Reputation Building under Observational Learning

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A patient seller faces a sequence of buyers and decides whether to build a reputation for supplying high quality products. Each buyer does not have access to the seller’s complete records, but can observe all previous buyers’ actions, and some informative private signal about the seller’s actions. I examine how the private signals the buyers receive affect the speed of social learning and the seller’s incentives to establish reputations. When each buyer privately observes a bounded (possibly stochastic) subset of the seller’s past actions, the speed of social learning is strictly positive but can vanish to zero as the seller becomes patient. As a result, reputation building leads to low payoff for the patient seller and low social welfare. When each buyer observes an unboundedly informative private signal about the seller’s current-period action, the speed of learning is uniformly bounded below and a patient seller can secure high returns from building reputations. My results provide an explanation to empirical findings of reputation failures in developing countries. I also discuss the effectiveness of various policies in accelerating social learning and encouraging sellers to establish good reputations.

Keywords: social learning, reputation, stochastic network, information aggregation.

JEL Codes: C73, D82, D83

1 Introduction

Recent empirical findings in developing economies suggest that reputation mechanisms fail to function in a variety of markets. In the markets for experience goods, such as malaria drugs (Nyqvist, Svensson and Yanagizawa-Drott 2018), fruits (Bai 2018), and milk powder (Bai, Gazze and Wang 2019), there is a lack of supply for high quality products despite consumers’ demands for high quality. A common reason behind these market failures is the persistent mistrust between sellers and consumers. Consumers believe that sellers are likely to supply low quality. This pessimistic belief lowers sellers’ returns from building reputations and makes consumers’ pessimistic beliefs self-fulfilling. These observations are at odds with the canonical reputation results in Fudenberg and Levine (1989,1992), which suggest that buyers’ mistrust cannot persist and patient sellers can secure high returns from building reputations.

This paper presents a reputation model, which, among other results, suggests a rationale for such reputation failures and persistent mistrust. I argue that when each consumer has limited access to a
seller’s past records (e.g., observes a bounded subset of the seller’s past actions), and learns primarily from previous consumers’ choices, consumers’ learning can be arbitrarily slow. This can wipe out the seller’s returns from building reputations no matter how patient he is. I also propose policy interventions that can accelerate social learning and restore seller’s incentives to sustain reputations.

My modeling assumptions, limited access to sellers’ past records and learning from others’ choices, fit into a number of retail markets in developing economies. Limited availability of formal records can be caused by inadequate record-keeping technologies. Even when official records are available, their credibility is undermined by institutional failures such as collusion between merchants and bureaucrats. As a result, information about the seller’s past actions is dispersed among consumers (e.g., each consumer observes the quality of product she bought from the seller), and can be passed on to future buyers via word-of-mouth communication. These patterns of learning are documented empirically in the markets for fertilizers (Conley and Udry 2010), drugs (Adhvaryu 2014), food (Bai 2018), etc.

Can consumers’ social learning provide adequate incentives for sellers to supply high quality? How does the speed of learning depend on consumers’ private signals? For example, does a seller have stronger reputational incentives when each consumer observes his actions in the last 10 periods (i.e., after selling a low quality product, a seller will be punished by the next 10 consumers), or when each consumer observes some informative signal about his current-period action? The answers to these questions not only help to understand the causes of reputation failures, but also inform policy-makers about which additional information to provide to consumers in order to restore efficiency.

Motivated by these questions, I study an infinitely repeated game between a patient player 1 (e.g., seller) and a sequence of player 2s (e.g., consumers), arriving one in each period and each plays the game only once. Player 1 is either a strategic type who maximizes his discounted average payoff, or a commitment type who plays his optimal (pure) commitment action in every period. I focus on situations in which the commitment type occurs with small but positive probability.

My modeling innovation is in the monitoring structure. In my baseline model, every player 2 observes the entire history of previous player 2s’ actions, and observes player 1’s actions in the last K periods. In Section 4 I extend my result when player 2s are connected via a bounded stochastic network, each player 2 observes player 1’s actions against her neighbors, in addition to player 2s’

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An obvious solution to the reputation failure problem is to let each consumer observe the seller’s complete records. However, such a policy is hard to implement in developing countries due to its violation of two constraints. First, budget and institutional constraints which rule out policies that require formidable implementation costs or those that require radical changes in the current institutions. Second, consumers’ inattention constraints which limit each consumer’s capacity to process detailed information about the seller’s actions. Policies that respect these constraints include: (1) marginal improvements in record-keeping technologies that allow each consumer to observe a longer history of the seller’s past actions; (2) inspect a small fraction of products currently sold on the market and provide quality certificates.
actions in all previous periods. This setup resembles many retail markets in developing countries. Given the geographic proximity, consumers can casually observe each other’s actions in these localized markets for food, drugs, and fertilizers. However, learning about the seller’s actions (i.e., the quality and attributes of his product) requires more time and effort. For example, a consumer needs to talk to his neighbors in person and learn about their personal experiences. It is usually the case that she has limited capacity to process such detailed information. Therefore, it is reasonable to assume that each consumer communicates with at most a bounded number of individuals before making her decision.

Theorem 1 shows that no matter how large $K$ is, there exist equilibria in which the patient player’s payoff is no more than his worst stage-game Nash Equilibrium payoff. In monotone-supermodular games (Condition 2) that fit into buyer-seller applications, there exist equilibria in which both players receive their respective minmax payoffs (Theorem 2). My reputation failure result generalizes when each short-run player randomly samples a bounded subset of her predecessors observes the patient player’s actions against the individuals in her sample, in addition to all of her predecessors’ actions (Theorem 3). These results contrast to the ones in Fudenberg and Levine (1989, 1992), which suggest that a patient player can secure his optimal commitment payoff in all equilibria.

My proofs of these results construct a class of sequential equilibria in which the speed of player 2’s learning is strictly positive, but vanishes to zero as player 1’s discount factor approaches unity. Section 2 illustrates the intuition behind my constructive proofs using the product choice game in Mailath and Samuelson (2001), and applies my results to a case study in the fruit market (Bai 2018).

In order to confirm that the driving force behind Theorem 1 is slow learning, rather than alternative mechanisms proposed in the social learning and reputation literature, I establish three properties that apply to all Bayes Nash Equilibria. These findings highlight the new economic forces that affect the speed of learning when the learning process is endogenously controlled by a strategic long-run player.

First, player 2 never herds on any action that does not best reply against player 1’s commitment action. Intuitively, player 1 has no intertemporal incentive after player 2 herds, and hence, plays a myopic best reply against that herding action. Suppose player 1’s myopic best reply is his commitment action, then player 2 will best reply against player 1’s commitment action. Suppose player 1’s best reply is not his commitment action, then player 2 believe that player 1 is the commitment type after observing him playing the commitment action, and therefore, will best reply against the commitment

\footnote{My results apply to every game that has a pure strategy Nash Equilibrium and satisfies a generic payoff assumption. I also provide sufficient conditions under which the patient player attains his minmax payoff.}

\footnote{Theorem 3 requires two additional assumptions on the stochastic network. First, the neighborhoods of different player 2s are independent random variables. This is a standard assumption in social learning models, which is also assumed in Banerjee and Fudenberg (2004) and Acemoglu, Dahleh, Lobel and Ozdaglar (2011). Second, the probability with which each player 2 can observe player 1’s action against her immediate predecessor is bounded from below.}
action in the next period. Both conclusions contradict the presumption on player 2’s herding.

Using similar ideas, I show that when players’ stage-game payoffs are monotone-supermodular, (1) player 2’s actions in the next $K$ periods are informative about player 1’s current-period action unless player 1 is guaranteed to receive his optimal commitment payoff in the next $K$ periods; (2) player 1’s asymptotic payoff from establishing a reputation is at least a fraction $\frac{K}{K+1}$ of his optimal commitment payoff. These findings suggest that Theorem 1 is not driven by low-payoff outcomes in the long run, or player 2’s actions being uninformative, which stands in contrast to models of bad reputations.

Section 5 examines an alternative specification of player 2’s private information: Every player 2 observes an informative private signal about player 1’s current period action, in addition to previous player 2’s actions, and possibly, player 1’s actions in the last $K \in \mathbb{N} \cup \{0\}$ periods. This is motivated by policy interventions in which a regulator randomly inspects a small fraction of products currently sold on the market and informs consumers about the quality of inspected products by issuing certificates.

Theorem 4 shows that in games where player 1’s action choice is binary, he can secure his commitment payoff in all equilibria if and only if player 2’s private signal is unboundedly informative, i.e., there exists a signal realization that occurs with positive probability only when player 1 plays his commitment action. This is reminiscent of a result in Smith and Sørensen (2000), that agents’ actions asymptotically match the state if and only if their private signals are unboundedly informative.

However, Smith and Sørensen (2000)’s result does not imply that player 1 can secure high payoffs from building reputations. This is because first, the myopic players in my model are learning about the endogenous actions of a strategic player rather than an exogenous state. Second, converging to a high-payoff outcome asymptotically does not imply that a patient patient receives a high discounted average payoff. This is demonstrated by the comparison between Theorem 1 and Proposition 3.

The key step to establish Theorem 4 is to show that in binary action games, unboundedly informative private signal guarantees a lower bound on the speed of social learning, which is independent of player 1’s discount factor. After observing previous player 2’s actions but before observing her private signal, if player 2 believes that she will not best reply against the commitment action with positive probability, then the probability with which she best replies against the commitment action is strictly higher when player 1 plays his commitment action. As a result, the informativeness of player 2’s action does not vanish as player 1 becomes arbitrarily patient, which implies that a patient player 1 can secure his commitment payoff by building a reputation.

When player 1 has three or more actions, player 2’s private signal being unboundedly informative is neither necessary nor sufficient for player 1 to secure high returns from building reputations. I show this by counterexample in which player 2’s action is uninformative about player 1’s action even when
she does not best reply against the commitment action with positive probability.

Nevertheless, the equivalence between securing high returns from building reputations and unboundedly informative signals is restored when players’ payoffs are monotone-supermodular and the distribution over private signals satisfies a monotone likelihood ratio property (i.e., the likelihood ratio between a high signal realization and a low signal realization is higher when the seller takes a higher action), both of which are reasonable assumptions in the buyer-seller application. Under these conditions, Theorem 4’ shows that player 2’s private signal being unboundedly informative is sufficient and almost necessary for a patient player to secure his commitment payoff from building reputations.

In terms of policy implications, Theorems 4 and 4’ suggest that in markets with lack of record keeping and consumers relying on social learning and word-of-mouth communication, providing consumers with informative signals about the quality of products currently sold on the market can effectively encourage sellers to supply high quality. For this intervention to be effective, the regulator needs to ensure that quality certificates cannot be forged, and in particular, the certificate for high quality products cannot be used on low quality ones (i.e., the signal is unboundedly informative). By contrast, Theorem 4’ suggests that allowing consumers to observe a longer history of the seller’s past actions (but remains bounded) is ineffective provided there is widespread mistrust between buyers and sellers.

**Related Literature:** This paper contributes to the social learning literature from two angles.

In terms of modeling, I study a social learning model in which a sequence of myopic players learn about a strategic long-run player’s endogenous behavior. This contrasts to the canonical social learning models of Banerjee (1992), Bikhchandani, Hirshleifer and Welch (1992), and Smith and Sørensen (2000), in which myopic players learn about an exogenous state.

In terms of research question, I examine the effects of social learning on a patient player’s discounted average payoff. This is novel compared to existing models that focus on players’ asymptotic beliefs (Banerjee 1992, Bikhchandani et al. 1992, and Smith and Sørensen 2000), asymptotic rates of learning (Gale and Kariv 2003, Harel, et al. 2019), and asymptotic payoffs (Rosenberg and Vieille 2019).

My model yields new insights and predictions. Due to the long-run player’s incentive constraints and the presence of commitment type, the myopic players never herd on any action that does not best reply against the commitment action. Moreover, whether the long-run player can secure high returns from building reputations depends on the minimal rate of learning that is consistent with his incentive constraints. In particular, whether this minimal rate goes to zero as he becomes arbitrarily patient.

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4Players are also learning about some exogenous state in the recent works of Mossel, Sly and Tamuz (2015), Rosenberg and Vieille (2019), and Harel, Mossel, Strack and Tamuz (2019). Jackson and Kalai (1997) study recurring games in which a sequence of myopic players learn about the exogenous distribution over types in some populations.
This paper contributes to the reputation literature by establishing reputation results in environments with social learning (Theorems 4 and 4). To the best of my knowledge, this has not been explored before. My Theorems 1 and 3 identify a new mechanism that accounts for several instances of reputation failures. In particular, reputation fails since the speed of learning vanishes as the reputation-building player becomes patient. This differs from existing theories that are based on the uninformed player’s forward-looking incentives, or the lack-of identification of the informed player’s actions.

Models with lack-of identification such as Ely and Välimäki (2003), Ely, Fudenberg and Levine (2008), and Deb, Mitchell and Pai (2019) focus on participation games, in which the uninformed player(s) can take a non-participating action under which the public signal is uninformative about the informed player’s current period action. In Levine (2019), the signals are less informative when the uninformed players do not participate. These features contrast to my model in which the uninformed players’ actions cannot prevent her successors from observing the informed player’s current period action, and their actions are informative about the informed player’s past actions (Proposition 2).

Cripps and Thomas (1997) and Chan (2000) construct low-payoff equilibria when players are equally patient and the uninformed player can observe the entire history of the informed player’s actions. In their models as well as other models with complete records, the informed player’s patience helps reputation building while the uninformed player’s patience hurts reputation building. In my model, the uninformed players are short-lived, and it is the informed player’s patience that decreases the rate of learning and causes reputations to fail.

My paper is also related to reputation models with bounded memories, such as Ekmekçi (2011), Liu (2011), Liu and Skrzypacz (2014), and Kaya and Roy (2020). These papers study reputation games in which every short-run player observes a bounded sequence of the long-run player’s past actions, but cannot observe each others’ actions. This assumption rules out the possibility of social learning. By contrast, my paper studies the effects of social learning on a patient player’s reputational incentives, as well as policy interventions that can accelerate social learning and motivate reputation building.

Logina, Lukyanov and Shamruk (2019) study a reputation model in which every myopic buyer observes an informative signal about a patient seller’s current period action, in addition to all previous buyers’ actions. They focus on stage games in which the seller’s optimal commitment payoff equals his minmax payoff. They compute Markov Perfect Equilibria in which the seller prefers to work when his reputation is intermediate, and prefers to shirk otherwise. Intuitively, a seller has no incentive to work when buyers’ belief about his type is sufficiently precise. By contrast, my results are driven by the low rate of social learning, which apply to games where player 1 can strictly benefit from commitment.
2 Illustrative Example: Product Choice Game

I illustrate my model using the product choice game in Mailath and Samuelson (2001). This game fits into retail markets with asymmetric information, widespread mistrust between buyers and sellers, and lack of formal records. As a result, buyers acquire information from other consumers, such as observing others’ choices and learning about others’ personal experiences.

I use the watermelon retail market studied in Bai (2018) as a concrete example to demonstrate the fitness of my model and to explain the implications of my results. Readers who are interested in the formal illustration of the model and results can jump to Section 3.

**Product Choice Game:** A patient seller with discount factor $\delta \in (0, 1)$ interacts with an infinite sequence of buyers, arriving one in each period and each has unit demand in the period she arrives. In every period, the seller chooses between high ($H$) and low ($L$) effort, that determines the quality of his product. Each buyer decides whether to trust the seller ($T$) or not ($N$). Players’ payoffs are:

|   | $T$ | $N$ |
|---|---|---|
| $H$ | 2, 1 | -1, 0 |
| $L$ | 3, -1 | 0, 0 |

The seller is either a commitment type who exerts high effort (or equivalently, supplies high quality) in every period, or a strategic type who chooses effort in order to maximize his payoff.

Each individual buyer cannot observe the quality of product sold in the current period, but can observe all previous buyers’ choices. Moreover, she can sample a bounded subset of previous buyers and learn about the quality of products they bought from the seller. Both deterministic sampling (Section 3) and stochastic sampling (Section 4) are considered.

**Case Study:** I map my model into the watermelon retail market studied in Bai (2018). My assumptions fit into this market since there is significant asymmetric information about the quality of melons between buyers and sellers, and quality varies considerably across melons sold by a given seller.

I interpret a consumer’s action $T$ as purchasing a melon from a premium pile that is sold at a higher price, and $N$ as purchasing from a standard pile that is sold at a lower price. Consumers cannot observe the quality of a melon before buying. However, they can observe quality after consumption, and can pass this information to future consumers via word-of-mouth communication.

5According to Bai (2018), watermelons are usually sold as a whole since cut melons are hard to preserve in hot weather. Consumers cannot detect the fruit’s true quality by inspecting the outside. As a result, the quality of a watermelon is a classic example of experience quality defined in Nelson (1970), which refers to aspects of quality that a consumer cannot observe before consumption but can learn after consumption.
In comparison, sellers have the expertise to distinguish the quality of melons. I interpret the seller’s action as his effort on sorting when he procures melons from the wholesale market, which is either high or low. High effort can improve the quality of his melons both in the premium pile and in the standard pile. Similar to other retail markets in developing countries, there is no official record that documents the seller’s behavior in each transaction. Since the retail market is highly localized and is frequently visited by people in the neighborhood, consumers can casually observe others’ purchasing decisions, and can learn about the quality of melons bought by some previous buyers.

**Benchmark:** When every buyer can observe the complete history of the seller’s actions, Fudenberg and Levine (1989)’s result suggests that in every equilibrium, a patient seller receives at least his commitment payoff from exerting high effort (equals 2). Whether previous buyers’ actions are observed is irrelevant. Intuitively, observing the seller’s past actions guarantees a minimal speed of learning: In every period where a consumer has an incentive to choose $N$, she believes that $H$ will be played with probability less than $1/2$. As a result, the probability she attaches to the commitment type is multiplied by 2 after she observes the seller choosing $H$. Therefore, as long as the seller chooses $H$ in every period, consumers will trust the seller in all except for a bounded number of periods.

**Reputation Failure Result:** Suppose each consumer does not have access to the seller’s complete records, but instead observes all previous consumers’ choices, and the seller’s actions in the last $K$ periods. Can social learning provide adequate incentives for a patient seller to supply high quality?

I show that reputation fails in the sense that there exist equilibria in which the seller’s payoff is zero no matter how patient he is and no matter how large $K$ is (Theorem 1). This result generalizes when buyers randomly sample among other buyers (Theorem 3). Those equilibria lead to low social welfare since buyers’ discounted average payoff is also close to their minmax payoff 0 (Theorem 2).

Interestingly, reputation fails not because buyers herd on the non-trusting action $N$ (Proposition 1), and moreover, buyers’ actions are informative about the seller’s past actions (Proposition 2). In fact, as long as the seller plays $H$ in every period, the asymptotic frequency of outcome $(H, T)$ is at least $\frac{K}{K+1}$ (Proposition 3). This suggests that in all equilibria, the seller’s asymptotic payoff from building a reputation is close to his optimal commitment payoff when $K$ is large enough.

Instead, reputation failure is caused by slow learning: the rate with which play converges to the trusting outcome $(H, T)$ goes to zero as the seller becomes patient. This suggests a rationale for a finding in Bai (2018), that in the baseline setting without policy interventions, mistrust between consumers and sellers persists for a long time, buyers’ choices respond slowly to their acquaintances’
past experiences, and sellers exert little effort on sorting to improve the quality of their melons\textsuperscript{6}.

The constructive proof of my result reflects the logic behind these empirical results. For an informal illustration, consider an equilibrium that consists of a \textit{non-trusting phase} and a \textit{trusting phase}. These phases translate into two self-fulfilling social norms: one in which buyers do not trust the seller and the seller supplies high quality with low probability, another one in which buyers trust the seller and the seller supplies high quality with high probability.

Play starts from the non-trusting phase, and enters the trusting phase after the seller has been trusted by some previous buyers\textsuperscript{7}. The probability with which play enters the trusting phase (which I call, the \textit{rate of trust building}) depends on the seller’s effort in the period before. This probability is zero if the seller supplied low quality, and is strictly positive if the seller supplied high quality.

What is the rate of trusting building when the seller supplied high quality in the period before? When this probability is too low, the strategic type finds it not worthwhile to exert high effort. Therefore, the seller will have a perfect reputation by exerting high effort for one period, which leads to a high rate of learning. When this probability is too high, the strategic type finds it strictly optimal to pool with the commitment type. As a result, the seller’s reputation remains unchanged even when she exerts high effort, which leads to a low rate of learning.

The above reasoning pins down the \textit{equilibrium rate of trust building}, which is the one that makes the strategic type seller indifferent between supplying low and high quality. As the seller becomes more patient, the equilibrium rate of learning decreases. Intuitively, when a buyer observes the seller supplying high quality in the non-trusting phase, he becomes more skeptical about the seller motives when the strategic type has stronger incentives to imitate the commitment type. This endogenously prolongs the trust building process and eliminates the seller’s returns from a good reputation.

**Policy Implications:** Which policies can motivate sellers to supply high quality? I compare the performance of two commonly used policies: (1) improve record keeping, which allows each buyer to observe a longer history of the seller’s past actions; (2) randomly inspect a \textit{small fraction of products} that are currently sold on the market, and provide quality certificates to the ones with high-quality. Under the second policy, the current-period buyer can observe the quality certificate and therefore, can identify high-quality products with low but positive probability.

\begin{itemize}
  \item Theorem\textsuperscript{4} suggests that random inspection successfully restores a patient seller’s reputation build-
\end{itemize}

\textsuperscript{6}Bai’s structural estimation of the seller’s discount factor rejects the hypothesis that reputation fails due to the seller’s impatience. In her randomized control trial, sellers who were assigned with a laser-tag machine have strong incentives to build reputations. This also rejects the hypothesis that reputation failure is caused by the seller’s impatience.

\textsuperscript{7}The presence of commitment type implies that play \textit{cannot} remain in the non-trusting phase in the long run. See Proposition\textsuperscript{3} for a formal statement.
ing incentives in all equilibria. This is the case despite each quality certificate is observed only by the current-period buyer, and each product receives the certificate with low probability. This contrasts to Theorem 1 which implies that improved record keeping (an increase in $K$) is ineffective when the initial trust level between buyers and sellers is low (i.e., prior probability of commitment type is low).

Interestingly, random inspections can restore patient seller’s reputational incentives if and only if the consumer’s private signal is unboundedly informative about the seller’s commitment action, i.e., the certificate given to the high quality products cannot be forged and cannot be used on low quality ones. Intuitively, when buyers’ prior belief attaches low enough probability to the commitment type seller, and entertains the adverse belief that the strategic-type seller is likely to exert low effort, they have an incentive not to trust the seller after they observe a boundedly informative signal.

