Abstract nonlinear evolution inclusions of second order with applications in visco-elasto-plasticity

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Abstract

Existence of strong solutions of an abstract Cauchy problem for a class of doubly nonlinear evolution inclusion of second order is established via a semi-implicit time discretization method. The principal parts of the operators acting on $u$ and $u'$ are multi-valued subdifferential operators and are discretized implicitly. A non-variational and non-monotone perturbation acting nonlinearly on $u$ and $u'$ is allowed and discretized explicitly in time. The convergence of a variational approximation scheme is established using methods from convex analysis. In addition, it is proven that the solution satisfies an energy-dissipation equality. Applications of the abstract theory to various examples, e.g., a model in visco-elastic-plasticity, are provided.

Keywords Evolution inclusion of second order · Nonlinear damping · Nonsmooth analysis · Variational approximation scheme · Rate-independent dissipation · Visco-Elasto-Plasticity · Martensitic transformation

Mathematics Subject Classification 34G25 · 35A15 · 35G31 · 35L70 · 49J52 · 74N20 · 74N30

1 Introduction

1.1 Problem setting

In this article, we investigate the abstract CAUCHY problem

$$\begin{cases} u''(t) + \partial\Psi_u(t)(u'(t)) + \partial\mathcal{E}_t(u(t)) + B(t, u(t), u'(t)) \ni f(t), & \text{for a.e. } t \in (0, T), \\ u(0) = u_0, & u'(0) = v_0, \end{cases}$$

(1.1)

where again $\Psi_u$ denotes the dissipation potential, $\mathcal{E}_t$ the energy functional, $B$ the perturbation, and $f$ the external force. Here, the dissipation potential $\Psi_u$ is, in general, nonlinear, non-quadratic, nonsmooth, and depends nonlinearly on the state $u$. The energy functional $\mathcal{E}_t = \mathcal{E}^1_t + \mathcal{E}^2_t$ is the sum of a functional $\mathcal{E}^1_t$ that is defined by a strongly positive, symmetric, and bounded bilinear form and a strongly continuous $\lambda$-convex functional $\mathcal{E}^2_t$. The perturbation $B$ is a strongly continuous perturbation of $\partial\Psi_u$ and $\partial\mathcal{E}_t$.

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1.2 Illustrative examples

In the following, we give some illustrative examples of evolution inclusions that can be solved with our abstract theory.

1. In the first example, we consider a visco-elasto-plastic model for the martensitic phase transformation in shape-memory alloys governed by the following system of equations:

\[
\begin{align*}
\rho \partial_{tt} u + \nu (-1)^n \Delta^n \partial_t u &- \nabla \cdot (\sigma_p + \sigma(\nabla u)) + \mu (-1)^m \Delta^m u = f, \quad \text{in } \Omega_T, \\
\sigma_p &\in \text{Sgn}(\lambda'(\nabla u(x)) : \nabla \partial_t u(x)) \lambda'(\nabla u(x)) \quad \text{a.e. in } \Omega_T,
\end{align*}
\]

2. In the second example, we consider an evolution inclusion with nonlinear damping given by

\[
\partial_{tt} u - \nabla \cdot p - \Delta u + b(u) = f \quad \text{in } \Omega_T,
\]

\[
p(x, t) \in \partial \psi(x, u(x, t), \nabla \partial_t u(x, t)) \quad \text{a.e. in } \Omega_T,
\]

that for \( \psi = 0 \) and \( b(u) = \gamma u, \gamma > 0 \), reduces to the classical KLEIN–GORDON equation, which is a relativistic wave equation with applications in relativistic quantum mechanics. For \( \psi(v) = \frac{1}{2} |\nabla v|^p \) the inclusion can be interpreted as a viscous regularization of the KLEIN–GORDON equation.

3. In the final example, we consider the evolution inclusion

\[
\begin{align*}
\partial_{tt} u + |\partial_t u|^{q-2} \partial_t u + p - \nabla \cdot (E \nabla u) + W''(u) &= f \quad \text{in } \Omega_T, \\
p(x, t) &\in \text{Sgn}(\partial_t u(x, t)) \quad \text{a.e. in } \Omega_T.
\end{align*}
\]

which can be interpreted as a model in ferro-magnetism [MRS13].

In the Applications (Section 4), we discuss the preceding examples more in details and show the existence of weak solutions satisfying an energy dissipation balance.

