Map of Witten’s $\star$ to Moyal’s $\star$

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Abstract

It is shown that Witten’s star product in string field theory, defined as the overlap of half strings, is equivalent to the Moyal star product involving the relativistic phase space of even string modes. The string field $\psi_A(x^\mu [\sigma])$ can be rewritten as a phase space field of the even modes $A(x^\mu_{2n}, x_0, p^\mu_{2n})$, where $x^\mu_{2n}$ are the positions of the even string modes, and $p^\mu_{2n}$ are related to the Fourier space of the odd modes $x^\mu_{2n+1}$ up to a linear transformation. The $p^\mu_{2n}$ play the role of conjugate momenta for the even modes $x^\mu_{2n}$ under the string star product. The split string formalism is used in the intermediate steps to establish the map from Witten’s $\star$-product to Moyal’s $\star$-star product. An ambiguity related to the midpoint in the split string formalism is clarified by considering odd or even modding for the split string modes, and its effect in the Moyal star product formalism is discussed. The noncommutative geometry defined in this way is technically similar to the one that occurs in noncommutative field theory, but it includes the timelike components of the string modes, and is Lorentz invariant. This map could be useful to extend the computational methods and concepts from noncommutative field theory to string field theory and vice-versa.

1 Moyal’s star product in half Fourier space

A long time ago deformation quantization was developed as a method for studying quantum mechanics. The methods for establishing the correspondence between deformation quantization and the traditional formulation of quantum mechanics was developed by Weyl, Wigner, Moyal and many others [1] - [3], leading the way to the modern ideas of noncommutative geometry [4]. Using Weyl’s correspondence [1] one establishes a map between functions of operators $\hat{A}(\hat{x}^M, \hat{p}_M)$ acting in a Hilbert space and their image in phase space $A(x^M, p_M)$.

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To preserve the quantum rules, phase space functions must be multiplied with each other by using the associative, noncommutative, Moyal star product

\[(A \star B) (x, p) = A (x, p) e^{\frac{\imath}{\hbar} \left( \frac{\partial M}{\partial x} \frac{\partial M}{\partial p} - \frac{\partial M}{\partial p} \frac{\partial M}{\partial x} \right)} B (x, p) \tag{1}\]

(or its deformed generalizations). Note that a spacetime metric does not enter in this expression because all positions have upper spacetime indices and all momenta have lower spacetime indices. If we use the notation \(X^m \equiv (x^M, p_M)\) with a single index \(m\) that takes \(D \times 2\) values, the Moyal star product takes a form which is more familiar in the recent physics literature on noncommutative geometry

\[(A \star B) (X) = A (X) e^{\imath \theta_{mn} \left( \frac{\partial X^m}{\partial x} \frac{\partial X^n}{\partial x} \right)} B (X), \tag{2}\]

where \(\theta_{mn} = \hbar \delta^M_N \varepsilon_{ij}\), with \(i = (1, 2)\) referring to \((x, p)\) respectively, and \(\varepsilon_{ij}\) the antisymmetric \(\text{Sp}(2, \mathbb{R})\) invariant metric. Henceforth we will set \(\hbar = 1\) for simplicity. The star commutator between any two phase space fields is defined by 

\[A, B \star \equiv A \star B - B \star A.\]

The phase space coordinates satisfy \([X^m, X^n]_\star = i\theta_{mn}\), which is equivalent to the Heisenberg algebra for \((x^M, p_M)\)

\[\left[ x^M, x^N \right]_\star = \left[ p_M, p_N \right]_\star = 0, \quad [x^M, p_N]_\star = i\delta^M_N. \tag{3}\]

Let us now consider the Fourier transform in the momentum variable \(p_M\). We will call this “half-Fourier space” since only one of the noncommutative variables is being Fourier transformed. So, the transform of \(A (x^M, p_M)\) is a bi-local function \(\psi_A (x^M, y^M)\) in position space, but we will write it in the form \(\tilde{A} (l^M, r^M) \equiv \psi_A (x^M, y^M)\) where

\[l^M = x^M + \frac{y^M}{2}, \quad r^M = x^M - \frac{y^M}{2}. \tag{4}\]

Thus, we define

\[A (x^M, p_M) = \int d^D y e^{-iy^M p_M} \tilde{A} \left( x^M + \frac{y^M}{2}, x^M - \frac{y^M}{2} \right) = \int d^D y e^{-iy^M p_M} \psi_A (x^M, y^M). \tag{5}\]

The Moyal star product of two functions \((A \star B) (x, p) = C (x, p)\) may now be evaluated in terms of these integral representations. The result can be written in terms of the Fourier transform of \(C (x, p) \rightarrow \tilde{C} (l^M, r^M) = \psi_C (x^M, y^M)\)

\[(A \star B) (x, p) = \int d^D y e^{-ip y} \tilde{C} \left( x + \frac{y}{2}, x - \frac{y}{2} \right). \tag{6}\]

