Fermion Clouds Around \( z = 0 \) Lifshitz Black Holes

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Abstract

The Dirac equation is solved in the \( z = 0 \) Lifshitz black hole (Z0LBH) spacetime. The set of equations representing the Dirac equation in the Newman-Penrose (NP) formalism is decoupled into a radial set and an angular set. The separation constant is obtained with the aid of the spin weighted spheroidal harmonics. The radial set of equations, which is independent of mass, is reduced to Zerilli equations (ZEs) with their associated potentials. In the near horizon (NH) region, these equations solved in terms of the Bessel functions of the first and second kinds arising from the fermionic perturbation on the background geometry. For computing the BQNMs instead of the ordinary quasinormal modes (QNMs), we first impose the purely ingoing wave condition at the event horizon. And then, Dirichlet boundary condition (DBC) and Newmann boundary condition (NBC) are applied in order to get the resonance conditions. For solving the resonance conditions we follow an iteration method. Finally, Maggiore’s method (MM) is employed to derive the entropy/area spectra of the Z0LBH which are shown to be equidistant.

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I. INTRODUCTION

The interaction of the BHs (depending on the couplings of spin-rotation, field mass-black hole (BH) mass, field charge- BH charge) with a fermionic (Dirac) field has attracted deep theoretical interest and it has motivated the researchers to have a better understanding of various physical fields, especially the spin 1/2 particles, in the vicinity of the BHs.

The seminal works of Bekenstein and Hawking [13, 44, 45] has shown that BH entropy ($S_{BH}$) so that BH area ($A_{BH}$) should be quantized in discrete and equidistant levels (see for a detailed discussion [14–17]). There is a proportionality between $S_{BH}$ and $A_{BH}$ ($S_{BH} = \frac{A_{BH}}{4}$), which is attested from the adiabatic invariance [46]. It has been proposed by Bekenstein that $A_{BH}$ should have the following discrete, equidistant spectrum (for the family of Schwarzschild BHs [15, 16]):

$$A_{BH} = \epsilon \hbar n, \quad n = 0, 1, 2, ...$$

(1)

where $\epsilon$ is known as the unknown fudge factor [40]. One can conclude from the above equation that the minimum change in the horizon area of the Schwarzschild BH ($\epsilon = 8\pi$) [34–36] is $\Delta A_{min} = 8\pi \hbar$. Among the methods that have been improved to obtain the entropy/area spectra of the numerous BHs inspired from Bekenstein (see [47] and references therein), MM [34], which was based on Kunstatter’s study [48] is in a full
agreement with Bekenstein’s original result (4). In Kunstatter’s study [48], the adiabatic invariant quantity \( I_{\text{adb}} \) was expressed by:

\[
I_{\text{adb}} = \int \frac{dM}{\Delta \omega},
\]

(2)

where \( \Delta \omega = \omega_{n-1} - \omega_n \) is the transition frequency of a BH having energy (mass) \( M \). Furthermore, \( I_{\text{adb}} \) was generalized to cover the BHs which are massive, charged, and rotating (hairy) (see [49] and references therein) as follows:

\[
I_{\text{adb}} = \int T_H dS_{\text{BH}} \frac{\Delta \omega}{\Delta \omega},
\]

(3)

in which the temperature of the BH is denoted by \( T_H \). The adiabatic invariant quantity acts as a quantized quantity \( (I_{\text{adb}} \simeq \hbar) \) for the highly excited states \( (n \to \infty) \) according to the Bohr–Sommerfeld quantization rule [38]. Therefore, the imaginary part of the frequency becomes dominant \( (\omega_I \gg \omega_R) \), and this yields that \( \Delta \omega \simeq \Delta \omega_I \). Meanwhile, Hod [50, 51] was the first physicist to argue that one can be use the QNMs [52, 53] in order for computing the transition frequency. Maggiore, inspired by Hod’s suggestion, considered the Schwarzschild BH as a highly damped harmonic oscillator and using a different method, re-derived Bekenstein’s original result (4). Recently, MM has been employed in numerous studies of BH quantization in the literature (see for instance [39, 41, 54–57]).

In this paper, we mainly investigate the entropy/area spectra of a four-dimensional Lifshitz BH (LBH) [5] possessing a particular dynamical exponent \( z = 0 \). The Lifshitz spacetimes have attracted attention from the researchers working on condensed matter and quantum field theories [4] for being invariant under anisotropic scale and characterizing gravitational dual of strange metals [6]. We start our analysis with the calculations of its quasilocal mass \( M_{QL} \) [9] and temperature by using the Wald’s entropy [58] and statistical temperature formula [1, 2], respectively. In fact, this problem was previously studied [18, 43, 59]. The QNMs were calculated and it was shown that the Z0LBH possesses a discrete and equidistant spectrum under massive scalar field perturbations. In this study, we reconsider the problem with fermionic perturbations and instead of the ordinary QNMs, we derive the boxed QNMs (BQNMs) [25–28, 39] that are the characteristic resonance spectra of the confined scalar fields in the Z0LBH geometry. To this end, we consider the Dirac equations with \( q = 0 \) and \( \mu^* = 0 \) spin-\( \frac{1}{2} \) test particles in the Z0LBH.
spacetime [5] and we show how the separation of the angular and the radial equations yields a Schrödinger-like wave equation (SLE) or the so-called Zerilli equation (ZE) [20] with its NH form.

