RICCI CURVATURE OF DOUBLE MANIFOLDS VIA ISOPARAMETRIC FOLIATIONS

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Abstract. Given a closed manifold \( M \) and a vector bundle \( \xi \) of rank \( n \) over \( M \), by gluing two copies of the disc bundle of \( \xi \), we can obtain a closed manifold \( D(\xi, M) \), the so-called double manifold.

In this paper, we firstly prove that each sphere bundle \( S_r(\xi) \) of radius \( r > 0 \) is an isoparametric hypersurface in the total space of \( \xi \) equipped with a connection metric, and for \( r > 0 \) small enough, the induced metric of \( S_r(\xi) \) has positive Ricci curvature under the additional assumptions that \( M \) has a metric with positive Ricci curvature and \( n \geq 3 \).

As an application, if \( M \) admits a metric with positive Ricci curvature and \( n \geq 2 \), then we construct a metric with positive Ricci curvature on \( D(\xi, M) \). Moreover, under the same metric, \( D(\xi, M) \) admits a natural isoparametric foliation.

For a compact minimal isoparametric hypersurface \( Y^n \) in \( S^{n+1}(1) \), which separates \( S^{n+1}(1) \) into \( S^{n+1} \) and \( S^{n+1} - \), one can get double manifolds \( D(S^{n+1}) \) and \( D(S^{n+1} -) \). Inspired by Tang, Xie and Yan’s work on scalar curvature of such manifolds with isoparametric foliations(cf. [TXY12]), we study Ricci curvature of them with isoparametric foliations in the last part.

1. Introduction

Let \( N \) be a connected complete Riemannian manifold. A non-constant smooth function \( f : N \to \mathbb{R} \) is called transnormal, if there exists a smooth function \( b : \mathbb{R} \to \mathbb{R} \) such that the gradient of \( f \) satisfies \( |\nabla f|^2 = b(f) \). Moreover, if there exists another function \( a : \mathbb{R} \to \mathbb{R} \) such that the Laplacian of \( f \) satisfies \( \Delta f = a(f) \), then \( f \) is said to be isoparametric. The two equations mean that regular level hypersurfaces are parallel and all have constant mean curvature. Each regular level hypersurface is said to be an isoparametric hypersurface. In [Wa87], Wang proved that singular level sets are also smooth submanifolds, the so-called focal submanifolds. Recall that an isoparametric foliation on a Riemannian manifold \( N \) is defined by the whole family of regular level hypersurfaces together with focal submanifolds of an isoparametric function(cf. [Wa87], [GT13] and [QT15]). Equivalently, an isoparametric foliation is a singular Riemannian foliation all of whose regular leaves are hypersurfaces with constant mean curvature(cf. [GQ15]). For instance, the orbits of an isometric cohomogeneity one action on \( N \) form an isoparametric foliation, the so-called homogeneous isoparametric foliation. Hence,
isoparametric foliations can be regarded as a geometric generalization of cohomogeneity one actions.

E. Cartan was the pioneer who made a comprehensive study of isoparametric hypersurfaces in real space forms, especially in the unit spheres. Particularly, he proved that isoparametric hypersurfaces in real space forms are equivalently hypersurfaces with constant principal curvatures and obtained the classification results for the Euclidean and hyperbolic cases. For the spherical case, if the number of distinct principal curvatures is no more than 3, then he showed that the isoparametric hypersurfaces must be homogeneous. Given an isoparametric hypersurface $Y^n$ in $S^{n+1}(1)$, let $\eta$ be a unit normal vector field, $g$ the number of distinct principal curvatures, $\cot \theta_\alpha (\alpha = 1, 2, \cdots, g; 0 < \theta_1 < \theta_2 < \cdots < \theta_g < \pi)$ the principal curvatures with respect to $\eta$ and $m_\alpha$ the multiplicity of $\cot \theta_\alpha$. Under this setting, H. F. Münzner [Mü80] proved that $\theta_\alpha = \theta_1 + (\alpha - 1) \pi / g$, $m_\alpha = m_\alpha + 2 \pmod{g}$, and there exists a homogeneous polynomial $F : \mathbb{R}^{n+2} \to \mathbb{R}$ of degree $g$, the so-called Cartan-Münzner polynomial, satisfying

$$|\nabla^E F|^2 = g^2 r^{2g-2}, \quad r = |x|,$$

$$\Delta^E F = \frac{m_1 - m_2}{2} g^2 r^{g-2},$$

where $m_1$ and $m_2$ are the two multiplicities, and $\nabla^E, \Delta^E$ are Euclidean gradient and Laplacian, respectively. It follows that $f = F|_{S^{n+1}(1)}$ is an isoparametric function on $S^{n+1}(1)$ with $\text{Im} f = [-1, 1]$, and $M_+ = f^{-1}(\pm)$ are two focal submanifolds with codimension $m_1 + 1$ and $m_2 + 1$ in $S^{n+1}(1)$ respectively. Moreover, based on this global structure, he obtained the splendid result that $g = 1, 2, 3, 4$ or 6. Owing to E. Cartan and H. F. Münzner, the classification of isoparametric hypersurfaces with 4 or 6 principal curvatures in unit spheres is an intriguing problem in submanifold geometry. Up to now, isoparametric hypersurfaces with 4 principal curvatures in unit spheres have been classified except for one case(cf. [CCJ07], [Im08] and [Ch13]). For recent progress and application in this subject, we refer to [GX10], [TY13], [QT14] and [TY15].

