The large-scale structure of passive scalar turbulence

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We investigate the large-scale statistics of a passive scalar transported by a turbulent velocity field. At scales larger than the characteristic lengthscale of scalar injection, yet smaller than the correlation length of the velocity, the advected field displays persistent long-range correlations due to the underlying turbulent velocity. These induce significant deviations from equilibrium statistics for high-order scalar correlations, despite the absence of scalar flux.

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Turbulent flows are systems far away from equilibrium, characterized by a flux of energy through a range of scales. This cascade process is often accompanied by strongly non-gaussian statistics and nontrivial scaling properties. Passive scalar turbulence is no exception, in this respect. Its understanding has recently undergone remarkable progress, with results thoroughly reviewed in Ref. 2. A passive scalar field \(\theta(x, t)\), like dilute dye concentration or temperature in appropriate conditions, transported by an incompressible velocity field \(v(x, t)\), evolves according to

\[
\partial_t \theta + v \cdot \nabla \theta = \kappa \Delta \theta + f,
\]

where \(f\) is a source of scalar fluctuations that acts at a lengthscale \(L\). The velocity field is assumed to be turbulent and characterized by a self-similar statistics (e.g. \(v(x + r) - v(x) \sim r^{1/3}\) according to Kolmogorov’s 1941 theory) in the range of scales delimitied above by the velocity correlation length \(L_v\) and below by the viscous scale \(\eta\). Passive scalar fluctuations generated at the scale \(L\) form increasingly finer structures due to velocity advection and this process results in a net flux of scalar variance to small scales, where it is eventually smeared out by molecular diffusivity at a scale \(r_d\). Here we will consider the case where these scales are ordered as follows: \(L_v \gg L \gg \eta, r_d\). In the range \(L \gg r \gg r_d\) the average scalar flux is constant and equals the average input rate: this is the well studied inertial-convective range where \(\theta\) displays non-gaussian statistics and anomalous scaling. Conversely, at scales larger than \(L\) there is no scalar flux. Accordingly, one would expect Gaussian statistics and equipartition of scalar variance, i.e. the hallmarks of statistical equilibrium. This expectation is correct at \(r \gg L_v\), where the dynamics of the large-scale passive scalar is ruled by an effective diffusion equation resulting in a Gaussian statistics. However, in the intermediate range \(L_v \gg r \gg L\), deviations from “thermal equilibrium” might arise as a consequence of turbulent transport. Indeed, Falkovich and Fouxon have recently shown — in the context of the Kraichnan model of passive scalar advection where the velocity field is gaussian, self-similar and short-correlated in time — that the scalar field shows a highly nontrivial behavior at scales larger than the pumping correlation length \(L\), with significant differences from Gibbs statistical ensemble. In this Letter we show that these results extend to passive scalar advection by a realistic flow, namely two-dimensional Navier-Stokes turbulence in the inverse cascade range. This flow has been studied in great detail both experimentally (in fast flowing soap films and in shallow layers of electromagnetically driven electrolyte solutions) and numerically. The velocity is statistically homogeneous and isotropic, scale-invariant with exponent \(1/3\) (no intermittency corrections to Kolmogorov scaling) and with dynamical correlation times. This flow has also been utilized to investigate passive scalar transport in the scalar flux range \(L \gg r \gg r_d\) and multi-particle dispersion, an intimately related subject. From this ensemble of studies it emerged that the lessons drawn from the study of the Kraichnan model are indeed relevant for the qualitative understanding of passive scalar transport by realistic flows; and this will turn out to be true for the present case as well.

Some basic, although incomplete, information about the scalar statistics in the supposed “thermal equilibrium range” can be gained by studying the isotropic spectrum \(E(k) = 2\pi k \langle|\theta(k)|^2 \rangle\) for \(kL \gtrsim 1\). Let us recall that the equilibrium statistics for the scalar field would be described by the Gibbs functional \(\mathcal{P}[\theta] = Z^{-1} \exp\left[-\beta \int \langle|\theta(k)|^2 \rangle d^3k\right]\), i.e. the Fourier modes should behave as independent Gaussian variables with equal variance \(1/(2\beta)\). As shown in Fig. we observe \(E(k) \sim k\), in agreement with equipartition arguments; moreover, the statistics of single Fourier modes is indistinguishable from Gaussian. However, we anticipate that from those findings alone one cannot state conclusively that large-scale passive scalar is in a thermal equilibrium state, given that they do not allow to rule out the possibility of long-range correlations for higher order observables (e.g. four-point scalar correlations).

