Global dynamics of the generalized fifth-order KdV equation with quintic nonlinearity

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Abstract. We prove global existence and scattering for the solutions of the generalized fifth-order KdV equation with quintic nonlinearity for small and localized initial data. The proof uses the space-time resonance method and the stationary phase argument.

1. Introduction

We consider the following generalized fifth-order KdV equation with quintic nonlinearity:

\[ u_t = \partial_x^5 u + \alpha u^4 u_x, \]  

(1.1)

where \( u \) is a real function which maps \( \mathbb{R} \times \mathbb{R} \) to \( \mathbb{R} \), and \( \alpha = 1 \) (defocusing case) or \( \alpha = -1 \) (focusing case). Equation (1.1) is a member of the general fifth-order KdV equations:

\[ u_t = \alpha_1 \partial_x^5 u + \alpha_2 \partial_x^3 u + \partial_x g(u, \partial_x u, \partial_x^2 u), \quad \alpha_1, \alpha_2 \in \mathbb{R}, \alpha_1 \neq 0, \]

which contains in particular some models of plasma waves and capillary–gravity waves and the fifth order equation in the KdV integrable hierarchy. (See the introduction of [2,12,13] for a useful survey.)

It is standard to show that the Cauchy problem of (1.1) is well posed in \( C([-T, T]; H^s(\mathbb{R})) \) for a short time \( T > 0 \), for instance, one may refer to [1,19]. Our aim in the present work is to study the global existence and scattering for the solutions of (1.1) with small and localized initial data, in the framework of the space-time resonance method [4,11] and the stationary phase argument [11]. The main ingredient is to study the evolutionary equation of the profile of the solutions in Fourier space which is unfolded by a careful stationary phase analysis based on an adaptation of the argument of [5]. Since we only focus on small solutions, the sign of \( \alpha \) will not matter and will be taken to be ‘\(-1\)’ in the rest of the paper. Our main result can be stated precisely as follows:

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Theorem 1.1. Given the initial data $u_0$ as

$$ u(x, 0) = u_0(x). $$

(1.2)

Assume that $u_0$ satisfies

$$ \|u_0\|_{H^2(\mathbb{R})} + \|xu_0\|_{L^2(\mathbb{R})} \leq \varepsilon_0 \leq \overline{\varepsilon}, $$

(1.3)

for some constant $\overline{\varepsilon}$ sufficiently small. Then, the Cauchy problem (1.1)–(1.2) admits a unique global solution $u \in C(\mathbb{R}; H^2(\mathbb{R}))$ satisfying the decay estimates for $t \geq 1$ and $x \in \mathbb{R}$

$$ \left| | \partial_x |^\beta u(x, t) \right| \lesssim \varepsilon_0 t^{- (\beta + 1)/5} (x/t^{1/5})^{-3/8 + \beta/2}, \quad \beta \in [0, 3]. $$

(1.4)

Moreover, the solution has the following asymptotics as $t \to +\infty$:

(Decaying region) When $x \geq t^{1/5}$, we have the decay estimate

$$ |u(x, t)| \lesssim \varepsilon_0 t^{-1/5} (x/t^{1/5})^{-7/20}. $$

(1.5)

(Self-similar region) When $|x| \leq t^{1/5} + 4\gamma$, with $\gamma = \frac{1}{5} (\frac{1}{10} - C\varepsilon_0^4)$, the solution is approximately self-similar:

$$ |u(x, t) - t^{-1/5} Q(x/t^{1/5})| \lesssim \varepsilon_0 t^{-1/5 - 2\gamma}, $$

(1.6)

where $Q$ is a bounded solution of the nonlinear ordinary differential equation

$$ Q^{(4)} - 5^{-1} x Q - Q^5 = 0 $$

(1.7)

with

$$ \|Q\|_{L^\infty(\mathbb{R})} \lesssim \varepsilon_0. $$

(1.8)

(Oscillatory region) When $x \leq -t^{1/5} + 4\gamma$, the leading order asymptotic behavior is linear: There exists $f_\infty \in L^\infty(\mathbb{R})$ with $\|f_\infty\|_{L^\infty(\mathbb{R})} \lesssim \varepsilon_0$ such that

$$ \left| u(x, t) - \frac{1}{\sqrt{5t\xi_0^3}} \Re \left\{ \exp \left( -4it\xi_0^5 + \frac{i\pi}{4} + \frac{i}{40t\xi_0^5} |f_\infty(\xi_0)|^4 \right) f_\infty(\xi_0) \right\} \right| \lesssim \varepsilon_0 t^{-1/5} (-x/t^{1/5})^{-9/20}, $$

(1.9)

where $\xi_0 := \sqrt{-x/(5t)}$, and $\Re$ denotes the real part.

Since equation (1.1) is time-reversible, the asymptotics for $t \to -\infty$ follows immediately. We mention that the proof presented in this work does not rely on the complete integrability, thus may be applied to a wider class with short range perturbations of the nonlinearity.

Throughout the paper, we will always use $f(t) = e^{-t\xi_0^5} u(t)$ to denote the profile of $u$. By time reversibility, we solely need to consider the existence for positive time. The
local well-posedness on the time interval \([0, 1]\) for (1.1)–(1.2) is standard provided \(\|u_0\|_{H^s} (s > \frac{3}{2})\) is sufficiently small, in particular under the smallness assumption (1.3). Then, the existence and uniqueness of global solutions may be constructed by a bootstrap argument which guarantees us to extend the local solutions. More precisely, assume that the following \(X\)-norm is a priori small:

\[
\|u\|_X = \sup_{t \geq 1} \left( \|u(t)\|_{H^2} + t^{-1/10} \|xf(t)\|_{L^2} + \|\hat{f}(t, \xi)\|_{L^\infty} \right) \leq \varepsilon_1. \tag{1.10}
\]

With \(\varepsilon_1 = \varepsilon_0^{1/5}\), we then aim to show that the above a priori assumption may be improved to

\[
\|u\|_X \leq C(\varepsilon_0 + \varepsilon_1^5), \tag{1.11}
\]

for some absolute constant \(C > 1\). Moreover, we may choose \(\bar{\varepsilon} := (4C)^{-5/4}\) as an upper bound of \(\varepsilon_0\) in Theorem 1.1.

There are other approaches dealing with asymptotics for large time solutions of dispersive PDEs: Via inverse scattering transform [3] for large solutions but relying on the complete integrability of the equation, and using PDE techniques [7–10, 16, 18] not relying on the complete integrability of the equation but restricting to small solutions, one can refer to [14] for a comparative survey on related results.

We learned that the large time behavior of solutions of (1.1) was also carefully studied recently [17] in which the author got a precise asymptotic behavior of the solutions following the testing by wave packets argument [7, 10]. For other studies on the fifth-order KdV equations, one may refer to [13] and references therein. Nevertheless, we prefer presenting an alternative approach to understand the asymptotics of the large time solutions of (1.1) since the argument we are using is also flexible for fractional dispersive models (e.g., [20]). Our proof is fully carried out in Fourier space whose main idea is to identify the ODE of \(\partial_t \hat{f}(t, \xi)\) precisely. Inspired by [5], we decompose the quintic nonlinearity into the stationary phase part and the non-stationary phase part in a crucial way, in which the stationary phase part contains the leading term of \(\partial_t \hat{f}(t, \xi)\) that determines the asymptotic behavior of the solutions. More precisely, we will derive

\[
\partial_t \hat{f}(t, \xi) = \left( \frac{-i}{40t^2 \xi^5} |\hat{f}(t, \xi)|^4 \hat{f}(t, \xi) + \frac{c_1 i}{t^2 \xi^5} e^{-\frac{624i}{625} \hat{f}(t, \xi/5)^5} \right) \mathbf{1}_{\|\xi\| > t^{-1/5}} + \{\text{reminders}\}.
\]

The first term on the right-hand side gives the leading order asymptotic behavior of (1.9) in the oscillatory region.

In comparison with the modified KdV equation [5], the space-time resonance analysis in our case is more complicated due to the structure of the phase function in the Duhamel’s formula. We mention that one can further study the stability of soliton solutions under small perturbations of (1.1) after Theorem 1.1 (the stability of the zero
solution under small perturbations) as [5], however, which is an independent interest and will be developed in a different work. We also mention that it is not clear whether one could improve the error estimate in (1.9) or not by this argument we are using. The Strichartz estimates might also apply to study the global well-posedness and scattering for (1.1). (See e.g., [15] for handing similar models.) However, the pointwise estimates in Theorem 1.1 cannot be derived by using Strichartz estimates, and this constitute the main contributions in this work.

In Sect. 2, we derive some crucial estimates on the semigroup generated by the linear part of equation (1.1). In Sect. 3, we estimate the Sobolev norm and weighted Sobolev norm in (1.10). Section 4 is devoted to controlling \( \hat{\mathcal{F}}g \) in \( L^\infty \)-norm in (1.10). We study asymptotic behavior in Sect. 5.

Notations. We finally list some notations frequently used throughout the paper. Let \( L^p(\mathbb{R}) \) \((p \in [1, \infty])\) be the standard Lebesgue spaces, in particular \( L^2(\mathbb{R}) \) is a Hilbert space with inner product
\[
(g, h)_2 := \int_{\mathbb{R}} gh \, dx.
\]
Similarly, let \( H^s(\mathbb{R}) \) \((s > 0)\) be the usual Sobolev spaces with norm
\[
\|g\|_{H^s(\mathbb{R})} := \|(1 - \partial_x^2)^{s/2} g\|_{L^2(\mathbb{R})},
\]
and let \( \mathcal{C}([0, T]; H^s(\mathbb{R})) \) be the space of all bounded continuous functions \( g : [0, T] \to H^s(\mathbb{R}) \) normed by
\[
\|g\|_{\mathcal{C}([0, T]; H^s(\mathbb{R}))} := \sup_{t \in [0, T]} \|g(t, \cdot)\|_{H^s(\mathbb{R})}.
\]
We denote by \( \mathcal{F}(g) \) or \( \hat{g} \) the Fourier transform of a Schwartz function \( g \) whose formula is given by
\[
\mathcal{F}(g)(\xi) = \hat{g}(\xi) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} g(x) e^{-ix\xi} \, dx
\]
with inverse
\[
\mathcal{F}^{-1}(g)(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} g(\xi) e^{ix\xi} \, d\xi,
\]
and by \( m(\partial_x) \) the Fourier multiplier with symbol \( m \) via the relation
\[
\mathcal{F}(m(\partial_x)g)(\xi) = m(i\xi)\hat{g}(\xi).
\]
Take \( \varphi \in C_0^\infty(\mathbb{R}) \) satisfying \( \varphi(\xi) = 1 \) for \(|\xi| \leq 1\) and \( \varphi(\xi) = 0 \) when \(|\xi| > 2\), and let
\[
\psi(\xi) = \varphi(\xi) - \varphi(2\xi), \quad \psi_j(\xi) = \psi(2^{-j}\xi), \quad \varphi_j(\xi) = \varphi(2^{-j}\xi),
\]
we then may define the Littlewood–Paley projections \( P_j, P_{\leq j}, P_{> j} \) via
\[
\hat{P}_jg(\xi) = \psi_j(\xi)\hat{g}(\xi), \quad \hat{P}_{\leq j}g(\xi) = \varphi_j(\xi)\hat{g}(\xi), \quad \hat{P}_{> j} = 1 - \hat{P}_{\leq j}.
\]
and also $P_{\sim j}, P_{\lesssim j}, P_{\ll j}$ by

$$P_{\sim j} = \sum_{2^k \sim 2^j} P_k, \quad P_{\lesssim j} = \sum_{2^k \lesssim 2^j + C} P_k, \quad P_{\ll j} = \sum_{2^k \ll 2^j} P_k.$$  

We will also denote $g_j = P_j g, g_{\lesssim j} = P_{\lesssim j} g$, and so on, for convenience.