One of the fundamental problems in Riemannian geometry is to investigate Riemannian manifolds with special curvature properties, especially manifolds with positive scalar, Ricci or sectional curvatures. For the scalar curvature case, due to the surgery theory of Schoen-Yau [SY79] and Gromov-Lawson [GL80], simply connected manifolds with positive scalar curvature are well understood. Motivated by Schoen-Yau and Gromov-Lawson surgery theory, Tang, Xie and Yan in [TXY12] obtained the following theorem based on the theory of isoparametric hypersurfaces in unit spheres.

**Theorem 1.1.** ([TXY12]) Let $Y^n$ be a compact minimal isoparametric hypersurface in $S^{n+1}(1)$, $n \geq 3$, which separates $S^{n+1}(1)$ into $S^+_n$ and $S^-_{n+1}$. Then each of the doubles $D(S^+_n)$ and $D(S^-_{n+1})$ admits a metric of positive scalar curvature. Moreover, there is still an isoparametric foliation in $D(S^+_n)$ (or $D(S^-_{n+1})$).

In this paper, we will study Ricci curvature properties of double manifolds $D(S^+_n)$ and $D(S^-_{n+1})$ together with isoparametric foliations. We start with a general setting. Let $\xi$ be a vector
bundle of rank \( n \) over a closed manifold \( M \). Given two copies of the disc bundle \( B(ξ) \) over \( M \) with boundary the sphere bundle \( S(ξ) \), we define the double manifold \( D(ξ, M) := B(ξ) \bigm/ id \ B(ξ) \), where \( id : S(ξ) \to S(ξ) \) is the identity map. For instance, \( D(S^m_{+1}) \equiv D(ν(M_+), M_+) \) and \( D(S^m_{−1}) \equiv D(ν(M_−), M_−) \), where \( ν(M_+) \) and \( ν(M_−) \) are the normal bundles over the focal submanifolds \( M_+ \) and \( M_− \) of isoparametric hypersurface \( Y^n \) in \( S^{n+1}(1) \), respectively.

To consider the Ricci curvature properties of double manifolds, we firstly obtain the following result.

**Theorem 1.2.** Let \( ξ \) be a Riemannian vector bundle of rank \( n \) over a closed Riemannian manifold \( (M, d^2_ξ M) \). Given a connection metric on \( E \), then each sphere bundle \( S_r(ξ) \) of radius \( r > 0 \) is an isoparametric hypersurface in \( E \). Moreover, assume \( (M, d^2_ξ M) \) has positive Ricci curvature and \( n \geq 3 \), then, for sufficient small \( r > 0 \), the sphere bundle \( S_r(ξ) \) of radius \( r > 0 \) with the induced metric in \( E \) has positive Ricci curvature.

**Remark 1.1.** Actually, for sufficient small \( r > 0 \), the sphere bundle \( S_r(ξ) \) of radius \( r \) with the induced metric in \( E \) has positive scalar curvature without the assumption that \( (M, d^2_ξ M) \) has positive Ricci curvature.

**Remark 1.2.** Theorem 1.2 and Remark 1.1 generalize Theorem 1 and Theorem 3 in [KS00] (also see [Na79]).

As an application of Theorem 1.2 we can prove

**Theorem 1.3.** Assume \( n \geq 2 \), and the closed manifold \( M \) has a metric with positive Ricci curvature. Then the double manifold \( D(ξ, M) \) admits a metric with positive Ricci curvature, and meanwhile a natural isoparametric foliation.

**Remark 1.3.** By the assumption that \( M \) has a metric with positive Ricci curvature, it follows from Bonnet-Myers theorem that \( π_1 M \) is finite. Then, by Van Kampen’s theorem, \( π_1 D(ξ, M) \) is isomorphic to \( π_1 M \), and is also finite.

At last, combining with Theorem A in [GZ02] and Theorem 1.3, we can infer the following result, which improves Theorem 1.1 in [TXY12].

**Theorem 1.4.** Let \( Y^n \) be a compact isoparametric hypersurface in a unit sphere \( S^{n+1}(1) \) with 4 distinct principal curvatures and multiplicities \( (m_1, m_2) \) (assume \( m_1 \leq m_2 \)), and \( M_± \) the focal submanifolds of \( Y^n \) with codimension \( m_1 + 1 \) and \( m_2 + 1 \) in \( S^{n+1}(1) \) respectively.

