INTERTWIXING OPERATOR ALGEBRAS,
GENUS-ZERO MODULAR FUNCTORS AND
GENUS-ZERO CONFORMAL FIELD THEORIES

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Abstract. We describe the construction of the genus-zero parts
of conformal field theories in the sense of G. Segal from represen-
tations of vertex operator algebras satisfying certain conditions. The
construction is divided into four steps and each step gives a math-
ematical structure of independent interest. These mathematical
structures are intertwining operator algebras, genus-zero modular
functors, genus-zero holomorphic weakly conformal field theories,
and genus-zero conformal field theories.

1. Introduction

Conformal field theory (see for example, [BPZ], [Wi] and [MS]) is a
physical theory related to many branches of mathematics, e.g., infinite-
dimensional Lie algebras and Lie groups, sporadic finite simple groups,
modular forms and modular functions, elliptic genera and elliptic co-
homology, Calabi-Yau manifolds and mirror symmetry, and quantum
groups and 3-manifold invariants. Recently there are efforts by mathe-
ematicians to develop conformal field theory as a rigorous mathematical
to see from it some very subtle but important properties
theories. Because of this, it is even more difficult to
construct directly conformal field theories satisfying Segal’s geometric
definition.

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In practice, physicists and mathematicians have studied concrete models of conformal field theories for many years and the methods used are mostly algebraic. These studies can also be summarized to give algebraic formulations of conformal field theory. The theory of vertex operator algebras is such an algebraic formulation. Many concrete examples of conformal field theories are formulated and studied in algebraic formulations and very detailed calculations can be carried out in these formulations. But on the other hand, algebraic formulations have a disadvantage that the higher-genus parts of conformal field theories cannot be even formulated, and thus in algebraic formulations, it is not easy to see the geometric and topological applications of conformal field theory.

Intuitively, it is expected that algebraic and geometric formulations be equivalent. But this equivalence, especially, the construction of conformal field theories satisfying the geometric definition from conformal field theories satisfying the algebraic definition is a highly nontrivial mathematical problem. In the present paper, we describe the construction of the genus-zero parts of conformal field theories from representations of vertex operator algebras satisfying certain conditions.

This construction can actually be divided into four steps and each step gives us a mathematical structure of independent interest. The first step is to construct an “intertwining operator algebra” from the irreducible representations of a vertex operator algebra satisfying certain conditions. The second step is to construct a “genus-zero modular functor” (a partial operad of a certain type) from an intertwining operator algebra. The third step is to construct a “genus-zero holomorphic weakly conformal field theory” (an algebra over the partial operad of the genus-zero modular functor satisfying certain additional properties) from the intertwining operator algebra. The last step is to construct a “genus-zero conformal field theory” from a genus-zero holomorphic weakly conformal field theory when the genus-zero holomorphic weakly conformal field theory is unitary in a certain sense. The first and the second steps are described in Sections 3 and 4, respectively. The third and the fourth are both described in Section 5.

We introduce the notions of intertwining operator algebra and genus-zero modular functor in Section 3 and 4, respectively. The notions of genus-zero holomorphic weakly conformal field theory and genus-zero conformal field theory are introduced in Section 5. The notion of intertwining operator algebra is new. The other three notions are modifications of the corresponding notions introduced by Segal in the genus-zero case. We also give a brief description of the operadic formulation of the notion of vertex operator algebra in Section 2.
The construction described in this paper depends on the solutions of two problems. The first is the precise geometric description of the central charge of a vertex operator algebra. This problem is solved completely in \[H4\] (see also \[H1\] and \[H2\]). The second is the associativity of intertwining operators for a vertex operator algebra satisfying certain conditions. This is proved in \[H3\] using the tensor product theory for modules for a vertex operator algebra (see \[HL3\], \[HL5\], \[HL6\]). All the results described in the present paper are consequences of the results obtained in \[H1\], \[H4\], \[HL3\], \[HL4\], \[HL6\], \[H3\] and \[H5\].

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Notations:
\(\mathbb{C}\): the (structured set of) complex numbers.
\(\mathbb{C}^\times\): the nonzero complex numbers.
\(\mathbb{R}\): the real numbers.
\(\mathbb{Z}\): the integers.
\(\mathbb{Z}_+\): The positive integers.
\(\mathbb{N}\): the nonnegative integers.

2. OPERADIC FORMULATION OF THE NOTION OF VERTEX OPERATOR ALGEBRA

We give a brief description of the operadic formulation of the notion of vertex operator algebra in this section. We assume that the reader is familiar with operads and algebras over operads. See, for example, \[M\] for an introduction to these notions. The material in this section is taken from \[H1\], \[HL1\], \[HL2\] and \[H4\]. Since these references contain detailed and precise definitions and theorems, here we only present the main ideas. The main purposes of this section is to convince those readers unfamiliar with vertex operator algebras that vertex operator algebras are in fact very natural mathematical objects, and to introduce informally some basic concepts and notations in preparation for later sections. For details, see the references above. Note that though the operadic formulation of the notion of vertex operator algebra described below is very natural from the viewpoint of operads and geometry, historically, vertex operator algebras occurred in mathematics and physics for completely different reasons. See the introduction of \[FLM\] for a detailed historical discussion. For the algebraic formulation of the notion of vertex (operator) algebra, see \[B\], \[FLM\] and \[FHL\].
As a motivation, we begin with associative algebras. Let $C(j)$, $j \in \mathbb{N}$, be the moduli space of circles (i.e., compact connected smooth one-dimensional manifolds) with $j + 1$ ordered points (called punctures), the zeroth negatively oriented, the others positively oriented, and with smooth local coordinates vanishing at these punctures. Then it is easy to see that $C(j)$ can be identified naturally with the set of permutations $(\sigma(1), \ldots, \sigma(j))$ of $(1, \ldots, j)$. Since this set can in turn be identified in the obvious way with the symmetric group $S_j$, the moduli space $C(j)$ can also be naturally identified with $S_j$, with the group $S_j$ acting on $C(j)$ according to the left multiplication action on $S_j$. That is, $S_j$ permutes the orderings of the positively oriented punctures. Given any two circles with punctures and local coordinates vanishing at the punctures, we can define the notion of sewing them together at any positively oriented puncture on the first circle and the negatively oriented puncture on the second circle by cutting out an open interval of length $2r$ centered at the positively oriented puncture on the first circle (using the local coordinate) and cutting out an open interval of length $2/r$ around the negatively oriented puncture on the second circle in the same way, and then by identifying the boundaries of the remaining parts using the two local coordinates and the map $t \mapsto 1/t$; we assume that the corresponding closed intervals contain no other punctures. The ordering of the punctures of the sewn circle is given by “inserting” the ordering for the second circle into that for the first. Note that in general not every two circles with punctures and local coordinates can be sewn together at a given positively oriented puncture on the first circle. But it is clear that such a pair of circles with punctures and local coordinates is equivalent to a pair which can be sewn together. Also, the equivalence class of the sewn circle with punctures and local coordinates depends only on the two equivalence classes. Thus we obtain a sewing operation on the moduli space of circles with punctures and local coordinates. Given an element of $C(k)$ and an element of $C(j_s)$ for each $j_s$, $s = 1, \ldots, k$, we define an element of $C(j_1 + \cdots + j_k)$ by sewing the element of $C(k)$ at its $s$-th positively oriented puncture with the element of $C(j_s)$ at its negatively oriented puncture, for $s = 1, \ldots, k$, and by “inserting” the orderings for the elements of the $C(j_s)$ into the ordering for the element of $C(k)$. The identity is the unique element of $C(1)$. It is straightforward to verify that $C = \{C(j)\}_{j \in \mathbb{N}}$ is indeed an operad.

Since in one dimension smooth structures and conformal structures are the same, the moduli space $C(j)$ can also be thought of as the moduli space of circles with conformal structures, with $j + 1$ ordered punctures with the zeroth negatively oriented and the others positively
oriented, and with local conformal coordinates vanishing at these punctures. In fact, we have just defined three operads, namely, $C$, the corresponding conformal moduli space and $\{S_j\}_{j \in \mathbb{N}}$, and these three operads are isomorphic.

It is well known that the category of algebras over the operad $\{S_j\}_{j \in \mathbb{N}}$ is isomorphic to the category of associative algebras. Thus the category of algebras over $C$ (or the category of algebras over the operad of the corresponding conformal moduli space) is isomorphic to the category of associative algebras.

