Research Article

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Existence of single peak solutions for a nonlinear Schrödinger system with coupled quadratic nonlinearity

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Abstract: We are concerned with the following Schrödinger system with coupled quadratic nonlinearity

\[
\begin{align*}
-\varepsilon^2 \Delta v + P(x)v &= \mu vw, & x \in \mathbb{R}^N, \\
-\varepsilon^2 \Delta w + Q(x)w &= \frac{\mu}{2}v^2 + \gamma w^2, & x \in \mathbb{R}^N, \\
v > 0, & w > 0, \\
v, w \in H^1(\mathbb{R}^N),
\end{align*}
\]

which arises from second-harmonic generation in quadratic media. Here \(\varepsilon > 0\) is a small parameter, \(2 \leq N < 6\), \(\mu > 0\) and \(\mu > \gamma\), \(P(x), Q(x)\) are positive function potentials. By applying reduction method, we prove that if \(x_0\) is a non-degenerate critical point of \(\Delta(P + Q)\) on some closed \(N - 1\) dimensional hypersurface, then the system above has a single peak solution \((v_\varepsilon, w_\varepsilon)\) concentrating at \(x_0\) for \(\varepsilon\) small enough.

Keywords: second-harmonic generation; non-degenerate; single peak solutions; reduction method

MSC: 35J10, 35B99, 35J60

1 Introduction and main result

In this paper, we consider the following Schrödinger system with coupled quadratic nonlinearity

\[
\begin{align*}
-\varepsilon^2 \Delta v + P(x)v &= \mu vw, & x \in \mathbb{R}^N, \\
-\varepsilon^2 \Delta w + Q(x)w &= \frac{\mu}{2}v^2 + \gamma w^2, & x \in \mathbb{R}^N, \\
v > 0, & w > 0, \\
v, w \in H^1(\mathbb{R}^N),
\end{align*}
\]

where \(\varepsilon > 0\) is a small parameter, \(2 \leq N < 6\), \(\mu > 0\) and \(\mu > \gamma\), \(P(x), Q(x)\) are positive potentials.

System (1.1) arises from the cubic nonlinear Schrödinger equation

\[
i \frac{\partial \phi}{\partial z} + r \nabla^2 \phi + \chi|\phi|^2 \phi = 0,
\]

which appears in the nonlinear optic theory and can be used to describe the formation and propagation of optical solutions in Kerr-type materials [6, 19]. Here \(\phi\) is a slowly varying envelope of electric field, the real-valued parameter \(r\) and \(\chi\) represent the relative strength and sign of dispersion/diffraction and nonlinearity respectively, and \(z\) is the propagation distance coordinate. The Laplacian operator \(\nabla^2\) can either be \(\frac{\partial^2}{\partial z^2}\) for...
temporal solitons with \( \tau \) is the normalized retarded time, or \( \tau^2 = \sum_{i=1}^{N} \frac{\delta^2}{\delta x_i^2} \), where \( x = (x_1, \cdots, x_N) \) is in the direction orthogonal to \( z \). Solitary wave solutions to (1.2) and its generations have been studied in [4, 18].

Also, (1.1) appears in the study of standing waves for the following nonlinear system

\[
\begin{aligned}
\frac{i}{\partial t} \phi_1 &= -\varepsilon^2 \Delta \phi_1 + (P(x) + \mu) \phi_1 - \mu |\phi_1|^2 \phi_2, \\
\frac{i}{\partial t} \phi_2 &= -\varepsilon^2 \Delta \phi_2 + (Q(x) + \mu) \phi_2 - \beta |\phi_1|^2 |\phi_2|^2, \\
(x, t) &\in \mathbb{R}^N \times \mathbb{R}^+, 
\end{aligned}
\]

(1.3)

with the form \( \phi_1(x, t) = v(x)e^{i\mu t}, \phi_2(x, t) = w(x)e^{i\mu t} \), where \( i \) is the imaginary unit and \( \varepsilon \) is the Planck constant. When \( \varepsilon = 1 \) and \( \mu = 0 \), the existence of ground state solutions of (1.3) was proved in [27]. Besides, (1.1) is closely related to the general parabolic system with coupled nonlinearity and the nonlinear evolution equations. For this information, we can refer to [11, 16, 23, 25, 26] and references therein.

By contrast with the coupled Schrödinger system (1.3) with \( \chi^{(2)} \) nonlinearities, the following \( \chi^2 \) nonlinear Schrödinger system

\[
\begin{aligned}
\frac{i}{\partial t} \phi_1 &= -\varepsilon^2 \Delta \phi_1 + V_1(x) \phi_1 - \mu_1 |\phi_1|^2 \phi_1 - \beta |\phi_2|^2 \phi_1, \\
\frac{i}{\partial t} \phi_2 &= -\varepsilon^2 \Delta \phi_2 + V_2(x) \phi_2 - \mu_2 |\phi_2|^2 \phi_2 - \beta |\phi_1|^2 \phi_2, \\
(x, t) &\in \mathbb{R}^N \times \mathbb{R}^+, 
\end{aligned}
\]

(1.4)

has been extensively investigated. There are many interesting results about (1.4) under various assumptions of \( V_1(x) \) and \( V_2(x) \), one can refer to [1–4, 7, 8, 12–15, 17, 21, 24] and their references therein.

In recent decades, system (1.1) and its related problems have attracted a lot of attention. When \( \varepsilon = 1 \), (1.1) reduces to

\[
\begin{aligned}
-\Delta v + P(x)v &= \mu v w, & x &\in \mathbb{R}^N, \\
-\Delta w + Q(x)w &= \frac{\varepsilon^2}{2} v^2 + \gamma w^2, & x &\in \mathbb{R}^N, \\
v &> 0, & w &> 0, \\
\nu, w &\in H^1(\mathbb{R}^N). 
\end{aligned}
\]

(1.5)

Applying the finite dimensional reduction method, Wang and Zhou [22] constructed the infinitely many non-radial positive solutions of (1.5) if the potential functions \( P(x), Q(x) \) are radial and satisfy some algebraic decay at infinity. Also, if \( \varepsilon \) is small, for any positive integer \( k \leq N + 1 \), Tang and Xie [20] proved that (1.1) has a \( k \) spikes solution concentrating at some strict local maximum of \( P(x) \) and \( Q(x) \) by using the finite dimensional reduction provided that \( |P(x) - P(y)| \leq L_1|x - y|^\beta_1 \) and \( |Q(x) - Q(y)| \leq L_2|x - y|^\beta_2 \) for some positive constants \( L_1, L_2, \theta_1, \theta_2 \).

Here we want to mention that, very recently, Luo, Peng and Yan [13] revisited the following Schrödinger equation

\[
-\varepsilon^2 \Delta u + V(x) u = u^{p-1}, \ u \in H^1(\mathbb{R}^N) 
\]

(1.6)

with \( 2 < p < 2^* \). Under the condition that \( V(x) \) obtains its local minimum or local maximum \( x_0 \) at a closed \( N - 1 \) dimensional hypersurface, they obtained the existence of a positive single peak solution for (1.6) concentrating at \( x_0 \) if \( x_0 \) is non-degenerate critical point of \( \Delta V \) and also verified the local uniqueness of single peak solutions by using local Pohazae type identity.

Motivated by [13, 20, 22], we want to apply the finite-dimensional reduction to study the existence of positive single peak solutions for (1.1). Our purpose here is to prove that (1.1) has a single peak solution concentrating at some non-degenerate critical point of \( \Delta (P + Q) \) on a closed \( N - 1 \) dimensional hypersurface.

To state our results, throughout this paper, we assume that \( P(x), Q(x) \) obtain their local minimum or local maximum at a closed \( N - 1 \) dimensional hypersurface \( \Gamma \). Without loss of generality, we suppose that \( P(x) = Q(x) = 1 \) if \( x \in \Gamma \) and more precisely, we assume that \( P(x), Q(x) \) satisfies the following conditions.

