Polytopal Estimate of Mirković-Vilonen polytopes
lying in a Demazure crystal

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Abstract

In this paper, we give a polytopal estimate of Mirković-Vilonen polytopes lying in a
Demazure crystal in terms of Minkowski sums of extremal Mirković-Vilonen polytopes.
As an immediate consequence of this result, we provide a necessary (but not sufficient)
polytopal condition for a Mirković-Vilonen polytope to lie in a Demazure crystal.

1 Introduction.

This paper is a continuation of our previous one [NS2], and our purpose is to give a polytopal
estimate of Mirković-Vilonen polytopes lying in a Demazure crystal in terms of Minkowski
sums of extremal Mirković-Vilonen polytopes. It should be mentioned that as an immediate
consequence of this result, we can provide an affirmative answer to a question posed in [NS2,
§4.6].

Following the notation and terminology of [NS2], we now explain our results more pre-
cisely. Let $G$ be a complex, connected, semisimple algebraic group with Lie algebra $\mathfrak{g}$, $T$ a

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maximal torus with Lie algebra (Cartan subalgebra) \( \mathfrak{h} \), \( B \) a Borel subgroup containing \( T \), and \( U \) the unipotent radical of \( B \); by our convention, the roots in \( B \) are the negative ones. Let \( X_s(T) \) denote the coweight lattice \( \text{Hom}(\mathbb{C}^*, T) \) for \( G \), which we regard as an additive subgroup of a real form \( \mathfrak{h}_\mathbb{R} := \mathbb{R} \otimes_{\mathbb{Z}} X_s(T) \) of \( \mathfrak{h} \). Denote by \( W \) the Weyl group of \( \mathfrak{g} \), with \( e \) the identity element and \( w_0 \) the longest element of length \( m \). Also, let \( \mathfrak{g}_\vee \) denote the (Langlands) dual Lie algebra of \( \mathfrak{g} \) with Weyl group \( W \), and let \( U_q(\mathfrak{g}_\vee) \) be the quantized universal enveloping algebra of \( \mathfrak{g}_\vee \) over \( \mathbb{C}(q) \).

For each dominant coweight \( \lambda \in X_s(T) \subset \mathfrak{h}_\mathbb{R} \), let us denote by \( \mathcal{MV}(\lambda) \) the set of Mirković-Vilonen (MV for short) polytopes with highest vertex \( \lambda \) that is contained in the convex hull \( \text{Conv}(W \cdot \lambda) \) in \( \mathfrak{h}_\mathbb{R} \) of the Weyl group orbit \( W \cdot \lambda \) through \( \lambda \), and by \( \mathcal{B}(\lambda) \) the crystal basis of the irreducible highest weight \( U_q(\mathfrak{g}_\vee) \)-module \( V(\lambda) \) of highest weight \( \lambda \). Recall that Kamnitzer \([\text{Kam1}], \text{Kam2}\) proved the existence of an isomorphism of crystals \( \Psi_\lambda \) from the crystal basis \( \mathcal{B}(\lambda) \) to the set \( \mathcal{MV}(\lambda) \) of MV polytopes, which is endowed with the Lusztig-Berenstein-Zelevinsky (LBZ for short) crystal structure; he also proved the coincidence of this LBZ crystal structure on \( \mathcal{MV}(\lambda) \) with the Braverman-Finkelberg-Gaitsgory (BFG for short) crystal structure on \( \mathcal{MV}(\lambda) \).

In \([\text{NS2}]\), for each \( x \in W \), we gave a combinatorial description, in terms of the lengths of edges of an MV polytope, of the image \( \mathcal{MV}_x(\lambda) \subset \mathcal{MV}(\lambda) \) (resp., \( \mathcal{MV}^F(\lambda) \subset \mathcal{MV}(\lambda) \)) of the Demazure crystal \( \mathcal{B}_x(\lambda) \subset \mathcal{B}(\lambda) \) (resp., opposite Demazure crystal \( \mathcal{B}^F(\lambda) \subset \mathcal{B}(\lambda) \)) under the isomorphism \( \Psi_\lambda : \mathcal{B}(\lambda) \to \mathcal{MV}(\lambda) \) of crystals. Furthermore, in \([\text{NS2}]\), we proved that for each \( x \in W \), an MV polytope \( P \in \mathcal{MV}(\lambda) \) lies in the opposite Demazure crystal \( \mathcal{MV}^F(\lambda) \) if and only if the MV polytope \( P \) contains (as a set) the extremal MV polytope \( P_{x,\lambda} \) of weight \( x \cdot \lambda \), which is identical to the convex hull \( \text{Conv}(W_{\leq x} \cdot \lambda) \) in \( \mathfrak{h}_\mathbb{R} \) of a certain subset \( W_{\leq x} \cdot \lambda \) of \( W \cdot \lambda \) (see \([25]\) for details). However, we were unable to prove an analogous statement for Demazure crystals \( \mathcal{B}_x(\lambda) \), \( x \in W \). Thus, we posed the following question in \([\text{NS2}] \S 4.6\):

**Question.** Let us take an arbitrary \( x \in W \). Are all the MV polytopes lying in the Demazure crystal \( \mathcal{MV}_x(\lambda) \) contained (as sets) in the extremal MV polytope \( P_{x,\lambda} = \text{Conv}(W_{\leq x} \cdot \lambda) \)?

Note that the converse statement fails to hold, as mentioned in \([\text{NS2}] \text{Remark 4.6.1}\).

In this paper, we provide an affirmative answer to this question. In fact, we considerably sharpen the polytopal estimate above of MV polytopes lying in a Demazure crystal as follows.

In what follows, for each dominant coweight \( \lambda \in X_s(T) \subset \mathfrak{h}_\mathbb{R} \), we denote by \( W_\lambda \subset W \) the stabilizer of \( \lambda \) in \( W \), and by \( W^\lambda_{\min} \subset W \) the set of minimal (length) coset representatives modulo the subgroup \( W_\lambda \subset W \).

**Theorem 1** (\(= \text{Theorem 3.3.1} \) combined with Proposition 3.2.1). Let \( \lambda \in X_s(T) \subset \mathfrak{h}_\mathbb{R} \) be a dominant coweight, and let \( x \in W^\lambda_{\min} \subset W \). If an MV polytope \( P \in \mathcal{MV}(\lambda) \) lies in the
Demazure crystal $\mathcal{M}V_x(\lambda)$, then there exist a positive integer $N \in \mathbb{Z}_{\geq 1}$ and minimal coset representatives $x_1, x_2, \ldots, x_N \in W^\lambda_{\min} \subset W$ such that

\[
\begin{cases}
x \geq x_k & \text{for all } 1 \leq k \leq N; \\
N \cdot P \subseteq P_{x_1, \lambda} + P_{x_2, \lambda} + \cdots + P_{x_N, \lambda},
\end{cases}
\]

where $N \cdot P := \{ Nv \mid v \in P \} \subset \mathfrak{h}_R$ is an MV polytope in $\mathcal{M}V(N\lambda)$, and $P_{x_1, \lambda} + P_{x_2, \lambda} + \cdots + P_{x_N, \lambda}$ is the Minkowski sum of the extremal MV polytopes $P_{x_1, \lambda}, P_{x_2, \lambda}, \ldots, P_{x_N, \lambda}$.

**Remark.** We see from Remark 3.2.2 and Theorem 3.3.1 below that the elements $x_1, x_2, \ldots, x_N \in W^\lambda_{\min} \subset W$ can be chosen in such a way that the vectors $x_1 \cdot \lambda, x_2 \cdot \lambda, \ldots, x_N \cdot \lambda \in W \cdot \lambda \subset \mathfrak{h}_R$ give the directions of the Lakshmibai-Seshadri path of shape $\lambda$ that corresponds to the MV polytope $P \in \mathcal{M}V(\lambda)$ under the (inexplicit) bijection via the crystal basis $B(\lambda)$. Hence it also follows that $x_1 \geq x_2 \geq \cdots \geq x_N$ in the Bruhat ordering on $W$.

From the theorem above, we can deduce immediately that for an arbitrary $P \in \mathcal{M}V_x(\lambda)$, there holds $N \cdot P \subset N \cdot P_{x, \lambda}$ and hence $P \subset P_{x, \lambda}$. Indeed, this follows from the inclusion $P_{x, \lambda} = \text{Conv}(W_{x} \cdot \lambda) \subset \text{Conv}(W_{x} \cdot \lambda) = P_{x, \lambda}$ for each $1 \leq k \leq N$, and the fact that the Minkowski sum $P_{x, \lambda} + P_{x, \lambda} + \cdots + P_{x, \lambda}$ ($N$ times) is identical to $N \cdot P_{x, \lambda}$ (see Remark 3.1.2).

The main ingredient in our proof of the theorem is the following polytopal estimate of tensor products of MV polytopes. Let $\lambda_1, \lambda_2 \in X_+(T) \subset \mathfrak{h}_R$ be dominant coweights. Since $\mathcal{M}V(\lambda) \cong B(\lambda)$ as crystals for every dominant coweight $\lambda \in X_+(T)$, the tensor product $\mathcal{M}V(\lambda_2) \otimes \mathcal{M}V(\lambda_1)$ of the crystals $\mathcal{M}V(\lambda_1)$ and $\mathcal{M}V(\lambda_2)$ decomposes into a disjoint union of connected components as follows:

\[
\mathcal{M}V(\lambda_2) \otimes \mathcal{M}V(\lambda_1) \cong \bigoplus_{\lambda \in X_+(T)} \mathcal{M}V(\lambda)^{\otimes m^\lambda_{\lambda_1, \lambda_2}},
\]

where $m^\lambda_{\lambda_1, \lambda_2} \in \mathbb{Z}_{\geq 0}$ denotes the multiplicity of $\mathcal{M}V(\lambda)$ in $\mathcal{M}V(\lambda_2) \otimes \mathcal{M}V(\lambda_1)$. For each dominant coweight $\lambda \in X_+(T)$ such that $m^\lambda_{\lambda_1, \lambda_2} \geq 1$, we take (and fix) an arbitrary embedding $\iota_\lambda : \mathcal{M}V(\lambda) \hookrightarrow \mathcal{M}V(\lambda_2) \otimes \mathcal{M}V(\lambda_1)$ of crystals that maps $\mathcal{M}V(\lambda)$ onto a connected component of $\mathcal{M}V(\lambda_2) \otimes \mathcal{M}V(\lambda_1)$, which is isomorphic to $\mathcal{M}V(\lambda)$ as a crystal.

**Theorem 2** (Theorem 4.1.1). Keep the notation above. Let $P \in \mathcal{M}V(\lambda)$, and write $\iota_\lambda(P) \in \mathcal{M}V(\lambda_2) \otimes \mathcal{M}V(\lambda_1)$ as: $\iota_\lambda(P) = P_2 \otimes P_1$ for some $P_1 \in \mathcal{M}V(\lambda_1)$ and $P_2 \in \mathcal{M}V(\lambda_2)$. We assume that the MV polytope $P_2 \in \mathcal{M}V(\lambda_2)$ is an extremal MV polytope $P_{x, \lambda_2}$ for some $x \in W$, then we have

$$P \subset P_1 + P_2,$$

where $P_1 + P_2$ is the Minkowski sum of the MV polytopes $P_1 \in \mathcal{M}V(\lambda_1)$ and $P_2 \in \mathcal{M}V(\lambda_2)$.
We have not yet found a purely combinatorial proof of the theorem above. In fact, our argument is a geometric one, which is based on results of Braverman-Gaitsgory in [BrG], where tensor products of highest weight crystals are described in terms of MV cycles in the affine Grassmannian; here we should remark that the convention on the tensor product rule for crystals in [BrG] is opposite to ours, i.e., to that of Kashiwara [Kas2], [Kas4]. Also, it seems likely that the theorem above still holds without the assumption of extremality on the MV polytope $P_2 \in \mathcal{MV}(\lambda_2)$.

This paper is organized as follows. In §2, we first recall the basic notation and terminology concerning MV polytopes and Demazure crystals, and also review the relation between MV polytopes and MV cycles in the affine Grassmannian. Furthermore, we obtain a few new results on extremal MV polytopes and MV cycles, which will be used in the proof of Theorem 2 (= Theorem 4.1.1). In §3, we introduce the notion of $N$-multiple maps from $\mathcal{MV}(\lambda)$ to $\mathcal{MV}(N\lambda)$ for a dominant coweight $\lambda \in X_*(T)$ and $N \in \mathbb{Z}_{\geq 1}$, which is given explicitly by: $P \mapsto N \cdot P$ in terms of MV polytopes, and also show that for each MV polytope $P \in \mathcal{MV}(\lambda)$, there exists some $N \in \mathbb{Z}_{\geq 1}$ such that $N \cdot P \in \mathcal{MV}(N\lambda)$ can be written as the tensor product of certain $N$ extremal MV polytopes in $\mathcal{MV}(\lambda)$. Furthermore, assuming Theorem 2, we prove Theorem 1 above, which provides an answer to the question mentioned above. In §4, after revisiting results of Braverman-Gaitsgory on tensor products of highest weight crystals in order to adapt them to our situation, we prove Theorem 2 by using the geometry of the affine Grassmannian. In the Appendix, we give a brief account of why Theorem 4.2.4 below is a reformulation of results of Braverman-Gaitsgory.

2 Mirković-Vilonen polytopes and Demazure crystals.

2.1 Basic notation. Let $G$ be a complex, connected, reductive algebraic group, $T$ a maximal torus, $B$ a Borel subgroup containing $T$, and $U$ the unipotent radical of $B$; we choose the convention that the roots in $B$ are the negative ones. Let $X_*(T)$ denote the (integral) coweight lattice $\text{Hom}(\mathbb{C}^*, T)$ for $G$, and $X_*(T)_+$ the set of dominant (integral) coweights for $G$; we regard the coweight lattice $X_*(T)$ as an additive subgroup of a real form $\mathfrak{h}_\mathbb{R} := \mathbb{R} \otimes \mathbb{Z} X_*(T)$ of the Lie algebra $\mathfrak{h}$ of the maximal torus $T$. We denote by $G^\vee$ the (complex) Langlands dual group of $G$.

For the rest of this paper, except in §2.3, §4.2, and the Appendix, we assume that $G$ is semisimple. Denote by $\mathfrak{g}$ the Lie algebra of $G$, which is a complex semisimple Lie algebra. Let

$$
\left( A = (a_{ij})_{i,j \in I}, \Pi := \{ \alpha_j \}_{j \in I}^\vee, \Pi^\vee := \{ h_j \}_{j \in I}^\vee, \mathfrak{h}^*, \mathfrak{h} \right)
$$

be the root datum of $\mathfrak{g}$, where $A = (a_{ij})_{i,j \in I}$ is the Cartan matrix, $\mathfrak{h}$ is the Cartan subalgebra, $\Pi := \{ \alpha_j \}_{j \in I} \subset \mathfrak{h}^* := \text{Hom}_\mathbb{C}(\mathfrak{h}, \mathbb{C})$ is the set of simple roots, and $\Pi^\vee := \{ h_j \}_{j \in I} \subset \mathfrak{h}$ is the
set of simple coroots; note that \( \langle h_i, \alpha_j \rangle = a_{ij} \) for \( i, j \in I \), where \( \langle \cdot, \cdot \rangle \) denotes the canonical pairing between \( k \) and \( k^* \). We set \( k_R := \sum_{j \in I} \mathbb{R}h_j \subset k \), which is a real form of \( k \); we regard the coweight lattice \( X_\cdot(T) = \text{Hom}(\mathbb{C}^*, T) \) as an additive subgroup of \( k_R \). Also, for \( h, h' \in k_R \), we write \( h' \geq h \) if \( h' - h \in Q_+ := \sum_{j \in I} \mathbb{Z}_{\geq 0}h_j \). Let \( W := \langle s_j \mid j \in I \rangle \) be the Weyl group of \( k \), where \( s_j, j \in I \), are the simple reflections, with length function \( \ell : W \to \mathbb{Z}_{\geq 0} \), the identity element \( e \in W \), and the longest element \( w_0 \in W \); we denote by \( \leq \) the (strong) Bruhat ordering on \( W \). Let \( k^\vee \) be the Lie algebra of the Langlands dual group \( G^\vee \) of \( G \), which is the complex semisimple Lie algebra associated to the root datum

\[
(tA = (a_{ji})_{i,j \in I}, \Pi' = \{h_j\}_{j \in I}, \Pi = \{\alpha_j\}_{j \in I}, k, k^*);
\]

note that the Cartan subalgebra of \( k \) is the complex semisimple Lie algebra associated to the root datum

2.1.1 Remark \((V \subset B \text{ Demazure crystal})\) where \( \tilde{G} \) characterizes by the inductive relations:

\[
\mathcal{B}_x(\lambda) = \bigoplus_{b \in \mathcal{B}_x(\lambda)} \mathbb{C}(q)G_\lambda(b),
\]

where \( G_\lambda(b), b \in \mathcal{B}(\lambda), \) form the lower global basis of \( \mathcal{B}(\lambda) \).

Remark 2.1.1. If \( x, y \in W \) satisfies \( x \cdot \lambda = y \cdot \lambda \), then we have \( V_x(\lambda) = V_y(\lambda) \) since \( V(\lambda)_{x \cdot \lambda} = V(\lambda)_{y \cdot \lambda} \). Therefore, it follows from (2.1.1) that \( \mathcal{B}_x(\lambda) = \mathcal{B}_y(\lambda) \).

We know from [Kas1] Proposition 3.2.3 that the Demazure crystals \( \mathcal{B}_x(\lambda), x \in W \), are characterized by the inductive relations:

\[
\mathcal{B}_e(\lambda) = \{u_\lambda\},
\]

\[
\mathcal{B}_x(\lambda) = \bigcup_{k \in \mathbb{Z}_{\geq 0}} f_j^k \mathcal{B}_{s_jx}(\lambda) \setminus \{0\} \quad \text{for} \ x \in W \text{ and } j \in I \text{ with } s_jx < x,
\]

where \( u_\lambda \in \mathcal{B}(\lambda) \) denotes the highest weight element of \( \mathcal{B}(\lambda) \), and \( f_j, j \in I \), denote the lowering Kashiwara operators for \( \mathcal{B}(\lambda) \).

