Cosymplectic and contact structures for time-dependent and dissipative Hamiltonian systems

M de León and C Sardón

Instituto de Ciencias Matemáticas, Campus Cantoblanco, Consejo Superior de Investigaciones Científicas, C/ Nicolás Cabrera, 13–15, 28049, Madrid, Spain

E-mail: cristinasardon@icmat.es

Received 14 November 2016, revised 21 April 2017
Accepted for publication 4 May 2017
Published 1 June 2017

Abstract

In this paper, we apply the geometric Hamilton–Jacobi theory to obtain solutions of classical hamiltonian systems that are either compatible with a cosymplectic or a contact structure. As it is well known, the first structure plays a central role in the theory of time-dependent hamiltonians, whilst the second is here used to treat classical hamiltonians including dissipation terms.

The interest of a geometric Hamilton–Jacobi equation is the primordial observation that if a hamiltonian vector field $X_H$ can be projected into a configuration manifold by means of a 1-form $dW$, then the integral curves of the projected vector field $X^{dW}_H$ can be transformed into integral curves of $X_H$ provided that $W$ is a solution of the Hamilton–Jacobi equation. In this way, we use the geometric Hamilton–Jacobi theory to derive solutions of physical systems with a time-dependent hamiltonian formulation or including dissipative terms. Explicit, new expressions for a geometric Hamilton–Jacobi equation are obtained on a cosymplectic and a contact manifold. These equations are later used to solve physical examples containing explicit time dependence, as it is the case of a unidimensional trigonometric system, and two dimensional nonlinear oscillators as Winternitz–Smorodinsky oscillators and for explicit dissipative behavior, we solve the example of a unidimensional damped oscillator.

Keywords: geometric Hamilton–Jacobi, cosymplectic geometry, contact geometry, dissipative terms, time dependent hamiltonian systems, Jacobi geometry
1. Introduction

In this paper we are concerned with almost cosymplectic structures and their application in classical Hamiltonian mechanics. By an almost cosymplectic structure we understand a $2n + 1$-dimensional manifold equipped with a one-form $\eta$ and a two-form $\Omega$ such that $\eta \wedge \Omega^n$ is a volume form. In particular, we will study two cases of almost cosymplectic manifolds. On one hand, the case of cosymplectic manifolds [5, 10, 42, 43], and on the other hand, the case of contact manifolds [5, 7, 17, 21]. Cosymplectic manifolds have shown their usefulness in theoretical physics, as in gauge theories of gravity, branes and string theory [4, 15, 25]. Among the early studies of cosymplectic manifolds we mention Lichnerowicz [45, 46], who studied the Lie algebra of infinitesimal automorphisms of a cosymplectic manifold, in analogy with the symplectic case. Posteriorly, some works have endowed cosymplectic manifolds with a Riemannian metric, the so-called coKähler manifolds [50]. These are the odd dimensional counterpart of Kähler manifolds. Another important role of the cosymplectic theory is the reduction theory to reduce time-dependent Hamiltonians by symmetry groups [2, 9, 16]. But since very foundational papers by Libermann, very sporadic papers have appeared on cosymplectic settings. It is our future intention to provide surveys on cosymplectic geometry due to their lack [10, 32]. Our particular interest in cosymplectic structures resides in their use in the description of time-dependent mechanics. They are present in numerous formulations of classical regular lagrangians [34], Hamiltonian systems [29] or Tulczyjew-like descriptions [36] in terms of lagrangian submanifolds [40].

However, there are more written monographs on contact geometry. The interest of contact structures roots in their applications in partial differential equations appearing in thermodynamics [55], geometric mechanics [26], geometric optics [13, 23, 24], geometric quantization [55] and applications to low dimensional topology, as it can be the characterization of Stein manifolds [53, 58]. Also, the theory of contact structures is linked to many other geometric backgrounds, as symplectic geometry, riemannian and complex geometry, analysis and dynamics [5, 21].

For both cosymplectic and contact approaches, the role of a vector field (said to be Hamiltonian) with a corresponding smooth function (the Hamiltonian function) with respect to its corresponding structure, is primordial to have dynamics. Furthermore, this vector field will be key in the construction of a geometric Hamilton–Jacobi theory. This theory has grown popular due to its simplicity and its equivalence to other theories of classical mechanics. It is based on a principal idea: a Hamiltonian vector field $X_H$ can be projected into the configuration manifold by means of a 1-form $\omega$, then the integral curves of the projected vector field $X_H^\omega$ can be transformed into integral curves of $X_H$ provided that $W$ is a solution of the Hamilton–Jacobi equation [3, 22, 27, 30, 44, 56]. In the last decades, the Hamilton–Jacobi theory has been interpreted in modern geometric terms [11, 12, 33, 48, 54] and has been applied in multiple settings: as nonholonomic [11, 12, 31, 38, 51, 52], singular lagrangian mechanics [39, 41] and classical field theories [37, 49]. The construction of a Hamilton–Jacobi theory often relies in the existence of lagrangian/legendrian submanifolds, a notion that has gained a lot of attention given its applications in dynamics since their introduction by Tulczyjew [60, 61]. We show how these submanifolds are a necessary condition for the obtainance of particular solutions through a geometric Hamilton–Jacobi equation. We will devise this fact by using particular cases of lagrangian/legendrian submanifolds in cosymplectic and contact geometry.

The paper is organized as follows: section 2 is dedicated to review fundamentals on geometric mechanics and the geometric Hamilton–Jacobi equation. In section 3, we recall some remarkable geometric structures of importance in mechanics. In particular, we focus on dynamics explained in geometric terms through contact and cosymplectic manifolds.
Section 4 contains the theory of lagrangian–legendrian submanifolds which will be key in the formulation of the Hamilton–Jacobi theory in subsequent sections. In section 5, we propose a geometric Hamilton–Jacobi theory on cosymplectic manifolds and illustrate our result with examples: a one-dimensional trigonometric system and two-dimensional nonlinear oscillators. One of the oscillators is the well-known Winternitz–Smorodinsky oscillator, for which we obtain an explicit expression for the solution $\gamma$ of the Hamilton–Jacobi equation on the cosymplectic manifold. Similarly, we devote section 6 to a geometric Hamilton–Jacobi equation on a contact manifold. We also illustrate our result through an example, a unidimensional damped oscillator with a dissipative term.

2. Geometric mechanics: fundamentals

We hereafter assume all mathematical objects to be $C^\infty$, globally defined and that all manifolds are connected. This permit us to omit technical details while highlighting the main aspects of our theory.

2.1. Hamiltonian mechanics

A classical hamiltonian system is given by a Hamilton function $H(q^i, p_i)$, where $(q^i)$ are the positions in a configuration manifold $Q$ and $(p_i)$ are the conjugated momenta, for $i = 1, \ldots, n$. The hamiltonian can be interpreted as total energy of the system $H = T + V$, where $H$ is a function on the cotangent bundle $T^*Q$ of $Q$. The vector field

$$X_H = \sum_{i=1}^{n} \left( \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i} \frac{\partial}{\partial p_i} \right)$$

(1)

is called a hamiltonian vector field and its integral curves $(q^i(t), p_i(t))$ satisfy the Hamilton equations

$$\begin{align*}
\dot{q}^i &= \frac{\partial H}{\partial p_i}, \\
\dot{p}_i &= -\frac{\partial H}{\partial q^i}
\end{align*}$$

(2)

for all $i = 1, \ldots, n$. We can define a Poisson bracket of two functions as

$$\{f, g\} = \sum_{i=1}^{n} \left( \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q^i} \right),$$

which is bilinear, skew symmetric and fulfills the Jacobi identity

$$\{f, \{g, h\}\} + \{f, \{h, g\}\} + \{h, \{f, g\}\} = 0, \quad \forall f, g, h \in C^\infty(Q).$$

The symplectic two-form is

$$\omega_Q = \sum_{i=1}^{n} dq^i \wedge dp_i$$

(3)

and we can rewrite the Hamilton equation (2) in a compact, geometric form

$$\iota_{X_H} \omega_Q = dH,$$

(4)

The pair $(T^*Q, \omega_Q)$ is the prototype for any symplectic manifold as we show in subsequent lines. A symplectic manifold is a pair $(M, \omega)$ such that the two-form $\omega$ is regular.
(that is, $\omega_n \neq 0$) and closed. Then, $M$ is even dimensional, say $2n$. The Darboux theorem states that given a symplectic manifold $(M, \omega)$ we can find Darboux coordinates $(q^i, p_i)$ around each point of $M$ such that the symplectic form is (3). Indeed, any symplectic manifold is locally equivalent to the cotangent bundle $T^* Q$ of a configuration manifold $Q$.

Given a configuration manifold $Q$, its cotangent bundle $T^* Q$ is the phase space. We consider the canonical projection $\pi_Q : T^* Q \rightarrow Q$. From the Poisson bracket, we can define a canonical two-contravariant tensor such that $\Lambda_Q(d f, d g) = \{ f, g \}$, for all $f, g \in C^\infty(T^* Q)$. This is what we call a Poisson bivector. In Darboux coordinates it reads

$$\Lambda_Q = \sum_{i=1}^{n} \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_i},$$

(5)

It is the contravariant version of the symplectic form (3). Furthermore, we consider the so-called Liouville form $\theta_Q = \sum_{i=1}^{n} p_i dq^i$ on $T^* Q$ such that $\omega_Q = -d \theta_Q$.

