Dynamics of oscillating relativistic tori around Kerr black holes

Olindo Zanotti\textsuperscript{1}, José A. Font\textsuperscript{1}, Luciano Rezzolla\textsuperscript{2,3}, Pedro J. Montero\textsuperscript{2}

\textsuperscript{(1)}Departamento de Astronomía y Astrofísica, Universidad de Valencia, Dr. Moliner 50, 46100 Burjassot (Valencia), Spain
\textsuperscript{(2)}SISSA, International School for Advanced Studies and INFN, Via Beirut, 2 34014 Trieste, Italy
\textsuperscript{(3)}Department of Physics and Astronomy, Louisiana State University, Baton Rouge, LA 70803 USA

\textbf{ABSTRACT}

We present a comprehensive numerical study of the dynamics of relativistic axisymmetric accretion tori with a power-law distribution of specific angular momentum orbiting in the background spacetime of a Kerr black hole. By combining general relativistic hydrodynamics simulations with a linear perturbative approach we investigate the main dynamical properties of these objects over a large parameter space. The astrophysical implications of our results extend and improve two interesting results that have been recently reported in the literature. Firstly, the induced quasi-periodic variation of the mass quadrupole moment makes relativistic tori of nuclear matter densities, as those formed during the last stages of binary neutron star mergers, promising sources of gravitational radiation, potentially detectable by interferometric instruments. Secondly, \(p\)-mode oscillations in relativistic tori of low rest-mass densities could be used to explain high frequency quasi-periodic oscillations observed in X-ray binaries containing a black hole candidate under conditions more generic than those considered so far.

\textbf{Key words:} accretion discs – general relativity – hydrodynamics – oscillations – gravitational waves

1 INTRODUCTION

Relativistic tori (i.e. geometrically thick discs) orbiting around black holes are expected to form in a number of different scenarios, such as after the gravitational collapse of the core of a rotating massive star \((M > 25M_\odot)\) or after a neutron star binary merger. State-of-the-art numerical simulations of these scenarios, both in Newtonian physics (Lee \textit{et al.} 2001; Ruffert & Janka 2001), as well as in a relativistic framework (Shibata \textit{et al.} 2003), have shown that under certain conditions, a massive disc may be produced. This disc surrounds the newly formed black hole, has large average rest-mass densities and highly super-Eddington mass fluxes, and its angular momentum obeys a sub-Keplerian distribution. Accretion tori of much smaller accretion rates have also been considered in general relativistic magnetohydrodynamic simulations of accretion flows onto Kerr black holes (De Villiers \textit{et al.} 2003; Gammie, McKinney & Tóth 2003, Gammie, McKinney & Tóth 2003; Gammie, Shapiro & McKinney 2004)

There are several reasons for considering these objects astrophysically interesting. One of the most important reason has to do with the idea that an accreting torus around a black hole generated after neutron star coalescence could be the progenitor source for short duration, \((i.e. < 2 \text{ s})\) gamma-ray bursts (GRBs), where the relativistic fireball expands along the rotation axis of the black hole (Woosley 2001) which subsequently grows accreting the surrounding envelope of the star through a geometrically thick disc. An energetic, long-duration \((i.e. > 2 \text{ s})\) GRB accompanied by a Type I\textsc{b}/I\textsc{c} supernova may be produced as a result of such (thick) disc accretion process. As in the neutron star merging scenario, part of the large amounts of the energy released by accretion is deposited in the low-density region along the rotation axis of the star, where the heated gas expands in a jet-like fireball (see MacFadyen & Woosley 1999; Aloy \textit{et al.} 2004; Aloy \textit{et al.} 2004 for numerical simulations of this process).

Even in the absence of other potential instabilities triggered by nonaxisymmetric perturbations or magnetic fields, the accretion torus that forms in the aforementioned catastrophic events is likely to be accreting on to the black hole with rates which largely exceed the Eddington rate, a property which may lead to the rapid destruction of the disc. Such a possibility was first suggested by Abramowicz \textit{et al.} (1983), who pointed out how the equilibrium solution of a barotropic fluid orbiting a black hole with a general, non-Keplerian rotation law might be hydrodynamically unstable.

The outermost closed equipotential surface of the torus has a distinctive cusp at the inner edge through which mass transfer can be driven by small changes in the pressure gradient there (in a stationary situation the pressure gradient at the cusp would be exactly zero). The instability is driven by the progressive penetration of the cusp into the disc as a result of the accretion of mass and angular
momentum and of the corresponding increase of mass and spin of the black hole. During the development of the instability the mass flux increases exponentially and leads to the complete disappearance of the disc into the black hole on a dynamical timescale, justifying why it is usually referred to as the “runaway instability”.

Axisymmetric, general relativistic hydrodynamical simulations of this process have been performed only recently (Font & Daigne 2002b; Zanotti et al. 2003; Daigne & Font 2004). In these works, the disc is assumed to be non-self-gravitating and the relativistic hydrodynamics equations are then evolved in time in the curved spacetime of the black hole. The latter is instantaneously modified to new stationary solutions as the mass and spin of the black hole gradually increase as a result of accretion. While this is clearly an approximation to solving the full Einstein equations, it is a reasonable one when the disc-to-hole mass ratio is small enough and the changes in the spacetime between two successive timelevels is correspondingly small. Within this approximation the numerical simulations have shown that while thick discs with a constant distribution of the specific angular momentum are subject to the runaway instability irrespective of the initial conditions (Font & Daigne 2002b; Zanotti et al. 2003), a small departure from a constant distribution of angular momentum (namely, a power-law increase with the radial distance with a small index) suffices to prevent the development of the instability (Font & Daigne 2002b; Daigne & Font 2004). This result is in agreement with perturbative calculations of radially oscillating tori which indicate that the oscillations have eigenfunctions that are significantly modified at the inner edge when the specific angular momentum is not constant (Rezzolla et al. 2003b). However, no definitive conclusion can be made yet on the occurrence of the runaway instability as there are at least two other physical processes that could play some role. One of these is provided by the self-gravity of the disc which, from studies based on sequences of stationary models has been shown to favour the instability (Masuda et al. 1998); the other one is brought up by at least two other physical processes that could play some role. One of these is the case in discs formed as a result of binary neutron star coalescence. In addition, when the thick discs are composed of the low-density material stripped from the secondary star in low-mass X-ray binaries (LMXBs), their oscillations could help explaining the high frequency quasi-periodic oscillations (HFQPOs) observed in the spectra of X-ray binaries (Rezzolla et al. 2003). In a model proposed recently and based on the evidence that tori around black holes have the fundamental mode of oscillation and the first overtones in the harmonic sequence 2 : 3 : 4 . . . to a good precision and in a very wide parameter space, the HFQPOs are explained in terms of p-mode oscillations of a small-size accretion torus orbiting around the black hole (see also Abramowicz & Kluźniak 2004 for a recent review on the phenomenology associated with HFQPOs and for alternative models).

