LINEAR PERIODS AND DISTINGUISHED LOCAL PARAMETERS

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Abstract. Let $F$ be a nonarchimedean local field of characteristic zero. Let $X$ be the $p$-adic symmetric space $X = H\backslash G$, where $G = \text{GL}_{2n}(F)$ and $H = \text{GL}_n(F) \times \text{GL}_n(F)$. We verify a conjecture of Sakellaridis and Venkatesh on the Langlands parameters of certain representations in the discrete spectrum of $X$.

1. Introduction

Let $F$ be a nonarchimedean local field of characteristic zero. Let $G = \text{GL}_{2n}(F)$ and let $H = \text{GL}_n(F) \times \text{GL}_n(F)$. The subgroup $H$ of $G$ is equal to the fixed points of an involution $\theta$ of $G$ and the quotient $X = H\backslash G$ is a $p$-adic symmetric space. We are interested in the study of harmonic analysis on $X$ and its relevance to the Local Langlands Correspondence (LLC). In particular, we aim to understand the discrete spectrum of $X$, as a representation of $G$, within the framework developed by Sakellaridis and Venkatesh [23]. The goal of this note is to understand the construction of relative discrete series for $X$ via [27, Theorem 6.3] (see Theorem 2.5) in terms of the associated Langlands parameters, and to provide evidence for the truth of the general conjectures on the discrete spectrum formulated by Sakellaridis and Venkatesh [23, Conjecture 16.2.2].

Let $(\pi, V)$ be an admissible representation of $G$ on a complex vector space $V$. If there is a nonzero element $\lambda \in \text{Hom}_H(\pi, 1)$, then we say that $(\pi, V)$ admits a (local) linear period and we refer to the $H$-invariant linear form $\lambda$ on $V$ as a linear period. This terminology parallels the language used in the global setting. If $(\pi, V)$ admits a nonzero linear period then we say that $\pi$ is $H$-distinguished. It is exactly the $H$-distinguished representations of $G$ that are relevant to the harmonic analysis on $X$. The irreducible direct summands of the space $L^2(X)$ of square integrable $\mathbb{C}$-valued functions on $X$ are referred to as relative discrete series representations. If $(\pi, V)$ is an $H$-distinguished discrete series representation of $G$, then $(\pi, V)$ is automatically a relative discrete series representation for $X$ [18]. The relative discrete series representations constructed in [27, Theorem 6.3] consist of certain tempered

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representations of $G$ that do not appear in the discrete spectrum of the group $G$.

For $H$-distinguished supercuspidal representations of $G$, the main result of this paper is due to Jiang, Nien and Qin [14, 15]. We refer the reader to [15, Theorem 1.1] for a complete statement of their characterization of $H$-distinction for supercuspidal representations of $G$ (in terms of $L$-functions, local Langlands functorial transfer, and the Shalika model). Our work relies heavily on the results of Jiang, Nien and Qin and the work of Matringe [21].

We now take a moment to briefly outline the contents of the paper. In the next section, we fix notation and conventions. We recall the notion of a distinguished parameter and the relevant conjecture of Sakellaridis–Venkatesh in Section 1.2. Linear periods and the discrete spectrum of $X$ are discussed in Section 2; in this section, we recall several known results including the work of Matringe [21] and the author’s construction of relative discrete series [27]. In the final part (Section 3), we study the $L$-parameters of the relative discrete series studied in [27] and prove the main result Theorem 3.3.

1.1. Notation. Let $W_F$ be the Weil group of $F$. The local Langlands group of $F$ is the Weil-Deligne group $L_F = W_F \times SL(2, \mathbb{C})$. For any integer $k \geq 1$, denote by $S(k)$ the unique (up to equivalence) $k$-dimensional irreducible representation of $SL(2, \mathbb{C})$. Recall that $S(k) \approx \text{Sym}^{k-1}(\mathbb{C}^2)$, where $\mathbb{C}^2$ is the standard representation of $SL(2, \mathbb{C})$. We denote the rank-$k$ general linear group $GL_k(F)$ by $G_k$. Let $H_{2k}$ be the subgroup $GL_k(F) \times GL_k(F)$ of $G_{2k}$.