1. \( (m_1, m_2) = (1, k) \): The double manifold \( D(S^+_n) \), associated with \( M_+ \) which is diffeomorphic to the Stiefel manifold \( V_{k+2,2} \), admits a metric with positive Ricci curvature and a homogeneous isoparametric foliation. However, the double manifold \( D(S^-_n) \), associated with \( M_- \) which is diffeomorphic to \( (S^1 \times S^{k+1})/\mathbb{Z}_2 \), only admits a metric with nonnegative Ricci curvature and a homogeneous isoparametric foliation.

2. \( 2 \leq m_1 \leq m_2 \): Both of the resulting double manifolds \( D(S^+_n) \) and \( D(S^-_n) \) admit metrics with positive Ricci curvature, and also natural isoparametric foliations.
Remark 1.4. For case \((m_1, m_2) = (1, k)\), by Van Kampen’s theorem, \(\pi_1(D(S^{n+1})) = \pi_1(S^1 \times S^k)/\mathbb{Z}_2 = \mathbb{Z} \) (cf. [CR85], [QT14a]). Hence, according to Bonnet-Myers theorem, \(D(S^{n+1})\) cannot admit a metric with positive Ricci curvature in this case.

Remark 1.5. The natural isoparametric foliations constructed in Theorem 1.4 are foliated diffeomorphic to the ones constructed in Theorem 1.1 of [TXY12]. For the definition of foliated diffeomorphism, we refer to [Ge14] and [GQ15].

2. Geometry of vector bundles and associated sphere bundles

In this paper, we shall make use of the following convention on the ranges of indices:

\[
1 \leq A, B, C, \ldots \leq m + n, 1 \leq i, j, k, \ldots \leq m,
\]

\[
m + 1 \leq \alpha, \beta, \gamma, \ldots \leq m + n, m + 1 \leq a, b, c, \ldots \leq m + n - 1.
\]

Generally, let \((N, ds^2_N)\) be a Riemannian manifold of dimension \(m + n\). Choose a local orthonormal frame \(e_1, \ldots, e_{m+n}\), and let \(\theta_1, \ldots, \theta_{m+n}\) be the dual frame such that \(ds^2_N = \sum_{A=1}^{m+n} (\theta_A)^2\). Then we have the first and second structural equations:

\[
\begin{align*}
\frac{d\theta_A}{\theta_{AB}} & = \theta_B + \theta_{BA} = 0, \\
\frac{d\theta_{AB}}{\Theta_{AB}} & = \theta_{AC} + \theta_{CB} - \Theta_{AB},
\end{align*}
\]

where \(\theta_{AB}\) are the connection 1-forms, and \(\Theta_{AB} = \frac{1}{n} K_{ABCD} \theta_C \wedge \theta_D\) are the curvature 2-forms with \(K_{ABCD}\) the components of the curvature tensor.

In this section, we will consider the geometry of vector bundles and sphere bundles. Let \((M, ds^2_M)\) be a \(m\)-dimensional closed Riemannian manifold, and \(\xi\) a Riemannian vector bundle of rank \(n\) over \(M\) with total space \(E\) and projection \(\pi : E \to M\). For our purpose, we fix a metric \(\langle \cdot, \cdot \rangle\) on \(\xi\) and choose a connection \(D\) compatible with \(\langle \cdot, \cdot \rangle\).

Let \(U\) be an open neighborhood in \(M\), and \(ds^2_M|_U = \sum_{i=1}^{m} (\omega_i)^2\), where \(\omega_1, \ldots, \omega_m\) are 1-forms on \(U\). Then the structural equations of \((M, ds^2_M)\) are given by

\[
\begin{align*}
\frac{d\omega_i}{\omega_i} & = \omega_j \wedge \omega_i, \omega_i + \omega_j = 0, \\
\frac{d\omega_{ij}}{\Omega_{ij}} & = \omega_{ik} \wedge \omega_{kj} - \Omega_{ij},
\end{align*}
\]

where \(\omega_i\) are connection 1-forms, and \(\Omega_{ij} = \frac{1}{2} R_{ijkl} \omega_k \wedge \omega_l\) are the curvature 2-forms with \(R_{ijkl}\) the components of the curvature tensor.