2.2 Mirković-Vilonen polytopes. In this subsection, following [Kam1], we recall a (combinatorial) characterization of Mirković-Vilonen (MV for short) polytopes; the relation between this characterization and the original (geometric) definition of MV polytopes given by Anderson [A] will be explained in §2.3.
As in (the second paragraph of) [2.1] we assume that \( \mathfrak{g} \) is a complex semisimple Lie algebra. Let \( \mu_* = (\mu_w)_{w \in W} \) be a collection of elements of \( \mathfrak{h}_R = \sum_{j \in I} \mathbb{R} h_j \). We call \( \mu_* = (\mu_w)_{w \in W} \) a Gelfand-Goresky-MacPherson-Serganova (GGMS) datum if it satisfies the condition that \( w^{-1} \cdot \mu_{w'} - w^{-1} \cdot \mu_w \in Q_+^\vee \) for all \( w, w' \in W \). It follows by induction with respect to the (weak) Bruhat ordering on \( W \) that \( \mu_* = (\mu_w)_{w \in W} \) is a GGMS datum if and only if

\[
\mu_{w^{-1}_i} - \mu_w \in \mathbb{Z}_{\geq 0} \left( w \cdot h_i \right) \quad \text{for every } w \in W \text{ and } i \in I. \tag{2.2.1}
\]

**Remark 2.2.1.** Let \( \mu_*^{(1)} = (\mu_w^{(1)})_{w \in W} \) and \( \mu_*^{(2)} = (\mu_w^{(2)})_{w \in W} \) be GGMS data. Then, it is obvious from the definition of GGMS data (or equivalently, from (2.2.1)) that the (componentwise) sum

\[
\mu_*^{(1)} + \mu_*^{(2)} := (\mu_w^{(1)} + \mu_w^{(2)})_{w \in W}
\]

of \( \mu_*^{(1)} \) and \( \mu_*^{(2)} \) is also a GGMS datum.

Following [Kam1] and [Kam2], to each GGMS datum \( \mu_* = (\mu_w)_{w \in W} \), we associate a convex polytope \( P(\mu_*) \subset \mathfrak{h}_R \) by:

\[
P(\mu_*) = \bigcap_{w \in W} \left\{ v \in \mathfrak{h}_R \mid w^{-1} \cdot v - w^{-1} \cdot \mu_w \in \sum_{j \in I} \mathbb{R}_{\geq 0} h_j \right\}; \tag{2.2.2}
\]

the polytope \( P(\mu_*) \) is called a pseudo-Weyl polytope with GGMS datum \( \mu_* \). Note that the GGMS datum \( \mu_* = (\mu_w)_{w \in W} \) is determined uniquely by the convex polytope \( P(\mu_*) \). Also, we know from [Kam1, Proposition 2.2] that the set of vertices of the polytope \( P(\mu_*) \) is given by the collection \( \mu_* = (\mu_w)_{w \in W} \) (possibly, with repetitions). In particular, we have

\[
P(\mu_*) = \text{Conv} \{ \mu_w \mid w \in W \}, \tag{2.2.3}
\]

where for a subset \( X \) of \( \mathfrak{h}_R \), \( \text{Conv} \ X \) denotes the convex hull in \( \mathfrak{h}_R \) of \( X \).

We know the following proposition from [Kam1] Lemma 6.1, which will be used later.

**Proposition 2.2.2.** Let \( P_1 = P(\mu_*^{(1)}) \) and \( P_2 = P(\mu_*^{(2)}) \) be pseudo-Weyl polytopes with GGMS data \( \mu_*^{(1)} = (\mu_w^{(1)})_{w \in W} \) and \( \mu_*^{(2)} = (\mu_w^{(2)})_{w \in W} \), respectively. Then, the Minkowski sum

\[
P_1 + P_2 := \{ v_1 + v_2 \mid v_1 \in P_1, v_2 \in P_2 \}
\]

of the pseudo-Weyl polytopes \( P_1 \) and \( P_2 \) is identical to the pseudo-Weyl polytope \( P(\mu_*^{(1)} + \mu_*^{(2)}) \) with GGMS datum \( \mu_*^{(1)} + \mu_*^{(2)} = (\mu_w^{(1)} + \mu_w^{(2)})_{w \in W} \) (see Remark [2.2.1]).

Let \( R(w_0) \) denote the set of all reduced words for \( w_0 \), that is, all sequences \( (i_1, i_2, \ldots, i_m) \) of elements of \( I \) such that \( s_{i_1} s_{i_2} \cdots s_{i_m} = w_0 \), where \( m \) is the length \( \ell(w_0) \) of the longest element \( w_0 \). Let \( i = (i_1, i_2, \ldots, i_m) \in R(w_0) \) be a reduced word for \( w_0 \). We set \( w_i := \)
s_{i_1}s_{i_2} \cdots s_{i_l} \in W$ for $0 \leq l \leq m$. For a GGMS datum $\mu_\bullet = (\mu_w)_{w \in W}$, define integers (called lengths of edges) $n^l_i = n^l_i(\mu_\bullet) \in \mathbb{Z}_{\geq 0}$, $1 \leq l \leq m$, via the following “length formula” (see [Kam1 Eq.(8)] and (2.2.1) above):

$$\mu_{w^l_i} - \mu_{w^l_{i-1}} = n^l_i w^l_{i-1} \cdot h_i.$$  \hspace{1cm} (2.2.4)

Now we recall a (combinatorial) characterization of Mirković-Vilonen (MV) polytopes, due to Kamnitzer [Kam1]. This result holds for an arbitrary complex semisimple Lie algebra $g$, but we give its precise statement only in the case that $g$ is simply-laced since we do not make use of it in this paper; we merely mention that when $g$ is not simply-laced, there are also conditions on the lengths $n^l_i$, $1 \leq l \leq m$, $i \in R(w_0)$, for the other possible values of $a_{ij}$ and $a_{ji}$ (we refer the reader to [BerZ, §3] for explicit formulas).

**Definition 2.2.3.** A GGMS datum $\mu_\bullet = (\mu_w)_{w \in W}$ is said to be a Mirković-Vilonen (MV) datum if it satisfies the following conditions:

1. If $i = (i_1, i_2, \ldots, i_m) \in R(w_0)$ and $j = (j_1, j_2, \ldots, j_m) \in R(w_0)$ are related by a 2-move, that is, if there exist indices $i, j \in I$ with $a_{ij} = a_{ji} = 0$ and an integer $0 \leq k \leq m - 2$ such that $i_l = j_l$ for all $1 \leq l \leq m$ with $l \neq k + 1, k + 2$, and such that $i_{k+1} = j_{k+2} = i$, $i_{k+2} = j_{k+1} = j$, then there hold

$$\begin{cases} n^l_i = n^l_j & \text{for all } 1 \leq l \leq m \text{ with } l \neq k + 1, k + 2, \\
n^l_{k+1} = n^l_{k+2}, & n^l_{k+2} = n^l_{k+1}. \end{cases}$$

2. If $i = (i_1, i_2, \ldots, i_m) \in R(w_0)$ and $j = (j_1, j_2, \ldots, j_m) \in R(w_0)$ are related by a 3-move, that is, if there exist indices $i, j \in I$ with $a_{ij} = a_{ji} = -1$ and an integer $0 \leq k \leq m - 2$ such that $i_l = j_l$ for all $1 \leq l \leq m$ with $l \neq k + 1, k + 2$, and such that $i_{k+1} = j_{k+2} = i$, $i_{k+2} = j_{k+1} = j$, then there hold

$$\begin{cases} n^l_i = n^l_j & \text{for all } 1 \leq l \leq m \text{ with } l \neq k + 1, k + 2, \\
n^l_{k+1} = n^l_{k+2}, & n^l_{k+2} = n^l_{k+1}. \end{cases}$$
m − 3 such that \( i_l = j_l \) for all \( 1 ≤ l ≤ m \) with \( l ≠ k + 1, k + 2, k + 3, \) and such that \( i_{k+1} = i_{k+3} = j_{k+2} = i, i_{k+2} = j_{k+1} = j_{k+3} = j, \) then there hold
\[
\begin{align*}
n^l_1 &= n^l_j, & \text{for all } 1 ≤ l ≤ m \text{ with } l ≠ k + 1, k + 2, k + 3, \\
n^1_{k+1} &= n^1_{k+2} + n^1_{k+3} - \min(n^1_{k+1}, n^1_{k+3}), \\
n^2_{k+2} &= \min(n^1_{k+1}, n^1_{k+3}), \\
n^1_{k+3} &= n^1_{k+1} + n^1_{k+2} - \min(n^1_{k+1}, n^1_{k+3}).
\end{align*}
\]

The pseudo-Weyl polytope \( P(μ_\bullet) \) with GGMS datum \( μ_\bullet = (μ_w)_{w ∈ W} \) (see (2.2.2)) is a Mirković-Vilonen (MV) polytope if and only if the GGMS datum \( μ_\bullet = (μ_w)_{w ∈ W} \) is a MV datum (see the proof of [Kam1, Proposition 5.4] and the comment following [Kam1, Theorem 7.1]). Also, for a dominant coweight \( λ ∈ X_*(T) \subset h_\mathbb{R} \) and a coweight \( ν ∈ X_*(T) \subset h_\mathbb{R}, \) an MV polytope \( P = P(μ_\bullet) \) with GGMS datum \( μ_\bullet = (μ_w)_{w ∈ W} \) is an MV polytope of highest vertex \( λ \) and lowest vertex \( ν \) if and only if \( μ_{w_0} = λ, \ μ_τ = ν, \) and \( P \) is contained in the convex hull \( \text{Conv}(W · λ) \) of the \( W \)-orbit \( W · λ \subset h_\mathbb{R} \) (see [A] Proposition 7); we denote by \( \mathcal{MV}(λ)_ν \) the set of MV polytopes of highest vertex \( λ \) and lowest vertex \( ν. \) For each dominant coweight \( λ ∈ X_*(T) \subset h_\mathbb{R}, \) we set
\[
\mathcal{MV}(λ) := \bigsqcup_{ν ∈ X_*(T)} \mathcal{MV}(λ)_ν.
\]

### 2.3 Relation between MV polytopes and MV cycles.

In this subsection, we review the relation between MV polytopes and MV cycles in the affine Grassmannian.

Let us recall the definition of MV cycles in the affine Grassmannian, following [MV2] (and [A]). Let \( G \) be a complex, connected, reductive algebraic group as in (the beginning of) §2.1. Let \( \mathcal{O} = \mathbb{C}[[t]] \) denote the ring of formal power series, and \( \mathcal{K} = \mathbb{C}(t) \) the field of formal Laurent series (the fraction field of \( \mathcal{O} \)). The affine Grassmannian \( \mathcal{G}r \) for \( G \) over \( \mathbb{C} \)
is defined to be the quotient space $G(\mathcal{K})/G(\mathcal{O})$, equipped with the structure of a complex, algebraic ind-variety, where $G(\mathcal{K})$ denotes the set of $\mathcal{K}$-valued points of $G$, and $G(\mathcal{O}) \subset G(\mathcal{K})$ denotes the set of $\mathcal{O}$-valued points of $G$; we denote by $\pi : G(\mathcal{K}) \twoheadrightarrow \mathcal{G}_r = G(\mathcal{K})/G(\mathcal{O})$ the natural quotient map, which is locally trivial in the Zariski topology. In the following, for a subgroup $H \subset G(\mathcal{K})$ that is stable under the adjoint action of $T$ and for an element $w$ of the Weyl group $W \cong N_G(T)/T$ of $G$, we denote by $w^H$ the $w$-conjugate $\bar{w}H\bar{w}^{-1}$ of $H$, where $\bar{w} \in N_G(T)$ is a lift of $w$ in $W$.

Since each coweight $\nu \in \pi^* X_s(T) = \text{Hom}(\mathbb{C}^*, T)$ is a regular map from $\mathbb{C}^*$ to $T \subset G$, it gives a point $t' \in G(\mathcal{K})$, which in turn, descends to a point $[t'] \in \mathcal{G}_r = G(\mathcal{K})/G(\mathcal{O})$. The following simple lemma will be used in the proof of Lemma 2.5.8.

**Lemma 2.3.1.** Let $L \subset G$ be a complex, connected, reductive algebraic group containing the maximal torus $T$ of $G$. Then, for each $\nu \in X_s(T)$, the inclusion $L(\mathcal{K})[t'] \hookrightarrow \mathcal{G}_r$ gives an embedding of the affine Grassmannian for $L$ into $\mathcal{G}_r$.

**Proof.** Observe that $(t'G(\mathcal{O})t^{-\nu}) \cap L(\mathcal{K}) = t'L(\mathcal{O})t^{-\nu}$. Hence the map $i_L : L(\mathcal{K}) \rightarrow \mathcal{G}_r$, $g \mapsto g[t']$, is factored through $L(\mathcal{K})/(t'L(\mathcal{O})t^{-\nu})$. Since $t' \in L(\mathcal{K})$, we conclude that the map $i_L : L(\mathcal{K}) \rightarrow \mathcal{G}_r$ descends to a map between the affine Grassmannian for $L$ and $\mathcal{G}_r$, as desired. (This construction is only at the level of sets, but we can indeed show that the map above commutes with the ind-variety structures.)

For each $\nu \in X_s(T)$, we set

$$\mathcal{G}^{w\nu} := G(\mathcal{O})[t'] \subset \mathcal{G}_r,$$

the $G(\mathcal{O})$-orbit of $[t']$, which is a smooth quasi-projective algebraic variety over $\mathbb{C}$. Also, for each $\nu \in X_s(T)$ and $w \in W$, we set

$$\mathcal{S}^w : = wU(\mathcal{K})[t'] \subset \mathcal{G}_r,$$

the $U(\mathcal{K})$-orbit of $[t']$, which is a (locally closed) ind-subvariety of $\mathcal{G}_r$; we write simply $\mathcal{S}_\nu$ for $\mathcal{S}^\nu$. Then, we know the following two kinds of decompositions of $\mathcal{G}_r$ into orbits. First, we have

$$\mathcal{G}_r = \bigsqcup_{\lambda \in X_s(T)_+} \mathcal{G}^{\nu\lambda} \quad \text{(Cartan decomposition)},$$

with $\mathcal{G}^{w\lambda} = \mathcal{G}^{\nu\lambda}$ for $\lambda \in X_s(T)_+$ and $w \in W$; note that (see, for example, [MV2, §2]) for each $\lambda \in X_s(T)_+$, the quasi-projective variety $\mathcal{G}^{\nu\lambda}$ is simply-connected, and of dimension $2\langle \lambda, \rho \rangle$, where $\rho$ denotes the half-sum of the positive roots $\alpha \in \Delta_+$ for $G$, i.e., $2\rho = \sum_{\alpha \in \Delta_+} \alpha$. Second, we have for each $w \in W$,

$$\mathcal{G}_r = \bigsqcup_{\nu \in X_s(T)} \mathcal{S}_\nu \quad \text{(Iwasawa decomposition).}$$
Moreover, the (Zariski-) closure relations among these orbits are described as follows (see [MV2 §2 and §3]):

\[
\overline{G^\lambda} = \bigcup_{\lambda' \in X_\ast(T)_+} \mathcal{G}^{\lambda'} \quad \text{for } \lambda \in X_\ast(T)_+; \tag{2.3.1}
\]

\[
\overline{S^w_\nu} = \bigcup_{\gamma \in X_\ast(T)} S^w_\gamma \quad \text{for } \nu \in X_\ast(T) \text{ and } w \in W. \tag{2.3.2}
\]

**Remark 2.3.2.** Let \( \mathcal{X} \subset \mathcal{G}_T \) be an irreducible algebraic subvariety, and \( \nu \in X_\ast(T) \), \( w \in W \). Then, it follows from (2.3.2) that the intersection \( \mathcal{X} \cap S^w_\nu \) is an open dense subset of \( \mathcal{X} \) if and only if \( \mathcal{X} \cap S^w_\nu \neq \emptyset \) and \( \mathcal{X} \cap S^w_\gamma = \emptyset \) for every \( \gamma \in X_\ast(T) \) with \( w^{-1} \cdot \gamma \nsubseteq w^{-1} \cdot \nu \).

For \( \lambda \in X_\ast(T)_+ \), let \( L(\lambda) \) denote the irreducible finite-dimensional representation of the (complex) Langlands dual group \( G^\vee \) of \( G \) with highest weight \( \lambda \), and \( \Omega(\lambda) \subset X_\ast(T) \) the set of weights of \( L(\lambda) \). We know from [MV2 Theorem 3.2 and Remark 3.3] that \( \nu \in X_\ast(T) \) is an element of \( \Omega(\lambda) \) if and only if \( G^\lambda \cap S_\nu \neq \emptyset \), and then the intersection \( G^\lambda \cap S_\nu \) is of pure dimension \( \langle \lambda - \nu, \rho \rangle \).

Now we come to the definition of MV cycles in the affine Grassmannian.

**Definition 2.3.3** ([MV2 §3]; see also [A §5.3]). Let \( \lambda \in X_\ast(T)_+ \) and \( \nu \in X_\ast(T) \) be such that \( G^\lambda \cap S_\nu \neq \emptyset \), i.e., \( \nu \in \Omega(\lambda) \). An MV cycle of highest weight \( \lambda \) and weight \( \nu \) is defined to be an irreducible component of the (Zariski-) closure of the intersection \( G^\lambda \cap S_\nu \).

We denote by \( Z(\lambda),_\nu \) the set of MV cycles of highest weight \( \lambda \in X_\ast(T)_+ \) and weight \( \nu \in X_\ast(T) \). Also, for each \( \lambda \in X_\ast(T)_+ \), we set

\[
Z(\lambda) := \bigcup_{\nu \in X_\ast(T)} Z(\lambda)_\nu,
\]

where \( Z(\lambda)_\nu := \emptyset \) if \( G^\lambda \cap S_\nu = \emptyset \).

**Example 2.3.4** (cf. [MV2 Eq.(3.6)]). For each \( \lambda \in X_\ast(T)_+ \), we have

\[
Z(\lambda)_\lambda = \{ [t^\lambda] \}, \quad \text{and} \quad Z(\lambda)_{\omega_0 \lambda} = \{ \overline{G^\lambda} \}.
\]

**Remark 2.3.5** ([NP Lemma 5.2], [MV2 Eq.(3.6)]). Let \( \lambda \in X_\ast(T)_+ \). If \( \nu \in X_\ast(T) \) is of the form \( \nu = x \cdot \lambda \) for some \( x \in W \), then

\[
b_{x \cdot \lambda} := U(\mathcal{O})[t^{x \cdot \lambda}] \subset \overline{G(\mathcal{O})[t^\lambda] \cap U(\mathcal{K})[t^{x \cdot \lambda}]} = \overline{G^\lambda \cap S_{x \cdot \lambda}}
\]

is the unique MV cycle of highest weight \( \lambda \) and weight \( x \cdot \lambda \) (extremal MV cycle of weight \( x \cdot \lambda \)). For an explicit (combinatorial) description of the corresponding extremal MV polytope, see §2.5 below.
Motivated by the discovery of MV cycles in the affine Grassmannian, Anderson [A] proposed considering the “moment map images” of MV cycles as follows: Let $\lambda \in X_*(T)_+$. For an MV cycle $b \in Z(\lambda)$, we set

$$P(b) := \text{Conv}\{\nu \in X_*(T) \subset h_R \mid [t^\nu] \in b\},$$

and call $P(b) \subset h_R$ the moment map image of $b$; note that $P(b)$ is indeed a convex polytope in $h_R$.

