All Singular Vectors of the $N=2$ Superconformal Algebra via the Algebraic Continuation Approach

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We give general expressions for singular vectors of the $N=2$ superconformal algebra in the form of monomials in the continued operators by which the universal enveloping algebra of $N=2$ is extended. We then show how the algebraic relations satisfied by the continued operators can be used to transform the monomials into the standard Verma-module expressions. Our construction is based on continuing the extremal diagrams of $N=2$ Verma modules to the states satisfying the twisted highest-weight conditions with complex twists. It allows us to establish recursion relations between singular vectors of different series and at different levels. Thus, the $N=2$ singular vectors can be generated from a smaller set of the so-called topological singular vectors, which are distinguished by being in a 1:1 correspondence with singular vectors in $\hat{sl}(2)$ Verma modules. The method of ‘continued products’ of fermions is a counterpart of the method of complex powers used in the constructions of singular vectors for affine Lie algebras.

1 Introduction

In this paper, we present a general algebraic construction of singular vectors in Verma modules over the $N=2$ superconformal algebra. This infinite-dimensional algebra underlies the construction of $N=2$ strings [1] and is also realized on the worldsheet of the bosonic non-critical string theory [2, 3]. Conformal field theory models with the $N=2$ supersymmetry are important in string compactifications [4]; the Kazama–Suzuki construction [5] provides a tool for constructing such models.

However, the $N=2$ algebra is not an affine Lie algebra, which is the source of several complications. Thus, the absence of a root system (hence, of a canonical triangular decomposition) results in the fact that there is no “canonical” way to impose highest-weight conditions on the vacuum vector in highest-weight-like representations. This obviously affects the definition of singular vectors. The conventional prescription is to define singular vectors by imposing the same annihilation conditions as on the highest-weight vector of the module. The thus defined singular vectors, however, do not generate maximal submodules and, therefore, one has to look for subsingular vectors in order to describe the structure of submodules. A more economical possibility, which we adopt in this paper, is to define singular vectors in such a way that they generate maximal submodules and, thus, no room is left for subsingular vectors. Since the significance of singular vectors, obviously, consists in generating submodules, it is rather natural to adapt the definition of singular vectors to the structure of submodules. The price to be paid for dealing with only maximal submodules is that the singular vectors satisfy twisted highest-weight conditions, i.e., the highest-weight conditions labelled by integers $\theta \in \mathbb{Z}$, all of which differ from one another by the spectral-flow transform [6, 7].

\footnote{That the subsingular vectors do exist in this approach was noticed in [6]; the structure of Verma modules, showing why subsingular vectors appear, how they can be constructed and why they are in fact unnecessary, is described in [8].}

\footnote{In the familiar case of the affine Lie algebra $\hat{sl}(2)$, every submodule is freely generated from a state satisfying the fixed set of annihilation conditions, therefore these same conditions are used to define singular vectors. In the $N=2$ algebra, as we will see, the situation is more complicated.}
In systematically dealing with the twisted highest-weight conditions, we extend the method of \[11, 12\], which elaborates on the idea of the spectral flow transform \[9, 10\] combined with the use of extremal vectors in the Verma modules. (Diagrams of) extremal vectors were introduced in \[13\], and their usefulness in representation theory has been demonstrated in \[11, 12, 14, 15, 8\]. The extremal states might be thought of as a generalization of different pictures \[16\] to the case of non-free fermions. Recall that for the free first-order bosonic systems, different pictures are inequivalent in the sense of Verma modules, while for free fermions, on the contrary, they are equivalent. For the interacting fermions, the situation in the general position is that the extremal states are still equivalent to each other in that the same module is generated from each of them, however the equivalence breaks down for some values of the parameters of the highest-weight vector. Recall also that the different pictures in the bosonic first-order system are changed by the exponential of a current that participates in bosonizing the system, \(\exp \phi\). By considering operators like \(\exp a\phi\), we can change the picture arbitrarily (at the expense of non-localities), and similarly for the bosonized free-fermion systems. However, when the fermions are non-free, such a bosonization no longer exists; instead, changing the picture by an arbitrary number is realized by the ‘continued’ products of modes of the fermionic generators. For the \(N=2\) superconformal algebra, we have two fermionic currents \(Q_m\) and \(G_n\), and the ‘continued’ operators \(q(a, b)\) and \(g(a, b)\) can heuristically be thought of as \(\prod_b a Q_{\mu}\) and \(\prod_b G_{\mu}\), respectively, with complex \(a\) and \(b\). The would-be bosonization rules are then replaced by a set of algebraic rules to deal with the new operators \(q(a, b)\) and \(g(a, b)\).

The ‘algebraic continuation’ scheme worked out along these lines allows us to construct the \(N=2\) singular vectors as \textit{monomial} expressions in the continued operators \(q(a, b)\) and \(g(a, b)\) with complex arguments \(a, b \in \mathbb{C}\). All of the \(N=2\) singular vectors will be written as monomials in \(q\) and \(g\), and a crucial ingredient of the proposed construction are the algebraic rules that extend the universal enveloping algebra so as to include the new operators \(g(a, b)\) and \(q(a, b)\). These rules, in particular, allow us to rewrite the \(q-g\)-monomials in the conventional form, i.e. as elements of the ordinary Verma module. All the algebraic rules can in fact be deduced as a continuation of certain relations in the universal enveloping algebra of \(N=2\). In the special cases (at certain ‘integral points’) when no continuation is actually required, our construction reduces to the one known previously to give a particular subset of the \(N=2\) singular vectors \[17\]. The algebraic setting corresponding to the continued operators is provided by the \textit{generalized Verma modules}, which are a continuation to non-integral (in fact, complex) twists of the twisted Verma modules — i.e., the Verma modules whose highest-weight vectors satisfy the spectral-flow-transformed (twisted) highest-weight conditions.

An important point is that this approach makes the structure of \(N=2\) Verma modules very transparent, which, eventually, leads one to realizing that this structure is the same (as regards the appearance and the embedding patterns of submodules) as the structure of the so-called relaxed Verma modules over the affine \(\hat{sl}(2)\) algebra \[15\], which are in a certain sense ‘larger’ than the standard \(\hat{sl}(2)\) Verma modules. On the \(N=2\) side, similarly, there exist the ‘larger’ and the ‘smaller’ Verma modules, which are called massive and topological, respectively. In general, a submodule of a massive Verma module can be either massive or topological, while submodules of a topological Verma module are again the (twisted) topological Verma modules (or a sum thereof). One thing that is impossible, is the \textit{embedding} a massive Verma module into a topological Verma module. Any morphism of a massive Verma module into a topological Verma module necessarily has a kernel that contains another topological submodule. This fact, which is concealed by the use of conventional singular vectors, appears to be the origin of the statements known\(^3\) See also \[18\] for another ‘continued’ scheme that follows the spirit of the approach developed previously for other infinite-dimensional algebras \[19, 20, 21\].
in the literature that a combination of mappings that seem to be embeddings actually vanishes.

This distinction between two classes of Verma-like modules carries over, obviously, to singular vectors: in the topological Verma modules, all submodules can be freely generated from the topological singular vectors, i.e., those states in the module that satisfy twisted topological highest-weight conditions. One could generate a given topological Verma module from other states than the topological highest-weight state, satisfying weaker highest-weight conditions. However, the module would not be freely generated from such states. In the massive Verma modules, on the other hand, there are two essentially different types of singular vectors, which we call massive and topological according to the type of the module they generate. For example, the so-called charged singular vectors always generate a topological Verma submodule.

The construction via the continuation of products of the fermionic generators provides monomial expressions for singular vectors and, hence, simplifies the analysis of what happens when one singular vector is built on top of the other — this now follows from the rules for multiplying the continued monomials. In addition, this approach also reveals several ‘hidden’ properties of the $N=2$ singular vectors, such as the relations between singular vectors of different series as well as certain recursion relations between singular vectors at different levels. As we will see, the massive singular vectors can be reconstructed from the topological ones: depending on a complex number $h$, two positive integers $r$ and $s$, and $t \in \mathbb{C}$ parametrizing the central charge, a massive $h$-dependent singular vector in the general position $|S(r, s, h, t)\rangle$ can be built out of the topological singular vectors $|E(r, s, t)\rangle^\pm$.

The picture that emerges in this way can be, rather heuristically, represented as follows. The topological singular vectors $|E^\pm(r, s, t)\rangle$ for $r, s \in \mathbb{N}$ are in a 1:1 correspondence with $\hat{sl}(2)$ singular vectors (see [15] for the details) and are related by a chain of recursion formulae (where we omit the parameter $t$ that determines the $N=2$ central charge and, likewise, omit the $\hat{sl}(2)$ level $k = t - 2$):

\begin{align}
\text{massive } N = 2 \\
\text{singly generated } N = 2 \\
\text{singly generated } sl(2)
\end{align}

Over every topological singular vector there grows a complex $h$-plane, and for any $h$ there exists a massive singular vector $|S(r, s, h, t)\rangle$. The dotted lines indicate a prescription to reconstruct the massive singular vector for any fixed $h$. Some points in the $h$ plane are special in that there may exist more singular vectors, depending on the $t$ parameter implicit in (1.1) that parametrizes the central charge of the $N=2$ algebra. For the highest-weight in the general position, diagram (1.1) gives a general idea of the structure of the set of $N=2$ singular vectors. An important circumstance which is omitted from the picture though, is that when constructing a massive singular vector $S(r, s, h, t)$ starting with either of the two topological ones, $E^+(r, s, t)$ or $E^-(r, s, t)$, one arrives at the same result as long as one does not descend to a higher codimension in the space of highest weights, where the highest weight satisfies additional equations.

This paper is organized as follows. In Sec. 2, we begin with reviewing the $N=2$ algebra, introduce the two types of $N=2$ Verma modules and their twists. We also define singular vectors taking into account
which type of submodules they generate. In Sec. 3, we introduce the continued fermionic generators and discuss their algebraic properties. Then, in Sec. 4, we use the continued operators to construct monomial expressions for \( N=2 \) singular vectors. In Sec. 3, we show how the singular vectors written in the “continued” form rewrite in the Verma form, i.e., as polynomials in the usual creation operators. We also give several characteristic examples of the interplay of different types of submodules appearing in degenerate Verma modules.

2 The \( N=2 \) algebra, spectral flow transform, and Verma modules

2.1 The algebra, its twisting and automorphisms

The \( N=2 \) superconformal algebra is spanned by Virasoro generators \( L_m \) with a central charge \( c \), two fermionic fields \( G^\pm_r \), and a \( U(1) \) current \( \mathcal{H}_n \). The nonvanishing commutation relations are:

\[
[L_n, L_m] = (n - m)L_{n+m} + \frac{\xi}{12}(n^3 - n)\delta_{n+m,0}, \quad [H_n, H_m] = \frac{\xi}{3}n\delta_{n+m,0},
\]

\[
[L_n, H_m] = -mH_{n+m}, \quad [L_n, G^\pm_r] = (\frac{1}{2}n - r)G^\pm_{n+r}, \quad [H_n, G^\pm_r] = \pm G^\pm_{n+r},
\]

\[
\{G^+_r, G^-_s\} = 2L_{r+s} + (r - s)H_{r+s} + \frac{\xi}{3}(r^2 - \frac{1}{4})\delta_{r+s,0}, \quad n, m \in \mathbb{Z}, \quad r, s \in \mathbb{Z} + \frac{1}{2},
\]

and \( c \) is the central charge. We find it more convenient to work with the twisted version of the algebra \( A \). The twisted \( N=2 \) algebra can be described by introducing new generators as

\[
\mathcal{L}_n = L_n + \frac{1}{2}(n + 1)\mathcal{H}_n, \quad G_r = G^+_r, \quad Q_r = G^-_{r+\frac{1}{2}}.
\]

The twisting does therefore affect two things: the modding of the fermions and the choice of Virasoro generators. The latter is due to the freedom of adding a derivative of the \( U(1) \) current to the energy-momentum tensor, and is nothing but a change of basis in the algebra. As to the different moddings, they label different members of a family of isomorphic algebras related by spectral flow transformations \( \xi \) (see Eq. (2.5) below). This family includes the Neveu–Schwarz and Ramond sectors, as well as algebras in which the fermion modes range over \( \pm \theta + \mathbb{Z}, \quad \theta \in \mathbb{C} \).

For definiteness, we pick out the \( N=2 \) algebra corresponding to \( \theta = 0 \) as our ‘reference’ algebra, and write the corresponding commutation relations in the basis that is the image of (2.1) under the mapping (2.2):

\[
[L_m, L_n] = (m - n)L_{m+n}, \quad [H_m, H_n] = \frac{\xi}{3}m\delta_{m+n,0},
\]

\[
[L_m, G_n] = (m - n)G_{m+n}, \quad [H_m, G_n] = G_{m+n},
\]

\[
[L_m, Q_n] = -nQ_{m+n}, \quad [H_m, Q_n] = -Q_{m+n}, \quad m, n \in \mathbb{Z}.
\]

Denote this algebra as \( \mathcal{A} \). It is this version of the \( N=2 \) superconformal algebra that we are going to work with in this paper. The generators \( \mathcal{L}_m, Q_m, H_m, \) and \( G^+_m \) are the Virasoro generators, the BRST current, the \( U(1) \) current, and the spin-2 fermionic current respectively.

\[\text{4Which is not the so-called twisted sector of the } N=2 \text{ algebra, where two fermions have different moddings.}\]
It will be convenient to parametrize the central charge \( c \) as

\[
c = 3 \frac{t-2}{t} \tag{2.4}
\]

with \( t \in \mathbb{C} \setminus \{0\} \). One should also add the point \( t = \{ \infty \} \), or \( c = 3 \), which deserves a special examination though.

