Isospectrality of quasinormal modes for black holes beyond Schwarzschild

Flora Moulin$^1$ and Aurélien Barrau$^1$

$^1$Laboratoire de Physique Subatomique et de Cosmologie, Université Grenoble-Alpes, CNRS/IN2P3
53, avenue des Martyrs, 38026 Grenoble cedex, France
(Dated: June 14, 2019)

The reason why the equations describing axial and polar perturbations of the Schwarzschild black hole have the same spectrum is far from trivial. In this article, we revisit the original proof and try to make it clearer. Still focusing on uncharged and non-rotating black holes, we extend the results to slightly more general metrics.

INTRODUCTION

The direct measurement of gravitational waves emitted by the coalescence of black holes (BHs) is now part of real experimental science. Since the seminal detection by LIGO [1], several other events were recorded and a catalogue is already available [2]. The recent improvement in sensitivity has even led to a dramatic increase in the detection rate. The recorded gravitational waves carry fundamental informations about the structure of spacetime – BHs are vacuum solutions of the Einstein field equations. Three phases can be distinguished during a coalescence: the inspiral, the merger and the ringdown. The later can be partially treated perturbatively as a superposition of damped oscillations with different complex frequencies, called quasinomal modes (QNMs). An intuitive introduction can be found in [3] and a review in [4]. The ringdown does not lead to pure “normal” modes because the system looses energy through the emission of gravitational waves. The equation for the metric perturbation is somehow unusual because of its boundary conditions: the waves have to be purely outgoing at infinity and purely ingoing at the event horizon. The radial part can schematically be written as $\phi \propto e^{-i\omega t} = e^{-i(\omega_R + i\omega_I) t}$ where $\omega_R$ is proportional to the frequency and $\omega_I$ is the inverse of the decaying time scale. The process is stable when $\omega_I < 0$.

Basically, QNMs are characterized by their overtone and multipole numbers: $n$ and $\ell$.

The determination of QNMs has driven a huge amount of efforts (see, e.g., [5] for a historical review, [6, 7] for an example of quite recent results based on a numerical approach and [8–13] for WKB treatments). This article is not about the calculation of the complex frequencies but about a remarkable – and quite strange – property. The perturbations of the metric are described by two different equations depending on their parity: whether polar or axial, they do not fulfill the same equation. They both obey a Schrödinger-like equation (Eq. 8) but with different potentials. For a spherical time-independent metric, one can write

$$ds^2 = e^{2\mu_t}dt^2 - e^{2\psi}(d\phi - q_t dt - q_r dr - q_\theta d\theta)^2 - e^{2\mu_r}dr^2 - e^{2\mu_\theta}d\theta^2. \quad (1)$$

For the special case such that

$$e^{2\mu_t} = e^{-2\mu_r} = B(r), \quad e^{2\mu_\theta} = r^2, \quad (2)$$

$$e^{2\psi} = r^2 \sin^2(\theta) \quad \text{and} \quad q_t = q_r = q_\theta = 0, \quad (3)$$

the perturbations will be described by $q_t$, $q_r$ and $q_\theta$, being first order small quantities, and $\mu_t$, $\mu_r$, $\mu_\theta$, and $\psi$ which receive small increments $\delta \mu_t$, $\delta \mu_r$, $\delta \mu_\theta$ and $\delta \psi$. The former lead to a non-static stationary distribution of mass-energy leading to a rotating BH. They are called the axial perturbations. The latter do not imply any rotation and are called the polar perturbations.

As an example in the Schwarzschild case, let us show the Regge-Wheeler potential (for the axial parity),

$$V_{\ell}^{RG}(r) = \left(1 - \frac{2M}{r}\right) \left[\frac{\ell(\ell + 1)}{r^2} - \frac{6M}{r^3}\right], \quad (4)$$

and the Zerilli potential (for the polar parity),

$$V_{\ell}^{Z}(r) = \frac{2}{r^2} \left(1 - \frac{2m}{r}\right) \times$$

$$\times \frac{9M^3 + 3L^2Mr^2 + L^2(1 + L)r^3 + 9M^2Lr}{(3M + Mr)^2}, \quad (5)$$

with $L = \ell(\ell + 1)/2 - 1$. The remarkable fact – known as isospectrality – is that those equations share the same spectrum of quasinormal modes. This is also true for the Reissner-Nordström and Kerr metrics. This might appear as a kind of “miracle” when using the standard tensor formalism where the axial and polar perturbations are treated independently. But when one actually works in the Newmann-Penrose (NP) formalism [14], isospectrality appears as quite natural. However, not all kinds of spacetimes do lead to isospectrality. It is not yet fully clear whether this property is generic or happens as an incredible “stroke of luck” for classical BHs.
In [15], it was shown that isospectrality is broken down for general \( f(R) \) gravity. In the case of Lovelock black holes, isospectrality is roughly recovered but not exactly [16]. Isospectrality also fails in Schwarzschild-anti-de Sitter spacetimes [17] and in Chern-Simons gravity [18]. The presence of a dilaton also breaks isospectrality [19, 20]. Actually, a perturbative analysis shows that isospectrality seems to be quite generically broken [21].

In this article, we do not derive spectacular new results. On the contrary, we try to make clearer the historical derivation from Chandrasekhar [22] and to slightly extend the proof. As it will be explained later, it seems hard (maybe impossible) to show isospectrality for a general metric of the form

\[
ds^2 = B(r)dt^2 - B(r)^{-1}dr^2 - r^2d\Omega^2. \tag{6}
\]

For reasons that will be made clearer in the next sections, we show in this work that the specific choice

\[
B(r) = (1 + \alpha) - \frac{2M}{r} + \beta r, \tag{7}
\]

with \( \alpha \) and \( \beta \) two arbitrary constants, leads to isospectrality. This metric corresponds to the exact vacuum solution to conformal Weyl gravity, with a null constant \( k \) [23]. The isospectrality proof can be straightforwardly extended to the case of a charged BHs (the steps are the same than for going from Schwarzschild to Reissner-Nordström). This is just a small generalization of the original result but the proof is already non-trivial. In the first section, we review sufficient conditions for isospectrality. Then, we introduce the NP formalism which will be used to determined the radial equation. We then prove isospectrality for the metric (6) with the function \( B(r) \) given by Eq. (7).

