On the initial value formulation of classical electrodynamics

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Abstract
We describe a seemingly un-noticed feature of the text-book Maxwell–Lorentz system of classical electrodynamics which challenges its formulation in terms of an initial value problem. For point-charges, even after appropriate renormalization, we demonstrate that most of the generic initial data evolves to develop singularities in the electromagnetic fields along the light cones of the initial charge positions. We provide explicit formulas for the corresponding fields, demonstrate how this phenomenon renders the initial value problem ill-posed, and show how such bad initial data can be ruled out by extra conditions in addition to the Maxwell constraints. These extra conditions, however, require knowledge of the history of the solution and, as we discuss, effectively turn the Maxwell–Lorentz system into a system of delay equations much like the Fokker–Schwarzschild–Tetrode equations. For extended charges such singular light fronts persist in a smoothened form and, as we argue, yield physically doubtful solutions. Our results also apply to some extent to expectation values of field operators in quantum field theory.

Keywords: classical electrodynamics, initial value problem, singular light fronts

(Some figures may appear in colour only in the online journal)
1. Introduction

In classical electrodynamics, the dynamics of $N$ charges and their corresponding electromagnetic fields is governed by the Lorentz equations

\[
\frac{d}{dt} \begin{pmatrix} q_{i,t} \\ p_{i,t} \end{pmatrix} = \begin{pmatrix} v_{i,t} \\ m \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{q_{i,t}}{\sqrt{p_{i,t}^2 + m^2}} \end{pmatrix},
\]

(1)

\[
L_{ij,t} := \int d^3x \ \rho(x - q_{i,t}) [E_{j,t}(x) + v_{i,t} \wedge B_{j,t}(x)],
\]

(2)

the Maxwell equations

\[
\partial_t \begin{pmatrix} E_{i,t} \\ B_{i,t} \end{pmatrix} = \begin{pmatrix} \nabla \wedge B_{i,t} - 4\pi v_{i,t} \rho(\cdot - q_{i,t}) \\ - \nabla \wedge E_{i,t} \end{pmatrix},
\]

(3)

and the Maxwell constraints

\[
\nabla \cdot E_{i,0} = 4\pi \rho(\cdot - q_{i,0}) \quad \text{and} \quad \nabla \cdot B_{i,0} = 0,
\]

(4)

for $i = 1, \ldots, N$. In our notation, $q_{i,t}, p_{i,t} \in \mathbb{R}^3$ denote the position and momentum of the $i$th charge at time $t \in \mathbb{R}$. For simplicity we give all charges the same mass $m > 0$ and rigid electric charge density $\rho(x)$ and use units such that the speed of light equals one and the vacuum permittivity equals $\varepsilon_0$. Note that by virtue of (3), the constraints (4) at $t = 0$ imply that they hold for all times $t$.

Contrary to the text-book presentation, see, e.g., [1, 2], in which one employs only one total electric and magnetic field, it will be convenient for our discussion to associate with each charge $i$ an individual electric and magnetic field $F_{i,t} = (E_{i,t}, B_{i,t})$. Thanks to the linearity of the Maxwell equations in the field degrees of freedom, the equations of motion (1)–(4) coincide with the one given in text-books when setting $e_{ij} = 1$. Other choices of $e_{ij}$ allow to switch on or off the interaction of the $j$th field on the $i$th charge.

For arbitrary $e_{ij}$ and smooth and compactly supported $\rho : \mathbb{R}^3 \to \mathbb{R}$ it has been proven that the coupled system of equations (1)–(4) has a well-posed initial problem for any initial data $(q_{i,0}, p_{i,0}, F_{i,0})_{1 \leq i \leq N}$ with reasonably regular fields $F_{i,0}$ fulfilling the constraints (4); see [3–5]. Spinning charges were discussed in [6] and the semi-relativistic system was considered in [7].

Very early, however, it was observed, e.g., in [8], that replacing the charge density $\rho$ by a Dirac delta distribution $\delta^3$ (for simplicity, setting the total electric charge equal one) renders the self-interaction summand $L_{ii,t}$ on the right-hand side of the Lorentz equation (1), and thereby, also the coupled system of equations (1)–(4), ill-defined. The reason for this is that, in the point-charge case $\rho = \delta^3$, the Maxwell fields $F_{i,t}$ are not entirely smooth anymore but have a second order pole at $q_{i,t}$ which is exactly where they would have to be evaluated in $L_{ii,t}$. In order to distinguish the case of general $\rho$ from the point-charge case of $\rho = \delta^3$, we use the convention that lower-case fields $f_{i,t}$ solve the equations (3) and (4) for $\rho = \delta^3$, which then implies the relation $F_{i,t} = \rho * f_{i,t} + F_{i,0}$, where $*$ denotes the convolution and $F_{i,0}$ is a solution to the free Maxwell equations, i.e. (3) and (4) for $\rho = 0$. To see the divergent behavior of $f_{i,t}$, thanks to the linearity, it suffices to regard a special solution to (3) and (4) for a fixed charge trajectory $(q_i, p_i) : t \mapsto (q_{i,t}, p_{i,t})$. In the following we drop the index $i$ to keep the notation slim. Two well-known solutions of (3) and (4) are the advanced and retarded Liénard–Wiechert fields $f^\pm_{i,t} [q, p] = (e^\pm_i, h^\pm_i)$, where the square bracket notation emphasizes the functional dependence on the charge trajectory $(q, p)$. They are given by
\begin{equation}
e_i^\pm(x) := \frac{(n \pm v)(1 - v^2)}{|x - q|^2 (1 \pm n \cdot v)^2} + \frac{n \wedge [(n \pm v) \wedge a]}{|x - q|(1 \pm n \cdot v)^3},
\end{equation}

where we have used the abbreviations

\begin{equation}\begin{aligned}
q^\pm &:= q_{, i}^\pm, & v^\pm &:= v(p_{, i}^\pm), & a^\pm &:= \frac{d}{dt} v(p_{, i=1}^\pm), \\
n^\pm &:= \frac{x - q^\pm}{|x - q^\pm|}, & \tau^\pm &:= \tau \pm |x - q^\pm|;
\end{aligned}\end{equation}

see [2, 9]. All other solutions \(f_i\) to (3) and (4) for the same trajectory \((q, p)\) can then be represented as

\begin{equation}f_i = \lambda f_i^\pm[q, p] + (1 - \lambda) f_i^0[q, p] + f_i^D(7)
\end{equation}

for \(\lambda \in [0, 1]\), where \(f_i^D\) is a solution to the corresponding homogeneous equations, i.e., (3) and (4) for \(\rho = 0\). For smooth \(f_i^D\), the explicit expressions in (5) imply that all corresponding fields \(f_i\) are smooth on \(\mathbb{R}^3 \setminus \{q_i\}\) where they admit the discussed singular behavior that renders the term \(L_{ii, \tau}\) in (2) ill-defined for \(\rho = \delta^3\).

To still make sense out of this ill-defined self-interaction, an informal mass renormalization argument is usually employed, see [10], which effectively replaces the problematic term \(L_{ii, \tau}\) with the finite Abraham–Lorentz–Dirac back reaction \(L_{ii, \tau}^{\text{ALD}}\). In the non-relativistic regime, the latter may be approximated by \(L_{ii, \tau}^{\text{ALD}} \approx \pm e^2 \bar{q}_{, i}\), with \(e\) denoting the electric charge. This procedure cures the original problem, however, introduces a dynamical instability as for almost all but very special initial accelerations, which now must be provided along with initial positions and momenta, the corresponding charge trajectories approach the speed of light exponentially fast. Nevertheless, it was shown that the subset of physically sensible solutions can be well approximated in certain regimes by a dynamically stable version that was suggested by Landau and Lifschitz; see [9].

