Research Article

On the Mean Values of Certain Character Sums

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1. Introduction

The sum

\[ S_\chi(n) = \frac{1}{q^2} \sum_{a=1}^{q} a^n \chi(a) \]  

(1)

appears frequently in number theory, where \( \chi \) is a nonprincipal primitive character modulo \( q \), and has been studied by several experts. For example, for \( q \equiv 3 \pmod{4} \) being a prime \( p \) and \( \chi \) being the Legendre symbol, Ayoub et al. [1] have proved that \( S_\chi(n) < 0 \) for \( n = 1, 2 \) and for \( n \geq p - 2 \). Fine [2] has showed that for \( n > 2 \), there exist infinitely many primes \( p \equiv 3 \pmod{4} \) with \( S_\chi(n) > 0 \) and infinitely many with \( S_\chi(n) < 0 \).

Williams [3] proved that

\[ S_\chi(n) = O \left( p^{1/2} \log p \right) \]  

(2)

for \( \chi \) being the Legendre symbol modulo \( p \). For primitive character \( \chi \) modulo \( q \), Toyoizumi [4] used the generalized Bernoulli numbers to express \( S_\chi(n) \) in terms of Gauss sums and Dirichlet \( L \)-functions as follows:

\[ \sum_{a=1}^{q} a^n \chi(a) = \begin{cases} 
2q^n \tau(\chi) & \text{if } \chi(-1) = 1, \\
2q^n \tau(\chi) \times \sum_{0 \leq m \leq (n-1)/2} \frac{(n^m)(2m)!L(2m+1, \chi)}{(-1)^{m+1}(2\pi)^{2m+1}i} & \text{if } \chi(-1) = -1,
\end{cases} \]  

(3)

where \( \tau(\chi) = \sum_{a=1}^{q} \chi(a)e(a/q) \) is the Gauss sum, \( e(y) = e^{2\pi iy} \), \( L(s, \chi) \) is the Dirichlet \( L \)-function corresponding to \( \chi \), and \( \binom{n}{m} \) denotes the binomial coefficient.

Toyoizumi [4] also gave explicit bounds for \( S_\chi(n) \).

**Proposition 1.** (a) Assume that \( \chi(-1) = 1 \) and \( n \geq 2 \). Then for any primitive character \( \chi \) mod \( q \), one has

\[ |S_\chi(n)| < C_1(n) q^{1/2}, \]  

(4)

where

\[ C_1(n) = \frac{2\zeta(2) \pi^{1/2}}{(2\pi)^{n+1}} \sum_{1 \leq m \leq n/2} \frac{(2\pi)^{n+1-2m}}{n+1-2m)!}, \]  

(5)

and \( \zeta(s) \) is the Riemann zeta function.
(b) Assume that \( \chi(-1) = -1 \) and \( n \geq 3 \). Then for any primitive character \( \chi \mod q \), one has
\[
|S_{\chi}(n)| < \left( C_2(n) + \frac{|L(1, \chi)|}{\pi} \right) q^{1/2},
\]
where
\[
C_2(n) = \frac{2 \zeta(3) n!}{(2\pi)^n} \sum_{1 \leq m \leq (n-1)/2} \frac{(2\pi)^{n-2m}}{(n-2m)!}.
\]

In [5], Peral used the Gauss sums and adequate Fourier expansion to greatly improve the result in Proposition 1.

**Proposition 2.** (a) Assume that \( \chi(-1) = 1 \) is a primitive nonprincipal character modulo \( q \), and then
\[
|S_{\chi}(n)| \leq q^{1/2} \left( \frac{n-1}{2(n+1)} \right).
\]

(b) Assume that \( \chi(-1) = -1 \) is a primitive character modulo \( q \); then,
\[
|S_{\chi}(n)| + \frac{\chi(n) L(1, \chi)}{\pi i} \leq q^{1/2} \left( \frac{n}{\pi} \int_{0}^{1} \frac{1}{2 \sin(\pi t)} t^{-1/2} dt \right).
\]

Furthermore, Liu and Zhang [6] gave an upper bound for \( S_{\chi}(n) \) when \( \chi \) is a nonprincipal character modulo \( q \).

It may be interesting to consider the mean value of certain character sums. For example, Burgess [7] proved that
\[
\sum_{\chi \mod q} \left| \sum_{1 \leq a \leq q/4} a \chi(a) \right|^4 \leq 8d(q) q^2 h^2,
\]
where \( \sum_{*} \) denotes the summation over primitive characters modulo \( q \), and \( d(q) \) is the Dirichlet divisor function. Xu and Zhang studied the power mean
\[
\sum_{\chi \mod q} \left| \sum_{1 \leq a \leq q/4} \chi(a) \right|^4
\]
in [8, 9] and obtained some sharper results.

In this paper, we study the fourth power mean of certain character sums
\[
\sum_{\chi \mod q} \left| \sum_{1 \leq a \leq q/4} a \chi(a) \right|^4,
\]
and give a few asymptotic formulae.

**Theorem 3.** Let \( q \geq 5 \) be an odd number. Then one has
\[
\sum_{\chi \mod q} \left| \sum_{1 \leq a \leq q/4} a \chi(a) \right|^4
\]
\[
= \frac{7}{2^{17} \cdot 3^2} q^6 J(q) \prod_{p \mid q} \left( \frac{1 - 1/p^2}{1 + 1/p^2} \right)^3
\]
\[
+ \frac{35}{2^{16} \cdot 3^2 \cdot 17} q^6 J(q) \prod_{p \mid q} \left( \frac{1 - 1/p^3}{1 + 1/p^3} \right)^3
\]
\[
- \frac{L^2(3, \chi_4)}{2^{14} \cdot 3^2 \cdot \pi \cdot L(5, \chi_4)}
\]
\[
\times q^6 J(q) \prod_{p \mid q} \left( \frac{1 - 1/p^2}{1 - \chi_4(p)/p^3} \right)^2
\]
\[
\times q^2 J(q) \prod_{p \mid q} \left( \frac{1 - 1/p^2}{1 + 1/p^2} \right)^3
\]
\[
\times \prod_{p \mid q} \left( 1 + \frac{1}{p^2 (1 - \chi_4(p)/p)} \right) + O(q^6 e),
\]
where \( J(q) \) is the number of primitive characters modulo \( q \), \( \chi_4 \) is the nonprincipal character modulo 4, and \( e \) is any fixed positive real number.

**Theorem 4.** Let \( q \geq 5 \) be an odd number. Then one has
\[
\sum_{\chi \mod q} \left| \sum_{1 \leq a \leq q/4} a \chi(a) \right|^4
\]
\[
= \frac{385}{2^{20} \cdot 51} \cdot q^6 J(q) \prod_{p \mid q} \left( \frac{1 - 1/p^4}{1 + 1/p^4} \right)^3
\]
\[
+ \frac{15 L^4(3, \chi_4)}{2^{12} \cdot 3^2 \cdot \pi^2} \cdot q^6 J(q) \prod_{p \mid q} \left( \frac{1 - \chi_4(p)/p^3}{1 + \chi_4(p)/p^3} \right)^3
\]
\[
- \frac{1423 \pi L^2(3, \chi_4)}{2^{16} \cdot 5^2 \cdot 3^4 \cdot L(7, \chi_4)}
\]
\[
\cdot q^6 J(q) \prod_{p \mid q} \left( \frac{1 - \chi_4(p)/p^3}{1 - \chi_4(p)/p^3} \right)^2
\]
\[
+ \frac{3}{2^{16}} \cdot q^6 J(q) \prod_{p \mid q} \left( \frac{1 - 1/p^2}{1 + 1/p^2} \right)^3
\]
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\[ \frac{L^2(3, \chi_4)}{2^{12} \cdot \pi \cdot L(5, \chi_4)} \cdot q^6 J(q) \prod_{p \nmid q} \left( 1 - \frac{1}{p^3} \right)^2 \left( 1 - \chi_4(p) / p^3 \right)^2 \]

\[ \frac{L(3, \chi_4)}{2^{10} \cdot 3 \cdot \pi^2} \cdot q^6 J(q) \times \prod_{p \nmid q} \left( 1 - \chi_4(p) / p^3 \right) \left( 1 - 1 / p^4 \right) \]

\[ \times \prod_{p \mid q} \left( 1 + \frac{1}{p^2(1 - \chi_4(p) / p)} \right) + O(q^{6+\epsilon}). \]

(14)

From Theorems 3 and 4, we immediately get the following corollaries.

**Corollary 5.** Let \( p \geq 5 \) be a prime. Then one has

\[ \left| \sum_{\chi \mod p} \left| \sum_{1 \leq a \leq q/4} a \chi(a) \right|^4 \right| = \frac{21}{2^{17} \cdot 17} q^7 + \frac{L^2(3, \chi_4)}{2^{14} \cdot 3^2 \cdot \pi \cdot L(5, \chi_4)} q^7 \]

\[ + \frac{L(3, \chi_4)}{2^8 \cdot 5 \cdot 3^2 \cdot \pi^2} q^7 \prod_{p_1} \left( 1 + \frac{1}{p_1^2(1 - \chi_4(p_1) / p_1)} \right) + O(q^{6+\epsilon}). \]

(15)

where \( \prod_{p_1} \) denotes the product over all primes.

