COMBINATORIAL GAP THEOREM
AND REDUCTIONS BETWEEN PROMISE CSPS

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Abstract. A value of a CSP instance is typically defined as a fraction of constraints that can be simultaneously met. We propose an alternative definition of a value of an instance and show that, for purely combinatorial reasons, a value of an unsolvable instance is bounded away from one; we call this fact a gap theorem.

We show that the gap theorem implies NP-hardness of a gap version of the Layered Label Cover Problem. The same result can be derived from the PCP Theorem, but a full, self-contained proof of our reduction is quite short and the result can still provide PCP-free NP-hardness proofs for numerous problems. The simplicity of our reasoning also suggests that weaker versions of Unique-Games-type conjectures, e.g., the $d$-to-$1$ conjecture, might be accessible and serve as an intermediate step for proving these conjectures in their full strength.

As the second, main application we provide a sufficient condition under which a fixed template Promise Constraint Satisfaction Problem (PCSP) reduces to another PCSP. The correctness of the reduction hinges on the gap theorem, but the reduction itself is very simple. As a consequence, we obtain that every CSP can be canonically reduced to most of the known NP-hard PCSPs, such as the approximate hypergraph coloring problem.

1. Introduction

Paul would like to find an assignment from $V$ to $A$ – an element of $A^V$ – that simultaneously satisfies a collection of local constraints. Each constraint demands that the restriction of the assignment onto a subset $W \subseteq V$ of size at most $m$ is in a prescribed subset of $A^W$. We call $V$ and $A$ together with such a collection of local constraints an instance of $m$-CSP over $A$ and denote it by $\Phi$; Paul is looking for a solution to $\Phi$.

Paul asks Carole to provide, for some specified $k \geq m$, a collection of partial assignments for $\Phi$: functions $I(U) \in A^U$, where $U$ runs through all $k$-element subsets of $V$ (we write $U \in \binom{V}{k}$), such that

1. each function $I(U)$ is a partial solution to $\Phi$, i.e., it satisfies every constraint defined on $W \subseteq U$, and
2. the partial solutions are consistent, i.e., for any $U_1$ and $U_2$, $\text{proj}_{U_1 \cap U_2} I(U_1) = \text{proj}_{U_1 \cap U_2} I(U_2)$ (where $\text{proj}_{U'} I(U)$ denotes the restriction of $I(U)$ to $U'$).

If Carole provides such a collection, must a solution exist? Can Paul find a solution given Carole’s answer?

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The answer to both questions is, trivially, “Yes”. Indeed, the consistency requirement ensures that all the $I(U)$ are restrictions of a single function $f : V \to A$ and $f$ satisfies all the local constraints since the $I(U)$ are partial solutions and $k \geq m$.

Let us fix $A$ and natural numbers $m, d, k_0 > k_1$, and make Carol’s task easier; she provides two collections $I_0, I_1$ such that

1. for every $U \in \binom{V}{k}$, the set $I_i(U) \subseteq A^U$ consists of partial solutions to $\Phi$,
2. every $I_i(U)$ has no more than $d$-elements, and
3. if $U_0 \supseteq U_1$ (of sizes $k_0, k_1$), then some elements of $I_0(U_0)$ and $I_1(U_1)$ are consistent, i.e., there is $f \in I_0(U_0)$ such that $\text{proj}_{U_1} f \in I_1(U_1)$.

If Carole provides such collections, must a solution exist? Can Paul find a solution given Carole’s answer?

Our main theorem provides a positive answer to these questions for each $A$, $m$, $d$ and suitable chosen $k_0$ and $k_1$. The property can be concisely stated in terms of a combinatorial measure of quality of an $m$-CSP instance defined as follows. The $(k_0, k_1)$-value of an instance $\Phi$, denoted $\text{val}_{k_0, k_1}(\Phi)$, is the smallest $d$ for which Carole can provide consistent collections, and $\infty$ if no such collections exists. The positive answer to the first question can now be stated as follows.

**Theorem 1 (Combinatorial Gap Theorem).** For every $A$, $m$ and $d$ there exists $k_0, k_1 \geq m$ such that for every instance $\Phi$ of $m$-CSP over $A$ either

- $\text{val}_{k_0, k_1}(\Phi) = 1$ (i.e., $\Phi$ is solvable) or
- $\text{val}_{k_0, k_1}(\Phi) > d$.

In fact, we prove in Corollary 1 (the Layered Combinatorial Gap Theorem) a stronger version that permits more than two collections and only requires a particularly weak form of consistency in the definition of the value of an instance. This fact is in turn a consequence of Theorem 8 (the Main Theorem) that does not require the underlying $m$-CSP instance $\Phi$ in the statement and provides an affirmative answer to the second question – Paul can compute the solution in polynomial time.

Our main application is in providing reductions between Promise Constraint Satisfaction Problems. However, let us first discuss the connection to simpler and more standard notions of value.

1.1. **Baby PCP Theorem.** The most straightforward notion of value of an instance $\Phi$ is the following [1]: the (standard) value of $\Phi$ is the largest $\varepsilon$ ($0 \leq \varepsilon \leq 1$) such that there exists a function $f : V \to A$ that satisfies $\varepsilon$ fraction of the constraints.

This is the standard measure in the area of optimization and approximation algorithms. The following theorem, which follows from the PCP Theorem [2, 3] and the Parallel Repetition Theorem [35], is then a starting point for many NP-hardness results in the area.

**Theorem 2 ([2, 3, 35]).** For every $1 \geq \varepsilon > 0$ there exists $A$ such that it is NP-hard to distinguish solvable instances of 2-CSP over $A$ from those whose value is smaller than $\varepsilon$.

As an immediate consequence of the Combinatorial Gap Theorem, we obtain a weaker version of Theorem 2, which we call the Baby PCP Theorem. Its formulation uses yet another notion of value: the combinatorial value of $\Phi$ is the smallest integer $d$ such that there exists a function $f$ from $V$ to the set of at most $d$-element subsets
of $A$ such that, for every local constraint $\varphi \subseteq A^W$, the projection of $f$ onto $W$ intersects $\varphi$.

**Theorem 3** (Baby PCP Theorem). For every integer $d \geq 1$ there exists $A$ such that it is NP-hard to distinguish solvable instances of 2-CSP over $A$ from those whose combinatorial value is greater than $d$.

Note that Theorem 3 is indeed a consequence of Theorem 2 by a probabilistic argument that goes as follows. If $f$ witnesses combinatorial value at most $d$ and we define $f'$ by choosing $f'(v)$ from $f(v)$ uniformly at random (independently for each $v$), then the probability that $f'$ satisfies a constraint is at least $1/d^2$ and so is then the expected fraction of satisfied constraints. Therefore, the trivial reduction (i.e., not changing the input) reduces the problem in Theorem 2 with $\varepsilon = 1/d^2$ to the problem in Theorem 3.

On the other, Theorem 3 is still sufficient for some NP-hardness results (such as many known NP-hard PCSPs to be discussed in later sections). Our proof of Theorem 3 is based on a very simple reduction from any NP-hard $m$-CSP and a proof of its correctness follows easily from the Main Theorem which is itself not excessively complex. Most importantly the result suggests that weaker, combinatorial versions of some refinements of Theorem 2 might be more accessible. We refer to Section 6 for further discussion.

### 1.2. Reductions Between CSPs

In order to state our main application of the Main Theorem (Theorem 8) we first give some background on the fixed template CSP (in this subsection) and fixed template Promise CSPs (in Section 1.3). Our contributions are then discussed in Section 1.4. The statements of theorems are informal in that we omit some obvious assumptions and we postpone some definitions to later sections.

The fixed template finite domain CSP is a framework for expressing many computational problems such as various versions of logical satisfiability, graph coloring, and systems of equations. A **template** can be specified as a relational structure $A = (A; R_1, \ldots, R_l)$, where $A$ is a finite set called the domain and each $R_i$ is a relation of some arity $\ar_i$, i.e., a subset of $A^{\ar_i}$. The **CSP over $A$**, denoted CSP($A$), is (in its search version) the problem of finding an assignment $V \rightarrow A$ that satisfies given local constraints as above, with the restriction that each constraint is of the form $\{ f \in A^W : (f(w_1), f(w_2), \ldots, f(w_{\ar_i})) \in R_i \}$, where $1 \leq i \leq l$ and $W = \{w_1, \ldots, w_{\ar_i}\} \subseteq V$. In the decision version of CSP($A$) we only want to decide whether such an assignment exists. Our results work for both versions and we do not carefully distinguish between them in the introduction.

Note that for $A$ consisting of all relations on $A$ of arity $m$, the CSP over $A$ is exactly the $m$-CSP over $A$. By choosing appropriate structures with a two-element domain we obtain various versions of satisfiability, such as $k$-SAT, HORN-$k$-SAT, NAE-$k$-SAT, etc. Important class of examples on larger domains is the CSP over $K_n$, the set $[n] = \{1, \ldots, n\}$ together with the disequality relation, which is (essentially) the $n$-coloring problem for graphs. More generally, the CSP over $\text{NAE}_n^k$, the set $[n]$ together with the $k$-ary not-all-equal relation, is the $n$-coloring problem for $k$-uniform hypergraphs. We refer to surveys in [33] for more details and examples, as well as many variants of the fixed template CSP framework.

