The effective field theory of inflation/dark energy and the Horndeski theory

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Abstract The effective field theory (EFT) of cosmological perturbations is a useful framework to deal with the low-energy degrees of freedom present for inflation and dark energy. We review the EFT for modified gravitational theories by starting from the most general action in unitary gauge that involves the lapse function and the three-dimensional geometric scalar quantities appearing in the Arnowitt-Deser-Misner (ADM) formalism. Expanding the action up to quadratic order in the perturbations and imposing conditions for the elimination of spatial derivatives higher than second order, we obtain the Lagrangian of curvature perturbations and gravitational waves with a single scalar degree of freedom. The resulting second-order Lagrangian is exploited for computing the scalar and tensor power spectra generated during inflation. We also show that the most general scalar-tensor theory with second-order equations of motion—Horndeski theory—belongs to the action of our general EFT framework and that the background equations of motion in Horndeski theory can be conveniently expressed in terms of three EFT parameters. Finally we study the equations of matter density perturbations and the effective gravitational coupling for dark energy models based on Horndeski theory, to confront the models with the observations of large-scale structures and weak lensing.

1 Introduction

The inflationary paradigm, which was originally proposed to solve a number of cosmological problems in the standard Big Bang cosmology [1, 2], is now widely accepted as a viable phenomenological framework describing the accelerated expansion in the early Universe. In particular, the Cosmic Mi-
Microwave Background (CMB) temperature anisotropies measured by COBE [3], WMAP [4], and Planck [5] satellites support the slow-roll inflationary scenario driven by a single scalar degree of freedom. Inflation generally predicts the nearly scale-invariant primordial power spectrum of curvature perturbations [6], whose property is consistent with the observed CMB anisotropies. In spite of its great success, we do not yet know the origin of the scalar field responsible for inflation (dubbed “inflaton”).

The observations of the type Ia Supernovae (SN Ia) [7, 8] showed that the Universe entered the phase of another accelerated expansion after the matter-dominated epoch. This has been also supported by other independent observations such as CMB [4] and Baryon Acoustic Oscillations (BAO) [9]. The origin of the late-time cosmic acceleration (dubbed “dark energy”) is not identified yet. The simplest candidate for dark energy is the cosmological constant \( \Lambda \), but if it originates from the vacuum energy appearing in particle physics, the theoretical value is enormously larger than the observed dark energy scale [10, 11]. There is a possibility that some scalar degree of freedom (like inflaton) is responsible for dark energy [12].

Although many models of inflation and dark energy have been constructed in the framework of General Relativity (GR), the modification of gravity from GR can also give rise to the epoch of cosmic acceleration. For example, the Starobinsky model characterized by the Lagrangian \( f(R) = R + R^2/(6M^2) \) [1], where \( R \) is a Ricci scalar and \( M \) is a constant, leads to the quasi de Sitter expansion of the Universe. The recent observational constraints on the dark energy equation of state \( w_{DE} = P_{DE}/\rho_{DE} \) (where \( P_{DE} \) and \( \rho_{DE} \) is the pressure and the energy density of dark energy respectively) imply that the region \( w_{DE} < -1 \) is favored from the joint data analysis of SN Ia, CMB, and BAO [13, 14, 5]. If we modify gravity from GR, it is possible to realize \( w_{DE} < -1 \) without having a problematic ghost state (see Refs. [15] for reviews).

Given that the origins of inflation and dark energy have not been identified yet, it is convenient to construct a general framework dealing with gravitational degrees of freedom beyond GR. In fact, the EFT of inflation and dark energy provides a systematic parametrization that accommodates possible low-energy degrees of freedom by employing cosmological perturbations as small expansion parameters about the Friedmann-Lemaître-Robertson-Walker (FLRW) background [16, 17, 18]. This EFT approach allows one to facilitate the confrontation of models with the cosmological data.

Originally, the EFT of inflation was developed to quantify high-energy corrections to the standard slow-roll inflationary scenario [19]. Expanding the action up to third order in the cosmological perturbations, it is also possible to estimate higher-order correlation functions associated with primordial non-Gaussianities [20]. The EFT formalism was applied to dark energy in connection to the large-distance modification of gravity [21, 33]. The advantage of this approach is that practically all the single-field models of inflation and dark energy can be accommodated in a unified way [20].
Starting from the most general action that depends on the lapse function and other geometric three-dimensional scalar quantities present in the ADM formalism, Gleyzes et al. expanded the action up to quadratic order in cosmological perturbations of the ADM variables. In doing so, the perturbation $\delta \phi$ of a scalar field $\phi$ can be generally present, but the choice of unitary gauge ($\delta \phi = 0$) allows one to absorb the field perturbation in the gravitational sector. Once we fix the gauge in this way, introducing another scalar-field perturbation implies that the system possesses at least two-scalar degrees of freedom. In fact, such a multi-field scenario was studied in Ref. \[33\] to describe both dark energy and dark matter.

By construction, the EFT formalism developed in Refs. \[16, 17, 27, 28\] keeps the time derivatives under control, while the spatial derivatives higher than second order are generally present. Imposing conditions to eliminate these higher-order spatial derivatives for the general theory mentioned above, Gleyzes et al. \[28\] derived the quadratic Lagrangian of cosmological perturbations with one scalar degree of freedom. If the scalar degree of freedom is responsible for inflation, for example, the resulting power spectrum of curvature perturbations can be computed on the quasi de Sitter background (along the same lines in Refs. \[34, 35, 36, 37, 38\]). In this review, we evaluate the inflationary power spectra of both scalar and tensor perturbations expressed in terms of the ADM variables.

In 1973, Horndeski derived the action of the most general scalar-tensor theories with second-order equations of motion \[39\]. This theory recently received much attention as an extension of (covariant) Galileons \[40, 41, 42\]. One can show that the four-dimensional action of “generalized Galileons” derived by Deffayet et al. \[43\] is equivalent to the Horndeski action after a suitable field redefinition \[35\]. Gleyzes et al. \[28\] expressed the Horndeski Lagrangian in terms of the ADM variables appearing in the EFT formalism (see also Ref. \[27\]). This allows one to have a connection between the Horndeski theory and the EFF of inflation/dark energy. In fact, it was shown that Horndeski theory belongs to a sub-class of the general EFT action \[28\].

For the background cosmology, the EFT of inflation/dark energy is characterized by three time-dependent parameters $f$, $\Lambda$, and $c$ \[16, 17, 18\]. This property is useful to perform general analysis for the dynamics of dark energy \[30\]. In the EFT of dark energy, Gleyzes et al. \[28\] obtained the equations of linear cosmological perturbations in the presence of non-relativistic matter (dark matter, baryons). This result reproduces the perturbation equations in Horndeski theory previously derived in Ref. \[44\]. We note that the perturbation equations in the presence of another scalar field (playing the role of dark matter) were also derived in Ref. \[33\]. These results are useful to confront modified gravitational models of dark energy with the observations of large-scale structures, weak lensing, and CMB.

In this lecture note, we review the EFT of inflation/dark energy following the recent works of Refs. \[28, 33\].
In Sec. 2 we start from a general gravitational action in unitary gauge and derive the background equations of motion on the flat FLRW background.

In Sec. 3 we obtain the linear perturbation equations of motion and discuss conditions for avoiding ghosts and Laplacian instabilities of scalar and tensor perturbations.

In Sec. 4 the inflationary power spectra of scalar and tensor perturbations are derived for general single-field theories with second-order linear perturbation equations of motion.

In Sec. 5 we introduce the action of Horndeski theory and express it in terms of the ADM variables appearing in the EFT formalism.

In Sec. 6 we discuss how the second-order EFT action accommodates Horndeski theory as specific cases and provide the correspondence between them.

In Sec. 7 we apply the EFT formalism to dark energy and obtain the background equations of motion in a generic way. In Horndeski theory, the equations of matter density perturbations and the effective gravitational coupling are derived in the presence of non-relativistic matter.

Sec. 8 is devoted to conclusions.

Throughout the paper we use units such that $c = \hbar = 1$, where $c$ is the speed of light and $\hbar$ is reduced Planck constant. The gravitational constant $G$ is related to the reduced Planck mass $M_{\text{pl}} = 2.4357 \times 10^{18}$ GeV via $8\pi G = 1/M_{\text{pl}}^2$. The Greek and Latin indices represent components in space-time and in a three-dimensional space-adapted basis, respectively. For the covariant derivative of some physical quantity $Y$, we use the notation $Y_\mu$ or $\nabla_\mu Y$. We adopt the metric signature $(-, +, +, +)$.

2 The general gravitational action in unitary gauge and the background equations of motion

The EFT of cosmological perturbations allows one to deal with the low-energy degree of freedom appearing for inflation and dark energy. In particular, we are interested in the minimal extension of GR to modified gravitational theories with a single scalar degree of freedom $\phi$. The EFT approach is based on the choice of unitary gauge in which the constant time hypersurface coincides with the constant $\phi$ hypersurface. In other words, this corresponds to the gauge choice

$$\delta \phi = 0,$$

(1)

where $\delta \phi$ is the field perturbation. In this gauge the dynamics of $\delta \phi$ is “eaten” by the metric, so the Lagrangian does not have explicit $\phi$ dependence about the flat FLRW background.

The EFT of cosmological perturbations is based on the 3 + 1 decomposition of the ADM formalism. In particular, the 3 + 1 splitting in unitary gauge allows one to keep the number of time derivatives under control, while
higher spatial derivatives can be generally present. As we will see later, this property is especially useful for constructing theories with second-order time and spatial derivatives. The ADM line element is given by

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -N^2 dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt),$$

(2)

where \(N\) is the lapse function, \(N^i\) is the shift vector, and \(h_{ij}\) is the three-dimensional metric. Then, the four-dimensional metric \(g_{\mu\nu}\) can be expressed as

$$g_{00} = -N^2 + N_i N^i, \quad g_{0i} = g_{i0} = N_i, \quad \text{and} \quad g_{ij} = h_{ij}.$$  

A unit vector orthogonal to the constant \(t\) hypersurface \(\Sigma_t\) is given by

$$n^\mu = (-N, 0, 0, 0),$$

and hence

$$n_\mu n^\mu = -1.$$  

The induced metric \(h_{\mu\nu}\) on \(\Sigma_t\) can be expressed as

$$h_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu,$$

so that it satisfies the orthogonal relation

$$n_\mu h_{\mu\nu} = 0.$$  

The extrinsic curvature is defined by

$$K_{\mu\nu} = h_{\lambda\mu} n_\nu;\lambda = n_\nu;\mu + n_\mu a_\nu,$$

(3)

where \(a_\nu = n^\lambda n_\nu;\lambda\) is the acceleration (curvature) of the normal congruence \(n_\nu\). Since there is the relation \(n_\mu K_{\mu\nu} = 0\), the extrinsic curvature is the quantity on \(\Sigma_t\). The internal geometry of \(\Sigma_t\) can be quantified by the three-dimensional Ricci tensor \(R_{\mu\nu}\) associated with the metric \(h_{\mu\nu}\). The three-dimensional Ricci scalar \(R = R_{\mu\nu} h^{\mu\nu}\) is related to the four-dimensional Ricci scalar \(R\), as

$$R = R + K_{\mu\nu} K^{\mu\nu} - K^2 + 2(K n^\mu - a_\mu)_\mu,$$

(4)

where \(K \equiv K_{\mu\mu}\) is the trace of the extrinsic curvature.

In the following we study general gravitational theories that depend on scalar quantities appearing in the ADM formalism \cite{26, 27, 28}. In addition to the lapse \(N\), we have the following scalars

$$K \equiv K_{\mu\mu}, \quad S \equiv K_{\mu\nu} K^{\mu\nu}, \quad R \equiv R_{\mu\nu} h^{\mu\nu}, \quad Z \equiv R_{\mu\nu} R^{\mu\nu}, \quad U \equiv R_{\mu\nu} K^{\mu\nu}. $$

(5)

The Lagrangian \(L\) of general gravitational theories depends on these scalars, so that the action is given by

$$S = \int d^4x \sqrt{-g} L(N, K, S, R, Z, U; t).$$

(6)

We do not include the dependence of the scalar quantity \(N' = N^\mu N_\mu\) coming from the shift vector, since such a term does not appear even in the most general scalar-tensor theories with second-order equations of motion (see Sec. \text{[5]}).
dom, such that $\phi = \phi(t)$. The field kinetic term

\[ X \equiv g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \]  

depends on the lapse $N$ and the time $t$. The field $\phi$ enters the equations of motion through the partial derivatives $L_N \equiv \partial L / \partial N$ and $L_{NN} \equiv \partial^2 L / \partial N^2$.