The aim of this paper is to extend the analysis of the dynamics of axisymmetric relativistic tori carried out by Zanotti et al. (2003) (henceforth paper I), in which the attention was limited to models with constant specific angular momentum orbiting around a Schwarzschild (nonrotating) black hole. Now we consider the more general case of nonconstant angular momentum thick discs in the rotating background spacetime provided by the Kerr metric. Furthermore, since the results reported recently by Daigne & Font (2004) pose limits on the occurrence of the runaway instability in the case of non self-gravitating, nonconstant angular momentum discs, we do not investigate this issue further here. Instead, we focus on the study of the oscillating behaviour of marginally stable discs. To this aim we combine two complementary tools: nonlinear hydrodynamic simulations and a linear perturbative approach. With the first one we can follow the long-term evolution of perturbed thick accretion discs, analyze their dynamics, and compute the gravitational wave emission in the Newtonian quadrupole approximation. Correspondingly, with the second approach we can investigate how the oscillation properties of these objects depend on their geometrical features, surveying a very large parameter space. When applied to vertically integrated relativistic tori, this second approach has already shown its utility for explaining the results of the numerical simulations of Font & Daigne (2002b) and of Zanotti et al. (2003).

The paper is organized as follows: In Section 2 we briefly review the main properties of relativistic tori which obey a power-law angular momentum distribution. Next, in Section 3 we present the two mathematical approaches used in our investigation, namely the techniques for the numerical solution of the hydrodynamic equations and the basic aspects related with the solution of the eigenvalue problem in the linear perturbative framework. Section 4 presents the initial models, while Section 5 contains the discussion on the results. Finally, Section 6 is devoted to the conclusions. Throughout the paper we use a space-like signature (−, +, +, +) and a system of geometrized units in which \( G = c = 1 \). The unit of length is chosen to be the gravitational radius of the black hole, \( r_g \equiv GM/c^2 \), where \( M \) is the mass of the black hole. Greek indices run from 0 to 3 and Latin indices from 1 to 3.

2 STATIONARY FLUID CONFIGURATIONS

The procedure for building a stationary and axisymmetric fluid configuration orbiting around a Kerr black hole and obeying a power-law distribution of the specific angular momentum in the equatorial
plane has been reviewed in great detail by Daigne & Font (2004). Here, for the sake of completeness, we will only recall those definitions and physical properties that are essential to the present study. In what follows we consider a perfect fluid described by the stress-energy tensor

\[ T^{\mu\nu} \equiv (e + p)u^\mu u^\nu + pg^{\mu\nu} = \rho hu^\mu u^\nu + pg^{\mu\nu} , \]

where \( g^{\mu\nu} \) are the coefficients of the Kerr metric in Boyer-Lindquist coordinates \((t, r, \theta, \phi)\) and \( e, p, \rho, \) and \( h = (e + p)/\rho \) are the energy density, the isotropic pressure, the rest-mass density, and the specific enthalpy, respectively, of each of them measured in the frame comoving with the fluid. The fluid is supposed to obey a polytropic equation of state \( p = \kappa \rho \) (EoS), where \( \kappa \) is the polytropic constant and \( \gamma \) is the adiabatic index. We use \( \Omega \equiv u^\phi/u^t \) to denote the angular velocity of the fluid and \( \ell \equiv -u_\theta/u_t \) to denote its specific angular momentum. Moreover, the rotation law on the equatorial plane is given by a power-law distribution of the specific angular momentum

\[ \ell(r, \theta = \pi/2) = Sr^q , \]

where \( S \) can be either positive or negative, according to the disc rotation being prograde or retrograde, respectively, with respect to the black hole rotation. When the motion of the fluid is just circular and \( \ell \) \( \pm \) \( \ell_0 \), the relativistic Euler equations in the \( r \) and \( \theta \) directions are given by (the Bernoulli-type) equilibrium conditions

\[ \frac{\nabla \rho}{e + p} = -\nabla_i \ln(u_e) + \frac{\Omega \nabla_i \ell}{1 - \Omega^2} \quad (i = r, \theta) , \]

where the right-hand-side is nothing but the opposite of the relativistic four acceleration \( a_i \equiv u^\alpha \nabla_\alpha u_i \), which vanishes in the case of a purely geodetic motion.\(^1\) The procedure for solving Eq. (3) exploits the fact that the surfaces of constant \( \Omega \) (the so-called von Zeipel cylinders) coincide with the surfaces of constant \( \ell \) (see Daigne & Font (2004) for details), a result which holds true only for a barotropic fluid (Abramowicz (1971)). One of the most relevant features of the equilibrium solution, irrespective of the distribution of the angular momentum, is that the resulting thick disc can have two Keplerian points on the equatorial plane (i.e. points where the rotation law is that of the Keplerian motion): the first one is the cusp, through which matter can accrete onto the black hole, and the second one marks the position of the maximum rest-mass density. However, while in the case of a constant angular momentum \( \ell \) the position of the cusp \( r_{\text{cusp}} \) is always smaller than the marginally stable orbit \( r_{\text{ms}} \), when the angular momentum is not constant, it may also happen that \( r_{\text{cusp}} > r_{\text{ms}} \), depending on the choice of the parameters \( S \) and \( q \) in Eq. (2).

Among the large family of initial configurations that are in principle possible, we only consider those that possess a cusp, a centre, and with a closed equipotential through the cusp. This only happens when \( 0 \leq q < 0.5 \) and when \( |S_{\text{ms}}| < |S| < |S_{\text{mb}}| \), where \( S_{\text{ms}} \) and \( S_{\text{mb}} \) are two limiting values defined in Daigne & Font (2004) (the subindex 'mb' refers to the marginally bound orbit). Also note that relativistic tori with \( \ell \) following a power-law generally have larger sizes than constant angular momentum tori, a fact which can produce initial models with central densities one or two orders of magnitude smaller.

\(^1\) Note that Eq. (2) implies that a stationary and axisymmetric circular motion of a perfect fluid is geodesic if and only if it follows a Keplerian rotation law.

