Decoherence effects on multiplayer cooperative quantum games

Salman Khan, M. Ramzan, and M. Khalid Khan

Department of Physics, Quaid-i-Azam University, Islamabad 45320, Pakistan.

Abstract

We study the behavior of cooperative multiplayer quantum games [35,36] in the presence of decoherence using different quantum channels such as amplitude damping, depolarizing and phase damping. It is seen that the outcomes of the games for the two damping channels with maximum values of decoherence reduce to same value. However, in comparison to phase damping channel, the payoffs of cooperators are strongly damped under the influence amplitude damping channel for the lower values of decoherence parameter. In the case of depolarizing channel, the game is a no-payoff game irrespective of the degree of entanglement in the initial state for the larger values of decoherence parameter. The decoherence gets the cooperators worse off.

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*Electronic address: sksafi@phys.qau.edu.pk
I. INTRODUCTION

Game theory provides a mathematical background for evaluating behavior of competing agents. Emerged from the work of von Neumann [1], theorists in various disciplines such as economics, biology, medical sciences, social sciences and physics utilize its concepts to maneuver competing situations [2-6]. Although technically difficult, quantum theory is conceptually very rich and quantum game theorists use it to study the behavior of classical games in this realm for more than a decade [7-16]. The quantum extension of classical games began from the seminal work of Meyer [17]. It is shown that the quantum mechanical treatment of classical games produces results that cannot be achieved in the classical formalism. Quantum strategies and quantum entanglement lead quantum players to harness the outcome of the game in their favour.

Quantum mechanically competing agents communicate with each other through quantum channels. Information can be encoded in qubits, qutrits or qudits. These sources of information while passing through the channels interact with the many degrees of freedom of the environment thereby creating entangled state with it. This leads to the distortion of system space and results in the loss of encoded information which is not inevitable [18]. The distortion of the system space through the interaction with environment is called decoherence. Quantum error correction [19] and entanglement purification [20] are the two methods developed to handle the problem of decoherence. Quantum games in the presence of decoherence have been studied by a number of authors [21-27] and many more. It is seen that the effect of decoherence on the payoff functions of players is different for different games setup. For example, in some cases it gets worse off the players while in other cases it makes better off one player over the other [22, 27].

In the field of quantum games most work in the beginning was done in studying two person games. Benjamin and Hayden [29] were the first to study multiplayer games. Few of many others who contributed to the study of multiplayer games are given in [30-36].

In this paper we investigate the effect of decoherence and entanglement on cooperative three and four players quantum game under the action of amplitude damping, depolarizing and phase damping channels. The amount of decoherence in the case of each channel is parameterized by the decoherence parameter $p$ which has values from the range 0 to 1. The lower and upper limits of $p$ correspond to undecohered and fully decohered cases respectively.
II. THREE-PLAYERS COOPERATIVE GAME

A classical three persons symmetric cooperative game consists of three players $A$, $B$, $C$ with strategy set $U_n, \{n = A, B, C\}$ for each player and three real valued payoff functions $P_A, P_B, P_C$, one each corresponds to a player. The strategy set of each player consists of two strategies denoted by 0 and 1. Players $A$ and $B$ are said to be cooperators if they choose the same strategy, different from $C$ in a play of the game. No one wins if they all choose the same strategy and the loser is one who chooses strategy different from the other two players. In each play of the game, the loser pay a fixed amount to the other two winners that is equally divided between them. Hence the game in its classical form is a zero sum game.

The quantum version of the game consists of three qubits, one for each player, that is, the game space is an eight dimensional Hilbert space. The strategy set of each player consists of two strategies $I$ and $\sigma_x$, where $I$ is the single qubit identity operator and $\sigma_x$ is the Pauli spin flip operator. The game starts from an initial three qubits entangled state, prepared by an arbiter. The initial state is sent to each player. The players execute their strategies on their own qubit and the final state is returned to the arbiter. The arbiter performs measurement in the computational basis and the corresponding payoffs of the players are declared.