Let us consider four scalar metric perturbations $A$, $\psi$, $\zeta$, and $E$ about the flat FLRW background with the scale factor $a(t)$. The general perturbed metric is given by

\[ ds^2 = -e^{2A} dt^2 + 2 \psi_i dx^i dt + a^2(t)(e^{2\zeta} \delta_{ij} + \partial_{ij} E) dx^i dx^j, \]

where $\psi_i$ represents a covariant derivative with respect to $h_{ij}$, and $\partial_{ij} \equiv \nabla_i \nabla_j - \delta_{ij} \nabla^2 / 3$. Under the transformation $t \to t + \delta t$ and $x^i \to x^i + \delta x^i$, the perturbations $\delta \phi$ and $E$ transform as

\[ \delta \phi \to \delta \phi - \dot{\phi} \delta t, \quad E \to E - \delta x, \]

where a dot represents a derivative with respect to $t$. Choosing the unitary gauge \([1]\), the time slicing $\delta t$ is fixed. The spatial threading $\delta x$ can be fixed with the gauge choice

\[ E = 0. \]

On the flat FLRW background with the line element $ds^2 = -dt^2 + a^2(t) \delta_{ij} dx^i dx^j$, the three-dimensional geometric quantities are given by

\[ \bar{K}_{\mu\nu} = H \bar{h}_{\mu\nu}, \quad \bar{K} = 3H, \quad \bar{S} = 3H^2, \quad \bar{R}_{\mu\nu} = 0, \quad \bar{R} = \bar{Z} = \bar{U} = 0, \]

where a bar represents background values and $H \equiv \dot{a} / a$ is the Hubble parameter. We define the following perturbed quantities

\[ \delta K_{\mu}^\nu = K_{\mu}^\nu - H h_{\mu}^\nu, \quad \delta K = K - 3H, \quad \delta S = S - 3H^2 = 2H \delta K + \delta K_{\mu}^\nu \delta K_{\mu}^\nu, \]

where the last equation arises from the first equation and the definition of $S$. Since $R$ and $Z$ vanish on the background, they appear only as perturbations. Up to quadratic order in perturbations, they can be expressed as

\[ \delta R = \delta_1 R + \delta_2 R, \quad \delta Z = \delta R_{\mu}^\nu \delta R_{\mu}^\nu, \]

where $\delta_1 R$ and $\delta_2 R$ are first-order and second-order perturbations in $\delta R$, respectively. The perturbation $Z$ is higher than first order. The first equality \([12]\) implies

\[ U = H R + R_{\mu}^\nu \delta K_{\mu}^\nu, \]

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1 We caution that the notation of the field kinetic energy is the same as that used in Refs. \([28, 33]\), but the notation of $X$ used in Refs. \([35, 36, 37, 38, 44]\) is $-1/2$ times different.
where the last term is a second-order quantity. In order to derive the background and perturbation equations of motion, we expand the action (11) up to quadratic order in perturbations, as

$$L = \bar{L} + L_N \delta N + L_K \delta K + L_S \delta S + L_\mathcal{R} \delta \mathcal{R} + L_Z \delta Z + L_{U} \delta U + \frac{1}{2} \left( \delta N \frac{\partial}{\partial N} + \delta K \frac{\partial}{\partial K} + \delta S \frac{\partial}{\partial S} + \delta \mathcal{R} \frac{\partial}{\partial \mathcal{R}} + \delta Z \frac{\partial}{\partial Z} + \delta U \frac{\partial}{\partial U} \right)^2 L, \tag{15}$$

where a lower index of the Lagrangian $L$ denotes the partial derivatives with respect to the scalar quantities represented in the index. From the second and third relations of Eq. (12), the expansion of the term $L_K \delta K + L_S \delta S$ up to second order reads

$$L_K \delta K + L_S \delta S = \mathcal{F}(K - 3H) + L_S \delta K^\mu_\nu \delta K^\nu_\mu \approx -\dot{\mathcal{F}} - 3HF + \mathcal{F} \delta N + L_S \delta K^\mu_\nu \delta K^\nu_\mu - \dot{\mathcal{F}} \delta N^2, \tag{16}$$

where

$$\mathcal{F} \equiv L_K + 2HL_S. \tag{17}$$

In the second line of Eq. (16), the term $\mathcal{F} K$ has been integrated by using the relation $K = n^\mu_\mu$, as

$$\int d^4x \sqrt{-g} \mathcal{F} K = - \int d^4x \sqrt{-g} n^\mu \mathcal{F}_{\mu} = - \int d^4x \sqrt{-g} \frac{\dot{\mathcal{F}}}{N}, \tag{18}$$

where the boundary term is dropped. Note that we have also expanded the term $N^{-1} = (1 + \delta N)^{-1}$ up to second order in Eq. (16).

The term $\mathcal{U}$ satisfies the relation

$$\alpha(t) \mathcal{U} = \frac{1}{2} \alpha(t) \mathcal{R} K + \frac{1}{2N} \dot{\alpha}(t) \mathcal{R}, \tag{19}$$

where $\alpha(t)$ is an arbitrary function of $t$. Using this relation and the fact that $\mathcal{U}$ is a perturbed quantity, it follows that

$$L_{U} \delta U = \frac{1}{2} \left( \dot{L_{U}} + 3HL_{U} \right) \delta \mathcal{R} + \frac{1}{2} \left( L_{U} \delta K - \dot{L_{U}} \delta N \right) \delta \mathcal{R} + \frac{1}{2} \left( \dot{L_{U}} + 3HL_{U} \right) \delta_2 \mathcal{R}, \tag{20}$$

where the first term on the r.h.s. is the first-order quantity, whereas the rest is second-order.

Summing up the terms discussed above, the zeroth-order and first-order Lagrangians of (15) are given, respectively, by

$$L_0 = \bar{L} - \dot{\mathcal{F}} - 3HF, \tag{21}$$

$$L_1 = (\mathcal{F} + L_N) \delta N + \mathcal{E} \delta_1 \mathcal{R}, \tag{22}$$
where
\[
\mathcal{E} = L_R + \frac{1}{2} \dot{L}_U + \frac{3}{2} H L_M .
\] (23)

Defining the Lagrangian density as \( \mathcal{L} = \sqrt{-g} L = N \sqrt{h} L \), where \( h \) is the determinant of the three-dimensional metric \( h_{ij} \), the zeroth-order and first-order terms read
\[
\mathcal{L}_0 = a^3 \left( \bar{L} - \dot{\mathcal{F}} - 3H \mathcal{F} \right) ,
\]
(24)
\[
\mathcal{L}_1 = a^3 \left( \bar{L} + L_N - 3H \mathcal{F} \right) \delta N + \left( \bar{L} - \dot{\mathcal{F}} - 3H \mathcal{F} \right) \delta h + a^3 \mathcal{E} \delta_1 R .
\]
(25)

The last term is a total derivative, so it can be dropped. Varying the first-order Lagrangian (25) with respect to \( \delta N \) and \( \delta \sqrt{h} \), we can derive the following equations of motion respectively:
\[
\bar{L} + L_N - 3H \mathcal{F} = 0 ,
\]
(26)
\[
\bar{L} - \dot{\mathcal{F}} - 3H \mathcal{F} = 0 .
\]
(27)

On using Eq. (27), the zero-th order Lagrangian density (24) vanishes. Subtracting Eq. (26) from Eq. (27), we obtain
\[
L_N + \dot{\mathcal{F}} = 0 .
\]
(28)

Two of Eqs. (26) - (28) determine the cosmological dynamics on the flat FLRW background.

As an example, let us consider the non-canonical scalar-field model given by [46, 47]
\[
L = \frac{M^2_{\text{pl}}}{2} R + P(\phi, X) ,
\]
(29)
where \( P \) is an arbitrary function with respect to \( \phi \) and \( X \). Using Eq. (21) and dropping the total divergence term, it follows that
\[
L = \frac{M^2_{\text{pl}}}{2} (\mathcal{R} + S - K^2) + P(\phi, X) ,
\]
(30)
where \( X = -N^{-2} \dot{\phi}^2 \). Since \( \bar{L} = -3M^2_{\text{pl}} H^2 + P \), \( L_N = 2\dot{\phi}^2 P_X \), and \( \mathcal{F} = -2M^2_{\text{pl}} H \) on the flat FLRW background, Eqs. (26) and (28) read
\[
3M^2_{\text{pl}} H^2 = -2P_X \dot{\phi}^2 - P ,
\]
(31)
\[
M^2_{\text{pl}} \dot{H} = \dot{\phi}^2 P_X ,
\]
(32)
which match with those derived in Ref. [46]. Taking the time derivative of Eq. (31) and using Eq. (32), we obtain the field equation of motion
\[
\frac{d}{dt} \left( a^3 P_X \dot{\phi}^2 \right) + \frac{1}{2} a^3 \dot{P} = 0 ,
\]
(33)
which is equivalent to \( \frac{d}{dt} \left( a^3 P \dot{\phi} \right) + \frac{1}{2} a^3 P \dot{\phi} = 0 \). For a canonical field characterized by the Lagrangian \( P = -X/2 - V(\phi) \), this reduces to the well-known equation \( \ddot{\phi} + 3H \dot{\phi} + V_{\phi} = 0 \).

3 Second-order action for cosmological perturbations

In order to derive the equations of motion for linear cosmological perturbations, we need to expand the action (6) up to quadratic order. The Lagrangian (15) reads

\[
L = \bar{L} - \mathcal{F} - 3HF + (\mathcal{F} + LN)\delta N + E\delta_1 \mathcal{R} \\
+ \left( \frac{1}{2} LN_N - \mathcal{F} \right) \delta N^2 + \frac{1}{2} A \delta K^2 + B \delta K \delta N + C \delta K \delta_1 \mathcal{R} + D \delta N \delta_1 \mathcal{R} \\
+ E \delta \mathcal{R} + \frac{1}{2} G \delta_1 \mathcal{R}^2 + L_S \delta K^\mu_\nu \delta K^\nu_\mu + L_Z \delta \mathcal{R}_\mu^\nu \delta \mathcal{R}^\mu_\nu ,
\]

(34)

where

\[
A = L_{KK} + 4HL_{SK} + 4H^2L_{SS}, 
B = L_{KN} + 2HL_{SN}, 
C = L_{KR} + 2HL_{SR} + \frac{1}{2} L_{tt} + H L_{Kt} + 2H^2 L_{tt}, 
D = LN_R - \frac{1}{2} L_{tt} + HL_{NN}, 
G = L_{RR} + 2HL_{Rt} + H^2 L_{tt}. 
\]

(35-39)

Then, we obtain the second-order Lagrangian density, as

\[
\mathcal{L}_2 = \delta \sqrt{h} \left[ (\mathcal{F} + LN)\delta N + E\delta_1 \mathcal{R} \right] \\
+ a^3 \left[ \left( LN + \frac{1}{2} LN_N \right) \delta N^2 + E \delta_2 \mathcal{R} + \frac{1}{2} A \delta K^2 + B \delta K \delta N + C \delta K \delta_1 \mathcal{R} \\
+ (D + E) \delta N \delta_1 \mathcal{R} + \frac{1}{2} G \delta_1 \mathcal{R}^2 + L_S \delta K^\mu_\nu \delta K^\nu_\mu + L_Z \delta \mathcal{R}_\mu^\nu \delta \mathcal{R}^\mu_\nu \right].
\]

(40)

For the gauge choice (10), the three-dimensional metric following from the metric (8) is \( h_{ij} = a^2(t)e^{2\zeta} \delta_{ij} \). Then, several perturbed quantities appearing in Eq. (40) can be expressed as

\[
\delta \sqrt{h} = 3a^3 \zeta, \quad \delta \mathcal{R}_{ij} = - \left( \delta_{ij} \partial^2 \zeta + \partial_i \partial_j \zeta \right), 
\delta_1 \mathcal{R} = -4a^{-2} \partial^2 \zeta, \quad \delta_2 \mathcal{R} = -2a^{-2} \left[ (\partial \zeta)^2 - 4 \zeta \partial^2 \zeta \right],
\]

(41)
where $\partial^2 \zeta \equiv \partial_i \partial_j \zeta = \sum_{i=1}^{3} \partial^2 / \partial (x^i)^2$ and $\partial \zeta)^2 = (\partial_i \zeta)(\partial_i \zeta) = \sum_{i=1}^{3} (\partial_i \zeta)^2$.

