INITIAL-BOUNDARY VALUE PROBLEMS IN A RECTANGLE
FOR TWO-DIMENSIONAL ZAKHAROV–KUZNETSOV
EQUATION

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Abstract. Initial-boundary value problems in a bounded rectangle with dif-
ferent types of boundary conditions for two-dimensional Zakharov–Kuznetsov
equation are considered. Results on global well-posedness in the classes of
weak and regular solution are established. As applications of the developed
technique results on boundary controllability and long-time decay of weak so-
lutions are also obtained.

1. Introduction. Description of main results

The two dimensional Zakharov–Kuznetsov equation (ZK)
\[ u_t + bu_x + u_{xxx} + u_{xyy} + uu_x = f(t, x, y) \] (1.1)
(b is a real constant) is one of the variants of multi-dimensional generalizations
of Korteweg–de Vries equation (KdV) \( u_t + bu_x + u_{xxx} + uu_x = f(t, x) \). For the
first time it was derived in the three-dimensional case in [37] for description of
ion-acoustic waves in magnetized plasma. The equation, considered in the present
paper, is known as a model of two-dimensional nonlinear waves in dispersive media
propagating in one preassigned (x) direction with deformations in the transverse
(y) direction. A rigorous derivation of the ZK model can be found, for example,
in [20, 22].

From the point of view of solubility and well-posedness the most signifi-
cant results for ZK equation and its generalizations were obtained for the initial value
problem. In the two-dimensional case the corresponding results in different func-
tional spaces can be found in [34, 5, 6, 2, 27, 28, 32, 16, 19, 31, 17, 18]. For
initial-boundary value problems such a theory is most developed for domains, where
the variable \( y \) is considered in the whole line, (7, 8, 10, 35, 12, 4).

Initial-boundary value problems posed on domains, where the variable \( y \) is con-
sidered on a bounded interval, are studied less, although from the physical point of
view they seem at least the same important. Certain technique developed for
the case \( y \in \mathbb{R} \) (especially related to the investigation of the corresponding linear
equation) up to this moment is extended to the case of bounded \( y \) only partially.
An initial-boundary value problem in a strip \( \mathbb{R} \times (0, L) \) with periodic boundary
conditions was considered in [29] for ZK equation and local well-posedness result
was established in the spaces \( H^s \) for \( s > 3/2 \). This result was improved in [31]
where $s \geq 1$, in addition, in the space $H^1$ appropriate conservation laws provided global well-posedness. Initial-boundary value problems in such a strip with homogeneous boundary conditions of different types – Dirichlet, Neumann or periodic – were considered in [1] [14] and results on global well-posedness in classes of weak solutions with power and exponential weights at $+\infty$ were established. Global well-posedness results for ZK equation with certain parabolic regularization also for the initial-boundary value problem in a strip $\mathbb{R} \times (0, L)$ with homogeneous Dirichlet boundary conditions can be found in [13, 14, 24, 25].

Similar results on global well-posedness in weighted spaces for initial-boundary value problems in a half-strip $\mathbb{R}_+ \times (0, L)$ were obtained in [26, 23, 15].

Initial-boundary value problems in a bounded rectangle were studied in [36, 4]. In [4] similar results in more regular classes for homogeneous boundary conditions of different types – Dirichlet, Neumann or periodic – were established. In [36] either homogeneous Dirichlet or periodic boundary conditions with respect to $y$ were considered and results on global existence and uniqueness of weak solutions were established. In [4] similar results in more regular classes for homogeneous Dirichlet boundary conditions were obtained. In both papers boundary conditions with respect to $x$ were homogeneous.

In the present paper we consider initial-boundary value problems in a domain $Q_T = (0, T) \times \Omega$, where $\Omega = (0, R) \times (0, L) = \{(x, y) : 0 < x < R, 0 < y < L\}$ is a bounded rectangle of given length $R$ and width $L$, $T > 0$ is arbitrary, for equation (1.1) with an initial condition

$$u(0, x, y) = u_0(x, y), \quad (x, y) \in \Omega,$$

boundary conditions for $(t, y) \in B_T = (0, T) \times (0, L)$

$$u(t, 0, y) = \mu_0(t, y), \quad u(t, R, y) = \nu_0(t, y), \quad u_x(t, R, y) = \nu_1(t, y)$$

and boundary conditions for $(t, x) \in (0, T) \times (0, R)$ of one of the following four types:

whether

1. $u(t, x, 0) = u(t, x, L) = 0,$
2. $u_y(t, x, 0) = u_y(t, x, L) = 0,$
3. $u(t, x, 0) = u_y(t, x, L) = 0,$
4. $u$ is an $L$-periodic function with respect to $y$.

We use the notation ”problem (1.1) – (1.4)” for each of these four cases.

The main results consist of theorems on global well-posedness in classes of weak and regular solutions. Besides that, certain results on large-time decay of small solutions and boundary controllability, when $\mu_0 = \nu_0 \equiv 0$, $f \equiv 0$, are established.

In what follows (unless stated otherwise) $j, k, l, m, n$ mean non-negative integers, $p \in [1, +\infty], s \in \mathbb{R}$. Let $[s]$ be the integer part of $s$ ($s - [s] \in [0, 1)$).

For any multi-index $\alpha = (\alpha_1, \alpha_2)$ let $\partial^\alpha = \partial^\alpha_x \partial^\alpha_y$, let

$$|D^k \varphi| = \left( \sum_{|\alpha| \leq k} (\partial^\alpha \varphi)^2 \right)^{1/2}, \quad |D \varphi| = |D^1 \varphi|.$$

Let $L_p = L_p(\Omega)$, $W_p^k = W_p^k(\Omega)$, $H^s = H^s(\Omega)$.

Introduce special function spaces taking into account boundary conditions (1.4).

Let $\Sigma = \mathbb{R} \times (0, L)$, $\tilde{S}(\Sigma)$ be a space of infinitely smooth on $\Sigma$ functions $\varphi(x, y)$ such that $(1 + |x|^n) |\partial^\alpha \varphi(x, y)| \leq c(n, \alpha)$ for any $n$, multi-index $\alpha$, $(x, y) \in \Sigma$ and $\partial^2 \varphi \big|_{y=0} = \partial^2 \varphi \big|_{y=L} = 0$ in the case a), $\partial^2 \varphi \big|_{y=0} = \partial^2 \varphi \big|_{y=L} = 0$ in the case d).
in the case b), \( \partial_g^2 \varphi|_{y=0} = \partial_g^3 \varphi|_{y=L} = 0 \) in the case c), \( \partial_g^m \varphi|_{y=0} = \partial_g^m \varphi|_{y=L} \) in the case d) for any \( m \).

Let \( \tilde{H}^s(\Sigma) \) be the closure of \( \tilde{S}(\Sigma) \) in the norm \( H^s(\Sigma) \) and \( \tilde{H}^s(I \times (0, L)) \) be the restriction of \( \tilde{H}^s(\Sigma) \) on \( I \times (0, L) \) for any interval \( I \subset \mathbb{R} \), \( \tilde{H}^s = H^s(\Omega) \).

It is easy to see, that \( \tilde{H}^0 = L_2 \); \( \tilde{H}^s = H^s \) if \( s < 0 \); for \( j \geq 1 \) in the case a) \( \tilde{H}^j = \{ \varphi \in H^j : \partial_y^m \varphi|_{y=0} = \partial_y^m \varphi|_{y=L} = 0, 2m < j \} \), in the case b) \( \tilde{H}^j = \{ \varphi \in H^j : \partial_y^m \varphi|_{y=0} = \partial_y^m \varphi|_{y=L} = 0, 2m + 1 < j \} \), in the case d) \( \tilde{H}^j = \{ \varphi \in H^j : \partial_y^m \varphi|_{y=0} = \partial_y^m \varphi|_{y=L}, m < j \} \).

We also use an anisotropic Sobolev space \( \tilde{H}^{(0,k)}(\Sigma) \) which is defined as the restriction on \( \Omega \) of a space \( \tilde{H}^{(0,k)}(\Sigma) \), where the last space is the closure of \( \tilde{S}(\Sigma) \) in the norm \( \sum_{m=0}^k \| \partial_y^m \varphi \|_{L_2(\Sigma)} \).

We construct solutions to the considered problems in spaces \( X^k(Q_T) \) for \( k = 0 \) and \( k = 3 \), consisting of functions \( u(t, x, y) \), such that

\[
\partial_t^3 u \in C([0, T]; \tilde{H}^{k-3j}) \cap L_2(0, T; \tilde{H}^{k-3j+1}) \tag{1.5}
\]

if \( k - 3j \geq 0 \), let \( X(Q_T) = X^0(Q_T) \).

For description of properties of the boundary data introduce anisotropic functional spaces. Let \( B = \mathbb{R}^t \times (0, L) \). Define the functional space \( \tilde{S}(\mathbb{R}) \) similarly to \( \tilde{S}(\Sigma) \), where the variable \( x \) is substituted by \( t \). Let \( \tilde{H}^{s/3, s}(B) \) be the closure of \( \tilde{S}(\mathbb{R}) \) in the norm \( H^{s/3, s}(B) \).

More exactly, let \( \psi(t, y) \), \( t = 1, 2, \ldots \), be the orthonormal in \( L_2(0, L) \) system of the eigenfunctions for the operator \( (-\psi'') \) on the segment \([0, L]\) with corresponding boundary conditions \( \psi(0) = \psi(L) = 0 \) in the case a), \( \psi'(0) = \psi'(L) = 0 \) in the case b), \( \psi(0) = \psi'(L) = 0 \) in the case c), \( \psi(0) = \psi(L), \psi'(0) = \psi'(L) \) in the case d), \( \lambda_\mu \) be the corresponding eigenvalues. Such systems are well-known and are written in trigonometric functions.

For any \( \mu \in \tilde{S}(\mathbb{R}), \theta \in \mathbb{R} \) and \( l \) let

\[
\hat{\mu}(\theta, l) = \int_B e^{-i\theta t} \psi(t, y) \mu(t, y) \, dt \, dy. \tag{1.6}
\]

Then the norm in \( H^{s/3, s}(B) \) is defined as

\[
\left( \sum_{1=1}^{+\infty} \left| \left( \frac{|\theta|^2 + l^2}{2} \right)^{s/2} \hat{\mu}(\theta, l) \right|^2 \right)^{1/2}
\]

and the norm in \( H^{s/3, s}(I \times (0, L)) \) for any interval \( I \subset \mathbb{R} \) as the restriction norm.

The use of these norms is justified by the following fact. Let \( v(t, x, y) \) be the appropriate solution to the initial value problem

\[
v_t + v_{xxx} + v_{xyy} = 0, \quad v|_{t=0} = v_0.
\]

Then according to [11] uniformly with respect to \( x \in \mathbb{R} \)

\[
\left| D^{1/3}_t v \|_{H^{s/3, s}(\mathbb{R})}^2 + \| D_x v \|_{H^{s/3, s}(\mathbb{R})}^2 + \| D_y v \|_{H^{s/3, s}(\mathbb{R})}^2 \right) \sim \| v_0 \|_{H^s(\mathbb{R})}^2. \tag{1.7}
\]

Introduce the notion of weak solutions to the considered problems.

**Definition 1.1.** Let \( u_0 \in L_2, \mu_0, v_0, v_1 \in L_2(B_T), f \in L_1((0, T); L_2) \). A function \( u \in L_\infty((0, T); L_2) \) is called a generalized solution to problem [1.1] if for
any function $\phi \in L_2(0,T;\bar{H}^2)$, such that $\phi_t, \phi_{xxx}, \phi_{xyy} \in L_2(Q_T)$, $\phi|_{t=T} \equiv 0$, $\phi|_{x=0} = \phi_x|_{x=0} = \phi_x|_{x=R} = 0$, the following equality holds:

$$
\iint_{Q_T} u(\phi_t + b\phi_x + \phi_{xxx} + \phi_{xyy}) + \frac{1}{2}u^2\phi_x + f\phi \, dx dy dt + \iint_{\Omega} u_0\phi|_{t=0} \, dx dy + \int_{B_T} [\mu_0\phi_{xx}|_{x=0} - \nu_0\phi_{xx}|_{x=R} + \nu_1\phi_x|_{x=R}] \, dt = 0. \quad (1.8)
$$

**Remark 1.2.** Note that the integrals in (1.8) are well defined (in particular, since $\phi_x \in L_2(0,T;H^2) \subset L_2(0,T;L_\infty)$).

Now we can formulate the main results of the paper concerning well-posedness, which means existence, uniqueness of solutions and Lipschitz continuity of the map $(u_0, \mu_0, \nu_0, \nu_1, f) \mapsto u$ in the corresponding norms on any ball in the space of the input data.

**Theorem 1.3.** Let $u_0 \in L_2$, $f \in L_1(0,T;L_2)$ for certain $T > 0$, $\mu_0, \nu_0 \in \bar{H}^{s/3,3}(B_T)$ for certain $s > 3/2$, $\nu_1 \in L_2(B_T)$. Then problem (1.1)–(1.3) is well-posed in the space $X(Q_T)$.

**Remark 1.4.** In the cases a) and d) for $\mu_0 = \nu_0 = \nu_1 \equiv 0$ similar result was established in [36]. In the last paper certain properties of traces of $u_x$ with respect to $x$ were also obtained.

**Theorem 1.5.** Let $u_0 \in \bar{H}^3$, $f \in C([0,T];L_2) \cap L_2(0,T;\bar{H}^{0,2})$, $f_t \in L_1(0,T;H^{-1})$ for certain $T > 0$, $\mu_0, \nu_0 \in \bar{H}^{4/3,4}(B_T)$, $\nu_1 \in \bar{H}^{1,3}(B_T)$, $\mu_0(0,y) \equiv u_0(0,y)$, $\nu_0(0,y) \equiv u_0(R,y)$, $\nu_1(0,y) \equiv u_0x(R,y)$. Then problem (1.1)–(1.4) is well-posed in the space $X^3(Q_T)$.

**Remark 1.6.** According to [14] the assumptions on the boundary data $\mu$ are natural. In [4] for construction of regular solutions only homogeneous Dirichlet boundary conditions were considered. Moreover, in that paper for $u_{yyy}$ was established only that $u_{yyy} \in L_2(Q_T)$.

Estimates on solutions, established in the proof of Theorem 1.3 provide the following result on the large-time decay of small solutions. Let $B_+ = \mathbb{R}_+^4 \times (0,L)$. 

**Theorem 1.7.** Let there exists $\delta \in (0,1)$ such that $\varkappa > 0$, where

$$
\varkappa = -b + \begin{cases} 
\pi^2(1-\delta)(\frac{3}{R^2} + \frac{1}{L^2}) & \text{in the case a)}, \\
\pi^2(1-\delta)(\frac{3}{R^2} + \frac{1}{4L^2}) & \text{in the case c)}, \\
\pi^2(1-\delta)\frac{3}{R^2} & \text{in the cases b) and d)}. 
\end{cases}
$$

Let

$$
\epsilon_0 = \frac{3^{5/4} \pi \delta}{4} \times \begin{cases} 
\max\left(\frac{\sqrt{3}}{R}, \frac{1}{L}\right) & \text{in the case a)}, \\
\max\left(\frac{\sqrt{3}}{R}, \frac{1}{2L}\right) & \text{in the case c)}, \\
\frac{\sqrt{3}}{R} \times \frac{3^{1/4}(\pi L)^{1/2}}{R^{1/2} + 3^{1/4}(\pi L)^{1/2}} & \text{in the cases b) and d)}. 
\end{cases}
$$

Let $u_0 \in L_2$, $\nu_1 \in L_2(B_+)$, $f \equiv 0$, $\mu_0 = \nu_0 \equiv 0$. Then the corresponding unique weak solution $u(t, x, y)$ to problem (1.11)–(1.14) from the space $X(Q_T)$ for $T > 0$ satisfies an inequality
\[
\|u(t, \cdot, \cdot)\|_{L_2}^2 \leq (1 + R)e^{-\varepsilon t/(1 + R)} \left[\|u_0\|_{L_2}^2 + \|\nu_1\|_{L_2(B_+)}^2\right] \forall t \geq 0.
\] (1.11)

Remark 1.8. In the case a) if $b = 1$, $\nu_1 \equiv 0$ a similar result for regular solutions in a slightly different form was previously established in [4].

