ON A MONGE-AMPÈRE TYPE EQUATION IN THE CEGRELL CLASS $\mathcal{E}_\chi$

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Abstract. We prove an existence and uniqueness result for a Monge-Ampère type equation in the Cegrell class $\mathcal{E}_\chi$.

1. Introduction

It is a classical problem in analysis to find, for given function $F$, solutions $u$ to the equation:

$$(dd^c u)^n = F(z, u(z))d\mu,$$  \hfill (1.1)

where $(dd^c u)^n$ is the complex Monge-Ampère operator. Equations of the type (1.1) has played a significant not only within the fields of fully nonlinear second order elliptic equations and pluripotential theory, but also in applications. We refer to [6, 7, 10, 11, 15, 17, 18] and the reference therein for further information about equations of Monge-Ampère type.

In the last decade the so called Cegrell classes has played a prominent role in working with the complex Monge-Ampère operator. For further information about the Cegrell classes see e.g. [1, 2, 3, 4, 5, 12, 13, 14]. Let $\mathcal{E}_0$, $\mathcal{E}_p$, $\mathcal{F}$, $\mathcal{N}$, and $\mathcal{E}$ be as in [12, 13, 14].

In [9], Benelkourchi, Guedj, and Zeriahi introduced the following notation of the Cegrell classes. Let $\chi : (-\infty, 0] \to (-\infty, 0]$ be a continuous and nondecreasing function and let $\mathcal{E}_\chi$ contain those plurisubharmonic functions $u$ for which there exists a decreasing sequence $u_j \in \mathcal{E}_0$, which converges pointwise to $u$ on $\Omega$, as $j$ tends to $+\infty$, and

$$\sup_j \int_{\Omega} -\chi(u_j)(dd^c u_j)^n < \infty.$$  

Note that if $\chi = -(t^p)$, then $\mathcal{E}_\chi = \mathcal{E}_p$, and if $\chi = -1$, then $\mathcal{E}_\chi = \mathcal{F}$.

The measure $(dd^c u)^n$ might have infinite total mass, i.e. $(dd^c u)^n(\Omega) = +\infty$. On the other hand, if $u \in \mathcal{E}_\chi(\Omega)$, then under certain assumption on the function $\chi$, the measure $(dd^c u)^n$ vanishes on all pluripolar sets in $\Omega$ and the following integral is always finite

$$\int_{\Omega} -\chi(u)(dd^c u)^n < +\infty,$$

which means that $-\chi(u)(dd^c u)^n$ is a positive and finite measure on $\Omega$. For this reason it is natural to consider the following Monge-Ampère type equation:

$$-\chi(u)(dd^c u)^n = d\mu,$$

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where \( \mu \) is a given positive and finite measure on \( \Omega \), which vanishes on pluripolar sets in \( \Omega \). We prove the following theorem.

**Main Theorem.** Assume that \( \Omega \) is a bounded hyperconvex domain in \( \mathbb{C}^n \), \( n \geq 1 \), and let \( \chi : (-\infty, 0] \to (-\infty, 0] \) be a continuous increasing function such that \( \chi(0) = 0 \) and \( \lim_{t \to -\infty} \chi(t) = -\infty \), such that \( \mathcal{E}_\chi \subset \mathcal{E} \). If \( \mu \) is a positive and finite measure in \( \Omega \), such that \( \mu(P) = 0 \), for all pluripolar sets \( P \subset \Omega \), then there exists a function \( u \in \mathcal{E}_\chi \) such that

\[
-\chi(u)(dd^c u)^n = d\mu.
\]

Furthermore, if \( \mathcal{E}_\chi \subset \mathcal{N} \), then the solution of above equation is uniquely determined.

\( \mathcal{E} \) denotes the Cegrell class of all negative plurisubharmonic functions on which the complex Monge-Ampère operator \((dd^c \cdot)^n\) is well defined and \( \mathcal{N} \subset \mathcal{E} \) denotes the Cegrell class for which the smallest maximal plurisubharmonic majorant is identically equal to 0. It follows from [12], [13], [14] that \( \mathcal{E} \subset \mathcal{N} \).

