The Chebyshev Exponent

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Abstract

The analogy between the nth power function and the nth Chebyshev polynomial is pursued, leading to consideration of Chebyshev radicals as analogous to ordinary radicals and Chebyshev exponents to ordinary exponents, and the cosine and hyperbolic cosine as analogs of the exponential function. We then discuss solving polynomial equations in Chebyshev radicals, and apply this to the construction of unramified extensions of quadratic number fields.

1 Basic Properties

1.1 Exponents

There are many well-known families of polynomials $P_n(x)$ consisting of a polynomial of degree $n$ for each non-negative integer $n$. Of all of these, the one which is at once the most important and the easiest to describe is the family of powers, $P_n(x) = x^n$. This family may be defined by means of a simple recurrence relationship, for we have $P_0(x) = 1$, $P_n(x) = xP_{n-1}(x)$. It has also the remarkable property that functional composition is multiplicative in the degree; that is

$$P_n(P_m(x)) = P_{nm}(x).$$

Suppose $E_n$ is a family of polynomials of the above sort, so that $E_n$ is of degree $n$, and $E_n(E_m(x)) = E_{nm}(x)$. We may immediately make several observations. First, functional composition commutes in such a family of polynomials, since

$$E_n(E_m(x)) = E_{nm}(x) = E_m(E_n(x)).$$

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Secondly, the family has a fixed point, namely, the constant value of $E_0$, since

$$E_n(E_0(x)) = E_0(x).$$

Thirdly, we have

$$E_1(E_1(x)) = E_1(x),$$

and hence

$$E_1(x) = x.$$

Let $\ell(x) = px + q$ be a linear transformation. We may then define a new family of polynomials by setting

$$\tilde{E}_n = \ell^{-1}(E_n(\ell(x))).$$

It is immediate that this family still has the functional composition property,

$$\tilde{E}_n(\tilde{E}_m(x)) = \tilde{E}_{nm}(x).$$

If we set $\ell(x) = (2x - b)/2a$, then

$$\tilde{E}_n(x) = aE_n\left(\frac{2x - b}{2a}\right) + \frac{b}{2}.$$

This transformed family has the property that if $E_2(x) = ax^2 + bx + c$, then

$$\tilde{E}_2 = x^2 - \frac{b^2 - 4ac - 2b}{4}.$$

Let us therefore define an exponent to be a family of polynomials $E_n(x)$, one for each degree $n$, such that $E_n(E_m(x)) = E_{nm}(x)$, normalized by the condition that $E_2(x)$ is a monic polynomial with trace term zero; i.e., a polynomial of the form $x^2 - c$. One can now ask, are there in fact any exponents other than the familiar one, $P_n(x) = x^n$? It turns out that there are; in fact it follows from the work of Julia ([1]) and Ritt ([4]) that there are two and only two such exponents. (See also [3] for a more elementary treatment.)

The “other” exponent, of course, is the family of Chebyshev polynomials of the first kind. Just as the family of power polynomials $P_n(x)$ can be defined by a recurrence relationship $P_n(x) = xP_{n-1}(x)$, so can these; we set $C_0(x) = 2$, $C_1(x) = x$, and

$$C_n(x) = xC_{n-1}(x) - C_{n-2}(x).$$
The reader should note that other normalizations of this family are in use, the most common one in fact being \( T_0(x) = 1, \ T_1(x) = x, \)
\[
T_n(x) = 2xT_{n-1}(x) - T_{n-2}.
\]
The relationship between these is simply
\[
T_n(x) = \frac{C_n(2x)}{2}.
\]
There are various advantages both conceptual and practical to the normalization we have adopted, however; especially from the point of view of an algebraist or number theorist.

### 1.2 Chebyshev Powers

**Definition 1** For any non-negative integer \( n \) and any \( x \) in a ring \( R \), we denote by \( x^{\circ n} \) the polynomial function defined by the recurrence relation \( x^{\circ 0} = 2, \ x^{\circ 1} = x \).
\[
x^{\circ n} = xx^{\circ n-1} - x^{\circ n-2}.
\]
(1)

For any negative integer \( n \), we set \( x^{\circ n} = x^{\circ -n} \).

We will term this the \( n \)-th Chebyshev power of \( x \). It is, of course, a new notation for the \( n \)-th Chebyshev polynomial map, so that \( x^{\circ n} = C_n(x) \). The point is to emphasize the close analogy to the ordinary power \( x^n \), and allow a more natural and suggestive notation for non-integral powers. The reader who would like a way of pronouncing \( x^{\circ n} \) is invited to pronounce it “\( x \) cheby \( n \)”.

The special symbol “\( \circ \)” will be used to denote the Chebyshev analogue of something expressed in terms of ordinary exponents. We will also use it as a formal operator.

**Definition 2** Let \( R \) be a commutative ring and \( R[x_1, \ldots, x_n] \) the \( R \)-algebra of formal polynomials in \( n \) indeterminates over \( R \). For \( P \in R[x_1, \ldots, x_n] \) we define \( P \mapsto P^{\circ} \) to be the \( R \)-linear map on \( R[x_1, \ldots, x_n] \) which takes each ordinary positive power \( x_i^k \) in each monomial term, and replaces it with the corresponding Chebyshev power \( x_i^{\circ k} \).
Note that while the constant term of a polynomial \( a_0x^n + \cdots + a_n \) might be thought of as \( a_nx^0 \), we do not replace it by \( a_nx^0 = 2a_n \). So, for example,
\[
(x^3y^2 + xy + 1)^\circ = x^3y^2 + xy + 1.
\]
Also, we must first expand the polynomial, so that for instance,
\[
((x - 1)(x^2 + x + 1))^\circ = (x^3 - 1)^\circ = x^3 - 1 = x^3 - 3x - 1,
\]
whereas
\[
(x - 1)^\circ(x^2 + x + 1)^\circ = (x - 1)(x^2 + x + 1) = x^3 - 2x - 1.
\]

When an expression (for instance in \( Q(x) \)) evaluates to a formal polynomial, we will allow ourselves to operate on it with “\( \circ \)”, so that for instance,
\[
\left( \frac{x^3 - 1}{x - 1} \right)^\circ = x^2 + x + 1.
\]

We will also make the usual identifications of formal polynomials with polynomial functions when appropriate.

We may now derive a formula of fundamental importance, which in fact characterizes Chebyshev powers.

**Proposition 1** Suppose \( z \in R \) has an inverse element \( z^{-1} \in R \) in a ring \( R \). For any integer \( n \), we then have the relation
\[
(z + z^{-1})^\circ^n = z^n + z^{-n}. \tag{2}
\]

Proof: By induction. \( \square \)

Note that in an algebraically closed field, such as \( \mathbb{C} \), we may always find a \( z \) satisfying \( x = z + z^{-1} \) for any \( x \), by solving \( z^2 - xz + 1 = 0 \). We can then get an expression
\[
x^\circ^n = z^n + z^{-n}
\]
for \( x^\circ^n \) in terms of the roots of \( z^2 - xz + 1 \), the characteristic polynomial for the recurrence relationship.

Equation (2) also allows us to give another definition of the “\( \circ \)” operator. If \( P(x) = a_0x^n + \cdots + a_n \) is a polynomial in \( R[x] \), where \( R \) is commutative, then we set
\[
Q(z) = z^n(P(z) + P(z^{-1}) - a_0).
\]
We may then eliminate \( z \) between two equations by taking the resultant of \( Q(z) \) with \( z^2 - xz + 1 \); because of (2) this will be \( (P^\circ)^2 \).
Proposition 2  For positive integer exponent \( n \),
\[
x^n = \sum_{i \leq n/2} \binom{n}{i} x^{n-2i}.
\]

Proof: We can prove this by induction. We may also begin by taking our
ring be the complex numbers \( \mathbb{C} \), and choosing \( z \) such that \( x = z + z^{-1} \). Expanding \((z + z^{-1})^n\) by the binomial theorem and collecting the terms \( z^i \) and \( z^{-i} \) together gives us the result over \( \mathbb{C} \). It then follows that the result is
true as an identity of formal polynomials over \( \mathbb{Z}[X] \). If we send \( X \) to \( x \in R \) and map \( \mathbb{Z} \) to \( R \), we have the result over any ring \( R \). \( \square \)

Proposition 3  For any exponent \( m = p^n \) which is a power of a prime \( p \), we have
\[
x^{\otimes m} \equiv x^m \pmod{p}.
\]

Proof: From the above, \( x^{\otimes p} \equiv x^p \pmod{p} \), which is to say, modulo the
ideal \( pR \). By induction,
\[
x^{\otimes n} = (x^{\otimes n/p})^{\otimes p} \equiv (x^{n/p})^p = x^n \pmod{p}.
\]
\( \square \)

Definition 3  We define a function \( K(n, m) \) by
\[
K(n, m) = \binom{n}{m} + \binom{n-1}{m-1}.
\]

\( K(n, m) \) can be expressed in a number of other ways. We have
\[
K(n, m) = (1 + \frac{n}{m}) \binom{n-1}{m-1} = \frac{n + m}{m} \binom{n-1}{m-1} = \\
\frac{1}{m} (n + m)(n-1)(n-2) \cdots (n-m+1) = \frac{n + m}{n} \binom{n}{m}.
\]

For our purposes, the most important property of \( K(n, m) \) is the follow-
ing:
Lemma 4

\[ K(n, m) = K(n - 1, m) + K(n - 1, m - 1) \]

Proof: Immediate from the corresponding formula for \( \binom{n}{m} \).

For positive \( n \) we have \( K(n, 0) = 1 \), \( K(n, n) = 2 \). From this and the lemma, we see that we can define \( K \) by means of a variant Pascal’s triangle, with 1 along the left side, and 2 along the right. We have

\[
\begin{array}{cccccc}
1 & 2 \\
1 & 3 & 2 \\
1 & 4 & 5 & 2 \\
1 & 5 & 9 & 7 & 2 \\
\end{array}
\]

and so forth.

Our interest in \( K(n, m) \) is a consequence of the following proposition.

Proposition 5 For positive integer exponent \( n \),

\[ x^{\odot n} = \sum_{i \leq n/2} (-1)^i K(n - i, i)x^{n-2i}. \]  

(4)

Proof: By an easy induction. \( \square \)

1.3 Chebyshev Radicals

By far the most familiar method for solving polynomial equations algebraically is the solution in radicals. A radical is an algebraic function defined by a choice of branch cut for the polynomial \( x^n - t \), giving \( x \) as a function of \( t \); so that

\[ x = \sqrt[n]{t}. \]

This allows us to extend the definition of the exponent, originally defined for positive integers, to positive rational numbers; we are able to do this because the powers are exponents; that is, because

\[(x^n)^m = x^{nm}. \]

The \( n \)-th Chebyshev power map is in many respects analogous to the ordinary \( n \)-th power map. If the ring \( R \) is the complex numbers, then according
to this analogy, an $n$-th Chebyshev root would be a solution of $x^{\circ n} - t$, and a Chebyshev radical would be a particular branch of this $n$-fold covering. We will make a canonical choice of a branch which contains the fixed point $2^{\circ n} = 2$.

**Definition 4** For $n$ any non-zero integer, we will denote by

$$\sqrt[n]{x}$$

that branch of the function of a complex variable $x$ satisfying

$$(\sqrt[n]{x})^{\circ n} = x$$

such that $\sqrt[n]{2} = 2$ and with a branch cut along $(-\infty, -2)$, choosing values along the cut to have positive imaginary part.

We will also denote this function by

$$x^{\circ \frac{1}{n}} = \sqrt[n]{x}.$$  

**Proposition 6** We have now the relation

$$\sqrt[n]{\sqrt[m]{x}} = \sqrt[m]{\sqrt[n]{x}} = \sqrt[nm]{x}. \quad (5)$$

Proof: Since $(y^{\circ n})^{\circ m} = (y^{\circ m})^{\circ n} = y^{\circ nm}$, the three expressions above all have the property that their $nm$-th Chebyshev powers are equal to $x$. They also all have the same branch cut $(-\infty, -2)$ and have positive imaginary part along the cut; hence they are equal. $\square$

We now may now extend the definition to rational exponents, in analogy to high school algebra.

**Definition 5** For complex $x$ and $p/q$ any rational number, define

$$x^{\circ \frac{p}{q}} = (\sqrt[q]{x})^{\circ p}.$$
1.4 Chebyshev Exponentials and Logarithms

In the case of ordinary powers, study of the powers of numbers close to the fixed point for this family of exponents, $1^n = 1$, is particularly illuminating. If we look at the powers

$$(1 + \frac{1}{n})^n$$

we find that they tend to a particular number $e$ as $n$ grows larger. Moreover, we find that if we define a function by

$$\exp z = \lim_{n \to \infty} (1 + \frac{z}{n})^n$$

we produce something which may be regarded as the giving powers of this number $e$, for any complex exponent.

We may therefore define

$$\exp x = e^x = \lim_{h \to 0} (1 + hx)^\frac{1}{h},$$

define a principal branch of the inverse function log, and discover that the relation

$$x^k = \exp(k \log x)$$

allows us to extend the definition of exponent to all complex exponents $k$.

These two observations are connected. The exponential function satisfies the functional equation

$$\exp nx = (\exp x)^n,$$

and this in turn leads to the functional composition property of the exponent.

Perhaps the most illuminating way of extending the definition of the Chebyshev exponent to complex arguments is to employ a similar procedure to discover the Chebyshev analogs of the exponential and logarithmic functions. We first define Chebyshev analogs to $e$ and $e^{-1}$, by looking at high powers of numbers near the fixed point for Chebyshev powers, $2^{\circ n} = 2$.

**Theorem 7** For a limit proceeding through rational values $h$, we have

$$\lim_{h \to 0} (2 + h^2)^\frac{1}{h} = 2 \cosh 1 = \hat{e}.$$
Proof: If we set
\[ g = \left( h^2 + \sqrt{h^4 + 4h^2} \right)/2 = h + \frac{1}{2}h^2 + \frac{1}{8}h^3 + \cdots \]
then
\[ 2 + h^2 = (1 + g) + (1 + g)^{-1} \]
and so
\[
\lim_{h \to 0^+} (2 + h^2)^{\frac{1}{h}} = \lim_{h \to 0^+} (1 + g + (1 + g)^{-1})^{\frac{1}{h}} = \lim_{h \to 0^+} (1 + h + O(h^2))^{\frac{1}{h}} + (1 + h + O(h^2))^{-\frac{1}{h}} = e + \frac{1}{e}.
\]
\[ \square \]

**Proposition 8** For \( x \) any positive real number and \( s \) any rational number, we have
\[ (x + x^{-1})^s = x^s + x^{-s}. \]

Proof: Let \( s = p/q \), where \( p \) and \( q \) are integers, and let \( u = \sqrt[2]{x} \). Then
\[ (u + u^{-1})^{q} = u^q + u^{-q} = x + x^{-1}. \]
Since \( u \) is positive and real, so is \( u + u^{-1} \); which therefore belongs to the correct branch and is the canonical \( n \)-th Chebyshev root. Hence
\[ u + u^{-1} = x^{\frac{1}{q}} + x^{-\frac{1}{q}} = x^{\frac{1}{q}}, \]
and so
\[ x^{\frac{s}{q}} = (x^{\frac{1}{q}} + x^{-\frac{1}{q}})^{\frac{p}{q}} = x^{\frac{p}{q}} + x^{-\frac{p}{q}} = x^s + x^{-s}. \]
\[ \square \]

We now have
\[ e^{\frac{s}{q}} = (e + e^{-1})^{\frac{1}{q}} = e^s + e^{-s} = 2 \cosh s \]
for any rational number \( s \). This suggests the following extended definition:
Definition 6 For any complex number \( z \),
\[
\hat{e}^{\odot z} = \exp_\odot z = 2 \cosh z.
\]

Despite the fact that it is essentially the hyperbolic cosine, we introduce the notation \( \exp_\odot z \) in order to underline the formal similarities at work here.

We have another limit analogous to the well-known
\[
\lim_{h \to 0} (1 - h)^{\frac{1}{h}} = e^{-1}
\]
which is also of interest.

Theorem 9 Let \( h \) approach 0 through rational values. Then
\[
\lim_{h \to 0} (2 - h^2)^{\frac{1}{h}} = 2 \cos 1 = \hat{c}.
\]
Proof: Let \( h = g/i \). Then
\[
\lim_{h \to 0} (2 - h^2)^{\frac{1}{h}} =
\lim_{g \to 0} (2 + g^2)^{\frac{1}{g}} =
(\lim_{g \to 0} (2 + g^2)^{\frac{1}{g}})^{\odot i} =
\hat{c}^{\odot i} = e^i + e^{-i} = 2 \cos 1 = \hat{c}.
\]

Now that we have identified \( \hat{e}^{\odot x} = 2 \cosh x \) and \( \hat{c}^{\odot x} = 2 \cos x \) as the Chebyshev analogs of the exponential function, we are led immediately to \( \text{arccosh} \frac{x}{2} \) and \( \frac{x}{2} \) as the Chebyshev versions of the logarithm. We make the following definition for the principal branch of the inverse function to \( \exp_\odot x \).

