David’s Trick

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In David [82] a method is introduced for creating reals \( R \) which not only code classes in the sense of Jensen coding but in addition have the property that in \( L[R] \), \( R \) is the unique solution to a \( \Pi^1_2 \) formula. In this article we cast David’s “trick” in a general form and describe some of its uses.

**Theorem.** Suppose \( A \subseteq \text{ORD} \), \( \langle L[A], A \rangle \models \text{ZFC}+0\# \) does not exist and suppose that for every infinite cardinal \( \kappa \) of \( L[A] \), \( H^L[A]_\kappa = L_\kappa[A] \) and \( \langle L_\kappa[A], A \cap \kappa \rangle \models \varphi \). Then there exists a \( \Pi^1_2 \) formula \( \psi \) such that:

(a) If \( R \) is a real satisfying \( \psi \) then there is \( A \subseteq \text{ORD} \) as above, definable over \( L[R] \) in the parameter \( R \).

(b) For some tame, \( \langle L[A], A \rangle \)-definable, cofinality-preserving forcing \( P \), \( P \models \exists R \psi(R) \).

Moreover if \( A \) preserves indiscernibles then \( \psi \) has a solution in \( L[A,0\#] \), preserving indiscernibles.

**Remark**

(1) We require that \( H^L[A]_\kappa \) equal \( L_\kappa[A] \) for infinite \( L[A] \)-cardinals solely to permit cofinality-preservation for \( P \); if cofinality-preservation is dropped then such a requirement is unnecessary, by coding \( A \) into \( A^* \) with this requirement and then applying our result to \( A^* \).

(2) A class \( A \) preserves indiscernibles if the Silver indiscernibles are indiscernible for \( \langle L[A], A \rangle \). It follows from the technique of Theorem 0.2 of Beller-Jensen-Welch [82] (see Friedman [98]) that if \( A \) preserves indiscernibles then \( A \) is definable from a real \( R \in L[A,0\#] \), preserving indiscernibles.

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Proof. Our plan is to create an \( \langle L[A], A \rangle \)-definable, tame, cofinality-preserving forcing \( P \) for adding a real \( R \) such that whenever \( L_\alpha [R] \models ZF^- \) there is \( A_\alpha \subseteq \alpha \), definable over \( L_\alpha [R] \) (via a definition independent of \( \alpha \)) such that \( L_\alpha [R] \models \) for every infinite cardinal \( \kappa \), \( H_\kappa = L_\kappa [A_\kappa] \) and \( \varphi \) is true in \( \langle L_\kappa [A_\alpha], A_\alpha \cap \kappa \rangle \). This property \( \psi \) of \( R \) is \( \Pi^1_2 \) and gives us (a), (b) of the Theorem. The last statement of the Theorem will follow using Remark (2) above.

\( P \) is obtained as a modification of the forcing from Friedman [97], used to prove Jensen’s Coding Theorem (in the case where \( 0^\# \) does not exist in the ground model). The following definitions take place inside \( L[A] \).

**Definition (Strings).** Let \( \alpha \in \text{Card} = \text{the class of all infinite cardinals}. \) \( S_\alpha \) consists of all \( s : [\alpha, |s|) \to 2 \) such that \( |s| \) is a multiple of \( \alpha \) and:

(a) \( \eta \leq |s| \to L_\delta[A \cap \alpha, s \upharpoonright \eta] \models \text{Card} \eta \leq \alpha \text{ for some } \delta < (\eta^+)^L \cup \omega_2. \)

(b) If \( \mathcal{A} = \langle L_\beta[A \cap \alpha, s \upharpoonright \eta], s \upharpoonright \eta \rangle \models (ZF^- \text{ and } \eta = \alpha^+) \) then over \( \mathcal{A}, s \upharpoonright \eta \) codes a predicate \( A(s \upharpoonright \eta, \beta) = A^* \subseteq \beta \text{ such that } A^* \cap \alpha = A \cap \alpha \text{ and for every cardinal } \kappa \text{ of } L_\beta[A^*], H^L_\kappa[A^*] = L_\kappa[A^*] \text{ and } \langle L_\kappa[A^*], A^* \cap \kappa \rangle \models \varphi. \)

**Remark** When in (b) above we say that \( s \upharpoonright \eta \) codes \( A^* \) we are referring to the canonical coding from the proof of Theorem 4 of Friedman [97] of a subset of \( \beta \) by a subset of \( (\alpha^+)^A = \eta \) (relative to \( A \cap \alpha \)).

