Entanglement Entropy of Compactified Branes and Phase Transition

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ABSTRACT
We first calculate the holographic entanglement entropy of M5 branes on a circle and see that it has phase transition during decreasing the compactified radius. In particular, it is shown that the entanglement entropy scales as $N^3$. Next, we investigate the holographic entanglement entropy of D0+D4 system on a circle and see that it scales as $N^2$ at low energy, likes as a gauge theory with instantons. However, at high energy it transforms to a phase which scales as $N^3$, like as M5 branes system. We also present the general form of holographic entanglement entropy of Dp, Dp + Dp+4 and M-branes on a circle and see some simple relations between them. Finally, we present an analytic method to prove that they all have phase transition from connected to disconnected surface during increasing the line segment of length $\ell$ which dividing the space.

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1 Introduction

The entanglement entropy $S_A$ in quantum field theories or quantum many body systems is a non-local quantity opposed to the correlation functions. It is defined as the von Neumann entropy $S_A$ of the reduced density matrix when we trace out (or smear out) the degrees of freedom inside a $d$-dimensional space-like submanifold $B$ in a given $(d+1)$-dimensional QFT, which is a complement of $A$. Physically, the observer who is only accessible to the subsystem $A$ will feel as if the total system is described by the reduced density matrix $\rho_A$

$$\rho_A = Tr_B [\rho_{total}] \quad (1.1)$$

The entanglement entropy of the subsystem $A$ is defined by

$$S_A = Tr_A [\rho_A \log \rho_A] \quad (1.2)$$

which measures how the subsystems $A$ and $B$ are correlated with each other. Intuitively, the entanglement entropy is defined as the entropy seen by an observer in $A$ who does not have access to the degrees of freedom in $B$, or vice versa. This origin of entropy looks analogous to the black hole entropy. Indeed, this was the historical motivation of considering the entanglement entropy in quantum field theories [1-6].

As the $d + 1$ dimensional conformal field theories have $AdS_{d+2}$ gravity duals [7-9], S. Ryu and T. Takayanagi [10-12] conjectured that the entanglement entropy between the regions $A$ and $B$ is proportional to the classical area of this surface,

$$S_A = \frac{1}{4G^{(d+2)}_N} \int_\gamma d^d \sigma \sqrt{G^{d}_{ind}} \quad (1.3)$$

where $G^{d+2}_N$ is the d+2 dimensional Newton constant and $G^{d}_{ind}$ is the induced string frame metric on $\gamma$. In non-conformal theories the dilaton are in general not constant. A natural generalization to the corresponding ten dimensional geometries was proposed by

$$S_A = \frac{1}{4G^{(10)}_N} \int_\gamma d^8 \sigma \ e^{-2\Phi} \sqrt{G^{8}_{ind}} \quad (1.4)$$

It is natural to generalize above relation to the eleven dimensional geometries

$$S_A = \frac{1}{4G^{(11)}_N} \int_\gamma d^8 \sigma \ \sqrt{G^{8}_{ind}} \quad (1.5)$$
The holographic entanglement entropy of “slab” (the subspace defined by \(-\ell/2 < x < \ell/2\), where \(x\) is one of the spacial coordinates) of Dp branes compactified on a circle had been studied by Klebanov, Kutasov and Murugan [13-15]. They had found that there are two types of minimal surfaces: connected and disconnected ones and the D4 and D3 systems have a phase transition between these two types of solutions. For small \(\ell\) the connected solution dominates the computation of entanglement entropy, while for large \(\ell\) the disconnected solution becomes preferred. In this paper we will follow their method to investigate the several M brane and Dp branes systems compactified on a circle and find some useful properties.

In section II we first investigate the holographic entanglement entropy M5 branes on a circle. We see that it scales as \(N^3\) [16,17], and has the phase transition during decreasing the compactified radius. Next, we investigate the holographic entanglement entropy of D0+D4 system on a circle and see that it scales as \(N^2\) at low energy, like as a gauge theory with instantons [18,19]. However, at high energy it scales as \(N^3\), like as a M5 system. In section III we present the general form of holographic entanglement entropy for Dp branes, Dp + Dp+4 and M-branes on a circle and see a simple relation between them. Finally, in section IV we present an analytic method to prove that they all have phase transition from connected to disconnected surface during increasing the line segment of length \(\ell\) which dividing the space. Last section is used to summarize our results.

After this paper is completed [20] appeared which has some overlaps with the material presented here. However, since we have calculated the entropy of D0+D4 we could study the phase transition of M5 branes. Also we find that the entangle entropy of compactified M5, M2, Dp and Dp + Dp+4 has similar mathematic form and thus study the universal phase transition.

