A simple study of the correlation effects in the superposition of waves of electric fields: the emergence of extreme events

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Abstract

In this paper, we study the effects of correlated random phases in the intensity of a superposition of $N$ wave-fields. Our results suggest that regardless of whether the phase distribution is continuous or discrete if the phases are random correlated variables, we must observe a heavier tail distribution and the emergence of extreme events as the correlation between phases increases. We believe that such a simple method can be easily applied in other situations to show the existence of extreme statistical events in the context of nonlinear complex systems.

1. Introduction

Rogue or freak waves appear in the deep sea. They have very high amplitudes when compared with the surrounding waves [1], but these “monsters of the sea” do not only appear in the ocean. Rogue waves have appeared in a variety of fields from photonics [2] to Economy [3], where a nonlinear wave model is a good alternative to the Black-Scholes model. Some authors have explored the emergence of such extremes events comparing situations occurring in hydrodynamics and optics [4].

The history begins with Rayleigh studies on the superposition of amplitudes of harmonic oscillations in the 1880s. He concluded that the distribution of such amplitudes was given by (probability density function) PDF

$$f(x) = \frac{x}{\sigma^2} \exp \left( -\frac{x^2}{2\sigma^2} \right).$$

This distribution - a particular case of chi-square distribution $P(x) = \frac{\text{exp}(-x^2/2\sigma^2)}{2^{n/2-1}\sigma^n\Gamma(n/2)}$, with $n = 2$ degrees of freedom - describes the sum of the squares of $n$ independent standard normal random variables. It is important to consider that Rayleigh distribution occurs in many contexts of Physics, for instance, in the spacing distribution of eigenvalues of symmetric random matrices and in the
known GOE (Gaussian orthogonal ensemble) to explain the spacing distribution of energy levels in heavy nucleus [5].

The idea of analysing the superposition of waves has motivated some authors to hypothesize that constructive interference will lead to extreme events, i.e., outliers events to a Rayleigh bulk. Rice in 1944, while exploring aspects linked to the relationship between the energy spectrum of a surface and its physical observables, considered that the elevation in a determined point - in connection with analysis of the electrical noise current - is given by a sum of different sine/cosine waves with different frequencies with phases uniformly distributed. It was then obtained an extension/generalization of Rayleigh distribution known as Rice distribution [6]:

$$f(x|\nu, \sigma) = \frac{x}{\sigma^2} \exp\left(-\frac{x^2 + \nu^2}{2\sigma^2}\right) I_0\left(\frac{\nu x}{\sigma^2}\right)$$

where \(I_0(x)\) is the 0-th order modified Bessel function of the first kind. For large \(x\), \(I_0(x)\) can be approximated as \(I_0(x) \approx \frac{e^{-x}}{\sqrt{2\pi x}}\), so that

$$f(x|\nu, \sigma) \approx \frac{1}{\sqrt{2\pi \sigma^2}} \exp\left(-\frac{(x-\nu)^2}{2\sigma^2}\right)$$

which in 0-th order is given by a Gaussian distribution: 

$$f(x|\nu, \sigma) \approx \frac{1}{\sqrt{2\pi \sigma^2}} \exp\left(-\frac{(x-\nu)^2}{2\sigma^2}\right).$$

More recently, several authors have been shown the existence of non-Rayleigh effects in many different contexts. In a recent contribution the authors [7] based on previous results [8] show that a “sudoku light phases sequence” which is built from a set of memory rules to correlate the phases can generate light rogue waves, i.e., intensities that large deviate from Rayleigh distribution. We believe that a simpler and more general correlation mechanism is enough to generate rogue waves, since the correlation coefficient seems to be the main parameter to reach this extreme events.

The question here is then if we can observe rogue waves/extreme events by controlling a correlation coefficient. As will be observed, the answer is positive! So in this work, we propose a simple and didactic method to show how the correlations imply deviation from Rayleigh distribution, by directly controlling the phases correlation. We show here that extreme events occur for both cases: discrete and continuous random variables.

In section 2 we present the details to generate correlated random variables from independent random variables keeping the same variance and we present a way to standardize our study for the different versions of distributions: discrete and continuous ones. We considered uniform random
variables since such distribution has compact support, and we can control with absolute accuracy, where the random phases will be generated in $[-\pi, \pi]$.

