PATH METHODS FOR STRONG SHIFT EQUIVALENCE OF POSITIVE MATRICES

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Abstract. In the early 1990’s, Kim and Roush developed path methods for establishing strong shift equivalence (SSE) of positive matrices over a dense subring $\mathcal{U}$ of $\mathbb{R}$. This paper gives a detailed, unified and generalized presentation of these path methods. New arguments which address arbitrary dense subrings $\mathcal{U}$ of $\mathbb{R}$ are used to show that for any dense subring $\mathcal{U}$ of $\mathbb{R}$, positive matrices over $\mathcal{U}$ which have just one nonzero eigenvalue and which are strong shift equivalent over $\mathcal{U}$ must be strong shift equivalent over $\mathcal{U}_+$. In addition, we show positive real matrices on a path of shift equivalent real matrices are SSE over $\mathbb{R}_+$; positive rational matrices which are SSE over $\mathbb{R}_+$ must be SSE over $\mathbb{Q}_+$; and for any dense subring $\mathcal{U}$ of $\mathbb{R}$, within the set of positive matrices over $\mathcal{U}$ which are conjugate over $\mathcal{U}$ to a given matrix, there are only finitely many SSE-$\mathcal{U}_+$ classes.

Contents

1. Introduction 1
2. Elementary splitting and strong shift equivalence 3
3. From strong shift equivalence to conjugacy 9
4. The Centralizer 17
5. From paths of similar matrices to strong shift equivalence 19
6. Finding positive paths: the case of one nonzero eigenvalue 20
7. The Connection Theorem 28
8. From SSE over $\mathbb{R}_+$ to SSE over $\mathcal{U}_+$ 32
Appendix A. Making SSE nondegenerate 33
Appendix B. Boolean matrices and positivity 35
Appendix C. Positive invariant tetrahedra 38
Appendix D. A local connectedness condition for nilpotent matrices 39
References 48

1. Introduction

The classification problem for shifts of finite type (SFTs) remains a central open problem for symbolic dynamics. In the foundational work of Williams [30, 31] over forty years ago, the problem was recast as the following question: when are two matrices strong shift equivalent (SSE) over $\mathbb{Z}_+$? (The definition of SSE for matrices over a semiring is recalled below in Definition 2.1.) In this paper, rings and semirings are always assumed to contain 1.) Since then, SSE (involving other semirings) has been used for classification of other symbolic dynamical systems: for example, SFTs with Markov measure [22], using matrices over Laurent polynomials; SFTs with a
finite group action \[8\], using matrices over the integral group ring of a finite group \[3\]; and sofic shifts \[4, 10, 13\], using a more complicated ring. Matsumoto has extended the ideas of SSE to a classification setting for arbitrary subshifts \[23, 24\].

The original, notoriously difficult question of Williams for \(\mathbb{Z}^+\) remains unanswered, and this is also a barrier to understanding the other classifications. One probe into this problem is to consider SSE over \(\mathcal{U}^+\) for primitive matrices over a dense subring \(\mathcal{U}\) of \(\mathbb{R}\).

Over a series of papers \[14, 15, 16, 17\] ending in 1992, Kim and Roush introduced path methods for the study of strong shift equivalence of positive matrices over the reals and certain subrings of it. One highlight of this work was the following theorem. For \(\mathcal{U}\) any subfield of \(\mathbb{R}\), if \(A\) and \(B\) are square matrices over \(\mathcal{U}^+\) which have eventual rank 1 (all large powers have rank 1) and have the same nonzero eigenvalue, then \(A\) and \(B\) are SSE over \(\mathcal{U}^+\). (As in Remark 6.5, there are in a sense no general results for greater eventual rank.) We extend this theorem to arbitrary dense subrings of \(\mathbb{R}\), under the additional necessary condition that \(A\) and \(B\) are SSE over the ring \(\mathcal{U}\). An additional condition cannot be avoided: for general \(\mathcal{U}\), matrices with eventual rank 1 and the same nonzero eigenvalue need not be even shift equivalent over \(\mathcal{U}\) to a 1 \(\times\) 1 matrix (see Remark 6.8). Whether the assumption of SSE over \(\mathcal{U}\) is equivalent to the more tractable condition of shift equivalence over \(\mathcal{U}\) remains an open question. However, our proof requires the assumption of SSE, not just SE, over \(\mathcal{U}\).

The central result of the path methods development was a Path Theorem for \(\mathbb{R}\): matrices on a path of positive conjugate (similar) matrices must be SSE over \(\mathbb{R}^+\). In this paper, we prove a generalized Path Theorem \(5.10\) which has application to arbitrary dense subrings of \(\mathbb{R}\). We also show that matrices on a path of positive matrices shift equivalent over \(\mathbb{R}\) must be SSE over \(\mathbb{R}^+\). This is a consequence of a more technical statement, the Connection Theorem \(7.2\), which relies in turn on a result which is pure linear algebra (Theorem \(D.4\)).

One indication of the power of the path method comes from the corollary due to Chuysurichay (Theorem \(5.12\)): the set of positive matrices in a given conjugacy (similarity) class over \(\mathbb{R}\) contains only finitely many SSE-\(\mathbb{R}^+\) classes. This holds even though (as shown by Chuysurichay) it may be impossible to connect matrices in the class with SSEs with uniformly bounded lags (see Remark 5.13). Using our Path Theorem, we generalize this finiteness result to arbitrary dense subrings of \(\mathbb{R}\) (Theorem \(5.16\)).

Using the Connection Theorem, we are able to show that positive real matrices SSE over \(\mathbb{R}^+\) are SSE over \(\mathbb{R}^+\) through positive matrices (Theorem \(8.1\)). As a consequence, primitive positive trace matrices over a subfield \(\mathcal{U}\) of \(\mathbb{R}\) which are SSE over \(\mathbb{R}^+\) must also be SSE over \(\mathcal{U}^+\) (Theorem \(8.2\)).

Altogether, for positive matrices SSE over a dense subring \(\mathcal{U}\) of \(\mathbb{R}\), the current paper reduces the gap between SSE over \(\mathbb{R}^+\) and SSE over \(\mathcal{U}^+\), and indicates the utility of investigating SSE of positive matrices over \(\mathbb{R}^+\). This is a problem to which more standard mathematics (e.g. fiber bundles, linear algebra) can be applied, as seen in \[13, 14, 15, 16, 17\] and the current paper.

We now say a little about the organization of the paper.

In Section 2, we explain the decomposition of SSE into row splittings, column splittings and diagonal refactorizations, and provide some basic technical results...
essential for the sequel. The results of this section hold over quite general rings, and refine the basic Williams theory.

In Section 3 given positive matrices $A, B$ which are SSE over $U$, we produce positive matrices $A', B'$ which are conjugate over $U$ such that $A$ is SSE-$U_+$ to $A'$ and $B$ is SSE-$U_+$ to $B'$. (This is the step for which we need matrices SSE-$U$, not just SE-$U$.)

In Section 4, we study $\text{Cent}_R(A)$, the group of invertible real matrices which commute with a given $n \times n$ real matrix $A$, and its group of connected components, $\pi_0(\text{Cent}_R(A))$. This group plays a key role in the formulation of obstructions to applying the Path Theorem to produce SSE-$U_+$.

In Section 5, we prove the Path Theorem 5.10 and some consequences. In Section 6, we prove the eventually rank 1 results. In Section 7, we prove the Connection Theorem. A large part of the proof is an independent result in linear algebra, which we relegate to Appendix D. In Section 8, we prove in particular that rational matrices SSE over $\mathbb{R}^+$ must be SSE over $\mathbb{Q}^+$. This is some supporting evidence for the conjecture [2, Conj. 5.1] that positive rational matrices shift equivalent over $\mathbb{Q}^+$ are SSE over $\mathbb{Q}^+$.

This paper is entirely devoted to matrices. For background on shifts of finite type and symbolic dynamics, see [20, 19].

2. Elementary splitting and strong shift equivalence

Bob Williams introduced shift equivalence and strong shift equivalence in his paper [30], which is the foundation of all future work on the topic. One of the fundamental contributions was a decomposition of an elementary strong shift equivalence using even more fundamental relations, splittings and amalgamations. In [30], Williams considered matrices over $\mathbb{Z}^+$ and $\{0, 1\}$. For our work with unital nondiscrete subrings of $\mathbb{R}$, we need some refinements to this work.

Definition 2.1. Let $U$ be a subset of a semiring containing 0 and 1 (additive and multiplicative identities). Matrices $A, B$ are elementary strong shift equivalent over $U$ (ESSE-$U$) if there exist matrices $R, S$ over $U$ such that $A = RS$ and $B = SR$. Matrices $A, B$ are strong shift equivalent over $U$ (SSE-$U$) if there exist matrices $A_0, A_1, \ldots, A_\ell$, and for $1 \leq i \leq \ell$ matrices $R_i, S_i$ over $U$ such that $A_{i-1} = R_i S_i$ and $A_i = S_i R_i$, with $A_0 = A$ and $A_\ell = B$. In this case the string $(R_i, S_i)$, $1 \leq i \leq \ell$, is a strong shift equivalence of lag $\ell$ from $A$ to $B$.

Although we do not use shift equivalence before Section 7 to clarify ideas we recall its basic features now.

Definition 2.2. Let $U$ be a subset of a semiring containing 0 and 1 (additive and multiplicative identities). Matrices $A, B$ are shift equivalent over $U$ (SE-$U$) if there exist matrices $R, S$ over $U$ and $\ell \in \mathbb{N}$ such that the following hold:

$$A^\ell = RS \quad B^\ell = SR \quad AR = RB \quad BS = SB.$$
There is a “conceptual” version of shift equivalence, in terms of isomorphism of associated dimension modules.

Williams asked whether the relatively tractable relation $\text{SE-} \mathbb{Z}_+$ implies $\text{SSE-} \mathbb{Z}_+$. Working in the framework of Wagoner’s algebraic topological framework the classification problem [23], Kim and Roush gave examples of primitive matrices over $\mathbb{Z}$ which are $\text{SSE over} \mathbb{Z}$ (equivalently, shift equivalent over $\mathbb{Z}$) but not $\text{SSE over} \mathbb{Z}_+$ [15]. There are also such examples over certain dense subrings of $\mathbb{R}$, in place of $\mathbb{Z}$ (see Remark 5.14). A feature of Wagoner’s framework is that it is built up out of elementary SSEs, not out of SE. We will see the same feature in the proof of Theorem 5.11. For a ring $\mathcal{U}$, can be useful to take $\text{SSE-} \mathcal{U}$ as a hypothesis, and leave the question of whether $\text{SE-} \mathcal{U}$ implies $\text{SSE-} \mathcal{U}$ as a separate issue.

We turn away now from shift equivalence, until Section 7. For a ring $\mathcal{U}$, a $\mathcal{U}$-

**Definition 2.3.** An amalgamation matrix is a matrix with entries from $\{0,1\}$ such that every row has exactly one 1 and every column has at least one 1. A subdivision matrix is the transpose of an amalgamation matrix.

**Definition 2.4.** An elementary row splitting is an elementary strong shift equivalence $UX = A, Xu = C$ in which $U$ is a subdivision matrix. In this case, $C$ is an elementary row splitting of $A$, and $A$ is an elementary row amalgamation of $C$.

**Definition 2.5.** An elementary column splitting is an elementary strong shift equivalence $XV = A, VX = C$ in which $V$ is an amalgamation matrix. In this case, $C$ is an elementary column splitting of $A$, and $A$ is an elementary column amalgamation of $C$.

Given $a_1 + a_2 + a_3 = a$, $b_1 + b_2 + b_3 = b$, $c_1 + c_2 = c$, $d_1 + d_2 = d$, here is an elementary row splitting $C$ of $A$:

$$UX = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} a \\ b \\ b \end{pmatrix} = A$$

$$XU = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \\ c_1 & d_1 \\ c_2 & d_2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} a_1 & a_1 & a_1 & b_1 \\ a_2 & a_2 & a_2 & b_2 \\ a_3 & a_3 & a_3 & b_3 \\ c_1 & c_1 & c_1 & d_1 \\ c_2 & c_2 & c_2 & d_2 \end{pmatrix} = C$$

For an elementary row splitting, rows of $A$ are split as sums of rows (as described by $X$), and then columns of $X$ are “copied” in such a way that indices of rows in $C$ with the same “parent” row in $A$ have equal columns in $C$. We say a row in $C$ in $A$ is sitting above its parent row.

Similarly, here is an example of an elementary column splitting.

$$YV = \begin{pmatrix} -3.4 \\ \pi \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -3.4 \\ \pi \end{pmatrix} = A$$

$$VY = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -3.4 \\ \pi \end{pmatrix} = \begin{pmatrix} -3.4 \\ \pi \end{pmatrix}$$
Here, columns 1 and 2 of $VY$ are sitting above column 1 of $A$ in the column splitting.

**Definition 2.6.** A matrix is nondegenerate if it has no zero row and it has no zero column.

**Definition 2.7.** A diagonal refactorization over a semiring $U$ is an elementary strong shift equivalence over $S$ of the form $A = DX$, $B = XD$, where $D$ is nondegenerate diagonal over $S$. In this case, $A$ is a diagonal refactorization of $B$ (and vice versa).

We now recall the canonical factorization of a nondegenerate matrix introduced by Williams [30].

Suppose $M$ is a nondegenerate matrix over a semiring $U$ containing $\{0, 1\}$, with rows indexed by the set $I$ and columns indexed by the set $J$. Let $E$ be the set of pairs $(i, j)$ such that $M(i, j) \neq 0$. Let $U_M$ be the $I \times E$ subdivision matrix such that $U_M(i', (i, j)) = 1$ iff $i' = i$. Let $V_M$ be the $E \times J$ amalgamation matrix such that $V_M((i, j), j') = 1$ iff $j = j'$. Let $D_M$ be the $E \times E$ diagonal matrix such that $D_M((i, j), (i, j)) = M(i, j)$. Then

$$M = U_M D_M V_M.$$  

Because $M$ is nondegenerate, $U_M$ and $V_M$ are defined (e.g., given $i$ there is at least one $j$ such that $D(i, j) \neq 0$, so row $i$ of $U_M$ has at least one 1), and $D$ has nonzero diagonal entries.

There is a graphical interpretation of the factorization $M = U_M D_M V_M$. The set $E$ can be viewed as the set of edges of a directed graph, in which there is an edge from $i$ to $j$ if $M(i, j)$ is nonzero. The matrices $U_M$ and $V_M$ attach (respectively) initial and terminal vertices to edges, and $D_M$ records the entry of $M$ labeling the edge.

**Definition 2.8.** We call the factorization $M = U_M D_M V_M$ above of a nondegenerate matrix $M$ the Williams factorization of $M$. It is well defined up to the choice of ordering of indices used for $D_M$ (and thus $U_M$ and $V_M$).

If $M$ is a nondegenerate matrix and $M = UDV$ with $U$ subdivision, $D$ nondegenerate diagonal and $V$ amalgamation, then $M = UDV$ must be the Williams factorization described above.

We may avoid the complications of defining a factorization for degenerate matrices, on account of the following proposition.

**Proposition 2.9.** Suppose $U$ is a ring which is torsion free as an additive group. Suppose nondegenerate matrices $A$ and $B$ are SSE over $U$. Then they are SSE through a chain of ESSEs $A_i-1 = R_i S_i, A_{i+1} = S_i R_i$ such that all the matrices $A_i, R_i, S_i$ are nondegenerate.

The proof of Proposition 2.9 is a digression, and we give it in Appendix A

**Proposition 2.10.** Suppose $A = RS, B = SR$ is a nondegenerate elementary strong shift equivalence over a semiring $U$ containing $\{0, 1\}$, and $U$ has no zero divisors. Then there are nondegenerate matrices $C_1, C_2, D$ over $U$ such that $D$ is diagonal and

1. $C_1$ is an elementary row splitting of $A$
2. There is a matrix $X$ over $U$ such that $DX = C_1$ and $XD = C_2$

(so, $C_1$ is a diagonal refactorization of $C_2$)
(3) $C_2$ is an elementary column splitting of $B$.

Proof. Using the Williams factorization above, we have
\[
A = (URDRV) (USDSVS) \\
B = (USDSVS) (URDRVR).
\]

Define
\[
C_1 = (DRVRUSDSVS) UR, \quad X_1 = DRVURDSVS
\]
\[
C_2 = VR (USDSVSRD), \quad X_2 = USDSVSURDR.
\]

Set $D = DR$ and $X = VRUSDSVSUR$. Then $DX = C_1$ and $XD = C_2$, proving (2). Also,
\[
A = URX_1 \quad B = X_2VR
\]
\[
C_1 = X_1UR \quad C_2 = VRX_2.
\]
This proves (1) and (3). It remains to prove the nondegeneracy claims.

The matrix $D = DR$ is nondegenerate by construction. The matrix $X_2$ has no zero row, because $B = X_2VR$ has no zero row. The matrix $(USDSVS)$ has no zero column because $A$ has no zero column. Because $UR$ is a subdivision matrix, the matrix $(USDSVS)UR$ then has no zero column. Because there are no zero divisors, the matrix $(USDSVS)URDR = X_2$ has no zero column. Thus $X_2$ is nondegenerate. Because $VR$ is an amalgamation matrix, the matrix $C_2 = VRX_2$ is also nondegenerate. Similarly, $C_1$ is nondegenerate. This proves the proposition. \qed

Remark 2.11. If (for example) $U$ is a subring of the reals, then in Proposition 2.10
the matrix $D^{-1}$ is defined over $\mathbb{R}$ and the matrices $D^{-1}C_1$ and $C_2D^{-1}$ have entries in $U$. Then we can summarize the proposition with a diagram
\[
\begin{array}{c}
\overset{(X,U)}{C_1} \quad \overset{(D,D^{-1}C_1)}{D^{-1}C_1D} \\
\overset{(V,Y)}{B}
\end{array}
\]

in which an arrow labelled $(J,K)$ from $M$ to $N$ represents an elementary strong shift equivalence $M = JK$, $N = KJ$; $U$ is a subdivision matrix; and $V$ is an amalgamation matrix. $C_1$ is an elementary row splitting of $A$ and $D^{-1}C_1D$ is an elementary column splitting of $B$.

For matrices over $\{0,1\}$, the next lemma is well known ([25], [20, Theorem 2.1.14]), and can be interpreted as a fiber product statement.

Lemma 2.12 (Fiber Lemma). Suppose over a ring $U$ there is an elementary row splitting of $A$ to a nondegenerate $C_1$ and an elementary column splitting of $A$ to a nondegenerate $C_2$. Then there is a nondegenerate matrix $F$ such that over $U$ there is an elementary column splitting of $C_1$ to $F$ and an elementary row splitting of $C_2$ to $F$.

If all entries of $C_1$ and $C_2$ are nonnegative, or positive, entries in a nondiscrete unital subring $U$ of $\mathbb{R}$, then all entries of $F$ can be chosen to have nonnegative, or positive, entries in $U$. 

Proof. Let $\mathcal{I}$ denote the set indexing the rows and columns of $A$. For $s = 1, 2$ let $\mathcal{I}_s$ be the index set for the rows and columns of $C_s$. The index set for the rows and columns of $F$ will be the set

$$\mathcal{V} := \{(i_1, i_2) \in \mathcal{I}_1 \times \mathcal{I}_2 : \overline{i_1} = \overline{i_2} = i\}$$

where $\overline{i}$ denotes the element of $\mathcal{I}$ associated to $i$ under the given elementary splitting of $A$ to $C_s$. For $i \in \mathcal{I}$, let $\mathcal{V}(i) = \{(i_1, i_2) \in \mathcal{V} : \overline{i_1} = \overline{i_2} = \overline{i}\}$. Let $F_{<i,j>}$ denote the submatrix of $F$ with index set $\mathcal{V}(i) \times \mathcal{V}(j)$. Given $i \in \mathcal{I}$, we let $\mathcal{I}_s(i)$ denote the set of indices $i_s$ in $\mathcal{I}_s$ such that $\overline{i}_s = i$.

We will define $F$ by defining $F_{<i,j>}$ for each $i, j$. So, consider now $i, j$ from $\mathcal{I}$. For notational simplicity, suppose for the definition of $F_{<i,j>}$ that $\mathcal{I}_1(i) = \{1, \ldots, m\}$ and $\mathcal{I}_2(j) = \{1, \ldots, n\}$. We will define an $m \times n$ matrix $M = M_{<i,j>}$ and then set $F_{<i,j>}((i_1, i_2), (j_1, j_2)) = M_{i_1, j_2}$.

Let $\alpha$ denote $A(i, j)$. By the nature of row splitting, there is a vector $\alpha = (\alpha_1, \ldots, \alpha_n)$ over $\mathcal{U}$ such that $C_1(s, k) = \alpha_s$ for every $k \in I_1(j)$ and $1 \leq s \leq m$. Likewise, there is a vector $\beta = (\beta_1, \ldots, \beta_n)$ over $\mathcal{U}$ such that $C_2(k, t) = \beta_t$ for every $k \in I_2(i)$ and $1 \leq t \leq n$. Also, $\sum_s \alpha_s = a = \sum_t \beta_t$.

We now arrange that the vector of row sums of $M$ is $\alpha$ and the vector of column sums of $M$ is $\beta$. (In the special case that $\mathcal{U}$ is a field, for $a \neq 0$ we could simply set $M(s, t) = (1/a)\alpha(s)\beta(t)$.) If $m = 1$, we necessarily set $M(1, t) = \beta(t)$, $1 \leq t \leq n$. If $n = 1$, we likewise set $M(s, 1) = \alpha(s)$, $1 \leq s \leq m$. If $m = n = 1$, the two definitions coincide. If $m$ and $n$ are greater than $1$, pick an $(m-1) \times (n-1)$ matrix $N$ over $\mathcal{U}$ and for $1 \leq s < m$ and $1 \leq t < n$ define $M(s, t) = N(s, t)$. Then for $1 \leq s < m$, define $M(s, n)$ so that the $st$th row sum is $\alpha(s)$, and for $1 \leq t < n$ define the entries $M(m, t)$ so that the $tt$th column sum is $\beta(t)$. These additional entries must lie in the ring $\mathcal{U}$. Finally define $M(m, n)$ so that the sum of the entries of $M$ is $a$. Necessarily $M(m, n)$ is in $\mathcal{U}$. The $mt$th row sum of $M$ is $\alpha(m)$ because it equals $a$ minus the sum of the other row sums $\alpha(1), \ldots, \alpha(m-1)$. Similarly the $nt$th column sum of $M$ is $\beta(n)$.

In the case that $C_1$ and $C_2$ have nonnegative real entries and $m > 1$ and $n > 1$, if $a = 0$ then set $M = 0$. If $a > 0$, then for notational convenience suppose $\alpha_n \beta_n > 0$. Then choose $N$ above such that $N(s, t) = 0$ whenever $\alpha(s)\beta(t) = 0$ and otherwise

$$0 < (1/a)\alpha(s)\beta(t) - N(s, t) < \epsilon$$

where $\epsilon$ is small enough to guarantee that $M(m, n) > 0$. Then $M$ will be nonnegative, and $M$ will be positive if $\alpha$ and $\beta$ are positive. This finishes the definition of $M$ and $F$.

