Gauge-natural field theories and Noether Theorems: canonical covariant conserved currents

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Abstract

Recently we found that canonical gauge-natural superpotentials are obtained as global sections of the reduced \((n - 2)\)-degree and \((2s - 1)\)-order quotient sheaf on the fibered manifold \(Y \times X \times K\), where \(K\) is an appropriate subbundle of the vector bundle of (prolongations of) infinitesimal right-invariant automorphisms \(\Xi\). In this paper, we provide an alternative proof of the fact that the naturality property \(L_j \Xi \omega(\lambda, K) = 0\) holds true for the new Lagrangian \(\omega(\lambda, K)\) obtained contracting the Euler–Lagrange form of the original Lagrangian with \(\Xi V \in K\). We use as fundamental tools an invariant decomposition formula of vertical morphisms due to Kolář and the theory of iterated Lie derivatives of sections of fibered bundles. As a consequence, we recover the existence of a canonical generalized energy–momentum conserved tensor density associated with \(\omega(\lambda, K)\).

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1 Introduction

Our general framework is the calculus of variations on finite order jets of gauge-natural bundles (i.e. jet prolongations of fiber bundles associated to some gauge-natural prolongation of a principal bundle \(P\) \([5, 16]\)). Such geometric structures have been widely recognized to suitably describe so-called gauge-natural field theories, i.e. physical theories in which right-invariant infinitesimal automorphisms of the structure bundle \(P\) uniquely define the transformation laws of the fields themselves (see e.g. \([5, 6, 13, 16]\) and references quoted therein). In particular, we shall work within the differential setting of finite order variational sequences on gauge-natural bundles. In fact, it become evident that the passage from Lagrangians to Euler–Lagrange equations can be seen as a differential of a complex (see e.g. \([27, 29, 30, 18]\)): the theory of finite order variational

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sequences provides then a suitable geometric framework for the Calculus of Variations. In this theory the Euler–Lagrange operator is a differential morphism in a sequence of sheaves of vector spaces. Geometric objects like Lagrangians, momenta, Poincaré–Cartan forms, Helmholtz conditions, Jacobi equations, find a nice interpretation in the quotient spaces of the sequence of a given order.

In the beginning of the second half of the past Century, to conveniently derive conserved quantities for covariant field theories, it appeared necessary to define in a functorial and unique way the lift of infinitesimal transformations of the basis manifolds to the bundle of fields (namely bundles of tensor fields or tensor densities as suitable representations of the action of infinitesimal spacetime transformations on frame bundles of a given order [23] [1, 3, 4]. Such theories were also called geometric or natural [28]. An important generalization of natural theories to gauge fields theories passed through the concept of jet prolongation of a principal bundle and the introduction of a very important geometric construction, namely the \textit{gauge-natural bundle functor} [5, 16].

In particular, P.G. Bergmann in [3] introduced what he called \textit{generalized Bianchi identities} for geometric field theories to get (after an integration by parts procedure) a consistent equation involving local divergences within the first variation formula. It is well known that, following the Noether theory [21], in the classical Lagrangian formulation of field theories the description of symmetries and conserved quantities consists in deriving from the invariance of the Lagrangian the existence of suitable conserved currents; in most relevant physical theories this currents are found to be the divergence of skew–symmetric (tensor) densities, which are called \textit{superpotentials} for the conserved currents themselves. It is also well known that the importance of superpotentials relies on the fact that they can be integrated to provide conserved quantities associated with the conserved currents \textit{via} the Stokes Theorem (see e.g. [6, 20] and references therein). Generalized Bergmann–Bianchi identities are in fact necessary and locally sufficient conditions for a Noether conserved current to be not only closed but also the divergence of a a superpotential along solutions of the Euler–Lagrange equations. However, the problem of the general covariance of such identities exists and it was already posed and partially investigated by Anderson and Bergmann in [1], where the invariance with respect to time coordinate transformations was studied. This problem reflects obviously on the covariance of conserved quantities (see Remark 4 below). Here we propose a way to deal with such open problems concerning globality aspects. For the relevance of the latter ones also in quantum field theories, see e.g. the preprints [2].

In [8] a representation of symmetries in finite order variational sequences was provided by means of the introduction of the \textit{variational Lie derivative}, \textit{i.e.} the induced quotient operator acting on equivalence classes of forms in the variational sequence. In [6], the theory of Noether conserved currents and superpotentials was tackled by using such representations for natural and gauge-natural Lagrangian field theories. Recently, further developments have been achieved concerning a canonical covariant derivation of Noether conserved quantities and global superpotentials [24, 25]. On the other hand the \textit{second variation} of the action functional can be conveniently represented in the finite order variational
sequence framework in terms of iterated variational Lie derivatives of the Lagrangian with respect to vertical parts of gauge-natural lifts of principal infinitesimal automorphisms. In particular, in [7, 9] the second variational derivative has been represented and related with the generalized Jacobi morphism. Furthermore, the gauge-natural structure of the theories under consideration enables us to define the generalized gauge-natural Jacobi morphism where the variation vector fields are Lie derivatives of sections of the gauge-natural bundle with respect to gauge-natural lifts.

In this paper we use representations of the Noether Theorems given in [8]; in particular we specialize in a new way the Second Noether Theorem for gauge-natural theories by means of the Jacobi morphism [24, 25] and show that the Second Noether Theorem plays a fundamental role in the derivation of canonical covariant conserved quantities in gauge-natural field theories (see Remark 5 below). In fact, the indeterminacy appearing in the derivation of gauge-natural conserved charges (for a review, see the interesting papers [11, 20]) - i.e. the difficulty of relating in a natural way infinitesimal gauge transformations with infinitesimal transformations of the basis manifold - can be solved by requiring the second variational derivative to vanish as well [25]. Moreover, for gauge-natural field theories, here we stress that generalized Bergmann–Bianchi identities hold true in a canonical covariant way if and only if the second variational derivative - with respect to vertical parts of gauge-natural lifts - of the Lagrangian vanishes [24]. As a quite strong consequence, for any gauge-natural invariant field theory we find that the above mentioned indeterminacy can be always solved canonically.

As a consequence of the Second Noether Theorem, we further show that there exists a covariantly conserved current associated with the Lagrangian obtained by contracting the Euler–Lagrange morphism with a gauge-natural Jacobi vector field.

2 Finite order jets of gauge-natural bundles

We recall some basic facts about jet spaces [16, 26]. Our framework is a fibered manifold \( \pi : Y \to X \), with \( \dim X = n \) and \( \dim Y = n + m \).