After replacing the ill-defined term \(L_{ii, \tau}\) appropriately or simply omitting it by setting \(e_i = 1 - \delta_i\), which often can be justified as its renormalized version is usually small (e.g., for small acceleration, jerk, and electric charge), one might hope that there are no further obstacles in arriving at a solution theory for the Maxwell–Lorentz system (1)–(4) in the point-charge limit \(\rho \to \delta^3\). A general proof of the well-posedness of the corresponding initial value problem, however, is difficult and remains open. The first two difficulties are obvious: (1) The charges must not collide, otherwise \(|x - q^\pm|^{-2}\) in (5) blows up; and (2) the charges must not approach the speed of light too fast, otherwise the factors \((1 \pm n^\pm \cdot v^\pm)^{-3}\) in (5) may blow up. Mathematically, difficulty (1) poses a similarly delicate problem as in the \(N\)-particle problem of gravitation, only now with the additional complication that the Coulomb potentials in (5) are Lorentz-boosted and to be evaluated at delayed or advanced times \(\tau^\pm\) as given in (6). Difficulty (2) is due to the accumulation of the escaping fields along the light cone and must be excluded with an \(a \text{ priori}\) bound on the charge velocities. When handled with care, it is reasonable to expect that at most only very few initial values \((q_{i, 0}, p_{i, 0}, f_{i, 0})_{1 \leq i \leq N}\) lead to catastrophic events due to these two difficulties. However, there is a third difficulty which is more subtle and, to our knowledge, has not received attention yet. Given a charge trajectory \((q, p)\), only rather special initial fields \(f_{i, 0}\) give rise to solutions \(f_{i, \tau}\) to (3) and (4) that are sufficiently regular outside a neighborhood of \(q_{i, \tau}\) in order to be evaluated in the terms \(L_{ii, \tau}\) in (2) for all times. Generic initial fields will generate singular fronts in the fields traveling at the
speed of light, and another charge \( j \) having velocities below the speed of light is bound to traverse such fronts in finite time.

In section 2 we explain the mathematical origin of this questionable artifact and discuss how solutions with singular light fronts can be ruled out by appropriate restrictions on the initial values. In section 3, for the point-charge case \( \rho = \delta^3 \), we give necessary conditions for global existence of piecewise as well as globally smooth solutions to the Maxwell–Lorentz system (1)–(4). We discuss that this point-charge phenomenon has a straight-forward analog in quantum field theory, and furthermore, implications on the case of extended charges \( \rho \). In the latter, the singular light fronts qualitatively persist in a mathematical procedure for finding admissible initial fields despite this fact. The latter, however, introduces an unwanted arbitrariness which, as we suggest in section 5, can be eliminated by physical reasoning. The resulting restrictions on the initial values naturally turn the equations of motion of the Maxwell–Lorentz system (1)–(4) into a class of delay differential equations that include the Fokker–Schwarzschild–Tetrode equations of motion of Wheeler–Feynman electrodynamics [11–15] and the Synge equations [16] as prime examples.

### 2. Singular light fronts in the electrodynamic fields

Let us assume for a moment that, at least for a neighborhood around \( t = 0 \), the Maxwell–Lorentz system (1)–(4) has a solution with actual charge trajectories \((q_i, p_i)\) and fields \( F_i : t \mapsto F_{i,t} = (E_{i,t}, B_{i,t}) \) for \( i = 1, \ldots, N \). The goal in this section is to introduce explicit formulas for those fields \( F_i \), depending on their corresponding trajectory \((q_i, p_i)\) and initial field \( F_{i,0} \), in order to infer the properties of general solutions to (3) and (4). Since we can always retrieve from the point-particle fields \( F_i \) the ones of the extended charges by convolution, \( F_{i,t} = \rho \ast f_{i,t} + F_{i,0} \), we will consider the point-particle case \( \rho = \delta^3 \) only. Furthermore, we drop the index \( i \) in this entire section because all the computations hold for any given charge \( i \). An explicit expression for \( f_i \) solving (3) and (4) for trajectory \((q, p)\) and initial field \( f_0 \) can be found by recasting the Maxwell equations in an integral form that reads

\[
f_i = f_i^{(1)} + f_i^{(2)},
\]

\[
f_i^{(1)} := \left( \frac{\partial}{\partial t} \right) \left( \nabla \wedge \frac{\partial}{\partial \nabla} \right) K_{t-t_0} \ast f_0,
\]

\[
f_i^{(2)} := 4\pi \int_{t_0}^{t} ds \left( \nabla \wedge \frac{\partial}{\partial t} \right) K_{t-s} \ast \left( \delta^3(\cdot - q_s) \right) \nabla \delta^3(\cdot - q_s).
\]
\[ K_t^- = K_t^+ - K_t^\pm \quad \text{for} \quad K_t^\pm = \frac{\delta(|t| \pm t)}{4\pi|t|}, \]

where \( K_t^\pm \) are the advanced and retarded Green’s functions of the d’Alembert operator. See appendix for details on the derivation of (8) from Kirchhoff’s formulas. Note that by virtue of the Maxwell equations (3), the Maxwell constraint (4) is preserved over time, which then also holds in this integral form (8).

Before we begin with an investigation of the properties of the general Maxwell field (8), it is illustrative to look at a simple example that shows how singular light fronts arise. Considering the Maxwell constraint (4), one might think that an obvious candidate for a good initial field \( f_0 \) is given by the Coulomb field

\[ f_0(x) = (v_0(x), b_0(x)) = \left( \frac{x - q_0}{|x - q_0|^3}, 0 \right). \]

Plugging the explicit form of the initial field (12) into (9) and the actual trajectory \((q, p)\) into (10) allows to compute (8). The corresponding solution \( f_t \) to (3) and (4) reads

\[ f_t = 1_{B_{t|1}(q_0)} f_t^{\sigma(t)}[q, p], \]

\[ + 1_{B_{0}(q_0)} F^t, \]

\[ + r_t^{\sigma(t)}[q_0, p_0], -r_t^{\sigma(t)}[q_0, 0], \]

using

\[ r_t^\pm[ q_0, p_0][x] = \frac{\delta(|t| - |x - q_0|)}{(1 \pm n_0 \cdot v_0)|x - q_0|} (n_0 \pm v_0), \]

\[ n_0 := \frac{x - q_0}{|x - q_0|}, \quad v_0 := v(p_0), \]

where, with slight abuse of the introduced square bracket notation, this time the arguments in the square brackets in (16) are not functions but just position and momentum \( q_0, p_0 \in \mathbb{R}^3 \), respectively. Furthermore, \( \sigma(t) \) denotes the sign of \( t \), i.e., \( f_t^{\sigma(t)} \) stands for \( f_t^t \) if \( t \geq 0 \) and for \( f_t^+ \) if \( t < 0 \), and \( 1_{B_{t|1}(q_0)}(x) \) denotes the characteristic function being one for \( x \) in the closed ball \( B_{t|1}(q_0) \) of radius \(|t|\) around \( q_0 \) and zero for \( x \) in the open set \( B_{t|1}(q_0) = \mathbb{R}^3 \setminus B_{t|1}(q_0) \). For the details regarding the computation we refer the reader to the appendix. The result shows that, according to (13), inside the light cone of space–time point \((t, x) = (0, q_0)\) the new advanced/retarded Liénard–Wiechert field generated by the charge trajectory \((q, p)\) builds up as expected while, according to (14), in this region the initial Coulomb field \( f_0 \) given in (12) is displaced. The field \( f_0 \) then persists only outside of that light cone. In addition, one finds two distributions in (15) that depend on the Newtonian initial data \((q_0, p_0)\) only and have support exclusively on the light cone. See figure 1 for an illustration.

