THICK COVERINGS FOR THE UNIT BALL OF A BANACH SPACE

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ABSTRACT. We study the behaviour of Whitley’s thickness constant of a Banach space with respect to \( \ell_p \)-products and we compute it for classical \( L_p \)-spaces.

1. Introduction and basic results

This paper contains a study of Whitley’s thickness constant and its computation in classical \( L_p \) spaces and \( \ell_p \)-products of Banach spaces. Unless otherwise stated, we shall assume that \( X \) is a real infinite-dimensional Banach space, but most results also hold in finite-dimensional spaces. We shall denote by \( B(x, r) \) the ball centered at \( x \), with radius \( r \). The symbols \( B_X \) and \( S_X \) will denote the unit ball and the unit sphere of \( X \). A finite set \( F \) is said to be an \( \varepsilon \)-net for a subset \( A \subset X \) if for any \( a \in A \) there exists \( f \in F \) such that \( ||a - f|| \leq \varepsilon \).

Whitley introduced in [16] the thickness constant \( T_W(X) \) as follows:

\[
T_W(X) = \inf \{ \varepsilon > 0 : \text{there exists an } \varepsilon\text{-net } F \subset S_X \text{ for } S_X \}.
\]

To study the thickness constant, it will be helpful to consider the following equivalent formulation (see [12, Prop. 3.4]):

\[
T(X) = \inf \{ \varepsilon > 0 : \exists \{x_1, \ldots, x_n\} \subset S_X : B_X \subset \bigcup_{i=1}^{n} B(x_i, \varepsilon) \}.
\]

**Lemma 1.** If \( X \) is an infinite dimensional Banach space then \( T(X) = T_W(X) \).

**Proof.** That \( T_W(X) \leq T(X) \) is clear. The converse inequality follows from [14] Prop. 2, which we reproduce here for the sake of completeness: Let \( A \) be a subset of a Banach space \( X \) which is weakly dense in its convex hull \( \operatorname{conv}(A) \). If a finite family of convex closed sets covers \( A \) they also cover the closed convex hull of \( A \). Indeed, assume \( A \subset \bigcup_{i=1}^{n} C_i \) for some closed convex sets \( C_i \). Taking the weak*-closures in \( X^* \) one gets \( \overline{\operatorname{conv}^w}(A) = \overline{A}^{w^*} \subset \bigcup_{i=1}^{n} \overline{C_i}^{w^*} \). Now, intersection with \( X \) yields

\[
\overline{\operatorname{conv}}(A) = X \cap \overline{\operatorname{conv}^w}(A) \subset X \cap \bigcup_{i=1}^{n} \overline{C_i}^{w^*} = \bigcup_{i=1}^{n} C_i.
\]

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This result can be considered a generalization (see [15]) of the antipodal theorem of Ljusternik and Šnirel’man (see [11, p. 180] or else [7]): Let $X$ be an infinite dimensional Banach space; if finitely many balls cover the unit sphere of $X$, then at least one of them must contain an antipodal pair $(y, -y)$.

Note that if $X$ is any finite-dimensional space, then one has $T(X) = 1$, while $T_{W}(X) = 0$ due to the compactness of $S_{X}$. It is also clear that $T(X) \in [1, 2]$ for every infinite-dimensional space. Generalizations of $T(\cdot)$ were considered and studied in [12, 11, 5]; while relations with other parameters can be seen in [14, 13, 3]. Spaces $X$ for which $T(X) = 2$ have been considered in [2, 9, 10]. In particular, a Banach space $X$ for which $T(X) = 2$ must contain $\ell_{1}$ (2); hence it cannot be reflexive (see also [9 Thm. 1.2]). Thus, reflexive spaces $X$ have $T(X) < 2$. Upper and lower estimates for $T(X)$ in uniformly convex spaces, as well as upper estimates in terms of the modulus of smoothness, follow from results in [13]. A reasonable characterization of the spaces $X$ with $T(X) = 1$ seems to be unknown.

