Long induced paths in graphs

Louis Esperet† Laetitia Lemoine‡ Frédéric Maffray§

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Abstract

We prove that every 3-connected planar graph on $n$ vertices contains an induced path on $\Omega(\log n)$ vertices, which is best possible and improves the best known lower bound by a multiplicative factor of $\log \log n$. We deduce that any planar graph (or more generally, any graph embeddable on a fixed surface) with a path on $n$ vertices, also contains an induced path on $\Omega(\sqrt{\log n})$ vertices. We conjecture that for any $k$, there is a positive constant $c(k)$ such that any $k$-degenerate graph with a path on $n$ vertices also contains an induced path on $\Omega((\log n)^{c(k)})$ vertices. We provide examples showing that this order of magnitude would be best possible (already for chordal graphs), and prove the conjecture in the case of interval graphs.

1 Introduction

A graph contains a long induced path (i.e., a long path as an induced subgraph) only if it contains a long path. However, this necessary condition is not sufficient, as shown by complete graphs and complete bipartite graphs. On the other hand, it was proved by Atminas, Lozin and Ragzon [2] that if a graph $G$ contains a long path, but does not contain a large complete graph or complete bipartite graph, then $G$ contains a long induced path. Their proof uses several applications of Ramsey theory, and the resulting bound on the size of a long induced path is thus quantitatively weak.

The specific case of $k$-degenerate graphs (graphs such that any subgraph contains a vertex of degree at most $k$) was considered by Nešetřil and Ossona de Mendez in [6]. These graphs clearly satisfy the assumption of the result of Atminas, Lozin and Ragzon [2], so $k$-degenerate graphs with long paths also contain long induced paths. Nešetřil and Ossona de Mendez [6, Lemma 6.4] gave the following more precise bound: if $G$ is $k$-degenerate

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†CNRS, Laboratoire G-SCOP, Université de Grenoble-Alpes, France.
‡Laboratoire G-SCOP, Université de Grenoble-Alpes, France.
§CNRS, Laboratoire G-SCOP, Université de Grenoble-Alpes, France.
and contains a path of size \( n \), then it contains an induced path of size \( \frac{\log \log n}{\log(k+1)} \). This result was then used to characterize the classes of graphs of bounded tree-length precisely as the classes of degenerate graphs excluding some induced path of fixed size. Nešetřil and Ossona de Mendez also asked \([6, \text{Problem } 6.1]\) whether their doubly logarithmic bound could be improved.

Arocha and Valencia \([1]\) considered the case of 3-connected planar graphs and 2-connected outerplanar graphs. An outerplanar graph is a graph that can be drawn in the plane without crossing edges and with all vertices on the external face. It was proved in \([1]\) that in any 2-connected outerplanar graph with \( n \) vertices, there is an induced path with \( \Omega(\sqrt{\log n}) \) vertices and, using this fact, that any 3-connected planar graph with \( n \) vertices contains an induced path with \( \Omega(\sqrt{\log n}) \) vertices. Note that in these results there is no initial condition on the size of a long path (the bounds only depend on the number of vertices in the graph). Di Giacomo, Liotta and Mchedlidze \([4]\) recently proved that any \( n \)-vertex 3-connected planar graph contains an induced outerplanar graph of size \( \sqrt{n} \), and that any \( n \)-vertex 2-connected outerplanar graph contains an induced path of size \( \frac{\log n}{2 \log \log n} \), and combining these two bounds, that any \( n \)-vertex 3-connected planar graph contains an induced path of size \( \frac{\log n}{12 \log \log n} \).

We will prove that if a \( k \)-tree (defined in the next section) contains a path of size \( n \), then it contains an induced path of size \( \frac{\log n}{k \log k} \). Using similar ideas, we will show that a partial 2-tree with a path of size \( n \) also contains an induced path of size \( \Omega(\log n) \). Outerplanar graphs are partial 2-trees, and 2-connected outerplanar graphs are Hamiltonian, so in particular this shows that any \( n \)-vertex 2-connected outerplanar graph contains an induced path of size \( \Omega(\log n) \). Using the results of Di Giacomo, Liotta and Mchedlidze \([4]\), this directly implies that any \( n \)-vertex 3-connected planar graph contains an induced path of size \( \Omega(\log n) \), improving their bound by a multiplicative factor of \( \log \log n \). Our bounds are tight up to a constant multiplicative factor.

We derive from our result on 3-connected planar graphs that any planar graph (and more generally, any graph embeddable on a fixed surface) with a path on \( n \) vertices contains an induced path of length \( \Omega(\sqrt{\log n}) \). We also construct examples of planar graphs with paths on \( n \) vertices in which all induced paths have size \( O\left(\frac{\log n}{\log \log n}\right) \). Our examples can be seen as special cases of a more general family of graphs: chordal graphs with maximum clique size \( k \), containing a path on \( n \) vertices, but in which every induced path has size \( O((\log n)^{\frac{2}{k-1}}) \). This shows that the doubly logarithmic bound of Nešetřil and Ossona de Mendez \([6]\) cannot be replaced by anything better than \( (\log n)^{c(k)} \), for some function \( c \). We believe that this is the correct order of magnitude.

**Conjecture 1.1.** There is a function \( c \) such that for any integer \( k \), any \( k \)-degenerate graph that contains a path of size \( n \) also contains an induced path of size \( (\log n)^{c(k)} \).

We prove this conjecture in the special case of interval graphs. More precisely, we show that any interval graph with maximum clique size \( k \) containing a path on \( n \) vertices contains an induced path of size \( \Omega((\log n)^{\frac{2}{k-1}+1}) \), where the hidden multiplicative constant depends...
on $k$.

We finish this section by recalling some definitions and terminology. In a graph $G$, we say that a vertex $x$ is complete to a set $S \subseteq V(G) \setminus x$ when $x$ is adjacent to every vertex in $S$. A block of a graph $G$ is any maximal 2-connected induced subgraph of $G$, where a bridge (a cut-edge) is also a block. It is well known that the intersection graph of the blocks and cut-vertices of $G$ can be represented by a tree $T$, which we call the block tree of $G$. The size of a path is the number of its vertices, and the length of a path is the number of its edges. A vertex is simplicial if its neighborhood is a clique. A simplicial vertex of degree $k$ is called $k$-simplicial. A graph is chordal if it contains no induced cycle of length at least four.

In this article, the base of the logarithm is always assumed to be 2.

2 Induced paths in $k$-trees

For any integer $k \geq 1$, the class of $k$-trees is defined recursively as follows:

- Any clique on $k$ vertices is a $k$-tree.
- If $G$ has a $k$-simplicial vertex $v$, and $G \setminus v$ is a $k$-tree, then $G$ is a $k$-tree.

Hence if $G$ is any $k$-tree on $p$ vertices, there is an ordering $x_1, \ldots, x_p$ of its vertices such that \{${x_1, \ldots, x_k}$\} induces a clique and, for each $i = k + 1, \ldots, p$, the vertex $x_i$ is a $k$-simplicial vertex in the subgraph induced by \{${x_1, \ldots, x_i}$\}. We call this a $k$-simplicial ordering, and we call \{${x_1, \ldots, x_k}$\} the basis of this ordering. We recall some easy properties of $k$-trees.

Lemma 2.1. Let $G$ be any $k$-tree. Then $G$ is chordal. Moreover, $G$ satisfies the following properties:

(i) If $G$ is not a $k$-clique, then every maximal clique in $G$ has size $k + 1$, and $G$ has exactly $|V(G)| - k$ maximal cliques.

(ii) If $G$ is not a clique, then $G$ has two non-adjacent $k$-simplicial vertices.

(iii) Any $k$-clique can be taken as the basis of a $k$-simplicial ordering of $G$.

(iv) For any $k$-clique $K$ of $G$, every component of $G \setminus K$ contains exactly one vertex that is complete to $K$.

Proof. Properties (i) and (ii) follow easily from the existence of a $k$-simplicial ordering, and we omit the details.

We prove (iii) by induction on $p = |V(G)|$. Consider any $k$-clique $K$ of $G$. If $G = K$, there is nothing to prove. So assume that $G$ is not a $k$-clique. By (ii), $G$ has two non-adjacent $k$-simplicial vertices $x$ and $y$. We may assume that $x \notin K$. By the induction
hypothesis, \( G \setminus x \) admits a \( k \)-simplicial ordering \( x_1, \ldots, x_{p-1} \) such that \( K \) is the basis of this ordering. Then \( x_1, \ldots, x_{p-1}, x \) is a \( k \)-simplicial ordering for \( G \), with \( K \) as a basis.