**CONDITIONS FOR ISOSPECTRALITY**

To study black hole perturbations, we separate the angular part so as to obtain a wave equation for radial and time variables. This equation has a Schrödinger-like form:

\[
\frac{d^2Z}{dr^*^2} + \omega^2Z - VZ = 0, \tag{8}
\]

with \( r_* \) the tortoise coordinate defined by \( dr_* = dr/B(r) \). In full generality, if \( Z_2 \) satisfies Eq. (8) with a potential \( V_2 \), then

\[
Z_1 = pZ_2 + q\frac{dZ_2}{dr^*}, \tag{9}
\]

with \( p \) and \( q \) two functions, also satisfies Eq. (8) with \( V_1 \) if

\[
V_1 = V_2 + \frac{2}{q}\frac{dp}{dr^*} + \frac{1}{q}\frac{dq}{dr^*}, \tag{10}
\]

and

\[
2p\frac{dp}{dr^*} + p\frac{d^2q}{dr^*^2} - \frac{d^2p}{dr^*^2}q - 2q\frac{dq}{dr^*}(V_2 - \omega^2) - q^2\frac{dV_2}{dr^*} = 0. \tag{11}
\]

Equation (11) is equivalent to

\[
p^2 + \left(p\frac{dq}{dr^*} - \frac{dp}{dr^*}q\right) - q^2(V_2 - \omega^2) = C^2 = \text{cte.} \tag{12}
\]

To show that Eq. (10) and Eq. (11) imply isospectrality, we use the fact that \( Z_2 \) satisfies Eq. (8), which implies

\[
\frac{d^3Z_2}{dr^*^3} + \omega^2\frac{dZ_2}{dr^*} - \frac{dV_2}{dr^*}Z_2 - V_2\frac{dZ_2}{dr^*} = 0. \tag{13}
\]

When replacing \( Z_1 \) and \( V_1 \) by their expressions given by Eq. (9) and Eq. (10), we are led to

\[
\frac{d^2Z_1}{dr^*^2} + \omega^2Z_1 - V_1Z_1 =
\left(\frac{d^2p}{dr^*^2} - 2p\frac{dp}{dr^*} - p\frac{d^2q}{dr^*^2}\right)Z_2
+ q\frac{dV_2}{dr^*}Z_2 + 2\frac{dq}{dr^*}\frac{d^2Z_2}{dr^*^2} = 0. \tag{14}
\]

Using Eq. (11) and Eq. (13), one can conclude that \( Z_1 \) satisfies Eq. (8) with \( V_1 \).

We will first establish the radial equation governing the gravitational perturbations, then it will be transformed into a wave equation. Finally we will show isospectrality by finding functions \( p \) and \( q \) satisfying Eq. (10) and Eq. (12) for the potentials of axial and polar perturbations.

**THE NEWMAN-PENROSE FORMALISM**

To go ahead, the perturbations need to be analyzed in the NP formalism [14]. This is a special case of the tetrad formalism (see, e.g., [24]). To guide the unfamiliar reader, we make every step leading to the result explicit. In this approach, one needs to set up a basis of four null vectors at each point of spacetime. This basis is made of a pair of real null vectors \( \mathbf{i} \) and \( \mathbf{n} \) and a pair of complex conjugate null vectors \( \mathbf{m} \) and \( \overline{\mathbf{m}} \).
1. \( n \cdot n = m \cdot m = \bar{m} \cdot \bar{m} = 0. \) (15)

Furthermore, these vectors satisfy the following orthogonality relations:

\[ l \cdot m = 1, m \cdot \bar{m} = n \cdot \bar{m} = 0. \] (16)

We also require the normalization:

\[ l \cdot l = 1 \quad \text{and} \quad m \cdot \bar{m} = -1, \] (17)

but this latter condition is less crucial in the NP formalism. The number of equations is conveniently reduced thanks to the use of complex numbers. Any basis with the properties given by Eqs (15), (16) and (17) can be used. For example, in the Schwarzschild case one usually works with the Kinnersley tetrad and sometimes the Carter one [25]. Here, we choose a Kinnersley-like tetrad:

\[ l^i = \left( \frac{1}{B(r)}, 1, 0, 0 \right), \] (18)

\[ n^i = \left( \frac{1}{2}, \frac{B(r)}{2}, 0, 0 \right), \] (19)

\[ m^i = \left( 0, 0, \frac{i}{\sqrt{2 r}}, \frac{1}{\sqrt{2 r} \sin \theta} \right), \] (20)

\[ \bar{m}^i = \left( 0, 0, \frac{i}{\sqrt{2 r}}, \frac{1}{\sqrt{2 r} \sin \theta} \right). \] (21)

In the NP formalism, the directional derivatives are usually denoted by the following symbols:

\[ D = l^i \partial_i; \quad \Delta = n^i \partial_i; \quad \delta = m^i \partial_i; \quad \delta^* = \bar{m}^i \partial_i. \] (22)

The equations will be written with the so-called spin coefficients [26] carrying (roughly speaking) the information on the Riemann tensor. To make things explicit, we switch, here, to the more general framework of the standard tetrad formalism. The four contravariant vectors of the basis are \( e^i_a \), where \( a, b, c, \ldots \) are the tetrad indices, indicating the considered vector and \( i, j, k, \ldots \) are the tensor indices, indicating the considered composant (alternatively, one can also think to the lower index as an internal Lorentz one and consider the upper index as a coordinate one). The correspondance reads as \( e_1 = l, e_2 = n, e_3 = m \) and \( e_4 = \bar{m} \) with \( e^1 = e_2, e^2 = e_1, e^3 = -e_4 \) and \( e^4 = -e_3 \). For example, \( e_{123} \) represents the second composant of \( I \), derived with respect to \( \theta \). We define the Ricci rotation coefficients

\[ \gamma_{cab} = e_c^b e_{ak;i}e^i_a. \] (23)

or equivalently

\[ e_{ak;i} = e^b_c \gamma_{cab} e^i_a. \] (24)

