Mixed-Integer Programming for the ROADEF/EURO 2020 challenge

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Abstract

The ROADEF 2020 challenge presents a maintenance scheduling problem from the French electricity grid company RTE. The modeling of uncertainty makes the problem highly nonconvex and apparently out of the reach of mathematical solvers. We present our approach for the challenge problem. It is based on a new family of cutting planes, coupled with a constraint generation approach. We present mathematical proofs and separation algorithms for the cutting planes. We then study the practical impact of our additions on the challenge instances, showing that our approach significantly reduces the optimality gap obtained by the solver.

1 Introduction

For 22 years and 12 editions, the ROADEF optimization challenge has provided a platform for industry practitioners and academic researchers to share and compete on practical optimization problems. The ROADEF 2020 challenge [1] presents a problem coming from RTE’s maintenance planning. RTE operates the French electric power grid. Reliability is paramount, and the maintenance must take into account the risk of the operation for the grid’s integrity. In order to model the uncertainty affecting the energy industry, RTE ran many different scenarios. The challenge problem is to find a maintenance schedule taking into account the risk for the given scenarios.

The challenge problems are, by design, difficult, and mathematical solvers struggle to find good solutions. All challenge winners since 2010 have used heuristic methods, with occasional use of exact methods or solvers for subproblems or simple instances [2–7]. Our code is no exception, and the best solutions from our team are obtained with a combination of local search and beam search. Still, while finding good solutions is the main goal, it is useful in practice to obtain bounds on the optimal solution value. If they are tight enough, they provide confidence in the solutions found by the method, which is crucial for decision makers and for actual use of the optimization solutions. Moreover, modeling a problem is much simpler than designing a custom algorithm, so that improving solvers can result in huge productivity improvements for industrial teams. This motivates us to research a pure solver-based method, which is able to prove lower-bounds on the possible objective value.

For this challenge, RTE considers two quality metrics across these scenarios: the mean risk and a quantile of the risk to take worse scenarios into account. The use of a quantile in the objective function transforms what would be a simple scheduling problem into a complex problem with a non-convex objective. In the field of constraint programming, the quantile function is known as the max constraint [8], and can be used to implement the more ubiquitous sort constraint [9], so that improvements in its modeling could have impact on many other problems.

A useful approach to solve such a two-stage stochastic problem with a mixed-integer programming (MILP) solver is the application of a constraint generation method i.e. Bender’s decomposition [10]. The classical Bender’s decomposition is limited to purely continuous linear problems, and a large body of the literature has been dedicated to its generalizations to integer problems [11].

In this paper, after stating the challenge problem, we will study the modeling of the quantile function and introduce a family of valid inequalities. We then design a constraint generation approach that takes into account the structure of the quantile function. This provides a simple algorithm, without the need for the solution of a mixed-integer programming subproblem. We extend this family of constraints in an attempt to improve the bounds proven by our models. Finally, we will study these
various methods in practice on the challenge instances, comparing their ability to find good solutions and their power in proving lower bounds on the objective value.

2 Problem statement

In this section, we will describe the problem and give a mathematical programming model. We refer the reader to the challenge subject in [1] for a more thorough description. The subject provides more explanations of the maintenance of electricity transmission lines and the implications of the problem from business and environmental perspectives.

2.1 Decision variables

The goal is to schedule interventions \( i \in I \) according to their starting date \( t \in H \), with a granularity of one day. The natural formulation for such discrete scheduling problems is to use boolean decision variables, \( x_{it} \), for each intervention and each possible start time. Since all interventions must be scheduled exactly once, we add the constraint

\[
\sum_t x_{it} = 1 \text{ for all interventions } i
\]

Additionally, interventions span multiple days, with a duration \( \Delta_{it} \) that depends on their start time. It is forbidden to schedule interventions beyond the planning horizon, which translates to the additional constraints:

\[
x_{it} = 0 \text{ if } t + \Delta_{it} > T + 1 \text{ for all intervention } i \text{ and time } t
\]

2.2 Resource constraints

The problem exhibits resource constraints, which can represent the availability of personnel and tooling. There are multiple resources \( c \in C \) and their availability varies with the date.