To prove (iv), consider any component \( A \) of \( G \setminus K \). By (iii), there is a \( k \)-simplicial ordering with \( K \) as a basis. The first vertex of \( A \) in the ordering has no neighbor in \( V(G) \setminus (K \cup A) \), so it must be complete to \( K \). Now suppose that \( A \) contains two vertices \( x, y \) that are complete to \( K \). Let \( x_0, \ldots, x_q \) be a shortest path in \( A \) with \( x = x_0 \) and \( y = x_q \). The vertex \( x_1 \) has a non-neighbor \( z \in K \), for otherwise \( K \cup \{x_0, x_1\} \) is a clique of size \( k + 2 \), contradicting (i). Let \( j \geq 2 \) be the smallest integer such that \( x_j \) is adjacent to \( z \); so \( 2 \leq j \leq q \). Then \( \{x_0, x_1, \ldots, x_j, z\} \) induces a cycle of length at least four in \( G \), contradicting the fact that \( G \) is chordal. \( \square \)

**Theorem 2.2.** Let \( k \) be a fixed integer, \( k \geq 2 \), and let \( G \) be a \( k \)-tree that contains an \( n \)-vertex path. Then \( G \) contains an induced path of size 

\[
\frac{\log(n-k-1)}{k \log k} = \frac{\log n}{k \log k} - O\left(\frac{1}{n}\right).
\]

**Proof.** Let \( P \) be a path on \( n \) vertices in \( G \). We may assume that \( G \) is minimal with these properties; in other words, if \( G \) has a vertex \( x \) such that \( G \setminus x \) is a \( k \)-tree and contains \( P \), then it suffices to prove the theorem for \( G \setminus x \); so we may assume that there is no such vertex. We claim that:

If \( K \) is any \( k \)-clique in \( G \), then \( G \setminus K \) contains at most \( k + 1 \) vertices that are complete to \( K \). \hfill (1)

Proof of (1): Whenever \( P \) goes from one component of \( G \setminus K \) to another component, it must go through at least one vertex of \( K \). This implies that \( P \) goes through at most \( k + 1 \) components of \( G \setminus K \). On the other hand, \( P \) must go through each component \( A \) of \( G \setminus K \), for otherwise we can restrict ourselves to \( G \setminus A \), which is a \( k \)-tree since we can take \( K \) as a basis of a \( k \)-simplicial ordering, and this contradicts the minimality of \( G \). Hence \( G \setminus K \) has at most \( k + 1 \) components; and by Lemma [2.1](iv), we deduce that (1) holds.

Let \( K_0 \) be a fixed \( k \)-clique in \( G \). We associate with \( G \) a labelled rooted tree \( T(G) \), where each node \( v \) has a label \( L(v) \) which is a set of \( k \)-cliques of \( G \), with the following properties:

- The root \( r \) of \( T(G) \) is a new node, and its label is \( \{K_0\} \);
- \( V(T(G)) \setminus \{r\} = V(G) \setminus K_0 \);
- The label of any non-root node \( v \) consists of \( k \) \( k \)-cliques of \( G \) that contain \( v \);
- Every \( k \)-clique of \( G \) is in the label of exactly one node of \( T(G) \);
- For any two nodes \( u, v \) in \( T(G) \) such that \( v \) is a child of \( u \), there is a \( k \)-clique \( K \in L(u) \) such that \( v \) is complete to \( K \) in \( G \) and each element of \( L(v) \) is a subset of \( K \cup \{v\} \).

We prove the existence of such a tree \( T(G) \) by induction as follows. If \( G \) is a \( k \)-clique, then \( K_0 = V(G) \), and \( T(G) \) is the tree with a unique node \( r \), whose label is \( \{V(G)\} \). Now
suppose that $G$ is not a $k$-clique. By Lemma 2.1 (iii), $G$ has a $k$-simplicial vertex $x$ with $x \notin K_0$. Let $K$ be the neighborhood of $x$, so $K$ is a $k$-clique of $G \setminus x$. By the induction hypothesis, $G \setminus x$ admits a labelled rooted tree $T(G \setminus x)$ that satisfies the properties above. Let $u$ be the unique node of $T(G \setminus x)$ whose label contains $K$. Then $T(G)$ is obtained from $T(G \setminus x)$ by adding $x$ as a child of $u$, and we set $L(x) = (K \cup \{x\}) \setminus y$ for all $y \in K$. Hence the $k$-cliques that contain $x$ are in $L(x)$. It follows that every $k$-clique of $G$ is in the label of exactly one node of $T(G)$. So all the required properties hold for $T(G)$.

Now we claim that:

Each node of $T(G)$ has degree at most $k^2 + 1$. \hfill (2)

Let $u$ be any node of $T(G)$ and $v$ be any child of $u$. By the properties of $T(G)$, the vertex $v$ is complete to a member $K$ of $L(u)$. By [1], at most $k + 1$ such vertices exist for each $K$. Hence if $u$ is the root, its degree is at most $k^2 + 1$. Now suppose that $u$ is not the root, and let $t$ be its parent node in $T(G)$. By the properties of $T(G)$, $u$ is complete (in $G$) to some $k$-clique $Q \in L(t)$, and we have $K \subseteq Q \cup \{u\}$. Note that the unique vertex in $Q \cup \{u\} \setminus K$ is complete to $K$ and is either an ancestor of $u$ in $T(G)$, or a vertex of $K_0$, and therefore does not appear among the children of $u$. So $u$ has at most $k(k + 1 - 1) = k^2$ children; hence the degree of $u$ in $T(G)$ is at most $k^2 + 1$. Thus (2) holds.

Let $\ell$ be the size of a longest path in $T(G)$, and let $P = v_1, \ldots, v_\ell$ be a path in $T(G)$. We claim that:

$$\ell \geq \frac{\log(n - k - 1)}{\log k}. \hfill (3)$$

Since $|V(G)| \geq n$, it follows that $T(G)$ has at least $n - k + 1$ nodes. First suppose that $\ell$ is odd. Let $m = (\ell + 1)/2$. So the vertex $v_m$ is the middle vertex of $P$, and every vertex of $T(G)$ is at distance at most $m - 1$ from $v_m$. It follows that $n - k + 1 \leq 1 + (k^2 + 1)(k^2)^{m - 2}$, so $n - k \leq k^{2m - 1}$, whence $\ell \geq \frac{\log(n - k - 1)}{\log k}$.

Now suppose that $\ell$ is even. Let $m = \ell/2$. So the edge $v_m v_{m+1}$ is the middle edge of $P$, and every vertex of $T(G)$ is at distance at most $m - 1$ from one of $v_m, v_{m+1}$. It follows that $n - k + 1 \leq 2 + (k^2)^{m - 1}$, so $n - k - 1 \leq k^{2m - 2}$, whence $\ell \geq \frac{\log(n - k - 1)}{\log k} + 2$. Thus (3) holds.

Using the path $P$ we construct, by induction on $i = 1, \ldots, \ell$, a collection of $k$ vertex-disjoint induced paths $P_1, \ldots, P_k$ in $G$, adding one vertex of $G$ at each step, so that the following properties hold at each step $i$, where $u_{j,i}$ is the last vertex of $P_j$:
Then, we take \( u \) the unique integer such that \( k \) is a child of \( v \) in \( G \), with \( K_i \in L(v_i) \); if \( i \geq 2 \) then \( |K_{i-1} \cap K_i| = k - 1 \).

We do this as follows. First pick one member \( K_1 \) of \( L(v_1) \), and for each \( j = 1, \ldots, k \) let the first vertex of \( P_j \) be the \( j \)-th vertex of \( K_1 \).

At step \( i + 1 \), we consider two cases since \( v_i \) is either a child of \( v_{i+1} \) or its parent.

Suppose that \( v_i \) is a child of \( v_{i+1} \). Let \( x_i \) be the vertex of \( G \) in \( L(v_i) \), and let \( K_{i+1} \) be the \( k \)-clique in \( L(v_{i+1}) \) such that \( x_i \) is complete to \( K_{i+1} \). Then \( |K_{i+1} \cap K_i| = k - 1 \). Let \( j \) be the unique integer such that \( u_{j,i} \in K_i \setminus K_{i+1} \), and consider the unique vertex \( y \in K_{i+1} \setminus K_i \). Then, we take \( u_{j,i+1} = y \) and \( u_{a,i+1} = u_{a,i} \) for all \( a \neq j \). In this case the vertices \( v_1, \ldots, v_{i-1} \) are all descendants of \( v_i \) in \( T(G) \), and have been added in the construction of \( T(G) \) as descendants of the clique \( K_i \). Since \( u_{j,i+1} = y \notin K_i \), among vertices of \( P_1, \ldots, P_k \) the vertex \( u_{j,i+1} \) is adjacent only to \( u_{1,i}, \ldots, u_{k,i} \); so the paths \( P_1, \ldots, P_k \) remain induced.

