CONVERGENCE ORDER OF THE GEOMETRIC MEAN ERRORS FOR MARKOV-TYPE MEASURES

SANGUO ZHU

Abstract. We study the quantization problem with respect to the geometric mean error for Markov-type measures $\mu$ on a class of fractal sets. Assuming the irreducibility of the corresponding transition matrix $P$, we determine the exact convergence order of the geometric mean errors of $\mu$. In particular, we show that, the quantization dimension of order zero is independent of the initial probability vector when $P$ is irreducible, while this is not true if $P$ is reducible.

1. Introduction

In this paper, we study the asymptotic geometric mean errors in the quantization for Markov-type measures on a class of fractal sets. We refer to [5, 7] for mathematical foundations of quantization theory and [8] for its background in engineering technology.

For every $n \geq 1$, we set $\mathcal{D}_n := \{ \alpha \subset \mathbb{R}^q : 1 \leq \text{card}(\alpha) \leq n \}$. Let $\nu$ be a Borel probability measure on $\mathbb{R}^q$. The $n$th quantization error for $\nu$ of order $r$ is defined by (see [5, 7]):

$$ e_{n,r}(\nu) := \inf_{\alpha \in \mathcal{D}_n} \left( \int d(x,\alpha)^r d\nu(x) \right)^{\frac{1}{r}}, \quad r > 0, $$

$$ e_{n,0}(\nu) := \inf_{\alpha \in \mathcal{D}_n} \exp \left( \int \log d(x,\alpha) d\nu(x) \right), \quad r = 0. $$

Here $d(\cdot,\cdot)$ is the metric induced by an arbitrary norm on $\mathbb{R}^q$. For $r > 0$, $e_{n,r}(\nu)$ agrees with the error in the approximation of $\nu$ by discrete probability measures supported on at most $n$ points, in the sense of $L_r$-metrics [5].

By [7, Lemma 3.5], the quantity $e_{n,0}(\nu)$—also called the $n$th geometric mean error for $\nu$, equals the limit of $e_{n,r}(\nu)$ as $r$ tends to zero. In this sense, the quantization with respect to the geometric mean error is a limiting case of that in $L_r$-metrics. As one of the main aims of the quantization problem, we are concerned with the asymptotic properties of the quantization errors, including the upper (lower) quantization coefficient (of order $r$) and the upper (lower) quantization dimension (of order $r$).

For $s > 0$, we define the $s$-dimensional upper and lower quantization coefficient for $\nu$ of order $r \in [0, \infty)$ by (cf. [5, 14])

$$ \overline{Q}_r^s(\nu) := \limsup_{n \to \infty} n^s e_{n,r}(\nu), \quad \underline{Q}_r^s(\nu) := \liminf_{n \to \infty} n^s e_{n,r}(\nu). $$

By [5, 14], the upper (lower) quantization dimension $\overline{D}_r(\nu)$ ($\underline{D}_r(\nu)$) of order $r$, as defined below, is exactly the critical point at which the upper (lower) quantization coefficient $Q_r^s(\nu)$ first becomes infinite (vanishes).

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coefficient jumps from zero to infinity:
\[
\overline{D}_r(\nu) := \limsup_{n \to \infty} \frac{\log n}{-\log \epsilon_{n,r}(\nu)}, \quad \underline{D}_r(\nu) := \liminf_{n \to \infty} \frac{\log n}{-\log \epsilon_{n,r}(\nu)}.
\]
If \( \overline{D}_r(\nu) = \underline{D}_r(\nu) \), the common value is denoted by \( D_r(\nu) \) and called the quantization dimension for \( \nu \) of order \( r \).

Compared with the upper (lower) quantization dimension of order \( r \), the upper (lower) quantization coefficient of order \( r \) provides us with more accurate information on the asymptotics of the geometric mean errors; accordingly, much more effort is required to examine the finiteness and positivity of the latter.

**Remark 1.1.** The upper (lower) quantization dimension of \( \nu \) of order zero is closely connected with the upper (lower) local dimension (cf. [4]) as defined by
\[
\dim_{\text{loc}}\nu(x) := \liminf_{\epsilon \to 0} \frac{\log \nu(B_\epsilon(x))}{-\log \epsilon}, \quad \dim_{\text{loc}}\nu := \limsup_{\epsilon \to 0} \frac{\log \nu(B_\epsilon(x))}{-\log \epsilon}.
\]
Here \( B_\epsilon(x) \) denotes the closed ball of radius \( \epsilon \) which is centered at a point \( x \in \mathbb{R}^q \). As we showed in [17], if the upper and lower local dimension are both equal to \( s \) for \( \nu \)-a.e. \( x \), then \( D_0(\nu) \) exists and equals \( s \).

Next, let us recall a result of Graf and Luschgy. Let \( (f_i)_{i=1}^N \) be a family of contractive similitudes on \( \mathbb{R}^q \) with contraction ratios \( (s_i)_{i=1}^N \). By [10], there exists a unique non-empty compact set \( K \) satisfying
\[
K = f_1(K) \cup f_2(K) \cdots \cup f_N(K).
\]
This set is called the self-similar set associated with \( (f_i)_{i=1}^N \). Also, By [10], for a probability vector \( (q_i)_{i=1}^N \), there exists a unique Borel probability measure \( \nu \) satisfying \( \nu = \sum_{i=1}^N q_i \nu \circ f_i^{-1} \). This measure is called the self-similar measure associated with \( (f_i)_{i=1}^N \) and \( (q_i)_{i=1}^N \).

We say that \( (f_i)_{i=1}^N \) satisfies the open set condition (OSC), if there exists a non-empty bounded open set \( U \) such that \( f_i(U) \cap f_j(U) = \emptyset \) for all \( 1 \leq i \neq j \leq N \) and \( f_i(U) \subset U \) for all \( 1 \leq i \leq N \). Let \( k_r, r \geq 0 \), be given by
\[
k_0 := \sum_{i=1}^N q_i \log q_i; \quad \sum_{i=1}^N (q_i s_i^r) = 1, \quad r > 0.
\]
Assume that \( (f_i)_{i=1}^N \) satisfies the OSC. Let \( \nu \) be the self-similar measure associated with \( (f_i)_{i=1}^N \) and \( (q_i)_{i=1}^N \). Graf and Luschgy proved [6, 7] that
\[
D_r(\nu) = k_r; \quad 0 < Q_r^{k_r}(\nu) \leq Q_r^r(\nu) < \infty.
\]

In the present paper, we study the finiteness and positivity of the upper and lower quantization coefficient of order zero for Markov-type measures. Some results on the quantization for such measures in \( L_r \)-metrics (with \( r > 0 \)) are contained in [12]. Recently, a complete treatment in this direction is given in [11], where the corresponding transition matrix is allowed to be reducible. Next, let us recall some definitions; we refer to [1, 3, 13] for more details.

Let \( P = (p_{ij})_{N \times N} \) be a row-stochastic matrix, i.e., \( p_{ij} \geq 0, 1 \leq i, j \leq N \); and \( \sum_{j=1}^N p_{ij} = 1, 1 \leq i \leq N \). It is easy to see that, 1 is an eigenvalue of \( P \) of largest absolute value (cf. [9, Theorem 8.1.22]). When \( P \) is irreducible, by the Perron-Frobenius theorem (cf. [9, Theorem 8.4.4]), there exists a unique normalized
positive left (row) eigenvector \( v = (v_1, \ldots, v_N) \) of \( P \) with respect to the eigenvalue 1. We will need the following notations:

\[
\theta := \text{empty word}, \ G_0 := \{\theta\}, \ G_1 := \{1, \ldots, N\}; \\
G_k := \{\sigma \in G_1^k : p_{\sigma_1\sigma_2} \cdots p_{\sigma_k \sigma_1} > 0\}, \ k \geq 2; \\
G^* := \bigcup_{k \geq 0} G_k, \ G_\infty := \{\sigma \in G_1^\infty : p_{\sigma_1\sigma_2} \cdots p_{\sigma_k\sigma_1} > 0 \text{ for all } h \geq 1\}.
\]