For the rest of this paper, except in §4.2 and the Appendix, we assume that $G$ (and hence its Lie algebra $g$) is semisimple. The following theorem, due to Kamnitzer [Kam1], establishes an explicit relationship between MV polytopes and MV cycles.

**Theorem 2.3.6.** (1) Let $\lambda \in X_*(T)_+$ and $\nu \in X_*(T)$ be such that $Gr^\lambda \cap S_\nu \neq \emptyset$. If $\mu_* = (\mu_w)_{w \in W}$ denotes the GGMS datum of an MV polytope $P \in MV(\lambda)_\nu$, that is, $P = P(\mu_*) \in MV(\lambda)_\nu$, then

$$b(\mu_*) := \bigcap_{w \in W} S^w_{\mu_w} \subset Gr^\lambda$$

is an MV cycle that belongs to $Z(\lambda)_\nu$.

(2) Let $\lambda \in X_*(T)_+$. For an MV polytope $P = P(\mu_*) \in MV(\lambda)$ with GGMS datum $\mu_*$, we set $\Phi_\lambda(P) := b(\mu_*)$. Then, the map $\Phi_\lambda : MV(\lambda) \rightarrow Z(\lambda)$, $P \mapsto \Phi_\lambda(P)$, is a bijection from $MV(\lambda)$ onto $Z(\lambda)$ such that $\Phi_\lambda(MV(\lambda)_\nu) = Z(\lambda)_\nu$ for all $\nu \in X_*(T)$ with $Gr^\lambda \cap S_\nu \neq \emptyset$. In particular, for each MV cycle $b \in Z(\lambda)$, there exists a unique MV datum $\mu_*$ such that $b = b(\mu_*)$, and in this case, the moment map image $P(b)$ of the MV cycle $b = b(\mu_*)$ is identical to the MV polytope $P(\mu_*) \in MV(\lambda)$.

**Remark 2.3.7** ([Kam1] §2.2]). For $\nu \in X_*(T)$ and $w \in W$, the “moment map image” $P(S^w_\nu)$ of $S^w_\nu$ is, by definition, the convex hull in $h_R$ of the set $\{\gamma \in X_*(T) \subset h_R \mid [t^\gamma] \in S^w_\nu\} \subset h_R$, which is identical to the (shifted) convex cone $\{v \in h_R \mid w^{-1} \cdot v - w^{-1} \cdot \nu \in \sum_{j \in I} \mathbb{R}_{\geq 0} h_j\}$.

2.4 Lusztig-Berenstein-Zelevinsky (LBZ) crystal structure. We keep the notation and assumptions of [2.2]. For an MV datum $\mu_* = (\mu_w)_{w \in W}$ and $j \in I$, we denote by $f_j \mu_*$ (resp., $e_j \mu_*$ if $\mu_e \neq \mu_s$; note that $\mu_s - \mu_e \in \mathbb{Z}_{\geq 0} h_j$ by (2.2.1)) a unique MV datum $\mu'_* = (\mu'_w)_{w \in W}$ such that $\mu'_e = \mu_e - h_j$ (resp., $\mu'_e = \mu_e + h_j$) and $\mu'_w = \mu_w$ for all $w \in W$ with $s_j w < w$ (see [Kam2] Theorem 3.5 and its proof); note that $\mu'_w = \mu_w$ and $\mu'_s = \mu_s$.

Let $\lambda \in X_*(T) \subset h_R$ be a dominant coweight. Following [Kam2] §6.2], we endow $MV(\lambda)$ with the Lusztig-Berenstein-Zelevinsky (LBZ) crystal structure for $U_q(g^\vee)$ as follows. Let $P = P(\mu_*) \in MV(\lambda)$ be an MV polytope with GGMS datum $\mu_* = (\mu_w)_{w \in W}$. The weight $wt(P)$ of $P$ is, by definition, equal to the vertex $\mu_e \in \lambda - Q^+_+$. For each $j \in I$, we define the
lowering Kashiwara operator $f_j : \mathcal{M} \mathcal{V} (\lambda) \cup \{0\} \to \mathcal{M} \mathcal{V} (\lambda) \cup \{0\}$ and the raising Kashiwara operator $e_j : \mathcal{M} \mathcal{V} (\lambda) \cup \{0\} \to \mathcal{M} \mathcal{V} (\lambda) \cup \{0\}$ by:

$$e_j \theta = f_j \theta := 0,$$

$$f_j P = f_j P(\mu_\bullet) := \begin{cases} P(f_j \mu_\bullet) & \text{if } P(f_j \mu_\bullet) \subset \text{Conv} (W \cdot \lambda), \\ 0 & \text{otherwise}, \end{cases}$$

$$e_j P = e_j P(\mu_\bullet) := \begin{cases} P(e_j \mu_\bullet) & \text{if } \mu_\bullet \neq \mu_j \ (\text{i.e., } \mu_{s_j} - \mu_e \in \mathbb{Z}_{\geq 0} h_j), \\ 0 & \text{otherwise}, \end{cases}$$

where $0$ is an additional element, not contained in $\mathcal{M} \mathcal{V} (\lambda)$. For $j \in I$, we set $\varepsilon_j (P) := \max \{ k \in \mathbb{Z}_{\geq 0} \mid e_j^k P \neq 0 \}$ and $\varphi_j (P) := \max \{ k \in \mathbb{Z}_{\geq 0} \mid f_j^k P \neq 0 \}$; note that for each $j \in I$, we have

$$\varphi_j (P) = \langle \text{wt} (P), \alpha_j \rangle + \varepsilon_j (P) \quad \text{for all } P \in \mathcal{M} \mathcal{V} (\lambda). \quad (2.4.1)$$

**Remark 2.4.1.** Let $P = P(\mu_\bullet) \in \mathcal{M} \mathcal{V} (\lambda)$ be an MV polytope with GGMS datum $\mu_\bullet = (\mu_w)_{w \in W}$. Then, we deduce from the definition of the raising Kashiwara operators $e_j$ (or, the MV datum $e_j \mu_\bullet$) that $\mu_{s_j} - \mu_e = \varepsilon_j (P) h_j$ for $j \in I$.

**Theorem 2.4.2** ([Kam2] Theorem 6.4). The set $\mathcal{M} \mathcal{V} (\lambda)$, equipped with the maps $\text{wt}$, $e_j$, $f_j$ ($j \in I$), and $\varepsilon_j$, $\varphi_j$ ($j \in I$) above, is a crystal for $U_q (\mathfrak{g}^\vee)$. Moreover, there exists a unique isomorphism $\Psi_\lambda : \mathcal{B} (\lambda) \cong \mathcal{M} \mathcal{V} (\lambda)$ of crystals for $U_q (\mathfrak{g}^\vee)$.

**Remark 2.4.3.** Kamnitzer [Kam2] proved that for each $\lambda \in X_+ (T)$, the bijection $\Phi_\lambda : \mathcal{M} \mathcal{V} (\lambda) \to \mathcal{Z} (\lambda)$ in Theorem 2.3.6 (2) also intertwines the LBZ crystal structure on $\mathcal{M} \mathcal{V} (\lambda)$ and the crystal structure on $\mathcal{Z} (\lambda)$ defined in [BrG] (and [BFG]).

For each $x \in W$, we denote by $\mathcal{M} \mathcal{V}_x (\lambda) \subset \mathcal{M} \mathcal{V} (\lambda)$ the image $\Psi_\lambda (\mathcal{B}_x (\lambda))$ of the Demazure crystal $\mathcal{B}_x (\lambda) \subset \mathcal{B} (\lambda)$ associated to $x \in W$ under the isomorphism $\Psi_\lambda : \mathcal{B} (\lambda) \cong \mathcal{M} \mathcal{V} (\lambda)$ in Theorem 2.4.2 for a combinatorial description of $\mathcal{M} \mathcal{V}_x (\lambda)$ in terms of the lengths $n^i_l \in \mathbb{Z}_{\geq 0}$, $i \in R (w_0)$, $0 \leq l \leq m$, of edges of an MV polytope, see [NS2] §3.2.

### 2.5 Extremal MV polytopes.

Let $\mathfrak{g}$ be a complex semisimple Lie algebra as in (the second paragraph of) §2.1. Let $\lambda \in X_+ (T) \subset \mathfrak{h}_R$ be a dominant coweight. For each $x \in W$, we denote by $P_{x \cdot \lambda}$ the image of the extremal element $u_{x \cdot \lambda} \in \mathcal{B} (\lambda)$ of weight $x \cdot \lambda \in X_+ (T) \subset \mathfrak{h}_R$ under the isomorphism $\Psi_\lambda : \mathcal{B} (\lambda) \cong \mathcal{M} \mathcal{V} (\lambda)$ in Theorem 2.4.2 we call $P_{x \cdot \lambda} \in \mathcal{M} \mathcal{V} (\lambda)$ the extremal MV polytope of weight $x \cdot \lambda$. We know the following polytopal description of the extremal MV polytopes from [NS2] Theorem 4.1.5 (2)].
Proposition 2.5.1. Let $\lambda \in X_*(T) \subset \mathfrak{h}_\mathbb{R}$ be a dominant coweight, and $x \in W$. The extremal MV polytope $P_{x,\lambda}$ of weight $x \cdot \lambda$ is identical to the convex hull $\text{Conv}(W_{\leq x} \cdot \lambda)$ in $\mathfrak{h}_\mathbb{R}$ of the set $W_{\leq x} \cdot \lambda$, where $W_{\leq x}$ denotes the subset $\{ z \in W \mid z \leq x \}$ of $W$.

Remark 2.5.2. In [NS2], we proved Proposition 2.5.1 above and Theorem 2.5.6 below in the case that $\mathfrak{g}$ is simply-laced. However, these results hold also in the case that $\mathfrak{g}$ is not simply-laced; for example, we can use a standard technique of “folding” by diagram automorphisms (see [NS1], [Ho], and also [Lu2]).

Remark 2.5.3. It follows from Theorem 2.3.6 that for each $\lambda \in X_*(T)_+ \subset \mathfrak{h}_\mathbb{R}$ and $x \in W$, the extremal MV polytope $P_{x,\lambda}$ is identical to the moment map image $P(b_{x,\lambda})$ of the extremal MV cycle $b_{x,\lambda}$ (see Remark 2.3.5). In particular, the highest weight element $P_{e,\lambda} = P_\lambda$ of $\mathcal{MV}(\lambda)$ is identical to the set $P([\ell^\lambda]) = \{ \lambda \}$, and the lowest weight element $P_{w_0,\lambda}$ of $\mathcal{MV}(\lambda)$ is identical to the set $P(G_{\mathfrak{r} \lambda}) = \text{Conv}(W \cdot \lambda)$.

The GGMS datum of an extremal MV polytope is given as follows (see [NS2 §4.1]). Let us fix a dominant coweight $\lambda \in X_*(T) \subset \mathfrak{h}_\mathbb{R}$ and $x \in W$ arbitrarily. Let $p$ denote the length $\ell(xw_0)$ of $xw_0 \in W$. For each $i = (i_1, i_2, \ldots, i_m) \in R(w_0)$, with $m = \ell(w_0)$, we set

$$S(xw_0, i) = \left\{ (a_1, a_2, \ldots, a_p) \in [1, m]_\mathbb{Z}^p \left| \begin{array}{c} 1 \leq a_1 < a_2 < \cdots < a_p \leq m, \\ s_{i_1} s_{i_2} \cdots s_{i_p} = xw_0 \end{array} \right. \right\},$$

where $[1, m]_\mathbb{Z} := \{ a \in \mathbb{Z} \mid 1 \leq a \leq m \}$. We denote by $\min S(xw_0, i)$ the minimum element of the set $S(xw_0, i)$ in the lexicographic ordering; recall that the lexicographic ordering $\succeq$ on $S(xw_0, i)$ is defined as follows: $(a_1, a_2, \ldots, a_p) \succeq (b_1, b_2, \ldots, b_p)$ if there exists some integer $1 \leq q_0 \leq p$ such that $a_q = b_q$ for all $1 \leq q \leq q_0 - 1$ and $a_{q_0} > b_{q_0}$. Now we define a sequence $y_0^i, y_1^i, \ldots, y_m^i$ of elements of $W$ inductively by the following formula (see [NS2 §4.2]):

$$y_m^i = e, \quad y_{l-1}^i = \begin{cases} y_l^i & \text{if } l \text{ appears in } \min S(xw_0, i), \\ s_{\beta} y_l^i & \text{otherwise} \end{cases} \quad (2.5.1)$$

for $1 \leq l \leq m$, where we set $\beta_l^i := w_{l-1}^i \cdot \alpha_i$ for $1 \leq l \leq m$, and denote by $s_\beta \in W$ the reflection with respect to a root $\beta$.

Remark 2.5.4. The element $y_l^i \in W$ above does not depend on the dominant coweight $\lambda \in X_*(T) \subset \mathfrak{h}_\mathbb{R}$.

Remark 2.5.5. Let $i = (i_1, i_2, \ldots, i_m) \in R(w_0)$. We define a sequence $v_0^i, v_1^i, \ldots, v_m^i$ of elements of $W$ inductively by the following formula:

$$v_m^i = e, \quad v_{l-1}^i = \begin{cases} s_{\alpha} v_l^i & \text{if } l \text{ appears in } \min S(xw_0, i), \\ v_l^i & \text{otherwise} \end{cases}$$
for $1 \leq l \leq m$; we see from the definition of the set $S(xw_0, i)$ that $\ell(v_{l-1}^i) = \ell(v_l^i) + 1$ if $l$ appears in $\min S(xw_0, i)$. Then we know from [NS2 Lemma 4.2.1] that $y_l^i = w_l^i v_l^0 w_{l-1}^i$ for every $0 \leq l \leq m$.

**Theorem 2.5.6.** Keep the notation and assumptions above. Let $\mu_\bullet = (\mu_w)_{w \in W}$ be the GGMS datum of the extremal MV polytope $P_{x, \lambda}$, i.e., $P_{x, \lambda} = P(\mu_\bullet)$. Let $w \in W$ be such that $w = w_1^i$ for some $i \in R(w_0)$ and $0 \leq l \leq m$. Then, we have $\mu_w = \mu_{w_l^i} = y_l^i \cdot \lambda$.

The following results on extremal MV polytopes and extremal MV cycles play an important role in the proof of Theorem 4.1.1 given in \([4.3]\).

**Lemma 2.5.7.** Keep the notation and assumptions of Theorem 2.5.6. For each $w \in W$ and $j \in I$ with $w < w_s j$, we have either (a) $\mu_{w s_j} = \mu_w$, or (b) $\mu_{w s_j} = w_s j w^{-1} \cdot \mu_w$. Moreover, in both of the cases (a) and (b), we have $\langle \mu_{w s_j}, w \cdot \alpha_j \rangle \geq 0$.

**Proof.** Take $i = (i_1, i_2, \ldots, i_m) \in R(w_0)$ such that $w_{l-1}^i = w$ and $w_l^i = w_s j$ for some $1 \leq l \leq m$; note that $i_l = j$, $\beta_l^i = w_l^i \cdot \alpha_{i_l} = w \cdot \alpha_j$, and hence $s_{\beta_l^i} = w_s j w^{-1}$. Since $\mu_w = \mu_{w_{l-1}^i} = y_l^i \cdot \lambda$ and $\mu_{w s_j} = \mu_{w_l^i} = y_l^i \cdot \lambda$ by Theorem 2.5.6, and since $y_{l-1}^i$ is equal to $y_l^i$ or $s_{\beta_l^i} y_l^i = w_s j w^{-1} y_l^i$ by definition, it follows immediately that either (a) $\mu_{w s_j} = \mu_w$ or (b) $\mu_{w s_j} = w_s j w^{-1} \cdot \mu_w$ holds.

We will show that $\langle \mu_{w s_j}, w \cdot \alpha_j \rangle \geq 0$. First, let us assume that $l$ does not appear in $\min S(xw_0, i)$. Then, we have $y_{l-1}^i = s_{\beta_l^i} y_l^i$ by definition, and hence $\mu_{w_{l-1}^i} = s_{\beta_l^i} \cdot \mu_{w_l^i}$ by Theorem 2.5.6. Also, it follows from the length formula (2.2.1) (or (2.2.4)) that $\mu_{w_l^i} - \mu_{w_{l-1}^i} \in \mathbb{Z}_{\geq 0}(w_{l-1}^i \cdot h_i) = \mathbb{Z}_{\geq 0}(\beta_l^i)^\vee$, where $(\beta_l^i)^\vee$ denotes the coroot corresponding to the root $\beta_l^i$. Combining these, we obtain

$$\mathbb{Z}_{\geq 0}(\beta_l^i)^\vee \ni \mu_{w_l^i} - \mu_{w_{l-1}^i} = \mu_{w_l^i} - s_{\beta_l^i} \cdot \mu_{w_l^i} = \langle \mu_{w_l^i}, \beta_l^i \rangle (\beta_l^i)^\vee,$$

and hence $\langle \mu_{w_l^i}, \beta_l^i \rangle \geq 0$. This implies $\langle \mu_{w s_j}, w \cdot \alpha_j \rangle \geq 0$ since $w_l^i = w_s j$ and $\beta_l^i = w \cdot \alpha_j$.