The notation $C$ always denotes a nonnegative universal constant which may be different from line to line but is independent of the parameters involved. Otherwise, we will specify it by the notation $C(a, b, \ldots)$. We write $f \lesssim g (f \gtrsim g)$ when $f \leq C g$ ($f \geq C g$), and $f \sim g$ when $f \lesssim g \gtrsim f$. We also write $\sqrt{1 + x^2} = \langle x \rangle$ for $x \in \mathbb{R}$ for simplicity.

### 2. Linear estimates

In this section, we aim to derive some crucial estimates on the semigroup generated by the linear part of equation (1.1). These estimates will be applied later in closing energy estimates and proving asymptotics. A similar argument was used in [5].

**Lemma 2.1.** Let $t \geq 1, x \in \mathbb{R}$ and $g$ be a real function. Then,

\[
\left| e^{t^\beta_5} \partial_x^\beta g(x, t) \right| \lesssim t^{-\beta/5} \langle x/t^{1/5} \rangle^{1/4} + t^{1/10} \|g\|_{L^\infty}, \quad \text{for } \beta \in [0, 3].
\]

Moreover, we have the refined estimates: When $x \geq t^{1/5}$,

\[
\left| e^{t^\beta_5} g(x, t) \right| \lesssim \left( t^{1/5} \|g\|_{L^\infty} + t^{-3/10} \|xg\|_{L^2} \right),
\]

and for $x \leq -t^{1/5}$,

\[
\left| e^{t^\beta_5} g(x, t) - \frac{1}{\sqrt{5t \xi_0^3}} \Im \exp \left( -4it\xi_0^5 + \frac{i\pi}{4} \right) \hat{g}(t, \xi_0) \right| \lesssim \left( -x/t^{1/5} \right)^{-9/20} \left( t^{-1/5} \|g\|_{L^\infty} + t^{-3/10} \|xg\|_{L^2} \right),
\]

where $\xi_0 := \sqrt{-x/(5t)}$, and $\Im$ denotes the real part.

**Proof.** (Proof of (2.1)) We write

\[
e^{t^\beta_5} \partial_x^\beta g(x, t) = \sqrt{\frac{2}{\pi}} \Im \int_0^\infty e^{it\Phi(\xi)} \xi^\beta \hat{g}(t, \xi) \, d\xi,
\]

where

\[
\Phi(\xi) = \Phi(\xi; x, t) := t^{-1} x \xi + \xi^5.
\]

Considering $\xi \geq 0$ in (2.4), so the phase function $\Phi$ has only one stationary point $\xi_0 = \sqrt{-x/(5t)}$ for $x \leq 0$. The estimate (2.1) is much easier as $x > 0$ due to the
absence of stationary points, and the following argument also applies. So for (2.1), it suffices to show
\[
\left| \int_0^\infty e^{i \tau \Phi(t, \xi)} \xi^\beta \hat{g}(t, \xi) \, d\xi \right| \lesssim t^{-\frac{1}{5} - \frac{\beta}{5}} \max(1, \xi_0 t^{1/5})^{-\frac{3}{2} + \beta},
\]  
(2.5)
for any \( t \geq 1, x \leq 0 \) and any function \( g \) satisfying
\[
\| \hat{g} \|_{L^\infty} + t^{-1/10} \| xg \|_{L^2} \leq 1.
\]  
(2.6)
There are two cases to consider depending on the size of \( \xi_0 \).

Case 1: \( \xi_0 \leq t^{-1/5} \). It reduces to obtain a bound of \( t^{-\frac{1}{5} - \frac{\beta}{5}} \) for RHS of (2.5). For this, we split the integral in (2.5) by small and large frequencies:

\[
\int_0^\infty e^{i \tau \Phi(t, \xi)} \xi^\beta \hat{g}(t, \xi) \, d\xi = \int_0^\infty e^{i \tau \Phi(t, \xi)} \xi^\beta \hat{g}(t, \xi) \varphi(2^{-10} t^{1/5} \xi) \, d\xi + \int_0^\infty e^{i \tau \Phi(t, \xi)} \xi^\beta \hat{g}(t, \xi) (1 - \varphi(2^{-10} t^{1/5} \xi)) \, d\xi =: A_1 + A_2.
\]
Using only the bound \( \| \hat{g} \|_{L^\infty} \leq 1 \) in (2.6), one immediately gets
\[
|A_1| \lesssim \| \hat{g} \|_{L^\infty} \int_0^\infty \xi^\beta \varphi(2^{-10} t^{1/5} \xi) \, d\xi \lesssim t^{-\frac{1}{5} - \frac{\beta}{5}}.
\]

For the second term \( A_2 \), we use an integration by parts to deduce
\[
|A_2| \lesssim |A_{21}| + |A_{22}|,
\]

\[
A_{21} = t^{-1} \int_0^\infty \left| \partial_\xi \left[ (\partial_\xi \Phi)^{-1} \xi^\beta (1 - \varphi(2^{-10} t^{1/5} \xi)) \right] \hat{g}(t, \xi) \right| \, d\xi,
\]
\[
A_{22} = t^{-1} \int_0^\infty \left| (\partial_\xi \Phi)^{-1} \xi^\beta (1 - \varphi(2^{-10} t^{1/5} \xi)) \partial_\xi \hat{g}(t, \xi) \right| \, d\xi.
\]
Observe that \( |\partial_\xi \Phi| \geq \xi^4 \geq t^{-4/5} \) on the support of the integral, we may estimate the resulting terms, by applying the first bound in (2.6) again to get
\[
A_{21} \lesssim t^{-1} \| \hat{g} \|_{L^\infty} \int_0^\infty \left( \xi^{\beta - 5} \left| 1 - \varphi(2^{-10} t^{1/5} \xi) \right| \right. \\
+ \left. \xi^{\beta - 4} \left| \varphi'(2^{-10} t^{1/5} \xi) t^{1/5} \right| \right) \, d\xi \lesssim t^{-1} t^{-\frac{1}{5} + \frac{\beta - 4}{5}} \lesssim t^{-\frac{1}{5} - \frac{\beta}{5}},
\]
(2.7)
and the second bound in (2.6) to obtain
\[
A_{22} \lesssim t^{-1} \| \partial_\xi \hat{g} \|_{L^2} \left( \int_0^\infty \left( \xi^{\beta - 4} (1 - \varphi(2^{-10} t^{1/5} \xi))^2 \right) \, d\xi \right)^{1/2} \lesssim t^{-1} t^{1/10} (t^{-\frac{1}{5} (2\beta - 7)})^{1/2} \lesssim t^{-\frac{1}{5} - \frac{\beta}{5}}.
\]
(2.8)
Case 2: $\xi_0 \geq t^{-1/5}$. We need to show a bound of $t^{-1/2} \xi_0^{-2} \xi_0^{\frac{3}{2}} +$ for RHS of (2.5) instead. Since the resonant contributions concentrate on $\xi \sim \xi_0$, we split the integral in (2.5) as follows:

$$
\int_0^\infty e^{i r(\xi)} \xi^\beta \hat{g}(t, \xi) \, d\xi = \int_0^\infty e^{i r(\xi)} \xi^\beta \hat{g}(t, \xi) (1 - \psi(\xi/\xi_0)) \, d\xi
$$

$$
+ \int_0^\infty e^{i r(\xi)} \xi^\beta \hat{g}(t, \xi) \psi(\xi/\xi_0) \, d\xi
$$

$$
=: A_3 + A_4.
$$

We first control the non-stationary contributions. By integration by parts, we bound $A_3$ by

$$
|A_3| \lesssim |A_{31}| + |A_{32}|,
$$

$$
A_{31} = t^{-1} \int_0^\infty |\partial_\xi \left[ (\partial_\xi \Phi)^{-1} \xi^\beta (1 - \psi(\xi/\xi_0)) \right]\hat{g}(t, \xi)| \, d\xi,
$$

$$
A_{32} = t^{-1} \int_0^\infty \left| (\partial_\xi \Phi)^{-1} \xi^\beta (1 - \psi(\xi/\xi_0)) \partial_\xi \hat{g}(t, \xi) \right| \, d\xi.
$$

Using the fact that $|\partial_\xi \Phi| \gtrsim \max(\xi^4, \xi_0^4)$ on the support of the integral, and (2.6), we estimate, respectively,

$$
|A_{31}| \lesssim t^{-1} \lVert \hat{g} \rVert_{L^\infty} \int_0^\infty \left( \max(\xi^4, \xi_0^4)^{-1} \xi^\beta - 1 - \psi(\xi/\xi_0) \right)
$$

$$
+ \max(\xi^4, \xi_0^4)^{-1} \xi^\beta |\psi'(\xi/\xi_0)| \xi_0^{-1} \right) \, d\xi \tag{2.9}
$$

$$
\lesssim t^{-1} \xi_0^{-\beta - 4},
$$

and

$$
|A_{32}| \lesssim t^{-1} \lVert \partial_\xi \hat{g} \rVert_{L^2} \left( \int_0^\infty \left( \max(\xi^4, \xi_0^4)^{-1} \xi^\beta - 1 - \psi(\xi/\xi_0) \right)^2 \, d\xi \right)^{1/2}
$$

$$
\lesssim t^{-9/10} \xi_0^{\beta - \frac{7}{2}}. \tag{2.10}
$$

Both bounds $t^{-1} \xi_0^{-\beta - 4}$ and $t^{-9/10} \xi_0^{\beta - \frac{7}{2}}$ are better than the desired bound $t^{-1/2} \xi_0^{-3/2} +$ due to $\xi_0 \geq t^{-1/5}$.

It remains to study the stationary contributions. Let $l_0$ be the smallest integer with the property that $2^{l_0} \geq t^{-1/2} \xi_0^{-3/2}$. Then, the term $A_4$ is dominated by

$$
|A_4| \leq \sum_{l=l_0}^{\log \xi_0 + 10} |A_{4l}|,
$$

where

$$
A_{4l_0} = \int_0^\infty e^{i r(\xi)} \xi^\beta \hat{g}(t, \xi) \psi(\xi/\xi_0) \varphi(2^{-l_0}(\xi - \xi_0)) \, d\xi,
$$

$$
A_{4l} = \int_0^\infty e^{i r(\xi)} \xi^\beta \hat{g}(t, \xi) \psi(\xi/\xi_0) \varphi(2^{-l}(\xi - \xi_0)) \, d\xi, \quad l \geq l_0 + 1.
$$
The desired bound for $A_{4l_0}$ is immediate from the definition of $l_0$, and the estimate is
\[ |A_{4l_0}| \lesssim \| \hat{g} \|_{L^\infty} \int_0^\infty \xi^\beta \psi(\xi/\xi_0) \varphi(2^{-l_0}(\xi - \xi_0)) \, d\xi \lesssim \xi_0^\beta 2^{l_0}. \]
We are left finally to handle the terms $A_{4l}$ for $l \geq l_0 + 1$. Integration by parts yields
\[ |A_{4l}| \lesssim |A_{4l,1}| + |A_{4l,2}| + \text{the boundary term}, \]

\[
A_{4l,1} = t^{-1} \int_0^\infty \left| \hat{g}(t, \xi) \left( \frac{\partial \hat{\Phi}}{\partial \xi} \right)^{-1} \xi^\beta \psi(\xi/\xi_0) \varphi(2^{-l}(\xi - \xi_0)) \right| \, d\xi, \\
A_{4l,2} = t^{-1} \int_0^\infty \left| \left( \frac{\partial \hat{\Phi}}{\partial \xi} \right)^{-1} \xi^\beta \psi(\xi/\xi_0) \varphi(2^{-l}(\xi - \xi_0)) \right| \partial_\xi \hat{g}(t, \xi) \, d\xi. 
\]

We first observe that $|\partial_\xi \Phi| \sim \xi_0^3 \xi^l$ on the support of the integrals and then estimate the integrations as (2.7)–(2.10) to obtain
\[
|A_{4l,1}| \lesssim t^{-1} \xi_0^{\beta - 3} 2^{-l}, \quad (2.11)
\]
and
\[
|A_{4l,2}| \lesssim t^{-9/10} \xi_0^{\beta - 3} 2^{-l/2}. \quad (2.12)
\]
The boundary term has the same bound as (2.11). Summing (2.11) over $l$ immediately gives the desired bound. One obtains a bound of $t^{-9/10} \xi_0^{\beta - 3} 2^{-l_0/2}$ by summing (2.12) in $l$ which is better than what we need.