### 3 Mathematical Framework

The results we report in this paper are based on two complementary approaches involving both nonlinear hydrodynamic simulations and the solution of the eigenvalue problem within a perturbative linear analysis. We recall in what follows the basic aspects related to both of them.

#### 3.1 Solution of the nonlinear hydrodynamics equations

The nonlinear, axisymmetric, general relativistic code used in the simulations we report is an extended version of the one presented by Zanotti et al. (2003) which incorporates the effects of a rotating background spacetime. In this code the hydrodynamic equations are implemented as a first-order, flux-conservative system according to the formulation developed by Banyuls et al. (1997). The explicit form of this system of equations for the Kerr metric can be found in Font & Daigne (2004). The equations are solved using a high-resolution shock-capturing scheme based on an approximate Riemann solver (either HLLE or Marquina’s). Second-order accuracy in both space and time is achieved by adopting a piecewise-linear cell reconstruction procedure and a second order, conservative Runge-Kutta scheme, respectively.

The computational grid consists of \( N_r \times N_\theta \) zones in the radial and angular directions, respectively. The innermost zone of the radial grid is placed at \( r_{\text{min}} = r_{\text{horizon}} + \epsilon \), while the outermost radial grid point is at a distance about 30% larger than the outer radius of the torus, \( r_{\text{out}} \).\(^2\) While \( \epsilon \) depends on the particular model considered, it is typically 0.1\( \, r_g \) or smaller. The radial grid is built by joining smoothly two patches obtained using two different algorithms. The first part extends from \( r_{\text{min}} \) to the outer radius of the torus. It is logarithmically spaced and has the maximum radial resolution at the innermost grid zone, \( \Delta r = 1 \times 10^{-5} \). The spacing of the second part of the radial grid is uniform and it extends up to \( r_{\text{max}} \). The total number of radial grid points \( N_r \) is selected so as to have a radial resolution at the outer boundary \( \Delta r \approx 0.2 \). Typically, we use \( N_r \approx 250 \) for the more compact tori (radius length \( L \leq 15 \)), while \( N_r \approx 350 \) for the more extended models. The angular grid, on the other hand, which consists of \( N_\theta = 84 \) zones in all of the simulations, covers the domain from 0 to \( \pi \).

As customary for finite difference hydrodynamical codes which cannot handle vacuum regions, a low density “atmosphere” is introduced in those parts of the numerical domain not occupied by the torus. In all our simulations we have chosen a special version of the spherically accreting solution described by Michel (1973), suitably modified to account for the rotation of the black hole (see Appendix A for a detailed description). Since this atmosphere is evolved as the rest of the fluid, one should simply take care that its dynamics does not interfere with that of the object being studied. We have verified that this is indeed the case if the maximum density of the atmosphere is \( 5 - 6 \) orders of magnitude smaller than the central density of the torus.

As mentioned in the Introduction we are not interested here in the study of the runaway instability but rather we focus on the oscillation properties of thick discs filling their outermost closed equipotential surface (we also refer to these discs as “marginally stable”). Hence, we assume that the background spacetime is simply the one provided by the Kerr metric and that it does not change during the evolution; this prevents the development of the instability (Font & Daigne 2004; Zanotti et al. 2003) by construction. This is a reasonable approximation in the present context for two different reasons. The first one is that most of the tori evolve

\(^2\) Note that the grid spacing is not constant. It is smaller near the center of the torus and increases with the radius. The grid resolution is further improved by refining the grid in regions of high density gradients.

\(^3\) The grid spacing is chosen such that it is smaller near the center of the torus and increases with the radius. The grid resolution is further improved by refining the grid in regions of high density gradients.
have rest masses $M_t$ which are much smaller than that of the black hole (i.e. $M_t/M = 0.1$). Hence, we can neglect their contribution to the overall gravitational field when compared to that created by the black hole. The second reason is that a small power-law index $q$ in the angular momentum distribution is enough to reduce significantly the amount of rest-mass accreted onto the black hole so that, effectively, the mass and spin of the black hole do not change significantly over the timescale of our simulations. This was shown numerically by Font & Daigne (2002b), Daigne & Font (2004) and explained in the perturbative analysis by Rezzolla et al. (2003b). To validate this approximation we monitor the rest-mass accreted by the black hole according to the formula
\begin{equation}
\dot m(r_{\text{min}}) \equiv -2\pi \int_0^\pi \sqrt{g_{\rho W} v^\prime} \, d\theta \bigg|_{r_{\text{min}}},
\end{equation}
where $g$ is the determinant of the metric and where $r_{\text{min}}$ is the radius of the innermost radial cell. Similarly, we monitor the angular momentum flux across the horizon as
\begin{equation}
\dot J(r_{\text{min}}) \equiv -2\pi \int_0^\pi \ell \sqrt{g_{\rho W} v^\prime} \, d\theta \bigg|_{r_{\text{min}}},
\end{equation}
Note that $W$ and $v^\prime$ in Eqs. (4) and (5) are the Lorentz factor and the coordinate radial velocity, respectively, as measured by the Zero Angular Momentum Observer. In all of the simulations performed here the relative changes in the black hole mass and spin resulting from accretion are $\Delta M/M \sim \Delta J/J \lesssim 10^{-4}$. Hence, the assumption of a fixed spacetime is accurate and justified.

### 3.2 Solution of the eigenvalue problem

In addition to the hydrodynamic simulations we also perform a linear perturbative analysis of the axisymmetric modes of oscillation of relativistic tori in a Kerr spacetime. This is done following the procedure outlined in Rezzolla et al. (2003b) for the Schwarzschild spacetime and recently extended to the Kerr case by Montero et al. (2004). The method, whose details can be found in the aforementioned references, consists in solving the perturbed relativistic continuity and Euler equations obtained after introducing perturbations with a harmonic time dependence in the velocity and the pressure. The system of perturbed hydrodynamical equations is then cast into a set of coupled ordinary differential equations and solved as an eigenvalue problem, where the perturbed quantities are treated as eigenfunctions and where the eigenvalues provide the eigenfrequencies of the system. A similar procedure but for thin discs has been described by Rodríguez et al. (2003).

