A. Introduction

After more than twenty years of low energy supersymmetry and grand unification the minimal supersymmetric Grand Unified Theory (GUT) is still commonly assumed to be based on the SU(5) gauge group with three generations of 5 and 10 dimensional matter supermultiplets, and $5_H, \bar{5}_H$ and 24$_H$ dimensional Higgs supermultiplets \footnote{The “Higgs” superpotential contains 4 complex parameters. It has two sets of Yukawa couplings, and as is well-known predicts equal down quark masses and 9 complex Dirac Yukawa couplings.} . The “Higgs” superpotential contains 4 complex parameters. It has two sets of Yukawa couplings, and as is well-known predicts equal down quark masses and 9 complex Dirac Yukawa couplings. It accounts correctly for all the fermion masses and mixings. The theory predicts at low energies the MSSM with exact R-parity. It is arguably the minimal consistent supersymmetric grand unified theory.

We show that the minimal renormalizable supersymmetric SO(10) GUT with the usual three generations of spinors has a Higgs sector consisting only of a “light” 10-dimensional and “heavy” 126, 210 and 210 supermultiplets. The theory has only two sets of Yukawa couplings with fifteen real parameters and ten real parameters in the Higgs superpotential. It accounts correctly for all the fermion masses and mixings. The theory predicts at low energies the MSSM with exact R-parity.

On top of that, this minimal GUT is plagued by R-parity violating couplings. If these are set to zero, neutrinos are predicted to be massless, which in view of atmospheric and solar neutrino data seems an untenable position. To correct for that, there are the following options:

(i) add the effective, non-renormalizable, couplings $f_{ij} \overline{5}_H \overline{5}_H / M$, where $M \gg M_{GUT}$. The natural choice $M = M_{Pl}$ fails, since the data strongly suggest $M < M_{GUT}$.

(ii) add the right-handed neutrinos, which means their 3 masses and 9 complex Yukawa couplings.

(iii) add $15_H$ and $\overline{15}_H$ dimensional Higgs superfields with 6 complex Yukawa couplings, $f_i \overline{5}_H \overline{15}_H$, and 4 more complex couplings among $15_H, \overline{15}_H, \overline{5}_H, 5_H$ and 24$_H$.

Let us count the number of parameters for any of these possible versions of the minimal realistic supersymmetric SU(5) theory. We can diagonalize down quark (charged leptons) Yukawas, which means 3 real parameters. The symmetric up-quark Yukawas with the freedom of a global U(1) rotation of $5_H$ give us $6 \times 2 - 1 = 11$ real parameters. With the $\overline{5}_H$ and 24 fields in $W_H$ we can redefine two phases, thus $4 \times 2 - 2 = 6$ real parameters in $W_H$. So, with massless neutrinos we have already

$$14(W_Y) + 6(W_H) + 1(gauge) = 21 \text{ real parameters.}$$

Add neutrino masses and you get

(i) $6 \times 2 = 12$ more real parameters in $W_Y$ \rightarrow 33 real parameters in total;

(ii) $3 + 9 \times 2 = 21$ more real parameters in $W_Y$ \rightarrow 42 real parameters in total;

(iii) $6 \times 2 = 12$ more real parameters in $W_Y$, $4 \times 2 - 2 = 6$ real parameters in $W_H$ and then the total is 39 real parameters.

Strictly speaking, (i) should be discarded; a realistic theory has at least 39 parameters. We keep (i), though, in order to emphasize the minimality of the SO(10) theory which we discuss below.

It is instructive to compare SU(5) with the minimal supersymmetric standard model (MSSM) with massive neutrinos. By this we mean effective neutrino mass operators of the type (i) mentioned in the case of SU(5), which amounts to effective Majorana masses for neutrinos. As usual, we assume R-parity (otherwise you have many more parameters). We have 6 quark masses, 3 quark mixings and 1 CP phase, 6 lepton masses, 3 lepton mixings and 3 CP phases, so in total 22 real parameters in $W_Y$. With 3 gauge couplings and 1 real parameter ($\mu$ term) in $W_H$, we get in total

$$26 \text{ real parameters in MSSM.}$$

Notice that we did not count the soft terms, they ought to be included separately, but they are present in any theory.