2 Holographic entanglement entropy of M5 and D0+D4 on circle

2.1 Holographic entanglement entropy of M5 on circle

Near-horizon M5 matric on a circle of coordinate \(w\) with radius \(R_w\) is

\[
d s^2 = \frac{U}{R_5} \left[ -dt^2 + dx^2 + \sum_{i=1}^{3} dx_i^2 \right] + \frac{U}{R_5} \left( 1 - \frac{b^3}{U^3} \right) dw^2 + \frac{R_3^2}{U^2} \frac{dU^2}{1 - \frac{b^3}{U^3}} + R_3^2 d\Omega_4^2
\]

(2.1)

in which \(b = \frac{4R_3^3}{9R_w^3}\) and \(R^3\) is proportional to number of M5 branes (see section 3). In terms of \(U = 4R_3^3/z^2\) and define

\[
z_c \equiv 3R_w
\]

(2.2)

then

\[
d s^2 = \frac{4R_3^2}{z^2} \left[ -dt^2 + dx^2 + \sum_{i=1}^{3} dx_i^2 + \left( 1 - \frac{z^6}{z_c^6} \right) dw^2 + \frac{dz^2}{1 - \frac{z^6}{z_c^6}} \right] + R_3^2 d\Omega_4^2
\]

(2.3)

On a time slice \(t = \text{constant}\) the induced metric of M5 is

\[
d s^2_{\text{ind}} = \frac{4R_3^2}{z^2} \left[ \sum_{i=1}^{3} dx_i^2 + \left( 1 - \frac{z^6}{z_c^6} \right) dx_3^2 + \left( \frac{z^2}{1 - \frac{z^6}{z_c^6}} + 1 \right) dx^2 \right] + R_3^2 d\Omega_4^2
\]

(2.4)
Using \( \int dw = 2\pi R_w \), \( \int dx_i = L \) and \( \int \Omega_4 = \omega_4 \) we find that surface is

\[
A = \int_0^{2\pi R_w} dw \int \Pi_{i=1}^3 dx_i \int \Omega_4 \int_{-\ell/2}^{\ell/2} dx \sqrt{g_{\text{ind}}}
\]

\[
= 32R_5^0 2\pi R_w L^3 2\omega_4 \int_{0}^{\ell/2} dx \sqrt{z'^2 + 1 - \frac{z^6}{z_c^6}}
\]

(2.5)

To find the profile of \( z(x) \) we can regard \( x \) as time and see that the Hamiltonian does not depend on \( x \). In this case we let \( z' = 0 \) at the turning point of the minimal surface, i.e. \( z = z_* \) then

\[
z' = \sqrt{\frac{z_1^0 (1 - \frac{z^6}{z_c^6})^2}{z_1^{10}(1 - \frac{z^6}{z_c^6})} - (1 - \frac{z^6}{z_c^6})}
\]

(2.6)

In considering the entanglement entropy of the “slab” in which the subspace is defined by \(-\ell/2 < x < \ell/2\) we have the following relation

\[
\ell/2 = \int_0^{\ell/2} dx = \int_0^{z_*} dz \frac{dx}{dz} = \int_0^{z_*} dz \frac{z^5 \sqrt{1 - \frac{z^6}{z_c^6}}}{\sqrt{z_1^{10}(1 - \frac{z^6}{z_c^6})^2 - z_1^{10}(1 - \frac{z^6}{z_c^6})(1 - \frac{z^6}{z_c^6})}}
\]

(2.7)

Above relation can be used to find function \( \ell(z_*) \). Note that \( z_* = 3R_w \) which is independent of number of M5 branes \( N \).

Now we can use the function of \( z' \) to find that the connection surface becomes

\[
A_C = (32R_5^0 2\pi R_w L^3 2\omega_4) \int_0^{z_*} dz \frac{z^5 \sqrt{1 - \frac{z^6}{z_c^6}}}{z^5 \sqrt{z_1^{10}(1 - \frac{z^6}{z_c^6})^2 - z_1^{10}(1 - \frac{z^6}{z_c^6})(1 - \frac{z^6}{z_c^6})}}
\]

(2.8)

To proceed we shall note that there is disconnection surface which is defined by taking \( z_* = z_c \equiv 3R_w \). In this case we can find the exact result:

\[
A_D = (32R_5^0 2\pi R_w L^3 2\omega_4) \int_\epsilon^{z_c} dz \frac{1}{z_c^6}
\]

\[
= (32R_5^0 2\pi R_w L^3 2\omega_4) \left( -\frac{1}{4z_c^4} + \frac{1}{4\epsilon^4} \right)
\]

(2.9)

in which \( \epsilon \) is a cutoff parameter. Let us now begin to analyze the above relation.

1. From (2.7) we see that \( z_c \) and \( z_* \) are independent of \( R_5 \). Also, from (2.8) and (2.9) we see that \( R_5 \) only appears as an overall factor \( R_5^0 \) before surface formula \( A \). As \( R_5^0 \) is propositional to number of M5, \( N \), the holographic entanglement entropy of compactified M5 has the simple property: \( S \sim N^3 \) [16,17].