Assume \(U\) is small enough such that \(\xi|_U\) is trivial, we can choose orthonormal cross sections \(e_{m+1}, \ldots, e_{m+n}\) on \(U\) for \(\xi\), i.e. \(\langle e_a, e_b \rangle = \delta_{ab}\), and \(\Psi_U : U \times \mathbb{R}^n \to \pi^{-1}(U)\) is the local trivialization given by \(\Psi_U(p,v_1,...,v_n) = \sum_{a=m+1}^{m+n} v_a e_a(p)\). Then \(De_a = \omega_{a\beta} e_\beta\), where \(\omega_{a\beta}\) is the matrix expression of the connection \(D\). Note that \(D\) is compatible with \(\langle \cdot, \cdot \rangle\) means that \((\omega_{a\beta})\) is skew-symmetric. The curvature 2-forms \(\Omega_{a\beta}\) of the connection \(D\) are determined by \(d\omega_{a\beta} = \omega_{a\gamma} \wedge \omega_{\gamma\beta} - \Omega_{a\beta}\), where \(\Omega_{a\beta} = \frac{1}{2} R_{a\beta ij} \omega_i \wedge \omega_j\).
Define $\theta_i := \pi^* \omega_i$ and $\theta_\alpha := dv_\alpha + \nu_\beta \omega_\beta$. Then $ds^2 = (\theta_i)^2 + (\theta_\alpha)^2$ is a well-defined Riemannian metric on $E$, i.e., the so-called connection metric (cf. Theorem 2.2 in [QT15]). For convenience, we give the formulae of connection 1-forms $\theta_{AB}$ and curvature 2-forms $\Theta_{AB}$ of the connection metric in the following by making use of the moving frame method.

**Proposition 2.1.** The connection 1-forms $\theta_{AB}$ are determined by

$$
\theta_{ij} = \pi^* \omega_{ij} + A_{ij\alpha} \theta_\alpha,
\theta_{i\alpha} = A_{ij\alpha} \theta_j,
\theta_{\alpha\beta} = \omega_{\alpha\beta},
$$

and the curvature 2-forms $\Theta_{AB}$ are given by

$$
\Theta_{ij} = \Omega_{ij} - A_{ik\alpha} A_{j\lambda\alpha} \theta_k \wedge \theta_j - A_{ij\alpha} A_{k\lambda\alpha} \theta_k \wedge \theta_\alpha - \frac{1}{2} R_{\alpha\beta\gamma\delta} \omega_{\alpha\beta} \wedge \omega_{\gamma\delta} + A_{ik\alpha} A_{j\lambda\beta} \theta_k \wedge \theta_\beta,
\Theta_{i\alpha} = \frac{1}{2} R_{\alpha\beta\gamma\delta} \omega_{\alpha\beta} \wedge \theta_{\gamma\delta} + A_{ik\alpha} A_{j\lambda\beta} \theta_k \wedge \theta_\beta,
\Theta_{\alpha\beta} = -A_{ij\alpha} A_{k\lambda\beta} \theta_j \wedge \theta_k + \Omega_{\alpha\beta},
$$

where $A_{ij\alpha} = \frac{1}{2} R_{\alpha\beta\gamma\delta} \omega_{\alpha\beta}$, and $R_{\alpha\beta\gamma\delta}$ is defined by $R_{\alpha\beta\gamma\delta} \omega_k := dR_{\alpha\beta\gamma} + R_{\gamma\alpha\rho} \omega_{\rho\delta} + R_{\gamma\delta\alpha} \omega_{\rho\rho} + R_{\gamma\rho\delta} \omega_{\rho\alpha}$. Define $S_r(\xi) := \{(p, v) \in E \mid v \in \pi^{-1}(p), \langle v, v \rangle = r^2\}$, the associated sphere bundle of radius $r > 0$. At present, we are in the position to prove Theorem [1.2].

**Proof of Theorem [1.2]**

At first, observe that the function $f : E \rightarrow \mathbb{R}, (p, v) \mapsto \langle v, v \rangle$ is a transnormal function (cf. Theorem 2.2 in [QT15]). Next, we need to compute the principal curvatures of $S_r(\xi) \subset E$ for any $r > 0$. On $\pi^{-1}(U), r^2 = \sum v_\alpha^2$. Define $u_\alpha := \frac{v_\alpha}{r}$. For $r > 0$, $\eta = u_\alpha \frac{\partial}{\partial v_\alpha}$ is
a natural global normal vector field of $S_r(\xi) \subset E$. In order to study the extrinsic geometry of $S_r(\xi)$, we introduce the orthogonal transformation $P : \mathbb{R}^n \rightarrow \mathbb{R}^n$ defined by

$$(P_{ab}) = \begin{pmatrix}
I_{n-1} - \frac{x^T x}{1-x_n} & x^T \frac{x}{x_n} \\
0 & x_n
\end{pmatrix},$$

where $x = (u_{m+1}, \ldots, u_{m+n-1})$ and $x_n = u_{m+n}$. Choose the moving frame $\{\varphi_i, \varphi_a, \varphi_{m+n}\}$ by $\varphi_i = \theta_i$ and $\varphi_a = P_{ab} \theta_b$. Particularly, $\varphi_{m+n} = u_a \theta_a = dr$. Therefore, $\{\varphi_i, \varphi_a, \varphi_{m+n}\}$ is exactly the adapted moving frame with respect to $S_r(\xi) \subset E$. The associated connection 1-forms $\psi$ are given by

