Planar Soap Bubbles

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The generalized soap bubble problem seeks the least perimeter way to enclose and separate $n$ given volumes in $\mathbb{R}^m$. We study the possible configurations for perimeter minimizing bubble complexes enclosing more than two regions. We prove that perimeter minimizing planar bubble complexes with equal pressure regions and without empty chambers must have connected regions. As a consequence, we show that the least perimeter planar graph that encloses and separates three equal areas in $\mathbb{R}^2$ using convex cells and without empty chambers is a “standard triple bubble” with connected regions.
Planar Soap Bubbles

By
RICHARD PAUL DEVEREAUX VAUGHN

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Professor Joel Hass
Dissertation Committee Chair
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Chapter 1

Introduction

The classic *isoperimetric problem* is to find the largest amount of area that can be enclosed using a simple closed curve of fixed length. The answer is of course a circle, although the proof is more difficult than some may realize. The ancient Greeks knew the problem and the solution. In fact, Pappus records that Zenodorus found the solution first \[9\]. The first mathematical proof, however, is credited to Steiner \[10\] in the 19th century. He proved that if a solution exists, then it must be a circle. Carathéodory\footnote{Blaschke\cite{4} credits Edler, Carathéodory and Study with existence results. Bandle\cite{3} claims Carathéodory was first. Schmidt and Weierstrauss completed the three dimensional analogue.} completed the proof by showing that a solution does exist. We refer to the excellent prefaces in Pólya and Szegö\cite{9} and in Bandle\cite{3} for more historical details.

\begin{footnotesize}
\begin{enumerate}
\item [1] Blaschke\cite{4} credits Edler, Carathéodory and Study with existence results. Bandle\cite{3} claims Carathéodory was first. Schmidt and Weierstrauss completed the three dimensional analogue.
\end{enumerate}
\end{footnotesize}
inequality
\[ A \leq \frac{1}{4\pi} (L)^2, \]
where \( L \) is the length of any simple closed curve in the plane and \( A \) is the enclosed area. Equality holds if and only if the simple closed curve is a circle.

A related problem is to find the simple closed curve with least perimeter that encloses a given area. For a single area in \( \mathbb{R}^2 \) or a single volume in \( \mathbb{R}^m \) the problem is equivalent to the classic isoperimetric problem. What if we wanted, however, to enclose and separate two different prescribed areas in the plane? The solution is not two disjoint circles as we can use less perimeter by letting the two different areas share some of their perimeter. The solution is a standard double bubble consisting of two chambers enclosed by three arcs of circles all meeting at angles of \( \frac{2\pi}{3} \). (See Figure 1.1.)

![Figure 1.1: A standard double bubble enclosing and separating areas \( A_1 \) and \( A_2 \)](image)

The problem that seeks the least perimeter closed curve, surface, or hypersurface that encloses and separates \( n \) given volumes in \( \mathbb{R}^m \) is called the generalized soap
bubble problem. The name comes from the fact that soap bubbles minimize surface
tension for the fixed volumes of air enclosed.

Another related problem is to separate or tile $\mathbb{R}^m$ into equal volume pieces as
efficiently as possible. In $\mathbb{R}^2$, the *honeycomb conjecture* states that a regular hexagonal tiling is the least perimeter tiling that separates the plane into unit area pieces, although it is not immediately clear what is meant by least perimeter in this infinite region context. One interpretation is to consider this problem as the limit of the $n$ soap bubble problem in $\mathbb{R}^2$ as $n \to \infty$.

Only recently has progress been made on even the smallest cases of the general-ized soap bubble problem. In 1976, Almgren [1] proved the existence and regularity of a solution to the generalized soap bubble problem in dimensions bigger than 2. Taylor[11] improved this result for dimension 3 in the same year. In 1992, Morgan [7] proved existence and regularity of a solution in dimension 2.

The double bubble problem in $\mathbb{R}^2$ (the least perimeter embedded planar graph that encloses and separates two given areas in the plane) was solved in 1994 by a group of undergraduates led by Frank Morgan [6]. The $n$ bubble problem in $\mathbb{R}^2$ for $n > 2$, i.e. the least perimeter graph that encloses and separates $n$ given areas in $\mathbb{R}^2$ is the focus of this paper. The general $n$ bubble conjecture is that the least perimeter solution will always have connected regions. In particular, we examine three regions and the corresponding triple bubble conjecture.

**Conjecture** [1] The least perimeter planar graph that encloses and separates three
finite areas $A_1$, $A_2$, and $A_3$ is a regular triple bubble complex with four vertices, six edges, and three connected regions.

Figure 1.2: The conjectured minimum graph enclosing areas $A_1$, $A_2$, and $A_3$

Although much of this work was completed in the context of the triple bubble problem, many results are true of planar bubbles in general. We will specify when a result is valid only for three regions.

We begin Chapter 2 by looking at some general considerations about perimeter minimizing planar bubble complexes. After some definitions, we examine Morgan’s existence and regularity theorem. We specifically look into the length $L(A_1, A_2, A_3)$ needed to enclose three areas $A_1$, $A_2$, and $A_3$. We prove

**Lemma 2.3** The length function $L(A_1, A_2, A_3)$ is continuous.

In an attempt to decrease perimeter, it is sometimes convenient to increase the area enclosed by a region. In Conjecture 2, we modify the triple bubble conjecture to allow these area increases.
Conjecture 2  Given three finite positive real numbers $A_1$, $A_2$, and $A_3$, the least perimeter graph that encloses and separates three finite areas $B_1$, $B_2$, and $B_3$ such that $B_i \geq A_i$ for all $i$, is a standard triple bubble.

Using Lemma 2.3, we prove that the two conjectures are equivalent in

Theorem 2.4 Let three positive areas $A_1$, $A_2$, and $A_3$ be given. There exists a least perimeter triple bubble complex $B$ that encloses and separates areas $B_1$, $B_2$, and $B_3$ with $B_i \geq A_i$. That is, if $C$ is any other complex enclosing areas $C_1$, $C_2$, and $C_3$ with $C_i \geq A_i$, then it must use at least as much perimeter ($\ell(B) \leq \ell(C)$). Furthermore, if the minimizer $B$ has connected regions (a standard triple bubble), then it must enclose the given areas $A_1$, $A_2$, and $A_3$ (i.e. $B_i = A_i$ for all $i$).

In Chapter 3, we will look at some restrictions on the shape of perimeter minimizing planar bubble complexes. In particular, we prove that connected portions of regions must have more than two sides, three sided pieces are determined by the curvature of their edges, and certain other connected pieces cannot touch the exterior more than once.

In Chapter 4, we examine some additional restrictions imposed on bubble complexes with equal pressure regions. We show that the number of edges that bound a portion of a region is limited to three, four, five or six edges. Then we show how a few parameters control the shapes of 3-gons, 4-gons and 5-gons. In addition, we examine the way in which these $n$-gons can meet in a perimeter minimizing complex.
For example, when a 3-gon shares an edge with another 3-gon, they are both adjacent to another 3-gon, thus creating a standard triple bubble component.

In Chapter 5, we use the results from Chapters 3 and 4 to solve a restricted case of the planar triple bubble problem. Specifically, we prove

**Theorem 5.1** A perimeter minimizing triple bubble complex with equal pressure regions and no empty chambers must be a standard triple bubble. In particular, it has connected regions.

Since there exists a triple bubble complex enclosing equal areas with connected, equal pressure regions, we have a partial solution to the triple bubble conjecture.

**Corollary 5.2** The least perimeter graph that encloses and separates three equal areas $A_1$, $A_2$, and $A_3$ without empty chambers using equal pressure regions is a standard triple bubble.

The restriction to equal pressure regions is analogous to the work of L. Fejes Tóth [12] in the 1940’s. Tóth proved that the hexagonal honeycomb is the least perimeter tiling of the plane with equal area polygonal cells. Since the pressure difference between two regions is measured by the curvature of the connecting edge, the restriction to polygonal cells is similar to an equal pressure restriction. We note this similarity in Corollary 5.3.

---

2 Least perimeter in this tiling context means that a limiting perimeter to area ratio is minimized.
Corollary 5.3  The least perimeter graph that encloses and separates three equal areas with convex cells and without empty chambers is a standard triple bubble.

In another corollary to Theorem 5.1, we show a bubble complex cannot be a solution to the triple bubble problem if it is close to a regular complex with equal pressure disconnected regions:

Corollary 5.4  Suppose \( \{A_i\} \) is a sequence of regular triple bubble complexes that converges in length and area to a triple bubble complex \( A \). If \( A \) does not have any empty chambers, has equal pressure regions and is not a standard triple bubble, then there exists an \( N \) such that for any \( i > N \), \( A_i \) is not a perimeter minimizer for the areas it encloses.

In Chapter 6, we extend the arguments used in Theorem 5.1 to \( n \) bubble complexes \( (n > 3) \) with disconnected, equal pressure regions. We point out several complexes that cannot be solutions to any \( n \) bubble problem. In particular, we severely restrict the possible configurations for a perimeter minimizing bubble complex that encloses and separates four prescribed areas with equal pressure regions.

All of our results agree with the general soap bubble conjecture. That is, we have not yet found a perimeter minimizing bubble complex with disconnected regions.
Chapter 2

Definitions and Preliminaries

A graph is called finite if it has a finite number of vertices. We will consider only finite, planar graphs such that every vertex has degree at least three. An embedded planar graph encloses areas $A_1, A_2, \ldots, A_n$ in $\mathbb{R}^2$ if it separates the plane into $n+1$ regions (not necessarily connected), $n$ of which contain the finite areas $A_1, A_2, \ldots, A_n$ respectively. Region $n+1$ is called the exterior region and contains infinite area. Any non-exterior region is called an interior region. Define an $n$ bubble complex to be an embedded planar graph that encloses some $n$ positive areas.

