A first order wave equation for the electromagnetic radiation and the Dirac equation: towards Pauli’s exclusion principle

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In the present work, it is constructed a heuristic, first-order differential equation for the electromagnetic waves in the vacuum, based on a phenomenological *ad hoc* argument, differing from the systematic derivation of the well-known second-order differential wave equation from the Maxwell equations. The formal similarity between this *ad hoc* wave equation and the Dirac equation is explored. The comparison between the two scenarios leaves to consider the issue of Pauli’s exclusion principle (EP), for what there is not yet a physical explanation beyond the mere mathematical requirement of the (anti)symmetrization of the wavefunction. It is proposed the existence of an underlying non-local physical interaction governing the EP.

1. Introduction

Following the line of the history, Riemann proposed\[1\] that the propagation of electric, magnetic, optic and even gravitational disturbances should be the consequence of compression resistance properties of aether, proposing the generalization of Poisson’s equation for static potentials $V$

\[
\nabla^2 V + \epsilon_0^{-1} \rho = 0,
\]

(1.1) \hspace{1cm} ($\rho$ is the charge density) to the most general time dependent equation,

\[
\Box V + \epsilon_0^{-1} \rho = 0,
\]

(1.2) \hspace{1cm} inspired on a Gauss idea, that forces between charges would be supplemented by other forces, such as would cause the electric (magnetic, gravitational, etc.) actions to be propagated between the charges with a constant velocity
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Soon after (1861-62), Maxwell obtained the second-order wave equation for the magnetic field vector $B$,

$$\partial_t^2 B = c^2 \nabla^2 B \iff □B = 0,$$

and in 1867, Lorenz started from Neumann’s theory in which the electric field $E$ was expressed by

$$E = \nabla φ - c^{-1} \partial_t A,$$

where $φ$ is the scalar electrostatic potential and $A$ is the vector potential, proposing the concept of retarded potentials requiring that

$$□φ = -\varepsilon_0^{-1} \rho; \quad □A = -\mu_0 J,$$

where $J$ is the current density vector. Writing in terms of the four-potential $φ_\mu$ ($φ_0 = φ, φ_i = A_i, i = 1, 2, 3$),

$$□φ_\mu = -\varepsilon_0^{-1} j_\mu,$$

where $j_\mu$ is the four-current. Finally, in special relativity theory, the relativistically invariant four-vector potential $φ_\mu$ plays the central role, while the electric $E$ and the magnetic $B$ fields appear sparsely in the components of the relativistically invariant electromagnetic tensor. The four-potential $φ_\mu$ is a basis for quantum electrodynamics (QED) instead of $E$ and $B$, since QED is based on the second quantization of fields, starting with the Fourier series decomposition of $φ_\mu$ in creation and annihilation operators.

We note that, along with these works, the focus of the attention changes between the potentials and the fields. Another issue considered yet on the 20th century was the question about the convenience of a first-order wave equation for the description of the electromagnetic waves, instead of the well-known second-order forms (as equations 1.3, 1.5 and 1.6). Yet in 1931, Oppenheimer argued that equation 1.6 is in several respects unsatisfactory, and tried the development of a first-order wave equation, reporting the previous first-order equation constructed by Jordan using the Pauli matrices $σ_i$,

$$\{(σ \cdot \nabla) + \partial_t\} φ = 0.$$
In this form, $\phi$ would not be invariant under spatial rotations and by this reason should refer not to vectors (as required) but to spinors. Then, Oppenheimer presented the equation

\[(1.8) \quad \{ (\tau \cdot \nabla) + \partial_t \} \phi = 0, \]

using $3 \times 3$ matrices $\tau_i$; and in order to satisfy Lorentz invariance, wrote

\[(1.9) \quad \{ (\rho \cdot \nabla) + \partial_t \} \phi = 0, \]

defining $4 \times 4$ matrices $\rho_i$, now with a four-component $\phi$, such that equation (1.9) reduces to equation (1.8) setting equal to zero the fourth component, $\phi_4 = 0$. This leads to null three-dimension divergences, whose Lorentz invariance is gained (and also gained by equation (1.8)) transforming $\phi$ not being a four-vector, but as the components of a self dual six vector $\Phi_{\mu \nu}$, completely determined just by:

\[(1.10) \quad \Phi_{41} = i/\sqrt{2}(\psi_1 + \psi_3); \quad \Phi_{42} = i/\sqrt{2}(\psi_1 - \psi_3); \quad \Phi_{43} = -i\psi_2; \quad \Phi_{44} = 0, \]

