SECTIONS, SELECTIONS AND PROHOROV’S THEOREM

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Abstract. The famous Prohorov theorem for Radon probability measures is generalized in terms of usco mappings. In the case of completely metrizable spaces this is achieved by applying a classical Michael result on the existence of usco selections for l.s.c. mappings. A similar approach works when sieve-complete spaces are considered.

1. Introduction

All spaces in this paper are assumed to be completely regular and Hausdorff. For a space $X$, let $\mathcal{B}(X)$ be the Borel $\sigma$-algebra associated to $X$, i.e. the smallest $\sigma$-algebra that contains all closed subsets of $X$. Thus, $\mathcal{B}(X)$ is closed with respect to complements and countable unions, its elements are often called Borel subsets of $X$.

A countably additive function $\mu : \mathcal{B}(X) \to [0, +\infty]$ is called a Radon measure on $X$ if

$$\mu(B) = \sup \{\mu(K) : K \subset B \text{ and } K \text{ is compact}\}, \quad B \in \mathcal{B}(X).$$

A Radon probability measure is a Radon measure $\mu$, with $\mu(X) = 1$. In the sequel, we will denote by $\mathcal{P}(X)$ the set of all Radon probability measures on $X$. Every measure $\mu \in \mathcal{P}(X)$ uniquely defines a positive linear functional $\mu(g) = \int gd\mu$, where $g$ runs over the bounded continuous functions on $X$. As a topological space, we consider $\mathcal{P}(X)$ endowed with the weakest topology with respect to which all these functionals are continuous. Thus, a net $\{\mu_{\alpha}\} \subset \mathcal{P}(X)$ converges to $\mu \in \mathcal{P}(X)$ if and only if $\{\mu_{\alpha}(g)\}$ converges to $\mu(g)$ for every bounded continuous function $g : X \to \mathbb{R}$. With respect to this topology, for every closed $F \subset X$ and $\varepsilon > 0$,

$$\{\mu \in \mathcal{P}(X) : \mu(F) < \varepsilon\}$$

is open in $\mathcal{P}(X)$.
The famous Prohorov theorem [13] states that if $X$ is a Polish space (i.e., a completely metrizable separable space), then for every compact $T \subset \mathcal{P}(X)$ and every $\varepsilon > 0$ there exists a compact $K \subset X$, with $\mu(X \setminus K) < \varepsilon$ for all $\mu \in T$. Spaces having this property, called Prohorov spaces, are widely investigated in the literature.

In this paper, we give a simple proof that all sieve-complete spaces are Prohorov (Theorem 3.1). In the special case of completely metrizable spaces, this result follows by the Michael theorem on the existence of usco selections for l.s.c. mappings, [10, Theorem 1.1]. The general case of arbitrary sieve-complete spaces follows by a selection-like result [5, Corollary 7.2] which utilizes “usco sections” instead of “usco selections”.

The idea to use some selection theorem for the proof of Prohorov’s theorem goes back to a question of Bouziad [2]. In fact, our approach provides a natural generalization of Prohorov’s theorem in which the compact subset $T \subset \mathcal{P}(X)$ is replaced by a paracompact one $Z \subset \mathcal{P}(X)$, and the compact $K \subset X$ — by an usco mapping from $Z$ into the compact subsets of $X$. This gives a solution to another problem of Bouziad [2] whether there is a “continuous” version of Prohorov’s theorem, see Corollary 3.2.

The paper is organized as follows. Section 2 is devoted to the main ingredient of our approach which is a construction of l.s.c. mappings generated by Radon probability measures (Proposition 2.1). Section 3 contains the proof of Theorem 3.1 which is preceded by that one for the special case of completely metrizable spaces.

