Graded components of local cohomology modules of invariant rings

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ABSTRACT
Let $A$ be a regular domain containing a field $K$ of characteristic zero, $G$ be a finite subgroup of the group of automorphisms of $A$ and $B = A^G$ be the ring of invariants of $G$. Let $S = A[X_1, \ldots, X_m]$ and $R = B[X_1, \ldots, X_m]$ be standard graded with $\deg A = 0, \deg B = 0$ and $\deg X_i = 1$ for all $i$. Extend the action of $G$ on $A$ to $S$ by fixing $X_i$. Note $S^G = R$. Let $I$ be an arbitrary homogeneous ideal in $R$. The main goal of this paper is to establish a comparative study of graded components of local cohomology modules $H_i(R)$ that would be analogs to those proven in the paper [5] for $H_i(S)$ where $J$ is an arbitrary homogeneous ideal in $S$.

1. Introduction

1.1. Standard assumption: Throughout this article, $A$ is a regular domain containing a field $K$ of characteristic zero, $G$ is a finite subgroup of the group of automorphisms of $A$ and $B = A^G$ is the ring of invariants of $G$. Let $S = A[X_1, \ldots, X_m]$ and $R = B[X_1, \ldots, X_m]$ be standard graded with $\deg A = 0, \deg B = 0$ and $\deg X_i = 1$ for all $i$. Extend the action of $G$ on $A$ to $S$ by fixing all $X_i$'s. Note that $S^G = R$. Set $M = T(R) = \bigoplus_{n \in \mathbb{Z}} M_n$ where $T(-) = H_i^S(H_s^R(\cdot \cdot \cdot H_s^R(-) \cdot \cdot \cdot))$ for some homogeneous ideals $I_1, \ldots, I_r$ in $R$ and $i_1, \ldots, i_r \geq 0$. Set $N = T'(S) = \bigoplus_{n \in \mathbb{Z}} N_n$ where $T'(-) = H_{i_1s}(H_{i_2s}(\cdot \cdot \cdot H_{i_rs}(-) \cdot \cdot \cdot))$.

Note. By [2, Theorem 6.4.5] we have that $B$ is Cohen-Macaulay.

In the article [5], we have seen that $T'(S)$ behaves nicely and has several good properties. In this paper we prove the following analogous results for $T(R)$.

I. (Bass numbers): The $j^{th}$ Bass number of an $R$-module $V$ with respect to a prime ideal $P$ is defined as $\mu_j(P, V) = \dim_{k(P)} \text{Ext}_{R_P}(k(P), V_P)$ where $k(P)$ is the residue field of $R_P$. We know that $\mu_j(P, V)$ is always a finite number (possibly zero) for all $j \geq 0$ if $V$ is a finitely generated $R$-module. But homogeneous components of $M = T(R)$ need not be finitely generated as $B$-modules. So $\mu_j(P, M_n)$ may not be a finite number. If $B_P$ is Gorenstein for some prime ideal $P$ of $B$ then we get the following result.

Theorem 1.2 (with hypotheses as in 1.1). Let $P$ be a prime ideal in $B$ such that $B_P$ is Gorenstein. Fix $j \geq 0$. Then EXACTLY one of the following holds:
(i) \( \mu_j(P,M_n) \) is infinite for all \( n \in \mathbb{Z} \).
(ii) \( \mu_j(P,M_n) \) is finite for all \( n \in \mathbb{Z} \). In this case EXACTLY one of the following holds:

(a) \( \mu_j(P,M_n) = 0 \) for all \( n \in \mathbb{Z} \).
(b) \( \mu_j(P,M_n) \neq 0 \) for all \( n \in \mathbb{Z} \).
(c) \( \mu_j(P,M_n) \neq 0 \) for all \( n \geq 0 \) and \( \mu_j(P,M_n) = 0 \) for all \( n < 0 \).
(d) \( \mu_j(P,M_n) \neq 0 \) for all \( n \leq -m \) and \( \mu_j(P,M_n) = 0 \) for all \( n > -m \).
(e) \( \mu_j(P,M_n) \neq 0 \) for all \( n \leq -m \), \( \mu_j(P,M_n) \neq 0 \) for all \( n \geq 0 \), and \( \mu_j(P,M_n) = 0 \) for all \( n \in \mathbb{Z} \) with \(-m < n < 0\).

In the following result, \( B \) is Cohen–Macaulay but not necessarily Gorenstein.

**Theorem 1.3** (with hypotheses as in 1.1). Assume \( m = 1 \). Fix \( j \geq 0 \). Let \( P \) be a prime ideal in \( B \). Then \( \mu_j(P,M_n) \) is finite for all \( n \in \mathbb{Z} \).

**II. (Growth of Bass numbers:)** Fix \( j \geq 0 \). Let \( P \) be a prime ideal in \( B \) such that \( B_P \) is Gorenstein and \( \mu_j(P,M_n) \) is finite for all \( n \in \mathbb{Z} \). We now investigate the growth of the function \( n \mapsto \mu_j(P,M_n) \) as \( n \to \infty \) and as \( n \to -\infty \).

**Theorem 1.4** (with hypothesis as in 1.1). Let \( P \) be a prime ideal in \( B \) such that \( B_P \) is Gorenstein. Fix \( j \geq 0 \). Suppose \( \mu_j(P,M_n) \) is finite for all \( n \in \mathbb{Z} \). Then there exist polynomials \( f^j_P(Z) \), \( g^j_M(Z) \in \mathbb{Q}[Z] \) of degree \( \leq m-1 \) such that

\[
 f^j_P(n) = \mu_j(P,M_n) \quad \text{for all } n \ll 0 \quad \text{AND} \quad g^j_M(n) = \mu_j(P,M_n) \quad \text{for all } n \gg 0 .
\]

Fix \( j \geq 0 \). If \( M_c = 0 \) for some \( c \), then for any prime ideal \( P \) in \( B \) we get \( \mu_j(P,M_c) = 0 \) is finite and hence by **Theorem 1.2** it follows that \( \mu_j(P,M_n) \) is finite for all \( n \in \mathbb{Z} \). For such cases, we prove the following result.

**Theorem 1.5** (with hypothesis as in 1.1). Let \( P \) be a prime ideal in \( B \) such that \( B_P \) is Gorenstein. Fix \( j \geq 0 \). Suppose \( \mu_j(P,M_c) = 0 \) for some \( c \) (this holds if for instance \( M_c = 0 \)). Then

\[
 f^j_P(Z) = 0 \quad \text{or} \quad \deg f^j_P(Z) = m-1 ,
\]

\[
 g^j_M(Z) = 0 \quad \text{or} \quad \deg g^j_M(Z) = m-1 .
\]

**III. (Dimension of Support and injective dimension:)** The support of a \( B \)-module \( V \) is defined as

\[
 \text{Supp}_B V = \{ P \mid P \text{ is a prime ideal in } B \text{ and } V_p \neq 0 \} .
\]

By \( \dim_B V \) we mean the dimension of \( \text{Supp}_B V \) as a subspace of \( \text{Spec } B \), the set of all prime ideals of \( B \). Let \( \text{injdim}_B V \) denote the injective dimension of \( V \). We show the following:

**Theorem 1.6** (with hypothesis as in 1.1). If \( B \) is Gorenstein then the following hold:

(i) \( \text{injdim } M_c \leq \dim M_c \) for all \( c \in \mathbb{Z} \).
(ii) \( \text{injdim } M_n = \text{injdim } M_{-m} \) for all \( n \leq -m \).
(iii) \( \text{injdim } M_n = \text{injdim } M_0 \) for all \( n \geq 0 \).
(iv) If \( m \geq 2 \) and \( -m < r, s < 0 \) then

(a) \( \text{injdim } M_r = \text{injdim } M_s \),
(b) \( \text{injdim } M_r \leq \min\{ \text{injdim } M_{-m}, \text{injdim } M_0 \} \).