$$(\psi_{AB}) = \begin{pmatrix}
\varphi_{ij} & \varphi_{ia} & \varphi_{a} \\
\varphi_{ai} & \varphi_{\alpha} & \varphi_{\alpha} \\
\varphi_{\alpha} & \varphi_{\beta} & \varphi_{\beta}
\end{pmatrix} = \begin{pmatrix}
\theta_{ij} & \theta_{ija} & \theta_{ijab} \\
P_{ab} \theta_{ji} & P_{ab} \theta_{jia} & P_{ab} \theta_{j} \\
P_{ab} \theta_{jiab} & P_{ab} \theta_{j} & P_{ab} \theta_{jbc}
\end{pmatrix}.$$
where
\[
\Psi_{ik,jk} = R_{ik,jk} - 3A_{ik}A_{jk},
\]
\[
\Psi_{ia,ja} = A_{ik}A_{jk}P_{aa}P_{\beta\alpha},
\]
\[
\Psi_{ik,ak} = \frac{1}{2}R_{a\beta,k}v_{\beta}P_{aa},
\]
\[
\Psi_{dbab} = 0,
\]
\[
\Psi_{ak,bk} = A_{km}A_{km}P_{aa}P_{\beta b},
\]
\[
\Psi_{ac,bc} = \frac{1}{r^{2}}\delta_{ab}(n - 2).
\]
Observe that \(A_{ij} = \frac{1}{2}R_{a\beta}v_{\beta} = \frac{1}{2}rR_{a\beta}u_{\beta}\), and the norms of \(R_{a\beta}v_{\beta}\), \(R_{a\beta}u_{\beta}\) are bounded on \(M\), since \(M\) is a closed Riemannian manifold. By the assumption \(M\) has positive Ricci curvature, it follows that, for \(r > 0\) small enough, the Ricci curvature tensor \(\text{Ric}^{S_{r}}\) is positive definite. And the proof is complete. \(\square\)

3. Ricci curvature and isoparametric foliation on double manifold

In this section, we will prove Theorem 1.3. The key observation is the following fact in \([TXY12]\).

Proposition 3.1. ([TXY12]) Let \(\xi\) be a vector bundle of rank \(n\) over \(M\) of dimension \(m\) and \(D(\xi, M)\) the double manifold by gluing two copies of disc bundle of \(\xi\). Then \(D(\xi, M)\) is diffeomorphic to \(S(\xi \oplus 1)\), where \(S(\xi \oplus 1)\) is the sphere bundle of the Whitney sum between \(\xi\) and a trivial line bundle \(1\).

For convenience, denote the total space of \(\xi \oplus 1\) by \(E(\xi \oplus 1)\) and the projection also by \(\pi: E(\xi \oplus 1) \to M\). As in Section 2, we fix a metric \(\langle \cdot, \cdot \rangle\) on \(\xi\) and choose a connection \(D\) compatible with \(\langle \cdot, \cdot \rangle\). At present, for the vector bundle \(\xi \oplus 1\), it is natural to extend the metric \(\langle \cdot, \cdot \rangle\) on \(\xi\) to \(\xi \oplus 1\) such that \(\xi \oplus 1\) is an orthogonal Whitney sum, denoted also by \(\langle \cdot, \cdot \rangle\). Moreover, choose a unit section \(e_{m+n+1}\) for the trivial line bundle \(1\). By demanding that \(e_{m+n+1}\) is parallel, we extend naturally the connection \(D\) on \(\xi\) to a connection on \(\xi \oplus 1\), denoted also by \(D\).

Assume \(U\) is small enough such that \(\xi|U\) is trivial, we can choose orthonormal cross sections \(e_{m+1}, \ldots, e_{m+n}\) on \(U\) for \(\xi\). Then \(e_{m+1}, \ldots, e_{m+n}, e_{m+n+1}\) is an orthonormal cross sections on \(U\) for \(\xi \oplus 1\). Moreover, for \(m + 1 \leq \alpha, \beta \leq m + n\), \(De_{\alpha} = \omega_{\alpha\beta}e_{\beta}\), and \(De_{m+n+1} = 0\), where \((\omega_{\alpha\beta})\) is the matrix expression of the connection \(D\). Consequently, \(\omega_{m+n+1}e_{\alpha} = -\omega_{\alpha m+n+1}e_{\beta} = 0\).

Proof of Theorem 1.3

**Proof.** By the assumption, choose the Riemannian metric \(ds_{M}^{2}\) on \(M\) with positive Ricci curvature. According to Theorem 1.2 for the connection metric on \(E(\xi \oplus 1)\), and for \(r_{0} > 0\) small enough, the sphere bundle \(S_{r_{0}}(\xi \oplus 1)\) of radius \(r_{0}\) with the induced metric has positive Ricci curvature.
Next, we will construct the natural isoparametric foliation on $S_{r_0}(\xi \oplus 1)$. Let $f : S_{r_0}(\xi \oplus 1) \to \mathbb{R}$ be a smooth function such that $f(p, v)$ is the $e_{m+n+1}$-component of $v$ for each point $(p, v) \in S_{r_0}(\xi \oplus 1)$. We will prove that $f$ is an isoparametric function.