For \( s \geq q \geq 0 \) integers we are concerned with the \( s \)-jet space \( J_s Y \) of \( s \)-jet prolongations of (local) sections of \( \pi \); in particular, we set \( J_0 Y \equiv Y \). We recall the natural fiberings \( \pi_q^s : J_s Y \to J_q Y \), \( s \geq q \), \( \pi^s : J_s Y \to X \), and, among these, the affine fiberings \( \pi_{s-1}^s \). We denote by \( VY \) the vector subbundle of the tangent bundle \( TY \) of vectors on \( Y \) which are vertical with respect to the fibering \( \pi \).

Greek indices \( \sigma, \mu, \ldots \) run from 1 to \( n \) and they label basis coordinates, while Latin indices \( i, j, \ldots \) run from 1 to \( m \) and label fibre coordinates, unless otherwise specified. We denote multi–indices of dimension \( n \) by boldface Greek letters such as \( \alpha = (\alpha_1, \ldots, \alpha_n) \), with \( 0 \leq \alpha_\mu, \mu = 1, \ldots, n \); by an abuse of notation, we denote by \( \sigma \) the multi–index such that \( \alpha_\mu = 0 \), if \( \mu \neq \sigma \), \( \alpha_\mu = 1 \), if \( \mu = \sigma \). We also set \(|\alpha| := \alpha_1 + \cdots + \alpha_n \) and \( \alpha! := \alpha_1! \cdots \alpha_n! \). The charts induced on \( J_s Y \) are denoted by \( (x^\sigma, y^\alpha) \), with \( 0 \leq |\alpha| \leq s \); in particular, we set \( y_0^i \equiv y^i \).
The local vector fields and forms of \( J_s Y \) induced by the above coordinates are denoted by \((\partial_\alpha^a)\) and \((d^a_\alpha)\), respectively.

For \( s \geq 1 \), we consider the natural complementary fibered morphisms over \( J_s Y \to J_{s-1} Y \) (see e.g. [18, 19, 31]):

\[
\mathcal{D} : J_s Y \times T X \to T J_{s-1} Y, \quad \partial : J_s Y \times T J_{s-1} Y \to V J_{s-1} Y,
\]

with coordinate expressions, for \( 0 \leq |\alpha| \leq s - 1 \), given by

\[
\mathcal{D} = d^\alpha \otimes \partial_\lambda = d^\alpha \otimes \left( \partial_\lambda + y_\alpha^j + \partial_\lambda \right), \quad \partial = \partial_\alpha^a \otimes \partial_\beta^b = (d^a_\alpha - y_\alpha^b \lambda) \otimes \partial_\beta^b.
\]

The morphisms above induce the following natural splitting (and its dual):

\[
J_s Y \times_{J_{s-1} Y} T^* J_{s-1} Y = \left( J_s Y \times_{J_{s-1} Y} T^* X \right) \oplus C^*_s[Y],
\]

(1)

where \( C^*_s[Y] := \text{im} \partial^s \) and \( \partial^s : J_s Y \times_{J_{s-1} Y} V^* J_{s-1} Y \to J_s Y \times_{J_{s-1} Y} T^* J_{s-1} Y \).

If \( f : J_s Y \to \mathbb{R} \) is a function, then we set \( D_\sigma f := D_\sigma \), \( D_{\alpha + \sigma} f := D_\sigma D_\alpha f \), where \( D_\sigma \) is the standard formal derivative. Given a vector field \( \Xi : J_s Y \to T J_s Y \), the splitting (1) yields

\[
\Xi \circ \pi_s^{s+1} = \Xi_H + \Xi_V
\]

where, if \( \Xi = \Xi^\gamma \partial_\gamma + \Xi^\alpha \partial_\alpha \), then we have \( \Xi_H = \Xi^\gamma D_\gamma \) and \( \Xi_V = (\Xi_\alpha - y_\alpha^b \Xi) \partial_\alpha \). We shall call \( \Xi_H \) and \( \Xi_V \) the horizontal and the vertical part of \( \Xi \), respectively.

The splitting (1) induces also a decomposition of the exterior differential on \( Y \), \( (\pi_s^{s+1})^* \circ d = d_H + d_V \), where \( d_H \) and \( d_V \) are defined to be the horizontal and vertical differential. The action of \( d_H \) and \( d_V \) on functions and 1-forms on \( J_s Y \) uniquely characterizes \( d_H \) and \( d_V \) (see, e.g., [26, 31] for more details). A projectable vector field on \( Y \) is defined to be a pair \((u, \xi)\), where \( u : Y \to T Y \) and \( \xi : X \to T X \) are vector fields and \( u \) is a fibered morphism over \( \xi \). If there is no danger of confusion, we will denote simply by \( u \) a projectable vector field \((u, \xi)\). A projectable vector field \((u, \xi)\) can be conveniently prolonged to a projectable vector field \((j_s u, j_s \xi)\); coordinate expression can be found e.g. in [18, 26, 31].

### 2.1 Gauge-natural bundles

Let \( P \to X \) be a principal bundle with structure group \( G \). Let \( r \leq k \) be integers and \( W^{(r,k)} P := J_r P \times L_k(X) \), where \( L_k(X) \) is the bundle of \( k \)-frames in \( X \). \[ 5 \] \[ 10 \]. \( W^{(r,k)} G := J_r G \circ GL_k(n) \) the semidirect product with respect to the action of \( GL_k(n) \) on \( J_r G \) given by the jet composition and \( GL_k(n) \) is the group of \( k \)-frames in \( \mathbb{R}^n \). Here we denote by \( J_r G \) the space of \((r,n)\)-velocities on \( G \). \[ 16 \]. The bundle \( W^{(r,k)} P \) is a principal bundle over \( X \) with structure group \( W^{(r,k)} G \). Let \( F \) be any manifold and \( \zeta : W^{(r,k)} G \times F \to F \) be a left action of \( W^{(r,k)} G \) on \( F \). There is a naturally defined right action of \( W^{(r,k)} G \) on \( W^{(r,k)} P \times F \) so that we can associate in a standard way to \( W^{(r,k)} P \) the bundle, on the given basis \( X, Y, \zeta := W^{(r,k)} P \times F \).
Definition 1 We say \((Y_\zeta, X, \pi_\zeta; F, G)\) to be the gauge-natural bundle of order \((r, k)\) associated to the principal bundle \(W^{(r,k)}P\) by means of the left action \(\zeta\) of the group \(W^{(r,k)}G\) on the manifold \(F\) [5, 16].