2.2. Hamilton–Jacobi equation

The Hamilton equation (2) can be equivalently be solved with the aid of the Hamilton–Jacobi theory. It consists of finding a principal function $S(t, q^i)$, that fulfills

$$\frac{\partial S}{\partial t} + H(q^i, \frac{\partial S}{\partial q^i}) = 0, \quad i = 1, \ldots, n$$

(6)

where $H = H(q^i, p_i)$ is the hamiltonian function of the system. Equation (6) is referred to as the Hamilton–Jacobi equation. If we set the principal function to be separable in time, $S = W(q^1, \ldots, q^n) - Et$, where $E$ is the total energy of the system, then (6) will now read [1, 22]

$$H\left(q^i, \frac{\partial W}{\partial q^i}\right) = E.$$  

(7)

The Hamilton–Jacobi equation is a useful instrument to solve the Hamilton equations for $H$. Indeed, if we find a solution $W$ of (7), then any solution of the Hamilton equations is retrieved by taking $p_i = \frac{\partial W}{\partial q^i}$, in case we can provide a complete solution.

Geometrically, the Hamilton–Jacobi theory can be reformulated as follows. Given a hamiltonian vector field $X_H : T^* Q \rightarrow T T^* Q$ and a one-form $d W$, we define the projected vector field $X_H^{\text{proj}} : Q \rightarrow TQ$. Then, the integral curves of $X_H^{\text{proj}}$ can be transformed into integral curves of $X_H$ provided that $W$ is a solution of (7). This explanation can be represented by the following diagram

This implies that $(d W)^* H = E$, with $d W$ a section of the cotangent bundle. In other words, we are looking for a section $\alpha$ of $T^* Q$ such that $\alpha^* H = E$. As it is well-known, the image of a one-form is a lagrangian submanifold of $(T^* Q, \omega_Q)$ if and only if $d \alpha = 0$ [1]. That is, $\alpha$ is locally exact, say $\alpha = d W$ on an open subset around each point.

1 By projected we do not refer to a projective vector field but to the restriction of a hamiltonian vector field on the phase space $T^* Q$ along the image of $d W$. 


3. Geometric structures and dynamics

A Jacobi structure is the triple \((M, \Lambda, Z)\), where \(Z\) is a vector field and \(\Lambda\) is a skew-symmetric bivector and such that they fulfill the following integrability conditions

\[
[\Lambda, \Lambda] = 2Z \wedge \Lambda, \quad \mathcal{L}_Z \Lambda = 0.
\]

If we drop the integrability conditions, we say we have an almost Jacobi manifold. The bivector \(\Lambda\) defines a Jacobi bracket [35]

\[
\{ f, g \} = \Lambda(df, dg) + fZ(g) - gZ(f), \quad f, g \in M
\]

that is skew-symmetric and satisfies the weaker Leibniz identity condition

\[
\text{supp}\{f, g\} \subseteq \text{supp}\{f\} \cap \text{supp}\{g\}.
\]

where supp refers to the support of a function in its usual topological meaning. This condition implies that the Jacobi bracket (9) is not a derivation in each argument, hence, the Leibniz derivation rule is weakened [45–47], but nevertheless, the Jacobi bracket satisfies the Jacobi identity if \((\Lambda, Z)\) is Jacobi. The space \(C^\infty(M, \mathbb{R})\) is a local Lie algebra in the Kirillov sense [28].

Consider the morphism \(\sharp_\Lambda : \Omega^1(M) \to \mathfrak{X}(M)\) that is the \(C^\infty(M, \mathbb{R})\) linear mapping induced by \(\Lambda\) between the \(C^\infty\) modules of one-forms \(\Omega^1(M)\) and vector fields \(\mathfrak{X}(M)\) defined on \(M\). The sharp-lambda morphism \(\sharp_\Lambda\) provides the pairing and components of the bivector \(\langle \sharp_\Lambda(\alpha), \beta \rangle = \Lambda(\alpha, \beta)\), for \(\alpha, \beta\) one-forms in \(\Omega^1(M)\). Vector fields associated with functions \(f\) on the algebra of smooth functions \(C^\infty(M, \mathbb{R})\) are defined as

\[
X_f = \sharp_\Lambda(df) + fZ.
\]

This vector field is said to be a Hamiltonian vector field with respect to a Jacobi structure.

The characteristic distribution \(\mathcal{C}\) of \((M, \Lambda, Z)\) is a subset of \(TM\) generated by the values of all the vector fields \(X_f\). This characteristic distribution \(\mathcal{C}\) is defined by \(\Lambda\) and \(Z\), that is,

\[
\mathcal{C}_p = \sharp_\Lambda(T_pM) + <Z_p>, \quad \forall p \in M
\]

where \(\sharp_\Lambda : T_pM \to TM\) is the restriction of \(\sharp_\Lambda\) to \(T_pM\) for every \(p \in M\) and \(<Z_p>\) denotes the subspace generated by the vector field \(Z_p\). Then, \(\mathcal{C}_p = \mathcal{C} \cap T_pM\) is the vector subspace of \(T_pM\) generated by \(Z_p\) and the image of the linear mapping \(\sharp_p\). The distribution is said to be transitive if the characteristic distribution is the whole tangent bundle \(TM\).

**Definition 1.** Given two Jacobi manifolds \((M_1, \Lambda_1, Z_1)\) and \((M_2, \Lambda_2, Z_2)\) we say that the map \(\phi : M_1 \to M_2\) is a Jacobi map if given two functions \(f, g \in C^\infty(M_2)\),

\[
\{ f \circ \phi, g \circ \phi \}_{M_1} = \{ f, g \}_{M_2} \circ \phi.
\]

A Jacobi manifold \((M, \Lambda, Z)\) is said to be be Poisson when \(Z = 0\); in that case we write \((M, \Lambda)\) instead of \((M, \Lambda, 0)\).

Next, we shall show several examples of Poisson and Jacobi manifolds.

3.1. Symplectic manifolds

Consider a pair \((M, \Omega)\), where \(\Omega\) is a symplectic two-form. We define the map

\[
b : TM \to T^*M \quad \text{such that} \quad b(X) = \iota_X \Omega
\]

which is an isomorphism. We can define its inverse as \(\sharp : T^*M \to TM\). The bracket

\[
\{ f, g \} = \Omega(\sharp(df), \sharp(dg)) = \langle dg, \sharp(df) \rangle = -\langle df, \sharp(dg) \rangle
\]
satisfies the Jacobi identity. The Hamiltonian vector field is \( X_\beta = \sharp(\beta) \). The pair \((M, \Lambda)\) is a Poisson manifold of necessarily even dimension with Poisson tensor given by

\[
\Lambda(\alpha, \beta) = \Omega(\sharp(\alpha), \sharp(\beta))
\]

for \( \alpha, \beta \) one-forms. In this case \( \sharp = \sharp_\Lambda \) and \( \sharp = \sharp^{-1} \).

### 3.2. Locally conformally symplectic structures

An almost symplectic manifold is a pair \((M, \Omega)\) where \( \Omega \) is a nondegenerate two-form and \( M \) is even dimensional. An almost symplectic manifold is said to be locally conformally symplectic if for each point \( x \in M \) there is an open neighborhood \( U \) such that \( d(e^s \Omega) = 0 \), for \( \sigma : U \to \mathbb{R} \), so \((U, e^s \Omega)\) is a symplectic manifold. If \( U = M \), then it is said to be globally conformally symplectic. An almost symplectic manifold is a locally (globally) conformally symplectic if there exists a one-form \( \eta \) that is closed \( d\eta = 0 \) and

\[
d\Omega = \eta \wedge \Omega.
\]

The one-form \( \eta \) is called the Lee one-form. Locally conformally symplectic manifolds (L.C.S.) with Lee form \( \eta = 0 \) are symplectic manifolds. We define a bivector \( \Lambda \) on \( M \) and a vector field \( Z \) given by

\[
\Lambda(\alpha, \beta) = \Omega(\sharp^{-1}(\alpha), \sharp^{-1}(\beta)) = \Omega(\sharp(\alpha), \sharp(\beta)), \quad Z = \sharp^{-1}(\eta)
\]

with \( \alpha, \beta \in \Omega^1(M) \) and \( \beta : \mathcal{X}(M) \to \Omega^1(M) \) is the isomorphism of \( C^\infty(M, \mathbb{R}) \) modules defined by \( b(X) = \iota_X \Omega \). Here \( \sharp = \sharp^{-1} \). In this case, we also have \( \sharp_\Lambda = \sharp \). The vector field \( Z \) satisfies \( i_Z \eta = 0 \) and \( \mathcal{L}_Z \Omega = 0 \). Then, \((M, \Lambda, Z)\) is an even dimensional Jacobi manifold.
Cosymplectic

\( (\mathcal{M}, \Omega, \eta) \)

\[ \theta = \sum_{i=1}^{n} p_i dq^i - H dt \quad \mathcal{R} = \frac{\partial}{\partial q^i} + \sum_{i=1}^{n} \frac{\partial H}{\partial p_i} \frac{\partial}{\partial p_i} \]

\[ \Lambda = \sum_{i=1}^{n} \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_i} \quad X_H = \mathcal{R}_H \]