One finds that \(\tilde{C} (l^M, r^M)\) is related to \(\tilde{A} (l^M, r^M)\) and \(\tilde{B} (l^M, r^M)\) by a matrix-like multiplication with continuous indices

\[\tilde{C} (l^M, r^M) = \int d^D y \tilde{A} (l^M, y^M) \tilde{B} (y^M, r^M). \tag{7}\]
This is verified explicitly by the following steps

\[
A \star B = \int d^D y d^D y' \tilde{A} \left( x + \frac{y}{2}, x - \frac{y}{2} \right) e^{-iyp} e^{\frac{i}{\hbar} \left( \frac{\partial}{\partial x} y' - \frac{\partial}{\partial p} y \right)} e^{-iy'p} \tilde{B} \left( x + \frac{y'}{2}, x - \frac{y'}{2} \right) \\
= \int d^D y d^D y' e^{-iyp} \tilde{A} \left( x + \frac{y}{2}, x - \frac{y}{2} \right) e^{\frac{i}{\hbar} \left( \frac{\partial}{\partial x} y' - \frac{\partial}{\partial p} y \right)} \tilde{B} \left( x - \frac{y}{2}, x - \frac{y}{2} \right) e^{-iy'p} \\
= \int d^D y d^D y' e^{-i(y+y')p} \tilde{A} \left( x + \frac{y+y'}{2}, x - \frac{y+y'}{2} \right) \tilde{B} \left( y, x - \frac{y}{2} \right) \\
= \int d^D y_+ d^D y_- e^{-ipy_+} \tilde{A} \left( x + \frac{y_+}{2}, x - \frac{y_+}{2} \right) \tilde{B} \left( y_-, x - \frac{y_-}{2} \right)
\]

(8)

where, in the second line the derivatives with respect to \( p \) are evaluated, in the third line the translation operators \( e^{\frac{i}{\hbar} \frac{\partial}{\partial x} y'} \), \( e^{-\frac{i}{\hbar} \frac{\partial}{\partial p} y} \) are applied on the \( x \) coordinates on the left and right respectively, in the fourth line one defines \( y_+ = y + y' \) and \( y_- = x - (y - y')/2 \), and finally the \( y_- \) integration is performed.

Hence, in our notation, the Moyal star product in Fourier space is equivalent to infinite matrix multiplication with the rules of Eq.(7): the right variable of \( \tilde{A} \) is identified with the left variable of \( \tilde{B} \) and then integrated. Eq.(7) is the key observation for establishing a direct relation between Witten’s star-product and Moyal’s star-product as we will see below.

The fact that the Moyal star-product is related to some version of matrix multiplication is no surprise, as by now a few versions of matrix representations have been used in the physics literature. The one used here is straightforward: after using the Weyl correspondence to derive an operator \( \hat{A} \) from the function \( A(x,p) \), the matrix representation \( A(l,r) \) is nothing but the matrix elements of the operator \( \hat{A} \) in position space: \( A(l,r) = \langle l | \hat{A} | r \rangle \).

## 2 Witten’s star product in split string space

We will show that the continuous matrix representation of the Moyal star product of the previous section is in detail related to Witten’s star product in string field theory. The rough idea is to replace the points \( l^M, r^M \) by left and right sides of a string \( x^\mu [\sigma] = l^\mu [\sigma] \oplus r^\mu [\sigma] \) (with \( l, r \) defined relative to the midpoint at \( \sigma = \pi/2 \)). If we consider the fields of two strings \( \tilde{A} \left( l_1^\mu [\sigma], r_1^\mu [\sigma] \right) \) and \( \tilde{B} \left( l_2^\mu [\sigma], r_2^\mu [\sigma] \right) \), then Witten’s string star product is formally given by the functional integral [3]

\[
\tilde{C} \left( l_1^\mu [\sigma], r_2^\mu (\sigma) \right) = \int [dz] \tilde{A} \left( l_1^\mu [\sigma], z^\mu [\sigma] \right) \tilde{B} \left( z^\mu [\sigma], r_2^\mu [\sigma] \right),
\]

(9)

where \( z^\mu [\sigma] = r_1^\mu [\sigma] = l_2^\mu [\sigma] \) corresponds to the overlap of half of the first string with half of the second string, and \( \tilde{C} \left( l_1^\mu [\sigma], r_2^\mu (\sigma) \right) \) is the field describing the joined half strings as a new full string \( x_3^\mu [\sigma] = l_1^\mu [\sigma] \oplus r_2^\mu (\sigma) \). Considering the close analogy to Eq.(7), morally we...
anticipate to be able to rewrite Witten’s star product as a Moyal star-product of the form (1) in a larger space. In the remainder of this paper the details of this map will be clarified, and will be shown that Witten’s ★ is indeed Moyal’s ★ in half of a relativistic phase space of the full string, involving only the even modes \((x_{2n}, p_{2n})\) or only the odd modes \((x_{2n-1}, p_{2n-1})\).