\( (H_1) \) There exist \( \delta > 0 \) and a closed smooth hypersurface \( \Gamma \) such that if \( y \in \Gamma, P(y), Q(y) = 1 \) and \( P(y), Q(y) > 1 \) (or \( P(y), Q(y) < 1 \)) for any \( y \in W_\delta \setminus \Gamma \), where \( W_\delta := \{ x \in \mathbb{R}^N, \text{dist}(x, \Gamma) < \delta \} \).

\( (H_2) \) The level set \( \Gamma = \{ x : P(x), Q(x) = t \} \) is a closed smooth hypersurface for \( t \in [1, 1 + \theta] \) (or \( t \in [1 - \theta, 1] \)) for some small \( \theta > 0 \). Also, for any \( x_i \in \Gamma \) and \( x_0 \in \Gamma \), there holds \( |v_i - v_i| \leq C|x_i - x_0| \) and \( |\zeta_i - \zeta_i| \leq C|x_i - x_0| \), \( i = 1, 2, \cdots, N - 1 \). Here, throughout this paper, we denote by \( v_i(x) \) the outward unit normal vector of \( \Gamma_i(\Gamma) \) at \( x_i(x_0) \), while we use \( \zeta_i(\zeta_i) \) to denote the \( i \)-th principal tangential unit vector of \( \Gamma_i(\Gamma) \) at \( x_i(x_0) \).

\( (H_3) \) For any \( x, y \in B_r(x_0) \), it holds \( P(x) = Q(x) \), where \( x_0 \in \Gamma \) and \( r \) is a small positive constant.
Remark 1.1. Let \( F(x) = \sum_{i=1}^{N} \frac{x_i^2}{a_i^2} - 1 \) with \( a_i > 0, a_i \neq a_j (i \neq j) \) and \( \Gamma = \{ x \in \mathbb{R}^N : F(x) = 0 \} \). Take
\[
P(x) = Q(x) = F^2 + 1, \text{ in } W_0,
\]
where \( W_0 = \{ x \in \mathbb{R}^N : |\sum_{i=1}^{N} \frac{x_i^2}{a_i^2} - 1| \leq \delta_0 \} \) for some small fixed \( \delta_0 > 0 \). Then we can use the above conditions to the potentials \( P(x), Q(x) \).

Let us point out that if \( \Gamma \) is a local minimum (or local maximum) set of \( P(x) \) and \( Q(x) \), then for any \( x \in \Gamma \),
\[
P(x) = 1, \quad \nabla P(x) = 0
\]
and
\[
Q(x) = 1, \quad \nabla Q(x) = 0.
\]
This implies that for any tangential vector \( \zeta \) of \( \Gamma \) at \( x \), one has
\[
(D_{\zeta} \nabla)P(x) = 0, \quad (D_{\zeta} \nabla)Q(x) = 0, \quad \forall x \in \Gamma,
\]
where \( D_{\zeta} \) denotes the directional derivative at the direction \( \zeta \).

Let \( U \) be the unique positive radial solution of the following problem
\[
\begin{cases}
\Delta u + u = u^2, & u > 0, \text{ in } \mathbb{R}^N, \\
u(0) = \max_{x \in \mathbb{R}^N} u(x), & u(x) \in H^1(\mathbb{R}^N).
\end{cases}
\]
(1.7)

It is well-known in [10] that \( U(x) \) is strictly decreasing and its \( s \) order derivative satisfies
\[
|D^s U(x)| e^{|x| \frac{L_k}{2}} \leq C
\]
for \( |s| \leq 1 \) and some constant \( C > 0 \).

For \( x_\varepsilon \) close to \( x_0 \), if we denote
\[
\tilde{V}_{x_\varepsilon} = P(x_\varepsilon) U(\sqrt{\frac{P(x_\varepsilon)}{\varepsilon}})(x - x_\varepsilon),
\]
then \( \tilde{V}_{x_\varepsilon} \) solves
\[
\begin{cases}
-\varepsilon^2 \Delta v + P(x_\varepsilon) v = v^2, & v > 0, \quad x \in \mathbb{R}^N, \\
v(x_\varepsilon) = \max_{x \in \mathbb{R}^N} v(x), & v \in H^1(\mathbb{R}^N).
\end{cases}
\]
(1.8)

Also, write
\[
\tilde{W}_{x_\varepsilon} = Q(x_\varepsilon) U(\sqrt{\frac{Q(x_\varepsilon)}{\varepsilon}})(x - x_\varepsilon),
\]
which is the solution of
\[
\begin{cases}
-\varepsilon^2 \Delta w + Q(x_\varepsilon) w = w^2, & w > 0, \quad x \in \mathbb{R}^N, \\
w(x_\varepsilon) = \max_{x \in \mathbb{R}^N} w(x), & w \in H^1(\mathbb{R}^N).
\end{cases}
\]
(1.9)

Take \( (V_{x_\varepsilon}, W_{x_\varepsilon}) = (a \tilde{V}_{x_\varepsilon}, \beta \tilde{W}_{x_\varepsilon}) \) with \( a = \frac{1}{\beta} \sqrt{\frac{2(\mu - 1)}{\mu}}, \beta = \frac{1}{\mu} \). Then \( (V_{x_\varepsilon}, W_{x_\varepsilon}) \) solves
\[
\begin{cases}
-\varepsilon^2 \Delta v + P(x_\varepsilon) v = \mu v w, & x \in \mathbb{R}^N, \\
-\varepsilon^2 \Delta w + Q(x_\varepsilon) w = \frac{\mu}{\beta} v^2 + \gamma w^2, & x \in \mathbb{R}^N, \\
v > 0, \quad w > 0,
\end{cases}
\]
(1.10)

since \( (H_2) \) holds.

For \( \varepsilon > 0 \) small, we will use \( (V_{x_\varepsilon}, W_{x_\varepsilon}) \) to construct the single peak solutions concentrating at \( x_0 \). First we give the following definitions.
**Definition 1.2.** We say that \((v_{\varepsilon}, w_{\varepsilon})\) is a single peak solution of (1.1) concentrating at \(x_0\) if there exist \(x_0 \in \mathbb{R}^N\) with \(|x_0 - x| = o(1)\) such that 
\[
\|(v_{\varepsilon} - V_{\varepsilon}, x_{\varepsilon}, w_{\varepsilon} - W_{\varepsilon}, x_{\varepsilon})\|_{\varepsilon} = o(\varepsilon^N),
\]
where \(\|(u, v)\|_{\varepsilon} = \int_{\mathbb{R}^N} (\varepsilon^2|\nabla u|^2 + P(x)u^2 + \varepsilon^2|\nabla v|^2 + Q(x)v^2) \, dx\).

**Definition 1.3.** We say that a critical point \(x_0 \in \Gamma\) of \(\Delta(P + Q)\) on \(\Gamma\) is non-degenerate if it holds
\[
\frac{\partial^2(P(x_0) + Q(x_0))}{\partial v^2} \neq 0, \quad \det \left( \frac{\partial^2 P(x_0)}{\partial \zeta_i \partial \zeta_j} \right)_{1 \leq i, j \leq N-1} \neq 0, \quad \det \left( \frac{\partial^2 Q(x_0)}{\partial \zeta_i \partial \zeta_j} \right)_{1 \leq i, j \leq N-1} \neq 0.
\]

The main result of this paper is the following.

**Theorem 1.4.** Assume that \((H_1) - (H_3)\) hold. If \(x_0 \in \Gamma\) is a non-degenerate critical point of \(\Delta(P + Q)\), then there exists \(\varepsilon_0 > 0\) such that (1.1) has a single peak solution \((v_{\varepsilon}, w_{\varepsilon})\) concentrating at \(x_0\) provided \(\varepsilon \in (0, \varepsilon_0]\).