**Proof of (2.2).** With the same assumption (2.6) here, recalling (2.4), it is enough to show
\[ \left| \int_0^\infty e^{it\Phi(\xi)} \hat{g}(t, \xi) \, d\xi \right| \lesssim t^{-1/5} (x/t^{1/5})^{-7/8}. \]
Using the fact that
\[ \partial_\xi \Phi = x/t + 5\xi^4 > 0, \quad \text{for } x > 0, \]
and integration by parts, we deduce
\[ \left| \int_0^\infty e^{it\Phi(\xi)} \hat{g}(t, \xi) \, d\xi \right| \lesssim t^{-1} \int_0^\infty \left| (\partial_\xi \Phi)^{-1} \partial_\xi \hat{g}(t, \xi) \right| \, d\xi \\
+ t^{-1} \int_0^\infty \left| (\partial_\xi \Phi)^{-2} \partial_\xi^2 \Phi \hat{g}(t, \xi) \right| \, d\xi \\
+ \{ \text{the boundary term} \} \\
= : B_1 + B_2 + \{ \text{the boundary term} \}. \]
To bound the first term, we use (2.6) to dominate
\[ |B_1| \lesssim t^{-1} \| \partial_\xi \hat{g} \|_{L^2} \left( \int_0^\infty (x/t + 5\xi^4)^{-2} \, d\xi \right)^{1/2} \lesssim t^{-3/10} (x/t^{1/5})^{-7/8}. \]
Similarly, we can estimate
\[ |B_2| \lesssim t^{-1} \| \hat{g} \|_{L^\infty} \int_0^\infty (x/t + 5\xi^4)^{-2} \xi^3 \, d\xi \lesssim t^{-1} (x/t)^{-1}. \]

This bound is better than the desired bound due to \( x \geq t^{1/5} \). The boundary term has the same bound as \( B_2 \).

Proof of (2.3). We will make the same assumption (2.6) again. We split the integral in (2.4) instead as follows:
\[
\int_0^\infty e^{i\tau \Phi(\xi)} \hat{g}(t, \xi) \, d\xi = \int_0^\infty e^{i\tau \Phi(\xi)} \left(1 - \psi\left(4(\xi - \xi_0)/\xi_0\right)\right) \hat{g}(t, \xi) \, d\xi + \int_0^\infty e^{i\tau \Phi(\xi)} \psi\left(4(\xi - \xi_0)/\xi_0\right) \hat{g}(t, \xi) \, d\xi =: B_3 + B_4.
\]

We first estimate the term \( B_3 \). By the fact that \( |\partial_\xi \Phi| \gtrsim \max(\xi^4, \xi_0^4) \) on the support of the integral, we may estimate \( B_3 \) in a similar fashion as (2.9)–(2.10) to obtain
\[
|B_3| \lesssim t^{-1} \xi_0^{-4} + t^{-9/10} \xi_0^{-7/2} \lesssim t^{-1/5} (-x/t^{1/5})^{-7/8},
\]
which is better than the desired bound \( t^{-1/5} (-x/t^{1/5})^{-9/20} \).

We next control the term \( B_4 \). Let \( \tilde{l}_0 \) be the smallest integer such that
\[
2\tilde{l}_0 \geq t^{-1/5} (-x/t^{1/5})^{-3/10}
\]
and bound the term \( B_4 \) as follows:
\[
|B_4| \leq \sum_{l=\tilde{l}_0}^{\log \xi_0 + 10} |B_{4l}|,
\]
\[
B_{4l} = \int_0^\infty e^{i\tau \Phi(\xi)} \psi\left(4(\xi - \xi_0)(\xi_0^{-1})\right) \varphi\left(2^{-\tilde{l}_0}(\xi - \xi_0)\right) \hat{g}(t, \xi) \, d\xi,
\]
\[
B_{4l} = \int_0^\infty e^{i\tau \Phi(\xi)} \psi\left(4(\xi - \xi_0)(\xi_0^{-1})\right) \varphi\left(2^{-l}(\xi - \xi_0)\right) \hat{g}(t, \xi) \, d\xi, \quad l \geq \tilde{l}_0 + 1.
\]

Using the fact that \( |\partial_\xi \Phi| \gtrsim 2^l \xi_0^3 \) and integrating by parts, similar to (2.11)–(2.12), we have
\[
|B_{4l}| \lesssim t^{-1} \left(1 + \| \hat{g} \|_{L^\infty} \xi_0^{-3} 2^{-l} + \| \partial_\xi^2 \hat{g} \|_{L^2} \xi_0^{-3} 2^{-l/2}\right), \quad l \geq \tilde{l}_0 + 1,
\]
which yields the desired bound \( t^{-1/5} (-x/t^{1/5})^{-9/20} \) after summing in \( l \).

To study the contributions from the term \( B_{4\tilde{l}_0} \), we split further
\[
B_{4\tilde{l}_0} = e^{i\tau \Phi(\xi_0)} \hat{g}(t, \xi_0) \int_0^\infty e^{10i\xi_0^2 (\xi - \xi_0)^2} \varphi\left(4(\xi - \xi_0)(\xi_0^{-1})\right) \varphi\left((\xi - \xi_0)2^{-\tilde{l}_0}\right) \hat{g}(t, \xi) \, d\xi + e^{i\tau \Phi(\xi_0)} \int_0^\infty e^{10i\xi_0^2 (\xi - \xi_0)^2} \varphi\left(4(\xi - \xi_0)(\xi_0^{-1})\right) \varphi\left((\xi - \xi_0)2^{-\tilde{l}_0}\right) \hat{g}(t, \xi) \, d\xi + \int_0^\infty \left(e^{i\tau \Phi(\xi)} - e^{i\tau \Phi(\xi_0) + 10i\xi_0^2 (\xi - \xi_0)^2}\right) \varphi\left(4(\xi - \xi_0)(\xi_0^{-1})\right) \varphi\left((\xi - \xi_0)2^{-\tilde{l}_0}\right) \hat{g}(t, \xi) \, d\xi =: D_1 + D_2 + D_3.
\]
In light of the bounds in (2.6), we estimate

\[ |D_2| \lesssim t^{1/10} 2^{3\tilde{l}_0/2} \lesssim t^{-1/5} (-x/t^{1/5})^{-9/20}, \]

and

\[ |D_3| \lesssim t^{4\tilde{l}_0} (\xi_0^2 + 2^{2\tilde{l}_0}) \lesssim t^{-1/5} (-x/t^{1/5})^{-7/10}, \]

where the bound is better than what we need. For the term \( D_1 \), we write

\[
D_1 = e^{it\Phi(\xi_0)} \hat{g}(t, \xi_0) \int_{-\xi_0}^{\infty} e^{10it\xi_0^3 \eta^2} \psi(4\eta \xi_0^{-1}) \varphi(\eta^2 - \tilde{l}_0) \, d\eta.
\]

Recalling the formula

\[
\int_{-\infty}^{\infty} e^{-bx^2} \, dx = \sqrt{\frac{\pi}{b}}, \quad b \in \mathbb{C}, \, \Re b > 0,
\]

one calculates that

\[
\int_{-\infty}^{\infty} e^{10it\xi_0^3 \xi^2} e^{-\xi^2/2\tilde{l}_0} \, d\xi = \sqrt{\frac{i\pi}{10t\xi_0^3}} + O(2^{-\tilde{l}_0} (t\xi_0^3)^{-3/2}).
\]

In conclusion, we have obtained

\[
D_1 = \sqrt{\frac{i\pi}{10t\xi_0^3}} e^{it\Phi(\xi_0)} \hat{f}(t, \xi_0) + O(t^{-1/5} (-x/t^{1/5})^{-9/20})
\]

which combines (2.4) to finish the proof of (2.3). \( \square \)

3. Estimates on \( \|u\|_{H^2} \) and \( \|xf\|_{L^2} \)

In this section, we will prove the uniform bounds for the energy part in (1.11) which may be stated precisely as follows:

**Proposition 3.1.** Let \( u \) be a solution of (1.1)–(1.2) satisfying the a priori bounds (1.10). Then, the following estimates hold true:

\[
\|u(t, \cdot)\|_{H^2} \leq C \varepsilon_0,
\]

and

\[
\|xf(t, \cdot)\|_{L^2} \leq C(\varepsilon_0 + \varepsilon_1^5) (t)^{1/10}.
\]

**Proof.** Recall the definition \( f = e^{-t\tilde{a}_5} u(t) \). The first estimate (3.1) is a consequence of the conservation of mass and Hamiltonian:

\[
\int_{\mathbb{R}} u^2(x) \, dx, \quad \int_{\mathbb{R}} \left( \frac{1}{2} |\partial_x|^2 u \right)^2 - \frac{1}{30} u^6 \right) (x) \, dx
\]
where we shall control \( \|u\|_{L^6}^6 \) by the two conservations via the interpolation inequality
\[
\|u\|_{L^6}^6 \lesssim \|u\|_{L^2}^{5/6} \|u_{xx}\|_{L^2}^{1/6}.
\]

We denote by \( S \) the scaling vector field \( S := 1 + x \partial_x + 5t \partial_t \), and by \( Ih \) the antiderivative of \( h \) vanishing at \( -\infty \), i.e., \( Ih(x) = \int_{-\infty}^x h \, dy \). Since \( u \) is a solution of (1.1)–(1.2), a direct calculation shows that \( ISu \) satisfies
\[
\partial_t ISu - \partial_x^5 ISu + u^4 \partial_x ISu = 0.
\]

Multiplying the above equation by \( ISu \), integrating in space and integrating by parts yield
\[
\frac{1}{2} \frac{d}{dt} \|ISu\|_{L^2}^2 = -2 \int u^3 u_x (ISu) dx \lesssim \|u^3 u_x\|_{L^\infty} \|ISu\|_{L^2}^2 \lesssim \epsilon_1^{-1} \|ISu\|_{L^2}^2,
\]
where we have used Lemma 2.1 (2.1) by taking \( g \) to be the profile \( f \) of \( u \) in the last inequality. We combine the above resulting inequality and Gronwall’s inequality to obtain
\[
\|ISu\|_{L^2} \lesssim \epsilon_0 t^{C\epsilon_1^4}.
\]