Our result may be summarized as follows: Define the Fourier transform of the string field \(\psi_A(x_0, x_{2n}, x_{2n-1}) \equiv \tilde{A}(l_1^\mu [\sigma], r_1^\mu [\sigma])\) in the odd modes only as follows

\[
A(x_{2n}, x_0, p_{2n}) = \int (\prod_{n=1}^\infty dx_{2n-1}) \psi_A(x_0, x_{2n}, x_{2n-1}) \exp \left(-2i \sum_{k,l=1}^\infty p_{2k}^\mu (T_{2k,2l-1}) x_{2l-1}^\nu \eta_{\mu\nu}\right)
\]

(10)

where \(T_{2k,2l-1}\) is a matrix to be defined below. Then Witten’s star product (9) for two string fields \((\psi_A \star_{\text{witten}} \psi_B)(x_0, x_{2n}, x_{2n-1})\) is equivalent to the Moyal star product for their Fourier transformed fields \((A \star B)(x_{2n}, x_0, p_{2n})\) with the usual definition of the Moyal star product involving the phase space of only the even modes

\[
\star \equiv \exp \left(\frac{i}{2} \sum_{n=1}^\infty \eta^{\mu\nu} \left(\frac{\partial}{\partial x_{2n}^\mu} \frac{\partial}{\partial x_{2n}^\nu} - \frac{\partial}{\partial p_{2n}^\mu} \frac{\partial}{\partial p_{2n}^\nu}\right)\right).
\]

(11)

Furthermore, the definition of trace is the phase space integration

\[
Tr(\psi) = Tr A \equiv \int (\prod_{n=1}^\infty \frac{dx_{2n} dp_{2n}}{2\pi}) \quad A(x_{2n}, x_0, p_{2n}).
\]

(12)

In the definitions of both the star \((A \star B)(x_{2n}, x_0, p_{2n})\) and trace \(Tr A\) either the center of mass mode \(x_0\) or the midpoint mode \(\bar{x} \equiv x(\frac{\pi}{2}) = x_0 + \sqrt{2} \sum_{n=1}^\infty x_{2n} (-1)^n\) is held fixed while taking derivatives or doing integration with respect to the \(x_{2n}\) modes. This is precisely related to the midpoint ambiguity in the split string formalism which will be clarified in the next section.

Witten’s star product (9) is more carefully defined in the split string formalism which was developed sometime ago [6][7] and was used in recent studies of string field theory [8][9][10]. As mentioned in the previous paragraph there is a dilemma involving the midpoint which so far has remained obscure in the literature. We will address this issue in the next section by considering the options that are available in the formulation of the split string formalism, namely odd versus even modding of the split string modes [6][7]. The choice affects the definition of the star product within the split string formalism. In turn this choice is related to whether the center of mass mode \(x_0\) or the midpoint mode \(\bar{x}\) is held fixed as described in the previous paragraph. In this section we begin with the odd modding that has been used in the recent literature.
The open string position modes are identified as usual by the expansion \( x(\sigma) = x_0 + \sqrt{2} \sum_{n=1}^{\infty} x_n \cos (n\sigma) \) (with \( 0 \leq \sigma \leq \pi \)), where \( x_0 \) is the center of mass position of the string. We will omit the spacetime index \( \mu \) on all vectors whenever there is no confusion. The position of the midpoint is given by \( \bar{x} \equiv x(\frac{\pi}{2}) = x_0 + \sqrt{2} \sum_{n=1}^{\infty} x_n \cos \left( \frac{n\pi}{2} \right) \). Therefore, instead of \( x_0 \) we may use \( \bar{x} \) as the independent degree of freedom and write the following mode expansions for the full string \( x(\sigma) \), as well as for the left side \( l(\sigma) \equiv \{ x(\sigma) \mid 0 \leq \sigma \leq \frac{\pi}{2} \} \) and the right side \( r(\sigma) \equiv \{ x(\pi - \sigma) \mid 0 \leq \sigma \leq \frac{\pi}{2} \} \) of the same string

\[
x(\sigma) = x_0 + \sqrt{2} \sum_{n=1}^{\infty} x_n \cos (n\sigma), \quad 0 \leq \sigma \leq \pi, \tag{13}
\]

\[
x(\sigma) = \bar{x} + \sqrt{2} \sum_{n=1}^{\infty} x_n \left( \cos (n\sigma) - \cos \left( \frac{n\pi}{2} \right) \right) \tag{14}
\]

\[
l(\sigma) = \bar{x} + \sqrt{2} \sum_{n=1}^{\infty} l_{2n-1} \cos ((2n-1)\sigma), \quad 0 \leq \sigma \leq \frac{\pi}{2}, \tag{15}
\]

\[
r(\sigma) = \bar{x} + \sqrt{2} \sum_{n=1}^{\infty} r_{2n-1} \cos ((2n-1)\sigma), \quad 0 \leq \sigma \leq \frac{\pi}{2}. \tag{16}
\]