Now suppose that \( v_i \) is the parent of \( v_{i+1} \). First suppose that \( i + 1 \neq \ell \). Then \( v_{i+2} \) is a child of \( v_{i+1} \). Let \( x_{i+2} \) be the vertex in \( L(v_{i+2}) \), and let \( K_{i+1} \) be the \( k \)-clique in \( L(v_{i+1}) \) such that \( x_{i+2} \) is complete to \( K_{i+1} \). Then \( |K_{i+1} \cap K_i| = k - 1 \). Let \( j \) be the unique integer such that \( u_{j,i} \in K_i \setminus K_{i+1} \), and let \( y \) be the unique vertex in \( K_{i+1} \setminus K_i \). Then we take \( u_{j,i+1} = y \) and \( u_{a,i+1} = u_{a,j} \) for all \( a \neq j \). Since the only neighbors of \( y \) are the vertices in \( K_i \) and vertices corresponding to descendants of \( v_{i+1} \) (which are not vertices in the paths \( P_1, \ldots, P_k \)), the paths \( P_1, \ldots, P_k \) remain induced.

Finally suppose that \( i + 1 = \ell \). Let \( x_\ell \) be the vertex of \( G \) that belongs to \( L(v_\ell) \). Then we can add \( x_\ell \) to any of the paths, say to \( P_1 \). Since \( x_\ell \) is adjacent to the vertices \( u_{1,i}, \ldots, u_{k,i} \) only, the paths \( P_1, \ldots, P_k \) remain induced. This completes the construction of these paths.

Since \( P \) has size \( \ell \), there are \( k + \ell - 1 \) vertices in \( P_1 \cup \cdots \cup P_k \). These paths are disjoint, so one of them has size at least \( \frac{\ell + k - 1}{k} \geq \frac{\ell}{k} \).

In summary, if \( G \) contains a path of size \( n \), then it contains an induced path of size \( \frac{\log(n - k - 1)}{k \log k} \). This completes the proof of the theorem.

This bound is optimal, up to a constant multiplicative factor of \( 2k \log k \). To see this, consider the family of graphs \( G_i \) depicted in Figure 2. These examples were found by Arocha and Valencia [1]. The graph \( G_0 \) is a triangle, and \( G_i \) is obtained from \( G_{i-1} \) by adding, for each edge \( uv \) created at step \( i - 1 \), a new vertex adjacent to \( u \) and \( v \). Clearly these graphs are outerplanar and 2-trees. Moreover they are Hamiltonian. The graph \( G_i \) has \( n = 3 \times 2^i \) vertices and therefore contains a path with the same number of vertices, while it is easy to check that the longest induced path in \( G_i \) has size \( 2(i+1) = 2 \log n + (2-2 \log 3) \).

Now, add \( k - 2 \) universal vertices to each \( G_i \). We obtain again Hamiltonian \( k \)-trees with \( n \) vertices in which all induced paths have size at most \( 2 \log n \), as desired. It would be interesting to construct examples such the size of the longest induced paths decreases as \( k \) grows (for instance of order \( \frac{\log n}{\log k} \)). We have not been able to do so.
2.1 Partial 2-trees

A partial $k$-tree is any subgraph of a $k$-tree. The tree-width of a graph $G$ is the least $k$ such that $G$ is a partial $k$-tree. Note that Theorem 2.2 has no direct corollary on the size of long induced paths in partial $k$-trees in general, but we can still deduce an asymptotically optimal bound for the class of partial 2-trees. Before doing so, we prove the following lemma which will be useful in several proofs. Recall that a class of graphs is called hereditary if it is closed under taking induced subgraphs.

Lemma 2.3. Let $\mathcal{F}$ be a hereditary family of graphs. Suppose that there are reals $\alpha, \beta > 0$ such that every 2-connected graph in $\mathcal{F}$ that contains an $n$-vertex path contains an induced path of size $\alpha(\log n)^\beta$. Let $G$ be any connected graph in $\mathcal{F}$ that contains an $n$-vertex path. Then $G$ contains an induced path of size $\alpha(\log n - \log(\alpha(\log n)^\beta))^{\beta} = (\alpha - o(1))(\log n)^\beta$.

Proof. Let $P$ be an $n$-vertex path in $G$. Let $T$ be the block tree of $G$. Let $k$ be the number of blocks intersecting $P$. If $k \leq \alpha(\log n)^\beta$, then there is a block that contains a subpath of $P$ with at least $\frac{n}{\alpha(\log n)^\beta}$ vertices. By the hypothesis this block has an induced path of size $\alpha \log(\frac{n}{\alpha(\log n)^\beta})^\beta = \alpha(\log n - \log(\alpha(\log n)^\beta))^{\beta} = (\alpha - o(1))(\log n)^\beta$. On the other hand, if $k > \alpha(\log n)^\beta$, then there is a path of $k$ blocks in $T$, which means that there are $k$ distinct blocks $B_1, \ldots, B_k$ such that $B_i$ has exactly one vertex $v_i$ in common with $B_{i+1}$, and the vertices $v_1, \ldots, v_{k-1}$ are pairwise distinct, and $v_iv_j$ is not an edge whenever $|i - j| \geq 2$. For each $i = 1, \ldots, k-1$ let $P_i$ be a shortest path between $v_i$ and $v_{i+1}$ in $B_{i+1}$. Then we obtain a path $v_1P_1v_2P_2\ldots v_{k-1}$ of size at least $\alpha(\log n)^\beta$. \qed

We now consider partial 2-trees. Recall that every 2-tree is a planar graph (because a simplicial vertex of degree 2 can be added to any planar graph in a way that preserves planarity).

Theorem 2.4. If $G$ is a 2-connected partial 2-tree that contains an $n$-vertex path, then $G$ contains an induced path of size $\frac{\log(n-3)}{2}$.

Proof. Let $P$ be an $n$-vertex path in $G$. We add edges to $G$ in order to obtain a 2-tree $G'$. As before, we can assume that $G'$ is a minimal 2-tree containing $P$. 

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Using the proof of Theorem 2.2, we can find in \( T(G') \) a path \( P' \) of size \( \ell \geq \log(n-3) \). We denote by \( U \) the set of vertices of \( G' \) corresponding to the vertices of \( P' \) (i.e., \( U \) consists of the union of the cliques \( K_i \) on 2 vertices defined in the proof of Theorem 2.2). The subgraph of \( G' \) induced by \( U \) is denoted by \( G'[U] \).

Given a planar embedding of a connected planar graph \( G \), we define the dual graph \( G^* \) of \( G \) as follows: the vertices of \( G^* \) are the faces of \( G \), and two vertices \( v_1^* \) and \( v_2^* \) of \( G^* \) are adjacent if and only if the corresponding faces of \( G \) share an edge. We call weak dual of \( G \) the graph obtained from \( G^* \) by deleting the vertex that represents the external face of \( G \).

We call path of triangles a 2-tree having a plane embedding whose weak dual is a path. It directly follows from our definition of the cliques \( K_i \) in the proof of Theorem 2.2 that \( G'[U] \) is a path of triangles. From now on we fix a planar embedding of \( G'[U] \) such that its weak dual is a path.

Since \( G'[U] \) is a path of triangles, it has exactly two simplicial vertices \( a, b \), both of degree 2, and the other vertices of \( G'[U] \) are not simplicial. Following the proof of Theorem 2.2, the outer face of \( G'[U] \) can be partitioned into two paths going from \( a \) to \( b \), and these two paths are induced paths in \( G' \), so one of them has size at least \( \log(n-3)^2 \). We denote this path by \( P_{G'} \).