On the basis of ideas and results from [33] as an application of the developed technique we obtain the following result on the controllability problem for system (1.4)–(1.4) with the unknown boundary control $\nu_1$ and with the condition of final overdetermination
\[
u(T, x, y) = u_T(x, y), \quad (x, y) \in \Omega.
\] (1.12)

Theorem 1.9. Let for any natural $l$, such that $\lambda_l < b$ (where $\lambda_l$ are the aforementioned eigenvalues of the operator $(-\psi''')$ on $(0, L)$ with corresponding boundary conditions),
\[
R \neq 2\pi \left(\frac{k^2 + km + m^2}{3(b - \lambda_l)}\right)^{1/2} \forall k, m \in \mathbb{N}.
\] (1.13)

Let $T > 0$, $f \equiv 0$, $\mu_0 = \nu_0 \equiv 0$, $u_0, u_T \in L_2$. Then there exists $\varepsilon > 0$, such that if $\|u_0\|_{L_2}, \|u_T\|_{L_2} < \varepsilon$ there exists a function $\nu_1 \in L_2(B_T)$, such that there exists a unique solution $u \in X(Q_T)$ to problem (1.11)–(1.14), satisfying (1.12).

Remark 1.10. In comparison with Theorem 1.7 the constant $\varepsilon$ is not evaluated explicitly.

Further, let $\eta(x)$ denotes a cut-off function, namely, $\eta$ is an infinitely smooth non-decreasing function on $\mathbb{R}$ such that $\eta(x) = 0$ when $x \leq 0$, $\eta(x) = 1$ when $x \geq 1$, $\eta(x) + \eta(1 - x) \equiv 1$.

We drop limits of integration in integrals over the rectangle $\Omega$.

The following interpolating inequality specifying the one from [21] is crucial for the study.

Lemma 1.11. Let $\varphi(x, y) \in H^1$ satisfy $\varphi|_{x=0} = 0$ or $\varphi|_{x=R} = 0$, then the following inequalities hold:
\[
\iint \varphi^4 dx dy \leq 4 \left(\iint \varphi_x^2 dx dy\right)^{1/2} \left(\iint \varphi_y dx dy\right)^{1/2} \left(\iint \varphi^2 dx dy\right)^3/2, \quad (1.14)
\]
\[
\iint |\varphi|^3 dx dy \leq 2 \left(\iint \varphi_x^2 dx dy\right)^{1/4} \left(\iint \varphi_y dx dy\right)^{1/4} \left(\iint \varphi^2 dx dy\right)^{3/4}.
\] (1.15)

where $\sigma = 0$ if $\varphi|_{y=0} = 0$ or $\varphi|_{y=L} = 0$ and $\sigma = 1$ in the general case.
Proof. We follow the argument from [21] and start with the following inequality:
\[ \iint \varphi^2 \, dx \, dy \leq \iint |\varphi_x| \, dx \, dy \left( \iint |\varphi_y| \, dx \, dy + \frac{2\sigma}{L} \iint |\varphi| \, dx \, dy \right). \] (1.16)
In fact,
\[ \sup_{x \in (0,R)} |\varphi(x, y)| \leq \int_0^R |\varphi_x(x, y)| \, dx; \]
in the general case \( \varphi(x, y) = \varphi(x, y) \frac{y}{L} + \varphi(x, y) \frac{L - y}{L} \equiv \varphi_1(x, y) + \varphi_2(x, y) \), where
\[ \sup_{y \in (0,L)} |\varphi_j(x, y)| \leq \int_0^L |\varphi_y(x, y)| \, dy + \frac{1}{L} \int_0^L |\varphi(x, y)| \, dy, \]
where either \( \alpha_j(y) \equiv y/L \), or \( \alpha_j(y) \equiv (L - y)/L \), therefore,
\[ \sup_{y \in (0,L)} |\varphi(x, y)| \leq \int_0^L |\varphi_y(x, y)| \, dy + \frac{2\sigma}{L} \int_0^L |\varphi(x, y)| \, dy. \]
Since
\[ \iint \varphi^2(x, y) \, dx \, dy \leq \int_0^L \sup_{x \in (0,R)} |\varphi(x, y)| \, dy \int_0^R \sup_{y \in (0,L)} |\varphi(x, y)| \, dx, \]
we obtain (1.16). Therefore,
\[ \iint \varphi^4 \, dx \, dy \leq \iint |(\varphi^2)_x| \, dx \, dy \left( \iint |(\varphi^2)_y| \, dx \, dy + \frac{2\sigma}{L} \iint \varphi^2 \, dx \, dy \right), \]
whence (1.14) succeeds. Inequality (1.15) obviously follows from (1.14) and Hölder’s inequality. \( \square \)

For the decay results, we need Steklov’s inequalities in the following form: for \( \psi \in H^1_0(0, L) \),
\[ \int_0^L \psi^2(y) \, dy \leq \frac{L^2}{\pi^2} \int_0^L \left( \psi'(y) \right)^2 \, dy, \] (1.17)
for \( \psi \in H^1(0, L) \), \( \psi \mid_{y=0} = 0 \),
\[ \int_0^L \psi^2(y) \, dy \leq \frac{4L^2}{\pi^2} \int_0^L \left( \psi'(y) \right)^2 \, dy. \] (1.18)

In the following obvious interpolating results values of constants are indifferent for our purposes: for \( \varphi \in H^1 \)
\[ \sup_{x \in [0,R]} \int_0^L \varphi^2(x, y) \, dy \leq c \left( \iint \varphi^2_x \, dx \, dy \int \varphi^2 \, dx \, dy \right)^{1/2} + c \iint \varphi^2 \, dx \, dy, \] (1.19)
\[ \|\varphi\|_{L_4} \leq c\|\varphi\|_H^{1/2}\|\varphi\|_L^{1/2} \] (1.20)
and for \( \varphi \in H^2 \)
\[ \|\varphi\|_{L_\infty} \leq c\|\varphi\|_{H^2}. \] (1.21)

Lemma 1.12. For \( k = 1 \) and \( k = 2 \) introduce functional spaces
\[ H^{(-k,0)} = \{ \varphi = \sum_{m=0}^k \partial_x^m \varphi_m : \varphi_m \in L_2 \} \]
endowed with the natural norms. Then for \( j = 1 \) and \( j = 2 \)
\[
\| \partial_x^j \psi \|_{L^2} \leq c(R)(\| \varphi_{x,x} \|_{H^{j-3,0}} + \| \varphi \|_{L^2}). \tag{1.22}
\]

Proof. First consider the case \( j = 2 \). For any \( \psi \in L^2 \) let \( a_0(y) = \int_0^R \psi(x, y) \, dx \),
then \( \| a_0 \|_{L^2(0, L)} \leq c \| \psi \|_{L^2} \). Let \( \omega(x) \in C_0^\infty(0, R) \), \( \| \omega \|_{L^2(0, R)} = 1 \).
Define \( \psi_0(x, y) = \int_0^x \psi(z, y) \, dz - a_0(y) \omega(x) \), then \( \| \psi_0 \|_{L^2}, \| \psi_{0x} \|_{L^2} \leq c \| \psi \|_{L^2} \),
\( \psi_0|_{x=0} = \psi_0|_{x=R} = 0 \), \( \psi = \psi_{0x} + a_0 \omega' \). We have:
\[
\langle \varphi_{xx}, \psi_0 \rangle = -\langle \varphi_{xxx}, \psi_0 \rangle \leq \| \varphi_{xxx} \|_{H^{(-1,0)}} (\| \psi_0 \|_{L^2} + \| \psi_{0x} \|_{L^2})
\leq c \| \varphi_{xxx} \|_{H^{(-1,0)}} \| \psi \|_{L^2},
\]
\[
\langle \varphi_{xx}, a_0 \omega' \rangle = \langle \varphi, a_0 \omega''' \rangle \leq c \| \varphi \|_{L^2} \| \psi \|_{L^2}.
\]
Therefore,
\[
\langle \varphi_{xx}, \psi \rangle \leq c (\| \varphi_{xxx} \|_{H^{(-1,0)}} + \| \varphi \|_{L^2}) \| \psi \|_{L^2}
\]
and (1.22) for \( j = 2 \) follows.

Now let \( j = 1 \). For \( \psi \in L^2 \) define \( a_1(y) = \int_0^R \psi_0(x, y) \, dx \), \( \psi_1(x, y) = \int_0^x \psi_0(z, y) \, dz - a_1(y) \omega(x) \). Then \( \psi = \psi_{1xx} + a_0 \omega' + a_1 \omega'' \) and similarly to the previous case
\[
\langle \varphi_x, \psi_{1xx} \rangle = \langle \varphi_{xxx}, \psi \rangle \leq \| \varphi_{xxx} \|_{H^{(-2,0)}} (\| \psi_1 \|_{L^2} + \| \psi_{1x} \|_{L^2} + \| \psi_{1xx} \|_{L^2})
\leq c \| \varphi_{xxx} \|_{H^{(-2,0)}} \| \psi \|_{L^2},
\]
\[
\langle \varphi_x, a_0 \omega' + a_1 \omega'' \rangle = -\langle \varphi, a_0 \omega''' + a_1 \omega''' \rangle \leq c \| \varphi \|_{L^2} \| \psi \|_{L^2}.
\]
Therefore,
\[
\langle \varphi_x, \psi \rangle \leq c (\| \varphi_{xxx} \|_{H^{(-2,0)}} + \| \varphi \|_{L^2}) \| \psi \|_{L^2},
\]
which finishes the proof. \( \Box \)

The paper is organized as follows. Auxiliary linear problems are considered in Section 2. Section 3 is devoted to the well-posedness results for the original problems. Decay of solutions is studied in Section 4 and boundary controllability in Section 5.

2. Auxiliary linear problems

Consider a linear equation
\[
u_t + b \nu_x + u_{xxx} + u_{xyy} = f(t, x, y). \tag{2.1}
\]
For any interval \( I \subset \mathbb{R}^x \) and \( k \) introduce functional spaces
\[
Y_k((0, T) \times I \times (0, L)) = \{ u(t, x, y) : \partial_t^j \partial_x^k u \in C([0, T]; \tilde{H}^{k-3j}(I \times (0, L))) \}
\]
if \( j \leq k/3 \),
\[
\partial_x^k u \in C_0^1(I; \tilde{H}^{(k-n+1)/3, k-n+1}(B_T)), \quad \text{if } n \leq k+1
\]
(here and further the lower index 'b' means a bounded map),
\[
M_k((0, T) \times I \times (0, L)) = \{ f(t, x, y) : \partial_t^j f \in L^2(0, T; \tilde{H}^{k-3j}(I \times (0, L))) \}
\]
if \( j \leq j_0 = [(k+1)/3] \).
Lemma 2.1. A unique solution \( u \) where the functions \( x, y \) with the initial profile (1.2) for \( (t, x, y) \) in (2.8) is achieved if

\[
\Phi_k(x, y) = \Phi_{k+1}(x, y) = \Phi_{k+2}(x, y).
\]

Solutions to an initial-boundary value problem in a domain \( \Pi_T = (0, T) \times \Sigma \) with the initial profile (1.2) for \( (x, y) \) in \( \Sigma \) and boundary conditions (1.4) for \( (t, x) \) in \( (0, T) \times \mathbb{R} \) for equation (2.1) can be constructed in a form (see [15])

\[
u(t, x, y; u_0) = S(t, x, y; u_0) + K(t, x, y; f),
\]

where potentials \( S \) and \( K \) are given by formulas

\[
S(t, x, y; u_0) = \sum_{l=1}^{+\infty} \frac{1}{2\pi} \int_{\mathbb{R}} e^{i(t\xi^2 - b\xi + \lambda_0 \xi_0)} e^{i\xi y} \tilde{u}_0(\xi, l) \, d\xi \psi_l(y),
\]

\[
K(t, x, y; f) = \int_{0}^{t} S(t - \tau, x, y; f(\tau, \cdot, \cdot)) \, d\tau,
\]

where the functions \( \tilde{u}_0(\xi, l) \) are defined similarly to (1.6).

**Lemma 2.1.** If \( u_0 \in \tilde{H}^k(\Sigma) \), \( f \in M_k(\Pi_T) \) for some \( T > 0 \) and \( k \geq 0 \), then a unique solution \( u(t, x, y) \) in \( Y_k(\Pi_T) \) to problem (2.1), (1.2), (1.4) exists and for any \( t_0 \in (0, T] \)

\[
\|u\|_{Y_k(\Pi_{t_0})} \leq c(T, k, b) \left( \|u_0\|_{\tilde{H}^k(\Sigma)} + t_0^{1/6} \|f\|_{M_k(\Pi_{t_0})} + \sum_{j=0}^{j_0-1} \|\phi_j f\|_{L_1(0, t_0; \tilde{H}^{k-3j}(\Sigma))} \right).
\]

**Proof.** First of all note that uniqueness of solutions to the considered problem in the space \( L_2(\Pi_T) \) (in fact, in a more wide class) was established in [11]. Next, note that

\[
\phi_j^T S(t, x, y; u_0) + \phi_j^T K(t, x, y; f) = S(t, x, y; \Phi_j) + K(t, x, y; \phi_j f).
\]

Then the corresponding estimates on \( \phi_j^T u \) in the norm \( C([0, t_0]; \tilde{H}^{k-3j}(\Sigma)) \) by \( \|\Phi_j\|_{\tilde{H}^{k-3j}(\Sigma)} \) and \( \|\phi_j f\|_{L_1(0, t_0; \tilde{H}^{k-3j}(\Sigma))} \) easily follow. In turn,

\[
\|\Phi_j\|_{\tilde{H}^{k-3j}(\Sigma)} \leq c(k, b) \left( \|u_0\|_{\tilde{H}^k(\Sigma)} + \sum_{m=0}^{j-1} \|\phi_m f\|_{L_1(0, t_0; \tilde{H}^{k-3(m+1)}(\Sigma))} \right),
\]

It was proved in [15] that for \( s \in [0, 3] \)

\[
\|u\|_{C_s(\mathcal{R}; \tilde{H}^{k+3s}(B_{t_0}))} \leq c(T, b) \left( \|u_0\|_{\tilde{H}^{k+1}(\Sigma)} + t_0^{1/6} \|f\|_{L_2(0, t_0; \tilde{H}^{k-1}(\Sigma))} \right).
\]

Applying (2.5)–(2.7) for \( j = [(k + 1 - n - 1)/3] \leq j_0 \), \( s = k + 1 - n - l - 3j \in [0, 3] \), we derive that

\[
\|\phi_j^T \partial_x^m \partial_y^n u\|_{C_s(\mathcal{R}; \tilde{H}^{k+3s}(B_{t_0}))} \leq \|S(\cdot, \cdot; \cdot, \cdot, \cdot)\|_{C_s(\mathcal{R}; \tilde{H}^{k+3s}(B_{t_0}))}
\]

\[+ \|K(\cdot, \cdot; \cdot, \cdot, \cdot)\|_{C_s(\mathcal{R}; \tilde{H}^{k+3s}(B_{t_0}))} \leq c(T, k, b) \left( \|u_0\|_{\tilde{H}^k(\Sigma)} \right.
\]

\[+ \sum_{m=0}^{j-1} \|\phi_m f\|_{L_1(0, t_0; \tilde{H}^{k-3(m+1)}(\Sigma))} + t_0^{1/6} \|\phi_j f\|_{L_2(0, t_0; \tilde{H}^{k-3j}(\Sigma))}. \]

Finally, it is suffice to note that the minimal value \( 1/6 \) for the degree \( (1/2 - s/6) \) in (2.8) is achieved if \( k + 1 - n - l = 3j + 2 \). \( \square \)
Next, consider an initial-boundary value problem in a domain \( \Pi_T^- = (0, T) \times \Sigma_- \), \( \Sigma_- = \mathbb{R}_- \times (0, L) \) = \{(x, y) : x < 0, 0 < y < L \}, for equation (2.1) with initial condition (1.1) for \((x, y) \in \Sigma_-\) , boundary conditions (1.2) for \((t, x) \in (0, T) \times \mathbb{R}_-\) and
\[
    u(t, 0, y) = \nu_0(t, y), \quad u_x(t, 0, y) = \nu_1(t, y), \quad (t, y) \in B_T. \tag{2.9}
\]
Weak solutions to this problem are understood similarly to Definition 1.1 with obvious changes, moreover, due to the absence of nonlinearity one can take solutions from the space \( L_2(\Pi_T^-) \).