The value of $T(\cdot)$ in many spaces is known (see [14]); in particular: $T(c_{0}) = 1$ and $T(\ell_{p}) = 2^{1/p}$ for $1 \leq p < \infty$. Our results in Section 3 can be considered the vector-valued generalization of these estimates.

2. Whitley Constant of $L_{p}$-Spaces

While it is known that $T(L_{1}) = 2$ (see [1] Ex. 3.6), to the best of our knowledge the thickness of $L_{p}[0, 1]$ for $p > 1$ is unknown.

Theorem 1. For $1 \leq p < \infty$ one has $T(L_{p}[0, 1]) = 2^{1/p}$.

Proof. Denote by $I$ the interval $[0, 1]$. Let $\{f_{1}, ..., f_{n}\}$ be a finite subset of $S_{X}$. Take $0 < \varepsilon < 1$. By the absolute continuity of integrals, there exists $\sigma > 0$ such that

\[
(1) \quad \int_{A} |f_{i}|^{p} < \varepsilon^{p} \quad \text{for } i = 1, ..., n \quad \text{whenever } \mu(A) < \delta.
\]

Take $A \subset I$ according to (1) and let $f = \frac{\chi(A)}{(\mu(A))^{1/p}}$ ($f \in S_{X}$). We have (for $i = 1, ..., n$):

\[
||f - f_{i}|^{p} = \int_{I - A} |f_{i}|^{p} + \int_{A} |\chi(A)(\frac{1}{(\mu(A))^{1/p}} - f_{i})|^{p} \geq 1 - \int_{A} |f_{i}|^{p} + ||f - \chi(A)f_{i}||^{p} > 1 - \varepsilon^{p} + ||f|| - ||\chi(A)f_{i}||^{p} > 1 - \varepsilon^{p} + (1 - \varepsilon)^{p}.
\]

Since $\varepsilon > 0$ is arbitrary, this shows that $T(L_{p}[0, 1]) \geq 2^{1/p}$.

Let $1 \leq p \leq 2$ and recall Clarkson’s inequality:

\[
||f + g||^{q} + ||f - g||^{q} \leq 2(||f||^{p} + ||g||^{p})^{q/p} \quad \text{where } 1/p + 1/q = 1.
\]

Taking $f_{0}, f \in S_{X}$ one has

\[
||f + f_{0}||^{q} + ||f - f_{0}||^{q} \leq 2(||f||^{p} + ||f_{0}||^{p})^{q/p} = 2^{1+q/p} = 2^{q},
\]

and thus

\[
\min\{||f + f_{0}||, ||f - f_{0}||\} \leq \left(\frac{2^{q}}{2}\right)^{1/q} = 2^{\frac{q-1}{q}} = 2^{1/p},
\]

so $T(X) \leq 2^{1/p}$ and the result is proved for $1 \leq p \leq 2$. 

Let now $2 \leq p < \infty$. For $i = 1, \ldots, n$ consider the norm one functions $\pm f_1, \ldots, \pm f_n$ with $f_i = n^{1/p} x |_{\left[\frac{i-1}{n}, \frac{i}{n}\right]}$. Take any $f \in S_X$; there exists $i$ such that $\int_{\left[\frac{i-1}{n}, \frac{i}{n}\right]} |f|^p \leq \frac{1}{n}$ (since $\sum_{i=1}^n \int_{\left[\frac{i-1}{n}, \frac{i}{n}\right]} |f|^p = 1$). Denote by $I_f = \left[\frac{i-1}{n}, \frac{i}{n}\right]$ the interval corresponding to $f$. Recall Hanner’s inequality (see [8]): for $p \geq 2$ one has
\[
\|f + g\|^p + \|f - g\|^p \leq (\|f\| + \|g\|)^p + \|\|f\| - \|g\|\|^p.
\]

