Capacitated Vehicle Routing in Graphic Metrics

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Capacitated vehicle routing problem (CVRP)

Input:
- depot $O$
- set of $n$ terminals
- capacity $k$

Minimize total length of tours

Fundamental problem in operations research
Capacitated vehicle routing problem (CVRP)

### Unit demands
- 1985: Haimovich and Rinnooy Kan
- 1990: Altinkemer and Gavish
- 1997: Asano, Katoh, Tamaki, and Tokuyama
- 1998: Hamaguchi and Katoh
- 2001: Asano, Katoh, and Kawashima
- 2006: Bompadre, Dror, and Orlin
- 2010: Adamaszek, Czumaj, and Lingas
- 2010: Das and Mathieu
- 2017: Becker, Klein, and Saulpic
- 2018: Becker, Klein, and Saulpic
- 2018: Becker
- 2019: Becker, Klein, and Schild
- 2019: Becker and Paul
- 2020: Cohen-Addad, Filtser, Klein, and Le
- 2021: Blauth, Traub, and Vygen
- 2021: Mathieu and Zhou
- 2022: Jayaprakash and Salavatipour
- 2022: Jayaprakash and Salavatipour
- 2022: Mathieu and Zhou

### Unsplittable demands
- 1981: Golden and Wong
- 1987: Altinkemer and Gavish
- 1991: Labbé, Laporte, and Mercure
- 2021: Blauth, Traub, and Vygen
- 2022: Friggstad, Mousavi, Rahgoshay, and Salavatipour
- 2023: Grandoni, Mathieu, and Zhou
- 2023: Mathieu and Zhou

- general metrics
- Euclidean plane
- planar graphs
- trees
- graphs of bounded treewidth
- graphs of bounded highway dimension
Results in general metrics

\(\alpha\): approximation ratio for TSP

### CVRP in general metrics

- \(1 + (1 - \frac{1}{k}) \cdot \alpha\) factor [Altinkemer and Gavish, 1990]
- \(1 + (1 - \frac{1}{k}) \cdot \alpha - \Omega\left(\frac{1}{k^3}\right)\) factor [Bompadre, Dror, Orlin, 2006]
- \(1 + \alpha - \frac{1}{3000}\) factor [Blauth, Traub, Vygen, 2022]
Let $G = (V, E)$ be a connected and unweighted graph.

For each $(u, v) \in V^2$, define

$$\delta(u, v) := \text{number of edges on a shortest } u\text{-to-}v \text{ path}.$$ 

$\delta$ is called a graphic metric.
## Results in graphic metrics

### TSP in graphic metrics
- $1.5 - \epsilon_0$ factor for some $\epsilon_0 > 0$  
  [Oveis Gharan, Saberi, Singh, 2011]
- 1.461 factor [Mömke and Svensson, 2011]
- $\frac{13}{9}$ factor [Mucha, 2012]
- 1.4 factor [Sebő and Vygen, 2014]

### CVRP in graphic metrics
- $2.4 - \frac{1}{3000}$ factor  
  [Sebő and Vygen, 2014] + [Blauth, Traub, Vygen, 2022]
- 1.95 factor [our result]
Iterated tour partitioning [Haimovich and Rinnooy Kan 1985]

1. Compute a TSP tour $S$ on all terminals

2. Partition $S$ into segments of $k$ terminals

3. For each segment, connect its endpoints to the depot

Fact: solution $\leq S + \text{rad}$, where $\text{rad} := \frac{2}{k} \cdot \sum_{v \in V} \delta(O, v)$. 
Algorithm for CVRP in graphic metrics

1. $S_1 \leftarrow$ TSP tour computed by Christofides algorithm
2. $S_2 \leftarrow$ TSP tour computed by Sebő-Vygen algorithm
3. $S \leftarrow$ cheaper one of $S_1$ and $S_2$
4. Apply iterated tour partitioning on $S$
Lemma

\[ \text{solution} \leq \text{rad} + 0.5 \cdot n + 0.95 \cdot \text{opt}. \]

**Proof.** Combining:

- **solution** \( \leq S + \text{rad} \)
- \( S \leq \frac{1}{2} (S_1 + S_2) \)
- \( S_1 \leq n + 0.5 \cdot \text{opt} \) [Christofides]
- \( S_2 \leq 1.4 \cdot \text{opt} \) [Sebő-Vygen]

Structure Theorem

\[ \text{opt} \geq \text{rad} + 0.5 \cdot n. \]

Main result

1.95-approximation for CVRP in graphic metrics.
Structure Theorem

\[
\text{opt} \geq \text{rad} + 0.5 \cdot n.
\]

\[
n = 48
\]
\[
k = 12
\]
\[
\text{opt} = n + \frac{n}{k}
\]
\[
\text{rad} = 0.5 \cdot n + \frac{n}{k}
\]
Proof of the Structure Theorem

Structure Theorem
\[ \text{opt} \geq \text{rad} + 0.5 \cdot n. \]

Structure Lemma
Consider any tour \( T \) visiting a subset \( U \) of \( m \) terminals.
Then \( \text{cost}(T) \geq 2L + 0.5 \cdot m \), where \( L := \frac{1}{m} \cdot \sum_{v \in U} \delta(O,v) \).
Proof of the Structure Lemma (1/2)

\[ H := \max_{v \in U} \{ \delta(O, v) \} \]

\[ a_i := \text{first terminal on } T \text{ s.t. } \delta(O, a_i) = i \]

\[ b_i := \text{last terminal on } T \text{ s.t. } \delta(O, b_i) = i \]

\[ W := \{ a_i \}_i \cup \{ b_i \}_i \]

Lemma 1
\[ \text{cost } (T) \geq 2H + |U \setminus W|. \]

Lemma 2
\[ |U \cap W| \leq 2 \sqrt{m \cdot (H - L)}. \]

Combining:
\[ \text{cost } (T) \geq 2H + m - 2 \sqrt{m \cdot (H - L)} \geq 2L + 0.5 \cdot m \]
Lemma 2

\[ |U \cap W| \leq 2 \sqrt{m \cdot (H - L)}. \]

Proof:

\[
\sum_{v \in U \cap W} \delta(O, v) \leq |U \cap W| \cdot H - \frac{|U \cap W|^2}{4}
\]

\[
\sum_{v \in U \setminus W} \delta(O, v) \leq (m - |U \cap W|) \cdot H
\]

Summing:

\[
m \cdot L = \sum_{v \in U} \delta(O, v) \leq m \cdot H - \frac{|U \cap W|^2}{4}
\]
Summary

Structure Theorem
\[ \text{opt} \geq \text{rad} + 0.5 \cdot n. \]

Main result
1.95-approximation for CVRP in graphic metrics.

Open question
Study the graphic TSP whose cost depends on \( n \) instead of \( \text{opt} \).

Thank you!