In [22], Feder and Vardi conjectured that each CSP($A$) is either solvable in polynomial time or NP-complete. Their conjecture inspired a very active research
A fundamental theorem, which initiated a rapid development of the subject, crystallized in the series of papers by Jeavons et al., e.g. [28, 27]. It gives a sufficient condition for the existence of a polynomial-time reduction between two CSPs in terms of multivariate functions that preserve relations of the templates, called polymorphisms (see Section 4). Denoting Pol(A) the set of all polymorphisms of A, the theorem can be stated as follows.

**Theorem 4 ([27]).** If Pol(A₁) ⊆ Pol(A₂), then CSP(A₂) is reducible to CSP(A₁).

This theorem was later made more applicable in [16] and then in [10] by replacing the inclusion by weaker requirements, thus providing more reductions. A modern formulation is in terms of minion homomorphisms (see Section 4) as follows.

**Theorem 5.** If Pol(A₁) has a minion homomorphism to Pol(A₂), then CSP(A₂) is reducible to CSP(A₁).

This theorem has a quite simple proof but it is surprisingly powerful: it follows from Bulatov’s and Zhuk’s complexity classification [17, 40] that for any CSP(A) either

- CSP(A) is solvable in polynomial time
- for every B, Theorem 5 provides a reduction from CSP(B) to CSP(A).

In other words, NP-hard CSPs form the largest equivalence class of the preorder given by the reducibility implied by minion homomorphisms. In this sense, Theorem 5 provides a single source of hardness and, in fact, the proof of this theorem gives a simple reduction from any CSP to any NP-hard CSP (assuming P ≠ NP). The theorem is interesting (but substantially weaker) on the algorithmic side as well; for instance, it gives a reduction of any width 1 CSP [22] to HORN-3-SAT (see [18, 8]).

### 1.3. Reductions between PCSPs.

The fixed template Promise CSP (PCSP) is a recently introduced generalization of the fixed template CSP, motivated by open problems about (in)approximability of SAT and graph coloring [5, 12, 13]. The idea is that each constraint has a strict version and a relaxed version and the problem is (in the search version) to find an assignment satisfying the relaxed constraints given an instance which is satisfiable under the strict constraints (this is the promise). More precisely, the template for PCSP is a pair of similar structures (A, B), where A specifies the allowed forms of strict constraints and B their relaxations (we again refer to Section 4 for precise definitions). Note that PCSP(A, A) is the same problem as CSP(A).

Important examples of PCSPs include graph coloring and hypergraph coloring problems, such as PCSP(Kₙ, Kₘ), m ≥ n – the problem to find an m-coloring of a given n-colorable graph, and approximate versions of satisfiability problems such as the (2+ε)-SAT problem from [5]. We refer to [11, 13, 14, 18, 8] for more examples.

The complexity classification of fixed template PCSPs beyond CSPs is largely unknown; indeed, even the complexity of PCSP(Kₙ, Kₘ) is a long-standing open problem [24] and it is known only for some choices of parameters n, m (see [34] for a recent account). However, an analogue of Theorem 4 [12] and even Theorem 5 [18, 8] is available with a natural generalization of polymorphisms. Denoting Pol(A, B)
the set of all polymorphisms of a template \((A, B)\), the latter theorem can be formulated as follows.

**Theorem 6** ([18, 8]). If \(\text{Pol}(A_1, B_1)\) has a minion homomorphism to \(\text{Pol}(A_2, B_2)\), then \(\text{PCSP}(A_2, B_2)\) is reducible to \(\text{PCSP}(A_1, B_1)\).

This theorem is still very useful in the more general PCSP setting. For instance, it gives a reduction of any CSP to \(\text{PCSP}(K_3, K_4)\) (as essentially shown in [11], cf. [18]) and gives a reduction from \(\text{PCSP}(\text{NAE}^3_2, \text{NAE}^3_n)\) (i.e., \(n\)-coloring a 2-colorable 3-uniform hypergraph), for a certain \(n\), to \(\text{PCSP}(K_3, K_5)\) as shown in [18, 8]. From NP-hardness of the former problem [21] one obtains NP-hardness of the latter problem. The algorithmic side of this theorem is discussed in Section 7 of [8].

However, **Theorem 6** is very much insufficient for proving NP-hardness of every NP-hard PCSP, e.g., one provably cannot apply it to reduce an NP-hard CSP (such as 3-SAT) to \(\text{PCSP}(\text{NAE}^3_2, \text{NAE}^3_n)\).

More widely applicable sufficient conditions for NP-hardness in terms of polymorphisms have been developed in [8, 15, 25], or follow from the results in these papers. They are all based on **Theorem 2** and its refinements. These conditions cover almost all known NP-complete PCSPs, a notable exception being [26].

On the other hand, these sufficient conditions are not quite satisfactory for two reasons. First, they are not based on a general reduction theorem such as **Theorem 6**, which limits their applicability and appeal. Second, they use complex NP-hardness results (**Theorem 2** and refinements), which, e.g., makes it difficult to reduce a standard NP-complete problem, like 3-SAT, to many NP-hard PCSPs. For instance, if we want to reduce 3-SAT to \(\text{PCSP}(K_3, K_5)\) using available theory, we first need to perform a sequence of reductions used in a proof of the PCP theorem, then another reduction for the Parallel Repetition Theorem, ending up in the situation of **Theorem 2**, then further reductions for an improved version of **Theorem 2** from [30], followed by reductions to approximate hypergraph coloring from [21], finally finishing with reductions provided by **Theorem 6** to \(\text{PCSP}(K_3, K_5)\) [18]. Such a long chain of reductions obscures the reasons why the problem is hard.

### 1.4. New reductions between PCSPs.

We define a concept of a minion \((d, r)\)-homomorphism (**Definition 3**) that weakens minion homomorphisms in the following sense: for \(d = 1, r = 1\) the concepts coincide, and increasing \(d\) or \(r\) makes the concept weaker. We then apply the Main Theorem (**Theorem 8**) to show that a generalization of **Theorem 6** remains true with this weaker concept, thus giving us more reductions between PCSPs.

**Theorem 7.** If \(\text{Pol}(A_1, B_1)\) has a minion \((d, r)\)-homomorphism to \(\text{Pol}(A_2, B_2)\), then \(\text{PCSP}(A_2, B_2)\) is reducible to \(\text{PCSP}(A_1, B_1)\).

This theorem partially resolves the shortcomings of the state-of-the-art discussed above. In particular, the theorem gives a reduction of any NP-hard CSP to many known NP-hard PCSPs, including

- all NP-hard Boolean symmetric PCSPs, which were classified in [23] (e.g., the \((2 + \varepsilon)\)-SAT from [5] and, more generally, NP-hard symmetric folded PCSPs classified in [13]),
- those NP-hard approximate coloring problems, i.e., PCSPs of the from \(\text{PCSP}(K_n, K_m)\), identified in [18] (e.g., \(\text{PCSP}(K_3, K_5)\)).
all approximate 3-uniform hypergraph coloring problems, i.e., PCSPs over $(\text{NAE}_n^3, \text{NAE}_m^3)$ [21] (for this we need an improvement of the proof by Wrochna [37]),

- those NP-hard PCSPs identified in [15] (promise SAT on non-Boolean domains), in [32] (graph 3-coloring with strong promises), and in [7] (variants of 3-uniform hypergraph coloring).

The examples of reductions that are not (known to be) covered include NP-hardness proofs in [25, 4, 26] and reductions that were used in [38] to improve [26]. These examples suggest directions for improving the Main Theorem and thus Theorem 7; we discuss these directions in the Conclusion.

The reduction in Theorem 7 is very simple and the proof of correctness essentially amounts to applying the Main Theorem, whose proof is itself quite short. We now explain the reduction in some detail.

We first observe that every PCSP($A$, $B$) is equivalent to a certain PCSP($A'$, $B'$), where $A'$ consists of all relations up to any fixed sufficiently large arity on any sufficiently large domain. This is a very simple consequence of [18] but still a remarkable observation: we can use any instance as an input to PCSP($A'$, $B'$) and thus effectively to PCSP($A$, $B$). In fact, the trivial reduction from CSP($A_2$, $B_2$) to CSP($A'_1$, $B'_1$) is correct in the situation of Theorem 6 (and, again, this was essentially proved in [18]). Our reduction is just the next most obvious one – in essence, we introduce a variable for every bounded arity subset of the original variables and include the obvious constraints coming from the requirement that values of variables form partial solutions.