As in [13, 20, 22], we mainly apply the finite dimensional reduction method to prove our main result. Compared with [13], we have to overcome much difficulties in the reduction process which involves some technical and careful computations due to the \(\chi^{(2)}\) nonlinearity appears. Moreover, to our best knowledge, our result exhibits a new phenomenon for the coupled Schrödinger system with \(\chi^{(2)}\) nonlinearity.

**Remark 1.5.** Combining the ideas from [9, 13], where in [9] the coupled nonlinear Gross-Pitaevskii system was studied, we guess that the following conclusions may hold.

1. On the basis of Theorem 1.4, further we can prove the local uniqueness of single peak solutions by using local Pohazaev type identity.

2. Under the conditions of Theorem 1.4, if \(\Delta(P + Q)\) has an isolated maximum or minimum point \(x_0 \in \Gamma\), then for any integer \(k > 0\), (1.1) has a \(k\)-peaks solution whose peaks cluster at \(x_0\).

The structure of this paper is organized as follows. In section 2, we do some preliminaries and then we carry out the finite dimensional reduction. We will prove our main result in section 3. In the sequel, for simplicity of notations we write \(\int f\) to mean the Lebesgue integral of \(f(x)\) in \(\mathbb{R}^N\).

## 2 the finite dimensional reduction

In this section, we mainly give some preliminaries and do the finite dimensional reduction. Hereafter, for any function \(K(x) > 0\), we define the Sobolev space
\[
H^1_{\varepsilon,K} = \{ u \in H^1(\mathbb{R}^N) : \int (\varepsilon^2|\nabla u|^2 + K(x)u^2) < \infty \},
\]
endowed with the standard norm
\[
\|u\|_{\varepsilon,K} = \left( \int (\varepsilon^2|\nabla u|^2 + K(x)u^2) \right)^{\frac{1}{2}},
\]
which is induced by the inner product
\[
\langle u, v \rangle_{\varepsilon,K} = \int (\varepsilon^2 \nabla u \nabla v + K(x)uv).
\]

Now we define \(H\) to be the product space \(H^1_{\varepsilon,P} \times H^1_{\varepsilon,Q}\) with the norm
\[
\|(u, v)\|_{\varepsilon}^2 = \|u\|_{\varepsilon,P}^2 + \|v\|_{\varepsilon,Q}^2.
\]
and set

\[ E_{\varepsilon, x_\varepsilon} = \{(\varphi, \psi) \in H : \langle (\varphi, \psi), \left(\frac{\partial V_{\varepsilon, x_\varepsilon}}{\partial x_i}, \frac{\partial W_{\varepsilon, x_\varepsilon}}{\partial x_i}\right)\rangle = 0 \} \]

for \( i = 1, \ldots, N \), where

\[ \langle (\varphi, \psi), \left(\frac{\partial V_{\varepsilon, x_\varepsilon}}{\partial x_i}, \frac{\partial W_{\varepsilon, x_\varepsilon}}{\partial x_i}\right)\rangle = \int \left( \varepsilon^2 \nabla^2 \varphi \frac{\partial V_{\varepsilon, x_\varepsilon}}{\partial x_i} + p(x) \frac{\partial V_{\varepsilon, x_\varepsilon}}{\partial x_i} \varphi + \varepsilon^2 \nabla \varphi \frac{\partial W_{\varepsilon, x_\varepsilon}}{\partial x_i} + Q(x) \frac{\partial W_{\varepsilon, x_\varepsilon}}{\partial x_i} \psi \right). \]

Note that the variational functional corresponding to (1.1) is

\[ I_\varepsilon(v, w) = \frac{1}{2} \int (\varepsilon^2 |\nabla v|^2 + p(x)v^2 + \varepsilon^2 |\nabla w|^2 + Q(x)w^2) - \frac{\mu}{2} \int v^2w - \frac{\gamma}{3} \int w^3. \tag{2.1} \]

Then \( I \in C^2(H, \mathbb{R}) \) and its critical points are solutions of (1.1).

Set

\[ J_\varepsilon(\varphi, \psi) = I_\varepsilon(V_{\varepsilon, x_\varepsilon} + \varphi, W_{\varepsilon, x_\varepsilon} + \psi), \quad (\varphi, \psi) \in E_{\varepsilon, x_\varepsilon}. \]

We can expand \( J_\varepsilon(\varphi, \psi) \) as follows:

\[ J_\varepsilon(\varphi, \psi) = J_\varepsilon(0, 0) + \ell_\varepsilon(\varphi, \psi) + \frac{1}{2} L_\varepsilon(\varphi, \psi) + R_\varepsilon(\varphi, \psi), \tag{2.2} \]

where

\[ \ell_\varepsilon(\varphi, \psi) = \int \left( \varepsilon^2 \nabla V_{\varepsilon, x_\varepsilon} \nabla \varphi + p(x) V_{\varepsilon, x_\varepsilon} \varphi + \varepsilon^2 \nabla W_{\varepsilon, x_\varepsilon} \nabla \psi + Q(x) W_{\varepsilon, x_\varepsilon} \psi \right) - \mu \int V_{\varepsilon, x_\varepsilon} W_{\varepsilon, x_\varepsilon} \varphi \]

\[ - \frac{\mu}{2} \int V_{\varepsilon, x_\varepsilon}^2 \psi - \gamma \int W_{\varepsilon, x_\varepsilon}^2 \psi \]

\[ = \int (p(x) - p(x_\varepsilon)) V_{\varepsilon, x_\varepsilon} \varphi + (Q(x) - Q(x_\varepsilon)) W_{\varepsilon, x_\varepsilon} \psi, \]

\[ L_\varepsilon(\varphi, \psi) = \int \left( \varepsilon^2 |\nabla \varphi|^2 + p(x) \varphi^2 + \varepsilon^2 |\nabla \psi|^2 + Q(x) \psi^2 \right) - \int (\mu W_{\varepsilon, x_\varepsilon} \varphi^2 + 2 \gamma W_{\varepsilon, x_\varepsilon} \psi^2) \]

\[ + 2 \mu V_{\varepsilon, x_\varepsilon} \varphi \psi \]

and

\[ R_\varepsilon(\varphi, \psi) = -\frac{\mu}{2} \int \varphi^2 \psi - \frac{\gamma}{3} \int \psi^3. \]

It follows from [5] that \( (V_{\varepsilon, x_\varepsilon} + \varphi, W_{\varepsilon, x_\varepsilon} + \psi) \) is a critical point of \( I_\varepsilon(v, w) \) if and only if \((\varphi, \psi)\) is a critical point of \( J_\varepsilon(\varphi, \psi)\). In order to find a critical point for \( J_\varepsilon(\varphi, \psi) \), we need to discuss each terms in the expansion (2.2).

**Lemma 2.1.** There exists \( C > 0 \) independent of \( \varepsilon \) such that

\[ ||R^i_\varepsilon(\varphi, \psi)|| \leq C \varepsilon^{-\frac{i}{2}} ||(\varphi, \psi)||^{\frac{i}{2}-1}, \quad i = 0, 1, 2, \]

where \( R^i_\varepsilon(\varphi, \psi) \) denotes the \( i \)-th derivative of \( R_\varepsilon(\varphi, \psi) \).