Now define a $\mathcal{V} \times \mathcal{I}_1$ amalgamation matrix $V$ and an $\mathcal{I}_1 \times \mathcal{V}$ matrix $X$ by the rules

$$V((i_1, i_2), i_3) = 1 \text{ iff } i_1 = i_3$$

$$X(i_1, (j_1, j_2)) = M_{<i,j>}((i_1, i_2), (j_1, j_2)) \text{, \ where } (i, j) = (\overline{i_1}, \overline{j_2})$$
Suppose Lemma 2.13.

If $A$ is a nondegenerate diagonal matrix and there is a positive integer $k$ such that $k < n$, then there are matrices $A'$ and $B$ such that $A' = AB$. Let $E'$ denote the $k \times k$ upper left corner of $E$. Then in block form, for some matrices $C_i$ over $U$ (with $1 \leq i \leq 4$ and $C_1 k \times k$) we have

$$A = \begin{pmatrix} E C_1 & E C_2 \\ C_3 & C_4 \end{pmatrix}, \quad B = \begin{pmatrix} C_1 E & C_2 \\ C_3 E & C_4 \end{pmatrix}.$$ 

An elementary row splitting of $A$ to an $(n + k) \times (n + k)$ matrix $A'$ is given by

$$A' = \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix} \begin{pmatrix} E C_1 & E C_2 \\ C_3 & C_4 \end{pmatrix} = \begin{pmatrix} E C_1 & E C_2 \\ C_3 & C_4 \end{pmatrix}.$$ 

Similarly,

$$(V X)((i_1, i_2), (j_1, j_2)) = X((i_1, j_1), (j_1, j_2))$$

$$= M_{<i,j>}(i_1, j_2), \text{ where } (i, j) = (\overline{i_1, j_2}),$$

$$= \alpha(i_1) = C_1(i_1, j_1).$$

Thus $C_1 = VX$ and $F = XV$, and $F$ is an elementary column splitting of $C_1$.

Likewise, $F$ is an elementary row splitting of $C_2$. Define an $I_2 \times V$ subdivision matrix $U$ and a $V \times I_2$ matrix $Y$ by the rules

$$U((i_3, (i_1, i_2)) = 1 \text{ iff } i_3 = i_2,$$

$$Y((i_1, i_2), j_2) = M_{<i,j>}(i_1, j_2), \text{ where } (i, j) = (\overline{i_1, j_2}).$$

Then $F = YU$ and $C = UY$, by a similar computation.

Finally, suppose $C_1$ and $C_2$ are nondegenerate. Then $F$ has no zero column (being a row splitting of $C_2$) and $F$ has no zero row (being a column splitting of $C_1$), so $F$ is nondegenerate.

Lemma 2.13. Suppose $U$ is a unital ring, $A$ and $B$ are $n \times n$ matrices over $U$, and there is a nondegenerate diagonal matrix $D$ and a matrix $C$ such that $A = DC$ and $B = CD$.

Then there are matrices $A', B'$ over $U$ such that $A'$ is an elementary row splitting of $A$, $B'$ is an elementary column splitting of $B$, and $A'$ is conjugate over $U$ to $B'$. If $A$ and $B$ are nondegenerate and the ring $U$ has no zero divisors, then the matrices $A', B'$ can be chosen nondegenerate.

Proof. If $D = I_n$, we are done, so suppose not. For notational simplicity, suppose there is a positive integer $k$ such that $D(i, i) = 1$ if $i > k$. Suppose $k < n$. Let $E$ denote the $k \times k$ upper left corner of $D$. Then in block form, for some matrices $C_i$ over $U$ (with $1 \leq i \leq 4$ and $C_1 k \times k$) we have

$$A = \begin{pmatrix} E C_1 & E C_2 \\ C_3 & C_4 \end{pmatrix}, \quad B = \begin{pmatrix} C_1 E & C_2 \\ C_3 E & C_4 \end{pmatrix}.$$ 

An elementary row splitting of $A$ to an $(n + k) \times (n + k)$ matrix $A'$ is given by

$$A' = \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix} \begin{pmatrix} E C_1 & E C_2 \\ C_3 & C_4 \end{pmatrix} = \begin{pmatrix} E C_1 & E C_2 \\ C_3 & C_4 \end{pmatrix}.$$ 

$$A' = \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix} \begin{pmatrix} E C_1 & E C_2 \\ C_3 & C_4 \end{pmatrix} = \begin{pmatrix} E C_1 & E C_2 \\ C_3 & C_4 \end{pmatrix}.$$
An elementary column splitting of $B$ to an $(n + k) \times (n + k)$ matrix $B'$ is given by

$$B = \begin{pmatrix} C_1 & C_1(-I_k + E) & C_2 \\ C_3 & C_3(-I_k + E) & C_4 \end{pmatrix} \begin{pmatrix} I_k & 0 \\ I_k & 0 \\ 0 & I_{n-k} \end{pmatrix} = \begin{pmatrix} C_1 E & C_2 \\ C_3 E & C_4 \end{pmatrix}$$

$$B' = \begin{pmatrix} I_k & 0 \\ I_k & 0 \\ 0 & I_{n-k} \end{pmatrix} \begin{pmatrix} C_1 & C_1(-I_k + E) & C_2 \\ C_3 & C_3(-I_k + E) & C_4 \end{pmatrix} = \begin{pmatrix} C_1 & C_1(-I_k + E) \\ C_1 & C_1(-I_k + E) \\ C_3 & C_3(-I_k + E) \end{pmatrix}.$$  

Define

$$W = \begin{pmatrix} 0 & I_k & 0 \\ I_k & E - 2I_k & 0 \\ 0 & 0 & I_{n-k} \end{pmatrix},$$

with inverse $W^{-1} = \begin{pmatrix} 2I_k - E & I_k & 0 \\ I_k & 0 & 0 \\ 0 & 0 & I_{n-k} \end{pmatrix}.$

A computation shows $A'W = WB'.$ If $A$ and $B$ are nondegenerate and $U$ has no zero divisors, then the constructed matrices $A'$ and $B'$ are nondegenerate. This finishes the proof for the case $k < n.$ If $k = n,$ then simply remove block rows and columns through $C_4$ from the proof above, and repeat the proof with $D$ in place of $E.$

\section{From strong shift equivalence to conjugacy}

Let $U$ be a nondiscrete unital subring of $\mathbb{R}.$ Two $n \times n$ matrices $A$ and $B$ with entries in $U$ are \textit{conjugate over $U$}, or \textit{similar over $U$}, if there exists $W$ in $GL(n, U)$ such that $W^{-1}AW = B.$

The purpose of this section is to prove the following theorem.

\textbf{Theorem 3.1.} Let $U$ be a nondiscrete unital subring of $\mathbb{R}.$ Suppose $A, B$ are positive matrices over $U$ and are strong shift equivalent over $U.$ Then $A$ and $B$ are strong shift equivalent over $U_+$ to positive matrices which are conjugate over $U.$

Moreover, the conjugating matrix can be chosen to have positive determinant and to send positive eigenvectors to positive eigenvectors.

We begin with the main lemma. We use the following notation: $I_k$ denotes the $k \times k$ identity matrix.

\textbf{Lemma 3.2 (Splitting Lemma).} Let $U$ be a nondiscrete unital subring of $\mathbb{R}.$ Suppose the following:

- $A$ and $C$ are $n \times n$ matrices over $U$
- $W$ is a matrix in $GL(n, U)$ such that $W^{-1}AW = C$
- $C'$ is obtained from $C$ by a finite sequence of row splittings over $U.$

Then the following hold.

1. There is a matrix $A'$ conjugate over $U$ to $C'$ such that $A'$ is obtained from $A$ by a finite sequence of row splittings over $U$; and such that, if $A$ and $C'$ are nondegenerate, then $A'$ is nondegenerate.
2. If $A$ is a positive matrix, then there is a positive matrix $A^+$ over $U$ such that $A^+$ is obtained from $A$ by a finite sequence of row splittings of positive matrices over $U_+,$ and $A^+$ is conjugate over $U$ to a matrix of the form $\begin{pmatrix} C' & 0 \\ 0 & 0 \end{pmatrix}.$
The lemma statement is also true with “row” replaced by “column”.

**Proof.** Any row splitting to a larger matrix is a composition of row splittings which increase the matrix size by exactly one. So, we have some positive integer $\ell$ and a finite sequence of elementary row splittings of matrices $C_i$ to $C_{i+1}$, $0 \leq i < \ell$, with $C_{i+1}$ obtained by splitting one row of $C_i$ to two rows, and with $C = C_0$ and $C' = C_\ell$.

**Proof of Claim (1)** We first consider the case that $\ell = 1$. For notational convenience, suppose row $n$ of $C$ is split into rows $n$ and $n + 1$ of $C'$. For any matrix $B$, we let $B_{\text{row}(i)}$ denote its $i$th row. We have matrices

$$X' = \begin{pmatrix} C_{\text{row}(1)} \\ \vdots \\ C_{\text{row}(n-1)} \\ s' \\ t' \end{pmatrix}, \quad U' = \begin{pmatrix} t_{n-1} & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

such that $U'X' = C$ and $X'U' = C'$. Set $t = t'W^{-1}$ and set $s = A_{\text{row}(n)} - t$ and define the $(n + 1) \times n$ matrix

$$Y' = \begin{pmatrix} A_{\text{row}(1)} \\ \vdots \\ A_{\text{row}(n-1)} \\ s \\ t \end{pmatrix}.$$}

Then $U'Y' = A$. Define $A' = Y'U'$, an elementary row splitting of $A$. Let $E$ be the $(n + 1) \times (n + 1)$ matrix equal to $I_{n+1}$ except that $E(n, n + 1) = -1$. Then we have matrix equations (in block forms)

$$E^{-1}A'E = \begin{pmatrix} A & 0 \\ t & 0 \end{pmatrix}, \quad E^{-1}C'E = \begin{pmatrix} C & 0 \\ t' & 0 \end{pmatrix}$$

$$\begin{pmatrix} A & 0 \\ t & 0 \end{pmatrix} \begin{pmatrix} W & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} W & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} C & 0 \\ t' & 0 \end{pmatrix}.$$}

Therefore $A'$ is conjugate over $U$ to $C'$.

At the inductive step, going from $\ell - 1$ to $\ell$, we apply the same argument to matrices $A_{\ell-1}$ and $C_{\ell-1}$ given by the induction hypothesis.

Now suppose $A$ is nondegenerate. Then no sequence of row splittings of $A$ can produce a matrix with a zero column. If $C'$ is nondegenerate, then we can choose all those row splittings $C_i$ to $C_{i+1}$, splitting some row as a sum $s_i + t_i$, such that $s_i \neq 0 \neq t_i$. Then the construction, splitting $A_i'$ to $A_{i+1}'$, never introduces a zero row, and in the end $A'$ will be nondegenerate. This completes the proof of (1).

**Proof of Claim (2).** As in part (1), we first consider the case $\ell = 1$. Let $s', t', s, t$
be as in part (1). Define matrices

\[ U = \begin{pmatrix} I_{n-1} & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}, \quad X'' = \begin{pmatrix} C_{\text{row}(1)} \\ \vdots \\ C_{\text{row}(n-1)} \\ s' + t' \\ 0 \end{pmatrix}, \quad Y'' = \begin{pmatrix} A_{\text{row}(1)} \\ \vdots \\ A_{\text{row}(n-1)} \end{pmatrix}. \]

Set \( A'' = Y''U \). Let \( F \) be the \((n + 2) \times (n + 2)\) matrix equal to \( I_{n+2} \) except that \( F(n, n + 1) = F(n, n + 2) = -1 \). Then

\[ F^{-1}A''F = \begin{pmatrix} A & 0 & 0 \\ t & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad F^{-1}C''F = \begin{pmatrix} C & 0 & 0 \\ t' & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \]

The matrix \( F^{-1}C''F \) (and therefore \( A'' \)) is conjugate over \( U \) to the \((n + 2) \times (n + 2)\) matrix \( \begin{pmatrix} C' & 0 \\ 0 & 0 \end{pmatrix} \).

It remains to conjugate \( A'' \) over \( U \) to a matrix \( A^+ \) which is the required row splitting of \( A \). For this we will pick a suitable invertible \( 3 \times 3 \) matrix \( M \) with all column sums equal to 1, define \( W^+ \) to be \( \begin{pmatrix} I_{n-1} & 0 \\ 0 & M \end{pmatrix} \) and set \( A^+ \) equal to \( W^+A''(W^+)^{-1} \).

Because \( M \) has all column sums 1 (i.e., fixes the row vector with every entry 1), it follows that \( M^{-1} \) has all column sums 1, and therefore \( U(W^+)^{-1} = U \). Consequently,

\[ A^+ = W^+Y''U(W^+)^{-1} = W^+Y''U = \begin{pmatrix} I_{n-1} & 0 \\ 0 & M \end{pmatrix} Y''U. \]

The matrix \( M \) will have the form

\[ (3.1) \begin{pmatrix} x_1 & x_1 + \epsilon_1 \\ x_2 & x_2 + \epsilon_2 \\ 1 - (x_1 + x_2) & 1 - (x_1 + x_2) - (\epsilon_1 + \epsilon_2) \\ 1 - (z_1 + z_2) \end{pmatrix} \]

and therefore the bottom three rows of \( W^+Y'' \) will equal

\[ (3.2) M \begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} x_1(s + t) + \epsilon_1 t \\ x_2(s + t) + \epsilon_2 t \\ 1 - (x_1 + x_2) (s + t) - (\epsilon_1 + \epsilon_2) t \end{pmatrix}. \]

These three rows sum to \( s + t \), which is row \( n \) of \( A \). Thus \( U(W^+Y'') = A \), and \( A^+ = (W^+Y'')U \) is an elementary row splitting of \( A \).

We now complete the definition of \( M \). Given \( \gamma > 0 \), pick positive numbers \( x_1, x_2, \epsilon_1 \) from \( U \) with \( |x_1 - 1/3|, |x_2 - 1/3| \) and \( \epsilon_1 \) all smaller than \( \gamma \). Pick \( K \in \mathbb{N} \) such that \( K \epsilon_1 \leq 1 < (K + 1) \epsilon_1 \) and set \( \epsilon_2 = 1 - K \epsilon_1 \). For small \( \gamma \), this guarantees that \( A_+ \) is positive (the rows in \( 3.2 \) are approximately \((1/3)A_{\text{row}(n)}\)).

Define \( z_1 = -1 + x_1 \) and \( z_2 = K + x_2 \). A computation shows

\[ \det(M) = \epsilon_1(z_2 - x_2) - \epsilon_2(z_1 - x_1) = \epsilon_1(K) - \epsilon_2(-1) = 1. \]

Therefore \( M \in SL(3, U) \), and \( W^+ \) gives a conjugacy of \( A^+ \) to \( A'' \) as required.

Let \( 0_i \) denote the \( i \times i \) zero matrix. At the inductive step, we begin with a conjugacy of a positive matrix \((A^+)_{n-1}\) to a matrix with block form \( \begin{pmatrix} C_{n-1} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} C_1 & 0 \\ 0 & 0 \end{pmatrix} \).
$C_{t-1} \oplus 0_{t-1}$, and a splitting of $C_{t-1}$ to $C_t$. The argument of the basic step produces a row splitting of $(A^+)_t$ to a positive matrix $(A^+)_t$ over $\mathcal{U}$ and a conjugacy over $\mathcal{U}$ of $(A^+)_t$ to $(C_t \oplus 0_{t-1}) \oplus 0_1$, which equals $C_t \oplus 0_t$.

The final claim of the lemma is clear by passing to transpose matrices. \qed

Remark 3.3. If the nondiscrete ring $\mathcal{U}$ is assumed to have a nontrivial unit, then in Lemma 3.2 the matrix $A^+$ can be chosen to have size equal to the matrix $C$ (the extra zero blocks can be avoided). For this, in the proof at the stage of splitting the row $s' + t'$ of $C$, pick $a, b$ from $\mathcal{U}$ such that $a$ closely approximates $1/2$ and $b$ is a sufficiently small unit. In place of the matrices $U, X, M$ in the proof use

$$U = \begin{pmatrix} I_{n-1} & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad X'' = \begin{pmatrix} * \\ s' \\ t' \end{pmatrix}, \quad M = \begin{pmatrix} a & a-b \\ 1-a & b+(1-a) \end{pmatrix}.$$

Then $\det M = b$, so $M$ is invertible over $\mathcal{U}$, and if $a$ is chosen close to $1/2$ and $b$ is sufficiently small, the positivity constraints will be satisfied.

Remark 3.4 (Matrices, module structures and splitting). Suppose $\mathcal{U}$ is a unital ring, and let $\mathcal{U}[t]$ denote the ring of polynomials with coefficients in $\mathcal{U}$. If $A$ is an $n \times n$ matrix $A$ over $\mathcal{U}$, then the free $\mathcal{U}$ module $\mathcal{U}^n$ of row vectors is a right $\mathcal{U}[t]$ module $\mathcal{M}_A$, where the action of $t$ is by $v \mapsto vA$. Two $n \times n$ matrices over $\mathcal{U}$ are conjugate over $\mathcal{U}$ if and only if their $\mathcal{U}[t]$ modules are isomorphic.

If a matrix $C'$ is obtained by an elementary row splitting from a matrix $C$, with associated subdivision matrix $U$, then there is an embedding of $\mathcal{M}_C$ into $\mathcal{M}_{C'}$, given by the rule $v \mapsto vU$. The conjugacy given by $W^+$ in Lemma 3.2 is constructed to extend the conjugacy of embedded copies of $\mathcal{M}_A$ and $\mathcal{M}_C$ obtained by lifting $W$. One finds an $A^+$ and $W^+$ by requiring further conditions on the vectors $e_n, e_{n+1}, e_{n+2}$.

The module viewpoint may give arguments which are easier and more conceptual, or help one find implementing matrices. On the other hand, it can be useful to have matrix arguments which can be verified by direct matrix computation.

It is worth noting that with any string of row splittings from a matrix $C$ to a matrix $C'$, the module $\mathcal{M}_C$ embeds as a $\mathcal{U}[t]$ submodule of $\mathcal{M}_{C'}$, and such that as a free $\mathcal{U}$ module (forgetting the $t$ action) $\mathcal{M}_{C'}$ is the internal direct sum of the embedded copy of $\mathcal{M}_C$ and another free $\mathcal{U}$ module.

We will also use the following easy lemma.

**Lemma 3.5.** Let $\mathcal{U}$ be a unital nondiscrete subring of $\mathbb{R}$. Suppose $C$ is a positive square matrix over $\mathcal{U}$ and $M$ is a square matrix over $\mathcal{U}$ of the (block) form $(\begin{smallmatrix} C & 0 \\ X & 0 \end{smallmatrix})$ or $(\begin{smallmatrix} C & 0 \\ 0 & X \end{smallmatrix})$. Then there is a positive matrix over $\mathcal{U}$ which is conjugate over $\mathcal{U}$ to $M$ and which is SSE over $\mathcal{U}^+$ to $C$.

**Proof.** Clearly it is sufficient to prove the lemma assuming $M = (\begin{smallmatrix} C & 0 \\ X & 0 \end{smallmatrix})$. Pick $\kappa > 0$ in $\mathcal{U}$ such that $X' := X + \kappa JC$ is positive, where $J$ denotes a matrix of appropriate size with every entry equal to $1$, and set $M' = (\begin{smallmatrix} C & 0 \\ X' & 0 \end{smallmatrix})$. Then $M'$ is SSE over $\mathcal{U}^+ \to C$, and also conjugate over $\mathcal{U}$ to $M$, since

$$M' = \begin{pmatrix} C & 0 \\ X' & 0 \end{pmatrix} = \begin{pmatrix} I & 0 \\ \kappa J & I \end{pmatrix} \begin{pmatrix} C & 0 \\ X & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ -\kappa J & I \end{pmatrix}.$$

Given \( \epsilon \) in \( U \), define another matrix conjugate over \( U \) to \( M \),
\[
M_\epsilon := \begin{pmatrix}
I & -\epsilon J \\
0 & I
\end{pmatrix} \begin{pmatrix}
C & 0 \\
X' & 0
\end{pmatrix} \begin{pmatrix}
I & \epsilon J \\
0 & I
\end{pmatrix}
\]
\[
\begin{pmatrix}
C - \epsilon JX' \\
X'
\end{pmatrix} \begin{pmatrix}
I & \epsilon J \\
0 & I
\end{pmatrix}
\]
Fix \( \epsilon > 0 \) in \( U \) sufficiently small to guarantee \( C - \epsilon JX' \) is positive. Then \( M_\epsilon \) is a positive matrix SSE over \( U_+ \) to \( M' \), and hence to \( C \). □

We are now prepared to prove the main result of this section.

Proof of Theorem 3.1. By assumption, for some \( \ell \) we have matrices
\[
A = A(0), A(1), \ldots A(\ell) = B \text{ and for } 0 \leq k < \ell \text{ an ESSE over } U,
\]
\[
A(i) = R(i)S(i) , \quad A(i+1) = S(i)R(i) .
\]
By Proposition 2.9 we may assume all the matrices \( A(i), R(i), S(i) \) are nondegenerate. For each \( i \), we can then by Proposition 2.10 associate to the ESSE (3.3) a diagram of splittings and a diagonal refactorization

\[
\begin{array}{c}
X_i \xrightarrow{(D_i, D_i^{-1}X_i)} Y_i \\
A(i-1) \quad A(i)
\end{array}
\]

as described in (2.1). By Lemma 2.13 we can lift each diagonal refactorization by a row and a column splitting to nondegenerate matrices conjugate over \( U \), giving a diagram of three levels,

\[
\begin{array}{c}
A(i-1) \quad A(i)
\end{array}
\]

For visual clarity, we use the horizontal “=” in diagrams to indicate conjugacy over \( U \) (not equality).

We consider an initial diagram (of three levels) formed by taking the union of the \( \ell \) diagrams above (one for each ESSE). Arrows point southwest for row amalgamations and southeast for column amalgamations. For visual clarity, we suppress matrix names and arrow tips. Here is the initial diagram for the case \( \ell = 3 \).

\[
\begin{array}{c}
A(i-1) \quad A(i)
\end{array}
\]

We apply the Fiber Lemma 2.12 to construct a nondegenerate common column/row splitting for each pair of matrices in the diagram with a common row/column amalgamation, and iterate this move as far as possible. For our case \( \ell = 3 \), this
produces the next diagram (with open circles and dotted lines reflecting additions to the diagram at this step).

\[(3.7)\]

Next, we apply part (1) of the Splitting Lemma 3.2 to lift conjugacies of non-degenerate matrices by row or column splittings. Where there is a choice, for definiteness (only) we choose to lift by row splittings. For \(\ell = 3\) this produces the following diagram. A nonhorizontal arrow here arising from the Splitting Lemma represents the composition of several splittings through nondegenerate matrices.

\[(3.8)\]

We iterate the application of Splitting Lemma and Fiber Lemma until we arrive at a final diagram of \(2\ell + 1\) levels whose top level consists of matrices which are all conjugate. For \(\ell = 3\), this happens at the next stage, and produces the diagram \[(3.9)\] below.

