CLOVER NIL RESTRICTED LIE ALGEBRAS OF QUASI-LINEAR GROWTH

VICTOR PETROGRADSKY

ABSTRACT. The Grigorchuk and Gupta-Sidki groups play fundamental role in modern group theory. They are natural examples of self-similar finitely generated periodic groups. The author constructed their analogue in case of restricted Lie algebras of characteristic 2. Shestakov and Zelmanov extended this construction to an arbitrary positive characteristic. Also, the author constructed a family of 2-generated restricted Lie algebras of slow polynomial growth with a nil $p$-mapping.

Now, we construct a family of so called clover 3-generated restricted Lie algebras $T(\Xi)$, where a field of positive characteristic is arbitrary and $\Xi$ an infinite tuple of positive integers. All these algebras have a nil $p$-mapping. We prove that $1 \leq \text{GKdim} T(\Xi) \leq 3$, moreover, the set of Gelfand-Kirillov dimensions of clover Lie algebras with constant tuples is dense on $[1,3]$. We construct a subfamily of non-isomorphic nil restricted Lie algebras $T(\Xi_{\kappa,q})$, where $q \in \mathbb{N}$, $\kappa \in \mathbb{R}^+$, with extremely slow quasi-linear growth of type: $\gamma_{T(\Xi_{\kappa,q})}(m) = m(\ln^q(m))^{\kappa+o(1)}$, as $m \to \infty$.

The present research is motivated by a construction by Kassabov and Pak of groups of oscillating growth. As an analogue, we construct nil restricted Lie algebras of intermediate oscillating growth in $[1,3]$. We call them Phoenix algebras because, for infinitely many periods of time, the algebra is "almost dying" by having "quasi-linear" growth as above, for infinitely many $n$ the growth function behaves like $\exp(n/(\ln n)^{\Lambda})$, for such periods the algebra is "resuscitating". The present construction of 3-generated nil restricted Lie algebras of quasi-linear growth is an important part of that result, responsible for the lower quasi-linear bound in that construction.

1. INTRODUCTION

1.1. Golod-Shafarevich algebras and groups. The General Burnside Problem asks whether a finitely generated periodic group is finite. The first negative answer was given by Golod and Shafarevich, they proved that there exist finitely generated infinite $p$-groups for each prime $p$. As an important instrument, they first construct finitely generated infinite dimensional associative nil-algebras. Using this construction, there are also examples of infinite dimensional 3-generated Lie algebras $L$ such that $(\text{ad} x)^{n(x,y)}(y) = 0$, for all $x,y \in L$, the field being arbitrary. Similarly, one easily obtains infinite dimensional finitely generated restricted Lie algebras $L$ with a nil $p$-mapping. This gives a negative answer to the question of Jacobson whether a finitely generated restricted Lie algebra $L$ is finite dimensional provided that each element $x \in L$ is algebraic, i.e. satisfies some $p$-polynomial $f_p(x) = 0$ ([21, Ch. 5, ex. 17]).

It is known that the construction of Golod yields associative nil-algebras of exponential growth. Using specially chosen relations, Lenagan and Smoktunowicz constructed associative nil-algebras of polynomial growth, there are more constructions including associative nil-algebras of intermediate growth. On further developments concerning Golod-Shafarevich algebras and groups see.

A close by spirit but different construction was motivated by respective group-theoretic results. A restricted Lie algebra $G$ is called large if there is a subalgebra $H \subset G$ of finite codimension such that $H$ admits a surjective homomorphism on a nonabelian free restricted Lie algebra. Let $K$ be a perfect at most countable field of positive characteristic. Then there exist infinite-dimensional finitely generated nil restricted Lie algebras over $K$ that are residually finite dimensional and direct limits of large restricted Lie algebras.

1.2. Grigorchuk and Gupta-Sidki groups. The construction of Golod is rather undirect, Grigorchuk gave a direct and elegant construction of an infinite 2-group generated by three elements of order 2. Originally, this group was defined as a group of transformations of the interval $[0,1]$ from which rational points of the form $\{k/2^n \mid 0 \leq k \leq 2^n, n \geq 0\}$ are removed. For each prime $p \geq 3$, Gupta and Sidki gave a direct construction of an infinite $p$-group on two generators, each of order $p$. This group was constructed as a subgroup of an automorphism group of an infinite regular tree of degree $p$.

2000 Mathematics Subject Classification. 16P90, 16N40, 16S32, 17B50, 17B65, 17B66, 17B70.

Key words and phrases. restricted Lie algebras, $p$-groups, growth, self-similar algebras, nil-algebras, graded algebras, Lie superalgebra, Lie algebras of differential operators.

The author was partially supported by grants CNPq 309542/2016-2, FAPDF 2019/01
The Grigorchuk and Gupta-Sidki groups are counterexamples to the General Burnside Problem. Moreover, they gave answers to important problems in group theory. So, the Grigorchuk group and its further generalizations are first examples of groups of intermediate growth [17], thus answering in negative to a conjecture of Milnor that groups of intermediate growth do not exist. The construction of Gupta-Sidki also yields groups of subexponential growth [12]. The Grigorchuk and Gupta-Sidki groups are self-similar. Now self-similar, and so called branch groups, form a well-established area in group theory [15-30]. There are also constructions of self-similar associative algebras [3, 51, 41].

1.3. Self-similar nil restricted Lie algebras. Fibonacci Lie algebra. Unlike associative algebras, for restricted Lie algebras, natural analogues of the Grigorchuk and Gupta-Sidki groups are known. Namely, over a field of characteristic 2, the author constructed an infinite dimensional restricted Lie algebra \( L \) generated by two elements, called a Fibonacci restricted Lie algebra [35]. Let \( K = p = 2 \) and \( R = K[t_i | i ≥ 0]/(t_i^2 | i ≥ 0) \) a truncated polynomial ring. Put \( \partial_i = \text{Alg}(v_1, v_2) \subset \text{End} R \). The Fibonacci restricted Lie algebra has a slow polynomial growth with Gelfand-Kirillov dimension \( \text{GKdim } \L = \log_2(\sqrt{5}+1)/2 ≈ 1.44 \) [35]. Further properties of the Fibonacci restricted Lie algebra are studied in [40, 42]. On background and some results on Lie algebras of differential operators in infinitely many variables see [47, 40, 39, 13].

Probably, the most interesting property of \( \L \) is that it has a nil \( p \)-mapping [35], which is an analog of the periodicity of the Grigorchuk and Gupta-Sidki groups. We do not know whether the associative hull \( \mathbf{A} \) is a nil-algebra. We have a weaker statement. The algebras \( \mathbf{L}, \mathbf{A} \), and the augmentation ideal of the restricted enveloping algebra \( \mathbf{u} = \omega \mathbf{L} \) are direct sums of two locally nilpotent subalgebras [40]. In case of arbitrary prime characteristic, Shestakov and Zelmanov suggested an example of a finitely generated restricted Lie algebra with a nil \( p \)-mapping [50]. An example of a \( p \)-generated nil restricted Lie algebra \( \mathbf{L} \), characteristic \( p \) being arbitrary, was studied in [13]. These infinite dimensional restricted Lie algebras can have different decompositions into a direct sum of two locally nilpotent subalgebras [49].

Observe that only the original example has a clear monomial basis [35, 40]. In other examples, elements of a Lie algebra are linear combinations of monomials, to work with such linear combinations is sometimes an essential technical difficulty, see e.g. [50, 13]. A family of nil restricted Lie algebras of slow growth having good monomial bases is constructed in [57]. These algebras are close relatives of a two-generated Lie superalgebra of [35].

1.4. Narrow groups and Lie algebras. Let \( G \) be a group and \( G = G_1 ≥ G_2 ≥ \cdots \) its lower central series. One constructs a related \( \mathbb{N} \)-graded Lie algebra \( L_K(G) = \bigoplus_{i≥1} L_i \), where \( L_i = G_i/G_{i+1} ⊕ \mathbb{K} K \), \( i ≥ 1 \). A product is given by \( [aG_{i+1}, bG_{j+1}] = (a, b)G_{i+j+1} \), where \( a ∈ G_i, b ∈ G_j \), and \( (a, b) = a^{-1}b^{-1}ab \) the group commutator.

A residually \( p \)-group \( G \) is said of finite width if all factors \( G_i/G_{i+1} \) are finite groups with uniformly bounded orders. The Grigorchuk group \( G \) is of finite width, namely, \( \dim_G G_i/G_{i+1} ∈ \{1, 2\} \) for \( i ≥ 2 \) [38, 4]. In particular, the respective Lie algebra \( L = L_K(G) = \bigoplus_{i≥1} L_i \) has a linear growth. Bartholdi presented \( L_K(G) \) as a self-similar restricted Lie algebra and proved that the restricted Lie algebra \( L_{K_2}(G) \) is nil while \( L_{K_1}(G) \) is not nil [3]. Also, \( L_K(G) \) is nil graded, namely, for any homogeneous element \( x ∈ L_i, i ≥ 1 \), the mapping \( ad x \) is nilpotent, because the group \( G \) is periodic.

A Lie algebra \( L \) is called of maximal class (or filiform), if the associated graded algebra with respect to the lower central series \( \text{gr } L = \bigoplus_{n=1}^{∞} \text{gr } L_n \), where \( \text{gr } L_n = L^n/L^{n+1}, n ≥ 1 \), satisfies

\[
\dim \text{gr } L_1 = 2, \quad \dim \text{gr } L_n ≤ 1, \quad n ≥ 2, \quad \text{gr } L_{n+1} = [\text{gr } L_1, \text{gr } L_n], \quad n ≥ 1,
\]

in particular, \( \text{gr } L \) is generated by \( \text{gr } L_1 \). An infinite dimensional filiform Lie algebra \( L \) has the smallest nontrivial growth function: \( γ_L(n) = n + 1, n ≥ 1 \). In case of positive characteristic, there are uncountably many such algebras [3]. Nevertheless, in case \( p > 2 \), they were classified in [9]. There are generalizations of filiform Lie algebras. Naturally \( \mathbb{N} \)-graded Lie algebras over \( \mathbb{R} \) and \( \mathbb{C} \) satisfying the condition \( \dim L_n + \dim L_{n+1} ≤ 3, n ≥ 1 \), are classified recently by Millionschikov [29]. More generally, an \( \mathbb{N} \)-graded Lie algebra \( L = \bigoplus_{n=1}^{∞} L_n \) is said of finite width \( d \) in the case that \( \dim L_n ≤ d, n ≥ 1 \), the integer \( d \) being minimal.
Pro-$p$-groups and $\mathbb{N}$-graded Lie algebras cannot be simple. Instead, appears an important notion of being just infinite, namely, not having non-trivial normal subgroups (ideals) of infinite index (codimension). A group (algebra) is said hereditary just infinite if and only if any normal subgroup (ideal) of finite index (codimension) is just infinite. The Gupta-Sidki groups were the first in the class of periodic groups to be shown to be just infinite [20]. The Grigorchuk group is also just infinite but not hereditary just infinite [18]. Concerning narrow Lie algebras and groups see survey [49].

1.5. Lie algebras in characteristic zero. Since the Grigorchuk group is of finite width, a right analogue of it should be a Lie algebra of finite width having ad-nil elements, in the next result the components are of bounded dimension and consist of ad-nil elements. Informally speaking, there are no "natural analogues" of the Grigorchuk and Gupta-Sidki groups in the world of Lie algebras of characteristic zero, strictly in terms of the following result.