These coefficients are antisymmetric with respect to the first pair of indices:

\[ \gamma_{cab} = -\gamma_{acb}. \] (25)

Let \( X, Y, Z \) be contravariant vector fields: \( X, Y, Z \in T^1_0 \). The Riemann tensor field \( R \) is of type \( (1, 3) \):

\[ R : T^1_0 \times T^1_0 \times T^1_0 \to T^1_0. \] (26)

It is defined as

\[ R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z, \] (27)

with the Ricci identity

\[ R^i_{jkl} Z_i = Z_{j;k:l} - Z_{j;l:k}. \] (28)

This leads, for \( Z = e_a \), to:

\[ R_{ijkl} e_a^j = e_a^{aj;kl} - e_a^{aj;lk}. \] (29)

We project this identity on the tetrad frame and use Eq. (23), Eq. (24) and Eq. (25). The projected Riemann tensor depends only of the rotation coefficients and their derivatives:

\[ R_{abcd} = R_{ijkl} e^i_a e^j_b e^k_c e^l_d \]
\[ = \left[ e_{ak;dl} - e_{a;k;l} \right] e^i_a e^j_b e^k_c e^l_d \]
\[ = \left( -\left[ \gamma_{afg} e^f_j e^g_k \right] + \left[ \gamma_{afg} e^f_j e^g_k \right] \right) e^i_a e^j_b e^k_c e^l_d \]
\[ = -\gamma_{abc,d} + \gamma_{abd,c} + \gamma_{bab}(\gamma_d^c - \gamma_d^f) \]
\[ + \gamma_{bac}(\gamma_d^c - \gamma_d^f). \] (30)

The spin coefficients of the NP formalism are also defined through the rotation coefficients:

\[ \kappa = \gamma_{311}, \quad \rho = \gamma_{314}, \quad \epsilon = \frac{1}{2}(\gamma_{211} + \gamma_{341}), \]
\[ \sigma = \gamma_{313}, \quad \mu = \gamma_{243}, \quad \gamma = \frac{1}{2}(\gamma_{212} + \gamma_{342}), \]
\[ \lambda = \gamma_{244}, \quad \tau = \gamma_{312}, \quad \alpha = \frac{1}{2}(\gamma_{214} + \gamma_{344}), \]
\[ \nu = \gamma_{242}, \quad \pi = \gamma_{241}, \quad \beta = \frac{1}{2}(\gamma_{213} + \gamma_{343}). \]
Obviously, the α and β coefficients appearing here are not the ones of Eq. (7), but no ambiguity will appear in the following. The 36 equations (31) can be written as 18 complex equations. The 10 independent components of the Weyl tensor $C_{abcd}$ are represented by five complex scalars:

\[ \Psi_0 = -C_{1313} = -C_{pqrs} i^p m^q i^r m^s, \]
\[ \Psi_1 = -C_{1213} = -C_{pqrs} i^p i^q m^r m^s, \]
\[ \Psi_2 = -C_{1342} = -C_{pqrs} i^p m^q m^r i^s, \]
\[ \Psi_3 = -C_{1242} = -C_{pqrs} i^p i^q i^r m^s, \]
\[ \Psi_4 = -C_{2424} = -C_{pqrs} i^p i^q i^r i^s, \]

and the 20 linearly independent Bianchi identities can be written as eight complex and four real equations.

**Preliminaries on Isospectrality**

**Derivation of the radial equation**

We assume that the perturbations have a $t$ and $φ$ dependence given by $e^{i(ωt+mφ)}$ and we define the following operators:

\[ D_n = \partial_t + \frac{iω}{B(r)} + n \left( \frac{B'(r)}{B(r)} + \frac{2}{r} \right), \]  
(33)

and

\[ L_n = \partial_φ + \frac{m}{\sin(θ)} + ncot(θ). \]  
(34)

The prime denotes the derivative with respect to $r$. Let $D_n^\dagger$ be the complex conjugate of $D_n$ and $L_n^\dagger(θ) = -L_n(π - θ)$. It is interesting to notice that

\[ Br^2 D_n D_{n+1} = D_n Br^2. \]  
(35)

The directional derivative given by Eq. (22) reads

\[ D = D_0, \quad Δ = \frac{B(r)}{2} D_0^\dagger, \]
\[ δ = \frac{1}{\sqrt{2r}} L_0^\dagger, \quad δ^* = \frac{1}{\sqrt{2r}} L_0. \]  
(36)

We calculate the five scalars:

\[ Ψ_2 = \frac{B''}{2}, \]  
(37)

\[ Ψ_0 = Ψ_1 = Ψ_3 = Ψ_4 = 0. \]  
(38)

As $Ψ_0$, $Ψ_1$, $Ψ_3$ and $Ψ_4$ vanish but not $Ψ_2$, the spacetime defined by Eq. (6) is a Petrov type D spacetime. A corollary of the Goldberg-Sachs theorem [27] shows that it implies that $κ$, $σ$, $λ$, and $ν$ do vanish. The explicit calculation indeed leads to:

\[ κ = σ = λ = ν = 0, \]  
(39)

\[ τ = π = ε = 0 \]  
(40)

and

\[ ρ = -\frac{1}{r}, \quad μ = -\frac{B}{2r}, \quad γ = \frac{B'}{4}, \]  
(41)

\[ α = -β = \frac{cotθ}{2√2}. \]  
(42)

