Many-body wave scattering by small bodies

A.G. Ramm
Mathematics Department, Kansas State University,
Manhattan, KS 66506-2602, USA
ramm@math.ksu.edu,
fax 785-532-0546, tel. 785-532-0580
http://www.math.ksu.edu/~ramm

Abstract

Scattering problem by several bodies, small in comparison with the wavelength, is reduced to linear algebraic systems of equations, in contrast to the usual reduction to some integral equations.

1 Introduction

Acoustic or electromagnetic (EM) wave scattering by one or several bodies is usually studied by reducing the problem to solving some integral equations. In this paper we show that if the bodies are small in comparison with the wavelength, then the scattering problem can be reduced to solving linear algebraic systems with matrices whose elements have physical meaning. These elements are electrical capacitances or elements of electric and magnetic polarizability tensors. The author has derived analytical explicit formulas allowing one to calculate these quantities for bodies of arbitrary shapes with arbitrary desired accuracy (see [1]).

We derive these linear algebraic systems and give formulas for the elements of the matrices of these systems. There is a large literature on wave scattering by small bodies, see [1] and references therein. The theory was originated by Lord Rayleigh [3], who understood that the main term in the scattered field is the dipole radiation if the body is small. Rayleigh did not give formulas for calculating the induced dipole moments for small bodies of arbitrary shapes. The dipole moments are uniquely defined by the

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polarizability tensors. Therefore, the formulas, derived by the author (see [1]), allow one to calculate the dipole radiation for acoustic and EM wave scattering by small bodies of arbitrary shapes.

2 Acoustic wave scattering by small bodies

Let us start with acoustic wave scattering. Consider the problem

\[(\Delta + k^2)u = 0 \text{ in } \mathbb{R}^3 \setminus \bigcup_{m=1}^{M} D_m \]  

\[u \mid_{S_m} = 0, \quad 1 \leq m \leq M, \quad S_m := \partial D_m \]

\[u = u_0 + v, \]  

\[\frac{\partial v}{\partial r} - ikv = o \left( \frac{1}{r} \right), \quad r := |x| \to \infty, \]

where \(\Delta\) is the Laplacean, \(u_0\) is an incident field which solves equation (1). Often, \(u_0 = e^{ik\alpha \cdot x}\), where \(\alpha \in S^2\) is a given vector and \(S^2\) is the unit sphere.

Let us look for the solution of the form

\[u = u_0 + \sum_{m=1}^{M} \int_{S_m} g(x,s)\sigma_m(s)ds, \quad g(x,y) := \frac{e^{ik|x-y|}}{4\pi|x-y|}, \]

where \(\sigma_m, 1 \leq m \leq M\), are to be chosen so that the boundary conditions (2) hold. The function (5) satisfies (1) and (3)- (4) for any \(\sigma_m \in L^2(S_m)\). The scattering amplitude is:

\[A(\alpha', \alpha) = \lim_{|x| \to \infty, \frac{|x|}{|x'|} = \alpha'} |x|e^{-ik|x|} v = \sum_{m=1}^{M} \frac{1}{4\pi} \int_{S_m} e^{-ik\alpha' \cdot s} \sigma_m ds, \quad \alpha' := \frac{x}{|x|}. \]

Let

\[a := \max_{1 \leq m \leq M} \text{diam } D_m, \]

and

\[d := \min_{m \neq j} \text{dist}(D_m, D_j). \]

We assume

\[ka \ll 1, \quad a \ll d. \]

Then

\[e^{-ik\alpha' \cdot (s-x_m)} \approx 1 \quad \text{if} \quad x_m \in D, \]
\[ A(\alpha', \alpha) = \sum_{m=1}^{M} \frac{e^{-ik\alpha' \cdot x_m}}{4\pi} \int_{S_m} \sigma_m ds := \sum_{m=1}^{M} \frac{Q_m}{4\pi} e^{-ik\alpha' \cdot x_m}, \quad Q_m := \int_{S_m} \sigma_m ds, \quad (8) \]

where \( x_m \in D_m \) and \( \alpha' \) is defined in (6). Since \( D_m \) is small, it does not matter which point \( x_m \) one takes in \( D_m \). The \( Q_m \) plays the role of the total charge on the surface \( S_m \).

