VALIDITY OF HYDROSTATIC EQUILIBRIUM IN GALAXY CLUSTERS FROM COSMOLOGICAL HYDRODYNAMICAL SIMULATIONS

Daichi Suto (須藤 大地), Hajime Kawahara (河原 創)²,³, Tetsu Kitayama (北山 哲)⁴, Shin Sasaki (佐々木 伸)², Yasushi Suto (須藤 謙)¹,⁵,⁶, and Renyue Cen⁶

¹ Department of Physics, The University of Tokyo, Tokyo 113-0033, Japan; daichi@utap.phys.s.u-tokyo.ac.jp
² Department of Physics, Tokyo Metropolitan University, Hachioji, Tokyo 192-0397, Japan
³ Department of Earth and Planetary Science, The University of Tokyo, Tokyo 113-0033, Japan
⁴ Department of Physics, Toho University, Funabashi, Chiba 274-8510, Japan
⁵ Research Center for the Early Universe, School of Science, The University of Tokyo, Tokyo 113-0033, Japan
⁶ Princeton University Observatory, Princeton, NJ 08544, USA

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ABSTRACT

We examine the validity of the hydrostatic equilibrium (HSE) assumption for galaxy clusters using one of the highest-resolution cosmological hydrodynamical simulations. We define and evaluate several effective mass terms corresponding to the Euler equations of gas dynamics, and quantify the degree of the validity of HSE in terms of the mass estimate. We find that the mass estimated under the HSE assumption (the HSE mass) deviates from the true mass by up to ∼30%. This level of departure from HSE is consistent with the previous claims, but our physical interpretation is rather different. We demonstrate that the inertial term in the Euler equations makes a negligible contribution to the total mass, and the overall gravity of the cluster is balanced by the thermal gas pressure gradient and the gas acceleration term. Indeed, the deviation from the HSE mass is well explained by the acceleration term at almost all radii. We also clarify the confusion of previous work due to the inappropriate application of the Jeans equations in considering the validity of HSE from the gas dynamics extracted from cosmological hydrodynamical simulations.

Key words: cosmology: theory – galaxies: clusters: general – methods: numerical – X-rays: galaxies: clusters

Online-only material: color figures

1. INTRODUCTION

Clusters of galaxies are important sources of various cosmological and astrophysical information, especially on the formation history of the large-scale structure and the estimates of cosmological parameters (Allen et al. 2011 for a recent review). Among other quantities, the mass of clusters is one of the most fundamental quantities in virtually all studies. The most conventional method is based on X-ray observations of the intracluster medium (ICM) combined with the assumption that the ICM is in hydrostatic equilibrium (HSE) with the total gravity of the cluster (we call the mass estimated by this method “the HSE mass”). It is unlikely, however, that the HSE assumption strictly holds, especially for unrelaxed clusters given their ongoing dynamical evolution. Therefore, it is important to examine the validity of the HSE assumption, which has mostly been assumed only for simplicity. The quantitative analysis of its validity and limitation is directly related to its applicability to future scientific opportunities, including upcoming X-ray missions such as the extended Röntgen Survey with Imaging Telescope Array⁶ and ASTRO-H,⁷ and observations of the Sunyaev–Zel’dovich effect performed by the Atacama Cosmology Telescope⁸ and the South Pole Telescope⁹.

The validity of the HSE assumption for observed clusters may be examined in a straightforward fashion by comparing the HSE mass with the cluster mass estimated by other methods. In this respect, gravitational lensing is particularly suited because it directly probes the total gravitational mass without any assumption on the dynamical state of dark matter. On the other hand, the lensing observations require a high angular resolution of the background galaxy images and are feasible only for a limited number of clusters located at z ≤ 0.5. In addition, the estimated lensing mass corresponds to the cylindrical mass along the line of sight, and may include an extra contribution not associated with the cluster itself. Previous studies (e.g., Mahdavi et al. 2008; Zhang et al. 2008, for recent ones) show that the HSE mass is smaller by approximately 20% on average than the lensing mass, suggesting that either HSE or lensing, or even both, should be systematically biased.

Another method to examine the validity of the HSE assumption that we pursue in this paper is to use numerical simulations, which enable us to make a detailed and critical comparison of the simulated data against the model prediction. Therefore, we can locate the origin of the systematic bias, if any, of the HSE assumption. This is useful because we may be able to apply the correction to the observational data eventually.

Of course, there are a number of previous studies of HSE using simulated clusters, but their results do not seem to be converged. For instance, let us focus on a couple of recent papers (Fang et al. 2009; Lao et al. 2009) that studied systematic errors in the HSE mass using the same set of 16 clusters simulated by Nagai et al. (2007). Fang et al. (2009) analyzed the gas particle data on the basis of the Euler equations, and evaluated the effective mass terms corresponding to several different terms in the equations. Lao et al. (2009) performed basically the same analysis, but used the Jeans equations instead of the Euler equations, despite the fact that they considered the gas particles in the simulated clusters. Both reached the same conclusion that the HSE mass underestimates the true mass of clusters systematically by ∼10%–20%. Nevertheless, their physical interpretations of the origin of the bias are very
different; Fang et al. (2009) claimed that the coherent rotation of gas plays a significant role as an additional support against gravity, while Lau et al. (2009) concluded that the random gas motion is responsible for the departure from HSE, and the gas rotation makes a relatively negligible contribution.

The purpose of the present paper is to clarify the theoretical method used to examine HSE of the simulated clusters, and then to revisit its validity using a high-resolution hydrodynamical simulation by Cen (2012). In particular, we compare the two different analysis formulations adopted by Fang et al. (2009) and Lau et al. (2009), and argue that the Euler equations, rather than the Jeans equations modified by a gas pressure gradient term (Rasia et al. 2004; Lau et al. 2009), should be used in analyzing the gas dynamics.