1.3 Literature review

Evolution equations of second order where the operator acting on the time derivative of the solution is nonlinear has been studied by very few authors. LIONS and STRAUSS [LiS65] showed in their seminal work the well-posedness of the CAUCHY problem for the doubly nonlinear evolution equation

\[
u''(t) + A(t)u'(t) + B(t)u(t) = f(t), \quad t \in (0, T),
\]

where \( B \) is an unbounded, self-adjoint, and linear operator and \( A \) is a nonlinear operator.

Under sufficient regularity conditions on the given data and the time dependence of \( A \), the authors show well posedness of the problem for two cases with two different methods: compactness and monotonicity methods. The peculiarity in both cases is the assumption that the operators \( A(t, u, \cdot) \) and \( B \) are, for each \( t \in [0, T] \), defined on different spaces whose intersection is densely and continuously embedded in both spaces. This implies that the solution \( u \) takes values in a different space than its time derivative \( u' \). Based on the techniques used in [LiS65], the authors in [EST15] showed the existence of solutions to the Cauchy problem for

\[
u''(t) + A(t)u'(t) + B(t)u(t) = f(t), \quad t \in (0, T),
\]
where for each \( t \in [0, T] \), \( A(t) : V_A \to V_A^* \) is a hemicontinuous operator that satisfy a suitable growth condition such that \( A + \kappa I \) is monotone and coercive, and the operator \( B(t) = B_0 + C(t) : V_B \to V_B^* \) is the sum of a linear, bounded, symmetric, and strongly positive operator and a strongly continuous perturbation \( C(t) \). As in [LiS65], the authors assume neither that \( V_A \) is continuously embedded in \( V_B \) nor the reverse case. The assumptions on \( A \) imply that \( A + \kappa I \) is maximal monotone and therefore not necessarily a potential operator. Therefore, the result obtained here only partially generalizes the above mentioned results. However, to the best of the authors’ knowledge, results on the existence of strong solutions for multivalued operators \( A \) which are nonlinear in \( u \) and \( u' \) do not exist in the literature. Evolution inclusions occur in many applications, e.g., physical phenomena where rate-independent responses of the body are typical such as in plasticity [MiR15], in ferromagnetic hysteresis [Vis00, MRS13] occurs or in Visco-Elasto-Plasticity [RaR03]. Applications are also find in optimal control theory [AuC84] or nonsmooth dynamical systems [Kun00]. Another motivation for this work is to complement the results obtained in B. [Bac20] where the principal part of the operator acting on \( u \) is nonlinear and multi-valued and the principal part of the operator acting on \( u \) is linear, symmetric and positive. Hence, our contributions concern the following:

- We allow the functionals that are acting on \( u \) and \( u' \) to be nonsmooth, hence generating a nonlinear multi-valued subdifferential in the equations.
- We allow the multi-valued operators to live on different spaces.
- The abstract theory for evolution inclusions has important applications in physics, e.g., in models for visco-elasto-plasticity, see Section 5.
- We allow non-variational and non-monotone perturbations of the subdifferential operators.

For further results on nonlinear abstract evolution inclusions, we refer to [Bac20, Bac22, Bar76, Zei90b, Rou13] and the references therein.

### 1.4 Organization of the paper

The paper is organized as follows. In Section 2, we set the analytical framework and briefly introduce some notions and results from the theory of convex analysis. In Section 3, we present and discuss the assumptions on the dissipation potential \( \Psi \), the energy functional \( \mathcal{E} \) and the perturbation \( B \) as well as the external force \( f \), and we state the main result. Section 4 is devoted to the proof and in Section 5, we apply the abstract theory developed here to physically relevant examples including a mathematical model for visco-elasto-plasticity. In Appendix, we collect certain results from the subdifferential calculus.

### 1.5 Notation and preliminaries

For a proper functional \( f : X \to (-\infty, +\infty] \) on a Banach space \( (X, \| \cdot \|_X) \), we denote with the multivalued map \( \partial f : X \rightrightarrows X^* \), the Fréchet subdifferential of \( f \) defined by

\[
\partial f(u) := \left\{ \xi \in X^* : \liminf_{v \to u} \frac{f(v) - f(u) - \langle \xi, v - u \rangle_{X^* \times X}}{\|v - u\|_X} \geq 0 \right\},
\]
where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between the Banach space $X$ and its topological dual space $X^*$. The elements of the subdifferential are also called subgradients. If the set of subgradients of $f$ at a given point $u$ is nonempty, we say that $f$ is subdifferentiable at $u$. The effective domain of $f$ and the domain of its subdifferential $\partial f$ are denoted by $D(f) := \{v \in X \mid f(v) < +\infty\}$ and $D(\partial f) := \{v \in X : \partial f(v) \neq \emptyset\}$, respectively.