**Lemma 2.2.** A generalized solution to problem (2.1), (1.2), (1.3), (2.9) is unique in the space \( L_2(\Pi_T^-) \).

**Proof.** According to [15] the backward problem in \( \Pi_T^- \) for equation (2.1) with boundary conditions \( u|_{t=T} = 0, \quad u|_{x=0} = 0\) and (1.2) for \( f \in C^0_0(\Pi_T^-) \) has a solution \( u \in C([0, T]; \dot{H}^3(\Sigma_-)), \quad u_t \in C([0, T]; L_2(\Sigma_-)) \), therefore, the desired result is obtained via the standard H"{o}lmgren’s argument. \( \square \)

**Lemma 2.3.** Let \( u_0 \equiv 0, \quad \nu_0, \nu_1 \in C_0^\infty(\mathbb{R}_+), \quad f \equiv 0. \) Then there exists a solution \( u(t, x, y) \) to problem (2.1), (1.2), (1.3), (2.9) such that \( \partial_t^j u \in C_b(\mathbb{R}_+; \dot{H}^n(\Sigma_-)) \) for any \( j \) and \( n \).

**Proof.** Let \( v(t, x, y) \equiv u(t, x, y) - \nu_0(t, y)\eta(x + 1) - \nu_1(t, y)x\eta(x + 1), \) then the original problem is equivalent to the problem of (2.1), (1.2), (1.3), (2.9) type for the function \( v \) with homogeneous initial-boundary conditions and \( f \equiv \nu_0\eta - \nu_1x\eta - b_0\eta' - b_1(x\eta)' - \nu_0\eta'' - \nu_1(x\eta)''' - \nu_0\eta'' - \nu_1\eta''(x\eta)' \).

Let \( \{\varphi_j(x) : j = 1, 2, \ldots\} \) be a set of linearly independent functions complete in the space \( \{\varphi \in H^3(\mathbb{R}_+) : \varphi(0) = 0\} \). We use the Galerkin method and seek an approximate solution in a form \( v_k(t, x, y) = \sum_{j=1}^{\infty} c_{kj}(t)\varphi_j(x)\psi_l(y) \) (remind that \( \psi_l \) are the orthonormal in \( L_2(0, L) \) eigenfunctions for the operator \((-\psi')\) on the segment \([0, L]\) with corresponding boundary conditions) via conditions for \( i, m = 1, \ldots, k, \quad t \in [0, T] \)
\[
    \iint_{\Sigma_-} (v_{kt}\varphi_i(x)\psi_m(y) - v_k(b\varphi_i'\psi_m + \varphi_i''\psi_m + \varphi_i'''\psi_m')) \, dx \, dy - \iint_{\Sigma_-} f\varphi_i\psi_m \, dx \, dy = 0, \tag{2.10}
\]
\( c_{kj}(0) = 0. \) In particular, \( v_k|_{t=0} = 0. \) Moreover, putting in (2.10) \( t = 0, \) multiplying by \( \partial_{kt}^j(0) \) and summing with respect to \( i, m \) we obtain that \( v_{kt}|_{t=0} = 0. \) Next, differentiating (2.10) \( j \) times with respect to \( t \) we derive that
\[
    \iint_{\Sigma_-} (\partial_{t}^{j+1}v_k\varphi_i\psi_m - \partial_{t}^{j}v_k(b\varphi_i'\psi_m + \varphi_i''\psi_m + \varphi_i'''\psi_m')) \, dx \, dy - \iint_{\Sigma_-} \partial_{t}^{j}f\varphi_i\psi_m \, dx \, dy = 0. \tag{2.11}
\]
Then by induction with respect to $j$ we find that $\partial_t^j v_k|_{t=0} = 0$ for all $j$. Since $\psi_m^{(2n)}(y) = (-\lambda_m)^n \psi_m(y)$ it follows from (2.10) and (2.11) that for all $j$ and $n$

$$
\int \int_{\Sigma_+} \left( \partial_t^{j+1} \partial_y^n v_k \varphi_j \psi_m^{(n)} - \partial_t^j \partial_y^n v_k (b \varphi'_m \psi_m^{(n)} + \varphi''_m \psi_m^{(n)} + \varphi'_m \psi_m^{(n+2)}) \right) dxdy = -\int \int_{\Sigma_-} \partial_t^j \partial_y^n f \varphi_j \psi_m^{(n)} dxdy = 0. \tag{2.12}
$$

Multiplying (2.12) by $2c_{km}^{(j)}(t)$ and summing with respect to $i, m$, we find that

$$
\frac{d}{dt} \int \int_{\Sigma_-} (\partial_t^j \partial_y^n v_k)^2 dxdy + \int_0^L (\partial_t^j \partial_y^n v_k x^2) \bigg|_{x=0} dy = 2 \int \int_{\Sigma_-} \partial_t^j \partial_y^n f \partial_t^j \partial_y^n v_k dxdy, \tag{2.13}
$$

and, therefore, for all $j$ and $n$

$$
\|\partial_t^j v_k\|_{L^\infty(\mathbb{R}_+; \tilde{H}^{0,n}((\Sigma_+))} \leq \|\partial_t^j f\|_{L^1(\mathbb{R}_+; \tilde{H}^{0,n}((\Sigma_+)))}. \tag{2.14}
$$

Estimate (2.14) provide existence of a weak solution $v(t, x, y)$ to the considered problem such that $\partial_t^j v \in C_b(\mathbb{R}_+; \tilde{H}^{0,n}((\Sigma_+))) \forall n, j$ in the following sense: for any $T > 0$ and a function $\phi \in L^2(0, T; \tilde{H}^2((\Sigma_-))$ such that $\phi_1, \phi_{xxx}, \phi_{xyy} \in L^2(P_T)$, $\phi|_{t=T} = 0$, $\phi|_{x=0} = 0$, the following equality holds:

$$
\int \int \int_{P_T} \left[ v(\phi_t + b \phi_x + \phi_{xxx} + \phi_{xyy}) + f \phi \right] dxdydt = 0. \tag{2.15}
$$

Note, that the traces of the function $v$ satisfy zero condition (1.2) and condition (1.4). Moreover, it follows from (2.15) that $\partial_t^j \partial_y^n v_{xxx} \in C_b(\mathbb{R}_+; \tilde{H}^{(-1, 0)}((\Sigma_-))) \forall n, j$, therefore, $\partial_t^j \partial_y^n v_x \in C_b(\mathbb{R}_+; L^2((\Sigma_-))) \forall n, j$ (see 14) and one more application of (2.10) yields that $\partial_t^j v_{xxx} \in C_b(\mathbb{R}_+; \tilde{H}^{0,n}((\Sigma_-))) \forall n$, the function $v$ satisfies the corresponding equation (2.1) a.e. in $P_T$ and its traces satisfy zero conditions (2.10). Finally, with the use of induction with respect to $m$ one can find that $\partial_t^j \partial^m v \in C_b(\mathbb{R}_+; \tilde{H}^{0,n})$ for all $m, j, n$.

In what follows, we need some properties of solutions to an algebraic equation

$$
z^3 + az + p = 0, \quad a \in \mathbb{R}, \quad p = \varepsilon + i\theta \in \mathbb{C}. \tag{2.16}
$$

For $\varepsilon > 0$ we denote by $z_1(p, a)$ and $z_2(p, a)$ two roots of this equation with positive real parts (the rest root has the negative real part). Let

$$
r_j(\theta, a) = \lim_{\varepsilon \rightarrow 0^+} z_j(\varepsilon + i\theta), \quad j = 1, 2. \tag{2.17}
$$

The values $r_j(\theta, a)$ are roots of the equation

$$
r^3 + ar + i\theta = 0 \tag{2.18}
$$

and $\Re r_j \geq 0$, $j = 1$ and 2. Moreover, it can be shown with the use of the Cardano formula, that for certain positive constants $c_0, c_1$ and all $\theta$ and $a$

$$
|r_j(\theta, a)| \leq c_1(|\theta|^{1/3} + |a|^{1/2}), \quad j = 1, 2, \tag{2.19}
$$

$$
|r_1(\theta, a) - r_2(\theta, a)| \geq c_0(|\theta|^{1/3} + |a|^{1/2}) \tag{2.20}
$$

(for more details see, for example, 9).
Now introduce special solutions of equation (2.1) for \( f \equiv 0 \) of "boundary potential" type.

**Definition 2.4.** Let \( \nu \in \tilde{S}(\mathcal{B}) \). Define for \( x \leq 0 \)

\[
J_0(t, x, y; \nu) = \sum_{l=1}^{+\infty} \mathcal{F}^{-1}_{t} \left[ \frac{r_1 e^{r_2 x} - r_2 e^{r_1 x}}{r_1 - r_2} \hat{\nu}(\theta, l) \right](t)\psi_l(y), \tag{2.21}
\]

\[
J_1(t, x, y; \nu) = \sum_{l=1}^{+\infty} \mathcal{F}^{-1}_{t} \left[ \frac{e^{r_1 x} - e^{r_2 x}}{r_1 - r_2} \hat{\nu}(\theta, l) \right](t)\psi_l(y), \tag{2.22}
\]

where \( \hat{\nu}(\theta, l) \) is given by formula (1.6) and \( r_j = r_j(\theta, b - \lambda_l) \) — by formula (2.17).

**Lemma 2.5.** For any \( s \in \mathbb{R} \) the notion of the function \( J_0(t, x, y; \nu) \) can be extended by continuity in the space \( C_0([0; \tilde{H}^{s/3, s}(\mathcal{B})]) \) to any function \( \nu \in \tilde{H}^{s/3, s}(\mathcal{B}) \). Moreover, for any \( n \)

\[
\| \partial_n^x J_0(\cdot, \cdot, \cdot; \nu) \|_{C_0([0; \tilde{H}^{s/3, s-n}(\mathcal{B})])} \leq c(n, b)\| \nu \|_{\tilde{H}^{s/3, s-n}(\mathcal{B})} \tag{2.23}
\]

and \( J_0|_{x=0} = \nu, J_{0x}|_{x=0} = 0 \).

**Proof.** Since

\[
\partial_n^x \hat{J}_0(\theta, x, l; \nu) = \frac{r_1 e^{r_2 x} - r_2 e^{r_1 x}}{r_1 - r_2} \hat{\nu}(\theta, l),
\]

and \( \Re(r_j x) \leq 0 \) the assertion of the lemma follows from (2.14), (2.20). \( \square \)

**Lemma 2.6.** For any \( s \in \mathbb{R} \) and \( R > 0 \) the notion of the function \( J_1(t, x, y; \nu) \) can be extended by continuity in the space \( C([-R, 0]; \tilde{H}^{s/3, s}(\mathcal{B})) \) to any function \( \nu \in \tilde{H}^{s/3, s}(\mathcal{B}) \). Moreover,

\[
\| x^{-1} J_1(\cdot, \cdot, \cdot; \nu) \|_{C_0([-R; \tilde{H}^{s/3, s}(\mathcal{B})])} \leq \| \nu \|_{\tilde{H}^{s/3, s}(\mathcal{B})}, \tag{2.24}
\]

for any \( n \geq 1 \)

\[
\| \partial_n^x J_1(\cdot, \cdot, \cdot; \nu) \|_{C_0([-R; \tilde{H}^{s/3, s-n+1}(\mathcal{B})])} \leq c(n, b)\| \nu \|_{\tilde{H}^{s/3, s-n+1}(\mathcal{B})} \tag{2.25}
\]

and \( J_1|_{x=0} = 0, J_{1x}|_{x=0} = \nu \).

**Proof.** Since

\[
\partial_n^x \hat{J}_1(\theta, x, l; \nu) = \frac{r_1 e^{r_1 x} - r_2 e^{r_2 x}}{r_1 - r_2} \hat{\nu}(\theta, l),
\]

and \( \Re(r_j x) \leq 0 \) (in particular, \( |\hat{J}_1(\theta, x, l; \nu)| \leq |x\hat{\nu}(\theta, l)| \)) the assertion of the lemma follows from (2.14), (2.20). \( \square \)

**Remark 2.7.** In the most important for us case \( s \geq 0 \) the values \( \hat{\nu}(\theta, l) \) can be defined directly as limits in \( L_2(\mathcal{B}) \), for example, of integrals

\[
\int_{-T}^{T} \int_{-L}^{L} e^{-i\theta t} \psi_l(t, y) \nu(t, y) dtdy, \quad T \to +\infty.
\]

Then the functions \( J_0(t, x, y; \nu) \) and \( J_1(t, x, y; \nu) \) can be equivalently defined simply by formulas (2.21), (2.22).