This result provides an explanation to Bai’s findings in her randomized control trial. Among the group of sellers who are provided with novel laser-cut labels, most of them exert high effort in sorting when procuring melons from the wholesale market, and trust is gradually built between these sellers and their buyers. Among the group of sellers who are provided with sticker labels that can be counterfeited, the outcomes are similar to the baseline setting in which sellers are reluctant to build reputations and consumers’ skepticism about these sellers’ product quality persists over time.

3 Reputation Failure under Social Learning

Primitives: Time is discrete, indexed by $t = 0, 1, 2, \ldots$. A long-lived player 1 (he, e.g., seller) with discount factor $\delta \in (0, 1)$ interacts with an infinite sequence of short-lived player 2s (she, e.g., buyers), arriving one in each period and each plays the game only once. In period $t$, players simultaneously choose their actions $a_t$ and $b_t$ from finite sets $A$ and $B$. Players have access to a public randomization device. Let $\xi_t$ be its realization in period $t$, which is uniformly distributed on $[0, 1]$.

Players’ stage-game payoffs are $u_1(a_t, b_t)$ and $u_2(a_t, b_t)$. Let $BR_1 : \Delta(B) \rightarrow 2^A \setminus \{\emptyset\}$ and $BR_2 : \Delta(A) \rightarrow 2^B \setminus \{\emptyset\}$ be player 1’s and player 2’s best reply correspondences in the stage-game. The set of player 1’s (pure) Stackelberg actions is $\arg \max_{a \in A} \{ \min_{b \in BR_2(a)} u_1(a, b) \}$. I introduce two assumptions on players’ stage-game payoffs:

**Assumption 1.** $BR_1(b)$ is a singleton for every $b \in B$. $BR_2(a)$ is a singleton for every $a \in A$. Player 1 has a unique pure Stackelberg action.

The difference between my result and the well-known result of Smith and Sørensen (2000) is explained in the introduction. The equivalence between unboundedly informative signal and securing high payoffs from building reputations holds when the long-run player has two actions, or when stage-game payoffs are monotone-supermodular and the signal distribution satisfies a monotone likelihood ratio condition. See Theorems 3 and 4’ in Section 5 for details.
A sufficient condition for Assumption 1 is that \( u_i(a,b) \neq u_i(a',b') \) for every \( i \in \{1,2\} \) and \( (a,b) \neq (a',b') \). This is satisfied for generic \( (u_1,u_2) \) given that \( A \) and \( B \) are finite sets. Let \( a^* \) be player 1’s pure Stackelberg action. Let \( b^* \in B \) be the unique element in \( \text{BR}_2(a^*) \), which I refer to as player 2’s Stackelberg best reply. Let \( u_1(a^*,b^*) \) be player 1’s Stackelberg payoff.

Assumption 2. There exists a pure-strategy Nash equilibrium in the stage-game.

Under Assumption 2, player 1’s pure Stackelberg payoff is weakly greater than his payoff in any pure-strategy Nash Equilibrium. This assumption is satisfied in most of the games studied in the reputation literature, such as product choice games, chain store games, and common interest games. It rules out rock-paper-scissors and other games in which commitment to pure actions is not beneficial.

Information & Monitoring Structure: Player 1 has persistent private information about his type, which is constant over time and is denoted by \( \omega \in \{\omega^s,\omega^c\} \). In particular, \( \omega^c \) stands for a commitment type who mechanically plays \( a^* \) in every period, and \( \omega^s \) stands for a strategic type who can flexibly choose his actions in order to maximize his payoff.

Player 1 can observe the all the actions taken in the past in addition to the current and past realizations of public randomization devices. Let \( h^1_t \) be a typical private history of the strategic-type player 1 in period \( t \), with \( h^1_t \equiv \{a_0,...,a_{t-1},b_0,...,b_{t-1},\xi_0,...,\xi_t\} \). Let \( \mathcal{H}_1^t \) be the set of \( h^1_t \) and let \( \mathcal{H}_1 \equiv \cup_{t=0}^{\infty} \mathcal{H}_1^t \). Strategic-type player 1’s strategy is \( \sigma_1 : \mathcal{H}_1 \to \Delta(A) \), with \( \sigma_1 \in \Sigma_1 \).

Player 2’s prior belief attaches probability \( \pi_0 \in (0,1) \) to the commitment type \( \omega^c \). Her private history coincides with the public history, which consists of all of her predecessors’ actions, player 1’s actions in the past \( K \in \{1,2,3,...\} \) periods, and the current realization of public randomization device.\(^9\) The exogenous parameter \( K \) measures player 2’s capacity to process detailed information about player 1’s past actions.\(^{10}\) Let \( h^t \) be the public history in period \( t \), with

\[
h^t \equiv \begin{cases} 
\{b_0,b_1,...,b_{t-1},a_{t-K},a_{t-K+1},...,a_{t-1},\xi_t\} & \text{if } t \geq K \\
\{b_0,b_1,...,b_{t-1},a_0,a_1,...,a_{t-1},\xi_t\} & \text{if } t < K.
\end{cases}
\]

Let \( \mathcal{H}^t \) be the set of \( h^t \) and let \( \mathcal{H} \equiv \cup_{t=0}^{\infty} \mathcal{H}^t \). Player 2’s strategy is \( \sigma_2 : \mathcal{H} \to \Delta(B) \), with \( \sigma_2 \in \Sigma_2 \). Let \( \pi(h^t) \) be the probability that player 2’s belief at \( h^t \) attaches to the commitment type, which I refer to as player 1’s reputation at \( h^t \).

\(^9\)My results apply when each player 2 observes all realizations of public randomization devices, or a subset of them.

\(^{10}\)In Section 4 I generalize my result to settings in which each player 2 observes a bounded stochastic subset of player 1’s past actions. In Section 5 I examine the game’s outcomes when each player 2 can also observe an informative signal about player 1’s action in the current period.
3 REPUTATION FAILURE UNDER SOCIAL LEARNING

Payoffs: For every strategy profile \((\sigma_1, \sigma_2)\), strategic-type player 1’s discounted average payoff is
\[
E_1^{(\sigma_1, \sigma_2)} \left[ \sum_{t=0}^{\infty} (1 - \delta)^t u_1(a_1, b_t) \right],
\]
where \(E_1^{(\sigma_1, \sigma_2)}[\cdot]\) is the expectation over histories when player 2 plays according to \(\sigma_2\) and player 1 plays according to \(\sigma_1\). Player 2’s welfare is \(E_2^{(\sigma_1, \sigma_2, \pi_0)} \left[ \sum_{t=0}^{\infty} (1 - \delta \pi)^t u_2(a_1, b_t) \right]\) where \(E_2^{(\sigma_1, \sigma_2, \pi_0)}[\cdot]\) is the expectation over histories when player 2 plays according to \(\sigma_2\), player 1 plays according to \(\sigma_1\) with probability \(1 - \pi_0\), and plays \(a_1^*\) in every period with probability \(\pi_0\), and \(\delta \pi \in (0, 1)\) is a discount factor that a planner uses to evaluate different generations of player 2’s welfare. Potentially, \(\delta\) and \(\delta \pi\) can be different.

Solution Concept: I use sequential equilibrium defined in Peški (2014) for results on reputation failures, i.e., the existence of equilibria in which a patient player 1 receives a low payoff. This ensures that the equilibria I construct are not driven by uninformed players’ unreasonable off-path beliefs. I use Bayes Nash Equilibrium for results that establish the common properties of all equilibria. This ensures the robustness of my findings against equilibrium selection. For a given parameter configuration \((\delta, \pi_0, K)\), let \(NE(\delta, \pi_0, K) \subset \Sigma_1 \times \Sigma_2\) be the set of Bayes Nash Equilibria, and let \(SE(\delta, \pi_0, K) \subset \Sigma_1 \times \Sigma_2\) be the set of strategy profiles that are part of some sequential equilibria.

3.1 Reputation Failure under Social Learning

I show that reputation building can result in low payoffs for both players, which contrasts to the conclusions of canonical reputation models. Let \((a', b') \in A \times B\) be the worst pure-strategy Nash equilibrium for player 1 in the stage-game. Let \(\underline{v}_1 \equiv u_1(a', b')\), which by definition, is weakly lower than \(u_1(a^*, b^*)\). Let
\[
\delta \equiv \begin{cases} 
\max \left\{ \max_{a' \in A} \frac{\max_{b' \in B} u_1(a^*, b') - u_1(a', b')}{u_1(a^*, b') - u_1(a^*, b')}, \frac{u_1(a', b') - u_1(a^*, b')}{u_1(a^*, b') - u_1(a^*, b')} \right\} & \text{if } \underline{v}_1 < u_1(a^*, b^*) \\
0 & \text{if } \underline{v}_1 = u_1(a^*, b^*).
\end{cases}
\]

**Theorem 1.** When stage-game payoffs satisfy Assumptions [1] and [2]. For every \(K \in \mathbb{N}\), there exists \(\pi_0 \in (0, 1)\), such that for every \(\pi_0 \in (0, \pi_0)\) and \(\delta \geq \delta\), there exists \((\sigma_1^*, \sigma_2^*) \in SE(\delta, \pi_0, K)\), such that:
\[
E_1^{(\sigma_1^*, \sigma_2^*)} \left[ \sum_{t=0}^{\infty} (1 - \delta)^t u_1(a_t, b_t) \right] = \underline{v}_1.
\]

According to Theorem [1], when the prior probability of commitment type is below some cutoff, there exist equilibria in which the long-run player receives his lowest stage-game Nash Equilibrium payoff regardless of his discount factor \(\delta\) and the short-run players’ capacity to process information \(K\). It demonstrates the failure of reputation effects when player 1’s pure Stackelberg payoff is strictly
greater than $u_1(a', b')$, i.e., player 1 can strictly benefit from commitment in the stage game:

**Condition 1** (Strict Benefit from Commitment). $u_1(a^*, b^*) > v_1$.\(^{11}\)

Under the strict benefit from commitment condition, Theorem 1 contrasts to the conclusions in Fudenberg and Levine (1989, 1992) and Gossner (2011): if player 2s have *unbounded observations* of player 1’s past actions (i.e., $K = \infty$), or more generally, unbounded observations of noisy signals that can statistically identify player 1’s actions, then a patient player 1 can *guarantee* his Stackelberg payoff $u_1(a^*, b^*)$ in *all Bayes Nash Equilibria* of the reputation game.

Intuitively, reputation fails since consumers’ learning is slow\(^{12}\). In particular, although player 2s’ actions are informative about player 1’s actions in the past, their informativeness vanishes endogenously as player 1 becomes patient. This is reflected in my constructive proof of Theorem 1 (Section 3.3): Player 2s’ actions converge to $b^*$ with probability 1 on the equilibrium path, but the rate with which their actions converge vanishes to 0 as $\delta$ goes to 1. The low speed of social learning wipes out player 1’s returns from reputation building, discourages him from building a reputation, and makes player 2s’ pessimistic beliefs about player 1’s actions in the early stages of the game self-fulfilling.

Two natural questions follow from Theorem 1. First, even when $v_1$ is strictly lower than $u_1(a^*, b^*)$, it may not equal to player 1’s minmax payoff. Second, Theorem 1 demonstrates the negative payoff consequences from the long-run player’s perspective, but a more important question in many applications is the short-run players’ welfare. I state two results that address these concerns:

1. I identify a class of games that fit into buyer-seller applications (i.e., games with monotone-supermodular payoffs defined in Condition 2), in which $v_1$ coincides with player 1’s minmax payoff. In Appendix A.3, I focus on games in which player 1’s minmax payoff is strictly lower than $v_1$ and provide sufficient conditions under which player 1’s lowest equilibrium payoff in the reputation game equals his minmax payoff.

2. I show that slow observational learning also results in low welfare for player 2s, regardless of the discount factor a social planner uses to evaluate different generations of player 2s’ payoffs. In games with monotone-supermodular payoffs, there exist equilibria in which both players attain their respective minmax payoffs.

\(^{11}\)Condition 1 is less demanding than the lack-of-commitment condition in Cripps, Mailath, and Sameulson (2004, Assumption 3), which requires that the Stackelberg action $a^*$ does not best reply against $b^*$. My strict benefit from commitment condition is satisfied not only in product choice games and entry deterrence games, but is also satisfied in games with conflicting interests such as chicken games, and coordination games in which player 1 receives different payoffs from different pure strategy Nash equilibria.

\(^{12}\)In order to confirm that slow learning is the driving force behind Theorem 1, I establish common properties of all Bayes Nash Equilibria in Section 3.2 which rule out other potential causes of reputation failures. The forces that are ruled out include player 2s’ herding, player 2s’ actions being uninformative, and player 1 receiving low asymptotic payoff.

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Attaining Long-Run Player’s Minmax Payoff: I review the notion of player 1’s minmax payoff in Fudenberg, Kreps and Maskin (1990) which takes player 2’s myopia into account. Let

\[ B^* \equiv \{ \beta \in \Delta(B) | \exists \alpha \in \Delta(A) \text{ s.t. } \beta \in BR_2(\alpha) \} \subset \Delta(B) \] (3.2)

be the set of player 2’s mixed actions that best reply against some mixed actions of player 1’s. Since player 2s are myopic, their actions belong to \( B^* \) at every on-path history. Player 1’s minmax payoff is:

\[ v_{1}^{\min} \equiv \min_{\beta \in B^*} \max_{a \in A} u_1(a, \beta), \] (3.3)

which is his lowest payoff in any Bayes Nash Equilibrium of the reputation game. Action \( \beta \in B^* \) is player 2’s \emph{minmax action} if and only if player 1’s payoff from best replying against \( \beta \) equals \( v_{1}^{\min} \).

I show that \( v_1 = v_{1}^{\min} \) in \emph{monotone-supermodular games}, which have been a primary focus of the reputation literature and are applicable to the study of business transactions (Mailath and Samuelson 2001, Liu 2011, Liu and Skrzypacz 2014), capital taxation (Phelan 2006), monetary policy (Barro and Gordon 1983), and so on.\(^\text{13}\)

\textbf{Condition 2.} Payoffs are monotone-supermodular if there exist complete orders on \( A \) and \( B \):

1. \( u_1(a,b) \) is strictly decreasing in \( a \) and is strictly increasing in \( b \).
2. \( u_1(a,b) \) has non-increasing differences, and \( u_2(a,b) \) has strictly increasing differences in \( (a,b) \).
3. \( a^* \) is not the lowest element in \( A \).

For example, the product choice game in Section\(^\text{2}\) satisfies monotone-supermodularity when player 1’s actions are ranked according to \( H \succ L \), and player 2’s actions are ranked according to \( T \succ N \). I provide economic interpretations of this condition in context of the buyer-seller application. Let player 1 be a seller with \( a \in A \) interpreted as his effort or the quality of his product. Let each player 2 be a buyer with \( b \in B \) interpreted as the quantity she buys. Monotone-supermodularity requires that (1) it is costly for the seller to exert high effort or to supply high quality, but he strictly benefits from buyers’ purchases; (2) buyers have stronger incentives to purchase larger quantities when the seller’s effort is higher or the quality of product is higher; (3) the seller receives more benefit from undercutting quality when a buyer purchases a larger quantity.\(^\text{14}\) (4) exerting the lowest effort is not

\(^{13}\) The monotone-supermodular condition is different from the one in Pei (2020) which incorporates interdependent values. Moreover, Pei (2020) does not require \( u_1(a,b) \) to have non-increasing differences. My condition is similar to the one in Liu and Skrzypacz (2014), which requires \( u_1(a,b) \) to have strictly decreasing differences.

\(^{14}\) \( u_1(a,b) \) having strictly decreasing differences in \( (a,b) \) is not required for the results in this section (i.e., Theorem\(^\text{2}\) Corollaries 1 and 2), but is needed for the results on stochastic sampling in Section\(^\text{4}\)
the seller’s optimal commitment action.

Under the first and second requirements in Condition 2 player 1’s minmax payoff coincides with his lowest pure strategy Nash equilibrium payoff in the stage game. Under the third requirement, his minmax payoff is strictly lower than his pure Stackelberg payoff.

**Corollary 1.** If players’ payoffs are monotone-supermodular, then $v_1 = v_1^{\min} < u_1(a^*, b^*)$.

**Short-Run Players’ Welfare:** My next result demonstrates the failure of reputation effects from player 2’s perspective. Let $(a'', b'') \in A \times B$ be a worst pure-strategy stage-game Nash equilibrium for player 2. Recall that $\delta_s$ is a planner’s discount factor when evaluating player 2’s welfare. Let

$$\delta' \equiv \begin{cases} \max \left\{ \frac{\max_{a', b'} \{u_1(a', b') - u_1(a^*, b^*)\}}{\max_{a, b} \{u_1(a, b) - u_1(a^*, b^*)\}} \right\} & \text{if } u_1(a'', b'') < u_1(a^*, b^*) \\ 0 & \text{if } u_1(a'', b'') = u_1(a^*, b^*) \end{cases}$$

I show that player 2’s welfare can be low regardless of the planner’s discount factor:

**Theorem 2.** Under Assumptions 1 and 2. For every $K \in \mathbb{N}$, $\delta_s \in (0, 1)$, and $\varepsilon > 0$, there exists $\pi_0 \in (0, 1)$ such that for every $\pi_0 \in (0, \pi_0)$ and $\delta \geq \delta'$, there exists $(\sigma_1^\delta, \sigma_2^\delta) \in SE(\delta, \pi_0, K)$, such that:

$$\mathbb{E}(\sigma_1^\delta, \sigma_2^\delta, \pi_0) \left[ \sum_{i=0}^{\infty} (1 - \delta_s)\delta_s^i u_2(a_t, b_t) \right] \leq u_2(a'', b'') + \varepsilon. \quad (3.4)$$

The proof is in Appendix A.1 Theorems 1 and 2 together have powerful implications on games with monotone-supermodular payoffs (Condition 2). Let $a \equiv \min A$ and $\{b\} \equiv BR_2(a)$. The third requirement in Condition 2 implies that $a^* > a$, and $(a, b)$ is a stage-game Nash equilibrium. This implies that games with monotone-supermodular payoffs automatically satisfy Assumption 2.

Theorems 1 and 2 imply the following corollary, that slow observational learning leads to sequential equilibria in which both players receive low payoffs. Under an additional requirement that $a$ minimizes $u_2(a, b)$ for every $b \in B$ (i.e., the seller’s lowest effort minimizes the buyer’s payoff), both players’ equilibrium payoffs are arbitrarily close to their respective minmax payoffs:

**Corollary 2.** If players’ payoffs are monotone-supermodular and satisfy Assumption 1, then there exists $\delta \in (0, 1)$. For every $K \in \mathbb{N}$, $\delta_s \in (0, 1)$, and $\varepsilon > 0$, there exists $\pi_0 \in (0, 1)$ such that for every $\pi_0 \in (0, \pi_0)$ and $\delta \geq \delta$, there exists $(\sigma_1^\delta, \sigma_2^\delta) \in SE(\delta, \pi_0, K)$ such that:

1. player 1’s discounted average payoff equals $u_1(a, b)$,

2. player 2’s discounted average welfare is at most $u_2(a, b) + \varepsilon$. 

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3.2 Common Properties of All Equilibria

To appreciate the subtlety of Theorem 1 and to distinguish its underlying mechanism from existing results on social learning and reputation failures, I establish three properties that apply to all Bayes Nash equilibria. I present these findings in decreasing level of generality.

First, I show that player 2s never herd on actions other than $b^*$. It clarifies the conceptual difference between my result and the ones in Banerjee (1992), Bikhchandani, Hirshleifer and Welch (1992), and Smith and Sørensen (2000), in which inefficiencies are caused by myopic players herding on an inefficient action. For every $\sigma_2 \in \Sigma_2$, $h^t \in H^t$, and $b \in B$, I say that player 2s herd on action $b$ at $h^t$ if $\sigma_2(h^s) = b$ for every on-path history $h^s \succeq h^t$.

**Proposition 1.** Under Assumptions 1 and 2. For every Bayes Nash equilibrium $(\sigma_1, \sigma_2)$, every on-path history $h^t$, and every $b \neq b^*$, if $\pi(h^t) > 0$, then player 2s cannot herd on $b$ at $h^t$.

**Proof of Proposition 1:** When future player 2s herd on action $b \neq b^*$ at $h^t$, strategic-type player 1 has no intertemporal incentive at all histories succeeding $h^t$. Therefore, strategic-type player 1 plays his myopic best reply against $b$ at every subsequent history in equilibrium. I consider two cases separately, depending on whether $a^*$ best replies against the herding action $b$ or not. First, suppose $\text{BR}_1(b) = \{a^*\}$, then both types of player 1 play $a^*$ in equilibrium with probability 1. As a result, player 2 has a strict incentive to play $b^*$ instead of $b$ at $h^t$. This contradicts the presumption that $b \neq b^*$. Next, suppose $\text{BR}_1(b) \neq \{a^*\}$. In equilibrium, strategic-type player 1 plays $a^*$ with probability 0 at $h^t$. Since $\pi(h^t) > 0$, player 2’s posterior belief attaches probability 1 to the commitment type after observing $a^*$ at $h^t$, and has a strict incentive to play $b^*$ in period $t + 1$ given that $K \geq 1$. This contradicts the presumption that player 2s herd on action $b \neq b^*$.

Next, I focus on games in which player 1 faces a strict lack-of-commitment problem, i.e., his pure Stackelberg action does not best reply against any action in $B^*$. This includes, but not limited to games with monotone-supermodular payoffs. It suggests that player 2s’ actions are informative signals about player 1’s actions whenever the latter receives a low payoff in the near future. It distinguishes my result from the ones on bad reputations such as Ely and Välimäki (2003) and Ely, Fudenberg and Levine (2008), in which the short-run players can take an action (i.e., refuse to participate) that stops future short-run players from learning while giving the long-run player his minmax payoff.

**Proposition 2.** If $a^*$ does not best reply against any action in $B^*$, then for every Bayes Nash Equilibrium and at every on-path history $h^t$ with $\pi(h^t) > 0$,

- either $(b_{t+1}, ..., b_{t+K})$ is informative about player 1’s action at $h^t$,
Moreover, there exists a sequential equilibrium with strategy profile \( \tau \pi_B \) for every \( 1 \leq \tau \leq K \) and \( h^{t+\tau} > h^t \) with \( a^* \) being played from \( t \) to \( t + \tau - 1 \).\(^{15}\)

Proposition 2 implies that in games where player 1 has a strict incentive to deviate from his pure Stackelberg action in the stage game, as long as the strategic type imitates the commitment type, either player 1 is guaranteed to receive his Stackelberg payoff \( u_1(a^*, b^*) \) in the next \( K \) periods, or player 2's actions in the next \( K \) periods are informative signals about player 1's action in the current period. The long-run player receives a high payoff in the first case, and information about his current-period behavior is communicated to all future player 2s in the second case. This is reminiscent of the logic behind Fudenberg and Levine (1989, 1992)'s reputation results, that by imitating the commitment type, the strategic long-run player either receives a high stage-game payoff, or can generate a public signal that is informative about his type.