**Theorem 1.1** (Martinez and Zelmanov [28]). Let $L = \oplus_{\alpha \in \Gamma} L_\alpha$ be a Lie algebra over a field $K$ of characteristic zero graded by an abelian group $\Gamma$. Suppose that

i) there exists $d > 0$ such that $\dim_K L_\alpha \leq d$ for all $\alpha \in \Gamma$,
ii) every homogeneous element $a \in L_\alpha$, $\alpha \in \Gamma$, is ad-nilpotent.

Then the Lie algebra $L$ is locally nilpotent.

1.6. Fractal nil graded Lie superalgebras. In the world of Lie superalgebras of an arbitrary characteristic, the author constructed analogues of the Grigorchuk and Gupta-Sidki groups [36]. Namely, two Lie superalgebras $\mathbf{R}$, $\mathbf{Q}$ were constructed. Both examples have clear monomial bases. They have slow polynomial growth, namely, GKdim $\mathbf{R} = \log_3 4 \approx 1.26$ and GKdim $\mathbf{Q} = \log_3 8 \approx 1.89$. In both examples, $\text{ad} a$ is nilpotent, $a$ being an even or odd element with respect to the $\mathbb{Z}_2$-gradings as Lie superalgebras. This property is an analogue of the periodicity of the Grigorchuk and Gupta-Sidki groups. The Lie superalgebra $\mathbf{R}$ is $\mathbb{Z}_2^2$-graded, while $\mathbf{Q}$ has a natural fine $\mathbb{Z}_2^1$-grading with at most one-dimensional components. In particular, $\mathbf{Q}$ is a nil finely graded Lie superalgebra, which shows that an extension of Theorem 1.1 for the Lie superalgebras of characteristic zero is not valid. Also, $\mathbf{Q}$ has a $\mathbb{Z}_2^2$-grading which yields a continuum of decompositions into sums of two locally nilpotent subalgebras $\mathbf{Q} = \mathbf{Q}_+ \oplus \mathbf{Q}_-$. Both Lie superalgebras are self-similar, they also contain infinitely many copies of itself, we call them fractal due to the last property.

We construct a more "handy" $2$-generated fractal Lie superalgebra $\mathbf{R}$ (the same notation as above but this is a different algebra) over an arbitrary field [10]. This Lie superalgebra $\mathbf{R}$ is $\mathbb{Z}_2^2$-graded by multidegree in the generators and the $\mathbb{Z}_2^2$-components are at most one-dimensional. As an analogue of periodicity, we establish that homogeneous elements of the $\mathbb{Z}_2$-grading $\mathbf{R} = \mathbf{R}_0 \oplus \mathbf{R}_1$ are ad-nilpotent. In case of $\mathbb{N}$-graded algebras, a close analogue to being simple is being just infinite. Unlike previous examples of Lie superalgebras [36], we are able to prove that $\mathbf{R}$ is just infinite. This example is close to the smallest possible one, because $\mathbf{R}$ has a linear growth with a growth function $\gamma_R(m) \approx 3m$, as $m \to \infty$. Moreover, its degree $\mathbb{N}_\mathbf{R}$-grading is of finite width $4$ (char $\mathbf{K} \neq 2$). In case char $\mathbf{K} = 2$, we obtain a Lie algebra of width $2$ that is not thin.

We also construct a just infinite fractal $3$-generated Lie superalgebra $\mathbf{Q}$ over arbitrary field, which gives rise to an associative hull $\mathbf{A}$, a Poisson superalgebra $\mathbf{P}$, and two Jordan superalgebras $\mathbf{J}$ and $\mathbf{K}$, the latter can be also considered as analogues of the Grigorchuk and Gupta-Sidki groups in respective classes of algebras [45].

2. Basic notions: restricted Lie algebras, Growth

As a rule, $\mathbf{K}$ is an arbitrary field of positive characteristic $p$, $\langle S \rangle_\mathbf{K}$ denotes a linear span of a subset $S$ in a $\mathbf{K}$-vector space. Let $L$ be a Lie algebra, then $U(L)$ denotes the universal enveloping algebra. Long commutators are right-normed: $[x, y, z] := [x, [y, z]]$. We use a standard notation $\text{ad} x(y) = [x, y]$, where $x, y \in L$. Also, we use notation $[x^k, y] := (\text{ad} x)^k(y)$, where $k \geq 1$, $x, y \in L$; in case $k = p^i$, we have also $[x^{p^i}, y] = [x^{[p^i]}, y]$, in terms of the $p$-mapping (see below).

2.1. Restricted Lie algebras. Let $L$ be a Lie algebra over a field $\mathbf{K}$ of characteristic $p > 0$. Then $L$ is called a restricted Lie algebra (or Lie $p$-algebra), if it is additionally supplied with a unary operation $x \mapsto x^{[p]}$, $x \in L$, that satisfies the following axioms [21] [1] [53] [54] [2]:

- $(\lambda x)^{[p]} = \lambda^p x^{[p]}$, for $\lambda \in \mathbf{K}$, $x \in L$;
- $\text{ad}(x^{[p]}) = (\text{ad} x)^p$, $x \in L$;
- $(x + y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{i=1}^{p-1} s_i(x, y)$, for all $x, y \in L$, where $s_i(x, y)$ is the coefficient of $t^{i-1}$ in the polynomial $\text{ad}(tx + y)^{p-1}(x) \in L[t]$.


This notion is motivated by the following construction. Let \( A \) be an associative algebra over a field \( K \). The vector space \( A \) is supplied with a new product \( [x,y] = xy - yx, x,y \in A \), one obtains a Lie algebra denoted by \( A^{(-)} \). In case char \( K = p > 0 \), the mapping \( x \mapsto x^p, x \in A^{(-)} \), satisfies three axioms above.

Suppose that \( L \) is a restricted Lie algebra. Let \( J(L) \) be an ideal of the universal enveloping algebra \( U(L) \) generated by \( \{ x^p \mid x \in L \} \). Then \( u(L) = U(L)/J \) is called a restricted enveloping algebra. In this algebra, the formal operation \( x^p \) coincides with the \( p \)th power \( x^p \) for any \( x \in L \). One has an analogue of Poincare-Birkhoff-Witt’s theorem yielding a basis of the restricted enveloping algebra [21] p. 213]. We shall use the following version of the formula above:

\[
(x + y)^p = x^p + y^p + (\text{ad}x)^p(y) + \sum_{i=1}^{p-2} s_i(x,y), \quad x, y \in L,
\]

where \( s_i(x,y) \) consists of commutators containing \( i \) letters \( x \) and \( p - i \) letters \( y \).

### 2.2. Growth

Let \( A \) be an associative (or Lie) algebra generated by a finite set \( X \). Denote by \( A^{(X,n)} \) the subspace of \( A \) spanned by all monomials in \( X \) of length not exceeding \( n \), \( n \geq 0 \). If \( A \) is a restricted Lie algebra, we define \( A^{(X,n)} = \langle x_1, \ldots, x_n \rangle^{(X)} \mid x_i \in X, \text{sp}^k \leq n \rangle_K \) [31]. One obtains a growth function:

\[
\gamma_A(n) = \gamma_A(X, n) := \dim_K A^{(X,n)}, \quad n \geq 0.
\]

Clearly, the growth function depends on the choice of the generating set \( X \). Let \( f, g : \mathbb{N} \to \mathbb{R}^+ \) be increasing functions. Write \( f(n) \lesssim g(n) \) if and only if there exist positive constants \( N, C \) such that \( f(n) \leq Cg(Cn) \) for all \( n \geq N \). Introduce equivalence \( f(n) \sim g(n) \) if and only if \( f(n) \lesssim g(n) \) and \( g(n) \lesssim f(n) \). Different generating sets of an algebra yield equivalent growth functions [29].

It is well known that the exponential growth is the highest possible growth for finitely generated Lie and associative algebras. A growth function \( \gamma_A(n) \) is compared with polynomial functions \( n^k, k \in \mathbb{R}^+ \), by computing the upper and lower Gelfand-Kirillov dimensions [23]:

\[
\text{GKdim} A := \lim_{n \to \infty} \frac{\ln \gamma_A(n)}{\ln n} = \inf\{ \alpha > 0 \mid \gamma_A(n) \lesssim n^\alpha \};
\]

\[
\text{GKdim} A := \lim_{n \to \infty} \frac{\ln \gamma_A(n)}{\ln n} = \sup\{ \alpha > 0 \mid \gamma_A(n) \gtrsim n^\alpha \}.
\]

Solvable finitely generated Lie algebras typically have intermediate growth, see [20] [32] [34].

Denote \( \ln^q(x) := (\cdots \ln(x)\cdots) \) and \( \exp^q(x) := \exp(\cdots \exp(x)\cdots) \) for all \( q \in \mathbb{N} \). In this paper, we study algebras of quasi-linear growth, growth functions of these algebras \( A \) behave as \( m \exp \left( (\ln m)^\beta \right) \), \( \beta \in (0,1) \), and even slower, like \( m(\ln^q(m))^\beta \), where \( q \in \mathbb{N} \), \( \beta \in \mathbb{R}^+ \). Clearly, \( \text{GKdim} A = \text{GKdim} A = 1 \). In order to specify parameters \( q, \beta \), define numbers:

\[
\text{Ldim}^0 A = \inf\{ \beta \in (0,1) \mid \gamma_A(n) \lesssim m \exp \left( (\ln m)^\beta \right) \};
\]

\[
\text{Ldim}^0 A = \sup\{ \beta \in (0,1) \mid \gamma_A(n) \gtrsim m \exp \left( (\ln m)^\beta \right) \};
\]

\[
\text{Ldim}^q A = \inf\{ \beta \in \mathbb{R}^+ \mid \gamma_A(n) \lesssim m(\ln^q(m))^\beta \}, \quad q \in \mathbb{N};
\]

\[
\text{Ldim}^q A = \sup\{ \beta \in \mathbb{R}^+ \mid \gamma_A(n) \gtrsim m(\ln^q(m))^\beta \}, \quad q \in \mathbb{N}.
\]

One checks that these numbers are invariants not depending on a generating set. Remark that notations are different from [37].

Assume that generators \( X = \{ x_1, \ldots, x_k \} \) are assigned positive weights \( wt(x_i) = \lambda_i, i = 1, \ldots, k \). Define a weight growth function:

\[
\tilde{\gamma}_A(n) = \text{dim}_K \langle x_{i_1} \cdots x_{i_m} \mid \text{wt}(x_{i_1}) + \cdots + \text{wt}(x_{i_m}) \leq n, x_{i_j} \in X \rangle_K, \quad n \geq 0.
\]

Set \( C_1 = \min\{ \lambda_i \mid i = 1, \ldots, k \}, C_2 = \max\{ \lambda_i \mid i = 1, \ldots, k \} \), then \( \tilde{\gamma}_A(C_1 n) \leq \gamma_A(n) \leq \tilde{\gamma}_A(C_2 n) \) for \( n \geq 1 \). Thus, we obtain an equivalent growth function \( \tilde{\gamma}_A(n) \sim \gamma_A(n) \). Therefore, we can use the weight growth function \( \tilde{\gamma}_A(n) \) in order to compute the Gelfand-Kirillov dimensions and \( \text{Ldim}^A, \text{Ldim}^A \) as well.

