds on $U_f$ for any $P \in I$, such that $M := D_n/I$ is a holonomic $D_n$-module. We denote by $u$ the residue class of $1 \in D_n$ modulo $I$. Let $L = \mathbb{C}[x, f^{-1}, s]f^s$ be the free $\mathbb{C}[x, f^{-1}, s]$-module generated by $f^s$, which has a natural structure of left $D_n[s]$-module. Let $N := D_n[s](f^s \otimes u)$ be the left $D_n$-submodule of $L \otimes_{\mathbb{C}[x]} M$ generated by $f^s \otimes u$.

As was established in the previous section, $f^\lambda \varphi$ is a $\mathcal{D}b(U)$-valued meromorphic function on $\mathbb{C}$ and is a solution of $N$. 

10
3.1 Mellin transform

Let us assume that $\varphi$ is real analytic on $U_f$ and set

$$\tilde{\varphi}(x, \lambda) := \int_{-\infty}^{\infty} t^\lambda \delta(t - f(x)) \varphi(x) \, dt.$$ 

This is well-defined and coincides with $f_+^\lambda \varphi$ as a distribution on $U_f \times \mathbb{C}_+$. Then we have

$$\int_{-\infty}^{\infty} t^\lambda \delta(t - f(x)) \varphi(x) \, dt = \tilde{\varphi}(x, \lambda + 1),$$

$$\int_{-\infty}^{\infty} t^\lambda \partial_t(\delta(t - f(x)) \varphi(x)) \, dt = -\int_{-\infty}^{\infty} \partial_t(t^\lambda) \delta(t - f(x)) \varphi(x) \, dt = -\lambda \tilde{\varphi}(x, \lambda - 1).$$

Let $D_{n+1} = D_n[t, \partial_t]$ be the $(n + 1)$-th Weyl algebra with $\partial_t = \partial/\partial t$. Let us consider the ring $D_n[s, E_s, E_s^{-1}]$ of difference-differential operators with the shift operator $E_s : s \mapsto s + 1$, where $s$ is an indeterminate corresponding to $\lambda$. In view of the identities above, let us define the ring homomorphism (Mellin transform of operators)

$$\mu : D_{n+1} \to D_n[s, E_s, E_s^{-1}]$$

by

$$\mu(t) = E_s, \quad \mu(\partial_t) = -sE_s^{-1}, \quad \mu(x_j) = x_j, \quad \mu(\partial x_j) = \partial x_j.$$ 

It is easy to see that $\mu$ is well-defined and injective since $[\partial_t, t] = [\mu(\partial_t), \mu(t)] = 1$. Hence we may regard $D_{n+1}$ as a subring of $D_n[s, E_s, E_s^{-1}]$. Since $\mu(\partial_t t) = -s$, we can also regard $D_n[s]$ as a subring of $D_{n+1}$. Thus we have inclusions

$$D_n[s] \subset D_{n+1} \subset D_n[s, E_s, E_s^{-1}]$$

of rings and $L \otimes_{\mathbb{C}[x]} M$ has a structure of left $D_n[s, E_s, E_s^{-1}]$-module compatible with that of left $D_n[s]$-module. Let $\mathcal{F}(U)$ be the $\mathbb{C}$-vector space of the $D_b(U)$-valued meromorphic functions on $\mathbb{C}$. Then $\mathcal{F}(U)$ has a natural structure of left $D_n[s, E_s, E_s^{-1}]$-module, which is compatible with that of $D_n[s]$-module. In particular, we can regard $\mathcal{F}(U)$ as a left $D_{n+1}$-module.