Apply this to the space $L_p(I_f)$: consider the restrictions of $f$ and the $f_i$ to $I_f$, that we still denote in the same way, to obtain
\[
\|f_i + f\|^p + \|f_i - f\|^p \leq (\|f_i\| + \|f\|)^p + \|\|f_i\| - \|f\|\|^p \leq (1 + \frac{1}{n^{1/p}})^p + (1 - \frac{1}{n^{1/p}})^p;
\]
thus
\[
\min \left\{ \int_{I_f} |f_i + f|^p, \int_{I_f} |f_i - f|^p \right\} \leq \frac{1}{2} \left[ (1 + \frac{1}{n^{1/p}})^p + (1 - \frac{1}{n^{1/p}})^p \right].
\]
Therefore:
\[
\min \left\{ \int_{f} |f_i + f|^p, \int_{f} |f_i - f|^p \right\} \leq \frac{1}{2} \left[ (1 + \frac{1}{n^{1/p}})^p + (1 - \frac{1}{n^{1/p}})^p \right] + 1;
\]
and then
\[
\min \{ \min(\|f_i + f\|, \|f_i - f\|) : i = 1, \ldots, n \} \leq \left( \frac{1}{2} \left[ (1 + \frac{1}{n^{1/p}})^p + (1 - \frac{1}{n^{1/p}})^p \right] + 1 \right)^{1/p}.
\]
Since we can take $n$ arbitrarily large, we obtain $T(L_p(0,1]) \leq 2^{1/p}$, which concludes the proof. \hfill \Box

3. Whitley’s constant in product spaces

Whitley’s constant is strongly geometric, hence it is not strange that thickness constants of $X \oplus_p Y$ can be different for different values of $p$. A bit more surprising is that the thickness constant of a product space $\ell_p(X_n)$ also depends on whether there is a finite or infinite number of factors: indeed, it follows from next theorem (part (1)) that $T(c_0 \oplus c_0) = 1$, while it follows from Corollary 1 below that $T(\ell_2(c_0)) = \sqrt{2}$.

**Theorem 2.** Let $1 \leq p \leq \infty$.

1. $T(X_1 \oplus_p \cdots \oplus_p X_N) \leq \max \{ T(X_n), 1 \leq n \leq N \}$;
2. $2^{1/p} \leq T(\ell_p(X_n)) \leq (\inf \{ T(X_n) \})^{1/p} + 1$, for $1 \leq p < \infty$. The upper estimate is also valid for finite sums.
3. $T(\ell_\infty(X_n)) = \inf \{ T(X_n) \}$.

**Proof.** To prove (1), assume $p < \infty$; indeed, for $p = \infty$ it is contained in (3). Let us call, just for simplicity, $Z = X_1 \oplus_p \cdots \oplus_p X_N$. Given $\varepsilon > 0$, let $\{x_1^n, \ldots, x_N^n\}$ ($n = 1, \ldots, N$) be a $(T(X_n) + \varepsilon)$-net for $B_{X_n}$ with $\|x_1^n\| = 1$. Take in $(\mathbb{R}^N, \|\cdot\|_p)$ a finite $\varepsilon$-net $\{\lambda_1^k, \ldots, \lambda_N^k\}$, $1 \leq k \leq M$ for its unit ball with $\sum_{j=1}^N |\lambda_j^k|^p = 1$ for every $k$. Consider all points $\{\lambda_1^k x_1^1, \ldots, \lambda_N^k x_N^1\}$ with $1 \leq k \leq M$, $1 \leq j_n \leq k_n$, $1 \leq n \leq N$. They form a finite subset of norm one points of $Z$. Let us show they form a $\max \{ T(X_n) + 2 \varepsilon : 1 \leq n \leq N \}$-net. Take $z = (z_1, \ldots, z_N) \in B_Z$. Choose an index $k(z)$ such that
\[
\|(\|z_1\|, \ldots, \|z_N\|) - (\|\lambda_1^k(z)\|, \ldots, \|\lambda_N^k(z)\|)\|_p \leq \varepsilon.
\]
Also, for each \( n \) choose some index \( i_n \) so that
\[
\| - x^n_{i_n} - T(X_n) + \varepsilon. \]

Thus
\[
\| - (\lambda^{k(z)} x^n_{i_n}) \|_Z \leq \| - (\| z_n \| x^n_{i_n}) \|_Z + \| (\| z_n \| x^n_{i_n}) - (\lambda^{k(z)} x^n_{i_n}) \|_Z \leq \| z_n \| (T(X_n) + \varepsilon)_p + \varepsilon \leq \max\{T(X_n)\} + 2\varepsilon.
\]

Since \( \varepsilon > 0 \) is arbitrary, this proves (1).

