EXTENDED FULLER INDEX, SKY CATASTROPHES AND THE SEIFERT
CONJECTURE

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Abstract. We extend the classical Fuller index, and use this to prove that for a certain general
class of vector fields $X$ on a compact smooth manifold, if a homotopy of smooth non-singular vector
fields starting at $X$ has no sky catastrophes as defined by the paper, then the time 1 limit of the
homotopy has periodic orbits. This class of vector fields includes the Hopf vector field on $S^{2k+1}$.
A sky catastrophe, is a kind of bifurcation originally discovered by Fuller. This answers a natural
question that existed since the time of Fuller’s foundational papers. We also put strong constraints
on the kind of sky-catastrophes that may appear for homotopies of Reeb vector fields.

1. Introduction

The original Seifert conjecture [13] asked if a non-singular vector field on $S^3$ must have a periodic
orbit. In this formulation the answer was shown to be no for $C^1$ vector fields by Schweitzer [12], for $C^2$
vector fields by Harrison [5] and later for $C^\infty$ vector fields by Kuperberg [10]. A $C^1$ volume preserving
counter-example is given by Kuperberg in [9]. The Hamiltonian analogue was given a counterexample
in Ginzburg and Gürel [4]. For a vector field $X$ $C^0$ close to the Hopf vector field it was shown to hold
by Seifert and later by Fuller [3] in his 1967 paper, using his Fuller index. Part of the importance
of the $C^0$ condition for Fuller is that it rules out “sky catastrophes” for an appropriate homotopy of
non-singular vector fields connecting $X$ to the Hopf vector field. The latter “sky catastrophes” are the
last discovered kind of bifurcations originally constructed by Fuller himself [2]. He constructs a smooth
family $\{X_t\}, t \in [0, 1]$ of vector fields on a solid torus, for which there is a continuous (and isolated)
family of $\{X_t\}$ periodic orbits $\{o_t\}$, with the period of $o_t$ going to infinity as $t \to 1$, and so that for
$t = 1$ the orbit disappears. We can make the following slightly more general preliminary definition.

Definition 1.1 (Preliminary). A sky catastrophe for a smooth family $\{X_t\}, t \in [0, 1]$, of vector
fields on a manifold $M$ is a continuous family of closed orbits $\tau \mapsto o_\tau$, $o_\tau$ is a non-constant periodic
orbit of $X_\tau$, $\tau \in [0, \infty)$, such that the period of $o_\tau$ unbounded from above.

These sky catastrophes (and their more robust analogues called blue sky catastrophes) turned out
to be common in many kinds of systems appearing in nature and have been studied on their own, see
for instance Shilnikov-Turaev [14].

However since the time of Fuller’s original papers it has not been understood if this the only thing
that can go wrong. That is if without existence of a “sky catastrophe” in an appropriate general sense,
the time 1 limit of a homotopy of smooth non-singular vector fields on $S^{2n+1}$ starting at the Hopf
vector field must have a periodic orbit. The difficulty in answering this is that although our orbits
cannot “disappear into the sky”, as there are infinitely many of them, they may “cancel each other
out” even if the Fuller index is “locally positive” - that is the index of isolated components in the
orbit space is positive. In the $C^0$ nearby case this cancellation is prevented as orbits from isolated
components of the orbit space may not interact. The reader may think of trying to make sense of the
infinite sum

\[(5 - 1) + (5 - 1) + \ldots + (5 - 1) + \ldots.\]

While generally meaningless it has some meaning if we are not allowed to move the terms out of the
parentheses. So one has to develop a version of Fuller’s index which precludes such total cancellation

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in general. This is what we do here and using this answer affirmatively the above question, and for more general kinds of manifolds and vector fields.

First we define our general notion of a “sky catastrophe”. Given a homotopy of smooth vector fields \( \{X_t\} \) on \( M \) we define the space of non-constant periodic orbits of \( \{X_t\} \):

\[
S = S(\{X_t\}) = \{(o, p, t) \in LM \times (0, \infty) \times [0, 1] \mid o : \mathbb{R}/\mathbb{Z} \to M \text{ is a non-constant periodic orbit of } \frac{1}{p}X_t \}.
\]

Here \( LM \) denotes the free loop space. We have an embedding

\[
\text{emb} : S \to M \times (0, \infty) \times [0, 1]
\]
given by \((o, p, t) \mapsto (o(0), p, t)\) and \( S \) is given the corresponding subspace topology. Further on the same kind of topology will be used on related spaces. (It is the same as the induced topology from compact-open or Frechet topology on \( LM \).)

**Definition 1.3.** Suppose that \( \{X_t\}, t \in [0, 1] \) is a smooth family of vector fields on \( M \). We shall say that \( \{X_t\} \) has a **sky catastrophe**, if there is an element

\[
y \in S \cap (LM \times (0, \infty) \times \{0\}),
\]

so that there is no open-closed and bounded from above subset of \( S \) containing \( y \), where bounded from above means that the projection to the component \((0, \infty)\) is bounded from above.

Then clearly Fuller’s catastrophe is a special case of the above definition.

**Remark 1.4.** The above general definition coincides with the preliminary definition if \( M \) is compact and the connected components of \( S(\{X_t\}) \) are open and path-connected, which likely happens generically.

We point out that when \( M \) is compact a subset of \( S \) is compact if and only if it is closed and its projection to \((0, \infty)\) has a non-zero bound from below. This of course is immediate by existence of the topological embedding \( \text{emb} \) above. Thus when \( \{X_t\} \) is a homotopy of non-singular vector fields, and \( M \) is compact, we may replace the open-closed and bounded from above condition by the more technically useful open and compact condition.

**Theorem 1.5.** Let \( X = X_1 \) be a smooth non-singular vector field on \( S^{2k+1} \) homotopic to the Hopf vector field \( H = X_0 \) through homotopy \( \{X_t\} \) of smooth non-singular vector fields. Suppose that \( \{X_t\} \) has no sky catastrophes then \( X \) has periodic orbits.

Let us call a homotopy \( \{X_t\} \) satisfying the conditions of the theorem above **partially admissible**.

**Lemma 1.6.** There exists a \( \delta > 0 \) so that whenever \( X \) is \( C^0 \) \( \delta \)-close to the Hopf vector field \( H \), \( X = X_1 \) for a partially admissible homotopy \( \{X_t\} \), with \( X_0 = H \). And in particular by the above \( X \) has periodic orbits.

Thus Theorem 1.5 may be understood as an extensive generalization of the theorem of Seifert giving existence of periodic orbits for non-singular vector fields \( C^0 \) close to the Hopf vector field, on which the Seifert conjecture was based.

It is interesting to consider the contrapositive of the theorem above, and which may be understood as a new phenomenon and one concrete application of our theory.

**Corollary 1.7.** Given any homotopy \( \{X_t\} \) of smooth non-singular vector fields from the Hopf vector field to a vector field with no periodic orbits, \( \{X_t\} \) has a sky catastrophe.

Note that by the main construction in Wilson [16], the Hopf vector field \( H \) is homotopic through smooth non-singular vector fields to a vector field with finitely many simple closed orbits. Combining this with the construction in [10] we find that there do exist homotopies of \( H \) through smooth non-singular vector fields to a vector field with no closed orbits.

More general and extended forms of the above theorem are stated in Section 3. To prove them we give a certain natural extension of the classical Fuller index, with the latter giving certain invariant
rational counts of periodic orbits of a smooth vector field in “dynamically isolated compact sets”. Our extension is \( \mathbb{Q} \cup \{ \pm \infty \} \) valued. One ingredient for this is a notion of perturbation system for a vector field, which will allow us to consider weighted in terms of index and multiplicity “infinite sums” of closed orbits of a vector field. For these sums to have any meaning we impose certain “positivity” or “negativity” conditions.

