Cones of lines having high contact with general hypersurfaces and applications

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Abstract

Given a smooth hypersurface $X \subset \mathbb{P}^{n+1}$ of degree $d \geq 2$, we study the cones $V^h_p \subset \mathbb{P}^{n+1}$ swept out by lines having contact order $h \geq 2$ at a point $p \in X$. In particular, we prove that if $X$ is general, then for any $p \in X$ and $2 \leq h \leq \min(n+1,d)$, the cone $V^h_p$ has dimension exactly $n+2-h$. Moreover, when $X$ is a very general hypersurface of degree $d \geq 2n+2$, we describe the relation between the cones $V^h_p$ and the degree of irrationality of $k$-dimensional subvarieties of $X$ passing through a general point of $X$. As an application, we give some bounds on the least degree of irrationality of $k$-dimensional subvarieties of $X$ passing through a general point of $X$, and we prove that the connecting gonality of $X$ satisfies

$$d - \left\lfloor \frac{\sqrt{16n+25} - 3}{2} \right\rfloor \leq \text{conn. gon}(X) \leq d - \left\lfloor \frac{\sqrt{8n+1} + 1}{2} \right\rfloor.$$ 

KEYWORDS

connecting gonality, degree of irrationality, hypersurface, tangent line

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1 INTRODUCTION

Let $X \subset \mathbb{P}^{n+1}$ be a smooth complex hypersurface of degree $d \geq 2$. Given a point $p \in X$ and an integer $h \geq 2$, we consider the cone $V^h_p \subset \mathbb{P}^{n+1}$ swept out by lines having intersection multiplicity at least $h$ with $X$ at $p$. These cones reflect the geometry of hypersurfaces and occur both in the local geometry of hypersurfaces (see, e.g., [11, 13, 14]) and in the study of their global properties, such as unirationality [6] and their covering gonality, that is, the least gonality of curves passing through a general point of $X$ [4].

In this paper, we study the cones $V^h_p$ of general hypersurfaces $X \subset \mathbb{P}^{n+1}$, and we apply our results to achieve some bounds concerning the degree of irrationality of $k$-dimensional subvarieties of $X$ passing through general points of $X$, where we recall that the degree of irrationality of an irreducible variety $Y$ of dimension $k$ is the least degree of a dominant rational map $Y \dashrightarrow \mathbb{P}^k$.

Given a smooth hypersurface $X \subset \mathbb{P}^{n+1}$ of degree $d \geq 2$, a point $p \in X$, and an integer $2 \leq h \leq d$, the cone $V^h_p \subset \mathbb{P}^{n+1}$ is defined by the vanishing of $h-1$ polynomials of degree $1, 2, \ldots, h-1$, respectively, where the linear polynomial defines the tangent hyperplane of $X$ at $p$ (cf. Section 2). When $p \in X$ is a general point, then $V^h_p$ is a complete intersection defined by those polynomials, that is, $\dim V^h_p = n + 2 - h$ (cf. [13]). However, it may happen that for some special point of $X$, the cone $V^h_p$ fails to be a complete intersection of multidegree $(1, 2, \ldots, h-1)$ and its dimension is larger than expected. We prove that when $X \subset \mathbb{P}^{n+1}$ is a general hypersurface of degree $d \geq 2$, this is not the case.
Theorem 1.1. Let $n \geq 2$ be an integer and let $X \subset \mathbb{P}^{n+1}$ be a general hypersurface of degree $d \geq 2$. Then, for any point $p \in X$ and for any integer $2 \leq h \leq \min\{n+1, d\}$, the cone $V^h_p$ has dimension

$$\dim(V^h_p) = n + 2 - h.$$ 

In recent years, there has been a great deal of interest concerning measures of irrationality of projective varieties, that is birational invariants, which somehow measure the failure of a given variety to be rational (see, e.g., [3, 5, 8, 10, 16, 18, 19]), and several interesting results have been obtained in this direction for very general hypersurfaces of large degree (cf. [2–4, 20]).

Given an irreducible complex projective variety $X$ of dimension $n$ and an integer $k$ such that $1 \leq k \leq n$, we are interested in the following birational invariants. According to [4, Section 5.3], we define the $k$-irrationality degree of $X$ as the integer

$$\text{irr}_k(X) := \min \left\{ c \in \mathbb{N} \mid \begin{array}{l}
given a general point $p \in X$, exists an irreducible subvariety $Z \subseteq X$ of dimension $k$ such that $p \in Z$ and there is a dominant rational map $Z \rightarrow \mathbb{P}^k$ of degree $c$
\end{array} \right\}.$$ 

and, in line with [3], we define the connecting gonality of $X$ as the integer

$$\text{conn. gon}(X) := \min \left\{ c \in \mathbb{N} \mid \begin{array}{l}
given two general points $q, q' \in X$, exists an irreducible curve $C \subset X$ such that $q, q' \in C$ and $\text{gon}(C) = c$
\end{array} \right\}.$$ 

Therefore, $\text{conn. gon}(X)$ can be thought as a measure of the failure of $X$ to be rationally connected, whereas $\text{irr}_k(X)$ measures how $X$ is far from being covered by $k$-dimensional rational varieties. We note further that $\text{irr}(X) := \text{irr}_n(X)$ is the degree of irrationality of $X$ and $\text{cov. gon}(X) := \text{irr}_1(X)$ is the covering gonality of $X$. Moreover, these invariants satisfy the obvious inequalities

$$\text{cov. gon}(X) \leq \text{conn. gon}(X) \leq \text{irr}(X) \quad \text{and} \quad \text{irr}_1(X) \leq \text{irr}_2(X) \leq \cdots \leq \text{irr}_n(X).$$ (1.1)

In [8], it has been proved that $\text{irr}_k(A) \geq k + \frac{1}{2}(\dim A + 1)$, provided that $A$ is a very general abelian variety of dimension at least 3 and $1 \leq k \leq \dim A$. Apart from this result and the cases $k \in \{1, n\}$, very little is known about the $k$-irrationality degrees and the connecting gonality of projective varieties. When $X \subset \mathbb{P}^{n+1}$ is a very general hypersurface of degree $d \geq 2n + 2$, it follows from [3, Theorem C] and [4, Theorem 1.1] that

$$\text{irr}_n(X) = d - 1 \quad \text{and} \quad d - \left[ \sqrt{16n + 9} - 1 \over 2 \right] \leq \text{irr}_1(X) \leq d - \left[ \sqrt{16n + 1} - 1 \over 2 \right],$$ (1.2)

where the latter relation is often a chain of equalities. Under the same assumption on $X \subset \mathbb{P}^{n+1}$, we are concerned with its connecting gonality and the $k$-irrationality degree, with $2 \leq k \leq n - 1$.

In this direction, we extend [4, Proposition 2.12], which relates curves of low gonality covering $X$ to the cones $V^h_p$ of lines having high contact with $X$. In particular, we prove that if $Z \subset X$ is a $k$-dimensional subvariety passing through a general point, endowed with a dominant rational map $\varphi : Z \rightarrow \mathbb{P}^k$ of degree $c \leq d - 3$, then $Z \subset V^h_p$ for some $p \in X$, and the map $\varphi$ is the projection from $p$ (cf. Proposition 4.1). Combining Theorem 1.1 and the latter result, we achieve bounds on the $k$-irrationality degrees of $X$.

Theorem 1.2. Let $n \geq 3$ and let $X \subset \mathbb{P}^{n+1}$ be a very general hypersurface of degree $d \geq 2n + 2$. Then,

$$\text{irr}_k(X) \geq d - 1 - n + k \quad \text{for} \ 1 \leq k \leq n.$$ (1.3)

Moreover, equality holds for $n - 2 \leq k \leq n$, that is,

$$\text{irr}_{n-2}(X) = d - 3, \quad \text{irr}_{n-1}(X) = d - 2 \quad \text{and} \quad \text{irr}_n(X) = d - 1.$$
We point out that the assertion for \( k = n \) is given by [3, Theorem C]. Furthermore, the larger the value of \( k \) is, the more significant Theorem 1.2 becomes, since for small values of \( k \) the bound (1.3) is superseded by (1.2) and (1.1).

In order to discuss the connecting gonality of \( X \), we prove further that for any pair of general points \( q, q' \in X \) and for any \( 2 \leq h \leq \frac{n}{2} + 1 \), there exists a general point \( p \in X \) such that \( q, q' \in V^h_p \) (cf. Lemma 3.2). Then, using the fact that for \( h = \left\lfloor \frac{\sqrt{8n+1}+1}{2} \right\rfloor \) the locus \( V^h_p \) is a cone over a rationally connected variety, we bound from above the connecting gonality of a very general hypersurface \( X \subset \mathbb{P}^{n+1} \) of large degree.

**Theorem 1.3.** Let \( n \geq 4 \) and let \( X \subset \mathbb{P}^{n+1} \) be a very general hypersurface of degree \( d \geq 2n + 2 \). Then,

\[
\text{conn.gon}(X) \leq d - \left\lfloor \frac{\sqrt{8n+1} + 1}{2} \right\rfloor.
\]

(1.4)

Finally, using our results and the Grassmannian techniques introduced in [17], we also obtain a lower bound on the connecting gonality of very general hypersurfaces, which slightly improves the bound descending from (1.2) and (1.1).

