Horizon Thermodynamics and Gravitational Tension

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We consider the thermodynamics of a horizon surface from the viewpoint of the vacuum tension $\tau = (c^4/4G)$. Numerically, $\tau \approx 3.026 \times 10^{43}$ Newton. In order of magnitude, this is the tension that has been proposed for microscopic string models of gravity. However, after decades of hard work on string theory models of gravity, there is no firm scientific evidence that such models of gravity apply empirically. Our purpose is thereby to discuss the gravitational tension in terms of the conventional Einstein general theory of relativity that apparently does explain much and maybe all of presently known experimental gravity data. The central result is that matter on the horizon surface is bound by the entropy-area law by tension in the closely analogous sense that the Wilson action-area law also describes a surface confinement.

I. INTRODUCTION

Several decades of difficult mathematical work on string theories of gravity have yielded no definitive results concerning experimental gravitational systems. Yet, conventional general relativity\[1, 2\] contains the notion of a vacuum tension $\tau = (c^4/4G)$ whose large magnitude $\tau \approx 3.026 \times 10^{43}$ Newton, determines the weak strength of the gravitational interaction. The vacuum tension is thought to determine the maximum force\[3–6\] that can be exerted on any material body

$$F \leq \left[ \tau = \left(\frac{c^4}{4G}\right) \right],$$

with equality taking place on bodies confined to a horizon surface.

In what follows we shall discuss the gravitational tension in terms of the Einstein gravitational field equations. The action-area result\[7\] of Wilson which follows from the much lower tension of strong interaction string fragmentation models may (or may not) follow from QCD field theory\[8\]. Strong interaction string-like confinement has been proposed for microscopic string models of gravity. However, after decades of hard work on string theory models of gravity, there is no firm scientific evidence that such models of gravity apply empirically. Our purpose is thereby to discuss the gravitational tension in terms of the conventional Einstein general theory of relativity that apparently does explain much and maybe all of presently known experimental gravity data. The central result is that matter on the horizon surface is bound by the entropy-area law by tension in the closely analogous sense that the Wilson action-area law also describes a surface confinement.

II. GRAVITATIONAL TENSION

To understand the notion of gravitational tension, one may start from the gravitational field equations,

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \left(\frac{8\pi G}{c^4}\right)T_{\mu\nu}. \quad (2)$$

The gravitational vacuum tension may be defined as

$$\tau = \frac{c^4}{4G} \quad [\tau \approx 3.026 \times 10^{43} \text{ Newton}]. \quad (3)$$

The physical meaning of the gravitational tension resides in the notion of gravitational stress; It is

$$T^{G}_{\mu\nu} = -\left(\frac{\tau}{2\pi}\right) \left( R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R \right). \quad (4)$$

Curvature induces gravitational stress. The Einstein field equations simply state that the total stress, gravitational plus material, is null;

$$T^{G}_{\mu\nu} + T_{\mu\nu} = 0. \quad (5)$$

The very large value of the gravitational tension in Eq.\[3\] means that a very small curvature gives rise to a very large gravitational stress in Eq.\[4\]. Since the total stress is null, i.e. $T^{\text{tot}}_{\mu\nu} = T^{G}_{\mu\nu} + T_{\mu\nu} = 0$, the total energy and total momenta are locally conserved in virtue of $0 = 0$. On the other hand, the matter stress obeys

$$D^{\mu}T_{\mu\nu} = 0, \quad (6)$$

wherein $D$ is a covariant derivative so that Eq.\[6\] is not a separate conservation law unless there is a Killing vector condition. Let us consider this in more detail.

III. MATTER ENERGY CONSERVATION

The matter stress tensor has precisely one time-like eigenvector,

$$u^\mu v_{\mu} = -c^2, \quad (7)$$
wherein

\[ T_{\mu\nu} v^\nu = -\varepsilon v^\mu, \]  

(8)

\( \varepsilon \) is the scalar energy per unit volume and \( v^\mu \) is the local material velocity four vector of the matter flow. Taking the covariant derivative of Eq. (5),

\[ D^\mu (T_{\mu\nu} v^\nu) = (D^\mu T_{\mu\nu}) v^\nu + T_{\mu\nu} D^\mu v^\nu, \]

\[ D^\mu (T_{\mu\nu} v^\nu) = T_{\mu\nu} D^\mu v^\nu, \]

\[ -D_\mu (\varepsilon v^\mu) = T_{\mu\nu} D^\mu v^\nu, \]  

(9)

in virtue of Eqs. (6) and (8). From Eqs. (9) one may prove the following

**Theorem:** If the material velocity is a Killing vector obeying

\[ D^\mu v^\nu + D^\nu v^\mu = 0, \]  

(10)

then a local material conservation law of energy exists of the form

\[ D_\mu (\varepsilon v^\mu) = 0 \implies \frac{1}{\sqrt{-g}} \partial_\mu \left( \sqrt{-g} \left( \varepsilon v^\mu \right) \right) = 0. \]  

(11)

Eq. (11) follows from Eqs. (6) and (10) and the stress tensor symmetry \( T_{\mu\nu} = T_{\nu\mu} \).