We have just seen that associative algebras are algebras over an operad obtained from one-dimensional objects. It is very natural to consider algebras over operads obtained from two-dimensional objects. The simplest two-dimensional objects are topological spheres, i.e., genus-zero compact connected smooth two-dimensional manifolds. If we consider the operad of the moduli spaces of genus-zero compact connected smooth two-dimensional manifolds with punctures and local coordinates vanishing at these punctures, then the category of algebras over this operad is isomorphic to the category of commutative associative algebras. We do not obtain new algebras. On the other hand, in dimension two, conformal structures are much richer than smooth structures. Since conformal structures in two real dimensions are equivalent to complex structure in one complex dimension, it is natural to consider operads constructed from genus-zero compact connected one-dimensional complex manifolds with punctures and local coordinates vanishing at these punctures.

Roughly speaking, vertex operator algebras are algebras over a certain operad constructed from genus-zero compact connected one-dimensional complex manifolds with punctures and local coordinates. But the operad in this case has an analytic structure and this analytic structure becomes algebraic in a certain sense when the operad is extended to a partial operad. Vertex operator algebras have properties which are not only the reflections of the operad structure above, but also the reflections of the analytic structure of the extended partial operad. These analyticity properties of vertex operator algebras are very subtle and important features of the theory of vertex operator algebras. They are essential to the constructions and many applications of conformal field theories.

We now begin to describe the notion of vertex operator algebra using the language of operads. A sphere with $1 + n$ tubes ($n \in \mathbb{N}$) is a genus-zero compact connected one-dimensional complex manifold $S$ with $n + 1$ distinct, ordered points $p_0, \ldots, p_n$ (called punctures) on $S$ with $p_0$ negatively oriented and the other punctures positively oriented,
and with local analytic coordinates \((U_0, \varphi_0), \ldots, (U_n, \varphi_n)\) vanishing at the punctures \(p_0, \ldots, p_n\), respectively, where for \(i = 0, \ldots, n\), \(U_i\) is a local coordinate neighborhood at \(p_i\) (i.e., an open set containing \(p_i\)) and \(\varphi_i : U_i \to \mathbb{C}\), satisfying \(\varphi_i(p_i) = 0\), is a local analytic coordinate map vanishing at \(p_i\). Let \(S_1\) and \(S_2\) be spheres with \(1 + m\) and \(1 + n\) tubes, respectively. Let \(p_0, \ldots, p_m\) be the punctures of \(S_1\), \(q_0, \ldots, q_n\) the punctures of \(S_2\), \((U_i, \varphi_i)\) the local coordinate at \(p_i\) for some fixed \(i\), \(0 < i \leq m\), and \((V_0, \psi_0)\) the local coordinate at \(q_0\). Assume that there exists a positive number \(r\) such that \(\varphi_i(U_i)\) contains the closed disc \(B^1_0\) centered at 0 with radius \(r\) and \(\psi_0(V_0)\) contains the closed disc \(\bar{B}^1_0\) centered at 0 with radius \(1/r\). Assume also that \(p_i\) and \(q_0\) are the only punctures in \(\varphi_i^{-1}(B^1_0)\) and \(\psi_0^{-1}(\bar{B}^1_0)\), respectively. In this case we say that the \(i\)-th tube of \(S_1\) can be sewn with the zeroth tube of \(S_2\). From \(S_1\) and \(S_2\), we obtain a sphere with \(1 + (m + n - 1)\) tubes by cutting \(\varphi_i^{-1}(B^1_0)\) and \(\psi_0^{-1}(\bar{B}^1_0)\) from \(S_1\) and \(S_2\), respectively, and then identifying the boundaries of the resulting surfaces using the map \(\varphi_i^{-1} \circ J \circ \psi_0\) where \(J\) is the map from \(\mathbb{C}^\times\) to itself given by \(J(z) = 1/z\).

The negatively oriented puncture of this sphere with tubes is \(p_0\) and the positively oriented punctures (with ordering) of this sphere with tubes are \(p_0, \ldots, p_{i-1}, q_1, \ldots, q_n, p_{i+1}, \ldots, p_m\). The local coordinates vanishing at these punctures are given in the obvious way. Thus we have a partial operation—the sewing operation—in the collection of spheres with tubes. We define the notion of conformal equivalence between two spheres with tubes in the obvious way except that two spheres with tubes are also said to be conformally equivalent if the only differences between them are local coordinate neighborhoods at the punctures. For any \(n \in \mathbb{N}\), the space of equivalence classes of spheres with \(1 + n\) tubes is called the moduli space of spheres with \(1 + n\) tubes. For \(n \in \mathbb{Z}_+\), the moduli space of spheres with \(1 + n\) tubes can be identified with \(K(n) = M^{n-1} \times H \times (\mathbb{C}^\times \times H)^n\) where \(M^{n-1}\) is the set of elements of \(\mathbb{C}^{n-2}\) with nonzero and distinct components and \(H\) is the set of all sequences \(A\) of complex numbers such that \((\exp(\sum_{j>0} A_j x^{j+1} dx/j))x\) is a convergent power series in some neighborhood of 0. We think of each element of \(K(n)\), \(n \in \mathbb{Z}_+\), as the sphere \(\mathbb{C} \cup \{\infty\}\) equipped with negatively oriented puncture \(\infty\) and positively oriented ordered punctures \(z_1, \ldots, z_{n-2}, 0\) with an element of \(H\) specifying the local coordinate at \(\infty\) and with \(n\) elements of \(\mathbb{C}^\times \times H\) specifying the local coordinates at the other punctures. Analogously, the moduli space of spheres with \(1 + 0\) tube can be identified with \(K(0) = \{B \in H \mid B_1 = 0\}\). From now on we shall refer to \(K(n)\) as the moduli space of spheres with \(1 + n\) tubes, \(n \in \mathbb{N}\). The sewing operation for spheres with tubes induces a partial
operation on the $\bigcup_{n \in \mathbb{N}} K(n)$. It is still called the sewing operation and is denoted $i_{\infty}$.

Let $K = \{K(n)\}_{n \in \mathbb{N}}$. For any $k \in \mathbb{Z}_+$, $j_1, \ldots, j_k \in \mathbb{N}$, and any elements of $K(k)$, $K(j_1), \ldots, K(j_k)$, by successively sewing the 0-th tube of elements of $K(j_i)$, $i = 1, \ldots, k$, with the $i$-th tube of the element of $K(k)$, we obtain an element of $K(j_1 + \cdots + j_k)$ if the conditions to perform the sewing operation are satisfied. Thus we obtain a partial map $\gamma_K$ from $K(k) \times K(j_1) \times \cdots K(j_k)$ to $K(j_1 + \cdots + j_k)$. Let $I \in K(1)$ be equivalence class of the standard sphere $\mathbb{C} \cup \{\infty\}$ with $\infty$ the negatively oriented puncture, 0 the only positively oriented puncture, and with standard local coordinates vanishing at $\infty$ and 0. For any $j \in \mathbb{N}$, $\sigma \in S_j$ and $Q \in K(j)$, $\sigma(Q)$ is defined to be the conformal equivalence class of spheres with $1 + j$ tubes obtained from members of the class $Q$ by permuting the orderings of their positively oriented punctures using $\sigma$. Thus $S_j$ acts on $K(j)$. It is easy to see that $K$ together with $\gamma_K$, $I$ and the actions of $S_j$ on $K(j)$, $j \in \mathbb{N}$ satisfies all the axioms for an operad except that the substitution (or composition) map $\gamma_K$ are only partially defined. Thus $K$ is a partial operad.

For any two element of the moduli space of spheres with tubes, in general the $i$-th puncture of the first element might not be able to be sewn with the 0-th puncture of the second element. But from the definition of sewing operation, we see that after rescaling the $i$-th puncture of the first element or the 0-th puncture of the second element, the $i$-th puncture of the first element can always be sewn with the 0-th puncture of the second element. So we see that though $K$ is partial, it is rescalable, that is, after rescaling, the substitution map is always defined. Since all rescalings of a local coordinate form a group isomorphic to $\mathbb{C}^\times$, $K$ is an example of $\mathbb{C}^\times$-rescalable partial operad and $\mathbb{C}^\times$ is the rescaling group of $K$.

For any $n \in \mathbb{N}$, $K(n)$ is an infinite-dimensional complex manifold. To be more precise, $K(n)$ is a complex (LB)-manifold, that is, a manifold modeled on an (LB)-space over $\mathbb{C}$ (the strict inductive limit of an increasing sequence of Banach spaces over $\mathbb{C}$) such that the transition maps are complex analytic (see, for example, [K] for the definition of (LB)-space and Appendix C of [H4] for the precise definition of complex (LB)-manifold). It is proved in [H1] and [H4] that the sewing operation and consequently the substitution map are analytic and even algebraic in a certain sense. So $K$ is an analytic $\mathbb{C}^\times$-rescalable partial operad.