We define \( |\sigma| := k \) for \( \sigma \in G_k \) and \( |\sigma| := \infty \) for \( \sigma \in G_1^\infty \). For every \( \sigma \in G^* \) with \( |\sigma| \geq k \) or \( \sigma \in G_\infty \), we write \( \sigma_k := (\sigma_1, \ldots, \sigma_k) \). For \( \sigma \in G^* \) and \( \omega \in G^* \cup G_\infty \) with \( (\sigma_{|\sigma|}, \omega_1) \in G_2 \), then we set

\[
\sigma \ast \omega = (\sigma_1, \sigma_2, \ldots, \sigma_{|\sigma|}, \omega_1, \ldots, \omega_{|\omega|}).
\]

Let \( J_i, i \in G_1 \), be non-empty compact subsets of \( \mathbb{R}^d \) with \( J_i = \overline{\text{int}(J_i)} \) for all \( i \in G_1 \), where \( \overline{B} \) and \( \text{int}(B) \) respectively denote the closure and interior in \( \mathbb{R}^d \) of a set \( B \subset \mathbb{R}^d \). We call these sets cylinders of order one. For each \( i \in G_1 \), let \( J_{ij}, (i, j) \in G_2 \), be non-overlapping subsets of \( J_i \) such that \( J_{ij} \) is geometrically similar to \( J_j \) and \( |J_{ij}|/|J_j| = c_{ij} \), where \( |A| \) denotes the diameter of a set \( A \) and \( c_{ij} \in (0, 1) \). We call these sets cylinders of order two. Assume that cylinders of order \( k \) are determined. Let \( J_{i+k}, \sigma \ast k+1 \in G_{k+1} \), be non-overlapping subsets of \( J_{\sigma} \) such that \( J_{i+k+1} \) is geometrically similar to \( J_{i+k} \). Hence, by induction, cylinders of order \( k \) are determined for all \( k \geq 1 \). Then, we get a Mauldin-Williams fractal set \( E \) (cf. [1, 13]):

\[
E := \bigcap_{k \geq 1} \bigcup_{\sigma \in G_k} J_\sigma.
\]

The set \( E \) need not be a self-similar set, and in general, \( E \) does not enjoy the nice invariance property as in (1.2). This will cause much difficulty in the study of the geometric mean error. For this reason, we assume the following separation property for \( E \): there exists some constant \( 0 < t < 1 \) such that for every \( \sigma \in G^* \) and \( j_l \) with \( (\sigma_{|\sigma|}, j_l) \in G_2, l = 1, 2, \)

\[
d(J_{\sigma \ast j_1}, J_{\sigma \ast j_2}) \geq t \max\{|J_{\sigma \ast j_1}|, |J_{\sigma \ast j_2}|\}.
\]

Let \( (q_i)_{i=1}^N \) be an arbitrary probability vector with \( q_i > 0 \) for all \( i \in G_1 \). By Kolmogorov consistency theorem, there exists a unique Markov-type measure \( \tilde{\mu} \) on \( G_\infty \) (cf. [15]) such that, for every \( k \geq 1 \) and \( \sigma = (\sigma_1 \ldots \sigma_k ) \in G_k \),

\[
\tilde{\mu}(\sigma) = q_{\sigma_1} p_{\sigma_1\sigma_2} \cdots p_{\sigma_{k-1}\sigma_k} \mu_\infty(\sigma_1), \quad \sigma_1, \ldots, \sigma_k \in G_k,
\]

where \( [\sigma] := \{ \sigma \ast \omega : \omega \in G_\infty, (\sigma_{|\sigma|}, \omega_1) \in G_2 \} \). With the assumption (1.3), we have the following bijection \( g : G_\infty \to E \):

\[
g(\sigma) := \bigcap_{k \geq 1} J_{\sigma_k}, \quad \sigma \in G_\infty.
\]

Let \( \mu := \tilde{\mu} \circ g^{-1} \). Then \( \mu \) is a Markov-type measure on \( E \) satisfying

\[
\mu(J_\sigma) = q_{\sigma_1} p_{\sigma_1\sigma_2} \cdots p_{\sigma_{k-1}\sigma_k} \mu_\infty(\sigma_1), \quad \sigma = (\sigma_1 \ldots \sigma_k ) \in G_k, \ k \geq 1.
\]

For each \( \sigma \in G^* \setminus \theta \), the way in which \( \mu \) distributes its measure among the subcylinders of \( J_\sigma \) depends on the last entry of \( \sigma \). This is different from that of the measures as considered in [18]. From now on, we assume

\[
\text{card}\{j \in G_1 : (i, j) \in G_2\} \geq 2 \text{ for all } i \in G_1.
\]
Under the assumption (1.5), for each \( \sigma \in G^* \), the cylinder \( J_\sigma \) has at least two sub-cylinders of order \(|\sigma| + 1\); in addition, we have that \( \max_{(i,j) \in G_2} p_{ij} < 1 \).

As the main result of the present paper, we will determine the exact convergence order of the geometric mean error for \( \mu \). That is,

**Theorem 1.2.** Assume that (1.3) and (1.5) are satisfied. Let \( \mu \) be as given in (1.4). Assume that the transition matrix \( P \) is irreducible. Then we have, \( D_0(\mu) = s_0 \) and \( 0 < Q^0_{s_0}(\mu) \leq \overline{Q}^0_{s_0}(\mu) < \infty \), where

\[
(1.6) \quad s_0 := \frac{\sum_{i=1}^N \sum_{j:(i,j) \in G_2} p_{ij} \log p_{ij}}{\sum_{i=1}^N \sum_{j:(i,j) \in G_2} p_{ij} \log c_{ij}},
\]

and \((v_i)_{i=1}^N\) is the normalized positive left eigenvector of \( P \) with respect to 1.

By Theorem 1.2, when the transition matrix \( P \) is irreducible, \( D_0(\mu) \) is independent of the initial probability vector \((q_i)_{i=1}^N\). In this case, according to [2, 3], \( D_0(\mu) \) coincides with the Hausdorff dimension of \( \mu \). At the end of the paper, we will give an example to show that, \((q_i)_{i=1}^N\) usually plays a role in the quantization with respect to the geometric mean error for \( \mu \) when the transition matrix is reducible.

2. A Characterization of the Geometric Mean Error

For every \( k \geq 2 \) and \( \sigma = (\sigma_1, \ldots, \sigma_k) \in G_k \), we write

\[
\sigma^\ominus := \sigma_{|\sigma|}^{-1} : \quad p_\sigma := p_{\sigma_1 \sigma_2} \cdots p_{\sigma_k \sigma_{k+1}}, \quad c_\sigma := c_{\sigma_1 \sigma_2} \cdots c_{\sigma_k \sigma_{k+1}}.
\]

If \(|\sigma| = 1\), we define \( p_\sigma = c_\sigma = 1 \) and \( \sigma^\ominus = \emptyset \). If \( \sigma, \omega \in G^* \) satisfy \(|\sigma| \leq |\omega|\) and \( \sigma = \omega|\sigma|\), then write \( \sigma < \omega \).

Set

\[
P := \min_{(i,j) \in G_2} p_{ij}, \quad \underline{c} := \min_{(i,j) \in G_2} c_{ij}, \quad \overline{p} := \max_{(i,j) \in G_2} p_{ij}, \quad \overline{c} := \max_{(i,j) \in G_2} c_{ij};
\]

(2.1) \[\Lambda_j := \{ \sigma \in G^* : p^\sigma_j \geq \overline{p}^j > p_{\sigma} \}, \quad \psi_j := \text{card}(\Lambda_j);\]

\[k_{ij} := \min_{\sigma \in \Lambda_j} |\sigma|, \quad k_{ij} := \max_{\sigma \in \Lambda_j} |\sigma|;\]

\[\overline{P}^j_0(\mu) := \liminf_{j \to \infty} \psi_j^+ e_{\psi_j}(\mu), \quad \overline{P}^j_0(\mu) := \limsup_{j \to \infty} \psi_j^+ e_{\psi_j}(\mu), \quad s > 0.\]

Without loss of generality, we assume that \(|J_i| = 1\) for all \( i \in G_1 \). Thus

\[|J_\sigma| = c_\sigma, \quad \sigma \in G_k, \quad k \geq 1.\]

**Lemma 2.1.** (i) There exist constants \( A_1, A_2 > 0 \) such that

\[A_1 j \leq k_{ij} \leq A_2 j.\]

(ii) \( Q^0_0(\mu) > 0 \) iff \( \overline{P}^j_0(\mu) > 0 \) and \( Q^0_0(\mu) < \infty \) iff \( \overline{P}^j_0(\mu) < \infty \).