**Theorem 1.4.** Let \( n \geq 4 \) and let \( X \subset \mathbb{P}^{n+1} \) be a very general hypersurface of degree \( d \geq 2n + 2 \). Then,

\[
\text{conn.gon}(X) \geq d - \left\lfloor \frac{\sqrt{16n+25} - 3}{2} \right\rfloor.
\]

(1.5)

In particular,

\[
\text{conn.gon}(X) > \text{cov.gon}(X)
\]

\( \forall n \in \mathbb{Z}_{\geq 4} \setminus \{4a^2 + 3a, 4a^2 + 5a, 4a^2 + 5a + 1, 4a^2 + 7a + 2, 4a^2 + 9a + 4, 4a^2 + 11a + 6| a \in \mathbb{N}\} \).

In Example 4.5, we also discuss the cases \( 1 \leq n \leq 3 \), which turn out to satisfy equality in (1.5). We note that the second part of the statement of Theorem 1.4 is obtained from (1.2) by determining the values of \( n \) such that \( \left\lfloor \frac{\sqrt{16n+1} - 1}{2} \right\rfloor \neq \left\lfloor \frac{\sqrt{16n+25} - 3}{2} \right\rfloor \). We believe that the bound (1.5) is far from being sharp. However, for any \( 4 \leq n \leq 16 \) with \( n \neq 9, 13, 14 \), the right-hand sides of (1.4) and (1.5) do coincide, hence Theorems 1.3 and 1.4 compute the connecting gonality of \( X \) in these cases (cf. Example 4.5).

The paper is organized as follows. In Section 2, we recall some basic facts on the cones of lines of high contact with a general hypersurface \( X \subset \mathbb{P}^{n+1} \) and we prove Theorem 1.1.

In Section 3, we consider polar hypersurfaces of a smooth hypersurface \( X \subset \mathbb{P}^{n+1} \), in order to discuss when any pair \( q, q' \in X \) lies on a cone \( V^h_p \) for some \( p \in X \).

Finally, Section 4 is concerned with the applications to measures of irrationality. In particular, after describing the relation between the cones \( V^h_p \) and the degree of irrationality of \( k \)-dimensional subvarieties of \( X \) passing through a general point, we prove Theorems 1.2, 1.3, and 1.4, and we discuss the behavior of the connecting gonality for small values of \( n = \dim X \).

### 1.1 Notation

We work throughout over the field \( \mathbb{C} \) of complex numbers. By *variety* we mean a complete reduced algebraic variety \( X \), and by *curve* we mean a variety of dimension 1. We say that a property holds for a *general* (resp. *very general*) point \( x \in X \) if it holds on a Zariski open nonempty subset of \( X \) (resp. on the complement of the countable union of proper subvarieties of \( X \)).
Let $n \geq 2$ be an integer, and let $X = V(F) \subset \mathbb{P}^{n+1}$ be a smooth hypersurface defined by the vanishing of a homogeneous polynomial $F \in \mathbb{C}[x_0, \ldots, x_{n+1}]$ of degree $d \geq 2$. Given a point $p \in X$ and an integer $2 \leq h \leq d$, we define the cone $V^h_p = V^h_p,X \subset \mathbb{P}^{n+1}$ of tangent lines of order $h$ at $p$ as the Zariski closure of the locus swept out by lines $\ell \subset \mathbb{P}^{n+1}$ such that either $\ell \subset X$ or $\ell \cdot X \geq h p$. Therefore, $V^h_p$ is a cone with vertex at $p$, defined by the vanishing of the following $h - 1$ polynomials occurring in the Taylor expansion of $F$ at $p$:

$$G_k(x_0, \ldots, x_{n+1}) := \left( x_0 \frac{\partial}{\partial x_0} + \cdots + x_{n+1} \frac{\partial}{\partial x_{n+1}} \right)^{(k)} F(p) = \sum_{l_0 + \cdots + l_{n+1} = k} \frac{k!}{l_0! \cdots l_{n+1}!} x_0^{l_0} \cdots x_{n+1}^{l_{n+1}} \frac{\partial^k F}{\partial x_0^{l_0} \cdots \partial x_{n+1}^{l_{n+1}}} (p), \quad (2.1)$$

where $(-)^{(k)}$ denotes the usual symbolic power, $1 \leq k \leq h - 1$, and $\deg G_k = k$ (cf. [6, p. 186]). In particular, the cone $V^2_p$ coincides with the (projective) tangent hyperplane $T_p X \subset \mathbb{P}^{n+1}$ to $X$ at $p$. When instead $h \geq 3$, we denote by $\Lambda^h_p = \Lambda^h_p,X$ the intersection of $V^h_p$ with a general hyperplane $H \subset \mathbb{P}^{n+1}$ not containing $p$, so that $\Lambda^h_p$ is defined in $H \cap T_p X \cong \mathbb{P}^{n-1}$ by $h - 2$ polynomial equations of degree $2, 3, \ldots, h - 1$, respectively, and $V^h_p$ is the cone over $\Lambda^h_p$ with vertex at $p$.

For any $2 \leq h \leq \min\{n+1, d\}$, it follows from this description of $V^h_p$ that

$$\dim V^h_p \geq n + 2 - h. \quad (2.2)$$

When $X$ is a general hypersurface of degree $d \geq 2$ and $p \in X$ is a general point, [4, Lemma 2.2] guarantees that $V^h_p \subset \mathbb{P}^{n+1}$ is a general complete intersection of multidegree $(1, 2, \ldots, h - 1)$ and, in particular, (2.2) is an equality.

According to the assertion of Theorem 1.1, we want to prove that equality in (2.2) holds for any point $p \in X$. The case $h = 2$ is trivial since $X$ is smooth and hence $V^2_p = T_p X \cong \mathbb{P}^n$. The assertion for $h = 3$ is implied by the following result.

**Lemma 2.1.** Let $n \geq 2$ be an integer and let $X \subset \mathbb{P}^{n+1}$ be a general hypersurface of degree $d \geq 2$. Then, for any point $p \in X$, the intersection of $X$ with the tangent hyperplane $T_p X$ has multiplicity 2 at $p$.

**Proof.** Let $\mathcal{L}$ be the linear system of all hypersurfaces of degree $d$ in $\mathbb{P}^{n+1}$, which has dimension

$$\dim \mathcal{L} = \binom{d + n + 1}{n + 1} - 1.$$ 

Consider the variety $\mathcal{Y}$ consisting of all triples $(p, \Pi, Y)$ where $\Pi \subset \mathbb{P}^{n+1}$ is a hyperplane, $p \in \Pi$, and $Y \subset \Pi$ is a hypersurface of degree $d$ with a point of multiplicity of at least 3 at $p$. Then, we define the variety

$$\mathcal{Z} := \{(p, \Pi, Y, X) \in \mathcal{Y} \times \mathcal{L}[Y \subset X]\}$$

endowed with the projection $\pi_1 : \mathcal{Z} \rightarrow \mathcal{Y}$, whose fibers are all isomorphic to linear systems of hypersurfaces of degree $d$ of the same dimension $\binom{d + n}{n+1}$. Looking at the map $\mathcal{Y} \rightarrow \mathbb{P}^{n+1} \times (\mathbb{P}^{n+1})^* \times \mathcal{L} \times \mathcal{L}$ given by $(p, \Pi, Y) \rightarrow (p, \Pi, Y, X)$, it is easy to see that $\mathcal{Y}$ is irreducible of dimension

$$\dim \mathcal{Y} = 2n + \binom{d + n}{n} - \binom{n + 2}{2}.$$ 

Hence $\mathcal{Z}$ is also irreducible, having dimension

$$\dim \mathcal{Z} = 2n + \binom{d + n}{n} - \binom{n + 2}{2} + \binom{d + n}{n+1} = 2n + \binom{d + n + 1}{n + 1} - \binom{n + 2}{2}.$$ 

Consider now the projection $\pi_2 : \mathcal{Z} \rightarrow \mathcal{L}$, whose image is the locus $\mathcal{T}$ of all hypersurfaces $X \subset \mathbb{P}^{n+1}$ of degree $d$ having a point $p \in X$ and a hyperplane $\Pi \subset \mathbb{P}^{n+1}$ such that the intersection scheme $X \cap \Pi$ has a point of multiplicity at least 3 at
p. Hence,

\[ \dim \mathcal{T} \leq \dim \mathcal{Z} = \dim \mathcal{L} + 2n + 1 - \left( \frac{n + 2}{2} \right) \]

and, since \((\frac{n+2}{2}) > 2n + 1\) as soon as \(n \geq 2\), we conclude that \(\mathcal{T}\) is a proper closed subset of \(\mathcal{L}\), as wanted. \(\Box\)

We notice that the double point at \(p \in X\) of the intersection of \(X\) with \(T_pX\) as in Lemma 2.1 does not need to be an ordinary double point, and the locus where the singularity is worse than an ordinary double point is the intersection of \(X\) with its Hessian hypersurface.

Now we prove Theorem 1.1.