**Proof of Proposition 2:** Let \((\sigma_1, \sigma_2)\) be a Bayes Nash equilibrium, and let \( h^t \) be an on-path history with \( \pi(h^t) > 0 \). Suppose \((b_{t+1}, ..., b_{t+K})\) is uninformative about player 1's action at \( h^t \), then strategic-type player 1 plays his myopic best reply against player 2's action at \( h^t \).

Since player 2's play some action in \( B^* \) at every on-path history, and \( a^* \) does not best reply against any action in \( B^* \), strategic-type player 1 has a strict incentive not to play \( a^* \) at \( h^t \). Given that \( \pi(h^t) > 0 \), player 2's posterior belief attaches probability 1 to type \( \omega^c \) after observing \( a^* \) at \( h^t \). For every \( \tau \in \{1, 2, ..., K\} \), player 2 plays \( b^* \) in period \( t + \tau \) when she observes \( a^* \) in the past \( K \) periods. \( \square \)

In games with monotone-supermodular payoffs, Proposition 2 implies a tight lower bound on player 1's asymptotic payoff when he plays \( a^* \) in every period, which is stated as Proposition 3. Let \( \mathbb{E}(a^*, \sigma_2) \) be the expectation when player 1 plays \( a^* \) in every period and player 2's use strategy \( \sigma_2 \).

**Proposition 3.** If stage-game payoffs are monotone-supermodular, then for every BNE \((\sigma_1, \sigma_2)\),

\[
\liminf_{t \to \infty} \frac{1}{t} \mathbb{E}(a^*, \sigma_2) \left[ \sum_{s=0}^{t-1} u_1(a_s, b_s) \right] \geq \frac{K}{K+1} u_1(a^*, b^*) + \frac{1}{K+1} \min_{b \in B} u_1(a^*, b).
\] (3.5)

Moreover, there exists a sequential equilibrium with strategy profile \((\sigma_1, \sigma_2)\) such that:

\[
\lim_{t \to \infty} \frac{1}{t} \mathbb{E}(a^*, \sigma_2) \left[ \sum_{s=0}^{t-1} u_1(a_s, b_s) \right] = \frac{K}{K+1} u_1(a^*, b^*) + \frac{1}{K+1} \min_{b \in B} u_1(a^*, b).^{16}
\] (3.6)

\(^{15}\)Compared to Proposition 1, Proposition 2 rules out a larger class of strategies, which includes but not limited to player 2's herding on a pure action. For example, Proposition 2 rules out situations in which player 2 ever played mixed actions, or their strategies depend nontrivially on calendar time or previous player 2's actions, but not on their observations of player 1's past actions. As a consequence, Proposition 2 also imposes an additional requirement on players' stage-game payoffs, namely, the Stackelberg action is strictly suboptimal for player 1 in the stage game.

\(^{16}\)Proposition 3 does not contradict the disappearing reputation result in Cripps, Mailath and Samuelson (2004) that

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The proof is in Appendix A.2. Inequality (3.5) implies that by imitating the commitment type, player 1’s asymptotic payoff is at least a fraction $\frac{K}{K+1}$ of his Stackelberg payoff, with the RHS of (3.5) converging to $u_1(a^*,b^*)$ as $K$ goes to infinity. This lower bound is tight in the sense that there exists an equilibrium that attains this asymptotic payoff. Proposition 3 contrasts to Theorem 1 which suggests that no matter how large $K$ is, player 1’s discounted average payoff equals his lowest stage-game Nash Equilibrium payoff. This comparison not only highlights the importance of player 1’s discount factor in models where the speed of learning is endogenous, but also shows that Theorem 1 is driven by the low speed of social learning, rather than low-payoff outcomes in the long run.

3.3 Proof of Theorem 1

Recall the definitions of $(a^{'},b^{'})$ and $(a^*,b^*)$. If $b^{' } = b^*$, then according to Assumption 1, $a^{' } = a^*$ and payoff $v_1$ is attained in an equilibrium where $(a^*,b^*)$ is played at every history.

In what follows, I consider the interesting case in which $b^{'} \neq b^*$. Assumption 1 implies that:

$$u_1(a^*,b^*) \geq u_1(a^{'},b^{'}) \geq u_1(a^*,b^{'})$$

(3.7)

Let $q^* \in (0,1)$ be small enough such that $b^{'}$ is player 2’s best reply against player 1’s mixed action $q^* \circ a^* + (1-q^*) \circ a^{' }$. Let $\pi_0 \in (0,1)$ be defined via:

$$\pi_0 = \left( \frac{q^*}{2-q^*} \right)^{K+1},$$

which is the upper bound on the prior probability of commitment type. For every $\pi_0 \in (0,\pi_0)$ and $\delta$ large enough, I construct the following three-phase equilibrium in which strategic-type player 1 attains payoff $v_1$. The current phase of play depends only on the history of player 2’s actions, which are commonly observed by both players. In period $t$,

1. play is in the reputation building phase if there exists no $s < t$ such that $b_s = b^*$;

2. play is in the reputation maintenance phase if (1) there exists $s < t$ such that $b_s = b^*$, and (2) there exists no $\tau \in \{s^* + 1, \ldots, t-1\}$ such that $b_\tau \neq b^*$, where $s^*$ is the smallest $s$ with $b_s = b^*$;

the long-run player’s asymptotic payoff equals to one of his payoffs in the repeated complete information game without commitment type. Notice that the LHS of (3.5) is player 1’s asymptotic payoff when he plays $a^*$ in every period, while Cripps, Mailath and Samuelson (2004)’s result examines player 1’s asymptotic payoff under his equilibrium strategy. In Cripps, Mailath and Samuelson (2004), the short-run players have unbounded observations of informative signals about the long-run player’s past actions. One can use the arguments in Fudenberg and Levine (1992) and Gossner (2011) to show that the long-run player’s asymptotic payoff by playing $a^*$ in every period is at least his pure Stackelberg payoff.
3. play is in the **punishment phase** if (1) there exists $s < t$ such that $b_s = b^*$, and (2) there exists $\tau \in \{s^* + 1, \ldots, t - 1\}$ such that $b_\tau \neq b^*$, where $s^*$ is the smallest $s$ such that $b_s = b^*$.

Play starts from the reputation building phase, and gradually reaches the reputation maintenance phase. Play reaches the punishment phase only at off-path histories\(^\text{17}\).

**Equilibrium Strategies:** Let $r \in (0, 1)$ be defined via the following equation:

\begin{equation}
(1 - \delta)u_1(a^*, b') + \delta ru_1(a^*, b^*) + \delta(1 - r)u_1(a', b') = u_1(a', b'). \tag{3.8}
\end{equation}

One can verify that when $\delta$ is large enough, $r$ is strictly between 0 and 1, and converges to 0 as $\delta$ goes to 1. At every history $h^t$ of the **reputation-building phase**,\(^\text{17}\)

- If (1) $t = 0$, or (2) $t \geq 1$ and $a_{t-1} \neq a^*$, or (3) $t \geq 1$, $a_{t-1} = a^*$ and $\xi_t > r$, then player 2 plays $b'$ and strategic-type player 1 plays $a^*$ with probability $q^*/2$ and plays $a'$ with probability $1 - q^*/2$.

- If $t \geq 1$, $a_{t-1} = a^*$ and $\xi_t \leq r$, then player 2 plays $b^*$ and player 1 plays $a^*$.

At every history of the **reputation maintenance phase**,\(^\text{17}\)

- If $a_{t-1} = a^*$, then player 1 plays $a^*$ and player 2 plays $b^*$.

- If $a_{t-1} \neq a^*$, then player 2 plays $b'$ and strategic-type player 1 plays $a'$.

At every history of the **punishment phase**, player 1 plays $a'$ and player 2 plays $b'$.

**Incentive Constraints:** I verify players’ incentives constraints. To start with, when $\delta > \tilde{\delta}$,

\[
u_1(a^*, b^*) \geq (1 - \delta) \max_{a \in A} u_1(a, b^*) + \delta u_1(a', b').\]

This implies that player 1 has an incentive to play $a^*$ in the reputation-maintenance phase. Next, given that player 1’s continuation value is $u_1(a^*, b^*)$ in the reputation-maintenance phase and is $u_1(a', b')$ in the reputation building phase, \([3.8]\) implies that player 1 is indifferent between $a'$ and $a^*$ in the reputation-building phase. Since $a'$ best replies against $b'$ in the stage game, player 1 strictly prefers $a'$ to actions other than $a^*$ and $a'$.

---

\^17The punishment phase only occurs off the equilibrium path when the seller knows each buyer’s sample and players have access to a public randomization device. It occurs on the equilibrium path either when each buyer’s sample is stochastic and is not observed by the seller (Section 4), or players do not have access to a public randomization device.
Next, I verify player 2’s incentive to play $b'$ in the reputation-building phase by showing that player 1’s reputation is at most $q^*/2$. If player 1 has played actions other than $a^*$ in at least one of the last $K$ periods, then player 2’s belief attaches probability 0 to the commitment type. Suppose $b^*$ has never been played before (i.e., play remains in the reputation-building phase) and $a^*$ was played in the last $K$ periods, $\pi_t$ satisfies the following equation:

$$
\frac{\pi_t}{1 - \pi_t} = \frac{\Pr(a^*, \sigma_1^2)(a^*, ..., a^*)}{\Pr(\sigma_1^2, \sigma_2^2)(a^*, ..., a^*)} \cdot \frac{\Pr(\sigma_1^*, \sigma_2^2)(b', ..., b', \xi | a^*, ..., a^*)}{\Pr(\sigma_1^2, \sigma_2^2)(b', ..., b', \xi | a^*, ..., a^*)},
$$

(3.9)

where $\Pr(\sigma_1^2, \sigma_2^2)(\cdot)$ is the probability measure induced by strategy profile $(\sigma_1^2, \sigma_2^2)$, and $\Pr(a^*, \sigma_2^2)(\cdot)$ is the probability measure when player 1 plays $a^*$ in every period and player 2’s strategy is $\sigma_2^2$.

Since the strategic type plays $a^*$ with probability $q^*/2$ in every period of the reputation building phase, we have:

$$
\frac{\Pr(\sigma_1^*, \sigma_2^2)(a^*, ..., a^* | \omega^c)}{\Pr(\sigma_1^2, \sigma_2^2)(a^*, ..., a^* | \omega^c)} \leq (\frac{q^*}{2 - q^*})^{-K}.
$$

(3.10)

In addition, $b'$ occurs with weakly lower probability when player 1 is the commitment type, which implies that:

$$
\frac{\Pr(\sigma_1^2, \sigma_2^2)(b', ..., b', \xi | a^*, ..., a^*, \omega^c)}{\Pr(\sigma_1^2, \sigma_2^2)(b', ..., b', \xi | a^*, ..., a^*, \omega^c)} \leq 1.
$$

(3.11)

Since $\frac{\pi_0}{1 - \pi_0} \leq (\frac{q^*}{2 - q^*})^{K+1}$, (3.9), (3.10) and (3.11) together imply that $\pi_t \leq \frac{q^*}{2}$. As a result, the unconditional probability with which player 2 believes that player 1 plays $a^*$ is at most $\frac{q^*}{2} + (1 - \frac{q^*}{2}) \frac{q^*}{2} < q^*$ at every history of the reputation-building phase. This verifies player 2’s incentive to play $b'$ in the reputation-building phase.

### 3.4 Connections to Existing Reputation Models

I reconcile Theorem 1 and the results in Fudenberg and Levine (1989, 1992) by applying Gossner (2011)’s arguments to my model. I explain why it leads to an uninformative payoff lower bound despite player 2’s actions are informative about player 1’s past actions. I also explain the conceptual differences between the low-payoff equilibria in my model with the ones in reputation models with two equally patient players, such as Cripps and Thomas (1997) and Chan (2000).

**Relative Entropy & Value of Reputations:** Recall that Gossner (2011) establishes the following upper bound on the expected sum of Kullback-Leibler divergence (hereafter, KL divergence) between the distribution over public signals generated by the commitment type and the distribution over public
signals generated by players’ equilibrium strategies:

$$
\mathbb{E}^{(a^*,\sigma^2)} \left[ \sum_{t=0}^{\infty} d \left( y_t(\cdot|a^*) \bigg| y_t(\cdot) \right) \right] \leq - \log \pi_0. \tag{3.12}
$$

where $\pi_0$ is the prior probability of the commitment type, $y_t(\cdot)$ is the distribution over period $t$ public signals according to players’ equilibrium strategies, $y_t(\cdot|a^*)$ is the distribution over period $t$ public signals when player 2s play their equilibrium strategy and player 1 plays $a^*$ in every period, and $d(\cdot||\cdot)$ is the relative entropy between the two probability distributions. I call $d \left( y_t(\cdot|a^*) \bigg| y_t(\cdot) \right)$ player 2’s one-step ahead prediction error in period $t$.

Inequality (3.12) applies to my setting once we take $y_t$ to be the distribution of $(b_{t+1},...,b_{t+K})$. According to Proposition 2, $(b_{t+1},...,b_{t+K})$ is an informative signal of $a_t$ unless player 1’s average payoff in the next $K$ periods is at least $u_1(a^*,b^*)$.

The difference arises when deriving the lower bound on player 1’s discounted average payoff from inequality (3.12). In the models of Fudenberg and Levine (1989, 1992) and Gossner (2011), if the public signals can statistically identify player 1’s actions and when player 2 does not have a strict incentive to play $b^*$, then $d \left( y_t(\cdot|a^*) \bigg| y_t(\cdot) \right)$ is bounded from below by a strictly positive number. Therefore, as long as player 1 imitates the commitment type, the expected number of periods in which player 2s’ myopic best reply is not $b^*$ is bounded from above. As player 1 becomes patient, the payoff consequence of this bounded number of periods goes to 0. As a result, a patient player 1 is guaranteed to receive his optimal commitment payoff.

In my model, the value of $d \left( y_t(\cdot|a^*) \bigg| y_t(\cdot) \right)$ is strictly positive whenever player 2 does not have a strict incentive to play $b^*$, and player 1’s average payoff from period $t$ to $t+K$ is less than

$$
\frac{K}{K+1} u_1(a^*,b^*) + \frac{1}{K+1} \min_{b \in B} u_1(a^*,b).
$$

However, the lower bound on $d \left( y_t(\cdot|a^*) \bigg| y_t(\cdot) \right)$ depends on $\delta$, and vanishes to 0 as $\delta \rightarrow 1$. Intuitively, future player 2s’ actions are responsive to player 1’s past actions in order to provide player 1 an incentive to play actions other than his myopic best reply (for example, player 1’s Stackelberg action). When player 1 becomes more patient, he puts more weight on his continuation value relative to his stage-game payoff. Therefore, he is willing to sacrifice his stage-game payoff even when his action affects player 2s’ future actions with low probability. This endogenously reduces the informativeness of player 2s’ actions, lowers the speed of learning, increases the amount of time required for player 1 to establish a reputation, which in turn, wipes out player 1’s returns from building reputations.

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In the constructed equilibrium in Section 3.2, player 2’s one-step ahead prediction error in the reputation building phase is:

$$\log (1 + (1 - q^*)(1 - \delta)).$$  \hfill (3.13)

Using the Taylor’s expansion, (3.13) is of magnitude $1 - \delta$ when $\delta$ is close to 1. As a result, when player 1 imitates the commitment type, the expected number of periods with which player 2’s belief about player 1’s action being far away from $a^*$ goes to infinity as $\delta \to 1$. As predicted by Theorem 1, the negative payoff consequence of such periods completely offsets the benefits from building reputations.

**Reputation Models with Two Equally Patient Players:** Cripps and Thomas (1997) study reputation games between an informed player and an *equally patient* uninformed player. They focus on *common interest games*, and assume that both players can perfectly observe each other’s actions in the past. When the prior probability of the commitment type is sufficiently low, they construct a sequential equilibrium in which both players’ payoffs are arbitrarily close to their minmax payoffs, regardless of their common discount factor. In the following example, suppose with small but positive probability, player 1 is a commitment type that plays $H$ in every period,

|   | A | B |
|---|---|---|
| $H$ | 1,1 | $-\epsilon, -\epsilon$ |
| $L$ | $-\epsilon, -\epsilon$ | 0,0 |

there *exist* equilibria in which both players’ discounted average payoffs are arbitrarily close to 0.

In the active learning phase of their constructed equilibrium, player 1 plays $H$ with probability close to 1, i.e., learning is slow. However, player 2 does not play her myopic best reply against $H$. The reason is: player 2 fears that once she plays $A$ while player 1 plays $L$, players will coordinate on the inefficient outcome ($L, B$) in all future periods. Similar to the equilibrium constructed in the proof of Theorem 1, the asymptotic play also converges to the Stackelberg outcome although the patient player’s discounted average payoff is low, regardless of the common discount factor shared by the two players. This finding is generalized by Chan (2000) to all games except for (1) games where player 1 has a strictly dominant action (such as the prisoner’s dilemma), and (2) games with strictly conflicting interests (such as the chain store game).

Compared to the reputation failure results of Cripps and Thomas (1997) and Chan (2000) that hinge on the uninformed player’s patience, I show that learning can be arbitrarily slow and reputation effects can fail even when the uninformed players are myopic. In terms of how players’ patience affects the informed player’s guaranteed payoff from building reputations, in models with *unbounded records*,

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such as Fudenberg and Levine (1989, 1992), Cripps and Thomas (1997) and Chan (2000), the informed player’s patience helps reputation building, while the uninformed player’s patience hurts reputation building. This contrasts to my model in which the informed player’s patience is self-defeating since it endogenously lowers the speed of social learning. This causes reputations to fail even when the uninformed players are myopic.

My reputation failure result also applies to games in which Chan (2000)’s folk theorem result fails. This includes games with strictly conflicting interests, such as the following entry deterrence game:

|     | Out | In  |
|-----|-----|-----|
| F   | 2, 0| 0, −1|
| A   | 2, 0| 1, 1|

When there exists a commitment type that plays $F$ in every period, Cripps, Dekel and Pesendorfer (2005) show that player 1 can secure his Stackelberg payoff 2 in a model with equally patient players.

In contrast, my Theorem 1 suggests that when player 2s are myopic, have unlimited observations of other player 2s’ actions, but have bounded observations about player 1’s past actions, there exist equilibria in which player 1’s payoff equals his minmax payoff 1.

4 Reputation Failure under Stochastic Sampling

In many applications of interest, consumers are connected via a social network. Each consumer communicates with her neighbors, learns about the seller’s actions against them before making a decision; or consumers stochastically sample their predecessors to learn about their personal experiences.

In contrast to the baseline model, a common feature in these scenarios is that the seller cannot observe who do each buyer samples and the realized social network among buyers. As a result, buyers are privately learning about the seller’s type and are privately monitoring the seller’s past actions.

Motivated by these practical concerns, this section studies games with monotone-supermodular stage-game payoffs (Condition 2) and generalizes the insights of Theorem 1 when each buyer samples a bounded stochastic subset of her predecessors and observes the seller’s actions against them.

---

18 Compared to reputation games with equal discounting, one can obtain positive reputation results in a larger class of games when the uninformed player is forward-looking, but is infinitely less patient relative to the informed player. See Schmidt (1993) and Evans and Thomas (1997) for positive reputation results.

19 The example in Cripps, Dekel and Pesendorfer (2005) violates Assumption 1 since both $F$ and $A$ are player 1’s stage-game best replies against player 2’s action Out. Nevertheless, one can use the same argument in the proof of Theorem 1 to find equilibria in which player 1’s discounted average payoff is 1. The details are available upon request.

20 In Online Appendix A, I further generalize my result by relaxing the monotone-supermodularity assumption on payoffs. I identify the assumptions that are needed on players’ payoffs. The general version of my results apply not only to monotone-supermodular games (e.g., product choice game, capital taxation game, and trust game), but also to common interest games, battle of sexes, chicken games, and so on.
Let \( \{N_t\}_{t=1}^{\infty} \) be a stochastic network among buyers, with \( N_t \in \Delta(2^{[0,1,...,t-1]}) \). The realization of \( N_t \) is \( N_t \subset \{0,1,...,t-1\} \), which is privately observed by the period \( t \) buyer and is unbeknownst to the seller. Period \( t \) buyer’s private history consists of \( N_t \), buyers’ actions from period 0 to \( t-1 \), and the seller’s actions against buyers in subset \( N_t \), i.e.,

\[
h_2^t \equiv \left\{ N_t, b_0, b_1, ..., b_{t-1}, \left( a_s \right)_{s \in N_t}, \xi_t \right\}.
\] (4.1)

Player 1’s private history remains the same as in the baseline model, i.e., \( h_1^t \equiv \{a_0, ..., a_{t-1}, b_0, ..., b_{t-1}, \xi_0, ..., \xi_t\} \). I introduce the following regularity condition on the stochastic network among buyers:

**Assumption 3.** For every \( s \neq t \), \( N_s \) and \( N_t \) are independent random variables. There exist \( K \in \mathbb{N} \) and \( \gamma \in (0,1) \) such that \( \Pr \left( |N_t| \leq K \right) = 1 \) and \( \Pr \left( t-1 \in N_t \right) \geq \gamma \) for every \( t \geq 1 \).

The first part of Assumption 3 requires different buyers’ neighborhoods to be independent. My result extends when each buyer’s sampling process depends on previous buyers’ actions, i.e., \( N_t \) depends on \( \{b_0, ..., b_{t-1}\} \), as long as for every \( s < t \), \( N_s \) and \( N_t \) are independent conditional on \( \{b_0, ..., b_{s-1}\} \). This independence assumption is standard in the observational learning literature, which is trivially satisfied when the network is deterministic (Banerjee 1992), and is also assumed in models with random sampling such as Banerjee and Fudenberg (2004) and Acemoglu, Dahleh, Lobel and Ozdaglar (2011). It implies that each buyer can learn about the seller’s type only through the seller’s actions against her neighbors \( \{a_s\}_{s \in N_t} \) and the previous buyers’ actions \( \{b_s\}_{s \leq t-1} \), i.e., she cannot obtain additional information from the realization of \( N_t \).

The second part \( \Pr \left( |N_t| \leq K \right) = 1 \) requires the existence of a *uniform upper bound* on the number of predecessors each buyer can sample. As in the baseline model, \( K \) is interpreted as a constraint on the buyers’ ability to process detailed information. The third part \( \Pr \left( t-1 \in N_t \right) \geq \gamma \) requires a *uniform lower bound* on the probability with which each buyer observes the seller’s action against her immediate predecessor. Without this requirement, the seller’s action in each period will be observed by future buyers with vanishingly low probability, and buyers’ actions will not be adequate to motivate the seller to exert high effort. This requirement is satisfied for sampling processes that exhibit recency bias.

It rules out uniform sampling (i.e., the agent samples \( K \) out of \( t \) predecessors, and each predecessor is sampled with equal probability) since the probability with which the immediate predecessor’s action being observed vanishes to zero as the sample size grows to infinity.