From Eq. (3) the extrinsic curvature can be expressed in the form

$$K_{ij} = \frac{1}{2N} \left( \dot{h}_{ij} - N_{ij} - N_{ji} \right). \quad (42)$$

For the perturbed metric (8), the first-order extrinsic curvature reads

$$\delta K^i_j = \left( \dot{\zeta} - H \delta N \right) \delta^i_j - \frac{1}{2a^2} \delta^{ik} (\partial_k N_j + \partial_j N_k), \quad (43)$$

where we have used the fact that the Christoffel symbols $\Gamma^k_{ij}$ are the first-order perturbations for non-zero values of $k, i, j$. Since the shift $N_i$ is related to the metric perturbation $\psi$ via $N_i = \partial_i \psi$, the trace of $\delta K_{ij}$ can be expressed as

$$\delta K = 3 \left( \dot{\zeta} - H \delta N \right) - \frac{1}{a^2} \partial^2 \psi. \quad (44)$$

On using the relations (41), (43), and (44), the second-order Lagrangian density (40), up to boundary terms, reduces to

$$L_2 = a^3 \left\{ \frac{1}{2} (2L_N + L_{NN} + 9AH^2 - 6BH + 6LSH^2) \delta N^2 
+ \left[ (B - 3AH - 2LSH) \left( 3 \zeta - \frac{\partial^2 \psi}{a^2} \right) + 4(3HC - D - E) \frac{\partial^2 \zeta}{a^2} \right] \delta N 
- (3A + 2LS) \dot{\zeta} \frac{\partial^2 \psi}{a^2} - 12 \zeta \frac{\partial^2 \psi}{a^2} + \left( \frac{9}{2} A + 3LS \right) \dot{\zeta}^2 + 2E \frac{(\partial \zeta)^2}{a^2} 
+ \frac{1}{2} (A + 2LS) \left( \frac{(\partial \psi)^2}{a^2} + 4C \frac{(\partial^2 \psi)(\partial^2 \zeta)}{a^4} + 2(4G + 3LZ) \frac{(\partial \zeta)^2}{a^4} \right) \right\}, \quad (45)$$

where we have used the background equation (28) to eliminate the term $3a^3(L_N + \dot{F})\zeta \delta N$. Variations of the second-order action $S_2 = \int d^4x L_2$ with respect to $\delta N$ and $\partial^2 \psi$ lead to the following Hamiltonian and momentum constraints, respectively:

$$\frac{2L_N + L_{NN} - 6HW - 3H^2 (3A + 2LS)}{\partial^2 \psi/a^2} + 3W \dot{\zeta} + 4(3HC - D - E) \frac{\partial^2 \zeta}{a^2} = 0, \quad (46)$$

$$W \delta N - (A + 2LS) \frac{\partial^2 \psi}{a^2} + (3A + 2LS) \dot{\zeta} - 4C \frac{\partial^2 \zeta}{a^2} = 0, \quad (47)$$

where

$$W \equiv B - 3AH - 2LSH. \quad (48)$$

From Eqs. (46) and (47) one can express $\delta N$ and $\partial^2 \psi/a^2$ in terms of $\dot{\zeta}$ and $\partial^2 \zeta/a^2$. The last three terms in Eq. (45) give rise to the equations of motion containing spatial derivatives higher than second order. If we impose
the three conditions
\[ A + 2L_S = 0, \quad (49) \]
\[ C = 0, \quad (50) \]
\[ 4G + 3L_Z = 0, \quad (51) \]
then such higher-order spatial derivatives are absent. Under the conditions (49)-(51), we obtain the following relations from Eqs. (46) and (47):
\[ \frac{\partial^2 \psi}{a^2} = \frac{3W^2 + 4L_S(2L_N + L_{NN} - 6HW + 12H^2L_S)}{W^2} \frac{\dot{\zeta}}{\zeta} - \frac{4(D + E)}{W} \frac{\partial^2 \zeta}{a^2}, \quad (52) \]
\[ \delta N = \frac{4L_S}{W} \frac{\dot{\zeta}}{\zeta}, \quad (53) \]
where \( W = L_{KN} + 2HL_{SN} + 4HL_S \). Substituting these relations into Eq. (45), we find that the second-order Lagrangian density can be written in the form
\[ L_2 = a^3 Q_s \left[ \dot{\zeta}^2 - \frac{c_s^2}{a^2} (\partial \zeta)^2 \right], \quad (54) \]
where
\[ Q_s \equiv \frac{2L_S[3B^2 + 4L_S(2L_N + L_{NN})]}{W^2}, \quad (55) \]
\[ c_s^2 \equiv \frac{2}{Q_s} \left( \mathcal{M} + HM - E \right), \quad (56) \]
and
\[ \mathcal{M} \equiv \frac{4L_S(D + E)}{W} = 4L_S \left( L_R + L_{NR} + HL_{NU} + \frac{3}{2} HL_{U} \right). \quad (57) \]
Varying the action \( S_2 = \int d^4x \mathcal{L}_2 \) with respect to the curvature perturbation \( \zeta \), we obtain the equation of motion for \( \zeta \):
\[ \frac{d}{dt} \left( a^3 Q_s \dot{\zeta} \right) - aQ_s c_s^2 \partial^2 \zeta = 0. \quad (58) \]
This is the second-order equation of motion with a single scalar degree of freedom. Provided that the conditions (49)-(51) are satisfied, the gravitational theory described by the action (49) does not involve derivatives higher than quadratic order at the level of linear cosmological perturbations. As we will see in Sec. 5, Horndeski theory satisfies the conditions (49)-(51).
While we have focused on scalar perturbations so far, we can also perform a similar expansion for tensor perturbations. The three-dimensional metric including tensor modes $\gamma_{ij}$ can be expressed as

$$h_{ij} = a^2(t)e^{2\hat{h}}\delta_{ij} + \gamma_{ij} + \frac{1}{2}\gamma_{il}\gamma_{lj}, \quad \det \hat{h} = 1,$$

(59)

where $\gamma_{ij}$ is traceless and divergence-free such that $\gamma_{ii} = \partial_i\gamma_{ij} = 0$. We have introduced the second-order term $\gamma_{il}\gamma_{lj}/2$ for the simplification of calculations [48]. On using the property that tensor modes decouple from scalar modes, we substitute Eq. (59) into the action (6) and then set scalar perturbations 0. We note that tensor perturbations satisfy the relations $\delta K_{ij} = 0$, $\delta K_{ij}^2 = \dot{\gamma}_{ij}^2/4$, $\delta_1 R = 0$, and $\delta_2 R = -\left(\partial_k\gamma_{ij}\right)^2/(4a^2)$. The second-order action for tensor perturbations reads

$$S_2^{(h)} = \int d^4x\, a^3\left[L_S\left(\delta K_{\mu}^\nu\delta K^\mu_{\mu} - \delta K^2\right) + \mathcal{E}\delta_2 R\right],$$

(60)

One can express $\gamma_{ij}$ in terms of two polarization modes, as $\gamma_{ij} = h^+_i e^+_j + h^-_i e^-_j$. In Fourier space, the transverse and traceless tensors $e^+_i$ and $e^-_i$ satisfy the normalization condition $e^+_i(k) e^+_j(-k)^* = 2$ for each polarization ($k$ is a comoving wavenumber), whereas $e_i^+(k) e_i^-(k)^* = 0$. The second-order Lagrangian [60] can be written as the sum of two polarizations, as

$$S_2^{(h)} = \sum_{\lambda=+,-} \int d^4x\, a^3 Q_{\lambda}\left[\hat{h}_{\lambda}^2 - \frac{c_t^2}{a^2}(\partial h_{\lambda})^2\right],$$

(61)

where

$$Q_{\lambda} = \frac{L_S}{2},$$

(62)

$$c_t^2 = \frac{\mathcal{E}}{L_S}.$$

(63)

Each mode $h_{\lambda}$ ($\lambda = +, -$) obeys the second-order equation of motion

$$\frac{d}{dt}\left(a^3 Q_{\lambda}\hat{h}_{\lambda}\right) - aQ_{\lambda}c_t^2\partial^2 h_{\lambda} = 0.$$

(64)

In order to avoid the appearance of ghosts, the coefficient in front of the term $\hat{h}_{\lambda}$ needs to be positive and hence $Q_{\lambda} > 0$. The small-scale instability associated with the Laplacian term $c_t^2\partial^2 h_{\lambda}$ is absent for $c_t^2 > 0$. Then, the conditions for avoidance of the ghost and the Laplacian instability associated with tensor perturbations are given, respectively, by [28, 33].
Similarly, the ghost and the Laplacian instability of scalar perturbations can be avoided for \( Q_s > 0 \) and \( c_s^2 > 0 \), respectively, i.e.,
\[
3(L_{KN} + 2HL_{SN})^2 + 4L_s(2L_N + L_{NN}) > 0 ,
\]
\[
\mathcal{M} + H\mathcal{M} - \mathcal{E} > 0 ,
\]
where we have used the condition (65). The four conditions (65)-(68) need to be satisfied for theoretical consistency.

4 Inflationary power spectra

The scalar degree of freedom discussed in the previous section can give rise to inflation in the early Universe. Moreover, the curvature perturbation \( \zeta \) generated during inflation can be responsible for the origin of observed CMB temperature anisotropies [6]. The tensor perturbation not only contributes to the CMB power spectrum but also leaves an imprint for the B-mode polarization of photons.

In this section we derive the inflationary power spectra of scalar and tensor perturbations for the general action (6). We focus on the theory satisfying the conditions (49)-(51). In this case, the equations of linear cosmological perturbations do not involve time and spatial derivatives higher than second order. Since the Hubble parameter \( H \) is nearly constant during inflation, the terms that do not contain the scale factor \( a \) slowly vary in time. Let us then assume that variations of the terms \( Q_s, Q_t, c_s, \) and \( c_t \) are small, such that the quantities
\[
\delta Q_s \equiv \frac{\dot{Q}_s}{HQ_s}, \quad \delta Q_t \equiv \frac{\dot{Q}_t}{HQ_t}, \quad \delta c_s \equiv \frac{\dot{c}_s}{Hc_s}, \quad \delta c_t \equiv \frac{\dot{c}_t}{Hc_t}
\]
are much smaller than unity.

We first study the evolution of the curvature perturbation \( \zeta \) during inflation. In doing so, we express \( \zeta \) in Fourier space, as
\[
\zeta(\tau, x) = \frac{1}{(2\pi)^3} \int d^3k \hat{\zeta}(\tau, k) e^{ik \cdot x} ,
\]
where
\[
\hat{\zeta}(\tau, k) = u(\tau, k)a(k) + a^*(\tau, -k)a^{\dagger}(-k) .
\]
Here, \( \tau \equiv \int a^{-1} dt \) is the conformal time, \( k \) is the comoving wavenumber, \( a(k) \) and \( a^{\dagger}(k) \) are the annihilation and creation operators, respectively, satisfying
the commutation relations

\[
[a(k_1), a(k_2)] = \langle 2\pi \rangle^3 \delta(k_1 - k_2),
\]
\[
[a(k_1), a(k_2)] = [a(k_1), a(k_2)]^\dagger = 0 .
\] (72)

On the de Sitter background where $H$ is constant, we have $a \propto e^{Ht}$ and hence $\tau = -1/(aH)$. Here, we have set the integration constant 0, such that the asymptotic past corresponds to $\tau \to -\infty$.

Using the equation of motion (58) for $\zeta$, we find that each Fourier mode $u$ obeys

\[\ddot{u} + \left( \frac{a^3 Q_s}{a^3 Q_s} \right)^2 \dot{u} + c_s^2 k^2 a^2 u = 0 .\] (73)

For large $k$, the second term on the l.h.s. of Eq. (73) is negligible relative to the third one, so that the field $u$ oscillates according to the approximate equation $\ddot{u} + c_s^2 (k^2/a^2) u \approx 0$. After the onset of inflation, the $c_s^2 (k^2/a^2) u$ term starts to decrease quickly. Since the second term on the l.h.s. of Eq. (73) is of the order of $H^2 u$, the third term becomes negligible relative to the other terms for $c_s k < aH$. In the large-scale limit ($k \to 0$), the solution to Eq. (73) is given by

\[u = c_1 + c_2 \int \frac{1}{a^3 Q_s} dt ,\] (74)

where $c_1$ and $c_2$ are integration constants. As long as the variable $Q_s$ changes slowly in time, $u$ approaches a constant value $c_1$. The field $u$ starts to be frozen once the perturbations with the wavenumber $k$ cross $c_s k = aH$ [6, 49, 50].

We recall that the second-order Lagrangian for the curvature perturbation $\zeta$ is given by Eq. (54). Introducing a rescaled field $v = zu$ with $z = a \sqrt{2Q_s}$, the kinetic term in the second-order action $S_2 = \int d^4x L_2$ can be rewritten as $\int d\tau d^3x v'^2/2$, where a prime represents a derivative with respect to $\tau$. This means that $v$ is a canonical field that should be quantized [34, 36]. Equation (58) can be written as

\[v'' + \left( c_s^2 k^2 - \frac{z''}{z} \right) v = 0 .\] (75)

On the de Sitter background with a slow variation of the quantity $Q_s$, we can approximate $z''/z \simeq 2/\tau^2$. In the asymptotic past ($k\tau \to -\infty$), we choose the Bunch-Davies vacuum characterized by the mode function $v = e^{-i c_s k \tau} / \sqrt{2c_s k}$. Then the solution to Eq. (75) is given by [49, 37, 36, 38]

\[u(\tau, k) = \frac{i H e^{-i c_s k \tau}}{2(c_s k)^{3/2} \sqrt{Q_s}} (1 + i c_s k \tau) .\] (76)

The deviation from the exact de Sitter background gives rise to a small modification to the solution (76), but this difference appears as a next-order slow-roll correction to the power spectrum [51, 52].
In the regime \( c_s k \ll aH \), the two-point correlation function of \( \zeta \) is given by the vacuum expectation value \( \langle 0 | \hat{\zeta}(\tau, k_1) \hat{\zeta}(\tau, k_2) | 0 \rangle \) at \( \tau \approx 0 \). We define the scalar power spectrum \( P_\zeta(k_1) \), as
\[
\langle 0 | \hat{\zeta}(0, k_1) \hat{\zeta}(0, k_2) | 0 \rangle = \frac{2\pi^2}{k_1^3} P_\zeta(k_1) (2\pi)^3 \delta^{(3)}(k_1 + k_2). \tag{77}
\]
Using the solution (76) in the \( \tau \to 0 \) limit, it follows that
\[
P_\zeta = \frac{H^2}{8\pi^2 Q_s c_s^3}. \tag{78}
\]
Since the curvature perturbation soon approaches a constant for \( c_s k < aH \), it is a good approximation to evaluate the power spectrum (78) at \( c_s k = aH \) during inflation. From the Planck data, the scalar amplitude is constrained as \( P_\zeta \approx 2.2 \times 10^{-9} \) at the pivot wavenumber \( k_0 = 0.002 \text{ Mpc}^{-1} \). 