**Remark 1.8.** These kind of summations are used in “positive topological quantum field theories” where infinite sums which are normally meaningless with coefficients in a ring like \( \mathbb{C} \) are made meaningful by working with a complete semi-ring in the sense of Samuel Eilenberg. But what we do here is of course much more basic in principle.

Can we use Theorem 1.5 and its analogues for more general manifolds in Section 3 to show existence of orbits? Let us assume some minimal regularity on the homotopy \( \{X_t\} \) so that connected components of \( S(\{X_t\}) \) are open. Then ideally we would like to have some a priori upper bounds for the period on connected components coming from some geometry-topology of the manifold and or the vector field. This then means that connected components of \( S(\{X_t\}) \) would be open and compact, and we can apply our theorems to get existence results. Fuller himself in [3] gives an example of such bounds for vector fields on the 2-torus, in fact his bounds are absolute for the whole \( S \) (for a fixed homotopy class of orbits) not just its connected components. He also speculates that if one works with divergence free vector fields one can do more to this effect. Below we examine the case of Reeb vector fields.

1.1. **Reeb vector fields and sky catastrophes.** Viterbo [15] shows that Reeb vector fields for the standard contact structure on \( S^{2k+1} \) always have periodic orbits. In the case of \( S^3 \) for an over-twisted contact structure this is shown by Hofer [6], using pseudo-holomorphic curve techniques. In the context of the Seifert conjecture, it is interesting to understand the most basic properties of qualitative-dynamical, or even just topological character, that Reeb vector fields posses, which makes them different from general non-singular vector fields. Indeed as a first step, in light of our results, we may ask if a homotopy of Reeb vector fields \( \{X_t\} \) on a closed manifold is necessarily free of sky catastrophes. The following theorem puts a very strong restriction on the kinds of sky catastrophes that can happen. It is likely, that if they exist, they must be pathological, and very hard to construct.

**Remark 1.9.** See however Kerman [8, Theorem 1.19], where a kind of partial Reeb plug is constructed, which is missing the matching condition, see for instance [10] for terminology of plugs, also see Kerman [7], Ginzburg and G"urel [4], where plugs are utilized in Hamiltonian context. If one had a plug with all conditions, then it is simple to construct a sky catastrophe. For we may deform such a plug through partial plugs satisfying all conditions except the trapping condition (condition 3 in [10]) to a trivial plug. This deformation then readily gives a sky catastrophe corresponding to the trapped orbit. Without matching, this argument does not obviously work.

The proof of the following uses only elementary geometry of the Reeb vector fields.

**Theorem 1.10.** Let \( \{X_t\}, t \in [0, 1] \) be a smooth homotopy through Reeb vector fields on a contact manifold \( M \). Let \( S = S(\{X_t\}) \) be defined as before, then there is no unbounded from above locally Lipschitz continuous \( p : [0, \infty) \to S \) whose composition with the projection \( \pi_3 : LM \times (0, \infty) \times [0, 1] \to [0, 1] \) has finite length.

The above rules out for example Fuller’s sky catastrophe that appears in [2], and described in the beginning of our paper.

**Conjecture 1.** Let \( \{X_t\}, t \in [0, 1] \) be a smooth homotopy through Reeb vector fields on a compact contact manifold \( M \). Then there is a \( C^0 \) nearby smooth family \( \{X_t\}, t \in [0, 1], X'_t = X_t, i = 0, 1, \) such that \( \{X'_t\} \) has no sky catastrophes.

Given this conjecture we may readily apply (general analogues of) Theorem 1.5 to get applications to existence of Reeb orbits.
2. Fuller index and its extension

The Fuller index is an analogue for orbits of the fixed point index, but with a couple of new ingredients: we must account for the symmetry groups of the orbits, and since the period is freely varying there is an extra compactness issue to deal with. Let us briefly recall the definition following Fuller’s original paper [3]. All vector vector fields from now on, everywhere in the paper, will be assumed to be smooth and non-singular and manifolds smooth and oriented (the last for simplicity).

Let $X$ be a vector field on $M$. Set

$$S(X) = S(X, \beta) = \{(o, p) \in L_\beta M \times (0, \infty) \mid o : \mathbb{R}/\mathbb{Z} \to M \text{ is a periodic orbit of } \frac{1}{p} X\},$$

where $L_\beta M$ denotes the free homotopy class $\beta$ component of the free loop space. Elements of $S(X)$ will be called orbits. There is a natural $S^1$ reparametrization action on $S(X)$, and elements of $S(X)/S^1$ will be called unparametrized orbits, or just orbits. Slightly abusing notation we write $(o, p)$ for the equivalence class of $(o, p)$. The multiplicity $m(o, p)$ of a periodic orbit is the ratio $p/l$ for $l > 0$ the least period of $o$. We want a kind of fixed point index which counts orbits $(o, p)$ with certain weights - however in general to get invariance we must have period bounds. This is due to potential existence of sky catastrophes as described in the introduction.

Let $N \subset S(X)$ be a compact open set (the open condition is the meaning in this case of “dynamically isolated” from before). Assume for simplicity that elements $(o, p) \in N$ are isolated. (Otherwise we need to perturb.) Then to such an $(N, X, \beta)$ Fuller associates an index:

$$i(N, X, \beta) = \sum_{(o, p) \in N/S^1} \frac{1}{m(o, p)} i(o, p),$$

where $i(o, p)$ is the fixed point index of the time $p$ return map of the flow of $X$ with respect to a local surface of section in $M$ transverse to the image of $o$. Fuller then shows that $i(N_t, X_t, \beta)$ is invariant for a deformation $\{X_t\}$ of $X$ if $N_t$ is dynamically isolated for all $t$, that is if $\bigcup_t N_t$ is open in $S(\{X_t\})$.

In the case where $X$ is the $R^\lambda$-Reeb vector field on a contact manifold $(C^{2n+1}, \xi)$, and if $(o, p)$ is non-degenerate, we have:

$$i(o, p) = \text{sign} \det(\text{Id} |_{\xi(x)} - F^\lambda_{p, *}|_{\xi(x)}) = (-1)^{CZ(o)} - n,$$

where $F^\lambda_{p,*}$ is the differential at $x$ of the time $p$ flow map of $R^\lambda$, and where $CZ(o)$ is the Conley-Zehnder index, (which is a special kind of Maslov index) see [11].

2.1. Extending Fuller index. We assume from now on, everywhere in the paper, that $M$ is compact, smooth, and oriented, although we sometimes reiterate for clarity. We will now describe an extension of the Fuller index allowing us to work with the entire Fuller phase space $LM \times \mathbb{R}_+$. We define an index $i(X, \beta)$ that depends only on $X, \beta$ but which is a priori defined and is invariant only for certain special vector fields and homotopies of vector fields. In fact we have already given the essence of the necessary condition on the homotopy in the special case of Theorem 1.5. Let

$$S(\{X_t\}, \beta) = \{(o, p, t) \in L_\beta M \times (0, \infty) \times [0, 1] \mid o : \mathbb{R}/\mathbb{Z} \to M \text{ is a periodic orbit of } \frac{1}{p} X_t\}.$$

**Definition 2.2.** For a smooth homotopy $\{X_t\}$ of smooth non-singular vector fields on $M$, we say that it is **partially admissible in free homotopy class** $\beta$, if every element of

$$S(\{X_t\}, \beta) \cap (L_\beta M \times (0, \infty) \times \{0\})$$

is contained in a compact open subset of $S(\{X_t\}, \beta)$. We say that $\{X_t\}$ is **admissible in free homotopy class** $\beta$ if every element of

$$S(\{X_t\}, \beta) \cap (L_\beta M \times (0, \infty) \times \partial[0, 1])$$

is contained in a compact open subset of $S(\{X_t\}, \beta)$. 
For $X$ a vector field, we set

$$S(X) = \{(o,p) \in LM \mid o : \mathbb{R}/\mathbb{Z} \to M \text{ is a periodic orbit of } \frac{1}{p}X\}.$$  