The game was initially quantized in two different ways of using the strategies $I$ and $\sigma_x$. In Ref. [37] the classical probability method, in which each player has the option to use $I$ with certain probability $x$ and $\sigma_x$ with probability $1 - x$, has been used. Whereas in Ref. [38] the quantum superposed operator method has been adopted in quantizing the three players game. In this method, each player has the option to execute his strategy as a linear combination of the two allowed strategies in the form $U_i = \sqrt{x}I + \sqrt{1-x}\sigma_x$. When this is applied to state $|0\rangle$, it gives $\sqrt{x}|0\rangle + \sqrt{1-x}|1\rangle$. This means that the player on measurement gets 0 with probability $x$ and 1 with probability $1 - x$. However for four players cooperative game, both methods are used in Ref. [38] to quantize the game. It is shown that the both methods produce the same outcome. Keeping this in mind, we proceed to incorporate decoherence effects in the quantum superposed formalism both for three and four players games.
A. Quantum channels

A quantum channel transfers information from one place (input) to another place (output). In the course of transformation, the source of information may interact with the many degrees of freedom of the channel and leads to the information damage. The effect of quantum channels on the state of a system is a completely positive trace preserving map that is described in terms of Kraus operators [39].

\[ \rho_f = \sum_k E_k \rho_i E_k^\dagger, \]  

where \( \rho_i = |\psi\rangle \langle \psi| \) is the initial density matrix of the system with \( |\psi\rangle \) being the initial state. The Kraus operators \( E_k \) satisfy the following completeness relation

\[ \sum_k E_k^\dagger E_k = I. \]  

The single qubit Kraus operators for different channels used in this paper are given in TABLE I: A single qubit Kraus operators for amplitude damping channel, phase damping channel and depolarizing channel.

| Amplitude damping | \( E_o = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{pmatrix} \), \( E_1 = \begin{pmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{pmatrix} \) |
|-------------------|---------------------------------|
| Phase damping     | \( E_o = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{pmatrix} \), \( E_1 = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{p} \end{pmatrix} \) |
| Depolarizing      | \( E_o = \sqrt{1-p} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \), \( E_1 = \sqrt{p/3} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \), \( E_2 = \sqrt{p/3} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \), \( E_3 = \sqrt{p/3} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \) |

The single qubit Kraus operators for different channels used in this paper are given in TABLE I: A single qubit Kraus operators for amplitude damping channel, phase damping channel and depolarizing channel.

The Kraus operators for three-players and four-players are of dimensions \( 2^3 \) and \( 2^4 \) respectively. These Kraus operators are constructed by taking the tensor product of all possible combinations of single qubit Kraus operators in the following way

\[ E_k = \bigotimes_i E_i \]  

where \( E_i \) represent the Kraus operators of a single qubit for a given channel and the index \( i \) stands for the number of single qubit Kraus operators for that particular channel.
For the three players game, we consider the initial state to be $|\psi\rangle = \cos\theta/2|000\rangle + \sin\theta/2|111\rangle$, where $\theta \in [0, \pi/2]$ is a measure of entanglement [38]. The final density matrix of the game after the players execute their moves is given by

$$\rho'_f = \frac{(U_{AB} \otimes U_C) \rho_f (U_{AB} \otimes U_C)^\dagger}{\text{Tr}((U_{AB} \otimes U_C) \rho_f (U_{AB} \otimes U_C)^\dagger)},$$

where the trace operation in the denominator ensures that the output is normalized and represents the final density matrix of the game. In Eq. [4] $\rho_f$ is given by Eq. [1]. The operator $U_{AB} = \sqrt{q}I \otimes I + \sqrt{1-q}\sigma_z \otimes \sigma_z$, represents the strategy of the two cooperators and $U_C = \sqrt{r}I + \sqrt{1-r}\sigma_z$, is the strategy of the third player. The payoff functions of the players are given by [37]