Let $P$ be a parabolic subgroup of $G$ with Levi subgroup $M$ and unipotent radical $N$. Given a smooth representation $(\rho, V_\rho)$ of $M$ we may inflate $\rho$ to a representation of $P$, also denoted $\rho$, by declaring that $N$ acts trivially. We define the representation $\iota_G^P \rho$ of $G$ to be the (normalized) parabolically induced representation $\text{Ind}_{G}^{P}(\delta^1_\rho \otimes \rho)$. We will also use the Bernstein–Zelevinsky [3, 28] notation $\pi_1 \times \ldots \times \pi_k$ for the (normalized) parabolically induced representation $\iota_{P_{(m_1, \ldots, m_k)}}^{G_n}(\pi_1 \otimes \ldots \otimes \pi_k)$ of $G_n$ obtained from the standard (block-upper triangular) parabolic subgroup $P_{(m_1, \ldots, m_k)}$ and representations $\pi_j$ of $G_{m_j}$, where $\sum_{j=1}^{k} m_j = n$.

Let $\rho$ be an irreducible supercuspidal representation of $G_r$ and let $k \geq 1$ be a positive integer. Let $\nu$ denote the character of $G_r$ given by $g \mapsto |\det(g)|_F$, where $|\cdot|_F$ is the normalized absolute value on $F$ and $r$ is understood from context. The unique irreducible quotient $\text{St}(k, \rho)$ of the induced representation

$$\nu^{\frac{1-k}{2}} \rho \times \ldots \times \nu^{\frac{k-1}{2}} \rho$$

is a discrete series representation of $G_{kr}$ [28, Theorem 9.3]. We refer to the representations $\text{St}(k, \rho)$ as generalized Steinberg representations. The usual Steinberg representation of $G_n$ is $\text{St}(n, 1)$. Note that $\text{St}(k_1, \rho_1)$ is equivalent to $\text{St}(k_2, \rho_2)$ if and only if $k_1 = k_2$ and $\rho_1$ is equivalent to $\rho_2$ [28, Theorem 9.7(b)].
1.2. **Distinguished parameters.** In this section, suppose that $G$ is an arbitrary connected reductive group that is defined and split over $F$. Let $G^\vee$ be the complex dual group of $G$.\(^1\) An $A$-parameter, or an Arthur parameter, for $G$ is a continuous homomorphism $\psi : \mathcal{L}_F \times \text{SL}(2, \mathbb{C}) \rightarrow G^\vee$ such that

- the restriction $\psi|_{\mathcal{W}_F}$ of $\psi$ to the Weil group $\mathcal{W}_F$ is bounded,
- the image of $\psi|_{\mathcal{W}_F}$ consists of semisimple elements of $G^\vee$,
- and the restriction of $\psi$ to each of the two $\text{SL}(2, \mathbb{C})$ factors is algebraic.\(^2\)

The $L$-parameter, or Langlands parameter, for $G$ associated to the $A$-parameter $\psi$ is the admissible homomorphism $\phi_\psi : \mathcal{L}_F \rightarrow G^\vee$ given by $\phi_\psi(w, g) = \psi(w, g, d_w)$, where

$$d_w = \begin{pmatrix} |w|^\frac{1}{2} & 0 \\ 0 & |w|^{-\frac{1}{2}} \end{pmatrix} \in \text{SL}(2, \mathbb{C}),$$

and $| \cdot | : \mathcal{W}_F \rightarrow \mathbb{R}_{>0}$ is the absolute value determined by the normalized absolute value $| \cdot |_F$ on $F^\times$ via local class field theory. Langlands parameters of the form $\phi_\psi$ are said to be of Arthur type \cite[Section 3.6]{7}.

Let $X$ be a homogeneous spherical variety for $G$, that is, $X$ is a (normal) variety defined over $F$ equipped with a transitive $G$ action such that a Borel subgroup $B$ of $G$ has a Zariski-dense orbit. Note that symmetric $G$-varieties are spherical. Let $X^\dagger$ denote the open $B$ orbit in $X$. Define $P_0$ to be the standard parabolic subgroup $G$ that stabilizes $X^\dagger$, that is, $P_0 = \{ g \in G : X^\dagger = X^\dagger g \}$. Let $x_0 \in X^\dagger(F)$ and let $H$ be the stabilizer of $x_0$ in $G$. Then $X \cong H \backslash G$. For a fixed $x_0$ there is a natural choice of Levi subgroup $M_0$ of $P_0$ \cite[§2.1]{23}.

