A study of two new generalized negative KdV type
equations

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Abstract
We give a simple geometric interpretation of the mapping of the negative KdV equation
\((\psi_{xx})_t = (\psi^2)_x\) as proposed by Qiao and Li \{arXiv:1101.1605 [math-ph],
Europhys. Lett., 94 (2011) 50003\} and the Fuchssteiner equation \((u_{xxx})_x + 4(u_x)_x + 2u_t = 0\)
using geometry of projective connection on \(S^1\) or stabilizer set of the Virasoro orbit. We propose a similar
connection between \((\psi_{xx})_t = (\psi^3)_x\) and \((\psi_4)_x\) with the higher-order negative
KdV equations of Fuchssteiner type described as \((u_{xxxx})_x + 10(u_{xx})_x + 3u_t = 0\) and
\((u_{xxxx})_x + 20(u_{xxxx})_x + 30(u_{xx})_x + 18(u_{xx}^2 + 64u_{xx}^2)_x + 4u_t = 0\) respectively.
We study the Painlevé and symmetry analyses of these newly found equations and show that they yield soliton solutions.

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1 Introduction

It has been shown by Qiao and his collaborators (cf. \cite{26, 27}) that the following integrable system
\[
\left( -\frac{\psi_{xx}}{\psi} \right)_t = 2\psi\psi_x,
\] (1)
is actually related to the first member of the negative KdV equation of Fuchssteiner type
\[
\left( \frac{u_{txx}}{u_x} \right)_x + 4 \left( \frac{uu_t}{u_x} \right)_x + 2u_t = 0.
\] (2)

Fuchssteiner \cite{8} gave a hodograph link from a Bäcklund transformation of the Camassa-Holm equation (CH)
\[
m_t + m_x u + 2mu_x = 0, \quad \text{where } m = u - u_{xx},
\] (3)
to this particular member of the (negative) KdV-hierarchy. Therefore the equation (1) yields a simpler reduced form of the CH equation. The Lax pair of (1) is derived to guarantee its integrability. Furthermore the equation is shown to have classical solitons, periodic solitons and kink solutions.

Inspired by Fuchssteiner’s work, Schiff \cite{28} introduced the associated Camassa-Holm (ACH) equation and showed that it is related the Camassa-Holm equation. He derived Bäcklund transformations by a loop group technique and used these to obtain some simple soliton and rational solutions. Hone \cite{16} showed that, because the hodograph transformation is essentially the same as in \cite{8}, the ACH equation is naturally related to the inverse or negative KdV equation and has a Lax pair of which one part is just a (time-independent) Schrödinger equation.

Qiao and Li \cite{26} unified the positive- and negative-order KdV hierarchies as
\[
v_{tk} = JH_k = KH_{k-1}, \quad \forall k \in \mathbb{Z},
\] (4)
where \( K = \frac{1}{4}u^{-2}\partial u^2\partial u^{-2} \) and \( J = \partial \). The Hamiltonians \( H_k \) are defined via recursion operators \( R = J^{-1}K \) and \( R^- = K^{-1}J \), which is given by \( H_k = R^kH_0 \) and \( H_k = R^{k+1}H_{-1} \). This yields the entire KdV hierarchy, the positive order (\( k \geq 0 \)) gives the regular KdV hierarchy, while the negative order (\( k < 0 \)) produces some interesting equations gauge-equivalent to the Camassa-Holm equation. This hierarchy possesses the bi-Hamiltonian structure because of the Hamiltonian properties of \( J \). The second positive member of the hierarchy is the well-known KdV equation
\[
v_{t2} = \frac{1}{2}v_{xxx} + \frac{3}{2}v v_x,
\]
where \( v = -u_{xx}/u \). When \( k = -1 \), this coincides with equation (1).

The stabilizer orbit of the coadjoint action of the Virasoro algebra on its dual has been studied and it is known that many integrable ODEs are connected to this set \cite{10, 11}. It has been shown that by using Kirillov’s superalgebra \cite{17, 18} it is possible to describe the solution of the integrable systems associated to the stabilizer orbit. We formulate the solutions of Ermakov-Pinney equation and equations of Painlevé II type. The vector
field $f(x) \frac{d}{dx} \in Vect(S^1)$ associated to the stabilizer orbit is called the projective vector field \cite{15} and the equation associated to this is called the projective vector field equation $\Delta^{(3)} f = f_{xxx} + 4u f_x + 2u_x f = 0$. The operator $\Delta^{(3)} = \partial_x^3 + 4u \partial_x + 2u_x$ associated to the projective vector field equation is called the projective connection. In the next section we give a definition of projective connection and its connection to the stabilizer set of the Virasoro orbit.

