On Frank’s conjecture on $k$-connected orientations

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Abstract

We disprove a conjecture of Frank [4] stating that each weakly $2k$-connected graph has a $k$-vertex-connected orientation. For $k \geq 3$, we also prove that the problem of deciding whether a graph has a $k$-vertex-connected orientation is NP-complete.

Introduction

An orientation of an undirected graph $G$ is a digraph obtained from $G$ by substituting an arc $uv$ or $vu$ for every edge $uv$ in $G$. We are interested in characterizing graphs admitting an orientation that satisfies connectivity properties. Robbins [10] proved that a graph $G$ admits a strongly connected orientation if and only if $G$ is 2-edge-connected. The following extension to higher connectivity follows from of a result of Nash-Williams [9]: a graph $G$ admits a $k$-arc-connected orientation if and only if $G$ is $2k$-edge-connected.

Little is known about vertex-connected orientations. Thomassen [12] conjectured that if a graph has sufficiently high vertex-connectivity then it admits a $k$-vertex-connected orientation.

Conjecture 1 (Thomassen [12]). For every positive integer $k$ there exists an integer $f(k)$ such that every $f(k)$-connected graph admits a $k$-connected orientation.

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The case \( k = 2 \) has been proved by Jordán [5] by showing \( f(2) \leq 18 \). Recently, it was shown in [2] that \( f(2) \leq 14 \). However, the conjecture of Thomassen remains open for \( k \geq 3 \).

A graph \( G = (V,E) \) is called \textit{weakly} \( 2k \)-\textit{connected} if \( |V| > k \) and for all \( U \subseteq V \) and \( F \subseteq E \) such that \( 2|U| + |F| < 2k \), the graph \( G - U - E \) is connected. It is easy to see that any graph admitting a \( k \)-connected orientation is weakly \( 2k \)-connected. Note that checking the weak \( 2k \)-connectivity of a graph can be done in polynomial time using a variation of the Max-flow Min-cut algorithm [3]. Frank [4] conjectured that this connectivity condition characterizes graphs admitting a \( k \)-connected orientation.

\textbf{Conjecture 2} (Frank [4]). A graph \( G \) admits a \( k \)-connected orientation if and only if \( G \) is weakly \( 2k \)-connected.

Berg and Jordán [1] proved this conjecture for the special case of Eulerian graphs and \( k = 2 \). For a short proof of this result, see [7]. In this article we disprove this conjecture for \( k \geq 3 \). For instance, the graph \( G_3 \) in Figure 1 is a counterexample for \( k = 3 \). We also prove that deciding whether a given graph has a \( k \)-connected orientation is NP-complete for \( k \geq 3 \). Both these results hold also for the special case of Eulerian graphs. Hence assuming \( P \neq NP \), there is no good characterisation of graphs admitting a \( k \)-connected orientation for \( k \geq 3 \). We mention that counterexamples can easily be derived from our NP-completeness proof, but we give simple self-contained counterexamples. Furthermore, the gadgets used in the NP-completeness proof are based on properties used in our first counterexample.

This paper is organized as follows. In Section 1 we establish the necessary definitions and some elementary results. In Section 2 we disprove Conjecture 2 for \( k \geq 3 \). For \( k \geq 4 \), we provide Eulerian counterexamples. In Section 3 for \( k \geq 3 \), we reduce the problem of \textsc{Not-All-Equal} 3-SAT to the problem of finding a \( k \)-connected orientation of a graph. This reduction leads to a Eulerian counterexample of Conjecture 2 for \( k = 3 \).

1 Preliminaries

Let \( k \) be a positive integer and let \( D = (V,A) \) be a digraph. We mention that digraphs may have multiple arcs. In \( D \) the indegree (respectively, the outdegree) of a vertex \( v \) is denoted by \( \rho_D(v) \) (respectively, by \( \delta_D(v) \)). The pair \( u,v \in V \) is called \textit{strongly connected} if there exist a dipath from \( u \) to \( v \) and a dipath from \( v \) to \( u \). The digraph \( D \) is called strongly connected if every pair of vertices is strongly connected. The pair \( u,v \in V \) is called \textit{k-connected} if, for all \( U \subseteq V \setminus \{u,v\} \) such that \( |U| < k \), \( u \) and \( v \) are strongly
connected in the digraph $D - U$. A set of vertices is called $k$-connected if every pair of vertices contained in this set is $k$-connected. The digraph $D$ is called $k$-connected if $|V| > k$ and $V$ is $k$-connected.