Our main results are presented in section 3 showing how the rogue waves can emerge with the introduction of correlated phases. We propose a simple way to measure the rogue-wave level obtained for different correlations. Finally, we summarize our results in section 4 where we also present our main conclusions.

2. The model

Let us consider a superposition of waves corresponding to electric fields on the Fraunhofer plane (far-field):

$$E = A_\beta \sum_{j=1}^{N} e^{i(\beta j + \phi_j)}$$

$$= A_\beta \sum_{j=1}^{N} \cos(\beta j + \phi_j) + iA_\beta \sum_{j=1}^{N} \sin(\beta j + \phi_j),$$

where $A_\beta = E_0 \text{sinc}(\beta/2)$, where $E_0$ is the amplitude of incident electric field, $\phi_j$ is a random phase of pixel $j$, and $\beta$ is related to the diffraction angle, and $\text{sinc}(x) = \frac{\sin(x)}{x}$.

We can calculate the intensity $I = |E|^2$, thus

$$I = A_\beta^2 \left\{ \left[ \sum_{j=1}^{N} \cos(\beta j + \phi_j) \right]^2 + \left[ \sum_{j=1}^{N} \sin(\beta j + \phi_j) \right]^2 \right\}$$

$$= A_\beta^2 \left[ N + \sum_{j \neq l} \cos(\beta(j - l) + (\phi_j - \phi_l)) \right]$$

Expanding the sum, we have:

$$\sum_{j \neq l} \cos(\beta(j - l) + (\phi_j - \phi_l)) =$$

$$= \sum_{j \neq l} \cos \beta(j - l) \left( \cos \phi_j \cos \phi_l + \sin \phi_j \sin \phi_l \right) +$$

$$-\sum_{j \neq l} \sin \beta(j - l) \left( \sin \phi_j \cos \phi_l - \sin \phi_l \cos \phi_j \right)$$

If the phases are independent random variables and identically distributed according to a pdf $p(\phi)$, we can write (after some algebra) and by the fact that $\sum_{j \neq l} \sin \beta(j - l) = 0$

$$\left\langle \sum_{j \neq l} \cos \beta(j - l) + (\phi_j - \phi_l) \right\rangle = 2(a^2 + b^2) \sum_{j < l} \cos \beta(j - l)$$

where $a = \int_{\phi_{\text{min}}}^{\phi_{\text{max}}} d\phi p(\phi) \cos \phi$, and $b = \int_{\phi_{\text{min}}}^{\phi_{\text{max}}} d\phi p(\phi) \sin \phi$, where are supposing that $\phi_{\text{min}}$ and $\phi_{\text{max}}$
are the extremal angles of the distribution.

Performing the sum, we have

$$\left\langle \sum_{j \neq l} \cos (\beta (j - l) + (\phi_j - \phi_l)) \right\rangle = (a^2 + b^2) \left[ \frac{\sin^2(\frac{BN}{2})}{\sin^2(\frac{B}{2})} - N \right]$$

(2)

And therefore the intensities $I$ on the screen are given by:

$$\langle I \rangle = A^2\beta \left[ N + (a^2 + b^2) \left( \frac{\sin^2(\frac{BN}{2})}{\sin^2(\frac{B}{2})} - N \right) \right]$$

(3)

Thus, in this paper, we will study the probability density function (PDF), $P(I)$, according to the phases distribution $p(\phi)$ which are not independent random variables, so that we can analyze the tail of $P(I)$ and deviations in relation to $\langle I \rangle$. Moreover we also analyze the possible effects on $P(I)$ considering that phases are discrete random variables with continuous limit to a uniform distribution.

