Duality, Superconvergence and the Phases of Gauge Theories

Reinhard Oehme

*Enrico Fermi Institute and Department of Physics*
*University of Chicago, Chicago, Illinois, 60637, USA*

and

*Max-Planck-Institut für Physik*
*Werner-Heisenberg-Institut - 80805 Munich, Germany*

Abstract

Results about the phase structure of certain \( N = 1 \) supersymmetric gauge theories, which have been obtained as a consequence of holomorphy and ‘electric-magnetic’ duality, are shown to be in quantitative agreement with corresponding consequences of analyticity and superconvergence of the gauge field propagator. This connection is of interest, because the superconvergence arguments for confinement are not restricted to theories with supersymmetry. The method of reduction in the space of coupling parameters is used in order to define, beyond the matching conditions, an asymptotically free, dual magnetic theory involving Yukawa couplings.

---

\(^1\)E-mail: oehme@control.uchicago.edu

\(^2\)Permanent Address
The phase structure of supersymmetric gauge theories has recently been elucidated with the help of holomorphy and duality [1, 2, 3]. These features seem to be characteristic for SUSY theories, where they provide insight into the non-perturbative structure. It is an important question, to what extent the results obtained for SUSY theories are generic, and have parallels in ordinary gauge theories like Quantum Chromodynamics.

We are mainly considering theories with the gauge groups $SU(N_C)$ or $SO(N_C)$, and with $N_F$ flavors of zero mass matter fields. For SUSY theories, duality arguments lead to confinement for values of the number of flavors $N_F$ which do not reach up to the point where asymptotic freedom is lost. There is a region in $N_F$, where the theories are in an interacting non-Abelian Coulomb phase. For these values of $N_F$, there is no confinement, neither of the ‘electric’ nor of the ‘magnetic’ excitations of the theory.

Some time ago, we have developed arguments for the confinement of gluons and quarks in theories like QCD [4, 5]. These arguments are based upon superconvergence relations of the gauge field propagator [6, 7, 8]. There are two approaches. One is more heuristic and considers the potential between static color charges [9, 10], the other is more formal and subtile, involving the definition of the physical state space $H$ in terms of the BRST algebra [4]. Both methods give confinement for values of $N_F$ below an upper limit, which is lower than the value where the one-loop $\beta$ function coefficient vanishes and asymptotic freedom is lost. We have a finite region with asymptotic freedom and no confinement.

Already in [11], we have applied the superconvergence arguments to SUSY theories in the Wess-Zumino representation. Hence we can now compare with the predictions of the new duality analysis. It is the main purpose of this note to show that there is quantitative agreement between both approaches,
and to further explore the rôle of analytic and asymptotic properties of the propagator for the problem of confinement. Since the superconvergence arguments are valid for SUSY and non-SUSY theories, the comparison underlines the generic character of the obtained phase structure.

An essential quantity in our superconvergence arguments is the ratio $\gamma_{00}/\beta_0$, where $\gamma_{00}$ is the one-loop anomalous dimension coefficient in the Landau gauge $\alpha = 0$, and $\beta_0$ is the one loop $\beta$-function coefficient. The asymptotic expansions

\[
\begin{align*}
\gamma(g^2, \alpha) &= (\gamma_{00} + \alpha \gamma_{01})g^2 + \cdots, \\
\beta(g^2) &= \beta_0 g^4 + \cdots
\end{align*}
\]

give the limits $g^2 \to 0$ of the anomalous dimension and the renormalization group function. From analyticity and renormalization group properties of the gauge field structure function $D(k^2, \kappa^2, g, \alpha)$, we find that it is the ratio $\gamma_{00}/\beta_0$, which essentially determines the asymptotic limit in all covariant gauges ($\alpha \geq 0$) and in all directions of the complex $k^2$-plane. Here $\kappa^2 < 0$ is the normalization point, and we have assumed that $\beta_0 < 0$. For the discontinuity of the function $D$ along the positive, real $k^2$-axis, we obtain

\[
-k^2 \rho(k^2, \kappa^2, g, \alpha) \simeq \frac{\gamma_{00}}{\beta_0} C_R(g^2, \alpha) \left(-\beta_0 \ln \frac{k^2}{|\kappa^2|}\right)^{-\gamma_{00}/\beta_0 - 1} + \cdots \tag{2}
\]