### The spectral flow transform

The spectral flow transform produces isomorphic images of the algebra \( \mathcal{A} \). When applied to the generators of (2.3) it acts as

\[
U_{\theta} : \begin{align*}
\mathcal{L}_n &\mapsto \mathcal{L}_n + \theta \mathcal{H}_n + \frac{\xi}{6}(\theta^2 + \theta)\delta_{n,0}, \\
\mathcal{Q}_n &\mapsto \mathcal{Q}_{n-\theta}, \\
\mathcal{H}_n &\mapsto \mathcal{H}_n + \frac{\xi}{3} \theta \delta_{n,0}, \\
\mathcal{G}_n &\mapsto \mathcal{G}_{n+\theta}
\end{align*}
\tag{2.5}
\]

This gives an isomorphic algebra \( \mathcal{A}_{\theta} \), whose generators \( \mathcal{L}_n^{\theta}, \mathcal{Q}_n^{\theta}, \mathcal{H}_n^{\theta} \) and \( \mathcal{G}_n^{\theta} \) can be taken as the RHSs of (2.5). Any two algebras \( \mathcal{A}_{\theta_1}, \mathcal{A}_{\theta_2} \) of this family are related by an isomorphism of type (2.5) with \( \theta = \theta_1 - \theta_2 \). Rather than working in the basis where the isomorphism \( \mathcal{A} \approx \mathcal{A}_{\theta} \) is transparent, however, we prefer choosing the basis where the respective generators \( \mathcal{L}_m \) and \( \mathcal{H}_m \) are identified between the different \( \mathcal{A}_{\theta} \) algebras. Thus, \( \mathcal{A}_{\theta} \) is spanned by the generators \( \mathcal{L}_m, \mathcal{H}_m, \mathcal{Q}_{-\theta + m}, \) and \( \mathcal{G}_{\theta + m}, m \in \mathbb{Z}, \) that satisfy the following commutation relations:

\[
\begin{align*}
[\mathcal{L}_m, \mathcal{L}_n] &= (m-n)\mathcal{L}_{m+n}, & [\mathcal{H}_m, \mathcal{H}_n] &= \frac{\xi}{3} \delta_{m+n,0}, \\
[\mathcal{L}_m, \mathcal{G}_\nu] &= (m-\nu)\mathcal{G}_{m+\nu}, & [\mathcal{H}_m, \mathcal{G}_\nu] &= \mathcal{G}_{m+\nu}, & m, n \in \mathbb{Z}, \\
[\mathcal{L}_m, \mathcal{Q}_\lambda] &= -\lambda \mathcal{Q}_{m+\lambda}, & [\mathcal{H}_m, \mathcal{Q}_\lambda] &= -\mathcal{Q}_{m+\lambda}, & \lambda \in -\theta + \mathbb{Z}, \\
[\mathcal{L}_m, \mathcal{H}_n] &= -n \mathcal{H}_{m+n} + \frac{\xi}{6}(m^2 + m)\delta_{m+n,0}, & & & (2.6) \\
\{\mathcal{G}_\nu, \mathcal{Q}_\lambda\} &= 2\mathcal{L}_{\nu+\lambda} - 2\lambda \mathcal{H}_{\nu+\lambda} + \frac{\xi}{3}(\nu^2 + \nu)\delta_{\nu+\lambda,0},
\end{align*}
\]

### The involutive automorphism

The \( N=2 \) algebra has an involutive automorphism

\[
\begin{align*}
\mathcal{G}_n &\mapsto \mathcal{Q}_n, & \mathcal{Q}_n &\mapsto \mathcal{G}_n, \\
\mathcal{H}_n &\mapsto -\mathcal{H}_n - \frac{\xi}{3} \delta_{n,0}, & \mathcal{L}_n &\mapsto \mathcal{L}_n - n \mathcal{H}_n.
\end{align*}
\tag{2.7}
\]

Together with the automorphisms (2.5) for \( \theta \in \mathbb{Z} \), transformations (2.7) span the group \( \mathbb{Z}_2 \times \mathbb{Z} \) which can be thought of as the would-be “\( N=2 \) affine Weyl group” acting on the entire \( N=2 \) algebra. As we will see, this group acts on the space of twisted topological highest-weight by reflections and does play the same role in the representation theory of the \( N=2 \) algebra as the affine Weyl group plays in the representation theory of affine Lie algebras.

#### 2.2 Topological Verma modules

We define a class of topological\(^5\) Verma modules over the \( N=2 \) algebra. Here and in what follows, \( \mathbb{N} = \{1,2,\ldots\} \) and \( \mathbb{N}_0 = \{0\} \cup \mathbb{N} \).

\(^5\)The name is inherited from the non-critical bosonic string, where the algebra (2.3) is realized \( \mathfrak{h} \) and matter vertices can be dressed into the \( N=2 \) primaries satisfying the highest-weight conditions (2.4); in that context, the algebra (2.3) is viewed as a topological algebra.
Definition 2.1 A topological Verma module $V_{h,t}$ is freely generated by
\[ L_m, \quad H_m, \quad Q_m, \quad G_m, \quad m \in \mathbb{N} \] (2.8)
from the state $|h,t\rangle_{\text{top}}$, called the topological highest-weight vector, that satisfies the following annihilation and eigenvalue conditions:
\[ Q_{\geq 0}|h,t\rangle_{\text{top}} = G_{\geq 0}|h,t\rangle_{\text{top}} = H_{\geq 1}|h,t\rangle_{\text{top}} = 0, \] (2.9)
\[ H_0|h,t\rangle_{\text{top}} = h|h,t\rangle_{\text{top}} \] (2.10)
(which are called the topological highest-weight conditions).

It follows that we also have
\[ L_{\geq 0}|h,t\rangle_{\text{top}} = 0. \] (2.11)

An important tool in our analysis are the extremal vectors in different $N=2$ modules. These can be arrived at as follows. Consider the first of the modes $G_m$ that do not annihilate the topological highest-weight state, namely $G_{-1}$. Acting with this mode we obtain a new state, which can be further acted upon with $G_{-2}$, etc.:

Definition 2.2 For every $\theta \in \mathbb{Z}$, the vectors $|x(\theta)\rangle = G_{\theta} \ldots G_{-1}|h,t\rangle_{\text{top}}$, for $\theta \leq -1$, and $|x(\theta)\rangle = Q_{-\theta} \ldots Q_{-1}|h,t\rangle_{\text{top}}$, for $\theta \geq 1$, are called the extremal vectors of the topological Verma module $V_{h,t}$.

Thus, the extremal vectors arrange into a diagram

![Diagram](https://via.placeholder.com/150)

(2.12)

called the extremal diagram of the topological Verma module $V_{h,t}$. It has the following meaning. Any module over the $N=2$ algebra is bigraded by the eigenvalues of $H_0$ and $L_0$. As we choose the highest-weight vector, we obtain a lattice in the plane of $H_0$- $L_0$-eigenvalues whose the sites correspond to the subspaces with definite bigradings. Diagram (2.12) represents the boundary of that part of the lattice whose sites correspond to at least a one-dimensional subspace (each point on the edge is such that there is precisely one, up to proportionality, state with that grade in the module): it is easy to see that all of the states in the module have the (charge, level) bigradings such that they lie in the interior of the extremal diagram.

Lemma 2.3 The extremal states in (2.12) satisfy the highest-weight conditions
\[ L_{\geq 1}|x(\theta)\rangle = 0, \quad H_{\geq 1}|x(\theta)\rangle = 0, \quad G_{\geq \theta}|x(\theta)\rangle = 0, \quad Q_{\geq -\theta+1}|x(\theta)\rangle = 0, \quad \theta \leq -1, \]
\[ G_{\geq \theta+1}|x(\theta)\rangle = 0, \quad Q_{\geq -\theta}|x(\theta)\rangle = 0, \quad \theta \geq 1. \] (2.13)
2.3 Twisted topological Verma modules and topological singular vectors

Definition 2.4 Any state satisfying the highest-weight conditions
\[
\begin{align*}
\mathcal{L}_m |h, t; \theta\rangle_{\text{top}} &= 0, \quad m \geq 1, \\
\mathcal{H}_m |h, t; \theta\rangle_{\text{top}} &= 0, \quad m \geq 1,
\end{align*}
\]
(2.14)
is called the twisted topological highest-weight state. In addition, \( h \) is chosen such that
\[
(\mathcal{H}_0 + \frac{\xi}{3} \theta) |h, t; \theta\rangle_{\text{top}} = h |h, t; \theta\rangle_{\text{top}}.
\]
(2.15)
Note that Eqs. (2.14) imply
\[
(L_0 + \theta \mathcal{H}_0 + \frac{\xi}{6} (\theta^2 + \theta)) |h, t; \theta\rangle_{\text{top}} = 0.
\]
(2.16)
Conditions (2.14)–(2.16) are called twisted topological highest-weight conditions. The parametrization of the eigenvalue of \( \mathcal{H}_0 \) in (2.15) is a matter of convention. We choose it so as to have \( h \) the same for all vectors related by the spectral flow transform; in [11], on the other hand, the (less convenient) notations were such that \( h \) was preserved for the entire extremal diagram.

Definition 2.5 The twisted topological Verma module \( \mathfrak{V}_{h,t;\theta} \) is a module freely generated from a twisted topological highest-weight state \( |h, t; \theta\rangle_{\text{top}} \) by the operators
\[
\mathcal{L}_m, \quad \mathcal{H}_m, \quad Q_{m-\theta}, \quad G_{m+\theta}, \quad m \in \mathbb{N}.
\]
The extremal diagram of a twisted topological Verma module \( \mathfrak{V}_{h,t;\theta} \) contains a cusp – a preferred point that satisfies annihilation conditions (2.14) with \( \theta = \theta_0 \), while all the other extremal vectors with \( \theta \in \mathbb{Z} \setminus \{\theta_0\} \) satisfy highest-weight conditions (2.13). These ‘cusp’ states in the extremal diagrams of submodules are the topological singular vectors (≡ singular vectors in topological Verma modules).

Definition 2.6 Topological singular vectors in a topological Verma module are vectors that are not proportional to the highest-weight vector and satisfy the twisted topological highest-weight conditions (2.14) with some \( \theta \in \mathbb{Z} \).

The point is that the twist parameter \( \theta \) that enters the highest-weight conditions satisfied by the singular vector may be different from the twist parameter of the module. One readily shows that acting with the \( N=2 \) generators on a topological singular vector generates a submodule.

It follows from [15, 3] that [27]

Lemma 2.7 A topological singular vector exists in \( \mathfrak{V}_{h,t} \) iff either \( h = h^+(r, s, t) \) or \( h = h^-(r, s, t) \), where
\[
\begin{align*}
h^+(r, s, t) &= -\frac{r-1}{t} + s - 1, \quad r, s \in \mathbb{N}, \\
h^-(r, s, t) &= \frac{r+1}{t} - s.
\end{align*}
\]
(2.17)
The “if” statement also follows from [23]: the determinant formula of that paper applies to massive Verma modules considered in the next subsection; a topological Verma module can be realized as either a submodule or a quotient of an appropriate massive Verma module. Then there is a submodule in the topological Verma module whenever Eqs. (2.17) are satisfied.
2.4 Massive Verma modules

In this subsection, we introduce another class of \(N=2\) modules. For the property of their highest-weight vectors to have a (generically non-zero) eigenvalue \(\ell\) of the Virasoro generator \(L_0\), they will be referred to as massive modules:

**Definition 2.8** The massive highest-weight vector \(|h, \ell, t\rangle\) satisfies the following set of highest-weight conditions:

\[
\begin{align*}
Q_\geq & \geq 1 \quad |h, \ell, t\rangle = 0, \\
H_0 & |h, \ell, t\rangle = h |h, \ell, t\rangle, \\
L_0 & |h, \ell, t\rangle = \ell |h, \ell, t\rangle.
\end{align*}
\]

A massive Verma module \(\mathcal{U}_{h, \ell, t}\) over \(\mathcal{A}\) is freely generated from a massive highest-weight vector \(|h, \ell, t\rangle\) by the generators

\[
L_m, m \in \mathbb{N}, \quad H_m, m \in \mathbb{N}, \quad Q_m, m \in \mathbb{N}_0, \quad G_m, m \in \mathbb{N},
\]

(2.19)

Extremal diagrams of massive Verma modules are defined similarly to the topological case:

\[
|X(\theta)\rangle = Q_{-\theta+1} \cdots Q_0 |h, \ell, t\rangle, \quad \theta \geq 1,
\]

\[
G_\theta \cdots G_{-1} |h, \ell, t\rangle, \quad \theta \leq -1,
\]

(2.20)

or, in more visual terms,

\[
\begin{aligned}
\bullet & \quad Q_0 \\
G_{-1} & \quad Q_{-1} \\
& \quad \vdots \\
& \quad \vdots
\end{aligned}
\]

(2.21)

**Lemma 2.9** Extremal states in a massive Verma module satisfy the annihilation conditions

\[
L_\geq |X(\theta)\rangle = 0, \quad H_\geq |X(\theta)\rangle = 0, \quad G_{\geq \theta} |X(\theta)\rangle = 0, \quad Q_{\geq -\theta+1} |X(\theta)\rangle = 0.
\]

(2.22)

Thus, all of the states in a massive extremal diagram satisfy, generically, the same type of annihilation conditions. This motivates the following

**Definition 2.10** Any state \(|h, \ell, t; \theta\rangle, \theta \in \mathbb{Z}\), from a massive Verma module that satisfies

\[
\begin{align*}
L_m |h, \ell, t; \theta\rangle & = 0, \quad m \geq 1, \quad Q_\lambda |h, \ell, t; \theta\rangle = 0, \quad \lambda = -\theta + p, \quad p = 1, 2, \ldots \\
H_m |h, \ell, t; \theta\rangle & = 0, \quad m \geq 1, \quad G_\nu |h, \ell, t; \theta\rangle = 0, \quad \nu = \theta + p, \quad p = 0, 1, 2, \ldots \\
(H_0 + \frac{\ell}{6}) |h, \ell, t; \theta\rangle & = h |h, \ell, t; \theta\rangle, \\
(L_0 + \theta H_0 + \frac{\ell}{6}(\theta^2 + \theta)) |h, \ell, t; \theta\rangle & = \ell |h, \ell, t; \theta\rangle
\end{align*}
\]

(2.23)

is called a twisted massive highest-weight state.
2.5 Branching of extremal diagrams and the topological highest-weight conditions

In the general position, every arrow in the extremal diagram of a massive Verma module can be, up to a factor, inverted by acting with the opposite mode of the other fermion:

\[
Q_{-\theta} |h, \ell, t; \theta\rangle = 2\ell \left| h - \frac{2}{\ell}, \ell + h - \frac{2}{\ell}, t; \theta + 1 \right>, \quad \theta < 0, \\
\mathcal{G}_{\theta-1} |h, \ell, t; \theta\rangle = 2(\ell - h) \left| h + \frac{2}{\ell}, \ell - h, t; \theta - 1 \right>, \quad \theta > 0.
\]  

(2.25)

It follows from (2.25) that, as soon as one of the factors on the right-hand sides vanishes, the respective state satisfies the twisted topological highest-weight conditions (2.14). At these ‘topological points’, the extremal diagram branches, e.g., choosing \( \ell = -2(h - \frac{2}{\ell}) \) we have

\[
\text{A crucial fact is that, once we are on the inner parabola, we can never leave it: none of the operators from the } N=2 \text{ algebra maps onto the remaining part of the big parabola from the small one. In other words, the inner diagram in (2.26) corresponds to an } N=2 \text{ subrepresentation.}
\]

These simple observations are summarized as a theorem on a class of singular vectors in the massive Verma modules. By iteratively applying relations (2.25), we arrive at the ‘if’ statement of the following Theorem:
Theorem 2.11 A massive Verma module $\mathcal{U}_{h,\ell,t}$ contains a twisted topological Verma submodule if and only if $\ell = l_{ch}(r, h, t)$, where

$$l_{ch}(r, h, t) = r(h + \frac{r-1}{r}), \quad r \in \mathbb{Z};$$

(2.27)

The corresponding singular vector reads

$$|E(r, h, t)\rangle_{ch} = \begin{cases} Q_r \ldots Q_0 |h, l_{ch}(r, h, t), t\rangle, & r \leq 0, \\ G_{-r} \ldots G_{-1} |h, l_{ch}(r, h, t), t\rangle, & r \geq 1 \end{cases}$$

(2.28)

It satisfies the twisted topological highest-weight conditions (2.14) with $\theta = -r$.

