CONFORMAL GAUGE GENERATORS IN LIOUVILLE THEORY *

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Abstract

The conformal symmetry in the Liouville theory is analysed by using the Hamiltonian light–front formalism. The boundary conditions of dynamical variables are seen to involve an arbitrary function of time, so that the standard methods for studying gauge symmetries do not work. We develop a general method for constructing the gauge generators, which enables a consistent treatment of the boundary conditions present in the case of the conformal symmetry.

* Work supported in part by the Serbian Research Foundation, Yugoslavia.
1. Introduction

Classical action for the bosonic string in Polyakov’s formulation is invariant under 2d reparametrizations and local Weyl rescalings. As a consequence, all three components of the metric $g_{\alpha\beta}$ can be completely gauged away and gravity is classically a nonphysical field. Quantization of the theory leads to the appearance of an anomaly, which means that not all classical symmetries are the symmetries of the quantum theory. We can use the reparametrization invariance to fix the gauge in the conformally flat form

$$g_{\alpha\beta}(\xi) = e^{\varphi(\xi)}\eta_{\alpha\beta},$$

(1)

where $\eta_{\alpha\beta} = (+, -)$. The quantum dynamics of the gravitational field in the conformal gauge is determined by the effective action

$$W[\varphi] = -\frac{D-26}{8\pi} I_L, \quad I_L[\varphi] \equiv \int d^2\xi \left( \frac{1}{2} \eta^{\alpha\beta} \partial_\alpha \varphi \partial_\beta \varphi - \mu^2 e^{\varphi} \right).$$

(2)

In $D \neq 26$ (noncritical string) this dynamics is nontrivial and is known as the Liouville theory [1,2].

It is interesting to study the behaviour of this theory under a subgroup of 2d reparametrizations, the group of conformal reparametrizations:

$$\delta_0 g_{\alpha\beta} = (\nabla \cdot \varepsilon) g_{\alpha\beta}. \quad (3a)$$

By using the conformal gauge one finds

$$\delta_0 \varphi = \varepsilon \cdot \partial \varphi + \partial \cdot \varepsilon, \quad (3b)$$

with $\varepsilon^{\pm} = \varepsilon^{\pm}(\xi^{\pm})$. The Liouville action is easily seen to be invariant under these transformations.

On the other hand, in the standard Hamiltonian approach the Liouville action in the conformal gauge is not degenerate. The nondegeneracy of the action (2) is a natural consequence of the gauge fixing procedure and implies the absence of first class constraints. As a consequence, the Hamiltonian origin of the conformal symmetry of $I_L[\varphi]$ remains a bit obscure. The situation is very similar to the case of the $SL(2, R)$ symmetry in the light–cone gauge [3,4,5].

The objective of the present paper is to study the Hamiltonian structure of the conformal symmetry in the Liouville theory. It is well known that gauge symmetries in the Hamiltonian framework are related to the presence of arbitrary multipliers in the total Hamiltonian [6]. To clarify the real meaning of this assertion let us consider a dynamical evolution of a system described by a phase-space trajectory starting from a given point at time $t = 0$. For different choices of arbitrary multipliers we can solve the Hamiltonian equations of motion and obtain different trajectories, all starting from the same point and describing the same physical state. At any time $t > 0$ we can pass from one trajectory to another, without changing the physical state. This unphysical transition between trajectories at a given time $t$ is called the gauge transformation. It is clear that the Hamiltonian
The definition of gauge symmetries is based on a *definite choice of time*. The absence of gauge symmetries in a given Hamiltonian formalism based on one specific choice of time does not mean that these symmetries are absent for any other choice. We shall show how the conformal symmetry of the Liouville theory can be detected by using the light–cone time variable $\xi^+$.

The conformal transformation with parameter $\varepsilon^+(\xi^+)$ contains the inhomogenious term $\partial_+\varepsilon^+$. The presence of this term suggests that the boundary conditions of $\varphi$ should involve an arbitrary function of time $\xi^+$, in order to be invariant under the symmetry transformation. A detailed investigation of the Liouville theory shows that this is indeed what happens. Standard Castellani’s method [7] is not general enough to treat gauge symmetries with such unusual boundary conditions. This motivated us to develop a general method for constructing the gauge generators, so that boundary conditions present in the case of conformal symmetry can also be consistently described. As a result, the Hamiltonian origin and structure of the conformal symmetry in the Liouville theory becomes much more clear.