Since \( G \) is a subgraph of \( G' \), some edges of \( P_{G'} \) may not be in \( G \); we call them missing edges. Consider any missing edge \( uu' \) of \( P_{G'} \), and let \( w \) be the third vertex of the triangular inner face of \( G'[U] \) incident with \( uu' \). Since \( G \) is 2-connected, there is a path \( P_{uw} \) in \( G \) between \( u \) and \( u' \) avoiding \( w \). The internal vertices of such a path are necessarily disjoint from \( P_{G'} \), since otherwise \( G' \) would contain \( K_4 \) as a minor (and it is well known that any 2-tree is \( K_4 \) minor-free). We can assume without loss of generality that \( P_{uw} \) is an induced path, by taking a shortest path with the aforementioned properties. Using again that \( G' \) does not contain \( K_4 \) as a minor, it is easy to see that if \( uu' \) and \( vv' \) are two missing edges, then the two paths \( P_{uw} \) and \( P_{vw} \) have no internal vertex in common, no edges between their internal vertices, and no edge from their internal vertices to \( P_{G'} \) (except possibly to the endpoints of their respective paths). Now, replacing every missing edge \( uw \) with the corresponding path \( P_{uw} \), we get an induced path \( P_G \) that is at least as long as \( P_{G'} \), and so \( G \) contains an induced path of size \( \log(n-3)^2 \). \( \square \)

As a direct consequence of Theorem 2.4 and Lemma 2.3, we obtain:

**Corollary 2.5.** If \( G \) is a partial 2-tree that contains an \( n \)-vertex path, then \( G \) contains an induced path of size \( (\frac{1}{2} - o(1)) \log n \).

### 3 Induced paths in planar and outerplanar graphs

Since an outerplanar graph is a partial 2-tree, we also obtain the following corollary.

**Corollary 3.1.** If \( G \) is an outerplanar graph with an \( n \)-vertex path, then \( G \) contains an induced path of size \( \frac{\log n}{2} (1 - o(1)) \).
We can give an alternative proof of this corollary. We give the proof in the case where the graph is 2-connected. If it is not, we can use the Lemma \[2.3\]. This proof is quite similar to the proof in \[1\].

**Theorem 3.2.** If \( G \) is a 2-connected outerplanar graph with \( n \) vertices, then \( G \) contains an induced path of size \( \frac{\log n}{2} \).

**Proof.** Let \( G \) be a 2-connected outerplanar graph with \( n \) vertices. We add edges to \( G \) in order to obtain a maximal outerplanar graph \( G' \). We denote by \( D \) and \( D' \) the weak duals of \( G \) and \( G' \), respectively. Note that \( D \) and \( D' \) are trees.

Each face of \( G \) with \( k \) vertices contains \( k - 2 \) triangular faces of \( G' \), and for each vertex in \( D \) corresponding to a \( k \)-vertex face, we have a tree with \( k - 2 \) vertices in \( D' \).

Let \( d \) be the diameter of \( D' \), and let \( m \) be the number of vertices of \( D' \). We have \( m \geq n - 2 \). Consider a leaf \( v \) of \( D' \): it has degree 1, and its neighbor \( u \) has degree at most 3 (since each vertex of \( D' \) has degree at most 3), and therefore at most 2 neighbors distinct from \( v \). Each vertex of \( D' \) is reachable from \( v \) with a path of at most \( d - 1 \) edges, so we have \( m \leq 2^{d-2} \). Then we have \( n \leq 2^{d-2} + 2 \), so \( d \geq \log n \), and so there is a path \( P' \) of size \( \ell' \geq \log n \) in \( D' \).

We associate with each vertex \( v \) of \( D \), corresponding to a \( k \)-vertex face of \( G \), a weight of \( k - 2 \) (which is the number of vertices in \( D' \) in the tree corresponding to \( v \)). The weight of a path \( P \) of \( D \) is defined as the sum of the weights of its vertices. Then a path \( P \) in \( D \) of weight \( w \) corresponds in \( G \) to a path of faces (a sequence of faces in which any two consecutive faces share an edge), with \( w - 2 \) vertices.

Let \( P' = u'_1 \cdots u'_p \). Each vertex \( u'_i \) corresponds to a face \( F'_i \) of \( G' \), which corresponds to a face \( F_j \) of \( G \) and a vertex \( u_j \) of \( D \). Then, we have \( P' = u'_1 \cdots u'_{i_1} \cdots u'_{i_2} \cdots u'_{i_3} \cdots u'_{s} \cdots u'_{i_{s+1}} \cdots u'_{i_{s+2}} \cdots u'_{i_{s+3}} \cdots \cdots u'_{i_{t}} \cdots u'_{i_{t+1}} \cdots u'_p \) with \( i_1 = 1 \), \( i_s + t = \ell' \) and for \( a = 1, \ldots, s \), the vertices \( u'_{i_a}, \ldots, u'_{i_{a+1}} \) corresponding to \( u_{j_a} \) in \( D \). For each \( a = 1, \ldots, s - 1 \), \( u_{ja} \) is adjacent to \( u_{ja+1} \) because \( u'_{i_a+1} \) and \( u'_{i_{a+1}} \) are adjacent in \( D' \). Moreover, we claim that each vertex is present only once in \( P \). Suppose not. Then we denote by \( u_{ja} \), corresponding to a face \( F_a \) of \( G \), a vertex which is present several times in \( P \), and by \( b \) the smallest index larger than \( a \) such that \( u_{ja} = u_{jb} \). In \( D' \), there is a path between \( u'_{i_{a+1}} - u'_{i_{a+1}} \) and \( u'_{i_{b}} \) in the tree corresponding to \( u_{ja} \), and a path \( u'_{i_{a+1}} - u'_{i_{a+1}} \cdots u'_{i_{b-1}} \cdots u'_{i_{b}} \), with no vertex from the tree corresponding to \( u_{ja} \). Then there is a cycle in the tree \( D' \), which is a contradiction. Therefore \( P \) is a path.

Let \( \ell \) be the weight of \( P \). Then we have \( \ell \geq \ell' \), because each vertex \( u \) of \( P \), corresponding to a \( k \)-vertex face, has a weight \( k - 2 \), and comes from \( k' \leq k - 2 \) vertices in \( D' \). Therefore, \( \ell \geq \log n \).

In \( G \), \( P \) corresponds to a path of faces separated by edges, with \( \ell + 2 \) vertices. If we remove a vertex from each extremal face of \( P \), we get two induced paths in \( G \), so one of them has size \( \frac{\ell}{2} \geq \frac{\log n}{2} \).

This bound is optimal up to a constant multiplicative factor, as we have seen before.
(recall that the graphs $G_i$ depicted in Figure 2 have $n = 3 \times 2^{i-1}$ vertices and their longest induced path have size $2(i + 1) = 2 \log n + (2 - 2 \log 3)$).

The following theorem, proved in [4], gives a bound on the largest induced outerplanar graph in a 3-connected planar graph. Their proof uses the existence of so-called “Schnyder woods” to define some partial orders, followed by an application of Dilworth theorem on these partial orders.

A bracelet is a connected outerplanar graph where each cut-vertex is shared by two blocks, and each block contains at most two cut-vertices.

**Theorem 3.3** ([4]). Any 3-connected planar graph with $n$ vertices contains an induced bracelet with at least $3 \sqrt{n}$ vertices.

Using our Theorem 3.2, we can prove the following bound for a bracelet with $n$ vertices.

**Lemma 3.4.** If $G$ is a bracelet containing $n$ vertices, then it contains an induced path of size $\frac{1}{2}(\log n - \log \log n) = \left(\frac{1}{2} - o(1)\right) \log n$.

**Proof.** Denote by $k$ the number of blocks of $G$.

If $k \leq \log n$, then there is a block with at least $\frac{n}{\log n}$ vertices. It follows from Theorem 3.2 that in this block we can find an induced path with $\frac{1}{2} \log\left(\frac{n}{\log n}\right) = \frac{1}{2}(\log n - \log \log n)$ vertices.

If $k > \log n$, then $G$ has at least $k - 1 \geq \log n - 1$ cut-vertices. Since $G$ is a bracelet, there are $k$ blocks $B_1, \ldots, B_k$, with $B_i$ having a cut-vertex $c_i$ with $B_{i+1}$. In each block $B_i$, we take a shortest path (which is induced) from $c_{i-1}$ to $c_i$; then the union of these paths is an induced path of length at least $\log n$.

Using Theorem 3.3, Di Giacomo et al. proved that a 3-connected planar graph with $n$ vertices contains an induced path of size $\Omega\left(\frac{\log n}{\log \log n}\right)$. Combining Theorem 3.3 with Lemma 3.4, we obtain the following theorem:

**Theorem 3.5.** If $G$ is a 3-connected planar graph with $n$ vertices, then $G$ contains an induced path of size $\frac{1}{2}\left(1 + \frac{1}{3} \log n - \log \log n\right) = \left(\frac{1}{6} - o(1)\right) \log n$.