Given $n$ positive real numbers $A_1, A_2, \ldots, A_n$, the generalized soap bubble problem in $\mathbb{R}^2$ is to find the least perimeter $n$ bubble complex that encloses those areas.

Define a half variation of an $n$ bubble complex $\mathcal{B}$ to be a continuous family $\{ \mathcal{B}_t \mid t \in [0, \epsilon) \}$ of $n$ bubble complexes such that $\mathcal{B}_0 = \mathcal{B}$. Let $\ell(\mathcal{B})$ be the function that returns the length of a bubble complex $\mathcal{B}$. 
Suppose $\mathcal{A}$ is an $n$ bubble complex enclosing areas $A_1, A_2, \ldots, A_n$. If there exists a half-variation of $\mathcal{A}$ such that the areas enclosed by each $\mathcal{A}_t$ is the same as the areas enclosed by $\mathcal{A}$ and $\frac{d\ell(\mathcal{A}_t)}{dt} \big|_{t=0} < 0$, then $\mathcal{A}$ is not the least perimeter way to enclose $A_1, A_2, \ldots, A_n$. The half-variation defines a deformation of $\mathcal{A}$ that preserves area and yet decreases perimeter.

Frank Morgan used variational arguments to prove that for any $n$ areas, a perimeter minimizer exists and must satisfy certain regularity conditions.

**Theorem 2.1** (Morgan[7]) For any positive real areas $A_1, A_2, \ldots, A_n$, there exists a perimeter-minimizing embedded graph that encloses those areas in $\mathbb{R}^2$. This least perimeter graph must satisfy the following conditions:

1. The graph consists of a finite number of vertices, edges, and faces;
2. Edges have constant curvature (arcs of circles or line segments);
3. Vertices are trivalent;
4. Edges meet at angles of $\frac{2\pi}{3}$;
5. Curves separating a specific pair of regions have the same curvature; and
6. Any half-variation $\mathcal{A}_t$ that preserves area must not initially decrease length.

That is, there does not exist a half-variation $\mathcal{A}_t$ such that

$$\frac{d\ell(\mathcal{A}_t)}{dt} \big|_{t=0} < 0.$$
We will call these six conditions the *regularity conditions*.

We define a *regular n bubble complex* to be an n bubble complex that satisfies all six of the regularity conditions. A connected portion of any interior region (i.e. a face of the embedded graph enclosing a piece of one of the given areas) will be called an *n-gon*, where n is the number of edges that enclose the connected piece. Edges that separate a part of the exterior region from another region will be called *exterior edges*. All non-exterior edges will be called *interior edges*.

Suppose we have a regular n bubble complex $\mathcal{A}$ that encloses areas $A_1, A_2, \ldots, A_n$. If we can find a non-regular n bubble complex $\mathcal{B}$ that contains the same areas such that $\ell(\mathcal{A}) \geq \ell(\mathcal{B})$, then $\mathcal{A}$ is not the least perimeter way to enclose $A_1, A_2, \ldots, A_n$. Non-regular bubble complexes yield half-variations that preserve area yet decrease perimeter. In other words, there exists a complex enclosing the same areas with length strictly less than the length of $\mathcal{B}$ and therefore less than the length of $\mathcal{A}$. This argument will be used regularly in the proof of Theorem 5.1 and throughout Chapter 6.

As stated above, Morgan established the existence of a minimum perimeter graph that encloses and separates any three areas. Let $L(A_1, A_2, A_3)$ be the function that gives the minimum length needed to enclose and separate areas $A_1$, $A_2$, and $A_3$. In Lemma 2.3 we show that this length function is continuous. First, however, we prove a lemma we will need.

**Lemma 2.2** For any $A_1 > 0$, $A_2 > 0$, $A_3 > 0$ and any $\epsilon > 0$, there exists a $\delta > 0$
such that if \(|x - A_1| \leq \delta, |y - A_2| \leq \delta, \text{ and } |z - A_3| \leq \delta\), then rescaling any complex enclosing \(x, y, \text{ and } z\) to get a complex enclosing \(x', y', \text{ and } z'\) where \(x' = A_1\) or \(y' = A_2\) or \(z' = A_3\) will result in at most an \(\epsilon\) change in the areas \(A_1, A_2\) and \(A_3\), i.e. \(|A_1 - x'| \leq \epsilon, |A_2 - y'| \leq \epsilon, \text{ and } |A_3 - z'| \leq \epsilon\).

**Proof:** This is simply a consequence of the continuity of rescaling. To be precise, suppose \(A_1, A_2, A_3, \text{ and } \epsilon\) are given. By the continuity of \(\frac{x A_2}{y A_1}\) for \(y \neq 0\), there exists a \(\delta_1 > 0\) such that for \(|x - A_1| \leq \delta_1\) and \(|y - A_2| \leq \delta_1\), we have \(|1 - \frac{x A_2}{y A_1}| \leq \frac{\epsilon}{A_1}\). Similarly, there exist \(\delta_2, \ldots, \delta_6 > 0\) such that

\[
|x - A_1| \leq \delta_2 \text{ and } |z - A_3| \leq \delta_2 \implies \left|1 - \frac{x A_3}{z A_1}\right| \leq \frac{\epsilon}{A_1},
\]

\[
|x - A_1| \leq \delta_3 \text{ and } |y - A_2| \leq \delta_3 \implies \left|1 - \frac{y A_1}{x A_2}\right| \leq \frac{\epsilon}{A_2},
\]

\[
|x - A_1| \leq \delta_4 \text{ and } |z - A_3| \leq \delta_4 \implies \left|1 - \frac{z A_1}{x A_3}\right| \leq \frac{\epsilon}{A_3},
\]

\[
|y - A_2| \leq \delta_5 \text{ and } |z - A_3| \leq \delta_5 \implies \left|1 - \frac{y A_3}{z A_2}\right| \leq \frac{\epsilon}{A_2}, \quad \text{and}
\]

\[
|y - A_2| \leq \delta_6 \text{ and } |z - A_3| \leq \delta_6 \implies \left|1 - \frac{z A_2}{y A_3}\right| \leq \frac{\epsilon}{A_3}.
\]

Let \(\delta = \min(\delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \delta_6)\).

Suppose \(\mathcal{A}\) is a complex enclosing \(x, y, \text{ and } z\) such that \(|x - A_1| \leq \delta, |y - A_2| \leq \delta, \text{ and } |z - A_3| \leq \delta\).

Rescale \(\mathcal{A}\) to get a complex \(\mathcal{B}\) enclosing \(x', y', \text{ and } z'\).

If \(x' = A_1\), then the complex was scaled by a factor of \(\frac{A_1}{x}\). So, \(y' = y \left(\frac{A_1}{x}\right)\) and \(z' = z \left(\frac{A_1}{x}\right)\). The first inequality is trivial \(|A_1 - x| = 0 < \epsilon\). But, we also get the
inequalities
\[ |A_2 - y'| = |A_2 \left( 1 - \frac{yA_1}{xA_2} \right)| \leq A_2 \left( \frac{\epsilon}{A_2} \right) = \epsilon \]

and
\[ |A_3 - z'| = |A_3 \left( 1 - \frac{zA_1}{xA_3} \right)| \leq A_3 \left( \frac{\epsilon}{A_3} \right) = \epsilon. \]

Similarly, if \( y' = A_2 \), the complex was scaled by a factor of \( \frac{A_2}{y} \) and we get the inequalities
\[ |A_1 - x'| = |A_1 \left( 1 - \frac{xA_2}{yA_1} \right)| \leq A_1 \left( \frac{\epsilon}{A_1} \right) = \epsilon, \]
\[ |A_2 - y| = 0 < \epsilon \]

and
\[ |A_3 - z'| = |A_3 \left( 1 - \frac{zA_2}{yA_3} \right)| \leq A_3 \left( \frac{\epsilon}{A_3} \right) = \epsilon. \]

Finally, if \( z' = A_3 \), then we scaled by a factor of \( \frac{A_3}{z} \) and we get the inequalities
\[ |A_1 - x'| = |A_1 \left( 1 - \frac{xA_3}{zA_1} \right)| \leq A_1 \left( \frac{\epsilon}{A_1} \right) = \epsilon, \]
\[ |A_2 - y'| = |A_2 \left( 1 - \frac{yA_3}{zA_2} \right)| \leq A_2 \left( \frac{\epsilon}{A_2} \right) = \epsilon, \]

and
\[ |A_3 - z| = 0 < \epsilon. \]

\[ \square \]

**Lemma 2.3** The length function \( L(A_1, A_2, A_3) \) is continuous for all \( A_i > 0 \).
**Proof:** Let \( A_1, A_2, A_3 \), and \( \epsilon > 0 \) be given.

Let \( \delta_1 = \frac{\epsilon^2}{36\pi} \). Let \( \delta_2 \) be the delta needed in Lemma 2.2 for \( \frac{\epsilon^2}{16\pi} \). That is, if 

\[
|x - A_1| \leq \delta_2, \quad |y - A_2| \leq \delta_2, \quad \text{and} \quad |z - A_3| \leq \delta_2,
\]

then when we rescale any complex containing \( x, y, \) and \( z \) to get one containing \( x', y', \) and \( z' \) with \( x' = A_1 \) or \( y' = A_2 \) or \( z' = A_3 \), we get the inequalities 

\[
|A_1 - x| \leq \frac{\epsilon^2}{16\pi}, \quad |A_2 - y| \leq \frac{\epsilon^2}{16\pi}, \quad \text{and} \quad |A_3 - z| \leq \frac{\epsilon^2}{16\pi}.
\]

Let \( \delta = \min\{\delta_1, \delta_2\} \). Suppose 

\[
|A_1 - x| \leq \delta, \quad |A_2 - y| \leq \delta, \quad \text{and} \quad |A_3 - z| \leq \delta.
\]

Let \( A_1 \) be a complex that uses \( L(A_1, A_2, A_3) \) perimeter to enclose areas \( A_1, A_2, \) and \( A_3 \). Similarly, let \( A \) be a complex enclosing areas \( x, y, \) and \( z \) with length \( L(x, y, z) \).