2. From (2.7) we see that the slab length \( \ell = 0 \) at \( z_* = 0 \) or \( z_* = z_C \). Thus, increasing the value of \( z_* \) the slab length \( \ell \) will be increasing too, which, however become zero at largest value of \( z_* = z_C \).

3. From (2.8) we see that at small value of \( z_* \approx 0 \) and the connection surface becomes

\[
A_C \approx (32R_5^0 2\pi R_w L^3 2\omega_4) \left( -\frac{1}{4z_*^4} + \frac{1}{4\epsilon^4} \right)
\]

(2.10)

Thus \( A_C - A_D < 0 \). However, increasing the value of \( z_* \) the connection surface \( A_C \) will be increasing too, which, will become the largest value at \( z_* = z_C \). The central problem to be studied is whether before \( z_* = z_C \) the connection surface could be larger then disconnection surface. If the answer is yes then the M5 will have phase transition.
4. In numerical, we have to integrate (2.7) to find the function $\ell(z_*)$ and integrate (2.8) to find the surface function $A(z_*)$. Then we use the two functions to obtain the surface function $A(\ell)$. May be between some regions of $\ell$ the disconnection surface is smaller then connection surface while between another regions of $\ell$ the disconnection surface is larger then connection surface $A_C < A_D$. In this case the system has phase transition at a critical value of $\ell$. This is the property that Klebanov et.el first found in the D4 and D3 systems [13].

5. To proceed let us quote following two properties: (a) As will be shown in following sections that the just replacing $L^3$ in (2.8) by $L^2$ we obtain the D4 on circle. (b) As that studied by Klebanov et.al [13] the phase transition is found at $\ell = 1.288R_w$. Thus, M5 system has phase transition at critical radius $R_c = R_w = \ell/1.288$. Also, using the property that $\ell$ playing the role of the inverse temperature [13] we thus conclude that M5 will has phase transition at critical compactified radius.

6. The next problem is that if compactified M5 has phase transition then what is its low energy phase? According to the recent discussion [18,19], the compactified M5 with Kaluza-Klein modes is just the 5D gauge theory with instanton. Thus, maybe the low energy phase of compactified M5 is the compactified D0+D4 which becomes 5D gauge theory with instanton at low energy. From this point of view we thus analyze the holographic entanglement entropy of D0+D4 system in below.

### 2.2 Holographic entanglement entropy of D0+D4 on circle

Near-horizon D0+D4 matic on a circle of coordinate $w$ with radius $R_w$ is

$$
ds^2 = \sqrt{\frac{R_0^3}{R_i^3}} \left( -dt^2 + dx^2 + \sum_{i=1}^{2} dx_i^2 \right) + \frac{U^3}{\sqrt{R_0^3 R_i^3}} \left( 1 - \frac{b^3}{U^3} \right) dw^2 + \frac{\sqrt{R_0^3 R_i^3}}{U^3} \left[ \frac{dU^2}{1 - \frac{b^3}{U^3}} + U^2 d\Omega_4^2 \right]$$

(2.11)

in which $b = \left( \frac{2}{\pi R_w} \right)^{1/2} \left( R_0^3 R_i^3 \right)^{1/4}$ and $R_0^3, R_i^3$ are proportional to number of D0, D4 branes respectively. In terms of $U = 4R_i^3 / z^2$ and define

$$z_c \equiv \left( \frac{64R_i^3}{b^3} \right)^{1/6} = 2 \left( \frac{3R_w}{2} \right)^{1/4} \frac{R_i^{9/8}}{R_0^{3/8}}$$

(2.12)

then the induced metric on a constant time slice is

$$ds^2_{ind} = \sqrt{\frac{R_0^3}{R_i^3}} \sum_{i=1}^{2} dx_i^2 + \left( 1 + \frac{z^2}{1 - \frac{z_c^2}{z^2}} \right) dx^2 + \left( 1 - \frac{z_c^6}{z^6} \right) \frac{64R_0^3}{R_i^3} \frac{dw^2}{\sqrt{R_0^3 R_i^3}} \frac{1}{z^6} + \frac{\sqrt{R_0^3 R_i^3}}{4R_i^3} z^2 d\Omega_4^2$$

Using $\int dx_w = 2\pi R_w$, $\int dx_i = L$ and $\int \Omega_4 = \omega_4$ we find that surface is

$$A = \int_0^{2\pi R_w} dx_w \int \Pi_{i=1}^{2} dx_i \int d\Omega_4 \int_{-\ell/2}^{\ell/2} dx_e^{-2\Phi} \sqrt{g_{ind}}$$

$$= 32 R_0^3 2\pi R_w L^3 2\omega_4 \int_0^{\ell/2} dx \frac{\sqrt{z^2 + 1 - \frac{z_c^6}{z^6}}}{z^5}$$

(2.13)

which is just that in M5 (i.e. (2.5)), except $R_0$ being replaced by $R_i$. Thus there is a phase transition and let us consider the phase at small $\ell$ and large $\ell$ respectively in below.