**Proof.** Let \( N_1 := \min \{ h \geq 1 : \overline{p}^h < \overline{p} \} \). Let \( \sigma^{(l)} \in G_{k_{ij}} \cap \Lambda_j, l = 1, 2 \). Then

\[\overline{p}^{k_{ij}} \leq p_{\sigma^{(i)}} < \overline{p}^j, \quad \overline{p}^{k_{ij}} \geq p_{\sigma^{(2)}} \geq \overline{p}^{j+1} \geq \overline{p}^{N_1(j+1)}.\]

It follows that \( j \leq k_{ij} \leq k_{ij} \leq N_1(j+1) + 1 \leq 3N_1 j \), for all \( j \geq 1 \). Hence (i) follows by setting \( A_1 := 1 \) and \( A_2 := 3N_1 \). As in [18], to see (ii), it suffices to show that for some constant \( N_2 > 0 \) such that \( \psi_j \leq \psi_{j+1} \leq N_2 \psi_j \). In fact, the first inequality is clear; to see the second, we note that, for every \( \sigma \in \Lambda_j \) and every \( \omega \in G_{N_1+1} \) with \((\sigma|\sigma|, \omega_1) \in G_2 \), we have \( p_{\sigma \omega} < \overline{p}^j \overline{P}^{N_1} < \overline{p}^{j+1} \). This implies that \( \psi_{j+1} \leq N^{N_1+1} \psi_j \). The lemma follows.
2.1. Push-forward and pull-back measures. For each $\sigma \in G^* \setminus \{\theta\}$, we take an arbitrary contracting similitude $f_\sigma$ on $\mathbb{R}^q$ of contraction ratio $c_\sigma$ and define $\nu_\sigma := \mu(\cdot | J_\sigma) \circ f_\sigma$. Then, since $f_\sigma$ is a Borel bijection, $\nu_\sigma$ is a measure supported on $K(\sigma) = f_\sigma^{-1}(J_\sigma)$ satisfying

\[ \mu(\cdot | J_\sigma) = \nu_\sigma \circ f_\sigma^{-1}, \quad \sigma \in G^* \setminus \{\theta\}. \]

For a finite set $\alpha \subset \mathbb{R}^q$ of cardinality $L$, by (2.2), we have

\[ \int_{J_\alpha} \log d(x, \alpha) d\mu(x) = \mu(J_\sigma) \int \log d(x, \alpha) d\nu_\sigma \circ f_\sigma^{-1}(x) \]

(2.3)

\[ = \mu(J_\sigma) \int \log d(f_\sigma(x), \alpha) d\nu_\sigma(x) \geq \mu(J_\sigma)(\log c_\sigma + \hat{e}_L(\nu_\sigma)). \]

**Lemma 2.2.** There exist constants $A_3, A_4 > 0$, such that

\[ \sup_{\sigma \in G^* \setminus \{\theta\}} \sup_{x \in \mathbb{R}^q} \nu_\sigma(B(x, \epsilon)) \leq A_3 \epsilon A_4 \text{ for all } \epsilon > 0. \]

**Proof.** Let $\sigma \in G^* \setminus \{\theta\}, x \in K(\sigma)$. By (1.3), there exists a unique word $\tau_\sigma \in G_\infty$ such that $\sigma \prec \tau_\sigma$ and $\bigcap_{k \geq 1} J_{\tau_\sigma i k} = \{x\}$. For every $\epsilon \in (0, \varnothing)$, we set

\[ C(\sigma) := \{\tau \in G^* : \sigma \prec \tau, c_\sigma^{-1} c_\tau - \epsilon > c_\sigma^{-1} c_\tau\}. \]

(2.5)

For each $i \in G_1$, there exists some $t_i \in (0, 1)$ such that $J_i$ contains a ball of radius $t_i |J_i| = t_i$ and is contained in a closed ball of radius 1. Set $\delta := \min_{1 \leq i \leq N} t_i$. Then, for each $\tau \in C(\sigma)$, $f_\sigma^{-1}(J_\tau)$ is contained in a ball of radius $\epsilon$ and contains a ball of radius $\delta \epsilon$. By (1.3), $J_\tau \cap C(\sigma)$ are pairwise disjoint, so are the sets $f_\sigma^{-1}(J_\tau), \tau \in C(\sigma)$ by the similarity of $f_\sigma$. Thus, by [10], there is a constant $M$ which is independent of $\epsilon$ such that

\[ \text{card}(\{\tau \in C(\sigma) : B(x, \epsilon) \cap f_\sigma^{-1}(J_\tau) \neq \emptyset\}) \leq M. \]

By (2.5), $\epsilon |\tau| - |\sigma| < \epsilon$ for $\tau \in C(\sigma)$, which implies $|\tau| - |\sigma| \geq \log \epsilon / \log \varnothing$. So,

\[ \nu_\sigma(B(x, \epsilon)) \leq M \epsilon^{\log \varnothing / \log \varnothing} = M \epsilon^{\log \varnothing / \log \varnothing}. \]

Let $A_4 := \log \varnothing / \log \varnothing$. Then by [5, Lemma 12.3], there is a constant $A_3 > 0$, independent of $\sigma$, such that $\nu_\sigma(B(x, \epsilon)) \leq A_3 \epsilon A_4$ for all $x \in \mathbb{R}^q$. Note that the above arguments holds true for any $\sigma \in G^* \setminus \{\theta\}$. The lemma follows. \qed

If the infimum in (1.1) is attained at some $\alpha$ with $1 \leq \text{card}(\alpha) \leq n$, we call $\alpha$ an $n$-optimal set for $\nu$ of order $r$. The collection of all $n$-optimal sets for $\nu$ of order $r$ is denoted by $C_{n,r}(\nu)$. We simply write $C_n(\nu)$ for $C_{n,0}(\nu)$. Note that $\nu_\sigma, \sigma \in G^*$, are compactly supported. By Lemma 2.4 and [5, Theorem 2.5], we conclude that $C_n(\nu_\sigma)$ is non-empty for every $\sigma \in G^* \setminus \{\theta\}$ and $n \geq 1$. Using similar arguments, one can show that $C_n(\mu)$ is non-empty for every $n \geq 1$.

**Lemma 2.3.** (see [7]) Let $\nu$ be a Borel probability measure on $\mathbb{R}^q$ with compact support $K$. Let $\hat{e}_n(\nu) := \log \hat{e}_{n,\sigma}(\nu)$. Assume that for some constants $d_1, d_2 > 0$ we have, $\sup_{x \in \mathbb{R}^q} \nu(B(x, \epsilon)) \leq d_1 \epsilon^{d_2}$. Then, we have

\[ \hat{e}_n(\nu) - \hat{e}_{n+1}(\nu) \leq (n + 1)^{-1} \log(3|K|) + d_1 \epsilon d_2 (n + 1)^{-1/p}, \quad n \geq 1. \]

where $p, q$ are real numbers satisfying $p > 1, p^{-1} + q^{-1} = 1$. 
Let $q := \min_{i \in G} q_i$ and $\overline{q} := \max_{i \in G} q_i$. As a consequence of (2.4) and Lemma 2.3, for given integers $k_1, k_2, k_3 \geq 1$, there exists an integer $A_3$ such that, for all $n \geq A_3$, we have

\[
\sup_{\sigma \in G^*} (\hat{\epsilon}_{n-k_1-k_3}(\nu_\sigma) - \hat{\epsilon}_{n+k_2}(\nu_\sigma)) < \frac{q-1}{2} p \log 2.
\]

**Remark 2.4.** Using (2.4) and the proof of Theorem 3.4 of [7], it is convenient to see, for every $k \geq 1$, there is a $B_k \in \mathbb{R}$ such that $\inf_{\sigma \in G^*} \hat{\epsilon}_k(\nu_\sigma) \geq B_k$.

