Identities for the $k$–generalized Fibonacci sequence with negative indices and its zero–multiplicity

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Abstract

In this paper, we prove identities for members of the $k$–generalized Fibonacci sequence with negative indices and we apply these identities to deduce an exact formula for its zero–multiplicity.

Key words and phrases: $k$–generalized Fibonacci sequence, zero–multiplicity of linear recurrences.

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1 Introduction

One of the most famous and curious numerical sequence, due to the large number of properties and relationships with other areas [8], is the Fibonacci sequence, denoted by $F := (F_n)_{n \geq 0}$. Its initial values are $F_0 = 0$, $F_1 = 1$ and obeys the recurrence $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 2$. This sequence has been generalized in many ways, some by preserving the initial conditions, and others by preserving the recurrence relation. In general, a sequence $(u_n)_{n \in \mathbb{Z}} \subseteq \mathbb{C}$ is a linear recurrence sequence of order $k \in \mathbb{Z}^+$ if it satisfies the recurrence relation $u_{n+k} = a_1 u_{n+k-1} + a_2 u_{n+k-2} + \cdots + a_k u_n$ for all $n \geq 0$ with coefficients $a_1, \ldots, a_k \in \mathbb{C}$ and $a_k \neq 0$. We assume that $k$ is minimal with the above property. Such a sequence $(u_n)_{n \in \mathbb{Z}}$ has an associated characteristic
Our identities are similar to the given by Ferguson [3], Gabai [4] or Cooper and Howard [1] for the Theorem 1.

2 The main results

H deduce an exact formula for the zero–multiplicity of $F_{n,k}$. We denote $H_{n,k}$ the term is the sum of the previous ones $F_{n,k} = F_{n-1,k} + F_{n-2,k} + \cdots + F_{n-k}$, for all $n, k \geq 2$. This sequence can be extended to all integer indices $n$. Since $F_{n,k} > 0$ for $n > 0$, all zeros of $F^{(k)}$ correspond to non–positive indices $n$. We denote $H^{(k)} := (H_{n,k}^{(k)})_{n \geq 0}$, where $H_{n,k}^{(k)} = F_{-n,k}^{(k)}$. This sequence obeys the recurrence relation

$$H_{n,k}^{(k)} = H_{n-k}^{(k)} - H_{n-(k-1)}^{(k)} - \cdots - H_{n-1}^{(k)} \quad \text{for all} \quad n \geq k, \quad (1)$$

with initial values $H_{i,k}^{(k)} = 0$ for $0 \leq i \leq k-2$ and $H_{k-1,k}^{(k)} = 1$.

In this paper, we find identities for members of $H^{(k)}$ and apply them to deduce an exact formula for the zero–multiplicity of $H^{(k)}$.

2.1 Identities for $H^{(k)}$

Our identities are similar to the given by Ferguson [3], Gabai [4] or Cooper and Howard [1] for the $k$–generalized Fibonacci sequence with positive indices.

Theorem 1. For all $k \geq 2$, the sequence $H^{(k)}$ satisfies

(i) For all $0 \leq m \leq r \leq k-2$, we have $H_{mk+r}^{(k)} = 0$.

(ii) For all $m \in [1, k-1]$, we have $H_{mk-1}^{(k)} = 2^{m-1}$.

(iii) For all $0 \leq r < m \leq k-1$,

$$H_{mk+r}^{(k)} = (-1)^{r+1} \left[ \binom{m-1}{r} + \binom{m}{r+1} \right] 2^{m-r-2}.$$  

(iv) For all $r \in [-1, k-2]$ and $m \geq k-1$,

$$H_{mk+r}^{(k)} = \sum_{i=0}^{\lfloor l \rfloor} (-1)^{ik+r+1} \left[ \binom{m-i-1}{ik+r} + \binom{m-i}{ik+r+1} \right] 2^{m-i-(k+1)-r-2},$$

where $l = m-1$ if $k = 2$, and $l = \lfloor m/(k-1) \rfloor$ if $k > 2$.  

2.2 Zero–multiplicity for $H^{(k)}$

One of the classical problems in the theory of linear recurrence sequences is the Skolem problem: Given the linear recurrence sequence $u := (u_n)_{n \in \mathbb{Z}}$, one wants to find $\mathcal{Z}(u) = \{n \in \mathbb{Z} : u_n = 0\}$. The cardinality of $\mathcal{Z}(u)$ (when this is finite) is called the zero–multiplicity of $u$. There is no known algorithm to find $\mathcal{Z}(u)$ in general. One of the more important results here is due to Skolem [11, 10]: If the coefficients $a_1, \ldots, a_k$ of the linear recurrence sequence $u$ are rational, then the set $\mathcal{Z}(u)$ is a union of finitely many arithmetical progressions together with a finite set. Hagedorn [7] showed that if the roots of $\Psi_k(z)$ are real then $\#\mathcal{Z}(u) \leq 2k - 3$.

Our current research is motivated by our previous work [5]. There we obtained that

$$\mathcal{A} := \bigcup_{m=0}^{k-2} [m(k+1), (m+1)k - 2] \subseteq \mathcal{Z}(H^{(k)}).$$

Thus, $\#\mathcal{Z}(H^{(k)}) \geq k(k-1)/2$. Furthermore, we checked that

$$\#\mathcal{Z}(H^{(2)}) = 1, \quad \#\mathcal{Z}(H^{(3)}) = 4 \quad \text{and} \quad \#\mathcal{Z}(H^{(k)}) = k(k-1)/2,$$

for $k \in [4, 500]$. Based on the above results, we proposed in [5] the following conjecture.

**Conjecture 1.** The zero–multiplicity $\#\mathcal{Z}(H^{(k)})$ of the $k$–generalized Fibonacci sequence with non-positive indices $H^{(k)}$ for $k \geq 4$ is the $(k-1)$st triangular number; i.e.

$$\#\mathcal{Z}(H^{(k)}) = k(k-1)/2.$$ 

Recently, relating the 2–adic valuation of $F^{(k)}$ with the Diophantine equation $H^{(k)}_n = 0$, Young [12] showed that for all $k > 500$,

$$\#\mathcal{Z}(H^{(k)}) \leq k(k+1)/2 + \lfloor k/2 \rfloor.$$ 

In this paper, we confirm the above conjecture. As a consequence of this, we get that $\mathcal{Z}(F^{(k)})$ is exactly $\mathcal{A}$.

3 Preliminary results

To simplify the notation, from now on we denote $H^{(k)}_n := H_n$, where $k$ is fixed. From the recurrence relation (1), it is easy to see that

$$H_n = 2H_{n-k} - (H_{n-k} + H_{n-(k-1)} + \cdots + H_{n-2} + H_{n-1})$$

$$= 2H_{n-k} - H_{n-k-1} \quad \text{for all} \quad n \geq k + 1. \quad (3)$$
The containment (2) is a direct consequence of the initial values of $H_n$ and the identity (3). Indeed, inductively we see that $H_n = 0$ for all $0 \leq n \leq k-2$. Further

$$H_n = 2H_{n-k} - H_{n-k-1} = 0 \quad \text{for all} \quad m(k+1) \leq n \leq mk + (k-2),$$

with $1 \leq m \leq k-2$. Moreover, the length of these intervals is given by the decreasing quantity

$$mk + (k-2) - m(k+1) = k - (m+2),$$

which becomes 0 at $m = k-2$.