The rest of the paper is organized as follows. Section 2 formulates our method to evaluate the validity of the HSE assumption. We pay particular attention to the comparison between the Euler and Jeans equations in analyzing the gas dynamics. The derivation of those sets of equations starting from the Boltzmann equations is summarized in Appendix A. Section 3 briefly describes the simulated cluster, and then presents our analysis results of the validity of HSE. Finally, our conclusion is summarized in Section 4. The analysis based on the Jeans equations using dark matter particles is shown in Appendix B.

2. THEORETICAL FORMULATION

2.1. Method to Examine the Validity of HSE based on the Euler Equations

Our analysis method to examine the validity of HSE using gas in a simulated cluster is based on the Euler equations:

$$\frac{\partial \rho}{\partial t} + (\mathbf{v} \cdot \nabla) \rho = - \frac{1}{\rho_{\text{gas}}} \nabla p - \nabla \phi, \tag{1}$$

where $\phi$ is the gravitational potential, and $\rho_{\text{gas}}, \mathbf{v}$, and $p$ are the density, velocity, and pressure of gas. While the Jeans equations do not describe the gas dynamics in their original form, Rasia et al. (2004) and Lau et al. (2009) added the gas pressure gradient term to the Jeans equations and adopted the resulting equations when analyzing simulated clusters. We argue in the next subsection that this is not justified, and present the result based on the Jeans equations but using collisionless dark matter particles in Appendix B, for reference.

We define the total mass $M_{\text{tot}}$ of a cluster inside a volume $V$ as

$$M_{\text{tot}} = \int_V d^3x \rho_{\text{tot}}, \tag{2}$$

where $\rho_{\text{tot}}$ is the total density of the cluster. For the simulated cluster considered throughout this paper, $\rho_{\text{tot}}$ consists of densities of gas, dark matter, and stars, i.e., $\rho_{\text{tot}} = \rho_{\text{gas}} + \rho_{\text{dm}} + \rho_{\text{star}}$. The total mass can be rewritten in terms of $p$ and $\mathbf{v}$ using Poisson’s equation and Gauss’s theorem:

$$M_{\text{tot}} = \frac{1}{4\pi G} \int_{3V} dS \cdot \nabla \phi \tag{3}$$

$$= \frac{1}{4\pi G} \int_{3V} dS \left[ -\frac{1}{\rho_{\text{gas}}} \nabla p - (\mathbf{v} \cdot \nabla) \rho - \frac{\partial \mathbf{v}}{\partial t} \right],$$

where $\partial V$ is the surface surrounding the volume $V$ and we have used Equation (1) in the second equality. Now the total mass is evaluated by the gas quantities alone, without any knowledge of dark matter and stars. This is why the present method is applicable, in principle, to the X-ray data of galaxy clusters.

If we adopt a spherical surface as $\partial V$, then the total mass can be decomposed into the following four effective mass terms:

$$M_{\text{tot}} = M_{\text{therm}} + M_{\text{rot}} + M_{\text{stream}} + M_{\text{accel}}, \tag{4}$$

where

$$M_{\text{therm}} = -\frac{1}{4\pi G} \int_{3V} dS \frac{1}{\rho_{\text{gas}}} \frac{\partial p}{\partial r}, \tag{5}$$

$$M_{\text{rot}} = \frac{1}{4\pi G} \int_{3V} dS \frac{v_\theta^2 + v_\phi^2}{r}, \tag{6}$$

$$M_{\text{stream}} = -\frac{1}{4\pi G} \int_{3V} dS \left[ \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_r}{\partial \varphi} \right], \tag{7}$$

and

$$M_{\text{accel}} = -\frac{1}{4\pi G} \int_{3V} dS \frac{\partial v_r}{\partial t}. \tag{8}$$

We emphasize here that the above set of equations does not assume spherical symmetry of the system; we just take a spherical surface as the integral surface and write down Equation (3) in spherical coordinates.

The first term, $M_{\text{therm}}$, originates from the thermal pressure gradient of gas. If the gas motion is negligible, then $M_{\text{therm}}$ should be equal to the total mass $M_{\text{tot}}$, and thus regarded as a cluster mass estimated under the HSE assumption. In other words, the difference between $M_{\text{tot}}$ and $M_{\text{therm}}$ is a quantitative measure of the departure from the HSE assumption.

The inertial term $(\mathbf{v} \cdot \nabla) \rho$ in the Euler equations reduces to $M_{\text{rot}}$ and $M_{\text{stream}}$. If there exists a coherent rotational motion around the center of the cluster, then $M_{\text{rot}}$ can be interpreted as the centrifugal force term. Without such a motion, however, the local tangential velocity of different directions at different locations on the sphere could make $M_{\text{rot}}$ very large. During the course of cluster evolution, gas generally falls toward the center of the cluster with larger streaming speed and increasing radius from the center. In this case, $M_{\text{stream}}$ becomes negative and cannot be neglected. On the other hand, it becomes positive and/or negligible, especially in the innermost region where the gas velocity is more randomized than that in the outer regions.

Finally, the acceleration term, $M_{\text{accel}}$, corresponds to the $-\frac{\partial v_r}{\partial t}$ term in the Euler equations and becomes positive/negative when gas is decelerating/accelerating.

All the mass terms are invariant with respect to the choice of the axis of the spherical coordinates, but are not necessarily positive. Note also that $M_{\text{rot}}$ and $M_{\text{stream}}$, corresponding to the inertial term, are not invariant with respect to the Galilean transformation. Thus, we evaluate those in the center-of-mass frame of the entire simulated cluster.