$$P_{A,B,C}(p,q,r) = \text{Tr}(P_{A,B,C}^{\text{oper}} \rho'_f),$$

where $P_{A,B,C}^{\text{oper}}$ are the payoff operators for players $A$, $B$ or $C$, which are given by

$$P_{A,B,C}^{\text{oper}} = \sum_{i=1}^{8} (\alpha_i, \beta_i, \gamma_i) \times \rho'_{ii},$$

with $\rho'_{ii}$ are the diagonal elements of the final density matrix $\rho'_f$ of the game. $\alpha_i$’s, $\beta_i$’s and $\gamma_i$’s are the elements of the payoff matrix of the three players game. In Eq. [6] $\alpha_i$’s correspond to the payoff operator $P_{A}^{\text{oper}}$ of player $A$, $\beta_i$’s correspond to the payoff operator $P_{B}^{\text{oper}}$ of player $B$ and $\gamma_i$’s correspond to the payoff operator $P_{C}^{\text{oper}}$ of player $C$ respectively. According to the rules of the game, the values of the matrix elements $\alpha_i$’s of player $A$ become

$$\begin{align*}
\alpha_1 &= \alpha_8 = 0, \\
\alpha_2 &= \alpha_3 = \alpha_6 = \alpha_7 = 1, \\
\alpha_4 &= \alpha_5 = -2.
\end{align*}$$

(7)

Similarly, the values of $\beta_i$’s and $\gamma_i$’s for players $B$ and $C$ are, respectively, given as

$$\begin{align*}
\beta_1 &= \beta_8 = 0, \\
\beta_2 &= \beta_4 = \beta_5 = \beta_7 = 1, \\
\beta_3 &= \beta_6 = -2, \\
\gamma_1 &= \gamma_8 = 0, \\
\gamma_3 &= \gamma_4 = \gamma_5 = \gamma_6 = 1, \\
\gamma_2 &= \gamma_7 = -2.
\end{align*}$$

(8)
B. Results and discussion for three players game

In this section, we present and discuss the results of our calculation obtained under the action of amplitude damping, depolarizing and phase damping channels for the three players game. In case of amplitude damping channel, the payoff function of cooperators $A$ and $B$ is obtained as

$$P_{AD}^{A,B} = \frac{(-q - r + 2qr) [1 - 2p (1 - p) (1 - \cos \theta)] - 2(1 - p)}{1 + 4(p - 1) \sqrt{qr (1 - p)(1 - q)(1 - r) \sin \theta}} \times \sqrt{qr (1 - p)(1 - q)(1 - r) \sin \theta}.$$

(10)

Maximizing $P_{AD}^{A,B}$ with respect to $q$, and $r$, we get $q = r = \frac{1}{2}$. This result is independent both from entanglement parameter $\theta$ and decoherence parameter $p$. Using these values of $q$ and $r$ in Eq. 10, the maximum payoff of cooperators becomes

$$P_{AD}^{A,B,\text{max}} = \frac{\left[1 - 2p (1 - p) (1 - \cos \theta) + (1 - p)^{3/2} \sin \theta\right]}{2(1 + (1 - p)^{3/2} \sin \theta)}.$$

(11)

Unlike the equilibrium payoff in the classical form of the three players game, this payoff depends both on entanglement parameter $\theta$ and decoherence parameter $p$. In figure 1, the dependence of cooperators’ payoff on both entanglement and decoherence parameters is shown in the form of a density plot. It is seen that for a maximally entangled initial state of the game, the payoffs of cooperators are minimum, when the decoherence parameter has values in the range from 0.45 to 0.72. Whereas for unentangled initial state, the presence of decoherence parameter does not affect the payoff considerably for the entire range of its values. For other values of entanglement parameter, the presence of decoherence damps the payoff as compared to undecohered case.