**Proof.** Recall that

\[ R_\varepsilon(\varphi, \psi) = -\frac{\mu}{2} \int \varphi^2 \psi - \frac{\gamma}{3} \int \psi^3. \]

By the direct computations, we get

\[ \langle R_\varepsilon(\varphi, \psi), (\xi, \eta) \rangle = -\frac{\mu}{2} \int (\varphi^2 \eta + 2 \varphi \psi \xi) - \gamma \int \psi^2 \eta \]
and
\[ R\varepsilon(\varphi, \psi)(\xi, \eta, g, h) = -\mu \int (\varphi g + \psi h \xi + \psi g \xi) - 2\gamma \int \psi \eta. \]

Observe that by letting \( u\varepsilon(x) = u(\varepsilon x) \), it is easy to check
\[ ||u||_{L^p(\mathbb{R}^N)} \leq C\varepsilon^{\frac{1}{2} - \frac{1}{p}} ||u||_e \tag{2.3} \]
with \( ||u||_e = (\int (\varepsilon^2 \nabla u^2 + u^2))^\frac{1}{2} \) for \( 2 \leq p \leq 2^* \) and some \( C > 0 \). Then we have
\[
|R\varepsilon(\varphi, \psi)| \leq C \int (|\varphi|^2 \eta + |\varphi||\psi||\xi| + |\psi|^2 \eta) \\
\leq C \int (|\varphi|^2 \eta + |\varphi||\psi||\xi| + |\psi|^2 \eta) \\
\leq C \varepsilon^{-\frac{q}{2}} |||\varphi||_e ||\psi||_e \\
\leq C \varepsilon^{-\frac{q}{2}} ||(\varphi, \psi)||^2_e ||(\xi, \eta)||_e
\]
and similarly,
\[
|R\varepsilon(\varphi, \psi)(\xi, \eta, g, h)| \\
\leq C \int (|\varphi||g| \eta + |\varphi||h||\xi| + |\psi||g||\xi| + |\psi||h||\eta|) \\
\leq C \varepsilon^{-\frac{q}{2}} ||\varphi||_e ||(\xi, \eta)||_e ||(g, h)||_e.
\]
This completes our proof. \(\square\)

**Lemma 2.2.** There exists \( C > 0 \) independent of \( \varepsilon \) such that
\[ ||\varepsilon|| \leq C \left((|\nabla P(x_\varepsilon)| + |\nabla Q(x_\varepsilon)|)\varepsilon^{\frac{q}{2} + 1} + \varepsilon^{\frac{q}{2} + 2}\right). \]

**Proof.** For any \( (\varphi, \psi) \in E_{e, x_\varepsilon} \), taking into account the decay property of \( U \), we find for any fixed \( d > 0 \),
\[
\varepsilon(\varphi, \psi) = \int (P(x) - P(x_\varepsilon))V_{e, x_\varepsilon} \varphi + (Q(x) - Q(x_\varepsilon)) \psi \\
= \varepsilon^{N} \left( \int (P(ey + x_\varepsilon) - P(x_\varepsilon))aP(x_\varepsilon)U(\sqrt{P(x_\varepsilon)})\varphi(ey + x_\varepsilon) \\
+ \int (Q(ey + x_\varepsilon) - Q(x_\varepsilon))\beta Q(x_\varepsilon)U(\sqrt{Q(x_\varepsilon)})\psi(ey + x_\varepsilon) \right)
\]
\[
= \varepsilon^{N} \left( \int_{B_\frac{d}{\varepsilon}(0)} (P(ey + x_\varepsilon) - P(x_\varepsilon))aP(x_\varepsilon)U(\sqrt{P(x_\varepsilon)})\varphi(ey + x_\varepsilon) \\
+ \int_{B_\frac{d}{\varepsilon}(0)} (Q(ey + x_\varepsilon) - Q(x_\varepsilon))\beta Q(x_\varepsilon)U(\sqrt{Q(x_\varepsilon)})\psi(ey + x_\varepsilon) \\
+ \int_{\mathbb{R}^N \setminus B_\frac{d}{\varepsilon}(0)} (P(ey + x_\varepsilon) - P(x_\varepsilon))aP(x_\varepsilon)U(\sqrt{P(x_\varepsilon)})\varphi(ey + x_\varepsilon) \\
+ \int_{\mathbb{R}^N \setminus B_\frac{d}{\varepsilon}(0)} (Q(ey + x_\varepsilon) - Q(x_\varepsilon))\beta Q(x_\varepsilon)U(\sqrt{Q(x_\varepsilon)})\psi(ey + x_\varepsilon) \right)
\]
Using the above result, we come to discuss the invertibility of \(L\). To this end, we choose \((V, W) = (aU, BU)\). Then \((V, W)\) solves

\[
-\Delta v + v = \mu v w, \quad v \in \mathbb{R}^N, \\
-\Delta w + w = \frac{\mu}{\mu_0} v^2 + \gamma w^2, \quad w \in \mathbb{R}^N.
\]  

(2.4)

It follows from Proposition 2.2 in [22] that

**Proposition 2.3.** For any \(\mu > 0\) and \(\mu > \gamma\), \((V, W)\) is non-degenerate for the system (2.4) in \(H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)\) in the sense that the kernel is given by span \(\{\lambda(\mu, \gamma) \frac{\partial V}{\partial x_i}, \frac{\partial W}{\partial x_i} | i = 1, 2, \ldots, N\}\), where \(\lambda(\mu, \gamma) \neq 0\).

Using the above result, we come to discuss the invertibility of \(L\) in \(E_{e, \epsilon}\).

**Lemma 2.4.** There exist constants \(\rho > 0\) and \(\epsilon_0 > 0\) such that for all \(\epsilon \in (0, \epsilon_0]\),

\[
\|L(v, w)\| \geq \rho \|(v, w)\|_{\epsilon}, \quad \forall (v, w) \in E_{e, \epsilon}.
\]
Proof. We argue by contradiction. Suppose that there exist \( \varepsilon_n \to 0 \), \( x_{\varepsilon_n} \to x_0 \) and \((v_n, w_n) \in E_{\varepsilon_n,x_{\varepsilon_n}}\) such that for any \((\varphi_n, \psi_n) \in E_{\varepsilon_n,x_{\varepsilon_n}}\),

\[
\langle L (v_n, w_n), (\varphi_n, \psi_n) \rangle = o_n(1) \|(v_n, w_n)\|_{E_{\varepsilon_n}} \|(\varphi_n, \psi_n)\|_{E_{\varepsilon_n}}.
\]

(2.5)

Without loss of generality, we can assume that \( \|(v_n, w_n)\|_{E_{\varepsilon_n}}^2 = \varepsilon_n^N \) and let

\[
\bar{v}_n(x) = \frac{1}{\sqrt{P(x_{\varepsilon_n})}} v_n \left( \frac{\varepsilon_n x}{\sqrt{P(x_{\varepsilon_n})}} + x_{\varepsilon_n} \right)
\]

and

\[
\bar{w}_n(x) = \frac{1}{\sqrt{Q(x_{\varepsilon_n})}} w_n \left( \frac{\varepsilon_n x}{\sqrt{Q(x_{\varepsilon_n})}} + x_{\varepsilon_n} \right).
\]

Then, in view of \( \|(v_n, w_n)\|_{E_{\varepsilon_n}}^2 = \varepsilon_n^N \), we get

\[
\|(\bar{v}_n, \bar{w}_n)\|_{H^1(\mathbb{R}^N)} = \int \left( |\nabla \bar{v}_n|^2 + |\nabla \bar{w}_n|^2 + |\bar{v}_n|^2 + |\bar{w}_n|^2 \right) \leq C,
\]

which implies that up to a subsequence, there exists \( v, w \in H^1(\mathbb{R}^N) \) such that as \( n \to +\infty \),

\[
\begin{align*}
\bar{v}_n \rightharpoonup v, & \quad \bar{w}_n \rightharpoonup w & \text{in } H^1(\mathbb{R}^N), \\
\bar{v}_n \to v, & \quad \bar{w}_n \to w & \text{in } L^2_{loc}(\mathbb{R}^N).
\end{align*}
\]