The spectral index of \( P_\zeta \) is defined by
\[
n_s - 1 \equiv \left. \frac{d \ln P_\zeta}{d \ln k} \right|_{c_s k = aH} = -2 \epsilon - \delta_{Q_s} - 3 \delta_{c_s}, \tag{79}
\]
where \( \delta_{Q_s} \) and \( \delta_{c_s} \) are given by Eq. (69), and
\[
\epsilon \equiv - \frac{\dot{H}}{H^2}. \tag{80}
\]
The slow-roll parameter \( \epsilon \) is much smaller than 1 on the quasi de Sitter background. Given that the variations of \( H \) and \( c_s \) are small during inflation, we can approximate the variation of \( \ln k \) at \( c_s k = aH \), as \( d \ln k = d \ln a = H dt \). Since we are considering the situation with \( |\delta_{Q_s}| \ll 1 \) and \( |\delta_{c_s}| \ll 1 \), the power spectrum is close to scale-invariant \( (n_s \approx 1) \).

We also define the running of the spectral index, as
\[
\alpha_s \equiv \left. \frac{d n_s}{d \ln k} \right|_{c_s k = aH}, \tag{81}
\]
which is of the order of \( \epsilon^2 \) from Eq. (79). With the prior \( \alpha_s = 0 \), the scalar spectral index is constrained as \( n_s = 0.9603 \pm 0.0073 \) at 68% confidence level (CL) from the Planck data. Since \( \epsilon \) is at most of the order \( 10^{-2} \), it is a good approximation to neglect the running \( \alpha_s \) in standard slow-roll inflation.

Let us also derive the spectrum of gravitational waves generated during inflation. The second-order action for tensor perturbations is given by Eq. (61), where \( h_\lambda \) obeys Eq. (64). A canonical field associated with \( h_\lambda (\lambda = +, \times) \) corresponds to \( v_t = z_t h_\lambda \) and \( z_t = a \sqrt{2 Q_t} \). Following a same procedure as that for scalar perturbations, the solution to the Fourier-transformed mode \( h_\lambda \), which recovers the Bunch-Davies vacuum in the asymptotic past, reads
\[ h_\lambda(\tau, k) = \frac{iH}{2(c_t k)^{3/2}\sqrt{Q_t}} (1 + i c_t k \tau). \]  
(82)

This solution approaches \( h_\lambda \to iH/[2(c_t k)^{3/2}\sqrt{Q_t}] \) in the \( \tau \to 0 \) limit.

We also define the tensor power spectrum \( \mathcal{P}_h \) in a similar way to (77). According to the chosen normalization for the tensors \( e^{\lambda}_{ij} \) explained in Sec. 3, we obtain

\[ \mathcal{P}_h = \frac{4}{2\pi^2} c_t^3 |h_\lambda(0, k)|^2 / (2\pi^2), \]

where \( h_\lambda(0, k) = iH/[2(c_t k)^{3/2}\sqrt{Q_t}] \).

It then follows that

\[ \mathcal{P}_h = \frac{H^2}{2\pi^2 Q_t c_t^3}. \]
(83)

The tensor spectral index, which is evaluated at \( c_t k = aH \), reads

\[ n_t \equiv \frac{d \ln \mathcal{P}_h}{d \ln k} \bigg|_{c_t k = aH} = -2\epsilon - \delta_{Q_t} - 3\delta_{c_t}, \]
(84)

where \( \delta_{Q_t} \) and \( \delta_{c_t} \) are given by Eq. (69). The tensor power spectrum is close to scale-invariant \( (n_t \simeq 0) \) provided that \( \epsilon \ll 1 \), \( |\delta_{Q_t}| \ll 1 \), and \( |\delta_{c_t}| \ll 1 \). The difference between the scalar and tensor spectral indices comes from the difference between \( (Q_s, c_s) \) and \( (Q_t, c_t) \).

For those times before the end of inflation \( (\epsilon \ll 1) \) when both \( \mathcal{P}_\zeta \) and \( \mathcal{P}_h \) are approximately constant, the tensor-to-scalar ratio can be estimated as

\[ r \equiv \frac{\mathcal{P}_h}{\mathcal{P}_\zeta} = 4 \frac{Q_s c_s^3}{Q_t c_t^3}. \]
(85)

The Planck data [5], combined with the WMAP large-angle polarization measurement [13] and ACT/SPT temperature data [53], showed that \( r \) is constrained as \( r < 0.11 \) (95\% CL). Recently, the Background Imaging of Cosmic Extragalactic Polarization (BICEP2) group [54] reported the first evidence for the primordial B-mode polarization of CMB photons and they derived the bound \( r = 0.20^{+0.07}_{-0.05} \) (68\% CL) with \( r = 0 \) disfavored at 7\sigma. There is a tension between the data of Planck and BICEP2, but future measurements of the B-mode polarization will place more precise bounds on \( r \).

The inflationary scalar and tensor power spectra (78) and (83) are valid for general theories given by the action (6), provided that the conditions (49)-(51) are satisfied. The quantities like \( Q_s \) and \( c_s^2 \) are written in terms of the partial derivatives of \( L \) with respect to the ADM variables such as \( K \) and \( N \). For a given theory, we need to express the Lagrangian \( L \) in terms of the three-dimensional quantities and the lapse \( N \) to derive concrete forms of the inflationary power spectra. In the next section, we will perform this procedure for the most general scalar-tensor theories with second-order equations of motion.
5 Horndeski theory

5.1 The Lagrangian of Horndeski theory

In this section we apply the EFT formalism advocated in Secs. 2 and 3 to the most general scalar-tensor theories with second-order equations of motion–Horndeski theory \[39\]. This theory is described by the action $S = \int d^4x \sqrt{-g} L$, with the Lagrangian \[43\]

$$L = \sum_{i=2}^{5} L_i,$$

where

$$L_2 = G_2(\phi, X),$$

$$L_3 = G_3(\phi, X) \Box \phi,$$

$$L_4 = G_4(\phi, X) R - 2G_{4X}(\phi, X) \left[ (\Box \phi)^2 - \phi^{\mu\nu} \phi_{\mu\nu} \right],$$

$$L_5 = G_5(\phi, X) G_{\mu\nu} \phi^{\mu\nu} + \frac{1}{3} G_{5X}(\phi, X) \left[ (\Box \phi)^3 - 3(\Box \phi) \phi^{\mu\nu} \phi_{\mu\nu} + 2\phi_{\mu\mu} \phi^{\mu\sigma} \phi_{\sigma\nu} \right].$$

Here $G_i (i = 2, 3, 4, 5)$ are functions in terms of a scalar field $\phi$ and its kinetic energy $X = \phi^{\mu\nu} \partial_\mu \phi \partial_\nu \phi$ with the partial derivatives $G_i X \equiv \partial G_i / \partial X$ and $G_i \phi \equiv \partial G_i / \partial \phi$, $R$ is the Ricci scalar, and $G_{\mu\nu}$ is the Einstein tensor. In 1973, Horndeski derived the Lagrangian of the most general scalar-tensor theories in a different form \[39\], but as shown in Ref. \[35\], it is equivalent to the above form. The Horndeski’s paper \[39\] has not been recognized much for a long time, but it was revived recently in connection to covariant Galileons \[40, 41\] and generalized Galileon theories \[42, 43\].

The Lagrangian \[86\] covers a wide variety of gravitational theories listed below.

- (1) General Relativity with a minimally coupled scalar field

  The minimally coupled scalar-field theory \[29\] is characterized by the functions \[46\].

---

2 When Horndeski wrote this paper, he was the PhD student of David Lovelock. In 1981, he was taking a sabbatical year in Netherlands as a tenured professor of applied mathematics at the University of Waterloo. When he saw a van Gogh exhibition, he was deeply moved. He stated “I was never that interested in art. Then I stumbled onto van Gogh. I never knew art could be like that. I had always thought of it as very representational and not very interesting. But then I thought, ‘This is something I eventually want to do.’ When I saw van Gogh I was sure I could paint.” After this, Horndeski left physics and became an artist.
\[ G_2 = P(\phi, X), \quad G_3 = 0, \quad G_4 = M_{\text{pl}}^2/2, \quad G_5 = 0. \] (91)

The canonical scalar field with a potential \( V(\phi) \) corresponds to the particular choice
\[ G_2 = -X/2 - V(\phi). \] (92)

• (2) Brans-Dicke theory

The Lagrangian of Brans-Dicke (BD) theory is given by
\[ G_2 = -M_{\text{pl}}\omega_{BD} X - V(\phi), \quad G_3 = 0, \quad G_4 = \frac{1}{2} M_{\text{pl}}\phi, \quad G_5 = 0, \] (93)
where \( \omega_{BD} \) is the so-called BD parameter. In the original BD theory \[55\], the field potential \( V(\phi) \) is absent. Dilaton gravity \[56\] corresponds to \( \omega_{BD} = -1 \).

• (3) \( f(R) \) gravity

This theory is characterized by the action
\[ S = \int d^4x \sqrt{-g} \frac{M_{\text{pl}}^2}{2} f(R), \] (94)
where \( f(R) \) is an arbitrary function of the Ricci scalar \( R \). The metric \( f(R) \) gravity corresponds to the case in which the action (94) is varied with respect to \( g_{\mu\nu} \). This can be accommodated by the Lagrangian (86) for the choice
\[ G_2 = -\frac{M_{\text{pl}}^2}{2}(RF - f), \quad G_3 = 0, \quad G_4 = \frac{1}{2} M_{\text{pl}}^2 F, \quad G_5 = 0, \] (95)
where \( F \equiv \partial f/\partial R \). There is a scalar degree of freedom \( \phi = M_{\text{pl}} F(R) \) with a gravitational origin. Comparing Eq. (93) with Eq. (95), we find that metric \( f(R) \) gravity is equivalent to BD theory with \( \omega_{BD} = 0 \) and the potential \( V = (M_{\text{pl}}^2/2)(RF - f) \).

In the Palatini formalism where the metric \( g_{\mu\nu} \) and the connection \( \Gamma^\alpha_{\beta\gamma} \) are treated as independent variables, the Ricci scalar is different from that in metric \( f(R) \) gravity. The Palatini \( f(R) \) gravity is equivalent to BD theory with the parameter \( \omega_{BD} = -3/2 \) \[13\].

• (4) Non-minimally coupled theory

This theory is described by the functions
\[ G_2 = \omega(\phi) X - V(\phi), \quad G_3 = 0, \quad G_4 = \frac{M_{\text{pl}}^2}{2} - \frac{1}{2} \xi \phi^2, \quad G_5 = 0. \] (96)
where $\omega(\phi)$ and $V(\phi)$ are functions of $\phi$. Higgs inflation \cite{57} corresponds to a canonical field ($\omega(\phi) = -1/2$) with the potential $V(\phi) = (\lambda/4)(\phi^2 - v^2)^2$ (see also Refs. \cite{58}).

- (5) Covariant Galileons

The covariant Galileons \cite{11}, in the absence of the field potential, are described by the functions

$$G_2 = c_2 X, \quad G_3 = c_3 X, \quad G_4 = \frac{M_{pl}^2}{2} + c_4 X^2, \quad G_5 = c_5 X^2,$$

where $c_i \ (i = 2, 3, 4, 5)$ are constants. The field equations of motion are invariant under the Galilean transformation $\partial_\mu \phi \rightarrow \partial_\mu \phi + b_\mu$ in the limit of Minkowski space-time \cite{40}.

- (6) Derivative couplings

A scalar field whose derivative couples to the Einstein tensor in the form $G_{\mu\nu} \partial^\mu \phi \partial^\nu \phi$ \cite{59, 60} corresponds to the choice

$$G_2 = -\frac{X}{2} - V(\phi), \quad G_3 = 0, \quad G_4 = 0, \quad G_5 = c\phi,$$

where $c$ is a constant and $V(\phi)$ is the field potential. In fact, integration of the term $c\phi G_{\mu\nu} \phi^\mu \phi^\nu$ by parts gives rise to the coupling $-cG_{\mu\nu} \partial^\mu \phi \partial^\nu \phi$.