Remark 1 A principal automorphism \(\Phi\) of \(W^{(r,k)}P\) induces an automorphism of the gauge-natural bundle by:

\[
\Phi_\zeta : Y_\zeta \rightarrow Y_\zeta : (j^x_\gamma, j^0_\kappa t), \hat{\bar{f}}_\zeta \mapsto \Phi(j^x_\gamma, j^0_\kappa t), \hat{\bar{f}}_\zeta,
\]

where \(\hat{\bar{f}} \in F\) and \([\cdot, \cdot]_\zeta\) is the equivalence class induced by the action \(\zeta\).

Definition 2 We define the vector bundle over \(X\) of right–invariant infinitesimal automorphisms of \(P\) by setting \(A = TP/G\).

We also define the vector bundle over \(X\) of right invariant infinitesimal automorphisms of \(W^{(r,k)}P\) by setting \(A^{(r,k)} := T\overline{W^{(r,k)}P}/W^{(r,k)}G\) \((r \leq k)\).

Denote by \(T\mathcal{X}\) and \(A^{(r,k)}\) the sheaf of vector fields on \(X\) and the sheaf of right invariant vector fields on \(W^{(r,k)}P\), respectively. A functorial mapping \(\mathfrak{G}\) is defined which lifts any right–invariant local automorphism \((\Phi, \phi)\) of the principal bundle \(W^{(r,k)}P\) into a unique local automorphism \((\Phi_\zeta, \phi)\) of the associated bundle \(Y_\zeta\). Its infinitesimal version associates to any \(\bar{\Xi} \in A^{(r,k)},\) projectable over \(\xi \in T\mathcal{X}\), a unique \(\text{projectable} \) vector field \(\hat{\bar{\Xi}} := \mathfrak{G}(\bar{\Xi})\) on \(Y_\zeta\), the gauge-natural lift, in the following way:

\[
\mathfrak{G} : Y_\zeta \times_X A^{(r,k)} \rightarrow TY_\zeta : (y, \bar{\Xi}) \mapsto \hat{\bar{\Xi}}(y),
\]

where, for any \(y \in Y_\zeta\), one sets: \(\hat{\bar{\Xi}}(y) = \frac{d}{dt}[(\Phi_\zeta t)(y)]_{t=0}\), and \(\Phi_\zeta t\) denotes the (local) flow corresponding to the gauge-natural lift of \(\Phi_\zeta\).

This mapping fulfills the following properties (see [16]):

1. \(\mathfrak{G}\) is linear over \(id_{Y_\zeta}\);
2. we have \(T\pi_\zeta \circ \mathfrak{G} = id_{T\mathcal{X}} \circ \pi^{(r,k)}\), where \(\pi^{(r,k)}\) is the natural projection \(Y_\zeta \times_X A^{(r,k)} \rightarrow T\mathcal{X}\);
3. for any pair \((\bar{\Lambda}, \bar{\Xi}) \in A^{(r,k)}\), we have \(\mathfrak{G}([\bar{\Lambda}, \bar{\Xi}]) = [\mathfrak{G}(\bar{\Lambda}), \mathfrak{G}(\bar{\Xi})]\).

2.2 Lie derivative of sections

Definition 3 Let \(\gamma\) be a (local) section of \(Y_\zeta\), \(\bar{\Xi} \in A^{(r,k)}\) and \(\hat{\bar{\Xi}}\) its gauge-natural lift. Following [16] we define the generalized Lie derivative of \(\gamma\) along the vector field \(\hat{\bar{\Xi}}\) to be the (local) section \(\pounds_{\hat{\bar{\Xi}}} \gamma : X \rightarrow VY_\zeta\), given by \(\pounds_{\hat{\bar{\Xi}}} \gamma = T\gamma \circ \xi - \hat{\bar{\Xi}} \circ \gamma\).

Remark 2 The Lie derivative operator acting on sections of gauge-natural bundles satisfies the following properties:
1. for any vector field $\Xi \in \mathcal{A}^{(r,k)}$, the mapping $\gamma \mapsto \mathcal{L}_{\Xi} \gamma$ is a first-order quasilinear differential operator;

2. for any local section $\gamma$ of $Y_\zeta$, the mapping $\Xi \mapsto \mathcal{L}_{\Xi} \gamma$ is a linear differential operator;

3. we can regard $\mathcal{L}_{\Xi} : \mathcal{J}^s Y_\zeta \to \mathcal{V} Y_\zeta$ as a morphism over the basis $X$. By using the canonical isomorphisms $\mathcal{V} \mathcal{J}^s Y_\zeta \simeq \mathcal{J}^s \mathcal{V} Y_\zeta$ for all $s$, we have $\mathcal{L}_{\Xi}[\mathcal{J}^s \gamma] = \mathcal{J}^s [\mathcal{L}_{\Xi} \gamma]$, for any (local) section $\gamma$ of $Y_\zeta$ and for any (local) vector field $\Xi \in \mathcal{A}^{(r,k)}$. Furthermore, for gauge-natural lifts, the fundamental relation hold true:

$$\hat{\Xi}_V := \mathfrak{g}(\hat{\Xi})_V = -\mathcal{L}_{\hat{\Xi}}.$$  \hspace{1cm} (4)

3 Variational sequences and Noether Theorems

For the sake of simplifying notation, sometimes, we will omit the subscript $\zeta$, so that all our considerations shall refer to $Y$ as a gauge-natural bundle as defined above.

For convenience of the reader, we sketch the connection of the purely differential setting of variational sequences with the classical integral presentation of Calculus of Variations, although the two approaches (differential and integral one) are completely independent, even if the latter provided the motivation to the former from an historical viewpoint.