The lower estimate in (2) is as follows. Let \( y_i = (y_i(n))_n \) be for \( i = 1, 2, ..., k \) a finite set of elements of the unit sphere of \( Y = \ell_p(X_n) \). Given \( \varepsilon > 0 \), let \( j \) be such that \( \| y_i(n) \| x_i < \varepsilon \) for \( n \geq j \) and all \( i \). Take a norm one element \( x \in X_j \) and form the element \( y = (0, ..., 0, x, 0, ..., 0) \) with \( x \) at the \( j \)-th position. One then has
\[
\| y_i - y \|_Y^p = \| (y_i^{(1)}, \ldots, y_i^{(j-1)}, y_i^{(j)} - x, y_i^{(j+1)}, \ldots, y_i^{(k)}) \|_Y^p = \| y_i^{(1)} \|_{X_1} + \ldots + \| y_i^{(j-1)} \|_{X_{j-1}} + \| y_i^{(j)} - x \|_{X_j} + \| y_i^{(j+1)} \|_{X_{j+1}} + \ldots \\
> 1 - \| y_i^{(j)} \|_{X_j} + \| y_i^{(j)} - x \|_{X_j}^p.
\]

This proves that \( (T(\ell_p(X_n)))^p \geq 1 - \varepsilon^p + |1 - \varepsilon|^p \). Since \( \varepsilon > 0 \) is arbitrary, the result follows.

To obtain the upper estimate in (2), given \( \varepsilon > 0 \) fix \( m \) and let \( \{u_1, \ldots, u_t\} \) be a \((T(X_m) + \varepsilon)-\)net for \( B_{X_m} \) with \( \|u_i\| = 1 \) for all \( 1 \leq i \leq t \). Consider as a net for the unit ball of \( Y = \ell_p(X_n) \) the points \( v_i = (0, \ldots, u_i, \ldots, 0) \), for \( 1 \leq i \leq t \) (\( u_i \) is in the \( m \)-th position). If \( (x_m) \in B_Y \) then in particular \( \|x_m\|_{X_m} \leq 1 \); fix \( i \) so that \( \|x_m - u_i\|_{X_m} = T(X_m) + \varepsilon \). If \( p < \infty \), then \( \| (x_m) - v_i \|_Y^p \leq \|x_m - u_i\|_Y^p + 1 \leq (T(X_m) + \varepsilon)^p + 1 \). This proves that \( T(\ell_p(X_n))^p \leq (T(X_m))^p + 1 \). Since \( m \) is arbitrary, the upper estimate follows.

The upper estimate in (3) is immediate from the arguments above since when \( p = + \infty \) one gets \( T(\ell_p(X_n)) \leq \max\{T(X_m), 1\} = T(X_m) \). For the lower estimate, assume that \( T(\ell_\infty(X_n)) < \inf\{T(X_n)\} \). Take \( \varepsilon' > 0 \) and \( \alpha \) such that
\[
T(\ell_\infty(X_n)) < \alpha - \varepsilon' < \alpha < \alpha + \varepsilon' < \inf\{T(X_n)\}
\]
and fix \( \varepsilon \) so that \( (1 - \varepsilon)(\alpha + \varepsilon') > \alpha \). Take a finite \( \alpha \)-net \( \{z_1, \ldots, z_t\} \) for \( B_{\ell_\infty(X_n)} \) verifying \( ||z_i|| = 1 \) for each \( i \). This in particular means that for each \( i, \) given \( \varepsilon > 0 \) there is some index \( n \) for which \( 1 - \varepsilon \leq ||z_i(n)||_{X_n} \leq 1 - \varepsilon \). Set \( I_n(\varepsilon) = \{i : ||z_i(n)||_{X_n} \geq 1 - \varepsilon\} \). The elements \( z_i(n)/||z_i(n)||, i \in I_n(\varepsilon) \) cannot form an \((\alpha + \varepsilon')\)-net for \( B_{X_n} \) and thus there must be \( x_n \in B_{X_n} \) such that \( ||z_i(n)/||z_i(n)|| - x_n|| > \alpha + \varepsilon' \) for all \( i \in I_n(\varepsilon) \). Since \( \bigcup I_n(\varepsilon) = \{1, \ldots, t\} \), for each \( i \in \{1, \ldots, t\} \) there is some \( n \) so that \( i \in I_n(\varepsilon) \). Form (for each \( i \)) one of the element(s) \( x \in \ell_\infty(X_n) \) as \( x(n) = ||z_i(n)||/x_n \) to get the contradiction:
\[
||z_i - x||_{\ell_\infty(X_n)} = \sup_n \left\{ ||z_i(n)|| \left( ||z_i(n)||/||x_n|| - x(n) \right) \right\} > (1 - \varepsilon)(\alpha + \varepsilon') > \alpha.
\]
As a consequence of (2) in the previous theorem we obtain.