Let $G = (V, E)$ be a graph. We mention that graphs may have multiple edges. In $G$ the degree of a vertex $v$ is denoted by $d_G(v)$ and the number of edges joining $v$ and a subset $U$ of $V - v$ is denoted by $d_G(v, U)$. The pair $u, v \in V$ is called connected if there is a path joining $u$ and $v$. The pair $u, v \in V$ is called weakly $2k$-connected if, for all $U \subseteq V \setminus \{u, v\}$ and $F \subseteq E$ such that $2|U| + |F| < 2k$, $u$ and $v$ are connected in the graph $G - U - F$. A set of vertices is called weakly $2k$-connected if every pair of vertices contained in this set is weakly $2k$-connected. So $G$ is weakly $2k$-connected if $|V| > k$ and $V$ is weakly $2k$-connected.

The constructions in this paper are based on the following facts.

**Proposition 1.** Let $G = (V, E)$ be a graph admitting a $k$-connected orientation $D$. Let $v$ be a vertex of degree $2k$ and $u \neq v$ be a vertex such that $d_G(u, v) = 2$. Then $\rho_D(v) = \delta_D(v) = k$ and the two parallel edges between $u$ and $v$ have opposite directions in $D$.

*Proof.* By $k$-connectivity of $D$, the indegree (respectively, the outdegree) of $v$ is at least $k$. Hence, since $2k = d_G(v) = \rho_D(v) + \delta_D(v)$ we have $\rho_D(v) = \delta_D(v) = k$. Now suppose for a contradiction that the two parallel edges between $u$ and $v$ have the same direction, say from $u$ to $v$. Then the set of vertices that have an outgoing arc to $v$ is smaller than $k$ and deleting this set results in a digraph that is not strongly connected, a contradiction. □

For $U \subseteq V$, a pair of dipaths of $D$ (respectively, paths of $G$) is called $U$-disjoint if each vertex of $U$ is contained in at most one dipath (respectively, path). Let $X$ and $Y$ be two disjoint vertex sets. A $k$-difan from $X$ to $Y$ (respectively, a $k$-fan joining $X$ and $Y$) is a set of $k$ pairwise $U$-disjoint dipaths from $X$ to $Y$ (respectively, paths joining $X$ and $Y$) where $U$ is defined by $U = V \setminus (X \cup Y)$ if $|X| = |Y| = 1$, $U = V \setminus X$ if $|X| = 1$ and $|Y| > 1$, $U = V \setminus Y$ if $|Y| = 1$ and $|X| > 1$, $U = V$ if $|X| > 1$ and $|Y| > 1$.

By Menger’s theorem [8], a pair $u, v$ of vertices of $D$ is $k$-connected if and only if there exist a $k$-difan from $u$ to $v$ and a $k$-difan from $v$ to $u$. Let $X$ be a $k$-connected set of at least $k$ vertices and let $v$ be a vertex in $V \setminus X$ such that there exist a $k$-difan from $X$ to $v$ and a $k$-difan from $v$ to $X$; then, it is easy to prove that $X \cup v$ is $k$-connected.

Kaneko and Ota [10] showed that a pair $u, v$ of vertices of $G$ is weakly $2k$-connected if and only if there exist $2$ edge-disjoint $k$-fans joining $u$ and $v$. Let $X$ be a weakly $2k$-connected set of at least $k$ vertices and let $v$ be a
vertex in $V \setminus X$ such that there exist 2 edge-disjoint $k$-fans joining $v$ and $X$; then, it is easy to prove that $X \cup v$ is weakly $2k$-connected. Let $X$ and $Y$ be two disjoint weakly $2k$-connected sets each of at least $k$ vertices such that there exist 2 edge-disjoint $k$-fans joining $X$ and $Y$; then, it is easy to prove that $X \cup Y$ is weakly $2k$-connected.

2 Counterexamples

We first disprove Conjecture 2 for $k = 3$ and then extend the idea of the proof to higher connectivity. We recall that $G_3$ is the graph defined in Figure 1.