**Case 1:** \( x > A_1, y > A_2, z > A_3, \) and \( L(x, y, z) \geq L(A_1, A_2, A_3) \).

Since \( x > A_1, y > A_2, \) and \( z > A_3 \), we can enclose areas \( x, y, \) and \( z \) by using \( A_1 \) together with three disjoint circles containing areas \( x - A_1, y - A_2, \) and \( z - A_3 \) respectively. These disjoint circles have perimeter \( 2\sqrt{\pi \sqrt{x - A_1}}, 2\sqrt{\pi \sqrt{y - A_2}}, \) and \( 2\sqrt{\pi \sqrt{z - A_3}} \). Since \( L(x, y, z) \) is the minimum length, it has shorter length than the perimeter used by \( A_1 \) together with the circles. (See Figure 2.1.)

We thus get

\[
L(x, y, z) < L(A_1, A_2, A_3) + 2\sqrt{\pi \sqrt{x - A_1}} + 2\sqrt{\pi \sqrt{y - A_2}} + 2\sqrt{\pi \sqrt{z - A_3}}.
\]

So,

\[
|L(x, y, z) - L(A_1, A_2, A_3)| = L(x, y, z) - L(A_1, A_2, A_3)
\]

\[
< 2\sqrt{\pi \sqrt{x - A_1}} + 2\sqrt{\pi \sqrt{y - A_2}} + 2\sqrt{\pi \sqrt{z - A_3}}
\]

\[
\leq 6\sqrt{\pi \delta}
\]
Figure 2.1: Adding three small circles to $A_1$ gives a complex enclosing areas $x$, $y$, and $z$ with more perimeter than $A$

\[
\leq 6\sqrt{\pi}\sqrt{\delta_1} 
\]

\[
= 6\sqrt{\pi}\sqrt{\frac{\epsilon^2}{36\pi}} = \epsilon.
\]

**Case 2:** $x < A_1$, $y < A_2$, $z < A_3$, and $L(x, y, z) \leq L(A_1, A_2, A_3)$.

We do exactly the same as case 1, but add little circles to $A$ instead of $A_1$. To be precise, we can enclose areas $A_1$, $A_2$, and $A_3$ by using $A$ together with three disjoint circles containing areas $A_1 - x$, $A_2 - y$, and $A_3 - z$ respectively. These disjoint circles have perimeter $2\sqrt{\pi}\sqrt{A_1 - x}$, $2\sqrt{\pi}\sqrt{A_2 - y}$, and $2\sqrt{\pi}\sqrt{A_3 - z}$. Since $L(A_1, A_2, A_3)$ is the minimum length, it has shorter length than the perimeter used by $A$ together with the circles. We thus get

\[
L(A_1, A_2, A_3) < L(x, y, z) + 2\sqrt{\pi}\sqrt{A_1 - x} + 2\sqrt{\pi}\sqrt{A_2 - y} + 2\sqrt{\pi}\sqrt{A_3 - z}.
\]

That is,

\[
|L(x, y, z) - L(A_1, A_2, A_3)| = L(A_1, A_2, A_3) - L(x, y, z)
\]
\[
< 2\sqrt{\pi A_1 - x} + 2\sqrt{\pi A_2 - y} + 2\sqrt{\pi A_3 - z} \\
\leq 6\sqrt{\pi \delta} \\
\leq 6\sqrt{\pi \delta_1} \\
= 6\sqrt{\pi \frac{\epsilon^2}{36\pi}} = \epsilon.
\]

**Case 3:** \( L(x, y, z) < L(A_1, A_2, A_3) \) and \( x > A_1 \) or \( y > A_2 \) or \( z > A_3 \)

Scale down \( \mathcal{A} \) to get a complex \( \mathcal{B} \) that encloses areas \( x', y', \) and \( z' \) such that one of areas equals an area from \( \mathcal{A}_1 \) (e.g. \( x' = A_1 \)) and the other two areas are smaller or equal to the remaining areas in \( \mathcal{A}_1 \) (e.g. \( y' \leq A_2 \) and \( z' \leq A_3 \)).

Let \( \ell(\mathcal{B}) \) be the length of \( \mathcal{B} \). Since scaled minimizers are still minimizers, \( \ell(\mathcal{B}) = L(x', y', z') \). Also, since \( \mathcal{B} \) was a scaled down copy of \( \mathcal{A} \), we have \( \ell(\mathcal{B}) < L(x, y, z) \).

Without loss of generality, assume that the first area is the one that is the same. In other words, \( x' = A_1, y' \leq A_2, \) and \( z' \leq A_3 \).

If we add two disjoint circles of area \( A_2 - y' \) and \( A_3 - z' \) to the complex \( \mathcal{B} \), we get a complex enclosing \( A_1, A_2, \) and \( A_3 \) again. The length of this complex is \( \ell(\mathcal{B}) + 2\sqrt{\pi A_2 - y'} + 2\sqrt{\pi A_3 - z'} \) and must be larger than \( L(A_1, A_2, A_3) \). (See Figure 2.2.)

The \( 2\sqrt{\pi A_2 - y} \) and \( 2\sqrt{\pi A_3 - z} \) terms are the perimeter needed to add back in the missing area. Since we have a bound on the missing area (by \( \delta_2 \) and lemma 2.2), we have a bound on the amount of perimeter needed. In fact, we get

\[
L(A_1, A_2, A_3) < \ell(\mathcal{B}) + 2\sqrt{\pi A_2 - y'} + 2\sqrt{\pi A_3 - z'}
\]
Figure 2.2: Scale down $\mathcal{A}$ to get $\mathcal{B}$. Then add two small circles to get a complex enclosing areas $A_1$, $A_2$, and $A_3$ with more perimeter than $A_1$. 
\begin{align*}
\leq \ell(B) + 2\sqrt{\pi} \sqrt{\frac{\epsilon^2}{16\pi}} + 2\sqrt{\pi} \sqrt{\frac{\epsilon^2}{16\pi}} \\
= \ell(B) + \epsilon
\end{align*}

In short, \( L(A_1, A_2, A_3) \leq \ell(B) + \epsilon \). Put this together with the previous inequalities to get

\[ \ell(B) < L(x, y, z) < L(A_1, A_2, A_3) < \ell(B) + \epsilon. \]

So, \(|L(x, y, z) - L(A_1, A_2, A_3)| = L(A_1, A_2, A_3) - L(x, y, z) < \epsilon.\]

**Case 4:** \( L(x, y, z) > L(A_1, A_2, A_3) \) and \( x < A_1 \) or \( y < A_2 \) or \( z < A_3 \)

This case is similar to case 3, except we scale up \( \mathcal{A} \) and add little circles to \( \mathcal{A}_1 \) to get the desired inequality.

Scale up \( \mathcal{A} \) to get a complex \( \mathcal{B} \) that encloses areas \( x', y', \) and \( z' \) such that one of these areas equals an area from \( \mathcal{A}_1 \) (e.g. \( x' = A_1 \)) and the other two areas are larger or equal to the remaining areas in \( \mathcal{A}_1 \) (e.g. \( y' \geq A_2 \) and \( z' \geq A_3 \)).

Let \( \ell(\mathcal{B}) \) be the length of \( \mathcal{B} \). Since scaled minimizers are still minimizers, \( \ell(\mathcal{B}) = L(x', y', z') \). Also, since \( \mathcal{B} \) was scaled up from \( \mathcal{A} \), we have \( \ell(\mathcal{B}) > L(x, y, z) \).

Without loss of generality, assume that the first area is the one that is the same. So, \( x' = A_1, y' \geq A_2, \) and \( z' \geq A_3. \)

If we add two disjoint circles of area \( y' - A_2 \) and \( z' - A_3 \) to the complex \( \mathcal{A}_1 \), we get a complex enclosing \( A_1, y', \) and \( z' \) again. The length of this complex is \( \ell(\mathcal{B}) + 2\sqrt{\pi} \sqrt{y' - A_2} + 2\sqrt{\pi} \sqrt{z' - A_3} \) and must be larger than \( L(A_1, x', y') = \ell(\mathcal{B}). \)
We then get the inequality
\[ \ell(\mathcal{B}) < L(A_1, A_2, A_3) + 2\sqrt{\pi} \sqrt{y' - A_2} + 2\sqrt{\pi} \sqrt{z' - A_3} \]
\[ \leq L(A_1, A_2, A_3) + 2\sqrt{\pi} \sqrt{\frac{\epsilon^2}{16\pi}} + 2\sqrt{\pi} \sqrt{\frac{\epsilon^2}{16\pi}} \]
\[ = L(A_1, A_2, A_3) + \epsilon. \]

In short, \( L(A_1, A_2, A_3) > \ell(\mathcal{B}) - \epsilon. \) Put this together with the previous inequalities to get
\[ \ell(\mathcal{B}) - \epsilon < L(A_1, A_2, A_3) < L(x, y, z) < \ell(\mathcal{B}) \]

So,
\[ |L(x, y, z) - L(A_1, A_2, A_3)| = L(x, y, z) - L(A_1, A_2, A_3) < \epsilon. \]

We have covered all the possibilities. Therefore, the length function \( L(A_1, A_2, A_3) \) is continuous. \( \square \)

Although presented in the context of three areas, Lemma 2.2 and Lemma 2.3 can be easily extended to any number of regions. In particular, the length function \( L(A_1, \ldots, A_n) \) is continuous for any \( n. \)

The triple bubble conjecture suggests a solution to the 3 bubble problem in \( \mathbb{R}^2. \)

**Conjecture 1** The least perimeter planar graph that encloses and separates three finite areas \( A_1, A_2, \) and \( A_3 \) is a regular triple bubble complex with four vertices, six edges, and three connected regions.
Figure 2.3: A standard triple bubble enclosing areas $A_1$, $A_2$, and $A_3$

Such a complex is called a *standard triple bubble* and has been proven to exist and be unique for any three areas [2]. (See Figure 2.3.) The standard triple bubble has also been shown to be the least perimeter way to enclose and separate any three areas using connected regions [4].