We introduce another vector field \( J = x + 5t \partial_x^4 \). A small calculation yields
\[
Ju = ISu + tu^5,
\]
which by applying Lemma 2.1 (2.1) again implies
\[
\|xf\|_{L^2} = \|Ju\|_{L^2} \lesssim \epsilon_0 t^{C\epsilon_1^4} + t \|u\|_{L^{10}}^{5} \lesssim \epsilon_0 t^{C\epsilon_1^4} + \epsilon_1^{5/10}.
\]

This completes the proof of (3.2). \( \square \)

4. Estimate on \( \|\hat{f}\|_{L^\infty} \)

The main purpose of this section is to show the following uniform bound on the profile \( f \) in Fourier space:

**Proposition 4.1.** Let \( u \) be a solution of (1.1)–(1.2) satisfying the a priori bounds (1.10). Then, we have
\[
\|\hat{f}(t, \cdot)\|_{L^\infty} \leq C(\epsilon_0 + \epsilon_1^5).
\]

4.1. Evolution of \( \hat{f} \)

To prove (4.1), we shall study the evolutionary equation of the profile \( f \) in Fourier space. Let
\[
\Psi(\xi, \eta, \sigma, \eta_1, \sigma_1) = \xi^5 - (\xi - \eta - \sigma - \eta_1 - \sigma_1)^5 - \eta^5 - \sigma^5 - \eta_1^5 - \sigma_1^5.
\]
Recall \( f(t) = e^{-it^5} u(t) \), then from (1.1) we see that the profile \( f \) obeys

\[
\partial_t \hat{f}(t, \xi) = \frac{-i\xi}{4\pi^2} \int_{\mathbb{R}^4} e^{-ir\Psi(\xi, \eta, \sigma, \eta_1, \sigma_1)} \hat{f}(t, \xi - \eta - \sigma - \eta_1 - \sigma_1) \hat{f}(t, \eta) \hat{f}(t, \sigma) \times \hat{f}(t, \eta) \hat{f}(t, \sigma_1) \text{d}\eta \text{d}\sigma \text{d}\eta_1 \text{d}\sigma_1.
\]

(4.2)

Given \( j \in \mathbb{Z} \) and \( |\xi| \in (2^j, 2^{j+1}) \), to separate the stationary phase, similar to [5], we decompose (4.2) as follows:

\[
\partial_t \hat{f}(t, \xi) = \sum_{k,l,k_1,l_1} \frac{-i\xi}{4\pi^2} \int_{\mathbb{R}^4} e^{-ir\Psi(\xi, \eta, \sigma, \eta_1, \sigma_1)} \hat{f}(t, \xi - \eta - \sigma - \eta_1 - \sigma_1) \hat{f}(t, \eta) \hat{f}(t, \sigma) \times \hat{f}(t, \eta) \hat{f}(t, \sigma_1) \text{d}\eta \text{d}\sigma \text{d}\eta_1 \text{d}\sigma_1
\]

\[
= \sum_{2^k, 2^l, 2^{k_1}, 2^{l_1} \leq 2^j, 2^k > t^{-1/5}} A_{k,l,k_1,l_1} + \sum_{2^k, 2^l, 2^{k_1}, 2^{l_1} \leq 2^j, 2^k > t^{-1/5}} A_{k,l,k_1,l_1}
\]

\[
+ \sum_{2^k \gg \mathcal{A} \setminus \mathcal{B}, 2^j} A_{k,l,k_1,l_1} + \sum_{2^k, 2^l, 2^{k_1}, 2^{l_1} < 2^j, 2^k > t^{-1/5}} A_{k,l,k_1,l_1}
\]

\[
+ \{ \text{similar terms to } R_1, R_2 \},
\]

(4.3)

where \( A_{k,l,k_1,l_1} \) stands for the expression inside of the summation of the first equality, and \( \{ \text{similar terms to } R_1, R_2 \} \) are those terms obtained by exchanging the orders of \( (2^k, 2^l, 2^{k_1}, 2^{l_1}) \) in the summation, and \( \mathcal{A} = \{2^l, 2^{k_1}, 2^{l_1}\} \) and \( \mathcal{B} \subseteq \mathcal{A} \). For simplicity, here we slightly make the abuse of notation \( 2^k \sim \mathcal{B} \gg \mathcal{A} \setminus \mathcal{B}, 2^j \) whose exact meaning is that \( R_2 \) contains the following 7 terms:

\[
\sum_{2^k \gg \mathcal{A} \setminus \mathcal{B}, 2^j} A_{k,l,k_1,l_1}, \sum_{2^k \gg \mathcal{A} \setminus \mathcal{B}, 2^j} A_{k,l,k_1,l_1}, \ldots, \sum_{2^k \gg \mathcal{A} \setminus \mathcal{B}, 2^j} A_{k,l,k_1,l_1}.
\]

We put these terms into \( R_2 \) since they can be treated in a same manner.

In this subsection, we aim to show the following crucial proposition:

**Proposition 4.2.** Let \( t > 1 \). Then, the profile \( f \) obeys the following ODE in Fourier space:

\[
\partial_t \hat{f}(t, \xi) = \left( \frac{-i}{40t^2 \xi^5} \hat{f}(t, \xi) \right)^4 \hat{f}(t, \xi) + \frac{c_1 i}{t^2 \xi^5} e^{-\frac{624t^5}{65}} \hat{f}(t, \xi/3)^5
\]

\[
+ \frac{c_2 i}{t^2 \xi^5} e^{-\frac{800t^5}{81}} \left| \hat{f}(t, \xi/3) \right|^2 \hat{f}(t, \xi/3)^3 \right) \chi_{|\xi| > t^{-1/5}} + R(t, \xi),
\]

(4.4)
where \( c_1 \) and \( c_2 \) are constants depending on sign \( \xi \), and the remainder \( R(t, \xi) \) is integrable in time and satisfies

\[
\int_0^\infty |R(t, \xi)| \, dt \lesssim \varepsilon_1^5.
\] (4.5)

We divide the proof of Proposition 4.2 into the following four steps.

**Step 1: Estimate of \( M \).** This is the main case. By change of variables

\[
(\xi, \eta, \sigma, \eta_1, \sigma_1) = 2^j (\xi', \eta', \sigma', \eta'_1, \sigma'_1),
\] (4.6)

we rewrite the term \( M \) into the following form:

\[
M = -i(4\pi^2)^{-1} \int \mathbb{R}^4 e^{-i\Psi(\xi, \eta, \sigma, \eta_1, \sigma_1)} f_{\leq j}(t, \xi - \eta - \sigma - \eta_1 - \sigma_1) f_{\leq j}(t, \sigma) \times f_{\leq j}(t, \eta_1) f_{\leq j}(t, \sigma_1) \, d\eta d\sigma d\eta_1 d\sigma_1
\]

\[
= -i(4\pi^2)^{-1} 2^{5j} \int \mathbb{R}^4 e^{-i2^{5j}\Psi(\xi', \eta', \sigma', \eta'_1, \sigma'_1)} \left| f_{\leq j}(t, 2^j(\xi' - \eta' - \sigma' - \eta'_1 - \sigma'_1)) \right| \times f_{\leq j}(t, 2^j\eta'_1) f_{\leq j}(t, 2^j\sigma'_1) \, d\eta' d\sigma' d\eta'_1 d\sigma'_1,
\] (4.7)

where \( C \sim 1 \). If we let

\[
F(\eta', \sigma', \eta'_1, \sigma'_1) = f_{\leq j}(t, 2^j(\xi' - \eta' - \sigma' - \eta'_1 - \sigma'_1)) \times f_{\leq j}(t, 2^j\eta'_1) f_{\leq j}(t, 2^j\sigma'_1),
\] (4.8)

then (4.7) becomes

\[
M = -i(4\pi^2)^{-1} 2^{5j} \int \mathbb{R}^4 e^{-i2^{5j}\Psi(\xi', \eta', \sigma', \eta'_1, \sigma'_1)} \xi' F(\eta', \sigma', \eta'_1, \sigma'_1) \times \varphi(C\eta') \varphi(C\sigma') \varphi(C\eta'_1) \varphi(C\sigma'_1) \, d\eta' d\sigma' d\eta'_1 d\sigma'_1,
\] (4.9)

where \( |\eta'|, |\sigma'|, |\eta'_1|, |\sigma'_1| \lesssim |\xi'| \sim 1 \).

To analyze \( M \), we need the following stationary phase in four dimensions:

**Lemma 4.3.** Assume \( \chi \in C^\infty_0 \) and \( |\nabla \chi| + |\nabla^2 \chi| \leq 1; \) and \( \phi \in C^\infty \) such that \( |\det \text{Hess} \phi| \geq 1 \) and \( |\nabla \phi| + |\nabla^2 \phi| + |\nabla^3 \phi| \leq 1 \). Let

\[
J = \int \mathbb{R}^4 e^{i\phi(\xi_1, \xi_2, \xi_3, \xi_4)} F(\xi_1, \xi_2, \xi_3, \xi_4) \chi(\xi_1, \xi_2, \xi_3, \xi_4) \, d\xi_1 d\xi_2 d\xi_3 d\xi_4.
\]

(i) If \( \nabla \phi \) only vanishes at \((\xi_{10}, \xi_{20}, \xi_{30}, \xi_{40})\), then the following asymptotic expansion holds true:

\[
J = \frac{4\pi^2 e^{i\phi(\xi_{10}, \xi_{20}, \xi_{30}, \xi_{40})}}{\sqrt{\Delta}} \frac{e^{i\phi(\xi_{10}, \xi_{20}, \xi_{30}, \xi_{40})}}{\chi^2} F(\xi_{10}, \xi_{20}, \xi_{30}, \xi_{40}) \chi(\xi_{10}, \xi_{20}, \xi_{30}, \xi_{40}) + O\left( \frac{\| \tilde{F} \|_{L^1}}{\lambda^2} \right),
\]

where

\[
\Delta = \left| \chi' \right|^2 + \left| \chi'' \right|^2 + \left| \chi''' \right|^2 + \left| \chi'''' \right|^2.
\]
where $\beta = \text{sign Hess } \phi(\xi_{0}, \xi_{20}, \xi_{30}, \xi_{40})$ and $\Delta = |\text{det Hess } \phi(\xi_{10}, \xi_{20}, \xi_{30}, \xi_{40})|$.