There are 6 linearized equations, 2 from the Ricci identities and 4 from the Bianchi identities. They read as

\[ (δ^* - 4α + π)Ψ_0 - (D - 2ε - 4ε)Ψ_1 = 3κΨ_2, \]
\[ (Δ - 4γ + μ)Ψ_0 - (δ - 4τ - 2β)Ψ_1 = 3σΨ_2, \]
\[ (D - ρ - ρ^* - 3ε + ε^*)σ - (δ - τ + π^* - α^* - 3β)κ = Ψ_θ, \]  
(43)

and

\[ (D + 4ε - ρ)Ψ_4 - (δ^* + 4π + 2α)Ψ_3 = -3λΨ_2, \]
\[ (δ + 4β - τ)Ψ_4 - (Δ + 2γ + 4μ)Ψ_3 = -3νΨ_2, \]
\[ (Δ + μ + μ^* + 3γ - γ^*)λ - (δ^* + 3α + β^* + π - τ^*)ν = -Ψ_4, \]  
(44)

where $Ψ_0$, $Ψ_1$, $Ψ_3$, $Ψ_4$, $κ$, $σ$, $λ$, and $ν$ are the perturbations. Using Eq. (37), Eq. (40) and Eq. (42), we obtain:

\[ \frac{1}{r \sqrt{2}} (L_0 + 2cotθ) Ψ_0 - (D_0 + \frac{4}{r}) Ψ_1 = \frac{3B''}{4} κ, \]  
(45)

\[ -\frac{B}{2} (D_0 + \frac{2B'}{B} + \frac{1}{r}) Ψ_0 - \frac{1}{r \sqrt{2}} (L_0 - cotθ) Ψ_1 = \frac{3B''}{4} σ, \]  
(46)

\[ (D_0 + \frac{2}{r}) δ - \frac{1}{r \sqrt{2}} (L_0 - cotθ) κ = Ψ_0, \]  
(47)

\[ (D_0 + \frac{1}{r}) Ψ_4 - \frac{1}{r \sqrt{2}} (L_0 - cotθ) Ψ_3 = \frac{3B''}{4} λ, \]  
(48)

\[ \frac{1}{r \sqrt{2}} (L_0 + 2cotθ) Ψ_4 + \frac{B}{2} (D_0^\dagger - \frac{B'}{B} + \frac{4}{r}) Ψ_3 = \frac{3B''}{4} ν, \]  
(49)

\[ -\frac{B}{2} (D_0^\dagger - \frac{B'}{B} + \frac{4}{r}) λ - \frac{1}{r \sqrt{2}} (L_0 - cotθ) ν = Ψ_4. \]  
(50)
We proceed to the following change of variables:

\[ \Phi_0 = \Psi_0, \quad \Phi_1 = \Psi_1 r \sqrt{2}, \quad k = \frac{\kappa}{r^2 \sqrt{2}}, \quad s = \frac{\sigma}{r}, \quad (54) \]
\[ \Phi_4 = \Psi_4 r^4, \quad \Phi_3 = \Psi_3 \frac{r^3}{\sqrt{2}}, \quad l = \frac{\lambda r}{2}, \quad n = \frac{\nu r^2}{\sqrt{2}} \quad (55) \]

This leads to:

\[ \mathcal{L}_2 \Phi_0 - \left( \mathcal{D}_0 + \frac{3}{r} \right) \Phi_1 = \frac{3B''}{2} r^3 k = -6 M k, \quad (56) \]
\[ B r^2 \left( \mathcal{D}_2 - \frac{3}{r} \right) \Phi_0 + \mathcal{L}_1^1 \Phi_1 = -\frac{3B''}{2} r^3 s = 6 M s, \quad (57) \]
\[ \left( \mathcal{D}_0 + \frac{3}{r} \right) s - \mathcal{L}_1^1 \Phi_0 = \frac{\Phi_0}{r}, \quad (58) \]
\[ \left( \mathcal{D}_0 - \frac{3}{r} \right) \Phi_4 - \mathcal{L}_1^1 \Phi_3 = -\frac{3B''}{2} r^3 l = 6 M l, \quad (59) \]
\[ \mathcal{L}_4^1 \Phi_4 + \frac{B r^2}{2} \left( \mathcal{D}_1^1 - \frac{3}{r} \right) \Phi_3 = -\frac{3B''}{2} r^3 n = 6 M n, \quad (60) \]
\[ B r^2 \left( \mathcal{D}_1^1 - \frac{3}{r} \right) l + \mathcal{L}_1^1 n = \frac{\Phi_4}{r}. \quad (61) \]

By applying \( \mathcal{L}_1^1 \) to Eq. (56) and \( \left( \mathcal{D}_0 + \frac{3}{r} \right) \) to Eq. (57), we are led to:

\[ \mathcal{L}_1^1 \mathcal{L}_2 \Phi_0 + \left( \mathcal{D}_0 + \frac{3}{r} \right) B r^2 \left( \mathcal{D}_2 - \frac{3}{r} \right) \Phi_0 \]
\[ = \frac{3B''}{2} r^3 \mathcal{L}_1^1 k - \frac{3}{2} \left( \mathcal{D}_0 + \frac{3}{r} \right) B'' r^3 s \]
\[ = 6 M \left( \mathcal{D}_0 + \frac{3}{r} \right) s - \mathcal{L}_1^1 k \]
\[ = 6 M \frac{\Phi_0}{r}. \quad (62) \]

It should be noticed that because \( B(r) \) is defined as in Eq. (7), \( B'' r^3 \) is a constant and can be factorized out in the previous equation. Then, the left part of Eq. (58) does appear and can be replaced by \( \Phi_0 / r \) which leads to a decoupled equation for \( \Phi_0 \). The key-point is the condition

\[ B''(r) r^3 = \text{cte.} \quad (63) \]

The constant is usually set to \( 6M \) but another value would just be a rescaling of the mass. In the same way, by applying \( \mathcal{L}_1 \) to Eq. (60) and \( B r^2 \left( \mathcal{D}_1^1 + \frac{3}{r} \right) \) to Eq. (59), we obtain:

\[ B r^2 \left( \mathcal{D}_1^1 + \frac{3}{r} \right) \left( \mathcal{D}_0 - \frac{3}{r} \right) \Phi_4 + \mathcal{L}_1 \mathcal{L}_2 \Phi_4 \]
\[ = B r^2 \left( \mathcal{D}_1^1 + \frac{3}{r} \right) \left( -\frac{3B''}{2} r^3 l \right) + \mathcal{L}_1 \left( \frac{3B''}{2} r^3 n \right) \]
\[ = 6 M \frac{\Phi_4}{r}. \quad (64) \]

where Eq. (61) has also been used. The condition (63) leads to two independent equations for \( \Phi_0 \) and \( \Phi_4 \). If \( B(r) \) were not fulfilling Eq. (7) – in particular if terms in \( r^n \) with \( n \geq 2 \) were present – it would not have been possible to obtain such decoupled equations.