2. Hamiltonian and constraints in the light–front formalism

There are several reasons to study relativistic field theories at fixed light–cone time [8]. Here, the Hamiltonian light–front formalism is used to clarify the nature of the conformal symmetry in the Liouville theory.

In the light–cone coordinates $\xi^\pm = (\xi^0 \pm \xi^1)/\sqrt{2}$ the Liouville action (2) takes the form

$$ I_L = \int d^2\xi \left( \partial_+ \varphi \partial_+ \varphi - \mu^2 e^\varphi \right). $$

(4)

If we choose $\xi^+$ as the time variable, the action becomes *degenerate*. The definition of the momentum $\pi_{\varphi}$ leads to the following primary constraint:

$$ \phi \equiv \pi_{\varphi} - \partial_\varphi \varphi \approx 0. $$

(5)

The canonical Hamiltonian is $H_c = \mu^2 \int d\xi^- e^\varphi$, while the total Hamiltonian takes the form

$$ H_T = \int d\xi^- \left( \mu^2 e^\varphi + u \phi \right), $$

(6)

where $u$ is, at this stage, an undetermined multiplier.

C. The presence of the term $u_0\phi_0$ in $\tilde{H}_T$, with $u_0$ an arbitrary multiplier, means that the dynamics of the system is characterized by a gauge symmetry. If the gauge transformation involves a gauge parameter and its time derivative, the gauge generator has the form

$$ G = \varepsilon G_0 + \dot{\varepsilon} G_1. $$

We could now try to construct $G$ by using Castellani’s algorithm [7] which asserts that the
phase–space functions $G_0, G_1$ are determined by the following set of conditions:

\[
G_1 = C_{PFC}, \\
G_0 + \{G_1, \tilde{H}_T\} = C_{PFC}, \\
\{G_0, \tilde{H}_T\} = C_{PFC},
\]

where $C_{PFC}$ denotes a primary first class (PFC) constraint, possibly modified by a surface term. It is natural to start with $G_1 = \tilde{\phi}_0$ and calculate $G_0$ from the second equation,

\[
G_0 = \frac{\mu^2}{2} \int d\xi e^{\varphi(\xi)} + \alpha \tilde{\phi}_0.
\]

However, the third equation, that represents a kind of consistency test, fails to be satisfied. Therefore, the application of this algorithm leads to contradiction. The resolution of the problem demands a generalization of Castellani’s approach, as will be seen in the exposition that follows.

The consistency requirements are calculated by using the Poisson brackets taken at the same time $\xi^+$. By demanding $\{\phi, H_T\} = 0$ one obtains a condition on $u$:

\[
2\partial_- u + \mu^2 e^\varphi \approx 0.
\] (7a)

A particular solution of this inhomogeneous equation can be chosen in the form

\[
\hat{u} = \frac{\mu^2}{4} \int dx \epsilon(x - \xi^-)e^{\varphi(x)} \equiv \int dx g(x, \xi^-),
\]

where the variable $x$ is of the $\xi^-$ type, the dependence on $\xi^+$ is, for simplicity, not explicitly displayed and $\epsilon(x)$ is the antisymmetric step function satisfying the relation $\partial_x \epsilon(x) = 2\delta(x)$. The quantity $\hat{u}$ obeys the antisymmetric boundary conditions:

\[
\hat{u}_+ = -\hat{u}_- , \quad \hat{u}_+ \equiv \hat{u}(\xi^- \to \pm \infty).
\]

General solution for $u$ is obtained by adding an arbitrary function of $\xi^+$ to $\hat{u}$:

\[
u(\xi^+, \xi^-) = \hat{u}(\xi^+, \xi^-) + u_0(\xi^+).
\] (7b)

After that the total Hamiltonian becomes

\[
H_T = H' + u_0 \phi_0 ,
\]

\[
H' \equiv \int d\xi^- (\mu^2 e^\varphi + \hat{u} \phi) , \quad \phi_0 \equiv \int d\xi^- \phi .
\] (8)

The Hamiltonian equations of motion are:

\[
\partial_+ \varphi = \hat{u} + u_0 , \quad 2\partial_+ \pi = -\mu^2 e^\varphi .
\] (9)
After differentiating the first equation with respect to $\xi^-$ one immediately obtains the Lagrangean equation of motion.