**Proof.** Let $G$ be a 3-connected planar graph. By Theorem 3.3, it contains an induced bracelet $H$ with $m \geq \sqrt{n}$ vertices. By Lemma 3.4, $H$ (and then $G$) contains an induced path of size $\frac{1}{2}(\log m - \log \log m) > \frac{1}{2}\left(\frac{1}{3} \log n - \log \log n\right)$.

Our bound is (asymptotically) optimal up to a constant multiplicative factor, as shown by the family of graphs $G_i$ depicted in Figure 4. The graph $G_0$ is a triangle, and we obtain $G_i$ from $G_{i-1}$ by adding a vertex adjacent to each triangle of $G_{i-1}$ that is not in $G_{i-2}$. The graph $G_i$ has $n = 3 + \frac{3^{i-1}}{2}$ vertices and its longest induced path contains $\ell = i + 1 \geq \frac{\log n}{\log 3}$ vertices.

From our bound for 3-connected planar graphs, we will now deduce a bound for 2-connected planar graphs. Since such graphs do not necessarily contain long paths (see for
example the complete bipartite graph $K_{2,n}$), we restrict ourselves to 2-connected planar graphs with long paths. We will use the so-called SPQR-trees [3], defined as follows.

Let $G$ be a 2-connected graph. One can represent the interaction of 3-connected induced subgraphs of $G$ by a tree $T_G$, in which each node is associated to a subgraph and has one of four types:

- Each node of type S is associated with a cycle on at least three vertices;
- Each node of type R is associated with a 3-connected simple subgraph;
- Each node of type P is associated with two vertices, with three or more edges between them (and two nodes of type P are not adjacent in $T_G$);
- Each node of type Q is associated with a single edge. This case is used only when the graph has only one edge.

If $x$ and $y$ are two adjacent nodes of $T_G$, and $G_x$ and $G_y$ are the associated graphs, then the edge $xy$ of $T_G$ is associated to one pair of adjacent vertices in $G_x$, and one pair of adjacent vertices in $G_y$ (the edges between these pairs of vertices are called virtual edges). Given an SPQR-tree $T$, we obtain the corresponding 2-connected graph as follows. For each edge $xy$ in $T_G$, we do the following: let $(a,b)$ be the pair of adjacent vertices associated to $xy$ in $G_x$, and let $(c,d)$ be the pair of adjacent vertices associated to $xy$ in $G_y$, then we identify the vertices $a$ and $c$, and the vertices $b$ and $d$, and remove the virtual edges between the two newly created vertices (see Figure 4). For any 2-connected graph $G$ the SPQR-tree $T_G$ is unique up to isomorphism.

Given a subtree $T'$ of $T_G$, we can define an induced subgraph $G_{T'}$ of $G'$ as described above by identifying vertices and then removing all virtual edges (including those that are not matched).

**Theorem 3.6.** If $G$ is a 2-connected planar graph containing a path with $n$ vertices, then $G$ contains an induced path of size at least $\frac{\sqrt{\log n}}{2\sqrt{6}} - \frac{1}{4} \log \log n - 1 = \frac{\sqrt{\log n}}{2\sqrt{6}} (1 - o(1))$.

**Proof.** Let $P$ be a path on $n$ vertices in $G$. We consider the smallest induced subgraph $G'$ of $G$ which contains $P$ and is 2-connected. Let $T_{G'}$ be the SPQR-tree corresponding to $G'$.
Let $\alpha = 2\sqrt{\log n \over 2}$. Note that either there is a node of $T_{G'}$ whose associated graph has size $\alpha$, or every graph associated with a node of $T_{G'}$ has less than $\alpha$ vertices.

Suppose first that there is a node $x$ of $T_{G'}$ whose associated graph has size $\alpha$. Then $x$ is a node of type $S$ or $R$. If $x$ is a node of type $S$, then there is an induced path of size $\alpha - 1$ in the associated graph (which is a cycle). If $x$ is a node of type $R$, then the associated graph is a 3-connected planar graph, and by Theorem 3.5, there is an induced path of size $1 \over 6 \log \alpha - 1 \over 2 \log \log \alpha$ in the associated graph. In both cases, we have an induced path $P_x$ of size at least $1 \over 6 \log \alpha - 1 \over 2 \log \log \alpha$ in the graph associated with the node $x$. If $P_x$ is also an induced path in $G$, we are done, so assume the contrary, which means that $P_x$ contains some virtual edges. Let $ab$ be any virtual edge in $P_x$. Then $ab$ corresponds to a virtual edge $cd$ in some node $y$ adjacent to $x$ in $T_{G'}$. Let $P_{ab}$ be a shortest path from $c$ to $d$ in the subgraph $G_{T_y}$, where $T_y$ is the subtree of $T_{G'}$ rooted at $y$ ($T_y$ contains the descendants of $y$, viewing $x$ as the root of $T_{G'}$). In $P_x$ we replace each virtual edge $ab$ with the corresponding path $P_{ab}$, so we denote by $P'$ the resulting path of $G'$. Observe that this path $P'$ is an induced path in $G'$, because all the replacement paths are in distinct subtrees of $T_{G'}$ (each of these subtrees corresponds to a distinct neighbor of $x$ in $T_{G'}$). Hence $P'$ is an induced path in $G'$, of size at least the size of $P_x$. So we have an induced path of size $1 \over 6 \log \alpha - 1 \over 2 \log \log \alpha$ in $G'$, which is an induced path of size $1 \over 6 \log \alpha - 1 \over 2 \log \log \alpha \geq \sqrt{\log n \over 2\sqrt{6}} - 1 \over 4 \log \log n - 1$ in $G$.

Suppose now that every graph associated with a node of $T_{G'}$ has less than $\alpha$ vertices. Then there are at least $2 \alpha$ nodes in $T_{G'}$. Since each graph corresponding to a node of type $R$ or $S$ is a planar graph with no multiple edge, and has at most $\alpha$ vertices, it contains at most $3\alpha - 6$ edges. So the degree of a node of type $R$ or $S$ is at most $3\alpha - 6$, since each edge contributes to at most 1 in the degree, if it is a virtual edge. Concerning the nodes of type $P$, we claim that their degree in $T_{G'}$ is at most 3. For suppose that there is a node $X$ of type $P$ of degree at least 4. Since the removal of the two vertices in the graph associated with $X$ disconnects the graph $G'$, there are edges of the path $P$ in at most three components associated with nodes adjacent to $X$. Since these nodes are adjacent to $X$, they are not of type $P$, and so they have at least three vertices. Removing in $G'$ the vertices of those components that do not intersect $P$, we obtain a smaller graph, induced, 2-connected, and containing the path $P$, contradicting the minimality of $G'$. So the claim holds. It follows that the degree of every node in $T_{G'}$ is at most $3\alpha$. Let $d$ be the diameter of $T_{G'}$. Then we
have \( \frac{\beta}{\alpha} \leq (3\alpha)^{(d-2)} \). So there is a path in \( T_{G'} \) of size \( d \geq \frac{\log n}{\log(3\alpha)} - \frac{\log \alpha}{\log(3\alpha)} + 2 \geq \frac{\log n}{\log(3\alpha)} \).

We claim that if there is a path \( P \) of size \( \ell \) in \( T_{G'} \), then there is an induced path in \( G' \) of size \( \frac{\ell}{4} \). First, since two nodes of type P cannot be adjacent, there are at most \( \frac{\ell}{4} \) nodes of type P in \( P \). The other nodes are of type R or S. Denote by \( p_1, \ldots, p_\ell \) the nodes of \( P \) and \( e_1, \ldots, e_{\ell-1} \) its edges, where \( e_i = p_ip_{i+1} \). Each edge \( e_i \) corresponds to one virtual edge in \( p_i \) and one virtual edge in \( p_{i+1} \), which correspond to two vertices \( x_i, y_i \) (adjacent or not) in the graph \( G' \). We have \( \{x_i, y_i\} = \{x_{i+1}, y_{i+1}\} \) if and only if the node \( p_{i+1} \) of \( P \) is of type P. Denote by \( p_1, \ldots, p_\ell \) the nodes that are not of type P. We have \( \ell \geq \frac{\ell}{4} \), since at most \( \frac{\ell}{4} \) nodes have type P. For each \( j \), there is at most one vertex in common between \( \{x_i, y_i\} \) and \( \{x_{i+1}, y_{i+1}\} \). Then we keep the name of \( \{x_i, y_i\} \), and rename the others vertices so that if \( \{x_i, y_i\} \) and \( \{x_{i+1}, y_{i+1}\} \) have a vertex in common, then we have either \( x_i = x_{i+1} \) or \( y_i = y_{i+1} \). In total, there are at least \( \frac{\ell}{4} \) vertices \( x_i \) and \( y_j \), so one of these two sets, say the set \( \{x_i \mid 1 \leq j \leq k\} \), contains at least \( \frac{\ell}{4} \) elements. We can then find an induced path in \( G' \) containing these vertices: we consider the induced subgraph of \( G \) corresponding to the subtree rooted at \( p_{i+1} \) and containing \( p_{i+1} \) and its descendance, except \( p_{i+1} \) and their descendance, and take a shortest path between \( x_i \) and \( x_{i+1} \) in this graph. The path obtained is induced since each path is taken in subtrees having no vertex in common. Then, we obtain an induced path of size \( \frac{\log n}{4\log(3\alpha)} \geq \sqrt{\frac{\log n}{2\sqrt{6}}} - 1 \) in \( G' \).