It was proved in [8] that if \( \chi : (-\infty, 0] \to (-\infty, 0] \) is a continuous increasing, convex or concave function such that \( \chi(0) = 0 \) and \( \lim_{t \to -\infty} \chi(t) = -\infty \), then \( \mathcal{E}_\chi \subset \mathcal{E} \).

It should be noted that uniqueness part is an immediate consequence of Cegrell’s work in [14] and the fact that \( \mathcal{E}_\chi \subset \mathcal{N} \). Let \( \chi(t) = -(t^p)^p \), \( p > 0 \), in the Main Theorem, then it follows that: If \( \mu \) is a positive and finite measure in \( \Omega \), such that \( \mu(P) = 0 \), for all pluripolar sets \( P \subset \Omega \), then there exists a unique function \( u \in \mathcal{E}_p \) such that

\[
(-u)^p(dd^c u)^n = d\mu.
\]

If \( \chi : (-\infty, 0] \to (-\infty, 0] \) is a continuous function such that \( \chi(0) < 0 \) and \( \lim_{t \to -\infty} \chi(t) = -\infty \), then the existence of solution to the Monge-Ampère type equation given by

\[
-\chi(u)(dd^c u)^n = d\mu
\]

is a consequence of [15] with the assumption that \( -\chi(t)^{-1} \) is bounded.

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2. **Proof of the Main Theorem**

**Lemma 2.1.** Let \( \Omega \) be a bounded hyperconvex domain in \( \mathbb{C}^n \). If a sequence \( u_j \in \mathcal{F} \) satisfies the condition

\[
\sup_j \int_{\Omega} (dd^c u_j)^n < +\infty,
\]

and if there exists \( u \in \text{PSH}(\Omega) \) such that \( u_j \to u \) weakly, then \( u \in \mathcal{F} \).

**Proof.** From [13] there exists \( w_j \in \mathcal{E}_0 \cap \mathcal{C}(\Omega) \) such that \( w_j \downarrow u, j \to \infty \). Note that since \( u_j \to u \) weakly, then \( u = \lim_{j \to \infty} v_j \), where

\[
v_j = \left( \sup_{k \geq j} u_k \right)^*.
\]
Observe that $v_j$ is a decreasing sequence, $v_j \geq u_j$, so $v_j \in \mathcal{F}$ and from [13] we have
\[
\int_{\Omega} (dd^c v_j)^n \leq \int_{\Omega} (dd^c u_j)^n.
\]
Define
\[
\varphi_j = \max(w_j, v_j).
\]
Then $\varphi_j \in \mathcal{E}_0$, $\varphi_j$ is a decreasing sequence $v_j \downarrow u_j$ and again from [13] we get
\[
\sup_j \int_{\Omega} (dd^c \varphi_j)^n \leq \sup_j \int_{\Omega} (dd^c v_j)^n \leq \sup_j \int_{\Omega} (dd^c u_j)^n < +\infty,
\]
which means that $u \in \mathcal{F}$.

First we prove our Main Theorem in the case of compactly supported measures.

**Lemma 2.2.** Let $\Omega$ be a bounded hyperconvex domain in $\mathbb{C}^n$, and let $\chi : (-\infty, 0] \rightarrow (-\infty, 0]$ be a continuous increasing function such that $\chi(0) = 0$ and $\lim_{t \to -\infty} \chi(t) = -\infty$. If $\mu$ is a positive, finite, and compactly supported measure in $\Omega$, such that $\mu(P) = 0$ for all pluripolar sets $P \subset \Omega$, then there exists a unique function $u \in \mathcal{F} \cap \mathcal{E}_\chi$ such that
\[
-\chi(u)(dd^c u)^n = d\mu.
\]  

**Proof.** If $\mu \equiv 0$, then it is clear that $u = 0$ is a solution of (2.1). Assume now that $\mu \not\equiv 0$. For $k \in \mathbb{N}$ consider the following equations
\[
(dd^c u_k)^n = \min \left( \frac{-1}{\chi(u_k)}, k \right) \ d\mu.
\]  