Definition 7 We define
\[
\log_\odot x = \text{arccosh} \frac{x}{2}
\]
for a branch such that if
\[
w = \log_\odot = r + i\theta,
\]
where \( r = \Re w \) and \( \theta = \Im w \), then \( r \geq 0 \), \(-\pi < \theta \leq \pi\), and when \( r = 0 \) then \( \theta \geq 0 \).
The reader may want to pronounce \( \exp \) as “cheby exp” and \( \log \) as “cheby log”.

We may now extend our definition of Chebyshev powers in a consistent way by setting

**Definition 8** For any complex \( x \) and \( a \), we set

\[
a^{\circ x} = e^{\circ \log \circ ax} = \exp (\log (a)x).
\]

In precalculus classes, students are told that for real and positive \( a \), \( a^x \) behaves in different ways depending on whether \( a \) is less than, equal to, or greater than one. In the same way, the behavior of \( a^{\circ x} \) for \( a \geq -2 \) depends on whether it is less than, equal to, or greater than two; but in this case, the difference is more interesting, since for \( a < 2 \) we obtain a periodic function.

We now check that we have the properties that we want. We note first that along the negative real axis from \(-2\) to \(-\infty\), \( \log \circ x \) is of the form \( r + i\pi \), where \( r \) is real and positive. This gives us a positive imaginary part for the \( n \)-th Chebyshev root, as desired.

We now have

**Theorem 10** For any complex \( x \), \( y \), and \( a \),

\[
(a^{\circ x})^{\circ y} = a^{\circ xy}.
\]  

Proof: Since \( f^{-1}(f(x)) = x \),

\[
f(yf^{-1}(fxf^{-1}(a))) = f(xfyf^{-1}(a)).
\]

Letting \( f(x) = e^{\circ x} \), \( f^{-1}(x) = \log \circ x \), we have \( a^{\circ x} = f(xf^{-1}(a)) \), and hence the theorem upon substituting and simplifying.

One consequence is that

\[
\tilde{e}^{\circ x} = (\tilde{e}^{\circ i})^{\circ x} = \tilde{e}^{\circ ix} = 2 \cosh ix = 2 \cos x,
\]

as we suggested ought to be the case.

We may also now evaluate

\[
\lim_{h \to 0} (2 + h^2)^{\frac{1}{h}}
\]

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by the same argument as before, with \( h \) now any non-zero complex number. If we substitute \( h = x/n \), we then get

\[
\lim_{{n \to \infty}} \left( 2 + \frac{x^2}{n^2} \right)^{n/x} = e.
\]

Raising each side to the “cheby x” power, we get

\[
\lim_{{n \to \infty}} \left( 2 + \frac{x^2}{n^2} \right)^n = e^x,
\]

and by restricting \( n \) to integer values, we get an alternative definition of \( \exp_{\Box} x \) analogous to the classical Cauchy definition of \( \exp x \), which simply requires us to take integral Chebyshev powers of numbers near to two.

We also have

**Theorem 11** For complex \( z \) and \( k \), let us define \( z^k \) by means of the principle branch of the logarithm, meaning the one such that

\[-\pi < \Im(\log x) \leq \pi.\]

Then except when \( z \) is a real number \(-1 < z \leq 0\), we have that

\[(z + z^{-1})^{\Box}k = z^k + z^{-k}.\]

Proof: Let \( u = \log_{\Box}(z + z^{-1}) \). Then

\[(z + z^{-1})^{\Box}k = \exp_{\Box} uk = e^{uk} + e^{-uk}.
\]

If \( z \) is not both real and negative, then one of \( e^{uk} \) or \( e^{-uk} \) will correspond to \( z^k \), and the other to \( z^{-k} \). If \( z \) is real and less than \(-1\), then \( z = e^u \) and \( z^k = e^{uk} \), giving us the result we want. \( \square \)

### 1.5 The Derivative

Throughout this section we will assume that the field of definition is the complex numbers unless stated otherwise.
Theorem 12 Let $x$ and $k$ be any complex numbers, and let $x = z + z^{-1}$, where $z$ is chosen so that it is not the case that $-1 < z < 0$. If $S_k(x)$ is the function defined by

$$S_k(x) = \frac{z^k - z^{-k}}{z - z^{-1}},$$

then

$$(x^{\odot k})' = kS_k(x).$$

Proof: If $x = z + z^{-1}$, then by Theorem 11 we have

$$x^{\odot k} = z^k + z^{-k}.$$ 

Taking derivatives and applying the chain rule, we get

$$(x^{\odot k})' = \frac{kz^{k-1} - kz^{-k-1}}{1 - z^{-2}} = k\frac{z^k - z^{-k}}{z - z^{-1}}.$$

$$\square$$

Note that when $n$ is an integer, $S_n(x)$ is a polynomial of degree $n - 1$ in $x$, since it is a multiple of the derivative of a polynomial of degree $n$ in $x$. Such polynomials are known as the Chebyshev polynomials of the second kind.

From the definition of $S_k$ we immediately obtain

$$S_k(e^{i\theta}) = \frac{\sinh k\theta}{\sinh \theta},$$

$$S_k(e^\theta) = \frac{\sin k\theta}{\sin \theta}.$$ 

We therefore have

$$S_k(x) = \frac{\sinh(k \log\odot x)}{\sinh \log\odot x} \quad \text{(7)}$$

Theorem 13

$$S_k(x) = xS_{k-1}(x) - S_{k-2}(x).$$
Proof:
\[
\frac{(z+z^{-1})(z^{k-1} - z^{-k+1}) - (z^{k-2} - z^{-k+2})}{z - z^{-1}} = \frac{z^k - z^{-k}}{z - z^{-1}}.
\]

For integer exponents \( n \), we may consider \( x^\otimes n \) and \( S_n(x) \) to be elements of the function field \( \mathbb{C}(x) \). Consider any linear recurrence
\[
P_n = xP_{n-1} - P_{n-2},
\]
where \( P_n(x) \) is a formal polynomial of degree \( n \) in \( \mathbb{C}(x) \). Since this is a linear recurrence of degree two, the solutions comprise a vector space of dimension two over \( \mathbb{C}(x) \). Hence \( x^\otimes n \) and \( S_n(x) \) form a basis for the solutions of the linear recurrence. Alternatively, we may write any such recurrence in terms of \( x^\otimes n \) and \( x^\otimes n-1 \), since these are also linearly independent.

So, for instance, we have
\[
x^\otimes n+1 = \frac{xx^\otimes n + (x^2 - 4)S_n(x)}{2},
\]
or
\[
S_n(x) = \frac{xx^\otimes n - 2x^\otimes n-1}{x^2 - 4}
\]
and so forth. Any of these may easily be derived if wanted by using undetermined coefficients.

### 1.6 Sum and Product Formulas

For various purposes, it is helpful to have product formulas for our functions. We start with the Chebyshev analog of the product formula for powers, \( x^n x^m = x^{nm} \). As before, arguments \( x \) and exponents \( n \) or \( m \) can be any complex number; results for integral exponents can then be exported to any ring.

**Proposition 14**
\[
x^\otimes n x^\otimes m = x^\otimes n+m + x^\otimes n-m.
\]

Proof: Writing \( x \) as \( x = z + z^{-1} \), we obtain
\[
x^\otimes n x^\otimes m = (z^n + z^{-n})(z^m + z^{-m}) = z^{n+m} + z^{-n-m} + z^{n-m} + z^{-n+m} = x^\otimes n+m + x^\otimes n-m.
\]

We can use (8) to derive one of the most familiar properties of the Chebyshev polynomials, their orthogonality.
Proposition 15 Let $n, m$ be integers. If $n \neq m$,
\[
\int_{-2}^{2} x^\circ n x^\circ m \frac{dx}{\sqrt{4 - x^2}} = 0,
\]
whereas if $n \neq 0$
\[
\int_{-2}^{2} (x^\circ n)^2 \frac{dx}{\sqrt{4 - x^2}} = 2\pi.
\]
Proof: Substituting $x = e^{\theta}$, we obtain
\[
\int_{0}^{\pi} e^{\circ n \theta} e^{\circ m \theta} d\theta = \int_{0}^{\pi} e^{\circ (n+m) \theta} + \int_{0}^{\pi} e^{\circ (n-m) \theta} d\theta.
\]
This integral is 0 unless $n = m$, when it reduces to
\[
\int_{0}^{\pi} 2 = 2\pi.
\]
We also have a nice formula for the product of $S_n$ with $x^m$.

Proposition 16
\[
S_n(x)x^\circ m = S_{n+m}(x) + S_{n-m}(x)
\]  
(9)

Proof:
\[
\frac{z^n - z^{-n}}{z - z^{-1}}(z^m + z^{-m}) = \frac{z^{n+m} - z^{-n-m} + z^{n-m} - z^{-n-m}}{z - z^{-1}} = S_{n+m}(x) + S_{n-m}(x).
\]

Finally, we have a formula for the product of $S_n$ and $S_m$.

Proposition 17
\[
(x^2 - 4)S_n(x)S_m(x) = x^{\circ n+m} - x^{\circ n-m}
\]
(10)

Proof:
\[
\frac{(z^n - z^{-n})(z^m - z^{-m})}{z - z^{-1}} = \frac{z^{n+m} + z^{-n-m} - z^{n-m} - z^{-n-m}}{(z - z^{-1})^2} = \frac{x^{\circ n+m} - x^{\circ n-m}}{x^2 - 4}.
\]

Corollary 18
\[
S_n(x) = \frac{x^{\circ n+1} - x^{\circ n-1}}{x^2 - 4}\]
(11)
Proof: From (10), we have

\[(x^2 - 4)S_n(x)S_1(x) = x^\circ n + 1 - x^\circ n - 1.\]

\[\square\]

This or other expressions for \(S_n(x)\) in terms of Chebyshev powers can be used to define it for any exponent.

We can use these product formulas to obtain formulas for the sum of the exponents.

**Proposition 19**

\[x^\circ n + m = \frac{x^\circ n \cdot x^\circ m + (x^2 - 4)S_n(x)S_m(x)}{2} \]  \hspace{1cm} (12)

\[S_{n+m}(x) = \frac{S_n(x)x^\circ m + S_m(x)x^\circ n}{2} \]  \hspace{1cm} (13)

Proof: From the product formulas.

These can be used to calculate high Chebyshev powers in any ring where two is invertible.

One way to regard these formulas is to see them as representing parts of the powers of

\[z = \frac{x + \sqrt{x^2 - 4}}{2},\]

so that

\[z^n = \frac{x^\circ n + S_n(x)\sqrt{x^2 - 4}}{2}.\]

We can find a similar formula without any divisions.

**Proposition 20**

\[z^n = S_n(x)z - S_{n-1}(x) \]  \hspace{1cm} (14)

\[(xz - 2)z^{n-1} = (2z - x)z^n = x^\circ n z - x^\circ n - 1 \]  \hspace{1cm} (15)

Proof: By induction.

Either of these propositions these can also be used to calculate Chebyshev powers by what is essentially the classic “square and multiply” algorithm for ordinary powers. However, we can also avoid the use of anything other than Chebyshev powers themselves to calculate high-order Chebyshev powers.
Lemma 21

\[
x^\circ 2n = (x^\circ n)^2
\]

\[
x^\circ 2n+1 = x^\circ n, x^\circ n+1 - x
\]

\[
x^\circ 2n+2 = xx^\circ 2n+1 - x^\circ 2n.
\]

Proof: The first is the composition property of (Chebyshev) exponents, (2). The second follows from the product formula (8). The third is the recurrence relationship, (1).

Using this, we may find any integral Chebyshev power by starting with the pair \((x^\circ 1, x^\circ 2)\), and setting \(l = n\). At each stage, If \(l\) is even, we replace \((x^\circ m, x^\circ m+1)\) with \((x^\circ 2m, x^\circ 2m+1)\), and replace \(l\) with \(l/2\). If \(l\) is odd we replace \((x^\circ m, x^\circ m+1)\) with \((x^\circ 2m+1, x^\circ 2m+2)\), and replace \(l\) with \((l - 1)/2\).

We continue in this way until \(l = 1\), when we arrive at \((x^\circ n, x^\circ n+1)\). If \(n - 1\) has fewer digits base two than does \(n\), we may start instead by setting \(l = n - 1\) and finish with \((x^\circ n-1, x^\circ n)\).

We now consider how, for integral exponents, we may express each of these polynomials in terms of the other.

Proposition 22

\[
S_{n+1}(x) - S_{n-1}(x) = x^\circ n
\]  

(16)

Proof:

\[
S_{n+1}(x) - S_{n-1}(x) = \frac{z^{n+1} - z^{n-1} + z^{1-n} - z^{-1-n}}{z - z^{-1}}
\]

\[
= \frac{(z - z^{-1})z^n + (z - z^{-1})z^{-n}}{z - z^{-1}} = z^n + z^{-n} = x^\circ n.
\]

Proposition 23 Let \(n\) be integral. For even \(n\),

\[
S_n(x) = \left(\frac{x^n - 1}{x^2 - 1}\right)^\circ = x^\circ n-1 + x^\circ n-3 + \ldots + x,
\]  

(17)

while for odd \(n\),

\[
S_n(x) = \left(\frac{x^{n+1} - 1}{x^2 - 1}\right)^\circ = x^\circ n-1 + x^\circ n-3 + \ldots + x^\circ 2 + 1.
\]  

(18)
Proof: On expressing the Chebyshev powers in terms of the $S_i$ using (16), we obtain a telescoping series.

We can use this to obtain a product formula for integral indices $n$ and $m$ of $S_nS_m$, solely in terms of Chebyshev polynomials of the second kind, $S_i$.

**Proposition 24** Let $n$ and $m$ be positive integers. We then have the following:

$$S_nS_m = \sum_{i=1}^{m} S_{n+m+1-2i}$$  \hspace{1cm} (19)

Proof: Suppose $m$ is even. Expanding $S_m$ by means of (17), we obtain

$$S_n(x)S_m(x) = S_n(x) \sum_{i=1}^{m/2} x^{\circ 2i-1}.$$  

Using (9), this becomes

$$m/2 \sum_{i=1}^{m/2} S_{n+2i-1} + S_{n-2i+1},$$

which we may rearrange and reindex to obtain the theorem.

Similarly, if $m > 1$ is odd, we may use (18) and obtain

$$S_n(x)S_m(x) = S_n(x)(1 + \sum_{i=1}^{(m-1)/2} x^{\circ 2i}).$$

Expanding this out gives

$$S_nS_m = S_n + \sum_{i=1}^{(m-1)/2} S_{n+2i} + S_{n-2i},$$

and we can rearrange and reindex this to obtain the theorem. Since it is plainly true when $m = 1$, we have a proof. $\square$

Finally, it is worth noting the connection of these to general second-order linear recurrences, and to Lucas and Fibonacci numbers and polynomials in particular.

Let

$$U_n = PU_{n-1} - QU_{n-2},$$

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\[ V_n = P V_{n-1} - Q V_{n-2}, \]

be the “fundamental” and “primordial” Lucas sequences respectively, initialized by \( U_0 = 0, U_1 = 1; V_0 = 2, V_1 = P \). Then

\[ U_n = Q^{n-1} S_n \left( \frac{P}{\sqrt{Q}} \right), \]

\[ V_n = Q^n \left( \frac{P}{\sqrt{Q}} \right)^{\circ n}. \]

In particular, if \( F_0 = 0, F_1 = 1, F_n = F_{n-1} + F_{n-2} \) are the Fibonacci numbers, and \( L_0 = 2, L_1 = 1, L_n = L_{n-1} + L_{n-2} \) are the Lucas numbers, then

\[ S_n(i) = i^{n-1} F_n, \]

\[ i^{\circ n} = i^n L_n, \]

so that

\[ F_n = i^{-n+1} S_n(i) = i^{n-1} S_n(-i), \]

\[ L_n = i^{-n} i^{\circ n} = i^n (-i)^{\circ n}. \]

We have in fact corresponding polynomials, where

\[ F_n(x) = i^{n-1} S_n(-ix) \]

are the Fibonacci polynomials, and

\[ L_n(x) = i^n (-ix)^{\circ n} \]

are the Lucas polynomials. These satisfy the recurrence relationship

\[ P_n(x) = x P_{n-1}(x) + P_{n-2}(x). \]

The Lucas polynomials also satisfy the functional equation

\[ L_n(z - z^{-1}) = z^n + (-z)^{-n}, \]

and can be used in place of the Chebyshev polynomials to solve algebraic equations.
2 Algebraic and Analytic Properties

2.1 Some Factorizations

For any univariate polynomial \( p \) of positive degree, we have that \( x - y \) is a factor of \( p(x) - p(y) \), since it is identically zero when \( x = y \). When \( p(x) = x^n \), we have the well-known expression for the difference of two \( n \)-th powers:

\[
x^n - y^n = (x - y)(x^{n-1} + x^{n-2}y + \cdots + y^{n-2}x + y^n).
\]

We seek the Chebyshev analogue to this; however we will find it useful to look first at the factorization for \( S_n(x) - S_n(y) \).