In summary, the reduction from PCSP($A_2$, $B_2$) to PCSP($A_1$, $B_1$) is a composition of a reduction from PCSP($A_2$, $B_2$) to PCSP($A'_1$, $B'_1$) (which is the “repetition” reduction describe above) and a reduction from the latter PCSP to PCSP($A_1$, $B_1$) (which is the “polymorphism” or “long code ” reduction). We remark that all the reductions in the PCSP/PCP area, which we are aware of, are variations of these two types of reductions. Is it a coincidence?

Finally, it seems unlikely that Theorem 6 is a single source of hardness for all PCSPs in the same sense as Theorem 5 is for CSPs. However, we hope that our result will serve as a useful step toward the goal of obtaining such a theorem, which would give a uniform reduction that completely replaces (and explain) a bit ad hoc intermediate problems and reductions that are still necessary for some PCSPs. Ideally, and this seems much more challenging even for CSPs, the theorem would also fully capture the tractability part. Another exciting direction is toward the more general Promise Valued CSP framework (see [6, 36]), which includes problems such as those in Theorem 2. Remarkably, an analogue of Theorem 6 is already available by an unpublished work of Kazda [29].

2. Main Theorem

This section is devoted to introducing notation and stating the main result of the paper in the full strength. First we formalize the notion of the information that is provided by Carole. For a set of variables $V$ and domain $A$, a partial assignment system (PAS) of arity $k$ ($k$-PAS) is a map from the set of all $k$-element subsets of $V$ such that, for each $U \in \binom{V}{k}$, we have $\emptyset \neq I(U) \subseteq A^U$. An assignment $f \in A^V$ is an $m$-solution of a $k$-PAS $I$, if every $U \in \binom{V}{m}$ can be extended to $W \in \binom{V}{k}$
while satisfying \( \text{proj}_f f \in \text{proj}_f \mathcal{I}(W) \). The value of a PAS \( \mathcal{I} \) is the maximal size of \( \mathcal{I}(U) \).

Let \( (\mathcal{I}_0, \ldots, \mathcal{I}_r) \) be partial assignment systems over common \( V \) and \( A \). We call such a sequence consistent if

- their arities \( k_0, \ldots, k_r \) form a non-increasing sequence, and
- for every \( U_0 \supseteq \cdots \supseteq U_r \) (of sizes \( k_0, \ldots, k_r \)) there exists \( i < j \) such that \( \mathcal{I}_j(U_j) \cap \text{proj}_{U_j} \mathcal{I}_i(U_i) \neq \emptyset \).

The value of such a sequence is the maximal among values of \( \mathcal{I}_i \)'s.

**Theorem 8 (Main Theorem).** For any \( A \) and any numbers \( m, r, d \in \mathbb{N} \) there exists a sequence \( k_0, \ldots, k_r \) such that if \( (\mathcal{I}_0, \ldots, \mathcal{I}_r) \) is a consistent sequence of arities \( k_0, \ldots, k_r \) and value \( \leq d \), then some \( \mathcal{I}_i \) has an \( m \)-solution. Additionally, for fixed \( m, r, d, A \), an \( m \)-solution can be computed in polynomial time.\(^1\)

A proof of this theorem is provided in Section 7.

Let us revisit the Paul/Carole interaction. Given an instance \( \Phi \), Carole is providing two consistent PASes containing local solutions to \( \Phi \). Clearly Paul can make Carole’s task easier, by asking for longer sequences. The Combinatorial Gap Theorem (Theorem 1) can be generalized to accommodate such extensions.

For \( \Phi \), an \( m \)-CSP instance over \( V \) and \( A \), and \( k_0 \geq k_1 \geq \cdots \geq k_r \geq m \) we can put the value \( \text{val}_{k_0, \ldots, k_r}(\Phi) \) to be the smallest value of a consistent sequence \( (\mathcal{I}_0, \ldots, \mathcal{I}_r) \) (over \( V \) and \( A \)) of arities \( k_0, \ldots, k_r \) such that every element of \( \mathcal{I}_i(U) \) is a partial solution to \( \Phi \). The following strengthening of the Combinatorial Gap Theorem follows immediately from Theorem 8.

**Corollary 1 (Layered Combinatorial Gap Theorem).** For every \( A \) and numbers \( m, r, d \in \mathbb{N} \) there exists \( k_0 \geq \cdots \geq k_r \geq m \) such that for every instance \( \Phi \) of \( m \)-CSP over \( A \) either

- \( \text{val}_{k_0, \ldots, k_r}(\Phi) = 1 \) (i.e. \( \Phi \) is solvable) or
- \( \text{val}_{k_0, \ldots, k_r}(\Phi) > d \).

**Proof.** Let \( k_0, \ldots, k_r \) be the numbers provided by Theorem 8 for \( A \) and \( m, r, d \). Let \( \Phi \) be an instance such that \( \text{val}_{k_0, \ldots, k_r}(\Phi) \leq d \) and let a sequence \( \mathcal{I}_0, \ldots, \mathcal{I}_r \) provides this value. By Theorem 8 there exists an \( m \)-solution to a \( k_i \)-PAS \( \mathcal{I}_i \) for some \( i \). Since \( \mathcal{I}_i \) consists of partial solutions to \( \Phi \) and \( \Phi \) is an \( m \)-CSP instance, the \( m \)-solution to \( \mathcal{I}_i \) is in fact a solution to \( \Phi \). Thus \( \Phi \) is solvable and \( \text{val}_{k_0, \ldots, k_r}(\Phi) = 1 \). \( \square \)

### 3. Baby Layered PCP Theorem

In this section we formulate an improvement of Theorem 2 that was essentially proved in [20] and adapted to this form in [15]. Then we show that a weaker, combinatorial version of this theorem (which is a stronger version of the Baby PCP Theorem from the introduction) is a straightforward consequence of the Layered Combinatorial Gap Theorem.

For convenience we define Layered Label Cover in a somewhat less standard way in that we allow different domains of variables. The difference is inessential.

An \( r \)-Layered Label Cover instance consists of

- a set \( X \) of variables, which is a disjoint union of sets \( X_0, \ldots, X_r \) (called layers),

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\(^1\)Our procedure is very much non-polynomial with respect to parameters \(|A|, m, r\) or \(d\).
• a set $A_x$ for each $x \in X$, called the domain of $x$,
• a set of constraints of the form $((x, y), \psi)$, where $x \in X_i$ and $y \in X_j$ for some $i < j$, and $\psi$ is a map $A_x \rightarrow A_y$. We refer to such a constraint as a constraint from $x$ to $y$ and require that there is at most one constraint from $x$ to $y$ for any pair of variables $x$, $y$.

An assignment for such an instance is a mapping $f$ with domain $X$ such that $f(x) \in A_x$ for every $x \in X$. It satisfies a constraint $((x, y), \psi)$ if $\psi(f(x)) = f(y)$. A chain is a sequence of variables $(x_0, \ldots, x_r)$, $x_i \in X_i$, such that there is a constraint from $x_i$ to $x_j$ for each $i < j$. It is weakly satisfied by an assignment $f$ if $f$ satisfies at least one of the constraints from $x_i$ to $x_j$, $i < j$. Finally, the layered value of an instance is the largest $\varepsilon$ such that there exists an assignment that weakly satisfies at least $\varepsilon$ fraction of all chains.

**Theorem 9** ([15], Layered PCP Theorem). For every $r \in \mathbb{N}$ and $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that, in the set of instances of $r$-Layered Label Cover with domain sizes at most $N$, it is NP-hard to distinguish solvable ones from those whose layered value is smaller than $\varepsilon$.

A combinatorial adaption of layered value goes as follows. For each variable we allow $d$-choices of values; formally a $d$-assignment for an $r$-Layered Label Cover instance is a mapping with domain $X$ such that each $f(x)$ is a subset of $A_x$ of size at most $d$. Then we generalize the notion of weak satisfiability in the most natural way: a chain $(x_0, \ldots, x_r)$ is weakly satisfied by a $d$-assignment $f$ if for some $i < j$ the constraint $((x_i, x_j), \psi)$ is such that $\psi(f(x_i)) \cap f(x_j) \neq \emptyset$. Finally, the combinatorial layered value of an instance is the smallest $d$ such that there exists a $d$-assignment that weakly satisfies all the chains.

**Theorem 10** (Baby Layered PCP Theorem). For every $r \in \mathbb{N}$ and $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that, in the set of instances of $r$-Layered Label Cover with domain sizes at most $N$, it is NP-hard to distinguish solvable ones from those whose combinatorial layered value is greater than $d$.

**Proof.** We fix $r$, $d$ and reduce from the $m$-CSP over $A$ for any fixed $m$ and $A$, which is enough since, e.g., 3-CSP over $\{0, 1\}$ is NP-hard.