The remainder of the definitions from the proof of Theorem 4 of Friedman [97] remain the same in the present context. We now verify that he proofs of the lemmas from Friedman [97] can successfully accommodate the new restriction (clause (b)) on elements of \( S_\alpha \).

**Lemma 1 (Distributivity for \( R^* \)).** Suppose \( \alpha \in \text{Card}, s \in S_{\alpha^+}. \) Then \( R^* \) is \( \alpha^+ \)-distributive in \( A^* \).

**Proof.** Proceed as in the proof of Lemma 5 of Friedman [97]. The only new point is to verify that in the proof of the Claim, \( t_\lambda \) satisfies clause (b) (of the new definition of \( S_\alpha \)). The fact that \( s \) belongs to \( S_{\alpha^+} \) and that \( t_\lambda \) codes \( H_\lambda \) imply that clause (b) holds for \( t_\lambda \) whenever \( \beta \) is at most \( \mu_\lambda = \text{the height of } H_\lambda \). But as \( |t_\lambda| \) is definably singular over \( L_{\mu_\lambda}[t_\lambda] \) these are the only \( \beta \)'s that concern us. \( \Box \)
Lemma 2 (Extendibility of $P^s$). Suppose $p \in P^s$, $s \in S_\alpha$, $X \subseteq \alpha$, $X \in A^s$. Then there exists $q \leq p$ such that $X \cap \beta \in A^q$ for each $\beta \in \text{Card} \cap \alpha$.

Proof. Proceed as in the proof of Lemma 6 of Friedman [97]. In the definition of $q$, the only instances of clause (b) to check are for $s_\beta$ when Even $(Y \cap \beta)$ codes $s_\beta$, $s_\beta$ satisfying clause (a) of the definition of membership in $S_\beta$. But the embedding $\bar{A}_\beta \to A$ is $\Sigma_1$-elementary and instances of clause (b) refer to ordinals less than the height of $A$; so the fact that $s$ belongs to $S_\alpha$ implies that $s_\beta$ belongs to $S_\beta$. □

Lemma 3 (Distributivity for $P^s$). Suppose $s \in S_{\beta^+}$, $\beta \in \text{Card}$.

(a) If $\langle D_i \mid i < \beta \rangle \in A^s, D_i$ $i^+$ dense on $P^s$ for each $i < \beta$ and $p \in P^s$ then there is $q \leq p$, $q$ meets each $D_i$.

(b) If $p \in P^s$, $f$ small in $A^s$ then there exists $q \leq p$, $q \in \Sigma_f^p$.

Proof. Proceed as in the proof of Lemma 7 of Friedman [97]. In the Claim we must verify that $p^\lambda_\mu$ satisfies clause (b). But once again this is clear by the $\Sigma_1$-elementary of $\bar{H}_\lambda(\gamma)$ and the fact that $L_{\bar{\mu}}[A \cap \gamma, p^\lambda_\mu] \models |p^\lambda_\mu|$ is $\Sigma_1$-singular, where $\bar{\mu} =$ height of $\bar{H}_\lambda(\gamma)$.

The argument of the proof of Lemma 3 can also be applied to prove the distributivity of $P$, observing that when building sequences of conditions $\langle p^i \mid i < \lambda \rangle$, $\lambda$ limit to meet an $\langle L[A], A \rangle$-definable sequence of dense classes, one has that $p^\lambda_\mu$ codes $\bar{H}^\lambda(\gamma)$ of height $\bar{\mu}$, where $L_{\bar{\mu}+1}[A \cap \gamma, p^\lambda_\mu] \models |p^\lambda_\mu|$ is not a cardinal. Thus there is no additional instance of clause (b) to verify beyond those considered in the proof of Lemma 3.