By inspecting (13)–(15) more closely one thus finds that in general the Maxwell field \( f_t \) is not smooth on \( \mathbb{R}^3 \setminus \{q_0\} \) although the initial field \( f_0 \) in (12) is. On the contrary, for most charge trajectories \((q, p)\) the field will express singular fronts on the light cone of \((0, q_0)\) because: (1) the distributions \( r^{\sigma(t)}[q_0, p_0] \) and \( r^{\sigma(t)}[q_0, 0] \) in (15) cancel only if \( p_0 = 0 \); and (2) the remaining terms (13) and (14) only connect continuously on this light cone if at least the acceleration \( \dot{v}(p_0) \) vanishes. Otherwise, the field \( f_t \) will have a discontinuity there.

At first sight this phenomenon may seem surprising. However, it has a rather simple explanation. Morally, the initial Coulomb field \( f_0 \) in (12) corresponds to the field generated by
a charge at rest at position \( q_0 \). More precisely, \( f_0 \) is the retarded Liénard–Wiechert field generated by an auxiliary charge trajectory \((q, \dot{q})\) fulfilling \( \ddot{q}_0 = q_0 \) and \( \dot{p}_0 = 0 \) for \( t \leq 0 \). If the actual charge trajectory \((q, p)\) does not connect smoothly to \((q, \dot{q})\) at \( t = 0 \) but admits a kink, this sudden change of acceleration will result in a radiation field traveling along the light cone of \((0, q_0)\). Should \( p_0 \) be non-zero, an infinite acceleration is necessary to change the momentum from \( \dot{p}_0 = 0 \) to \( p_0 \), and the corresponding radiation gives rise to the distributions (16), whereas a step in the acceleration merely causes a discontinuity on the light cone. This simple example demonstrates that in order to prevent singular light fronts, a compatibility condition between the initial field \( f_0 \) and the actual trajectory \((q, p)\) has to be met.

In the rest of this section, the objective is to identify a necessary compatibility condition for the general case. According to the general splitting in (7), also any relevant initial field \( f_0 \), obeying the Maxwell constraint (4), can be written in the form

\[
f_0 = \lambda f_0^r [q, \dot{p}] + (1 - \lambda) f_0^a [q, \dot{p}] + f_0^0,
\]

for some \( \lambda \in [0, 1] \), \( f_0^s [\dot{q}, \dot{p}] \) being the Liénard–Wiechert fields (5) generated by a smooth auxiliary charge trajectory \((q, \dot{q})\) fulfilling \( \ddot{q}_0 = q_0 \). Note that, given any general initial field \( f_0 \), equation (18) is merely a definition of \( f_0^0 \) which must then be a homogeneous field, i.e., one fulfilling the Maxwell constraint (4) for \( \rho = 0 \). As this free field \( f_0^0 \) propagates independently of the charges, nevertheless, influences them, it is reasonable to assume its initial value \( f_0^0 \) to be smooth (which implies \( f_0^0 \) to be smooth) to avoid additional difficulties—less regularity of \( f_0^0 \) and \((q, \dot{q})\) suffices, but this is not our focus here. Plugging the actual trajectory \((q, p)\) and the initial field \( f_0 \) in the form of (18) into the explicit expressions (8)–(10) above, one finds

\[
f_i = \mathbb{1}_{\rho_0(q_0)} \lambda f_i^s [q, \dot{p}] + \mathbb{1}_{\rho_0(q_0)} (1 - \lambda) (f_i^r [q, \dot{p}] - f_i^{-s}(q, \dot{p}]),
\]

\[
+ \mathbb{1}_{\rho_0(q_0)} \lambda (f_i^r [q, \dot{p}] - f_i^{-s}(q, \dot{p})],
\]

\[
+ \mathbb{1}_{\rho_0(q_0)} (1 - \lambda) f_i^s [q, \dot{p}] + (1 - \lambda) f_i^r [q, \dot{p}] + f_i^0.
\]

The details are again given in the appendix. The first three terms have support inside and on the light cone of \((0, q_0)\). The term (19) describes the field that is generated by the actual charge trajectory \((q, p)\) between time 0 and \( t \), and terms (20) and (21) describe how the initial advanced and retarded Liénard–Wiechert fields encoded in (18) are propagated inside the light cone. Depending on the sign of \( t \), one of the terms (20) and (21) will vanish and the respective other will be proportional to the difference \( \sigma(t)(f_i^r [q, \dot{p}] - f_i^{-s}(q, \dot{p})] \), which according to Dirac [10] can be interpreted as the radiation emitted or absorbed by the auxiliary charge trajectory \((\dot{q}, \dot{p})\) between time 0 and \( t \). Moreover, the term (22) is the propagated remainder of the initial retarded and advanced Liénard–Wiechert fields, and therefore, only has support outside the light cone. The terms in (23) are again the distributions given in (16) having support on the light cone, and \( f_i^0 \) in (24) is simply the field \( f_i^0 \) propagated from 0 to \( t \) by the free Maxwell equations, i.e., (3) and (4) for \( \rho = 0 \). Note that \( f_i^0 \) is as regular as \( f_0^0 \). See figure 2 for an illustration of the trajectories and supports of the terms (19)–(24). The solution \( f_i \) in (19)–(24) can be recast in a more compact form
\[ f_t = 1_{\mathbb{R}^3\setminus\{q\}_t}(\mathcal{F}^{-\sigma(t)}[q, p] - \mathcal{F}^{-\sigma(t)}[\tilde{q}, \tilde{p}]), \]  
\[ + \lambda f_t^{-}[\tilde{q}, \tilde{p}] + (1 - \lambda)f_t^+[\tilde{q}, \tilde{p}], \]  
\[ + r_t^{-\sigma(t)}[q_0, p_0] - r_t^{-\sigma(t)}[\tilde{q}_0, \tilde{p}_0], \]  
\[ + f_t^0, \]  

from which one can read off necessary compatibility conditions between the initial field \( f_0 \) and the charge trajectory \((q, p)\) that prevent the development of singular light fronts:

(C1) The distributions (27) must cancel each other because neither (25), (26), nor (28) contain Dirac delta distributions. This is the case if and only if \((\tilde{q}_0, \tilde{p}_0) = (q_0, p_0)\), where \(\tilde{q}_0 = q_0\) was already assumed in order to fulfill the Maxwell constraint (4).

(C2) Provided (C1) is fulfilled, the field \( f_t \) is continuous on \(\mathbb{R}^3\setminus\{q\}_t\) if and only if term (25) vanishes on the light cone of \((0, q_0)\). This can be seen as follows: by virtue of (5), for all times \( t \) the terms (25) and (26) are smooth everywhere except maybe on the light cone of \((0, q_0)\) as well as the points \(q_{\pm}\) and \(\tilde{q}_{\pm}\). However, since these terms coincide with (19)–(22), they must be smooth in \(\tilde{q}_{\pm}\) as (20) and (21) are free fields and (22) has only support outside of the light cone of \((0, q_0)\). As the free field \( f_t^0 \) is smooth, and by (C1) terms (27), (23) vanish, the field \( f_t \) is continuous on \(\mathbb{R}^3\setminus\{q\}_t\) if and only if (25) vanishes on the light cone of \((0, q_0)\). This is the case if and only if the accelerations \(\tilde{q}_{\pm} \) and \(\tilde{q}_{\pm} \) coincide at \( t = 0 \). Furthermore, if and only if all \(k\)th derivatives of \(\tilde{q}_{\pm} \) and \(q_{\pm} \), for \( l = 1, \ldots, k + 2 \) coincide at \( t = 0 \), the field \( f_t \) has \( k \) spatial derivatives on \(\mathbb{R}^3\setminus\{q\}_t\). Finally, if and only if the trajectories \(\tilde{q}_{\pm} \) and \(q_{\pm} \) connect smoothly at time \( t = 0 \), the field \( f_t \) is smooth on \(\mathbb{R}^3\setminus\{q\}_t\).