(2.3) $$S(X,\beta) = \{(o,p) \in L_\beta M \mid o : \mathbb{R}/\mathbb{Z} \to M \text{ is a periodic orbit of } \frac{1}{p}X\}.$$  

$$S(X,a,\beta) = \{(o,p) \in S(X,\beta) \mid p \leq a\}.$$  

**Definition 2.4.** Suppose that $S(X,\beta)$ has open connected components. And suppose that we have a collection of vector fields $\{X^a\}$, for each $a > 0$, satisfying the following:

- $S(X^a,a,\beta)$ consists of isolated orbits for each $a$.
- $S(X^a,a,\beta) = S(X^b,a,\beta)$, (equality of subsets of $L_\beta M \times \mathbb{R}_+$) if $b > a$, and the index and multiplicity of the orbits corresponding to the identified elements of these sets coincide.
- There is a prescribed homotopy $\{X^a\}$ of each $X^a$ to $X$, called structure homotopy, with the property that for every
  $$y \in S(\{X^a\}) \cap (L_\beta M \times (0,a] \times \partial[0,1])$$
  there is an open compact subset $C_y \ni y$ of $S(\{X^0\})$ which is non-branching which means that
  $$C_y \cap (L_\beta M \times (0,\infty) \times \{i\})$$
  for $i = 0,1$ are connected.
- $S(\{X^a\},\beta) \cap (L_\beta M \times (0,a] \times [0,1]) = S(\{X^b\},\beta) \cap (L_\beta M \times (0,a] \times [0,1])$, (equality of subsets of $L_\beta M \times \mathbb{R}_+ \times [0,1]$) if $b > a$ is sufficiently large.

We will then say that $\{X^a\}$ is a perturbation system for $X$ in the class $\beta$, (keeping track of structure homotopies and of $\beta$ implicitly).

We shall see shortly that a Morse-Bott Reeb vector field always admits a perturbation system.

**Definition 2.5.** Suppose that $X$ admits a perturbation system $\{X^a\}$ so that there exists an $E = E(\{X^a\})$ with the property that

$$S(X^a,a,\beta) = S(X^E,a,\beta)$$

for all $a > E$, where this as before is equality of subsets of $L_\beta M \times \mathbb{R}_+$, and the index and multiplicity of identified elements are also identified. Then we say that $X$ is finite type and set:

$$i(X,\beta) = \sum_{(o,p) \in S(X^E,\beta)/\mathbb{S}^1} \frac{1}{m(o,p)}i(o,p).$$

**Definition 2.6.** Otherwise, suppose that $X$ admits a perturbation system $\{X^a\}$ and there is an $E = E(\{X^a\}) > 0$ so that the index $i(o,p)$ of elements $(o,p) \in S(X^a,\beta)$ with $E \leq p \leq a$ is positive, respectively negative for every $a > E$, and s.t.

$$\lim_{a \to \infty} \sum_{(o,p) \in S(X,a,\beta)/\mathbb{S}^1} \frac{1}{m(o,p)}i(o,p) = \infty,$$

respectively

$$\lim_{a \to \infty} \sum_{o \in S(X,a,\beta)/\mathbb{S}^1} \frac{1}{m(o,p)}i(o,p) = -\infty.$$

Then we say that $X$ is positive infinite type, respectively negative infinite type and set $i(X,\beta) = \infty$, respectively $i(X,\beta) = -\infty$. We say it is infinite type if it is one or the other.

**Definition 2.7.** We say that $X$ is definite type if it is infinite type or finite type.

With the above definitions

$$i(X,\beta) \in \mathbb{Q} \cup \infty \cup -\infty,$$

when it is defined.
Remark 2.8. It is an elementary exercise that the condition that
\[
\lim_{a \to \infty} \sum_{o \in S(X^*, a, \beta)/S^1} \frac{1}{m(o)} i(o) = \infty, \text{ respectively } \lim_{a \to \infty} \sum_{o \in S(X^*, a, \beta)/S^1} \frac{1}{m(o)} i(o) = -\infty,
\]
actually follows if the other conditions are satisfied, so is only stated for emphasis.

Definition 2.9. A vector field \( X \) is admissible if it admits a perturbation system, and if it is definite type.

Remark 2.10. One may be tempted to extend the finite type to include the case when
\[
\lim_{a \to \infty} \sum_{o \in S(X^*, a, \beta)/S^1} \frac{1}{m(o)} i(o),
\]
exists and the associated series is absolutely convergent. This definitely works for our arguments later, but it is an elementary exercise to show that there are no such \( X \) unless it is of the previous finite type.

2.1.1. Perturbation systems for Morse-Bott Reeb vector fields.

Definition 2.11. A contact form \( \lambda \) on \( M \), and its associated flow \( R^\lambda \) are called Morse-Bott if the \( \lambda \) action spectrum \( \sigma(\lambda) \) - that is the space of critical values of \( o \mapsto \int_{S^1} o^* \lambda \), is discreet and if for every \( a \in \sigma(\lambda) \), the space
\[
N_a := \{ x \in M | F_a(x) = x \},
\]
\( F_a \) the time \( a \) flow map for \( R^\lambda \) - is a closed smooth manifold such that rank \( d\lambda|_{N_a} \) is locally constant and \( T_2 N_a = \ker(dF_a - I) \).

Proposition 2.12. Let \( \lambda \) be a contact form of Morse-Bott type, on a closed contact manifold \( C \). Then the corresponding Reeb vector field \( R^\lambda \) admits a perturbation system \( \{ X^\alpha \} \), for every class \( \beta \), with each \( X^\alpha \) Reeb so that all the structure homotopies are through Reeb vector fields.

The above very likely extends to more general “Morse-Bott type” and beyond vector fields of non Reeb type, however one must take care to give the right definitions.

Proof. Let
\[
O_{\leq E} = O_{\leq E}(R^\lambda) \simeq S(R^\lambda, E)
\]
denote the set of points \( x \in C \), s.t. \( F^\lambda_p(x) = x \), for \( F^\lambda_p \) the time \( p \leq E \) flow map for \( R^\lambda \). Given an \( a \) take an \( E > a \) s.t. the set \( O_{\leq E} = O_{\leq E}(R^\lambda) \) is a union of closed manifolds (of varying dimension), call such an \( E \) appropriate. Let \( O_{\leq E} \) be the natural \( S^1 \)-quotient of \( O_{\leq E} \). By [1, Section 2.2] we may find a smooth function \( f_E \) on \( C \) with support in a normal neighborhood of \( O_{\leq E} \), with \( Df_E(R^\lambda) = 0 \) on \( O_{\leq E} \) descending to a Morse function on the union of closed orbifolds \( O_{\leq E} \).