3.2 Computation of $N = D_n[s](f^s \otimes u)$

The inclusion $D_{n+1}f^s \subset L = \mathbb{C}[x, f^{-1}, s]f^s$ induces a natural $D_{n+1}$-homomorphism

$$D_{n+1}f^s \otimes_{\mathbb{C}[x]} M \xrightarrow{\iota'} L \otimes_{\mathbb{C}[x]} M$$

$$\cup$$

$$N' \xrightarrow{\iota'} N$$
where $N'$ is the left $D_n[s]$-submodule of $D_{n+1}f^s \otimes_{C[x]} M$ generated by $f^s \otimes u$ and $N$ is the left $D_n[s]$-submodule of $L \otimes_{C[x]} M$ generated by $f^s \otimes u$. The homomorphism $\iota$ induces a surjective $D_n[s]$-homomorphism $\iota' : N' \rightarrow N$.

**Proposition 3.1** The homomorphism $\iota$ is injective if and only if $M$ is $f$-saturated; i.e., the homomorphism $f : M \rightarrow M$ is injective.

Proof: First note that $D_{n+1}f^s$ is isomorphic to the first local cohomology group $C[x,t,(t - f)^{-1}] / C[x,t]$ of $C[x,t]$ supported in the non-singular hypersurface $t - f(x) = 0$ since

$$(t - f)^i f^s = 0, \quad (\partial_{x_i} + f_i \partial_t) f^s = 0 \quad (i = 1, \ldots, n).$$

In particular, $D_{n+1}f^s$ is a free $C[x]$-module generated by $\partial^j f^s$ with $j \geq 0$. Hence an arbitrary element $w$ of $D_{n+1}f^s \otimes_{C[x]} M$ is uniquely written in the form

$$w = \sum_{j=0}^{k} (\partial^j f^s) \otimes u_j$$

with $u_j \in M$ and $k \in N$. Then

$$\iota(w) = \sum_{j=0}^{k} (-1)^j s(s - 1) \cdots (s - j + 1) f^{s-j} \otimes u_j$$

vanishes if and only if $f^{s-j} \otimes u_j = 0$, which is equivalent to $f^{m_j} u_j = 0$ with some $m_j \in N$ by Lemma 2.3 for all $j = 0, 1, \ldots, k$. This completes the proof. \hfill \Box

Let $\tilde{M}$ be the left $D_n$-submodule of the localization $M[f^{-1}] := C[x,f^{-1}] \otimes_{C[x]} M$ which is generated by $1 \otimes u$. Then $\tilde{M}$ is $f$-saturated and the natural homomorphism

$$L \otimes_{C[x]} M \rightarrow L \otimes_{C[x]} \tilde{M}$$

is an isomorphism by Lemma 2.3.

An algorithm to compute $M[f^{-1}]$ was presented in [7] under the assumption that $M$ is holonomic on $\mathbb{C}^n \setminus \{ f = 0 \}$. It provides us with an algorithm to compute $\tilde{M}$, i.e., the annihilator of $1 \otimes u \in M[f^{-1}]$. Hence we may assume, from the beginning, that $M$ is holonomic and $f$-saturated. Then $\iota' : N' \rightarrow N$ is an isomorphism by Proposition 3.1. The $f$-saturatedness is equivalent to the vanishing of the zeroth local cohomology group of $M$ with support in $f = 0$, which can be computed by algorithms presented in [3], [8], [6].

Thus we have only to give an algorithm to compute the structure of $N'$ assuming $M$ to be $f$-saturated. We follow an argument introduced by Walther [8]. Note that we gave in [3] an algorithm based on tensor product computation which is less efficient.
**Definition 3.2** For a differential operator \( P = P(x, \partial) \in D_n \), set
\[
\tau(P) := P(x, \partial_{x_1} + f_1 \partial_t, \ldots, \partial_{x_n} + f_n \partial_t) \in D_{n+1}
\]
with \( f_j = \partial f / \partial x_j \). This substitution is well-defined since the operators \( \partial_{x_j} + f_j \partial_t \) commute with one another and \([\partial_{x_j} + f_j \partial_t, x_i] = \delta_{ij}\) holds.

Moreover, for a left ideal \( I \) of \( D_{n+1} \), let \( \tau(I) \) be the left ideal of \( D_{n+1} \) which is generated by the set \( \{\tau(P) \mid P \in I\} \).