First, for small $\ell$ the connection surface dominates and

$$A_C \approx -32R_i^3 2\pi R_w L^3 2\omega_4 \frac{1}{4z_c^4} + \text{div.}$$

(2.14)
As before, because \( z_c \) and \( z_s \) are independent of \( R_4 \) and \( R_4 \) only appears as an overall factor \( R_5^0 \) before surface formula \( A_C \). Thus the holographic entanglement entropy of compactified D0+D4 has the simple property:

\[ S \sim N^3, \quad \text{at small } \ell \] (2.15)

Since that in this case \( R_5^0 \) is proposition to number of D4, \( N \). Note that after considering the \( g_s \) factor in \( G_N^{(10)} \) we see that above result just produce the M5 brane entanglement entropy because the \( g_s \) is the M-theory radius of circle [16,17].

Now, increasing the value of \( z_s \) the connection surface \( A_C \) will be increasing too. And eventually, the disconnection surface will dominate and

\[
A_D = -(32R_4^0 2\pi R_w L^3 2\omega_4) \frac{1}{4z_c^4} + \text{div.}
\]

\[
= -(32R_4^0 2\pi R_w L^3 2\omega_4) \frac{R_0^{3/2}}{42(\frac{3R_p}{2}) R_4^{9/2}} + \text{div.} \quad (2.16)
\]

In the case of \( k \) smeared D0 per unit D4 we see that at large \( \ell \)

\[ S \sim N^3 \frac{(kN)^{1/2}}{(N^{3/2})} = k^{1/2}N^2, \quad \text{at large } \ell \] (2.17)

In conclusion, we have investigated the holographic entanglement entropy of D0+D4 system on a circle and see that it scales as \( N^2 \) at low energy, likes as a 5D gauge theory with instantons. However, at high energy it will transform to a phase which scales as \( N^3 \), like as a M5 system. Note that paper [21] had performed the similar computation on the background dual to the theory at finite theta angle, which is very similar to the D0-D4 one.

## 3 Holographic Entanglement Entropy of Compactified Dp and M Branes

The metric of black Dp branes, M5 and M2 branes are

\[
ds^2_{Dp} = H_p^{-1/2} \left[ - F_p dt^2 + dx^2 + dx_0^2 + \sum_{i=1}^{p-2} dx_i^2 \right] + H_p^{1/2} \left[ \frac{du^2}{F_p} + U^2 d\Omega_{8-p}^2 \right]
\] (3.1)

\[ e^{-2\Phi} = H_p^{(p-3)/2}, \quad H_p = 1 + \frac{R_0^{p-1}}{U^{1-p}}, \]

\[ F_p = 1 - \frac{b^{p-3}}{U^{1-p}}, \quad T_H = \frac{7 - p + \frac{b^{p-3}}{R_p}}{4\pi} R_{p-2}^{p-2} \] (3.2)

\[
ds^2_{M5} = H_{M5}^{-1/3} \left[ - F_{M5} dt^2 + dx^2 + dx_0^2 + \sum_{i=1}^{3} dx_i^2 \right] + H_{M5}^{2/3} \left[ \frac{du^2}{F_{M5}} + U^2 d\Omega_2^2 \right]
\] (3.4)

\[ H_{M5} = 1 + \frac{R_{M5}^{3}}{U^3}, \quad F_{M5} = 1 - \frac{b^3}{U^3}, \quad T_H = \frac{3}{4\pi} \frac{1}{b} R_{M5}^{3/2} \] (3.5)

\[
ds^2_{M2} = H_{M2}^{-2/3} \left[ - F_{M2} dt^2 + dx^2 + dx_0^2 \right] + H_{M2}^{1/3} \left[ \frac{du^2}{F_{M2}} + U^2 d\Omega_7^2 \right]
\] (3.6)

\[ H_{M2} = 1 + \frac{R_{M2}^6}{U^6}, \quad F_{M2} = 1 - \frac{b^6}{U^6}, \quad T_H = \frac{3}{2\pi} \frac{b^2}{2} R_{M2}^{3/2} \] (3.7)
in which the Hawking temperature $T_H$ is determined at near-horizon limit. Using above relations we can obtain metric of associated branes compactified on coordinate $w$ with radius $R_w = 1/2\pi T_H$

$$
\begin{align*}
\frac{d}{s_D^{p}} &= H^{-1/2}_p \left[- dt^2 + F_p dw^2 + dx^2 + \sum_{i=1}^{p-2} dx_i^2 \right] + H^{1/2}_p \left[ \frac{dU^2}{F_p} + U^2 d\Omega^{2-p}_{5-p} \right] \\
\frac{d}{s_D^{M5}} &= H^{-1/3}_M \left[- dt^2 + F_{M5} dw^2 + dx^2 + \sum_{i=1}^{3} dx_i^2 \right] + H^{2/3}_M \left[ \frac{dU^2}{F_{M5}} + U^2 d\Omega^1_4 \right] \\
\frac{d}{s_D^{M2}} &= H^{-2/3}_M \left[- dt^2 + F_{M2} dw^2 + dx^2 \right] + H^{1/3}_M \left[ \frac{dU^2}{F_{M2}} + U^2 d\Omega^2_4 \right]
\end{align*}
$$