It was called to our attention that also in [17, 18], where a rigorous electrodynamic point-charge limit was studied in the dipole approximation, a condition relating the initial fields and initial momenta similar to (C1) was needed to ensure convergence.

### 3. Implications on the Maxwell–Lorentz system

In this chapter we discuss the implications of the observations made in section 2 on the fully coupled system of Maxwell’s and Lorentz’s equations (1)–(4). Our main interest, which will be discussed first, lies in the case of \( N \geq 2 \) point-like charges, i.e., \( \rho = \delta^3 \), either with a properly renormalized self-interaction term \( \mathcal{L}_{\text{int}} \) or without it, i.e., \( \epsilon_{ij} = 1 - \delta_{ij} \). The implications on the Maxwell–Lorentz system for smooth extended charges \( \rho \) are considered in the end.

First and foremost, we observe that in a system of at least two charges, one charge, say number 2, will inevitably cross the light cone of the initial space–time point of another charge, say number 1, at a time \( t^* \), which is bounded from below by the minimal distance divided by speed of light; see figure 3. Thus, at \( t = t^* \) the Lorentz force (1) felt by charge 2 must evaluate the field \( f_{\mathbb{R}^3\setminus\{q\}} \) at some point on the light cone of \((0, q_{0})_t \). Recall that for an initial field \( f_{\mathbb{R}_t} \) of the form (18) with auxiliary charge trajectory \((\tilde{q}_t, \tilde{p}_t) \), the propagated field \( f_{\mathbb{R}^3\setminus\{q\}} \) is given by (19)–(24). Should condition (C1) of section 2 not be satisfied, this evaluation is ill-defined because of the presence of the distributions (23). In this case, the dynamics will cease to exist beyond the time instant \( t^* \). Hence, (C1) is a necessary condition for global existence of solutions to the Maxwell–Lorentz system (1)–(4). Should condition (C1) hold but not (C2), then the force on charge 2 will undergo a discontinuous jump when traversing the light cone.
at time $t^*$. Therefore, (C2) is a necessary condition for having continuous or smooth solutions to the Maxwell–Lorentz system (1)–(4).

The following two arguments illustrate that (C1) and (C2) are violated for generic initial data $(q_{0,i}, p_{0,i}, f_{0,i})_{1 \leq i \leq N}$ obeying the Maxwell constraints (4) only. Precisely, they show that global existence is not stable under arbitrarily small perturbations of the initial data. For this purpose, let us assume that $(q_{0}, p_{0}, f_{0})_{1 \leq i \leq N}$ is a global solution to the Maxwell–Lorentz system (1)–(4) for some initial value $(q_{0,i}, p_{0,i}, f_{0,i})_{1 \leq i \leq N}$ such that the initial fields $f_{0,i}$ are of the form (18) for some smooth auxiliary trajectory $(\hat{q}_{i}, \hat{p}_{i})$ and some smooth initial free field $\hat{f}_{0,i}$. Recall from our discussion in section 2 that any relevant initial field can be written in this form, and then, it automatically fulfills the Maxwell constraint (4).

No-go argument (A1): By Maxwell constraints (4) and necessary condition (C1) we have $(\hat{q}_{i,0}, \hat{p}_{i,0}) = (q_{0,i}, p_{0,i})$ for $i = 1, \ldots, N$. Then, perturbing the initial momentum of charge 1 by $p_{1,0} \to p'_{1,0} = p_{1,0} + \delta$ for any vector $\delta$ of arbitrarily small norm $|\delta| > 0$ leads to a corresponding local solution $(q'_{i,0}, p'_{i,0}, f'_{i,0})_{1 \leq i \leq N}$ with $f'_{1,0}$ taking the form of (19)–(24), whereas the contribution (23) equals the distribution $(r_{1}^{-\pi(0)}[q_{0}, p_{0}] - r_{1}^{-\pi(0)}[q_{0}, p_{0}])$, which does not vanish. In other words (C1) is violated, and $f'_{1,0}$ manifests a singular light front with support on the light cone of space-time point $(0, q_{0,0})$, as discussed in section 2. By virtue of (1)–(4), this perturbation in the initial momentum propagates not faster than the speed of light. In particular, the perturbed field $f'_{1,t}$ of charge 1 and the perturbed trajectory $(q'_{1,t}, p'_{1,t})$ of charge 2 remain identical on $B_{in}(q_{0}, 0)$ for $t \in \mathbb{R}$. In consequence, charge 2 is bound to touch the light cone of $(0, q_{1,0})$ at the very same time $t^*$ as in the unperturbed solution, only now the perturbed field $f'_{1,t}$ contains a singular light front consisting of distributions. In conclusion, the

![Figure 1](https://example.com/figure1.png)

Figure 1. This figure illustrates the supports of the terms in (13)–(15) making up the solution $f_{t}$ to the Maxwell equations (3) and (4) for an initial Coulomb field $f_0$ given in (12) and some fixed charge trajectory $(q, p)$. Inside the light cone of $(0, q_{0})$ (blue area) the initial field $f_0$ is displaced by the retarded/advanced Liénard–Wiechert field $F^{-\pi(0)}(q, p)$ in (13). Outside that light cone (white area) the initial Coulomb field (14) persists. The distribution valued terms (15) are located only on the light cone.
dynamics will cease to exist beyond time \( t^* \), as discussed above. The argument is depicted in figure 4.

**No-go argument (A2):** This time, let us assume the global solution is also smooth and \( \lambda > 0 \) (for \( \lambda = 0 \) a similar argument can be found). Due to condition (C2), \( \tilde{q}_{1,t} \) and \( \tilde{q}_{2,t} \) coincide at time \( t = 0 \). Now, we perturb a little bit the trajectory \((\tilde{q}_{2,t}, \tilde{p}_{2,t})\) that defined the initial field \( f_{2,0} \) given in (18) in an arbitrarily small neighborhood of the retarded time \( \tau^* \) belonging to space–time point \((0, q_{2,0})\). Due to (19)–(24) this causes a small perturbation \( f_{2,0} \to f_{2,0}' \), and we tune this perturbation such that the Lorentz force (1) on charge 1 at \( t = 0 \) changes its value. In consequence, the potential local solution \((q_{1,t}', \rho_1', f_{1,0}')_{t \leq \tau} \) corresponding to this perturbed initial data violates (C2) as the accelerations \( \dot{q}_{1,t} \) and \( \dot{q}_{2,t}' \) do not match anymore at \( t = 0 \). As discussed in section 2, this leads to a discontinuity on the light cone of \((0, q_{2,0})\). However, by virtue of (1)–(4) the perturbed field \( f_{1,t}' \) of charge 1 and the perturbed trajectory \((q_{2,0}', \rho_{2,0}', f_{2,0}') \) of charge 2 remain identical on \( B_{\tau}(q_{1,0}) \) for \( t \in \mathbb{R} \). Therefore, charge 2 is bound to hit the light cone of \((0, q_{2,0})\) at the very same time \( t^* \) as in the unperturbed solution.

At this instant, due to the discontinuity of \( f_{1,t}' \), the acceleration of charge 2 will undergo a likewise discontinuous jump. Hence, should the perturbed solution exist globally it can only be piecewise smooth. Furthermore, the discontinuity in the acceleration of charge 2 will give rise to a corresponding discontinuity in the field \( f_{2,t} \) on the light cone of \((t^*, q_{2,t})\), which charge 1 is bound to cross eventually. By this mechanism, a whole network of singular light fronts is developed. The argument is depicted in figure 5.