The presence of an arbitrary multiplier ($u_0$) in the total Hamiltonian is a signal of the existence of a gauge symmetry in the theory [6]. It is important to note that $u_0$ is a function of only one coordinate, the time $\xi^+$.

In field theory all kinds of generators, in particular the Hamiltonian and symmetry generators, are generally nonlocal functionals of the phase-space variables. Owing to this, the role of surface terms becomes important in establishing the finiteness and differentiability of the generators, which is needed to properly define their action [9].

We shall begin the construction of the gauge generators by adopting a definite asymptotic behaviour for the basic dynamical variables $\varphi$ and $\pi_\varphi$.

### 3. Asymptotic behaviour and surface terms

The choice of asymptotics is always guided by some physical requirements. By demanding the finiteness of the Hamiltonian (finite energy condition) we easily find that $\exp \varphi$ must decrease faster than $(\xi^-)^{-1}$ for large $\xi^-$. The general solution of the Liouville equation is given by

$$e^\varphi = -\frac{16}{\mu^2} \frac{A'(\xi^+)B'(\xi^-)}{(1 - A(\xi^+)B(\xi^-))^2},$$

where $A(\xi^+)$ and $B(\xi^-)$ are differentiable functions. The solution is regular if the following conditions are satisfied:

$$AB \neq 1 : \quad AB < 1 \text{ or } AB > 1,$$
$$A' \neq 0, B' \neq 0 : \quad A \text{ and } B \text{ are monotonous}.$$

The second statement assumes that $A$ and $B$ are continuous functions. From these relations it follows that $B$ (and $A$) must have at least one horizontal asymptote.

If $B(\xi^-)$ has two horizontal asymptotes, say $B \sim a_\pm + \beta_\pm/(\xi^-)^{s_1}$ ($s_1 > 0$), then one finds

$$e^\varphi \sim \frac{a_\pm(\xi^+)}{(\xi^-)^{s_1+1}}, \quad \text{when } \xi^- \to \pm \infty,$$

where $a_\pm$ are two different functions characterized by the behaviour of $B(\xi^-)$ at $\xi^- \sim \pm \infty$.

The case when $B(\xi^-)$ has only one horizontal asymptote, say when $\xi^- \to -\infty$, while for $\xi^- \to +\infty$ it behaves like $(\xi^-)^{s_2}, s_2 > 0$, leads to

$$e^\varphi \sim \begin{cases} b_+/((\xi^-)^{s_2+1}), & \xi^- \to +\infty, \\ b_-/((\xi^-)^{s_1+1}), & \xi^- \to -\infty. \end{cases}$$

Thus we see that both cases satisfy the finite energy requirement. To simplify further exposition we shall consider the case $s_1 = s_2 = 1$, having in mind that the other cases
can be treated in a completely analogous manner. This leads us to adopt the following asymptotic behaviour for the field $\phi$:

$$\varphi \sim -2 \ln |\xi^-| + C_{\pm} + O_1, \quad \xi^- \to \pm \infty,$$

(11)

where $C_{\pm} = C_{\pm}(\xi^+)$ are two generally different functions, independent of $\xi^-$, and $O_n$ denotes a term that decreases as $(\xi^-)^{-n}$ or faster when $\xi^- \to \infty$.

To define the asymptotic behaviour of the momentum variable we shall use the fact that one can demand an arbitrarily fast decrease for those expressions that vanish on shell, as no solution of the equations of motion is thereby lost. In accordance with this we define

$$\pi \sim -2 \frac{\xi^-}{\xi^-} + O_2,$$

(12)

which ensures the $O_2$ behaviour of the constraint $\phi$. It is now easy to verify that both the Hamiltonian $H'$ and the constraint $\phi_0$ are well defined, finite quantities.