**Lemma 3.3** \( \tau(P)(f^* \otimes v) = f^* \otimes (Pv) \) holds in \( L \otimes_{\mathbb{C}[x]} M \) for any \( P \in D_n \) and \( v \in M \).

**Proof:** By the definition of the action of \( D_{n+1} \) on \( L \otimes_{\mathbb{C}[x]} M \) via the Mellin transform, we have
\[
(\partial_{x_j} + f_j \partial_t)(f^* \otimes v) = sf^{-1}f_j f^* \otimes v + f^* \otimes (\partial_{x_j} v) - sf_j f^{-1} f^* \otimes v = f^* \otimes (\partial_{x_j} v).
\]
This implies the conclusion of the lemma. \( \square \)

**Proposition 3.4** Let \( I \) be a left ideal of \( D_n \) and set \( M = D_n/I \) with \( u \in M \) being the residue class of 1 modulo \( I \). Let \( J \) be the left ideal of \( D_{n+1} \) which is generated by \( \tau(I) \cup \{t - f(x)\} \). Then \( J \) coincides with the annihilator \( \text{Ann}_{D_{n+1}}(f^* \otimes u) \) of \( f^* \otimes u \in D_{n+1}f^* \otimes_{\mathbb{C}[x]} M \).

**Proof:** We have only to show that for \( P \in D_{n+1} \) the equivalence
\[
P \in J \iff P(f^* \otimes u) = 0 \text{ in } D_{n+1}f^* \otimes_{\mathbb{C}[x]} M.
\]
Suppose \( Q \) belongs to \( J \). Then \( P \) annihilates \( f^* \otimes u \) by Lemma 3.3.

Conversely, suppose \( P(f^* \otimes u) = 0 \) in \( D_{n+1}f^* \otimes_{\mathbb{C}[x]} M \). We can rewrite \( P \) in the form
\[
P = \sum_{\alpha \in \mathbb{N}^n, \nu \in \mathbb{N}} p_{\alpha, \nu}(x) \left( \partial_{x_1} + \frac{\partial f}{\partial x_1} \partial_t \right)^{\alpha_1} \cdots \left( \partial_{x_n} + \frac{\partial f}{\partial x_n} \partial_t \right)^{\alpha_n} \partial_t^\nu + Q \cdot (t - f(x))
\]
with \( p_{\alpha, \nu}(x) \in \mathbb{C}[x] \) and \( Q \in D_{n+1} \). Setting \( P_{\nu} := \sum_{\alpha \in \mathbb{N}^n} p_{\alpha, \nu}(x) \partial_x^\alpha \), we get
\[
0 = P(f^* \otimes u) = \sum_{\nu = 0}^{\infty} (\partial_t^\nu f^*) \otimes P_{\nu} u \in D_{n+1}f^* \otimes_{\mathbb{C}[x]} M.
\]
It follows that each \( P_{\nu} \) belongs to \( I \) since \( \{\partial_t^\nu f^*\} \) constitutes a free basis of \( D_{n+1}f^* \) over \( \mathbb{C}[x] \). Hence we have
\[
P = \sum_{\nu = 1}^{\infty} \partial_t^\nu \tau(P_{\nu}) + Q \cdot (t - f(x)) \in J.
\]
This completes the proof. □

In order to compute the structure of the $D_n[s]$-submodule $N' = D_n[s](f^s \otimes u)$ of $D_{n+1}f^s \otimes \mathbb{C}[s]M$, we have only to compute the annihilator

$$\text{Ann}_{D_n[s]}(f^s \otimes u) = D_n[s] \cap J,$$

where we regard $D_n[s]$ as a subring of $D_{n+1}$. This can be done as follows:

Introducing new variables $\sigma$ and $\tau$, for $P \in D_{n+1}$, let $h(P) \in D_{n+1}[\sigma, \tau]$ be the homogenization of $P$ with respect to the weights

$$x_j \quad \partial_{x_j} \quad t \quad \partial_t \quad \tau$$

$$0 \quad 0 \quad -1 \quad 1 \quad -1$$

Let $J'$ be the left ideal of $D_{n+1}[\sigma, \tau]$ generated by

$$\{h(P) \mid P \in \tilde{G}\} \cup \{1 - \sigma \tau\},$$

where $\tilde{G}$ is a set of generators of $J$.