The appearance of the coefficient $\gamma_{00}$ is due to the fact that the gauge parameter $\alpha$ is renormalized, and $\alpha = 0$ is a UV-fixed point. For this reason, $\gamma_{00}$ can be of physical relevance, similar to the coefficient $\beta_0$. In fact, for $N = 1$ SUSY theories, we will show in the following that $2\gamma_{00}$ is given by the negative of $\beta_0^d$, the one-loop $\beta$-function coefficient of the dual theory.

\[3\]A preliminary report about these results has been presented at the International Workshop on High Energy Physics, Novi Svit, Crimea, September 1995 \[12\].
In the following we work in the Landau gauge. Although we have the asymptotic properties of propagators in all covariant gauges [8], the renormalization group equations are most easy to handle in the Landau gauge. In any case, since confinement is a physical aspect of the theory, it is easy to show that it is sufficient to argue in a particular gauge. We have the superconvergence relation [6, 7]

\[
\int_{-\infty}^{\infty} dk^2 \rho(k^2, \kappa^2, g) = 0 ,
\]

which is valid provided the ratio \(\gamma_{00}/\beta_0 > 0\), and hence for \(\gamma_{00} < 0\) in the presence of asymptotic freedom. This relation gives a connection between high- and low-energy properties of the theory.

In [4] we have described in detail how one can make use of the superconvergence relation in order to show, that states involving transverse gauge field excitations are not elements of the physical state space \(\mathcal{H}\), as defined using the BRST algebra. Together with other unphysical states of the theory, they form quartet representations of the algebra, whereas physical states should be singlets. In the more heuristic approach to confinement [9], we use a dipole representation of the structure function \(D(k^2) = \int_{-\infty}^{\infty} dk'^2 \sigma(k'^2)(k'^2 - k^2)^{-2}\), with the weight function \(\sigma(k^2) = \int_{-\infty}^{\kappa^2} dk'^2 \rho(k'^2)\). For \(\gamma_{00}/\beta_0 > 0\), we have \(\sigma(\infty) = 0\), \(\sigma(k^2) > 0\), \(\sigma'(k^2) = \rho(k^2) < 0\) for sufficiently large values of \(k^2\). Together with \(C_R(g^2) > 0\), we use these properties in order to argue for an approximately linear confining potential. There is no indication of such a potential if \(\gamma_{00}/\beta_0 < 0\). In this case, the dipole representation is still valid, but \(\sigma(k^2) \to \infty\) for \(k^2 \to \infty\).

If we consider theories with a gauge group like \(SU(N_C)\), with massless matter fields in the fundamental representation, the coefficient \(\gamma_{00}\) generally has a zero as a function of \(N_F\) which is below the point where \(\beta_0\) vanishes. For
example, let us take $SU(N_C)$ gauge theory with the matter $N_F \times (N_C + N_C)$. Then

$$\frac{\gamma_{00}}{\beta_0} = \frac{-\frac{13}{6}N_C + \frac{2}{3}N_F}{-\frac{11}{3}N_C + \frac{2}{3}N_F}. \quad (4)$$

We have superconvergence, and hence confinement, for $N_F < \frac{13}{4}N_C$, because $\gamma_{00}/\beta_0 > 0$. Furthermore, there is an interval

$$\frac{13}{4}N_C < N_F < \frac{22}{4}N_C, \quad (5)$$

where $\gamma_{00}/\beta_0 < 0$. In this region, superconvergence is lost, but not asymptotic freedom. Our arguments for confinement do not apply, and the study of the potential suggests that there is no confinement. There must be a phase transition around the point $N_F = \frac{13}{4}N_C$ where $\gamma_{00}$ changes sign, and where we still have $\beta_0 < 0$.

In the article [11], we have applied our arguments for confinement to SUSY theories. The appropriate quantity is the gauge field propagator in the Wess-Zumino gauge, in particular if we ask the question whether the elementary, transverse gauge field excitations are excluded from the physical state space $\mathcal{H}$. For $N = 1$ supersymmetric gauge theories, we write the one-loop coefficients of the function $\beta(g^2)$ and the anomalous dimension $\gamma(g^2, 0)$ in the form

$$\beta_0 = (16\pi^2)^{-1} \left( -3C_2(G) + \sum_i n_i T(R_i) \right), \quad (6)$$

and

$$\gamma_{00} = (16\pi^2)^{-1} \left( -\frac{3}{2}C_2(G) + \sum_i n_i T(R_i) \right), \quad (7)$$

where $n_i$ is the number of $N = 1$ chiral superfields in the representation $R_i$. These coefficients are determined by the elementary field content of the
theory. We have

\[ \gamma_{00} < 0, \quad \beta_0 < 0 \quad \text{for} \]
\[ \sum_i n_i T(R_i) < \frac{3}{2} C_2(G), \quad (8) \]