Next, let us assume that $l$ appears in $\min S(xw_0, i)$. Because $\mu_{w_l^i} = y_l^i \cdot \lambda = w_l^i v_l^0 w_{l-1}^i \cdot \lambda$ by Theorem 2.5.6 and Remark 2.5.3, we see, by noting $w_l^i = w_s j$ and $i_l = j$, that

$$\langle \mu_{w s_j}, w \cdot \alpha_j \rangle = \langle \mu_{w_l^i}, w \cdot \alpha_j \rangle = \langle w_l^i v_l^0 w_{l-1}^i \cdot \lambda, w \cdot \alpha_j \rangle = \langle w_s j v_l^0 w_{l-1}^i \cdot \lambda, w \cdot \alpha_j \rangle = -\langle v_l^0 w_{l-1}^i \cdot \lambda, w \cdot \alpha_j \rangle = -\langle \lambda, w_0(v_l^0)^{-1} \cdot \alpha_{i_l} \rangle.$$ 

Also, since $l$ appears in $\min S(xw_0, i)$ by assumption, we have $v_{l-1}^i = s_{i_l} v_l^i$ with $\ell(v_{l-1}^i) = \ell(v_l^i) + 1$ (see Remark 2.5.3). It follows from the exchange condition that $(v_l^i)^{-1} \cdot \alpha_{i_l}$ is a positive root, and hence $w_0(v_l^i)^{-1} \cdot \alpha_{i_l}$ is a negative root. Therefore, we conclude that

$$\langle \mu_{w s_j}, w \cdot \alpha_j \rangle = -\langle \lambda, w_0(v_l^i)^{-1} \cdot \alpha_{i_l} \rangle \geq 0$$

\[\leq 0\]
since \( \lambda \in X_*(T) \subset \mathfrak{h}_\mathbb{R} \) is a dominant coweight, This proves the lemma. \( \square \)

Let \( G \) be a complex, connected, semisimple algebraic group with Lie algebra \( \mathfrak{g} \). Take \( \lambda \in X_*(T)_+ \) and \( x \in W \) arbitrarily, and let \( \mu_* = (\mu_w)_{w \in W} \) denote the GGMS datum of the extremal MV polytope \( P_{x, \lambda} \in \mathcal{MV}(\lambda) \) of weight \( x \cdot \lambda \), i.e., \( P_{x, \lambda} = P(\mu_*) \); recall from Theorem 2.5.6 that \( \mu_w \in W \cdot \lambda \) for all \( w \in W \). Now, for each \( w \in W \), we consider the irreducible algebraic variety

\[
b^w := \overline{Gr^\lambda \cap S_{\mu_w}^w},
\]

which is the \( \dot{w} \)-translate of the extremal MV cycle \( b_{w^{-1}, \mu_w} \) of weight \( w^{-1} \cdot \mu_w \) since \( Gr^{w^{-1} \cdot \lambda} = Gr^\lambda \) (see Remark 2.3.5); note that \( b^\epsilon = b_{x, \lambda} \) since \( \mu_\epsilon = x \cdot \lambda \).

For each \( j \in I \), we set \( P_j := B \sqcup (B s_j B) \), which is the minimal parabolic subgroup (containing \( B \)) of \( G \) corresponding to \( s_j \in W \). Also, let \( P_j = L_j U_j \) be its Levi decomposition such that \( T \subset L_j \).

**Lemma 2.5.8.** Keep the notation above. For each \( w \in W \) and \( j \in I \) with \( ws_j < w \), we have \( w L_j(O) b^{ws_j} \subset b^w \).

**Proof.** For simplicity of notation, we write \( N_+ \) for \( w(L_j \cap U) \); the root in \( N_+ \) is \( -w \cdot \alpha_j \) by our convention. Because \( \mu_* = (\mu_w)_{w \in W} \) is the GGMS datum of the extremal MV polytope \( P_{x, \lambda} \), it follows from Lemma 2.5.7 that we have either (a) \( \mu_{ws_j} = \mu_w \) or (b) \( \mu_{ws_j} = ws_j w^{-1} \cdot \mu_w \), and that in both of the cases (a) and (b), we have \( \langle \mu_w, w \cdot \alpha_j \rangle \leq 0 \). Consequently, by taking into account Lemma 2.3.4 applied to \( w L_j \subset G \) and \( \mu_w \in X_*(T) \), we deduce from Example 2.3.4 that

\[
N_+(O)[t^{\mu_w}] = w L_j(O)[t^{\mu_w}] \quad \text{(2.5.3)}
\]

Also, by Remark 2.3.3 applied to the extremal MV cycle \( \dot{w}^{-1} \cdot b^w \) of weight \( w^{-1} \cdot \mu_w \), we obtain \( \dot{w}^{-1} \cdot b^w = U(O)[t^{w^{-1} \cdot \mu_w}] \), and hence

\[
b^w = \overline{w U(O)[t^{\mu_w}]} \quad \text{(2.5.4)}
\]

Similarly, we obtain

\[
b^{ws_j} = \overline{ws_j U(O)[t^{\mu_{ws_j}}]} \quad \text{(2.5.5)}
\]

Here we note that \( w L_j U = w U w L_j \) since \( L_j U = UL_j \) and \( s_j \in L_j \). It follows that

\[
w L_j(O) w s_j U(O)[t^{\mu_{ws_j}}] = w U(O) w L_j(O)[t^{\mu_{ws_j}}].
\]

Because \( [t^{\mu_{ws_j}}] = [t^{\mu_w}] \) in case (a), and \( [t^{\mu_{ws_j}}] = \dot{w} s_j \dot{w}^{-1} [t^{\mu_w}] \) in case (b) as above, we deduce that in both of the cases (a) and (b),

\[
w U(O) w L_j(O)[t^{\mu_{ws_j}}] = w U(O) w L_j(O)[t^{\mu_w}].
\]
Thus, we get
\[ wL_j(\mathcal{O})^{w} \mathcal{O}[t^{\mu_w}] = wU(\mathcal{O})^w L_j(\mathcal{O})[t^{\mu_w}] \]
In addition, we have
\[
wU(\mathcal{O})^w L_j(\mathcal{O})[t^{\mu_w}] \subset wU(\mathcal{O})^w U(\mathcal{O})[t^{\mu_w}] = wU(\mathcal{O}) N_+(\mathcal{O})[t^{\mu_w}] \quad \text{by (2.5.3)}
\]
\[
\subset wU(\mathcal{O})^w U(\mathcal{O})[t^{\mu_w}] \quad \text{since } N_+ \subset wU \text{ by definition}
\]
\[
= wU(\mathcal{O})[t^{\mu_w}] = b_w \quad \text{by (2.5.4)}.
\]
Hence we obtain \[ wL_j(\mathcal{O})^{w} \mathcal{O}[t^{\mu_w}] \subset b_w \]. From this, we conclude, by using (2.5.5), that
\[ wL_j(\mathcal{O})^{w} \mathcal{O}[t^{\mu_w}] \subset b_w \].

This proves the lemma. \( \square \)

3 \textbf{N-multiple maps for MV polytopes and their applications.}

As in (the second paragraph of) §2.1, we assume that \( \mathfrak{g} \) is a complex semisimple Lie algebra. Let \( \lambda \in X_*(T) \subset \mathfrak{h}_\mathbb{R} \) be an arbitrary (but fixed) dominant coweight.

3.1 \textbf{N-multiple maps for MV polytopes.} Let \( N \in \mathbb{Z}_{\geq 1} \). For a collection \( \mu_* = (\mu_w)_{w \in W} \) of elements of \( \mathfrak{h}_\mathbb{R} \), we set
\[ N \cdot \mu_* := (N \mu_w)_{w \in W} \].

Also, for a subset \( P \subset \mathfrak{h}_\mathbb{R} \), we set
\[ N \cdot P := \{ Nv \mid v \in P \} \subset \mathfrak{h}_\mathbb{R} \].

The next lemma follows immediately from the definitions.

\textbf{Lemma 3.1.1.} Let \( N \in \mathbb{Z}_{\geq 1} \) be a positive integer.

1. If \( \mu_* = (\mu_w)_{w \in W} \) is a GGMS datum, then \( N \cdot \mu_* \) is also a GGMS datum.
2. If \( \mu_* = (\mu_w)_{w \in W} \) is an MV datum, then \( N \cdot \mu_* \) is also an MV datum.
3. Let \( P = P(\mu_*) \in \mathcal{M} \) be an MV polytope with GGMS datum \( \mu_* \). Then, \( N \cdot P \) is the MV polytope with GGMS datum \( N \cdot \mu_* \), that is, \( N \cdot P = P(N \cdot \mu_*) \). Moreover, if \( P \in \mathcal{M}(\lambda) \), then \( N \cdot P \in \mathcal{M}(N \lambda) \).
Remark 3.1.2. Let \( N \in \mathbb{Z}_{\geq 1} \) be a positive integer. If \( P = P(\mu_\bullet) \) is a pseudo-Weyl polytope with GGMS datum \( \mu_\bullet \), then the set \( N \cdot P \) is identical to the Minkowski sum \( P + P + \cdots + P \) (\( N \) times). Indeed, we see that

\[
N \cdot P = P(N \cdot \mu_\bullet) \quad \text{by Lemma 3.1.1(3)}
\]

\[
= P(\mu_\bullet + \mu_\bullet + \cdots + \mu_\bullet) \quad \text{(see Remark 2.4.1)}
\]

\[
= P(\mu_\bullet) + P(\mu_\bullet) + \cdots + P(\mu_\bullet) \quad \text{by Proposition 2.2.2}
\]

\[
= P + P + \cdots + P.
\]

By Lemma 3.1.1(3), we obtain an injective map \( S_N : \mathcal{MV}(\lambda) \hookrightarrow \mathcal{MV}(N\lambda) \) that sends \( P \in \mathcal{MV}(\lambda) \) to \( N \cdot P \in \mathcal{MV}(N\lambda) \); we call the map \( S_N \) an \( N \)-multiple map. Note that \( S_N(P_\lambda) = P_{N\lambda} \) (see Remark 2.5.3).

Proposition 3.1.3. Let \( N \in \mathbb{Z}_{\geq 1} \). For \( P \in \mathcal{MV}(\lambda) \), we have

\[
\text{wt}(S_N(P)) = N \cdot \text{wt}(P),
\]

\[
S_N(e_j P) = e_j^N(S_N(P)), \quad S_N(f_j P) = f_j^N(S_N(P)) \quad \text{for } j \in I,
\]

\[
\varepsilon_j(S_N(P)) = N\varepsilon_j(P), \quad \varphi_j(S_N(P)) = N\varphi_j(P) \quad \text{for } j \in I,
\]

where it is understood that \( S_N(0) = 0 \).

Proof. Let \( \mu_\bullet = (\mu_w)_{w \in W} \) be the GGMS datum of \( P \in \mathcal{MV}(\lambda) \). It follows from the definition of \( \text{wt} \) and Lemma 3.1.1(3) that \( \text{wt}(P) = \mu_e \) and \( \text{wt}(S_N(P)) = N \cdot \text{wt}(P) = N \mu_e \). Hence we have \( \text{wt}(S_N(P)) = N \cdot \text{wt}(P) \).

Next, let us show that \( \varepsilon_j(S_N(P)) = N\varepsilon_j(P) \) and \( \varphi_j(S_N(P)) = N\varphi_j(P) \) for \( j \in I \). Let \( j \in I \). By Remark 2.4.1, we have \( \mu_{s_j} - \mu_e = \varepsilon_j(P) h_j \). Also, since \( S_N(P) = N \cdot P = P(N \cdot \mu_\bullet) \) by Lemma 3.1.1(3), it follows from Remark 2.4.1 that \( N\mu_{s_j} - N\mu_e = \varepsilon_j(N \cdot P) h_j = \varepsilon_j(S_N(P)) h_j \). Combining these equations, we have

\[
\varepsilon_j(S_N(P)) h_j = N\mu_{s_j} - N\mu_e = N(\mu_{s_j} - \mu_e) = N\varepsilon_j(P) h_j,
\]

which implies that \( \varepsilon_j(S_N(P)) = N\varepsilon_j(P) \). In addition, we have

\[
\varphi_j(S_N(P)) = \langle \text{wt}(S_N(P)), \alpha_j \rangle + \varepsilon_j(S_N(P)) \quad \text{by 2.4.1}
\]

\[
= \langle N \cdot \text{wt}(P), \alpha_j \rangle + N\varepsilon_j(P) \quad \text{by the equations shown above}
\]

\[
= N(\langle \text{wt}(P), \alpha_j \rangle + \varepsilon_j(P))
\]

\[
= N\varphi_j(P) \quad \text{by 2.4.1}.
\]
Finally, let us show that $S_N(e_j P) = e_j^N(S_N(P))$ and $S_N(f_j P) = f_j^N(S_N(P))$ for $j \in I$; we give a proof only for the equality $S_N(e_j P) = e_j^N(S_N(P))$ since the equality $S_N(f_j P) = f_j^N(S_N(P))$ can be shown similarly. Let $j \in I$. First observe that

$$S_N(e_j P) = 0 \iff e_j P = 0 \quad \text{by Remark 2.4.1}$$

$$\iff \varepsilon_j(P) = 0 \quad \text{by the definition of } \varepsilon_j(P)$$

$$\iff e_j^N(S_N(P)) = 0 \quad \text{since } \varepsilon_j(S_N(P)) = N \varepsilon_j(P).$$

Now, assume that $S_N(e_j P) \neq 0$, or equivalently, $e_j P \neq 0$. Recall from the definition of the raising Kashiwara operator $e_j$ that the GGMS datum of the MV polytope $e_j P \in \mathcal{MV}(\lambda)$ is equal to $e_j \mu \ast = (\mu'_w)_{w \in W}$, which is the unique MV datum such that $\mu'_e = \mu_e + h_j$ and $\mu'_w = \mu_w$ for all $w \in W$ with $sjw < w$. Hence we see from Lemma 3.1.1(3) that the GGMS datum of the MV polytope $S_N(e_j P) = N \cdot (e_j P) \in \mathcal{MV}(N\lambda)$ is equal to the unique MV datum $\mu''_w = (\mu''_w)_{w \in W}$ such that $\mu''_e = N\mu_e + Nh_j$ and $\mu''_w = N\mu_w$ for all $w \in W$ with $sjw < w$. Because the GGMS datum of the MV polytope $S_N(P) = N \cdot P \in \mathcal{MV}(N\lambda)$ is equal to $N \cdot \mu \ast = (N\mu_w)_{w \in W}$, we deduce from the definition of the raising Kashiwara operator $e_j$ that the GGMS datum $\mu''_w = (\mu''_w)_{w \in W}$ of $e_j^N(S_N(P)) = e_j^N(N \cdot P) \in \mathcal{MV}(N\lambda)$ also satisfies the condition that $\mu''_e = N\mu_e + Nh_j$ and $\mu''_w = N\mu_w$ for all $w \in W$ with $sjw < w$. Hence, by the uniqueness of such an MV datum, we obtain $\mu''_w = \mu''_w$, which implies that $S_N(e_j P) = P(\mu''_w) = P(\mu''_w) = e_j^N(S_N(P))$. This completes the proof of Proposition 3.1.3.

3.2 Application of $N$-multiple maps. Let $\lambda \in X_+(T) \subset \mathfrak{h}_\mathbb{R}$ be an arbitrary (but fixed) dominant coweight as in §3.1. We denote by $W_\lambda \subset W$ the stabilizer of $\lambda$ in $W$, and by $W_{\min} \subset W$ the set of minimal (length) coset representatives modulo the subgroup $W_\lambda \subset W$.

For a positive integer $N \in \mathbb{Z}_{\geq 1}$, let us denote by

$$K_N : \mathcal{MV}(\lambda) \hookrightarrow \mathcal{MV}(\lambda)^{\otimes N} = \underbrace{\mathcal{MV}(\lambda) \otimes \mathcal{MV}(\lambda) \otimes \cdots \otimes \mathcal{MV}(\lambda)}_{N \times \text{times}}$$

the composite of the $N$-multiple map $S_N : \mathcal{MV}(\lambda) \hookrightarrow \mathcal{MV}(N\lambda)$ with the canonical embedding $G_N : \mathcal{MV}(N\lambda) \hookrightarrow \mathcal{MV}(\lambda)^{\otimes N}$ of crystals that sends the highest weight element $P_{N\lambda}$ of $\mathcal{MV}(N\lambda)$ to the highest weight element $P_{\lambda}^{\otimes N} = P_{\lambda} \otimes P_{\lambda} \otimes \cdots \otimes P_{\lambda}$ ($N$ times) of $\mathcal{MV}(\lambda)^{\otimes N}$.

**Proposition 3.2.1.** Let $\lambda \in X_+(T) \subset \mathfrak{h}_\mathbb{R}$ be a dominant coweight, and let $x \in W_{\min}$. If an MV polytope $P \in \mathcal{MV}(\lambda)$ lies in the Demazure crystal $\mathcal{MV}_x(\lambda)$, then there exist a positive integer $N \in \mathbb{Z}_{\geq 1}$ and minimal coset representatives $x_1, x_2, \ldots, x_N \in W_{\min}^\lambda$ such that

$$\begin{cases}
  x \geq x_k & \text{for all } 1 \leq k \leq N; \\
  K_N(P) = P_{x_1 \lambda} \otimes P_{x_2 \lambda} \otimes \cdots \otimes P_{x_N \lambda}.
\end{cases}$$

(3.2.1)
Remark 3.2.2. Keep the notation and assumption above. A positive integer \( N \in \mathbb{Z}_{\geq 1} \) and minimal coset representatives \( x_1, x_2, \ldots, x_N \in W^\lambda_{\min} \) satisfying the conditions (3.2.1) are, in a sense, determined by the “turning points” and “directions” of the Lakshmibai-Seshadri path of shape \( \lambda \) that corresponds to the MV polytope \( P \in \mathcal{MV}(\lambda) \) under the (inexplicit) bijection via the crystal basis \( B(\lambda) \).

We will explain the remark above more precisely. Let \( B(\lambda) \) denote the set of Lakshmibai-Seshadri (LS for short) paths of shape \( \lambda \), which is endowed with a crystal structure for \( U_q(\mathfrak{g}^\vee) \) by the root operators \( e_j, f_j, j \in I \) (see [Li1] and [Li2] for details). We know from [Kas3, Theorem 4.1] (see also [Kas4, Théorème 8.2.3]) and [J, Corollary 6.4.27] that \( B(\lambda) \) is isomorphic to the crystal basis \( B(\lambda) \) as a crystal for \( U_q(\mathfrak{g}^\vee) \). Thus, we have \( B(\lambda) \cong B(\lambda) \cong \mathcal{MV}(\lambda) \) as crystals for \( U_q(\mathfrak{g}^\vee) \).