(ii) If $|\nabla \phi| \geq 1$, then we have

$$J = \mathcal{O}\left(\frac{\|\hat{F}\|_{L^{1}}}{\lambda^{2}}\right).$$

Proof. The results follow from similar arguments as in Lemma A.1 in [5].\hfill\Box

A direct computation shows that the stationary points of $\Psi$ are given by

$$(\eta_1,\sigma_1,\eta_{11},\sigma_{11}) = (\xi/5,\xi/5,\xi/5,\xi/5),$$

$$\bigcup_{a=2}^{6}(\eta_a,\sigma_a,\eta_{1a},\sigma_{1a}) = \{(\xi/3,\xi/3,\xi/3,\xi/3), (\xi/3, -\xi/3, \xi/3, \xi/3), (\xi/3, \xi/3, -\xi/3, \xi/3), (\xi/3, \xi/3, \xi/3, -\xi/3)\},$$

$$\bigcup_{a=7}^{16}(\eta_a,\sigma_a,\eta_{1a},\sigma_{1a}) = \{(-\xi, -\xi, \xi, \xi), (-\xi, \xi, -\xi, \xi), (-\xi, \xi, -\xi, -\xi), (-\xi, -\xi, -\xi, -\xi), (\xi, -\xi, -\xi, \xi), (\xi, -\xi, \xi, -\xi), (\xi, -\xi, -\xi, -\xi), (\xi, -\xi, -\xi, -\xi)\}. $$

Moreover, we have

$$\Psi(\eta_1,\sigma_1,\eta_{11},\sigma_{11}) = (624/625)\xi^{5}, \quad \Delta_1 = 5^{-7} \cdot 4^{4}|\xi|^{12},$$

$$\Psi(\eta_a,\sigma_a,\eta_{1a},\sigma_{1a}) = (80/81)\xi^{5}, \quad \Delta_a = 3^{-11} \cdot 4^{4}\cdot 5^{4}|\xi|^{12}, \quad 2 \leq a \leq 6,$$

$$\Psi(\eta_a,\sigma_a,\eta_{1a},\sigma_{1a}) = 0, \quad \Delta_a = 4^{4}\cdot 5^{4}|\xi|^{12}, \quad 7 \leq a \leq 16,$$

and

$$\beta_a = 1 - \text{sign } \xi, \quad \text{for } 1 \leq a \leq 6,$$

$$\beta_a = 0, \quad \text{for } 7 \leq a \leq 16.$$

Observing (4.10)–(4.11) and (4.6), applying Lemma 4.3 (i) to (4.9), and recalling $|\xi| > t^{-1/5}$ by the definition of $M$, we obtain

$$M = \left(\frac{-i}{40t^{2}\xi^{5}}|\hat{f}(t, \xi)|^{4} \hat{f}(t, \xi) + \frac{c_1 i}{t^{2}\xi^{5}}e^{-\frac{624\xi^{5}}{625}} \hat{f}(t, \xi/5)^{5} + \frac{c_2 i}{t^{2}\xi^{5}}e^{-\frac{80\xi^{5}}{81}}|\hat{f}(t, \xi/3)|^{5} \hat{f}(t, \xi/3)^{3}\right)1_{|\xi| > t^{-1/5}} + 2^{5j}O\left(\frac{\|\hat{F}\|_{L^{1}}}{(2^{5j}t)^{2}}\right),$$

$$R_{0}$$

where $c_1$ and $c_2$ are constants depending on sign $\xi$. The rest of this step is to show that $R_{0}$ is integrable in time and satisfies

$$\int_{2^{-5j}}^{\infty}|R_{0}|ds \lesssim \varepsilon_{1}^{j}.$$

As a prerequisite, we give the following estimate:
Lemma 4.4. Suppose $2^j > t^{-1/5}$, then we have
\[ \| f_{\leq j} \|_{L^1} \lesssim \epsilon_1 2^{j/2} t^{1/30}. \]

Proof. We first see that
\[ \| f_a \|_{L^2} \leq \| \psi_a \|_{L^2} \| \hat{f} \|_{L^\infty} \lesssim \epsilon_1 2^{a/2}, \]
and
\[ \| x f_a \|_{L^2} = \| \partial f_a \|_{L^2} \lesssim 2^{-a} \| \psi_a \|_{L^2} \| \hat{f} \|_{L^\infty} + \| \psi_a \|_{L^\infty} \| \partial \hat{f} \|_{L^2} \lesssim \epsilon_1 (2^{-a/2} + t^{1/10}). \]
Hence, we have
\[ \| f_a \|_{L^1} \lesssim \| f_a \|_{L^2}^{1/3} \| x f_a \|_{L^2}^{2/3} \lesssim \epsilon_1 \left( 1 + 2^{a/2} t^{1/30} \right). \tag{4.14} \]
We next estimate
\[ \| f_{\leq -\frac{1}{2} \log_t} \|_{L^2} \leq \| \varphi (t^{1/5} \xi) \|_{L^2} \| \hat{f} \|_{L^\infty} \lesssim \epsilon_1 t^{-1/10}, \tag{4.15} \]
and
\[ \| x f_{\leq -\frac{1}{2} \log_t} \|_{L^2} = \| \partial \xi [\varphi (t^{1/5} \xi) \hat{f} (\xi)] \|_{L^2} \leq t^{1/5} \| \varphi' (t^{1/5} \xi) \|_{L^2} \| \hat{f} \|_{L^\infty} + \| \varphi (t^{1/5} \xi) \|_{L^\infty} \| \partial \hat{f} \|_{L^2} \lesssim \epsilon_1 t^{1/10}. \tag{4.16} \]
It finally results from (4.14)–(4.16) that
\[ \| f_{\leq j} \|_{L^1} \leq \| f_{\leq -\frac{1}{2} \log_t} \|_{L^1} + \sum_{t^{-1/5} \leq 2^a \leq 2^j} \| f_a \|_{L^1} \lesssim \epsilon_1 t^{-1/30} \times 2^{a/2} t^{1/30} + \epsilon_1 \sum_{t^{-1/5} \leq 2^a \leq 2^j} \left( 1 + 2^{a/2} t^{1/30} \right) \lesssim \epsilon_1 + \epsilon_1 2^{j/2} t^{1/30} \lesssim \epsilon_1 2^{j/2} t^{1/30}, \]
where we have used $2^j > t^{-1/5}$ in the last two inequalities. \qed

From (4.8), we calculate
\[ \hat{F}(x, y, x_1, y_1) = \frac{2^{-5j}}{4\pi^2} \int e^{-iz\xi} f_{\leq j} (t, 2^{-j} (z - x)) f_{\leq j} (t, 2^{-j} (z - x_1)) \times f_{\leq j} (t, 2^{-j} z) f_{\leq j} (t, 2^{-j} (y - z)) d\zeta, \tag{4.17} \]
which gives
\[ \| \hat{F} \|_{L^1} \lesssim \| f_{\leq j} \|_{L^4}^5. \tag{4.18} \]
Applying Lemma 4.4 to (4.18), one has
\[ \| R_0 \| \lesssim 2^{-5j} t^{-2} \| f_{\leq j} \|_{L^1}^5 \lesssim \epsilon_1^5 2^{-15j} t^{-7}. \tag{4.19} \]
It follows that
\[
\int_{2^{-5j}}^{\infty} |R_0| \, ds \lesssim \epsilon_1^5 2^{-\frac{15}{4}j} \int_{2^{-5j}}^{\infty} s^{-\frac{7}{4}} \, ds \lesssim \epsilon_1^5.
\]
This completes the proof of (4.13).

**Step 2: Estimate of \( R_1 \).** There is no stationary point in this case. We use change
variables \( (\eta, \sigma, \eta_1, \sigma_1) = 2^k (\eta', \sigma', \eta'_1, \sigma'_1) \) to write
\[
R_1 = - \sum_{2^k \gg 2^j \atop 2^k > t^{-1/5}} i(4\pi^2)^{-1} 2^{4k} \xi \int_{\mathbb{R}^4} e^{-i2^{5k} \psi(2^{-k} \xi, \eta', \sigma', \eta'_1, \sigma'_1)} F_k(\eta', \sigma', \eta'_1, \sigma'_1)
\times \psi(\eta') \psi(\sigma') \psi(\eta'_1) \psi(\sigma'_1) \, d\eta' \, d\sigma' \, d\eta'_1 \, d\sigma'_1,
\]
where
\[
F_k(\eta', \sigma', \eta'_1, \sigma'_1) = \hat{f}_{\lesssim k}(t, \xi) \, 2^{-k} (\eta' + \sigma' + \eta'_1 + \sigma'_1) \hat{f}_{\lesssim k}(t, 2k \eta')
\times \hat{f}_{\lesssim k}(t, 2k \sigma') \hat{f}_{\lesssim k}(t, 2k \eta'_1) \hat{f}_{\lesssim k}(t, 2k \sigma'_1).
\]
From Lemma 4.3 (ii), it follows that
\[
|R_1| \lesssim 2^j \sum_{2^k \gg 2^j \atop 2^k > t^{-1/5}} 2^{4k} \frac{\|\hat{F}_k\|_{L^1}}{(2^{5k} t)^{\frac{7}{4}}}.
\]
(4.20)

Proceeding as (4.17)–(4.18), the following estimate holds true:
\[
\|\hat{F}_k\|_{L^1} \lesssim \|\hat{f}_{\lesssim k}\|_{L^1}^5.
\]
(4.21)

Inserting (4.21) into (4.20), one then uses Lemma 4.4 to estimate
\[
|R_1| \lesssim 2^j \sum_{2^k \gg 2^j \atop 2^k > t^{-1/5}} 2^{-6k} t^{-2} \|\hat{f}_{\lesssim k}\|_{L^1}^5 \lesssim \epsilon_1^5 2^j \sum_{2^k \gg 2^j \atop 2^k > t^{-1/5}} 2^{-19 \frac{k}{4}} t^{-\frac{7}{4}}
\lesssim \epsilon_1^5 2^j t^{-\frac{3}{4}} \max \left( 2^j, t^{-1/5} \right)^{-\frac{19}{4}}.
\]
(4.22)

This implies that \( R_1 \) is integrable in time and satisfies
\[
\int_0^{\infty} |R_1| \, ds \lesssim \epsilon_1^5.
\]
(4.23)

**Step 3: Estimate of \( R_2 \).** Recall that there are 7 terms in the summands in \( R_2 \), we first
handle the case \( 2^k \gg 2^j, 2^{k_1}, 2^{j_1}, 2^j \), i.e., \( B = \emptyset \), and denote by \( R_{21} \) the corresponding
term after summation in \( k \), and then explain how to treat the other cases in a same
manner. We are going to show
\[
\int_0^{\infty} |R_2| \, ds \lesssim \epsilon_1^5.
\]
(4.24)
Case 1: \( \mathcal{B} = \emptyset \). Notice that \(|\eta|\) is the largest variable among \{\( \xi, \eta, \sigma, \eta_1, \sigma_1 \}\), one thus may rewrite

\[
R_{21} = - \sum_{2^k > 2^j} i(4\pi^2)^{-1} \xi \int_{\mathbb{R}^4} e^{-i\Psi(\xi, \eta, \sigma, \eta_1, \sigma_1)} f_{\sim k}(t, \xi - \eta - \sigma - \eta_1 - \sigma_1) \\
\times f_{\ll k}(t, \eta) f_{\ll k}(t, \sigma) f_{\ll k}(t, \eta_1) f_{\ll k}(t, \sigma_1) \psi_k(\eta) \psi_k(\sigma) \psi_k(\eta_1) \\
\times \varphi_k(\sigma_1) d\eta d\sigma d\eta_1 d\sigma_1.
\]

(4.25)

Let \( m_1, m_2, m_3, m_4 \in \{0, \ldots, 10\} \). On the support of the resulting integrand, one calculates that

\[
|\partial_\sigma \Psi| = 5 \left| (\xi - \eta - \sigma - \eta_1 - \sigma_1)^4 - \sigma^4 \right| \sim 2^{4k},
\]

(4.26)

and

\[
\left| \partial_{m_1} \partial_{m_2} \partial_{m_3} \partial_{m_4} \left((\partial_\sigma \Psi)^{-1}\right) \right| \leq 2^{-4k} 2^{-(m_1 + m_2 + m_3 + m_4)k}.
\]

(4.27)