Resources are mobilized each day during an intervention. In order to plan an intervention, all the required resources must be available. We note \( r_{it}^{ct} \) the amount of resource \( c \) required at time \( t \) if the intervention \( i \) is scheduled at time \( t' \). \( u_i^c \) is the amount of resource \( c \) available at time \( t \), and \( l_i^c \) is a minimal amount to be used - usually 0. The resource constraints to model the available resources are the following linear constraints:

\[
l_i^c \leq \sum_{i,t'} r_{it'}^{ct} x_{it'} \leq u_i^c \text{ for all timestep } t \text{ and resource } c
\]

RTE’s planners may define additional scheduling constraints, so that two interventions are forbidden to take place simultaneously for a given time period. We can simply express these mutual exclusions as linear constraints of the form \( x_{it} + x_{i't'} \leq 1 \).

2.3 Risk and problem objective

For each timestep \( t \), we are given a set of scenarios \( s \in S_t \), and the risk associated with the schedule of intervention \( i \) at time \( t' \), \( \text{risk}_{it'}^{st} \). The risk associated with a planning for a given scenario is simply the sum of the risks of the scheduled interventions i.e. \( \sum_{i,t'} \text{risk}_{it'}^{st} x_{it'} \), and is noted \( \text{risk}^{st} \).

We define the \( \tau \)-quantile of a set as the smallest value such that a proportion \( \tau \) of the elements are lower or equal to it. It is noted \( Q_\tau \). Mathematically, \( Q_\tau(E) = \min\{\max(X), X \in E, |X| \leq \tau \times |E|\} \).

The goal is to minimize a blended objective. The first part of this blended objective is the mean risk across all scenarios and timesteps, with coefficient \( \alpha \). The second part averages the \( \tau \)-quantile of the risk across all scenarios if it is larger than the mean, with coefficient \( (1 - \alpha) \).
In mathematical terms:

\[ \overline{risk^t} = \frac{1}{|S_t|} \sum_{s \in S_t} risk^{st} \]

is the mean of the risk at time \( t \)

\[ Q^t_\tau = Q_\tau(\{risk^{st}\}_{s \in S_t}) \]

is the \( \tau \)-quantile of the risk at time \( t \)

\[ obj_1 = \frac{1}{|T|} \sum_{t \in H} risk^t \]

is the mean risk

\[ obj_2 = \frac{1}{|T|} \sum_{t \in H} \max(Q^t_\tau - \overline{risk^t}, 0) \]

is the mean excess risk

\[ \text{minimize } \alpha \text{ } obj_1 + (1 - \alpha) \text{ } obj_2 \]

is our blended objective

The only non-linear term is the computation of the quantile. Without this term, the model is actually easy to solve for mixed-integer programming solvers. This is exemplified by the challenge instance A_04, which has only one scenario but is otherwise one of the largest: a solver proves the optimal solution in a matter of seconds. We will see in the next section that the quantile function can be modeled with the introduction of auxiliary variables.

3 MILP formulation and constraint programming equivalent

In the rest of this paper, we will use the following definitions of \( Q_k(x), x \in \mathbb{R}^n \), which are equivalent:

\[ Q_k(x) \text{ is the } k^{th} \text{ largest element of } x \]

or

\[ Q_k(x) \text{ is the } n - k^{th} \text{ smallest element of } x \]

or

\[ Q_k(x) = \min \{ \max(v), v \subseteq x, |v| \leq n - k \} \]

or

\[ Q_k(x) = \max \{ \min(v), v \subseteq x, |v| \leq k \} \]

The quantile function can be modeled straightforwardly for a mixed-integer programming (MILP) solver with additional binary variables \( \kappa_i \). To model \( y \geq Q_k(x) \), we may use indicator constraints as follows:

\[ y \geq x_i \text{ if } \kappa_i \text{ for } 1 \leq i \leq n \]

\[ \sum_{i=1}^{n} \kappa_i = n - k \]

In practice we assume that the solver is able to work well with indicator constraints, and rely on their expressive power if available. For solvers that do not support them, we can express them using the big-M method as long as we know bounds \( l \) and \( u \) such that \( l \leq x \leq u \). The constraints become, with \( M \geq u - l \):

\[ y \geq x_i + M_i(1 - \kappa_i) \text{ for } 1 \leq i \leq n \]

\[ \sum_{i=1}^{n} \kappa_i = n - k \]

Or alternatively:

\[ y \geq x_i - M_i\kappa_i \text{ for } 1 \leq i \leq n \]

\[ \sum_{i=1}^{n} \kappa_i = k \]
4 Polyhedral analysis of the quantile function

**Theorem 4.1** (Non-convexity). \( Q_k \) is non-convex except in the trivial case \( Q_1 = \max \).

**Proof.** Suppose \( k \geq 2 \). Then \( Q_k(2, \ldots, 2, 0, \ldots) = Q_k(2, \ldots, 2, 0, 2, \ldots) = 0 \). However, for the average of these two points, \( Q_k(2, \ldots, 2, 1, 1, 0, \ldots) = 1 \): \( Q_k \) is non-convex.