To this end, we consider a mirror (confining cavity) surrounding the Z0LBH which is located at a constant radial coordinate. Next, we impose that the fermionic field should terminate at the mirror’s location, which requires two boundary conditions to be used: DBC and NBC [25, 31, 32, 39]. With this scenario, we focus our analysis of the ZE in the NH region [25].

After getting the NH form of the ZE, we show that the radial equation has the solution in terms of the Bessel functions of the first and second kind [29]. Finally, we use an iteration scheme and show how the Dirac BQNMs and the quantum spectra of entropy and area of the Z0LBH are obtained.

The present paper is organized as follows. In section 2, we review the background geometry (Z0LBH) and we obtain the quasilocal mass ($M_{QL}$) via the Brown-York (BY) formalism [9, 10]. In section 3, we solve the Dirac equations for the uncharged massless spin $1/2$ particles, in the framework of NP formalism. In particular, we analyze the effective potential for the ZE. We show the existence of the Dirac BQNMs of Z0LBH in section 4. Finally, the paper ends with a discussion of our findings in section 5. Throughout this study we use units with $G = c = \hbar = 1$.

II. Z0LBH SPACETIME

In this section we introduce the Lifshitz spacetimes [4, 18] and the special case Z0LBH in four dimensions [5] on which we focus our analysis, briefly. Lifshitz spacetimes are described by the following line element [4, 18]

$$ds^2 = -\frac{r^2 z}{l^2 z} dt^2 + \frac{l^2}{r^2} dr^2 + \frac{r^2}{l^2} d\vec{x}^2,$$

(4)

where $l$ is the length scale in the geometry, $z$ is the dynamical exponent and $\vec{x}$ is the D-2 spatial vector. The action corresponding to the Einstein-Weyl gravity is given by [5]

$$S = \frac{1}{2\kappa^2} \int \sqrt{-g} d^4x (R - 2\Lambda + \frac{1}{2} \alpha |Weyl|^2),$$

(5)
where $\kappa^2 = 8\pi G$ , $|\text{Weyl}|^2 = R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} - 2R^{\mu\nu}R_{\mu\nu} + \frac{1}{3}R^2$ and $\alpha = \frac{z^2 + 2z + 3}{4z(z-4)}$ is a constant that corresponds to $\alpha = \infty$ for $z = 0$ or $z = 4$.

It is shown that the static asymptotically LBH solutions for both $z = 4$ and $z = 0$ are received by the CG [5, 8] and also $z = 3$ and $z = 4$ LBH solutions exist in the Hořava-Lifshitz gravity [5, 7, 18]. Now we turn our attention to a special case of CG, the four dimensional Z0LBH, with its metric given by [5]

$$ ds^2 = f(r) dt^2 - \frac{4}{r^2 f(r)} dr^2 - r^2 d\Omega^2_{2,k} $$  \hspace{1cm} (6) $$

with [7]

\begin{align*}
  d\theta^2 + d\varphi^2, & \quad k = 0 \\
  d\Omega^2_{2,k} = d\theta^2 + \sin^2(\theta) d\varphi^2, & \quad k = 1 \\
  d\theta^2 + \sinh^2(\theta) d\varphi^2, & \quad k = -1.
\end{align*} \hspace{1cm} (7) $$

The metric function $f(r)$ is defined as

$$ f(r) = 1 + \frac{c}{r^2} + \frac{c^2 - k^2}{3r^4} \hspace{1cm} \text{(8)} $$

In this metric $d\Omega^2_{2,k}$ becomes 2-D sphere in accordance with our choice $k = 1$. One may choose $k = -1$ and get the unit hyperbolic plane or $k = 0$ to get 2-torus [7]. The metric given in Eq. (6) conformally describes an AdS BH if one sets $k = -1$, on the other hand, it describes a dS BH if one choose $k = 1$ [5]. The solution has a singularity at $r = 0$ however this singularity becomes naked for $k = 0$. There is an horizon for $k = \pm 1$ solution of the form [5, 7]

$$ r_h^2 = \frac{1}{6} \left( \sqrt{3(4 - c^2)} - 3c \right). \hspace{1cm} \text{(9)} $$

Since $r_h^2$ is positive ($r_h^2 = 1$), it is required to be $-2 \leq c < 1$. Our focus is on $k = 1$ solution with $c = -1$, so now the metric is given by

$$ ds^2 = f(r) dt^2 - \frac{4}{r^2 f(r)} dr^2 - r^2 \left[ d\theta^2 + \sin^2(\theta) d\varphi^2 \right], \hspace{1cm} \text{(10)} $$

with

$$ f(r) = 1 - \frac{r_h^2}{r^2}. \hspace{1cm} \text{(11)} $$
The corresponding Ricci scalar and Kretschmann scalar change in the following proportions at spatial infinity, respectively

\[
R = R_{\mu}^{\mu} \sim \frac{5(c^4 - 2c^2 + 1)}{12r^8},
\]

\[
K = R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} \sim \frac{25(c^4 - 2c^2 + 1)}{12r^8}.
\]

(12)

A. Mass Computation of Z0LBH via BY Formalism

The spacetime, as a union of space, time and gravitation in GR, has a curvature singularity which stands for the gravitational force. In GR, energy conservation is fundamental. Our spacetime is a four dimensional manifold, so that, one should be able to get the information from 2-D boundary surface \(d\Omega_{2,k}^2\) (hypersurface \(\Sigma\)). Quasilocal mass \(M_{QL}\) [9, 10] measures the density of matter.