Let \( SE(\delta, \pi_0, N) \) be the set of sequential equilibria in a repeated game with social network \( N \equiv \text{Electronic copy available at: https://ssrn.com/abstract=3626511} \)

\[21\text{A notable exception is Lobel and Sadler (2015), in which they present examples where social learning fails when agents’ neighborhoods are correlated.}\]
{N_t}_{t \in \mathbb{N}}$, discount factor $\delta$, and prior belief $\pi_0$. Theorem 3 generalizes the finding of Theorem 1 to reputation games with stochastic network monitoring:

**Theorem 3.** If stage-game payoffs are monotone-supermodular and satisfy Assumption 1, and the stochastic network $\mathcal{N}$ satisfies Assumption 3, then there exist $\pi_0 \in (0, 1)$ and $\delta \in (0, 1)$, such that for every $\pi_0 \in (0, \pi_0)$ and $\delta > \delta$, there exists $(\sigma_1^\delta, \sigma_2^\delta) \in SE(\delta, \pi_0, \mathcal{N})$, such that:

$$E_1^{(\sigma_1^\delta, \sigma_2^\delta)} \left[ \sum_{t=0}^{\infty} (1 - \delta)^t u_1(a_t, b_t) \right] = v_1. \quad (4.2)$$

A constructive proof is in Appendix B. Several features in the construction of Section 3.3 extends. First, play starts from a phase with active learning in which player 1 receives a low payoff, and gradually enters a phase in which learning stops and player 1 receives a high payoff. Second, the transition probability between the two phases depends endogenously on player 1’s discount factor, which vanishes to zero as $\delta \to 1$. Third, the asymptotic play converges to player 1’s Stackelberg outcome $(a^*, b^*)$ with probability close to 1. However, the speed with which play converges to this high-payoff phase is low. This wipes out player 1’s gains from reputation building and provides him an incentive to play the stage-game Nash Equilibrium action rather than his Stackelberg action.

The technical challenge stems from private monitoring and private learning, both of which arise when the seller does not observe the realized network. The belief-free approach to construct equilibria in Ely, Hörner and Olszewski (2005), Hörner and Lovo (2009), and Hörner, Lovo and Tomala (2011) does not apply in my model, since buyers are myopic and do not have intertemporal incentives. In equilibria where active learning takes place, buyers’ actions are sensitive to their posterior beliefs about the seller’s type, making it hard to sustain belief-free incentives.

To illustrate the idea, consider the product choice game in Section 2. Let $q^*$ be the minimal probability with which $H$ needs to be played in order to provide player 2 an incentive to play $T$. Let $\{a_0, ..., a_{t-1}, b_0, ..., b_{t-1}\}$ be a complete history in period $t$, consisting of all actions taken in the past.

First, consider providing belief-free incentives such that (1) conditional on each complete history, player 2 believes that $H$ will be played with probability $q^*$, and (2) each player 2 mixes between $T$ and $N$ with probability that makes player 1 indifferent between $H$ and $L$. Under this arrangement, both $T$ and $N$ are player 2’s best replies, regardless of her belief about player 1’s type and private history.

However, this belief-free construction is feasible in period $t$ only if after observing each period $t$ complete history, player 2’s posterior belief attaches probability less than $q^*$ to the commitment type. Since player 2’s play $N$ in the active learning phase, the probability with which player 1 plays $H$ is bounded away from 1. Therefore, a hypothetical observer’s posterior belief attaches probability
arbitrarily close to 1 to the commitment type after observing a long string of \( H \). This implies the existence of a cutoff calendar time, such that the above belief-free arrangement is feasible only if calendar time is below this cutoff.

In light of this observation, I use the following belief-based construction when calendar time is above the aforementioned cutoff. In particular, player 1’s action depends on his private history, which is chosen such that each player 2 is indifferent under her posterior belief about player 1’s private history. This is equivalent to establishing the existence of solution to a system of linear equations, in which the number of player 1’s private histories is the number of free variables, and the number of player 2’s private histories is the number of linear constraints. In period \( t \), the number of free variables is \( 2^t \). Given that the sample size is bounded above by \( K \), the number of constraints is at most \( 2^K \sum_{j=0}^{K} \binom{t}{j} \). An important observation is that the linear system is under-determined if and only if \( t \) is large relative to \( K \). This explains why the belief-free construction is used when calendar time is low, and the belief-based construction is used when calendar time is large.

5 Private Signals about Current Period Action

This section studies a variant of the baseline model in which every uninformed player observes an informative private signal about the informed player’s current-period action, in addition to the entire history of her predecessors’ actions, and possibly, the informed player’s actions in the last \( K \) periods (or the informed player’s actions against her neighbors in some stochastic network). I provide necessary and sufficient conditions under which the patient player can secure his Stackelberg payoff in all equilibria. I also discuss policy implications based on the comparison between the results in this section and the ones in Section 3.

Players move sequentially in the stage game. In period \( t \), player 1 chooses \( a_t \in A \) after observing \( h_t^1 \). Before choosing \( b_t \in B \), player 2 observes a noisy private signal about \( a_t \) denoted by \( s_t \in S \), the entire history of player 2’s past actions, and potentially, player 1’s actions in the past \( K \) periods, i.e.,

\[
h_t^2 \equiv \{s_t, b_0, ..., b_{t-1}, a_{t-K}, ..., a_{t-1}\} \tag{22}
\]

Let \( f(\cdot | a_t) \) be the conditional distribution of \( s_t \), with \( f \equiv \{f(\cdot | a)\}_{a \in A} \).

Let \( \text{NE}(\delta, \pi_0, K, f) \) be the set of Bayes Nash Equilibria. Let \( \text{SE}(\delta, \pi_0, K, f) \) be the set of strategy

\[\text{My results in this section applies to all values of } K, \text{ including } K = 0. \text{ They also apply when player 2 observes player } 1\text{'s past actions according to a stochastic network that satisfies Assumption 3. Whether player 1 observes } s_t \text{ or not is irrelevant for my results.} \]
profiles that are part of some sequential equilibria. Let
\[
V_1(\delta, \pi_0, K, f) \equiv \inf_{(\sigma_1, \sigma_2) \in \text{NE}(\delta, \pi_0, K, f)} \mathbb{E}_1^{(\sigma_1, \sigma_2)} \left[ \sum_{t=0}^{\infty} (1 - \delta^t) u_1(a_t, b_t) \right].
\] (5.1)
be player 1’s worst equilibrium payoff in Bayes Nash Equilibrium.

I start from games in which player 1’s action choice is binary, i.e., $|A| = 2$, while $|B|$ and $|S|$ can be any finite integer. I characterize the set of $f$ such that a patient player 1 can secure his Stackelberg payoff in all BNEs. For given $a \in A$, I say that $f$ is unboundedly informative about $a$ if there exists $s \in S$ such that $f(s|\bar{a}) > 0$ if and only if $\bar{a} = a$. Otherwise, $f$ is boundedly informative about $a$.

**Theorem 4.** Suppose $|A| = 2$.

1. If $f$ is unboundedly informative about $a^*$, then for every $K \in \mathbb{N} \cup \{0\}$ and $\pi_0 > 0$,
\[
\liminf_{\delta \to 1} V_1(\delta, \pi_0, K, f) \geq u_1(a^*, b^*). \tag{5.2}
\]

2. If $f$ is boundedly informative about $a^*$ and players’ stage-game payoffs satisfy Assumptions $[3]$ and $[2]$ then for every $K \in \mathbb{N} \cup \{0\}$, there exists $\pi_0 \in (0, 1)$ such that for every $\pi_0 \in (0, \pi_0)$ and $\delta$ large enough, there exists $(\sigma_1^\delta, \sigma_2^\delta) \in \text{SE}(\delta, \pi_0, K, f)$, such that:
\[
\mathbb{E}_1^{(\sigma_1^\delta, \sigma_2^\delta)} \left[ \sum_{t=0}^{\infty} (1 - \delta^t) u_1(a_t, b_t) \right] = v_1. \tag{5.3}
\]

Theorem 4 suggests that in games where $|A| = 2$ and player 1’s Stackelberg payoff is strictly greater than $v_1$, player 1 can secure his Stackelberg payoff if and only if player 2’s private signal about his action is unboundedly informative. Intuitively, observing an unboundedly informative signal about $a_t$ guarantees a lower bound on the speed of social learning. It accelerates the process of reputation building, making it worthwhile for player 1 to establish a good reputation. When $s_t$ is boundedly informative about $a^*$, the speed with which player 1 builds his reputation can be arbitrarily low, and vanishes to zero as player 1 becomes patient. Similar to Theorem 1, the prolonged learning process wipes out a patient player 1’s gains from building reputations, and consumers’ social learning cannot provide adequate incentives for a patient seller to establish good reputations.

The requirement of unboundedly informative signals is reminiscent of a well-known result in Smith and Sørensen (2000), that in canonical social learning models, myopic players’ actions are asymptotically efficient if and only if the private signals they observe are unboundedly informative about the
payoff-relevant state. Compared to their results, Theorem 4 is conceptually different for two reasons.

First, as hinted by Theorem 1 and Proposition 3, converging to a high-payoff outcome asymptotically is insufficient for player 1 to receive a high discounted average payoff, no matter how close his discount factor is to 1. This is demonstrated by the constructed equilibria in the proof of Theorem 1 in which a patient player’s discounted average payoff is low despite his asymptotic payoff is high.

Second, in my model, the short-run players learn about the endogenous actions of a strategic long-run player, while in Smith and Sørensen (2000), they learn about an exogenous state. Even when \( f \) is unboundedly informative about \( a^* \), there is no guarantee that \( b_t \) is informative about \( a_t \), or \( b_t \) is informative about player 1’s type in periods where player 1 receives a low stage-game payoff. An example is provided later in this section, in which \( f \) is unboundedly informative about \( a^* \), but \( b_t \) is uninformative about \( a_t \) and \( \omega \) despite \( b^* \) is played with low ex ante probability.

In addition, a technical difficulty arises when \( K \geq 1 \). Player 2s in my model can entertain heterogeneous beliefs about the informativeness of a public signal (i.e., previous short-run players’ actions), while in Smith and Sørensen (2000), the informativeness of each public signal is commonly known among players. Intuitively, each short-run player privately observes the long-run player’s actions in the last \( K \) periods, which are not observed by other short-run players. Due to the potential serial correlation in player 1’s actions, the informativeness of \( b_t \) about \( a_t \) can be different under the private beliefs of different short-run players.

My proof in Appendix C addresses these concerns using three observations. First, regardless of player 2’s prior belief about player 1’s action, she has a strict incentive to play \( b^* \) after observing the signal realization that occurs only when \( a = a^* \). This highlights the role of unboundedly informative private signals, which contrasts to private signals with bounded informativeness and the baseline model in which player 2s do not receive any private signal about player 1’s current-period action.

Second, when player 1’s action choice is binary and \( f \) is unboundedly informative about \( a^* \),

\[
\frac{\Pr(b_t = b^*|a_t = a^*)}{\Pr(b_t = b^*|a_t \neq a^*)}
\]

is strictly bounded above 1 whenever the ex ante probability of \( b_t = b^* \) is bounded away from 1. From the perspective of period \( t \) short-run player, it bounds the informativeness of \( b_t \) about \( a_t \) from below.

---

23My result does not follow from Mirrlees (1976) who shows that in principal-agent models, the principal can implement the first best outcome when there exists a signal realization that occurs with zero probability when the agent takes the first-best action and occurs with positive probability otherwise. This is because in my model, the rewards and punishments to player 1 are dictated by future player 2s’ behaviors. Depending on the equilibrium being played, there are multiple ways that the signal realizations are mapped into player 1’s continuation payoffs. When \( \delta \) is close to 1, it is unclear whether player 1 has an incentive to play \( a^* \) and attain his Stackelberg payoff \( u_1(a^*, b^*) \) in all equilibria.
Third, if the ex ante probability that $b_t = b^*$ is bounded away from 1 and $b_t$ is informative about player 1’s type according to player 2’s private belief in period $t$, then the informativeness of $b_t$ about $\omega$ is also uniformly bounded from below according to the private beliefs of all future player 2s. To understand why this is true, suppose in period $t$, player 2 observes that $a^*$ has been played in the last $K$ periods, and believes that $b^*$ will be played with probability at most $1 - \epsilon$, then the probability with which $(a_{t-K}, ..., a_{t-1}) = (a^*, ..., a^*)$ under the equilibrium strategy profile is bounded from below. Otherwise, period $t$ player 2 believes that the commitment type occurs with probability close to 1 after observing $(a_{t-K}, ..., a_{t-1}) = (a^*, ..., a^*)$, so the probability that she plays $b^*$ in period $t$ cannot be bounded away from 1. Therefore, the probability with which player 2 in period $s$ believes that $(a_{t-K}, ..., a_{t-1}) = (a^*, ..., a^*)$ occurring with very low probability is uniformly bounded from above. As a result, any lower bound on $b_t$’s informativeness about $\omega$ from the perspective of period $t$ player 2 leads to a lower bound on its informativeness from the perspectives of all future player 2s.

**Games with $|A| \geq 3$:** In games where player 1 has three or more actions, the equivalence between high returns from building reputations and unboundedly informative signals breaks down. To illustrate, consider the following $2 \times 3$ game:

|    | $b^*$ | $b'$ |
|----|------|------|
| $\pi$ | 1, 4 | -2, 0 |
| $a^*$ | 2, 1 | -1, 0 |
| $a$  | 3, -2| 0, 0 |

Let $S \equiv \{s, s^*, s\}$, with $f(s^*|a^*) = 2/3$, $f(s|a^*) = 1/3$, $f(s|\pi) = 1$, $f(s|a) = 1/3$, and $f(s|a) = 2/3$.

One can verify that players’ payoffs are monotone-supermodular when player 1’s actions are ranked according to $\pi \succ a^* \succ a$ and player 2’s actions are ranked according to $b^* \succ b'$. These payoffs also satisfy Assumptions [1] and [2], and $f$ is unboundedly informative about $a^*$. Player 1’s pure Stackelberg action is $a^*$, and his pure Stackelberg payoff is 2.

Consider the following strategy profile. Strategic-type player 1 plays a mixed action that depends only on player 2’s posterior belief about his type. If player 2’s posterior belief assigns probability $\pi$ to the commitment type, then the strategic-type player 1 plays $\alpha(\pi) \in \Delta(A)$, which is pinned down by:

$$(1 - \pi) \circ \alpha(\pi) + \pi \circ a^* = 0.5 \circ a^* + 0.25 \circ \pi + 0.25 \circ a.$$

Player 2 plays $b^*$ if $s_t \in \{s^*, s\}$ and plays $b'$ if $s_t = s$. 

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Player 1’s payoff under this strategy profile is 1, which is strictly below his pure Stackelberg payoff 2. This strategy profile is an equilibrium since player 1’s expected stage-game payoff is 1 no matter which action he plays, and his continuation payoff is independent of the action he takes. Player 2 has a strict incentive to play $b^*$ after observing $\overline{s}$ or $s^*$, and has an incentive to play $b'$ after observing $s$. Conditional on each type of player 1, the probability with which player 2 plays $b^*$ is $2/3$.

In this example, $b_t$ is uninformative about $a_t$ despite $f$ is unboundedly informative about $a^*$ and the ex ante probability that $b_t = b^*$ is bounded away from 1. This is driven by the heterogeneity in player 2’s incentive to play $b^*$ against different actions of player 1’s. In particular, player 2 has a stronger incentive to play $b^*$ when player 1 plays $\overline{a}$ than when player 1 plays $a^*$. As a result, player 2 has an incentive to play $b^*$ following a signal realization $\overline{s}$ that occurs with lower probability under $a^*$, and has an incentive to play $b'$ following a signal realization $s$ that occurs with higher probability under $a^*$. This concern never arises when $|A| = 2$ given there is only one action in $A$ that is not the Stackelberg action, but can happen in games where $|A| \geq 3$.

Motivated by the applications in retail markets, I focus on games with monotone-supermodular payoffs and extend Theorem 4 to games in which player 1 can have any number of actions, and the distribution of $s$ satisfies a standard monotone likelihood ratio property (or MLRP):

\begin{equation}
\frac{f(s|a)}{f(s'|a)} \geq \frac{f(s|a^*)}{f(s'|a^*)}. \tag{5.4}
\end{equation}

In applications to retail markets where a patient seller chooses the quality he supplies and each buyer along a sequence chooses whether to trust the seller after observing an informative private signal about the seller’s action in the current period, MLRP requires the buyers’ private signal realizations to be ranked such that a higher signal realization indicates that the product is of higher quality.

When player 1’s actions are ranked according to $\overline{a} > a^* > a$, the signal distribution $f$ in the previous example violates MLRP regardless of the complete order on $S$. This is because $\overline{s}$ occurs with strictly positive probability under $\overline{a}$ and $a$, but occurs with zero probability under $a^*$.

**Theorem 4’**. Suppose players’ payoffs are monotone-supermodular and $f$ satisfies MLRP.

1. If $f$ is unboundedly informative about $a^*$, then for every $K \in \mathbb{N} \cup \{0\}$ and $\pi_0 > 0$,
\[ \liminf_{\delta \to 1} V_{\overline{l}}(\delta, \pi_0, K, f) \geq u_1(a^*, b^*). \]

\[2^4\text{General necessary and sufficient conditions for positive reputation results are provided in a companion paper.} \]
2. If \( f(\cdot|a) \) has full support for every \( a \in A \) and players’ payoffs satisfy Assumption \([7]\) then for every \( K \in \mathbb{N} \cup \{0\} \), there exists \( \pi_0 \in (0,1) \) such that for every \( \pi_0 \in (0,\pi_0) \) and \( \delta \) large enough, there exists \( (\sigma^1,\sigma^2) \in \text{SE}(\delta,\pi_0,K,f) \), such that:

\[
E_1^{(\sigma^1,\sigma^2)} \left[ \sum_{t=0}^{\infty} (1-\delta^t)\delta^tu_1(a_t,b_t) \right] = v_1.
\]

Theorem 4’ suggests that in games with monotone-supermodular payoffs and the signal distributions satisfy the MLRP, \( f \) being unboundedly informative about \( a^* \) is sufficient and almost necessary for player 1 to secure his Stackelberg payoff in all equilibria of the reputation game. The 2 × 3 game example earlier in this section demonstrates why the MLRP requirement is indispensable. In Appendix [C.4] I use an example to explain why the full support condition in statement 2 of Theorem 4’ cannot be replaced by bounded informativeness when player 1 has three or more actions.

The proof is similar to that of Theorem [4] with the differences explained in Appendix [C.3]. For statement 1, the key is to show that under monotone-supermodular payoffs, unboundedly informative signals, and MLRP, for every prior belief about player 1’s action \( \alpha \in \Delta(A) \) with \( a^* \in \text{supp}(\alpha) \), and every best reply \( \beta : S \to \Delta(B) \) of player 2’s against \( \alpha \) after observing the realization of \( s \), if the ex ante probability that \( b^* \) is played is bounded away from 1, then the relative entropy between the distribution over \( b \) induced by \((\alpha,\beta)\) and that induced by \((a^*,\beta)\) is bounded away from 0.

I explain the role of unbounded informativeness and MLRP in deriving this reputation result. Since players’ payoffs are monotone-supermodular and \( f \) satisfies MLRP, player 2 has an incentive to play \( b^* \) only when the realization of \( s \) belongs to some interval \([s_*,s]\). Since \( f \) is unboundedly informative, there exists \( s^* \) such that \( f(s^*|a) > 0 \) if and only if \( a = a^* \), i.e., the probability that \( s \neq s^* \) is strictly lower when player 1 plays \( a^* \) compared to any other action. MLRP also implies that the likelihood ratio between \( s \in (s^*,s]\) and \( s > s^* \) is decreasing in \( a \), and the likelihood ratio between \( s \in [s^*,s^*) \) and \( s < s^* \) is increasing in \( a \). As a result, for any \( \alpha' \in \Delta(A\backslash\{a^*\}) \) and \( \beta : S \to \Delta(B) \) that is non-decreasing in \( s \) and is not constantly \( b^* \) in the support of \( f(\cdot|\alpha') \), the probability with which player 2 plays \( b^* \) is strictly higher when player 1 plays \( a^* \) relative to \( \alpha' \). This implies that player 2’s action is informative about player 1’s type as long as her ex ante probability of playing \( b^* \) is bounded away from 1.

6 Concluding Remarks

This paper examines a long-run player’s incentive to build reputations when his opponents have limited observations of his past actions, and instead, learn primarily from other short-run players’ actions. I
identify a new mechanism that accounts for reputation failures based on slow learning, which arises when each short-run player observes a bounded subset of the long-run player’s past actions. I also establish a reputation result when each short-run player observes an unboundedly informative private signal about the long-run player’s action in the current period.

My results provide an explanation to instances of reputation failures observed in developing economies. They also shed light on the effectiveness of various policies in accelerating social learning and encouraging sellers to build reputations.

For example, Theorems 1 and 3 suggest that marginal improvements in record-keeping technologies that allow each buyer to observe a longer history of the seller’s past actions (i.e., increasing $K$ to another finite number) is ineffective. This is especially the case when there is widespread mistrust between buyers and sellers before the policy intervention (i.e., $\pi_0$ is low). In particular, the market may get stuck in a bad equilibrium in which the rate of trust building is slow, and in response, patient sellers have weak incentives to build reputations, making buyers’ pessimistic beliefs self-fulfilling.

Theorems 4 and 4’ suggest that in markets where reputation mechanisms break down due to the lack of record keeping, randomly inspecting a small fraction of products that are currently sold on the market and issuing quality certificates to the inspected products can restore the patient seller’s incentives to supply high quality. As long as the quality certificate can be observed by one buyer, its informational content affects future buyers’ decisions via this buyer’s action. For such a policy to be effective, the regulator needs to ensure that the certificate given to the high quality products cannot be forged and cannot be used on low-quality products (i.e., the quality certificate is unboundedly informative about the seller’s Stackelberg action).
A  Proofs in Section 3

A.1 Proof of Theorem 2 & Corollary 2

I show Theorem \[2\] For Corollary \[2\] notice that \((a, b)\) is the unique Nash Equilibrium of the stage game, one can obtain a constructive proof to Corollary \[2\] by replacing \((a'', b'')\) with \((a, b)\).

First, consider the case in which \(u_1(a'', b'') = u_1(a^*, b^*)\). Assumption \[1\] implies that \((a'', b'') = (a^*, b^*)\), and the discounted average welfare of player 2 equals \(u_2(a'', b'')\) in a pooling equilibrium in which \((a^*, b^*)\) is played at every on-path history.

