Light propagation in a birefringent plate with topological charge

Ebrahim Karimi, Bruno Piccirillo, Lorenzo Marrucci, and Enrico Santamato

1 Dipartimento di Scienze Fisiche, Università degli Studi di Napoli “Federico II”, Complesso di Monte S. Angelo, via Cintia, 80126 Napoli, Italy
2 Consiglio Nazionale delle Ricerche-INFM Coherentia, Napoli, Italy
*Corresponding author: enrico.santamato@na.infn.it

Compiled May 5, 2009

We calculated the Fresnel paraxial propagator in a birefringent plate having topological charge \( q \) at its center, named “q-plate”. We studied the change of the beam transverse profile when it traverses the plate. An analytical closed form of the beam profile propagating in the “q-plate” can be found for many important specific input beam profiles. We paid particular attention to the plate having a topological unit charge and we found that if small losses due to reflection, absorption and scattering are neglected, the plate can convert the photon spin into orbital angular momentum with up to 100% efficiency, provided the thickness of the plate is less than the Rayleigh range of the incident beam.

OCIS codes: 050.1960, 260.1960, 260.6042.

Light beams carrying orbital angular momentum (OAM) are receiving increasing attention as a resource in quantum and classical optics, since OAM exists in an inherently multidimensional space. Information can thus be encoded in higher dimensional OAM-alphabets [1, 2] for its use in free-space communication systems [3] or to increase the dimensionality of the working Hilbert space in quantum communications systems [4]. The main characteristics of a light beam carrying OAM is the presence of a topological charge and we found that if small losses due to reflection, absorption and scattering are neglected, the plate can convert the photon spin into orbital angular momentum with up to 100% efficiency, provided the thickness of the plate is less than the Rayleigh range of the incident beam.

In this approximation, the wave equation reduces to Helmholtz’s vector equation \( \nabla \times \mathbf{E} + \kappa_{\perp}^2 \mathbf{E} = 0 \) for the transverse part \( \mathbf{E}_\perp \) of the field. In view of the cylindrical symmetry of the problem, it is convenient to find the eigenmodes of the Helmholtz’s vector equation in the circular polarization basis \( \mathbf{E}_\pm = (\mathbf{E}_+ \pm i \mathbf{E}_-)/\sqrt{2} \) and in the cylindrical coordinates \( (r, \phi, z) \), by setting \( \mathbf{E}_\pm (r, \phi, z) = (\mathbf{E}_+ (r) e^{i(m+q)\phi}, \mathbf{E}_- (r) e^{i(m-q)\phi}, 0) e^{-i k_0 \gamma z + i \omega t} \), where \( \gamma \) is the longitudinal spatial frequency and \( z = 0 \) is the input-face of the QP. Inserting this field into Helmholtz’s equation, yields a pair of coupled radial equations

\[
\begin{align*}
 f''(r) + \frac{f'(r)}{r} + \left( k_0^2 (n_2^2 - \gamma^2) - \frac{\mu^2}{r^2} \right) f(r) &= \frac{\nu g(r)}{r^2} \\
 g''(r) + \frac{g'(r)}{r} + \left( k_0^2 (n_2^2 - \gamma^2) - \frac{\mu^2}{r^2} \right) g(r) &= \frac{\nu f(r)}{r^2}
\end{align*}
\]

where \( f(r) = (\mathbf{E}_+ + \mathbf{E}_-)/\sqrt{2}, g(r) = (\mathbf{E}_+ - \mathbf{E}_-)/\sqrt{2}, \mu = \sqrt{m^2 + q^2} \) and \( \nu = 2 m q \). Equations (1) are exact
The Fresnel kernels $K^o$ and $K^e$ in Eq.(4) are characterized by the presence of Bessel function of irrational order. Although, $K^o$ and $K^e$ cannot be obtained in a closed form, they permit to evaluate analytically the field transmitted by the QP in important cases as, for instance, for Laguerre-Gauss incident beams. Here we consider only the case of a LG$_0l$ beam impinging onto the QP. Setting $E_\perp(\rho, \phi, 0) = e^{i\delta} \text{LG}_l^1(\rho) \left[ \begin{array}{c} a \\ b \end{array} \right]$ in the circular polarization basis, where $\text{LG}_l^1(\rho)$ is the radial amplitude of Laguerre-Gauss modes, we obtain

$$\begin{bmatrix} E_+ \\ E_- \end{bmatrix} = e^{i(\phi - k_0 n_o z)} \begin{bmatrix} K^+ \mu^{2i\phi} \\ K^- e^{-2i\phi} \mu^{2i\phi} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

(5)

where $K^\pm = (\text{HyGG}_{l-\mu,\mu}(r, z/n_o) \pm e^{-ik_0 n_o z}\text{HyGG}_{l-\mu,\mu}(r, z/n_o))/2$, $\Delta n = n_e - n_o$, $\mu^\pm = \mu(\pm q)$ and $\text{HyGG}_{p,m}(r, z)$ is the Hypergeometric-Gaussian mode [8], viz.

$$\text{HyGG}_{p,m}(\rho, \zeta) = C_{pm} \text{ζ}(\zeta + i)\left((1 + |m| + \frac{1}{2})\rho^{|m|}\right) \times e^{-\zeta^2/2} F_1 \left(-\frac{p}{2}, 1 + |m|; \frac{\rho^2}{\zeta^2} \right)$$