2.1. Correlated random phases from non-correlated random phases

In this section we will show that we can generate correlated random variables from non-correlated random phases considering that both (correlated and non-correlated) have the same variance and average. Let us consider two random variables

$$\phi_1 = \alpha_1 \varphi_1 + \alpha_2 \varphi_2$$
$$\phi_2 = \beta_1 \varphi_1 + \beta_2 \varphi_2$$

(4)

where $\varphi_1$ and $\varphi_2$ are i.i.d. random variables, which means: $\langle \varphi_1 \rangle = \langle \varphi_2 \rangle = \langle \varphi \rangle$, and $\langle \varphi_1 \varphi_2 \rangle = \langle \varphi_1 \rangle \langle \varphi_2 \rangle = \langle \varphi \rangle^2$. The variance of variable $\phi_1$, for example, can be calculated - after some cancellations - according to

$$\langle (\Delta \phi_1)^2 \rangle = \langle \phi_1^2 \rangle - \langle \phi_1 \rangle^2$$

$$= \langle (\alpha_1 \varphi_1 + \alpha_2 \varphi_2)^2 \rangle - \langle \alpha_1 \varphi_1 + \alpha_2 \varphi_2 \rangle^2$$

$$= (\alpha_1^2 + \alpha_2^2) \langle (\Delta \varphi)^2 \rangle$$

(5)

where $\langle \varphi_1^2 \rangle - \langle \varphi_1 \rangle^2 = \langle \varphi_2^2 \rangle - \langle \varphi_2 \rangle^2 = \langle (\Delta \varphi)^2 \rangle$, and similarly

$$\langle (\Delta \phi_2)^2 \rangle = (\beta_1^2 + \beta_2^2) \langle (\Delta \varphi)^2 \rangle.$$
Now we want the condition
\[
\langle (\Delta \phi_1)^2 \rangle = \langle (\Delta \phi_2)^2 \rangle = \langle (\Delta \varphi)^2 \rangle , \tag{7}
\]
which implies that \( \alpha_1^2 + \alpha_2^2 = \beta_1^2 + \beta_2^2 = 1 \).

It is worth noting that although \( \varphi_1 \) and \( \varphi_2 \) are not non-correlated random variables, \( \phi_1 \) and \( \phi_2 \) are, and the correlation between these random variables can be calculated:

\[
\rho = \frac{\langle (\phi_1 - \langle \phi_1 \rangle) (\phi_2 - \langle \phi_2 \rangle) \rangle}{\sqrt{\langle (\Delta \phi_1)^2 \rangle \langle (\Delta \phi_2)^2 \rangle}} \tag{8}
\]

Thus, again after some cancellations and combinations:

\[
\langle (\phi_1 - \langle \phi_1 \rangle) (\phi_2 - \langle \phi_2 \rangle) \rangle = \langle \phi_1 \phi_2 \rangle - \langle \phi_1 \rangle \langle \phi_2 \rangle = (\alpha_1 \beta_1 + \alpha_2 \beta_2) \langle (\Delta \varphi)^2 \rangle \tag{9}
\]

Thus, we can conclude:

\[
\rho = (\alpha_1 \beta_1 + \alpha_2 \beta_2) \tag{10}
\]

Given the properties previously considered, we can denote \( \alpha_1 = \beta_2 = \sin \theta \) and \( \alpha_2 = \beta_1 = \cos \theta \), and therefore \( \sin 2\theta = \rho \), so that

\[
\theta = \frac{1}{2} \sin^{-1}(\rho) \tag{11}
\]

Thus the random variables

\[
\phi_1 = \sin \left( \frac{1}{2} \sin^{-1}(\rho) \right) \varphi_1 + \cos \left( \frac{1}{2} \sin^{-1}(\rho) \right) \varphi_2 \tag{12}
\]

and

\[
\phi_2 = \cos \left( \frac{1}{2} \sin^{-1}(\rho) \right) \varphi_1 + \sin \left( \frac{1}{2} \sin^{-1}(\rho) \right) \varphi_2 \tag{13}
\]

have the same average that are given by:

\[
\langle \phi_1 \rangle = \langle \phi_2 \rangle = \frac{1}{\sqrt{2}} \left[ \left( 1 - \sqrt{1-\rho^2} \right)^{1/2} + \left( 1 + \sqrt{1-\rho^2} \right)^{1/2} \right] \langle \varphi \rangle
\]
and the same dispersion of $\varphi_1$ and $\varphi_2$ according to Eq. (7). It is important to emphasize that $\varphi_1$ and $\varphi_2$ have the same variance of $\varphi_1$ and $\varphi_2$ since $\langle (\Delta \varphi_1)^2 \rangle = \langle (\Delta \varphi_2)^2 \rangle = \langle (\Delta \varphi_1)^2 \rangle = \langle (\Delta \varphi_2)^2 \rangle = \langle (\Delta \varphi)^2 \rangle$, but the averages of $\varphi_1$ and $\varphi_2$ are not the same of $\varphi_1$ and $\varphi_2$ that are equal to $\langle \varphi \rangle$, unless $\langle \varphi \rangle = 0$. So if one considers $\varphi_1$ and $\varphi_2$ as $\rho$-correlated random variables generated from two independent random variables $\varphi_1$ and $\varphi_2$ – with average zero and variance $\sigma^2 = \langle (\Delta \varphi)^2 \rangle$–, $\varphi_1$ and $\varphi_2$ also have average zero and variance $\sigma^2$.

Here one has to be careful if we want to study the effects on the wave amplitudes by introducing phase correlations, since if one works with different average and dispersion between the cases $\rho = 0$ and $\rho \neq 0$ the results can be misleading since there will be no parameter for a fair comparison.

2.2. Discrete and continuous random variables

In this paper we consider phases which are uniformly distributed, since there is no reason for some phases to be more probable than others. So to study possible effects or similarities between discrete and continuous version of this distribution on the $P(I)$ shape, we generalize our method considering an equiprobable distribution for the phases, starting from a few number of states $Q$ up to $Q \to \infty$ which corresponds to a continuous and uniformly distributed random variable.

Our phases $\varphi_j$, $j = 1, 2, \ldots, N$, must be taken from the interval $[-\pi, \pi]$ with symmetric PDF $p_\rho(\varphi)$, since $\langle \varphi \rangle = 0$. So we propose a simple discrete random variable:

$$\varphi_k = \left( \frac{2k}{Q} - 1 \right) \pi$$

where $Q > 1$, an odd number of states by construction with $k = 0, 1, 2, \ldots, Q - 1$. As an example, for $Q = 3$ we have $\varphi_0 = -\pi$, $\varphi_1 = 0$, $\varphi_2 = \pi$. For $Q = 5$, we have $\varphi_0 = -\pi$, $\varphi_1 = -\pi/2$, $\varphi_2 = 0$, $\varphi_3 = \pi/2$, $\varphi_4 = \pi$. If we define $p_k = p(\varphi = \varphi_k) = \frac{1}{Q}$, i.e., all phases occur with same probability, so we have

$$\langle \varphi_k \rangle = \frac{\pi}{Q} \sum_{k=0}^{Q-1} \left( \frac{2k}{Q} - 1 \right) = 0$$

(15)
and we want to calculate

\[ \langle \phi_k^2 \rangle = \sum_{k=0}^{Q-1} \phi_k^2 p_k \]

\[ = \frac{\pi^2}{Q} \sum_{k=0}^{Q-1} \left( \frac{2k}{Q-1} - 1 \right)^2 \]

\[ = \frac{\pi^2}{Q} \left[ -\frac{4}{(Q-1)^2} \sum_{k=0}^{Q-1} k^2 \right. \]

\[ - \left. \frac{4}{(Q-1)} \sum_{k=0}^{Q-1} k + \sum_{k=0}^{Q-1} 1 \right] \]

\[ = \frac{\pi^2 Q}{3 (Q-1)} \]  

(16)

The importance of this distribution is twofold:

1. The distribution is discrete and \( p_k \) becomes the uniform distribution when \( Q \to \infty \).

\[ p(\phi) = \begin{cases} \frac{1}{2\pi} & \text{if } |\phi| \leq \pi \\ 0 & \text{if } |\phi| > \pi \end{cases} \]  

(17)

Also, \( \langle \phi_k \rangle = \langle \phi \rangle = 0 \), and from Eq. (16), \( \langle \phi_k^2 \rangle = \langle \phi^2 \rangle = \frac{\pi^2}{3} \), i.e., we can study a system with discrete phases and analyze what occur when \( Q \to \infty \);

2. It is more interesting to study a distribution with compact support for the phases since \( -\pi \leq \phi \leq \pi \). Thus, one makes that the boundary is respected which results necessarily in considering \( \sigma = \pi/\sqrt{3} \).