We will consider BY formalism, in order to calculate \(M_{QL}\). In this formalism, a spherically symmetric four dimensional metric solution is given by [9, 10]

\[
ds^2 = -F(R)^2dt^2 + \frac{dR^2}{G(R)^2} + R^2d\Omega_{2,k}^2,
\]

which adopts the \(M_{QL}\) with the following definition [11, 12]

\[
M_{BY} = \frac{N - 2}{2}R^{N-3}F(R) \left\{ G_{\text{ref}}(R) - G(R) \right\},
\]

(14)

where \(R, N,\) and \(G_{\text{ref}}(R)\) stand for the radius of hypersurface \(\Sigma\), the dimension and an optional non-negative reference function, respectively. \(G_{\text{ref}}(R)\) assures zero energy for the spacetime. The functions of the metric (13) read

\[
F(R) = \sqrt{f(r)},
\]

\[
G(R) = \frac{RF(R)}{2} = \frac{r\sqrt{f(r)}}{2},
\]

(15)

such that

\[
G_{\text{ref}}(R) = G|_{z=0},
\]

(16)

in which \(z = r_{h}^2/r^2\).
After applying these to the solution \( f(r) = 1 - z \) and making a straightforward calculation, the quasilocal mass of the 4-D Z0LBH (10) can be obtained as

\[
M_{BY} = \frac{r_h^2}{4},
\]

or

\[
r_h^2 = 4M_{BY} = 4M_{QL}.
\]

One may check if this result is in consistent with the mass derived from \( dM = T dS \).

One should first calculate the surface gravity [1] with the timelike Killing vector \( \xi^\mu = (1, 0, 0, 0) \)

\[
\kappa = \lim_{r \to r_h} \sqrt{\xi^\mu \nabla_\mu \xi^\nu \nabla_\rho \xi^\rho - \xi^2},
\]

which corresponds to the following expression [3]

\[
\kappa = \frac{r_h f'(r_h)}{4} = \frac{1}{2}.
\]

Therefore, the Hawking temperature [1, 2] reads

\[
T_H = \frac{\kappa}{2\pi} = \frac{1}{4\pi}.
\]

As it can be deduced from that constant Hawking temperature, Z0LBH radiates with isothermal process. After substituting Eq. (17) into \( dM = T_H dS_{BH} \), we derive the entropy as

\[
S_{BH} = \pi r_h^2 = \frac{A_{BH}}{4},
\]

which agrees with the Bekenstein-Hawking entropy [14–17].

III. DIRAC FIELD ON Z0LBH

In GR, in order to study the Dirac equation in curved spacetimes, many formalisms have been developed such as NP and spinor formalisms [19]. In order to solve Dirac equation in Z0LBH geometry, we use NP formalism [19, 22] and consider a massive Dirac field. In order for solving the Dirac equation, we separate the Dirac equation and give the solution of angular equation with the separation constant \( \lambda \). Finally, we show how the radial equation reduces to the ZE [20] with an effective potential in the NH.
The basis vectors of the null tetrad [19] in the geometry defined by the line element Eq. (10) are chosen as

\[
\begin{align*}
  l_\mu &= \sqrt{\frac{f(r)}{2}} \left[ 1, -\frac{2}{rf(r)}, 0, 0 \right], \\
  n_\mu &= \sqrt{\frac{f(r)}{2}} \left[ 1, \frac{2}{rf(r)}, 0, 0 \right], \\
  m_\mu &= -\frac{r}{\sqrt{2}} [0, 0, 1, i \sin(\theta)], \\
  \overline{m}_\mu &= \frac{r}{\sqrt{2}} [0, 0, -1, i \sin(\theta)].
\end{align*}
\] (23)

The nonzero NP spin coefficients [19] regarding the above covariant null tetrad are found as

\[
\begin{align*}
  \alpha &= -\beta = -\frac{\sqrt{2} \cot(\theta)}{4}, \\
  \varepsilon &= \gamma = \frac{\sqrt{2}}{16} \frac{r}{\sqrt{f(r)}} [\partial_r f(r)], \\
  \rho &= \mu = -\frac{\sqrt{2}}{4} \sqrt{f(r)}.
\end{align*}
\] (24)