2.2. Comparison with Analysis Methods Adopted by Previous Work

The set of basic equations that we adopt in this paper is essentially identical to that of Fang et al. (2009), except for the fact that they interpreted the difference between $M_{\text{tot}}$ and $M_{\text{therm}} + M_{\text{rot}} + M_{\text{stream}}$ as “turbulent gas motion” while we call it the acceleration term $M_{\text{accel}}$, as directly implied from Equation (1), and evaluate it from the residual, $M_{\text{accel}} = M_{\text{tot}} - M_{\text{therm}} - M_{\text{rot}} - M_{\text{stream}}$. 

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We are not sure why they ascribed the term to the turbulent motion. It is true that part of the gas acceleration would be due to the turbulent gas motion, but not entirely. Furthermore, the numerical simulation does not include any physical processes directly related to the turbulent motion. Even if the turbulent motion might be important for real clusters, it should come from some physics below the subgrid scales that cannot be properly resolved in the numerical simulation. The effects of gas random motion above the resolved scales should be included in the numerical simulation. The effects of gas random motion might be important for real clusters, it should come from such physics.

In previous literature, the Jeans equations are sometimes used to analyze the gas motion in simulated clusters. For that purpose, a thermal pressure gradient term is added by hand to the basic tensor in the Jeans equations. The right-hand side of the above equation originate from “random gas motion.” We are not sure why they ascribed the term to the turbulent motion.

3. APPLICATION TO A SIMULATED CLUSTER

We briefly describe the main features of the current simulation and refer readers to Cen (2012) for more detail. The simulation was run first with a low-resolution mode in a periodic box of 120 $h^{-1}$ Mpc on a side. Then, a region centered on a cluster with a mass of $\sim 2 \times 10^{14} M_\odot$ was resimulated with a higher resolution in an adaptively refined manner. The size of the refined region is $21 \times 24 \times 20 (h^{-1} \text{Mpc})^3$. The mean interparticle separation and the dark matter particle mass in the refined region are $117 h^{-1} \text{kpc}$ comoving and $1.07 \times 10^8 h^{-1} M_\odot$, respectively.

Star particles are created according to the prescription of Cen & Ostriker (1992). Their typical mass is $\sim 10^6 M_\odot$. The simulation includes a metagalactic UV background (Haardt & Madau 1996), shielding of UV radiation by neutral hydrogen (Cen et al. 2005), and metallicity-dependent radiative cooling (Cen et al. 1995). While supernova feedback is modeled following Cen et al. (2005), active galactic nucleus feedback is not included in this simulation. The cosmological parameters used in this simulation (Hubble constant, dark energy, and baryon density) are $(H_0, \Omega_M, \Omega_b, h, n_s, \sigma_8) = (0.046, 0.28, 0.72, 0.70, 0.96, 0.82)$, following the WMAP7-normalized ΛCDM model (Komatsu et al. 2011).

Then, a cluster is identified and the cubic box of a side of $3.8 h^{-1}$ Mpc surrounding the entire cluster is extracted from the simulation data. The dark matter and stars are represented by particles, and the temperature and density of gas are given on the $520^3$ grids (the grid length is $7.324 h^{-1} \text{kpc}$).

The radius $r_{500}$ of the cluster is $\sim 640 h^{-1} \text{kpc}$ ($r_{500}$ is defined so that the mean density inside $r_{500}$ is 500 times the critical density of the universe). The center-of-mass velocity of the cluster within $r_{500}$ is set to vanish. The total mass $M_{500}$ within $r_{500}$ is $\sim 2 \times 10^{14} M_\odot$. The average ICM temperature at $r_{500}$ is $\sim 2 \text{keV}$, and the circular speed there is $v_{500} = \sqrt{GM_{500}/r_{500}} \sim 1000 \text{ km s}^{-1}$.

Projected surface densities of gas, dark matter, and stars on the $x-z$ plane are plotted in the left panels of Figure 1. The right panels of Figure 1 show the three-dimensional view of the three components. The left (right) plots are color-coded according to the surface (space) densities normalized by the fraction of each component averaged over the box, $\Omega_k$ (k = gas, dark matter, and stars). Note that the fraction $\Omega_k$ is different from the density parameter $\Omega_k$ because the box is selected preferentially around the cluster.

As is clear from Figure 1, the gas distribution is smoother but traces the underlying dark matter distribution very well. In contrast, stars are more significantly concentrated in high density regions, and exhibit numerous small clumps, most of which are not identified/resolved in the gas distribution.

Figure 2 plots the radial density and mass profiles of the cluster. The stellar fraction in the inner region ($r < 200 h^{-1} \text{kpc}$) is significantly higher than the typical observed value. This is a well-known common problem among current high-resolution cosmological simulations, and implies that some important
baryon physics including high-energy phenomena and star formation is still missing in the simulation. We perform the analysis of cluster gas, assuming that the excess star densities in the inner region do not affect our conclusions at the outer radius.

Figure 3 represents velocity fields in the x–y, y–z, and z–x planes passing through the center of the cluster. The red/blue arrows have negative/positive radial velocity, showing that the gas in the outer regions ($r \gtrsim 1 \, h^{-1} \text{Mpc}$) falls toward the center while its direction is randomized in the inner region.

3.2 Results

We evaluate the effective mass terms defined in Section 2 for the simulated cluster, which is plotted in Figure 5; the left panel shows the mass profiles, while the right panel indicates their fractional contribution to the total mass within the radius.

The total mass $M_{\text{tot}}(r)$ is computed by directly summing all the dark matter and star particles and gas of grids within the sphere of $r$. The other terms, $M_{\text{therm}}$, $M_{\text{rot}}$, and $M_{\text{stream}}$, require the pressure and velocity fields evaluated at $r$. For that
Figure 2. Radial profiles of densities (left) and masses (right) are shown for gas (red), dark matter (green), and stars (blue). The black line in the right panel shows the total gravitational mass; $M_{\text{tot}} = M_{\text{gas}} + M_{\text{dm}} + M_{\text{star}}$. The analysis is performed on the 50 logarithmically equal radial bins.