These singular vectors are from the ‘charged’ series of [23]. To be more precise, the charged singular vectors of Ref. [23] are the top-level representatives of the extremal diagram of the submodule built on the charged singular vector. This representative satisfies the highest-weight conditions (2.9) and is constructed as a descendant of (2.28) as follows:

$$|s(r, h, t)\rangle_{ch} = \begin{cases} G_0 \ldots G_{-r-1} Q_r \ldots Q_0 |h, l_{ch}(r, h, t), t\rangle, & r \leq -1, \\ Q_1 \ldots Q_{-r-1} G_{-r} \ldots G_{-1} |h, l_{ch}(r, h, t), t\rangle, & r \geq 1 \end{cases}$$

(2.29)

In the generic situation, the same submodule is generated by either of the representatives – the one that satisfies topological highest-weight conditions and the top-level one. In some degenerate cases, however, the top-level representative may not generate the entire submodule, in which case subsingular vectors would have to be considered, see [8] for the details. On the other hand, the topological representative (which we will simply refer to as the charged singular vector) generates the maximal submodule.

An important remark is in order regarding the structure of submodules in $N = 2$ Verma modules. As can be seen from (2.26), the submodule generated from $C$ is a twisted topological Verma module, its characteristic property being that the state $C$ satisfies stronger annihilation conditions than the other states in the extremal diagram; in the language of extremal diagrams, this state is represented by a ‘cusp’. A similar ‘cusp’ necessarily exists in the extremal diagram of any submodule of a topological Verma module. Consider, again, the diagram (2.12) and a submodule generated from a twisted topological singular vector (the ‘cusp’ in the extremal diagram of the submodule):

When such a submodule is viewed as being generated from the top-level vector (marked with a cross in the diagram), it might be taken for a massive Verma module. This is the reason behind the confusion existing in the literature as regards the embedding diagrams and other properties of submodules (and, in fact, singular vectors) in $N = 2$ Verma modules.

A similar situation exists with submodules generated from the charged singular vectors in massive Verma modules: any such submodule is a twisted topological Verma module, which is concealed by the
fact that the top-level representative of the extremal diagram of the submodule satisfies the massive, not topological, highest-weight conditions; it is only some distance along the extremal diagram that one encounters a state satisfying twisted topological highest-weight conditions (2.14) with some $\theta$. However, the presence of such a state (a ‘cusp’ in the diagram, shown with a circle) drastically changes the nature of the submodule (e.g., the topological and the massive Verma modules have different characters, etc.).

2.6 ‘Massive’ singular vectors

As we have seen, the “charged” series of singular vectors in massive Verma modules are constructed very easily, in fact they immediately follow from the analysis of extremal diagrams. This does not exhaust all singular vectors in massive Verma modules however.

Our strategy in finding explicit expressions for all singular vectors in $N=2$ Verma modules will be to develop the observation that a singular vector appears as soon as there is a topological point in the diagram (2.24). We extend the notion of extremal vectors to arbitrary complex $\theta$, then any diagram (2.24) can be considered as having a branching point, albeit a ‘non-integral’ one. This essentially reduces the problem of constructing the general massive singular vectors to constructing topological singular vectors.

However, we should first of all define precisely which singular vectors are massive (i.e., generate massive Verma modules) and which are topological (i.e., generate twisted topological Verma modules). In the definition of massive singular vectors, we have to ensure the following three points: that a given state, indeed, generate a submodule, and that this not be a topological submodule. On the other hand, a given submodule can be generated from different vectors, all of which are then referred to as representatives of the singular vector (in fact, of the extremal diagram of the submodule).

**Definition 2.12** Let $|Y\rangle$ be a state in an $N=2$ Verma module that satisfies twisted massive highest-weight conditions with some $\theta \in \mathbb{Z}$. Then $|X\rangle$ is said to be a dense $G/Q$-descendant of $|Y\rangle$ if either

$$|X\rangle = \alpha G_{\theta-N} \ldots G_{\theta-1} |Y\rangle, \quad N \in \mathbb{N},$$

or

$$|X\rangle = \alpha Q_{-\theta-M} \ldots Q_{-\theta} |Y\rangle, \quad M \in \mathbb{N}_0.$$

for some $\alpha \in \mathbb{C} \setminus \{0\}$.

**Definition 2.13** A representative of a massive singular vector in the massive Verma module $U_{h,\ell,t}$ is any element of $U_{h,\ell,t}$ such that

i) it satisfies twisted massive highest-weight conditions, i.e., it is annihilated by the operators $L_m, H_m, m \in \mathbb{N}, Q_{\lambda}, \lambda \in -\theta + \mathbb{N},$ and $G_\nu, \nu = \theta + \mathbb{N}_0$ with some $\theta \in \mathbb{Z},$

ii) none of its dense $G/Q$-descendants vanishes,

iii) the highest-weight state $|h, \ell, t\rangle$ is not one of its descendants.

In order to choose a representative of the extremal subdiagram, we, again, have to fix its relative charge. It will be useful to choose the representatives $|S(r, s, h, t)^-\rangle$ and $|S(r, s, h, t)^+\rangle$ such that (charge, level) = $(-rs, \frac{1}{2}(rs + 1)(rs + 2) - 1)$ and $(rs, \frac{1}{2}rs(rs + 1))$, respectively. In what follows, we construct these singular vectors explicitly and discuss their properties. Another choice could be the top-level
representative that satisfies the $\theta = 0$ case of highest-weight conditions (2.22), namely (with $\approx$ standing for the equalities that hold on the highest-weight state)

$$Q_{\geq 1} \approx G_{\geq 0} \approx L_{\geq 1} \approx H_{\geq 1} \approx 0,$$

(2.31)

with $(\text{charge, level}) = (0, rs)$. In the general position, all the three vectors $|S(r, s, h, t)^+\rangle$, $|S(r, s, h, t)^-\rangle$, and the top-level representative generate the same submodule, and it is therefore irrelevant which one to select. In many degenerate cases, however, the top-level representative generates only a submodule of the module generated from $|S(r, s, h, t)^+\rangle$ or $|S(r, s, h, t)^-\rangle$ (see [8] for the classification of degenerations of $N=2$ Verma modules).

Applying to the above definitions the spectral flow transform would produce the necessary modifications for the twisted modules $\mathcal{U}_{h,\ell,t;\theta}$. We will thus explicitly construct singular vectors only in untwisted massive Verma modules.

Lemma 2.14 ([23]) A massive Verma submodule exists in $\mathcal{U}_{h,\ell,t}$ if and only if $\ell = l(r, s, h, t)$, where

$$l(r, s, h, t) = -\frac{1}{4}(h-h^-(r, s, t))(h-h^+(r, s, t+1)),$$

(2.32)

$$\forall (r, s, h, t) \in (N \times N \times \mathbb{C} \times \mathbb{C})$$

### 3 Continued extremal states, and continued products of fermionic generators

In this section, we extend the definition of twisted highest-weight vectors to the case of non-integral (in fact, complex) twist parameter $\theta$. Thus the generalized topological highest-weight vectors $|h, t; \theta\rangle_{\text{top}}$ and generalized massive highest-weight vectors $|h, \ell, t; \theta\rangle$ are defined by the same annihilation conditions as the twisted highest-weight vectors, Eqs. (2.14)-(2.16) and (2.22)-(2.23) respectively, with arbitrary $\theta \in \mathbb{C}$. Accordingly, generalized topological Verma modules $\mathcal{V}_{h, t; \theta}$ and generalized massive Verma modules $\mathcal{U}_{h,\ell,t;\theta}$ are freely generated from the generalized highest-weight states by the respective creation operators, where it is understood that any $\theta'$-module is considered over the algebra $\mathcal{A}_\theta$, which is the image of the algebra (2.3) under the spectral flow transform $\mathcal{U}_\theta$; thus the modes of $\mathcal{G}$ are running over $\theta + n, n \in \mathbb{Z}$, and those of $\mathcal{Q}$, over $-\theta + n, n \in \mathbb{Z}$ (of which $\mathcal{G}_\mu$ with $\mu \in \theta - 1 - \mathbb{N}_0$ and $\mathcal{Q}_\nu$ with $\nu \in -\theta - \mathbb{N}_0$ are the creation operators in the generalized massive Verma modules). Obviously, the generalized Verma modules become the twisted Verma modules whenever $\theta \in \mathbb{Z}$.

In the thus defined generalized modules, one considers the extremal states, e.g. $\mathcal{G}_{\theta-N} \ldots \mathcal{G}_{\theta-1} \cdot |h, \ell, t; \theta\rangle, \theta \in \mathbb{C}$. These are further continued in $\theta - N \sim \theta'$, with $\theta' \in \mathbb{C}$ not necessarily differing from $\theta$ by an integer. Such generalized extremal states are in fact elements of another generalized Verma module, namely the one characterized by the parameter $\theta'$ (therefore, in accordance with the above, the states are acted upon with $\mathcal{G}_\mu$ with $\mu \in \theta' + \mathbb{Z}$, and $\mathcal{Q}_\nu$, $\nu \in -\theta' + \mathbb{Z}$).

Rather than working with states, we introduce the operators $q(-\theta-n, -\theta) = \mathcal{Q}_{-\theta-N} \mathcal{Q}_{-\theta-N+1} \ldots \mathcal{Q}_{-\theta}$ and $g(\theta - N, \theta) = \mathcal{G}_{\theta-N} \mathcal{G}_{\theta-N+1} \ldots \mathcal{G}_{\theta}$ and then continue them to arbitrary complex arguments,

$$g(\theta', \theta) \quad \text{and} \quad q(\theta', \theta), \quad \theta', \theta \in \mathbb{C},$$

(3.1)

by postulating their algebraic properties in such a way that, whenever $\theta - \theta' \in \mathbb{N}$, the properties become those of the above products of modes. This approach of dealing with relative objects (operators) rather
than with the vectors on which they act, proves to be very fruitful because the new operators possess rich algebraic properties.

The fact that the new operators $g(a, b)$ and $q(a, b)$ can be thought of as an extension of the products of modes to a non-integral (in fact, complex) number of factors is formalized as follows:

Define the length of $g(a, b)$ or $q(a, b)$ as $b - a + 1$. Then, the ‘tautological’ property of the new operators is

**Positive integral length reduction.** Whenever the length is a non-negative integer, the operator $g(a, b)$ or $q(a, b)$ becomes the product of the corresponding modes:

$$
g(a, b) = \prod_{i=0}^{L-1} G_{a+i}, \quad q(a, b) = \prod_{i=0}^{L-1} Q_{a+i}, \quad \text{iff} \quad L \equiv b - a + 1 = 0, 1, 2, \ldots \tag{3.2}
$$

(in the case $L = 0$ the product evaluates as 1). By definition, these products are ordered as $\ldots \cdot G_a \cdot G_{a+1} \cdot \ldots$.

**Gluing rules.**

$$
g(a, b - 1) g(b, \theta - 1) |\theta\rangle_g = g(a, \theta - 1) |\theta\rangle_g \quad a, b, \theta \in \mathbb{C}. \tag{3.3}
$$

where $|\theta\rangle_g$ is any state that satisfies $G_{\theta+n} |\theta\rangle_g = 0$ for $n \in \mathbb{N}_0$, and $|\theta\rangle_q$, similarly, satisfies $Q_{-\theta+n} |\theta\rangle_q = 0$ for $n \in \mathbb{N}_0$.

**Under the spectral flow transform** (2.5), the operators $g(a, b)$ and $q(a, b)$ behave in the manner inherited from the behaviour of the products (3.2):

$$
U_{\theta} : \begin{aligned}
g(a, b) &\mapsto g(a + \theta, b + \theta), \\
q(a, b) &\mapsto q(a - \theta, b - \theta).
\end{aligned} \tag{3.4}
$$

Further properties of the new operators originate in the fact that, the $N = 2$ generators $Q$ and $G$ being fermions, they satisfy the vanishing formulae such as, e.g.,

$$
G_a \cdot \prod_{i=a}^{a+N} G_i = 0, \quad N \in \mathbb{N}_0, \quad a \leq n \leq a + N.
$$

‘Pauli principle’.

$$
G_a \cdot g(b, c) = 0, \quad Q_a \cdot g(b, c) = 0, \quad a - b \in \mathbb{N}_0 \quad \text{and} \quad (a - c \notin \mathbb{N} \quad \text{or} \quad b - c - 1 \in \mathbb{N}) \tag{3.5}
$$

Similarly, the ‘left-handed’ annihilation properties are expressed by the relations

$$
g(a, b) \cdot G_c = 0, \quad q(a, b) \cdot Q_c = 0, \quad b - c \in \mathbb{N}_0 \quad \text{and} \quad (a - c \notin \mathbb{N} \quad \text{or} \quad a - b - 1 \in \mathbb{N}). \tag{3.6}
$$

We will also need some vanishing conditions with respect to the bosonic operators $L \geq 1$ and $H \geq 1$. These can be derived from the following basic commutation relations for the continued operators:
Positive-moded bosons. The bosonic generators $L_p$ and $H_p$ with $p = 1, 2, \ldots$, commute with operators \([3.1]\) as

$$
[K_n, g(a, b)] = \sum_{p=0}^{d(n,a,b)} g(a, b - p - 1) [K_n, G_{b-p}] G_{b-p+1} \ldots G_b, \quad n \in \mathbb{N},
$$

$$
[K_n, q(a, b)] = \sum_{p=0}^{d(n,a,b)} q(a, b - p - 1) [K_n, Q_{b-p}] Q_{b-p+1} \ldots Q_b,
$$

where $K = L$ or $H$ and

$$
d(n,a,b) = \begin{cases} 
  b - a, & n - b + a \in \mathbb{N}_0 \text{ and } b - a + 1 \in \mathbb{N}_0, \\
  n - 1, & \text{otherwise}
\end{cases}
$$

All the commutators $[K_n, G_{b-l}]$ and $[K_n, Q_{b-l}]$ are to be taken from \([2.6]\). In the case of a positive integral length $b - a + 1 \in \mathbb{N}$, the relations \([3.7]\) turn into identities in the universal enveloping of the $N=2$ algebra. In fact, Eqs. \([3.7]\) are the algebraic continuation of those identities to the case of complex $a$ and $b$. The main point is that, even though the length $b - a + 1$ may not be an integer, there is always an integral number of terms on the RHS of \([3.7]\) (this is reconciled with a naive application of the Leibnitz rule, according to which $K_n$ should be commuted with each of the modes $G_\mu$ 'virtually contained' in $g(a, b)$; according to the Pauli principle, the application of the Leibnitz rule is restricted to the last $n$ modes of the $G_\mu$).

Eigenvalues. Application of the operators $g$ and $q$ changes the eigenvalues of $L_0$ and $H_0$. The effect can be expressed by the commutation relations

$$
[L_0, g(a, b)] = -\frac{1}{2} (a + b)(b - a + 1) g(a, b), \quad [H_0, g(a, b)] = (b - a + 1) g(a, b),
$$

$$
[L_0, q(a, b)] = -\frac{1}{2} (a + b)(b - a + 1) q(a, b), \quad [H_0, q(a, b)] = (-b + a - 1) q(a, b).
$$

Underlying the commutation properties \([3.9]\) is the same intuitive idea as above, that $g(a, b)$ represents the states from $a$ to $b$ filled with fermions $G_\mu$. Namely, the eigenvalues on the RHSs of \([3.9]\) are simply the number of such fermionic factors in the case of $H_0$ and (minus) the sum of their modes in the case of $L_0$.