Set $J'' = J' \cap D_{n+1}$. Since each element $P$ of $J''$ is homogeneous with respect to the above weights, there exists $P'(s) \in D_n[s]$ such that $P = SP'(-\partial_t)$ with $S = t^\nu$ or $S = \partial_t^\nu$ with some integer $\nu \geq 0$. We set $P'(s) = \psi(P)(s)$. Then $\{\psi(P) \mid P \in J''\}$ generates the left ideal $J \cap D_n[s]$ of $D_n[s]$. This procedure can be done by using a Gröbner basis in $D_{n+1}[\sigma, \tau]$. In conclusion, we have a set of generators of $J \cap D_n[s]$. Then $N'$, and hence $N$ also if $M$ is $f$-saturated, is isomorphic to $D_n[s]/(J \cap D_n[s])$ as left $D_n[s]$-module.

The generalized $b$-function for $f$ and $u$ can be computed as the generator of the ideal

$$\mathbb{C}[s] \cap (\text{Ann}_{D_n[s]}f^s \otimes u + D_n[s]f)$$

of $\mathbb{C}[s]$ by elimination via Gröbner basis computation in $D_n[s]$.

### 3.3 Holonomic systems for the Laurent coefficients of $f^\lambda \varphi$

Let $\lambda_0$ be an arbitrary complex number. Our purpose is to compute a holonomic system of which each coefficient of the Laurent expansion of $f^\lambda \varphi$ is a solution.

Take $m \in \mathbb{N}$ such that $\text{Re} \lambda_0 + m > 0$. Let $b_0(s)$ be the $b$-function of $f$ and $u$. We can find a $P_0(s) \in D_n[s]$ such that

$$P_0(s)(f^{s+1} \otimes u) = b_0(s)f^s \otimes u$$
holds in \( N \) by, e.g., syzygy computation. By using this functional equation, we can find a nonzero polynomial \( b(s) \) and \( P(s) \in D_n[s] \) such that

\[
b(\lambda)f^+_\lambda = P(\lambda)f^+_{\lambda+m}.
\]

In fact, we have only to set

\[
P(s) := P_0(s)P_0(s+1)\cdots P_0(s+m-1), \quad b(s) := b_0(s)b_0(s+1)\cdots b_0(s+m-1).
\]

Factorize \( b(s) \) as \( b(s) = c(s)(s-\lambda_0)^l \) with \( c(\lambda_0) \neq 0 \). Then \( f^+_\lambda \varphi \) has a Laurent expansion of the form

\[
f^+_\lambda \varphi = \sum_{k=-l}^\infty (\lambda - \lambda_0)^k \varphi_k
\]

around \( \lambda_0 \), where \( \varphi_k \in Db(U) \) is given by

\[
\varphi_k = \frac{1}{(l+k)!} \lim_{\lambda \to \lambda_0} \left( \frac{\partial}{\partial \lambda} \right)^{l+k} (c(\lambda)^{-1} P(\lambda)f^+_{\lambda+m}) = \sum_{j=0}^{l+k} Q_{kj}(f^+_{\lambda_0+m}(\log f)^j)
\]

with

\[
Q_{kj} := \frac{1}{j!(l+k-j)!} \left[ \left( \frac{\partial}{\partial \lambda} \right)^{l+k-j} (c(\lambda)^{-1} P(\lambda)) \right]_{\lambda=\lambda_0}.
\]