Proof of Theorem 1.1. We already discussed the trivial case \(h = 2\). Moreover, Lemma 2.1 ensures that for any \(p \in X\), a general line \(\ell \subset T_pX\) tangent to \(X\) at \(p\) intersects \(X\) at \(p\) with multiplicity exactly 2. Hence, \(V^3_p\) is a proper subvariety of \(T_pX\), and (2.2) is an equality. Thus, we assume hereafter \(h \geq 4\).

Let \([x_0 : \ldots : x_{n+1}]\) be homogeneous coordinates in \(\mathbb{P}^{n+1}\) and, for any positive integer \(k\), we set

\[ S_k := \mathbb{C}[x_0, \ldots, x_{n+1}]^k \quad \text{and} \quad S^*_k := \mathbb{C}[x_0, \ldots, x_{n+1}]^k \setminus \{0\}. \quad (2.3) \]

For any \(F \in S^*_d\), we denote by \(V(F) \subset \mathbb{P}^{n+1}\) the hypersurface defined by the vanishing of \(F\), and for any \(G := (G_1, \ldots, G_{h-1}) \in \prod_{k=1}^{h-1} S_k\), we denote by \(V(G)\) the intersection scheme of the hypersurfaces \(V(G_k)\) for \(1 \leq k \leq h-1\).

For \(F \in S^*_d\) and \(p \in V(F) \subset \mathbb{P}^{n+1}\), let \(G_k = G_{p,k}(F)(x_0, \ldots, x_{n+1})\) be the homogeneous polynomial of degree \(k\) defined in (2.1), where \(1 \leq k \leq h-1\), and let

\[ G = G_p(F) := (G_{p,1}(F), \ldots, G_{p,h-1}(F)) \in \prod_{k=1}^{h-1} S_k. \]

Therefore, \(V^h_p\) is the cone \(V(G_p(F))\) with vertex at \(p\), and (2.2) fails to be an equality if and only if \(G_p(F)\) is not a regular sequence. In order to prove the assertion, we show that if \(F \in S^*_d\) is general, then the sequence \(G_p(F)\) is regular for all points \(p \in V(F)\).

To this aim, let \(U_d \subset S^*_d\) be the open dense subset parameterizing those \(F \in S^*_d\) such that \(V(F)\) is smooth and let

\[ J := \left\{ (p, F, G) \in \mathbb{P}^{n+1} \times U_d \times \prod_{k=1}^{h-1} S_k \mid p \in V(F) \text{ and } G = G_p(F) \right\}, \]

which is endowed with the two projections \(\pi_1 : J \rightarrow U_d\) and \(\pi_2 : J \rightarrow \mathbb{P}^{n+1} \times \prod_{k=1}^{h-1} S_k\). The map \(\pi_1\) is surjective, and for any \(F \in U_d\), the fiber \(\pi_1^{-1}(F)\) is isomorphic to \(V(F)\), which is irreducible of dimension \(n\). Thus, \(J\) is irreducible of dimension

\[ \dim J = \dim(U_d) + n = \left(\frac{n + 1 + d}{d}\right) + n. \quad (2.4) \]

Let us define \(W := \pi_2(J) \subset \mathbb{P}^{n+1} \times \prod_{k=1}^{h-1} S_k\), which is irreducible too.

Claim 2.2. All fibers of \(\pi_2 : J \rightarrow W\) have dimension

\[ f = \left(\frac{n + h}{h}\right) + \left(\frac{n + h + 1}{h + 1}\right) + \cdots + \left(\frac{n + d}{d}\right) = \left(\frac{n + d + 1}{d}\right) - \left(\frac{n + h}{h - 1}\right). \]

Proof of Claim 2.2. Let \((p, G) = \pi_2(p, F, G) \in W\), with \((p, F, G) \in J, G = (G_1, \ldots, G_{h-1})\) and \(G_k = G_{p,k}(F)\) for \(1 \leq k \leq h-1\). Up to projective transformations, we may assume \(p = [1 : 0 : \ldots : 0]\) and \(T_pV(F) = V(x_{n+1})\). Then, \(F\) is of the form

\[ F(x_0, \ldots, x_{n+1}) = cx_n x_0^{d-1} + F_2(x_1, \ldots, x_{n+1}) x_0^{d-2} + \cdots + F_d(x_1, \ldots, x_{n+1}), \quad (2.5) \]
where $c$ is a non-zero constant and each $F_i \in \mathbb{C}[x_1, \ldots, x_{n+1}]$ is homogeneous of degree $i$. Easy computations show that

\begin{align*}
G_1 &= cx_{n+1} \\
G_2 &= 2(d - 1)cx_{n+1}x_0 + 2F_2 \\
G_3 &= 3(d - 1)(d - 2)cx_{n+1}x_0^2 + 6(d - 2)x_0F_2 + 6F_3 \\
&\vdots \\
G_{h-1} &= (h - 1)! (d - 1)! (d - h + 1)! cx_{n+1}x_0^{h-2} + \sum_{i=2}^{h-1} \frac{(h - 1)! (d - i)!}{(h - i - 1)! (d - h + 1)!} x_0^{h-i-1}F_i.
\end{align*}

(2.6)

To determine the fiber of $\pi_2$ over $(p, G)$, we have to find all forms $F' \in U_d$ such that $(p, F', G) \in J$. As in (2.5), we have $F' = c'x_{n+1}x_0^{d-1} + F_2'x_0^{d-2} + \cdots + F_d'$, which satisfies the corresponding equations in (2.6). Therefore, Equations (2.6) imply $c = c'$ and $F_k' = F_k$ for any $2 \leq k \leq h - 1$. Thus, $F'$ may differ from $F$ by the terms $F_i'$ for $h \leq i \leq d$, which can be chosen arbitrarily in $\mathbb{C}[x_1, \ldots, x_{n+1}]$. \hfill \square

We deduce from the claim and (2.4) that

$$\dim \mathcal{W} = \dim J - f = \binom{n + h}{h - 1} + n.$$  

Moreover, according to the description of (2.6), this equality implies that $\mathcal{W}$ coincides with the set of all $h$-tuples $(p, G_1, \ldots, G_{h-1})$, where $p \in \mathbb{P}^{n+1}$ is arbitrary, $G_1$ is an arbitrary homogeneous polynomial of degree 1 vanishing at $p$, and for any $k = 2, \ldots, h - 1$, $G_k$ is a homogeneous polynomial of degree $k$ such that modulo $G_1, \ldots, G_{k-1}$ is arbitrary.

Then, we set

$$\mathcal{W}_0 := \{(p, G) \in \mathcal{W} | G \text{ is not a regular sequence}\} \subset \mathcal{W},$$

so that proving the assertion is equivalent to showing that $\pi_1(\pi_2^{-1}(\mathcal{W}_0))$ is a proper closed subset of $U_d$. In particular, it suffices to prove that

$$\text{codim}_{\mathcal{W}} \mathcal{W}_0 > n,$$

(2.7)

because in this case Claim 2.2 and (2.4) yield

$$\dim \pi_1(\pi_2^{-1}(\mathcal{W}_0)) \leq \dim \pi_2^{-1}(\mathcal{W}_0) = \dim \mathcal{W}_0 + f = \dim \mathcal{W}_0 + \dim J - \dim \mathcal{W} =$$

$$= \dim J - \text{codim}_{\mathcal{W}} \mathcal{W}_0 < \dim J - n = \dim U_d,$$

so that the assertion holds. Hence, we focus on proving (2.7).

To this aim, let $Z$ be an irreducible component of $\mathcal{W}_0$ and let $(p, G) \in Z$ be a general point, with $G = (G_1, \ldots, G_{h-1})$. Then, $G$ is not a regular sequence, and we may define $\alpha = \alpha_Z$ to be the greatest integer such that $(G_1, \ldots, G_{\alpha})$ is a regular sequence. Since we already showed that $V_p^3 = V(G_1, G_2)$ has dimension $n - 1$, we have $2 \leq \alpha < h - 1$. By maximality of $\alpha$, there exists an irreducible component $Y$ of $V(G_1, G_2, \ldots, G_{\alpha})$ of dimension $m := \dim Y = n - \alpha + 1$ and contained in $V(G_{\alpha+1})$. Therefore, in view of the above interpretation of $\mathcal{W}$ in terms of $h$-tuples $(p, G_1, \ldots, G_{h-1})$, in order to prove (2.7), it is enough to show that the Hilbert function $h_Y : \mathbb{N} \rightarrow \mathbb{N}$ of $Y$ satisfies $h_Y(\alpha + 1) > n$.