Suppose that \( L \) is a Lie algebra and \( X \subset L \). By \( \text{Lie}(X) \) denote the subalgebra of \( L \) generated by \( X \). In case \( L \) is a restricted Lie algebra \( \text{Lie}_p(X) \) denotes the restricted subalgebra of \( L \) generated by \( X \). Similarly, assume that \( X \) is a subset in an associative algebra \( A \). Write \( \text{Alg}(X) \subset A \) to denote an associative subalgebra (without unit) generated by \( X \).
3. Main results: Clover restricted Lie algebras of Quasi-linear Growth

3.1. Clover restricted Lie algebras. Recently, the author introduced a large class of drosophila Lie algebras \([38]\), that generalized some examples of (restricted) Lie (super)algebras considered before \([37, 39]\). In particular, it includes a family of 2-generated restricted Lie algebras studied in \([37]\), now we call such algebras as **duplex Lie algebras**.

**Example 1** (family of restricted Lie algebras \(\mathbf{L}(\Xi)\) in \([37]\)). Let \(\text{char } K = p > 0\). Consider integers \(\Xi = (S_n, R_n| n \geq 0)\), which determine a divided power series ring \(\Omega(\Xi) = \langle x_i^{(\alpha_i)} y_i^{(\beta_i)} \cdots x_i^{(\alpha_i)} y_i^{(\beta_i)} z_i^{(\gamma_i)} | 0 \leq \alpha_i < p^{S_i}, 0 \leq \beta_i, \gamma_i < p^{R_i}, i \geq 0 \rangle\). Define pivot elements recursively:

\[
\begin{align*}
  a_i &= \partial x_i + x_i^{(p^{S_i} - 1)} y_i^{(p^{R_i} - 1)} a_{i+1}; \\
  b_i &= \partial y_i + x_i^{(p^{S_i} - 1)} y_i^{(p^{R_i} - 1)} b_{i+1};
\end{align*}
\]

Define the restricted Lie algebra \(\mathbf{L}(\Xi) := \text{Lie}_p(a_0, b_0) \subset \text{Der } \Omega(\Xi)\).

The following example is the main object to study in the present work.

**Example 2.** Fix the same tuple of integers \(\Xi = (S_n, R_n| n \geq 0)\) and consider another formal divided power series ring:

\[
R = R(\Xi) := \langle x_0^{(\alpha_0)} y_0^{(\beta_0)} z_0^{(\gamma_0)} \cdots x_i^{(\alpha_i)} y_i^{(\beta_i)} z_i^{(\gamma_i)} | 0 \leq \alpha_i < p^{S_i}, 0 \leq \beta_i, \gamma_i < p^{R_i}, i \geq 0 \rangle.
\]

Define pivot elements recursively:

\[
\begin{align*}
  v_i &= \partial x_i + x_i^{(p^{S_i} - 1)} y_i^{(p^{R_i} - 1)} v_{i+1}; \\
  w_i &= \partial y_i + x_i^{(p^{S_i} - 1)} y_i^{(p^{R_i} - 1)} w_{i+1}; \\
  u_i &= \partial z_i + x_i^{(p^{S_i} - 1)} y_i^{(p^{R_i} - 1)} u_{i+1};
\end{align*}
\]

We define the **clover restricted Lie algebra** \(\mathbf{T}(\Xi) := \text{Lie}_p(v_0, w_0, u_0) \subset \text{Der } R(\Xi)\).

Remark 1. Let us draw attention that there is a symmetry between \(v_i\) and \(w_i\), while the remaining \(u_i\) stay separate because there is no \(\mathbb{Z}_3\)-**cyclic symmetry** unlike the second example of Lie superalgebras in \([36]\).

Remark 2. In terminology of \([38]\), species of flies having two flies in some generation either have two flies in all subsequent generations or go extinct. The goal in introducing the clover specie is to have three flies in each generation (yielding respective three pivot elements \([4]\)), so that at some moment three flies can produce a wild specie and the constructed Lie algebra can return to a respective fast intermediate growth. To this end we extend the duplex specie in a specific ”skew” way and obtain the clover specie. This idea enables us to construct restricted Lie algebras with an oscillating growth in \([38]\) using two theorems below.

3.2. Main results. As a specific case, we construct restricted Lie algebras of quasi-linear growth. The proof is rather technical and close to that for the 2-generated duplex Lie restricted Lie algebras \([37]\) (Example 1 above). The main goal of the paper is to prove the following two theorems, which are an important part of the construction of nil restricted Lie algebras of oscillating intermediate growth in \([38]\), namely, the algebras constructed below are responsible for periods of quasi-linear growth of that algebras. In comparison with \([37]\), the asymptotic of the next theorem is a little bit more precise.

**Theorem 3.1.** Let \(K\) be a field, \(\text{char } K = p > 0\), fix \(\kappa \in (0, 1)\). There exists a tuple of integers \(\Xi_\kappa\) such that the 3-generated clover restricted Lie algebra \(\mathbf{T} = \mathbf{T}(\Xi_\kappa) = \text{Lie}_p(v_0, w_0, u_0)\) has the following properties.

i) \(\gamma_T(m) = m \exp\left((C + o(1))(\ln m)^\kappa\right)\) as \(m \to \infty\), where \(C := 2(\ln p)^{1-\kappa}/\kappa\);

ii) \(\text{GKdim } T = \text{GKdim } T = 1\);

iii) \(\text{Ldim } T = \text{Ldim } T = \kappa\);

iv) the growth function \(\gamma_T(m)\) is not linear;

v) algebras \(\mathbf{T}(\Xi_\kappa)\) for different \(\kappa \in (0, 1)\) are not isomorphic.

In comparison with \([37]\), algebras with even slower quasi-linear growth are constructed in the next theorem.

**Theorem 3.2.** Let \(\text{char } K = p > 0\), fix \(q \in \mathbb{N}\), \(\kappa \in \mathbb{R}^+\). There exists a tuple of integers \(\Xi_{q, \kappa}\) such that the 3-generated clover restricted Lie algebra \(\mathbf{T} = \mathbf{T}(\Xi_{q, \kappa}) = \text{Lie}_p(v_0, w_0, u_0)\) has the following properties.

i) \(\gamma_T(m) = m\left(\ln(q)m\right)^{\kappa + o(1)}\) while \(m \to \infty\);
ii) GKdim $T = GKdim T = 1$;
iii) $\text{Ldim}^q T = \text{Ldim}^q T = \kappa$;
iv) the growth function $\gamma_T(m)$ is not linear;
v) algebras $T(\mathbb{N}_{q,\kappa})$ for different pairs $(q, \kappa)$ are not isomorphic.

**Remark 3.** Similar to [37], we can consider also the associative algebra $A = \text{Alg}(v_0, w_0, u_0) \subset \text{End} R(\Xi)$ and prove that it has a quasi-quadratic growth, namely $\gamma_A(m) = m^2 (\ln q, m)^{\kappa+o(1)}$, $m \to \infty$.

**Theorem 3.3** ([38]). Fix char $K = p > 0$ and a tuple of integers $\Xi = (S_n, R_n | n \geq 0)$. Consider the respective clover restricted Lie algebra $T = T(\Xi)$. Then $T$ has a nil $p$-mapping.

**Proof.** The nillity is proved in a more general setting of so called drosophila Lie algebras with uniform parameters [38]. The proof is a modification of that for the 2-generated duplex Lie algebra $L(\Xi)$ [37]. \hfill \Box

Let us describe some more interesting results and ideas of the paper.

- We describe the structure and construct a clear monomial basis for all clover restricted Lie algebras (Theorem 4.7).
- An important instrument is a notion of a weight function, using which we prove that $T(\Xi) = T(v_0, w_0, u_0)$ is $\mathbb{N}_{S}^3$-graded by the multidegree in the generators (Theorem 4.3).
- We prove that $1 \leq \text{GKdim} T(\Xi) \leq 3$ for any tuple $\Xi$ (Theorem 6.2).
- In case $S_i = S$ and $R_i = R$ for all $i \geq 0$ we say that $\Xi$ is a constant tuple and denote $T(S, R) := T(\Xi)$. In this case, $T(S, R)$ is a self-similar restricted Lie algebra and $\{\text{GKdim} T(S, R) | S, R, R \in \mathbb{N}\}$ is dense on $[1, 3]$ (Corollary 6.5).

**Remark 4.** We suggest that the clover restricted Lie algebras $T(\Xi)$ are analogues of the family of the Grigorchuk groups $G_\omega$ constructed and studied in [17].

### 3.3. Nil Lie algebras of slow polynomial growth.

The Gelfand-Kirillov dimension of an associative algebra cannot belong to the interval $(1, 2)$ [24]. Bergman. One has the same gap for finitely generated Jordan algebras [27]. Martin and Zelmanov. The author showed that a similar gap does not exist for Lie algebras, the Gelfand-Kirillov dimension of a finitely generated Lie algebra can be an arbitrary number $(0) \cup [1, +\infty)$ [33]. The same fact is also established for Jordan superalgebras [14]. Also, an interesting direction of research is constructing associative nil algebras of different kinds of growth, in particular, of slow polynomial growth, see [24, 7, 25, 52].

Now we get a stronger version of [33], the gap $(1, 2)$ can be filled with nil Lie $p$-algebras. Namely, using constant tuples, we get clover nil restricted Lie algebras which Gelfand-Kirillov dimensions are dense on $[1, 3]$ (Corollary 6.5).

### 4. Structure of Clover Lie algebras $T(\Xi)$

#### 4.1. Basic relations.

We start with establishing basic relations in clover restricted Lie algebras. In what follows, we assume that a field $K$ of characteristic char $K = p > 0$ and a tuple of integers $\Xi = (S_n, R_n | n \geq 0)$ are fixed, and we consider 3-generated clover restricted Lie algebra $T = \text{Lie}_p(v_0, w_0, u_0)$.

**Lemma 4.1.** Let $i \geq 0$. Then

$$
\begin{align*}
\partial_i^m v_i &= \partial_i^m + x_i^{(p^m - p^n)} y_i^{(p^n - 1)} v_i, & 0 \leq m &\leq S_i; \\
\partial_i^m w_i &= \partial_i^m + y_i^{(p^m - p^n)} x_i^{(p^n - 1)} w_i, & 0 \leq m &\leq R_i; \\
\partial_i^m u_i &= \partial_i^m + x_i^{(p^m - p^n)} x_i^{(p^n - 1)} u_i, & 0 \leq m &\leq R_i,
\end{align*}
$$

where $\partial_i^{p^n} = \partial_i^0 = \partial_i^{p^n} = 0$ above.

**Proof.** Let us prove the first equality by induction on $m$. The base of induction $m = 0$ is trivial by (1). Assume that the claim is valid for $0 \leq m < S_i$. The summation in (2) is trivial because the second term cannot be used more than once:

$$
\begin{align*}
v_i^{m+1} &= (v_i^m)^p = (\partial_i^m + x_i^{(p^m - p^n)} y_i^{(p^n - 1)} v_i)^p \\
&= (\partial_i^m)^p + (\text{ad} \partial_i^m)^{p-1} (x_i^{(p^m - p^n)} y_i^{(p^n - 1)} v_i) \\
&= \partial_i^{m+1} + x_i^{(p^m - p^n)} y_i^{(p^n - 1)} v_i, & 0 \leq m &\leq S_i.
\end{align*}
$$

\hfill \Box
Lemma 4.2. Let char $K = p > 0$ and a tuple $\Xi$ be fixed. Consider the clover restricted Lie algebra $T(\Xi) = \text{Lie}_p(v_0, w_0, u_0)$. Then

i) \( v_i^{p^R_i} = y_i^{(p^R_i - 1)} w_{i+1}, \quad w_i^{p^R_i} = x_i^{(p^S_i - 1)} w_{i+1}, \quad u_i^{p^R_i} = x_i^{(p^S_i - 1)} u_{i+1}, \) for all $i \geq 0$.

ii) \( [v_i^{p^R_i - 1}, v_i^{p^S_i}] = w_{i+1}, \quad [w_i^{p^R_i - 1}, w_i^{p^S_i}] = w_{i+1}, \quad [v_i^{p^S_i - 1}, u_i^{p^R_i}] = u_{i+1}, \) for all $i \geq 0$.

iii) $v_i, w_i, u_i \in T(\Xi), i \geq 0$.