Furthermore, we recall an important tool from the theory of convex analysis. For a proper, lower semicontinuous and convex function $f : X \to (-\infty, +\infty]$, we define the so-called Legendre–Fenchel transform (or convex conjugate) $f^* : X^* \to (-\infty, +\infty]$ by

$$f^*(\xi) := \sup_{u \in V} \{\langle \xi, u \rangle - f(u)\}, \quad \xi \in X^*.$$ 

By definition, we directly obtain the Fenchel–Young inequality

$$\langle \xi, u \rangle \leq f(u) + f^*(\xi), \quad v \in X, \xi \in X^*.$$ 

It is easily checked that the transform itself is proper, lower semicontinuous and convex, see, e.g., [EkT99, Section 4, pp. 16]. If, in addition, we assume $f(0) = 0$, then $f^*(0) = 0$ holds as well.

We recall also the following fact: let $(X, \| \cdot \|_X)$ and $(Y, \| \cdot \|_Y)$ a Banach spaces such that both $X$ and $Y$ are continuously embedded into another Banach space $Z$, and such that $X \cap Y$, equipped with the norm $\| \cdot \|_{X \cap Y} = \| \cdot \|_X + \| \cdot \|_Y$, is dense in both $X$ and $Y$. Then, the space $X \cap Y$ becomes a Banach space itself and if $X$ and $Y$ are separable and reflexive Banach spaces, the dual space can be identified as $X^* + Y^*$ with the dual norm $\|\xi\|_{X^* + Y^*} = \inf_{\xi_1 \in X^*, \xi_2 \in Y^*} \max\{\|\xi_1\|_{X^*}, \|\xi_2\|_{Y^*}\}$, and the duality pairing between $X \cap Y$ and $X^* + Y^*$ is given by

$$\langle f, v \rangle_{(X^* + Y^*)' \times (X \cap Y)} = \langle f_1, u \rangle_{X^* \times X} + \langle f_2, u \rangle_{Y^* \times Y}, \quad u \in X \cap Y,$$

for all $v \in X \cap Y$ and any decomposition $f = f_1 + f_2$ with $f_1 \in X$ and $f_2 \in Y$ see, e.g., [GGZ74, Kapitel 1, §5]. Furthermore, it is easily shown that $L^p(0, T; X) \cap L^p(0, T; Y)$ is $L^p(0, T; X \cap Y)$ for any $p \in [1, +\infty[$, where the measurability follows from the Pettis theorem, see, e.g., [DiU77, Theorem 2, p. 42], and that if $X$ is separable and reflexive, the spaces $L^p(0, T; X)$ for $1 < p < \infty$ are also separable and reflexive, whereas $L^\infty(0, T; X)$ is the dual of the separable space $L^1(0, T; X^*)$. If the continuous embedding $X \hookrightarrow Y$ holds, then

$$\langle f, v \rangle_{X^* \times X} = \langle f, v \rangle_{Y^* \times Y} \quad \text{if } v \in X \text{ and } f \in Y^*.$$ 

see, e.g., [Bré11, Remark 3, pp. 136] and [GGZ74, Kapitel 1, §5].

## 2 Topological assumptions and main result

### 2.1 Function space setting

We assume that $(U, \| \cdot \|_U)$, $(V, \| \cdot \|_V)$, $(W, \| \cdot \|_W)$ and $(\overline{W}, \| \cdot \|_{\overline{W}})$ are real, reflexive, and separable Banach spaces such that $U \cap V$ is separable and reflexive and that $(H, \| \cdot \|, (\cdot, \cdot))$
is a HILBERT space with norm $|\cdot|$ induced by the inner product $(\cdot, \cdot)$.