If \( \min_m |x - x_m| \gg a \) and \( x_m \in D_m \), then

\[ u(x) = u_0(x) + \sum_{m=1}^{M} \frac{e^{ik|x - x_m|}}{4\pi |x - x_m|} Q_m \left[ 1 + O(ka + \frac{a}{d}) \right]. \quad (9) \]

Let us derive a formula for \( Q_m \). Using the boundary condition (2), one gets:

\[ 0 = u_0(s_m) + \sum_{j \neq m} g(s_m, x_j)Q_j + \int_{S_m} g(s_m, s)\sigma_m(s)ds, \quad (10) \]

where \( s_m \in S_m \).

Since \( ka \ll 1 \), one has

\[ g(s_m, s) = g_0(s_m, s) + O(ka), \]

where

\[ g_0(s, t) := \frac{1}{4\pi |x - t|}. \]

Therefore equation (10) is the equation for the electrostatic charge distribution \( \sigma_m \) on the surface \( S_m \) of a perfect conductor \( D_m \), charged to the potential

\[ u_m := -u_0(s_m) - \sum_{j \neq m} g(s_m, x_j)Q_j. \]

The total charge on \( S_m \) is:

\[ Q_m = C_m U_m, \]

where \( C_m \) is the electrical capacitance of the conductor with the shape \( D_m \). The total charge is defined as:

\[ Q_m := \int_{S_m} \sigma_m ds. \]

Therefore, one gets:

\[ Q_m = C_m \left( -u_0(s_m) - \sum_{j \neq m} g(s_m, x_j)Q_j \right), \quad 1 \leq m \leq M, \quad (11) \]

where \( C_m \) is the electrical capacitance of the perfect conductor with the boundary \( S_m \).
Linear algebraic system (11) allows one to find $Q_j$, $1 \leq j \leq M$. If

$$\max_{1 \leq m \leq M} \sum_{j \neq m} \frac{C_m}{4\pi |s_m - x_j|} < 1,$$  \hspace{1cm} (12)$$

then the matrix of the system (11) has diagonally dominant elements and, consequently, can be solved by iterations.

The approximate solution to the many-body scattering problem (1) – (4) is given by formula (9), where $Q_m$ are determined from linear algebraic system (11).

Let us give a formula from [1] for the capacitance of a perfect conductor $D$ with the boundary $S$. Denote the area of $S$ by $|S|$. We assume that the conductor is placed in the medium with the dielectric permittivity $\varepsilon_0 = 1$. In this case the approximate formula for the capacitance is (see [1], p. 26):

$$C^{(n)} = 4\pi |S|^2 \left\{ \frac{-1}{2\pi} \int_S \int_S \frac{dsdt}{r_{st}} \int_S \cdots \int_S \psi(t, t_1) \cdots \psi(t_{n-1}, t_n) dt_1 \cdots dt_n \right\}^{-1},$$  \hspace{1cm} (13)$$

and the error estimate of formula (13) is:

$$|C^{(n)} - C| = O(q^n), \quad 0 < q < 1,$$  \hspace{1cm} (14)$$

where $q$ depends on the geometry of $S$, and $n = 1, 2, 3, \ldots$ is the approximation order.

If the boundary condition

$$u_N = \zeta u \quad \text{on} \quad S_m$$  \hspace{1cm} (15)$$

is imposed in place of the Dirichlet condition (2), and $\zeta$ is the impedance, then $C_m$ in (11) is replaced by

$$C_m \zeta := \frac{C_m}{1 + C_m (\zeta |S|)^{-1}},$$  \hspace{1cm} (16)$$

see [1], p. 97.