**Corollary 1.** Let \( X_n \) be a family of Banach spaces so that \( T(X_n) = 1 \) for at least one index \( i \). Then \( T(\ell_p(X_n)) = 2^{1/p} \).

It is simple to see that estimates (1) and (2) (right inequality) in Theorem 2 are independent for \( 1 < p < \infty \): for example, consider a pair of spaces \( X, Y \) with \( T(X) = 1 \): if \( T(Y) = 1 \), then (1) is better; if \( T(Y) = 2 \), then (2) is better. The upper estimate in (2) is meaningful only if \( \min\{T(X), T(Y)\} < (2^p - 1)^{1/p} \). Both estimates are sharp: see Proposition 1 below (where they coincide if we take \( p \) in (2) is meaningful only if \( \min\{T(X), T(Y)\} < (2^p - 1)^{1/p} \). The same corollary shows that, in general, one can have \( T(\ell_p) = 2^{1/p} \) and \( T(\mathbb{R}) = 1 \). The same corollary shows that, in general, one can have \( T(Y) > \sup\{T(X_n) : n \in N\} \). Corollary 1 is not true for the sum of two spaces: for example, according to (1) in Theorem 2, \( T(c_0 \oplus c_0) = 1 \). This also shows that the lower estimate in (2) of Theorem 2 does not apply in general to finite sums. The same corollary shows that, in general, one can have \( T(Y) > \sup\{T(X_i) : i \in N\} \).

The aim of the following example is twofold: first, it shows that for \( 1 < p < \infty \) one can have \( T(X \oplus_p Y) > 2^{1/p} \). Then, it shows that it is possible to have \( T(X \oplus_p Y) < \min\{T(X), T(Y)\} \).

**Lemma 2.** \( T(\ell_1 \oplus_2 \ell_1) = \sqrt{2 + \sqrt{2}} < 2 \).

**Proof.** Let \( Z = \ell_1 \oplus_2 \ell_1 \). Consider the first element, \( e_1 \), of the natural basis in \( \ell_1 \); take in \( Z \) the four points \( z_1 = (e_1, 0) \); \( z_2 = (-e_1, 0) \); \( z_3 = (0, e_1) \); \( z_4 = (0, -e_1) \). Let \( z = (x, y) \in S_Z \); \( \|x\|_1 = a \); \( \|y\|_1 = b \); \( a^2 + b^2 = 1 \); this implies \( 1 \leq a + b \leq \sqrt{2} \). We want to prove that \( \min_{i=1,2,3,4} \|z - z_i\|_Z \leq \sqrt{2 + \sqrt{2}} \). One has

\[
\min_{i=1,2} \|z - z_i\|_Z^2 \leq (1 + a)^2 + b^2; \quad \min_{i=3,4} \|z - z_i\|_Z^2 \leq a^2 + (1 + b)^2;
\]

therefore

\[
\min_{i=1,2,3,4} \|z - z_i\|_Z^2 \leq ((1 + a)^2 + b^2 + a^2 + (1 + b)^2)/2 = 2 + a + b
\]

and thus

\[
T(Z) \leq \sup_{z \in S_Z} \min_{i=1,2,3,4} \|z - z_i\|_Z \leq \sqrt{2 + a + b} \leq \sqrt{2 + \sqrt{2}}.
\]