For any $\ell = 0, \ldots, m$, let $Y_\ell$ denote a general linear section of $Y$ of dimension $\ell$. In particular, $Y_m = Y$ and $Y_{m-1}$ is a general hyperplane section of $Y$. Then, for any positive integer $t$, we have

$$h_Y(t) - h_Y(t - 1) \geq h_{Y_{m-1}}(t)$$

where $c$ is a non-zero constant and each $F_i \in \mathbb{C}[x_1, \ldots, x_{n+1}]$ is homogeneous of degree $i$. Easy computations show that

\begin{align*}
G_1 &= cx_{n+1} \\
G_2 &= 2(d - 1)cx_{n+1}x_0 + 2F_2 \\
G_3 &= 3(d - 1)(d - 2)cx_{n+1}x_0^2 + 6(d - 2)x_0F_2 + 6F_3 \\
&\vdots \\
G_{h-1} &= (h - 1)! (d - 1)! (d - h + 1)! cx_{n+1}x_0^{h-2} + \sum_{i=2}^{h-1} \frac{(h - 1)! (d - i)!}{(h - i - 1)! (d - h + 1)!} x_0^{h-i-1}F_i.
\end{align*}

(2.6)

To determine the fiber of $\pi_2$ over $(p, G)$, we have to find all forms $F' \in U_d$ such that $(p, F', G) \in J$. As in (2.5), we have $F' = c'x_{n+1}x_0^{d-1} + F_2'x_0^{d-2} + \cdots + F_d'$, which satisfies the corresponding equations in (2.6). Therefore, Equations (2.6) imply $c = c'$ and $F_k' = F_k$ for any $2 \leq k \leq h - 1$. Thus, $F'$ may differ from $F$ by the terms $F_i'$ for $h \leq i \leq d$, which can be chosen arbitrarily in $\mathbb{C}[x_1, \ldots, x_{n+1}]$. \hfill \square

We deduce from the claim and (2.4) that

$$\dim \mathcal{W} = \dim J - f = \binom{n + h}{h - 1} + n.$$  

Moreover, according to the description of (2.6), this equality implies that $\mathcal{W}$ coincides with the set of all $h$-tuples $(p, G_1, \ldots, G_{h-1})$, where $p \in \mathbb{P}^{n+1}$ is arbitrary, $G_1$ is an arbitrary homogeneous polynomial of degree 1 vanishing at $p$, and for any $k = 2, \ldots, h - 1$, $G_k$ is a homogeneous polynomial of degree $k$ such that modulo $G_1, \ldots, G_{k-1}$ is arbitrary.

Then, we set

$$\mathcal{W}_0 := \{(p, G) \in \mathcal{W} | G \text{ is not a regular sequence}\} \subset \mathcal{W},$$

so that proving the assertion is equivalent to showing that $\pi_1(\pi_2^{-1}(\mathcal{W}_0))$ is a proper closed subset of $U_d$. In particular, it suffices to prove that

$$\text{codim}_{\mathcal{W}} \mathcal{W}_0 > n,$$

(2.7)

because in this case Claim 2.2 and (2.4) yield

$$\dim \pi_1(\pi_2^{-1}(\mathcal{W}_0)) \leq \dim \pi_2^{-1}(\mathcal{W}_0) = \dim \mathcal{W}_0 + f = \dim \mathcal{W}_0 + \dim J - \dim \mathcal{W} =$$

$$= \dim J - \text{codim}_{\mathcal{W}} \mathcal{W}_0 < \dim J - n = \dim U_d,$$

so that the assertion holds. Hence, we focus on proving (2.7).

To this aim, let $Z$ be an irreducible component of $\mathcal{W}_0$ and let $(p, G) \in Z$ be a general point, with $G = (G_1, \ldots, G_{h-1})$. Then, $G$ is not a regular sequence, and we may define $\alpha = \alpha_Z$ to be the greatest integer such that $(G_1, \ldots, G_{\alpha})$ is a regular sequence. Since we already showed that $V_p^3 = V(G_1, G_2)$ has dimension $n - 1$, we have $2 \leq \alpha < h - 1$. By maximality of $\alpha$, there exists an irreducible component $Y$ of $V(G_1, G_3, \ldots, G_{\alpha})$ of dimension $m := \dim Y = n - \alpha + 1$ and contained in $V(G_{\alpha+1})$. Therefore, in view of the above interpretation of $\mathcal{W}$ in terms of $h$-tuples $(p, G_1, \ldots, G_{h-1})$, in order to prove (2.7), it is enough to show that the Hilbert function $h_Y : \mathbb{N} \rightarrow \mathbb{N}$ of $Y$ satisfies $h_Y(\alpha + 1) > n$.

For any $\ell = 0, \ldots, m$, let $Y_\ell$ denote a general linear section of $Y$ of dimension $\ell$. In particular, $Y_m = Y$ and $Y_{m-1}$ is a general hyperplane section of $Y$. Then, for any positive integer $t$, we have

$$h_Y(t) - h_Y(t - 1) \geq h_{Y_{m-1}}(t)$$
(see [12, Lemma 3.1]). Analogously, for any $1 \leq \ell \leq m$ and for any integer $t > 0$, we obtain

$$h_{Y,\ell}(t) \geq h_{Y,\ell}(t-1) + h_{Y,\ell-1}(t),$$

(2.8)

and we deduce by iteration that for any integer $t > 0$,

$$h_Y(t) \geq \sum_{\ell=1}^{m} h_{Y,\ell}(t-1) + h_{Y,0}(t) \geq \sum_{\ell=1}^{m} h_{Y,\ell}(t-1) + 1.$$  

(2.9)

Claim 2.3. For any $1 \leq \ell \leq m$ and for any integer $t > 0$,

$$h_{Y,\ell}(t) \geq \binom{\ell + t}{t}.$$  

Proof of Claim 2.3. We recall that $Y_\ell$ is a general linear section of $Y \subset V(G_1) \cong \mathbb{P}^n$ of dimension $\ell$, so that $Y_\ell$ is irreducible and it sits in a projective space $\Lambda_\ell \cong \mathbb{P}^{n+\ell-m}$, with $1 \leq \ell \leq m$. If $t = 1$, the claim is true as the linear span of $Y_\ell \subset \Lambda_\ell$ has dimension at least $\ell = \dim Y_\ell$, that is, $Y_\ell$ contains at least $\ell + 1$ independent points of $\Lambda_\ell$. Then, we argue by induction on $t$, and using inequality (2.9) applied to $Y_\ell$, we obtain

$$h_{Y,\ell}(t) \geq \sum_{j=1}^{\ell} h_{Y,j}(t-1) + 1 \geq \sum_{j=1}^{\ell} \binom{j + t - 1}{t - 1} + 1 = \sum_{j=0}^{\ell} \binom{j + t - 1}{t - 1} = \binom{\ell + t}{t}$$

as desired. \hfill \Box

Finally, setting $t = \alpha + 1$ and $\ell = m = n - \alpha + 1$, Claim 2.3 ensures that

$$h_Y(\alpha + 1) \geq \binom{m + \alpha + 1}{\alpha + 1} = \binom{n + 2}{\alpha + 1}.$$  

By assumption, we have $\alpha < h - 1 \leq n$, and hence $\binom{n+2}{\alpha+1} \geq n + 1$. Thus, $h_Y(\alpha + 1) > n$, which concludes the proof of Theorem 1.1. \hfill \Box

3 | POLAR HYPERSURFACES AND CONES OF LINES HAVING HIGH CONTACT

Let $n \geq 2$ be an integer and let $X := V(F) \subset \mathbb{P}^{n+1}$ be a smooth hypersurface of degree $d \geq 2$. Given a point $q = [q_0 : \ldots : q_{n+1}] \in \mathbb{P}^{n+1}$ and an integer $0 \leq s \leq d$, we introduce the $s$-th polar hypersurface of $X$ with respect to $q$ as the hypersurface $\Delta_q^s = \Delta_q^s(X) \subset \mathbb{P}^{n+1}$ defined by the vanishing of the polynomial of degree $d-s$

$$\text{Pol}_q^s(F)(x_0, \ldots, x_{n+1}) := \left( q_0 \frac{\partial}{\partial x_0} + \cdots + q_{n+1} \frac{\partial}{\partial x_{n+1}} \right)^{(s)} F(x_0, \ldots, x_{n+1}),$$

(3.1)

where $(-)^{(s)}$ denotes the usual symbolic power and $\text{Pol}_q^0(F) = F$, that is, $\Delta_q^0 = X$ for any $q \in \mathbb{P}^{n+1}$. Furthermore, we define the intersection scheme

$$\Delta_{q,\cdot,h}(X) := \bigcap_{s=0}^{h-1} \Delta_q^s.$$  

(3.2)

In this section, we use polar hypersurfaces of $X$ and Theorem 1.1 to show that for general $q, q' \in X$ and for any $2 \leq h \leq \frac{n}{2} + 1$, there exists a general point $p \in X$ such that $q, q' \in V_p^h$. In particular, this fact is crucial in order to prove Theorem 1.3. To start, we prove the following.
Lemma 3.1. Let \( n \geq 2 \) be an integer and let \( X \subset \mathbb{P}^{n+1} \) be a smooth hypersurface of degree \( d \geq 2 \). For any integer \( 2 \leq h \leq \frac{n}{2} + 1 \) and for any \( q, q' \in X \), there exists a point \( p \in X \) such that \( q, q' \in V^h_p \).

Proof. We point out that for \( q \in \mathbb{P}^{n+1} \), the intersection \( X \cap \Delta^1_q \) consists of the points \( p \in X \) such that \( q \in T_p X \), that is, the line \( \langle q, p \rangle \) intersects \( X \) with multiplicity of at least 2 at \( p \), provided that \( p \neq q \). Similarly, given two points \( q \in \mathbb{P}^{n+1} \) and \( p \in X \) with \( p \neq q \), the line \( \langle q, p \rangle \) intersects \( X \) with multiplicity of at least \( h \) at \( p \)—that is, \( q \in V^h_p \)—if and only if \( p \) belongs to \( \Delta_{q,h}(X) \) defined in (3.2).

