ABELIANIZATION OF LOW ENERGY SU(2)
EFFECTIVE ACTION

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Abstract

Recently Faddeev and Niemi proposed a low energy effective action for pure SU(2) Yang Mills theory in 4 dimension to describe its long distance physics. The effective action is O(3) $\sigma$ model with a mass parameter, a dimensionless coupling constant $e$ and a topological term. In this work, choosing a new set of variables, we relate this $\sigma$ model to a U(1) gauge theory with electric and magnetic charges of charge $e$ and $4\pi e^{-1}$ respectively. In the new formulation the connection of the mass parameter with the monopole condensate is discussed. This theory after lattice regularisation is the compact U(1) gauge theory coupled to electric charges.

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1. INTRODUCTION

Perhaps one of the most outstanding problems in physics is to understand the mechanism of color confinement in non-abelian gauge theories. The idea of color confinement through dual Meissner effect was proposed by t’Hooft and Mandelstam more than twenty years back and is well accepted by now. This conjecture has been tested in the simpler toy model case of compact lattice quantum electrodynamics (CLQED). However, in the non-abelian gauge theories we still lack a general consensus on even the type of dynamical variables appropriate to describe its low energy physics. This can be contrasted with CLQED where in the Villain form of the action the variables are the photons and the magnetic monopoles described by the vector field $A_\mu$ and an anti-symmetric tensor field $H_{\mu\nu}$ respectively. In terms of these variables the condensation mechanism of magnetic monopoles has been well understood by both analytical as well as Monte Carlo simulations [2]. In the strong coupling region the monopole condensation leads to dual Meissner effect confining all the electric charges. Motivated by this simple observation in the abelian case, in [3] we mapped the SU(2) gauge theory coupled to adjoint Higgs to $U(1)$ gauge theory by a change of variables in the partition function. The final dynamical variables were SU(2) gauge invariant and included a “photon” field topologically coupled to magnetic monopoles (like in CLQED) and minimally to electrically charged matter. Later, these results were generalised to SU(N) gauge theories [4].

Recently, Faddeev and Niemi [5] proposed a reformulation of pure SU(2) Yang Mills theory in terms of new variables suitable for its low energy content. In terms of these variables the long distance physics is described by an effective action which is O(3) $\sigma$ model with a a topological term and a mass parameter which appeared invoking the renormalisation group arguments. It was argued in [5] that this effective model describing infrared SU(2) physics is unique in the sense that it contains all the infrared relevant and marginal local Lorentz invariant operators of the $\sigma$ model field which are atmost quadratic in time derivative. However, in order to understand the picture of color confinement in this theory, it is desirable to explore its connection with the magnetic monopoles. In particular, an important issue is whether the mass parameter introduced in [5] related to the monopole condensate. In this work we address these questions and show that the low energy action of Faddeev and Niemi for SU(2) Yang Mills theory can be further mapped into a U(1) gauge theory with magnetic monopoles and electrically charged particles. This also establishes a connection between [5] and the work done in [3, 4]. Moreover, using duality transformations, we give a qualitative argument regarding the connection between the mass parameter and monopole condensate. The plan of the paper is as follow: In the first part of the paper we will briefly discuss the kinematical issues related to the new set of variables which map $O(3) \sigma$ model of [5] to the abelian gauge model. A more detailed discussion on the kinematical properties of these variables can be found in [3, 4]. In the second part we will study the dynamical content of the reformulated U(1) gauge theory and argue that the new variables should be more appropriate to address the issues discussed above.

The effective action describing pure SU(2) Yang Mills theory in [5] is:

\[ \text{This U(1) gauge group has nothing to do with the initial SU(2) gauge group.} \]
\[
S(\hat{n}) = \int_{\mathbb{R}^4} \left[ m^2 (\partial_\mu \hat{n}(x))^2 + \frac{1}{\epsilon^2} (\hat{n}(x), \partial_\mu \hat{n}(x) \times \partial_\nu \hat{n}(x))^2 \right].
\] (1)

In (1) \(\hat{n}(x)\) is a 3-dimensional unit vector in the internal SO(3) space, \(\epsilon\) is dimensionless coupling constant and \(m\) is the mass scale introduced invoking the renormalisation group arguments. While the details of (1) can be found in [5], an important point for our purpose is the global invariance of (1) under the SO(3) transformations:

\[
\hat{n}(x) \to \hat{n}(x) + \vec{\lambda} \times \hat{n}(x).
\] (2)

The invariance of (1) under the transformation (2) corresponds to the global SU(2) invariance of the initial standard Yang Mills action in terms of the gluon fields [5].