Figure 3. Velocity fields in $x$–$y$ (left), $y$–$z$ (middle), and $z$–$x$ (right) planes passing through the center of the cluster. The arrow is red if $v_r < 0$, and blue if $v_r > 0$. The length of the arrow is proportional to the magnitude of the velocity. An arrow with a speed of 1000 km s$^{-1}$ is shown for reference.

In this way, $M_{\text{therm}}(r)$, $M_{\text{rot}}(r)$, and $M_{\text{stream}}(r)$ are computed by integrating the corresponding integrands evaluated above. Finally, we estimate $M_{\text{accel}}$ simply from the residual of $M_{\text{accel}} = M_{\text{tot}} - M_{\text{therm}} - M_{\text{rot}} - M_{\text{stream}}$, since we have the cluster data at $z = 0$ alone. This estimate for $M_{\text{accel}}$ may be different from the original definition, i.e., Equation (8). Indeed, when we attempted to compute $M_{\text{accel}}$ directly from the box mentioned in Section 3, it turned out to be too small to obtain a correct gravitational potential for the entire cluster. Thus, we go back to a larger simulation box with a side of 22.5 $h^{-1}$ Mpc and a grid length of 29.34 $h^{-1}$ kpc that encloses our cluster. Then, we compute the gravitational potential using a fast Fourier transform to obtain the gas acceleration at each grid point. This enables us to directly calculate $M_{\text{accel}}$. Figure 4 is a comparison of $M_{\text{accel}}$s calculated by two methods. The directly calculated $M_{\text{accel}}$ (magenta line) is in good agreement with $M_{\text{tot}} - M_{\text{therm}} - M_{\text{rot}} - M_{\text{stream}}$ (black line), although there is a large difference between the two within 200 $h^{-1}$ kpc. Also, we make sure that the sum $M_{\text{therm}} + M_{\text{rot}} + M_{\text{stream}} + M_{\text{accel}}$ reproduces $M_{\text{tot}}$ within $\sim 2\%$, except for the innermost region ($r < 200 h^{-1}$ kpc), where it deviates from $M_{\text{tot}}$ by up to $\sim 9\%$. Thus, the estimation of $M_{\text{accel}}$ by $M_{\text{tot}} - M_{\text{therm}} - M_{\text{rot}} - M_{\text{stream}}$ is sufficiently good given the quoted errors of our conclusion below. Although it seems better to use $M_{\text{accel}}$ directly calculated from Equation (8), the grid size of the larger box is so coarse that we cannot take advantage of the high resolution of the simulations in this study. Therefore, we decided to use the smaller box explained in Section 3 and $M_{\text{accel}}$ is calculated by $M_{\text{tot}} - M_{\text{therm}} - M_{\text{rot}} - M_{\text{stream}}$ in the following analysis.

The left panel of Figure 5 implies that $M_{\text{therm}}$ agrees with $M_{\text{tot}}$ reasonably well. Each of the other three terms contributes less than 10% of the total mass (the dotted curves correspond to the case in which each term becomes negative and its absolute value is plotted instead).
In order to consider the validity of HSE more quantitatively, we plot the fractional contribution of each mass term in the right panel of Figure 5. The rotation term, $M_{\text{rot}}$, is almost independent of radius. In contrast, the streaming velocity term, $M_{\text{stream}}$, is mostly negative, and varies a lot at different radii. As a result, the difference of the total mass $M_{\text{tot}}$ and the HSE mass $M_{\text{therm}}$ is mostly explained by the acceleration term $M_{\text{accel}}$ alone; compare the black and magenta curves in the right panel of Figure 5. At $r = r_{500}$ and $r_{200}$, the deviation from HSE in terms of the mass difference $(M_{\text{therm}} - M_{\text{tot}})/M_{\text{tot}}$ is about 10%. Nevertheless, the value significantly varies at different radii and it is safe to conclude that $(M_{\text{therm}} - M_{\text{tot}})/M_{\text{tot}}$ ranges approximately 10%–20% at $r < r_{200}$. Also, there is no systematic trend of the validity of the HSE assumption as a function of radius. Even though the reliability of the simulation is suspicious for $r < 200 \ h^{-1} \ \text{kpc}$ due to the excessive stellar concentration (Section 3.1), $(M_{\text{therm}} - M_{\text{tot}})/M_{\text{tot}}$ fluctuates between −10% and +25% for $300 \ h^{-1} \ \text{kpc} < r < r_{500}$. Thus, there is no guarantee that HSE becomes a better approximation toward the inner central region.

Physically speaking, $M_{\text{tot}} + M_{\text{stream}} + M_{\text{accel}} (= M_{\text{tot}} - M_{\text{therm}})$ corresponds to the term integrating the Lagrangian derivative of the gas velocity over the sphere. Therefore, the fact that it is small compared with $M_{\text{therm}}$ and $M_{\text{rot}}$ is simply translated into the condition of HSE that the gas acceleration from a Lagrangian point of view is negligible compared with the pressure gradient and the total gravity.