Now assume that \( (x_1, y_1), \ldots, (x_n, y_n) \) is a finite net of norm one elements for \( B_Z \). Given \( \varepsilon > 0 \), there exists \( k \) large enough such that all sequences \( (x_i), (y_i) \) have the \( k^{th} \) component, in modulus, smaller than or equal to \( \varepsilon \). Take \( (x, y) \in S_Z \), such that both \( x \) and \( y \) have all
components equal to 0, except the $k^{th}$ component equal to $1/\sqrt{2}$. One has that for all $i$:

$$\|(x, y) - (x_i, y_i)\|^2_Z = \|x - x_i\|^2_1 + \|y - y_i\|^2_1$$

$$= (a - |x_k| + \frac{1}{\sqrt{2}} - x_k)^2 + (b - |y_k| + \frac{1}{\sqrt{2}} - y_k)^2$$

$$\geq (a - \varepsilon + \frac{1}{\sqrt{2}} - \varepsilon)^2 + (b - \varepsilon + \frac{1}{\sqrt{2}} - \varepsilon)^2.$$ 

Since $\varepsilon$ is arbitrary, this proves that

$$T(Z) \geq \sqrt{a^2 + b^2 + 1 + 2(a + b)} \geq \sqrt{2 + \sqrt{2}},$$

and the assertion follows. □

**Proposition 1.** Let $Z = \ell_p \oplus_p Y$, $1 \leq p < \infty$; then $T(Z) = 2^{1/p}$.

**Proof.** Set $X = \ell_p$; let $z = (x, y) \in B_Z : x \in \ell_p$; $y \in Y$; $a^p + b^p = 1$ where $a^p = \|x\|^p_X$; $b^p = \|y\|^p_Y$. Consider the net given by the two points in $S_Z : z_1 = (e_1, 0)$; $z_2 = -z$. In $\ell_p$ we have either $||e_1 - x||^p_X \leq 1 + a^p$ or $||e_1 - x||^p_X \leq 1 + a^p$. Thus $||z_1 - z||^p_Z = ||\pm e_1 - x||^p_X + ||y||^p_Y \leq 1 + a^p + b^p$ for either $i = 1$ or $i = 2$. This proves that $T(Z) \leq 2^{1/p}$.

Let $z_1 = (x_1, y_1), ..., z_n = (x_n, y_n)$ be a net for $S_Z$ from $S_Z$: we have $||x_i||^p_X + (||y_i||^p_Y)^p = 1$ for all $i$. Given $\varepsilon > 0$, there is an index $j$ such that $\|x_j\| < \varepsilon$ for $i = 1, ..., n$. Consider the point $z_j = (e_j, 0) \in Z$. Then we have, for every $i : ||z_i - z_j||^p_Z = (||x_i - e_j||^p_X + (||y_i||^p_Y)^p \geq (||x_i||^p_X)^p - \varepsilon^p + (1 - \varepsilon)^p + (||y_i||^p_Y)^p$. Since $\varepsilon$ is arbitrary, this proves that $T(Z) \geq 2^{1/p}$, so the equality. □

4. Further Remarks and Open Questions

The core of the strange behaviour of $T(\cdot)$ is the following result:

**Lemma 3.** Every Banach space $X$ can be embedded as a 1-complemented hyperplane in a space $Y$ with $T(Y) = 1$.

**Proof.** Set $Y = X \oplus_{\infty} \mathbb{R}$ and consider in $Y$ the points $\pm y_0 = (0, \pm 1)$. Clearly $||\pm y_0|| = 1$ and, for $y = (x, c) \in B_Y$, we have $||y \pm y_0|| = \max\{||x||, ||c| \pm 1||\}$, and so $\min\{||y - y_0||, ||y + y_0||\} \leq 1$.