Contact

\( (\mathcal{M}, \eta) \)

\[ \eta = dt - \sum_{i=1}^{n} p_i dq^i \quad \mathcal{R} = \frac{\partial}{\partial \eta} \]

\[ \Lambda = \sum_{i=1}^{n} \left( \frac{\partial}{\partial q^i} + p_i \frac{\partial}{\partial p_i} \right) \wedge \frac{\partial}{\partial t} \quad X_H = \sum_{i=1}^{n} \left( p_i \frac{\partial}{\partial q^i} - H \right) \frac{\partial}{\partial t} \]

\[ \Omega = d\eta \quad - \sum_{i=1}^{n} \left( p_i \frac{\partial}{\partial q^i} + \frac{\partial}{\partial t} \right) \frac{\partial}{\partial \eta} + \frac{\partial}{\partial \eta} \wedge \frac{\partial}{\partial \eta} \]

There is a classical Darboux theorem that states the following. Around every point \( x \in \mathcal{M} \), there exist coordinates and a local function \( \sigma \) such that \( d\sigma = \eta \). Then,

\[ \Omega = e^\sigma \sum_{i=1}^{n} dq^i \wedge dp_i, \quad \eta = d\sigma = \sum_{i=1}^{n} \left( \frac{\partial \sigma}{\partial q^i} dq^i + \frac{\partial \sigma}{\partial p_i} dp_i \right) \]

and, in consequence, we have

\[ \Lambda = e^{-\sigma} \sum_{i=1}^{n} \left( \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_i} \right), \quad Z = e^{-\sigma} \sum_{i=1}^{n} \left( \frac{\partial \sigma}{\partial p_i} \frac{\partial}{\partial q^i} - \frac{\partial \sigma}{\partial q^i} \frac{\partial}{\partial p_i} \right) \]

The Hamiltonian vector field corresponding with the function \( f \) is

\[ X_f = i_\Lambda (df) + f Z \]

We say that this vector field is Hamiltonian with respect to a L.C.S. structure.

### 3.3. Almost cosymplectic structure

An almost cosymplectic manifold is a \( 2n + 1 \)-dimensional manifold \( M \) equipped with \( (\eta, \Omega) \), where \( \eta \) is a one-form and \( \Omega \) is a two-form such that \( \eta \wedge \Omega^n \neq 0 \). Therefore, we have an isomorphism of \( C^\infty \)-modules \( \mathfrak{b} : \mathfrak{X}(M) \rightarrow \Omega^1(M) \) defined by

\[ b(X) = i_X \Omega + \eta(X) \eta. \]

**Theorem 2.** If \( (\mathcal{M}, \eta, \Omega) \) is an almost cosymplectic structure, then there exists a unique vector field \( \mathcal{R} \), the so-called Reeb vector field such that

\[ \iota_\mathcal{R} \eta = 1, \quad \iota_\mathcal{R} \Omega = 0. \]

In other words, \( \mathcal{R} = b^{-1}(\eta) \). Now, we will consider two particular classes of almost cosymplectic manifolds.

#### 3.3.1. Cosymplectic structure

An almost cosymplectic structure \( (\mathcal{M}, \Omega, \eta) \) is a cosymplectic structure if \( d\eta = 0, \Omega^n = 0 \); recall that \( \Omega^n \wedge \eta \neq 0 \). A cosymplectic manifold is equipped with the \( b \) isomorphism in (19) and the Reeb vector field is retrieved as \( \mathcal{R} = b^{-1}(\eta) = \mathcal{Z}(\eta) \) and satisfies (20). In this case \( \mathcal{Z} = b^{-1} \). A cosymplectic manifold is a particular case of odd dimensional Poisson manifold. It gives rise to a Poisson bivector given by
There exist Darboux coordinates \{t, q^i, p_i\} on \(T^*Q \times \mathbb{R}\) with \(i = 1, \ldots, n\) such that

\[
\Omega = \sum_{i=1}^{n} dq^i \wedge dp_i, \quad \eta = dt, 
\]

and therefore,

\[
\mathcal{R} = \frac{\partial}{\partial t}, \quad \Lambda = \sum_{i=1}^{n} \frac{\partial}{\partial q_i} \wedge \frac{\partial}{\partial p_i}. 
\]

To define a Poisson structure, we consider the bivector \(\Lambda(df, dg) = df(\sharp(\Lambda(dg))) = \{f, g\}\).

In this case, \(\sharp \Lambda \neq \sharp\).

### 3.3.2. Contact structure

An almost cosymplectic structure \((M, \Omega, \eta)\) is a contact structure if \(\Omega = d\eta\). From here, we refer to \(\eta\) as a contact form and \((M, \eta)\) a contact manifold. It is satisfied that \(\eta \wedge (d\eta)^n \neq 0\) for all \(x \in M\).

A contact manifold is a Jacobi manifold whose associated bivector \(\Lambda\) is given by

\[
\Lambda(\alpha, \beta) = d\eta(\flat^{-1}(\alpha), \flat^{-1}(\beta)) = d\eta(\sharp(\alpha), \sharp(\beta))
\]

for all \(\alpha, \beta \in \Omega^1(M)\) and \(\flat: \mathfrak{X}(M) \to \Omega^1(M)\) is the isomorphism given by

\[
\flat(X) = \iota_X d\eta + \eta(X)\eta.
\]

Here \(\sharp = \flat^{-1}\). The Reeb vector field is \(\mathcal{R} = \flat^{-1}(\eta) = \sharp(\eta)\) and it is the unique vector field that satisfies (20). Then \((M, \Lambda, \mathcal{R})\) is a Jacobi manifold. Then, the bracket on a contact manifold is defined as

\[
\{f, g\} = \Lambda(df, dg) + f\mathcal{R}(g) - g\mathcal{R}(f).
\]

There exist coordinates \(\{t, q^i, p_i\}\), with \(i = 1, \ldots, n\), such that

\[
\eta = dt - \sum_{i=1}^{n} p_i dq^i, \quad \Lambda = \sum_{i=1}^{n} \left(\frac{\partial}{\partial q^i} + p_i \frac{\partial}{\partial t}\right) \wedge \frac{\partial}{\partial p_i}, \quad \mathcal{R} = \frac{\partial}{\partial t}.
\]

In this case, \(\sharp \Lambda = \sharp\).

### 3.4. Structure theorem for Jacobi manifolds

**Theorem 3.** The characteristic distribution of a Jacobi manifold \((M, \Lambda, Z)\) is completely integrable in the sense of Stefan–Sussmann [57, 59], thus \(M\) defines a foliation whose leaves are not necessarily of the same dimension, and it is called a characteristic foliation. Each leaf has a unique transitive Jacobi structure such that its canonical injection into \(M\) is a Jacobi map. Each leaf defines

1. A locally conformally symplectic manifold if the dimension is even.
2. A manifold equipped with a contact one-form if its dimension is odd.

See [45, 46] for proof of this theorem.

**Remark 4.** Tables 1 and 2 provide a summary of the different classes of Jacobi and Poisson manifolds.
4. Lagrangian–legendrian submanifolds

Let \((M, \Lambda, Z)\) be a Jacobi manifold with characteristic distribution \(C\).

**Definition 5.** A submanifold \(N\) of a Jacobi manifold \((M, \Lambda, Z)\) is said to be a lagrangian-legendrian submanifold if the following equality holds
\[
\sharp(TN^\circ) = TN \cap C,
\]
where \(TN^\circ\) denotes the annihilator of \(TN\).

If \((M, \Lambda)\) is a Poisson manifold, the lagrangian-legendrian submanifold of \(M\) will simply be called lagrangian.

4.1. Particular cases

1. A submanifold \(N\) of a symplectic manifold \((M, \Omega)\) is lagrangian if
\[
\sharp_{\Lambda}(TN^\circ) = TN.
\]

2. As a consequence, we deduce that a submanifold \(N\) of a cosymplectic manifold \((M, \eta, \Omega)\) is lagrangian if
\[
\sharp_{\Lambda}(TN^\circ) = TN \cap C.
\]

3. A submanifold \(N\) of a contact manifold \((M, \eta)\) is legendrian if the following condition is fulfilled
\[
\sharp_{\Lambda}(TN^\circ) = TN.
\]

The following result gives a characterization of legendrian submanifolds of contact manifolds.

**Proposition 6.** A submanifold \(N\) of a contact manifold \((M, \eta)\) is a legendrian submanifold if and only if it is an integral manifold of maximal dimension \(n\) of the distribution \(\eta = 0\). In this case, \(C\) is the whole tangent space to \(N\).

**Proof.** Assume that \(M\) has dimension \(2n + 1\). If a submanifold \(N\) of \(M\) is legendrian then the condition
\[
\sharp_{\Lambda}(TN^\circ) = TN
\]
implies that \(\eta|_{N} = 0\). Moreover, \(N\) has necessary dimension \(n\), since \(T_{x}N\) will be a lagrangian subspace of the symplectic vector space \((\ker \eta, (d\eta)_{x})\) for all \(x \in N\). The converse is proved reversing the arguments.

5. Hamilton–Jacobi theory on cosymplectic manifolds

5.1. Geometric approach

Consider the extended phase space \(T^*Q \times \mathbb{R}\) and its canonical projections of the first and second factor, \(\rho : T^*Q \times \mathbb{R} \to T^*Q\) and \(t : T^*Q \times \mathbb{R} \to \mathbb{R}\), respectively and a time-dependent hamiltonian \(H : T^*Q \times \mathbb{R} \to \mathbb{R}\). Let us depict the problem with a diagram.
Here \( t \) is an abuse of notation in which it is understood as a coordinate and as a coordinate function (a projection).