To consider the entanglement entropy of the “slab” in which the subspace is defined by $-\ell/2 < x < \ell/2$ we need to investigate the following induced metric

$$
\begin{align*}
\frac{d}{s_{Dp,\text{ind}}} &= H^{-1/2}_p \left[ F_p dw^2 + \sum_{i=1}^{p-2} dx_i^2 \right] + (H^{-1/2}_p + H^{1/2}_p \frac{U^2}{F_p}) dx^2 + U^2 H^{1/2}_p d\Omega^{2-p}_{5-p} \\
\frac{d}{s_{M5,\text{ind}}} &= H^{-1/3}_M \left[ F_{M5} dw^2 + \sum_{i=1}^{3} dx_i^2 \right] + (H^{-1/3}_M + H^{2/3}_M \frac{U^2}{F_{M5}}) dx^2 + U^2 H^{2/3}_M d\Omega_4 \\
\frac{d}{s_{M2,\text{ind}}} &= H^{-2/3}_M \left[ F_{M2} dw^2 + dx^2 \right] + (H^{-2/3}_M + H^{1/3}_M \frac{U^2}{F_{M2}}) dx^2 + U^2 H^{1/3}_M d\Omega^2_4
\end{align*}
$$

Thus, the associated entanglement entropy is

$$
\begin{align*}
S^{(Dp)}_A &= \frac{1}{4G_N^{(10)}} \int d^8 \sigma \sqrt{H_p F_p U^{8-p}} \sqrt{1 + H_p \frac{U^2}{F_p}} \\
S^{(M5)}_A &= \frac{1}{4G_N^{(11)}} \int d^8 \sigma \sqrt{H_{M5} F_{M5} U^4} \sqrt{1 + H_{M5} \frac{U^2}{F_{M5}}} \\
S^{(M2)}_A &= \frac{1}{4G_N^{(11)}} \int d^8 \sigma \sqrt{H_{M2} F_{M2} U^7} \sqrt{1 + H_{M2} \frac{U^2}{F_{M2}}}
\end{align*}
$$

From the definitions of $H_p$, $H_{M5}$, $H_{M2}$, $F_p$, $F_{M5}$ and $F_{M2}$ we see the property that the area of surfaces of M5 and D4 branes have the similar function form and that of M2 and D1 branes have the similar function form too. Thus, M5 has the same phase structure as D4 and M2 has the same phase structure as D1. Note that in the case of $F_p = F_{M5} = F_{M2} = 1$, which is the zero-temperature system without compactification, above property also useful. Also, remaining $F_A$ in $U^2/F_A$ while let $F_A = 1$ in $H_A F_A$, which is the finite-temperature system without compactification, above property also useful. Thus, in following section we merely analyze the property of $S^{(Dp)}_A$.

In a similar way, the metric of Dp branes smeared on black $D_{p+4}$ is

$$
\begin{align*}
\frac{d}{s} &= H^{-1/2}_p H^{-1/4}_p \left[- F dt^2 + \sum_{i=1}^{2} dy_i^2 \right] + H^{1/2}_p H^{-1/2}_p \left[ dx^2 + dx_5^2 + \sum_{i=1}^{2} dx_i^2 \right] \\
&+ H^{1/2}_p H^{1/2}_p \left[ \frac{dU^2}{F_p} + U^2 d\Omega^{2-p}_{3-p} \right] \\
H_{p,p+4} &= 1 + \frac{R_{p,p+4}}{U^{3-p}}, \quad F = 1 - \frac{1}{U^{3-p}}, \\
T_H &= \frac{3 - p}{4\pi} \left( R_{p} R_{p+4} \right)^{3-p} e^{-2\Phi} = H^{(p-3)/2}_p H^{(p+1)/2}_p
\end{align*}
$$

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in which the Hawking temperature $T_H$ is determined at near-horizon limit. Using above relations we can obtain metric of associated branes compactified on coordinate $w$ with radius $R_w = 1/2\pi T_H$

$$ds^2 = -H_p^{1/2}H_{p+4}^{-1/2}dt^2 + H_p^{1/2}H_{p+4}^{-1/2}Fdw^2 + H_p^{1/2}H_{p+4}^{-1/2}\left(dx^2 + \sum_{i=1}^2 dx_i^2\right)$$

$$+ H_p^{1/2}H_{p+4}^{1/2}\left(\frac{dU^2}{F} + U^2d\Omega_{p+4}^2\right)$$

(3.20)