At the top of the left side of the final diagram is a nondegenerate matrix \(C\) which is obtained from \(A\) by a sequence of row splittings through nondegenerate matrices over \(U\). By Part (2) of the Splitting Lemma 3.2 there is a matrix \(A^+\) SSE over \(U_+\) to \(A\) and conjugate over \(U\) to a matrix of the form \((C_0 0)\). Similarly, if \(E\) is the matrix on the right side of the top level of the final diagram, then there is a positive matrix \(B^+\) which is SSE over \(U_+\) and conjugate over \(U\) to a matrix of the form \((E_0 0)\). If \(A^+\) and \(B^+\) are not of the same size, then we may apply Lemma 3.5.
to enlarge one of them, and assume they have the same size. Because $C$ and $E$ are conjugate over $U$, it then follows that $A^+$ and $B^+$ are conjugate over $U$.

This concludes the proof, apart from the “moreover” claim for the conjugating matrix. This is perhaps already clear from previous work, but we will give a self contained proof in the following lemma. □

**Lemma 3.6.** Suppose $U$ is a nondiscrete unital subring of $\mathbb{R}$, and $A, B$ are $n \times n$ positive matrices conjugate over $U$. Then there are positive matrices $A', B'$ SSE over $U_+$ to $A, B$ respectively and a matrix $U$ invertible over $U$ such that $U^{-1}A'U = B'$, $\det U > 0$, and $U$ sends positive eigenvectors to positive eigenvectors.

**Proof.** We are given $U \in \text{GL}(n, U)$ such that $U^{-1}AU = B$. We may assume (if necessary after replacing $U$ with $-U$) that $U$ sends positive eigenvectors to positive eigenvectors. In the case $\det U < 0$, it would suffice to have some $W$ invertible over $U$ such that $AW = WA$, $\det W < 0$ and $W$ respects positive eigenvectors of $A$. (We could then replace $U$ with $WU$.)

If no such $W$ exists, then for some small $\epsilon > 0$ we will define $A'$ of size $n + 1$ as a row splitting of $A$, by splitting the first row $A_1$ of $A$ as $\epsilon A_1 + (1 - \epsilon)A_1$. Here $\epsilon > 0$, $\epsilon \in U$ and $\epsilon$ is small enough that $A' > 0$. Let $E_{ij}(s)$ denote the size $n + 1$ matrix equal to $I$ except that the $ij$ entry is $s$. Let $F = E_{12}(-\epsilon)E_{21}(1)$. Then

$$F A' F^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix} = A''.$$ 

Because the matrix $K = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ has negative determinant, commutes with $A''$ and fixes eigenvectors for nonzero eigenvalues, the matrix $W = F^{-1}KF$ will have the same properties with respect to $A' = F^{-1}A''F$.

Now define $B'$ from $B$ in the same way. The matrices $A', B'$ are positive, SSE over $U_+$ to $A, B$ respectively, and conjugate by a matrix with positive determinant which sends positive eigenvectors to positive eigenvectors. □

We prepare for the last result of this section with the next lemma.

**Lemma 3.7.** Suppose $A, B, D, B'$ are matrices over a field $U$ such that $D$ is diagonal nonsingular, $A = D^{-1}BD$ and $B'$ is a row splitting of $B$. Then there is a row splitting $A'$ of $A$ and a diagonal matrix $D'$ over $U$ such that $A' = (D')^{-1}BD'$.

The same statement is true if “row” is replaced with “column”.

**Proof.** Clearly it suffices to prove the row statement. Let $U$ be the subdivision matrix for the assumed row splitting, $BU = UB'$. Define the diagonal matrix $D'$ in the obvious way, $D'(i, i) = D(\overrightarrow{i}, \overrightarrow{i})$ where $\overrightarrow{i}$ satisfies $U(\overrightarrow{i}, i) = 1$. Define $A'$ to be $(D')^{-1}BD'$. Then $U$ is also the subdivision matrix for a row splitting of $A$ to $A'$.

For an example of this, take

$$A = \begin{pmatrix} a & b_{\frac{\delta_1}{\delta_2}} \\ \frac{\delta_2}{\delta_1} & c \\ \frac{\delta_1}{\delta_2} & d \end{pmatrix}, \quad B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad D = \begin{pmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{pmatrix},$$

$$A' = \begin{pmatrix} a_1 & a_2 \\ a_2 & a_1 \\ \frac{\delta_1}{\delta_2} & \frac{\delta_2}{\delta_1} \\ \frac{\delta_1}{\delta_2} & \frac{\delta_2}{\delta_1} \\ c & d \end{pmatrix}, \quad B' = \begin{pmatrix} a_1 & a_2 \\ a_2 & a_1 \\ b_1 & b_2 \\ b_2 & b_1 \\ c & d \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$ 

}
If \( \mathcal{U} \) is not a field, then the definition of \( A' \) in the proof of Lemma 3.7 above might not give a matrix over \( \mathcal{U} \).
We will need the following result from [15].

**Theorem 3.8.** Suppose \( A, B \) are nondegenerate matrices SSE over \( \mathcal{U}_+ \), where \( \mathcal{U} \) is a subfield of \( \mathbb{R} \). Then there is a nondegenerate matrix \( C \) over \( \mathcal{U}_+ \) and a nonsingular diagonal matrix \( D \) over \( \mathcal{U}_+ \) such that \( C \) is reached from \( A \) by finitely many row splittings through matrices over \( \mathcal{U}_+ \), and \( D^{-1}CD \) is reached from \( B \) by finitely many column splittings through matrices over \( \mathcal{U}_+ \). If \( A \) is primitive, then \( C \) is primitive.

**Proof.** The proof is a simplification of part of the proof of Theorem 3.1. We begin with a lag \( \ell \) SSE from \( A \) to \( B \) through nondegenerate matrices over \( \mathcal{U}_+ \). As in (3.4), from each elementary SSE we produce a row splitting, diagonal refactorization and column splitting. For example, with \( \ell = 4 \) we get a diagram

\[
(3.10)
\]

A southwest pointing arrow represents a row amalgamation. A southeast pointing arrow represents a column amalgamation. A horizontal arrow represents a diagonal refactorization over \( \mathcal{U} \).

Above each \( A_{(i)} \), we apply Lemma 2.12 to produce a common splitting: With this move, we have added another level to the top of the diagram:

\[
(3.11)
\]

Then we apply Lemma 3.7 to lift each diagonal refactorization on the old top row by a splitting to the new top row:

\[
(3.12)
\]

When there is a choice, for definiteness (only) we make the choice to lift with a row splitting.

Iterating this pair of moves \( \ell - 1 \) additional times, we produce a diagram with \( \ell + 1 \) horizontal levels. For \( \ell = 4 \), this is the following diagram, in which we insert some matrix names (whose generalizations to arbitrary \( \ell \) should be clear) and for
visual simplicity suppress arrowheads and bullets.

\[(3.13)\]

\[
\begin{tikzpicture}
  \node (A) at (0,0) {$A$};
  \node (B) at (3,0) {$B$};
  \node (C1) at (-1.5,-1) {$C_1$};
  \node (C2) at (1.5,-1) {$C_2$};
  \node (C3) at (0,-2) {$C_3$};
  \node (C4) at (0,-3) {$C_4$};
  \node (E1) at (1.5,-4) {$E_1$};
  \node (E2) at (2.25,-3) {$E_2$};
  \node (E3) at (2.25,-2) {$E_3$};
  \node (E4) at (2.25,-1) {$E_4$};
  \draw (A) -- (C1); \draw (A) -- (C2); \draw (A) -- (C3); \draw (A) -- (C4); \draw (C1) -- (E1); \draw (C2) -- (E1); \draw (C3) -- (E2); \draw (C4) -- (E2); \draw (C4) -- (E3); \draw (C4) -- (E4);
\end{tikzpicture}
\]

Define $A' = C_\ell$ and $B' = E_\ell$. The diagonal matrix $D$ such that $D^{-1}A'D = B'$ is produced by composing the $\ell$ diagonal refactorizations on the top level of the diagram. If $A$ is primitive, then $C$ is primitive, because $C$ is nondegenerate and $\text{SSE over } \mathbb{R}_+$ to $A$. \hfill $\square$

4. The Centralizer

**Definition 4.1.** Given an $n \times n$ matrix $A$ over $\mathbb{R}$, we let $\text{Cent}_{\mathbb{R}}(A)$ denote $\{ B \in GL(n, \mathbb{R}) : AB = BA \}$, the centralizer of $A$ in $GL(n, \mathbb{R})$. Let $GL_+(n, \mathbb{R})$ denote the connected component of the identity in $GL(n, \mathbb{R})$ (the matrices with positive determinant). Define $\text{Cent}^+_{\mathbb{R}}(A)$ to be $\text{Cent}_{\mathbb{R}}(A) \cap GL_+(n, \mathbb{R})$.

This section provides some background and notation for $\text{Cent}_{\mathbb{R}}(A)$, needed for the results to come on strong shift equivalence over nondiscrete unital subrings of $\mathbb{R}$.

In the next lemma, $\mathcal{U}$ could be for example a field, or it could be obtained from a real algebraic number ring by inverting all but finitely many primes.

**Lemma 4.2** (Centralizer Lemma). Suppose that $\mathcal{U}$ is a dense unital subring of $\mathbb{R}$ which contains an ideal $J \neq \mathcal{U}$ such that every element of $\mathcal{U}$ outside $J$ is a unit in $\mathcal{U}$. Suppose that $A$ is an $n \times n$ matrix over $\mathcal{U}$.

Then every connected component of $\text{Cent}_{\mathbb{R}}(A)$ contains an element of $GL(n, \mathcal{U})$.

**Proof.** Let $\mathbb{F}$ denote the field of fractions of $\mathcal{U}$. The set of (not necessarily invertible) real matrices which commute with $A$, as the solution set of $AX -XA = 0$, is a real vector space $V_{\mathbb{R}}$ in which the matrices over $\mathbb{F}$ are an $\mathbb{F}$ vector space $V_{\mathbb{F}}$ of equal dimension. Thus $V_{\mathbb{F}}$ is dense in $V_{\mathbb{R}}$. It suffices to show $GL(n, \mathcal{U}) \cap V_{\mathbb{R}}$ is dense in $V_{\mathbb{F}}$ (in which case it is also dense in every connected component of $GL(n, \mathbb{R}) \cap V_{\mathbb{R}}$, which is $\text{Cent}_{\mathbb{R}}(A)$).

If $J = \{ 0 \}$, then $\mathcal{U}$ is a field. Then any element of $V_{\mathbb{F}}$ close to an element of $GL(n, \mathbb{R}) \cap V_{\mathbb{R}}$ has nonzero determinant and thus lies in $GL(n, \mathcal{U})$.

So, let $J$ be a proper ideal of $\mathcal{U}$. Then $J$ is dense in $\mathbb{R}$. Suppose $X \in V_{\mathbb{F}}$, and $\epsilon > 0$. Pick a nonzero $d_1 \in \mathcal{U}$ such that $d_1X$ has all entries in $\mathcal{U}$. Pick $d_2 \in \mathcal{U}$ such that $|d_1d_2 - 1| < \epsilon$. Let $Y = d_1d_2X$. Then $Y$ has entries in $\mathcal{U}$ and $||Y - X|| < \epsilon ||X||$. Pick $c$ in $J$ such that $|1 - c| < \epsilon$. Let $M = (1 - c)I + cY$. Then $M \in V_{\mathbb{F}}$ and $||Y - M|| = ||((1 - c)(Y - I))|| \leq \epsilon ||Y - I||$.

Since $\det(M) = \det(I + c(Y - I)) \equiv 1 \mod J$, we also have $M \in GL(n, \mathcal{U})$. \hfill $\square$

For completeness we recall an example from [16].
Example 4.3. Let $U = \mathbb{Z}[1/p]$, where $p$ is a prime (e.g., 5) which does not split in the algebraic number ring $\mathbb{Z}[\sqrt{3}]$. Let $A$ be a nonzero $2 \times 2$ matrix of the form $c_1I + c_2B$, with $c_1, c_2$ in $U$, where $B = \left( \begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} \right)$.

Then $GL(2, U)$ does not intersect every connected component of $Cent_{\mathbb{R}}(A)$. If $C$ is the $3 \times 3$ matrix $B \oplus 1$, then $GL(3, U)$ does not intersect every connected component of $Cent_{\mathbb{R}}^+(C)$.

Proof. There is an isomorphism of fields from $\mathbb{Q}[\sqrt{3}]$ to $\mathbb{Q}[A]$ induced by $\sqrt{3} \mapsto B$. A fundamental unit for the algebraic number ring $\mathbb{Z}[\sqrt{3}]$ is $2 + \sqrt{3}$, which has positive norm 1. If $p$ is an odd prime and $p > 3$ and 3 is not a square mod $p$ (for example $p = 5$), then $p$ does not split in $\mathbb{Z}[\sqrt{3}]$ [21 p.74].

The matrix $A$ has distinct real eigenvalues and can be diagonalized over the reals. When diagonalized, its centralizer becomes all diagonal matrices and is all linear combinations over $U$ of $I, A$. So that is also true when it is not diagonalized. The real centralizer will be a direct sum of two copies of the reals and has four components. Some components have negative determinant. The centralizer of $A$ within the $2 \times 2$ matrices over $U$ consists of linear combinations of $I, A$ over $U$, and is isomorphic as a ring to the quadratic number ring $R = \mathbb{Z}[\sqrt{3}]$ with the prime $p$ inverted. By assumption $p$ is prime in $R$, so any unit of $U$ has the form $p^m(2 + \sqrt{3})^n$ for some integers $m, n$. The norms of all units in $U$ are positive, which translates to the determinants of all elements of $GL(2, U)$ in the centralizer of $A$ being positive. Therefore the negative determinant components of $Cent_{\mathbb{R}}(A)$ will not intersect $GL(2, U)$.

A matrix in the centralizer of $C$ will act like a matrix in the centralizer of $A$ together with multiplication by a scalar on the fixed direction. A negative-determinant component of $Cent_{\mathbb{R}}(A)$, with multiplication by a negative number on the fixed direction, will yield a positive-determinant component of $Cent_{\mathbb{R}}(C)$ which does not intersect $GL(2, U)$.

Definition 4.4. Given a square real matrix $A$, let $\mathcal{J}(A)$ denote the set of pairs $(\lambda, j)$ such that $\lambda \in \mathbb{R}, j \in \mathbb{N}$ and the Jordan form of $A$ contains a $j \times j$ Jordan block for $\lambda$. Then define

$$\gamma(A) = |\mathcal{J}(A)|.$$ 

For $A$ $n \times n$ over $\mathbb{R}$ and $(\lambda, j) \in \mathcal{J}(A)$, define the vector space

$$V(A, \lambda, j) = \{ x(A - \lambda I)^{j-1} \in ker(A) : x \in \mathbb{R}^n \setminus \text{image}(A) \} \cup \{0\}.$$ 

A matrix $B$ in $Cent_{\mathbb{R}}(A)$ maps each $V(A, \lambda, j)$ to itself. Let $\sigma(\lambda, j)$ be the sign of the determinant of this map determined by $B$.

Recall, $\pi_0(X)$ is the set of connected components of a topological space $X$.

Proposition 4.5. Let $A$ be an $n \times n$ real matrix. Then with $\gamma$ as defined above,

$$|\pi_0(Cent_{\mathbb{R}}(A))| = 2^\gamma(A).$$

Two matrices lie in the same component if and only if they have the same sign $\sigma(\lambda, j)$ for all $(\lambda, j)$ in $\mathcal{J}(A)$.

Proof. If $A$ is zero, the claim holds because $|\pi_0(GL(n, \mathbb{R}))| = 2$. Now suppose $A \neq 0$. $A$ is a sum of commuting nonzero real matrices $A_\lambda$, where $\lambda$ denotes a root of $\chi_A$ with nonnegative imaginary part, and $A_\lambda - \lambda I$ is nilpotent if $\lambda \in \mathbb{R}$, and $(A_\lambda - \lambda I)(A_\lambda - \lambda \overline{I})$ is nilpotent if $\lambda \in \mathbb{R} \setminus \mathbb{C}$. The centralizer $Cent_{\mathbb{R}}(A)$ is
homeomorphic to the product $\prod_{\lambda} \text{Cent}_{\mathbb{R}}(A_{\lambda})$. So it suffices to show the claim for each $A_{\lambda}$.

If $\lambda$ is real, then $\text{Cent}_{\mathbb{R}}(A_{\lambda}) = \text{Cent}_{\mathbb{R}}(A_{\lambda} - \lambda I)$, and we can consider $\text{Cent}_{\mathbb{R}}(M)$ for $M$ nilpotent. Let $C_j$ denote a matrix of the form $J \otimes I$, where $J$ is a Jordan block of size $j$ with zero diagonal. Then $\text{Cent}_{\mathbb{R}}(C_j)$ has a banded form, e.g. for $C_3$ of size $3k \times 3k$,

$$C_3 = \begin{pmatrix} 0 & I & 0 \\ 0 & 0 & I \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{Cent}_{\mathbb{R}}(C_3) = \left\{ \begin{pmatrix} X & Y & Z \\ 0 & X & Y \\ 0 & 0 & X \end{pmatrix} : X \in GL(k, \mathbb{R}) \right\}.$$

Each $\text{Cent}_{\mathbb{R}}(C_j)$ has exactly two connected components, depending on the sign of the determinant of the repeated diagonal block, which is $\sigma(\lambda, j)$. Up to conjugacy, the nilpotent matrix $M$ will be block diagonal of the form $\text{diag}(M_1, M_2, \ldots, M_j)$, i.e.

$$M = \begin{pmatrix} M_1 & 0 & 0 & \cdots & 0 \\ 0 & M_2 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 0 & \cdots & M_j \end{pmatrix}$$

where $M_i = C_{n(i)}$, $n(1) > n(2) > \cdots > n(j)$ and $j = \gamma(M)$. $\text{Cent}_{\mathbb{R}}(M)$ is a subset of the set of block upper triangular matrices such that an element of $\text{Cent}_{\mathbb{R}}(M_i)$ occupies the $i$th diagonal block. There is a homotopy from $\text{Cent}_{\mathbb{R}}(M)$ to the set of its block diagonal matrices, which is homeomorphic to $\prod_{\lambda} \text{Cent}_{\mathbb{R}}(M_j)$, which has $2^j$ connected components.

If $\lambda$ is complex and $A_{\lambda}$ has size $2k$, then let $M$ be the size $k$ complex matrix which is the direct sum of the $\lambda$-Jordan blocks in the Jordan form of $A_{\lambda}$ (or $A$). Then $\text{Cent}_{\mathbb{R}}(A_{\lambda})$ is homeomorphic to $\text{Cent}_{\mathbb{C}}(M)$, the centralizer of $M$ in $GL(k, \mathbb{C})$. The triangular structure described earlier applies to $\text{Cent}_{\mathbb{C}}(M)$. However, because $GL(n, \mathbb{C})$ is connected for every $n$, we have that $\text{Cent}_{\mathbb{C}}(M)$ is connected.

5. From paths of similar matrices to strong shift equivalence

In this section, we will see how to pass from a path of positive conjugate matrices to a strong shift equivalence through positive matrices. (The problem of finding such a path we consider later.) For completeness, we begin with a proof for the path lifting Proposition 5.3 for which we make some preparation. Below, the particular choice of norm for $\mathbb{R}^n$ is unimportant. The next lemma was proved in [13] with a citation to [9]; we include a proof for completeness.

**Lemma 5.1.** Suppose $B$ is a matrix over $\mathbb{R}$ and $\epsilon > 0$. Then there exists $\delta > 0$ such that if $||B - B'|| < \delta$ and $B'$ is conjugate over $\mathbb{R}$ to $B$, then there exists $U$ in $GL(n, \mathbb{R})$ such that $U^{-1}BU = B'$ and $||U - I|| < \epsilon$.

**Proof.** We begin with a Claim: Suppose $\epsilon > 0$ and $M$ is an $n \times n$ matrix of rank $r$ over scalar field $\mathbb{C}$ or $\mathbb{R}$, and $u_1, \ldots, u_{n-r}$ is a basis of $\ker(M)$. Then there is $\delta > 0$ such that for any $M'$ with $||M - M'|| < \delta$ and $\text{rank}(M') = r$, there is a basis $u'_1, \ldots, u'_{n-r}$ of $\ker(M')$ with $||u_j - u'_j|| < \epsilon ||u_j||$, for each $j$.

**Proof of Claim.** Without loss of generality, suppose $0 < r < n$. To set notation, we use row vectors for $\ker M = \{v : vM = 0\}$. Let $\text{proj}_W$ denote orthogonal projection onto $W$. Let $\text{col} M$ denote the vector space generated by column vectors of $M$. 

Within the set of \( n \times n \) matrices \( M \) of rank \( r \), the map \( \text{proj}_{\text{col}}M \) varies continuously with \( M \). (For \( M' \) near a given \( M \), the same \( r \) linearly independent columns can be used to construct an orthonormal basis with the Gram-Schmidt algorithm.) So we may suppose \( \delta \) is small enough that \( ||\text{proj}_{\text{col}}A'(v)|| \leq ||\text{proj}_{\text{col}}A(v)|| + \epsilon ||v|| \), for all \( v \). Set \( u_j' = \text{proj}_{\ker A'}(u_j) \). Considering \((\ker A')^\perp = \{v^\perp : v \in \text{col} A'\}\), we have
\[
||u_j - u_j'|| = ||\text{proj}_{(\ker A')^\perp}(u_j)|| = ||\text{proj}_{\text{col}}A'(u_j')||
\leq ||\text{proj}_{\text{col}}A'(u_j')|| + \epsilon ||u_j'|| = \epsilon ||u_j'||.
\]
This proves the claim.

Now suppose \( \lambda \) is an eigenvalue of \( B \) and \( J_\lambda = \{u_1, \ldots, u_s\} \) is a Jordan basis for the restriction of \( B - \lambda I \) to \( \ker(B - \lambda I)^n \). (\( \cup \lambda J_\lambda \) is a Jordan basis for \( B \).) Define
\[
J_\lambda(t) = \{u_i \in J_\lambda : u_i(B - \lambda I)^t = 0 \text{ and } u_i \notin \text{Image}(B - \lambda I)\}.
\]
(The number of vectors in \( J_\lambda(t) \) equals the number of Jordan \( \lambda \)-blocks of size \( t \) in the Jordan form of \( B \).) For \( B' \) close to \( B \), let \( \{u_1', \ldots, u_s'\} \) be the nearby basis of \( \ker(B' - \lambda I)^n \), given by the Claim. With \( B' \) close enough to \( B \),
\[
u_i \in J_\lambda(t) \implies u_i(B - \lambda I)^{t-1} \neq 0.
\]
Consider the \( t \) in decreasing order, we then deduce from the conjugacy of \( B \) and \( B' \) that also
\[
u_i \in J_\lambda(t) \implies u_i(B - \lambda I)^t = 0.
\]
For \( \lambda \) real, those vectors \( u_i, u_i' \) can be chosen to be real, and the map on \( J_\lambda \) defined by
\[
u_i(B - \lambda I)^j \mapsto u_i'(B' - \lambda I)^j,
\]
determines a map \( \ker(B - \lambda I)^n \to \ker(B' - \lambda I)^n \) which conjugates the restrictions of \( B \) and \( B' \) to these invariant subspaces. For \( \lambda \) not real, say with positive imaginary part, pull back the complex conjugacy to define a map from a real Jordan form basis for \( B \) for \( \ker(B - \lambda I)^n \) to a corresponding nearby real Jordan basis for \( B' \) for \( \ker(B' - \lambda I)^n \).