2. A CONSTRUCTION OF L.S.C. MAPPINGS

For a space $X$, let $2^X$ be the family of all nonempty subsets of $X$, and let $\mathcal{C}(X)$ be the subfamily of $2^X$ which consists of all compact members of $2^X$. A part of our considerations will involve $\mathcal{C}(X)$ endowed with the Vietoris topology $\tau_V$. Recall that $\tau_V$ is generated by all collections of the form

$$\langle \mathcal{V} \rangle = \left\{ S \in \mathcal{C}(X) : S \subset \bigcup \mathcal{V} \quad \text{and} \quad S \cap V \neq \emptyset, \text{ whenever } V \in \mathcal{V} \right\},$$

where $\mathcal{V}$ runs over the finite families of open subsets of $X$. For convenience, for an open subset $V \subset X$, we write $\langle V \rangle$ rather than $\langle \{V\} \rangle$.

Another topology on $\mathcal{C}(X)$ that will play an important role in this paper is the upper Vietoris topology $\tau_V^+$, i.e. the topology generated by the family

$$\{ \langle V \rangle : V \subset X \text{ is open} \}.$$

Clearly, $\tau_V^+$ is a coarser topology than the Vietoris one $\tau_V$, i.e. $\tau_V^+ \subset \tau_V$. In this regard, let us make the explicit agreement that if $\tau$ is a topology on $\mathcal{C}(X)$, then
the prefix “τ-” will be used to express properties related to the topology τ, say τ-open sets, τ-closure, etc.

Finally, let us recall that a set-valued mapping Φ : Z → 2^Y is lower semi-continuous, or l.s.c., if the set

\[ \Phi^{-1}(U) = \{ z \in Z : \Phi(z) \cap U \neq \emptyset \} \]

is open in Z for every open U ⊂ Y.

**Proposition 2.1.** Let X be a space, and let ε ∈ (0,1). Define a set-valued mapping Ψε : ℙ(X) → 2^{ε(X)} by

\[ \Psi_{\varepsilon}(\mu) = \{ K \in \mathcal{C}(X) : \mu(X \setminus K) < \varepsilon \}, \mu \in \mathcal{P}(X). \]

Then, Ψε is a nonempty-valued τ_V-l.s.c. mapping.

**Proof.** Take μ ∈ ℙ(X). Since μ(X) = 1 > 1 − ε, by (1.1), there is K ∈ ℂ(X) such that μ(K) > 1 − ε, so Ψ_{\varepsilon}(μ) ̸= ∅. Let K ∈ Ψε(μ) and let \( \mathcal{V} \) be a finite family of open subsets of X, with K ∈ (\mathcal{V}). Then, X \( \setminus \bigcup \mathcal{V} \) ⊂ X \( \setminus K \), it is closed in X and μ(X \( \setminus \bigcup \mathcal{V} \)) < ε. Hence, by (1.2), there exists a neighbourhood U of μ such that ν(X \( \setminus \bigcup \mathcal{V} \)) < ε for every ν ∈ U. If ν ∈ U, then ν(\( \bigcup \mathcal{V} \)) > 1 − ε and, by (1.1), there is a compact subset H ⊂ ∪ \mathcal{V}, with ν(H) > 1 − ε. We now have that H \( \cup K \) ∈ (\mathcal{V}), while H \( \cup K \) ∈ Ψε(ν) because ν(X \( \setminus (H \cup K) \)) ≤ ν(X \( \setminus H \)) < ε. □

**Proposition 2.2.** Let X be a space, ε ∈ (0,1), Ψε : ℙ(X) → 2^{ε(X)} be defined as in Proposition 2.1 and let \( \Phi_{\varepsilon}(\mu) \) be the τ_V^+ -closure of Ψε(μ), for each μ ∈ ℙ(X). Then, μ(X \( \setminus K \)) ≤ ε for every K ∈ Φε(μ) and μ ∈ ℙ(X).

**Proof.** Take μ ∈ ℙ(X) and K ∈ ℂ(X) such that μ(X \( \setminus K \)) > ε. By (1.1), there exists a compact subset H ⊂ X \( \setminus K \), with μ(H) > ε. Let V = X \ H. We now have that K ∈ (\mathcal{V}), while ε < μ(H) = μ(X \( \setminus V \)) ≤ μ(X \( \setminus S \)) for every S ∈ (\mathcal{V}). Consequently, K ∉ \( \Phi_{\varepsilon}(\mu) \) because Ψε(μ) ⊂ ℂ(\( X \setminus (\mathcal{V}) \)). □

We conclude this section with a well-known property of compact sets in the upper Vietoris topology.