Now, take an element \( P \in \mathcal{MV}_d(\lambda) \subset \mathcal{MV}(\lambda) \), and assume that a positive integer \( N \in \mathbb{Z}_{\geq 1} \) and elements \( x_1, x_2, \ldots, x_N \in W^\lambda_{\min} \) satisfy conditions (3.2.1). Consider a piecewise linear, continuous map \( \pi: [0, 1] \to \mathfrak{h}_\mathbb{R} \) by:

\[
\pi(t) = \sum_{l=1}^{k-1} \frac{1}{N} x_l \cdot \lambda + \left(t - \frac{k-1}{N}\right) x_k \cdot \lambda \quad \text{for } t \in \left[\frac{k-1}{N}, \frac{k}{N}\right], \quad 1 \leq k \leq N;
\]

note that for each \( 1 \leq k \leq N \), the vector \( x_k \cdot \lambda \in W \cdot \lambda \subset \mathfrak{h}_\mathbb{R} \) gives the direction of \( \pi \) on the interval \([ (k-1)/N, k/N]\), and the point \( t = k/N \) is a turning point of \( \pi \) if \( x_{k-1} \cdot \lambda \neq x_k \cdot \lambda \). Then, we deduce from the proof of [Kas3, Theorem 4.1] (or [Kas4, Théorème 8.2.3]), together with the commutative diagram (3.2.2) below, that this map \( \pi: [0, 1] \to \mathfrak{h}_\mathbb{R} \) is precisely the LS path of shape \( \lambda \) that corresponds to the \( P \in \mathcal{MV}(\lambda) \) under the isomorphism \( B(\lambda) \cong B(\lambda) \cong \mathcal{MV}(\lambda) \) of crystals. The argument above implies, in particular, that for a fixed positive integer \( N \in \mathbb{Z}_{\geq 1} \), the elements \( x_1, x_2, \ldots, x_N \in W^\lambda_{\min} \) are determined uniquely by the MV polytope \( P \) via the corresponding LS path. Also note that by the definition of LS paths, if a positive integer \( N \in \mathbb{Z}_{\geq 1} \) and elements \( x_1, x_2, \ldots, x_N \in W^\lambda_{\min} \) satisfy conditions (3.2.1), then we necessarily have \( x_1 \geq x_2 \geq \cdots \geq x_N \).

Proof of Proposition 3.2.1. Let \( N \in \mathbb{Z}_{\geq 1} \). We know from [Kas3, Theorem 3.1] and [Kas4, Corollarie 8.1.5] that there exists an injective map \( S_N : \mathcal{B}(\lambda) \hookrightarrow \mathcal{B}(N\lambda) \) which sends the highest weight element \( u_\lambda \in \mathcal{B}(\lambda) \) to the highest weight element \( u_{N\lambda} \in \mathcal{B}(N\lambda) \), and which has the same properties as the \( N \)-multiple map \( S_N : \mathcal{MV}(\lambda) \hookrightarrow \mathcal{MV}(N\lambda) \) given in Proposition 3.1.3. Let us denote by \( K_N : \mathcal{B}(\lambda) \hookrightarrow \mathcal{B}(\lambda)^{\otimes N} \) the composite of \( S_N : \mathcal{B}(\lambda) \hookrightarrow \mathcal{B}(N\lambda) \) with the canonical embedding \( G_N : \mathcal{B}(N\lambda) \hookrightarrow \mathcal{B}(\lambda)^{\otimes N} \) of crystals that sends the highest weight element \( u_{N\lambda} \in \mathcal{B}(N\lambda) \) to the highest weight element \( u_\lambda^{\otimes N} \in \mathcal{B}(\lambda)^{\otimes N} \) (see [Kas3].
Similarly, we have 
\[ K_N(\Psi_\lambda(b)) = K_N(\Psi_\lambda(f_s u_\lambda)) = K_N(f_s P_\lambda) = G_N(S_N(f_s P_\lambda)) = G_N(f_s^N P_{\lambda N}) \quad \text{by Proposition 3.1.3} \]

Thus, we obtain \( K_N(\Psi_\lambda(b)) = \Psi_\lambda^{\otimes N}(K_N(b)) \) for all \( b \in \mathcal{B}(\lambda) \).

Now, let \( P \in \mathcal{MV}(\lambda) \). Applying [Kas4, Proposition 8.3.2] to \( \Psi_\lambda^{-1}(P) \in \mathcal{B}(\lambda) \), we see that if \( N \in \mathbb{Z}_{\geq 1} \) contains “sufficiently many” divisors, then
\[ K_N(\Psi_\lambda^{-1}(P)) = u_{x_1,\lambda} \otimes u_{x_2,\lambda} \otimes \cdots \otimes u_{x_N,\lambda} \quad (3.2.3) \]

for some \( x_1, x_2, \ldots, x_N \in W_{\min}^\lambda \). Then, we have
\[ K_N(P) = \Psi_\lambda^{\otimes N}(K_N(\Psi_\lambda^{-1}(P))) \quad \text{by (3.2.2)} \]
\[ = \Psi_\lambda^{\otimes N}(u_{x_1,\lambda} \otimes u_{x_2,\lambda} \otimes \cdots \otimes u_{x_N,\lambda}) \]
\[ = P_{x_1,\lambda} \otimes P_{x_2,\lambda} \otimes \cdots \otimes P_{x_N,\lambda}. \]

It remains to show that \( x \geq x_k \) for every \( 1 \leq k \leq N \). In view of [Kas1, Proposition 4.4] (see also [Kas4, §9.1]), it suffices to show that \( u_{x_k,\lambda} \in \mathcal{B}_x(\lambda) \) for every \( 1 \leq k \leq N \). Let \( x = s_{j_1} s_{j_2} \cdots s_{j_r} \) be a reduced expression of \( x \in W \). We know from [Kas4, Proposition 9.1.3(2)] that
\[ \mathcal{B}_x(\lambda) = \left\{ f_{c_1}^{j_1} f_{c_2}^{j_2} \cdots f_{c_r}^{j_r} u_\lambda \mid c_1, c_2, \ldots, c_r \in \mathbb{Z}_{\geq 0} \right\} \setminus \{0\} \quad (3.2.4) \]

Since \( P \in \mathcal{MV}_x(\lambda) \) by our assumption, \( \Psi_\lambda^{-1}(P) \) is contained in \( \mathcal{B}_x(\lambda) \), and hence \( \Psi_\lambda^{-1}(P) \) can be written as:
\[ \Psi_\lambda^{-1}(P) = f_{c_1}^{j_1} f_{c_2}^{j_2} \cdots f_{c_r}^{j_r} u_\lambda \quad \text{for some } c_1, c_2, \ldots, c_r \in \mathbb{Z}_{\geq 0}. \]
Here we recall from §3.3 Main results.

In this subsection, we prove the following theorem, by using a polytopal estimate (Theorem 4.1.1 below) of tensor products of MV polytopes. We use the definition of crystals that sends \( f \) to \( f^{\lambda} \) for some \( b, c, \lambda, \). Therefore, we have 

\[
K_N(P^{\lambda} \cdot P) = K_N(f_{j_1}^{c_1} f_{j_2}^{c_2} \cdots f_{j_r}^{c_r} u_\lambda) = G_N(S_N(f_{j_1}^{c_1} f_{j_2}^{c_2} \cdots f_{j_r}^{c_r} u_\lambda)) = G_N(f_{j_1}^{Nc_1} f_{j_2}^{Nc_2} \cdots f_{j_r}^{Nc_r} u_{\lambda}^{\otimes N}).
\]

It follows from the tensor product rule for crystals that

\[
f_{j_1}^{Nc_1} f_{j_2}^{Nc_2} \cdots f_{j_r}^{Nc_r} u_\lambda^{\otimes N} = (f_{j_1}^{b_{1,1}} f_{j_2}^{b_{1,2}} \cdots f_{j_r}^{b_{1,r}} u_\lambda) \otimes (f_{j_1}^{b_{2,1}} f_{j_2}^{b_{2,2}} \cdots f_{j_r}^{b_{2,r}} u_\lambda) \otimes \cdots \otimes (f_{j_1}^{b_{N,1}} f_{j_2}^{b_{N,2}} \cdots f_{j_r}^{b_{N,r}} u_\lambda)
\]

for some \( b_{k,t} \in \mathbb{Z}_{\geq 0}, 1 \leq k \leq N, 1 \leq t \leq r, \) with \( \sum_{k=1}^{N} b_{k,t} = Nc_t \) for each \( 1 \leq t \leq r. \)

Combining these equalities with (3.2.3), we obtain

\[
u_{x_1, \lambda} \otimes \nu_{x_2, \lambda} \otimes \cdots \otimes \nu_{x_N, \lambda} = K_N(P^{\lambda}) = (f_{j_1}^{b_{1,1}} f_{j_2}^{b_{1,2}} \cdots f_{j_r}^{b_{1,r}} u_\lambda) \otimes (f_{j_1}^{b_{2,1}} f_{j_2}^{b_{2,2}} \cdots f_{j_r}^{b_{2,r}} u_\lambda) \otimes \cdots \otimes (f_{j_1}^{b_{N,1}} f_{j_2}^{b_{N,2}} \cdots f_{j_r}^{b_{N,r}} u_\lambda),
\]

from which it follows that \( \nu_{x_k, \lambda} = f_{j_1}^{b_{k,1}} f_{j_2}^{b_{k,2}} \cdots f_{j_r}^{b_{k,r}} u_\lambda \) for \( 1 \leq k \leq N. \) This implies that \( \nu_{x_k, \lambda} \in B_\lambda(\lambda) \) for each \( 1 \leq k \leq N \) since \( f_{j_1}^{b_{k,1}} f_{j_2}^{b_{k,2}} \cdots f_{j_r}^{b_{k,r}} u_\lambda \in B_\lambda(\lambda) \) by (3.2.3). Thus, we have proved Proposition 3.3.1.

\[\square\]

3.3 Main results. In this subsection, we prove the following theorem, by using a polytopal estimate (Theorem 4.1.1 below) of tensor products of MV polytopes. We use the setting of 3.2.

Theorem 3.3.1. Let \( \lambda \in X_*(T) \subset h_\mathbb{R} \) be a dominant coweight, and let \( x \in W_\lambda^{\min}. \) If a positive integer \( N \in \mathbb{Z}_{\geq 1} \) and minimal coset representatives \( x_1, x_2, \ldots, x_N \in W_\lambda^{\min} \) satisfy the condition (3.2.3) in Proposition 3.2.1, then

\[ N \cdot P \subseteq P_{x_1, \lambda} + P_{x_2, \lambda} + \cdots + P_{x_N, \lambda}, \]

where \( P_{x_1, \lambda} + P_{x_2, \lambda} + \cdots + P_{x_N, \lambda} \) is the Minkowski sum of the extremal MV polytopes \( P_{x_1, \lambda}, P_{x_2, \lambda}, \ldots, P_{x_N, \lambda}. \)

Proof. By our assumption, we have

\[
\begin{cases}
x \geq x_k \text{ for all } 1 \leq k \leq N; \\
K_N(P) = G_N(N \cdot P) = P_{x_1, \lambda} \otimes P_{x_2, \lambda} \otimes \cdots \otimes P_{x_N, \lambda}.
\end{cases}
\]

Here we recall from 3.2 that \( G_N : \mathcal{MV}(N\lambda) \to \mathcal{MV}(\lambda)^{\otimes N} \) denotes the canonical embedding of crystals that sends \( P_{N\lambda} \in \mathcal{MV}(N\lambda) \) to \( P_{\lambda}^{\otimes N} \in \mathcal{MV}(\lambda)^{\otimes N}. \) Therefore, by using Theorem 4.1.1 (or rather, Corollary 4.1.3) successively, we can show that

\[ N \cdot P \subseteq P_{x_1, \lambda} + P_{x_2, \lambda} + \cdots + P_{x_N, \lambda}. \]

This completes the proof of Theorem 3.3.1.

\[\square\]
This theorem, together with Proposition 3.2.1 yields Theorem 1 in the Introduction. As an immediate consequence, we can provide an affirmative answer to a question posed in [NS2 §4.6].

**Corollary 3.3.2.** Let $\lambda \in X_*(T) \subset h_\mathbb{R}$ be a dominant coweight, and let $x \in W$. All the MV polytopes lying in the Demazure crystal $MV_x(\lambda)$ are contained (as sets) in the extremal MV polytope $P_{x,\lambda} = \text{Conv}(W_{\leq x} \cdot \lambda)$ of weight $x \cdot \lambda$. Namely, for all $P \in MV_x(\lambda)$, there holds

$$P \subset P_{x,\lambda} = \text{Conv}(W_{\leq x} \cdot \lambda).$$

**Remark 3.3.3.** (1) The assertion of Theorem 3.3.1 is not obvious, as explained in [NS2 Remark 4.6.2 and Example 4.6.3].

(2) The converse statement fails to hold; see [NS2 Remark 4.6.1].

**Proof of Corollary 3.3.2.** We know from Remark 2.1.1 that if $x, y \in W$ satisfies $x \cdot \lambda = y \cdot \lambda$, then $MV_x(\lambda) = MV_y(\lambda)$ and $P_{x,\lambda} = P_{y,\lambda}$. Hence we may assume that $x \in W^\lambda_{\min}$. Let us take an arbitrary $P \in MV_x(\lambda)$. By Proposition 3.2.1 there exist $N \in \mathbb{Z}_{\geq 1}$ and $x_1, x_2, \ldots, x_N \in W^\lambda_{\min}$ satisfying the condition (3.2.1). Also, for each $1 \leq k \leq N$, it follows from Proposition 2.5.1 and the inequality $x \geq x_k$ that

$$P_{x_k,\lambda} = \text{Conv}(W_{\leq x_k} \cdot \lambda) \subset \text{Conv}(W_{\leq x} \cdot \lambda) = P_{x,\lambda}.$$  (3.3.1)

Therefore, we have

$$N \cdot P \subset P_{x_1,\lambda} + P_{x_2,\lambda} + \cdots + P_{x_N,\lambda} \quad \text{by Theorem 3.3.1}$$

$$\subset \underbrace{P_{x,\lambda} + P_{x,\lambda} + \cdots + P_{x,\lambda}}_{N \text{ times}} \quad \text{by (3.3.1)}$$

$$= N \cdot P_{x,\lambda} \quad \text{by Remark 3.1.2}$$

Consequently, we obtain $N \cdot P \subset N \cdot P_{x,\lambda}$, which implies that $P \subset P_{x,\lambda}$. This proves Corollary 3.3.2.

---

### 4 Polytopal estimate of tensor products of MV polytopes.

The aim of this section is to state and prove a polytopal estimate of tensor products of MV polytopes.

**4.1 Polytopal estimate.** As in (the second paragraph of) §2.1 we assume that $\mathfrak{g}$ is a complex semisimple Lie algebra. Let $\lambda_1, \lambda_2 \in X_*(T) \subset h_\mathbb{R}$ be dominant coweights. Because $MV(\lambda) \cong B(\lambda)$ as crystals for every dominant coweight $\lambda \in X_*(T) \subset h_\mathbb{R}$, the tensor product...
$\mathcal{MV}(\lambda_2) \otimes \mathcal{MV}(\lambda_1)$ of the crystals $\mathcal{MV}(\lambda_1)$ and $\mathcal{MV}(\lambda_2)$ decomposes into a disjoint union of connected components as follows:

$$\mathcal{MV}(\lambda_2) \otimes \mathcal{MV}(\lambda_1) \cong \bigoplus_{\lambda \in X_+(T)} \mathcal{MV}(\lambda)^{\otimes m^\lambda_{\lambda_1,\lambda_2}},$$

where $m^\lambda_{\lambda_1,\lambda_2} \in \mathbb{Z}_{\geq 0}$ denotes the multiplicity of $\mathcal{MV}(\lambda)$ in $\mathcal{MV}(\lambda_2) \otimes \mathcal{MV}(\lambda_1)$. For each dominant coweight $\lambda \in X_+(T) \subset \mathfrak{h}_\mathbb{R}$ such that $m^\lambda_{\lambda_1,\lambda_2} \geq 1$, we take (and fix) an arbitrary embedding $\iota_\lambda : \mathcal{MV}(\lambda) \hookrightarrow \mathcal{MV}(\lambda_2) \otimes \mathcal{MV}(\lambda_1)$ of crystals that maps $\mathcal{MV}(\lambda)$ onto a connected component of $\mathcal{MV}(\lambda_2) \otimes \mathcal{MV}(\lambda_1)$, which is isomorphic to $\mathcal{MV}(\lambda)$ as a crystal.

**Theorem 4.1.1.** Keep the notation above. Let $P \in \mathcal{MV}(\lambda)$, and write $\iota_\lambda(P) \in \mathcal{MV}(\lambda_2) \otimes \mathcal{MV}(\lambda_1)$ as: $\iota_\lambda(P) = P_2 \otimes P_1$ for some $P_1 \in \mathcal{MV}(\lambda_1)$ and $P_2 \in \mathcal{MV}(\lambda_2)$. We assume that the MV polytope $P_2 \in \mathcal{MV}(\lambda_2)$ is an extremal MV polytope $P_{x,\lambda_2}$ for some $x \in W$. Then, we have

$$P \subset P_1 + P_2,$$

where $P_1 + P_2$ is the Minkowski sum of the MV polytopes $P_1 \in \mathcal{MV}(\lambda_1)$ and $P_2 \in \mathcal{MV}(\lambda_2)$.

**Remark 4.1.2.** It should be mentioned that in the theorem above, the $\iota_\lambda(P)$ may lie in an arbitrary connected component of $\mathcal{MV}(\lambda_2) \otimes \mathcal{MV}(\lambda_1)$ that is isomorphic to $\mathcal{MV}(\lambda)$ as a crystal.

The proof of this theorem will be given in §4.3 below; it seems likely that this theorem still holds without the assumption of extremality on the MV polytope $P_2 \in \mathcal{MV}(\lambda_2)$.

For dominant coweights $\lambda_1, \lambda_2 \in X_+(T) \subset \mathfrak{h}_\mathbb{R}$, there exists a unique embedding

$$\iota_{\lambda_1,\lambda_2} : \mathcal{MV}(\lambda_1 + \lambda_2) \hookrightarrow \mathcal{MV}(\lambda_1) \otimes \mathcal{MV}(\lambda_2)$$

of crystals, which maps $\mathcal{MV}(\lambda_1 + \lambda_2)$ onto the unique connected component of $\mathcal{MV}(\lambda_1) \otimes \mathcal{MV}(\lambda_2)$ (called the Cartan component) that is isomorphic to $\mathcal{MV}(\lambda_1 + \lambda_2)$ as a crystal; note that $\iota_{\lambda_1,\lambda_2}(P_{\lambda_1+\lambda_2}) = P_{\lambda_1} \otimes P_{\lambda_2}$. Applying Theorem 4.1.1 to the case $\lambda = \lambda_1 + \lambda_2$, we obtain the following corollary; notice that the ordering of the tensor factors $\mathcal{MV}(\lambda_1), \mathcal{MV}(\lambda_2)$ is reversed.

**Corollary 4.1.3.** Let $\lambda_1, \lambda_2 \in X_+(T) \subset \mathfrak{h}_\mathbb{R}$ be dominant coweights. Let $P \in \mathcal{MV}(\lambda_1 + \lambda_2)$, and write $\iota_{\lambda_1,\lambda_2}(P) \in \mathcal{MV}(\lambda_1) \otimes \mathcal{MV}(\lambda_2)$ as: $\iota_{\lambda_1,\lambda_2}(P) = P_1 \otimes P_2$ for some $P_1 \in \mathcal{MV}(\lambda_1)$ and $P_2 \in \mathcal{MV}(\lambda_2)$. We assume that the MV polytope $P_1 \in \mathcal{MV}(\lambda_1)$ is an extremal MV polytope $P_{x,\lambda_1}$ for some $x \in W$. Then, we have

$$P \subset P_1 + P_2.$$
The following is a particular case in which the equality holds in (4.1.2).