**Corollary 6.** Let \( p \geq 5 \) be a prime. Then

\[ \left| \sum_{\chi \mod p} \left| \sum_{1 \leq a \leq q/4} a \chi(a) \right|^4 \right| \]

\[ \left( 1 \chi(-1) = 1 \chi \neq \chi_0 \right)^4 \]

\[ = \frac{2833}{2^{20} \cdot 51} q^7 + \frac{15L^4(3, \chi_4)}{\pi^{12}} q^7 \]

\[ - \frac{1423 \pi L^2(3, \chi_4)}{2^{16} \cdot 5^2 \cdot 3 \cdot L(7, \chi_4)} q^7 \]

\[ - \frac{L^2(3, \chi_4)}{2^{12} \cdot \pi \cdot L(5, \chi_4)} q^7 \]

\[ + \frac{L(3, \chi_4)}{2^{10} \cdot 3 \cdot \pi^2} q^7 \prod_{p_1} \left( 1 + \frac{1}{p_1^2(1 - \chi_4(p_1) / p_1)} \right) \]

\[ + O(q^{6+\epsilon}). \]

(16)

Remark 7. It seems that the contributions of odd and even primitive characters to the fourth power moment of character sums over \([1, q/4]\) are very different.

## 2. Express the Character Sum in terms of Gauss Sums and L-Functions (I)

Let \( \chi \) be an odd primitive character modulo \( q \). In this section, we will express \( \sum_{1 \leq a \leq q/4} a \chi(a) \) in terms of Gauss sums and Dirichlet \( L \)-functions. We need the following lemmas.

**Lemma 8.** Suppose that \( q \geq 5 \) is an odd number, and \( \chi \) is an odd character modulo \( q \).

(i) For \( q \equiv 1 \pmod{4} \), one has

\[ \sum_{1 \leq a \leq q/4} a \chi(a) = \sum_{1 \leq a \leq q} a \chi(a) + q \sum_{1 \leq a \leq (q-1)/4} \chi(a) \]

\[ - \sum_{1 \leq a \leq (q-1)/4} a \chi(a) \]

\[ = \sum_{1 \leq a \leq (3q-3)/4} a \chi(a) \]

\[ - \sum_{1 \leq a \leq (q-1)/4} a \chi(a) \]

(17)

(ii) For \( q \equiv 3 \pmod{4} \), one has

\[ \sum_{1 \leq a \leq q/4} a \chi(a) = \sum_{1 \leq a \leq q} a \chi(a) + q \sum_{1 \leq a \leq (q-3)/4} \chi(a) \]

\[ - \sum_{1 \leq a \leq (q-3)/4} a \chi(a) \]

\[ = \sum_{1 \leq a \leq (3q-3)/4} a \chi(a) \]

\[ - \sum_{1 \leq a \leq (q-3)/4} a \chi(a) \]

(18)

**Proof.** It is easy to show that

\[ \sum_{1 \leq a \leq (3q-3)/4} a \chi(a) \]

\[ = \sum_{1 \leq a \leq (3q+1)/4} a \chi(a) \]

\[ = \sum_{1 \leq a \leq (3q+1)/4} a \chi(a) \]

\[ - \sum_{1 \leq a \leq (q-1)/4} a \chi(a) \]

\[ = \sum_{1 \leq a \leq (3q-3)/4} a \chi(a) \]

\[ - \sum_{1 \leq a \leq (q-1)/4} a \chi(a) \]
\[
\sum_{1 \leq a \leq (3q-3)/4} \chi(a) = \sum_{1 \leq a \leq q} \chi(a) - \sum_{(3q+1)/4 \leq a \leq q} \chi(a) = \sum_{1 \leq a \leq (q-1)/4} \chi(a).
\]

(19)

This proves (i). Similarly, we can deduce (ii).

Lemma 9. Suppose that \( q \geq 5 \) is an odd number, and \( \chi \) is an odd character modulo \( q \). Let \( \chi_4 \) be the nonprincipal character modulo 4. For \( q \equiv 1 \pmod{4} \), one has

\[
\sum_{a=1}^{4q} a^2 \chi(a) \chi_4(a)
= 16 \chi(4) \sum_{a=1}^{q} a \chi(a) + 16 \chi(4) q^2 \sum_{1 \leq a \leq (q-1)/4} \chi(a)
- 64 \chi(4) q \sum_{1 \leq a \leq (q-1)/4} a \chi(a).
\]

(20)

Proof. Note that \( \chi_4(1) = 1 \) and \( \chi_4(3) = -1 \), and we get

\[
\sum_{a=1}^{4q} a^2 \chi(a) \chi_4(a) = \sum_{a=0}^{q-1} (4a + 1)^2 \chi(4a + 1)
- \sum_{a=0}^{q-1} (4a + 3)^2 \chi(4a + 3).
\]

First we have

\[
\sum_{a=0}^{q-1} (4a + 1)^2 \chi(4a + 1)
= 16 \sum_{a=0}^{q-1} a^2 \chi(4a + 1) + 8 \sum_{a=0}^{q-1} a \chi(4a + 1)
+ \sum_{a=0}^{q-1} \chi(4a + 1)
= 16 \chi(4) \sum_{a=0}^{q-1} a^2 \chi(a + 4) + 8 \chi(4) \sum_{a=0}^{q-1} a \chi(a + 4)
= 16 \chi(4) \sum_{a=0}^{q-1} (a + 4)^2 \chi(a + 4)
+ 8 \chi(4) (1 - 4 \cdot 4) \sum_{a=0}^{q-1} (a + 4) \chi(a + 4),
\]

(22)

where \( 4 \) is the inverse of 4 modulo \( q \) with \( 4 \cdot 4 \equiv 1 \pmod{q} \) and \( 1 \leq 4 \leq q \). Since \( q \equiv 1 \pmod{4} \), we get \( 4 = (3q + 1)/4 \). Then from Lemma 8, we have

\[
\sum_{a=0}^{q-1} (a + 4)^2 \chi(a + 4)
= \sum_{0 \leq a \leq (q-1)/4} (a + 4)^2 \chi(a + 4)
+ \sum_{(q-1)/4 < a \leq q-1} (a + 4)^2 \chi(a + 4)
= \sum_{0 \leq a \leq (q-1)/4} (a + 4)^2 \chi(a + 4)
+ 2q \sum_{(q-1)/4 < a \leq q-1} (a + 4 - q) \chi(a + 4 - q)
+ q^2 \sum_{(q-1)/4 < a \leq q-1} \chi(a + 4 - q)
= \sum_{a=1}^{4} a^2 \chi(a) + 2q \sum_{1 \leq \alpha \leq (3q-3)/4} a \chi(a)
+ q^2 \sum_{1 \leq \alpha \leq (3q-3)/4} \chi(a)
- 2q \sum_{1 \leq \alpha \leq (q-1)/4} a \chi(a),
\]

(21)
\[ \sum_{a=1}^{q} a \chi(a) + q \sum_{1 \leq a \leq (3q-1)/4} \chi(a) = \chi\left( a + \frac{q+1}{4} \right) \]
\[ + 24 \chi(4) \sum_{a=0}^{q-1} \chi\left( a + \frac{q+3}{4} \right) \]
\[ = 16 \chi(4) \sum_{a=0}^{q-1} \left( a + \frac{q+3}{4} \right)^2 \chi\left( a + \frac{q+3}{4} \right) \]
\[ + 24 \left( 1 - \frac{q+3}{3} \right) \chi(4) \times \sum_{a=0}^{q-1} \left( a + \frac{q+3}{4} \right) \chi\left( a + \frac{q+3}{4} \right). \]  
(23)

Therefore
\[ \sum_{a=0}^{q-1} (4a+1)^2 \chi(4a+1) \]
\[ = 16 \chi(4) \sum_{a=0}^{q-1} \left( a + \frac{q+3}{4} \right)^2 \chi\left( a + \frac{q+3}{4} \right) \]
\[ + 8 \chi(4) \left( 1 - \frac{q+3}{4} \right) \sum_{a=0}^{q-1} \chi(a) \]
\[ = 16 \chi(4) \sum_{a=1}^{q} a^2 \chi(a) + 8 \chi(4) \sum_{a=0}^{q-1} a \chi(a) \]
\[ + 24 \chi(4) q^2 \sum_{1 \leq a \leq (q-1)/4} \chi(a) \]
\[ - 32 \chi(4) q \sum_{1 \leq a \leq (q-1)/4} a \chi(a). \]  
(24)

On the other hand, we get
\[ \sum_{a=0}^{q-1} (4a + 3)^2 \chi(4a + 3) \]
\[ = 16 \chi(4) \sum_{a=0}^{q-1} \left( a + \frac{q+3}{4} \right)^2 \chi\left( a + \frac{q+3}{4} \right) \]
\[ + 8 \chi(4) \left( 1 - \frac{q+3}{4} \right) \sum_{a=0}^{q-1} \chi(a) \]
\[ = 16 \chi(4) \sum_{a=1}^{q} a^2 \chi(a) + 8 \chi(4) \sum_{a=0}^{q-1} a \chi(a) \]
\[ + 24 \chi(4) q^2 \sum_{1 \leq a \leq (q-1)/4} \chi(a) \]
\[ - 32 \chi(4) q \sum_{1 \leq a \leq (q-1)/4} a \chi(a). \]  
(25)