We have canonical coordinates \( \{ q^i, p_i, t \} \) with \( i = 1, \ldots, n \), where \( (q^i, p_i) \) are fibered coordinates in \( T^* Q \) and \( t \in \mathbb{R} \). We consider the two-form on \( T^* Q \times \mathbb{R} \) as \( \Omega_H = -d\theta_H \) and

\[
\theta_H = \theta_Q - H dt \tag{32}
\]

where \( \theta_Q \) is the canonical Liouville one-form. We abuse notation by identifying the pullbacks of the one-forms with the one-forms themselves. That is, \( \rho^*(\theta_Q) = \theta_Q \). Hence,

\[
\Omega_H = \sum_{i=1}^n dq^i \wedge dp_i + dH \wedge dt. \tag{33}
\]

Let us consider the cosymplectic structure \( (dt, \Omega_H) \). The corresponding Reeb vector field needs to satisfy

\[
\iota_{\mathcal{R}_H} dt = 1, \quad \iota_{\mathcal{R}_H} \Omega_H = 0. \tag{34}
\]

The unique Reeb vector field that satisfies (34) has the following expression in coordinates

\[
\mathcal{R}_H = \frac{\partial}{\partial t} + \sum_{i=1}^n \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q^i} - \sum_{i=1}^n \frac{\partial H}{\partial q^i} \frac{\partial}{\partial p_i}. \tag{35}
\]

The corresponding classical Hamilton–Jacobi equations are

\[
\begin{cases}
q^i = \frac{\partial H}{\partial p_i}, \\
p_i = -\frac{\partial H}{\partial q^i}, \\
i = 1.
\end{cases} \tag{36}
\]

Since \( i = 1 \), we can consider \( t \) a time-parameter (up to an affine change). We consider the fibration \( \pi : T^* Q \times \mathbb{R} \to Q \times \mathbb{R} \) and a section \( \gamma \) of \( \pi : T^* Q \times \mathbb{R} \to Q \times \mathbb{R} \), i.e. \( \pi \circ \gamma = \text{id}_{Q \times \mathbb{R}} \). Also, we assume that \( \text{Im}(\gamma_t) \) with \( \gamma_t : Q \to T^* Q \times \mathbb{R} \) such that \( \gamma_t(q^i) \) in coordinates \( (q^i, \gamma_t(q^i), t) \) is a lagrangian submanifold for a fixed time \( t \) of the cosymplectic manifold \( (T^* Q \times \mathbb{R}, dt, \Omega_H) \) for a fixed time, that is \( d\gamma_t = 0 \).

We can use \( \gamma \) to project \( \mathcal{R}_H \) on \( Q \times \mathbb{R} \) just defining a vector field \( \mathcal{R}_H^\gamma \), the denominated projected vector field on \( Q \times \mathbb{R} \) by

\[
\mathcal{R}_H^\gamma = T\pi \circ \mathcal{R}_H \circ \gamma \tag{37}
\]

The following diagram summarizes the above construction
**Definition 7.** If \( \alpha \) is a one-form, locally expressed as \( \alpha = \sum_{i=1}^{n} \alpha_i dq^i \), we designate by \( \alpha^V \) the vertical lift on \( Q \) [63] of \( \alpha \) to \( T^*Q \) defined by

\[
\iota_{\alpha^V} \omega_Q = \alpha
\]

Hence, the vector field \( \alpha^V \) has the local expression

\[
\alpha^V = -\sum_{i=1}^{n} \alpha_i \frac{\partial}{\partial p_i}.
\] (38)

**Theorem 8.** The vector fields \( \mathcal{R}_H \) and \( \mathcal{R}_H^{\gamma} \) are \( \gamma \)-related if and only if the following equation is satisfied

\[
[d(H \circ \gamma)]^V = \dot{\gamma}_q
\] (39)

where \([\ldots]^V\) denotes the vertical lift of a one-form on \( Q \) to \( T^*Q \). Now \( \dot{\gamma}_q \) is the tangent vector in a point \( q \) associated with the curve

\[
\gamma_q : \mathbb{R} \rightarrow Q \times \mathbb{R} \rightarrow T^*Q \times \mathbb{R} \xrightarrow{\mathcal{L}} T^*Q
\]

Notice that these applications are given for a fixed point \( q \rightarrow (q,t,\gamma) \).

**Proof.** The vector fields \( \mathcal{R}_H \) and \( \mathcal{R}_H^{\gamma} \) are \( \gamma \)-related if \( T\gamma(\mathcal{R}_H^{\gamma}) = \mathcal{R}_H \). That is,

\[
T\gamma(\mathcal{R}_H^{\gamma}) = T\gamma \left( \frac{\partial}{\partial t} + \sum_{i=1}^{n} \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q^i} \right)
\] (40)

We choose a section \( \gamma(q',t) \) in coordinates \( (q',\gamma^j(q',t),t) \) with \( i,j = 1, \ldots, n \) such that the lift in the tangent bundle reads,

\[
T\gamma \left( \frac{\partial}{\partial t} \right) = \frac{\partial}{\partial t} + \sum_{j=1}^{n} \frac{\partial \gamma^j}{\partial t} \frac{\partial}{\partial p_j}, \quad T\gamma \left( \frac{\partial}{\partial q^j} \right) = \frac{\partial}{\partial q^j} + \sum_{j=1}^{n} \frac{\partial \gamma^j}{\partial q^j} \frac{\partial}{\partial p_j}
\] (41)

Introducing equations (41) in (40), it is straightforward to retrieve condition (39) if we use that \( \gamma_t \) is closed. The closedness condition is necessary for the permutation of indices in intermediate steps to obtain (39).

Equation (39) is known as a Hamilton–Jacobi equation on a cosymplectic manifold. In local coordinates \( (q',p_i,t) \), we have

\[
\frac{\partial \gamma^j}{\partial t} + \sum_{i=1}^{n} \frac{\partial H}{\partial p_i} \frac{\partial \gamma^j}{\partial q^i} + \frac{\partial H}{\partial q^j} = 0.
\] (42)
This equation retrieves (6) through the following arguments. In principle, time is frozen, that implies that \( \gamma(q', t) = (q', \gamma(q', t), t) \) is considered as a frozen \( \gamma \) in time, \( \gamma_t(q') = \gamma(q', t) \).

In this situation, the usual condition of closedness for \( \gamma \), that is \( d\gamma = 0 \) is no longer true, but \( d\gamma_t = 0 \), for a fixed time. Hence, \( \gamma_t \) is closed. By the lemma of Poincare [6], we can express \( \gamma_t = \frac{\partial}{\partial t} S(q', t) = S_t(q') \) for a fixed time. If we differentiate expression (6) with respect to \( q^j \) for a fixed time \( t \) and use the lemma of Poincare, we recover expression (42).

5.2. Complete solutions

**Definition 9.** A complete solution of the Hamilton–Jacobi equation for a time-dependent system on the cosymplectic manifold \((T^*Q \times \mathbb{R}, \eta, \Omega)\) is a diffeomorphism \( \Phi : \mathbb{R}^n \to T^*Q \times \mathbb{R} \) such that for a set of parameters \( \lambda \in \mathbb{R}^n \), \( \lambda = (\lambda_1, \ldots, \lambda_n) \), the mapping

\[
\Phi_{\lambda} : Q \times \mathbb{R} \to T^*Q \times \mathbb{R} \\
\Phi_{\lambda}(q, t) \mapsto \Phi_{\lambda}(q, \gamma(q, t), t)
\]

is a solution of the Hamilton–Jacobi equation.

We have the following diagram

\[
\begin{array}{ccc}
Q \times \mathbb{R} \times \mathbb{R}^n & \xrightarrow{\Phi} & T^*Q \times \mathbb{R} \\
\downarrow \alpha & & \downarrow f_i \\
\mathbb{R}^n & \xrightarrow{\pi_i} & \mathbb{R}
\end{array}
\]

with \( \pi_i : \mathbb{R}^n \to \mathbb{R} \) the projection of \((\lambda_1, \ldots, \lambda_n)\) to \( \lambda_i \). We define functions \( f_i \) such that for a point \( p \in T^*Q \times \mathbb{R} \), it is satisfied

\[
f_i(p) = \pi_i \circ \alpha \circ \Phi_{\lambda}^{-1}(p).
\]

(44)

and \( \alpha : Q \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n \) is the canonical projection.

**Theorem 10.** If \( \Phi \) is a complete solution of the Hamilton–Jacobi problem on \( T^*Q \times \mathbb{R} \), then the functions defined in (44) commute with respect to a Poisson bracket, that is,

\[
\{f_i, f_j\} = 0, \quad \forall i, j = 1, \ldots, n.
\]

(45)

**Proof.** It is immediate that

\[
\text{Im}(\Phi_{\lambda}) = \bigcap_{i=1}^n f_i^{-1}(\lambda_i)
\]

An element in \( \text{Im}(\Phi_{\lambda}) \) will be \( \Phi_{\lambda}(x) \), for a point \( x \in Q \times \mathbb{R} \) and it happens

\[
f_i(\Phi_{\lambda}(x)) = f_i(\Phi(x, \lambda)) = \lambda_i.
\]

If \( f_i \) is constant on \( \text{Im}(\Phi_{\lambda}) \), then \( df_i \) vanishes on \( T(\text{Im}(\Phi_{\lambda})) \). If \( \Phi_{\lambda} \) is a solution of the Hamilton–Jacobi equation, then \( \text{Im}\Phi_{\lambda} \) is a lagrangian submanifold and we have that

\[
\sharp_{\lambda}(T(\text{Im}(\Phi_{\lambda})))^\circ = T(\text{Im}(\Phi_{\lambda})) \cap C
\]

that implies
\{ f_i, f_j \} = 0, \quad \forall i, j = 1, \ldots, n. \quad (47)

Because of the definition of the bracket \( \{ f_i, f_j \} = df_i(\partial_{\lambda}(df_j)) \) and the definition of \( \Lambda \) in (22), we deduce that it is zero since \( df_i \) is in the annihilator of \( T(\text{Im}(\Phi_\lambda)) \) and \( \partial_{\lambda}(df_j) \) is in \( T(\text{Im}(\Phi_\lambda)) \cap \mathcal{C} \).