Let $k_0, \ldots, k_r$ be the numbers provided by Corollary 1 and let $\Phi$ be an instance of $m$-CSP over $A$ with a set of variables $V$. We define an instance $\Psi$ of $r$-Layered Label Cover as follows. The $i$-th layer variable set is defined as $X_i = \binom{V}{k_i}$ and the domain $A_{U_i}$ of $U \in X_i$ is defined as the set of all partial solutions $U \rightarrow A$ of $\Phi$ (i.e., a variable $U \in X_i$ is a $k_i$-element set of the original variables and its domain is a subset of $A^U$). For each $U \supseteq W$ we include a constraint $((U, W), \psi)$ with $\psi : A_U \rightarrow A_W$ defined by $\psi(g) = \text{proj}_W(g)$. Notice that the definition of $\psi$ makes sense since a restriction of a partial solution to $\Phi$ is a partial solution to $\Psi$. This finishes the construction.

Soundness of this reduction is immediate: if $h : V \rightarrow A$ is a solution to $\Phi$, then $f(U) = \text{proj}_W h$ defines a solution to $\Psi$. To prove completeness, assume that $f$ is a $d$-assignment for $\Psi$ that weakly satisfies all the chains. For $0 \leq i \leq r$ and $U \in X_i$ define $I_i(U) = f(U)$ and note that, by construction of $\Psi$, $I_i$ is a $k_i$-PAS for $\Phi$ and, since $f$ is a $d$-assignment that weakly satisfies chains, the sequence $(I_0, \ldots, I_r)$ is consistent. Therefore $\text{val}_{k_0, \ldots, k_r}(\Phi) \leq d$ and the Layered Combinatorial Gap Theorem (Corollary 1) finishes the proof by showing that $\Phi$ is solvable. $\square$
4. Promise Constraint Satisfaction Problems

In this section we formally define fixed template PCSPs, their polymorphism minions, and minion homomorphisms – the concepts that are necessary to fully understand the statement of Theorem 6.

We start by defining homomorphisms between relational structures. We will only work with finite relational structures of finite signature, therefore we can use the formalism from the introduction, that is, a relational structure $A = (A; R_1, \ldots, R_l)$, where $A$ is a finite domain and $R_i \subseteq A^{ar_i}$ is a nonempty relation of arity $ar_i$. Two structures are similar if they have the same number of relations and corresponding relations have the same arity. For two similar structures $A = (A; R_1, \ldots, R_l)$ and $B = (B; S_1, \ldots, S_l)$, a homomorphism for $A$ to $B$ is a map $h : A \rightarrow B$ that preserves the relations, i.e., for any $i$ and any tuple $a \in R_i$, the tuple $h(a)$, obtained by component-wise application of $h$, is in $S_i$.

A CSP template is a relational structure. The CSP over $A$ is defined by allowing only the CSP instances over $A$ such that each constraint is, in essence, one of the $R_i$. Formally, each constraint $\varphi \subseteq A^W$ is equivalent to $\{ f \in A^W : (f(w_1), f(w_2), \ldots, f(w_{ar_j})) \in R_j \}$, where $1 \leq j \leq l$ and $W = \{w_1, \ldots, w_{ar_j}\}$. For notation’s sake, we identify $\varphi$ with the pair $((w_1, \ldots, w_{ar_j}), R_j)$.

4.1. Promise CSPs. A PCSP template is a pair $(A, B)$ of similar relational structures such that there exists a homomorphism from $A$ to $B$. Denoting $A = (A; R_1, \ldots, R_l)$ and $B = (B; S_1, \ldots, S_l)$, the PCSP over such a template is defined as follows.

**Promise CSP: PCSP($A, B$)**

**INSTANCE:** a set of formal constraints of the form $((w_1, \ldots, w_{ar_j}), R_j/S_j)$

**Promise:** instance with constraints $((w_1, \ldots, w_{ar_j}), R_j)$ is solvable

**Goal:** find a solution to the instance with constraints $((w_1, \ldots, w_{ar_j}), S_j)$

Given an instance $\Phi$ of PCSP($A, B$), the instance of CSP($A$) appearing in the promise is denoted $\Phi^A$ and referred to as the strict version of $\Phi$. Similarly, the instance of CSP($B$) in the goal is the relaxed version of $\Phi$, denoted $\Phi^B$.

The existence of a homomorphism $h : A \rightarrow B$ is sufficient (and necessary) to guarantee that PCSP($A, B$) makes sense: if the promise is fulfilled, i.e., $\Phi^A$ has a solution $f : V \rightarrow A$, then the goal can be reached, i.e., $\Phi^B$ has a solution, namely $hf$.

We have defined the fixed template PCSP in its search version. In the decision version of PCSP($A, B$), the task is to distinguish instances $\Phi$ solvable in $A$ (i.e., $\Phi^A$ is solvable) from those that are not even solvable in $B$. We present our reductions for the official, search version of the problem, which clearly gives us reductions for the decision version as well.

4.2. Polymorphism minions. Let $t : A^n \rightarrow B$ and $\pi : [n] \rightarrow [m]$. We say that $s : A^m \rightarrow B$ is a minor (or $\pi$-minor, if $\pi$ matters) of $t$ and write $t \xrightarrow{\pi} s$ if $s(a_1, \ldots, a_m) = t(\pi(a_1), \ldots, \pi(a_n))$ for any $(a_1, \ldots, a_m) \in A^m$.

**Definition 1** (minion). A minion $M$ on a pair of sets $(A, B)$ is a subset of $\bigcup_{i \geq 1} B^{A^i}$ such that

---

2 Note that neither the sequence $(w_1, \ldots, w_{ar_j})$ nor the relation $R_j$ needs to be uniquely determined by $\varphi_i$. On the other hand $\varphi_i$ is determined by $(w_1, \ldots, w_{ar_j})$ and $R_j$. 
Theorem 11

by this weaker concept. A proof is in

\[ \text{Theorem 6} \]

is minion homomorphism. It is obtained by replacing minion

\[ \text{homomorphism} \]

15 \[ \text{is} \]

Theorem 7

Section 8

\[ \text{homomorphisms from either side} \]

d,r

increase. We also remark that \((d,r)\)-homomorphism and that the concept of \((d,r)\)-homomorphism is defined by requiring a weak form of preservation of chains as follows.

Definition 3 \((d,r)\)-minon homomorphism.\)

Let \(\mathcal{M}, \mathcal{N}\) be two minions and \(d, r \in \mathbb{N}\). A mapping \(\xi : \mathcal{M} \to \mathcal{N}\) is called a \((d,r)\)-homomorphism if

1. it preserves arities, i.e., \(\text{arity of } \xi(t) = \text{arity of } t\) for all \(t \in \mathcal{M}\), and
2. it preserves taking minors i.e. if \(t \xrightarrow{\pi} s\) then \(\xi(t) \xrightarrow{\pi} \xi(s)\).

We are ready to formally state Theorem 6.

Theorem 11 (Theorem 3.1 \[8\]). Let \((\mathbf{A}_1, \mathbf{B}_1)\) and \((\mathbf{A}_2, \mathbf{B}_2)\) be two PCSP templates, and let \(\mathcal{M}_i = \text{Pol}(\mathbf{A}_i, \mathbf{B}_i)\) for \(i = 1, 2\). If there exists a minion homomorphism \(\xi : \mathcal{M}_1 \to \mathcal{M}_2\) then PCSP\((\mathbf{A}_2, \mathbf{B}_2)\) is log-space reducible to PCSP\((\mathbf{A}_1, \mathbf{B}_1)\).

5. New reduction for PCSPs

In this section we formally state our main application of the Layered Combinatorial Gap Theorem, Theorem 7 and mention some consequences.

A chain of minors, which is a useful notion we borrow from \[15\], is a sequence of minors \(t_0 \xrightarrow{\pi_{0,1}} t_1 \xrightarrow{\pi_{1,2}} t_2 \xrightarrow{\pi_{2,3}} \cdots \xrightarrow{\pi_{r-1,r}} t_r\). For such a sequence and \(i < j\) we denote by \(\pi_{i,j}\) the composition \(\pi_{j-1,j} \circ \cdots \circ \pi_{i+1,i}\); observe that \(t_i \xrightarrow{\pi_{i,j}} t_j\).

The new concept of \((d,r)\)-minion homomorphism is defined by requiring a weak form of preservation of chains as follows.

Definition 3 ((d,r)-minion homomorphism). Let \(\mathcal{M}, \mathcal{N}\) be two minions and \(d, r \in \mathbb{N}\). A mapping \(\xi\) from \(\mathcal{M}\) to the set of all at most \(d\)-element subsets of \(\mathcal{N}\) is called a minion \((d,r)\)-homomorphism if

1. it preserves arities, i.e., every \(g \in \xi(t)\) has the same arity as \(t\); and
2. for any chain of minors \(t_0 \xrightarrow{\pi_{0,1}} t_1 \xrightarrow{\pi_{1,2}} t_2 \xrightarrow{\pi_{2,3}} \cdots \xrightarrow{\pi_{r-1,r}} t_r\) there exists \(i < j\) and \(g \in \xi(t_i)\ h \in \xi(t_j)\) satisfying \(g \xrightarrow{\pi_{i,j}} h\).