The quantile function is non-convex in general, with the minimum function and the maximum function as special cases when \( k = n \) and \( k = 1 \) respectively. However, in many practical instances of risk mitigation, \( k \ll n \), and the function is intuitively “close” to the convex maximum function. It is indeed the case for many of the challenge’s instances. We could expect the convex relaxation to be better in this case, possibly good enough for practical models. This prompts us to study the lower convex envelope \( \text{cv} Q_k \) of the quantile function \( Q_k \).

**Theorem 4.2** (Unbounded domain). Except in the trivial case \( Q_1 = \max \), the lower convex envelope of \( Q_k \) on \( \mathbb{R}^n_+ \) is 0, and there is no lower convex envelope on \( \mathbb{R}^n \).

**Proof.** Trivially, \( \text{cv} Q_k \geq 0 \) on \( \mathbb{R}^n_+ \). A point in \( \mathbb{R}^n_+ \) is a convex combination of points on the coordinate axes, for which all components except one are 0. \( Q_k \) is 0 on such points if \( k \geq 2 \) so that \( \text{cv} Q_k \leq 0 \). Similarly, can write any element of \( \mathbb{R}^n \) as a convex combination of points with arbitrarily low images.

As the lower convex envelope in the unbounded case is useless to tighten the model, we will only study \( Q_k \) on a bounded domain, where we can hope to obtain a tighter convex envelope.

4.1 Symmetric case

For the sake of simplicity, we first assume uniform lower and upper bounds on all variables, respectively 0 and 1. This allows us to derive a simple formula for a family of linear constraints.

**Theorem 4.3** (Symmetric lower bound). Suppose \( x \in [0, 1]^n \). Then, given a subset \( W \) of \([1 \ldots n] \) with at least \( k \) elements:

\[
Q_k(x) \geq \frac{1}{|W| + k + 1} \left( \sum_{i \in W} x_i - k + 1 \right)
\]

**Proof.** We assume without loss of generality that \( x_i \geq x_{i+1} \) for \( 1 \leq i < n \). Since \( |\{i \in W, i \geq k\}| \geq |W| - k + 1 \):

\[
Q_k(x) = x_k = \frac{1}{|W| - k + 1} \left( x_k(|W| - k + 1) \right) = \frac{1}{|W| - k + 1} \left( |\{i \in W, i < k\}| + |\{i \in W, i \geq k\}| x_k - k + 1 \right. \\
+ \left. (1 - x_k)(|\{i \in W, i \geq k\}| - |W| + k - 1) \right) \\
\geq \frac{1}{|W| - k + 1} \left( |\{i \in W, i < k\}| + |\{i \in W, i \geq k\}| x_k - k + 1 \right) \\
\geq \frac{1}{|W| - k + 1} \left( \sum_{i \in W, i < k} 1 + \sum_{i \in W, i \geq k} x_k - k + 1 \right) \\
\geq \frac{1}{|W| - k + 1} \left( \sum_{i \in W} x_i - k + 1 \right)
\]

While this constraint family is of exponential size in general, the most violated constraint is trivially found in linearithmic time \( O(n \log n) \) by sorting the coordinates of \( x \) and iterating on the size of \( |W| \), as implemented by Algorithm 1.
Algorithm 1 Constraint separation on a symmetric domain from a sorted vector

```plaintext
function SYMMETRIC SEPARATION(x ∈ [0, 1]^n, x_i ≥ x_{i+1} ∀ i, 1 ≤ i < n)
    c ← ∑_{i=1}^{n} x_i
    best bound ← −∞
    best w ← k
    for w ∈ [k..n] do
        c ← c + x_w
        bound ← c/k + 1
        if bound > best bound then
            best bound ← bound
            best w ← w
        end if
    end for
    return [1.. best w]
end function
```

4.2 General case

Let’s consider the general, non-symmetric case with arbitrary finite lower and upper bounds.