**Lemma 2.8.** If \( \nu \in \tilde{H}^{(s+1)/3, s+1}(\mathcal{B}) \) for certain \( s \geq 0 \), then for any \( j \leq s/3 \) there exists \( \partial_j^x J_0(t, x, y; \nu) \in C_0(\mathbb{R}^3; \tilde{H}^{s-3j}(\Sigma_\nu)) \) and uniformly with respect to \( t \in \mathbb{R} \)

\[
\| \partial_j^x J_0(t, \cdot, \cdot; \nu) \|_{\tilde{H}^{s-3j}(\Sigma_\nu)} \leq c(b, s, L)\| \nu \|_{\tilde{H}^{(s+1)/3, s+1}(\mathcal{B})}. \tag{2.26}
\]
If $\nu \in \widetilde{H}^{s/3,s}(B)$ for certain $s \geq 0$, then for any $j \leq s/3$ there exists
\[
\vartheta^J_1(t, x, y; \nu) \in C_b(\mathbb{R}; \widetilde{H}^{s-3j}(\Sigma_\nu))
\]
and uniformly with respect to $t \in \mathbb{R}$,
\[
\|\vartheta^J_1(t, \cdot, \cdot; \nu)\|_{\widetilde{H}^{s-3j}(\Sigma_\nu)} \leq c(b, s, L)\|\nu\|_{\widetilde{H}^{s/3,s}(B)}.
\] (2.27)

**Proof.** The proof is based on the following inequality, established in [9]: let
\[
I(t, x) = \int_{\mathbb{R}} e^{i\theta t} e^{r_j(\theta, a)x} w(\theta) d\theta,
\]
where $r_j(\theta, a)$, $j = 1$ and $2$, are the roots of equation (2.18), defined in (2.17).
Then there exists a positive constant $c$, such that uniformly with respect to $t \in \mathbb{R}$
\[
\|I(t, \cdot)\|_{L_2(\mathbb{R}^s)} \leq c(\|\theta\|^{1/3} + |a|^{1/2})w(\theta)\|_{L_2(\mathbb{R})}.
\] (2.28)

Now let
\[
J(t, x, y) = \sum_{l = 1}^{+\infty} \int_{\mathbb{R}} e^{i\theta t} e^{r_j(\theta, b-x)l} w(\theta, l) d\theta \psi_l^{(m)}(y).
\]
Then it follows from (2.28) that uniformly with respect to $t \in \mathbb{R}$ since the system
\[
\{\psi_l^{(m)}\}
\]
is also orthogonal in $L_2(0, L)$ and $\|\psi_l^{(m)}\|_{L_2(0, L)} \leq c(l/L)^m$
\[
\|J(t, \cdot, \cdot)\|_{L_2(\Sigma_{\nu})} = \left(\sum_{l = 1}^{+\infty} \left\|\int_{\mathbb{R}} e^{i\theta t} e^{r_j(\theta, b-x)l} w(\theta, l) d\theta\right\|_{L_2(\mathbb{R}^\nu)}^2 \right)^{1/2}
\]
\[
\leq c(m, L) \left(\sum_{l = 1}^{+\infty} \left\|\left(\|\theta\|^{1/3} + |b|^{1/2}\right)w(\theta, l)\right\|_{L_2(\mathbb{R}^\nu)}^2 \right)^{1/2}.
\] (2.29)

Without loss of generality one can assume that $\nu \in \overline{S(B)}$. Let $s$ be integer. Then for $3j + n + m = s$
\[
\vartheta^J_1(t, x, y) = \sum_{l = 1}^{+\infty} \frac{1}{2\pi} \int_{\mathbb{R}} \left(\vartheta^J_1(t, x, y; \nu) \right) \right|_{L_2(\Sigma_{\nu})} \leq c\left(\sum_{l = 1}^{+\infty} \left\|\left(\|\theta\|^{1/3} + |b|^{1/2}\right)w(\theta, l)\right\|_{L_2(\mathbb{R}^\nu)}^2 \right)^{1/2}
\]
\[
\leq c\|\nu\|_{\widetilde{H}^{s/3,s}(B)}.
\] (2.30)

Similarly,
\[
\vartheta^J_1(t, x, y; \nu) = \sum_{l = 1}^{+\infty} \frac{1}{2\pi} \int_{\mathbb{R}} \left(\vartheta^J_1(t, x, y; \nu) \right) \right|_{L_2(\Sigma_{\nu})} \leq c\left(\sum_{l = 1}^{+\infty} \left\|\left(\|\theta\|^{1/3} + |b|^{1/2}\right)w(\theta, l)\right\|_{L_2(\mathbb{R}^\nu)}^2 \right)^{1/2}
\]
\[
\leq c\|\nu\|_{\widetilde{H}^{s/3,s}(B)}.
\] (2.31)

Finally, use interpolation. \[\Box\]
Lemma 2.9. Let $\nu \in \tilde{\mathcal{H}}^{s/3,s}(B)$, then for any $T > 0$

$$\|J_1(\cdot, \cdot; \cdot; \cdot; \nu)\|_{C^m(B)} \leq c(T, b, s, L)\|\nu\|_{\tilde{\mathcal{H}}^{s/3,s}(B)}.$$ \hfill (2.32)

Proof. Without loss of generality one can assume that $\nu \in \tilde{\mathcal{S}}(B)$. There exists $l_0$

such that for $l > l_0$ and all $\theta$ and there exists $\theta_0 \geq 1$ such that for $|\theta| \geq \theta_0$ and all $l$

$$|r_1(\theta, b - \lambda_1) - r_2(\theta, b - \lambda_1)| \geq c_0(|\theta|^{1/3} + l).$$ \hfill (2.33)

Divide $\nu$ into two parts:

$$\nu_0(t, y) \equiv \sum_{l=1}^{l_0} \int_t^{\infty} \hat{v}(\theta, l) \xi(\theta_0 + 1 - |\theta|)\mathcal{F}_t \psi_l(y), \quad \nu_1(t, y) \equiv \nu(t, y) - \nu_0(t, y).$$

For $\nu_0$ inequality \hfill (2.27)

yields, that for any $j$ and $m$

$$\|\partial_t^j \partial_y^m J_1(\cdot, \cdot; \cdot; \cdot; \nu_0)\|_{C^m(B(T))} \leq T^{1/2} \sup_{t \in [0, T]} \|\partial_t^j J_1(\cdot, \cdot; \cdot; \cdot; \nu_0)\|_{H^{m+1}(\Sigma_-)} \leq c(T, b, j, m, L)\|\nu\|_{\tilde{\mathcal{H}}^{s/3,s}(B)}.$$ \hfill (2.34)

For $\nu_1$ by virtue of \hfill (2.33)

$$\|J_1(\cdot, \cdot; \cdot; \cdot; \nu_0)\|_{C^m(B(T))} \leq c \left( \sum_{l=1}^{l_0} \|\partial_t^j \partial_y^m J_1(\cdot, \cdot; \cdot; \cdot; \nu_0)\|_{L^2(\mathbb{R}^n)} \right)^{1/2} \leq c_1(b, s)\|\nu\|_{\tilde{\mathcal{H}}^{s/3,s}(B)}.$$ \hfill \Box

Lemma 2.10. Let $\nu_0 \in \tilde{\mathcal{H}}^{1/3,1}(B)$, $\nu_1 \in L_2(B)$ and $\nu_0(t, y) = \nu_1(t, y) = 0$ for $t < 0$, then the function $u(t, x, y) \equiv J_0(0, t, x, y; \nu_0) + J_1(0, t, y; \nu_1)$ for any $T > 0$

is a weak solution from the space $Y_0(\Pi_T)$ to problem \hfill (2.7) \hfill (for $f \equiv 0$), \hfill (1.4) \hfill (for $u_0 \equiv 0$), \hfill (2.9)

Proof. First let $\nu_0, \nu_1 \in C_0^\infty(B_+)$. Consider the smooth solution $u(t, x, y)$ to the considered problem constructed in Lemma 2.3. For any $p = \epsilon + i\theta$, where $\epsilon > 0$, define the Laplace–Fourier transform-coefficients

$$\tilde{u}(p, x, l) \equiv \int_{\mathbb{R}^n} \int_0^T e^{-pt}\mathcal{F}_t \psi_l(y)u(t, x, y) dy dt.$$ \hfill (2.35)

The function $\tilde{u}(p, x, l)$ solves a problem

$$\varepsilon \overline{u}(p, x, l) + \overline{u}_x(p, x, l) + \overline{u}_{xx}(p, x, l) - \lambda \overline{u}_x(p, x, l) = 0,$$

where $z_j = z_j(p, b - \lambda_1)$ are defined in \hfill (2.10) \hfill for $a = b - \lambda_1$. Using the formula of inversion of the Laplace transform we find, that the Fourier coefficients of the function $u(t, x, \cdot)$ are the following:

$$\tilde{u}(t, x, l) = e^{\epsilon t} \mathcal{F}_t^{-1} \left( \frac{z_1 e^{z_2 x} - z_2 e^{z_1 x}}{z_1 - z_2} \tilde{v}_0(\varepsilon + i\theta, l) + \frac{z_1 e^{z_1 x} - z_2 e^{z_2 x}}{z_1 - z_2} \tilde{v}_1(\varepsilon + i\theta, l) \right)(t).$$
Lemma 2.11. Let $\mu \in s$ and for any $j < k/3$, $\mu_j \equiv \tilde{\Phi}_{jx}(0, y)$ for $j < (k - 1)/3$. Then there exists a unique solution $u(t, x, y) \in Y_k(\Pi_T)$ to problem (2.1), (1.2), (1.4) in the class $\tilde{Y}_k(\Pi_T)$ respectively and consider the solution $U(t, x, y)$ to the initial value problem (2.1), (1.2), (1.4) in the class $Y_k(\Pi_T)$ given by Lemma 2.1. Note that

$$
\tilde{v}_0 \equiv v_0 - U|_{x=0} \in \tilde{H}^{(k+1)/3,k+1}(B_T), \quad \tilde{v}_1 \equiv v_1 - U_x|_{x=0} \in \tilde{H}^{k/3,k}(B_T),
$$

and by virtue of the compatibility conditions $\partial_t^j \tilde{v}_0|_{t=0} = 0$ for $j < k/3$, $\partial_t^j \tilde{v}_1|_{t=0} = 0$ for $j < (k - 1)/3$, so the functions $\tilde{v}_0, \tilde{v}_1$ can be extended in the same spaces to the whole strip $B$, such that $\tilde{v}_0(t, y) = \tilde{v}_1(t, y) = 0$ for $t < 0$. Then Lemmas 2.1, 2.10 for the function

$$
u(t, x, y) \equiv U(t, x, y) + J_0(t, x, y; \tilde{v}_0) + J_1(t, x, y; \tilde{v}_1)
$$

provide the desired result. \hfill \square

Lemma 2.12. Let $u_0 \in \tilde{H}^k$, $\mu_0, v_0 \in \tilde{H}^{(k+1)/3,k+1}(B_T)$, $\nu_1 \in \tilde{H}^{k/3,k}(B_T)$, $f \in M_k(Q_T)$ for certain $T > 0$, $k \geq 0$. Assume also that $\partial_t^j \mu_0(0, y) \equiv \tilde{\Phi}_{jx}(0, y)$ for $j < k/3$, $\partial_t^j \nu_1(0, y) \equiv \tilde{\Phi}_{jx}(R, y)$ for $j < (k - 1)/3$. Then there exists a unique solution $u(t, x, y) \in Y_k(Q_T)$ to problem (2.1), (1.2), (1.3) and for any $t_0 \in (0, T)$

$$
\|u\|_{Y_k(Q_{t_0})} \leq c(T, k, b, R, L) \left( \|u_0\|_{\tilde{H}^k} + \|\mu_0\|_{\tilde{H}^{(k+1)/3,k+1}(B_T)} + \|v_0\|_{\tilde{H}^{k/3,k}(B_T)} + \|\nu_1\|_{\tilde{H}^{k/3,k}(B_T)} + \frac{1}{6} \|f\|_{M_k(Q_{t_0})} + \sum_{j=0}^{j_0 - 1} \|\partial_t^j f\|_{t=0} \right),
$$

(2.35)
Proof. Solutions to the considered problem (similarly to the corresponding problem in [10]) are constructed in the form

\[ u(t, x, y) = w(t, x, y) + v(t, x, y), \]  

where \( w(t, x, y) \) is a solution to an initial-boundary value problem in \( \bar{\Pi}^-_T = (0, T) \times \Sigma_-, \Sigma_- = (-\infty, R) \times (0, L) \) for equation (2.1) with initial and boundary conditions (1.2) for \((x, y) \in \Sigma_-\), (1.4) for \((t, x) \in (0, T) \times (-\infty, R)\) and (2.9) (where \( x = 0 \) is substituted by \( x = R \)) in the class \( Y_k(\bar{\Pi}^-_T) \). Then according to (2.34), \( u_0 \) and \( f \) are extended to \( x < 0 \) in a appropriate way)

\[ \|w\|_{Y_k(\bar{\Pi}^-_0)} \leq c(T, k, b, L) \left( \|u_0\|_{\bar{H}^k} + \|v_0\|_{\bar{H}^{(k+1)/3,k+1}(B_T)} + \|\nu_1\|_{\bar{H}^{k/3,k}(B_T)} + f_0^{1/6} \|f\|_{M_k(Q_{\nu_0})} + \sum_{j=0}^{j_0-1} \|\partial_t^j f\|_{t=0} \right) \]  

Moreover,

\[ \bar{\mu}_0(t, y) \equiv \mu_0(t, y) - w(t, 0, y) \in H^{(k+1)/3,k+1}(B_T), \]

by virtue of the compatibility conditions on the line \((0, 0, y)\) \( \partial_t^j \bar{\mu}_0(0, y) = 0 \) for \( j < k/3 \) and

\[ \|\bar{\mu}_0\|_{\bar{H}^{(k+1)/3,k+1}(B_T)} \leq c(T, k, b) \left( \|u_0\|_{\bar{H}^k} + \|\mu_0\|_{\bar{H}^{(k+1)/3,k+1}(B_T)} + \|\nu_1\|_{\bar{H}^{k/3,k}(B_T)} + f_0^{1/6} \|f\|_{M_k(Q_{\nu_0})} + \sum_{j=0}^{j_0-1} \|\partial_t^j f\|_{t=0} \right) \]  

In particular, the function \( \bar{\mu}_0 \) can be considered as extended in the same class to the whole strip \( B \) such that \( \bar{\mu}_0(t, y) = 0 \) for \( t < 0 \).

Consider in \( Q_T \) a problem for the function \( v \):

\[ v_t + b v_x + v_{xxx} + v_{xyy} = 0, \]  

\[ v|_{t=0} = 0, \quad v|_{x=0} = \bar{\mu}_0, \quad v|_{x=R} = v_x|_{x=R} = 0 \]

also with corresponding boundary conditions (1.4). In order to construct a solution to this problem we consider for \( x \geq 0 \) the boundary potential \( J(t, x, y; \mu) \) for an arbitrary function \( \mu \in \bar{H}^{(k+1)/3,k+1}(B), \mu(t, y) = 0 \) for \( t < 0 \). Such a potential was introduced in [15] as a solution to an initial-boundary value problem in \( \Pi^+_T = (0, T) \times \Sigma_+, \Sigma_+ = \mathbb{R}_+ \times (0, L) \), for equation (2.1) in the case \( f \equiv 0 \) with zero initial condition (1.2) for \((x, y) \in \Sigma_+\), boundary condition (1.4) for \((t, x) \in (0, T) \times \mathbb{R}_+ \) and boundary condition

\[ u(t, 0, y) = \mu(t, y), \quad (t, y) \in B_T. \]  

According to [15] the function \( J \) is infinitely differentiable for \( x > 0 \) and for any \( \delta \in (0, T] \)

\[ \|J(\cdot, R, \cdot; \mu)\|_{\bar{H}^{(k+1)/3,k+1}(B_k)} + \|\partial_x J(\cdot, R, \cdot; \mu)\|_{\bar{H}^{k/3,k}(B_k)} \leq c(T, k, b, R, L) \delta^{1/2} \|\mu\|_{L_2(B_k)}. \]  

Moreover, \( \partial_t^j J(0, R, y; \mu) = \partial_t^j J_x(0, R, y; \mu) \equiv 0 \) for all \( j \).
Moreover, for

\[ V(\cdot, 0, \cdot) \|_{\tilde{H}^{(k+1)/3,k+1}(B_k)} \leq c(T, k, b, L) \left( \| J(\cdot, R, \cdot; \mu) \|_{\tilde{H}^{(k+1)/3,k+1}(B_k)} + \| \partial_x J(\cdot, R, \cdot; \mu) \|_{\tilde{H}^{k/3,k}(B_k)} \right). \]

Moreover, it is obvious that \( \partial_t^j V(0,0,y) \equiv 0 \) if \( j < k/3 \).

Consider a linear operator \( \Gamma : \mu \mapsto V(\cdot, 0, \cdot) \) in the space \( \tilde{H}^{(k+1)/3,k+1}(B_k) \), \( \partial_t^0 \mu(0,y) \equiv 0 \) if \( j < k/3 \). For small \( \delta = \delta(T, k, b, L) \) estimates \((2.42)\) and \((2.43)\) provide that the operator \( (E + \Gamma) \) is invertible (\( E \) is the identity operator) and setting \( \mu \equiv (E + \Gamma)^{-1}\mu_0 \) we obtain the desired solution to problem \((2.39), (2.40), (1.4)\).

Thus, the solution \( v(t,x,y) \) to problem \((2.1), (1.2)\) in the domain \( Q\delta \) is constructed and according to \((2.36), (2.38)\) and \((2.44)\) is evaluated in the space \( Y_k(\tilde{Q}\delta) \) by the right part of \((2.35)\). Moving step by step (\( \delta \) is constant) we obtain the desired solution in the whole domain \( Q_T \).