which is the condition for the validity of the superconvergence relation (3) for the structure function, and hence for the confinement of the elementary transverse gauge field excitations. For \( G = SU(N_C) \), and matter fields in the fundamental representation \( N_F \times (N_C + \overline{N}_C) \), we have \( \gamma_{00} < 0, \beta_0 < 0 \) for \( N_F < \frac{3}{2} N_C \), where \( N_F \) refers to four-component spinors in contrast to \( n_i \). As mentioned before, the superconvergence argument implies confinement for \( N_F < \frac{3}{2} N_C \), and we have indications of a de-confining phase transition \[ 4 \]
at \( N_F = \frac{3}{2} N_C \), as \( N_F \) increases \[ 11 \]. Above this transition point, there is the region

\[ \frac{3}{2} N_C < N_F < 3 N_C, \quad (9) \]

where there is no superconvergence, and hence no confinement. It corresponds to the interval given in Eq.(5) for the non-SUSY case.

After these preliminaries, we would like to connect the results obtained using superconvergence, with the picture which emerges from electric magnetic duality of \( N = 1 \) SUSY gauge theories. As the electric theory, we use again the gauge group \( G = SU(N_C) \), with massless matter fields in the representation \( N_F \times (N_C + \overline{N}_C) \). For appropriate values of \( N_C \) and \( N_F \), the corresponding dual magnetic theory has the gauge group \( G_d = SU(N_F - N_C) \) with \( N^d_F = N_F \) flavors of magnetic chiral superfields and a certain number of gauge-singlet massless superfields.

\[ 4 \text{See page 450 of } [11], \text{ where the existence of a phase transition at } N_F = \frac{3}{2} N_C \text{ has already been pointed out, and } [11] \text{ for the corresponding non-SUSY result.} \]
The one loop coefficients of both theories are given by:

\[ G = SU(N_C) \quad \text{"electric"} \quad N = 1 \text{ SUSY} \]

\[
\begin{align*}
\beta_0 &= (16\pi^2)^{-1}(-3N_C + N_F) \\
\gamma_{00} &= (16\pi^2)^{-1}\left(-\frac{3}{2}N_C + N_F\right),
\end{align*}
\] (10)

and

\[ G = SU(N_F - N_C) \quad \text{"magnetic"} \quad N = 1 \text{ SUSY} \]

\[
\begin{align*}
\beta_0^d &= (16\pi^2)^{-1}(-2N_F + 3N_C) \\
\gamma_{00}^d &= (16\pi^2)^{-1}\left(-\frac{1}{2}N_F + \frac{3}{2}N_C\right). \\
\end{align*}
\] (11)

Here the coefficients of the dual map have been evaluated at the same number of flavors \( N_F^d = N_F \) as the original, electric theory. However, these flavors refer to representations of the magnetic gauge group.

From the above equations, we can extract the following important relationships between electric and magnetic coefficients [12]:

\[
\begin{align*}
\beta_0^d(N_F) &= -2\gamma_{00}(N_F), \\
\beta_0(N_F) &= -2\gamma_{00}^d(N_F),
\end{align*}
\] (12, 13)

where it is again understood, that the variable \( N_F \) on both sides refers to matter fields with different quantum numbers in the electric and magnetic functions respectively. The appearance of the factor ‘two’ in Eqs. (12, 13) is due to our definition of the anomalous dimension by \( u\frac{\partial R}{\partial u} = \gamma R \), where \( R = -k^2 D(k^2) \) and \( u = k^2/\kappa^2 \).
The duality relationships (12, 13) are not restricted to the particular model considered. For example, \( N = 1 \) supersymmetric gauge theory with the group \( G = SO(N_C) \) and with \( N_F \) flavors in the representation \( N_F \times N_C \), has a dual map with the group \( G^d = SO(N_F - N_C + 4) \) [13]. The duality relations are again given by equations (12) and (13). For this supersymmetric theory with \( G = SO(N_C) \), the coefficient \( \gamma_{00}(N_F) \) changes sign at \( N_F = \frac{3}{2}(N_C - 2) \). It is certainly of interest to study further models. We plan to discuss \( SO(N_C) \) and other gauge theories elsewhere.