A more refined description of the large-scale properties of passive scalar can be obtained in terms of the coarse-grained field

\[
\theta_r(x, t) = \int G_r(x - y) \theta(y, t) dy
\]
where $G_r$ acts as a low-pass filter in Fourier space (for instance, the top-hat filter $G_{\rho}(x-y) = 1/(\pi r^2)$ if $|x-y| < r$ and zero otherwise; or the Gaussian filter $G_r(x-y) = (2\pi r^2)^{-1} \exp(-|x-y|^2/(2r^2))$. For $r \to 0$ the filter reduces to a two-dimensional $\delta$-function and therefore $G_r \to \theta$. The statistics of $\theta$ is typically supergaussian [17,18]; its probability density function has exponential-like tails even for a gaussian driving force $f$. Indeed, in the latter case it can be shown that $\theta$ is the product of two independent random variables $\theta = \phi \sqrt{F_0 T}$ where $\phi$ is a gaussian variable of zero mean and unit variance, $F_0$ is the average injection rate of scalar fluctuations, and $T$ is a positive-defined random variable, independent from $\phi$. The variable $T$ is essentially the time taken by a spherical blob of minute initial size to disperse along a length $L$ for a given flow configuration [17,18]. For example, assuming a poissonian distribution for the rare events when $T$ exceeds its average value yields exponential tails for the pdf of $\theta$ [19]. The distribution of $\theta_r$ is supergaussian as well; however, as $r$ increases above the forcing correlation length, the probability density of $\theta_r$ tends to a gaussian distribution, as it is clearly seen by the scale-dependence of the distribution flatness and hyperflatness (see Fig. 2).

Within the framework of Gibbs statistical equilibrium, the scalar field has vanishingly small correlations above the scale $L$: therefore one could view $\theta_r$ as the sum of $N \approx (r/L)^2$ independent random variables (identically distributed as $\theta$) divided by $N$. By central limit theorem arguments [20], the moments of order $2n$ of the coarse-grained scalar field (odd-order moments vanish by symmetry) should then scale as $N^{-n}$, giving $\langle (\theta_r)^{2n} \rangle \sim \langle (\theta_r^2)^n (r/L)^{2n} \rangle$. This is a very good estimate for $n = 1$: indeed, as shown in Fig. 3, the product $(r/L)^2 \langle \theta_r^2 \rangle$ has a very neat plateau. This is consistent with the fast decay of the two-point scalar correlation $\langle \theta(x,t) \theta(x+r,t) \rangle$ at $r \gtrsim L$. Indeed, in this case the second-order moment $\langle \theta_r^2 \rangle = \int \langle \theta(y_1) \theta(y_2) \theta_r(y_1-x) \theta_r(y_2-x) \rangle \propto (r/L)^{-2}$ is dominated by contributions with $|y_1-y_2| \lesssim L$ yielding $\langle \theta_r^2 \rangle \sim (\theta^2)(r/L)^{-2}$. Alternatively, by Fourier transforming the coarse-grained field one obtains $\langle \theta_r^2 \rangle = \int \langle \hat{G}_r(k) \rangle^2 \langle |\hat{\theta}(k,t)|^2 \rangle dk \simeq \langle \theta^2 \rangle (r/L)^{-2}$, since the transformed filter $\hat{G}_r(k)$ is close to unity for $kr \ll 1$ and falls off very rapidly for $kr \gtrsim 1$, and $\langle |\hat{\theta}(k,t)|^2 \rangle \simeq \langle \theta^2 \rangle / (\pi L^2)$. In summary, two-point statistics appears to be consistent with Gibbs equilibrium ensemble. The situation for multi-point correlations will turn out to be different.