Proof. Follows from computations of [35]. But let us check the formulas directly. The first claim is a partial case of Lemma 4.4. The third claim follows from the second. Finally, let us check the second claim.

\[
[w_i^{p^R_i - 1}, v_i^{p^S_i}] = (\text{ad } w_i)^{p^R_i - 1} [w_i, v_i^{p^S_i}]
= (\text{ad } w_i)^{p^R_i - 2} \left[ \partial_i v_i + y_i^{(p^R_i - 1)} x_i^{(p^S_i - 1)} w_{i+1}, y_i^{(p^R_i - 1)} u_{i+1} \right] = v_{i+1}. \]

\[\square\]

Lemma 4.3. The subalgebra of $T(\Xi)$ generated by $v_0, w_0$ is isomorphic to $L(\Xi)$ defined by (3).

Proof. We observe that $v_0, w_0$ have the same presentation as $a_0, b_0 \in L(\Xi)$.

\[\square\]

4.2. Head elements of two types. We construct a clear monomial basis for $T(\Xi)$ similar to that for its subalgebra $L(\Xi) \cong \text{Lie}_p(v_0, w_0) \subset T(\Xi)$ found in [27]. In case of a clover algebra, a generation of a pivot element is also referred to as its length. Consider products of two pivot elements of the same length 4:

\[
h_{i+1} := [w_i, v_i] = x_i^{(p^S_i - 1)} y_i^{(p^R_i - 2)} v_{i+1} - x_i^{(p^R_i - 2)} y_i^{(p^R_i - 1)} w_{i+1},
\]

\[
g_{i+1} := [v_i, u_i] = x_i^{(p^S_i - 2)} y_i^{(p^R_i - 1)} u_{i+1},
\]

\[\text{with } w_i, u_i = 0, \quad i \geq 0. \quad (5)\]

Lemma 4.4 ([27], Lemma 2.2). For all $i \geq 0$ we have the following elements:

i) For all $0 \leq \xi < p^S_i, 0 \leq \eta < p^R_i$ (except the case $\xi = p^S_i - 1$ and $\eta = p^R_i - 1$) we get:

\[
k_{i+1}^{\xi, \eta} := [v_i^{\xi}, w_i^{\eta}, h_{i+1}] = x_i^{(p^S_i - 1 - \xi)} y_i^{(p^R_i - 2 - \eta)} v_{i+1},
\]

\[
= x_i^{(p^R_i - 2 - \xi)} y_i^{(p^R_i - 1 - \eta)} u_{i+1}; \quad (6)\]

ii) The order of the multiplication above is not essential. As partial cases, we get:

\[
h_{i+1}^{0, 0} = h_{i+1} = [w_i, v_i];
\]

\[
h_{i+1}^{p^S_i - 1, 0} = y_i^{(p^R_i - 2)} v_{i+1}, \quad \text{for } 0 \leq \xi = p^R_i - 2; \quad \text{as a particular case}:
\]

\[
h_{i+1}^{p^R_i - 1, 0} = -x_i^{(p^R_i - 2)} u_{i+1}; \quad \text{as a particular case}:
\]

\[
h_{i+1}^{p^S_i - 2, p^R_i - 1} = -w_{i+1}; \quad (8)\]

Thus, for all $i \geq 0$, we obtain elements (8), called heads of first type of length $i + 1$:

\[
\left\{ h_{i+1}^{\xi, \eta} \mid 0 \leq \xi < p^S_i, 0 \leq \eta < p^R_i, \quad \text{except } (\xi = p^S_i - 1 \text{ and } \eta = p^R_i - 1) \right\}. \quad (7)\]

Consider (7) as a table of size $p^S_i \times p^R_i$, rows and columns being indexed by $\xi, \eta$, the lower right corner is empty. The table contains $v_{i+1}$ and $-w_{i+1}$ in respective cells.

We multiply (5) by $v_i, u_i$ (the order is not essential) we get heads of second type of length $i + 1$:

\[
g_{i+1}^{\xi, \eta} := [v_i^{\xi}, w_i^{\eta}, [v_i, u_i]] = y_i^{(p^S_i - 2 - \xi)} x_i^{(p^R_i - 1 - \eta)} u_{i+1} \quad 0 \leq \xi \leq p^S_i - 2, \quad 0 \leq \xi \leq p^R_i - 1; \quad (8)\]

We put (8) in table of size $p^S_i \times p^R_i$, rows and columns being indexed by $\xi, \eta$, the lower right corner containing $u_{i+1}$.

4.3. Monomial basis of clover restricted Lie algebras $T(\Xi)$. Define tails of first and second types:

\[
r_n(x, y) = x_0^{(\xi_0)} y_0^{(\eta_0)} \cdots x_n^{(\xi_n)} y_n^{(\eta_n)} \in R, \quad 0 \leq \xi_i < p^S_i, 0 \leq \eta_i < p^R_i; \quad n \geq 0;
\]

\[
r_n(x, y, z) = x_0^{(\xi_0)} y_0^{(\eta_0)} z_0^{(\zeta_0)} \cdots x_n^{(\xi_n)} y_n^{(\eta_n)} z_n^{(\zeta_n)} \in R, \quad 0 \leq \xi_i < p^S_i, 0 \leq \eta_i, \zeta_i < p^R_i; \quad n \geq 0. \quad (9)\]

For $n < 0$ we assume that $r_n = 1$. Another elements of type (9) will be denoted as $r'_n, \tilde{r}_n, r_n^{(k)}$, etc., while the lower index denotes the biggest index of variables it depends on.
Define standard monomials of first type of length $n, n \geq 1$:

$$r_{n-2}(x, y, z)h_n^{\xi_n-1} \eta_n-1 = r_{n-2}(x, y)\left(x_n^{(p_n^{\xi_n-1} - 1 - \xi_{n-1})} (p_n^{R_n-1} - 2- \eta_{n-1}) v_n - x_n^{(p_n^{\xi_n-1} - 2 - \xi_{n-1})} (p_n^{R_n-1} - 1 - \eta_{n-1}) w_n\right),$$

(10)

where $0 \leq \xi_{n-1} < p_n^{S_n-1}$, $0 \leq \eta_{n-1} < p_n^{R_n-1}$, except $(\xi_{n-1}=p_n^{S_n-1}$ and $\eta_{n-1}=p_n^{R_n-1})$.

Recall that the heads $h_n^{\xi_n-1} \eta_n-1$ are described by [3], while the tails $r_{n-2}(x, y)$ are [4]. We call $x_n-1, y_n-1$ neck letters. By Lemma [4], we get the pivot elements $v_n, w_n$, for $n \geq 1$, as particular cases of such monomials. So, we consider that $v_0, w_0$ are also the standard monomials of first type of length 0.

Define standard monomials of second type of length $n, n \geq 1$:

$$r_{n-2}(x, y, z)g_n^{\xi_n-1} \zeta_n-1 = r_{n-2}(x, y, z)x_n^{(p_n^{\xi_n-1} - 2 - \xi_{n-1})} z_n^{(p_n^{R_n-1} - 1 - \zeta_{n-1})} u_n,$$

(11)

where $0 \leq \zeta_{n-1} \leq p_n^{S_n-1} - 2$, $0 \leq \zeta_{n-1} \leq p_n^{R_n-1} - 1$.

Recall that the heads $g_n^{\xi_n-1} \zeta_n-1$ are described by [5], while the tails $r_{n-2}(x, y, z)$ are [6]. We call $x_n-1, z_n-1$ neck letters. By definition, consider that $u_0$ is a standard monomial of second type of length 0.

Theorem 4.5. A basis of the Lie algebra $L = \text{Lie}(v_0, w_0, u_0)$ (i.e., we use only the Lie bracket) is given by

i) the standard monomials of first type of length $n \geq 0$;

ii) the standard monomials of second type of length $n \geq 0$.

Proof. The standard monomials of first type form a basis of Lie($v_0, w_0$) [37, Theorem 5.1]. Let us prove that the standard monomials of second type belong to $L$. We proceed by induction on length $n$. We have $u_0 \in L$. By [5], we get $g_0^{\xi_0-1} \zeta_0 = x_0^{(p_0^{R_0-1} - 1 - \xi_0)} z_0^{(p_0^{S_0-1} - 1 - \zeta_0)} u_0 = [v_0, u_0, [v_0, u_0]] \in R$, for all $0 \leq \xi_0 \leq p_0^{R_0-1}$, $0 \leq \zeta_0 \leq p_0^{R_0-1} - 1$. Thus, we have the base of induction for $n = 0, 1$. Now let $n \geq 1$. By induction hypothesis, $r_{n-2}(x, y, z)z_n^{(p_n^{R_n-1} - 1)} u_n \in L$. We use recurrence formula for $v_n$ and [5]:

$$[v_{n-1}, r_{n-2}(x, y, z)z_n^{(p_n^{R_n-1} - 1)} u_n] = \left[\partial x_n + x_n^{(p_n^{S_n-1})} y_n^{(p_n^{R_n-1})} v_n, r_{n-2}(x, y, z)z_n^{(p_n^{R_n-1} - 1)} u_n\right]$$

$$= r_{n-2}(x, y, z)z_n^{(p_n^{R_n-1} - 1)} y_n^{(p_n^{R_n-1})} \partial x_n z_n^{(p_n^{R_n-1} - 1)} [v_n, u_n]$$

$$= r_{n-2}(x, y, z)z_n^{(p_n^{R_n-1} - 1)} y_n^{(p_n^{R_n-1})} x_n^{(p_n^{S_n-2})} z_n^{(p_n^{R_n-1})} u_{n+1}.$$  

By assumption, $r_{n-2}(x, y, z)$ can have arbitrary powers of its variables. Multiplying by $v_n = \partial x_n + x_n^{(p_n^{S_n-1})} y_n^{(p_n^{R_n-1})} u_{n+1}$, we can reduce the power of $x_n$ above to any desired value. The same argument applies to the remaining variables. Thus, all standard monomials of second type belong to $L$.

It remains to show that products of standard monomials are expressed via standard monomials. This is true for monomials of first type because they form a basis of Lie($v_0, w_0$) [37]. Take two standard monomials of second type, short written as $d_1 = r_{m-1} u_m$, and $d_2 = r_{m-1} u_m$, divided variables will be short written as $x_i^*$, $y_i^*$, $z_i^*$. Assume that $n \leq m$. Using recurrence presentation, we get

$$[d_1, d_2] = \left[r_{n-1} (\partial x_n + x_n^{a_n^{*} z_n^{*}} (\partial z_n) + \cdots + x_n^{a_{m-2} z_m^{*} - 2} (\partial z_{m-1} + x_n^{a_{m-1} z_m^{*} - 1} u_m)) , r_{m-1} u_m\right]$$

$$= r_{n-1} \partial x_n (r_{m-1} u_m) + r_n (r_{n+1} (r_{m-1} u_m) + \cdots + r_m^{m-2} (\partial z_{m-1} (r_{m-1} u_m)).$$

Observe that we get standard monomials of second type above.