In an analogous manner we construct in this letter the minimal renormalizable supersymmetric SO(10) theory. We find, much to our surprise, that the theory has 26 real parameters, equal to the MSSM, and much less than the minimal SU(5). And yet it consistently describes all the low-energy phenomena, in particular the fermionic masses and mixings. It is also a theory of R-parity and it predicts its conservation to all orders in perturbation theory. In short, it is the minimal supersymmetric grand unified theory based on a single gauge group without additional ad-hoc symmetries. The further tests of the theory will be provided by $d = 5$ proton decay, lepton genesis and flavour violation processes.

At the first glance, this result of SO(10) being a minimal theory, more economical, simpler than SU(5), may appear as a contradiction in terms. After all, SU(5) is a smaller gauge group. Here, it may be useful to recall a lesson from the SO(3) model of leptons of Georgi and...
In spite of being a smaller gauge group than the SU(2)×U(1) theory, it is a much more complicated theory with many more parameters in the Yukawa sector. In this sense, the SM is the minimal electro-weak theory, not just the minimal correct one.

In this work, we describe the salient features of the minimal SO(10) theory. We pay special attention to the symmetry breaking analysis and the study of fermionic masses and mixings.

B. The theory

We construct now a renormalizable SO(10) theory. The reader may ask why necessarily renormalizable. Why not take 1/Mπ terms? After all they could play a nontrivial role in fermion masses and mixings. However in this case we lose both minimality and predictivity. We have a nice example of the principle of renormalizability: the minimal standard model itself. Both its great phenomenological success and its eventual failure in predicting massless neutrinos are very useful. At the same time you have a precise starting point and a window toward the physics at high energies. We believe thus that it is rather important to extend this to the theory of grand unification and test it to the bitter end. Let experiment decide its fate and hopefully open a window for new physics even beyond, be it strongly interacting new dynamics, quantum gravity, superstrings, whatever.

It is well-known that this requires 126 (Σ) and 126 (Σ) supermultiplets: 126 alone produce right-handed neutrino masses and does not lead to anomalies, but its vacuum expectation value (VEV) leads to a non-vanishing D-term, and therefore to supersymmetry breaking; adding 126, supersymmetry can be preserved at a high scale by assuming (Σ) = (Σ). These VEVs are SU(5) singlets, so they leave SU(5) unbroken: thus more Higgs fields are needed. The minimal choice is the four-index antisymmetric representation 210 (Φ), that was first considered in [2]. Notice that neither 45 nor 54 by themselves can do the job at the renormalizable level, both are needed [3], leading to more parameters in the superpotential. In short, the minimal theory consists of Φ, Σ and Σ. The superpotential involving those heavy fields is

\[ W_H = \frac{m}{4!} \Phi_{ijkl} \Phi_{ijkl} + \frac{\lambda}{4!} \Phi_{ijkl} \Phi_{klmn} \Phi_{mnij} + M \Sigma_{ijklm} \Sigma_{ijklm} + \frac{\eta}{4!} \Phi_{ijkl} \Sigma_{ijmn} \Sigma_{klmno} \]  

and it has only 4 independent parameters. In other words, although the representations are large, the number of parameters is small; the theory is simple and economical. We argue in passing against the usual phobia of large representations: it is not size that matters!

It is convenient to decompose the Higgs superfields under the Pati-Salam maximal subgroup SU(2)_L×SU(2)_R×SU(4)_C (for a useful connection between SO(10) and Pati-Salam see [2]):

\[ \Phi \equiv 210 = (1, 1, 15) + (1, 1, 1) + (1, 3, 15) + (3, 1, 15) + (2, 2, 6) + (2, 2, 10) + (2, 2, \overline{10}) \]

\[ \Sigma \equiv 126 \equiv (1, 3, \overline{10}) + (3, 1, 10) + (1, 1, 6) + (2, 2, 15) \]