**Theorem 4.4** (Asymmetric lower bound). Suppose x ∈ ℝ^n, l_i ≤ x_i ≤ u_i. We write L = Q_k(l). Then, given a subset W of [1.. n] with at least k elements and U > L,

\[ Q_k(x) ≥ L + \frac{U - L}{|W| - k + 1} \left( \sum_{i \in W} \frac{x_i - L}{\max(U, u_i) - L} - k + 1 \right) \]

**Proof.** Consider a point x ∈ ℝ^n, l_i ≤ x_i ≤ u_i. We assume without loss of generality that x_i ≥ x_{i+1}, 1 ≤ i < n. Then:

\[ Q_k(x) = x_k \]

\[ Q_k(x) = L + \frac{U - L}{|W| - k + 1} (|W| - k + 1) \frac{x_k - L}{U - L} \]

\[ Q_k(x) ≥ L + \frac{U - L}{|W| - k + 1} (|W| - k + 1) \frac{x_k - L}{\max(U, u_i) - L} \]

since \( x_k ≥ L \)

\[ Q_k(x) ≥ L + \frac{U - L}{|W| - k + 1} \left( |\{ i \in W, i < k \}| + |\{ i ∈ W, i ≥ k \}| \frac{x_k - L}{\max(U, u_i) - L} - k + 1 \right) \]

\[ Q_k(x) ≥ L + \frac{U - L}{|W| - k + 1} \left( \sum_{i \in W, i < k} \frac{u_i - L}{\max(U, u_i) - L} + \sum_{i \in W, i ≥ k} \frac{x_k - L}{\max(U, u_i) - L} - k + 1 \right) \]

\[ Q_k(x) ≥ L + \frac{U - L}{|W| - k + 1} \left( \sum_{i \in W} \frac{x_i - L}{\max(U, u_i) - L} - k + 1 \right) \]

This family of lower bounds is dominated by the cases where \( U = u_i \). We can repeat the separation algorithm used in the symmetric case for every possible \( U \), sorting by \( \frac{x_i - L}{\max(U, u_i) - L} \). Once the vector is sorted, the separation algorithm runs in linear time for each \( U \). If we perform two initial sorts, by \( x_i \) and \( \frac{x_i - L}{u_i - L} \), they can be used to sort the vector by \( \frac{x_i - L}{\max(U, u_i) - L} \) in linear time with one merge-sort step for any \( U \). This gives a constraint separation algorithm that runs in quadratic time (\( \mathcal{O}(n^2) \)) in the general case.

5 Constraint generation method

The above formulation as a mixed-integer linear problem, while complete, is not efficient. The model of the quantile function introduces one binary variable and one indicator constraint for each
element of the population. For the RTE challenge, the indicator constraints involve one variable per intervention for multiple timesteps, making the model grow even faster. This repeated subproblem is a natural target for decomposition methods.

We propose a constraint generation method with the subproblem \( y \geq Q_k(Ax) \). On the challenge problem, \( y \) would be the risk’s \( \tau \)- quantile for a given timestep, \( x \in \{0, 1\}^n \) are decision variables for the intervention dates, and \( A \geq 0 \) is the associated risk per scenario.

**Theorem 5.1** (Simple generated constraint). Assume \( A \geq 0 \). Given an incumbent solution \( \bar{x} \in \{0, 1\}^n \), the following inequality is valid for all \( x \in \{0, 1\}^n \) and is tight for \( x = \bar{x} \):

\[
Q_k(Ax) \geq Q_k(A\bar{x}) \left( \sum_i \bar{x}_i x_i - \sum_i \bar{x}_i + 1 \right)
\]

**Proof.** If there is a \( j \) such that \( x_j < \bar{x}_j \), the factor on the right-hand side is negative or zero. Since \( Q_k(Ax) \) is non-negative, the inequality is valid. Otherwise, since \( A \) takes non-negative values, \( Ax \geq A\bar{x} \). \( Q_k \) is non-decreasing, so that \( Q_k(Ax) \geq Q_k(A\bar{x}) \).