The matrix \( U \) in \( GL(n, \mathbb{R}) \) implementing these maps on invariant subspaces induces a conjugacy of \( B \) and \( B' \) and is close to the identity. \( \Box \)

Below, we suppose \( A \) is an \( n \times n \) real matrix, \( \text{Cent}(A) = \{U \in GL(n, \mathbb{R}) : UA = AU\} \); \( \text{Conj}(A) = \{U^{-1}AU : U \in GL(n, \mathbb{R})\} \); \( \gamma : U \mapsto U^{-1}AU \); the topology of \( \text{Conj}(A) \) is by the metric induced by a matrix norm, and the image of \( \pi \) has the quotient topology.

**Proposition 5.2.** The map \( \phi \) which makes the following diagram commute
\[
\begin{array}{ccc}
GL(n, \mathbb{R}) & \xrightarrow{\pi} & GL(n, \mathbb{R})/\text{Cent}(A) \\
\text{Conj}(A) & \xleftarrow{\phi} & \end{array}
\]
is a homeomorphism.

**Proof.** For \( U, V \) in \( GL(n, \mathbb{R}) \), we have \( \gamma(U) = \gamma(V) \) if and only if \( VU^{-1} \in \text{Cent}(A) \). So, \( \phi \) is a well defined bijection. The map \( \phi \) is continuous because \( \gamma \) is continuous, \( \pi \) is open and \( GL(n, \mathbb{R})/\text{Cent}(A) \) has the quotient topology.
It remains to show that $\phi$ is an open map. This holds if $\gamma$ is an open map. Suppose $V$ is an open subset of $GL(n, \mathbb{R})$ and $V' \subseteq V$. Choose $\epsilon > 0$ such that $V'$ contains $\{ VU \in GL(n, \mathbb{R}) : |U-I| < \epsilon \}$. Then $\gamma(V')$ contains $\{ U^{-1}CU : |U-I| < \epsilon \}$, which by Lemma 5.1 contains some neighborhood of $C$ in $Conj(C) = Conj(A)$. This shows the map $\gamma$ is open and finishes the proof. $\Box$

Above, $((GL(n, \mathbb{R}), \pi, GL(n, \mathbb{R})/Cent(A))$ is a principal bundle [11, Ch. 4.2]. The projection $\pi$ is locally trivial: for every $x \in GL(n, \mathbb{R})$, there is a neighborhood $U$ of $x$ and a neighborhood $V$ of $\pi x$ and a homeomorphism $h : U \to V \times Cent(A)$ such that on $U$, $\pi$ is equal to $h$ followed by projection onto $V$, $(v, c) \mapsto v$.

**Proposition 5.3.** Suppose $(A_t)_{t=0}^1$ is a path of conjugate $n \times n$ real matrices, $U \subseteq GL(n, \mathbb{R})$ and $U^{-1}A_0U = A_0$. Then there is a path $G_t$ in $GL(n, \mathbb{R})$ such that $G_0 = U$ and $G_t^{-1}A_0G_t = A_t$ for $0 \leq t \leq 1$.

**Proof.** The map $\gamma$ has the topological properties of the principal bundle projection $\pi$ in Proposition 5.2. The proposition translates to $\gamma$ the path lifting property which $\pi$ enjoys on account of its local triviality as a projection. $\Box$

Let $\mathcal{H}(Id, Cent_R^+(A))$ denote the homotopy classes of paths in $GL_+(n, \mathbb{R})$ from the identity to $Cent_R^+(A)$ (by homotopy through paths with initial point $Id$ and terminal points in $Cent_R^+(A)$). In a topological space $X$, $\pi_0(X)$ denotes the set of connected components and $\pi_1(x, X)$ denotes the fundamental group at basepoint $x$ in $X$.

**Proposition 5.4.** Suppose $(A_t)_{t=0}^1$ is a loop from $A$ to $A$ in $Conj(n, \mathbb{R})$. Then there is a path $(G_t)_{t=0}^1$ in $SL(n, \mathbb{R})$ such that $G_0 = Id$ and $A_t = (G_t)^{-1}AG_t$, $0 \leq t \leq 1$. Moreover, the homotopy class of the loop $(A_t)$ determines both the element of $\mathcal{H}(Id, Cent_R^+(A))$ containing $(G_t)$ and the connected component of $Cent_R^+(A)$ which contains $G_1$. The induced maps

$$\pi_1(A, Conj(A)) \to \mathcal{H}(Id, Cent_R^+(A)),$$

$$\pi_1(A, Conj(A)) \to \pi_0(Cent_R^+(A))$$

are bijections.

**Proof.** Proposition 5.3 explains the existence of the lift of $(A_t)$ to a path $G_t$ in $GL(n, \mathbb{R})$ beginning at $G_0 = Id$. By continuity, each $G_t$ has positive determinant, and we may replace $G_t$ with $(\det G_t)^{-1/n}G_t$ to put the conjugating path into $SL(n, \mathbb{R})$. It is straightforward to check the remaining claims about well defined induced bijections. $\Box$

**Definition 5.5.** Suppose $U$ is a nondiscrete unital subring of the reals. Given $A$ an $n \times n$ matrix over $U$. We say the centralizer condition holds for $(A, U)$ if every connected component of $Cent_R^+(A)$ has nonempty intersection with $GL(n, U)$. We say that $U$ satisfies the centralizer condition if the centralizer condition holds for $(A, U)$ for every square matrix $A$ over $U$.

One equivalent statement of the Centralizer Condition is that $Cent_R^+(A)$ is generated by $U \cap Cent_R^+(A)$ and the connected component of the identity in $Cent_R^+(A)$.

**Lemma 5.6.** Suppose $A$ is a positive $n \times n$ real matrix. There is an $\epsilon > 0$ such that for $U \in SL(n, \mathbb{R})$ with $||U - Id|| < \epsilon$, $U$ can be written as a product of
Proof. The given \( U \) can be made upper triangular by \((n-1) + (n-2) + \cdots + 1 = n(n-1)/2 \) operations of adding multiples of rows successively to lower rows. For \( U \) close to the identity, at each stage the diagonal terms will remain positive, and the multiples of row \( i \) added to lower rows to zero out entries in column \( i \) will depend continuously on \( U \). Likewise, the same number of additions of lower rows to upper rows will diagonalize \( U \). Finally, for \( a \neq 0 \),

\[
\begin{pmatrix}
a & 0 \\
0 & 1/a
\end{pmatrix} = \begin{pmatrix} 1 & a(a-1) \\
0 & 1
\end{pmatrix} \begin{pmatrix} 1 & 0 \\
1/a & 1
\end{pmatrix} \begin{pmatrix} 1 & 1-a \\
0 & 1
\end{pmatrix} \begin{pmatrix} 1 & 0 \\
-1 & 1
\end{pmatrix}.
\]

So, we can multiply a diagonal determinant 1 matrix by 4\((n-1)\) elementary matrices to produce the identity. In total we have factored \( U \) as a product of \( 2[n(n-1)/2] + 4(n-1) = (n+4)(n-1) \) basic elementary matrices. We may fix the order of operations. Then in each \( E_k \), there is a single offdiagonal element which is allowed to be nonzero (or zero), and it varies continuously as a function of \( U \). \( \square \)

**Lemma 5.7.** Given \( 0 < \epsilon < \kappa \) and \( n \in \mathbb{N} \), there is a \( \delta > 0 \) such that for every \( n \times n \) real matrix \( M \) with all entries bounded below by \( \epsilon \) and above by \( \kappa \), for every \( U \in SL(n, \mathbb{R}) \) with \( ||U-I|| < \delta \), the matrix \( U^{-1}MU \) is positive and SSE over \( \mathbb{R}_+ \) to \( M \).

**Proof.** Pick \( \delta > 0 \) small enough that for every \( n \times n \) positive matrix \( M \) with entries bounded below by \( \epsilon \) and above by \( K \), with \( m = (n+4)(n-1) \) we have

- for \( U \in SL(n, \mathbb{R}) \) with \( ||U-I|| < \delta_1 \), there is a continuous factorization \( U = E_1 \cdots E_m \) of \( U \) into basic elementary matrices, as in Lemma 5.6 and
- with \( E_0 = Id \) and \( V_i = E_0 \cdots E_{i-1} \), for \( 0 \leq i < m \) all of the matrices \( V_i^{-1}MV_i \), \( V_i^{-1}MV_iE_{i+1} \), and \( E_{i+1}V_i^{-1}MV_i \) are positive.

Given such a matrix \( M \), set \( B_i = V_i^{-1}MV_i \); then \( B_0 = M \) and \( B_m = U^{-1}MU \). For \( 1 \leq i \leq m \), \( B_i \) is positive, and one of the pairs \( (E_i, E_{i}^{-1}B_i) \), \( (B_i, E_{i}^{-1}) \) will give an ESSE over \( \mathbb{R}_+ \) from \( B_{i-1} \) to \( B_i \). Thus \( M \) and \( U^{-1}MU \) are positive matrices which are SSE over \( \mathbb{R}_+ \). \( \square \)

**Lemma 5.8.** Suppose \( U \) is a nondiscrete unital subring of \( \mathbb{R} \), \( B \) is an \( n \times n \) positive matrix over \( U \) and \( d > 0 \).

Then there is \( \epsilon > 0 \) such that \( V \in GL(n, U) \) with \( ||V - dI|| < \epsilon \) implies the matrix \( V^{-1}BV \) is positive and is SSE over \( U_+ \) to \( B \).

**Proof.** If \( ||V - dI|| < \epsilon \) with \( \epsilon \) sufficiently small, then we may add small positive multiples of a row \( i \) of \( V \) to other rows to make all off diagonal entries of column \( i \) positive and still small. Iterating, we may find nonnegative elementary matrices \( E_1, \ldots, E_k \) over \( U \), with \( k \leq n(n-1) \), such that with \( E = E_kE_{k-1} \cdots E_1 \), the matrix \( EV \) is a positive matrix in \( GL(n, U) \). For small \( \epsilon \), we can take \( E \) close enough to \( I \) that \( BE^{-1} \) is positive and \( V \) is close enough to \( dI \) that \( V^{-1}(BE^{-1}) \) is positive.

Then the pair \( (V^{-1}BE^{-1}, EV) \) gives an ESSE over \( U_+ \) from \( V^{-1}BV \) to \( EBE^{-1} \), and \( V^{-1}BV \) is positive.

Set \( V_0 = I \) and \( B_0 = B \). For \( 1 \leq i \leq k \) set \( V_i = E_iV_{i-1} \) and \( B_i = E_iB_{i-1}E_i^{-1} \). We take the \( E_i \) close enough to the identity that for \( 1 \leq i \leq k \), \( B_{i-1} \) is SSE over \( U_+ \) to \( B_i \). This gives an SSE over \( U_+ \) from \( B \) to \( B_k = EBE^{-1} \). \( \square \)
Definition 5.9. Let $\mathcal{U}$ be a semiring in $\mathbb{R}$. Matrices $A, B$ are SSE over $\mathcal{U}_+$, through positive matrices of size $n$, if for some $\ell \in \mathbb{N}$ there are $n \times n$ positive matrices $A = A_0, A_1, \ldots, A_\ell = B$ such that $A_{i-1}$ is ESSE over $\mathcal{U}_+$ to $A_i$, $1 \leq i \leq \ell$.

In the next theorem, part (1) was proved in [15]. Parts (2) and (3) were proved in [16] under the condition that elements of $GL(n, \mathbb{U})$ are dense in $GL(n, \mathbb{R})$. Here we remove this condition by working with the special linear group and scalar matrices.

Theorem 5.10 (Path Theorem). Let $\mathcal{U}$ be a unital nondiscrete subring of $\mathbb{R}$, and suppose $(A_t)_{0 \leq t \leq 1}$ is a path of positive, real $n \times n$ matrices, all in the same conjugacy class over $\mathbb{R}$, from $A = A_0$ to $B = A_1$. Then the following hold

1. $A$ and $B$ are SSE over $\mathbb{R}_+$, through positive matrices of size $n$.
2. Suppose there is a path $(G_t)$ in $GL(n, \mathbb{R})$ such that $G_0 = \text{Id}$ and $G_t^{-1}AG_t = A_t$ for all $t$, and there is a $W$ in $GL(n, \mathcal{U})$ such that $W^{-1}AW = B$ and $WG_t^{-1}$ is in the connected component of the identity in $\text{Cent}_\mathbb{R}(A)$. Then $A$ and $B$ are SSE over $\mathcal{U}_+$, through positive matrices of size $n$.
3. Suppose $A$ and $B$ are conjugate matrices over $\mathcal{U}$ such that every connected component of $\text{Cent}_\mathbb{R}(A)$ contains a matrix from $GL(n, \mathcal{U})$. Then $A$ and $B$ are SSE over $\mathcal{U}_+$, through positive matrices of size $n$.
4. Suppose $A$ and $B$ are conjugate matrices over $\mathcal{U}$ and $\mathcal{U}$ is a field, or more generally contains an ideal $J \neq \mathcal{U}$ such that every element of $\mathcal{U}$ outside $J$ is a unit in $\mathcal{U}$. Then $A$ and $B$ are SSE over $\mathcal{U}_+$, through positive matrices of size $n$.

Proof. (1) Proposition 5.3 gives us a path $(G_t)$ in $SL(n, \mathbb{R})$ with $G_0 = \text{Id}$ and $G_t^{-1}AG_t = A_t$. The entries of the positive matrices $A_t$ are by compactness of the path uniformly bounded below by some positive $\epsilon$ and above by some $\kappa$. Let $\delta > 0$ be chosen as in Lemma 5.7 for $\epsilon, \kappa$. Now by uniform continuity pick $k \in \mathbb{N}$ such that $s \leq t \leq s + \frac{1}{k}$ implies $||G_t(G_s^{-1}) - \text{Id}|| < \delta$. By Lemma 5.7 $A_{i/k}$ is SSE over $\mathbb{R}_+$ to $A_{(i+1)/k}$, for $0 \leq i < k$. Therefore $A = A_0$ and $B = A_1$ are SSE over $\mathbb{R}_+$.

(2) Let $X_t$, $0 \leq t \leq 1$, be a path from the identity to $WG_t^{-1}$ in the centralizer of $A$ in $GL(n, \mathbb{R})$. Then $(X_tG_t)$ is a path from the identity to $W$ in $GL(n, \mathbb{R})$, and $W^{-1}AW = B$. For all $t$, $(X_tG_t)^{-1}A(X_tG_t) = G_t^{-1}AG_t$. So we may assume $G_1 \in GL(n, \mathcal{U})$.

Next define the path $H_t$ in $SL(n, \mathbb{R})$ by $H_t = c_tG_t$, $0 \leq t \leq 1$, where $c_t = (\det(G_t))^{-1/n}$. We have $H_t^{-1}AH_t = G_t^{-1}AG_t$, $0 \leq t \leq 1$. As in (1), from Lemma 5.7 we get an SSE through positive real matrices from $A$ to $B$. We denote these matrices in order as $A = B_0, B_1, \ldots, B_\ell = A_1 = B$. For $1 \leq i \leq \ell$, we have from Lemma 5.7 a conjugacy $B_t = E_t^{-1}B_1E_t$, with $E_t$ close enough to $\text{Id}$ that $E_t^{-1}B_1$ and $B_1E_t$ are positive, which guarantees that there is an ESSE from $B_{i-1}$ to $B_i$. We also have $E_1E_2 \cdots E_\ell = H_1$.

Now for $1 \leq i \leq \ell$ we will choose a basic elementary matrix $E_t'$ over $\mathcal{U}$, set $B_0' = A$, and recursively define $B_i' = E_iB_{i-1}E_i$, $1 \leq i \leq \ell$. Define $H_t' = E_t'E_2' \cdots E_\ell'$ and $A_1' = B_\ell'$. Set $V = (H_1')^{-1}G_1 \in GL(n, \mathcal{U})$. Then $V^{-1}A_1'V = B$. Let $d = [\det(G_1)]^{1/n}$. Then $H_1 = (1/d)G_1$ and

$$V = \left((H_1')^{-1} - (H_1)^{-1}\right)G_1 + (H_1)^{-1}G_1$$

$$= \left((H_1')^{-1} - (H_1)^{-1}\right)G_1 + dI.$$
Now choose $\epsilon > 0$ for $B$ as in the statement of Lemma 5.8. We choose the $E'_i$ sufficiently close to the $E_i$ to guarantee

- $B'_i$ is positive and ESSE over $\mathcal{U}_+$ to $B'_{i-1}$, for $1 \leq i \leq \ell$, and
- $\|V - dI\| < \epsilon$.

We have $A'_i$ SSE over $\mathcal{U}_+$ through positive matrices to $A$. It remains now to show $A'_i$ is SSE over $\mathcal{U}_+$ through positive matrices to $B$. This now follows from Lemma 5.8.

(3) Again find a path $G_t$ in $GL(n, \mathbb{R})$ such that $A_t = G_t^{-1}AG_1$. By assumption, there is a $Y$ in $GL(n, \mathcal{U})$ such that $Y^{-1}AY = B$. Therefore $YG_1^{-1} \in \text{Cent}_{\mathbb{R}}(A)$. By assumption there is a matrix $Q$ in $GL(n, \mathcal{U})$ such that $Q^{-1}AYG_1^{-1}$ are in the same connected component of $\text{Cent}_{\mathbb{R}}(A)$. Let $W = Q^{-1}Y \in GL(n, \mathcal{U})$. Then $WG_1^{-1}$ is in the connected component of the identity in $\text{Cent}_{\mathbb{R}}(A)$, and $W^{-1}AW = B$. Therefore (3) follows from (2).

(4) This claim follows from (3) and Lemma 4.2. 

\[ \square \]

Remark 5.11. For a given $n \times n$ matrix $A$, the set of all positive matrices conjugate to $A$ has only finitely many connected components. This is an observation made by Sompong Chuysurichay [6, Theorem 1.4.2], made in the language of invariant tetrahedra (discussed in the appendix C). It holds because the set of matrices conjugate to a given matrix can be defined by finitely many inequalities in finitely many variables, and a semialgebraic set has only finitely many connected components [1, Theorem 2.4.4]. Chuysurichay [6, Introduction] pointed out the following corollary of this fact and Theorem 5.10(1) (which was proved in [14]). We record this fact as the following theorem.

Theorem 5.12 ([6, 14]). Suppose $A$ is a positive $n \times n$ matrix. The collection of positive $n \times n$ matrices conjugate over $\mathbb{R}$ to $A$ contains only finitely many SSE-$\mathbb{R}_+$ classes.

Remark 5.13. Note, the set of matrices of a given size which are SSE-$\mathbb{R}_+$ to a given matrix is not a priori semialgebraic when the lag is unbounded. Indeed, in contrast to Corollary 5.12 Chuysurichay gave an example [6, Theorem 1.9.1] of a connected component $C$ in a conjugacy class of positive $2 \times 2$ real matrices such that the lag of the SSE over $\mathbb{R}_+$ guaranteed to exist between any two matrices in $C$ by Theorem 5.10(1) above (which was proved in [14]), cannot be uniformly bounded in $C$. (The unboundedness of the lag arises for a component of positive conjugate matrices whenever there is a matrix on its boundary with more than one irreducible component.)

The fact that the components method produces the finiteness result (5.12) despite the possibility of unbounded lag is an indication of the power of the method.

Remark 5.14. There are examples [2, Appendix E] of primes $p$ in $\mathbb{Z}$ and primitive matrices over $\mathcal{U} = \mathbb{Z}[1/p]$ which are SE over $\mathcal{U}_+$ (and hence SSE over $\mathcal{U}$, since $\mathcal{U}$ is a principal ideal domain) but are not SSE over $\mathcal{U}_+$. (We do not know whether these examples are SSE over $\mathbb{Q}_+$ or $\mathbb{R}_+$.) There are no positive matrix examples known, for any nondiscrete unital subring $\mathcal{U}$ of $\mathbb{R}$, of matrices which are SSE over $\mathcal{U}$ but not SSE over $\mathcal{U}_+$. The examples [2, Appendix E], based on the work over $\mathbb{Z}$ in [18], are matrices with zero trace, and the general method relies in a fundamental way on the existence of certain matrix powers having zero trace.
Unfortunately, if \( p \) is a prime in \( \mathbb{Z} \), then the ring \( \mathbb{Z}[1/p] \) does not satisfy the ideal hypothesis of Theorem 5.10(4), and the Centralizer Condition 5.5 is not satisfied by \( \mathbb{Z}[1/p] \) (Example 4.3). Therefore Theorem 5.11 does not rule out the possibility that for some \( p \) there are positive matrices SSE over \( \mathbb{Z}[1/p] \) which are not SSE over \( (\mathbb{Z}[1/p])_+ \), even in the case the matrices are connected by a path of positive conjugate matrices.

The rest of this section is devoted to generalizing Theorem 5.12 to arbitrary dense subrings of \( \mathbb{R} \). To prepare, we need more definitions. Let \( \mathcal{U} \) be a dense subring of \( \mathbb{R} \). Suppose \( A \) and \( B \) are matrices over \( \mathcal{U} \); \( W \in \text{GL}(n, \mathcal{U}) \); and \( W^{-1}AW = B \). Given a path \( P = (A_i)_{0 \leq t \leq 1} \) of positive conjugate matrices from \( A \) to \( B \), let \( G \) be a matrix such that there is a path \( G(t) \) in \( \text{GL}(n, \mathbb{R}) \) such that \( G_{t(i)}^{-1}AG_{t(i)} = A_i, \) \( 0 \leq t \leq 1 \), with \( G_{(0)} = \text{Id} \) and \( G_{(1)} = G \). Let \( \pi_0^U(\text{Cent}_R(A)) \) denote the subgroup of \( \pi_0(\text{Cent}_R(A)) \) consisting of those connected components which contain a matrix with all entries in \( \mathcal{U} \). Define \( \pi_0(\mathcal{P}, W) \) to be the connected component of \( \pi_0(\text{Cent}_R(A)) \) containing \( WG^{-1} \). This component is uniquely determined by \( \mathcal{P} \) and \( W \). Finally, let \( \pi_0^U(\mathcal{P}, W) \) be the coset of \( \pi_0^U(\text{Cent}_R(A)) \) in \( \pi_0(\text{Cent}_R(A)) \) which contains \( WG^{-1} \). (We remark as an aside that the coset space \( \pi_0(\text{Cent}_R(A))/\pi_0^U(\text{Cent}_R(A)) \) is a group, because the group \( \pi_0(\text{Cent}_R(A)) \) is abelian, because all its elements have order two.)

**Lemma 5.15.** Let \( \mathcal{U} \) be a dense subring of \( \mathbb{R} \). Suppose \( A, A_1, A_2 \) are positive matrices over \( \mathcal{U} \) and for \( i = 1,2 \)

- \( W_i \) is a matrix in \( \text{GL}(n, \mathcal{U}) \) such that \( (W_i)^{-1}AW_i = A_i \).
- \( \mathcal{P}_i \) is a path of positive conjugate matrices from \( A \) to \( A_i \).

Suppose \( \pi_0^U(\mathcal{P}_1, W) = \pi_0^U(\mathcal{P}_2, W) \).

Then \( A_1 \) and \( A_2 \) are SSE over \( \mathcal{U} \).