The payoff of player $C$ is obtained as

$$P_{AD}^{C} = - \frac{2 [(2qr - q - r) (-1 + 2p (1 - p) (1 - \cos \theta))] + 4 (1 - p)}{+1 + 4 (1 - p) \sqrt{qr (1 - p)(1 - q)(1 - r) \sin \theta}} \times \sqrt{qr (1 - p)(1 - q)(1 - r) \sin \theta}.$$

(12)

Maximizing player $C$’s payoff with respect to $q$ and $r$, and using their values in Eq. 12, the
FIG. 1: The payoff of a player $A(B)$ is plotted against the decoherence parameter $p$ and entanglement parameter $\theta$ for amplitude damping channel with $q = r = 0.2$.

equilibrium payoff becomes

$$P_{\text{AD}}^{\text{C, max}} = \frac{-1 - 2p(1 - p)(1 - \cos \theta) + (1 - p)^{3/2} \sin \theta}{1 + (1 - p)^{3/2} \sin \theta}. \quad (13)$$

In case of depolarizing channel, the payoff function of the cooperators becomes

$$P_{\text{DP}}^{\text{A,B}} = \frac{(3 - 4p)^2(3q + 3r - 6qr + 2(3 - 4p)\sqrt{qr(1 - q)(1 - r)\sin \theta})}{27 - 4(-3 + 4p)^2\sqrt{qr(1 - q)(1 - r)\sin \theta}}. \quad (14)$$

The maximization of $P_{\text{DP}}^{\text{A,B}}$ with respect to $q$ and $r$, leads to $q = r = \frac{1}{2}$, and the maximum payoff of cooperators becomes

$$P_{\text{DP}}^{\text{A,B, max}} = \frac{(3 - 4p)^2(3 + (3 - 4p)\sin \theta)}{54 - 2(-3 + 4p)^3 \sin \theta}. \quad (15)$$
The dependence of the payoff on decoherence and entanglement parameters shows that the

behavior of the game is different both from undecohered and unentangled initial state cases.
In this case, the payoff of cooperators against decoherence and entanglement parameters is
illustrated in figure 2 as a density plot. In the range of large values of decoherence parameter,
the payoffs of the players vanish irrespective of the degree of entanglement in the initial state
of the game. Thus for a fully decohered depolarizing channel the advantage of entanglement,
contrary to small values of decoherence parameter, in the initial state of the game vanishes.

The payoff of player $C$ is given by

$$P_C^{\text{DP}} = -\frac{2(3 - 4p)^2(3q + 3r - 6qr + 2(3 - 4p)\sqrt{qr(1 - q)(1 - r)}\sin \theta)}{27 - 4(-3 + 4p)^3\sqrt{qr(1 - q)(1 - r)}\sin \theta}. \tag{16}$$
The maximized payoff of player $C$ becomes

$$P_{C,\text{max}}^{\text{DP}} = \frac{(3 - 4p)^2(3 + (3 - 4p)\sin \theta)}{27 - (-3 + 4p)^3 \sin \theta}. \quad (17)$$

The payoff of cooperators under the action of phase damping channel is given by

$$P_{A,B}^{\text{PD}} = \frac{q + r - 2qr + 2(1 - p)\sqrt{qr(1 - p)(1 - q)(1 - r)} \sin \theta}{1 + 4(1 - p)\sqrt{qr(1 - p)(1 - q)(1 - r)} \sin \theta}. \quad (18)$$

The maximized payoff of the cooperators happens at $q = r = 1/2$ and is given by

$$P_{A,B,\text{max}}^{\text{PD}} = \frac{1}{2}. \quad (19)$$

In figure 3, we plot the payoff of cooperators as a function of decoherence and entanglement parameters for phase damping channel. It can be seen that in the absence of entanglement in the initial state, the payoff is minimum for the entire range of decoherence parameter. Similarly, for highly decohered channel, the degree of entanglement does not effect the outcome of the game and the payoff of cooperators in this range of decoherence parameter remains minimum.

