FOURIER-MUKAI TRANSFORMS AND SEMI-STABLE SHEAVES ON NODAL WEIERSTRASS CUBICS

IGOR BURBAN AND BERND KREUSSLER

Dedicated to Gert–Martin Greuel on his sixtieth birthday

Abstract. We completely describe all semi-stable torsion free sheaves of degree zero on nodal cubic curves using the technique of Fourier-Mukai transforms. The Fourier-Mukai images of such sheaves are torsion sheaves of finite length, which we compute explicitly. We show that the twist functors, which are associated to the structure sheaf $\mathcal{O}$ and the structure sheaf $k(p_0)$ of a smooth point $p_0$, generate an $\text{SL}(2,\mathbb{Z})$-action (up to shifts) on the bounded derived category of coherent sheaves on any Weierstrass cubic.

1. Introduction

In recent years, derived categories of coherent sheaves on smooth projective varieties and their groups of auto-equivalences attracted a lot of interest. This was mainly driven by Kontsevich’s homological mirror symmetry conjecture [23]. It turns out [5] that interesting auto-equivalences can exist only if the canonical sheaf and its dual are not ample. From a mirror symmetry perspective, the most interesting case is the case with trivial canonical sheaf. One particular class of varieties with trivial canonical bundle are the abelian varieties. If $X$ is an abelian variety, $X^\vee$ its dual, $\mathcal{P}$ a Poincaré bundle on $X \times X^\vee$ and $\pi_1, \pi_2$ are the two projections, Mukai [28] has shown that the functor

$$\Phi_\mathcal{P} : D^b(X) \to D^b(X^\vee), \quad \Phi_\mathcal{P}(F) = R\pi_{2*}(\mathcal{P} \otimes \pi_1^*F)$$

is an exact equivalence of categories. Nowadays, such functors are called Fourier-Mukai transforms. If $X$ is a smooth elliptic curve (or more generally: a principally polarised abelian variety), one has an isomorphism $X \cong X^\vee$. In this case, the above functor induces an auto-equivalence of the derived category of $X$. In general, a result of Orlov [30] says that any equivalence between bounded derived categories of

2000 Mathematics Subject Classification. 18E30, 14H60, 14H20, 16G20.

Both authors were partially supported by DFG Schwerpunkt “Globale Methoden in der komplexen Geometrie”.

1
coherent sheaves on smooth projective varieties $X$ and $Y$ is a Fourier-
Mukai transform, where we allow $P$ to be replaced by an arbitrary
object of $D^b(X \times Y)$.

On the other hand, not so much is known in the singular case. In
this paper, we study the first non-trivial example: singular Weierstraß
cubics. Our principal tool is a Fourier-Mukai transform $F : D^b(E) \to
D^b(E)$, which we define and study with techniques introduced by Seidel
and Thomas [31] (see also [21], [24] and [27]). Our results are of interest
in the study of relative Fourier-Mukai transforms on elliptic fibrations,
see for example [2] and [7]. Interesting applications of Fourier-Mukai
transforms on Weierstraß cubics to Calogero-Moser systems can be
found in [3].

In a series of papers, Friedman, Morgan and Witten [19], [17], [18]
studied, among other things, semi-stable vector bundles on Weierstraß
cubics. We extend their results and give a complete and explicit de-
scription of all semi-stable torsion free sheaves of degree zero on a nodal
Weierstraß cubic $E$. A description of all stable vector bundles (of any
degree) on $E$ can be found in [14]. Although a description of all tor-
sion free sheaves on $E$ is available, it is a non-trivial problem to find
all semi-stable torsion free sheaves among them. With the aid of the
Fourier-Mukai transform $F$ we are able to translate this problem into
a problem which is known as a matrix problem of Gelfand type in represen-
tation theory. We use the solution of this problem which was
given in [20] and provide an explicit description of the functor $F$ on the
sheaves in question. An explicit description of $F$ on semi-stable torsion
free sheaves seems to be worthwhile, because the interplay between
such sheaves and torsion sheaves was used as an efficient technical tool
by Friedman, Morgan and others. As a result, we obtain a clear de-
scription of all semi-stable torsion free sheaves of degree zero and their
Fourier-Mukai images.

It is instructive to associate to such a sheaf (or their Fourier-Mukai
image) the so-called band and string diagram from representation the-
ory. They are particularly useful in the study of the Fourier-Mukai
image of the dual of a sheaf. This leads us to Matlis duality over lo-
cal Gorenstein rings. In the nodal case, Matlis duality is described by
reversing the direction of all arrows in the band and string diagrams.

The structure and more detailed content of this paper is the follow-
ing. In section 2 we use twist functors which are defined by spherical
objects to give a definition of the Fourier-Mukai functor $F : D^b(E) \to
D^b(E)$. One of the main results in this section is (theorem 2.18):

$$F \circ F = i^*[1],$$
where \( i : E \to E \) is the involution which is induced by taking the inverse in the group structure on the regular locus of \( E \).

This theorem and a braid group relation between two particular twist functors allow us to define an action (up to translation and in a weak sense) of \( \text{SL}(2, \mathbb{Z}) \) on \( D^b(E) \). If we let auto-equivalences of \( D^b(E) \) act on pairs formed by rank and degree, which can be defined for complexes as well, we obtain a surjection from the group of exact auto-equivalences onto \( \text{SL}(2, \mathbb{Z}) \). This is parallel to the smooth case. But, in contrast, we do not know whether the whole group of auto-equivalences of \( D^b(E) \) is generated by this \( \text{SL}(2, \mathbb{Z}) \)-action and automorphisms of \( E \). Clearly, all shifts and twists by line bundles are obtained this way.

If \( E \) is a reducible curve of arithmetic genus one, for example a cycle of rational curves, most of the techniques which are used so far are applicable as well. However, in this case it is more complicated to describe \( F \circ F \). Furthermore, if \( E \) has \( n > 1 \) components, the rank lives in \( \mathbb{Z}^n \), because it needs to be constant on the irreducible components only. Hence, the auto-equivalences of \( D^b(E) \) naturally act on \( \mathbb{Z}^{n+1} \) and we expect to be forced to replace \( \text{SL}(2, \mathbb{Z}) \) by another group.

The second main result (theorem 2.21) says that \( F \) induces an equivalence of categories:

\[
\begin{align*}
\{ & \text{semi-stable torsion free} \\
& \text{sheaves of degree zero} \} \\
\rightarrow & \{ \text{coherent torsion sheaves} \}
\end{align*}
\]

On this category, our functor \( F \) coincides with the functor introduced by Teodorescu [33], see also [17].

In section 3 we describe vector bundles on \( E \) as direct images of line bundles under cyclic Galois covers, provided \( E \) is a nodal Weierstrass cubic. The results are similar to Atiyah’s classical results for the smooth case [11]. In addition, we describe torsion free sheaves which are not locally free as direct images of line bundles via a finite morphism from chains of rational curves. In the smooth case, such sheaves do not exist. It can be derived from [14] that the sheaves described this way comprise all torsion free sheaves on \( E \). To do so, one has to relate their description of sheaves via gluing of bundles to the description by direct images of line bundles. This is possible in characteristic zero only. To prove results which are similar to ours over a field of arbitrary characteristic, one had to avoid the use of this description by direct images.

In section 4 we recall the results of Gelfand, Ponomarev [20] and Nazarova, Roiter [29] on the classification of \( R \)-modules of finite length. In contrast to the case of a regular local ring of dimension one, where indecomposable modules of finite length are determined by their length,
over the complete local Gorenstein ring $R = k[[x, y]]/(x \cdot y)$ there are plenty of such modules. As a useful tool to describe them, we introduce the band and string diagrams, which were already used in \[20\].

The main result of this paper (theorem \[5,1\]) is proven in section \[5\]. We explicitly compute the Fourier-Mukai images of certain torsion free sheaves. Using the results of sections \[2\] and \[4\] we are able to conclude that these sheaves are precisely the semi-stable torsion free sheaves of degree zero on a nodal Weierstraß cubic $E$.

Finally, in section \[6\] we study the relationship between $F(E)$ and $F(E^\vee)$, where $E$ is a semi-stable torsion free sheaf of degree zero on $E$. After briefly recalling Matlis duality we show (theorem \[6,11\]):

\[ F(E^\vee) \cong F(E)^*, \]

where $M^*$ denotes the twisted Matlis dual (definition \[6,10\]) of a module $M$ of finite length.

Throughout this paper, $k$ denotes an algebraically closed field of characteristic zero. All schemes are schemes over $k$.

Acknowledgement. Both authors like to thank G.-M. Greuel for his constant support and Yu. Drozd, H. Krause, H. Lenzing and R. Rouquier for helpful discussions about results of this paper.

\[2\]. Fourier-Mukai functors on Weierstrass cubics

For abelian varieties $X$, Mukai \[28\], Thm. 2.2, has shown that the integral transform $\Phi_P : D^b(X) \to D^b(X^\vee)$ which is given by

\[ \Phi_P(F) := R\pi_2^*(P \otimes \pi_1^*F) \]

is an equivalence of categories. Here, $P \in \text{Coh}(X \times X^\vee)$ is a Poincaré bundle and $X^\vee$ is the dual abelian variety. Nowadays, such an integral transform with arbitrary $P \in D^b(X \times Y)$ is called a Fourier-Mukai transform if it is an equivalence of categories.

We are dealing here with the one dimensional case (elliptic curves) and attempt to generalise this result to singular curves. In doing so, one encounters difficulties which are caused by the fact that a Poincaré sheaf $P$ is not locally free if the curve is singular. To circumvent such problems, we use techniques and results from P. Seidel and R. Thomas \[31\]. They study twist functors defined by spherical objects and relate them to Fourier-Mukai transforms. These twist functors resemble the mutations which were used in the study of exceptional vector bundles (see \[10, 21\]). In a more algebraic framework such functors were studied by H. Meltzer in \[27\], where they are called tubular mutations (see also \[24\]).
Definition 2.1. A Weierstraß cubic $E$ is a plane cubic curve which is given in $\mathbb{P}^2$ by an equation
\[ y^2z = 4x^3 - g_2xz^2 - g_3z^3, \]
where $(x : y : z)$ are homogeneous coordinates on $\mathbb{P}^2$ and $g_2, g_3 \in k$ are constants.

Remark 2.2. Weierstraß cubics are precisely the reduced and irreducible curves of arithmetic genus one. A Weierstraß cubic $E$ has at most one singular point. It is singular if and only if $g_3^2 = 27g_2^3$. Unless $g_2 = g_3 = 0$, the singularity is a node (ordinary double point), whereas in the case $g_2 = g_3 = 0$ the singularity is a cusp.

If $E$ is singular, its normalisation is a morphism $n : \mathbb{P}^1 \to E$ and the canonical morphism of sheaves $\mathcal{O} \to n_*n^*\mathcal{O}$ on $E$ is injective with cokernel $k(s)$, the structure sheaf of the singular point $s \in E$.

Because any Weierstraß cubic $E$ is given by a single equation in $\mathbb{P}^2$, it is Gorenstein. Its dualising sheaf is $\omega_E \cong \mathcal{O}_E$, because the equation has degree three. In particular, Serre duality \[22\], II.7, holds, this means for any integer $i \in \mathbb{Z}$ there exists a functorial isomorphism:
\[ \text{Ext}^i(E, F) \cong \text{Ext}^{1-i}(F, E)^\vee, \]
where $F \in \text{Coh}(E)$ is a coherent sheaf and $E$ is locally free of finite rank.

We use $D^b(E)$ to denote the bounded derived category of coherent sheaves on $E$. Complexes $F^\bullet$ are considered to be cochain complexes: $\cdots \to F^{-2} \to F^{-1} \to F^0 \to F^1 \to F^2 \to \cdots$ and the shift functor is $(F^\bullet[1])^k = F^{k+1}$. In particular, if $E$ is a coherent sheaf, $E[-k]$ denotes the complex with $E$ at position $k$ and zero elsewhere.

Using induction on the length of the complex and the standard notation $\text{Hom}^i(E^\bullet, F^\bullet) = \text{Hom}_{D^b(E)}(E^\bullet, F^\bullet[i])$, we deduce the following from Serre-duality in its formulation above: for any $i \in \mathbb{Z}$ and $F \in D^b(E)$ and any bounded complex of locally free sheaves $E$ there exist functorial isomorphisms
\[ \text{Hom}^i(E, F) \cong \text{Hom}^{1-i}(F, E)^\vee. \]

On a smooth curve this is true for arbitrary $E \in D^b(E)$, because any coherent sheaf on a smooth projective variety has a finite resolution by locally free sheaves. If $E$ is singular, this is no longer true.

The definition of spherical objects which was given by P. Seidel and R. Thomas \[31\], Definition 2.14, reads in our context as follows:

Definition 2.3. An object $E \in D^b(E)$ is called spherical, if the following conditions are satisfied:
(S1) $E$ has a finite resolution by injective quasi-coherent sheaves;
(S2) for any $F \in D^b(E)$ the total morphism spaces $\text{Hom}^*(E, F)$ and $\text{Hom}^*(F, E)$ are of finite dimension;
(S3) $\text{Hom}^i(E, E) \cong \begin{cases} k & \text{if } i = 0, 1 \\ 0 & \text{otherwise} \end{cases}$;
(S4) the composition map $\text{Hom}^i(F, E) \times \text{Hom}^{1-i}(E, F) \to \text{Hom}^1(E, E) \cong k$

is a non-degenerate pairing for any $i \in \mathbb{Z}$ and any $F \in D^b(E)$.

**Proposition 2.4.** If $E$ is a Weierstraß cubic and $E \in D^b(E)$ is isomorphic to a bounded complex of locally free sheaves, then:

$$E \text{ is spherical } \iff \text{Hom}^i(E, E) \cong \begin{cases} k & \text{if } i = 0 \\ 0 & \text{if } i < 0 \end{cases}$$

*Proof.* Serre duality gives (S4). The assumption and (S4) imply (S3). Now, (S2) follows from (S4), $\text{Hom}^*(E, F) \cong \mathbb{H}^*(E^\vee \otimes F)$ and the standard finiteness theorem on projective schemes. Finally, (S1) is true because $E$ is Gorenstein and hence any locally free sheaf has finite injective dimension (see e.g. [10], Section 3).  

**Lemma 2.5.** Let $X$ be a Gorenstein scheme, which is projective over $k$. If $E$ is a coherent sheaf which has a finite resolution by injective quasi-coherent sheaves, then $E$ has a finite resolution by locally free sheaves.

*Proof.* A finitely generated module over a local Gorenstein ring has finite injective dimension if and only if it has finite projective dimension [25], Thm. 2.2. Hence, our assumptions imply: $E_x$ has finite projective dimension for any $x \in X$. Using the Auslander-Buchsbaum formula we can bound these projective dimensions by $\dim(X)$. This means

$$\text{Ext}^i(E_x, -) = 0 \quad \text{if } i > \dim(X) \quad \text{and } x \in X.$$  

On the other hand, because $X$ is projective, any coherent sheaf has a (not necessarily finite) resolution $\cdots \to L^{-n} \to L^{-n+1} \to \cdots \to L^0$ by locally free sheaves $L^i$ of finite rank. If this complex has length greater than $d = \dim(X)$, we replace $L^{-d}$ by the kernel $K \subset L^{-d+1}$ of the following map. This produces a bounded acyclic complex of coherent sheaves $0 \to K \to L^{-d+1} \to L^{-d+2} \to \cdots \to L^0 \to E \to 0$. By standard arguments from homological algebra, we obtain

$$\mathcal{E}xt^1(K, -) \cong \mathcal{E}xt^{d+1}(E, -) = 0.$$
This implies that all the stalks $K_x$ are projective, hence, free. Therefore, we have got a finite resolution of $E$ consisting of coherent locally free sheaves.