**Proposition 4.1.4.** Let \( \lambda_1, \lambda_2 \in X_*(T) \subset \mathfrak{h}_\mathbb{R} \) be dominant coweights, and let \( x \in W \). Then, 
\[
\ell_{\lambda_1,\lambda_2}(P_{x-(\lambda_1+\lambda_2)}) = P_{x-\lambda_1} \otimes P_{x-\lambda_2}, \quad \text{and} \quad P_{x-(\lambda_1+\lambda_2)} = P_{x-\lambda_1} + P_{x-\lambda_2}.
\]

**Proof.** First we show the equality \( \ell_{\lambda_1,\lambda_2}(P_{x-(\lambda_1+\lambda_2)}) = P_{x-\lambda_1} \otimes P_{x-\lambda_2} \) (which may be well-known to experts) by induction on \( \ell(x) \). If \( x = e \), then we have \( \ell_{\lambda_1,\lambda_2}(P_{\lambda_1+\lambda_2}) = P_{\lambda_1} \otimes P_{\lambda_2} \) as mentioned above. Assume now that \( \ell(x) > 0 \). We take \( j \in I \) such that \( \ell(s_jx) < \ell(x) \). Set 
\[
k := \langle s_jx \cdot (\lambda_1 + \lambda_2), \alpha_j \rangle, \quad k_1 := \langle s_jx \cdot \lambda_1, \alpha_j \rangle, \quad k_2 := \langle s_jx \cdot \lambda_2, \alpha_j \rangle;
\]
and note that we have \( k = k_1 + k_2 \), with \( k_1, k_2, k \in \mathbb{Z}_{\geq 0} \), since \( \ell(s_jx) < \ell(x) \). We see from [Kas4, Lemme 8.3.1] that \( f_j^k P_{s_jx-(\lambda_1+\lambda_2)} \) is equal to \( P_{x-(\lambda_1+\lambda_2)} \). Hence we have
\[
\ell_{\lambda_1,\lambda_2}(P_{x-(\lambda_1+\lambda_2)}) = \ell_{\lambda_1,\lambda_2}(f_j^k P_{s_jx-(\lambda_1+\lambda_2)}) = f_j^k \ell_{\lambda_1,\lambda_2}(P_{s_jx-(\lambda_1+\lambda_2)}) \]
\[
= f_j^k (P_{s_jx-\lambda_1} \otimes P_{s_jx-\lambda_2}) \quad \text{by the induction hypothesis.}
\]
Here, by the tensor product rule for crystals,
\[
f_j^k (P_{s_jx-\lambda_1} \otimes P_{s_jx-\lambda_2}) = (f_j^{k_1} P_{s_jx-\lambda_1}) \otimes (f_j^{k_2} P_{s_jx-\lambda_2})
\]
for some \( l_1, l_2 \in \mathbb{Z}_{\geq 0} \) with \( k = l_1 + l_2 \). It follows from [Kas4, Lemme 8.3.1] that \( l_1 = k_1 \) and \( l_2 = k_2 \). Therefore, we deduce that
\[
\ell_{\lambda_1,\lambda_2}(P_{x-(\lambda_1+\lambda_2)}) = f_j^k (P_{s_jx-\lambda_1} \otimes P_{s_jx-\lambda_2}) = (f_j^{k_1} P_{s_jx-\lambda_1}) \otimes (f_j^{k_2} P_{s_jx-\lambda_2})
\]
\[
= P_{x-\lambda_1} \otimes P_{x-\lambda_2} \quad \text{by [Kas4, Lemme 8.3.1].}
\]

This proves the first equality.

Next we show the equality \( P_{x-(\lambda_1+\lambda_2)} = P_{x-\lambda_1} + P_{x-\lambda_2} \). Let us denote by
\[
\mu_{x-(\lambda_1+\lambda_2)} = (\mu_{x-(\lambda_1+\lambda_2)})_{w \in W}, \quad \mu_{x-\lambda_1} = (\mu_{x-\lambda_1})_{w \in W}, \quad \mu_{x-\lambda_2} = (\mu_{x-\lambda_2})_{w \in W}
\]
the GGMS data of the extremal MV polytopes \( P_{x-(\lambda_1+\lambda_2)} \in \mathcal{MV}(\lambda_1 + \lambda_2), P_{x-\lambda_1} \in \mathcal{MV}(\lambda_1), \) and \( P_{x-\lambda_2} \in \mathcal{MV}(\lambda_2) \), respectively. We verify the equality
\[
\mu_{x-(\lambda_1+\lambda_2)} = \mu_{x-\lambda_1} + \mu_{x-\lambda_2} \quad \text{for every } w \in W.
\]

(4.1.3)

Let \( w \in W \), and take \( i \in R(w_0) \) such that \( w = w_i^l \) for some \( 0 \leq l \leq m \). Then it follows from Theorem 2.5.6 that
\[
\mu_{w-\lambda_1} = \mu_{w-\lambda_2} = y_i^l \cdot \lambda_1, \quad \mu_{w-\lambda_2} = y_i^l \cdot \lambda_2,
\]
and
\[
\mu_{w-\lambda_1} = \mu_{w-\lambda_2} = y_i^l \cdot (\lambda_1 + \lambda_2),
\]
for some \( 0 \leq l \leq m \). Then it follows from Theorem 2.5.6 that
\[ y_i^l \text{ is defined as } (2.5.1); \text{ recall from Remark 2.5.4 that } y_i^l \in W \text{ does not depend on the dominant coweights } \lambda_1 + \lambda_2, \lambda_1, \text{ and } \lambda_2. \text{ Therefore, we deduce that } \]
\[ \mu_w^{x(\lambda_1 + \lambda_2)} = y_i^l \cdot (\lambda_1 + \lambda_2) = y_i^l \cdot \lambda_1 + y_i^l \cdot \lambda_2 = \mu_w^{x\lambda_1} + \mu_w^{x\lambda_2}, \]
as desired. Hence it follows that
\[ \mu_x^{(\lambda_1 + \lambda_2)} = (\mu_w^{x(\lambda_1 + \lambda_2)})_{w \in W} = (\mu_w^{x\lambda_1} + \mu_w^{x\lambda_2})_{w \in W} = \mu_x^{\lambda_1} + \mu_x^{\lambda_2}. \quad (4.1.4) \]

Consequently, we have
\[ P_{x\lambda_1} + P_{x\lambda_2} = P(\mu_x^{\lambda_1}) + P(\mu_x^{\lambda_2}) = P(\mu_x^{\lambda_1} + \mu_x^{\lambda_2}) \quad \text{by Proposition 2.2.2} \]
\[ = P(\mu_x^{(\lambda_1 + \lambda_2)}) \quad \text{by (4.1.4)} \]
\[ = P_{x(\lambda_1 + \lambda_2)}. \]
This proves the second equality, thereby completes the proof of the proposition.

4.2 Reformation of Braverman-Gaitsgory’s result on tensor products. In this subsection, we revisit results of Braverman-Gaitsgory on tensor products of highest weight crystals, and provide a reformation of it, which is needed in our proof of Theorem 4.1.1 given in \S 4.3.

We now recall the construction of certain twisted product varieties. Let \( G \) be a complex, connected, reductive algebraic group as in (the beginning of) \( \S 2.3 \). The twisted product variety \( G_{\mathcal{E}} \times G_{\mathcal{E}} \) is defined to be the quotient space \( G(\mathcal{K}) \times G(\mathcal{O}) \mathcal{E} \times \mathcal{E} = G(\mathcal{K}) \times G(\mathcal{O}) \) / \( \sim \), where \( \sim \) is an equivalence relation on \( G(\mathcal{K}) \times G(\mathcal{E}) \) given by: \( (a, bG(\mathcal{O})) \sim (ah^{-1}, hbG(\mathcal{O})) \) for \( a, b \in G(\mathcal{K}), \) and \( h \in G(\mathcal{O}) \); for \( (a, bG(\mathcal{O})) \in G(\mathcal{K}) \times G(\mathcal{E}), \) we denote by \( [(a, bG(\mathcal{O}))] \) the equivalence class of \( (a, bG(\mathcal{O})) \). This variety can be thought of as a fibration over \( G_{\mathcal{E}} \) (the first factor) with its typical fiber \( G_{\mathcal{E}} \) (the second factor). Let \( \pi_1 : G_{\mathcal{E}} \times G_{\mathcal{E}} \to G_{\mathcal{E}}, \) \( [(a, bG(\mathcal{O}))] \mapsto aG(\mathcal{O}), \) be the projection onto the first factor, and \( m : G_{\mathcal{E}} \times G_{\mathcal{E}} \to G_{\mathcal{E}}, \) \( [(a, bG(\mathcal{O}))] \mapsto abG(\mathcal{O}), \) the multiplication map. If \( X \subset G_{\mathcal{E}} \) is an algebraic subvariety and \( Y \subset G_{\mathcal{E}} \) is an algebraic subvariety that is stable under the left \( G(\mathcal{O}) \)-action on \( G_{\mathcal{E}}, \) then we can form an algebraic subvariety
\[ X \times Y := \tilde{X} \times G(\mathcal{E}) Y \subset G_{\mathcal{E}} \mathcal{E} G_{\mathcal{E}}, \]
where \( \tilde{X} := \pi_1^{-1}(X) \) is the pullback of \( X \subset G_{\mathcal{E}} = G(\mathcal{K}) / G(\mathcal{O}) \) to \( G(\mathcal{K}). \)
Proposition 4.2.1 ([Lu1], [MV2 Lemma 4.4]). The multiplication map \( m : \Gr \times \Gr \to \Gr \), when restricted to \( \Gr^{\lambda_1} \times \Gr^{\lambda_2} \) for \( \lambda_1, \lambda_2 \in X_*(T)_+ \), is projective, birational, and semi-small with respect to the stratification by \( G(O) \)-orbits. In particular, for \( \lambda_1, \lambda_2 \in X_*(T)_+ \) and \( \lambda \in X_*(T)_+ \),

\[
m^{-1}(\Gr^\lambda) \cap \left( \Gr^{\lambda_1} \times \Gr^{\lambda_2} \right) \neq \emptyset \quad \text{if and only if} \quad \lambda_1 + \lambda_2 \geq \lambda;
\]

if \( \lambda_1 + \lambda_2 \geq \lambda \), then

\[
\dim \left( m^{-1}(\Gr^\lambda) \cap \left( \Gr^{\lambda_1} \times \Gr^{\lambda_2} \right) \right) \leq \dim \Gr^\lambda + (\lambda_1 + \lambda_2 - \lambda, \rho).
\]

Let us introduce another kind of twisted products. For each \( \nu_1, \nu_2 \in X_*(T) \) and \( w \in W \), we define \( S^w_{\nu_1, \nu_2} \) to be the quotient space

\[
^wU(K)t^{\nu_1} \times ^wU(O)wU(K)[t^{\nu_2}] = \left( ^wU(K)t^{\nu_1} \times ^wU(K)[t^{\nu_2}] \right) / \sim,
\]

where \( \sim \) is an equivalence relation on \( ^wU(K)t^{\nu_1} \times ^wU(K)[t^{\nu_2}] \) given by: \( (a, bG(O)) \sim (au^{-1}, ubG(O)) \) for \( a \in ^wU(K)t^{\nu_1} \subset G(K), b \in ^wU(K)[t^{\nu_2}] \subset Gr \), and \( u \in ^wU(O) \subset G(O) \).

Since \( ^wU(O) = G(O) \cap ^wU(K) \) and \( t^\nu(^wU(K)) = (^wU(K))t^\nu \) for \( \nu \in X_*(T) \), we have a canonical embedding

\[
S^w_{\nu_1, \nu_2} \hookrightarrow G(K) \times ^G(O) \Gr = \Gr \times \Gr.
\]

Lemma 4.2.2. For each \( \nu \in X_*(T) \) and \( w \in W \), we have

\[
m^{-1}(S^w_\nu) = \bigsqcup_{\nu_1, \nu_2 \in X_*(T)} S^w_{\nu_1, \nu_2}
\]

under the canonical embedding (4.2.1).

Proof. It follows from the Iwasawa decomposition \( \Gr = \bigsqcup_{\nu \in X_*(T)} S^w_\nu \) that

\[
\Gr \times \Gr = \bigsqcup_{\nu \in X_*(T)} \pi^{-1}(S^w_\nu),
\]

where \( \pi^{-1}(S^w_\nu) = ^wU(K)_t^{\nu_1}G(O) \times ^G(O) \Gr \). Therefore it suffices to show that

\[
\pi^{-1}(S^w_\nu) \cap m^{-1}(S^w_\nu) = S^w_{\nu_1, \nu_2 - \nu_1} \quad \text{for each} \quad \nu_1 \in X_*(T).
\]

Now, for each \( \nu_1 \in X_*(T) \), let us take \( [y] \in \pi^{-1}(S^w_\nu) \cap m^{-1}(S^w_\nu) \), where \( y \in ^wU(K)t^{\nu_1}G(O) \times \Gr \), and write it as: \( y = (u_1 t^{\nu_1} g_1, g_2 G(O)) \) for \( u_1 \in ^wU(K), g_1 \in G(O), \) and \( g_2 \in G(K) \). Since \( m([y]) = u_1 t^{\nu_1} g_1 g_2 G(O) \in S^w_\nu \), we have

\[
g_1 g_2 G(O) \in (u_1 t^{\nu_1})^{-1}S^w_\nu = ^wU(K)t^{\nu_2 - \nu_1}G(O).
\]

26
Consequently, using the equivalence relation \( \sim \) on \( G(K) \times G r \), we see that

\[
[y] = [(u_1 t^{\mu_1} g_1, g_2 G(O))] = [(u_1 t^{\mu_1}, g_1 g_2 G(O))] = [(u_1 t^{\mu_1}, u_2 t^{\mu_2} G(O))]
\]

for some \( u_2 \in wU(K) \). This implies that

\[
[y] \in wU(K)t^{\mu_1} \times wU(O) \cong S_{(\mu_1, \nu_1)}^w \cap \pi^{-1}(S_{\nu_1}^w) \subset S_{(\nu_1, \nu_1)}^w.
\]

and hence \( \pi_1^{-1}(S_{\nu_1}^w) \cap m^{-1}(S_{\nu_1}^w) \subset S_{(\nu_1, \nu_1)}^w \). The opposite inclusion is obvious. Thus, we obtain

\[
\pi_1^{-1}(S_{\nu_1}^w) \cap m^{-1}(S_{\nu_1}^w) = S_{(\nu_1, \nu_1)}^w.
\]

This proves the lemma.

For \( \nu_1, \nu_2 \in X_*(T) \), we set \( S_{\nu_1, \nu_2} := S_{\nu_1}^w \). If we take (and fix) an element \( t \in T(\mathbb{R}) \) such that

\[
\lim_{k \to \infty} Ad(t^k)u = e \quad \text{for all} \quad u \in U,
\]

then we have (by \[MV2\] Eq.(3.5))

\[
S_{\nu} = \left\{ [y] \in Gr \mid \lim_{k \to \infty} t^k[y] = [t^\nu] \right\} \quad \text{for} \quad \nu \in X_*(T).
\]

From this, by using Lemma 4.2.2, we have

\[
S_{\nu_1, \nu_2} = \left\{ [y] \in Gr \times Gr \mid \lim_{k \to \infty} t^k[y] = (t^{\nu_1}, t^{\nu_2}) \right\} \quad \text{for} \quad \nu_1, \nu_2 \in X_*(T); \tag{4.2.2}
\]

in particular, these strata of \( Gr \times Gr \) are simply-connected.

For each \( \nu \in X_*(T) \) and \( w \in W \), we set

\[
S_{\nu}^w := wU(K)t^\nu(wU(O))/wU(O),
\]

which is canonically isomorphic to \( wU(K)t^\nu G(O)/G(O) = S_{\nu}^w \) since \( wU(K) \cap G(O) = wU(O) \); note that \( wU(K)t^\nu(wU(O)) = wU(K)t^\nu \) since \( t^\nu(wU(K)) = (wU(K))t^\nu \). Also, for a subset \( X \subset Gr \), we define the intersection \( X \cap S_{\nu}^w \) to be the image of \( X \cap S_{\nu}^w \subset S_{\nu}^w \) under the identification \( S_{\nu}^w = S_{\nu}^w \) above.

In the sequel, for an algebraic variety \( X \), we denote by Irr(\( X \)) the set of irreducible components of \( X \). Let \( \lambda \in X_*(T)_+ \) and \( \nu \in X_*(T) \) be such that \( Gr^\lambda \cap S_{\nu} \neq \emptyset \), i.e., \( \nu \in \Omega(\lambda) \); note that \( Gr^\lambda \cap S_{\nu} \neq \emptyset \) if and only if \( Gr^\lambda \cap S_{w^{-1}\nu} \neq \emptyset \), i.e., \( w^{-1} \cdot \nu \in \Omega(\lambda) \) for each \( w \in W \). Let us take an arbitrary \( w \in W \). Because \( Gr^\lambda \cap S_{\nu}^w = \hat{w}(Gr^\lambda \cap S_{w^{-1}\nu}^w) \) for \( w \in W \), we have a bijection

\[
\text{Irr}(\hat{w}(Gr^\lambda \cap S_{w^{-1}\nu}^w)) \to \text{Irr}(\hat{w}(Gr^\lambda \cap S_{\nu}^w)), \quad b \mapsto \hat{w}b.
\]

Thus, each element of \( \text{Irr}(\hat{w}(Gr^\lambda \cap S_{\nu}^w)) \) can be written in the form \( \hat{w}b \) for a unique MV cycle \( b \in Z(\lambda)_{\nu} \). The variety \( b^w \) defined by (2.5.2) is a special case of such elements, in which \( b \in Z(\lambda) \) is an extremal MV cycle with GGMS datum \( \mu_\nu \) and \( \nu = \mu_\nu \).
Lemma 4.2.3. With the notation as above, let us take an arbitrary element 
\[ \hat{\nu} \in \text{Irr} \left( \mathcal{G}_r^\lambda \cap S_{\nu}^w \right), \text{ where } \nu \in \mathcal{Z}(\lambda)_{w-1} \nu. \]

Then, the intersection \( \hat{\nu} \mathcal{S}_w \) (and hence \( \hat{\nu} \mathcal{S}_w \)) is stable under the action of \( \mathcal{U}(\mathcal{O}) \) for all \( \gamma \in \mathcal{X}_s(T) \).