Notice than minion \((1,1)\)-homomorphism is essentially the same as minion homomorphism and that the concept of \((d,r)\)-homomorphism gets weaker as \(d\) or \(r\) increase. We also remark that \((d,r)\)-homomorphisms can be composed with \((1,1)\)-homomorphisms from either side.\)

The following formal statement of Theorem 7 is obtained by replacing minion homomorphisms in Theorem 11 by this weaker concept. A proof is in Section 8.

\[ \text{It is, however, unclear to us whether the composition of two \((d,r)\)-homomorphisms (for some \(d, r\) is a \((d', r')\)-homomorphism.} \]
Theorem 12. Let \((A_1, B_1)\) and \((A_2, B_2)\) be two PCSP templates, and let \(M_i = Pol(A_i, B_i)\) for \(i = 1, 2\). If there is a minion \((d,r)\)-homomorphism \(\xi : M_1 \to M_2\) (for some \(d\) and \(r\)) then \(\text{PCSP}(A_2, B_2)\) is P-time reducible\(^4\) to \(\text{PCSP}(A_1, B_1)\).

The condition for NP-hardness of \(\text{PCSP}(A, B)\) stated as Corollary 4.2. in [15] is equivalent to requiring that \(M = Pol(A, B)\) has a \((d, r)\)-homomorphism to the trivial minion \(T\) consisting of all the dictators on some (any) set \(C\) of size at least 2 (a dictator is the function \((c_1, \ldots, c_n) \mapsto c_i\) for some \(i \leq n\)). Theorem 12 additionally provides a reduction from \(\text{PCSP}(A', B')\) to \(\text{PCSP}(A, B)\) for any template \((A', B')\) whose polymorphism minion has a homomorphism to \(T\), such as any NP-hard CSP.

A special situation when a \((d, r)\)-homomorphism with \(r = 1\) from \(M\) to \(T\) exists is when \(M\) does not contain a constant map and all members of \(M\) depend on at most \(d\) coordinates (the homomorphism assigns to \(t\) the dictators corresponding to the coordinates that \(t\) depends on). This special situation is already quite useful for NP-hardness results (see [8, 34]).

A new general consequence we can derive from Theorem 12 is that, roughly, the complexity of a PCSP does not depend on low arity polymorphisms. More precisely, if two polymorphism minions differ only in functions that depend on bounded number of coordinates, then the corresponding PCSPs have the same complexity.

6. Conclusion

We have shown that solutions to CSP instances can be reconstructed from weakly consistent small systems of partial solutions.

The first application was in showing a combinatorial version of (Layered) PCP Theorem, the Baby (Layered) PCP Theorem. One open question is whether there is a combinatorial analogue of the Parallel Repetition Theorem [35], in particular, whether the tame dependence of domain size on the value in Raz’s result can be achieved in the combinatorial version (note that the dependence in the presented version is rather wild). Another direction is exploring combinatorial versions of known improvements of the PCP Theorem, in particular the Smooth Label Cover of Khot [30] (cf. [25]). Finally, the most interesting direction seems to be in exploring combinatorial versions of conjectural improvements of the PCP Theorem, e.g., the \(d\)-to-1 Conjecture [31]. One of the combinatorial versions of this conjecture is the problem in Theorem 10 restricted to \(r = 1\) and instances where every constraint \(\psi\) is given by a \(d\)-to-1 map.

The second, main application of the main result was in providing a general condition for the existence of a polynomial time reduction between two PCSPs in terms of polymorphisms – symmetries of the template. This, and similar such results should not be regarded as heavy hammers that are giving us reductions for free. They rather serve as tools that enable one to disregard the inessential layers and concentrate on the core of the problem, which can then be attacked using various methods (such as algebraic [17, 40], topological [21, 8, 34], or analytic [25]). As such tools, they are indeed useful.

Moreover, such a general condition seems necessary for a prospective dichotomy result for PCSPs, since “the non-existence of [some specific kind of a] homomorphism to the trivial minion \(T\)” can potentially be translated to a positive property.

\(^4\)We believe that the reduction can be done in log space, but do not include the details here.
that can be exploited by an algorithm (as was done in the CSP context [17, 40]), whereas “the non-existence of series of tricks proving NP-hardness” lacks this potential.

We do not believe that $(d, r)$-homomorphism is already the right, sufficiently weak concept. A concrete direction for an improvement is, besides the directions mentioned above, to incorporate the reduction in [34] via an adjunction, which was used to significantly enlarge the NP-hardness region for the approximate graph coloring problem. It is interesting that the reduction in the proof of Theorem 12 works, but it does not seem to be explained by $(d, r)$-homomorphisms.

Another appealing direction for generalizing the reduction theorem is to “let the Baby PCP Theorem grow up”, i.e. to consider weighted relations (cost functions), where tuples can have weights instead of just being present or absent. It may be challenging to obtain such a generalization (as a satisfactory analogue would cover, e.g., the PCP Theorem) but there are some indications that such a result is not out of reach: an analogue of Theorem 11 is available [29] and Dinur’s proof of the PCP Theorem [19] uses essentially the same two reductions as Theorem 12 — they are substantially fine-tuned and repeated more times, but the essence is the same.

7. Appendix: Proof of Theorem 8

For reader’s convenience we recall the basic definitions and notations that appear in the proof. The set of variables is denoted by $V$, while the domain is $A$. By $\binom{V}{k}$ we denote the set of all $k$-element subsets of $V$, and for a function $f : V \rightarrow A$ and $U \subseteq V$, the restriction of $f$ to $U$ will be denoted by $\text{proj}_U f$; the same notation applies to sets of functions.

A partial assignment system (PAS) of arity $k$ ($k$-PAS) is a map such that for each $U \in \binom{V}{k}$ we have $\emptyset \neq \mathcal{I}(U) \subseteq A^U$. An $f \in A^V$ is an $m$-solution of a $k$-PAS $\mathcal{I}$, if every $U \in \binom{V}{k}$ can be extended to $W \in \binom{V}{r}$ satisfying $\text{proj}_U f \in \text{proj}_U \mathcal{I}(W)$. The value of a PAS $\mathcal{I}$ is the maximal size of $\mathcal{I}(U)$.

Let $(\mathcal{I}_0, \ldots, \mathcal{I}_k)$ be a sequence of partial assignment systems over common $V$ and $A$. We call such a sequence consistent if

- their arities $k_0, \ldots, k_r$ form a non-increasing sequence, and
- for every $U_0 \supseteq \cdots \supseteq U_r$ (of sizes $k_0, \ldots, k_r$) there exists $i < j$ such that $\mathcal{I}_j(U_j) \cap \text{proj}_U \mathcal{I}_i(U_i) \neq \emptyset$.

The value of such a sequence is the largest among values of $\mathcal{I}_i$. Finally, we recall the theorem we are proving:

**Theorem 8 (Main Theorem).** For any $A$ and any numbers $m, r, d \in \mathbb{N}$ there exists a sequence $k_0, \ldots, k_r$ such that if $(\mathcal{I}_0, \ldots, \mathcal{I}_k)$ is a consistent sequence of arities $k_0, \ldots, k_r$ and value $\leq d$, then some $\mathcal{I}_i$ has an $m$-solution. Additionally, for fixed $m, r, d, A$, an $m$-solution can be computed in polynomial time.\(^5\)

7.1. Working with a single PAS. Let $\mathcal{I}$ be a $k$-PAS over $V$ and $A$. Let $X \subseteq V$ satisfy $|X| \leq k$ and $f \in A^X$; for $l \leq k$ we consider two $l$-properties a pair $(X, f)$ can have:

- \((P)\) \quad \forall W \in \binom{V}{l} \exists U \in \binom{V}{k} \ X \cup W \subseteq U \text{ and } \text{proj}_X g = f \text{ for some } g \in \mathcal{I}(U)
- \((Q)\) \quad \forall W \in \binom{V}{l} \exists U \in \binom{V}{k} \ X \cup W \subseteq U \text{ and } \text{proj}_X g \neq f \text{ for all } g \in \mathcal{I}(U)

\(^5\)Our procedure is very much non-polynomial with respect to parameters $|A|, m, r$ or $d$.\]
For convenience, we state their negations as well:
\[
\begin{align*}
&\neg P \quad \exists W \in \binom{V}{k} \forall U \in \binom{V}{k} \ X \cup W \subseteq U \text{ implies } \text{proj}_X g \neq f \text{ for all } g \in \mathcal{I}(U) \\
&\neg Q \quad \exists W \in \binom{V}{k} \forall U \in \binom{V}{k} \ X \cup W \subseteq U \text{ implies } \text{proj}_X g = f \text{ for some } g \in \mathcal{I}(U)
\end{align*}
\]
If \( X = \{ v \} \) and \( f(v) = a \) we say that the property holds for \( (v, a) \) instead of \( (X, f) \); the sets \( W \) in the definitions of \( \neg Q, \neg P \) are called witnesses.