**Proposition 25** Let

\[
R_n(x, y) = \sum_{i=1}^{n-1} S_i(x)S_{n-i}(y).
\]

Then

\[
S_n(x) - S_n(y) = (x - y)R_n(x, y)
\]  \( (20) \)

Proof: Since

\[
xS_n(x) = S_{n+1}(x) + S_{n-1}(x),
\]

we have

\[
(x-y)S_i(x)S_j(y) = S_{i+1}(x)S_j(y) - S_i(x)S_{j+1}(y) + S_{i-1}(x)S_j(y) - S_i(x)S_{j-1}(y).
\]

Hence

\[
(x - y)R_n(x, y) = t_1 - t_2 + t_3 - t_4,
\]

where

\[
t_1 = \sum_{i=1}^{n-1} S_{i+1}(x)S_{n-i}(y) = S_n(x) + \sum_{i=2}^{n-1} S_i(x)S_{n+1-i}(y)
\]

\[
t_2 = \sum_{i=1}^{n-1} S_i(x)S_{n+1-i}(y) = S_n(y) + \sum_{i=2}^{n-1} S_i(x)S_{n+1-i}(y)
\]

\[
t_3 = \sum_{i=1}^{n-1} S_{i-1}(x)S_{n-i}(y) = \sum_{i=1}^{n-2} S_i(x)S_{n-i}(y)
\]

\[
t_4 = \sum_{i=1}^{n-1} S_i(x)S_{n-1-i}(y) = \sum_{i=1}^{n-2} S_i(x)S_{n-1-i}(y)
\]
After cancelling the summation terms in the second expressions given for the \( t_i \), we are left with \( S_n(x) - S_n(y) \).

**Theorem 26**

\[
x^\oplus n - y^\oplus n = (x - y)(R_{n+1}(x, y) - R_{n-1}(x, y)) \tag{21}
\]

Proof: By (16), we have

\[
x^\oplus n - y^\oplus n = S_{n+1}(x) - S_{n+1}(y) - S_{n-1}(x) + S_{n-1}(y).
\]

Factoring this using (20) we obtain the proposition.

**Proposition 27**

\[
x^\oplus n - y^\oplus n = (x - y)\left(\sum_{i=1}^{n-1} x^{\oplus n-i} S_i(y) + S_n(y)\right) \tag{22}
\]

Proof: Substituting \( S_{i+1}(x) - S_{i-1}(x) \) for \( x^\oplus i \) in the above and expanding, we obtain the result.

If \( y = a \) is a constant, we can rewrite this as

\[
x^\oplus n - a^\oplus n = (x - a)\left(\sum_{i=1}^{n} S_i(a)x^{n-i}\right)^\oplus.
\]

Special cases of this are of interest. If \( a = -1 \), then we have

\[
S_{3i}(-1) = 0, S_{3i+1}(-1) = 1, S_{3i+2}(-1) = -1.
\]

From this we get

\[
x^\oplus n - (-1)^\oplus n = (x + 1)(x^{n-1} - x^{n-2} + x^{n-4} - x^{n-5} + \cdots)^\oplus,
\]

where the series on the right continues through all the non-negative exponents.

In a similar way, we can derive

\[
x^\oplus n - 0^\oplus n = x(x^{n-1} - x^{n-3} + x^{n-5} - \cdots)^\oplus,
\]

\[
x^\oplus n - 1^\oplus n = (x - 1)(x^{n-1} + x^{n-2} - x^{n-4} - x^{n-5} + \cdots)^\oplus,
\]
and also
\[ x^{\odot n} - (-2)^{\odot n} = (x + 2)(x^{n-1} - 2x^{n-2} + 3x^{n-3} - \ldots)^{\odot}, \]
\[ x^{\odot n} - 2^{\odot n} = x^{\odot n} - 2 = (x - 2)(x^{n-1} + 2x^{n-2} + 3x^{n-3} + \ldots)^{\odot}. \]

We also have formulas containing Fibonacci and Lucas numbers, as for instance
\[ x^{\odot n} - 3^{\odot n} = x^{\odot n} - L_{2n} = (x - 3)(x^{n-1} + F_4x^{n-2} + F_6x^{n-3} + \ldots)^{\odot}. \]

The factorization has some properties which correspond to the analogous factorization of the difference of ordinary powers. In particular, we have the following.

**Theorem 28** The factorization given by (21) is irreducible over \( \mathbb{Q} \) if and only if the exponent \( n \) is prime.

Proof: If \( n = lm \) is not prime, then substituting \( x^{\odot m} \) for \( u \) and \( y^{\odot m} \) for \( v \) into the factorization of \( u^{\odot l} - v^{\odot l} \) will further factorize \( x^{\odot n} - y^{\odot m} \).

On the other hand, if \( n \) is an odd prime, then \( 0^{\odot n} = 0 \) shows that 0 is a Chebyshev \( n \)-th power, and hence
\[ x^{\odot n} - 0^{\odot n} = x(x^{\odot n-1} - K(n-1,1)x^{\odot n-3} + \ldots) \]
is the specialization obtained by setting \( y = 0 \). Since \( n \) is a prime, \( x^{\odot n} = x^n \) (mod \( n \)), so all the coefficients in the second factor aside from the leading term are divisible by \( n \). The constant term is \( (-1)^{(n-1)/2}K(n+1,1,-1) \) by (4). Since
\[ K(m+1,m) = \binom{m+1}{m} + \binom{m}{m-1} = \binom{m+1}{1} + \binom{m}{1} = 2m + 1, \]
we have
\[ K\left(\frac{n+1}{2},\frac{n-1}{2}\right) = 2\frac{n-1}{2} + 1 = n. \]
Hence the constant term is \( (-1)^{\frac{n-1}{2}}n \), so that the second factor is an Eisenstein polynomial, and hence irreducible. Since a specialization is irreducible, the factorization in the theorem must be irreducible also, and since the theorem is plainly true for the case \( n = 2 \), we have a proof.
2.2 Chebyshev Roots of Two

For solving polynomial equations and for many other purposes, the roots of unity \( \zeta \) such that \( \zeta^n = 1 \) for some positive integer \( n \) are of fundamental importance. If the field in question is the complex numbers, we have that \( \zeta \) is a root of unity if and only if

\[
\zeta = \exp(2\pi ir)
\]

for a rational number \( r \). Here, if \( r = \frac{m}{n} \) we have \( \zeta^n = \exp(2\pi im) = 1 \).

The Chebyshev analog of this turns out to play a similar role when solving equations using Chebyshev exponents. We first make the following definition.

**Definition 9** For any \( a \in R \) in a ring \( R \) and any integer \( n \), we define an \( n \)-th Chebyshev root of \( a \) to be any element of \( \mu \in R \) such that \( \mu \circ n = a \).

In particular, a Chebyshev root of two is an element \( \mu \) such that

\[
\mu \circ n = 2.
\]

In close analogy with the situation for roots of unity, we have that when the ring in question is the field of complex numbers,

\[
\mu = \check{e} \circ 2\pi r = 0 \circ 4r = \exp(2\pi ir)
\]

is a Chebyshev root of two if and only if \( r \) is a rational number. It follows that \( \mu \) is real, and that \( \mu = \zeta + \zeta^{-1} \), where \( \zeta \) is a root of unity. If the order is the least positive \( n \) such that \( \zeta^n = 1 \) (for roots of unity) or \( \mu \circ n = 2 \) (for Chebyshev roots of two) we also have that the order of \( \mu \) and the order of \( \zeta \) are identical. If the order of \( \zeta \) is \( n \), we say that \( \zeta \) is a primitive \( n \)-th root of unity; in the same way, we can call \( \mu \) a primitive \( n \)-th Chebyshev root of two if \( n \) is the order of \( \mu \).

If \( x \) is an \( n \)-th Chebyshev root of two, then \( x \circ n - 2 = 0 \). If we factor this using (22), we obtain

\[
x \circ n - 2 = (x - 2)(\sum_{i=1}^{n} ix^{n-i}) \circ.
\]

If \( n > 2 \) this factorization is further reducible. We have already remarked that composite exponents such as \( x \circ 2m - y \circ 2m \) (for \( m > 1 \)) can be factored further, and in fact we have the following.
Proposition 29 For any complex $x$ and $k$, we have
\[ x^{\oplus 2k} - 2 = (x^2 - 4)S_k^2(x). \] (23)

Proof: Immediate from (10). \(\square\)

For odd integer exponents, we have the following.

Theorem 30 For any positive integer $n$,
\[ x^{\oplus 2n+1} - 2 = (x-2)((\sum_{i=0}^{n} x^{n-i})^{\oplus})^2 = (x-2)(x^{\oplus n} + x^{\oplus n-1} + \cdots + x+1)^2 \] (24)

Proof: If $x$ is a $2n+1$-th Chebyshev root of two other than two, and if $x = z + z^{-1}$, then $z$ is a $2n+1$-th root of unity other than one, and so
\[ z^{2n} + z^{2n-1} + \cdots + 1 = 0. \]

Dividing by $z^n$ and collecting terms we obtain
\[ z^n + z^{-n} + z^{n-1} + z^{-n+1} + \cdots + 1 = x^{\oplus n} + x^{\oplus n-1} + \cdots + 1 = 0. \]

Since we obtain the root $x$ for each of $z$ and $z^{-1}$, it appears with multiplicity two, and hence the factor is squared. \(\square\)

We can rewrite this factorization as the Chebyshev analog of a geometric series, for if $x \neq 2$ we have
\[ (1 + x + x^{\oplus 2} + \cdots + x^{\oplus n})^2 = \frac{x^{\oplus 2n+1} - 2}{x - 2}. \]

As is so often the case, we can also express this factorization more compactly using Chebyshev polynomials of the second kind, $S_n$. To this end and others, we introduce a new function.

Definition 10 For any complex $k$ and $x$ we define $U_k(x)$ by
\[ U_k(x) = S_{(k+1)/2}(x) + S_{(k-1)/2}(x). \]

Proposition 31 We have a recurrence relationship
\[ U_{k+4}(x) = xU_{k+2}(x) - U_k(x) \] (25)
Proof: From the recurrence relationship for $S_k$. □

**Corollary 32** For positive integral $n$, $U_{2n+1}(x)$ is given by the recurrence relationship $U_1(x) = 1, U_3(x) = x + 1,$

$$U_{2n+3}(x) = xU_{2n+1}(x) - U_{2n-1}(x).$$

**Proposition 33** For any complex $k$ and $x$, we have

$$U_k(x) = \sqrt[\circ]{x}S_{k/2}(x) = \sqrt{x + 2S_{k/2}(x)}. \quad (26)$$

Proof: Apply (9) to the right-hand side. □

We now have

**Proposition 34** For any complex $x$ and $k$, we have

$$x^{\circ2k+1} - 2 = (x - 2)U_{2k+1}^2(x). \quad (27)$$

Proof: From (23), we obtain

$$x^{\circ2k+1} - 2 = (x^2 - 4)S_{k+1/2}^2(x).$$

From (26), we have

$$(x^2 - 4)S_{k+1/2}^2(x) = (x + 2)U_{2k+1}(x),$$

and hence the theorem. □

For integral $n$, we have a nice characterization of $U_{2n+1}$ in terms of its roots.

**Proposition 35** For positive integers $n$, we have

$$U_{2n+1}(x) = \prod_{i=1}^{n}(x - \mu_i);$$

where the $\mu_i$ are the $n$ 2n + 1-th Chebyshev roots of two other than two.

Proof: From (27), we know a root of $U_{2n+1}(x) = 0$ must be a 2n + 1-th Chebyshev root of two. From the recurrence relationship we have that $U_{2n+1}(2) = 2n + 1$, so two is not a root; the other $n$ roots are distinct from two and give the factors. □

We may do something similar for $S_n(x)$.
Proposition 36 If \( n > 1 \) is integral, then
\[
S_n(x) = \prod_{i=1}^{n-1} (x - \mu_i);
\]
where the \( \mu_i \) are the \( n - 1 \) 2n-th Chebyshev roots of two other than ±2.

Proof: From (23), we know a root of \( S_n(x) = 0 \) must be a 2n-th root of two. Since \( S_n(\pm 2) = \pm n \), all of the roots of \( S_n(x) \) must be distinct from ±2. □

Finally, we may characterize \( x^\odot n \) in the same way.

Proposition 37 For positive integers \( n \),
\[
x^\odot n = \prod_{i=1}^{n} (x - \mu_i);
\]
where the \( \mu_i \) are the \( n \) 4n-th Chebyshev roots of two which are not also 2n-th roots of two.

Proof: If \( \mu^\odot n = 0 \), then \( \mu^\odot 4n = 0^\odot 4 = 2 \), and hence any root of \( x^\odot n = 0 \) must be a 4n-th root of two. However if \( x^\odot 2n = 2 \), so that \( x \) is also a 2n-th root of two, then \( x^\odot n = \pm 2 \). Hence \( \mu \) must be a 4n-th root of two which is not a 2n-th root of two; there are \( n \) of these, which is the degree of \( x^\odot n \), so it must have all of them as roots. □

Theorem 38 For positive integral \( n \), we have the factorizations
\[
S_{2n}(x) = S_n(x)x^\odot n
\]
\[
S_{2n+1}(x) = (-1)^n U_{2n+1}(x)U_{2n+1}(-x)
\]

Proof: The first formula is an immediate consequence of (9). The second follows from the fact that since \( x^\odot 4n+2 - 2 = (x^\odot 2n+1 + 2)(x^\odot 2n+1 - 2) \), the roots of \( U_{2n+1}(-x) \) are the \( 2n + 1 \)-th roots of \(-2\) other than \(-2\), which is to say the \( 4n + 2 \)-th roots of two which are not also ±2 or 2n + 1-th roots of two. □

For Chebyshev powers, we have the following.

Theorem 39 For positive integral \( n \),
\[
x^\odot 2n+1 = (-1)^n xU_{2n+1}(2 - x^2)
\]
Proof: As before, the roots of $U_{2n+1}(-x)$ are $2n+1$-th Chebyshev roots of $-2$ other than $-2$; hence the roots of $U_{2n+1}(-x^2)$ are $2n+1$-th Chebyshev roots of 0 other than 0. Adding the factor $x$ to get the root $x = 0$ and multiplying by $(-1)^n$ to ensure the polynomial is monic, we obtain the theorem. □

For exponents which are powers of two, we have the following.

**Theorem 40** If $n = 2^m$ is a power of two, the polynomial $x^\otimes n$ is irreducible over $\mathbb{Q}$.

Proof: We have $$x^\otimes n \equiv x^n \pmod{2}$$
by Proposition 3. Hence each of the coefficients of $x^\otimes n$ other than the leading term is divisible by two. Since $x^\otimes n = (x^\otimes n/2)^2 - 2$, the constant term is divisible by two only once, and so the polynomial is Eisenstein, and hence irreducible. □

Putting (28) and (40) together, we have the following.

**Corollary 41** If $n = lm$, where $l$ is a power of two and $m$ is odd, then $$x^\otimes n = x^\otimes l U_m(-x^\otimes l).$$

We therefore may conclude that the problem of factoring our various polynomials reduces to the problem of factoring $U_{2n+1}(x)$.

**Corollary 42** The $2n+1$-th Chebyshev roots of two other than two are units in the maximal real subfield of $2n+1$-th roots of unity.

**Proposition 43** If $n$ is not divisible by four, the $n$-th Chebyshev roots of unity other than $\pm 2$ are units in the maximal real subfield $\mathbb{Q}(\mu)$ of the cyclotomic field $\mathbb{Q}(\zeta)$, where $\zeta$ and $\mu$ are primitive $n$-th roots of unity and primitive Chebyshev $n$-th roots of two, respectively.

The irreducible factors of $x^n - 1$ are the cyclotomic polynomials. The cyclotomic polynomial (of degree $\phi(n)$) whose roots are the primitive roots of unity of order $n$ is called the $n$-th cyclotomic polynomial, and is ordinarily denoted $\Phi_n$. If $\zeta$ is a primitive $n$-th root of unity, then so is $\zeta^{-1}$. Hence the coefficients of $\Phi_n$ for $n > 1$ are palindromic; if

$$\Phi_n(x) = \sum_{i=0}^{\phi(n)} a_i x^{\phi(n)-i}$$
then $a_i = a_{\phi(n)-i}$. Since $\phi(n)$ is even when $n > 2$, we may make the following definition.

**Definition 11** For $n > 2$ we define the $n$-th Chebyshev-cyclotomic polynomial to be

$$\Psi_n(x) = \left( \sum_{i=0}^{\phi(n)/2} a_i x^{\phi(n)/2-i} \right) \circ \Phi_n(x) = \sum_{i=0}^{\phi(n)} a_i x^{\phi(n)-i},$$

where

$$\Phi_n(x) = \sum_{i=0}^{\phi(n)} a_i x^{\phi(n)-i}$$

is the $n$-th cyclotomic polynomial. We also set $\Psi_1(x) = 1$.

We then have

**Theorem 44** For $n > 2$, $\Psi_n$ is an irreducible polynomial of degree $\phi(n)/2$ whose roots are the $\phi(n)/2$ primitive $n$-th Chebyshev roots of two.