- (7) Gauss-Bonnet couplings

The Gauss-Bonnet couplings of the from $-\xi(\phi)R_{GB}^2$, where $R_{GB}^2 = R^2 - 4R_{\alpha\beta}R^{\alpha\beta} + R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$, can be accommodated by the choice \cite{35}

$$G_2 = -2\xi^{(4)}(\phi)X^2[3 - \ln(-X/2)], \quad G_3 = 2\xi^{(3)}(\phi)X[7 - 3\ln(-X/2)], \quad G_4 = 2\xi^{(2)}(\phi)X[2 - \ln(-X/2)], \quad G_5 = 4\xi^{(1)}(\phi)\ln(-X/2),$$

where $\xi^{(n)}(\phi) = \partial^n \xi(\phi)/\partial \phi^n$.

### 5.2 Horndeski Lagrangian in terms of ADM variables

Let us express the Horndeski Lagrangians \cite{57}--\cite{60} in terms of the lapse $N$ and the three-dimensional quantities introduced in Sec. 2. In unitary gauge, the unit vector $n_\mu$ orthogonal to the constant $\phi$-hypersurface is given by \cite{28}

$$n_\mu = -\gamma \dot{\phi} n_\mu, \quad \gamma = \frac{1}{\sqrt{-X}}.$$

Taking the covariant derivative of Eq. \cite{28} and using the relation \cite{41}, we obtain
\[ \phi_{\mu\nu} = -\frac{1}{\gamma} (K_{\mu\nu} - n_\mu a_\nu - n_\nu a_\mu) + \frac{\gamma^2}{2} \phi^\sigma X_{\sigma n_\mu n_\nu}. \]  

(101)

The trace of Eq. (101) gives

\[ \Box \phi = -\frac{1}{\gamma} K + \frac{\phi^\sigma X_{\sigma}}{2X}. \]  

(102)

First of all, the Lagrangian \( L_2 \) depends on \( N \) through the field kinetic energy, i.e.,

\[ L_2 = G_2(\phi, X(N)). \]  

(103)

On using the property \( X(N) = -\dot{\phi}^2/N^2 \) on the flat FLRW background, the quantity like \( L_{2N} = 2\dot{\phi}^2 G_{2X} \).

For the computation of \( L_3 = G_3 \Box \phi \), it is convenient to introduce an auxiliary function \( F_3(\phi, X) \), as

\[ G_3 = F_3 + 2XF_{3X}. \]  

(104)

After integration by parts, the term \( F_3 \Box \phi \) reduces to \( -(F_{\phi\phi} + F_{3X}X_{\mu})\phi^{\mu} \) up to a boundary term. On using the relation (102) for the term \( 2XF_{3X} \Box \phi \), it follows that

\[ L_3 = 2(-X)^{3/2}F_{3X}K - XF_{3\phi}. \]  

(105)

Although the auxiliary function \( F_3 \) is present in the expression of \( L_3 \), the combination of quantities appearing in the background and linear perturbation equations of motion can be expressed in terms of \( G_3 \).

Substituting Eqs. (101) and (102) into Eq. (89), the term \( L_4 \) reads

\[ L_4 = G_4 R + 2XG_{4X}(K^2 - S) + 2G_{4X}X_{\mu}(Kn_\mu - a_\mu), \]  

(106)

where we have used the property \( a_\mu = -h_\mu^\nu X_{\nu}/(2X) \). Substituting Eq. (11) into Eq. (106) and employing the relations \( G_{4X}X_{\mu} = G_{4\mu} + \gamma^{-1}G_{4\phi}n_\mu \) and \( n_\mu a_\mu = 0 \), we obtain

\[ L_4 = G_4 R + (2XG_{4X} - G_4)(K^2 - S) - 2\sqrt{-X}G_{4\phi}K. \]  

(107)

The Lagrangian \( L_5 \) is most complicated to be dealt with. We refer readers to Ref. \[28\] for detailed calculations. Introducing an auxiliary function \( F_5(\phi, X) \) such that

\[ G_{5X} \equiv \frac{F_5}{2X} + F_{5X}, \]  

(108)

the final expression of \( L_5 \) is given by

\[ L_5 = \sqrt{-X}F_5 \left( \frac{1}{2} K R - U \right) - H(-X)^{3/2}G_{5X}(2H^2 - 2KH + K^2 - S) + \frac{1}{2} X(G_{5\phi} - F_{5\phi}) R + \frac{1}{2} XG_{5\phi}(K^2 - S), \]  

(109)
which is valid up to quadratic order in the perturbations.

Summing up the contributions (103), (105), (107), and (109), the Lagrangian (86) can be expressed as

\[
L = G_2 + 2(-X)^{3/2} F_{3X} K - XF_{3\phi} \\
+ G_4 \mathcal{R} + (2XG_{4X} - G_4)(K^2 - S) - 2\sqrt{-X}G_{4\phi}K \\
+ \sqrt{-X} F_5 \left( \frac{1}{2} K \mathcal{R} - \mathcal{U} \right) - H(-X)^{3/2}G_{5X}(2H^2 - 2KH + K^2 - S) \\
+ \frac{1}{2} X (G_{5\phi} - F_{5\phi}) \mathcal{R} + \frac{1}{2} XG_{5\phi}(K^2 - S) ,
\]

(110)

where \(G_{2,3,4,5}\) and \(F_{3,5}\) are functions of \(\phi\) and \(X(N)\). The Lagrangian (110) depends on \(N, K, S, \mathcal{R}, \mathcal{U}\), but not on \(\mathcal{Z}\). We evaluate the partial derivatives of the Lagrangian (110) with respect to \(N, K\) e.t.c. and finally set \(N = 1, K = 3H, S = 3H^2, \mathcal{R} = 0, \mathcal{U} = 0\).

Among the terms appearing in Eqs. (49)-(51), the non-vanishing ones are given by

\[
L_{KK} = -2L_S = 2(2XG_{4X} - G_4) - 2H(-X)^{3/2}G_{5X} + XG_{5\phi},
\]

(111)

\[
L_{KR} = -\frac{1}{2} L_{\mathcal{U}} = \frac{1}{2} \sqrt{-X}F_5 ,
\]

(112)

so that all the three conditions (49)-(51) are satisfied. In Horndeski theory, there are no spatial derivatives higher than second order.

### 5.3 Conditions for the avoidance of ghosts and Laplacian instabilities

The conditions (65) and (66) for avoiding the ghost and the Laplacian instability of tensor perturbations translate to

\[
L_S = G_4 - 2XG_{4X} - H \dot{\phi} X G_{5X} - \frac{1}{2} X G_{5\phi} > 0 ,
\]

(113)

\[
\mathcal{E} = G_4 + \frac{1}{2} XG_{5\phi} - XG_{5X} \ddot{\phi} > 0 ,
\]

(114)

respectively. In the presence of the terms \(G_4(X)\) and \(G_5(\phi, X)\), the tensor propagation speed square \(c_t^2 = \mathcal{E}/L_S\) is generally different from 1.

On using the properties \(\mathcal{B} = L_{KN} + 2H L_{SN}\) and \(\mathcal{W} = L_{KN} + 2H L_{SN} + 4HL_S\), the quantity \(Q_s\) in Eq. (55) can be expressed as

\[
Q_s = \frac{2L_S}{3W^2} (9W^2 + 8L_S w) ,
\]

(115)
\[ w \equiv 3L_N + 3L_{NN}/2 - 9H(L_{KK} + 2HL_{SN}) - 18L_S H^2 \]

\[ = -18H^2G_4 + 3(XG_{2X} + 2X^2G_{2XX}) - 18H \phi(2XG_{3X} + X^2G_{3XX}) \]

\[ - 3X(G_{3\phi} + XG_{3\phi}) + 18H^2(7XG_{4X} + 16X^2G_{4XX} + 4X^3G_{4XXX}) \]

\[ - 18H \dot{\phi}(G_{4\phi} + 5XG_{4\phi X} + 2X^2G_{4\phi XX}) + 6H^3 \phi(15XG_{5X} + 13X^2G_{5XX}) \]

\[ + 2X^3G_{5XXX}) + 9H^2 X(6G_{5\phi} + 9XG_{5\phi X} + 2X^2G_{5\phi XX} ), \]

\[ (116) \]

\[ W = 4HG_4 + 2\dot{\phi}XG_{3X} - 16H(XG_{4X} + X^2G_{4XX}) + 2\phi(G_{4\phi} + 2XG_{4\phi X}) \]

\[ - 2H^2 \phi(5XG_{5X} + 2X^2G_{5XX}) - 2HX(3G_{5\phi} + 2XG_{5\phi X}). \]

\[ (117) \]

Taking into account the requirement (113), the no-ghost condition for scalar perturbations reads

\[ 9W^2 + 8L_S w > 0. \]

\[ (118) \]

In Horndeski theory (110), we notice that there is the following relation

\[ L_S = D + E = L_R + L_{NR} + \frac{3}{2}HL_A + HL_{NU}, \]

so that the quantity (57) reduces to

\[ M = \frac{4L_S^2}{\mathcal{W}}. \]

\[ (120) \]

Then, the condition (98) for avoiding the Laplacian instability of scalar perturbations reads

\[ \frac{d}{dt} \left( \frac{4L_S^2}{\mathcal{W}} \right) + 4HL_S^2 - \mathcal{E} > 0, \]

\[ (121) \]

where \( L_S, \mathcal{E}, \) and \( \mathcal{W} \) are given by Eqs. (113), (114), and (117) respectively.

As an example, let us consider BD theory described by the functions (93).

Since \( L_S = \mathcal{E} = G_4 = M_{pl} \phi/2 \) in this case, the conditions (113) and (114) are satisfied for

\[ \phi > 0, \]

\[ (122) \]

with the tensor propagation speed square \( c_t^2 = 1. \) Since \( \mathcal{W} = M_{pl}(\dot{\phi} + 2H \phi) \)

and \( w = -3M_{pl}(6H^2 \phi^2 - \omega_{BD} \phi_0^2 + 6H \phi \dot{\phi})/(2\phi), \) the quantity (115) reads

\[ Q_s = \frac{(3 + 2\omega_{BD})M_{pl} \phi_0^2}{(\phi + 2H \phi)^2}. \]

\[ (123) \]

On using the condition (122), we find that the scalar ghost is absent for

\[ \omega_{BD} > -3/2. \]

\[ (124) \]

\(^\text{3}\) The four quantities \( w_1, w_2, w_3, w_4 \) introduced in Ref. [38] are related to \( L_S, \mathcal{W}, w, \) and

\( \mathcal{E}, \) as \( w_1 = 2L_S, w_2 = \mathcal{W}, w_3 = w, \) and \( w_4 = 2\mathcal{E}. \)
The quantity $\mathcal{M}$ can be expressed as

$$\mathcal{M} = -\frac{M_{\text{pl}}^2 \phi^2}{\mathcal{F}},$$  \hspace{1cm} (125)$$

where we have used the fact that the term $\mathcal{F}$ in Eq. (17) is given by $\mathcal{F} = -M_{\text{pl}}(\dot{\phi} + 2H\phi)$. From the background equation (28), it follows that

$$\dot{\mathcal{F}} = -L_N = -M_{\text{pl}}\dot{\phi}(3H\phi - \omega_{\text{BD}}\dot{\phi})/\phi.$$  \hspace{1cm} (126)$$

Then, the condition (68) for avoiding the Laplacian instability of scalar perturbations translates to

$$\dot{\mathcal{M}} + H\mathcal{M} - \mathcal{E} = \frac{(3 + 2\omega_{\text{BD}})M_{\text{pl}}\phi^2}{2(\phi + 2H\phi)^2} > 0,$$  \hspace{1cm} (127)$$

which is satisfied under (122) and (124). In fact, from Eq. (56), the scalar propagation speed square $c_s^2$ is equivalent to 1 in BD theory.

### 5.4 Primordial power spectra in $k$-inflation

Let us consider a non-canonical scalar-field theory described by the Lagrangian (29). This theory can be expressed in terms of the ADM variables as Eq. (30). Since $L_S = \mathcal{E} = G_4 = M_{\text{pl}}^2/2$, $Q_t = M_{\text{pl}}^2/4$ and $c_s^2 = 1$, the tensor mode is not plagued by any ghosts and Laplacian instabilities. From Eq. (83), the tensor power spectrum is given by

$$P_h = \frac{2H^2}{\pi^2 M_{\text{pl}}^2},$$  \hspace{1cm} (128)$$

which depends only on $H$. Therefore, if the amplitude of primordial gravitational waves is measured, the energy scale of inflation can be explicitly known.