In this paper we show an elegant geometrical connection between equations (1) and (2) in terms of higher-order projective connections $\Delta^{(n)}$ on $S^1$ \cite{12} \cite{13} and solution curves associated to $\Delta^{(n)} f = 0$. Using the same kind of map we then construct two new negative KdV equations of Fuchssteiner type, 

$$
\left(\frac{u_{txxx}}{3u^2 + u_{xx}}\right)_x + 10 \left(\frac{u t_x}{3u^2 + u_{xx}}\right)_x + 3u_t = 0
$$

and

$$
\left(\frac{u_{txxxx}}{u'''' + 16uu'}\right)_x + 20 \left(\frac{u t_{xxx}}{u'''' + 16uu'}\right)_x + 30 \left(\frac{u u_{xxx}}{u'''' + 16uu'}\right)_x + 18 \left(\frac{u_{xxx} u_t + 64u^2 u_x}{u'''' + 16uu'}\right)_x + 4u_t = 0,
$$

and show these are associated to some equations of Qian type, $(-\psi_{xxx}/\psi)_t = (\psi^3)_x$ and $(-\psi_{xxx}/\psi)_t = (\psi^4)_x$, respectively.

The projective connections appear most naturally in 2-D conformal field theory (CFT) and integrable systems. In 2-D CFT there exist several differential operators of various orders which transform covariantly under the coadjoint action of $Diff(S^1)$. Moreover, at least for $n \leq 4$, all these operators depend only upon $u$ and its derivatives. These are also known as the Adler-Gelfand-Dikii (or AGD) operators. Mathieu has listed several extended conformal operators in \cite{20} and some of the members of this family are $\Delta^{(0)} = 1$, $\Delta^{(1)} = \partial_x$, $\Delta^{(2)} = \partial_x^2 + u$, $\Delta^{(3)} = \partial_x^3 + 4u \partial_x + 2u'$, where $\Delta^{(2)}$ is the famous Hill’s operator and $\Delta^{(3)}$ is the second Hamiltonian operator for the KdV equation. Notice that the $\Delta^{(3)}$ operator also plays an important role in the inverse or negative KdV equation. In this article we focus on the next two higher-order operators,

$$
\Delta^{(4)} = \partial_x^4 + 9u^2 + 3u''^x + 10u' \partial_x + 10u \partial_x^2
$$

and

$$
\Delta^{(5)} = \partial_x^5 + 20u \partial_x^3 + 30u' \partial_x^2 + 18u'' \partial_x + 64u^2 \partial_x + 4u''' + 64uu'.
$$

One must note that the operator $\Delta^{(5)}$ can be rescaled to

$$
\tilde{\Delta}^{(5)} = \partial_x^5 + 10u \partial_x^3 + 15u' \partial_x^2 + 9u'' \partial_x + 16u^2 \partial_x + 2u''' + 16uu'
$$

for $u \to u/2$. We work with the second version of $\Delta^{(5)}$ operator.

The paper is organized as follows. In Section 2 we firstly introduce the stabilier set of the Virasoro orbit and projective connection on $S^1$ and then we give the derivations of the equations of negative KdV type. We establish geometrically the correspondences between the negative KdV equation of Qiao type and the KdV equation of Fuchssteiner type. Using this geometric correspondence we derive two new negative KdV equations in Section 3. In Section 4 we study the Painlevé properties and symmetry analysis of these two newly found equations. We obtain soliton solutions of these new equations in Section 5.
2 Stabilizer Set of Virasoro orbit, projective connection on $S^1$ and negative KdV equation

Initially we give a description of the stabilizer set of the Virasoro algebra and projective connection on $S^1$. Then we explore their roles in the derivation of the equations of negative KdV type. In particular we describe projective connections on $S^1$ and show their roles for the construction of the KdV equations of Qian and Fuchssteiner type.