If

$$u_N |_{S_m} = 0, \quad 1 \leq m \leq M,$$  \hspace{1cm} (17)$$

then the formula for the solution to problem (1), (17), (3), (4), is

$$u(x) = u_0(x) + \sum_{m=1}^{M} q(x, x_m) V_m \left[ \Delta u(x_m) + \sum_{p,q=1}^{M} \beta_{pq,m} k \frac{\partial u(x_m)}{\partial x_{m,q}} \frac{(x - x_m)_p}{|x - x_m|} \right],$$  \hspace{1cm} (18)$$
where \((x - x_m)_p\) is the \(p\)-th coordinate of the vector \(x - x_m\), \(\frac{\partial}{\partial x_{m,q}}\) is the derivative with respect to the \(q\)-th coordinate of \(x\) calculated at the point \(x_m\), and \(\beta_{pq,m}\) is the magnetic polarizability tensor of \(D_m\), defined by the formula ([1], p.98):

\[
V_m \beta_{pq,m} = \int_S s_p \sigma(s) ds,
\]

where \(V_m\) is the volume of \(D_m\), the function \(\sigma\) solves the equation

\[
\sigma = A\sigma - 2N_q,
\]

\(N\) is the exterior unit normal to \(S_m\), and

\[
A\sigma = \int_{S_m} \frac{\partial}{\partial N} \frac{1}{2\pi r_{st}} \sigma(t) dt, \quad r_{st} = |s - t|.
\]

The formulas for the tensor \(\beta_{pq,m}\), analogous to the formulas (13)-(14) for the capacitance, are derived in [1, p.55, formula (5.15)]. The unknown quantities \(\Delta u(x_m)\) and \(\frac{\partial u(x_m)}{\partial x_q}\), \(1 \leq m \leq M, 1 \leq q \leq 3\), in (18) can be found from the following linear algebraic system, analogous to (11):

\[
\Delta u(x_m) = \Delta u_0(x_m) - k^2 \sum_{j \neq m,j=1}^{M} g(x_m, x_j) V_j [\Delta u(x_j) + \sum_{p,q=1}^{3} \beta_{pq,j} \frac{k}{|x_m - x_j|}] 
\]

\[
\frac{\partial u(x_m)}{\partial x_{m,q}} = \frac{\partial u_0(x_m)}{\partial x_{m,q}} + \sum_{j \neq m,j=1}^{M} \frac{\partial g(x_m, x_j)}{\partial x_{m,q}} V_j [\Delta u(x_j) + \sum_{p,q=1}^{3} \beta_{pq,j} \frac{k}{|x_m - x_j|}] 
\]

In (19) we have used the equation

\[
\Delta g(x, y) = -k^2 g(x, y),
\]

which holds if \(x \neq y\).

From the linear algebraic system (19) - (20) one finds the unknowns \(\Delta u(x_m)\) and \(\frac{\partial u(x_m)}{\partial x_{m,q}}\), \(1 \leq m \leq M, 1 \leq q \leq 3\).

If conditions (7) hold, then system (19) - (20) has a unique solution which can be obtained by iterations.

This completes the description of our method for solving many-body scattering problem for small bodies and acoustic (scalar) waves.
3 Electromagnetic wave scattering by small bodies

In the problem of electromagnetic (EM) wave scattering by many small bodies we assume

\[ a \ll \lambda \ll d. \]  

(21)

This assumption is more restrictive than \([7]\). The reason is: in EM theory the fields are obtained by an application of first order differential operators, for instance \(\nabla \times\), to potentials, such as the vector potential. Applying this operator and calculating the field in the far zone one neglects the term \(\frac{1}{x-x_m}\) compared with the term \(k\). This means that the following inequality is assumed:

\[ \frac{1}{d} \ll \frac{1}{\lambda}, \]

or

\[ d \gg \lambda. \]

In the acoustic wave theory the potential itself \(\int_S g(x, s)\sigma ds\) has physical meaning, it is the acoustic pressure, and this pressure is studied. Therefore, the condition \(d \gg \lambda\) does not appear.