Choose an open neighborhood $U$ in $M$ such that $ds_{\xi|U}^2 = \sum_i (\omega_i)^2$, where $\omega_1, \ldots, \omega_m$ are 1-forms on $U$. Let $\omega_{ij}$ and $\Omega_{ij}$ be the connection 1-forms and curvature 2-forms, respectively. Moreover, assume $U$ is small enough such that $\xi|U$ is trivial, and choose orthonormal cross sections $e_{m+1}, \ldots, e_{m+n}$ for $\xi|U$, with connection 1-forms $\omega_{ij}$ and curvature 2-form $\Omega_{ij}$. For the vector bundle $\xi \oplus 1$, we assume $e_{m+n+1}$ is parallel as before, i.e., $\omega_{m+n+1} = - \omega_{m+n+1} = 0$.

Under this local trivialization of $(\xi \oplus 1)|_U$, for each point $(p, v) \in (\xi \oplus 1)|_U$, we have $v = v_0 e_0 + v_{m+n+1} e_{m+n+1}$. Define $\theta_0 := \pi^* \omega_0$, $\theta_a := dv_0 + v_0 \omega_{0a}$ and $\theta_{m+n+1} := dv_{m+n+1}$. The connection metric $ds_{\xi \oplus 1}^2$ can be expressed as $ds_{\xi \oplus 1}^2 = (\theta_0)^2 + (\theta_a)^2 + (\theta_{m+n+1})^2$. Write $u_a = \frac{\theta_a}{\theta_0}$ and $u_{m+n+1} = \frac{v_{m+n+1}}{r_0}$, where $r^2 = (v_0)^2 + (v_{m+n+1})^2$. Then, for the sphere bundle $S_{r_0}(\xi \oplus 1)$, the induced metric $ds_{\xi \oplus 1}^2$ metric can be expressed as

$$ds_{\xi \oplus 1}^2 = (\theta_0)^2 + r_0^2 (du_a + u_0 \omega_{0a})^2 + r_0^2 (du_{m+n+1})^2.$$  

Under this local coordinate system, $f(p, v) = ru_{m+n+1}$ for $(p, v) \in S_{r_0}(\xi \oplus 1)$. It follows that $|\nabla f|^2 = 1 - \frac{r^2}{r_0}$.

To complete the proof, it is sufficient to show the regular hypersurfaces of $f$ have constant principal curvatures. Write $w_a = \frac{u_a}{1 - (u_{m+n+1})^2}$, then

$$ds_{\xi \oplus 1}^2 = (\theta_0)^2 + r_0^2 \left(\frac{(du_{m+n+1})^2}{1 - (u_{m+n+1})^2}\right) + r_0^2 (1 - (u_{m+n+1})^2) (dw_0 + w_0 \omega_{0a})^2.$$  

Using the substitution $t = \text{arccos} u_{m+n+1}$ for $-1 < u_{m+n+1} < 1$, we obtain that

$$ds_{\xi \oplus 1}^2 = (\theta_0)^2 + r_0^2 (dt)^2 + r_0^2 \sin^2 t (dw_0 + w_0 \omega_{0a})^2.$$  

Now, we can choose $\varphi_i = \theta_i$, $\varphi_0 = ru_0 dt$, and $\varphi_a = ru_0 \sin \varphi_a$ such that

$$\sum_{a=m+1}^{m+n-1} (\varphi_a)^2 = (dw_0 + w_0 \omega_{0a})^2.$$  

Then $\{\varphi_0, \varphi_i, \varphi_a\}$ is a moving frame for $S_{r_0}(\xi \oplus 1)$ with $\varphi_0|f^{-1}(c) = 0$ for $-r_0 < c < r_0$. Let $\varphi_i, \varphi_0, \varphi_i, \varphi_0, \varphi_{ab}$ be the connection 1-forms. To obtain the principal curvatures of $f^{-1}(c) \subset S_{r_0}(\xi \oplus 1)$, it is sufficient to determine the forms $\varphi_0$ and $\varphi_0$. Define

$$\varphi_i = h_i \varphi_i + h_0 \varphi_a, \\
\varphi_0 = h_0 \varphi_i + h_0 \varphi_0, \\
\varphi_i = \lambda_{ij} \varphi_j \mod \{\varphi_i, \varphi_a\}, \\
\varphi_i = \lambda_{ia} \varphi_0 \mod \{\varphi_i, \varphi_a\}, \\
\varphi_ab = \lambda_{ab} \varphi_0 \mod \{\varphi_i, \varphi_a\}.$$
From the first structural equations, we have

\[
\begin{align*}
    d\varphi_0 &= \varphi_0 \wedge \varphi_i + \varphi_{0a} \wedge \varphi_i, \\
    d\varphi_i &= \varphi_{i0} \wedge \varphi_0 + \varphi_{ij} \wedge \varphi_j + \varphi_{ia} \wedge \varphi_a, \\
    d\varphi_a &= \varphi_{a0} \wedge \varphi_0 + \varphi_{ai} \wedge \varphi_i + \varphi_{ab} \wedge \varphi_b.
\end{align*}
\]