Since we analyze a single simulated cluster, it is not clear to what extent our interpretation that $M_{\text{accel}}$ determines the departure from HSE holds in general. Thus, we use a different set of smoothed particle hydrodynamic (SPH) simulation clusters (Dolag et al. 2009) kindly provided by Klaus Dolag. For these clusters, we assume spherical symmetry and do not divide the spherical surface at a given radius just for simplicity. Klaus Dolag also provides us with $M_{\text{accel}}$ directly computed from the acceleration data. We find that in most cases, $|M_{\text{accel}}|$ is larger than $|M_{\text{rot}}|$ and |$M_{\text{stream}}$|, especially where $M_{\text{therm}}$ deviates from $M_{\text{tot}}$. We also confirm that $M_{\text{therm}} + M_{\text{rot}} + M_{\text{stream}} + M_{\text{accel}}$ reproduces $M_{\text{tot}}$ within ~5% for most regions (see Appendix C).

The above results basically support our conclusion for the cluster from the AMR simulations, but we do not have a statistical discussion including clusters from the SPH simulations because of the differences in resolutions and simulation methods. Instead, we divide the AMR cluster into two regions; upper and lower hemispheres with respect to the $x$–$y$ plane. Then, we duplicate each hemisphere into one cluster. We call the synthetic cluster constructed from the $z > 0$ ($z < 0$) hemisphere “$z+$” (“$z−$”). Although these clusters are of course not independent of the original cluster and we cannot make a statistical argument on their properties, we can briefly look at the effects of substructures or the inhomogeneity of temperature and velocity field.

**Figure 4.** Comparison of the acceleration mass $M_{\text{accel}}$ calculated directly from the acceleration data (magenta) and $M_{\text{tot}} - M_{\text{therm}} - M_{\text{rot}} - M_{\text{stream}}$ (black). The dotted line means that its sign is inverted. The analysis is performed on the 25 logarithmically equal radial bins.

(A color version of this figure is available in the online journal.)

**Figure 5.** Effective mass terms in Equation (4) for the gas in the simulated cluster are shown in the left panel: $M_{\text{tot}}$ (black), $M_{\text{therm}}$ (red), $M_{\text{rot}}$ (green), $M_{\text{stream}}$ (blue), and $M_{\text{accel}}$ (magenta). Here, $M_{\text{accel}}$ is calculated by $M_{\text{accel}} = M_{\text{tot}} - M_{\text{therm}} - M_{\text{rot}} - M_{\text{stream}}$. The dotted line means that its sign is inverted. Ratios of mass terms to $M_{\text{tot}}$ are shown in the right panel. The black line shows $(M_{\text{tot}} - M_{\text{therm}})/M_{\text{tot}}$ and colored lines represent the same things as the left panel. The analysis is performed on the 50 logarithmically equal radial bins and 10 linearly equal bins both in polar and azimuthal angles.

(A color version of this figure is available in the online journal.)
We repeat the same analysis on these two synthetic clusters, and the results are plotted in Figure 6. The amplitudes of the different terms vary significantly between the two clusters, and the degree of the validity of HSE is also very different. Nevertheless, the generic trend is clear; the relation of $M_{\text{accel}} \approx M_{\text{tot}} - M_{\text{therm}}$ holds almost independently of $r$.

It is not clear, however, why the two hemispheres have such different values of $(M_{\text{tot}} - M_{\text{therm}})/M_{\text{tot}}$; HSE holds very well for “$z+$,” while this is not the case for “$z-$.” The visual inspection of Figure 1 does not reveal any significant difference between the two. It may be because some local concentrations of dark matter enhance the acceleration/deceleration of gas and influence the overall non-sphericity of the gas density. Thus, an analysis that accounts for the ellipticity may provide a deeper insight into the validity of HSE, but it is beyond the scope of the present paper.

4. CONCLUSION

We have examined the validity of HSE that has been conventionally assumed in estimating the mass of galaxy clusters from X-ray observations. We use a simulated cluster and evaluate several mass terms directly corresponding to the Euler equations that govern the gas dynamics. We find that the mass estimated under the HSE assumption, $M_{\text{therm}}$ in the present study, deviates from the true mass $M_{\text{tot}}$ on average by $\sim 10\% - 20\%$ fractionally for $r < r_{200}$. There is no clear tendency that the HSE becomes a better approximation toward the inner region. More importantly, we find that $M_{\text{tot}} - M_{\text{therm}}$ is nearly identical to $M_{\text{accel}}$. In other words, the validity of HSE is controlled by the amount of gas acceleration. This trend is confirmed by the separate analysis of the different hemispheres of the same simulated cluster.

Our current analysis is limited to a single simulated cluster, but the overall conclusion that the HSE mass agrees with the total mass within $10\% - 20\%$ is consistent with previous results by Fang et al. (2009) and Lau et al. (2009). Nevertheless, the interpretation of the origin of the departure from HSE is very different. Fang et al. (2009) concluded that the gas rotation term $M_{\text{rot}}$ makes a significant contribution and that $M_{\text{therm}} + M_{\text{rot}}$ well reproduces the total mass, especially for relaxed clusters. This is not the case, however, for our simulated cluster at least. Similarly, Lau et al. (2009) found a similar degree for the departure from HSE, but they ascribed the discrepancy to the random gas motion. Their analysis, however, is based on
the modification of the Jeans equations, which does not appear
to be justified for the analysis of the gas dynamics, and thus
their conclusion should be interpreted with caution.