Uniqueness of weak solutions to problem \((2.1), (1.2)\) in \( L_2(Q_T) \) succeeds from existence of smooth solutions to the adjoint problem

\[
\begin{align*}
\phi_t + b\phi_x + \phi_{xxx} + \phi_{xxy} &= f \in C_0^\infty(Q_T), \\
\phi|_{t=T} &= 0, \quad \phi|_{x=0} = \phi|_{x=R} = 0
\end{align*}
\]

and with the corresponding boundary conditions of \((1.4)\) type, which after simple change of variables transforms to the original one. \( \square \)

Remark 2.13. In further lemmas of this section all intermediate argument is performed for smooth solutions constructed in Lemma 2.12 with consequent pass to the limit on the basis of obtained estimates due to linearity of the problem.

Lemma 2.14. Let \( u_0 \in L_2, \mu_0 = \nu_0 \equiv 0, \nu_1 \in L_2(B_T), f \equiv f_0 + f_1, \) where \( f_0 \in L_1(0,T;L_2), \) \( f_1 \in L_2(Q_T) \). Then there exist a (unique) weak solution to problem \((2.1), (1.2)\) from the space \( X(Q_T) \) and a function \( \mu_1 \in L_2(B_T) \), such that for any function \( \phi \in L_2(0,T;H^2), \phi_t, \phi_{xxx}, \phi_{xxy} \in L_2(Q_T), \phi|_{t=T} = 0, \phi|_{x=0} = \phi|_{x=R} = 0, \) the following equality holds:

\[
\begin{align*}
\iint_{Q_T} & u(\phi_t + b\phi_x + \phi_{xxx} + \phi_{xxy}) + f_0 \varphi - f_1 \phi_x \ dx\,dy\,dt \\
& + \iint_{B_T} [\nu_1 \phi_x|_{x=R} - \mu_1 \phi_x|_{x=0}] \ dy\,dt = 0.
\end{align*}
\]

Moreover, for \( t \in (0,T) \)

\[
\| u \|_{X(Q_t)} + \| \mu_1 \|_{L_2(B_t)} \leq c(T,b,R) \left( \| u_0 \|_{L_2} + \| \nu_1 \|_{B_T} + \| f_0 \|_{L_1(0,T;L_2)} \\
+ \| f_1 \|_{L_2(Q_t)} \right),
\]

(2.46)
and if either \( \rho(x) \equiv 1 \) or \( \rho(x) \equiv 1 + x \)

\[
\int_0^t \int_0^L u_x^2(t, x, y) \rho(x) \, dx \, dy + \int_0^t \int_0^L (3u_x^2 + u_y^2) \rho'(x) \, dx \, dy \, d\tau + \rho(0) \int_B \mu_1^2 \, dy \, d\tau
\]

\[
= \int_0^t \int_0^L u_0^2 \rho(x) \, dx \, dy + \rho(R) \int_B \nu_1^2 \, dy + 2 \int_0^t \int_0^L f_0 u_0 \rho(x) \, dx \, dy \, d\tau
\]

\[
- 2 \int_0^t \int_0^L f_1(u \rho(x))_x \, dx \, dy \, d\tau.
\]

**Proof.** Multiplying (2.1) by \( 2u(t, x, y) \rho(x) \) and integrating over \( \Omega \), we find that

\[
\frac{d}{dt} \int_0^L u_0^2 \rho \, dx + \rho(0) \int_0^L u_0^2 \big|_{x=0} \, dx + \int_0^L (3u_x + u_y - bu_0) \rho' \, dx
\]

\[
= \rho(R) \int_0^L \nu_1^2 \, dy + 2 \int_0^L f_0 u_0 \rho \, dx \, dy - 2 \int_0^t \int_0^L f_1 u \rho \, dx \, dy.
\]  

Note that

\[
\left| \int_0^t \int_0^L f_1(u \rho)_x \, dx \, dy \right| \leq c \| f_1 \|_{L_2} \left( \| u_x \| + |u| \right)_{L_2}
\]

\[
\leq \varepsilon \int_0^L \int_0^L (u_x^2 + u_y^2) \, dx \, dy + c(\varepsilon) \| f_1 \|_{L_2}^2.
\]

where \( \varepsilon > 0 \) can be chosen arbitrarily small. Equality (2.48) for \( \rho \equiv 1 + x \) and inequality (2.49) imply that for smooth solutions

\[
\| u \|_{X(Q_T)} + \| u_x \|_{L_2(B_T)} \leq c,
\]

The end of the proof is standard. \( \square \)

**Remark 2.15.** The method of construction of weak solution in Lemma 2.17 via closure ensures that \( u \big|_{x=0} = u \big|_{x=R} = 0 \) in the trace sense (this fact can be also easily derived from equality (2.35), since \( u_x \in L_2(Q_T) \)). Moreover, if \( f \in L_2(Q_T) \) then according to Lemma 2.12 \( u \in Y_0(Q_T) \) and, in particular, \( \mu_1 \equiv u_x \big|_{x=0} \).

**Lemma 2.16.** Let \( u_0 \in \tilde{H}^{(0,1)}, \ \mu_0 = \nu_0 = \nu_1 = 0, \ f \in L_2(Q_T) \). Then for the unique weak solution \( u(t, x, y) \in X(Q_T) \) to problem (2.1), (1.2)–(1.4) \( u_y \in C([0, T]; L_2), \ |Du_y| \in L_2(Q_T) \) and for any \( t \in (0, T] \)

\[
\int_0^t \int_0^L u_y^2(t, x, y) \, dx \, dy + \int_0^t \int_0^L |Du_y|^2 \, dx \, dy \, d\tau
\]

\[
\leq (1 + R) \int_0^L u_y^2 \big|_{x=0} \, dx + b \int_0^t \int_0^L u_y^2 \, dx \, dy \, d\tau - 2 \int_0^t \int_0^L (1 + x) f u_{yy} \, dx \, dy \, d\tau.
\]

**Proof.** Multiply (2.1) by \(-2(1 + x) u_{yy}(t, x, y)\) and integrate over \( \Omega \), then

\[
\frac{d}{dt} \int_0^L (1 + x) u_y^2 \, dx \, dy + \int_0^L u_y^2 \big|_{x=0} \, dx + \int_0^L (3u_{xy}^2 + u_{yy}^2 - bu_x^2) \, dx \, dy
\]

\[
= -2 \int_0^L (1 + x) f u_{yy} \, dx \, dy,
\]

whence the assertion of the lemma obviously follows. \( \square \)
Lemma 2.17. Let $u_0 \in \tilde{H}^2$, $u_0|_{x=0} = u_0|_{x=R} = u_0x|_{x=R} \equiv 0$ and $u_{0xx}, u_{0xy} \in L_2$; $\mu_0 = v_0 = \nu_1 \equiv 0$, $f \in C([0,T]; L_2)$, $f_x \in L_2(0,T; H^{-1})$. Then for the (unique) weak solution to problem (2.1), (1.2)–(1.4) from the space $X(Q_T)$ there exists $u_t \in X(Q_T)$, which is the weak solution to problem of (2.1), (1.2)–(1.4) type, where $f$ is substituted by $f_t$, $u_0 \equiv 0$ (by $(f|_{t=0} - b u_{0x} - u_{0xx} - u_{0xy})$, $\mu_0 = v_0 = \nu_1 \equiv 0$.

Proof. The proof for the function $v \equiv u_t$ is similar to Lemma 2.14. □

Lemma 2.18. Let the hypothesis of Lemma 2.17 be satisfied and, in addition, $f \in L_1(0, T; \tilde{H}^{(0,2)})$. Then there exists a (unique) solution to problem (2.1), (1.2)–(1.4) from the space $X^2(Q_T)$ and for any $t \in [0, T]$

$$
\|u\|_{X^2(Q_t)} \leq c(T, b, R) \left(\|u_{0yy}\|_{L_2} + \|f\|_{C([0,t]; L_2)} + \|u\|_{C([0,t]; L_2)} + \|u_t\|_{C([0,t]; L_2)} + \sup_{\tau \in (0,t)} \left|\int_0^\tau \int_0^T (1 + x) f_{yy} u_{yy} \, dx \, dy \, ds\right|\right). \quad (2.52)
$$

Proof. For smooth solutions differentiating equality (2.1) twice with respect to $y$, multiplying the obtained equality by $2 u_{yy}(t, x, y) \rho(x)$, $\rho(x) \equiv (1 + x)$, and integrating over $\Omega$ we derive, that

$$
\frac{d}{dt} \int \int u_{yy}^2 \rho \, dx \, dy + \int_0^L u_{xyy}^2 \big|_{x=0} \, dy + \int \int (3 u_{xyy}^2 + u_{yy}^2 - b u_{yy}^2) \, dx \, dy = 2 \int \int f_{yy} u_{yy} \rho \, dx \, dy, \quad (2.53)
$$

whence obviously follows that

$$
\|u_{yy}\|_{X(Q_T)} \leq c. \quad (2.54)
$$

Hence, for the weak solution also $u_{yy} \in X(Q_T)$. Lemmas 2.14 and 2.17 provide, that $u, u_t \in X(Q_T)$. Write equality (2.1) in the form

$$
u_{xxx} = f - u_t - bu_x - u_{xyy}. \quad (2.55)
$$

Then, inequality (1.22) for $j = 2$ and (2.55) yield that

$$
\|u_{xx}\|_{L_2} \leq c(R) \left(\|u_{xxx}\|_{H^{-1,0}} + \|u\|_{L_2}\right) \leq c(b, R) \left(\|f\|_{L_2} + \|u_t\|_{L_2} + \|u_{yy}\|_{L_2} + \|u\|_{L_2}\right). \quad (2.56)
$$

Since

$$
\int \int u_{xy}^2 \, dx \, dy = \int \int u_{xx} u_{yy} \, dx \, dy,
$$

estimates (2.54) and (2.56) yield that $u \in C([0,T]; \tilde{H}^2)$ and

$$
\|u(t, \cdot ; y)\|_{\tilde{H}^2} \leq c \left(\|f\|_{L_2} + \|u_t\|_{L_2} + \|u_{yy}\|_{L_2} + \|u\|_{L_2}\right). \quad (2.57)
$$

Next,

$$
\int \int u_{xy}^2 \, dx \, dy = \int \int u_{xxx} u_{xyy} \, dx \, dy + \int_0^L (u_{xyy} u_{xx}) \big|_{x=0} \, dy + \int \int u_{xyy}^2 \, dx \, dy \quad (2.58)
$$

and inequality (1.19) provides that

$$
\int \int u_{xy}^2 \, dx \, dy \leq \int \int (u_{xx}^2 + u_{xy}^2) \, dx \, dy + \int_0^L u_{xyy}^2 \big|_{x=0} \, dy + c \int \int u_{xx}^2 \, dx \, dy. \quad (2.58)
$$

From equality (2.55) we derive that
\[
\iint u_{xxx}^2 \, dxdy \leq c \iint (f^2 + u_x^2 + b^2 u_y^2 + u_{xyy}^2) \, dxdy,
\] (2.59)
and combining (2.53), (2.57)–(2.59) finish the proof. \(\square\)

**Lemma 2.19.** Let the hypothesis of Lemma 2.17 be satisfied and, in addition, \(u_0 \in \dot{H}^3\), \(f \in L_2(0,T; \dot{H}^{0.2})\). Then there exists a (unique) solution to problem (2.1), (1.2)–(1.4) from the space \(X^3(Q_T)\) and for any \(t \in (0,T]\)
\[
\|u\|_{X^3(Q_t)} \leq c(T, b, R, L)(\|u_0\|_{\dot{H}^3} + \|f\|_{C([0,t];L_2)} + \|f\|_{L_2(0,t;\dot{H}^{0.2})})
+ \|f_t\|_{L_2(0,t;\dot{H}^{-1})}.\] (2.60)

**Proof.** First of all note that hypotheses of Lemmas 2.14 (for \(f_1 \equiv 0\)), 2.17 and 2.18 are satisfied. Therefore, taking into account also Remark 2.15 we derive for smooth solutions that
\[
\|u\|_{X^3(Q_T)} + \|u_x\|_{L_2(B_T)} + \|u_t\|_{X(Q_T)} + \|u_{tt}\|_{L_2(B_T)} \leq c.\] (2.61)
Next, differentiating equality (2.1) twice with respect to \(y\), multiplying the obtained equality by \(-2u_{yxyy}(t,x,y)\rho(x)\), \(\rho(x) \equiv (1+x)\) and integrating over \(\Omega\) we derive similarly to (2.53) that
\[
\frac{d}{dt} \iint u_{yxyy}^2 \rho \, dxdy + \int_0^L u_{yxyy}^2 \big|_{x=0} \, dy + \iint (3u_{xxyy} + u_{yxyy}^2 - bu_{xyy}^2) \, dxdy
= -2 \iint f_{yy} u_{yxyy} \rho \, dxdy.\] (2.62)
Here
\[
\left| 2 \iint f_{yy} u_{yxyy} \rho \, dxdy \right| \leq \varepsilon \iint u_{yxyy}^2 \rho \, dxdy + \frac{(1 + R)^2}{\varepsilon} \iint f_{yy}^2 \, dxdy,
\]
where \(\varepsilon > 0\) can be chosen arbitrarily small, and equality (2.62) yields that
\[
\|u_{xyy}\|_{X(Q_T)} + \|u_{xxyy}\|_{L_2(B_T)} \leq c.\] (2.63)
Again apply equality (2.53). Then it follows from (2.63) that we have the suitable estimate on \(u_{xxx}\) in the space \(L_2(Q_T)\). Similarly to (2.58)
\[
\iint u_{xxy}^2 \, dxdy \leq \iint (u_{xxy}^2 + u_{xyy}^2) \, dxdy + \int_0^L u_{xyy}^2 \big|_{x=0} \, dy + c \iint u_{xxy}^2 \, dxdy,
\]
whence follows the suitable estimate on \(u_{xyy}\) in \(L_2(Q_T)\) and, as a result, on \(u_y\) in \(L_2(0,T; \dot{H}^3)\). One more application of (2.55) yields the estimate on \(u_{xxx}\) in \(L_2(Q_T)\). Therefore,
\[
\|u\|_{L_2(0,T;\dot{H}^4)} \leq c.\] (2.64)
Consider the extensions of the functions \(u\) and \(f\) for \(y \in (L,2L] \) and \(y \in [-L,0)\) in the case a) by the even reflections through \(y = L\) and \(y = 0\), in the case b) – by the odd ones, in the case c) – by the corresponding combination of these methods, in the case d) – by the periodic extension. Then the functions \(u\) and \(f\) remain smooth in the more wide domain \([0,T] \times [0,R] \times [-L,2L],\) and equality (2.1) also remains valid. Let \(\eta_L(y) \equiv \eta(1+y/L)\eta(2-y/L), \bar{u}(t,x,y) \equiv u(t,x,y)\eta_L(y)\),
\( \tilde{f}(t, x, y) \equiv f(t, x, y)\eta_L(y) \). Now we apply the inequality (see, e.g. \cite{30}) for the domain \( \tilde{\Omega} = (0, R) \times \mathbb{R}^y \)

\[
\|g\|_{H^2(\tilde{\Omega})} \leq c(\|\Delta g\|_{L_2(\tilde{\Omega})} + \|g\|_{H^{3/2}(\mathbb{R})} + \|g\|_{H^1(\tilde{\Omega})})
\]

for the function \( g \equiv \tilde{u}_x \). Note that \( g|_{x=R} = 0 \) and

\[
\Delta_{x,y} \tilde{u}_x = \tilde{f} - \tilde{u}_t - b\tilde{u}_x + 2u_{xy}\eta_L + u_x\eta''_L.
\]

It follows from (2.61) that

\[
\|\Delta_{x,y} \tilde{u}_x\|_{C((0,T];L_2(\tilde{\Omega})))} \leq c.
\]

Moreover, by virtue of (2.61), (2.63) and embedding \( H^2(\tilde{\Omega}) \subset H^{3/2}(\{x = 0\} \times \mathbb{R}^y) \) (see \cite{30})

\[
\|u_x\|_{x=0} \leq \|u_0\|_{x=0} \leq \|H^{3/2}(\mathbb{R})
\]

\[
+ 2\|u_t\|_{x=0} \leq 1/2\|u_{x0}\|_{L_2(0,T)\times\mathbb{R}}\|u_x\|_{x=0} \leq 1/2 \leq c.
\]

Therefore,

\[
\|u_x\|_{C((0,T];H^2)} \leq c. \tag{2.65}
\]

Estimates (2.61), (2.63)–(2.65) provide the desired result. \( \square \)

At the end of this section consider the particular case of problem (2.1), (1.2)–(1.4) in \( Q_T \) for \( \mu_0 = \nu_0 = \nu_1 \equiv 0, f \equiv 0 \). Denote its solution by \( \tilde{u}_0 \), then it succeeds from Lemma 2.12 that the operator \( P \) is linear and bounded from \( L_2 \) to \( Y_0(Q_T) \). Moreover, it easily follows from (2.47) that

\[
\|\partial_x(P\tilde{u}_0)\|_{x=0} \leq \|u_0\|_{L_2}. \tag{2.66}
\]

For the controllability purposes we need the following observability result.