In the formulation of variational problems on jet spaces of a fibered manifold $Y \to X$, with $n = \text{dim} X$ and $m = \text{dim} Y - n$ (see e.g. [12, 14, 22, 26]), it is well known that, given an $s$-th order Lagrangian $\lambda \in \mathcal{H}_s^n$, the action of $\lambda$ along a section $\gamma : U \to Y$, on an oriented open subset $U$ of $X$ with compact closure and regular boundary, is defined to be the real number

$$\int_U (\mathcal{J}^s \gamma)^* \lambda.$$  

A variation vector field is a vertical vector field $u : Y \to VY$ defined along $\gamma(U)$. A local section $\gamma : U \to Y$ is said to be critical if, for each variation vector field with flow $\phi_t$, we have

$$\delta \int_U (\mathcal{J}^s \phi_t \circ \mathcal{J}^s \gamma)^* \lambda = 0,$$

where $\delta$ is the Fréchet derivative with respect to the parameter $t$, at $t = 0$. It is easy to see that the previous integral expression is equal to $\int_U (\mathcal{J}^s \gamma)^* L_{\mathcal{J}^s u} \lambda = 0$ for each variation vector field $u$, where $L_{\mathcal{J}^s u}$ is the Lie derivative operator. For each variation vector field $u$ satisfying suitable boundary conditions, since $L_{\mathcal{J}^s u} \lambda = i_{\mathcal{J}^s u} d\lambda$, as an application of the Stokes Theorem, we find that the above equation is equivalent to $\int_U (\mathcal{J}^{2s} \gamma)^* (i_u E_{d\lambda}) = 0$, where $E_{d\lambda}$ is the generalized
Euler–Lagrange operator associated with \( \lambda \) (see later). Finally, by virtue of the fundamental Lemma of the Calculus of Variations the above condition is equivalent to \( E_{\Delta j} \circ j_{2s}\gamma = 0 \), known as the Euler–Lagrange equations (see e.g. the review in [18]).

Let us now construct the Krupka’s finite order variational sequence.

According to [18, 31], the fibered splitting [1] yields the sheaf splitting
\[
\mathcal{H}^p_{(s+1,s)} = \bigoplus_{t=0}^p C^p_{(s+1,s)} \wedge \mathcal{H}^p_{s+1},
\]
which restricts to the inclusion \( \Lambda_s^p \subset \bigoplus_{t=0}^p C^{p-t}_{s} \wedge \mathcal{H}^{t,h}_{s+1} \), where \( \mathcal{H}^{p,h}_{s+1} := h(\Lambda_s^p) \) for \( 0 < p \leq n \) and the surjective map \( h \) is defined to be the restriction to \( \Lambda_s^p \) of the projection of the above splitting onto the non–trivial summand with the highest value of \( t \). By an abuse of notation, let us denote by \( d\ker h \) the sheaf generated by the presheaf \( d\ker h \) in the standard way. We set \( \Theta_s^\ast := \ker h + d\ker h \).

In [18] it was proved that the following \( s \)–th order variational sequence associated with the fibered manifold \( Y \to X \) is an exact resolution of the constant sheaf \( \mathcal{R}_Y \) over \( Y \):
\[
0 \to \mathcal{R}_Y \to \Lambda_s^0 \xrightarrow{\mathcal{E}_0} \Lambda_s^1 \to \Lambda_s^2 \to \cdots \to \Lambda_s^{I-1} \to \Lambda_s^I \xrightarrow{d} 0,
\]
where the integer \( I \) depends on the dimension of the fibers of \( y \) (see [18]).

For practical purposes we shall limit ourselves to consider the truncated variational sequence introduced by Vitolo in [31]:
\[
0 \to \mathcal{R}_Y \to \mathcal{V}_s^0 \xrightarrow{\mathcal{E}_0} \mathcal{V}_s^1 \xrightarrow{\mathcal{E}_1} \cdots \xrightarrow{\mathcal{E}_{n+1}} \mathcal{V}_s^{n+1} \xrightarrow{\mathcal{E}_{n+1}} \mathcal{V}_s^{n+2} \xrightarrow{d} 0,
\]
where, following [31], the sheaves \( \mathcal{V}_s^p := C^{p-n}_s \wedge \mathcal{H}^{n,h}_{s+1}/h(d\ker h) \) with \( 0 \leq p \leq n+2 \) are suitable representations of the corresponding quotient sheaves in the variational sequence by means of sheaves of sections of tensor bundles.

Let \( \alpha \in C^1_s \wedge \mathcal{H}^{n,h}_{s+1} \subset \mathcal{V}_s^{n+1} \). Then there is a unique pair of sheaf morphisms [14, 17, 31]
\[
E_\alpha \in C^1 (2s,0) \wedge \mathcal{H}^{n,h}_{2s+1}, \quad F_\alpha \in C^1 (2s,s) \wedge \mathcal{H}^{n,h}_{2s+1},
\]
such that \( (\pi_{s+1}^{2s+1})^* \alpha = E_\alpha - F_\alpha \) and \( F_\alpha \) is locally of the form \( F_\alpha = d_H p_\alpha \), with \( p_\alpha \in C(2s-1,s-1) \wedge \mathcal{H}^{n-2,2s} \).

We shall now introduce a - for our purposes - fundamental morphism, denoted by \( K_\eta \), represented by Vitolo in [31] and further studied by Kolář and Vitolo in [17].

Let then \( \eta \in C^1_s \wedge C^1 (s,0) \wedge \mathcal{H}^{n,h}_{s+1} \subset \mathcal{V}_s^{n+2} \); then there is a unique morphism
\[
K_\eta \in C^1 (2s,s) \wedge C(2s,0) \wedge \mathcal{H}^{n,h}_{2s+1}
\]
such that, for all \( \Xi : Y \to VY \), \( E_{j_s,\Xi} \eta = C_1 (j_{2s} \Xi \otimes K_\eta) \), where \( C_1 \) stands for tensor contraction on the first factor and \( \otimes \) denotes inner product (see [17, 31]).

Furthermore, there is a unique pair of sheaf morphisms
\[
H_\eta \in C^1 (2s,s) \wedge C^1 (2s,0) \wedge \mathcal{H}^{n,h}_{2s+1}, \quad G_\eta \in C^2 (2s,s) \wedge \mathcal{H}^{n,h}_{2s+1},
\]
such that \((\pi^2_{s+1})^* \eta = H_\eta - G_\eta\) and \(H_\eta = \frac{1}{2} A(K_\eta)\), where \(A\) stands for antisymmetrisation. Moreover, \(G_\eta\) is \textit{locally} of the type \(G_\eta = d_H q_\eta\), where \(q_\eta \in C^2_{(2s-1,s-1)} \wedge H^{n-12s}\); hence \([q_\eta]\) = \([H_\eta]\) [17, 31].

**Remark 3** A section \(\lambda \in V^n_s\) is just a Lagrangian of order \((s+1)\) of the standard literature. Furthermore \(E_{n+1}(\lambda) \in V^{n+1}_s\) coincides with the standard higher order Euler–Lagrange morphism associated with \(\lambda\). Let \(\gamma \in \Lambda^{n+1}_s\). The morphism \(H_{hd\gamma} = H[\varepsilon_{n+1}(\gamma)]\), where square brackets denote equivalence class, is called the \textit{generalized Helmholtz morphism}; its kernel coincides with Helmholtz conditions of local variationality. We shall integrate by parts the morphism \(K_\eta\) to provide a suitable representation of the \textit{generalized Jacobi morphism} associated with \(\lambda\) [7, 9, 24, 25].