Let us now check if $\phi_0$ and $H'$ have well defined functional derivatives. A functional $G[q,p]$ has well defined functional derivatives if its variation can be written as

$$\delta G = \int dx \left[ A(x)\delta q(x) + B(x)\delta p(x) \right],$$

where $\delta q_{,\alpha}$ and $\delta p_{,\alpha}$ are absent [9]. In general, when the adopted asymptotics does not make surface terms disappear, this requirement may not be satisfied. For example,

$$\delta \phi_0 = \int d\xi^- (\delta \pi - \partial_- \delta \varphi) = \int d\xi^- \delta \pi - \delta (C_+ - C_-).$$

Obviously, $\phi_0$ is not a differentiable functional but it can be improved by adding a suitable surface term. The quantity

$$\tilde{\phi}_0 \equiv \phi_0 + C_+ - C_-$$

(13)

has well defined functional derivatives,

$$\frac{\delta \tilde{\phi}_0}{\delta \varphi(x)} = 0, \quad \frac{\delta \tilde{\phi}_0}{\delta \pi(x)} = 1.$$

(14)

Note that its action on local quantities coincides with that of $\phi_0$.

The variation of the Hamiltonian $H'$ has a similar structure:

$$\delta H' = \int dx \left\{ \left[ \frac{\mu^2}{2} e^{\varphi(x)} + \int dy g(x,y) \phi(y) \right] \delta \varphi(x) + \hat{u}(x) \delta \pi(x) \right\} + \hat{u}_- \delta (C_+ + C_-).$$

(15)

We see that $H'$ is not differentiable either, but, as opposed to the case of $\phi_0$, there is no suitable surface term to improve its differentiability, since $\hat{u}_- \delta (C_+ + C_-)$ cannot be put into the form $\delta$(something). Consequently, we are forced to further specify the asymptotic
behaviour of the field \( \varphi \) in order to get rid of the troublesome term. The appropriate restriction of (11) is given by

\[
C_+ + C_- = 2v(\xi^+),
\]

where \( v(\xi^+) \) is an arbitrarily fixed function of time, so that \( \delta(C_+ + C_-) = 0 \). Using this condition we easily find:

\[
\frac{\delta H'}{\delta \varphi(x)} = \frac{\mu^2}{2} e^{\varphi(x)} + \int dy \, g(x, y) \phi(y), \quad \frac{\delta H'}{\delta \pi(x)} = \hat{u}(x).
\]

Before we definitely adopt the new asymptotics we should chec k if any important physical solution is thereby lost. The consistency require ment on the restriction (16),

\[
\partial_+ v = \frac{1}{2} \partial_+(C_+ + C_-) = \frac{1}{2} [(\partial_+ \varphi)_\infty + (\partial_+ \varphi)_-\infty] = u_0,
\]

shows that \( v \) is the arbitrary multiplier and, consequently, it does not constrain the theory any further. The simpler choice \( v = \text{const.} \) would obviously be too restrictive, and would completely destroy the gauge symmetry of the theory.

The asymptotic behaviour of the basic dynamical variables, defined by (11), (12) and (16), ensures the finiteness and differentiability of the improved total Hamiltonian

\[
\tilde{H}_T \equiv H' + u_0 \tilde{\varphi}_0.
\]

It has now well defined Poisson brackets with other well defined, nonlocal quantities.

We now wish to make a few interesting observations.

A. The first one is related to the question of how it is seen that the quantity \( \varphi_0 \), which is a “linear combination” of the local constraints \( \phi(\xi) \), is also conserved during the time evolution of the system. The temporal development of \( \varphi_0 \) can not be calculated through the Poisson bracket \( \{\varphi_0, \tilde{H}_T\} \) since \( \varphi_0 \) is not a differentiable functional. Instead, we can employ well defined \( \tilde{\varphi}_0 \) so that

\[
\frac{d\tilde{\varphi}_0}{d\xi^+} = \frac{\tilde{\varphi}_0}{d\xi^+} - \partial_+(C_+ - C_-) = \{\tilde{\varphi}_0, \tilde{H}_T\} - \partial_+(C_+ - C_-).
\]

Using

\[
\{\tilde{\varphi}_0, \tilde{H}_T\} = \{\tilde{\varphi}_0, H'\} \approx -\frac{\mu^2}{2} \int d\xi^- e^{\varphi(\xi)},
\]

one finds that the time conservation of \( \varphi_0 \) is equivalent to the condition

\[
\partial_+(C_+ - C_-) + \frac{\mu^2}{2} \int d\xi^- e^{\varphi(\xi)} \approx 0.
\]

On the other hand, the Hamiltonian equations of motion yield

\[
\partial_+(C_+ - C_-) = (\partial_+ \varphi)_\infty - (\partial_+ \varphi)_-\infty = \hat{u}_+ - \hat{u}_-,
\]
so that the above condition is automatically satisfied. Therefore, \( \partial_+ \phi_0 \approx 0 \), as expected.