An alternate version of the triple bubble conjecture allows increasing the areas enclosed in an attempt to minimize perimeter:

**Conjecture 2** *Given three positive real numbers $A_1$, $A_2$, and $A_3$, the least perimeter graph that encloses and separates three finite areas $B_1$, $B_2$, and $B_3$ such that $B_i \geq A_i$ for all $i$, is a standard triple bubble.*

Conjecture 2 eliminates the possibility of empty chambers. If a complex has an empty chamber, it could be filled in with any one of the adjacent areas and at least one edge could be eliminated. The resulting complex encloses more area, but uses less perimeter. (See Figure 2.4.)
Theorem 2.4 establishes that the two versions of the triple bubble conjecture are equivalent.

**Theorem 2.4** Let three positive areas $A_1$, $A_2$, and $A_3$ be given. There exists a least perimeter triple bubble complex $B$ that encloses and separates areas $B_1$, $B_2$, and $B_3$ with $B_i \geq A_i$. That is, if $C$ is any other complex enclosing areas $C_1$, $C_2$, and $C_3$ with $C_i \geq A_i$, then it must use at least as much perimeter ($\ell(B) \leq \ell(C)$). Furthermore, if the minimizer $B$ has connected regions (a standard triple bubble), then it must enclose the given areas $A_1$, $A_2$, and $A_3$ (i.e. $B_i = A_i \ \forall i$).

**Proof:**

To optimize the length function $L$ for areas greater than or equal to $A_1$, $A_2$, and $A_3$, the domain we need to consider is bounded below (by $A_1$, $A_2$, and $A_3$) and bounded above as well. The upper bound can be chosen to be $B_1$, $B_2$, $B_3$ where $B_i = (2\pi + 3)\sqrt[3]{\frac{A_1 + A_2 + A_3}{\sqrt{\pi}}}$. This is the area needed to enclose and separate the three
areas with a circle and three radii. Since the total perimeter needed to enclose even one \( B_i \) is larger than a known way to enclose \( A_1, A_2, \) and \( A_3, \) the total perimeter used in any attempt to enclose areas bigger than \( B_1, B_2, \) and \( B_3 \) must be larger than the minimum way to enclose \( A_1, A_2, \) and \( A_3. \) Since a continuous function on a compact set achieves its maximum and minimum value, there is a minimum value for the length function.

If the minimum is always a standard triple bubble (i.e. Conjecture 2 is correct), then the minimum must enclose exactly \( A_1, A_2, \) and \( A_3. \) If it encloses some \( B_1, B_2, \) and \( B_3 \) with \( B_i > A_i \) for some \( i, \) we could reduce perimeter by replacing a small portion of the exterior arc of \( B_i \) by a straight line. (See Figure 2.5.) The line can be chosen small enough so that the area enclosed by the region is still larger than \( A_i, \) and yet we’ve used less perimeter. This contradicts the assumption that the complex was the minimum. \( \square \)

![Figure 2.5: Cut off a little bit to save perimeter.](image)

We define the *pressure* of a region in a regular \( n \) bubble complex to be 0 for the exterior region. For any other region, we pick a path from the exterior to that region
such that the path intersects the edges of the complex transversely in a finite number of points. The pressure is then the sum of the signed curvatures of the edges at these finite number of intersection points. We use the sign convention as shown in Figure 2.6. When exterior edges bulge outward (as in soap bubbles), the choice of sign guarantees that regions adjacent to the exterior have positive pressure. It also makes the sign of the curvature agree with the standard definition of curvature when the edges are given a counter-clockwise orientation.

![Figure 2.6: Sign convention for curvature](image)

Cox, Harrison, Hutchings, et al. [3] proved that for any closed path intersecting a regular bubble transversely, the sum of the signed curvatures along that path must be zero. In Lemma 2.5, we generalize this result to any path that starts and ends in the same (possibly disconnected) region. It also guarantees that pressure is well defined.

**Lemma 2.5** Let $\mathcal{A}$ be a regular $n$ bubble complex. Let $\gamma$ be any path that intersects the edges of the complex transversely such that $\gamma$ starts and stops in portions of the same region (not necessarily connected). Then, the sum of the signed curvatures of
the edges crossed is zero.

**Proof:** Suppose that \( \gamma \) goes through regions \( R_1, R_2, \ldots, R_n, R_1 \) and crosses edges with curvatures \( \kappa_1, \kappa_2, \ldots, \kappa_n \) at points \( p_1, p_2, \ldots, p_n \). Define a half-variation \( A_t \) that transfers \( t \) area from each \( R_i \) to \( R_{i+1} \) by adjusting each edge in a neighborhood about \( p_i \). The initial change in length by this half-variation is just the sum of the signed curvatures (see e.g. Morgan\[8\]). That is,

\[
\frac{d\ell(A_t)}{dt} \bigg|_{t=0} = \sum_{i=1}^{n} \kappa_i.
\]

By regularity (condition six), this sum must be greater than or equal to 0. If, however, the sum is greater than zero, we can traverse \( \gamma \) in the opposite direction to get the same curvatures with opposite orientation. Therefore, the half-variation defined by adjusting area along \(-\gamma\) has negative initial change in length which violates regularity. Thus, the sum of the curvatures must be zero. \( \square \)

In Chapter \( \ref{ch3} \), we show that Conjecture 2 is true in the case of equal pressure regions, or that Conjecture 1 is true in the case of equal pressure regions with no empty chambers. The restriction that the regions have equal pressures guarantees that the inner edges (edges that don’t touch the connected exterior region) are all line segments (0 curvature) and the outer edges all have the same curvature. In particular, every \( n \)-gon is convex.
Chapter 3

Structure of Perimeter Minimizing Bubbles

We begin with some observations about the possible configurations for perimeter minimizing bubble complexes. Theorem 2.1 guarantees that they must be regular bubble complexes. The restrictions we discuss in this chapter are applicable to arbitrary regular bubble complexes enclosing any number of regions.

We first note that perimeter minimizing complexes must be connected. If a complex has two disconnected components, they can be pushed together until a vertex of degree at least four is created. This new complex violates regularity and therefore there exists a complex enclosing the same areas with less perimeter.

**Lemma 3.1** Perimeter minimizing regular $n$ bubble complexes ($n > 2$) have no 2-gons.
Proof: Suppose there is a 2-gon. By regularity, every vertex must be trivalent. In particular, a 2-gon will have two vertices and two edges with an additional edge leading away from each vertex $\alpha$ and $\beta$. (See Figure 3.1.)

![Figure 3.1: A 2-gon in a bubble complex](image)

Case I: edge $\alpha = \text{edge } \beta$ (See Figure 3.2)

![Figure 3.2: A disconnected double bubble region](image)

The 2-gon and the adjacent 2-gon form a double bubble disconnected from the rest of the complex. Move this disconnected piece until it touches another component of the bubble complex. A four valent vertex would be created at the point of intersection thus violating regularity.

Case II: edge $\alpha \neq \text{edge } \beta$ (See Figure 3.1)
The data from a single vertex is enough to completely determine a connected double bubble complex. In other words, if three arcs of circles meet at a vertex at equal angles \( \left( \frac{2\pi}{3} \right) \) and the sum of the signed curvatures of the arcs around the vertex is zero, then the arcs will extend to a standard double bubble complex. All three arcs meet again at some other point and with the same angles as the angles at which they leave \( \left( \frac{2\pi}{3} \right) \).

By regularity, the curvatures of \( \alpha \) and \( \beta \) are determined by the curvatures of the 2-gon and therefore must be the same. Furthermore, edge \( \alpha \) and edge \( \beta \) must be arcs of the same circle since the data from one vertex is enough to determine the other. The 2-gon can be *slid* along this circle without changing perimeter or area. That is, we can remove the 2-gon and extend edge \( \alpha \) and \( \beta \) to get a continuous arc of a circle. The 2-gon can then be reinserted anywhere along this arc. (See Figure 3.3.) To finish the slide move, we erase the portion of the circle inside of the new 2-gon. We then have a bubble complex enclosing equivalent areas with exactly the same amount of perimeter. The slide move was introduced by the SMALL Geometry Group [6] when they proved the planar double bubble conjecture.

![Figure 3.3: Slide a 2-gon along a circle](image-url)
We continue sliding this 2-gon until it either touches another edge or the edge $\alpha$ disappears. In either case, a 4-valent vertex is created and regularity is violated. □

**Lemma 3.2** If there exists a 3-gon with edges of curvature $\kappa_1$, $\kappa_2$, and $\kappa_3$, then its shape is unique (up to orientation and isometry) and is determined by the triangle of its vertices.

**Proof:** When two arcs of circles of radius $r_1$ and $r_2$ meet at an angle of $\frac{2\pi}{3}$, the centers of the two circles are at distance $d = \sqrt{r_1^2 + r_2^2 - r_1 r_2}$. Suppose that there exists a 3-gon with curvatures $\kappa_1$, $\kappa_2$, and $\kappa_3$.

If two curvatures are zero, the 3-gon is determined by the curvature of the third arc. Uniqueness is guaranteed by the Gauss-Bonnet Theorem (See Lemma 4.2).