Our constraint generation procedure adds this constraint for each incumbent solution, which enforces the correct value of \( y \) at this point. As the set of possible values for \( x \) is finite, this guarantees eventual convergence. This is similar to the addition of a “no-good” cut in [12] and [13], with the addition of the continuous variable \( y \).

With large coefficients and a huge negative constant term, this first constraint is very weak and leads to no improvements in the linear relaxation in most cases. Moreover, if \( x_i = 0 \) and \( \bar{x}_i = 1 \), the right-hand side is always non-positive: the constraint is only active on solutions where no coordinate has decreased, so it is unlikely to tighten the value of \( y \) at other integral points. Our goal in this section will be to strengthen these constraints to obtain tighter bounds and a faster solution process.

### 5.1 Bounds on the quantile value

As a first step to improve our formulation, we will attempt to derive simple inequalities for \( Q_k(Ax) \) to strengthen the master problem. These constraints can be added at the root node or as cutting planes, and will help us obtain tighter generated constraints later. Compared to the convex relaxation of \( Q_k \) alone, these methods will be able to take into account the similarities between scenarios, and should yield to tighter constraints.

**Theorem 5.2** (Subset constraint). The following inequality is valid for any subset \( P \) of \([1 \ldots n]\) with at least \( k \) elements:

\[
Q_k(Ax) \geq \sum_{j=1}^m \min_{i \in P} a_{ij} x_j
\]

**Proof.** Since the quantile function can be stated as \( Q_k(x) = \max\{\min(v), v \subseteq x, |v| \leq k\} \), we obtain:

\[
Q_k(Ax) \geq \min_{i \in P} \sum_{j=1}^m a_{ij} x_j
\]

\[
Q_k(Ax) \geq \sum_{j=1}^m \min_{i \in P} (a_{ij}) x_j
\]

**Theorem 5.3** (NP-hardness of separation). The optimization problem of maximizing \( \sum_{j=1}^m \min_{i \in P} a_{ij} \) over \( P \subseteq [1 \ldots n] \), subject to \( P \) having at least \( k \) elements, is NP-hard. As a corollary, given a solution to the relaxed problem, finding a violated constraint is NP-hard.

**Proof.** We show a reduction from the MINIMUM PARTIAL VERTEX COVER problem, which is NP-hard [14]. Given a graph, the MINIMUM PARTIAL VERTEX COVER problem is to find a subset of at least \( k \) vertices that covers the smallest number of edges.

Given an undirected graph with \( n \) vertices and \( m \) edges, we set \( a_{ij} = 0 \) if the edge \( j \) is incident to node \( i \), 1 otherwise. With this setup, \( \min_{i \in P} (a_{ij}) = 1 \) if and only if \( P \) does not cover edge \( j \), and the problem is equivalent to the MINIMUM PARTIAL VERTEX COVER.
While the problem is hard in theory, heuristic choices for $P$ achieve good results. We attempt to add these constraints as cutting planes with a separation heuristic, either a best-first search with restarts, or based on an auxiliary MILP model. However, it is difficult to adjust the subproblem heuristic to obtain good running times in practice. For our benchmarks, we chose to add the constraints corresponding to the optimal set for each decision variable $x_j$ separately i.e. $P_j = \arg \max_{i \in P_j} (a_{ij})$, directly to the master problem.

5.2 Strengthened constraints

As seen earlier, the previous constraints alone are bad at strengthening and pruning the master problem. To improve the resolution of the master problem, we need to strengthen it. This results in the following goals:

- Keeping the constraint tight for the current assignment $\bar{x}_j$, guaranteeing eventual convergence
- Decreasing the coefficient associated with $x_j$ when $\bar{x}_j = 1$
- Increasing the coefficient associated with $x_j$ when $\bar{x}_j = 0$

**Theorem 5.4** (Generated subset constraint). Given $\bar{x} \in \{0,1\}^n$, we pick a subset $P$ of $[1..n]$ with at least $k$ elements so that $Q_k(A\bar{x}) = \min_{i \in P} \sum_j a_{ij} \bar{x}_j$. Then the following inequality is valid for all $x \in \{0,1\}^n$ and is tight for $x = \bar{x}$:

$$Q_k(Ax) \geq Q_k(A\bar{x}) + \sum_{j: \bar{x}_j = 1} (x_j - 1) \max_{i \in P} a_{ij} + \sum_{j: \bar{x}_j = 0} x_j \min_{i \in P} a_{ij}$$