Proof. By definition, each MV cycle is an irreducible component of the (Zariski-) closure of the intersection of a \( \mathcal{G}(\mathcal{O}) \)-orbit and a \( \mathcal{U}(\mathcal{K}) \)-orbit. Since \( \mathcal{U}(\mathcal{O}) = \mathcal{G}(\mathcal{O}) \cap \mathcal{U}(\mathcal{K}) \) is connected, such an irreducible component is stable under the action of \( \mathcal{U}(\mathcal{O}) \). Therefore, the variety \( \hat{\nu} \mathcal{S}_w \) (and hence its intersection with an arbitrary \( \mathcal{U}(\mathcal{K}) \)-orbit) is stable under the action of \( \mathcal{U}(\mathcal{O}) \). This proves the lemma. \( \square \)

Let \( \lambda_1, \lambda_2 \in \mathcal{X}_s(T)_+ \) and \( \nu_1, \nu_2 \in \mathcal{X}_s(T) \) be such that \( \mathcal{G}_r^{\lambda_i} \cap \mathcal{S}_{\nu_i} \neq \emptyset \) for \( i = 1, 2 \), and let \( w \in \mathcal{W} \). Let us take \( \mathcal{B}_1 \in \mathcal{Z}(\lambda_1)_{\nu_1} \), \( \hat{\nu}_1 \mathcal{B}_2 \in \text{Irr} \left( \mathcal{G}_r^{\lambda_2} \cap \mathcal{S}_{\nu_2} \right) \), and \( \gamma_1, \gamma_2 \in \mathcal{X}_s(T) \). Then, by virtue of the lemma above, we can form the twisted product

\[ (\mathcal{B}_1 \cap \mathcal{S}_{\nu_1}^w)^{\sim} \times \mathcal{U}(\mathcal{O}) \mathcal{B}_2 \cap \mathcal{S}_{\nu_2}^w \subset \mathcal{U}(\mathcal{K})^{\gamma_2} \times \mathcal{U}(\mathcal{O}) \mathcal{S}_{\nu_2}^w = \mathcal{S}_{\nu_1}^\gamma \]

where \( (\mathcal{B}_1 \cap \mathcal{S}_{\nu_1}^w)^{\sim} \) denotes the pullback of 
\( \mathcal{B}_1 \cap \mathcal{S}_{\nu_1}^w \subset \mathcal{S}_{\nu_1}^w = \mathcal{U}(\mathcal{K})^{\gamma_2} \mathcal{U}(\mathcal{O})/\mathcal{U}(\mathcal{O}) \)

to \( \mathcal{U}(\mathcal{K})^{\gamma_2} \mathcal{U}(\mathcal{O}) = \mathcal{U}(\mathcal{K})^{\gamma_2} \subset \mathcal{G}(\mathcal{K}) \). By \( \mathcal{B}_1 \mathcal{B}_2 \), we denote the image of this algebraic subvariety of \( \mathcal{G}_r^{\gamma} \mathcal{G}_r \) under the map \( \mathcal{G}_r^{\gamma} \mathcal{G}_r \rightarrow \mathcal{G}_r \); note that \( \mathcal{B}_1 \mathcal{B}_2 \subset \mathcal{M}(\mathcal{S}_{\nu_1}^w) = \mathcal{S}_{\nu_1}^{\gamma_2} \).

The following is a reformulation of Braverman-Gaitsgory’s result on tensor products of highest weight crystals in \( \mathcal{B}_2 \mathcal{G} \) (see also \( \mathcal{B}_2 \mathcal{G} \)); we will give a brief account of the relationship in the Appendix. Here we should warn the reader that the convention on the tensor product rule for crystals in \( \mathcal{B}_2 \mathcal{G} \) is opposite to ours, i.e., to that of Kashiwara (see, for example, \( \mathcal{K}_2 \mathcal{G} \) and \( \mathcal{K}_4 \)).

Theorem 4.2.4. Let \( \lambda_1, \lambda_2 \in \mathcal{X}_s(T)_+ \). There exists a bijection

\[ \Phi_{\lambda_1, \lambda_2} : \mathcal{MV}(\lambda_1) \times \mathcal{MV}(\lambda_2) \rightarrow \bigcup_{\nu \in \mathcal{X}_s(T)} \text{Irr} \left( \mathcal{M}^{-1}(\nu) \cap (\mathcal{G}_r^{\lambda_1} \times \mathcal{G}_r^{\lambda_2}) \right) \]

given as follows: for \( P_1 \in \mathcal{MV}(\lambda_1) \) and \( P_2 \in \mathcal{MV}(\lambda_2) \),

\[ \Phi_{\lambda_1, \lambda_2}(P_1, P_2) = \left( \Phi_{\lambda_1}(P_1) \cap \mathcal{S}_{\nu_1} \right)^{\sim} \mathcal{U}(\mathcal{O}) \left( \Phi_{\lambda_2}(P_2) \cap \mathcal{S}_{\nu_2} \right), \quad (4.2.3) \]

where \( \nu_1 := \text{wt}(P_1) \), \( \nu_2 := \text{wt}(P_2) \), and \( \left( \Phi_{\lambda_1}(P_1) \cap \mathcal{S}_{\nu_1} \right)^{\sim} \) denotes the pullback of \( \Phi_{\lambda_1}(P_1) \cap \mathcal{S}_{\nu_1} \) to \( \mathcal{U}(\mathcal{K})^{\gamma_2} \subset \mathcal{G}(\mathcal{K}) \). Moreover, the bijection \( \Phi_{\lambda_1, \lambda_2} \) has the following properties.
(i) For each \( P_1 \in \mathcal{MV}(\lambda_1) \) and \( P_2 \in \mathcal{MV}(\lambda_2) \), the image of \( \Phi_{\lambda_1, \lambda_2}(P_1, P_2) \) under the map \( \mu \) is equal to \( \Phi_{\lambda}(P) \) for a unique \( \lambda \in X_*(T)_+ \) and a unique \( P \in \mathcal{MV}(\lambda) \) such that \( \iota_{\lambda}(P) = P_2 \otimes P_1 \), where \( \iota_{\lambda} : \mathcal{MV}(\lambda) \hookrightarrow \mathcal{MV}(\lambda_2) \otimes \mathcal{MV}(\lambda_1) \) is an embedding of crystals;

(ii) \( \pi_1(\Phi_{\lambda_1, \lambda_2}(P_1, P_2)) = \Phi_{\lambda_1}(P_1) \) for each \( P_1 \in \mathcal{MV}(\lambda_1) \) and \( P_2 \in \mathcal{MV}(\lambda_2) \);

(iii) \( [(t^w, \Phi_{\lambda_2}(P_2))] \subset \Phi_{\lambda_1, \lambda_2}(P_1, P_2) \) for each \( P_1 \in \mathcal{MV}(\lambda_1) \) with \( \nu_1 = \omega_1(P_1) \) and \( P_2 \in \mathcal{MV}(\lambda_2) \).

4.3 Proof of the polytopal estimate. This subsection is devoted to the proof of Theorem 4.1.1. Let \( G \) be a complex, connected, semisimple algebraic group with Lie algebra \( \mathfrak{g} \). We keep the setting of §4.1. Let \( \mu^{(1)} = (\mu^{(1)}_w)_{w \in W} \) and \( \mu^{(2)} = (\mu^{(2)}_w)_{w \in W} \) be the GGMS datum of \( P_1 \in \mathcal{MV}(\lambda_1) \) and \( P_2 \in \mathcal{MV}(\lambda_2) \), respectively. Also, let \( \mu_* = (\mu_w)_{w \in W} \) be the GGMS datum of \( P \in \mathcal{MV}(\lambda) \); note that

\[
\mu_\ast = \omega_1(P) = \omega_1(P_1) + \omega_1(P_2) = \mu^{(1)}_\ast + \mu^{(2)}_\ast.
\]

Recall from Proposition 2.2.2 that the Minkowski sum \( P_1 + P_2 \) is a pseudo-Weyl polytope \( P(\mu^{(1)}_* + \mu^{(2)}_* ) \) with GGMS datum \( \mu^{(1)}_* + \mu^{(2)}_* = (\mu^{(1)}_w + \mu^{(2)}_w)_{w \in W} \). Therefore, it follows from equation (2.2.2) together with Remark 2.3.7 that

\[
P_1 + P_2 = \bigcap_{w \in W} \{ v \in \mathfrak{h}_\mathbb{R} \mid w^{-1} \cdot v - w^{-1} \cdot (\mu^{(1)}_w + \mu^{(2)}_w) \in \sum_{j \in I} \mathbb{R}_{\geq 0} h_j \}
\]

\[
= \bigcap_{w \in W} \text{Conv} \{ \gamma \in X_*(T) \subset \mathfrak{h}_\mathbb{R} \mid [t^w] \in \overline{S_{\mu^{(1)}_w + \mu^{(2)}_w}} \}.
\]

Also, recall from Theorem 2.3.6 that \( \Phi_{\lambda}(P) \in Z(\lambda) \) and

\[
P = \text{Conv} \{ \gamma \in X_*(T) \subset \mathfrak{h}_\mathbb{R} \mid [t^w] \in \Phi_{\lambda}(P) \}.
\]

Hence, in order to prove that \( P \subset P_1 + P_2 \), it suffices to show that

\[
\Phi_{\lambda}(P) \subset \overline{S_{\mu^{(1)}_w + \mu^{(2)}_w}} \quad \text{for all} \ w \in W.
\]

We set \( b^{(1)} := \Phi_{\lambda_1}(P_1) \in Z(\lambda_1) \) and \( b^{(2)} := \Phi_{\lambda_2}(P_2) \in Z(\lambda_2) \). Because \( P_2 = P(\mu^{(2)}_* ) \) is the extremal MV polytope of weight \( x \cdot \lambda \) for some \( x \in W \) by our assumption, we know from Theorem 2.5.6 that \( \mu^{(2)}_w \in W \cdot \lambda_2 \) for all \( w \in W \). Hence the algebraic variety \( \overline{G_{T \lambda_2} \cap S^{w^{(2)}_{\mu^{(2)}_w}}} \) is irreducible, and is the \( w \)-translate of the extremal MV cycle \( b^{(2),w} := b^{(2)} \) of weight \( w^{-1} \cdot \mu^{(2)}_w \) (see Remark 2.3.4; note that \( b^{(2),w} = b^{(2)} \)).

Now suppose, contrary to our assertion (4.3.1), that \( \Phi_{\lambda}(P) \not\subset \overline{S_{\mu^{(1)}_w + \mu^{(2)}_w}} \) for some \( w \in W \); we take and fix such a \( w \in W \). Then, by equation (2.3.2) (see also Remark 2.3.4), there exists some \( \nu \in X_*(T) \) such that

\[
\Phi_{\lambda}(P) \cap S_{\nu}^w \neq \emptyset \quad \text{and} \quad w^{-1} \cdot \nu \geq w^{-1} \cdot (\mu^{(1)}_w + \mu^{(2)}_w).
\]
Claim. For the (fixed) $w \in W$ above, we have the following inclusion of varieties when they are regarded as subvarieties of $\text{Gr}$:

\[
\mathbf{b}^{(1)} \ast_{\mu_e^{(1)}, \mu_e^{(2)}} \mathbf{b}^{(2)} \subset \mathbf{b}^{(1)} \ast_{\mu_{w_k}^{(1)}, \mu_{w_k}^{(2)}} \mathbf{b}^{(2), w}.
\]

Proof of Claim. Let $w = s_{i_1}s_{i_2}\cdots s_{i_k}$ be a reduced expression, and set $w_k = s_{i_1}s_{i_2}\cdots s_{i_k}$ for $0 \leq k \leq \ell$. For simplicity of notation, we set for $0 \leq k \leq \ell$

\[
b_k^{(2)} := \mathbf{b}^{(2), w_k},
\]

\[
b^{(1)} \ast_k \mathbf{b}^{(2)} := \mathbf{b}^{(1)} \ast_{\mu_{w_k}^{(1)}, \mu_{w_k}^{(2)}} \mathbf{b}^{(2), w_k},
\]

and

\[
\mu_k^{(1)} := \mu_{w_k}^{(1)}, \quad \mu_k^{(2)} := \mu_{w_k}^{(2)};
\]

note that

\[
b^{(1)} \ast_{0} \mathbf{b}^{(2)} = \mathbf{b}^{(1)} \ast_{\mu_e^{(1)}, \mu_e^{(2)}} \mathbf{b}^{(2), e} = \mathbf{b}^{(1)} \ast_{\mu_e^{(1)}, \mu_e^{(2)}} \mathbf{b}^{(2)}.
\]

If we can show the inclusion

\[
b^{(1)} \ast_k \mathbf{b}^{(2)} \subset \mathbf{b}^{(1)} \ast_{k+1} \mathbf{b}^{(2)}
\]

for each $0 \leq k \leq \ell - 1$, (4.3.3)

then we will obtain the following sequence of inclusions:

\[
b^{(1)} \ast_{\mu_e^{(1)}, \mu_e^{(2)}} \mathbf{b}^{(2)} = \mathbf{b}^{(1)} \ast_{0} \mathbf{b}^{(2)} \subset \mathbf{b}^{(1)} \ast_{1} \mathbf{b}^{(2)} \subset \mathbf{b}^{(1)} \ast_{2} \mathbf{b}^{(2)} \subset \cdots
\]

\[
\cdots \subset \mathbf{b}^{(1)} \ast_{\ell} \mathbf{b}^{(2)} = \mathbf{b}^{(1)} \ast_{\mu_{e_k}^{(1)}, \mu_{e_k}^{(2)}} \mathbf{b}^{(2), w_k} = \mathbf{b}^{(1)} \ast_{\mu_{w_k}^{(1)}, \mu_{w_k}^{(2)}} \mathbf{b}^{(2), w_k},
\]

as desired. In order to show the inclusion (4.3.3), take an element $[(y, gG(e))] \in \mathbf{b}^{(1)} \ast_k \mathbf{b}^{(2)}$, where

\[
y \in \mathbf{b}^{(1)} \cap \mathbf{S}_{\mu_k^{(1)}}^{w_k}, \quad gG(e) \in \mathbf{b}^{(2)}_{k+1} \cap \mathbf{S}_{\mu_k^{(2)}}^{w_k},
\]

and write the element $y \in w_k U(K) e^{(1)}$ as: $y = u_k \mu_k^{(1)}$ for $u_k \in w_k U(K)$. Since $\mathbf{b}^{(1)} = \bigcap_{e \in W} S_{\mu_{e_k}^{(2)}}^{w_k}$ by Theorem 2.3.6, we may (and do) assume that $y G(e) \in S_{\mu_k^{(1)}}^{w_k} \cap S_{\mu_{e_k}^{(2)}}^{w_k}$ to show the inclusion (4.3.3). Therefore, we can take $u_{k+1} \in w_{k+1} U(K)$ and $g_{k+1} \in G(O)$ such that

\[
y = u_k \mu_k^{(1)} = u_{k+1} \mu_{k+1}^{(1)} g_{k+1};
\]

note that

\[
g_{k+1} \in T(K)(w_{k+1} U(K)) T(K).
\]

Here, since $w_{k+1} = w_k s_{i_{k+1}}$ by definition, it follows that

\[
T(K)(w_{k+1} U(K)) T(K) = w_k (s_{i_{k+1}} U(K) s_{i_{k+1}} U(K)) w_k^{-1} \subset w_k P_{k+1}(K),
\]

and hence that $g_{k+1} \in w_k P_{k+1}(K)$. Moreover, since $g_{k+1} \in G(O)$, we get $g_{k+1} \in w_k P_{k+1}(K) \cap G(O) = w_k P_{k+1}(O)$. Therefore, we obtain

\[
g_{k+1} \mathbf{b}^{(2)}_k \subset w_k P_{k+1}(O) \mathbf{b}^{(2)}_k = w_k L_{i_{k+1}}(O) w_k U_{i_{k+1}}(O) \mathbf{b}^{(2)}_k.
\]
Since the extremal MV cycle \( \mathring{w}_k^{-1}b_k^{(2)} \) is stable under \( U_{i_{k+1}}(O) \subset U(O) = G(O) \cap U(K) \) (see the proof of Lemma 4.2.3), we have \( w_k U_{i_{k+1}}(O) b_k^{(2)} \subset b_k^{(2)} \), and hence
\[
g_{k+1}b_k^{(2)} \subset w_k L_{i_{k+1}}(O)b_k^{(2)}.\]

Also, we see that
\[
w_k L_{i_{k+1}}(O)b_k^{(2)} = w_k L_{i_{k+1}}(O)b_k^{(2)} \quad \text{since} \quad w_{k+1} = w_k s_{i_{k+1}} \quad \text{and} \quad s_{i_{k+1}} \in L_{i_{k+1}}\]
\[
\subset b_{k+1}^{(2)} \quad \text{by Lemma 2.5.8 since} \quad w_k < w_k s_{i_{k+1}} = w_{k+1}.
\]

Combining these, we obtain \( g_{k+1}b_k^{(2)} \subset b_{k+1}^{(2)} \), and hence \( g_{k+1}G(O) \subset g_{k+1}b_k^{(2)} \subset b_{k+1}^{(2)} \).

Furthermore, we have
\[
[(y, gG(O))] = [(u_k t_k^{(1)}, gG(O))] = [(u_k t_k^{(1)} g_{k+1}, gG(O))] = [(u_k t_k^{(1)}, g_{k+1} gG(O))]
\]
by the equivalence relation \( \sim \) on \( G(K) \times Gr, \) and
\[
yG(O) = u_k t_k^{(1)} g_{k+1} G(O) \subset b^{(1)} \cap S_{\nu_k+1}^{\mu_k+1}
\]
by the choice above of \( y \). Consequently, we conclude that
\[
[(y, gG(O))] \in m \left( \left( b^{(1)} \cap S_{\nu_k+1}^{\mu_k+1} \right) \times w_{k+1} U(O) b_{k+1}^{(2)} \right)
\]
\[
= m \left( \left( b^{(1)} \cap S_{\nu_k+1}^{\mu_k+1} \right) \times w_{k+1} U(O) \left( b_{k+1}^{(2)} \cap S_{\nu_k+1}^{\mu_k+1} \right) \right)
\]
\[
= b^{(1)} \ast_{k+1} b^{(2)},
\]
since \( b_{k+1}^{(2)} \cap S_{\nu_k+1}^{\mu_k+1} = b_{k+1}^{(2)} \) (see Remark 2.3.3). This proves the inclusion (4.3.3), and hence the claim. 