Note that
\[ \sum_{a=0}^{q-1} \left( a + \frac{q+3}{4} \right)^2 \chi\left( a + \frac{q+3}{4} \right) \]
\[ = \sum_{0 \leq a \leq (3q-3)/4} \left( a + \frac{q+3}{4} \right)^2 \chi\left( a + \frac{q+3}{4} \right) \]
\[ + \sum_{(3q+1)/4 \leq a \leq q-1} \left( a + \frac{q+3}{4} \right)^2 \chi\left( a + \frac{q+3}{4} \right) \]
\[ = \sum_{0 \leq a \leq (3q-3)/4} \left( a + \frac{q+3}{4} \right)^2 \chi\left( a + \frac{q+3}{4} \right) \]
\[ + \sum_{(3q+1)/4 \leq a \leq q-1} \left( a + \frac{q+3}{4} - q \right)^2 \chi\left( a + \frac{q+3}{4} - q \right) \]
\[ + 2q \sum_{(3q+1)/4 \leq a \leq q-1} \left( a + \frac{q+3}{4} - q \right) \chi\left( a + \frac{q+3}{4} - q \right) \]
\[ + q^2 \sum_{(3q+1)/4 \leq a \leq q-1} \chi\left( a + \frac{q+3}{4} - q \right) \]
\[ = \sum_{a=1}^{q} a^2 \chi(a) + q^2 \sum_{1 \leq a \leq (q-1)/4} \chi(a) \]
\[ + 2q \sum_{1 \leq a \leq (q-1)/4} a \chi(a), \]
\[ \sum_{a=0}^{q-1} \left( a + \frac{q+3}{4} \right)^2 \chi\left( a + \frac{q+3}{4} \right) \]
\[ = \sum_{0 \leq a \leq (3q-3)/4} \left( a + \frac{q+3}{4} \right)^2 \chi\left( a + \frac{q+3}{4} \right) \]
\[ + \sum_{(3q+1)/4 \leq a \leq q-1} \left( a + \frac{q+3}{4} \right)^2 \chi\left( a + \frac{q+3}{4} \right) \]
\[ = \sum_{0 \leq a \leq (3q-3)/4} \left( a + \frac{q+3}{4} \right)^2 \chi\left( a + \frac{q+3}{4} \right) \]
\[ + \sum_{(3q+1)/4 \leq a \leq q-1} \left( a + \frac{q+3}{4} \right)^2 \chi\left( a + \frac{q+3}{4} \right) \]  
(26)
\[ + \sum_{(3q+1)/4 \leq a \leq q-1} (a + \frac{q+3}{4} - q) \chi (a + \frac{q+3}{4} - q) \]
\[ + q \sum_{(3q+1)/4 \leq a \leq q-1} \chi (a + \frac{q+3}{4} - q) \]
\[ = \sum_{a=1}^{q} a \chi (a) + q \sum_{1 \leq a \leq (q-1)/4} \chi (a) , \tag{27} \]

so we get
\[ \sum_{a=0}^{q-1} (4a + 3)^2 \chi (4a + 3) \]
\[ = 16 \chi (4) \sum_{a=0}^{q-1} \left( a + \frac{q+3}{4} \right)^2 \chi \left( a + \frac{q+3}{4} \right) \]
\[ + 24 \left( 1 - \frac{q+3}{4} \right) \chi (4) \]
\[ \times \sum_{a=0}^{q-1} \left( a + \frac{q+3}{4} \right) \chi \left( a + \frac{q+3}{4} \right) \]
\[ = 16 \chi (4) \sum_{a=1}^{q} a^2 \chi (a) - 8 \chi (4) q \sum_{a=1}^{q} a \chi (a) \]
\[ + 8 \chi (4) q^2 \sum_{1 \leq a \leq (q-1)/4} \chi (a) + 32 \chi (4) q \sum_{1 \leq a \leq (q-1)/4} a \chi (a) . \tag{28} \]

Now combine (21)–(28); we have
\[ \sum_{a=1}^{4q} a^2 \chi (a) \chi_4 (a) \]
\[ = \sum_{a=0}^{q-1} (4a + 1)^2 \chi (4a + 1) \]
\[ - \sum_{a=0}^{q-1} (4a + 3)^2 \chi (4a + 3) \]
\[ = 16 \chi (4) q \sum_{a=1}^{q} a \chi (a) \]
\[ + 16 \chi (4) q^2 \sum_{1 \leq a \leq (q-1)/4} \chi (a) \]
\[ - 64 \chi (4) q \sum_{1 \leq a \leq (q-1)/4} a \chi (a) . \tag{29} \]

**Lemma 10.** Suppose that \( q \geq 5 \) is an odd number, and \( \chi \) is an odd character modulo \( q \). Let \( \chi_4 \) be the nonprincipal character modulo 4. For \( q \equiv 3 \pmod{4} \), one has
\[ \sum_{a=1}^{q} a^2 \chi (a) \chi_4 (a) = -16 \chi (4) q \sum_{a=1}^{q} a \chi (a) \]
\[ - 16 \chi (4) q^2 \sum_{1 \leq a \leq (q-1)/4} \chi (a) \]
\[ + 64 \chi (4) q \sum_{1 \leq a \leq (q-1)/4} a \chi (a) . \tag{30} \]

**Proof.** For \( q \equiv 3 \pmod{4} \), we get \( \bar{4} = (q + 1)/4 \) and \( 3 \cdot \bar{4} = (3q + 3)/4 \). Using the methods of proving Lemma 9, we have
\[ \sum_{a=0}^{q-1} (4a + 1)^2 \chi (4a + 1) \]
\[ = \sum_{a=0}^{q-1} (4a + 1)^2 \chi (4a + 1) - \sum_{a=0}^{q-1} (4a + 3)^2 \chi (4a + 3) , \tag{31} \]
\[ \sum_{a=0}^{q-1} (4a + 1)^2 \chi (4a + 1) \]
\[ = 16 \chi (4) \sum_{a=0}^{q-1} (a + \bar{4})^2 \chi (a + \bar{4}) \]
\[ + 8 \chi (4) (1 - 4 \cdot \bar{4}) \sum_{a=0}^{q-1} (a + \bar{4}) \chi (a + \bar{4}) , \tag{32} \]
\[ \sum_{a=0}^{q-1} (4a + 3)^2 \chi (4a + 3) \]
\[ = 16 \chi (4) \sum_{a=0}^{q-1} a^2 \chi (4a + 3) + 24 \sum_{a=0}^{q-1} a \chi (4a + 3) \]
\[ + 9 \sum_{a=0}^{q-1} \chi (4a + 3) \]
\[ = 16 \chi (4) \sum_{a=0}^{q-1} a^2 \chi (a + 3 \cdot \bar{4}) \]
\[ + 24 \chi (4) \sum_{a=0}^{q-1} a \chi (a + 3 \cdot \bar{4}) \]
\[ = 16 \chi (4) \sum_{a=0}^{q-1} \left( a + \frac{3q + 3}{4} \right)^2 \chi \left( a + \frac{3q + 3}{4} \right) \]
\[ - 24 \chi (4) q \sum_{a=0}^{q-1} \left( a + \frac{3q + 3}{4} \right) \chi \left( a + \frac{3q + 3}{4} \right) . \tag{33} \]
It is not hard to show that
\[
\sum_{a=0}^{q-1} (a + 4) \chi (a + 4)
= \sum_{0 \leq a \leq (3q-1)/4} (a + 4) \chi (a + 4)
\]
\[
+ \sum_{(3q-1)/4 < a \leq q-1} (a + 4) \chi (a + 4)
= \sum_{0 \leq a \leq (3q-1)/4} (a + 4) \chi (a + 4)
\]
\[
+ 2q \sum_{(3q-1)/4 < a \leq q-1} (a + 4 - q) \chi (a + 4 - q)
\]
\[
+ q^2 \sum_{(3q-1)/4 < a \leq q-1} \chi (a + 4 - q)
\]
\[
= \sum_{a=1}^{q} a^2 \chi (a + 4)
+ q \sum_{1 \leq a \leq (q-3)/4} \chi (a + 4 - q)
\]
\[
= \sum_{0 \leq a \leq (3q-1)/4} (a + 4) \chi (a + 4)
\]
\[
+ \sum_{(3q-1)/4 < a \leq q-1} (a + 4 - q) \chi (a + 4 - q)
\]
\[
+ q \sum_{(3q-1)/4 < a \leq q-1} \chi (a + 4 - q)
\]
\[
= \sum_{a=1}^{q} a^2 \chi (a) + \sum_{1 \leq a \leq (q-3)/4} \chi (a)
\]
\[
+ 2q \sum_{1 \leq a \leq (q-3)/4} a \chi (a)
\]
\[
+ q^2 \sum_{1 \leq a \leq (q-3)/4} \chi (a)
\]
\[
+ 3q \sum_{1 \leq a \leq (q-3)/4} \chi (a) - 2q \sum_{1 \leq a \leq (q-3)/4} a \chi (a)
\]
\[
= \sum_{0 \leq a \leq (3q-1)/4} (a + 4) \chi (a + 4)
\]
\[
+ 3q^2 \sum_{1 \leq a \leq (q-3)/4} \chi (a) - 2q \sum_{1 \leq a \leq (q-3)/4} a \chi (a)
\]
\[
= \sum_{0 \leq a \leq (3q-1)/4} (a + 4) \chi (a + 4)
\]
\[
+ 3q^2 \sum_{1 \leq a \leq (q-3)/4} \chi (a) - 2q \sum_{1 \leq a \leq (q-3)/4} a \chi (a)
\]
\[
+ \frac{q}{4} \sum_{0 \leq a \leq (3q-1)/4} \chi (a + 4)
\]
\[
- \frac{q}{4} \sum_{1 \leq a \leq (q-3)/4} \chi (a + 4)
\]
\[
+ 2q \sum_{1 \leq a \leq (q-3)/4} a \chi (a)
\]
\[
+ \frac{q}{4} \sum_{1 \leq a \leq (q-3)/4} \chi (a + 4)
\]
\[
+ 3q \sum_{1 \leq a \leq (q-3)/4} \chi (a) - 2q \sum_{1 \leq a \leq (q-3)/4} a \chi (a)
\]
\[
= \sum_{0 \leq a \leq (3q-1)/4} (a + 4) \chi (a + 4)
\]
\[
+ 8 \chi (4) \left(1 - 4 \times \frac{3q}{4} \right) \sum_{a=0}^{q-1} (a + 4) \chi (a + 4)
\]
\[
= 16 \chi (4) \sum_{a=1}^{q} a^2 \chi (a)
\]
\[
- 8 \chi (4) q \sum_{a=1}^{q} a \chi (a) + 8 \chi (4) q^2 \sum_{1 \leq a \leq (q-3)/4} \chi (a)
\]
\[
+ 32 \chi (4) q \sum_{1 \leq a \leq (q-3)/4} a \chi (a)
\].