Using the identity

\[
e^{-i\Psi(\xi, \eta, \sigma, \eta_1, \sigma_1)} = i \int \partial_\sigma e^{-i\Psi(\xi, \eta, \sigma, \eta_1, \sigma_1)},
\]

and integrating by parts in \( \sigma \) in (4.25), we obtain

\[
R_{21} = \sum_{2^k > 2^j} \left( R_{k21} + R_{k22} + R_{k23} \right),
\]

(4.28)

where the terms under summation are given by

\[
R_{k21} := (4\pi^2 t)^{-1} \xi \int_{\mathbb{R}^4} e^{-i\Psi(\xi, \eta, \sigma, \eta_1, \sigma_1)} \partial_{\sigma} (-1) \partial_{\sigma} f_{\sim k}(t, \xi - \eta - \sigma - \eta_1 - \sigma_1) \\
\times f_{\ll k}(t, \eta) f_{\ll k}(t, \sigma) f_{\ll k}(t, \eta_1) f_{\ll k}(t, \sigma_1) \psi_k(\eta) \psi_k(\sigma) \\
\times \varphi_k(\eta_1) \varphi_k(\sigma_1) d\eta d\sigma d\eta_1 d\sigma_1.
\]

\[
R_{k22} := (4\pi^2 t)^{-1} \xi \int_{\mathbb{R}^4} e^{-i\Psi(\xi, \eta, \sigma, \eta_1, \sigma_1)} \partial_{\sigma} (-1) \partial_{\sigma} f_{\sim k}(t, \xi - \eta - \sigma - \eta_1 - \sigma_1) \\
\times f_{\sim k}(t, \eta) \partial_{\sigma} f_{\ll k}(t, \sigma) f_{\ll k}(t, \eta_1) f_{\ll k}(t, \sigma_1) \psi_k(\eta) \psi_k(\sigma) \\
\times \varphi_k(\eta_1) \varphi_k(\sigma_1) d\eta d\sigma d\eta_1 d\sigma_1.
\]

\[
R_{k23} := (4\pi^2 t)^{-1} \xi \int_{\mathbb{R}^4} e^{-i\Psi(\xi, \eta, \sigma, \eta_1, \sigma_1)} \partial_{\sigma} (-1) \psi_k(\sigma) \\
\times f_{\ll k}(t, \sigma_1) \psi_k(\eta) \psi_k(\eta_1) \psi_k(\sigma_1) d\eta d\sigma d\eta_1 d\sigma_1.
\]

To control \( R_{21} \), we need the following estimate on pseudo-product operators satisfying certain strong integrability conditions:
Lemma 4.5. If \( m \in L^1(\mathbb{R}^4) \) satisfies
\[
\left\| \int_{\mathbb{R}^4} m(\eta, \sigma, \eta_1, \sigma_1) e^{i(x \eta + y \sigma + x_1 \eta_1 + y_1 \sigma_1)} \, d\eta d\sigma d\eta_1 d\sigma_1 \right\|_{L^1_{x,y,x_1,y_1}} := \| m \|_{S^\infty} < \infty,
\]
then the following estimate holds:
\[
\left| \int_{\mathbb{R}^4} m(\eta, \sigma, \eta_1, \sigma_1) \hat{g}_1(-\eta - \sigma - \eta_1 - \sigma_1) \hat{g}_2(\eta) \hat{g}_3(\sigma) \hat{g}_4(\eta_1) \hat{g}_5(\sigma_1) \, d\eta d\sigma d\eta_1 d\sigma_1 \right|
\lesssim \| m \|_{S^\infty} \| g_1 \|_{L^{p_1}} \| g_2 \|_{L^{p_2}} \| g_3 \|_{L^{p_3}} \| g_4 \|_{L^{p_4}} \| g_5 \|_{L^{p_4}},
\]
for any \( p_1, p_2, p_3, p_4, p_5 \in [1, \infty] \) satisfying \( \sum_{a=1}^5 p_a^{-1} = 1 \).

Proof. The same argument as in Lemma A.2 in [5] applies here.

Corresponding to the estimate in Lemma 4.4, we also have:

Lemma 4.6. Let \( 2^k > t^{-1/5} \) and \( \delta > 0 \), then the following estimates hold
\[
\| u_{\ll k} \|_{L^\infty} \lesssim \varepsilon_1 2^k, \quad \| f_{\ll k} \|_{L^2} \lesssim \varepsilon_1 2^{k/2}, \quad \| \partial f_{\ll k} \|_{L^2} \lesssim \varepsilon_1 t^{1/10 + \delta/2} 2^k.
\]

Proof. The above three inequalities can be proven in a similar fashion, so we only show the last one:
\[
\| \partial f_{\ll k} \|_{L^2} \leq \| \partial \mathcal{F}(f_{\ll -1/2 \log t}) \|_{L^2} + \sum_{t^{-1/5} \leq 2^a \ll 2^k} \| \partial f_a \|_{L^2} \lesssim \varepsilon_1 t^{1/10} + \varepsilon_1 \sum_{t^{-1/5} \leq 2^a \ll 2^k} (2^{-a/2} + t^{1/10}) \lesssim \varepsilon_1 t^{1/10} + \varepsilon_1 t^{1/10} \sum_{t^{-1/5} \leq 2^a \ll 2^k} 1 \lesssim \varepsilon_1 t^{1/10} + \varepsilon_1 t^{1/10} \delta/5 \sum_{t^{-1/5} \leq 2^a \ll 2^k} 2^{\delta a} \lesssim \varepsilon_1 t^{1/10 + \delta/2} 2^k,
\]
for any \( \delta > 0 \).

Applying Lemma 4.5 and Lemma 4.6, we obtain
\[
|R_{k21}| \lesssim t^{-1} 2^{j} 2^{-4k} \| \partial \hat{f}_{\ll k} \|_{L^2} \| f_{\ll k} \|_{L^2} \| u_{\ll k} \|_{L^\infty}^2 \| u_{\ll k} \|_{L^\infty} \lesssim \varepsilon_1^5 2^j 2^{-11/2} 2^{-3k/2}, \tag{4.29}
\]
\[
|R_{k22}| \lesssim t^{-1} 2^{j} 2^{-4k} \| f_{\ll k} \|_{L^2} \| \partial \hat{f}_{\ll k} \|_{L^2} \| u_{\ll k} \|_{L^\infty}^2 \| u_{\ll k} \|_{L^\infty} \lesssim \varepsilon_1^5 2^j 2^{1/10 + \delta/2} 2^{-3k}, \tag{4.30}
\]
and
\[
|R_{k23}| \lesssim t^{-1} 2^{j} 2^{-5k} \| f_{\ll k} \|_{L^2} \| f_{\ll k} \|_{L^2} \| u_{\ll k} \|_{L^\infty}^2 \| u_{\ll k} \|_{L^\infty} \lesssim \varepsilon_1^5 2^j 2^{-6/5} 2^{-2k}. \tag{4.31}
\]
Choosing $\delta \in (0, 1/2)$, substituting (4.29)–(4.31) into (4.28), and summing over $k$, we integrate in time to estimate
\[
\int_0^\infty |R_{21}| \, ds \lesssim \epsilon_1^5 2^j \int_0^\infty s^{-11/10} \max \left( 2^j, s^{-1/5} \right)^{-3/2} \, ds 
+ \epsilon_1^5 2^j \int_0^\infty s^{-11/10 + \delta} \max \left( 2^j, s^{-1/5} \right)^{-3/2 + \delta} \, ds 
+ \epsilon_1^5 2^j \int_0^\infty s^{-6/5} \max \left( 2^j, s^{-1/5} \right)^{-2} \, ds 
\] (4.32)
\[
\lesssim \epsilon_1^5.
\]

Case 2: $B$ is a singleton. Due to symmetry (three cases totally), we only analyze the case $2^k \sim 2^l \gg 2^{l_1}, 2^{j_1}, 2^j$, and denote by $R_{22}$ the corresponding term after summation in $k$, which may be written as
\[
R_{22} = - \sum_{2^k \gg 2^l} \left( 4\pi^2 \right)^{-1/2} \int e^{-it\Psi(\xi, \eta, \sigma, \eta_1, \sigma_1)} f_{\sim k}(t, \xi - \eta - \sigma - \eta_1 - \sigma_1)
\]
\[
\times f_{\sim k}(t, \eta) f_{\sim k}(t, \sigma) \hat{\varphi}(t, \eta_1, \sigma_1) \hat{\varphi}(\sigma) \hat{\varphi}_k(\eta) \hat{\varphi}_k(\sigma) \, d\eta d\sigma d\eta_1 d\sigma_1.
\]

There are two cases: $k = l$ and $|k - l| \geq 1$. For the former, (4.26) and (4.27) still hold, and thus one can also apply the argument of $R_{21}$ to deduce the desired bound. As for the latter, we have
\[
|\partial_{\eta_1} \Psi| \sim 2^{4k}, \quad \left| \partial_{\eta_1}^{m_1} \partial_\sigma^{m_2} \partial_{\eta_1}^{m_3} \partial_{\sigma_1}^{m_4} \left( (\partial_{\eta_1} \Psi)^{-1} \right) \right| \lesssim 2^{-4k} 2^{-(m_1 + m_2 + m_3 + m_4)k}.
\]

This is enough to prove the desired estimate in a similar fashion as $R_{21}$ after integration by parts in $\eta_1$ instead.

Case 3: $B$ is a binary set. By symmetry (three cases totally), we only also consider the case $2^k \sim 2^l \sim 2^{l_1} \gg 2^{l_1}, 2^j$, and denote by $R_{23}$ the corresponding term after summation in $k$, which reads as
\[
R_{23} = - \sum_{2^k \gg 2^l} \left( 4\pi^2 \right)^{-1/2} \int e^{-it\Psi(\xi, \eta, \sigma, \eta_1, \sigma_1)} f_{\sim k}(t, \xi - \eta - \sigma - \eta_1 - \sigma_1)
\]
\[
\times f_{\sim k}(t, \eta) f_{\sim k}(t, \sigma) \hat{\varphi}(t, \eta_1, \sigma_1) \hat{\varphi}(\sigma) \hat{\varphi}_k(\eta) \hat{\varphi}_k(\sigma) \, d\eta d\sigma d\eta_1 d\sigma_1.
\]

We shall discuss $k = l = l_1$ and $\max(|k - l|, |k - k_1|, |l - k_1|) \geq 1$, separately. Considering the former, it holds that
\[
|\partial_{\sigma_1} \Psi| \sim 2^{4k}, \quad \left| \partial_{\eta}^{m_1} \partial_\sigma^{m_2} \partial_{\eta_1}^{m_3} \partial_{\sigma_1}^{m_4} \left( (\partial_{\sigma_1} \Psi)^{-1} \right) \right| \lesssim 2^{-4k} 2^{-(m_1 + m_2 + m_3 + m_4)k},
\]
which suffices to yield the desired bound via integration by parts in \( \sigma \) as \( R_{21} \). For the latter, we further divide two sub cases: Either there are only two numbers no less than one, or all of the three numbers are no less than one. By symmetry again, we only consider \(|k - l|, |k - k_1| \geq 1, l = k_1\) for the first sub-case and may assume \( k > l > k_1\) for the second sub-case. Both cases satisfy (4.26) and (4.27) which give the desired bound by repeating the argument of \( R_{21} \).

We finish the proof of (4.24) by summarizing the above three cases.