Thus, while $T(\ell_\infty) = 1$, $T(L_\infty([0, 1})) = 2$ since $T(C(K)) = 2$ whenever $K$ is an infinite compact Hausdorff space without isolated points (see [16]) and thus $\ell_\infty$ can be renormed to have $T(\cdot) = 2$. This also follows from the following result proved in [21], Thm. 1.2: A space $Y$ admits a renorming with $T(Y) = 2$ if and only if it contains an isomorphic copy of $\ell_1$. Which also means that there is a renorming of $Y = \ell_1 \oplus_{\infty} \mathbb{R}$ for which $T(Y) = 2$. Since $T(X) < 2$ for every reflexive space, no renorming of $Y = X \oplus \mathbb{R}$ with $T(Y) = 2$ exists when $X$ is reflexive.

Recall that a Banach space $X$ is said to be polyhedral if the unit ball of any two-dimensional subspace is a polyhedron. Obviously, $c_0$ is polyhedral and $T(c_0) = 1$. Moreover, every subspace of a polyhedral space contains almost-isometric copies of $c_0$. Nevertheless, there are polyhedral renormings of $c_0$ with $T(\cdot)$ as close to 2 as desired (it cannot
be 2 by the comments above). Consider the following renorming of \( c_0 \): for \( k \in \mathbb{N} \) set
\[
\|(x_n)_n\|_k = \max_k \left\{ \frac{1}{k} \sum_{j=1}^{k} |x_{n_j}| \right\}
\]
where the maximum is taken over all choices of \( k \) different indexes \( n_1, \ldots, n_k \). It is easy to check that this space is polyhedral (see [6, p.873]). Moreover, given a finite net from its unit sphere, let \( j \) be an index such that every element of the net have all components in modulus less than \( \varepsilon \) from \( j \) onwards. We see that the distance from \( ke_j \) to all elements of the net is at least \((k - \varepsilon + k - 1)/k \); thus, \( T(X) \geq \frac{2k-1}{k} \). Since \( k \) can be as large as we like, \( T(X) \) can approach 2 as much as one wants.

An interesting class of Banach spaces with \( 1 < T(X) < 2 \) is formed by the uniformly nonsquare (UNS is short) spaces. Recall that a Banach space \( X \) is said to be (UNS), if sup \{ \min\{||x - y||, ||x + y||\} : x, y \in S_X \} < 2. If a space is (UNS), then \( 1 < T(X) < 2 \) (see [12, Cor. 5.4 and Thm. 5.10]). Next example shows that the converse fails.

Example The space \( X = \mathbb{R} \oplus \ell_p \) \((1 \leq p < \infty)\) is not (UNS); we want to show that \( T(X) = 2^{1/p} \). By Theorem 2 (1), \( T(X) \leq 2^{1/p} \); now take a finite net in \( S_X \) and \( \varepsilon > 0 \). Let the modulus of the \( j \)-th component, for the part in \( \ell_p \), be smaller than or equal to \( \varepsilon \) for all elements in the net. Assume that an element of the net \((c_i, x_i)\) has \( ||x_i|| = b \), so \( |c_i| = 1 - b \); for \( z = (0, e_j) \), the distance from it is at least \( 1 - b + (b^p - \varepsilon^p + (1 - \varepsilon)^p)^{1/p} \), so \( T(X) \geq 1 - b + (b^p + 1)^{1/p} \). In \( \mathbb{R}^2 \), for any \( x \) we have \( \|x\|_p/\|x\|_1 \geq 2^{1/p-1} \); so, by taking \( x = (1, b) \), we see that \((b^p + 1)^{1/p} \geq (b + 1)(\frac{2^{1/p}}{2}) \). An easy computation then shows that \( T(X) \geq 2^{1/p} \).

The equalities \( T(\ell_p) = T(L_p) = 2^{1/p} \) for \( 1 \leq p \leq \infty \) suggest that spaces with the same “isometric local structure” -whatever this may mean- have the same thickness. A trying question posed in [3] is whether \( T(X) = T(X^{**}) \).

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