**Proof.** For each path \( \mathcal{P}_i \), let \( G_i \) be as in the preceding definitions, with \( W_iG_i^{-1} \) in the coset \( \pi_0(\mathcal{P}_i, W) \). We get a path \( \mathcal{P} \) of positive conjugate matrices from \( A_1 \) to \( A_2 \) by composing the reversal of \( \mathcal{P}_1 \) with \( \mathcal{P}_2 \). Let \( W = W_1^{-1}W_2; \) then \( W^{-1}A_1W = A_2 \). For the path \( \mathcal{P} \), we have \( G = G_1^{-1}G_2 \). We compute

\[
WG^{-1} = W_1^{-1}W_2G_2^{-1}G_1 = \left(W_1^{-1}(G_1W_1^{-1})W_1\right)\left(W_1^{-1}(W_2G_2^{-1})W_1\right)
\]

\[
= W_1^{-1}CW_1
\]

where \( C = (W_1G_1^{-1})^{-1}(W_2G_2^{-1}) \). There is a matrix \( V \) over \( \mathcal{U} \) which lies in the connected component of \( \text{Cent}_R(A) \) containing \( C \), and therefore the connected component of \( \text{Cent}_R(A_1) \) containing \( WG^{-1} \) contains the matrix \( V' = W_1^{-1}CW_1 \) from \( \text{GL}(n, \mathcal{U}) \). Now \( (V')^{-1}W\) is in the connected component of the identity in \( \text{Cent}_R(A_1) \) and \( (V')^{-1}W \in \text{GL}(n, \mathcal{U}) \). It follows from the Path Theorem 5.10(2) that \( A_1 \) and \( A_2 \) are SSE over \( \mathcal{U} \).

The number \( \gamma(A) \) below was defined at (4.3).

**Theorem 5.16.** Let \( \mathcal{U} \) be a dense subring of \( \mathbb{R} \). Suppose \( C \) is a path connected set of positive, conjugate \( n \times n \) matrices containing a matrix \( A \) over \( \mathcal{U} \). Then the number of distinct SSE-\( \mathcal{U}_+ \) classes of matrices which contain a matrix in \( C \) which is conjugate to \( A \) over \( \mathcal{U} \) is finite and cannot exceed

\[
|\pi_0(\text{Cent}_R(A))|/|\pi_0^U(\text{Cent}_R(A))|
\]
which is not greater than $2^{\gamma(A)-1}$.

Consequently, the set of positive matrices conjugate over $\mathcal{U}$ to $A$ intersects only finitely many SSE-$\mathcal{U}_t$ classes.

Proof. The upper bound by the displayed ratio follows from the lemma and the pigeonhole principle. The bound $2^{\gamma(A)-1}$ follows from Proposition 4.5 and the observation that $|r_0^d(\text{Cent}_R(A))| \geq 2$ (since $-I \in \text{Cent}_R(A)$). The final claim follows from the lemma and the fact that the set of positive matrices conjugate ver $\mathbb{R}$ to $A$ contains only finitely many connected components. □

6. Finding positive paths: the case of one nonzero eigenvalue

Definition 6.1. A matrix $A$ is eventually rank $m$ if it is square and rank($A^k$) = $m$ for all large $k$. That $A$ has eventual rank 1 means that its characteristic polynomial has the form $\lambda_A(t) = t^n(t - \lambda)$ with $\lambda$ nonzero.

Lemma 6.2. Suppose $A$ and $B$ are positive $n \times n$ real matrices with spectral radius $\lambda$. Let $\ell_A, r_A$ be positive left, right eigenvectors of $A$. Suppose there is $U \in \text{GL}(n, \mathbb{R})$ with positive determinant such that $U^{-1}AU = B$ and the eigenvector $\ell_AU$ of $B$ is positive.

Then there is a path $\{U_t\}_{0 \leq t \leq 1}$ in $\text{GL}(n, \mathbb{R})$ with $U_0 = I$ and $U_1 = U$ such that for $0 \leq t \leq 1$, the vectors $\ell_AU_t$ and $U_t^{-1}r_A$ are positive eigenvectors for eigenvalue $\lambda$ for the matrix $A_t = U_t^{-1}AU_t$.

Proof. After passing to $(1/\lambda)A$ and $(1/\lambda)B$, without loss of generality we can suppose $\lambda = 1$. Let $D$ be the diagonal matrix such that $D(i, i) = r_A(i)$. Then $D^{-1}AD$ is stochastic (the right eigenvector has every entry 1). Let $D_t = (1-t)I + tD$. Then $\{D_t^{-1}AD_t\}_{0 \leq t \leq 1}$ is a path of positive matrices from $A$ to a positive stochastic matrix. The same argument holds for $B$, so without loss of generality we may suppose $A$ and $B$ are stochastic, with positive right eigenvector $r = r_A$ having every entry 1.

Because the subspace of row vectors $W = \{v \in \mathbb{R}^n : vr = 0\}$ is the annihilator of $r$, the matrix $U$ maps $W$ to $W$. From the assumptions, if $B$ is a basis of $W$, then the matrix representing the restriction to $W$ of $U$ with respect to $B$ must have positive determinant. Because there is a path from the identity to this matrix in $\text{SL}(n-1, \mathbb{R})$, there is a path $\{T_t\}_{0 \leq t \leq 1}$ of invertible linear transformations $T_t : W \to W$ such that $T_0 = I$ and $T_1 = U|_W$.

Now we determine the required path of matrices, $\{U_t\}_{0 \leq t \leq 1}$, by specifying the corresponding linear transformations. For $w \in W$, set $wU_t = T_t(w)$. Also require $\ell_AU_t = (1-t)\ell_A + t\ell_AU := \ell_t$. Then $\ell_t > 0$ for all $t$. Because $W$ contains no positive vector and $W$ has codimension one, the matrices $U_t$ are well defined and invertible. The vectors $\ell_AU_t$ and $U_t^{-1}r_A := r_t$ are eigenvectors of $A_t$ for the eigenvalue 1. If $w \in W$, then there is a $w'$ in $W$ such that $w'U_t = w$, and therefore $wr_t = w'U_tU_t^{-1}r_A = w'r_A = 0$. Since $W$ has codimension 1, there must be a constant $c_t$ such that $r_t = c_t r$. Because $0 < \ell_Ar = \ell_AU_tU_t^{-1}r = \ell_tr = c_tr$, we conclude $c > 0$. Consequently, both $\ell_t$ and $r_t$ are positive, as required. Clearly, $U_0 = I$ and $U_1 = U$. □
For the next lemma, we note that if \( M \) is a nilpotent real matrix and \( c \neq 0 \), then \( M \) is conjugate to \( cM \). For a concrete example,

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & c^{-1} & 0 \\
0 & 0 & c^{-2}
\end{pmatrix} \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & c^2
\end{pmatrix} = c \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix}.
\]

**Lemma 6.3.** Suppose \( A \) and \( B \) are positive eventually rank one matrices with nonzero eigenvalue \( \lambda \), and there is a path \( \{U_t\}_{0 \leq t \leq 1} \) in \( GL(n, \mathbb{R}) \) such that \( U_0 = Id \), \( U_1^{-1}AU_1 = B \) and for each \( A_t = U_t^{-1}AU_t \), the left and right eigenvectors of \( A_t \) are positive.

Then there is a path \( \{V_t\}_{0 \leq t \leq 1} \) in \( GL(n, \mathbb{R}) \) such that \( V_0 = Id \), \( V_1 = U_1 \) and each matrix \( V_t^{-1}AV_t \) is positive.

**Proof.** Without loss of generality, suppose \( \lambda = 1 \). Let \( \ell_t \) and \( r_t \) be the left and right positive eigenvectors of \( A_t \), normalized so that \( \ell_t r_t = (1) \). Let \( P_t = r_t \ell_t \). Let \( Q_t \) be the nilpotent matrix such that \( A_t = P_t + Q_t \). Then \( P_t > 0 \), \( P_t Q_t = Q_t A_t = 0 \), \( U_t^{-1}P_0 U_t = P_t \) and \( U_t^{-1}Q_0 U_t = Q_t \).

Along the path, the entries of the \( P_t \) have a positive lower bound \( m \) and the absolute values of entries of the \( Q_t \) have a positive upper bound \( \lambda \). Choose a positive \( \epsilon < m/\lambda \). Then we have a path of positive conjugate matrices \( P_t + \epsilon Q_t \) from \( P_0 + \epsilon Q_0 \) to \( P_1 + \epsilon Q_1 \), taking \( \epsilon \) from \( \epsilon \) to \( 1 \), we get a path of positive conjugate matrices from \( P + \epsilon Q_0 \) to \( P + Q_0 = A \), and likewise from \( P_1 + \epsilon Q_1 \) to \( P_1 + Q_1 = B \). Composing paths, we get a path of positive conjugate matrices from \( A \) and \( B \). Reparametrizing, we get the path \( \{V_t\}_{0 \leq t \leq 1} \) such that \( V_0 = Id \) and \( V_1 = U_1 \). \( \square \)

The next result was proved in [14] for the case \( \mathcal{U} = \mathbb{Q} \) or \( \mathbb{R} \). The positive matrix path construction below is a matrix version of the invariant tetrahedra argument in [14]. We describe the approach from [14] of “positive invariant tetrahedra” in Appendix C.

**Theorem 6.4.** Suppose \( \mathcal{U} \) is a nondiscrete unital subring of \( \mathbb{R} \), and \( A \) and \( B \) are nonnegative eventually rank one matrices which are SSE over \( \mathcal{U} \).

Then \( A \) and \( B \) are SSE over \( \mathcal{U}_+ \).

**Proof.** After passing to matrices SSE over \( \mathcal{U}_+ \), we may assume that \( A \) and \( B \) are primitive (using the eventually rank one assumption), and then positive (by Proposition 5.2). By Theorem 5.1 we may assume also we have \( U \in GL(n, \mathcal{U}) \) such that \( AU = UB \). Let \( U > 0 \) and \( U \) sends a positive eigenvector of \( A \) to a positive eigenvector of \( B \). By Lemmas 6.2 and 6.3 there is a path \( \{U_t\}_{0 \leq t \leq 1} \) in \( GL(n, \mathbb{R}) \) such that \( U_0 = I, U_1 = U \) and each \( U_t^{-1}AU_t \) is positive. By Theorem 5.10(2), it follows that \( A \) and \( B \) are SSE over \( \mathcal{U}_+ \). \( \square \)

**Remark 6.5.** The one eigenvalue result above looks better in contrast to the lack of other general results. For every subring \( \mathcal{U} \) of \( \mathbb{R} \), for every primitive matrix \( A \) over \( \mathcal{U} \), it is unknown whether there exists an algorithm which given \( B \) primitive and SSE over \( \mathcal{U} \) to \( A \) decides whether \( B \) is SSE over \( \mathcal{U}_+ \) to \( A \).

Theorem 6.4 is not a complete solution to the problem of classifying eventually rank one positive matrices over \( \mathcal{U} \) (for an arbitrary dense subring of \( \mathbb{R} \)). It is complete with regard to addressing positivity, but we do not understand in general how SSE refines SE over \( \mathcal{U} \). Especially,
Problem 6.6. Suppose \( \mathcal{U} \) is a nondiscrete unital subring of \( \mathbb{R} \), and \( A, B \) are eventually rank one matrices which are shift equivalent over the ring \( \mathcal{U} \). Must they be strong shift equivalent over \( \mathcal{U} \)?

However, we are able to handle some classes of rings, as follows.

Theorem 6.7. Suppose \( \mathcal{U} \) is a nondiscrete unital subring of \( \mathbb{R} \), and \( A \) is a nonnegative eventually rank one matrix over \( \mathcal{U} \), with nonzero eigenvalue \( \lambda \). Then the following hold.

1. If \( \mathcal{U} \) is a Dedekind domain and \( A \) is shift equivalent over \( \mathcal{U} \) to the matrix \( (\lambda) \), then \( A \) is SSE over \( \mathcal{U} \) to \( (\lambda) \).
2. If \( \mathcal{U} \) is a principal ideal domain (e.g., a field), then \( A \) is SSE over \( \mathcal{U} \) to \( (\lambda) \).

Proof. (1) Over a Dedekind domain \( \mathcal{U} \), SE-\( \mathcal{U} \) implies SSE-\( \mathcal{U} \) [3, Prop. 2.4], so Theorem 6.4 applies.

(2) Over the principal ideal domain \( \mathcal{U} \), \( A \) is SSE-\( \mathcal{U} \) to a nonsingular matrix [7, 31]. This matrix can only be \( (\lambda) \), so again Theorem 6.4 applies. \( \square \)

Remark 6.8. Example 2.2 in [3] provides a \( 2 \times 2 \) positive matrix \( A' \) with eigenvalues 0 and 1 over the Dedekind domain \( \mathcal{U} = \mathbb{Z}[\sqrt{15}] \) which is not shift equivalent over \( \mathcal{U} \) to a nonsingular matrix.

Whenever a Dedekind domain \( \mathcal{U} \) is not a principal ideal domain, there will be matrices over \( \mathcal{U} \) which are not SSE-\( \mathcal{U} \) to a nonsingular matrix [3].

Remark 6.9. The proofs above easily adapt to prove the result stated next, which is one version of the “positive models” result in [16]).

Theorem 6.10. Suppose \( A \) and \( B \) are \( n \times n \) positive real matrices, and there are matrices \( P, Q_0, Q_1, U \) such that the following hold.

- \( P \) is a positive matrix
- \( A \) and \( B \) are internal direct sums, \( A = P + Q_0 \) and \( B = P + Q_1 \), with the matrices \( Q_0, Q_1 \) nilpotent
- There is \( U \in \text{SL}(n, \mathbb{R}) \) such that \( UP = PU \) and \( UQ_0 = Q_1U \).

Then there is a path of positive matrices \( A_t = U_t^{-1}AU_t \) from \( A = A_0 \) to \( B = A_1 \), such that \( U_0 = Id \) and \( U_1 = U \).

Theorem 6.10 looks like a powerful tool, but so far it has not led to a general result.

Problem 6.11. Suppose \( A \) is a positive real matrix. Must there exist a positive matrix \( P \), with \( \text{rank}(P) = \text{rank}(P^2) \), and a nilpotent matrix \( Q \), such that \( PQ = QP = 0 \) and \( A \) is strong shift equivalent over \( \mathbb{R}_+ \) to \( P + Q \)?

7. The Connection Theorem

Definition 7.1. For an \( n \times n \) real matrix \( A \) and \( \epsilon > 0 \), \( A_{\text{SE}}^{\mathcal{N}}(A) \) denotes the set of \( n \times n \) matrices \( B \) which are shift equivalent over \( \mathbb{R} \) to \( A \) and satisfy \( \|B - A\|_{\text{max}} < \epsilon \).

Theorem 7.2 (Connection Theorem). Suppose \( A \) is an \( n \times n \) positive matrix. Then there is a \( \delta > 0 \) such that for any \( B, C \) in \( A_{\text{SE}}^{\mathcal{N}}(A) \) and \( m \geq n^2/2 \), there are row splittings of \( B, C \) to positive conjugate matrices \( B', C' \) such that there exists a path of positive conjugate matrices from \( B' \) to \( C' \), and therefore the matrices \( B, C \) are SSE-\( \mathbb{R}_+ \), through positive matrices of size at most \( n^2/2 \).
Moreover, if $B$ and $C$ have their entries in a nondiscrete subring $\mathcal{U}$ of $\mathbb{R}$, then the splittings to $B'$ and $C'$ can be done through matrices over $\mathcal{U}_+$. If in addition $\mathcal{U}$ is a field, then the matrices $B, C$ are SSE-$\mathcal{U}_+$, through positive matrices of size at most $n^2/2$.

In the Connection Theorem, $A = B$ is allowed. Before proving the theorem, we record some immediate consequences.

**Corollary 7.3.** If $A$ is a positive $n \times n$ matrix and $\dim(\ker(A)) \geq 1$, then $A$ is SSE over $\mathbb{R}_+$ to a positive $n \times n$ matrix $B$ such that $\dim(\ker(B)) = 1$.

**Proof.** Given $\dim(\ker(A)) > 1$, there are positive matrices shift equivalent to $A$ which are arbitrarily close to $A$ such that $\dim(\ker(A)) = 1$, as one can see by replacing superdiagonal zeros in the Jordan form of the nilpotent part of $A$ with $\epsilon$s. Therefore Theorem [7.3] applies to prove the corollary. $\square$

The Path Theorem [5,10] produced SSE’s over $\mathbb{R}_+$ from paths of positive matrices which are conjugate. The following consequence of the Connection Theorem shows we only need those matrices to be shift equivalent.

**Theorem 7.4.** Suppose $A_t, 0 \leq t \leq 1$, is a path of positive shift equivalent $n \times n$ matrices. Then $A_0$ and $A_1$ are SSE over $\mathbb{R}_+$.

**Proof of Theorem 7.4.** It follows from compactness that for $0 \leq t \leq 1$, the Connection Theorem holds for $A_t$ in place of $A$, for a uniform $\epsilon$ (independent of $t$). Consequently $A_0$ and $A_1$ are SSE over $\mathbb{R}_+$. $\square$

The rest of this section is devoted to the proof of the Connection Theorem, which relies also on Theorem [D.4]. The consequences of the Connection Theorem in later sections can be read independent of the proof of Theorem [D.3].

We prepare for the proof with two lemmas. The idea behind Lemma [7.5] apart from the generality of $\mathcal{U}$, can be found in [17] and [14] Lemma 1. $J_k$ denotes the standard $k \times k$ Jordan block matrix (zero except for entries 1 in positions $(i, i+1)$, $1 \leq i < k$). $J_0(A)$ denotes the nilpotent part of the Jordan form of a matrix $A$.

**Lemma 7.5.** Suppose $\mathcal{U}$ is a nondiscrete subring of $\mathbb{R}$, $n \in \mathbb{N}$, $A$ and $B$ are positive $n \times n$ matrices over $\mathcal{U}$, $||A - B|| < \epsilon$, $0 < \alpha < 1$ and the $i$th rows of $A$ and $B$ are denoted $w_A$ and $w_B$. Let $A'$ be the $(n+1) \times (n+1)$ matrix obtained by a splitting of its $i$th row corresponding to $w = \alpha w + (1 - \alpha)w$. Suppose $B$ is an $n \times n$ positive matrix. Then there is an $(n+1) \times (n+1)$ positive matrix $B'$, obtained by a splitting of row $i$ of $B$, such that the following hold.

1. $||B' - A'|| < \epsilon$
2. $J_0(B')$ can be chosen to be either (i) $J_0(B) \oplus [0]$ or, (ii) for any $k$ such that $J_0(A)$ has a Jordan block of size $k$, $J_0(B')$ can be obtained from $J_0(B)$ by replacing a Jordan block of size $k$ with a Jordan block of size $k + 1$.
3. If $B$ has entries in $\mathcal{U}$, then the splitting to $B'$ can be done over $\mathcal{U}$.

**Proof.** Without loss of generality, we may assume $\alpha \in \mathcal{U}$. For (i), use the splitting of row $i$ by $\beta v'$. For (ii), we have by assumption that there is a vector $v$ such that $\beta v B^k = 0$, $\beta v B^{k-1} \neq 0$ and $v$ is not in the image of $A$. Let $\mathcal{F}$ denote the field of fractions of $\mathcal{U}$. The kernel of $B^k$ contains a dense subset of vectors from $\mathcal{F}^n$. Pick $v'$ in $\ker(B^k) \cap \mathcal{U}^n$ with $||u - v'||$ small enough that $\beta v B^{k-1} \neq 0$ and $v'$ is not in the image of $B$. Pick $\beta \neq 0$ from $\mathcal{U}$ such that $\beta v' \in \mathcal{U}^n$. Pick $\gamma > 0$ in
Since \( U \) arbitrarily small, and small enough that for \( s := \alpha w_B + \gamma \beta v \) we have \( s > 0 \) and \( w_B - s > 0 \). Now form \( B' \) by splitting row \( i \) of \( B \) according to \( w_B = s + (w_B - s) \).

Then \( B' \) is conjugate to \( \left( \begin{array}{cc} B & 0 \\ s & 0 \end{array} \right) \) and to \( \left( \begin{array}{cc} B & 0 \\ \gamma \beta v' & 0 \end{array} \right) \). This last matrix has the required Jordan form. For small \( \gamma \), we have \( \|B' - A'\| < \epsilon \).

For the next lemma we establish some notation. For a square matrix \( M \), we let \( G_M, H_M \) denote the unique matrices \( G, H \) such that \( M = G + H, GH = HG = 0, H \)
is nilpotent and rank \( (G) = \text{rank}(G^2) \). We also use \( U_M, N_M, F_M \) to denote matrices such that \( U_M^{-1} M U_M = F_M \oplus N_M \), where \( F_M \) is nonsingular and \( N_M \) is nilpotent, and for concreteness \( N_M \) is in Jordan form. In this case, \( U_M^{-1} G_M U_M = F_M \oplus 0 \) and \( U_M^{-1} H_M U_M = 0 \oplus N_M \), where 0 denotes a zero matrix of appropriate size. The definition of the local connectedness condition in Lemma 7.6 below is Definition 7.2. The norm used below is the max norm.

**Lemma 7.6.** Suppose \( \epsilon > 0 \) and \( A, B \) are positive \( n \times n \) matrices such that \( G_A \) and \( G_B \) are conjugate, and the conjugacy class \( C \) of \( N_B \) is locally connected at \( N_A \).

Then there is a \( \delta > 0 \) such that any two matrices in \( N_\delta(A) \) which are conjugate to \( B \) are connected by a path of positive conjugate matrices in \( N_\epsilon(C) \).

**Proof.** Fix a matrices \( U, F_A, N_A \) such that \( U^{-1} A U = F_A \oplus N_A \). The idea of the proof is the following. For \( \delta \) small enough, given two matrices conjugate to \( B \) inside \( N_\delta(A) \), we show there are paths from them in \( N_\delta(A) \) to matrices \( C_1, C_2 \) which are
conjugated by \( U \) to matrices \( C_1' = \left( \begin{array}{cc} F_A & 0 \\ 0 & N_1 \end{array} \right) \) and \( C_2' = \left( \begin{array}{cc} F_A & 0 \\ 0 & N_2 \end{array} \right) \) such that there is a path of conjugate matrices from \( N_1 \) to \( N_2 \) which induces a path from \( C_1' \) to \( C_2' \) which \( U^{-1} \) conjugates to the desired path from \( C_1 \) to \( C_2 \). We spell out quantifiers for this next, for a matrix \( C \) conjugate to \( B \).

Take \( \epsilon \) to be smaller than \( \|A\| \). Pick \( \epsilon_1 > 0 \) such that \( \|N' - N_A\| < \epsilon_1 \) implies \( \|U(F_A \oplus N') U^{-1} A\| < \epsilon \). Pick \( \epsilon_2 > 0 \) such that if conjugate nilpotent matrices \( N_1, N_2 \) are in \( N_\epsilon_2(N_A) \), then there is a path of conjugate matrices in \( N_\epsilon_2(N_A) \) from \( N_1 \) to \( N_2 \). Pick \( \epsilon_3 > 0 \) such that \( \|X - A\| < \epsilon_3 \), then \( \|U^{-1}(X - A)U\| < \epsilon_2 \).