5.3. Examples

5.3.1. A trigonometric system

Let us consider the time-dependent hamiltonian on \( T^*Q \times \mathbb{R} \) with the local set of coordinates \( \{ q, p, t \} \)

\[
H = \frac{p^2}{2} + \frac{q^2}{2} + \alpha \sin (wt) \frac{q^2p^2}{2}. \quad (48)
\]

In our setting, we consider the cosymplectic manifold \( (T^*Q \times \mathbb{R}, \Omega_H, \theta_H) \) where \( \Omega_H \) and \( \theta_H \) are those given in (33) in (32), correspondingly. The Reeb vector field according to the conditions (20) has the expression

\[
R_H = \partial_{\partial_t} + (p + \alpha \sin (wt)q^2p) \frac{\partial}{\partial q} - (q + \alpha \sin (wt)p^2q) \frac{\partial}{\partial p}, \quad (49)
\]

We choose a lagrangian section \( \gamma(q, t) \) whose components are \( \langle q, \gamma(q, t), t \rangle \). The \( R_H^\gamma \) field on \( Q \times \mathbb{R} \) is

\[
R_H^\gamma = \frac{\partial}{\partial t} + (p + \alpha \sin (wt)q^2p) \frac{\partial}{\partial q}. \quad (50)
\]

If we impose (37) to be fulfilled, we need to compute the terms

\[
T_\gamma \left( \frac{\partial}{\partial t} \right) = \frac{\partial}{\partial t} + \frac{\partial \gamma}{\partial t} \frac{\partial}{\partial p}, \quad T_\gamma \left( \frac{\partial}{\partial q} \right) = \frac{\partial}{\partial q} + \frac{\partial \gamma}{\partial q} \frac{\partial}{\partial p}, \quad (51)
\]

and the arising equation reads

\[
\frac{\partial \gamma}{\partial t} + (p + \alpha \sin (wt)q^2p) \frac{\partial \gamma}{\partial q} = - (q + \alpha \sin (wt)p^2q). \quad (52)
\]

This equation is a quasi-linear first-order PDE for a function \( \gamma(q, t) \). It can be solved with the aid of the method of characteristics [18]

\[
dt = \frac{dq}{p + \alpha \sin (wt)q^2p} = - \frac{d\gamma}{q + \alpha \sin (wt)p^2q} \quad (53)
\]

which turns in the following system of equations

\[
\frac{dq}{dt} = p(1 + \alpha \sin (wt)q^2), \quad \frac{d\gamma}{dt} = -q(1 + \alpha \sin (wt)p^2). \quad (54)
\]

Integrating the equations along the section \( \gamma \), we have that \( p = \gamma \), then, we can solve system (54) whose solutions result in

\[
\gamma(q, t) = \frac{C - q^2}{1 + \alpha \sin (wt)q^2} \quad (55)
\]

where \( C \) is a constant of integration.
Equation (55) is a particular solution of the Hamilton–Jacobi equation corresponding with a nonlinear trigonometric system on a cosymplectic manifold.

For the complete solution, we need $\gamma(q,t)$ expressed in terms of one single parameter $C$ as in (55), according to the theory explained in (43). We construct the diffeomorphism that provides the complete solution

$$\Phi : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow T^* \mathbb{R} \times \mathbb{R}
$$

such that

$$\Phi(q, \gamma(q,t), C) = \left( q, \gamma(q,t), \frac{C - q^2}{1 + \alpha \sin (w t) q^2} \right)$$

(57)

5.3.2. Nonlinear oscillators

Fris et al [20] studied in 1965 systems that admit separability in two different coordinates and obtained families of superintegrable potentials with constants of motion linear or quadratic in the velocities (momenta). The two first families can be considered as the more general euclidean deformations [14] with strengths $k_2$ and $k_3$

$$V_a = \frac{1}{2} \omega_0(t)^2(x^2 + y^2) + \frac{k_2}{x^2} + \frac{k_3}{y^2}, \quad V_b = \frac{1}{2} \omega_0(t)^2(4x^2 + y^2) + k_2x + k_3$$

(58)

of the 1 : 1 and 2 : 1 harmonic oscillators preserving quadratic superintegrability. The superintegrability of $V_a$ is known as Winternitz–Smorodinsky oscillator [62] studied by Evans [19] for the general case of degrees of freedom. For models in two dimensions, we have to choose two sections $\gamma(x,t), \gamma(y,t) : Q \times \mathbb{R} \rightarrow T^* Q \times \mathbb{R}$ as a solution of the Hamilton–Jacobi equation.

(1) For the first potential $V_a$, the hamiltonian reads

$$H = \frac{1}{2} p_x^2 + \frac{1}{2} p_y^2 + \frac{1}{2} \omega_0(t)^2(x^2 + y^2) + \frac{k_2}{x^2} + \frac{k_3}{y^2}.$$ (59)

In this case, the Reeb vector field according to (20) is

$$R_H = \frac{\partial}{\partial t} + p_x \frac{\partial}{\partial x} + p_y \frac{\partial}{\partial y} - \left( \omega_0(t)^2 x - \frac{2k_2}{x^2} \right) \frac{\partial}{\partial p_x} - \left( \omega_0(t)^2 y - \frac{2k_3}{y^2} \right) \frac{\partial}{\partial p_y}.$$ (60)

The projected vector field on $Q \times \mathbb{R}$ is

$$R^\gamma_H = \frac{\partial}{\partial t} + p_x \frac{\partial}{\partial x} + p_y \frac{\partial}{\partial y}.$$ (61)

Now,

$$T^\gamma(R^\gamma_H) = \frac{\partial}{\partial t} + \frac{\partial \gamma^x}{\partial t} \frac{\partial}{\partial p_x} + \frac{\partial \gamma^y}{\partial t} \frac{\partial}{\partial p_y} + p_x \left( \frac{\partial}{\partial x} + \frac{\partial \gamma^x}{\partial x} \frac{\partial}{\partial p_x} \right) + p_y \left( \frac{\partial}{\partial y} + \frac{\partial \gamma^y}{\partial y} \frac{\partial}{\partial p_y} \right)$$

M de León and C Sardón

J. Phys. A: Math. Theor. 50 (2017) 255205
that compared with $\mathcal{R}_H$, it results in the Hamilton–Jacobi equations

$$
\gamma^{[i]}_t + \frac{1}{m} \gamma^{[i]}_y \gamma^{[i]}_x = -\left(\omega_0(t)^2 x - \frac{2k_2}{x^3}\right),
$$

$$
\gamma^{[i]}_y + \frac{1}{m} \gamma^{[i]}_x \gamma^{[i]}_y = -\left(\omega_0(t)^2 y - \frac{2k_3}{y^3}\right).
$$

The two equations for the sections $\gamma^{[i]}$ and $\gamma^{[b]}$ are the same. They are quasilinear PDEs that can be solved with the method of the characteristics. So, the associated characteristic system for $\gamma^{[i]}$ is

$$
dt = \frac{dx}{\gamma^{[i]}_x} = -\frac{d\gamma^{[i]}_y}{(\omega_0(t)^2 x - \frac{2k_2}{x^3})}.
$$

If we need $\gamma^{[i]}$ in terms of $(x, t)$, we have that

$$
\gamma^{[i]} = \sqrt{-\omega_0(t)^2 x^2 - 2k_2 x^2 + C}
$$

including one parameter $C$. For $\gamma^{[b]}$ we obtain equivalent expressions to (64),

$$
\gamma^{[b]} = \sqrt{-\omega_0(t)^2 y^2 - 2k_2 y^2 + K}
$$

The sections $\gamma^{[i]}$ and $\gamma^{[b]}$ are a solution of the Hamilton–Jacobi equation corresponding with a nonlinear oscillator with potential $V_a$. To contemplate the complete solutions, we make use of (64) in terms of the parameter $C$. We construct a diffeomorphism

$$
\Phi : \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^2 \rightarrow \mathbb{T}^* \mathbb{R}^2 \times \mathbb{R}
$$

such that

$$
\Phi(x, y, t, C, K) \mapsto (x, y, \gamma^{[i]}(C), \gamma^{[b]}(K), t)
$$

(2) For the second potential $V_b$, the hamiltonian on $\mathbb{T}^* Q \times \mathbb{R}$ reads

$$
H = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2} \omega^2(4x^2 + y^2) + k_2 x + \frac{k_3}{y^3}.
$$

The Reeb vector field (20) on $\mathbb{T}^* Q \times \mathbb{R}$ for this example reads

$$
\mathcal{R}_H = \frac{\partial}{\partial t} + p_x \frac{\partial}{\partial x} + p_y \frac{\partial}{\partial y} - (4\omega(t)^2 x + k_2) \frac{\partial}{\partial p_x} \left(\omega_0(t)^2 y - \frac{2k_3}{y^3}\right) \frac{\partial}{\partial p_y}
$$

and the projected Reeb vector field on $Q \times \mathbb{R}$ is

$$
\mathcal{R}_H^\gamma = \frac{\partial}{\partial t} + p_x \frac{\partial}{\partial x} + p_y \frac{\partial}{\partial y}
$$