Therefore, proving the statement is equivalent to showing that for any \( q, q' \in X \), the intersection of \( \Delta_{q,h}(X) \) and \( \Delta_{q',h}(X) \) is not empty. Since both \( \Delta_{q,h}(X) \) and \( \Delta_{q',h}(X) \) lie on \( X = \Delta_0 = \Delta_0^q = \Delta_0^{q'} \), then \( \Delta_{q,h}(X) \cap \Delta_{q',h}(X) \) is the intersection of \( 2h - 1 \) hypersurfaces of \( \mathbb{P}^{n+1} \), which is nonempty because of the assumption \( h \leq \frac{n}{2} + 1 \). \( \square \)

In addition, when both \( X \subset \mathbb{P}^{n+1} \) and the points \( q, q' \in X \) are assumed to be general, the following holds.

Lemma 3.2. Let \( n \geq 2 \) be an integer and let \( X \subset \mathbb{P}^{n+1} \) be a general hypersurface of degree \( d \geq 2 \). Then, for any integer \( 2 \leq h \leq \frac{n}{2} + 1 \) and for general \( q, q' \in X \), there exists a general point \( p \in X \) such that \( q, q' \in V^h_p \).

Proof. Set \( 2 \leq h \leq \frac{n}{2} + 1 \) and consider the variety

\[
P_h := \{(q, q', p) \in X \times X \times X | q \neq q' \text{ and } p \in \Delta_{q,h} \cap \Delta_{q',h}\}
\]

endowed with the projections

\[
X \times X \overset{\pi_{12}}{\longrightarrow} P_h \overset{\pi_3}{\longrightarrow} X.
\]

Thanks to Lemma 3.1, the map \( \pi_{12} \) is surjective. Let \( Z \subset P_h \) be an irreducible component dominating \( X \times X \). Therefore, the proof of Lemma 3.1 gives that for any \( (q, q') \in X \times X \),

\[
\dim (\Delta_{q,h}(X) \cap \Delta_{q',h}(X)) \geq n + 2 - 2h \quad \text{and hence} \quad \dim Z \geq 3n + 2 - 2h. \tag{3.3}
\]

Moreover, given any point \( p \in \pi_3(Z) \) and setting \( Z_p := V^h_p \cap X \), we have

\[
\left( \pi_{3|Z} \right)^{-1}(p) \subseteq \pi_3^{-1}(p) \cong Z_p \times Z_p. \tag{3.4}
\]

In order to prove that for general \( q, q' \in X \), there exists a general point \( p \in X \) such that \( q, q' \in V^h_p \), it is enough to prove that \( Z \) dominates \( X \) via \( \pi_3 \). We assume by contradiction that \( \pi_{3|Z} \) is not dominant, that is, \( \dim \pi_3(Z) < n = \dim X \). If \( p \in \pi_3(Z) \) is a general point, then (3.3) and (3.4) give

\[
\dim(Z_p \times Z_p) \geq \dim \left( \left( \pi_{3|Z} \right)^{-1}(p) \right) = \dim Z - \dim \pi_3(Z) > 2n + 2 - 2h. \tag{3.5}
\]

It follows that \( \dim Z_p > n + 1 - h \) and hence \( \dim V^h_p > n + 2 - h \). Since the latter inequality contradicts Theorem 1.1, we conclude that \( \pi_{3|Z} \) is dominant. \( \square \)

4 \ k-IRRATIONALITY DEGREE AND CONNECTING GONALITY OF GENERAL HYPERSURFACES

In this section, we apply the previous results in order to bound the \( k \)-irrationality degree and the covering gonality of a very general hypersurface \( X \subset \mathbb{P}^{n+1} \) of degree \( d \geq 2n + 2 \).
According to Section 2, we recall that if $V^h_p \subset \mathbb{P}^{n+1}$ is the cone of tangent lines of order $h$ at $p \in X$, we denote by $\Lambda_p^h$ a general hyperplane section of $V^h_p$. The link between the cones $V^h_p \subset \mathbb{P}^{n+1}$ and the invariants $\text{irr}_k(X)$ and conn. gon$(X)$ is expressed by the following result, which extends [4, Proposition 2.12] to higher dimensional subvarieties of $X$.

**Proposition 4.1.** Let $n \geq 3$ and $1 \leq k \leq n-1$ be integers. Let $X \subset \mathbb{P}^{n+1}$ be a very general hypersurface of degree $d \geq 2n+2$. Suppose that for a general point $q \in X$, there exist a $k$-dimensional irreducible subvariety $Z \subset X$ containing $q$ and a dominant rational map $\varphi : Z \to \mathbb{P}^k$ of degree $c \leq d-3$. Then,

(i) there exists a point $p \in Z$ such that $Z \subset V^{d-c}_p \cap X$;

(ii) the map $\varphi : Z \to \mathbb{P}^k$ of degree $c$ is the projection from $p$.

In particular, the image of $Z$ under $\varphi$ is a $k$-dimensional rational variety $R \subset \Lambda_p^{d-c}$.

**Proof.** The case $k = 1$ is covered by [4, Proposition 2.12], so we assume $2 \leq k \leq n-1$. Since $d \geq 2n+2$, we have that $c \leq d-3 < 2d-2n-1$. Let $z \in Z$ be a general point and let $\ell \subset \mathbb{P}^k$ be a general line passing through $\varphi(z)$. Consider the curve $C_\ell \subset \varphi^{-1}(\ell)$, which is the union of all irreducible components of curves in $Z$, which dominate $\ell$ via $\varphi$. We claim that $C_\ell$ is irreducible. Indeed, if $C'$ and $C''$ were two irreducible components, then $\text{gon}(C'_{\text{red}}) + \text{gon}(C''_{\text{red}}) \leq \text{deg}(\varphi|_{C_\ell})$. Being $\varphi|_{C_\ell} : C_\ell \to \ell \cong \mathbb{P}^1$ a map of degree $c$, either $C'_\text{red}$ or $C''_\text{red}$ would have gonality at most $\frac{c}{2} < d-n - \frac{1}{2}$. By varying $\ell \subset \mathbb{P}^k$, $z \in Z$, and $q \in X$, we conclude that $X$ is covered by curves of gonality smaller than $\frac{d-n}{2}$, which is impossible (cf. [3, Theorem A]). The same argument shows that $C_\ell$ is reduced.

Therefore, we may define a family $C \to U \subset \mathbb{G}(1,k)$ of curves with a map of degree $c$ to $\mathbb{P}^1$, where $U \cong \mathbb{P}^{k-1}$ parameterizes lines through $\varphi(z)$, and for any $[\ell] \in U$, the corresponding curve is $C_\ell$. As we vary $q \in X$ (and hence $Z$), we may define a family of curves covering $X$, each endowed with a $c$-gonal map. Thus, [4, Proposition 2.12] ensures that for general $[\ell] \in U$, there exists a point $x_\ell \in C_\ell$ such that $C_\ell \subset X \cap V^{d-c}_x$ and the degree $c$ map $\varphi|_{C_\ell} : C_\ell \to \ell \cong \mathbb{P}^1$ is the projection from $x_\ell$.

Next, we need to show that all the points $x_\ell$ coincide with some fixed point $p \in Z$. For this, we consider the map $\psi : U \to Z \subset X$ sending $[\ell] \in U$ to the corresponding point $x_\ell$. Since $U \cong \mathbb{P}^{k-1}$, the image of $\psi$ is unirational. As $X$ does not contain rational curves (see, e.g., [9]), we conclude that $\psi(U)$ is a point $p \in Z$. Thus, $Z$ is covered by the curves $C_\ell \subset V^{d-c}_p$ for general $[\ell] \in U$, and the degree $c$ map $\varphi|_{C_\ell} : C_\ell \to \ell \cong \mathbb{P}^1$ is the projection from $p$. The assertion follows. \hfill $\square$

**Remark 4.2.** Let $X$ be an irreducible projective variety of dimension $n$ and let $Z \to T$ be a family of $k$-dimensional subvarieties of $X$. If $Z \to T$ is a covering family (i.e., for general $q \in X$, there exists $t \in T$ such that $q \in Z_t = \pi^{-1}(t)$), then $\dim(T) \geq n - k$. Indeed, the map $f : Z \to X$ must be dominant and hence $\dim(Z) = \dim(T) + k \geq n$.

If in addition $Z \to T$ is a connecting family (i.e., for general $q, q' \in X$, there exists $t \in T$ such that $q, q' \in Z_t$), then $\dim(T) \geq 2n - 2k$. Indeed the map $Z \times_T Z \to X \times X$ induced by $f : Z \to X$ must be dominant, hence $\dim(\pi_T Z) = 2k + \dim(T) \geq 2n$.

**Remark 4.3.** If $X \subset \mathbb{P}^{n+1}$ and $Y \subset \mathbb{P}^{n+1}$ are very general hypersurfaces of degree $d$, with $n \leq m$, then

\[
\text{irr}_k(Y) \leq \text{irr}_k(X) \quad \text{for any } 1 \leq k \leq n \quad \text{and} \quad \text{conn. gon}(Y) \leq \text{conn. gon}(X).
\]

Indeed, the section of $Y$ by a general $(n+1)$-plane of $\mathbb{P}^{n+1}$ is a very general hypersurface of $\mathbb{P}^{n+1}$.