Positive-moded fermions. Annihilation properties with respect to the fermionic generators can be arrived at as follows. For a negative integral $\theta$ we have a relation in the Verma module $U_{h,l,t}$

$$
Q_{-\theta+n} g(\theta, -1) |h, \ell, t\rangle = \sum_{i=0}^{-\theta-1} (-1)^i g(\theta, \theta + i - 1) \{Q_{-\theta+n}, G_{\theta+i}\} g(\theta + i + 1, -1) |h, \ell, t\rangle,
$$

$$
n \in \mathbb{N}_0, \quad \theta \in -\mathbb{N}.
$$

In this form, the formula does not continue to $\theta \in \mathbb{C}$. However, when we insert the commutators $\{Q_{-\theta+n}, G_{\theta+i}\}$ from \([2.6]\) and evaluate the resulting modes of $L$ and $H$ according to \([3.7]\), we see that \([3.10]\) vanishes for $n \geq 1$. We continue this to

$$
Q_{-\theta+n} g(\theta, -1) |h, \ell, t\rangle = 0, \quad \theta \in \mathbb{C}, \quad n \in \mathbb{N},
$$
while in the case of \( n = 0 \) we are left with
\[
\mathcal{Q}_{-\theta} g(\theta, -1) |h, \ell, t\rangle = \mathcal{Q}_{-\theta} \mathcal{G}_\theta g(\theta + 1, -1) |h, \ell, t\rangle \quad \text{by (3.3)}
\]
\[
= \{\mathcal{Q}_{-\theta}, \mathcal{G}_\theta\} g(\theta + 1, -1) |h, \ell, t\rangle \quad \text{by (3.11)}
\]
\[
= (2\mathcal{L}_0 + 2\theta \mathcal{H}_0 + 2(\theta^2 + \theta)) g(\theta + 1, -1) |h, \ell, t\rangle \quad \text{by (2.6)}
\]

Recalling now Eq. (3.9), we arrive at
\[
\mathcal{Q}_{-\theta} g(\theta, -1) |h, \ell, t\rangle = 2(\ell + \theta h - \frac{1}{\ell}(\theta^2 + \theta)) g(\theta + 1, -1) |h, \ell, t\rangle
\]
\[
(3.13)
\]

Similarly, for the \( q \)-operators, we have the following properties:
\[
\mathcal{G}_{\theta+n} q(-\theta, 0) |h, \ell, t\rangle = 0, \quad n \in \mathbb{N},
\]
\[
\mathcal{G}_\theta q(-\theta, 0) |h, \ell, t\rangle = 2(\ell + \theta h - \frac{1}{\ell}(\theta^2 + \theta)) q(-\theta + 1, 0) |h, \ell, t\rangle.
\]
\[
(3.14)
\]
The formulae (3.11), (3.13), (3.14) are now viewed as continued to \( \theta \in \mathbb{C} \).

Similarly to (3.13) and (3.14), we derive the following relations for integral parameters and then postulate for all \( \theta, \theta' \in \mathbb{C} \):
\[
\mathcal{Q}_{-\theta'} g(\theta', -1) |h, \ell, t\rangle_{\text{top}} = 2(\theta' - \theta)(h + \frac{1}{\ell}(\theta - \theta' - 1)) g(\theta' + 1, \theta - 1) |h, \ell, t\rangle_{\text{top}}
\]
\[
(3.15)
\]
and
\[
\mathcal{G}_{\theta'} q(-\theta', -1) |h, \ell, t\rangle_{\text{top}} = 2(\theta' - \theta)(h + 1 + \frac{1}{\ell}(\theta - \theta' - 1)) q(\theta' + 1, \theta - 1) |h, \ell, t\rangle_{\text{top}}
\]
\[
(3.16)
\]

We have thus given the list of those algebraic properties of the \( q \) and \( g \) operators that are related to the highest-weight/annihilation conditions. The following property is of a somewhat different spirit, but it can be derived in a direct analogy with (3.3):

**Negative-moded bosons.** In addition to the various types of the highest-weight conditions, which apply to essentially the ‘positive’-moded generators, we will also need to commute the negative-moded \( \mathcal{H} \) and \( \mathcal{L} \) operators through \( q(a, b) \) and \( g(a, b) \). The corresponding formulae read
\[
[g(a, b), \mathcal{K}_p] = \sum_{l=0}^{d(-p,a,b)} g_a \cdots g_{a+l-1} \{g_{a+l}, \mathcal{K}_p\} g(a + l + 1, b),
\]
\[
(3.17)
\]
\[
[q(a, b), \mathcal{K}_p] = \sum_{l=0}^{d(-p,a,b)} q_a \cdots q_{a+l-1} \{q_{a+l}, \mathcal{K}_p\} q(a + l + 1, b),
\]
\[
(3.18)
\]
where \( p \in -\mathbb{N} \) and \( d(p, a, b) \) is given by formula (3.3). As before, \( \mathcal{K} = \mathcal{H} \) or \( \mathcal{L} \).

**Highest-weight conditions.** The above annihilation properties (3.7), (3.11), and (3.14) allow us to formulate the following assertion regarding the highest-weight properties of continued states \( g(\theta, -1) |h, \ell, t\rangle \) and \( q(-\theta, 0) |h, \ell, t\rangle \), which turn out to be the generalized highest-weight states:
Lemma 3.1

1. The objects $g(\theta, -1) | h, \ell, t \rangle$ and $q(-\theta, 0) | h, \ell, t \rangle$ satisfy the following annihilation conditions:

\[
L_m \, g(\theta, -1) | h, \ell, t \rangle = 0, \quad \mathcal{H}_m \, g(\theta, -1) | h, \ell, t \rangle = 0, \quad m \in \mathbb{N},
\]

\[
\mathcal{G}_a \, g(\theta, -1) | h, \ell, t \rangle = 0, \quad a \in \theta + \mathbb{N}_0,
\]

\[
\mathcal{Q}_a \, g(\theta, -1) | h, \ell, t \rangle = 0, \quad a \in -\theta + \mathbb{N}.
\]

\[
L_m \, q(-\theta, 0) | h, \ell, t \rangle = 0, \quad \mathcal{H}_m \, q(-\theta, 0) | h, \ell, t \rangle = 0, \quad m \in \mathbb{N},
\]

\[
\mathcal{G}_a \, q(-\theta, 0) | h, \ell, t \rangle = 0, \quad a \in \theta + \mathbb{N},
\]

\[
\mathcal{Q}_a \, q(-\theta, 0) | h, \ell, t \rangle = 0, \quad a \in -\theta + \mathbb{N}_0.
\]

2. In the topological case, we have

\[
\mathcal{G}_a \, g(\theta', -1) | h, t; \theta \rangle_{\text{top}} = 0, \quad \mathcal{Q}_a \, g(\theta', -1) | h, t; \theta \rangle_{\text{top}} = 0, \quad a - \theta' \in \mathbb{N}_0,
\]

\[
\mathcal{G}_a \, g(\theta', -1) | h, t; \theta \rangle_{\text{top}} = 0, \quad \mathcal{G}_a \, q(\theta', -1) | h, t; \theta \rangle_{\text{top}} = 0, \quad a + \theta' \in \mathbb{N}
\]

\[
L_m \, g(\theta', -1) | h, t; \theta \rangle_{\text{top}} = 0, \quad \mathcal{H}_m \, q(\theta', -1) | h, t; \theta \rangle_{\text{top}} = 0, \quad m \in \mathbb{N},
\]

\[
\mathcal{H}_m \, g(\theta', -1) | h, t; \theta \rangle_{\text{top}} = 0, \quad \mathcal{L}_m \, q(\theta', -1) | h, t; \theta \rangle_{\text{top}} = 0, \quad m \in \mathbb{N}.
\]

Now, we can prove the following

Theorem 3.2 Up to a normalization factor, the objects $g(\theta, -1) | h, \ell, t \rangle$ and $q(-\theta, 0) | h, \ell, t \rangle$ represent generalized massive highest-weight vectors with the following parameters:

\[
|h', \ell', t; \theta'\rangle \sim g(\theta', -1) | h, \ell, t \rangle,
\]

\[
h' = h + \frac{2}{\ell}(\theta - \theta'),
\]

\[
\ell' = \ell + (\theta' - \theta)(h - \frac{1}{\ell}(\theta' - \theta + 1))
\]

and

\[
|h'', \ell'', t; \theta' + 1\rangle \sim q(-\theta', -\theta) | h, \ell, t \rangle,
\]

\[
h'' = h + \frac{2}{\ell}(\theta - \theta' - 1),
\]

\[
\ell'' = \ell + (\theta' - \theta + 1)(h - \frac{1}{\ell}(\theta' - \theta + 2))
\]

Indeed, the annihilation properties follow by applying (3.4) to the relations of the previous Lemma. The actual parameters of the states follows by a direct calculation, which we demonstrate for $h'$. From (2.23), we have

\[
\mathcal{H}_0 | h', \ell', t; \theta'\rangle = (h' - \frac{3}{\ell}\theta') | h', \ell', t; \theta'\rangle.
\]

On the LHS of (3.21), we evaluate $\mathcal{H}_0$ using (3.9), whence (3.21) follows. Note also that whenever $\ell + (\theta' - \theta)h - \frac{1}{\ell}(\theta' - \theta)^2 + \theta' - \theta = 0$, Eqs. (3.14) also allow us to show that, in addition to (3.22),

\[
q(-\theta', -\theta) | h, \ell, t; \theta\rangle \sim | h + \frac{2}{\ell}(\theta - \theta' - 1), t; \theta'\rangle_{\text{top}}.
\]
Transitivity identities. We saw in the previous section that in the general positions, the arrows in the extremal diagrams can be inverted by the action of the ‘opposite’ fermion. The following formulae formalize the procedure of ‘canceling’ several fermionic generators from the left of the $g$ and $q$ operators

1. for the twisted topological highest-weight vectors,
   \[ g(\theta' + 1, \theta - 1) |h, t; \theta\rangle_{\text{top}} = \frac{1}{\Lambda_g(\theta, \theta', h, t)} Q_{-\theta'} g(\theta' \theta - 1) |h, t; \theta\rangle_{\text{top}} \]  
   \[ \text{where} \]
   \[ \Lambda_g(\theta, \theta', h, t) = 2(\theta' - \theta)(h + \frac{1}{t}(\theta - \theta' - 1)), \]  
   and
   \[ g(-\theta' + 1, -\theta - 1) |h, t; \theta\rangle_{\text{top}} = \frac{1}{\Lambda_g(\theta, \theta', h, t)} G_{\theta'} q(-\theta' - \theta - 1) |h, t; \theta\rangle_{\text{top}} \]  
   \[ \text{where} \]
   \[ \Lambda_g(\theta, \theta', h, t) = 2(\theta' - \theta)(h + 1 + \frac{1}{t}(\theta - \theta' - 1)), \]

2. for the massive highest-weight vectors,
   \[ g(\theta + 1, -1) |h, \ell, t\rangle = \frac{1}{\Lambda(\theta, h, \ell, t)} Q_{-\theta} g(\theta, -1) |h, \ell, t\rangle, \]
   and
   \[ q(-\theta + 1, 0) |h, \ell, t\rangle = \frac{1}{\Lambda(\theta, h, \ell, t)} G_{\theta} q(-\theta, 0) |h, \ell, t\rangle, \]
   with
   \[ \Lambda(\theta, h, \ell, t) = 2(\ell + \theta h - \frac{1}{t}(\theta^2 + \theta)) \]

in both cases.

It follows that ‘violations of transitivity’ occur whenever the corresponding $\Lambda$-factor vanishes. Then the state on the RHS of the respective formula satisfies stronger annihilation conditions, namely it becomes a (generalized) topological highest-weight state (while in the general position it is just a (generalized) state on the RHS of the respective formula satisfies stronger annihilation conditions, namely it becomes a (generalized) topological highest-weight state (while in the general position it is just a (generalized) massive highest-weight state). These ‘enhanced’ highest-weight conditions occur in the following cases:

\[ \theta' = ht + \theta - 1 \iff g(\theta', \theta - 1) |h, t; \theta\rangle_{\text{top}} = |h', t; \theta'\rangle_{\text{top}} \iff \Lambda_g(\theta, \theta', h, t) = 0 \text{ in } (3.24) \]
\[ \theta' = (h + 1)t + \theta - 1 \iff q(-\theta', -\theta - 1) |h, t; \theta\rangle_{\text{top}} = |h', t; \theta'\rangle_{\text{top}} \iff \Lambda_g(\theta, \theta', h, t) = 0 \text{ in } (3.26) \]
\[ \ell + h\theta - \frac{1}{t}(\theta^2 + \theta) = 0 \iff g(\theta', -1) |h, t; \ell\rangle = |h', t; \theta'\rangle_{\text{top}} \iff \Lambda(\theta, h, \ell, t) = 0 \text{ in } (3.28) \]
\[ q(-\theta', 0) |h, t; \ell\rangle = |h', t; \theta'\rangle_{\text{top}} \iff \Lambda(\theta, h, \ell, t) = 0 \text{ in } (3.27) \]

4 Algebraic constructions of $N=2$ singular vectors

4.1 Constructing the topological singular vectors

The $N=2$ singular vectors are constructed in this section in a monomial form, as a product of the $q$ and $g$ operators acting on the corresponding highest-weight vector. To begin with, observe that the mappings of generalized topological highest-weight vectors by the $g$- and $q$- operators can be combined as follows. We apply $q$, $g$, $q$, . . . , to the highest-weight vector $|h, t\rangle_{\text{top}}$ either as

\[
\mathcal{W}_{h,t;\theta} \xrightarrow{g(\theta_{1}, \theta_{1} - 1)} \mathcal{W}_{h_{1}, t;\theta_{1}} \xrightarrow{q(-\theta_{2} - \theta_{1} - 1)} \ldots
\]  
(4.1)
or as
\[ \mathcal{V}_{h,t,\theta} q^{(-\theta_{-1} - \theta_{-1} - \theta^{-1})} \mathcal{V}_{h_{-1},t,\theta_{-1}} q^{(\theta_{-2} - \theta_{-1} - \theta^{-1})} \cdots \] (4.2)
and requiring that the generalized topological highest-weight conditions be preserved at each step, we see that the \( h \) and \( \theta \) parameters should take on the values

\[ \begin{align*}
\theta_i &= \begin{cases}
(h - j)t - 1 + \theta & i = 2j + 1, \\
jt + \theta & i = 2j.
\end{cases}
\] (4.3)

\[ \begin{align*}
h_i &= \begin{cases}
\frac{2}{t} - h + 2j & i = 2j + 1, \\
h - 2j & i = 2j
\end{cases}
\] (4.3)

We thus take a chain of the \( q \) and \( g \) operators and act with it on the topological highest-weight state \( |h, t\rangle_{\text{top}} \) of the topological Verma module \( \mathcal{V}_{h,t} = \mathcal{V}_{h,0} \); it follows that the \( \theta \) parameters in \((4.1)\) are

\[ \begin{align*}
\theta_i(h) &= \begin{cases}
(h - j)t - 1 & i = 2j + 1, \\
jt & i = 2j
\end{cases}
\] (4.4)

This is in fact similar to the singular vector construction for affine Lie algebras [1], however there is no affine Weyl group in the \( N=2 \) case, while in the affine case it is this group that governs the exponents in the MFF formulae. As in the case of affine Lie algebras, the condition for a Verma module to contain a singular vector is that after \( 2s - 1 \) “reflections” \((4.1) - (4.2)\), we return to \( \mathcal{V}_{h,t} \). Consider for definiteness the case when we start from the action of \( g \), hence \( i > 0 \):

\[
\begin{array}{c}
\mathcal{V}_{h,t} \\
\vdots \\
\end{array}
\]
\[ q \]
\[ g \]
\[ q \]

The condition for this to happen is that \( \theta_{2s-1}(h) = (h - s + 1)t - 1 \) be a negative integer, say \(-r\). We thus rederive the formulae \((2.17)\) expressing \( h^\pm \) in terms of two positive integers \( r \) and \( s \). The formula for \( h^\pm \) is recovered by considering a similar loop starting and ending with a \( q \) operator.