By a direct computation, we have

\[
\begin{align*}
    d\varphi_0 &= 0, \\
    d\varphi_i &= \pi^* \omega_{ij} \wedge \varphi_j, \\
    d\varphi_a &= \frac{\cot t}{r_0} \varphi_0 \wedge \varphi_a \mod [\varphi_0 \wedge \varphi_j, \varphi_i \wedge \varphi_a, \varphi_a \wedge \varphi_b].
\end{align*}
\]

It follows that

\[
\begin{align*}
    h_{ij} &= h_{ji}, h_{ia} = h_{ai}, h_{ab} = h_{ba}, \\
    h_{ij} &= -\lambda_{ij}, h_{ia} = -\lambda_{ai}, \\
    h_{ai} &= \lambda_{ia}, h_{ab} = -\lambda_{ab} + \frac{\cot t}{r_0} \delta_{ab}.
\end{align*}
\]

Therefore, \( \varphi_{0i} = 0 \) and \( \varphi_{0a} \equiv \frac{\cot t}{r_0} \varphi_a \). That is to say, the hypersurface \( f^{-1}(c) \) has constant principal curvatures 0 and \( -\frac{\cot t}{r_0} \) with \( t = \arccos \frac{c}{r_0} \) for the unit normal vector field determined by \( \varphi_0 \). The proof is complete. \( \square \)

4. Applications to isoparametric foliation of unit spheres

In this section, based on Theorem [1.3] we will study the Ricci curvature and isoparametric foliation on double manifolds \( D(S^n_{n+1}) \), and prove Theorem [1.4]

**Proof of Theorem [1.4]**

*Proof*. Let \( Y^n \) be a closed isoparametric hypersurface with 4 distinct principal curvatures in \( S^{n+1}(1) \) with two focal submanifolds \( M_+ \) and \( M_- \) of codimension \( m_1 + 1 \) and \( m_2 + 1 \) in \( S^{n+1}(1) \) respectively(cf. [CR83]). And \( n = 2(m_1 + m_2) \). Without loss of generality, we can assume \( m_1 \leq m_2 \).

**Case (1).** \( (m_1, m_2) = (1, k) \): According to [1a] [6], the isoparametric hypersurfaces in this case must be homogeneous. More precisely, they are the principal orbits of isotropy representation of the symmetric pair \( (\text{SO}(k+4), \text{SO}(2) \times \text{SO}(k+2)) \). For the symmetric pair \( (\text{SO}(k+4), \text{SO}(2) \times \text{SO}(k+2)) \), we have the Cartan decomposition \( \mathfrak{o}(k+4) = (\mathfrak{o}(2) + \mathfrak{o}(k+2)) \oplus \mathfrak{p} \), where \( \mathfrak{p} \) is the orthogonal complement of \( (\mathfrak{o}(2) + \mathfrak{o}(k+2)) \) with respect to the Killing form \( B \) of \( \mathfrak{o}(k+4) \), and \( \mathfrak{p} \) has the canonical induced inner product. Let \( S(\mathfrak{p}) \) be the unit sphere in \( \mathfrak{p} \). Then \( G = \text{SO}(2) \times \text{SO}(k+2) \) acts on \( \mathfrak{p} \) by conjugation and it induces the cohomogeneity one action on \( S(\mathfrak{p}) \), the principal orbits of which are isoparametric hypersurfaces in this case. Let \( K_+ = \Delta \text{SO}(2) \times \text{SO}(k) \), \( K_- = \mathbb{Z}_2 \cdot \text{SO}(k+1) \) and \( H = \mathbb{Z}_2 \cdot \text{SO}(k) \) be the closed subgroups of \( G \).

Under the notations of [GZ02], \( (S(\mathfrak{p}), G) \) is a homogeneity one manifold determined by the group diagram \( H \subset K_+, K_- \subset G \). The two focal submanifolds(singular orbits) \( M_+ \) and \( M_- \) are
determined by $M_+ = G/K_+ \equiv V_{k+2,2}$, the Stiefel manifold of $2$-orthonormal frames in $\mathbb{R}^{k+2}$, and

$$M_- = G/K_- \equiv (S^1 \times S^{k+1})/\mathbb{Z}_2 = (S^1 \times S^{k+1})/(\theta, x) \sim (\theta + \pi, -x).$$