(6)

where $C_{pm} = i^{|m|+1} \sqrt{\frac{2^{2|p|+|m|} \Gamma(|p|+|m|+\frac{3}{2})}{\Gamma(|1+p|+|m|+\frac{3}{2})}}$, $\rho = r/w_0$, $\zeta = z/z_R$ and $z_R = k_0 w_0^2/2$ is the beam Rayleigh range. Because $n_o \simeq n_e$, the arguments of the function $\text{HyGG}_{p,m}$ in Eq.(5) are very close, so that when $\Delta n z = j\lambda$ ($j = 1, 2, \ldots$) the matrix in Eq.(5) is almost diagonal, the beam in the QP has the same value of OAM, i.e. $\ell h$ per photon. When $\Delta n z = (2j - 1)\lambda/2$, instead, only the off-diagonal elements survive, the right and left circular components of transmitted field assume a phase factor $e^{\pm 2i\phi}$ and the beam OAM change by $\pm 2\hbar h$ per photon, depending on the input circular polarization helicity. As the beam propagates in the QP, its transverse profile, spin and OAM change. From Eq.(5), we may calculate the average SAM and OAM carried by the beam at the plane $z$ in the QP, obtaining

$$S_z(z) = \frac{1}{\omega} \mathbb{R} \left[ e^{-ik_0 n_o z} \left(|b|^2 I_{|\ell|+\mu^+,\mu^+}(z) - |a|^2 I_{|\ell|+\mu^-,\mu^-(z)} \right) \right]$$

(7)

$$L_z(z) + \frac{q}{\omega} S_z(z) = \frac{1}{\omega} \left((\ell - q)|a|^2 + (\ell + q)|b|^2 \right)$$

(8)

where

$$I_{p,m}(\zeta) = \frac{2^{p+|m|+1} \Gamma \left(\frac{p}{2} + |m| + 1\right) \Gamma \left(|m| + 1\right) \Gamma \left(p + |m| + 1\right) \chi^{-p/2}(\zeta)}{\Gamma \left(|p| + |m| + \frac{3}{2}\right)} \times \left(\frac{n_o n_e}{\sqrt{2}} \right)^{p+|m|+1} \times \left(\frac{n_o n_e - i(n_e - n_o)}{2}\right) \times F_1 \left(-\frac{p}{2}, \frac{p}{2}; |m| + 1; \chi(\zeta) \right)$$

(9)

and $\chi(\zeta) = \left(\frac{n_o n_e}{n_o n_e - i(n_e - n_o)} \right)^2$. As expected, we have $I_{p,m}(0) = 1$ so that Eqs.(8) and (7) yield $S_z(0) = (|b|^2 - |a|^2) - 1$ and $L_z(0) = \ell h$.
\[ |a|^2 \sim 1 / \omega \text{ and } L_z(0) = \frac{(|a|^2 + |b|^2) \ell}{\omega}. \]

In Fig. (1) the photon STOC [5] is shown as a function of the propagation depth in the 1-plate for LG_{00} and LG_{01} input beams. The conversion efficiency is practically 100% and its maximum occurs at optical retardation \[ \Delta n z = (2j - 1)\lambda/2 \]

with integer \( j \). When the optical retardation is \( j\lambda \), no conversion occurs and the beam has no OAM. Changing the optical retardation of the 1-plate provides a good way to control the STOC process. However, when the thickness of the 1-plate becomes very large (much larger than the beam Rayleigh range) the conversion efficiency slowly decays. According to Eqs. (5) and (6), the field intensity profile inside the 1-plate (and at its exit face) vanishes as \( r^\sqrt{7} \) along the beam axis so that the intensity profile has the characteristic doughnut shape irrespective of the OAM carried by the beam. Fig. 2 shows the intensity profiles for (a) full STOC (b) no STOC. For the sake of comparison, the results obtained in the GOA [7] are also shown. We can deduce that the GOA approximation is fairly good for the case of full STOC, but is very bad in the near field and in the case of no STOC. Dramatic changes of the intensity profile depending on the final OAM are seen, however, in the far-field after free-air propagation. When the STOC is maximum, in fact, we observe the doughnut profile, while when no conversion occurs, the far-field pattern has again a maximum at its center. This is shown in Fig. (3).

In conclusion, we calculated the Fresnel propagator of an optical beam in a birefringent plate having a topological charge \( q \) in the paraxial approximation and for normal incidence. We considered in the some details the propagation of a LG_{00} beam in the QP. As the beam traverses the plate, its transverse profile changes from Laguerre-Gaussian to Hypergeometric-Gaussian and STOC occurs. We paid particular attention to the 1-plate and we found that the conversion efficiency is almost 100% when the thickness of the plate is much smaller than the beam Rayleigh range and slowly decreases when the thickness of the 1-plate is increased. The free propagation of the HyGG modes generated by the QP is not stable and the characteristic transition from dot-to doughnut profile when the OAM changes from 0 is observed only in the far field. The possibility offered by the azimuthally oriented plate in manipulating entanglement among several degrees of freedom of the light may be of great interest for quantum information, quantum communications and quantum computing.

References

1. G. Molina-Terriza, J. P. Torres, and L. Torner, Phys. Rev. Lett. 88, 013601 (2002).
2. G. Molina-Terriza, J. P. Torres, and L. Torner, Nat. Phys. 3, 305 (2007).
3. G. Gibson, J. Courtial, M. J. Padgett, M. Vasnetsov, V. Pasko, S. M. Barnett, and S. Franke-Arnold, Opt. Express 12, 5448 (2004).
4. A. Mair, A. Vaziri, G. Welhs, and A. Zeilinger, Nature (London) 412, 313 (2001).
5. L. Marrucci, C. Manzo, and D. Paparo, Phys. Rev. Lett. 96, 163905 (2006).
6. L. Marrucci, C. Manzo, and D. Paparo, Appl. Phys. Lett. 88, 221102 (2006).
7. G. F. Calvo and A. Picón, Opt. Lett. 32, 838 (2007).
8. E. Karimi, G. Zito, B. Piccirillo, L. Marrucci, and E. Santamato, Opt. Lett. 32, 3053 (2007).