Let \( \lambda_{E, \mu} = (1 + \mu f_E)\lambda \). By [1, Section 2.2] we may choose \( \mu_0 > 0 \) so that elements of \( O_{\leq E}(R^{\lambda_{E, \mu}}) \) are non-degenerate and correspond to critical points of \( f_E \), for \( 0 < \mu \leq \mu(E) \). Let \( \{ E_n \} \) be an increasing sequence of appropriate levels as above. Since the action spectrum of \( \lambda \) is discreet by the Morse-Bott assumption, we may take \( \{ f_{E_n} \} \) so that \( f_{E_n} \mid_{E_n} \) coincides with \( f_{E_n} \) if \( E_n > E_n \). (Note however that the cutoff value \( \mu(E_n) \) needs to in general be smaller then \( \mu(E_n) \).) Given this, we set \( X^\alpha = R^{\lambda_{E_n, \mu(E_n)}} \), for any \( E_n > a \). For the structure homotopies we take the obvious homotopies induced by the homotopies
\[
t \mapsto (1 + (1 - t)\mu(E_n) f_{E_n})\lambda, \quad t \in [0, 1],
\]
of the contact forms. \[\square\]

Lemma 2.13. The Hopf vector field \( H \) on \( S^{2k+1} \) is infinite type.
Proof. Pick a perfect Morse function $f$ on $\mathbb{CP}^k$. This induces a perfect Morse function $f$ on $\mathcal{O}_{\leq 2\pi n}$, upon identifying $\mathcal{O}_{\leq 2\pi n}$ with the $n$-fold disjoint union of copies of $\mathbb{CP}^k$ (forgetting the totally non-effective orbifold structure). Use the construction above, to obtain a perturbation system \{H^{2n\pi}\} for $H$, $n \in \mathbb{Z}_{+}$, so that the space $\mathcal{O}_{\leq 2\pi n}^{pert} = \mathcal{O}_{\leq 2\pi n}(H^{2n\pi})$, be identified with critical points of $f$ on the space $\mathcal{O}_{\leq 2\pi n}$. Given a critical point $p$ of $f$ on the component $\mathcal{O}_{2\pi i} \simeq \mathbb{CP}^k$, $0 < i \leq 2\pi n$, of $\mathcal{O}_{\leq 2\pi n}$, let $o_p$ denote the corresponding orbit in $\mathcal{O}_{2\pi i}^{pert}$. By [1, Lemma 2.4],

$$\mu_{CZ}(o_p) = \mu_{CZ}(\mathcal{O}_{2\pi i}) - \frac{1}{2} \dim_{\mathbb{R}} \mathcal{O}_{2\pi i} + \text{morseindex}_f(p),$$

where $\mu_{CZ}(\mathcal{O}_{2\pi i})$ is the generalized Maslov index for an element of $\mathcal{O}_{2\pi i}$, see for instance [1, Section 5.2.2]. Let us slightly elaborate as our reader may not be familiar with this. We pick a representative for a class of an orbit $o$ in $\mathcal{O}_{2\pi i}$, pick a bounding disk for the orbit and using this choice trivialize the contact distribution along $o$. Given this trivialization the $R^k$ Reeb flow induces a path of symplectic matrices to which we apply the generalized Maslov index. Since $\mu_{CZ}(\mathcal{O}_{2\pi i})$ has even parity for all $k$, it follows that $\mu_{CZ}(o_p)$ has the same parity for all $n$, and so by (2.1) $H$ is even type. \qed

3. Extensions of theorem 1.5 and their proofs

**Theorem 3.1.** Suppose we have an admissible vector field $X_0$, with $i(X_0, \beta) \neq 0$ on a closed, oriented manifold $M$, which is joined to $X_1$ by a partially admissible homotopy $\{X_t\}$, then $X_1$ has periodic orbits.

Theorem 1.5 clearly follows by the above and by Lemma 2.13. What follows is a more precise result.

**Theorem 3.2.** If $M$ is closed, oriented and $X_0, X_1$ and $\{X_t\}$ are admissible then $i(X_0, \beta) = i(X_1, \beta)$.

3.1. Fuller correspondence. We need a beautiful construction of Fuller [3], which converts contractible orbits of $X$ into non-contractible orbits in an associated space, for an associated vector field, so that the correspondence between the periodic orbits is particularly suitable. Let $M'$ be the subset of the $k$-fold product $M \times \ldots \times M$, for $k$ a prime, consisting of points all of whose coordinates $(x_1, \ldots, x_k)$ are distinct. Let $M$ be the quotient of $M'$ by the permutation action of $\mathbb{Z}_k$, generated by $P(x_1, x_2, \ldots, x_k) = (x_2, \ldots, x_k, x_1)$. As this is a free action the projection map $M' \to M$ is a regular $k$-sheeted covering. And so we have a homomorphism

$$\mu : \pi_1(M) \to \mathbb{Z}_k,$$

which extends to $H_1(M)$ since $\mathbb{Z}_k$ is abelian.

A vector field $X$ on $M$ determines a vector field $\mathbf{X}$ on $M$ given by

$$\mathbf{X}(x) = [(X(x_1), \ldots, X(x_k)],$$

$x = [(x_1, \ldots, x_k)]$. It is easy to see $\mathbf{X}$ is complete when $X$ is complete, which holds in our case by compactness of $M$. Now for an orbit $(o, p) \in S(X, \beta)$ with multiplicity $m < k$ define a multiplicity $m$ orbit $(o, \frac{P}{k}) \in S(\mathbf{X})$.

$$o(t) = [o(t/k), o(t/k + 1/k), \ldots, o(t/k + (k - 1)/k)] \in LM.$$  

Clearly $\mu([o]) = 1$, (1 corresponding to the generator $P$ of the permutation group $\mathbb{Z}_k$, with $P$ as above) and $i(o) = i(o)$ by Fuller [3, Lemma 4.5]. If we work over all classes $\beta$ then it is easy to see that

$$Ful_k : (o, p) \mapsto (o, \frac{P}{k}),$$

is a bijection from the set of all (unparametrized) period $p$ orbits of $X$ with multiplicity less than $k$ to the set of all (unparametrized) period $\frac{p}{k}$ orbits of $\mathbf{X}$ with multiplicity less than $k$.

**Lemma 3.5.** Let $\{X_t\}$ be usual and let $m(\{X_t\}, a)$ denote the least upper bound for the set of multiplicities of elements of

$$S(\{X_t\}, \beta) \cap (L_\beta M \times (0, a] \times [0, 1])$$

then $m(\{X_t\}, a) < \infty$. 

Proof. This is a version of [3, Lemma 4.2]. The proof is as follows. Since
\[ S(\{X_t\}, \beta) \cap (L_\beta M \times (0, a] \times [0, 1]) \]
is compact as it is identified under the embedding \( emb : S(\{X_t\}, \beta) \rightarrow M \times (0, \infty) \times [0, 1] \), \((a, p, t) \mapsto (a(0), p, t)\) with a closed bounded subset of a finite dimensional manifold, we would otherwise have a convergent sequence \( \{t_k\} \) in \( S(\{X_t\}, \beta) \) with \( \{t_k\} \) also convergent, and so that \( p_k \) converges to 0. But this contradicts the assumption that \( X_t \) are non-singular. \( \square \)

We set
\[ (3.6) \quad S(X, a, \beta) = \text{Fal}_k(S(X, a, \beta)), \]
for \( k > m(X, a) \).

3.2. Preliminaries on admissible homotopies.

Definition 3.7. Let \( \{X_t\} \) be a smooth homotopy of non-singular vector fields. For \( b > a > 0 \) we say that \( \{X_t\} \) is partially \( a, b \)-admissible, respectively \( a, b \)-admissible (in class \( \beta \)) if for each
\[ y \in (S = S(\{X_t\}, \beta)) \cap (L_\beta M \times (0, a) \times \{0\}), \]
there is a compact open subset \( C_y \ni y \) of \( S \) contained in \( M \times (0, b) \times [0, 1] \). Respectively, if for each
\[ y \in S \cap (L_\beta M \times (0, a) \times \partial[0, 1]), \]
there is a compact open subset \( C_y \ni y \) of \( S \) contained in \( M \times (0, b) \times [0, 1] \).

Lemma 3.8. Suppose that \( \{X_t\} \) is partially admissible, then for every \( a \) there is a \( b > a \) so that
\[ \{\tilde{X}^b_t\} = \{X_t\} \cdot \{X^b_t\} \text{ is partially } a, b \text{-admissible}, \]
where \( \{X_t\} \cdot \{X^b_t\} \) is the (reparametrized to have \( t \) domain \([0, 1]\)) concatenation of the homotopies \( \{X_t\}, \{X^b_t\} \), and where \( \{X^b_t\} \) is the structure homotopy from \( X^b \) to \( X_0 \).