In NP formalism, in the case of test \(q = 0\) Dirac particles, the Dirac equations can be expressed by the well known Chandrasekhar-Dirac equations (CDEs) [20] with the aid of the spin coefficients as follows

\[
\begin{align*}
  (D + \varepsilon - \rho) F_1 + (\bar{\delta} + \pi - \alpha) F_2 &= i\mu_p G_1, \\
  (\Delta + \mu - \gamma) F_2 + (\delta + \beta - \tau) F_1 &= i\mu_p G_2, \\
  (D + \bar{\varepsilon} - \bar{\rho}) G_2 - (\delta + \bar{\pi} - \bar{\alpha}) G_1 &= i\mu_p F_2, \\
  (\Delta + \bar{\mu} - \bar{\gamma}) G_1 - (\bar{\delta} + \bar{\beta} - \bar{\tau}) G_2 &= i\mu_p F_1,
\end{align*}
\] (25)

where \(\mu^* = \sqrt{2}\mu_p\) is the mass of the uncharged Dirac particles. The directional derivatives corresponding to the null tetrad are defined as [20].
\[ D = i l j, \]
\[ \Delta = n l j, \]
\[ \delta = m l j, \]
\[ \bar{\delta} = \bar{m} l j, \]  

(26)

where a bar over a quantity denotes complex conjugation.

The wave functions \( F_1, F_2, G_1, \) and \( G_2 \) which represent the Dirac spinors are assumed to be [20]

\[
\begin{align*}
F_1 &= f_1 (r) A_1 (\theta) \exp [i (\omega t + m \phi)], \\
G_1 &= g_1 (r) A_2 (\theta) \exp [i (\omega t + m \phi)], \\
F_2 &= f_2 (r) A_3 (\theta) \exp [i (\omega t + m \phi)], \\
G_2 &= g_2 (r) A_4 (\theta) \exp [i (\omega t + m \phi)],
\end{align*}
\]

(27)

where \( \omega \) (positive and real) and \( m \) are the frequency of the incoming wave related to the energy of the Dirac particle and the azimuthal quantum number of the wave, respectively.

Substituting Eq. (27) and Eq. (24) into Eq. (25), one can simplify the Dirac equations to get

\[
\begin{align*}
\frac{\tilde{Z} f_1 (r)}{f_2 (r)} + \frac{L A_3 (\theta)}{A_1 (\theta)} - i \mu^* r g_1 (r) A_2 (\theta) \frac{f_1 (r)}{f_2 (r)} A_1 (\theta) &= 0, \\
\frac{\tilde{Z} f_2 (r)}{f_1 (r)} - \frac{L^* A_1 (\theta)}{A_3 (\theta) f_1 (r)} + i \mu^* r g_2 (r) A_4 (\theta) \frac{f_2 (r)}{f_1 (r)} A_3 (\theta) &= 0, \\
\frac{\tilde{Z} g_2 (r)}{g_1 (r)} - \frac{L^* A_2 (\theta)}{A_4 (\theta) g_1 (r)} - i \mu^* r f_2 (r) A_3 (\theta) \frac{g_2 (r)}{g_1 (r)} A_4 (\theta) &= 0, \\
\frac{\tilde{Z} g_1 (r)}{g_2 (r)} + \frac{L A_4 (\theta)}{A_2 (\theta) g_2 (r)} + i \mu^* r f_1 (r) A_1 (\theta) \frac{g_1 (r)}{g_2 (r)} A_2 (\theta) &= 0,
\end{align*}
\]

(28)

with the radial and the angular operators, respectively.
\[ \tilde{Z} = \frac{i\omega r}{\sqrt{f(r)}} + \frac{1}{2} r^2 \sqrt{f(r)} \partial_r + \frac{1}{8} \frac{r^2}{\sqrt{f(r)}} [\partial_r f(r)] + \frac{1}{2} r \sqrt{f(r)}, \]

\[ \overline{Z} = -\frac{i\omega r}{\sqrt{f(r)}} + \frac{1}{2} r^2 \sqrt{f(r)} \partial_r + \frac{1}{8} \frac{r^2}{\sqrt{f(r)}} [\partial_r f(r)] + \frac{1}{2} r \sqrt{f(r)}, \]

and

\[ L = \partial_\theta + \frac{m}{\sin(\theta)} + \frac{\cot(\theta)}{2}, \]

\[ L^\dagger = \partial_\theta - \frac{m}{\sin(\theta)} + \frac{\cot(\theta)}{2}. \]

As it is obvious from Eq. (28) that \{f_1, f_2, g_1, g_2\} and \{A_1, A_2, A_3, A_4\} are the functions of two distinct variables \(r\) and \(\theta\), respectively, one can introduce a separation constant \(\lambda\) and assume that

\[ f_1 (r) = g_2 (r), \]
\[ f_2 (r) = g_1 (r), \]
\[ A_1 (\theta) = A_2 (\theta), \]
\[ A_3 (\theta) = A_4 (\theta). \]

(31)

to split Eq. (28) into two sets of radial and angular equations

\[ \overline{Z} g_1 (z) = (\lambda - i\mu^* r) g_2 (z), \]
\[ \tilde{Z} g_2 (z) = (\lambda + i\mu^* r) g_1 (z), \]