We consider the Lie algebra of vector fields on $S^1$, $\text{Vect}(S^1)$. The dual of this algebra is identified with the space of quadratic differential forms $u(x)dx^2$ by the pairing

$$<u(x), f(x)> = \int_{0}^{2\pi} u(x)f(x)dx,$$

where $f(x)dx \in \text{Vect}(S^1)$. The Virasoro algebra $Vir$ has a unique nontrivial central extension by means of $R$

$$0 \longrightarrow R \longrightarrow Vir \longrightarrow \text{Vect}(S^1)$$
described by the Gelfand-Fuchs cocycle $\omega_1(f,g) = \frac{1}{2} \int_{S^1} f'g''dx$.

The elements of $Vir$ can be identified with the pairs ($2\pi$ periodic function, real number). The commutator in $Vir$ takes the form

$$\left( f(x)\frac{d}{dx}, a \right), \left( g(x)\frac{d}{dx}, b \right) = \left( fg' - gf', \frac{d}{dx} \int_{S^1} \frac{1}{2} f'g''dx \right).$$

The dual space $Vir^*$ can be identified with the set $\{(\mu, u dx^2) | \mu \in R\}$.

A pairing between a point $(\lambda, f(x)dx/dx) \in Vir$ and a point $(\mu, u dx^2)$ is given by $\lambda \mu + \int_{S^1} f(x)u(x) dx$.

Lemma 1

$$\text{ad}^{\ast}_{(\lambda, f(x)dx/dx)}(\mu, u dx^2) = \frac{1}{2} \mu f''' + 2f'u + 2fu'. \quad (7)$$

Proof: It follows from the definition

$$<\text{ad}^{\ast}_{(\lambda, f)}(\mu, (\nu, g))> = \left< (\mu, u), \text{ad}_{(\lambda, f)}(\nu, g) \right>$$

$$= \left< (\mu, u), \left( \frac{1}{2} \int_{S^1} f'g''dx, \left[ f' \frac{d}{dx}, g' \frac{d}{dx} \right] \right) \right>$$

$$= \int_{S^1} u (fg' - f'g') dx + \frac{1}{2} \mu \int_{S^1} f'g''.$$
Definition-Proposition 1 A vector field $v = f(x)\frac{d}{dx}$ is a called projective vector field which keeps fixed a given projective connection $\Delta = \frac{d^2}{dx^2} + u(x)$

$$\mathcal{L}_v \Delta s = \Delta (\mathcal{L}_v s),$$ (10)

for all $s \in \Gamma(\Omega^{-\frac{n-1}{2}})$, where $\mathcal{L}_v$ is the Lie derivative of $v$. A projective vector field $v = fd/dx \in \Gamma(\Omega^{-1})$ satisfies

$$f''' + 4f'u + 2fu' = 0.$$ (11)

2.1 Higher-order projective connections and Fuchssteiner’s negative KdV equation

We start with the definitions of projective connections.

**Definition 1 (Projective Connection)** An extended projective connection on the circle is a class of differential (conformal) operators,

$$\Delta^{(n)} : \Gamma(\Omega^{-\frac{n-1}{2}}) \rightarrow \Gamma(\Omega^{\frac{n+1}{2}}),$$

such that

1. the symbol of $\Delta^{(n)}$ is the identity,

2. \[\int_{S^1} (\Delta^{(n)} s_1) s_2 = \int_{S^1} s_1 (\Delta^{(n)} s_2)\]

for all $s_1 \in \Gamma(\Omega^{-\frac{n-1}{2}})$.

It is known that the symbol of an $n$th-order operator from a vector bundle $U$ to $V$ is a section of $\text{Hom}(U, V \otimes \text{Sym}^nT)$, where

$$U = \Omega^{-\frac{n-1}{2}} V = \Omega^{\frac{n+1}{2}}.$$ (12)

Because $T = \Omega^{-1}$, we have

$$V \otimes \text{Sym}^nT \cong U,$$

thereby giving an invariant meaning to the first condition.

If $s_2 \in \Gamma(\Omega^{-\frac{n+1}{2}})$, then $s_1 \Delta^{(n)} s_2 \in \Gamma(\Omega)$ is a one-form to integrate.