Proof: Since $n > 2$, $\phi(n)$ is even. If $x = z + z^{-1}$ and $\Phi_n(z) = 0$, we may divide by $z^{\phi(n)/2}$ and collect terms, obtaining

$$\sum_{i=0}^{\phi(n)/2} a_i z^{\phi(n)/2-i} + a_{n-i} z^{i-\phi(n)/2}.$$ 

Since $a_i = a_{\phi(n)-i}$, this can be rewritten $\Psi_n(x)$. If $i$ is relatively prime to $n$, $z^i$ is a Galois conjugate of $z$; the roots of $\Phi_n(z)$ are the $\phi(n)$ Galois conjugates $z^i$ for $0 < i < n$ prime to $n$, and so $\Phi_n$ is irreducible. In the same way, $x^{\phi(n)/2}$ for $i$ prime to $n$ is a root of $\Psi_n$. If we take all $i$ prime to $n$ such that $0 < i < n/2$, we obtain a full orbit of conjugates in number equal to the degree $\phi(n)/2$ of $\Psi_n$, and hence $\Psi_n$ must be irreducible. $\square$

**Theorem 45** For any positive integer $n$, we may factor $U_{2n+1}(x)$ by

$$U_{2n+1}(x) = \prod_{d|2n+1} \Psi_d(x) \quad (29)$$

Proof: Each side of the equation is a monic polynomial whose roots are all of the $2n+1$-th Chebyshev roots of two other than two. $\square$
2.3 Differential Equations

The product formula (10) for the product of two Chebyshev functions of the second kind allows us as an immediate consequence to find differential equations satisfied by Chebyshev exponents.

**Theorem 46** If \( y = x^{\oplus k} \), then
\[
(x^2 - 4)(y')^2 - k^2(y^2 - 4) = 0
\]  
(30)

Proof: From (23) we obtain
\[
(x^2 - 4)S_k^2(x) = x^{\oplus 2k} - 2 = (x^{\oplus k})^2 - 4.
\]
Substituting \( y'/k \) for \( S_k(x) \), we obtain the theorem. □

From this nonlinear equation of order one we can now find a linear equation of order two.

**Theorem 47** If \( y = x^{\oplus k} \), then
\[
(x^2 - 4)y'' + xy' - k^2y = 0
\]  
(31)

Proof: Take the derivative of (30) and divide by \( 2y' \). □

These equations are familiar as the differential equations satisfied by Chebyshev polynomials for integral \( k \); here we are assuming \( k \) may be any complex number.

We now may find a recursive expression for higher derivatives.

**Theorem 48** If \( y = x^{\oplus k} \), then
\[
(x^2 - 4)y^{(n+2)} + (2n + 1)xy^{(n+1)} + (n^2 - k^2)y^{(n)}
\]  
(32)

By an induction hypothesis, we may assume
\[
(x^2 - 4)y^{(n+1)} + (2n - 1)xy^{(n)} + ((n - 1)^2 - k^2)y^{(n-1)} = 0.
\]
Taking the derivative, we obtain the theorem. □

Since (31) is a second-order linear equation, it must have a second linearly independent solution. When \( k \) is not an integer we have the following.

**Proposition 49** For any \( k \), \( y = (-x)^{\oplus k} \) satisfies the differential equation (31). When \( k \) is not an integer, it is linearly independent of \( x^{\oplus k} \).
Proof: Substituting $-x$ for $x$ and $-y'$ for $y'$ in (31) leaves it unchanged, and hence $(-x)^\circ k$ is a solution.

By Taylor’s theorem, we may express $x^\circ k$ in a power series of radius of convergence two around $x = 0$, since the nearest singularity is at $x = -2$. We then have

$$x^\circ k = 0^\circ k \mathcal{E}(x) + k\mathcal{S}_k(0)\mathcal{O}(x),$$

where $\mathcal{E}$ is an even function, and $\mathcal{O}$ is an odd function. If $k$ is an odd integer, then $0^\circ k = 0$, whereas if it is an even integer, $k\mathcal{S}_k(0) = 0$; otherwise it is non-zero. Hence assuming $k$ is not an integer,

$$(-x)^\circ k = 0^\circ k \mathcal{E}(x) - k\mathcal{S}_k(0)\mathcal{O}(x)$$

is linearly independent of $x^\circ k$. □

This theorem is illuminating in connection with the problem of defining the branches of the Chebyshev exponent, since any solution of the equation (31) for non-integral $k$ must be expressible at least locally as a linear combination of $x^\circ k$ and $(-x)^\circ k$.

We may approach the matter in another way, which will also cover the case where $k$ is a non-zero integer.

**Proposition 50** Let

$$y = \sqrt{x^2 - 4} \mathcal{S}_k(x).$$

Then $y$ satisfies the second order differential equation (31) for the Chebyshev exponent, namely

$$(x^2 - 4)y'' + xy' - k^2 y = 0.$$

Proof: Let $w = \mathcal{S}_k(x)$. From (32) we have

$$(x^2 - 4)w'' + 3xw' + (1 - k^2)w = 0.$$ 

By differentiating, we find that

$$\sqrt{x^2 - 4}w = y,$$

$$\sqrt{x^2 - 4}w' = y' - \frac{xy}{x^2 - 4},$$

$$\sqrt{x^2 - 4}w'' = y'' - \frac{y + 2xy'}{x^2 - 4} + \frac{3x^2 y}{(x^2 - 4)^2}.$$ 

Substituting, we obtain (31). □
\textbf{Proposition 51} For all nonzero $k$, $\sqrt{x^2 - 4S_k(x)}$ is linearly independent of $x^{\circ k}$

Proof: The Chebyshev power function $x^{\circ k}$ is ramified only at $x = -2$, whereas $\sqrt{x^2 - 4S_k(x)}$ ramifies when $x = 2$. $\Box$

Note that this second solution can be expressed in other ways, up to a possible change of sign, for we have

$$\sqrt{x^2 - 4S_k(x)} = \pm \sqrt{x - 2U_{2k}(x)} = \pm \sqrt{x^{\circ 2k} - 2};$$

for most purposes $\sqrt{x - 2U_{2k}(x)}$ is the preferred version.

It should also be noted that we could use instead the function $\sqrt{x^2 - 4S_k(x)}/k$ in all cases, for we have the following limit.

\textbf{Proposition 52} We have

$$\lim_{k \to 0} \frac{1}{k} \sqrt{x^2 - 4S_k(x)} = \pm 2 \log_c x.$$

Proof: Let $x = z + z^{-1}$. From the definition of $S_k(x)$, we then have

$$\sqrt{x^2 - 4S_k(x)} = \pm (z^k - z^{-k}).$$

By l'Hôpital’s rule,

$$\lim_{k \to 0} \frac{z^k - z^{-k}}{k} = 2 \log z = \pm 2 \log_c x,$$

and hence the proposition. $\Box$

\section*{2.4 Power Series Expansions}

Despite any appearances to the contrary, $x^{\circ k}$ is analytic in a neighborhood of $x = 2$. In fact, in analogy with the binomial expansion of $(1 + x)^k$, we have the following.

\textbf{Theorem 53} For any complex $k$ and any $x$ such that $|x| < 4$, $(2 + x)^{\circ k}$ can be expressed as a power series around $x = 0$ by

\begin{equation}
(2 + x)^{\circ k} = 2 + k \sum_{i=1}^{\infty} \left( \frac{k + i - 1}{2i - 1} \right) x^i
= 2 + k^2 x + \frac{2}{4!} k^2 (k^2 - 1)x^2 + \frac{2}{6!} k^2 (k^2 - 1)(k^2 - 4)x^3 + \cdots.
\end{equation}
Proof: From (32), we have that when $x = 2$, if $y = x^{c_x}$, then

$$y^{(n+1)} = \frac{k^2 - n^2}{4n + 2} y^{(n)}.$$ 

From this we deduce that if

$$y = \sum_{n=0}^{\infty} a_n x^n$$

then

$$a_{n+1} = \frac{1}{n+1} \frac{k^2 - n^2}{4n + 2} a_n = \frac{k^2 - n^2}{(2n + 1)(2n + 2)} a_n.$$ 

Since $a_0 = 2$, the series follows by induction.

Since

$$\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = -\frac{1}{4},$$

the series converges absolutely when $|x| < 4$, as expected, and does not converge anywhere on the circle $|x| = 4$. \(\square\)

We now also have

**Corollary 54** For $|x| < 4$, we have

$$S_k(2 + x) = \sum_{i=0}^{\infty} \left( \frac{k}{2i + 1} \right) x^i$$ 

(34)

Proof: Termwise differentiation. \(\square\)

The reader familiar with hypergeometric functions will not be surprised to learn that the above power series expansions are special cases of the classical hypergeometric series of Gauss.

**Theorem 55** Let

$$F(a, b, c, x) = \sum_{i=0}^{\infty} \binom{a+i-1}{i} \binom{b+i-1}{c+i-1} x^i$$

be the hypergeometric series of Gauss, convergent (at least) when $|x| < 1$. Then for $|x| < 4$,

$$(2 + x)^c = 2F(k, -k, \frac{1}{2}, -\frac{x}{4})$$ 

(35)
Proof: Comparison of the two series shows they are identical. □

With a corresponding choice of branch, we might now define the Chebyshev exponent as

\[ x^{\otimes k} = 2F(k, -k, \frac{1}{2}, \frac{2 - x}{4}) \]  \hspace{1cm} (36)

where \( F \) is now the classical hypergeometric function, defined even where the hypergeometric series does not converge by analytic continuation. We might then use the varied and sometimes complicated properties of the hypergeometric function to produce a variety of formulas and series expansions. We will not pursue the matter, as only a few series will be important for later sections.

We will also note that one property of the hypergeometric function is that its derivatives to all orders are also hypergeometric functions. This gives us a formula for the higher derivatives of Chebyshev powers.

**Proposition 56** For any positive integer \( n \), we have

\[ (x^{\otimes k})^{(n)} = (n - 1)!k \left( \frac{k + n - 1}{2n - 1} \right) F(n + k, n - k, n + \frac{1}{2}, \frac{2 - x}{4}) \]  \hspace{1cm} (37)

**Proof:** It is immediate from the power series expansion that

\[ F'(a, b, c, x) = \frac{ab}{c} F(a + 1, b + 1, c + 1, x). \]

Applying this to (36) gives us the theorem. □

We also have a nice expression for \( U_k \) in terms of a power series expansion around \( x = 2 \).

**Theorem 57** For any complex \( k \) and for \( |x| < 4 \), we have

\[ U_k(2 + x) = k \sum_{i=0}^{\infty} \frac{(k - 1)/2 + i}{2i} \frac{x^i}{2i + 1} \]

\[ = k + \frac{k(k^2 - 1^2)}{3!} \left( \frac{x}{4} \right) + \frac{k(k^2 - 1^2)(k^2 - 3^2)}{5!} \left( \frac{x}{4} \right)^2 + \cdots \]  \hspace{1cm} (38)

**Proof:** From 54, we have

\[ U_k(2 + x) = \sum_{i=0}^{\infty} \left( \frac{(k - 1)/2 + i}{2i + 1} \right) + \left( \frac{(k + 1)/2 + i}{2i + 1} \right) x^i. \]
For any non-negative integer $n$, we define a polynomial $p_n$ of degree $2n + 1$ by setting

$$p_n(x) = \frac{(x - 1)/2 + n}{2n + 1} + \frac{(x + 1)/2 + n}{2n + 1}.$$ 

We then have $p_n(m) = 0$ for any odd integer $-2n < m < 2n$, and we also have $p_n(0) = 0$. Hence we have $2n + 1$ distinct roots for a polynomial of degree $2n + 1$, which must therefore all be simple roots. We also have a coefficient on the leading term of $1/(4^n(2n+1))$. Since these conditions determine $p_n$, we have

$$p_n(x) = \frac{x}{4^n(2n+1)} \frac{(x - 1)/2 + n}{2n},$$

since they are also plainly true of the right-hand side. Hence the $i$-th term of the series is $p_i(k)x^i$, and we have the theorem. \(\square\)

**Corollary 58** For an appropriate choice of branch cut, we have

$$U_k(x) = kF\left(\frac{1+k}{2}, \frac{1-k}{2}, \frac{3}{2}, \frac{2-x}{4}\right).$$

Proof: Comparison of the power series expansion around $x = 2$. \(\square\)

We may use the expansion for $U$ in expressing $x^\circ k$ in power series; we began with the expansion around $x = 0$.

**Theorem 59** The power series for $x^\circ k$ expanded around $x = 0$ and valid for $|x| < 2$ can be obtained by expanding

$$x^\circ k = \cos(\frac{\pi}{2} k)(2 - x^2)^{\circ k/2} + \sin(\frac{\pi}{2} k)xU_k(2 - x^2)$$

(40)

Proof: Substituting $x = 0$ into (32), we obtain

$$y^{(n+2)} = \frac{n^2 - k^2}{4} y^{(n)};$$

so that if $a_n$ is the coefficient of $x^n$, we have

$$a_{n+2} = \frac{n^2 - k^2}{4(n+1)(n+2)} a_n.$$
Using this, we find the even part of the power series expansion is
\[ 0 \circ^k (1 - \frac{k^2}{2!} \left(\frac{x}{2}\right)^2 + \frac{k^2(k^2 - 2^2)}{4!} \left(\frac{x}{2}\right)^4 - \frac{k^2(k^2 - 2^2)(k^2 - 4^2)}{6!} \left(\frac{x}{2}\right)^6 + \ldots). \]

Noting that
\[ \frac{0 \circ^k}{2} = \frac{i^k + i^{-k}}{2} = \cos \frac{\pi}{2} k, \]
and comparing power series expansions, we find that this is
\[ \cos \frac{\pi}{2} k (2 - x^2) \circ^{k/2}. \]

In the same way, we find that the odd part of the power series expansion is
\[ kS_k(0)x(1 - \frac{k^2 - 1^2}{3!} \left(\frac{x}{2}\right)^2 + \frac{(k^2 - 1^2)(k^2 - 3^2)}{5!} \left(\frac{x}{2}\right)^4 + \ldots), \]
by noting that
\[ S_k(0) = \frac{i^k - i^{-k}}{i - i^{-1}} = \sin \frac{\pi}{2} k, \]
and comparing power series expansions with
\[ \sin(\frac{\pi}{2} k)xU_k(2 - x^2) \]
we see that they are identical. \(\square\)

Of course, having established this identity when \(|x| \leq 2\) we may continue it analytically. We then also get the following identities, valid everywhere.

**Corollary 60** When \(k\) is not an odd integer, we have
\[ (2 - x^2) \circ^{k/2} = \frac{x \circ^k + (-x) \circ^k}{2 \cos(\pi k/2)}, \]
while if \(k\) is not an even integer we have
\[ xU_k(2 - x^2) = \frac{x \circ^k - (-x) \circ^k}{2 \sin(\pi k/2)}. \]

We may also use this identity to derive the Puiseux series expansion (in powers of \(\sqrt{x + 2} = \circ\sqrt{x}\)) around \(x = -2\).
Theorem 61  The Puiseux series expansion of $x \odot^k$ around the ramified place $x = -2$ is given by expanding each part of
\[ x \odot^k = \cos(\pi k)(-x)^\odot^k + \sin(\pi k)\sqrt{x + 2U_{2k}}(-x) \]
in power series.

Proof: We may write (40) as
\[ x \odot^k = \cos(\frac{\pi}{2})k(-x^2)^\odot^{k/2} + \sin(\frac{\pi}{2})kU_k(-x^2). \]

If rewrite $x \odot^k$ in this expression as $(\sqrt[2]{x})^\odot^{2k}$, we obtain
\[ x \odot^k = \cos(\pi k)(-x)^\odot^k + \sin(\pi k)\sqrt[2]{x}U_k(-x). \]
\[ \blacksquare \]

3  Polynomial Equations

3.1  The $n$ Chebyshev $n$-th Roots

In any field containing $n$-th roots of unity, if we know an element $r_0$ such that $r_0^n = t$, it is easy to find the other $n - 1$ roots of $x^n - t = 0$; for we simply take a primitive $n$-th root of unity $\zeta$, and the other roots will be $r_i = \zeta^i r_0$.

The situation is similar though a bit more complicated for Chebyshev roots.

Definition 12  Let $n$ be a positive integer, and let $r$ be $n$ elements of an algebraically closed field of characteristic not dividing $n$, indexed by residue classes modulo $n$. Let $\zeta$ be a primitive $n$-th root of unity. Noting if $i$ is a residue class modulo $n$ that $\zeta^n$ is well-defined, we will call $r$ an indexed set of $n$-th Chebyshev roots of $t$ if $r_0 = u + u^{-1}$, $r_0^\odot^n = t$, and
\[ r_i = \zeta^i u + \zeta^{-i} u^{-1}. \]

Proposition 62  Let $r$ be an indexed set of $n$-th Chebyshev roots of $t$. Then the $n$ roots of $x \odot^n - t = 0$ are in fact given by $r$.

Proof:
\[ r_i^\odot^n = u^n + u^{-n} = r_0^\odot^n = t. \]
\[ \blacksquare \]
Proposition 63 Let \( r \) be an indexed set of \( n \)-th Chebyshev roots of \( t \). Then

\[
r_{k+i} + r_{k-i} = \mu^i r_k,
\]

where \( \mu = \zeta + \zeta^{-1} \) is a primitive \( n \)-th Chebyshev root of two corresponding to the primitive \( n \)-th root of unity \( \zeta \).