In writing the one-loop coefficients for the magnetic theory, we see that there is no contribution from the Yukawa coupling of the singlet meson fields \( M_i^j \) with the \( N_F \) flavors of magnetic quark fields \( q_i \) and \( \bar{q}_j \). The corresponding superpotential is of the form \( \sqrt{\lambda}M_i^jq_i\bar{q}_j \) [3, 13]. By itself, the Yukawa coupling \( \lambda \) is not asymptotically free. As the dual theory, however, we should consider one which is obtained by the method of the ‘reduction of couplings’ [14], [15]. With this method, the Yukawa coupling is expressed as a function of the gauge coupling, \( \lambda = \lambda(g^2) \), so that the resulting theory depends upon the gauge coupling only, satisfies the appropriate renormalization group equations in this coupling, and preserves supersymmetry. As a solution of the reduction equations, we get a theory which is UV-asymptotically free above \( N_F = \frac{3}{2}N_C \), or IR-free for \( N_F \) in an interval below the transition point, with \( \lambda(g^2) \) being proportional to \( g^2 \) for \( g^2 \to 0 \). The Green’s functions of the reduced theory have asymptotic power series expansions in \( g^2 \) in this limit. We see that the one-loop coefficient \( \beta^d \) is not affected by the Yukawa coupling, nor is the coefficient \( \gamma_{00}^d \) in view of the singlet character of the \( M \) fields. For our results about the de-confining phase transition, we need only Eq. (12).

To be more specific about the construction of the dual map, using re-
duction in coupling parameter space, we briefly describe some aspects of the
procedure for the particular system considered here. The result is also
relevant for the infrared fixed point to be mentioned later.

For the gauge group, we take a generic $G = SU(N)$, where $N = N_F - N_C$ in the case of the magnetic theory described above. For the following
discussion, we omit the label $d$ indicating the dual system. In the usual
renormalization group equations for the effective couplings $g^2(u)$ and \(\lambda(u)\),
with $u = k^2/\kappa^2$, we eliminate the scaling variable $u$, and obtain the reduction
equations

$$
\beta(g^2, \lambda(g^2)) \frac{d\lambda(g^2)}{dg^2} = \beta_\lambda(g^2, \lambda(g^2)) \, ,
\tag{14}
$$

This is an equation for the Yukawa coupling $\lambda$ as a function of $g^2$. The $\beta$-
functions for the original two-parameter theory are assumed to have asymptotic power series expansions. General two-loop results for SUSY theories
may be found in the references [18]. For the model considered, the $\beta$-functions
are of the form

$$
\beta(g^2, \lambda) = \beta_0 g^4 + (\beta_1 g^6 + \beta_{1,\lambda} g^4 \lambda) + \cdots
\beta_\lambda(g^2, \lambda) = c_\lambda g^2 \lambda + c_{\lambda\lambda} \lambda^2 + \cdots \, .
\tag{15}
$$

The coefficients are given by

$$
\begin{align*}
\beta_1 &= (16\pi^2)^{-2} \left( 2N(-3N + N_F) + 4N_F \frac{N^2 - 1}{2N} \right) \\
\beta_{1,\lambda} &= (16\pi^2)^{-2} (-2N_F^2) \\
c_{\lambda\lambda} &= (16\pi^2)^{-1} (N + 2N_F) \\
c_\lambda &= (16\pi^2)^{-1} \left( -4 \frac{N^2 - 1}{2N} \right) .
\end{align*}
\tag{16}
$$

\footnote{For a recent survey of the reduction method, see [19]. The case of two couplings has been discussed in detail in [17].}
With the expansions of the $\beta$-functions as given, we see that the differential equation (14) is singular at $g^2 = 0$. By a uniformization transformation, we can remove the singularity, and show that all solutions in the neighborhood of the origin are given by asymptotic series expansions [17, 19, 20]. They may contain non-integer powers (and logarithms, in general). However, we are mainly interested in special solutions [17, 20], which have a power series expansion. For the system considered, we write these solutions in the form

$$\lambda(g^2) = g^2 f(g^2), \quad \text{with} \quad f(g^2) = f_0 + \sum_{m=1}^{\infty} \chi^{(m)} g^{2m}. \quad (17)$$

Substitution into the reduction equation yields the fundamental relation

$$\beta_0 f_0 = (c_{\lambda\lambda} f_0 + c_{\lambda}) f_0. \quad (18)$$

There are two solutions:

$$f_0 = f_{00} = 0 \quad \text{and} \quad f_0 = \frac{\beta_0 - c_{\lambda}}{c_{\lambda\lambda}}. \quad (19)$$