The consequence of the first condition is that all the differential operators are monic, that is, the coefficient of the highest derivative is always one. The second condition says that the term $u_{n-1} = 0$.

The weights, $-\frac{1}{2}(n-1)$ and $\frac{1}{2}(n+1)$, related to the space of operators $\Delta^{(n)}$ are known to physicists and mathematicians [19,37] but not from the point view of projective connections. Consider a one-parameter family of $\text{Vect}(S^1)$ acting on the space of smooth functions $a(x) \in C^\infty(S^1)$ [31]

$$\mathcal{L}_v^\lambda a(x) := f(x)a'(x) - \lambda f'(x)a(x),$$ (11)

where $\mathcal{L}_v^\lambda$ is the Lie derivative with respect to $v = f(x)\frac{d}{dx} \in \text{Vect}(S^1)$, given by

$$\mathcal{L}_v^\lambda := f(x)\frac{d}{dx} - \lambda f'(x).$$ (12)
**Definition 2** The action of $\text{Vect}(S^1)$ on the space of Hill’s operator $\Delta \equiv \Delta^{(2)}$ is defined by the commutator with the Lie derivative

$$[L_v, \Delta] := L_v^{-3/2} \circ \Delta - \Delta \circ L_v^{1/2}. \quad (13)$$

This action can be identified with the coadjoint action of Virasoro algebra on its dual. Here we discuss this briefly.

Similarly we can generalize this action on $\Delta^{(n)}$. The $\text{Vect}(S^1)$ action on $\Delta^{(n)}$ is defined by

$$[L_v, \Delta^{(n)}] := L_v^{-(n+1)/2} \circ \Delta^{(n)} - \Delta^{(n)} \circ L_v^{(n-1)/2}. \quad (14)$$

**Proposition 1** A vector field is called a projective vector field if it keeps fixed a given projective connection $\Delta$, $L_v \Delta^{(n)} s = \Delta^{(n)}(L_v s)$, for all $s \in \Gamma(\Omega^{-\frac{n-1}{2}})$. It satisfies

$$f''' + 4f'u + 2fu' = 0. \quad (15)$$

**Illustration**

$n = 3$: In this case the projective connection is defined by

$$\Delta^{(3)} = \partial_x^3 + 4u\partial_x + 2u'$$

and $s \in \Gamma(\Omega^{-1})$. It is easy to check that $L_v \Delta^{(3)} = \Delta^{(3)} L_v$ yields

$$(f''' + 4f'u + 2fu')' \phi + (f''' + 4f'u + 2fu') \phi' = 0.$$  

$n = 4$: The 4th-order projective connection $\Delta^{(4)} = \partial_x^4 + 9u^2 + 3u'' + 10u'\partial_x + 10u\partial_x^2$ maps $\Delta^{(4)} : \Gamma(\Omega^{-\frac{1}{2}}) \to \Gamma(\Omega^\frac{1}{2})$ and this immediately yields

$$(\partial_x^2 + 6u)(f''' + 4f'u + 2fu') + 5(f''' + 4f'u + 2fu')' \phi' + (f''' + 4f'u + 2fu') \phi''' = 0.$$  

A similar result is found for other higher-order projective connections.

**Lemma 2** If $\psi_1$ and $\psi_2$ are solutions of Hill’s equation

$$\Delta \psi = \left( \frac{d^2}{dx^2} + u \right) \psi = 0, \quad (16)$$

then the product $\psi_i \psi_j \in \Gamma(\Omega^{-1})$ satisfies equation $f''' + 2u'f + 4uf' = 0$ and traces out a three-dimensional space of solution.

**Remark** The sections of $\Gamma(\Omega^{-\frac{1}{2}})$ which satisfy equation (16) are not functions, but the square root of a projective vector field, because $\psi \in \Omega^{-1/2}$, the space of scalar densities of weight $-1/2$, is the square root of $f \in \text{Vect}(S^1)$. 