Next, consider the nontrivial case in which \(u_1(a'', b'') < u_1(a^*, b^*)\). Consider a similar equilibrium as the proof of Theorem \[1\] except for two differences: first, replace \((a', b')\) with \((a'', b'')\), and second, calibrate the probability with which strategic-type player 1 playing \(a^*\) in the reputation-building phase such that the unconditional probability of \(a^*\) equals \(q^*\) at every reputation-building phase history. This is feasible given that player 1 knows player 2’s belief about his type at every on-path history. Let \(V_2\) be player 2’s discounted average welfare in the reputation-building phase (with discount factor \(\delta_s\)):

\[
V_2 = (1 - \delta_s) \left\{ q^* u_2(a^*, b^*) + (1 - q^*) u_2(a'', b'') \right\} + \delta_s \left\{ (1 - q^*) V_2 + q^* (1 - r) V_2 + q^* ru_2(a^*, b^*) \right\}, \tag{A.1}
\]

where

\[
r = \frac{1 - \delta}{\delta} \frac{u_1(a'', b'') - u_1(a^*, b^*)}{u_1(a^*, b^*) - u_1(a'', b'')}, \tag{A.2}
\]

is the transition probability between phases that makes strategic-type player 1 indifferent between playing \(a^*\) and \(a''\) in the reputation-building phase. Equation \(A.1\) yields:

\[
V_2 \left\{ 1 - \delta_s (1 - q^*) - \delta_s q^* (1 - r) \right\} = \delta_s q^* ru_2(a^*, b^*) + (1 - \delta_s) \left\{ q^* u_2(a^*, b^*) + (1 - q^*) u_2(a'', b'') \right\}. \tag{A.3}
\]

Since \(q^*\) can be arbitrarily low, when \(q^*\) converges to 0, \(A.3\) reduces to \(V_2 (1 - \delta_s) = u_2(a'', b'')(1 - \delta_s)\), which implies that \(V_2\) is arbitrarily close to \(u_2(a'', b'')\) as \(q^*\) and \(\pi_0\) become arbitrarily small.

A.2 Proof of Proposition 3

Lower Bound: I establish inequality \[3.5\]. For every \(\beta \in \Delta(B)\) and \(a' < a^*\), players’ payoffs being monotone-supermodular implies that \(u_1(a^*, \beta) < u_1(a', \beta)\). Strategic-type player 1 has a strict incentive to play \(a'\) instead of \(a^*\) at public history \(h^t\) if

\[
||y_t(\cdot | a^*, h^t) - y_t(\cdot | a', h^t)|| \leq \frac{1 - \delta}{2\delta(\pi_1 - u_1)} \left( u_1(a', \beta) - u_1(a^*, \beta) \right), \tag{A.4}
\]
where $y_t(\cdot|a, h^t) \in \Delta(B^K)$ is the distribution over $(b_{t+1}, \ldots, b_{t+K})$ conditional on player 1 playing $a$ at $h^t$, $\bar{u}_1$ and $u_1$ are player 1’s highest and lowest feasible stage-game payoffs, and $|| \cdot ||$ is the total variation norm. Let

$$\Delta \equiv \frac{1 - \delta}{2\delta((\bar{u}_1 - u_1))} \min_{\beta \in \Delta(B), a' < a} \left\{ u_1(a', \beta) - u_1(a^*, \beta) \right\}. \quad (A.5)$$

Let $H^{(a^*, \sigma_2)}$ be the set of public histories that occur with positive probability when player 1 plays $a^*$ in every period and player 2 plays $\sigma_2$. I partition $H^{(a^*, \sigma_2)}$ into two subsets, $H^{(a^*, \sigma_2)}_0$ and $H^{(a^*, \sigma_2)}_1$:

1. if there exists $a' < a^*$ such that $||y_t(\cdot|a^*, h^t) - y_t(\cdot|a', h^t)|| \leq \Delta$, then $h^t \in H^{(a^*, \sigma_2)}_0$;
2. if $||y_t(\cdot|a^*, h^t) - y_t(\cdot|a', h^t)|| \geq \Delta$ for every $a' < a^*$, then $h^t \in H^{(a^*, \sigma_2)}_1$.

For every $h^t \in H^{(a^*, \sigma_2)}_0$, strategic-type player 1 has a strict incentive not to play $a^*$ at $h^t$, and according to Proposition 2, $\sigma_2(h^{t+\tau}) = b^*$ for every $\tau \in \{1, 2, \ldots, K\}$ and $h^{t+\tau} \succ h^t$ with $a^*$ being played from period $t$ to $t + \tau - 1$. This implies that for every $h^t \in H^{(a^*, \sigma_2)}_0$, we have:

$$\frac{1}{t} \mathbb{E}^{(a^*, \sigma_2)} \left[ \sum_{s=t}^{t+K} u_1(a_s, b_s) \big| h^t \right] \geq \frac{K}{K+1} u_1(a^*, b^*) + \frac{1}{K+1} \min_{(a,b) \in (A,B)} u_1(a,b). \quad (A.6)$$

For every $h^t \in H^{(a^*, \sigma_2)}_1$, there exists a constant $\gamma > 0$ such that for every $a \in \Delta(A)$ such that $b^* \succ b^*$ best replies against $a$, we have $||y_t(\cdot|a^*, h^t) - y_t(\cdot|a, h^t)|| \geq \gamma \Delta$. The Pinsker’s Inequality implies that

$$d \left( y_t(\cdot|a, h^t) \bigg| y_t(\cdot|a^*, h^t) \right) \geq 2\gamma^2 \Delta^2. \quad (A.7)$$

for every such $a \in \Delta(A)$. For every Bayes Nash Equilibrium $(\sigma_1, \sigma_2)$ and every $\tau \in \{0, 1, \ldots, K\}$,

$$\mathbb{E}^{(a^*, \sigma_2)} \left[ \sum_{s=0}^{\infty} d \left( y_{s(K+1)+\tau}(\cdot|\sigma_1(h^{s(K+1)+\tau}), h^{s(K+1)+\tau}) \bigg| y_{s(K+1)+\tau}(\cdot|a^*, h^{s(K+1)+\tau}) \right) \right] \leq -\log \pi_0. \quad (A.8)$$

Inequalities (A.7) and (A.8) together imply that:

$$\mathbb{E}^{(a^*, \sigma_2)} \left[ \sum_{s=0}^{\infty} \mathbf{1} \left( h^{s(K+1)+\tau} \in H^{(a^*, \sigma_2)}_1 \text{ and } \sigma_2(h^{s(K+1)+\tau}) \prec b^* \right) \right] \leq -\frac{\log \pi_0}{2\gamma^2 \Delta^2}. \quad (A.9)$$

I derive a lower bound for $\liminf_{t \to \infty} \frac{1}{t} \mathbb{E}^{(a^*, \sigma_2)} \left[ \sum_{s=0}^{t-1} u_1(a_s, b_s) \right]$ using (A.6) and (A.9). For every $\tau \in \{0, 1, \ldots, K\}$, let

$$H^\tau_0 \equiv \left\{ h^t \exists h^{s(K+1)+\tau} \in H^{(a^*, \sigma_2)}_0 \text{ such that } h^t \succeq h^{s(K+1)+\tau} \text{ and } t \in [s(K+1), s(K+1)+K] \right\},$$

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let
\[ \mathcal{H}_1^\tau \equiv \left\{ h^{s(K+1)+\tau} \in \mathcal{H}_1^{(a^*,\sigma_2)} \mid s \in \mathbb{N} \right\}, \]
and let \( \mathcal{H}^\tau \equiv \mathcal{H}_0^\tau \cup \mathcal{H}_1^\tau \). By definition, \( \mathcal{H}^{(a^*,\sigma_2)} = \bigcup_{\tau=0}^K \mathcal{H}^\tau \). An important observation is that for every \( \tau, \tau' \in \{0, 1, \ldots, K\} \) with \( \tau \neq \tau' \),
\[ \mathcal{H}_1^\tau \cap \mathcal{H}_1^{\tau'} = \{ \emptyset \} \text{ and } \mathcal{H}_0^\tau \cap \mathcal{H}_0^{\tau'} = \{ \emptyset \}. \tag{A.10} \]
The former is straightforward. For the latter, suppose toward a contradiction that \( h^\tau \in \mathcal{H}_0^\tau \cap \mathcal{H}_0^{\tau'} \) with \( \tau < \tau' \), there exist \( h^s \) and \( h^{s+\tau'-\tau} \) such that \( h^\tau \succ h^{s+\tau'-\tau} \succ h^s \), \( h^s \in \mathcal{H}_0^\tau \), \( t-s \leq K \), and \( s-\tau \) is divisible by \( K+1 \). On one hand \( h^s \in \mathcal{H}_0^\tau \) and \( \tau'-\tau \leq K \) implies that \( \sigma_1(h^{s+\tau'-\tau}) = a^* \). On the other hand \( h^{s+1} \in \mathcal{H}_0^{\tau'} \) implies that \( \sigma_1(h^{s+\tau'-\tau}) \neq a^* \). This leads to a contradiction.

For every \( \tau \in \{0, 1, \ldots, K\} \), inequality (A.6) implies that player 1’s expected average payoff at histories in \( \mathcal{H}_0^\tau \) is at least \( K \frac{1}{K+1} u_1(a^*, b^*) + \frac{1}{K+1} \min_{(a,b) \in (A,B)} u_1(a,b) \). Since \( \mathcal{H}_0^\tau \cap \mathcal{H}_0^{\tau'} = \{ \emptyset \} \) for every \( \tau \neq \tau' \), it implies that player 1’s expected average payoff at histories in \( \bigcup_{\tau=0}^K \mathcal{H}_0^\tau \) is at least \( K \frac{1}{K+1} u_1(a^*, b^*) + \frac{1}{K+1} \min_{(a,b) \in (A,B)} u_1(a,b) \). For every \( \tau \in \{0, 1, \ldots, K\} \), (A.9) implies that player 1’s expected average payoff at histories in
\[ \mathcal{H}_1^\tau \setminus \bigcup_{s=0}^K \mathcal{H}^s_0 \]
is at least \( u_1(a^*, b^*) \). Since \( \mathcal{H}_1^\tau \cap \mathcal{H}_1^{\tau'} = \{ \emptyset \} \) for every \( \tau \neq \tau' \), it implies that player 1’s expected average payoff at histories in
\[ \bigcup_{s=0}^K \mathcal{H}_1^s \setminus \bigcup_{s=0}^K \mathcal{H}^s_0 \]
is at least \( u_1(a^*, b^*) \). The two parts together imply that
\[ \liminf_{t \to \infty} \frac{1}{t} \mathbb{E}^{(a^*,\sigma_2)} \left[ \sum_{s=0}^{t-1} u_1(a_s, b_s) \right] \geq \frac{K}{K+1} u_1(a^*, b^*) + \frac{1}{K+1} \min_{b \in B} u_1(a^*, b) . \]

**Tightness:** I construct a sequential equilibrium in which player 1’s asymptotic payoff equals the RHS of (3.6). At every on-path history (the set of which can be derived recursively using players’ strategies at on-path histories specified below):

- if \( t \) is divisible by \( K+1 \), then player 1 plays \( a' \) and player 2 plays \( b' \) in period \( t \);
- if \( t \) is not divisible by \( K+1 \), then player 1 plays \( a^* \) and player 2 plays \( b^* \) in period \( t \).

I partition off-path histories into three subsets. For every period \( t \) public history such that:
• (1) there exists no \( r < t \), such that \( b_r \neq b^* \) and \( r \) is not divisible by \( K+1 \); (2) there exists no \( s < t \) such that \( b_s \neq b' \) and \( s \) is divisible by \( K+1 \); (3) player 2 observes player 1’s off-path deviation in period \( t-1 \), then players play \((a^*,b^*)\) if \( t \) is divisible by \( K+1 \), and play \((a',b')\) if \( t \) is not divisible by \( K+1 \).

• (1) there exists no \( r < t \), such that \( b_r \neq b^* \) and \( r \) is not divisible by \( K+1 \), but (2) there exists \( s < t \) such that \( b_s \neq b' \) and \( s \) is divisible by \( K+1 \). Let \( s^* \) be the smallest of such \( s \). If \( t-1 \) is not divisible by \( K+1 \), then players play \((a^*,b^*)\) in period \( t \). If \( t-1 \) is divisible by \( K+1 \), then players play \((a^*,b^*)\) in period \( t \) if and only if \( \xi_t > 1/2 \) and play \((a',b')\) in period \( t \) otherwise.

• there exists \( r < t \), such that \( b_r \neq b^* \) and \( r \) is not divisible by \( K+1 \), then players play \((a',b')\).

Such an arrangement is incentive compatible when \( \delta \) is above some cutoff.

A.3 Sufficient Conditions for Attaining Minmax Payoff

Focusing on games in which \( v_1^{min} < v_1 \), I identify a sufficient condition under which player 1’s lowest equilibrium payoff in the reputation game coincides with his minmax payoff.

Condition 4. There exists a minmax action \( \beta \in \mathcal{B}^* \) such that:

1. \( b^* \notin \text{supp}(\beta) \),

2. there exists \( \alpha \in \Delta(A) \) with \( a^* \in \text{supp}(\alpha) \) such that \( \beta \) best replies against \( \alpha \),

3. \( u_1(\alpha,\beta) \geq u_1(a^*,\beta) \).

Condition 3 requires that first, there exists a minmax action that excludes the Stackelberg best reply \( b^* \) in its support. Second, \( \beta \) best replies against a (potentially mixed) action of player 1’s that includes the Stackelberg action in its support. Third, playing the Stackelberg action against \( \beta \) yields player 1 a weakly lower payoff compared to playing \( \alpha \). The first and third part of this condition is satisfied when the Stackelberg action is costly for player 1 regardless of player 2’s action, and player 2s’ Stackelberg best reply is always beneficial to player 1. The second part of condition 3 is satisfied for generic \((u_1,u_2)\), since it only requires \( \beta \) to be a strict best reply against some \( \alpha \in \Delta(A) \).

I provide an example in which \( v_1^{min} < v_1 \) and players’ stage-game payoffs satisfy Condition 3 and Assumptions 1 and 2:
Action $R$ belongs to $B^*$ since $R$ is a strict best reply against $M$, and therefore, best replies against any mixed action in which the probability of $M$ is close to 1. Player 1’s worst stage-game Nash equilibrium payoff is $1/2$, his pure Stackelberg payoff is 1, and his minmax payoff is 0. When player 2 plays her minmax action $R$, player 1’s payoff from playing $M$ is strictly greater than his payoff from playing $U$.

The following theorem extends Theorem 1, with proof in Appendix A.3:

**Theorem 1’.** When players’ stage-game payoffs satisfy Assumptions 1 and 2 and Condition 3. For every $K \in \mathbb{N}$, there exists $\pi_0 \in (0, 1)$, such that for every $\pi_0 \in (0, \pi_0)$ and $\delta \geq \hat{\delta}$, there exists $(\sigma_1^0, \sigma_2^0) \in \text{SE}(\delta, \pi_0, K)$, such that:

$$E_1(\sigma_1^0, \sigma_2^0) \left[ \sum_{t=0}^{\infty} (1-\delta)^{\delta} u_1(a_t, b_t) \right] = v_{1, \text{min}}.$$  

**Proof.** When $u_1(a^*, b^*) = v_{1, \text{min}}$, $(a^*, b^*)$ is the unique pure strategy Nash Equilibrium of the stage game. An equilibrium that attains payoff $v_{1, \text{min}}$ is that both players play $(a^*, b^*)$ at every history.

Next, I focus on the interesting case in which $u_1(a^*, b^*) > v_{1, \text{min}}$. Recall the definitions of $\alpha$ and $\beta$ in Condition 3. Let $q^*$ be the probability that $\alpha$ attaches to $a^*$. Let $\pi_0 \in (0, 1)$ be small enough such that:

$$\frac{\pi_0}{1-\pi_0} \leq \left( \frac{q^*}{2-q^*} \right)^{K+1}. \tag{A.11}$$

For every $\pi_0 \in (0, \pi_0)$ and $\delta$ large enough, I construct the following three-phase equilibrium in which strategic-type player 1 attains payoff $v_{1, \text{min}}$. The current phase of play depends only on the history of player 2’s actions, which are commonly observed by both players. In period $t$,

1. play is in the reputation building phase if there exists no $s < t$ such that $b_s = b^*$;
2. play is in the reputation maintenance phase if (1) there exists $s < t$ such that $b_s = b^*$, and (2) there exists no $\tau \in \{s^* + 1, \ldots, t - 1\}$ such that $b_\tau \neq b^*$, where $s^*$ is the smallest $s$ with $b_s = b^*$;
3. play is in the punishment phase if (1) there exists $s < t$ such that $b_s = b^*$, and (2) there exists $\tau \in \{s^* + 1, \ldots, t - 1\}$ such that $b_\tau \neq b^*$, where $s^*$ is the smallest $s$ such that $b_s = b^*$.

Play starts from the reputation building phase, and gradually reaches the reputation maintenance phase. Play reaches the punishment phase only at off-path histories.
Equilibrium Strategies: At every history $h^t$ of the reputation-building phase,

- If $t = 0$, then player 2 plays $\beta$ and strategic type player 1 plays $\alpha_0$ that satisfies:
  \[(1 - \pi_0)\alpha_0 + \pi_0 a^* = \alpha, \tag{A.12}\]
  Such $\alpha_0$ exists given that $\pi_0 < q^*/2$, which have been assumed in (A.11).

- If $t \geq 1$ and $\xi_t > r(a_{t-1})$, then player 2 plays $\beta$ and strategic type player 1 plays $\alpha(h^t)$ that satisfies:
  \[(1 - \pi(h^t))\alpha(h^t) + \pi(h^t)a^* = \alpha, \tag{A.13}\]
  where $\pi(h^t)$ is the probability player 2’s belief at $h^t$ attaches to the commitment type. Such $\alpha(h^t)$ exists given that $\pi(h^t) < q^*/2$, which I will verify by induction by the end of this proof.

- If $t \geq 1$ and $\xi_t \leq r(a_{t-1})$, then player 2 plays $b^*$ and strategic type player 1 plays $a^*$.

The transition probability to the reputation maintenance phase is a function of player 1’s action in the previous period $r : A \rightarrow [0,1]$, which is pinned down by the following equation:

\[(1 - \delta)u_1(a, \beta) + \delta r(a)u_1(a^*, b^*) + \delta (1 - r(a)) \max_{a \in A} u_1(a, \beta) = \max_{a \in A} u_1(a, \beta). \tag{A.14}\]

Given that $u_1(a^*, b^*) > \underline{\pi}_1^{\min}$, one can verify that (1) for every $a, a' \in A$, $r(a) \geq r(a')$ if and only if $u_1(a, \beta) \leq u_1(a', \beta)$, and (2) for every $a \in A$, $r(a)$ is strictly between 0 and 1 when $\delta$ is large enough, and as $\delta \rightarrow 1$, the value of $r(a)$ converges to 0.

At every history of the reputation maintenance phase, If $a_{t-1} = a^*$, then player 1 plays $a^*$ and player 2 plays $b^*$. If $a_{t-1} \neq a^*$, then player 2 plays $\beta$ and strategic-type player 1 plays $\alpha$, after which the continuation play enters the punishment phase. Player 1’s continuation value in the last period of the reputation maintenance phase equals $\underline{\pi}_1^{\min}$.

Incentive Constraints: I verify players’ incentives constraints. To start with, when $\delta$ is large enough,

\[u_1(a^*, b^*) \geq (1 - \delta) \max_{a \in A} u_1(a, b^*) + \delta \underline{\pi}_1^{\min}.\]

This implies that player 1 has an incentive to play $a^*$ in the reputation maintenance phase when $\delta$ is large enough. Next, given that player 1’s continuation value is $u_1(a^*, b^*)$ in the reputation maintenance
phase and is \( \omega_1 \) in the reputation building phase, (A.14) implies that player 1 is indifferent between all of his actions in the reputation building phase.

Next, I verify player 2’s incentive to play \( \beta \) at the reputation building phase by showing that (1) player 1’s reputation at every history of the reputation building phase is less than \( q^* / 2 \), and (2) \( \alpha(h^t) \) defined in (A.13) attaches probability at least \( q^* / 2 \) to action \( a^* \). According to (A.11), player 1’s reputation in period 0 is less than \( q^* / 2 \) and according to (A.12), \( \alpha(h^0) \) attaches probability more than \( q^* / 2 \) to action \( a^* \).

Suppose the conclusion holds for all reputation-building phase histories \( h^s \) with \( s < t \). If \( h^t \) belongs to the reputation building phase, then \( b^* \) has never been played before given that player 2 plays \( \beta \) in every period of the past given that \( b^* \notin \text{supp}(\beta) \). Player 2’s posterior belief about player 1 being committed is 0 unless \( a^* \) was played in the last \( \min\{K, t\} \) periods, in which case her posterior belief \( \pi_t \) satisfies the following equation:

\[
\frac{\pi_t}{1-\pi_t}/\frac{\pi_0}{1-\pi_0} = \frac{\text{Pr}(a^*, a^*_2)(a^*, ..., a^*)}{\text{Pr}(a^*, a^*_2)(a^*, ..., a^*)} \cdot \frac{\text{Pr}(\alpha^*, \omega_2)(b_0, ..., b_{t-1}, \xi_t|a^*, ..., a^*)}{\text{Pr}(\sigma_1^t, \sigma_2^t)(b_0, ..., b_{t-1}, \xi_t|a^*, ..., a^*)},
\]

where \( \text{Pr}(\sigma_1^t, \sigma_2^t)(\cdot) \) is the probability measure induced by strategy profile \( (\sigma_1^t, \sigma_2^t) \), and \( \text{Pr}(a^*, \omega_2)(\cdot) \) is the probability measure when player 1 plays \( a^* \) in every period and player 2s’ strategy is \( \omega_2 \).

Since the strategic type plays \( a^* \) with probability \( q^* / 2 \) in the reputation building phase, we have:

\[
\frac{\text{Pr}(\sigma_1^t, \sigma_2^t)(a^*, ..., a^*|\omega_2)}{\text{Pr}(\sigma_1^t, \sigma_2^t)(a^*, ..., a^*|\omega_2)} \leq \left( \frac{q^*}{2 - q^*} \right)^{\min\{t, K\}} \leq \left( \frac{q^*}{2 - q^*} \right)^K.
\]

Since \( u_1(\alpha, \beta) \geq u_1(a^*, \beta) \), we have:

\[
r(a^*) \geq \mathbb{E}[r(\bar{a})|\alpha].
\]

This implies that for every \( (b_0, ..., b_{t-1}) \) with \( b_s \in \text{supp}(\beta) \) for every \( s \), we have:

\[
\frac{\text{Pr}(\sigma_1^t, \sigma_2^t)(b_0, ..., b_{t-1}, \xi_t|a^*, ..., a^*, \omega_2)}{\text{Pr}(\sigma_1^t, \sigma_2^t)(b_0, ..., b_{t-1}, \xi_t|a^*, ..., a^*, \omega_2)} \leq 1.
\]

Since \( \frac{\pi_0}{1-\pi_0} \leq \frac{\pi_0}{1-\pi_0} = \left( \frac{q^*}{2 - q^*} \right)^{K+1} \), (A.15), (A.16) and (A.17) together imply that \( \pi_t \leq \frac{q^*}{2} \). As a result, the probability with which player 2 believes player 1 playing \( a^* \) is at most \( \frac{q^*}{2} + (1 - \frac{q^*}{2})^q \leq q^* \). This completes the verification of player 2s’ incentives.