Proof. More explicitly
\[ (3.9) \quad \{\tilde{X}^b_t\} = \{X^b_{2t}\} \text{ for } t \in [0, 1/2] \]
\[ \{\tilde{X}^b_t\} = \{X^b_{2(t-1/2)}\} \text{ for } t \in [1/2, 1]. \]

Let \( y \in S(\{X^b_t\}) \cap (L_\beta M \times (0, a) \times \{0\}) \). Let \( C'_y \) be a non-branching open compact subset of \( S(\{X^b_t\}) \) containing \( y \). Let
\[ y' \in K_y = C'_y \cap (L_\beta M \times (0, \infty) \times \{1\}), \]
with \( K_y \) by assumptions connected and since it is an open and closed subset of
\[ S(\{X^b_t\}) \cap (L_\beta M \times (0, \infty) \times \{1\}), \]
\( \pi(K_y) \) coincides with one of the connected components of \( S(X_0) \), for
\[ \pi : L_\beta M \times (0, \infty) \times [0, 1] \rightarrow L_\beta M \times (0, \infty) \]
the projection.

Let \( C_{y'} \) be a compact open subset of \( S(\{X_t\}) \) containing \( \pi(y') \times \{0\} \in L_\beta M \times (0, \infty) \times [0, 1] \), \( C_{y'} \) exists by partial admissibility assumption on \( \{X_t\} \). Let
\[ M_y = C_{y'} \cap (L_\beta M \times (0, \infty) \times \{0\}) \]
then \( M_y \) must contain \( \pi(K_y) \times \{0\} \) as these sets are open and closed in \( S(\{X_t\}) \cap (L_\beta M \times (0, \infty) \times \{0\}) \) and \( \pi(K_y) \times \{0\} \) is connected. And \( M_y - \pi(K_y) \times \{0\} \) is a finite union (possibly empty) of compact open connected components \( \{W^y_k\} \), by the assumption that connected components of \( S(X) \) are open, and since \( M_y \) is compact.

Let
\[ S_a = \bigcup_y C_{y'} \]
for \( y, y' \) as above. Then since there are only finitely many such \( y \) \( S_a \) is compact and so is contained in \( L_\beta M \times (0, b') \times [0, 1] \), for some \( b' > a \) and sufficiently large as in the last axiom for a perturbation system.

For each \( W_y \) as above let \( C_y \) be a non-branching open compact subset of \( S(\{X^b_t\}) \) intersecting \( \pi(W_y) \times \{1\} \), and hence so that
\[
C_y \cap (L_\beta M \times (0, \infty) \times \{1\}) = \pi(W_y) \times \{1\}.
\]
This equality again follows by \( C_y \cap (L_\beta M \times (0, \infty) \times \{1\}) \) and \( \pi(W_y) \times \{1\} \) being open and connected subsets of \( S(\{X^b_t\}) \cap (L_\beta M \times (0, \infty) \times \{1\}) \).

Let
\[
T_y = \bigcup_j C_y \cup C_y \cup S_a,
\]
where this union is taken in
\[
(3.10) \quad U = U_- \cup U_+/\sim,
\]
where \( U_\pm \) are two names for \( L_\beta M \times (0, \infty) \times [0, 1] \) and the equivalence relation the identification map of \( M \times (0, \infty) \times \{1\} \) in the first component with \( M \times (0, \infty) \times \{0\} \) in the second component. And here \( \bigcup_j C_y \cup C_y \) is understood as being a subset of \( U_- \) and \( S_a \) of \( U_+ \).

Let
\[
Q = [0, 1]_- \cup [0, 1]_+/\sim,
\]
be the quotient space, where \([0, 1]_\pm\) are two names for \([0, 1]\), and \( \sim \) identifying \( 1 \in [0, 1]_- \) with \( 0 \in [0, 1]_+ \). Let
\[
\phi : Q \to [0, 1],
\]
be the “linear” (linear, if one naturally identifies \( Q \) with \([0, 2]\)) homeomorphism with \( \phi(0) = 0 \), for \( 0 \in [0, 1]_- \), \( \phi(1) = 1/2 \) for \( 1 \in [0, 1]_- \), \( \phi(0) = 1/2 \), for \( 0 \in [0, 1]_+ \), and \( \phi(1) = 1 \), for \( 1 \in [0, 1]_+ \). Then by the above discussion \( T_y \) is a compact subset of \((3.10)\), and \( C_y = \tilde{\phi}(T_y) \) is a compact and open subset of \( S(\tilde{X}_y^b) \), containing \( y \), where
\[
\tilde{\phi} : U \to L_\beta M \times (0, \infty) \times [0, 1],
\]
is induced by \( \phi \). Again since there are only finitely many such \( y \)
\[
\bigcup_y C_y
\]
is contained in \( L_\beta M \times (0, b) \times [0, 1] \), for some \( b \) sufficiently large. So our assertion follows. \( \square \)

The analogue of Lemma 3.8 in the admissible case is the following:

**Lemma 3.11.** Suppose that \( X_0, X_1 \) and \( \{X_t\} \) are admissible, then for every \( a \) there is an \( b > a \) so that \( \{\tilde{X}^b_t\} = \{X^b_t\}^{-1} \cdot \{X_t\} \cdot \{X^b_{0, t}\} \) is \( a,b \)-admissible, where \( \{X^b_{0, t}\} \) are the structure homotopies from \( X^b_t \) to \( X_t \).

The proof of this is completely analogous to the proof of Lemma 3.8.

### 3.3. Proof of Theorem 3.1

Suppose that \( X_0 \) is admissible with \( i(X_0, \beta) \neq 0 \), \( \{X_t\} \) is partially admissible and \( X_t \) has no periodic orbits. Let \( a \) be given and \( b \) determined so that \( \{\tilde{X}^b_t\} \) is a partially \((a,b)\)-admissible homotopy. Set \( m = m(\tilde{X}^b_t, b) \). Take a prime \( k = k(a) > m \) and define \( M \) as above with respect to \( k \). As \( M \) is compact the flow \( \tilde{X}^b_t \) is complete for every \( t \), and consequently as previously observed the flow of \( X_t \) is complete for every \( t \).

Let \( F_{t,p} \) denote the time \( p \) flow map of \( X_t \). Define
\[
F : M \times (0, b] \times [0, 1] \to M \times M,
\]
by
\[
F(x, p, t) = (F_{t,p}(x), x).
\]
Set
\[ (3.12) \quad \vec{S} = F^{-1}(\Delta), \]
for \( \Delta \) the diagonal. And set
\[ S = \text{emb} \circ \text{Ful}_k(S(\{X^r\}, \beta)) \subset M \times (0, \infty) \times [0, 1], \]
for
\[ (3.13) \quad \text{emb} : \text{Ful}_k(S(\{X^r\}, \beta)) \rightarrow M \times (0, \infty) \times [0, 1], \]
the map \((o, p, t) \mapsto (o(0), p, t)\). Let
\[ S^i = (S \cap M \times \mathbb{R}_+ \times \{i\}). \]

By assumptions \( S^1 \) is empty.

Let \( \{X_t\} \) be partially \( a, b \)-admissible and let
\[ (3.14) \quad S_{a,\beta,0} = \bigcup_{y \in S(\{X_t\}, \beta) \cap (M \times (0, a) \times \{0\})} C_y. \]
Here \( C_y \) is as in the Definition 3.7 with respect to \( a, b \). Then \( S_{a,\beta,0} \) is an open compact subset of \( S(\{X_t\}, \beta) \cap (L_\beta M \times (0, b) \times [0, 1]) \). Define
\[ S_{a,0} = S_{a,\beta,0} = \text{emb} \circ \text{Ful}_k(S_{a,\beta,0}). \]

A given orientation on \( M \) orients \( M, M \times M \), and the diagonal \( \Delta \subset M \times M \). Let \( \{\nabla_r\} \) be a sequence of forms \( C^\infty \) dual to the diagonal \( \Delta \), with support of \( \nabla_r \) converging to the diagonal as \( r \rightarrow \infty \), uniformly on compact sets. \( \{\nabla_r\} \) are characterized by the condition that \( C \cdot \Delta = \int_C f^* \nabla_r \), where \( f : C \rightarrow M \times M \) is a chain whose boundary is disjoint from \( \Delta \), and from the support of \( \nabla_r \) and where \( C \cdot \Delta \) is the intersection number.