So far, we have only summarized what we obtained in [5]. This is fundamental for what follows, since we note that the sequence $H_n$ can be represented as lists matrices, starting with a rectangular matrix of size $(k - 1) \times (k + 1)$ and continuing with square matrices of size $k \times k$. Then, we observed interesting patterns in $H_n$ that we were able to formulate and prove.

We start by ordering the first $k^2-1$ elements of the sequence $H_n$ in matrix form $(H_{(i-1)(k+1)+(j-1)})_{ij}$ with $1 \leq i \leq k-1$ and $1 \leq j \leq k+1$. This is

$$
\begin{pmatrix}
H_0 & \cdots & H_{k-2} & H_{k-1} & H_k \\
H_{(k+1)+0} & \cdots & H_{(k+1)+k-2} & H_{(k+1)+k-1} & H_{(k+1)+k} \\
H_{2(k+1)+0} & \cdots & H_{2(k+1)+k-2} & H_{2(k+1)+k-1} & H_{2(k+1)+k} \\
& \ddots & \ddots & \ddots & \ddots \\
H_{(k-2)(k+1)+0} & \cdots & H_{(k-2)(k+1)+k-2} & H_{(k-2)(k+1)+k-1} & H_{(k-2)(k+1)+k}
\end{pmatrix}
$$

and we notice that the “upper triangular” part, which is in bold, is composed of zeros. We can also write this matrix as

$$
\begin{pmatrix}
0 & 0 & \cdots & 0 & 0 & H_{k-1} & H_k \\
0 & 0 & \cdots & 0 & H_{2k} & H_{2k-1} & H_{2k+1} \\
0 & 0 & \cdots & H_{3k-1} & H_{3k} & H_{3k+1} & H_{3k+2} \\
& \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & H_{(k-1)k-1} & \cdots & H_{(k-1)k+k-5} & H_{(k-1)k+k-4} & H_{(k-1)k+k-3} & H_{(k-1)k+k-2}
\end{pmatrix},
$$

and we observe the following behavior in the non-zero diagonals.

**Lemma 1.**

(a) For all $m \in [1, k-1]$, we have $H_{mk-1} = 2^{m-1}$.

(b) For all $m \in [1, k-1]$, we have $H_{mk} = -(m+1)2^{m-2}$.

(c) For all $1 \leq r < m \leq k-1$,

$$H_{mk+r} = -\sum_{j=r}^{m-1} 2^{m-1-j} H_{jk+r-1}.$$
Proof. To show that the items of this lemma are fulfilled, we proceed by induction on \( m \), taking into account that the upper bound for \( m \) is \( k - 1 \geq 1 \).

(a) If \( m = 1 \), given the initial values, we have \( H_{k-1} = 1 = 2^0 \). Suppose that \( H_{(m-1)k-1} = 2^{m-2} \) for \( m \in [2, k-1] \). Then, using identity \((3)\), it follows that

\[
H_{mk-1} = 2H_{(m-1)k-1} - H_{(m-1)k-2} = 2^{m-1},
\]

since \( 0 \leq m - 2 \leq k - 3 \) and therefore \( H_{((m-2)+1)k-2} = 0 \) (see \((2)\)).

(b) When \( m = 1 \), we have

\[
H_k = H_0 - H_1 - \cdots - H_{k-1} = -1 = -2 \times 2^{-1}.
\]

Assume \( H_{(m-1)k} = -m2^{m-3} \) and we obtain from identity \((3)\) and item (a), that

\[
H_{mk} = 2H_{(m-1)k} - H_{(m-1)k-1} = -m2^{m-2} - 2^{m-2} = -(m + 1)2^{m-2}.
\]

(c) Assume \( m = r + 1 \). We have by identity \((3)\) that

\[
H_{(r+1)k+r} = 2H_{rk+r} - H_{rk+r-1} = -H_{rk+r-1} = -\sum_{j=r}^{r} 2^{r-j} H_{jk+r-1},
\]

because \( H_{rk+r} = H_{r(k+1)} = 0 \) with \( 1 \leq r \leq k - 2 \) (see \((2)\)). We take

\[
H_{(m-1)k+r} = -\sum_{j=r}^{m-2} 2^{m-2-j} H_{jkr-1}
\]

as the inductive hypothesis. Then

\[
H_{mk+r} = 2H_{(m-1)k+r} - H_{(m-1)k+r-1}
\]

\[
= -\left( \sum_{j=r}^{m-2} 2^{m-1-j} H_{jkr-1} \right) - H_{(m-1)k+r-1}
\]

\[
= -\sum_{j=r}^{m-1} 2^{m-1-j} H_{jkr-1}.
\]

Matrix \((4)\) is the first element of the list of matrices that we use to organize all the elements of \( H_n \), as we mentioned before. It is rectangular of size \((k - 1) \times (k + 1)\). The matrix that follows is of size \( k \times k \) and includes all the non-zero elements of the last row of matrix \((4)\), this is

\[
\begin{pmatrix}
H_{(k^2-k-1)+0} & H_{(k^2-k-1)+1} & \cdots & H_{(k^2-k-1)+k-1} \\
H_{(k^2-k-1)+1} & H_{(k^2-k-1)+k+1} & \cdots & H_{(k^2-k-1)+k+(k-1)} \\
\vdots & \vdots & \ddots & \vdots \\
H_{(k^2-k-1)+(k-1)k} & H_{(k^2-k-1)+(k-1)k+1} & \cdots & H_{(k^2-k-1)+(k-1)k+(k-1)}
\end{pmatrix}.
\]

\((5)\)
In general, all square matrices after matrix \( \mathbf{M} \) form the sequence \( \{M_b\}_{b \in \mathbb{Z}^+} \), where

\[
M_b := \begin{pmatrix}
H_{(bk^2-bk-1)+0} & H_{(bk^2-bk-1)+1} & \cdots & H_{(bk^2-bk-1)+k-1} \\
H_{(bk^2-bk-1)+k} & H_{(bk^2-bk-1)+k+1} & \cdots & H_{(bk^2-bk-1)+(k-1)} \\
\vdots & \vdots & \ddots & \vdots \\
H_{(bk^2-bk-1)+(k-1)k} & H_{(bk^2-bk-1)+(k-1)k+1} & \cdots & H_{(bk^2-bk-1)+(k-1)k+(k-1)}
\end{pmatrix}.
\]

Furthermore, by calling \( M_0 \) the matrix \( \mathbf{M} \), we have arrived at

\[
(H^{(k)}_n)_{n \geq 0} = \bigcup_{b \geq 0} \{h : h \text{ is an entry of } M_b\}.
\]

From now on, we simplify the notation by setting

\[
H_{b,jk+r} := H_{(bk^2-bk-1)+jk+r} \text{ for all } b \geq 0.
\]

Note that matrix \( \mathbf{M} \) is exactly \( M_1 \). In fact, these matrices satisfy that their first row is exactly the last row of the immediately preceding matrix. Indeed,

\[
H_{b,r} = H_{b-1,(k-1)k+r} \text{ for all } 0 \leq r \leq k-1 \text{ and } b \geq 1. \tag{6}
\]

Also,

\[
H_{b,1} = H_{b-1,(k-2)k+(k-1)} \text{ for all } b \geq 1 \tag{7}
\]

and in particular, by containment \( \mathbf{2} \),

\[
H_{1,1} = H_{0,(k-2)k+(k-1)} = H_{(k-2)(k+1)} = 0.
\]

Moreover, we observe that identity \( \mathbf{3} \) is still satisfied with this notation

\[
H_{b,n} = 2H_{b,n-k} - H_{b,n-k-1} \tag{8}
\]

for all \( b \geq 1 \) and \( n \geq 2 \), or for \( b = 0 \) and \( n \geq k + 2 \).