These mode expansions are obtained by imposing Neumann boundary conditions at the ends of the string \( \partial_{\sigma} x|_{0,\pi} = \partial_{\sigma} l|_{0} = \partial_{\sigma} r|_{0} = 0 \), and Dirichlet boundary conditions at the midpoint \( x(\frac{\pi}{2}) = l(\frac{\pi}{2}) = r(\frac{\pi}{2}) = \bar{x} \). Using the completeness and orthogonality of the trigonometric functions in these expansions one can easily extract the relationship between the left/right modes and the full string modes

\[
l_{2n-1} = \frac{2\sqrt{2}}{\pi} \int_{0}^{\frac{\pi}{2}} d\sigma \ (l(\sigma) - \bar{x}) \cos (2n-1)\sigma = \frac{2\sqrt{2}}{\pi} \int_{0}^{\frac{\pi}{2}} d\sigma \ (x(\sigma) - \bar{x}) \cos (2n-1)\sigma
\]

\[
r_{2n-1} = \frac{2\sqrt{2}}{\pi} \int_{0}^{\frac{\pi}{2}} d\sigma \ (r(\sigma) - \bar{x}) \cos (2n-1)\sigma = \frac{2\sqrt{2}}{\pi} \int_{0}^{\frac{\pi}{2}} d\sigma \ (x(\pi - \sigma) - \bar{x}) \cos (2n-1)\sigma
\]

\[
x_{n \neq 0} = \frac{\sqrt{2}}{\pi} \int_{0}^{\pi} d\sigma \ x(\sigma) \ \cos (n\sigma) = \frac{\sqrt{2}}{\pi} \int_{0}^{\frac{\pi}{2}} d\sigma \ [l(\sigma) + (-1)^{n} r(\sigma)] \ \cos (n\sigma).
\]

The result is

\[
x_{2n-1} = \frac{1}{2} (l_{2n-1} - r_{2n-1}) \tag{17}
\]

\[
x_{2n \neq 0} = \frac{1}{2} \sum_{m=1}^{\infty} \mathcal{T}_{2n,2m-1} (l_{2m-1} + r_{2m-1}) \tag{18}
\]

\[
x_{0} = \bar{x} + \frac{1}{4} \sum_{m=1}^{\infty} \mathcal{T}_{0,2m-1} (l_{2m-1} + r_{2m-1}) \tag{19}
\]

where

\[
\mathcal{T}_{2n,2m-1} = \frac{4}{\pi} \int_{0}^{\frac{\pi}{2}} d\sigma \ \cos ((2n)\sigma) \ \cos ((2m-1)\sigma) \tag{20}
\]

\[
= \frac{2(-1)^{m+n+1}}{\pi} \left( \frac{1}{2m-1+2n} + \frac{1}{2m-1-2n} \right). \tag{21}
\]
The inverse relations are

\[ l_{2n-1} = x_{2m-1} + \sum_{n=1}^{\infty} R_{2m-1,2n} x_{2n} \]  

(22)

\[ \bar{x} = x_0 + \sqrt{2} \sum_{n=1}^{\infty} (-1)^n x_{2n} \]  

(23)

\[ r_{2m-1} = -x_{2m-1} + \sum_{n=1}^{\infty} R_{2m-1,2n} x_{2n} \]  

(24)

where

\[ R_{2m-1,2n} = \frac{4}{\pi} \int_{0}^{\pi/2} d\sigma \cos (2m-1) \sigma \left[ \cos 2n\sigma - (-1)^n \right] \]  

(25)

\[ = T_{2m,2m-1} - (-1)^n T_{0,2m-1} \]  

(26)

\[ = \frac{4n}{\pi (2m-1)} \left( \frac{1}{2m-1+2n} - \frac{1}{2m-1-2n} \right). \]  

(27)

We note that \( T_{0,2m-1} \) is given by Eq.(21), but it also satisfies \( T_{0,2m-1} = -2 \sum_{k=1}^{\infty} (-1)^k T_{2k,2m-1} \). It must be mentioned that \( R_{2k-1,2m} \) is the inverse of \( T_{2m-1,2n} \) on both sides

\[ (RT)_{2n-1,2k-1} = \delta_{n,k}, \quad (TR)_{2m,2l} = \delta_{m,l}. \]  

(28)