![Graph G_3](image)

Figure 1: $G_3$ every thick and red edge represents a pair of parallel edges and black edges represent simple edges.

**Proposition 2.** The graph $G_3$ is weakly 6-connected and has no 3-connected orientation.

**Proof.** First we show that $G_3$ is weakly 6-connected. Observe that there exist 2 edge-disjoint 3-fans joining any pair of vertices in $A \setminus w_a$. Then, note that there exist 2 edge-disjoint 3-fans joining $w_a$ and $A \setminus w_a$. Hence $A$ is weakly 6-connected. Symmetrically $B$ is also weakly 6-connected. There exist 2 edge-disjoint 3-fans joining $A$ and $B$ so $A \cup B$ is weakly 6-connected. There exists 2 edge-disjoint 3-fans joining $x$ (respectively, $y$) and $A \cup B$. It follows that $G_3$ is weakly 6-connected.

Suppose for a contradiction that $G_3$ has a 3-connected orientation $D$. Note that every pair of parallel edges is incident to a vertex of degree 6 and the maximal edge multiplicity is 2. Hence, by Proposition 1, the two edges in every parallel pair have opposite directions in $D$. Thus, in $D$ the orientation of the edges of the path $u_a v_a w_b y x w_a v_b u_b$ results in a directed path from $u_a$...
to $u_b$ or from $u_a$ to $u_b$. In particular both $v_aw_b$ and $v_bw_a$ are directed from $A$ to $B$ or from $B$ to $A$. In both cases $D - \{x, y\}$ is not strongly connected, a contradiction.

We mention that $G_3$ is not a minimal counterexample. Indeed the graph $H_3$ obtained from $G_3$ by deleting the two vertices $t_a$ and $t_b$ and adding the new edges $u_av_a, v_ay, yu_a, u_bv_b, v_bx$ and $xu_b$ is weakly 6-connected but has no 3-connected orientation. (Suppose that $H_3$ has a 3-connected orientation $D$. Then, by Proposition 1 in $D$ the orientation of the edges of the two triangles $v_ayw_b$ and $v_bxw_a$ results in circuits. Considering the cut $\{x, y\}$, we see that those circuits must be either both clockwise or both counterclockwise, say clockwise. Hence, by Proposition 1 in $D$ the orientation of the path $u_axu_b$ results in a dipath from $u_a$ to $u_b$ or from $u_b$ to $u_a$. In the first case $D - \{y, v_b\}$ is not strongly connected, in the other case $D - \{x, v_a\}$ is not strongly connected.)

We now extend this construction to higher connectivity. Let $k \geq 4$ be an integer. We define the graph $G_k = (V, E)$ as follows (see Figure 2). Let $n \geq k^2$ be an odd integer. The vertex set $V$ is the union of the pairwise disjoint sets $A, B, C$ and $\{w, x, y, z\}$ where $|A| = |B| = n$ and $|C| = k - 3$. Now we add simple edges such that each of $A$ and $B$ induces a complete simple graph. Choose arbitrarily one vertex from each of $A, B$ and $C$, say $a \in A, b \in B$ and $c \in C$ and add the cycle $azywbc$. By the choice of $n$, we can now add pairs of parallel edges between vertices in $A \cup B \setminus \{a, b\}$ and

![Diagram](image-url)

Figure 2: $G_k$ every thick and red edge represents a pair of parallel edges and black edges represent simple edges.
Let $C \cup \{w, x, y, z\}$ such that each vertex of $A \cup B$ is incident to at most one pair of parallel edges,

\[
d_{G_k}(v, A) = d_{G_k}(v, B) = 2\lceil \frac{k}{2} \rceil \text{ for all } v \in C - c,
\]

\[
d_{G_k}(c, A) = d_{G_k}(c, B) = 2\lceil \frac{k}{2} \rceil + 1,
\]

\[
d_{G_k}(w, A) = d_{G_k}(z, B) = 2k - 2,
\]

\[
d_{G_k}(y, A) = d_{G_k}(x, B) = 2 \text{ and}
\]

\[
d_{G_k}(x, A) = d_{G_k}(y, B) = 2k - 4.
\]

**Proposition 3.** Let $k \geq 4$ be an integer. The graph $G_k$ is Eulerian, weakly $2k$-connected and has no $k$-connected orientation.