The standard Lie derivative of fibered morphisms with respect to a projectable vector field \(j_\Xi\) passes to the quotient in the variational sequence, so defining a new quotient operator (introduced in [8]), the \textit{variational Lie derivative} \(L_{j_\Xi}\), acting on equivalence classes of fibered morphisms which are sections of the quotient sheaves in the variational sequence. Thus variational Lie derivatives of generalized Lagrangians or Euler–Lagrange morphisms can be conveniently represented as equivalence classes in \(V^n_s\) and \(V^{n+1}_s\). In particular, the following two results hold true [8], to which for evident reasons we will refer as the First and the Second Noether Theorem, respectively.

**Theorem 1** Let \([\alpha] = h(\alpha) \in V^n_s\). Then we have locally (up to pull-backs)

\[
L_{j_\Xi}(h(\alpha)) = \Xi_V | E_n(h(\alpha)) + d_H(j_{2s}\Xi_V | p_{dv;h(\alpha)} + \xi | h(\alpha)).
\]

**Theorem 2** Let \(\alpha \in \Lambda^{n+1}_s\). Then we have globally (up to pull-backs)

\[
L_{j_\Xi}[\alpha] = E_n(j_{s+1}\Xi_V | h(\alpha)) + C_1^1(j_s\Xi_V \otimes K_{hd\alpha}).
\]

Notice that the Second Noether Theorem as formulated above, is represented in terms of the morphism \(K_{hd\alpha}\).

### 3.1 Noether conserved currents

In the following we assume that the field equations are generated by means of a variational principle from a Lagrangian which is gauge-natural invariant, \textit{i.e.} invariant with respect to any gauge-natural lift of infinitesimal right invariant vector fields. Both the Noether Theorems take a quite particular form in the case of gauge-natural Lagrangian field theories (see \textit{e.g.} [6, 20]) due to the fact that the generalized Lie derivative of sections of the gauge-natural bundles has specific linearity properties recalled in Subsection 2.2 and it is related with the vertical part of gauge-natural lifts by Eq. (4).

**Definition 4** Let \((\tilde{\Xi}, \xi)\) be a projectable vector field on \(Y_\zeta\). Let \(\lambda \in V^n_s\) be a generalized Lagrangian. We say \(\tilde{\Xi}\) to be a \textit{symmetry} of \(\lambda\) if \(L_{j_{s+1}\tilde{\Xi}} \lambda = 0\).
We say \( \lambda \) to be a \textit{gauge-natural invariant Lagrangian} if the gauge-natural lift \((\hat{\Xi}, \xi)\) of any vector field \(\bar{\Xi} \in \mathcal{A}(r,k)\) is a symmetry for \(\lambda\), i.e. if \(\mathcal{L}_{j_{n+1}} \hat{\Xi} \lambda = 0\). In this case the projectable vector field \(\hat{\Xi} \equiv \mathcal{G}(\bar{\Xi})\) is called a \textit{gauge-natural symmetry} of \(\lambda\).

In the following we rephrase the First Noether Theorem in the case of gauge-natural Lagrangians.

**Proposition 1** Let \(\lambda \in V^n_s\) be a gauge-natural Lagrangian and \((\hat{\Xi}, \xi)\) a gauge-natural symmetry of \(\lambda\). Then we have

\[
0 = -E_{\hat{\Xi}}(\mathcal{E}_n(\lambda)) + d\mathcal{H}(\mathcal{L}_{j_{n+1}} \mathcal{E}_n(\lambda)) + \xi[\lambda] \text{ fulfills the equation } d((j_2 \sigma)^*(\varepsilon)) = 0.
\]

If \(\sigma\) is a critical section for \(E_n(\lambda)\), i.e. \((j_2 \sigma)^*E_n(\lambda) = 0\), the above equation admits a physical interpretation as a so-called \textit{weak conservation law} for the density associated with \(\varepsilon\) and the associated sheaf morphism \(\varepsilon: J_2^s \rightarrow \mathcal{C}_{2s}^s[A(r,k)] \otimes \mathcal{C}_{0}^s[A(r,k)] \wedge (n-1)\wedge T^*X\) is said to be a \textit{gauge-natural weakly conserved current}.

**Remark 4** We stress that such a Noether conserved current is not uniquely defined, even up to divergences. In fact, it depends on the choice of \(p_\omega\), which in general is not unique - even up to divergences - depending on the fixing of suitable connections used to derive it in an invariant way (see [14, 31] and references quoted therein).

4 Variations and generalized Jacobi morphisms

We consider \textit{formal variations} of a morphism as \textit{multiparameter deformations} and relate the second variational derivative of the Lagrangian \(\lambda\) to the Lie derivative of the associated Euler–Lagrange morphism and in turn to the generalized Bergmann–Bianchi morphism; see [24] for details.

Let \(\alpha: J_s Y \rightarrow J^p T^*J_s Y\) and let \(\mathcal{L}_{j_i} \mathcal{Z}_i\) be the Lie derivative operator acting on differential fibered morphism. Let \(\mathcal{Z}_k, 1 \leq k \leq i,\) be (vertical) variation vector fields on \(Y\) in the sense of [7, 9, 24]. We define the \(i\)-th formal variation of the morphism \(\alpha\) to be the operator:

\[
\delta^i \alpha = \mathcal{L}_{j_i} \mathcal{Z}_i \ldots \mathcal{L}_{j_1} \mathcal{Z}_1 \alpha.
\]

**Definition 5** Let \(\alpha \in (V^n_s)_Y\) and \(\mathcal{L}_{\mathcal{Z}_i}\) the \textit{variational Lie derivative} operator with respect to the variation vector field \(\mathcal{Z}_i\).

We define the \(i\)-th \textit{variational derivative} operator as follows:

\[
\delta^i [\alpha] := [\delta^i \alpha] = [\mathcal{L}_{\mathcal{Z}_i} \ldots \mathcal{L}_{\mathcal{Z}_1} \alpha] = \mathcal{L}_{\mathcal{Z}_i} \ldots \mathcal{L}_{\mathcal{Z}_1} [\alpha].
\]

It is clear that the first variational derivative is noting but the variational Lie derivative with respect to vertical parts of (gauge-natural lifts) of vector fields. Analogously, the second variational derivative is nothing but the iterated (twice) variational Lie derivative; thus it can be expressed by means of the Noether
Theorems. As a straightforward consequence the following characterization of the second variational derivative of a generalized Lagrangian in the variational sequence holds true [24].