Using Theorem 3.6, we deduce the following corollary for connected planar graphs using Lemma 2.3.

**Corollary 3.7.** If \( G \) is a connected planar graph containing a path with \( n \) vertices, then \( G \) contains an induced path of size \( \sqrt{\frac{\log n}{2\sqrt{6}}} \) \( (1 - o(1)) \).

In this paper, a surface is a non-null compact connected 2-manifold without boundary. A surface is either orientable or non-orientable. The orientable surface \( S_h \) of genus \( h \) is obtained by adding \( h \geq 0 \) handles to the sphere, and the non-orientable surface \( N_k \) of genus \( k \) is formed by adding \( k \geq 1 \) cross-caps to the sphere. The Euler genus of a surface \( \Sigma \) is defined as twice its genus if \( \Sigma \) is orientable, and as its genus if \( \Sigma \) is non-orientable. We refer the reader to the monograph of Mohar and Thomassen [5] for background on graphs on surfaces.

Using Corollary 3.7, we easily deduce a similar bound for graphs embedded on a fixed surface.

**Theorem 3.8.** For any surface \( \Sigma \), any graph \( G \) embedded in \( \Sigma \), and containing a path with \( n \) vertices, also contains an induced path of size \( \frac{1}{6\sqrt{2}} \) \( (1 - o(1)) \sqrt{\log n} \) (where the \( o(1) \) depends on \( \Sigma \)).

Proof. Let \( f_g \) be the function defined as follows: \( f_0 \) is the \( o(1) \) defined in Corollary 3.7 and for each \( g > 0 \), \( f_g(n) = \frac{1}{2\sqrt{6}} - \frac{1}{2\sqrt{6}} - f_{g-1}(\frac{n}{\log n})(1 - \frac{\log \log n}{\log n}) \). It is not difficult to prove by induction on \( g \) that for fixed \( g \), \( f_g = o(1) \). We prove by induction on the Euler genus \( g \)
of $\Sigma$ that every graph embeddable in $\Sigma$ with a path $P$ on $n$ vertices has an induced path on $(\frac{1}{2\sqrt{6}} - f_g(n))\sqrt{\log n}$ vertices.

If $g = 0$, the result follows from Corollary 3.7, so assume that $g > 0$. Let $C$ be a shortest non-contractible cycle of $G$. Note that $C$ is an induced cycle, therefore, if $C$ has size at least $\log n$, then $G$ contains an induced path of size $\log n - 1 \geq (\frac{1}{2\sqrt{6}} - f_g(n))\sqrt{\log n}$ and we are done. Hence, we can assume that $C$ contains at most $\log n$ vertices.

The path $P$ and the cycle $C$ can have at most $\log n$ vertices in common. Let us denote by $p_1, \ldots, p_k$ these common vertices, in order of appearance in $P$. Then we have $P = P_0 - p_1P_1 - p_2 \cdots - p_k - P_k$, where each $P_i$ is a path (possibly empty). Since $P$ has $n$ vertices, there is one $P_i$ with at least $\frac{n}{\log n}$ vertices. If we remove the vertices of $C$ from $G$, we obtain a graph $G'$ such that each connected component is embeddable on a surface of Euler genus at most $g - 1$, and at least one such component contains a path on $\frac{n}{\log n}$ vertices. Then, by induction, $G'$ (and therefore $G$) contains an induced path of size

$$
\left(\frac{1}{2\sqrt{6}} - f_{g-1}(\frac{n}{\log n})\right)\sqrt{\log \frac{n}{\log n}} \\
\geq \left(\frac{1}{2\sqrt{6}} - f_{g-1}(\frac{n}{\log n})(1 - \frac{\log \log n}{\log n})\sqrt{\log n} \\
= \left(\frac{1}{2\sqrt{6}} - f_g(n)\right)\sqrt{\log n},
$$

as desired. \hfill \square

We do not know if the bound in Theorem 3.6 and Corollary 3.7 is optimal. We now construct a family of planar graphs containing a path with $n$ vertices in which the longest induced path has size $3\frac{\log n}{\log \log n}$. Let $G_1$ be the graph obtained by taking a path $P = p_1 - \cdots - p_k$ on $k$ vertices and adding two adjacent vertices $u$ and $v$ that are adjacent to each vertex of the path (see Figure 5). The graph $G_1$ has $k + 2$ vertices and a Hamiltonian path $u, p_1, \ldots, p_k, v$, unique up to symmetry. We define $G_{i+1}$ by induction: we start with a copy of $G_1$ (called the original copy of $G_1$) and replace each edge $p_j p_{j+1}$ of the path $P$ in $G_1$ by a copy of $G_{i}$, identifying $u$ of $G_i$ with $p_j$ of $G_1$ and $v$ of $G_i$ with $p_{j+1}$ of $G_1$. The vertices $u$ and $v$ in $G_{i+1}$ are then defined to be the vertices $u$ and $v$ or the original copy of $G_1$. We claim that $G_i$ has at least $(k - 2)^{i-1}$ vertices, a Hamiltonian path, and that the longest induced path in $G_i$ has $2i + (k - 2)$ vertices.

First, note that in $G_1$, the longest induced path starting from $u$ or $v$ has size $i + 1$. This is trivial for $G_1$, and if it is true for $G_{i-1}$, then in $G_i$, the longest path starting by $u$ (or $v$) is obtained by taking an edge from $u$ (or $v$) to a vertex of $P$, and then taking the longest induced path starting by this vertex in the copy of $G_{i-1}$, which by induction has $i - 1$ vertices.

Now, observe that an induced path in $G_i$ consists of an induced path in some copy of $G_{i-1}$, followed by an induced path in $G_1$, followed by an induced path in some copy of $G_{i-1}$. Note that the two copies might coincide, and if the induced path is not completely contained in a unique copy of $G_{i-1}$, then it contains some vertex $u$ or $v$ of the copies of $G_{i-1}$ it intersects. In any case, any induced path in $G_i$ contains at most $2i + (k - 2)$ vertices.
For $k = i$, we have a 2-connected planar graph with a path on $n \geq (k - 2)^{k-1}$ vertices and a longest induced path of size $3k - 2 \leq 3 \frac{\log n}{\log \log n}$.

We can use a similar construction to find a family of Hamiltonian chordal graphs of maximum clique size $2t + 1$ (and therefore tree-width $2t$) with $n$ vertices and no longest induced path of length more than $2t (\log n)^{\frac{1}{2}}$.

We first deal with graphs of tree-width 4. We consider the outerplanar graphs of Figure 2. We build $G_{1,4}$ by taking some $G_i$ of Figure 2 (a graph with $n$ vertices and a longest induced path of length $2 \log n$), and we add two adjacent vertices $u, v$ that are adjacent to every vertex of $G_i$. The graph $G_{1,4}$ is a 4-tree and contains a Hamiltonian path $P$ starting at $u$ and ending at $v$. Then we obtain $G_{k+1,4}$ by replacing each edge $a,b$ in the Hamiltonian path $P$ in $G_{1,4}$ by $G_{k,4}$, identifying $a$ with $u$ and $b$ with $v$ (there is still a Hamiltonian path starting by $u$ and ending by $v$ in $G_{k+1,4}$, and the graph has tree-width 4). We claim that $G_{k,4}$ has at least $n^k$ vertices and a longest induced path of length at most $2(\log n + k - 1)$.