One has the following Theorem:

**Theorem 4.1 ([1])** All singular vectors in the topological Verma module \( \mathcal{V}_{h=(r,s,t),t} \) over the \( N=2 \) superconformal algebra are given by the explicit construction:

\[ |E(r,s,t)^+\rangle = g(-r, (s-1)t - 1) q(-(s-1)t, r - 1 - t) \cdots g((s-2)t - r, t - 1) q(-t, r - 1 - t(s-1)) \cdot g((s-1)t - r, -1) |h^+(r,s,t), t\rangle_{\text{top}}, \] (4.6)

\[ |E(r,s,t)^-\rangle = q(-r, (s-1)t - 1) g(-(s-1)t, r - 1 - t) \cdots q((s-2)t - r, t - 1) g(-t, r - 1 - (s-1)t) \cdot q((s-1)t - r, -1) |h^-(r,s,t), t\rangle_{\text{top}} \] (4.7)

\[ r, s \in \mathbb{N} \]

where the factors in the first line of each formula are

\[ g(-r - t - mt + st, -1 + mt) q(-mt, r - 1 + mt - st), \quad s - 1 \geq m \geq 1 \] (4.8)

and

\[ q(-r - t - mt + st, -1 + mt) g(-mt, r - 1 + mt - st), \quad s - 1 \geq m \geq 1 \] (4.9)
respectively. The $|E(r, s, t)|^{\pm}$ singular vectors satisfy twisted topological highest-weight conditions with the ‘spectral’ parameter $\theta = \mp r$ and are on level $rs + \frac{1}{r}(r-1)$ over the corresponding topological highest-weight state and have relative charge $\pm r$.

In a direct analogy with the well-known affine Lie algebra case \cite{M2,AS}, “all singular vectors” applies literally to non-rational $t$, while for rational $t$, a singular vector may be given already by a subformula of Eqs. (4.6), (4.7) as soon as that subformula (obtained by dropping several $g$- and $q$-operators from the left) produces an element of the Verma module.

It will be shown in section 5.1 how the states (4.6), (4.7) can be rewritten as polynomials in the topological singular vector $V$ and $\varphi_1(r,s,t)$ thereby effectively decreasing the value of $s$:

\begin{align*}
\mathcal{Q}_{h^+(r,s,t),t}^{(1)} & \quad \mathcal{Q}_{h^-(r,s,t),t}^{(-1)} \\
\mathcal{Q}_{h^-(r,s-1,t),t}^{(1)} & \quad \mathcal{Q}_{h^+(r,s-1,t),t}^{(-1)}
\end{align*}

(4.10)

(\text{the horizontal arrows being the spectral flow transform mappings), where we have rewritten $\theta_{\pm 1}$ from (14) as}

$$\theta_{\pm 1}(r, s, t) = \theta_{\pm 1}(h^\pm(r, s, t)) = \pm(t(s-1)-r)$$

Thus, introducing the singular vector operators $\mathcal{E}$ as

$$|E(r, s, t)|^{\pm} = \mathcal{E}^{\pm}(r, s, t) |h^{\pm}(r, s, t), t\rangle_{\text{top}},$$

we have the recursion relations

\begin{align*}
\mathcal{E}^+(r, s, t) &= g(-r, (s-1)t-1) \mathcal{E}^-(s-1)t-r(r, s-1, t) g((s-1)t-r, -1), \\
\mathcal{E}^-(r, s, t) &= q(-r, (s-1)t-1) \mathcal{E}^+(s-1)t+r(r, s-1, t) q((s-1)t-r, -1).
\end{align*}

(4.11)

where $\mathcal{E}^{\pm, \theta}$ is the spectral flow transform of $\mathcal{E}^{\pm}$. These relations involve the continued operators, however the combination in which these operators appear allows us to evaluate Eqs. (4.11) in terms of the usual $N=2$ generators. This will be shown in Sec. 5.

A remark is in order regarding the top-level representatives \cite{S, M} of topological singular vectors, discussed in (2.30). The top-level representatives satisfy the ‘untwisted’ annihilation conditions (2.31). Such vectors are easily constructed from the above $|E(r, s, t)|^{\pm}$ by traveling along the extremal diagram as follows:

\begin{align*}
|s(r, s, t)|^{+} &= \mathcal{Q}_0 \ldots \mathcal{Q}_{r-1} |E(r, s, t)|^{+}, \\
|s(r, s, t)|^{-} &= \mathcal{G}_0 \ldots \mathcal{G}_{r-1} |E(r, s, t)|^{-}.
\end{align*}

(4.12)
They have relative charge zero and level \(rs\) with respect to the corresponding topological highest-weight state \(|h^\pm(r, s, t), t\rangle_{\text{top}}\). However, there are two factors that make working with \(|s(r, s, t)\rangle^\pm\) quite complicated. Firstly, as we have already discussed, the \(|s(r, s, t)\rangle^\pm\) vectors satisfy the massive highest-weight conditions and, thus, conceal the fact that the submodule generated from them is actually topological. Secondly, these vectors may not even generate the entire of the topological submodule generated from the respective \(|E(r, s, t)\rangle^\pm\). Indeed, while in the general position we have

\[ G_{-r+1} \ldots G_0 |s(r, s, t)\rangle^\pm = C^\pm(r, s, t) |E(r, s, t)\rangle^+ \]

(where \(C\) is a scalar factor) and, thus, the respective singular vectors (4.12) and (4.7) generate the same submodule, yet in degenerate cases the action of one of the \(G_n\) in the last formula may give the vanishing result, in which case \(|s(r, s, t)\rangle^\pm\) would generate only a submodule in the submodule generated from \(|E(r, s, t)\rangle^+\). Therefore, working with the singular vectors that satisfy highest-weight conditions (2.31) makes it necessary to introduce subsingular vectors in topological Verma modules in order to completely describe the structure of submodules. On the other hand, having the \(|E(r, s, t)\rangle^\pm\) singular vectors at our disposal makes the subsingular vector superfluous, namely an artifact of having chosen \(|s(r, s, t)\rangle^\pm\) to represent singular vectors [8].

### 4.2 Singular vectors in massive Verma modules

The prescription to construct singular vectors in a massive Verma module \(U\) is to map the highest-weight vector of the module into a generalized topological highest-weight vector by means of the continued operators \(g\) or \(q\). In the generalized topological Verma module, one then requires that a (topological!) singular vector exist. Further, one uses the \(g\) or \(q\) operator to map this topological singular vector back into the original module \(U\). This program is implemented as follows.

Given a massive highest-weight vector \(|h, \ell, t\rangle\), let \(\theta'\) and \(\theta'' = -\theta' + ht - 1\) be two roots of the equation

\[ \ell = -\theta h + \frac{1}{t}(\theta^2 + \theta). \]  

Then the states

\[ g(\theta', -1)|h, \ell, t\rangle, \quad q(-\theta', 0)|h, \ell, t\rangle \]

and

\[ g(\theta'', -1)|h, \ell, t\rangle, \quad q(-\theta'', 0)|h, \ell, t\rangle, \]

are the generalized topological highest-weight vectors. Whenever one of the states (4.15) (or (4.16)) belongs to \(U_{h,\ell,t}\), it gives a charged singular vector in \(U_{h,\ell,t}\) and we, thus, recover Theorem 2.11. Note that the condition for this to be the case reproduces Eq. (2.27) for the parameters of the highest-weight vector. Other singular vectors in massive Verma modules can be constructed using the following trick. We require one of the states in (4.15) or (4.16) to admit a topological singular vector. Using (3.21)–(3.23), we, thus, require that \(h'\) or \(h''\) take one of the values \(h^\pm(r, s, t)\), see Lemma 2.7. Therefore, \(\theta'\) and \(\theta''\) must be equal to

\[ \theta_1(r, s, h, t) = \frac{1}{2}(h - h^-(r, s, t)), \]

\[ \theta_2(r, s, h, t) = \frac{1}{2}(h - 1 - h^+(r, s, t)). \]

Note that using these values of \(\theta\) in (4.14), we recover Eq. (2.32). Thus, along with (2.27), we have recovered all of the zeros of the Kač determinant [23]. The charged singular vector are already given by
the explicit construction (2.28), while the above derivation of (2.32) suggests how the massive singular vectors, too, can be constructed explicitly.

Let, for definiteness, the first of the states (4.15) admit the topological singular vector $|E(r, s, t)\rangle^-$, Eq. (4.7). Then $\theta' = \theta_1(r, s, h, t)$ and, as is easy to see, the second state from (4.15) admits a $|E(r, s, t)\rangle^+$ topological singular vector. We thus obtain two singular vectors on the generalized topological highest-weight states (4.15):

$$E^{-, \theta_1(r,s,h,t)}(r, s, t) g(\theta_1(r, s, h, t), -1) |h, l(r, s, h, t), t\rangle,$$
$$E^{+, \theta_2(r,s,h,t)}(r, s + 1, t) q(-\theta_2(r, s, h, t), 0) |h, l(r, s, h, t), t\rangle$$

(4.18)

Mapping these singular vectors back to the original massive Verma module $\mathcal{U}_{h,l(r,s,h,t),t}$ by the appropriate $g$- or $q$-operator, we obtain

**Theorem 4.2**

1. For generic $h$ and $t$, representatives of the massive singular vector in $\mathcal{U}_{h,l(r,s,h,t),t}$ read

$$|S(r, s, h, t)\rangle^- = g(-rs, r + \theta_1(r, s, h, t) - 1) E^{-, \theta_1(r,s,h,t)}(r, s, t) g(\theta_1(r, s, h, t), -1) |h, l(r, s, h, t), t\rangle,$$
$$|S(r, s, h, t)\rangle^+ = q(1 - rs, r - \theta_2(r, s, h, t) - 1) E^{+, \theta_2(r,s,h,t)}(r, s, t) q(-\theta_2(r, s, h, t), 0) |h, l(r, s, h, t), t\rangle,$$

(4.19)

where $\theta_1(r, s, h, t)$ and $\theta_2(r, s, h, t)$ are given by (4.14) and $E^{\pm, \theta}(r, s, t)$ denotes the spectral flow transform (2.3) of the topological singular vector operator $E^{\pm}(r, s, t)$.

2. The RHSs of (4.19) evaluate as elements of $\mathcal{U}_{h,l(r,s,h,t),t}$ and satisfy the twisted massive highest-weight conditions

$$Q_{\geq 1, \mp rs} |S(r, s, h, t)\rangle^\pm = H_{\geq 1} |S(r, s, h, t)\rangle^\pm = L_{\geq 1} |S(r, s, h, t)\rangle^\pm = G_{\geq \mp rs} |S(r, s, h, t)\rangle^\pm = 0,$$
$$L_0 |S(r, s, h, t)\rangle^\pm = 1^\pm (r, s, h, t) |S(r, s, h, t)\rangle^\pm,$$
$$H_0 |S(r, s, h, t)\rangle^\pm = (h \mp rs) |S(r, s, h, t)\rangle^\pm,$$

(4.20)

with

$$1^- (r, s, h, t) = l(r, s, h, t) + \frac{1}{2} (rs + 1)(rs + 2) - 1,$$
$$1^+ (r, s, h, t) = l(r, s, h, t) + \frac{1}{2} rs (rs + 1).$$

(4.21)

In the generic case, either of the states $|S(r, s, h, t)\rangle^\pm$ generates the entire massive Verma submodule; in particular, all of the dense $G/Q$-descendants of $|S(r, s, h, t)\rangle^+$ and $|S(r, s, h, t)\rangle^+$ are on the same extremal subdiagram (the extremal diagram of the submodule) and coincide up to numerical factors whenever they are in the same grade:

$$c_- (i, h, t) Q_{i+1- \mp rs} \ldots Q_{\pm rs} |S(r, s, h, t)\rangle^- = c_+(i, h, t) G_{\pm rs-1} \ldots G_{\pm rs-1} |S(r, s, h, t)\rangle^+, \quad i = 0, \ldots, 2rs,$$
$$c_- (i, h, t) G_{-\pm rs+i} \ldots G_{-\pm rs-1} |S(r, s, h, t)\rangle^- = c_+(i, h, t) G_{-\pm rs-1} \ldots G_{-\pm rs-1} |S(r, s, h, t)\rangle^+, \quad i \leq -1,$$
$$c_- (i, h, t) Q_{\mp rs-i} \ldots Q_{\pm rs} |S(r, s, h, t)\rangle^- = c_+(i, h, t) Q_{\pm rs-1} \ldots G_{\pm rs} |S(r, s, h, t)\rangle^+, \quad i \geq 2rs + 1,$$

(4.22)

where the numerical coefficients $c_\pm (i, h, t)$ are ($r$- and $s$-dependent) polynomials in $h$ and $t$.

Thus, whenever the $h$ and $t$ parameters of the massive Verma module $\mathcal{U}_{(r,s,h,t),h,t}$ are in the general position, the maximal submodule can be generated from $|S(r, s, h, t)\rangle^-$ as well as from $|S(r, s, h, t)\rangle^+$. In
the generic case, moreover, the top-level representative of the extremal diagram of the massive submodule generates that very same submodule as $|S(r, s, h, t)|^\pm$. In a number of degenerate cases, however, the situation changes. The complete classification of the degenerate cases will be given in [8], while here, in Sec. 5, we consider two basic cases of such a degeneration.

5 Evaluating $N=2$ singular vectors in the Verma form

In this section, we use the algebraic properties of the continued operators from section 3 in order to formulate the recipe of rewriting the general formulae for the massive and topological singular vectors as polynomials in the usual Verma module creation operators acting on the highest-weight state.

5.1 The topological singular vectors

In this subsection, we show how topological singular vectors (4.6) and (4.7) can be recast into the Verma form, i.e., into polynomials in the standard creation operators $L_{\leq -1}$, $H_{\leq -1}$, $G_{\leq -1}$, and $Q_{\leq -1}$ in the topological Verma module.

To this end, we formulate a recursive procedure which is based on relations (4.11). We concentrate on the $E^+$ vector for definiteness. Assume that $E^-(r, s - 1, t)$ does already have the conventional Verma form (i.e., is a polynomial in creation operators). Then, for any $\theta$ (which we will actually take to be $(s - 1)t - r$), the spectral-flow-transformed singular vector operator $E^{-, \theta}(r, s - 1, t)$ also rewrites as a polynomial in $L_{-m}$, $H_{-m}$, $m \in \mathbb{N}$, $G_{-m+\theta}$, and $Q_{-m-\theta}$, with $m \in \mathbb{N}$.