Proof: By the previous theorem,

\[
r_{k+i} + r_{k-i} = (\zeta^{k+i} + \zeta^{k-i})u + (\zeta^{-k-i} + \zeta^{-k-i})u^{-1}
\]

\[
= (\zeta^i + \zeta^{-i})\zeta^k u + (\zeta^i + \zeta^{-i})\zeta^{-k} u^{-1} = \mu^i r_k.
\]

\[\square\]

We may express all of the roots of \( t \) in terms of just one of them, by means of a quadratic equation.

Proposition 64 Let \( r \) be an indexed set of \( n \)-th Chebyshev roots of \( t \), with \( \mu \) as before. Then we have that \( r_{k+i} \) and \( r_{k-i} \) are the two roots of

\[
x^2 - \mu^i r_k x + r_k^{\circ} 2 + \mu^{\circ} 2i = 0.
\]

Proof: The trace term (the coefficient of \( x \)) is given by the previous proposition. The constant term we may find by multiplying out

\[
r_{k+i} r_{k-i} = (\zeta^{k+i} + \zeta^{-k-i} u^{-1})(\zeta^{k-i} u + \zeta^{-k} u^{-1})
\]

\[
= \zeta^{2k} u^2 + z^{-2k} u^{-2} + \zeta^{2i} + \zeta^{-2i} = r_k^{\circ} 2 + \mu^{\circ} 2i.
\]

\[\square\]

We may also express all of the roots as a linear combination of two of them.

Proposition 65 Let \( \mu \) be as before, let \( r \) be an indexed set of \( n \)-th Chebyshev roots of \( t \), and let \( r_i \) and \( r_j \) be two of the roots. Then if \( e = j - i \) and if \( k \) is an integer, we have that

\[
r_{i+ke} = S_k(\mu^{\circ} e)r_j - S_{k-1}(\mu^{\circ} e)r_i.
\]

\[41\]
Proof: The proposition is clearly true when \( k = 0 \) or \( k = 1 \), and by induction we may assume it to be true up to \( k - 1 \). Then

\[
ri + ke + r_i + (k-2)e = \mu \odot e r_i + (k-1)e.
\]

Hence

\[
ri + ke = \mu \odot e (Sk-1(\mu \odot e) r_j - Sk-2(\mu \odot e) r_i) - Sk-2(\mu \odot e) r_j + Sk-3(\mu \odot e) r_i
\]

\[
= (\mu \odot e Sk-1(\mu \odot e) r_j - (\mu \odot e Sk-2(\mu \odot e) r_j - Sk-3(\mu \odot e) r_i)
\]

\[
= Sk(\mu \odot e) r_j - Sk-1(\mu \odot e) r_j.
\]

The theorem for negative values of \( k \) now follows immediately. \( \Box \)

**Theorem 66** Let \( r \) be an indexed set of \( n \)-th Chebyshev roots of \( t \), and let \( i \) and \( j \) be indices such that \( e = j - i \) is prime to \( n \). This last condition means that any element of the residue class \( e \) is prime to \( n \), and entails \( e \neq 0 \), so that \( i \neq j \). If \( \mu \) is as before, then if the base field is of characteristic 0, any root \( r_i \) of \( t \) can be written as a \( \mathbb{Z}[\mu] \)-linear combination of \( r_i \) and \( r_j \); while if it is of characteristic \( p \), with \( n \) not divisible by \( p \), then it is an \( \mathbb{F}_p(\mu) \)-linear combination of \( r_i \) and \( r_j \), where \( \mathbb{F}_p \) is the field of \( p \) elements.

Proof: That \( e \) is prime to \( n \) means that \( e \) is invertible, which is to say, \( ke \) for values of \( k \) from 0 to \( n - 1 \) traverse a complete set of residue classes modulo \( n \). Hence \( i + ke \) traverses a complete set of residue classes, and so \( ri + ke \) is a complete set of Chebyshev roots of \( t \), taken in another order.

In characteristic 0, \( \mathbb{Q}(\mu) \) is cyclic and since \( e \) is prime to \( n \), \( \mathbb{Q}(\mu \odot e) = \mathbb{Q}(\mu) \), and any polynomial in \( \mu \odot e \) can also be expressed as a polynomial (of degree less than \( \phi(n)/2 \)) in \( \mu \). Hence any root is a \( \mathbb{Z}[\mu] \)-linear combination of \( r_i \) and \( r_j \). The theorem in characteristic \( p \) follows immediately on reduction modulo \( p \). \( \Box \)

### 3.2 Branches of the Chebyshev Radical

When considering ordinary \( n \)-th roots, it is of course of considerable utility to have in mind the representation of complex numbers in terms of polar coordinates. If we wish for something similar in the case of Chebyshev \( n \)-th roots, we may began by considering the curves which arise by setting the real or imaginary part of \( \log \odot z \) to a constant value.
If \( \log z = r + i\theta \), then

\[
z = \exp(z) = 2 \cosh r \cos \theta + 2 \sinh r \sin \theta.
\]

If \( x = 2 \cosh r \cos \theta \) and \( y = 2 \sinh r \sin \theta \), then

\[
\frac{x^2}{4 \cosh^2 r} + \frac{y^2}{4 \sinh^2 r} = 1 \tag{42}
\]

\[
\frac{x^2}{4 \cos^2 \theta} - \frac{y^2}{4 \sin^2 \theta} = 1 \tag{43}
\]

as we may check by substituting.

Hence if \( r \) is a constant, (42) gives the equation of an ellipse which we may take as the analog of a circle centered on the origin in polar coordinates. The analog of a ray from the origin of constant angular argument \( \theta \) would be either the top or the bottom part of one branch of the hyperbola (43). To get the analog of a line through the origin, we take a complete branch of the hyperbola, and hence half a hyperbola instead of one quarter of one.

We may view the matter more schematically by representing any complex number \( z \) which is not a real number of absolute value less than or equal to two by the polar coordinates \( r \) and \( \theta \), where \( \log z = r + i\theta \). Real \( z \) between \(-2\) and \(2\) correspond to the origin; however, since \( \theta \) varies, we may “blow up” the origin into a circle which doubly covers the line segment \(-2 \leq z \leq 2\) by taking \( \theta \) in the range \(-\pi < \theta \leq \pi\), so that we have a complete circle, and then identifying the point represented by \( \theta \) with that represented by \(-\theta\).

Bearing the above representations in mind, we may make the following definition.

**Definition 13** Let \( t \) be any complex number, \( n \) a positive integer, \( l \) any integer, and \( z_0 = \sqrt[n]{t} \) the \( n \)-th Chebyshev root of \( t \). If

\[
\log z_0 = r + i\theta_0,
\]

we define an indexed set of complex branches of the \( n \)-th Chebyshev root by

\[
z_l = \sqrt[n]{t} = \exp(r + i\theta_l),
\]

where either \( r \geq 0 \) and

\[
\frac{l\pi}{n} \leq \theta_l < \frac{(l+1)\pi}{n} \tag{44}
\]
or $r > 0$ and

$$-\frac{(l+1)\pi}{n} \leq \theta_l < -\frac{l\pi}{n}$$

(45)

obtains; and such that

$$\theta_l \equiv \theta_0 \pmod{\frac{2\pi}{n}}.$$

**Definition 14** For any rational exponent $p/q$, we define $x^{p/q}_c$ as

$$x^{p/q}_c = \left( x^{c/q}_c \right)^p.$$

**Proposition 67** The functions $c^{n}_c$ and $c^{n}_j$ are identical if and only if $i \equiv j \pmod{2n}$ or $i + j + 1 \equiv 0 \pmod{2n}$.

**Proof:** If $i \equiv j \pmod{2n}$, then (44) and (45) differ by a multiple of $2\pi$, which is the period of $\exp_c$. Substituting $-l - 1$ for $l$ in (44) gives us (45), and vice-versa. Putting these together, we get the two conditions of the theorem. No more conditions are possible, since $i + j + 1 \equiv 0 \pmod{2n}$ identifies pairs of congruence classes modulo $2n$, and hence leads to $n$ classes, which may not be further identified since we have $n$ branches. □

**Theorem 68** The $n$ functions defined above do in fact constitute $n$ distinct branches of the $n$-th Chebyshev root.

**Proof:** It is immediate that

$$\left( c^{n}_c \right)^n = \exp_c(nr + in\theta_l) = \exp(nr + in\theta_0 + 2\pi m) = z_0^n = t,$$

since $m$ is an integer; and hence each $c^{n}_c$ is an $n$-th Chebyshev root of $t$.

Since $\exp_c$ is an even function, we are at liberty to replace $\log_c$ by $-\log_c$, and so (45) by the condition $r < 0$ and

$$\frac{l\pi}{n} < \theta_l \leq \frac{(l+1)\pi}{n}.$$

We therefore have that an equivalent condition is merely (44) with $r$ allowed to take any real value.

Multiplication by $n$ now gives a continuous bijection between the pathwise connected region so defined and the region (for all $r$)

$$l\pi < \theta \leq (l+1)\pi,$$
and Chebyshev exponentiation by \( \exp_\otimes \) gives a continuous bijection from this to the whole complex plane. Hence \( x = \sqrt[n]{t} \) gives a bijection from the complex plane to a pathwise connected region of the complex plane, whose inverse map is the \( n \)-th Chebyshev power map \( t = x^{\otimes n} \). Since there are \( n \) non-overlapping regions defined by these branches which together cover the complex plane, we have defined a complete set of branches. \( \square \)

Since \( r_i = \sqrt[n]{t} \) gives us an indexed set of \( n \)-th Chebyshev roots of \( t \), we could use Theorem 66 to write all of the roots in terms of two of them. However, it seems better to refer back instead to the linear differential equation (31) satisfied by Chebyshev powers, and to note that any solution must, at least locally, be a linear combination of \( \sqrt[n]{t} \) and \( \sqrt[n]{-t} \); we therefore seek to write the branches as a combination of these functions, and hence in terms of the principal branch.

**Theorem 69** Let \( \mu = \exp_\otimes (\pi i/n) = \sqrt[n]{-2} \). Then for even integers \( 0 \leq i < n \) we have

\[
\sqrt[n]{t} = S_{i+1}(\mu) \otimes \sqrt[n]{t} - S_i(\mu) \otimes \sqrt[n]{-t},
\]  

while for odd integers \( 0 \leq i < n \) we have

\[
\sqrt[n]{t} = -S_i(\mu) \otimes \sqrt[n]{t} + S_{i+1}(\mu) \otimes \sqrt[n]{t}.
\]

**Proof:** Let \( F(t) = \sqrt[n]{t} \), and let \( E(t) \) and \( O(t) \) be the even and odd parts of \( \sqrt[n]{t} \), so that

\[
E(t) = \frac{\sqrt[n]{t} + \sqrt[n]{-t}}{2}, \quad O(t) = \frac{\sqrt[n]{t} - \sqrt[n]{-t}}{2}.
\]

Since \( F \) satisfies (31) and \( E \) and \( O \) form a basis for the solutions, we have, at least in a neighborhood of zero,

\[
F(t) = \frac{F(0)}{E(0)} E(t) + \frac{F'(0)}{O'(0)} O(t).
\]

We now find

\[
E(0) = \sqrt[n]{0} = 2 \cos(\frac{\pi}{2n}), \quad F(0) = 0^{(2i+1)/n} = 2 \cos(\frac{\pi}{2n}(2i + 1)),
\]
\[ O'(0) = \frac{1}{n} S_{1/n}(0) = \sin\left(\frac{\pi}{2n}\right)/n, \]

\[ F'(0) = \frac{1}{n S_{n}(0)\circ(2i+1)/n} = (-1)^i \sin\left(\frac{\pi}{2n}(2i+1)\right)/n. \]

Hence we have

\[ F(t) = (\cos(\frac{\pi}{2n}(2i+1))/\cos(\frac{\pi}{2n})) E(t) + (-1)^i(\sin(\frac{\pi}{2n}(2i+1))/\sin(\frac{\pi}{2n})) O(t) \]

We now write everything with a common denominator of

\[ 2\sin(\frac{\pi}{2n})\cos(\frac{\pi}{2n}) = \sin(\frac{\pi}{n}), \]

and apply the angle-sum trigonometrical identities to write this in terms of \( \circ\sqrt{i} \) and \( \circ\sqrt{-i} \), obtaining for even values of \( i \)

\[ F(t) = (\sin(\frac{\pi}{n}(i+1))/\sin(\frac{\pi}{n})) \circ\sqrt{i} - (\sin(\frac{\pi}{n}i)/\sin(\frac{\pi}{n})) \circ\sqrt{-i}, \quad (49) \]

and for odd values of \( i \)

\[ F(t) = -(\sin(\frac{\pi}{n}i)/\sin(\frac{\pi}{n})) \circ\sqrt{i} + (\sin(\frac{\pi}{n}(i+1))/\sin(\frac{\pi}{n})) \circ\sqrt{-i}. \quad (50) \]

Putting this into the more algebraic language we used for general fields, we obtain the identities in the theorem.

All that remains is to check where these identities are valid; but since they mesh together in the correct way along the branch cuts from \((2, \infty)\) and \((-\infty, -2)\) they are in fact valid everywhere. \( \square \)

### 3.3 Polynomial Equations and Radicals

Perhaps the best known of all the uses to which the extraction of roots has been put is in the solution of algebraic equations in terms of radicals. This has been productive not so much for what it accomplished, but for what it failed to accomplish. It failed to express real roots in terms of real numbers, and in so doing it lead to the discovery of complex numbers. It failed to express the roots of all polynomials with integer coefficients, and in so doing it led to the discovery of Galois theory. With the notable exception of the quadratic formula, it has also failed to be of much practical use.
Some of these failures can be ameliorated by the simple expedient of employing Chebyshev radicals in the place of ordinary ones. In particular, while expressions for roots in terms of Chebyshev radicals will still grow more complicated with increasing degree, they are nearly always less complicated. It is in fact rather remarkable with what persistence people have insisted on radicals as the only canonical family of algebraic functions, given the problems that this sometimes entails.

In practice, a solution in radicals usually means an expression for the roots of a polynomial in terms of the function $\sqrt[n]{z}$. In a general algebraic context, it is taken to mean a solution in terms of a tower of extensions, each of which is given by an $n$-th root, for various $n$. The following definition is standard.

**Definition 15** Let $P$ be a univariate polynomial with coefficients in a field $F_0$, and suppose that $P$ may be completely factored into linear factors in a field $F_n$, where we have for each $i$ with $0 < i \leq n$ that $F_i = F_{i-1}(r_i)$ with $r_i^{m_i} \in F_{i-1}$. We then say that $P$ is solvable in radicals relative to the field $F_0$.

It is of course obvious what the Chebyshev version of this should be.

**Definition 16** Let $P$ and $F_0$ be as before, and suppose that $P$ may be completely factored into linear factors in a field $F_n$, where we have for each $i$ with $0 < i \leq n$ that $F_i = F_{i-1}(r_i)$ with $r_i^{m_i} \in F_{i-1}$. We then say that $P$ is solvable in Chebyshev radicals relative to the field $F_0$.

We now may prove the following fundamental theorem.

**Theorem 70** A polynomial with coefficients in a field whose characteristic is not two is solvable in radicals if and only if it is solvable in Chebyshev radicals.

Proof: Let us suppose that a polynomial $P$ has been solved in radicals with respect to a field $F_0$. We wish to see if it can also therefore be solved in Chebyshev radicals. Since the field $F_n$ is obtained as a tower of radical extensions over $F_0$, it suffices to consider whether any polynomial of the form $x^n - t$, with $t$ in a field $K$ of characteristic equal to that of $F_0$, may be solved in terms of Chebyshev radicals. Because of the composition property of exponents, we may reduce to the case where the exponent $x^a - t$ is a prime
$q$; and since in characteristic $p$, $x^p = x^{\circ p}$, we may further assume without loss of generality that $q$ is not equal to the characteristic.

We wish therefore to create a tower of Chebyshev radical extensions which will factor the polynomial $x^q - t$, where $t \in K_0$, and where we may assume $t \neq 0$ since we have trivially $0 \in K_0$.

We first set $K_1 = K_0(\mu)$, where $\mu \neq 2$ and $\mu^{\circ q} = 2$. Then we have $K_2 = K_1(\lambda)$, where $\lambda^{\circ q} = \mu^{\circ q} - 4$. Now set $\zeta = (\mu + \lambda)/2$. The pair of equations $\mu^2 + \lambda^2 = 4$, $\mu + \lambda = 2$ have solutions $\mu = 2$, $\lambda = 0$ and $\mu = 0$, $\lambda = 2$. The first solution is ruled out since we have assumed $\mu \neq 2$, and the second is ruled out since $0^{\circ q} \neq 2$ for any prime $q$. Hence $\zeta \neq 1$. However, we have

$$\left(\frac{\mu + \lambda}{2}\right)\left(\frac{\mu - \lambda}{2}\right) = \frac{\mu^{\circ q} - \lambda^{\circ q}}{4} = 1,$$

and so $\zeta^{-1} = (\mu - \lambda)/2$. Also,

$$\zeta^q + \zeta^{-q} = (\zeta + \zeta^{-1})^{\circ q} = \mu^{\circ q} = 2,$$

and so

$$\zeta^{2q} - 2\zeta^q + 1 = (\zeta^q - 1)^2 = 0.$$

It follows that $\zeta$ is a $q$-th root of unity not equal to one.