The function defined by
\[
F_k(t) = \min \left( \frac{-1}{\chi(t)}, k \right)
\]
is bounded and continuous. Therefore it follows from [13] that there exists $u_k \in \mathcal{F}$, which satisfies (2.2). We also have that
\[
(dd^c u_k)^n = \min \left( \frac{-1}{\chi(u_k)}, k \right) \ d\mu \leq \frac{-1}{\chi(u_k)} \ d\mu,
\]
so
\[
\sup_k \int_{\Omega} (-\chi(u_k))(dd^c u_k)^n \leq \mu(\Omega) < +\infty.
\]
We shall next prove that there exist $\alpha \in \mathcal{E}_0, \beta \in \mathcal{F}$ such that
\[
\beta \leq u_k \leq \alpha
\]  
almost everywhere $d\mu$, for $k \geq 2$.

By Cegrell decomposition theorem (see [13]) there exist $\phi \in \mathcal{E}_0$ and $f \in L^1((dd^c \phi)^n)$, $f \geq 0$ such that
\[
\mu = f(dd^c \phi)^n.
\]
Fix $a > 0$ such that $\chi(-a) \geq -\frac{1}{2}$. Then there exists $\alpha \in \mathcal{E}_0$ such that (see [16])
\[
(dd^c \alpha)^n = \min \left( f, \frac{a^n}{\|\phi\|^n} \right) (dd^c \phi)^n,
\]
where $\|\phi\| = \sup_{z \in \Omega} |\phi(z)|$. The comparison principle (see [7]) yields that
\[
\alpha \geq \frac{a}{\|\phi\|} \phi \geq -a
\]
and 
\[
\int_{\{\alpha < u_k\}} (dd^c u_k)^n \leq \int_{\{\alpha < u_k\}} (dd^c \alpha)^n \leq \int_{\{\alpha < u_k\}} d\mu.
\]
Observe that on the set \(\{\alpha < u_k\}\) we have \(u_k > -a\) and 
\[
(dd^c u_k)^n = \min \left( -\frac{1}{\chi(u_k)}, k \right) d\mu \geq \min \left( -\frac{1}{\chi(-a)}, k \right) d\mu \geq 2 d\mu,
\]
for \(k \geq 2\), which implies that \(\mu(\{\alpha < u_k\}) = 0\), \(k \geq 2\).

There exists \(\psi \in F\) such that \((dd^c \psi)^n = d\mu\) (see \([13]\)). Fix \(w \in \mathcal{E}_0\) and \(b > 0\) such that 
\[
\chi(\text{sup} \supp \mu \psi + bw) < -2.
\]
Let \(\beta = \psi + bw\). Note that 
\[
(dd^c \beta)^n \geq d\mu.
\]
By the comparison principle (see \([7]\)) we obtain 
\[
\int_{\{u_k < \beta\}} d\mu \leq \int_{\{u_k < \beta\}} (dd^c \beta)^n \leq \int_{\{u_k < \beta\}} (dd^c u_k)^n,
\]
but on the set \(\{u_k < \beta\} \cap \supp \mu\) we have \(u_k < \beta \leq \sup_{\supp \mu} \beta\) and 
\[
(dd^c u_k)^n = \min \left( -\frac{1}{\chi(u_k)}, k \right) d\mu \leq \frac{1}{2} d\mu,
\]
which means that \(\mu(\{u_k < \beta\}) = 0\), for all \(k\).

Now it follows from \([23]\) that there exists a plurisubharmonic function \(u \neq 0\) and a subsequence (also denoted by \(u_k\)) such that \(u_k \to u\) almost everywhere \([d\mu]\). Since \(u \neq 0\) then 
\[
-\frac{1}{\chi(\text{sup} \supp \mu u)} < +\infty.
\]
By Hartog’s lemma, functions 
\[
F_k(u_k) = \min(-\chi(u_k)^{-1}, k)
\]
are uniformly bounded on \(\supp \mu\) and therefore 
\[
\sup_k \int_{\Omega} (dd^c u_k)^n \leq \sup_k \int_{\Omega} F_k(u_k) d\mu < +\infty.
\]
Lemma \([2.1]\) yields that \(u \in F\).