Note that in the operadic formulation of the notion of associative algebra, the nontrivial element $\sigma_{12}$ of $S_2$ generate the operad $\{S_j\}_{j \in \mathbb{N}}$ for associative algebras. In fact, the associativity of the product in an associative algebra is the reflection of a property of $\sigma_{12}$. This element
$\sigma_{12}$ is called an *associative element* of the operad $\{S_j\}_{j \in \mathbb{N}}$ and $\{S_j\}_{j \in \mathbb{N}}$ is an example of *associative operads*.

For any $z \in \mathbb{C}^\times$, let $P(z) \in K(2)$ be the conformal equivalence class of the sphere $\mathbb{C} \cup \{\infty\}$ with $\infty$ the negatively oriented puncture, $z$ and 0 the first and second positively oriented punctures, respectively, and with the standard local coordinates vanishing at these punctures. Then $P(z)$ together with $K(0)$ and $K(1)$ also generates the partial operad $K$ and $P(z)$ also has a property similar to that of $\sigma_{12}$ (see [H4]). We call $P(z)$ an *associative element* of $K$ and $K$ is an example of *associative analytic* $\mathbb{C}^\times$-rescalable partial operads.

In general, vertex operator algebras have nonzero central charges. To incorporate central charges, we need determinant line bundles and their complex powers. For any $n \in \mathbb{N}$, there is a holomorphic line bundle $\tilde{K}(n)$ over $K(n)$ called the *determinant line bundle*. The family $\tilde{K} = \{\tilde{K}(n)\}_{n \in \mathbb{N}}$ is an associative analytic $\mathbb{C}^\times$-rescalable partial operad such that the projections from $\tilde{K}(n)$ to $K(n)$ for all $n \in \mathbb{N}$ give a morphism of associative analytic $\mathbb{C}^\times$-rescalable partial operads and such that when restricted to the fibers, the substitution maps are linear maps. For any $c \in \mathbb{C}$ and $n \in \mathbb{N}$, the complex power $\tilde{K}^c = \{\tilde{K}^c(n)\}_{n \in \mathbb{N}}$ of $\tilde{K}$ is a well-defined associative analytic $\mathbb{C}^\times$-rescalable partial operad such that for any $n \in \mathbb{N}$, $\tilde{K}^c(n)$ is a holomorphic line bundle over $K(n)$ equal to $\tilde{K}(n)$, such that the projections from $\tilde{K}^c(n)$ to $K(n)$ for all $n \in \mathbb{N}$ give a morphism of associative analytic $\mathbb{C}^\times$-rescalable partial operads and such that when restricted to the fibers, the substitution maps are linear maps. For detailed descriptions of determinant line bundles and their complex powers, see [H4].

To define algebras over a $\mathbb{C}^\times$-rescalable partial operad, we need to define a $\mathbb{C}^\times$-rescalable partial operad constructed from a vector space. Let $V = \bigoplus_{n \in \mathbb{R}} V(n)$ be a $\mathbb{R}$-graded vector space such that $\dim V(n) < \infty$ for all $n \in \mathbb{R}$ and $V' = \bigoplus_{n \in \mathbb{R}} V(n)$. For any $n \in \mathbb{N}$, let $\mathcal{H}_V(\langle \rangle)$ be the space of linear maps from $V^{\otimes n}$ to $V = \bigoplus_{n \in \mathbb{R}} V(n) = V^*$ where $'$ and $*$ denote the functors of taking restricted duals and duals of $\mathbb{R}$-graded vector spaces. Since images of elements of $\mathcal{H}_V(\langle \rangle)$ is in general in $V$ but not in $V$, the substitutions are not defined in general. For example, for $f \in \mathcal{H}_V(\langle \rangle)$, $g_1, g_2 \in \mathcal{H}_V(\infty)$ and $v_1, v_2 \in V$, $f(g_1(v_1), g_2(v_2))$ is not defined in general. For any $n \in \mathbb{C}$, let $P_n$ be the projection from $V$ to $V(n)$. Then for any $m, n \in \mathbb{C}$, $f(P_m(g_1(v_1)), P_n(g_2(v_2)))$ is an element of $\overline{V'} = V'^*$. If for any $v_1, v_2 \in V$ and $v' \in V'$, the series $\langle v', \sum_{m, n \in \mathbb{C}} f(P_m(g_1(v_1)), P_n(g_2(v_2))) \rangle$ is absolutely convergent, then we obtain an element of $\mathcal{H}_V(\langle \rangle)$. In this way, we obtain a partial map from $\mathcal{H}_V(\langle \rangle) \times \mathcal{H}_V(\infty) \times \mathcal{H}_V(\infty)$ to $\mathcal{H}_V(\langle \rangle)$. In general we can define partial
substitution maps for the family \( \{ \mathcal{H}_V(\lambda) \}_{\lambda \in \mathbb{N}} \) in this way. There is also an identity \( I_V \in \mathcal{H}_V(\infty) \) which is the identity map from \( V \) to \( V \subset \nabla \). The symmetric group \( S_n \) obviously acts on \( \mathcal{H}_V(\lambda) \). In general, however, the substitution maps do not satisfy the associativity even when both sides exist. We call a structure like \( \mathcal{H}_V \) a partial pseudo-operad. In particular, partial operads are partial pseudo-operads. Morphisms between partial pseudo-operads are defined in the obvious way.

For any \( c \in \mathbb{C} \), a pseudo-algebra over \( \tilde{K}^c \) is a \( \mathbb{Z} \)-graded vector space \( V \) and a morphism of partial pseudo-operads from \( \tilde{K}^c \) to \( \mathcal{H}_V \). An algebra over \( \tilde{K}^c \) is a pseudo-algebra \( V \) over \( \tilde{K}^c \) such that the image of the morphism from \( \tilde{K}^c \) to \( \mathcal{H}_V \) is a partial operad.

A vertex associative algebra of central charge \( c \) is a pseudo-algebra \( V \) over \( \tilde{K}^{c/2} \) such that \( V(n) = 0 \) for \( n \) sufficiently small and the morphism from \( \tilde{K}^{c/2} \) to \( \mathcal{H}_V \) is meromorphic in a certain sense (see [HL1], [HL2] and [H4] for the precise definition). It turns out that the condition that \( V(n) = 0 \) for \( n \) sufficiently small and the meromorphicity of the pseudo-algebra imply that \( V \) is actually an algebra over \( \tilde{K}^{c/2} \).

The following theorem announced in [HL1]–[HL2] and proved in [H4] is the main result of the operadic formulation of the notion of vertex operator algebra:

**Theorem 2.1.** The category of vertex operator algebra of central charge \( c \) is isomorphic to the category of vertex associative algebra of central charge \( c \).

3. **INTERTWINING OPERATOR ALGEBRAS**

In this section, the properties of the algebra of intertwining operators for a vertex operator algebra satisfying certain conditions are summarized to formulate the notion of intertwining operator algebra. The axioms in the definition can be relaxed or modified to give many variants of this notion. Since we are interested in constructing genus-zero modular functors and weakly holomorphic conformal field theories from intertwining operator algebras, we shall only discuss the version introduced below in this paper.

Let \( A \) be an \( n \)-dimensional commutative associative algebra over \( \mathbb{C} \). Then for any basis \( \mathcal{A} \) of \( A \), there are structure constants \( \mathcal{N}^{a_3}_{a_1 a_2} \in \mathbb{C} \), \( a_1, a_2, a_3 \in \mathcal{A} \), such that

\[
a_1 a_2 = \sum_{a_3 \in \mathcal{A}} \mathcal{N}^{a_3}_{a_1 a_2} a_3
\]

for any \( a_1, a_2 \in \mathcal{A} \). Assume that \( A \) has a basis \( \mathcal{A} \subset \mathcal{A} \) containing the identity \( e \in A \) such that all the structure constants \( \mathcal{N}^{a_3}_{a_1 a_2} \), \( a_1, a_2, a_3 \in \mathcal{A} \),
\(A\), are in \(\mathbb{N}\). Note that in this case, for any \(a_1, a_2 \in A\),

\[
N_{\{a_1, a_2\} = \delta_{a_1 = a_2} = \begin{cases} 1 & a_1 = a_2, \\ 0 & a_1 \neq a_2. \end{cases}
\]

The commutativity and the associativity of \(A\) give the following identities:

\[
\sum_{a \in \mathcal{A}} N_{a_1 a_2} = \sum_{a \in \mathcal{A}} N_{a_1 a_2},
\]

for \(a_1, a_2, a_3, a_4 \in \mathcal{A}\).