Similarly, we assume again the following dense, continuous and compact embeddings

\[
\begin{cases}
U \cap V \xrightarrow{d} U \xrightarrow{d} \widetilde{W} \xrightarrow{d} H \cong H^* \xrightarrow{d} \widetilde{W}^* \xrightarrow{d} U^* \xrightarrow{d} V^* + U^*, \\
U \cap V \xrightarrow{d} V \xrightarrow{d} W \xrightarrow{d} H \cong H^* \xrightarrow{d} W^* \xrightarrow{d} V^* \xrightarrow{d} V^* + U^*,
\end{cases}
\]

and if the perturbation does not explicitly depend on $u$ or $u'$, then we do not assume $U \xhookrightarrow{d} \widetilde{W}$ or $V \xhookrightarrow{d} W$, respectively. We further assume $V \hookrightarrow W$ if $\xi^0 \neq 0$, see Condition (2.Ea). We note that we neither assume $U \hookrightarrow V$ nor $V \hookrightarrow U$. Since in this case the subdifferential of $\Psi_u$ is nonlinear, we refer to the inclusion (1.1) in the given framework as nonlinearly damped inertial system $(U, V, W, \tilde{W}, H, E, \Psi, B, f)$.

### 2.2 Assumptions on the functionals and operators

We first collect all the assumptions for the energy functional $\mathcal{E}_t$, the dissipation potential $\Psi_u$, the perturbation $B$ as well as the external force $f$, and discuss them subsequently. We start with the assumptions for the dissipation potential $\Psi$.

(2.2.a) **Dissipation potential.** For every $u \in U$, let $\psi_u : V \to [0, +\infty)$ be a lower semicontinuous and convex functional with $\psi_u(0) = 0$ such that the mapping $(u, v) \mapsto \psi_u(v)$ is $\mathcal{B}(U) \otimes \mathcal{B}(V)$-measurable.

(2.2.b) **Superlinearity.** The functional $\psi$ satisfies the following growth condition, i.e., there exists a positive real number $q > 1$ such that for all $R > 0$ there exist positive constants $c_R, C_R > 0$ such that for all $u \in U$ with $\sup_{t \in [0, T]} \mathcal{E}_t(u) \leq R$, there holds

\[
c_R(\|v\|_V^q - 1) \leq \psi_u(v) \leq C_R(\|v\|_V^q + 1)
\]

for all $v \in V$. \hspace{1cm} (2.1)

(2.2.c) **Lower semicontinuity of $\psi_u + \psi_u^*$.** For all sequences $v_n \to v$ in $L^q(0, T; V)$, $\eta_n \to \eta$ in $L^{q'}(0, T; V^*)$, and $u_n(t) \to u(t)$ in $U$ for all $t \in [0, T]$ as $n \to \infty$, there exists a positive real number $q > 1$ such that $\sup_{t \in [0, T]} \mathcal{E}_t(u(t)) < +\infty$ and $\eta_n(t) \in \partial \psi_u(u_n(t))(v_n(t))$ a.e. in $t \in (0, T)$ for all $n \in \mathbb{N}$, there holds

\[
\int_0^T \left( \psi_{u(t)}(v(t)) + \psi_{u(t)}^*(\xi(t)) \right) dt \leq \liminf_{n \to \infty} \int_0^T \left( \psi_{u_n(t)}(v_n(t)) + \psi_{u_n(t)}^*(\eta_n(t)) \right) dt.
\]

For the solvability of problem (1.1), only the previous assumptions are required. If we additionally assume the uniform monotonicity of $\partial \psi_u$, we obtain stronger convergence of the discrete time-derivatives $\nabla_{\tau_n}$ in the space $L^q(0, T; V)$, see Lemma 3.3.

(2.2.d) **Uniform monotonicity of $\partial \psi$.** For all $R > 0$, there exists a constant $\mu_R > 0$ such that

\[
\langle \xi - \eta, v - w \rangle_{V^* \times V} \geq \mu_R \|v - w\|_V^{\max\{2, q\}}
\]

for all $\xi \in \partial \psi_u(v), \eta \in \partial \psi_u(w)$ and $u, v, w \in \{ \tilde{v} \in V : \mathcal{E}_t(\tilde{v}) \leq R \}$.

Remark 2.1.
\textit{i)} We recall that the conjugate \( \Psi^*: V^* \to \mathbb{R} \) is lower semicontinuous and convex itself and that the growth condition (2.1) implies the following growth condition for the conjugate \( \Psi^* \): for all \( R > 0 \), there exist positive numbers \( \bar{c}_R, \bar{C}_R > 0 \) such that for all \( u \in U \) with \( \sup_{t \in [0,T]} E_t(u) \leq R \), there holds
\[
\bar{c}_R(\|\xi\|_{V^*}^q - 1) \leq \Psi^*_u(\xi) \leq \bar{C}_R(\|\xi\|_{V^*}^q + 1) \quad \text{for all } \xi \in V^*,
\]
where \( q^* = q/(q-1) \).