The payoff of player $C$ is negative and twice the payoff function of a cooperator, that is,

$$P_{C}^{\text{PD}} = -2P_{A,B}^{\text{PD}}. \quad (20)$$

The superscripts AD, DP and PD in the above relations stand for amplitude damping, depolarizing and phase damping channels respectively. As the sum of payoffs of the players under all the three channels is zero, the game in its quantum form with decoherence is a zero sum game. In the classical form of the game, the maximum values of payoffs that define the Nash equilibrium of the game is a fixed point. Whereas in the presence of decoherence, the Nash equilibrium under the action of amplitude and depolarizing channels is a function of both decoherence parameter $p$ and entanglement parameter $\theta$. The effect of decoherence on the payoff of player $A$ ($B$) for the maximally entangled initial state for all the three channels is shown in figure 4. It is seen that for a highly decohered case, the amplitude damping and phase damping channels reduce the outcome of the game to the same value. However, heavy damping is observed in the case of amplitude damping channel as compare to damping in the case of phase damping channel for $p$ lesser than 1. The depolarizing channel ends the game with no payoffs around a 75% decoherence. In figure 5, we plot the payoff of cooperators with and without decoherence against entanglement angle for a 50% decoherence. It is
FIG. 3: The payoff of player A(B) is plotted against the decoherence parameter $p$ and entanglement parameter $\theta$ for phase damping channel with $q = r = 0.2$.

seen that the channels damp the payoff for the entire range of entanglement parameter in comparison to undecohered case. However, the phase damping channel makes better off the cooperators than the other two channels in the range of large values of entanglement parameter. The amplitude damping channel results in high degradation in the range of large values of entanglement parameter. It can also be seen that under the influence of depolarizing channel, the decoherence results in heavy damping of the payoff. Furthermore, the effect of entanglement on the payoff function for depolarizing channel almost vanishes. It can also be shown that the game becomes a no-loss no-gain game for the entire range of entanglement parameter when the channel is highly decohered.
FIG. 4: The payoff of player $A(B)$ for all the three channels is plotted against the decoherence parameter $p$ when the initial state is maximally entangled with $q = r = 0.2$. The labels AD, DP, PD and ND stand for amplitude damping, depolarizing, phase damping and no damping cases respectively.

III. DECOHERENCE IN FOUR PLAYERS COOPERATIVE GAME

In this section, we study the effect of decoherence on four players cooperative game, using quantum superposed operator method, by using the three quantum channels as in the case of three players cooperative game. The game space in this case is a sixteen dimensional Hilbert space. We consider the initial state of the game to be $|\psi\rangle = \cos \theta/2 |0000\rangle + \sin \theta/2 |1111\rangle$. The strategy of the two cooperating players is $U_{AB} = \sqrt{q} I \otimes I + \sqrt{1-q} \sigma_x \otimes \sigma_x$, whereas for the other two players the strategies are respectively given by $U_C = \sqrt{r} I + \sqrt{1-r} \sigma_x$ and $U_D = \sqrt{s} I + \sqrt{1-s} \sigma_x$. The final density matrix of the game after the players execute their strategies is given by

$$
\rho'_f = \frac{(U_{AB} \otimes U_C \otimes U_D) \rho_f (U_{AB} \otimes U_C \otimes U_D)^\dagger}{\text{Tr}((U_{AB} \otimes U_C \otimes U_D) \rho_f (U_{AB} \otimes U_C \otimes U_D)^\dagger)}, \quad (21)
$$
FIG. 5: The payoff of player $A(B)$ for all the three channels is plotted against the entanglement parameter $\theta$ for decoherence parameter $p = 0.5$ with $q = r = 0.2$. The labels AD, DP, PD and ND stand for amplitude damping, depolarizing, phase damping and no damping cases respectively.

where $\rho_f$ is given by Eq. 1. The payoff functions of the players are given by Eq. 5 with payoff operator given by

$$P_{A,B,C,D}^{\text{oper}} = \sum_{i=1}^{16} (\alpha_i, \beta_i, \gamma_i, \delta_i) \times \rho'_{ii}. \quad (22)$$