For a vector space \(W = \bigoplus_{a \in \mathcal{A}} \bigoplus_{n \in \mathbb{R}} W_{(n)}^a\) doubly graded by \(\mathbb{R}\) and \(\mathcal{A}\), let

\[
W_{(n)} = \prod_{a \in \mathcal{A}} W_{(n)}^a,
\]

\[
W^a = \prod_{n \in \mathbb{R}} W_{(n)}^a.
\]

Then

\[
W = \prod_{n \in \mathbb{R}} W_{(n)} = \prod_{a \in \mathcal{A}} W^a.
\]

The graded dual of \(W\) is the vector space \(W' = \bigoplus_{a \in \mathcal{A}} \bigoplus_{n \in \mathbb{R}} (W_{(n)}^a)^*\) doubly graded by \(\mathbb{R}\) and \(\mathcal{A}\), where for any vector space \(V\), \(V^*\) denotes the dual space of \(V\). We shall denote the canonical pairing between \(W'\) and \(W\) by \(\langle \cdot, \cdot \rangle_W\).

For a vector space \(W\) and a formal variable \(x\), we shall in this paper denote the space of all formal sums of the form \(\sum_{n \in \mathbb{R}} w_n x^n\) by \(W\{x\}\). Note that we only allow real powers of \(x\). For any \(z \in \mathbb{C}\), we shall always choose \(\log z\) so that

\[
\log z = \log |z| + i \arg z \quad \text{with} \quad 0 \leq \arg z < 2\pi.
\]

**Definition 3.1.** An intertwining operator algebra of central charge \(c \in \mathbb{C}\) consists of the following data:

1. A finite-dimensional commutative associative algebra \(A\) and a basis \(\mathcal{A}\) of \(A\) containing the identity \(e \in A\) such that all the structure constants \(N_{\{a_1, a_2, a_3\} = \{\delta_{a_1 = a_2} = \begin{cases} 1 & a_1 = a_2, \\ 0 & a_1 \neq a_2. \end{cases}\}
\]

2. A vector space

\[
W = \prod_{a \in \mathcal{A}, n \in \mathbb{R}} W_{(n)}^a, \quad \text{for } w \in W_{(n)}^a, \quad n = \text{wt } w, \quad a = \text{clr } w
\]

doubly graded by \(\mathbb{R}\) and \(\mathcal{A}\) (graded by weight and by color, respectively).
3. For each triple \((a_1, a_2, a_3) \in \mathcal{A} \times \mathcal{A} \times \mathcal{A}\), an \(N_{\infty}^{\mathcal{A}}\)-dimensional subspace \(\mathcal{V}_{\infty}^{\mathcal{A}}\) of the vector space of all linear maps \(W_{a_1} \otimes W_{a_2} \rightarrow W_{a_3}\{x\}\), or equivalently, an \(N_{\infty}^{\mathcal{A}}\)-dimensional vector space \(\mathcal{V}_{\infty}^{\mathcal{A}}\) whose elements are linear maps

\[ \mathcal{V} : \mathcal{W}_{\infty} \rightarrow \text{Hom}(W_{a_2}, W_{a_3})\{x\} \]

\[ w \mapsto \mathcal{V}(\Xi, \xi) = \sum_{\xi \in \mathbb{R}} \mathcal{V}(\Xi, \xi) \xi^{-\infty} (\text{where } \mathcal{V}(\Xi, \xi) \in \text{End } \mathcal{W}). \]

4. Two distinguished vectors \(1 \in \mathcal{W}^{\text{even}} \) (the vacuum) and \(\omega \in \mathcal{W}^{\text{even}} \) (the Virasoro element).

These data satisfy the following conditions for \(a_1, a_2, a_3, a_4, a_5, a_6 \in \mathcal{A}\), \(w_i \in W_{a_i}, i = 1, 2, 3\), and \(w' \in W_{a_4}\):

1. The grading-restriction conditions:

\[ \dim W_{(n)}^{a} < \infty \text{ for } n \in \mathbb{Z}, a \in \mathcal{A}, \]

\[ W_{(n)}^{a} = 0 \text{ for } n \text{ sufficiently small and for all } a \in \mathcal{A} , \]

2. The single-valuedness condition: for any \(\mathcal{V} \in \mathcal{V}_{\infty}^{\mathcal{A}}\),

\[ \mathcal{V}(\Xi, \xi) \in \text{Hom}(\mathcal{W}_{\infty}, \mathcal{W}_{\mathcal{B}_{\infty}})[[\xi, \xi^{-\infty}]]. \]

3. The lower-truncation property for vertex operators: for any \(\mathcal{V} \in \mathcal{V}_{\infty}^{\mathcal{A}}\), \(\mathcal{V}(\Xi, \xi) = 1\) for \(n \) sufficiently large.

4. The identity property: for any \(\mathcal{V} \in \mathcal{V}_{\infty}^{\mathcal{A}}\), there is \(\lambda_{\mathcal{V}} \in \mathbb{C}\) such that \(\mathcal{V}(1, 1) = \lambda_{\mathcal{V}} I_{W_{a_1}}\) where \(I_{W_{a_1}}\) on the right is the identity operator on \(W_{a_1}\).

5. The creation property: for any \(\mathcal{V} \in \mathcal{V}_{\infty}^{\mathcal{A}}\), there is \(\mu_{\mathcal{V}} \in \mathbb{C}\) such that \(\mathcal{V}(\Xi, \xi) 1 \in \mathcal{W}[[x]]\) and \(\lim_{x \rightarrow 0} \mathcal{V}(\Xi, \xi) 1 = \mu_{\mathcal{V}} w_1\) (that is, \(\mathcal{V}(\Xi, \xi) 1\) involves only nonnegative integral powers of \(x\) and the constant term is \(\mu_{\mathcal{V}} w_1\)).

6. The convergence properties: for any \(m \in \mathbb{Z}_+, a_i, b_i, c_i \in \mathcal{A}\), \(w_i \in W_{a_i}\), \(\mathcal{V} \in \mathcal{V}_{\infty}^{\mathcal{A}}\), \(i = 1, \ldots, m\), \(w' \in (W^{c_1})'\) and \(w \in W^{b_m}\), the series

\[ \langle w', \mathcal{V}_{\infty}(\Xi, \xi) \cdots \mathcal{V}_{\infty}(\Xi, \xi) \rangle_{\mathcal{W}^{\infty}} \bigg|_{\xi_i = 1, j = 0, \ldots, \xi_i \in \mathbb{R}} \]

is absolutely convergent when \(|z_1| > \cdots > |z_m| > 0\), and for any \(\mathcal{V}_{\infty} \in \mathcal{V}_{\infty}^{\mathcal{A}}\) and \(\mathcal{V}_{\xi} \in \mathcal{V}_{\infty}^{\mathcal{A}}\), the series

\[ \langle w', \mathcal{V}_{\infty}(\Xi, \xi) \cdots \mathcal{V}_{\infty}(\Xi, \xi) \rangle_{\mathcal{W}^{\infty}} \bigg|_{\xi_i = 1, j = 0, \ldots, \xi_i \in \mathbb{R}} \]

is absolutely convergent when \(|z_2| > |z_1 - z_2| > 0\).
7. The associativity: for any $\mathcal{Y}_\infty \in \mathcal{V}_{i\infty,i\infty}^{\infty}$ and $\mathcal{V}_{i\infty,i\infty}^{\infty}$, there exist $\mathcal{Y}_\infty \in \mathcal{V}_{i\infty,i\infty}^{\infty}$ and $\mathcal{Y}_\infty \in \mathcal{V}_{i\infty,i\infty}^{\infty}$ for all $a \in A$ such that the (multi-valued) analytic function

$$\langle w', \mathcal{Y}_\infty(\exists_\infty, \infty_\infty)\mathcal{Y}_\infty(\exists_\infty, \infty_\infty)\rangle_w \bigg|_{\exists_\infty=\infty_\infty, \infty_\infty=\infty_\infty}$$

on $\{(z_1, z_2) \in \mathbb{C} \times \mathbb{C} \mid |z_1| > |z_2| > 0\}$ and the (multi-valued) analytic function

$$\sum_{a \in A} \langle w', \mathcal{Y}_\infty(\exists_\infty, \infty_\infty)\mathcal{Y}_\infty(\exists_\infty, \infty_\infty)\rangle_w \bigg|_{\exists_\infty=\infty_\infty, \infty_\infty=\infty_\infty}$$

on $\{(z_1, z_2) \in \mathbb{C} \times \mathbb{C} \mid |z_2| > |z_1 - z_2| > 0\}$ are equal on $\{(z_1, z_2) \in \mathbb{C} \times \mathbb{C} \mid |z_1| > |z_2| > |z_1 - z_2| > 0\}$.

8. The Virasoro algebra relations: Let $Y$ be the element of $\mathcal{V}_{i\infty,i\infty}^{\infty}$ such that $Y(1, x) = I_{W_{a1}}$ and let $Y(w, x) = \sum_{n \in \mathbb{Z}} L(n)x^{-n-2}$. Then

$$[L(m), L(n)] = (m - n)L(m + n) + \frac{c}{12}(m^3 - m)\delta_{m+n,0}$$

for $m, n \in \mathbb{Z}$.