In the split string notation the string field is \( \psi_{\lambda}(x_0, x_{2n}, x_{2n-1}) \equiv \tilde{A} \{l_{2n-1} \}, \bar{x}, \{r_{2n-1} \} \). Note that the midpoint \( \bar{x} \) is treated as the independent degree of freedom rather than the center of mass mode \( x_0 \). Witten’s star product takes the form (no integration over \( \bar{x} \))

\[ \tilde{C} \{l_{2n-1} \}, \bar{x}, \{r_{2n-1} \} = \int \tilde{A} \{l_{2n-1} \}, \bar{x}, \{z_{2n-1} \} \tilde{B} \{z_{2n-1} \}, \bar{x}, \{r_{2n-1} \} \prod_k dz_{2k-1}. \]  

(29)

By analogy to section-1 we see that we should compare \( \{l_{2n-1}^\mu \}, \{r_{2n-1}^\mu \} \) to \( (l^M, r^M) \) and therefore via Eqs.(1,23) we should establish the following correspondence

\[ \sum_{n=1}^{\infty} R_{2m-1,2n} x_{2n}^\mu \sim x^M; \quad x_{2m-1}^\mu \sim \frac{y^M}{2}. \]  

(30)

This suggests that we define a Fourier transform in twice the odd modes \( (2x_{2m-1}^\mu) \sim y^M \) to obtain the string field in phase space. The Fourier parameters would play the role of conjugate momenta to the following combination of even modes \( R_{x_e} = \sum_{n=1}^{\infty} R_{2m-1,2n} x_{2n}^\mu \sim x^M \). We will use the symbol \( R_{x_e} \) as a short hand notation to indicate that the even modes (denoted by the subscript \( e \)) are transformed by the matrix \( R \). Therefore it is convenient to choose the Fourier parameters in the combination \( p_e T = \{ \sum_{n=1}^{\infty} x_{2n}^\mu T_{2n,2m-1}; \quad m = 1, 2, \cdots \} \sim p_M \). We will also use \( x_{\text{odd}} \) as a short hand notation for \( x_{\text{odd}} = \{x_{2n-1}^\mu \} \). Then we define the string field in phase space by complete analogy to Eq.(3)

\[ A(R_{x_e}, \bar{x}, p_e T) = \int [dx_{\text{odd}}] \exp (-i (p_e T \cdot 2x_{\text{odd}})) \tilde{A} (R_{x_e} + x_{\text{odd}}, \bar{x}, R_{x_e} - x_{\text{odd}}) \]  

(31)
The right hand side is the same expression given in (11) since $\tilde{A}(R \delta_t + x_{even}, \bar{x}, R \delta_t - x_{odd}) = \psi_A(x_0, x_{2n}, x_{2n-1})$. This construction guarantees that Witten’s star product in the split string notation of Eq.(29) will be precisely reproduced by a Moyal star product in which $R \delta_t$ and $p_e T$ are the conjugate phase space variables. This is seen by retracing the steps of the computations that lead to Eq.(8). Therefore, the Moyal star should satisfy $[R \delta_t, p_e T]_\star \sim i$, or in more detail

$$\left[ \left( \sum_{n=1}^{\infty} R_{2k-1,2n} x_{2n} \right), \left( \sum_{m=1}^{\infty} T_{2m,2l-1} p_{2m} \right) \right]_\star = i \delta_{k,l} \eta^{\mu\nu}. \quad (32)$$

However, using the fact that $R$ and $T$ are each other’s inverses we see that this is equivalent to the simple Heisenberg commutation relations under the Moyal star product given in Eq.(11)

$$[x_{2n}^{\mu}, p_{2n}^{\nu}]_\star = i \eta^{\mu\nu}. \quad (33)$$

Therefore, the Witten star-product reduces just to the usual Moyal product in the phase space of only the even modes. Since the Moyal $\star$ and $\psi_A(x_0, x_{2n}, x_{2n-1})$ are both independent of $R$ and $T$ we see that $R$ and $T$ can be removed from the phase space string fields in comparing the left hand sides of Eqs.(10,31). Therefore we may write the string field in (31) simply as $A(x_e, \bar{x}, p_e)$ and define string field theory using the Moyal star product given in Eq.(11)

$$A \star B (x_e, \bar{x}, p_e). \quad (34)$$

The net effect of the intermediate steps involving the split string formalism with odd modes $l_{2n-1}, r_{2n-1}$ is to keep the midpoint $\bar{x}$ fixed while evaluating string overlaps in Eq.(29). Therefore, in the Moyal basis $x_{2n}, p_{2n}$, the star product of Eq.(11) must be evaluated by first writing all string fields $\psi_{A,B}(x_0, x_{2n}, x_{2n-1})$ in terms of $\bar{x}$ instead of $x_0$, and then applying the derivatives with respect to $x_{2n}$. Other than this relic of split strings, the relation between the original string field $\psi_A(x_0, x_{2n}, x_{2n-1})$ and its Fourier transform $A(x_{2n}, x_0, p_{2n})$ given in Eq.(11), or the computation of star products, do not involve the split string formalism.