A careful inspection of higher-order moments shows a less good agreement with central-limit theorem estimates (see Fig. 3): this points to the existence of subleading contributions to the moments $\langle \theta_r^{2n} \rangle$ for $n > 1$ arising from long-range correlations of multiple scalar products. In order to quantify more precisely the rate of convergence to gaussianity and its relationship to long-range correlations, it is useful to consider the cumulants of the random variable $\theta_r$. According to central limit theorem [20], the

![FIG. 1: (a) Passive scalar and velocity spectra. The data result from the time integration of the two-dimensional Navier-Stokes equations $\partial_t v + \nabla \cdot \mathbf{v} = -\nabla p + \nu \nabla^2 \mathbf{v} + \mathbf{F} - \alpha \mathbf{v}$ and eq. (1) by a pseudospectral method on a 1024$^2$ grid. The passive scalar is injected by a Gaussian, $\delta$-correlated in time, statistically homogeneous and isotropic forcing restricted to a narrow band of wavenumbers. The initial condition for the velocity field is a configuration taken from a previous long-time integration and thus already at the statistically stationary state. The passive scalar starts from a zero field configuration, and after a transient of a few large-eddy turnover times $\tau_v = L_v/v_{rms}$ where $L_v$ is the integral scale of the velocity field, it reaches its own statistically steady state as well. Time averages are taken after this relaxation time has elapsed, for a total duration of more than $10^4$ scalar correlation times $\tau_L \approx \tau_v (L/L_v)^{5/3}$. Here $L/L_v \approx 0.02$. The velocity spectrum agrees with the Kolmogorov prediction $k^{-5/3}$ and the passive scalar one follows very closely the equipartition spectrum in two dimensions $E(k) \sim k$ (see also the inset of panel (b)). (b) The marginal probability density function of a single Fourier amplitude $\theta(k)$ is indistinguishable from a Gaussian (dotted curve) for all wavenumbers in the range $L^{-1} \gtrsim k \gtrsim L_v^{-1}$. Here are shown three wavenumbers with $kL = 0.5, 0.25, 0.12$. In the inset is shown the spectral density $\langle |\hat{\theta}(k)|^2 \rangle$ that shows a neat plateau at $kL \ll 1$ (notice the linear scale on the vertical axis).]
cumulant of order $2n$ should vanish with $N^{-2n+1}$ leading to an expected scaling $\langle (\theta^{2n}_r) \rangle \sim (\langle (\theta^{2n}_r) \rangle (r/L)^{-2n+2}$. Let us reiterate that the former expression is expected to be valid in absence of scalar correlations across length scales $r \gtrsim L$.

For $n = 1$ we have $\langle (\theta^2_r) \rangle = \langle \theta^2 \rangle$ whose behavior has been already detailed above. In Fig. 2 we show the behavior of $\langle (\theta^2_r) \rangle = \langle \theta^2 \rangle - 3\langle \theta^2 \rangle^2$ and $\langle (\theta^6_r) \rangle = \langle \theta^6 \rangle - 15\langle \theta^2 \rangle^3\langle \theta^4 \rangle + 30\langle \theta^2 \rangle^2\langle \theta^4 \rangle^2$. For the fourth-order cumulant, we observe a scaling law very close to the theoretical expectation $\langle (\theta^2_r) \rangle \sim \langle (\theta^2) \rangle (r/L)^{-16/3}$.

This has to be contrasted with the scaling law $(r/L)^{-6}$ given by central limit arguments. The breakdown of central limit theorem is due to the existence of long-range dynamical correlations in the range $r \gg L$. These exclude the possibility of a true Gibbs statistical equilibrium at large scales. The leading contribution to the fourth-order cumulant $\langle (\theta^2_r) \rangle = \int d\mathbf{y}_1 d\mathbf{y}_2 d\mathbf{y}_3 d\mathbf{y}_4 G_r(\mathbf{y}_1 - \mathbf{x})G_r(\mathbf{y}_2 - \mathbf{x})G_r(\mathbf{y}_3 - \mathbf{x})G_r(\mathbf{y}_4 - \mathbf{x})\langle \theta(\mathbf{y}_1, t)\theta(\mathbf{y}_2, t)\theta(\mathbf{y}_3, t)\theta(\mathbf{y}_4, t) \rangle$ comes from configurations with the four points arranged in two pairs of close particles (e.g. $|\mathbf{y}_1 - \mathbf{y}_2| \lesssim L$ and $|\mathbf{y}_3 - \mathbf{y}_4| \lesssim L$) separated by a distance $r$ (e.g. $|\mathbf{y}_1 - \mathbf{y}_3| \approx r$). Otherwise stated, two-point correlators of the squared scalar field $\langle (\theta^2(x, t)\theta^2(x + r, t)) \rangle$ must display a nontrivial scaling $(r/L)^{-4/3}$. We will get back to the issue of the statistics of $\theta^2$ momentarily. The sixth-order cumulant $\langle (\theta^6_r) \rangle$ is extremely difficult to measure because of the strong cancellations between various terms. Upon collecting the statistics over about ten thousand scalar correlation times, we can conclude that the results are consistent with the power-law decay $\langle (\theta^6_r) \rangle \sim (\langle (\theta^6) \rangle (r/L)^{-22/3}$ suggested by the theory, and arising from terms like $\langle (\theta^2_r) \rangle\langle (\theta^4_r) \rangle$ that appear in the expansion of the sixth-order cumulant. The actual exponent for $\langle (\theta^6_r) \rangle$ cannot be determined with great precision, yet it lies within the range between $-7$ and $-8$, thus definitely different from the central-limit-theorem expectation $-10$.