9. The $L(0)$-grading property: $L(0)w = nw = (wt w)w$ for $n \in \mathbb{R}$ and $w \in W(n)$.

10. The $L(-1)$-derivative property: For any $\mathcal{Y} \in \mathcal{V}_{i\infty,i\infty}^{\infty}$,

$$\frac{d}{dx}\mathcal{Y}(\exists_\infty, \infty_\infty) = \mathcal{Y}(\mathcal{L}(\infty_\infty)\exists_\infty, \infty_\infty).$$

11. The skew-symmetry: There is a linear map $\Omega$ from $\mathcal{V}_{i\infty,i\infty}^{\infty}$ to $\mathcal{V}_{i\infty,i\infty}^{\infty}$ such that for any $\mathcal{Y} \in \mathcal{V}_{i\infty,i\infty}^{\infty}$,

$$\mathcal{Y}(\exists_\infty, \infty_\infty) = \frac{\mathcal{L}(\infty_\infty)}{\mathcal{L}(\infty_\infty)}(\Omega(\mathcal{Y}))(\exists_\infty, \infty_\infty)\bigg|_{\exists_\infty=\infty_\infty, \infty_\infty=\infty_\infty}. $$

We shall denote the intertwining operator algebra defined above by $(W, A, \{\mathcal{V}_{i\infty,i\infty}^{\infty}\}, 1, \omega)$ or simply by $W$. The commutative associative algebra $A$ is called the Verlinde algebra or the fusion algebra of $W$. The linear maps in $\mathcal{V}_{i\infty,i\infty}^{\infty}$ are called intertwining operators of type $(a_3^{a_1, a_2})$.

Remark 3.2. In the definition above, the second convergence property can be derived from the first one using the skew-symmetry. We include the second convergence property in the definition in order to state the associativity without proving this fact. The associativity or the skew-symmetry can be replaced by the following commutativity: for any
\[ Y_\infty \in \mathcal{V}_{4\infty + t\infty}^{\Lambda_+} \quad \text{and} \quad Y_\infty \in \mathcal{V}_{4\infty + t\infty}^{\Lambda_-}, \quad \text{there exist} \quad Y^+_{\mathcal{V}} \in \mathcal{V}_{4\infty + t\infty}^{\Lambda_+} \quad \text{and} \quad Y^+_{\mathcal{V}} \in \mathcal{V}_{4\infty + t\infty}^{\Lambda_-} \quad \text{for all} \quad a \in A \quad \text{such that the (multi-valued) analytic function} \]

\[
\langle w', Y_\infty (\Xi_{\infty}, \Xi_e) Y_{\mathcal{V}} (\Xi_{\infty}, \Xi_e) \rangle_W \bigg|_{\Xi_{\infty} = t\infty, \Xi_e = t\infty}
\]
on \{(z_1, z_2) \in \mathbb{C} \times \mathbb{C} \mid |z_1| > |z_2| > 0\} and the (multi-valued) analytic function

\[
\langle w', Y_{\mathcal{V}} (\Xi_{\infty}, \Xi_e) Y_{\mathcal{V}} (\Xi_{\infty}, \Xi_e) \rangle_W \bigg|_{\Xi_{\infty} = t\infty, \Xi_e = t\infty}
\]
on \{(z_1, z_2) \in \mathbb{C} \times \mathbb{C} \mid |z_2| > |z_1| > 0\} are analytic extensions of each other.

**Remark 3.3.** The associativity gives a linear map from \( \mathcal{V}_{4\infty + t\infty}^{\Lambda_+} \otimes \mathcal{V}_{4\infty + t\infty}^{\Lambda_-} \) to \( \oplus_{a \in A} \mathcal{V}_{4\infty + t\infty}^{\Lambda_+} \otimes \mathcal{V}_{4\infty + t\infty}^{\Lambda_-} \) for any \( a_1, a_2, a_3, a_4, a_5 \in A \). Thus we obtain a linear map from \( \oplus_{a \in A} \mathcal{V}_{4\infty + t\infty}^{\Lambda_+} \otimes \mathcal{V}_{4\infty + t\infty}^{\Lambda_-} \) to \( \oplus_{a \in A} \mathcal{V}_{4\infty + t\infty}^{\Lambda_+} \otimes \mathcal{V}_{4\infty + t\infty}^{\Lambda_-} \) for any \( a_1, a_2, a_3, a_4 \in A \). It is easy to show that these linear maps are in fact linear isomorphisms. These linear isomorphisms are called the *fusing isomorphisms* and the associated matrices under any basis are called *fusing matrices*. Similarly, the commutativity in Remark 3.2 gives a linear isomorphism from \( \oplus_{a \in A} \mathcal{V}_{4\infty + t\infty}^{\Lambda_+} \otimes \mathcal{V}_{4\infty + t\infty}^{\Lambda_-} \) to \( \oplus_{a \in A} \mathcal{V}_{4\infty + t\infty}^{\Lambda_+} \otimes \mathcal{V}_{4\infty + t\infty}^{\Lambda_-} \) for any \( a_1, a_2, a_3, a_4 \in A \). These linear isomorphisms are called the *braiding isomorphisms* or the associated matrices under any basis are called *braiding matrices*.

In the rest of this section, we assume that the reader is familiar with the notions of abelian intertwining algebra, module for a vertex operator algebra, and intertwining operator among three modules for a vertex operator algebra. We also assume that the reader knows the basic concepts and results in the representation theory of vertex operator algebras, for example, the rationality of vertex operator algebras and the conditions to use the tensor product theory for modules for a vertex operator algebra. See [DL], [FLM], [FHL], [HL3]–[HL6] and [H3].

It is easy to verify the following:

**Proposition 3.4.** An abelian intertwining algebra in the sense of Dong and Lepowsky [DL] satisfying in addition the grading-restriction conditions is an intertwining operator algebra whose Verlinde algebra is the group algebra of the abelian group associated with the abelian intertwining algebra. In particular, vertex operator algebras are intertwining operator algebras.
The results in [FHL], [HL3], [HL4], and [HL6] imply the following result:

**Theorem 3.5.** Let $V$ be a rational vertex operator algebra satisfying the conditions to use the tensor product theory for $V$-modules, $\mathcal{A} = \{-\infty, \ldots, -1\}$ the set of all equivalence classes of irreducible $V$-modules, and $W^{a_1}, \ldots, W^{a_m}$ representatives of $a_1, \ldots, a_m$, respectively. Then there is a natural commutative associative algebra structure on the vector space $\mathcal{A}$ spanned by $\mathcal{A}$ such that $W = \bigsqcup_{i=1}^{m} W^{a_i}$ has a natural structure of intertwining operator algebra whose Verlinde algebra is $A$.

We know that minimal Virasoro vertex operator algebras are rational (see [DMZ] and [Wa]). In [H5], it is proved that any vertex operator algebra containing a vertex operator subalgebra isomorphic to a tensor product algebra of minimal Virasoro vertex operator algebras satisfies the conditions to use the tensor product theory. The proof uses the representation theory of the Virasoro algebra and the Belavin-Polyakov-Zamolodchikov equations (see [BPZ]) for minimal models. We also know that the vertex operator algebras associated to Wess-Zumino-Novikov-Witten models (WZNW models) are rational (see [FZ]). It is easy to see that if the representation theory of the Virasoro algebra and the Belavin-Polyakov-Zamolodchikov equations for minimal models are replaced by the representation theory of affine Lie algebras and the Knizhnik-Zamolodchikov equations (see [KZ]) for WZNW models, respectively, the methods in [H5] works for the vertex operator algebras associated to WZNW models. Thus the vertex operator algebras associated to WZNW models also satisfy the conditions to use the tensor product theory. So we have:

**Corollary 3.6.** Let $V$ be a rational vertex operator algebra containing a vertex operator subalgebra isomorphic to a tensor product algebra of minimal Virasoro vertex operator algebras or a vertex operator algebras associated to a WZNW model, $\mathcal{A} = \{-\infty, \ldots, -1\}$ the set of all equivalence classes of irreducible $V$-modules, and $W^{a_1}, \ldots, W^{a_m}$ representatives of $a_1, \ldots, a_m$, respectively. Then there is a natural commutative associative algebra structure on the vector space $\mathcal{A}$ spanned by $\mathcal{A}$ such that $W = \bigsqcup_{i=1}^{m} W^{a_i}$ has a natural structure of intertwining algebra whose Verlinde algebra is $A$.