These two arguments indicate that the initial value problem of the Maxwell–Lorentz system (1)–(4) with renormalized (or without) self-interaction term is ill-posed for general initial values \((q_{2,0}, \rho_{2,0}, f_{2,0})_{t \leq \tau} \) only fulfilling the Maxwell constraints (4): even if a global solution is found, only a small perturbation in the initial values suffices to prevent either global existence, by (A1), or global smoothness, by (A2), of the potential solution corresponding to the perturbed initial values.

One might tend to think that these are all problems connected to the point-like nature of the charges, a concept that could even be considered questionable in the classical regime.

Indeed, as discussed in the introduction, it is true that for the Maxwell–Lorentz system of smoothly extended charges those mathematical problems do not show up. Nevertheless, the qualitative behavior of generation of singular light fronts for initial conditions that violate (C1) or (C2) remains the same. As the fields of the extended charges are of the form \( F_{i,t} = \rho \ast f_{i,t} + F_{i,0} \), the discussed singular fronts are now only smeared out by the charge density \( \rho \). For \( \rho \) supported on the scale of the classical electron radius, i.e., \( r_e \sim 10^{-15} \text{ m} \), the singular fronts will still result in sharp—though smooth—steps in the fields on the respective light cones. Other charges are bound to eventually traverse such steps and will suddenly—on time scales of \( r_e \) divided by their respective speed—start or stop to radiate, thus, leading to potentially observable though physically questionable phenomena.

Furthermore, it is interesting to note that the singular light fronts persist also in quantum field theory. This can readily be observed in the following toy model in which a fixed source at \( q \in \mathbb{R}^3 \) interacts with a second-quantized and massless scalar field

\[
\varphi(t, x) = \int d^3k \frac{(2\pi)^{-\frac{1}{2}}}{\sqrt{2|k|}} (a_k e^{ik\cdot x - i\omega_k t} + \text{c.c.})
\]

by means of the interaction Hamiltonian \( H_I(t) = g\varphi(t, q) \), where \( g \in \mathbb{R} \) and \( a_k, a_k^\dagger \) are the bosonic creation and annihilation operators fulfilling the CCR \([a_k, a_{k'}^\dagger] = \delta^3(k - k')\). Formally, \( H_I(t) \) is not well-defined without an ultraviolet cut-off such as \( \rho \) but for the sake of
the argument it is sufficient to continue informally. Let \( U_t(t) \) be the time evolution generated by \( H(t) \), then for any initial unit Fock state \( \Psi \) we get

\[
\langle \Psi | U_t(t)^\dagger \varphi(t, \mathbf{x}) U_t(t) | \Psi \rangle = -g \delta^3(\mathbf{x} - \mathbf{q}).
\]

The expectation value of this scalar field can therefore be represented by means of Kirchhoff’s formulas as it was done for the Maxwell field in the appendix. In the simplest case of an initial vacuum \( |\Psi\rangle = |0\rangle \) one finds

\[
\langle 0 | U_t(t)^\dagger \varphi(t, \mathbf{x}) U_t(t) | 0 \rangle = -g \int_0^t ds \langle K_{s-x} \ast \delta^3(-q) \rangle(\mathbf{x}) = -\frac{g}{4\pi} \frac{1_{ \mathbb{R}^3/q}(\mathbf{x})}{|\mathbf{x} - \mathbf{q}|},
\]

where the discontinuity on the light front shows up again as the field is built up over time starting from an initial vacuum. This behavior only disappears for special initial \( |\Psi\rangle \), precisely, the ground state of this toy model, which can be computed explicitly, plus smooth additional free fields. If we further allow the charge \( \mathbf{q} \) to move, very similar scenarios as discussed in section 2 can be constructed; but this shall not be our focus here.

In conclusion, for any choice of \( \rho \), and be it for mathematical or physical reasons, it seems desirable to restrict the space of initial values of the Maxwell–Lorentz system \( (1) \) beyond the Maxwell constraints \( (4) \).

4. Admissible initial values

If for a moment we also admit piecewise smooth solutions to the Maxwell–Lorentz system \( (1) \)–\( (4) \), a sensible restriction on the space of initial values can be taken from condition \( (C1) \). If we require the initial value \( (q_{l,0}, p_{l,0}, f_{l,0})_{l \in \mathbb{N}} \) to comprise fields \( f_{l,0} \) of the form \( (18) \) for...
piecewise smooth auxiliary trajectories \((\tilde{q}, \tilde{p})\) fulfilling \((\tilde{q}_0, \tilde{p}_0) = (q_0, p_0)\), condition (C1) as well as the Maxwell constraints (4) are fulfilled by definition and there seems to be no further obstacle concerning mathematical well-posedness of the respective initial value problem.

If, however, we demand smooth global solutions, we would also need to comply with condition (C2). In order to do so we would have to know the derivatives of the charge trajectories \((q, p_i)\) at initial time \(t = 0\). But those are unknown as they already require knowledge of a local solution in a neighborhood of \(t = 0\). Hence, there is no possibility to restrict the space of initial fields a priori in order to ensure well-posedness.

As a workaround one may consider the following approach: given initial data \((q_{i,0}, p_{i,0}, f_{i,0})\) \(\in \mathbb{N}\) fulfilling (4) and (C1), it is possible to compute the solution of the Maxwell–Lorentz equations in a sufficiently small time interval \([0, \tau]\). This can be done as the singular fronts live only on the light cones of the initial space–time points \((0, q_{i,0})\) so that \(\tau\) only has to be chosen smaller than the smallest time \(t^*\) when some charge hits a singular front. This preliminary local solution allows to compute all derivatives of the charge trajectories \((q, p_i)\) at \(t = 0\), and hence, it would allow to adapt the auxiliary trajectories \((\tilde{q}, \tilde{p})\) in a neighborhood of \(t = 0\) to connect smoothly to \((q, p_i)\) such that (C2) is fulfilled. This procedure changes the initial fields \(f_{i,0} \rightarrow f'_{i,0}\) in a spatial neighborhood around the initial positions \(q_{i,0}\). If self-interaction is excluded, the adapted initial values \((q_{i,0}, p_{i,0}, f'_{i,0})\) \(\in \mathbb{N}\), however, fulfill the Maxwell constraints (4), (C1), and (C2), and therefore, should not bare any further obstacles concerning smooth global solutions. If self-interaction is included, the above procedure would have to be iterated until a fixed-point is found as the change in \(f'_{i,0}\) implies again a change in the initial acceleration of charge \(i\).

Though mathematically sound, physically, this is a rather opaque procedure. It is not anymore a formulation of classical electrodynamics in terms of an initial value problem for (1)–(4) but in terms of an initial guess, that, first, has to be adapted in a quite arbitrary way before a global solution can be inferred at all.

So what is overlooked when naively regarding the Maxwell–Lorentz system (1)–(4) as an initial value problem? Any inhomogeneous solution \(f_{i,t}\) to the Maxwell equations (3) and (4) is of the form (7), which implies that the entire history of the charge trajectory \((q, p_i)\) is already encoded in the spatial dependence of the field \(f_{i,t}\); recall the \(t^+\) dependence in (5). Now, if we set some initial field \(f_{i,0}\) by hand, for which the Maxwell constraint (4) only requires that we choose it of the form (18) with some auxiliary trajectory \((\tilde{q}, \tilde{p})\) fulfilling
\( \tilde{q}_0 = q_0 \), the Maxwell time evolution is fooled to believe that the history of the charge trajectory is given by \( \tilde{q}_0, \tilde{p}_0 \). But except for \( \tilde{q}_0 = q_0 \), the history of the auxiliary trajectory \( (\tilde{q}_0, \tilde{p}_0) \) may have nothing in common with the actual one \( (q_0, p_0) \), which is to be computed. As a matter of fact, the Maxwell equations propagate such an initial field \( f_{0,0} \) as if it was generated by the auxiliary charge trajectory \( (\tilde{q}_0, \tilde{p}_0) \) outside the light cone of \( (q_0, p_0) \) while, inside, a new field is generated according to the actual trajectory \( (q_0, p_0) \). It is therefore not surprising that the incompatibilities between the actual charge trajectories \( (q_0, p_0) \) and the initial fields \( f_{0,0} \) of the solution (1)–(4) discussed in section 2 occur during the dynamics and that any mismatch between the actual and auxiliary charge trajectories in the sense of (C1) and (C2) expresses itself as a singular light front.