(35)

On the other hand, by Lemma 8, we get
\[
\sum_{a=0}^{q-1} (a + 3q + 3/4) \chi (a + 3q + 3/4)
\]
\[
= \sum_{0 \leq a \leq (3q-3)/4} (a + 3q + 3/4) \chi (a + 3q + 3/4)
\]
\[
+ \sum_{(q+1)/4 \leq a \leq q-1} (a + 3q + 3/4) \chi (a + 3q + 3/4)
\]
\[
= \sum_{0 \leq a \leq (3q-3)/4} (a + 3q + 3/4) \chi (a + 3q + 3/4)
\]
\[
+ 2q \sum_{(q+1)/4 \leq a \leq q-1} (a + 3q + 3/4 - q) \chi (a + 3q + 3/4 - q)
\]
\[
+ q^2 \sum_{(q+1)/4 \leq a \leq q-1} \chi (a + 3q + 3/4 - q)
\]
\[
= \sum_{a=1}^{q} a^2 \chi (a) + 2q \sum_{1 \leq a \leq (3q-3)/4} a \chi (a)
\]
\[
+ q^2 \sum_{1 \leq a \leq (3q-3)/4} \chi (a)
\]
\[
= \sum_{a=1}^{q} a^2 \chi (a) + 2q \sum_{a=1}^{q} a \chi (a)
\]
\[
+ 3q^2 \sum_{1 \leq a \leq (q-3)/4} \chi (a) - 2q \sum_{1 \leq a \leq (q-3)/4} a \chi (a)
\]
\[
= \sum_{0 \leq a \leq (3q-1)/4} (a + 4) \chi (a + 4)
\]
\[
+ 3q^2 \sum_{1 \leq a \leq (q-3)/4} \chi (a) - 2q \sum_{1 \leq a \leq (q-3)/4} a \chi (a)
\]
\[
+ \frac{q}{4} \sum_{0 \leq a \leq (3q-1)/4} \chi (a + 4)
\]
\[
- \frac{q}{4} \sum_{1 \leq a \leq (q-3)/4} \chi (a + 4)
\]
\[
+ 2q \sum_{1 \leq a \leq (q-3)/4} a \chi (a)
\]
\[
+ \frac{q}{4} \sum_{1 \leq a \leq (q-3)/4} \chi (a + 4)
\].

(34)
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\[ + \sum_{(q+1)4 \leq a \leq q} \left( a + \frac{3q+3}{4} \right) \chi \left( a + \frac{3q+3}{4} \right) \]

\[ = \sum_{0 \leq a \leq (q-3)/4} \left( a + \frac{3q+3}{4} \right) \chi \left( a + \frac{3q+3}{4} \right) \]

\[ + q \sum_{(q+1)4 \leq a \leq q} \chi \left( a + \frac{3q+3}{4} - q \right) \]

\[ = \frac{q}{4} \sum_{a=1} q \chi(a) + \frac{q}{4} \sum_{1 \leq a \leq (q-3)/4} \chi(a). \quad (36) \]

Then from (33), we have

\[ \sum_{a=0}^{q-1} (4a + 3)^2 \chi(4a + 3) \]

\[ = 16 \chi(4) q \sum_{a=0}^{q-1} \left( a + \frac{3q+3}{4} \right)^2 \chi \left( a + \frac{3q+3}{4} \right) \]

\[ - 24 \chi(4) q \sum_{a=0}^{q-1} \left( a + \frac{3q+3}{4} \right) \chi \left( a + \frac{3q+3}{4} \right) \]

\[ = 16 \chi(4) q \sum_{a=1} q \chi(a) + 8 \chi(4) \sum_{a=1} q \chi(a) \]

\[ + 24 \chi(4) q^2 \sum_{1 \leq a \leq (q-3)/4} \chi(a) \]

\[ - 32 \chi(4) q \sum_{1 \leq a \leq (q-3)/4} \chi(a). \quad (37) \]

Combining (31), (35), and (37), we have

\[ \sum_{a=1}^{4q} a^2 \chi(a) \chi_4(a) \]

\[ = \sum_{a=0}^{q-1} (4a + 1)^2 \chi(4a + 1) \]

\[ - \sum_{a=0}^{q-1} (4a + 3)^2 \chi(4a + 3) \]

\[ = -16 \chi(4) \sum_{a=1} q \chi(a) \]

\[ - 16 \chi(4) q^2 \sum_{1 \leq a \leq (q-3)/4} \chi(a) + 64 \chi(4) q \sum_{1 \leq a \leq (q-3)/4} \chi(a). \quad (38) \]

Now we can express \( \sum_{1 \leq a \leq q/4} a \chi(a) \) in terms of Gauss sums and Dirichlet L-functions.

**Theorem 11.** Let \( \chi \) be an odd primitive character modulo odd integer \( q \geq 5 \), and let \( \chi_4 \) be the nonprincipal character modulo 4. Then one has

\[ \sum_{1 \leq a \leq q/4} a \chi(a) \]

\[ = \frac{q}{8\pi^2} \tau(\chi) \left( \overline{\chi}(2) L(1, \overline{\chi}) - \overline{\chi}(4) L(1, \overline{\chi}) \right) + \frac{4}{\pi} L(2, \overline{\chi} \chi_4). \quad (39) \]

**Proof.** By Lemmas 9 and 10, we get

\[ 16 \chi(4) q \sum_{a=1} q a \chi(a) + 16 \chi(4) q^2 \sum_{1 \leq a \leq q/4} \chi(a) \]

\[ - 64 \chi(4) q \sum_{1 \leq a \leq q/4} a \chi(a) \]

\[ = \left\{ \begin{array}{ll}
\frac{4q}{\pi} \sum_{a=1}^{4q} a^2 \chi(a) \chi_4(a), & \text{if } q \equiv 1 \pmod{4} \\
- \sum_{a=1}^{4q} a^2 \chi(a) \chi_4(a), & \text{if } q \equiv 3 \pmod{4}
\end{array} \right. \]

\[ = \chi_4(q) \sum_{a=1}^{4q} a^2 \chi_4(a). \quad (40) \]

From the Fourier expansion for primitive character sums (see [10] or [11])

\[ \sum_{a \leq \lambda q} \chi(a) = \frac{\tau(\chi) \cos(2\pi n \lambda)}{2\pi i} \sum_{n=1}^\infty \frac{\chi(n) \sin(2\pi n \lambda)}{n}, \quad \text{if } \chi(-1) = 1, \]

\[ = \frac{\tau(\chi) \pi}{2\pi i} \sum_{n=1}^\infty \frac{\chi(n) \cos(2\pi n \lambda)}{n}, \quad \text{if } \chi(-1) = -1, \]

we easily have

\[ \sum_{1 \leq a \leq q/4} \chi(a) = 2 + \overline{\chi}(2) - \overline{\chi}(4) \tau(\chi) L(1, \overline{\chi}). \quad (42) \]