We can now prove Theorem 1.2.

**Proof of Theorem 1.2.** When $k = n$, the assertion is covered by [3, Theorem C] and $\text{irr}_n(X) = d - 1$.

If $k = n-1$, we claim that $\text{irr}_{n-1}(X) \leq d - 2$. Indeed, tangent hyperplane sections $Z = X \cap T_p X$ of $X$ are $(n-1)$-dimensional varieties of degree $d$ having a double point at $p$ (see Lemma 2.1), so that the projection from $p$ is a dominant rational map $Z \to \mathbb{P}^{n-1}$ of degree $d - 2$. On the other hand, suppose by contradiction that $\text{irr}_{n-1}(X) = c \leq d - 3$. Proposition 4.1 ensures that any $(n-1)$-dimensional subvariety $Z \subset X$ computing $\text{irr}_{n-1}(X)$ is contained in $X \cap V^{d-c}_p$ for some $p \in X$. Thanks to (1.1) and [3, Theorem A], we have $c \geq \text{irr}_1(X) \geq d - n$, so that $3 \leq d - c \leq n$. Then, Theorem 1.1 gives...
that \( \dim V^{d-c}_p = n + 2 - (d - c) \). In order to cover \( X \) by \((n-1)\)-dimensional varieties cut out by the cones \( V^{d-c}_p \), we must have that \( \dim (X \cap V^{d-c}_p) = n + 1 - (d - c) \geq n - 1 \) and hence \( c \geq d - 2 \), a contradiction. Thus, \( \text{irr}_{n-1}(X) = d - 2 \).

If \( 1 \leq k \leq n - 2 \), we claim that \( \text{irr}_k(X) \leq d - 3 \). To see this, we note that for any \( p \in X \), Theorem 1.1 ensures that \( V_{3} p \) is a cone in \( T_p X \cong \mathbb{P}^n \) over a quadric \( \Lambda_3^p \subset \mathbb{P}^{n-1} \) (cf. Section 2). Then, the variety \( Z = X \cap V_{3} p \) has dimension \( n - 2 \) and the projection from \( p \) is a dominant map \( Z \to \Lambda_3^p \) of degree \( d - 3 \) to a rational variety. Thus, \( \text{irr}_1(X) \leq \cdots \leq \text{irr}_{n-2}(X) \leq d - 3 \).

Finally, any \( k \)-dimensional subvariety \( Z \subset X \) computing \( c = \text{irr}_k(X) \) is contained in some \( X \cap V^{d-c}_p \) by Proposition 4.1. As above, we deduce \( d - c \leq n \) and for any \( p \in X \), Theorem 1.1 gives \( \dim (X \cap V^{d-c}_p) = n + 1 - (d - c) \). Thus, in order to cover \( X \) by \( k \)-dimensional varieties in \( X \cap V^{d-c}_p \), we must have that \( n + 1 - (d - c) \geq k \), that is, \( c \geq d - 1 - n + k \).

For \( k = n - 2 \), the latter inequality gives \( \text{irr}_{n-2}(X) \geq d - 3 \), so the assertion follows. \( \square \)

Now, we prove Theorem 1.3.

**Proof of Theorem 1.3.** By [4, Lemma 2.2], if \( p \in X \) is a general point and \( 3 \leq h \leq \min\{n+1,d\} \), then \( \Lambda^h_p \) is a general complete intersection of type \((2,3,\ldots,h-1)\) in \( \mathbb{P}^{n-1} \). If \( \Lambda^h_p \) is a Fano variety, then it is rationally connected (see [15]). The canonical bundle of \( \Lambda^h_p \subset \mathbb{P}^{n-1} \) is \( \Omega_{\Lambda^h_p} = \sum_{i=2}^{h-1} i - n \). Therefore, \( \Lambda^h_p \) is a Fano variety if and only if

\[
\frac{h(h-1)}{2} \leq n - 1 \iff h \leq \left\lfloor \frac{\sqrt{8n+1}+1}{2} \right\rfloor. \tag{4.1}
\]

We note that \( d > n + 1 \geq \left\lfloor \frac{\sqrt{8n+1}+1}{2} \right\rfloor \geq 3 \) and we assume hereafter \( h = \left\lfloor \frac{\sqrt{8n+1}+1}{2} \right\rfloor \), so that the general \( \Lambda^h_p \) is a smooth, rationally connected, complete intersection, whose dimension is \( n + 1 - h \).

Setting \( Z_p := V^h_p \cap X \), the projection \( \varphi : Z_p \to \Lambda^h_p \) from \( p \) has degree \( d - h \leq d - 3 \). Given two general points \( q, q' \in Z_p \), let \( D \subset \Lambda^h_p \) be a rational curve connecting \( \varphi(q) \) and \( \varphi(q') \), and let \( C := \varphi^{-1}(D) \). By arguing as for the curves \( C^l \) in the proof of Proposition 4.1, we deduce that \( C \) is integral. Then, \( \Lambda^h_p \) is an irreducible curve passing through two general points \( q, q' \in Z_p \) endowed with a map \( \varphi|_C : C \to D \) of degree \( d - h \). Thus, \( \text{conn.gon}(Z_p) \leq d - h = d - \left\lfloor \frac{\sqrt{8n+1}+1}{2} \right\rfloor \).

Since \( n \geq 4 \), we have \( h = \left\lfloor \frac{\sqrt{8n+1}+1}{2} \right\rfloor \leq \frac{n}{2} + 1 \). So Lemma 3.2 ensures that for general \( q, q' \in X \), there exists a general point \( p \in X \) such that \( q, q' \in Z_p \), that is, the varieties \( Z_p \) produce a connecting family. Thus, \( \text{conn.gon}(X) \leq \text{conn.gon}(Z_p) \leq d - \left\lfloor \frac{\sqrt{8n+1}+1}{2} \right\rfloor \). \( \square \)

Let us consider integers \( n, d \geq 2 \) and \( 2 \leq h \leq \min\{n+1,d\} \). Before proving Theorem 1.4, we aim at introducing a suitable parameter space \( \Theta^{n+1}_h \) for \( 4 \)-tuples \( (p,\ell_1,\ell_2,X) \), where \( X \subset \mathbb{P}^{n+1} \) is a hypersurface of degree \( d \), \( p \in X \) and \( \ell_1, \ell_2 \subset V^h_p \) are lines having intersection multiplicity of at least \( h \) with \( X \) at \( p \).

To this aim, we define \( S_d := \mathbb{C}[x_0,\ldots,x_{n+1}]_d \) and \( S^*_d := S_d \setminus \{0\} \) as in (2.3), and we set

\[
P := \mathbb{P}^{n+1}, \quad \mathbb{G} := \mathbb{G}(1,n+1) \quad \text{and} \quad N+1 := \dim_{\mathbb{C}}(S_d) = \binom{d + n + 1}{d}.
\]

Let \( P \subset P \times \mathbb{G} \) be the universal family of lines over \( \mathbb{G} \), endowed with the projections \( P \xrightarrow{\pi_1} P \xrightarrow{\pi_2} \mathbb{G} \). The morphism \( \pi_1 \) makes \( P \) a \( \mathbb{P}^{n+1} \)-bundle over \( P \), whereas \( \pi_2 \) makes \( P \) a \( \mathbb{P}^1 \)-bundle over \( \mathbb{G} \), so that \( \dim(P) = 2n + 1 \). Consider the fibered product

\[
P \times_{\mathbb{G}} P := \{ (p, [\ell_1],[\ell_2]) \mid p \in \ell_1 \cap \ell_2 \} \subset P \times \mathbb{G} \times \mathbb{G}
\]

and its diagonal locus

\[
\Delta := \{ (p, [\ell],[\ell]) \in P \times_{\mathbb{G}} P \mid p \in \ell \} \cong P.
\]
Let $\overline{\mathcal{P}}$ denote the blow-up of $P \times_{\mathbb{P}} P$ along $\Delta$ and let $\overline{\Delta}$ be the exceptional divisor. Thus,

$$\dim(\overline{\mathcal{P}}) = 3n + 1 = \dim(\overline{\Delta}) + 1.$$ 

Moreover, as a set, we have $\overline{\Delta} = \{(p, [\mathcal{E}], [\Pi]) | p \in \mathcal{E} \subset \Pi \} \subset P \times \mathbb{G} \times \mathbb{G}(2, n + 1)$.