First, note that in $G_{k,4}$ the longest induced path starting from $u$ or $v$ has size $k + 1$. Then observe as above that induced paths in $G_{k,4}$ are the concatenation of an induced path in a copy of $G_{k-1,4}$, an induced path in $G_{1,4}$, and an induced path in a copy of $G_{k-1,4}$. As before, if the induced path of $G_{k,4}$ is not contained in a copy of $G_{k-1,4}$, then it contains a vertex $u$ or $v$ of each of the at most copies of $G_{k-1,4}$ it intersects. Again, we conclude that any induced path in $G_{k,4}$ has size at most $2(\log n + k - 1)$.

For $k = \log n$, we obtain a graph with $N \geq n^{\log n}$ vertices, with a longest induced path of length at most $4 \log n \leq 4(\log N)^{\frac{1}{2}}$.

If we have a family of Hamiltonian graphs of tree-width $2t$ with $N$ vertices and a longest induced path of length $(\log N)^{\frac{1}{2}}$, then we can build the family for tree-width $2(t + 1)$. We take a Hamiltonian graph $G$ of tree-width $2t$ with $n$ vertices and a longest induced path of length $(\log n)^{\frac{1}{2}}$, and we add two adjacent vertices $u, v$ that are adjacent to every vertex of $G$: denote by $G_{1,2(t+1)}$ this graph. Then we obtain $G_{k+1,2(t+1)}$ by replacing each edge $ab$ in the Hamiltonian path in $G_{1,2(t+1)}$ by a copy of $G_{k,2(t+1)}$, identifying $a$ with $u$ and $b$ with $v$, which gives a graph of tree-width $2(t + 1)$. 

Figure 5: The graph $G_1$
Similarly, \(G_{k,2(t+1)}\) has \(N \geq n^k\) vertices, and a longest path in \(G_{k,2(t+1)}\) has size at most \(2(k + t(\log n)^{\frac{1}{t+1}})\). For \(k = (\log n)^{\frac{1}{t+1}}\), we have \(N \geq n^{(\log n)^{\frac{1}{t+1}}}\) vertices and the longest path has size at most \(2(t + 1)(\log N)^{\frac{1}{t+1}}\).

4 Induced paths in interval graphs

An important class of chordal graphs is the class of interval graphs. An interval graph is the intersection graph of a family of intervals on the real line. We will use the following notation. Let \(G\) be an interval graph. For every vertex \(v \in V(G)\), let \(I(v) = [l(v), r(v)]\) be the corresponding interval in an interval representation of \(G\). We may assume without loss of generality that the real numbers \(l(v), r(v)\ (v \in V(G))\) are all different. We call left ordering the ordering \(v_1 < \cdots < v_n\) of the vertices of \(G\) such that \(v < w\) if and only if \(l(v) < l(w)\). It is easy to see that for each \(i\) the vertex \(v_i\) is a simplicial vertex in the subgraph induced by \(v_1, \ldots, v_i\).

We now prove that interval graphs satisfy Conjecture \([I.1]\)

**Theorem 4.1.** For any integer \(k\), there is a constant \(c_k > 0\) such that if \(G\) is an interval graph with \(n\) vertices containing a Hamiltonian path and \(G\) has maximum clique size \(k\), then \(G\) has an induced path of size at least \(c_k(\log n)^{\frac{1}{k-1}}\).

The proof of this theorem is divided into three lemmas.

**Lemma 4.2.** Let \(G\) be an interval graph with \(n\) vertices, containing a Hamiltonian path. Let \(k \geq 2\) be the maximum clique size in \(G\) and let \(v_1 < \cdots < v_n\) be a left ordering of the vertices of \(G\). Then \(G\) contains an induced subgraph \(H\) of size at least \(f_1(n,k) = \log_{k+2}(\frac{n}{k+1})\) containing \(v_n\) where, in the induced left ordering, each vertex is adjacent to its successor.

**Proof.** We prove the lemma by induction on \(n\) and \(k\). If \(k = 2\), then the hypothesis implies that \(G\) is a path on \(n\) vertices, and the desired result holds with \(H = G\), and the inequality \(n \geq f_1(n,2) = \log_4(\frac{n}{6})\) holds for all \(n \geq 2\). If \(n \leq 3\), then either \(k = 2\) or \(k = 3\) and since \(f_1(3,3) < 1\) the result holds trivially. Now assume that \(k \geq 3\) and \(n \geq 4\). Let \(v_1 < \cdots < v_n\) be a left ordering of \(V(G)\). Let \(P\) be a Hamiltonian path in \(G\). Let \(i\) be the largest integer such that \(v_i\) is a neighbor of \(v_n\). and define the sets

\[
L = \{v \in V(G) \mid r(v) < l(v_n)\},
\]

\[
R = \{v \in V(G) \mid l(v_i) < l(v)\},\text{ and}
\]

\[
K = \{v \in V(G) \mid l(v) \leq l(v_i) \leq r(v)\}.
\]

So \(L, R\) and \(K\) form a partition of \(V(G)\). Clearly \(K\) is a clique, so the subgraph \(G_{RK}\) of \(G\) induced by \(R \cup K\) also has a Hamiltonian path. Every vertex \(v\) in \(R\) satisfies \(l(v_i) < l(v) \leq l(v_n) \leq r(v_i)\), so every vertex of \(G_{RK} \setminus \{v_i\}\) is adjacent to \(v_i\). It follows that \(G_{RK} \setminus \{v_i\}\).
has maximum clique size at most \( k - 1 \). Observe that \( v_n \) is the last vertex of \( G_{RK} \setminus \{v_i\} \) in its induced left ordering. Therefore, if \( G_{RK} \setminus \{v_i\} \) contains at least \( \frac{n}{k+2} \) vertices, then by the induction hypothesis \( G_{RK} \setminus \{v_i\} \) (and then \( G \)) contains an induced subgraph that satisfies the desired property and has size at least \( f_1(\frac{n}{k+2}, k - 1) = \log_{k+1}(\frac{n}{(k+2)(k+1)}) = \log_{k+1}(\frac{n}{(k+2)(k+1)}) \geq f_1(n, k) \).

Assume now that \( G_{RK} \setminus \{v_i\} \) contains strictly less than \( \frac{n}{k+2} \) vertices. Then \( L \) contains at least \( \frac{k+1}{k+2} n \) vertices, and the restriction of \( P \) to \( L \) consists of at most \( k + 1 \) subpaths (because \( |K| \leq k \)), so one of these subpaths has size at least \( \frac{n}{k+2} \). Let \( G_L \) be the graph induced by the vertices of this subpath, together with \( v_i \) and a vertex of \( K \) adjacent to an endpoint of the subpath. Note that \( G_L \) is Hamiltonian, and \( v_i \) is by definition the last vertex of \( G_L \) in its induced left ordering. By the induction hypothesis \( G_L \) (and then \( G \)) contains an induced subgraph \( H \) that satisfies the desired property and has size at least size \( f_1(\frac{n}{k+2}, k) \). In particular, the last vertex in the induced left ordering of \( H \) is \( v_i \). Appending \( v_i \) to \( H \) yields an induced subgraph of \( G \) that satisfies the desired property and has size at least \( f_1(\frac{n}{k+2}, k) + 1 = \log_{k+2}(\frac{n}{(k+2)(k+2)}) + 1 = f_1(n, k) \). \( \square \)

**Lemma 4.3.** Let \( G \) be an interval graph with \( n \) vertices, and let \( k \geq 2 \) be the maximum clique size in \( G \). Suppose that in the left ordering each vertex is adjacent to its successor. Then \( G \) contains an induced subgraph \( H \) of size \( f_2(n, k) = n^{1/k} \) where in the left ordering each vertex is adjacent to its successor and where there is no simplicial vertices, except the last and the first.

**Proof.** First, note that when we remove a simplicial vertex, we still have a graph where in left ordering, each vertex is adjacent to its successor.

At each step, we will remove a simplicial vertex which is not the first or the last, until there is no simplicial vertices other than the last and the first.

Let \( v \) be a vertex which is not the first or the last. Denote by \( w \) the first neighbor of \( v \) in left ordering. We claim that:

\[
\text{w is never removed}. \quad (4)
\]

Indeed if \( w \) is the first vertex in the ordering, it cannot be removed; so let us assume that \( w \) is not the first vertex, and let \( w' \) be its predecessor. So \( w' \) is adjacent to \( w \), and \( w' \) is not adjacent to \( v \) by the definition of \( w \). We claim that at each step there is a non-edge \( ab \) such that \( a \) and \( b \) are neighbors of \( w \) and \( l(a) < l(w) < l(v) \leq l(b) \). This is true at the first step with \( a = w' \) and \( b = v \). Assume that at step \( i \), there is such a non-edge \( ab \).