□

**Corollary 2.6.** A coherent sheaf $E$ on $E$ is spherical, if and only if

1. $E$ has a finite resolution by locally free sheaves and
2. $\text{Hom}(E, E) \cong k$.

**Example 2.7.**

(a) The structure sheaf $k(x)$ of a regular point $x \in E$ is spherical.

(b) If $L \in \text{Pic}(E)$ is a locally free sheaf of rank one, we have $\text{Hom}(L, L) \cong H^0(L^\vee \otimes L) \cong H^0(\mathcal{O}_E) \cong k$. Hence, $L$ is spherical.

(c) More generally, any simple vector bundle is spherical.

**Remark 2.8.** The structure sheaf $k(y)$ of a singular point $y \in E$ does not have a finite locally free resolution, hence it is not spherical.

Seidel and Thomas carefully define a *twist functor*

$$T_E : D^b(E) \to D^b(E)$$

which is an exact equivalence, if $E$ is spherical. Basically, for any $F \in D^b(E)$, the object $T_E(F) \in D^b(E)$ is the cone over the “evaluation map”

$$R \text{Hom}(E, F) \otimes E \to F.$$

More precisely, they replace $E$ and $F$ by injective resolutions $I_E$ and $I_F$ which consist of quasi-coherent sheaves and define $T_E(F)$ to be the cone over the map of complexes

$$\text{hom}(I_E, I_F) \otimes I_E \to I_F$$

whose non-zero components are the usual evaluation maps

$$\text{Hom}(I_E^k, I_F^{k+i}) \otimes I_E^i \to I_F^{k+i}.$$

Here, $\text{hom}(I_E, I_F)$ denotes the complex of vector spaces which has $\oplus_i \text{Hom}(I_E^i, I_F^{k+i})$ at place $k$. If $E, F$ are coherent sheaves, the $k$-th cohomology of $\text{hom}(I_E, I_F)$ is $\text{Ext}^k(E, F)$.

Because the objects in $I_F$ are injective, basic homological algebra shows that the (genuine) evaluation map

$$\text{hom}(E, I_F) \otimes E \to I_F$$

is quasi-isomorphic to the above map (see [31], Prop. 2.6 and Lemma 3.2). Now, $R \text{Hom}(E, F)$ is the complex $\text{hom}(E, I_F)$, hence this map defines the above “evaluation map” as a morphism in $D^b(E)$.

Most of our computations will be based on the following lemma.
Lemma 2.9. Suppose $E, F$ are coherent sheaves on $E$, $E$ is locally free and $\text{Ext}^1(E, F) = 0$. Then:

$$T_E(F) \cong \text{cone}(\text{Hom}(E, F) \otimes E \xrightarrow{ev} F).$$

Proof. According to the definition of the twist functor $T_E$, the object $T_E(F)$ is quasi-isomorphic to the cone over the evaluation map $\text{hom}(E, I_F) \otimes E \to I_F$, where $F \to I_F$ is an injective resolution. Because $E$ is a curve, our assumption implies that the natural inclusion $\text{Hom}(E, F) \to \text{hom}(E, I_F)$ is a quasi-isomorphism. Because $E$ is locally free, this is true for the inclusion $\text{Hom}(E, F) \otimes E \to \text{hom}(E, I_F) \otimes E$ as well. The claim follows now from the commutative diagram:

$$
\begin{array}{ccc}
\text{Hom}(E, F) \otimes E & \xrightarrow{ev} & F \\
\downarrow & & \downarrow \\
\text{hom}(E, I_F) \otimes E & \xrightarrow{ev} & I_F,
\end{array}
$$

in which the horizontal maps are ordinary evaluation maps and the vertical maps are quasi-isomorphisms. $\square$

Proposition 2.10. If $E \in D^b(E)$ is a spherical object which is isomorphic to a bounded complex of locally free sheaves, the twist functor $T_E$ is isomorphic to the Fourier-Mukai functor $\Phi_P$, whose kernel $P$ is the cone of the natural map $R\text{Hom}(\pi_2^*E, \pi_1^*E) \to O_\Delta$.

The proof of [31], Lemma 3.2, carries over to our situation literally.

Example 2.11. If $E = O_E$ we obtain $R\text{Hom}(\pi_2^*E, \pi_1^*E) \cong O_{E \times E}$ and $P \cong I_{E \times E}$, where $I_{E \times E}$ is the ideal sheaf of the diagonal $\Delta \subset E \times E$. This implies:

$$T_O \cong \Phi_{I_{E \times E}[1]}.$$

Example 2.12. If $k(x)$ is the structure sheaf of a regular point $x \in E$ and $O_E(x)$ is the locally free sheaf which corresponds to the Cartier divisor $x$ on $E$, [31], 3.11, shows:

$$T_{k(x)}(F) \cong F \otimes O_E(x)$$

for any $F \in D^b(E)$. In particular, $T_{k(x)}$ is isomorphic to the Fourier-Mukai transform whose kernel is the sheaf $\pi_2^*O_E(x) \otimes O_\Delta$.

We shall use the following lemma in the proof of theorem 2.18 below.

Lemma 2.13. Let $x, y \in E$ be closed points and suppose $x$ is a regular point. By $I_y \subset O$ we denote the ideal sheaf of $y$. Then there are
isomorphisms:

\[ T_\mathcal{O}(k(y)) \cong \mathcal{I}_y[1] \]
\[ T_\mathcal{O}(\mathcal{O}(x)) \cong k(x) \]
\[ T_\mathcal{O} (\mathcal{O}) \cong \mathcal{O}. \]

If \( E \) is singular with singular point \( s \), normalisation \( n : \mathbb{P}^1 \to E \) and \( \mathcal{O} = n_*(\mathcal{O}_{\mathbb{P}^1}) \), we have:

\[ T_\mathcal{O}(\mathcal{O}) \cong k(s). \]

**Proof.** The first two isomorphisms are obtained from lemma 2.9 with \( E = \mathcal{O}, F = k(y) \) and \( F = \mathcal{O}(x) \) respectively.

To compute \( T_\mathcal{O}(\mathcal{O}) \), we use an injective resolution \( \mathcal{O} \to I_\mathcal{O} \) of \( \mathcal{O} \) and the exact sequence of complexes which is obtained from the definition of the mapping cone:

\[ 0 \to I_\mathcal{O} \to T_\mathcal{O}(\mathcal{O}) \to \text{hom}(\mathcal{O}, I_\mathcal{O}) \otimes \mathcal{O}[1] \to 0 \]

and its exact cohomology sequence

\[ 0 \to H^{-1}(T_\mathcal{O}(\mathcal{O})) \to \text{Hom}(\mathcal{O}, \mathcal{O}) \otimes \mathcal{O} \xrightarrow{\delta} \mathcal{O} \to H^0(T_\mathcal{O}(\mathcal{O})) \to \text{Ext}^1(\mathcal{O}, \mathcal{O}) \otimes \mathcal{O} \to 0. \]

The connecting homomorphism \( \delta \) is in fact the \( H^0 \) of the evaluation map \( \text{hom}(\mathcal{O}, I_\mathcal{O}) \otimes \mathcal{O} \to I_\mathcal{O} \) whose cone is the complex \( T_\mathcal{O}(\mathcal{O}) \). Hence, \( \delta \) is the evaluation map \( \text{Hom}(\mathcal{O}, \mathcal{O}) \otimes \mathcal{O} \to \mathcal{O} \) which is an isomorphism. This implies: \( T_\mathcal{O}(\mathcal{O}) \) is isomorphic to a complex which is concentrated in degree zero and \( H^0(T_\mathcal{O}(\mathcal{O})) \cong \text{Ext}^1(\mathcal{O}, \mathcal{O}) \otimes \mathcal{O} \cong \mathcal{O} \), which is the claim.

Finally, if \( E \) is singular, we have \( H^0(\mathcal{O}) \cong H^0(\mathcal{O}_{\mathbb{P}^1}) \cong k \) and \( H^j(\mathcal{O}) = 0 \) for \( j \neq 0 \). The evaluation map \( H^0(\mathcal{O}) \otimes \mathcal{O} \to \mathcal{O} \) is the canonical map \( \mathcal{O} \to \mathcal{O} \). It is injective with cokernel \( k(s) \). Hence, by lemma 2.9 we obtain \( T_\mathcal{O}(\mathcal{O}) \cong k(s). \)

The key ingredient for the proof of the theorem below is the following result, versions of which can be found in [9], Section 3.3, and [19], Lemma 0.3.

**Lemma 2.14.** If \( X \) is a projective variety and \( \mathcal{G} : D^b(X) \to D^b(X) \) an integral transform, then the following conditions are equivalent:

(i) \( \mathcal{G} \) is isomorphic to the identity functor;
(ii) \( \mathcal{G}(\mathcal{O}_X) \cong \mathcal{O}_X \) and for all \( x \in X : \mathcal{G}(k(x)) \cong k(x). \)

**Proof.** Denote by \( P \in D^b(X \times X) \) the kernel of \( \mathcal{G} \), i.e. \( \mathcal{G}(E) \cong R\pi_2^*(P \otimes \pi_1^*E) \) for any \( E \in D^b(X) \). With \( E = k(x) \) we obtain
\[ R_{\pi_2} ( P \otimes \pi_1^* k(x)) \cong L_i^* P, \] where \( i_x : X \to X \times X \) denotes the embedding which satisfies \( \pi_2 \circ i_x = id_X \) and \( \pi_1 \circ i_x \) is the constant map with image \( x \). Our assumption implies \( L_i^* P \cong k(x) \) is a sheaf for any \( x \in X \). By \( [\ref{remark}] \), Lemma 4.3, this implies that \( P \) is a sheaf which is \( \pi_1 \)-flat. In particular, \( L_i^* P \cong i_i^* P \cong k(x) \).

Because \( X \) is projective, we can choose a very ample line bundle \( A \in \text{Pic}(X) \). Hence, for large positive \( m \), the canonical mapping

\[ (1) \quad \pi_1^* (P \otimes \pi_2^* A^\otimes m) \to P \otimes \pi_2^* A^\otimes m \]

is surjective \( [\ref{remark}] \), Theorem III.8.8. Observe, \( h^j((i_i^* (P \otimes \pi_2^* A^\otimes m)) = h^j(i_i^* (P) \otimes A^\otimes m) = h^j(k(x) \otimes A^\otimes m) = h^j(k(x)) = 1 \), if \( j = 0 \) and zero, if \( j > 0 \). This implies: \( L_m := \pi_1^* (P \otimes \pi_2^* A^\otimes m) \) is locally free of rank one \( [\ref{remark}] \), Corollary III.12.9. From \( [\ref{remark}] \) we obtain a surjection

\[ \mathcal{O}_{X \times X} \to P \otimes \pi_1^* L_m^\vee \otimes \pi_2^* A^\otimes m. \]

Because \( i_i^* (P \otimes \pi_1^* L_m^\vee \otimes \pi_2^* A^\otimes m) \cong k(x) \), there exists a unique morphism \( \varphi : X \to \text{Hilb}^1(X) \) which satisfies \( \varphi^* (\mathcal{U}) \cong P \otimes \pi_1^* L_m^\vee \otimes \pi_2^* A^\otimes m \), where \( \mathcal{O}_{X \times \text{Hilb}^1(X)} \to \mathcal{U} \) denotes the universal quotient sheaf on \( \text{Hilb}^1(X) \). More precisely, \( (\text{Id} \times \varphi)^* (\mathcal{O}_{X \times \text{Hilb}^1(X)}) \to \mathcal{U} \) is isomorphic to the above surjection. In our situation, \( \text{Hilb}^1(X) \cong X \) and \( \mathcal{U} \cong \mathcal{O}_\Delta \) and the morphism must be \( \varphi = \text{Id}_X \). Hence, \( P \otimes \pi_1^* L_m^\vee \otimes \pi_2^* A^\otimes m \cong \mathcal{O}_\Delta \), that is \( P \cong \mathcal{O}_\Delta \otimes \pi_1^* (L_m \otimes A^\otimes m) \). From \( \mathcal{G}(\mathcal{O}) \cong \mathcal{O} \) we obtain \( P \cong \mathcal{O}_\Delta \) which means \( \mathcal{G} \cong \text{Id} \), as desired.

In the sequel we fix a regular point \( p_0 \in E \). Because \( E \) is irreducible, there is the structure of an abelian group on its regular part and we can choose this structure in such a way that \( p_0 \) will be the neutral element in this group structure. More specifically, the choice of \( p_0 \) allows to define a bijection between the smooth points on \( E \) and the Picard group of invertible sheaves of degree zero on \( E \) which sends a point \( x \in E_{\text{reg}} \) to the sheaf \( \mathcal{O}(x - p_0) \). Via this isomorphism, the involution \( \mathcal{L} \mapsto \mathcal{L}^\vee \) on the Picard group corresponds to an involution \( i \) on the regular part of \( E \). It extends to an involution \( i : E \to E \) which fixes the singular point. If \( x \in E \) is a regular point, \( i(x) \) is characterised by the linear equivalence \( i(x) = 2p_0 - x \).

**Definition 2.15.** We define \( \mathbb{F} := T_{k(p_0)} T_{\mathcal{O}_E} T_{k(p_0)} : D^b(E) \to D^b(E) \).

**Remark 2.16.** Since \( \mathcal{O}_E \) and \( k(p_0) \) are spherical objects, the functor \( \mathbb{F} \) is an exact equivalence. Because the two spherical objects \( \mathcal{O}_E \) and \( k(p_0) \) satisfy \( \text{Hom}^i(\mathcal{O}_E, k(p_0)) \cong k \), they form an \( A_2 \) configuration in the sense of \( [\ref{remark}] \). From \( [\ref{remark}] \), Prop. 2.13, we obtain

\[ T_{k(p_0)} T_{\mathcal{O}_E} T_{k(p_0)} \cong T_{\mathcal{O}_E} T_{k(p_0)} T_{\mathcal{O}_E}. \]
Remark 2.17. Using the description of $T_O$ as a Fourier-Mukai transform, we obtain $R \pi_{2*}(I_{\Delta}[1] \otimes \pi_1^{-1}(E \otimes O(p_0))) \otimes O(p_0) \cong R \pi_{2*}(I_{\Delta}[1] \otimes \pi_1^*O(p_0) \otimes \pi_2^*O(p_0)) \otimes \pi_1^*E$. This is the Fourier-Mukai transform with kernel $I_{\Delta} \otimes \pi_1^*O(p_0) \otimes \pi_2^*O(p_0)[1]$. If $E$ is smooth, up to the shift, this is the dual of a Poincaré line bundle. From [2], Lemma 1.2, we obtain that, in the smooth case, our functor $F$ is a quasi-inverse of the usual Fourier-Mukai functor as studied by Mukai [28]. This explains why the sign of the shift in theorem 2.18 is different from Mukai’s.

Theorem 2.18. It holds $F \circ F \cong i^*[1]$. Consequently, $F^4 \cong [2].$

Proof. Because $F \circ F$ and $i^*[1]$ are Fourier-Mukai transforms, by Lemma 2.14 we just need to show that these two functors give isomorphic objects, if applied to the structure sheaf of $E$ and to structure sheaves of closed points on $E$.

Let us compute $F(F(O_E))$. From $T_O(O) \cong O$ and remark 2.16 we see: $F(F(O_E)) \cong T_{k(p_0)}T_O T_{k(p_0)}(O_E)$. Hence, and because $i^*O \cong O$, we just need to show $T_O T_{k}(E(p_0)) \cong O_E(-p_0)[1]$, but this follows from lemma 2.13.

Next, we compute $F(F(k(x)))$, where $x \in E$ is a closed point. Again, from $T_{k(p_0)}(k(x)) \cong k(x)$ and remark 2.16 we deduce: $F(F(k(x))) \cong T_O T_{k(p_0)}T_{k(p_0)}(k(x))$. If $x$ is a regular point, lemma 2.13 shows that this is isomorphic to $k(i(x))[1]$. Here we used $O(2p_0 - x) \cong O(i(x))$, which follows from the definition of $i$.