Given a polynomial $F \in S_d^+$, let $X_F := V(F) \subset \mathbb{P}$ denote its vanishing locus. Then, we define the variety $\Theta^{n+1}_h \subset \overline{\mathcal{P}} \times P(S_d)$ as

$$\Theta^{n+1}_h := \left\{(p, [\mathcal{E}_1], [\mathcal{E}_2], [F]) \in \overline{\mathcal{P}} \times P(S_d) \mid \mathcal{E}_1 \neq \mathcal{E}_2 \text{ and for } 1 \leq i \leq 2, \text{ either } \mathcal{E}_i \subset X_F \text{ or } X_F \cdot \mathcal{E}_i \geq h p \right\}. \quad (4.2)$$

**Lemma 4.4.** For any $2 \leq h \leq \min\{n + 1, d\}$, $\Theta^{n+1}_h$ is smooth, irreducible, of dimension $3n + 2 + N - 2h$, dominating both $\overline{\mathcal{P}}$ and $P(S_d)$ via the projection maps

$$\overline{\mathcal{P}} \xleftarrow{\Psi} \Theta^{n+1}_h \xrightarrow{\Phi} P(S_d).$$

**Proof.** Let $(p, [\mathcal{E}_1], [\mathcal{E}_2]) \in \overline{\mathcal{P}} \setminus \overline{\Delta}$ and $F \in S_d^+$. Requiring that $(p, [\mathcal{E}_1], [\mathcal{E}_2], [F]) \in \Theta^{n+1}_h$ amounts to impose $2h - 1$ independent linear conditions to $F$, corresponding to the conditions $\mathcal{E}_1 \cdot X_F \geq h p$ and $\mathcal{E}_2 \cdot X_F \geq h p$.

We claim that the same happens at $(p, [\mathcal{E}], [\Pi]) \in \overline{\Delta}$, when we require $(p, [\mathcal{E}], [\Pi], [F]) \in \Theta^{n+1}_h$. In fact, choose affine coordinates $(\eta, \xi)$ on $\Pi$ such that $p = (0, 0)$ and $\mathcal{E} = V(\eta)$. Write $F_\Pi = F_0 + F_1 + \cdots + F_d$, where $F_i = \sum_{0 \leq j \leq i} a_{ij} \eta^i \xi^j$ is a homogeneous polynomial of degree $i$. Imposing the condition $X_F \cdot \mathcal{E} \geq h p$ gives $a_{ij} = 0$, for any $0 \leq i \leq h - 1$. Now, consider a general line $\mathcal{E}' := V(\eta - \zeta)$ in $\pi$ through $p = (0, 0)$. Imposing the condition $X \cdot \mathcal{E}' \geq h p$ and letting $\mathcal{E}'$ approach $\mathcal{E}$, that is, letting $t$ approach zero, gives $a_{i,i-1} = 0$ for any $1 \leq i \leq h - 1$. Therefore, there are again $2h - 1$ independent conditions for the coefficients of $F$ in order to have $(p, [\mathcal{E}], [\Pi], [F]) \in \Theta^{n+1}_h$.

Hence, the projection $\Psi : \Theta^{n+1}_h \rightarrow \overline{\mathcal{P}}$ is onto, and its fibers are parameterized by $(2h - 1)$-codimensional linear subspaces of $P(S_d)$. Therefore, $\Theta^{n+1}_h$ is smooth, irreducible, of dimension

$$\dim(\Theta^{n+1}_h) = \dim(\overline{\mathcal{P}}) + \dim(P(S_d)) - (2h - 1) = 3n + 1 + N - 2h + 1.$$ 

The surjectivity of $\Phi : \Theta^{n+1}_h \rightarrow P(S_d)$ is clear; indeed, for any $F \in S_d^+$ and any $p \in X_F$, we have that $\dim(V^h_{p,X_F}) \geq n + 2 - h > 0$, as $h \leq n + 1$ by assumption. \qed

For a general polynomial $F \in S_d^+$, we set

$$\Theta^{n+1}_{h,F} := \Phi^{-1}([F]),$$

which is smooth, equidimensional, of dimension $3n + 2 - 2h$.

Now, we argue as in [17, Proof of Theorem 2.3] and we prove Theorem 1.4.

**Proof of Theorem 1.4.** Let $X \subset P^{n+1}$ be a very general hypersurface of degree $d \geq 2n + 2$ and let $h \in \mathbb{N}$ such that $\text{conn. gon}(X) = d - h$. It follows from Theorem 1.3 that $h \geq \left\lfloor \frac{1 + \sqrt{8n + 1}}{2} \right\rfloor$. We note that if $h = \left\lfloor \frac{1 + \sqrt{8n + 1}}{2} \right\rfloor$, we are done because $\left\lfloor \frac{1 + \sqrt{8n + 1}}{2} \right\rfloor \leq \left\lfloor \frac{\sqrt{16n + 25} - 3}{2} \right\rfloor$ for any $n \geq 4$. Hence, we assume hereafter

$$h > \left\lfloor \frac{1 + \sqrt{8n + 1}}{2} \right\rfloor. \quad (4.3)$$
Given two general points \(x_1, x_2 \in X\), there exists an irreducible curve \(C \subset X\) containing \(x_1\) and \(x_2\) such that \(\text{gon}(C) = \text{gon}(X) = d - h\). Since \(h \geq 3\), by Proposition 4.1 there exists a point \(p \in X\) such that \(C \subset V^h_p\), and the projection \(\pi_p : V^h_p \to \Lambda^h_p\) from \(p\) maps \(C\) to a rational curve \(D \subset \Lambda^h_p\). In particular, if \(\ell_1, \ell_2 \subset V^h_p\) denote the lines connecting the vertex \(p\) to the points \(x_1, x_2 \in X\), respectively, then \(D\) passes through the corresponding points \(\ell_1 \cap \Lambda^h_p\) and \(\ell_2 \cap \Lambda^h_p\). So, for any integer \(i \geq n + 1\), we consider the variety \(\Theta_t^i\), defined as in (4.2) (using the obvious modifications \(\mathbb{P} = \mathbb{P}^t, \mathbb{G} = \mathbb{G}(1, t)\), etc.), and we introduce the locus \(R_t \subset \Theta_t^i\) given by

\[
R_t := \left\{ (p, [\ell_1], [\ell_2], [F]) \in \Theta_t^i \left| \begin{array}{l}
\ell_1 \neq \ell_2 \text{ and } \exists \text{ a rational curve } D \subset \Lambda^h_{p,X_F} \text{ passing through the points } \\
\ell_1 \cap \Lambda^h_{p,X_F} \text{ and } \ell_2 \cap \Lambda^h_{p,X_F}
\end{array} \right. \right\}.
\]

(4.4)

In particular, we are interested in the case \(t = n + 1\).

It follows from [4, Lemma 2.2] that for \(F \in S^*_d\) general and \(p \in X_F\) general, the variety \(\Lambda^h_p = \Lambda^h_{p,X_F}\) is a general complete intersection of type \((1, 1, 2, ..., h - 1)\) in \(\mathbb{P}^{m+2}\). Thus, its canonical bundle is isomorphic to \(\mathcal{O}_{\Lambda^h_p}(1)\), which is effective by (4.1) and (4.3). In particular, \(\Lambda^h_p\) is not covered by rational curves, so that \(R_{n+1}\) consists of (at most) countably many proper closed subsets of \(\Theta_{n+1}^i\).

Let \(F \in S^*_d\) be the polynomial defining the very general hypersurface \(X \subset \mathbb{P}^{m+1}\). According to the discussion above, for general \(x_1, x_2 \in X\), we may find (at least) one 4-tuple \((p, [\ell_1], [\ell_2], [F]) \in R_{n+1}\), where \(\ell_i = \langle p, x_i \rangle\) for \(i = 1, 2\). Since \(\ell_1 \cap X\) consists of finitely many points, as we vary the pair \((x_1, x_2) \in X \times X\), the corresponding 4-tuples \((p, [\ell_1], [\ell_2], [F])\) describe a subset of \(R_{n+1} \cap \Theta_{n+1}^{i+1}\) having dimension at least \(2n\). In particular, \(\dim (R_{n+1} \cap \Theta_{n+1}^{i+1}) \geq 2n\) and as \(F \in S^*_d\) is very general, we deduce that \(\dim R_{n+1} \geq 2n + N\). Therefore,

\[
\text{codim}_{\Theta_{n+1}^i} R_{n+1} \leq n + 2 - 2h.
\]

(4.5)

We point out that for any subfamily \(F \subset \Theta_{n+1}^{i+1}\) such that \((p, [\ell_1], [\ell_2], [F]) \in F\), we have \(\text{codim}_F (R_{n+1} \cap F) \leq \text{codim}_{\Theta_{n+1}^{i+1}} R_{n+1}\). Hence, (4.5) gives

\[
\text{codim}_F (R_{n+1} \cap F) \leq n + 2 - 2h.
\]

(4.6)

We construct a subfamily \(F \subset \Theta_{n+1}^{i+1}\) with \((p, [\ell_1], [\ell_2], [F]) \in F\), as follows. Let

\[
m := \frac{h(h - 1)}{2} - 2
\]

and let \((p', [\ell'_1], [\ell'_2], [F']) \in \Theta_{n+1}^{i+2}\) be a 4-tuple such that \(Y' := V(F') \subset \mathbb{P}^{m+2}\) is a very general hypersurface of degree \(d\), \(p' \in Y'\) is a very general point, \(\ell'_1\) is very general among lines in \(V_{p', Y'}^h\) passing through \(p'\), and \(\ell'_2 \neq \ell'_1\). Moreover, we deduce from (4.3) that \(m + 2 \geq n + 1\).

Let \(M \geq m + 2 \geq n + 1\) and let \((p'', [\ell''_1], [\ell''_2], [F'']) \in \Theta_{n+1}^M\), where \(Y'' := V(F'') \subset \mathbb{P}^M\) is a hypersurface of degree \(d\) such that \(X\) is an \((n+1)\)-plane section and \(Y''\) is an \((m+2)\)-plane section, with \(p = p' = p''\), \(\ell_1 = \ell''_1 = \ell'_1\), and \(\ell_2 = \ell''_2 = \ell'_2\).