The stability theorem proved in \([15]\) implies that the weak convergence, \(u_k \to u\), is equivalent to convergence in capacity. Using Xing’s theorem in \([19]\) we get that \((dd^c u_k)^n \to (dd^c u)^n\) weakly. Therefore using the dominated convergence theorem we get that 
\[
(dd^c u)^n = \lim_{k \to +\infty} (dd^c u_k)^n = \lim_{k \to +\infty} F_k(u_k) d\mu = \frac{-1}{\chi(u)} d\mu.
\]
So we have proved that there exists a solution \(u \in F\) to \((2.1)\). Then we have that 
\[
\int_{\Omega} (\chi(u))(dd^c u)^n < +\infty
\]
and therefore it follows from \([9]\) that \(u \in \mathcal{E}_\chi\).

It will be proved in the proof of the Main Theorem that if \(u, v \in F\) are solutions of \((2.1)\) then \((dd^c u)^n = (dd^c v)^n\) and therefore \(u = v\) (see \([13]\)). \(\square\)
Proof of the Main Theorem. Assume that \( \mu \) is a positive and finite measure in \( \Omega \) such that \( \mu(P) = 0 \) for all pluripolar sets \( P \subset \Omega \). Let \( \Omega_j \) be a fundamental sequence of strictly pseudoconvex domains, i.e., \( \Omega_j \subset \Omega_{j+1} \subset \Omega \) and \( \bigcup_{j=1}^{\infty} \Omega_j = \Omega \) (see [14]). Let us define \( d\mu_j = 1_{\Omega_j}d\mu \), where \( 1_{\Omega_j} \) is a characteristic function for \( \Omega_j \). By Lemma 2.2 there exists a sequence \( u_j \in F \cap \mathcal{E}_\chi \) such that

\[
-\chi(u_j)(dd^c u_j)^n = d\mu_j.
\]

We now shall prove that \( u_j \) is a decreasing sequence. Let \( A = \{ z \in \Omega : u_j(z) < u_{j+1}(z) \} \). On the set \( A \), we have that

\[
(dd^c u_j)^n = -\chi(u_j)^{-1}d\mu_j \leq -\chi(u_{j+1})^{-1}d\mu_j \leq -\chi(u_{j+1}^{-1})d\mu_{j+1} = (dd^c u_{j+1})^n
\]

and by the comparison principle (see [7]) we get that

\[
\int_A (dd^c u_{j+1})^n \leq \int_A (dd^c u_j)^n.
\]

Hence,

\[
(dd^c u_j)^n = (dd^c u_{j+1})^n \tag{2.4}
\]

on \( A \). Similarly on the set \( B = \Omega_j \setminus A = \{ z \in \Omega_j : u_j(z) \geq u_{j+1}(z) \} \) we obtain that

\[
(dd^c u_j)^n = -\chi(u_j)^{-1}d\mu_j \geq -\chi(u_{j+1})^{-1}d\mu_j = -\chi(u_{j+1}^{-1})d\mu_{j+1} = (dd^c u_{j+1})^n.
\]

From the equalities (2.4) and (2.5) we get that \( (dd^c u_j)^n \geq (dd^c u_{j+1})^n \) on \( \Omega_j \). This implies that \( -\chi(u_j)^{-1}d\mu_j \geq -\chi(u_{j+1})^{-1}d\mu_j \) and then \( \chi(u_j) \geq \chi(u_{j+1}) \) a.e. \([d\mu_j]\), so \( u_j \geq u_{j+1} \) a.e. \([d\mu_j]\). Hence \( \mu_j(\{ u_j < u_{j+1} \}) = 0 \) and \( (dd^c u_j)^n = 0 \) on \( A \cap \Omega_j \). Since \( (dd^c u_j)^n = d\mu_j = 0 \) on \( \Omega \setminus \Omega_j \) we finally obtain that \( (dd^c u_j)^n = 0 \) on \( A = \{ u_j < u_{j+1} \} \).