The physically allowed VEVs we call \( p \) for (1, 1, 1), \( a \) for (1, 1, 15), \( \omega \) for (1, 3, 15) and \( \sigma \) for (1, 3, \overline{10}) and (1, 3, 10). After some straightforward computation, the superpotential as a function of these VEVs becomes

\[ W_H = m (p^2 + 3a^2 + 6\omega^2) + 2\lambda (a^3 + 3p\omega^2 + 6a\omega^2) + M \sigma \sigma \sigma \sigma \sigma (p + 3a + 6\omega) . \]  

(2)

Vanishing of the D-terms (\( \Rightarrow \sigma = \sigma \) and the F-terms determines the supersymmetric vacua. The SU(5) symmetric vacuum \( p = a = \omega = \sigma = 0 \), \( a = -m/\lambda \) have simple expressions, while the standard model vacuum is

\[ p = \frac{m}{\lambda} \cdot x(1 - 5 x^2), \quad a = -\frac{m}{\lambda} \cdot (1 - 2x - x^2), \quad \omega = \frac{m}{\lambda} \cdot x, \quad \sigma = 2 \cdot \frac{m^2}{\eta \lambda} \cdot \frac{x (1 - 3x) (1 + x^2)}{(1 - x)^2} , \]

(3)

where \( x \) is a solution of the cubic equation:

\[ 3 - 14 + 15x^2 - 8x^3 = (x - 1)^2 \lambda \pi \eta \]

(4)

The most typical case has single step breaking. E.g., when \( \lambda \pi \eta \pi \eta \pi \eta \), we get \( p \sim -0.27 m/\lambda, a \sim -0.67 m/\lambda, \omega \sim -0.21 m/\lambda, \sigma \sim 0.51 m/\sqrt{\eta \lambda} \), that corresponds to an approximate single step breaking of SO(10)→MSSM. However, in some cases \( \sigma \), and thus the scale of right handed neutrino masses, is smaller than the other VEVs. E.g., this happens when \( x \sim 1/3 \), that is \( 3\lambda M \sim -2\pi \).

Before proceeding with the inclusion of the light Higgs and matter superfields, we discuss the situation regarding R-parity. Since it is equivalent to matter parity, \( M \equiv (-1)^{3-(H-\bar{H})} \), we discuss \( M \) in what follows. Under \( M \), 16 is odd and the rest even. On the other hand, SO(10) has a Z_4 center: 16 → 16, 10 → -10, 210 → 210, 126 → -126, 126 → -126. Clearly, \( M \in Z_4 \) and thus R-parity is an exact symmetry of gauge SO(10). Since \( \Phi, \Sigma \) and \( \Sigma \) are even under \( M \), R-parity continues to be an exact symmetry so far.

The “light” Higgs field we choose to be 10-dimensional (H), and the superpotential gets an additional piece

\[ \Delta W_H = h_H H^2 + \frac{1}{4!} \Phi_{ijkl} H_m (\alpha \Sigma_{ijklm} + \beta \Sigma_{ijklm}) \]

(5)

with three more complex parameters, seven in total.

With the matter fields being 16-dimensional spinors \( \Psi_a \) (\( a=1,2,3 \) is a generation index), we have two symmetric Yukawa couplings in generation space

\[ W_Y = \Psi (Y_{10} H + Y_{126} \Sigma) \Psi . \]  

(6)

We diagonalize one of them, which amounts to 3 real couplings; the other one then has \( 6 \times 2 = 12 \) more real parameters.
With four Higgs superfields we can redefine four phases of the seven complex couplings and so we have \(7 \times 2 - 4 = 10\) real parameters in the Higgs superpotential \(W_H\). Together with the Yukawa and gauge sector, this means \(15 + 10 + 1\), that is:

26 real parameters in minimal SO(10),

just as in the MSSM. This is a remarkable result. Furthermore, the theory is in perfect accord with all the quark and lepton masses and mixings, as we now discuss.