2.2. **An estimate of the geometric mean error.** For $\epsilon > 0$, let $(A)_\epsilon$ denote the closed $\epsilon$-neighborhood in $\mathbb{R}^q$ of a set $A \subset \mathbb{R}^q$. Let $t$ be the same as in (1.3). For a finite subset $\alpha$ of $\mathbb{R}^q$ and $\sigma \in G^*$, we write $\alpha_\sigma := \alpha \cap (J_\sigma)_{t-1+t_\sigma}$ and

\[
L_\sigma := \text{card}(\alpha_\sigma), I_\sigma(\alpha) := \int_{J_\sigma} \log d(x, \alpha) d\mu(x).
\]

By (1.4), we have, $\mu(J_\sigma) = q_\sigma p_\sigma$ for every $\sigma \in G^* \setminus \{\emptyset\}$. We set

\[
J_\emptyset := E; \ m_\sigma := q_\sigma p_\sigma, \ \sigma \in G^* \setminus \{\emptyset\}.
\]

**Lemma 2.5.** There exists a constant $L_1$ which is independent of $j$ such that

\[
\sup_{\alpha \in C_{\psi_j}(\mu)} \max_{\sigma \in \Lambda_j} L_\sigma \leq L_1
\]

**Proof.** Since all $\nu_\sigma, \sigma \in G^*$, share the properties in (2.6) and (2.4), it suffices to follow the induction in [16, Proposition 3.4] by using (2.2). \hfill \Box

**Remark 2.6.** For the reader’s convenience, let us explain the main idea of the induction in [16] by following 3.4 by using (2.2).

Let $H_1$ be the smallest integer such that $(J_\sigma)_{t-1+t_\sigma}$ can be covered by $H_1$ closed balls of radii $8^{-1} t_\sigma$, which are centered in $(J_\sigma)_{t-1+t_\sigma}$. Let $\gamma_1(\sigma)$ the centers of such $H_1$ closed balls. Let $H_2$ be the smallest integer such that $(J_\sigma)_{t-1+t_\sigma}$ can be covered by $H_2$ closed balls of radii $8^{-1} t_\sigma$. Let us denote by $\gamma_2(\sigma)$ the centers of such $H_2$ closed balls. Then by (2.2), we have

\[
I_\sigma(\alpha) \geq I_\sigma(\alpha \cup \gamma_2(\sigma)) \geq m_\sigma (\log c_\sigma + \hat{\epsilon}_{L_\alpha+H_2}(\nu_\sigma)).
\]

Let $H_3$ be the smallest integer such that $J_\tau$ can be covered by $H_3$ closed balls of radii $8^{-1} t_\tau$, which are centered in $J_\tau$ and we denote by $\gamma_3(\tau)$ the centers of such $H_3$ closed balls. Let $L_0$ be the smallest integer such that (2.4) holds with $k_i = H_i, i = 1, 2, 3$. Set $L_1 := L_0 + H_1 + H_3$.

Suppose that $L_\sigma = \text{card}(\alpha_\sigma) > L_1$. By (1.3), $\alpha_\sigma \cap \alpha_\omega = \emptyset$ for distinct words $\sigma, \omega \in \Lambda_j$. So, there is a $\tau \in \Lambda_j$ with $L_\tau = 0$. Let $\gamma_4(\sigma) \in C_{L_\alpha-H_1-H_3}(\nu_\sigma)$. Set

\[
\beta := (\alpha \setminus \alpha_\sigma) \cup \gamma_1(\sigma) \cup \gamma_3(\tau) \cup \gamma_4(\sigma).
\]

Then $\text{card}(\alpha) \geq \text{card}(\beta)$. The set $\gamma_1(\sigma)$ ensures that $J_\omega, \omega \in \Lambda_j \setminus \{\sigma, \tau\}$, are not affected unfavorably while we try to adjust the “optimal points” between $\alpha_\sigma$ and $\alpha_\tau$. In fact, by triangle inequality, we have, $d(x, \alpha_\sigma) \geq d(x, \gamma_1(\sigma))$ for $x \in J_\omega, \omega \in \Lambda_j \setminus \{\sigma, \tau\}$. It follows that

\[
I_\omega(\alpha) > I_\omega(\beta) \text{ for all } \omega \in \Lambda_j \setminus \{\sigma, \tau\}.
\]

Thus, it suffices to estimate the following differences separately:

\[
\Delta_1 := I_\sigma(\beta) - I_\sigma(\alpha); \ \Delta_2 := I_\tau(\alpha) - I_\tau(\beta).
\]
Using (2.2), (2.6) and (2.8), it is easy to show that \( \Delta_1 < \Delta_2 \). This, together with (2.9), implies that \( I_\theta(\alpha) > I_\theta(\beta) \), contradicting the optimality of \( \alpha \).

**Lemma 2.7.** There exists a constant \( C_0 \) such that for all large \( j \in \mathbb{N} \),

\[
\sum_{\tau \in \Lambda_j} m_\tau \log c_\tau + C_0 \leq \hat{\epsilon}_{\psi_j}(\mu) \leq \sum_{\tau \in \Lambda_j} m_\tau \log c_\tau.
\]

**Proof.** Let \( \alpha \in C_{\psi_j}(\mu) \) and let \( \gamma_3(\tau) \) be as defined in Remark 2.6. By (2.7), \( \text{card}(\alpha_\tau \cup \gamma_3(\tau)) \leq \tilde{L}_1 + H_3 \) for every \( \tau \in \Lambda_j \). One can see that,

\[ d(x,\alpha) \geq d(x,\alpha_\tau \cup \gamma_3(\tau)) \text{ for all } x \in J_\tau. \]

Set \( C_0 := B_{L_1 + H_3} \). By (2.3) and Remark 2.4,

\[ \hat{\epsilon}_{\psi_j}(\mu) \geq \sum_{\tau \in \Lambda_j} I_\tau(\alpha_\tau \cup \gamma_3(\tau)) \geq \sum_{\tau \in \Lambda_j} m_\tau \log c_\tau + C_0. \]

For each \( \tau \in \Lambda_j \), let \( b_\tau \) be an arbitrary point in \( J_\tau \) and set \( \gamma := \{b_\tau\}_{\tau \in \Lambda_j} \). Then we have, \( \hat{\epsilon}_{\psi_j}(\mu) \leq I_\theta(\gamma) \leq \sum_{\tau \in \Lambda_j} m_\tau \log c_\tau \). The lemma follows. \( \square \)

For \( i \in G_1 \), let \( \mu_i \) denote the conditional probability measure \( \mu(\cdot | J_i) \), namely, for every Borel set \( B \subset \mathbb{R}^n, \mu_i(B) = \mu(B \cap J_i)/\mu(J_i) \). We define

\[
G_k(i) := \{ \sigma \in G_k : \sigma_1 = i \}, \quad k \geq 1; \quad G^*(i) := \bigcup_{k \geq 1} G_k(i);
\]

\[
\Lambda_k(i) := \{ \sigma \in G^*(i) : p_{\sigma} \geq p^k \} = \psi_k(i), \quad \psi_k(i) := \text{card}(\Lambda_k(i)), \quad k \geq 1.
\]

For \( \sigma \in G^*(i) \), we have, \( \mu_i(J_\sigma) = p_\sigma \). As we did for \( \mu \), one can show that, there exists a constant \( C_0(\iota) \) such that for all large \( k \in \mathbb{N} \),

\[
(2.10) \quad \sum_{\sigma \in \Lambda_k(i)} p_\sigma \log c_\sigma + C_0(i) \leq \hat{\epsilon}_{\psi_k(i)}(\mu) \leq \sum_{\sigma \in \Lambda_k(i)} p_\sigma \log c_\sigma.
\]

### 3. Proof of Theorem 1.2

For the proof of Theorem 1.2, we need to establish several lemmas. For \( k, n \geq 1 \), and \( \sigma \in G_k \), let \( \Gamma(\sigma, n) := \{ \omega \in G_{k+n} : \sigma \prec \omega \}; \) we define

\[
\xi(i, n) := \sum_{\tau \in \Gamma(i, n)} p_\tau \log p_\tau; \quad \lambda(i, n) := \sum_{\tau \in \Gamma(i, n)} p_\tau \log c_\tau, \quad i \in G_1;
\]

\[
\Delta_n(i, j) := |\xi(i, n) - \xi(j, n)|, \quad \bar{\Delta}_n(i, j) := |\lambda(i, n) - \lambda(j, n)|, \quad i, j \in G_1.
\]

For \( 1 \leq i, j \leq N \) with \( (i, j) \notin G_2 \), we have, \( p_{ij} = c_{ij} = 0 \). In the following, we take the convention that \( 0 \log 0 := 0 \), so that we may take the sums from \( i = 1 \) to \( N \), instead of considering words in \( G^* \). We always denote by \( v = (v_i)_{i=1}^N \) the normalized positive left (row) eigenvector of \( P \) with respect to the Perron-Frobenius eigenvalue 1, when \( P \) is irreducible. Let \( u_0 \) and \( l_0 \) denote the numerator and denominator in the definition of \( s_0 \) (see (1.6)).