**Lemma 3.15.** We may choose an \( r \) so that \( F^* \nabla_r \) breaks up as the sum:
\[ \gamma^r + \sum_y \alpha^r_y, \]
for \( y \) as in (3.14), where each \( \alpha^r_y \) has compact support in open sets \( U_y \),
\[ M \times (0, \frac{b}{R}) \times [0, 1] \supset U_y \supset \text{emb} \circ \text{Ful}_k(C_y), \]
and where \( \gamma^r \) has support which does not intersect any of the \( U_y \).

**Proof.** Note that \( S_{a,0} \) is an open and compact subset of \( \vec{S} \cap (M \times (0, \frac{b}{R}) \times [0, 1]) \) by construction, and by the fact that the maps (3.13) are open. (The latter, by the definition of the topology on these spaces as discussed following (1.2)). Likewise
\[ \vec{S} - S_{a,0} \]
is compact as \( \vec{S} \) is compact, and \( S_{a,0} \) is open.

Let \( \text{sup} \) denote the support of \( F^* \nabla_r \). Then for \( r \) sufficiently large \( \text{sup} \) is contained in an \( \epsilon \)-neighborhood of \( \vec{S} \), for \( \epsilon \) arbitrarily small (just by compactness). Since \( S_{a,0} \) and \( \vec{S} - S_{a,0} \) are both compact and disjoint they have disjoint metric \( \epsilon \)-neighborhoods for \( \epsilon \) sufficiently small. The lemma then clearly follows.

As \( \gamma^r + \sum_y \alpha^r_y \) is closed, \( \omega^r = \sum_y \alpha^r_y \) is closed and has compact support. Let \( \omega_0^r, \omega_1^r \) be the restrictions of \( \omega^r \) to \( M \times \mathbb{R}_+ \times \{0\} \), respectively \( M \times \mathbb{R}_+ \times \{1\} \), with \( \omega_1^r \) by assumption identically vanishing, for \( r \) sufficiently large.

Let \( S(X^b, b, \beta)/S^1 \), be as in (3.6), whose elements by slight abuse of notation we denote by \( o = (o, p) \). Let \( [(\omega_0^r)] \) denote the Poincaré dual class of \( \omega_0^r \) and given \( o \) as above we denote by \( [o] \) the class of the 1-cycle in \( M \times \mathbb{R}_+ \) represented by the (strictly speaking \( S^1 \)-equivalence class of) map \( t \mapsto (o(t), p) \), \( t \in [0, p]/0 \sim p \). If \( r \) is taken to be large, then by construction and since all elements of \( S(X^b, b, \beta)/S^1 \),
are isolated, the support of $\omega_0^i$ breaks up as the disjoint union of sets contained in $\epsilon_r$-neighborhoods of the images of the orbits $o \in S(X^b, b, \beta)/S^1$, for $\epsilon_r \to 0$ as $r \to \infty$. We shall say for short that the support is localized at these images. So we may write $\omega_0^i = \sum_l \omega_{0,l}$, with $\omega_{0,l}$ having support localized at the image of $o_0^i$, where $\{o_0^i\}$ is the enumeration of $S(X^b, b, \beta)/S^1$.

By (proof of) [3, Theorem 1] (for $r$ sufficiently large) the Poincare dual class $[\omega_{0,l}^i]$ is given by the class

$$i(o_0^i) \frac{1}{\text{mult}(o_0^i)} [o_0^i],$$

and by Fuller’s correspondence $i(o_0^i) = i(o_0^l)$, and $\text{mult}(o_0^i) = \text{mult}(o_0^l)$, where $o_0^l$ is the corresponding element of $S(X^b, b, \beta)$. So

$$[(\omega_0^i)^*] = c_a + \sum_{o^0 \in S(X^a, a, \beta)/S^1} i(o^0) \frac{1}{\text{mult}(o^0)} [o^0],$$

where

$$c_a = \sum_{o \in Q_a} i(o) \frac{1}{\text{mult}(o)} [o],$$

for

$$Q_a \subset S(X^b, b, \beta)/S^1 - S(X^a, a, \beta)/S^1$$

(possibly empty). Thus

$$c_a + \sum_{o^0 \in S(X^a, a, \beta)/S^1} i(o^0) \frac{1}{\text{mult}(o^0)} [o^0] = 0,$$

as for $r$ sufficiently large $\omega_0^a, \omega_1^l$ are cohomologous with compact support in $M \times (0, \frac{b}{a}) \times [0, 1]$ and as $\omega_1^l$ is identically vanishing. Applying $\mu$ we get

$$(3.16) \quad 1 (\sum_{o \in Q_a} i(o) \frac{1}{\text{mult}(o)} + \sum_{o^0 \in S(X^a, a, \beta)/S^1} i(o^0) \frac{1}{\text{mult}(o^0)}) = 0 \mod k,$$

where $l$ is the least common denominator for all the fractions, and this holds for all $a, k = k(a)$ (going higher in the perturbation system and adjusting the least common denominator).

3.4. Case I, $X_0$ is finite type. Let $E = E(\{X^a\})$ be the corresponding cutoff value in the definition of finite type, and take any $a > E$. Then $Q_a = \emptyset$ and

$$\sum_{o \in Q_a} i(o) \frac{1}{\text{mult}(o)} + \sum_{o^0 \in S(X^a, a, \beta)/S^1} i(o^0) \frac{1}{\text{mult}(o^0)} = i(X_0, \beta) \neq 0.$$

Clearly this gives a contradiction to (3.16).

3.5. Case II, $X_0$ is infinite type. We may assume that $i(X_0, \beta) = \infty$, and take $a > E$, where $E = E(\{X^a\})$ is the corresponding cutoff value in the definition of infinite type. Then

$$\sum_{o \in Q_a} i(o) \frac{1}{\text{mult}(o)} \geq 0,$$

as $a > E(\{X^0\})$. While

$$\lim_{a \to \infty} \sum_{o^0 \in S(X^a, a, \beta)/S^1} i(o^0) \frac{1}{\text{mult}(o^0)} = \infty,$$

by $i(X_0, \beta) = \infty$. This also contradicts (3.16).

□
3.6. **Proof of Theorem 3.2.** The proof is very similar to the proof of Theorem 3.1, and all the same notation is used. Suppose that $X_0$, $X_1$, and $\{X_i\}$ are admissible. Let $a$ be given and $b$ determined so that $\{\tilde{X}^b_i\}$ is a $(a,b)$-admissible homotopy. Let $F$, $\tilde{S}$, $S$ and $S^1$, be as before.

Let

$$S_{a,\beta} = S_{a,\beta}(F) = \bigcup_{y \in S_{b}(F) \cap (M \times (0, a) \times \partial(0, 1))} C_y,$$

where $C_y$ are as in the Definition 3.7. Then $S_{a,\beta,0}$ is an open compact subset of $S(\{X_i\}, \beta) \cap (L_{\beta} M \times (0, b) \times [0, 1])$. Define

$$S_a = S_{a,\beta} = \text{emb} \circ \text{Ful}_k(S_{a,\beta}).$$

Let $\{\nabla_r\}$ be a sequence of forms $C^\infty$ dual to the diagonal $\Delta \subset M \times M$, as before.