C. Fermion masses and mixings  
Naively, one imagines only the field \((2, 2, 1)\) in \(H\) to have a VEV at the electroweak scale. The situation is more subtle. It is easy to see that \((2, 2, 1)\) in \(H\) to have a VEV as it couples to fermions. The idea to use \(\mathbf{210}\) for this purpose was suggested long time ago \(\mathbf{\overline{16}}\) and the predictions for fermion masses were studied extensively \(\mathbf{\overline{3}}\). It was never realized though that \(\mathbf{210}\) alone is sufficient. What seemed to be an ugly model building, where one chooses the fields and interactions for a particular scope, turns out to be the minimal supersymmetric SO(10), or, better to say, the minimal supersymmetric grand unified theory.

What is really surprising is that the theory is still consistent with all the data even when one specifies the nature of the see-saw mechanism \(\mathbf{3}, \mathbf{11}, \mathbf{1}, \mathbf{12}, \mathbf{15}, \mathbf{14}\). As is well known, the see-saw may proceed through the VEV of \((1, 1, 15)\) in \(\mathbf{210}\), which is of order \(M_{\text{GUT}}\). The usual minimal fine tuning implies that one combination of the two L-R bidoublets is light and thus both \((2, 2, 1)\) and \((2, 2, 15)\) contribute to fermion masses. The effective state \(\mathbf{210}\) for this purpose was suggested long time ago \(\mathbf{\overline{16}}\) and the predictions for fermion masses were studied extensively \(\mathbf{\overline{3}}, \mathbf{\overline{\overline{3}}, \mathbf{\overline{\overline{3}}}\overline{\overline{3}}}\). It was never realized though that \(\mathbf{210}\) alone is sufficient. What seemed to be an ugly model building, where one chooses the fields and interactions for a particular scope, turns out to be the minimal supersymmetric SO(10), or, better to say, the minimal supersymmetric grand unified theory.

D. The fate of R-parity  
The theory is even more predictive. It uniquely determines the low-energy effective theory: it is MSSM with exact R-parity. As we have seen, R-parity remains unbroken at the first stage of symmetry breaking. The question is, what happens at the low energy supersymmetry breaking or electroweak scale. More precisely, one must know whether light sneutrinos get a VEV and thus break R-parity. This seems a hopeless task since it should depend on the theory of supersymmetry breaking and soft masses. Fortunately, a spontaneous breakdown of R-parity through the sneutrino VEV would result in the existence of a pseudo-Majoron with its mass inversely proportional to the right-handed sneutrino mass. This is ruled out by the Z decay width \(\mathbf{18}, \mathbf{19}\). This is completely analogous to the impossibility of breaking R-parity spontaneously in the MSSM, where the Majoron is strictly massless.

E. Proton decay  
Another important aspect of the theory regards proton decay. For simplicity, here we assume a single step breaking from the GUT symmetry down to the MSSM. The GUT scale then is, neglecting the GUT threshold effects, close to \(10^{16}\) GeV \(\mathbf{20}, \mathbf{21}, \mathbf{22}, \mathbf{23}\). In this case \(d = 6\) induced proton decay is on the slow side and will be a long time before being verified (for a recent attempt to increase this rate see for example \(\mathbf{24}\) ). On the other hand the \(d = 5\) decay tends to be quite fast \(\mathbf{25}, \mathbf{26}\). For example in minimal SU(5) the situation is dramatic \(\mathbf{27}, \mathbf{28}, \mathbf{29}\), although there are ways out \(\mathbf{30}, \mathbf{31}, \mathbf{32}\). This will clearly be one of the central tests of the theory. What are the ingredients that go into the analysis? First of all, there are a number of colour triplet supermultiplets responsible for the decay, as in other SO(10) model \(\mathbf{33}\). The effective rate depends on their masses and mixings, and also the mixings to other colour triplets, which do not directly couple to fermions. Those colour triplets that contribute directly live in \(\mathbf{10}\) and \(\mathbf{126}\), whereas the others that just mix with them live in \(\mathbf{210}\) and \(\mathbf{126}\). The detailed discussion is beyond the scope of this note, but it is extremely important and is being performed in detail. A related issue is the existence of intermediate symmetry breaking scales which in principle affect the triplet masses.