Consider products of monomials of different types. Write shortily $d_1 = r_{m-1}(x, y, z) u_m$, $d_2 = r_{m-2}(x, y) h_n^{\xi_n-1} = r_{m-1}(x, y) v_m - r_{m-1}(x, y) w_m$. Consider the case $n \leq m$. Using recurrence presentation and [6],

$$[d_1, d_2] = \left[r_{n-1}(x, y, z) \partial x_n + x_n^{a_n^{*} z_n^{*}} (\partial z_n) + \cdots + x_n^{a_{m-2} z_m^{*} - 2} (\partial z_{m-1} + x_n^{a_{m-1} z_m^{*} - 1} u_m)) , r_{m-1}(x, y) v_m - r_{m-1}(x, y) w_m\right]$$

$$= r_{m-1}(x, y, z) [u_m, v_m] - r_{m-1}(x, y, z) [u_m, w_m] = - r_{m-1}(x, y, z) x_n^{(p_n^{S_n-2})} z_n^{(p_n^{R_n-1})} u_{m+1},$$
yielding a standard monomial of second type. Consider the case \( m < n \). Then

\[
d_2 = \sum_{j=m}^{n-1} \left( p_j^{(j)}(x,y)\partial_{x_j} - r_j^{(j)}(x,y)\partial_{y_j}\right) + q_j^{(j)}(x,y)v_n - r_j^{(j)}(x,y)w_n;
\]

\[
[d_2, d_1] = \sum_{j=m}^{n-1} \left( q_j^{(j)}(x,y)\partial_{x_j} - r_j^{(j)}(x,y)\partial_{y_j}\right) r_n(x,y,z)u_n + r_j^{(j)}(x,y,z)v_n, \quad j = 0, \ldots, n-1.
\]

The last term yielding a standard monomial of second type by \([5]\). In the preceding sum, the action on variables with indices \( n-1 \) appears in case \( j = n-1 \). Recall that \( d_1 = r_{n-1}(x,y,z)v_n = r_{n-2}(x,y,z)x_{n-1}^{\alpha\gamma}y_{n-1}u_n \), where \( 0 \leq \alpha < p^{S_{n-1}} - 1 \), \( 0 \leq \gamma < p^{R_{n-1}} - 1 \). After the action in the sum above, we again get standard monomials of second type.

By Lemma \([4,1]\)

\[
v_n^p = \begin{cases} 
\partial_{x_n}^p + x_n(y_n^{p^{S_n} - p^p} - y_n^{p^{R_n} - 1})v_{n+1}, & 1 \leq i < S_n; \\
\partial_{y_n}^p + y_n(x_n^{p^{S_n} - 1})w_{n+1}, & i = S_n; \\
\partial_{z_n}^p + z_n(x_n^{p^{S_n} - 1})w_{n+1}, & 1 \leq i < R_n; \\
\partial_{z_n}^p + z_n(x_n^{p^{S_n} - 1})w_{n+1}, & i = R_n.
\end{cases}
\]

(12)

We refer to nonzero powers of \( v_n, w_n \) as power standard monomials of first type of length \( n + 1 \), powers of \( u_n \) are power standard monomials of second type. One checks that they are linearly independent with the standard monomials.

**Lemma 4.6.** Let \( n \geq 0 \). There are \( S_n + R_n \) power standard monomials of first type and \( R_n \) power standard monomials of second type of length \( n + 1 \).

**Theorem 4.7.** Let \( T(\Xi) = \text{Lie}_p(v_0, w_0, u_0) \) be the clover restricted Lie algebra. Then

i) A basis of \( T(\Xi) \) is given by the standard and power standard monomials of first and second types.

ii) We have a semidirect product

\[
T(\Xi) = \text{Lie}_p(v_0, w_0) \ltimes J, \quad \text{Lie}_p(v_0, w_0) \cong L(\Xi),
\]

the subalgebra \( \text{Lie}_p(v_0, w_0) \) is spanned by the standard and power standard monomials of first type, the ideal \( J \) is spanned by the standard and power standard monomials of second type.

**Proof.** To get a basis of the \( p \)-hull of a Lie algebra we need to add \( p^m \)-powers, \( m \geq 1 \), of its basis \([54]\). Observe that the standard monomials contain non-trivial tails except for the pivot elements. Thus, to get a basis of \( \text{Lie}_p(v_0, w_0, u_0) \) we add nontrivial powers of the pivot elements, i.e. power standard monomials. The second claim follows from computations of Theorem \( 4.5 \).

---

5. **Weight function, \( \mathbb{Z}^3 \)-grading, bounds on weights**

5.1. **Weights.** By pure monomials we call products of divided powers and pure derivations. In particular, if a monomial contains one pure derivation, we get a pure Lie monomial. Set \( \alpha_n = \text{wt}(\partial_{x_n}) = -\text{wt}(x_n) \in \mathbb{C} \), \( \beta_n = \text{wt}(\partial_{y_n}) = -\text{wt}(y_n) \in \mathbb{C} \), \( \gamma_n = \text{wt}(\partial_{z_n}) = -\text{wt}(z_n) \in \mathbb{C} \), for all \( n \geq 0 \). This values are easily extended to a weight function on pure monomials, additive on their (Lie or associative) products. Next, consider weight functions such that all terms in recurrence relation \([13]\) have the same weight, thus, attaching the same value as a weight for the pivot element as well. We get a recurrence relation:

\[
\begin{pmatrix} \alpha_{n+1} \\ \beta_{n+1} \\ \gamma_{n+1} \end{pmatrix} = \begin{pmatrix} p^{S_n} \\ p^{R_n} - 1 \\ 0 \end{pmatrix} \begin{pmatrix} \alpha_n \\ \beta_n \\ \gamma_n \end{pmatrix}, \quad n \geq 0.
\]

(13)

Recurrence relation \([13]\) expresses weights of the pivot elements of generation \( n + 1 \) via weights of the pivot elements of generation \( n \). Hence, any weight function satisfying \([13]\) is determined by its values on the zero generation, namely, by \( \text{wt}(v_0), \text{wt}(w_0), \text{wt}(u_0) \). Also, let \( \text{wt}_i(*) \) be a weight function which is equal to zero for all but \( i \)-th element in the list \( \{v_0, w_0, u_0\}, \) for \( i = 1, 2, 3 \). Compose the multidegree weight
function $\text{Gr}(v) := (\text{wt}_1(v), \text{wt}_2(v), \text{wt}_3(v))$, where $v$ is a pure monomial. By definition, $\text{Gr}(v_0) = (1, 0, 0)$, $\text{Gr}(w_0) = (0, 1, 0)$, $\text{Gr}(u_0) = (0, 0, 1)$. Thus, the space of weight functions satisfying $[13]$ is 3-dimensional with a basis $\text{wt}_1(*)$, $\text{wt}_2(*)$, $\text{wt}_3(*)$. Using $[13]$ we see that $\text{Gr}(v_n) \in \mathbb{N}_0^3$ for all $n \geq 0$. Finally, define a total degree weight function $\text{wt}(v) := \sum_{j=1}^3 \text{wt}_j(v)$.

**Lemma 5.1.** $\text{wt}(v_n) = \text{wt}(w_n) = \text{wt}(u_n) = \prod_{i=0}^{n-1} (p^{S_i} + p^{R_i} - 1)$ for all $n \geq 0$.

*Proof.* Follows by induction from $[1]$. \qed

Recall that $\text{wt}(*)$ is the total degree function determined by $\text{wt}(v_0) = \text{wt}(w_0) = \text{wt}(u_0) = 1$. Let $\text{wt}_{12}(*)$ be determined by the initial values $\text{wt}_{12}(v_0) = \text{wt}_{12}(w_0) = 1$ and $\text{wt}_{12}(u_0) = 0$. Observe that $\text{wt}_{12}(v) = \text{wt}_1(v) + \text{wt}_2(v)$ and $\text{wt}(v) = \text{wt}_{12}(v) + \text{wt}_3(v)$ for any monomial $v$. Thus, $\text{wt}_{12}(v)$ counts multiplicity of $v$ with respect to $v_0, w_0$ and $\text{wt}_3(v)$ counts multiplicity with respect to $u_0$ only.

**Lemma 5.2.** For all $n \geq 0$ we have
\[
\begin{align*}
\text{wt}_3(u_n) &= p^{R_0 + \cdots + R_{n-1}}, & \text{wt}_3(v_n) &= \text{wt}_3(w_n) = 0; \\
\text{wt}_{12}(v_n) &= \text{wt}_{12}(w_n) = \prod_{i=0}^{n-1} (p^{S_i} + p^{R_i} - 1); \\
\text{wt}_{12}(u_n) &= \prod_{i=0}^{n-1} (p^{S_i} + p^{R_i} - 1) - p^{R_0 + \cdots + R_{n-1}}.
\end{align*}
\]

*Proof.* Three formulas are checked by induction. Using $\text{wt}(*) = \text{wt}_{12}(*) + \text{wt}_3(*)$ and Lemma 5.1 we get the last formula. \qed

5.2. $\mathbb{N}_0^3$-gradings. By a generalized monomial $a \in \text{End } R$ we call any (Lie or associative) product of pure monomials and pivot elements. By construction, actual pivot elements and their products are generalized monomials. Observe that generalized monomials are written as infinite linear combinations of pure monomials. Our construction implies that these pure monomials have the same weight, we call this value the weight of a generalized monomial. Thus, the weight functions are well-defined on generalized monomials as well. Also, $\text{Gr}(v) \in \mathbb{N}_0^3$ for any generalized monomial $v$.

In many examples studied before $[10, 22, 36, 37, 11, 15]$ we were able, as a rule, to compute explicitly basis functions for the space of weight functions and study multigradings in more details. Using that base weight functions and multigradings we were able to get more information about our algebras. In a general setting of the present paper it is not possible.

**Theorem 5.3.**

i) the multidegree weight function $\text{Gr}(v)$ is additive on products of generalized monomials $v, w \in \text{End } R$:
\[
\text{Gr}([v, w]) = \text{Gr}(v) + \text{Gr}(w), \quad \text{Gr}(v \cdot w) = \text{Gr}(v) + \text{Gr}(w).
\]

ii) $T = \text{Lie}_p(v_0, w_0, u_0)$, $A = \text{Alg}(v_0, w_0, u_0)$ are $\mathbb{N}_0^3$-graded by multidegree in the generators $\{v_0, w_0, u_0\}$:
\[
T = \bigoplus_{(n_1, n_2, n_3) \in \mathbb{N}_0^3} T_{n_1, n_2, n_3}, \quad A = \bigoplus_{(n_1, n_2, n_3) \in \mathbb{N}_0^3} A_{n_1, n_2, n_3}.
\]

iii) $\text{wt}(*)$ counts the degree of $v \in T, A$ in $\{n_1, n_2, n_3\}$ yielding gradings:
\[
T = \bigoplus_{n=1}^{\infty} T_n, \quad A = \bigoplus_{n=1}^{\infty} A_n.
\]

*Proof.* Claim i) follows from the additivity of the weight function on products of pure monomials. Consider ii). Recall that $\text{Gr}(v) \in \mathbb{N}_0^3$ for any generalized monomial $v$ and $\text{Gr}(*)$ is additive on their products. Thus, we get $\mathbb{N}_0^3$-gradings on $T, A$.