**Proposition 2.3.** Let \( \mathcal{K} \subset ℂ(\mathcal{X}) \) be a τ_V^+ -compact set. Then, \( \bigcup \mathcal{K} \) is compact in X.

**Proof.** Take an open in X cover \( \mathcal{V} \) of \( \bigcup \mathcal{K} \). Then, \( \mathcal{V} = \{ \cup \mathcal{\mathcal{E}} : \mathcal{E} \subset \mathcal{V} \text{ is finite} \} \) is a τ_V^+ -open cover of \( \mathcal{K} \). Hence, \( \mathcal{V} \) contains a finite subcover of \( \mathcal{K} \), so there exists a finite \( \mathcal{V}' \subset \mathcal{V} \), with \( \mathcal{K} \subset \cup \{ \cup \mathcal{\mathcal{E}} : \mathcal{E} \subset \mathcal{V} \text{ is finite} \} \). This \( \mathcal{V}' \) is a finite cover of \( \bigcup \mathcal{K} \). □
3. USCO Mappings and Prohorov’s Theorem

Recall that a set-valued mapping \( \psi : Z \to 2^X \) is upper semi-continuous, or u.s.c., if the set

\[
\psi^u(U) = \{ z \in Z : \psi(z) \subset U \}
\]

is open in \( Z \) for every open \( U \subset X \). We say that \( \psi : Z \to 2^X \) is usco if it is u.s.c. and compact-valued. Let us explicitly mention that if \( \psi : Z \to \mathcal{C}(X) \) is usco, then \( \psi(T) = \bigcup \{ \psi(z) : z \in T \} \) is compact for every compact \( T \subset Z \).

A space \( X \) is sieve-complete [3] if it has an open complete sieve. Every Čech-complete space is sieve-complete, and it was shown in [3] (see, also, [11]) that the two concepts are equivalent in the presence of paracompactness.

**Theorem 3.1.** Let \( X \) be a sieve-complete space, and let \( Z \subset \mathcal{P}(X) \) be paracompact. Then, for every \( \varepsilon > 0 \) there is an usco mapping \( \varphi : Z \to \mathcal{C}(X) \) such that \( \mu(X \setminus \varphi(\mu)) < \varepsilon \) for every \( \mu \in Z \).

Turning to the proof of Theorem 3.1, let us first demonstrate the special case of a completely metrizable \( X \). In this case, let \( \Psi_\varepsilon : \mathcal{P}(X) \to 2^{\mathcal{C}(X)} \) be defined as in Proposition 2.1, and let \( \Phi(\mu) \) be the \( \tau_V \)-closure of \( \Psi_\varepsilon(\mu) \), for each \( \mu \in \mathcal{P}(X) \). By Proposition 2.1 and [9] Proposition 2.3, \( \Phi : \mathcal{P}(X) \to 2^{\mathcal{C}(X)} \) is \( \tau_V \)-l.s.c. Also, \( (\mathcal{C}(X), \tau_V) \) is completely metrizable because so is \( X \), [6] [7] [8]. Hence, by [10] Theorem 1.1, \( \Phi | Z \) has a \( \tau_V \)-usco selection \( \theta : Z \to 2^{\mathcal{C}(X)} \). That is, \( \theta \) is a \( \tau_V \)-usco mapping such that \( \theta(\mu) \subset \Phi(\mu) \) for every \( \mu \in Z \). Then, define \( \varphi : Z \to \mathcal{C}(X) \) by letting \( \varphi(\mu) = \bigcup \theta(\mu), \mu \in Z \). This \( \varphi \) is as required. Indeed, each \( \theta(\mu), \mu \in Z \), is \( \tau_V \)-compact, hence \( \tau_V^+ \)-compact as well, and, by Proposition 2.3 each \( \varphi(\mu), \mu \in Z \), is a compact subset of \( X \). If \( V \) is a neighbourhood of \( \varphi(\mu) \) for some \( \mu \in Z \), then \( \langle V \rangle \) is a neighbourhood of \( \theta(\mu) \). This implies that \( \varphi \) is u.s.c. Finally, take \( \mu \in Z \) and \( K \in \theta(\mu) \subset \Phi(\mu) \). Since \( \tau_V^+ \subset \tau_V \), we have that \( \Phi(\mu) \) is a subset of the \( \tau_V^+ \)-closure of \( \Psi_\varepsilon(\mu) \). Therefore, by Proposition 2.2 \( \mu(X \setminus \varphi(\mu)) \leq \mu(X \setminus K) \leq \varepsilon \) because \( K \subset \varphi(\mu) \).