To study the asymptotics of the geometric mean errors, we will naturally need the following estimate which reflects some hereditary information of \( \mu \). One may see [18, (2.11)] for a comparison.

**Lemma 3.1.** Assume that \( P \) is irreducible. There exists a \( C_1 > 0 \) such that

\[
(3.1) \quad \sup_{n \geq 1} \max_{i,j \in G_1} \max \{ \Delta_n(i, j), \bar{\Delta}_n(i, j) \} \leq C_1.
\]
Proof. For $h \geq 1$ and $l, p \in G_1$, let $b_{lp}^{(h)}$ denote the $(l, p)$-entry of $P^h$. For $h \geq 3$ and $l \in G_1$, we have, $\sum_{\tau \in G_{h-2}\{i\}} p_{\tau+1} = b_{il}^{(h-2)}$ (cf. [13, (30)]). In addition, we have, $G_h(i) = \Gamma(i, h - 1)$. One can see

$$\xi(i, h - 1) = \sum_{\omega \in G_{h-1}\{i\}} \sum_{j=1}^N p_{\omega \omega_{h-1,j}} \log p_{\omega} + \sum_{\tau \in G_{h-2}\{i\}} \sum_{l=1}^N \sum_{j=1}^N p_{\tau+1} p_{ij} \log p_{ij}$$

$$= \xi(i, h - 2) + \sum_{l=1}^N \sum_{j=1}^N \xi_{ij}(l, p) \log p_{ij}, h \geq 3.$$

Write $d_h(i) := \sum_{l=1}^N \sum_{j=1}^N b_{il}^{(h-2)} p_{ij} \log p_{ij}, h \geq 3$. By induction, we have

$$\xi(i, k - 1) = \xi(i, 1) + \sum_{h=3}^k d_h(i), \ k \geq 3.$$

Set $w_{k,i} := \xi(i, 1) + (k-2)u_0$. Then we have

$$\xi(i, k - 1) = w_{k,i} + \sum_{l=1}^N \sum_{j=1}^N p_{ij} \log p_{ij} \sum_{h=3}^k b_{il}^{(h-2)} - v_l.$$ 

Similarly, set $z_{k,i} := \lambda(i, 1) + (k - 2)l_0$. Then, for $k \geq 3$, we have

$$\lambda(i, k - 1) = z_{k,i} + \sum_{l=1}^N \sum_{j=1}^N p_{ij} \log c_{ij} \sum_{h=3}^k b_{il}^{(h-2)} - v_l.$$ 

Let $u = (\chi_i)_{i=1}^N$ be the column vector with $\chi_i = 1$ for all $1 \leq i \leq N$. Then $u$ is a right eigenvector of $P$ with respect to 1 and $\sum_{i=1}^N \chi_i v_i = 1$. We have

$$L := uv =: (l_{ij})_{N \times N}, \ l_{ij} = v_j, 1 \leq i, j \leq N.$$ 

Applying [9, Theorem 8.6.1] with the above matrix $L$, there exists a constant $C(P)$ such that for all $k \geq 3$,

$$\frac{1}{k-2} \left| \sum_{h=3}^k \left( b_{il}^{(h-2)} - v_l \right) \right| = \frac{1}{k-2} \sum_{h=3}^k b_{il}^{(h-2)} - v_l \leq \frac{C(P)}{k-2}, p, l \in G_1.$$ 

This, together with (3.2), yields

$$\frac{1}{k-2} |\xi(i, k - 1) - w_{k,i}| \leq \frac{C(P)}{k-2} \sum_{l=1}^N \sum_{j=1}^N |p_{ij} \log p_{ij}| =: \frac{\delta_0}{k-2}, k \geq 3.$$ 

Hence, $|\xi(i, k - 1) - w_{k,i}| \leq \delta_0$ for all $k \geq 3$. Note that, the above argument is true for all $i \in G_1$. Set $\delta_1 := \max_{i,j \in G_1} |\xi(i, 1) - \xi(j, 1)|$. Then for $n = k - 1$,

$$\Delta_n(i, j) := |\xi(i, n) - w_{k,i}| + |w_{k,i} - w_{k,j}| + |\xi(j, n) - w_{k,j}| \leq 2\delta_0 + \delta_1 =: \delta_2.$$ 

Analogously, for some constant $\delta_3 > 0$, we have, $\Delta_n(i, j) \leq \delta_3$ for $i, j \in G_1$ and $h \geq 1$. Thus, the lemma follows by setting $C_1 := \max\{\delta_2, \delta_3\}$. \hfill \Box

The following two number sequences $(t_k)_{k=1}^\infty$ and $(s_k)_{k=1}^\infty$ are closely connected with the asymptotic geometric mean errors:

$$t_k := \frac{\sum_{\sigma \in \Lambda_1} m_\sigma \log m_\sigma}{\sum_{\sigma \in \Lambda_k} m_\sigma \log c_\sigma}; \ s_k := \frac{\sum_{\sigma \in \Lambda_1} m_\sigma \log m_\sigma}{\sum_{\sigma \in \Lambda_k} m_\sigma \log c_\sigma}, k \geq 1.$$
Let \( u_k, l_k \) denote the numerator and denominator in the definition of \( s_k \). Then

\[
\begin{align*}
(3.6) \quad u_1 &= \sum_{i=1}^{N} q_i \log q_i, \quad u_2 = \sum_{i=1}^{N} \sum_{j=1}^{N} q_i p_{ij} \log(q_p p_{ij}) = u_1 + \sum_{i=1}^{N} q_i \xi(i,1). 
\end{align*}
\]

**Lemma 3.2.** Assume that \( P \) is irreducible. There exists a constant \( C_2 \) such that for all large \( k \in \mathbb{N} \), we have \( |s_k - s_0| \leq C_2 k^{-1} \).

**Proof.** Let \( w_{k,i}, z_{k,i} \) be as defined in the proof of Lemma 3.1. We write

\[
x_k := \sum_{i=1}^{N} q_i (\xi(i,k-1) - w_{k,i}), \quad y_k := \sum_{i=1}^{N} q_i (\lambda(i,k-1) - z_{k,i}), \quad k \geq 3.
\]

For \( k \geq 3 \), by (3.2), (3.6) and the definitions of \( u_k \) and \( \xi(i,k-1) \), we deduce

\[
\begin{align*}
u_k &= \sum_{i=1}^{N} \sum_{\omega \in G_k(i)} q_i p_{i\omega} \log q_i + \sum_{i=1}^{N} \sum_{\omega \in G_k(i)} q_i p_{i\omega} \log p_{i\omega} \\
&= u_1 + \sum_{i=1}^{N} q_i \xi(i, k-1) \\
&= u_1 + \sum_{i=1}^{N} q_i (\xi(i,1) + (k-2)u_0 + \xi(i, k-1) - w_{k,i}) \\
&= u_2 + (k-2)u_0 + x_k.
\end{align*}
\]

Note that \( l_1 = 0 \). By replacing \( \log p_{i\omega} \) in (3.7) with \( \log c_{ij} \), we have

\[
l_k = l_2 + (k-2)l_0 + y_k.
\]

By (3.4), we have \( |\xi(i,k-1) - w_{k,i}| \leq \delta_0 \) for all \( i \in G_1 \). Thus,

\[
\begin{align*}
(3.9) \quad |x_k| &\leq \sum_{i=1}^{N} q_i |\xi(i,k-1) - w_{k,i}| \leq \delta_0 \quad \text{and} \quad |y_k| \leq \delta_3.
\end{align*}
\]