For $\beta_0 < 0$, we find that, in both cases, the coefficients $\chi^{(m)}$ in Eq. (17) are uniquely determined by the expansions of the $\beta$-functions. For the solution $f_{00}$, it follows directly that $\chi^{(m)} = 0$ for all $m$. It corresponds to the $SU(N)$ theory without the Yukawa coupling. Of main interest as a dual map is the second solution. Using regular reparametrizations of the theory, we can remove the coefficients $\chi^{(m)}$ for this solution. Then we have the reduced theory with

$$\lambda(g^2) = g^2 f_0 \quad \text{with} \quad f_0 = \frac{\beta_0 - c_{\lambda}}{c_{\lambda\lambda}}, \quad (20)$$

modulo terms which vanish faster than any power. The solution (20) represents an asymptotically free theory, which depends only upon the gauge
coupling $g^2$, and contains no further arbitrary parameters. In order to use this solution as a physical theory, we must require that the coefficient $f_0$ is positive. From Eqs. (14) and (24) we see that $f_0 > 0$ provided $N_F > N + 2/N$ for the $SU(N)$ theory with Yukawa coupling. With $N = N_F - N_C$, as required for the dual theory, the positivity condition is $N_F > N_C + 2/N_C$.

Interestingly, it is of the same form as for the $SU(N)$ theory. It follows, that we can certainly use this solution in the window $\frac{3}{2}N_C < N_F < 3N_C$ and above; only for $N_C = 2$ does the bound touch the lower end of the window.

It remains to discuss the general solution of the reduction equation (14) for the case $\beta_0 < 0$. In the models considered here, this solution contains rational powers in the asymptotic expansion, with exponents given by

$$\xi = \frac{\beta_0 - c_\lambda}{-\beta_0}$$

and multiples thereof. Note that $\xi > 0$ in the window $[3]$ and above. It can also be written in the form $\xi = \frac{c_{\lambda\lambda}}{3\beta_0}f_0$, where $c_{\lambda\lambda} > 0$. In all cases, even if $\xi$ should happen to be an integer, the leading term in the asymptotic expansion of the general solution is

$$\lambda(g^2) = A g^{2(1+\xi)} + \cdots,$$

where $A$ is an undetermined coefficient. Once the factor $A$ is fixed, all higher order coefficients in the expansion are determined. If expressed as a function of $N_F$ and $N_C$ for the dual map with $SU(N_F - N_C)$, the exponent $\xi$ is given by

$$\xi = \frac{N_C(N_F - N_C - 2/N_C)}{2(N_F - N_C)(N_F - \frac{3}{2}N_C)},$$

and it becomes large as $N_F$ approaches the lower end of the window at $N_F = \frac{3}{2}N_C$. We see that the general solution (22) does not approach the
special power series solution (20) in the limit $g^2 \to 0$. The latter is therefore isolated or unstable with respect to it’s embedding into the two-parameter theory. The ratio of both solutions does not approach ‘one’. This is a feature, which is quite common for SUSY theories [20, 21]. Within the framework of theories with asymptotic freedom and renormalized power series expansions, we consider the special solution with $N = N_F - N_C$ as the appropriate dual of the original $SU(N_C)$ theory. As we have mentioned, the solution (20) should be amended by non-perturbative contributions, which vanish faster than any power in the limit $g^2 \to 0$. These should be present, provided there are corresponding additions to the $\beta$-functions.

So far, we have assumed that $\beta_0 < 0$. But the unique power series solution (17, 20) is also valid for $\beta_0 > 0$, provided certain conditions are satisfied. One is the positivity condition $f_0 > 0$. It requires $N_C + 2/N_C < N_F < \frac{3}{2}N_C$, an interval which is non-empty for $N_C > 2$. The other is, that the quantity $\xi$, as defined in (21), is not a negative integer. (We refer to [14] for the details of the special case where $\xi = -n, \ n = 1, 2, ...$). Assuming that the conditions stated above are satisfied, we have a unique solution for $\beta_0 > 0$. This special solution represents a renormalized, IR-asymptotically free theory. As the dual map, it describes the low energy excitations of the system.

Given the conditions described above, the general solution for $\beta_0 > 0$ has an asymptotic expansion for $g^2 \to 0$, which can be brought into the form

$$\lambda(g^2) \simeq f_0 g^2 + B g^{2(1+|\xi|)} + \cdots,$$

with an undetermined coefficient $B$. For all $B$, this general solution approaches the unique special solution (20), which therefore is stable with respect to the embedding into the two-parameter theory.

There remains the boundary case where $N_F = \frac{3}{2}N_C$, and hence $\beta_0 = 0$ for the magnetic theory. In this situation, we have $f_0 > 0$ for $N_C > 2$. There
is again a unique power series solution of the form (17, 20).