The low energy partition function is

\[
Z = \int \mathcal{D}(\hat{n}) \exp - S(\hat{n}).
\] (3)

In (3) \(\mathcal{D}(\hat{n})\) is the SO(3) Haar measure and can be written as \(\sin \theta d\theta d\psi\), where \(\theta(x)\) and \(\psi(x)\) are the polar and azimuthal angles of \(\hat{n}\) in the internal space. We can also characterise \(\hat{n}\) in terms of its Euclidean co-ordinates in an internal frame rigidly attached in the space time. We call it space fixed frame (SFF) and denote its orthonormal unit basis vectors by \(\hat{e}^a\) (\(a = 1, 2, 3\)). Generally, the dynamics is described in the SFF. Given \(\hat{n}(x)\), we now construct a local body fixed frame (BFF) characterised by an orthonormal set of unit vectors \(\hat{\xi}^a(x)\) (\(a = 1, 2, 3\)) with the identification \(\hat{\xi}^3(x) \equiv \hat{n}(x)\). The other two basis vectors \(\hat{\xi}^\pm(x) \equiv 2^{-1}(\hat{\xi}^1(x) \pm i\hat{\xi}^2(x))\) are arbitrary up to U(1) local rotations around \(\hat{\xi}^3(x)\) axis. The BFF and SFF are related by a SO(3) matrix \(O(x)\):

\[
\hat{\xi}^a(x) = O(x)^b_\alpha \hat{e}^b.
\] (4)

Any vector \(\vec{v}(x)\) can also be expanded in the BFF: \(\vec{v} \equiv v^3 \hat{\xi}^3 + v^+ \hat{\xi}^- + v^- \hat{\xi}^+\). The inbuilt local U(1) gauge invariance of the BFF in turn induces a local U(1) rotations on the BFF components of the vector \(\vec{v}\):

\[
\hat{\xi}^\pm(x) \to \exp(\pm i\alpha(x))\hat{\xi}^\pm(x) \quad \Rightarrow \quad v^\pm(x) \to \exp(\pm i\alpha(x))v^\pm(x).
\] (5)

In (3), the local transformations being rotation, \(\alpha(x)\) is compact. This construction of the BFF and the description of the dynamics (4) in the BFF can be motivated by the two simple observations:

3. In what follows, the components \(v^\pm\) will be called the chiral components of the vector \(\vec{v}\).
4. Note that the U(1) local transformations are change of basis in the internal space. Therefore all vectors are untouched and only their chiral BFF components undergo induced rotations.
1. The dynamics (3) when described in the BFF will have an inbuilt local U(1) gauge invariance. In the sequel, we will show that this reformulated dynamics and the associated local U(1) gauge invariance is that of “photons” minimally (topologically) coupled to electric (magnetic) charged matter fields.

2. The components of all vectors in the BFF are explicitly invariant under (9). This SO(3) invariance also implies invariance under the initial SU(2) global transformations.

In order to describe the dynamics of (3) in the BFF, we now make a new change of variables from the description $\hat{n}(x)(\equiv \xi^3(x))$ to the set of vector fields $\vec{\omega}_\mu$ defined by:

$$\partial_\mu \xi^1 \equiv \omega^{2}_\mu \xi^3 - \omega^{3}_\mu \xi^2, \quad \partial_\mu \xi^2 \equiv \omega^{3}_\mu \xi^1 - \omega^{1}_\mu \xi^3, \quad \partial_\mu \xi^3 \equiv \omega^{1}_\mu \xi^2 - \omega^{2}_\mu \xi^1.$$ (6)