**Proof.**

$$Q_k(Ax) \geq \min_{i \in P} \sum_{j=1}^n a_{ij} x_j$$

$$Q_k(Ax) \geq \min_{i \in P} \sum_{j=1}^n a_{ij} (x_j - \bar{x}_j) + \min_{i \in P} \sum_{j=1}^n a_{ij} \bar{x}_j$$

$$Q_k(Ax) \geq \min_{i \in P} \sum_{j=1}^n a_{ij} (x_j - \bar{x}_j) + Q_k(A\bar{x})$$

$$Q_k(Ax) \geq \sum_{i \in P} \min_{j} a_{ij} (x_j - \bar{x}_j) + Q_k(A\bar{x})$$

Like the constraint from Section 5.1, this inequality is tight for the incumbent solution, which guarantees eventual convergence of the constraint generation procedure. It obtains non-zero coefficients for variables that are not part of the incumbent solution, and smaller coefficients for variables that are part of it. As a result, we expect a much larger effect on the linear relaxation.

5.3 General subset constraints

Finally, we present the more generic version of the constraints above, that we can apply as cutting planes during the solution process.

**Theorem 5.5** (General subset constraint). Given bounds $l_i \leq u_i$, a subset $P$ of $[1..n]$ with at least $k$ elements and $\beta \in [0,1]^n$. Then the following inequality is valid for all $l_i \leq x_i \leq u_i$:

$$Q_k(Ax) \geq \sum_{j=1}^n (1 - \beta_j) \min_{i \in P} (a_{ij})(x_j - l_j) + \sum_{j=1}^n \beta_j \max_{i \in P} (a_{ij})(x_j - u_j)$$

$$+ \min_{i \in P} \sum_{j=1}^n (1 - \beta_j) a_{ij} l_j + \min_{i \in P} \sum_{j=1}^n \beta_j a_{ij} u_j$$
The proof is trivial following the exact same steps. This new formulation supersedes the generated subset constraints above, which are found with \( l_j = 0 \), \( u_j = 1 \) and \( \beta_j = \bar{x}_j \). It still does not supersede the valid inequalities derived in Section 4, as can be seen when \( A \) is the identity matrix, \( l_j = 0 \) and \( u_j = 1 \): only the case of \( |W| = k \) is covered by this new constraint. Finding a more general formula is left to future work.

We implement the optimal constraint separation problem as a MILP subproblem, but the running time is too slow to use as a cutting-plane generation algorithm: more work is needed to obtain efficient separation heuristics for these constraints.

6 Numerical results

We compare the models using the following methods:

- The full formulation with indicator constraints from Section 3 (Full), and our constraint generation method introduced in Section 5.2 (CGen).
- The above formulations with subset constraints added to the model as defined in Section 5.1 (Full+S/CGen+S). The constraints are added at model creation time, without taking the linear relaxation into account.
- The full formulation with the valid inequalities for the quantile function, following Section 4 (Full+C). As the separation algorithm is efficient, we add the most violated inequality as a cutting plane at the root node until no such inequality can be found. The algorithm requires bounded variables, and to keep reasonable running times we use the simple bound on the risk derived from the sum of the maximum risk for each intervention. We validated that these bounds are close to the best bounds obtained with a MILP model.
- An idealized method with the most general constraint family in Section 5.3 and a near-perfect separation algorithm at the root node (CGen+O). We perform 20 rounds of constraint separation at the root node using a MILP subproblem. The time required to solve this subproblem is large and is not included in the measurement.

We implement our models in Python and solve them using CPLEX 11.0. The code is made available on Github\(^1\). We run our models on the instance sets A, B and C from RTE, for one hour not including model creation time. Our benchmarking machine has a 4-core CPU with 10GB of memory. In the results presented here, we remove instances where all the models reach the optimal solution (8 from A set), as well as instances where all models crash or finish without a feasible solution (4 from B and C sets). We report the optimality gap found by the solver in Table 1. A comparison of the bounds and solutions to the best known results is useful to understand the behaviour of the methods, and is available in Appendix A.