Finally, if $x = s \in E$ is the singular point, we have:

$$T_O T_{k(p_0)} T_{k(p_0)}(k(s)) \cong T_O T_{k(p_0)} T_{k(p_0)}(\hat{O}(-2)[1]) \cong T_O(\hat{O}[1]) \cong k(s)[1].$$

Here we used $I_s \cong \hat{O}(-2) \cong n_*(O(-2) \otimes O(2p_0) \cong n_*(O(-2) \otimes n^*O(2p_0)) \cong n_*O \cong \hat{O}$. Because $i(s) = s$ and $i^*k(s) \cong k(s)$ the proof is complete.

Remark 2.19. As in the smooth case, we can use this result to obtain an action of the group $\text{SL}(2, \mathbb{Z})$ on the derived category $D^b(E)$. A presentation of $\text{SL}(2, \mathbb{Z})$ is given by (see [31], section 3d):

$$\langle A, B, T \mid ABA = BAB, \ (AB)^6 = T^2, \ [A, T] = [B, T] = 1 \rangle.$$

This is a central extension of $\text{SL}(2, \mathbb{Z})$ by $\mathbb{Z}$, where the normal subgroup is generated by $T$ and the projection to $\text{SL}(2, \mathbb{Z})$ sends the generators
A, B and T to
\[
\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]
respectively.

The action of $\tilde{\text{SL}}(2, \mathbb{Z})$ on $D^b(E)$ is obtained by letting the generators $A, B$ and $T$ act as $T_{O, k(p_0)}$ and as the translation functor $[1]$.

For later use, we formulate a consequence of lemma 2.9 explicitly.

**Remark 2.20.** Suppose $E$ is a coherent sheaf on $E$ which satisfies $H^1(E(p_0)) = 0$. If the evaluation map

\[
ev : H^0(E(p_0)) \otimes O \to E(p_0)
\]

(i) is injective, then $\mathcal{F}(E) \cong \text{coker}(\ev) \otimes O(p_0)$;

(ii) is surjective, then $\mathcal{F}(E) \cong \text{ker}(\ev)[1] \otimes O(p_0)$.

Recall that the degree of a torsion free sheaf $E$ on $E$ is by definition $\deg(E) = \chi(E) = h^0(E) - h^1(E)$. Such a sheaf $E$ is called semi-stable if for any subsheaf $F \subset E$ with $0 < \text{rk}(F) < \text{rk}(E)$ it holds:

\[
\frac{\deg(F)}{\text{rk}(F)} \leq \frac{\deg(E)}{\text{rk}(E)}.
\]

Therefore, any torsion free sheaf of rank one is automatically semi-stable. If $E$ is semi-stable torsion free and $d > 0$ then:

\[
\begin{align*}
\deg(E) = d, & \quad \Rightarrow \quad h^0(E) = d, \quad h^1(E) = 0; \\
\deg(E) = -d, & \quad \Rightarrow \quad h^0(E) = 0, \quad h^1(E) = d.
\end{align*}
\]

With the aid of theorem 2.18, we are able to give a new proof of a theorem of T. Teodorescu [33]. An alternate proof can also be found in [17], Cor. 1.2.9.

**Theorem 2.21.** For any semi-stable torsion free sheaf $E$ on $E$ of degree zero, the evaluation map

\[
ev : H^0(E(p_0)) \otimes O \to E(p_0)
\]

is a monomorphism with cokernel of rank zero.

The functor which sends the sheaf $E$ to the cokernel of this evaluation map is the restriction of the functor $\mathcal{F}$. It is an exact equivalence between the category of semi-stable torsion free sheaves of degree zero and the category of coherent torsion sheaves on $E$.

**Proof.** The injectivity of the evaluation map was shown in [19], Theorem 1.2. Because $E$ is semi-stable, $E(p_0)$ is semi-stable as well. From $\deg(E) = 0$ we obtain $\deg(E(p_0)) = \text{rk}(E)$. This implies $H^1(E(p_0)) = 0$. Hence, by remark 2.20 $\mathcal{F}(E) \cong \text{coker}(\ev) \otimes O(p_0)$. Since $h^0(E(p_0)) =\]
deg\(E(p_0)\) = rk\(E\), the cokernel of the evaluation map is a torsion sheaf. This implies coker\((ev) \cong coker(ev) \otimes \mathcal{O}(p_0) \cong \mathbb{F}(E)\).

From theorem 2.18 we know that \(\mathbb{F}\) is an exact equivalence with quasi-inverse \(i^* \circ \mathbb{F}[-1]\). In order to complete the proof we just need to show for any torsion sheaf \(T\) on \(E\) that \(\mathbb{F}(T)[-1] \cong K(p_0)\), where \(K\) is the kernel of the evaluation map, sitting in the exact sequence

\[
0 \to K \to H^0(T) \otimes \mathcal{O} \to T \to 0.
\]

This implies that \(K\) is torsion free and of rank \(d = h^0(T)\) and degree \(-d\). Hence, \(\deg(K(p_0)) = 0\).

In order to prove semi-stability of \(K\), we proceed by induction on \(d\). If \(d = 1\) the rank of \(K(p_0)\) is one and \(K(p_0)\) is semi-stable. If \(d > 1\), we suppose \(\mathbb{F}(T')[-1]\) is semi-stable for any torsion sheaf \(T'\) with \(h^0(T') < d\). There exists a torsion subsheaf \(T' \subset T\) with \(0 < h^0(T') < h^0(T) = d\). This gives an exact commutative diagram

\[
\begin{array}{cccccc}
0 & \longrightarrow & H^0(T') \otimes \mathcal{O} & \longrightarrow & H^0(T) \otimes \mathcal{O} & \longrightarrow & H^0(T'') \otimes \mathcal{O} & \longrightarrow & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & \longrightarrow & T' & \longrightarrow & T & \longrightarrow & T'' & \longrightarrow & 0
\end{array}
\]

where \(T''\) is another torsion sheaf with \(h^0(T'') < d\). The vertical maps are evaluation maps and their kernels are the torsion free sheaves whose twist with \(\mathcal{O}(p_0)\) gives the value of \(\mathbb{F}[-1]\) on the corresponding torsion sheaf. The snake lemma provides therefore an exact sequence of torsion free sheaves

\[
0 \longrightarrow \mathbb{F}(T')[-1] \longrightarrow \mathbb{F}(T)[-1] \longrightarrow \mathbb{F}(T'')[-1] \longrightarrow 0
\]

in which two members are semi-stable by our assumption. Since all three sheaves are of degree zero, the middle one is semi-stable as well. \(\square\)

Because an indecomposable torsion sheaf on \(E\) is supported at precisely one point, any indecomposable semi-stable torsion free sheaf \(E\) of degree zero has the property that supp\((\mathbb{F}(E))\) is one point. If supp\((\mathbb{F}(E)) = \{x\}\), Friedman, Morgan and Witten [10] call the sheaf \(E\) to be concentrated at \(x\).

In section 5 we shall use theorem 2.21 as our main tool to give an explicit description of all semi-stable torsion free sheaves of degree zero (of finite rank) on a nodal Weierstraß cubic \(E\). This extends results of R. Friedman, J. Morgan [18] who described all semi-stable locally free
sheaves of degree zero on $E$. With our methods we obtain a new proof of their result.

The following two sections provide the necessary background knowledge about torsion free sheaves on $E$ as well as torsion sheaves on $E$ which are supported at the singularity $s \in E$, which is supposed to be a node.

3. Torsion free sheaves

Throughout this section, $E$ denotes a Weierstraß cubic with one node (ordinary double point) $s \in E$. The classification of indecomposable vector bundles in the smooth case is now classical, see [1]. In [14], Drozd and Greuel applied methods from representation theory to the classification of indecomposable torsion free sheaves on $E$. It is possible to deduce from their results that the sheaves described below comprise all indecomposable torsion free sheaves on $E$. They describe these sheaves by gluing fibres of sheaves on the normalisation, a method which was already used by Seshadri [32]. We prefer a description of such sheaves as direct images of line bundles under finite morphisms. Friedman, Morgan [18], Section 2.4, and Teodorescu [33] used similar methods to describe stable and semi-stable vector bundles.

3.1. Unipotent vector bundles. As in [1], for any integer $m \geq 1$ an indecomposable vector bundle $F_m$ of rank $m$ which satisfies $F_m^\vee \cong F_m$ and $\text{Ext}^1(F_m, \mathcal{O}) \cong k$ is defined as follows: $F_1 \cong \mathcal{O}$ and for $m \geq 1$ the bundle $F_{m+1}$ is the unique one which appears in a non-split extension

$$0 \to \mathcal{O} \to F_{m+1} \to F_m \to 0.$$ 

3.2. Étale covers. Let $E_n$ be a cycle of $n$ rational curves. This means, $E_n$ is a connected reduced curve with $n$ ordinary double points whose normalisation has $n$ connected components $D_1, D_2, \ldots, D_n$ each of which is isomorphic to $\mathbb{P}^1$. In addition, the singularities $s_1, \ldots, s_n$ are the points where these components are glued together. We choose notation in such a way that $s_\nu$ is the point where $D_\nu$ and $D_{\nu+1}$ are glued together. Here and below, we use cyclic subscripts, i.e. $D_{n+k} = D_k$ for any integer $k$. Up to isomorphism, there is only one such curve for any $n \geq 1$. It is a reduced curve of arithmetic genus one. If $n = 1$ we obtain $E_1 = E$.

Once and for all, we fix projective coordinates $(x_\nu : y_\nu)$ on each $D_\nu \cong \mathbb{P}^1$ in such a way that $s''_{\nu-1} = (1 : 0)$ and $s'_{\nu} = (0 : 1)$ are the points on $D_\nu$ which are mapped to $s_{\nu-1} \in E_n$ and $s_\nu \in E_n$ respectively.

Furthermore, we fix a normalisation map $\pi_1 : D_1 \to E_1$ which sends the point with coordinates $(1 : -1)$ to the point $p_0 \in E$ (see p. 10). The
morphism $\coprod D_\nu \to D_1$ which is the identity map on each component with respect to the coordinates chosen above, descends to an étale morphism $\pi_n : E_n \to E$. For any $n \geq 1$, this is a cyclic Galois cover of degree $n$. Our choices imply that the point $p_{0\nu} \in E_n$ which corresponds to $(1 : -1) \in D_\nu$ satisfies $\pi_n(p_{0\nu}) = p_0$.

We obtain local coordinates $x_\nu$ and $y_{\nu+1}$ for the two branches of $E_n$ which intersect at $s_\nu$. Therefore, the completion $\hat{R}_\nu$ of the local ring of $E_n$ at $s_\nu$ is isomorphic to

$$R_\nu \cong k[[x_\nu, y_{\nu+1}]]/(x_\nu \cdot y_{\nu+1}).$$

We adjust these isomorphisms in such a way that

$$k[[x, y]]/(x \cdot y) \xrightarrow{\sim} k[[x_\nu, y_{\nu+1}]]/(x_\nu \cdot y_{\nu+1})$$

which sends $x$ to $x_\nu$ and $y$ to $y_{\nu+1}$, is the isomorphism induced by $\pi_n$ between the completion $\hat{R}$ of the local ring of $E$ at the singular point $s$ and $\hat{R}_\nu$.

3.3. Line bundles on $E_n$. We denote $d_{p_0} := \sum_{i=1}^n d_i p_{0i}$, which is a divisor on $E_n$ supported in the regular locus. It is well known that $\text{Pic}(E_n) \cong \mathbb{Z}^n \times k^\times$.

We choose these isomorphisms such that the line bundle $L = L(d, \lambda)$, which corresponds to $(d, \lambda) = ((d_1, \ldots, d_n), \lambda) \in \mathbb{Z}^n \times k^\times$, satisfies:

$$d_\nu = \deg(L|_{D_\nu}).$$

We fix notation by the two requirements:

$$L(d, 1) \cong \mathcal{O}(d_{p_0}) \quad \text{and} \quad L(0, \lambda^n) \cong \pi_n^*(\mathcal{L}(0, \lambda)).$$

In case $n = 1$ all line bundles of degree one are of the form $\mathcal{O}(p)$ with a regular point $p \in E$. Hence, we obtain a bijection $P : k^\times \to E_{\text{reg}}$ which satisfies $P(1) = p_0$ and $L(1, \lambda) \cong \mathcal{O}(P(\lambda))$. Thus, for any $(d, \lambda) \in \mathbb{Z} \times k^\times$, one has

$$L(d, \lambda) \cong \mathcal{O}(P(\lambda) + (d - 1)p_0).$$

With this notation, the involution $i : E_{\text{reg}} \to E_{\text{reg}}$ is described as $i(P(\lambda)) = P(\lambda^{-1})$, i.e. $P$ is a homomorphism.

The above conditions fix our choices, because

$$L(d, \lambda) \cong \mathcal{L}(d, 1) \otimes \mathcal{L}(0, \lambda)$$

$$\cong \mathcal{O}(d_{p_0}) \otimes \pi_n^*(\mathcal{L}(0, \xi))$$

$$\cong \mathcal{O}(d_{p_0}) \otimes \pi_n^*(\mathcal{O}(P(\xi) - p_0))$$

$$\cong \mathcal{O}(\pi_n^{-1}(P(\xi)) + (d - 1)p_0)$$

where $\xi \in k^\times$ is arbitrary with $\xi^n = \lambda$. 
Our choices are made in such a way that \( \mathcal{L}(d, \lambda) \) can be described by gluing the line bundles \( \mathcal{O}(d) \). Using the coordinates chosen above and the standard local trivialisation of \( \mathcal{O}(d) \) on \( D_\nu \), the line bundle \( \mathcal{L}(d, \lambda) \) is obtained by gluing with the identity at \( s_1, \ldots, s_{n-1} \) but with \( \lambda \) at \( s_n \). This will be made more precise below.

Using the standard trivialisations means that the section \( x^a y^{d-a} \) of the line bundle \( \mathcal{O}(d) \) on \( \mathbb{P}^1 \) has a local description as \( x^a \) about the point \((0 : 1)\) where \( x \) is a local coordinate. About the point \((1 : 0)\), where \( y \) is a local coordinate, the local description is \( y^{d-a} \). More generally, the section of \( \mathcal{O}(d) \), which is given by a homogeneous polynomial \( f(x, y) \) of degree \( d \), has local descriptions \( f(x, 1) \) about the point \((0 : 1)\) and \( f(1, y) \) about the point \((1 : 0)\). The bundles \( \mathcal{O}(d) \) are glued together to give \( \mathcal{L}(d, \lambda) \) in such a way that homogeneous polynomials \( f_\nu \in H^0(D_\nu, \mathcal{O}(d)) \) represent a section of \( \mathcal{L}(d, \lambda) \) if and only if they satisfy the gluing condition:

\[
\begin{align*}
f_\nu(0 : 1) &= f_{\nu+1}(1 : 0) \\
f_n(0 : 1) &= \lambda f_1(1 : 0).
\end{align*}
\]

To see that this matches our conventions, observe that \( \mathcal{L}(d, 1) \) has a global section which is given by \( f_\nu = (x_\nu + y_\nu)^d \in H^0(D_\nu, \mathcal{O}(d)) \). This section vanishes on each component precisely at \( \rho_\nu \), because \( f_\nu \) vanishes on \( D_\nu \) precisely at the point with coordinates \((1 : -1)\). On the other hand, denoting \( 1 := (1, 1, \ldots, 1) \in \mathbb{Z}^n \), \( \mathcal{L}(1, \lambda) \cong \pi_* \mathcal{L}(1, \lambda) \) has a global section which is given by \( f_\nu = \lambda_{\nu-1} \) \( (x_\nu + \lambda y_\nu) \in H^0(D_\nu, \mathcal{O}(1)) \). This section vanishes on each component \( D_\nu \) at the point with coordinates \((\lambda : -1)\). This means in particular \( P(\lambda) = \pi_1(\lambda : -1) \) and \( \lambda(P(\lambda)) = \pi_1(-1 : \lambda) \), hence \( \lambda : \mathcal{E} \to \mathcal{E} \) is the morphism which is induced by the map \((x : y) \mapsto (y : x)\) on the normalisation \( D_1 \cong \mathbb{P}^1 \).