Let $v$ be a monomial in the generators $\{n_1, n_2, n_3\}$ each number $n_i$ counting entrees of $v_0, v_0, u_0$, respectively. By additivity, $\text{Gr}(v) = n_1 \text{Gr}(v_0) + n_2 \text{Gr}(u_0) + n_3 \text{Gr}(u_0) = (n_1, n_2, n_3)$. Hence, $T, A$ are $\mathbb{N}_0^3$-graded by multidegree in the generators. Now, the last claim is evident. \qed
5.3. Bounds on weights.

Lemma 5.4 ([37], Lemma 6.5). Let \( w > 0 \) be a (power) standard monomial of first type of length \( n \geq 0 \). Then
\[
\mathrm{wt}(v_{n-1}) + 1 \leq \mathrm{wt}(w) \leq \mathrm{wt}(v_n).
\]

We shall also write a standard monomial \( w > 0 \) of second type (see (11), (12)) as:
\[
w = r_{n-3}(x, y, z)x_{n-2}^{(p^{n-2} - 1 - \alpha)p_{n-2}^{n-2} - 1 - \beta} y_{n-2}^{(p^{n-2} - 1 - \gamma)p_{n-2}^{n-2} - 1 - \gamma} z_{n-2}^{(p^{n-2} - 2 - \xi)p_{n-2}^{n-2} - 2 - \xi} v_{n-1}^{(p^{n-1} - 1 - \zeta)p_{n-1}^{n-1} - 1 - \zeta} u_n,
\]
where \( 0 \leq \alpha < p^{S^{n-2}} - 0 \leq \beta, \gamma < p^{R^{n-2}}, 0 \leq \xi < p^{R^{n-1}}, 0 \leq \xi \leq p^{S^{n-1}} - 2. \)

Lemma 5.5. Let \( w > 0 \) be a (power) standard monomial of second type of length \( n \geq 0 \).

i) Let \( w > 0 \) be a power standard monomial. Then
\[
\mathrm{wt}(v_{n-1}) + 1 \leq \mathrm{wt}(w) \leq \mathrm{wt}(v_n).
\]

ii) Let \( w > 0 \) be a standard monomial of second type of length \( n \geq 2 \), then
\[
(p^{S^{n-2}} - 1) \mathrm{wt}(v_{n-2}) < \mathrm{wt}(w) \leq \mathrm{wt}(v_n).
\]

iii) Let \( w > 0 \) of second type be presented as (14), we get more precise bounds:
\[
\begin{align*}
\mathrm{wt}(w) > (p^{S^{n-2}} - 1 + \alpha + \beta + \gamma) \mathrm{wt}(v_{n-2}) + (\xi + \zeta) \mathrm{wt}(v_{n-1}); \\
\mathrm{wt}(w) \leq \mathrm{wt}(v_{n-1})(\xi + \zeta + 2). 
\end{align*}
\]

iv) Let \( w > 0 \) be of second type \( (14) \) and assume that \( \xi > 0 \) or \( \zeta > 0 \), then
\[
\mathrm{wt}(v_{n-1}) + 1 \leq \mathrm{wt}(w) \leq \mathrm{wt}(v_n).
\]

Proof. i) Weights of power standard monomials of second type (i.e. powers of \( u_{n-1} \)) are equal to weights of powers of \( u_{n-1} \), and we apply Lemma 5.4.

ii) Let \( w > 0 \) be a standard monomial of second type \( (11) \). Clearly, weight is bounded by \( \mathrm{wt}(u_n) = \mathrm{wt}(v_n) \).

We get a lower bound by taking the maximal allowed powers of variables. Below we get homogeneous components of partial recurrence expansions for \( u_0 \) and \( v_0 \), and use that \( \mathrm{wt}(v_0) = \mathrm{wt}(u_0) = 1. \)

\[
\begin{align*}
\mathrm{wt}(w) \geq \mathrm{wt}(\left( \prod_{i=0}^{n-2} x_i^{(p^{S_i} - 1)} y_i^{(p^{R_i} - 1)} z_i^{(p^{R_i} - 1)} \right) x_{n-1}^{(p^{S_{n-1}} - 2)} z_{n-1}^{(p^{R_{n-1}} - 2)} u_n) \\
= \mathrm{wt}(\left( \prod_{i=0}^{n-1} x_i^{(p^{S_i} - 1)} z_i^{(p^{R_i} - 1)} \right) u_n) - \mathrm{wt}(x_{n-1}^{(1)}) + \mathrm{wt}(\left( \prod_{i=0}^{n-2} y_i^{(p^{R_i} - 1)} \right)) \\
= \mathrm{wt}(u_0) + \mathrm{wt}(v_{n-1}) + \mathrm{wt}(\left( \prod_{i=0}^{n-2} y_i^{(p^{R_i} - 1)} \right)) = 1 + \mathrm{wt}(\left( \prod_{i=0}^{n-2} x_i^{(p^{S_i} - 1)} y_i^{(p^{R_i} - 1)} v_{n-1} \right) - \sum_{i=0}^{n-2} \mathrm{wt}(x_i^{(p^{S_i} - 1)})) \\
= 1 + \mathrm{wt}(v_0) + \sum_{i=0}^{n-2} (p^{S_i} - 1) \mathrm{wt}(v_i) \geq 2 + (p^{S^{n-2}} - 1) \mathrm{wt}(v_{n-2}).
\end{align*}
\]

iii. The preceding lower bound is given by the maximal allowed powers of the divided variables. In comparison with that bound we get additional terms \((\alpha + \beta + \gamma) \mathrm{wt}(v_{n-2}) \) and \((\xi + \zeta) \mathrm{wt}(v_{n-1}) \). Recall that by \([3] \) the head of \( w \) is \( g_n^{S} = [v_{n-1}^{S}, u_{n-1}^{S}, v_{n-1}, u_{n-1}] \), the latter multiplicands having the same weight, we get \( \mathrm{wt}(g_n^{S}) = \mathrm{wt}(v_{n-1})(\xi + \zeta + 2) \). Since tail variables only decrease the weight, we obtain (15).

iv. We use iii) and \((\xi + \zeta) \mathrm{wt}(v_{n-1}) \geq \mathrm{wt}(v_{n-1}) \). 

\[ \square \]

6. Growth of general clover restricted Lie algebras \( T(\Xi) \)

6.1. Arbitrary tuple \( \Xi \).

Lemma 6.1. Fix numbers \( p > 1 \) and \( r, s > 0 \). Then \( p^s + p^r - 1 > p^{(s+2r)/3} \).

Proof. Assume that \( s \geq r \), then \( s \geq (s + 2r)/3 \geq r \) and \( p^s + p^r - 1 \geq p^{(s+2r)/3} + (p^r - 1) > p^{(s+2r)/3} \). Consider the case \( s < r \), then \( s < (s + 2r)/3 < r \) and \( p^s + p^r - 1 > p^{(s+2r)/3} + (p^s - 1) > p^{(s+2r)/3} \). 

\[ \square \]

Theorem 6.2. Let \( \Xi \) be an arbitrary tuple of parameters and \( T(\Xi) \) the respective clover restricted Lie algebra. Then \( 1 \leq \text{GKdim } T(\Xi) \leq 3 \).
Proof. By Theorem 4.7, \( T(\Xi) \) is a semidirect product of \( L(\Xi) \) with the ideal \( J \), where bases of \( L(\Xi) \) and \( J \) consist of monomials of first and second types, respectively. By [37] Theorem 7.2, \( \text{GKdim } L(\Xi) \leq 2 \), yielding an upper bound on the number of the (power) standard monomials of first type. Since power standard monomials of second type (i.e. powers of \( u_n, n \geq 0 \)) behave like powers of \( w_n \), the same estimate on growth of \( L(\Xi) \) applies to them.

Fix a number \( m > 1 \). It remains to derive an upper bound on the number of standard monomials of second type of weight at most \( m \). Let \( n = n(m) \) be such that

\[
\text{wt}(v_{n-1}) < m \leq \text{wt}(v_n).
\]

(17)

Put \( m_0 := \text{wt}(v_{n-1}) \) and \( m_1 := \lfloor m/m_0 \rfloor \). By (5.1) and Lemma 6.1

\[
m_0 = \text{wt}(v_{n-1}) = \prod_{i=0}^{n-2} \left( p^{S_i} + p^{R_i} - 1 \right) > p^{(S_0 + \cdots + S_{n-2} + 2(R_0 + \cdots + R_{n-2}))}/3, \quad n \geq 2.
\]

(18)

Let \( w \) be a standard monomial of second type of length \( n' \) and \( \text{wt}(w) \leq m \). Assume that \( n' \geq n + 2 \). By Claim ii) of Lemma 5.5 \( m \geq \text{wt}(w) > \text{wt}(v_{n'-2}) \geq \text{wt}(v_n) \), a contradiction with (17). Hence, \( w \) is of length at most \( n + 1 \).

1) We evaluate a number \( f_1(m) \) of standard monomials \( w \) of second type of length \( n + 1 \) satisfying \( \text{wt}(w) \leq m \). By claim iv) of Lemma 5.5 \( \xi = \zeta = 0 \) (i.e. the neck variables reach the maximal values). Thus, we get monomials

\[
w = r_{n-2}(x, y, z)\alpha_{n-1}^{p^{S_{n-1}+1-\alpha}} y_{n-1}^{p^{R_{n-1}+1-\beta}} z_{n-1}^{p^{R_{n-1}+1-\gamma}} x_n^{p^{S_n-2}} z_n^{p^{R_n-1}} u_{n+1}.
\]

Using (21) and (28), we estimate a number of tails \( r_{n-2}(x, y, z) \) in (29) as:

\[
p^{S_0 + \cdots + S_{n-2} + 2(R_0 + \cdots + R_{n-2})} < m_0^3.
\]

(20)

Using estimate (15) (the indices are shifted by one!), \( m \geq \text{wt}(w) > (1 + \alpha + \beta + \gamma) \text{wt}(v_{n-1}) \). We get estimates

\[
0 \leq \alpha + \beta + \gamma \leq \left( \frac{m}{\text{wt}(v_{n-1})} \right) - 1 = m_1 - 1, \quad \alpha, \beta, \gamma \geq 0.
\]

(21)

A number of possibilities for variables with indices \( n-1 \) in (19) is bounded by a number of triples of integers \( \alpha, \beta, \gamma \) satisfying (21), which is equal to

\[
\left( \frac{m_1 + 2}{3} \right) = \frac{m_1(m_1 + 1)(m_1 + 2)}{6} \leq m_1^3.
\]

(22)

where the last estimate is checked directly for all \( m_1 \geq 1 \). Using (20) and (22), we get

\[
f_1(m) < m_0^3 m_1^3 = (m_0 m_1)^3 \leq m_1^3.
\]

(23)

2) Let \( f_2(m) \) be a number of standard monomials of second type \( \xi \) of length \( n \) satisfying \( \text{wt}(w) \leq m \). Using (28), a number of possibilities for divided powers with indices \( 0, \ldots, n-2 \) in (27) is evaluated by

\[
p^{S_0 + \cdots + S_{n-2} + 2(R_0 + \cdots + R_{n-2})} < m_0^3.
\]

(24)

Using estimate (15), \( m \geq \text{wt}(w) > (\xi + \zeta) \text{wt}(v_{n-1}) \). We get estimates

\[
0 \leq \xi + \zeta \leq \left( \frac{m}{\text{wt}(v_{n-1})} \right) = m_1, \quad \xi, \zeta \geq 0.
\]

(25)