Next, we find the following patterns in all entries below the main diagonal of \( M_b \) with \( b \geq 1 \).

**Lemma 2.** Let \( b \geq 1 \), then

(I) \( H_{b,jk} = \begin{cases}
2H_{b,0} - H_{b-1,(k-2)k+(k-1)} & \text{if } j = 1, \\
2^{j-1}H_{b,k} - \sum_{i=0}^{j-2}2^{j-1-2}H_{b,(i+1)k-1} & \text{if } 2 \leq j \leq k-1.
\end{cases} \)

(II) For all \( r, j \in [1, k-1] \), it holds that

\[
H_{b,jk+r} = 2^{j-1}H_{b,k+r} - \sum_{i=1}^{j-1}2^{j-1-i}H_{b,ik+r-1}.
\]
Let $r, j \in [0, k - 1]$ be fixed. Then

\[ H_{b,jk+r} = \sum_{i=0}^{t} (-1)^i \binom{t}{i} 2^{t-i} H_{b,(j-t)k+r-i}, \]

for all $t \in \left[0, \left\lfloor \frac{bk^2 - bk - 1 + jk + r}{k+1} \right\rfloor \right]$. 

Proof. We consider each item separately.

(I) The case $j = 1$, follows from identities (7) and (8). Let $2 \leq j \leq k - 1$. We argue by induction on $j$. If $j = 2$, item (I) is immediate by identity (8). Suppose it is satisfied for $j - 1$; i.e.,

\[ H_{b,(j-1)k} = 2^{j-2} H_{b,k} - \sum_{i=0}^{j-3} 2^{j-i-3} H_{b,(i+1)k-1}. \]  

(9)

Then, by (8) and (9),

\[ H_{b,jk} = 2H_{b,(j-1)k} - H_{b,(j-1)k-1} \]

\[ = \left( 2^{j-1} H_{b,k} - \sum_{i=0}^{j-3} 2^{j-i-2} H_{b,(i+1)k-1} \right) - H_{b,(j-1)k-1} \]

\[ = 2^{j-1} H_{b,k} - \sum_{i=0}^{j-2} 2^{j-i-2} H_{b,(i+1)k-1}, \]

and thus we conclude item (I) for $j$.

(II) If $j = 1$, the identity is trivial. For $j \geq 2$, we apply recursively, $j - 1$ times, identity (8) on $H_{b,jk+r}$. That is,

\[ H_{b,jk+r} = 2H_{b,(j-1)k+r} - H_{b,(j-1)k+r-1} \]

\[ = 2^2 H_{b,(j-2)k+r} - 2H_{b,(j-2)k+r-1} - H_{b,(j-1)k+r-1} \]

\[ = 2^3 H_{b,(j-3)k+r} - 2^2 H_{b,(j-3)k+r-1} - 2H_{b,(j-2)k+r-1} - H_{b,(j-1)k+r-1} \]

\[ \vdots \]

\[ = 2^{j-1} H_{b,(j-(j-1))k+r} - 2^{j-2} H_{b,(j-(j-1))k+r-1} - \cdots - H_{b,(j-1)k+r-1}. \]

(III) We proceed by induction on $t$. If $t = 0$, the identity is trivial. Suppose it is satisfied for $t - 1 \geq 0$; i.e.,

\[ H_{b,jk+r} = \sum_{i=0}^{t-1} (-1)^i \binom{t-1}{i} 2^{t-i-1} H_{b,(j-t+1)k+r-i}. \]
Then, by the inductive hypothesis and identity (8), it follows that

\[ H_{b,j+rk} = \sum_{i=0}^{t-1} (-1)^i \binom{t-1}{i} 2^{t-i-1} (2H_{b,(j-t)k+i} - H_{b,(j-t)k+i-1}) \]

\[ = (-1)^0 \binom{t-1}{0} 2^t H_{b,(j-t)k+r} + \sum_{i=1}^{t-1} (-1)^i \binom{t-1}{i} 2^{t-i} H_{b,(j-t)k+r-i} \]

\[ + \sum_{i=1}^{t} (-1)^i \binom{t-1}{i-1} 2^{t-i} H_{b,(j-t)k+r-i} \]

\[ = 2^t H_{b,(j-t)k+r} + \sum_{i=1}^{t-1} (-1)^i \left( \binom{t-1}{i} + \binom{t-1}{i-1} \right) 2^{t-i} H_{b,(j-t)k+r-i} \]

\[ + (-1)^t \binom{t-1}{t-1} 2^{t-t} H_{b,(j-t)k+r-t} \]

\[ = 2^t H_{b,(j-t)k+r} + \sum_{i=1}^{t-1} (-1)^i \binom{t}{i} 2^{t-i} H_{b,(j-t)k+r-i} + (-1)^t H_{b,(j-t)k+r-t} \]

\[ = \sum_{i=0}^{t} (-1)^i \binom{t}{i} 2^{t-i} H_{b,(j-t)k+r-i}, \]

and the identity is satisfied for \( t \). We must assume that \( t \leq \left\lfloor \frac{bk^2-bk+1+jk+r}{k+1} \right\rfloor \),

because in this case

\[ bk^2 - bk - 1 + (j - t)k + r - i \geq bk^2 - bk - 1 + (j - t)k + r - t \geq 0, \]

for all \( 0 \leq i \leq t \). Therefore, \( H_{b,(j-t)k+r-i} \) is well defined for all \( 0 \leq i \leq t \).

This completes the proof of this lemma. \( \square \)

The above lemmas will allow us, in Section 4, to characterize all entries of the matrices \( M_b \) and therefore all elements of \( H_n \). For this purpose we introduce the following notation:

\[ \psi(v,w) := \binom{v}{w} + \binom{v+1}{w+1}. \] (10)

Note that

\[ \psi(v,-1) = 1 \text{ for all } v \neq -1 \quad \text{and} \quad \psi(v,w) = 0 \text{ for all } w > v \geq 0. \] (11)

We prove the following properties of \( \psi(v,w) \).

**Lemma 3.** The function \( \psi \) satisfies:

1. \( \psi(v,w) + \psi(v,w+1) = \psi(v+1,w+1) \).
2. \( \sum_{i=0}^{n} \psi(v+i,v) = \psi(v+n+1,v+1) \).
(3) $\sum_{i=1}^{n} \psi(v + i, i) = \psi(v + n + 1, n)$.

(4) $\sum_{i=w}^{v} \psi(i - 1, w - 1) = \psi(v, w)$.

(5) $\sum_{i=0}^{w} \binom{u}{i} \psi(v, w - i - 1) = \psi(u + v, w - 1)$.

Proof. To show that all items are satisfied, we use basic properties of binomial coefficients. Item (1) is immediate.