6.1 Direct convex relaxation

The quality of the constraints that can be deduced from the convex relaxation of the quantile function alone is highly dependent on the tightness of the lower and upper bounds on the risk values. In our experiments, the convex relaxation of the quantile function leads to an actually worse model than the natural MILP model, presumably due to the time spent in the constraint separation algorithm and the computational cost of the additional constraints. This is expected in hindsight, as the interventions tend to be spread over time due to the resource constraints, and the risk for any given time tends to be much lower than the theoretical upper bound, where the constraint is tighter.

6.2 Constraint generation method

Looking at the objective value only, the constraint generation method is better than the full model for 31 of 33 instances. For 19 instances, one of the two constraint generation methods obtains a solution within 1% of the best known solution for the challenge, against only 7 for the full model. The solution of instance A_08 is even proven optimal.

\(^1\) https://github.com/Coloquinte/Roadef2020
Table 1: Optimality gap obtained by the solver with various methods after one hour.

| Instance | Full   | Full+C | Full+S | CGen   | CGen+S | CGen+O |
|----------|--------|--------|--------|--------|--------|--------|
| A_02     | 54.65% | 62.64% | 2.41%  | 2.70%  | 2.06%  | 1.79%  |
| A_05     | 9.27%  | 9.90%  | -      | 7.70%  | 5.99%  | 5.19%  |
| A_08     | 6.31%  | 6.98%  | 2.77%  | 0.00%  | 0.00%  | 0.00%  |
| A_11     | 8.72%  | 10.17% | 8.63%  | 4.35%  | 4.63%  | 4.16%  |
| A_13     | 0.10%  | 0.10%  | 0.12%  | 0.10%  | 0.12%  | 0.08%  |
| A_14     | 9.15%  | 9.82%  | 9.07%  | 6.72%  | 6.44%  | 6.17%  |
| A_15     | 12.78% | 14.42% | 12.10% | 7.37%  | 7.53%  | 6.44%  |
| B_01     | -      | -      | 8.24%  | 14.09% | 7.45%  |        |
| B_02     | 60.65% | -      | 62.09% | 60.79% | 61.97% |        |
| B_03     | 68.50% | -      | -      | -      | -      |        |
| B_04     | -      | 62.90% | -      | 9.48%  | -      |        |
| B_05     | 51.15% | 52.86% | -      | 16.62% | 5.70%  |        |
| B_06     | 59.86% | 61.81% | 58.98% | 60.94% | 62.05% |        |
| B_07     | 47.45% | -      | -      | 46.31% | 46.60% |        |
| B_08     | 68.08% | -      | 4.02%  | 7.92%  | 6.12%  |        |
| B_09     | 67.44% | -      | 3.46%  | 22.37% | 4.59%  |        |
| B_10     | -      | -      | -      | 47.81% | -      |        |
| B_11     | 58.58% | -      | 56.05% | 56.57% | 57.61% |        |
| B_12     | -      | -      | -      | 12.40% | -      |        |
| B_13     | -      | -      | -      | 12.62% | 7.13%  |        |
| B_14     | -      | -      | -      | 39.17% | -      |        |
| C_01     | 70.65% | 68.46% | 5.47%  | 77.31% | 6.45%  |        |
| C_02     | -      | -      | -      | 39.34% | 39.63% |        |
| C_03     | 70.03% | 67.23% | -      | 71.80% | -      |        |
| C_04     | 64.85% | 64.29% | -      | 9.38%  | -      |        |
| C_05     | 59.77% | 60.70% | 4.30%  | 14.44% | 4.20%  |        |
| C_06     | 54.12% | -      | -      | 47.55% | 47.61% |        |
| C_07     | 54.23% | -      | 3.77%  | 16.50% | 6.06%  |        |
| C_08     | 50.28% | -      | -      | 44.77% | 44.80% |        |
| C_09     | 69.92% | -      | 70.30% | 70.03% | 71.10% |        |
| C_10     | -      | -      | -      | 70.37% | -      |        |
| C_11     | 59.58% | -      | 9.32%  | 12.64% | 7.03%  |        |
| C_12     | -      | -      | -      | 47.35% | -      |        |

The effect on the proven lower bound is less clear: while the lower bound is better for all small A instances, it is frequent for the full model to get better lower bounds on B and C instances.

6.3 Subset constraints

The effect of adding subset constraints at the root node varies between instances. For some of them, the gap between the lower bound and the best known solution is reduced by a factor of 20, while for others it is virtually unchanged. It is particularly efficient when combined with the full model, where it yields some of the best bounds.