Now we claim that \( v = w = 0 \). Considering (2.5), we find

\[
\begin{align*}
\int \left[ P^2(x_{\varepsilon_n}) & \left( \nabla \bar{v}_n \nabla \bar{\varphi}_n + \nabla \bar{w}_n \nabla \bar{\psi}_n - \mu(\beta U) \bar{v}_n \bar{\varphi}_n - \mu(\alpha U) \bar{w}_n \bar{\psi}_n \right) - 2\gamma(\beta U) \bar{w}_n \bar{\psi}_n \right] P(x_{\varepsilon_n}) \left( \frac{\varepsilon_n x}{\sqrt{P(x_{\varepsilon_n})}} + x_{\varepsilon_n} \right) \bar{v}_n \bar{\varphi}_n \\
& + Q(x_{\varepsilon_n}) Q \left( \frac{\varepsilon_n x}{\sqrt{Q(x_{\varepsilon_n})}} + x_{\varepsilon_n} \right) \bar{w}_n \bar{\psi}_n \left( \frac{\varepsilon_n x}{\sqrt{Q(x_{\varepsilon_n})}} + x_{\varepsilon_n} \right) \bar{\varphi}_n^2 \\
& + Q(x_{\varepsilon_n}) Q \left( \frac{\varepsilon_n x}{\sqrt{Q(x_{\varepsilon_n})}} + x_{\varepsilon_n} \right) \bar{\psi}_n^2 \right] \\
& = (P(x_{\varepsilon_n}))^\frac{2}{\varepsilon_n} o_n(1) \left[ \int P^2(x_{\varepsilon_n}) (|\nabla \bar{\varphi}_n|^2 + |\nabla \bar{\psi}_n|^2)^2 + Q^2(x_{\varepsilon_n}) (|\nabla \bar{\psi}_n|^2)^2 + P(x_{\varepsilon_n}) \left( \frac{\varepsilon_n x}{\sqrt{P(x_{\varepsilon_n})}} + x_{\varepsilon_n} \right) \bar{\varphi}_n^2 \\
& + Q(x_{\varepsilon_n}) Q \left( \frac{\varepsilon_n x}{\sqrt{Q(x_{\varepsilon_n})}} + x_{\varepsilon_n} \right) \bar{\psi}_n^2 \right] \frac{1}{\varepsilon_n}.
\end{align*}
\]

(2.6)

where

\[
\begin{align*}
\bar{\varphi}_n(x) &= \frac{1}{\sqrt{P(x_{\varepsilon_n})}} \varphi_n \left( \frac{\varepsilon_n x}{\sqrt{P(x_{\varepsilon_n})}} + x_{\varepsilon_n} \right), \\
\bar{\psi}_n(x) &= \frac{1}{\sqrt{Q(x_{\varepsilon_n})}} \psi_n \left( \frac{\varepsilon_n x}{\sqrt{Q(x_{\varepsilon_n})}} + x_{\varepsilon_n} \right)
\end{align*}
\]

and \((\bar{\varphi}_n, \bar{\psi}_n) \in \bar{E}_{\varepsilon_n,x_{\varepsilon_n}}\) with

\[
\bar{E}_{\varepsilon_n,x_{\varepsilon_n}} = \left\{ (\bar{\varphi}, \bar{\psi}) : \left( \bar{\varphi} \left( \frac{\sqrt{P(x_{\varepsilon_n})}}{\varepsilon_n} (x - x_{\varepsilon_n}) \right), \bar{\psi} \left( \frac{\sqrt{Q(x_{\varepsilon_n})}}{\varepsilon_n} (x - x_{\varepsilon_n}) \right) \right) \in E_{\varepsilon_n,x_{\varepsilon_n}} \right\}.
\]

On the other hand, being \( \|(v_n, w_n)\|_{E_{\varepsilon_n}}^2 = \varepsilon_n^N \), we have

\[
\begin{align*}
\int \left[ P(x_{\varepsilon_n}) (|\nabla \bar{v}_n|^2 + |\nabla \bar{w}_n|^2)^2 + P \left( \frac{\varepsilon_n x}{\sqrt{P(x_{\varepsilon_n})}} + x_{\varepsilon_n} \right) \bar{v}_n^2 \\
& + Q \left( \frac{\varepsilon_n x}{\sqrt{Q(x_{\varepsilon_n})}} + x_{\varepsilon_n} \right) \bar{w}_n^2 \right] = (P(x_{\varepsilon_n}))^\frac{2}{\varepsilon_n} - 1.
\end{align*}
\]

(2.7)

So, taking \( \varphi_n = v_n, \psi_n = w_n \), (2.6) gives

\[
\begin{align*}
\int \left[ P(x_{\varepsilon_n}) (|\nabla \bar{v}_n|^2 + |\nabla \bar{w}_n|^2 - \mu(\beta U) \bar{v}_n^2 - 2\mu(\alpha U) \bar{w}_n - 2\gamma(\beta U) |\bar{w}_n|^2) \\
& + P \left( \frac{\varepsilon_n x}{\sqrt{P(x_{\varepsilon_n})}} + x_{\varepsilon_n} \right) \bar{v}_n^2 + Q \left( \frac{\varepsilon_n x}{\sqrt{Q(x_{\varepsilon_n})}} + x_{\varepsilon_n} \right) \bar{w}_n^2 \right] \\
& = o_n(1) (P(x_{\varepsilon_n}))^\frac{2}{\varepsilon_n} - 1.
\end{align*}
\]