(2) We know that

$$\sum_{i=0}^{n} \psi(v + i, v) = \sum_{i=v}^{v+n} \binom{i}{v} + \sum_{i=v+1}^{v+n+1} \binom{i}{v+1} = \psi(v + n + 1, v + 1).$$

(3) It is clear that

$$\sum_{i=-1}^{n} \psi(v + i, i) = \sum_{i=0}^{n} \binom{v+i}{v} + \sum_{i=-1}^{n} \binom{v+i+1}{i+1}$$

$$= \sum_{i=v}^{v+n} \binom{i}{v} + \sum_{i=v}^{v+n+1} \binom{i}{v}$$

$$= \psi(v + n + 1, n).$$

(4) We have

$$\sum_{i=w}^{v} \psi(i - 1, w - 1) = \sum_{i=w-1}^{v-1} \binom{i}{w-1} + \sum_{i=w}^{v} \binom{i}{w} = \psi(v, w).$$

(5) We obtain

$$\sum_{i=0}^{w} \binom{u}{i} \psi(v, w - i - 1) = \sum_{i=0}^{w-1} \binom{u}{i} \binom{v}{w - i - 1} + \sum_{i=0}^{w} \binom{u}{i} \binom{v+1}{w - i}$$

$$= \psi(u + v, w - 1).$$

4 Proof of Theorem \[ \square \]

For items (i) and (ii), see containment \[ \square \] and Lemma \[ \square \] item (a), respectively. The remaining items are shown below.

4.1 Item (iii)

With notation \[ \square \], we must prove that

$$H_{mk+r} = (-1)^{r+1} \psi(m - 1, r)2^{m-r-2} \quad \text{for all} \quad 0 \leq r < m \leq k - 1. \quad (12)$$
For this, we apply induction on $r$. If $r = 0$, by Lemma [1] item (b), we get that identity (12) is satisfied. Indeed,
\[ H_{mk} = -(m + 1)2^{m-2} = (-1)^1\psi(m - 1, 0)2^{m-2} \quad \text{for all} \quad 1 \leq m \leq k - 1. \]
We assume by the inductive hypothesis that identity (12) is satisfied for $r - 1$ and therefore $0 \leq r - 1 < m \leq k - 1$; i.e.,
\[ H_{mk+r-1} = (-1)^r\psi(m - 1, r - 1)2^{m-r-1} \quad \text{for all} \quad r \leq m \leq k - 1. \]
In particular,
\[ H_{jk+r-1} = (-1)^r\psi(j - 1, r - 1)2^{j-r-1} \quad \text{for all} \quad r \leq j \leq m - 1. \quad (13) \]
Then, by Lemma [1] item (c), Lemma [3] item (4) and identity (13), we obtain
\[
H_{mk+r} = -\sum_{j=r}^{m-1} 2^{m-1-j}((-1)^r\psi(j - 1, r - 1)2^{j-r-1})
= (-1)^{r+1}(\sum_{j=r}^{m-1}\psi(j - 1, r - 1))2^{m-r-1}
= (-1)^r\psi(m - 1, r)2^{m-r-1} \quad \text{for all} \quad 1 \leq r < m \leq k - 1.
\]
This completes the proof of item (iii).

4.2 Item (iv)

4.2.1 Case $k = 2$:

Here we must prove that for all $m \geq 1$,
\[
H_{2m+r} = \begin{cases} 
\sum_{i=0}^{m-1}\psi(m - i - 1, 2i - 1)2^{m-3i-1}, & \text{if } r = -1, \\
-\sum_{i=0}^{m-1}\psi(m - i - 1, 2i)2^{m-3i-2}, & \text{if } r = 0.
\end{cases} \quad (14)
\]
For this we proceed by induction on $m$. Suppose that identity (14) is satisfied up to $m - 1 \geq 2$ (the cases $m = 1$ and $m = 2$ are easily verified). Then using (3), (11), Lemma [3] and the fact that $\psi(0, 2m - 2) = \psi(0, 2m - 3) = 0$ (since $m \geq 2$), we obtain
\[
H_{2m-1} = 2H_{2(m-1)-1} - H_{2(m-2)}
= \sum_{i=0}^{m-2}\psi(m - i - 2, 2i - 1)2^{m-3i-1} + \sum_{i=0}^{m-3}\psi(m - i - 3, 2i)2^{m-3i-4}
= \psi(m - 2, -1)2^{m-1} + \sum_{i=1}^{m-2}(\psi(m - i - 2, 2i - 1) + \psi(m - i - 2, 2i - 2))2^{m-3i-1}
= \psi(m - 1, -1)2^{m-1} + \sum_{i=1}^{m-2}\psi(m - i - 1, 2i - 1)2^{m-3i-1} + \psi(0, 2m - 3)2^{-2(m-1)}
= \sum_{i=0}^{m-1}\psi(m - i - 1, 2i - 1)2^{m-3i-1},
\]
and

\[ H_{2m} = 2H_{2(m-1)} - H_{2(m-1)-1} \]

\[ = - \sum_{i=0}^{m-2} \psi(m - i - 2, 2i)2^{m-3i-2} - \sum_{i=0}^{m-2} \psi(m - i - 2, 2i - 1)2^{m-3i-2} \]

\[ = -\psi(0, 2m - 2)2^{-2m+1} - \sum_{i=0}^{m-2} (\psi(m - i - 2, 2i) + \psi(m - i - 2, 2i - 1))2^{m-3i-2} \]

\[ = - \sum_{i=0}^{m-1} \psi(m - i - 1, 2i)2^{m-3i-2}. \]

### 4.2.2 Case \( k > 2 \):

To treat this case, we present the following two lemmas.

**Lemma 4.** For all \( k > 2 \) and \( j, r \in [0, k - 1] \), it holds that

\[ H_{1, jk+r} = (-1)^r 2^{k+j-r-2} \psi(k + j - 2, r - 1) \]

\[ + (-1)^{k+r} 2^{j-r-3} \psi(k + j - 3, k + r - 1). \]  \( \text{(15)} \)

**Proof.** We apply double induction on \( j, r \in [0, k - 1] \).

**(A)** Let \( j = r = 0 \). Since \( k > 2 \), by observation (11), it follows that

\[ \psi(k - 2, -1) = 1 \quad \text{and} \quad \psi(k - 3, k - 1) = 0. \]

Then, using item \((ii)\), we obtain

\[ H_{1,0} = H_{(k-1)k-1} = 2^{k-2} = 2^{k-2} \psi(k - 2, -1) + (-1)^{k+2} 2^{k-3} \psi(k - 3, k - 1) \]

fulfilling identity \((15)\).

**(B)** Let \( j = 0 \) and \( r \in [1, k - 1] \). Then, by item \((iii)\), we get that

\[ H_{1,r} = H_{k(k-1)+r-1} = (-1)^r \psi(k - 2, r - 1) 2^{k-r-2} \]

and by observation (11) it follows that

\[ H_{1,r} = (-1)^r 2^{k-r-2} \psi(k - 2, r - 1) \]

\[ = (-1)^r 2^{k-r-2} \psi(k - 2, r - 1) + (-1)^{k+r} 2^{-r-3} \psi(k - 3, k + r - 1). \]  \( \text{(16)} \)

Thus, identity \((15)\) is satisfied.