We now set $K_3 = K_2(s)$, where $s^{\circ q} = t + t^{-1}$. Finally, we set $K_4 = K_3(r)$, where $r^{\circ q} = s^{\circ q} - 4$.

We now proceed just as before, noting first that

$$\left(\frac{s + r}{2}\right)\left(\frac{s - r}{2}\right) = \frac{s^{\circ q} - r^{\circ q}}{4} = 1,$$

and so if $w = (s + r)/2$, $w^{-1} = (s - r)/2$. Then we have

$$w^q + w^{-q} = (w + w^{-1})^{\circ q} = s^{\circ q} = t + t^{-1},$$

and so

$$w^{2q} - (t + t^{-1})w^q + 1 = (w^q - t)(w^q - t^{-1}) = 0.$$

Hence either $w^q = t$ or $(w^{-1})^q = t$. We now may factor $x^q - t$ by taking its roots to be $\zeta^i w$ or $\zeta^i w^{-1}$, depending on which of $w$ or $w^{-1}$ has $q$-th powers equal to $t$.

To show the contrary direction, we need to create a tower of ordinary radical extensions which will factor $x^{\circ q} - t$, since we may make precisely the
same reductions as before, and assume without loss of generality that $q$ is a prime not equal to the characteristic of $F_0$.

We began by setting $K_1 = K_0(\zeta)$, where $\zeta^q = 1$ but $\zeta \neq 1$. Now set $K_2 = K_1(s)$, where $s^2 = t^2 - 4$. Finally, set $K_3 = K_2(r)$, where $r^q = (s + t)/2$. Noting as before that if $r^q = (s + t)/2$ we have $r^{-q} = (s - t)/2$, we have

$$(\zeta^i r + \zeta^{-i}r^{-1})^q = r^q + r^{-q} = s,$$

which therefore gives us the $q$ roots of $x^q = t$.

In characteristic two, Chebyshev radicals turn out to solve a wider class of polynomial equations than do ordinary radicals. We first prove the following lemma.

**Lemma 71** Let $F$ be a field of characteristic two, and let $a \neq 1$ be an element of $F$. If $b^3 = a/(a + 1)$ and if $b \neq a/(a + 1)$, then if we set $c = (a + 1)b$ then $c$ satisfies the equation

$$c^2 + ac + 1 = 0.$$

**Proof:** Since

$$\left(\frac{c}{a + 1}\right)^3 = \left(\frac{c}{a + 1}\right)^3 + \left(\frac{c}{a + 1}\right) = \frac{a}{(a + 1)^3},$$

we have on multiplying through by $(a + 1)^3$ that

$$c^3 + (a + 1)^2 c + a = 0.$$

Since $b \neq a/(a + 1)$ we have $c \neq a$; hence we may divide by $c + a$ and obtain

$$c^2 + ac + 1 = 0.$$

We now have the following.

**Theorem 72** If $P$ is a polynomial with coefficients in a field of characteristic two, then if $P$ can be solved in radicals it can be solved in Chebyshev radicals, but not conversely.
Proof: As before, we may reduce the problem to factoring $x^q + t$, where $q$ is an odd prime, over a field obtained as a tower of Chebyshev radical extensions of $K_0$. We first show that we may introduce $q$-th roots of unity.

If $\zeta \neq 1$ and $\zeta^3 = 1$, then $\zeta^{\circ 5} = \zeta(\zeta^2 + \zeta + 1)^2$, and so we may define $\zeta$ as a Chebyshev radical by setting $\zeta^{\circ 5} = 0$, $\zeta \neq 0$. In case $q = 3$, we define $K_1 = K_0(\zeta)$, and $K_2 = K_1$.

Now suppose $q \neq 3$ and we want to express $\zeta$ such that $\zeta^q = 1$ while $\zeta \neq 1$ by a Chebyshev radical extension. Let $\mu \neq 0$, while $\mu \neq 0$. We now set $F_1 = F_0(\mu)$, and let $\lambda^{\circ 3} = \mu/(\mu + 1)^3$ with $\lambda \neq \mu$. Since $q \neq 3$, $\mu \neq 1$ and this is well-defined. We now set $K_2 = K_1(\lambda)$. Invoking the lemma, if $\zeta = (\mu + 1)\lambda$ then $\zeta^2 + \mu\zeta + 1 = 0$, and hence $\mu = \zeta + \zeta^{-1}$, from which it follows that $\zeta^q = 1$ while $\zeta \neq 1$, and hence $K_2$ contains the $q$-th roots of unity.

We now set as before $K_3 = K_2(s)$, where $s^{\circ q} = t + t^{-1}$. Next we define $r$ by $r^{\circ q} = s/(s + 1)^3$, $r \neq s$, and set $K_4 = K_3(r)$. If $w = (s + 1)r$, we invoke the lemma and conclude that $w^2 + sw + 1 = 0$. Hence just as before we have

$$w^q + w^{-q} = (w + w^{-1})^{\circ q} = s^{\circ q} = t + t^{-1},$$

and so

$$w^{2q} - (t + t^{-1})w^q + 1 = (w^q - t)(w^q - t^{-1}) = 0,$$

so that either $w^q = t$ or $(w^{-1})^q = t$. Again we conclude that with the roots of unity in place, we may completely factor $P$ over the field $K_4$.

Let us now suppose that $F$ is a field of characteristic two containing the $q$-th roots of unity for every odd prime $q$, that $t$ is an element transcendental over $F$, and let $K_0 = F(t)$. From the lemma, the polynomial $x^2 + tx + 1$ may be solved in Chebyshev radicals over $K_0$. We will show it cannot be solved in ordinary radicals over $K_0$.

Suppose $K_n$ is a field obtained as a tower of radical extensions of prime degree over $K_0$ in which $x^2 + tx + 1$ may be factored. If $K_n/K_{n-1}$ is a radical extension, obtained by adjoining a square root, then the polynomial can be factored over $K_{n-1}$, since $x^2 + tx + 1$ is separable. On the other hand, if $K_n = K_{n-1}(r)$ where $r^q \in K_{n-1}$ and $q$ is an odd prime, then either $r \in K_{n-1}$ and so $K_n = K_{n-1}$, or $K_n$ is cyclic of degree $q$ over $K_{n-1}$, in which case $x^2 + tx + 1$ must already be factorable over $K_{n-1}$, since there exists no subextension. Iterating this process, we conclude that $x^2 + tx + 1$ is factorable over $K_0$, which is a contradiction. \(\square\)
3.4 Galois Theory

Before discussing how to go about solving polynomial equations in Chebyshev radicals, we will recall for the reader of some of the basic facts of Galois theory; the reader who wishes something more than this should consult any of the standard textbooks.

If $P$ is a polynomial with coefficients in $F$ for which we seek a solution in radicals, we may always first factor $P$ over $F$, and hence we may without loss of generality assume $P$ is irreducible over $F$. In characteristic $p$, any irreducible $P$ may be written as $P(x) = Q(x^q) = Q(x^q)$, where $q = p^n$ is a power of $p$ and where $Q$ is irreducible and separable; that is, $(Q, Q') = 1$, so that $Q$ has no repeated factors. Hence we may without loss of generality assume that $P$ is both irreducible and separable, since the $q$-th root extension is both an ordinary and a Chebyshev radical.

If $P$ is separable, its roots when adjoined to $F$ determine a Galois extension $K/F$. The automorphisms of $K$ fixing $F$ constitute finite group called the Galois group of the extension $K/F$; this group acts faithfully on the roots of $P$, giving a faithful permutation representation of the Galois group as a permutation group, which is transitive if $P$ is irreducible.

If the Galois group $G$ has a normal subgroup $H_1$ such that the quotient group $G/H_1$ is cyclic of prime degree, and $H_1$ in turn has a normal subgroup $H_2$ such that $H_1/H_2$ is cyclic of prime degree, and if we may continue in this way until we finally reach a normal subgroup $H_n$ which is itself cyclic of prime degree, we say that the group $G$ is solvable.

There are now two cases to be considered. The group $G$ may have an order which is prime to the characteristic of $F$; this will always be the case when $F$ is of characteristic 0. Or it may be divisible by the finite characteristic $p$ of $F$, in which case some of the composition factors will be of degree $p$.

In the first case, for each composition factor of order $q$, we may by definition solve the equation $x^q - 1$ in radicals, and hence we may add $q$-th roots of unity whenever they are not present. It is a basic fact of algebra (“Kummer theory”) that in a field $L$ of characteristic other than $p$ containing the $q$-th roots of unity, any cyclic extension of degree $q$ can be given by a radical; that is, by the roots of an equation $x^q = t$, with $t \in L$. Each of the composition factors can therefore be related to a corresponding cyclic extension defined by a radical, and in this way we can construct an extension containing the field $K$ derived by adjoining the roots of $P$ to $F$, and so factor $P$ in terms of
radicals; which is to say, solve it in terms of radicals. Since any polynomial which can be solved in radicals can be solved in Chebyshev radicals, it follows that any polynomial with solvable Galois group prime of order prime to the characteristic of the base field can be solved in Chebyshev radicals as well.

If the group $G$ is divisible by the characteristic $p$ of $F$, then $G$ will have composition factors which are cyclic of order $p$. These will correspond to cyclic field extensions of degree $p$. It is a basic fact of algebra ("Artin-Schreier theory") that in a field $L$ of characteristic $p$, any cyclic extension of degree $p$ can be given by a root (and hence all roots) of an equation $x^p - x = t$, with $t \in L$. It follows that if this equation can be solved in radicals (ordinary or Chebyshev), then any solvable extension in characteristic $p$ can be solved in radicals.

In this connection we have the following theorem.

**Theorem 73** In a field of characteristic two, we may solve the polynomial $x^2 + x + t$ in Chebyshev radicals.

Proof: As before, we note that if $x^2 + x + 1 = 0$ then $x^{\circ 5} = x(x^2 + x + 1)^2 = 0$, and hence $x^2 + x + 1$ is solvable in Chebyshev radicals. Hence we may assume that $t \neq 1$.

Suppose that $F_0$ is a field of characteristic two with $t \in F_0$. We define $F_1 = F_0(s)$, where $s^{\circ 2} = t + 1$. Following this, we define $F_2 = F_1(r)$, where $r \neq 1/s$ and $r^{\circ 3} = t/s^3$.

Now let $w = sr$. We claim that $w^2 + w + t = 0$. Since $r \neq 1/s$, we may multiply by $rs + 1$ without introducing any zeros. We then obtain

$$(w^2 + w + 1)(rs + 1) = s^3r^3 + (t + 1)sr + t,$$

and on dividing by $s^3$ we obtain

$$r^3 + r + t/s^3 = r^{\circ 3} + t/s^3 = 0.$$

□

It follows that in characteristic two, any extension $K/F$ for which the separable closure of $F$ in $K$ has a solvable Galois group over $F$ may be expressed in terms of Chebyshev radicals.

We also have the following negative result.
Theorem 74 If $F$ is a field of characteristic $p$, where $p$ is an odd prime, and if $t$ is transcendental over $F$, then $x^p - x - t$ cannot be solved in either ordinary or Chebyshev radicals over $F(t)$.

Proof: We essentially duplicate an argument already given. Let $F$ be a field of characteristic $p$ containing the $q$-th roots of unity for every prime $q/nep$, let $t$ be an element transcendental over $F$, and let $K_0 = F(t)$.

Supposing $K_n$ is a field obtained as a tower of $q$-th root extensions of prime degree over $K_0$ in which $x^p - x - t$ may be factored. If $K_n/K_{n-1}$ is a radical extension, so that $K_n = K_{n-1}(r)$ with $r^p \in K_{n-1}$, then the polynomial can be factored over $K_{n-1}$, since $x^p - x - t$ is separable. On the other hand, if $K_n = K_{n-1}(r)$ where $r^q \in K_{n-1}$ and $q \neq p$, then either $r \in K_{n-1}$ and so $K_n = K_{n-1}$, or $K_n$ is cyclic of degree $q$ over $K_{n-1}$, in which case $x^p - x - t$ must already be factorable over $K_{n-1}$, since there exists no subextension. Iterating this process, we conclude that $x^p - x - t$ is factorable over $K_0$, which is a contradiction. ✷

We defined solvability for a group $G$ in terms of a composition series; that is, a series of subgroups, each of which is normal in the preceding subgroup, which descends to the trivial group, and which has quotients which are cyclic groups of prime power order. An equivalent definition relaxes the condition on the quotient to merely require that these be abelian, and it is often advantageous to consider abelian extensions (corresponding to these abelian quotients) in general.

Let us suppose that $P$ is a separable polynomial with coefficients in a field $F$ and that $L$ is a splitting field for $P$ obtained by adjoining all of the roots of $P$ to $F$. Let $G$ be the Galois group of the extension $L/F$, $H$ a normal subgroup with an abelian quotient $E = G/H$, and $K$ the subfield of $L$ left fixed by $H$. Let $n$ be the annihilator of $E$, that is, the least integer such that $ne = 0$ for every $e \in E$. Let $n$ be prime to the characteristic of $F$, and let $\zeta$ be a primitive $n$-th root of unity.

In these circumstances, the Galois group of $L(\zeta)/K(\zeta)$ is abelian, being a quotient of $E$. If we can solve $P$ in radicals over $K(\zeta)$, we will have completed the first layer of a complete solution in radicals. We may begin by factoring $P$ over $K(\zeta)$; hence we may assume that we wish to solve an irreducible polynomial with abelian Galois group over a field whose characteristic does not divide the order of the group and which contains the $n$-th roots of unity, where $n$ is the annihilator of the group. If we can do that, we may iterate
the process until the equation is completely solved.

Let us therefore reassign our names, so that $G$ is now an abelian group with annihilator $n$, and $F$ is a field with characteristic prime to $n$ containing the $n$-th roots of unity. We make the following definition.

**Definition 17** Let $F[G]$ be the set of maps $r : F \rightarrow G$. We make this into a vector space over $F$ by defining addition and scalar multiplication in the obvious way, so that $(r + s)(g) = r(g) + s(g)$ for $g \in G$ and $r, s \in F[G]$; and $(\lambda r)(g) = \lambda r(g)$ for $\lambda \in F$. We then define a product on $F[G]$ by setting

$$(r \ast s)(g) = \sum_{e + f = g} r(e)s(f).$$

The $F$-algebra so defined is called the group algebra of $G$ over $F$.

We may define another product on $F[G]$, making it an algebra in another way, by means of coordinatewise multiplication.

**Definition 18** For $r, s \in F[G]$, we define

$$(r \cdot s)(g) = r(g)s(g).$$

Of particular interest are the elements of $F[G]$ which are homomorphisms of $G$ to the multiplicative group $F^\times$ of $F$.

**Definition 19** An element $\chi \in F[G]$ is called a character of $G$ if $\chi(g + h) = \chi(g) \cdot \chi(h)$.

The characters of $G$ together with the product “$\cdot$” form an abelian group denoted by $\hat{G}$. We have a natural and nondegenerate pairing between $G$ and $\hat{G}$ given by

$$(g, \chi) \mapsto \chi(g).$$

Using this, we may conclude that $G$ is isomorphic to $\hat{G}$, and canonically isomorphic to $\hat{\hat{G}}$. This is therefore a duality, and $\hat{G}$ is called the dual group to $G$.

To understand the structure of a commutative algebra, it is helpful to identify idempotent elements. In the case of $F[G]$ we have the following.
4 Class Field Theory

4.1 The $p$-adic Chebyshev Radical

We have seen the great utility of the Chebyshev exponent as a means of expressing the roots of solvable polynomials. The solution of polynomial equations in Chebyshev exponents is interesting in another way: it sometimes happens that arithmetical questions are more easily analysed in Chebyshev terms. In particular, it is helpful when analyzing the unramified extensions which class field theory associates with the ideal class group.

When studying arithmetical questions, it is often of great benefit to work locally, over the completions of the number field. We will recall for the reader of some of the basic facts about $p$-adic fields; the reader who requires more should consult the standard references, such as [2].

We make the following standard definition.

**Definition 20** Let $p$ be a prime number, and let $\lvert\cdot\rvert_p$ denote a function from $\mathbb{Q}$ to $\mathbb{R}$ such that $\lvert x\rvert_p = 0$ if and only if $x = 0$, and such that if $q = p^n r$ is a rational number, where $r$ is a rational number such that neither the numerator nor the denominator is divisible by $p$, then $\lvert q\rvert_p = p^{-n}$. Then the $p$-adic completion of $\mathbb{Q}$, $\mathbb{Q}_p$, is the topological field $C_p/N_p$, where $C_p$ is the ring of Cauchy sequences for the metric defined by $\lvert x - y\rvert_p$, and $N_p$ is the maximal ideal of sequences converging to 0 under this metric.

By the way this is defined, all Cauchy sequences converge, and hence $\mathbb{Q}_p$ is complete. It is not hard to see that $N_p$ is maximal, and hence that $\mathbb{Q}_p$ is indeed a field. It differs from the field of real numbers in that the algebraic closure of $\mathbb{Q}_p$ is not topologically complete. To find a $p$-adic field which plays the same role as the complex numbers do, it is usual to complete the topological closure, obtaining thereby a field both algebraically closed and topologically complete.