(2.8)
Also, since \((\tilde{v}_n, \tilde{w}_n) \in \tilde{E}_{\varepsilon_n, \varepsilon_n}\), we obtain
\[
\int \left[ P(x) \nabla \tilde{v}_n \nabla \frac{\partial (aU)}{\partial y_i} + P\left( \frac{e_n x}{\sqrt{P(x)}} + x_n \right) \tilde{v}_n \frac{\partial (aU)}{\partial y_i} + \frac{Q(x) \nabla \tilde{w}_n \nabla \frac{\partial (\beta U)}{\partial y_i}}{\partial y_i} + Q\left( \frac{e_n x}{\sqrt{Q(x)}} + x_n \right) \tilde{w}_n \frac{\partial (\beta U)}{\partial y_i} \right] = 0,
\]
which gives that by letting \(n \to +\infty\),
\[
\int \left[ \nabla \tilde{v} \nabla \frac{\partial (aU)}{\partial y_i} + v(x) \frac{\partial (aU)}{\partial x_j} + \nabla w \nabla \frac{\partial (\beta U)}{\partial x_i} + w(x) \frac{\partial (\beta U)}{\partial x_i} \right] = 0. \tag{2.9}
\]
Now we take \((\tilde{\varphi}, \tilde{\psi}) \in C_0^\infty(\mathbb{R}^N) \times C_0^\infty(\mathbb{R}^N)\) satisfying
\[
\int \left[ \nabla \tilde{\varphi} \nabla \frac{\partial (aU)}{\partial y_i} + \tilde{\varphi} \frac{\partial (aU)}{\partial x_j} + \nabla \tilde{\psi} \nabla \frac{\partial (\beta U)}{\partial x_i} + \tilde{\psi} \frac{\partial (\beta U)}{\partial x_i} \right] = 0. \tag{2.10}
\]
Meanwhile, we can decompose \((\tilde{\varphi}, \tilde{\psi}) \in \tilde{E}_{\varepsilon_n, \varepsilon_n}\) as follows
\[
(\tilde{\varphi}, \tilde{\psi}) = (\tilde{\varphi}_n, \tilde{\psi}_n) - \sum_{i=1}^N a_{i,n} \left( \frac{\partial V_{\varepsilon_n, \varepsilon_n}}{\partial x_i} + \frac{\partial W_{\varepsilon_n, \varepsilon_n}}{\partial x_i} \right).
\]
Then from \((\tilde{\varphi}_n, \tilde{\psi}_n) \in \tilde{E}_{\varepsilon_n, \varepsilon_n}\) and the definition of \(\tilde{E}_{\varepsilon_n, \varepsilon_n}\), we get
\[
\int \left[ P(x) \nabla \tilde{\varphi}_n \nabla \frac{\partial (aU)}{\partial y_i} + P\left( \frac{e_n x}{\sqrt{P(x)}} + x_n \right) \tilde{\varphi}_n \frac{\partial (aU)}{\partial y_i} + \frac{Q(x_n) \nabla \tilde{\psi}_n \nabla \frac{\partial (\beta U)}{\partial y_i}}{\partial y_i} + Q\left( \frac{e_n x}{\sqrt{Q(x)}} + x_n \right) \tilde{\psi}_n \frac{\partial (\beta U)}{\partial y_i} \right]
\]
\[
+ \int \left[ \sum_{i=1}^N a_{i,n} \nabla \frac{\partial (aU)}{\partial x_i} + \sum_{i=1}^N a_{i,n} \nabla \frac{\partial (\beta U)}{\partial x_i} \right] = 0. \tag{2.11}
\]
But by decay property of \(U\), there exists \(C > 0\) such that
\[
\int \left[ |\nabla \frac{\partial (aU)}{\partial x_i}|^2 + |\nabla \frac{\partial (\beta U)}{\partial x_i}|^2 + |\frac{\partial (aU)}{\partial x_i}|^2 + |\frac{\partial (\beta U)}{\partial x_i}|^2 \right] = C > 0. \tag{2.12}
\]
Hence, combining (2.10)-(2.12), we can easily check that \(a_{i,n} \to 0\) as \(n \to +\infty\) and then it follows from (2.6) that
\[
\int \left[ P(x) \nabla \tilde{\varphi} \nabla \frac{\partial (aU)}{\partial y_i} + v \frac{\partial (aU)}{\partial x_j} + \nabla w \nabla \frac{\partial (\beta U)}{\partial x_i} + w \frac{\partial (\beta U)}{\partial x_i} \right] = 0.
\]
This implies that for any \((\varphi, \psi) \in C_0^\infty(\mathbb{R}^N) \times C_0^\infty(\mathbb{R}^N) \cap \tilde{E}_{\varepsilon_n, \varepsilon_n}\),
\[
\int \left[ \nabla \varphi \nabla \psi + \nabla w \nabla \psi + v \varphi + w \psi - \mu (\beta U) \nu \varphi - \mu (aU) \nu \psi \right] = 0. \tag{2.13}
\]
On the other hand, from the fact that \((aU, \beta U)\) solves (2.4), we see
\[
\int \left[ \nabla \varphi \nabla \frac{\partial (aU)}{\partial x_i} + \nabla w \nabla \frac{\partial (\beta U)}{\partial x_i} + v \frac{\partial (aU)}{\partial x_i} + w \frac{\partial (\beta U)}{\partial x_i} \right]
\]
\[
= \mu (aU) \nu \psi - 2 \gamma (\beta U) \mu (aU) \nu \psi = 0. \tag{2.14}
\]
(2.15)
which, together with (2.14), yields that for any \((\varphi, \psi) \in C_0^\infty(\mathbb{R}^N) \times C_0^\infty(\mathbb{R}^N)\)

\[
\int \left[ \nabla v \nabla \varphi + \nabla w \nabla \psi + v \varphi + w \psi - \mu(\beta U) \varphi - \mu(aU)w \varphi - \mu(aU)v \psi - 2\gamma(\beta U)w \psi \right] = 0.
\]

So \((v, w)\) is a solution of

\[
\begin{align*}
-\Delta v + v &= \mu(\beta U)v + \mu(aU)w, \quad x \in \mathbb{R}^N, \\
-\Delta w + w &= \mu(aU)v + 2\gamma(\beta U)w, \quad x \in \mathbb{R}^N.
\end{align*}
\]

Using Proposition 2.3, there exist \(b_i \in \mathbb{R}, i = 1, 2, \ldots, N\) such that

\[
(v, w) = \sum_{i=1}^{N} b_i \left( \frac{\partial(aU)}{\partial x_i}, \frac{\partial(\beta U)}{\partial x_i} \right).
\]

But (2.9) gives that \(b_i = 0, i = 1, 2, \ldots, N\). That is \((v, w) = (0, 0)\), which is exactly our claim. Finally, taking into account that \(\bar{v}_n \to 0\) in \(L^2_{\text{loc}}(\mathbb{R}^N)\) and the exponential decay of \(U\), we have

\[
\int_{B_R(0)} U|\bar{v}_n|^2 + \int_{\mathbb{R}^N \setminus B_R(0)} U|\bar{v}_n|^2 = o_R(1) + O(e^{-R}),
\]

where \(o_R(1) \to 0\) as \(R \to +\infty\).

As a result, from (2.8) and (2.7), we deduce that

\[
o_n(1)(P(x_{e_n}))^{2\frac{N}{2} - 1} = \int P(x_{e_n})(|\nabla \bar{v}_n|^2 + |\nabla \bar{w}_n|^2 - \mu(\beta U)\bar{v}_n^2 - 2\mu(aU)\bar{v}_n\bar{w}_n - 2\gamma(\beta U)|\bar{w}_n|^2)
\]

\[
+ P \left( \frac{\epsilon_n x_{\bar{x}_n}}{\sqrt{P(x_{e_n})}} + x_{\bar{x}_n} \right) \bar{w}_n^2 + Q \left( \frac{\epsilon_n x_{\bar{x}_n}}{\sqrt{Q(x_{e_n})}} + x_{\bar{x}_n} \right) \bar{w}_n^2 = (P(x_{e_n}))^{2\frac{N}{2} - 1} + o_R(1) + O(e^{-R}),
\]

which is impossible for large \(n\) and \(R\). So we complete this proof. \(\square\)

**Proposition 2.5.** For \(\varepsilon > 0\) sufficiently small, there is \((\varphi_\varepsilon, \psi_\varepsilon) \in E_{\varepsilon,x}\), such that

\[
(f_\varepsilon(\varphi_\varepsilon, \psi_\varepsilon), (g, h)) = 0, \quad \forall (g, h) \in E_{\varepsilon,x}\.
\]

Moreover,

\[
\|(\varphi_\varepsilon, \psi_\varepsilon)\|_\varepsilon \leq C \left[ \left( |\nabla P(x_{e})| + |\nabla Q(x_{e})| \right) e^{\frac{\lambda}{\varepsilon^{\frac{1}{2}}} + \varepsilon^{\frac{\lambda}{2} + \frac{2}{3}}} \right]
\]

for some constant \(C > 0\) independent of \(\varepsilon\).

**Proof.** We will use the contraction mapping theorem to prove the wanted result. By Lemma 2.2, \(\ell_\varepsilon(\varphi, \psi)\) is a bounded linear function in \(E_{\varepsilon,x}\). So using Riesz representation theorem, we obtain that there is an \(\hat{\ell}_\varepsilon \in E_{\varepsilon,x}\), such that

\[
\ell_\varepsilon(\varphi, \psi) = \langle \hat{\ell}_\varepsilon, (\varphi, \psi) \rangle.
\]

Hence finding a critical point for \(f_\varepsilon(\varphi, \psi)\) is equivalent to solving

\[
\ell_\varepsilon + L(\varphi, \psi) + R_\varepsilon(\varphi, \psi) = 0.
\]

It follows from Lemma 2.4 that (2.16) can be rewritten as

\[
(\varphi, \psi) = A(\varphi, \psi) := -L^{-1}(\ell_\varepsilon + R_\varepsilon(\varphi, \psi)).
\]