Let

\[
\tilde{L} = \mathbb{C}[x, f^{-1}, s]f^s \oplus \mathbb{C}[x, f^{-1}, s]f^s \log f \oplus \mathbb{C}[x, f^{-1}, s]f^s(\log f)^2 \oplus \cdots
\]

be the free \( \mathbb{C}[x, f^{-1}, s] \)-module with a natural structure of left \( D_n(s, \partial_s) \)-module. Consider the left \( D_n[s] \)-submodule

\[
N[k] = D_n[s](f^s \otimes u) + D_n[s]((f^s \log f) \otimes u) + \cdots + D_n[s]((f^s(\log f)^k) \otimes u)
\]

of \( \tilde{L} \otimes_{\mathbb{C}[x]} M \). For a complex number \( \lambda_0 \), set

\[
N_{\lambda_0}[k] = N[k]/(s - \lambda_0)N[k].
\]

Let us first give an algorithm to compute the structure of \( N[k] \).

**Proposition 3.5** Let \( G_0 \) be a set of generators of the annihilator \( \text{Ann}_{D_n[s]}(f^s \otimes u) = J \cap D_n[s] \). Let \( e_1 = (1, 0, \ldots, 0) \), \( \ldots \), \( e_{k+1} = (0, \ldots, 0, 1) \) be the canonical basis of \( \mathbb{Z}^{k+1} \). For each \( Q(s) \in G_0 \) and an integer \( j \) with 0 \( \leq j \leq k \), set

\[
Q^{(j)}(s) := \sum_{i=0}^j \binom{j}{i} \frac{\partial^{j-i} Q(s)}{\partial s^{j-i}} e_{i+1} \in (D_n[s])^{k+1}.
\]

Let \( J_k \) be the left \( D_n[s] \)-submodule of \((D_n[s])^{k+1}\) generated by \( G_1 := \{ Q^{(j)}(s)(\lambda_0) \mid Q(s) \in G_0, 0 \leq j \leq k \} \). Then \((D_n[s])^{k+1}/J_k\) is isomorphic to \( N[k] \).
Proof: Let $\varpi : (D_n[s])^{k+1} \to N[k]$ be the canonical surjection. Let $Q(s)$ belong to $G_0$. Differentiating the equation $Q(s)(f^s \otimes u) = 0$ in $N[k]$ with respect to $s$, one gets

$$
\sum_{i=0}^{j} \binom{j}{i} \frac{\partial^{j-i} Q(s)}{\partial s^{j-i}} ((f^s (\log f)^i) \otimes u) = 0.
$$

Hence $J_k$ is contained in the kernel of $\varpi$. Conversely, assume that $\vec{Q}(s) = (Q_0(s), Q_1(s), \ldots, Q_k(s))$ belongs to the kernel of $\varpi$. This implies $Q_k(s)(f^s \otimes u) = 0$ since $N[k]/N[k-1]$ is isomorphic to $N = D_n[s](f^s \otimes u)$. Hence $\vec{Q}(s) - Q_k(s)$ belongs to the kernel of $\varpi$, the last component of which is zero. We conclude that $\vec{Q}(s)$ belongs to $J_k$ by induction. $\square$

Thus we have

$$
N_{\lambda_0}[k] = (D_n)^{k+1}/J_k|_{s=\lambda_0}, \quad J_k|_{s=\lambda_0} := \{ Q(\lambda_0) \mid Q(s) \in J_k \}.
$$

Set

$$
w := \sum_{j=0}^{l+k} Q_{kj}((f^{\lambda_0+m}(\log f)^j) \otimes u), \quad M_k := D_n w.
$$

Then we have

$$
P w = 0 \iff P(Q_0, Q_1, \ldots, Q_{l+k}) \in J_{l+k}|_{s=\lambda_0+m}.
$$

Thus we can find a set of generators of $\text{Ann}_{D_n} w$ by computation of syzygy or intersection. As was shown in §2.4, $\varphi_k$ is a solution of the holonomic system $M_k$.