On the counterpart, Mignani et al. [5] stressed the importance of the magnitude $E - icB$, on the context of the electrodynamics, which is just the complex conjugate of well-known Riemann-Silberstein vector $G = E + icB$. Then, they appeal to an idea of Majorana of a Dirac-like equation for the photon [6, 7], who expressed a probability quantum wave of a photon in terms of $E$, $B$, again conducting to a first-order wave equation, and where the components of the fields $E$, $B$ are the relevant magnitudes instead of the potentials. In such a formalism, two of the Maxwell equations are condensed on the eigenvalue equation $\hat{H}G = ic\hat{p} \times G$, with energy eigenvalues $E = cp$. This leads to the electromagnetic radiation (ER) energy density $|G|^2 \propto u$ equivalent to the photon probability density. In this formalism, the four potential $\phi^\mu$ again loses importance and $E$, $B$ are the most significant magnitudes. Finally, Giannetto [9] even generalized the idea of a first-order equation to a non-Abelian charged field [10] for potentials.

These initial considerations were presented to stress two points: 1. that the traditional wave equations are of second-order, but there were attempts to construct a first-order equation; and 2. there was relativism of the hierarchy of the importance of the potential $\phi^\mu$ and the electric and magnetic fields, $E$, $B$, alongside diverse tentative models.

The present work was developed in a different way if compared to these resources of the literature. While they stand in more theoretical argumentation, the basis for the construction of a first-order wave equation for the ER
the present work has, in some sense, a more realistic nature, for being constructed with a basis in a phenomenological reality, as will be shown on the starting part of the next Section. However, these resources of the literature serve, in some way, as a support for the present work, in the sense that they reveal that other researchers on the past yet considered the possibility of a first-order equation for the ER to be plausible. Of course, the second-order equation is nowadays preferred and currently adopted, and the purpose of the present article is not to advocate the replacement of second-order by first-order equations, neither developing an alternative mathematical formalism to replace the current models for the ER. Instead, the purpose is to open the possibility of a direct comparison between the mathematical description of the ER and the Dirac equation, constructing an ad hoc, heuristic first-order differential equation for the electromagnetic four-potential $\phi_{\mu}$, that emerges directly from the phenomenology of the propagation of the ER. From this comparison, which is the most significant part of the article, it is made a discussion about the essential nature of Pauli’s exclusion principle (EP). It is argued that the mandatory request of the antisymmetrization of the wavefunction may lie on a fermion-fermion interaction.

2. The heuristic ER wave equation

In this section, we show the construction of a first-order wave equation for the ER, which differs from the previous works of the literature. The construction is very simple and based on the spatial arrays of the electric and the magnetic fields concerning the direction of propagation of the wave, taking as the base the plane electromagnetic waves.

Let us consider plane electromagnetic waves in the vacuum, for what the electric and the magnetic fields are mutually perpendicular, writing:

\begin{equation}
E \cdot B = 0 \iff E_x B_x + E_y B_y + E_z B_z = 0.
\end{equation}

Note that $E \cdot B$ is an invariant of the electromagnetism. Consequently, (2.1) is Lorentz invariant.

We will consider a reference frame where the projections of $E$ and $B$ on the plane $xy$ are mutually perpendicular. On this frame,

\begin{equation}
E_x = B_y, \quad E_y = -B_x.
\end{equation}

Objection could be made with respect to this restriction. However, this does not prejudice the Lorentz invariance. In fact, once it is chosen a second reference frame $S'$ moving with a velocity $v$ with respect to the given reference
frame $S$, we may guide the orientation of the new axis $x', y', z'$ respectively parallel to $x, y, z$, to arrive to similar relations as (2.2) for the new components of the electric $E'$ and the magnetic $B'$ fields. Note that the arbitrary choice of orientation of axis is neither incorrect nor incompatible with the special relativity, nor a new issue, being commonly applied on the context of angular momentum and spin in relativistic quantum mechanics. Expressions (2.1) and (2.2) imply that

\begin{equation}
E_z = 0 \quad \text{and} \quad B_z = 0.
\end{equation}

Conditions (2.2) and (2.3) can be written explicitly as a function of the four-potential. Doing this and grouping the resulting expressions from (2.2) and the first (for example) of (2.3) we obtain (adopting Einstein summation rule)

\begin{equation}
\begin{aligned}
-\partial_0 \phi_1 - \partial_1 \phi_0 - \epsilon_{2jk} \partial_j \phi_k &= 0 \\
-\partial_0 \phi_2 - \partial_2 \phi_0 + \epsilon_{1jk} \partial_j \phi_k &= 0 \\
-\partial_0 \phi_3 - \partial_3 \phi_0 &= 0
\end{aligned}
\end{equation}