Two approximations are adopted in the perturbative analysis. First, we neglect the perturbations of the background spacetime, something which is usually referred to as the Cowling approximation (Cowling 1941). This choice is consistent with that of neglecting the contribution of mass and angular momentum fluxes to the mass and spin of the black hole in the nonlinear hydrodynamic simulations. The second approximation is that of adopting a vertically integrated description of the tori. While this approach simplifies the numerical treatment considerably (the angular dependence is integrated description of the tori. While this approach simplifies the numerical treatment considerably (the angular dependence is

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\frac{\partial \psi}{\partial \theta} = 0,
$$
resulting from accretion are $\Delta M/M \sim \Delta J/J \lesssim 10^{-4}$. Hence, the assumption of a fixed spacetime is accurate and justified.

### 4 INITIAL MODELS

The initial models for the numerical simulations are a sequence of relativistic tori which fill their outermost closed equipotential surface and thus their inner radius coincide with the position of the cusp, i.e. $r_{\text{in}} = r_{\text{cusp}}$. Table I reports the main properties of the unperturbed models. All of the models but two (i.e. A5b and A5c) are built with an adiabatic index $\gamma = 4/3$. More specifically, models A5b and A5c have $\gamma = 5/3$ and $\gamma = 2$, respectively.

The value of the constant $S$ in the power-law distribution, Eq. (2), spans the allowed range between $S_{\text{ms}}$ and $S_{\text{mb}}$, and is selected via the additional parameter $\lambda$ as $S = \min(S_{\text{ms}}, S_{\text{mb}}) + \lambda(S_{\text{mb}} - S_{\text{ms}})$. The resulting value of $S$, together with the power-law index $q$, are the most important factors in determining the geometrical properties of the torus, in particular for the location of the Keplerian points (i.e. $r_{\text{cusp}}$ and $r_{\text{ms}}$). Table I and for the radial size of the disc $L = r_{\text{out}} - r_{\text{in}}$.

Figure 1 provides a graphical view of the distribution of the initial models in the plane $(r_{\text{cusp}}(e) - r_{\text{ms}}, L)$. We recall that the position of the marginally stable orbit, $r_{\text{ms}}$, is a decreasing function of the black hole spin parameter $a$, while increasing the power law index $q$. The inner radius of the tori gets larger. As a result it is a natural consequence of not constant angular momentum tori in the Kerr metric to have $r_{\text{cusp}}(e) - r_{\text{ms}}$ usually larger than zero, making accretion more difficult than in the case of constant angular momentum tori. Only for small values of $a$ and $q$ can the torus penetrate deeper in the potential well, pushing the cusp below the marginally stable orbit.

It is worth underlining that, once perturbed, the numerical evolution of the stationary models reported in Table I can indeed provide an insight into the dynamics of astrophysical thick discs.
5 NUMERICAL RESULTS

The disc-to-hole mass ratio considered in our initial models ($M_t/M = 0.1$) as well as the power-law index of the angular momentum distribution chosen, make these objects stable with respect to the runaway instability. Indeed, as reported recently by Daigne & Font (2004) for discs with $M_t/M = 0.1$, the critical power-law index separating stable and unstable models appears to be $q_{\text{cr}} \sim 0.05 - 0.06$ for mass accretion rates of about a few times

![Figure 2. Time evolution of the mass accretion rate for representative tori orbiting around a Schwarzschild black hole. The figure shows the stabilizing effect of increasing the power law index from $q = 0$ (model A1a) to $q = 0.1$ (model A3a), which reflects in progressively smaller accretion rates. In all of the models the perturbation factor $q = 0.08$.](image-url)
$M_0\dot{M}_{\text{acc}}^{-1}$. Hence, when perturbed, these models will only respond with oscillations that are harmonic as long as the amplitude of the perturbation is sufficiently small (see paper I for a discussion of the transition to nonlinear oscillations). As a result, with the exception of model A1a, which has a power law index $q = 0$, we expect the models selected in Table 1 to be particularly suitable for studying their response to perturbations and, in particular, to compute the associated mode eigenfrequencies (Section 5.1) and gravitational wave emission (Section 5.2).

## 5.1 Oscillation properties

### 5.1.1 Dynamics of representative models

Figure 2 shows the time evolution of the accretion rate of the first three models of Table 1, namely models A1a, A2a, A3a, all of them referring to a Schwarzschild black hole. The time evolution displayed in Fig. 2 corresponds to models having an initial perturbation in the radial velocity with an amplitude $\eta = 0.08$. The final time plotted in the figure corresponds to 100 orbital periods, which are measured with respect to the orbital period of the rest-mass density maximum in the unperturbed torus.

From model A1a to model A3a the power law index changes from 0 to 0.1, thus providing a progressive deviation from a constant angular momentum distribution toward a steeper power law distribution. As it is obvious from the figure when the power law index $q$ increases while maintaining $\lambda$ constant, the mass accretion rate decreases correspondingly. Already at $q = 0.05$ (model A2a) the behaviour of $\dot{m}$ is quite different from that of a constant angular momentum model, and at $q = 0.1$ the behaviour of the mass flux is entirely dominated by the behaviour of the accreting atmosphere.

We have analyzed the time evolution of the accretion rate for the rest of models reported in Table 1, finding that apart from an early intense burst of accretion due to the response of the system to the initial perturbation (this is evident e.g. in model A3a in Fig. 2), none of them shows a significant variation of the position of the inner radius when compared to the initial one, thus having an essentially suppressed mass flux. This behaviour does not depend in a substantial manner on whether the inner radius is smaller or larger than the marginally stable orbit. This is the case, for instance, of model C1a which although has $r_{\text{cusp}} - r_{\text{in}} \sim -1.0$ is very stable and barely accretes on to the black hole, with $\dot{m} \sim 10^{-6} M_\odot\text{s}^{-1}$. On the contrary, this behaviour strongly depends on the value of the power law index $q$. This reflects the fact that tori with nonconstant distributions of specific angular momentum have oscillations which are significantly damped at the inner edge (Rezzolla et al. 2003a).

We also recall that the amplitude of the accretion rates reported in Fig. 2 depends on the perturbation factor $\eta$, and that, as shown in paper I, this dependence is linear for $\eta \lesssim 0.04$ (cf. Fig. 9 of paper I).

Despite a distinctive quasi-periodic behaviour is quite apparent from Fig. 2 we note that for most models the evolution of the accretion rate reflects mainly the response of the atmosphere to the perturbations propagating from the torus. For this reason the periodic character of the dynamics of the discs becomes more evident in the time evolution of the maximum rest-mass density rather than in the mass flux. This is reported in Fig. 3, for four representative models, two of them corresponding to a Kerr black hole of spin $a = 0.7$ (D1a, D2a), and the other two corresponding to a Kerr black hole of spin rate $a = 0.9$ (E3a, E4a). After the tori relax from the initial perturbation at $t/t_{\text{orb}} \approx 2$, they start oscillating at regular, quasi-periodic intervals. This is a feature which discs with power-law distributions of angular momentum share with constant angular momentum discs as those investigated in paper I. The harmonic variation of the hydrodynamical quantities is analyzed in more detail next.