Now, for any \(r \geq n + 1\), let \(Z_r \subset \text{Hom}(\mathbb{P}^r, \mathbb{P}^M)\) be the set of parameterized \(r\)-planes in \(\mathbb{P}^M\) containing the plane \(\langle \ell_1, \ell_2 \rangle\), and let \(Z'_r \subset Z_r\) be the subset of parameterized \(r\)-planes \(\Lambda \subset \mathbb{P}^M\) such that \((p, [\ell_1], [\ell_2], [F_{\Lambda}]) \in R_{n+1}\), where \(F_{\Lambda}\) is a polynomial defining the section of \(Y''\) by \(\Lambda\) as a hypersurface in \(\Lambda\).\(^2\)

We point out that for any \(A \in Z_r\), we have \((p, [\ell_1], [\ell_2], [F_{\Lambda}]) \in \Theta_{n+1}^h\). To see this fact, consider the hypersurface \(Y := V(F_{\Lambda}) = \Lambda \cap Y''\) of degree \(d\) in \(\Lambda \cong \mathbb{P}^r\). For \(i = 1, 2\), \(\text{we have } \ell_i \subset \Lambda\), so the intersection schemes \(\ell_i \cdot Y\) and \(\ell_i \cdot Y''\) are supported on the same \(0\)-cycle of degree \(d\), that is, \(\text{mult}_q(\ell_i \cdot Y) = \text{mult}_q(\ell_i \cdot Y'')\) for any \(q \in Y \cap \ell_i\). In particular, \(Y \cdot \ell_i \geq hp\) for \(i = 1, 2\), so that \((p, [\ell_1], [\ell_2], [F_{\Lambda}]) \in \Theta_{n+1}^h\).

As in [17], let \(F\) be the image of \(Z_{n+1}\) in \(\Theta_{n+1}^{i+1}\) under the map sending an \((n+1)\)-plane \(\Lambda \in Z_{n+1}\) to the point \((p, [\ell_1], [\ell_2], [F_{\Lambda}]) \in \Theta_{n+1}^{i+1}\). Thus, \(R_{n+1} \cap F\) is the image of \(Z_{n+1}^{i+1}\). According to (4.6), we have

\[
\text{codim}_{Z_{n+1}^{i+1}} Z'_{n+1} \leq n + 2 - 2h.
\]

(4.7)
Let $\varepsilon_r := \operatorname{codim}_{\mathcal{Z}_r} \mathcal{Z}_r'$ Since $Y' \subseteq \mathbb{P}^{m+2}$ and $p \in Y'$ are very general, then $\Lambda^h_{p,Y'}$ is a smooth complete intersection of type $(1,1,2,...,h-1)$ in $\mathbb{P}^{m+2}$ by [4, Lemma 2.2]. Hence its canonical bundle is $O_{\Lambda^h_{p,Y'}} \left( \sum_{i=2}^{h-1} i - m - 1 \right)$, which is trivial by the choice of $m$. In particular, $\Lambda^h_{p,Y'}$ is not covered by rational curves and as $\ell_1 \subseteq V^h_{p,Y'}$ is a very general line through $p$, there are no rational curves of $\Lambda^h_{p,Y'}$ passing through the point $\ell_1 \cap \Lambda^h_{p,Y'}$. Thus, $(p, [\ell_1], [\ell_2], [F']) \notin R_{m+2}$ and $\varepsilon_{m+2} \geq 1$.

Applying [17, Proposition 2.5], we obtain

$$
\varepsilon_{m+1} = \operatorname{codim}_{\mathcal{Z}_m} \mathcal{Z}_m' \geq \varepsilon_{m+2} + 1 \geq 2,
$$

and by recursion

$$
\varepsilon_{n+1} = \operatorname{codim}_{\mathcal{Z}_n} \mathcal{Z}_n' \geq m - n + 2.
$$

By (4.7), we must have $m - n + 2 \leq n + 2 - 2h$, and as $m := \frac{h(h-1)}{2} - 2$, we deduce

$$
\frac{h(h-1)}{2} - n \leq n + 2 - 2h, \quad \text{so that} \quad h \leq \frac{-3 + \sqrt{16n + 25}}{2}.
$$

Thus, the connecting gonality of $X$ satisfies $\operatorname{conn.gon}(X) \geq d - \left\lfloor \frac{-3 + \sqrt{16n + 25}}{2} \right\rfloor$.

The final part of the statement is achieved by using (1.2) and noting that $\left\lfloor \frac{-1 + \sqrt{16n + 1}}{2} \right\rfloor = \left\lfloor \frac{-3 + \sqrt{16n + 25}}{2} \right\rfloor$ if and only if $n$ belongs to the set

$$
\{ 4a^2 + 3a, 4a^2 + 5a, 4a^2 + 5a + 1, 4a^2 + 7a + 2, 4a^2 + 9a + 4, 4a^2 + 11a + 6 | a \in \mathbb{N} \}.
$$

Finally, we discuss the values of $\operatorname{conn.gon}(X)$, when the hypersurface $X$ has small dimension.

**Example 4.5.** Let $X \subseteq \mathbb{P}^{n+1}$ be a very general hypersurface of degree $d \geq 2n + 2$, with $1 \leq n \leq 16$ and $n \neq 9, 13, 14$.

**Case** $n = 1$. When $X$ is a plane curve, $\operatorname{conn.gon}(X)$ equals the gonality of $X$, which is $\operatorname{gon}(X) = d - 1$ (cf. [7, Theorem 3.14]).

**Case** $n = 2$. The connecting gonality of very general surfaces $X \subseteq \mathbb{P}^3$ of degree $d \geq 5$ is computed by tangent hyperplane sections $X \cap T_pX$, so that $\operatorname{conn.gon}(X) = d - 2$ (see, e.g., [1]).

**Case** $n = 3$. When $n = 3$, we have $\operatorname{conn.gon}(X) = d - 2$. To see this fact, notice that $\operatorname{conn.gon}(X) \leq d - 2$ by Remark 4.3 and case $n = 2$ above. On the other hand, $\operatorname{conn.gon}(X) \geq \operatorname{cov.gon}(X) = d - 3$ by (1.1) and (1.2). Suppose by contradiction that there exists a connecting family $C \xrightarrow{\pi} T$ of $(d - 3)$-gonal curves. Then, Proposition 4.1 ensures that the general curve $C_t := \pi^{-1}(t)$ lies on $X \cap V^3_p$ for some $p \in X$. By Theorem 1.1, the varieties $Z_p := X \cap V^3_p$ are curves and, as we vary $p \in X$, we obtain a 3-dimensional family. However, according to Remark 4.2, the family $C \xrightarrow{\pi} T$ should have dimension at least 4, a contradiction.

**Cases** $4 \leq n \leq 16$ with $n \neq 9, 13, 14$. For all these values of $n$, we may apply Theorems 1.3 and 1.4, and the bounds included therein coincide. Thus,

$$
\operatorname{conn.gon}(X) = \begin{cases} 
  d - 3 & \text{if } n = 4, 5 \\
  d - 4 & \text{if } n = 6, 7, 8 \\
  d - 5 & \text{if } n = 10, 11, 12 \\
  d - 6 & \text{if } n = 15, 16.
\end{cases}
$$
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ENDNOTES
1This fact follows from a standard argument, but we sketch it for the sake of completeness. The exceptional divisor $\Delta$ is the projectivization of the normal bundle of a divisor in $P^r \times P$. Let $\Gamma$ denote the algebraic set $\{(p, [\ell], [\Pi]) | p \in \ell \subset \Pi \}$ on the right-hand side. There is an obvious morphism $\xi : \Gamma \to \Delta$, that maps a triple $(p, [\ell], [\Pi]) \in \Gamma$ to the point of $\Delta$ corresponding to the deformation of $(p, [\ell], [\ell'])$ to a pair $(p, [\ell], [\ell'])$, where $\ell'$ moves in the pencil of lines passing through $p$ inside the plane $\Pi$. The map $\xi$ is clearly injective. It is also surjective, because it has an inverse $\eta : \Delta \to \Gamma$ defined as follows. A point $x \in \Delta$ that lies over $(p, [\ell], [\Pi])$ corresponds to a deformation $\{(p, [\ell_{1,t}], [\ell_{2,t}])\}$ of $(p, [\ell], [\ell])$, with $t$ moving in a disc with center 0. Then, $\eta$ associates to $x$ the point $(p, [\ell], [\Pi]) \in \Gamma$, where $\Pi$ is the flat limit of the plane spanned by $\ell_{1,t}$ and $\ell_{2,t}$, when $t \to 0$. This shows that $\Gamma$ and $\Delta$ are set-theoretically the same.

2As in [17], we consider the locus $Z \subset \text{Hom}(P^r, P^M)$—rather than its counterpart in $G(r,M)$—because we need to fix homogeneous coordinates $[y_0 : \ldots : y_r]$ on each $r$-plane $\Lambda \subset P^M$, in order to define properly the polynomial $F_\Lambda \in \mathbb{C}[y_0, \ldots, y_r] \setminus \{0\}$ (up to scalar multiplication).

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