Now,

$$
T\gamma(\mathcal{R}_H^\gamma) = \frac{\partial}{\partial t} + \frac{\partial \gamma^{[i]}_y}{\partial t} \frac{\partial}{\partial p_x} + \frac{\partial \gamma^{[b]}_y}{\partial t} \frac{\partial}{\partial p_y}
$$

$$
+ \frac{p_x}{m} \left(\frac{\partial}{\partial x} + \frac{\partial \gamma^{[i]}_x}{\partial x} \frac{\partial}{\partial p_x} + \frac{\partial \gamma^{[b]}_x}{\partial x} \frac{\partial}{\partial p_y}\right)
$$

$$
+ \frac{p_y}{m} \left(\frac{\partial}{\partial y} + \frac{\partial \gamma^{[i]}_y}{\partial y} \frac{\partial}{\partial p_x} + \frac{\partial \gamma^{[b]}_y}{\partial y} \frac{\partial}{\partial p_y}\right)
$$
whose difference with respect to \( R_H \) gives us the Hamilton–Jacobi equations for the sections \( \gamma^{[i]} \) and \( \gamma^{[j]} \). These are

\[
\gamma^{[i]}_t + \frac{1}{m} \gamma^{[i]} \gamma^{[i]}_x = - \left( 4x \omega_0(t)^2 + k_2 \right),
\]

\[
\gamma^{[j]}_t + \frac{1}{m} \gamma^{[j]} \gamma^{[j]}_y = - \left( \omega_0(t)^2 y - \frac{2k_3}{y^3} \right).\]

(69)

The equation for \( \gamma^{[j]} \) in (69) has the same solution as (62), that is equation (64),

\[
\gamma^{[j]} = \sqrt{-\omega_0(t)^2 y^2 - 2k_3 y^2 + K}.
\]

(70)

The equation for \( \gamma^{[i]} \) in (69) can be solved by proposing the associated characteristic system

\[
dt = \frac{dx}{\pi} = -\frac{d\gamma}{(4x \omega_0(t)^2 + k_2)}.
\]

(71)

If we want to express \( \gamma^{[i]} \) in terms of \((x,t)\), we have

\[
\gamma^{[i]} = m \sqrt{-\left( \frac{4 \omega_0(t)^2}{m} x^2 + \frac{2kx}{m} \right) + C}
\]

(72)

with \( C \) a parameter of integration.

According to the theory exposed in (43), if we aim at obtaining complete solutions, we need to construct a diffeomorphism based on the one parameter solutions given in (72).

\[
\Phi : \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^2 \rightarrow T^* \mathbb{R}^2 \times \mathbb{R}
\]

\[
\Phi(x, y, t, C, K) \mapsto \Phi(x, y, \gamma^{[i]}(C), \gamma^{[j]}(K), t)
\]

such that

\[
\Phi(x, y, t, C, K) = \left( x, y, \gamma^{[i]} = m \sqrt{-\left( \frac{4 \omega_0(t)^2}{m} x^2 + \frac{2kx}{m} \right) + C}, \gamma^{[j]} = \sqrt{-\omega_0(t)^2 y^2 - 2k_3 y^2 + K}, t \right)
\]

(73)

(74)

6. Hamilton–Jacobi theory on contact manifolds

We consider the extended phase space \( T^* Q \times \mathbb{R} \) with canonical projections of the first and second variables \( \rho : T^* Q \times \mathbb{R} \rightarrow T^* Q \) and \( t : T^* Q \times \mathbb{R} \rightarrow \mathbb{R} \). The hamiltonian function is \( H : T^* Q \times \mathbb{R} \rightarrow \mathbb{R} \). It can be illustrated through the following diagram

\[
\begin{array}{ccc}
T^* Q \times \mathbb{R} & \xrightarrow{\rho} & T^* Q \\
\vee t & & \vee \theta_Q \\
T^* Q & \xrightarrow{H} & \mathbb{R}
\end{array}
\]

Here \( t \) is an abuse of notation in which it is understood as a coordinate and as a coordinate function (a projection). We have local canonical coordinates \( \{q^i, p_i, t\}, i = 1, \ldots, n \).

The one-form is \( \eta = dt - \rho^* \theta_Q \), which reads
\[ \eta = dt - \sum_{i=1}^{n} p_i dq^i. \]  

(75)

The pair \((T^*Q \times \mathbb{R}, \eta)\) is a contact manifold. The Reeb vector field \(\mathcal{R} = \frac{\partial}{\partial t}\) satisfies

\[ \iota_\mathcal{R} \eta = 1, \quad \iota_\mathcal{R} d\eta = 0. \]

Consider the fibration \(\pi : T^*Q \times \mathbb{R} \to Q \times \mathbb{R}\). To have dynamics, we consider the vector field

\[ X_H = i_\Lambda (dH) + H\mathcal{R}. \]

(76)

In coordinates \([8]\), it reads

\[ X_H = \sum_{i=1}^{n} \left( p_i \frac{\partial H}{\partial p_i} - H \right) \frac{\partial}{\partial t} - \sum_{i=1}^{n} \left( p_i \frac{\partial H}{\partial t} + \frac{\partial H}{\partial q^i} \right) \frac{\partial}{\partial p_i} + \sum_{i=1}^{n} \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q^i}, \]

(77)

that is compatible with (27) and furthermore it that satisfies the conditions

\[ \flat(X_H) = -(\mathcal{R}(H) + H)\eta + dH. \]

where \(\flat\) is the isomorphism defined in (19) and

\[ \eta(X_H) = -H. \]

(78)

Recall that \((T^*Q \times \mathbb{R}, \Lambda, \mathcal{R})\) is a Jacobi manifold with \(\Lambda\) given in (27). The proposed contact structure provides us with the dissipation Hamilton equations \([8]\].

\[
\begin{cases}
\dot{q}^i = \frac{\partial H}{\partial p_i}, \\
\dot{p}_i = -\frac{\partial H}{\partial q^i} - p_i \frac{\partial H}{\partial t}, \\
i = p_i \frac{\partial H}{\partial p_i} - H.
\end{cases}
\]

(79)

for all \(i = 1, \ldots, n\).

Consider \(\gamma\) a section of \(\pi : T^*Q \times \mathbb{R} \to Q \times \mathbb{R}\), i.e. \(\pi \circ \gamma = \text{id}_{Q \times \mathbb{R}}\). We can use \(\gamma\) to project \(X_H\) on \(Q \times \mathbb{R}\) just defining a vector field \(X_H^\gamma\) on \(Q \times \mathbb{R}\) by

\[ X_H^\gamma = T_\pi \circ X_H \circ \gamma. \]

(80)

The following diagram summarizes the above construction

\[ \begin{align*}
T^*Q \times \mathbb{R} & \xrightarrow{T} T(T^*Q \times \mathbb{R}) \\
\gamma \circ & \xrightarrow{X_H} TQ \times \mathbb{R} \\
& \xrightarrow{X_H^\gamma} T(Q \times \mathbb{R})
\end{align*} \]

Assume that \(\gamma(Q \times \mathbb{R})\) is a legendrian submanifold of \((T^*Q \times \mathbb{R}, \eta)\), such that \(\gamma_t\) is closed.

**Definition 11.** The operation \([\ldots]_{\Delta}\) denotes the vertical lift of a one-form

\[ \alpha_{\Delta} = \alpha - \alpha_0 \Delta \]

(81)

where \(\alpha\) is the vertical lift defined in (38),
\[ \alpha_0 = \frac{\partial H}{\partial t} + \frac{\partial H}{\partial p} \frac{\partial \gamma}{\partial t} \]

and \( \vec{\alpha} \) is the Liouville vector field.

**Theorem 12.** The vector fields \( X_H \) and \( X_\gamma \) are \( \gamma \)-related if and only if the following equation is satisfied

\[ [\delta (H \circ \gamma)]^\gamma = -H \dot{\gamma}_q \quad (82) \]

Here \( \dot{\gamma}_q \) is the tangent vector in a point \( q \) associated with the curve

\[ \mathbb{R} \xrightarrow{\gamma} Q \times \mathbb{R} \xrightarrow{T^* \gamma} T^* Q \times \mathbb{R} \xrightarrow{p} T^* Q \]

**Proof.** The vector fields \( X_H \) and \( X_\gamma \) are \( \gamma \)-related if \( T_\gamma (X^\gamma_H) = X_H \). That is,

\[ T_\gamma (X^\gamma_H) = \left( p_i \frac{\partial H}{\partial q_i} - H \right) T_\gamma \left( \frac{\partial}{\partial t} \right) + \frac{\partial H}{\partial p_i} T_\gamma \left( \frac{\partial}{\partial q_i} \right) = X_H \circ \gamma \quad (83) \]

The section \( \gamma \) in local coordinates has components \( (q^i, \gamma^i(q^i, t), t) \) with \( i, j = 1, \ldots, n \) such that

\[ T_\gamma \left( \frac{\partial}{\partial t} \right) = \frac{\partial}{\partial t} + \sum_{j=1}^n \gamma^j \frac{\partial}{\partial p_j}, \quad T_\gamma \left( \frac{\partial}{\partial q^i} \right) = \frac{\partial}{\partial q^i} + \sum_{j=1}^n \frac{\partial \gamma^j}{\partial q^i} \frac{\partial}{\partial p_j}. \quad (84) \]

Introducing (84) in equation (83), it is straightforward to retrieve condition (82) if a further condition on the one-form \( \gamma \) is imposed. It is

\[ d\gamma_t = 0. \quad (85) \]

This means that \( \gamma_t \) is closed and fulfills the legendrian submanifold condition.