**Step 4: Estimate of** \( R_{3} \). This is the easiest case. By Young’s inequality, we obtain

\[
|R_{3}| \lesssim 2^j \sum_{2^k, 2^{k_1}, 2^{l_1} \leq t^{-1/5}} 2^k 2^l 2^{k_1} 2^{l_1} \|F(f_{t-1/5})\|_{L^\infty}^5
\]

\[
\lesssim \epsilon_j^5 2^j t^{-4/5} 1_{t \leq 2^{-5} j}.
\]

Therefore, \( R_{3} \) is integrable in time and satisfies

\[
\int_0^\infty |R_{3}| \, ds \lesssim \epsilon_j^5 2^j \int_0^{2^{-5} j} s^{-4/5} \, ds \lesssim \epsilon_j^5.
\]

\( \square \)

### 4.2. Proof of Proposition 4.1

To bound \( \|\hat{f}(t, \cdot)\|_{L^\infty} \), we introduce the new profile \( w \) as follows:

\[
\tilde{w}(t, \xi) := e^{iB(t, \xi)} \hat{f}(t, \xi)
\]

with

\[
B(t, \xi) := \frac{1}{40 \xi^5} \int_1^t |\hat{f}(s, \xi)|^4 \frac{ds}{s^2}.
\]

In view of (4.4) and (4.33)–(4.34), we then deduce

\[
\begin{aligned}
\partial_t \tilde{w}(t, \xi) &= e^{iB(t, \xi)} \left( \partial_t \hat{f}(t, \xi) + i\partial_t B(t, \xi) \hat{f}(t, \xi) \right) \\
&= e^{iB(t, \xi)} \left[ \left( \frac{c_1 i}{t^2 \xi^5} e^{-\frac{624 \xi^5}{65}} \hat{f}(t, \xi/5)^5 + \frac{c_2 i}{t^2 \xi^5} e^{-\frac{800 \xi^5}{81}} \left|\hat{f}(t, \xi/3)\right|^2 \hat{f}(t, \xi/3)^3 \right] 1_{|\xi| > t^{-1/5}} \\
&\quad + R(t, \xi) \right], \quad \text{for } t > 1.
\end{aligned}
\]

Observing that \( B \) is real, integrating in time in (4.35), and taking account of (4.5), we get

\[
\begin{aligned}
|\hat{f}(t, \xi)| &\leq |\hat{u}_0(\xi)| + \left| \frac{\xi^{-5}}{t^2} \int_{|\xi| = 5}^t e^{iB(s, \xi)} e^{-\frac{624 \xi^5}{65}} \hat{f}(s, \xi/5)^5 s^{-2} \, ds \right| \\
&\quad + \left| \frac{\xi^{-5}}{t^2} \int_{|\xi| = 5}^t e^{iB(s, \xi)} e^{-\frac{800 \xi^5}{81}} \left|\hat{f}(t, \xi/3)\right|^2 \hat{f}(t, \xi/3)^3 s^{-2} \, ds \right| + \epsilon_j^5.
\end{aligned}
\]
To complete the proof, we need to show that
\[
\left| \xi^{-5} \int_{\xi^{-5}}^{t} e^{iB(s, \xi)} e^{-\frac{624i\pi^5}{625} \hat{f}(s, \xi/5)^5 s^{-2}} ds \right|
+ \left| \xi^{-5} \int_{\xi^{-5}}^{t} e^{iB(s, \xi)} e^{-\frac{80i\pi^5}{81} |\hat{f}(t, \xi/3)|^2 \hat{f}(t, \xi/3)^3 s^{-2}} ds \right| \lesssim \varepsilon_1^5. \tag{4.36}
\]
Indeed, applying the a priori assumption \( \|\hat{f}(t, \cdot)\|_{L^\infty} \leq \varepsilon_1 \), one finds that both of the two terms on the left-hand side of (4.36) may be bounded by
\[
\varepsilon_1^5 |\xi|^{-5} \int_{|\xi|^{-5}}^{t} s^{-2} ds \lesssim \varepsilon_1^5.
\]

5. Asymptotics

In Sects. 3–4, we have shown (1.11). Choosing \( \varepsilon_1 = \varepsilon_0^{1/5} \), then (1.11) becomes
\[
\|u\|_X \leq 2C \varepsilon_0. \tag{5.1}
\]
The asymptotics (1.5) in decaying region is a consequence of (5.1), and (2.1) by taking \( g \) to be the profile \( f \). So this section is devoted to studying the asymptotics in self-similar region and oscillatory region.

5.1. Asymptotics in self-similar region

We introduce the following self-similar change of variables of the solution \( u \) to (1.1)–(1.2):
\[
v(t, x) = t^{1/5} u(t, xt^{1/5}), \tag{5.2}
\]
and then calculate that
\[
\partial_t v = t^{-1} \partial_x (5^{-1} x v - \partial_x^4 v - v^5). \tag{5.3}
\]
Recalling \( \gamma = \frac{1}{5} \left( \frac{1}{10} - C \varepsilon_1^4 \right) \), we will show that as \( |x| \leq t^{4\gamma} \) the following estimates hold:
\[
|P_{\geq 2^{20}t^{\gamma}} v(t, x)| \lesssim \varepsilon_0 t^{-7\gamma/2}, \tag{5.4}
\]
and
\[
|\partial_t P_{\leq 2^{20}t^{\gamma}} v(t, x)| \lesssim \varepsilon_0 t^{-\frac{11}{10} + \frac{3\gamma}{2} + C\varepsilon_1^4}. \tag{5.5}
\]
To estimate (5.4), we first write
\[
P_{\geq 2^{20}t^{\gamma}} v(t, x) = t^{1/5} \int_{-\infty}^{\infty} e^{i\tilde{\Phi}(\xi)} (1 - \varphi(\xi t^{1/5} - 2^{-20})) \hat{f}(t, \xi) d\xi,
\]
where
\[
\tilde{\Phi}(\xi) = \tilde{\Phi}(\xi; x, t) := x \xi t^{1/5} + t \xi^5.
\]
Observing that $t\xi^4 \gg |x|t^{1/5}$ on the support of the above integral, so that $|\partial_\xi \hat{\Phi}| \gtrsim t^{\frac{1}{5} + 4\gamma}$, we then use integration by parts to bound
\[
|P \geq \frac{220}{t^{1/5} v(t, x)}| \leq G_1 + G_2,
\]
\[
G_1 \lesssim t^{1/5} \int_{-\infty}^{\infty} \left| \partial_\xi \left[ (\partial_\xi \hat{\Phi})^{-1} (1 - \varphi(\xi t^{\frac{1}{5} - \gamma} 2^{-20})) \right] \hat{f}(t, \xi) \right| d\xi,
\]
\[
G_2 \lesssim t^{1/5} \int_{-\infty}^{\infty} \left| (\partial_\xi \hat{\Phi})^{-1} (1 - \varphi(\xi t^{\frac{1}{5} - \gamma} 2^{-20})) \partial_\xi \hat{f}(t, \xi) \right| d\xi.
\]
(5.6)

Using the bounds on $\|\hat{f}\|_{L^\infty}$ and $\|\partial \hat{f}\|_{L^2}$ in (5.1), we, respectively, estimate
\[
G_1 \lesssim t^{1/5} \|\hat{f}\|_{L^\infty} \int_{-\infty}^{\infty} \left( t^{-1} |\xi|^{-5} |1 - \varphi(\xi t^{\frac{1}{5} - \gamma} 2^{-20})| + t^{-1} |\xi|^{-4} |\varphi'(\xi t^{\frac{1}{5} - \gamma} 2^{-20})| \right) d\xi \lesssim \varepsilon_0 t^{-4\gamma},
\]
and
\[
G_2 \lesssim t^{1/5} \|\partial \hat{f}\|_{L^2} \left( \int_{-\infty}^{\infty} t^{-2} |\xi|^{-8} |1 - \varphi(\xi t^{\frac{1}{5} - \gamma} 2^{-20})|^2 d\xi \right)^{1/2} \lesssim \varepsilon_0 t^{-7\gamma/2}.
\]
These two estimates together complete the proof of (5.4).

We now turn to the proof of (5.5). One first computes
\[
\partial_t P \leq \frac{220}{t^{1/5} v(t, x)} = \mathcal{F}^{-1} \left( \partial_t \varphi(\xi t^{\gamma} 2^{-20}) \hat{v}(t, \xi) \right) + P \leq \frac{220}{t^{1/5} \partial_t v(t, x)}.
\]
We rewrite the first term in the form
\[
\mathcal{F}^{-1} \left( \partial_t \varphi(\xi t^{\gamma} 2^{-20}) \hat{v}(t, \xi) \right) = t^{1/5} \int_{-\infty}^{\infty} e^{i\xi \cdot (\xi t^{\frac{1}{5} - \gamma} 2^{-20})} \varphi'(\xi t^{\frac{1}{5} - \gamma} 2^{-20}) \hat{f}(t, \xi) d\xi,
\]
and then by a similar fashion to (5.6) estimate
\[
\|\mathcal{F}^{-1} \left( \partial_t \varphi(\xi t^{\gamma} 2^{-20}) \hat{v}(t, \xi) \right) \|_{L^\infty} \lesssim \varepsilon_0 t^{-1 - \frac{7}{2}}.
\]
(5.7)

For the second term, we write
\[
P \leq \frac{220}{t^{1/5} \partial_t v(t, x)} = \int_{-\infty}^{\infty} e^{i\xi \cdot x} \varphi(\xi t^{\gamma} 2^{-20}) (i\xi). \mathcal{F}(\partial_x^{-1} \partial_t v)(t, \xi) d\xi.
\]
Recalling that $S = 1 + x \partial_x + 5it \partial_t$, it is straightforward to show
\[
\partial_t v(t, x) = t^{-1} S u(t, xt^{1/5}).
\]
(5.8)
Therefore, we may estimate
\[
|P_{\leq 220^2} \partial_t v(t, x)| \lesssim \|\partial_x^{-1} \partial_t v\|_{L^2} \left( \int_{-\infty}^{\infty} |\varphi(\xi t^{-\gamma} 2^{-20})\xi|^{2} \, d\xi \right)^{1/2} \\
\lesssim t^{3\gamma/2} \|\partial_x^{-1} \partial_t v\|_{L^2} \lesssim t^{3\gamma/2-1} \|IS u\|_{L^2} t^{-1/10} \\
\lesssim \epsilon_0 t^{-\frac{11}{10} + \frac{3\gamma}{2}} + C \epsilon_0^4.
\] (5.9)

The desired estimate is a consequence of (5.7) and (5.9).

We decompose \( v \) into the following form:
\[
v(t, x) = v(t, x) \left( 1 - \psi(x/t^{4\gamma}) \right) + P_{\geq 220^2} v(t, x) \psi(x/t^{4\gamma}) \\
+ P_{\leq 220^2} v(t, x) \psi(x/t^{4\gamma}).
\] (5.10)

Via the definition (5.2) and the decay estimate (2.2), we have
\[
\left| v(t, x) \left( 1 - \psi(x/t^{4\gamma}) \right) \right| \lesssim \epsilon_0 t^{-3\gamma/2}.
\]

This together with (5.4)–(5.5) implies that \( v(t, x) \) is a Cauchy sequence in time in \( L^\infty \)-norm. Let
\[
Q(x) := \lim_{t \to \infty} v(t, x).
\] (5.11)

Thus, (1.8) immediately follows. For \( |x| \leq t^{4\gamma} \), from (5.4), (5.5), and (5.11), it follows that
\[
|v(t, x) - Q(x)| \lesssim \epsilon_0 t^{-7\gamma/2} + \epsilon_0 \int_{t}^{\infty} s^{-\frac{11}{10} + \frac{3\gamma}{2}} + C \epsilon_0^4 \, ds \\
\lesssim \epsilon_0 t^{-7\gamma/2},
\]
which implies (1.6).