Now we also consider the Lorentz gauge $\partial_\mu \phi^\mu = 0$, and group it to the set of equations (2.4) constructing the matrix equation:

\begin{equation}
\varepsilon^\mu \partial_\mu \phi = 0
\end{equation}

which is our heuristic, first-order linear homogeneous differential equation for the ER (being $\phi = (\phi^0, \phi^1, \phi^2, \phi^3)$ expressed as a column vector), where we have defined the four $4\times4$ matrices:

\begin{equation}
\varepsilon^0 = \begin{pmatrix} 0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 \end{pmatrix}, \quad \varepsilon^1 = \begin{pmatrix} -1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \end{pmatrix},
\end{equation}

\begin{equation}
\varepsilon^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \end{pmatrix}, \quad \varepsilon^3 = \begin{pmatrix} 0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \end{pmatrix}.
\end{equation}

whose asymmetry arises from the chosen reference frame orientation. Such an equation could be constructed in other way (e.g. choosing the second expression of (2.3) instead of the first), leading to different matrices $\varepsilon^\mu$, though this is not relevant for our purposes. In the following, the most important result is that the ER can be described by a first-order wave equation (equation (2.5) or the cited from past works mentioned in Section 1). Strictly under the mathematical aspect, the solution of a linear equation provides directly
the four-potential $\phi^\mu$ from a set of boundary conditions (integration constants), while the second-order wave equation $\Box \phi^\mu = 0$ requires two sets of integration constants, and in this sense, it has some “redundancy”.

3. The Dirac equation: a parallel

Let us consider the Dirac equation for massless fermions ($\hbar = 1$),

\[(3.1)\quad i\gamma^\mu \partial_\mu \Psi = 0.\]

We are led to apply the inverse route of mathematical manipulation we followed to obtain 2.5. For that, we write explicitly the spinor as $\Psi = (\psi_n + i\chi_n)$, where $\psi_n, \chi_n \in \mathbb{R}, n = 0, \ldots, 3$ are respectively the real and the imaginary parts of each component of $\Psi$. By direct substitution on 3.1 we obtain the following set of equations:

\[(3.2)\]
\[
\begin{pmatrix}
\partial_2 \chi_3 + \partial_0 \psi_0 + \partial_3 \psi_2 + \partial_1 \psi_3 \\
-\partial_2 \chi_2 + \partial_0 \psi_1 + \partial_1 \psi_2 - \partial_3 \psi_3 \\
-\partial_2 \chi_1 - \partial_3 \psi_0 - \partial_1 \psi_1 - \partial_0 \psi_2 \\
\partial_2 \chi_0 - \partial_1 \psi_0 + \partial_3 \psi_1 - \partial_0 \psi_3
\end{pmatrix} = 0
\]

and

\[(3.3)\]
\[
\begin{pmatrix}
\partial_0 \chi_0 + \partial_3 \chi_2 + \partial_1 \chi_3 - \partial_2 \psi_3 \\
\partial_0 \chi_1 + \partial_1 \chi_2 - \partial_3 \chi_3 + \partial_2 \psi_2 \\
-\partial_3 \chi_0 + \partial_1 \chi_1 - \partial_0 \chi_2 + \partial_2 \psi_1 \\
-\partial_1 \chi_0 + \partial_3 \chi_1 - \partial_0 \chi_3 - \partial_2 \psi_0
\end{pmatrix} = 0
\]

Let us make the change of labeling:

\[(3.4)\]
\[
\begin{pmatrix}
\psi_0 \\
\psi_1 \\
\psi_2 \\
\psi_3
\end{pmatrix} \rightarrow \begin{pmatrix}
\chi_0 \\
\chi_1 \\
\chi_2 \\
\chi_3
\end{pmatrix} ; \quad \begin{pmatrix}
\chi_0 \\
\chi_1 \\
\chi_2 \\
\chi_3
\end{pmatrix} \rightarrow \begin{pmatrix}
\psi_0 \\
-\chi_2 \\
\psi_3 \\
\psi_1
\end{pmatrix}.
\]