We now use gluing rules (3.3) to rewrite the corresponding formula from (4.11) as

$$E^+(r, s, t) = g(-r, (s - 1)t - r - 1) G_{(s-1)t-r} \cdots G_{(s-1)t-1} E^{-, (s-1)t-r}(r, s - 1, t) g((s - 1)t - r, -1).$$  (5.1)

Recall also that this singular vector operator is to be applied to the highest-weight vector $|h^+(r, s, t), t\rangle_{\text{top}}$. Then, in accordance with Lemma 3.1, we observe that each of the operators $G_{(s-1)t-r}, \ldots, G_{(s-1)t-1}$ annihilates the generalized topological highest-weight state $g((s - 1)t - r, -1) |h^+(r, s, t), t\rangle_{\text{top}}$. We, thus, can commute all of these operators in (5.1) through the topological singular vector operator $E^{-, (s-1)t-r}(r, s - 1, t)$ and kill them as soon as they reach the $g$-operator on the right. The topological singular vector thus rewrites as

$$|E(r, s, t)\rangle^+ = g(-r, (s - 1)t - r - 1) \mathcal{P}(L, H, G, Q) g((s - 1)t - r, -1) |h^+(r, s, t), t\rangle_{\text{top}}$$  (5.2)

where $\mathcal{P}$ is a polynomial in $L_{-m}$, $H_{-m}$, $G_{-m+(s-1)t-r}$, and $Q_{-m-(s-1)t+r}$, with $m \in \mathbb{N}$. Since the topological singular vector $|E(r, s, t)\rangle^{-}$ has relative charge $-r$, the result of commuting its singular vector operator with $r$ modes of $G$ is $H_0$-neutral; in other words, $\mathcal{P}$ has zero charge.

Observe further that any mode $G_\mu$ that can be encountered in $\mathcal{P}$ is necessarily such that formulae (5.6) apply to the product $g(-r, (s - 1)t - r - 1) G_\mu$. In this way, all of the $G$-modes can be commuted to the left, after which they give the vanishing contribution to (5.2). Therefore, the singular vector rewrites as

$$|E(r, s, t)\rangle^+ = g(-r, (s - 1)t - r - 1) \mathcal{P}(L, H) g((s - 1)t - r, -1) |h^+(r, s, t), t\rangle_{\text{top}}.$$  (5.3)

In what follows, we refer to the thus obtained operator polynomial $\mathcal{P}(L, H)$ as the skeleton of the corresponding singular vector. In (5.3), had it not been for the skeleton in the middle, we would have
used the formula
\[ g(-r, (s-1)t - r - 1) \cdot g((s-1)t - r, -1) \mid h^{+}(r, s, t), t \rangle_{\text{top}} = g(-r, -1) \mid h^{+}(r, s, t), t \rangle_{\text{top}} = \prod_{i=-r}^{-1} G_{i} \mid h^{+}(r, s, t), t \rangle_{\text{top}} \]
to glue two \( g \)-operators together, after which they reduce to the Verma form. In fact, Eqs. (3.17) allow us to commute the \( L \)- and \( H \)-modes on the left, after which the two \( g \)-operators meet and one can use
\[ g(j, (s-1)t - r - 1) \cdot g((s-1)t - r, -1) \mid h^{+}(r, s, t), t \rangle_{\text{top}} = g(j, -1) \mid h^{+}(r, s, t), t \rangle_{\text{top}} \]
which is the whole expression (5.1) is in turn evaluated in the Verma form. The argument now applies recursively, until we reach the respective simplest singular vector. Thus, having assumed that the inner singular vector operators in (5.1) are already in the Verma form,
\[ |E(r, s, t)\rangle^{+} = \mathcal{P}(\mathcal{L}, \mathcal{H}) + \sum_{j=1}^{r(s-2)} (\prod_{i=1}^{r+j} \mathcal{P}_{L}(\mathcal{L}, \mathcal{H}) g(j, -1)) \mid h^{+}(r, s, t), t \rangle_{\text{top}}, \quad j \geq 2 \quad (5.4) \]
where \( \mathcal{P}_{j} \) are polynomials in \( \mathcal{L}_{-m}, \mathcal{H}_{-m}, \) and \( \mathcal{G}_{-m}, m \in \mathbb{N} \). To the terms that involve the \( g \)-operators of a negative integral length, we now apply Eqs. (3.24)–(3.30).

Starting with higher \( j \), we, thus, replace
\[ g(j, -1) \mid h^{+}(r, s, t), t \rangle_{\text{top}} = \frac{t}{2(j-1)(h^{+}(r, s, t)t-j)} Q_{1-j} g(j-1, -1) \mid h^{+}(r, s, t), t \rangle_{\text{top}}, \quad j \geq 2 \quad (5.5) \]
and commute the product \( (\prod \mathcal{G} \rangle \) in (5.4) to the right. Then, in some terms the combination \( (\prod \mathcal{G} \rangle \) \( g(j-1, -1) \) would allow us to apply (3.3), which would give one of the \( g(j-2, -1), g(j-3, -1), \ldots, g(0, -1) \) operators. In the latter case \( (g(0, -1) = 1 \) the \( g \)-operator will have disappeared. To the terms that would still contain \( g(j-1, -1), \ldots, g(2, -1) \), we apply (5.3) and the corresponding rearrangements again, until we end up having, on top of a state from the Verma module \( V_{h^{+}(r, s, t), t} \), only the terms that contain \( g(1, -1) \). However, \textit{all the latter cancel against the different terms}, and we are therefore left with a state in \( V_{h^{+}(r, s, t), t} \).

This vanishing property is a non-trivial feature of the whole scheme; a related fact is that applying formula (3.24) to the evaluation of topological singular vectors does always preserve the coefficients in front of the different terms in the ring \( t^{-M} \cdot \mathbb{Q}[t] \) (polynomials with rational coefficients times a certain negative power of \( t \) that may come out of the highest-weights (2.17)), i.e. no rational dependence on \( t \) arises, apart from a possible \( t^{-M} \).

Thus, having assumed that the inner singular vector operators in (5.1) are already in the Verma form, we see that the whole expression (5.1) is in turn evaluated in the Verma form. The argument now applies recursively, until we reach the respective simplest singular vector
\[ |E(r, 1, t)\rangle^{+} = \mathcal{G}_{-r} \ldots \mathcal{G}_{-1} \mid h^{+}(r, 1, t), t \rangle_{\text{top}}, \quad \mathcal{E}^{+}(r, 1, t) = \mathcal{G}_{-r} \ldots \mathcal{G}_{-1} \]
\[ |E(r, 1, t)\rangle^{-} = \mathcal{Q}_{-r} \ldots \mathcal{Q}_{-1} \mid h^{-}(r, 1, t), t \rangle_{\text{top}}, \quad \mathcal{E}^{-}(r, 1, t) = \mathcal{Q}_{-r} \ldots \mathcal{Q}_{-1} \]
Evaluation in the \( \mathcal{E}^{-} \)-case is completely similar.

5.2 The massive singular vectors

As follows from (4.19), the representatives \( |S^{\pm}(r, s, t)\rangle \) of a massive singular vector can be derived from the appropriate topological singular vectors \( |E^{\pm}(r, s, t)\rangle \). As before, an important point is that, at a
certain stage in the evaluation, all that remains of the topological singular vector operator is the skeleton – a polynomial in modes of only $\mathcal{L}_m$ and $\mathcal{H}_m$.

Consider for definiteness how the massive singular vectors are evaluated as Verma module elements using the form $|S(r,s,h,t))\rangle$ from Eq. (4.19). First of all, we rewrite

$$g(-rs, r + \theta_1(r, s, h, t) - 1) \mathcal{E}^{-\theta_1(r,s,h,t)}(r, s, t) g(\theta_1(r, s, h, t), -1)$$

as

$$g(-rs, \theta_1(r, s, h, t) - 1) \mathcal{G}_{\theta_1(r,s,h,t)} \cdots \mathcal{G}_{\theta_1(r,s,h,t) + r - 1} \mathcal{E}^{-\theta_1(r,s,h,t)}(r, s, t) g(\theta_1(r, s, h, t), -1).$$

We now observe that the modes $\mathcal{G}_{\theta_1(r,s,h,t)}, \ldots, \mathcal{G}_{\theta_1(r,s,h,t) + r - 1}$ annihilate the state $g(\theta_1(r, s, h, t), -1) \cdot |h, l(r, s, h, t), t\rangle$. We, thus, commute all these modes to the right in every operator monomial. Then, the remaining $\mathcal{G}$-modes are annihilated by $g(-rs, r + \theta_1(r, s, h, t) - 1)$ in accordance with (4.6). After commuting the $\mathcal{G}$-modes to the left and then dropping the $\mathcal{G}$-dependent monomials, the operator $g(0, \theta_1(r, s, h, t) - 1)$ is separated from the right one, $g(\theta_1(r, s, h, t), -1)$, by modes of only $\mathcal{L}$ and $\mathcal{H}$ (the skeleton). We now apply Eqs. (4.17) repeatedly. Then the two g-operators meet and produce, in accordance with (3.3) and (3.2), either the product of $\mathcal{G}$ modes or the identity operator. Thus we are left with a singular vector in the Verma module $\mathcal{U}_{h, l(r, s, h, t), t}$.

The $+\text{-}$representative of the massive singular vector is evaluated in a similar way starting with the topological singular vector $|S(r, s, t))\rangle^+$. Here, one first commutes $r$ modes $\mathcal{Q}_{-\theta_2(r,s,h,t)}, \ldots, \mathcal{Q}_{r - \theta_2(r,s,h,t) - 1}$ to the right until they annihilate the state $q(-\theta_2(r, s, h, t), 0) |h, l(r, s, h, t), t\rangle$. Then all the remaining $q$-modes are commuted to the left, where, again, they are annihilated by $q(1 - rs, -\theta_2(r, s, h, t) - 1)$.

In this way, we obtain massive singular vectors as elements of the corresponding Verma module.

### 5.3 Evaluating massive singular vectors: an example

Here, we give an example of the evaluation of massive singular vectors. As the length of the expressions grows rapidly with $r$ and $s$, we restrict ourselves to level-3 singular vector $|S(1,3, h, t))\rangle^+$ and $|S(1,3, h, t))\rangle^-$. We follow the strategy of section 1.2: deriving the massive singular vector from the corresponding topological one. Then, we demonstrate that $|S(1,3, h, t))\rangle^+$ and $|S(1,3, h, t))\rangle^-$ generate the same submodule.

Consider first $|S(1,3, h, t))\rangle^-$. The starting point is the topological singular vector $|E(1,3, t))\rangle$ (see (1.7)) written in the polynomial form (see (1) for the examples of calculations of topological singular vectors):

$$|E(1,3, t))\rangle^- = \left( (4t + 16t^2)Q_{-3} + (-4 + 8t^2)H_{-2}Q_{-1} + (-12t + 16t^2)H_{-1}Q_{-2} - 4tL_{-2}Q_{-1} \right) |3 + \frac{t}{2}, t\rangle_{\text{top}}$$

Now, in accordance with the procedure described in Sec. 5.2, we calculate the skeleton of $|E(r, s, t))\rangle^-$. To this end, we write the massive singular vector to be evaluated as

$$|S(1,3, h, t))\rangle^- = g(-3, \frac{t}{2}(h - h^-(1,3,t) - 1)) \cdot \mathcal{G}_{\frac{t}{2}(h - h^-(1,3,t))} \mathcal{E}^{-\frac{t}{2}(h - h^-(1,3,t))} g(\frac{t}{2}(h - h^-(1,3,t)), -1) |h, l(1,3, h, t), t\rangle,$$
To evaluate the underbraced factors, where $G_{\frac{1}{2}(h-h^{-1}(1,3,t))}$ has to be commuted on the right (after which it vanishes), we can consider the expression $G_1 \mid E(r,s,t)\rangle$ before the spectral flow transform and rearrange each monomial in such a way that every mode of $G$ is moved on the right of every mode of $Q$; then, dropping all the monomials that contain $G$ or $Q$ modes, we evaluate the underbraced factors as

$$
(16 + 48t + 32t^2)\mathcal{H}_{-3} + (-16t + 32t^2)\mathcal{L}_{-3} + (-24 + 48t^2)\mathcal{H}_{-2}\mathcal{H}_{-1} + (-32t + 32t^2)\mathcal{L}_{-2}\mathcal{H}_{-1} - 32t\mathcal{L}_{-2}\mathcal{L}_{-1} + (-24 - 24t + 16t^2)\mathcal{L}_{-1}\mathcal{H}_{-2} + (8 - 24t + 16t^2)\mathcal{H}_{-3}^3 + (24 - 48t + 16t^2)\mathcal{L}_{-1}\mathcal{H}_{-1}^2 + (24 - 24t)\mathcal{L}_{-1}^2\mathcal{H}_{-1} + 8\mathcal{L}_{-1}^3.
$$

(5.9)

To obtain the skeleton $\mathcal{P}(\mathcal{L},\mathcal{H})$, this must be subjected to the spectral flow transform with the spectral parameter $\frac{1}{2}(h-h^{-1}(1,3,t))$. This gives

$$
|S(1,3,h,t)\rangle = g(-3, \frac{1}{2}(h-h^{-1}(1,3,t)) - 1) \cdot \left( (-16t + 32t^2)\mathcal{L}_{-3} - 32t\mathcal{L}_{-2}\mathcal{L}_{-1} + (24t + 8ht + 72t^2 + 48t^3 + 16ht^3)\mathcal{H}_{-3} + (-3t^3 - ht^3 + 3ht^3 + h^3t^3)\mathcal{H}_{-1}^3 - (6t^2 + 20ht^2 + 6ht^2 + 24ht^3 + 8h^2t^3)\mathcal{H}_{-2}\mathcal{H}_{-1} - (16t^2 + 16ht^2)\mathcal{L}_{-2}\mathcal{H}_{-1} - (28t + 12ht + 32t^2 + 16ht^2)\mathcal{L}_{-1}\mathcal{H}_{-2} - (2t^2 - 12ht^2 - 6ht^2)\mathcal{L}_{-1}\mathcal{H}_{-1}^2 + (12t + 12ht)\mathcal{L}_{-1}^2\mathcal{H}_{-1} + 8\mathcal{L}_{-1}^3 \right) \cdot g\left( \frac{1}{2}(h-h^{-1}(1,3,t)), -1 \right) |h, l(1,3,h,t)\rangle
$$

(5.10)