It remains to verify that \( Q \) satisfies the ODE (1.7). In view of (5.3) and (5.8), we have
\[
\|\partial_x^4 v - 5^{-1} x v + v^5\|_{L^2} = t^{-1/10} \|IS u\|_{L^2} \lesssim \epsilon_0 t^{-\frac{11}{10} + C \epsilon_0^4}.
\] (5.12)

This together with (5.11) completes the proof of (1.7).

5.2. Asymptotics in oscillatory region

This subsection is devoted to determining the leading order asymptotic term for the solution \( u \) of (1.1)–(1.2) in the oscillatory region which can be stated precisely as follows:

**Lemma 5.1.** Assume \( t \gg 1, |\xi| \geq 2^{-10} t^{-\frac{1}{5} + \gamma} \) and \( \delta \in (0, 1/2) \). Let \( w \) be the new profile defined by (4.33)–(4.34), then there exists \( w_\infty \in L^\infty(\mathbb{R}) \) with \( \|w_\infty\|_{L^\infty(\mathbb{R})} \lesssim \epsilon_0 \) such that
\[
|\tilde{w}(t, \xi) - w_\infty(\xi)| \lesssim \epsilon_0 (|\xi| t^{1/5})^{-\left(\frac{1}{2} - \delta\right)}.
\] (5.13)

Moreover, there exists \( f_\infty \in L^\infty(\mathbb{R}) \) with \( \|f_\infty\|_{L^\infty(\mathbb{R})} \lesssim \epsilon_0 \) such that
\[
\left| \hat{f}(t, \xi) - \exp \left( \frac{i}{40t \xi^5} |f_\infty(\xi)|^4 \right) f_\infty(\xi) \right| \lesssim \epsilon_0 (|\xi| t^{1/5})^{-\left(\frac{1}{2} - \delta\right)}.
\] (5.14)
We postpone proving Lemma 5.1 and first show how to use Lemma 5.1 to complete the proof of (1.9).

**Proof of (1.9).** In the oscillatory region, we first see that

\[
\xi_0 = \sqrt[4]{-x/(5t)} = 5^{-1/4}t^{-1/5}(-x/t^{1/5})^{1/4} \leq 5^{-1/4}t^{-\frac{1}{5} + \frac{1}{5}}.
\]

Recall that \(u(t) = e^{it\xi_0^5}f(t)\), we then insert (5.14) with \(\xi = \xi_0\) into (2.3) with \(g = f\) to obtain

\[
\left| u(x, t) - (5t\xi_0^3)^{-1/2}g \right| \lesssim \varepsilon_0(t\xi_0^3)^{-1/2}(t^{1/5}\xi_0)^{-\left(\frac{1}{2} - \delta\right)} + \varepsilon_0 t^{-1/5}(-x/t^{1/5})^{-9/20}
\]

\[
\lesssim \varepsilon_0 t^{-1/5}(-x/t^{1/5})^{-\frac{9}{20}} + \varepsilon_0 t^{-1/5}(-x/t^{1/5})^{-9/20}
\]

where we have chosen \(\delta > 0\) sufficiently small in the last inequality. This completes the proof of (1.9). □

We are now going to prove Lemma 5.1.

**Proof of Lemma 5.1.** Given \(t_2 \geq t_1 \gg 1\) and \(|\xi| \in (2^j, 2^{j+1})\) with \(j \in \mathbb{Z}\) such that \(2^j \geq t_1^{-1/5 + \gamma}\). For (5.13), we only need to show

\[
|\tilde{w}(t_1, \xi) - \tilde{w}(t_2, \xi)| \lesssim \varepsilon_0 (2^j t_1^{1/5})^{-\left(\frac{1}{2} - \delta\right)}.
\]

Go back to the decomposition (4.3), and notice that \(|\xi| \geq 2^{-10}t^{-\frac{1}{5} + \gamma} \gg t^{-1/5}\), we then utilize Proposition 4.2 to obtain

\[
\partial_t \hat{f}(t, \xi) = \frac{-i}{40t^2\xi^5} \hat{f}(t, \xi)^4\hat{f}(t, \xi) + \frac{c_1i}{t^2\xi^5} e^{-\frac{624\varepsilon_0^5}{65}} \hat{f}(t, \xi/5)^5
\]

\[
+ \frac{c_2i}{t^2\xi^5} e^{-\frac{80\varepsilon_0^5}{81}} \left| \hat{f}(t, \xi/3) \right|^2 \hat{f}(t, \xi/3)^3 + \bar{R}(t, \xi),
\]

where \(\bar{R}(t, \xi)\) may include \(R_0(t, \xi), R_1(t, \xi)\) and \(R_2(t, \xi)\), but not \(R_3(t, \xi)\).

Let \(t \in [t_1, t_2]\). Similar to (4.35), we have

\[
\partial_t \tilde{w}(t, \xi) = e^{iB(t, \xi)} \left( \frac{c_1i}{t^2\xi^5} e^{-\frac{624\varepsilon_0^5}{65}} \hat{f}(t, \xi/5)^5 + \frac{c_2i}{t^2\xi^5} e^{-\frac{80\varepsilon_0^5}{81}} \left| \hat{f}(t, \xi/3) \right|^2 \hat{f}(t, \xi/3)^3 + \bar{R}(t, \xi) \right).
\]

To show (5.15), it is enough to estimate

\[
\left| \xi^{-5} \int_{t_1}^{t_2} e^{iB(s, \xi)} e^{-\frac{624\varepsilon_0^5}{65}} \hat{f}(s, \xi/5)^5 s^{-2} ds \right|
\]

\[
+ \left| \xi^{-5} \int_{t_1}^{t_2} e^{iB(s, \xi)} e^{-\frac{80\varepsilon_0^5}{81}} \left| \hat{f}(t, \xi/3) \right|^2 \hat{f}(t, \xi/3)^3 s^{-2} ds \right| \lesssim \varepsilon_0 (2^j t_1^{1/5})^{-\left(\frac{1}{2} - \delta\right)},
\]

(5.16)
and

\[ \int_{t_1}^{t_2} |\hat{R}(s, \xi)| \, ds \lesssim \varepsilon_0 (2^j t_1^{1/5})^{-(\frac{1}{2} - \delta)}. \]  

(5.17)

Using the bound on \( \| \hat{f} \|_{L^\infty} \) in (5.1), we observe that both of the two terms on the left-hand side of (5.16) can be controlled by

\[ \varepsilon_0^5 |\xi|^{-\frac{5}{2}} \left| \int_{t_1}^{t_2} s^{-2} \, ds \right| \lesssim \varepsilon_0^5 (2^j t_1^{1/5})^{-\frac{15}{4}}, \]

which is stronger than the desired one since \( 2^j t_1^{1/5} \gg 1 \) and \( \varepsilon_0 \in (0, 1) \).

We turn to prove (5.17). Here, it should be clear that we now use the bounds in (5.1) instead of (1.10), so the small coefficient \( \varepsilon_1 \) of the estimates in Lemma 4.4 and Lemma 4.6 should be replaced by \( \varepsilon_0 \). Go back to (4.19), we then estimate

\[ \int_{t_1}^{t_2} |R_0| \, ds \lesssim \varepsilon_0^5 2^{-\frac{15}{4} j} \int_{t_1}^{t_2} s^{-\frac{7}{4}} \, ds \lesssim \varepsilon_0^5 (2^j t_1^{1/5})^{-\frac{15}{4}}, \]

Recalling (4.22), we have

\[ \int_{t_1}^{t_2} |R_1| \, ds \lesssim \varepsilon_0^5 2^j \int_{t_1}^{t_2} s^{-\frac{7}{4}} \max \left( 2^j, s^{-\frac{1}{2}} \right)^{-\frac{19}{4}} \, ds \lesssim \varepsilon_0^5 (2^j t_1^{1/5})^{-\frac{15}{4}}, \]

where we have used the fact \( 2^j t_1^{1/5} \gg 1 \) in the last inequality. These two estimates are more than sufficient for the desired bound in (5.17). It remains to bound \( R_2 \); we only consider \( R_{21} \) since other cases can be analyzed like Step 3 in the proof of Proposition 4.2. In view of (4.32), using \( 2^j t_1^{1/5} \gg 1 \) again, we have

\[ \int_{t_1}^{t_2} |R_{21}| \, ds \lesssim \varepsilon_0^5 2^j \int_{t_1}^{t_2} s^{-11/10} 2^{-3j/2} \, ds + \varepsilon_0^5 2^j \int_{t_1}^{t_2} s^{-\frac{11}{10} + \frac{5}{2} j} 2^{-\frac{3}{2} \delta j} \, ds \]

\[ + \varepsilon_0^5 2^j \int_{t_1}^{t_2} s^{-6/5} 2^{-2j} \, ds \lesssim \varepsilon_0^5 (2^j t_1^{1/5})^{-\frac{1}{2} - \delta}. \]

We now come to the proof of (5.14). Since \( B \) is real, it follows from (5.13) that

\[ ||\hat{f}(t, \xi)| - |w_\infty(\xi)|| \leq \varepsilon_0 (||\xi||^{-1/5} - (\frac{1}{2} - \delta)). \]  

(5.18)

Let

\[ A(t, \xi) := B(t, \xi) + \frac{1}{40 r \xi^5} |\hat{f}(t, \xi)|^4. \]  

(5.19)

A direct calculation shows that

\[ A(t_2, \xi) - A(t_1, \xi) = \frac{1}{40 \xi^5} \int_{t_1}^{t_2} \left( |\hat{f}(s, \xi)|^4 - |\hat{f}(t_2, \xi)|^4 \right) \frac{ds}{s^2} \]

\[ - \frac{1}{40 t_1 \xi^5} \left( |\hat{f}(t_1, \xi)|^4 - |\hat{f}(t_2, \xi)|^4 \right). \]
This together with (5.18) implies that $A(t, \xi)$ is a Cauchy sequence in time. Therefore, there exists $A_\infty \in L^\infty$ such that

$$|A(t, \xi) - A_\infty(\xi)| \lesssim \varepsilon_0(|\xi| t^{1/5})^{-\left(\frac{1}{2} - \delta\right)}.$$  \hspace{1cm} (5.20)

Combining (5.18), (5.19), and (5.20), we obtain

$$\left|B(t, \xi) - \left(A_\infty(\xi) - \frac{1}{40 t^{1/5}} |w_\infty(\xi)|^4\right)\right| \lesssim \varepsilon_0(|\xi| t^{1/5})^{-\left(\frac{1}{2} - \delta\right)}.$$  

So this resulting estimate together with (4.33)–(4.34) yields

$$\left|\hat{f}(t, \xi) - w_\infty(\xi) \exp\left(-i A_\infty(\xi) + \frac{i}{40 t^{1/5}} |w_\infty(\xi)|^4\right)\right| \lesssim \varepsilon_0(|\xi| t^{1/5})^{-\left(\frac{1}{2} - \delta\right)}.$$  

We finally define $f_\infty(\xi) := w_\infty(\xi) \exp\left(-i A_\infty(\xi)\right)$ to conclude (5.14). \hfill \Box

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