3 Split strings with even modes

It seems puzzling that $\bar{x}$ was distinguished since $x_0$ appears to be more natural in the Moyal basis. Furthermore, $x_0$ is gauge invariant under world sheet reparametrizations, unlike $\bar{x}$. In fact, there is another split string formalism [7] that favors fixing $x_0$ rather than $\bar{x}$ as explained below. First we note the following properties of trigonometric functions when $0 \leq \sigma \leq \pi$ for
integers $m, n \geq 1$

$$\cos((2n - 1)\sigma) = \text{sign}(\frac{\pi}{2} - \sigma) \sum_{m=1}^{\infty} [\cos(2m\sigma) - (-1)^m] T_{2m, 2n-1}$$  \hspace{1cm} (35)

$$[\cos(2m\sigma) - (-1)^m] = \text{sign}(\frac{\pi}{2} - \sigma) \sum_{n=1}^{\infty} \cos((2n - 1)\sigma) R_{2n-1, 2m}.$$  \hspace{1cm} (36)

Both sides of these equations satisfy Neumann boundary conditions at $\sigma = 0$ and Dirichlet boundary conditions at $\sigma = \frac{\pi}{2}$, and both are equivalent complete sets of trigonometric functions for the range $0 \leq \sigma \leq \frac{\pi}{2}$. In the previous section we made the choice of expanding $l(\sigma), r(\sigma)$ in terms of the odd modes. Now we see that we could also expand them in terms of the even modes as follows

$$l(\sigma) = \bar{x} + \sqrt{2} \sum_{m=1}^{\infty} l_{2m} [\cos(2m\sigma) - (-1)^m] = l_0 + \sqrt{2} \sum_{m=1}^{\infty} l_{2m} \cos(2m\sigma)$$  \hspace{1cm} (37)

and similarly for $r(\sigma)$. Comparing to the expressions in the previous section, and using (36) we can find the relation between the odd modes $(l_{2n-1}, r_{2n-1})$ and the even modes $(l_{2n}, r_{2n})$

$$l_{2n-1} = \sum_{m=1}^{\infty} R_{2n-1, 2m} l_{2m}, \quad l_{2m} = \sum_{n=1}^{\infty} T_{2m, 2n-1} l_{2n-1}$$  \hspace{1cm} (38)

$$r_{2n-1} = \sum_{m=1}^{\infty} R_{2n-1, 2m} r_{2m}, \quad r_{2m} = \sum_{n=1}^{\infty} T_{2m, 2n-1} r_{2n-1}$$  \hspace{1cm} (39)

Furthermore, by using the relation between the odd string modes $(l_{2n-1}, \bar{x}, r_{2n-1})$ and the full string modes $(x_0, x_{2n}, x_{2n-1})$ in Eqs.(22-24) or by direct comparison to $x(\sigma)$, we derive the relation between the even split string modes and the full string modes.

$$l_{2m} = x_{2m} + \sum_{n=1}^{\infty} T_{2m, 2n-1} x_{2n-1}, \quad r_{2m} = x_{2m} - \sum_{n=1}^{\infty} T_{2m, 2n-1} x_{2n-1}.$$  \hspace{1cm} (40)

The inverse relation is

$$x_{2m} = \frac{1}{2}(l_{2m} + r_{2m}), \quad x_{2m-1} = \sum_{n=1}^{\infty} R_{2m-1, 2n} \frac{1}{2}(l_{2n} - r_{2n}).$$  \hspace{1cm} (41)

Furthermore, the relations between the zero modes are

$$l_0 = x_0 + \frac{1}{\sqrt{2}} \sum_{n=1}^{\infty} T_{0, 2n-1} x_{2n-1}, \quad r_0 = x_0 - \frac{1}{\sqrt{2}} \sum_{n=1}^{\infty} T_{0, 2n-1} x_{2n-1}.$$  \hspace{1cm} (42)

Note that the matching condition at the midpoint $l(\pi/2) = r(\pi/2) = x(\pi/2) = \bar{x}$ is satisfied by the even modes, because $l_0, r_0$ automatically obey the relation

$$l_0 - r_0 + \sqrt{2} \sum_{n=1}^{\infty} (l_{2n} - r_{2n}) (-1)^n = 0$$  \hspace{1cm} (43)
thanks to the property \( \mathcal{T}_{0,2n-1} + 2 \sum_{n=1}^{\infty} (-1)^m \mathcal{T}_{2m,2n-1} = 0 \).