Using formulæ (3.17) and then (3.3) and (3.2), we evaluate (5.10) in the polynomial form

$$
|S(1,3,h,t)\rangle = \left( -(6 + 5t + ht)(32 + 20t + 12ht + 9t^2 + 6ht^2 + h^2t^2)G_{-6}G_{-2}G_{-1} + 2(-48 - 44t - 4ht + 16ht^2 - 20ht^2 - 12ht^2 + 27ht^3 + 9ht^3 - 3ht^3 - h^3t^3)G_{-5}G_{-3}G_{-1} + (3 + h)t(-2 + 3t - ht)(4 - 5t + ht)G_{-4}G_{-3}G_{-2} + 8(3 + h)t(1 + t)(1 + 2t)H_{-3}G_{-3}G_{-2}G_{-1} + 2t(28 + 12t + 35t + 26ht + 3ht^2 + 12ht^2 + 4ht^2)H_{-2}G_{-4}G_{-3}G_{-1} + t(72 + 72h + 14t + 8ht + 30ht^2 - 9t^2 + 21ht^2 + 17ht^2 + 3ht^2)H_{-1}G_{-5}G_{-2}G_{-1} + t(24 + 24h - 14t + 28ht - 18ht^2 - 9t^2 + 27ht^2 + h^2t^2 + 3ht^2)H_{-1}G_{-4}G_{-3}G_{-1} + (72 + 54t + 30ht + 15ht^2 + 14ht^2 + 3ht^2)\mathcal{L}_{-1}G_{-5}G_{-2}G_{-1} + 16t(4 + t + ht)\mathcal{L}_{-2}G_{-4}G_{-2}G_{-1} + 2(24 + 10t + 18ht - 17t^2 - 2ht^2 + 3ht^2)\mathcal{L}_{-1}G_{-4}G_{-3}G_{-1} + 16t(-1 + 2t)\mathcal{L}_{-3}G_{-3}G_{-2}G_{-1} - 2(3 + h)t^2(1 + 3h + 4ht)H_{-2}G_{-2}G_{-1}G_{-1} - 32\mathcal{L}_{-2}\mathcal{L}_{-1}G_{-3}G_{-2}G_{-1} + t^2(4 - 24h - 12ht^2 + 9ht^2 - 9ht^2 - 3ht^2)H_{-2}G_{-4}G_{-2}G_{-1} - 16(1 + h)t^2L_{-2}G_{-1}G_{-3}G_{-2}G_{-1} - 4t(7 + 3h + 8t + 4ht)\mathcal{L}_{-1}G_{-2}G_{-3}G_{-2}G_{-1} + 4t(-12 - 12h + t - 6ht - 3ht^2)L_{-1}G_{-1}G_{-4}G_{-2}G_{-1} - 12(4 + t + ht)L_{-2}^2G_{-4}G_{-2}G_{-1} + (-1 + h)(1 + h)(3 + h)t^3G_{-1}G_{-3}G_{-2}G_{-1} + 2(-1 + 6h + 3ht^2)G_{-1}G_{-2}G_{-1} + 12(1 + h)tL_{-2}^3G_{-1}G_{-3}G_{-2}G_{-1} + 8L_{-1}^3G_{-3}G_{-2}G_{-1}) |h, l(1,3,h,t)\rangle
$$

(5.11)

In the same way, the vector $|S(1,3,h,t)\rangle^+$ can be written in the polynomial form

$$
|S(1,3,h,t)\rangle^+ = \left( -(6 - 5t + ht)(32 + 20t - 12ht + 9t^2 - 6ht^2 + h^2t^2)Q_{-5}Q_{-1}Q_0
$$

---

6 Note that the order of operations is irrelevant; one can subject the vector $|E(r,s,t)\rangle$ to the spectral flow transform and then calculate the skeleton or calculate the skeleton of $|E(r,s,t)\rangle$ and then subject it to the spectral flow with the same spectral parameter; the result is the same in both cases.
+2(-48 - 44t + 44ht + 16t^2 + 20ht^2 - 12h^2t^2 + 27t^3 - 9ht^3 - 3h^2t^3 + h^3t^3) Q_{-4} Q_{-2} Q_0
+(-3 + h)t(-2 + 3t + ht)(-4 + 5t + ht) Q_{-3} Q_{-2} Q_{-1} + 8t(1 + 2t)(1 + h - 3t + ht) H_{-3} Q_{-2} Q_{-1} Q_0
+2t(4 + 12h - 27t + 18ht - 3h^2t + 12h^2t^2 - 4h^2t^2) H_{-2} Q_{-3} Q_{-1} Q_0
+t(-72 + 72h - 14t + 84ht - 30ht^2 + 9t^2 + 21ht^2 - 17h^2t^2 + 3h^3t^2) H_{-1} Q_{-4} Q_{-1} Q_0
+t(-24 + 24h + 14t + 28ht - 18ht^2 + 9t^2 - 27ht^2 - h^2t^2 + 3h^3t^2) H_{-1} Q_{-3} Q_{-2} Q_0
+2(72 + 54t - 30ht + 15t^2 - 14ht^2 + 3h^2t^2) L_{-1} Q_{-4} Q_{-1} Q_0 + 16t(-1 + 2t) L_{-3} Q_{-2} Q_{-1} Q_0
+2(24 + 10t - 18ht - 17t^2 + 2ht^2 + 3h^2t^2) L_{-1} Q_{-3} Q_{-2} Q_0 + 16t(4 + t - ht) L_{-2} Q_{-3} Q_{-1} Q_0
+2t^2(5 + 2h - 3h^2 + 12ht - 4h^2t) H_{-2} H_{-1} Q_{-2} Q_{-1} Q_0 + 16(1 - h)t^2 L_{-2} H_{-1} Q_{-2} Q_{-1} Q_0
+t^2(4 + 24h - 12h^2 + 9t - 9ht + 9h^2t + 3h^3t) H_{1} Q_{-3} Q_{-1} Q_0 - 32t L_{-2} L_{-2} Q_{-2} Q_{-1} Q_0
+4t(-1 - 3h + 8t - 4ht) L_{-1} H_{-2} Q_{-2} Q_{-1} Q_0 + 4t(12 - 12h - t - 6ht + 3h^2t) L_{-1} H_{-1} Q_{-3} Q_{-1} Q_0
+12(-4 - t + ht) L_{-1}^2 Q_{-3} Q_{-1} Q_0 + (-3 + h)(-1 + h)(1 + h)t^3 H_{-1} Q_{-2} Q_{-1} Q_0
+2(-1 - 6ht + 3h^2t)^2 L_{-1} H_{-1} H_{-1} Q_{-2} Q_{-1} Q_0 + 12(-1 + h)t L_{-1}^2 H_{-1} Q_{-2} Q_{-1} Q_0
+8L_{-1}^2 Q_{-2} Q_{-1} Q_0) ) h, l(1, 3, h, t, t) \text{(5.12)}

We now illustrate (4.22) by an explicit calculation:

\[ Q_{-3} Q_{-2} Q_{-1} Q_0 Q_1 Q_2 Q_3 |S(1, 3, h, t)\rangle^\text{\(h\)} = \frac{1}{87} (3 + h)(-2 + 3t - ht)(4 - 3t + ht)(2 + 3t + ht)(4 + 3t + ht)(6 + 3t + ht)|S(1, 3, h, t)\rangle \text{(5.13)} \]

Whenever one of the factors on the RHS vanishes, there is a charged singular vector simultaneously with the massive one, and \(|S(1, 3, h, t)\rangle^\text{\(h\)}\) is inside the twisted topological Verma submodule generated from the charged singular vector. In particular, \(|S(1, 3, h, t)\rangle^\text{\(h\)}\) does not generate the massive Verma submodule then. For example, in the case where \(h = -3\), we have

\[ Q_{-1} Q_0 Q_1 Q_2 Q_3 |S(1, 3, -3, t)\rangle^\text{\(h\)} = 0 \text{(5.14)} \]

whereas \(Q_0 Q_1 Q_2 Q_3 |S(1, 3, -3, t)\rangle^\text{\(h\)} \neq 0\). This means that the state \(Q_0 Q_1 Q_2 Q_3 |S(1, 3, -3, t)\rangle^\text{\(h\)}\) is the (twisted) topological highest-weight state and \(|S(1, 3, -3, t)\rangle^\text{\(h\)} \neq 0\) generates the (twisted) topological Verma submodule rather than massive one. However, the entire massive Verma submodule is generated from \(|S(1, 3, -3, t)\rangle^\text{\(h\)}\). As regards \(Q_0 Q_1 Q_2 Q_3 |S(1, 3, -3, t)\rangle^\text{\(h\)}\), we can apply to this state the operator \(g(2, 0)\) of length \(-1\). The state \(g(2, 0) Q_0 Q_1 Q_2 Q_3 |S(1, 3, -3, t)\rangle^\text{\(h\)}\) is to be evaluated in accordance with the rules of Sec. 3. In our example, we then continue acting with the corresponding modes of \(Q\), and eventually recover the vector \(|S(1, 3, -3, t)\rangle^\text{\(h\)}\):

\[ Q_{-3} Q_{-2} g(2, 0) Q_0 Q_1 Q_2 Q_3 |S(1, 3, -3, t)\rangle^\text{\(h\)} = \frac{24(1 - 3t)(-2 + 3t)}{t^2(1 + 3t)} |S(1, 3, -3, t)\rangle^\text{\(h\)} \text{(5.15)} \]

The case where \(t = -\frac{1}{3}\) is the one of a yet higher codimension, where there is yet another charged singular vector in the module, and the structure of submodules is more complicated, as described in 3. An even simpler example is provided by the massive singular vector \(|S(1, 1, h, t)\rangle^\text{\(h\)}\). In this case, the topological singular vector from which we start reads

\[ |E(1, 1, t)\rangle = Q_{-1} \left| -1 + \frac{2}{7}, t \right\rangle_{\text{top}} \text{(5.16)} \]
The skeleton is simply $2\mathcal{L}_{-1} + 2\mathcal{H}_{-1}$, and one easily obtains
\[
|S(1,1,h,t)^-\rangle = \left(-(1+h)t\mathcal{G}_{-2} + (1+h)t\mathcal{H}_{-1}\mathcal{G}_{-1} + 2\mathcal{L}_{-1}\mathcal{G}_{-1}\right)|h,\frac{1}{2} + \frac{2}{T} - \frac{h}{2}h^2 t, t\rangle \tag{5.17}
\]
One can also obtain, in a similar way,
\[
|S(1,1,h,t)^+\rangle = \left(-(1+h)t\mathcal{Q}_{-1} + (1+h)t\mathcal{H}_{-1}\mathcal{Q}_{0} + 2\mathcal{L}_{-1}\mathcal{Q}_{0}\right)|h,\frac{1}{2} + \frac{2}{T} - \frac{h}{2}h^2 t, t\rangle \tag{5.18}
\]
In order to “compare” $|S(1,1,h,t)^-\rangle$ with $|S(1,1,h,t)^+\rangle$ we evaluate $\mathcal{Q}_0\mathcal{Q}_0|S(1,1,h,t)^-\rangle$. We find
\[
\mathcal{Q}_0|S(1,1,h,t)^-\rangle = \left(\frac{1}{2}(1-h)(1+h)(2+t+ht)\mathcal{H}_{-1} + (1-h)(2+t+ht)\mathcal{L}_{-1} - (2+t+ht)\mathcal{G}_{-1}\mathcal{Q}_0\right)\cdot
\]
and finally,
\[
\mathcal{Q}_1\mathcal{Q}_0|S(1,1,h,t)^-\rangle = -\frac{1}{2}(1+h)(2+t+ht)|S(1,1,h,t)^+\rangle. \tag{5.20}
\]

5.4 Examples of singular vectors in codimension $\geq 2$

For the complete classification of degenerate cases, the reader is referred to [8]; here, we give most characteristic examples, which would at the same time illustrate how our general construction for singular vectors works. We consider some simplest cases where charged and massive singular vectors coexists in a massive Verma module and a simple case where two singular vectors lie in the same grade.

Coexistence of a massive and a charged singular vector Let us consider the case where a charged singular vector $|E(n,h,t)\rangle_{ch}$ exists in $\mathcal{U}_{h,1,t}$ simultaneously with the massive singular vector labelled by two positive integers $r$ and $s = 1$ (we choose $s = 1$ for simplicity, in particular to simplify the diagrams with which we illustrate this case). This happens when the highest-weight parameters of the massive Verma module $\mathcal{U}_{h,1,t}$ are $\ell = \text{lcm}(r,1,h,t)$ and $h = \text{hcm}(r,1,h,t)$, with
\[
\begin{align*}
\text{lcm}(r,1,h,t) &= n\left(-\frac{2}{T} + \frac{2}{T} - 1\right), \\
\text{hcm}(r,1,h,t) &= -\frac{2m+1}{T} + \frac{1}{T} - 1, \quad n \in \mathbb{Z}, \quad r \in \mathbb{N}.
\end{align*} \tag{5.21}
\]
In this case the module $\mathcal{U}_{h,1,t}$ contains a massive Verma submodule $\mathcal{U}'$ and a submodule $\mathcal{C}$ generated from the charged singular vector (with $n$ chosen to be positive, for definiteness)
\[
|E_{ch}(n,\text{hcm}(r,1,h,t),t)\rangle = \mathcal{G}_{-n} \ldots \mathcal{G}_{-1} |\text{hcm}(r,1,h,t),\text{lcm}(r,1,h,t),t\rangle. \tag{5.22}
\]
$\mathcal{C}$ is the twisted topological Verma module $\mathcal{G}_{-n}\mathcal{T}_{-r+1,1,t,\ell-n}$. Also, we have $\mathcal{C}' = \mathcal{U}' \cap \mathcal{C} = \mathcal{G}_{-n+1,1,t,\ell-r-n}$, which is generated from the state
\[
|T\rangle = \mathcal{Q}_{n-r} \ldots \mathcal{Q}_{n-1} \mathcal{G}_{-n} \ldots \mathcal{G}_{-1} |\text{hcm}(r,1,h,t),\text{lcm}(r,1,h,t),t\rangle. \tag{5.23}
\]
Now, states (5.19) take the form
\[
\begin{align*}
|S(r,1,\text{hcm}(r,1,h,t),t)\rangle^- &= g(-r, r - n - 1) \mathcal{Q}_{n-r} \ldots \mathcal{Q}_{n-1} \mathcal{G}_{-n} \ldots \mathcal{G}_{-1} |\text{hcm}(r,1,h,t),\text{lcm}(r,1,h,t),t\rangle, \\
|S(r,1,\text{hcm}(r,1,h,t),t)\rangle^+ &= g(1-r, n + t - 1) \mathcal{G}_{n-t} \ldots \mathcal{G}_{n+r-t} q(n-r+t,0) \\
&\quad |\text{hcm}(r,1,h,t),\text{lcm}(r,1,h,t),t\rangle, \tag{5.24}
\end{align*}
\]
whence we see that the $|S(r, 1, h_{cm}(r, 1, h, t), t)\rangle^+$ representative — which is evaluated as an element of the module $U_{h,t,t}$ as described in Sec. 5.2 — generates the massive submodule $U'$. On the other hand, whenever $2r \geq n$, the representative $|S(r, 1, h_{cm}(r, 1, h, t), t)\rangle^-$ does not, because $|S(r, 1, h_{cm}(r, 1, h, t), t)\rangle^-$ is a descendant of the state $(5.23)$:

$$|S(r, 1, h_{cm}(r, 1, h, t), t)\rangle^- = \mathcal{G}_{r-r-n-1} \mathcal{Q}_{n-r} \mathcal{Q}_{n-1} \mathcal{G}_{n-1} \mathcal{G}_{r-n} |h_{cm}(r, 1, h, t), l_{cm}(r, 1, h, t), t\rangle$$

$$= \mathcal{G}_{r-r-n-1} |T\rangle .$$

Thus, whenever $2r \geq n$, the vector $|S(r, 1, h_{cm}(r, 1, h, t), t)\rangle^-$ generates the submodule $\mathcal{C}'$ rather than $U'$. The maximal submodule in this case is $U' \cup \mathcal{C}$; it is generated by two singular vectors $|E_{ch}(n, h_{cm}(r, 1, h, t), t)\rangle$ and $|S(r, 1, h_{cm}(r, 1, h, t), t)\rangle^+$.