Proposition 2 Let \( \lambda \in (V^n_s)_Y \) and let \( \Xi \) be a variation vector field; then we have

\[
\delta^2 \lambda = \left[ \mathcal{E}_n(j_{2s}\Xi)h\delta\lambda + C^1_1(j_{2s}\Xi \otimes K_{h\delta\lambda}) \right].
\] (7)

4.1 Generalized gauge-natural Jacobi morphisms

Let \( \lambda \) be a Lagrangian and \( \Xi \) a variation vector field. Let us set \( \chi(\lambda, \Theta(\Xi)_V) := C^1_1(j_{2s}\Xi \otimes K_{h\delta\lambda}) \equiv E(j_{2s}\Xi \otimes h\delta\lambda) \).

Lemma 1 We have:

\[
s^2 \chi(\lambda, \Theta(\Xi)_V) = E\chi(\lambda, \Theta(\Xi)_V) + F\chi(\lambda, \Theta(\Xi)_V),
\] where

\[
E\chi(\lambda, \Theta(\Xi)_V) : J_{4\times} Y_\zeta \times VJ_{4\times} A^{(r,k)} \to C^*_0[A^{(r,k)}] \otimes C^*_0[A^{(r,k)}] \Lambda (\wedge T^* X),
\] and locally, \( F\chi(\lambda, \Theta(\Xi)_V) = D_H M\chi(\lambda, \Theta(\Xi)_V) \), with

\[
M\chi(\lambda, \Theta(\Xi)_V) : J_{4\times-1} Y_\zeta \times VJ_{4\times-1} A^{(r,k)} \to C^*_1[A^{(r,k)}] \otimes C^*_0[A^{(r,k)}] \Lambda (\wedge^{n-1} T^* X).
\]

Definition 6 We call the morphism \( J(\lambda, \Theta(\Xi)_V) := E\chi(\lambda, \Theta(\Xi)_V) \) the gauge-natural generalized Jacobi morphism associated with the Lagrangian \( \lambda \) and the variation vector field \( \Theta(\Xi)_V \).

The morphism \( J(\lambda, \Theta(\Xi)_V) \) is a linear morphism with respect to the projection \( J_{4\times} Y_\zeta \times VJ_{4\times} A^{(r,k)} \to J_{4\times} Y_\zeta \).

As a consequence of Theorem 2 and Proposition 2 we have the following characterization of the Second Noether Theorem for gauge-natural invariant Lagrangian field theories in terms of the second variational derivative (see [24] for the proof in detail).

Theorem 3 Let \( \delta^2 \lambda \) be the variation of \( \lambda \) with respect to vertical parts of gauge-natural lifts of infinitesimal principal automorphisms. We have:

\[
\mathcal{L}_{\Theta(\Xi)_V} \mathcal{L}_{\Theta(\Xi)_V} := \delta^2 \lambda = J(\lambda, \Theta(\Xi)_V).
\] (8)

Furthermore:

\[
J(\lambda, \Theta(\Xi)_V) = \Theta(\Xi)_V | \mathcal{E}_n(\Theta(\Xi)_V | \mathcal{E}_n(\lambda)) = \mathcal{E}_n(\Theta(\Xi)_V | h(\delta\lambda)).
\] (9)
This result generalizes a classical result due to Goldschmidt and Sternberg relating the Hessian with the Jacobi morphism for first order field theories; in addition here the gauge-natural structure of the theories under consideration enables us to define the \textit{generalized gauge-natural Jacobi morphism} where the \textit{variation vector fields} are Lie derivatives of sections of the gauge-natural bundle with respect to gauge-natural lifts.

### 4.2 The Bergmann–Bianchi morphism

It is a well known procedure to perform suitable integrations by parts to decompose the conserved current $\epsilon$ into the sum of a conserved current $\tilde{\epsilon}$ vanishing along solutions of the Euler–Lagrange equations, the so–called \textit{reduced current}, and the formal divergence of a skew–symmetric (tensor) density $\nu$ called a super-potential (which is then defined modulo a divergence). Within such a procedure, the generalized Bergmann–Bianchi identities are in fact necessary and (locally) sufficient conditions for the conserved current $\epsilon$ to be not only closed but also the divergence of a skew-symmetric (tensor) density along solutions of the Euler–Lagrange equations. In [24, 25], for the first time, the relation of the kernel of the gauge-natural Jacobi morphism with the kernel of the Bergmann–Bianchi morphism has been explicated in order to characterize generalized Bianchi identities in terms of a special class of gauge-natural lifts, namely those which have their vertical part in the kernel of the generalized gauge-natural Jacobi morphism.

Let now consider the term $\omega(\lambda, \mathfrak{g}(\bar{\xi})_V) := -\mathcal{L}_{\bar{\xi}}|E_\beta(\lambda)$ appearing in the formulation of the First Noether Theorem given in Proposition 1. We stress that along sections which are not critical such a term is not vanishing, in general. In the following we shall manipulate it to derive - under precise conditions - a strongly conserved current, \textit{i.e.} a current satisfying a Noether conservation law also along non critical sections, considered for the first time by Bergmann in [3].

In fact, as a further application of the global decomposition formula of vertical morphisms due to Kolář [14], following essentially the procedure proposed by Bergmann in [3], we can integrate by parts $\omega(\lambda, \mathfrak{g}(\bar{\xi})_V)$ to define the \textit{generalized Bergmann–Bianchi morphism}.

**Lemma 2** We have globally

$$(\pi^{A+1}_+)^*\omega(\lambda, \mathfrak{g}(\bar{\xi})_V) = \beta(\lambda, \mathfrak{g}(\bar{\xi})_V) + F_{\omega(\lambda, \mathfrak{g}(\bar{\xi})_V)}$$

where $\beta(\lambda, \mathfrak{g}(\bar{\xi})_V) \equiv E_{\omega(\lambda, \mathfrak{g}(\bar{\xi})_V)}$, and locally, $F_{\omega(\lambda, \mathfrak{g}(\bar{\xi})_V)} = D_H M_{\omega(\lambda, \mathfrak{g}(\bar{\xi})_V)}$.