A number of possibilities for the neck letters \( x_{n-1}, z_{n-1} \) in (21) is bounded by a number of pairs of integers \( \xi, \zeta \) satisfying (21). We get a bound

\[
\left( \frac{m_1 + 2}{2} \right) \leq 3m_1^3, \quad m_1 \geq 1,
\]

(26)

where one checks the last estimate directly. Using (23) and (25) we obtain an estimate

\[
f_2(m) \leq 3m_0^3 m_1^3 \leq 3m^3.
\]

(27)

3) Let \( f_3(m) \) be a number of all standard monomials of second type \( \xi \) of length \( n - 1 \). Using (28), a number of possibilities for all divided powers (having indices \( 0, \ldots, n-2 \)) is evaluated by

\[
f_3(m) \leq p^{S_0 + \cdots + S_{n-2} + 2(R_0 + \cdots + R_{n-2})} < m_0^3 \leq m^3.
\]

(28)
A similar estimate on the number of standard monomials of second type of length \( n - 2 \) is smaller at least by factor \( p^{-3} \) than the estimate above. The same applies to lengths \( n - 3, \ldots, 0 \). Let \( f_3(m) \) be the number of standard monomials of second type \( \mathbf{[14]} \) of length at most \( n - 1 \). We get a bound
\[
f_3(m) \leq \sum_{i=0}^{n-1} p^{-3i} \cdot f_3(m) \leq \frac{m^3}{1 - p^{-3}} \leq \frac{8}{7} m^3 < 2m^3.
\]
Finally, the obtained bounds yield a desired estimate on the number of standard monomial of second type and weight at most \( m \):
\[
f_1(m) + f_2(m) + f_3(m) \leq 6m^3.
\]
\[
\Box
\]

6.2. Constant tuple \( \Xi \).

**Theorem 6.3.** Let a tuple \( \Xi = (S_i, R_i|i \geq 0) \) be constant: \( S_i = S, R_i = R \) for \( i \geq 0 \), where \( S, R \geq 1 \).

Denote \( \lambda = \frac{(S + 2R) \ln p}{\ln(p^3 + p^R - 1)} \). Consider the clover restricted Lie algebra \( T = T(S, R) := T(\Xi) \). Then
\[ \begin{align*}
\text{i)} & \quad \text{GKdim } T = \text{GKdim } T = \lambda, \\
\text{ii)} & \quad C_1 m^\lambda < \gamma_T(m) < C_2 m^\lambda \text{ for } m \geq 1, \text{ and } C_1, C_2 \text{ being positive constants.} \\
\text{iii)} & \quad \lambda \in [1, 3].
\end{align*} \]

**Proof.** Fix a number \( m > 1 \). Using \( \mathbf{[5,1]} \), we choose \( n = n(m) \) satisfying
\[
\text{wt}(v_{n-1}) = (p^S + p^R - 1)^{n-1} \leq \text{wt}(v_n) = (p^S + p^R - 1)^n.
\]
Then
\[
n < \log_{p^S + p^R - 1}(m) + 1.
\]

Consider a standard monomial \( w \) of second type such that \( \text{wt}(w) \leq m \). Assume that \( w \) has length \( n' \geq n + 2 \). By claim ii) of Lemma \( \mathbf{[5,6]} \), \( \text{wt}(w) \geq \text{wt}(v_{n-1}) \) for \( n \), a contradiction. Hence, \( w \) is of length at most \( n + 1 \). Let \( f_1(n) \) be a number of standard monomials of second type of length at most \( n + 1 \). By \( \mathbf{[11], [10], \text{ and } 27} \), we get
\[
f_1(n) < p^{(S+2R)(n+1)} \leq p^{(S+2R)(\log_{p^S + p^R - 1}(m)+2)} \leq p^{2(S+2R)\frac{(S+2R)\ln p}{\ln(p^3 + p^R - 1)}}.
\]

Let \( w \) be a standard monomial of first type with \( \text{wt}(w) \leq m \). By the lower bound in Lemma \( \mathbf{[5,4]} \) and the upper bound in \( \mathbf{[20]} \), \( w \) is of length at most \( n \). Let \( f_2(n) \) be the number of standard monomials of first type of length at most \( n \). By \( \mathbf{[10], [10], \text{ and } 27} \), we get \( f_2(n) < p^{S+R}n \), yielding a smaller bound than above.

Let \( f_3(n) \) be the number of all power standard monomials of weight at most \( m \). By Lemmas \( \mathbf{[5,4]} \) and \( \mathbf{[5,5]} \) they are of length at most \( n \). We apply Lemma \( \mathbf{[14]} \) and \( \mathbf{[27]} \);
\[
f_3(n) \leq n(S + 2R) \leq (\log_{p^S + p^R - 1}(m) + 1)(S + 2R).
\]

Now, the upper bound follows using that \( \gamma_T(m) \leq f_1(n) + f_2(n) + f_3(n) \).

By \( \mathbf{[20]} \), \( n \geq \log_{p^S + p^R - 1}(m) \). Consider standard monomials \( w \) of second type \( \mathbf{[11]} \) of length \( n - 1 \). By the lower bound in \( \mathbf{[20]} \), \( \text{wt}(w) \leq \text{wt}(v_{n-1}) < m \). We evaluate the number of such monomials by counting their different tails \( r_{n-3}(x, y, z) \) (see \( \mathbf{[3]} \)), yielding a lower bound:
\[
\gamma_T(m) \geq p^{(S+2R)(n-2)} \geq p^{(S+2R)(\log_{p^S + p^R - 1}(m)-2)} = p^{-2(S+2R)\frac{(S+2R)\ln p}{\ln(p^3 + p^R - 1)}}.
\]

The second claim follows by our estimates. The last claim follows from Theorem \( \mathbf{[6,2]} \). \( \Box \)

**Lemma 6.4.** Let a tuple \( \Xi \) be constant or, more generally, periodic. Then the clover restricted Lie algebra \( T(\Xi) \) is self-similar.

**Proof.** The notion of self-similarity for Lie algebras was introduced by Bartholdi \( \mathbf{[5]} \). Both Lie superalgebras of \( \mathbf{[36]} \) are self-similar. The self-similarity of the two-generated subalgebra \( \mathbf{L(\Xi)} \cong \text{Lie}_p(v_0, v_0) \subset T(\Xi) \) in case of a periodic tuple \( \Xi \) was observed in \( \mathbf{[37]} \). We refer a reader to that paper for details. \( \Box \)

**Corollary 6.5.** Let char \( K = p \geq 2 \). Consider self-similar clover restricted Lie algebras \( T(S, R) \) given by constant tuples \( \Xi \) determined by two integers \( S, R \). Then \( \{\text{GKdim } T(S, R) | S, R \in \mathbb{N}\} \) is dense on \([1, 3]\).

**Proof.** By choosing the numbers \( R = S \) sufficiently large, we can obtain \( \text{GKdim } T(S, R) \) arbitrarily close to 3. By fixing \( R = 1 \) and choosing \( S \) sufficiently large we can obtain \( \text{GKdim } T(S, R) \) arbitrarily close to 1.

Consider a large positive integer \( S \) and \( R \in \{1, \ldots, S\} \). One checks that for \( R, R' = R + 1 \) the respective Gelfand-Kirillov dimensions differ by \( O(1/S), S \to +\infty \). \( \Box \)
7. Proof of main results: Clover restricted Lie algebras of quasi-linear growth

Proof of Theorem 5.7. All claims follow from the first one. Recall that we consider the tuple of integers \( \Xi := (S_i := [(i + 1) \div \kappa - 1], R_i := 1 \mid i \geq 0) \), and the clover Lie algebra \( T = T(\Xi) \). By Theorem 4.7, \( T(\Xi) \) is a semidirect product of \( L(\Xi) \) with the ideal \( J \), where bases of \( L(\Xi) \) and \( J \) consist of monomials of first and second types, respectively.

We start with general estimates used in the proof of the next theorem as well. By Lemma 5.1,

\[
\text{wt}(v_n) = \prod_{i=0}^{n-1} (p^{S_i} + p - 1) > p^{S_0 + \cdots + S_{n-1}}, \quad n \geq 1;
\]

\[
\text{wt}(v_n) = \prod_{i=0}^{n-1} (p^{S_i} + p - 1) < \theta p^{S_0 + \cdots + S_{n-1}}, \quad \theta := \prod_{i=0}^{\infty} (1 + p^{1 - S_i}), \quad n \geq 1.
\]

Indeed, it is well known that convergence of the infinite product is equivalent to convergence of the sum \( \sum_{i=0}^{\infty} p^{1 - S_i} \). We have \( S_i > 2 \log_p i \) for \( i \geq N \). Thus, \( \sum_{i=0}^{\infty} p^{-S_i} \leq \sum_{i=N}^{\infty} 1/i^2 < \infty \). We shall use the following well-known estimates:

\[
(\kappa + o(1)) \kappa^{1/\kappa} = \sum_{i=0}^{n-2} S_i < \sum_{i=0}^{n} (i + 1)^{1/\kappa - 1} = (\kappa + o(1)) \kappa^{1/\kappa}, \quad n \to \infty.
\]

Let us prove the desired upper bound on the standard monomials of second type. Fix a number \( m > 1 \). Choose \( n = n(m) \) such that

\[
\text{wt}(v_{n-1}) < m \leq \text{wt}(v_n).
\]

Put \( m_0 := \text{wt}(v_{n-1}) \) and \( m_1 := [m/m_0] \). By (28) and lower estimate in (30), we get

\[
m_0 = \text{wt}(v_{n-1}) > p^{S_0 + \cdots + S_{n-2}} \geq p^{(1 + o(1)) \kappa^{1/\kappa}}, \quad n \to \infty;
\]

\[
n \leq \left( (1/\kappa + o(1)) \log_p m_0 \right) \kappa \leq \left( 1 + o(1) \right) \kappa \ln p,
\]

\[
m \to \infty.
\]

Let \( w \) be a standard monomial of second type with \( \text{wt}(w) \leq m \). Suppose that it has length \( n' \geq n + 2 \). By claim ii) of Lemma 5.5, \( \text{wt}(w) > \text{wt}(v_{n'-2}) \geq \text{wt}(v_n) \geq m \geq \text{wt}(w) \). The contradiction proves that \( w \) is of length at most \( n + 1 \).