### 3.4. Vector bundles via étale covers.

\[ \mathcal{B}(d, m, \lambda) := \pi_{n*} \mathcal{L}(d, \lambda) \otimes \mathcal{F}_m \]

is a vector bundle of rank \( mn \) and degree \( m \sum_{\nu=1}^{n} d_\nu \).

It is indecomposable if and only if \( d \) is non-periodic. This means that there does not exist an integer \( n' < n \) and a vector \( e \in \mathbb{Z}^{n'} \) such that \( d = (e, e, \ldots, e) \). Equivalently, \( \mathcal{L}(d, \lambda) \) is not the pull back \( \pi_{n', n*}(\mathcal{L}(e, \lambda')) \) of a line bundle under an étale morphism \( \pi_{n', n} : \mathcal{E}_{n'} \to \mathcal{E}_n \).

### 3.5. Chains of lines. Let \( I_n \subset \mathcal{E}_{n+1} \) be the chain of projective lines which is obtained from \( \mathcal{E}_{n+1} \) by removing one component (say \( D_{n+1} \)). The singularities of \( I_n \) are the points \( s_1, s_2, \ldots, s_{n-1} \). We use the same coordinates and conventions as before, with the exception that the
subscripts are not cyclic. We denote the restriction of $\pi_{n+1}$ to $I_n$ by $p_n : I_n \to E$. The two smooth points in $p_n^{-1}(s)$ are denoted by $s_0$ (corresponding to $(1 : 0) \in D_1$) and $s_n$ (corresponding to $(0 : 1) \in D_n$). If $n = 1$ we have $I_1 \cong \mathbb{P}^1$ and $p_1 : I_1 \to E$ is the normalisation. In this case, $s_0$ and $s_1$ are the two preimages of the singular point $s \in E$. We have $\text{Pic}(I_n) \cong \mathbb{Z}^n$ and $L = L(d) \in \text{Pic}(I_n)$ is determined by the degrees $d_\nu = \deg(L|_{D_\nu})$.

The description of the completed local rings at the singularities $s_\nu$ does not differ from the case $E_{n+1}$. If $1 \leq \nu \leq n - 1$ the mappings $R \to R_\nu$ between the completed local rings which are induced by $p_n$ have the same description as above. As before, $R = k[[x, y]]/(x \cdot y)$.

The completed local rings at the regular points $s_0$ and $s_n$ are isomorphic to $R_0 = k[[y_1]]$ and $R_n = k[[x_n]]$. We choose these isomorphisms such that the mapping $R \to R_0$, which is induced by $p_n$, sends $x$ to $0$ and $y$ to $y_1$. Similarly, $R \to R_n$ sends $x$ to $x_n$ and $y$ to $0$.

3.6. Non-locally free sheaves.

$$S(d) := p_{n*}L(d)$$

is an indecomposable torsion free sheaf of rank $n$ and degree $1 + \sum_{\nu=1}^{n} d_\nu$ on $E$. The sheaf $S(d)$ is not locally free. If $n = 1$ and $d \in \mathbb{Z}$, $S(d)$ is the sheaf which was denoted $\mathcal{O}(d)$ in section 2.

3.7. Useful results.

**Lemma 3.1.** Let $m \geq 1$, $\lambda \in k^\times$ and $d \in \mathbb{Z}$ which satisfies $d_\nu \geq 0$ for any $\nu$ and $\sum_{\nu=1}^{n} d_\nu > 0$. Then, the following holds:

- $h^0(B(d, m, \lambda)) = m \sum_{\nu=1}^{n} d_\nu$
- $h^0(S(d)) = 1 + \sum_{\nu=1}^{n} d_\nu$
- $h^1(B(d, m, \lambda)) = 0$
- $h^1(S(d)) = 0$

**Proof.** We defined $B(d, 1, \lambda) = \pi_{n*}L(d, \lambda)$. Finiteness of $\pi_n : E_n \to E$ implies $H^i(B(d, 1, \lambda)) \cong H^i(L(d, \lambda))$. Let $\eta : \bigsqcup_{\nu=1}^{n} D_\nu \to E_n$ denote the normalisation. There is an exact sequence

$$0 \longrightarrow L(d, \lambda) \longrightarrow \eta_* \eta^* L(d, \lambda) \longrightarrow \bigoplus_{\nu=1}^{n} k(s_\nu) \longrightarrow 0$$

Because $H^0(\eta_* \eta^* L(d, \lambda)) \cong H^0(\eta^* L(d, \lambda)) \cong \bigoplus_{\nu=1}^{n} H^0(D_\nu, \mathcal{O}(d_\nu))$, $H^0(\alpha) = 0$ is precisely our gluing condition. With the choices made above, an explicit description of $H^0(\alpha)$ is the following: the $\nu$-th component of $H^0(\alpha)(f_1, \ldots, f_n)$ is

- $f_\nu(0 : 1) - f_{\nu+1}(1 : 0)$ if $\nu < n$
- $f_n(0 : 1) - \lambda f_1(1 : 0)$ if $\nu = n$. 
Because all $d_n \geq 0$ and at least one of them is positive, it is now easy to see that $H^0(\alpha)$ is surjective. Using $H^1(\eta, \eta^* L(d, \lambda)) \cong H^1(\eta^* L(d, \lambda)) \cong \bigoplus_{\nu=1}^n H^1(D_n, O(d_n)) = 0$, the exact sequence implies $H^1(L(d, \lambda)) = 0$ and $h^0(L(d, \lambda)) = h^0(\bigoplus O(d_{\nu})) - n = \sum_{\nu=1}^n d_{\nu} - n = \sum_{\nu=1}^n d_{\nu}$.

Using induction and the exact sequences

$$0 \longrightarrow \mathcal{B}(d, 1, \lambda) \longrightarrow \mathcal{B}(d, m + 1, \lambda) \longrightarrow \mathcal{B}(d, m, \lambda) \longrightarrow 0$$

the remaining statements about $\mathcal{B}(d, m, \lambda)$ follow. In particular, we have shown $\deg(\mathcal{B}(d, m, \lambda)) = m \sum_{\nu=1}^n d_n$ under the assumptions made.

The sheaves $S(d)$ are studied in the same way. This time we use the finite morphism $p_n : I_n \to E$ and a normalisation $\eta : \prod_{\nu=1}^n D_{\nu} \to I_n$.

We obtain an exact sequence

$$0 \longrightarrow L(d) \longrightarrow \eta^* \eta^* L(d) \longrightarrow \bigoplus_{\nu=1}^{n-1} k(s_\nu) \longrightarrow 0$$

The description of $H^0(\beta)$ is as above but without the $n$-th component involving $\lambda$. The surjectivity is seen easily. We might allow all $d_{\nu}$ to be zero here. As above, we conclude with $H^1(S(d)) = 0$ and $h^0(S(d)) = h^0(\bigoplus_{\nu=1}^n O(d_{\nu})) - (n - 1) = 1 + \sum_{\nu=1}^n d_{\nu}$. $\square$

For $\mathcal{B}(d, m, \lambda)$ with arbitrary $d$ this result is contained in [15]. We include a proof here, because we did not show that our definition of the sheaves $\mathcal{B}(d, m, \lambda)$ coincides with theirs.

The following lemma will be used at the end of section 6.

**Lemma 3.2.** If $d = (d_1, \ldots, d_n) \in \mathbb{Z}^n, \lambda \in k^\times, m \geq 1$, we have:

1. $\mathcal{B}(d, m, \lambda)^\vee \cong \mathcal{B}(-d, m, \lambda^{-1})$
2. $S(d)^\vee \cong S(\kappa - d)$ with $\kappa = \begin{cases} \{-1, 0, \ldots, 0, -1\} & \text{if } n \geq 2 \\ -2 & \text{if } n = 1. \end{cases}$

**Proof.** If $f : X \to E$ is a finite morphism, $F$ a coherent sheaf on $X$ and $G$ a coherent sheaf on $E$, there is a natural $f_* \mathcal{O}_X$-morphism

$$f_* \mathcal{H}om_X(F, f^! G) \cong \mathcal{H}om_E(f_* F, G).$$

The coherent $\mathcal{O}_X$-module $f^! G$ is characterised by the isomorphism of $f_* \mathcal{O}_X$-modules $f_* f^! G \cong \mathcal{H}om_E(f_* \mathcal{O}_X, G)$. Recall that $f^! \omega_E$ is a dualising sheaf on $X$ if $\omega_E$ is one on $E$. In our situation $\omega_E \cong \mathcal{O}_E$ and we obtain an isomorphism

$$f_* \mathcal{H}om_X(F, \omega_X) \cong \mathcal{H}om_E(f_* F, \mathcal{O}_E) \cong (f_* F)^\vee.$$

To show (i), we consider $X = E_n$ and $f = \pi_n$. The claim follows now from $\omega_{E_n} \cong \mathcal{O}_{E_n}$, $F_m^\vee \cong F_m$ and $L(d, \lambda)^\vee \cong L(-d, \lambda^{-1})$ on $E_n$.

For the proof of (ii) we let $X = I_n$ and $f = p_n$. Now $\omega_{I_n} \cong L(\kappa)$ and the result follows from $L(d)^\vee \cong L(-d)$ on $I_n$. $\square$
Based on the results collected above, we easily obtain:

**Corollary 3.3.** Any locally free sheaf on $E$ which is isomorphic to $L \otimes F^m$ with $L \in \text{Pic}(E)$ is semi-stable. For any $d \in \mathbb{Z}$, the rank one torsion free sheaves $\mathcal{S}(d)$ are semi-stable. On the sheaves of degree zero among those, the functor $\mathbb{F}$ has the following description:

$$\mathbb{F}(\mathcal{S}(-1)) \cong k(s) \quad \text{and} \quad \mathbb{F}(\mathcal{B}((0), m, \lambda)) \cong \mathcal{O}_{\mathcal{P}(\lambda)}/m_{\mathcal{P}(\lambda)}^m.$$ 

**Proof.** The semi-stability is clear from the definitions. The description of $\mathbb{F}$ follows easily from theorem 2.21 if the rank is one. If $m > 1$ we use the extensions which define the vector bundles $\mathcal{F}_m$ and proceed by induction. \qed

In section 5 we shall describe the semi-stable torsion free sheaves of degree zero and arbitrary rank as well as their images under $\mathbb{F}$. This requires some knowledge about torsion sheaves supported at the node $s \in E$ which will be collected in section 4.

### 4. Torsion Sheaves

In this section we study indecomposable modules of finite length over the complete local ring $R = k[[x, y]]/(x \cdot y)$. Such a module is given by a finite dimensional vector space $V$ over $k$ and two commuting $k$-linear nilpotent endomorphisms $X, Y : V \to V$ which satisfy $XY = 0$. Two modules $(V, X, Y)$ and $(V', X', Y')$ are isomorphic if there is an isomorphism of $k$-vector spaces $S : V \to V'$ such that $SX = X'S, SY = Y'S$. Hence, the classification of $R$-modules of finite length is equivalent to the classification of pairs of commuting nilpotent matrices $(X, Y)$ with $XY = 0$ modulo simultaneous conjugation. This problem was solved by Gelfand and Ponomarev \[20\] (see also \[29\], \[6\]). They gave a complete classification of such modules.

Let us recall their results. There are two types of indecomposable $R$-modules of finite length: the so-called bands (forming a family depending on one continuous and some discrete parameters) and strings (a discrete family). In \[20\] strings were called modules of the first kind and bands were called modules of the second kind. For our purposes the following description of indecomposable objects is the most appropriate. Using the band and string diagrams introduced below, it is not hard to derive this description from the explicit description which was given in \[6\], see \[12\].

**Bands.** A band $\mathcal{M}(q, m, \lambda)$ depends on an integer $m \geq 1$, a parameter $\lambda \in k^\times$ and a non-periodic sequence of pairs of integers

$$q = (n_1, m_1)(n_2, m_2)\ldots(n_N, m_N)$$
with \( n_i, m_i \geq 1 \) and \( N \geq 1 \). Its minimal free resolution is

\[
0 \longrightarrow R^{mN} \xrightarrow{M(q,m,\lambda)} R^{mN} \longrightarrow \mathcal{M}(q,m,\lambda) \longrightarrow 0,
\]

where

\[
M(q,m,\lambda) = \begin{pmatrix}
x^{n_1} I_m & y^{m_1} I_m & 0 & \cdots & 0 \\
0 & x^{n_2} I_m & y^{m_2} I_m & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & y^{m_{N-1}} I_m \\
y^{m_N} J_m(\lambda) & 0 & 0 & \cdots & x^{m_N} I_m
\end{pmatrix}.
\]

If \( N = 1 \) this reads as

\[
M((n_1, m_1), m, \lambda) = (x^{n_1} I_m + y^{m_1} J_m(\lambda)).
\]

Here, by \( I_m \) we denote the identity matrix of size \( m \times m \) and by \( J_m(\lambda) \) we denote the Jordan block of size \( m \times m \) and with eigenvalue \( \lambda \). We obtain the same module, if we place \( J_m(\lambda) \) as a factor at any other power of \( y \), whereby we ensure that it appears only once in the matrix. Because \( J_m(\lambda)^{-1} \) and \( J_m(\lambda^{-1}) \) are similar matrices, instead we could place \( J_m(\lambda^{-1}) \) at precisely one position as a factor of a power of \( x \).

Any band module has finite injective dimension. A cyclic permutation of the pairs which constitute \( q \) does not change the band module.

**Strings.** A *string* module \( \mathcal{N}(q) \) depends on a sequence of integers

\[
q = n_0(m_1, n_1)(m_2, n_2) \cdots (m_N, n_N)m_{N+1},
\]

where \( n_0, m_{N+1} \geq 0 \) and \( n_i, m_i \geq 1 \) for \( 1 \leq i \leq N \). It is convenient to write their resolution in the form

\[
0 \longrightarrow R^{N+1} \xrightarrow{N(q)} k[[y]] \oplus R^N \oplus k[[x]] \longrightarrow \mathcal{N}(q) \longrightarrow 0,
\]

where

\[
N(q) = \begin{pmatrix}
y^{n_0} & 0 & 0 & \cdots & 0 \\
x^{m_1} & y^{n_1} & 0 & \cdots & 0 \\
0 & x^{m_2} & y^{n_2} & \cdots & \vdots \\
0 & \ddots & \ddots & \ddots & \vdots \\
\vdots & 0 & 0 & x^{m_N} & y^{n_N} \\
0 & 0 & 0 & 0 & x^{m_{N+1}}
\end{pmatrix}.
\]

If \( N = 0 \) we obtain \( q = n()m \) and the above exact sequence becomes

\[
0 \longrightarrow R \xrightarrow{(y^n_x)} k[[y]] \oplus k[[x]] \longrightarrow \mathcal{N}(n()m) \longrightarrow 0.
\]

If \( q = 0()0 \), this means \( \mathcal{N}(0()0) \cong k(s) \). In all other cases the module has length at least two. String modules do not have finite injective dimension.
According to [20], any indecomposable $R$-module of finite length is isomorphic to one of the bands or strings described above. Hence, any indecomposable $R$-module of finite length and of finite injective dimension is isomorphic to a module $\mathcal{M}(q, m, \lambda)$. Whereas, if such a module has infinite injective dimension, it must be isomorphic to a module $\mathcal{N}(q)$.