(32)

and

\[ L^\dagger A_1 (\theta) = \lambda A_3 (\theta), \]
\[ LA_3 (\theta) = -\lambda A_1 (\theta). \]

(33)

At this point it would me more convenient to make the choice of massless Dirac particles \((\mu^* = 0)\) to deal with above sets of equations.
In the spherical case, the angular operators (or the so-called laddering operators) $L^\dagger$ and $L$ lead the spin weighted spheroidal harmonics $sY^m_l(\theta)$ [19, 21–24] and they are governed by

$$
\left( \partial_\theta - \frac{m}{\sin \theta} - s \cot \theta \right) \left[ sY^m_l(\theta) \right] = -\sqrt{(l'-s)(l'+s+1)} \left[ s_{+1}Y^m_l(\theta) \right],
$$

$$
\left( \partial_\theta + \frac{m}{\sin \theta} + s \cot \theta \right) \left[ sY^m_l(\theta) \right] = \sqrt{(l'+s)(l'-s+1)} \left[ s_{-1}Y^m_l(\theta) \right].
$$

(34)

The spin weighted spheroidal harmonics $sY^m_l(\theta)$ has the following generic form [24]

$$
sY^m_l(\theta) = \exp \left( im\varphi \right) \sqrt{\frac{2l'+1}{4\pi} \frac{(l'+m)!}{(l'+s)! (l'-s)!}} \left[ \sin \left( \frac{\theta}{2} \right) \right]^{2l'} \times \sum_{r=-l'}^{l'} (-1)^{l'+m-r} \left( \begin{array}{c} l' - s \\ r - s \end{array} \right) \left( \begin{array}{c} l' + s \\ r - m \end{array} \right) \left[ \cot \left( \frac{\theta}{2} \right) \right]^{2r-m-s} \sin \left( \frac{\theta}{2} \right) (\sin \theta)^{l'} \sin \phi \sin \phi^2 \sin^2 \frac{\theta}{2},
$$

(35)

where $l'$ is the angular quantum number and $s$ is the spin weight with $l' = |s|, |s| + 1, |s| + 2...$ and $-l' < m < +l'$. Thus, having $s = \pm \frac{1}{2}$, and comparing Eq. (33) with Eq. (34) one can identify

$$
A_1(\theta) = -\frac{1}{2} Y^m_l(\theta),
$$

$$
A_3(\theta) = \frac{1}{2} Y^m_l(\theta),
$$

(36)

and obtain the separation constant $\lambda$ which is the eigenvalue of the spin weighted spheroidal harmonic [24] equation as

$$
\lambda = - \left( l' + \frac{1}{2} \right).
$$

(37)

A. Effective Potential

Now, we give the effective potential and obtain the solution to the radial part of the Dirac equation (32). In fact, it is appropriate to alter the radial equations in a Schrödinger-type with an effective potential, in order to investigate the BQNMs [25–28].

Defining a new function in terms of the metric function
and substituting the scalings

\begin{align*}
g_1 (r) &= G_1 (r) \exp \left[ -2 \int \frac{Y (r)}{r^2 \sqrt{f (r)}} dr \right], \\
g_2 (r) &= G_2 (r) \exp \left[ -2 \int \frac{Y (r)}{r^2 \sqrt{f (r)}} dr \right],
\end{align*}

(39)

into the radial equations Eq. (32) and setting \( \mu^* = 0 \), one gets

\begin{align*}
[A (r) \partial_r - i \omega] G_1 (r) &= B (r) G_2 (r), \\
[A (r) \partial_r + i \omega] G_2 (r) &= B (r) G_1 (r),
\end{align*}

(40)

where the functions \( A(r) \) and \( B(r) \) read

\begin{align*}
A (r) &= \frac{r f (r)}{2}, \\
B (r) &= \frac{\lambda \sqrt{f (r)}}{r},
\end{align*}

(41)

and

\begin{align*}
\partial_r &= \frac{1}{A (r)} dr,
\end{align*}

(43)

leads us to express Eq. (40) as follows

\begin{align*}
[\partial_{r^*} - i \omega] G_1 (r^*) &= B (r) G_2 (r^*), \\
[\partial_{r^*} + i \omega] G_2 (r^*) &= B (r) G_1 (r^*).
\end{align*}

(44)

In order to decouple the above equations, we consider the solutions of the form
\[ G_1 (r_*) = P_1 (r_*) + P_2 (r_*), \]
\[ G_2 (r_*) = P_1 (r_*) - P_2 (r_*), \]  
(45)

which yields the two radial equations of Schrödinger-type (or the so-called ZE [20])

\[ \partial^2_{r_*} P_1 (r_*) + \left[ \omega^2 - V_1 \right] P_1 (r_*) = 0, \]
\[ \partial^2_{r_*} P_2 (r_*) + \left[ \omega^2 - V_2 \right] P_2 (r_*) = 0, \]  
(46)

whose associated Zerilli potentials are given by

\[ V_1 = B^2 (r) + \partial_{r_*} B (r) = \lambda \sqrt{r^2 - r_h^2} \left( 2 \lambda \sqrt{r^2 - r_h^2} + 2r_h^2 - r^2 \right), \]
\[ V_2 = B^2 (r) - \partial_{r_*} B (r) = \lambda \sqrt{r^2 - r_h^2} \left( 2 \lambda \sqrt{r^2 - r_h^2} - 2r_h^2 + r^2 \right). \]  
(47)