The induced metric in considering the entanglement entropy of the “slab” in which the subspace is defined by $-\ell/2 < x < \ell/2$ is

$$ds^2_{ind} = H_0^{1/2}H_4^{-1/2}\left(\sum_{i=1}^2 dx_i^2\right) + H_0^{-1/2}H_4^{1/2}Fdw^2 + \left(H_0^{1/2}H_4^{-1/2} + H_0^{1/2}H_4^{1/2}\frac{U^2}{F}\right)dx^2$$

$$+ H_0^{1/2}H_4^{1/2}U^2d\Omega_4^2$$

(3.21)

Thus, the associated entanglement entropy is

$$S_A^{(D_p+D_{p+4})} = \frac{1}{4G_N^{(10)}}\int d^8\sigma \sqrt{H_p F_p} U^{8-p} \sqrt{1 + H_p \frac{U^2}{F_p}}$$

(3.22)

Which is just the function of $S_A^{(D_p)}$ as claimed before in the case of D0+D4. However we shall notice that the Hawking temperature $T_H$ and the associated radius of compactified coordinate $R_w$ between D_p brane system and D_p+D_{p+4} brane system are different from each other.

### 4 Phase Transition of Compactified Branes

#### 4.1 General formulation

From above section we see that in near-horizon limit the entanglement entropy of Dp brane can be expressed as

$$S_A^{(D_p)} \approx \frac{1}{4G_N^{(10)}}\int d^8\sigma \sqrt{R^{7-p} \frac{U^7}{U^{7-p}}} U^{8-p} \sqrt{1 + \frac{b^{7-p}}{U^{7-p}} + \frac{R^{(7-p)^2}}{U^{7-p}} \frac{R^{(7-p)^2}}{b^{7-p}}}$$

$$= \frac{1}{4G_N^{(10)}} R^{(7-p)^2} b^{7-p} \int d^8\sigma \left(\frac{5-p}{2}\right) \left(\frac{2}{z_c} \frac{2(7-p)}{z_c} \frac{R^{(7-p)^2}}{b^{7-p}}\right) \sqrt{1 - \frac{z}{z_c}}$$

(4.1)

in which we define

$$U^{-\frac{7-p}{2}} = \frac{5-p}{2} R^{-\frac{7-p}{2}} z$$

(4.2)

$$z_{c-\frac{2(7-p)}{z_c}} = \left(\frac{5-p}{2}\right) \frac{2(7-p)}{b^{7-p}} R^{(7-p)^2} = \left(\frac{7-p}{5-p}\right) R^{(7-p)^2}$$

(4.3)

We now need to find the minimal surface by minimizing the above area functional. Let $z' = 0$ at the turning point of the minimal surface, i.e. $z = z_c$, then

$$z' = \sqrt{\frac{\frac{z_c}{z_{c-\frac{2(7-p)}{z_c}}}}{\frac{2}{z_{c-\frac{2(7-p)}{z_c}}}} \left(1 - \left(\frac{z}{z_c}\right)^\alpha\right)^2 - \left(1 - \left(\frac{z}{z_c}\right)^\alpha\right)^2}$$

(4.4)
Thus
\[ \ell = 2 \int_0^{\ell/2} dx = 2 \int_0^{z_*} \frac{dz}{dz} \]
\[ = 2 \int_0^{z_*} \frac{z^\beta \sqrt{1 - \left( \frac{z}{z_c} \right)^{\alpha}}}{\sqrt{z_c^{2\beta} \left( 1 - \left( \frac{z}{z_c} \right)^{\alpha} \right)^2 - z^{2\beta} \left( 1 - \left( \frac{z}{z_c} \right)^{\alpha} \right) \left( 1 - \left( \frac{z}{z_c} \right)^{\alpha} \right)}} \]  
\[ (4.5) \]

and
\[ A = \frac{1}{4 G_N^{(10)}} \left( \frac{5 - p}{2} \right)^{\frac{2-p}{p-\pi}} 2 \pi R_w \, L^{p-2} \, \omega_{8-p} \int_{-\ell/2}^{\ell/2} dx \, \frac{1}{z^\beta} \sqrt{1 - \left( \frac{z}{z_c} \right)^{\alpha} + z^2} \]
\[ = \frac{1}{4 G_N^{(10)}} \left( \frac{5 - p}{2} \right)^{\frac{2-p}{p-\pi}} 4 \pi R_w \, L^{p-2} \, \omega_{8-p} \int_0^{z_*} \frac{dz}{z_c^{2\beta} \left( 1 - \left( \frac{z}{z_c} \right)^{\alpha} \right)^2 - z^{2\beta} \left( 1 - \left( \frac{z}{z_c} \right)^{\alpha} \right) \left( 1 - \left( \frac{z}{z_c} \right)^{\alpha} \right)}} \]
\[ (4.6) \]
in which
\[ \alpha = \frac{2(7-p)}{5-p}, \quad \beta = \frac{9-p}{5-p}, \quad \int_{i=1}^{p-2} dx \, L^{p-2}, \quad \int d\Omega_{8-p} = \omega_{8-p} \]  
\[ (4.7) \]

In numerical, we can use (4.5) to find the function \( \ell(z_*) \) and use (4.6) to find the surface function \( A(z_*) \). Then we use the two functions to obtain the surface function \( A(\ell) \). However, we will use the analytic method to prove that there always is phase transition in these systems.