### 5.1.2 Comparisons with the perturbative analysis

Additional information on the quasi-periodic behaviour of the hydrodynamics variables discussed in the preceding section can be extracted through a Fourier analysis of the corresponding time evolutions. For this purpose we have calculated the Fourier transforms of the time evolution of the $L_2$ norm of the rest-mass density for all models, defined as $|\rho|^2 \equiv \sum_{j=1}^{N_r} \sum_{i=1}^{N_{\lambda}} \rho_{ij}^2$. As this is a global quantity it is particularly useful for comparisons with the results of the eigenvalue problem in the linear perturbative analysis. Furthermore, we can also compare our findings with those reported in paper I for the case of constant angular momentum discs.

A first result that emerges from the Fourier analysis is that the overall dynamics of nonconstant angular momentum discs is more complex than in the case of constant distributions of the angular momentum. This is reflected in particular in the fact that the power spectra of the $L_2$ norm of the rest-mass density show, in general, a richer structure. This can be seen in Fig. 4, where we plot the power spectra of the $L_2$ norm of $\rho$, obtained with a Hanning filter (Press et al. 1992) for one representative model, namely E3a.

Although not all of the computed spectra show the features of Fig. 4 with the same clarity, all of them present the same essential characteristics, namely a fundamental mode $f$ and a series of overtones. Among them we can recognize the two first overtones predicted by the linear analysis and indicated as $o_1$ and $o_2$. In addition to these, further modes appear as linear combinations of the observed ones (Press et al. 1992).

| Model | $(o_1/\bar{f})_{\text{num}}$ | $(o_1/\bar{f})_{\text{linear}}$ |
|-------|-------------------------------|---------------------------------|
| A1a   | 1.45                          | 1.47                            |
| A2a   | 1.46                          | 1.41                            |
| A4a   | 1.36                          | 1.36                            |
| A5a   | 1.31                          | 1.26                            |
| D1a   | 1.37                          | 1.32                            |
| D2a   | 1.40                          | 1.36                            |
| D3a   | 1.35                          | 1.28                            |
| D4a   | 1.35                          | 1.28                            |
| E1a   | 1.33                          | 1.40                            |
| E3a   | 1.31                          | 1.25                            |
| E4a   | 1.36                          | 1.29                            |
| E5a   | 1.38                          | 1.32                            |

Table 2. Frequency ratio $o_1/\bar{f}$ as extracted from the power spectra of the $L_2$ norm of the rest mass density (second column) and as computed from the solution of the eigenvalue problem (third column).
Figure 3. Time evolution of the central rest-mass density normalized to its initial value for models D1a, D2a (black hole spin $a = 0.7$) and E3a, E4a (black hole spin $a = 0.9$). The perturbation factor $\eta = 0.08$ for models of both panels.

Figure 4. Power spectrum of the $L_2$ norm of the rest-mass density for model E3a. The units in the vertical axis are arbitrary and the power spectrum was obtained using a Hanning filter.

Figure 5. Power spectra of the $L_2$ norm of the rest-mass density for model E3a. The different lines refer to different initial perturbations: the solid line corresponds to an initial perturbation given by a parametrized spherically symmetric radial velocity, the dotted line to a global initial perturbation in the density, and the dashed line to an initial perturbation given by the eigenfunction of mode $\nu_1$. Vertical units are arbitrary. The values have been rescaled in order to match the power of the fundamental frequency.

shown to differ from the one predicted by the linear analysis by less than 7%.

It is worth emphasizing that the excitation of a particular overtone depends sensibly on the choice of the initial perturbation. In particular, we can excite selectively a specific mode by perturbing a given equilibrium model through the use of the vertically integrated eigenfunctions for the velocity and rest-mass density that have been calculated for that model through the perturbative analysis. Exploiting this possibility we have investigated in detail the dynamics of model E3a, considering initial data which consist either of generic perturbations in the radial velocity (with $\eta = 0.08$) or in the rest-mass density (or in both), or of specific perturbations obeying the eigenfunctions for the velocity and the rest-mass density for the...
o_1\) overtone (the fundamental mode also would be excited in this case). The results of this investigation are summarized in Fig. 5, which shows the power spectra of the L_2 norm of \(\rho\) for the three different initial perturbations. As it is clear from this figure the two generic initial perturbations, represented by a global perturbation in the radial velocity (solid line) and by a global perturbation in the rest-mass density (dotted line), are less efficient in exciting the mode \(o_1\) than when the mode eigenfunction is used (dashed line). In the latter case the power channeled in that mode increases by a factor \(\sim 20\) with respect to initial perturbations in the global velocity and by a factor \(\sim 5\) with respect to the initial perturbations in the global density.

We note that exciting overtones of the fundamental mode above \(o_1\) is increasingly difficult. The tests that we have performed using the eigenfunctions of the mode \(o_2\), for instance, could not provide a clear signature of a selective excitation, in contrast with the mode \(o_1\). We believe this is probably due to the approximation made in the linear perturbative approach in treating thick discs as vertically integrated objects. While we expect this approximation to ally less accurate as the mode number increases and the small-scale features of the eigenfunctions become more important.

We have not yet commented on the presence of the modes as linear combinations of \(f\), \(o_1\) and \(o_2\) shown in Fig. 4 and in particular on the overtone appearing at twice the fundamental frequency, which is, after the fundamental mode, where most of the power is concentrated. A plausible interpretation of the peak at \(2f\), and of the other linear combinations shown in the spectrum of Fig. 4 is that they are the result of a nonlinear coupling effect. It is, in fact, a general property of nonlinear systems in the limit of small oscillations that of showing nonharmonic oscillations, i.e. linear combinations of their normal modes of oscillations. Thus, if the system has eigenfrequencies \(\omega_i\), the nonlinearity of the equations will also produce modes at frequencies \(\omega_i \pm \omega_j\) (cf. Landau & Lifschitz 1976, §28), with amplitudes which are proportional to the product of the amplitudes of the combining frequencies. It should be noted that this nonlinear coupling is particular evident in the present calculations which have been performed with an initial perturbation amplitude \(\eta = 0.08\), at which the system’s response is already nonlinear \(^2\).