This setup allows us to define split string fields that distinguish the center of mass mode \( x_0 \) rather than the midpoint \( \bar{x} \) as follows

\[
\hat{A}(l_{2n}, x_0, r_{2n}) \equiv \psi_A(x_0, x_{2n}, x_{2n-1}) = \hat{A}(l_{2n-1}, \bar{x}, r_{2n-1})
\]

where \((l_{2n}, r_{2n})\) are given above in terms of the full string modes. For this case we define the Witten star product in the form (no integration over \( x_0 \))

\[
\hat{A}((l_{2n}), x_0, (r_{2n})) = \int \hat{A}((l_{2n}), x_0, (z_{2n})) \hat{B}((z_{2n}), x_0, (r_{2n})) \prod_k dz_{2k}.
\]

Evidently, this product is different than the one in Eq.(29) since the mode that is held fixed during integration is different (i.e. \( x_0 \) rather than \( \bar{x} \)).

We now repeat the arguments that made analogies to section-1 to rewrite this overlap of half strings in terms of a Moyal product. We find again that the even modes in phase space \( x_{2n}, p_{2n} \) are the relevant ones. Furthermore, the expressions for the string field in phase space \( A(x_{2n}, x_0, p_{2n}) \) in terms of the original \( \psi_A(x_0, x_{2n}, x_{2n-1}) \) is identical to the one given in Eq.(10), and the expression for the star product is also the same as Eq.(11). The only difference is that now \( x_0 \) must be held fixed while evaluating the derivatives with respect to \( x_{2n} \). We see that the relic of the split string formalism with even modes is to hold \( x_0 \) fixed (rather than \( \bar{x} \)) while performing computations in the Moyal basis.

More work is required in order to decide which of these procedures is the correct definition of string field theory. In particular, the symmetries of the full action, including ghosts, will be relevant in distinguishing them from each other. Of course, the computation of string amplitudes will also play a role. We hope to report on further work along these lines in a future publication.

4 Remarks

It seems puzzling that only the even modes appear in the Moyal star product. Although the theory in position space contains both even and odd position modes \( x_{2n}, x_{2n-1} \), the mapping of the Witten \( \star \) to the Moyal \( \star \) necessarily requires that the Fourier space for the odd positions be named as the even momenta since \((x_e, p_e)\) are canonical under the Moyal star-product. Likewise, the Fourier space for the even positions should be named as the odd momenta. Therefore the double Fourier transform of the string field \( A(x_e, \bar{x}, p_e) \), with Fourier kernels of the form \([10]\) (with \( T \) or \( R \) as needed) that mix odd-even phase space variables, would be written purely in terms of odd phase space variables \( a(p_{odd}, \bar{x}, x_{odd}) \).

In usual phase space quantization all positions and all momenta enter directly in the Moyal product, however in the present case, which is designed to be equivalent to Witten’s
open string field theory, only half of the phase space variables enter into the definition of the Moyal star product in Eq. (11).

It is straightforward to extend the star product to the ghost sector in the bosonized ghost formalism. Then the ghost field $\phi(\sigma)$ plays the role of one extra dimension. If we follow the standard wisdom, our Moyal star product, including ghosts, would be modified only by inserting a phase at the midpoint, $\exp(\frac{2i}{\pi}\phi(\frac{\pi}{2}))$, after evaluating the Moyal product. In view of the discussion in the previous sections one should analyse this phase insertion more carefully. To construct a string field theory one would also need to define a BRST operator with the usual properties. The study of string field theory takes a new form with the new star product. It would be interesting to see where this leads.

The new Moyal star product in string field theory Eq. (11) is Lorentz invariant, in contrast to the Moyal product used in recent studies of noncommutative field theory (in the presence of a Magnetic field). The string field formalism includes gauge symmetries that remove ghosts. Since string theory makes sense, and is ghost free (unitary), it implies that there is at least one way of making sense of noncommutative field theory when timelike components of coordinates are included (see also [11] in this respect).

In the presence of a large background $B$-field the midpoint coordinates $\vec{x}^\mu$ become non-commuting as well. In that case, the Moyal star product in Eq. (11) is easily modified to accomodate the midpoint noncommutativity in the usual way. If the $B$-field is small the noncommutative structure is considerably more complicated.

For the study of $D_p$-brane solutions in the vacuum string field theory approach of [12][13][14][15] one seeks solutions (independent of $x_0$ or $\vec{x}$) to the projector equation

$$(A \star A)(x_e, p_e) = A(x_e, p_e)$$

in the pure matter sector. Such projectors, involving the Moyal product in phase space, have been studied for a long time in the literature; they are known as Wigner functions [2] and they have applications in various branches of physics. It would be simple to generalize the known Wigner functions to the multi-dimensional string mode space needed in string theory, and then study their interpretation in string theory. In particular, the recent solutions obtained in the split string formalism can be easily Fourier transformed to the phase space formalism. For example, the solution for the sliver state in our formalism becomes a Gaussian of the form

$$A(x_e, p_e) = (\det(2 \times 1_e))^d \exp(-x_e^T M x_e - p_e^T M^{-1} p_e),$$

where $d$ is the number of spacetime dimensions, $M$ is a matrix in even mode space, $1_e$ is the identity matrix in that space, and $det(2 \times 1_e) = (\prod_{n=1}^{\infty} 2)$. The phase space integral over