We remove a simplicial vertex \( s \). Clearly \( s \neq w \). If \( s \notin \{a, b\} \), then \( ab \) remains a non-edge in the neighborhood of \( w \). Suppose that \( s = b \). Let \( b' \) be the successor of \( b \) (note that \( b' \) exists since \( s \) is not the last vertex). Then \( b' \) is adjacent to \( w \) since \( b \) is simplicial, and \( l(v) \leq l(b) < l(b') \). Also \( b' \) is not adjacent to \( a \), for that would force \( b \) to be adjacent to \( a \) (because \( l(a) < l(b) < l(b') \)). Hence \( ab' \) is a non-edge with the desired property. Finally suppose that \( s = a \). Let \( a' \) be its predecessor (note that \( a' \) exists since \( s \) is not the first vertex). Then \( a' \) is adjacent to \( w \) since \( a \) is simplicial. Also \( a' \) is not adjacent to \( b \), for
otherwise $a'$ is adjacent to $v$ (because $l(a') < l(w) < l(v) \leq l(b)$), and so $a'$ contradicts the choice of $w$. Hence $a'b$ is a non-edge with the desired property. Thus [1] holds.

Now, we can prove the lemma by induction on $k$. For $k = 2$, the graph $G$ is a path and we can take $H = G$. Assume that $k \geq 3$. For a vertex $w$, we denote by $S_w$ the set of vertices having $w$ as their first neighbor. Suppose that for some vertex $w$, the set $S_w$ has size at least $n^{\frac{k-2}{k-1}}$. Note that all the vertices of $S_w$ are consecutive, and the subgraph $G[S_w]$ of $G$ induced by $S_w$ has clique size at most $k - 1$; hence, by the induction hypothesis, $G[S_w]$ has an induced subgraph of size $n^{\frac{k-2}{k-1}}$ with the desired property. Assume now that for every $w$, $S_w$ has size at most $n^{\frac{k-2}{k-1}}$. It follows from [1] that for every removed vertex $v$, the first neighbor of $v$ is preserved. Each first neighbor is counted at most $n^{\frac{k-2}{k-1}}$ times, so at least $n^{\frac{k-2}{k-1}}$ vertices are preserved, as desired. □

**Lemma 4.4.** Let $G$ be an interval graph with $n$ vertices and with maximal clique of size $k \geq 2$ where, in left ordering, each vertex is adjacent to its successor and where there are no simplicial vertices, except the last and the first. Then $G$ contains an induced path of size $f_3(n,k) = \left(\frac{n}{k}\right)^{\frac{1}{k-1}}$.

**Proof.** Note that the hypothesis implies that $G$ is connected. Let $S = \{s_1,\ldots,s_q\}$ be a maximal stable set, with $s_1 < \cdots < s_q$, and such that each interval $I(s_i)$ ($i \in \{1,\ldots,q\}$) is minimal by inclusion and (with respect to these two conditions) the numerical vector $l = (l(s_1),\ldots,l(s_q))$ is minimal in lexicographic order. We claim that for each $i = 2,\ldots,q - 1$, there is a vertex $t_{i-1}$ with $r(s_{i-1}),l(s_i) \in I(t_{i-1})$. This follows from the fact that $s_i$ is not simplicial and therefore has a non-edge $ab$, say with $a < b$, in its neighborhood. By inclusion-wise minimality of $I(s_i)$, $I(a)$ intersects $l(s_i)$, and if $a$ is not adjacent to $s_{i-1}$ then $I(a)$ (or an interval contained in $I(a)$ which is minimal with this property) contradicts the lexicographic minimality of $S$. Moreover we claim that there is a vertex $t_{q-1}$ with $r(s_{q-1}),l(s_q) \in I(t_{q-1})$. Indeed, since $G$ is connected, there is a chordless path $u_0\cdots u_p$ with $u_0 = s_{q-1}$ and $u_p = s_q$. Let $j$ be the smallest integer in $\{0,\ldots,p\}$ such that $l(u_j) > r(u_0)$; note that $j$ exists since $l(u_p) > r(u_0)$. Moreover, $l(u_j) \leq l(u_p)$ since the path is chordless. In fact $j = p$, for otherwise we have $l(u_j) < l(u_p)$ and the set $\{s_1,\ldots,s_{q-1},u_j\}$ would contradict the choice of $S$. Now the vertex $u_{p-1}$ can play the role of $t_{q-1}$ and satisfies the claim.

Let $U = S \cup \{t_1,\ldots,t_{q-1}\}$. We prove by induction on $k$ that if $U$ has size $N$, then there is an induced path of size $N^{\frac{1}{k-1}}$ in the subgraph of $G$ induced by $U$. If $k = 2$ then the subgraph induced by $U$ is a path. Now assume that $k \geq 3$. We consider the number of vertices of $U$ intersected by $t_1,\ldots,t_{q-1}$. Suppose that one of these vertices intersects more than $N^{\frac{1}{k-1}}$ vertices. In the graph induced by these vertices (and the corresponding vertices of $S$), the maximal clique has size at most $k - 1$, so by the induction hypothesis it contains an induced path of size $N^{\frac{1}{k-1}}$. Now suppose that each of $t_1,\ldots,t_{q-1}$ intersects at most $N^{\frac{1}{k-1}}$ vertices of $U$. Then we can build a path, starting with $s_q$, such that after each vertex $s_i$ with $i \neq 1$ we take the vertex $t_{i-1}$, and after each vertex $t_i$, we take the smallest
vertex of $S$ adjacent to $t_i$. Thus we obtain an induced path with at least $N^{\frac{1}{k-1}}$ vertices.

Since an interval graph with $n$ vertices and of maximum clique size $k$ is properly $k$-colorable, it contains a stable set of size at least $\frac{n}{k}$. It follows that $N \geq q \geq \frac{n}{k}$, and therefore $G$ contains an induced path of size $(\frac{n}{k})^{\frac{1}{k-1}}$, as desired. \hfill $\square$

It follows from the preceding three lemmas that any interval graph of maximum clique size $k$ containing a path on $n$ vertices also contains an induced path of size

$$\left(\frac{\log_{k+2}(n) - \log_{k+2}((k+2)!)^{k-1}}{k}\right)^{\frac{1}{k-1}} \leq c_k (\log n)^{\frac{1}{(k-1)^2}},$$

for some constant $c_k$. This proves Theorem 4.1.

This result shows that interval graphs satisfy Conjecture 1.1, but unfortunately we do not have a construction showing that our lower bound has the correct order of magnitude in the specific case of interval graphs. It might still be the case that interval graphs with long paths and bounded clique number have induced paths of polynomial size. Improving Lemma 4.2 might be the key in proving such a result (since the other two lemmas gives polynomial bounds).

## 5 Conclusion

We proved that $k$-trees with long paths have induced paths of logarithmic size. However, this does not give any clue whether partial $k$-trees with paths of size $n$ have induced paths of size polylogarithmic in $n$. We only proved that one cannot hope to obtained a bound exceeding $\Omega((\log n)^{\frac{1}{k-1}})$.

We believe that proving Conjecture 1.1 for partial $k$-trees will also imply (with a reasonable amount of work, based on our result for graphs embedded on fixed surfaces) that Conjecture 1.1 holds for any proper minor-closed class, and any proper class closed under topological minor (using the corresponding structure theorems).

## References

[1] J.L. Arocha and P. Valencia, *Long induced paths in 3-connected planar graphs*, Discuss. Math. Graph Theory **20** (2000), 105–107.

[2] A. Atminas, V. Lozin and I. Ragzon, *Linear time algorithm for computing a small biclique in graphs without long induced paths*, 13th International Scandinavian Symposium and Workshops on Algorithm Theory (SWAT 2012), LNCS, Springer, Vol. 7357 (2012), 142–152.
[3] G. Di Battista and R. Tamassia, *Incremental planarity testing*, Proc. 30th Annual Symposium on Foundations of Computer Science (1989), 436–441.

[4] E. Di Giacomo, G. Liotta and T. Mchedlidze, *Lower and upper bounds for long induced paths in 3-connected planar graphs*, Proc. 39th International Workshop on Graph-Theoretic Concepts in Computer Science (WG 2013), LNCS, Springer, Vol. 8165, 213–224.

[5] B. Mohar and C. Thomassen, *Graphs on Surfaces*. Johns Hopkins University Press, Baltimore, 2001.

[6] J. Nešetřil and P. Ossona de Mendez, *Sparsity – Graphs, Structures, and Algorithms*, Algorithms and Combinatorics Vol. 28, Springer, 2012.