**(C)** Let \( r = 0 \) and \( j \in [1, k - 1] \). If \( j = 1 \), by Lemma 2, item \((I)\) and observation (11), we have that identity \((15)\) is satisfied. In fact,

\[ H_{1,k} = 2H_{1,0} - H_{0,(k-2)k+(k-1)} = 2H_{1,0} = 2^{k-1} \]

\[ = 2^{k-1} \psi(k - 1, -1) + (-1)^k 2^{-2} \psi(k - 2, k - 1). \]  \( \text{(17)} \)
If \( j \in [2, k - 1] \), we note that by Lemma 2, item (I), combined with item (III) of this same Lemma and items (A), (B) above, we get

\[
H_{1,jk} = 2^{j-1}H_{1,k} - \sum_{i=0}^{j-2} 2^{j-i-2}H_{1,jk+i+1}
\]

\[
= 2^{j-1}H_{1,k} + \sum_{i=0}^{j-2} (-1)^{\ell+1} \binom{\ell+1}{\ell} 2^{j-\ell}H_{1,k-\ell}
\]

\[
= 2^{j-1}H_{1,k} + \sum_{i=0}^{j-2} (-1)^{\ell+1} \left( \sum_{\ell=0}^{j-2} \binom{j-1}{\ell+1} 2^{j-\ell}H_{1,k-\ell} \right)
\]

\[
= 2^{j-1}H_{1,k} + (-1)^{k-2} \sum_{\ell=1}^{j-1} \binom{j-1}{\ell} \psi(k-2, k-1-\ell).
\]  (18)

In addition,

\[
\psi(k-2, k-1) = 0 \quad \text{and} \quad \binom{j-1}{\ell} = 0 \quad \text{for} \quad j \leq \ell \leq k.
\]  (19)

Then, by identities (17), (18), (19), observation (11) and Lemma 3 (item 5), we obtain

\[
H_{1,jk} = 2^{k+j-2} + (-1)^{k-2} \sum_{\ell=1}^{j-1} \binom{j-1}{\ell} \psi(k-2, k-1-\ell)
\]

\[
= 2^{k+j-2} \psi(k+j-2, -1) + (-1)^{k} 2^{j-3} \psi(k+j-3, k-1).
\]

Thus, identity (15) is satisfied.

(D) Let \( r \in [1, k - 1] \) be fixed and assume as inductive hypothesis that identity (15) is satisfied up to \( j - 1 \geq 0 \). Then replacing \( r \) by \( r - 1 \) in the inductive hypothesis, we have

\[
H_{1,i,k+r-1} = (-1)^{r-1} 2^{k+i-r-1} \psi(k + i - 2, r - 2)
\]

\[
+ (-1)^{k+r-1} 2^{i-r-2} \psi(k + i - 3, k + r - 2),
\]

for all \( 1 \leq i \leq j - 1 \) and \( r - 1 \in [0, k - 2] \) fixed. In particular, this also implies (see (11)) that

\[
H_{1,k+r} = (-1)^{r} 2^{k-r-1} \psi(k-1, r-1) + (-1)^{k+r-1} 2^{r-2} \psi(k-2, k+r-1)
\]

\[
= (-1)^{r} 2^{k-r-1} \psi(k-1, r-1)
\]

*This item is used with \( j = i, r = k - 1 \) and \( t = i \) for the elements \( H_{1,i,k+i} \).

†The case \( r - 1 = 0 \) is obtained from Item (C) above.
for $r \in [1, k - 1]$ fixed. Thus, by the above two identities and Lemma 2, item (II), we get

$$H_{b,jk+r} = 2^{j-1}H_{b,k+r} - \sum_{i=1}^{j-1}2^{j-1-i}H_{b,ik+r}$$

$$= (-1)^r2^{k+j-r-2}\psi(k-1, r-1)$$

$$+ (-1)^r2^{k+j-r-2}\sum_{i=1}^{j-1}\psi(k+i-2, r-2)$$

$$+ (-1)^{k+r}2^{j-r-3}\sum_{i=1}^{j-1}\psi(k+i-3, k+r-2) \quad (20)$$

for all $r, j \in [1, k - 1]$, with $r$ fixed. We note that using the identity (1) of Lemma 3, it follows that

$$\psi(k - 1, r - 1) + \sum_{i=1}^{j-1}\psi(k+i-2, r-2)$$

$$= \psi(k, r - 1) + \sum_{i=2}^{j-1}\psi(k+i-2, r-2)$$

$$= \psi(k+1, r - 1) + \sum_{i=3}^{j-1}\psi(k+i-2, r-2)$$

$$\vdots$$

$$\psi(k + j - 3, r - 1) + \sum_{i=j-1}^{j-1}\psi(k+i-2, r-2)$$

$$= \psi(k + j - 2, r - 1). \quad (21)$$

Furthermore, by observation (11),

$$\psi(k + i - 3, k + r - 2) = 0 \quad \text{for} \quad i < r + 1. \quad (22)$$

Then, by the identities (20), (21), (22) and Lemma 3 item (2), we obtain

$$H_{b,jk+r} = (-1)^r2^{k+j-r-2}\left(\psi(k-1, r-1) + \sum_{i=1}^{j-1}\psi(k+i-2, r-2)\right)$$

$$+ (-1)^{k+r}2^{j-r-3}\sum_{i=1}^{j-1}\psi(k+i-3, k+r-2)$$

$$= (-1)^r2^{k+j-r-2}\psi(k+j-2, r-1)$$

$$+ (-1)^{k+r}2^{j-r-3}\psi(k+j-3, k+r-1).$$

Therefore, identity (15) is satisfied.
(E) We fix \( j \in [1,k-1] \) and take as inductive hypothesis the identity (15) for \( r-1 \geq 0 \). In items (A), (C) and (D) above, we proved that identity (15) is satisfied on \( H_{ik+r-1} \), for fixed \( r-1 \geq 0 \) and all \( i \in [0,k-1] \). Then

\[
H_{1,ik+r-1} = (-1)^{r-1}2^{k+i-r-1}\psi(k+i-2,r-2)
+ (-1)^{k+r-1}2^{i-r-2}\psi(k+i-3,k+r-2)
\]

holds for all \( 0 \leq i \leq j-1 \). Thus, by the above identity, identity (16) and Lemma 2, item (II), it follows that

\[
H_{1,jk+r} = 2^j H_{1,r} - \sum_{i=0}^{j-1} 2^{j-1-i}H_{1,ik+r-1}
= (-1)^{2}2^{k+j-r-2}\psi(k-2,r-1) + (-1)^{r}2^{k+j-r-2}\sum_{i=0}^{j-1}\psi(k+i-2,r-2)
+ (-1)^{k+r}2^{j-r-3}\sum_{i=0}^{j-1}\psi(k+i-3,k+r-2)
\]

holds for all \( r, j \in [1,k-1] \), with \( j \) fixed. Now, by identity (1) of Lemma 3 and identity (21), we deduce that

\[
\psi(k-2,r-1) + \sum_{i=0}^{j-1}\psi(k+i-2,r-2)
= \psi(k-1,r-1) + \sum_{i=1}^{j-1}\psi(k+i-2,r-2) = \psi(k+j-2,r-1).
\]

Therefore, identities (22), (23), (24) and Lemma 3 item (2), lead us to

\[
H_{1,jk+r} = (-1)^{r}2^{k+j-r-2}\psi(k+j-2,r-1)
+ (-1)^{k+r}2^{j-r-3}\sum_{i=r+1}^{j-1}\psi(k+i-3,k+r-2)
= (-1)^{r}2^{k+j-r-2}\psi(k+j-2,r-1)
+ (-1)^{k+r}2^{j-r-3}\sum_{i=0}^{j-r-2}\psi(k+r-2+i,k+r-2)
= (-1)^{r}2^{k+j-r-2}\psi(k+j-2,r-1)
+ (-1)^{k+r}2^{j-r-3}\psi(k+j-3,k+r-1),
\]

thus satisfying identity (15).