Now take \( \psi \in \mathcal{E}_\chi \) such that \((dd^c \psi)^n = d\lambda\), where \( d\lambda \) is the Lebesgue measure, and consider \( A_k = \{ z \in \Omega : u_j < u_{j+1} + \frac{1}{k}\psi \} \). Observe that \( u_j + \frac{1}{k}\psi \in F \) and \( A_k \subset A \). By the comparison principle (see [7]) we obtain that

\[
\int_{A_k} (dd^c(u_{j+1} + \frac{1}{k}\psi))^n \leq \int_A (dd^c u_j)^n \leq \int_A (dd^c u_j)^n = 0,
\]

and then

\[
0 = \int_{A_k} (dd^c(u_{j+1} + \frac{1}{k}\psi))^n \geq \frac{1}{k} \int_{A_k} (dd^c \psi)^n = \frac{1}{k} \chi(A_k),
\]

which means that \( \lambda(A_k) = 0 \). Hence \( \lambda(A) = 0 \), since \( A = \bigcup_{k=1}^{\infty} A_k \). We have proved that \( u_j \geq u_{j+1} \) a.e. \([d\lambda]\), but since functions \( u_j, u_{j+1} \) are plurisubharmonic we obtain that \( u_j \geq u_{j+1} \) on \( \Omega \), so \( u_j \) is a decreasing sequence.

Note also that

\[
\sup_j \int_\Omega -\chi(u_j)(dd^c u_j)^n \leq \int_\Omega d\mu < +\infty.
\]

Moreover from [9] we obtain

\[
\int_\Omega -\chi(u_j)(dd^c u_j)^n = \int_0^{+\infty} \chi'(t)(dd^c u_j)^n(\{ u_j < -t \})dt \geq \int_0^{+\infty} \chi'(t)t^n C_n(\{ u_j < -2t \})dt,
\]

where \( C_n \) is Bedford-Taylor capacity (see [7]). Therefore

\[
\sup_j \int_0^{+\infty} \chi'(t)t^n C_n(\{ u_j < -t \})dt \leq \sup_j \int_\Omega -\chi(u_j)(dd^c u_j)^n \leq \int_\Omega d\mu < +\infty,
\]
which implies that there exists \( \lim_{j \to -\infty} u_j = u \in \mathcal{E}_\chi \), and \( (dd^cu_j)^n \) tends weakly to \( (dd^cu)^n \). Therefore using the monotone convergence theorem we get that

\[
(dd^cu)^n = \lim_{j \to -\infty} (dd^cu_j)^n = \lim_{j \to -\infty} -\chi(u_j)^{-1}1_{\Omega} d\mu = -\chi(u)^{-1}d\mu.
\]

This ends the proof of the existence part of this theorem.

Now we will proceed with the uniqueness part. Assume that \( u, v \in \mathcal{E}_\chi \) such that \(-\chi(u)(dd^cu)^n = -\chi(v)(dd^cv)^n = d\mu\). Observe that on the set \( \{ z \in \Omega : u(z) < v(z) \} \) we have

\[
(dd^cu)^n = -\chi(u)^{-1}d\mu \leq -\chi(v)^{-1}d\mu = (dd^cv)^n.
\]

Using the comparison principle (see [7]) we obtain

\[
\int_{\{u<v\}} (dd^cv)^n \leq \int_{\{u<v\}} (dd^cu)^n,
\]

so \( (dd^cu)^n = (dd^cv)^n \) on \( \{ z \in \Omega : u(z) < v(z) \} \). Similarly, we get that \( (dd^cu)^n = (dd^cv)^n \) on the set \( \{ z \in \Omega : u(z) > v(z) \} \). Since \( \mu \) does not put mass on pluripolar sets and \( \{ u = -\infty \} = \{ \chi(u) = -\infty \} \) and \( \{ v = -\infty \} = \{ \chi(v) = -\infty \} \), then

\[
(dd^cu)^n = (dd^cv)^n = 0 \quad \text{on the set} \quad C = \{ u = -\infty \} \cup \{ v = -\infty \}.
\]

On the set \( \{ u = v \} \setminus C \) we also have

\[
(dd^cu)^n = -\chi(u)^{-1}d\mu = -\chi(v)^{-1}d\mu = (dd^cv)^n.
\]

Thus, \( (dd^cu)^n = (dd^cv)^n \) on \( \Omega \), which implies that \( u = v \) (see [14]). \( \square \)

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