Finally, pick \( \delta > 0 \) such that if \( C \) is conjugate to \( B \) and \( \|A - C\| < \delta \), then the conjugate matrices \( A^n \) and \( C^n \) are sufficiently close that there is a \( V \) in \( \text{SL}(n, \mathbb{R}) \) such that \( V^{-1} A^n V = C^n \) (which means \( V^{-1} (G_A)^n V = (G_C)^n \)) and \( \|V - I\| \) is sufficiently small that the following hold:

1. \( V^{-1} G_A V = G_C \).
2. \( \|V^{-1} CV - A\| < \epsilon_3 \).
3. There is a path \((V_t)\) in \( \text{SL}(n, \mathbb{R}) \) from \( I = V_0 \) to \( V = V_1 \) remaining sufficiently close to \( I \) that \( V_t^{-1} CV_t \in N_\epsilon(A), 0 \leq t \leq 1 \).

Because \( V \) maps \( \ker(C^n) \) onto \( \ker(A^n) \), the matrix \( U^{-1}(V^{-1} BV)U \) has the form \( \left( \begin{array}{cc} F_A & 0 \\ 0 & N_A \end{array} \right) \), with

\[
\left( \begin{array}{cc} 0 & 0 \\ 0 & N' \end{array} \right) - \left( \begin{array}{cc} 0 & 0 \\ 0 & N_A \end{array} \right) = \left( \begin{array}{cc} F_A & 0 \\ 0 & N' \end{array} \right) - \left( \begin{array}{cc} F_A & 0 \\ 0 & N \end{array} \right) = U^{-1}(V^{-1} BV - A)U.
\]

Since \( \|V^{-1} BV - A\| < \epsilon_3 \), we have

\[
\|N' - N_A\| = \left\| \left( \begin{array}{cc} 0 & 0 \\ 0 & N' \end{array} \right) - \left( \begin{array}{cc} 0 & 0 \\ 0 & N_A \end{array} \right) \right\| < \epsilon_2
\]

which shows our \( \delta \) is small enough to establish the conclusion of the lemma. \( \square \)
Proof of the Connection Theorem. If $B$ and $A$ are conjugate, then the theorem follows from the Path Theorem. So we assume $B$ and $A$ are not conjugate, which implies $n \geq 3$.

The strategy of the proof is to take a row splitting of $A$ to a suitable positive matrix $A'$ chosen independent of $B$ and $C$; pick a suitable class $C$ of nilpotent matrices which is locally connected at $N_{A'}$; and then for $\delta > 0$ taken from Lemma 7.6 perform row splittings of $B$ and $C$ to matrices $B'$ and $C'$ in $C \cap N_{A'}(A')$.

To begin the proof, we choose $A'$ and a sequence of matrices $A_i$, $n \leq i \leq m$, such that $A = A_n$, $A' = A_m$ and for $n \leq i < m$, the matrix $A_{i+1}$ is obtained by splitting a row of $A_i$ into two proportional rows. For example, if $m = sn + p$ with $s \geq 1$ and $1 \leq p < n$, then we could split each of the first $p$ rows into $s$ equal rows and split each of the remaining rows into $s - 1$ equal rows. Then $\text{rank}(A') = \text{rank}(A)$, and $A'$ is conjugate to $A \oplus 0_{m-n}$.

Using notations from Definition 12.4 we define $h = \max\{h(A), h(B), h(C)\}$ and $\beta = \max\{\beta(A), \beta(B), \beta(C)\}$. (If $B$ is in $N^\text{se}_c(A)$ and $\epsilon$ is sufficiently small, then $h(B) \geq h(A)$ must hold.) Let $r$ be the rank of $F_A$. For any $C$ shift equivalent to $A$ over $\mathbb{R}$, we have $F_C$ conjugate to $F_A$ and $\text{rank}(F_C) = r$. So, if $C$ is $m \times m$ and shift equivalent to $A$, then $C$ is conjugate to $F_A \oplus N_C$, and the nilpotent matrix $N_C$ is $(m-r) \times (m-r)$. We will define $C$ to be the conjugacy class of $(m-r) \times (m-r)$ matrices which contains the matrix $(J_h)^{\beta} \oplus 0_q$, where $q = m - r - \beta$. We need to check this makes sense ($q \geq 0$) and also that $q > 0$. Because $n \geq 3$ and

$$(\beta)(h) \leq \frac{n^2 - r}{2}(n - r - 1) = \frac{n^2}{2} - \frac{r}{2}(3n - 1),$$

we have

$$(\beta)(h) + r \leq \frac{n^2}{2} - \frac{r}{2}(3n - 3)$$

$$\leq \frac{n^2}{2} - \frac{1}{2}(3(3) - 1) = \frac{n^2}{2} - 3r < \frac{n^2}{2} \leq m,$$

and therefore $q > 0$. Clearly

1. $C$ has a zero Jordan block (because $q > 0$).
2. $h(C) = h \geq h(A) = h(A')$.
3. $\beta^{\text{top}}(C) = \beta \geq \beta(A) = \beta(A')$.

It follows from Theorem 12.4 that $C$ is locally connected at $N_{A'}$. Therefore, we can specify any $\epsilon > 0$ and for that $\epsilon$ pick $\delta > 0$ in the statement of Lemma 7.6. This will be the $\delta$ in the statement of the Connection Theorem.

Now we describe the splitting of $B$ to $B'$ (the argument to split $C$ to $C'$ is the same), such that $N_{B'} \in C$. Starting with $B_0 = B$ and $||B - A|| < \delta$, for $n \leq i < m$ we inductively appeal to Lemma 7.3 to split $B_i$ to $B_{i+1}$, with $||A_{i+1} - B_{i+1}|| < \delta$. We use condition (2) of Lemma 7.3 at each stage as follows, applying the first listed criterion for which $B_i$ satisfies the required condition.

1. If $N_{B_i}$ has no zero Jordan block, then $N_{B_{i+1}} = N_{B_i} \oplus 0_1$.
2. If $N_{B_i}$ has fewer than $\beta$ Jordan blocks of size at least $2$, then $N_{B_{i+1}}$ is $N_{B_i}$ with a zero block replaced by a size 2 block.
3. If $N_{B_i}$ has a Jordan block of size $k$ with $2 \leq k < h$, then for a maximum such $k$, $N_{B_{i+1}}$ is $N_{B_i}$, with a Jordan $k$-block replaced by a $(k+1)$-block.
4. If $N_{B_i}$ has $\beta$ Jordan blocks of size $h$, then $N_{B_{i+1}} = N_{B_i} \oplus 0_1$. 

PATH METHODS FOR STRONG SHIFT EQUIVALENCE OF POSITIVE MATRICES 31
Clearly $B'$ has the required form, and there is a path of positive conjugate matrices from $B'$ to $C'$. By the Path Theorem (5.10), $B'$ and $C'$ are SSE over $\mathbb{R}_+$, through positive matrices of size $m$.

The “Moreover” condition of keeping splittings over $\mathcal{U}$ can be achieved by condition (3) of Lemma 7.4. If $\mathcal{U}$ is a field, then the conjugacy of $B'$ and $C'$ over $\mathbb{R}$ implies their conjugacy over $\mathcal{U}$, and (again using that $\mathcal{U}$ is a field) by the Path Theorem we have an SSE-$\mathcal{U}$ through positive matrices from $B'$ to $C'$, and hence also from $B$ to $C$. This finishes the proof. □

8. FROM SSE OVER $\mathbb{R}_+$ TO SSE OVER $\mathbb{U}_+$

**Theorem 8.1.** Suppose $A, B$ are positive matrices over $\mathbb{R}$ which are SSE-$\mathbb{R}_+$. Then $A, B$ are SSE over $\mathbb{R}_+$ through positive matrices: there are positive matrices $A = A_0, A_1, \ldots, A_\ell = B$, with $A_i$ elementary SSE over $\mathbb{R}_+$ to $A_{i+1}$, $0 \leq i < \ell$.

**Proof.** Appealing to Theorem 5.8, choose a primitive matrix $C$ over $\mathbb{R}_+$ and a nonsingular diagonal matrix $D$ over $\mathbb{R}_+$ such that $C$ is reached from $A$ by finitely many row splittings through primitive matrices and $D^{-1}CD$ is reached from $B$ by finitely many column splittings through primitive matrices. Then choose a construction, by the procedure described in Appendix B of a positive matrix $\tilde{C}$ SSE over $\mathbb{R}_+$ to $C$. Appealing to the Connection Theorem, choose $\delta > 0$ such that matrices shift equivalent to $\tilde{C}$ and $\delta$ close to $\tilde{C}$ are SSE to each other through positive matrices. Appealing to Lemma B.3 pick $\nu > 0$ such that for any positive matrix $M$ with $||M - C|| < \nu$ there is a strong shift equivalence over $\mathbb{R}_+$ through positive matrices from $M$ to a positive matrix $\tilde{M}$ such that $||\tilde{M} - \tilde{C}|| < \delta$.

Now, perform row splittings from $A$ through positive matrices to a positive matrix $A^*$ such that $||A^* - C|| < \nu$. This is done simply by approximating the string of splittings from $A$ to $C$ over $\mathbb{R}_+$ by a string of splittings from $A$ to $C$ through positive matrices. By composition, we have an SSE over $\mathbb{R}_+$ from $A$ to a matrix $\tilde{A}$ within $\delta$ of $\tilde{C}$.

The argument to obtain an SSE over $\mathbb{R}_+$ through positive matrices from $B$ to a matrix $\tilde{B}$ within $\delta$ of $\tilde{C}$ is similar. We obtain a positive matrix $B^*$ near $D^{-1}CD$ by approximating the given column splittings from $B$ to $D^{-1}CD$. There is an elementary SSE over $\mathbb{R}_+$ from $B^*$ to the positive matrix $D^{-1}B^*D := B^{**}$. We take $B^*$ close enough to $D^{-1}CD$ to guarantee $||B^{**} - C|| < \nu$. Then we apply Lemma B.3 again to obtain the SSE through positive matrices from $B^{**}$ to the desired $\tilde{B}$ near $C$. By the Connection Theorem, $\tilde{A}$ and $\tilde{B}$ are SSE over $\mathbb{R}_+$ through positive matrices. By composing the assembled SSEs, the theorem is proved. □

Theorem 8.2 below was proved in [17] for $\mathcal{U} = \mathbb{Q}$ under the additional assumption that $A$ and $B$ are SSE over $\mathbb{R}_+$ through positive matrices.

**Theorem 8.2.** Let $\mathcal{U}$ be a subfield of $\mathbb{R}$. Suppose $A, B$ are positive matrices over $\mathcal{U}$ which are SSE over $\mathbb{R}_+$. Then $A$ and $B$ are SSE over $\mathcal{U}_+$, through positive matrices.

**Proof.** We examine the proof of Theorem 8.1 and check that the SSEs constructed in the various steps can be taken through positive matrices over $\mathcal{U}$.

The splittings to $A^*$ and $B^*$ can be done over $\mathcal{U}$.

Approximate $D$ by a diagonal matrix $D'$ over $\mathcal{U}_+$. In place of $B^{**}$, use $(D')^{-1}B^*D' := B'$. Because $\mathcal{U}$ is a field, the positive matrix $B'$ has its entries in $\mathcal{U}$ and is ESSE over $\mathcal{U}_+$ to $B^*$ (by the matrices $(D')^{-1}$ and $B^*D'$).
The matrices $\tilde{A}$ and $\tilde{B}$ are constructed from $A^*$ and $B'$ (matrices over $U$) by appeal to Proposition 3.2 and therefore they can be taken over $U$. Lemma 3.3 allows the approximating SSE through positive matrices to be taken over $U$. Because $U$ is a field, the Connection Theorem then gives an SSE through positive matrices over $U$ from $\tilde{A}$ to $\tilde{B}$.

This completes the proof. □

**Theorem 8.3.** Suppose $A$ is a positive $n \times n$ matrix. The collection of positive $n \times n$ matrices conjugate over $\mathbb{R}$ to $A$ contains only finitely many SSE-$\mathbb{R}_+$ classes.

**Proof.** This follows immediately from Theorem 5.12 and Theorem 8.2. □

**Problem 8.4.** Let $U$ be a dense subring of $\mathbb{R}$. Suppose $A$ and $B$ are positive matrices which are strong shift equivalent over $U$ and also over $\mathbb{R}_+$. Must they be strong shift equivalent over $U_+$?

**Appendix A. Making SSE nondegenerate**

The purpose of the appendix is to prove Proposition 2.9, which we now restate.

**Proposition 2.9** Suppose $U$ is a ring which is torsion free as an additive group. Suppose nondegenerate matrices $A$ and $B$ are SSE over $U$. Then they are SSE through a chain of ESSEs $A_{i-1} = R_iS_i, A_{i+1} = S_iR_i$ such that all the matrices $A_i, R_i, S_i$ are nondegenerate.

Recall, a matrix is nondegenerate if it has no zero row and no zero column. Below, by a nonzero matrix we mean a matrix which is not the zero matrix.

We will prove Proposition 2.9 after proving three lemmas.

**Lemma A.1.** Suppose $U$ is a ring and there are matrices $A, B, C, R, S, R', S'$ over $U$ satisfying the following conditions $A = RS$, $B = SR$; $B = R'S'$, $C = S'R'$; $A \neq 0$, $B = 0$, $C \neq 0$. Then there are nonzero matrices $A = A_0, A_1, \ldots, A_6 = C$ such that $A_{i-1}$ is ESSE over $U$ to $A_i$, $1 \leq i \leq 6$.

**Proof.** In block form, define

$$
A_1 = \begin{pmatrix} RS & 0 \\ S & 0 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 0 & 0 \\ S' & 0 \end{pmatrix}, \quad A_5 = \begin{pmatrix} 0 & 0 \\ S' & S'R' \end{pmatrix},
$$

$$
A_2 = \begin{pmatrix} 0 & 0 \\ S & SR \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & 0 \\ S & 0 \end{pmatrix}, \quad A_6 = (S'R') = B.
$$

The matrices $S$ and $S'$ cannot be zero (because $A$ and $C$ are not zero), so the $A_i$ are not zero. An ESSE from $A \rightarrow A_1$ is given by

$$
A = (I \ 0) \begin{pmatrix} RS \\ S \end{pmatrix} , \quad A_1 = \begin{pmatrix} RS \\ S \end{pmatrix} (I \ 0).
$$
There are ESSEs $A_2 \to A_3, A_3 \to A_4, A_5 \to A_6$ of this type or a transpose type. An ESSE $A_1 \to A_2$ is given by
\[
A_1 = \begin{pmatrix} RS & 0 \\ S & 0 \end{pmatrix} = \begin{pmatrix} RS & RSR \\ S & SR \end{pmatrix} \begin{pmatrix} I & -R \\ 0 & I \end{pmatrix}
\]
\[
A_2 = \begin{pmatrix} 0 & 0 \\ S & SR \end{pmatrix} = \begin{pmatrix} I & -R \\ 0 & I \end{pmatrix} \begin{pmatrix} RS & RSR \\ S & SR \end{pmatrix}
\]
The remaining ESSE $A_4 \to A_5$ is of the same type. \hfill \Box

**Lemma A.2.** Suppose $\mathcal{U}$ is a unital semiring and $A$ is ESSE over $\mathcal{U}$ to $0_m$, the $m \times m$ zero matrix. Then $A$ is ESSE over $\mathcal{U}$ to $0_{m+k}$, for all $k \in \mathbb{N}$.

**Proof.** We are given $A = RS, 0_M = SR$. Then $A = (r \circ \delta)(\delta \circ 0)$ and $0_{m+k} = (\delta \circ 0)(r \circ \delta)$ where $0$ denotes a zero block of the necessary size. \hfill \Box

**Lemma A.3.** Let $\mathcal{U}$ be a unital ring which is torsion free as an additive group. Suppose $A$ is an $n \times n$ matrix over $\mathcal{U}$ which is not the zero matrix. Then there is $V$ in $SL(n, \mathbb{Z})$ such that $V^{-1}AV$ is nondegenerate.

**Proof.** We can assume $n > 1$. For example, suppose row 1 of $A$ is nonzero and row $n$ of $A$ is zero. Given $M \in \mathbb{N}$, let $E$ be the basic elementary matrix such that $E(n, 1) = M$ and set $C = EAE^{-1}$. Then
\[
C(i, 1) = A(i, 1) - MA(i, n) \quad \text{if} \; i < n
\]
\[
C(n, 1) = MA(1, 1) - M^2A(1, n)
\]
\[
C(n, j) = MA(1, j) \quad \text{if} \; j > 1
\]
\[
C(i, j) = A(i, j) \quad \text{if} \; i < n \; \text{and} \; j > 1
\]
Appealing to the torsion free assumption, choose $M$ such that
\[
1 \leq i < n \; \text{and} \; A(i, 1) \neq 0 \implies A(i, 1) \neq MA(i, n)
\]
Then $A(i, j) \neq 0$ implies $C(i, j) \neq 0$. In addition, row $n$ of $C$ is not zero, as follows. If there exists $j > 1$ with $A(1, j) \neq 0$, then $C(n, j) \neq 0$; otherwise, $A(1, 1)$ is the only nonzero entry of row 1 of $A$, and $C(n, 1) = MA(1, 1) \neq 0$.

Iterating this move as needed, with other indices $(i, j)$ in place of $(1, n)$, and interchanging the role of column and row as needed, we produce $V \in SL(n, \mathbb{Z})$ such that $V^{-1}AV$ is nondegenerate. \hfill \Box

**Remark A.4.** We are not concerned in this paper with finding the sharpest version of Proposition 2.9. However, we note that Lemma A.3 would be false if the “torsion free” assumption were simply dropped. Over the field $\mathbb{Z}/2$, let $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Then $A \neq 0$ but $A$ is not conjugate over $\mathbb{Z}/2$ to a nondegenerate matrix, because $B$ is the only rank one nondegenerate $2 \times 2$ matrix over $\mathbb{Z}/2$, and $A^2 \neq 0 = B^2$.

We are now ready to prove Proposition 2.9.

**Proof of Proposition 2.9.** We are given some string of ESSEs over $\mathcal{U}$ from $A = A_0$ to $B = A_\ell$,
\[
A = A_0 \to A_1 \to A_2 \to \cdots \to A_{\ell-1} \to A_\ell = B
\]
with matrices $R_i, S_i$ over $\mathcal{U}$ such that $A_{i-1} = R_iS_i$, $A_i = S_iR_i$, for $1 \leq i \leq \ell$. 

Suppose for some \( i \) and some \( k > 2 \) that \( A_i \) and \( A_{i+k} \) are not zero, but \( A_j \) is a zero matrix for \( i < j < i + k \). By Lemma \[A.2\] there is a zero matrix \( Z \) ESSE to \( A_i \) and to \( A_{i+k} \). We replace the ESSEs \( A_i \rightarrow A_{i+1} \rightarrow \cdots \rightarrow A_{i+k} \) with ESSEs \( A_i \rightarrow Z \rightarrow A_{i+k} \). After iterating this move as necessary, we may assume that \( A_i = 0 \) implies \( A_{i-1} \neq 0 \) and \( A_{i+1} \neq 0 \).

Then, by Lemma \[A.1\] if \( A_i = 0 \), we may replace the ESSEs \( A_{i-1} \rightarrow A_i \rightarrow A_{i+1} \) with a string of ESSEs from \( A_{i-1} \) to \( A_{i+1} \) through nondegenerate matrices. After iterating as needed, we may assume every \( A_i \) is not zero.

If \( 0 < i < \ell \) and \( U^{-1}A_iU = A'_i \), then we can replace

\[
(A.1) \quad A_{i-1} \xrightarrow{(R_i,S_i)} A_i \xrightarrow{(R_{i+1},S_{i+1})} A_{i+1}
\]

with

\[
(A.2) \quad A_{i-1} \xrightarrow{(R_i,U^{-1}S_i)} A'_i \xrightarrow{(U^{-1}R_{i+1},S_{i+1}U^{-1})} A_{i+1}.
\]

Thus by repeated application of this move, with \( U^{-1}A_iU = A'_i \) nondegenerate by Lemma \[A.3\], we can pass to an SSE through nondegenerate matrices as required. \( \square \)

**Appendix B. Boolean matrices and positivity**

Let \( \mathcal{U} \) be a nondiscrete unital subring of \( \mathbb{R} \).

We will include in this section a proof of the result of [12] that every primitive matrix over \( \mathcal{U} \) is SSE over \( \mathcal{U}_+ \) to a positive matrix. As in [12], this is done by proving the result for Boolean matrices and then carrying it over. We can then prove the approximation result Lemma \[B.3\] which we need in Section \[S\].

Boolean matrices are matrices with entries in the Boolean semiring \( B = \{0, 1\} \), in which \( 1 + 1 = 1 \). The usual row and column splitting and amalgamations can be used to produce SSEs over \( B \). In particular, if a row \( i \) of \( A \) is less than or equal to row \( j \) of \( A \), then adding column \( j \) of \( A \) to column \( i \) produces a matrix \( B \) SSE over \( B \) to \( A \); for the corresponding elementary matrix \( E \), we have \( A = EA \) and \( B = AE \).

An example, assuming row 1 of \( A \) is less than or equal to row 2, is

\[
A = \begin{pmatrix}
a_1 & a_2 & a_3 \\
b_1 & b_2 & b_3 \\
c_1 & c_2 & c_3
\end{pmatrix} = EA = \begin{pmatrix}
1 & 0 & 0 \\
1 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
a_1 & a_2 & a_3 \\
b_1 & b_2 & b_3 \\
c_1 & c_2 & c_3
\end{pmatrix}
\]

\[
B = \begin{pmatrix}
a_1 + a_2 & a_2 & a_3 \\
b_1 + b_2 & b_2 & b_3 \\
c_1 + c_2 & c_2 & c_3
\end{pmatrix} = AE = \begin{pmatrix}
a_1 & a_2 & a_3 \\
b_1 & b_2 & b_3 \\
c_1 & c_2 & c_3
\end{pmatrix} \begin{pmatrix}
1 & 0 & 0 \\
1 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]

If \( A \) is the Boolean image of a matrix \( A' \) over \( \mathcal{U}_+ \), then there are \( E', B' \) over \( \mathcal{U}_+ \) with Boolean images \( E, B \) such that \( A' = E'A' \) and \( B' = A'E' \). Here, \( E' \) is an elementary matrix whose off diagonal entry can be chosen arbitrarily close to zero, and \( B' \) is conjugate over \( \mathcal{U} \) to \( A' \). In the example (using the letter entries in \( A \)
An example with $C$

For any sufficiently small and positive $\epsilon$ in through vertex $i$, we have

\[ B' = (E')^{-1} A' E' = \begin{pmatrix} 1 & 0 & 0 \\ -\epsilon & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ \epsilon & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \]

\[ = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 - \epsilon a_1 & b_2 - \epsilon a_2 & b_3 - \epsilon a_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ \epsilon & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \]

\[ = \begin{pmatrix} a_1 + \epsilon a_2 & a_2 & a_3 \\ b_1 - \epsilon a_1 + \epsilon b_2 - \epsilon^2 a_2 & b_2 - \epsilon a_2 & b_3 - \epsilon a_3 \\ c_1 + \epsilon c_2 & c_2 & c_3 \end{pmatrix} \]

For any sufficiently small and positive $\epsilon$ from $\mathcal{U}$, we have $(E')^{-1} A' \geq 0$, and therefore an ESSE over $\mathcal{U}_+$ between $A'$ and $(E')^{-1} A' E'$.

The next result is proved in [12] and we take that proof.

**Proposition B.1.** Suppose $A$ is a primitive Boolean matrix with positive trace. Then $A$ is SSE over $\mathcal{B}$ to $[1]$.