The proof of Theorem 3.1 for the general case of arbitrary sieve-complete spaces follows exactly the same idea but is now based on the upper Vietoris topology and another selection-like result for usco mappings.

**Proof of Theorem 3.1.** Let \( X \) and \( Z \subset \mathcal{P}(X) \) be as in that theorem, and let \( \varepsilon \in (0, 1) \). Also, for each \( \mu \in \mathcal{P}(X) \), let \( \Phi_\varepsilon(\mu) \) be the \( \tau_V^+ \)-closure of \( \Psi_\varepsilon(\mu) \), where \( \Psi_\varepsilon : \mathcal{P}(X) \to 2^{\mathcal{C}(X)} \) is defined as in Proposition 2.1. By Proposition 2.1 and [9] Proposition 2.3, \( \Phi_\varepsilon : \mathcal{P}(X) \to 2^{\mathcal{C}(X)} \) is \( \tau_V^+ \)-l.s.c. because \( \tau_V^+ \subset \tau_V \). By [12] Lemma 3.1, \( (\mathcal{C}(X), \tau_V^+) \) is sieve-complete because so is \( X \). Hence, by [5] Corollary 7.2, \( \Phi_\varepsilon | Z \) has a \( \tau_V^+ \)-usco section \( \theta : Z \to 2^{\mathcal{C}(X)} \). That is, \( \theta \) is a \( \tau_V^+ \)-usco mapping such that \( \theta(\mu) \cap \Phi_\varepsilon(\mu) \neq \emptyset \) for every \( \mu \in Z \). Finally, define the required \( \varphi : Z \to \mathcal{C}(X) \) by \( \varphi(\mu) = \bigcup \theta(\mu), \mu \in Z \). By Proposition 2.3 each \( \varphi(\mu), \mu \in Z \), is a compact
subset of $X$. Just like before $\varphi$ is u.s.c. because if $V$ is a neighbourhood of $\varphi(\mu)$ for some $\mu \in Z$, then $\langle V \rangle$ is a neighbourhood of $\theta(\mu)$. Finally, if $\mu \in Z$ and $K \in \theta(\mu) \cap \Phi(\mu)$, then, by Proposition 2.2, $\mu(X \setminus \varphi(\mu)) \leq \mu(X \setminus K) \leq \varepsilon$ because $K \subset \varphi(\mu)$. The proof is completed. □

It is well-known that $\mathcal{P}(X)$ is paracompact (and Čech-complete) whenever $X$ is so, \[1, 14, 15\], see also \[4\]. This gives the following immediate consequence.

**Corollary 3.2.** Let $X$ be a paracompact Čech-complete space, and $\varepsilon > 0$. Then, there is an usco mapping $\varphi : \mathcal{P}(X) \to \mathcal{C}(X)$ such that $\mu(X \setminus \varphi(\mu)) < \varepsilon$ for every $\mu \in \mathcal{P}(X)$. In particular, $\Phi(T) = \bigcup \{ \varphi(\mu) : \mu \in T \}$, $T \in \mathcal{C}(\mathcal{P}(X))$, defines a continuous map $\Phi : \left( \mathcal{C}(\mathcal{P}(X)), \tau^+ \right) \to \left( \mathcal{C}(X), \tau^+ \right)$ such that $\mu(X \setminus \Phi(T)) < \varepsilon$ for every $T \in \mathcal{C}(\mathcal{P}(X))$ and $\mu \in T$.

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