For studying the fermion BQNMs analytically, we follow a recent study of [25]. Taking

\[ r = x r_h + r_h, \]  
(48)

and substituting into Eq. (41) gives

\[ A (x) = r_h x \frac{(x + 2)}{2(x + 1)}, \]  
(49)

with its NH form

\[ A_{NH} (x) \approx r_h x + O \left( x^2 \right) = 2 \kappa r_h x. \]  
(50)

This allows us to write the NH forms of the Zerilli potentials (47) as

\[ V_1^{NH} (x) = \frac{\lambda}{\sqrt{2} r_h} \sqrt{x} x + \frac{2 \lambda^2}{r_h^2} x + O \left( x^{3/2} \right), \]
\[ V_2^{NH} (x) = - \frac{\lambda}{\sqrt{2} r_h} \sqrt{x} \frac{2 \lambda^2}{r_h^2} x + O \left( x^{3/2} \right). \]  
(51)
Setting a new coordinate

\[ y = \int \frac{\kappa r_h}{A_{NH}(x)} \, dx = \frac{1}{2} \ln (x) = \kappa r_*, \]  

(52)

with its limits

\[
\lim_{r \to r_h} y = -\infty, \\
\lim_{r \to \infty} y = \infty, 
\]  

(53)

one can express the radial coordinate \( x \) in terms of the surface gravity as

\[ x = \exp (2y) = \exp (2\kappa r_*). \]  

(54)

Therefore, we can recast the NH Zerilli potentials (or the so-called effective potentials) (51), in the leading order terms as follows

\[
V_{1NH}(y) = \frac{\lambda}{\sqrt{2r_h}} \exp(y), \\
V_{2NH}(y) = -\frac{\lambda}{\sqrt{2r_h}} \exp(y). 
\]  

(55)

The NH ZEs in which \( \tilde{\omega} = \omega / \kappa \)

\[
\frac{d^2}{dy^2} P_1(y) + \left( \tilde{\omega}^2 - \frac{V_{1NH}(y)}{\kappa^2} \right) P_1(y) = 0, \\
\frac{d^2}{dy^2} P_2(y) + \left( \tilde{\omega}^2 - \frac{V_{2NH}(y)}{\kappa^2} \right) P_2(y) = 0, 
\]  

(56)

can be solved after inserting the value of the separation constant. The solutions of these equations are obtained in terms of the Bessel functions of the first and the second kind [29] as follows

\[
P_j(y) = C_{j1} J_{-2i\tilde{\omega}} \left[ \kappa 2^{5/4} \sqrt{\frac{2l^l + 1}{r_h}} \exp(y/2) \right] + \\
C_{j2} Y_{-2i\tilde{\omega}} \left[ \kappa 2^{5/4} \sqrt{\frac{2l^l + 1}{r_h}} \exp(y/2) \right], 
\]  

(57)
where $C_{j1}$ and $C_{j2}$ are constants and

$$
\mathbb{K} = (i)^{j-1}, \quad j = 1, 2. \quad (58)
$$

**IV. BQNMS AND QUANTUM SPECTRA OF Z0LBH**

In this section, our interest is to compute the Dirac BQNMs. First, we impose the purely ingoing wave condition at the event horizon for QNMs to appear [20]. Then, we impose the DBC and the NBC in order for getting the resonance conditions. At this point, we use an iteration in order to solve the resonance conditions [25, 31].

The solutions of the NH ZEs can be rewritten as

$$
P_j(x) = C_{j1} J_{-2i\tilde{\omega}} \left( 4\mathbb{K} \sqrt{\Omega \sqrt{x}} \right) + C_{j2} Y_{-2i\tilde{\omega}} \left( 4\mathbb{K} \sqrt{\Omega \sqrt{x}} \right) \quad (59)
$$

with the parameter

$$
\Omega = \frac{2l' + 1}{2\sqrt{2}r_h}. \quad (60)
$$

Using the limiting forms of the Bessel functions [29, 30] given by

$$
J_\nu(z) \sim \frac{[\left(1/2\right) z]^\nu}{\Gamma(1 + \nu)}, \quad \nu \neq -1, -2, -3, ...
$$

$$
Y_\nu(z) \sim -\frac{1}{\pi} \Gamma(v) \left( \frac{1}{2} z \right)^{-\nu}, \quad \Re \nu > 0 \quad (61)
$$

the NH ($\exp(y/2) \ll 1$) behavior of the solution can be obtained as

$$
P_j \sim C_{j1} \frac{\left( 2\mathbb{K} \sqrt{\Omega} \right)^{2i\tilde{\omega}}}{\Gamma(1 - 2i\tilde{\omega})} \exp(-i\tilde{\omega}y) - C_{j2} \frac{1}{\pi} \Gamma(-2i\tilde{\omega}) \left( 2\mathbb{K} \sqrt{\Omega} \right)^{2i\tilde{\omega}} \exp(i\tilde{\omega}y)
$$

$$
= C_{j1} \frac{\left( 2\mathbb{K} \sqrt{\Omega} \right)^{-2i\tilde{\omega}}}{\Gamma(1 - 2i\tilde{\omega})} \exp(-i\omega r_s) - C_{j2} \frac{1}{\pi} \Gamma(-2i\tilde{\omega}) \left( 2\mathbb{K} \sqrt{\Omega} \right)^{2i\tilde{\omega}} \exp(i\omega r_s). \quad (62)
$$