Thus, the double manifold $H$ diagram $\pi$ and homogeneous isoparametric hypersurfaces. For the case of $M_+$ admits an invariant metric with positive Ricci curvature. Meanwhile, the principal orbits are one action by $\pi$. The principal curvatures of the shape operator $\alpha_i$ for $1 \leq i \leq m_1 + 2m_2$ and $m_1 + 2m_2 + 1 \leq \alpha \leq n + 1$, where $e_i$’s are tangent to $M_+$ and $e_i$’s are normal to $M_+$. Then, by the Gauss equation, for each unit tangent vector $X$ of $M_+$, the Ricci curvature $\text{Ric}^{M_+}(X, X)$ is given by

$$\text{Ric}^{M_+}(X, X) = 2m_2 + m_1 - 1 - \sum_{\alpha} \langle A_\alpha X, A_\alpha X \rangle.$$

For any unit normal vector $\eta$ of $M_+$, the principal curvatures of the shape operator $A_\eta$ are $1, -1, 0$. It follows that $\langle A_\alpha X, A_\alpha Y \rangle \leq 1$. Therefore, $\text{Ric}^{M_+}(X, X) \geq 2(m_2 - 1) > 0$ if $m_2 \geq 2$. Now, the proof is complete.

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REFERENCES

[CCKJ07] T. E. Cecil, Q. S. Chi and G. R. Jensen, Isoparametric hypersurfaces with four principal curvatures, Ann. Math., 166 (2007), 1–76.

[CR85] T. E. Cecil and P. T. Ryan, Tight and taut immersions of manifolds, Research Notes in Math. 107, Pitman, London, (1985).

[Ch13] Q. S. Chi, Isoparametric hypersurfaces with four principal curvatures, III, J. Diff. Geom., 94 (2013), 469–504.

[Ge14] J. Q. Ge, Isoparametric foliations, diffeomorphism groups and exotic smooth structures, arXiv:1404.6194, 2014.
[GQ15] J. Q. Ge and C. Qian, Differential topology interacts with isoparametric foliations, to appear in Geometry and Topology of Manifolds, The 10th Geometry Conference for the Friendship of China and Japan 2014, Springer Proceedings in Mathematics & Statistics.

[GT13] J. Q. Ge and Z. Z. Tang, Isoparametric functions and exotic spheres, J. Reine Angew. Math., 683 (2013), 161–180.

[GX10] J. Q. Ge and Y. Q. Xie, Gradient map of isoparametric polynomial and its application to Ginzburg-Landau system, J. Func. Anal., 258 (2010), 1682–1691.

[GL80] M. Gromov and H. B. Lawson, The classification of simply connected manifolds of positive scalar curvature, Ann. Math., 111 (1980), 423–434.

[GZ02] K. Grove and W. Ziller, Cohomogeneity one manifolds with positive Ricci curvature, Invent. Math., 149 (2002), 619–646.

[Im08] S. Immervoll, On the classification of isoparametric hypersurfaces with four distinct principal curvatures in spheres, Ann. Math., 168 (2008), 1011–1024.

[Je73] G. R. Jensen, Einstein metrics on principal fibre bundles, J. Diff. Geom., 8 (1973), 599–614.

[KS00] O. Kowalski and M. Sekizawa, On tangent sphere bundles with small or large constant radius, Ann. Glob. Anal. Geom., 18 (2000), 207–219.

[Mu80] H. F. Münzner, Isoparametric hyperflächen in sphären, I and II, Math. Ann., 251 (1980), 57–71 and 256 (1981), 215–232.

[Na79] J. C. Nash, Positive Ricci curvature on fibre bundles, J. Diff. Geom., 14 (1979), 241–254.

[QT14] C. Qian and Z. Z. Tang, Recent progress in isoparametric functions and isoparametric hypersurfaces, Real and Complex Submanifolds, Volume 106 of the series Springer Proceedings in Mathematics & Statistics, 65–76.

[QT14a] C. Qian and Z. Z. Tang, Isoparametric foliations, a problem of Eells-Lemaire and conjectures of Leung, arXiv:1407.0539, 2014.

[QT15] C. Qian and Z. Z. Tang, Isoparametric functions on exotic spheres, Adv. Math.,272 (2015), 611–629.

[SY79] R. Schoen and S. T. Yau, On the structure of manifolds with positive scalar curvature, Manuscripta Math.,28 (1979), 159–183.

[Ta76] R. Takagi, A class of hypersurfaces with constant principal curvatures in a sphere, J. Diff. Geom., 11 (1976), 225–233.

[TXY12] Z. Z. Tang, Y. Q. Xie and W. J. Yan, Schoen-Yau-Gromov-Lawson theory and isoparametric foliations, Comm. Aanl. Geom., 20(2012), 989–1018.

[TY13] Z. Z. Tang and W. J. Yan, Isoparametric foliation and Yau conjecture on the first eigenvalue, J. Diff. Geom., 94(2013), 521–540.

[TY15] Z. Z. Tang and W. J. Yan, Isoparametric foliation and a problem of Besse on generalizations of Einstein condition, Adv. Math., 285(2015), 1970–2000.

[Wa87] Q. M. Wang, Isoparametric functions on Riemannian manifolds. I, Math. Ann.,277 (1987), 639–646.

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