1) We evaluate a number \( f_1(m) \) of standard monomials \( w \) of second type of length \( n + 1 \) satisfying \( \text{wt}(w) \leq m \). By Claim iv) of Lemma 5.5, the head variables in \( w \) have the maximal degrees and we get monomials of the form:

\[
w = r_{n-2}(x, y, z)x_n^{(p^{n-1} - 1 - \alpha)}y_n^{(p^{n-1} - 1 - \beta)}z_n^{(p^{n-1} - 1 - \gamma)}x_n^{(p^{n-2} - 1)}z_n^{(p^{n-1} - 1)}u_{n+1}.
\]

Using (32), we evaluate the number of tails \( r_{n-2}(x, y, z) \) in (34) as:

\[
p^{S_0 + \cdots + S_{n-2} + 2(R_n + \cdots + R_{n-2})} < m_0 p^{2n}.
\]

By estimate (14), \( m \geq \text{wt}(w) > (1 + \alpha + \beta + \gamma) \text{wt}(v_{n-1}) \). We get estimates

\[
0 \leq \alpha + \beta + \gamma \leq \left[ \frac{m}{\text{wt}(v_{n-1})} \right] - 1 = m_1 - 1, \quad \text{where } 0 \leq \beta, \gamma < p^{R_{n-1}} = p.
\]

So, both \( \beta, \gamma \) have at most \( p \) choices. Now the number of possibilities for variables with indices \( n - 1 \) in (34) is equal to the number of integers \( \alpha, \beta, \gamma \) satisfying (34), which is bounded by \( p^2 m_1 \). Combining with the bound on the number of tails (34), we get

\[
f_1(m) \leq p^2 m_1 \cdot m_0 p^{2n} \leq p^2 m p^{2n}.
\]

2) We evaluate a number \( f_2(m) \) of standard monomials of second type and length \( n \) such that \( \text{wt}(w) \leq m \). Using (32), a number of possibilities for variables with indices \( 0, \ldots, n-2 \) in (14) is evaluated by:

\[
p^{S_0 + \cdots + S_{n-2} + 2(R_0 + \cdots + R_{n-2})} < m_0 p^{2n}.
\]

Using estimate (14), we get \( m \geq \text{wt}(w) > (\xi + \zeta) \text{wt}(v_{n-1}) \) and

\[
0 \leq \xi + \zeta \leq \left[ \frac{m}{\text{wt}(v_{n-1})} \right] = m_1, \quad \text{where } 0 \leq \zeta < p^{R_{n-1}} = p.
\]
Thus, the number of possibilities for the neck letters \( x_{n-1}, z_{n-1} \) in (14) is bounded by the number of integers \( \xi, \zeta \) satisfying (39), which is bounded by \( p(m_1 + 1) \). Using the bound on the number of tails \( \Xi \), we get
\[
f_2(m) \leq p(m_1 + 1) \cdot m_0 \theta^{2n} \leq 2p \cdot mp^{2n}.
\] (40)

3) We evaluate a number \( f_3(m) \) of standard monomials of second type and length \( n - 1 \). Using (32), a number of possibilities for all divided powers, now having indices 0, \ldots, n - 2 in (14) is evaluated by
\[
f_3(m) \leq p^{S_0 + \cdots + S_{n-2} + 2(R_0 + \cdots + R_{n-2})} < m_0 \theta^{2n} \leq mp^{2n}.
\]
The number of standard monomials of second type of length \( n - 2 \) is smaller at least by factor \( p^{-3} \) than estimate above. The same applies to lengths \( n - 3, \ldots, 0 \). Let \( f_3(m) \) be the number of standard monomials of second type \( (14) \) of length at most \( n - 1 \). Using (33) we get
\[
f_3(m) \leq \sum_{i=0}^{n-2} p^{-3i} f_3(m) \leq \frac{mp^{2n}}{1 - p^{-3}} \leq 2mp^{2n}.
\] (41)

Let \( f_4(m) \) be the number of power standard monomials \( w \) of second type of weight at most \( m \). By (31) and claim i) of Lemma 5.5, \( f_4(m) \) is of length at most \( n \). By (12), \( f_4(m) \leq R_0 + \cdots + R_{n-1} = n \). Combining (32), (40), (41), and using (33), the number of all standard monomials of second type and weight at most \( m \) is evaluated by
\[
f_1(m) + f_2(m) + f_3(m) + f_4(m) \leq (p^2 + 2p + 2)mp^{2n} + n
\]
\[
\leq (p^2 + 2p + 2)m \exp \left( 2 \ln p \left( \frac{1/\kappa + o(1)}{\ln m} \ln m \right)^\kappa \right)
\]
\[
= m \exp \left( \frac{2(\ln p)^{1/\kappa} + o(1)}{\kappa} \ln m \right), \quad m \to \infty.
\]

We have an upper bound on the growth of the subalgebra \( L(\Sigma) \) given by (31) Theorem 9.2], which actually yields upper bounds on the number of (power) standard monomials of first type. But now we are proving a little bit stronger bounds, a reader can either trace that computations or modify more lengthy computations obtained for the monomials of second type.

Finally, let us establish the lower bound. We keep notations (31). Similar to (29)
\[
m \leq \text{wt}(v_n) \leq \prod_{i=0}^{n-1} \left( p^{(i+1)/\kappa - 1} + p - 1 \right) < \theta \prod_{i=0}^{n-1} p^{(i+1)/\kappa - 1}, \quad \theta := \prod_{i=0}^{\infty} (1 + p^{1-(i+1)/\kappa - 1}).
\] (43)

Using (13) and the upper bound (30), we get \( m \leq \theta p^{(\kappa + o(1))n^{1/\kappa}} \). Hence
\[
n \geq \left( \frac{\log_p(m/\theta)}{\kappa + o(1)} \right)^\kappa \geq \left( \frac{1 + o(1)}{\kappa \ln p} \ln m \right)^\kappa, \quad m \to \infty.
\] (44)

By (29),
\[
m_0 = \text{wt}(v_{n-1}) = \prod_{i=0}^{n-2} \left( p^{S_i} + p - 1 \right) < \theta p^{S_0 + \cdots + S_{n-2}}.
\] (45)

Consider standard monomials of second type \( w = r_{n-2}(x, y, z)\hat{g}_x^{\xi_{n-1}, \xi_{n-1}}(11) \) of length \( n \). We evaluate the number of their tails \( r_{n-2}(x, y, z) \) using (15)
\[
p^{S_0 + \cdots + S_{n-2}} p^{2(R_0 + \cdots + R_{n-2})} > \frac{m_0}{\theta} p^{2(n-1)}.
\] (46)

By our construction and Lemma 5.1,
\[
m_1 = \left[ \frac{m}{m_0} \right] \leq \frac{\text{wt}(v_n)}{\text{wt}(v_{n-1})} = p^{S_{n-1} + p - 1}.
\] (47)

Consider standard monomials of second type \( w \) which heads satisfy \( \xi_{n-1} \in \{0, \ldots, m_1 - p\} \), \( \xi_{n-1} = 0 \). Using (47), we have \( 0 \leq \xi_{n-1} < p^{S_{n-1}} \), so, we get standard monomials indeed. Also, using (10), these monomials are of weight not exceeding \( m \):
\[
\text{wt}(w) \leq \text{wt}(v_{n-1})(\xi_{n-1} + \xi_{n-1} + 2) \leq \text{wt}(v_{n-1})m_1 = m_0 m_1 \leq m.
\]
There are $m_1 - p + 1$ such heads. We multiply this number by the number of different tails, using estimate (46), we obtain the desired lower bound:

$$\left( m_1 - p + 1 \right) \frac{m_0}{\theta} p^{2(n-1)} \geq \frac{m}{2p^2 \theta} p^{2n}$$

$$\geq \frac{m}{2p^2 \theta} p^{\left( \frac{\ln p}{\alpha p} + 1 \right) \ln m} = m \exp \left( \frac{2(\ln p)^{1-\kappa} + o(1)}{\kappa} (\ln m)^\kappa \right), \quad m \to \infty. \quad \square$$

**Proof of Theorem 3.2.** We use estimates and notations of the previous proof. It is sufficient to prove the first claim. Fix the constant $\lambda := (\ln p^2)/\kappa \in \mathbb{R}^+$. Now we consider the tuple $\Xi_{q, \kappa} = (S_i, R_i | i \geq 0)$, where $R_i := 1$ for all $i \geq 0$, and define integers $S_i$ by induction: $S_0 = 1$ and

$$S_n := \left[ \exp^{(q)}(\lambda(n + 2)) \right] + 1 - S_0 - \cdots - S_{n-1}, \quad n \geq 1. \quad (49)$$

Let us prove the desired upper bound on the standard monomials of second type. Fix a number $m > 1$. Choose $n = n(m)$ such that

$$\text{wt}(v_{n-1}) < m \leq \text{wt}(v_n). \quad (50)$$

Put $m_0 := \text{wt}(v_{n-1})$ and $m_1 := [m/m_0]$. By (28) and (19) we get

$$m > m_0 = \text{wt}(v_{n-1}) > p^{S_0 + \cdots + S_{n-2}} \geq \exp^{(q)}(\lambda n);$$

$$n < \frac{1}{\lambda} \ln^{(q)} \log_p(m);$$

$$p^{2n} < \exp \left( \frac{\ln^2 \exp^{(q)}(\log_p(m)) (\ln^2 / \lambda)}{\lambda} \right) = \left( \ln^{(q-1)} \log_p(m) \right)^{(\ln^2 / \lambda)}, \quad m \to \infty.$$  

Using estimate (12) on the number of all standard monomials of second type of weight at most $m$, we get the desired upper asymptotic on the number of these monomials. Similar bounds are valid for monomials of first type.

Let us check the lower bound. We use notations (49). By estimate (29) and (19)

$$m \leq \text{wt}(v_m) \leq \theta p^{S_0 + \cdots + S_{n-1}} \leq \theta p^{\exp^{(q)}(\lambda(n+1)) + 1};$$

$$n > \frac{1}{\lambda} \ln^{(q)} \left( \log_p(m/\theta) - 1 \right) = 1 + o(1) \frac{\ln^{(q+1)}(m)}{\lambda}, \quad m \to \infty;$$

$$p^{2n} > \exp \left( \frac{\ln^2 + o(1)}{\lambda} \ln^{(q+1)}(m) \right) = \left( \ln^{(q)} m \right)^{\kappa + o(1)}, \quad m \to \infty.$$  

Finally, using the lower bound on the number of standard monomials of second type (18) and the bound above, we obtain the desired lower bound on the growth of $T$. \quad \square

**References**

[1] Bahturin Yu. A., Identical relations in Lie algebras. VNU Science Press, Utrecht, 1987.

[2] Bahturin Yu.A., Mikhalev A.A., Petrogradsky V.M., and Zaicev M.V., Infinite dimensional Lie superalgebras, de Gruyter Exp. Math., vol. 7, de Gruyter, Berlin, 1992.

[3] Bahturin, Yu.A.; Olshanskiii, A., Large restricted Lie algebras, J. Algebra (2007) 310, No. 1, 413–427.

[4] Bartholdi L., Branch rings, thinned rings, tree enveloping rings. Israel J. Math. 154 (2006), 93–139.

[5] Bartholdi L., Self-similar Lie algebras. J. Eur. Math. Soc. (JEMS) 17 (2015), no. 12, 3113–3151.

[6] Bartholdi L., Grigorchuk R.I., Lie methods in growth of groups and groups of finite width. Computational and geometric aspects of modern algebra,1–27, London Math. Soc. Lecture Note Ser., 275, Cambridge Univ. Press, Cambridge, 2000.

[7] Bell J., Young A.A., On the Kurosh problem for algebras of polynomial growth over a general field. J. Algebra 342 (2011), 265–281.

[8] Caranti, A.; Mattarei, S.; Newman, M.F. Graded Lie algebras of maximal class. Trans. Am. Math. Soc. 349 (1997) No. 10, 4021–4051.

[9] Caranti, A.; Newman, M.F. Graded Lie algebras of maximal class. II. J. Algebra 229 (2000), No. 2, 750–784.

[10] de Morais Costa O.A., Petrogradsky, V., Fractal just infinite nil Lie superalgebra of finite width, J. Algebra, 504, (2018), 291–335.

[11] Ershov M., Golod-Shafarevich groups: a survey, Int. J. Algebra Comput. 22, (2012) No. 5, Article ID 1230001.

[12] Fabrykowski, J., Gupta, N., On groups with sub-exponential growth functions. J. Indian Math. Soc. (N.S.) 49 (1985), no. 3-4, 249–256.

[13] Futorny V., Kochloukova D.H., Sidki S.N., On Self-Similar Lie Algebras and Virtual Endomorphisms, Math. Z. 292 (2019). No.3-4, 1123–1156.