Thus $P$ is tame and cofinality-preserving. The final statement of the Theorem also follows, using Remark (2) immediately after the statement of the Theorem. □

Applications

(1) Local $\Pi_2^1$-Singletons. David [82] proves the following: There is an $L$-definable forcing $P$ for adding a real $R$ such that $R$ is a $\Pi_2^1$-singleton in every set-generic extension of $L[R]$ (via a $\Pi_2^1$ formula independent of the set-generic extension). This is accomplished as follows: One can produce an
$L$-definable sequence $\langle T(\kappa) \ | \ \kappa \text{ an infinite } L\text{-cardinal} \rangle$ such that $T(\kappa)$ is a $\kappa^{++}$-Suslin tree in $L$ for each $\kappa$ and the forcing $\prod T(\kappa)$ for adding a branch $b(\kappa)$ through each $T(\kappa)$ (via product forcing, with Easton support) is tame and cofinality-preserving. Now for each $n$ let $X_n \subseteq \omega^L$ be class-generic over $L$, $X_n$ codes a branch through $T(\kappa)$ iff $\kappa$ is of the form $(\aleph_L^\lambda + n)$, $\lambda$ limit. The forcing $\prod P_n$, where $P_n$ adds $X_n$, can be shown to be tame and cofinality-preserving. Finally over $L[\langle X_n \ | \ n \in \omega \rangle]$ add a real $R$ such that $n \in R$ iff $R$ codes $X_n$. Then one has that in $L[R]$, $n \in R$ iff $T(\aleph_L^{\lambda+n})$ is not $\aleph_L^{\lambda+n}$-Suslin for sufficiently large $\lambda$. Clearly this characterization will still hold in any set-generic extension of $L[R]$. David’s trick is used to strengthen this to a $\Pi^1_2$ property of $R$.

(2) A Global $\Pi^1_2$-Singleton. Friedman \cite{Friedman} produces a $\Pi^1_2$-singleton $R$, $0 <_L R < L 0^\#$. This is accomplished as follows: assume that one has an index for a $\Sigma_1(L)$ classification $(\alpha_1 \cdots \alpha_n) \mapsto r(\alpha_1 \cdots \alpha_n)$ that produces $r(\alpha_1 \cdots \alpha_n) \in 2^{<\omega}$ for each $\alpha_1 < \cdots < \alpha_n$ in ORD such that $R = \cup \{r(i_1 \cdots i_n) \ | \ i_1 < \cdots < i_n \text{ in } I = \text{Silver indiscernibles }\}$. For each $r \in 2^{<\omega}$ there is a forcing $Q(r)$ for “killing” all $(\alpha_1 \cdots \alpha_n)$ such that $r(\alpha_1 \cdots \alpha_n)$ is incompatible with $r$. No $(i_1 \cdots i_n)$ from $I^n$ can be killed. Now build $R$ such that $r \subseteq R$ iff $R$ codes a $Q(r)$-generic. Then $R$ is the unique real with this property. David’s trick is used to strengthen this to a $\Pi^1_2$ property.

(3) New $\Sigma^1_3$ facts. Friedman \cite{Friedman} shows that if $M$ is an inner model of ZFC, $0^\# \notin M$, then there is a $\Sigma^1_3$ sentence false in $M$ yet true in a forcing extension of $M$. This is accomplished as follows: let $\langle C_\alpha | \alpha \text{ L-singular} \rangle$ be a $\Box$-sequence in $L$; i.e., $C_\alpha$ is CUB in $\alpha$, $\ot C_\alpha < \alpha$, $\bar{\alpha} \in \lim C_\alpha$ $\Rightarrow C_\alpha = C_\alpha \cap \bar{\alpha}$. Define $n(\alpha) = 0$ if $\ot C_\alpha$ is $L$-regular and otherwise $n(\alpha) = n(\ot C_\alpha) + 1$. Then for some $n$, $\{\alpha \ | \ n(\alpha) = n\}$ is stationary in $M$. And for each $n$, there is a tame forcing extension of $M$ in which $\{\alpha \ | \ n(\alpha) \leq n\}$ is non-stationary, and is in fact disjoint from the class of limit cardinals. David’s trick is used to strengthen the latter into a $\Sigma^1_3$ property.
References

[82] R. David, A Very Absolute \( \Pi^1_2 \)-Singleton, *Annals of Pure and Applied Logic* **23** pp. 101-120.

[82] A. Beller, R. Jensen, P. Welch, *Coding the Universe*, book, *Cambridge University Press*

[90] S. Friedman, The \( \Pi^1_2 \) Singleton Conjecture, *Journal of the American Mathematical Society*, Vol.3, No.4, pp. 771-791.

[97] S. Friedman, Coding without Fine Structure, *Journal of Symbolic Logic*, Vol.62, No.3, pp. 808-815.

[98] S. Friedman, New \( \Sigma^1_3 \) Facts, to appear.

[99] S. Friedman, *Fine Structure and Class Forcing*, book, in preparation.