**Remark 1.1.** If $X = G^\theta \backslash G$ is a symmetric variety, defined by an $F$-involution $\theta$ of $G$, then there is a natural choice of $x_0 = G^\theta$-e. The parabolic subgroup $P_0$ is a minimal $\theta$-split parabolic subgroup and the Levi subgroup $M_0 = P \cap \theta(P_0)$ is $\theta$-stable. Recall that a parabolic subgroup $P$ of $G$ is $\theta$-split if $\theta(P)$ is opposite to $P$.

Before stating their conjectures on the parameters of relative discrete series, we recall Sakellaridis and Venkatesh’s definition of a distinguished morphism. We refer the reader to \cite[§2.1–2.2, 3.2]{23} for more detail and in particular for the definition of the spherical roots. Fix a maximal $F$-split torus $A_0$ of $G$ contained in $B \cap M_0$. Let $A^\vee \subset G^\vee$ be the complex dual torus of $A_0$. Define $A_X$ to be the torus $A_0/(A_0 \cap H)$. Let $A^*_X$ be the complex torus dual to $A_X$. Dualizing the surjective map $A_0 \rightarrow A^*_X$ gives rise to a finite-to-one map $A^*_X \rightarrow A^\vee$. Let $G^\vee_X$ be the complex dual group of the spherical variety $X$ defined by Sakellaridis and Venkatesh, defined under the technical assumption of \cite[Proposition 2.2.2]{23} on the spherical roots. The

\(^1\)Since $G$ is split over $F$, $\mathcal{W}_F$ acts trivially on $G^\vee$ and without loss of generality the $L$-group $^L G$ coincides with the dual group $G^\vee$.

\(^2\)The second $\text{SL}(2, \mathbb{C})$ factor is referred to as the Arthur $\text{SL}(2)$. 
torus $A^*_X$ is a maximal torus of $G'_X$. The aim of Definition 1.2 is to produce an appropriate notion of extending the map $A^*_X \to A^*$ to a homomorphism from $G'_X$ to $G'$. Let $\Sigma_X$ denote the set of spherical roots of $X$. Let $\gamma \in \Sigma_X$. It is known [1, 5] that either

- $\gamma$ is proportional to a positive root $\alpha$ of $A_0$ in $G$, or;
- $\gamma$ is proportional to the sum $\alpha + \beta$ of two positive roots $\alpha$ of $A_0$ in $G$ such that $\alpha$ is orthogonal to $\beta$ and both $\alpha$ and $\beta$ are contained in a system of simple roots (but not necessarily the simple roots corresponding to $B$),

In the first case, set $\gamma_0 = \alpha$ and in the second case set $\gamma_0 = \alpha + \beta$ (note that in the second case $\alpha + \beta$ is not a root of $A_0$ in $G$; moreover, $\alpha$ and $\beta$ are not unique, but there is a canonical choice [23, Corollary 3.1.4]). The roots $\alpha$ and $\beta$ are referred to as the associated roots of $\gamma_0$ (if $\gamma_0 = \alpha$, then $\alpha$ is the associated root). Sakellaridis and Venkatesh call the set $\Delta_X = \{\gamma_0 : \gamma \in \Sigma_X\}$ the simple normalized spherical roots of $X$.

**Definition 1.2.** A distinguished morphism $\xi : G'_X \times \text{SL}(2, \mathbb{C}) \to G'$ is a group homomorphism such that

1. the restriction of $\xi$ to $G'_X$ extends the canonical map of tori $A^*_X \to A^*$
2. for every simple normalized spherical root $\gamma_0 \in \Delta_X$, the corresponding root space of the Lie algebra $\mathfrak{g}'_X$ maps into the sum of root spaces of its associated roots under the differential of $\xi$
3. the restriction of $\xi$ to the $\text{SL}(2, \mathbb{C})$ factor is a principal morphism into $M'_0 \subset G'$ with weight $2\rho_{M_0} : \mathbb{G}_m \to G'$, where $\mathbb{G}_m$ is identified with the maximal torus of $\text{SL}(2, \mathbb{C})$ via $a \mapsto \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ and $2\rho_{M_0}$ is the sum of the positive roots of $A_0$ in $M_0$.

**Remark 1.3.** Sakellaridis and Venkatesh proved that distinguished morphisms are unique up to $A^*$ conjugacy [23, Proposition 3.4.3]. Knop and Schalke have proved the existence of distinguished morphisms in full generality [20].