Set \( s_2 := u_2 + l_2 \) and \( A_6 := |s_2(l_0 + u_0)| \). By (3.7)-(3.9), for large \( k \), we deduce

\[
\begin{align*}
|s_k - s_0| &= \frac{|u_k + (k-2)u_0 + x_k - u_0|}{l_2 + (k-2)l_0 + y_k - l_0} \\
&\leq \frac{A_6 + |x_k l_0 - y_k u_0|}{|l_0(l_2 + (k-2)l_0 + y_k)|} \\
&\leq \frac{2A_6 + 2|\delta_0 + \delta_3||u_0| + |l_0|}{(k-2)l_0^2} \\
&\leq \frac{4A_6 + 4|\delta_0 + \delta_3||u_0| + |l_0|}{kl_0^2} =: C_2 k^{-1}.
\end{align*}
\]

This completes the proof of the lemma. \( \square \)

**Remark 3.3.** If \( (q_i)_{i=1}^{N} \) agrees with \( v \), then \( \sum_{i=1}^{N} q_i b_{ij}^{(k-2)} = v_1 \) since \( vP^k = v \) for all \( k \geq 1 \). Hence, \( x_k = 0 \) and (3.7) becomes: \( u_k = u_1 + (k-2)u_0 \). This was calculated in [15, Theorem 4.27].

Next we establish a connection between \((t_j)_{j=1}^{\infty}\) and \((s_k)_{k=1}^{\infty}\). We have
Lemma 3.4. Assume that $P$ is irreducible. There exist a constant $C_3$ and two integers $k_j^{(1)} \in [k_1, k_2]$, $i = 1, 2$ such that

$$s_{k_j^{(1)}} - C_3 j^{-1} \leq t_j \leq s_{k_j^{(2)}} + C_3 j^{-1}. $$

Proof. For $k \geq 1$ and $\sigma \in G_k$, we have

$$\sum_{\tau \in \Gamma(\sigma, k_2, -|\sigma|)} m_\tau \log m_\tau = \sum_{\tau \in \Gamma(\sigma, k_2, -|\sigma|)} m_\tau (\log m_\sigma + \log \frac{m_\tau}{m_\sigma})$$

$$= m_\sigma \log m_\sigma + m_\sigma \sum_{\tau \in \Gamma(\sigma, k_2, -|\sigma|)} \frac{m_\tau}{m_\sigma} \log \frac{m_\tau}{m_\sigma}$$

$$= m_\sigma \log m_\sigma + m_\sigma \xi(\sigma, k_2 - |\sigma|).$$

(3.10)

By (3.10) and Lemma 3.1, for every $k \in [k_1, k_2]$ and $\omega \in G_k$, we have

$$u_{k_2} - u_k = \sum_{\sigma \in G_k} \sum_{\tau \in \Gamma(\sigma, k_2, -k)} m_\tau \log m_\tau - u_k = \sum_{\sigma \in G_k} m_\sigma \xi(\sigma, k_2 - k)$$

(3.11)

Similarly, we have that $u_{k_2} - u_k \geq \xi(\omega, k_2 - k) - C_1$. We define

$$\zeta(\sigma) := m_\sigma (\log m_\sigma - u_{|\sigma|}), \quad \sigma \in G^* \setminus \{\emptyset\}.$$

Then, by (3.10) and (3.11), we deduce

$$\sum_{\tau \in \Gamma(\sigma, k_2, -|\sigma|)} \zeta(\tau) = \sum_{\tau \in \Gamma(\sigma, k_2, -|\sigma|)} m_\tau (\log m_\tau - u_{k_2})$$

$$= m_\sigma \log m_\sigma + m_\sigma \xi(\sigma, k_2 - |\sigma|) - m_\sigma u_{k_2}$$

$$\leq m_\sigma \log m_\sigma + m_\sigma (u_{k_2} - u_{|\sigma|} + C_1) - m_\sigma u_{k_2} = \zeta(\sigma) + m_\sigma C_1.$$

$$\geq m_\sigma \log m_\sigma + m_\sigma (u_{k_2} - u_{|\sigma|} - C_1) - m_\sigma u_{k_2} = \zeta(\sigma) - m_\sigma C_1.$$

This is equivalent to

$$\sum_{\tau \in \Gamma(\sigma, k_2, -|\sigma|)} \zeta(\tau) - m_\sigma C_1 \leq \zeta(\sigma) \leq \sum_{\tau \in \Gamma(\sigma, k_2, -|\sigma|)} \zeta(\tau) + m_\sigma C_1.$$

Note that $\sum_{\tau \in G_{k_2}} \zeta(\tau) = 0$. We further deduce

$$\sum_{\sigma \in \Lambda_j} m_\sigma (\log m_\sigma - u_{|\sigma|}) = \sum_{\sigma \in \Lambda_j} \zeta(\sigma)$$

$$\leq \sum_{\sigma \in \Lambda_j} (\sum_{\tau \in \Gamma(\sigma, k_2, -|\sigma|)} \zeta(\tau) + m_\sigma C_1) = C_1.$$

$$\geq \sum_{\sigma \in \Lambda_j} (\sum_{\tau \in \Gamma(\sigma, k_2, -|\sigma|)} \zeta(\tau) - m_\sigma C_1) = -C_1.$$

As an immediate consequence, we have

(3.12) $\sum_{\sigma \in \Lambda_j} m_\sigma u_{|\sigma|} - C_1 \leq \sum_{\sigma \in \Lambda_j} m_\sigma \log m_\sigma \leq \sum_{\sigma \in \Lambda_j} m_\sigma u_{|\sigma|} + C_1.$

Similarly, one can show that

(3.13) $\sum_{\sigma \in \Lambda_j} m_\sigma l_{|\sigma|} - C_1 \leq \sum_{\sigma \in \Lambda_j} m_\sigma \log c_\sigma \leq \sum_{\sigma \in \Lambda_j} m_\sigma l_{|\sigma|} + C_1.$
Thus, for large $j$, by (3.12) and (3.13), we have
\begin{equation}
\frac{\sum_{\sigma \in \Lambda_j} m_\sigma u_{|\sigma|} + C_1}{\sum_{\sigma \in \Lambda_j} m_\sigma l_{|\sigma|} - C_1} \leq \frac{\sum_{\sigma \in \Lambda_j} m_\sigma \log m_\sigma}{\sum_{\sigma \in \Lambda_j} m_\sigma \log c_\sigma} \leq \frac{\sum_{\sigma \in \Lambda_j} m_\sigma u_{|\sigma|} - C_1}{\sum_{\sigma \in \Lambda_j} m_\sigma l_{|\sigma|} + C_1}.
\end{equation}

Let $A_7 := 2^{-1}|l_0|$. Note that $l_k, k \geq 2$, are all negative. By (3.8) and (3.9),
\begin{equation}
|l_k| \geq (k-2)|l_0| - \delta_0 \geq 2^{-1}k|l_0| = A_7 k, \quad k \geq 4 + \delta_0 |l_0|^{-1}.
\end{equation}

This, together with the definition of $k_{1j}$, implies
\begin{equation}
\left| \sum_{\sigma \in \Lambda_j} m_\sigma l_{|\sigma|} - C_1 \right| \geq \sum_{\sigma \in \Lambda_j} m_\sigma |l_{\sigma}| \geq A_7 k_{1j}.
\end{equation}

By Lemma 3.2, we have, $s_k \leq 2s_0$ for all large $k$. Hence,
\begin{equation}
\frac{\sum_{\sigma \in \Lambda_j} m_\sigma u_{|\sigma|}}{\sum_{\sigma \in \Lambda_j} m_\sigma l_{|\sigma|}} \leq \max_{k_{1j} \leq k \leq k_{2j}} s_k \leq 2s_0.
\end{equation}