**Lemma 3.17.** We may choose an $r$ so that $F^*\nabla_r$ breaks up as the sum:

$$\gamma^r + \sum_{y} \alpha^r_y,$$

for $y$ as in (3.14), where each $\alpha^r_y$ has compact support in open sets $U_y$,

$$M \times (0, \frac{b}{k}) \times [0, 1] \supset U_y \supset \text{emb} \circ \text{Ful}_k(C_y),$$

and where $\gamma^r$ has support which does not intersect any of the $U_y$.

**Proof.** Analogous to the proof of Lemma 3.15. \hfill \Box

As $\gamma^r + \sum_{y} \alpha^r_y$ is closed, $\omega^r = \sum_{y} \alpha^r_y$ is closed and has compact support. Let $\omega^0_0$, $\omega^1_1$ be the restrictions of $\omega^r$ to $M \times \mathbb{R}_+ \times \{0\}$, respectively $M \times \mathbb{R}_+ \times \{1\}$. Let $\omega^0_i$, $\omega^1_i$ be the restrictions of $\omega^r$ to $M \times \mathbb{R}_+ \times \{0\}$, respectively $M \times \mathbb{R}_+ \times \{1\}$. We may write $\omega^r_i = \sum_{i,l} \omega^r_{i,l}$, with $\omega^r_{i,l}$ having support localized at the image of $o^r_i$, where $\{o^r_i\}$ is the enumeration of $S(X^b_i, \beta, b)/S^1$.

By (proof of) [3, Theorem 1] (for $r$ sufficiently large) the Poincare dual class $[\omega^r_{i,l}]$ is given by the class

$$i(o^r_i) \frac{1}{\text{mult}(o^r_i)}[o^r_i],$$

and by Fuller’s correspondence $i(o^r_i) = i(o^r_i)$, and $\text{mult}(o^r_i) = \text{mult}(o^r_i)$, where $o^r_i$ is the corresponding element of $S(X^b_i, \beta, b)$.

So

$$[(\omega^0_0)^r] = c_a + \sum_{\rho^0 \in S(X^b_0, \alpha, \beta)/S^1} i(\rho^0) \frac{1}{\text{mult}(\rho^0)}[o^0],$$

where

$$c_a = \sum_{\rho \in Q_a} i(\rho) \frac{1}{\text{mult}(\rho)}[\rho],$$

for

$$Q_a \subset S(X^b_0, \beta, b)/S^1 - S(X^a_0, \alpha, \beta)/S^1$$

(possibly empty).

Likewise

$$[(\omega^1_1)^r] = c'_a + \sum_{\rho^1 \in S(X^b_1, \alpha, \beta)/S^1} i(\rho^1) \frac{1}{\text{mult}(\rho^1)}[o^1],$$

where

$$c'_a = \sum_{\rho \in Q'_a} i(\rho) \frac{1}{\text{mult}(\rho)}[\rho],$$

for

$$Q'_a \subset S(X^b_1, \beta, b)/S^1 - S(X^a_1, \alpha, \beta)/S^1$$
\[c_a + \sum_{\alpha \in Q_a} i(\alpha)^0 \frac{1}{\text{mult}(\alpha)}[\alpha^0] = c'_a + \sum_{\alpha \in E(X_0^a, a, \beta)} i(\alpha) \frac{1}{\text{mult}(\alpha)}[\alpha^1],\]

as \(\omega_0^a, \omega_1^a\) are cohomologous with compact support in \(M \times (0, \frac{1}{k}) \times [0, 1]\). Applying \(\mu\) we get

\[
(3.18) \quad l \left( \sum_{\alpha \in Q_a} i(\alpha) \frac{1}{\text{mult}(\alpha)} + \sum_{\alpha \in E(X_0^a, a, \beta) / S^1} i(\alpha)^0 \frac{1}{\text{mult}(\alpha)^0} \right)
\]

\[
= l \left( \sum_{\alpha \in Q_a'} i(\alpha) \frac{1}{\text{mult}(\alpha)} + \sum_{\alpha \in E(X_1^a, a, \beta) / S^1} i(\alpha) \frac{1}{\text{mult}(\alpha)} \right) \mod k
\]

where \(l\) is the least common denominator for all the fractions, and this holds for all \(a, k\) (changing \(l\) appropriately and going higher in the perturbation system).

Suppose by contradiction that \(i(X_0, \beta) \neq i(X_1, \beta)\).

**Case I, \(X_i\) are finite type.** Let \(E_i = E(\{X_i^a\})\) be the corresponding cutoff values, and take \(a > \max(E_0, E_1)\). Then \(Q_a = Q_a' = \emptyset\) and

\[
\sum_{\alpha \in Q_a} i(\alpha) \frac{1}{\text{mult}(\alpha)} + \sum_{\alpha \in E(X_0^a, a, \beta) / S^1} i(\alpha)^0 \frac{1}{\text{mult}(\alpha)^0} = i(X_0, \beta),
\]

and

\[
\sum_{\alpha \in Q_a'} i(\alpha) \frac{1}{\text{mult}(\alpha)} + \sum_{\alpha \in E(X_1^a, a, \beta) / S^1} i(\alpha) \frac{1}{\text{mult}(\alpha)} = i(X_1, \beta).
\]

This gives a contradiction to (3.18).

**Case II, \(X_i\) are infinite type.** Let \(E_i = E(\{X_i^a\})\) be the corresponding cutoff values, and take \(a > \max(E_0, E_1)\). Suppose in addition (WLOG) that \(i(X_0) = \infty, i(X_1) = -\infty\). Then

\[
\sum_{\alpha \in Q_a} i(\alpha) \frac{1}{\text{mult}(\alpha)} \geq 0,
\]

and

\[
\sum_{\alpha \in Q_a'} i(\alpha) \frac{1}{\text{mult}(\alpha)} \leq 0,
\]

while

\[
\lim_{a \to \infty} \sum_{\alpha \in E(X_0^a, a, \beta) / S^1} i(\alpha)^0 \frac{1}{\text{mult}(\alpha)^0} = \infty,
\]

and

\[
\lim_{a \to \infty} \sum_{\alpha \in E(X_1^a, a, \beta) / S^1} i(\alpha) \frac{1}{\text{mult}(\alpha)} = -\infty.
\]

Clearly this also gives contradiction.

**Case III, \(X_0\) is infinite type and \(X_1\) is finite type.** Let \(E_i = E(\{X_i^a\})\) be the corresponding cutoff values, and take \(a > \max(E_0, E_1)\). Suppose in addition that \(i(X_0) = \infty\). Then

\[
\sum_{\alpha \in Q_a} i(\alpha) \frac{1}{\text{mult}(\alpha)} \geq 0,
\]

while

\[
\lim_{a \to \infty} \sum_{\alpha \in E(X_0^a, a, \beta) / S^1} i(\alpha)^0 \frac{1}{\text{mult}(\alpha)^0} = \infty,
\]

and \(Q_a' = \emptyset\) so
\[
\sum_{\omega \in Q^i_n} \frac{i(\omega)}{\text{mult}(\omega)} + \sum_{\omega^i \in S(X^i_n, \alpha, \beta)/S^i} \frac{i(\omega^i)}{\text{mult}(\omega^i)} = i(X_1, \beta).
\]

Again this is a contradiction. \qed

4. Proof of Lemma 1.6

**Lemma 4.1.** Let \( X \) be a smooth a non-singular vector field on a manifold \( M \), and \( F_p \) denote the time \( p \) flow map of \( X \). Set

\[ S(X) = \{(x, p) \in M \times (0, \infty) \mid F_p(x, p) = x\}. \]

Suppose that \( S \) consists of compact isolated components \( \{S_i\} \), meaning that for every \( S_{i_0} \in \{S_i\} \) there exists an \( \kappa > 0 \) so that there is a neighborhood \( U_{i_0} \) of \( S_{i_0} \) in \( M \times (0, \infty) \), whose closure \( \overline{U_i} \) is compact and so that \( \overline{U_i} \cap S(X) = S_i \). Then for every \( \epsilon > 0 \) there exists an \( \delta \) so that whenever \( X_1 \) is a smooth vector field \( C^0 \) \( \delta \) close to \( X \), and \( p \in S(X_1) \) is contained in \( \overline{U_i} \), then \( p \) is in the \( \epsilon \)-neighborhood of \( S_i \).