In the Reissner-Nordström case, the condition (63) does not hold but it is possible to find new variables that mix the \( \Phi_i \) functions with the spin coefficients so that a separation is possible [22]. In this case it works because \( \Psi_1 \) does not vanish and this implies two more equations which lead to a radial equation of the form of Eq. (79) such that \( P \) and \( Q \) lead to isospectrality. As far as our argument is concerned, the extension from the Schwarzschild case to the charged case is therefore straightforward.

**PROOF OF ISOSPECTRALITY**

The equations (62) and (64) read

\[ \left[ \mathcal{L}_1 \mathcal{L}_2 + B r^2 \mathcal{D}_1 \mathcal{D}_1^1 + (3\beta - 6i\omega) r \right] \Phi_0 = 0, \quad (65) \]
\[ \left[ \mathcal{L}_1^1 \mathcal{L}_2 + B r^2 \mathcal{D}_1^1 \mathcal{D}_0 + (3\beta + 6i\omega) r \right] \Phi_4 = 0. \quad (66) \]

If we set

\[ \Phi_0 = R_{+2}(r) S_{+2}(\theta), \quad \Phi_4 = R_{-2}(r) S_{-2}(\theta), \quad (67) \]

they are separable with a separation constant \( \mu^2 \). This leads to:

\[ \mathcal{L}_1 \mathcal{L}_2 S_{+2} = -\mu^2 S_{+2}, \quad (68) \]
\[ \left[ B r^2 \mathcal{D}_1 \mathcal{D}_1^1 + (3\beta - 6i\omega) r \right] R_{+2} = \mu^2 R_{+2}, \quad (69) \]
\[ \mathcal{L}_1^1 \mathcal{L}_2 S_{-2} = -\mu^2 S_{-2}, \quad (70) \]
\[ \left[ B r^2 \mathcal{D}_1^1 \mathcal{D}_0 + (3\beta + 6i\omega) r \right] R_{-2} = \mu^2 R_{-2}. \quad (71) \]

The separation constant is calculated with Eq. (68) or Eq. (70) – by required the regularity of \( S_{+2} \) at \( \theta = 0 \) and \( \theta = \pi \). The angular equation is the same
as in the Schwarzschild case, which gives $\mu^2 = l(l+1) - 2$.

We set

$$D_0 = \frac{1}{B} \Lambda_+ , \quad D_0^\dagger = \frac{1}{B} \Lambda_-.$$  \hfill (72)

Using the the tortoise coordinate $r^*$ (with $\frac{d}{dr^*} = B \frac{d}{dr}$), we are led to

$$\Lambda_+ = \frac{d}{dr^*} + i\omega, \quad \Lambda_- = \frac{d}{dr^*} - i\omega \quad \text{and} \quad \Lambda^2 = \Lambda_+ \Lambda_-,$$

that is

$$\Lambda_+ = \Lambda_+ \pm 2i\omega.$$  \hfill (74)

It should be pointed that the equation

$$\left[ Br^2 D_{-1} D_0^\dagger + (3\beta - 6i\omega) r \right] B^2 r^4 R_{+2} = \mu^2 B^2 r^4 R_{+2}$$  \hfill (75)

is the same than Eq. (69). Using the properties of Eq. (35), we obtain

$$Br^2 D_{-1} D_0^\dagger = (Br^2)^2 D_0 \frac{1}{B r^2} D_0^\dagger = r^4 B \Lambda_+ \left( \frac{1}{B^2} \frac{d}{dr^*} \Lambda_- \right).$$  \hfill (76)

Defining $Y$ as

$$Y = B^2 r R_{+2},$$  \hfill (77)

we are led to

$$\Lambda_+ \left( \frac{1}{B^2} \Lambda_-(r^3 Y) \right) = \frac{r}{B^2} \Lambda^2 Y + \left( \frac{r}{B^2} \right) \Lambda_+ Y + 3 \left( \frac{3}{B} \right) Y.$$

By calculating the derivative and replacing $\Lambda_+$ by $\Lambda_- + 2i\omega$ in Eq. (78), we find that Eq. (75) is equivalent to

$$\Lambda^2 Y + P \Lambda_- Y - Q Y = 0,$$  \hfill (79)

with

$$P = \left( \frac{4B}{r} - 2B' \right) = \frac{B^2 d}{dr^*} \left( \frac{r^4}{B^2} \right) = \frac{d}{dr^*} \left( \log \left( \frac{r^4}{B^2} \right) \right).$$  \hfill (80)

and

$$Q = \left( \frac{2}{r} B B' - 3\beta \frac{B}{r} + \mu^2 \frac{B}{r^2} \right).$$  \hfill (81)

For the same reasons, $Y_{-2} = r^{-3} R_{-2}$, satisfies

$$\Lambda^2 Y_{-2} + P \Lambda_+ Y_{-2} - Q Y_{-2} = 0.$$  \hfill (82)

Equation (79) needs to be transformed into a wave equation in one dimension:

$$\Lambda^2 Z = V Z.$$  \hfill (83)

The functions $Y$ and $Z$ both satisfying a second order equation, we write $Y$ as is a linear combination of $Z$ and its derivative:

$$Y = \zeta \Lambda_+ \Lambda_+ Z + W \Lambda_+ Z = \zeta V Z + (W + 2i\omega \zeta) \Lambda_+ Z,$$  \hfill (84)

with $\zeta$ and $W$ two functions of $r^*$. Applying $\Lambda_-$ to Eq. (84) yields

$$\Lambda_- Y = \left[ \frac{d}{dr^*} (\zeta V) + W V \right] Z + \left[ \zeta V + \frac{d}{dr^*} (W + 2i\omega \zeta) \right] \Lambda_+ Z$$

$$= - \gamma \frac{B^2}{r^4} Z + R \Lambda_+ Z,$$  \hfill (85)

with

$$R = \zeta V + \frac{d}{dr^*} (W + 2i\omega \zeta),$$  \hfill (86)

$$\gamma = - \frac{r^4}{B^2} \left( \frac{d}{dr^*} (\zeta V) + W V \right).$$  \hfill (87)