**Remark 4.1.** With $q$ as above, the matrix

$$
N'(q) := \begin{pmatrix}
1 & 0 & \ldots & \ldots & \ldots & 0 \\
1 & y^{n_0} & 0 & 0 & \ldots & 0 \\
0 & x^m & y^{n_1} & 0 & \ldots & 0 \\
0 & 0 & x^{m_2} & x^{n_2} & \ldots & \vdots \\
0 & 0 & \ldots & \ldots & \ldots & 0 \\
0 & \vdots & 0 & 0 & x^{m_N} & y^{n_N} \\
0 & 0 & 0 & 0 & 0 & x^{m_{N+1}}
\end{pmatrix}
$$

defines a linear mapping $R^{N+2} \to k[[y]] \oplus R^{N+1} \oplus k[[x]]$ whose cokernel is isomorphic to $\mathcal{N}(q)$. To see this, we subtract $y^{n_0}$ times the first column from column two and add row one to row two in $N'(q)$. This corresponds to applying $R$-linear automorphisms to $R^{N+2}$ and $k[[y]] \oplus R^{N+1} \oplus k[[x]]$ and transforms $N'(q)$ to:

$$
\tilde{N}'(q) := \begin{pmatrix}
0 & y^{n_0} & \ldots & \ldots & \ldots & 0 \\
1 & 0 & 0 & 0 & \ldots & 0 \\
0 & x^m & y^{n_1} & 0 & \ldots & 0 \\
0 & 0 & x^{m_2} & x^{n_2} & \ldots & \vdots \\
0 & 0 & \ldots & \ldots & \ldots & 0 \\
0 & \vdots & 0 & 0 & x^{m_N} & y^{n_N} \\
0 & 0 & 0 & 0 & 0 & x^{m_{N+1}}
\end{pmatrix}.
$$

Hence, we can split $\tilde{N}'(q) = N(q) \oplus \text{id}_R$, which proves the claim.

In the same way, we can show that the cokernel of

$$
N''(q) := \begin{pmatrix}
y^{n_0} & 0 & 0 & \ldots & 0 & 0 \\
x^m & y^{n_1} & 0 & \ldots & 0 & 0 \\
0 & x^{m_2} & y^{n_2} & \ldots & \vdots & 0 \\
0 & \ldots & \ldots & \ldots & 0 & 0 \\
0 & 0 & 0 & x^{m_N} & y^{n_N} & \vdots \\
0 & 0 & 0 & 0 & x^{m_{N+1}} & 1 \\
0 & \ldots & \ldots & \ldots & 0 & 1
\end{pmatrix}
$$

is isomorphic to $\mathcal{N}(q)$. To see this, we subtract $y^{n_0}$ times the first column from column two and add row one to row two in $N''(q)$. This corresponds to applying $R$-linear automorphisms to $R^{N+2}$ and $k[[y]] \oplus R^{N+1} \oplus k[[x]]$ and transforms $N''(q)$ to:

$$
\tilde{N}''(q) := \begin{pmatrix}
0 & y^{n_0} & \ldots & \ldots & \ldots & 0 \\
1 & 0 & 0 & 0 & \ldots & 0 \\
0 & x^m & y^{n_1} & 0 & \ldots & 0 \\
0 & 0 & x^{m_2} & x^{n_2} & \ldots & \vdots \\
0 & 0 & \ldots & \ldots & \ldots & 0 \\
0 & \vdots & 0 & 0 & x^{m_N} & y^{n_N} \\
0 & 0 & 0 & 0 & 0 & x^{m_{N+1}}
\end{pmatrix}.
$$

Hence, we can split $\tilde{N}''(q) = N(q) \oplus \text{id}_R$, which proves the claim.
is isomorphic to $N(q)$. In this sense,

$$
\begin{align*}
q &= n_0(m_1, n_1) \ldots (m_N, n_N) m_{N+1} \\
q' &= 0(0, n_0)(m_1, n_1) \ldots (m_N, n_N) m_{N+1} \\
q'' &= n_0(m_1, n_1) \ldots (m_N, n_N)(m_{N+1}, 0)0 \\
q''' &= 0(0, n_0)(m_1, n_1) \ldots (m_N, n_N)(m_{N+1}, 0)0.
\end{align*}
$$

all define the same $R$-module $N(q)$.

Having these resolutions it is easy to compute the matrices of the multiplication with $X$ and $Y$. But it is much more instructive to visualise these $R$-modules through a directed graph. We refer to them as band and string diagrams. Such diagrams were introduced by Gelfand and Ponomarev [20]. They contain $\dim(V)$ vertices in the case of strings and $\frac{1}{m} \dim(V)$ vertices in the case of bands. Each vertex corresponds to a subspace $V_i$ of $V$. In the case of strings $\dim(V_i) = 1$, whereas in the case of bands one has $\dim(V_i) = m$. The $V_i$ are quotients of subspaces of the direct summands of the middle term in the resolution which defines the band or string module. These subspaces satisfy $X(V_i) \subset V_{i-1}, Y(V_i) \subset V_{i+1}$ and the restrictions of $X, Y$ on any $V_i$ are either isomorphisms or zero. If such a restriction is an isomorphism it is represented by an arrow in the diagram. In the case of strings, one can choose a basis vector of each of the $V_i$ such that all these isomorphisms map these basis vectors onto each other. In this case, we label the arrows with $x$ respectively $y$ if it represents the restriction of $X$ or $Y$, respectively. In the case of bands, in addition, we have $Y(V_n) \subset V_1$. After choosing appropriate bases all but one of these isomorphisms can be represented by the identity matrix $I_m$. Usually, we normalise the restriction of $Y$ to the space $V_n$ which corresponds to the “last” vertex to be the Jordan block $J_m(\lambda)$. Equivalently, this could be done with any other non-zero restriction of $Y$. Instead, we could normalise one of the non-zero restrictions of $X$ to be $J_m(\lambda^{-1})$. In any case, the arrows in a band diagram are labelled by $x$ or $y$ as above and the matrix which represents the corresponding isomorphism.

Details and proofs can be found in [20]. We give here two examples.

**Example 4.2.** The string $N(2(3, 2)1)$ is represented by the diagram:
Example 4.3. The band $\mathcal{M}(((2, 2)(3, 4)(1, 3)), m, \lambda)$ is represented by:

5. Semi-Stable torsion free sheaves of degree zero

The aim of this section is twofold. First, we generalise results of [18] on semi-stable vector bundles to semi-stable torsion free sheaves. The second main achievement will be an explicit description of the functor $\mathcal{F}$ (see section 2) on semi-stable torsion free sheaves of degree zero. We use the description of torsion sheaves supported at the singular point which was given in section 4. Our main result is the following theorem.

Theorem 5.1. Let $E$ be a nodal Weierstraß cubic and $E$ a semi-stable indecomposable torsion free sheaf of degree zero on $E$ which is not of the form $\mathcal{L} \otimes \mathcal{F}_m$ with $\mathcal{L} \in \text{Pic}(E)$.

(a) If $E$ is locally free, $E \cong \mathcal{B}(d, m, \lambda)$ with $n \geq 2$ and non-periodic $d \in \mathbb{Z}^n$ of the form

$$d = (n_1, 0, \ldots, 0, -1, 0, \ldots, 0, \ldots, 1, 0, \ldots, 0, -1, 0, \ldots, 0),$$
where \( N \geq 1, m_i \geq 1, n_i \geq 1 \). Because \( E \) is semi-stable, \( \mathbb{F}(E) \) is a torsion sheaf. It is supported at \( s \in E \) and its stalk is the \( R \)-module \( \mathcal{M}(q, m, (-1)^{n+\lambda}) \), where
\[
q = (n_1, m_1)(n_2, m_2) \ldots (n_N, m_N).
\]

(b) If \( E \) is not locally free, \( E \cong S(d) \) with \( n \geq 1 \) and \( d \in \mathbb{Z}^n \) of the form
\[
d = (0, \ldots, 0, -1, 0, \ldots, 0, 1, 0, \ldots, 0, -1, \ldots, 1, 0, \ldots, 0, -1, 0, \ldots, 0),
\]
where \( N \geq 1, n_i \geq 0, m_i \geq 0 \). Because \( E \) is semi-stable, \( \mathbb{F}(E) \) is a torsion sheaf. It is supported at \( s \in E \) and its stalk is the \( R \)-module \( \mathcal{N}(q) \) with
\[
q = 0(n_1, m_1 + 1)(n_2 + 1, m_2 + 1) \ldots (n_N + 1, m_N)0.
\]
If \( n_1 = 0 \) or \( m_N = 0 \), we apply remark 4.1 and cancel some zeroes at the end. If \( N = 1, 2 \) this might lead to \( q = n(m) \).

The indecomposable semi-stable torsion free sheaves of degree zero which are excluded in the theorem were considered in corollary 3.3. The sheaves \( E \) which were considered in corollary 3.3 all satisfy \( \ell(\mathbb{F}(E)_s) \leq 1 \). The only sheaf which is treated in corollary 3.3 and in theorem 5.1 is the sheaf \( S(-1) \) which satisfies \( \ell(\mathbb{F}(S(-1))_s) = 1 \).

For strongly indecomposable vector bundles, this is the case \( N = 1 \) of theorem 5.1 (a), the result was known to Friedman and Morgan, see [18], Cor. 2.3.2.

Proof of Theorem 5.1. Throughout the proof we use the notation introduced in section 3. The idea of the proof is the following. For the sheaves \( E \) which are described in the theorem, we calculate \( \hat{\text{ev}}_s^\wedge \). This is the completion of the germ at the singular point \( s \in E \) of the evaluation map \( \text{ev} : H^0(E(p_0)) \otimes \mathcal{O} \to E(p_0) \). It turns out that \( \hat{\text{ev}}_s^\wedge \) is injective. This suffices to prove the injectivity of the evaluation map. Using \( h^0(E(p_0)) = \text{rk}(E(p_0)) \), this implies that \( \text{coker}(\text{ev}) \) is a torsion sheaf. By remark 2.20 this cokernel is isomorphic to \( \mathbb{F}(E) \). Finally, using corollary 3.3 theorem 2.21 and the results collected in section 4, we can prove the theorem.

From lemma 3.3
\[
\mathcal{B}(d, m, \lambda) \otimes \mathcal{O}(p_0) \cong \mathcal{B}(d + 1, m, \lambda) \quad \text{and} \quad \mathcal{S}(d) \otimes \mathcal{O}(p_0) \cong \mathcal{S}(d + 1),
\]
we obtain \( H^1(E(p_0)) = 0 \) and \( h^0(E(p_0)) = \text{rk}(E(p_0)) = mn \) for the sheaves \( E \) described in the theorem. Here we let \( m = 1 \) if \( E = S(d) \).
Our first goal is the computation of $\ker(ev_s)$ and $\operatorname{coker}(ev_s)$. To identify these, it is sufficient to know its completion, because the completion of a Noetherian local ring is faithfully flat ([20], Thm. 8.14, see also lemma 6.4).

If $L$ is a line bundle on $E_n$ and $\sigma \in H^0(L)$ a global section, the completion $\hat{ev}_s$ at $s$ of the evaluation map

$$ev : H^0(\pi_n \ast L) \otimes \mathcal{O} \to \pi_n \ast L$$

is described as follows.

Because $\pi_n$ is étale, we have an isomorphism $(\pi_n \ast L)_s^\sim \cong \bigoplus_{\nu=1}^n L_{s_{\nu}}^\sim$. Using trivialisations of $L$ about each $s_{\nu}$, we obtain an isomorphism $(\pi_n \ast L)_s^\sim \cong \bigoplus_{\nu=1}^n R_{\nu}$. The $R$-module structure on it is given by [2] (see section 3.2), so that we identify it with $R^n$. The completed evaluation map sends the section $\sigma$ to the vector in $R^n$ whose $\nu$-th component is obtained by localising $\sigma$ at $s_{\nu}$.

More explicitly, suppose $\sigma$ is represented by homogeneous polynomials $f_\nu(x_{\nu}, y_{\nu}) \in H^0(D_{\nu}, \mathcal{O}(d_{\nu} + 1))$ which satisfy the gluing condition (3) in section 3.3. If $\nu \neq n$, the local description of $\sigma$ at the two preimages of the singularity $s_{\nu}$ is $(f_\nu(x_{\nu}, 1), f_{\nu+1}(1, y_{\nu+1})) \in k[[x_{\nu}]] \oplus k[[y_{\nu+1}]]$. The gluing condition ensures that these elements are indeed in $R_{\nu}$. The corresponding element in $R$, which is the $\nu$-th component of $\hat{ev}_s(\sigma)$, is $g_\nu = f_\nu(x, 1) + f_{\nu+1}(1, y) - f_\nu(0, 1) \in R$. At $s_n$, the gluing condition is $f_n(0, 1) = \lambda f_1(1, 0)$. The corresponding section on the normalisation is locally given by $(f_n(x_n, 1), \lambda f_1(1, y_1)) \in k[[x_n]] \oplus k[[y_1]]$. The corresponding element in $R_n$ is represented by $g_n = f_n(x, 1) + \lambda f_1(1, y) - f_n(0, 1) \in R$.

If $L$ is a line bundle on $I_n$, the completion of

$$ev : H^0(p_n \ast L) \otimes \mathcal{O} \to p_n \ast L$$

at $s \in E$ has a similar description. The main difference appears at $s_0$ and $s_n$. Again, $(p_n \ast L)_s^\sim \cong \bigoplus_{\nu=0}^n R_{\nu}$, but now we have $R_0 \cong k[[y]]$, $R_n \cong k[[x]]$ and all other $R_{\nu} \cong R$. The components $g_1, \ldots, g_{n-1}$ are computed as above. But now, $g_0 = f_1(1, y) \in R_0 \cong k[[y]]$ and $g_n = f_n(x, 1) \in R_n \cong k[[x]]$.

Below, we use these descriptions to calculate a matrix representation of $\hat{ev}_s$, which will be used to determine $\mathbb{F}(E)$.

**The case $B(d, 1, \lambda)$.**

Because $d_{\nu} \in \{-1, 0, 1\}$ for all $\nu$, it is easy to describe a basis of $H^0(E(p_0))$. Such a basis should have $n$ elements, because $m = 1$. Observe that the vectors $d \in \mathbb{Z}^n$ we study, satisfy $n \geq 2, d_1 = 1$ and $d_2, d_3 \in \{-1, 0\}$. Below, we give all non-zero components $f_\nu$ for the basis elements we have chosen. If $n \geq 3$ we have:
type A. Any \( \nu \) with \( d_\nu = 1 \) gives a basis vector with components:

\[
f_\nu = x_\nu y_\nu
\]

type B. Any \( \nu \) with \( d_\nu = -1 \) gives a basis vector with components:

\[
\begin{align*}
\nu \neq 1, n & \quad f_{\nu - 1} = y_{\nu - 1}^{d_{\nu - 1} + 1} & \quad f_n = y_{n - 1}^{d_{n - 1} + 1} \\
f_\nu & = 1 & \quad f_n = 1 \\
f_{\nu + 1} & = x_{\nu + 1}^{d_{\nu + 1} + 1} & \quad f_1 = \lambda^{-1} x_1^2
\end{align*}
\]

type C. Any \( \nu \) with \( d_\nu \neq -1 \neq d_{\nu + 1} \) gives a basis vector:

\[
\begin{align*}
\nu \neq n & \quad f_\nu = y_\nu^{d_\nu + 1} & \quad \nu = n & \quad f_n = y_n \\
f_{\nu + 1} & = x_{\nu + 1}^{d_{\nu + 1} + 1} & \quad f_1 = \lambda^{-1} x_1^2
\end{align*}
\]

It is easy to see that we obtained precisely \( n = \text{rk}(E) \) basis vectors. In the case \( n = 2 \), there is no basis vector of type C and the only change to be made occurs in type B, where the basis vector is \((\lambda^{-1} x_1^2 + y_1^2, 1)\).