A relatively small systematic error of the HSE mass inferred
from current numerical simulations may be partly ascribed
to the assumptions inherent in the Euler equations, i.e., local
thermal equilibrium and negligible viscosity (Appendix A). This
is supported by the fact that the error in the mass estimated
from the random motion of collisionless particles tends to be
much greater at large radii (Appendix B) because the relaxation
time scale for collisionless particles is appreciably longer than
that for collisional gas. We should also note that the HSE
mass can be influenced by other physical processes that are not
included in the numerical simulations, such as pressure support
from micro-turbulence, the magnetic field, and accelerated
particles (e.g., Lagana et al. 2010). The neglected components
mentioned above are closely linked with one another (e.g.,
viscosity can play a role in generating turbulence and the
magnetic field can affect both thermalization and acceleration
of gas particles) and will be investigated in the near future by
X-ray missions such as NuSTAR10 and ASTRO-H as well as by
radio telescopes including EVLA11 and LOFAR.12

The present analysis is fairly idealized in the sense that the
evaluation of all the mass terms has been done from the
full three-dimensional data of simulated clusters. In reality,
observational data of X-ray clusters are basically projected
along the line of sight. Therefore, additional uncertainties and
systematic errors may become important as well. Also, it is
necessary to carry out the current analysis for a number of
clusters in order to obtain the statistically robust conclusion.
These issues will be discussed elsewhere.

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APPENDIX A

RELATION BETWEEN THE EULER EQUATIONS
AND THE JEANS EQUATIONS

From a microscopic point of view, both the Euler equations
and the Jeans equations can be derived from the Boltzmann
equation under different assumptions. In the following, we
explicitly compare the two sets of equations in both Cartesian
and spherical coordinates.

A.1. Cartesian Coordinates

We define the distribution function $f$ such that

$$f(x, v, t) d^3 x d^3 v$$

is the probability that a randomly chosen particle
in the system lies in the phase space volume $d^3 x d^3 v$
at position $(x, v)$ and time $t$. The motion of such particles
under the gravitational potential $\phi$ is described by the Boltzmann
equation:

$$\frac{\partial f}{\partial t} + v_i \frac{\partial f}{\partial x_i} - \frac{\partial \phi}{\partial x_i} \frac{\partial f}{\partial v_i} = \left( \frac{\delta f}{\delta t} \right)_{\text{coll}},$$

where the collision term on the right-hand side takes into account
the collisions between particles. Note that $v_i (i = 1, 2, 3)$
represents a coordinate in the phase space and should not be
confused with the velocity field at the spatial point $x'$. For
simplicity, we assume that all particles have the same mass
$m$ in the following.

First, we consider a collisionless case with $(\delta f/\delta t)_{\text{coll}} = 0$.
Multiplying Equation (A1) by $m$ and integrating it over the
velocity space yields the continuity equation:

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho \bar{v})}{\partial x_i} = 0,$$

where

$$\rho(x, t) = \int \, d^3 v \, m f(x, v, t)$$

and we introduce the average over the velocity space:

$$\bar{q}(x, t) = \frac{1}{\rho(x, t)} \int \, d^3 v \, m q(x, v, t) f(x, v, t)$$

for an arbitrary variable $q$ such as $v_i$. Multiplying Equation (A1)
by $m v_i$ and integrating over the velocity space gives the momen-
tum equations:

$$\frac{\partial (\rho \bar{v}_i)}{\partial t} + \frac{\partial \tau_{ij}}{\partial x_j} = -\frac{\partial \rho}{\partial x_j}$$

where

$$\tau_{ij} = \rho \bar{v}_i \bar{v}_j = \rho \sigma^{2,ij} + \rho \bar{v}_i \bar{v}_j$$

and $\sigma^{2,ij}$ is the velocity dispersion tensor. Equations (A2)
and (A5) reduce to the Jeans equations:

$$\frac{\partial \bar{v}_i}{\partial t} + \bar{v}_j \frac{\partial \bar{v}_i}{\partial x_j} = -\frac{1}{\rho} \frac{\partial (\rho \sigma^{2,ij})}{\partial x_j} - \frac{\partial \phi}{\partial x_i}.$$ (A7)

Next, we consider a collisional case. Rigorous handling of the
collisional term is rather complicated and simplified models are
often used. A conventional one is the Bhartnagar–Gross–Krock
equation, which employs a linearized collisional term:

$$\frac{\partial f}{\partial t} + v_i \frac{\partial f}{\partial x_i} - \frac{\partial \phi}{\partial x_i} \frac{\partial f}{\partial v_i} = - \frac{f - f_0}{\tau},$$

where $\tau$ is the relaxation time of the system considered, $f_0$ is
the Maxwellian distribution function characterized by the local
temperature $T(x, t)$:

$$f_0(x, v, t) = \left[ \frac{m}{2 \pi k_B T(x, t)} \right]^{3/2} \exp \left[ -\frac{m(v - \bar{v}(x, t))^2}{2 k_B T(x, t)} \right],$$

where $\bar{v}(x, t)$ is the time-averaged velocity at position $x$. This
expression is obtained by using the statistical mechanics of
a gas and assuming the gas is in local thermal equilibrium.
and $k_B$ is the Boltzmann constant. If we assume that mean values of conservatives such as mass and momentum are the same as those in local thermal equilibrium, then the collision term vanishes in the continuity and momentum equations. In local thermal equilibrium, pressure is defined from the diagonal components of $\sigma_{ij}^2$ by $p \delta_{ij} \equiv \rho \sigma_{ij}^2$ and the dispersion tensor can be written as

$$
\tau_{ij} = p \delta_{ij} + \rho \vec{v} \cdot \vec{\sigma}_{ij}^i.
$$

(A10)

If we replace $\tau_1$ in Equation (A5) with $\tau_E$ and combine them with Equation (A2), then we obtain the Euler equations:

$$
\frac{\partial \vec{v}_i}{\partial t} + \vec{v} \cdot \nabla \vec{v}_i = -\frac{1}{\rho} \frac{\partial p}{\partial x_i} - \frac{\partial \phi}{\partial x_i}.
$$

(A11)

The difference between the Euler and the Jeans equations resides only in the form of the dispersion tensor.

If we retain the off-diagonal components of the dispersion tensor, then they can be interpreted as viscosity, and the equations reduce to the Navier–Stokes equations (Choudhuri 1998; Chapman & Cowling 1970).