**Proof.** Since $n$ is odd, both of the complete graphs induced by $A$ and $B$ are Eulerian. Hence $G_k$, which is obtained from those graphs by adding a cycle and parallel edges, is Eulerian. Since $k \geq 4$, $n \geq k^2 \geq 2k + 2$ thus both of the complete graphs induced by $A$ and $B$ are weakly $2k$-connected. Note that there exist 2 edge-disjoint $k$-fans joining $A$ and $B$ (one uses $C \cup \{w, x, y\}$ the other one uses $C \cup \{x, y, z\}$), thus $A \cup B$ is weakly $2k$-connected. Note also that, for any vertex $v \in C \cup \{w, x, y, z\}$, there exist 2 edge-disjoint $k$-fans joining $v$ and $A \cup B$. Hence, $G_k$ is weakly $2k$-connected.

Suppose for a contradiction that $G_k$ has a $k$-connected orientation $D$. Since $d_{G_k}(w) = d_{G_k}(x) = d_{G_k}(y) = d_{G_k}(z) = 2k$ and by Proposition 1 the orientation of the set of simple edges of the path $azyxwb$ results in the dipath $azyxwb$ or the dipath $bwxyza$. In both cases, $D - (C \cup \{x, y\})$ is not strongly connected, a contradiction.

Note that with a slightly more elaborate construction we can obtain a counterexample such that $|V| = O(k)$.

### 3 NP-completeness

In this section we prove the following result.

**Theorem 1.** Let $k \geq 3$ be an integer. Deciding whether a graph has a $k$-connected orientation is NP-complete. This holds also for Eulerian graphs.

A reorientation of a digraph $D$ is a digraph obtained from $D$ by reversing a subset of arcs. Obviously, the problem of finding a $k$-connected orientation of a graph and the problem of finding a $k$-connected reorientation of a digraph are equivalent. For convenience we prove the NP-completeness of the
second problem by giving a reduction from the problem of NOT-ALL-EQUAL 3-SAT which is known to be NP-complete [11].

Let $\Pi$ be an instance of NOT-ALL-EQUAL 3-SAT and let $k \geq 3$ be an integer. We define a directed graph $D_k = D_k(\Pi) = (V, A)$ such that there exists a $k$-connected reorientation of $D_k$ if and only if there is an assignment of the variables which satisfies $\Pi$.

The construction of $D_k$ associates to each variable $x$ a circuit $\Delta_x$ and to each pair $(C, x)$ where $x$ is a variable that appears in the clause $C$ a special arc $e_x^C$ (see Figure 3). A reorientation of $D_k$ is called consistent if the orientation of parallel edges is preserved and, for each variable $x$, the orientations of the special arcs of type $e_x^C$ and the circuit $\Delta_x$ are either all preserved or all reversed. A consistent reorientation of $D_k$ defines a natural assignment of the variables in which a variable $x$ receives value true if $\Delta_x$ is preserved and false if $\Delta_x$ is reversed. We define reciprocally a natural consistent reorientation from an assignment of the variables.

![Figure 3: Representation of the circuits and the special arcs of $D_3(\Pi)$ where $\Pi$ is composed of the clauses $C = (x, y, z)$ and $C' = (x, y, z)$. The dashed boxes represent the clause-variable gadgets.](image-url)

For each clause $C$ we construct a $C$-gadget (see Figure 4) that uses the special arcs associated to $C$. The purpose of the $C$-gadgets is to obtain the following property.

**Proposition 4.** An assignment of the variables satisfies $\Pi$ if and only if it defines a natural consistent $k$-connected reorientation.

For each pair $(C, x)$ where $C$ is a clause and $x$ is a variable that appears
in \( C \) we define a \((C, x)\)-gadget (see Figure 5) which links the orientation of \( \Delta_x \) to the orientation of \( e^C_x \). We will prove the following fact.

**Proposition 5.** If there exists a \( k \)-connected reorientation of \( D_k \) then there exists a consistent \( k \)-connected reorientation of \( D_k \).

Figure 4: A clause gadget for \( k = 3 \) and \( C = (x, y, z) \). Each red and thick edge represents a pair of parallel arcs in opposite directions.

Figure 5: A \((C, x)\)-gadget for \( k = 3 \) and \( x \in C \). Each red and thick edge represents a pair of parallel arcs in opposite directions.