The first proposition states that if the parameters are suitable chosen, property \( P \) will appear.

**Proposition 1.** Let \( \mathcal{I} \) be a \( k \)-PAS over \( V \) and \( A \). If \( k \geq |A|^{|X|} l + |X| \) for \( X \subseteq V \), then there is \( f \) so that \( (X, f) \) has \( l \)-property \( P \).

**Proof.** Suppose, for a contradiction, that every \( f \in A^X \) has property \( \neg P \) and let \( W_f \) be a witness of \( \neg P \) for \( f \). Choose \( W \in \binom{V}{k} \) so that \( X \subseteq W \) and moreover \( W \subseteq W_f \) for every \( f \in A^X \) (which is possible by the assumed inequality). Fix an arbitrary \( g \in \mathcal{I}(W) \) and note the contradiction: since \( W_{\text{proj}_X g} \subseteq W \) we derived \( \text{proj}_X g \neq \text{proj}_X f \). \( \square \)

The next proposition concerns PASes of special form and will serve as a base for an inductive proof. It says that, given suitable parameters, if \( \neg Q \) is present throughout the PAS of value 1, then an \( m \)-solution can be found.

**Proposition 2.** Let \( \mathcal{I} \) be a \( k \)-PAS over \( V \) and \( A \) and \( \text{val}(\mathcal{I}) = 1 \). If for every \( v \in V \) there exists \( a \in A \) such that \( (v, a) \) has \( l \)-property \( \neg Q \), then \( \mathcal{I} \) is \( \lfloor \frac{1}{l+1} \rfloor \)-solvable.

**Proof.** Let \( s \) be the function mapping \( v \) to the associated \( a \) with \( l \)-property \( \neg Q \). Choose \( W \subseteq V \) so that \( |W| \leq \frac{1}{l+1} \) and let \( W' \in \binom{V}{k} \) include \( W \) as well as a witness for every \( a \in W \). By property \( \neg Q \), the projection \( \text{proj}_W \mathcal{I}(W') \) is equal to \( \{ \text{proj}_W s \} \) and the proposition is proved. \( \square \)

### 7.2. Refining PASes.
We will be repeatedly performing a construction called refining a PAS: given
- a \( k \)-PAS denoted by \( \mathcal{I} \),
- a number \( l \leq k \), and
- a mapping \( \text{ex} : \binom{V}{k} \rightarrow \binom{V}{k} \) satisfying \( U \subseteq \text{ex}(U) \),
we define an \( l \)-PAS \( \mathcal{J} \) by putting \( \mathcal{J}(U) \) to be \( \text{proj}_U(\mathcal{I}(\text{ex}(U))) \). That is, to define the value of \( \mathcal{J} \) on \( U \) we extend it to \( \text{ex}(U) \), use \( \mathcal{I} \) to obtain an associated set of functions, and restrict these functions to \( U \). It follows from the definition that every \( m \)-solution of \( \mathcal{J} \) is an \( m \)-solution of \( \mathcal{I} \).

The next proposition is very similar to Proposition 2, but will be applied if a PAS has value greater than one. It states that if property \( \neg Q \) can be found "everywhere", then the PAS can be turned to a consistent sequence of two PASes.

**Proposition 3.** Let \( \mathcal{I} \) be a \( k \)-PAS. If every \( X \in \binom{V}{k} \) has an \( f \) with \( l \)-property \( \neg Q \) and \( k \geq k' + \binom{k''}{k'} l \), then there exists a \( k'' \)-PAS \( \mathcal{I}' \) of value 1 and a \( k' \)-PAS \( \mathcal{I}'' \), a refinement of \( \mathcal{I} \), so that \( (\mathcal{I}'', \mathcal{I}') \) is compatible.

**Proof.** Define \( \mathcal{I}' \) by putting \( \mathcal{I}'(X) = \{ f \} \) where \( f \) has \( l \)-property \( \neg Q \) for \( X \) witnessed by \( W_X \). For each \( Y \in \binom{V}{k''} \) put \( \text{ex}(Y) \) to be any set of size \( k \) including \( Y \cup \bigcup_{X \in \binom{V}{k'}} W_X \). Define a refinement of \( \mathcal{I} \) according to \( \text{ex} \) and call it \( \mathcal{I}'' \). The definition of property \( \neg Q \) provides compatibility of \( (\mathcal{I}'', \mathcal{I}') \). \( \square \)
7.3. Putting things together, i.e., a proof of Theorem 8. We fix $A$ and $m$ and the sequence of values $(d_0, \ldots, d_r)$ ($r \geq 1$). We claim that there exist $k_0, \ldots, k_r$ such that every sequence of compatible PASes $(I_0, \ldots, I_r)$ such that $I_i$ is a $k_i$-PAS and $\text{val}(I_i) \leq d_i$ is $m$-solvable.

The general idea is to transform the sequence into another compatible sequence. This is achieved in two steps. In the first step we look at every, except for $I_0$, PAS $I_i$ separately. If the $-Q$ property “can be found everywhere” in the PAS, then either Proposition 2 provides an $m$-solution, or Proposition 3 offers a reduction to a sequence $(I'', I')$ with $\text{val}(I') = 1$. In the remaining case, we refine $(I_0, \ldots, I_r)$ one by one, to obtain a new sequence and then add a twist that makes value of the new PAS at position zero at most $d_0 - 1$. This finishes the reduction. Note that the second case cannot happen when $d_0 = 1$, and that Proposition 3 can be applied at most once during the procedure.

Formally, we proceed by induction on the sequence of values $(d_0, \ldots, d_r)$. While working on $(d_0, \ldots, d_r)$ we need the result established for

- sequence of values $(d_i, 1)$ for each $i \geq 1$ with $d_i \geq 2$, and
- the sequence of values $(d_0 - 1, d_1, \ldots, d_r)$, if $d_0 \geq 2$.

In particular, to establish the base of induction, one needs to prove the result for sequence of values $(1, 1)$.

Let us fix a sequence $(d_0, \ldots, d_r)$ and begin the proof. If $d_0 \geq 2$ we let $p_0, \ldots, p_r$ to be the sequence provided by an inductive assumption, i.e., if $(J_0, \ldots, J_r)$ is a compatible sequence, $J_i$ is a $p_i$-PAS, and $\text{val}(J_i) \leq d_i$ while $\text{val}(J_0) \leq d_0 - 1$, then some $J_i$ has an $m$-solution. If $d_0 = 1$ we put $p_1 = \cdots = p_r = 1$.

Next, we will construct sequence $(k_0, \ldots, k_r)$ and an auxiliary sentence $(l_0, \ldots, l_r)$. Both sentences are constructed simultaneously from their last elements, $k_r$ and $l_r$, to the first ones. The sequences are defined as follows.

- For $i$ equal to $r, r - 1, \ldots, 1$ we put $l_i = p_i + \sum_{j \geq i+1}(p_j)(k_j - p_j)$ (if $i = r$ the sum contributes nothing) and compute $k_i$ from $l_i$:
  - if $d_i = 1$ we put $k_i = (l_i + 1)m$ (we also fix $k'_i = 1$ to be used later),
  - otherwise we set $k''_i, k'_i$ to be the arities, which work for the sequence of values $(d_i, 1)$, and put $k_i = k''_i + (k'_i)l_i$.
- Finally, $l_0 = p_0 + \sum_{1 \leq j}(p_j)(k_j - p_j)$ (i.e., exactly as above) and let $k_0 = \sum_{1 \leq j}k'_j + |A|\sum_{1 \leq j}k''_j$.

In the first step of the proof, we assume $i \geq 1$ and work with every $k_i$-PAS $I_i$ separately (we ignore $I_0$ in this step). If $d_i = 1$ and Proposition 2 can be applied to $I_i$ with the parameter $l_i$ we obtain an $m$-solution and the proof is done. From now on we assume this is not the case and thus, if $d_i = 1$, there exists $v_i$ so that for all $a_i$ the pair $(v_i, a_i)$ has $l_i$-property $Q$ for $I_i$. We put $X_i = \{v_i\}$ for later reference.

If $d_i \geq 2$, the numbers $k''_i, k'_i$ provide an $m$-solution for $(I'', I')$ whenever $I''$ is a $k''$-PAS, $I'$ is a $k'$-PAS, and $\text{val}(I'') = d_i$ while $\text{val}(I') = 1$. If Proposition 3 can be applied to $I_i$ with parameters $l_i, k'_i,$ and $k''_i$ (in places of $l, k', k''$ respectively), we can reduce the problem to the pair of PASes provided by Proposition 3 and a solution exists by inductive assumption. From now on we assume this is not the case and thus there exists $X_i$ of size $k'_i$ with all $f \in A^{X_i}$ having $l_i$-property $Q$ in $I_i$.