Further insight on the deviations from statistical equilibrium at large scales can be gained by studying the statistics of the coarse-grained squared scalar field

$$\theta^2_r(x, t) = \int G_r(x - y)\theta^2(y, t) dy.$$ (3)

The cumulants of $\theta^2_r$ give useful information about the presence of long-range correlations of the field $\theta^2$. The first-order cumulant $\langle (\theta^2_r) \rangle \equiv \langle \theta^2_r \rangle$ is trivially equal to $\langle \theta^2 \rangle$. The second-order cumulant $\langle (\theta^2_r)^2 \rangle = \langle \theta^2_r \rangle^2$ -
This is not the case for $T_1$ and $T_2$ because of the underlying velocity field. Therefore, the long power-law tail for $\langle (\theta^2)^2 \rangle$ arises from events where $\langle T_1 T_2 \rangle \gg \langle T \rangle^2$. This amounts to say that two blobs of initial size smaller than $L$, released at a distance $r \gg L$ in the same flow, do not spread considerably by turbulent diffusion (i.e. $T_{1,2} \gg \langle T \rangle$) with a probability $\sim (r/L)^{-4/3}$.

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[20] The statistics of the random variable $z_N = (\sum_{i=1}^{N} w_i)/N$, where $w_i$ are independent and identically distributed random variables with zero mean and finite variance $\sigma^2$, is completely characterized by the generating function $G(s) = \langle \exp(sz_N) \rangle$. By virtue of statistical independence and identity in distribution of the $w_i$, we have

$G(s) = \langle \prod_{i=1}^{N} \exp(sw_i/N) \rangle = \langle \exp(sw/N) \rangle^N = g(s/N)^N$

where $g$ is the generating function for $w$. For $N \gg 1$ we have $g(s/N) \simeq (1 + s^2\sigma^2/(2N^2))^N \approx \exp(s^2\sigma^2/(2N))$ (a version of the central limit theorem) and therefore $\langle z_N^2 \rangle \simeq (2n-1)!s^{2n}N^{-n}$. The cumulants $\langle z_N^{2n} \rangle$ are defined in terms of the Taylor series of $\ln G(s)$ around $s = 0$: since $\ln G(s) = N \ln g(s/N)$ we have $\langle z_N^{2n} \rangle \sim \langle (w^{2n}) \rangle N^{-(2n+1)}$.\n
\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig5.png}
\caption{Cumulants of the coarse-grained, squared scalar field $\theta^2$. The definitions for the low-order cumulants of a generic random variable $z$ are: $\langle \langle z \rangle \rangle = \langle z \rangle$, $\langle \langle z^2 \rangle \rangle = \langle z^2 \rangle - \langle z \rangle^2$, $\langle \langle z^3 \rangle \rangle = \langle z^3 \rangle - 3\langle z \rangle \langle z^2 \rangle + 2\langle z \rangle^3$, $\langle \langle z^4 \rangle \rangle = \langle z^4 \rangle - 4\langle z \rangle \langle z^3 \rangle - 3\langle z^2 \rangle^2 + 12\langle z \rangle\langle z^2 \rangle - 6\langle z \rangle^4$.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig6.png}
\caption{A snapshot of the squared scalar field $\theta^2$. Remark the inhomogeneous distribution of scalar intensity originating from long-range correlations $\langle \langle \theta^2 \rangle \rangle \sim r^{-4/3}$.}
\end{figure}