\( \square \)
Proof of Theorem 3

In Appendix B.1, I relax the monotone-supermodularity condition on payoffs, and identify weaker sufficient conditions for my result. My sufficient conditions are satisfied not only in games with monotone-supermodular payoffs, but also in coordination games, common interest games, and many other games studied in the reputation literature. I state a Theorem 3' that generalizes Theorem 3 to a larger class of payoff structures. The proof of Theorem 3' is in Appendices B.2 and B.3.

B.1 Relax Monotone-Supermodularity

Recall that \( a^* \) is player 1’s pure Stackelberg action and \( b^* \) is player 2’s unique best reply against \( a^* \). Let \( a'' \) be the unique element in \( \text{BR}_1(b^*) \). If \( a'' \neq a^* \), then \( b^* \notin \text{BR}_2(a'') \). This is because \( u_1(a'', b^*) > u_1(a^*, b^*) \), and \( b^* \) best replying against \( a'' \) implies that committing to \( a'' \) yields player 1 a strictly higher payoff, contradicting the definition of \( a^* \). Let \( p^* \) be the largest \( p \in [0, 1] \) such that:

\[
\{b^*\} \neq \text{BR}_2(pa^*_1 + (1-p)a'').
\]

This suggests the existence of \( b'' \neq b^* \) such that \( b'' \in \text{BR}_2(p^*a^*_1 + (1-p^*)a'') \). My first requirement is:

\[
u_1(a'', b'') \geq u_1(a^*, b''). \tag{B.1}\]

Recall that \((a', b')\) is player 1’s worst stage-game Nash equilibrium, which is strict under Assumption 1. When \( a' \neq a^* \), \( b' \notin \text{BR}_2(a^*) \). This is because otherwise, \( b' \in \text{BR}_2(a^*) \), and given that \( b^* \in \text{BR}_2(a^*) \), Assumption 1 implies that \( b^* = b' \). As a result, \((a', b^*)\) is a stage-game Nash Equilibrium, and Assumption 1 implies that \( u_1(a', b^*) > u_1(a^*, b^*) \), i.e., player 1 obtains strictly higher payoff by committing to \( a' \) compared to committing to \( a^* \). This contradicts the presumption that \( a^* \) is player 1’s pure Stackelberg action. Let \( q^* \) be the smallest \( q \in [0, 1] \) such that:

\[
\{b'\} \neq \text{BR}_2(qa^*_1 + (1-q)a').
\]

This suggests the existence of \( b^{**} \neq b' \) such that \( b^{**} \in \text{BR}_2(q^*a^*_1 + (1-q^*)a') \). My second and third requirements are:

\[
u_1(a', b^{**}) \geq u_1(a^*, b^{**}), \tag{B.2}\]

and

\[
u_1(a'', b^*) - u_1(a^*, b^*) \geq u_1(a', b^{**}) - u_1(a^*, b^{**}). \tag{B.3}\]
I introduce two classes of games, starting from games with strict lack-of-commitment.

**Definition 1.** \((u_1,u_2)\) is a game with strict lack-of-commitment if \((a^*,b^*)\) is not a Nash Equilibrium, and players’ payoffs satisfy \((B.1),(B.2),(B.3)\).

Under Assumption 1, the requirement that \((a^*,b^*)\) is not a Nash Equilibrium implies that \(a^* \neq a'\) and \(a^* \neq a''\), i.e., the Stackelberg action is *strictly suboptimal* for player 1 both when player 2 plays her Stackelberg best reply \(b^*\) and when she plays her Nash equilibrium action \(b'\). Actions \(a'\) and \(a''\) are player 1’s best replies to these player 2’s actions, where \(a'\) and \(a''\) can potentially coincide. Inequality \((B.1)\) requires that player 1 benefits from deviating to \(a''\) not only when player 2 plays \(b^*\), but also when player 2 plays \(b''\), her best reply when she faces uncertainty about whether player 1 will play \(a^*\) or \(a''\). Inequality \((B.2)\) requires that player 1 benefits from deviating to \(a'\) not only when player 2 plays \(b'\), but also when player 2 plays \(b^{**}\), her best reply when she faces uncertainty about whether player 1 will play \(a^*\) or \(a'\). Inequality \((B.3)\) requires that player 1’s benefit from cheating is larger when player 2 plays her Stackelberg best reply. Lemma \(B.1\) shows that games with strict lack-of-commitment contains games with monotone-supermodular payoffs.

**Lemma B.1.** When a game’s stage-game payoffs are monotone-supermodular, then it is a game with strict lack-of-commitment.

*Proof of Lemma B.1:* Since \(u_1(a,b)\) is strictly decreasing in \(a\), we have \(a' = a'' = a\). Since \(a^* \neq a\), we have \(a^* \neq a'\) and \(a^* \neq a''\). Since \(a^* \succ a'\) and \(a^* \succ a''\), we obtain \((B.1)\) and \((B.2)\). By construction, \(p^* \geq q^*\). Since \(u_2(a,b)\) has strictly increasing differences in \(a\) and \(b\), we have \(b^* \succeq b^{**}\). Given that \(a^* \succ a = a' = a''\) and \(u_1(a,b)\) has strictly decreasing differences in \(a\) and \(b\), which yields \((B.3)\).

Next, I define generalized coordination games.

**Definition 2.** \((u_1,u_2)\) is a general coordination game if \((a^*,b^*)\) is a Nash Equilibrium.

When player 1’s Stackelberg outcome is a Nash Equilibrium, either \(a^* = a'\) or \(a^* = a''\) or both. This can be further categorized into two subclasses: trivial games in which the pure Stackelberg outcome coincides with the worst pure strategy Nash equilibrium (for example, prisoner’s dilemma), games that have at least two pure-strategy Nash Equilibria. When \((a^*,b^*)\) is a Nash Equilibrium, it must be player 1’s favorite Nash Equilibrium, while \((a',b')\) is player 1’s least favorite Nash Equilibrium. This includes for example, battle of sexes, chicken games, and common interest coordination games in which different equilibria can be Pareto ranked. Theorem 3’ generalizes Theorem 3 to games with strict lack-of-commitment and generalized coordination games.
Theorem 3’. If the monitoring structure \( N \) satisfies Assumption 3 and the stage game satisfies Assumption 1, and is either a game with strict lack-of-commitment or a generalized coordination game, then there exists \( \pi_0 \in (0,1) \), such that for every \( \pi_0 \in (0,\pi_0) \) and \( \delta \) large enough, there exists \( (\sigma^1_1,\sigma^2_2) \in SE(\delta,\pi_0,N) \), such that:

\[
E_1^{(\sigma^1_1,\sigma^2_2)} \left[ \sum_{t=0}^{\infty} (1-\delta)^{t} u_1(a_t, b_t) \right] = v_1.
\]

B.2 Proof of Theorem 3’: Games with Strict Lack-of-Commitment

The constructed equilibrium consists of three phases: a reputation building phase, a reputation maintenance phase, and a punishment phase, which depends only on the history of player 2’s actions that is commonly observed by both players. In period \( t \),

- play is in the reputation building phase if \( t = 0 \) or \( (b_0,\ldots,b_{t-1}) = (b',\ldots,b') \);
- play is in the reputation maintenance phase if first, there exists \( s \leq t-1 \) such that \( b_{s} = b^{**} \), and second, \( (b_{s+1},\ldots,b_{t-1}) = (b^*,\ldots,b^*) \) where \( s^* \) is the smallest \( s \in \mathbb{N} \) such that \( b_{s} = b^{**} \).
- play is in the punishment phase if first, there exists \( s \leq t-1 \) such that \( b_{s} = b^{**} \), and second, \( (b_{s+1},\ldots,b_{t-1}) \neq (b^*,\ldots,b^*) \) where \( s^* \) is the smallest \( s \in \mathbb{N} \) such that \( b_{s} = b^{**} \).

Play starts from the reputation building phase, and eventually ends up in the reputation maintenance phase or the punishment phase. Different from the construction in Theorem 1, the punishment phase is reached with strictly positive probability due to private monitoring and private learning. In what follows, I describe players’ strategies and verify their incentive constraints in each of the three phases. By the end of this proof, I verify the promise keeping condition for player 1.

**Punishment Phase:** At every punishment phase history, player 1 plays \( a' \) and player 2 plays \( b' \). As will become clear after describing the reputation maintenance phase, play never reaches the punishment phase conditional on player 1 being committed. This implies the existence of an assessment that is consistent with the equilibrium strategy profile and attaches probability 1 to the strategic type at every punishment-phase history. This verifies players’ incentive constraints in the punishment phase.

**Reputation Maintenance Phase:** Let \( s^* \) be the smallest \( s \) such that \( b_{s} = b^{**} \). In period \( t \geq s^*+2 \),

- If \( a_{t-1} = a^* \), then strategic-type player 1 plays \( a^* \).

  If \( a_{t-1} \neq a^* \), then strategic-type player 1 plays \( p^*a_1 + (1-p^*)a'' \).
• If \( t - 1 \notin N_t \) or \( a_{t-1} = a^* \), then player 2 plays \( b^* \).

If \( t - 1 \in N_t \) and \( a_{t-1} \neq a^* \), then player 2 plays \( \tilde{\beta}b^* + (1 - \tilde{\beta})b'' \). Let \( \beta \) be the probability with which player 2 plays \( b^* \) conditional on \( a_{t-1} \neq a^* \) but unconditional on realization of \( N_t \). It is also the unconditional probability that play remains in the reputation maintenance phase in period \( t + 1 \) given that \( a_{t-1} \neq a^* \). One can compute \( \beta \) and player 1’s continuation value in period \( t \) when \( a_{t-1} \neq a^* \), denoted by \( V_1 \), by solving the following system of quadratic equations:

\[
V_1 = (1 - \delta)u_1(a^*, \beta b^* + (1 - \beta)b'') + \delta u_1(a^*, b^*) + \delta(1 - \beta_t)u_1(a', b'),
\]

\[
V_1 = (1 - \delta)u_1(a'', \beta b^* + (1 - \beta)b'') + \delta \beta V_1 + \delta(1 - \beta_t)u_1(a', b').
\]

To understand (B.4) and (B.5), note that \( u_1(a^*, b^*) \) is player 1’s continuation value in period \( t + 1 \) when \( a_t = a^* \) and play remains in the reputation maintenance phase, and \( u_1(a', b') \) is player 1’s continuation value in period \( t + 1 \) when play reaches the punishment phase. Player 2’s mixing probability conditional on \( t - 1 \in N_t \) and \( a_{t-1} \neq a^* \) satisfies:

\[
1 - \beta = (1 - \tilde{\beta}_t) \Pr(t - 1 \in N_t).
\]

Since \( \Pr(t - 1 \in N_t) \) is uniformly bounded from below by \( \gamma > 0 \), \( \tilde{\beta}_t \in (0, 1) \) when \( \beta > 1 - \gamma \). Lemma [B.2] verifies that \( \beta \) converges to 1 as \( \delta \to 1 \), i.e., \( \beta > 1 - \gamma \) when \( \delta \) is large enough.

In period \( s^* + 1 \),

• If \( a_{s^*} = a^* \), then strategic-type player 1 plays \( a^* \).

If \( a_{s^*} \neq a^* \) and

\[
\xi_{s^*} > \xi \equiv \frac{u_1(a', b'^*) - u_1(a^*, b'^*)}{u_1(a'', b'') - u_1(a', b'') + \frac{1 - \beta}{\beta} (u_1(a'', b'') - u_1(a^*, b''))},
\]

then strategic-type player 1 plays \( a^* \).

If \( a_{s^*} \neq a^* \) and \( \xi_{s^*} \leq \xi \), then strategic-type player 1 plays \( p^*a_1^* + (1 - p^*)a'' \).

• If \( s^* \notin N_{s^* + 1} \), or \( a_{s^*} = a^* \), or \( \xi_{s^*} \) satisfies (B.7), then player 2 plays \( b^* \).

If \( s^* \in N_{s^* + 1} \), \( a_{s^*} \neq a^* \), and \( \xi_{s^*} \) does not satisfy (B.7), then player 2 plays \( \tilde{\beta}b^* + (1 - \tilde{\beta})b'' \), where \( \tilde{\beta} \) can be solved via (B.4), (B.5) and (B.6).
To verify players’ incentive constraints in this phase, I only need to verify their incentives from period $s^* + 2$ and onwards. This is because the equilibrium play in period $s^* + 1$ is a randomization between the two automaton states from period $s^* + 2$ and onwards.

I start from verifying player 2’s incentives. According to the definition of $p^*$ in Appendix B.1 player 2 is indifferent between $b^*$ and $b''$ when her belief about player 1’s action is $p^*a_1^* + (1 - p^*)a''$. In the constructed strategy for player 2s in the reputation maintenance phase,

- Player 2 randomizes between $b^*$ and $b'$ in period $t$ only when she has observed $a_{t-1} \neq a^*$, after which her posterior belief attaches probability 0 to the commitment type, and therefore, believes that player 1’s action is $p^*a_1^* + (1 - p^*)a''$.

- Player 2 plays $b^*$ at other histories. This is incentive compatible since $b^*$ best replies against $pa_1^* + (1 - p^*)a''$ for every $p \in [p^*, 1]$, and given that the strategic type plays $p^*a_1^* + (1 - p^*)a''$, the unconditional probability with which player 1 plays $a^*$ is between $p$ and 1.

Next, I verify player 1’s incentives. I show that first, when $a_{t-1} = a^*$, player 1 has an incentive to play $a^*$. This requires:

$$u_1(a^*, b^*) \geq \max_{a \neq a^*} \left\{ (1 - \delta)u_1(a, b^*) + \delta V_1 \right\},$$

where $V_1$ solves (B.4) and (B.5). Intuitively, $V_1$ is player 1’s continuation value in the reputation maintenance phase conditional on his previous period action is not $a^*$. Given that $a''$ is player 1’s stage-game best reply against $b^*$, the above inequality reduces to:

$$u_1(a^*, b^*) - V_1 \geq \frac{1 - \delta}{\delta} (u_1(a'', b^*) - u_1(a^*, b^*)) \quad (B.8)$$

Deduct the RHS of (B.4) from the RHS of (B.5), we obtain:

$$\delta (u_1(a^*, b^*) - V_1) = (1 - \delta) \beta (u_1(a'', b^*) - u_1(a^*, b^*)) + (1 - \delta) (1 - \beta) (u_1(a'', b'') - u_1(a^*, b'')) \quad (B.9)$$

Equation (B.9) implies that $\beta > 0$. This is because otherwise, $u_1(a'', b'') - u_1(a^*, b'') = 0$, and given the presumption that $a^* \neq a''$, this violates Assumption 1. According to (B.1), $u_1(a'', b'') - u_1(a^*, b'') \geq 0$, and therefore, (B.9) implies (B.8).

Next, I show that when $a_{t-1} \neq a^*$, player 1 has an incentive to mix between $a^*$ and $a''$ in period $t$. Since $\mathcal{N}_t$ is independent of $\{\mathcal{N}_s\}_{s=0}^{t-1}$, and $\beta$ is the probability with which player 2 plays $b^*$ in period $t$ conditional on $a_{t-1} \neq a^*$ but unconditional on the realization of $\mathcal{N}_t$, player 1 believes that player 2 plays $b^*$ with probability $\beta$ conditional on $a_{t-1} \neq a^*$. Equations (B.4) and (B.5) imply that player 1 is
indifferent between $a^*$ and $a''$. What remains to be shown is that player 1 prefers $a''$ to actions other than $a^*$ and $a''$. This hinges on the following lemma, implying that $\beta$ is close to 1 when $\delta$ is close to 1.

**Lemma B.2.** For every $\gamma \in (0, 1)$, there exists $\delta \in (0, 1)$, such that for every $\delta > \delta$, there exists $\beta \in (1 - \gamma, 1)$ that solves (B.4) and (B.5).

The proof requires some algebra and is relegated to Appendix B.4. Since $a''$ best replies against $b^*$, Assumption 1 implies that $a''$ is a strict best reply against $b^*$. Lemma B.2 implies that there exists $\delta$ large enough such that $a''$ best replies against $\beta a^* + (1 - \beta) a''$. This suggests that player 1 receives higher payoff by playing $a''$ compared to actions other than $a^*$ and $a''$.

**Reputation Building Phase:** First, I describe player 2’s equilibrium strategy and verifies strategic-type player 1’s incentive to mix between $a^*$ and $a'$ at every private history of the reputation building phase. Second, I describe strategic-type player 1’s equilibrium strategy, and verifies player 2’s incentives to mix between $b^{**}$ and $b'$ at every private history of the reputation building phase.

**Player 2’s Strategy:** Player 2 plays $b'$ if (1) $t = 0$, or (2) $t - 1 \notin N_t$, or (3) $t - 1 \in N_t$ but $a_{t - 1} \neq a^*$. Player 2 plays $\tilde{\rho}_t b^{**} + (1 - \tilde{\rho}_t) b'$ if $t - 1 \in N_t$ and $a_{t - 1} = a^*$. Let $\rho$ be the probability with which player 2 plays $b^{**}$ conditional on $a_{t - 1} = a^*$ but unconditional on the realization of $N_t$. This probability $\rho$ is pinned down by the following equation:

$$V_1'(t) = (1 - \delta) u_1(a^*, \rho b^{**} + (1 - \rho) b') + \delta \rho u_1(a^*, b^*) + \delta (1 - \rho) V_1'. \tag{B.10}$$

where $V_1'$ is given by:

$$u_1(a', b') = (1 - \delta) u_1(a^*, b') + \delta V_1'. \tag{B.11}$$

Player 2's mixing probability conditional on $t - 1 \in N_t$ and $a_{t - 1} = a^*$, which is $\tilde{\rho}_t$, satisfies:

$$\rho = \tilde{\rho}_t \Pr(t - 1 \in N_t), \tag{B.12}$$

Since $\Pr(t - 1 \in N_t)$ is uniformly bounded from below by $\gamma$, $\tilde{\rho}_t \in (0, 1)$ as long as $\rho \in (0, \gamma)$. Lemma B.3 shows that $\rho \to 0$ as $\delta \to 1$. In what follows, I show that such mixing probabilities give strategic-type player 1 an incentive to mix between $a^*$ and $a'$ at every reputation-building phase history.

First, $u_1(a', b')$ is player 1’s continuation value in the reputation building phase when $t = 0$ or $a_{t - 1} \neq a^*$. Equation (B.11) implies that player 1 is indifferent between $a^*$ and $a'$ when $t = 0$ or
$a_{t-1} \neq a^*$, and moreover, given that $a'$ is player 1’s stage-game best reply against $b'$, $a'$ yields strictly higher payoff compared to actions other than $a^*$ and $a'$.

Next, recall that $u_1(a^*, b^*)$ is player 1’s continuation payoff in the reputation maintenance phase conditional on player 1 playing $a^*$ in the period before moving to the reputation maintenance phase; and $\xi V_1 + (1 - \xi)u_1(a^*, b^*)$ is player 1’s continuation payoff in the reputation maintenance phase conditional on player 1 not playing $a^*$ in the period before moving to the reputation maintenance phase, where $V_1$ solves (B.4) and (B.5) and $\xi$ is given by (B.7). At every reputation-building phase history with $a_{t-1} = a^*$, player 1’s expected payoff from playing $a^*$ is given by the RHS of (B.10). His expected payoff from playing $a'$ is:

$$(1 - \delta)u_1(a', \rho b^* + (1 - \rho)b') + \delta \rho \left( \xi V_1 + (1 - \xi)u_1(a^*, b^*) \right) + \delta(1 - \rho)u_1(a', b').$$

(B.13)

Let $V_1^{**} \equiv \xi V_1 + (1 - \xi)u_1(a^*, b^*)$. Subtracting the RHS of (B.10) from (B.13), we obtain:

$$(1 - \delta)\rho(u_1(a^*, b^*) - u_1(a', b^*)) + (1 - \delta)(1 - \rho)(u_1(a^*, b') - u_1(a', b')) + \delta \rho(u_1(a^*, b^*) - V_1^{**}) + \delta(1 - \rho)(V_1' - u_1(a', b'))$$

(B.14)

According to (B.11), (B.14) reduces to:

$$(1 - \delta)\rho(u_1(a^*, b^*) - u_1(a', b^*)) + \delta \rho(u_1(a^*, b^*) - V_1^{**}),$$

which equals 0 according to (B.7). This suggests that (B.13) equals the RHS of (B.10), and that the strategic-type player 1 is indifferent between $a^*$ and $a'$. To show that player 1 strictly prefers $a'$ to actions other than $a^*$ and $a'$, I establish the following lemma:

**Lemma B.3.** For every $\gamma \in (0, 1)$, there exists $\delta \in (0, 1)$, such that for every $\delta > \delta$, there exists $\rho \in (0, \gamma)$ that solves (B.10) and (B.11).

**Proof of Lemma B.3:** According to (B.11), $V_1'$ converges to $u_1(a', b')$ as $\delta \to 1$. According to (B.10)

$$(1 - \delta + \delta \rho)V_1' = (1 - \delta)u_1(a^*, \rho b^* + (1 - \rho)b') + \delta \rho u_1(a^*, b^*).$$

(B.15)

Suppose toward a contradiction that there exists a sequence of $\{\delta_n\}_{n=1}^\infty$ with $\lim_{n \to \infty} \delta_n = 1$ such that $\lim_{n \to \infty} \rho_n = \rho^* > 0$. Then the LHS of (B.15) converges to $\rho^* V_1'$ while the RHS converges to $\rho^* u_1(a^*, b^*)$. Since $V_1' \to u_1(a', b')$ and $u_1(a', b') < u_1(a^*, b^*)$ in games with strict lack-of-commitment. This yields a contradiction.

\[\square\]
Since $a'$ is a strict best reply against $b'$, there exists $\varepsilon > 0$ such that $a'$ best replies against $\varepsilon b^{**} + (1 - \varepsilon)b'$ for every $\varepsilon \in (0, \pi)$. Lemma [B.3] implies that when $\delta$ is close to 1, $a'$ is player 1’s strict best reply against $\rho b^{**} + (1 - \rho)b'$, i.e., $a'$ yields player 1 a strictly higher payoff compared to actions other than $a^*$ and $a'$.