The reduction described above gives a more detailed picture of the dual, magnetic theory, which is defined, a priori, on the basis of the matching conditions [2, 3]. It is important to have this picture for our comparison with the results obtained on the basis of analyticity and superconvergence, which uses the functions $\beta(g^2, \lambda(g^2))$ and $\gamma(g^2, \lambda(g^2))$ of the reduced theory. We note here, that it is of interest to consider the reduction equations away from the fixed point $g^2 = 0$, wherever one has some information about the $\beta$-functions [20, 22]. We hope to discuss the general solutions, and non-perturbative aspects of the reduction, at another occasion.

Now we return to the phase transition at $N_F = \frac{3}{2}N_C$. We revert to the notation where the label $d$ indicates the dual map. It follows from Eqs. (12, 13), that the zero of the one-loop anomalous dimension coefficient $\gamma_{00}(N_F)$, which has emerged as a critical point in our superconvergence arguments for confinement, corresponds to a zero of the $\beta$-function coefficient $\beta_0^d$ of the dual theory. There is an analogous relationship between $\gamma_{00}^d$ and $\beta_0$. These connections show, that the coefficient $\gamma_{00}$ is a characteristic quantity for the structure of the gauge theory, on the same level as $\beta_0$.

Let us assume that $N_F \geq N_C + 2$, so that we have a non-Abelian dual map of the $SU(N_C)$ theory. The point $N_F = N_C + 2$ is below the critical value $N_F = \frac{3}{2}N_C$ for $N_C > 4$. In the region $N_C + 2 < N_F < \frac{3}{2}N_C$ we have $\gamma_{00}/\beta_0 > 0$, and the superconvergence arguments imply confinement of the electric excitations, by showing that they are not elements of the physical state space $\mathcal{H}$. According to Eq. (13), the dual magnetic theory has $\beta_0^d > 0$ in this region of $N_F$. It is not asymptotically free. Rather, we have non-interacting magnetic quanta in the infrared, which should be viewed as
composites of the elementary electric quanta. The magnetic theory looses asymptotic freedom at \( N_F = 3N_C - N_F \), which exactly corresponds to the point \( N_F = \frac{3}{2}N_C \) where the coefficient \( \gamma_{00} \) vanishes. For even smaller values of \( N_F \), like \( N_F = N_C + 1 \) and \( N_F = N_C \), the Higgs mechanism has removed the massless magnetic quanta of the dual gauge theory. We have massive composites as physical states in \( \mathcal{H} \).

Of particular interest is the window \( \frac{3}{2}N_C < N_F < 3N_C \) for the electric theory. With \( N_F = N_F \), it exactly corresponds to the region \( 3(N_F - N_C) > N_F > \frac{3}{2}(N_F - N_C) \) for the dual magnetic formulation. In this region, we have \( \gamma_{00} > 0 \) and \( \beta_0 < 0 \), and for the dual system, with our relations \( \frac{12}{13} \), \( \gamma_{00}^d > 0 \) and \( \beta_0^d < 0 \). Both, electric and magnetic versions, are asymptotically free and have no superconvergence. Our superconvergence arguments indicates that there is no confinement. The gauge field propagator is not compatible with an approximately linear potential, and the BRST arguments do not prevent elementary quanta from being elements of the physical state space.

In the window for \( N_F \) discussed above, the UV-asymptotic behavior of the gauge field propagator for the electric \( SU(N_C) \) theory is given by

\[
-k^2 D(k^2) \simeq C_R \left(-\beta_0 \ln \frac{k^2}{\kappa^2}\right)^{-\gamma_{00}/\beta_0} + \cdots,
\]

which diverges since \( \gamma_{00}/\beta_0 < 0 \). Although this result is valid in all covariant gauges \( \alpha \geq 0 \), we consider here only the Landau gauge for simplicity. For \( N_F \) near the upper limit \( 3N_C \) of the region considered, where \( \beta_0 \) approaches zero from below, the exponent in Eq. \( (14) \) is very large and we have strong divergence. In contrast, for \( N_F \) near the lower limit \( \frac{3}{4}N_F \), the coefficient \( \gamma_{00} \) vanishes, and we have only small modifications of an asymptotic \( 1/k^2 \) behavior of the function \( D(k^2) \). As we continue to values of \( N_F \) below
the critical point $\frac{3}{2}N_C$, where $\frac{\gamma_{00}}{\beta_0} > 0$, the structure function is superconvergent, and we have confinement. We see that the phase transition of the theory at $N_F = \frac{3}{2}N_C$ is reflected in an essential change of the asymptotic behavior.