Let \( L \) be a set of \( k - 1 \) vertices. We construct a clause gadget as follows. For a clause \( C \) composed of the variables \( x, y, z \) we add the vertices \( w^C, u^C_x, u^C_y, u^C_z \). We add arcs such that \( L \cup w^C \) induces a complete digraph. We add the special arc \( w^C u^C_x \) if \( x \in C \) and the special arc \( u^C_x w^C \) if \( x \in C \). This special arc is denoted by \( e^C_x \). We define similarly the special arcs \( e^C_y \) and \( e^C_z \). This ends the construction of the \( C \)-gadget. Let \( W \) denote the set of all vertices of type \( w^C \).

Let \( M \) be a set of \( k - 2 \) new vertices and choose arbitrarily one vertex \( m \in M \). For each pair \((C, x)\) where \( C \) is a clause and \( x \) is a variable that appears in \( C \) we add the new vertices \( t^C_x, u^C_x, u^C_{mx}, u^C_{mx}, v^C_x \) and denote \( U^C_x = \{ u^C_x, u^C_{mx}, u^C_{mx}, u^C_{mx} \} \). We add arcs such that \( M \cup (U^C_x \setminus u^C_x) \) induces a complete digraph. We add pairs of parallel arcs in opposite directions between the pairs of vertices \((v^C_x, t^C_x)\), \((t^C_x, u^C_{mx})\), \((u^C_{mx}, u^C_x)\), \((u^C_x, u^C_{mx})\) and all the pairs of type \((t^C_x, m')\) and \((u^C_x, m')\) for each \( m' \in M \setminus m \). Note that, so far, the undirected degree of \( t^C_x \) and \( u^C_x \) is \( 2k - 2 \). We add an arc \( t^C_x u^C_x \) if \( x \in C \) and an arc \( u^C_x t^C_x \) if \( x \in C \). Call this arc \( f^C_x \). The definition of the \((C, x)\)-gadget is concluded by the following definition of the circuit \( \Delta_x \).
For each variable $x$ define a new vertex $v_x$ and add arcs such that $v_x$ and the set of vertices of type $t^C_x$ and $u^C_x$ induce a circuit $\Delta_x$ that traverses (in arbitrary order) all the $(C, x)$-gadgets such that $C$ is a clause containing $x$. In this circuit connect a $(C, x)$-gadget to the next $(C', x)$-gadget by adding an arc leaving the head of $f^C_x$ and entering the tail of $f^C_{x'}$ (see Figure 3). Note that now the undirected degree of $t^C_x$ and $u^C_x$ is $2k$.

We denote by $N$ the union of $L$, $M$ and all the vertices of type $v_x$ or $v^C_x$. To conclude the definition of $D_k$ we add edges such that $N$ induces a complete digraph.

The proof of Proposition 5 follows from the construction of the $(C, x)$-gadgets.

Proof of Proposition 5. Let $D'$ be a $k$-connected reorientation of $D_k$ and let $x$ be a variable. Observe that all the vertices incident to $\Delta_x$ except $v_x$ are of degree $2k$ and incident to $k - 1$ pairs of parallel edges. Hence, by Proposition 1 $\Delta_x$ is either preserved or reversed. Let $C$ be a clause in which $x$ appears. In $D' - (M \cup t^C_x)$ exactly one arc enters $U^C_x$ and exactly one arc leaves $U^C_x$ (see Figure 3). One of these arcs belongs to $\Delta_x$ and the other is the special arc $e^C_x$. Hence, by $k$-connectivity of $D'$, $e^C_x$ is reversed if and only if $\Delta_x$ is reversed.

If there exists a pair of parallel arcs in the same direction in $D'$ then reversing the orientation of one arc of this pair preserves the $k$-connectivity. Hence we may assume that in $D'$ the orientation of parallel edges is preserved.

The following fact follows easily from the definition of $D_k$. We recall that $W$ is the set of vertices of type $w^C$.

Proposition 6. In every consistent reorientation of $D_k$ the set $V \setminus W$ is $k$-connected.

Proof. Let $D'$ be a consistent reorientation of $D_k$. Clearly $N$ is $k$-connected. Let $C$ be a clause and $x$ be a variable that appears in $C$. The circuit $C_x$ contains a dipath from (respectively, to) $t^C_x$ to (respectively, from) $v_x$ that is disjoint from $M \cup v^C_x$. Hence $N \cup t^C_x$ is $k$-connected.