**Proof.** $A$ is the adjacency matrix of a directed graph. Take a closed walk through the graph which passes through every vertex at least once. Suppose the walk passes through vertex $i$ $n_i$ times. Define a matrix $V$ which has, for each $i$, $n_i$ copies of row $i$ of $A$. (Over $\mathcal{B}$, a row copying is an example of a row splitting.) Let $U$ be the subdivision matrix such that $UV = A$, and set $VA = B$, SSE over $\mathcal{B}$ to $A$. Then there is a closed walk through the graph of $B$ which hits every vertex exactly once.

Without loss of generality, then, suppose $B$ is $m \times m$ and $B(1,1) = 1$ and $P \leq B$, where $P$ is the matrix with positive entries at $(1,1), (1,2), (2,3), \ldots, (m,1)$.

Next, define an ESSE from $B$ to a $2m \times 2m$ matrix $C$, by

\[ C = \begin{pmatrix} P & B \\ P & B \end{pmatrix} = \begin{pmatrix} I_m & P \\ I_m & B \end{pmatrix} \begin{pmatrix} P & B \\ I_m & I_m \end{pmatrix} \]

An example with $C$ $10 \times 10$ is

\[ C = \begin{pmatrix} P & B \\ P & B \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 1 & 1 & \bullet & \bullet & \bullet \\ 0 & 0 & 1 & 0 & 0 & \bullet & \bullet & 1 & \bullet & \bullet \\ 0 & 0 & 0 & 1 & 0 & \bullet & \bullet & \bullet & 1 & \bullet \\ 0 & 0 & 0 & 0 & 1 & \bullet & \bullet & \bullet & \bullet & \bullet \\ 1 & 0 & 0 & 0 & 0 & 1 & \bullet & \bullet & \bullet & \bullet \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & \bullet & \bullet & \bullet \\ 0 & 0 & 1 & 0 & 0 & \bullet & \bullet & 1 & \bullet & \bullet \\ 0 & 0 & 0 & 1 & 0 & \bullet & \bullet & \bullet & 1 & \bullet \\ 0 & 0 & 0 & 0 & 1 & \bullet & \bullet & \bullet & \bullet & \bullet \\ 1 & 0 & 0 & 0 & 0 & 1 & \bullet & \bullet & \bullet & \bullet \end{pmatrix} \]

in which a bullet denotes an entry which could be 0 or 1, depending on $A$.

Note, column 1 of $C$ is greater than or equal to column 2. So, we may add row 1 of $C$ to row 2 (to produce an SSE matrix). Now in the order $i = 2, 3, \ldots, m - 1$, add row $i$ to row $i + 1$. At the point row $i$ is added, column $i$ will be greater than or equal to column $i + 1$, so the addition will give an ESSE. After these moves,
row $m$ has every entry 1. In order, for $i = m, m - 1, \ldots, 2, 1$, add column $i$ to every other column. At the point column $i$ is added, row $i$ will be all 1’s, so SSE is respected. Because every row has an entry 1 in one of the columns $1, 2, 3, \ldots, m$, at the conclusion of this $C$ will be transformed to a matrix with every entry 1. Such a matrix equals $vv^t$, where $v$ is a column vector with every entry 1, and then $v^tv = [1]$. □

The next result extends a result in [15], with essentially the same proof. Let $\omega(n)$ denote the maximum size of a minimal length closed walk which hits all vertices in a strongly connected directed graph with exactly $n$ vertices. Clearly, $\omega(n) \leq n^2$ by composition of shortest paths $i \rightarrow i + 1$ to get $1 \rightarrow 2 \rightarrow \cdots \rightarrow n \rightarrow 1$. On the other hand (we thank Richard Brualdi for this remark), for $n = 2k \geq 4$ consider the directed graph on vertices $\{1, 2, \ldots, 2k\}$ for which the set of nonzero entries of the adjacency matrix is the union of the following sets:

\[
\{(1, j) : 1 \leq j \leq k\}, \quad \{(i, k + 1) : 1 \leq j \leq k\},
\{(i, i + 1) : k \leq i < 2k\}, \quad \{(2k, 1)\}.
\]

We see that $\omega(2k) \geq (k - 1)(k + 2)$ and therefore $\omega(n) \geq \frac{1}{4}n^2$.

**Proposition B.2.** Suppose $U$ is a unital nondiscrete subring of $\mathbb{R}$ and $A$ is an $n \times n$ primitive matrix over $U_+$ with positive trace. Then $A$ is SSE over $U_+$ to a positive matrix $\tilde{A}$, of size at most $2n^2$.

**Proof.** The matrix operations used in the proof of Proposition B.1 give elementary SSEs over $B$, which can be mimicked over $U_+$ by matrices with the same zero/nonzero pattern to give the SSE over $U_+$ to a positive matrix $\tilde{A}$. The size of $\tilde{A}$ is at most $2\omega(n) \leq 2n^2$. □

**Lemma B.3.** Suppose $U$ is a unital nondiscrete subring of $\mathbb{R}$, $A$ is an $n \times n$ primitive real matrix positive trace and $\tilde{A}$ is a positive matrix SSE over $\mathbb{R}$ to $A$ and constructed from $A$ using the algorithm of the proof of Proposition B.3. Suppose $\delta > 0$. Then there is $\nu > 0$ such that the following holds. If $A'$ is a positive matrix over $U$ and $||A' - A|| < \nu$, then $A'$ is SSE over $U_+$, through positive matrices over $U$, to a matrix $\tilde{A}'$ such that $||\tilde{A}' - \tilde{A}|| < \delta$.

**Proof.** The construction of $\tilde{A}$ in proposition B.2 proceeds by mimicking over $\mathbb{R}_+$ the Boolean construction in the proof of Proposition B.1. There are three steps in the construction of $\tilde{A}$.

1. Mimicking the row splittings of the Boolean image of $A$ to a Boolean matrix $B$, there are row splittings over $\mathbb{R}_+$ of $A$ to some matrix $B^*$.
2. Mimicking the splitting of the $m \times m$ Boolean $B$ to the $2m \times 2m$ Boolean $C$, there is a real $2m \times 2m$ matrix $C^*$ split from $B^*$.
3. Mimicking the Boolean SSE from $C$ to a matrix with no zero entry, there is a string of elementary matrices $E_t$ which produces an SSE from $C^*$ to $\tilde{A}$. Let $\ell$ be the lag of this SSE, and use a notation $C^* = M_0$, $\tilde{A} = M_\ell$, $M_t = (E_t)^{-1}M_{t-1}E_t$. For $1 \leq t \leq \ell$, there is an $\epsilon_t > 0$ which without loss of generality we assume is in $U$, such that
   - if $1 \leq t \leq m - 1$, then $(E_t)^{-1} \geq 0$ and $M_{t-1}E_t$ is obtained by subtracting $\epsilon_t$ times column $t + 1$ from column $t$. 


• if \( m \leq t \leq \ell \), then \( E_t \geq 0 \) and there are \( i, j \) such that \( (E_t)^{-1}M_{t-1} \) is obtained from \( M_{t-1} \), in which row \( j \) has no zero entry, by subtracting \( \epsilon_t \) times row \( i \) from row \( j \).

First consider step 3. Suppose \( M'_0 \) is a positive \( 2m \times 2m \) matrix. We recursively define \( M'_t = (E_t)^{-1}M'_{t-1}E_t \), for \( t = 1, 2, \ldots, \ell \). Then define \( \delta' \) to be the minimum of \( \delta \) and the smallest positive entry in a matrix of the form \( M_t, E_t, (E_t)^{-1}M_{t-1} \) or \( M_{t-1}E_t, 1 \leq t \leq \ell \). Suppose the following hold:

(i) \( M'_0 \) is close enough to \( M_0 \) that for \( 1 \leq t \leq \ell \), if \( G \) is a matrix in one of the four forms above, and \( G' \) is defined by replacing \( M_t \) with \( M'_t \) wherever it appears in the definition of \( G \), then \( ||G' - G||_{\text{max}} < \delta' \).

(ii) For \( 1 \leq t < m \) and \( 1 \leq i \leq 2m \), if \( C(i, t) = C(i, t + 1) = 0 \), then \( M'_0(i, t + 1) < (1/\epsilon_t)M'_0(i, t) \).

We claim that the matrices \( M'_t \) are then positive, and for \( 1 \leq t \leq \ell \), there is an ESSE over \( U_t \) from \( M'_{t-1} \) to \( M'_t \). For \( 1 \leq t < m - 1 \), the conditions (i) and (ii) imply that \( M'_{t-1}E_t \geq 0 \) and \( M'_t > 0 \), and the matrices \( (E_t)^{-1} \) and \( M'_{t-1}E_t \) give the required ESSE over \( U_t \). For \( t \geq m \), condition (i) implies the matrices \( (E_t)^{-1}M'_{t-1} \) will be nonnegative, and again we get the ESSE over \( U_t \). It also follows from (i) that \( ||M'_t - A||_{\text{max}} < \delta \).

To finish, we first note that by taking \( \nu \) sufficiently small we can approximate the splittings from \( A \) to \( B^* \) in the first step arbitrarily closely by splittings from \( A' \) to a matrix \( B'^* \) through positive matrices over \( U \); and for the second step, given a positive matrix over \( U \) close to \( B \), we can split \( B'^* \) to a positive matrix \( M'_0 = (P \gamma' B'') \) over \( U \) close to \( C^* \) and also satisfying the inequalities listed in condition (ii). \( \square \)

Appendix C. Positive invariant tetrahedra

Suppose \( A \) and \( B \) are positive \( n \times n \) matrices over \( \mathbb{R} \), and are conjugate over \( \mathbb{R} \). In this section we describe the approach introduced in [14] for finding a positive path \( A_t \) of conjugate matrices from \( A \) to \( B \). Without loss of generality, we suppose that \( A \) and \( B \) have spectral radius 1.

First we give some terminology from [14]. A positive tetrahedron is an \( n \)-tuple \( \tau \) of vectors in an \( n - 1 \) dimensional real vector space such that the origin is contained in the interior of its convex hull \( C(\tau) \). With respect to a given linear endomorphism \( T \), an invariant positive tetrahedron is a positive tetrahedron \( \tau \) such that \( C(\tau) \) is mapped into its interior by \( T \).

Now, suppose that \( A_t = G_t^{-1}AG_t \) is a path of positive matrices from \( A = A_0 \) to \( B = A_1 \). We can deform such a path to a path of positive stochastic matrices, so we assume now that the \( A_t \) are positive and stochastic (so, letting \( r \) denote the column vector with every entry 1, we have \( A_t(r) = r \)). Let \( e_i \) denote the row vector which is the \( i \)th canonical basis vector. Let \( \ell \) be the stochastic left eigenvector of \( A_t \) and set \( v_i = e_i - \ell \), the projection of \( e_i \) along \( \mathbb{R}r \) to the \( A_t \) invariant subspace \( W \) of vectors whose entries sum to zero. The tuple \( (v_1, \ldots, v_n) \) has convex hull which contains the origin in its interior. Set \( w_i = v_iG_t^{-1} \). Then \( \tau_t = (w_1, \ldots, w_n) \) is a positive invariant tetrahedron with respect to the linear transformation \( T : W \to W \) defined by \( w \mapsto wA \).

The path \( (A_t) \) gives rise to the path \( \tau_t \). Given \( t \), the matrix \( A_t \) is recovered from the action of \( A \) on \( \tau_t \), as follows. For each \( i \), the vector \( w_iA \) is a unique convex combination of the \( w_i \) (which are the extreme points of \( C(\tau_t) \)), and the coefficients
for this convex combination are provided by row $i$ of the matrix $A_t$, as follows:

$$w_i^t A = (v_i^t G_i^{-1} )(G_t A_t G_t^{-1} ) = v_i^t A_t G_t^{-1} = \sum_j A_{ij}^t w_j^t .$$

Conversely, starting from a path of positive invariant tetrahedra from $\tau_0$ to $\tau_1$, we have a path $(A_t)$ of positive stochastic matrices, with the $A_t$ defined as above. Given $t$, there is a unique matrix $G_t$ such that $v_i^0 G_t = v_i^t$ for $1 \leq i \leq n$ and $G_t r = r$, and for this matrix we have $A_t = G_t^{-1} A G_t$.

Now, to find a path of positive invariant tetrahedra, one passes (for example, see Lemmas 6.2, 6.3) to considering a path of matrices $A_t = G_t^{-1} A G_t$ with $A_t r = r$ for every $t$. As before, define the vectors $v_i^t$ and $w_i^t$ to get a path of positive tetrahedra $\tau_t$. Now, if $A_t$ is not positive, then $\tau_t$ will not be an invariant positive tetrahedron. The problem of deforming the path $(A_t)$ to a path of conjugate positive matrices is replaced with the problem of deforming the path $(\tau_t)$ to a path of invariant positive tetrahedra. So, one is led to study the set of connected components of invariant positive tetrahedra for a positive matrix.

There is more information about these components in the thesis [6] of Chuy-surichay.

**Appendix D. A Local Connectedness Condition for Nilpotent Matrices**

Below, $||C||_{\text{max}}$ denotes the maximum absolute value of an entry of $C$.

**Definition D.1.** For an $n \times n$ real matrix $A$, and $\epsilon > 0$, $\mathcal{N}_\epsilon(A)$ denotes the set of $n \times n$ matrices $B$ such that $||B - A||_{\text{max}} < \epsilon$.

**Definition D.2.** Suppose $N$ is an $n \times n$ nilpotent matrix and $C$ is a conjugacy class of $n \times n$ nilpotent matrices. We say $C$ is locally connected at $N$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that any two matrices in $C \cap \mathcal{N}_\delta(A)$ are connected by a path in $C \cap \mathcal{N}_\epsilon(A)$.

We introduce some notation. Given a matrix $M$, $||M||_{\text{max}}$ is the maximum of the absolute values of its entries. $I_t$ is the $t \times t$ identity matrix, $0_t$ is the $t \times t$ zero matrix, and $e_i$ denotes a zero-one row vector whose only nonzero entry is in coordinate $i$. The direct sum of square matrices $A, B$ is the matrix $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$. $J_n$ is the Jordan block matrix of size $n$: the $n \times n$ zero-one matrix $J$ such that $J(i, j) = 1$ iff $1 \leq i < n$ and $j = i + 1$. A matrix in Jordan form is a direct sum of Jordan blocks; the matrix has a zero Jordan block iff its kernel is not contained in its image.

**Definition D.3.** Suppose $M$ is a nilpotent matrix in Jordan form. Then

- $h(M)$ is the maximum size of a Jordan block summand of $M$.
- $\beta(M)$ is the number of Jordan block summands of $M$ of size at least 2.
- $\beta^{\text{top}}(M)$ is the number of Jordan block summands of $M$ of size $h(M)$.

For $N$ nilpotent in a conjugacy class $C$, we define $h(N)$ and $h(C)$ to be $h(M)$ for any $M$ in Jordan form conjugate to $N$. Similarly for $\beta$ and $\beta^{\text{top}}$.

**Theorem D.4.** Suppose $N$ is a nilpotent $n \times n$ matrix and $C$ is a conjugacy class of nilpotent $n \times n$ matrices, such that the following hold:

1. The Jordan form of a matrix in $C$ has a zero block.
(2) $h(\mathcal{C}) \geq h(N)$.
(3) $\beta^{top}(\mathcal{C}) \geq \beta(N)$.

Then $\mathcal{C}$ is locally connected at $N$.

The necessity of condition (2) in the statement is clear. Without condition (1), $N$ can be the limit of matrices from different connected components of $\mathcal{C}$, as happens with $t > 0$ in the following example:

$$N = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad M_t = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & t & 0 \\ 0 & 0 & 0 & t \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad P_t = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & t & 0 \\ 0 & 0 & 0 & -t \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$ 

Question D.5. Does Theorem [1, 4] remain true if the assumption (3) is removed?

Our partial result and the structure of the nilpotent matrices as a stratified space [26, 27, 29] suggest the answer may be yes.

It is clear that the theorem holds for $N$ if and only if it holds for some matrix conjugate to $N$. For the proof, we will make explicit constructions using a matrix of a specific form. We will formulate the constructive result below as a technical lemma, for which we make some preparations.

Theorem [1, 4] is true if $N \in \mathcal{C}$ (Lemma [5, 1]) and it is vacuously true for $\mathcal{C}$ if $N$ is not a limit of matrices from $\mathcal{C}$. So, we assume from here that $N \not\in \mathcal{C}$ and $N$ is a limit of matrices from $\mathcal{C}$, which implies for $M$ in $\mathcal{C}$ that $\text{rank}(M^k) \geq \text{rank}(N^k)$, $0 \leq k \leq n$. If $N = 0$, then Theorem [1, 4] can be proved quickly with the argument of Step 2 of Stage 4 below. So we also assume from here that $N \neq 0$, which means $\beta(N) \geq 1$. Set $\beta = \beta(N)$ and $h = h(\mathcal{C})$.

Given $k$ with $1 \leq k < h$, we define the $h \times h$ matrix $N_k$ by the rule

$$N_k(i, j) = \begin{cases} 1 & \text{if } 1 \leq i \leq k \text{ and } j = k + 1 \\ 0 & \text{otherwise} \end{cases}.$$ 

The first $k$ rows of $N_k$ equal those of $J_h$ and the remaining rows of $N_k$ are zero.

We also fix a list $k_1, \ldots, k_\beta$ with $k_i \geq 2$ for each $i$, such that $N$ is conjugate to the direct sum of $J_{k_1} \oplus J_{k_2} \oplus \cdots \oplus J_{k_\beta}$ and a zero matrix. Then we fix the form we will use for our matrix $N$:

$$N = N_{k_1} \oplus N_{k_2} \oplus \cdots \oplus N_{k_\beta} \oplus 0_{n-\beta h}$$

and define some associated subsets of $\{1, 2, \ldots, n\}$,

$$\mathcal{J} = \{i : \text{row } i \text{ of } N \text{ is nonzero}\}$$

$$\mathcal{K} = \{i : \text{row } i \text{ of } N \text{ is zero}\}$$

$$\mathcal{T} = \{1 + rh : 0 \leq r < \beta\}.$$ 

Given $M = J_h \oplus J_h \oplus \cdots \oplus J_h$, let $J$ be an $(n - \beta h) \times (n - \beta h)$ matrix in Jordan form such that $(\begin{smallmatrix} M_{0,0} \\ 0 & j \end{smallmatrix})$ is in $\mathcal{C}$ and has row $n$ and column $n$ zero (which is possible by the assumption (1) of a zero block). The set $\mathcal{T}$ indexes the rows of $M$ through the top rows of its first $\beta$ Jordan blocks, each of which is $J_h$. These are also the top rows of the diagonal blocks $N_{k_1}$ in $N$.

Given $\epsilon > 0$, define $M(\epsilon)$ to be the $n \times n$ matrix such that

$$M(\epsilon)(i, j) = N(i, j) \quad \text{if } i \in \mathcal{J}$$
$$= \epsilon M(i, j) \quad \text{if } i \notin \mathcal{J}.$$
Given δ > 0, $\mathcal{M}_δ$ denotes the set of $n \times n$ matrices $C$ such that the following hold:

(i) $C$ is conjugate to $M$.
(ii) $||C - N||_{\max} < \delta$.

We say $C \in \mathcal{M}_δ^0$ if $C \in \mathcal{M}_δ$ and in addition

(iii) If $i \in J$, then row $i$ of $C$ equals row $i$ of $N$.

We will use $\mathcal{M}$ and $\mathcal{M}_0$ to denote the union over $\delta > 0$ of $\mathcal{M}_δ$ and $\mathcal{M}_δ^0$ (respectively).

**Example D.6.** For the matrix arguments to follow, it may be helpful to have the block structure of an example in view. For this example, we take

$$h = 5, \quad \beta = 2, \quad n = 12, \quad k_1 = 2, \quad k_2 = 1,$$

$$M = J_5 \oplus J_5 \oplus 0_2,$$

$$N = N_2 \oplus N_1 \oplus 0_2.$$

Now a matrix $C$ in $\mathcal{M}_δ^0$ has a block structure:

$$C = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

in which each $\bullet$ has absolute value less than $\delta$. If $C$ is only in $\mathcal{M}_δ$, then the entries marked 0 and 1 above are only approximated to within $\delta$. Continuing the example, we have

$$M(\epsilon) = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
0 & 0 & 0 & \epsilon & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
0 & 0 & 0 & 0 & \epsilon & 0 & 0 & 0 & 0 & 0 & 0 & 0
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$
The example is somewhat special in that the summand \( 0_2 \) of \( M \) could have been much more complicated. However, it turns out that this possible complication doesn’t matter in the proof below until the last stage, where it is not a big problem.

We are finally ready to state the technical lemma, from which Theorem 4.3 follows immediately.

**Lemma D.7.** Given \( 0 < \gamma < 1/49 \), there exists \( \delta > 0 \) such that for all \( C \) in \( M_{\delta} \) and all \( \epsilon \) such that \( 0 < \epsilon < \delta \), there is a path in \( M_\epsilon \) from \( C \) to \( M(\epsilon) \).

**Proof.** The path will be a concatenation of paths constructed in four stages. Combining the estimates, given \( 0 < \gamma < 1/49 \), the lemma will hold for

\[
\delta = \frac{n^{1/2} \gamma^2}{4(3n + 3)^n}.
\]

We do not claim this estimate or the requirement \( \gamma < 1/49 \) are sharp.

Below, subscripted matrices \( C \) in different stages are dummy variables not related to subscripted matrices \( C \) in other stages.

**Stage 1.** Given \( C \) in \( M_\mu \), we produce a path in \( M_{\kappa} \) to a matrix \( C_S \) in \( M_{\kappa}^0 \), where \( \kappa = [3(n + 1)]^S \) and \( S = \#J < n \). For this stage, let \( i_1 < i_2 < \cdots < i_S \) denote the elements of \( J \). Set \( C_0 = C \). For \( 1 \leq s \leq S \), given \( C_{s-1} \), we will define inductively \( C_s \) and a path \( \tilde{C}_s \), \( 0 \leq t \leq 1 \) such that there is a \( \kappa > 0 \), independent of \( C \), such that the following hold whenever \( 0 < \mu < 1/2 \).

1. For \( i = i_s \), \( e_t(\tilde{C}_1) = e_{i+1} \).
2. For \( i = i_s < i_s \), \( e_t(\tilde{C}_1) = e_{i+1} \).
3. If \( C_{s-1} \notin M_{\kappa} \), then \( \tilde{C}_1 \in M_{\kappa} \), \( 0 \leq t \leq 1 \).

The path will be a concatenation of paths. The first is simply \( C'_s = D_s C_{s-1} D_s^{-1}, 0 \leq t \leq 1 \) where \( D_s \) is diagonal and equal to \( I \) except at \( D_s(i + 1, i + 1) = (C_{s-1})^t \). (Note \( C_{s-1} \) greater than 0 given \( \mu < 1/2 \).) Now \( D_s C_{s-1} D_s^{-1}(i + 1, i + 1) = 1 \), and property (2) holds. Given \( \mu < 1/2 \), we have \( C_{s-1}(i + 1, i + 1) \in (1/2, 3/2) \), so an entry \( C'_s(i, j) \) cannot exceed \( 2C_{s-1}(i, j) \). Considering the worst case, \( C_{s-1}(i + 1, i + 1) \) near 1 + \( \mu \), we have (3) holding for \( \kappa = 3 \).