We also prove the following result.

**Theorem 1.7** (with hypotheses as in 1.1). Assume \( m = 1 \). Let \( P \) be a prime ideal in \( B \) such that \( B_P \) is not Gorenstein. Fix \( n \in \mathbb{Z} \). Then EXACTLY one of the following holds:
there exists \( c \) such that \( \mu_j(P, M_n) = 0 \) for \( j < c \) and \( \mu_j(P, M_n) > 0 \) for \( j \geq c \).

Let \( M_n \neq 0 \) and \( \operatorname{injdim}_B M_n < \infty \) for some \( n \). It should be noted that if \( \mu_j(P, M_n) \neq 0 \) for some \( j \), then \( B_P \) is Gorenstein.

IV. (Associate primes:) A prime ideal \( P \) is associated to \( V \) if there is some element \( x \) of \( V \) such that \( \operatorname{ann}(x) = P \). The set of all such prime ideals is denoted by \( \operatorname{Ass}_V \). In this paper, we investigate finiteness and the asymptotic behavior of the set of primes associated to the \( B \)-module \( M_n \). We prove the following:

**Theorem 1.8** (with hypothesis as in 1.1). Further assume that either \( A \) is local or a smooth affine algebra over a field \( K \) of characteristic zero. Then \( \bigcup_{n \in \mathbb{Z}} \operatorname{Ass}_B M_n \) is a finite set.

Moreover, if \( B \) is Gorenstein then

1. \( \operatorname{Ass}_B M_n = \operatorname{Ass}_B M_{-m} \) for all \( n \leq -m \).
2. \( \operatorname{Ass}_B M_n = \operatorname{Ass}_B M_0 \) for all \( n \geq 0 \).

V. (Infinite generation:) Let \( R = \bigoplus_{n \geq 0} R_n \) be a positively graded ring, and \( M \) be a finitely generated graded \( R \)-module. Then by [1, Theorem 15.1.5], the \( R_0 \)-module \( H^i_{R_0}(M)_n \) is finitely generated for all \( i \geq 0 \) and \( n \in \mathbb{Z} \). By [5, Theorem 1.7] we have one sufficient condition for infinite generation of a component of graded local cohomology module over \( R \). In this paper, we give another one as follows.

**Theorem 1.9** (with hypotheses as in 1.1). Let \( J \) be a homogeneous ideal in \( R \) such that \( J \cap B \neq 0 \). If \( B \) is Gorenstein and \( H^i_J(R)_e \neq 0 \), then \( H^i_J(R)_e \) is NOT finitely generated as a \( B \)-module.

We begin Section 2 with some basic properties of skew group rings and the action of \( G \) on the graded components of \( N \) that we need later on. In Section 3, we discuss for a fixed \( j \), how Bass numbers \( \mu_j(P, M_n) \) relate with each other for all \( n \), when \( P \) is a prime ideal in \( B \) such that \( B_P \) is Gorenstein. In Section 4, we study the behavior of the function \( n \mapsto \mu_j(P, M_n) \) as \( n \to \infty \) and as \( n \to -\infty \). In Section 5, we talk about Bass numbers of \( M_n \) when \( B_P \) is NOT Gorenstein and \( m = 1 \). In Sections 6 and 7, we study finiteness of injective dimensions and associated primes respectively. Finally in Section 8, we give a sufficient condition under which \( M_n \) is not finitely generated as a \( B \)-module.

### 2. Skew group rings and graded local cohomology

#### 2.1. Recall: Let \( A \) be a ring (not necessarily commutative) and \( G \) be a finite subgroup of the group of automorphisms of \( A \). Assume that \( |G| \) is invertible in \( A \).

The skew-group ring of \( A \) (with respect to \( G \)) is

\[
A \rtimes G = \left\{ \sum_{\sigma \in G} a_{\sigma} \sigma \mid a_{\sigma} \in A \text{ for all } \sigma \right\},
\]

with multiplication defined as

\[
(a_{\sigma} \sigma)(a_{\tau} \tau) = a_{\sigma} \sigma(a_{\tau} \tau).
\]

An \( A \rtimes G \) module \( M \) is an \( A \)-module on which \( G \) acts such that for all \( \sigma \in G \),

\[
\sigma(am) = \sigma(a)\sigma(m) \quad \text{for all } a \in A \text{ and } m \in M.
\]

**Definition 2.2.** Let \( M \) be an \( A \rtimes G \)-module. Then

\[
M^G = \left\{ m \in M \mid \sigma(m) = m \text{ for all } \sigma \in G \right\}.
\]

Let us set \( A^G \) to be the ring of invariants of \( G \). Let \( M, N \) be \( A \rtimes G \)-modules. It can be easily checked that
(1) $M^G$ is an $A^G$-module.
(2) If $u : M \to N$ is $A \ast G$-linear then $u(M^G) \subseteq N^G$ and the restriction map $\bar{u} : M^G \to N^G$ is $A^G$-linear. Thus we have a functor
\[-^G : \text{Mod}(A \ast G) \to \text{Mod}(A^G).\]
(3) $(-)^G = \text{Hom}_{A \ast G}(A, -)$ and hence it is left exact.

2.3. Reynolds operator: For any $A \ast G$-module $M$ we have a Reynolds operator
\[
\rho^M : M \to M^G
\]
\[m \mapsto \frac{1}{|G|} \sum_{g \in G} \sigma_m.
\]
It is easy to check that $\rho^M$ is $A^G$-linear and $\rho^M(m) = m$ for all $m \in M^G$. From [4, Lemma 2.5] we get that $(-)^G$ is an exact functor.

2.4. Let $A$ be a commutative Noetherian ring and $G$ be a finite subgroup of the group of automorphisms of $A$ with $|G|$ invertible in $A$. Let $B = A^G$ be the ring of invariants of $G$. Let $S = A[X_1, \ldots, X_m]$ and $R = B[X_1, \ldots, X_m]$. Let $G$ act on $A$ and fix $X_i$. Then clearly $R = S^G$. Set $M = T(R) = \bigoplus_{n \in \mathbb{Z}} M_n$ where $T(-) = H^0_i(H^0_{1j} \cdots H^0_{ij}(-) \cdots)$ for some ideals $I_1, \ldots, I_r$ in $R$ and $i_1, \ldots, i_r \geq 0$. Set $N = T'(S) = H^0_{IjS}(H^0_{Ij,S}(\cdots \cdot H^0_{Ij,S}(S) \cdots) = \bigoplus_{n \in \mathbb{Z}} N_n$. Then by [4, Corollary 3.3] we get that $T'(S)$ is a $S \ast G$-module and $T'(S)^G = T(R)$.