The same mechanism leads to the failure of the top-level representative of the singular vectors to generate the maximal submodule whenever $r \geq n$. In this case, the top-level representative of the massive singular vector labelled by $(r, 1)$ is the descendant of the state $|T\rangle$:

$$|s\rangle = \mathcal{G}_0 \ldots \mathcal{G}_{r-n-1} \mathcal{Q}_{n-r} \mathcal{Q}_{n-1} \mathcal{G}_{n-1} \mathcal{G}_{r-n} |h_{cm}(r, 1, h, t), l_{cm}(r, 1, h, t), t\rangle = \mathcal{G}_0 \ldots \mathcal{G}_{r-n-1} |T\rangle .$$

and therefore belongs to the module $\mathcal{C}'$. Now, it is obvious that the states of the form

$$\mathcal{Q}_{-r-m} \ldots \mathcal{Q}_{-r} |S(r, 1, h_{cm}, t)\rangle^+ \quad m \geq 0$$

cannot be generated from $|s\rangle$. The top-level representative generates the submodule $\mathcal{C}$ only. Thus, in the conventional approach, where one uses only the top-level representatives of the extremal diagrams as ‘singular vectors’, one has to use a subsingular vector in order to generate the entire submodule $U'$. Of course, the subsingular vector is a descendant of the state $|S(r, 1, h_{cm}, t)\rangle^+$ since the latter generates the submodule $U'$. One can check that the state

$$|\text{Sub}\rangle = \mathcal{G}_0 \ldots \mathcal{G}_{r-n-1} \mathcal{G}_{r-n+1} \ldots \mathcal{G}_{r-1} |S(r, 1, h_{cm}(r, 1, h, t), t)\rangle^+$$

$$= \mathcal{G}_0 \ldots \mathcal{G}_{r-n-1} g(r + n + 1, r - n - 1) |T\rangle$$

is the subsingular vector.\footnote{The vector $|\text{Sub}\rangle$ exists as an element of the massive Verma module because $|S(r, 1, h_{cm}(r, 1, h, t), t)\rangle^+$ is an element of the Verma module as described in Sec. 5.2. However, one can be interested in how the expression $g(r + n + 1, r - n - 1) |T\rangle$ can be rewritten as an element of the Verma module. To do this, one represents the expression

$$g(r + n + 1, r - n - 1) \mathcal{Q}_{n-r} \ldots \mathcal{Q}_{n-1} \mathcal{G}_{n-1} \mathcal{G}_{r-n} |h_{cm}(r, 1, h, t), l_{cm}(r, 1, h, t), t\rangle$$

in the form

$$g(r + n + 1, -n - 1) \mathcal{G}_{n-r} \ldots \mathcal{G}_{r-n-1} \mathcal{Q}_{n-r} \ldots \mathcal{Q}_{n-1} g(-n - 1) |h_{cm}(r, 1, h, t), l_{cm}(r, 1, h, t), t\rangle$$

\footnote{Indeed,}

and other conditions can be checked by similar direct calculations. Thus, $|\text{Sub}\rangle$ satisfies the conventional annihilation conditions (2.31) as soon as $|s\rangle$ is set to zero.

$$\mathcal{H}_1 |\text{Sub}\rangle = \mathcal{G}_0 \ldots \mathcal{G}_{r-n-2} |T\rangle = (-1)^{r-n-1} \frac{t}{2(r+t-1)} \mathcal{Q}_{n-r+1} |s\rangle$$

and other conditions can be checked by similar direct calculations. Thus, $|\text{Sub}\rangle$ satisfies the conventional annihilation conditions (2.31) as soon as $|s\rangle$ is set to zero.
This can be illustrated in the following extremal diagram:

\[
\begin{array}{c}
|S\rangle - |S\rangle +

\end{array}
\]

For brevity, we have denoted \(|S\rangle^\pm = |S(r, 1, h_{\text{cm}}(r, 1, h, t), t)\rangle^\pm\) and \(|E\rangle_{\text{ch}} = |E(n, h_{\text{cm}}(r, 1, h, t), t)\rangle_{\text{ch}}\). Then, \(|S\rangle - |S\rangle^+\) is the extremal diagram of the massive Verma submodule \(U'\), the crosses denote top-level representatives, \(\bullet\)s are (twisted) topological highest-weight states, and \(* = g(r - n + 1, r - n - 1) |T\rangle\) is the state in \(U'\) from which the action of \(G_{r - n}\) produces the \(|T\rangle\) state. The line \(|T\rangle - |S\rangle^+\) cannot be reached by the action of elements of the \(N = 2\) algebra on \(|T\rangle\), since the arrow in the diagram that represents the action of \(G_{r - n}\) cannot be inverted because of the topological highest-weight conditions satisfied by \(|T\rangle\). Instead, acting with the highest of modes of \(Q\) that produces a non-vanishing result, one spans out the branch \(|T\rangle - 1\).

The subsingular vector emerges whenever it happens that the top-level representative \(|s\rangle\) does not generate all of the \(U'\) submodule. Namely, assume that \(r > n\) (this is actually the case in the diagram drawn above). Then, neither \(|T\rangle\) nor the top-level representative \(|s\rangle\) generate the states on the line \(|T\rangle - |S\rangle^+\) and, therefore, the corresponding part of the massive Verma submodule. Restricting oneself to only the top-level representatives of extremal diagrams as singular vectors, one is limited, therefore, to the submodule generated from the top-level vector \(|s\rangle\). After taking the quotient with respect to the singular vector \(|S(r, 1, h_{\text{cm}}, t)\rangle^-\) (or, equivalently, \(|s\rangle\)), one is left with the submodule whose extremal diagram is precisely the line \(2 - |S\rangle^+\). The * state becomes a twisted topological highest-weight state — the cusp in the dotted extremal diagram. Therefore, \(|\text{Sub}\rangle\) becomes a subsingular vector. We see, however, that the appearance of a subsingular vector is entirely due to choosing an inconvenient definition of singular vectors: the problem with top-level representatives of (extremal diagrams of) submodules is that they do not necessarily generate maximal submodules.

**Two singular vectors in the same grade.** As another instructive example of degeneration of massive Verma modules, we consider the case where two linearly independent singular vectors exist in the same grade. This effect was first observed in [18]. We consider here the simplest explicit example (see [8] for the complete description).

In the module \(U_{h, t, t}\), two linearly independent singular vectors exist in the same grade when \(h = \frac{(1 + m + n)s}{m - n + r}, \ t = \frac{m - n + r}{s} \neq 0, \ m, n \in \mathbb{N}, \ 2r - m - n \geq 0\) (5.31)
Then,
\[ \theta_1(r, s, h, t) \big|_{(5.31)} = -m, \quad \theta_2(r, s, h, t) \big|_{(5.31)} = n \]
and the above formulas (4.19) take the following form:
\[ |S(r, s, h, t)^{-}\rangle_{(5.31)} = G_{-r} \cdots G_{-m-1} E^{-m}(r, s, t) G_{-m} \cdots G_{-1} |h, l(r, s, h, t), t\rangle_{(5.34)} \cdot |h, l(r, s, h, t), t\rangle_{(5.31)}, \]
\[ |S(r, s, h, t)^{+}\rangle_{(5.31)} = Q_{1-r} \cdots Q_{-n-1} E^{+n}(r, s, t) Q_{-n} \cdots Q_{0} |h, l(r, s, h, t), t\rangle_{(5.31)} \cdot |h, l(r, s, h, t), t\rangle_{(5.31)}, \] (5.33)

Obviously, now that the \( g \) and \( q \) operators from (1.19) have become the products of the \( N = 2 \) generators, one observes that \( |S(r, s, h, t)^{\pm}\rangle \) are dense \( G/Q \)-descendants of the respective vectors \( |T^{\pm}\rangle \) given by
\[ |T^{-}\rangle = E^{-m}(r, s, t) G_{-m} \cdots G_{-1} |h, l(r, s, h, t), t\rangle_{(5.31)} \cdot |h, l(r, s, h, t), t\rangle_{(5.31)}, \]
\[ |T^{+}\rangle = E^{+n}(r, s, t) Q_{-n} \cdots Q_{0} |h, l(r, s, h, t), t\rangle_{(5.31)} \cdot |h, l(r, s, h, t), t\rangle_{(5.31)}, \] (5.34)
and each of which satisfies the twisted topological highest-weight conditions:
\[ G_{r-m}|T^{-}\rangle = 0, \quad Q_{m-r}|T^{-}\rangle = 0, \]
\[ G_{m-r}|T^{+}\rangle = 0, \quad Q_{r-n}|T^{+}\rangle = 0. \] (5.35)

Let \( C_- \) and \( C_+ \) be the submodules generated from the respective vectors (5.34). There exist \( 2r - m - n \) states \( |x_i\rangle^{-} \) from the extremal diagram of the \( C_- \) submodule and the same number of states \( |x_i\rangle^{+} \) from the extremal diagram of \( C_+ \) such that, for each \( i \), \( |x_i\rangle^{-} \) and \( |x_i\rangle^{+} \) are in the same grade. As all of the states on the extremal diagram of a (twisted) topological Verma module, these states satisfy twisted massive highest-weight conditions.

As a simple example, consider the case \( r = 2, s = 1, m = 2, n = 1 \). There are two charged singular vectors in the corresponding massive Verma module \( U_{0, -2, -1} \). In each of the respective twisted topological Verma modules, there exist topological singular vectors directly on the extremal diagrams. These two topological singular vectors have identical \( L_0 \) and \( H_0 \)-gradings (the dimension and the \( U(1) \) charge) and are linearly independent. They are shown by the double bullet in the following diagram:

(5.36)

(where the filled dots show twisted topological highest-weight states).

From this diagram we read off what the formulae (5.34) give us in this particular case:
\[ |T^{-}\rangle = Q_0 Q_1 G_{-2} G_{-1} |0, -2, -1\rangle, \]
\[ |T^{+}\rangle = G_{-1} G_0 Q_{-1} Q_0 |0, -2, -1\rangle. \] (5.37)
or, after some rearrangements,
\[ |T\rangle^- = \left(8\mathcal{L}_{-2} - 4\mathcal{G}_{-2} Q_0 - 4\mathcal{H}_{-1} \mathcal{L}_{-1} + 4\mathcal{L}_{-1} \mathcal{L}_{-1} - 2\mathcal{Q}_{-1} \mathcal{G}_{-1} + 2\mathcal{H}_{-1} \mathcal{G}_{-1} Q_0 - 2\mathcal{L}_{-1} \mathcal{G}_{-1} Q_0 \right) |0, -2, -1\rangle, \]
\[ |T\rangle^+ = \left(8\mathcal{H}_{-2} + 8\mathcal{L}_{-2} - 4\mathcal{Q}_{-1} \mathcal{G}_{-1} + 2\mathcal{G}_{-1} \mathcal{H}_{-1} Q_0 + 2\mathcal{G}_{-1} \mathcal{L}_{-1} Q_0 \right) |0, -2, -1\rangle. \]

(5.38)

6 Conclusions and an outlook

The construction of the $N=2$ singular vectors has been presented in a way that makes it parallel to the known constructions for singular vectors of the affine Lie algebras. It is the topological singular vectors of the $N=2$ algebra with central charge $c \neq 3$ that are in a 1:1 correspondence with singular vectors of the affine $s\ell(2)$ algebra of level $k$, $c = 3k/(k + 2)$.

The analysis of the monomial expressions for the singular vectors suggests that rather than identifying the three classes of singular vectors (as in section 4) and then analyzing their intersections, it may be useful to view the set of the $N=2$ singular vectors as ‘stratified’ according to which lengths of the continued operators become integers in either of the formulae (6.10), as we move from right to left. This approach is further developed in [8], with the embedding diagrams of $N=2$ Verma modules classified and constructed in [24].

It would be interesting to apply the present construction to the various realizations of the $N=2$ algebra. In that respect, it would be quite useful to have the conformal field-theoretic counterparts of the operators (3.1). The $N=2$ algebra will then be represented by conformal fields $\mathcal{T}(z)$, $\mathcal{H}(z)$, $\mathcal{G}(z)$, and $\mathcal{Q}(z)$. Denoting the respective ‘continued’ field operators as $G_{\mu,\nu}(z)$ and $Q_{\mu,\nu}(z)$, with complex $\mu$ and $\nu$, we would have the following counterpart of the ‘reduction’ rule (6.2):

\[ G_{\lambda+N,\lambda}(z) = G^{(\lambda+N)}(z) \ldots G^{(\lambda)}(z) \text{ for } N \in \mathbb{N}_0, \]

where $G^{(\mu)}(z)$ is the $\mu$-th derivative of $G(z)$; for arbitrary complex $\mu$ it can be defined by analytically continuing the integral

\[ f^{(N)}(z) = \frac{1}{N!} \frac{1}{2\pi i} \oint \frac{f(u) \, du}{(z - u)^{N+1}}. \]

(6.2)

The operator products of the bosonic fields $\mathcal{T}$ and $\mathcal{H}$ with $G$ are easily found, e.g.,

\[ \mathcal{H}(z) G_{\mu,\nu}(w) = (\mu - \nu + 1) \frac{G_{\mu,\nu}(w)}{z - w}, \]
\[ \mathcal{T}(z) G_{\mu,0}(w) = \frac{1}{2}(\mu + 1)(\mu + 4) \frac{G_{\mu,0}(w)}{(z - w)^2} + \frac{\partial G_{\mu,0}(w)}{z - w}. \]

(6.3)

Therefore $G_{\mu,0}$ are primary fields, an obvious translation of the fact that the operators (3.1) map highest-weight states into extremal states. Further, we have the following version of the gluing rules (3.3):

\[ G_{\mu,\nu}(z) \, G_{\nu-1,\lambda}(z) = G_{\mu,\lambda}(z). \]

(6.4)

The positive powers in the short-distance expansion are controlled by the following property of the $G$ operators:

\[ \partial G_{\mu,\nu}(w) = G^{(\mu+1)}(w) \, G_{\mu-1,\nu}(w), \]

(6.5)
which allows one to evaluate the higher-order derivatives recursively. Thus, it is easily seen that
\[ \partial^N G_{\mu,\nu}(w) = \partial^N (\mathcal{G}^{(\mu)}(w) \ldots \mathcal{G}^{(\mu-N+1)}(w)) G_{\mu-N,\nu}(w). \] (6.6)

The fields \(Q_{\mu,\nu}(z)\) satisfy the obvious analogues of the above equations.

When considering various special realizations of the \(N=2\) algebra, the behaviour of \(g(a, b)\) and \(q(a, b)\) can be very complicated, while evaluating the corresponding realizations of the field operators like \(G_{\mu,\nu}(z)\) can be much easier. One can thus consider the Kazama–Suzuki mapping relating the affine \(s\ell(2)\) and \(N=2\) algebras and extend it to include the continued operators. This will then serve to find the pullbacks of the \(N=2\) singular vectors in the monomial forms, without having to rewrite them in the conventional Verma forms. Thus, in particular, the continued Kazama–Suzuki mapping relates the topological singular vectors in the monomial form \([4.6], [4.7]\) to the respective MFF monomials \([6]\). The construction for \(N=2\) singular vectors given in this paper is further be applied in \([15]\) to constructing a functor between categories of highest-weight-type modules over affine \(s\ell(2)\) and \(N=2\) algebras.

It is interesting whether the continued operators \(g\) and \(q\) can be represented in the spirit of \([26]\), as integral operators acting on an appropriate manifold. Such a representation would then give an explicit integral representation for the \(N=2\) correlation functions.

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