In the second step, we put $X = \bigcup_i X_i$ and use Proposition 1 to find $f \in A^X$ so that $(X, f)$ has $l_0$-property $P$ in $I_0$. Then, for every $i \geq 1$ we put $f_i = \text{proj}_{X_i} f$. 


The last part is direct if a bit technical. We will define sequence \((J_0, \ldots, J_r)\) such that \(J_i\) is a \(p_i\)-PAS and a refinement of \(I_i\) for \(i \geq 1\) (in particular \(\text{val}(J_i) \leq \text{val}(I_i)\)). The \(p_0\)-PAS \(J_0\) is a refinement of \(I_0\) with enough functions removed so that \(\text{val}(J_0) \leq d_0 - 1\). To fix these refinements, we need to define a map \(e_{\chi}\) for every PAS \(I_{\chi}\).

We start with \(i = r\) and progressively define \(e_{\chi}\) for smaller \(i\). For \(Y \in \binom{V}{p_i}\) we put \(e_{\chi}(Y)\) to be a set \(U\) provided by property \(Q\) for \((X_i, f_i)\) and

\[
W = Y \cup \bigcup_{j \geq i+1} \bigcup_{Z \in \binom{V}{p_j}} \text{ex}_j(Z).
\]

In the PAS \(I_0\) we additionally remove everything that arose from functions extending \(f\), that is, \(J_0(Y) = \text{proj}_{\chi} I_0\{g \in \text{ex}_0(Y) \mid \text{proj}_{X} g \neq f\}\).

It remains to confirm that the sequence \((J_0, \ldots, J_r)\) is compatible. Let \(Y_0 \supseteq Y_1 \supseteq \cdots \supseteq Y_r\) be a sequence (for the \(J_i\), i.e., of sizes \(p_0, \ldots, p_r\), respectively) and consider the sequence \(\text{ex}_1(Y_0), \ldots, \text{ex}_r(Y_r)\) (for the \(I_i\), i.e., of sizes \(k_0, \ldots, k_r\), respectively). Note that, by the definition of \(W\) in the previous paragraph, we have \(\text{ex}_0(Y_0) \supseteq \cdots \supseteq \text{ex}_r(Y_r)\).

By the compatibility of the original sequence, we get \(i < j\) and \(g \in I_j(\text{ex}_j(Y_j)) \cap \text{proj}_{\chi} I_j(\text{ex}(Y_j))\). If \(1 \leq i < j\), the conclusion is now immediate: \(\text{proj}_{\chi} (g) \in J_j(\text{ex}_j(Y_j)) \cap \text{proj}_{\chi} J_i(Y_i)\). If \(0 = i < j\), we additionally need to make sure that that the element \(g' \in I_0(\text{ex}_0(Y_0))\) satisfying \(\text{proj}_{\chi}(g') = g\) satisfies \(\text{proj}_{X}(g') \neq f\). This fact follows from the choice of \(f_j\) and \(X_j: X_j \subseteq \text{ex}_j(Y_j)\) and, by property \(Q\), \(\text{proj}_{X_j} g \neq f\). Clearly \(\text{proj}_{X_j}(g') = \text{proj}_{X_j}(g) \neq \text{proj}_{X_j} f\) as required, and the proof is complete (noticing that the procedure in the proof gives a polynomial time algorithm).

Note that in the case \(d_0 = 1\), we would obtain a compatible sequence containing a PAS with value 0. This is clearly impossible and shows that the second case of the proof cannot happen if \(d_e = 1\). In particular, in the base case of induction (i.e., with the sequence of values \((1, 1)\)), the only possible scenario is that the application of Proposition 2 in the first step provides an \(m\)-solution.

8. APPENDIX: PROOF OF THEOREM 12

8.1. Polymorphisms of general arity. It is convenient to slightly extend the notion of minions and polymorphism so that the arity can be any set, not just a natural number. This way we can avoid ad hoc (and confusing) choices of bijections between \(X\) and \([|X|]\).

An \(X\)-ary polymorphism of a PCSP template \((A, B)\), where \(X\) is a finite nonempty set, is a map \(t: A^X \to B\) such that for any relation \(R_i\) of \(A\) and any \([|X|] \times X\) matrix \(Z \in A^{[|X|] \times X}\) whose each column \(Z[j, x]\) is in \(R_i\), the tuple \((Z[j, x])\) obtained by applying \(t\) to the rows \((Z[j, x])\) is in the corresponding relation \(S_i\) of \(B\). Note that an \(n\)-ary polymorphism as defined in Section 4 is the same as an \([n]\)-ary polymorphism according to this extended definition.

Let \(t: A^X \to B\) and \(\pi: X \to Y\). We say that \(s: A^Y \to B\) is a minor (or \(\pi\)-minor, if \(\pi\) matters) of \(t\) and write \(t \overset{\pi}{\rightarrow} s\) if \(s(f) = t(f \pi)\) for every \(f \in A^Y\). For the sake of clarity we extend the minor notation: instead \(t \overset{\pi}{\rightarrow} s\) (as above) we will sometimes write \(t \overset{\pi}{\rightarrow} X^Y s\) to stress the fact that \(t\) is \(X\)-ary, \(s\) is \(Y\)-ary and \(\pi\) is viewed as mapping \(X\) into \(Y\). Note that polymorphisms of a PCSP template (of
general arity) are still closed under taking minors. We also extend the definitions of minion, polymorphism minion, and minion homomorphism in the obvious way to accommodate functions of any arity.6

A simple but crucial property of polymorphisms is that it maps tuples of (partial) solutions of the strict version of an instance to (partial) solutions of the relaxed instance. More precisely, if \( \Phi \) is an instance of PCSP(\( \mathbf{A}, \mathbf{B} \)), \( t \) is an \( X \)-ary polymorphism of \( \mathbf{A}, \mathbf{B} \) and \( Z \) is a \( U \times X \) matrix whose each column \( Z[x_0, x] \) is a partial solution (a map \( U \to A \)) to \( \Phi^X \), then \( t(Z) \) (a map \( U \to B \)) is a partial solution to \( \Phi^B \).

8.2. Free PCSP templates. Let \( \mathcal{M} \) be any minion, \( C \) any nonempty finite set, and \( R \subseteq C^m \) any nonempty relation. We follow [8] and define an \( m \)-ary relation \( F^C_M(R) \) over the set of \( C \)-ary functions of \( \mathcal{M} \): denoting \( \text{proj}_1, \ldots, \text{proj}_m \) the projection maps \( R \to C \), we define

\[
(s_1, \ldots, s_m) \in F^C_M(R) \text{ iff } \exists t \in \mathcal{M} \text{ (arity } R \text{) : } t \overset{\text{proj}_i}{R \to C} s_i \text{ for all } i.
\]

(i.e. the arity of \( s_i \) does not depend on the set of elements that actually appear on position \( i \) in the tuples in \( R \)).

For a minion \( \mathcal{M} \), a fixed positive integer \( m \), and a (finite nonempty) set \( C \) we define the \( m \)-ary free PCSP template on \( C \) by \( (F^m(C), F^m_{\mathcal{M}}(C)) \), where

- \( F^m(C) = (C; R_1, \ldots, R_l) \) where the relations list every relation on \( A \) of arity at most \( m \) and
- \( F^m_{\mathcal{M}}(C) \) is build on the set of \( C \)-ary members of \( \mathcal{M} \) and the relation \( S_i \) corresponding to \( R_i \) in \( F^m(C) \) is \( S_i = F^m_{\mathcal{M}}(R_i) \) in \( F^m_{\mathcal{M}}(C) \).

The following reduction will serve as the second reduction in the proof of Theorem 12.

**Theorem 13** ([8]). Let \( \mathbf{A}, \mathbf{B} \) be a PCSP template, \( m \in \mathbb{N} \), and \( C \neq \emptyset \) be finite. Then \( \text{PCSP}(F^m(C), F^m_{\text{Pol}(\mathbf{A}, \mathbf{B})}(C)) \) is log-space reducible to \( \text{PCSP}(\mathbf{A}, \mathbf{B}) \).

**Comments on the proof.** The mapping \( \xi : \text{Pol}(\mathbf{A}, \mathbf{B}) \to \text{Pol}(F^m(C), F^m_{\text{Pol}(\mathbf{A}, \mathbf{B})}(C)) \) defined for an \( X \)-ary polymorphism \( t \) of \( \mathbf{A}, \mathbf{B} \) by \( \xi(t)(f)(g) = t(gf) \) for every \( f \in C^X \), \( g \in A^C \) is a minion homomorphism (this is the minion homomorphism \( \phi \) from Section 4.1 of [8]), so the claim follows from Theorem 11.

For the interested readers, we mention that the reduction is the standard long code reduction. It works as follows. For each original variable we introduce a cloud of \( A^C \) variables (that are meant to provide the long code of the original variable) and for each original constraint involving relation \( R \) we introduce a cloud of \( A^R \) variables (meant to provide the long code of a member of \( R \)). We introduce constraints which say that each cloud determines a polymorphism of \( \mathbf{A}, \mathbf{B} \) and finally we merge suitable variables to ensure satisfaction of the original constraints.