By applying again $\Lambda_-$ to Eq. (85), we obtain

$$\Lambda_- \Lambda_- Y = \left[ - \frac{B^2}{r^4} \frac{d R}{dr^*} \right] \Lambda_+ Z + \left[ 2i\omega \gamma \frac{B^2}{r^4} - \frac{d \gamma}{dr^*} \frac{B^2}{r^4} - \gamma \frac{d}{dr^*} \left( \frac{B^2}{r^4} \right) + R V \right] Z.$$  \hfill (88)

On the other hand, one can notice that Eq. (79) leads to:

$$\Lambda_- \Lambda_- Y = -(P + 2i\omega) \Lambda_- Y + Q Y$$

$$= \left[ - (P + 2i\omega) R + Q (W + 2i\omega \zeta) \right] \Lambda_+ Z$$

$$+ \left[ (P + 2i\omega) \frac{\gamma B^2}{r^4} + Q V \right] Z.$$  \hfill (89)

Identifying Eq. (88) and Eq. (89), and by using the definition of $P$ given by Eq. (80), we find:
\[-\frac{d^2 B^2}{dr^* r^4} - \gamma \frac{d B^2}{dr^* r^4} + RV\]
\[= \frac{d}{dr^*} \left( \log \left( \frac{r^4}{B^2} \right) \right) \gamma B^2 + Q \zeta V, \tag{90}\]
which gives

\[-\frac{d^2 B^2}{dr^* r^4} = (Q - R)V, \tag{91}\]

and

\[\frac{dR}{dr^*} - \frac{B^2}{r^4} \gamma = Q(W + 2i\omega \zeta) - (P + 2i\omega)R, \tag{92}\]

\[\frac{r^4}{B^2} \frac{dR}{dr^*} + r^4 \frac{d}{dr^*} \left( \log \left( \frac{r^4}{B^2} \right) \right) R = \gamma + \frac{r^4}{B^2} Q(W + 2i\omega \zeta) - 2i\omega r^4 B^2 R, \tag{93}\]

\[\frac{d}{dr^*} \left( \frac{r^4}{B^2} R \right) = \gamma + \frac{r^4}{B^2} \left( Q(W + 2i\omega \zeta) - 2i\omega R \right). \tag{94}\]

The combination \(\zeta V \times \text{Eq. (94)} + R \times \text{Eq. (87)} - \gamma \times \text{Eq. (86)} - \frac{r^4}{B^2}(W + 2i\omega \zeta) \times \text{Eq. (91)}\) leads to

\[\zeta V \frac{d}{dr^*} \left( \frac{r^4}{B^2} R \right) + \frac{r^4}{B^2} R \frac{d(\zeta V)}{dr^*} + \gamma \frac{d}{dr^*} (W + 2i\omega \zeta) + (W + 2i\omega \zeta) \frac{d^2}{dr^*} = 0,\]

that is to say

\[\frac{r^4}{B^2} R \zeta V + \gamma (W + 2i\omega \zeta) = K = \text{cte.} \tag{96}\]

As we have written \(Y\) as a linear combination of \(Z\) and \(\Lambda + Z\) in Eq. (84), it is possible to write \(Z\) as a linear combination of \(Y\) and \(\Lambda + Y\). Using Eq. (84) and Eq. (85):

\[KZ = \frac{r^4}{B^2} R \zeta V Z + \gamma (W + 2i\omega \zeta) Z\]
\[= \frac{r^4}{B^2} RY - \frac{r^4}{B^2} (W + 2i\omega \zeta)(\Lambda - Y + \gamma B^2 \frac{Z}{r^4} Z) + \gamma (W + 2i\omega \zeta) Z\]
\[= \frac{r^4}{B^2} RY - \frac{r^4}{B^2} (W + 2i\omega \zeta) \Lambda - Y, \tag{97}\]

and

\[K\Lambda + Z = \frac{r^4}{B^2} R \zeta V \Lambda + Z + \gamma (W + 2i\omega \zeta) \Lambda + Z\]
\[= \frac{r^4}{B^2} \zeta V \Lambda - Y + \gamma B^2 \frac{Z}{r^4} Z + \gamma Y - \gamma V Z\]
\[= \frac{r^4}{B^2} \zeta V \Lambda - Y + \gamma Y. \tag{98}\]

By requiring \(\gamma = \text{cte}\) and \(\zeta = 1\), Eq. (91) leads to

\[R = Q, \tag{99}\]

and from Eq. (86) one obtains

\[V = Q - \frac{dW}{dr^*}. \tag{100}\]

Equation (94) then leads to

\[\frac{d}{dr^*} \left( \frac{r^4}{B^2} R \right) = \gamma + \frac{r^4}{B^2} QW, \tag{101}\]

and Eq. (96) yields

\[\frac{r^4}{B^2} QV + \gamma W = K - 2i\omega \gamma = \kappa = \text{cte.} \tag{102}\]

Defining

\[F = \frac{r^4}{B^2} Q, \tag{103}\]

Eq. (101) and Eq. (102) lead to

\[W = \frac{1}{F} \left( \frac{dF}{dr^*} - \gamma \right), \tag{104}\]

and

\[FV + \gamma W = F \left( Q - \frac{dW}{dr^*} \right) + \gamma W = \kappa, \tag{105}\]

\[FQ - \frac{d}{dr^*} \left[ \frac{1}{F} \frac{dF}{dr^*} - \gamma \right] + \gamma \left( \frac{dF}{dr^*} - \gamma \right) = \kappa, \tag{106}\]

which gives

\[\frac{1}{F} \left( \frac{dF}{dr^*} \right)^2 - \frac{d^2 F}{dr^* 2} + \frac{B^2}{r^4} F^2 = \frac{\kappa^2}{F} + \kappa. \tag{107}\]
There exist constants $\gamma$ and $\kappa$ such that Eq. (107) is satisfied by the function (103). Depending on the square root of $\gamma^2$ chosen ($-\gamma$ or $+\gamma$), one is led to the equation for axial or polar perturbations. With