Proposition 2. Let \( \psi \) be a solution of Hill’s equation. If the flow on the immersion space satisfies \((-\frac{\psi_{xx}}{\psi})_t = (\psi^2)_x\), then the stabilizer set of the Virasoro orbit satisfies the negative KdV equation of Fuchssteiner’s type, namely
\[
\left( \frac{u_{xxx}}{u_x} \right)_x + 4 \left( \frac{uu_t}{u_x} \right)_x + 2u_t = 0.
\]

**Proof** Because \( \psi \) satisfies Hill’s equation, we find that \( u_t = (\psi^2)_x \). We know that, if \( \psi \) is a solution of the Hill equation, \( \psi^2 \) satisfies \( f''' + 2uf' + 4uf = 0 \). Thus we obtain our desired result. \( \square \)

2.1.1 Connection with the nonholonomic deformed KdV equation

In an interesting paper Kupershmidt [19] constructed a nonholonomic deformation of the KdV equation. By rescaling \( v \) and \( t \) he further modified this to
\[
\begin{align*}
\ u_t - 6uu_x - u_{xxx} + w_x &= 0, \\
\ w_{xxx} + 4uw_x + 2u_xw &= 0.
\end{align*}
\]
This can be converted into bi-Hamiltonian form
\[
\frac{\partial}{\partial t} = \mathcal{O}^2 \left( \frac{\delta H_n}{\delta u} \right) - \mathcal{O}^1(w), \quad \mathcal{O}^2(w) = 0,
\]
where
\[
\mathcal{O}^1 = \frac{\partial}{\partial x}, \quad \mathcal{O}^2 = \partial^3 + 2(u\partial + \partial u)
\]
are the two standard Hamiltonian operators of the KdV hierarchy, \( n = 2 \), and \( H_1 = u \), \( H_2 = u^2/2 \), \( H_3 = u^3/3 - u_x^2/2, \cdots \) are the conserved densities. It is known that the KdV6 equation always appears as a pair of equations, an evolution equation of \( u \) and a constraint equation of \( w \). In [14] we have studied various equivalent forms of the nonholonomic deformation of the KdV equation.

Equations (11) and (2) follow from the reduction of the nonholonomic deformation of the KdV equation. If we assume \( H_n = 0 \) and set \( w = -\psi^2 \), then the evolution \( u \) becomes \( u_t = (\psi^2)_x \) and the constraint equation becomes \( (\psi^2)_{xxx} + 4u(\psi^2)_x + 2u_x(\psi^2) = 0 \), which in turn is related to the negative KdV equation for \( u_t = (\psi^2)_x \).

3 A new flow on immersion space and generalized flows of negative KdV type

We define \( n \) independent solutions \((\psi_1, \psi_2, \cdots, \psi_n)\). The map
\[
x \mapsto (\psi_1(x), \psi_2(x), \cdots, \psi_n(x)), \quad \mathbb{R} \to \mathbb{R}P^{n-1},
\]
defines an immersion in homogeneous coordinates.
Lemma 3 There is a one-to-one correspondence between
(1) the nth-order equation on $S^1 \Delta^{(n)}\psi = 0$, where $\psi$ is the unknown function, and
(2) smooth orientation-preserving immersions $g : S^1 \to \mathbb{R}P^{n-1}$, modulo the equivalence
upto $PSL(n, \mathbb{R})$.

This proof goes as follows. Given the $n$ independent solutions, $\psi_1, \cdots, \psi_2$, of the equation
$\Delta^{(n)}\psi = 0$, then $x \mapsto (\psi_1(x), \psi_2(x), \cdots, \psi_n(x))$ defines a curve in the projective space
$\mathbb{R}P^{n-1}$. Because the Wronskian of the solution curve is constant upto multiplication by a
matrix in $SL(n, \mathbb{R})$, then the Wronskian of any immersion can be expressed by one. $\square$

Thus we obtain a solution curve associated to $\Delta^{(n)}$. As the coefficients are periodic,
hence, if $\psi(x)$ is a solution, then $\psi(x + 2\pi)$ is also a solution. This implies that
$$\psi(x + 2\pi) = M_\psi \psi(x),$$
where
$$M_\psi = \psi(2\pi)\psi(0)^{-1}$$
is a monodromy matrix. This matrix preserves the skew form given by the Wronskian so that $det(M_\psi) = 1$, i.e., $M_\psi \in SL(n, \mathbb{R})$. If one chooses a different solution curve, then a
new monodromy matrix appears. This is the conjugate of $M_\psi$ by an element of $SL(n, \mathbb{R})$.
This means that for each Lax operator we can associate a projective curve the monodromy
of which is an element of the conjugacy class $[M_\psi]$. This curve is unique up to the projective
action of $SL(n, \mathbb{R})$.