**Definition 1.4.** An $A$-parameter $\psi : \mathcal{L}_F \times \text{SL}(2, \mathbb{C}) \to G'$ is $X$-distinguished if it factors through the distinguished morphism $\xi : G'_X \times \text{SL}(2, \mathbb{C}) \to G'$, that is, there exists a tempered (that is, bounded on $\mathcal{W}_F$) $L$-parameter $\psi_X : \mathcal{L}_F \to G'_X$ such that $\psi(w, g) = \xi(\psi_X(w), g)$.

We are now in a position to state the conjecture of Sakellaridis and Venkatesh on the parameters of relative discrete series representations (see [23, Conjectures 16.2.2, 16.5.1] for statements of the full local conjectures). Recall that an $L$-parameter $\phi : \mathcal{L}_F \to G'$ is elliptic if and only if the image of $\phi$ is not contained in any proper parabolic subgroup of $G'$.

**Conjecture 1.5** (Sakellaridis and Venkatesh). A relative discrete series representation $\pi$ in $L^2(X)$ is contained in an Arthur packet corresponding to an $X$-distinguished $A$-parameter $\psi : \mathcal{L}_F \times \text{SL}(2, \mathbb{C}) \to G'$ such that the $L$-parameter $\psi_X : \mathcal{L}_F \to G'_X$ is elliptic.
Our ultimate aim is to prove that the relative discrete series representations for $X = \text{GL}_n(F) \times \text{GL}_n(F) \backslash \text{GL}_{2n}(F)$ considered in [27] satisfy Conjecture 1.5. In the case that $G = \text{GL}_{2n}$, Arthur packets are $L$-packets and are singleton sets. This fact greatly simplifies the situation; moreover, we ultimately only need to consider $L$-parameters because the relative discrete series produced in [27] are all tempered.

2. Linear periods and relative discrete series

For the remainder of the paper, we let $G = \text{GL}_{2n}(F)$ and let $H = \text{GL}_n(F) \times \text{GL}_n(F)$. Let $X = H \backslash G$ be the $p$-adic symmetric space associated to the symmetric pair $(G, H)$. In this section, we recall the main result of [27] and some background on local linear periods. For the symmetric pair $(G, H)$, multiplicity-one is known due to work of Jacquet and Rallis [13].

Theorem 2.1 (Jacquet–Rallis). Let $(\pi, V)$ be an irreducible admissible representation of $G$. The dimension of the space $\text{Hom}_H(\pi, 1)$ is at most one. If $\dim \text{Hom}_H(\pi, 1) = 1$, then $\pi$ is self-contragredient, that is, $\tilde{\pi} \cong \pi$.

Remark 2.2. Jacquet and Rallis also proved that if an irreducible admissible representation $(\pi, V)$ of $G$ admits a nonzero Shalika model, then $\pi$ is $H$-distinguished. The converse for irreducible supercuspidal representations was obtained by Jiang, Nien, and Qin [14, Theorem 5.5]. Sakellaridis and Venkatesh [23, Section 9.5], and independently Matringe [21, Theorem 5.1], proved the converse for irreducible $H$-relatively integrable and relative discrete series representations.

As above, for a positive integer $m$ we write $G_m = \text{GL}_m(F)$ and, if $m$ is even, $H_m = \text{GL}_{m/2}(F) \times \text{GL}_{m/2}(F)$. Let $\pi$ be a discrete series representation of $G_m$. Let $L(s, \pi \times \pi)$ be the local Rankin–Selberg convolution $L$-function. Shahidi [25, Lemma 3.6] proved the local identity

$$L(s, \pi \times \pi) = L(s, \pi, \wedge^2)L(s, \pi, \text{Sym}^2),$$

where $L(s, \pi, \wedge^2)$, respectively $L(s, \pi, \text{Sym}^2)$, denotes the exterior square, respectively symmetric square, $L$-function of $\pi$ defined via the Local Langlands Correspondence (LLC). The $L$-function $L(s, \pi \times \pi)$ has a simple pole at $s = 0$ if and only if $\pi$ is self-contragredient [12]. Note that $L(s, \pi, \wedge^2)$ cannot have a pole when $m$ is odd (see, for instance, [6, Section 9], [19, Theorem 4.5]). For all discrete series representations of $G_m$, and all irreducible generic representations of $G_{2m}$, the associated Jacquet–Shalika and Langlands–Shahidi local exterior square $L$-functions agree with with the exterior square $L$-function defined via the LLC [11, Theorem 4.3 in §4.2], [19, Theorems 1.1 and 1.2].