Combining this with (3.16), we have
\begin{equation}
\frac{\sum_{\sigma \in \Lambda_j} m_\sigma u_{|\sigma|} + C_1}{\sum_{\sigma \in \Lambda_j} m_\sigma l_{|\sigma|} - C_1} - \frac{\sum_{\sigma \in \Lambda_j} m_\sigma l_{|\sigma|}}{\sum_{\sigma \in \Lambda_j} m_\sigma l_{|\sigma|}} \leq \frac{C_1 |\sum_{\sigma \in \Lambda_j} m_\sigma l_{|\sigma|} + \sum_{\sigma \in \Lambda_j} m_\sigma u_{|\sigma|}|}{\sum_{\sigma \in \Lambda_j} m_\sigma l_{|\sigma|}} \leq \frac{C_1 (1 + 2s_0)}{A_7 k_{1j}}.
\end{equation}

For large $j$, we have, $C_1 \leq 2^{-1}|\sum_{\sigma \in \Lambda_j} m_\sigma l_{|\sigma|}|$. Hence, we similarly get
\begin{equation}
\frac{\sum_{\sigma \in \Lambda_j} m_\sigma u_{|\sigma|} - C_1}{\sum_{\sigma \in \Lambda_j} m_\sigma l_{|\sigma|} + C_1} - \frac{\sum_{\sigma \in \Lambda_j} m_\sigma l_{|\sigma|}}{\sum_{\sigma \in \Lambda_j} m_\sigma l_{|\sigma|}} \leq \frac{2C_1 (1 + 2s_0)}{A_7 k_{1j}}.
\end{equation}

Set $A_8 := 2C_1 (1 + 2s_0)A_7^{-1}$. By (3.14), (3.17) and (3.18), we deduce
\begin{equation}
\frac{\sum_{\sigma \in \Lambda_j} m_\sigma u_{|\sigma|}}{\sum_{\sigma \in \Lambda_j} m_\sigma l_{|\sigma|}} \leq \frac{A_8}{k_{1j}} \leq \frac{\sum_{\sigma \in \Lambda_j} m_\sigma \log m_\sigma}{\sum_{\sigma \in \Lambda_j} m_\sigma \log c_\sigma} \leq \frac{\sum_{\sigma \in \Lambda_j} m_\sigma u_{|\sigma|}}{\sum_{\sigma \in \Lambda_j} m_\sigma l_{|\sigma|}} + \frac{A_8}{k_{1j}}.
\end{equation}

Now one can see that, there exist some $k_{j}^{(i)} \in [k_{1j}, k_{2j}], i = 1, 2$, such that
\begin{equation}
s_{k_{j}^{(1)}} = \min_{k_{1j} \leq k \leq k_{2j}} \frac{u_k}{l_k} \leq \frac{\sum_{\sigma \in \Lambda_j} m_\sigma u_{|\sigma|}}{\sum_{\sigma \in \Lambda_j} m_\sigma l_{|\sigma|}} \leq \max_{k_{1j} \leq k \leq k_{2j}} \frac{u_k}{l_k} = s_{k_{j}^{(2)}}.
\end{equation}

Thus, in view of (2.1), the lemma follows by setting $C_3 := A_8/A_1$. \hfill \Box

With the above analysis, we obtain the convergence order of $(t_j)_{j=1}^{\infty}$:

**Lemma 3.5.** Assume that $P$ is irreducible. There exists a constant $C_4$ such that $|t_j - s_0| < C_4 j^{-1}$ for all large $j$.

**Proof.** By Lemmas 3.2, 3.4 and Lemma 2.1, we have
\begin{equation}
t_j - s_0 \begin{cases} 
\leq s_{k_{j}^{(2)}} - s_0 + C_3 j^{-1} \leq (C_2 A_1^{-1} + C_3) j^{-1} \\
\geq s_{k_{j}^{(1)}} - s_0 - C_3 j^{-1} \geq -(C_2 A_1^{-1} + C_3) j^{-1}.
\end{cases}
\end{equation}

Hence, the lemma follows by setting $C_4 := C_2 A_1^{-1} + C_3$. \hfill \Box
Now we are able to give the proof of Theorem 1.2. For a Borel probability measure \( \nu \) on \( \mathbb{R}^2 \) and every \( n \geq 1 \), we write
\[
Q_n(\nu, a) := a^{-1} \log n + \hat{\epsilon}_n(\nu), \quad a > 0.
\]

**Proof of Theorem 1.2** By (2.1), we easily see
\[
\mu^j = \mu^j \leq \mu^j \quad \text{for } j = 0, 1, 2, \ldots
\]

Thus, for the above Lemma 3.6. Assume that
\[
\text{Set } a = (3.19) \quad P^{j+1} \leq P^j, \quad \psi_j \leq \psi_{j+1}.
\]

Since \( t_j \to s_0 \), we have, \( 2^{-1}s_0 \leq t_j \leq 2s_0 \) for all large \( j \). By (3.19),
\[
\sum_{\sigma \in \Lambda_j} m_{\sigma} \log c_{\sigma} = t_j^{-1} \sum_{\sigma \in \Lambda_j} m_{\sigma} \log m_{\sigma} = t_j^{-1} \sum_{\sigma \in \Lambda_j} m_{\sigma} \log (q_{\sigma} p_{\sigma})
\]
\[
= t_j^{-1} \sum_{\sigma \in \Lambda_j} m_{\sigma} \log q_{\sigma} + t_j^{-1} \sum_{\sigma \in \Lambda_j} m_{\sigma} \log p_{\sigma} \quad \{ \leq (2s_0)^{-1} \log Q + t_j^{-1} \log P^j
\]
\[
\geq 2s_0^{-1} \log q + t_j^{-1} \log P^j \}
\]

This, together with Lemma 2.7, yields
\[
2s_0^{-1} \log q + t_j^{-1} \log P^j + C_0 \leq \hat{\epsilon}_{\psi_j}(\mu) \leq t_j^{-1} \log P^j + (2s_0)^{-1} \log Q.
\]

Thus, by Lemma 3.4, (3.19) and the fact that \( 2^{-1}s_0 \leq t_j \leq 2s_0 \), we deduce
\[
Q_{\psi_j}(\mu, s_0) = s_0^{-1} \log \psi_j + \hat{\epsilon}_{\psi_j}(\mu)
\]
\[
\{ \leq (s_0^{-1} - t_j^{-1}) \log P^j - s_0^{-1} \log (qp) + (2s_0)^{-1} \log Q
\]
\[
\geq (s_0^{-1} - t_j^{-1}) \log P^j - 2s_0^{-1} \log q + 2s_0^{-1} \log P + C_0 \}.
\]

Finally, using Lemma 3.5, we obtain
\[
Q_{\psi_j}(\mu, s_0) \quad \{ \leq 2C_4s_0^{-2} \log P^j - s_0^{-1} \log (qp) + (2s_0)^{-1} \log Q
\]
\[
\geq -2C_4s_0^{-2} \log P^j + 2s_0^{-1} \log q + 2s_0^{-1} \log P + C_0 \}.
\]

Hence, \( 0 < P^{s_0}(\mu) \leq Q_{\psi_j}(\mu) < \infty \). By Lemma 2.1, the theorem follows.

In the following, we study the asymptotic geometric mean error for the conditional probability measures \( \mu_i \). For every \( i \in G_1 \), we write
\[
t_k(i) := \sum_{\sigma \in \Lambda_k(i)} \log p_{\sigma}, \quad s_k(i) := \sum_{\sigma \in \Lambda_k(i)} \log c_{\sigma},
\]
\[
s_k := \sum_{\sigma \in \Lambda_k(i)} \log p_{\sigma}, \quad k \geq 2.
\]

Let us denote by \( u_k(i), l_k(i) \) the numerator and denominator in the definition of \( s_k(i) \). We have the following estimate for the convergence order of \( (t_j(i))_{j=1}^\infty \).