The eqns. (6) are nothing but the orthonormality of the BFF and can be re-written in compact form $D_\mu(\vec{\omega}) \xi^3(x) \equiv 0$ where the “covariant derivative” $D^a_\mu(\vec{\omega}) \equiv \delta^{ac} \partial_\mu - e^{abc} \omega^b_\mu$ is defined with respect to the new variables $\vec{\omega}$. Infact, these new variables have a simple geometrical interpretation of the “angular velocities” of the BFF with respect to the SFF [3]. The $U(1)$ gauge transformations (5) on the BFF components of $\vec{\omega}(x)$ are given by:

$$\omega^3_\mu(x) \rightarrow \omega^3_\mu(x) + \partial_\mu \alpha(x), \quad \omega^\pm_\mu(x) \rightarrow \exp(\pm i \alpha(x)) \omega^\pm_\mu(x).$$ (7)

From (7) we see that $\omega^3_\mu$ transforms like “photon” and the orthogonal chiral components of the angular velocities transform like electrically charged matter fields. Therefore, we will denote $\omega^3_\mu$ by $eA_\mu$. We can now define the U(1) co-variant derivative of $\omega^\pm_\mu$:

$$D_\mu(A) \omega^\pm_\mu \equiv (\partial_\mu \pm i e A_\mu) \omega^\pm_\mu.$$ (8)

We also have the original SO(3) global transformations:

$$\vec{\omega}_\mu(x) \rightarrow \tilde{\lambda} \times \vec{\omega}_\mu(x)$$ (9)

The final dynamical fields in the reformulated model will be the angular velocities $\vec{\omega}_\mu$. The eqn. (3) implies that in the BFF we will have a formulation in terms of explicitly SU(2) color neutral fields. In (6), we have replaced 2 compact degrees of freedom of the initial variable $\hat{n}$ by a set of 12 non-compact angular velocities. Therefore, not all of them are independent. The corresponding constraints are easily obtained by defining the “field strength tensor” $\vec{F}_{\mu\nu}(\vec{\omega}(x)) \equiv \partial_\mu \vec{\omega}_\nu - \partial_\nu \vec{\omega}_\mu + \vec{\omega}_\mu \times \vec{\omega}_\nu$ and considering the commutator:

5The coupling constant $e$ in the definition of $A_\mu$ has been introduced for later convenience.
\[ [D_\mu(\omega), D_\nu(\omega)]\hat{\xi}^a = 0 \Rightarrow \vec{F}_{\mu\nu}(\vec{\omega}(x)) = \vec{L}\cos\theta(x) (\partial_\mu \partial_\nu - \partial_\nu \partial_\mu) \psi(x) \]  

(10)

In (10) \( \vec{L} \equiv (\sin\theta, 0, -\cos\theta) \). The space time points where the azimuthal angle \( \psi(x) \) is single valued the right hand side of (10) vanishes and the solutions are pure gauges. From (4) and (6) we find that \( \omega_\mu(x) = O(x)\partial_\mu O^{-1}(x) \). This orthogonal matrix \( O(x) \) defined in (4) can be characterised by the 3 Euler angles \( (\alpha, \theta, \psi) \). Thus we recover the original 2 angular degrees of freedom of \( \hat{n} \) along with the U(1) gauge angle. We now come to the hidden topological degrees of freedom in (3). These degrees of freedom are at the space time points \([x_0]\) where the R.H.S of (10) is non-vanishing. These points can be characterised by the topological index:

\[ N = \frac{1}{2\pi} \int_{\Sigma = \partial \Sigma_{x_0}} dx_\mu \partial_\mu \psi(x) = 2\pi Z \quad Z \in \text{Integers}. \]  

(11)

In (11) \( \Sigma_{x_0} \) is an infinitesimal area enclosing the point \( x_0 \) and \( C \) is its boundary. The necessary condition for the R.H.S of (11) to be non-zero is \( \theta(x_0) = 0/\pi \). Equivalently the unit vector \( \hat{n} \) is rotated along either the +ve or -ve polar axis in the internal space. Therefore, \([x_0]\) form a one dimensional string. We will see that these strings correspond to monopoles attached with Dirac strings carrying \( 4\pi e^{-1} Z \) unit of magnetic flux towards the monopoles. To study this aspect further we re-write the constraints (10) in a form which will be more convenient at later stage:

\[ F^\pm(\omega)_{\mu\nu} = D_\mu(A)\omega^\pm_\nu - D_\nu(A)\omega^\pm_\mu = 0 \]

\[ \left( \partial_\mu A_\nu - \partial_\nu A_\mu - \frac{\cos\theta}{e} (\partial_\mu \partial_\nu - \partial_\nu \partial_\mu) \psi \right) = \frac{2i}{e} \left( \omega^+_\mu \omega^-_\nu - h.c. \right). \]  