As explained above, the completed evaluation map sends these sections to vectors in \( R^n \) with components

\[
\begin{align*}
g_\nu & = f_\nu(x, 1) + f_{\nu + 1}(1, y) - f_\nu(0, 1) & \quad \text{if } \nu < n, \\
g_n & = f_n(x, 1) + \lambda f_1(1, y) - f_n(0, 1).
\end{align*}
\]

If \( n \geq 4 \) the basis above yields the following elements \((g_1, g_2, \ldots, g_n) \in R^n\). Again, we write down the non-zero components only:

**type A:** \((d_\nu = 1)\)

\[
\begin{align*}
(\nu \neq 1) & \quad g_{\nu - 1} = y & \quad (\nu = 1) & \quad g_n = \lambda y \\
(\nu = 1) & \quad g_\nu = x & \quad g_1 = x
\end{align*}
\]

**type B:** \((d_\nu = -1)\)

\[
\begin{align*}
(\nu \neq 1, 2, n) & \quad g_{\nu - 2} = y_{\nu - 1}^{d_{\nu - 1} + 1} & \quad (\nu = n) & \quad g_{n - 2} = y_{n - 1}^{d_{n - 1} + 1} & \quad (\nu = 2) & \quad g_n = \lambda y^2 \\
(\nu = n) & \quad g_{\nu - 1} = 1 & \quad g_{n - 1} = 1 & \quad g_1 = 1
\end{align*}
\]

\[
\begin{align*}
(\nu = 2) & \quad g_\nu = 1 & \quad g_n = 1 & \quad g_2 = 1 \\
(\nu = 1) & \quad g_{\nu + 1} = x_{\nu + 1}^{d_{\nu + 1} + 1} & \quad g_1 = \lambda^{-1} x_1^2 & \quad g_3 = x^{d_3 + 1}
\end{align*}
\]
type C: \(d_\nu \neq -1 \neq d_{\nu+1}\)

\[
\begin{align*}
\nu \neq n, 1: & & g_{\nu-1} = y^{d_\nu+1} & & (\nu = n) & & g_{n-1} = y & & (\nu = 1) & & g_n = \lambda y^2 \\
g_\nu = 1 & & g_n = 1 & & g_1 = 1 & & g_1 = \lambda^{-1} x^2 & & g_2 = x
\end{align*}
\]

The completed evaluation map as a mapping \(R^n \to R^n\) is described by the matrix whose columns are the vectors \((g_1, \ldots, g_n)\). Our computation below aims at showing that this mapping is injective and its cokernel is isomorphic to the cokernel of the matrix \(M(q, 1, (-1)^{n+N}\lambda)\). We might reduce our matrix using elementary operations of rows and columns (with coefficients from \(R\)). In addition, we might erase a row and a column if the entry which is in both of them is a unit in \(R\) and all other entries in this row and this column are equal to zero. This operation corresponds to splitting an isomorphism \(R \to R\) as a direct summand from \(R^n \to R^n\). We call two matrices equivalent if they have isomorphic kernels and cokernels.

Observe that any vector \(g\) of type A has the property that \(x \cdot g\) has precisely one non-zero component and this is equal to \(x^2\). Similarly, \(y \cdot g\) has \(y^2\) (or \(\lambda y^2\), if \(\nu = 1\)) as its only non-zero component. The position of the value \(x^2\) in \(x \cdot g\) is the component with number \(\nu\) precisely if \(d_\nu = 1\). The value \(y^2\) sits in \(y \cdot g\) at the component with number \(\nu\) precisely if \(d_{\nu+1} = 1\). On the other hand, values \(x^2\), \(\lambda x^2\), \(y^2\) or \(\lambda^{-1} y^2\) occur as components \(g_\mu\) of vectors of type B and type C precisely at the same positions. Hence, we can annihilate in our matrix all entries which involve \(x^2\) or \(y^2\).

If \(n = 2\) we have \(d = (1, -1)\) and the two vectors are \((x, \lambda y)\) and \((\lambda^{-1} x^2 + 1, \lambda y^2 + 1)\). If \(n = 3\) and \(d = (1, 0, -1)\), the three image vectors are \((x, 0, \lambda y), (\lambda^{-1} x^2 + y, 1, 1), (1, x, \lambda y^2)\). If \(d = (1, -1, 0)\), the three image vectors are \((x, 0, \lambda y), (1, 1, x + \lambda y^2), (\lambda^{-1} x^2, y, 1)\). These three cases are dealt with easily by hand and are left to the reader. Suppose \(n \geq 4\) for the rest of the proof.

Each part of \(d\) of the form \(\overbrace{1, 0, \ldots, 0}^{n_k}, \overbrace{-1, 0, \ldots, 0}^{m_k}, 1\) gives rise to the following part of our matrix:
If $N = 1$, this has to be changed slightly. In this case the matrix consists just of the part inside the large box. Moreover, the last column inside the box has to be erased and its content must be added to the first column. In this way, the encircled entry $y$ in the last row appears in the first column. With these changes, the procedure described below applies to the case $N = 1$ as well.

Because the rows of our matrix correspond to singular points, the space between two rows corresponds to a component of $E_n$ and so to one of the components of $d$. In this sense, the double lines correspond to $d_\nu = 1$ and the single horizontal line in the middle corresponds to $d_\nu = -1$. The number of rows inside the upper part of the box is $n_k$. If $n_k = 1$ only the last row of the upper part is present. Similarly, if $m_k = 1$ the only row which is present in the lower part of the depicted portion of the matrix is the first row of the lower part. The lower part contains $m_k$ rows. The underlined zero entries indicate the positions where we deleted (multiples of) $x^2$ or $y^2$. If $k = N$, the encircled entry $y$ in the last row above the lower double line is to be replaced by $\lambda y$. If $k = 1$, the encircled entry $y$ in first row has to be replaced by $\lambda y$, but this does not influence our reduction of the part inside the boxes. All non-zero entries are visible, if they belong to a row or column, a part of which is contained inside the depicted box. Outside this box, these are just the entry $y$ in the first row and the entry $x$ in the last row.

We use that we have $xy = 0$ in $\mathcal{R}$ and subtract successively $x$ or $y$ times a column from one of its neighbours. In this way, we find that our original matrix is equivalent to one where the portion above is replaced by
Of course, here and below, $\lambda$ must be replaced by 1 if $k \neq N$. If $k = 1$, the encircled entry $y$ in the first row should be replaced by $\lambda y$. We can now erase the $n_k - 1$ rows and columns which meet at the unit matrix in the upper part and similarly $m_k - 1$ rows from the lower part with their corresponding columns. We obtain an equivalent matrix, if we replace the original portion of our matrix by:

\[
\begin{array}{c|ccc}
1 & y & 0 & 0 \\
\hline
2 & 0 & 1 & 0 \\
& 0 & 1 & 0 \\
& & \ddots & \ddots \\
& & 0 & 1 & 0 \\
& & & 0 & 1 & 0 \\
(-x)^{n_k} & 0 & 1 & 0 & -\lambda(-y)^{m_k} \\
\hline
0 & 1 & 0 & \cdots & \cdots & 0 & 1 & 0 \\
0 & 1 & 0 & \cdots & \cdots & 0 & 1 & 0 \\
0 & 1 & 0 & \cdots & \cdots & 0 & 1 & 0 \\
0 & 1 & 0 & \cdots & \cdots & 0 & 1 & 0 \\
\hline
0 & x & 1 \\
\end{array}
\]

In a final step we reduce this to

\[
\begin{array}{c|ccc}
1 & y \\
\hline
-(-x)^{n_k} & 1 & 0 \\
\hline
0 & 1 & -\lambda(-y)^{m_k} \\
\hline
0 & x & 1 \\
\end{array}
\]

If $N = 1$, there is just one column left and the entry is $(-x)^n_1 - \lambda(-y)^m_1$ or, equivalently, $x^{n_1} + (-1)^{n_1} + 1 \lambda y^{m_1}$.

If $N > 1$, this process does not change any non-zero entry outside the area between the two double lines. Moreover, we can perform the same procedure even if the four non-zero entries $1, y, x, 1$ in the picture, which are outside the box in which the changes take place, are replaced by other values. Therefore, each of the $N$ blocks as described above can be replaced by the corresponding $1 \times 2$ matrix which was obtained at the end of the procedure. Up to this point we think of the numbers of the rows of this matrix as cyclic subscripts. In other words, we
do not fix a preference which subscript is chosen as the first row in our matrix. But now, at the end, when we write down the complete matrix, we chose to write the row which corresponds to \( s_1 \) as our first row. In this way, we obtain that the original matrix is equivalent to the following matrix of size \( N \times N \):

\[
\begin{pmatrix}
(-x)^{n_1} & (-y)^{m_1} & 0 \\
0 & (-x)^{n_2} & (-y)^{m_2} \\
0 & 0 & (-x)^{n_3} \\
& \ddots & \\
& & & (-x)^{n_{N-1}} & (-y)^{m_{N-1}} \\
& & & 0 & (-x)^{n_N} \\
\end{pmatrix}.
\]

Using \( n = \sum_{\nu=1}^{N} (m_\nu + n_\nu) \) this is easily seen to be equivalent to

\[
\begin{pmatrix}
x^{n_1} & y^{m_1} & 0 \\
0 & x^{n_2} & y^{m_2} \\
0 & 0 & x^{n_3} \\
& \ddots & \\
& & & x^{n_{N-1}} & y^{m_{N-1}} \\
& & & 0 & x^{n_N} \\
\end{pmatrix},
\]

which is \( M(q, 1, (-1)^{n+N} \lambda) \). Hence, we have shown

\[
\text{coker}(ev_s)^\sim \cong M(q, 1, (-1)^{n+N} \lambda) \quad \text{and} \quad \ker(ev_s)^\sim = 0.
\]

This implies ker\((ev_s) = 0 \) and, because \( E \) is irreducible, ker\((ev) \) has rank zero. But, as a subsheaf of a torsion free sheaf, ker\((ev) \) is torsion free itself, hence zero. Remark \( 2.20 \) implies now \( F(B(d, 1, \lambda)) \cong \text{coker}(ev) \). Because \( h^0(E(p_0)) = \text{rk}(E(p_0)) \), this cokernel is a torsion sheaf and, with \( d \) and \( q \) as described in the theorem, we have

\[
F(B(d, 1, \lambda))^\sim \cong M(q, 1, (-1)^{n+N} \lambda),
\]

which is an indecomposable module of finite length. Theorem \( 2.21 \) implies now that \( B(d, 1, \lambda) \) is semi-stable, if \( d \) is of the form which is described in the theorem. To see that \( B(d, 1, \lambda) \) is indecomposable, we have to exclude that the support of \( F(B(d, 1, \lambda)) \) contains a regular point \( p \in E \). To see this, it is sufficient to show

\[
\text{Hom}(k(p), F(B(d, 1, \lambda))) = 0.
\]

By remark \( 2.16 \), \( F \) is an equivalence of categories and by corollary \( 3.3 \), \( k(p) \cong F(L) \), where \( L \) is a line bundle of degree zero on \( E \). Thus,

\[
\text{Hom}(k(p), F(B(d, 1, \lambda))) \cong \text{Hom}(L, B(d, 1, \lambda)) \cong \\
\text{Hom}(L, \pi_n^*(L(d, \lambda))) \cong \text{Hom}(\pi_n^*(L), L(d, \lambda)).
\]
As \( L \) has degree zero, \( \pi_n^s(L) \) is of degree zero on each component of \( E_n \). Hence, \( \text{Hom}(\pi_n^s(L), L(d, \lambda)) \cong H^0(L(d, \lambda')) \), with some \( \lambda' \in k^\times \). But \( H^0(L(d, \lambda')) = 0 \) (same proof as lemma 3.1), hence \( \mathbb{F}(B(d, 1, \lambda)) \) is supported at \( s \) only. This shows that we identified all indecomposable semi-stable torsion free sheaves of degree zero whose Fourier-Mukai image is one of the band modules \( \mathcal{M}(q, 1, \lambda) \).

The case \( B(d, m, \lambda) \).

Using induction on \( m \) and the exact sequences

\[
0 \to B(d, 1, \lambda) \to B(d, m + 1, \lambda) \to B(d, m, \lambda) \to 0,
\]

we obtain the semi-stability of \( B(d, m, \lambda) \) for any \( m \geq 1 \), provided \( d \) is one of the vectors appearing in the theorem. Theorem 2.21 implies that \( \mathbb{F}(B(d, m, \lambda)) \) is isomorphic to the cokernel of the evaluation map. Using the same exact sequences, exactness of \( \mathbb{F} \) and induction on \( m \) imply that the support of \( \mathbb{F}(B(d, m, \lambda)) \) is the singular point \( s \).

If \( m > 1 \) we have \( B(d, m, \lambda) \cong \pi_n^s(L(d, \lambda) \otimes \pi_n^s F_m) \). The vector bundle \( L(d, \lambda) \otimes \pi_n^s F_m \) on \( E_n \) is obtained by gluing the bundles \( \mathcal{O}(d)_{\mathcal{O}}^{\oplus m} \) in a similar way as we obtained the bundles \( L(d, \lambda) \): with the same convention and coordinates as before, over \( s_1, \ldots, s_{n-1} \) we glue with the identity matrix \( I_m \) and over \( s_n \) we glue with the matrix \( J_m(\lambda) \).

Actually, the definition of the bundle implies that the gluing matrix over \( s_n \) should be \( \lambda J_m(1)^n \), but this matrix is similar to \( J_m(\lambda) \). (The assumption \( \text{char}(k) = 0 \) is essential here!) Sections of \( B(d, m, \lambda) \) are now given by \( n \)-tuples \( (f_1, \ldots, f_n) \in \oplus H^0(\mathcal{O}(d))^{\oplus m} \) which satisfy the gluing condition \( \hat{\mathcal{B}} \) with \( \lambda \) replaced by \( J_m(\lambda) \) in the second equation.

As a result, we obtain a basis of \( H^0(B(d+1, m, \lambda)) \) if we interpret the description of the basis of \( H^0(B(d+1, 1, \lambda)) \) on page 26 in the following way: each basis vector in the case \( m = 1 \) gives rise to \( m \) basis vectors by replacing \( x_k, y_k, \lambda y_k \) by \( x_k I_m, y_k I_m \) and \( y_k J_m(\lambda) \) respectively. The \( m \) columns produced this way are the new basis vectors.

We can now carry out the same procedure as in the case \( m = 1 \). The only differences are now that the entries in our matrix are blocks of size \( m \times m \) and that the factor \( \lambda \) has to be replaced by the matrix \( J_m(\lambda) \). This does not influence any of the reduction steps, so that we finally obtain the matrix \( M(q, m, (-1)^{n+\lambda}) \). This proves

\[
\mathbb{F}(B(d, m, \lambda))_s \cong \mathcal{M}(q, m, (-1)^{n+\lambda}).
\]

As seen above, for any \( m \geq 1 \), the support of the sheaf \( \mathbb{F}(B(d, m, \lambda)) \) is the point \( s \). Using the results from section 4 this shows that any indecomposable \( R \)-module \( \mathcal{M}(q, m, \lambda) \) is isomorphic to a module \( \mathbb{F}(E) \) with \( E \cong B(d, m, (-1)^{n+\lambda}) \) as given in the theorem.

The case \( S(d) \).
If $E = S(d)$, the computations are very similar to the computations in the case $E = B(d, 1, \lambda)$. We confine ourselves to describe the differences. First of all, an $n$-tuple $(f_1, f_2, \ldots, f_n) \in \bigoplus_{\nu=1}^{n} H^0(D_\nu, \mathcal{O}(d_\nu+1))$ is an element of $H^0(E(p_0))$ if and only if

$$f_\nu(0 : 1) = f_{\nu+1}(1 : 0) \quad 1 \leq \nu \leq n-1.$$ 

We start with an explicit description of a basis of $H^0(E(p_0))$. As before, only the non-zero entries of the vectors $(f_1, \ldots, f_n)$ are written down.