**Player 1’s Strategy:** I start from the following Lemma:

**Lemma B.4.** For every $K \in \mathbb{N}$, there exists $M \in \mathbb{N}$, such that

$$2^K \sum_{j=0}^{K} \binom{n}{j} < 2^n, \text{ for every } n \geq M. \tag{B.16}$$

**Proof of Lemma B.4:** To start with, $\sum_{j=0}^{n} \binom{n}{j} = 2^n$ for every $n \in \mathbb{N}$. Moreover, $\binom{n}{j}$ is increasing in $j$ when $j < n/2$ and is decreasing in $j$ when $j > n/2$. This suggests that for every $m \geq 2$ and $n > mK$,

$$\sum_{j=0}^{K} \binom{n}{j} \leq \frac{2^n}{m}.$$

Let $M = 2^K K$, we have:

$$2^K \sum_{j=0}^{K} \binom{n}{j} < \frac{2^{K+n}}{2^K} = 2^n.$$

Let $\eta \in (0, 1/2)$ be a small enough real number which will be determined by the end of the construction (Lemma [B.5]). Pick $\bar{\pi}_0(\eta) \in (0, 1)$ small enough such that:

$$\frac{\bar{\pi}_0(\eta)}{1 - \bar{\pi}_0(\eta)} \left(\frac{1}{\eta q^*}\right)^M < \frac{\eta q^*}{1 - \eta q^*}. \tag{B.17}$$

Recall the definitions of $a'$, $b'$, $b^{**}$, and $q^*$ in Appendix [B.1]. Player 1’s strategy in the reputation building phase depends on calendar time, and in particular, the comparison between $t$ and $M$. Let $\bar{\pi}_t \in \Delta(\Omega)$ be the posterior belief of a hypothetical observer who shares the same prior belief as player 2s, but can observe the entire history of actions and public randomization devices, i.e., he observes $\{a_0, ..., a_{t-1}, b_0, ..., b_{t-1}, \xi_0, ..., \xi_t\}$. Let $\bar{\pi}_t$ be the probability $\bar{\pi}_t$ attaches to the commitment type. Player 1 can compute $\bar{\pi}_t$ based on his private history, but player 2 cannot.

At every reputation-building phase history $h^t$ with $t \leq M$, strategic-type player 1 mixes between
Let \( q(h_t^1) \equiv q(h_t^1_t) \), with the probability of playing \( a^* \) being \( q(h_t^1) \), satisfying:

\[
(1 - \pi_t)q(h_t^1) + \pi_t = q^*
\]  

(B.18)

i.e., player 2 is indifferent between \( b^{**} \) and \( b' \) when she observes the entire history of actions and public randomization devices.

I show that when \( \pi_0 < \pi_0(\eta) \), we have \( \pi_t < \eta q^* \) for every \( \{a_0, \ldots, a_{t-1}, b_0, \ldots, b_{t-1}, \xi_0, \ldots, \xi_t\} \) with \( t \leq M \). This is because \( \pi_t \) is bounded from above by an observer’s belief who observes \( \{a_0, \ldots, a_{t-1}\} = \{a^*, a^*, \ldots, a^*\} \). According to (B.18), \( q(h_t^1) > \eta q^* \) when \( \pi_t < \eta q^* \), and therefore, \( \pi_{t+1} \leq \frac{\pi_t}{\eta q^*} \). Applying (B.17), one can then show inductively that \( \pi_t < \eta q^* \) for every \( t \leq M \). The above conclusion also suggests that \( q(h_t^1) > \eta q^* \) for every \( h_t^1 \) with \( t \leq M \).

At every reputation-building phase history \( h_t^1 \) with \( t > M \), strategic-type player 1 mixes between \( a^* \) and \( a' \). The probability with which he plays \( a^* \) depends on his private history only through \( \chi_t \equiv \{\chi_0, \ldots, \chi_{t-1}\} \in X^t \equiv \{0, 1\}^t \), with

\[
\chi_s = \begin{cases} 
1 & \text{if } a_s = a^* \\
0 & \text{if } a_s \neq a^*.
\end{cases}
\]

Let \( q(\chi_t) \) be the probability with which strategic-type player 1 plays \( a^* \), and let \( q_t \equiv \{q(\chi_t)\}_{\chi_t \in \{0, 1\}^t} \).

For every \( h_t^2 \), let \( \kappa(h_t^2) \in \Delta(X_t) \) be player 2’s belief about \( \chi_t \) conditional on player 1 being the strategic type. For every \( t > M \), recall that \( \pi(h_t^2) \) is the probability player 2’s posterior belief attaches to the commitment type when her private history is \( h_t^2 \). Let \( q_t \) be such that:

\[
\pi(h_t^2) + (1 - \pi(h_t^2))\kappa(h_t^2) \cdot q_t = q^* \text{ for every } h_t^2 \in H_t^2.
\]  

(B.19)

When (B.19) is satisfied, player 2 believes that player 1 plays \( a^* \) with probability \( q^* \) and \( a' \) with probability \( 1 - q^* \) at private history \( h_t^2 \), and is therefore, indifferent between \( b^{**} \) and \( b' \). I show the following lemma, which verifies player 2’s incentive constraints in the reputation building phase.

**Lemma B.5.** There exists \( \eta \in (0, 1/2) \) such that for every \( t > M \), if \( q(\chi^s) \in [\eta q^*, 1 - \eta q^*] \) for every \( \chi^s \in \{0, 1\}^s \) with \( s \leq t - 1 \), then there exists \( q_t \in [0, 1]^2 \) that solves (B.19) and satisfies \( q(\chi^t) \in [\eta q^*, 1 - \eta q^*] \) for every \( \chi^t \in \{0, 1\}^t \).

**Proof of Lemma B.5:** First, I show that linear system (B.19), which is equivalent to

\[
\kappa(h_t^2) \cdot q_t = \frac{q^* - \pi(h_t^2)}{1 - \pi(h_t^2)} \text{ for every } h_t^2 \in H_t^2.
\]
admits a solution. To start with, it is without loss to focus on \( h^t_2 \) with \(|N_t| = K\). This is because for every \(|N'_t| < K\), there exists \(|N''_t| = K\) such that an agent who observes \( \{a_s\}_{s \in N''_t} \) has a Blackwell more informative information structure compared to an agent who observes \( \{a_s\}_{s \in N'_t} \). As a result, \((B.20)\) is satisfied for all \( h^t_2 \) with \(|N_t| = K\) implies that \((B.20)\) is satisfied for all \( h^t_2 \) with \(|N_t| \leq K\).

Second, the definition of \( M \) suggests that \( 2^t > 2^K (\frac{t}{K}) \) for every \( t \geq M \). This suggests that linear system:

\[
\kappa(h^t_2) \cdot q_t = \frac{q^* - \pi(h^t_2)}{1 - \pi(h^t_2)} \quad \text{for every } h^t_2 \in \mathcal{H}_2^t \text{ with } |N_t| = K
\]  

(B.20)

is underdetermined. An important observation is that the following set of vectors:

\[
\{\kappa(h^t_2)\}_{h^t_2 \in \mathcal{H}_2^t \text{ with } |N_t| = K}
\]

are convex independent. This is because (1) when \( s \in N_t \), player 2 knows the value of \( \chi_s \) and (2) when \( s \notin N_t \), player 2’s belief about \( \chi_s \) is not degenerate. Let \( \kappa_t \) be the coefficient matrix of linear system \((B.20)\). Convex independence of \( \kappa(h^t_2) \) suggests that the rank of the \( \kappa_t \) is \( \left( \frac{t}{K} \right) \). Let

\[
\tilde{q}_t \equiv \left\{ \frac{q^* - \pi(h^t_2)}{1 - \pi(h^t_2)} \right\}_{h^t_2 \in \mathcal{H}_2^t \text{ with } |N_t| = K}.
\]

(B.21)

Since \((B.20)\) is underdetermined, the rank of the augmented matrix \((\kappa_t, \tilde{q}_t)\) is also \( \left( \frac{t}{K} \right) \). The Rouché-Capelli theorem implies that \((B.20)\) admits at least one solution.

Third, I show there exists a solution where each entry of \( q_t \) belongs to the interval \([\eta q^*, 1 - \eta q^*] \). This is because the RHS of \((B.20)\) converges to \( q^* \) as \( \pi(h^t_2) \to 0 \), and linear system \((B.20)\) admits a solution \( q_t = (q^*, ..., q^*) \) when \( \pi(h^t_2) = 0 \). Proposition 10 in Fefferman and Kollár (2013) suggests that the solution correspondence is continuous with respect to \( \pi(h^t_2) \). This suggests the existence of \( \eta \in (0, 1/2) \) such that for every \( \pi(h^t_2) < \eta q^* \), there exists a solution to \((B.20)\) \( q_t \) such that

\[
||q_t - (q^*, ..., q^*)||_{L^\infty} < \eta.
\]

When \( \eta \) is small enough, all entries of \( q_t \) belong to the interval \([q^* \eta, 1 - q^* \eta] \).

Fourth, I show by induction that \( \pi(h^t_2) < \eta q^* \). In period \( t = M \), since the strategic-type player 1 plays \( a^* \) with probability at least \( \eta q^* \) at every history of the reputation-building phase before period \( M \) and player 2’s prior belief attaches probability no more than \( \overline{\pi}(\eta) \) to the commitment type, her posterior belief attaches probability at most \( \eta \pi^* \) to the commitment type given inequality \((B.17)\). The third step then implies the existence of \( q_t \) such that \( q(\chi^t) \in [\eta q^*, 1 - \eta q^*] \) for every \( \chi^t \in \{0, 1\}^t \). Suppose
\[ \pi(h_t^2) < \eta q^* \quad \text{for every} \quad h_t^2 \in \mathcal{H}_2^t \quad \text{and every} \quad t \leq T. \]

The same reasoning implies that \( \pi(h_{T+1}^2) < \eta q^* \) for every \( h_{T+1}^2 \in \mathcal{H}_2^{T+1} \) given that each player 2 observes at most \( K \) realizations among \( \{a_t\}_{t=0}^T \). This also suggests the existence of \( q_{T+1}^* \) with \( q(\chi_{T+1}) \in [\eta q^*, 1 - \eta q^*] \) for every \( \chi_{T+1} \in \{0, 1\}^{T+1} \).

**Promising Keeping Constraint:** I conclude the proof by verifying player 1’s promise keeping constraint, i.e., the continuation play delivers strategic-type player 1 his continuation value in period 0, which is \( u_1(a', b') \). Since strategic-type player 1 plays \( a^* \) with probability at least \( \eta q^* \) in the reputation building phase, and conditional on \( a_{t-1} = a^* \), play transits to the reputation maintenance phase with probability \( \rho \), the equilibrium play belongs to the reputation maintenance phase or punishment phase with probability 1 as \( t \to \infty \). This suggests that on the equilibrium path, play either converges to \((a^*, b^*)\) in every period, or \((a', b')\) in every period, and player 1’s continuation value in those cases are \( u_1(a^*, b^*) \) and \( u_1(a', b') \), respectively. This verifies the promise keeping constraints.

### B.3 Proof of Theorem 3': Generalized Coordination Games

First, consider the trivial case in which there exists a unique pure strategy Nash Equilibrium, i.e., \( a' = a^* \). Player 1’s payoff is \( u_1(a', b') \) in an equilibrium where player 1 plays \( a^* \) at every history and player 2 plays \( b^* \) at every history.

Next, consider the nontrivial case in which \( a'' = a^* \) but \( a' \neq a^* \), i.e., the Stackelberg outcome is a pure-strategy Nash Equilibrium in the stage game, but there also exists another stage-game Nash Equilibrium which results in strictly lower payoff for player 1.

I construct an equilibrium that consists of three phases: a *reputation building phase*, a *reputation maintenance phase*, and a *punishment phase*, which depends only on the history of player 2’s actions that is commonly observed by both players. I only highlight the differences between this construction and the one in Appendix [B.2] in order to avoid repetition. In period \( t \),

- play is in the reputation building phase if \( t = 0 \) or \( (b_0, ..., b_{t-1}) = (b', ..., b') \);
- play is in the reputation maintenance phase if (1) there exists \( s \leq t - 1 \) such that \( b_s = b^{**} \), and (2) \( b_{s^*+1} = b^* \) where \( s^* \) is the smallest \( s \in \mathbb{N} \) such that \( b_s = b^{**} \).
- play is in the punishment phase if (1) there exists \( s \leq t - 1 \) such that \( b_s = b^{**} \), and (2) \( b_{s^*+1} \neq b^* \) where \( s^* \) is the smallest \( s \in \mathbb{N} \) such that \( b_s = b^{**} \).

Play starts from the reputation building phase, and eventually ends up in the reputation maintenance phase or the punishment phase. Different from the construction in Appendix [B.2].
1. The set of reputation maintenance phase histories is larger in the current construction. In particular, player 2 only checks her predecessor’s action in the next period after $b^{**}$ occurs, and as long as it is $b^*$, play remains in the reputation maintenance phase regardless of players’ actions after period $s^* + 1$.

2. Players’ strategies differ in the reputation maintenance phase, and player 2’s strategy is different in the reputation building phase.

Players’ strategies in the punishment phase and player 1’s strategy in the reputation building phase remain the same as in Appendix B.2, which I omit to avoid repetition. The differences are in players’ strategies in the reputation maintenance phase.

**Reputation Maintenance Phase:** Let $s^*$ be the smallest $s$ such that $b_s = b^{**}$. At every history of the reputation maintenance phase with $t \geq s^* + 2$, strategic-type player 1 plays $a^*$ and player 2 plays $b^*$. In period $s^* + 1$,

- If $a_{s^*} = a^*$, then strategic-type player 1 plays $a^*$ and player 2 plays $b^*$.

If $a_{s^*} \neq a^*$

$$\xi_{s^*} > \xi' \equiv \frac{1 - \delta}{\delta} \cdot \frac{u_1(a', b') - u_1(a^*, b^*)}{u_1(a^*, b^*) - u_1(a', b')}.$$  \(\text{(B.22)}\)

then strategic-type player 1 plays $a^*$.

If $a_{s^*} \neq a^*$ and $\xi_{s^*} \leq \xi'$, then strategic-type player 1 plays $a'$ and player 2 plays $b'$.

**Reputation Building Phase:** Player 1’s strategy remains the same as Appendix B.2. Player 2s’ strategy differs in terms of $\rho$, the probability of playing $b^{**}$ in period $t$ conditional on $a_{t-1} = a^*$ but unconditional on the realization of $N_t$. This $\rho$ is pinned down by the following equality:

$$V'_1 = (1 - \delta)u_1(a^*, \rho b^* + (1 - \rho)b') + \delta \rho u_1(a^*, b^*) + \delta (1 - \rho)V'_1,$$ \(\text{(B.23)}\)

with

$$u_1(a', b') = (1 - \delta)u_1(a^*, b') + \delta V'_1.$$

The rest of the construction remains the same.
B.4 Proof of Lemma B.2

**Proof of Lemma B.2:** According to (B.9), we have:

\[ V_1 = u_1(a^*, b^*) - \frac{1 - \delta}{\delta} (u_1(a^*, b^*) - u_1(a^*, b^*)) - \frac{1 - \delta}{\beta} \left( u_1(a^*, b^*) - u_1(a^*, b^*) \right). \]

Plugging this into (B.4), we obtain:

\[
\frac{1}{1 - \beta} \left( u_1(a'', b^*) - u_1(a^*, b^*) \right) + \frac{1}{\beta} \left( u_1(a'', b'') - u_1(a^*, b'') \right) = \frac{\delta}{1 - \delta} \left( u_1(a^*, b^*) - (1 - \delta)u_1(a^*, b'') - \delta u_1(a', b') \right).
\]

The two roots are given by:

\[ \beta = \frac{X + Z - Y \pm \sqrt{(X + Z - Y)^2 - 4XZ}}{2X} \quad (B.24) \]

When the stage-game features strict lack-of-commitment (see Definition 1), \( X > 0 \) and \( Y, Z \geq 0 \). As \( \delta \to 1 \), \( X \to \infty \) while \( Y \) and \( Z \) remain constant, and therefore, both roots are real. Moreover, the larger of the two roots converge to 1. To show that it is strictly less than 1, it is equivalent to show that:

\[ \sqrt{(X + Z - Y)^2 - 4XZ} < 2X - (X + Z - Y), \]

which is equivalent to:

\[ X^2 + Y^2 + Z^2 - 2XZ - 2YZ - 2XY < X^2 + Y^2 + Z^2 - 2XZ - 2YZ + 2XY. \]

The last inequality is true given that both \( X \) and \( Y \) are strictly positive. This establishes the existence of a real root \( \beta \) that is strictly less than 1 and converges to 1 as \( \delta \to 1 \).

\[ \square \]

C Proof of Theorem 4

C.1 Proof of Theorem 4: Statement 1

For every public history \( h^t \), let \( g(h^t) \) be the probability that player 2 plays \( b^* \) at \( h^t \). Let \( g(h^t, \omega^e) \) be the probability that player 2 plays \( b^* \) at \( h^t \) conditional on player 1 is the commitment type. For any public history \( h^t \equiv \{a_{\max\{0, t-K\}}, \ldots, a_{t-1}\} = \{a^*, \ldots, a^*\} \), I derive a lower bound on: \[ \frac{g(h^t, \omega^e)}{g(h^t)} \] as a
function of $g(h^t)$, or equivalently, an upper bound on

$$\frac{1 - g(h^t, \omega^c)}{1 - g(h^t)}. \quad (C.1)$$

Let $A \equiv \{a^*, a'_1\}$ and $S \equiv \{s^*, s_1, s_2, ..., s_m\}$. Let $r(h^t)$ be the probability that $a^*$ is played at $h^t$, let $\tau(s_i)(h^t)$ be the probability that signal $s_i$ occurs at $h^t$, and let $p(s_i)(h^t)$ be the posterior probability of $a^*$ conditional on observing $s_i$ at $h^t$. I suppress the dependence on $h^t$ in order to simplify notation.

Since $\{b^*\} = \text{BR}_2(a^*)$ and $|A| = 2$, we have the following two implications:

1. there exists a cutoff belief $p^* \in (0, 1)$ such that player 2 has a strict incentive to play $b^*$ after observing $s_i$ if and only if $p(s_i) > p^*$.

2. there exists a constant $C \in \mathbb{R}_+$ such that $1 - r \geq C(1 - g)$.

According to the first implication, it is without loss of generality to label the signal realizations such that $p(s_1) \geq p(s_2) \geq ... \geq p(s_m)$, and moreover, there exists $k \in \{1, 2, ..., m\}$ such that player 2 plays $b^*$ for sure after observing $s_1, ..., s_{k-1}$, and does not play $b^*$ otherwise.\footnote{Ignoring the possibility that player 2 plays a mixed action following certain signal realizations is without loss of generality in proving the current theorem. This is because when player 2 mixes between $n$ actions after one signal realization, we can split this signal realization into $n$ signal realizations with the same posterior belief, such that player 2 plays a pure action following each of these signal realizations.}

Therefore,

$$r(1 - f(s^*|a^*)) = \sum_{i=1}^{m} \tau(s_i)p(s_i), \quad 1 - r = \sum_{i=1}^{m} \tau(s_i)(1 - p(s_i)),$$

and

$$\sum_{i=k}^{m} \tau(s_i)(1 - p(s_i)) \geq \frac{1 - r}{1 - r f(s^*|a^*)} (1 - g). \quad (C.2)$$

As a result,

$$\sum_{i=k}^{m} \tau(s_i)p(s_i) \leq \frac{r(1 - f(s^*|a^*))}{1 - r f(s^*|a^*)} (1 - g), \quad (C.3)$$

and

$$\sum_{i=k}^{m} \tau(s_i)(1 - p(s_i)) \geq \frac{1 - r}{1 - r f(s^*|a^*)} (1 - g). \quad (C.4)$$

Therefore,

$$\frac{1 - g(\omega^c)}{1 - g} \leq \frac{1 - f(s^*|a^*)}{1 - r f(s^*|a^*)}. \quad (C.5)$$
Using the second implication, namely, \( r \leq 1 - C(1 - g) \), we have:

\[
\frac{1 - g(\omega^c)}{1 - g} \leq \frac{1 - f(s^*|a^*)}{1 - f(s^*|a^*) + Cf(s^*|a^*)(1 - g)}. \tag{C.6}
\]

Similarly, the lower bound on the likelihood ratio with which \( b^* \) occurs is given by:

\[
\frac{g(\omega^c)}{g} \geq 1 + \frac{f(s^*|a^*)(1 - g(h_t))}{g - rf(s^*|a^*)} \geq 1 + \frac{f(s^*|a^*)(1 - g)}{g - f(s^*|a^*)(1 - C(1 - g))}. \tag{C.7}
\]

Let \( \beta(h_t) \in \Delta(B) \) be the distribution over player 2’s action at \( h_t \), and let \( \beta(h_t, \omega^c) \in \Delta(B) \) be the distribution over player 2’s action at \( h_t \) conditional on player 1 being the commitment type. Inequalities (C.6) and (C.7) imply the following lower bound on the KL divergence between \( \beta(h_t) \) and \( \beta(h_t, \omega^c) \):

\[
d\left( \frac{\beta(h_t)}{\beta(h_t, \omega^c)} \right) \leq \mathcal{L}(1 - g(h_t)). \tag{C.8}
\]

The Pinsker’s inequality implies that \( \mathcal{L}(\cdot) \) is of the magnitude \((1 - g(h_t))^2\).

This lower bound on the KL divergence bounds the speed of learning at \( h_t \) from below, as a function of the probability with which player 2 at \( h_t \) does not play \( b^* \). This implies a lower bound on the speed of learning when player 2 in the future observes \( b^* \) in period \( t \), given that he knew that the probability with which player 2 plays \( b^* \) at \( h_t \) is no more than \( g(h_t) \). However, unlike models with unbounded memory, future player 2’s information does not nest that of player 2’s in period \( t \). This is because future player 2s may not observe \( \{a_{t-K}, ..., a_{t-1}\} \), and hence, cannot interpret the meaning of \( b_t \) in the same way as player 2 in period \( t \) does.

For every \( s, t \in \mathbb{N} \) with \( s > t \), I provide a lower bound on the informativeness of \( b_t \) about player 1’s type from the perspective of player 2 who arrives in period \( s \), as a function of the informativeness of \( b_t \) (about player 1’s type) from the perspective of player 2 who arrives in period \( t \). This together with (C.8) establishes a lower bound on the informativeness of \( b_t \) from the perspective of future player 2s as a function of the probability with which \( b^* \) is not being played. Applying the result in Gossner (2011), one obtains the commitment payoff theorem.