(12)

We now consider the dynamical issues related to (3). The action (4) in terms of angular velocities is:

\[ S(\hat{n}) = 4m^2 \omega^+_\mu \omega^-_\mu + \left( \frac{2i}{e} \right)^2 \left( \omega^+_\mu \omega^-_\nu - h.c. \right)^2. \]  

(13)

We notice that the topological term in (13) is just the square of the left hand side of the constraint (12). Therefore, we can write:

\[ S(\hat{n}) = S(A_\mu, \omega^\pm_\mu, \psi) = \left[ \partial_\mu A_\nu - \partial_\nu A_\mu - \mathcal{F}_{\mu\nu}^{np} \right]^2 + 4m^2 \omega^+_\mu \omega^-_\mu \]  

(14)

In (14) \( \mathcal{F}_{\mu\nu}^{np} \equiv e^{-1}\cos\theta (\partial_\mu \partial_\nu - \partial_\nu \partial_\mu) \psi \) and the superscript (np) is to emphasize that it is a non-perturbative term. We can now define the abelian field strength tensor:

\[ F_{\mu\nu}(A) \equiv \partial_\mu A_\nu - \partial_\nu A_\mu - \mathcal{F}_{\mu\nu}^{np} \]  

(15)

\(^6\text{This implies } \vec{L} \text{ in (10) is effectively } (0, 0, \pm 1)\)
Along the one dimensional space time point where $\theta = 0$ or $\pi$ and $\psi$ is multivalued, we get a singular magnetic flux from $\mathcal{F}_{\mu\nu}$ in (13). On these singular magnetic strings the value of $\theta$ can flip from 0 to $\pi$ and thus changing the direction of the magnetic flux. As the polar angle $\theta$ of $\hat{n}$ is not invariant under the transformations (2), the $\theta = 0$ and $\theta = \pi$ parts of the strings are unphysical. Their locations can be changed by the transformation (2). However, the discrete space time points on a particular string where $\theta$ flips between 0 and $\pi$ are left invariant under global $SO(3)$ transformations and are the sources of magnetic flux. More explicitly, defining the topological magnetic current $K_\mu \equiv \partial_\nu \tilde{\mathcal{F}}_{\mu\nu}$ ($\tilde{\mathcal{F}}_{\mu\nu} \equiv 2^{-1} \varepsilon_{\mu\nu\rho\sigma} F_{\rho\sigma}$). The Bianchi identities are:

$$\partial_\nu \tilde{\mathcal{F}}_{\mu\nu} \equiv K_\mu.$$ 

Thus the first term in the action (14) has the simple physical interpretation of photons interacting with $SO(3)$ invariant magnetic monopoles of charge $(4Z\pi e^{-1})$ attached with two unphysical Dirac strings characterised by $\theta = 0$ and $\theta = \pi$ respectively. Each of them carry $2\pi Ze^{-1}$ unit of the magnetic flux towards the monopole. It is interesting to note that the origin of these topological magnetic monopoles is the topological term in (1).

The partition function (3) in terms of the U(1) gauge and charged matter is:

$$Z = \int dA_\mu d\omega^\pm_\mu d\theta d\psi \delta(C^3(x)) \delta(C^+(x)) \delta(C^-(x)) \exp - S(A_\mu, \omega^\pm_\mu, \psi)$$

$$S(A_\mu, \omega^\pm_\mu, \psi) = \int_x \left[ 4m^2 \omega^+_\mu \omega^-_\mu + \left( \partial_\mu A_\nu - \partial_\nu A_\mu - F_{\mu\nu}^{np} \right)^2 \right]$$ (16)

In (16) we have used $C^\pm$ and $C^3$ to denote the three constraints in (12) respectively. We once again mention the two important properties of (16): a) It is manifestly invariant under local U(1) gauge transformations, b) All the fields appearing in (16) are SU(2) color neutral. Both these properties were expected and were emphasized in the beginning.