**Lemma 2.20.** If condition (1.13) holds, then there exists a constant \( c = c(T, b, R, L) > 0 \), such that

\[
\|u_0\|_{L_2} \leq c\|\partial_x(Pu_0)\|_{x=0} \|L_2(B_T). \tag{2.67}
\]

**Proof.** In the smooth case multiplying (2.1) by \( 2(T-t)u(t, x, y) \) and integrating over \( Q_T \) we find, that

\[
\iint_{Q_T} u^2 \, dx \, dy \, dt - T \iint_{B_T} u^2_0 \, dx \, dy + \iint_{B_T} (T-t)u^2_x \, dx \, dy = 0,
\]

whence follows, that

\[
\iint_{B_T} u^2_0 \, dx \, dy \leq \frac{1}{T} \iint_{Q_T} (Pu_0)^2 \, dx \, dy + \iint_{B_T} (\partial_x(Pu_0)\big|_{x=0})^2 \, dy \, dt. \tag{2.68}
\]

By continuity this estimate can be extended to any \( u_0 \in L_2 \).

Now assume, that inequality (2.67) is not true. Then there exists a sequence \( \{u_{0n}\} \subset L_2 \) such that

\[
\|u_{0n}\|_{L_2} = 1 \quad \forall \, n, \quad \lim_{n \to +\infty} \|\partial_x(Pu_{0n})\|_{x=0} \|L_2(B_T) = 0. \tag{2.69}
\]

It follows from (2.47) that the sequence \( \{Pu_{0n}\} \) is bounded in \( L_2(0,T; H^1) \). Moreover, equality (2.41) provides that the sequence \( \{\partial_x Pu_{0n}\} \) is bounded in \( L_1(0,T; H^{-2}) \) and the standard argument provides that \( \{Pu_{0n}\} \) is precompact in \( L_2(Q_T) \). Extract the subsequence \( n' \), such that \( \{Pu_{0n'}\} \) converges in \( L_2(Q_T) \).
Lemma 3.1. The notion of a weak solution to problem (3.1), (1.2)–(1.4) is similar to Definition
Proof. We apply the contraction principle. For any function \( \phi \in L_2(0, T; \hat{H}^2) \), \( \phi_t, \phi_{xxx}, \phi_{xyy} \in L_2(Q_T) \), \( \phi|_{t=T} = 0 \), \( \phi|_{x=R} = 0 \), the following equality holds:
\[
\iint_{Q_T} \phi \, dx \, dy + \int_0^T \iint_{Q_T} P(u)(\phi_t + b_0 \phi_x + \phi_{xxx} + \phi_{xyy}) \, dx \, dy \, dt = 0.
\] (2.70)

For any natural \( l \) let
\[
\psi(t, x) \equiv \int_0^L (P(u))(t, x, y) \psi_l(y) \, dy, \quad \psi_0(x) \equiv \int_0^L u_0(x, y) \psi_l(y) \, dy.
\] (2.71)

Let \( \psi(t, x) \) be an arbitrary function, such that \( \psi \in L_2(0, T; H^3(0, R) \cap H_0^1(0, R)) \), \( \psi_t \in L_2((0, T) \times (0, R)) \), \( \psi|_{t=T} = 0 \). Choose \( \phi(t, x, y) \equiv \psi(t, x) \psi_l(y) \), then it follows from (2.70), (2.71), that
\[
\iint_{(0, T) \times (0, R)} \psi_t \partial_x + \partial_{xxx} dx dt + \int_0^R \psi_0 \partial|_{t=0} dx = 0.
\] (2.72)

It means, that the function \( \psi \in C([0, T]; L_2(0, R)) \), \( \psi \in C([0, R]; L_2(0, T)) \) is a weak solution in the rectangle \( (0, T) \times (0, R) \) to an initial-boundary value problem
\[
\psi_t + (b - \lambda) \psi_x + \psi_{xxx} = 0, \quad \psi|_{t=0} = \psi_0, \quad \psi|_{x=0} = \psi|_{x=R} = 0.
\] (2.73)

But the obvious generalization of results from 33 (in that paper the case of the equation \( \psi_t + \psi_x + \psi_{xxx} = 0 \) was considered) shows that under condition (1.13) (if \( b - \lambda \leq 0 \) there are no restrictions on \( R \) \( \psi_0 \equiv 0 \) and, therefore, \( \psi_0 \equiv 0 \), which contradicts the fact, that \( \| \psi_0 \|_{L_2} = 1 \).

3. Existence of solutions

Consider an auxiliary equation
\[
u_t + b \nu_x + \nu_{xxx} + \nu_{xyy} + (g(u))_x + (\psi(t, x, y) u)_x = f(t, x, y).
\] (3.1)

The notion of a weak solution to problem (3.1), (1.2)–(1.4) is similar to Definition 1.1.

Lemma 3.1. Let \( g \in C^1(\mathbb{R}) \), \( g(0) = 0 \), \( |g'(u)| \leq c \forall u \in \mathbb{R} \), \( \psi \in L_2(0, T; L_\infty) \), \( u_0 \in L_2 \), \( f \in L_1(0, T; L_2) \), \( \mu_0 = v_0 \equiv 0 \), \( \nu_1 \in L_2(2\Omega) \). Then problem (3.1), (1.2)–(1.4) has a unique weak solution \( u \in X(Q_T) \).

Proof. We apply the contraction principle. For \( \nu_0 \in (0, T] \) define a mapping \( \Lambda \) on \( X(Q_{\nu_0}) \) as follows: \( u = \Lambda v \in X(Q_{\nu_0}) \) is a weak solution to a linear problem
\[
u_t + b \nu_x + \nu_{xxx} + \nu_{xyy} = f - (g(u))_x - (\psi v)_x
\] (3.2)
in \( Q_{\nu_0} \) with initial and boundary conditions (1.2)–(1.4).

Since
\[
\| g(v) \|_{L_2(Q_{\nu_0})} \leq c \| v \|_{C([0, \nu_0]; L_2)} < \infty,
\]
\[
\| \psi v \|_{L_2(Q_{\nu_0})} \leq c \| \psi \|_{L_2(0, \nu_0; L_\infty)} \| v \|_{C([0, \nu_0]; L_2)} < \infty,
\]
Lemma 2.12 provides that the mapping \( \Lambda \) exists. Moreover, for functions \( v, \bar{v} \in X(Q_{t_0}) \)
\[
\|g(v) - g(\bar{v})\|_{L^2(Q_{t_0})} \leq c\|v - \bar{v}\|_{L^2(Q_{t_0})} \\
\|\psi(v) - \psi(\bar{v})\|_{L^2(Q_{t_0})} \leq c\|\psi\|_{L^2(0,t_0;L^\infty)}\|v - \bar{v}\|_{C([0,t_0];L^2)}.
\]
As a result, according to inequality (2.46)\[
\|\Lambda v - \Lambda \bar{v}\|_{X(Q_{t_0})} \leq c(T)\|v - \bar{v}\|_{X(Q_{t_0})},
\]
proof of existence Part of Theorem 1.3. First of all we make zero the boundary \( u_{\nu} \) where \( \omega \) with initial and boundary conditions and the same boundary conditions on \( (0, t_0) \) as (1.4). Note also that the functions \( U_0, F, V_1 \) satisfy the same assumptions as the corresponding functions \( u_0, f, v_1 \) in the hypothesis of the theorem.

For \( h \in (0, 1) \) consider a set of initial-boundary value problems in \( Q_T \) for an equation
\[
U_t + bU_x + U_{xxx} + U_{xyy} + (g_h(U))_x + (\psi U)_x = F
\]
with boundary conditions (1.4) and (3.7).
Note that $g_h(u) = u^2/2$ if $|u| \leq 1/h$, $|g'_h(u)| \leq 2/h \forall u \in \mathbb{R}$ and $|g'_h(u)| \leq 2|u|$ uniformly with respect to $h$.

According to Lemma 3.1, there exists a unique solution to this problem $U_h \in X(Q_T)$.

Next, establish appropriate estimates for functions $U_h$ uniform with respect to $h$ (we drop the index $h$ in intermediate steps for simplicity). First, note that $g'(U)U_x, \psi U_x, \psi_x U, F \in L_1(0, T; L_2)$ and so the hypothesis of Lemma 2.14 is satisfied (for $f_1 \equiv 0$). Write down the analogue of equality (2.47) for $\rho \equiv 1$, then:

$$
\int\int U^2 \, dx \, dy \leq \int\int U_0^2 \, dx \, dy + \int_0^t \int\int (2F - 2(g(U))_x - \psi_x U) \, dx \, dy \, d\tau. \tag{3.9}
$$

Since

$$(g(U))_x U = \partial_x \left( \int_0^U g'(\theta) d\theta \right) \tag{3.10}$$

we derive that

$$
\int\int (g(U))_x U \, dx \, dy = 0. \tag{3.11}
$$

Therefore, since $\psi_x \in L_2(0, T; L_\infty)$ uniformly with respect to $h$

$$
\|u_h\|_{C([0,T]; L_2)} \leq c. \tag{3.12}
$$

Next, equalities (2.47) and (3.10) provide that for $\rho(x) \equiv (1 + x)$

$$
\int\int U^2 \, dx \, dy + \int_0^t \int\int (3U_x^2 + U^2_x) \, dx \, dy \, d\tau \leq (1 + R) \int\int U_0^2 \, dx \, dy
+ b \int_0^t \int\int U^2 \, dx \, dy \, d\tau + (1 + R) \int\int V_t^2 \, dy \, d\tau + 2 \int_0^t \int\int FU \rho \, dx \, dy \, d\tau
+ \int_0^t \int\int (\psi - \psi_x \rho)U^2 \, dx \, dy \, d\tau + 2 \int_0^t \int\int \left( \int_0^U g'(\theta) \, d\theta \right) \, dx \, dy \, d\tau. \tag{3.13}
$$

Note that

$$
\left| \int_0^U g'(\theta) d\theta \right| \leq c|U|^3. \tag{3.14}
$$

Applying interpolating inequality (1.15) (here the exact value of the constant is indifferent), we obtain that

$$
\int\int |U|^3 \, dx \, dy \leq c \int\int U^2 \, dx \, dy \left( \int\int (|DU|^2 + U^2) \, dx \, dy \right)^{1/2} \tag{3.15}
$$

Since the norm of the functions $u_h$ in the space $L_2$ is already estimated in (3.12), it follows from (3.13)–(3.15) that uniformly with respect to $h$

$$
\|u_h\|_{X(Q_T)} \leq c. \tag{3.16}
$$

From equation (3.8) itself, estimate (3.14) and the well-known embedding $L_1 \subset H^{-2}$, it follows that uniformly with respect to $h$

$$
\|u_h\|_{L_1(0, T; H^{-3})} \leq c. \tag{3.17}
$$

Estimates (3.16), (3.17) by the standard argument provide existence of a weak solution to problem (1.1)–(1.4) $u \in L_\infty(0, T; L_2) \cap L_2(0, T; \tilde{H}^1)$, as a limit of functions $u_h$ when $h \to +0$. 

Finally, since by virtue of (1.20) (here the exact value of the constant is again indifferent)

\[ \iint_{Q_T} U^4 \, dx \, dy \, dt \leq c \int_0^T \| U(t, \cdot , \cdot ) \|_{H^1}^2 \| U(t, \cdot , \cdot ) \|_{L^2}^2 \, dt \]

\[ \leq c \| U \|_{L^2(0,T;H^1)}^2 \| U \|_{L^\infty(0,T;L^2)}^2 \leq \infty \quad (3.18) \]

and

\[ \iint_{Q_T} \psi^2 U^2 \, dx \, dy \, dt \leq \| \psi \|_{L^2(0,T;L^\infty)}^2 \| U \|_{L^\infty(0,T;L^2)}^2 \leq \infty, \quad (3.19) \]

it follows from Lemma 2.14 (where \( f_1 \equiv U^2/2 + \psi U \), that after possible modification on a set of zero measure \( U \in C([0,T];L^2) \).

\[ \square \]

Result on uniqueness and continuous dependence of weak solutions succeeds from the following theorem.

**Theorem 3.2.** For any \( T > 0 \) and \( M > 0 \) there exist constant \( c = c(T, M, b, R, L) \), such that for any two weak solutions \( u(t, x, y) \) and \( \tilde{u}(t, x, y) \) to problem (1.1)-(1.4), satisfying \( \| u \|_{X(Q_T)}, \| \tilde{u} \|_{X(Q_T)} \leq M \), with corresponding data \( u_0, \tilde{u}_0 \in L^2, \mu_0, \tilde{\mu}_0, \nu_0, \tilde{\nu}_0 \in \dot{H}^{1/3, 1}(B_T) \), \( \nu_1, \tilde{\nu}_1 \in L^2(B_T) \) \( f, \tilde{f} \in L^1(0, T; L^2) \) the following inequality holds:

\[ \| u - \tilde{u} \|_{X(Q_T)} \leq c \left( \| u_0 - \tilde{u}_0 \|_{L^2} + \| \mu_0 - \tilde{\mu}_0 \|_{H^{1/3, 1}(B_T)} + \| \nu_0 - \tilde{\nu}_0 \|_{H^{1/3, 1}(B_T)} \right) \quad (3.20) \]

Proof. Let the function \( \psi \) is defined by formula (3.23), the function \( \tilde{\psi} \) in a similar way for \( \tilde{\mu}_0, \tilde{\nu}_0 \) and \( \tilde{\Psi} \equiv \psi - \tilde{\psi} \). Then, in particular,

\[ \| \tilde{\Psi} \|_{X(Q_T)} \leq c \left( \| \mu_0 - \tilde{\mu}_0 \|_{H^{1/3, 1}(B_T)} + \| \nu_0 - \tilde{\nu}_0 \|_{H^{1/3, 1}(B_T)} \right) \quad (3.21) \]

Let \( U_0 \equiv u_0 - \tilde{u}_0 - \tilde{\Psi}_{|_{t=0}}, \quad F \equiv f - \tilde{f} - (\Psi_t + b \tilde{\Psi}_x + \Psi_{xxx} + \Psi_{xyy}), \quad V_1 \equiv \nu_1 - \tilde{\nu}_1 - \tilde{\Psi}_x |_{x=R}, \quad \text{then} \]

\[ \| U_0 \|_{L^2} \leq \| u_0 - \tilde{u}_0 \|_{L^2} + c \left( \| \mu_0 - \tilde{\mu}_0 \|_{H^{1/3, 1}(B_T)} + \| \nu_0 - \tilde{\nu}_0 \|_{H^{1/3, 1}(B_T)} \right), \quad (3.22) \]

\[ \| F \|_{L^1(0,T;L^2)} \leq \| f - \tilde{f} \|_{L^1(0,T;L^2)} \]

\[ \quad + c \left( \| \mu_0 - \tilde{\mu}_0 \|_{H^{1/3, 1}(B_T)} + \| \nu_0 - \tilde{\nu}_0 \|_{H^{1/3, 1}(B_T)} \right), \quad (3.23) \]

\[ \| V_1 \|_{L^2(B_T)} \leq \| \nu_1 - \tilde{\nu}_1 \|_{L^2(B_T)} + c \left( \| \mu_0 - \tilde{\mu}_0 \|_{H^{1/3, 1}(B_T)} + \| \nu_0 - \tilde{\nu}_0 \|_{H^{1/3, 1}(B_T)} \right). \quad (3.24) \]