**Definition 21** We denote by $C_p$ be the topological completion of the algebraic closure of the $p$-adic field $\mathbb{Q}_p$. The function from $C_p$ to the reals extending the absolute value on $\mathbb{Q}_p$ to $C_p$ we will again denote by $\lvert\cdot\rvert$; we retain the normalizing condition $\lvert p\rvert_p = \frac{1}{p}$.

We began by defining Chebyshev powers over arbitrary rings, so of course they are defined over any $p$-adic field. By using the same power series we
derived over $\mathbb{C}$, we may also define Chebyshev functions for $p$-adic values of the exponent.

**Definition 22** For any $x$ and $k$ in $\mathbb{C}_p$, we define
\[
(2 + x)\circ = 2 + k^2 x + \frac{2}{4!} k^2 (k^2 - 1) x^2 + \frac{2}{6!} k^2 (k^2 - 1) (k^2 - 4) x^3 + \cdots,
\]
whenever the series above, which is (33), converges.

**Theorem 75** Let $F$ be a topologically complete subfield of $\mathbb{C}_p$, and let $k$ be an element of $F$ such that $|k|_p > 1$. Then for any $x \in F$ the power series (33) defines an element $x\circ^k \in F$ if and only if
\[
|x - 2|_p < 1/(|k|_p^{p-\sqrt{p}})^2.
\]

Proof: If $|k|_p > 1$ then $|k + i|_p = |k|_p$ for any integer $i$. Hence (33) converges precisely when
\[
2 + \frac{2}{2!} k^2 x + \frac{2}{4!} k^2 x^2 + \cdots
\]
converges. However, this is the power series expansion of $2 \cosh(k\sqrt{x})$. Since ([2]) $\cosh x$ converges if and only if
\[
|x|_p < 1/\sqrt{p},
\]
(33) will converge if and only if
\[
|k\sqrt{x}|_p < 1/\sqrt{p},
\]
which proves (51). □

**Theorem 76** Let $F$ be as before. Then for any $x \in F$ and $k \in \mathbb{Z}_p$, (33) converges and defines $x\circ^k \in F$ when
\[
|x - 2|_p < 1.
\]

Proof: Since (33) gives a polynomial with integer coefficients when $k$ is a positive integer, the coefficients of (33) are integers for positive integer values of $k$. Hence they are less than or equal to one in absolute value if $k \in \mathbb{Z}_p$ by the density of the positive integers in $\mathbb{Z}_p$. Hence if $|x|_p < 1$ the terms of (33) form a null sequence, and so the series is convergent. □

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Theorem 77 Let $F$ be as before, and let $x$, $n$, and $m$ be elements of $F$. If $(x^n)^m$ is defined, then $x^{nm}$ is defined and $(x^n)^m = x^{nm}$.

Since this is an identity over $\mathbb{C}$ when $|x - 2| < 2$, it is an identity in formal power series. Hence when the series converge, it must remain true in $F$. □

The corresponding results for $S_k(x)$ follow immediately, and may easily prove corresponding results for $U_k(x)$.

Definition 23 For $x$ and $k$ in $\mathbb{C}_p$, we define

$$U_k(2 + x) := k + \frac{k(k^2 - 1^2)}{3!} \frac{x}{4} + \frac{k(k^2 - 1^2)(k^2 - 3^2)}{5!} \frac{x}{4}^2 + \cdots,$$

whenever the power series above, which is (38), converges.

Theorem 78 Let $F$ be as before, and let $k$ be an element of $F$ such that $|k|_p > 1$. Then $U_k(x)$ is defined if and only if

$$|x - 2|_p < |4|_p/(|k|_p \sqrt{p})^2.$$  \hspace{1cm} (52)

Proof: We argue as before; the series converges precisely when the series for $\sinh(k\sqrt{x/4})/\sqrt{x/4}$ converges, which entails

$$|k\sqrt{x/4}|_p < 1/ \sqrt{p},$$

and hence (52). □

Theorem 79 Let $F$ be as before. Then for any $x \in F$ and $k \in \mathbb{Z}_p$, (38) converges and defines $U_k(x) \in F$ when

$$|x - 2|_p < |4|_p.$$ 

Proof: We argue as before, except now we note that (38) gives a polynomial with integer coefficients when $k$ is an odd positive integer, and odd positive integers are dense in $\mathbb{Z}_p$ when $p$ is odd. When $p = 2$, the odd positive integers are dense in the odd 2-adic integers; since these are larger than the even 2-adic integers the result follows in this case also. □
4.2 Showing an Extension is Unramified

We are interested in finding unramified abelian extensions. One means for accomplishing this is to use of pure radical extensions. In this instance, we have the following well-known criterion: a pure radical extension $F(\sqrt[n]{a})$ of a number field $F$ is unramified outside of $n\infty$ if and only if the principal ideal $(a)$ generated by $a$ is an $n$-th power as an ideal. In particular, this includes extensions generated by roots of units of $F$. This is easily seen to be the case by working locally, where $a$ becomes an $n$-th power times a unit.

We will find it convenient to work with Chebyshev radical extensions as well. Similar conditions to the above in terms of the trace of an ideal from a certain field can be developed.

We will be particularly interested in split unramified extensions. Suppose we have a Galois extension $L/F$ of a number field $F$ whose Galois group is split with an abelian kernel $A$. We then have a subextension $K/F$ with Galois group $G$, and the Galois group of $L/F$ is $\text{Gal}(L/F) \cong G \rtimes A$ of $L/F$. The extension $L/F$ can be given as the splitting field of a polynomial $g$ of degree $|A|$ with coefficients in $F$, and $K/F$ as the splitting field of a polynomial $f$ of lesser degree.

In such a situation, the extension $L/K$ is unramified at primes lying over the prime $\wp$ of $F$ if and only if the inertia groups for the commutative algebra given by the roots of $f$ over $F_{\wp}$ are the same as those for the roots of $g$ over $F_{\wp}$.

$\text{Gal}(K/F)$ as a permutation group on the roots of $f$ injects into $\text{Gal}(L/F)$ as a permutation group on the roots of $g$. If $r_f$ is a root of $f$ and $r_g$ is a root of $g$, for $\wp$ a prime in $F$ which ramifies totally in $F(\sqrt{r_f})$, we have a correspondingly that the factorization of $\wp$ in $F(\sqrt{r_g})$ will consist of unramified primes, and $m$ primes which ramify with inertia corresponding to the inertia of $\wp$ in $F(\sqrt{r_f})$. Hence locally at $\wp$, we have that the $\wp$ factor in the discriminant of the field $F(\sqrt{r_g})$ is the $m$-th power of what it is for $F(\sqrt{r_f})$.

4.3 Unramified Families For Degree Three

The simplest example of the situation described above is where $G$ is cyclic of degree two, and $A$ cyclic of degree three. In this case we are seeking extensions of $L/F$ with Galois group $S_3$, such that if $K$ is the quadratic subfield, then $L/K$ is unramified. Looking at degree three polynomials, this
means that at all primes \( \wp \) of \( F \) which ramify in \( L \) we do not get total ramification in the degree three field \( F(r_g) \), but rather a product of a prime of ramification index one and one of index two, the latter corresponding to the (total) ramification of \( \wp \) in \( K \). Hence, \( L/K \) will be unramified if and only if the field discriminants of \( F(r_f) \) and of \( F(r_g) \) are the same. This happens if and only if no prime ramifies with index three in \( F(r_g) \).

The polynomial
\[
x^3 + bx + c
\]
is generic for the Galois group \( S_3 \). By this I mean that over \( \mathbb{Q}(a, b) \) its splitting field has group \( S_3 \), and (in characteristic other than 3) any Galois extension with group \( S_3 \) can be obtained as the splitting field of a polynomial of this form.

Let us therefore consider a polynomial
\[
x^3 + bx + c,
\]
with coefficients in a number field \( F \).

If we set \( z = \sqrt{-3/bx} \), then we find that \( z \) satisfies the polynomial
\[
z^3 - 3z + \sqrt{-27/b^3c},
\]
so that if \( h = -\sqrt{-27/b^3c} \), we have a solution in Chebyshev radicals, given by
\[
z_1 = \sqrt[3]{h},
\]
\[
z_2 = -\sqrt[3]{-h},
\]
and
\[
z_3 = -z_1 - z_2.
\]

In order not to trouble ourselves with square roots we can eliminate them by using instead the polynomial for
\[
z^{\circ 2} = -3x^2 - 2.
\]

Up to a change of scale, this transformation is what results from squaring the Lagrange resolvents and finding roots in terms of these.
If the polynomial is irreducible, then the splitting field for \( z^{\sqrt{2}} \) gives the same extension as for \( x \). On the other hand,

\[
z^{\sqrt{2}} = (\sqrt[3]{-2-\frac{27c^2}{b^3}}) = \sqrt[3]{\frac{27c^2}{b^3}},
\]

which gives us a root when it is defined.

If we denote the discriminant of the original polynomial in \( x \) by

\[
\delta = -4b^3 - 27c^2,
\]

then we can write

\[-2 - \frac{27c^2}{b^3} \]

as

\[2 + \frac{\delta}{b^3},\]

so that we have a root given by

\[\sqrt[3]{2 + \frac{\delta}{b^3}},\]

when this is defined.

If \( \wp \) is a prime in a number field \( F \) lying over the rational integral prime \( p \), we normalize the associated valuation in the usual way, by setting

\[|p|_{\wp} = |p|_p = 1/p.\]

We now can state a criterion for determining unramified extensions.

**Theorem 80** If \( x^3 + bx + c \) with \( b \neq 0 \) has coefficients in an algebraic number field \( F \) and is irreducible over \( F \), and if \( \epsilon = -2 - 27c^2/b^3 \), \( \delta = -4b^3 - 27c^2 \), and \( r \) is any root, then \( F(r, \sqrt{\delta}) \) gives an unramified extension of \( F(\sqrt{\delta}) \) if one of the following three conditions is true for each prime \( \wp \) which ramifies in \( F(\sqrt{\delta}) \), and gives an extension unramified outside of primes lying over \( 3 \) if and only if one of the three conditions always holds. If the field \( F \) contains no subextension which is unramified over \( \mathbb{Q} \) at 3, then the conditions are both necessary and sufficient.
1. 
\[ |\frac{\epsilon + 2}{27}|_\wp = |\frac{c^2}{b^3}|_\wp < 1, \]

2. 
\[ |\frac{\epsilon - 2}{27}|_\wp = |\frac{\delta}{27b^3}|_\wp < 1, \]

3. 
\[ |\frac{c^2}{27}|_\wp = |\frac{(\delta + 2b^3)^2}{27b^3}|_\wp < 1. \]

Proof: Since we have complex embeddings of $F(x)$ only if we have corresponding complex embeddings of $F(\sqrt{\delta})$, we never have ramification at an infinite place. Hence we will have an unramified extension if and only if there is a $\wp$-adic root of the above polynomial for each prime $\wp$ of $F$ which ramifies in $F(\sqrt{\delta})$. And we will have such a root if and only if we have a root of $x^2 - 3x - \epsilon$, where $\epsilon = -2 - 27c^2/b^3$.

If $\wp$ is prime to 3, then $-T_1(-\epsilon)$ will give us a root if

\[ |\epsilon + 2|_\wp = |\frac{27c^2}{b^3}|_\wp = |\frac{c^2}{b^3}|_\wp < 1 \]

by the lemma on the power series expansion of $T_n$. On the other hand, if $\wp$ lies over 3, then we will have a root if

\[ |\epsilon + 2|_\wp = |\frac{27c^2}{b^3}|_\wp < 1/27, \]

which entails

\[ |\frac{c^2}{b^3}|_\wp < 1. \]

So if the first condition is true for a prime $\wp$, then we do not have ramification at $\wp$.

Since

\[ \epsilon = -2 - \frac{27c^2}{b^3} = 2 + \frac{\delta}{b^3}, \]

we will have a root $\sqrt[\wp]{\epsilon}$ when

\[ |\frac{\delta}{b^3}|_\wp = |\frac{27\delta}{b^3}|_\wp < 1 \]
for \( \wp \) prime to \( p \), and
\[
|\frac{\delta}{b^3}|_{\wp} < \frac{1}{27}.
\]
for \( \wp \) lying over \( p \). Putting these together, we get the condition
\[
|\frac{\delta}{27b^3}|_{\wp} < 1.
\]
So if the second condition is true for a prime \( \wp \), we do not have ramification at \( \wp \).

Finally, if \( |\epsilon|_{\wp} < 1 \) and \( \wp \) does not lie over 3, then \( \sqrt[3]{2 - \epsilon^2} \) will give a root of the polynomial transformed for the second time by \( x \mapsto x^{\wp^2} \). And if \( |\epsilon^2|_{\wp} < \frac{1}{27} \) for a prime \( \wp \) lying over 3, then \( T_{\frac{1}{3}}(2 - \epsilon^2) \) will likewise give a root. Putting these conditions together, we have a root of the transformed polynomial when \( |\epsilon^2/27|_{\wp} < 1 \), so the third condition entails we do not have ramification at \( \wp \).

If one of these conditions is true for each prime \( \wp \) which ramifies in \( F(\delta) \), then we have an unramified extension at every prime, and hence a globally unramified extension. This is the first part of the theorem.

For the second part of the theorem, it is helpful to assume that the polynomial is \( \wp \)-reduced. The conditions are in terms of \( \frac{c^2}{b^3} \) and \( \frac{\delta}{b^3} \), and so remain the same if we make the transformation \( b \mapsto k^2b, \ c \mapsto k^3c \). Hence we may assume the polynomial is \( \wp \)-reduced, meaning that \( |b|_{\wp} \leq 1 \), \( |c|_{\wp} \leq 1 \), and either \( |b|_{\wp} > |\wp|_{\wp}^2 \), or \( |c|_{\wp} > |\wp|_{\wp}^3 \), or both.

Assuming this reduction, we can have \( b \) a unit and \( c \) a non-unit, \( b \) and \( c \) both units, or \( c \) a unit and \( b \) not a unit.

If \( b \) is a unit and \( c \) is not, then the first condition holds.

If both \( b \) and \( c \) are units, then since \( |\delta|_{\wp} < 1 \) and \( p \nmid 3 \), the second condition holds.

If \( c \) is a unit and \( b \) is not, then since \( \wp \) divides \( \delta = -4b^3 - 27\epsilon^2 \), \( \wp \) must divide \( 27c^2 \), and since \( c \) is a unit, \( \wp \) must lie over 3. If \( F/\mathbb{Q} \) contains no subextension which is unramified at 3, then if \( r \) is a root of the \( \wp \)-reduced polynomial \( x^3 + bx + c \) which does not lie in \( \mathbb{Z}_p \), then one of the conditions \( |r + 2|_{\wp} < \frac{1}{3}, |r|_{\wp} < \frac{1}{3}, \) or \( |r - 2|_{\wp} < \frac{1}{3} \) must hold. Since \( \epsilon = r^{\wp^3} \), then this entails one of the conditions \( |\epsilon + 2|_{\wp} < \frac{1}{27}, |\epsilon|_{\wp} < \frac{1}{9}, \) or \( |\epsilon - 2|_{\wp} < \frac{1}{27} \) must hold; which then shows that one of \( |(\epsilon + 2)/27|_{\wp} < 1, |\epsilon^2/81|_{\wp} < 1 \) or \( |(\epsilon - 2)/27|_{\wp} < 1 \) must hold. Hence one of the conditions holds, and this is the second part of the theorem. \( \square \)
We may now employ this theorem to construct families of unramified extensions.

**Corollary 81** Let

\[ x^3 + bx + b^2 t \]

be an irreducible polynomial with \( b \) and \( t \) in the ring of integers of a number field \( F \). Let \( r \) be a root and let \( d = -27bt^2 - 4 \). Then \( F(r, \sqrt{bd}) \) is an unramified extension of \( F(\sqrt{bd}) \).

Proof: The first condition holds if \( |tb^2|_{\wp} < 1 \), and hence holds if \( b \) is not a \( \wp \)-unit. If \( b \) is a \( \wp \)-unit, then \( |d|_{\wp} < 1 \), since \( \delta = b^3d \). Hence if \( \wp \) is prime to 3,

\[ \left| \frac{\delta}{27b^3} \right|_{\wp} = \left| \frac{d}{27} \right|_{\wp} < 1. \]

But we cannot have \( \wp \) lying over 3, since \( \delta \equiv -4b^3 \pmod{27} \) and \( b \) is a \( \wp \)-unit, and so \( \delta \) is prime to 3. Hence the second condition holds if the first condition fails, and we have an unramified extension in all cases. \( \Box \)

**Theorem 82** Let \( F \) be a number field, and \( b \) an non-zero integer in \( F \). Then

\[ x^3 + bx + c \]

is a congruence family.

Proof:

Let \( c \) be an integer of \( F \) such that \( x^3 + bx + c \) produces an unramified extension. By the previous corollary, values of \( c \) of the form \( tb^2 \) will produce unramified extensions, so such \( c \) exist.