Imposing the boundary condition at the event horizon for BQNMs requires us to vanish the outgoing waves by choosing $C_{j2} = 0$. Therefore the proper solution of Eq. (56) becomes
\[ P_j (x) = C_j J_{-2i\tilde{\omega}} \left( 4\mathcal{N} \sqrt{\Omega \sqrt{x}} \right). \] (63)

Taking into account the DBC at the horizon (confining cage) \[25, 31, 32, 39\]

\[ P_j (x) \big|_{x=x_m} = 0, \] (64)

one gets the condition

\[ J_{-2i\tilde{\omega}} \left( 4\mathcal{N} \sqrt{\Omega \sqrt{x_m}} \right) = 0. \] (65)

With the aid of the relation \[29\],

\[ Y_\nu (z) = J_\nu (z) \cot (\nu \pi) - J_{-\nu} (z) \csc (\nu \pi) \] (66)

the boundary condition Eq. (65) can be stated as

\[ \tan (2i\tilde{\omega} \pi) = \frac{J_{2i\tilde{\omega}} \left( 4\mathcal{N} \sqrt{\Omega \sqrt{x_m}} \right)}{Y_{2i\tilde{\omega}} \left( 4\mathcal{N} \sqrt{\Omega \sqrt{x_m}} \right)} \] (67)

The boundary of the cage is at the event horizon Thus, one can use the limiting forms of the Bessel functions \[29\] in the above condition and obtain the resonance condition as follows

\[ \tan (2i\tilde{\omega} \pi) \sim -\frac{\pi \left( 2\mathcal{N} \sqrt{\sqrt{z_m}} \right)^{2i\tilde{\omega}}}{\Gamma (2i\tilde{\omega}) \Gamma (2i\tilde{\omega} + 1)} \]

\[ \quad = i \frac{\pi \exp \left( -4\pi \tilde{\omega} / j \right)}{2\tilde{\omega} \Gamma^2 (2i\tilde{\omega})} \left( 4\sqrt{z_m} \right)^{2i\tilde{\omega}}, \] (68)

where \( z_m = \Omega^2 x_m \).

Imposing the NBC \[25, 31, 32, 39\] given by

\[ \frac{dR(x)}{dx} \big|_{x=x_m} = 0, \] (69)

we find

\[ J_{-2i\tilde{\omega}+1} \left( 4\mathcal{N} \sqrt{\sqrt{z_m}} \right) - J_{-2i\tilde{\omega}-1} \left( 4\mathcal{N} \sqrt{\sqrt{z_m}} \right) = 0 \] (70)

from Eq. (66) we obtain the following relation
\[ Y_{\nu+1}(z) - Y_{\nu-1}(z) = \cot(\nu \pi) [J_{\nu+1}(z) - J_{\nu-1}(z)] - \csc(\nu \pi) [J_{-\nu-1}(z) - J_{-\nu+1}(z)] \quad (71) \]

We combine the equations (70) and (71) and express the NBC’s resonance condition as

\[
\tan(2i\bar{\omega}\pi) = \frac{J_{2i\bar{\omega}-1} \left( 4N\sqrt{z_m} \right)}{J_{2i\bar{\omega}+1} \left( 4N\sqrt{z_m} \right)} \left[ \frac{-1 + J_{2i\bar{\omega}+1} \left( 4N\sqrt{z_m} \right) / J_{2i\bar{\omega}-1} \left( 4N\sqrt{z_m} \right)}{1 - Y_{2i\bar{\omega}-1} \left( 4N\sqrt{z_m} \right) / Y_{2i\bar{\omega}+1} \left( 4N\sqrt{z_m} \right)} \right]. \quad (72)
\]

Using the limiting forms (61), one finds

\[
\frac{J_{2i\bar{\omega}+1} \left( 4N\sqrt{z_m} \right)}{J_{2i\bar{\omega}-1} \left( 4N\sqrt{z_m} \right)} \equiv \frac{Y_{2i\bar{\omega}-1} \left( 4N\sqrt{z_m} \right)}{Y_{2i\bar{\omega}+1} \left( 4N\sqrt{z_m} \right)} \sim O(z_m), \quad (73)
\]

and reads the resonance condition (72) in the NH as

\[
\tan(2i\bar{\omega}\pi) \sim -\frac{J_{2i\bar{\omega}-1} \left( 4N\sqrt{z_m} \right)}{Y_{2i\bar{\omega}+1} \left( 4N\sqrt{z_m} \right)} = -i \frac{\pi (8^2)^{2i\bar{\omega}}}{2\bar{\omega} \Gamma^2 (2i\bar{\omega})} (4\sqrt{z_m})^{2i\bar{\omega}}.
\]