The function \( U(t, x, y) \equiv u(t, x, y) - \tilde{u}(t, x, y) - \tilde{\Psi}(t, x, y) \) is a weak solution to an initial-boundary value problem in \( Q_T \) for an equation

\[ U_t + bU_x + U_{xxx} + U_{xyy} = F - (uu_x - \tilde{u}\tilde{u}_x) \]

with initial and boundary conditions (1.4),

\[ U|_{t=0} = U_0, \quad U|_{x=0} = U|_{x=R} = 0, \quad U_x|_{x=R} = V_1. \]
Apply Lemma 2.14 where $f_1 \equiv -(u^2 - \bar{u}^2)/2$. Note that similarly to (3.16) $f_1 \in L_2(Q_T)$. Therefore, we derive from (2.47) that for $t \in (0, T]$ and $\rho(x) \equiv (1 + x)$

$$
\iint U^2 \, dx dy + \int_0^t \iint (3U_x^2 + U_y^2) \, dx dy \, dt \leq (1 + R) \int_0^T U_0^2 \, dx dy + b \int_0^t \iint U^2 \, dx dy \, dt + (1 + R) \int_0^t \iint V_t^2 \, dy \, dt + 2 \int_0^t \iint F \, dx dy \, dt + \int_0^t \iint (u^2 - \bar{u}^2)(U \rho)_x \, dx dy \, dt.
$$

(3.25)

Here $u^2 - \bar{u}^2 = (u + \bar{u})(U + \Psi)$ and by virtue of (1.20)

$$
\iint |u(U + \Psi) U_x| \, dx dy \leq c \left( \iint u^4 \, dx dy \right)^{1/4} \left( \iint U_x^2 \, dx dy \right)^{1/2} \leq c_1 \|u\|_{H^1}^{1/2} \|\Psi\|_{L^2}^{1/2} \left( \iint |DU|^2 \, dx dy \right)^{3/4} \left( \iint U^2 \, dx dy \right)^{1/4} + \iint U^2 \, dx dy + \left( \iint |DU|^2 \, dx dy \right)^{1/2} \|\Psi\|_{H^1} \|\Psi\|_{L^2}
$$

and, therefore,

$$
\int_0^t \iint |u(U + \Psi) U_x| \, dx dy \leq \varepsilon \int_0^t \iint |DU|^2 \, dx dy \, dt + \int_0^t \|\Psi\|_{H^1}^2 \, dt + c(\varepsilon) \int_0^t \gamma(\tau) \iint (U^2 + \Psi^2) \, dx dy \, dt,
$$

(3.26)

where $\gamma(t) \equiv 1 + \|u(t, \cdot, \cdot)\|_{H^1}^2 \|u(t, \cdot, \cdot)\|_{L^2}^2 \in L_1(0, T)$ and $\varepsilon > 0$ can be chosen arbitrarily small. Then estimates (3.21), (3.24), (3.26) and inequality (3.25) provide the desired result.

Finally, consider regular solutions.

**Lemma 3.3.** Let $g(u) \equiv u^2/2$, $\mu_0 = \nu_0 = \nu_1 \equiv 0$, the functions $u_0$ and $f$ satisfy the hypothesis of Theorem 1.7, $\psi \in X^3(Q_T)$. Then problem (3.1), (1.2)–(1.4) has a unique solution $u \in X^3(Q_T)$.

**Proof.** For $t_0 \in (0, T]$, $v \in X^3(Q_{t_0})$ let $u = \Delta v \in X^3(Q_{t_0})$ be a solution to a linear problem (3.2) (for $g(v) \equiv v^2/2$), (1.2)–(1.4).

Apply Lemma 2.14. We have:

$$
\|v v_x + \psi v_x + \psi x v\|_{C[0, t_0]; L^2} \leq \|u_0 u_{0x} + \psi|_{t=0} u_{0x} + \psi_{x|_{t=0}} u_0\|_{L^2} + \|(v v_x)_x + (\psi v)_x\|_{L^1(0, t_0; L^2)} \quad (3.27)
$$

and with the use of (1.21) derive that

$$
\|u_0 u_{0x}\|_{L^\infty} \leq c \|u_0\|_{L^\infty} \|u_{0x}\|_{L^2} \leq c_1 \|u_0\|_{H^{3}}, \quad (3.28)
$$

$$
\|\psi|_{t=0} u_{0x} + \psi_{x|_{t=0}} u_0\|_{L^2} \leq c \|\psi|_{t=0} \|_{H^{1}} \|u_0\|_{W^{1, \infty}} \leq c \|\psi\|_{X^3(Q_T_0)} \|u_0\|_{H^{3}} \quad (3.29)
$$

next,

$$
\|v v_{t2}\|_{L^1(0, t_0; L^2)} \leq \int_0^{t_0} \|v\|_{L^\infty} \|v_{t2}\|_{L^2} \, dt \leq c t_0^{1/2} \|v\|_{X^2(Q_{t_0})} \|v\|_{X^3(Q_{t_0})}, \quad (3.30)
$$

and
\[\|v_xv_t\|_{L^1(0,t_0;L^2)} \leq \int_0^{t_0} \|v_x\|_{L^4} \|v_t\|_{L^4} dt \leq c t_0^{1/2} \|v\|_{X^2(Q_{t_0})} \|v\|_{X^3(Q_{t_0})} \] (3.31)

and similarly
\[\|v(x_t)\|_{L^1(0,t_0;L^2)} \leq c t_0^{1/2} \|v\|_{X^3(Q_{t_0})} \] (3.32)

Next,
\[\|v_xv_t\|_{L^2(Q_{t_0})} \leq (\int_0^{t_0} \|v\|_{L^2}^2 \|v_t\|_{L^2}^2 dt)^{1/2} \leq c t_0^{1/2} \|v\|_{X^2(Q_{t_0})} \|v\|_{X^3(Q_{t_0})}, \] (3.33)
\[(v_xv_y)_y = v_{xxy} + 2v_y v_{xy} + v_x v_{yy}, \] where similarly to (3.33)
\[\|v_xv_y\|_{L^2(Q_{t_0})} \leq c t_0^{1/2} \|v\|_{X^2(Q_{t_0})} \|v\|_{X^3(Q_{t_0})}, \] (3.34)
\[\|v_yv_{xy}\|_{L^2(Q_{t_0})} \leq (\int_0^{t_0} \|v_y\|_{L^2}^2 \|v_{xy}\|_{L^2}^2 dt)^{1/2} \leq c t_0^{1/2} \|v\|_{X^2(Q_{t_0})} \|v\|_{X^3(Q_{t_0})} \] (3.35)
and similar estimate holds for \(v_xv_{yy}. \) Finally, similarly to (3.33)–(3.35)
\[\|v\|_{L^2(Q_{t_0})} + \|v(x_t)\|_{L^2(Q_{t_0})} \leq c t_0^{1/2} \|v\|_{X^3(Q_{t_0})} \] (3.36)

Moreover, the assumptions on the function \(\psi\) ensure that the corresponding boundary conditions on the function \(v_x + (\psi)v\) are satisfied for \(y = 0 \) and \(y = L. \)
Therefore, the mapping \(\Lambda\) exists and one can use estimate (2.60) to derive inequalities
\[\|\Lambda v\|_{X^3(Q_{t_0})} \leq \bar{c} + c t_0^{1/2} (\|\psi\|_{X^3(Q_{t_0})} \|v\|_{X^3(Q_{t_0})} + \|v\|_{X^3(Q_{t_0})}^2) \] (3.37)
\[\|\Lambda v - \bar{\Lambda} v\|_{X^3(Q_{t_0})} \leq c t_0^{1/2} (\|\psi\|_{X^3(Q_{t_0})} \|v - \bar{v}\|_{X^3(Q_{t_0})} + \|\psi\|_{X^3(Q_{t_0})} \|v - \bar{v}\|_{X^3(Q_{t_0})}) \] (3.38)
where the constant \(c\) depends on the parameters \(T, b, R, \) and the constant \(\bar{c}\) also on the properties of functions \(u_0, f, \psi. \) Hence, existence of the unique solution to the considered problem in the space \(X^3(Q_{t_0})\) on the time interval \([0, t_0]\), depending on \(\|u_0\|_{\tilde{H}^3}\), follows by the standard argument.

Now establish the following a priori estimate: if \(u \in X^3(Q_{t_0})\) is a solution to the considered problem for some \(T' \in (0, T]\), then
\[\|u\|_{X^3(Q_{t_0})} \leq c, \] (3.39)
where the constant \(c\) depends on \(T, b, R, \) and the properties of the functions \(u_0, f, \psi\) from the hypothesis of the present lemma.

It is already known, that (see (3.10))
\[\|u\|_{X(Q_T')} \leq c. \] (4.0)

Apply Lemma 2.16 then by virtue of (2.30) for \(\rho(x) \equiv 1 + x\)
\[\int_0^t \int u_x^2 dx dy + \int_0^t \int |D u_x|^2 dx dy \leq (1 + R) \int_0^t u_{yy}^2 dx dy + b \int_0^t \int u_y^2 dx dy - 2 \int_0^t \int (f - uu_x - (\psi u)_x) u_y \rho dy dy. \] (3.41)
Here for arbitrary $\varepsilon > 0$

$$2\int_0^t \int \int u_{xy}u_{yy}\rho \, dx dy \, dt = \int_0^t \int \int (u - u_x\rho)u_y^2 \, dx dy \, dt$$

$$\leq c \int_0^t \left( \int \int (u_x^2 + u_y^2) \, dx dy \int \int u_y^2 \, dx dy \right)^{1/2} \, dt$$

$$\leq \varepsilon \int_0^t \int \int (|Du_y|^2 + u_y^2) \, dx dy \, dt + c(\varepsilon) \int_0^t \int \int u_y^2 \, dx dy \, dt,$$  \hspace{1cm} (3.42)

where $\gamma \equiv \|u(t, \cdot, \cdot)\|_{H^1}^2 \in L_1(0, T')$.

$$2\int_0^t \int \int (\psi u)_{y} u_{yy}\rho \, dx dy \, dt$$

$$\leq \sup_{t \in [0, T]} \|\psi(t, \cdot, \cdot)\|_{W^1_\infty} \int_0^t \int \int u_{yy}^2 \, dx dy \int \int (u_x^2 + u_y^2) \, dx dy \, dt^{1/2} \, dt$$

$$\leq \varepsilon \int_0^t \int \int u_{yy}^2 \, dx dy \, dt + c(\varepsilon) \|\psi\|_{X_1(0, T')}^2 \|u\|_{X_1(0, T')},$$ \hspace{1cm} (3.43)

Therefore, inequality (3.41) yields that

$$\|u_y\|_{C([0, T'; L_2)} + \||Du_y||_{L_2(Q_T)} \leq c.$$ \hspace{1cm} (3.44)

Next, since the hypothesis of Lemma 2.17 is fulfilled, write down the corresponding analogue of equality (2.41) for the function $u_t$ and $\rho(x) \equiv 1 + x$:

$$\int \int u_t^2 \, dx dy + \int_0^t \int \int (3u_{tx}^2 + u_{ty}^2) \, dx dy \, dt$$

$$\leq (1 + R) \int \int (f - bu_x - u_{xxx} - u_{xyy} - u_{xx} - (\psi u)_x)^2 \big|_{t=0} \, dx dy + b \int_0^t \int \int u_t^2 \, dx dy \, dt$$

$$+ 2\int_0^t \int \int f u_t \rho \, dx dy \, dt + 2\int_0^t \int \int (u_t + (\psi u)_t) u(t, \cdot) \, dx dy \, dt.$$ \hspace{1cm} (3.45)

Here similarly to (3.42), (3.43) for arbitrary $\varepsilon > 0$

$$2\int_0^t \int \int u_{t}(u_t\rho)_x \, dx dy \, dt = \int_0^t \int \int (u - u_x\rho) u_{t}^2 \, dx dy \, dt$$

$$\leq \varepsilon \int_0^t \int \int (|Du_t|^2 + u_t^2) \, dx dy \, dt + c(\varepsilon) \int_0^t \int \int u_t^2 \, dx dy \, dt,$$

where $\gamma \equiv \|u(t, \cdot, \cdot)\|_{H^1}^2 \in L_1(0, T')$.

$$2\int_0^t \int \int \psi u_{t}(u_t\rho)_x \, dx dy \, dt$$

$$\leq c \int_0^t \left( \int \int (u_{tx}^2 + u_{ty}^2) \, dx dy \right)^{1/2} \left( \int \int \psi_{t}^4 \, dx dy \int \int u^4 \, dx dy \right)^{1/4} \, dt$$

$$\leq \varepsilon \int_0^t \int \int (u_{tx}^2 + u_{ty}^2) \, dx dy \, dt + c(\varepsilon) \|\psi\|_{X_1(0, T')}^2 \|u\|_{X_1(0, T')},$$
and
\[ 2 \int_0^t \int \psi u_t(u_t \rho) dx dy dt = \int_0^t \int (\psi - \psi \rho) u_t^2 dx dy dt \]
\[ \leq c \|\psi\|_{X^3(Q_T)} \int_0^t \int u_t^2 dx dy dt. \]

Consequently, it follows from (3.45), that
\[ \|u_t\|_{X(Q_{T'})} \leq c. \quad (3.46) \]

Now apply Lemma 2.18, then inequality (2.52) and estimates (3.40), (3.44) and (3.46) yield that for any \( t \leq T' \) and \( \rho(x) \equiv 1 + x \)
\[ \|u\|_{X^2(Q_T)} \leq c + c\|uu_x\|_{C([0,t];L^2)} + c\|\psi u\|_{C([0,t];L^2)} \]
\[ + c \sup_{t \in (0,1)} \left| \int_0^t \int \left( uu_x + (\psi u)_x \right) yy u_{yy} \rho dx dy ds \right|. \quad (3.47) \]