F. Leptogenesis  
An important appealing feature of the see-saw mechanism in general and SO(10) in particular is the celebrated mechanism of leptogenesis \(\mathbf{34}\). It would be nice to see how it works in this theory. The situation is quite complicated though. In most works one assumes the type I see-saw mechanism being responsible both for neutrino masses and leptogenesis. This of course does not have to be true at all and we have a number of possibilities depending of what dominates the neutrino masses and what is responsible for leptogenesis. Although the constraints from leptogenesis can not be taken as seriously as those from proton decay and even more fermion masses, certainly they are worth studying.

G. Conclusions  
In this letter we have presented what is according to us the minimal supersymmetric grand unified theory. In defining minimality we have used the criterion of the economy of the parameters or, equivalently, the power of predictability. We have stuck to the gauge principle, and demanded the simple gauge group to be the only symmetry of the theory. This enables one to have a well defined framework, which can test the idea of supersymmetric grand unification. In a sense, this minimal theory should be viewed as the GUT analogue of the standard model as the minimal theory of low-energy electro-weak phenomena.

The theory is based on the SO(10) gauge symmetry and contains the usual three generations of \(\mathbf{16}\)-dimensional matter superfields, a (light) \(\mathbf{10}\)-dimensional Higgs superfield and the (heavy) \(\mathbf{210}, \mathbf{126}\) and \(\mathbf{126}\) representation Higgses. The theory is blessed by a small number of parameters, 26 real ones in total, which makes it quite predictable. One often fears large representations, according to us for no valid reason. In fact large representations often mean less parameters. They also have an important characteristic of predicting the Landau pole above the physical scale in question. In this particular
case this happens at the scale $\Lambda_F$ and order of magnitude or so above $M_{\text{GUT}}$. Someone could be unhappy about this, but to us this is an interesting prediction of the theory: there is new, presumably strongly coupled dynamics not far from the unification scale. The scale $\Lambda_F$ as much as $M_P$ could easily leave imprints at the GUT scale. We decided to ignore them in order to be concrete. The theory makes clear predictions and experiment will tell us how good they are. One could also imagine a scenario with $\Lambda_F \sim M_{\text{GUT}}$, in which case the theory becomes strongly coupled.

Contrast this with what is often coined the minimal supersymmetric SO(10) theory based on $16_H, 16_H, 45_H, 10$ and non-renormalizable superpotentials $36, 34, 38, 39, 40$. At the order $1/M_P$ this theory has many more parameters; for example, it has 4 sets of Yukawa couplings (33 real parameters in total). This is clearly much less economical than the theory we presented. Obviously choosing particular textures is against the spirit of extracting information from the principle of grand unification; it has to do much more with flavour physics, which is outside grand unification. On top of that, it has to be augmented with extra discrete (or continuous) symmetries in order not to break R-parity at $M_{\text{GUT}}$.

We have shown that there is consistent single-step symmetry breaking down to the MSSM. The main outcome is the exact R-parity which guarantees the stability of the lightest supersymmetric partner and its possible role as the dark matter. Furthermore, the theory is completely in accord with all the data on fermion masses and mixings. In the limiting cases of type I or type II see-saw it predicts the 1-3 leptonic mixing angle to be about 0.16; whether this remains true for the general see-saw formula remains still to be seen. If it survives, it could be the smoking gun of the model. As we discussed above, another important test of the theory will be provided by the careful evaluation of proton decay, baryon asymmetry of the universe and flavour violating processes. We plan to address these questions in a future publication.

Acknowledgments

The work of A.M., G.S. and B.B. is supported by CDCHT-ULA (Project C-1073-01-05-A), EEC (TMR contracts ERBFMRX-CT960090 and HPRN-CT-2000-00152) and the Ministry of Education, Science and Sport of the Republic of Slovenia, respectively. A.M. and F.V. thank ICTP, and INFN exchange program (F.V.), for hospitality during the course of this work. We thank Z. Berezhiani, G. Dvali and R.N. Mohapatra for useful discussions.