All of our results depend on the following statement.

Proposition 2.5. $N_n$ is an $A \ast G$-module and $N^G_n = M_n$ for all $n$.

Proof. As $N$ is a graded $S$-module we have $N_n$ is a $S_0 = A$-module. We also have $N = T'(S)$ is a $S \ast G$-module. Now $A \subseteq S$ is a sub-ring. So $N$ is an $A \ast G$-module. Notice $G$ acts linearly on $S$ (fixes $X_i$). Thus $\sigma(N_n) \subseteq N_n$ for all $\sigma \in G$. Moreover, $N_n \subseteq A$ and hence $N_n$ is an $A \ast G$-module. Clearly $\rho^N : N \to N^G$ is a degree zero $R$-module homomorphism. It follows that $N^G_n = M_n$. \qed

3. Bass numbers

3.1. Setup: Let $A$ be a regular domain containing a field of characteristic zero. Let $G$ be a finite subgroup of the group of automorphisms of $A$. Let $B = A^G$ be the ring of invariants of $G$. Let $S = A[X_1, \ldots, X_m]$ and $R = B[X_1, \ldots, X_m]$ be standard graded with deg $A = 0$, deg $B = 0$ and deg $X_i = 1$ for all $i$.

Set $M = T(R) = \bigoplus_{n \in \mathbb{Z}} M_n$ where $T(-) = H^0_i(H^0_{1j} \cdots H^0_{ij}(-) \cdots)$ for some ideals $I_1, \ldots, I_r$ in $R$ and $i_1, \ldots, i_r \geq 0$ and $N = T'(S) = \bigoplus_{n \in \mathbb{Z}} N_n$ where $T'(S) = H^0_{IjS}(H^0_{Ij,S}(\cdots \cdot H^0_{Ij,S}(S) \cdots) = \bigoplus_{n \in \mathbb{Z}} N_n$.

Note that $\mu_i(P, M) = \mu_i(PR_P, M_P)$. So by [4, Lemma 3.1, Lemma 3.4], it is enough to prove any result of Bass numbers only for maximal ideals after localizing. In this section we assume that $(B, \mathfrak{m})$ is a local ring. Set $E = E_B(B/\mathfrak{m})$ and $l = B/\mathfrak{m}$. Let $\eta_1, \eta_2, \ldots, \eta_r$ be all the maximal ideals of $A$ lying over $\mathfrak{m}$. Since $A$ is a normal domain and $N_n$ is an $A \ast G$-module by Proposition 2.5, so from [4, Theorem 4.1] we get that $H^0_{Iji}(N_n) = \bigoplus_{l = 1}^r H^0_{Ij}(N_{\eta_l})$.

Lemma 3.2. Let height $P = g$. Then
\[
(H^0_{Ij}(N_{\eta_l})_{P}) = H^0_{Ij}(B_{P})^{s_l(n)} \text{ for some } s_l(n) \geq 0.
\]
Here $s_l$ is some cardinal (possibly infinite).
Proof. It suffices to prove this result only for maximal ideal considering \((B, \mathfrak{m})\) is a local ring with \(\dim B = g\). We have \(H^i_{\mathfrak{m}A}(N_n) = \bigoplus_{i=1}^t H^i_{\mathfrak{m}A}(N_n)\). Again \(H^i_{\mathfrak{m}A}(N_n) = (H^i_{\mathfrak{m}A}(N_n))_{\mathfrak{m}}\) by [4, Proposition 5.2]. From the proof of [5, Proposition 9.4] we get that \(H^i_{\mathfrak{m}A}(N_n)\) is an injective module and \(H^i_{\mathfrak{m}A}(N_n) \cong E_A(A/\mathfrak{m})^{\beta(n)}\) where \(\beta(n)\) is some cardinal (possibly infinite). So \(H^i_{\mathfrak{m}A}(N_n) = \bigoplus_{i=1}^t E_A(A/\mathfrak{m})^{\beta(n)}\). Since \(A_{\mathfrak{m}}\) is a Gorenstein local ring,

\[
H^i_{\mathfrak{m}A}(A) = (H^i_{\mathfrak{m}A}(A))_{\mathfrak{m}} = H^i_{\mathfrak{m}A}(A_{\mathfrak{m}}) \cong E_A(A/\mathfrak{m}).
\]

Thus we get \(H^i_{\mathfrak{m}A}(N_n) = \bigoplus_{i=1}^t H^i_{\mathfrak{m}A}(A)^{\beta(n)}\). Moreover, by [4, Theorem 4.1] we have \(\sigma^k(H^i_{\mathfrak{m}A}(N_n)) = H^i_{\mathfrak{m}A}(N_n)\) where \(\sigma^l \in G\) such that \(\sigma^l(n) = n_k\) for all \(l, k\). Therefore \(\alpha(j(n) = \cdots = \alpha(j(n) = \beta(j(n)\) (say). Thus we have an isomorphism

\[
H^i_{\mathfrak{m}A}(N_n) = \left(\bigoplus_{i=1}^t H^i_{\mathfrak{m}A}(A)^{\beta(n)}\right) \cong H^i_{\mathfrak{m}A}(A)^{\beta(n)} \tag{3.2.1}
\]

of \(A \ast G\)-modules. Again by Proposition 2.5 we have \(M_n = N_n^G\). Now applying \((-)^G\) on both sides of (3.2.1) and by that is, [4, Theorem 3.2(2)], we get \(H^i_{\mathfrak{m}A}(N_n^G) \cong H^i_{\mathfrak{m}A}(A^G)^{\beta(n)} = H^i_{\mathfrak{m}A}(B)^{\beta(n)}\). \hfill \Box

As an application of the above lemma we prove the following.

**Proposition 3.3.** Let \(P\) be a prime ideal in \(B\) such that \(B_P\) is Gorenstein. Set \(V = M_n\). Then \((H^p_p(V))_p\) is injective \(B\)-module for all \(j \geq 0\).

Proof. Since for any prime ideal \(P\) in \(B\), \((B_P, PB_P)\) is a Gorenstein local ring so we have \(H^2_{PB_P}(B_P) = E_B(B/P)\) where \(g = \text{height} P\). Thus by Lemma 3.2 we get \((H^p_p(V))_p \cong E_B(B/P)^{\beta(n)}\) for some \(\beta(n) \geq 0\) and hence it is injective. \hfill \Box

We need the following lemma from [3, 1.4].