It is easy to see that in general the Verlinde algebras for intertwining operator algebras obtained from these vertex operator algebras are not group algebras. Thus these intertwining operator algebras are not abelian intertwining algebras. So we do have many examples of intertwining operator algebras which are not abelian intertwining algebras.
4. Genus-Zero Modular Functors

In the definition of intertwining operator algebra, one of the data is the collections of the vector spaces $V_{\pi,\log,\log}^{a_1,a_2,a_3}$, $a_1, a_2, a_3 \in A$. From these vector spaces, without using the properties of the operators in them, we can construct a geometric object. The properties of these geometric objects are in fact the axioms in the notions of genus-zero modular functors and rational genus-zero modular functor.

We first give the definition of genus-zero modular functor.

Definition 4.1. A genus-zero modular functor is an analytic $C^\infty$-rescalable partial operad $\mathcal{M}$ together with a morphism $\pi : \mathcal{M} \to \mathcal{K}$ of $C^\infty$-rescalable partial operads satisfying the following axioms:

1. For any $j \in \mathbb{N}$ the triple $(\mathcal{M}(\|), \mathcal{K}(\|), \pi)$ is a finite-rank holomorphic vector bundle over $K(j)$.
2. For any $Q \in \mathcal{M}(\|), Q_\infty \in \mathcal{M}(\|_\infty), \ldots, Q_k \in \mathcal{M}(\|_k)$, $j_1, \ldots, j_k \in \mathbb{N}$, the substitution $\gamma_M(Q; Q_\infty, \ldots, Q_k)$ in $\mathcal{M}$ exists if (and only if)
\[ \gamma_M(\pi(Q); \pi(Q_\infty), \ldots, \pi(Q_k)) \]
exists;
3. Let $Q \in K(k), Q_1 \in K(j_1), \ldots, Q_k \in K(j_k), j_1, \ldots, j_k \in \mathbb{N}$, such that $\gamma(Q; Q_1, \ldots, Q_k)$ exists. The map from the Cartesian product of the fibers over $Q, Q_1, \ldots, Q_k$ to the fiber over $\gamma_M(Q; Q_1, \ldots, Q_k)$ induced from the substitution map of $\mathcal{M}$ is multilinear and gives an isomorphism from the tensor product of the fibers over $Q, Q_1, \ldots, Q_k$ to the fiber over $\gamma_M(Q; Q_1, \ldots, Q_k)$.
4. The actions of the symmetric groups on $\mathcal{M}$ are isomorphisms of holomorphic vector bundles covering the actions of the symmetric groups on $K$.

Homomorphisms and isomorphisms from genus-zero modular functors to genus-zero modular functors are defined in the obvious way.

Let $\mathcal{M}$ be a genus-zero modular functor. For any $k \in \mathbb{Z}_+, j_1, \ldots, j_k \in \mathbb{N}$, let $\mathcal{M}_i(\|), \mathcal{M}_i(\|_\infty), \ldots, \mathcal{M}_i(\|_k)$ be vector subbundles of $\mathcal{M}(\|), \mathcal{M}(\|_\infty), \ldots, \mathcal{M}(\|_k)$, respectively, for $i = 1, 2$, and $P_i(k), P_i(j_1), \ldots, P_i(j_k)$ the projections from $\mathcal{M}(\|), \mathcal{M}(\|_\infty), \ldots, \mathcal{M}(\|_k)$ to $\mathcal{M}_i(\|), \mathcal{M}_i(\|_\infty), \ldots, \mathcal{M}_i(\|_k)$, respectively, for $i = 1, 2$. Then $\gamma_M \circ (P_1(k) \times P_1(j_1) \cdots \times P_1(j_k))$ and $\gamma_M \circ (P_2(k) \times P_2(j_1) \cdots \times P_2(j_k))$ are both homomorphisms of vector bundles from $\mathcal{M}(\|) \times \mathcal{M}(\|_\infty) \times \cdots \times \mathcal{M}(\|_k)$ to $\mathcal{M}(\|_\infty + \cdots + \|_k)$. So the sum of $\gamma_M \circ (P_1(k) \times P_1(j_1) \cdots \times P_1(j_k))$ and $\gamma_M \circ (P_2(k) \times P_2(j_1) \cdots \times P_2(j_k))$ is well-defined. Similarly, sums of more than two such homomorphisms is also well-defined.

We next give the definition of rational genus-zero modular functor:
Definition 4.2. A rational genus-zero modular functor is a genus-zero modular functor $M$ and a finite set $A$ satisfying the following axioms:

5. For any $n \in \mathbb{N}$ and $a_0, a_1, \ldots, a_n \in A$, there are finite-rank holomorphic vector bundles $M^{a_n \cdots a_1 \langle n \rangle}$ over $K(n)$ such that $M(\langle n \rangle) = \bigoplus_{a_n, \cdots, a_1} M^{a_n \cdots a_1 \langle n \rangle} \langle n \rangle$ where $\bigoplus$ denotes the direct sum operation for vector bundles.

6. For any $n \in \mathbb{N}$, $b_0, b_1, \ldots, b_n \in A$, let $P_{b_0}(\langle n \rangle)$ be the projection from $M^{\langle n \rangle}$ to $\bigoplus_{a_1, \ldots, a_n} M^{a_n \cdots a_1 \langle n \rangle}$ and $P_{b_1} \cdots b_n(\langle n \rangle)$ the projection from $M^{\langle n \rangle}$ to $\bigoplus_{a_0} M^{a_0 \langle n \rangle}$. Then for any $k \in \mathbb{Z}_+, j_1, \ldots, j_k \in \mathbb{N}$,

$$\gamma_M = \sum_{b_1, \ldots, b_k \in A} \gamma_M \circ (P_{b_1} \cdots b_k)(k) \times P_{b_1}(j_1) \times \cdots \times P_{b_k}(j_k).$$

We shall denote the rational genus-zero modular functor just defined by $(M, A)$.

The simplest examples of rational genus-zero modular functors $\tilde{K}^c$, $c \in \mathbb{C}$.

Using the methods developed in [H4], we have:

Theorem 4.3. Let $(W, \mathcal{A}, \{\mathcal{V}^{\tilde{t}_3}_{i_{\infty \cdots i_{\infty}}}, 1, \omega\})$ be an intertwining operator algebra of central charge $c$. Then there is a canonical rational genus-zero modular functor $(M_W, \mathcal{A})$ such that for any $a_1, a_2, a_3 \in \mathcal{A}$, $M^{\tilde{t}_3}(i) = \tilde{K}^1/\varepsilon(i)$ and the fiber of $M^{\tilde{t}_3}_{i_{\infty \cdots i_{\infty}}}(\varepsilon)$ is isomorphic to $\mathcal{V}^{\tilde{t}_3}_{i_{\infty \cdots i_{\infty}}}$.

In particular, by Theorem 4.3 and Corollary 3.6, we have:

Corollary 4.4. Let $V$ be a rational vertex operator algebra containing a vertex operator subalgebra isomorphic to a tensor product algebra of minimal Virasoro vertex operator algebras or a vertex operator algebras associated to a WZNW model, $\mathcal{A} = \{-\infty, \ldots, -\tilde{t}_3\}$ the set of all equivalence classes of irreducible $V$-modules, and $W^{a_1}, \ldots, W^{a_m}$ representatives of $a_1, \ldots, a_m$, respectively. Then there is a canonical rational genus-zero modular functor $(M_V, \mathcal{A})$ such that for any $a_1, a_2, a_3 \in \mathcal{A}$, $M^{\tilde{t}_3}_{i_{\infty \cdots i_{\infty}}}(\varepsilon)$ is isomorphic to $\mathcal{V}^{\tilde{t}_3}_{i_{\infty \cdots i_{\infty}}}$, where $c \in \mathbb{C}$ is the central charge of $V$ and $\mathcal{V}^{\tilde{t}_3}_{i_{\infty \cdots i_{\infty}}}$ is the space of all intertwining operators for $V$ of type $(W^{a_3}_{W^{a_1}W^{a_2}})$.

Genus-zero modular functors contain almost all of the topological information one can obtain from an intertwining operator algebra. To see this, we need a result of Segal [S]:

Proposition 4.5. Let $M$ be a genus-zero modular functor. Then there are canonical projectively flat connections on the vector bundles $M(\langle n \rangle)$,
For any \( n \in \mathbb{N} \), such that they are compatible with the substitution maps for the partial operad \( \mathcal{M} \).