Finally, we complete the proof of Theorem 4.1.1 By the claim above, we obtain
\[
S'_w \cap \left( b^{(1)} \ast_{\mu_1^{(1)}, \mu_2^{(2)}} b^{(2)} \right) \subset S'_w \cap \left( b^{(1)} \ast_{\mu_1^{(1)}, \mu_2^{(2)}} b^{(2),w} \right). \tag{4.3.4}
\]

Also, we have
\[
S'_w \cap \left( b^{(1)} \ast_{\mu_1^{(1)}, \mu_2^{(2)}} b^{(2),w} \right) \subset S'_w \cap S_{\nu_k+1}^{\mu_1^{(1)}+\mu_2^{(2)}} \tag{4.3.5}
\]
by the inclusion \( b^{(1)} \ast_{\mu_1^{(1)}, \mu_2^{(2)}} b^{(2),w} \subset S_{\nu_k+1}^{\mu_1^{(1)}+\mu_2^{(2)}} \), which is an immediate consequence of the definition. Since \( w^{-1} \cdot \nu \geq w^{-1} \cdot (\mu_1^{(1)} + \mu_2^{(2)}) \) by (3.5.2), it follows from equation (2.3.2) that \( S'_w \cap S_{\nu_k+1}^{\mu_1^{(1)}+\mu_2^{(2)}} = \emptyset \), and hence by (1.3.4) together with (4.3.4) that
\[
S'_w \cap \left( b^{(1)} \ast_{\mu_1^{(1)}, \mu_2^{(2)}} b^{(2)} \right) = \emptyset. \tag{4.3.6}
\]
Now, we know from Theorem [4.2.4] that
\[ \Phi_{\mu_e(1) + \mu_e(2)} \cap m(\Phi_{\lambda_1, \lambda_2}(P_1, P_2)) \subset \Phi_{\lambda}(P) \]
is an open dense subset, where \( \text{wt}(P) = \mu_e(1) + \mu_e(2) \); for an MV polytope \( P \in MV(\lambda) \), choosing
an embedding \( \iota_{\lambda} : MV(\lambda) \hookrightarrow MV(\lambda_2) \otimes MV(\lambda_1) \) of crystals so that \( \iota_{\lambda}(P) = P_2 \otimes P_1 \) for some \( P_1 \in MV(\lambda_1) \) and \( P_2 \in MV(\lambda_2) \) corresponds, via Theorems 2.3.6 and 4.2.4, to choosing an
irreducible component \( X \in \text{Irr}(Gr^{\lambda_1, \lambda_2} \cap S_{\nu_1, \nu_2}) \) such that \( \Phi_{\lambda}(P) \in Z(\lambda) \) is the (Zariski-)
closure of the image of \( X \) under the map \( \star \) in the commutative diagram of Theorem A.1.4 in the Appendix, where \( \nu_1 = \text{wt}(P_1) = \mu_e(1) \) and \( \nu_2 = \text{wt}(P_2) = \mu_e(2) \). Also, we see from the explicit construction of \( \Phi_{\lambda_1, \lambda_2}(P_1, P_2) \) given in Theorem [4.2.3] that
\[ b^{(1)} \otimes b^{(2)}(1) \subset S_{\mu_e(1) + \mu_e(2)} \cap m(\Phi_{\lambda_1, \lambda_2}(P_1, P_2)) \]
is an open dense subset. Therefore, \( b^{(1)} \otimes b^{(2)} \subset \Phi_{\lambda}(P) \) is an open dense subset. Combining this fact with (4.3.6), we conclude that \( \Phi_{\lambda}(P) \cap S_{\nu} = \emptyset \), which contradicts (4.3.2). Thus, we have completed the proof of Theorem 4.1.1.

A Appendix: On Braverman-Gaitsgory’s bijection.

The aim of this appendix is to explain why Theorem 4.2.4 is a reformulation of results on tensor products of highest weight crystals in [BrG]. Keep the setting of 4.2. In addition, we generally follow the notation of [CG, Chapter 8].

For \( \lambda \in X_+(T) \), we denote by \( IC_\lambda \) the intersection cohomology complex of \( Gr^\lambda \) (and also its extension by zero to the whole \( Gr \)). Similarly, for \( \lambda_1, \lambda_2 \in X_+(T) \), we denote by \( IC_{\lambda_1, \lambda_2} \) the intersection cohomology complex of \( Gr^{\lambda_1} \times Gr^{\lambda_2} \) (and also its extension by zero to the whole \( Gr \)).

Theorem A.1.1 ([Lu1], [Gi], [MV2], [BeiD]). (1) Let \( \lambda \in X_+(T) \). Then,
\[ H^\bullet(IC_\lambda) \cong L(\lambda) \quad \text{as } G^{\nu}-\text{modules}. \tag{A.1.1} \]

(2) Let \( \lambda_1, \lambda_2 \in X_+(T) \). The direct image \( R^\bullet m_* IC_{\lambda_1, \lambda_2} \) is isomorphic to a direct sum of simple perverse sheaves as follows:
\[ R^\bullet m_* IC_{\lambda_1, \lambda_2} \cong \bigoplus_{\lambda \in X_+(T)} IC^\lambda \cong \bigoplus_{\lambda \in X_+(T)} IC_\lambda, \tag{A.1.2} \]
where \( \mathbb{C}^{m_{\lambda_1, \lambda_2}} \) denotes the vector space of dimension \( m_{\lambda_1, \lambda_2} \in \mathbb{Z}_{\geq 0} \) over \( \mathbb{C} \) for \( \lambda \in X_+(T) \).

(3) Let \( \mathcal{P} \) be the full subcategory of the category of perverse sheaves on \( Gr \) whose objects are direct sums of \( IC_\lambda \)'s. Then, the assignment
\[ (IC_{\lambda_1}, IC_{\lambda_2}) \mapsto IC_{\lambda_1, \lambda_2} \mapsto R^\bullet m_* IC_{\lambda_1, \lambda_2} \]

is an equivalence of categories.
defines on \( \mathcal{P} \) the structure of a tensor category with a fiber functor \( IC_\lambda \mapsto H^\bullet(IC_\lambda) \).

(4) We have an equivalence \( \mathcal{P} \cong \text{Rep}(G^\vee) \) of tensor categories with fiber functors, where \( \text{Rep}(G^\vee) \) denotes the category of finite-dimensional rational representations of \( G^\vee \).

Let us fix arbitrarily \( \lambda_1, \lambda_2 \in X_*(T)_+ \), and \( \nu_1, \nu_2 \in X_*(T) \). In the sequel, we assume that \( Gr^{\lambda_i} \cap S_{\nu_i} \neq \emptyset \) for \( i = 1, 2 \), i.e., \( \nu_1, \nu_2 \in X_*(T) \) are weights of the \( G^\vee \)-modules \( L(\lambda_1) \) and \( L(\lambda_2) \), respectively; namely, we assume that \( \nu_i \in \Omega(\lambda_i) \) for \( i = 1, 2 \) in the notation of \( \S 2.3 \) For simplicity of notation, we set

\[
S_{\nu_1, \nu_2} := S_{e_{\nu_1}, e_{\nu_2}}, \quad Gr^{\lambda_1, \lambda_2} := Gr^{\lambda_1} \times Gr^{\lambda_2}.
\]

The following Lemma is essentially due to Ngo and Polo [NP, Corollary 9.2]; the assumption of this corollary can be dropped by using [NP, Lemme 9.1] along with [MV2, Theorem 3.2 (i)].

**Lemma A.1.2.** Keep the setting above. The intersection \( Gr^{\lambda_1, \lambda_2} \cap S_{\nu_1, \nu_2} \) is a union of irreducible components of dimension \( \langle \lambda_1 + \lambda_2 - (\nu_1 + \nu_2), \rho \rangle \).

By virtue of (4.2.2) and Lemma A.1.2, we can imitate all the constructions in [MV2, Theorem 3.5 and Proposition 3.10] to obtain:

**Theorem A.1.3** (cf. [MV2, §3]; see also [BrG]). Keep the setting above.

1. The cohomology group \( H^k_c(S_{\nu_1, \nu_2}, IC_{\lambda_1, \lambda_2}) \) vanishes except for \( k = -2(\nu_1 + \nu_2, \rho) \).
2. There is an isomorphism of vector spaces

\[
H^{-2(\nu_1 + \nu_2, \rho)}_c(S_{\nu_1, \nu_2}, IC_{\lambda_1, \lambda_2}) \cong \mathbb{C} \text{Irr}(Gr^{\lambda_1, \lambda_2} \cap S_{\nu_1, \nu_2}).
\]

The following result is implicit in [A, §8] and [BrG].

**Theorem A.1.4.** Keep the setting above. For each \( \nu \in X_*(T) \), there exists the following
Here, \( \mathfrak{tr} \) is the map obtained by summing up the isotypical component of \( IC_\lambda \) for each \( \lambda \in X_*(T)_+ \) inside \( \mathfrak{m}IC_{\lambda_1, \lambda_2} \) via Theorem A.1.1(2), and \( \ast \) is the map induced by \( \mathfrak{m} \).

**Proof.** We write simply \( \mathbb{C} \) for the constant sheaf \( \mathbb{C}_X \) of an algebraic variety \( X \).

Consider the restriction of the multiplication map \( \mathfrak{m} : \mathcal{G}_r \times \mathcal{G}_r \to \mathcal{G}_r \) (see Proposition 4.2.1) to \( \mathcal{G}_{r^{\lambda_1, \lambda_2}} \), and write it as:

\[
\mathfrak{m}' : \mathcal{G}_{r^{\lambda_1, \lambda_2}} \to \mathcal{G}_{r^{\lambda_1+\lambda_2}}.
\]

Then, there exists the Leray spectral sequence

\[
E_2^{p,q} := H_c^q(S_\nu \cap \mathcal{G}_{r^{\lambda_1+\lambda_2}}, R^p\mathfrak{m}'\mathbb{C}) \Rightarrow \bigoplus_{\nu_1 + \nu_2 = \nu} H_c^{q+p}(S_{\nu_1, \nu_2} \cap \mathcal{G}_{r^{\lambda_1, \lambda_2}}, \mathbb{C}). \tag{A.1.3}
\]

Here, from Theorem A.1.3(2) together with Theorem A.1.1(3), we see that each direct summand of the right-hand side of (A.1.3), with \( q + p = 2(\lambda_1 + \lambda_2 - \nu, \rho) \), is isomorphic to the tensor product of the \( \nu_1 \)-weight space of \( L(\lambda_1) \) and the \( \nu_2 \)-weight space of \( L(\lambda_2) \). Also, note that the map \( \mathfrak{m}' \) above is a \( G(\mathcal{O}) \)-equivalent fibration. Hence, by virtue of the simply-connectedness of \( \mathcal{G}_{r^{\lambda}} \) for \( \lambda \in X_*(T)_+ \), we conclude that the proper direct image \( R^p\mathfrak{m}'\mathbb{C} \), when restricted to \( \mathcal{G}_{r^{\lambda}} \) with \( \lambda \leq \lambda_1 + \lambda_2 \), decomposes into a direct sum of constant sheaves (with degree shifts). Moreover, we know from [BrG, §3.4] that the number of irreducible components of (generic) fiber of an element of \( \mathcal{Z}(\lambda)_\nu \) with its dimension \( q + p = 2(\lambda_1 + \lambda_2 - \nu, \rho) \) along \( \mathfrak{m}' \) is \( m^\lambda_{\lambda_1, \lambda_2} \) for \( \lambda \in X_*(T)_+ \) (see Theorem A.1.1(2)); note that this is the largest possible dimension of an irreducible component of (generic) fiber of an element of \( \mathcal{Z}(\lambda)_\nu \) by Lemma A.1.2. Combining all these, we deduce that

\[
\sum_{p+q=2(\lambda_1+\lambda_2-\nu, \rho)} \dim E_2^{p,q} = \sum_{\nu_1 + \nu_2 = \nu} \dim H_c^{2(\lambda_1+\lambda_2-\nu, \rho)}(S_{\nu_1, \nu_2} \cap \mathcal{G}_{r^{\lambda_1, \lambda_2}}, \mathbb{C}).
\]
As a result, the spectral sequence \((E_2^{p,q})\) stabilizes at \(E_2\)-terms when \(q + p = 2(\lambda_1 + \lambda_2 - \nu, \rho)\) (top degree cohomology groups). In addition, the stalk \((R^n m(C))_x\) at \(x \in G r^\lambda\) for \(\lambda \in X_*(T)_+\) vanishes whenever \(p > 2(\lambda_1 + \lambda_2 - \lambda, \rho)\) since the inequality

\[
\langle \lambda_1 + \lambda_2 - \lambda, \rho \rangle \geq \dim \left( m^{-1}(x) \cap \left( \overline{G r^\lambda \times G r^{\lambda_2}} \right) \right)
\]

holds by Proposition 4.2.1. It follows that

\[
H^q_c(S_r \cap \overline{G r^{\lambda_1+\lambda_2}}, R^n m(C)) \cong \bigoplus_{\lambda \in X_*(T)_+} H^q_c((S_r \cap G r^{\lambda}, R^n m(C))
\]

for each \(q \in \mathbb{Z}\) with \(p = 2(\lambda_1 + \lambda_2 - \nu, \rho) - q\). Consequently, again by comparing the dimensions, we get an isomorphism

\[
\bigoplus_{\lambda \in X_*(T)_+} H^2c^{(\lambda-\nu, \rho)}(S_r \cap G r^\lambda, R^n m(C)) \cong \bigoplus_{\nu_1 + \nu_2 = \nu} H^2c^{(\lambda_1+\lambda_2-\nu, \rho)}(S_{\nu_1, \nu_2} \cap G r^{\lambda_1, \lambda_2}, \mathbb{C}) \quad (A.1.4)
\]

where \(n_\lambda := 2(\lambda_1 + \lambda_2 - \lambda, \rho)\).

Now we note that the stalk \((R^n m(C))_x\) at \(x \in G r^\lambda\) for \(\lambda \in X_*(T)_+\) admits a basis corresponding to top-dimensional irreducible components of \(m^{-1}(x) \cap \overline{G r^{\lambda_1, \lambda_2}}\). If we normalize each class of such components to 1 in \(\mathbb{C}\), then we get

\[
H^2c^{(\lambda-\nu, \rho)}(S_r \cap G r^\lambda, R^n m(C)) \cong H^2c^{(\lambda-\nu, \rho)}(S_r \cap G r^\lambda, \mathbb{C}) \quad (A.1.5)
\]

for \(\lambda \in X_*(T)_+\).

By putting together the maps \(\text{tr}\) in \((A.1.5)\), Theorem \(A.1.3\) [MV2 Theorem 3.5], and the isomorphism \((A.1.4)\), we obtain a commutative diagram

\[
\begin{array}{ccc}
\bigoplus_{\nu_1 + \nu_2 = \nu} H^2c^{(\lambda_1+\lambda_2-\nu, \rho)}(G r^{\lambda_1, \lambda_2} \cap S_{\nu_1, \nu_2}, \mathbb{C}) & \cong & \mathbb{C} \text{ Irr}(G r^{\lambda_1, \lambda_2} \cap S_{\nu_1, \nu_2}) \\
\bigoplus_{\lambda \in X_*(T)_+} H^2c^{(\lambda-\nu, \rho)}(S_r \cap G r^\lambda, R^n m(C)) & \text{tr} & \bigoplus_{\lambda \in X_*(T)_+} \mathbb{C} \text{ Irr}(G r^\lambda \cap S_r) \\
\end{array}
\]

where the maps \(\text{cl}\) are isomorphisms obtained by taking the corresponding cycles. Here we note that by comparison of dimensions, each of \(\mathbb{C}\) on \(G r^\lambda\) (resp., \(G r^{\lambda_1, \lambda_2}\)) can be replaced by
Therefore, inverting the isomorphism $\text{(A.1.4)}$ in the commutative diagram above yields the desired commutative diagram.

**Remark A.1.5.** In the theorem above, the proper direct image $R^m_! IC_{\lambda_1, \lambda_2}$ and the direct image $R^m_\ast IC_{\lambda_1, \lambda_2}$ are isomorphic since $m$ is a projective map when restricted to $G_{r_{\lambda_1}} \times G_{r_{\lambda_2}}$.

*(Sketch of) Proof of Theorem 4.2.4.* Note that $\Phi_{\lambda_2}(P_2) \cap S_{\nu_2}$ is $U(\mathcal{O})$-stable by Lemma 4.2.3. It follows that

$$\dim((\Phi_{\lambda_1}(P_1) \cap S_{\nu_1}) \sim \times U(\mathcal{O}) \left(\Phi_{\lambda_2}(P_2) \cap S_{\nu_2}\right)) = \dim \Phi_{\lambda_1}(P_1) + \dim \Phi_{\lambda_2}(P_2)$$

from the isomorphism (induced by the map $m$)

$$(\Phi_{\lambda_1}(P_1) \cap S_{\nu_1}) \times (\Phi_{\lambda_2}(P_2) \cap S_{\nu_2}) \sim (\Phi_{\lambda_1}(P_1) \cap S_{\nu_1}) \sim \times U(\mathcal{O}) (\Phi_{\lambda_2}(P_2) \cap S_{\nu_2}).$$

Therefore, using Lemma [A.1.2] and the statement preceding Definition 2.3.3 we deduce that

$$(\Phi_{\lambda_1}(P_1) \cap S_{\nu_1}) \sim \times U(\mathcal{O}) (\Phi_{\lambda_2}(P_2) \cap S_{\nu_2}) \in \text{Irr} \left(m^{-1}(S_{\nu_1+\nu_2}) \cap G_{r_{\lambda_1}, \lambda_2}\right)$$

by comparison of dimensions. Thus, by defining as in [4.2.3], we obtain the desired map $\Phi_{\lambda_1, \lambda_2}$ with properties (ii) and (iii). Moreover, properties (ii) and (iii) uniquely determine this map.

In the case that $G$ has semisimple rank 1, we see that the map $\Phi_{\lambda_1, \lambda_2}$ is indeed the canonical bijection that respects the BFG crystal structure by comparing Theorem A.1.4 with [BrG, §5.4]; note the (unusual) convention on the tensor product rule for crystals in [BrG]. Also, it is straightforward to see that the commutative diagram of Theorem A.1.4 is compatible with the restriction to $P_j(\mathcal{K})$-orbits for each $j \in I$. Since a crystal structure is determined uniquely by weights and the behavior of Kashiwara operators for all simple roots, in view of [BrG, §5.2], we conclude that the map $\Phi_{\lambda_1, \lambda_2}$ is the canonical bijection that respects the BFG crystal structure for a general reductive $G$. Hence, by taking into account the commutative diagram of Theorem A.1.4, property (i) can also be verified, as desired.

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