Lemma 4 Let $\psi_1$ and $\psi_2$ be solutions of Hill’s equation. The equation

\[(a) \quad f^{\prime\prime\prime\prime} + 10uf'' + 10uf' + (9u^2 + 3u'')f = 0 \quad (21)\]

traces out a four-dimensional space of solutions spanned by \{$\psi_1^3, \psi_1^2\psi_2, \psi_1\psi_2^2, \psi_2^3$\}.

\[(b) \quad f^{\prime\prime\prime\prime} + 20uf'' + 30uf' + 18u''f' + 64u^2f' + 4u'''f + 64uu'f = 0 \quad (22)\]

traces out a five-dimensional space of solutions spanned by \{$\psi_1^4, \psi_1^3\psi_2, \psi_1^2\psi_2^2, \psi_1\psi_2^3, \psi_2^4$\}.

Proof: By direct lengthy computation. $\square$

Proposition 3 Let $\psi$ be a solution of Hill’s equation $\psi_{xx} + u\psi = 0$.

1. Suppose that $\psi$ satisfies the flow equation

$$ \left( -\frac{\psi_{xx}}{\psi} \right)_t = 3\psi^2\psi_x = \left( \psi^3 \right)_x. \quad (23) $$

Then $u$ satisfies

$$ \left( \frac{u_{xxxx}}{3u^2 + u_{xx}} \right)_x + 10 \left( \frac{(uu_t)_x}{3u^2 + u_{xx}} \right)_x + 3u_t = 0. \quad (24) $$
2. If $\psi$ satisfies
\[
\left( \frac{-\psi_{xx}}{\psi} \right)_t = 4\psi^3 \psi_x = \left( \psi^4 \right)_x,
\]
then $u$ satisfies
\[
\left( \frac{u_{txxx}}{u'' + 16uu'} \right)_x + 20 \left( \frac{uu_{tx}}{u'' + 16uu'} \right)_x + 30 \left( \frac{u_xu_{tx}}{u'' + 16uu'} \right)_x + 4u_t = 0. \tag{25}
\]

**Proof** Because $\psi$ satisfies Hill’s equation, hence $u = (\psi^4)_x$. Substituting this expression into (21) we obtain our result. Similarly, when we substitute $u = (\psi^4)_x$ into (22), we obtain (26).

\[\Box\]

4 Painlevé and symmetry analyses for the newly derived equations

The equation
\[
\left( \frac{u_{txx}}{u_x} \right)_x + 4 \left( \frac{uu_t}{u_x} \right)_x + 2u_t = 0, \tag{27}
\]
in which $u = u(t, x)$, possesses the symmetries
\[
\Gamma_1 = \partial_x,
\Gamma_2 = x\partial_x - 2u\partial_u \quad \text{and} \quad \Gamma_3 = a(t)\partial_t. \tag{28, 29, 30}
\]
The function $a(t)$ is arbitrary, apart from the requirement that it be differentiable, and reflects the fact that (27) is homogeneous in the first derivative with respect to time.

The algebra is $A_2 \oplus A_\infty$.

Equation (27) can be expanded by the denominator removed to give
\[
27u^4u_t - 60uu_x^2u_t + 48u^2u_{xxx}u_t + 13u_{xx}^2u_t - 10u_xu_{xxx}u_t
+ 20u_xu_{xx}u_{tx} - 10uu_{xxx}u_{tx} + 30u_x^3u_{txx} + 10uu_{xx}u_{txx}
- 6uu_xu_{txxx} - u_{xxx}u_{txxx} + 3u^2u_{ttxxx} + u_{xxx}u_{txxxx} = 0. \tag{31}
\]

To seek a travelling-wave solution of (31) we make the substitution $u(t, x) \rightarrow w(v)$, where $v = x - ct$. The fifth-order equation subsequent upon this substitution and division by $-c$ is
\[
27w^4w' - 60ww'^3 + 48w^2w''w' + 33ww'w'' + 30w^3w''' - 10w'^2w''' - 6ww'w'''
- w'''w''' + 3w^2w^{(5)} + w^{(5)} = 0. \tag{32}
\]