Matringe proved the following results which characterize the $H$-distinction of discrete series representations of $G$. The first result appears as [21, Proposition 6.1], and the second appears as [21, Theorem 6.1].
**Theorem 2.3** (Matringe). Suppose that $\pi$ is a square integrable representation of $G$, then $\pi$ is $H$-distinguished if and only if the exterior square $L$-function $L(s, \pi, \wedge^2)$ has a pole at $s = 0$.

**Theorem 2.4** (Matringe). Suppose that $m = kr$ is even. Let $\rho$ be an irreducible supercuspidal representation of $G_r$. Let $\pi = St(k, \rho)$ be a generalized Steinberg representation of $G_m$.

1. If $k$ is odd, then $r$ must be even, and $\pi$ is $H_m$-distinguished if and only if $L(s, \rho, \wedge^2)$ has a pole at $s = 0$ if and only if $\rho$ is $H_r$-distinguished.
2. If $k$ is even, then $\pi$ is $H_m$-distinguished if and only if $L(s, \rho, \text{Sym}^2)$ has a pole at $s = 0$.

The author has studied relative discrete series for several $p$-adic symmetric spaces and carried out a systematic construction of relative discrete series in the papers [26, 27]. The following result forms part of [27, Theorem 6.3]. As the work [27] relies on [17], we further assume that the residual characteristic of $F$ is odd.

**Theorem 2.5.** Suppose that $F$ has odd residual characteristic. Let $(m_1, \ldots, m_d)$ be a partition of $2n$ such that each $m_i$, $1 \leq i \leq d$ is an even integer. Let $\tau_i$, $1 \leq i \leq d$, be pairwise inequivalent $H_{m_i}$-distinguished discrete series representations of $G_{m_i}$. The parabolically induced representation $\pi = \tau_1 \times \ldots \times \tau_d$ is a relative discrete series representation. That is, $\pi$ occurs in the discrete spectrum of $X = H \backslash G$.

The aim of the current work is to prove that the local parameters of the relative discrete series representations described in Theorem 2.5 satisfy Conjecture 1.5.

### 3. $X$-distinguished $X$-elliptic parameters

Next we record a description of the dual group $G_X^\vee$ attached to $X = H \backslash G$ and choose a representative for the distinguished morphism $\xi : G_X^\vee \times \text{SL}(2, \mathbb{C}) \to G^\vee$. The existence of the distinguished morphism is now known in full generality by the work of Knop and Schalke [20]. Lemma 3.1 is well known, thus we omit the proof which amounts to a routine calculation of the restricted root system attached to $X$ (see [27, Sections 3.1 and 5.1], cf. [20, Table 3]). We note that the fact that the distinguished morphism $\xi : G_X^\vee \times \text{SL}(2, \mathbb{C}) \to G^\vee$ is trivial on the $\text{SL}(2, \mathbb{C})$-factor follows from the fact that a minimal $\theta$-split parabolic subgroup of $G$ is a Borel subgroup $P_0$, with $\theta$-stable Levi $A_0 = M_0$, and the principal morphism of $\text{SL}(2, \mathbb{C})$ into $M_0^\vee = A_0^\vee$ is trivial (cf. Definition 1.2).

**Lemma 3.1.** The complex dual group $G_X^\vee$ associated to $X$ is the symplectic group $\text{Sp}(2n, \mathbb{C})$. The distinguished morphism $\xi : G_X^\vee \times \text{SL}(2, \mathbb{C}) \to G^\vee$ is trivial on the $\text{SL}(2, \mathbb{C})$ factor and is given by the inclusion map into $G^\vee = \text{GL}(2n, \mathbb{C})$ on the $G_X^\vee$ component.
Since the distinguished morphism $\xi : G^\vee_X \times \text{SL}(2, \mathbb{C}) \to G^\vee$ is trivial on the $\text{SL}(2, \mathbb{C})$-factor, Conjecture 1.5 predicts that an $X$-distinguished $A$-parameter $\psi : \mathcal{L}_F \times \text{SL}(2, \mathbb{C}) \to G^\vee$ must also be trivial on the Arthur $\text{SL}(2)$.