We define the following three-dimensional, three-component magnitudes $E_{\psi(\chi)}$ and $B_{\psi(\chi)}$:

\[(3.5)\]
\[
E_\psi = -\partial_0 \psi - \nabla \psi_0; \quad B_\psi = \nabla \times \psi; \\
E_\chi = -\partial_0 \chi - \nabla \chi_0; \quad B_\chi = \nabla \times \chi,
\]

which satisfy the same algebra of the usual electromagnetic fields $E$ and $B$ with respect to the four-vector electromagnetic potential $\phi^\mu$. Despite until
now without physical meaning, these magnitudes $E_{\psi(\chi)}$ and $B_{\psi(\chi)}$ have components on the three dimensional space as well as $\psi$ and $\chi$.

An objection can be made with respect to making to these magnitudes the category of “vectors”, and in fact they are not, since $\psi$ and $\chi$ do not transform as genuine vectors, being components of real and the imaginary parts of the spinor $\Psi$ in SU(2), so $\psi$ and $\chi$ do not belong to SO(3). Though $E_{\psi(\chi)}$ and $B_{\psi(\chi)}$ are fields in the space, they are not of vector nature.

The substitution of 3.5 in 3.2 and 3.3 gives:

\[
\begin{pmatrix}
\partial_\mu \psi^\mu \\
E_{\chi,2} - B_{\psi,2} \\
E_{\psi,3} + B_{\chi,3} \\
E_{\psi,1} + B_{\chi,1}
\end{pmatrix} = 0 \quad \text{and} \quad \begin{pmatrix}
E_{\chi,3} - B_{\psi,3} \\
E_{\chi,1} - B_{\psi,1} \\
\partial_\mu \chi^\mu \\
E_{\psi,2} + B_{\chi,2}
\end{pmatrix} = 0,
\]

which can be written as

\[
\begin{align*}
E_\psi &= -B_\chi \\
E_\chi &= B_\psi,
\end{align*}
\]

together with the conditions

\[
\partial_\mu \chi^\mu = 0 \quad \text{and} \quad \partial_\mu \psi^\mu = 0.
\]

As can be seen, relations 3.7 between the fields $E_{\psi(\chi)}$, $B_{\psi(\chi)}$ are the equivalent, for the massless fermion, to the photon relations 2.2, and relations 3.8 are “gauge like” conditions (analogous to the Lorentz gauge condition $\partial_\mu \phi^\mu$).

The spinor field $\Psi$ for massless fermions leads to the set of (nonvector) fields $E_{\psi(\chi)}$, $B_{\psi(\chi)}$ of the three dimensional space.

In particular, it was verified that, for other alternative permuted relabelings (instead of 3.4), we arrive basically to the same results, apart from irrelevant changes of signal.

Now we treat the most general case of massive fermions, for which Dirac equation 3.1 has the mass term:

\[
i\gamma^\mu \partial_\mu \Psi = m\Psi.
\]

Explicitly,

\[
\begin{pmatrix}
\partial_\mu \psi^\mu \\
E_{\chi,2} - B_{\psi,2} \\
E_{\psi,3} + B_{\chi,3} \\
E_{\psi,1} + B_{\chi,1}
\end{pmatrix} = m \begin{pmatrix}
\chi_3 \\
\chi_1 \\
\chi_0 \\
\psi_2
\end{pmatrix}
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and

\[
\begin{pmatrix}
E_{\chi,3} - B_{\psi,3} \\
E_{\chi,1} - B_{\psi,1} \\
\partial_{\mu}\chi^\mu \\
E_{\psi,2} + B_{\chi,2}
\end{pmatrix} = m \begin{pmatrix}
\psi_0 \\
-\chi_2 \\
\psi_3 \\
-\psi_1
\end{pmatrix},
\]

which were obtained respectively from the real and the imaginary parts of the Dirac massive fermion equation after the application of the relabeling \[3.3\]. It is interesting to consider the components of the four-vector fermion current density \[j^\mu = \psi_{\gamma^\mu}\gamma^0\psi\], that can be written as

\[
\begin{align*}
    j^0 &= m^{-2}(C^2 + D^2) + \psi_3^2 + \chi_3^2 \\
    j^\mu &= 2m^{-2}(C \times D) + \Lambda
\end{align*}
\]