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\[2\] A recent application of deformation quantization produced the proper approach for discussing two time physics in a field theory setting. This has conceptual and technical similarities to string field theory, especially with the new form of string theory based on the Moyal product in phase space [11].
this function gives the rank of the projector, and this is easily seen to be rank one for any matrix $M$.

$$\text{Tr} A = \int \prod_{n=1}^{\infty} \prod_{\mu=0}^{d-1} \frac{dx_\mu^n dp_\mu^n}{2\pi} \, A(x_e, p_e) = 1. \quad (48)$$

In more general computations we anticipate that it would be useful to evaluate the star product for phase space functions of the form

$$A_{M,\lambda,N}(x_e,p_e) = N \exp^{-\eta_{\mu\nu}(x_e^a ax_e^b + x_e^a bp_e^b + p_e^a b^T x_e^a + p_e^a d x_e^a)} - (x_e^a v_\mu + p_e^a w_\mu), \quad (49)$$

where in even mode space, $(a, d)$ are symmetric matrices, $b$ is a general square matrix with $b^T$ its transpose, $(v_\mu, w_\mu)$ are column matrices, and $N$ is a normalization factor. In general these parameters are complex numbers. With the definition of trace given in Eq.(12), we have

$$\text{Tr} (A_{M,\lambda,N}) = N \exp^{i\lambda M^{-1} \lambda}/(\det (2M))^{d/2}. \quad (51)$$

We record the result of our computation of the star product for use in future applications

$$(A_{M_1,\lambda_1,N_1} \star A_{M_2,\lambda_2,N_2})(x_e, p_e) = A_{M_{12},\lambda_{12},N_{12}}(x_e, p_e), \quad (52)$$

with

$$M_{12} = (M_1 + M_2 \sigma M_1)(1 + \sigma M_2 \sigma M_1)^{-1} + (M_2 - M_1 \sigma M_2)(1 + \sigma M_1 \sigma M_2)^{-1} \quad (53)$$

$$\lambda_{12}^\mu = (1 + M_2 \sigma)(1 + M_1 \sigma M_2^{-1} \lambda_1^\mu + (1 - M_1 \sigma)(1 + M_2 \sigma M_1 \sigma)^{-1} \lambda_2^\mu \quad (54)$$

$$N_{12} = N_1 N_2 (\det (1 + M_2 \sigma M_1 \sigma))^{-d/2} e^{i\pi \mu \nu \lambda_1^\mu \lambda_2^\nu} \quad (55)$$

$$K^{ij} = \left( \begin{array}{cc} (M_1 + \sigma M_2^{-1} \sigma)^{-1} & (\sigma + M_2 \sigma M_1)^{-1} \\ - (\sigma + M_1 \sigma M_2)^{-1} & (M_2 + \sigma M_1^{-1} \sigma)^{-1} \end{array} \right) \quad (56)$$

where $\sigma$ is the purely imaginary matrix that results from the star commutation rules of $(x_e, p_e)$ in even mode space $[x_e^\mu, p_e^\nu] = i\eta^{\mu\nu} 1_e$ and $[x_e^\mu, x_e^\nu] = 0$, $[p_e^\mu, p_e^\nu] = 0$

$$\sigma = i \left( \begin{array}{cc} 0 & 1_e \\ -1_e & 0 \end{array} \right). \quad (57)$$

Eq.(52) may be used as a generating function for the star products of more general functions. For example, the star products of more general functions, such as polynomials multiplied by exponentials of the form (49), can be obtained by taking derivatives of both sides of (52) with respect to the parameters in $A_{M_1,\lambda_1,N_1}, A_{M_2,\lambda_2,N_2}$. We see that Eqs.(46,47)
for the projector follow from the more general multiplication rule \( (52) \). We also see that a more general projector is given by Eq.\((49) \) when \( M \) satisfies \( \sigma M \sigma = M^{-1} \) and \( N = (\det (2 \times 1_e))^{26} \exp (-\lambda Mu^{-1} \lambda_\mu/4) \), since according to \((53-56) \), one gets \( M = M_1 = M_2 = M_3 \), and \( \lambda = \lambda_1 = \lambda_2 = \lambda_1 \) and \( N = N_1 = N_2 = N_12 \). The trace of the more general projector is still 1, according to \((51) \).

It has long been known that Witten’s star product defines a noncommutative geometry for strings, but its relation to other forms of noncommutative geometry has remained obscure. By making the present bridge to the Moyal star product one may expect new progress, as well as cross fertilization between studies in string field theory and noncommutative field theory, and perhaps even other fields of physics that utilize Wigner functions.

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