B. The second observation is related to the question of energy conservation. Since \( \{ \tilde{H}_T, H_T \} = 0 \), we have

\[
\frac{d\tilde{H}_T}{d\xi^+} = \frac{\partial \tilde{H}_T}{\partial \xi^+} + \{ \tilde{H}_T, H_T \} = \frac{\partial H'}{\partial \xi^+} + \dot{u}_0 \tilde{\phi}_0 \approx 2u_0 \dot{u}_- + \dot{u}_0 (C_+ - C_-) .
\]

We see that the energy is conserved only in the gauge \( u_0 = 0 \). This is a consequence of the fact that the asymptotic behaviour of \( H' \) is time dependent and \( \tilde{\phi}_0 \) is not a constraint due to the presence of surface terms. As a consequence, the explicit time dependence of \( \tilde{H}_T \) is absent only in the gauge \( u_0 = 0 \).

4. Construction of gauge generators

Let us observe that Castellani’s algorithm for the construction of gauge generators is not general enough to treat the cases in which the Hamiltonian may nontrivially depend on the arbitrary parameters of the theory. This is exactly the case with our \( \tilde{H}_T \) which explicitly depends not only on the multiplier \( \dot{v} \), but also on \( v \):

\[
\tilde{H}_T = H' + \dot{v} \tilde{\phi}_0 , \quad \frac{\partial H'}{\partial v} = 2\dot{u}_- ,
\]

as the relations (15), (16) and (18) show. Thus, we are led to generalize Castellani’s method to include a wider class of theories.

**Generalized conditions for symmetry generators.** Let us consider a system with finite number of degrees of freedom whose Hamiltonian explicitly depends on an arbitrary parameter \( v(t) \), as well as on its time derivative \( \dot{v} \):

\[
H_T = H_T(q, p; v, \dot{v}) .
\]

If there exists a gauge symmetry of the equations of motion involving only gauge parameter \( \varepsilon(t) \) and its time derivative \( \dot{\varepsilon}(t) \), we assume that the corresponding gauge generator has the form

\[
G(\varepsilon) \equiv \varepsilon G_0 + \dot{\varepsilon} G_1 , \quad G_a = G_a(q, p; v, \dot{v}) \quad (a = 0, 1) ,
\]

so that

\[
\delta q = \{ q, G[\varepsilon] \} , \quad \delta p = \{ p, G[\varepsilon] \} .
\]

What conditions the functions \( G_a \) should satisfy in order that \( G[\varepsilon] \) represents a symmetry generator of the theory for arbitrary \( \varepsilon(t) \)? Obviously, one must demand that the transformed trajectories \( q(t) + \delta q(t), p(t) + \delta p(t) \) also satisfy the Hamiltonian equations of motion with possibly different parameter \( v(t) + \delta v(t) \). Thus, the equations

\[
\frac{d}{dt}(\delta q) = \delta\{ q, H_T \} , \quad \frac{d}{dt}(\delta p) = \delta\{ p, H_T \}
\]

are
must have a solution for $\delta v(t)$. The calculation of the left-hand side of the first equation yields

$$L = \frac{d}{dt}\{q, G(\varepsilon)\} = \dot{\varepsilon}\{q, G_1\} + \dot{\varepsilon}\left[\{q, H_T\}, G_1\right] + \frac{d}{dt}\left[\{q, H_T\}, G_1\right] + \frac{d}{dt}\{q, \partial G_1/\partial t\}$$

$$+ \varepsilon\left[\{q, H_T\}, G_0\right] + \{q, \{G_0, H_T\}\} + \{q, \partial G_0/\partial t\},$$

while the calculation of the right-hand side leads to

$$R = \left[\dot{\varepsilon}\{q, H_T\}, G_1\right] + \varepsilon\left[\{q, H_T\}, G_0\right] + \{q, \partial H_T/\partial v\} \delta v + \{q, \partial H_T/\partial \dot{v}\} \delta \dot{v}.$$ 