The expressions in \[3.12\] were written in a compact form in terms of the three-dimensional field \[\Lambda\] with the components

\[
\begin{align*}
    \Lambda_{1,2} &= - (\psi \times \chi)_{1,2} \\
    \Lambda_3 &= (\chi_0\chi_3 + \psi_0\psi_3)
\end{align*}
\]

and of the three-dimensional fields,

\[
\begin{align*}
    C &= E_{\chi} - B_{\psi}; \\
    D &= E_{\psi} + B_{\chi}.
\end{align*}
\]

Note that on the particular case of massless fermions, \[C\] and \[D\] vanish, resulting on the equalities \[3.7\] and also in this case, \[j^0 = \psi_3^2 + \chi_3^2\] and \[j = \Lambda\].

Expressions \[3.12\] are strikingly similar to, respectively, the expressions of the electromagnetic energy density and the Poynting vector \[11\],

\[
\begin{align*}
    u &= (2\varepsilon_0)^{-1}(E^2 + B^2) \\
    S &= \varepsilon_0^{-1}(E \times B)
\end{align*}
\]

which can be seen replacing \[C \rightarrow E, D \rightarrow B\] and \[m^2 \rightarrow 2\varepsilon_0\], apart from the new additive terms \[\psi_3^2 + \chi_3^2\] and \[\Lambda\], which have not any corresponding similar on the electromagnetic theory. The similarity between the magnitude \[j^0\] in expression \[3.12\] and the energy density \[u\] in expression \[3.15\] may be considered more than a simple mathematical similarity, since the time flux \[j^0\] represents, apart from a multiplicative factor, the energy flux associated strictly with the fermion rest mass \[m\] (that corresponds to the total amount of energy \[mc^2 = m\], in our system of units).

This parallel between the first-order wave equation for the ER and the Dirac equation bringing the fields \[C, D\], and the mathematical similarity
of the expressions of the current and energy densities (eqs. 3.12 and their
correspondents in ER) leaves to consider that, while the electric and the
magnetic fields $E$ and $B$ are real measurable physical magnitudes that act
on charged particles, the fields $E_{\psi(\chi)}, B_{\psi(\chi)}$ or either $C, D$ may also cor-
respond to a physical interaction and be experimentally detected as the
signature of fermions. If this is valid, while the electric and the magnetic
fields satisfy globally the Maxwell equations, equations 3.7 or 3.14 (and con-
sequently the Dirac equation itself) correspond to a “fermionic radiation
field” globally governed by a set of “Maxwell-like” equations. In fact, from
3.5 we obtain

\begin{align}
    \nabla \times E_\psi &= -\partial_0 B_\psi; & \nabla \times B_\psi &= \nabla \pi - \nabla^2 \psi; \\
    \nabla \times E_\chi &= -\partial_0 B_\chi; & \nabla \times B_\chi &= \nabla \kappa - \nabla^2 \chi;
\end{align}

(3.16)

\begin{align}
    \nabla \cdot E_\psi &= -\partial_0 \pi - \nabla^2 \psi_0; & \nabla \cdot B_\psi &= 0; \\
    \nabla \cdot E_\chi &= -\partial_0 \kappa - \nabla^2 \chi_0; & \nabla \cdot B_\chi &= 0;
\end{align}

which are the equivalent to the Maxwell equations, with $\pi = \nabla \cdot \psi$, and $\kappa = \nabla \cdot \chi$.

It can be argued that there is not any experimental evidence to justify
the distinction of these fields in nature, and the present work should be con-
sidered only of formal value. On the other case, it could be wondered if the
fields $\psi, \chi$ are associated to a real physical interaction. Here we consider that
these fields underlie the mechanism that governs Pauli’s exclusion principle
– whose origin stays yet on the category of a ‘principle’ without physical
explanation. Eventually, a pathway is to consider the Dirac equation with
electromagnetic coupling

\begin{equation}
    \gamma^\mu \left( i \partial_\mu - e \phi_\mu \right) \Psi = m \Psi,
\end{equation}

(3.17)

that contains the interaction term $-e \phi_\mu \Psi$. From the parallel shown on the
present work, we could conceive interaction terms of the form

\begin{equation}
    -\sigma \psi'_\mu \Psi, \quad -\sigma \chi'_\mu \Psi
\end{equation}