**Lemma 3.4.** Let \(C\) be a Noetherian ring, and let \(V\) be a \(C\)-module (\(V\) need not be finitely generated). Let \(P\) be a prime ideal in \(C\). If \((H^p_p(V))_p\) is injective for all \(j \geq 0\), then \(\mu_j(P, V) = \mu_0(P, H^p_p(V))\).

As an immediate consequence we get the following.

**Lemma 3.5.** Let \(P\) be a prime ideal in \(B\) such that \(B_P\) is Gorenstein. Then \(\mu_j(P, M_n) = \mu_0(P, H^p_p(M_n))\) for all \(j \geq 0\).

Proof. The result follows from Proposition 3.3 and Lemma 3.4. \hfill \Box

We are now ready to prove the main result of this section. This shows that what the first author observed in [5] also holds in our case.

**Theorem 3.6.** Let \(P\) be a prime ideal in \(B\) such that \(B_P\) is Gorenstein. Fix \(j \geq 0\). Then EXACTLY one of the following holds:

(i) \(\mu_j(P, M_n)\) is infinite for all \(n \in \mathbb{Z}\).
(ii) \(\mu_j(P, M_n)\) is finite for all \(n \in \mathbb{Z}\). In this case EXACTLY one of the following holds:
   (a) \(\mu_j(P, M_n) = 0\) for all \(n \in \mathbb{Z}\).
   (b) \(\mu_j(P, M_n) \neq 0\) for all \(n \in \mathbb{Z}\).
   (c) \(\mu_j(P, M_n) \neq 0\) for all \(n \geq 0\) and \(\mu_j(P, M_n) = 0\) for all \(n < 0\).
   (d) \(\mu_j(P, M_n) \neq 0\) for all \(n \leq -m\) and \(\mu_j(P, M_n) = 0\) for all \(n > -m\).
   (e) \(\mu_j(P, M_n) \neq 0\) for all \(n \leq -m\), \(\mu_j(P, M_n) \neq 0\) for all \(n \geq 0\), and \(\mu_j(P, M_n) = 0\) for all \(n \in \mathbb{Z}\) with \(-m < n < 0\).

Proof. If \(M_n = 0\) then \(\mu(P, M_n) < \infty\). So without loss of generality we can take \(M_n \neq 0\). It is enough to prove this result only for maximal ideal considering \((B, \mathfrak{m})\) is Gorenstein local. Then
from the proof of [4, Theorem 4.1] we get \( H^j_{m_A}(N_n) = \bigoplus_{i=1}^r H^j_{n_i}(N_n) = \bigoplus_{i=1}^r E_A(A/n_i)^{s(n)} \) for all \( j \geq 0 \), where \( n_1, ..., n_r \) are all the maximal ideals of \( A \) lying over \( m \).

Claim. \( \mu_j(m, M_n) \) is finite if and only if \( s_j(n) \) is finite.

By Lemma 3.5 we have \( \mu_j(m, M_n) = \mu_0(m, H^j_m(M_n)) \) for all \( j \geq 0 \). Now by Lemma 3.2 we get \( H^j_m(M_n) = H^j_m(B)^{s_j(n)} \) where \( g = \dim B \). Furthermore, as \( (B, m) \) is a Gorenstein local ring so \( H^j_m(B) \cong E \). Therefore \( \mu_0(m, H^j_m(M_n)) = s_j(n) \) and hence \( \mu_j(m, M_n) = s_j(n) \). The claim follows.

If \( \mu_j(m, M_{n_0}) \) is finite for some \( n_0 \), then by the above claim we get that \( s_j(n_0) \) is finite. Therefore \( \mu_0(n_0, H^j_{m_A}(N_{n_0})) \equiv s_j(n) \) is finite for any \( l \). Fix \( l \). Note that \( H^j_{m_A}(N_{n_0}) = (H^j_{m_S}(N))_{n_0} = (H^j_{m_S}(T^l(S)))_{n_0} \) and \( H^j_{m_S}(T^l(\cdots)) \) is a graded Lyubeznik functor on \( \ast Mod(S) \). Then by [5, Theorem 9.2] we have \( \mu_0(n_0, H^j_{m_A}(N_{n_0})) = s_j(n) \) is finite for all \( n \in \mathbb{Z} \) and satisfies one of \( (a), (b), (c), (d), (e) \). Therefore by the above claim \( \mu(m, M_n) = s_j(n) \) is finite for all \( n \) and satisfies one of \( (a), (b), (c), (d), (e) \).

\[ \square \]

4. Growth of Bass numbers

In this section, we study the behavior of the function \( n \mapsto \mu_j(P, M_n) \) as \( n \to \infty \) and as \( n \to -\infty \).

**Theorem 4.1** (with hypothesis as in 3.1). Let \( P \) be a prime ideal in \( B \) such that \( B_P \) is Gorenstein. Fix \( j \geq 0 \). Suppose \( \mu_j(P, M_n) \) is finite for all \( n \in \mathbb{Z} \). Then there exist polynomials \( f_{M}^{j,P}(Z), g_{M}^{j,P}(Z) \in \mathbb{Q}[Z] \) of degree \( \leq m-1 \) such that

\[ f_{M}^{j,P}(n) = \mu_j(P, M_n) \quad \text{for all } n < 0, \quad \text{AND} \quad g_{M}^{j,P}(n) = \mu_j(P, M_n) \quad \text{for all } n \geq 0. \]

**Proof.** As in Section 3, it is enough to prove this result only for maximal ideal considering \( (B, m) \) is local. By the claim in the proof of Theorem 3.6 we have \( \mu_j(m, M_n) = s_j(n) = \mu_0(n_0, H^j_{m_A}(N_{n_0})) \) for any \( 1 \leq l \leq r \) where \( n_1, ..., n_r \) are all the maximal ideals of \( A \) lying over \( m \). Set \( V = H^j_{m_S}(N) \). Clearly \( V_n = H^j_{m_A}(N_n) \). Fix \( l \). Since \( \mu_j(m, M_n) \) is finite for all \( n \in \mathbb{Z} \) so we get \( \mu_0(n_0, H^j_{m_A}(N_{n_0})) \) is finite for all \( n \in \mathbb{Z} \). Therefore by [5, Theorem 1.11] there exist polynomials \( f_{V}^{0,n}(Z), g_{V}^{0,n}(Z) \in \mathbb{Q}[Z] \) of degree \( \leq m-1 \) such that \( f_{V}^{0,n}(n) = \mu_0(n_0, H^j_{m_A}(N_{n_0})) \) for all \( n \leq 0 \) and \( g_{V}^{0,n}(n) = \mu_0(n_0, H^j_{m_A}(N_{n_0})) \) for all \( n \geq 0 \). Take \( f_{M}^{j,m}(Z) = f_{V}^{0,n}(Z) \) and \( g_{M}^{j,m}(Z) = g_{V}^{0,n}(Z) \). The result follows.

The following result gives some properties of the polynomials appeared in the foregoing Theorem.