Observe, $d_1 \neq 1, d_n \neq 1, n \geq 2$.

**type A.** Any $\nu$ with $d_\nu = 1$ gives a basis vector with components:

$$f_\nu = x_\nu y_\nu$$

**type B:** Any $\nu$ with $d_\nu = -1$ gives a basis vector with components:

- $(\nu \neq 1, n)$
- $(\nu = n)$
- $(\nu = 1)$

$$f_{\nu - 1} = y_{\nu - 1}^{d_{\nu - 1} + 1}$$
$$f_\nu = 1$$
$$f_{\nu + 1} = x_{\nu + 1}^{d_{\nu + 1} + 1}$$
$$f_{n - 1} = y_{n - 1}^{d_n + 1}$$
$$f_n = 1$$
$$f_1 = 1$$
$$f_2 = x_2^{d_2 + 1}$$

**type C:** Any $\nu$ with $d_\nu \neq -1 \neq d_{\nu + 1}$ gives a basis vector:

- $(\nu \neq n)$
- $(\nu = n)$

$$f_\nu = y_\nu^{d_\nu + 1}$$
$$f_n = y_n$$
$$f_{\nu + 1} = x_{\nu + 1}^{d_{\nu + 1} + 1}$$

**type D:** If $d_1 = 0$ there is a basis vector with:

$$f_1 = x_1$$

Again, the number of these vectors is equal to $n$ and their images $g = (g_0, \ldots, g_n)$ under the completed evaluation map are computed by the procedure explained above.

We shall study parts of the matrix which represents the completed evaluation map $R^n \to k[[y]] \oplus R^{n-1} \oplus k[[x]]$. As before, we look at those parts which correspond to portions of $d$ the form

$$1, 0, \ldots, 0, -1, 0, \ldots, 0, 1$$

with $1 \leq k \leq N$. If $k = 1$ there is no leading 1, whereas the trailing 1 is missing in case $k = N$. If $1 \neq k \neq N$, the corresponding portion of the matrix looks exactly like before. The number of rows in the upper part inside the box is now equal to $n_k + 1$ and in the lower part equal to $m_k + 1$. The same reduction steps as in the locally free case lead to
However, if $k = 1$ the row above the upper double line is not present in the portion of the matrix depicted on page 27. Furthermore, the first column inside the box is to be deleted as well. So, the upper part of the box contains $n_1 + 1$ rows and the upper left corner looks like:

\[
\begin{array}{cccc}
1 & y & 0 & \cdots \\
x & 1 & y & \\
0 & x & 1 & y \\
\end{array}
\]

and (even if $n_1 = 0$) we reduce it to

\[
\begin{array}{cccc}
1 & 0 & \cdots & \\
-x & 0 & \cdots & \\
x & 1 & y & \\
0 & x & 1 & y \\
\end{array}
\]

Similarly, if $k = N$, we have $n_N + 1$ rows in the upper part and $m_N + 1$ rows in the lower part and deal with a lower right corner or the form

\[
\begin{array}{cccc}
x & 1 & y & 0 \\
x & 1 & y & \\
\cdots & 0 & x & 1 \\
\end{array}
\]

and (even if $m_N = 0$) the reduction leads to

\[
\begin{array}{cccc}
1 & y & 0 & \\
-x & 0 & \cdots & \\
x & 1 & y & \\
0 & x & 1 & y \\
\end{array}
\]

We need to keep in mind here that the entry in the first row is an element of $k[[y]]$ with trivial multiplication by $x \in R$ and the entry in the last row is in $k[[x]]$ with trivial multiplication by $y \in R$. Multiplying rows and columns by appropriate powers of $(\pm 1)$, we arrive at the following matrix, which is equivalent to the completed evaluation map:

\[
\begin{pmatrix}
1 & 0 & \cdots & 0 \\
x^{n_1} & y^{m_1+1} & 0 & \cdots & 0 \\
0 & x^{n_2+1} & y^{m_2+1} & 0 & \cdots & 0 \\
0 & 0 & x^{n_3+1} & \ddots & \vdots \\
\vdots & \ddots & \ddots & y^{m_{N-1}+1} & 0 \\
0 & \cdots & \cdots & 0 & x^{n_N+1} & y^{m_N} \\
0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 \\
\end{pmatrix}
\]
This matrix defines a mapping $R^{N+1} \to k[[y]] \oplus R^N \oplus k[[x]]$.

If $n_1 > 0$ and $m_N > 0$ this is precisely the matrix $N(q)$. Whereas, if $n_1 = 0$ or $m_N = 0$, we obtained one of the matrices $N'(q)$ or $N''(q)$ of remark 4.1. This shows that the cokernel of this matrix is isomorphic to the module $N(q)$, with $q$ being the sequence described in the theorem. In addition, we see that $ev_s$ is injective.

As before, we conclude $F(S(d)) \cong \text{coker}(ev)$. Our computation implies now $F(S(d))_s \cong N(q)$ for $d$ and $q$ as in the theorem. Together with the results from section 4 this shows that any indecomposable $R$-module $N(q)$ has the form $F(S(d))_s$ with $d$ and $q$ as given in the theorem. To see that $F(S(d))$ and, hence, $S(d)$ are indecomposable, we proceed as before in the case $B(d, 1, \lambda)$. We have to verify $\text{Hom}(p^n_\ast(L), \mathcal{L}(d)) = 0$ for any line bundle $L$ of degree zero on $E$. But $p^n_\ast(L) \cong \mathcal{O}$ on $I_n$. Hence, $\text{Hom}(p^n_\ast(L), \mathcal{L}(d)) \cong H^0(\mathcal{L}(d)) = 0$. This proves the indecomposability of $S(d)$ and that $F(S(d))$ is supported at $s$ only.

The conclusion now is that any coherent indecomposable torsion module on $E$ with support in $s$ has the form $F(E)$ where $E$ is a sheaf as described in the theorem. On the other hand, any indecomposable torsion sheaf of finite length which is supported at a regular point $x \in E$ is isomorphic to $\mathcal{O}_E \otimes m_x^m$ with $m \geq 1$ being the length of the sheaf. We have shown in corollary 3.3 that these sheaves are isomorphic to $F(\mathcal{L}(0, \lambda) \otimes F_m)$. Using theorem 2.21 this implies that we described all indecomposable semi-stable torsion free sheaves of rank zero.

Remark 5.2. Indecomposable semi-stable vector bundles of degree zero were characterised in [18]. We do not use their result here and produce a new proof of the fact that the vectors $d$ as described in the theorem correspond precisely to the semi-stable vector bundles. If $E$ is not locally free, semi-stability of these sheaves was not known before.

Remark 5.3. To avoid any possible confusion or misinterpretation, the structure of the vectors $d \in \mathbb{Z}^n$ in both parts of theorem 5.1 is the following: if all the zero components are removed, we obtain an alternating sequence of numbers $-1$ and $1$.

Remark 5.4. Although the description of the sequences $q$ in part (b) of theorem 5.1 seems to be a bit awkward, the description of the string diagram of $F(S(d))$ in terms of $d$ is straightforward: if $d$ is of the form described in part (b) of theorem 5.1, the string diagram contains a vertex for each component $d_\nu$ of $d$, the arrows connect neighbours only and form sequences pointing in the same direction which start at vertices corresponding to $d_\nu = -1$ and end at vertices which correspond
to $d_\nu = 1$. This applies to the sheaf $S(-1)$ as well. In this case, the

diagram consists of one vertex and no arrows.

Literally the same description applies to give a description of the

band diagram for $F(B(d, m, \lambda))$. Here we require $d \neq 0$ to obtain a

sheaf supported at the singularity. The only difference to the string
case is that the components of $d$ are considered to be in cyclic order,
so that $d_n$ and $d_1$ are neighbours, hence connected by an arrow.

6. Matlis Duality

In this section we study the relationship between $F(E)$ and $F(E^\vee)$,

where $E$ is a semi-stable torsion free sheaf of degree zero on a Weier-

straß cubic $E$. The answer (theorem 6.11) involves Matlis duality.

We briefly recall the features of Matlis duality which will be used

later. Details and a more comprehensive treatment can be found in

[26] and [10].

First, recall that a local Noetherian ring $(A, \mathfrak{m}, k)$ is called Goren-

stein, if its injective dimension is finite. Further, the injective hull of an

$A$-module $M$ is an injective $A$-module $E(M)$ which contains $M$ such

that $M \subseteq E(M)$ is an essential extension. To be an essential exten-
sion means here that for any non-zero submodule $N \subseteq E(M)$ one has

$M \cap N \neq 0$. The indecomposable injective $A$-modules are precisely the

modules $E(A/\mathfrak{p})$, where $\mathfrak{p} \subseteq A$ is a prime ideal [26], Thm. 18.4.

Lemma 6.1. Let $(A, \mathfrak{m}, k)$ be a local Gorenstein ring of dimension one.

By $K := \text{Quot}(A)$ we denote its total quotient ring. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$ be

the minimal prime ideals of $A$.

(i) One has $K/A \cong E(k)$ and $K \cong E(A) \cong \bigoplus_{\nu=1}^m E(A/\mathfrak{p}_\nu)$.

(ii) If $M$ is an $A$-module of finite length (or, equivalently, a torsion

module, i.e. $M \otimes_A K = 0$), then

$$\text{Hom}_A(M, K/A) \cong \text{Ext}_A^1(M, A).$$

Proof. (i) This follows easily from the standard theory, see [26], Ch.

18.

(ii) The assumption that $M$ is torsion implies $\text{Hom}_A(M, E(A/\mathfrak{p}_\nu)) = 0$

for all $\nu$. Hence, applying the functor $\text{Hom}_A(M, -)$ to the minimal

injective resolution of $A$ (see [26], Thm. 18.8)

$$0 \to A \to \bigoplus_{\nu=1}^m E(A/\mathfrak{p}_\nu) \to E(k) \to 0,$$

yields the desired isomorphism. \qed
Definition 6.2. If \((A, m, k)\) is a local Noetherian ring, the functor
\[ \mathbb{M}_A(-) := \text{Hom}_A(-, E(k)) \]
is called the Matlis functor.

The main result of Matlis (see [26], Thm. 18.6) states:

(a) For any \(A\)-module \(M\), the canonical map
\[ M \to \mathbb{M}_A(\mathbb{M}_A(M)) \]
is injective.

(b) If \(M\) is an \(A\)-module of finite length \(\ell(M)\), one has \(\ell(\mathbb{M}_A(M)) = \ell(M)\) and the canonical map in (a) is an isomorphism.

(c) If \(A\) is complete, \(\mathbb{M}_A\) is an anti-equivalence between the categories of Artinian and Noetherian \(A\)-modules. For both types of modules, the canonical map in (a) is an isomorphism.

These results can be used to prove Grothendieck’s local duality theorem, see [10], Sect. 3.5.

Lemma 6.3. Let \((A, m, k)\) be a local Gorenstein ring of dimension one. Suppose \(E\) is an injective \(A\)-module which satisfies \(\dim_k \text{Hom}_A(k, E) = 1\). Then there is an isomorphism of functors on the category of \(A\)-modules of finite length:
\[ \mathbb{M}_A(-) \cong \text{Hom}_A(-, E). \]

Proof. Using [26], Thm. 18.5, our assumption implies \(E \cong E(k) \oplus E'\), where \(E'\) is a direct sum of injective modules \(E(A/p_{\nu})\), where the \(p_{\nu}\) denote the minimal prime ideals of \(A\), as above. The projection \(E \to E(k)\) induces a morphism of functors \(\text{Hom}_A(-, E) \to \text{Hom}_A(-, E(k))\). This is an isomorphism on modules \(M\) of finite length, because, for such \(M\), \(\text{Hom}_A(M, A/p_{\nu}) = 0\).

Lemma 6.4. Let \((A, m, k)\) be a local Noetherian ring and \(M\) an \(A\)-module of finite length. Then, there are isomorphisms
\[ M \cong \widehat{M} \quad \text{and} \quad \mathbb{M}_A(M) \cong \mathbb{M}_{\widehat{A}}(\widehat{M}) \]

Proof. We know \(A \to \widehat{A}\) is flat and \(\widehat{A}\) is a local Noetherian ring with maximal ideal \(m\widehat{A}\) and residue field \(\widehat{A}/m\widehat{A} \cong k\). In particular, \(k \otimes_A \widehat{A} \cong k\). Using these facts, induction on the length \(\ell(M)\) shows easily that the canonical map \(M \to \widehat{M} \cong M \otimes_A \widehat{A}\) is an isomorphism.

From [25], Thm. 18.6, we know that the canonical map \(E(k) \to E(k) \otimes_A \widehat{A}\) is an isomorphism and that \(E(k)\) is the injective hull of \(k\) as an \(\widehat{A}\)-module as well. This implies that \(\text{Hom}_A(M, E(k))\) and \(\text{Hom}_{\widehat{A}}(\widehat{M}, E(k))\) are isomorphic, which gives the claim.
Remark 6.5. Let \((A, m, k)\) be a local Gorenstein ring of dimension one, which contains \(k\). If \(M\) is an \(A\)-module, we can consider it as a \(k\)-module and equip \(\text{Hom}_k(M, k)\) with the structure of an \(A\)-module by \((a \cdot f)(m) = f(am)\). There is a functorial isomorphism of \(A\)-modules

\[
\text{Hom}_A(M, \text{Hom}_k(A, k)) \sim - \rightarrow \text{Hom}_k(M, k),
\]

which sends \(\varphi : M \rightarrow \text{Hom}_k(A, k)\) to the mapping \(\Phi : M \rightarrow k\) defined by \(\Phi(m) := \varphi(m)(1)\). Because \(\text{Hom}_k(\_ , k)\) is exact on the category of \(k\)-modules it is exact on the category of \(A\)-modules as well, hence \((4)\) implies that the \(A\)-module \(\text{Hom}_k(A, k)\) is injective. If we substitute \(M = k\) in \((4)\), we obtain

\[
\text{Hom}_A(k, \text{Hom}_k(A, k)) \cong \text{Hom}_k(k, k) \cong k.
\]

Now, we can apply lemma 6.3 to \(E = \text{Hom}_k(A, k)\) and obtain

\[
\mathbb{M}_A(\_ ) \cong \text{Hom}_k(\_ , k)
\]

on the category of \(A\)-modules of finite length. We shall use this functorial isomorphism in the case \(A = R := k[[x, y]]/(x \cdot y)\).

This isomorphism allows a very nice description of Matlis duality on \(R\)-modules of finite length in terms of the band and string diagrams: all the vector spaces are to be replaced by its dual and the mappings by its transposed. Because the transposed of \(J_m(\lambda)\) is similar to \(J_m(\lambda)\), this means: the diagram of the Matlis dual is obtained by reversing the direction of all arrows.

We are interested in a global version of the Matlis functor on a curve. As usual, if \(X\) is a scheme, we denote by \(\mathcal{K}\) its sheaf of total quotient rings. Its stalk \(\mathcal{K}_x\) is the total quotient ring of the local ring \(\mathcal{O}_{X,x}\).

Lemma 6.6. Let \(X\) be a Noetherian scheme of dimension 1.

(i) For any closed point \(x \in X\) and any \(\mathcal{O}_{X,x}\)-module \(W\) we denote by \(i_x(W)\) the \(\mathcal{O}_X\)-module with stalk \(W\) at \(x\) and 0 elsewhere. There exists an exact sequence of \(\mathcal{O}_X\)-modules

\[
0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{K} \rightarrow \bigoplus_{x \in X} i_x(\mathcal{K}_x/\mathcal{O}_{X,x}) \rightarrow 0,
\]

where the direct sum ranges over all closed points \(x \in X\).

(ii) If \(X\) is Gorenstein, there is an isomorphism of \(\mathcal{O}_X\)-modules

\[
\mathcal{K}/\mathcal{O}_X \cong \bigoplus_{x \in X} i_x(E(k(x))).
\]

The direct sum is taken over closed points of \(X\), \(E(\_ )\) denotes the injective hull of an \(\mathcal{O}_{X,x}\)-module and \(k(x) \equiv \mathcal{O}_{X,x}/m_{X,x}\).