For any \( n \in \mathbb{Z}_+ \), we can embed the configuration space
\[
F(n) = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid z_i \neq z_j \text{ for } i \neq j\}
\]
into \( K(n) \) in the obvious way. Thus given any genus-zero modular functor \( \mathcal{M} \), for any \( n \in \mathbb{Z}_+ \), \( \mathcal{M}(\emptyset) \) is pulled back to a vector bundle over \( F(n) \) and the canonical connection on \( \mathcal{M}(\emptyset) \) is pulled back to a connection on the pull-back vector bundle over \( F(n) \). Since the actions of symmetric groups on \( \mathcal{M} \) are isomorphisms of holomorphic vector bundles covering the actions of the symmetric actions on \( K \), the pull-back vector bundle and the pull-back connection over \( F(n) \) induce a vector bundle over \( F(n)/S_n \) and a connection over this vector bundle for any \( n \in \mathbb{Z}_+ \). It is not difficult to prove the following:

**Proposition 4.6.** Let \( \mathcal{M} \) be a genus-zero modular functor. For any \( n \in \mathbb{Z}_+ \), the connections on the pull-back vector bundle over \( F(n) \) and on the induced vector bundle over \( F(n)/S_n \) are flat.

We know that a flat connection on a vector bundle over a connected manifold gives a structure of a representation of the fundamental group of the manifold to the fiber of the vector bundle at any point on the manifold. We also know that the braid group \( B_n \) of \( n \) strings is by definition the fundamental group of \( F(n)/S_n \). So we obtain:

**Theorem 4.7.** Let \( \mathcal{M} \) be a genus-zero modular functor, \( n \in \mathbb{Z}_+ \) and \( V \) the fiber at any point in \( F(n) \subset K(n) \) of the vector bundle \( \mathcal{M}(\emptyset) \). Then \( V \) has a natural structure of a representation of the braid group \( B_n \) of \( n \) strings.

In particular, from a vertex operator algebra containing a subalgebra isomorphic to a tensor product algebra of minimal Virasoro vertex operator algebras or from a vertex operator algebra associated to WZNW models, we obtain representations of braid groups. In the case that the vertex operator algebras are associated to WZNW models based on the Lie group \( SU(2) \), the corresponding representations of the braid groups are the same as those obtained by Tsuchiya and Kanhe \( [1,2] \) and the corresponding knot invariants are the Jones polynomials for knots \( [3] \). (In fact, from a genus-zero modular functor, we obtain not only representations of the braid groups, but also representations of braid groups with twists.)

Let \( \mathcal{M} = \{\mathcal{M}(\emptyset)\}_{\mathbb{C}^n} \) be a genus-zero modular functor and \( \overline{\mathcal{M}}(n) \), \( n \in \mathbb{N} \), the complex conjugates of the holomorphic vector bundles \( \mathcal{M}(\emptyset) \). We define the **complex conjugate partial operad of** \( \mathcal{M} \) to be
the sequence $\overline{M} = \{\overline{M}(n)\}_{n \in \mathbb{N}}$ with the obvious partial operad structure induced from that of $\mathcal{M}$. Let
\[ \mathcal{M} \otimes \overline{\mathcal{M}} = \{\mathcal{M}(\underline{m}) \otimes \overline{\mathcal{M}}(\underline{m})\}_{\underline{m} \in \mathcal{M}} \]
where on the right-hand side, $\otimes$ is the tensor product operation for vector bundles over $K(n)$, $n \in \mathbb{N}$. Then the partial operad structures on $\mathcal{M}$ and $\overline{\mathcal{M}}$ induce a partial operad structure on $\mathcal{M} \otimes \overline{\mathcal{M}}$. In particular, for any $c \in \mathbb{C}$, since $\overline{K}^{c/2}$ is a genus-zero modular functor, we obtain partial operads $\overline{K}^{c/2}$ and $\overline{K}^{c/2} \otimes \overline{K}^{c/2}$. Using these partial operads, we have the following notion:

**Definition 4.8.** A genus-zero modular functor $\mathcal{M}$ is unitary if there is a complex number $c \in \mathbb{C}$ and a morphism of analytic partial operads from $\mathcal{M} \otimes \overline{\mathcal{M}}$ to $\overline{K}^{c/2} \otimes \overline{K}^{c/2}$ satisfying the following conditions:

1. For any $n \in \mathbb{N}$ and $Q \in K(n)$, the morphism from $\mathcal{M} \otimes \overline{\mathcal{M}}$ to $\overline{K}^{c/2} \otimes \overline{K}^{c/2}$ maps the fiber of $\mathcal{M}(\underline{m}) \otimes \overline{\mathcal{M}}(\underline{m})$ at $Q$ linearly to the fiber of $\overline{K}^{c/2}(n) \otimes \overline{K}^{c/2}(n)$ at $Q$.
2. The linear map above from the fiber of $\mathcal{M}(\underline{m}) \otimes \overline{\mathcal{M}}(\underline{m})$ at $Q$ to the fiber of $\overline{K}^{c/2}(n) \otimes \overline{K}^{c/2}(n) = \mathbb{C}$ at $Q$ when viewed as a linear map from the fiber of $\mathcal{M}(\underline{m}) \otimes \overline{\mathcal{M}}(\underline{m})$ at $Q$ to $\mathbb{C}$ gives a positive-definite Hermitian form on the fiber of $\mathcal{M}(\underline{m})$ at $Q$.

The complex number $c$ is called the central charge of the unitary genus-zero modular functor $\mathcal{M}$.

Let $(W, \mathcal{A}, \{V^\pm_{i_{-\infty}}\}, 1, \omega)$ be an intertwining algebra of central charge $c$. Assume that there are positive-definite Hermitian forms on $V^\pm_{i_{-\infty}}$ for all $a_1, a_2, a_3 \in \mathcal{A}$. These positive-definite Hermitian forms induce positive-definite Hermitian forms on $\bigoplus_{a \in \mathcal{A}} V^\pm_{i_{-\infty}} \otimes V^\pm_{i_{-\infty}}$ and $\bigoplus_{a \in \mathcal{A}} V^\pm_{i_{-\infty}} \otimes V^\pm_{i_{-\infty}}$ for any $a_1, a_2, a_3, a_4 \in \mathcal{A}$.

We have:

**Proposition 4.9.** Let $(W, \mathcal{A}, \{V^\pm_{i_{-\infty}}\}, 1, \omega)$ be an intertwining operator algebra of central charge $c$. If there are positive-definite Hermitian forms on $V^\pm_{i_{-\infty}}$ for all $a_1, a_2, a_3 \in \mathcal{A}$ such that the fusing isomorphism from $\bigoplus_{a \in \mathcal{A}} V^\pm_{i_{-\infty}} \otimes V^\pm_{i_{-\infty}}$ to $\bigoplus_{a \in \mathcal{A}} V^\pm_{i_{-\infty}} \otimes V^\pm_{i_{-\infty}}$ pulls the induced positive-definite Hermitian form on $\bigoplus_{a \in \mathcal{A}} V^\pm_{i_{-\infty}} \otimes V^\pm_{i_{-\infty}}$ back to that on $\bigoplus_{a \in \mathcal{A}} V^\pm_{i_{-\infty}} \otimes V^\pm_{i_{-\infty}}$ for any $a_1, a_2, a_3, a_4 \in \mathcal{A}$, then these positive-definite Hermitian forms give a unitary structure to the rational modular functor $(\mathcal{M}_W, \mathcal{A})$.

In particular, we have:
Proposition 4.10. The genus-zero modular functors obtained from minimal Virasoro vertex operator algebras and from vertex operator algebras associated to WZNW models are unitary.

5. Genus-zero conformal field theories

Modular functors only reflects the geometric objects constructed from the vector spaces $V^{a_1, a_2, a_3}$, $a_1, a_2, a_3 \in A$, without using the properties of the operators in these spaces. Also an intertwining operator algebra has the vacuum and the Virasoro element which are not reflected in the definition of (rational) genus-zero modular functor. These data and the properties they satisfied are reflected in the following definition of genus-zero weakly holomorphic conformal field theory:

Definition 5.1. Let $\mathcal{M}$ be a genus-zero modular functor. An genus-zero weakly holomorphic conformal field theory over $\mathcal{M}$ is a $R$-graded vector space $W$ and a morphism of partial pseudo-operads from $\mathcal{M}$ to the endomorphism partial pseudo-operad $\mathcal{H}_W$ (see Section 2) satisfying the following additional axioms:

1. The image of the morphism from $\mathcal{M}$ to $\mathcal{H}_W$ is a partial operad.
2. The morphism from $\mathcal{M}$ to the $C^\times$-rescalable endomorphism partial pseudo-operad $\mathcal{H}_W$ is linear on the fibers of the bundles $\mathcal{M}(\mathcal{A}), n \in \mathbb{N}$.
3. This morphism is holomorphic.

A genus-zero weakly holomorphic conformal field theory over a rational genus-zero modular functor is called a rational genus-zero holomorphic weakly conformal field theory.

Using the method developed in [H4], we have:

Theorem 5.2. Let $(W, A, \{V^{a_1, a_2, a_3}_{\leq \infty}, 1, \omega\})$ be an intertwining operator algebra and $(\mathcal{M}_W, A)$ the corresponding rational genus-zero modular functor. Then $W$ has a canonical structure of a rational holomorphic genus-zero weakly conformal field theory.