We now replace \( C_{s-1} \) with \( C'_s \), and for notational simplicity denote it as \( C \). For \( 0 \leq t \leq 1 \), define an \( n \times n \) matrix \( V_t \) by setting

\[
V_t(i + 1, i) = -tC(i, i + 1) \quad \text{if } i \neq i_s + 1
\]

\[
V_t(i, j) = I(i, j) \quad \text{otherwise}.
\]

Define \( \tilde{C}_t = V_t^{-1}CV_t \). Then \( V_t \) acts to add multiples of column \( i_s + 1 \) of \( C \) to other columns so that row \( i_s \) of \( CV_t \) equals \( e_{i+1} \). The rows of \( \tilde{C}_t \) and \( CV_t \) must be equal, except for row \( i_s + 1 \). It follows that (1) and (2) hold. Also,

\[
||N - V_t^{-1}CV_t|| \leq ||N - C|| + ||C - CV_t|| + ||CV_t - V_t^{-1}CV_t||
\]

\[
\leq \mu + \mu^2 + (n - 1)\mu < (n + 1)\mu.
\]

So, combining this path together with the diagonal conjugation path, property (3) holds with \( \kappa = 3(n + 1) \). We now pass from \( C_0 \) to \( C_S \). This completes the proof for Stage 1.

**Stage 2.** Given \( C \) in \( M_{\mu}^0 \), with \( \mu > 0 \), we produce a path in \( M_{\mu}^0 \) to a new \( C \) satisfying the following condition:

4. If \( i \in I \), then \( e_i C^{h-1} \neq 0 \).
So, suppose $C \in \mathcal{M}_\mu^0$, and let $\mathcal{U} = \{i : e_i C^{h-1} \neq 0\}$. Suppose $\mathcal{T} \not\subset \mathcal{U}$. We have $\#\mathcal{U} \geq \#\mathcal{T}$, because $\beta^{\text{op}}(C) > \beta(N) = \beta = \#\mathcal{T}$. Therefore, we can choose an injection $\mathcal{T} \setminus (\mathcal{T} \cap \mathcal{U}) \rightarrow \mathcal{U} \setminus (\mathcal{T} \cap \mathcal{U})$, $i \mapsto \xi(i)$. Let $\{i_1, \ldots, i_R\}$ now denote the set $\mathcal{T} \setminus (\mathcal{T} \cap \mathcal{U})$. Beginning with $C_0 = C$, for $1 \leq r \leq R$ in order we will define inductively a path $\tilde{C}_t$, $0 \leq t \leq \nu$, such that $\tilde{C}_0 = C_{r-1}$ and such that the following hold for each $t$.

1. For $t \neq 0$ and $i = i_r$, $e_i(\tilde{C}_t)^{h-1} \neq 0$.
2. If $i \in \mathcal{T}$ and $e_i(C_{r-1})^{h-1} \neq 0$, then $e_i(\tilde{C}_t)^{h-1} \neq 0$.
3. If $i \in \mathcal{J}$, then $e_i \tilde{C}_t = e_{i+1}$.
4. $||\tilde{C}_t - C_{r-1}||_{\text{max}} < \mu - ||C_{r-1}||_{\text{max}}$.

(The conditions (3) and (4) keep the path in $\mathcal{M}_\mu^0$.) Then we pass from $C$ to $C_R$.

For notational simplicity, we let $C'$ denote $C_{r-1}$; $j$ denote $\xi(i_r)$; $k$ be the $k_i$ such that row $i_r$ is the top row of $N_k$; and let $i_r$ be 1. Because $e_i$ is in the image of $C$ if $1 < i \leq k$, we have $j \notin \{1, \ldots, k\}$. Given a scalar $t$, let $V_t$ be the $n \times n$ matrix such that

$$
\text{row } i \text{ of } V_t = e_i + t e_j C^{i-1} \quad 1 \leq i \leq k + 1 \n$$

$$
= e_i, \quad \text{otherwise}.
$$

We keep $t$ small enough that $V_t$ is invertible, and define

$$
\tilde{C}_t = V_t C V_t^{-1}.
$$

Now we verify the induction conditions. The proof of (1) is a computation:

$$
e_1(V_t C V_t^{-1})^{h-1} = e_1 V_t (C^{h-1} V_t^{-1}) = (e_1 + t e_j)(C^{h-1} V_t^{-1})
$$

$$
= t e_j C^{h-1} V_t^{-1} \neq 0.
$$

The proof for (2) is similar. If $i \in \mathcal{T}$ and $e_i C^{h-1} \neq 0$, then

$$
e_i(V_t C V_t^{-1})^{h-1} = e_i V_t (C^{h-1} V_t^{-1}) = e_i (C^{h-1} V_t^{-1}) \neq 0.
$$

For (3), given $i \in \mathcal{J}$, we must show that $e_i \tilde{C}_t = e_{i+1}$. We do this for two cases. If $1 \leq i \leq k$, then

$$
e_i (V_t C V_t^{-1}) = (e_i V_t)(C V_t^{-1}) = (e_i + t e_j C^{i-1})(C V_t^{-1})
$$

$$
= (e_{i+1} + t e_j C^i V_t^{-1} = e_{i+1}.
$$

If $i \in \mathcal{J} \setminus \{1, \ldots, k\}$, then $\{i, i+1\} \cap \{1, \ldots, k\} = \emptyset$; so, if $e_i C = e_{i+1}$, then

$$
e_i V_t C V_t^{-1} = e_i C V_t^{-1} = e_{i+1} V_t^{-1} = e_{i+1}.
$$

It is clear that (4) holds if $\nu$ is sufficiently small. This completes the proof for Stage 2.

**Stage 3.** We begin with $C$ in $\mathcal{M}_\mu^0$, with $0 < \mu < 1/49$, with $C$ (from Stage 2) satisfying condition (iv) of Stage 2. We produce a path in $\mathcal{M}_\nu^0$ from $C$ to a matrix $C_G$ whose first $\beta h$ rows have the form of $M(\nu)$, but with the $\nu$’s replaced perhaps by various positive numbers, with $\nu = (2n)^2 \mu^{1/2n}$. Then, given $0 < \mu < 1$, we will have $C_G \in \mathcal{M}_\nu^0$ if

$$
\mu < \frac{n(1/2) c^n}{4}.
$$

For the proof, we will inductively produce a finite sequence of matrices $C_g$ and index sets $\mathcal{S}_g$, $0 \leq g \leq G$, with $G < \beta h$, beginning with $C_0 = C$. Property (iv)
combination of the vectors $e_i$.

Let $C$ be nonempty. We will show that given all this, we define $G$ to be the index $g$ at which $S_g = \{1, 2, \ldots, \beta h\}$. Given all this, we define $G$ to be the index $g$ at which $S_g = \{1, 2, \ldots, \beta h\}$.

(A1) If $i \in S$, then
\[
\begin{align*}
\text{row } i \text{ of } M &= e_{i+1} \\
&= 0 & \quad \text{if } i \in S \\
&= 0 & \quad \text{if } h \text{ divides } i \\
&= \text{a positive multiple of } e_{i+1}, & \quad \text{otherwise}.
\end{align*}
\]

(A2) If $0 \leq r < \beta$ and $1 \leq j \leq h$ and $rh + j \in S$, then $S$ contains $\{rh + i : 1 \leq i \leq j\}$.

Note, $S(M)$ contains $\mathcal{J}$. Also define
\[
\begin{align*}
\mathcal{R}(M) &= \{i : i \notin S(M), \ i - 1 \in S(M)\} \\
\mathcal{P}(M) &= \{(i, j) : i \in \mathcal{R}(M), \ j \notin S(M) \cup \mathcal{R}, \ M(i, j) \neq 0\}.
\end{align*}
\]

Let $S_g = S(C_g)$, $\mathcal{R}_g = \mathcal{R}(C_g)$, $\mathcal{P}_g = \mathcal{P}(C_g)$. Continuing Example 4.6 with $S_{g-1} = \{1, 2, 3, 6, 7\}$ we would see the matrix $C_{g-1}$ having the following form:

\[
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \varepsilon_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \varepsilon_1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \varepsilon_2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \varepsilon_3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \varepsilon_4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \varepsilon_5 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

In this example, $\mathcal{R}_{g-1} = \{4, 8\}$; $(i, j) \in \mathcal{P}_{g-1}$ iff $i \in \{4, 8\}$ and $j \in \{5, 9, 10, 11, 12\}$.

We will arrange by induction that the following hold for $g \geq 1$.

(B1) If $\#S_{g-1} \neq \beta h$, then $S_{g-1}$ is properly contained in $S_g$.

(B2) If $C_{g-1} \in \mathcal{M}_0^\mu$, with $\mu < 1$, then there is a path in $\mathcal{M}_0^\mu$ from $C_{g-1}$ to $C_g$.

Given all this, we define $G$ to be the index $g$ at which $S_g = \{1, 2, \ldots, \beta h\}$.

Now, suppose we are given $C_{g-1}$ and $S_{g-1}$ with $\#S_{g-1} < \beta h$ (i.e., $\mathcal{R}_{g-1}$ is nonempty). We will show $\mathcal{P}_{g-1}$ is nonempty. Pick $i \in \mathcal{R}_{g-1}$.

If $i$ is divisible by $h$, then (by property A2) let $t$ in $T$ be such that row $i$ of $C_{g-1}$ (which is $e_iC_{g-1}$) is a positive multiple of $e^t C^h$, and therefore is zero. Since $i - 1 \in S_{g-1}$, it follows that $i \in S_{g-1}$, a contradiction. So $i$ is not divisible by $h$.

Let $k$ be the positive integer in $[1, h - 1]$ such that $e_iC_{g-1} = e_iC_g^{k}$.

Now suppose $C_{g-1}(i, j) \neq 0$ implies $j \in S_{g-1} \cup \mathcal{R}_{g-1}$. Then $e_iC_{g-1}$ is a linear combination of the vectors $e_{\tau}C^\tau$ such that $\tau \in T$ and $0 \leq k < h$. This is a
contradiction, because the set \( \{ e_t C^g_{j-1} \mid t \in \mathcal{T}, 0 \leq j < h \} \) is linearly independent, by property (iv). Therefore there is a \( j \) such that \((i, j) \in \mathcal{P}_{g-1}\).

From here, we will handle the inductive transition from \( g-1 \) to \( g \) in three steps, given \( C_{g-1} \) and \( S_{g-1} \) with \#\( S_{g-1} \) \( < \beta h \). By a signed transposition matrix for indices \( i, j \) we mean a matrix \( Q \) which is equal to the permutation matrix \( P \) for the transposition exchanging \( i \) and \( j \), except that one of the entries \( Q(i, j) \) or \( Q(j, i) \) is \(-1\).

STEP 1. Given \( C_{g-1} \in M^0_n \), we produce an index \( i \) in \( \mathcal{R} \), and a matrix \( Q \) which is either \( I \) or is a signed transposition matrix \( Q \) for indices outside \( \mathcal{R} \cup S_{g-1} \), such that the following hold for the matrix \( \overline{C} = Q^{-1}C_{g-1}Q \):

\[ \begin{align*}
&\text{(D1)} \ (i, i+1) \in \mathcal{P}(\overline{C}) . \\
&\text{(D2)} \ \overline{C}(i, i+1) = \max\{|\overline{C}(i', j')| : (i', j') \in \mathcal{P}(\overline{C})\} . \\
&\text{(D3)} \ \text{There is a path in } M^0_{\sqrt{\mu}} \text{ from } C_{g-1} \text{ to } \overline{C} . \\
&\text{(D4)} \ S(\overline{C}) = S_{g-1} , \text{ and } ||N - \overline{C}||_{\max} = ||N - C_{g-1}||_{\max} < \mu .
\end{align*} \]

STEP 2. For the matrix \( \overline{C} \) produced in Step 1, defining \( \alpha = ||N - \overline{C}||_{\max} \), we produce a matrix \( C' \) and \((i, i+1) \in \mathcal{P} \) such that the following hold.

\[ \begin{align*}
&\text{(C1)} \ ||N - C'||_{\max} = |C'(i, i+1)| = \sqrt{\alpha} . \\
&\text{(C2)} \ \text{If } r \in S_{g-1} , \text{ then } |C'(i, r)| < \mu . \\
&\text{(C3)} \ \text{There is a path in } M^0_{\sqrt{\mu}} \text{ from } \overline{C} \text{ to } C' .
\end{align*} \]

STEP 3. Given \( C' \) from Step 2, we produce a path in \( M^0_{2n, \sqrt{\mathcal{P}}} \) from \( C' \) to the desired matrix \( C_g \).

PROOF FOR STEP 1.

Choose \((i, j)\) from the nonempty set \( \mathcal{P} \) such that \( |C(i, j)| = \max\{|C(i', j')| : (i', j') \in \mathcal{P} \} \). There are two cases.

CASE 1: \( j \neq i + 1 \). Both \( j \) and \( i + 1 \) are outside \( \mathcal{R} \cup S_{g-1} \). Let \( Q \) denote the \( n \times n \) signed transposition matrix for indices \( i + 1 \) and \( j \) such that

\[ \begin{align*}
Q(j, i + 1) &= -1 , \quad \text{if } C_{g-1}(i, j) < 0 \\
Q(i + 1, j) &= -1 , \quad \text{if } C_{g-1}(i, j) > 0 .
\end{align*} \]

Set \( \overline{C} = Q^{-1}C_{g-1}Q \). We have \( Q \in \text{SL}(n, \mathbb{R}) \), and for a path from \( C_{g-1} \) to \( \overline{C} \) we can use \( U_t^{-1}C_{g-1}U_t \), \( 0 \leq t \leq 1 \), with \((U_t)\) a path from \( I \) to \( Q \). E.g., for \( Q = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \), we may use (in the principal submatrix on coordinates \{\( i + 1, j \}\})

\[ U_t = \begin{pmatrix}
1 & 0 \\
-t & 1
\end{pmatrix}
\begin{pmatrix}
1 & t \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
-t & 1
\end{pmatrix}
= \begin{pmatrix}
1 - t^2 & t \\
-2t + t^3 & 1 - t^2
\end{pmatrix} .
\]

Then for \( 0 \leq t \leq 1 \),

\[ ||U_t^{-1}C_{g-1}U_t||_{\max} \leq 4||U_t||_{\max}||C_{g-1}||_{\max} < 5||C_{g-1}||_{\max} . \]

CASE 2: \( j = i + 1 \). If \( C_{g-1}(i, i + 1) > 0 \), then set \( Q = I \) and \( \overline{C} = C \). If \( C_{g-1}(i, i + 1) < 0 \), then let \( W \) be the matrix in \( \text{SL}(n, \mathbb{R}) \) obtained from \( I_n \) by multiplying rows \( i + 1 \) and \( n \) by \(-1\), and define \( \overline{C} = W^{-1}C_{g-1}W \). As in Case 1, we may produce a path from \( C_{g-1} \) to \( \overline{C} \) by conjugating with a path \((W_t)\),
0 \leq t \leq 1$, from $I$ to $W$. One such path is given (on the principal submatrix on indices $\{i + 1, n\}$) by

$$W_t = \begin{pmatrix} 1 & 0 \\ -2t & 1 \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -2t & 1 \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 - 2t^2 & 2t - 2t^3 \\ -4t + 4t^4 & 1 - 6t^2 + 4t^4 \end{pmatrix}$$

Then

$$||W_t^{-1}C_{g-1}W_t||_{\max} \leq 4||W_t||_{\max}||C_{g-1}||_{\max} < 7||C_{g-1}||_{\max}.$$
We have \( ||N - C||_{\text{max}} < \sqrt{\mu} \), since \( C \in \mathcal{M}_{\mu}^0 \).

Because \( C - CV_t = C(I - V_t) \), column \( r \) of \( C - CV_t \) is zero if \( r = i + 1 \) and otherwise equals column \( i + 1 \) of \( C \) (whose entries are smaller in absolute value than \( \sqrt{\mu} \), since \( i \notin S_{g-1} \cup J \)) multiplied by \( -tC(i, r)/C(i, i + 1) \) (which by (C1) has absolute value at most \( 1 \)). Therefore \( ||C - CV_t||_{\text{max}} < \sqrt{\mu} \) (and then \( ||N - CV_t||_{\text{max}} < 2\sqrt{\mu} \)).

The last term in (D.1) is the maximum over \((i', j')\) of the absolute value of
\[
(CV_t - V_t^{-1}CV_t)(i', j') = (I - V_t^{-1})(CV_t)(i', j') .
\]
This quantity is zero if \( i' \neq i + 1 \), since row \( i + 1 \) is the only nonzero row of \((I - V_t^{-1})\).

For \( i' = i + 1 \), we have
\[
(I - V_t^{-1})(CV_t)(i + 1, j') = \sum_r (I - V_t^{-1})(i + 1, r)(CV_t)(r, j')
\]
\[
= \sum_{r \neq i + 1} \left( \frac{-tC(i, r)}{C(i, i + 1)} \right)(CV_t)(r, j') .
\]
Suppose \( r \in J \). If \( j' \neq r + 1 \), then \((CV_t)(r, j') = 0 \), and for \( j' = r + 1 \) we have \((CV_t)(r, r + 1) = 1 \) and by (C1) and (C2)
\[
||CV_t(r, r + 1)|| = \left| \frac{-tC(i, r)}{C(i, i + 1)} \right|(CV_t)(r, r + 1)
\]
\[
< \frac{\mu}{\sqrt{\alpha}} < \frac{\mu}{\sqrt{\mu}} = \sqrt{\mu} .
\]

If \( r \notin J \), then \( N(r, j') = 0 \) and
\[
\left| \frac{-tC(i, r)}{C(i, i + 1)} \right||(CV_t)(r, j')|| \leq (1)||N - CV_t||_{\text{max}} < 2\sqrt{\mu} .
\]
Using the estimates above to bound the third term of (D.1), and then substituting bounds into (D.1), we get
\[
||N - V_t^{-1}CV_t||_{\text{max}} < \sqrt{\mu} + \sqrt{\mu} + 2(n - 1)\mu = 2n\mu .
\]
This finishes the proof for Step 3, and for Stage 3.

**Stage 4.**

Given \( 0 < \epsilon < \mu \) and a matrix \( C \) in \( \mathcal{M}_\mu^0 \) with the first \( \beta h \) rows agreeing with those of \( M(\epsilon) \) (except that the epsilons are allowed to be different positive numbers), we produce a path in \( \mathcal{M}_\mu^0 \) from \( C \) to \( M(\epsilon) \).

The \( n \times n \) matrix \( C \) has a block form \( C = (X \ Y) \), in which \( X \) is \( \beta h \times \beta h \).

**STEP 1:** Defining \( C_t = (X \ Y) \), \( t \geq 0 \), we show \( (C_t) \) is a path of conjugate matrices. Conjugacy is clear for \( 0 < t \leq 1 \), where
\[
C_t = \begin{pmatrix} (1/t)I & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} X & 0 \\ Y \ Z \end{pmatrix} \begin{pmatrix} tI & 0 \\ 0 & I \end{pmatrix} .
\]
For \( 1 \leq t \leq h \), let \( Z_t \) denote the set of indices from \( \{1, \ldots, \beta h\} \) (which indexes rows and columns of \( X \)) which are congruent to \( i \) mod \( h \). To check conjugacy of \( C_0 \) and \( C \), we first note there is a conjugacy of the form
\[
C' = \begin{pmatrix} X & 0 \\ Y' & Z \end{pmatrix} = \begin{pmatrix} I & 0 \\ L & I \end{pmatrix} \begin{pmatrix} X & 0 \\ Y \ Z \end{pmatrix} \begin{pmatrix} I & 0 \\ -L & I \end{pmatrix} .
\]
such that column $i$ of $Y'$ is zero if $i \notin Z_1$. This result from a composition of conjugations arising from elementary row and column operations as follows. In decreasing order for $i = h - 1, h - 2, \ldots, 1$: for each $j \in Z_i$, and for each $i'$ in $[\beta h + 1, n]$ such that position $(i', j + 1)$ of the current matrix has a nonzero entry, we add a multiple of row $j$ to row $i'$, and then subtract the same multiple of column $i'$ from column $j$.

We will show $Y'$ must be zero. For this, consider row vectors in the form $(u, v)$, where $u$ has $\beta h$ entries and $v$ has $n - \beta h$ entries. For $1 \leq t \leq h$, let $W_t$ be the subspace of $\mathbb{R}^{\beta h}$ spanned by the vectors $e_j$ such that $j \in Z_t$ and $1 \leq i \leq t$. Let $W_0$ be the trivial space $\{0\}$. Given $(0, v)$ and $k \geq 1$, define $(u^{(k)}, v^{(k)}) = (0, v)C^k$. We claim for $1 \leq k \leq h$ that

1. $u^{(k)} \in W_k$.
2. If $u^{(1)} \neq 0$, then $u^{(k)} \in W_k \setminus W_{k-1}$.

The claim is clear for $k = 1$, because column $j$ of $Y'$ is zero if $j \notin Z_1$. Suppose $1 \leq k < h$, and the claim holds for $k$. Then

$$
(0, v)C^{k+1} = (u^{(k)}, v^{(k)})C = (u^{(k)}, 0)C + (0, v^{(k)})C
= (u^{(k)}X, 0) + (u'', v'')
$$

for some $u'' \in W_1$. The claim then holds for $k + 1$ because $X$ maps $W_k \setminus W_{k-1}$ injectively into $W_{k+1} \setminus W_k$ (since $k < h$) and $W_k$ contains $W_1$.

If $Y' \neq 0$, then there must be some $v$ for which $(0, v)C = (u^{(1)}, v^{(1)})$ with $u^{(1)} \neq 0$. It follows from the claim that $(0, v)C^h$ is nonzero. That contradicts $C^h = 0$, So, $Y' = 0$, and $C_0$ is conjugate to $C$.

**STEP 2.** We begin with $C = \begin{pmatrix} X & 0 \\ 0 & Z \end{pmatrix}$ in $\mathcal{M}_\mu^0$ and define a conjugacy in $\mathcal{M}_\mu^0$ to $M(\epsilon)$. Clearly, after applying a path in $\mathcal{M}_\mu^0$, we can suppose that $X$ exactly equals the corresponding $\beta h \times \beta h$ block in $M(\epsilon)$, and write $M(\epsilon)$ as $\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}$. Let $m = n - \beta h$.

The conjugacy of $C$ and $M(\epsilon)$ and the fact that $Z$ has a Jordan zero block imply that there is a matrix $U \in \text{SL}(m, \mathbb{R})$ such that $U^{-1}ZU = J$. Then there is a path $(U_t)$ in $\text{SL}(m, \mathbb{R})$ from $I$ to $U$, giving a path of conjugate matrices from $Z$ to $J$, $Z_t = U_t^{-1}ZU_t$. Given $\delta > 0$, we can follow a path $\begin{pmatrix} X & 0 \\ 0 & sZ \end{pmatrix}$, $1 \geq s \geq \delta$, with a path $\begin{pmatrix} X & 0 \\ 0 & sZ_Y \end{pmatrix}$, $1 \geq s \geq \delta$, and then a path $\begin{pmatrix} X & 0 \\ 0 & s \end{pmatrix}$, $\delta \leq s \leq 1$. This gives a path of conjugate matrices from $C$ to $M(\epsilon)$, and with $\delta$ small enough, the path is contained in $\mathcal{M}_\mu^0$. This completes the proof of Stage 4, and the lemma.

\[\square\]

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