Now we set

\[
S_\varepsilon = \{ (\varphi, \psi) \in E_{\varepsilon,x}, \|(\varphi, \psi)\|_\varepsilon \leq \left[ (|\nabla P(x_{e})| + |\nabla Q(x_{e})|) e^{\frac{\lambda}{\varepsilon^{\frac{1}{2}}} + \varepsilon^{\frac{\lambda}{2} + \frac{2}{3}}} \right] \}
\]
with \( \theta \in (0, 1) \). Then for any \((\varphi, \psi) \in S_e\),
\[
\|A(\varphi, \psi)\| \leq C(\|\tilde{e}\|_e + \|R_e^\nu(\varphi, \psi)\|) \\
\leq C\|\tilde{e}\|_e + Ce^{-\frac{N}{2}}\|\varphi, \psi\|_e^2 \\
\leq \left[\left(\|\nabla P(x_e)\| + |\nabla Q(x_e)\|\right)e^{\frac{N}{2} + 1 - \theta} + e^{\frac{N}{2} + 2 - \theta}\right].
\]

Then \(A\) maps \(S_e\) to \(S_e\). On the other hand, for any \((\varphi_1, \psi_1), (\varphi_2, \psi_2) \in S_e\), form Lemma 2.1,
\[
\|A(\varphi_1, \psi_1) - A(\varphi_2, \psi_2)\| = \|L^{-1}(R^\nu_e(\varphi_1, \psi_1) - R^\nu_e(\varphi_2, \psi_2))\| \\
\leq C\|\tilde{e}\|_e + Ce^{-\frac{N}{2}}\|\varphi_1 - \varphi_2\|_e + Ce^{-\frac{N}{2}}\|\psi_1 - \psi_2\|_e \\
\leq \frac{1}{2}\|\varphi_1 - \varphi_2\|_e + \frac{1}{2}\|\psi_1 - \psi_2\|_e,
\]
where \(t \in (0, 1)\). This gives that \(A\) is a contraction map from \(S_e\) to \(S_e\). Applying the contraction mapping theorem, we can find a unique \((\varphi_\varepsilon, \psi_\varepsilon) \in S_e\) satisfying (2.17) and
\[
\|\varphi_\varepsilon, \psi_\varepsilon\|_e \leq \left[\left(\|\nabla P(x_e)\| + |\nabla Q(x_e)\|\right)e^{\frac{N}{2} + 1 - \theta} + e^{\frac{N}{2} + 2 - \theta}\right].
\]

Furthermore, in view of (2.17), we get
\[
\|\varphi_\varepsilon, \psi_\varepsilon\|_e = \|L^{-1}(\tilde{e}) + L^{-1}(R^\nu_e(\varphi_\varepsilon, \psi_\varepsilon))\|_e \\
\leq C\|\tilde{e}\|_e + Ce^{-\frac{N}{2}}\|\varphi_\varepsilon, \psi_\varepsilon\|_e^2 \\
\leq C\|\tilde{e}\|_e + C\left[\|\nabla P(x_e)\| + |\nabla Q(x_e)\|\right)e^{1 - \theta} + e^{2 - \theta}\|\varphi_\varepsilon, \psi_\varepsilon\|_e,
\]
which implies that
\[
\|\varphi_\varepsilon, \psi_\varepsilon\|_e \leq C\|\tilde{e}\|_e \leq C\left[\|\nabla P(x_e)\| + |\nabla Q(x_e)\|\right]e^{\frac{N}{2} + 1} + e^{\frac{N}{2} + 2}.
\]

\[\square\]

### 3 Proof of our main result

In this section, we assume that \(x_0 \in \Gamma\) is a non-degenerate critical point of \(\Delta(P + Q)\) and we will construct a single peak solution \((v_\varepsilon, w_\varepsilon)\) of (1.1) concentrating at \(x_0\).

From Proposition 2.5, we can get the following result.

**Proposition 3.1.** There exists an \(\varepsilon_0 > 0\) such that for any \(\varepsilon \in (0, \varepsilon_0)\) and \(y\) close to \(x_0\), there is \((\varphi_{\varepsilon, y}, \psi_{\varepsilon, y}) \in E_{\varepsilon, y}\) such that for any \((g, h) \in E_{\varepsilon, y}\),
\[
\int \left[\varepsilon^2 \nabla(V_{\varepsilon, y} + \varphi_{\varepsilon, y}) \nabla g + P(x)(V_{\varepsilon, y} + \varphi_{\varepsilon, y}) g + \varepsilon^2 \nabla(W_{\varepsilon, y} + \psi_{\varepsilon, y}) \nabla h \\
+ Q(x)(W_{\varepsilon, y} + \psi_{\varepsilon, y}) h\right] - \int \left[\mu(V_{\varepsilon, y} + \varphi_{\varepsilon, y}) g(W_{\varepsilon, y} + \psi_{\varepsilon, y}) + \frac{\mu}{2}(V_{\varepsilon, y} + \varphi_{\varepsilon, y})^2 h \\
+ \gamma (W_{\varepsilon, y} + \psi_{\varepsilon, y})^2 h\right] = 0.
\]

Moreover,
\[
\|\varphi_{\varepsilon, y}, \psi_{\varepsilon, y}\|_e = O((\|\nabla P(y)\| + |\nabla Q(y)\|) e^{\frac{N}{2}} + \varepsilon^{\frac{N}{2} + 2}).
\]

To get a true solution of (1.1), we need to choose \(y\) such that
\[
\int \left[\varepsilon^2 \nabla V_{\varepsilon} \nabla \frac{\partial \varphi_{\varepsilon}}{\partial x_i} + P(x) v_e \frac{\partial \varphi_{\varepsilon}}{\partial x_i} + \varepsilon^2 \nabla W_{\varepsilon} \nabla \frac{\partial \psi_{\varepsilon}}{\partial x_i} + Q(x) w_e \frac{\partial \psi_{\varepsilon}}{\partial x_i} - \int \left[\mu v_e \frac{\partial \varphi_{\varepsilon}}{\partial x_i} w_e \\
+ \frac{\mu}{2} v_e^2 \frac{\partial w_e}{\partial x_i} + \gamma w_e^2 \frac{\partial \psi_{\varepsilon}}{\partial x_i}\right] = 0,
\]

(3.1)
where \(v_\varepsilon = V_{\varepsilon,y} + \varphi_{\varepsilon,y}, w_\varepsilon = W_{\varepsilon,y} + \psi_{\varepsilon,y}\) and \(i = 1, \ldots, N\). It is easy to check that (3.1) is equivalent to

\[
\int \left[ \frac{\partial P(x)}{\partial x_i} v_\varepsilon^2 + \frac{\partial Q(x)}{\partial x_i} w_\varepsilon^2 \right] = 0, \quad i = 1, \ldots, N, \tag{3.2}
\]

which is exact the Pohozaev type identity.

For \(y\) close to \(x_0\), \(y \in \Gamma_t\) for some \(t\) close to 1. In the following, we denote by \(\nu\) the unit normal vector of \(\Gamma_t\) at \(y\) and we use \(c_i, i = 1, \ldots, N - 1\), to denote the principal direction of \(\Gamma_t\) at \(y\). Then, at \(y\), one has

\[
D_{\nu} P(y) = 0, \quad |\nabla P(y)| = |D_{\nu} P(y)|
\]

and

\[
D_{\nu} Q(y) = 0, \quad |\nabla Q(y)| = |D_{\nu} Q(y)|.
\]

First, we prove the following results.