The UV-behavior described above for the $N = 1$ SUSY theory with $G = SU(N_C)$ in the window (9) is completely analogous to that of the corresponding non-Susy gauge theory in the Coulomb phase comprising the interval (3).

Let us now consider the UV-limit of the gauge field structure function for the dual theory with $G^d = SU(N_F - N_C)$. From Eqs. (12,13), we obtain the relation $\frac{\gamma_{00}^d}{\beta_0^d} = \frac{\beta_0}{4\gamma_{00}}$, so that

$$-k^2 D^d(k^2) \simeq C_R^d \left( \frac{1}{2} \frac{\gamma_{00}}{\ln \frac{k^2}{\kappa_d^2}} \right) \frac{\beta_0}{4\gamma_{00}} + \cdots .$$

The relation between $\kappa_d^2$ and $\kappa^2$ is given in [13]. As expected, the UV-behavior for the magnetic theory in the Coulomb region is just the opposite of the one described above for the electric case, interchanging the upper and the lower end of the window (9). The transition from the Coulomb to the confining phase is always associated with the loss of asymptotic freedom of the dual theory, the excitation of which then give the observable low energy composites.

It has been argued, that the SUSY gauge theories we consider here, have an IR-fixed point in the Coulomb interval (11) [3, 23]. In the neighborhood of this fixed point $g_\ast^2$, we write the $\beta$-function of the magnetic theory in the
form

\[ \beta^d(g^2) = \beta_*(g^2 - g_*^2) + \cdots, \]  

(26)

where \( \beta_* = \beta^d(g_*^2) > 0 \). Here \( \beta^d(g^2) = \beta^d(g^2, \lambda(g^2)) \) is the renormalization group function of the theory corresponding to the special solution of the reduction equation \( \text{(14)} \) in the limit of small values of \( g^2 \). With \( \lambda(g^2) \) being a solution of this differential equation, we see that \( \beta^d(g_*^2) = 0 \) implies \( \beta^d_\lambda(g_*^2) = \beta^d_\lambda(g_*^2, \lambda(g_*^2)) = 0 \), provided we require that the solution is bounded at \( g^2 = g_*^2 \).

Assuming that the anomalous dimension \( \gamma^d(g^2) \) is regular at \( g^2 = g_*^2 \), we find for the IR-limit of the gauge field structure function for \( SU(N_F - N_C) \)

\[-k^2 D^d(k^2) \simeq C_* \left( \frac{k^2}{\kappa_d^2} \right)^{\gamma^d(g_*^2)}. \]  

(27)

The IR-fixed point \( g_*^2 \) is a function of \( N_C \) and \( N_F \). Using the asymptotic expansion \( \text{(15)} \) in the special frame where \( \lambda = g^2 f_0 \), we can obtain an approximate expression for \( g_*^2 \) in the limit where \( N_F \) approaches the lower end \( N_F = \frac{3}{2} N_C \) of the window. Here \( g_*^2 \) is proportional to \( (N_F - \frac{3}{2} N_C) \), and hence vanishes in this limit. With \( \gamma^d(g_*^2) \simeq \gamma^d_{00} g_*^2 \), and because \( \gamma^d_{00} > 0 \) in the window, we see that there is a slight softening of the \( k^{-2} \)-behavior of the structure function \( D^d(k^2) \). The modification of the \( k^{-2} \)-behavior disappears at the transition point \( \frac{3}{2} N_C \), below which the theory looses asymptotic freedom, and the IR-limit contains free magnetic gauge field excitations. In a similar fashion, we can discuss the structure function of the electric \( SU(N_C) \) gauge theory at the upper end of the window near \( N_F = 3 N_C \).

To sum up, for the SUSY model considered, criteria for confinement, which are based upon analyticity and supercovergence of the gauge field
propagator, are in exact agreement with the results of the duality approach. However, the superconvergence arguments are also applicable to non-SUSY gauge theories like QCD, for which they were introduced originally. The method of the reduction of coupling parameters has been used in order to define the dual theory beyond the matching conditions. Even though it involves Yukawa couplings, in the window and above, the reduced magnetic theory is UV-asymptotically free, with a renormalized asymptotic power series expansion. In an interval below the window, it is IR-free and describes the low energy excitations of the system.

ACKNOWLEDGMENTS

For conversations or communications. I am indebted to D.R.T. Jones, D. Kutasov, K. Sibold, W. Zimmermann, G. Zoupanos and, in particular, to Jisuke Kubo. It is a pleasure to thank Wolfhart Zimmermann, and the theory group of the Max Planck Institut für Physik - Werner Heisenberg Institut -, for their kind hospitality in München.