In view of this, it would be desirable to find a formulation of classical electrodynamics that automatically avoids any such incompatibilities. This is possible and in section 5 we discuss a whole class of such formulations having two representatives that are well-known since the beginning of classical electrodynamics. But first, we end this section with a short example that illustrates quantitatively (1) that the phenomenon of singular light fronts can lead to significant radiation effects and (2) how the initial fields encode the histories of their respective charge trajectories:

Quantitative example: We reconsider the introductory example from section 2 of a charge, referred to as charge 1, having an initial position and momentum \( (q_{1,0}, p_{1,0}) \) and an initial Coulomb field \( f_{1,0} \), i.e., (12) Lorentz-boosted w.r.t. \( \nu(p_{1,0}) \). Whatever its future trajectory \( (q_1, p_0) \) may be, its field \( f_{1'0} \) will be of the form (13)–(15) as depicted in figure 1. In the following we will use SI units and charges smeared out by \( \rho \) instead of point-charges, hence we regard \( F_{1,0} = \rho * f_{1,0} \). Let us suppose that the other initial fields \( F_{j,0} \) of all \( j = 2, \ldots, N \) other charges do not comprise free fields, i.e., \( F^0_{j,0} = 0 \), and are such that charge 1 experiences a large initial acceleration, say, \( a_{1,0} \sim (10^{17}, 0, 0) \) m s\(^{-2}\). Charge 2 is assumed to have initial position and momentum \( (q_{2,0}, p_{2,0}) \) at a sufficiently large distance, say, \( q_{2,0} = (0, 10^2, 0) \) m, so that, initially, it moves almost freely with velocity, say, \( |\nu(p_{2,0})| \sim 10^4 \) m s\(^{-1}\). Eventually, it will reach the vicinity of the light cone of \( (0, q_{1,0} = 0) \) smeared out by \( \rho \). Now, if charge 1 and 2 are made out of clouds of, say, \( Z = 10^{13} \) electrons produced by electron guns and collimated to balls of diameter \( (\rho) \sim 10^{-2} \) m, the acceleration...
of charge 2 is

\[ a_{2,t^*} \approx \left( \frac{e}{m} E_{1,t^*}^x, 0, 0 \right) \sim (-10^{14}, 0, 0) \text{ m s}^{-2}, \]

where \( t^* \) is the time of arrival inside the smeared out light cone and \( E_{1,t^*}^x \) is the \( x \)-component of the electric field \( E_{1,t^*} \) computed with the help of (13) and (5):

\[ E_{1,t^*}^x \approx \frac{eZ}{4\pi\varepsilon_0} \frac{-|a_{1,0}|}{c^2|q_{2,t^*} - q_{1,0}|}. \]

Note that, since for simplicity we assumed that the initial Coulomb field \( f_{1,0} \) was already properly Lorentz-boosted, the distributions (15) cancel in contrast to the introductory example in section 2. In other words, (C1) is fulfilled which makes (13) the only contribution to the field. As a result of this analysis, we find that within the time of traversal of the smeared out light cone, \( \Delta t \approx \frac{\text{diameter}(\rho)}{|v_{2,0}|} \sim 1 \mu\text{s} \), there must be a sudden rise in emission of radiation of charge 2. According to Larmor’s formula, the resulting increase in power goes from almost zero to

\[ P_2 = \frac{2}{3} \frac{Z^2 e^3 |a_{2,t^*}|^2}{6\pi\varepsilon_0 c^3} \sim 1 \text{ W}. \] (29)

One may now wonder, why the flank in radiation power increase is so steep. As we discussed, the initial Lorentz-boosted Coulomb field of charge 1 encodes the history of a
charge with constant momentum \( p_{1,0} \). This tells the Maxwell dynamics that there must be a sudden change in acceleration at time \( t = 0 \) from zero to \( a_{1,0} \) in order to fit the initial data, and therefore, that a step in increase of radiation power (smeared out by \( \rho \)) must be produced. In a more realistic scenario, however, charge 1 would first have to acquire the initial acceleration \( a_{1,0} \) in the past \( t < 0 \); to match the above numbers, for instance, by entering a capacitor and falling through a voltage of \( U = 10^4 \text{ V} \) over a distance of \( s = 10^{-2} \text{ m} \). Depending on the duration of the acceleration process which may take considerably longer than \( \Delta t \approx 1 \text{ \mu s} \), a quite different initial field \( F_{1,0} \) is produced. This time, it consists of the former Lorentz-boosted Coulomb field plus a radiation part that was emitted during the process of acceleration. In contrast to the above scenario, this additional radiation will hit charge 2 much earlier before entering the light cone region, and instead of generating a steep flank in radiation power of charge 2 from zero to \( P_2 \) there will be a respectively smoother increase.

In other words, and independently of the fictitious numbers we have used above, this example shows that if the choice of initial values to the Maxwell–Lorentz system (1)–(4) imply a large initial acceleration for charge 1, which may later on enforce charge 2 to generate significant radiation (29), the cause that led to this large initial acceleration lies in the history of charge 1 which is encoded in the initial field \( F_{1,0} \). Hence, an un-natural choice for \( F_{1,0} \) such as the Lorentz-boosted Coulomb field above, which would imply that charge 1 did not accelerate for \( t < 0 \) but then suddenly does at \( t = 0 \), leads to the peculiar effect of the steep flank in radiation power increase of charge 2. Modeling more accurately how charge 1 acquired the initial acceleration in the past will eliminate this effect.

This demonstrates the intimate connection between the history of a charge trajectory and its generated field, which is the starting point of our discussion of the formulation of classical electrodynamics in section 5.

5. Conclusion

As demonstrated, in the case of point-charges, the restriction of the solution space of the Maxwell–Lorentz system (1)–(4) to smooth solutions does not allow a formulation in terms of an initial value problem. Though a potential global solution is uniquely identified by its initial data \( (q_{0,0}, p_{0,0}, f_{0,0}) \), only very special initial fields fulfilling the necessary condition (C2) lead to smooth global solutions. Furthermore, the information needed to restrict the initial data according to (C2) would already require knowledge of the unknown solution. Even for smooth charge distributions \( \rho \), neglect in matching the initial fields \( f_{0,i} \) to the history of the charge trajectories yields rather arbitrary differences in the predictions as the example in the preceding section illustrates. These circumstances suggest that we might need to change the way we look at the solution theory for the Maxwell–Lorentz system.