$$W^\pm = \frac{1}{F} \left( \frac{dF}{dr^*} \mp \gamma \right),$$

then

$$V^\pm = Q - \frac{d}{dr^*} \left( \frac{dF}{dr} \mp \gamma \right).$$

Defining $f \equiv \frac{1}{F}$,

$$V^\pm = \pm \gamma \frac{df}{dr^*} + \gamma^2 f^2 + \kappa f,$$

and

$$Y = V^\pm Z^\pm + (W^\pm + 2i\omega)\Lambda_+ Z^\pm,$$

$$\Lambda_- Y = \mp \frac{B^2}{r^2} Z^\pm + Q \Lambda_+ Z^\pm,$$

$$K^\pm = \kappa \mp 2i\omega \gamma,$$

$$K^\pm Z^\pm = \frac{r^4}{B^2} (Q Y - (W^\pm + 2i\omega)\Lambda_- Y),$$

$$K^\pm \Lambda_+ Z^\pm = \frac{r^4}{B^2} V^\pm \Lambda_- Y \pm \gamma Y.$$  

By inserting eq. (111) and Eq. (112) in Eq. (114), one obtains

$$K^- Z^- = \frac{r^4}{B^2} \left[ Q [V^+ Z^+ + (W^+ + 2i\omega)\Lambda_+ Z^+] ight. - (W^- + 2i\omega) [\gamma B^2 \frac{2}{r^3} Z^+ + Q \Lambda_+ Z^+] \left. \right] + \frac{r^4}{B^2} Q V^+ + \gamma (W^+ + 2i\omega) - \gamma (W^- - W^-) \right] Z^+,\n$$

$$+ F[W^+ - W^-] \Lambda_+ Z^+,\n$$

which simplifies to

$$(\kappa - 2i\omega \gamma) Z^- = (\kappa + 2\gamma^2 f) Z^+ - 2\gamma \frac{dZ^+}{dr^*}.$$  

Equivalently, one can show that

$$(\kappa + 2i\omega \gamma) Z^+ = (\kappa + 2\gamma^2 f) Z^- + 2\gamma \frac{dZ^-}{dr^*}.$$  

By identification with the previously given condition we are led to

$$q = 2\gamma \text{ and } p = \kappa + 2\gamma^2 f.$$  

Conditions given by Eq. (9), Eq. (10) and Eq. (11) are therefore respected.

The values of $\kappa$ and $\gamma$ when the metric function $B(r)$ is defined by (7) now need to be determined. First, one can notice that:

$$-F \left( \frac{d^2}{dr^* \log F} \right) = \frac{1}{F} \left( \frac{dF}{dr^*} \right)^2 - \frac{d^2 F}{dr^*},$$

which implies that Eq. (107) reads as

$$-F \left[ \frac{d^2}{dr^*} \left( \log F \right) - \frac{B^2}{r^4} \right] = \gamma^2 F + \kappa.$$  

Moreover, $F$ is given by

$$F = \frac{r}{B} (\mu^2 r + 6M).$$

This leads to

$$\frac{B}{F} \frac{dF}{dr} = \frac{\mu^2 r}{F} + \frac{1}{r^3} \left( 3Br^2 - \frac{d(Br^2)}{dr} r \right) \frac{F}{r^3} = \mu^2 r + \frac{1}{r^3} ((1 + \alpha) r^2 - 4Mr),$$

and

$$\frac{d}{dr} \left( \frac{B}{F} \frac{dF}{dr} \right) \frac{d}{dr} = -\frac{1}{r^2} + \frac{8M}{r^3} + \frac{\mu^2}{F} \left( 1 - \frac{r}{F} \frac{dF}{dr} \right) \frac{d}{dr} = -\frac{1}{r^2} + \frac{8M}{r^3} + \frac{\mu^2}{FBr^2} \left( -2Br^2 - \frac{d(Br^2)}{dr} r - \frac{\mu^2 r^4}{F} \right) \frac{d}{dr} = -\frac{1}{r^2} + \frac{8M}{r^3} + \frac{\mu^2}{FBr^2} \left( 2Mr + \beta r^3 - \frac{\mu^2 r^4}{F} \right),$$

(124)

together with

$$\frac{FB}{dF} \left( \frac{B}{F} \frac{dF}{dr} \right) \frac{d^2 F^2}{dr^*} = (1 + \alpha) \mu^2 + \mu^2 + \frac{6M}{r} (1 + \alpha) + \frac{2\mu^2}{r} - \frac{12M^2}{r^2} - \beta \gamma r^2 + \frac{\mu^2 r^2}{F} \frac{dF}{dr} = (1 + \alpha) \mu^2 + \mu^2 + \frac{1}{FB} \left( (1 + \alpha) \mu^2 r (6M + \mu^2 r) + (1 + \alpha) 36M^2 - 72 \frac{M^3}{r} - 6M \beta \mu^2 r^2 \right) \frac{dF}{dr}$$

(125)
We therefore obtain
\[ -F \left[ \frac{d^2}{dr^2} \left( \log F \right) - \frac{B^2}{r^4} F \right] = \mu^2 (2(1+\alpha) + \mu^2) - 6\mu^2 + \frac{36M^2}{F}. \] (126)

Identifying with Eq. (121), this means
\[ \gamma^2 = 36M^2 \quad \kappa = \mu^2 (2(1+\alpha) + \mu^2) - 6M\beta. \] (127)

We have the found \( p \) and \( q \) with Eq. (119) and Eq.(127), which proves the isospectrality for a metric such that \( B(r) \) satisfies Eq. (7).

The potentials can also be explicitly determined (from Eq. (110)) for both perturbations. We have \( \mu^2 = 2L \).

The axial perturbation are described by:
\[ V^- = \left( 1 + \alpha - \frac{2M}{r} + \beta r \right) \left[ \frac{l(l+1)}{r^2} - \frac{6M}{r^3} + \frac{2\alpha}{r^2} \right], \] (128)

while the polar perturbations feel the potential
\[ V^+ = \frac{2}{r^3} \left( 1 + \alpha - \frac{2M}{r} + \beta r \right) \times \frac{9M^3 + 3L^2Mr^2 + L^2(1+\alpha + L)r^3 + 9M^2Lr - 3M\beta Lr^3}{(Lr + 3M)^2}. \] (129)

One recovers the Regge-Wheeler and Zerilli potentials when \( \alpha \) and \( \beta \) vanish.