\textit{ii)} It has been shown in \textsc{Stefanelli} [Ste08, Lemma 4.1] that the following convergence in the sense of MOSCO (we write \( \Psi|_\alpha \mapsto \Psi|_n \)) implies Condition (2.\Psi_c): Let \( u_n \to u \in V \). Then, for all \( v \in V \), there holds
\[
\begin{cases}
\alpha) & \Psi(u(v) \leq \liminf_{n \to \infty} \Psi|_{u_n}(v_n) \quad \text{for all } v_n \to v \text{ in } V, \\
b) & \exists v_n \to v \text{ in } V \text{ such that } \Psi(u(v) \geq \limsup_{n \to \infty} \Psi|_{u_n}(v_n).
\end{cases}
\]
(2.2)

Now, we proceed with the assumptions for the energy functional.

\textbf{(2.Ea) Basic properties.} For all \( t \in [0,T] \), the functional \( E_t : U \to \mathbb{R} \) is the sum of functionals \( E^1_t : U \to \mathbb{R} \) and \( E^2_t : \tilde{W} \to \mathbb{R} \). The functional \( E^1(\cdot) = \frac{1}{2} b(\cdot, \cdot) \) is induced by a bounded, symmetric, and strongly positive bilinear form \( b : U \times U \to \mathbb{R} \), i.e., there exist constants \( \mu, \alpha > 0 \) such that
\[
b(u,v) \leq \alpha \|u\|_U \|v\|_U \quad \text{for all } u,v \in U
\]
\[
\mu \|u\|_U^2 \leq b(u,u) \quad \text{for all } u \in U.
\]

\textbf{(2.Eb) Bounded from below.} \( E_t \) is bounded from below uniformly in time, i.e., there exists a constant \( C_0 \in \mathbb{R} \) such that
\[
E_t(u) \geq C_0 \quad \text{for all } u \in U \text{ and } t \in [0,T].
\]

Since a potential is uniquely determined up to a constant, we assume without loss of generality \( C_0 = 0 \).

\textbf{(2.Ec) Coercivity.} For every \( t \in [0,T] \), \( E_t \) has bounded sublevel sets in \( U \).

\textbf{(2.Ed) Control of the time derivative.} For all \( u \in U \), the mapping \( t \mapsto E^2_t(u) \) is in \( C([0,T]) \cap C^1(0,T) \) and its derivative \( \partial_t E^2_t \) is controlled by the function \( E^2_t \), i.e., there exists \( C_1 > 0 \) such that
\[
|\partial_t E^2_t(u)| \leq C_1 E^2_t(u) \quad \text{for all } t \in (0,T) \text{ and } u \in V.
\]

Furthermore, for all sequences \( (u_n)_{n \in \mathbb{N}}, u \subset D \) with \( u_n \to u \) as \( n \to \infty \) and \( \sup_{n \in \mathbb{N}, t \in [0,T]} E_t(u_n) < +\infty \), there holds
\[
\limsup_{n \to \infty} \partial_t E^2_t(u_n) \leq \partial_t E^2_t(u) \quad \text{for a.e. } t \in (0,T).
\]

\textbf{(2.Ee) Fréchet differentiability.} For all \( t \in [0,T] \), the mapping \( u \mapsto E^2_t(u) \) is Fréchet differentiable on \( \tilde{W} \) with derivative \( \text{D} E^2_t \) such that the mapping \( (t,u) \mapsto \text{D} E^2_t(u) \) is continuous as a mapping from \( [0,T] \times \tilde{W} \) to \( U^* \) on sublevel sets of the energy, i.e., for all \( R > 0 \) and sequences \( (u_n)_{n \in \mathbb{N}}, u \subset \tilde{W} \) and \( (t_n)_{n \in \mathbb{N}}, t \subset [0,T] \) with \( \sup_{t \in [0,T], n \in \mathbb{N}} E_t(u_n) < +\infty, u_n \to u \) in \( \tilde{W} \), and \( t_n \to t \) as \( n \to \infty \), there holds
\[
\lim_{n \to \infty} \|\text{D} E^2_t(u_n) - \text{D} E^2_t(u)\|_{U^*} = 0.
\]
(2.Ed) \(\lambda\)-convexity. There exists a non-negative real number \(\lambda \geq 0\) such that
\[
\mathcal{E}_t(\vartheta u + (1 - \vartheta)v) \leq \vartheta \mathcal{E}_t(u) + (1 - \vartheta)\mathcal{E}_t(v) + \vartheta(1 - \vartheta)\lambda |u - v|^2
\]
for all \(t \in [0, T]\), \(\vartheta \in [0, 1]\) and \(u, v \in U\).