Finally, items (A), (B), (C), (D) and (E) above prove the lemma.

The following result generalizes the previous one.
Lemma 5. For all $k > 2; j, r \in [0, k - 1]$ and $b \in \mathbb{Z}^+$, it holds that

$$H_{b,jk+r} = \sum_{i=0}^{b} (-1)^i 2^{b-i} \psi((bk + j - b - i - 1, ik + r - 1)).$$

(25)

Proof. Here, we proceed by induction in $b$. The case $b = 1$ follows from Lemma [4]. We state as inductive hypothesis that

$$H_{b-1,jk+r} = \sum_{i=0}^{b-1} (-1)^i 2^{b-i} \psi((b-1)k + j - b - i - 1, ik + r - 1),$$

(26)

for $b - 1 \geq 1$, $k > 2$ and $j, r \in [0, k - 1]$. Next, we consider the following cases.

(A′) Let $j \leq r$. Then, by Lemma [2] (item III) with $t = j$ and the identities (26), (6), we have that

$$H_{b,jk+r} = \sum_{\ell=0}^{j} (-1)^\ell \binom{j}{\ell} 2^{j-\ell} H_{b-1,(k-1)k+r-\ell}$$

$$= \sum_{i=0}^{b-1} (-1)^i 2^{b-i} \psi((b-1)k + j - b - i - 1, ik + r - j).$$

(27)

We note that $j \leq r < ik + r$ for all $i \geq 0$, and

$$\binom{j}{\ell} = 0 \quad \text{for } j + 1 \leq \ell \leq ik + r.$$ 

(28)

We use the identities (27), (28) and Lemma [3], item (5), to obtain

$$H_{b,jk+r} = \sum_{i=0}^{b-1} (-1)^i 2^{b-i} \psi((bk - b - i - 1, ik + r - j).$$

(29)

In fact, $b(k-2) + j - 1 < bk + r - 1$ and therefore (see (11))

$$\psi(b(k-2) + j - 1, bk + r - 1) = 0.$$ 

So, the sum in identity (29) can be extended to $b$, fulfilling identity (25) in this case.

(B′) Let $r < j$. Here, we proceed by induction on $r$. 

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• For \( r = 0 \), we apply induction on \( j > 0 \). If \( j = 1 \), by the identities (6), (7) and (26) we obtain that

\[
H_{b,k} = 2H_{b,0} - H_{b,-1} = 2H_{b-1,(k-1)k} - H_{b-1,(k-2)k+(k-1)}
= \sum_{i=0}^{b-1} (-1)^{ik}2^{(b-i)k-b-i}\psi(bk - b - i - 1, ik - 1)
- \sum_{i=0}^{b-1} (-1)^{ik+k-1}2^{(b-i-1)k-b-i-1}\psi(bk - b - i - 2, ik + k - 2).
\]

In addition,

\[
\psi(bk - b - b - 1, bk - 1) = 0,
\psi(bk - b - 1, -1) = \psi(bk - b, -1) = 1
\]
(see (11)) and by Lemma 3, item (1) we obtain

\[
H_{b,k} = \sum_{i=0}^{b-1} (-1)^{ik}2^{(b-i)k-b-i}\psi(bk - b - i - 1, ik - 1)
+ \sum_{i=1}^{b} (-1)^{ik}2^{(b-i)k-b-i}\psi(bk - b - i - 1, ik - 2)
= 2^{bk-b}\psi(bk - b - 1, -1)
+ \sum_{i=1}^{b} (-1)^{ik}2^{(b-i)k-b-i} (\psi(bk - b - i - 1, ik - 1) + \psi(bk - b - i - 1, ik - 2))
= \sum_{i=0}^{b} (-1)^{ik}2^{(b-i)k-b-i}\psi(bk - b - i, ik - 1)
\]

satisfying identity (25).

• Now, we assume that identity (25) is satisfied for \( j - 1 \), when \( r = 0 \). We obtain that

\[
H_{b,jk} = 2H_{b,(j-1)k} - H_{b,(j-1)k-1} = 2H_{b,(j-1)k} - H_{b,(j-2)k+k-1}
= -H_{b,(j-2)k+k-1} + \sum_{i=0}^{b} (-1)^{ik}2^{(b-i)k+j-b-i-1}\psi(bk + j - b - i - 2, ik - 1).
\]

We also note that \( j - 2 < k - 1 \), so item (A') above implies that

\[
H_{b,(j-2)k+k-1} = \sum_{i=0}^{b} (-1)^{ik+k-1}2^{(b-i-1)k+j-b-i-2}\psi(bk+j-b-i-3, ik+k-2).
\]

Then, since

\[
\psi(bk + j - 2b - 3, bk + k - 2) = 0
\psi(bk + j - b - 2, -1) = \psi(bk + j - b - 1, -1) = 1
\]
(see (11)), we use identities (31), (32) and Lemma 3, item (1) to obtain

\[ H_{b,jk} = 2^{bk+j-b-1} \psi(bk + j - b - 2, -1) \]

\[ + \sum_{i=1}^{b} (-1)^i 2^{(b-i)k+j-b-i-1} \psi(bk + j - b - i - 2, ik - 2) \]

\[ + \sum_{i=1}^{b} (-1)^i 2^{(b-i)k+j-b-i-1} \psi(bk + j - b - i - 2, ik - 1) \]

\[ = \sum_{i=0}^{b} (-1)^i 2^{(b-i)k+j-b-i-1} \psi(bk + j - b - i - 1, ik - 1). \]

Therefore identity (25) holds in this case. This concludes the induction on \( j \) when \( r = 0 \). Thus, the base case of induction on \( r \) is done.

- Let \( j = r + 1 \). Item \( (A') \) above implies that

\[ H_{b,k+\ell} = \sum_{i=0}^{b} (-1)^i 2^{(b-i)k-k-b-i} \psi(bk - b - i, ik + \ell - 1) \]

for \( 1 \leq \ell \leq r \). By Lemma 2, item (III) with \( t = r \), and the identities (30), (33), we obtain

\[ H_{b,(r+1)k+r} = \sum_{\ell=0}^{r} (-1)^{r-\ell} \binom{r}{r-\ell} 2^\ell H_{b,k+\ell} \]

\[ = \sum_{\ell=0}^{r} \binom{r}{\ell} \left( \sum_{i=0}^{b} (-1)^i 2^{(b-i)k-k-b-i} \psi(bk - b - i, ik + \ell - 1) \right) \]

\[ = \sum_{i=0}^{b} (-1)^i 2^{(b-i)k-k-i} \sum_{\ell=0}^{r} \binom{r}{r-\ell} \psi(bk - b - i, ik + r - \ell - 1). \]

Furthermore, \( r \leq ik + r \) for \( i \geq 0 \) and

\[ \binom{r}{r-\ell} = 0, \quad \text{for } r+1 \leq \ell \leq ik + r \quad \text{when } i \geq 1. \]

Then, by identities (34), (35) and Lemma 3, item (5), we arrive at

\[ H_{b,(r+1)k+r} = \sum_{i=0}^{b} (-1)^i 2^{(b-i)k-k-i} \sum_{\ell=0}^{r} \binom{r}{r-\ell} \psi(bk - b - i, ik + r - \ell - 1) \]

\[ = \sum_{i=0}^{b} (-1)^i 2^{(b-i)k-k-i} \psi(bk - b - i + r, ik + r - 1). \]