**Theorem 4.2** (with hypothesis as in 3.1). Let \( P \) be a prime ideal in \( B \) such that \( B_P \) is Gorenstein. Fix \( j \geq 0 \). Suppose \( \mu_j(P, M_c) = 0 \) for some \( c \) (this holds if for instance \( M_c = 0 \)). Then

\[ f_{M}^{j,P}(Z) = 0 \quad \text{or} \quad \deg f_{M}^{j,P}(Z) = m-1, \]

\[ g_{M}^{j,P}(Z) = 0 \quad \text{or} \quad \deg g_{M}^{j,P}(Z) = m-1. \]

**Proof.** We have \( \mu_j(m, M_c) = 0 = \mu_0(n_0, H^j_{m_A}(N_{n_0})) \). Set \( V = H^j_{m_S}(N) \). Clearly \( V_n = H^j_{m_A}(N_n) \). Fix \( l \). As \( f_{M}^{j,m}(Z) = f_{V}^{0,n}(Z) \) and \( g_{M}^{j,m}(Z) = g_{V}^{0,n}(Z) \) so by [5, Theorem 1.12] it follows that \( f_{M}^{j,m}(Z) = 0 \) or \( \deg f_{M}^{j,m}(Z) = m-1 \) and \( g_{M}^{j,m}(Z) = 0 \) or \( \deg g_{M}^{j,m}(Z) = m-1 \). \[ \square \]

5. Bass numbers when \( B_P \) is NOT Gorenstein and \( m = 1 \)

We now concentrate on the case when \( m = 1 \). The following result gives us a sufficient condition under which for any fixed \( j \) and prime ideal \( P \) in \( B \) the Bass number \( \mu_j(P, M_n) \) is finite for all \( n \in \mathbb{Z} \). Recall that under our assumption \( B \) is always Cohen-Macaulay.
**Theorem 5.1** (with standard assumption 1.1). Assume \( m = 1 \). Fix \( j \geq 0 \). Let \( B \) be Cohen-Macaulay but not necessarily Gorenstein and \( P \) be a prime ideal in \( B \). Then \( \mu_j(P, M_n) \) is finite for all \( n \in \mathbb{Z} \).

**Proof.** It is enough to prove this result only for the maximal ideal considering \((B, m)\) is local with \( \dim B = d \). We have \( H^i_{mA}(N_n) = \bigoplus_{i=1}^r E_i(A/\mathfrak{n}_i)^{s_i(n)} \) where \( \mathfrak{n}_1, \ldots, \mathfrak{n}_r \) are all the maximal ideals of \( A \) lying over \( m \). Since \( H^i_{mA}(N_n) = (H^i_{mn}(N_n))_n \) so by [5, Theorem 1.9] we get \( \mu_0(\mathfrak{n}_1, H^i_{mA}(N_n)) = s_j(n) \) is finite for all \( n \in \mathbb{Z} \). Thus if \( B \) is Gorenstein then by the claim of Theorem 3.6 we are done.

Otherwise let \( G \) be a minimal injective resolution of \( M_n \) and
\[
\Gamma_j^G = \Gamma_j^G \oplus E^0 \quad \text{with} \quad m \notin \text{Ass} \, \Gamma_j^G.
\]
Now \( \mu_j(m, M_n) = \dim \Gamma_j^B(l, M_n) \) for \( j \geq 0 \) and \( \text{Ext}^j_B(l, M_n) = \text{Hom}_B(l, \Gamma_j^G) = \text{Hom}_B(l, \Gamma_j^G \oplus E^0) \)
\[
= \text{Hom}_B(l, \Gamma_j^G) \oplus \text{Hom}_B(l, E^0) = \text{Hom}_B(l, E^0) = E(l)^{r_j} = l^r. \]
Hence \( \mu_j(m, M_n) = r_j. \) So we have to prove that \( r_j \) is finite for all \( j \geq 0 \).

Set \( E = \Gamma_m(G) \). Since \( m \notin \text{Ass} \, \Gamma_j^G \) so we have \( \Gamma_j^m(G) = 0 \). Now it is well known that \( \Gamma_j^m(E^r) = E^r \). Thus \( E' = E^r \) for all \( j \geq 0 \). Furthermore by Lemma 3.2 we have
\[
H^j(E) = H^j_m(M_n) \cong H^j_m(B)^{s_j(n)} \quad \text{for some finite} \quad s_j(n) \geq 0.
\]
Note that finiteness of \( s_j(n) \) follows from the first part of this proof.

**Claim.** \( \dim \Gamma_j^B(l, H^j(E)) \) is finite for all \( j \geq 0 \).

Let \( \hat{B} \) be the completion of \( B \) at \( m \). Then \((\hat{B}, \hat{m})\) is a local ring of dimension \( d \) with maximal ideal \( \hat{m} = m \hat{B} \) and \( E = E_{\hat{B}}(B/\hat{m}) = E_{\hat{B}}(l) \). By [2, 3.5.4(d)] we get \( H^d_m(B) = H^d_m(\hat{B}) \) and by [2, 3.5.4(a)] we get \( H^d_m(B) \) is Artinian. Therefore \( H^d_m(B) = H^d_m(\hat{B}) \subseteq E^t \) where \( t = \dim_n \text{soc}(H^d_m(B)) < \infty \). The finiteness comes as \( \text{soc}(H^d_m(B)) \subset H^d_m(B) \) is Artinian (and hence is a finite dimensional \( \ell \) vector space). Hence
\[
\dim_l \text{Hom}_B(l, H^j(E)) = \dim_l \text{Hom}_B(l, H^d_m(B))^{s_j(n)} \leq \dim_l \text{Hom}_B(l, E^t)^{s_j(n)} = \dim_l l^{s_j(n)} = ts_j(n)
\]
is finite and the claim follows.

We now prove by induction that \( r_j \) is finite for all \( j \geq 0 \). Since \( G \) is a minimal injective resolution of \( M_0 \) so we have \( \text{Hom}_B(l, \Gamma_j^G) \rightarrow \text{Hom}_B(l, \Gamma_j^{G+1}) \) is a zero map for all \( j \). As \( \text{Hom}_B(l, \Gamma_j^G) = 0 \) for all \( j \geq 0 \) so we get \( \text{Hom}_B(l, E^n) \rightarrow \text{Hom}_B(l, E^{n+1}) \) is a zero map for all \( j \geq 0 \).

For \( j = 0 \), we have an exact sequence
\[
0 \rightarrow H^0(E) \rightarrow E^0 \rightarrow E^1.
\]
Applying \( \text{Hom}_B(l, -) \) we get another exact sequence
\[
0 \rightarrow \text{Hom}_B(l, H^0(E)) \rightarrow l^0 \rightarrow l^1.
\]
Thus \( l^0 \cong \text{Hom}_B(l, H^0(E)) \) and hence \( r_0 = \dim_l \text{Hom}_B(l, H^0(E)) \) is finite by our claim.