(Note that this equation replaces (23) in the text.)

\[\text{[1] Courtesy of the mathematica add-on Sym [4, 5, 6].}\]
Table 1: Listing of the resonances corresponding to the three possible values of the coefficient of the leading-order term [23]

| Coefficient | Set of Resonances                      |
|-------------|----------------------------------------|
| −12         | −1, 2, 14, \( \frac{1}{2} (1 - \sqrt{337}) \), \( (1 + \sqrt{337}) \)     |
| −2          | −2, −1, 6                              |
| −\( \frac{10}{9} \) | −1, 2, \( \frac{10}{3} \), \( \frac{14}{3} \), 7 |

Equation (32) has just the two symmetries \( \Gamma_1 = \partial_v \) and \( \Gamma_2 = 2v\partial_v - 3w\partial_w \) with the algebra \( A_2 \) in the Mubarakzyanov Classification Scheme [21, 22, 23, 24]. The number of Lie point symmetries is insufficient to reduce the equation completely.

We investigate (32) from the point of view of singularity analysis. The exponent of the leading-order term is −2 and the equation for the coefficient is the solution of

\[
-1440a^2 - 2136a^3 - 816a^4 - 54a^5 = 0, \tag{33}
\]

namely \( a = -12, -2 \) and −10/9 in addition to a double zero. The equation to be satisfied by the resonances is

\[
-2880a - 6408a^2 - 3264a^3 - 270a^4 + 3048as + 3712a^2s + 828a^3s
+ 27a^4s - 1416as^2 - 1320a^2s^2 - 366a^3s^2 + 474as^3 + 257a^2s^3 + 30a^3s^3
- 96as^4 - 48a^2s^4 + 6as^5 + 3a^2s^5 = 0. \tag{34}
\]

The resonances corresponding to the three values of the coefficient of the leading-order term are listed in Table 1.

Some observations are in order. Firstly only the third resonance, −10/9, has the full number of acceptable resonances. The first resonance, −12 certainly has five resonances, but two of them are quite irrational. The second coefficient does not even have the right number of resonances. The Check Sum for the first and third resonances is 22 whereas that for the second resonance is 3. The Laurent expansion is a Right Painlevé Series in \( (v - v_0)^{1/9} \). Note that the expansion would be valid only on a section of the punctured disc due to the presence of powers which are multiples of 1/9.

There has been no success in attempts to integrate the equation directly.
Table 2: Listing of the resonances corresponding to the three possible values of the coefficient of the leading-order term (24)

| Coefficient                  | Set of Resonances                                      |
|------------------------------|--------------------------------------------------------|
| $-\frac{1}{4}$              | $-2, -1, 4, 6$                                        |
| $\frac{3}{76}(-6 - i\sqrt{154})$ | $-1, 2, 6, 8, \frac{1}{133}\left(133 - \sqrt{\frac{917 + 72i\sqrt{154}}{19}}\right)$, $\frac{1}{133}\left(133 + \sqrt{\frac{917 + 72i\sqrt{154}}{19}}\right)$ |
| $\frac{3}{76}(-6 + i\sqrt{154})$ | $-1, 2, 6, 8, \frac{1}{133}\left(133 - \sqrt{\frac{917 - 72i\sqrt{154}}{19}}\right)$, $\frac{1}{133}\left(133 + \sqrt{\frac{917 - 72i\sqrt{154}}{19}}\right)$ |

Equation (26) has the travelling-wave form

$$
9456w^2w'^3 - 768w^w^w + 3200ww'^2w'' + 832w^2w''w''' + 48w''^2w''' \\
+ 72w'w'''^2 - 832w^2w'(4) - 48w'w''(4) - 16w'^2w(5) \\
- 16ww'''w(5) - w(4)w(5) + 16ww'w(6) + w''w(6) = 0.
$$

(35)

Again the exponent of the leading-order term is $-2$. The coefficient of the leading-order term satisfies the equation

$$
-34560a^2 - 101376a^3 - 190464a^4 - 155648a^5 = 0.
$$

(36)