If none of the curvatures are zero, the radii of the respective arcs are $r_i = \frac{1}{\kappa_i}$. Consider the 2-gon formed by intersecting a circle of radius $r_1$ with a circle of radius $r_2$ at an angle of $\frac{2\pi}{3}$. Let $C_1$ be the circle obtained by extending the arc with curvature $\kappa_1$. Similarly, let $C_2$ denote the circle obtained by extending the arc with curvature $\kappa_2$. To get any 3-gon with the same curvature edges, we need to add a third circle of radius $r_3$ so that the angles made with both circles $C_1$ and $C_2$ is again $\frac{2\pi}{3}$. Since there is a 3-gon with these curvatures, we know it is possible. The center of this circle must be at distance $d_1 = \sqrt{r_1^2 + r_3^2 - r_1 r_3}$ from the center of $C_1$ and distance $d_2 = \sqrt{r_2^2 + r_3^2 - r_2 r_3}$ from the center of $C_2$. Construct a circle of radius $d_1$ around the center of $C_1$ and a circle of radius $d_2$ around the center of $C_2$. The center of the third circle...
Figure 3.4: Two oppositely oriented 3-gons can be created from a 2-gon by adding an arc of given curvature $\kappa_3$.

circle must lie on the intersection of these circles. The circles are not equivalent (since they have different centers) and intersect at least once (since there is a solution). The only other possibility is that the circles intersect twice. If so, the two choices for the center of the third circle give the same intersection pattern with the 2-gon, but on opposite sides. (See Figure 3.4.) Generically, when an arc can be added to a 2-gon to form a 3-gon using one of the original vertices of the 2-gon, an arc of the same curvature can be added as well with opposite orientation. This creates a 3-gon with the opposite vertex of the 2-gon. The two different 3-gons created are equivalent but have opposite orientation.

If only one curvature is zero, we can build a 2-gon with a straight line and a circle of curvature $\kappa_2 \neq 0$. Consider arcs of curvature $\kappa_3$ leaving a point on the line segment between the vertices of our 2-gon at an angle of $\frac{2\pi}{3}$. When such an arc meets the given arc of curvature $\kappa_2$, the angle made is strictly increasing between 0 and $\pi$ as the point of departure varies from one vertex to the other. At only one point is the angle exactly $\frac{2\pi}{3}$. 
To prove that 3-gons are determined by triangles, we will establish a map from triangles to 3-gons and show that it is bijective. Suppose we have a triangle \( \triangle ABC \) with side lengths \( a, b, \) and \( c \) (opposite side from the appropriately labeled vertex) and angles \( \alpha = \angle CAB, \beta = \angle ABC, \) and \( \gamma = \angle ACB. \) (See Figure 3.5.)

![Figure 3.5: A generic triangle](image)

Given any angle \( \theta_\alpha, \) there is a unique arc of a circle that passes through \( B \) and \( C \) that makes angle \( \theta_\alpha \) with the line segment \( BC. \) In fact, since \( a \) is the length of \( BC, \) the curvature \( \kappa_\alpha \) of the arc through \( BC \) with angle \( \theta_\alpha \) is given by the formula \( \kappa_\alpha = \frac{2 \sin(\theta_\alpha)}{a}. \)

We consider angles exterior to the triangle to be positive. (See Figure 3.6.) Similarly, angles \( \theta_\beta \) and \( \theta_\gamma \) uniquely determine arcs of circles through \( AC \) and \( AB \) respectively.

To get a valid 3-gon, the internal angles should all be \( \frac{2\pi}{3}. \) A 3-gon must then satisfy the linear equations

\[
\theta_\beta + \alpha + \theta_\gamma = \frac{2\pi}{3},
\]

\[
\theta_\alpha + \beta + \theta_\gamma = \frac{2\pi}{3}.
\]
Figure 3.6: A triangle with arcs of circles attached

and

$$\theta_\alpha + \gamma + \theta_\beta = \frac{2\pi}{3}.$$ 

From the triangle we also get the equation

$$\alpha + \beta + \gamma = \pi.$$ 

The unique solution to these equations is

$$\theta_\alpha = \alpha - \frac{\pi}{6}$$

$$\theta_\beta = \beta - \frac{\pi}{6}$$

$$\theta_\gamma = \gamma - \frac{\pi}{6}.$$ 

Given any 3-gon, we can get a triangle by connecting the vertices of the 3-gon. So, the map from triangles to 3-gons is surjective. Suppose now that two different
triangles produce the same 3-gon. Since the vertices of the 3-gon coincide with the vertices of the triangle that produced it, the two triangles must be identical. Therefore the map is also injective. □

**Lemma 3.3** Any 4-gon or 5-gon in a perimeter minimizing regular bubble complex without empty chambers shares at most one edge with the exterior region.

![Figure 3.7: A region with two exterior edges](image)

**Proof:** A 4-gon or a 5-gon is distinguished by the fact that any pair of edges are separated by at most one edge. Suppose a 4-gon or 5-gon shares two edges $\alpha$ and $\beta$ with the exterior. Then there is a single edge $\gamma_1$ that connects the two exterior edges. (See Figure 3.7.) Let $p$ be the vertex shared by $\gamma_1$ and $\alpha$ and $q$ be the vertex shared by $\gamma_1$ and $\beta$. Pick a point $r$ at distance $\epsilon$ from $q$ on $\beta$. Let $\gamma_2$ be an arc of a circle or line segment from $p$ to $r$ that does not intersect any edges of the $n$-gon. (See
Figure 3.8) If the $n$-gon is convex, the arc $\gamma_2$ can always be chosen to be a straight line segment.

![Diagram of a polygon with arcs $\gamma_1$ and $\gamma_2$, points $p$, $q$, $r$, and labels $A$, $B$, $C$.]

Figure 3.8: Cut out this wedge shaped region

Cut out the triangular wedge formed by $\gamma_1$, $\gamma_2$, and the arc from $q$ to $r$. Label the corners of this triangular wedge as follows: $A$ for the corner that came from the point $p$, $B$ for the corner that came from the point $q$, and $C$ for the corner that came from the point $r$. Then separate the remaining complex into three disjoint pieces by splitting $p$ into two points $p_1$ and $p_2$ such that $p_1$ is connected to the point $q$ and $p_2$ is connected to the point $r$. (See Figure 3.9)

Now re-attach the triangular wedge with an opposite orientation by identifying the point $p_1$ with $B$, $p_2$ with $C$, and the points $q$, $r$, and $A$ with each other. (See Figure 3.10)

The resulting complex has identical perimeter and encloses the same areas. The edge $\alpha$, however, either has a corner at $p_2$ (if angle $\omega$ differs from angle $\rho$) or is still...
smooth but longer. If the edge has a corner, we have a complex that encloses the same areas with the same perimeter that violates regularity. If the angles agree and no corner is created, the effect is that the complex has slid distance $\epsilon$ along the edge $\alpha$. Continue sliding (i.e. repeat this procedure) until either the complex bumps into itself somewhere or the edge $\beta$ disappears. In either case, a complex that encloses the same areas with identical perimeter is created. But, this new complex contains a four-valent vertex and therefore violates regularity. $\square$

The same argument can be used to show that many other kinds of $n$-gon’s in a perimeter minimizing regular bubble complex cannot touch the exterior region more than once. All that is needed to extend the argument is the existence of symmetric arcs $\gamma_1$ and $\gamma_2$ that intersect one exterior edge at a point and the other exterior edge at two points distance $\epsilon$ apart such that the symmetric arcs do not touch any other portion of the boundary of the $n$-gon.
In a paper regarding triple bubbles with connected regions, Cox, Harrison, Hutchings et. al. [5] made an interesting remark. They found a nice relationship between the perimeter $L$ of a regular $n$ bubble complex, the areas enclosed $A_1, A_2, \ldots, A_n$, and the pressure of each region $p_1, p_2, \ldots, p_n$:

$$L = 2 \sum_{i=1}^{n} p_i A_i$$

(3.1)

Using this relationship, we easily prove the following lemma:

**Lemma 3.4** If a regular $n$ bubble complex $C$ encloses areas $A_1, A_2, \ldots, A_n$ and has pressure $p_1, p_2, \ldots, p_n$ respectively, then any regular $n$ bubble complex $B$ enclosing the same areas with pressures $q_i < p_i$ for all $i$ is not a minimizer.

**Proof:** Let $\ell(B)$ be the length of $B$ and $\ell(C)$ be the length of $C$. Then, equation (3.1)
gives us

$$\ell(B) = 2 \sum_{i=1}^{n} q_i A_i > 2 \sum_{i=1}^{n} p_i A_i = \ell(C)$$

$C$ uses less perimeter and therefore $B$ is not a perimeter minimizer. □
Chapter 4

Restrictions Imposed By Equal Pressure Regions

In this chapter we consider regular $n$ bubble complexes that have equal pressure interior regions. Since there is no pressure change from one interior region to another, the curvature of the interior edges must be zero. In addition, by following a closed path that touches only two distinct exterior edges (any number of interior edges), we get that the curvature of the exterior edges must all be the same.

Lemma 4.1  If a regular $n$ bubble complex has no empty chambers and has positive equal pressure regions, then $n$-gons have at most 6 sides. In addition, $n$-gons that share an edge with the exterior region have at most 5 sides.

Proof: By regularity conditions, the internal angles of each $n$-gon must be $\frac{2\pi}{3}$. In addition, their edges have either 0 curvature (edges that separate two $n$-gons) or one
fixed curvature (edges that separate an $n$-gon from the exterior). Furthermore, since regions have positive pressure, edges separating an $n$-gon from the exterior must bulge outward (i.e. have positive curvature).