The above identity verifies identity (25) in this case, and we finish the base step of induction on \( j \) and on \( r \).
• We assume by the inductive hypothesis that identity (25) is fulfilled for \( r - 1 \) and up to \( j - 1 \); i.e.,

\[
H_{b,\ell k + r - 1} = \sum_{i=0}^{b} (-1)^{ik + r - 1} 2^{(b-i)k + \ell - r - b - i} \psi(bk + \ell - b - i - 1, ik + r - 2)
\]

for \( r \leq \ell \leq j - 1 \). Also, by item \((A')\) above, we obtain that identity (36) holds for \( 0 \leq \ell \leq r - 1 \). In fact, it implies that

\[
H_{b,r} = \sum_{i=0}^{b} (-1)^{ik + r} 2^{(b-i)k - r - b - i - 1} \psi(b - b - i - 1, ik + r - 1).
\]

Then, by Lemma \(2\) item \((II)\) and identity (36), we arrive at

\[
H_{b,jk + r} = 2^j H_{b,r} - \sum_{\ell=0}^{j-1} 2^{j-1-\ell} H_{b,\ell k + r - 1}
\]

\[
= \sum_{i=0}^{b} (-1)^{ik + r} 2^{(b-i)k + j - r - b - i - 1} \psi(b - b - i - 1, ik + r - 1)
\]

\[
+ \sum_{i=0}^{b} (-1)^{ik + r} 2^{(b-i)k + j - r - b - i - 1} \sum_{\ell=0}^{j-1} \psi(bk + \ell - b - i - 1, ik + r - 2)
\]

for all \( r, j \in [1, k - 1] \). We observe, by Lemma \(3\), item \((1)\), that

\[
\psi(bk - b - i - 1, ik + r - 1) + \sum_{\ell=0}^{j-1} \psi(bk + \ell - b - i - 1, ik + r - 2)
\]

\[
= \psi(bk - b - i, ik + r - 1) + \sum_{\ell=1}^{j-1} \psi(bk + \ell - b - i - 1, ik + r - 2)
\]

\[
= \psi(bk - b - i + 1, ik + r - 1) + \sum_{\ell=2}^{j-1} \psi(bk + \ell - b - i - 1, ik + r - 2)
\]

\[
\vdots
\]

\[
= \psi(bk - b - i + j - 2, ik + r - 1) + \psi(bk + j - b - i - 2, ik + r - 2)
\]

\[
= \psi(bk - b - i + j - 1, ik + r - 1).
\]

\(^{\dagger}\)The case \( r = 0 \) is made in the first item of this item \((B')\).
Then, by the identities (37) and (38), we obtain

\[
H_{b,j+k+r} = \sum_{i=0}^{b} (-1)^{ik+r+2(b-i)k+j+r-b-i-1} \left( \psi(bk - b - i - 1, ik + r - 1) + \sum_{\ell=0}^{j-1} \psi(bk + \ell - b - i - 1, ik + r - 2) \right) \\
= \sum_{i=0}^{b} (-1)^{ik+r+2(b-i)k+j+r-b-i-1} \psi(bk - b - i + j - 1, ik + r - 1)
\]

and identity (25) is satisfied. This finishes the induction on both \(j\) and \(r\) thus completing the proof of this lemma.

\[\square\]

Now we return to the proof of item \((iv)\). Replacing \(b(k - 1) + j\) by \(m\) and \(r - 1\) by \(r\) in Lemma 5, we obtain that

\[
H_{mk+r} = H_{b(k-1)+j+k+r} = H_{b,j+k+r+1}
\]

(39)

\[
= \sum_{i=0}^{b} (-1)^{ik+r+1} \psi(m - i - 1, ik + r) 2^{m-i(k+1)-r-2},
\]

(40)

for all \(k > 2, j \in [0, k - 1], r \in [-1, k - 2]\) and \(b \in \mathbb{Z}^+\). By identity (6), we have

\[H_{b,r+1} = H_{b-1,(k-1)k+r+1} \quad \text{for all} \quad -1 \leq r \leq k - 2 \quad \text{and} \quad b \in \mathbb{Z}^+,
\]

so we can assume that \(j \in [0, k - 1)\) (in order not to repeat elements of the sequence in (39)). Then

\[
\frac{m}{k-1} - 1 < \frac{m-j}{k-1} \leq \frac{m}{k-1} \quad \text{for} \quad k - 1 > 1,
\]

therefore \(\left\lfloor \frac{m}{k-1} \right\rfloor - 1 < b \leq \left\lfloor \frac{m}{k-1} \right\rfloor\) and \(b = \left\lfloor \frac{m}{k-1} \right\rfloor\). In conclusion, item \((iv)\) follows from replacing \(b\) by \(\left\lfloor \frac{m}{k-1} \right\rfloor\) in identity (40)\(^8\).

This completes the proof of Theorem 1.

5 Proof of Conjecture 1

We distinguish two cases according to the parity of \(k\). Furthermore, since item \((i)\) of Theorem 1 determines the set of zeros of \(H^{(k)}\) in (2) and items \((ii)\)–\((iii)\) don’t provide zeros for \(H^{(k)}\), we can assume that \(m \geq k - 1\).

\(^8\)Note that \(m = b(k - 1) + j \geq k - 1\).
5.1 Case \( k \) even

In this case, by item (iv) in Theorem 1, we get that for all \( r \in [-1, k - 2] \) and \( m \geq k - 1 \),

\[
H_{mk + r} = (-1)^{r+1} \sum_{i=0}^{l} \left[ \left( \frac{m - i - 1}{ik + r} \right) + \left( \frac{m - i}{ik + r + 1} \right) \right] 2^{m-i(k+1)-r-2},
\]

where \( l = m - 1 \) if \( k = 2 \), and \( l = \lfloor m/(k-1) \rfloor \) if \( k > 2 \). So, for \( n = mk + r \geq k^2 - k - 1 \), we have \( H_n < 0 \) if \( n \) is even, while \( H_n > 0 \) if \( n \) is odd, given that \( n \) and \( r \) have same parity because \( k \) is even.

Thus, item (i) of Theorem 1 implies that in this case the zero–multiplicity of \( H^{(k)} \) is exactly \( k(k - 1)/2 \), confirming Conjecture 1 in this case.

5.2 Case \( k \) odd

Since we verified in our previous work \([5]\) that \( \#Z(H^{(k)}) = k(k - 1)/2 \) for \( k \in [4, 500] \), we will assume that \( H_n = 0 \), for \( n = mk + r \), with \( m \geq k - 1 \), \( r \in [-1, k - 2] \) and \( k > 500 \) odd.

5.2.1 A lower bound for \( n \) in terms of \( k \)

By item (iv) of Theorem 1, we have after simplifying a factor of \( 2^{m-r-2} \), that

\[
\sum_{i=0}^{l} (-1)^i \psi(m - i - 1, ik + r) 2^{-i(k+1)} = 0, \quad \text{for } l = \lfloor m/(k-1) \rfloor. \tag{41}
\]

We note that in the sum of equation (41), the term for \( i = 0 \) is non–zero since \( r \leq k - 2 < k - 1 \leq m \), which implies that \( r \leq m - 1 \) and therefore \( \psi(m-1, r) \neq 0 \). If the case \( i = 0 \) were the only for which \( \psi(m-i-1, ik+r) \) is non–zero, we would have that equation (41) has no solution and our problem of zero multiplicity for the odd \( k \) case would be solved. So, we must assume that at least two terms of the above sum (the first with \( i = 0 \) and the second with \( i \) odd) are non–zero.