Note that \( \chi \chi_4 \) is a primitive character modulo \( 4q \) satisfying \( \chi \chi_4(-1) = 1 \), and

\[ \tau(\chi_4) = \sum_{a=1}^{4q} \chi_4(a) e \left( \frac{a}{4q} \right) \]

\[ = \sum_{a=1}^{4q} \chi(4b + qa) \chi_4(4b + qa) e \left( \frac{4b + qa}{4q} \right) \]

\[ = \sum_{a=1}^{4q} \chi(4b) \chi_4(qa) e \left( \frac{b + a}{q} \right) \]

\[ = \frac{4}{\pi} L(2, \chi \chi_4). \]
\[= \chi(4) \chi_4(q) \left( \sum_{a=1}^{q} \chi_4(a) e\left(\frac{a}{q}\right) \right) \]
\[\times \left( \sum_{b=1}^{q} \chi(b) e\left(\frac{b}{q}\right) \right) \]
\[= 2i \chi(4) \chi_4(q) \tau(\chi). \quad (43)\]

then from (3) we have
\[\sum_{a=1}^{q} a \chi(a) = -\frac{q}{\pi i} \tau(\chi) L(1, \chi), \]
\[\sum_{a=1}^{q} a^2 \chi_4(a) = \frac{16q^2}{\pi^2} \tau(\chi_4) L(2, \chi_4) \quad (44)\]
\[= \frac{32i \chi(4) \chi_4(q)}{\pi^2} q^2 \tau(\chi) L(2, \chi_4). \]

Therefore
\[- \frac{16 \chi(4)}{\pi i} q^2 \tau(\chi) L(1, \chi) \]
\[+ \frac{8 \chi(4)}{\pi i} \left( 2 + \chi_4(2) - \chi_4(4) \right) q^2 \tau(\chi) L(1, \chi) \]
\[- 64 \chi(4) q \sum_{1 \leq a \leq q/4} a \chi(a) \]
\[= \frac{32i \chi(4) \chi_4(q)}{\pi^2} q^2 \tau(\chi) L(2, \chi_4). \quad (45)\]

Then we have
\[\sum_{1 \leq a \leq q/4} a \chi(a) \]
\[= \frac{q}{8\pi i} \tau(\chi) \left( \chi_4(2) L(1, \chi) - \chi_4(4) L(1, \chi) \right) \]
\[+ \frac{4}{\pi} L(2, \chi_4) \quad (46)\]

\[\square\]

3. Express the Character Sum in terms of Gauss Sums and L-Functions (II)

Let \(\chi\) be an even primitive character modulo \(q\). In this section, we express \(\sum_{1 \leq a \leq q/4} a \chi(a)\) in terms of Gauss sums and Dirichlet L-functions.

Lemma 12. Let \(q > 2\) be an odd number, and let \(\chi\) be a nonprincipal character modulo \(q\). Then
\[4 \chi(2) q \sum_{a=1}^{\frac{q-1}{2}} a \chi(a) \]
\[= (2 \chi(2) + 1) q \sum_{a=1}^{q} a \chi(a) - (4 \chi(2) - 1) \sum_{a=1}^{q} a^2 \chi(a), \quad (47)\]

Proof. We have
\[\sum_{a=1}^{\frac{q-1}{2}} (2a)^2 \chi(2a) = \sum_{a=1}^{\frac{q-1}{2}} (2a)^2 \chi(2a) + \sum_{a=1}^{\frac{q+1}{2}} (2b)^2 \chi(2b) \]
\[+ \sum_{b=1}^{\frac{q+1}{2}} (2b + q - 1)^2 \chi(2b + q - 1) \quad (48)\]
\[= \sum_{a=1}^{\frac{q-1}{2}} (2a)^2 \chi(2a) + \sum_{b=1}^{\frac{q+1}{2}} (2b - 1)^2 \chi(2b - 1) \]
\[+ 2q \sum_{b=1}^{\frac{q+1}{2}} (2b - 1) \chi(2b - 1) \]
\[+ q^2 \sum_{a=1}^{\frac{q+1}{2}} \chi(2a - 1) \quad (49)\]

Since
\[\sum_{a=1}^{\frac{q+1}{2}} \chi(2a - 1) + \sum_{a=1}^{\frac{q-1}{2}} \chi(2a) = \sum_{a=1}^{q} \chi(a) = 0, \]
\[\sum_{a=1}^{\frac{q+1}{2}} (2a - 1) \chi(2a - 1) + \sum_{a=1}^{\frac{q-1}{2}} 2a \chi(2a) = \sum_{a=1}^{q} a \chi(a), \quad (49)\]

we have
\[\sum_{a=1}^{q} (2a)^2 \chi(2a) = \sum_{a=1}^{q} a^2 \chi(a). \]
\[ + 2q \sum_{a=1}^{q} a \chi(a) - 4 \chi(2) q \sum_{a=1}^{(q-1)/2} a \chi(a) \]
\[ - \chi(2) q^{2} \sum_{a=1}^{(q-1)/2} \chi(a). \]  
(50)

It is not hard to show that
\[ \chi(2) q \sum_{a=1}^{(q-1)/2} \chi(a) = (1 - 2 \chi(2)) \sum_{a=1}^{q} a \chi(a), \]
\[ \chi(2) q \sum_{a=(q+1)/2}^{q} \chi(a) = (2 \chi(2) - 1) \sum_{a=1}^{q} a \chi(a). \]  
(51)

Therefore
\[ 4 \chi(2) q \sum_{a=1}^{(q-1)/2} a \chi(a) \]
\[ = (2 \chi(2) + 1) q \sum_{a=1}^{q} a \chi(a) \]
\[ \times \sum_{a=1}^{q} a \chi(a) - (4 \chi(2) - 1) \sum_{a=1}^{q} a^{2} \chi(a). \]  
(52)

Note that
\[ \sum_{a=1}^{q} a \chi(a) + \sum_{a=(q+1)/2}^{q} a \chi(a) = \sum_{a=1}^{q} a \chi(a), \]  
(53)

we have
\[ 4 \chi(2) q \sum_{a=(q+1)/2}^{q} a \chi(a) \]
\[ = (2 \chi(2) - 1) q \sum_{a=1}^{q} a \chi(a) \]
\[ \times \sum_{a=1}^{q} a \chi(a) + (4 \chi(2) - 1) \sum_{a=1}^{q} a^{2} \chi(a). \]  
(54)

\[ \text{Lemma 13. Let } q \geq 5 \text{ be an odd number, and let } \chi \text{ be an nonprincipal even character modulo } q. \text{ If } q \equiv 1 \mod 4, \text{ then} \]
\[ 16 \chi(4) \sum_{a=1}^{q-1} a^{2} \chi(a) \]
\[ = (8 \chi(4) - 2 \chi(2) + 1) \sum_{a=1}^{q} a^{2} \chi(a) - 16 q \chi(4) \]
\[ \times \sum_{a=1}^{(q-1)/4} \chi(a). \]  
(55)

While if \( q \equiv 3 \mod 4 \), we have
\[ 16 \chi(4) \sum_{a=1}^{q-1} a^{2} \chi(a) \]
\[ = (8 \chi(4) - 2 \chi(2) + 1) \sum_{a=1}^{q} a^{2} \chi(a) \]
\[ - 16 q \chi(4) \]
\[ \times \sum_{a=1}^{(q-1)/4} \chi(a). \]  
(56)

Proof. First suppose that \( q \equiv 1 \mod 4 \). Then \( q = (3q + 1)/4 \).
We have
\[ 16 \chi(4) \sum_{a=1}^{q-1} a^{2} \chi(a) \]
\[ = \sum_{a=1}^{q-1} (4a)^{2} \chi(4a) \]
\[ - 16 q \chi(4) \]
\[ \times \sum_{a=1}^{(q-1)/4} \chi(a). \]  
(57)
\[ + \frac{(q-1)/4}{2} \sum_{a=1}^{(q-1)/4} (4a - 1) \chi(4a - 1) \]
\[ + 4q \sum_{a=1}^{(q-1)/4} (4a - 2) \chi(4a - 2) \]
\[ + 6q \sum_{a=1}^{(q-1)/4} (4a - 3) \chi(4a - 3) \]
\[ + q^2 \sum_{a=1}^{(q-1)/4} \chi(4a - 1) + 4q^2 \sum_{a=1}^{(q-1)/4} \chi(4a - 2) \]
\[ + 9q^2 \sum_{a=1}^{(q-1)/4} \chi(4a - 3) \]
\[ = \sum_{a=1}^{q-1} a^2 \chi(a) + 8q \chi(4) \sum_{a=1}^{(q-1)/4} \left( a + \frac{q-1}{4} \right) \chi(a + \frac{q-1}{4}) \]
\[ - 2q^2 \chi(4) \sum_{a=1}^{(q-1)/4} \chi \left( a + \frac{q-1}{4} \right) \]
\[ + 16q \chi(4) \sum_{a=1}^{(q-1)/4} \left( a + \frac{q-1}{2} \right) \chi \left( a + \frac{q-1}{2} \right) \]
\[ - 8q^2 \chi(4) \sum_{a=1}^{(q-1)/4} \chi \left( a + \frac{q}{4} \right) \]
\[ + 24q \chi(4) \sum_{a=1}^{(q-1)/4} \left( a + \frac{3q-3}{4} \right) \chi \left( a + \frac{3q-3}{4} \right) \]
\[ - 18q^2 \chi(4) \sum_{a=1}^{(q-1)/4} \chi \left( a + \frac{3q-3}{4} \right) \]
\[ + q^2 \chi(4) \sum_{a=1}^{(q-1)/4} \chi \left( a + \frac{q-1}{4} \right) \]
\[ + 4q^2 \chi(4) \sum_{a=1}^{(q-1)/4} \chi \left( a + \frac{q-1}{2} \right) \]
\[ + 9q^2 \chi(4) \sum_{a=1}^{(q-1)/4} \chi \left( a + \frac{3q-3}{4} \right) \]
\[ = \sum_{a=1}^{q-1} a^2 \chi(a) + 8q \chi(4) \sum_{a=1}^{(q-1)/2} a \chi(a) \]
\[ + 16q \chi(4) \sum_{a=1}^{(q-1)/2} a \chi(a) \]
\[ + 24q \chi(4) \sum_{a=1}^{(3q-3)/4} a \chi(a) \]
\[ + 16q \chi(4) \sum_{a=1}^{(q-1)/2} a \chi(a) \]
\[ - q^2 \chi(4) \sum_{a=1}^{(q-1)/2} a \chi(a) - 4q^2 \chi(4) \sum_{a=1}^{(q-1)/2} \chi(a) \]
\[ - 9q^2 \chi(4) \sum_{a=1}^{q-1} \chi(a) \].