As a final remark, let us mention that a reduction in the opposite direction works as well, provided \( m \) and \( C \) are sufficiently large [8].

In the proof of Theorem 12 we will use a 2-ary free PCSP template and only use relations that are graphs of maps from a subset of \( C \) to \( C \). The following notation and observation will come in handy. We denote the identity map by id, independent

6If one defines \( \mathcal{M}X \) as the set of all \( X \)-ary polymorphisms and \( \mathcal{M}\pi(t) = s \) for \( t \overset{s}{X} \) as above, \( \mathcal{M} \) becomes a functor from the category of nonempty finite sets to itself. Minor homomorphisms then exactly correspond to natural transformations. However, we follow the more standard notation in this paper.
on its domain or co-domain, and if is a function with domain and co-domain satisfying and , we treat as a subset of .

**Lemma 1.** Let be a minion, a (finite nonempty) set, and subsets of , and a map. Then if and only if there exist members of , where , such that , and .

**Proof.** Straightforward. □

Note that for any injective : and there exists at most one -ary with . In particular, the in the lemma are unique.

### 8.3. The proof.

**Theorem 12.** Let and be two PCSP templates, and let be such that for some and (if is fixed and does not depend on ). Then and is -time reducible \(^7\) to PCSP(A, B).

**Proof.** Given an instance of PCSP(A, B) we produce , which is an instance of PCSP(F\(^2\)(C), F\(^2\)(M\(_{1}\))(C)) (for which is fixed and does not depend on ). Then we use the reduction from Theorem 13 to produce an instance of PCSP(A, B).

Let , be the number provided by Corollary 1 for (in place of ) and let be the maximal arity of a relation in or (or ). The last thing we need to fix is : it would be most convenient to have a different domain for each variable of since then we could define the reduction in essentially the same way as in the proof of Theorem 10. However, we do not have such a freedom (see the remarks in Section 8.4) and we set to be an arbitrary set of size at least \(|A^{k_0}_2|\).

Our reduction transforms an instance of PCSP(A, B) with a set of variables to an auxiliary instance and then to an instance of PCSP(F\(^2\)(C), F\(^2\)(M\(_{1}\))(C)).

The set of variables, of both and , is \(X = \bigcup X_i\) where \(X_i = \binom{V}{k_i}\). For each \(U \in X\) we put \(D_U \subseteq A^{k_0}_U\) to be the set of partial solutions to \(\Phi^A_2\) (the set needs to be non-empty as a solution of \(\Phi^A_2\) is promised). The constraints of \(\Psi'\) are introduced for each pair of elements of \(X\) satisfying \(U \supseteq W\); we put \(((U, W), \pi_{U,W})\) where \(\pi_{U,W} = \{(f, g) \in D_U \times D_W \mid \text{proj}_W f = g\}\). Note that \(\pi_{U,W}\) is in fact a function from \(D_U\) into \(D_W\) (as a restriction of a partial solution is a partial solution). The only problem with the instance \(\Psi'\) is that its domain is huge, and the reduction requires a domain of constant (i.e., independent on ) size. This problem is resolved in a rather pedestrian fashion.

For each \(U \in X\) we fix \(\sigma_U\) as a bijection between \(D_U\) and some \(C_U \subseteq C\). Then for each constraint \(\pi_{U,W}\) of \(\Psi'\) we introduce into \(\Psi\) the constraint \(\pi_{U,W}\) where \(\pi_{U,W} = \{(\sigma_U(f), \sigma_W(g)) \mid (f, g) \in \pi_{U,W}\}\). In essence, the last transformation renames the elements of the domain without altering the structure of the instance. Thus we obtain an instance with domain and reduction is finished.

Soundness of the reduction is, again, immediate: if \(h : V \rightarrow A\) is a solution to \(\Phi^A_2\), then \(s\) defined by \(s'(U) = \text{proj}_U h\) is a solution to \(\Psi'\) and \(s(U) = \sigma_U(s'(U))\) a solution to .

For the completeness part, take a solution \(s\) of \(\Psi^F_{M_1}(C)\), that is, for each \(U \in X\), \(s(U)\) is a \(C\)-ary member of \(M_1\), and \(s\) satisfies all the constraints, that is,
(s(U), s(W)) ∈ F^M_1(σ(π_U,W)) for any U ⊇ W in X. Lemma 1 now delivers two pieces of information:

- For each U ∈ X there exist unique t'_U ∈ M_1 such that t'_U \xrightarrow{\text{id}} C_U \rightarrow C U \rightarrow C W \rightarrow s(U).
- For any U ⊇ W in X we have t_U \xrightarrow{σ(π_U,W)} t'_U .

By defining a D_U-ary t_U by t_U \xrightarrow{σ^{-1}_U} t_U we finally obtain

\begin{equation}
\tag{1}
t_U \xrightarrow{π_{U,W}} t_W \quad \text{for any } W; U \in X \text{ such that } U \supseteq W.
\end{equation}

It remains to decode the t_U into a sequence of PASes. To this end we first define (for any U ∈ X) a U × D_U matrix Z_U by Z_U[u, f] = f(u) for u ∈ U and f ∈ D_U. Observe that each column Z_U[\_ \_ \_ f] of this matrix is a partial solution of \(Φ^A\), namely, f.

For 0 ≤ i ≤ r and U ∈ X_i define \(I_i(U) = \{ q(Z_U) \mid q \in ξ(t_U) \}\), and note that every element of \(I_i(U)\) is a partial solution to \(Φ^B\) (recall the remark in the final paragraph of Section 8.1), that the size of \(I_i(U)\) is at most d (by the definition of (d, r)-homomorphism), and that \(ζ_i\) is a \(k_i\)-PAS.

It remains to verify consistency. Let \(U_0 \supseteq U_1 \supseteq \cdots \supseteq U_r\) be subsets of V (of sizes \(k_0, k_1, \ldots, k_r\)) and consider the chain of minors \(t_{U_0} = t_{U_1} \xrightarrow{π_{U_1,U_2}} \cdots \xrightarrow{π_{U_{r-1},U_r}} t_{U_r}\) that we have from Eq. (1). By the definition of (d, r)-homomorphism, there exist i < j and \(q_i \in ξ(t_{U_i}), q_j \in ξ(t_{U_j})\) such that \(q_i \xrightarrow{π_{U_i,U_j}} q_j\).

We claim that \(\text{proj}_{U_i} q_i(Z_{U_i}) = q_j(Z_{U_j})\) − then this element witnesses \(I(U_j) \cap \text{proj}_{U_i} I(U_i)\) and consistency is established. To prove the claim, we need to verify that, for each u ∈ U_j, \(q_i\) applied to the u-th row of \(Z_{U_j}\) (i.e. a map \(w_i : D_{U_i} \rightarrow A_2\) mapping \(f → f(u)\)) gives the same element of \(B_2\) as \(q_j\) applied to the u-th row (denoted \(w_j\)) of the matrix \(Z_{U_j}\).

Since \(q_i \xrightarrow{π_{U_i,U_j}} q_j\), we have \(q_j(w_j) = q_i(w_j π_{U_i,U_j})\) by definition of minors. It is enough to verify \(q_i(w_i) = q_i(w_j π_{U_i,U_j})\). But this is clear − for any \(f \in D_{U_i}\) we have \(w_i(f) = f(u)\) (by definition of the matrix) and \(w_j π_{U_i,U_j}(f) = (π_{U_i,U_j}(f))(u) = (\text{proj}_{U_i} f)(u) = f(u)\).

We have shown that the value \(\text{val}_{k_1, \ldots, k_r}(Φ^B)\) is at most d, and by Corollary 1 it must be 1, which makes \(Φ^B\) solvable. This finishes the proof of soundness and of Theorem 12. \(□\)

### 8.4. Multisorted PCSP

There are two phenomena apparent from the proof (among other contexts) worth a short note. The first one is that it would be convenient to allow multiple domains for variables, e.g., to work with multi-sorted relational structures. The second one is that we have only used relations that are graphs of functions (the functions were partial, but they would become proper had we multiple sorts). If we do these modification to the definition of a CSP template (i.e., allow multiple sorts but only binary constraints that are graphs of functions\(^8\)), the framework we get would become richer: we could still express all the (P)CSPs (by replacing relations by projection maps) and, moreover, Layered Label Cover would become a CSP, the gap version from Theorem 10 with \(r = 1\)

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\(^8\) so a template for CSP can be defined as a finite subcategory of the category of finite sets – this is perhaps a nicer definition than via structures
would become a PCSP, and the gap version from Theorem 9 with \( r = 1 \) would become a Valued PCSP. Note, however, that the gap versions with \( r > 1 \) would still not be \((V)\)PCSPs. Is this because Gap Layered Label Cover is an unnatural problem which will eventually become obsolete, or is it hinting us toward a better framework?

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