Since the explicit time dependence of the generators is given only through the parameters $v$ and $\dot{v}$, we have

$$\frac{\partial G_a}{\partial t} = \frac{\partial G_a}{\partial v} \dot{v} + \frac{\partial G_a}{\partial \dot{v}} \ddot{v}.$$ 

Similar results are obtained for $p$, too. Combining these relations the requirements (23) can be written in the form:

$$\varepsilon G_1 + \dot{\varepsilon}\left[\{G_0, H_T\} + \partial G_1/\partial t\right]$$

$$+ \varepsilon\left[\{G_0, H_T\} + \partial G_0/\partial t\right] = \frac{\partial H_T}{\partial v} \delta v + \frac{\partial H_T}{\partial \dot{v}} \delta \dot{v}.$$  

Here, the equality means an equality up to quantities that act trivially on $q$ and $p$, i.e., whose Poisson brackets with $q$ and $p$ weakly vanish. The equation represents a condition for $\delta v$ and must hold for every $\varepsilon(t)$ and $v(t)$. Consequently, it implies:

$$G_1 = \alpha_1 \frac{\partial H_T}{\partial v} + \beta_1 \frac{\partial H_T}{\partial \dot{v}},$$

$$G_0 + \{G_1, H_T\} + \frac{\partial G_1}{\partial t} = \alpha_0 \frac{\partial H_T}{\partial v} + \beta_0 \frac{\partial H_T}{\partial \dot{v}},$$

$$\{G_0, H_T\} + \frac{\partial G_0}{\partial t} = \alpha \frac{\partial H_T}{\partial v} + \beta \frac{\partial H_T}{\partial \dot{v}}.$$  

If the total Hamiltonian depends on $\dot{v}$ as in Eq.(20) but $H'$ does not depend on $v$, the above conditions reduce to those of Castellani. The dependence of $H_T$ on both $\dot{v}$ and $v$ is the property which demands the generalization of Castellani’s method. It is now clear why the naive application of Castellani’s conditions to the case of conformal symmetry of the Liouville theory did not lead to the correct answer.

**Solution of the symmetry conditions.** After using the result (25) for $G_a$, the left–hand side of Eq.(24) takes the form

$$(\ddot{\varepsilon}\alpha_1 + \dot{\varepsilon}\alpha_0 + \varepsilon\alpha) \frac{\partial H_T}{\partial v} + (\ddot{\varepsilon}\beta_1 + \dot{\varepsilon}\beta_0 + \varepsilon\beta) \frac{\partial H_T}{\partial \dot{v}}.$$ 

Assuming that neither $\partial H_T/\partial v$ nor $\partial H_T/\partial \dot{v}$ vanish, we can solve Eq.(24) to obtain:

$$\delta v = \ddot{\varepsilon}\alpha_1 + \dot{\varepsilon}\alpha_0 + \varepsilon\alpha,$$

$$\delta \dot{v} = \ddot{\varepsilon}\beta_1 + \dot{\varepsilon}\beta_0 + \varepsilon\beta.$$
These equations are not consistent unless we require

\[ \ddot{\varepsilon}_1 + \varepsilon_0 + \varepsilon = \frac{d}{dt}(\dot{\varepsilon}_1 + \dot{\varepsilon}_0 + \varepsilon), \]

wherefrom one easily finds the following relations among the coefficients:

\[ \alpha_1 = 0, \quad \beta_1 = \alpha_0, \quad \beta_0 = \dot{\alpha}_0 + \alpha, \quad \beta = \dot{\alpha}. \]