**Proof.** Let

\[ F(X) : M \times (0, \infty) \to M \times M, \]

be the map

\[ F(X)(x, p) = (F_p(x), x), \]

then \( S(X) \) is the preimage of the diagonal \( \Delta \) by \( F(X) \). Let \( \epsilon \) be given, so that the \( \epsilon \)-neighborhood \( V_i \) of \( S_i \) in \( M \times (0, \infty) \) is contained in \( U_i \). Suppose otherwise by contradiction, then there exists a sequence \( \{X_i\} \) of vector fields \( C^0 \) converging to \( X \), and a sequence \( \{p_i\} \subset U_i - V_i, p_i \in S(X_i) \). We may then find a convergent subsequence \( \{p_{i_k}\} \to p \in \overline{U_i} - V_i \). But \( \{F(X_{i_k})\} \) is uniformly on \( \overline{U_i} \) convergent to \( F(X) \), so that \( F(X)(p) \in \Delta \) and so \( p \in S(X) \). But this is a contradiction to the hypothesis that \( \overline{U_i} \cap S(X) = S_i \). \qed

**Lemma 1.6** then readily follows. Let us leave out the details. \qed

5. Proof of Theorem 1.10

Suppose that \( \lambda_t = f_t \lambda, f_t > 0 \), and let \( X_t \) be the Reeb vector field for \( \lambda_t \). Let \( p : [0, \infty) \to S \) be a locally Lipschitz path, so that \( p_t = \pi_3 \circ p \) has finite length \( L \). This means that the metric derivative function \( \frac{d}{d\tau} p_3 \):

\[ \frac{d}{d\tau} p_3(\tau_0) = \lim_{h \to \tau_0} \frac{|p_3(\tau_0) - p_3(s)|}{|\tau_0 - s|}, \]

satisfies

\[ \int_0^\infty \frac{d}{d\tau} p_3(\tau) d\tau = L < \infty. \]

Note that \( \frac{d}{d\tau} p_3 \) is always locally Riemann integrable by the Lipschitz condition. From now on if we write expression of the form \( \frac{d}{dt} f(x_0) \) we shall mean the metric derivative evaluated at \( x_0 \).

Consequently, for any \( D > 0 \), we may reparametrize (with same domain) \( \rho = p|_{[0,D]} \) so that the reparametrized path (notationally unchanged) satisfies:

\[ \frac{d}{d\tau} p_3(\tau_0) \leq \frac{L}{D} \text{ for almost all } \tau_0, \quad \rho_3 = \pi_3 \circ \rho. \]

Set

\[ K = \max_{t \in [0,1], M} \frac{df_t}{dt} \cdot \max_{t, M} f_t. \]

Let \( \rho_t = \pi_t \circ \rho \), for \( \pi_t \) the projections of \( LM \times (0, \infty) \times [0,1] \) onto the \( i \)’th factor. Clearly our theorem follows by the following lemma.

**Lemma 5.1.** The metric length of the path \( \rho_2 = \pi_2 \circ \rho \) is bounded from above by \( \exp(L \cdot K) \) for any \( D \).
Proof. After reparametrization of $\rho$ to have domain in $[0,1]$ and keeping the same notation for the path, we have

$$\frac{d}{d\tau} \rho_{3}(\tau_{0})| \leq L, \text{ for almost all } \tau_{0}. \tag{5.2}$$

For every $\tau$ we have a loop $\gamma_{\tau}: [0, 1] \rightarrow M$ given by

$$\gamma_{\tau}(t) = F_{\rho_{2}(\tau)}(\rho_{1}(\tau)),\tag{5.3}$$

where the flow maps $F$ are defined as before. In other words $\gamma_{\tau}$ is the closed orbit of $\rho_{2} \cdot R^{\rho_{2}(\tau)}$ a Reeb orbit. So we get a locally Lipschitz path $\tilde{\rho}$ in $LM \times [0,1]$, $\tilde{\rho}(\tau) = (\gamma_{\tau}, \rho_{3}(\tau))$, where $LM$ is the free loop space of $M$ with uniform metric.

We have the smooth functional:

$$\Lambda: LM \times [0,1] \rightarrow \mathbb{R}, \quad \Lambda(\gamma, t) = \langle \lambda_{t}, \gamma \rangle,$$

where $\langle, \rangle$ denotes the integration pairing, and clearly $\Lambda(\tilde{\rho}(\tau)) = \rho_{2}(\tau)$.

We also have the restricted functionals $\lambda_{t}: LM \rightarrow \mathbb{R}, \quad \lambda_{t}(\gamma) = \langle \lambda_{t}, \gamma \rangle$, called the $\lambda_{t}$ functionals. Let $\xi(\tau) = \pi_{1, \tau} \frac{d}{d\tau} \tilde{\rho}(\tau)$, for $\pi_{i}$ the projections of $LM \times [0,1]$ onto the $i$th factor. The differential of $\Lambda$ at $(o, \tau_{0}) \in LM \times [0,1]$ is

$$D\Lambda(o, \tau_{0})(\xi, \frac{d}{d\tau}) = D\lambda_{t}(o)(\xi) + \frac{d}{dt} \bigg|_{\tau_{0}} \lambda_{t}(o),$$

where $\xi \in T_{o}LM$, and if $o$ is a $R^{\lambda_{\tau_{0}}}$ Reeb orbit the first term vanishes. Consequently,

$$|\frac{d}{d\tau} \Lambda \circ \tilde{\rho}(\tau_{0})| = |\frac{d}{d\tau} \lambda_{t}(o)(\pi_{1, \tau} \circ \tilde{\rho}(\tau_{0}))|. \tag{5.4}$$

On the other hand

$$|\frac{d}{d\tau} \lambda_{t}(o)(\pi_{1, \tau} \circ \tilde{\rho}(\tau_{0}))| \leq |\frac{d}{d\tau} \lambda_{t}(\tilde{\rho}(\tau_{0}))| \cdot |\frac{d}{d\tau} \rho_{3}(\tau_{0})|.$$

We have by direct calculation for any $\tau_{0} \in [0,1]$

$$|\frac{d}{dt} \lambda_{t}(\tilde{\rho}(\tau_{0}))| \leq \max_{\tau \in [0,1], M} \frac{|df_{t}|}{d\tau} \cdot \frac{|d \rho_{3}(\tau_{0})|}{d\tau}. \tag{5.5}$$

On the other hand

$$\int_{\tilde{\rho}(\tau_{0})} \lambda \leq \max_{M, t} f_{t} \cdot \Lambda(\tilde{\rho}(\tau_{0})).$$

Consequently:

$$\frac{d}{d\tau} \rho_{2}(\tau_{0}) = \frac{d}{d\tau} (\Lambda \circ \tilde{\rho}(\tau)) (\tau_{0}) \leq \max_{\tau \in [0,1], M} \frac{|df_{t}|}{d\tau} \cdot \max_{\tau, M} f_{t} \cdot \Lambda(\tilde{\rho}(\tau_{0})) \cdot L = \max_{\tau \in [0,1], M} \frac{|df_{t}|}{d\tau} \cdot \max_{\tau, M} f_{t} \cdot L \cdot \rho_{2}(\tau_{0}),$$

for almost all $\tau_{0} \in [0,1]$.

\[
\square
\]

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