(57)

Note that \(\sum_{1 \leq a \leq (q-1)/2} \chi(a) = 0\) and \(\sum_{a=1}^{q} a \chi(a) = 0\) for even character \(\chi\). By Lemma 12 we have

\[ 16\chi(4) \sum_{a=1}^{q-1} a^2 \chi(a) \]
\[ = \sum_{a=1}^{q} a^2 \chi(a) - 16q \chi(4) \]
\[ \times \sum_{a=1}^{(q-1)/4} a \chi(a) - 8q \chi(4) \sum_{a=1}^{(q-1)/2} \chi(a) \]
\[ + 4q^2 \chi(4) \sum_{a=1}^{(q-1)/4} \chi(a) \]
\[ = (8\chi(4) - 2\chi(2) + 1) \]
\[ \times \sum_{a=1}^{q} a^2 \chi(a) - 16q \chi(4) \sum_{a=1}^{(q-1)/4} \chi(a) \]
\[ + 4q^2 \chi(4) \sum_{a=1}^{(q-1)/4} \chi(a) \].

(58)

Now assume that \(q \equiv 3 \pmod{4}\). Then \(q = (q + 1)/4\). We have

\[ 16\chi(4) \sum_{a=1}^{q-1} a^2 \chi(a) \]
\[ = \sum_{a=1}^{q-1} (4a)^2 \chi(4a) \]
\[ = \sum_{a=1}^{(q-3)/4} (4a)^2 \chi(4a) + \sum_{a=1}^{(2q-2)/4} (4a)^2 \chi(4a) \]
\[ + \sum_{a=1}^{(3q-3)/4} (4a)^2 \chi(4a) + \sum_{a=1}^{q-1} (4a)^2 \chi(4a) \]
\[ = \sum_{a=1}^{(q-3)/4} (4a)^2 \chi(4a) \]
\[ + \sum_{a=1}^{(q-1)/4} (4a + q - 3)^2 \chi(4a - 3) \]
\[ + \sum_{a=1}^{(q-1)/4} (4a + 2q - 2)^2 \chi(4a - 2) \]
\begin{align*}
+ \sum_{a=1}^{(q-3)/4} (4a + 3q - 1)^2 \chi(4a - 1) \\
= \sum_{a=1}^{(q-3)/4} (4a)^2 \chi(4a) + \sum_{a=1}^{(q-3)/4} (4a - 1)^2 \chi(4a - 1) \\
+ \sum_{a=1}^{(q+1)/4} (4a - 2)^2 \chi(4a - 2) \\
+ \sum_{a=1}^{(q+1)/4} (4a - 3)^2 \chi(4a - 3) \\
+ 6q \sum_{a=1}^{(q-3)/4} (4a - 1) \chi(4a - 1) \\
+ 4q \sum_{a=1}^{(q+1)/4} (4a - 2) \chi(4a - 2) \\
+ 2q \sum_{a=1}^{(q+1)/4} (4a - 3) \chi(4a - 3) \\
+ 9q^2 \sum_{a=1}^{(q-3)/4} \chi(4a - 1) + 4q^2 \sum_{a=1}^{(q+1)/4} \chi(4a - 2) \\
+ q^2 \sum_{a=1}^{(q+1)/4} \chi(4a - 3) \\
= \sum_{a=1}^{q-1} a^2 \chi(a) \\
+ 24q \chi(4) \sum_{a=1}^{(q-3)/4} \left( a + \frac{3q - 1}{4} \right) \chi \left( a + \frac{3q - 1}{4} \right) \\
- 18q^2 \chi(4) \sum_{a=1}^{(q-3)/4} \chi \left( a + \frac{3q - 1}{4} \right) \\
+ 16q \chi(4) \sum_{a=1}^{(q+1)/4} \left( a + \frac{q - 1}{2} \right) \chi \left( a + \frac{q - 1}{2} \right) \\
- 8q^2 \chi(4) \sum_{a=1}^{(q+1)/4} \chi \left( a + \frac{q - 1}{2} \right) \\
+ 8q \chi(4) \sum_{a=1}^{(q+1)/4} \left( a + \frac{q - 3}{4} \right) \chi \left( a + \frac{q - 3}{4} \right) \\
- 2q^2 \chi(4) \sum_{a=1}^{(q+1)/4} \chi \left( a + \frac{q - 3}{4} \right) \\
+ 9q^2 \chi(4) \sum_{a=1}^{(q-3)/4} \chi \left( a + \frac{3q - 1}{4} \right) \\
+ 4q^2 \chi(4) \sum_{a=1}^{(q-3)/4} \chi \left( a + \frac{q - 1}{2} \right) \\
+ \frac{q^2 \chi(4)}{4} \sum_{a=1}^{(q+1)/4} \chi \left( a + \frac{q - 3}{4} \right) \\
= \sum_{a=1}^{q-1} a^2 \chi(a) + 24q \chi(4) \sum_{a=(3q+3)/4}^{q-1} \chi(a) \\
+ 16q \chi(4) \sum_{a=(q+1)/4}^{(3q-1)/2} \chi(a) \\
+ 8q \chi(4) \sum_{a=(q+1)/4}^{(q-1)/2} \chi(a) \\
- 9q^2 \chi(4) \sum_{a=(3q+3)/4}^{q-1} \chi(a) \\
- 4q^2 \chi(4) \sum_{a=(q+1)/2}^{(3q-1)/4} \chi(a) \\
- q^2 \chi(4) \sum_{a=(q+1)/4}^{(q-1)/2} \chi(a)
\end{align*}

Note that \( \sum_{1 \leq a \leq (q-1)/2} \chi(a) = 0 \) and \( \sum_{a=1}^{q} a \chi(a) = 0 \) for even character \( \chi \). By Lemma 12 we have

\begin{align*}
16 \chi(4) \sum_{a=1}^{q-1} a^2 \chi(a) \\
= \sum_{a=1}^{q} a^2 \chi(a) - 16q \chi(4) \sum_{a=(3q+3)/4}^{q-1} a \chi(a) \\
- 8q \chi(4) \sum_{a=(q+1)/2}^{(3q-1)/4} a \chi(a) \\
= (8 \chi(4) - 2 \chi(2) + 1) \sum_{a=1}^{q} a^2 \chi(a) \\
- 16q \chi(4) \sum_{a=1}^{(q-3)/4} a \chi(a) \\
+ 4q^2 \chi(4) \sum_{a=1}^{(q-3)/4} \chi(a).
\end{align*}

Now we express \( \sum_{1 \leq a \leq q/4} a \chi(a) \) in terms of Gauss sums and Dirichlet \( L \)-functions.

**Theorem 14.** Let \( \chi \) be an even primitive character modulo odd integer \( q \geq 5 \). Then one has

\begin{align*}
\sum_{1 \leq a \leq q/4} a \chi(a) = \frac{q}{16 \pi^2} \left( \bar{\chi}(4) - 2 \bar{\chi}(2) - 8 \right) \tau(\chi) L(2, \bar{\chi}) \\
+ \frac{q}{4 \pi} \tau(\chi) L(1, \bar{\chi} \chi_a).
\end{align*}
Proof. By Lemma 13, (3), and (37), we have

\[
16q \chi (4) \sum_{1 \leq a \leq q/4} a \chi (a) = (1 - 2 \chi (2) - 8 \chi (4)) \sum_{a=1}^{q/4} a \chi (a)
\]

\[
= \frac{q^2}{\pi^2} (1 - 2 \chi (2) - 8 \chi (4)) L(2, \bar{\chi})
\]

\[
+ \frac{4q^2}{\pi} \chi (4) \tau (\chi) L(1, \bar{\chi}).
\]

Therefore

\[
\sum_{1 \leq a \leq q/4} a \chi (a) = \frac{q}{16\pi} (\bar{\chi} (4) - 2 \chi (2) - 8)
\]

\[
\times \tau (\chi) L(2, \bar{\chi}) + \frac{q}{4\pi} \tau (\chi) L(1, \bar{\chi}).
\]

Therefore

\[
\sum_{1 \leq a \leq q/4} a \chi (a) = \frac{q}{16\pi} (\bar{\chi} (4) - 2 \chi (2) - 8)
\]

\[
\times \tau (\chi) L(2, \bar{\chi}) + \frac{q}{4\pi} \tau (\chi) L(1, \bar{\chi}).
\]  

(62)

4. Mean Values of Dirichlet L-Functions

In this section, we will study the mean values of Dirichlet L-functions, which will be used to prove Theorems 3 and 4.