Now let us assume that \( r_0, \ldots, r_{u-1} \) are finite and consider the following part of \( E; \)
\[
E^{u-1} \rightarrow E^u \rightarrow E^u \rightarrow \cdots.
\]
Set \( Z^u = \ker d^u \) and \( B^u = \text{image } d^{u-1} \). Then we have an exact sequence
\[
0 \rightarrow B^u \rightarrow Z^u \rightarrow H^u(E) \rightarrow 0.
\]
Since $E$ is Artinian and by induction hypothesis $r_{n-1} < \infty$ so we get $E^*_{n-1}$ is Artinian. Therefore $B^*$ is also Artinian and hence is an $m$-torsion module. So $B^* \subseteq E^*$ for some $s > 0$ (can take $s = \dim_\B soc(B^*)$). Thus $\dim_\B \Hom_B(I, B^*) \leq \dim_\B \Hom_B(I, E^*) = s < \infty$. Since 
$$0 \to \Hom_B(I, B^*) \to \Hom_B(I, Z^u) \to \Hom_B(I, H^u(\E))$$
is an exact sequence, by our claim it follows that $\Hom_B(I, Z^u)$ is a finite dimensional $l$-vector space.

Again we have an exact sequence 
$$0 \to Z^u \to E^u \xrightarrow{d^u} E^{u+1}.$$ Applying $\Hom_B(I, -)$ we get 
$$0 \to \Hom_B(I, Z^u) \to \Hom_B(I, E^u) \xrightarrow{\theta} \Hom_B(I, E^{u+1}).$$ Therefore $r_u = \dim_\B \Hom_B(I, E^u) = \dim_\B \Hom_B(I, Z^u)$ is finite and the result follows. \hfill \Box

**Remark 5.2.** The fact that $\dim_\B \Hom_B(I, H^t(\E)) = t_0(n) < \infty$ for some $t < \infty$, plays an important role in proving Theorem 5.1. To prove the finiteness we use the result [5, Theorem 1.9] which says that $s_j(n) = \mu_0(\nu, (H^0_{\text{nil}}(\N))_n) < \infty$ for all $n \in \Z$ when $m = 1$. So the assumption $"m = 1"$ is needed.

The next result gives us a sufficient condition when $\text{injdim } M_n$ is infinite if $0 \neq M_n$ is not injective.

**Theorem 5.3** (with standard assumption 1.1). Assume $m = 1$. Let $P$ be a prime ideal in $B$ such that and $B_P$ is not Gorenstein. Fix $n \in \Z$. Then EXACTLY one of the following holds:

(i) $\mu_j(P, M_n) = 0$ for all $j$.
(ii) there exists $c$ such that $\mu_j(P, M_n) = 0$ for $j < c$ and $\mu_j(P, M_n) > 0$ for all $j \geq c$.

**Proof.** As noted earlier, it is sufficient to prove this result only for the maximal ideal considering $(B, m)$ is a local ring with $\dim B = d$. Fix $n$. Let $G$ be a minimal injective resolution of $M_n$ and 
$$G^l = \overline{G^l} \oplus \mathcal{E}^l$$ with $m \notin \text{Ass } \overline{G^l}$.

Now $\mu_j(m, M_n) = \dim_\B \Ext^j_B(I, M_n)$ for $j \geq 0$ and $\Ext^j_B(I, M_n) = \Hom_B(I, G^j) = \Hom_B(I, E^j)$. Hence $\mu_j(m, M_n) = r_j$. So by Theorem 5.1 we get that $r_j$ is finite for all $j \geq 0$. Let 
$$c = \min\{j \mid r_j > 0\}.$$ We have to prove that $r_j > 0$ for all $j \geq c$.

Set $E = \Gamma_m(G)$. Since $m \notin \text{Ass } \overline{G^l}$ so we have $\Gamma_m(\overline{G^l}) = 0$. Now it is well known that $\Gamma_m(E^l) = E^l$. Thus $\mathcal{E}^l = E^l$ for all $j \geq 0$. Furthermore by Lemma 3.2 we have

$$H^l(\mathcal{E}) = H^l_m(M_n) \cong H^l_m(B)^{s_j(n)}$$ for some $s_j(n) \geq 0$,

By the first part of this proof of Theorem 5.1 we have $s_j(n)$ is finite for all $n$. Let $T$ be the completion of $B$ at $m$. Then $(T, \mathfrak{n})$ is a local ring of dimension $d$ with maximal ideal $\mathfrak{n} = mT$ and $E = E_T(T/\mathfrak{n}T) = E_T(I)$. By [2, 3.5.4] we have $H^d_m(B) = H^d_m(T)$ and $H^d_m(B)$ is Artinian.

Set $Z' = \ker d^l$ and $B' = \text{image } d^{l+1}$. Let $(-)^\vee$ denote the Matlis dual of $T$. Now we prove the following assertions by induction on $j \geq c$:

(i) $Z' \neq 0$;
(ii) $\text{injdim } Z' = \infty$;
(iii) $(Z')^\vee$ is a non-free maximal Cohen-Macaulay $T$-module;
(iv) \( B^{i+1} \neq 0 \);
(v) \( \text{injdim } B^{i+1} = \infty \);
(vi) \((B^{i+1})^\vee\) is a non-free maximal Cohen-Macaulay \( T \)-module;

Although (i) proves our assertion, we prove all the above assertion together for \( j \geq c \).

Since \( G \) is a minimal injective resolution of \( M \) so we have \( \text{Hom}_B(l, G^j) \rightarrow \text{Hom}_B(l, G^{j+1}) \) is a zero map for all \( j \). As \( \text{Hom}_B(l, G^j) = 0 \) for all \( j \geq 0 \) so we get \( \text{Hom}_B(I, E^j) \rightarrow \text{Hom}_B(I, E^{j+1}) \) is a zero map for all \( j \geq 0 \). Now we have an exact sequence \( 0 \rightarrow Z^j \rightarrow E^j \rightarrow B^{j+1} \rightarrow 0 \). Applying \( \text{Hom}_B(l, -) \) we get another exact sequence
\[
0 \rightarrow \text{Hom}_B(l, Z^j) \rightarrow \text{Hom}_B(l, E^j) \rightarrow \text{Hom}_B(l, B^{j+1}).
\]
Thus \( \text{dim } \text{Hom}_B(l, Z^j) = r_j \) and hence \( Z^j \not\cong 0 \). As \( Z^j \subseteq E^j \) and \( r_j = 0 \) for all \( j < c \) it follows that \( Z^j = 0 \) for all \( j < c \). Since \( B^j \cong E^{j-1}/Z^{j-1} = 0 \) so we have \( Z^c = H^n(E) = H^n_m(B)^{(s_c(n))} \) for some finite \( s_c(n) \geq 0 \). As \( B \) is not Gorenstein so \( T \) is not Gorenstein and hence \( H^n_m(T) \) is not injective. Moreover, \( H^n_m(T) = \omega \), the canonical module of \( T \). If \( \omega \) is free then \( \text{projdim } \omega = 0 \). But in that case as \( \omega^\vee = H^n_m(T) \), it follows that \( \text{injdim } H^n_m(T) = 0 \), a contradiction. Therefore \( \omega \) is a non-free maximal Cohen-Macaulay \( T \)-module. Hence \((Z^c)^\vee\) is a non-free maximal Cohen-Macaulay \( T \)-module. Therefore \( Z^c \) has infinite injective dimension. As otherwise \( \text{injdim } Z^c < \infty \) implies \( \text{projdim } (Z^c)^\vee < \infty \) and hence by Auslander–Buchsbaum formula we get \( \text{projdim } (Z^c)^\vee = 0 \), i.e., \((Z^c)^\vee \) is free (as \( T \) is local), a contradiction.