Condition \([7]\) allows one to have many small particles on the distance of order \(\lambda\), while condition \([21]\), namely the inequality \(d \gg \lambda\), does not allow this. Recall that \(d\) is the minimal distance between two neighboring particles. The formula for the scattering amplitude, analogous to \([8]\), for EM wave scattering by small bodies is (see \([2]\)):

\[ A(\theta', \theta) = \frac{1}{4\pi} \sum_{m=1}^{M} S_m U_m e^{-ik\theta' \cdot x_m}. \]

(22)

Here \(U = (E, H)\) is a 6-component vector, \(S_m\) is a 6x6 matrix, the scattering matrix, \(\varepsilon_0\) and \(\mu_0\) are dielectrical and magnetic parameters of the medium, in which the body \(D_m\) is placed, and \(\theta, \theta'\) are the unit vectors in the direction of the incident and scattered waves, respectively. These vectors were denoted \(\alpha\) and \(\alpha'\) in Section \([2]\). We have changed the notations because in EM theory \(\alpha\) denotes the polarizability tensor.

The formula for \(S\) is (cf. \([2]\))

\[ S_m = \left( \begin{array}{c} E_m \\ H_m \end{array} \right) = \frac{k^2 V_m}{4\pi} \left( \begin{array}{cc} \alpha E - \theta'(\theta', \alpha E) & -\frac{\mu_0^{3/2}}{\varepsilon_0^{1/2}} [\theta', \beta H] \\ \left(\frac{\varepsilon_0}{\mu_0}\right)^{1/2} [\theta', \alpha E] & \mu_0 (\tilde{\beta} - \theta'(\theta', \tilde{\beta} H)) \end{array} \right). \]

(23)

Here \(V_m\) is the volume of \(D_m\), \(\alpha\) is the electric polarizability tensor of \(D_m\), \(\tilde{\beta}\) is the magnetic polarizability tensor of \(D_m\). In \([1]\) pp. 54–55] the author derives analytical formulas for calculation of the polarizability tensors \(\alpha\) and \(\beta\),

\[ \tilde{\beta} := \alpha(\tilde{\gamma}) + \beta, \quad \beta := \alpha_{ij} (-1), \quad \tilde{\gamma} := \frac{\mu - \mu_0}{\mu + \mu_0}, \quad \gamma := \frac{\varepsilon - \varepsilon_0}{\varepsilon + \varepsilon_0}. \]

(24)
Tensors $\beta$ and $\tilde{\beta}$ are expressed through the polarizability tensor $\alpha = \alpha(\gamma)$. One has $\gamma = -1$ if $\varepsilon = 0$. Here $[\cdot, \cdot]$ is the vector product, $(\cdot, \cdot)$ is the scalar product.

The analytic formula from [1], p. 54, formula (5.9), for the tensor $\alpha = \alpha_{ij}(\gamma)$, $1 \leq i, j \leq 3$, that we referred to above, is analogous to formulas (13)–(14) for the electrical capacitance. The incident direction $\theta$ enters via the vectors $E$ and $H$, which depend on $\theta$. These vectors are calculated in formula (23) at the point $x_m$. The values of these vectors are determined from a linear algebraic system of equations. This system is derived similarly to the derivation of the systems (11) and (19)–(20). We do not write down this system since it would take much space, but the ideas are the same as the ones used in the derivations of (11) and (19)–(20).

4 Conclusions

In this paper it is shown how to reduce rigorously the many-body scattering problem to linear algebraic system in the case when the bodies are small in comparison with the wavelength. The theory is constructed for acoustic and EM wave scattering. The basic physical assumptions are (7) for acoustic scattering, and (21) for EM scattering.

References

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