The starting point for such a consideration is the fact that the Maxwell field at one time instant and the entire trajectory of the charge that generated it are intimately intertwined beyond the Maxwell constraint (4). This can be observed best when imagining a single charge \( i \) incoming from the remote past \( t = -\infty \). Considering, e.g., the case \( \lambda = 1 \), any auxiliary trajectory \( (\bar{q}_i, \bar{p}_i) \) in the expression of the field \( f_{i,t} \) in (19)–(24) is forgotten during a time evolution from \( i = -\infty \) to any finite time \( t \) and so any potential singular fronts as they escape to spatial infinity with the speed of light. Concerning point-wise evaluation in any finite region of space-time, the Maxwell field in (19)–(24) reduces to the expression

\[
\begin{align*}
    f_{i,t} &= f^0_{i,t} + f^0_{i,t}.
\end{align*}
\]
Nothing changes in this argument and in the form of (30) when the charge trajectory \((q, p)\) is not restricted but also develops simultaneously to the evolution of the Maxwell fields, i.e., according to the fully coupled system (1)–(4). Hence, stopping the dynamics at time \(t = 0\) and starting it again in an initial value problem fashion dictates the natural choice (30) for the initial field at \(t = 0\). This means that the initial field \(f_{t,0}\) should be of the form (18) for an auxiliary trajectory \((\tilde{q}, \tilde{p})\) that coincides with the actual one \((q, p)\) and that the free field \(f_{t,0}^0\), as it evolves independently of the charges, equals the incoming free field evolved from \(t = -\infty\) to \(t = 0\).

Hence, in the general case for any \(\lambda \in [0, 1]\), where also advanced Liénard–Wiechert fields may occur, one would expect the Maxwell field to take the form

\[
f_{t,i} = \lambda f_{t,i}^+ [q, p, i] + (1 - \lambda) f_{t,i}^- [q, p, i] + f_{t,i}^0. \tag{31}
\]

Any compatibility condition, such as the Maxwell constraint (4), (C1), and (C2), is now naturally fulfilled for all times \(t\). But this comes at a high price. By (31), the fields \(f_{t,0}\) at time \(t = 0\) depend on the entire history of the charge trajectories which consequently means letting go of the initial value formulation of classical electrodynamics.

In view of the above, however, such a step seems well grounded. In section 3, it was already indicated when insisting on the merely mathematical property of smoothness of solutions. But there, one might even have been tempted to accept potential kinks in the charge trajectories, say, as long as they decay fast enough. However, the discussion above and in section 4 shows that there is also a physical reason why the initial value formulation is questionable, namely the fact that at each time instant the entire history of a charge trajectory is already encoded in the spatial dependence of its field. Therefore, when entertaining the thought that charges are incoming from the remote past, the form of the Maxwell fields is already presupposed by (31) and the space of potential solutions \((q, p, f)_{t \leq 0}^\infty\) of the Maxwell–Lorentz system (1)–(4) should consequently be restricted to solutions having Maxwell fields \(f_{t,i}\) that fulfill (31).

Such a restriction is easily implemented in the fundamental equations of motion (1)–(4). It simply means replacing the Maxwell fields on the right-hand side of (2) with the explicit form given in (31). This makes the Maxwell equations and constraints (3) and (4) redundant and turns the coupled system of the ODEs (1) and PDEs (3) and (4), only consisting of terms that are all evaluated at the same time instant \(t\), into the following system of ODEs that involve terms depending on advanced or delayed times \(t^\pm\) as given in (6):

\[
\frac{d}{dt} \begin{pmatrix} q_{t,i} \\ p_{t,i} \end{pmatrix} = \begin{pmatrix} v_{t,i} \\ \sum_{j=1}^N e_{ij} \mathbf{L}_{ij,t} \end{pmatrix},
\]

\[
\mathbf{L}_{ij,t} = \int d^3 x \rho(x - q_{t,i}) [E_{t,i}(x) + v_{t,i} \times B_{t,i}(x)],
\]

\[
\mathbf{F}_{t,i} = (E_{t,i}, B_{t,i}) = \rho * (\lambda f_{t,i}^+ [q, p, i] + (1 - \lambda) f_{t,i}^- [q, p, i]) + F_{t,i}^0. \tag{32}
\]

Here, \(F_{t,i}^0\) denotes any given solution of the free Maxwell equations. It is interesting to note that by virtue of (8)–(10) the free fields \(F_{t,i}^0\), when prescribed in the remote past, are forgotten should they have some spatial decay at spatial infinity [3, 19]. In this case, for \(\lambda = 1/2\), no self-interaction \(e_{ij} = 1 - \delta_{ij}\) or \(\mathbf{L}_{ii,t} = 0\), and for point charges \(\rho = \delta^3\), the system of equation (32) is equivalent to the Fokker–Schwarzschild–Tetrode equations [11–13] as used in Wheeler’s and Feynman’s investigation of classical radiation reaction [14, 15]. They can be derived from a simple action principle [11, 15], and furthermore, allow a derivation of Dirac’s radiation damping term \(\mathbf{L}_{ii,t}^{ALD}\) without the need of a mass renormalization procedure [14, 20].
Moreover, for λ = 1, point charges ρ = δ³, and no initial free fields, the resulting equations are equivalent to the Synge equations [16].

The nature of these equations, involving a priori unbounded state-dependent delays t±, cf (6), in the definition of the Liénard–Wiechert fields (5), renders a general classification of solutions very difficult. In mathematics, this problem is known as the electrodynamic N-body problem. To this day, global existence has only been established when considering two repelling charges and restricting the motion of the charges to a straight line [21–25, 30]. When constraining the charge trajectories at times |t| ≥ T, for arbitrary large but finite T, existence of solutions on [−T, T] was shown for N smoothly extended charges in three-dimensions [5, 26]. However, except for very special situations [22], almost nothing is known about uniqueness of solutions; see [20, 27]. It may turn out that solutions can only be identified uniquely when whole stripes of trajectories are specified. Nevertheless, such types of state-dependent delay differential equations are currently heavily under investigation in the contemporary mathematics literature (see, e.g., [28] and the references therein) and there is good reason to expect that their solution theory will soon be better understood.

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Appendix. Kirchhoff’s formulas and explicit expression for the Maxwell fields

In this appendix we explain how the solution formula (8)–(10) for Maxwell’s equations (3) and (4) can be derived from Kirchhoff’s formulas. Afterwards we demonstrate the main steps in the computation of the explicit expression (25)–(28) from which (19)–(24) and also (13)–(15) can be inferred. Again we omit the charge index and restrict ourselves to the point charge case ρ = δ³ from which the corresponding result for general ρ can be inferred; all equalities are meant in distribution sense.

The Maxwell equations (3), taking into account the constraint (4), imply the following inhomogeneous wave equation

\[
\Box f_t = 4\pi \left(-\nabla - \delta_t\right) \left(\delta^3(\cdot - q_t)\right)
\]

(A1)

Thanks to Kirchhoff’s formulas [29], the unique solution t ↦ A_t to the initial value problem

\[
\Box A_t = 0, \quad A_0 = A_{t_0 = 0}, \quad A_0 = \partial_t A_{t_0 = 0}
\]

(A2)

with \(K_t\) as given in (11). Applying this formula to system (A1) with initial values \(f_0\) and \(\partial_t f_{t_0 = 0} = (\nabla \wedge b_0 - 4\pi v_0 \delta^3(\cdot - q_0) \cdot \nabla \wedge e_0)\) at initial time \(t_0\) we find an expression for the unique solution given by

\[
f_t = f_t^{(1)} + f_t^{(2)},
\]

(A3)
\begin{equation}
\begin{aligned}
f^{(1)}_t &:= \left( \frac{\partial_t}{-\nabla \cdot} \nabla \cdot \right) K_{1, t_0} * f_0 \\
f^{(2)}_t &:= 4\pi \int_{t_0}^{t'} ds \left( \frac{-\nabla}{0} - \frac{\partial_t}{-\nabla \cdot} \right) K_{1, x} * \left( \delta^3(- q_s) \right).
\end{aligned}
\end{equation}

These formulas can also be found in [3, 9, 26]. Note, that in general \( f^{(1)}_t \) and \( f^{(2)}_t \) do not solve Maxwell’s equations (3) individually. In the limit \( t_0 \to \pm \infty \) and for \( f_0 \) having some spatial decay \( f^{(1)}_t \) vanishes and \( f^{(2)}_t \) coincides with the advanced/retarded Liénard–Wiechert field \( f^\pm [q, p] = (e^\pm, b^\pm) \) as given in (5), which are solutions of the Maxwell equations (3) and (4); see [19, 26].