Next, we take \( l' \) to be the largest index \( i > 0 \) for which the \( i \)-th term of the sum in equation (41) is non–zero and see that equation (41) takes the form

\[
\sum_{i=0}^{l'} (-1)^i \psi(m - i - 1, ik + r) 2^{-i(k+1)} = 0, \quad \text{for } 0 < l' \leq \lfloor m/(k-1) \rfloor,
\]

where \( \psi(m - l' - 1, l'k + r) \) is non–zero.

Separating the case \( i = l' \), we get

\[
\psi(m - l' - 1, l'k + r) = 2^{k+1} \sum_{i=0}^{l'-1} (-1)^{i+l'} \psi(m - i - 1, ik + r) 2^{(l'-i-1)(k+1)},
\]

where \( l' = m - 1 \) if \( k = 2 \), and \( l' = \lfloor m/(k-1) \rfloor \) if \( k > 2 \).
with \( l' - i - 1 \geq 0 \) for all \( i = 0, \ldots, l' - 1 \). Hence,
\[
2^{k+1} \mid \psi(m' - l - 1, l'k + r), \quad \text{where} \quad l' \leq \lfloor m/(k - 1) \rfloor.
\]

Now,
\[
\psi(v, w) = \binom{v}{w} + \binom{v + 1}{w + 1} = \binom{v}{w} \left( 1 + \frac{v + 1}{w + 1} \right).
\]

Kummer [9] proved that \( \nu \left( \binom{v}{w} \right) \) equals the number of carries when adding \( w \) with \( v - w \) in base 2. Here, for a nonzero integer \( m \), \( \nu(m) \) is the exponent of 2 in the factorization of \( m \). In particular,
\[
\nu \left( \binom{v}{w} \right) \leq \frac{\log v}{\log 2} + 1.
\]

Hence,
\[
\nu_2(\psi(v, w)) \leq \nu_2 \left( \binom{v}{w} \right) + \nu_2(v + w + 2) \leq \frac{\log v}{\log 2} + 1 + \frac{\log(v + w + 2)}{\log 2} \leq 2 \frac{\log v}{\log 2} + 2,
\]

where for the last inequality we used the fact that we may assume that \( w \leq v - 2 \) (indeed, if \( w = v \), then \( \psi(v, w) = 2 \), so the above bound holds while if \( w = v - 1 \), then \( \psi(v, w) = 2v + 1 \) is odd so the above bound again holds). Since
\[
2^{k+1} \mid \psi(v, w), \quad \text{for} \quad (v, w) = (m' - l - 1, l'k + r),
\]
we get \( \nu_2(\psi(v, w)) \geq k + 1 \). Hence,
\[
k + 1 \leq 2 \frac{\log v}{\log 2} + 2, \quad \text{therefore} \quad v \geq 2^{(k-1)/2}.
\]

Since \( m - l' - 1 \leq n \), we get that
\[
2^{(k-1)/2} \leq n. \quad (42)
\]

### 5.2.2 An upper bound for \( n \) in terms of \( k \)

We now review the preliminary work [5] on zero–multiplicity of \( H^{(k)} \), to obtain a better upper bound for \( n \) on \( k \), which we then combine with the above lower bound (42). In [5], we used a Binet–type formula of \( H_n \), namely
\[
H_n = f_k(\alpha_1^{-(n+1)}) + \cdots + f_k(\alpha_{k-1}^{-(n+1)}) + f_k(\alpha_k)^{-(n+1)}
\]
where \( \alpha_1, \ldots, \alpha_{k-1}, \alpha_k \) are all the roots of the characteristic polynomial of \( F^{(k)} \), with \( \alpha_1 > 1 \) being the only real root and the remaining \( k - 1 \) roots lie inside the unit disk. Furthermore,
\[
\alpha_1 > |\alpha_2| \geq \cdots \geq |\alpha_{k-1}| \geq |\alpha_k| \quad \text{and} \quad f_k(z) := (z-1) / (2 + (k + 1)(z - 2)).
\]
Thus, if $H_n = 0$, then

$$\left| \Lambda \right| = \left| 1 + \left( \frac{f_k(\alpha_{k-1})}{f_k(\alpha_k)} \right) \left( \frac{\alpha_k}{\alpha_{k-1}} \right)^{n+1} \right| = \sum_{i=1}^{k-2} \left| \frac{f_k(\alpha_i)}{f_k(\alpha_k)} \right| \left| \frac{\alpha_i}{\alpha_k} \right|^{n+1}$$

$$< \frac{3(k - 2)(k - 1)}{\alpha_{k-2}/\alpha_k^{n+1}} \left| \frac{1}{f_k(\alpha_k)} \right|$$

$$< 13(k(3k + 1)}{\alpha_{k-2}/\alpha_k^{n+1}}.$$

Using an argument involving lower bounds for nonzero linear forms in logarithms of complex algebraic numbers, we found

$$\left| \Lambda \right| > \exp(-5 \cdot 10^{13} \times k^7 \log(n + 1)(\log k)^2). \quad (44)$$

The following result will be fundamental to combine inequalities (43) and (44), in order to find a better upper bound for $n$ on $k$ that one given in [5]. The next lemma is Theorem 1 in [6] and represents an improvement over the analogous inequality in [2].

**Lemma 6.** The inequality

$$\frac{|\alpha_i|}{|\alpha_j|} > 1 + \frac{1}{10k^{9.6}(\pi/e)^k} \quad \text{holds for all} \quad 1 \leq i < j \leq (k - 1)/2$$

and all $k \geq 4$.

Hence, by inequalities (43), (44) and Lemma 6

$$n < 3 \cdot 10^{14}k^{1.76}(\log k)^3(\pi/e)^k. \quad (45)$$

### 5.2.3 Absolute bounds for $n, k$ and final conclusion

Combining (42) and (45), we get

$$\left( \frac{\sqrt{2}}{(\pi/e)} \right)^k < 3\sqrt{2} \cdot 10^{14}k^{1.76}(\log k)^3,$$

showing that $k \leq 790$.

By (43) and (44), we have

$$5 \cdot 10^{13} \times k^7 \log(n + 1)(\log k)^2 > \log \left| \frac{f_k(\alpha_k)}{3} \right| + (n + 1) \log \left| \frac{\alpha_{k-2}}{\alpha_k} \right|. \quad (46)$$

Thus, using the fact that $k \in [501, 789]$, we obtain computationally an upper bound on $n$ in each case:

If $H_n = 0$ and $k \in [501, 789]$, then $n \in 2.5 \cdot 10^{45}$.
Returning to inequality (42), we have \(2^{(k-1)/2} < n < 2.5 \cdot 10^{45}\) which leads to \(k \leq 517\). Finally, we return once again to (46) where now we get \(n < 3.5 \cdot 10^{43}\) for all odd \(k \in [501, 517]\). Then, by (42), we conclude that \(k < 500\), contradicting our initial assumption about \(k\).

This completes the proof of Conjecture 1 for the case of \(k\) odd.

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