Equation (82) is known as a Hamilton–Jacobi equation for a dissipative Hamiltonian system on the contact structure \((T^* Q, \eta)\). In local coordinates,

\[ p_i \frac{\partial H}{\partial t} + \frac{\partial H}{\partial q^i} + \sum_{i=1}^n \left( p_i \frac{\partial H}{\partial p_i} - H \right) \frac{\partial \gamma^i}{\partial t} + \sum_{i=1}^n \frac{\partial H}{\partial p_i} \frac{\partial \gamma^i}{\partial q^i} = 0 \quad (86) \]

### 6.1. Complete solutions

**Definition 13.** A complete solution of the Hamilton–Jacobi equation for a dissipative Hamiltonian system on the contact manifold \((T^* Q, \eta)\) is a diffeomorphism \( \Phi : Q \times \mathbb{R} \times \mathbb{R}^n \to T^* Q \times \mathbb{R} \) such that for a set of parameters \( \lambda \in \mathbb{R}^n, \lambda = (\lambda_1, \ldots, \lambda_n) \), the mapping

\[ \Phi_\lambda : Q \times \mathbb{R} \to T^* Q \times \mathbb{R} \]

\[ \Phi_\lambda (q, t) \mapsto \Phi_\lambda (q, \gamma(q, t), t) \quad (87) \]

is a solution of the Hamilton–Jacobi equation.
We have the following diagram

\[
\begin{array}{ccc}
Q \times \mathbb{R} \times \mathbb{R}^n & \xrightarrow{\Phi} & T^*Q \times \mathbb{R} \\
\downarrow & & \downarrow \pi_i \\
\mathbb{R}^n & \xrightarrow{f_i} & \mathbb{R}
\end{array}
\]

where we define functions \( f_i \) such that for a point \( p \in T^*Q \times \mathbb{R} \), it is satisfied

\[
f_i(p) = \pi_i \circ \alpha \circ \Phi^{-1}(p).
\]

(88)

and \( \alpha : Q \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n \) is the canonical projection.

**Theorem 14.** There exist no linearly independent commuting set of first-integrals in involution (88) for a complete solution of the Hamilton–Jacobi equation for a dissipative system on the contact manifold \((T^*Q \times \mathbb{R}, \eta)\).

**Proof.** Consider the bracket

\[
\{ f_i, f_j \} = \Lambda(df_i, df_j) + f_i \mathcal{R}(f_j) - f_j \mathcal{R}(f_i)
\]

(89)

On the other hand, recall the definition of the hamiltonian vector field associated with \( f_i \in C^\infty \) as

\[
X_{f_i} = \sharp \Lambda(df_i, \cdot) + f_i \mathcal{R},
\]

then

\[
X_{f_j}(f_i) = df_j(X_{f_i}) = df_j(\sharp \Lambda(df_i) + f_i \mathcal{R}) = df_j(\sharp \Lambda(df_i)) + df_j(f_i \mathcal{R})
\]

(90)

Knowing that \( \sharp \Lambda(df_i) \in T \text{Im} \Phi \), for all \( df_i \) the terms \( \Lambda(df_i, df_j) \) and \( df_j(\sharp \Lambda(df_i)) \) vanish. Therefore,

\[
f_i \mathcal{R}(f_j) - f_j \mathcal{R}(f_i) = 0, \quad \forall i, j = 1, \ldots, n.
\]

(91)

From here, we can discuss two possibilities

1. \( \mathcal{R}(f_i) = 0 \)
2. \( f_i \mathcal{R}(f_j) - f_j \mathcal{R}(f_i) = 0 \)

From 1. we have that \( f_i \) are constants. From 2. we have that \( \mathcal{R}(f_i/f_j) = 0 \) that implies that \( f_i \) and \( f_j \) are not linearly independent.

**Remark.** If we calculate the evolution of the functions along the Hamiltonian flow, that is

\[
X_H(f_i) = df_i(X_H) = df_i(\sharp \Lambda(dH) + H\mathcal{R}) = H(\mathcal{R}(f_i))
\]

(92)

since \( df_i(\sharp \Lambda(dH)) = 0 \) because it is in \((T \text{Im} \Phi)^\circ\). We conclude that \( f_i \) are not constants, given \( X_H(f_i) \neq 0 \).

6.2. Example

Let us consider the hamiltonian on \( T^*Q \times \mathbb{R} \) with local coordinates \((q, p, S)\),

\[
H = \frac{p^2}{2m} + V(q) + \alpha S
\]

(93)
This is the corresponding Hamiltonian of a damped oscillator [8] which is retrieved by (79). Taking the Hamiltonian vector field as in (77), we have

\[ X_H = \left( \frac{p^2}{2m} - V(q) - \alpha S \right) \frac{\partial}{\partial S} - (\alpha p + V'(q)) \frac{\partial}{\partial p} + \frac{p}{m} \frac{\partial}{\partial q} \tag{94} \]

We choose a legendrian section \( \gamma \) with local components \((q, \gamma(q, S), S)\). And \( X^\gamma_H \) on \( Q \times \mathbb{R} \) reads

\[ X^\gamma_H = \left( \frac{p^2}{2m} - V(q) - \alpha S \right) \frac{\partial}{\partial S} + \frac{p}{m} \frac{\partial}{\partial q} \tag{95} \]

Using (80), we need to perform the computations

\[ T_\gamma \left( \frac{\partial}{\partial S} \right) = \frac{\partial}{\partial S} + \frac{\partial \gamma}{\partial S} \frac{\partial}{\partial p}, \quad T_\gamma \left( \frac{\partial}{\partial q} \right) = \frac{\partial}{\partial q} + \frac{\partial \gamma}{\partial q} \frac{\partial}{\partial p} \tag{96} \]

The Hamilton-Jacobi equation of the damped oscillator reads

\[ \left( \frac{p^2}{2m} - V(q) - \alpha S \right) \frac{\partial \gamma}{\partial S} + \frac{p}{m} \frac{\partial \gamma}{\partial q} + (p \alpha + V'(q)) = 0 \tag{97} \]

with \( dS = 0 \), that is \( \gamma_S = \text{constant} \). Integrating the equations along the section \( \gamma \), we have that \( p = \gamma \), and setting the constant \( \gamma_S = 1 \), then (97) can be rewritten as

\[ \frac{\partial \gamma}{\partial q} + \frac{1}{2} \gamma + \alpha m + \frac{m}{\gamma} \left( V'(q) - V(q) - \alpha S \right) = 0, \tag{98} \]

which can be solved as

\[ q = \frac{c_1}{\sqrt{c_1^2 - 2c_2}} \ln \left( \frac{\gamma + c_1 - \sqrt{c_1^2 - 2c_2}}{\gamma + c_1 + \sqrt{c_1^2 - 2c_2}} \right) - \ln \left( \frac{1}{2} \gamma^2 + c_1 \gamma + c_2 \right) + C \tag{99} \]

when \( c_1^2 > 2c_2 \), and

\[ q = \frac{2}{\sqrt{2c_2 - c_1^2}} \ln \left( \frac{\gamma + c_1}{\sqrt{2c_2 - c_1^2}} \right) - \ln \left( \frac{1}{2} \gamma^2 + c_1 \gamma + c_2 \right) + C \tag{100} \]

when \( c_2 > \frac{c_1^2}{2} \), with

\[ c_1 = \alpha m, \quad c_2 = -m^2 \alpha S. \tag{101} \]

where \( V(q) = V'(q) = 0 \) have conveniently been chosen equal to zero for the possible analytical integration. This solution is complemented by \( \gamma_S = \text{constant} \). The solution \( \gamma \) of the Hamilton-Jacobi equation on a contact manifold for a damped oscillator is provided by the implicit equations (99) and (100).

For a complete solution, we need to construct the diffeomorphism

\[
\Phi : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to T^* \mathbb{R} \times \mathbb{R} \\
\Phi(q, t, C) \mapsto \Phi(q, \gamma(C), t) \tag{102}
\]

with \( \gamma(q, t) \) derived from the implicit equations (99) and (100) for \( c_1^2 > 2c_2 \) and \( c_2 > \frac{c_1^2}{2} \), correspondingly.
7. Conclusions

We have developed a two-fold geometric Hamilton–Jacobi theory: for time-dependent hamiltonians through a cosymplectic geometric formalism and for dissipative hamiltonians through a contact geometry formalism. We have derived an explicit, new expression for the Hamilton–Jacobi equation on a cosymplectic manifold to find solutions of a unidimensional trigonometric system and two superintegrable potentials in two dimensions, one of them corresponds with the Winternitz–Smorodinsky oscillator. Furthermore, we have developed a geometric Hamilton–Jacobi theory on a contact manifold for hamiltonians containing a dissipation term. We have derived an explicit, new expression for the Hamilton–Jacobi equation on a contact manifold to find solutions of a one-dimensional damped oscillator.

Acknowledgments

This work has been partially supported by MINECO MTM 2013-42-870-P and the ICMAT Severo Ochoa project SEV-2011-0087.