To handle the singular fluxes in $\mathcal{F}_{\mu\nu}^{np}$, the most convenient regularization is on the lattice. On lattice the minimal length curve is the boundary of a plaquette. Therefore the multivalued field $\psi$ and their derivatives are well defined. In other words the singular integer valued flux (in the units of $4\pi e^{-1}$) along the strings gets spread over a plaquette. This integer flux can be characterised by an anti-symmetric tensor field $H_{\mu\nu}$ [6] and the partition function can be written as:

$$Z = \sum_{H_{\mu\nu} = -\infty}^{+\infty} \int dA_\mu d\omega^\pm_\mu \delta(C^\pm) \delta(C^3) \exp - S(A_\mu, H_{\mu\nu})$$

$$S(A_\mu, H_{\mu\nu}) = \int_x \left[ \left( \partial_\mu A_\nu - \partial_\nu A_\mu - \frac{4\pi}{\hbar} H_{\mu\nu} \right)^2 + 4m^2 \omega^+_\mu \omega^-_\mu \right]$$ (17)

In (17) the integer valued anti-symmetric field reflects the multi-valued nature of the field $\psi$. 

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All the fields in (17) are SU(2) color neutral by construction. However, now we have the new U(1) symmetry whose charges must be unobservable. They should be confined via dual Meissner effect. At this stage it is interesting to compare (17) with the results of [6] where a manifestly Lorentz co-variant and local quantum field theory for magnetic monopoles and electric charges was proposed by exploiting duality transformations. The starting theory was dual abelian Higgs model with a complex scalar fields \( \phi_m(x), \phi^*_m(x) \) carrying magnetic charge \( g \) and coupled minimally to the dual vector potential \( \tilde{A}_\mu(x) \) in the presence of a potential \( V(\phi_m\phi^*_m) \). The action is:

\[
S(\phi_m, \phi^*_m, \tilde{A}) = D_\mu(\tilde{A})\phi_m D_\mu(\tilde{A})\phi^*_m + \frac{1}{4}(\partial_\mu \tilde{A}_\nu - \partial_\nu \tilde{A}_\mu)^2 + (\phi_m\phi^*_m - m^2)^2. \tag{18}
\]

The co-variant derivatives in (18) are defined in the standard way. Decomposing the complex scalar field into its radial and angular parts \( \phi_m \equiv \rho(x) \exp(i\psi(x)) \) and then performing a duality transformations, we found that the partition function of (18) could be re-written as [6]:

\[
Z = \sum_{H_{\mu\nu} = -\infty}^{+\infty} \int \rho d\rho dA_\mu exp(-S) \tag{19}
\]

\[
S = \int_{x^4} \left[ \frac{1}{4}(\partial_\mu \tilde{A}_\nu - \partial_\nu \tilde{A}_\mu - gH_{\mu\nu})^2 + \frac{1}{4\rho^2}(\partial_\mu \tilde{H}_{\mu\nu})^2 + (\partial_\mu \rho)^2 + (\rho^2 - m^2)^2 \right].
\]

Note that the role of the multivalued field \( \psi \), representing the phase of complex Higgs \( \phi(x) \) in the abelian case is similar to its role in SO(3) \( \sigma \) model case where it represented the azimuthal angle of \( \mathbf{n} \). In both abelian and non-abelian cases its dual field is the antisymmetric tensor field representing the magnetic monopoles. This correspondence is also valid for SU(N) gauge theories [4].

We expect that the integration of charged matter field \( \omega_\mu^{\pm} \) in (17) will generate a term proportional to \((4m^2)^{-1}(\partial_\mu \tilde{H}_{\mu\nu})^2\). This term is consistant with the local U(1) gauge invariance of the theory. If true, comparing it with (19) at \( \rho = m \), this effective theory derived from the original Yang Mills action via (3) will describe an explicit SU(2) color neutral dual superconductor (18) in dual language (i.e in terms of \( A_\mu \) and \( H_{\mu\nu} \)). This important issue is currently under investigation and will be reported elsewhere. Finally it will be interesting to study the massive knotlike solitons [7] in terms of the new variables discussed in this paper.

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The subscript \( m \) on the complex scalar field is the emphasize that these fields carry the magnetic charges. The electric field in terms of the dual vector potential is given by \( E_i \equiv \epsilon_{ijk}\partial_j A_k \).
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