Let \( \pi(h_t) \) be player 2’s belief about \( \omega \) at \( h_t \) before observing the period \( t \) signal \( s_t \). By definition, \( \pi(h^0) = \pi_0 \). For every strategy profile \( \sigma \), let \( \mathcal{P}^\sigma \) be the probability measure over \( \mathcal{H} \) induced by \( \sigma \), let \( \mathcal{P}^\sigma,\omega^c \) be the probability measure induced by \( \sigma \) conditional on player 1 being the commitment type, and let \( \mathcal{P}^\sigma,\omega^s \) be the probability measure induced by \( \sigma \) conditional on player 1 being the strategic type. One can the write the posterior likelihood ratio as the product of the likelihood ratio of the signals
observed in each period:

\[
\frac{\pi(h^t)}{1 - \pi(h^t)} / \frac{\pi_0}{1 - \pi_0} = \frac{\mathcal{P}^{\sigma,\omega^c}(b_0)}{\mathcal{P}^{\sigma,\omega^c}(b_1|b_0)} \cdot \frac{\mathcal{P}^{\sigma,\omega^c}(b_{t-1}|b_{t-2}, \ldots, b_0)}{\mathcal{P}^{\sigma,\omega^c}(b_t|b_{t-2}, \ldots, b_0)} \cdot \frac{\mathcal{P}^{\sigma,\omega^c}(a_{t-K}, \ldots, a_{t-1}|b_t, b_{t-1}, \ldots, b_0)}{\mathcal{P}^{\sigma,\omega^c}(a_{t-K}, \ldots, a_{t-1}|b_t, b_{t-1}, \ldots, b_0)}
\]  

(C.9)

Furthermore, for every \( \epsilon > 0 \) and every \( t \), we know that:

\[
\mathcal{P}^{\sigma,\omega^c}(\pi^\sigma(b_0, b_1, \ldots b_{t-1}) < \epsilon \pi_0) \leq \frac{1 - \epsilon \pi_0}{1 - \epsilon \pi_0 \epsilon},
\]

(C.10)

in which \( \pi^\sigma(b_0, b_1, \ldots b_{t-1}) \in \Delta(\Omega) \) is player 2’s belief about player 1’s type after observing \((b_0, \ldots, b_{t-1})\) but before observing player 1’s actions and \( s_t \). For every \( \epsilon > 0 \), let \( \rho^*(\epsilon) \) be defined as:

\[
\rho^*(\epsilon) \equiv \frac{\epsilon \pi_0}{1 - C\epsilon}.
\]

(C.11)

Next, if \( \pi^\sigma(b_0, b_1, \ldots b_{t-1}) \geq \epsilon \pi_0 \), and player 2 in period \( t \) believes that \( b_t = b^* \) occurs with probability less than \( 1 - \epsilon \) after observing \((a_{t-K}, \ldots, a_{t-1}) = (a^*, \ldots, a^*)\), then under probability measure \( \mathcal{P}^\sigma \), the probability of \( \{a_{t-K}, \ldots, a_{t-1}\} = \{a^*, \ldots, a^*\} \) conditional on \((b_0, \ldots, b_{t-1})\) is at least \( \rho^*(\epsilon) \).

To see this, suppose towards a contradiction that the probability with which \((a_{t-K}, \ldots, a_{t-1}) = (a^*, \ldots, a^*)\) is strictly less than \( \rho^*(\epsilon) \) conditional on \((b_0, \ldots, b_{t-1})\). According to (C.11), after observing \((a_{t-K}, \ldots, a_{t-1}) = (a^*, \ldots, a^*)\) in period \( t \) and given that \( \pi^\sigma(b_0, b_1, \ldots b_{t-1}) \geq \epsilon \pi_0 \), \( \pi(h^t) \) attaches probability strictly more than \( 1 - C\epsilon \) to the commitment type. As a result, player 2 in period \( t \) believes that \( a^* \) is played with probability at least \( 1 - C\epsilon \) at \( h^t \). This contradicts presumption that she plays \( b^* \) with probability less than \( 1 - \epsilon \).

Next, I study the believed distribution of \( b_t \) from the perspective of player 2 in period \( s \) in the event that \( \pi^\sigma(b_0, b_1, \ldots b_{t-1}) \geq \epsilon \pi_0 \). Let \( \mathcal{P}(\sigma, t, s) \in \Delta(\Delta(A^K)) \) be player 2’s signal structure in period \( s(\geq t) \) about \( \{a_{t-K}, \ldots, a_{t-1}\} \) under equilibrium \( \sigma \). For every small enough \( \eta > 0 \), given that \( \mathcal{P}(\sigma, t) \) attaches probability at least \( \rho^*(\epsilon) \) to \( \{a_{t-K}, \ldots, a_{t-1}\} = \{a^*, \ldots, a^*\} \), the probability with which \( \mathcal{P}(\sigma, t, s) \) attaches to the event that \( \{a_{t-K}, \ldots, a_{t-1}\} = \{a^*, a^*, \ldots, a^*\} \) occurs with probability less than \( \eta \rho^*(\epsilon) \) conditional on \( \{a_{t-K}, \ldots, a_{t-1}\} = \{a^*, a^*, \ldots, a^*\} \) is bounded from above by:

\[
\frac{\eta \rho^*(\epsilon)(1 - \rho^*(\epsilon))}{(1 - \eta \rho^*(\epsilon)) \rho^*(\epsilon)} = \frac{1 - \rho^*(\epsilon)}{1 - \rho^*(\epsilon) \eta}.
\]

(C.12)

Let \( g(t|h^s) \) be player 2’s belief about the probability with which \( b^* \) is played in period \( t \) when she observes \( h^s \). Let \( g(t, \omega^c|h^s) \) be her belief about the probability with which \( b^* \) is played in period \( t \)
conditional on player 1 being committed. The conclusions in (C.6) and (C.7) also apply in this setting, namely,

\[
\frac{1 - g(t, \omega^c| h^s)}{1 - g(t| h^s)} \leq \frac{1 - f(s^*|a^*)}{1 - f(s^*|a^*) + Cf(s^*|a^*)(1 - g(t| h^s))}
\]  

(C.13)

and

\[
\frac{g(t, \omega^c| h^s)}{g(t| h^s)} \geq 1 + \frac{f(s^*|a^*)(1 - g(t| h^s))}{g(t| h^s) - f(s^*|a^*)(1 - C(1 - g(t| h^s)))}
\]  

(C.14)

Whenever player 2 in period \(s\) believes that \(\{a_{t-K}, ..., a_{t-1}\} = \{a^*, a^*, ..., a^*\}\) occurs with probability more than \(\eta \cdot \rho^*(\epsilon)\), we have:

\[
g(t| h^s) \leq 1 - \epsilon \eta \rho^*.
\]  

(C.15)

Applying (C.15) to (C.13) and (C.14), we obtain a lower bound on the KL divergence between \(g(t, \omega^c| h^s)\) and \(g(t| h^s)\). This is the lower bound on the speed with which player 2 at \(h^s\) will learn through \(b_t = b^*\) about player 1’s type, which applies to all events except for one that occurs with probability less than \(\eta \cdot \rho^*(\epsilon)\). Therefore, for every \(\epsilon\) and \(\pi_0\), there exists \(\delta\) such that when \(\delta > \delta\), the strategic player 1’s payoff by playing \(a^*\) in every period is at least:

\[
\left(1 - \epsilon - \epsilon \frac{1 - \pi_0}{1 - \pi_0 \epsilon}\right) u_1(a^*, b^*) + \left(\epsilon + \epsilon \frac{1 - \pi_0}{1 - \pi_0 \epsilon}\right) \min_{a,b} u_1(a,b) - \epsilon.
\]  

(C.16)

Taking \(\epsilon \to 0\) and \(\delta \to 1\), (C.16) implies the commitment payoff theorem.

**C.2 Proof of Theorem 4: Statement 2**

Recall the definitions of \((a^*, b^*)\) and \((a', b')\) in the proof of Theorem 1. I omit the trivial case in which \(a^* = a'\), i.e., \(a^*\) is player 1’s action in his worst pure-strategy Nash Equilibrium. I focus on the interesting case in which \(a^* \neq a'\). Since \(|A| = 2\), we have \(A = \{a^*, a'\}\). Let

\[
l^*(f) \equiv \max_{s \in S} f(s|a^*) / f(s|a')
\]  

(C.17)

Consider the construction in the proof of Theorem 1 with one modification: the overall probability with which player 1 plays \(a^*\) is:

\[
\hat{q} \equiv \frac{q^*}{q^* + (1 - q^*)l^*(f)},
\]  

(C.18)

and the probability with which he plays \(a'\) is \(1 - \hat{q}\). Let \(\pi_0 = \frac{K}{q^*}\). Player 2 has an incentive to play \(b'\) in the reputation building phase, regardless of her observation of player 1’s action in the past \(K\) periods, and regardless of the signal she receives about player 1’s action in the current period. When
$K \geq 1$, the rest of the constructive proof follows from that of Theorem 4. When $K = 0$, strategic-type player 1 plays $a'$ at every history and player 2 plays $b'$ at every history. Such a strategy profile is an equilibrium when the prior probability of commitment type is small enough.

### C.3 Proof of Theorem 4'

For every $\alpha, \beta : S \rightarrow \Delta (B)$, let $\pi (\alpha, \beta) \in \Delta (B)$ be the distribution over $b$ induced by $(\alpha, \beta)$. When players’ payoffs are monotone-supermodular and $f$ satisfies MLRP, I establish the existence of $C > 0$ such that for every $\alpha \in \Delta (A)$ with $a^* \in \text{supp}(\alpha)$, and every $\beta : S \rightarrow \Delta (B)$ that is player 2’s stage-game best reply against $\alpha$ after observing the realization of $s$, if $\pi (\alpha, \beta) [b^*] < 1 - \varepsilon$, then

$$d (\pi (\alpha, \beta) \bigg| \pi (a^*, \beta)) > C \varepsilon^2. \quad (C.19)$$

The rest of the proof follows from that of Theorem 4 in Appendix C.1 and is omitted to avoid repetition.

Let $\overline{A}$ be the set of actions that are strictly higher than $a^*$ and let $\underline{A}$ be the set of actions that are strictly lower than $a^*$. Since $f$ is unboundedly informative about $a^*$, there exists $s^* \in S$ such that $f(s^*|a) > 0$ if and only if $a = a^*$. Let $\overline{S}$ be the set of signal realizations that are strictly higher than $s^*$ and let $\underline{S}$ be the set of signal realizations that are strictly lower than $s^*$. Since $f$ satisfies MLRP, for every $s \in \overline{S}$, $f(s|a) > 0$ only if $a \succeq a^*$, and for every $s \in \underline{S}$, $f(s|a) > 0$ only if $a \preceq a^*$.

For every $\alpha \in \Delta (A)$, let $\alpha' \in \Delta (A)$ be the distribution over $A$ conditional on $a \neq a^*$. If $\text{supp}(\alpha) \cap \overline{A} \neq \emptyset$, then let $\overline{\alpha} \in \Delta (A)$ be the distribution over $A$ conditional on $a \in \text{supp}(\alpha) \cap \overline{A}$; if $\text{supp}(\alpha) \cap \underline{A} \neq \emptyset$, then let $\underline{\alpha} \in \Delta (A)$ be the distribution over $A$ conditional on $a \in \text{supp}(\alpha) \cap \underline{A}$. By definition, there exists $\lambda \in [0, 1]$ such that $\alpha' = \lambda \overline{\alpha} + (1 - \lambda) \underline{\alpha}$. When $\pi (\alpha, \beta) [b^*] < 1 - \varepsilon$, either $\overline{\alpha}$ is well-defined and $\pi (\overline{\alpha}, \beta) [b^*] < 1$, or $\underline{\alpha}$ is well-defined and $\pi (\underline{\alpha}, \beta) [b^*] < 1$, or both. Since players’ payoffs are monotone-supermodular, and $\beta$ best replies against $\alpha$, there exist $\overline{s} \in \overline{S}$ and $\underline{s} \in \underline{S}$ such that $\beta (s) [b^*] = 1$ only if $\overline{s} > s > \underline{s}$, and $\beta (s) [b^*] > 0$ only if $\overline{s} \succeq s \succeq \underline{s}$.

Suppose toward a contradiction that $d (\pi (\alpha, \beta) \bigg| \pi (a^*, \beta)) = 0$, then $d (\pi (\alpha', \beta) \bigg| \pi (a^*, \beta)) = 0$, which suggests:

$$\lambda \left( \beta (\overline{s}) [b^*] \cdot f(\overline{s}|\overline{\alpha}) + \sum_{s > \overline{s}} f(s|\overline{\alpha}) \right) = \beta (\overline{s}) [b^*] \cdot f(\overline{s}|a^*) + \sum_{s > \overline{s}} f(s|a^*) \quad (C.20)$$

and

$$(1 - \lambda) \left( \beta (\underline{s}) [b^*] \cdot f(\underline{s}|\underline{\alpha}) + \sum_{s < \underline{s}} f(s|\underline{\alpha}) \right) = \beta (\underline{s}) [b^*] \cdot f(\underline{s}|a^*) + \sum_{s < \underline{s}} f(s|a^*). \quad (C.21)$$
Since \( f(s|a) > 0 \) if and only if \( a = a^* \), (C.20) and (C.21) together imply that either

\[
\lambda \left( (1 - \beta(s)[b^*]) \cdot f(s|\overline{a}) + \sum_{s>s^*} f(s|s) \right) > (1 - \beta(s)[b^*]) \cdot f(s|a^*) + \sum_{s>s^*} f(s|a^*) \quad (C.22)
\]

or

\[
(1 - \lambda) \left( (1 - \beta(s)[b^*]) \cdot f(s|a) + \sum_{s<s^*} f(s|s) \right) > (1 - \beta(s)[b^*]) \cdot f(s|a^*) + \sum_{s<s^*} f(s|a^*). \quad (C.23)
\]

If (C.22) is true, then (C.20) and (C.22) violate MLRP of \( f \). If (C.23) is true, then (C.21) and (C.23) violate MLRP of \( f \). Since the number of signal realizations is finite, there exists \( C > 0 \) such that \( ||\pi(\alpha, \beta) - \pi(a^* , \beta)|| > \frac{C}{2} \varepsilon \) whenever \( \pi(\alpha, \beta)[b^*] < 1 - \varepsilon \). The Pinsker’s inequality then implies that \( d\left(\pi(\alpha, \beta)\|\pi(a^*, \beta)\right) > C\varepsilon^2 \).

### C.4 Bounded Informativeness vs Full Support

I provide an example that explains why the full support condition in statement 2 of Theorem 4’ cannot be replaced by \( f \) being boundedly informative about \( a^* \). Players’ stage-game payoffs are given by:

| \( - \) | \( b^* \) | \( b' \) |
|---|---|---|
| \( \overline{a} \) | 1, 4 | -2, 0 |
| \( a^* \) | 2, 1 | -1, 0 |
| \( a \) | 3, -2 | 0, 0 |

Let \( S \equiv \{ s, s^*, \overline{s} \} \), with \( f(s|\overline{a}) = 2/3, f(s^*|\overline{a}) = 1/3, f(s|a^*) = 1/3, f(s^*|a^*) = 2/3, \) and \( f(s|a) = 1 \). One can verify that players’ stage-game payoffs are monotone-supermodular when player 1’s actions are ranked according to \( \overline{a} \succ a^* \succ a \), and player 2’s actions are ranked according to \( b^* \succ b' \). When signal realizations are ranked according to \( \overline{s} \succ s^* \succ s \), \( f \) satisfies MLRP. It is easy to show that player 1 can guarantee payoff 2 in every Bayes Nash Equilibrium. The reason is: if player 1 plays \( a^* \) in every period, player 2 observes signal \( s^* \) or \( \overline{s} \), and has a strict incentive to play \( b^* \). As a result, player 1’s payoff from playing \( a^* \) in every period is at least 2.

### References

[1] Abreu, Dilip, David Pearce and Ennio Stacchetti (1990) “Toward a Theory of Discounted Repeated Games with Imperfect Monitoring,” Econometrica, 58(5), 1041-1063.

[2] Acemoglu, Daron, Munther Dahleh, Ilan Lobel and Asu Ozdaglar (2011) “Bayesian Learning in Social Networks,” Review of Economic Studies, 78(6), 1201-1236.
[3] Adhvaryu, Achyuta (2014) “Learning, Misallocation, and Technology Adoption: Evidence from New Malaria Therapy in Tanzania,” Review of Economic Studies, 81(4), 1331-1365.

[4] Bai, Jie (2018) “Melons as Lemons: Asymmetric Information, Consumer Learning and Quality Provision,” Working Paper.

[5] Bai, Jie, Ludovica Gazze, and Yukun Wang (2019) “Collective Reputation in Trade: Evidence from the Chinese Dairy Industry,” Working Paper.

[6] Banerjee, Abhijit (1992) “A Simple Model of Herd Behavior,” Quarterly Journal of Economics, 107(3), 797-817.

[7] Banerjee, Abhijit and Drew Fudenberg (2004) “Word-of-mouth Learning,” Games and Economic Behavior, 46, 1-22.

[8] Bar-Isaac, Heski and Steven Tadelis (2008) “Seller Reputation,” Foundations and Trends in Microeconomics.

[9] Barro, Robert (1986) “Reputation in a Model of Monetary Policy with Incomplete Information,” Journal of Monetary Economics, 17(1), 3-20.

[10] Bikhchandani, Sushil, David Hirshleifer, and Ivo Welch (1992) “A Theory of Fads, Fashion, Custom, and Cultural Change as Information Cascades,” Journal of Political Economy, 100, 992-1026.

[11] Chan, Jimmy (2000) “On the Non-Existence of Reputation Effects in Two-Person Infinitely-Repeated Games,” Working Paper.

[12] Cripps, Martin, Eddie Dekel and Wolfgang Pesendorfer (2005) “Reputation with Equal Discounting in Repeated Games with Strictly Conflicting Interests,” Journal of Economic Theory, 121(2), 259-272.

[13] Cripps, Martin, George Mailath and Larry Samuelson (2004) “Imperfect Monitoring and Impermanent Reputations,” Econometrica, 18(2), 141-158.

[14] Cripps, Martin and Jonathan Thomas (1997) “Reputation and Perfection in Repeated Common Interest Games,” Games and Economic Behavior, 28, 433-462.

[15] Conley, Timothy and Christopher Udry (2010) “Learning about a New Technology: Pineapple in Ghana,” American Economic Review, 100(1), 35-69.

[16] Deb, Rahul, Matthew Mitchell, and Mallesh Pai (2020) “(Bad) Reputation in Relational Contracting,” Working Paper.

[17] Ely, Jeffrey and Junso Välimäki (2003) “Bad Reputation,” Quarterly Journal of Economics, 118(3), 785-814.

[18] Ely, Jeffrey, Johannes Hörner and Wojciech Olszewski (2005) “Belief-Free Equilibria in Repeated Games,” Econometrica, 73(2), 377-415.

[19] Ely, Jeffrey, Drew Fudenberg and David Levine (2008) “When is Reputation Bad?” Games and Economic Behavior, 63(2), 498-526.

[20] Ekmekci, Mehmet (2011) “Sustainable Reputations with Rating Systems,” Journal of Economic Theory, 146(2), 479-503.
REFERENCES

[21] Evans, Robert and Jonathan Thomas (1997) “Reputation and Experimentation in Repeated Games With Two Long-Run Players,” Econometrica, 65(5), 1153-1173.

[22] Fefferman, Charles and János Kollár (2013) “Continuous Solutions of Linear Equations,” Fourier Analysis and Number Theory to Radon Transforms and Geometry. Developments in Mathematics, vol 28. Springer.

[23] Fudenberg, Drew, David Kreps and Eric Maskin (1990) “Repeated Games with Long-Run and Short-Run Players,” Review of Economic Studies, 57(4), 555-573.

[24] Fudenberg, Drew and David Levine (1989) “Reputation and Equilibrium Selection in Games with a Patient Player,” Econometrica, 57(4), 759-778.

[25] Fudenberg, Drew and David Levine (1992) “Maintaining a Reputation when Strategies are Imperfectly Observed,” Review of Economic Studies, 59(3), 561-579.

[26] Gale, Douglas and Shachar Kariv (2003) “Bayesian Learning in Social Networks,” Games and Economic Behavior, 45(2), 329-346.

[27] Gossner, Olivier (2011) “Simple Bounds on the Value of a Reputation,” Econometrica, 79(5), 1627-1641.

[28] Harel, Matan, Elchanan Mossel, Philipp Strack and Omer Tamuz (2019) “Rational Groupthink,” Working Paper.

[29] Hörner, Johannes and Stefano Lovo (2009) “Belief-Free Equilibria in Games With Incomplete Information,” Econometrica, 77(2), 453-487.

[30] Hörner, Johannes, Stefano Lovo and Tristan Tomala (2011) “Belief-free Equilibria in Games with Incomplete Information: Characterization and Existence,” Journal of Economic Theory, 146(5), 1770-1795.

[31] Jackson, Matthew and Ehud Kalai (1997) “Social Learning in Recurring Games,” Games and Economic Behavior, 21, 102-134.

[32] Kaya, Ayça and Santanu Roy (2020) “Market Screening with Limited Records,” Working Paper.

[33] Levine, David (2019) “The Reputation Trap,” Working Paper.

[34] Liu, Qingmin (2011) “Information Acquisition and Reputation Dynamics,” Review of Economic Studies, 78(4), 1400-1425.

[35] Liu, Qingmin and Andrzej Skrzypacz (2014) “Limited Records and Reputation Bubbles,” Journal of Economic Theory 151, 2-29.

[36] Lobel, Ilan and Evan Sadler (2015) “Information Diffusion in Networks Through Social Learning,” Theoretical Economics, 10(3), 807-851.

[37] Logina, Ekaterina, Georgy Lukyanov and Konstantin Shamruk (2019) “Reputation and Social Learning,” Working Paper.

[38] Mailath, George and Larry Samuelson (2001) “Who Wants a Good Reputation?” Review of Economic Studies, 68(2), 415-441.

[39] Mossel, Elchanan, Allan Sly and Omer Tamuz (2015) “Social Learning Equilibrium,” Econometrica, 83(5), 1755-1794.
[40] Nyqvist, Martina Björkman, Jakob Svensson and David Yanagizawa-Drott (2018) “Can Competition Reduce Lemons? A Randomized Intervention in the Antimalarial Medicine Market in Uganda,” Working Paper.

[41] Pei, Harry (2020) “Reputation Effects under Interdependent Values,” *Econometrica*, forthcoming.

[42] Pęski, Marcin (2014) “Repeated Games with Incomplete Information and Discounting,” *Theoretical Economics*, 9, 651-694.

[43] Phelan, Christopher (2006) “Public Trust and Government Betrayal,” *Journal of Economic Theory*, 130(1), 27-43.

[44] Rosenberg, Dinah, and Nicolas Vieille (2019) “On the Efficiency of Social Learning,” *Econometrica*, 87(6), 2141-2168.

[45] Schmidt, Klaus (1993) “Reputation and Equilibrium Characterization in Repeated Games with Conflicting Interests,” *Econometrica*, 61(2), 325-351.

[46] Smith, Lones and Peter Norman Sørensen (2000) “Pathological Outcomes of Observational Learning,” *Econometrica*, 68(2), 371-398.

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