A.2. Spherical Coordinates

One can rewrite the continuity and momentum equations in the previous section in general coordinates

$$
\frac{\partial \rho}{\partial t} + \nabla_i(\rho \vec{v}_i) = 0
$$

(A12)

and

$$
\frac{\partial (\rho \vec{v}_i)}{\partial t} + \nabla_j \tau_{ij} = -\rho g_{ij} \nabla_j \phi,
$$

(A13)

by using the covariant derivative operator $\nabla_i$. For spherical coordinates $(x^1 = r, x^2 = \theta, x^3 = \phi)$, the non-zero components of the metric tensor $g^{ij}$ are

$$
g_{11} = 1, \quad g_{22} = r^2, \quad g_{33} = r^2 \sin^2 \theta
$$

(A14)

and the corresponding non-zero connection coefficients are

$$
\Gamma^1_{22} = -r, \quad \Gamma^1_{33} = -r \sin^2 \theta, \quad \Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{r},
$$

$$
\Gamma^2_{33} = -\sin \theta \cos \theta, \quad \Gamma^1_{31} = \frac{1}{r}, \quad \Gamma^2_{23} = \Gamma^3_{23} = \cot \theta.
$$

(A15)

The velocity vector is now given by $\vec{v} = (\vec{v}_r, \vec{v}_\theta/r, \vec{v}_\phi/r \sin \theta)$.

In spherical coordinates, the continuity equation reads

$$
\frac{\partial \rho}{\partial t} + \frac{1}{r^2} \frac{\partial (r^2 \rho \vec{v}_r)}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial (r \sin \theta \rho \vec{v}_\theta)}{\partial \theta} + \frac{1}{r \sin \theta \sin \phi} \frac{\partial (\rho \vec{v}_\phi)}{\partial \phi} = 0.
$$

(A16)

Setting $\tau_{ij} = \tau_{ij}^E = \rho \vec{v} \cdot \vec{\sigma}_{ij}^i + p g_{ij}^\rho$ gives the Euler equations:

$$
\left[ \frac{\partial}{\partial t} + \vec{v}_r \frac{\partial}{\partial r} + \vec{v}_\theta \frac{\partial}{\partial \theta} + \frac{\vec{v}_\phi}{r \sin \theta} \frac{\partial}{\partial \phi} \right] \vec{v}_r - \frac{\vec{v}_r^2 + \vec{v}_\theta^2 + \vec{v}_\phi^2}{r} = \frac{1}{\rho} \frac{\partial p}{\partial r} + \frac{1}{\rho} \frac{\partial \phi}{\partial r}
$$

(A17)

$$
\left[ \frac{\partial}{\partial t} + \vec{v}_r \frac{\partial}{\partial r} + \vec{v}_\theta \frac{\partial}{\partial \theta} + \frac{\vec{v}_\phi}{r \sin \theta} \frac{\partial}{\partial \phi} \right] \frac{\vec{v}_\theta}{r} - \frac{\vec{v}_r \vec{v}_\theta - \vec{v}_\phi \cot \theta}{r} = \frac{1}{\rho} \frac{\partial p}{\partial \theta} - \frac{1}{\rho} \frac{\partial \phi}{\partial \theta}.
$$

(A18)

APPENDIX B

SYSTEMATIC ERRORS IN MASS ESTIMATES FOR COLLISIONLESS SYSTEMS

Similar to Section 2.1, we can compute the gravitational mass using the Jeans equations:

$$
\frac{\partial \rho}{\partial t} + (\vec{v} \cdot \nabla) \rho = -\frac{1}{\rho_{dm}} \nabla \rho_{dm} (\vec{\sigma}^2) - \nabla \phi.
$$

(B1)

$$
M_{tot} = \frac{1}{4\pi G} \int_S dS \cdot \left[ -\frac{1}{\rho_{dm}} \nabla \rho_{dm} (\vec{\sigma}^2) - (\vec{v} \cdot \nabla) \vec{v} - \frac{\partial \vec{v}}{\partial t} \right],
$$

(B2)

where $\vec{v}$ and $\vec{\sigma}^2$ are the velocity field of particles and the velocity dispersion tensor, respectively. We here represent the collisionless component by dark matter, but the same formulation is readily applicable to galaxies. We decompose the right-hand side of Equation (B2) into the following terms by means of Equation (A20):

$$
M_{tot} = M_{\text{rand}} + M_{\text{aniso}} + M_{\text{rot}} + M_{\text{stream}} + M_{\text{cross}} + M_{\text{accel}}.
$$

(B3)
Figure 7. Effective mass terms in Equation (B3) for the dark matter in the simulated cluster are shown in the left panel: \( M_{\text{tot}} \) (black), \( M_{\text{rand}} \) (red), \( M_{\text{rot}} \) (green), \( M_{\text{stream}} \) (blue), \( M_{\text{aniso}} \) (cyan), \( M_{\text{cross}} \) (orange), and \( M_{\text{accel}} \) (magenta). Here \( M_{\text{accel}} \) is calculated by \( M_{\text{accel}} = M_{\text{tot}} - M_{\text{rand}} - M_{\text{rot}} - M_{\text{stream}} - M_{\text{aniso}} - M_{\text{cross}} \). Dotted line means that its sign is inverted. Ratios of mass terms to \( M_{\text{tot}} \) are shown in the right panel; the black line shows \( (M_{\text{tot}} - M_{\text{rand}})/M_{\text{tot}} \) and other mass terms are colored in the same colors as the left panel.