We also have the relations $W = 2HM_{\text{pl}}^2$, $w = -9H^2M_{\text{pl}}^2 + 3X(P_X + 2XP_{XX})$, and

$$Q_s = -\frac{\dot{\phi}^2(P_X + 2XP_{XX})}{H^2},$$  \hspace{1cm} (129)$$

so the scalar ghost is absent for $P_X + 2XP_{XX} < 0$. Since $\mathcal{F} = -2M_{\text{pl}}^2\dot{H}$ and $L_N = 2\dot{\phi}^2P_X$, the background equation of motion (28) gives $M_{\text{pl}}^2\dot{H} = \dot{\phi}^2P_X$. Taking the time derivative of the quantity $\mathcal{M} = M_{\text{pl}}^2/(2H)$, it follows that

$$\dot{\mathcal{M}} + H\mathcal{M} - \mathcal{E} = -\frac{M_{\text{pl}}^2\dot{H}}{2H^2} = -\frac{\dot{\phi}^2P_X}{2H^2}.$$  \hspace{1cm} (130)$$

Title Suppressed Due to Excessive Length
To avoid the instability of scalar perturbations, we require that $P_X < 0$. Substituting Eqs. (129) and (130) into Eq. (56), we obtain

$$c_s^2 = \frac{P_X}{P_X + 2XP_{XX}}.$$ (131)

In standard slow-roll inflation driven by the potential energy $V(\phi)$ of a canonical scalar field ($P = -X/2 - V(\phi)$), $c_s^2$ is equivalent to 1. If the Lagrangian $P$ contains a non-linear term in $X$, the scalar propagation speed is generally different from 1.

From Eqs. (129) and (131), we find that the slow-roll parameter $\epsilon = -\dot{H}/H^2$ is related to $Q_s$ and $c_s^2$, as

$$\epsilon = \frac{Q_s c_s^2}{M^2_{\text{pl}}}.$$ (132)

Then, the scalar power spectrum (78) reads

$$P_\zeta = \frac{H^2}{8\pi^2 M^2_{\text{pl}} c_s^3}.$$ (133)

From Eqs. (128) and (133), the tensor-to-scalar ratio is given by (49)

$$r = 16c_s \epsilon.$$ (134)

Since $\epsilon \ll 1$ during inflation, it follows that $r \ll 1$ for $c_s \leq 1$.

6 Horndeski theory in the language of EFT

In this section, we relate the variables introduced in Sec. 2 with those employed in the EFT language of Refs. [17, 25, 26]. The action expanded up to quadratic order in the perturbations can be written in the following form

$$S = \int d^4x \sqrt{-g} \left[ \frac{M^2}{2} f R - \Lambda - c g^{00} + \frac{M^4}{2} (\delta g^{00})^2 - \frac{m_1^3}{2} \delta K \delta g^{00} - \frac{M^2}{2} \delta K^2 \right. - \frac{M^2}{2} \delta K^\mu \delta K^\mu + \frac{\mu^2}{2} \mathcal{R} \delta g^{00} + \frac{\bar{m}_5}{2} \mathcal{R} \delta K + \frac{\lambda_1}{2} \mathcal{R}^2 + \frac{\lambda_2}{2} \mathcal{R}_\mu \mathcal{R}^\mu \right],$$ (135)

where $g^{00} = -1/N^2$, $M_*$ is a constant, and other coefficients such as $f, \Lambda, c, M^4$ depend on time. We note that the four-dimensional Ricci scalar $R$ can be written in terms of the three-dimensional quantities as Eq. (4). After integration by parts, the first term in Eq. (135) reads
Comparing the terms up to the second line of Eq. (137) with those in Eq. (22), in terms of the scalar variables. The conditions (49)-(51) reduce, respectively, to

\[
\frac{M^2}{2} f R = \frac{M^2}{2} \left( f R + f S - f K^2 - 2 \dot{f} K + \dot{K} \right). \tag{136}
\]

Now we substitute \( \mathcal{R} = \delta \mathcal{R} + \tilde{\mathcal{R}} \), \( K = 3H^2 + \delta K \), and \( \mathcal{S} = 3H^2 + 2H \delta K + \delta K^\nu_\mu \delta K^\mu_\nu \) into Eq. (136) and then expand the action (135) up to quadratic order in the perturbations. In doing so, we use the similar property to Eq. (18), i.e., \( \int d^4 x \sqrt{-g} \beta(t) \delta \mathcal{K} = \int d^4 x \sqrt{-g} (\beta - 3H \delta \beta + \beta \delta N - \beta \delta N^2) \), where \( \beta(t) \) is an arbitrary function in terms of \( t \). Then, the resulting Lagrangian reads

\[
L = M^4_2 (\ddot{f} + 2H \dot{f} + 2Hf + 3H^2 f) - \Lambda + c
\]

\[
+ [M^2_2 (\dddot{f} + H \dddot{f} - 2 \dddot{f}) - 2c] \delta \mathcal{N} + \frac{M^2_2}{2} f \delta \mathcal{R}
\]

\[
+ \left[ M^2_2 (\dddot{f} - H \dddot{f} + 2 \dddot{f}) + 3c + 2M^4_2 \right] \delta \mathcal{N}^2 - \left( \frac{M^2_2}{2} f + \frac{M^2_2}{2} \right) \delta K^2
\]

\[
+ (M^2_2 \dddot{f} - \frac{\dot{M}^3_2}{2}) \delta \mathcal{K} \delta \mathcal{N} + \frac{\dot{M}^2_2}{2} \delta \mathcal{K} \dot{\mathcal{R}}_1 + \mu_2 \delta \mathcal{N} \delta \mathcal{R}_1 + \frac{M^2_2}{2} f \delta \mathcal{R}_2^\mu_\nu \delta \mathcal{R}^\nu_\mu.
\tag{137}
\]

Comparing the terms up to the second line of Eq. (137) with those in Eq. (22), it follows that

\[
M^4_2 (\dddot{f} + 2H \dddot{f} + 2Hf + 3H^2 f) - \Lambda + c = \overset{\text{L}}{\dddot{f}} - \overset{\text{L}}{\mathcal{F}} - 3H \overset{\text{L}}{\mathcal{F}} \tag{138}
\]

\[
M^2_2 (\dddot{f} - H \dddot{f} + 2 \dddot{f}) - 2c = \overset{\text{L}}{f} + \overset{\text{L}}{N} \tag{139}
\]

\[
f = \frac{2}{M^4_2} \overset{\text{L}}{E} = \frac{1}{M^4_2} \left( 2 \overset{\text{L}}{R} + \overset{\text{L}}{U} + 3H \overset{\text{L}}{U} \right). \tag{140}
\]

From Eqs. (24) and (25), the r.h.s. of Eq. (138) and (139) vanish in the absence of matter. The background equations of motion are characterized by the three parameters \( f, \Lambda, \text{and} \ c \). Comparing the second-order terms in Eq. (137) with those in Eq. (54), we obtain the following relations

\[
M^4_2 = \frac{1}{4} (2 \overset{\text{L}}{N} + \overset{\text{L}}{N} + 2c), \quad \dot{m}^3_2 = \frac{1}{2} \overset{\text{L}}{E} - L_{KNK} - 2H \overset{\text{L}}{R} \overset{\text{L}}{N} \overset{\text{L}}{N},
\]

\[
\dot{M}^2_2 = -2 \overset{\text{L}}{E} - \overset{\text{L}}{K} - 4H \overset{\text{L}}{R} - 4H^2 \overset{\text{L}}{S}, \quad \dot{M}^3_2 = 2 \overset{\text{L}}{E} - 2 \overset{\text{L}}{S},
\]

\[
\mu^2_2 = L_{NR} - \frac{1}{2} \overset{\text{L}}{R} + H \overset{\text{L}}{R} \overset{\text{L}}{N} \overset{\text{L}}{U},
\]

\[
\dot{m}^3_3 = 2L_{K} + 4H \overset{\text{L}}{R} + L_{U} + 2H \overset{\text{L}}{R} \overset{\text{L}}{K} + 4H^2 \overset{\text{L}}{R} \overset{\text{L}}{U},
\]

\[
\lambda_1 = L_{R} + 2H \overset{\text{L}}{R} \overset{\text{L}}{K} + 2H \overset{\text{L}}{R} \overset{\text{L}}{U}, \quad \lambda_2 = 2 \overset{\text{L}}{Z}, \tag{141}
\]

where we have used Eq. (139) to derive \( M^4_2 \). In Horndeski theory, the r.h.s. of Eq. (141) can be evaluated by taking partial derivatives of the Lagrangian (110) in terms of the scalar variables.

The conditions (49)-(61) reduce, respectively, to
\[ M_2^2 + M_4^2 = 0, \quad \bar{m}_5 = 0, \quad 8\lambda_1 + 3\lambda_2 = 0, \quad (142) \]

under which the spatial derivatives higher than second order are absent. On using these conditions, the Lagrangian can be expressed as

\[ S = \int d^4x\sqrt{-g}\left[\frac{M_2^2}{2} fR - \Lambda - cg^{00} + \frac{M_4^2}{2}(\delta g^{00})^2 - \bar{m}_5^4 \delta K\delta g^{00} - m_4^4 (\delta K^2 - \delta K^\mu_\nu \delta K^\nu_\mu) + \frac{\mu_1^4}{2} \delta R\delta g^{00}\right], \quad (143) \]

where

\[ m_4^2 \equiv \frac{1}{4} (M_2^2 - M_4^2) = \frac{1}{4} (-4E + 2L_S - L_{KK} - 4HL_{SK} - 4H^2L_{SS}). \]

The terms containing \( R^2 = 16(\partial^2\zeta)^2/a^4 \) and \( R_{ij}R^{ij} = [5(\partial^2\zeta)^2 / a^4] \) are absent in Eq. (143) because they only involve spatial derivatives of \( \zeta \) higher than second order.

In Horndeski theory described by the action (110), the coefficients in the action can be computed by using Eqs. (138)-(141). They are given by

\[ M_2^2 f = 2G_4 - G_{5\phi}\dot{\phi}^2 + 2G_{5X}\dot{\phi}^2, \quad (145) \]

\[ A = XG_{2X} - G_2 + \dot{\phi}^3(\dot{\phi} + 3H\dot{\phi})G_{3X} + \dot{F}_4/2 + 3H\dot{X}G_{4X} - 18H^2G_{4X}\dot{\phi}^3 + 6HG_{4\phi X}\dot{\phi}^3 + 12H^2G_{4XX}\dot{\phi}^4 + \dot{F}_{5}/2 + 3M_2^2H^2f_5 + 3M_4^2Hf_5/2 \]

\[ -6H^2G_{5\phi}\dot{\phi}^2 - 7H^3G_{5\phi X}\dot{\phi}^3 + 3H^2G_{5\phi X}\dot{\phi}^4 + 2H^2_3G_{5XX}\dot{\phi}^5, \quad (146) \]

\[ c = XG_{2X} + \dot{\phi}^3(-\dot{\phi} + 3H\dot{\phi})G_{3X} + \dot{\phi}^2G_{3\phi} - \dot{F}_4/2 + 3H\dot{X}G_{4X} \]

\[ -6H^2G_{4\phi X}\dot{\phi}^2 + 6HG_{4\phi X}\dot{\phi}^3 + 12H^2G_{4XX}\dot{\phi}^4 - \dot{F}_{5}/2 + 3M_2^2H^2f_5/2 \]

\[ -3H^2G_{5\phi}\dot{\phi}^2 - 3H^2G_{5XX}\dot{\phi}^3 + 3H^2G_{5\phi X}\dot{\phi}^4 + 2H^2G_{5XX}\dot{\phi}^5, \quad (147) \]

\[ M_4^2 = X^2G_{2XX} + (\dot{\phi} + 3H\dot{\phi})G_{3XX}\dot{\phi}^2/2 - 3HG_{3XX}\dot{\phi}^5 - G_{3\phi X}\dot{\phi}^4/2 \]

\[ +\dot{F}_4/4 - 3H\dot{X}G_{4XX}/2 + 6HG_{4\phi X}\dot{\phi}^3 + 18H^2G_{4XX}\dot{\phi}^4 - 6HG_{4\phi XX}\dot{\phi}^5 \]

\[ -12H^2G_{4\phi XX}\dot{\phi}^3 + \dot{F}_{5}/4 - 3M_2^2H^2f_5/4 - 3H^3G_{5XX}\dot{\phi}^3/2 \]

\[ +6H^2G_{5\phi XX}\dot{\phi}^4 + 6H^3G_{5XX}\dot{\phi}^5 - 3H^2G_{5\phi XX}\dot{\phi}^6 - 2H^3G_{5XX}\dot{\phi}^7, \quad (148) \]

\[ \bar{m}_5^4 = 2G_{3XX}\dot{\phi}^3 + 2\dot{X}G_{4X} - 8HG_{4X}\dot{\phi}^2 + 4G_{4\phi X}\dot{\phi}^3 + 4HG_{3XX}\dot{\phi}^3 + 16HG_{4XX}\dot{\phi}^4, \]

\[ +M_2^2f_5 - 4HG_{5\phi X}\dot{\phi}^2 - 6H^2G_{5XX}\dot{\phi}^3 + 4HG_{5\phi X}\dot{\phi}^3 + 4H^2G_{5XX}\dot{\phi}^5, \quad (149) \]

\[ m_4^2 = \mu_1^2 = 2G_{4X}\dot{\phi}^2 + G_{5\phi X}\dot{\phi}^2 - G_{5XX}\dot{\phi}^2, \quad (150) \]

where

\[ F_4 = 2\dot{X}G_{4X} - 8HG_{4X}\dot{\phi}^2, \quad (151) \]

\[ F_5 = 2M_2^2Hf_5 + M_4^2f_5 - 2HG_{5\phi X}\dot{\phi}^2 - 2H^2G_{5XX}\dot{\phi}^3, \quad (152) \]

\[ M_2^2f_5 = -G_{5\phi X}\dot{\phi}^2 + 2G_{5XX}\dot{\phi}^2, \quad (153) \]
We stress that Horndeski theory satisfies the additional relation $m_4^2 = \mu_1^2$. The time and spatial derivatives for the theory (143) are kept up to second order for linear cosmological perturbations. If $m_4^2 \neq \mu_1^2$, then higher-order spatial derivatives should appear beyond linear order. For the computation of primordial non-Gaussianities of curvature perturbations generated during inflation, we need to expand the action (6) higher than quadratic order. In such cases, the presence of higher-order spatial derivatives can modify the shape of non-Gaussianities relative to that derived for Horndeski theory.