Using arcs of constant positive curvature in the Gauss-Bonnet Theorem, we get that

$$\sum_{i=1}^{n} \kappa_i l_i + \sum_{i=1}^{n} \frac{\pi}{3} = 2\pi$$

where $n$ is the number of edges. Note that since each interior angle of an $n$-gon is $\frac{2\pi}{3}$, each exterior angle is $\frac{\pi}{3}$. We solve for the sum of the exterior angles to get

$$n\left(\frac{\pi}{3}\right) = 2\pi - \sum_{i=1}^{n} \kappa_i l_i \leq 2\pi.$$ 

So, $n \leq 6$ with equality only when the edges all have 0 curvature. That is, 6-gon’s are internal since they do not share an edge with the exterior.

Recall that there also can’t be any 2-gons in a perimeter minimizing complex (by Lemma 3.1). Therefore, $n$-gons that share an edge with the exterior must be 3-gons, 4-gons, or 5-gons. □

**Lemma 4.2** In an $n$ bubble complex with equal pressure regions, there is a unique shape for a 3-gon region, a one parameter family of possible 4-gons, and a two parameter family of possible 5-gons (up to orientation preserving isometry). 4-gons are determined by the length of a side adjacent to an exterior edge. 5-gons are determined by the lengths of any two of the non-curved edges.
**Proof:** For a 3-gon, the Gauss-Bonnet theorem says that

$$\sum_{i=1}^{3} \left( \kappa_i l_i + \frac{\pi}{3} \right) = 2\pi.$$  

Since two of the sides have no curvature, we get $\kappa l_1 = \pi$ or $l_1 = \frac{\pi}{\kappa}$ where $\kappa$ is the curvature of the outside arcs and $l_1$ is the length of that curved arc in a 3-gon. In particular, the length of the curved arc in a 3-gon with fixed curvature $\kappa$ is constant.

![Figure 4.1: The unique shape for a 3-gon](image)

Now, two straight lines that leave the ends of such an arc at an angle of $\frac{2\pi}{3}$ will then meet in only one point, also at an angle of $\frac{2\pi}{3}$. This is the unique configuration for a 3-gon. (See Figure 4.1.)

For a 4-gon, the length of the curved arc is again completely determined by the Gauss-Bonnet theorem. Two lines that leave the end points of this arc at $\frac{2\pi}{3}$ will meet at some point at an angle of $\frac{\pi}{3}$. Draw the line segment connecting the ends of the arc. The fourth side of the 4-gon (opposite the curved arc) must be parallel to this line segment. A choice of how far along a side edge to place this opposite edge completely determines the 4-gon. This choice also corresponds to the size of equilateral triangle
Figure 4.2: Cut out an equilateral triangle to determine a 4-gon that is cut off of the bottom. (See Figure 4.2)

For a 5-gon, the length of the arc is again fixed. This time the adjacent edges will be parallel to each other. Choose any length for one edge, and draw a line segment from the end of this edge such that the internal angle is $\frac{2\pi}{3}$. This edge will meet the other adjacent edge at an angle of $\frac{\pi}{3}$. (See Figure 4.3)

Figure 4.3: Two choices determine a 5-gon

Now, the vertex opposite the curved edges can be chosen to be any point on this line segment. Once this choice is made, the 5-gon is completely determined. \(\square\)
Regular bubble complexes with equal pressure, disconnected regions have a lot of symmetry that can be used to find non-regular complexes that enclose the same areas using equal or less total perimeter. Since 3-gons are unique, it is convenient to consider what can possibly be next to a 3-gon.

If a 3-gon shares an edge with another 3-gon in a minimizing complex, then they both share an edge with a third 3-gon. Otherwise, the region adjacent to both 3-gons would have two exterior edges which violates Lemma 3.3. The complex is either disconnected and not a minimizer or is a standard triple bubble.

Suppose a 3-gon is adjacent to two 4-gons (one on each side). Let $P$ be the vertex shared by the 3-gon and both 4-gons. Let $Q$ be the other interior vertex shared by the 4-gons. Let $R$ and $S$ be the vertices of the 4-gons not shared with the original 3-gon. (See Figure 4.4.) Edges $RQ$ and $SQ$ separate the 4-gons from another $n$-gon.
This $n$-gon (opposite the original 3-gon) must also be a 3-gon. If not, then the $n$-gon has two different exterior edges (One coming from vertex $R$ and one from vertex $S$) in violation of Lemma \ref{lem:unique_3gon}. Since the complex is connected, there aren’t any more regions. The complex must have just these four chambers.

Lemma \ref{lem:4gon} below, together with the fact that 3-gons are unique, proves that the complex is actually completely determined, i.e. it is just a scaled copy of Figure \ref{fig:4gon_simplified}.

If a 3-gon shares an edge with a 4-gon, the 4-gon is unique. That is, there is only one possible shape for a 4-gon adjacent to a 3-gon since the one parameter has been determined for the 4-gon. (See Figure \ref{fig:3gon_4gon}.) Lemma \ref{lem:4gon} gives a nice relationship between the side lengths of the 4-gon.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig3gon_4gon.png}
\caption{A 3-gon adjacent to a 4-gon}
\end{figure}

**Lemma 4.3** Let $a$ be the length of the central edge (opposite the curved arc) of a 4-gon adjacent to a 3-gon. Let $b$ be the length of the shared edge. Then $a = \frac{1}{2}b$. 
Proof: We use the relationship between the radius $r$, angle $\theta$ and chord length $C$ of a section of a circle $r = \frac{C}{2\sin(\theta)}$. (See Figure 4.6.)

\[ r = \frac{C}{2\sin(\theta)} \]

Figure 4.6: A section of a circle

Let $r$ be the radius of the exterior edges. For a 3-gon, the chord made by connecting the endpoints of the exterior arc makes a right angle with the exterior arc. For a 4-gon, the corresponding angle is $\frac{\pi}{3}$ as in Figure 4.7. Simple trigonometry gives us $r = \frac{\sqrt{3}}{2}b$. In addition, the chord of the 4-gon gives us the relationship $r = \frac{\frac{a+b}{\sqrt{3}}}{2\sin(\frac{\pi}{3})} = \frac{a+b}{\sqrt{3}}$. Eliminate $r$ to get $\frac{a+b}{\sqrt{3}} = \frac{\sqrt{3}b}{2}$ or $\frac{3b}{2} = a + b$. Finally, we isolate $a$ to get the desired equality $a = \frac{1}{2}b$. □

There is a one-parameter family of 5-gons that can share an edge with a 3-gon since only one of the two parameters has been determined. There still is, however, a nice relationship between the side lengths of a 3-gon and an adjacent 5-gon.

Lemma 4.4 Suppose a 5-gon shares an edge with a 3-gon. The sum of the lengths of
the inside edges of the 5-gon (edges that don’t meet the exterior arc) equals the length of the shared edge between the 3-gon and 5-gon.

**Proof:** Let $a$ be the length of the inside edge of the 5-gon adjacent to the shared edge. Let $b$ be the length of the remaining inside edge. Let $c$ be the length of the shared edge. Let $P$ and $R$ be the vertices of the shared edge, with $R$ the inner vertex. Let $Q$ be the remaining vertex of the 3-gon. Let $S$ be the vertex of the 5-gon opposite the curved arc and let $T$ be the remaining interior vertex of the 5-gon. Extend the adjacent edge of the 5-gon (edge of length $a$) and the edge of the 5-gon opposite the shared edge until they meet at an angle of $\frac{\pi}{3}$ at a point $U$. Let $V$ be a point on the edge of the 5-gon opposite of the shared edge such that the angle $\angle RPV$ is $\frac{\pi}{3}$. (See Figure 4.8.)
Triangle $\triangle STU$ is equilateral and so segment $SU$ has length $b$ and segment $RU$ has length $a+b$. $PV$ is parallel to $RU$ by construction. $PR$ is parallel to $UV$ since the opposite flat sides of a 5-gon are parallel. Therefore, $RPVU$ is a parallelogram. The diagonal $PU$ must then bisect the angle $\angle RPV$ and the angle $\angle RPU$ is $\frac{\pi}{6}$. Finally, we notice that triangles $PRU$ and $PRQ$ are similar. Therefore edge $RQ$ has the same length as edge $RU$, i.e. $a + b = c$. $\square$

Since the interior edges of a 5-gon that meet the exterior vertices are parallel for any 5-gon (not necessarily adjacent to a 3-gon), the sum of the lengths of the other two interior edges (opposite the curved arc) must be a constant. In other words, for any 5-gon, the sum of the lengths of the two innermost edges is equal to the length
of the flat side of a 3-gon created with the same curvatures.
Chapter 5

Triple Bubbles with Equal Pressure Regions

In this chapter, we solve the triple bubble conjecture in the case of equal pressure regions without empty chambers. In the figures, we will label the interior of each $n$-gon with an integer (1, 2, or 3) to denote the region to which they contribute area. For example, an $n$-gon labeled 1 contributes area towards $A_1$, where $A_1$, $A_2$, and $A_3$ are the areas enclosed by the triple bubble complex. Choices for numbering is done arbitrarily and without loss of generality.

Our main result is that unless we have a standard triple bubble, a triple bubble complex with equal pressure regions cannot be a perimeter minimizer.

Theorem 5.1 A perimeter minimizing triple bubble complex with equal pressure regions and without empty chambers must be a standard triple bubble. In particular, it
has connected regions.

Proof: Suppose not.

Case 1: There is a 3-gon.

Consider the $n$-gons adjacent to the 3-gon. If one of them is a 3-gon, then the other must also be a 3-gon. Otherwise, the $n$-gon adjacent to both 3-gons would have two exterior edges which violates Lemma 3.3. The complex must then be a standard triple bubble with connected regions.