Proof. Part (i) is well-known and is obtained by taking stalks. Part (ii) follows from (i) and lemma 6.1 \(\square\)
Definition 6.7. If $X$ is a Gorenstein curve, we call
\[ M(-) := \mathcal{H}om_X(-, K/O_X) \]
the (global) Matlis functor.

Remark 6.8. Let $F$ be a coherent torsion sheaf on a Gorenstein curve $X$ and suppose $\text{supp}(F) = \{ x \}$. From lemma 6.6 we obtain: $M(F)$ is a sky-scraper sheaf with stalk $M_{O_X,x}(F_x)$ at $x$. More generally, for any coherent torsion sheaf $F$ and any $x \in X$ one has:
\[ M(F)_x \cong M_{O_X,x}(F_x). \]
From the results quoted above, we obtain that the canonical map $F \to M(M(F))$ is an isomorphism if $F$ is a coherent torsion sheaf. This is not true for general sheaves. From lemma 6.1 we obtain for any torsion sheaf $F$ on $X$:
\[ M(F) \cong \mathcal{E}xt^1_X(F, O_X). \]

Lemma 6.9. Let $E$ be a semi-stable torsion free sheaf of degree zero on $E$ (a Weierstraß cubic) and define $F := \mathcal{E}(E)$. This is a coherent torsion sheaf on $E$. With notation as in section 2, we have
\[ T_O(M(F)) \cong E^\vee(-p_0)[1]. \]

Proof. By theorem 2.21 $F$ sits in an exact sequence
\[
\begin{array}{ccc}
0 & \longrightarrow & H^0(E(p_0)) \otimes O \\
& & \overset{ev}{\longrightarrow} E(p_0) \\
& & \longrightarrow F \\
& & \longrightarrow 0.
\end{array}
\]
Since $F$ is a torsion sheaf, $M(F) \cong \mathcal{E}xt^1_X(F, O_X)$ is a torsion sheaf as well. In particular, $H^1(M(F)) = 0$ and the evaluation map $H^0(M(F)) \otimes O \to M(F)$ is surjective. Hence, by lemma 2.9, we have:
\[ T_O(M(F)) \cong \ker(H^0(M(F)) \otimes O \to M(F))[1]. \]
Observe $\mathcal{H}om(F, O) = 0$ and $\mathcal{E}xt^1(E(p_0), O) = 0$. The latter is true because $\mathcal{E}xt^1_A(M, A) = 0$ if $A$ is a local Gorenstein ring of dimension 1 and $M$ is a finite $A$-module with $\text{depth}(M) = 1$. This follows easily from Nakayama’s lemma. Hence, application of the functor $\mathcal{H}om(-, O)$ to the exact sequence (5) gives the exact sequence
\[ 0 \to E^\vee(-p_0) \to \mathcal{H}om(H^0(E(p_0)) \otimes O, O) \to \mathcal{E}xt^1(F, O) \to 0. \]
Application of $H^0(-) \otimes O$ gives the following commutative diagram:
\[
\begin{array}{ccc}
\mathcal{H}om(H^0(E(p_0)) \otimes O, O) & \overset{\delta}{\longrightarrow} & \mathcal{E}xt^1(F, O) \\
\upsilon \downarrow & & \upsilon \\
\mathcal{H}om(H^0(E(p_0)) \otimes O, O) \otimes O & \overset{H^0(\delta) \otimes \text{Id}_O}{\longrightarrow} & H^0(\mathcal{E}xt^1(F, O)) \otimes O.
\end{array}
\]
Obviously, the left vertical evaluation map is an isomorphism. Furthermore, the exact sequence

\[ H^0(E^\vee(-p_0)) \rightarrow \text{Hom}(H^0(E(p_0) \otimes O, O)) \xrightarrow{H^0(\delta)} H^0(\text{Ext}^1(F, O)) \]

shows that \( H^0(\delta) \) is injective, because \( \text{deg}(E(p_0)) > 0 \) and semi-stability of \( E(p_0) \) imply \( H^0(E^\vee(-p_0)) \cong \text{Hom}(E(p_0), O) = 0 \). As seen before, \( \dim \text{Hom}(H^0(E(p_0)) \otimes O, O) = h^0(E(p_0)) = \text{rk}(E) \) as well as \( h^0(\text{Ext}^1(F, O)) = \dim \text{Ext}^1(F, O) = \dim \text{Hom}(O, F) = h^0(F) = \text{rk}(E) \). Hence, \( H^0(\delta) \) is an isomorphism. Now, the conclusion is that \( \delta \) and the right vertical evaluation map in the diagram above have isomorphic kernels, that is: \( E^\vee(-p_0)[1] \cong T_O(M(F)) \).

**Definition 6.10.** If \( F \) is a torsion sheaf on \( E \), we define:

\[ F^\star := i^*\mathbb{M}(F) \]

and call it the *twisted Matlis dual* of \( F \).

**Theorem 6.11.** If \( E \) is a semi-stable torsion free sheaf of degree zero on \( E \), there is an isomorphism:

\[ \mathbb{F}(E^\vee) \cong \mathbb{F}(E)^\star. \]

**Proof.** Using lemma [6.9] we obtain \( E^\vee[1] \cong T_{k(p_0)}T_O(M(F)) \). As \( F = \mathbb{F}(E) \) and \( M(F) \) are torsion sheaves, \( T_{k(p_0)}M(F) \cong M(F) \) and we obtain \( E^\vee[1] \cong FM(F) \cong FMF(E) \). This implies \( \mathbb{F}(E^\vee)[1] \cong FMF(E) \). From theorem [2.18] we deduce \( \mathbb{F}(E^\vee) \cong i^*\mathbb{M}(F(E)) \cong \mathbb{F}(E)^\star. \)

**Remark 6.12.** If \( E \) is a smooth elliptic curve, our result is a special case of a result of Mukai [28], (3.8).

**Remark 6.13.** As before, \( R \) denotes the complete local Gorenstein ring \( k[[x, y]]/(x \cdot y) \). If the singular point \( s \in E \) is a node, the ring map \( R \to \tilde{R} \), which is induced by \( i : E \to \tilde{E} \) at the fixed point \( s \in E \), interchanges \( x \) and \( y \). Hence, if \( M \) is an \( R \)-module of finite length, the diagram for \( i^*(M) \) is obtained from the diagram of \( M \) by interchanging \( x \) and \( y \). Let now \( E \) be an indecomposable semi-stable torsion free sheaf of degree zero. Using lemma [3.2] and theorem [5.1] it is not hard to verify that the diagram of \( i^*\mathbb{F}(E^\vee) \) is obtained from that of \( \mathbb{F}(E) \) by reversing all arrows. Because, by theorem [6.11] \( M_R(\mathbb{F}(E)) \cong i^*\mathbb{F}(E^\vee) \), this gives another verification of the description of the Matlis dual of an \( R \)-module of finite length which was given in remark [6.13]. We illustrate this by the two examples below.

**Example 6.14.** The Matlis dual module of the string module

\[ \mathcal{N}(2(3, 2)1) \cong \mathbb{F}(\mathcal{S}(-1, 0, 1, 0, 0, -1, 0, 1, -1)) \]
is the module
\[ i^* \mathbb{F}(S(-1, 0, 1, 0, 0, 1, -1, 0, 1, -1)) \cong i^* \mathbb{F}(S(0, 0, -1, 0, 0, 1, 0, -1, 0)) \]
\[ \cong \mathbb{F}(S(0, -1, 0, 1, 0, 0, -1, 0, 0)) \cong N(0(1, 2)(3, 2)0) \]

This module is represented by the diagram which is obtained from the diagram in example 4.2 by reversing all arrows:

Example 6.15. The Matlis dual module of the band module
\[ \mathcal{M}(((2, 2)(3, 4)(1, 3)), m, \lambda) \]
\[ \cong \mathbb{F}(B((1, 0, -1, 0, 1, 0, 0, -1, 0, 0, 1, -1, 0, 0), m, \lambda)) \]

is the module
\[ i^* \mathbb{F}(B((1, 0, -1, 0, 1, 0, 0, -1, 0, 0, 1, -1, 0, 0), m, \lambda)) \]
\[ \cong i^* \mathbb{F}(B((-1, 0, 1, 0, -1, 0, 0, 1, 0, 0, -1, 1, 0, 0), m, \lambda^{-1})) \]
\[ \cong \mathbb{F}(B((0, 0, 1, -1, 0, 0, 0, 1, 0, 0, -1, 0, 1, 0, -1), m, \lambda)) \]
\[ \cong \mathbb{F}(B((1, -1, 0, 0, 1, 0, 0, -1, 0, 1, 0, -1, 0, 0), m, \lambda)) \]
\[ \cong \mathcal{M}((1, 4)(3, 2)(2, 3)), m, \lambda) \]

This module is represented by the following diagram which is obtained from the diagram in example 4.3 by reversing all arrows (and moving the Jordan block \( J_m(\lambda) \) two arrows forward):
REFERENCES

[1] Atiyah, M.F. *Vector bundles over an elliptic curve*, Proc. Lond. Math. Soc., III. Ser., 7 (1957) 414–452.

[2] Bartocci, C., Bruzzo, U., Hernández Ruipérez, D. & Muñoz Porras, J.M. *Mirror symmetry on K3 surfaces via Fourier-Mukai transform*, Comm. Math. Phys. 195 (1998) 79–93, arXiv:alg-geom/9704023.

[3] Ben-Zvi, D. & Nevins, T. *From solitons to many-body systems*, arXiv:math.AG/0310490.

[4] Bondal, A. & Orlov, D. *Semiorthogonal decomposition for algebraic varieties*, arXiv:alg-geom/9506012.

[5] Bondal, A. & Orlov, D. *Reconstruction of a variety from the derived category and groups of autoequivalences*, Compositio Mathematica 125 (2001) 327–344, arXiv:alg-geom/9712029.

[6] Nazarova, L.A., Roiter, A.V., Sergeicuk, V.V. & Bondarenko, V.M. *Application of modules over a dyad for the classification of finite p-groups possessing an Abelian subgroup of index p and of pairs of mutually annihilating operators*, J. Sov. Math. 3 (1975) 636–653; translation from Zapiski naucn. Sem. Leningrad Otd. Mat. Inst. Steklov 28 (1972) 69–92.

[7] Bridgeland, T. *Fourier-Mukai transforms for elliptic surfaces*, J. Reine Angew. Math. 498 (1998) 115–133, arXiv:alg-geom/9705002.

[8] Bridgeland, T. *Equivelances of triangulated categories and Fourier-Mukai transforms*, Bull. Lond. Math. Soc. 31 (1999) 25–34, arXiv:math.AG/9809114.

[9] Bridgeland, T. & Maciocia, A. *Fourier-Mukai transforms for quotient varieties*, arXiv:math.AG/9811101.

[10] Bruns, W. & Herzog, J. *Cohen-Macaulay Rings*, Cambridge Studies in Advanced Mathematics 39 (1998).

[11] Burban, I. *Stable Bundles on a Rational Curve with One Simple Double Point*, Ukrainian Math. J. 55 (2003) 1043–1053.

[12] Burban, I. & Drozd, Yu. *Coherent sheaves on rational curves with simple double points and transversal intersections*, to appear in Duke Math. J., arXiv:math.AG/0101140.

[13] Burban, I., Drozd, Yu. & Greuel, G.-M. *Vector bundles on singular projective curves*, Ciliberto (ed.) et al., Applications of algebraic geometry to coding theory, physics and computation (Eilat, Israel, 2001), Dordrecht: Kluwer Academic Publishers, NATO Sci. Ser. II, Math. Phys. Chem. 36 (2001) 1–15.

[14] Drozd, Yu. & Greuel, G.-M. *Tame and wild projective curves and classification of vector bundles*, J. Algebra 246 (2001) 1–54.

[15] Drozd, Yu., Greuel, G.-M. & Kashuba, I. *On Cohen-Macaulay Bundles on Surface Singularities*, Moscow Math. J. 3 No. 2 (2003).

[16] Drezet, J.-M. *Fibrés exceptionnels et suite spectrale de Beilinson généralisée sur P^n(C)*, Math. Ann. 275 (1986) 25–48.

[17] Friedman, R. & Morgan, J. *Minuscule representations, invariant polynomials, and spectral covers*, Cütkosky (ed.) et al., Vector bundles and representation theory. (Columbia, MO, USA, 2002) Providence, RI: AMS, Contemp. Math. 322 (2003) 1–41, arXiv:math.AG/0011082.

[18] Friedman, R. & Morgan, J. *Holomorphic Principal Bundles Over Elliptic Curves III: Singular Curves and Fibrations*, arXiv:math.AG/0108104.
[19] Friedman, R., Morgan, J. & Witten, E. Vector bundles over elliptic fibrations J. Algebr. Geom. 8 (1999) 279–401, arXiv:alg-geom/9709029.
[20] Gelfand, I.M. & Ponomarev, V.A. Indecomposable representations of the Lorentz group, Russ. Math. Surv. 23 (1968) 1–58; translation from Usp. Mat. Nauk 23 (1968) no. 2 (140), 3–60.
[21] Gorodentsev, A.L. & Rudakov, A.N. Exceptional vector bundles on projective spaces, Duke Math. J. 54 (1987) 115–130.
[22] Hartshorne, R. Algebraic Geometry, Springer, GTM 52 (1983).
[23] Kontsevich, M. Homological algebra of mirror symmetry, Chatterji (ed.), International congress of mathematicians, ICM ’94 (Zürich, Switzerland, 1994) Basel: Birkhäuser, Vol. I (1995) 120–139, arXiv:alg-geom/9411018.
[24] Lenzing, H. & Meltzer, H. Sheaves on a weighted projective line of genus one, and representations of a tubular algebra, Representations of algebras (Ottawa, ON, Canada, 1992) Providence, RI: AMS, CMS Conf. Proc. 14 (1993) 313–337.
[25] Levin, G. & Vasconcelos, W. Homological dimensions and Macaulay rings, Pacific J. Math. 25 (1968) 315–353.
[26] Matsumura, H. Commutative Ring Theory, Cambridge Studies in Advanced Mathematics 8 (1989).
[27] Meltzer, H. Tubular mutations, Colloq. Math. 74 (1997) 267–274.
[28] Mukai, S. Duality between $D(X)$ and $D(\hat{X})$ with its application to Picard sheaves, Nagoya Math. J. 81 (1981) 153–175.
[29] Nazarova, L.A. & Roiter, A.V. Finitely generated modules over a dyad of two local Dedekind rings, and finite groups with an Abelian normal divisor of index $p$, Math. USSR Izv. 3 (1969) 65–86; translation from Izv. Akad. Nauk SSSR, Ser. Mat. 33 (1969) 65–89.
[30] Orlov, D. Equivalences of derived categories and K3 surfaces, J. Math. Sci., New York 84 (1997) 1361–1381, arXiv:alg-geom/9606006.
[31] Seidel, P. & Thomas, R. Braid group actions on derived categories of coherent sheaves, Duke Math. J. 108 (2001) 37–108, arXiv:math.AG/0001043.
[32] Seshadri, C.S. Fibrés vectoriels sur les courbes algébriques, Astérisque 96 (1982).
[33] Teodorescu, T. Semistable Torsion-Free Sheaves over Curves of Arithmetic Genus One, PhD Thesis, Columbia University, New York, NY, USA (1999).

Institut de Mathématiques de Jussieu, Université Pierre et Marie Curie – Paris VI, 4 Place Jussieu, 75252 PARIS Cedex 05, France
E-mail address: burban@math.jussieu.fr

Mary Immaculate College, South Circular Road, Limerick, Ireland
E-mail address: bernd.kreussler@mic.ul.ie