A consequence of this nonlinear coupling among modes is that, if \(A_f\) and \(A_{2f}\) denote the power spectra amplitudes of a generic quantity \(A\) (e.g. the rest-mass density) at the frequencies \(f\) and \(2f\), respectively, we then expect that \(A_f \gg A_{2f}\) and the two amplitudes to scale like \(A_{2f} \propto A_f^2\). We have tested that this is indeed the case by considering one representative model, D2a. For this model we have computed from the power spectrum a sequence of pairs \((\log A_f, \log A_{2f})\) for different values of the perturbation parameter \(\eta\), verifying that to a good approximation they settle along a straight line of slope 2 in the corresponding plane.

In [Rezzolla et al. 2003b], it was shown through linear analysis that \(p\)-modes in a thick disc with a constant distribution of angular momentum have frequencies which stay in the harmonic sequence \(2 : 3 : 4 \ldots\). Since this same sequence was also found in the non-linear numerical simulations reported in paper I it was thus natural to interpret them as an evidence of the excitation of \(p\)-modes. In particular, the mode at twice the fundamental frequency was identified as the mode \(o_2\), for which \(o_2/f = 2\). However, it is now clear that in the case of constant angular momentum discs the peak at \(2f\) receives contributions both from the mode \(o_2\) and from the nonlinear coupling \(f + f\). As a result, part (if not most) of the power at the peak at twice the fundamental frequency in Fig. 7 of paper I could be the result of the nonlinear coupling discussed here.

More recently, [Montero et al. 2004] have investigated \(p\)-modes in thick discs with more generic angular momentum distributions and found that in this case deviations from the simple sequence \(2 : 3 : 4 \ldots\) can be large. We show in Fig. 6 the frequency ratios \(o_1/f\) and \(o_2/f\) for a number of models of our sample as a function of the penetration of the inner radius of the discs in the black hole potential well. All of the values in Fig. 6 have been computed with the linear code and some of the unperturbed models coincide with those evolved with the nonlinear code. These are E3a, E4a, E5a and are indicated with the geometric symbols. Note that both \(o_1/f\) and \(o_2/f\) differ from \(3/2\) and \(2\), values which they approach only in the limit of a torus penetrating deeply in the potential well. Furthermore, while the spectra obtained from the numerical simulations show a clear peak at the position predicted by the linear analysis for the mode \(o_1\), it is difficult to find a peak at the position predicted by the perturbative code for the \(o_2\) overtone.

In summary, the comparison between the results of the non-linear numerical simulations and those coming from the linear perturbation analysis indicate that the fundamental mode \(f\) and its first overtone in the numerical simulations do represent the first two \(p\)-modes of the system and that these are in ratio \(o_1/f\) close to \(3/2\), with deviations that can be as large as \(\sim 15\%). Additional \(p\)-modes are also probably excited in the simulations, namely the mode \(o_2\) as shown in Fig. 4 for model E3a, but the corresponding power is in general too small to be visible in the spectra of the rest of the models. On the other hand, the computed spectra show overtones at integer multiples of the fundamental frequency (most notably at

\(^2\) The maximum perturbation amplitude triggering linear perturbations has been estimated to be \(\eta \sim 0.04\) in paper I (cf. Fig. 9 of that paper).
2f), plus additional modes which are linear combinations of f and \(v_0\), and that are all the result of nonlinear coupling effects.

The present results also shed some additional light on the physical mechanism proposed by \cite{rezzolla2003} to explain HFOQPs in black hole candidates. The main idea is that a particular class of observations showing frequencies in the ratio 2 : 3 and 1 : 2 \cite{remillard2002,abramowicz2003} could be explained in terms of the excitation of the p-modes of a thick accretion disc orbiting around a black hole. According to the numerical simulations presented here, in the linear regime the \(o_2\) p-mode will be little excited, since very little power is in general channeled in this mode. However, a mode at 2f will be present and strongly excited in the nonlinear regime. As a result, the ratio 1 : 2 is satisfied with very good precision, is independent of the distribution of the angular momentum and appears as soon as the nonlinear coupling is triggered. The ratio of the fundamental mode and of the first overtone, on the other hand, is more sensitive on the distribution of the angular momentum and may differ from 2 : 3 up to 15%.

We conclude this section by recalling that the fundamental modes are closely related to the epicyclic oscillations at the location of the maximum rest mass density of the disc and that, as shown by \cite{rezzolla2003} and by \cite{montero2004}, the eigenfrequency of the fundamental p-mode of the disc tends in the limit of vanishing size to the radial epicyclic frequency at the disc rest-mass density maximum.

### 5.2 Gravitational wave emission

The oscillating behaviour of the perturbed tori that we have discussed in the previous section is responsible of significant changes of the quadrupole moment of the resultant gravitational radiation. As a consequence, the emission of potentially detectable gravitational radiation results in the emission of potentially detectable gravitational radiation. This property is closely related to the toroidal topology of these objects, which have their maximum rest mass density off-centered and thus have intrinsically high quadrupole moments. In particular, the sources at the distance of 20 Mpc (i.e. for extragalactic sources at a distance of 20 Mpc) are due to the differences in the average mass of the black holes considered.

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The resulting signal-to-noise ratio is then computed as

\[
\frac{S}{N} = \frac{h_c}{h_{\text{rms}}(f_c)},
\]

where \(h_c\) is the characteristic frequency, \(S(f_c)\) is the power spectral density of the gravitational wave emission detected by a given particular instrument and reads

\[
h_c \equiv \left[ 3 \int_0^\infty \frac{S_h(f)}{S_h(f)} |\langle \dot{h}(f) \rangle|^2 df \right]^{1/2}.
\]

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where \(k = 16\pi^{3/2}/\sqrt{15}, z \equiv \cos \theta\) and \(\Phi\) is the gravitational potential, approximated to the first Post-Newtonian order as \(\Phi = (1 - g_\nu)/2\). The transverse traceless (TT) wave amplitude is then computed as \cite{zwerger1997}

\[
h_{\text{TT}}(t) = F_+ \left( \frac{1}{8} \sqrt{\frac{15}{\pi}} \right) \frac{A_{20}^{E2} \omega}{R},
\]

where \(R\) is the distance to the source and where the detector’s beam pattern function, \(F_+ = F_\times(R, \theta, \phi)\), is set to one if optimal conditions of detectability were met. In order to investigate the possibility that pulsating relativistic tori can be effectively detected by the interferometric instruments presently in operation or under construction, we have computed the signal-to-noise ratio with respect to LIGO I, Advanced LIGO, and VIRGO. For doing this we need an estimate of the frequency where most of the gravitational wave emission is channeled. This is provided by the characteristic frequency \(f_c\), which is a detector dependent quantity.