We have an exact sequence \( 0 \rightarrow Z^c \rightarrow E^c \rightarrow B^{c+1} \rightarrow 0 \). As \( \text{injdim } Z^c = \infty \) it follows that \( B^{c+1} \neq 0 \) and has infinite injective dimension. By taking \((-1)^\vee\) we get an exact sequence
\[
0 \rightarrow (B^{c+1})^\vee \rightarrow T^c \rightarrow (Z^c)^\vee \rightarrow 0.
\]
It follows that \((B^{c+1})^\vee\) is a non-free maximal Cohen-Macaulay \( T \)-module.

We now assume the result is true for \( j = u \) and prove it for \( j = u+1 \). We have an exact sequence
\[
0 \rightarrow B^{u+1} \rightarrow Z^{u+1} \rightarrow H^{u+1}(E) \rightarrow 0.
\]
By induction hypothesis \( B^{u+1} \neq 0 \) and it satisfies (v) and (vi). Therefore \( Z^{u+1} \neq 0 \). If \( H^{u+1}(E) = 0 \) then clearly \( Z^{u+1} \cong B^{u+1} \) satisfies (ii) and (iii). If \( H^{u+1}(E) \neq 0 \) then taking Matlis duals we get an exact sequence
\[
0 \rightarrow \omega^{s_{u+1}(n)} \rightarrow (Z^{u+1})^\vee \rightarrow (B^{u+1})^\vee \rightarrow 0.
\]
Now for any maximal Cohen-Macaulay \( T \)-module \( N \) we have \( \text{Ext}_T^1(N, \omega) = 0 \). In particular, \( \text{Ext}_T^1((B^{u+1})^\vee, \omega^{s_{u+1}(n)}) = 0 \). Therefore
\[
(Z^{u+1})^\vee \cong \omega^{s_{u+1}(n)} \oplus (B^{u+1})^\vee.
\]
Thus \((Z^{u+1})^\vee\) is a non-free maximal Cohen-Macaulay \( T \)-module. Since \( r_i \) is finite so we have \( Z^i \subseteq E^i \) is Artinian and hence \( B^i \subseteq Z^i \) is Artinian for all \( i \). Therefore by taking Matlis duals again we get that
\[
Z^{u+1} \cong H^n_m(T)^{s_{u+1}(n)} \oplus B^{u+1};
\]
has infinite injective dimension.

Again we have an exact sequence
\[
0 \rightarrow Z^{u+1} \rightarrow E^{u+1} \rightarrow B^{u+2} \rightarrow 0.
\]
As \( \text{injdim } Z^{u+1} = \infty \) it follows that \( B^{u+2} \neq 0 \) and has infinite injective dimension. Taking Matlis duals we get an exact sequence
\[
0 \rightarrow Z^{u+1} \rightarrow E^{u+1} \rightarrow B^{u+2} \rightarrow 0.
\]
Thus \((B^{a+2})^\vee\) is a non-free maximal Cohen-Macaulay \(T\)-module and satisfies (v) and (vi). \(\square\)

6. Dimension of support and injective dimension

We begin with the following relation which shows that \(\text{injdim } M_n\) is finite for any \(n \in \mathbb{Z}\) if \(B\) is Gorenstein.

**Lemma 6.1** (with standard assumption 1.1). Let \(c \in \mathbb{Z}\). If \(B\) is Gorenstein, then

\[\text{injdim } M_c \leq \dim M_c.\]

**Proof.** Let \(P\) be a prime ideal in \(B\). By Proposition 3.3 along with [3, Lemma 1.4] we get

\[\mu_j(P, M_c) = \mu_0(P, H^j_P(M_c)).\]

Moreover, by Grothendieck’s Vanishing Theorem \(H^j_P(M_c) = 0\) for all \(j > \dim M_c\). So \(\mu_j(P, M_c) = 0\) for all \(j > \dim M_c\). \(\square\)

The following example shows that Lemma 6.1 does not hold true if \(B\) is not Gorenstein.

**Example 6.2.** Let \(A = \mathbb{C}[[Y_1, \ldots, Y_n]]\) and \(G \subseteq \text{GL}_n(\mathbb{C})\) acting linearly with \(A^G\) NOT Gorenstein. Let \(\mathfrak{m}\) and \(\mathfrak{m}^G\) be maximal ideals of \(A\) and \(B := A^G\) respectively. As \(B\) is NOT Gorenstein we have \(\text{injdim}_B H^a_{\mathfrak{m}^G}(B) = \infty\). Set \(S = A[X_1, \ldots, X_m]\) and \(R = B[X_1, \ldots, X_m]\). Set \(M = H^a_{\mathfrak{m}^G}(R) = H^a_{\mathfrak{m}^G}(B) \otimes_B R\). It follows that \(\text{injdim}_B M_0 = \infty\).

We now establish the following under the extra hypothesis that \(B\) is Gorenstein.

**Theorem 6.3** (with standard assumption 1.1). If \(B\) is Gorenstein then the following hold:

(i) \(\text{injdim } M_n = \text{injdim } M_{-m}\) for all \(n \leq -m\).

(ii) \(\text{injdim } M_n = \text{injdim } M_0\) for all \(n \geq 0\).

(iii) If \(m \geq 2\) and \(-m < r, s < 0\) then

(a) \(\text{injdim } M_r = \text{injdim } M_s\).

(b) \(\text{injdim } M_r \leq \min\{\text{injdim } M_{-m}, \text{injdim } M_0\}\).

**Proof.** Let \(B\) be Gorenstein, and \(P\) be a prime ideal in \(B\).

(i) Fix \(j \geq 0\). By Theorem 3.6(ii)(d) we get that \(\mu_j(P, M_c) > 0\) if and only if \(\mu_j(P, M_{-m}) > 0\) for any \(c \leq -m\). The result follows.

(ii) Follows by Theorem 3.6(ii)(c) with similar arguments as in (ii).

(iii)(a) and (iv)(b) clearly follow from Theorem 3.6. \(\square\)

7. Associated primes

In this section, we maintain our general assumptions and give a sufficient condition under which the collection of all associated primes of any graded component of local cohomology module is finite. We also establish relations among the sets of associated primes of graded components under certain condition.

**Theorem 7.1** (with standard assumption 1.1). Further assume that \(A\) is a regular local domain or a smooth affine algebra over a field \(K\) of characteristic zero. Then \(\cup_{n \in \mathbb{Z}} \text{Ass}_B M_n\) is a finite set. Moreover, if \(B\) is Gorenstein then
(1) \( \text{Ass}_B M_n = \text{Ass}_B M_{-m} \) for all \( n \leq -m \).
(2) \( \text{Ass}_B M_n = \text{Ass}_B M_0 \) for all \( n \geq 0 \).