The roots of (36) are given by

$$
\left\{ a \to -\frac{3}{4} \right\}, \left\{ a \to 0 \right\}, \left\{ a \to \frac{3}{76}(-6 - i\sqrt{154}) \right\}, \left\{ a \to \frac{3}{76}(-6 + i\sqrt{154}) \right\}.
$$

(37)

The equation for the resonances is

$$
-69120a - 304128a^2 - 761856a^3 - 778240a^4 + 87552as + 22184a^2s \\
+ 427520a^3s + 233472a^4s - 47136as^2 - 98176a^2s^2 \\
- 55808a^3s^2 + 16320as^3 + 29056a^2s^3 - 5504a^3s^3 - 3960as^4 - 5664a^2s^4 \\
+ 1664a^3s^4 + 528as^5 + 704a^2s^5 - 24as^6 - 32a^2s^6 = 0.
$$

(38)

The resonances for the nonzero values of $a$ are in Table 2. In the case of the first resonance the check sum is 7 whereas for the second and third resonances it is 22.

None of the possible values of the coefficient of the leading-order term gives a satisfactory set of resonances. In the cases of the second and third one could opine that the complex
values completely destroy any possible results from the singularity analysis. The values of the resonances for the first coefficient are acceptable, albeit of insufficient number. In the past there has been talk of possessing the Partial Painlevé Property, but the idea has not been accepted by the experts in the area of singularity analysis.

5 Solitonic solutions of generalised negative KdV equations

We consider the possibility of the existence of solitonic solutions to the three partial differential equations of interest. The first is (23)

\[ \left( \psi^3(t, x) \right)_x + \left( \frac{\psi(t, x)_{xx}}{\psi(t, x)} \right)_t = 0. \] (39)

We make the substitution

\[ \psi(t, x) = A \operatorname{sech}^n(x - pt), \] (40)

where the three parameters \( A, n \) and \( p \) are to be determined. After some simplification we have

\[ 2A^2 n(1 + n)p \operatorname{sech}^{3+2n}(x - pt) + 3A^5 p \operatorname{sech}^{1+5n}(x - pt) = 0. \] (41)

We achieve balance by equating the two exponents which means that \( n = 2/3 \). It then follows that the ‘speed of propagation’ is given by \( p = -9A^3/10 \).

The second equation (25) is

\[ \left( \psi^4(t, x) \right)_x + \left( \frac{\psi(t, x)_{xx}}{\psi(t, x)} \right)_t = 0. \] (42)

We make the same substitution as in (40) and the simplified equation is

\[ p(1 + n) \left( 1 - \tanh^2(x - pt) \right) + 2A^4 \operatorname{sech}^{4n}(x - pt) = 0. \] (43)

Evidently balance is achieved by setting \( n = \frac{1}{2} \) and \( p = -4A^4/3 \).

The third equation is

\[ \left( \frac{u(t, x)_{xx}}{u(t, x)} \right)_x + 4 \left( \frac{u(t, x)u(t, x)_x}{u(t, x)_x} \right)_x + 2u(t, x)_t = 0. \] (44)

After the substitution of (40) and some simplification we have

\[
2A^2 n^2 p \operatorname{sech}^{3+2n}(x - pt) \sinh^{3}(x - pt) \left\{ 2 + 3n + n^2 + 3An \operatorname{sech}^n(x - pt) \right. \\
\left. + (2 + 3n + n^2) \tanh^{2}(x - pt) \right\} = 0
\] (45)

in which balance is achieved by setting \( n = 2 \) and, as a consequence, \( A = 2 \).

So the three equations all have solitonic solutions.
6 Conclusion

We have given a brief description of the projective connections on $S^1$ and their role for the construction of generalised equations of negative KdV type. Using our method we established a connection between the negative KdV equation of Fuchssteiner and that of Qiao. In this paper we studied two equations of generalised Qiao type, namely, $(-\frac{\psi_{xx}}{\psi})_t = 3\psi^2\psi_x$ and $(-\frac{\psi_{xx}}{\psi})_t = 4\psi^3\psi_x$ and mapped these two generalised negative KdV equations to generalized equations of Fuchssteiner type. We have studied symmetries and Painlevé properties of these two latter equations. We also showed that they admit solitonic solutions.

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