### 3.4 Difference equations for the local zeta function

In the sequel, we assume that $\varphi$ is a locally integrable function on $\mathbb{R}^n$. As we have seen so far, $f_+^n \varphi \in \mathcal{F}(\mathbb{R}^n)$ is a solution of the holonomic $D_{n+1}$-module $D_{n+1}/J$. Hence if the local zeta function $Z(\lambda) := \int_{\mathbb{R}^n} f_+^n \varphi dx$ is well-defined, e.g., if $\varphi$ has compact support, or else is smooth on $\mathbb{R}^n$ with all its derivatives rapidly decreasing on the set $\{ x \in \mathbb{R}^n \mid f(x) \geq 0 \}$, then $Z(\lambda)$ is a solution of the integral module

$$
D_{n+1}/(J + \partial_{x_1} D_{n+1} + \cdots + \partial_{x_n} D_{n+1})
$$

of $D_{n+1}/J$, which is a holonomic module over $D_1 = \mathbb{C}(t, \partial_t)$. This $D_1$-module can be computed by the integration algorithm which is the 'Fourier
transform’ of the restriction algorithm given in [6] (see [5] for the integration algorithm). Then by Mellin transform we obtain linear difference equations for \( Z(\lambda) \). Thus we get

**Theorem 3.6** Under the above assumptions, \( Z(\lambda) \) satisfies a non-trivial linear difference equation with polynomial coefficients in \( \lambda \).

**Example 3.7** \( \Gamma(\lambda+1) = \int_0^\infty x^\lambda e^{-x} \, dx = \int_{-\infty}^\infty x^\lambda e^{-x} \, dx \) satisfies the difference equation

\[
(E_\lambda - (\lambda + 1))\Gamma(\lambda + 1) = 0,
\]

where \( E_\lambda : \lambda \mapsto \lambda + 1 \) is the shift operator.

### 3.5 Examples

Let us present some examples computed by using algorithms introduced so far and their implementation in the computer algebra system Risa/Asir.

**Example 3.8** Set \( f = x^3 - y^2 \in \mathbb{R}[x,y] \) and \( \varphi = 1 \). Since the \( b \)-function of \( f \) is \( b_f(s) = (s + 1)(6s + 5)(6s + 7) \), possible poles of \( f_+^\lambda \) are \(-1 - \nu, -5/6 - \nu, -6/7 - \nu \) with \( \nu \in \mathbb{N} \) and they are at most simple poles. The residue \( \text{Res}_{\lambda = -1} f_+^\lambda \) is a solution of

\[
D_2/(D_2(2x \partial_x + 3y \partial_y + 6) + D_2(2y \partial_x + 3x^2 \partial_y) + D_2(x^3 - y^2)).
\]

\( \text{Res}_{\lambda = -5/6} f_+^\lambda \) is a solution of \( D_2/(D_2x + D_2y) \). Hence it is a constant multiple of the delta function \( \delta(x,y) = \delta(x)\delta(y) \). \( \text{Res}_{\lambda = -7/6} f_+^\lambda \) is a solution of \( D_2/(D_2x^2 + D_2(x \partial_x + 2) + D_2y) \). Hence it is a constant multiple of \( \delta'(x)\delta(y) \).

**Example 3.9** Set \( f = x^3 - y^2 \) and \( \varphi(x,y) = \exp(-x^2 - y^2) \). Then \( \varphi \) is a solution of a holonomic system \( M := D_2/(D_2(\partial_x + 2x) + D_2(\partial_y + 2y)) \) on \( \mathbb{R}^2 \), which is \( f \)-saturated since it is a simple \( D_2 \)-module. The generalized \( b \)-function for \( f \) and \( u := [1] \in M \) is \( b_f(s) = (s + 1)(6s + 5)(6s + 7) \). The local zeta function \( Z(\lambda) := \int_{\mathbb{R}^2} f_+^\lambda \varphi \, dx dy \) is annihilated by the difference operator