7 Application to dark energy

In this section, we study the dynamics of dark energy based on Horndeski theory in the presence of matter (cold dark matter, baryons, photons e.t.c.). The action in such a theory is given by

$$S = \int d^4x \sqrt{-g} \sum_{i=2}^5 L_i + \int d^4x L_m,$$

(154)

where $L_{2,3,4,5}$ are given by Eqs. (87)-(90) and $L_m$ is the matter Lagrangian of a barotropic perfect fluid. The scalar degree of freedom is responsible for the late-time cosmic acceleration. We assume that matter does not have a direct coupling to $\phi$.

7.1 Background equations of motion

On the flat FLRW background, the energy-momentum tensor of the barotropic perfect fluid is given by $T_0^0 = -\rho_m$ and $T_i^j = P_m \delta_i^j$, where $\rho_m$ is the energy density and $P_m$ is the pressure. This satisfies the continuity equation $T_{\mu}^{\mu} = 0$, i.e.,

$$\dot{\rho}_m + 3H(\rho_m + P_m) = 0.$$

(155)

In the presence of matter, the background equations of motion are modified to

$$\dot{\bar{L}} + L_N - 3H \bar{F} = \rho_m,$$

(156)

$$\dot{\bar{F}} + L_N = \rho_m + P_m.$$

(157)

Substituting Eqs. (156)-(157) into Eqs. (138)-(139), we obtain

$$A + c = 3M_*^2 (fH^2 + \dot{f}H) - \rho_m,$$

(158)
\[ A - c = M_2^2(2 f \ddot{H} + 3 f H^2 + 2 \dot{f} H + \dot{f}) + P_m. \]  

(159)

In Horndeski theory, the functions \( f, A, c \) are given, respectively, by Eqs. (145), (146), and (147). Among the four functions \( G_2, G_3, G_4, G_5 \), the three combinations of them (i.e., \( f, A, c \)) determine the cosmological dynamics.

Taking the time derivative of Eq. (158) and using Eqs. (155) and (159), we obtain

\[ \dot{\Lambda} + \dot{c} + 6 H c = 3 M_2^2 \dot{f}(2 H^2 + \dot{H}). \]  

(160)

The background equations of motion (158) and (159) can be expressed as

\[ 3 M_2^2 H^2 = \rho_{DE} + \rho_m, \]  

(161)

\[ M_2^2(2 \dot{H} + 3 H^2) = -P_{DE} - P_m, \]  

(162)

where

\[ \rho_{DE} = c + 3 H^2(M_2^2 - M_2^2 f) - 3 M_2^2 \dot{f} H, \]  

(163)

\[ P_{DE} = c - A - (2 \dot{H} + 3 H^2)(M_2^2 - M_2^2 f) + M_2^2(2 \dot{H} + \dot{f}). \]  

(164)

On using Eq. (160), we find that the “dark” component satisfies the standard continuity equation

\[ \dot{\rho}_{DE} + 3 H (\rho_{DE} + P_{DE}) = 0. \]  

(165)

Then, we can define the equation of state of dark energy, as

\[ w_{DE} = \frac{P_{DE}}{\rho_{DE}} = -1 + \frac{2c - 2H(M_2^2 - M_2^2 f) - M_2^2(\dot{H} \dot{f} - \dot{f})}{c + A + 3 H^2(M_2^2 - M_2^2 f) - 3 M_2^2 \dot{f} H}. \]  

(166)

For quintessence described by the Lagrangian \( G_2 = P(\phi, X), G_3 = 0, G_4 = M_2^2/2, \) and \( G_5 = 0 \), we have \( M_2^2 f = M_2^2 \phi, A = V(\phi) \), and \( c = \dot{\phi}^2/2 \). Since \( w_{DE} = [\dot{\phi}^2/2 - V(\phi)/[\dot{\phi}^2/2 + V(\phi)] \) in this case, it follows that \( w_{DE} > -1 \). For a non-canonical scalar field with the Lagrangian (29) we have \( w_{DE} < -1 \) for \( P_X > 0 \), but the scalar ghost is present. For the theories in which the quantity \( f \) varies in time (i.e., \( G_4 \) or \( G_5 \) varies), it is possible to realize \( w_{DE} < -1 \) under the condition

\[ 2c - 2H(M_2^2 - M_2^2 f) - M_2^2(\dot{H} \dot{f} - \dot{f}) < 0, \]  

(167)

where we have assumed \( \rho_{DE} > 0 \). In \( f(R) \) gravity \[62\] \[63\] \[64\] \[65\] \[66\] and Galileons \[67\], the dark energy equation of state can be smaller than \(-1\), while avoiding the appearance of ghosts.
7.2 Matter density perturbations and effective gravitational couplings

Let us proceed to discuss the equations of motion for linear cosmological perturbations. The discussion in Sec. 2 is based on unitary gauge, but for the study of dark energy, the Newtonian gauge is commonly used. The general metric in the presence of scalar perturbations $\Psi$, $\psi$, $\Phi$, and $E$ can be written as

$$ds^2 = -(1 + 2\Psi)dt^2 + 2\psi_i dx^i dt + a^2(t) \left[(1 + 2\Phi)\delta_{ij} + \partial_i\partial_j E\right] dx^i dx^j. \quad (168)$$

The Newtonian gauge corresponds to $\psi = 0$ and $E = 0$.

Since the Horndeski action is equivalent to the EFT action (143) in unitary gauge with $m_4 = \mu_1^2$ (up to second order), it is possible to derive the perturbation equations in general gauge by reintroducing the scalar perturbation $\delta \phi$ via the Stueckelberg trick [16, 17, 27, 28]. The quantities appearing in the action (143) transform under the time coordinate change $t \rightarrow t + \delta \phi(t, x)$, e.g., $\delta K_{ij} \rightarrow \delta K_{ij} - H \delta \phi h_{ij} - \partial_i \partial_j \delta \phi$, $R_{ij} \rightarrow R_{ij} + H (\partial_i \partial_j \delta \phi + \delta_{ij} \partial^2 \delta \phi)$. This transformation allows one to write the action up to quadratic order in the perturbations for the general metric (168). Varying the resulting action $S$ with respect to $\Psi$, $\psi$, $\Phi$, $\delta \phi$ and finally setting $\psi = 0 = E$, we can derive the perturbation equations in the Newtonian gauge. This is the approach taken in Ref. [28].

As performed in Ref. [14], the perturbation equations can be also derived by directly expanding the Horndeski action (154) for the metric (168). In the following we assume that the matter Lagrangian $L_m$ is described by a barotropic perfect fluid of non-relativistic matter with the energy-momentum tensor

$$T^0_0 = -(\rho_m + \delta \rho_m), \quad T^0_i = -\rho_m \partial_i v_m, \quad T^i_j = 0. \quad (169)$$

Since there is no direct coupling between matter and the field $\phi$, the perturbed energy-momentum tensor obeys the continuity equation

$$\delta T^{\mu \nu} ; \mu = 0. \quad (170)$$

From the $\nu = 0$ and $\nu = i$ components of Eq. (170), we obtain the following equations in Fourier space respectively,

$$\dot{\delta \rho_m} + 3H \delta \rho_m + 3\rho_m \dot{\Phi} + \frac{k^2}{a^2} \rho_m v_m = 0, \quad (171)$$
$$\dot{v}_m = \Psi, \quad (172)$$

where $k$ is a comoving wavenumber. We introduce the gauge-invariant density contrast

$$\delta_m = \frac{\delta \rho_m}{\rho_m} + 3H v_m. \quad (173)$$
Taking the time derivative of (171) and using Eq. (172), the density contrast satisfies
\[\ddot{\delta}_m + 2H\dot{\delta}_m + \frac{k^2}{a^2}\Psi = 3\ddot{Q} + 6H\dot{Q},\] (174)
where \(Q \equiv H\nu_m - \Phi\).

Expanding the action (154) for the metric (168) up to quadratic order in the perturbations, varying the resulting action with respect to \(E, \Psi, \delta\phi,\) and finally setting \(\psi = E = 0,\) we obtain the following perturbation equations respectively:
\[B_6\Phi + B_7\delta\phi + B_8\Psi = 0,\] (175)
\[A_1\dot{\Phi} + A_2\dot{\delta}\phi - \rho_m\Psi + B_9\frac{k^2}{a^2}\Phi + A_4\Psi + \left(A_5\frac{k^2}{a^2} - \mu\right)\delta\phi - \delta\rho_m = 0,\] (176)
\[D_1\ddot{\Phi} + D_2\ddot{\delta}\phi + D_3\dot{\Phi} + D_4\dot{\delta}\phi + D_5\Psi + \left(B_7\frac{k^2}{a^2} + D_8\right)\Phi
+ \left(D_9\frac{k^2}{a^2} - M^2\right)\delta\phi + \left(A_6\frac{k^2}{a^2} + D_{11}\right)\Psi = 0,\] (177)
where
\[B_6 = 4E = 4G_4 + 2XG_{5\phi} - 4XG_{5XX\phi},\] (178)
\[B_7 = 4\phi \left[\dot{L}_S + H(L_S - E)\right],\]
\[= 8G_{4X}H\dot{\phi} + 8(G_{4X} + 2XG_{4XX})\ddot{\phi} + 4G_{4\phi} - 8XG_{4\phi X} + 4(G_{5\phi} + XG_{5\phi X})\ddot{\phi} + 4H\left[2(G_{5X} + XG_{5XX})\ddot{\phi} + G_{5\phi} - XG_{5\phi X}\right]\dot{\phi}
- 2XG_{5\phi\phi} - 4(H^2 + \dot{H})XG_{5X},\] (179)
\[B_8 = 4L_S = 4G_4 - 4XG_{4X} - 4H\phi XG_{5X} - 2XG_{5\phi}.\] (180)

Explicit forms of the time-dependent coefficients \(A_i\) and \(D_i\) as well as other perturbations equations (derived by the variations of \(\Phi\) and \(\psi\)) are given in Ref. [44]. The definition of the term \(\mu\) in Eq. (170) \(\mu = \mathcal{H}_\phi,\) where \(\mathcal{H} \equiv -(\dot{L} + L_N - 3H\mathcal{F}).\) The term \(M\) in Eq. (177) is defined by
\[M^2 \equiv [\mu + 3H(\mu + \nu)]/\dot{\phi},\] (181)
where \(\nu = \mathcal{P}_\phi\) with \(\mathcal{P} \equiv \dot{L} - \dot{\mathcal{F}} - 3H\mathcal{F}.\) The mass square \(M^2\) involves the second derivative of \(-G_2\) with respect to \(\phi\) [44]. For a canonical field with the potential \(V(\phi),\) this means that the second derivative \(V_{\phi\phi}\) is present in the expression of \(M^2.\) For dark energy models in which the so-called chameleon mechanism [68] works to suppress the fifth force mediated by the field \(\phi,\) the models are designed to have a large mass \(M\) in the region of high density [62, 63, 64, 65, 66, 69]. In the low-energy regime where the late-time cosmic acceleration comes into play, the mass \(M\) should be as small as \(H_0.\)
The perturbations related to the observations of large-scale structures and weak lensing have been deep inside the Hubble radius in the low-redshift regime. In the following we use the quasi-static approximation on sub-horizon scales, under which the dominant contributions to Eqs. (176) and (177) are those involving the terms $k^2/a^2$, $\delta \rho_m$, and $M^2$. In doing so, we neglect the contribution of the oscillating term of the field perturbation $\delta \phi$ relative to the one induced from the matter perturbation $\delta \rho_m$. Under this approximation scheme, the variations of the gravitational potentials $\Phi$ and $\Psi$ are small such that $|\Phi| < |H\Phi|$ and $|\Psi| < |H\Psi|$. Then, Eqs. (170) and (177) read

$$B_6 \frac{k^2}{a^2} \Phi + A_6 \frac{k^2}{a^2} \delta \phi - \delta \rho_m \simeq 0,$$

$$B_7 \frac{k^2}{a^2} \Phi + \left( D_9 \frac{k^2}{a^2} - M^2 \right) \delta \phi + A_6 \frac{k^2}{a^2} \psi \simeq 0,$$

where

$$A_6 = 2XG_{3X} + 8H(G_{4X} + 2XG_{4XX}) \dot{\phi} + 2G_{4\phi} + 4XG_{4\phi X} + 4H(G_{5\phi} + XG_{5\phi X}) \dot{\phi} - 2H^2 X (3G_{5\phi X} + 2XG_{5XX}).$$

$$D_9 = 2G_{2X} - 4(G_{3X} + XG_{3XX}) \ddot{\phi} - 8HG_{3X} \ddot{\phi} - 2G_{3\phi} + 2XG_{3\phi X} + [-16H(3G_{4XX} + 2XG_{4XX}) \ddot{\phi} - 8H(3G_{4\phi X} - 2XG_{4\phi XX}) \ddot{\phi} - 4(3G_{4\phi X} + 2XG_{4\phi XX}) \ddot{\phi} + 40H^2 XG_{XX} + 4XG_{4\phi XX} + 8H(G_{4X} + 2XG_{4XX}) + 12H^2 G_{4X} + \{-8H(2G_{5\phi X} + XG_{5\phi XX}) \ddot{\phi} + 8H^2 + H\}(G_{5X} + XG_{5XX}) + 4H XG_{5\phi XX} \ddot{\phi} - 4H^2 X^2 G_{5\phi XX} + 4H^2 (G_{5X} + 5XG_{5XX} + 2X^2 G_{5XX}) \ddot{\phi} + 2(3H^2 + 2H)G_{5\phi} + 4H XG_{5\phi XX} + 10H^2 XG_{5\phi XX}.$$