**IV. ACTION AND ENTROPY**

Let us now consider the action and entropy on a horizon area element from the viewpoint of a quantum loop. We presume a Killing vector field over the horizon surface as in Eq. (10). In space-time, a loop consists of a particle going forward in time and an anti-particle going backwards in time as well as space-like sections. The motion of a particle in a loop is surely not a classically allowed path. But in space-time, the loop action \( \int_{\partial\Sigma} \mathcal{W} = \int_{\Sigma} d\mathcal{W} \) is virtual and occurs only in amplitudes as \( e^{i\int \mathcal{W}/\hbar} \). For a purely spatial loop, Euclidean field theory dictates an entropy \( S \) such that

\[ \left[ \frac{\mathcal{W}}{\hbar} \right] \rightarrow - \left[ \frac{S}{k_B} \right]. \]  

(12)

Finally, the quantum gravitational length scale \( \Lambda \) is determined by

\[ \Lambda^2 = \left( \frac{\hbar G}{c^3} \right) \quad [\Lambda \approx 1.616 \times 10^{-33} \text{ cm}] \]  

(13)

that represents a natural area for discussing a horizon surface.

**A. Action**

The action differential form for a displacement in the \( x \) direction is

\[ \mathcal{W} = p_x dx. \]  

(14)

**B. Entropy**

From Eqs. (12), (16) and (18) we go from the Wilson action area law for tension \( \tau \) to the entropy area law on the horizon

\[ \frac{dS}{k_B} = \left( \frac{\tau}{\hbar c} \right) dA = \left( \frac{dA}{4\Lambda^2} \right). \]  

(19)

Equivalently, the entropy per unit area of horizon is given by

\[ \tilde{s} = \left( \frac{dS}{dA} \right) = \left( \frac{k_B \tau}{\hbar c} \right) = \left( \frac{k_B}{4\Lambda^2} \right). \]  

(20)

Let us consider the meaning of the entropy-area law for a spherical black hole of radius

\[ R = \left( \frac{2GM}{c^2} \right) = \left( \frac{2GE}{c^4} \right) = \left( \frac{\mathcal{E}}{2\tau} \right), \]  

(21)
wherein $\mathcal{E} = M c^2$ is the black hole energy. The area of the horizon of the black hole is thereby

$$A = 4\pi R^2 = \pi \left( \frac{\mathcal{E}}{\tau} \right)^2$$

(22)

with the entropy

$$S = \pi k_B \left( \frac{\mathcal{E}^2}{\hbar c \tau} \right),$$

(23)

and temperature

$$\frac{1}{T} = \frac{dS}{d\mathcal{E}} \Rightarrow k_B T = \left( \frac{\tau}{2\pi} \right) \left( \frac{\hbar c}{\mathcal{E}} \right) = \left( \frac{\hbar c}{4\pi R} \right).$$

(24)

The free energy may be written as

$$\mathcal{F} = \mathcal{E} - T S = T S = \frac{1}{2} \mathcal{E}.$$  

(25)

Finally, the surface tension of the black hole is $\sigma = \mathcal{F}/A$; i.e. with $\dot{\mathcal{E}} = \mathcal{E}/A$

$$\sigma = T \dot{\mathcal{S}} = \left( \frac{k_B T}{\hbar c} \right) \tau = \frac{1}{2} \dot{\mathcal{E}}.$$  

(26)

Eq. (26) may also be written as the purely classical equation

$$\sigma = \left[ \frac{\tau}{4\pi R} \right] = \left[ \frac{c^3}{16\pi G R} \right] = \frac{1}{2} \dot{\mathcal{E}}.$$  

(27)

If one draws an equator around the sphere, then the resulting surface energy force $2\pi R \dot{\mathcal{E}} = \tau$ so that the two halves of the sphere attract each other via the vacuum gravitational tension $\tau$. Eq. (27) is central to our discussion. The surface tension of the horizon is from the confinement of both energy and entropy on the surface of the black hole. There is no need to discuss what is “internal” to the black hole. All of the physical quantities are confined to the horizon surface by the gravitational tension.

V. CONCLUSION

We have discussed the notion of gravitational tension and the Killing vector associated with local conservation of material energy. The action and entropy per unit area associated with gravitational horizon surfaces have been computed as a Wilson loop with an action-area law employing gravitational tension. For a black hole horizon, the surface tension of the horizon is computed. All energy and entropy is thereby confined to the horizon surface. Horizon confinement is thereby proved to be a consequence of gravitational tension. There is no need to consider what happens on the inside of a black hole.

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