Now let \( h \) be an integer of \( F \) such that

\[ \left| \frac{h}{27b^3} \right|_{\wp} < 1 \]

for every prime \( \wp \) of \( F \) dividing \((3b)\). Let \( \delta = -4b^3 - 27c^2 \) be the discriminant of the polynomial \( x^3 + bx + c \), and \( \delta' = -4b^3 - 27(c + h)^2 \) the discriminant of \( x^3 + bx + c + h \). For any prime \( \wp \) dividing \( \delta' \), we will have a local root if

\[ \left| \frac{\delta'}{27b^3} \right|_{\wp} < 1, \]
so we may confine our attention to primes where

$$\left| \frac{\delta}{27b^3} \right|_\nu \geq 1.$$  

In this case, since

$$\left| \frac{\delta - \delta'}{27b^3} \right|_\nu = \left| \frac{h(h + 2c)}{b^3} \right|_\nu < 1,$$

we have also

$$\left| \frac{\delta}{27b^3} \right|_\nu \geq 1,$$

and hence $|h|_\wp < |\delta|_\wp$.

The $\wp$-adic root $r$ of $x^3 + bx + c$ will converge by Newton’s method to a $\wp$-adic root of $f(x) = x^3 + bx + c + h$ if

$$|f(r)|_\wp < |f'(r)|_\wp^2,$$

which means precisely if

$$|h|_\wp < |\delta|_\wp.$$  

and hence we get a $\wp$-adic root in all cases, and hence an unramified extension. ✷

While eg. the modulus $81b^4$ will always work, it is by no means the case that so large a large modulus is always needed. For instance, we have the following, when the base field $F$ is $\mathbb{Q}$:

**Corollary 83** Let $b$ be $\pm p$, for $p$ a rational integral prime different from 3. Then $x^3 + bx + c$ for integral $c$ gives an unramified extension of its quadratic subfield if it is irreducible, and if $c$ is either prime to $p$, or divisible by $p^2$.

**Proof:** Let $\ell$ be a prime dividing $\delta = -4b^3 - 27c^2$. Then since $\delta \equiv -4b^3 \pmod{3}$, $\ell \neq 3$. If $\ell \neq p$, we have

$$\left| \frac{\delta}{27b^3} \right|_\ell < 1.$$  

Hence we will only possibly have ramification if $\ell = p$. If $c$ is divisible once by $p$, then we have total ramification at $p$. If $c$ is divisible more than once, we have

$$\left| \frac{c^2}{b^3} \right|_p < 1,$$

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and hence no ramification. □

In this case, we see that the minimum modulus defining the congruence family is \( p^2 \). If \( p = 3 \), we can do a similar analysis; however as we shall see there is another way of looking at this case which has an interesting generalization.

**Theorem 84** Let \( b \) be a fixed nonzero rational integer. Then the polynomial

\[
x^3 + bx + c
\]

for rational integer values of \( c \) such that the above polynomial is irreducible will give an unramified extension of \( \mathbb{Q}(\sqrt{\delta}) \) if and only if mod \( b^3 \) we have

\[
c^2 \equiv 0 \pmod{b^3}
\]

or

\[
c^2 \equiv -\frac{4b^3}{27} \pmod{b^3}.
\]

### 4.4 Unramified Families for Degree Four

Just as

\[
x^3 + bx + c
\]

is generic in characteristic other than 3 for the triangle group \( D_3 = S_3 \), the polynomial

\[
x^4 + bx^2 + c
\]

is generic for the square group \( D_4 \) in characteristic other than 2. It has a splitting field with Galois group \( D_4 \) over \( \mathbb{Q}(a, b) \), and outside of characteristic 2, any Galois extension with group \( D_4 \) can be obtained as the splitting field of a polynomial in this form.

The reduction to this form is a little less trivial. If we have a polynomial of degree four with roots \( r_0, r_1, r_2, r_3 \) giving a Galois extension with group \( D_4 \), we may obtain another polynomial with roots \( r_0 - r_2 \) and its conjugates, if the roots are correctly ordered. One way to obtain this polynomial is to factor a resolvent polynomial of degree twelve.

**Lemma 85** If

\[
x^4 + a_1 x^3 + a_2 x^2 + a_3 x + a_4
\]
is a polynomial with coefficients in a field $F$ of characteristic other than 2, whose splitting field gives a Galois extension of $F$ with group $D_4$, then

$$z^{12} - 3a1^2z^{10} + (-16a1^2a2 + 8a4 + 3a1^4 - 2a3a1 + 22a2^2)z^8$$

$$+ (26a3^2 + 8a1^4a2 + 16a4a2^2 - a1^6 - 24a1^2a2^2 - 30a3a1a2$$

$$+ 28a2^3 + 8a3a1^3 - 6a4a1^2)z^6$$

$$+ (17a2^4 + 48a3^2a2 - 25a3^2a1^2 + 6a4a1^4 - 54a3a2^2a1$$

$$- 112a4^2 - 12a1^2a2^3 + 38a3a1^3a2 - 6a3a1^5 + 24a2^2a4$$

$$- 32a4a1^2a2 + 2a1^4a2^2 + 56a3a1a4)z^4$$

$$+ (6a3a1^3a2^2 + 216a3^2a4 - 120a3a1a4a2 + 72a4^2a1^2 + 42a3^2a1^2a2$$

$$+ 18a3^2a2^2 + 18a3a1^3a4 - 9a3^2a1^4 - 54a3^3a1 + 32a4a2^3$$

$$- 26a3a1a2^3 - a2^4a1^2z^2 - 192a4^2a2 + 4a2^5z^2 - 6a4a2^2a1^2)z^2$$

$$- 27a1^4a4^2 - 192a3a1a4^2 - 4a3^3a1^3 - 80a3a1a2^2a4 - 128a2^2a4^2$$

$$+ a2^3a3^2a1^2 + 256a4^3 - 4a2^3a1^2a4 - 27a3^4 + 16a2^4a4 - 6a3^2a1^2a4$$

$$- 4a2^3a3^2 + 144a2a1^2a4^2 + 144a4a3^2a2 + 18a3a1^3a2a4 + 18a3^3a1a2$$

will factor into either a single factor of degree four and a single factor of degree eight, or into a single factor of degree four and a repeated factor of degree four. The single factor of degree four is of the form

$$z^4 + bz^2 + c$$

and is such that if $r$ is a root of our original polynomial, then there is a root $s$ of this polynomial such that $F(s)$ defines the same field extension as $F(r)$.

Proof:

If the Galois group is $D_4$, it will contain a generator $\sigma$ for the cyclic subgroup of order four. Applying this to a root $r_0$, we obtain $r_i = r_0^i$. Then $s_0 = r_0 - r_2$ and its conjugates generate the same extension, and it is trivial to check that the polynomial derived from these roots has a zero $x$ and $x^3$ term. Moreover, $F(s_0)$ generates the same extension as $F(r_0)$, since this is the same extension as $F(r_2)$.
The polynomial of degree twelve in the statement is the polynomial for \( r_i - r_j \); it is readily calculated using e.g. resultants. Hence it will have a factor of degree four corresponding to the polynomial considered in the previous paragraph. □

Let \( \delta = b^2 - 4c \). In some cases, the field \( L = F(r_0, r_1, r_2, r_3) \) obtained by adjoining the roots of

\[
x^4 + bx^2 + c
\]

will define an unramified extension of \( K = F(\sqrt{c\delta}) \), where \( b, c \in F \). We consider first infinite places.

**Theorem 86** The extension \( L/K \) is unramified at any real infinite place iff either \( b < 0 \) and \( c > 0 \), or \( \delta < 0 \) at that place.

Proof: If \( \delta < 0 \) at the place in question, \( K \) is already complex there and the extension is unramified trivially. The polynomial \( x^2 + bx + c \) has two positive real roots if and only if \( b < 0 \) and \( c > 0 \), and in this case, \( L \) is real at this place and hence the extension is unramified there. □

**Theorem 87** If \( x^4 + bx^2 + c \) is a polynomial with coefficients in a number field \( F \) whose roots generate a \( D_4 \) extension of \( F \), and if \( r \) is any root and \( \delta = b^2 - 4c \), then \( L = K(r) \) is unramified at a prime \( \wp \) of \( K = F(\sqrt{c\delta}) \) lying over a prime \( p \) of \( F \) if one of the following conditions holds:

1. The prime \( p \) divides \( c\delta \) an odd number of times, and

\[
\frac{c}{b^2} |_{p} < 1.
\]

2. The prime \( p \) divides \( c\delta \) an odd number of times, and

\[
\frac{\delta}{2b^2} |_{p} < 1.
\]

Proof: If \( |x| < 1 \) in any valuation, we have that

\[
\sqrt{1 - 4x} = 1 - 2x - 2x^2 \ldots
\]
converges. Hence if $|\frac{c}{b^2}|_p < 1$, we have that

$$\sqrt{b^2 - 4c} = b\sqrt{1 - 4\frac{c}{b^2}}$$

has a $p$-adic root, and hence the roots of

$$x^4 + bx^2 + c$$

are either elements of $F_p$ of square roots of such elements. The ramification is thus locally no more than degree two. If $p$ divides $c\delta$ an odd number of times, the ramification of $F(\sqrt{c\delta})$ is also two, and hence no further ramification can occur.

If we set $s_0 = r_0 - r_1$ together with its conjugates, we obtain the four roots of the polynomial

$$x^4 + 2bx^2 + \delta.$$

Hence the second part of the theorem follows from applying the conclusions of the first part to this polynomial, which defines different stem fields but the same splitting field. □

**Corollary 88** Let

$$x^4 + bx^2 + b^3t$$

or

$$x^4 - bx^2 + b^3t$$

be a polynomial with $b \neq 0$ and $b$ and $t$ in the ring of integers of a number field $F$, defining a $D_4$ extension of $F$. Let $r$ be a root and let $\delta = 1 - 4bt$. Then $F(r, \sqrt{b\delta})$ is an unramified extension of $F(\sqrt{b\delta})$ at all non-infinite places.

**Proof:**

We have

$$(\pm b)^2 - 4b^3t = b^2\delta.$$ 

If $p$ is a finite prime of $F$ which divides $b$, then it does not divide $\delta$.

$$|\frac{b^3t}{b^2}|_p = |bt|_p < 1,$$

and hence $S_3$ extensions of any number field $F$ (this is obvious both from its derivation and from the fact that specializing to $u = 1$ leads us to a
polynomial which is still generic.) And it is well-adapted to the purpose of analyzing when a degree extension is unramified over the quadratic subfield, which is given by the roots of \(x^2 + u(4s^3 + 27t^2u)\). We may analyze these by fixing \(s\), and we begin by setting \(s = 1\).

In this case, the basis \([1, x, x^2/u]\) consists of elements integral over \(\mathbb{Z}[u, t]\), and the trace form of this basis has discriminant \(-u(27ut^2 + 4)\). The polynomial giving the quadratic subfield is \(x^2 + u(27ut^2 + 4)\). If \(u\) is even, then \([1, x/2]\) removes a factor of 4, whereas if \(u\) is odd, \([1, (x + t)/2]\) removes a factor of 4, so we are left with \(-u(27ct^2 + 4)\) in all cases.

If this is square free we are done, and we have the same discriminant for the degree three field as for the degree two field. If the discriminant contains a factor which is the square of a prime \(p\) which is prime to \(2u\), we remove may remove the square term from both the degree two and degree field discriminants, by \([1, x/p]\) for the degree two field, and \([1, x, (x_1 - x_2)^2/p]\) (where \(x_1\) and \(x_2\) are the two conjugates of \(x\)) for the degree three case. In case \(p\) divides \(2u\) we modify this a bit, but in any case we simply remove the prime factor from the degree two and degree three extension in just the same way, which we can do since the ramified primes and the trace forms correspond.

Since the field will be totally real if the quadratic subfield is real, we need not concern ourselves with the place at infinity; and for this particular polynomial we are dealing with negative discriminants in any case. Hence we may conclude that substituting integer values for \(t\) and \(u\) into \(x^3 + ux + tu^2\) gives an unramified extension of the quadratic subfield given by \(x^2 + u(27ut^2 + 4)\), in case both polynomials are irreducible.

While this result is very satisfactory in that it constructs an infinity of unramified extensions, it is limited to doing so over imaginary quadratic fields. For this reason it is worth our while to investigate other values for \(s\). Since substituting \(u \mapsto su, t \mapsto st, x \mapsto sx\) into \(x^3 + ux + tu^2\) leads to \(x^3 + sxu + tu^2\), we can conclude that this is unramified outside the primes dividing \(s\) (where \(s\) is a local unit) by our previous analysis. Hence all we need to do is consider values of \(t\) and \(u\) \(s\)-adically; and we may began by noting that the above substitution lets us conclude immediately that if both \(u\) and \(t\) are divisible by \(s\), we have an unramified extension.

If \(s = 2\), we obtain \(x^3 + 2ux + tu^2\), with quadratic subfield \(x^2 + u(27t^2u + 32)\). We know that if both \(u\) and \(t\) are even the extension is unramified. If \(u\) is even and \(t\) is odd, then we have total ramification at 2 unless \(u\) is divisible
by 8. However, $u \mapsto 8u, x \mapsto 4x$ leads us back to $x^2 + ux + tu^2$, and so in this case also the extension is unramified.

If $u$ is odd, then modulo 4 the polynomial becomes $x^3 + 2x + t$. This is totally ramified at 2 if $t \equiv 2 \pmod{4}$, and is unramified at 2 if $t$ is odd. If $t \equiv 0 \pmod{4}$, then it is ramified at 2 but not of degree three, and so the ramification corresponds to the ramification of the quadratic subfield.

The conclusion is that we have an unramified extension if and only if $u$ is divisible by 8, or if it is even and $t$ is even, or if it is odd and $t$ is not congruent to 2 mod 4.

If $s = 3$, then we will obtain an unramified extension if $u$ is divisible by three and

If $u$ is not divisible by three, we will obtain an unramified extension if $t \equiv 0 \pmod{9}$; or if $u^9 \equiv -1 \pmod{27}$ and $ut^2 \equiv -4 \pmod{27}$; or finally if $u^9 \equiv 1$ and $ut^2 \equiv -2 \pmod{9}$.

If $c \equiv \pm 3 \pmod{9}$, then we get a polynomial with integer coefficients if $t = t'/9$, where $t'$ is integral. Under these conditions, we obtain an unramified extension if $t' \equiv 0 \pmod{27}$, or if $t' \equiv \pm c/3 \pmod{9}$.

Finally, if $c \equiv 0 \pmod{9}$, then transforming by $c = 9c', t = t'/3$, and $x = 3x'$ leads to the same polynomial, so we have already done this case.

In conclusion, we get the following

**Theorem 89** There exist an infinite number of real quadratic fields as well as an infinite number of imaginary quadratic fields with 3-cycles in the non-genus class group.

To find families of unramified extensions corresponding to 4-cycles in the class group of quadratic extensions, we need to proceed a little differently, since a degree four dihedral extension has three quadratic subfield rather than one. We could proceed as before by analyzing a generic polynomial for dihedral extensions (such as $x^4 - 4acx^2 - a^2c(b^2 - 4c)$ for instance), but we will content ourselves with something simpler.

Consider the polynomial

$$x^4 - x^3 - tx^2 - x + 1.$$ 

Over $\mathbb{Q}(t)$ this has dihedral Galois group, and three quadratic subfields given by $x^2 - t(t - 4)$, $x^2 - 4t - 9$, and $x^2 - t(t - 4)(4t + 9)$. The discriminant of the polynomial is $t(t - 4)(4t + 9)^2$. 

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The for any integral specialization of $t$, the ideal $(t, t - 4)$ divides 4, the ideal $(t, 4t + 9)$ divides 9, and the ideal $(t - 4, 4t + 9)$ divides 25. Hence for any prime $p > 5$ we have that it divides only one of these three. We can write our polynomial in three different ways:

\[
(x^2 + x + 1)(x - 1)^2 - tx^2,
\]
\[
(x^2 - 3x + 1 + 1)(x + 1)^2 - (t - 4)x^2,
\]
\[
(x^2 - x/2 + 1)^2 - (t + 4/9)x^2.
\]

This corresponds to ramification of order two and with multiplicity one if $p$ divides $t$ or $t - 4$ an odd number of times, and with multiplicity two if $t$ divides $4t + 9$ an odd number of times.

If $t$ is odd then $t(t - 4)(4t + 9)$ is odd. If $t \equiv -1 \pmod{3}$ then $t(t - 4)(4t + 9)$ is not divisible by three. And if $t$ is congruent to 1 or $\pm 2$ mod 5, then $t(t - 4)(4t + 9)$ is not divisible by five. Hence for the combined congruence condition mod 30, if $t$ is of the form $30n - 13$, $30n - 7$ or $30n + 11$ we have that $t(t - 4)(4t + 9)$ is not divisible by 30, and hence that the degree four polynomial for this specialization is unramified over the quadratic field $\mathbb{Q} (\sqrt{t(t - 4)(4t + 9)})$

Hence we have

**Theorem 90** There exist an infinite number of quadratic fields with both real and imaginary discriminant and with a 4-cycle in the class group, such that the corresponding portion of the Hilbert class field is generated by

\[x^4 - x^3 - tx^2 - x + 1\]

for values of $t$ such that $t \equiv -13, -7, 11 \pmod{30}$.

**References**

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