That is, Conjecture 1.5 predicts that the relative discrete series representations for $X$ should be tempered representations of $G$. This is in agreement with the fact that the symmetric space $X$ is known to be tempered, that is, the Plancherel measure of $L^2(X)$ is supported on tempered representations of $G$ [4, Proposition 2.7.1], [8, Corollary 1.2]. Moreover, we may restrict our attention to tempered $L$-parameters for $G$ instead of dealing with more general Arthur parameters.

Let $\phi : \mathcal{L}_F \to G^\vee$ be an $X$-distinguished $L$-parameter, i.e., $\phi \otimes 1 : \mathcal{L}_F \times \text{SL}(2, \mathbb{C}) \to G^\vee$ is an $X$-distinguished $A$-parameter. Thus $\phi$ is an $L$-parameter for $G$ that has image contained in $G^\vee_X = \text{Sp}(2n, \mathbb{C})$ (up to conjugacy), and since $\text{SO}(2n + 1)^\vee = \text{Sp}(2n, \mathbb{C})$, we may regard $\phi$ as an $L$-parameter for the split special orthogonal group $\text{SO}_{2n+1}(F)$. We say that the parameter $\phi$ is $X$-elliptic if the image of $\phi$ is not contained in any proper parabolic subgroup of $G^\vee_X$. In particular, if $\phi$ is $X$-elliptic, then the parameter $\phi : \mathcal{L}_F \to G^\vee_X$ corresponds to a finite set (an $L$-packet) $\Pi_\phi$ of essential discrete series representation of $\text{SO}_{2n+1}(F)$ [2, Proposition 6.6.5]. Conjecture 1.5 predicts that the discrete spectrum of $X$ is contained in the image of the local Langlands functorial transfers, determined by the inclusion $\text{Sp}(2n, \mathbb{C}) \hookrightarrow \text{GL}(2n, \mathbb{C})$, of representations in the discrete spectrum of $\text{SO}_{2n+1}(F)$. In fact, it is known that an irreducible tempered representation $\pi$ of $G$ is $H$-distinguished if and only if $\pi$ is the local Langlands functorial lift of a generic discrete series representation of $\text{SO}_{2n+1}(F)$ [16, Theorem 2.1], [22, Theorem 3.1].

Let $\tau = \text{St}(k, \rho)$ be a generalized Steinberg representation, where $\rho$ is an irreducible unitary supercuspidal representation of $G_r$. Let $m = kr$. By the LLC for the general linear group [9, 10, 24], the $L$-parameter $\phi_r : \mathcal{L}_F \to \text{GL}(m, \mathbb{C})$ of the generalized Steinberg representation $\tau = \text{St}(k, \rho)$ is equal to $\phi_r = \phi_\rho \otimes \mathcal{S}(k)$, where $\phi_\rho : \mathcal{W}_F \to \text{GL}(r, \mathbb{C})$ is the (irreducible) $L$-parameter of the supercuspidal representation $\rho$ and $\mathcal{S}(k)$ is the unique irreducible $k$-dimensional representation of $\text{SL}(2, \mathbb{C})$. The following proposition is a consequence of Theorem 2.3 [21, Theorem 6.1].

**Proposition 3.2.** Let $k, r \geq 2$ be integers such that $m = kr$ is even. Let $\rho$ be an irreducible self-contragredient supercuspidal representation of $G_r$. Let $\tau = \text{St}(k, \rho)$ be the generalized Steinberg representation of $G_m$ attached to $k$ and $\rho$. If $\tau$ is $H_m$-distinguished, then the image of the $L$-parameter $\phi_r$ is contained in the complex symplectic group $\text{Sp}(m, \mathbb{C})$.