If either of the adjacent $n$-gons is a 4-gon, then we can reflect it into the 3-gon. Consider a 3-gon with an adjacent 4-gon. Let $A$ be the line segment of the 4-gon opposite the shared edge. Let $\alpha$ be the remaining straight edge (opposite the curved arc) of the 4-gon, and $\beta$ be the shared edge.

By lemma 4.3, $\alpha$ is half as long as $\beta$.

Figure 5.1: Reflect a 4-gon into an adjacent 3-gon
Add an edge $B$ inside the 3-gon halfway between the vertices of $\beta$ at an angle of $\frac{2\pi}{3}$, and then remove edge $A$. The edge $B$ creates a 4-gon of the appropriate size inside of the original 3-gon. Then, we can renumber the areas and erase the top half of edge $\beta$ to get a non-regular complex that encloses the same areas and uses less perimeter. Therefore, the original complex is not a minimizer. (See Figure 5.1.)

If both adjacent $n$-gons are 5-gons, then they have an edge in common and they both share an edge with the 3-gon. Since both interior edges of a 3-gon have the same length, Lemma 4.2 tells us that the adjacent 5-gons must be identical. Switch the numbering of the areas enclosed by these 5-gons to get two sets of adjacent $n$-gons that enclose portions of the same areas. (See Figure 5.2.)

Eliminating the common edges gives a complex that encloses the same areas with
less total perimeter. Therefore, the original complex is not a minimizer.

Case 2: There are no 3-gons.

Suppose that the complex has two 4-gons that share an edge next to an exterior arc. By Lemma 4.2, they would be identical. The \( n \)-gons could be swapped (i.e. renumbered) yielding disconnected portions of the same area sharing an edge. Eliminate these shared edges to get a non-regular complex that encloses the same areas with less perimeter. The original complex is thus not a minimizer. (See Figure 5.3.)

![Figure 5.3: Swap adjacent 4-gons](image)

Suppose now that the complex has a pair of adjacent 5-gons that share an edge next to an exterior arc.

If they are identical 5-gons (i.e. they enclose the same areas), they can be swapped as in the 4-gon case above. The shared edge can again be eliminated thus reducing perimeter. (See Figure 5.4.)

If they are different sizes, the smaller one can be reflected into the larger one. To be precise, let the interior edges adjacent to the shared edge be \( \alpha \) and \( \beta \). Since the
5-gons are not identical, $\alpha$ and $\beta$ must be of different length. Assume $\alpha < \beta$. Add a line segment $B$ that makes an angle of $\frac{2\pi}{3}$ with $\beta$ at distance equal to the length of $\alpha$ from the vertex shared by $\alpha$ and $\beta$. (See Figure 5.5.)

$B$ will intersect the opposite interior edge, will have the same length as $A$, and will form a 5-gon identical to the one with edge $\alpha$. We can then renumber the $n$-gons.
and eliminate edge $A$. The perimeter is unchanged, but the resulting complex is not regular and therefore cannot be a minimizer.

The only remaining possibility, then, is that the $n$-gons that share edges with the exterior must alternate 4-gons and 5-gons.

Consider any 4-gon containing without loss of generality region number 1. Adjacent to it on each side is a 5-gon containing a different region. Assume (without loss of generality) that one adjacent 5-gon contains region number 2. The next 4-gon (adjacent to the same 5-gon) must then enclose region number 3 since it shares an edge with a 5-gon of region 2. The $n$-gons that the 5-gon shares edges with must alternate 1,3,1,3. In other words, the 4-gons must alternate in the regions that they enclose, and the 5-gons all enclose the same region. (See Figure 5.6.)

Figure 5.6: Structure of alternating 4-gons and 5-gons
By looking at the interior edges that meet exterior vertices, we see that there must be exactly six 4-gons. Each 4-gon rotates the angle of this edge by $\frac{\pi}{3}$ and 5-gons do not rotate them at all.

Consider two 4-gons that contain different regions. If they are exactly the same size, they can be swapped. In other words, the complex is renumbered such that areas and perimeter are preserved. The swapped 4-gons each now share their central edge (opposite the exterior arc) with another $n$-gon that encloses a portion of the same area. These edges can then be erased, thus reducing perimeter. (See Figure 5.7.)

Figure 5.7: Swap any two identical 4-gons containing different areas

If the 4-gons are not the same size, the smaller one can be reflected into the larger one. The edge of the smaller 4-gon opposite the curved arc is erased and inserted inside the larger 4-gon parallel to the edge opposite its curved arc. Renumbering the $n$-gons creates a non-regular complex with identical perimeter. (See Figure 5.8.)
Figure 5.8: Reflect small 4-gons into larger 4-gons

The only case that did not produce a contradiction was the standard triple bubble with three adjacent 3-gons. Any other complex with equal pressure regions is not a perimeter minimizer. □

**Corollary 5.2** The least perimeter graph that encloses and separates three equal areas $A_1$, $A_2$, and $A_3$ without empty chambers using equal pressure regions is a standard triple bubble.

**Proof:** The only standard triple bubble with equal pressure regions is the one that encloses and separates three equal area regions. (See Figure 5.9). Any other complex enclosing those same areas with equal pressure regions must have disconnected regions and therefore is not a minimizer by Theorem 5.1. □
Corollary 5.3  The least perimeter graph that encloses and separates three equal areas with convex cells and without empty chambers is a standard triple bubble.

Proof: To be a perimeter minimizer, the graph must be a regular triple bubble complex. In order to have convex cells, it must have equal pressure regions. The result then follows from Theorem 5.1. □

Since triple bubble complexes with disconnected, equal pressure regions are not minimizers, complexes close to such complexes cannot be minimizers either. To make this idea precise, suppose we have a sequence of regular $n$ bubble complexes $\{A_i\}$. Let $A_{i,m}$ for $m = 1 \ldots n$ be the area of the $m$th region enclosed by $A_i$. We say that the sequence $\{A_i\}$ converges in length and area to a regular $n$ bubble complex $A$ enclosing areas $A_1, A_2, \ldots, A_n$ if

1. $\lim_{i \to \infty} A_{i,m} = A_m \forall m$ and

2. $\lim_{i \to \infty} \ell(A_i) = \ell(A)$,
where \( \ell(A) \) is the length of the complex \( A \).

**Corollary 5.4** Suppose \( \{A_i\} \) is a sequence of regular triple bubble complexes that converges in length and area to a triple bubble complex \( A \). If \( A \) does not have any empty chambers, has equal pressure regions and is not a standard triple bubble, then there exists an \( N \) such that for any \( i > N \), \( A_i \) is not a perimeter minimizer for the areas it encloses.

**Proof:** \( A \) is not a minimizer by Theorem 5.1. So, there must exist a perimeter minimizing complex \( B \) that encloses the same areas but uses at least \( \epsilon \) less perimeter for some \( \epsilon \). Since \( B \) is the minimizer, \( \ell(B) = L(A_1, A_2, A_3) \) and we get the inequality

\[
|\ell(A) - \ell(B)| = |\ell(A) - L(A_1, A_2, A_3)| > \epsilon. \tag{5.1}
\]

Recall that Lemma 2.3 guarantees that the minimum length function \( L \) is continuous. Since the lengths converge, we can find an \( N_1 \) such that \( |\ell(A_i) - \ell(A)| < \frac{\epsilon}{2} \) for all \( i > N_1 \). In addition, since the areas converge, we can find \( N_2 \) such that \( |L(A_{i,1}, A_{i,2}, A_{i,3}) - L(A_1, A_2, A_3)| < \frac{\epsilon}{2} \) for all \( i > N_2 \). Let \( N = \max\{N_1, N_2\} \).

Suppose \( \ell(A_i) = L(A_{i,1}, A_{i,2}, A_{i,3}) \) for some \( i > N \). By the triangle inequality,

\[
|\ell(A) - L(A_1, A_2, A_3)| \leq |\ell(A) - \ell(A_i)| + |\ell(A_i) - L(A_1, A_2, A_3)|
\]

\[
= |\ell(A) - \ell(A_i)| + |L(A_{i,1}, A_{i,2}, A_{i,3}) - L(A_1, A_2, A_3)|
\]

\[
< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\]
This, of course contradicts inequality 5.1. Therefore, \( \ell(A_i) \neq L(A_{i,1}, A_{i,2}, A_{i,3}) \) for any \( i > N \).

\[ \square \]

**Corollary 5.5** Let \( A \) be a regular triple bubble complex enclosing areas \( A_1, A_2, \) and \( A_3 \) with pressures \( p_1, p_2, \) and \( p_3 \). If there exists a complex \( B \) with equal pressure regions that encloses the same areas but with less pressure (i.e. \( p \leq p_i \forall i \)), then \( A \) is not a minimizer.

**Proof:** Follows directly from Theorem 5.1 and Lemma 3.4.
Chapter 6

General Bubbles with Equal Pressure Regions

The results we achieved for triple bubbles do not depend on the fact that only three areas are being enclosed. The symmetry of the $n$-gons and the numbering of adjacent $n$-gons was important. Indeed, most of the moves presented in Chapter 5 can be generalized.

In this chapter, we assume that all complexes are regular $n$ bubble complexes with equal pressure regions and without empty chambers. We will continue to use integers inside of $n$-gons to denote the region number that the area is counted towards. We assume that every edge separates two differently numbered $n$-gons.

If there is a 3-gon and it has a 3-gon adjacent to it, the whole complex is a standard triple bubble. If a 3-gon has 4-gons adjacent to it on both sides, the complex is either
a non-minimizing triple bubble (if only three integers are used as labels) or a four chamber bubble we call a standard quadruple bubble. (See Chapter 4 and Figure 4.4.)