(3.18)

where $\sigma$ would be a constant, or even a $4 \times 4$ matrix, and the four-component
“potential-like” magnitudes $\psi'_\mu$ and $\chi'_\mu$ correspond to the real and the imag-
inary part of the wavefunction $\Psi'$ of a second identical fermion. We simply
generalize 3.18 to the reduced single term

\begin{equation}
    -\sigma \Psi' \Psi,
\end{equation}

(3.19)
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with \( \Psi' = \psi' + i\chi' \). This interaction term \(3.19\) constructed with the product of two different wavefunctions, should be the basis for a fermion pair wavefunction \(\Theta\). However, the interaction term \(-e\phi_\mu(x)\Psi(x)\) in \(3.17\) is referred to a single set of coordinates \(x\) as well as \(\Psi'\Psi\) in \(3.19\). On the other side, the Slater determinantal wavefunction requires mixed coordinates, \(\Psi'(x')\Psi(x)\).

Secondly, the above formalism does not seem to offer a reason for the required antisymmetrization of terms \(\Psi'(x')\Psi(x) - \Psi(x')\Psi'(x)\).

In order to solve this, we propose the following argumentation. Firstly, we write the Dirac equation as

\[
(3.20) \quad i\gamma^\mu \partial_\mu \Psi(x) - \sigma \Psi(x) = m\Psi(x),
\]

where the term \(-\sigma \Psi(x)\) brings an interaction to be defined. Now, we multiply this equation by the wavefunction \(\Psi'(x')\) of the second fermion in another coordinate \(x'\),

\[
(3.21) \quad i\gamma^\mu \partial_\mu \left[\Psi'(x')\Psi(x)\right] - \sigma \Psi'(x')\Psi(x) = m\Psi'(x')\Psi(x),
\]

where we note the interaction term \(3.19\) on the first member. The product \(\Psi(x)\Psi'(x')\) is the first step towards the collective wavefunction \(\Theta(x, x')\). However, since interaction is mutual, \(\Psi'(x')\) must also be governed by \(3.20\):

\[
(3.22) \quad i\gamma^\mu \partial'_\mu \Psi'(x') - \sigma \Psi'(x') = m\Psi'(x').
\]

We multiply this equation by \(\Psi(x)\) to obtain

\[
(3.23) \quad i\gamma^\mu \partial'_\mu \left[\Psi(x)\Psi'(x')\right] - \sigma \Psi(x)\Psi'(x') = m\Psi(x)\Psi'(x'),
\]

and again, we have the same interaction term \(3.19\). Making the coordinate relabeling \(x \leftrightarrow x'\), the last equation turns to

\[
(3.24) \quad i\gamma^\mu \partial'_\mu \left[\Psi'(x')\Psi'(x)\right] - \sigma \Psi'(x')\Psi'(x) = m\Psi'(x')\Psi'(x),
\]

From the ambiguity between \(3.21\) and \(3.24\), we conclude that the most general collective wavefunction must be of the form

\[
(3.25) \quad \Theta(x, x') = c_1\Psi(x)\Psi'(x') + c_2\Psi'(x')\Psi(x).
\]

The symmetry of the problem requires that \(|c_1| = |c_2|\), so the wavefunction of the two identical fermions is:

\[
(3.26) \quad \Theta(x, x') = \Psi(x)\Psi'(x') \pm \Psi(x')\Psi'(x).
\]
On the present treatment, the interaction between the identical fermions shows close analogies with respect to the interaction with the electromagnetic field. In both cases, the interaction term is linear on the external field. The dependence with respect to the electric charge $e$ on $-e\phi_\mu$ suggests considering that $\sigma$ is a characteristic charge, and the most natural choice is the spin. On the other side, note that the interaction term (3.19) depends on two different coordinates, which can be considered the signature of the non-locality of the interaction between identical fermions.

In general, for $N$ identical (indistinguishable) fermions, the interaction term will be

\begin{equation}
\sigma\Psi_1\Psi_2...\Psi_N,
\end{equation}

and similarly, the collective wavefunction will be

\begin{equation}
\Theta(x,x') = \sum_{\{i_1,i_2;..,i_N\}} \pm \Psi_1(x_{i_1})\Psi_2(x_{i_2})...\Psi_N(x_{i_N}).
\end{equation}