### 4.2 Analysis

First, from (4.6) we see that in the cases of \( z_* \approx 0 \) the connected surface is
\[ A_{con} \approx \frac{1}{4 G_N^{(10)}} \left( \frac{5 - p}{2} \right)^{\frac{2-p}{p-\pi}} 4 \pi R_w \, L^{p-2} \, \omega_{8-p} \int_0^{z_*} \frac{dz}{z^{\beta}} \]
\[ = \frac{1}{4 G_N^{(10)}} \left( \frac{5 - p}{2} \right)^{\frac{2-p}{p-\pi}} 4 \pi R_w \, L^{p-2} \, \omega_{8-p} \left[ \frac{1}{-\beta + 1} \left( z_*^{-\beta + 1} - \epsilon^{-\beta + 1} \right) \right] \]  
\[ (4.8) \]

As the disconnected surface is defined by
\[ A_{dis} = \frac{1}{4 G_N^{(10)}} \left( \frac{5 - p}{2} \right)^{\frac{2-p}{p-\pi}} 4 \pi R_w \, L^{p-2} \, \omega_{8-p} \int_0^{z_*} \frac{dz}{z^{\beta}} \]
\[ = \frac{1}{4 G_N^{(10)}} \left( \frac{5 - p}{2} \right)^{\frac{2-p}{p-\pi}} 4 \pi R_w \, L^{p-2} \, \omega_{8-p} \left[ \frac{1}{-\beta + 1} \left( z_*^{-\beta + 1} - \epsilon^{-\beta + 1} \right) \right] \]  
\[ (4.9) \]

thus
\[ A_{con} - A_{dis} = \frac{1}{4 G_N^{(10)}} \left( \frac{5 - p}{2} \right)^{\frac{2-p}{p-\pi}} 4 \pi R_w \, L^{p-2} \, \omega_{8-p} \left[ \frac{1}{-\beta + 1} \left( z_*^{-\beta + 1} - \epsilon^{-\beta + 1} \right) \right] \]
\[ = \frac{1}{4 G_N^{(10)}} \left( \frac{5 - p}{2} \right)^{\frac{2-p}{p-\pi}} 4 \pi R_w \, L^{p-2} \, \omega_{8-p} \, z_*^{-\beta + 1} \left[ \frac{1 - (z_c/z_*)^{-\beta + 1}}{-\beta + 1} \right] \]  
\[ (4.10) \]

Note that \( z_c > z_* \) and \( -\beta + 1 = \frac{4}{p-5} \) which may be positive or negative. However, in any case the value \( A_{con} - A_{dis} \) is always negative and we conclude that the connected surface dominates at small value of \( z_* \).
Now, when we increasing $z_*$ to $z_c$ then connected surface will be increasing to the area of $A_{\text{dis}}$. If at $z_*=z_c$ the slop of $A_{\text{con}}$ is a decreasing function then we can conclude that $A_{\text{con}}-A_{\text{dis}}>0$ in some region of $z_* \approx z_c$. Thus, the disconnected surface will dominate in this region and we have a phase transition. This is the case we will prove.

Through the calculation we find that

$$\frac{dA_{\text{con}}}{dz_*} = \frac{1}{4G_N^{(10)}} \left( \frac{|5-p|}{2} \right)^{\frac{9-p}{2}} 4\pi R_w L^{p-2} \omega_{S-p} \left[ \frac{1}{z_*^{\beta}} - \frac{2}{2z_*^{2\beta+1}} \int_0^{z_*} dz \frac{z^{\beta}}{1-z^{\alpha z_*^{-\alpha}}} \right]$$

which implies that

$$\Rightarrow \quad \frac{dA_{\text{con}}}{dz_*} \bigg|_{z_*=z_c} = -\infty,$$

Thus, near the region $z_* \approx z_c$, the area of connected surface is larger than that of the disconnected surface and the system will transform to disconnected surface.