Coordinate expressions for the morphisms $\beta(\lambda, \mathfrak{g}(\bar{\xi})_V)$ and $M_{\omega(\lambda, \mathfrak{g}(\bar{\xi})_V)}$ can be found by a backwards procedure (see \textit{e.g.} [14]). In particular, $\beta(\lambda, \mathfrak{g}(\bar{\xi})_V)$ is nothing but the Euler–Lagrange morphism associated with the \textit{new} Lagrangian $\omega(\lambda, \mathfrak{g}(\bar{\xi})_V)$ defined on the fibered manifold $J_2xY_\xi \times V J_2xA^{(r,k)} \rightarrow X$. In particular, we get the following local decomposition of $\omega(\lambda, \mathfrak{g}(\bar{\xi})_V)$:

$$\omega(\lambda, \mathfrak{g}(\bar{\xi})_V) = \beta(\lambda, \mathfrak{g}(\bar{\xi})_V) + D_H \tilde{\epsilon}(\lambda, \mathfrak{g}(\bar{\xi})_V),$$

where we put $\tilde{\epsilon}(\lambda, \mathfrak{g}(\bar{\xi})_V) \equiv M_{\omega(\lambda, \mathfrak{g}(\bar{\xi})_V)}$. 

\[\text{(10)}\]
Definition 7 We call the global morphism $\beta(\lambda, \mathcal{G}(\bar{\Xi})_V) := E_\omega(\lambda, \mathcal{G}(\bar{\Xi})_V)$ the generalized Bergmann–Bianchi morphism associated with the Lagrangian $\lambda$ and the variation vector field $\mathcal{G}(\bar{\Xi})_V$.

As mentioned in the Introduction, the problem of the general covariance of generalized Bianchi identities for field theories was posed by Anderson and Bergmann already in 1951 (see [1]). This problem reflects obviously on the covariance of conserved quantities (see Remark 4 above). Here we propose a way to deal with such open problems concerning globality aspects.

In fact, let now $\mathcal{K} := \text{Ker} J(\lambda, \mathcal{G}(\bar{\Xi})_V)$ be the kernel of the generalized gauge-natural morphism $J(\lambda, \mathcal{G}(\bar{\Xi})_V)$. As a consequence of Theorem 3 and of Lemma 2, we have the following covariant characterization of the kernel of generalized Bergmann–Bianchi morphism, the detailed proof of which will appear in [24].

Theorem 4 The generalized Bianchi morphism is globally vanishing if and only if $\delta^2 G_\lambda \equiv J(\lambda, \mathcal{G}(\bar{\Xi})_V) = 0$, i.e. if and only if $\mathcal{G}(\bar{\Xi})_V \in \mathcal{K}$.

The gauge-natural invariance of the variational principle in its whole enables us to solve the intrinsic indeterminacy in the conserved charges associated with gauge-natural symmetries of Lagrangian field theories (in [20], for example, the special case of the gravitational field coupled with fermionic matter is considered and the Kosmann lift is then invoked as an ad hoc choice to recover the well known expression of the Komar superpotential). This is well known to be of great importance within the theory of Lie derivative of sections of a gauge-natural bundle and notably for the Lie derivative of spinors (see e.g. the review given in [20]). As a quite strong consequence of the above Theorem, for any gauge-natural invariant field theory we find that the above mentioned indeterminacy can be always solved canonically as shown by the following.

Corollary 1 Let $\lambda \in \mathcal{V}_n$ be a gauge-natural invariant generalized Lagrangian and let $\mathcal{G}(\bar{\Xi})$ be a gauge-natural lift of the principal infinitesimal automorphism $\bar{\Xi} \in \mathcal{A}^{r,k}$, i.e. a gauge-natural symmetry of $\lambda$. Then $\mathcal{G}(\bar{\Xi})_V$ is the generator of a canonical global conserved quantity, if and only if $\mathcal{G}(\bar{\Xi})_V$ satisfies the invariant condition

\[ (-1)^{s|\sigma|} D_\sigma \left( D_\mu \hat{\Xi}^i_V \left( \partial_j (\partial^\mu \lambda) - \sum_{|\alpha|=0}^{s-|\mu|} (-1)^{\mu+\alpha} \frac{(\mu+\alpha)!}{\mu!\alpha!} D_\alpha \partial^\alpha (\partial^\mu \lambda) \right) \right) = 0. \]

In particular, the condition $j_s \hat{\Xi}_V = D_\alpha (\hat{\Xi}_V) \partial^\alpha \in \mathcal{K}$ implies, of course, that the components $\hat{\Xi}^i_\alpha$ and $\hat{\Xi}^\gamma$ are not independent, but they are related in such a way that $j_s \hat{\Xi}_V = D_\alpha (\hat{\Xi}_V - y^i \hat{\Xi}_V^i) \partial^\alpha$ must be a solution of generalized gauge-natural Jacobi equations for the Lagrangian $\lambda$.

According with the above Corollary, we shall refer to canonical covariant currents or to corresponding superpotentials by stressing their dependence on $\mathcal{K}$; i.e. by Theorem 4 in correspondence of gauge-natural lifts satisfying covariant Bergmann–Bianchi identities.
Remark 5 Let then $\lambda \in V^1_s$ be a gauge-natural Lagrangian and $j_s\hat{\Xi}_V \in \mathfrak{r}$ a gauge-natural symmetry of $\lambda$. Being $\beta(\lambda, \mathfrak{r}) \equiv 0$, we have, globally, $\omega(\lambda, \mathfrak{r}) = D_H\epsilon(\lambda, \mathfrak{r})$, then from the First Noether Theorem we have $D_H(\epsilon(\lambda, \mathfrak{r}) - \tilde{\epsilon}(\lambda, \mathfrak{r})) = 0$, which is a so-called gauge-natural ‘strong’ conservation law for the global canonical density $\epsilon(\lambda, \mathfrak{r}) - \tilde{\epsilon}(\lambda, \mathfrak{r})$.

As an important application, we recall that recently the existence of canonical gauge-natural superpotential associated with $\lambda$ and $\mathfrak{r}$ has been accordingly established in the framework of variational sequences [24, 25]. In fact, let $\lambda \in V^1_s$ be a gauge-natural Lagrangian and $(j_s\hat{\Xi}, \xi)$ a gauge-natural symmetry of $\lambda$. Then there exists a canonical global sheaf morphism $\nu(\lambda, \mathfrak{r}) \in (V^1_{2s-1})_{Y \times \mathfrak{r}}$ such that $D_H\nu(\lambda, \mathfrak{r}) = \epsilon(\lambda, \mathfrak{r}) - \tilde{\epsilon}(\lambda, \mathfrak{r})$. Notice that by the exactness of the variational sequence, the existence of a local superpotential can be deduced as a section of $(V^1_{2s-1})_{Y \times \mathfrak{r}}$. This local section can be always globalized by choosing (prolongations of) principal connections on $P$. However, such a globalization depends on the choice of the connection itself. Furthermore, although this choice can be always done geometrically, connections are generally the unknown to be determined in field theories and then they should not be fixed a priori in a consistent truly covariant field theory. Our result enables us to get global sections of the reduced sheaf $(V^1_{2s-1})_{Y \times \mathfrak{r}}$ without fixing any connection a priori.