We see that there are only two out of six parameters in (25) which remain undetermined. The generalized conditions for the existence of gauge generators are therefore found to be:

\[ G_1 = \alpha_0 \frac{\partial H_T}{\partial \dot{v}}, \]

\[ G_0 + \{G_1, H_T\} + \frac{\partial G_1}{\partial t} = \alpha_0 \frac{\partial H_T}{\partial v} + (\dot{\alpha}_0 + \alpha) \frac{\partial H_T}{\partial \dot{v}}, \]

\[ \{G_0, H_T\} + \frac{\partial G_0}{\partial t} = \alpha \frac{\partial H_T}{\partial v} + \dot{\alpha} \frac{\partial H_T}{\partial \dot{v}}. \]

Note that this holds only when \( H_T \) nontrivially depends on both \( v \) and \( \dot{v} \). If one of the quantities \( \partial H_T/\partial v \) or \( \partial H_T/\partial \dot{v} \) vanishes, the conditions (26) become much simpler, and boil down to Castellani’s conditions if \( v \) (or \( \dot{v} \)) stands for the usual arbitrary multiplier of the theory. In the case when both \( \partial H_T/\partial v \) and \( \partial H_T/\partial \dot{v} \) vanish the theory does not possess any gauge symmetry at all (in this case there are no arbitrary parameters in \( H_T \)).

5. Conformal symmetry in Liouville theory

Let us, now, apply the new method to the Liouville Hamiltonian (20), which is obtained in the light–front formalism with time \( \tau = \xi^+ \). As \( \tilde{\phi}_0 \) does not depend on either \( v \) or \( \dot{v} \) and \( H' \) depends on \( v \) alone, the conditions (26) simplify to

\[ G_1 = \alpha_0 \tilde{\phi}_0, \]

\[ G_0 + \{G_1, \tilde{H}_T\} + \frac{\partial G_1}{\partial \tau} = \alpha_0 \frac{\partial H'}{\partial v} + (\dot{\alpha}_0 + \alpha) \tilde{\phi}_0, \]

\[ \{G_0, \tilde{H}_T\} + \frac{\partial G_0}{\partial \tau} = \alpha \frac{\partial H'}{\partial v} + \dot{\alpha} \tilde{\phi}_0. \]

The first two equations are easily solved to give

\[ G_1 = \alpha_0 \tilde{\phi}_0, \quad G_0 = \alpha_0 \left( \frac{\partial H'}{\partial v} - \{\tilde{\phi}_0, H'\} \right) + \alpha \tilde{\phi}_0, \]

where the parameters \( \alpha \) and \( \alpha_0 \) should be determined, if possible, from the third requirement. Using Eqs.(14), (17) and (20) we find

\[ G_1 = \alpha_0 \tilde{\phi}_0, \quad G_0 = \alpha_0 H' + \alpha \tilde{\phi}_0, \]

\[ 10 \]
with
\[ \{G_0, \tilde{H}_T\} + \frac{\partial G_0}{\partial \tau} = \alpha \frac{\partial H'}{\partial v} + \dot{\alpha}_0 \tilde{\phi}_0 + (\dot{\alpha}_0 + \alpha_0 \dot{v} - \alpha)H'. \]

Thus, the third requirement in (27) will be satisfied if
\[ \dot{\alpha}_0 + \alpha_0 \dot{v} - \alpha = 0, \]
the simplest solution of which is
\[ \alpha_0 = 1, \quad \alpha = \dot{v}. \]

These values determine the final form of the gauge generators:
\[ G_1 = \tilde{\phi}_0, \quad G_0 = \tilde{H}_T. \quad (28a) \]

It is easily checked that the generator
\[ G[\varepsilon] \equiv \int d\xi \left[ \varepsilon \tilde{H}_T + (\partial_+ \varepsilon) \tilde{\phi}_0 \right] \quad (28b) \]
indeed produces the conformal gauge transformations:
\[ \delta \varphi = \{\varphi, G[\varepsilon]\} = \varepsilon \partial_+ \varphi + \partial_+ \varepsilon, \]
\[ \delta \pi = \{\pi, G[\varepsilon]\} = \varepsilon \partial_+ \pi. \quad (29) \]

In conclusion, we studied the conformal symmetry of the Liouville theory in the Hamiltonian formalism by going over to the light–front time. The corresponding gauge generators were found after generalizing the existing methods, so that the dependence of \( H_T \) on both \( v \) and \( \dot{v} \), stemming from the specific boundary conditions, could be consistently taken into account.
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