5 Conclusion

In this paper we use the entanglement entropy to study the phase transition of M-branes and $D_p$ branes systems. We first show that the entanglement entropy of M5 branes on a circle scales as $N^3$ [16,17] and has the phase transition during decreasing the compactified radius. We next show that the holographic entanglement entropy of D0+D4 system on a circle will scale as $N^2$ at low energy, likes as a gauge theory with instantons. However, at high energy it transforms to a phase which scales as $N^3$, like as the M5 branes system. The property is consistent with [18,19] which showed that the compactified M5 with Kaluza-Klein modes is just the 5D gauge theory with instanton. Thus the low energy phase of compactified M5 is the compactified D0+D4 which becomes 5D gauge theory with instanton at low energy. Finally, we have seen that the entangle entropy of compactified M5, M2 and $D_p + D_{p+4}$ has similar mathematic form likes as that of $D_p$ branes. Use this property we then prove that they all have phase transition from connected to disconnected surface during increasing the line segment of length $\ell$ which dividing the space.

REFERENCES

1. G."t Hooft, “On The Quantum Structure Of A Black Hole,” Nucl. Phys. B 256, 727 (1985).
2. L. Bombelli, R. K. Koul, J. H. Lee and R. D. Sorkin, “A Quantum Source Of Entropy For Black Holes,” Phys. Rev. D 34, 373 (1986).
3. M. Srednicki, “Entropy and area,” Phys. Rev. Lett. 71, 666 (1993) [arXiv:hep-th/9303048].
4. P. Calabrese and J. L. Cardy, “Entanglement entropy and quantum field theory: A non-technical introduction,” Int. J. Quant. Inf. 4 (2006) 429 [arXiv:quant-ph/0505193].
5. L. Amico, R. Fazio, A. Osterloh and V. Vedral, “Entanglement in many-body systems,” Rev. Mod. Phys. 80 (2008) 517 [quant-ph/0703044 [QUANT-PH]].
6. J. Eisert, M. Cramer and M. B. Plenio, “Area laws for the entanglement entropy - a review,” Rev. Mod. Phys. 82 (2010) 277 [arXiv:0808.3773 [quant-ph]].
7. J. M. Maldacena, “The large N limit of superconformal field theories and supergravity,” Adv. Theor. Math. Phys. 2, 231 (1998) [Int. J. Theor. Phys. 38, 1113 (1999)] [arXiv:hep-th/9711200].

8. S. S. Gubser, I. R. Klebanov and A. M. Polyakov, “Gauge theory correlators from noncritical string theory,” Phys. Lett. B 428, 105 (1998) [arXiv:hep-th/9802109].

9. E. Witten, “Anti-de Sitter space and holography,” Adv. Theor. Math. Phys. 2, 253 (1998) [arXiv:hep-th/9802150].

10. S. Ryu and T. Takayanagi, “Holographic derivation of entanglement entropy from AdS/CFT,” Phys. Rev. Lett. 96, 181602 (2006) [arXiv:hep-th/0603001].

11. S. Ryu and T. Takayanagi, “Aspects of holographic entanglement entropy,” JHEP 0608, 045 (2006) [arXiv:hep-th/0605073].

12. T. Takayanagi, “Entanglement entropy from a holographic viewpoint,” [arXiv:1204.2450 [hep-th]].

13. I. R. Klebanov, D. Kutasov and A. Murugan, “Entanglement as a Probe of Confinement,” Nucl. Phys. B 796 (2008) 274 [arXiv:0709.2140 [hep-th]].

14. A. Pakman and A. Parnachev, , “Topological Entanglement Entropy and Holography,” JHEP 0807 (2008) 097 [arXiv:0805.5918 [hep-th]].

15. O. Ben-Ami, D. Carmi and J. Sonnenschein, “Holographic Entanglement Entropy of Multiple Strips,” JHEP 11(2014)144 [arXiv:1409.6305 [hep-th]].

16. S.S. Gubser, I.R. Klebanov and A.W. Peet, “Entropy and Temperature of Black 3-Branes,” Phys. Rev. D 54 (1996) 3915 [arXiv:hep-th/9602135].

17. I.R. Klebanov and A.A. Tseytlin, “Entropy of Near-Extremal Black p-branes,” Nucl. Phys. B 475 (1996) 164 [arXiv:hep-th/9604089].

18. M. R. Douglas, “On D=5 super Yang-Mills theory and (2,0) theory,” JHEP 1102 (2011) 011 [arXiv:1012.2880 [hep-th]].

19. N. Lambert, C. Papageorgakis, M. Schmidt-Somerfeld, “M5-Branes, D4-Branes and Quantum 5D super-Yang-Mills,” JHEP 1101 (2011) 083 [arXiv:1012.2882 [hep-th]].

20. E. Quijada and H. Boschi-Filho, “Entanglement Entropy for D3-, M2- and M5-brane backgrounds,” [arXiv:1711.08505 [hep-th]].

21. F. Bigazzi, A. L. Cotrone and R. Sisca, “Notes on Theta Dependence in Holographic Yang-Mills,” JHEP 08 (2015) 090 [arXiv:1506.03826 [hep-th]].