**Proof.** As above, the $L$-parameter of $\tau$ is equal to $\phi_r = \phi_\rho \otimes \mathcal{S}(k)$.

1. If $k$ is odd, then $r$ is even. By assumption $\tau$ is $H_m$-distinguished; therefore, by Theorem 2.4, $\rho$ is $H_r$-distinguished. By [14, Theorem 5.5], $\rho$ is a local Langlands functorial transfer from $\text{SO}_{r+1}(F)$. In particular, $\rho$ is of symplectic type and its $L$-parameter $\phi_\rho$ has image
contained in \( \text{Sp}(r, \mathbb{C}) \). When \( k \) is odd, the image of \( \text{SL}(2, \mathbb{C}) \) under \( S(k) \) is contained in the complex orthogonal group \( \text{O}(k, \mathbb{C}) \). It follows that the image of \( \phi_\tau \) preserves a non-degenerate skew-symmetric bilinear form on \( \mathbb{C}^r \otimes \mathbb{C}^k \cong \mathbb{C}^m \) given by the tensor product of the non-degenerate skew-symmetric bilinear form on \( \mathbb{C}^r \) preserved by \( \text{Im} \varphi_\rho \) and the non-degenerate symmetric bilinear form on \( \mathbb{C}^k \) preserved by \( \text{Im} S(k) \). That is, \( \text{Im} \phi_\tau \) is contained in \( \text{Sp}(m, \mathbb{C}) \).

(2) If \( k \) is even, then the image of \( \text{SL}(2, \mathbb{C}) \) is contained in the symplectic group \( \text{Sp}(k, \mathbb{C}) \). Since \( \tau \) is \( H_m \)-distinguished; it follows from Theorem 2.3, that the symmetric square \( L \)-function \( L(s, \rho, \text{Sym}^2) \) has a pole at \( s = 0 \). In this case, \( \rho \cong \tilde{\rho} \) is self-contragredient and the exterior square \( L \)-function \( L(s, \rho, \wedge^2) \) does not have a pole at \( s = 0 \) \([12], [25, Lemma 3.6]\). It follows that \( \phi_\tau \) is an irreducible self-dual \( r \)-dimensional representation of \( W_F \) such that \( \text{Im} \phi_\tau \) is not contained in a symplectic group \([14, Theorem 5.5]\). Following \([2, Section 1.2]\), one may readily observe that \( \text{Im} \phi_\tau \) must then be contained in the complex orthogonal group \( \text{O}(r, \mathbb{C}) \). As above, the image of \( \phi_\tau \) preserves a non-degenerate skew-symmetric bilinear form on \( \mathbb{C}^m \) and \( \text{Im} \phi_\tau \) is contained in \( \text{Sp}(m, \mathbb{C}) \).

□

**Theorem 3.3.** Let \( \pi \) be an \( H \)-distinguished relative discrete series representation of \( G \) constructed via Theorem 2.5. The image of the \( L \)-parameter \( \phi_\pi \) of \( \pi \) is contained in the symplectic group \( \text{Sp}(2n, \mathbb{C}) \); moreover, the image of \( \phi_\pi \) is not contained in any proper parabolic subgroup of \( \text{Sp}(2n, \mathbb{C}) \).

**Proof.** By assumption, \( \pi = \tau_1 \times \ldots \times \tau_d \), where \( 2n = \sum_{i=1}^d m_i \) and each \( m_i = k_i r_i \) is an even integer, and \( \tau_i \) are pairwise inequivalent \( H_{m_i} \)-distinguished discrete series representation of \( G_{m_i} \), for all \( 1 \leq i \leq d \). Moreover, each \( \tau_i = \text{St}(k_i, \rho_i) \) is a generalized Steinberg representation, where \( \rho_i \) is an irreducible unitary self-contragredient supercuspidal representation of \( G_{r_i} \). The compatibility of the LLC with parabolic induction gives that \( L \)-parameter of \( \pi \) is equal to the direct sum

\[
\phi_\pi = \phi_{\tau_1} \oplus \ldots \oplus \phi_{\tau_d}.
\]

By Proposition 3.2, \( \text{Im} \phi_{\tau_i} \subset \text{Sp}(m_{i}, \mathbb{C}) \), for all \( 1 \leq i \leq d \). Each of the parameters \( \phi_{\tau_i} \) are elliptic in \( \text{GL}(m_i, \mathbb{C}) \) and thus are also elliptic in \( \text{Sp}(m_i, \mathbb{C}) \) \([8, Lemma 3.1]\). Moreover, since \( \tau_i \) is determined by \( \phi_{\tau_i} \) up to conjugacy in \( \text{GL}(m_i, \mathbb{C}) \), we may assume that \( \text{Im} \phi_{\tau_i} \) is contained in the symplectic group determined by the non-singular skew-symmetric matrix \( J'_{m_i} \), where

\[
J'_m = \begin{pmatrix} 0 & J_{m/2} \\ -J_{m/2} & 0 \end{pmatrix} \quad \text{and} \quad J_k = \begin{pmatrix} 1 \\ & \ddots \\ \vdots & \ddots & 1 \end{pmatrix},
\]
for any even integer $m \geq 2$ and any $k \geq 1$. That is, for an even integer $m$, we realize the symplectic group $\text{Sp}(m, \mathbb{C})$ as the subgroup