To solve the signal ambiguity, we appeal to the idea behind the Aharonov-Bohm effect (ABE) [12], where the momentum operator $\hat{p}$ is replaced by $(\hat{p} - e\phi)$ due to the external electromagnetic field ($\phi$ is the vector potential). In the ABE, when the electron follows a path $P$ immersed on a magnetic field $B = \nabla \times \phi$, its wavefunction is changed by a phase factor $e^{i\varphi}$, where $\varphi = e \int_P \phi \cdot dx$. We hypothesize that the interaction term (3.19) gives rise to a similar effect. For that, let us consider two identical fermions 1 and 2, respectively attributed to the coordinates $x$ and $x'$. Making a coordinate permutation $x \leftrightarrow x'$, the fermions 1 and 2 are displaced through the paths $P$ and $P'$, respectively. For any choice of $P$ and $P'$, the relative displacement of one fermion with respect to the other is the same, due to the symmetry of the problem, and 1 “sees” 2 circling it by 180 degrees (counter)clockwise, and the same happens with respect to what to 2 “sees”. Consequently, both quantum phases are equal in moduli and signal, changing by the same amount $e^{i\varphi}$. If we assume that the mutual interaction is nonlocal, the phase factors will be independent of the path. Further coordinate permutation leaving the particles to their original positions will amount the total phase $e^{2i\varphi} \equiv 1$, and consequently $e^{i\varphi} = \pm 1$.

Similarly as the electric charge $e$ appears on the ABE phase, the spin parameter $\sigma$ will appear on the fermion phase factor $e^{i\varphi}$. We distinguish two different situations presented in Figure 1. In Figure 1A, the spins are parallel, and each fermion will “see” the another one following the same clockwise rotation during the coordinate permutation. However, if the spins
are antiparallel (Figure B1), the particle 1 will "see" the particle 2 moving clockwise while the particle 2 will "see" the particle 1 moving counter clockwise. This can be seen also inspecting the Figure B2 (which is exactly the Figure B1 turned upside down and rotated, in order that now the spin of particle 2 is upside). Consequently, when the spins are parallel, the phases are equal, but when the spins are antiparallel the phases rotate by opposite angles. Considering that these exponential phase factors are multiplied on the products $\Psi\Psi'$ that enter on the total wavefunction $\Theta$, we conclude that the coordinate permutation results in a global signal "−" in $\Theta$ for the case of parallel spins, thus requiring the antisymmetric spatial part of the global wavefunction. Inversely, when the spins are antiparallel, the opposite phases cancel in the product, and the coordinate permutation must remain the global wavefunction unchanged, requiring the symmetric spatial wavefunction. Apparently this reasoning exhausts the problem of signal ambiguity and enters in accordance with the expected.

Figure 1: The permutation of fermions 1 and 2 from coordinates $x$ and $x'$.

We may consider that for fermions (half integer spin), $2\varphi$ is an odd multiple of $\pi$, so for parallel spins the signal of the product of the phase factors changes with the particle permutation. On the other side, for bosons (integer spin), $2\varphi$ would be an even multiple of $\pi$, and there must not be change of signal. This reasoning is sufficient to justify the well-known choice of signals in 3.28, for either bosons or fermions. As a corollary, we conceive that there must be a photon-photon interaction with no electromagnetic character.
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Despite the limitations of the idea, the considerations of the present work may serve as a starting point to the comprehension of Pauli’s EP, which has been considered without any mechanism of physical interaction and remain unexplored though it left to extensive reflections[13–15]. Following Margenau[13], there are some paradoxical aspects on the EP, though not so appealing as the twins paradox of relativity, from which philosophers “remained uninterested, and physicists often looked upon the principle as a curious but handy tool, unaware of its philosophical importance”. In fact, we may wonder that the mere existence of a non-zero spin magnetic moment may lead us to consider a kind of magnetic interaction mechanism between the particles – which may be represented by terms like[3.19] or [3.27].

Yet following Margenau, while the general relativity replaced the concept of force by metric, the EP brought the concept of asymmetry, to account for the concept of “exchange forces”, i.e., interaction created by a mechanism of mathematical symmetry. Maybe the obscure, essentially mathematical character of the EP come to be explained by an intrinsic interaction mechanism, as suggested on the present work.

As a final minor observation, note the four-component magnitudes $\eta^\mu = (\eta_0, \eta)$ associated with three-dimensional space magnitudes $e = -\nabla \eta_0 - \partial_0 \eta$ and $b = \nabla \times \eta$, appearing on different physical contexts. On the electromagnetic theory, $\eta = \phi$, and $e = E$, $b = B$. The present work showed the same mathematical structure inside the Dirac equation. This structure is also inherent to the Ricci curvature tensor, in first-order of approximation[16, 17].

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