**Lemma 3.6.** Assume that \( P \) is irreducible. There exists a constant \( C_5 \) such that
\[
|t_j(i) - s_0| < C_5j^{-1} \quad \text{for every } i \in G_1 \text{ and all large } j.
\]

**Proof.** Fix an arbitrary \( i \in G_1 \). By Lemma 3.2, for every pair \( l, h \in G_1 \),
\[
u_k(h) - C_1 \leq u_k(l) = \xi(i, k - 1) \leq \xi(h, k - 1) + C_1 = u_k(h) + C_1.
\]

Thus, for the above \( i \) and \( k \geq 2 \), we have
\[
Nu_k(i) - NC_1 \leq u_k = \sum_{j=1}^N u_k(j) \leq Nu_k(i) + NC_1.
\]

Set \( a_k := u_k(i) - N^{-1}u_k \) and \( b_k := l_k(i) - N^{-1}l_k \). Then \( |a_k|, |b_k| \leq C_1 \). Note that \( |l_k| \to \infty \) as \( k \to \infty \). Hence, for large \( k \), we have,
\[
|N^{-1}l_k + b_k| \geq 2^{-1}|N^{-1}l_k|, \quad s_k \leq 2s_0.
\]
Using these facts and (3.15), we deduce

\[
|s_k(i) - s_k| \leq \left| \frac{u_k(i)}{l_k(i)} - \frac{u_k}{l_k} \right| \leq \frac{C_1}{|l_k(N^{-1}l_k + b_k)|} \leq \frac{C_1}{|l_k(N^{-1}l_k + b_k)|} + \frac{C_1|u_k|}{|l_k(N^{-1}l_k + b_k)|}
\]

(3.20)

Let \( k_{1j}(i) := \min_{\sigma \in A_j(i)} |\sigma| \) and \( k_{2j}(i) := \max_{\sigma \in A_j(i)} |\sigma| \). By Lemma 2.1,

\[ A_{1j} \leq k_{1j} \leq k_{2j} \leq A_{2j} \]

Along the line in the proof of Lemma 3.4, one can show, there exist a constant \( C_3(i) \) and two integers \( k_j^{(h)} \in [k_{1j}(i), k_{2j}(i)], h = 1, 2, \) such that

\[ s_{k_j}^{(1)}(i) - C_3(i)j^{-1} \leq t_j(i) \leq s_{k_j}^{(2)}(i) + C_3(i)j^{-1} \]

Combining this and (3.20), we have

\[
t_j(i) - s_0 \leq s_{k_j}^{(2)}(i) + C_3(i)j^{-1} - s_0 \leq A_0k_j^{(2)}(i)N^{-1} + C_3(i)j^{-1} \leq A_0A_1^{-1}j^{-1} + C_3(i)j^{-1} = (A_0A_1^{-1} + C_3(i))j^{-1}.
\]

Similarly, one can show that \( t_j(i) - s_0 \geq -(A_0A_1^{-1} + C_3(i))j^{-1} \). Thus, the lemma follows by setting \( C_5 := A_0A_1^{-1} + \max_{\sigma \in G_1} C_3(i) \).

**Proposition 3.7.** Assume that \( P \) is irreducible. Then, we have

\[ 0 < Q_0^{(i)}(\mu_i) \leq Q_0^{(i)}(\mu_i) < \infty, \quad i \in G_1. \]

**Proof.** It suffices to follow the proof of Theorem 1.2 by using (2.10) and Lemma 3.6. We omit the details. \( \square \)

Next, we show that, when the transition matrix \( P \) is reducible, \( D_0(\mu) \) is dependent on the initial probability vector. For this, we need the following observation which is a consequence of the arguments in [7, Example 4.1].

**Proposition 3.8.** Let \( \nu_i, 1 \leq i \leq N \), be Borel probability measures on \( \mathbb{R}^d \) of compact support such that, for all \( \epsilon > 0 \), we have

\[ \max_{1 \leq i \leq N} \sup_{x \in \mathbb{R}^d} \nu_i(B(x, \epsilon)) \leq d_1 \epsilon^{d_2}. \]

Assume that \( D_0(\nu_i) = t_i > 0, \quad i \in G_1 \). Let \( (q_i)_{i=1}^N \) be a probability vector with \( q_i > 0 \) for all \( i \in G_1 \). Then for \( \nu = \sum_{i=1}^N q_i \nu_i \), we have \( D_0(\nu) = t_0 \), where

\[ t_0 = \frac{t_1t_2 \cdots t_N}{q_1t_2 \cdots t_N + \cdots + q_Nt_1 \cdots t_{N-1}}. \]

Moreover, we have \( Q_0^{(i)}(\nu) > 0 \) if \( Q_0^{(i)}(\nu_i) > 0 \) for all \( 1 \leq i \leq N \); and \( Q_0^{(i)}(\nu) \leq Q_0^{(i)}(\nu_i) < \infty \) if \( Q_0^{(i)}(\nu_i) \leq Q_0^{(i)}(\nu) < \infty \) for all \( 1 \leq i \leq N \).

**Proof.** We denote by \([x]\) the largest integer not greater than \( x \in \mathbb{R} \). By the arguments in [7, Example 4.1], we have

\[
\hat{e}_n(\nu) \left\{ \begin{array}{l}
\geq q_1\hat{e}_n(\nu_1) + \cdots + q_N\hat{e}_n(\nu_N) \\
\leq q_1\hat{e}_n(\nu_1) + \cdots + q_N\hat{e}_n(\nu_N)
\end{array} \right.
\]
By (1.1) and (3.21), we easily see that $D_0(\nu) = t_0$. Furthermore, by (3.21),

$$Q_n(\nu, t_0) \geq t_0^{-1} \log n + \sum_{i=1}^{N} q_i \hat{c}_n(\nu_i) = \sum_{i=1}^{N} q_i Q_n(\nu_i, t_i).$$

(3.22)

Set $C_5 := (q_1 t_1^{-1} + \ldots + q_N t_N^{-1}) \log(2N)$. Then, for all $n \geq 2N$, we have

$$Q_n(\nu, t_0) \leq t_0^{-1} \log n + \sum_{i=1}^{N} q_i \hat{c}_n(\nu_i) = \sum_{i=1}^{N} q_i Q_n(\nu_i, t_i) + C_5.$$

(3.23)

By (3.22) and (3.23), we conclude

$$\prod_{i=1}^{N} Q_n^{(i)}(\nu_i)^{q_i} \leq Q_n^{(h)}(\nu) \leq \prod_{i=1}^{N} C_5^{(i)}(\nu_i)^{q_i}.$$

This implies the second assertion of the proposition. \hfill \square

**Example 3.9.** Let $Q_1 = (q_{ij})_{i,j=1}^{4}$, $Q_2 = (t_{ij})_{i,j=3}^{4}$ be positive row-stochastic matrices $(q_{ij} > 0, 1 \leq i,j \leq 2$ and $t_{ij} > 0, 3 \leq i,j \leq 4)$. Let $P$ denote the block diagonal matrix $\text{diag}(Q_1, Q_2)$. Then $P$ is reducible. Let $(c_{ij})_{4 \times 4}$ be given and assume that (1.3) holds. Let $\mu$ be the Markov-type measure associated with $P$ and initial probability vector $(q_{ij})_{i=1}^{4}$. Clearly, $Q_1$, $Q_2$ are both irreducible. Let $(v_{i}^{(h)})_{i=1}^{2}$ be the normalized positive left eigenvector of $Q_h$ for $h = 1, 2$. Write

$$t_1 := \frac{\sum_{i=1}^{2} q_i \log q_{ij}}{\sum_{i=1}^{2} v_{i}^{(1)} \sum_{j=1}^{2} q_{ij} \log c_{ij}}, \quad t_2 := \frac{\sum_{i=3}^{4} t_{ij} \log t_{ij}}{\sum_{i=3}^{4} v_{i}^{(2)} \sum_{j=3}^{4} t_{ij} \log c_{ij}}.$$

By Proposition 3.7, we have, $0 < Q_0^{(1)}(\mu_i) \leq \prod_{i=1}^{N} Q_0^{(i)}(\mu_i) < \infty$ for $i = 1, 2$; and for $i = 3, 4$, $0 < Q_0^{(3)}(\mu) \leq \prod_{i=1}^{N} Q_0^{(i)}(\mu_i) < \infty$. Set

$$t_0 := \frac{t_1 t_2}{(q_1 + q_2) t_2 + (q_3 + q_4) t_1}.$$

Then by Proposition 3.8, we have, $0 < Q_0^{(h)}(\mu) \leq \prod_{i=1}^{N} Q_0^{(i)}(\mu_i) < \infty$. In this example, $D_0(\mu)$ depends on the initial probability vector provided that $t_1 \neq t_2$.

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School of Mathematics and Physics, Jiangsu University of Technology, Changzhou 213001, China.

E-mail address: sgzhu@jsut.edu.cn