Solving Eqs. (173), (182), and (183) for $\Psi$ and $\Phi$, it follows that

$$k^2 \frac{\Psi}{a^2} \simeq \frac{(B_6 D_9 - B_7^2)(k/a)^2 - B_6 M^2}{(A_6 B_6 + B_7^2 D_9 - 2A_6 B_7 B_8)(k/a)^2 - B_8 M^2} \delta \rho_m,$$

$$k^2 \frac{\Phi}{a^2} \simeq \frac{(A_6 B_7 - B_8 D_9)(k/a)^2 + B_8 M^2}{(A_6 B_6 + B_7^2 D_9 - 2A_6 B_7 B_8)(k/a)^2 - B_8 M^2} \delta \rho_m.$$

From Eq. (177), we find that the term $H v_m$ is at most of the order of $(aH/k)^2 \delta \rho_m/\rho_m$. For the modes deep inside the Hubble radius $(k \gg aH)$, we then have $\delta_m \simeq \delta \rho_m/\rho_m$ in Eq. (170). Under the quasi-static approximation on sub-horizon scales, the r.h.s. of Eq. (174) is negligible relative to the l.h.s. of it. On using Eq. (180), the linear matter perturbation obeys

$$\ddot{\delta}_m + 2H \dot{\delta}_m - 4\pi G \rho_m \delta_m \simeq 0.$$
where
\[ G_{\text{eff}} = \frac{2M_{\text{pl}}^2[(B_6D_9 - B_7^2)(k/a)^2 - B_6M^2]}{(A_6^2B_9 + B_9^2D_9 - 2A_6B_7B_8)(k/a)^2 - B_6^2M^2} G. \] (189)

Note that \( G \) is the bare gravitational constant related with the reduced Planck mass \( M_{\text{pl}} \) via the relation \( 8\pi G = M_{\text{pl}}^{-2} \). Since the effective gravitational coupling \( G_{\text{eff}} \) is different depending on gravitational theories, it is possible to discriminate between different modified gravity models from the growth of matter perturbations.

In order to quantify the difference between the two gravitational potentials \( \Psi \) and \( \Phi \), we define
\[ \eta = -\frac{\Phi}{\Psi}. \] (190)

On using the solutions (186) and (187), the anisotropy parameter reads
\[ \eta \simeq \frac{(B_6D_9 - A_6B_7)(k/a)^2 - B_6M^2}{(B_6D_9 - B_7^2)(k/a)^2 - B_6M^2}. \] (191)

The effective gravitational potential associated with deviation of the light rays in CMB and weak lensing observations is given by [71]
\[ \Phi_{\text{eff}} \equiv \frac{(\Psi - \Phi)}{2}, \] (192)

From Eqs. (186), (189), and (190), we obtain
\[ \Phi_{\text{eff}} \simeq -4\pi G_{\text{eff}} \frac{1 + \eta}{2} \left( \frac{a}{k} \right)^2 \rho_m \delta_m, \] (193)
which is related to both \( \delta_m \) and \( \eta \).

### 7.3 Growth of matter perturbations

Introducing the matter density parameter \( \Omega_m = \rho_m / (3M_{\text{pl}}^3H^2) \), we can write the matter perturbation equation (188) in the form
\[ \delta_m'' + \left( 2 + \frac{H'}{H} \right) \delta_m' - \frac{3G_{\text{eff}}}{G} \Omega_m \delta_m \simeq 0, \] (194)
where a prime represents a derivative with respect to ln \( a \).

Let us first consider a non-canonical scalar field described by the Lagrangian
\[ L = \frac{M_{\text{pl}}^2}{2} R + P(\phi, X), \] (195)
in which case \( G_2 = P(\phi, X) \), \( G_3 = 0 \), \( G_4 = M_{\text{pl}}^2/2 \), and \( G_5 = 0 \). Since \( B_6 = B_8 = 2M_{\text{pl}}^2 \), \( B_7 = A_6 = 0 \), and \( D_9 = 2P_X \), it follows that \( G_{\text{eff}} = G \).
and $\eta = 1$ from Eqs. (189) and (191). During the matter-dominated epoch characterized by $\Omega_m = 1$ and $H'/H = -3/2$, there is the growing-mode solution to Eq. (194):

$$\delta_m \propto t^{2/3}. \quad (196)$$

In this regime, the effective gravitational potential (193) is constant. After the Universe enters the epoch of cosmic acceleration, the growth rate of $\delta_m$ becomes smaller than that given in Eq. (196), so $\Phi_{\text{eff}}$ starts to decay. Since $G_{\text{eff}}$ is equivalent to $G$ for the models in the framework of GR, the difference of the growth rate between the models comes from the different background expansion history. In the $\Lambda$CDM model characterized by $P = -\Lambda$, the growth rate $f \equiv \dot{\delta}_m / (H \delta_m)$ can be estimated as $f \approx (\Omega_m)^{\gamma}$ with $\gamma \approx 0.55$ in the low-redshift regime ($z < 1$) [72]. As long as the dark energy equation of state does not significantly deviate from $-1$, $\gamma$ is close to the value 0.55 for the models in the framework of GR [73, 74].

As an example of modified gravity models, we consider BD theory described by the action (93). Since $B_6 = 2 M_{\text{pl}} \phi$, $B_7 = 2 M_{\text{pl}}$, $B_8 = 2 M_{\text{pl}} \phi$, $A_6 = M_{\text{pl}}$, and $D_9 = -M_{\text{pl}} \omega_{\text{BD}} / \phi$, Eqs. (189) and (191) reduce to

$$G_{\text{eff}} = \frac{M_{\text{pl}} 4 + 2 \omega_{\text{BD}} + 2(\phi/M_{\text{pl}})(Ma/k)^2}{\phi} G, \quad (197)$$

$$\eta = \frac{1 + \omega_{\text{BD}} + (\phi/M_{\text{pl}})(Ma/k)^2}{2 + \omega_{\text{BD}} + (\phi/M_{\text{pl}})(Ma/k)^2}, \quad (198)$$

where

$$M^2 = V_{\phi\phi} + \omega_{\text{BD}} M_{\text{pl}} [\dot{\phi}^2 - \phi (\ddot{\phi} - 3H \dot{\phi})]. \quad (199)$$

In the $\omega_{\text{BD}} \rightarrow \infty$ limit with $\phi \rightarrow M_{\text{pl}}$, we obtain $G_{\text{eff}} \rightarrow G$ and $\eta \rightarrow 1$, so the General Relativistic behavior can be recovered. The same property also holds for $M \rightarrow \infty$, as the scalar field does not propagate.

In the massless limit $M^2 \rightarrow 0$, it follows that $G_{\text{eff}} \approx (M_{\text{pl}}/\phi)(4 + 2 \omega_{\text{BD}})G/(3 + 2 \omega_{\text{BD}})$ and $\eta \approx (1 + \omega_{\text{BD}})/(2 + \omega_{\text{BD}})$, so the growth rates of $\delta_m$ and $\Phi_{\text{eff}}$ are different from those in GR. Since $\omega_{\text{BD}} = 0$ in metric $f(R)$ gravity, we have $G_{\text{eff}} \approx (M_{\text{pl}}/\phi)(4/3)G$ and $\eta \approx 1/2$. The viable dark energy models based on $f(R)$ gravity [24, 63, 64, 65, 66] are constructed in a way that the mass $M$ is large for $R \gg H_0^2$ and that $M$ decreases to the similar order to $H_0$ by today. There is a transition from the “massive” regime $M > k/a$ to the “massless” regime $M < k/a$, depending on the wavenumber $k$ [64, 65, 75]. If this transition happens in the deep matter era characterized by $H'/H \approx -3/2$ and $\dot{\Omega}_m = \rho_m / (3M_{\text{pl}} \phi H^2) \approx 1$, the growing-mode solution during the “massless” regime of metric $f(R)$ gravity is given by [64]

$$\delta_m \propto t^{(\sqrt{3} - 1)/6}, \quad (200)$$

whose growth rate is larger than that in GR. This leaves an imprint for the measurement of red-shift space distortions in the galaxy power spectrum.
From Eq. (193), the effective gravitational coupling evolves as $\Phi_{\text{eff}} \propto t^{(\sqrt{33} - 5)/6}$. This modification affects the weak lensing power spectrum as well as the ISW effect in CMB [77, 78].

In other modified gravity models like covariant Galileons [79], the growth rate of perturbations is different from that in GR and $f(R)$ gravity. Although the current observations are not enough to discriminate between different models precisely, we hope that future observations will allow us to do so.

8 Conclusions

We have reviewed a framework for studying the most general four-dimensional gravitational theories with a single scalar degree of freedom. The EFT of cosmological perturbations is useful for the unified description of modified gravitational theories in that it can be describe practically all single-field models proposed in the literature. This unified scheme can allow one to provide model-independent constraints on the properties of inflation/dark energy and to put constraints on individual models consistent with observations.

Starting from the general action (6) that depends on the lapse $N$ and other three-dimensional scalar ADM variables, we have expanded the action up to quadratic order in cosmological perturbations about the FLRW background. The choice of unitary gauge allows one to absorb dynamics of the field perturbation $\delta \phi$ into the gravitational sector. Provided that the three conditions (49)-(51) are satisfied, the second-order Lagrangian density reduces to the simple form (54) with a single scalar degree of freedom characterized by the curvature perturbation $\zeta$. We have also shown that the quadratic action for tensor perturbations is given by Eq. (60). In order to avoid ghosts and Laplacian instabilities of scalar and tensor perturbations, we require the conditions $Q_s > 0$, $c_s^2 > 0$, $Q_t > 0$, and $c_t^2 > 0$.

The most general scalar-tensor theories with second-order equations of motion–Horndeski theory–belong to a sub-class of the action (6) in the framework of EFT. The Horndeski Lagrangian can be expressed in terms of the ADM scalar quantities in the form (110). Using the relations (138)-(141) between the EFT variables appearing in the action (135) and the partial derivatives of the Lagrangian $L$ with respect to the ADM variables, we have shown that, up to quadratic order in perturbations, Horndeski theory corresponds to the action (143) with the additional condition $m_4^2 = \mu_1^2$. The dictionary between the EFT variables and the functions $G_i(\phi, X)$ in Horndeski theory is given by Eqs. (145)-(150).

In Sec. 4 we have also derived the power spectra of scalar and tensor perturbations generated during inflation for general second-order theories satisfying the conditions (49)-(51). The formulas (78) and (83) cover a wide variety of modified gravitational theories presented in Sec. 5.1, so they can be used for constraining each inflationary model from the CMB observations.
(along the lines of Ref. [80]). In particular, it will be of interest to discriminate between a host of single-field inflationary models from the precise B-mode polarization data available in the future.

In Sec. 7 we have applied the EFT of cosmological perturbations to dark energy in the presence of a barotropic perfect fluid. The background cosmology is described by three time-dependent functions \( f, A, \) and \( c \), with which different models can be distinguished from the evolution of the dark energy equation of state. In Horndeski theory, we have obtained the effective gravitational coupling appearing in the matter perturbation equation under the quasi-static approximation on sub-horizon scales. Together with the effective gravitational potential given in Eq. (193), it will be possible to discriminate between different modified gravity models from the observations of large-scale structures, weak lensing, and CMB.

While we have studied the effective single-field scenario in unitary gauge, another scalar degree of freedom can be also taken into account in the action. Such a second scalar field can be potentially responsible for dark matter. It will be of interest to provide a unified framework for understanding the origins of inflation, dark energy, and dark matter.

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