4.3 Generalized symmetries and Bergmann–Bianchi identities

As well known, the Second Noether Theorem deals with invariance properties of the Euler-Lagrange equations (so-called generalized symmetries or also Bessel-Hagen symmetries, see e.g. the fundamental papers [28]). Although symmetries of a Lagrangian turn out to be also symmetries of the Euler–Lagrange morphism (Second Noether Theorem) impose some constraints on the conserved quantities associated with gauge-natural symmetries of $\lambda$ (see e.g. [1]).

In particular, although for a gauge-natural invariant Lagrangian $\lambda$ we always have $\mathcal{L}_{j_s\hat{\Xi}}\lambda = 0$, $\mathcal{L}_{j_s\hat{\Xi}_V}\lambda$ does not need to be zero in principle; however when the second variation $\delta_2^2\lambda$ is required to vanish then $\mathcal{L}_{j_s\hat{\Xi}_V}\mathcal{E}_n(\lambda)$ surely vanishes, i.e. $j_s\hat{\Xi}_V$ is a generalized or Bessel–Hagen symmetry. The symmetries of the Euler–Lagrange morphism (Second Noether Theorem) impose some constraints on the conserved quantities associated with gauge-natural symmetries of $\lambda$ (see e.g. [1]).

Symmetries of the Euler–Lagrange morphism are clearly related with invariance properties of $\omega(\lambda, \mathfrak{G}(\hat{\Xi})_V) \coloneqq -\mathcal{L}_{\hat{\Xi}}\mathcal{E}_n(\lambda)$. We stress that, because of linearity properties of $\mathcal{L}$, $\omega(\lambda, \mathfrak{G}(\hat{\Xi})_V)$ can be considered as a new Lagrangian, defined on an extended space; thus Theorems 3 and 4 can provide us with some kind of Noether conservation law associated with the induced invariance properties of $\omega(\lambda, \mathfrak{G}(\hat{\Xi})_V)$.

First of all let us make the following important consideration.
Proposition 3 For each $\Xi \in \mathcal{A}^{(r,k)}$ such that $\Xi_V \in \mathfrak{r}$, we have
\[
\mathcal{L}_{j_s} \Xi_V \omega(\lambda, \mathfrak{r}) = - D_H (- j_s \mathcal{L}_{\Xi_V} | p_{D_V \omega(\lambda, \mathfrak{r})} ).
\] (11)

Proof. The horizontal splitting gives us $\mathcal{L}_{j_s} \Xi_V \omega(\lambda, \mathfrak{r}) = \mathcal{L}_{j_s} \Xi_V \omega(\lambda, \mathfrak{r}) + \mathcal{L}_{j_s} \Xi_V \omega(\lambda, \mathfrak{r})$. Furthermore, $\omega(\lambda, \mathfrak{r}) = - \mathcal{L}_\Xi \xi(\lambda) = \mathcal{L}_{j_s} \Xi_V - d_H (- j_s \mathcal{L}_{\Xi_V} | p_{D_V \omega(\lambda, \mathfrak{r})} + \xi(\lambda))$; so that $\mathcal{L}_{j_s} \Xi_V \omega(\lambda, \mathfrak{r}) = \mathcal{L}_{j_s} \Xi_V \mathcal{L}_{j_s} \Xi_V \omega(\lambda, \mathfrak{r})$. On the other hand we have $\mathcal{L}_{j_s} \Xi_V \omega(\lambda, \mathfrak{r}) = \mathcal{L}_{j_s} [\Xi_V, \Xi_V] \omega(\lambda, \mathfrak{r})$. Recall now that $\Xi_V \in \mathfrak{r}$ if and only if $\beta(\lambda, \mathfrak{r}) = 0$. Since
\[
\mathcal{L}_{j_s} \Xi_V \omega(\lambda, \mathfrak{r}) = - \mathcal{L}_{\Xi_V} [\omega(\lambda, \mathfrak{r})] + D_H (- j_s \mathcal{L}_{\Xi_V} | p_{D_V \omega(\lambda, \mathfrak{r})} ) =
\]
\[
\beta(\lambda, \mathfrak{r}) + D_H (- j_s \mathcal{L}_{\Xi_V} | p_{D_V \omega(\lambda, \mathfrak{r})} ),
\]
we get the assertion. \qed

It is easy to realize that, because of the gauge-natural invariance of the generalized Lagrangian $\lambda$, the new generalized Lagrangian $\omega(\lambda, \mathfrak{r})$ is gauge-natural invariant too, i.e. $\mathcal{L}_{j_s} \Xi_V \omega(\lambda, \mathfrak{r}) = 0$. However, a stronger result holds true. In fact, we can state the following naturality property for $\omega(\lambda, \mathfrak{r})$, which provides some more information concerning the Hamiltonian structure of gauge-natural field theories [10].

Proposition 4 Let $\Xi_V \in \mathfrak{r}$. We have $\mathcal{L}_{j_s} \Xi_V \omega(\lambda, \mathfrak{r}) = 0$.

Proof. In fact, when $\Xi_V \in \mathfrak{r}$, by the theory of iterated Lie derivatives of sections [16], we have $\mathcal{L}_{j_s} \Xi_V \omega(\lambda, \mathfrak{r}) = [\Xi_V, \Xi_V] \omega(\lambda) = 0$. Thus $\mathcal{L}_{j_s} \Xi_V \omega(\lambda, \mathfrak{r}) = \mathcal{L}_{j_s} \Xi_V \omega(\lambda, \mathfrak{r}) + \mathcal{L}_{j_s} \Xi_V \omega(\lambda, \mathfrak{r}) = 0$. \qed

As a consequence of Propositions 3 and 4, corresponding to any $\Theta(\Xi)_H$, we get the existence of a generalized Noether conserved current (which could be interpreted as a generalized energy–momentum tensor for $\omega(\lambda, \mathfrak{r})$).

Corollary 2 Let $\Xi_V \in \mathfrak{r}$. We have the covariant conservation law
\[
D_H (- j_s \mathcal{L}_{\Xi_V} | p_{D_V \omega(\lambda, \mathfrak{r})} ) = 0 .
\] (12)

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