Lemma 15. Let \( q \) and \( r \) be integers with \( q \geq 2 \) and \( (r, q) = 1 \).
Then one has the identities

\[
\sum_{\chi \mod q} \chi (r) = \sum_{d|(q-r)} \mu \left( \frac{q}{d} \right) \phi (d),
\]

\[
f (q) = \sum_{d|(q-r)} \mu \left( \frac{q}{d} \right) \phi (d),
\]

where \( \sum_{\chi \mod q} \) denotes the summation over all primitive characters modulo \( q \), and \( f (q) \) is the number of primitive characters modulo \( q \).

Proof. This is Lemma 3 of [12].

Lemma 16. Let \( q \geq 2 \) be an odd number, and let \( k \geq 0 \) be an integer. Then one has

\[
\sum_{n=1}^{\infty} \frac{d (n) d (2^k n)}{n^2} = \frac{27}{80} \cdot \frac{\zeta (2)}{\zeta (4)} \prod_{p \nmid q} \left( 1 - 1/p^2 \right)^3,
\]

\[
\sum_{n=1}^{\infty} \frac{d (n) d (2^k n)}{n^4} = \frac{3375}{4352} \cdot \frac{\zeta (4)}{\zeta (8)} \prod_{p \nmid q} \left( 1 - 1/p^2 \right)^3,
\]

\[
\sum_{n=1}^{\infty} \frac{d (n) \chi_4 (n)}{n^3} = \frac{1}{(1 - 1/2^6)} \cdot \frac{L (5, \chi_4)}{\zeta (6)} \prod_{p \nmid q} (1 - \chi_4 (p)/p^3),
\]

\[
\sum_{n=1}^{\infty} \frac{\tau_1 (n) \tau_1 (2^k n)}{n^2}
\]

\[
= \frac{\zeta (2)}{2} \frac{\zeta (4)}{\zeta (6)} \prod_{p \nmid q} (1 - \chi_4 (p)/p^3) \left( 1 - \frac{1}{p^4} \right),
\]

\[
\sum_{n=1}^{\infty} \frac{d (n) \tau_2 (2^k n)}{n^3}
\]

\[
= \frac{15}{16} \cdot \frac{\zeta (2)}{2} \frac{\zeta (4)}{\zeta (6)} \prod_{p \nmid q} (1 - \chi_4 (p)/p^3) \left( 1 - \frac{1}{p^4} \right)^2,
\]

\[
\sum_{n=1}^{\infty} \frac{d (n) \tau_2 (2^k n)}{n^4}
\]

\[
= \frac{15}{16} \cdot \frac{\zeta (2)}{2} \frac{\zeta (4)}{\zeta (6)} \prod_{p \nmid q} (1 - \chi_4 (p)/p^3) \left( 1 - \frac{1}{p^4} \right)^2,
\]
\[
\sum_{m=1}^{\infty} \frac{\chi_4(n) d(n) \tau_2(n)}{n^2} = \frac{9}{16} \zeta(2) L^2(3, \chi_4) L(5, \chi_4)
\times \prod_{p \mid q} \left(1 - \frac{1}{p^2}\right) \left(1 - \frac{1}{\chi_4(p)/p^3}\right),
\]
\[
\sum_{n=1}^{\infty} \frac{\tau_2(n) \tau_2(2^k n)}{n^2} = \frac{4}{2^k 5} \cdot \zeta(4) L(3, \chi_4)
\times \prod_{p \mid q} \left(1 - \frac{1}{X_4(p)/p^3}\right) \left(1 - \frac{1}{p^4}\right)
\times \prod_{p \nmid q} \left(1 + \frac{1}{p^2 - \chi_4(p)/p}\right),
\]
(65)

where \(d(n) = \sum_{d \mid n} 1\), \(\tau_1(n) = \sum_{d \mid n} \chi_4(d)/d\), and \(\tau_2(n) = \sum_{d \mid n} \chi_4(n/d)/d\).

**Proof.** By the Euler product, we have
\[
\sum_{n=1}^{\infty} \frac{d(n) d(2^k n)}{n^2} = \sum_{i=0}^{\infty} \sum_{m=1}^{\infty} \frac{d(2^i m) d(2^{i+k} n)}{2^i m^2}
= \sum_{i=0}^{\infty} \left(\frac{d(2^i)}{2^i}\right) \sum_{m=1}^{\infty} \frac{d^2(m)}{m^2}
= \left(\sum_{i=0}^{\infty} \frac{(i+1)(k+i+1)}{2^i}\right) \prod_{p \mid 2q} \left(\sum_{j=0}^{\infty} \frac{(j+1)^2}{p^{2j}}\right)
= \frac{5/4 + (3/4)k}{(1 - 1/2^2)^3} \prod_{p \mid 2q} \left(1 + \frac{1}{p^2}\right)
= \frac{5 + 3k}{5} \prod_{p} \left(1 + \frac{1}{p^2}\right) \prod_{p \mid 2q} \left(1 - \frac{1}{p^2}\right)^3
= \frac{5 + 3k}{5} \cdot \zeta(2) \left(\sum_{p \mid q} \left(1 - \frac{1}{p^2}\right)^3 \right) \prod_{p \mid q} \left(1 + \frac{1}{p^2}\right),
\]
(66)

Similarly, we can deduce the other identities.

\[\]
For \((r, q) = 1\), from Lemma 15, we have

\[
\sum_{\chi \mod q} \chi(r) = 1 - \frac{1}{2} \sum_{\chi \mod q} (1 - \chi(-1)) \chi(r) \tag{69}
\]

Then from Lemma 16 we get

\[
\sum_{\chi \mod q} \chi(2^k) \left| L(1, \chi) \right|^4 = \left( \sum_{\chi \mod q} \chi(2^k) \right)^4 \sum_{1 \leq n \leq N} \frac{\chi(n) d(n)}{n} + O \left( \frac{q^{1/2} \log q \log N}{N^{1/2}} \right)^2
\]

\[
= \frac{1}{2} \sum_{d \mid (q, r-1)} \mu \left( \frac{q}{d} \right) \phi(d)
\]

\[
- \frac{1}{2} \sum_{d \mid (q, r+1)} \mu \left( \frac{q}{d} \right) \phi(d).
\]
Now taking $N = q^4$, we immediately get
\[
\sum_{\chi \text{ mod } q} * \chi (2^k) |L(1, \chi)|^4
= \frac{5 + 3k}{2^{k+1} \cdot 5} \cdot \frac{\zeta(2)}{\zeta(4)} J(q)
\times \prod_{p|q} \left(1 - \frac{1}{1 + p^2}\right)^3
+ O(q^\epsilon).
\] (72)

Lemma 18. Let $q \geq 2$ be an odd number. For integers $k \geq 0$ and $l \geq 1$, one has
\[
\sum_{\chi \text{ mod } q} * \chi (2^k) |L(1, \chi)|^4
= \frac{15k + 17}{2^{2k+1} \cdot 17} \cdot \frac{\zeta(4)}{\zeta(8)} J(q) \prod_{p|q} \left(1 - \frac{1}{1 + p^4}\right)^3
+ O(q^\epsilon),
\]
\[
\sum_{\chi \text{ mod } q} * \chi (2^k) L^2(2, \chi) L^2(1, \chi \chi^4)
= \frac{1}{2} \cdot \frac{\zeta(4)}{\zeta(6)} \cdot \frac{\zeta(2)}{\zeta(8)} J(q) \prod_{p|q} \left(1 - \frac{1}{1 + \chi_4(p) / p^3}\right)^3
+ O(q^\epsilon),
\]
\[
\sum_{\chi \text{ mod } q} * \chi (2^k) L^2(2, \chi) L(1, \chi \chi_4) L(2, \chi)
= \frac{(15/16)k + 1}{2^{2k+1}} \cdot \frac{\zeta(2)}{\zeta(3)} \cdot \frac{\zeta(4)}{\zeta(6)} \cdot \frac{\zeta(2)}{\zeta(4)} J(q) \prod_{p|q} \left(1 - \frac{1}{1 + \chi_4(p) / p^3}\right)^3
+ O(q^\epsilon),
\]
\[
\sum_{\chi \text{ mod } q} * \chi (2^k) L^2(2, \chi) L(1, \chi \chi_4) L(1, \chi \chi_4)
= \frac{9}{32} \cdot \frac{\zeta(2)}{\zeta(3)} \cdot \frac{\zeta(4)}{\zeta(6)} \cdot \frac{\zeta(2)}{\zeta(4)} \cdot \frac{\zeta(4)}{\zeta(8)} \cdot J(q) \prod_{p|q} \left(1 - \frac{1}{1 + \chi_4(p) / p^3}\right)^3
+ O(q^\epsilon).
\] (73)