$$\text{Sp}(J'_m) = \{ g \in \text{GL}(m, \mathbb{C}) : J'_m g = J'_m \}$$

of $\text{GL}(m, \mathbb{C})$. It follows that the image of the $L$-parameter $\phi_\pi$ is contained in the subgroup $\prod_{i=1}^{d} \text{Sp}(J'_m)$ of the symplectic group $\text{Sp}(J'_{2n}) = \text{Sp}(2n, \mathbb{C})$ in $G^\prime = \text{GL}(2n, \mathbb{C})$, where $\text{Sp}(J'_{2n})$ is determined by the non-singular skew-symmetric matrix $J'_{2n} = J'_{m_1} \oplus \ldots \oplus J'_{m_d}$. In fact, $J'_{2n} = w_{+}^{-1}J'_{2n}w_{+}$, where $w_{+}$ is the permutation matrix corresponding to the permutation of $2^{i-1}$, and $2i \mapsto 2n + 1 - i$, for $1 \leq i \leq n$. Thus $\text{Sp}(J'_{2n}) = w_{+}^{-1}\text{Sp}(J'_{2n})w_{+}$; moreover, $w_{+}(\text{Im } \phi_\pi)w_{+}^{-1} \subset \text{Sp}(J'_{2n})$.

It now remains to show that $\phi_\pi$ is elliptic in $\text{Sp}(2n, \mathbb{C})$. We argue by contradiction. Suppose that $\phi_\pi$ factors through a (proper) maximal parabolic subgroup $P$ of $\text{Sp}(2n, \mathbb{C})$. Since each $\tau_i$ is a discrete series representation, the representations $\phi_{\tau_i}$ of $L_{F}$ are irreducible; in particular, $\phi_\pi$ is semisimple and must factor through a Levi component $L$ of $P$. Up to conjugacy, $L$ has the form

$$L = \left\{ \begin{pmatrix} x & y \\ J'_{m_1}^{-1} & J'_{m_2} \end{pmatrix} : x \in \text{GL}(m, \mathbb{C}), y \in \text{Sp}(2k, \mathbb{C}) \right\},$$

for some integers $m, k$ so that $n = m + k$ and $m \geq 1$. It follows that $\phi_\pi$ can be decomposed as the direct sum $\phi_\pi = \phi_{1} \oplus \phi_{0} \oplus \phi_{1}^\vee$, where $\phi_{1}$ and $\phi_{0}$ are finite dimensional representations of $L_{F}$, $\phi_{1} \neq 0$, and $\phi_{1}^\vee$ denotes the contragredient of $\phi_{1}$. By assumption, $\phi_\pi$ is the direct sum of self-dual non-isomorphic irreducible representations (3.1); therefore, $\phi_\pi$ cannot be decomposed as $\phi_\pi = \phi_{1} \oplus \phi_{0} \oplus \phi_{1}^\vee$ and this is a contradiction. We conclude that $\text{Im } \phi_\pi$ cannot be contained in a proper parabolic subgroup of $\text{Sp}(2n, \mathbb{C})$, that is, $\phi_\pi$ is elliptic in $\text{Sp}(2n, \mathbb{C})$.

**Theorem 3.3** says precisely that the $L$-parameters of the known relative discrete series representations for $X = H \backslash G$ are $X$-distinguished and $X$-elliptic. That is, the relative discrete series for $X$ produced via **Theorem 2.5** satisfy **Conjecture 1.5**.

**Remark 3.4.** The last paragraph of the proof of **Theorem 3.3**, together with **Conjecture 1.5**, suggests that we cannot relax the regularity condition imposed in **Theorem 2.5** (cf. [27, Remark 6.6]). The author expects that the representations constructed in **Theorem 2.5** exhaust the discrete spectrum of $X$, as predicted by **Conjecture 1.5**; however, the author does not have a proof of this fact.

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3Recall that $\pi$ is determined by $\phi_\pi$ up to $G^\vee$-conjugacy. The choice to work with $\text{Sp}(2n, \mathbb{C}) = \text{Sp}(J'_{2n})$ in what follows is convenient for working with block-upper triangular parabolic subgroups.
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