**PHANTOM GAUGE**

In this section, we briefly discuss the Phantom gauge. As we deal with six equations, namely Eqs. (56-61), and eight unknown variables, the solutions involve two arbitrary functions. This comes from the degrees of freedom associated with the rotation of the chosen tetrad. If first order infinitesimal rotations of the tetrad basis are performed, \( \Psi_0 \) and \( \Psi_4 \) are affected at the second order level while \( \Psi_1 \) and \( \Psi_3 \) are affected in the first order level (the interested reader can find a clear proof in [22], Chapter 17.(g) or through Eq. (7.79) in [28]). At the linear order which is considered here, \( \Psi_0 \) and \( \Psi_4 \) are therefore gauge invariant (not affected by infinitesimal rotations), contrarily to \( \Psi_1 \) and \( \Psi_3 \). We have chosen a gauge such that
\[ \Psi_1 = \Psi_3 = 0. \] (130)

The vanishing of \( \Psi_1 \) and \( \Psi_3 \) does not affect the behavior of \( \Psi_0 \) and \( \Psi_4 \). This gauge leads to the radial equations (62) and (64).

Another meaningful choice could have been done: the so-called “Phantom Gauge”. The previous gauge was useful to separate the equations when the condition given by Eq. (63) is fulfilled. However, if this condition is not respected it is still possible to obtain two decoupled equations. Thanks to the freedom associated with the rotation of the tetrad, one can imposed two additional (ad hoc) constraints. By applying \( Br^2 \left( D^1_2 - \frac{3}{r} \right) \) to Eq. (56) and \( L_2 \) to Eq.(57), it is possible eliminate \( \Phi_0 \). Then the condition
\[ -Br^2 \left( D^1_2 - \frac{3}{r} \right) \left( \frac{3}{2} B'' r^3 k \right) - L_2 \left( \frac{3}{2} B'' r^3 s \right) = 6\pi B' \Phi_1, \] (131)
gives
\[ [Br^2 D^1_2 D_0 - 6i\omega r + L_2 L^1_1] \Phi_1 = 0, \] (132)
and therefore
\[ [Br^2 D^1_2 D_0 - 6i\omega r] R_1 = 0. \] (133)

The same procedure can be followed for \( \Phi_3 \). This gauge might have appeared to be well suited to derive isospectrality for more general metrics, that is beyond the condition (63). The radial equation (133) can be written in the form (79) with \( Y \) defined by
\[ Y = rBR_1, \] (134)
as well as \( P \) and \( Q \) expressed by:
\[ P = \frac{d}{dr^2} \log \left( \frac{r^2}{B} \right), \] (135)
and
\[ Q = \frac{B}{r^2} (4\pi B' + r^2 B'' + 2B + \mu^2). \] (136)

However, in that case, it seems difficult (if not impossible) to find \( p \) and \( q \) so that Eq. (10) and Eq. (11) are fulfilled. One could follow the same procedure than previously and replace Eq. (86) and (87) with
\[ R_{PG} = \zeta_{PG} V + \frac{d}{dr^2} (W + 2i\omega \zeta_{PG}), \] (137)
\[ \gamma_{PG} = -\frac{r^2}{B} \left( \frac{d}{dr^2} (\zeta_{PG} V) + WV \right), \] (138)
where \( \frac{r^2}{B} \) appears instead of \( \frac{r^4}{Br} \). Then, Eq. (96) is replaced by
\[
\frac{r^2}{\mathcal{E}} R_{\mathcal{PG}} \zeta_{\mathcal{PG}} V + \gamma_{\mathcal{PG}} (W + 2i\omega_{\mathcal{PG}}) = K_{\mathcal{PG}} = \text{cte}.
\]  

(139)

However, it is not anymore possible to require \( \gamma_{\mathcal{PG}} = \text{cte} \) and \( \zeta_{\mathcal{PG}} = 1 \) as it has been done for \( \gamma \) and \( \zeta \) previously. Indeed, if \( \zeta_{\mathcal{PG}} = 1 \), then \( \gamma_{\mathcal{PG}} = \frac{1}{r^2} \gamma \) which cannot be constant. The phantom gauge does not seem to bring any new convenient way to go ahead.

**SUMMARY AND CONCLUSION**

Let us summarize the main ingredients of the calculation. The condition (63) allows to decoupled the equations (56-58) in the form of Eq. (79) with functions \( P \) and \( Q \) so that Eq. (96) is fulfilled. Thanks to Eq. (63), we are led to the expression of \( B \) given by Eq. (7) and this allows to write

\[
F \left[ \frac{d}{dr} \left( \log F \right) - \frac{B}{r^2} F \right]
\]

as \( \text{cte} + \frac{c^2}{r^2} \), leading to explicit expressions for \( p \) and \( q \) which show the isospectrality.

The main aim of this article was not to derive spectacular results but, instead, to go a bit deeper into the original argument from Chandrasekhar so as to make it accessible to the reader who wants to apply the method to a specific spacetime structure.

We took this opportunity to show the isospectrality of quasinormal modes for a metric slightly more general than the Schwarzschild one. Although this is in itself a little step, the path is far from trivial. It should be once again noticed that the proof would not have been possible if terms like \( r^n \) with \( n > 1 \) were present in the metric. This agree with the breaking down of isospectrality numerically shown in many different cases, including in S(A)dS spacetimes.

Isospectrality is a beautiful property which seems to be true only for very specific spacetimes. As far as we know, no analytical proof of isospectrality (or of the breakdown of isospectrality) as been produced yet in full generality. This article just made clearer the main ingredients required for isospectrality and pointed out the difficulties one has to face when trying to extend the proof to more general spacetimes.

[1] B. P. Abbott et al. (LIGO Scientific, Virgo), Phys. Rev. Lett. 116, 061102 (2016), 1602.03837.