We may assume without loss of generality that, in $D' - (M \cup t^C_x)$, the special arc $e^C_x$ enters $U^C_x$ and an arc of $\Delta_x$ leaves $U^C_x$. Let $u$ be a vertex of $U^C_x$. Observe that there is a $k$-difan from $u$ to $M \cup t^C_x \cup v_x$ (the dipath to $v_x$ uses arcs of $\Delta_x$). Observe that there is a $k$-difan from $M \cup t^C_x \cup L$ to $u$ (the dipath from $L$ uses the arc $e^C_x$). Hence, since $M$ and $L$ are subsets of $N$, $N \cup t^C_x \cup U^C_x$ is $k$-connected and the proposition follows.
We can now prove Proposition 4.

Proof of Proposition 4. Let $\Omega$ be an assignment of the variables and $D'$ the natural consistent reorientation of $D_k$ defined by $\Omega$. Let $e^C_x$ be a special arc associated to a clause $C$ and a variable $x$. In $D'$, the arc $e^C_x$ leaves $w_C$ if and only if $x = \text{true}$ and $x \in C$ or $x = \text{false}$ and $\bar{x} \in C$. And, in $D'$, the arc $e^C_x$ enters $w_C$ if and only if $x = \text{true}$ and $\bar{x} \in C$ or $x = \text{false}$ and $x \in C$. Hence $C$ contains a true (respectively, false) value if and only if there exists a special arc leaving (respectively, entering) $w_C$ in $D'$. Thus a clause $C$ is satisfied by $\Omega$ if and only if $w_C$ is left by at least one special arc and entered by at least one special arc. ($\star$)

Observe that, for each clause $C$, the only arcs incident to $w_C$ in $D' - L$ are special. Since $|L| = k - 1$, if $D'$ is $k$-connected then ($\star$) holds for all clauses thus $\Omega$ satisfies $\Pi$. Conversely, if $\Omega$ satisfies $\Pi$ then for every clause $C$ ($\star$) holds and $w_C$ has at least $k$ out-neighbors and at least $k$ in-neighbors. Thus by Proposition 6 $D'$ is $k$-connected.

Denote by $G'_k = G'_k(\Pi)$ the underlying undirected graph of $D_k(\Pi)$. We can now prove the main theorem of this section.

Proof of Theorem 1. By Propositions 5 and 4, $G'_k(\Pi)$ has a $k$-connected orientation if and only if there exists an assignment satisfying $\Pi$. Since the order of $G'(\Pi)$ is a linear function of the size of $\Pi$ and $\text{NOT-ALL-EQUAL}$ $3$-$\text{SAT}$ is NP-complete [11] this proves the first part of Theorem 1.

Observe that in $G'_k$ the only vertices of odd degree are of type $u^C_x$ and $w^C$. Let $l$ be an arbitrary vertex of $L$. We can add a set $F$ of edges of type $u^C_xm, ml, lw^C$ such that $G'_k + F$ is Eulerian. Observe that for any orientation of $F$, Propositions 5 and 4 still hold for $D_k + F$. This proves the second part of Theorem 1.

The following fact shows that $G'_k(\Pi)$ is a counterexample to Conjecture 2 if $\Pi$ is not satisfiable.

Proposition 7. The graph $G'_k(\Pi)$ is weakly $2k$-connected.

Proof. By Proposition 6, $V \setminus W$ is $k$-connected in $D_k$, thus $V \setminus W$ is weakly $2k$-connected in $G'_k$. Since there exist 2 edge-disjoint $k$-fans from $w^C$ to $V \setminus W$ for every clause $C$, $G'_k$ is weakly $2k$-connected. ■
We now construct an Eulerian counterexample to Conjecture \(^2\) for \(k = 3\). Let \(x\) be a variable and \(C = (x, x)\) be a clause. Let \(H'_3\) be the Eulerian graph obtained from \(G'_3(\{C\})\) by adding an edge \(u_C^m\) in each of the two copies of the \((C, x)\)-gadget. The next result follows from the discussion above.

**Proposition 8.** \(H'_3\) is an Eulerian weakly 6-connected graph that has no 3-connected orientation.

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