(A color version of this figure is available in the online journal.)

\[
\begin{align*}
M_{\text{rand}} & = -\frac{1}{4\pi G} \int_{\partial V} dS \frac{1}{\rho_{\text{dm}}} \frac{\partial (\rho_{\text{dm}} \sigma_{rr}^2)}{\partial r}, \quad (B4) \\
M_{\text{aniso}} & = -\frac{1}{4\pi G} \int_{\partial V} dS \frac{2\sigma_{rr}^2 - \sigma_{\theta\theta}^2 - \sigma_{\phi\phi}^2}{r}, \quad (B5) \\
M_{\text{rot}} & = \frac{1}{4\pi G} \int_{\partial V} dS \frac{v_r^2 + v_\theta^2}{r}, \quad (B6) \\
M_{\text{stream}} & = -\frac{1}{4\pi G} \int_{\partial V} dS \left[ v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi} \right], \quad (B7) \\
M_{\text{cross}} & = -\frac{1}{4\pi G} \int_{\partial V} dS \left[ \frac{1}{\rho_{\text{dm}}} \frac{\partial (\rho_{\text{dm}} \sigma_{rr}^2)}{\partial \rho} \\
& \quad + \frac{1}{\rho_{\text{dm}} \sin \theta} \frac{\partial (\rho_{\text{dm}} \sigma_{r\phi}^2)}{\partial \varphi} + \frac{\sigma_{r\phi}^2 \cot \theta}{r} \right], \quad (B8) \\
M_{\text{accel}} & = -\frac{1}{4\pi G} \int_{\partial V} dS \frac{\partial v_r}{\partial t}. \quad (B9)
\end{align*}
\]

The physical interpretation of each mass term is as follows. The term \( M_{\text{rand}} \) comes from the gradient of velocity dispersion in the \( r \)-direction and corresponds to \( M_{\text{therm}} \) for a collisional gas. The meaning of \( M_{\text{tot}} \), \( M_{\text{stream}} \), and \( M_{\text{accel}} \) is similar to the corresponding terms for the collisional gas (Equations (6)–(8)). The terms that have no counterpart in the Euler equations are \( M_{\text{aniso}} \) and \( M_{\text{cross}} \); the former represents anisotropy of the velocity dispersion, whereas the latter arises from the off-diagonal components of velocity dispersion tensor and vanishes if velocities of different directions are uncorrelated.

We apply the above formulation to dark matter particles in the simulated cluster described in Section 3 to quantify intrinsic systematic errors of the mass estimation using collisionless particles. Note that one can apply the same method to galaxies, but with a much larger impact of statistical errors. Therefore, we do not do so here because we are interested in intrinsic systematic errors independent of observational complexities. Each term is computed in a manner similar to the case of collisional gas described in Section 3.

Figure 7 shows that the difference between \( M_{\text{tot}} \) and \( M_{\text{rand}} \) increases toward the outer envelope, mainly owing to the presence of \( M_{\text{aniso}} \). This is because the relaxation timescale of collisionless particles is much longer than that of the collisional gas. Once this term is subtracted, \( M_{\text{tot}} - M_{\text{rand}} - M_{\text{aniso}} \) closely matches \( M_{\text{accel}} \), whose absolute value is limited to within \( \sim 0.3M_{\text{tot}} \). The amount of \( M_{\text{accel}} \) is similar to that for the collisional gas (Figure 5). The other mass terms such as \( M_{\text{cross}} \) are less important.

The above results imply that properly accounting for velocity anisotropies is essential for the mass reconstruction using a collisionless component. We stress that \( M_{\text{aniso}} \) is irrelevant to the collisional fluid as long as local thermal equilibrium is established (Appendix A).

APPENDIX C

VALIDITY OF HYDROSTATIC EQUILIBRIUM
IN CLUSTERS FROM SMOOTHED PARTICLE HYDRODYNAMICS SIMULATION

We have shown the behavior of the mass terms for the cluster from the AMR simulation in Section 4. Specifically, the HSE mass deviates from the total mass by up to \( \sim 30\% \) and the difference is explained chiefly by the gas acceleration. In order to determine whether such behavior is common for simulated clusters, we analyze the other simulated clusters. Since our AMR simulation includes only a single cluster, we analyze clusters extracted from an SPH simulation performed by Dolag et al. (2009) using GADGET-2 to confirm the results in Section 4. We refer Dolag et al. (2009) for the detail of the simulation.

We use five regions named g1, g72, g1542, g3344, and g914 extracted from the simulation. Each region has a cluster near the center and the cluster is labeled “a” (g1, g72, g1542, g3344, and g914). For each cluster, we create \( 256^3 \) mesh data centered on the cluster’s center of mass within the radius \( r_{500} \). The size of
the data box is determined so that it contains the entire sphere of radius $3r_{500}$ centered on the cluster’s center of mass.

We calculate the mass terms defined in Section 2 for the five simulated clusters. Since we have the gas acceleration data, we calculate $M_{\text{accel}}$ directly from the data without using $M_{\text{tot}} - M_{\text{therm}} - M_{\text{rot}} - M_{\text{stream}}$.

Figure 8 shows the result for the cluster g914a. This figure basically supports the results in Section 4 in that $M_{\text{therm}}$ deviates from $M_{\text{tot}}$ by $\sim 30\%$ at most and that $M_{\text{accel}}$ becomes large where the difference between $M_{\text{tot}}$ and $M_{\text{therm}}$ is large. The fact that the sum $M_{\text{therm}} + M_{\text{rot}} + M_{\text{stream}} + M_{\text{accel}}$ is approximately $M_{\text{tot}}$ means that the estimation of $M_{\text{accel}}$ by $M_{\text{tot}} - M_{\text{therm}} - M_{\text{rot}} - M_{\text{stream}}$ is good, as confirmed in Section 4 for the AMR cluster.

Although not graphically shown, the mass terms for the other simulated clusters also exhibit similar behavior.

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