Uniformly with respect to \( t \in [0,T'] \) for arbitrary \( \varepsilon > 0 \)
\[ \|uu_x\|_{L^2}^2 \leq \varepsilon \|u\|_{H^1}^2 u_x^2 + c(\varepsilon)(\|u_t\|_{X(Q_{T'})}^2 + \|u\|_{X(Q_{T'})}^2), \]
\[ \|\psi u\|_{L^2}^2 \leq \|u\|_{W_{1,\infty}}^2 c(\|u_t\|_{X(Q_{T'})}^2 + \|u\|_{X(Q_{T'})}^2); \]
then,
\[ \int_0^t \int (uu_x)_{yy} u_{yy} \rho dx dy dy = \frac{1}{2} \int_0^t \int (u_x \rho - u) u_{yy} dx dy + \int_0^t \int u_y u_x u_{yy} \rho dx dy, \]
where
\[ \int_0^t \int |u_y u_x| \rho dx dy dt \leq \sup_{t \in [0,1]} \int_0^t \int u_{yy}^2 dx dy \left( \int_0^t \int (u_{yy}^4 + u_{yy}^4) dx dy \right)^{1/2} dt \]
\[ \leq \varepsilon \int_0^t \int |D^3 u|^2 dx dy dt + c(\varepsilon)(\|u_t\|_{X(Q_{T'})}^2 + \|u\|_{X(Q_{T'})}^2) \int_0^t \int |D^2 u|^2 dx dy dt, \]
\[ \int_0^t \int |u - u_x| \rho u_{yy} \rho dx dy dt \leq \varepsilon \sup_{t \in [0,1]} \|u\|_{H^1}^2 \left( \int_0^t \int u_{yy}^4 dx dy \right)^{1/2} dt \]
\[ \leq \varepsilon \int_0^t \int |D u_{yy}|^2 dx dy dt + c(\varepsilon)(\|u_t\|_{X(Q_{T'})}^2 + \|u\|_{X(Q_{T'})}^2) \int_0^t \int u_{yy}^2 dx dy dt; \]
finally, \( (\psi u)_{xy} = \psi_{xy} u + 2\psi_{xy} u_y + \psi_{yy} u_x + \psi_x u_{yy} + 2\psi_y u_{xy} + \psi_{xy} u_y \), where
\[ \int_0^t \int |\psi_{xy} u_{yy}| dx dt \]
\[ \leq \sup_{t \in [0,1]} \|\psi_{xy}\|_{L^2} \int_0^t \left( \int u^4 dx \int u_{yy}^4 dx \right)^{1/4} dt \]
\[ \leq \varepsilon \int_0^t \int \left( |D^3 u|^2 + |D^2 u|^2 \right) dx dy dt + c(\varepsilon)\|\psi\|_{X^3(Q_{T'})}^2 \|u\|_{X(Q_{T'})}^2; \]
\[ \int_0^t \int \left| \psi_{xy} u_y u_{yy} \right| dx dt d\tau \]

\[ \leq \varepsilon \int_0^t \int \left( |D^3 u|^2 + |D^2 u|^2 \right) dx dy dt + c(\varepsilon) \| \psi \|_{X^3(Q_T)}^2 \| u \|_{X^3(Q_T)}^2 \]

and similar estimate holds for the integral of \( \psi_{yy} u_x u_{yy} \). The result is immediate.

\[ \| u \|_{X^3(Q_T)} \leq c. \quad (3.48) \]

Finally, apply Lemma 2.19 on the basis of the already obtained estimates (3.46), (3.48), then inequality (2.60) and estimates (3.59) applied to \( v \equiv u \) provide similarly to (3.27) that for any \( t_0 \in (0, T'] \)

\[ \| u \|_{X^3(Q_{t_0})} \leq \tilde{c} + c t_0^{1/2} \left( \| \psi \|_{X^3(Q_T)} + \| u \|_{X^3(Q_{t_0})} \right) \| u \|_{X^3(Q_{t_0})}, \]

whence (3.39) follows.

Proof of Theorem 1.3. Let \( \psi \in Y^3(Q_T) \subset X^3(Q_T) \) be the solution to problem (2.1), (1.2)–(1.4) for \( f \equiv 0 \) (see Lemma 2.10). Introduce the function \( U \) by formula (3.5) and consider problem (3.6), (3.7), (1.4) (here \( \tilde{\psi} \equiv 0 \), \( V_1 \equiv 0 \)). Then the functions \( \psi, F \sim f \) and \( U_0 \sim u_0 \) satisfy the hypothesis of Lemma 3.3 and the result is immediate.

4. LARGE-TIME DECAY OF SMALL SOLUTIONS

Proof of Theorem 1.7. Consider the solution to problem (1.1)–(1.4) \( u \in X(Q_T) \forall T \). Note that \( u^2 \in L^2(Q_T) \) (see, for example, (3.18)). Apply Lemma 2.14 then equality (4.1) for \( f_1 \equiv u^2/2, \rho \equiv 1 \) and equality (3.10) for \( g(u) \equiv u^2/2 \) yield similarly to (4.1), that

\[ \| u(t, \cdot, \cdot) \|_{L^2}^2 \leq \| u_0 \|_{L^2}^2 + \| u_1 \|_{L^2(B_t)}^2 \leq c_0^2 \quad \forall t \geq 0. \quad (4.1) \]

Next, it follows from equality (2.47) for \( \rho \equiv 1 + x, \) that

\[ \int u^2 \rho dx dy + \int \mu_1^2 dy dt + \int_0^t \int (3u_x^2 + u_y^2 - bu^2) dx dy dt \]

\[ = \int u^2 \rho dx dy + (1 + R) \int \mu_1^2 dy dt + \frac{2}{3} \int_0^t \int u^3 dx dy dt. \quad (4.2) \]

Since \( u^3 \in L^1(Q_T) \) equality (4.2) provides the following inequality in a differential form: for a.e. \( t > 0 \)

\[ \frac{d}{dt} \int u^2 \rho dx dy + \int (3u_x^2 + u_y^2 - bu^2) dx dy \leq (1 + R) \int_0^L \nu_1^2 dy + \frac{2}{3} \int u^3 dx dy. \quad (4.3) \]

Next, we show that inequality (4.3) implies the following one:

\[ \frac{d}{dt} \int u^2 \rho dx dy + \frac{\kappa}{1 + R} \int u^2 \rho dx dy \]

\[ + \delta \int \left[ 1 - \frac{1}{\varepsilon_0} \| u(t, \cdot, \cdot) \|_{L^2} \right] (3u_x^2 + u_y^2) dx dy \leq (1 + R) \int_0^L \nu_1^2 dy. \quad (4.4) \]
where $\delta$, $\varkappa$ and $\epsilon_0$ are from the hypothesis of the theorem. First of all note, that in all cases inequality (1.17) implies, that

$$
\iint u_x^2 \, dx \, dy \geq \frac{\pi^2}{R^2} \iint u^2 \, dx \, dy.
$$

(4.5)

Further consider different cases separately.

In the cases b) and d) it follows from inequality (4.5), that

$$
(1 - \delta) \iint (3u_x^2 + u_y^2) \, dx \, dy - b \iint u^2 \, dx \, dy \geq \frac{\varkappa}{1 + R} \iint u^2 \, dx \, dy.
$$

(4.6)

Moreover, by virtue of (1.15) and (4.5)

$$
\frac{2}{3} \iint u^3 \, dx \, dy \leq \frac{4R}{3\pi} \left( \iint u_x^2 \, dx \, dy \right)^{3/4} \left( \iint u_y^2 \, dx \, dy \right)^{1/4} \left( \iint u^2 \, dx \, dy \right)^{1/2}
$$

$$
+ \frac{4R^{3/2}}{3L^{1/2} \pi^{3/2}} \iint u_x^2 \, dx \, dy \left( \iint u^2 \, dx \, dy \right)^{1/2} \leq \frac{\delta}{\epsilon_0} \|u(t, \cdot, \cdot)\|_{L_2} \iint (3u_x^2 + u_y^2) \, dx \, dy,
$$

(4.7)

and (4.4) follows.

In the case a) we also use an inequality

$$
\iint u_y^2 \, dx \, dy \geq \frac{\pi^2}{L^2} \iint u^2 \, dx \, dy
$$

(4.8)

and, therefore, obtain (4.6) with the corresponding $\varkappa$. Then we can alternatively derive, that either similarly to (4.7)

$$
\frac{2}{3} \iint u^3 \, dx \, dy \leq \frac{4R}{3\pi} \left( \iint u_x^2 \, dx \, dy \right)^{3/4} \left( \iint u_y^2 \, dx \, dy \right)^{1/4} \left( \iint u^2 \, dx \, dy \right)^{1/2}
$$

$$
\leq \frac{4R}{3^{7/4} \pi} \|u(t, \cdot, \cdot)\|_{L_2} \iint (3u_x^2 + u_y^2) \, dx \, dy,
$$

or

$$
\frac{2}{3} \iint u^3 \, dx \, dy \leq \frac{4L}{3\pi} \left( \iint u_x^2 \, dx \, dy \right)^{1/4} \left( \iint u_y^2 \, dx \, dy \right)^{3/4} \left( \iint u^2 \, dx \, dy \right)^{1/2}
$$

$$
\leq \frac{4L}{3^{5/4} \pi} \|u(t, \cdot, \cdot)\|_{L_2} \iint (3u_x^2 + u_y^2) \, dx \, dy,
$$

(4.9)

whence (4.3) follows.

In the case c) inequality (4.8) must be substituted by the following one:

$$
\iint u_y^2 \, dx \, dy \geq \frac{\pi^2}{4L^2} \iint u^2 \, dx \, dy.
$$

Similar modification must be done in (4.9) and (4.4) in this case also follows.

Inequalities (4.1) and (4.4) imply, that

$$
\frac{d}{dt} \iint u^2 \, dx \, dy + \frac{\varkappa}{1 + R} \iint u^2 \, dx \, dy \leq (1 + R) \int_0^L \nu_1^2 \, dy,
$$

whence (1.11) easily succeeds.
5. Boundary controllability

First establish the result on boundary controllability for the linear equation.

**Theorem 5.1.** Let condition (1.13) be satisfied for any natural \( l \), such that \( \lambda_1 < b \).

Let \( T > 0, f \equiv 0, \mu_0 = \nu_0 \equiv 0 \). Then for any \( u_0, u_T \in L_2 \) there exists a function \( \nu_1 \in L_2(B_T) \), such that there exists a unique solution \( u \in Y_0(Q_T) \) to problem (2.1), (1.2)–(1.4), satisfying (1.12).

**Proof.** Assume first that \( u_0 \equiv 0 \). In the case \( \nu_1 \in L_2(B_T), u_0 \equiv 0, \mu_0 = \nu_0 \equiv 0, f \equiv 0 \) denote the solution \( u \in Y_0(Q_T) \) to problem (2.1), (1.2)–(1.4) by \( P_1 \nu_1 \).

Then Lemma 2.12 provides, that \( P_1 \) is the linear bounded operator from \( L_2(B_T) \) to \( Y_0(Q_T) \).

Let \( P_{1T} \nu_1 \equiv P_1 \nu_1 \big|_{t=T} \), then \( P_{1T} \) is the linear bounded operator from \( L_2(B_T) \) to \( L_2 \).

Consider also the backward problem in \( Q_T \)

\[
\phi_t + b \phi_x + \phi_{xxx} + \phi_{xxy} = 0, \quad \phi \big|_{t=T} = \phi_0(x,y), \quad \phi \big|_{x=0} = \phi \big|_{x=R} = 0 \tag{5.1}
\]

with corresponding boundary conditions of (1.4) type, which after change of variables \( (t,x,y) \to (T-t,R-x,y) \) transforms to the corresponding problem of (2.1), (1.2)–(1.4) type. In particular, if we denote \( \phi = \tilde{P} \phi_0 \), then \( \tilde{P} \) is the linear bounded operator from \( L_2 \) to \( Y_0(Q_T) \). Moreover estimates (2.66), (2.67) yield, that for \( \Lambda \phi_0 \equiv \partial_x (\tilde{P} \phi_0) \big|_{x=R} \)

\[
\| \Lambda \phi_0 \|_{L_2(B_T)} \leq \| \phi_0 \|_{L_2} \leq c \| \Lambda \phi_0 \|_{L_2(B_T)}. \tag{5.3}
\]

In the smooth case multiplying \( (5.1) \) by \( P_1 \nu_1 \) and integrating over \( Q_T \) one can easily derive an equality

\[
\iint_{B_T} P_{1T} \nu_1 \cdot \phi_0 \, dx \, dy = \iint_{B_T} \nu_1 \cdot \Lambda \phi_0 \, dy \, dt. \tag{5.4}
\]

By continuity this equality can be extended to the case \( \nu_1 \in L_2(B_T), \phi_0 \in L_2 \).

Let \( A \equiv P_{1T} \circ \Lambda \), then according to (5.3) and the aforementioned properties of the operator \( P_{1T} \) the operator \( A \) is bounded in \( L_2 \). Moreover, (5.3) and (5.4) provide, that

\[
(A \phi_0, \phi_0) = \iint (P_{1T} \circ \Lambda) \phi_0 \cdot \phi_0 \, dx \, dy = \iint_{B_T} (\Lambda \phi_0)^2 \, dy \, dt \geq \frac{1}{c^2} \| \phi_0 \|_{L_2}^2. \tag{5.5}
\]

Application of Lax–Milgram theorem implies, that \( A \) is invertible and \( A^{-1} = \Lambda^{-1} \circ P_{1T}^{-1} \) is bounded in \( L_2 \). Let

\[
\Gamma \equiv \Lambda \circ A^{-1} = P_{1T}^{-1} \tag{5.5}
\]

(linear bounded operator from \( L_2 \) to \( L_2(B_T) \)), then \( \nu_1 \equiv \Gamma u_T \) and \( u \equiv P_1 \nu_1 \) provide the desired solution in the case \( u_0 \equiv 0 \).

In the general case the solution is given by the formula

\[
\nu_1 \equiv \Gamma (u_T - Pu_0) \big|_{t=T}, \quad u \equiv Pu_0 + P_1 \nu_1 \tag{5.6}
\]

(remind that \( Pu_0 \) is the solution to problem (2.1), (1.2)–(1.4) for \( \mu_0 = \nu_0 = \nu_1 \equiv 0, f \equiv 0 \)).

Now we can prove Theorem 1.9.
Proof of Theorem 7.3. Consider first linear problem (2.1), (1.2)–(1.4). Let $u_0 \equiv 0$, $\mu_0 = \nu_0 = \nu_1 \equiv 0$, $f \equiv f_{1x}$, $f_1 \in L^2(Q_T)$. Let $P_2 f_1 \in X(Q_T)$ be the solution to this problem, existing by virtue of Lemma 2.14. In particular, estimate (2.46) yields, that $P_2$ is the linear bounded operator from $L^2(Q_T)$ to $X(Q_T)$.

Obviously, a solution $\nu_1 \in L^2(B_T)$, $u \in X(Q_T)$ to the controllability problem

$$
\begin{equation}
\begin{aligned}
&u_t + bu_x + u_{xxx} + u_{yyy} = f_{1x}, \quad f_1 \in L^2(Q_T), \\
&u|_{t=0} = u_0 \in L^2, \quad u|_{t=T} = u_T \in L^2, \quad u|_{x=0} = u|_{x=R} = 0, \quad u_x|_{x=R} = \nu_1
\end{aligned}
\end{equation}
$$

is given by the formula

$$
\begin{equation}
\begin{aligned}
\nu_1 &\equiv \Gamma (u_T - P u_0|_{t=T} - P_2 f_1|_{t=T}), \\
u &\equiv P u_0 + P_1 \nu_1 + P_2 f_1.
\end{aligned}
\end{equation}
$$

The solution to the original problem is constructed as a fixed point of the map $u = \Theta v \equiv P u_0 + (P_1 \circ \Gamma) (u_T - P u_0|_{t=T} + P_2 (v^2/2)|_{t=T}) - P_2 (v^2/2)$, defined on $X(Q_T)$. Similarly to (3.13)

$$
\begin{equation}
\begin{aligned}
&\|v^2\|_{L^2(Q_T)} \leq c\|v\|^2_{X(Q_T)}, \\
&\|v^2 - \bar{v}^2\|_{L^2(Q_T)} \leq c\left(\|v\|_{X(Q_T)} + \|\bar{v}\|_{X(Q_T)}\right)\|v - \bar{v}\|_{X(Q_T)}.
\end{aligned}
\end{equation}
$$

Therefore,

$$
\begin{equation}
\begin{aligned}
\|\Theta v\|_{X(Q_T)} &\leq c\left(\|u_0\|_{L^2} + \|u_T\|_{L^2} + \|v\|^2_{X(Q_T)}\right), \\
\|\Theta v - \Theta \bar{v}\|_{X(Q_T)} &\leq c\left(\|v\|_{X(Q_T)} + \|\bar{v}\|_{X(Q_T)}\right)\|v - \bar{v}\|_{X(Q_T)}
\end{aligned}
$$

and the standard contraction argument provides the desired result. \qed

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ZAKHAROV–KUZNETSOV EQUATION

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