This work has been supported in part by the National Science Foundation, grant PHY 91-23780.

References

[1] N. Seiberg and E. Witten, Nucl. Phys. B426 (1994) 19; ibid. B431 (1994) 484.

[2] N. Seiberg, Phys. Rev. D49 (1994) 6857.
[3] N. Seiberg, Nucl. Phys. B435 (1995) 129.

[4] R. Oehme, Phys. Rev. D42 (1990) 4209; Phys. Lett. B155 (1987) 60.

[5] K. Nishijima, Prog. Theor. Phys. 75 (1986) 22, Nucl. Phys. B238 (1984) 601; K. Nishijima in Symmetry in Nature, Festschrift for Luigi A. Radicati di Brozolo (Scuola Normale Superiore, Pisa, 1989) pp. 627-655.

[6] R. Oehme and W. Zimmermann, Phys. Rev. D21 (1980) 474, 1661.

[7] R. Oehme, Phys. Lett. B252 (1990) 641.

[8] R. Oehme and W. Xu, Phys. Lett. B333, (1994) 172; ibid. B384 (1996) 269.

[9] R. Oehme, Phys. Lett. B232 (1989) 489.

[10] K. Nishijima, Prog. Theor. Phys. 77 (1987) 1053.

[11] R. Oehme, in Leite Lopes Festschrift, edited by N. Fleury, S. Joffily, J. A. M. Simões, and A. Troper (World Scientific, Singapore, 1988) pp. 443 - 457; University of Tokyo Report UT 527 (1988).

[12] R. Oehme, ‘Supercovergence, Confinement and Duality’, (Talk presented at the International Workshop on High Energy Physics, Novy Svit, Crimea, September 1995), Proceedings, edited by G.V. Bugrij and L. Jenkovsky (Bogoliubov Institute, Kiev, 1995) pp. 107-116; EFI 95-45, MPI-Ph/95-102, hep-th/9511014.

[13] K. Intriligator and N. Seiberg, Nucl. Phys BC45 1, hep-th/9503179; Nucl. Phys. B444 (1995) 125, hep-th/9503179.
[14] R. Oehme and W. Zimmermann, Max Planck Institut Report MPT-PAE/Pth 60/82 (1982), Commun. Math. Phys. 97 (1985) 569; R. Oehme, K. Sibold and W. Zimmermann, Phys. Lett. B147 (1984) 115; W. Zimmermann, Commun. Math. Phys. 97 (1985) 211; R. Oehme, K. Sibold and W. Zimmermann, Phys. Lett. B153 (1985) 142; R. Oehme, Prog. Theor. Phys. Suppl. 86 (1986) 215.

[15] N.-P. Chang, Phys. Rev. D10 (1974) 2706; N.-P. Chang, A. Das and J. Perez-Mercader, Phys. Rev. D22 (1980) 1829.

[16] R. Oehme, ‘Reduction of Coupling Parameters’, (Plenary talk at the XVIIIth International Workshop on High Energy Physics and Field Theory, June 1995, Moscow-Protvino, Russia), Proceedings, edited by V.A. Petrov, A.P. Samokhin and R.N. Rogalyov, pp. 251-270; EFI 95-47, MPI-Ph/95-81, hep-th/9511006.

[17] R. Oehme and W. Zimmermann, Commun. Math. Phys. 97 (1985) 569.

[18] D.R.T. Jones, Nucl. Phys. B87 (1975) 127; R. Barbieri et al, , Phys. Lett. B115 (1982) 212; A.J. Parkes and P.C. West, Nucl. Phys. B256 (1985) 340; P. West, Phys. Lett. B137 (1984) 371; D.R.T. Jones and L. Mezincescu, Phys. Lett. B136 (1984) 293; I.I. Kogan, M. Shifman, and A. Vainstein, Phys. Rev. D53 (1996) 4526.

[19] W. Zimmermann, in the Vladimir Glaser Memorial Volume (1985), MPI-PAE-PTH 85-17 (June 1985).

[20] R. Oehme, Prog. Theor. Phys. Suppl. 86 (1986) 215.

[21] M. Suzuki, Nucl. Phys. B83 (1978) 269.
[22] M. Strassler, in YKIS’95, hep-ph/9602024. R.G. Leigh and M. Strassler, Nucl. Phys. B447 (1999) 95, hep-th/9503121.

[23] T. Banks and A. Zaks, Nucl. Phys. B196 (1982) 189.