(74)

To solve the resonance conditions, we use an iteration method [25, 31, 39], since the obtained resonance conditions are small quantities. The zeroth order resonance condition [25, 31, 39] has the form

\[
\tan(2i\bar{\omega}_n^{(0)} \pi) = 0, \quad (75)
\]

which means that

\[
\bar{\omega}_n^{(0)} = -i \frac{n}{2}, \quad (n = 0, 1, 2, ...). \quad (76)
\]

The first order resonance condition can be obtained by substituting Eq. (76) into the r.h.s. of (68) and (74), as follows

\[
\tan(2i\bar{\omega}_n^{(1)} \pi) = \pm i \frac{\pi \exp \left( 2i\pi n / j \right)}{(-in) \Gamma^2 (n)} (4\sqrt{z_m})^n, \quad (77)
\]

which reduces to

\[
\tan(2i\bar{\omega}_n^{(1)} \pi) = \mp n \frac{\pi}{(n!)^2} \left[ (-1)^{2/j} \ 4\sqrt{z_m} \right]^n. \quad (78)
\]

\[17\]
Using \( \tan(x + n\pi) = \tan(x) \approx x \) for \( x \ll \) yields

\[
2i\tilde{\omega}_n\pi = n\pi \left\{ 1 \mp \frac{1}{(n!)^2} \left[a \right]^n \right\}
\]  

(79)

Hence we find

\[
\tilde{\omega}_n = -i \frac{n}{2} \left\{ 1 \mp \frac{1}{(n!)^2} \left[a \right]^n \right\},
\]  

(80)

and read the Dirac BQNMs as follows

\[
\omega_n = -i\kappa \frac{n}{2} \left\{ 1 \mp \frac{1}{(n!)^2} \left[a \right]^n \right\}, \quad (n = 0, 1, 2, \ldots).
\]  

(81)

where \( n \) stands for the overtone quantum number (resonance parameter) \[33\].

For the highly excited states \( (n \to \infty) \), BQNM frequencies read

\[
\omega_n \approx -i\kappa \frac{n}{2}, \quad (n \to \infty).
\]  

(82)

The transition frequency from MM \[34\] can be obtained as

\[
\Delta \omega_I = \frac{\kappa}{2} = \frac{\pi T_H}{\hbar}.
\]  

(83)

Therefore, the adiabatic invariant quantity \[35–37\] becomes

\[
I_{adb} = \frac{\hbar}{\pi} S_{BH}.
\]  

(84)

With Bohr-Sommerfeld quantization rule \( (I_{adb} = n\hbar) \) \[38\], the entropy spectrum of Z0LBHs is determined as

\[
S_{BH}^{n} = \pi n,
\]  

(85)

and using \( S_{BH} = A_{BH}^{n}/4\hbar \) one may obtain the area spectrum

\[
A_{BH}^{n} = 4\pi n\hbar,
\]  

(86)

with the minimum spacing given by

\[
\Delta A_{min} = 4\pi \hbar.
\]  

(87)
As such, one concludes that the entropy/area spectra of the Z0LBHs are evenly spaced and are independent from the BH parameters whereas the spacing coefficient reads $\epsilon = 4\pi$, which is half the Bekenstein’s result for the Schwarzschild BH [34–36].

V. DISCUSSION

In the present study, based on the adiabatic invariant formulation in MM (3), the quantum entropy/area spectra of the Z0LBH are investigated. The massless Dirac equation for the test particles on a Z0LBH is solved by the separation of variables and it is decoupled with an eigenvalue $\lambda$ into the radial and the angular equations. Particularly, we have obtained the ZEs with their associated potentials and limited them to the NH region in order to show analytically the existence of the BQNMs in the presence of a Dirac field on Z0LBH by using an iteration method. For this purpose, we have considered a confining mirror that is placed in the NH region of the Z0LBH. We have therefore showed that the Dirac BQNMs are purely imaginary and negative which guarantees the stability of these BHs under massless fermionic field perturbations. After imposing the appropriate boundary conditions, we have computed the resonant frequencies of the Z0LBH which is confined in a finite-volume cavity (mirror). Later on, MM is applied to the highly damped Dirac BQNMs to derive the entropy/area spectra of the Z0LBH which is found to be equally spaced and independent of the BH parameters as stated in [13–17, 41, 42], although the spacing coefficient is half of the Bekenstein’s original result [34–36].

The QNMs of $z=0$ and $z=2$ LBHs were previously studied by [18] and [43], respectively. Our solution enables us to investigate a similar results with the Dirac fields.

Extremal BHs are believed to have connection with the ground states of quantum gravity which indicates the significance of the spin- $1/2$ particles on such backgrounds. Therefore, the analysis of a Dirac field interacting with a rotating or a charged rotating BH would be our next interest to obtain the possible Dirac BQNMs.

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