To prove this theorem we use the following fact.

Observation: Let \( C \) be a commutative Noetherian ring and \( G \) be a finite subgroup of the group of automorphisms of \( C \) with \( |G| \) invertible in \( C \). Let \( D \) be the ring of invariants of \( G \). Let \( T \) be a Lyubeznik functor on \( \text{Mod}(D) \). Now we have a \( D \)-linear Reynolds operator \( p^C : C \to D \) which splits the inclusion map \( i : D \to C \). Thus \( C = D \oplus L \) as \( D \)-modules. So \( T(C) = T(D) \oplus T(L) \) as \( D \)-modules. It follows that if \( \text{Ass}_D T(C) \) is a finite set then so is \( \text{Ass}_D T(D) \).

We also need the following result from [5, Proposition 12.1].

Proposition 7.2. Let \( f : D \to C \) be a homomorphism of commutative Noetherian rings. Let \( L \) be a \( C \)-module. Then

\[
\text{Ass}_D L = \{ P \cap D \mid P \in \text{Ass}_C L \}.
\]

In particular, if \( \text{Ass}_C L \) is a finite set then so is \( \text{Ass}_D L \).

Proof of Theorem 7.1. If \( A \) is a smooth affine algebra over a field \( K \), then so is \( S = A[X_1, \ldots, X_m] \).

By [3, Remark 3.7] we get \( \text{Ass}_S T'(S) \) is a finite set. Since \( T'(S) = T(S) \) as \( R \)-modules and \( \text{Ass}_R T(S) = (\text{Ass}_S T'(S)) \cap R \) is a finite set by Proposition 7.2 so by the above observation it follows that \( \text{Ass}_S T(R) \) is a finite set. Again \( i : B \to R \) is a ring homomorphism. So by Proposition 7.2 we get \( \text{Ass}_B T(R) \) is also a finite set. Moreover, \( T(R) = \bigoplus_{n \in \mathbb{Z}} M_n \) as \( B \)-module. Therefore \( \bigcup_{n \in \mathbb{Z}} \text{Ass}_B M_n = \text{Ass}_B T(R) \) is a finite set.

Now assume that \( A \) is local with maximal ideal \( \mathfrak{n} \). It is easy to see that \( B \) is local with maximal ideal \( \mathfrak{m} = \mathfrak{n} \cap B \). So \( M = (\mathfrak{m}, X_1, \ldots, X_m) \) is the homogeneous maximal ideal of \( R \). As \( T(R) \) is graded \( R \)-module, by [2, 1.5.6] all its associated primes are homogeneous and hence contained in \( M \). Moreover we have a bijection \( \text{Ass}_R T(R) \to \text{Ass}_R M(T(R)_M) \). Note that \( T(R)_M = \bigoplus_{n \geq 0} (\bigcap_{i} H_i^G(R_M)_M) = G(M) \) where \( G \) is a Lyubeznik functor on \( \text{Mod}(R_M) \). By [4, Lemma 3.1] we have \( R_M = S^G_M = (S_M)^G \). Notice \( G(S_M) = \bigoplus_{n \geq 0} (\bigcap_{i} H_i^G(S_M)_M) = G(S) \) where \( G \) is a Lyubeznik functor on \( \text{Mod}(S_M) \). So \( \text{Ass}_S M \) is finite by [3, Theorem 3.4]. From Proposition 7.2 we get that \( \text{Ass}_B T(R) \) is finite. Then by the observation stated earlier, \( \text{Ass}_B M_n \) is finite and hence \( \text{Ass}_B T(R) \) is finite. Thus \( \text{Ass}_B T(R) = \bigcup_{n \in \mathbb{Z}} \text{Ass}_B M_n \) is finite by Proposition 7.2.

Now let \( B \) be Gorenstein and

\[
\bigcup_{n \in \mathbb{Z}} \text{Ass}_B M_n = \{ P_1, \ldots, P_i \}.
\]

(1) Let \( P = P_i \) for some \( i \). If \( r \leq -m \) then by Theorem 3.6 we get that \( \mu_0(P, M_r) > 0 \) if and only if \( \mu_0(P, M_{-m}) > 0 \). It follows that \( P \in \text{Ass}_B M_r \) if and only if \( P \in \text{Ass}_B M_{-m} \). Hence the result follows.

(2) Let \( P = P_i \) for some \( i \). Let \( s \geq 0 \). Then by Theorem 3.6 we get that \( \mu_0(P, M_s) > 0 \) if and only if \( \mu_0(P, M_0) > 0 \). It follows that \( P \in \text{Ass}_B M_s \) if and only if \( P \in \text{Ass}_B M_0 \). Hence the result follows. \( \square \)

8. Infinite generation

Our aim in this section is to give a sufficient condition under which \( M_n \) is not finitely generated as a \( B \)-module. Note that \( B \) is a domain.

Theorem 8.1 (with standard assumption 1.1). Let \( f \) be a homogeneous ideal in \( R \) such that \( f \cap B \neq 0 \). If \( B \) is Gorenstein and \( H^f_1(R)_c \neq 0 \), then \( H^f_1(R)_c \) is NOT finitely generated as a \( B \)-module.
Proof. Set $M = H^j_I(R)$. Let $P \in \text{Supp}_B M_c$. Notice $R_P = B_P[X_1, ..., X_m]$ for any prime ideal $P$ in $B$. Again $(M_c)_P = H^j_I(R)_c \otimes_B B_P = H^j_I(R \otimes_B B_P)_c = H^j_I(R_P)_c$. Clearly if $H^j_I(R_P)_c$ is not finitely generated, then $M_c$ is not finitely generated. Thus it is enough to prove this result considering $B$ is a local ring.

We prove by contradiction. If possible let $0 \neq H^j_I(R)_c$ be a finitely generated $B$-module. Then we have $\text{depth } M_c \leq \dim M_c$. Since $B$ is Gorenstein so by Theorem 6.3 we have $\text{injdim } M_c \leq \dim M_c < \infty$. Thus by [2, Theorem 3.1.17] we get $\dim M_c \leq \text{injdim } M_c = \text{depth } B$. As $B$ is Cohen-Macaulay so $\text{depth } B = \dim B$. Together we have $\dim M_c \leq \dim B = \text{injdim } M_c \leq \dim M_c$, i.e., $\dim M_c = \dim B$. It follows that $M_c$ is a maximal Cohen-Macaulay $B$-module. Moreover, $B$ is a domain. Therefore $M_c$ is torsion-free. Let $0 \neq a \in J \cap B$. Then $0 \neq a^i \in J \cap B$ for all $i \geq 1$. Clearly $a^i$ is $B$-regular. Since $M_c$ is $J$-torsion and $a \in J$ so we get $(0:_{M_c}a^i) \neq 0$ for some $j$, a contradiction. \hfill \square

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