In what follows, we present the results for the distribution of amplitudes \( I \) considering random correlated phases given by Eqs. (12) and (13) from random independent variables drawn according to Eq. (14) and also by studying the limit case \( (Q \to \infty) \) according to Eq. (17).

3. Results

We computed the intensities distribution \( I \), i.e., the diffraction patterns. We used \( N = 1024 \) random waves considering 300 values of \( \beta \) equally spaced in the interval \([0, 2\pi]\). We repeated the procedure for an ensemble of \( N_{\text{run}} = 10000 \) realizations.

The \( N = 1024 \) random phases generated for each realization are shuffled since the correlated random variables are generated by pairs. Figure shows the probability density function (PDF) of light intensities scaled with total average light intensity sampled (i.e., including all values of \( \beta \)).
We used a mono-log scale. We can observe that tails become heavier as $\rho$ increases indicating the existence of rogue waves.

For each value of $\rho$ one measures

$$\eta(\rho) = \frac{I_{\text{max}}(\rho) - \bar{I}(\rho)}{\sigma(\rho)}$$

which measures the rogue-wave level. The inset plot in Fig. 1 shows that for $\rho = 1$, we have $\eta = 28$ while $\eta \approx 18$ for $\rho = 0$ (open balls). However we should calculate $\eta = \frac{I_{\text{max}}(\rho) - \bar{I}(\rho)}{\sigma(\rho=0)}$, take into account the standard deviation in relation to our base: $\rho = 0$ (non correlated). In this case we can obtain $\eta \approx 34$.

We also calculated $\langle I(\beta) \rangle$ as function of $\beta$ for different values of $\rho$. We performed two dif-

Figure 1: Distribution of intensities for different values of $\rho$ considering phases uniformly generated in $[-\pi, \pi]$ (shuffled case). The inset plot show the variable $\eta = (I_{\text{max}} - \bar{I})/\sigma$ for each correlation. We can observe that $\eta$ reaches 28 for $\rho = 1$. 
different studies. First, we analyze the effects of not shuffling or shuffling the sample of correlated random variables according to Figs. 2 (a) and (b). In Fig. 2 (b), the inset plot shows that results for $\rho = 0$, according to Eq. 3 (where $a = b = 0$ according with our distribution) perfectly agrees with simulation results.

After, we study the effects of the discretization of random variables on $\langle I(\beta) \rangle$ as observed in Fig. 2 (c). We do not have sensitive differences in comparison with results obtained with continuous distribution described in Fig. 2(b). The inset plot in Fig. 2 (c) shows that regardless of $Q$ ($Q \geq 3$) the behavior is similar to that one obtained in 2(b), which shows that discretization is not indeed an important factor in the average $\langle I(\beta) \rangle$.

Finally, we analyzed the effects of the partitioning of the interval $[0, 2\pi]$ for $\beta$, since the PDFs
of the intensities should depend on the granulation of the interval. So we partitioned the interval in $N_\beta = 100, 200, 300$, and 400 parts, and we calculated the PDF for all these cases. These results are shown in Fig. 3.

![PDF considering different number of repartitions of the interval $[0, 2\pi]$ for $\beta$.](image)

We used 300 subdivisions for the interval $[0, 2\pi]$ in all previous cases studied in this manuscript. We can observe that there is no visual difference between $N_\beta = 300$ and 400, and qualitatively we have no differences for all studied cases, by showing that we need no greater values of $N_\beta$ in order to build the histograms of PDFs in our results.

4. **Summaries, conclusions, and discussions**

In this work we show the existence of rogue waves for PDF of the intensities associated to the superposition of waves of electric fields with the introduction of random correlated phases controlled by the correlation coefficient $\rho$. We can observe that heavier tail distributions are obtained
as the correlation increases. Our results also suggest that discretization is not an important ingredient to change the PDF shape. The average intensity is also studied as function of $0 \leq \beta \leq 2\pi$, which corroborates the analytical results. Shuffling procedure leads to a reduction of the oscillatory effects on the $\langle I \rangle (\beta)$ (see Eq. [3]). Our results are simple and easy to be applied, and deserve future investigations in other problems involving extreme statistics.

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