**Lemma 3.2.** If \((H_1) - (H_3)\) hold, then

\[
\int (D_{\nu} P(x) v_\varepsilon^2 + D_{\nu} Q(x) w_\varepsilon^2) = 0
\]

is equivalent to

\[
D_{\nu} P(y) + D_{\nu} Q(y) = O(\varepsilon^2).
\]

**Proof.** By the direct computations, we have

\[
\int (D_{\nu} P(x) v_\varepsilon^2 + D_{\nu} Q(x) w_\varepsilon^2)
\]

\[
= - \int \left[ 2D_{\nu} P(x) V_{\varepsilon,y} \varphi_{\varepsilon,y} + D_{\nu} P(x) \varphi_{\varepsilon,y}^2 + 2D_{\nu} Q(x) W_{\varepsilon,y} \psi_{\varepsilon,y} + D_{\nu} Q(x) \psi_{\varepsilon,y}^2 \right] + O((|D_{\nu} P(y)| + |D_{\nu} Q(y)|) \varepsilon^2).
\]

On the other hand, using Taylor’s expansion, we get

\[
\int (D_{\nu} P(x) v_\varepsilon^2 + D_{\nu} Q(x) w_\varepsilon^2)
\]

\[
e^N \left[ \int_{B_\varepsilon(0)} \left( D_{\nu} P(y) + \frac{\varepsilon^2 |x|^2}{2!} (\Delta D_{\nu} P)(y) \right) \alpha^2 P^2(y) U^2(\sqrt{P(y)}x) \right]
\]

\[
+ \int_{B_\varepsilon(0)} \left( D_{\nu} Q(y) + \frac{\varepsilon^2 |x|^2}{2!} (\Delta D_{\nu} Q)(y) \right) \beta^2 Q^2(y) U^2(\sqrt{Q(y)}x) \right] + O(e^{N+4})
\]

\[
= O((|D_{\nu} P(y)| + |D_{\nu} Q(y)|) e^N + (|\Delta D_{\nu} P(y)| + |\Delta D_{\nu} Q(y)|) e^{N+2} + e^{N+4}).
\]

Combining (3.3) and (3.4), we find

\[
D_{\nu} P(y) + D_{\nu} Q(y) = O(\varepsilon^2).
\]

\[\square\]

**Lemma 3.3.** Under the conditions \((H_1) - (H_3)\),

\[
\int (D_{\nu} P(x) v_\varepsilon^2 + D_{\nu} Q(x) w_\varepsilon^2) = 0
\]

is equivalent to

\[
(\Delta D_{\nu} P)(y) + (\Delta D_{\nu} Q)(y) + B_1 K(y) e^2
\]

\[
= O((|\nabla G_1(y)| + |\nabla G_1(y)| + |\nabla Q(y)| + |\nabla Q(y)| + |\nabla G_2(y)|) e^2),
\]

where \(B_1\) is some constant, \(K(y)\) is a smooth function, and \(G_1(x) = \langle \nabla P(x), \zeta \rangle, G_2(x) = \langle \nabla Q(x), \zeta \rangle\).
Proof. Since for any fixed $d > 0$ and $j = 1, 2$

$$G_j(x) = \sum_{i=1}^{N} \frac{\partial G_j(y)}{\partial y_i}(x_i - y_i) + \frac{1}{2} \sum_{i=1}^{N} \sum_{\ell=1}^{N} \frac{\partial^2 G_j(y)}{\partial y_i \partial y_\ell}(x_i - y_i)(x_\ell - y_\ell) + o(|x - y|^2), \text{ in } B_d(y),$$

we have

$$\int (G_1(x)V_{e,y} + G_2(x)W_{e,y}^2) = -2 \int G_1(x)V_{e,y} \varphi_{e,y} - \int G_1(x)\varphi_{e,y}^2 - 2 \int G_2(x)W_{e,y} \psi_{e,y} - \int G_2(x)\psi_{e,y}^2.$$

This is also equivalent to

$$= O\left( \varepsilon^2 \left| \nabla G_1(y) \right| + \varepsilon \left| \nabla G_2(y) \right| + \varepsilon^2 \left( \| \varphi_{e,y} \|_e + \varepsilon \left| \nabla G_1(y) \right| \right) + \left| \nabla G_2(y) \right| \| \psi_{e,y} \|_e^2 \right)$$

$$= O\left( \left( \varepsilon N^2 \left| \nabla G_1(y) \right| + \varepsilon^2 \left| \nabla G_2(y) \right| \right) + \left( \varepsilon^2 \left| \nabla G_1(y) \right| + \varepsilon^2 \left| \nabla G_2(y) \right| \right) \right) + O\left( \varepsilon^N + \varepsilon^N \right).$$

On the other hand, from $G_j(y) = 0$, $j = 1, 2$, we get

$$\int (G_1(x)V_{e,y} + G_2(x)W_{e,y}^2) = O\left( \varepsilon^{N-2} (\Delta G_1(y) + \Delta G_2(y)) + B_1 K(y) \varepsilon^{N+4} + \varepsilon^{N+6} \right).$$

As a result, the result follows.

\[\Box\]

**Proof of Theorem 1.4:** Now, by Lemmas 3.2 and 3.3, (3.2) is equivalent to

$$D_vP(y) + D_vQ(y) = O(\varepsilon^2), \quad (D_c\Delta P)(y) + (D_c\Delta Q)(y) = O(|\nabla P(y)| + |\nabla Q(y)| + \varepsilon^2),$$

which is also equivalent to

$$D_vP(y) + D_vQ(y) = O(\varepsilon^2), \quad (D_c\Delta P)(y) + (D_c\Delta Q)(y) = O(\varepsilon^2).$$

Let $\tilde{y} \in \Gamma$ be the point such that $y - \tilde{y} = \kappa v$ for some $\kappa \in \mathbb{R}$. We have $D_vP(\tilde{y}) = 0$ and $D_vQ(\tilde{y}) = 0$. As a result,

$$D_vP(y) + D_vQ(y) = D_vP(y) - D_vP(\tilde{y}) + D_vQ(y) - D_vQ(\tilde{y})$$

$$= (D_v^2 P)(\tilde{y}) + (D_v^2 Q)(\tilde{y})(y - \tilde{y}, v) + O(|y - \tilde{y}|^2),$$

which, together with the non-degenerate assumption, yields that $D_vP(y) + D_vQ(y) = O(\varepsilon^2)$ can be written as

$$(y - \tilde{y}, v) = O(\varepsilon^2 + |y - \tilde{y}|^2).$$

Let $\zeta_i$ be the $i$-th tangential unit vector of $\Gamma$ at $\tilde{y}$. It follows from the assumption $(H_2)$ that

$$(D_c\Delta P)(y) + (D_c\Delta Q)(y) = (D_c\Delta P)(\tilde{y}) + (D_c\Delta Q)(\tilde{y}) + O(|y - \tilde{y}|)$$

$$= (D_{c,0}\Delta P)(\tilde{y}) + (D_{c,0}\Delta Q)(\tilde{y}) + O(\varepsilon^2)$$

and

$$(D_{c,0}\Delta P)(\tilde{y}) + (D_{c,0}\Delta Q)(\tilde{y})$$

$$= (D_{c,0}\Delta P)(\tilde{y}) - (D_{c,0}\Delta P)(x_0) + (D_{c,0}\Delta Q)(\tilde{y}) - (D_{c,0}\Delta Q)(x_0)$$

$$= O\left( (\nabla T D_{c,0}\Delta P)(x_0), \tilde{y} - x_0 \right) + O\left( (\nabla T D_{c,0}\Delta Q)(x_0), \tilde{y} - x_0 \right),$$

where $\nabla T$ is the tangential gradient on $\Gamma$ at $x_0$ and $c_{i,0}$ is the $i$-th tangential unit vector of $\Gamma$ at $x_0$. Hence

$$(D_{c,0}\Delta P)(y) + (D_{c,0}\Delta Q)(y) = O(\varepsilon^2)$$

can be rewritten as

$$(\nabla T D_{c,0}\Delta P)(x_0) + (\nabla T D_{c,0}\Delta Q)(x_0), \tilde{y} - x_0) = O(\varepsilon^2 + |\tilde{y} - x_0|^2).$$

So we can solve (3.7) and (3.8) to get $y = x_\varepsilon$ with $x_\varepsilon \to x_0$ as $\varepsilon \to 0$. \[\Box\]
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