\[
32E_s^4 + 16(4s + 13)E_s^3 - 4(s + 3)(27s^2 + 154s + 211)E_s^2 - 6(s + 2)(s + 3)(36s^2 + 162s + 173)E_s - 3(s + 1)(s + 2)(s + 3)(6s + 5)(6s + 13),
\]

where \( s \) is an indeterminate corresponding to \( \lambda \). From this we see that \(-7/6\) is not a pole of \( Z(\lambda) \).
Example 3.10 Set $\varphi(x) = \exp(-x - 1/x)$ for $x > 0$ and $\varphi(x) = 0$ for $x \leq 0$. Then $\varphi(x)$ belongs to the space $S(\mathbb{R})$ of rapidly decreasing functions on $\mathbb{R}$ and satisfies a holonomic system

$$M := D_1/D_1(x^2\partial_x + x^2 - 1),$$

which is $x$-saturated. The generalized $b$-function for $f = x$ and $u = [1] \in M$ is $s + 1$. The local zeta function $Z(\lambda) := \int_\mathbb{R} x^\lambda \varphi(x) \, dx$ is entire (i.e., without poles) and satisfies a difference equation

$$(E_\lambda^2 - (\lambda + 2)E_\lambda - 1)Z(\lambda) = 0.$$

This can also be deduced by integration by parts.

Example 3.11 Set $\varphi_1(x) = \exp(-x - 1/x)$ for $x > 0$ and $\varphi_1(x) = 0$ for $x \leq 0$. Set $\varphi(x, y) = \varphi_1(x)e^{-y}$. Then $\varphi$ satisfies a holonomic system

$$M := D_2/(D_2(x^2\partial_x + x^2 - 1) + D_2(\partial_y + 1)).$$

The generalized $b$-function for $f := y^2 - x^2$ and $u = [1] \in M$ is $s + 1$. Moreover, we can confirm that $M$ is $f$-saturated by using the localization algorithm in [7]. The local zeta function $Z(\lambda) := \int_{\mathbb{R}^2} f^\lambda \varphi \, dx \, dy$ is well-defined since $f(x, y) < 0$ if $y < 0$. It is annihilated by a difference operator of the form

$$E_s^{11} + a_{10}(s)E_s^{10} + \cdots + a_1(s)E_s + a_0(s),$$

$$a_0(s) = c(s + 1)(s + 2)(s + 3)(s + 4)(s + 5)(s + 6)(s + 7)(s + 8)(s + 9),$$

where $c$ is a positive rational number and $a_1(s), \ldots, a_{10}(s)$ are polynomials of $s$ with rational coefficients. Possible poles of $f_+^\lambda \varphi$ are the negative integers. For example, $-1$ is at most a simple pole of $f_+^\lambda \varphi$ and $\text{Res}_{\lambda = -1} f_+^\lambda \varphi$ is a solution of a holonomic system

$$D_2/(D_2(3x^2\partial_x + 2xy\partial_y + 3x^2 + (2y + 6)x - 3) + D_2(y^2 - x^2)).$$

Example 3.12 Set $f = x^3 - y^2z^2$. The $b$-function of $f$ is $(s + 1)(3s + 4)(3s + 5)(6s + 5)^2(6s + 7)^2$. For example, its maximum root $-5/6$ is at most a pole of order 2 of $f_+^\lambda$. Let

$$f_+^\lambda = \left(\lambda + \frac{5}{6}\right)^{-2} \varphi_2 + \left(\lambda + \frac{5}{6}\right)^{-1} \varphi_1 + \varphi_0 + \cdots$$

be the Laurent expansion. Then $\varphi_2$ satisfies

$$x\varphi_2 = y\varphi_2 = z\varphi_2 = 0.$$

Hence $\varphi_2$ is a constant multiple of $\delta(x, y)$. On the other hand, $\varphi_1$ satisfies a holonomic system

$$x\varphi_1 = (y\partial_y - z\partial_z)\varphi_1 = yz\varphi_1 = (z^2\partial_z - z)\varphi_1 = 0.$$
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