\[
f_c \equiv \left[ \int_0^\infty \frac{\langle |\dot{h}(f)|^2 \rangle f df}{S_h(f)} \right]^{1/2}.
\]

where \(\langle |\dot{h}(f)|^2 \rangle\), which is an average over randomly distributed angles of the Fourier transforms \(\dot{h}(f)\) of \(h_{\text{TT}}(t)\), has been approximated as \(\langle |\dot{h}(f)|^2 \rangle \approx |\dot{h}(f)|^2\). Strictly related to the characteristic frequency is the characteristic amplitude, that gives a typical measurement of the gravitational wave emission detected by a given particular instrument and reads

\[
h_{\text{rms}}(f_c) \equiv \sqrt{f_c S_h(f_c)}
\]

The oscillating behaviour of the perturbed tori that we have discussed in the previous section is responsible of significant changes of the quadrupole moment of the disc and that, as shown by \cite{rezzolla2003} and by \cite{montero2004}, the eigenfrequency of the fundamental p-mode of the disc tends in the limit of vanishing size to the radial epicyclic frequency at the disc rest-mass density maximum.
We have studied the dynamics and the gravitational wave emission from nonconstant specific angular momentum tori undergoing axisymmetric oscillations while orbiting around Kerr black holes. The angular momentum distribution has been chosen to be increasing outward with the radial distance following a power-law dependence. The self-gravity of the discs has been neglected and the accretion of mass and angular momentum has been assumed not to affect the underlying background metric. We have also removed the possible role played by the runaway instability by selecting indices in the power-law distribution of the specific angular momentum that are above the critical value for the instability (cf. Daigne & Font (2004)). The results presented in this paper have thus extended the previous investigation of Zanotti et al. (2003) where constant specific angular momentum discs around Schwarzschild black holes were considered.

A comprehensive sample of initial (marginally stable) equilibrium tori has been built. These tori have been subsequently perturbed in various ways in order to study their dynamical response to the perturbations during long-term time evolutions (extending up to 100 orbital periods for each model). The time evolutions are characterized by a distinctive quasi-periodic pattern present in all fluid quantities as well as in the mass and angular momentum accretion rates and in the gravitational waveforms (which have been extracted using the Newtonian quadrupole formula). Our study has been carried out using two complementary tools: an axisymmetric nonlinear hydrodynamics code and a linear perturbative code for vertically integrated axisymmetric tori. While the first one has proven useful to investigate nonlinear effects in the axisymmetric oscillation, the second approach has been useful in confirming and interpreting the numerical simulations.

The main results of our study are as follows: The power-law angular momentum distribution results in more stable tori than in the uniform angular momentum case, yielding systematically smaller values of the mass accretion rate as the power-law index increases. This is in agreement with the previous studies of Rezzolla et al. (2003b) and of Daigne & Font (2004). The comparison of the eigenfrequencies calculated through the perturbative analysis and those obtained from the power spectra of the hydrodynamical variables in the numerical simulations have shown a good agreement between the two approaches for the first two modes $f$ and $o_1$. They have been found close to a simple sequence of integers $2/3$, but with deviations that become significant as the power-law index $q$ increases. Namely, for $q = 0.2$ the ratio $f/o_1$ differs from $2/3$ by $\sim 15\%$. On the other hand, the second overtone $o_2$ predicted by the perturbative analysis could be detected only in two cases among the several power spectra calculated from the hydrodynamical models. This is either due to the limitations of the vertically integrated assumption in the perturbative analysis in capturing higher frequency oscillations (something which is also confirmed when performing selective excitations of modes), or by the very small power confined in these modes which prevents them to become visible in the power spectra.

The comparison has also shown that the peak found at $2f$ in the power spectra should not be interpreted as a proper eigenmode of the system. Rather, it represents the result of a nonlinear coupling of the lower-order modes as it is common in nonlinear systems subject to nonharmonic oscillations.

Finally, we have reported on the detectability of the oscillations of perturbed tori via their gravitational wave emission. In good agreement with what we already found for constant angular
Table 3. Computed estimates regarding the detection of the gravitational wave signals emitted by relativistic tori around Kerr black holes with power-law distributions of the angular momentum. From left to right the table reports the characteristic frequency, the characteristic amplitude, and the signal-to-noise ratio computed for three detectors, LIGO I, VIRGO, and Advanced LIGO, assuming a galactic distance for the first two, and an extragalactic distance for the Advanced LIGO detector. \( \tau_{\text{life}} = 100 \) orbital periods.