Proof. By Lemma 17 and the methods proving Lemma 18, we can get this lemma. □

5. Proof of Theorems 3 and 4

First we prove Theorem 3. By Theorem 11 and Lemma 18, we have
\[
\sum_{\chi \text{ mod } q} * \chi (2^k) |L(1, \chi \chi_4)|^4
= \frac{27}{160} \cdot \frac{\zeta(2)}{\zeta(4)} \cdot \frac{\zeta(4)}{\zeta(8)} \cdot J(q) \prod_{p|q} \left(1 - \frac{1 - p^2}{1 + p^2}\right)^3
+ O(q^\epsilon),
\]
\[
\sum_{\chi \text{ mod } q} * \chi (2^k) L^2(1, \chi \chi_4) L(1, \chi_4) L(2, \chi)
= \frac{9}{32} \cdot \frac{\zeta(2)}{\zeta(3)} \cdot \frac{\zeta(4)}{\zeta(6)} \cdot \frac{\zeta(2)}{\zeta(4)} \cdot \frac{\zeta(4)}{\zeta(8)} \cdot J(q) \prod_{p|q} \left(1 - \frac{1 - p^2}{1 + p^2}\right)^3
+ O(q^\epsilon),
\]
\[
\sum_{\chi \text{ mod } q} * \chi (2^k) |L(1, \chi \chi_4)|^4
= \frac{9}{32} \cdot \frac{\zeta(2)}{\zeta(3)} \cdot \frac{\zeta(4)}{\zeta(6)} \cdot \frac{\zeta(2)}{\zeta(4)} \cdot \frac{\zeta(4)}{\zeta(8)} \cdot J(q) \prod_{p|q} \left(1 - \frac{1 - p^2}{1 + p^2}\right)^3
+ O(q^\epsilon).
\] (74)

Proof. By Lemma 17 and the methods proving Lemma 18, we can get this lemma. □
\[
\begin{align*}
&= \frac{q^6}{8^4 \pi^4} \sum_{\chi \equiv 1 \mod q} L(1, \chi) L^2(1, \chi) + \frac{1}{16} L^2(2, \chi \chi_4) + \frac{8}{\pi^2} \sum_{\chi \equiv 1 \mod q} L(1, \chi) L(2, \chi \chi_4) \\
&+ \frac{8}{\pi} L(4, \chi) L(1, \chi) L(2, \chi \chi_4) + \frac{8}{\pi^2} \sum_{\chi \equiv 1 \mod q} \left| L(1, \chi) \right|^4 \left| L(2, \chi \chi_4) \right|^2 \\
&= \frac{3}{2048 \pi^4} \sum_{\chi \equiv 1 \mod q} L(1, \chi) L^2(1, \chi) + \frac{1}{512 \pi^4} \sum_{\chi \equiv 1 \mod q} L^2(2, \chi \chi_4) \\
&+ \frac{1}{16 \pi^4} \sum_{\chi \equiv 1 \mod q} \left| L(1, \chi) \right|^4 \left| L(2, \chi \chi_4) \right|^2 \\
&= \frac{7}{2^{17} \cdot 3^2 \pi^4} J(q) \prod_{p \mid q} \left( 1 - \frac{1}{p^2} \right)^3 \\
&+ \frac{35}{2^{16} \cdot 3^2 \cdot \pi^4} \sum_{p \mid q} \left( 1 - \frac{1}{p^4} \right)^3 \\
&- \frac{L^2(3, \chi_4)}{2^{14} \cdot 3^2 \cdot \pi \cdot \nu(3, \chi_4)} \\
&\times q^6 J(q) \prod_{p \mid q} \left( 1 - \frac{1}{p^2} \right)^3 \left( 1 - \frac{\chi_4(p) / p^3}{p^2} \right)^2 \\
&+ \frac{\nu(3, \chi_4)}{2^{28} \cdot 3^2 \cdot \pi^2} \sum_{p \mid q} \left( 1 - \frac{\chi_4(p) / p^3}{p^2} \right)^2 \\
&\times \prod_{p \mid q} \left( 1 - \frac{1}{p^2} \right)^3 \\
&\times \prod_{p \mid q} \left( 1 + \frac{1}{p^2 (1 - \chi_4(p) / p^3)} \right) + O(q^{6+\epsilon}).
\end{align*}
\]

This proves Theorem 3.

On the other hand, by Theorem 14 and Lemma 18, we have

\[
\begin{align*}
&= \frac{q^6}{16^4 \pi^8} \sum_{\chi \equiv 1 \mod q} L(1, \chi) L^2(2, \chi \chi_4) \\
&= \frac{q^6}{16^4 \pi^8} \sum_{\chi \equiv 1 \mod q} \left| L(1, \chi) \right|^4 \\
&= \frac{q^6}{16^4 \pi^8} \sum_{\chi \equiv 1 \mod q} \left| L(2, \chi \chi_4) \right|^2
\end{align*}
\]
\[ + 16\pi^2 L^2 (1, \chi \chi_4) + 8\pi (\chi(4) - 2\chi(2) - 8) \times L (1, \chi \chi_4) L (2, \chi) \]
\[ - \frac{59}{1024\pi^2} q^6 \sum_{\chi \mod q \chi(-1) = 1} \chi(2) L^2 (2, \chi) L (1, \chi \chi_4) L (2, \chi) \]
\[ + \frac{105}{4096\pi^2} q^6 \sum_{\chi \mod q \chi(-1) = 1} \chi(4) L^2 (2, \chi) L (1, \chi \chi_4) L (2, \chi) \]
\[ + \frac{15}{2048\pi^2} q^6 \sum_{\chi \mod q \chi(-1) = 1} \chi(8) L^2 (2, \chi) L (1, \chi \chi_4) L (2, \chi) \]
\[ - \frac{1}{512\pi^2} q^6 \sum_{\chi \mod q \chi(-1) = 1} \chi(16) L^2 (2, \chi) L (1, \chi \chi_4) L (2, \chi) \]
\[ + \frac{1}{256\pi^2} q^6 \sum_{\chi \mod q \chi(-1) = 1} \chi(4) L^2 (1, \chi \chi_4) L (1, \chi \chi_4) L (2, \chi) \]
\[ - \frac{1}{128\pi^2} q^6 \sum_{\chi \mod q \chi(-1) = 1} \chi(2) L^2 (1, \chi \chi_4) L (1, \chi \chi_4) L (2, \chi) \]
\[ - \frac{1}{32\pi^2} q^6 \sum_{\chi \mod q \chi(-1) = 1} L^2 (1, \chi \chi_4) L (1, \chi \chi_4) L (2, \chi) \]
\[ + \frac{69}{1024\pi^2} q^6 \sum_{\chi \mod q \chi(-1) = 1} |L(1, \chi \chi_4)|^2 |L(2, \chi)|^2 \]
\[ + \frac{7}{256\pi^2} q^6 \sum_{\chi \mod q \chi(-1) = 1} \chi(2) |L(1, \chi \chi_4)|^2 |L(2, \chi)|^2 \]
\[ - \frac{1}{64\pi^6} q^6 \sum_{\chi \mod q \chi(-1) = 1} \chi(4) |L(1, \chi \chi_4)|^2 |L(2, \chi)|^2 \]
\[ = \frac{385}{2^{10} \cdot 51} \cdot q^6 f (q) \prod_{p \nmid q} \left( \frac{1 - 1/p^4}{1 + 1/p^4} \right)^3 \]
\[ + \frac{151 L^4 (3, \chi_4)}{\pi^{12}} \cdot q^6 f (q) \prod_{p \mid q} \left( \frac{1 - \chi_4 (p) / p^3}{1 + \chi_4 (p) / p^3} \right)^3 \]
\[ + \frac{1423 n L^2 (3, \chi_4)}{2^{16} \cdot 5^2 \cdot 3^4 \cdot L (7, \chi_4)} \cdot q^6 f (q) \prod_{p \mid q} \left( \frac{1 - \chi_4 (p) / p^3}{1 - \chi_4 (p) / p^2} \right)^2 \]
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\[ + \frac{3}{2^{12}} \cdot q^6 f(q) \prod_{\text{p}|q} \left( \frac{1 - 1/p}{1 + 1/p^3} \right)^3 \]
\[ - \frac{L^2(3, \chi_4)}{2^{12} \cdot \pi \cdot L(5, \chi_4)} \]
\[ \cdot q^6 f(q) \prod_{\text{p}|q} \left( \frac{1 - 1/p^2}{1 - \chi_4(p)/p^3} \right)^2 \]
\[ + \frac{L(3, \chi_4)}{2^{10} \cdot 3 \cdot \pi^2} \cdot q^6 f(q) \prod_{\text{p}|q} \left( 1 - \chi_4(p)/p^3 \right) \left( 1 - 1/p^4 \right) \]
\[ \times \prod_{\text{p}|q} \left( 1 + \frac{1}{p^2 (1 - \chi_4(p)/p)} \right) \]
\[ + O \left( q^{6+\epsilon} \right). \]

(75)

This completes the proof of Theorem 4.

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