References

[1] Abraham R and Marsden J E 1978 Foundations of Mechanics 2nd edn (Reading: Benjamin-Cumming)
[2] Albert C 1989 Le theoreme de reduction de Marsden-Weinstein en geometrie cosymplectique et de contact J. Geom. Phys. 6 627–249
[3] Arnold V I 1997 Mathematical Methods of Classical Mechanics vol 60 (New York: Springer)
[4] Becker K, Becker M and Strominger A 1995 Fivebranes, membranes and non-perturbative string theory Nucl. Phys. B 456 130–52
[5] Blair D E 1976 Contact Manifolds in Riemannian Geometry (Lecture Notes in Mathematics) (New York: Springer)
[6] Boothby W M 1975 An Introduction to Differentiable Manifolds and Riemannian Geometry (New York: Academic)
[7] Boothby W H and Wang H C 1958 On contact manifolds Ann. Math. 68 721–34
[8] Bravetti A, Cruz H and Tapia D 2017 Contact Hamiltonian mechanics Ann. Phys. 376 17–39
[9] Cariñena J F, de León M, Marrero J C and de Diego D M 1998 Reduction of nonholonomic mechanical systems with symmetries Rep. Math. Phys. 42 25–45
[10] Cappelletti-Montano B, de Nicola A and Yadin I 2013 A survey on cosymplectic geometry Rev. Math. Phys. 25 1343002
[11] Cariñena J F, Gracia X, Marmo G, Martínez E, Muñoz Lecanda M C and Román-Roy N 2010 Geometric Hamilton–Jacobi theory for nonholonomic dynamical systems Int. J. Geom. Methods Mod. Phys. 7 431–54
[12] Cariñena J F, Gracia X, Marmo G, Martínez E, Muñoz Lecanda M C and Román-Roy N 2006 Geometric Hamilton–Jacobi theory Int. J. Geom. Methods Mod. Phys. 3 1417–58
[13] Cariñena J F and Na˘ srarre J 1996 On the symplectic structures arising in geometric optics Fortschr. Phys. 44 181–98
[14] Cariñena J F, Ra˜ nada M F and Santander M 2007 A superintegrable two-dimensional nonlinear oscillator with an exactly solvable quantum analog Symmetry Integrability Geom.: Methods Appl. 3 030, 23p
[15] Chamseddine A H 1989 Topological gauge theory of gravity in five and all odd dimensions Phys. Lett. B 233 291–4
[16] Chinea D, de León M and Marrero J C 1994 The constraint algorithm for time-dependent lagrangians J. Math. Phys. 35 3410–47
[17] Etnyre J B 2003 Introductory lectures on contact manifolds (Introductory Lectures on Contact Geometry) Proc. Sympos. Pure Math. 71 81–107
[18] Evans L C 1998 Partial Differential Equations (Providence: American Mathematical Society)
[19] Evans N W 1990 Superintegrability of the Winternitz–Smorodinsky system Phys. Lett. 147 483–6
[20] Fris T I, Mandrosior V, Smorodinsky Y A, Uhlir M and Winternitz P 1965 On higher symmetries in quantum mechanics Phys. Lett. 16 354–6
[21] Godbillon G 1989 Geometrie Differentielle et Mecanique Analytique (Hermann: Collection Methodes)
[22] Goldstein H 1979 Mecanica Clasica 4a edn (Iasi: Aguilar SA Madrid)
[23] Hamilton W 1833 On a General Method of Expressing the Paths of Light, and of the Planets, by the Coefficients of a Characteristic Function (PD Hardy: Dublin University Review) pp 795–826
[24] Hamilton W 1834 On the application to dynamics of a general mathematical method previously applied to Optics Br. Assoc. Rep. 513–8
[25] Hitchin N 2003 Generalized Calabi-Yau manifolds Q. J. Math. 54 281–308
[26] Ibort A, de León M and Sosa D 1997 Reduction of Jacobi manifolds J. Phys. A: Math. Gen. 30 2783–98
[27] Kibble T W and Berkshire F H 2004 Classical Mechanics 5th edn (London: Imperial College Press)
[28] Kirillov A 1976 Local Lie algebras Russ. Math. Surv. 31 55–75
[29] Lacroix J, Marrero J C and Padron E 2012 Reduction of symplectic principal R-bundles J. Phys. A: Math. Theor. 45 325202
[30] Landau L D and Lifshitz E M 1988 On higher symmetries in quantum mechanics Phys. Lett. 16 354–6
[31] Leok M, Ohsawa T and Sosa D 2012 Hamilton J. Math. Phys. 53 072905
[32] de León M, Chinea D and Marrero J C 1993 Topology of cosympletic manifolds J. Math. Pures Appl. 72 567–91
[33] de León M, Iglesias-Ponte D and Martín de Diego D 2009 Towards a Hamilton–Jacobi theory for nonholonomic mechanical systems J. Phys. A: Math. Gen. 42 015205
[34] de León M, Martín Solano J and Marrero J C 1996 The constraint algorithm in the jet formalism Differ. Geom. Appl. 6 275–300
[35] de León M, Marrero J C and Padrón E 1997 On the geometric quantization of Jacobi manifolds J. Math. Phys. 38 605
[36] de León M and Marrero J C 1993 Constrained t-dependent lagrangian systems and lagrangian submanifolds J. Math. Phys. 34 622–44
[37] de León M, Marrero J C and Martín de Diego D 2009 A geometric Hamilton–Jacobi theory for classical field theories Variations, Geometry and Physics pp 129–40 (New York: Nova Sci. Publ.)
[38] de León M, Marrero J C and Martín de Diego D 2010 Linear almost Poisson structures and Hamilton–Jacobi equation: applications to nonholomic mechanics J. Geom. Mech. 2 159–98
[39] de León M, Marrero J C, Martín de Diego D and Vaquero M 2013 A Hamilton–Jacobi theory for singular lagrangian systems J. Phys. A: Math. Theor. 46 032902, 32p
[40] de León M, Marrero J C and Martínez E 2005 Lagrangian submanifolds and dynamics on Lie algebroids J. Math. Phys. Math. Gen. 38 R241–308
[41] de León M, Martín de Diego D and Vaquero M 2012 A Hamilton–Jacobi theory for singular lagrangian systems in the Skinner and Ruck setting Int. J. Geom. Methods Mod. Phys. 9 125007, 24p
[42] de León M and Saralegi M 1993 Cosymplectic reduction for singular momentum maps J. Phys. A: Math. Gen. 26 5033–43
[43] de León M and Tuynman G M 1996 A universal model for cosymplectic manifolds J. Geom. Phys. 20 77–86
[44] Libermann P and Marle C M 1987 Symplectic geometry and analytical mechanics Mathematics and its Applications vol 35 (Dordrecht: D. Reidel Publishing Company)
[45] Lichnerowicz A 1961 Sur la reductivite de certaines algebres d’automorphismes C. R. Acad. Sci. Paris 253 1302–4
[46] Lichnerowicz A 1978 Les varietes de Jacobi et leurs algebres de Lie associees J. Math. Pures Appl. 57 453–88
[47] Marle C M 1991 On Jacobi manifolds and Jacobi bundles Symplectic Geometry, Groupoids and Integrable systems vol 20 (Berkeley, CA: Mathematics Sciences Research Institute Publications) pp 227–46
[48] Marrero J C and Sosa D 2006 The Hamilton–Jacobi equation on Lie Affgebroids Int. J. Geom. Methods Mod. Phys. 3 605–22
[49] Martínez Campos C, de León M, Martín de Diego D and Vaquero M 2015 Hamilton–Jacobi theory on Cauchy data space Rep. Math. Phys. 76 359–87
[50] Okumura M 1965 Cosymplectic hypersurfaces in Kahlerian manifolds of constant holomorphic sectional curvature Ködai Math. Sem. Rep. 17 63–73
[51] Ohsawa T and Bloch A M 2009 Nonholonomic Hamilton–Jacobi equation and integrability J. Geom. Mech. 1 461–81
[52] Ohsawa T, Fernandez O E, Bloch A M and Zenkov D V 2011 Nonholonomic Hamilton–Jacobi theory via Chaplygin Hamiltonization J. Geom. Phys. 61 1263–91
[53] Otto F 1981 Lectures on Riemannian surfaces Graduate Texts in Mathematics vol 81 (New York: Springer)
[54] Pauffer C and Romer H 2002 Geometry of Hamiltonian n-vector fields in multisymplectic field theory J. Geom. Phys. 44 52–69
[55] Rajeev S G 2008 Quantization of contact manifolds and thermodynamics Ann. Phys. 323 768–82
[56] Rund H 1973 The Hamilton–Jacobi Theory in the Calculus of Variations ed R E Krieger (New York: Publ. Co. Nuntington)
[57] Stefan P 1974 Accessibility and foliations with singularities Bull. Am. Math. Soc. 80 1142–5
[58] Stein K 1951 Analytische funktionen mehrer komplexer Veränderlichen zu vorgegebenen Periodizitätmodul und das zweite cousinche problem Math. Ann. 123 201–22
[59] Sussmann H J 1973 Orbits of families of vector fields and integrability of distributions Trans. Am. Math. Soc. 180 171–80
[60] Tulczyjew W M 1976 Les sous-varietes lagrangiennes et la dynamique hamiltonienne C. R. Acad. Sci. Paris. Ser. A-B 283 A15–8
[61] Tulczyjew W M 1976 Les sous-varietes lagrangiennes et la dynamique hamiltonienne C. R. Acad. Sci. Paris. Ser. A-B 283 A675–8
[62] Winternitz P, Smorodinsky A, Uhir M and Fris J 1967 Symmetry groups in classical and quantum mechanics Sov. J. Nucl. Phys. 4 444–50
[63] Yano K and Ishihara S 1973 Tangent and Cotangent Bundles: Differential Geometry (New York: Dekker)