| Model | \( f_c \) (Hz) | \( f_c \) (Hz) | \( h_c \) | \( h_c \) | \( h_c \) | \( h_c \) | \( S/N \) | \( S/N \) | \( S/N \) |
|-------|---------------|---------------|---------|---------|---------|---------|--------|--------|--------|
|       | LIGO I (10 Kpc) | VIRGO (10 Kpc) | ADV. LIGO (20 Mpc) | LIGO I (10 Kpc) | VIRGO (10 Kpc) | ADV. LIGO (20 Mpc) | LIGO I (10 Kpc) | VIRGO (10 Kpc) | ADV. LIGO (20 Mpc) |
| A1a   | 223           | 236           | 223     | 1.5 \times 10^{-20} | 1.5 \times 10^{-20} | 7.1 \times 10^{-24} | 29.1   | 23.5   | 0.26   |
| A2a   | 267           | 274           | 264     | 3.8 \times 10^{-20} | 3.8 \times 10^{-20} | 1.9 \times 10^{-23} | 62.6   | 55.2   | 0.59   |
| A3a   | 296           | 308           | 291     | 1.3 \times 10^{-20} | 1.4 \times 10^{-20} | 6.3 \times 10^{-24} | 18.9   | 17.7   | 0.17   |
| A4a   | 121           | 117           | 122     | 1.4 \times 10^{-21} | 1.7 \times 10^{-21} | 8.3 \times 10^{-25} | 4.1    | 3.2    | 0.03   |
| A5a   | 205           | 207           | 206     | 4.6 \times 10^{-21} | 4.7 \times 10^{-21} | 2.3 \times 10^{-24} | 10.1   | 7.5    | 0.09   |
| C1a   | 269           | 388           | 237     | 3.3 \times 10^{-22} | 4.2 \times 10^{-22} | 1.1 \times 10^{-25} | 0.5    | 0.5    | <10^{-2} |
| D1a   | 446           | 454           | 436     | 1.2 \times 10^{-19} | 1.2 \times 10^{-19} | 5.6 \times 10^{-23} | 100    | 122    | 0.6    |
| D2a   | 366           | 393           | 349     | 4.3 \times 10^{-20} | 4.5 \times 10^{-20} | 1.8 \times 10^{-22} | 48     | 52     | 0.3    |
| D3a   | 344           | 367           | 336     | 1.6 \times 10^{-20} | 1.7 \times 10^{-20} | 7.3 \times 10^{-24} | 19.5   | 20.4   | 0.14   |
| D4a   | 498           | 505           | 487     | 2.3 \times 10^{-19} | 2.4 \times 10^{-19} | 1.1 \times 10^{-22} | 175    | 225    | 1.0    |
| E1a   | 272           | 341           | 258     | 1.6 \times 10^{-20} | 1.9 \times 10^{-20} | 6.4 \times 10^{-24} | 26.4   | 23.8   | 0.21   |
| E2a   | 249           | 323           | 236     | 8.5 \times 10^{-21} | 1.1 \times 10^{-20} | 3.3 \times 10^{-24} | 15.1   | 13.2   | 0.12   |
| E3a   | 538           | 556           | 506     | 6.5 \times 10^{-20} | 6.7 \times 10^{-20} | 3.0 \times 10^{-23} | 44     | 58     | 0.23   |
| E4a   | 454           | 493           | 406     | 2.3 \times 10^{-20} | 2.4 \times 10^{-20} | 1.0 \times 10^{-23} | 20     | 24     | 0.12   |
| E5a   | 412           | 491           | 347     | 1.1 \times 10^{-20} | 1.3 \times 10^{-20} | 4.3 \times 10^{-24} | 10.9   | 12.3   | 0.07   |

momentum discs, the chances for detection of gravitational radiation from nonconstant angular momentum tori, when sufficiently compact and dense, are good and within the sensitivity curves of LIGO and VIRGO for galactic sources, but only marginal even for Advanced LIGO for extragalactic sources located at 20 Mpc.

Extensions of the work reported here include the incorporation of additional physics in the models such as viscosity and magnetic fields and will be reported in forthcoming papers.

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APPENDIX A: DESCRIPTION OF THE ATMOSPHERE USED IN THE NUMERICAL SIMULATIONS

We model the low density atmosphere surrounding the tori in terms of a semi-analytic solution of the relativistic stationary accretion of a rotating fluid onto a Kerr black hole. In this sense, it represents the extension of the relativistic spherical accretion solution onto a Schwarzschild black hole [Michel1973] to account for the rotation of both the infalling fluid and of the background spacetime.

In practice, in addition to the continuity equation, \( \nabla \cdot (\rho u^\alpha) = 0 \), and to the energy conservation equation, \( \nabla \cdot T^\alpha_\beta = 0 \), one also has to ensure the conservation of the angular momentum, \( \nabla \times T^\alpha_\phi = 0 \). During its motion, a fluid with a general four velocity \( u^\alpha \equiv (u^t, u^r, u^\theta, u^\phi) \) follows a spiral ending at the black hole horizon along cones of constant \( \theta \). This motion can be derived from the following three constraints coming directly from the conservation equations

\[
\sqrt{-g} u^r = C_1, \tag{A1}
\]
\[
\sqrt{-g} u^r u_t = C_2, \tag{A2}
\]
\[
\sqrt{-g} \partial_\phi u^r = C_3, \tag{A3}
\]

where \( C_1, C_2 \) and \( C_3 \) can only depend on the polar angle \( \theta \). Dividing Eqs. (A1) and (A3), gives the condition of constant specific angular momentum \( \ell = -u_\phi/u_t \), while the division of Eqs. (A2) and (A1) provides the relativistic Bernoulli equation

\[
h u_r = C, \tag{A4}
\]

where \( C = C_2/C_1 \). Following Michel [1973], Eq. (A5) is squared and then differentiated to obtain, with the help of the continuity equation, Eq. (A1), the so-called “wind equation”

\[
\frac{2}{u} \frac{du}{dr} \left[ -V^2 G(r, u) + \frac{g_{rr}}{H(r, u)} u^2 \right] + \frac{d}{dr} \left[ -4V^2 G(r, u) + S(r, u) \right] = 0, \tag{A5}
\]

where \( u \) is a shorthand notation for \( u^r \), while

\[
F(r, u) = 1 + g_{rr} u^2, \tag{A6}
\]
\[
H(r, u) = g^{tt} - 2g^{t\phi} + \ell^2 g^{\phi\phi}, \tag{A7}
\]
\[
G(r, u) = F(r, u)/H(r, u), \tag{A8}
\]
\[
S(r, u) = r [u^2 \partial_r g_{rr} / H(r, u) - F(r, u)] \tag{A9}
\]
\[
(\partial_r g^{tt} - 2\ell \partial_\phi g^{t\phi} + \ell^2 g^{\phi\phi})/H(r, u), \tag{A10}
\]

and \( V^2 = d\ln(hp)/d\ln p - 1 \) is the square of the sound speed. Imposing the vanishing of the terms in the square brackets of Eq. (A5) allows to determine the position of the critical point of the flow, thus guaranteeing that the derivative \( du/dr \) is always finite. Note that according to Eq. (A5) the sound speed at the critical point is given by

\[
V^2 = \frac{g_{rr} u^2}{1 + g_{rr} u^2}. \tag{A11}
\]
which is different from the velocity of the fluid $v^2 = g_{ij} v^i v^j$. This means that the critical point is not a transonic point as in the case of purely spherical accretion.

In the numerical implementation of this solution we further impose the simplifying assumption $u_\phi / u_t = 0$, thus reducing considerably the algebra. Once the density $\rho_c$ at the critical point has been given as a free parameter, the rest of the solution is computed as follows:

(i) Compute $V_c$ from the equation of state.
(ii) Compute the relation between $u_c$ and $r_c$ by equating the terms in the square brackets of Eq. (A5).
(iii) Compute the position of the critical point $r_c$ by solving Eq. (A11).
(iv) Compute the constants $C_1, C_2, C_3$ from the known values of the solution at the critical point.
(v) Solve the relativistic Bernoulli equation as an algebraic equation in the unknown $u(r)$. Complete the solution by computing $\rho(r)$ from the continuity equation and all of the other thermodynamic quantities from the equation of state.

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