**Theorem 6.1** Suppose a perimeter minimizing complex with equal pressure regions and without empty chambers has a 3-gon with a 4-gon adjacent on one side and a 5-gon adjacent on the other. The $n$-gon opposite the 3-gon (adjacent to both the 4-gon and 5-gon) is not numbered the same as the 3-gon. In addition, the $n$-gon adjacent to the 5-gon but not adjacent to the 4-gon is not numbered the same as the 3-gon or the 4-gon.

**Proof:** Assume without loss of generality that the 3-gon is part of region 1, the 4-gon is part of region 2, and the 5-gon is part of region 3.

Suppose first that the $n$-gon opposite the 3-gon is also numbered with a 1. Let $A$ be the edge shared by the 4-gon and the $n$-gon opposite the 3-gon. The 4-gon can be reflected into the 3-gon as in the proof of Theorem 5.1. That is, we add a line segment $B$ at angle of $\frac{2\pi}{3}$ halfway between the vertices of the edge shared by the 3-gon and the 4-gon, then erase the top half of this same shared edge. The new complex is not regular and therefore the original complex was not a minimizer. (See Figure 6.1.)

Now assume that the opposite $n$-gon is numbered something else and assume that the $n$-gon adjacent to the 5-gon is numbered with a 1. We can reflect the 5-gon into the 3-gon to get a non-regular complex with identical perimeter. To be precise, let $2a$ be the length of the 3-gon. The edge shared by the 4-gon and 5-gon must have length $a$ by Lemma 4.3. By Lemma 4.4 the other interior edge of the 5-gon must
then also have length $a$. The 5-gon is actually symmetric and its remaining edge will again have length $2a$.

If we add a line segment of length $a$ at an angle of $\frac{2\pi}{5}$ halfway between the vertices of the edges shared by the 3-gon and the 5-gon and connect this line segment with another line segment of length $2a$ at angle $\frac{2\pi}{3}$, we have constructed an identical 5-gon inside of the 3-gon. We can renumber the areas, erase the top half of the edge between the old 3-gon and old 5-gon, and erase the other edge of length $2a$ from the old 5-gon. We now have no change in perimeter or areas enclosed, but the complex is clearly not regular. (See Figure 6.2.)

For the last case, suppose that the $n$-gon adjacent to the 5-gon and not adjacent to the 3-gon or 4-gon is numbered the same as the 4-gon. This time, we can reflect the 5-gon into the 4-gon. We add an edge of length $2a$ at the midpoint of either
long edge of the 4-gon at an angle of $\frac{2\pi}{3}$. When this segment hits the external curved arc, we have created a 5-gon inside the 4-gon identical to the 5-gon we started with. Renumber areas and delete the edge of length $2a$ between the 5-gon and the $n$-gon adjacent to it. Once again, we have a non-regular complex using identical perimeter. (See Figure 6.3.) $\square$

**Corollary 6.2** A perimeter minimizing 4 bubble complex with equal pressure regions and no empty chambers is either the standard quadruple bubble or has no 3-gons adjacent to 4-gons.

**Proof:** Suppose a 4-bubble complex has a 3-gon adjacent to a 4-gon. If the other $n$-gon adjacent to the 3-gon is also a 4-gon, the complex is the standard quadruple bubble. If the other $n$-gon adjacent to the 3-gon is a 5-gon, we look at the number of the $n$-gon on the other side of the 5-gon. Without loss of generality, assume that
the 3-gon is labeled with a 1, the 4-gon is labeled 2, and the 5-gon is labeled 3. (See Figure 6.4) By Theorem 6.1 the $n$-gon adjacent to the 4-gon and 5-gon (opposite the 3-gon) must be assigned a 4. But then, the $n$-gon adjacent to it and the 5-gon must be labeled either 2 or 1. Neither of these is possible again by Theorem 6.1. □

We next consider what can happen when a 3-gon is adjacent to two 5-gons. First we note that the 5-gons will be identical since they share an edge and the edges that they each share with the 3-gon have the same length. That is, the two parameters that determine a 5-gon have been chosen and are the same. The 5-gons, however, do not have to be symmetric. The non-shared interior edge could be longer or shorter than the shared edge.

**Theorem 6.3** If a 3-gon is adjacent to two 5-gons in a perimeter minimizing $n$ bubble complex with equal pressure regions and no empty chambers, then the $n$-gons
Figure 6.4: A 3-gon adjacent to a 4-gon in a 4 bubble.

adjacent to each 5-gon and the exterior are not numbered the same as the 3-gon.

Proof: When an n-gon adjacent to a 5-gon contains the same number as the 3-gon, we can reflect that 5-gon into the 3-gon. Let the innermost edges of the 5-gon have length $a$ and $b$. (See Figure 6.3.) By Lemma 4.4, the length of the shared edge between the 3-gon and 5-gon must have length $a + b$. Let $c$ be the length of the remaining interior edge of the 5-gon. We build a 5-gon inside of the 3-gon identical to our 5-gon by adding a line segment of length $b$ at an angle of $\frac{2\pi}{3}$ at a distance of $a$ from the inner vertex of the 3-gon along the shared edge. We then add a line segment of length $c$ at an angle of $\frac{2\pi}{3}$ to the end of the first line segment. Renumbering the areas, we can delete the top side of the shared edge (length $a$) and the edge between the old 5-gon and the adjacent n-gon (length $c$). We have a non-regular complex using the same perimeter which contradicts the assumption that the complex was a perimeter.
Corollary 6.4  Suppose a perimeter minimizing 4 bubble complex with equal pressure regions and without empty chambers has a 3-gon. The complex is either the standard quadruple bubble or has an interior hexagon containing the same number as the 3-gon.

Proof: By Corollary 6.2 the complex is either the standard quadruple bubble or the 3-gon is adjacent to 5-gons. Assume the 3-gon is numbered with a 1 and the 5-gons are numbered 2 and 3. (See Figure 6.5.) By Theorem 5.3 the n-gons adjacent to the 3-gon are numbered with 4, 5, and 6. Since the 3-gon is adjacent to 5-gons it cannot have a 4 as a neighbor, so the 4-gon is adjacent to 5-gons. This means that the 3-gon is the interior hexagon containing the same number as the 3-gon.
5-gons that are also adjacent to the exterior cannot be numbered 1. They cannot be numbered 2 or 3 either, since the 5-gons can be renumbered (swapped) to yield two adjacent \(n\)-gons enclosing portions of the same area. They must therefore be labeled with a 4. The \(n\)-gon that shares edges with both 5-gons must then be labeled with a 1. It has at least four flat edges and thus is at least a 5-gon.

If it is a 5-gon, then the \(n\)-gons adjacent to the 5-gons and the exterior must be 3-gons. Either of these 3-gons could be swapped with the 3-gon we started with. In this case, the 5-gon we just numbered 1 would now be adjacent to a 3-gon numbered with a 1. We could eliminate this edge and save perimeter. So, the \(n\)-gon adjacent to both 5-gons must be a hexagon. \(\square\)

We can put one additional restriction on the number of 3-gons in a complex with equal pressure regions.

**Theorem 6.5** A perimeter minimizing triple bubble complex with equal pressure regions can have at most one 3-gon enclosing portions of any given region.

**Proof:** If there are two 3-gons enclosing portions of the same region, we can pop a 3-gon (remove the exterior arc) and recover more than the lost area with less total perimeter by expanding the curvature of another 3-gon. (See Figure 6.7.)

We use the formulas 

\[
L(\theta, C) = \frac{C \theta}{\sin(\theta)} \quad \text{and} \quad A(\theta, C) = \frac{C^2 (\theta - \sin(\theta) \cos(\theta))}{\left(2 \sin(\theta)\right)^2}
\]

that relate the arc length \(L\) and area \(A\) of a section of a circle to the chord length \(C\) and angle \(\theta\). (See Figure 4.6.)
Figure 6.6: A 3-gon adjacent to two 5-gons in a 4 bubble

By joining the vertices of a 3-gon in a bubble complex with equal pressure regions, we can decompose the 3-gon into a triangle and a half circle. Let \( C \) be the diameter of this half circle. The triangle has base \( C \), height \( \frac{C}{2\sqrt{3}} \) and area \( \frac{C^2}{4\sqrt{3}} \). The area of the half circle is of course \( \frac{\pi C^2}{4} \). Therefore, the area enclosed by a 3-gon is \( C^2(\frac{\pi}{4} + \frac{1}{4\sqrt{3}}) \).

We also note that the arc length is \( \frac{\pi C}{2} \).

Increase the curvature of another 3-gon (so \( C \) is the same) until the angle is 2.3 radians. The new arc length is \( L(2.3, C) = \frac{C(2.3)}{\sin(2.3)} \). Which is approximately \( C(3.08) \), but is definitely less than \( 2L = C(2\pi) \). In other words, we’ve used less total perimeter.
Now we compute the area inside the section $A(2.3, C) = \frac{C^2(2.3 - \sin(2.3) \cos(2.3))}{(2 \sin(2.3))^2}$ which is approximately $C^2(1.2574)$. We need to recover the area from the other 3-gon as well as the area from the section we increased. The area we need to recover is $C^2(\frac{\pi}{2} + \frac{1}{4\sqrt{3}})$ which is approximately $C^2(.9297)$. Since $C^2(.9297) > C^2(\frac{\pi}{4} + \frac{1}{4\sqrt{3}})$ and therefore we have enclosed more area with less perimeter.

Since 2.3 radians is less than $\pi$ and the tangent to an adjacent exterior edge makes an angle of $\frac{7\pi}{6}$ radians with the chord $C$, the increased 3-gon will not intersect any portion of the existing complex.

We chose $\theta = 2.3$ as an approximation to the solution of $\theta = \pi \sin \theta$ which is difficult to solve explicitly. This is the value of $\theta$ needed to use exactly the same amount of perimeter. By decreasing curvature, we could enclose exactly the same areas and use strictly less perimeter. □
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