The payoff operators $P_A^{\text{oper}}, P_B^{\text{oper}}, P_C^{\text{oper}}$ and $P_D^{\text{oper}}$ correspond to the matrix elements $\alpha_i$’s, $\beta_i$’s, $\gamma_i$’s and $\delta_i$’s respectively. According to the rules of the game, the matrix elements $\alpha_i$’s of player $A$ become

$$\begin{align*}
\alpha_1 &= \alpha_4 = \alpha_6 = \alpha_7 = \alpha_{10} = \alpha_{11} = \alpha_{13} = \alpha_{16} = 0, \\
\alpha_2 &= \alpha_3 = \alpha_5 = \alpha_{12} = \alpha_{14} = \alpha_{15} = 1, \\
\alpha_8 &= \alpha_9 = -3. \quad (23)
\end{align*}$$
For the other three players, the matrix elements $\beta_i$'s, $\gamma_i$'s and $\delta_i$'s are given by

\begin{align*}
\beta_1 = \beta_4 = \beta_6 = \beta_7 = \beta_{10} = \beta_{11} = \beta_{13} = \beta_{16} &= 0, \\
\beta_2 = \beta_3 = \beta_8 = \beta_9 = \beta_{14} = \beta_{15} &= 1, \\
\beta_5 = \beta_{12} &= -3,
\end{align*}

(24)

\begin{align*}
\gamma_1 = \gamma_4 = \gamma_6 = \gamma_7 = \gamma_{10} = \gamma_{11} = \gamma_{13} = \gamma_{16} &= 0, \\
\gamma_2 = \gamma_5 = \gamma_8 = \gamma_9 = \gamma_{12} = \gamma_{15} &= 1, \\
\gamma_3 = \gamma_{14} &= -3,
\end{align*}

(25)

\begin{align*}
\delta_1 = \delta_4 = \delta_6 = \delta_7 = \delta_{10} = \delta_{11} = \delta_{13} = \delta_{16} &= 0, \\
\delta_3 = \delta_5 = \delta_8 = \delta_9 = \delta_{12} = \delta_{14} &= 1, \\
\delta_2 = \delta_{15} &= -3.
\end{align*}

(26)

The payoffs of players for the case of amplitude damping channel become

\begin{align*}
P_{A,B}^{AD} &= \frac{(r + s - 2rs) [1 - 2p(1 - p)(1 - \cos \theta)] + 4(1 - p)^2 \times \sqrt{qrs(1 - q)(1 - r)(1 - s)\sin \theta}}{1 + 8(1 - p)^2 \sqrt{qrs(1 - q)(1 - r)(1 - s)\sin \theta}}, \\
&P_{C}^{AD} = \frac{[s - 4qs + r(-3 + 4q + 2s)][(1 - 2p(1 - p)(1 - \cos \theta)] - 4(1 - p)^2 \times \sqrt{qrs(1 - q)(1 - r)(1 - s)\sin \theta}}{1 + 8(1 - p)^2 \sqrt{qrs(1 - q)(1 - r)(1 - s)\sin \theta}}, \\
&P_{D}^{AD} = \frac{[r(-1 + 4q - 2s) + (3 - 4q)s][1 + 2p(1 - p)(1 - \cos \theta)] - 4(1 - p)^2 \times \sqrt{qrs(1 - q)(1 - r)(1 - s)\sin \theta}}{1 + 8(1 - p)^2 \sqrt{qrs(1 - q)(1 - r)(1 - s)\sin \theta}}.
\end{align*}

(27)

The maximization of these payoffs gives $q = r = s = \frac{1}{2}$ and the corresponding maximum payoffs of the players become

\begin{align*}
P_{A,B,\text{max}}^{AD} &= \frac{1 - 2p(1 - p)(1 - \cos \theta) + (1 - p)^2 \sin \theta}{2[1 + (1 - p)^2 \sin \theta]}, \\
P_{C,D,\text{max}}^{AD} &= -\frac{1 - 2p(1 - p)(1 - \cos \theta) + (1 - p)^2 \sin \theta}{2[1 + (1 - p)^2 \sin \theta]}
\end{align*}