We see that the “ideal” case, with perfect constraint separation at the root node, is still significantly better at reducing the optimality gap. With appropriate fast heuristics to find the cutting planes, there is certainly room for improvement. Another consequence of our crude heuristic is that the model is much larger. The solver runs out of memory or is unable to find a solution on our machine for some instances, making the pure constraint generation method a credible alternative if finding solutions is more important than the quality of the bounds.
7 Conclusion

In this paper, we presented a mathematical programming method for the ROADEF 2020 challenge. The modeling of uncertainty using a quantile function makes the problem hard to model, and the natural mixed integer programming model is large and slow to converge. We followed two paths to improve its modeling. First, we studied valid inequalities for the quantile function and developed separation algorithms and heuristics; this led to a stronger linear relaxation of the model. Second, we introduced a constraint generation approach; this method made the model much smaller, making the solution process faster and tackling larger instances.

We then studied the practical performance of our methods on the challenge instances, showing that the constraint generation approach obtains much better solutions and the cutting planes better optimality bounds compared to the natural model. Our work on this approach doesn’t end here, as neither the constraint generation algorithm nor the cutting planes are restricted to the challenge problem. In particular, the quantile function we studied can be used to implement and strengthen the ubiquitous sort constraint in constraint programming.

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## A Numerical results

| Instance | Full   | Full+S  | CGen   | CGen+S  | CGen+O  |
|----------|--------|---------|--------|---------|---------|
| A,02     | 54.47% | 2.05%   | 2.38%  | 1.91%   | 1.61%   |
| A,05     | 6.76%  | -       | 6.22%  | 5.43%   | 4.33%   |
| A,08     | 6.25%  | 2.74%   | 0.00%  | 0.00%   | 0.00%   |
| A,11     | 6.33%  | 5.89%   | 4.18%  | 4.45%   | 3.88%   |
| A,13     | 0.08%  | 0.09%   | 0.05%  | 0.06%   | 0.05%   |
| A,14     | 7.44%  | 7.06%   | 5.55%  | 5.76%   | 4.91%   |
| A,15     | 7.76%  | 7.60%   | 6.66%  | 6.74%   | 5.19%   |
| B,01     | -      | 6.22%   | 13.22% | 7.28%   |         |
| B,02     | 50.24% | 50.11%  | 55.12% | 55.13%  |         |
| B,03     | 63.58% | -       | -      | -       |         |
| B,04     | -      | -       | 9.34%  | -       |         |
| B,05     | 47.12% | -       | 15.24% | 5.48%   |         |
| B,06     | 50.04% | 50.13%  | 55.04% | 55.09%  |         |
| B,07     | 41.50% | -       | 41.58% | 41.63%  |         |
| B,08     | 64.09% | 3.88%   | 7.90%  | 6.11%   |         |
| B,09     | 66.03% | 3.24%   | 15.73% | 4.59%   |         |
| B,10     | -      | -       | 46.04% | -       |         |
| B,11     | 48.25% | 49.65%  | 52.51% | 52.45%  |         |
| B,12     | -      | -       | 11.18% | -       |         |
| B,13     | -      | -       | 12.25% | 7.12%   |         |
| B,14     | -      | -       | 37.61% | -       |         |
| C,01     | 66.61% | 4.73%   | 76.44% | 6.40%   |         |
| C,02     | -      | -       | 37.75% | 37.86%  |         |
| C,03     | 65.14% | -       | 37.57% | -       |         |
| C,04     | 63.97% | -       | 9.29%  | -       |         |
| C,05     | 57.35% | 2.68%   | 11.70% | 3.83%   |         |
| C,06     | 44.85% | -       | 45.31% | 45.39%  |         |
| C,07     | 51.70% | 3.45%   | 15.06% | 5.95%   |         |
| C,08     | 41.45% | -       | 42.66% | 42.65%  |         |
| C,09     | 59.58% | 59.53%  | 65.24% | 65.23%  |         |
| C,10     | -      | -       | 70.18% | -       |         |
| C,11     | 54.45% | 7.42%   | 11.96% | 7.00%   |         |
| C,12     | -      | -       | 45.51% | -       |         |

Table 2: Gap between the best lower bound proven by the solver in one hour and the best solution reported for the challenge after 90 min.
Table 3: Gap between the solution found by the solver in one hour and the best solution reported for the challenge after 90 min.