Next, we discuss the computation of (25)–(28). Without loss of generality we set \( t_0 = 0 \), and exemplary compute the term in (A5) coming from the upper left entry in the matrix of differential operators evaluated at spacial point \( x \in \mathbb{R}^3 \). The other two terms can be inferred analogously. We compute

\begin{equation}
\begin{aligned}
4\pi \int_{t_0}^{t'} \! ds \left( \frac{-\nabla}{0} \delta^3(- q_s) \right)(x) \\
= -4\pi \int_{t_0}^{t'} \! ds \frac{1}{4\pi(t-s)} \int_{\partial B_{r_0}(0)} \! d\sigma(y) \nabla_y \delta^3(x - y - q_s) \\
= -\int_{t_0}^{t'} \! dr \frac{1}{r} \int_{\partial B_{r}(0)} \! d\sigma(y) \nabla_y \delta^3(x - y - q_{s, \pm}) \\
= -\int_{B_0(0)} \! d^3y \frac{1}{|y|} \nabla_y \delta^3(x - y - q_{r, \pm}).
\end{aligned}
\end{equation}

In the next step, we employ the identity

\[ \nabla \delta^3(y - y - q_{s, \pm}) = L(y)^\pm \cdot \nabla \delta^3(x - y - q_{s, \pm}), \]

for the matrix \( L(y)^\pm \) with entries

\[ L(y)^{ij}_{\pm} := -\delta_{ij} \pm \frac{n_i v_j}{|n|}, \]

where the indices \( i, j \) denote the components of the respective vectors and we have used the abbreviations

\[ n := n(y) = \frac{y}{|y|}, \quad v := v(p_{r, \pm}), \]

not to be confused with the notations \( n^\pm, v^\pm \) in (6) and \( n_0, v_0 \) in (17). For the \( i \)th component of the vector (A6) we then obtain

\begin{equation}
\begin{aligned}
(A6)_i = -\int_{B_0(0)} \! d^3y \frac{1}{|y|} L(y)^{ij}_{\pm} \partial_j \delta^3(x - y - q_{r, \pm}) \\
= \int_{B_0(0)} \! d^3y \partial_j \left[ \frac{1}{|y|} L(y)^{ij}_{\pm} \right] \delta^3(x - y - q_{r, \pm}) \\
- \int_{\partial B_{r}(0)} \! d\sigma(y) n_j \frac{1}{|y|} L(y)^{ij}_{\pm} \delta^3(x - y - q_{r, \pm}).
\end{aligned}
\end{equation}

by partial integration and Stokes theorem, where \( \partial B_{r_0}(0) \) denotes the surfaces of the ball of radius \( |t| \) around the origin and \( d\sigma \) the respective surface measure. Furthermore, we adopt the
Einstein summation convention regarding the Latin indices. Next, it is convenient to carry out a change of variables according to transformation

\[ T(y) = y + q_{i \pm |y|}, \]

having Jacobi determinant \( |DT(x)| = 1/(1 \pm n \cdot v^\pm) \). Note that, as the trajectory \((q, p)\) is time-like, \(T\) has an inverse, and therefore, \(T(B_0(0)) = B_0(q_0)\) holds. After this transformation, (A7) turns into

\[
\int_{B_0(q_0)} \frac{d^3 z}{\sqrt{g}} \left[ \frac{1}{|y|} L(y)^2 \right] \delta^3(x - z)
= \frac{1}{1 \pm n \cdot v} \left[ \frac{1}{|y|} L(y)^2 \right] \bigg|_{y = x - q^\pm},
\]

where we used the abbreviations introduced in (6). The derivatives can now be carried out in a straightforward manner. The boundary term (A8) of the partial integration gives

\[
(A8) = \int_{\partial B_0(0)} d\sigma(y) \frac{n_i}{1 \pm n \cdot v} \frac{\delta^3(x - y - q_0)}{|y|}
= \int_{\partial B_0(q_0)} d\sigma(z) \frac{n_i(z - q_0)}{1 \pm n(z - q_0) \cdot v_0} \frac{\delta^3(x - z)}{|z - q_0|}
= \delta(|t| - |x - q_0|) \frac{n_{0,i}}{1 \pm n_0 \cdot v_0} \frac{|x - q_0|}{|x - q_0|}.
\]

Carrying out the analogous computations as for (A6) and (A8) for the remaining two terms in (A5) gives the following structure:

\[
(f_1^{(2)} = \frac{1}{1 \pm n \cdot v} \sum_{\sigma} \left[ f_0^{-\sigma}(q, p) + r_0^{-\sigma}(q_0, p_0) \right],
\]

where we have used the notation (5) and (16). The first summand in (A9) comprises all \(d^3 y\) integrals while the \(d\sigma(y)\) integrals make up the second summand.

Finally, in order to compute \(f_1^{(1)}\), given in (A4), we assume an initial field \(f_0\) as in (18) given in terms of a smooth auxiliary trajectory \((\tilde{q}, \tilde{p})\), a homogeneous field \(f_0^0\), and a parameter \(\lambda \in [0, 1]\). Plugging this choice for \(f_0\) into (A4) gives

\[
\begin{align*}
(f_1^{(1)} &= \lambda \left( \frac{\partial_t}{-\nabla \wedge} \frac{\nabla \wedge}{\partial_t} \right) K_0 * f_0^{-}[\tilde{q}, \tilde{p}] + (1 - \lambda) \left( \frac{\partial_t}{-\nabla \wedge} \frac{\nabla \wedge}{\partial_t} \right) K_0 * f_0^{+}[\tilde{q}, \tilde{p}] + \lambda \left( \frac{\partial_t}{-\nabla \wedge} \frac{\nabla \wedge}{\partial_t} \right) K_0 * f_0^0 .
\end{align*}
\]

Making use of the fact that the Liénard–Wiechert fields \(f_0^{\pm}[\tilde{q}, \tilde{p}]\) solve Maxwell’s equations and that \(f_0^0\) is a homogeneous field, the three summands (A10)–(A12) can be simplified according to
\[
\begin{align*}
\left( \frac{\partial_t}{-\nabla_\lambda} \right) K_\lambda \ast f_0^\pm [\tilde{q}, \tilde{p}] &= f_t^\pm [\tilde{q}, \tilde{p}] - f_t^{(2)} [\tilde{q}, \tilde{p}]
\end{align*}
\]

where \( f_t^0 \) denotes the unique solution for the free Maxwell equations to the initial value \( f_0^0 \).

Thanks to (A9) we get
\[
\begin{align*}
\quad f_t^{(1)} &= \lambda(f_t^+ [\tilde{q}, \tilde{p}] - r_t^{-\sigma(t)} [\tilde{q}_0, \tilde{p}_0]) - 1_{B(t)}(\tilde{q}_0, r_t^{-\sigma(t)} [\tilde{q}, \tilde{p}]) \\
&\quad + (1 - \lambda)(f_t^- [\tilde{q}, \tilde{p}] - r_t^{-\sigma(t)} [\tilde{q}_0, \tilde{p}_0]) - 1_{B(t)}(\tilde{q}_0, r_t^{-\sigma(t)} [\tilde{q}, \tilde{p}]) \\
&\quad + f_t^0.
\end{align*}
\]

Plugging this result together with (A9) into (A3), we arrive at the representation (25)–(28) for the field \( f_t \).

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