(28)
The payoffs of players under the action of depolarizing channel are given as

\[ P_{DP}^{A,B} = \frac{(3 - 4p)^2[9(r + s - 2rs) + 4(3 - 4p)^2 \sqrt{qrs(1 - q)(1 - r)(1 - s) \sin \theta}]}{81 + 8(3 - 4p)^4 \sqrt{qrs(1 - q)(1 - r)(1 - s) \sin \theta}}, \]

\[ (3 - 4p)^2[9(s - 4qs - 3r + 4qr + 2rs) - 4(3 - 4p)^2 \times \sqrt{qrs(1 - q)(1 - r)(1 - s) \sin \theta}]\]

\[ P_{DP}^{C} = \frac{(3 - 4p)^2[9(r + s - 2rs) + 4(3 - 4p)^2 \sqrt{qrs(1 - q)(1 - r)(1 - s) \sin \theta}]}{81 + 8(3 - 4p)^4 \sqrt{qrs(1 - q)(1 - r)(1 - s) \sin \theta}}, \]

\[ (3 - 4p)^2[9(s - 4qs - 3r + 4qr + 2rs) + 4(3 - 4p)^2 \times \sqrt{qrs(1 - q)(1 - r)(1 - s) \sin \theta}]\]

\[ P_{DP}^{D} = \frac{(3 - 4p)^2[9(r + s - 2rs) + 4(3 - 4p)^2 \sqrt{qrs(1 - q)(1 - r)(1 - s) \sin \theta}]}{81 + 8(3 - 4p)^4 \sqrt{qrs(1 - q)(1 - r)(1 - s) \sin \theta}}. \]

A similar behavior of the players’ payoffs is seen as in the case of three-player game under decoherence.

**IV. CONCLUSION**

Cooperative three and four player quantum games influenced by different noise channels are analyzed. The advantage of quantum entanglement in the initial state of the game for cooperators is adversely affected. For a given decoherence level, the cooperators are better off under the action of phase damping channel in the range of larger values of entanglement angle as compared to the other two channels. In the case of amplitude damping channel, for a fixed value of decoherence parameter, a decrease in payoff of cooperators is observed with the increasing value of entanglement parameter. The game becomes a no-payoff game around a decoherence of 75% irrespective of the degree of entanglement in the case
of depolarizing channel. For a fully decohered case, the amplitude damping and phase damping channels reduce the outcome of the game to the same value. Furthermore, for a maximally entangled initial state under the action of amplitude damping channel the payoffs of cooperators reaches to a minimum at $p = 0.7$ and increase again till the channel becomes fully decohered. In brief, the decoherence makes the cooperators’ payoffs worse off both in three players and four players cooperative game.

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Figures Captions

Figure 1: The payoff of player $A(B)$ is plotted against the decoherence parameter $p$ and entanglement parameter $\theta$ for amplitude damping channel with $q = r = 0.2$.

Figure 2: The payoff of player $A(B)$ is plotted against the decoherence parameter $p$ and entanglement parameter $\theta$ for depolarizing channel with $q = r = 0.2$.

Figure 3: The payoff of player $A(B)$ is plotted against the decoherence parameter $p$ and entanglement parameter $\theta$ for phase damping channel with $q = r = 0.2$.

Figure 4: The payoff of player $A(B)$ for all the three channels is plotted against the decoherence parameter $p$ when the initial state is maximally entangled with $q = r = 0.2$. The labels AD, DP, PD and ND stand for amplitude damping, depolarizing, phase damping and no damping cases respectively.

Figure 5: The payoff of player $A(B)$ for all the three channels is plotted against the entanglement parameter $\theta$ for decoherence parameter $p = 0.5$ with $q = r = 0.2$. The labels AD, DP, PD and ND stand for amplitude damping, depolarizing, phase damping and no damping cases respectively.

Table Caption

Table 1. A single qubit Kraus operators for amplitude damping channel, phase damping channel and depolarizing channel.