In particular, by Theorem 5.2 and Corollary 3.6, we have:

Corollary 5.3. Let $V$ be a rational vertex operator algebra containing a vertex operator subalgebra isomorphic to a tensor product algebra of minimal Virasoro vertex operator algebras or a vertex operator algebras associated to a WZNW model, $A = \{-\infty, \ldots, -q\}$ the set of all equivalence classes of irreducible $V$-modules, and $W^{a_1, \ldots, W^{a_m}}$ representatives of $a_1, \ldots, a_m$, respectively. Then there is a canonical rational genus-zero weakly holomorphic conformal field theory structure on $W = \bigsqcup_{i=1}^m W^{a_i}$. 
Let $H_\infty$ be the Banach space of all analytic functions on the closed unit disk $\{z \in \mathbb{C} \mid |z| \leq 1\}$ and $K_{H_\infty}(n)$ for any $n \in \mathbb{N}$ the subset of $K(n)$ consisting of conformal equivalence classes containing spheres with $n + 1$ tubes such that the inverses of the local coordinates at punctures can be extended to the closed unit disk and the images of the closed unit disks under these inverses are disjoint. Let $K_{H_\infty} = \{K_{H_\infty}(n)\}_{n \in \mathbb{N}}$. Then the sewing operation is always defined in $K_{H_\infty}$. Also the identity of $K$ is in $K_{H_\infty}$ and the symmetric groups act on $K_{H_\infty}$. So $K_{H_\infty}$ is a suboperad of $K$ (not just a partial suboperad of $K$).

Let $\tilde{K}_{c/2}^{H_\infty}$ be the complex conjugate operad of $\tilde{K}_{c/2}^{H_\infty}$ defined in the obvious way and $\tilde{K}_{c/2}^{H_\infty} \otimes \overline{\tilde{K}_{c/2}^{H_\infty}}$ the tensor product operad of $\tilde{K}_{c/2}^{H_\infty}$ and $\overline{\tilde{K}_{c/2}^{H_\infty}}$. In general, algebras over the operad $\tilde{K}_{c/2}^{H_\infty} \otimes \overline{\tilde{K}_{c/2}^{H_\infty}}$ might not have any topological structure. But since $\tilde{K}_{c/2}^{H_\infty} \otimes \overline{\tilde{K}_{c/2}^{H_\infty}}$ consists of infinite-dimensional Banach manifolds, we are interested in algebras over it with topological structures. Consider the abelian category of Hilbert spaces over $\mathbb{C}$. Then the tensor product operation for Hilbert spaces gives a tensor category structure to this abelian category. Thus for any Hilbert space $H$ over $\mathbb{C}$, we have the endomorphism operad of $H$. This endomorphism operad has a topological structure. A Hilbert algebra over $\tilde{K}_{c/2}^{H_\infty} \otimes \overline{\tilde{K}_{c/2}^{H_\infty}}$ is a Hilbert space $H$ over $\mathbb{C}$ and a continuous morphism of topological operads from $\tilde{K}_{c/2}^{H_\infty} \otimes \overline{\tilde{K}_{c/2}^{H_\infty}}$ to the endomorphism operad of $H$.

**Definition 5.4.** A genus-zero conformal field theory of central charge $c$ is a Hilbert algebra over $\tilde{K}_{c/2}^{H_\infty} \otimes \overline{\tilde{K}_{c/2}^{H_\infty}}$ such that the morphism from $\tilde{K}_{c/2}^{H_\infty} \otimes \overline{\tilde{K}_{c/2}^{H_\infty}}$ to the endomorphism operad of $H$ is linear on the fibers of $\tilde{K}_{c/2}^{H_\infty}(n) \otimes \overline{\tilde{K}_{c/2}^{H_\infty}(n)}$, $n \in \mathbb{N}$.

Let $\mathcal{M}$ be a genus-zero modular functor and $\mathcal{M}_{H_\infty}(\cdot)$, $n \in \mathbb{N}$, the restrictions of $\mathcal{M}(\cdot)$ to $K_{H_\infty}(n)$. Then $\mathcal{M}_{H_\infty} = \{\mathcal{M}_{H_\infty}(\cdot)\}_{\mathbb{N}}$ is a suboperad (not partial) of $\mathcal{M}$. We also have the notion of Hilbert algebra over $\mathcal{M}_{H_\infty}$ defined in the obvious way.

**Definition 5.5.** Let $\mathcal{M}$ be a unitary genus-zero modular functor. A genus-zero weakly holomorphic conformal field theory $W$ over $\mathcal{M}$ is unitary if there is a positive-definite Hermitian form on $W$ such that
the morphism of partial pseudo-operads $M$ to $\mathcal{H}_W$ induces a Hilbert algebra structure on the completion $H^h_W$ ($h$ means holomorphic) of $W$ over $M_{H_\infty}$.

We have:

**Proposition 5.6.** The genus-zero weakly holomorphic conformal field theories constructed from the minimal Virasoro vertex operator algebras and from the vertex operator algebras associated to WZNW models are unitary.

Let $M$ be a unitary genus-zero modular functor. Then for any $n \in \mathbb{N}$ and $Q \in K(n)$, the corresponding positive-definite Hermitian form on the fiber of $\overline{M}(n)$ at $Q$ identifies the dual space of this fiber with the fiber of $\overline{\mathcal{M}}(n)$ at $Q$ and thus also identifies the dual space of the fiber of $\overline{M}(n)$ at $Q$ with the fiber of $\overline{\mathcal{M}}(n)$ at $Q$. Thus the adjoint of the morphism from $\mathcal{M} \otimes \overline{\mathcal{M}}$ to $\tilde{K}^{c/2} \otimes \overline{\tilde{K}}^{c/2}$ gives a morphism from $(\tilde{K}^{c/2} \otimes \overline{\tilde{K}}^{c/2})^{-1}$ to $\mathcal{M} \otimes \overline{\mathcal{M}}$ where $(\tilde{K}^{c/2} \otimes \overline{\tilde{K}}^{c/2})^{-1} = \{(\tilde{K}^{c/2} \otimes \overline{\tilde{K}}^{c/2})^{-1}(n)\}_{n \in \mathbb{N}}$, and for any $n \in \mathbb{N}$, $(\tilde{K}^{c/2} \otimes \overline{\tilde{K}}^{c/2})^{-1}(n)$ is the line bundle whose fibers are the duals of the fibers of $\tilde{K}^{c/2}(n) \otimes \overline{\tilde{K}}^{c/2}(n)$. It is clear that $(\tilde{K}^{c/2} \otimes \overline{\tilde{K}}^{c/2})^{-1}$ is canonically isomorphic to $\tilde{K}^{c/2} \otimes \overline{\tilde{K}}^{c/2}$. Thus we obtain a morphism from $\tilde{K}^{c/2} \otimes \overline{\tilde{K}}^{c/2}$ to $\mathcal{M} \otimes \overline{\mathcal{M}}$.

Let $W$ be a unitary genus-zero weakly holomorphic conformal field theory over a unitary genus-zero modular functor $M$. Let $\overline{H}_W^h$ be the complex conjugate of $H^h_W$ and $\mathcal{M}_{H_\infty}$ the complex conjugate operad of $M_{H_\infty}$. Then $\overline{H}_W^h$ has a structure of Hilbert algebra over $\mathcal{M}_{H_\infty}$. Let $H_W = H^h_W \otimes \overline{H}_W^h$ where $\otimes$ is the Hilbert space tensor product. Then the Hilbert algebra structure on $H^h_W$ over $\mathcal{M}_{H_\infty}$ and the Hilbert algebra structure on $\overline{H}_W$ over $\overline{\mathcal{M}}_{H_\infty}$ induce a Hilbert algebra structure on $H_W$ over $\mathcal{M} \otimes \overline{\mathcal{M}}$. Since $M$ is unitary, we have a morphism from $\tilde{K}^{c/2} \otimes \overline{\tilde{K}}^{c/2}$ to $\mathcal{M} \otimes \overline{\mathcal{M}}$ by the discussion above. Combining this morphism and the Hilbert algebra structure over $\mathcal{M} \otimes \overline{\mathcal{M}}$ on $H_W$, we obtain:

**Proposition 5.7.** Let $W$ be a unitary genus-zero weakly holomorphic conformal field theory over a unitary genus-zero modular functor $M$. Then $H_W$ is a genus-zero conformal field theory.

In particular, we have:
Corollary 5.8. Let $W$ be the unitary genus-zero weakly holomorphic conformal field theory constructed from a minimal Virasoro vertex operator algebra or from a vertex operator algebras associated to WZNW model. Then $H_W$ is a genus-zero conformal field theory.

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