(2.Eg) Control of \(\mathcal{D}\mathcal{E}_t^2\). There exist positive constants \(C_2 > 0\) and \(\sigma > 0\) such that
\[
\|\mathcal{D}\mathcal{E}_t^2(u)\|_{\tilde{W}^*}^2 \leq C_3(1 + \mathcal{E}_t^2(u) + \|u\|_{\tilde{W}^*}) \text{ for all } t \in [0, T], u \in \tilde{W}.
\]

Again, several remarks are in order.

**Remark 2.2.**

i) The assumptions on the quadratic form \(\mathcal{E}^1\) imply that the Fréchet derivative \(\mathcal{D}\mathcal{E}^1\)

is given by a linear, bounded, symmetric and strongly positive operator \(E \in \mathcal{L}(V, V^*)\)
such that \(\mathcal{E}^1(u) = \frac{1}{2}\langle Eu, u \rangle\) is strongly convex and therefore sequentially weakly

lower semicontinuous. Furthermore, the corresponding Nemitskiĭ operator is a linear and bounded map from \(L^2(0, T; V)\)
to \(L^2(0, T; V^*)\) and hence weak-to-weak continuous from \(L^2(0, T; V)\) to \(L^2(0, T; V^*)\).

ii) From Assumption (2.Ed), it follows after integration
\[
\sup_{t \in [0, T]} \mathcal{E}_t^2(u) \leq e^{C_2 T} \inf_{t \in [0, T]} \mathcal{E}_t^2(u),
\]
\[
|\mathcal{E}_t^2(u) - \mathcal{E}_s^2(v)| \leq e^{C_2 T} \sup_{r \in [0, T]} \mathcal{E}_r^2(u) |s - t| \text{ for all } u \in U, s, t \in [0, T].
\]

iii) The derivative of the \(\lambda\)-convex energy functional is characterized by the inequality
\[
\mathcal{E}_t^2(u) - \mathcal{E}_t^2(v) \leq \langle \mathcal{D}\mathcal{E}_t^2(u), u - v \rangle_{U^* \times U} + \lambda |u - v|^2
\]
for all \(t \in [0, T], u, v \in U\). In fact, the \(\lambda\)-convexity can be replaced by the latter
inequality, since we only make use of (2.3) in order to obtain a priori estimates, see
Lemma 3.2.

We recall that the Fréchet differentiability of \(\mathcal{E}_t\) implies the subdifferentiability of
\(\mathcal{E}_t\) and the subdifferential is a singleton with \(\partial \mathcal{E}_t(u) = \{\mathcal{D}\mathcal{E}_t(u)\}\).

Finally, we collect the assumptions concerning the perturbation \(B\) and the external force \(f\).

(2.Ba) Continuity. The mapping \(B : [0, T] \times \tilde{W} \times W \to V^*\) is continuous on sublevel sets
of \(\mathcal{E}_t\), i.e., for every converging sequence \((t_n, u_n, v_n) \to (t, u, v)\) in \([0, T] \times \tilde{W} \times W\)
with \(\sup_{n \in \mathbb{N}} G(u_n) < +\infty\), there holds \(B(t_n, u_n, v_n) \to B(t, u, v)\) in \(V^*\) as \(n \to \infty\).

(2.Bb) Control of the growth. There exist positive constants \(\beta > 0\) and \(c, \nu \in (0, 1)\) such that
\[
\frac{c \Psi_u^*}{\nu} \left(\frac{B(t, u, v)}{c}\right) \leq \beta(1 + \mathcal{E}_t(u) + |v|^2 + \Psi_{u - rv}(v)^\nu)
\]
for all \(u \in U, v \in V, t \in [0, T]\), and all \(r \in [0, 1]\).

(8.f) External force. There holds \(f \in L^1(0, T; H)\).

**Remark 2.3.** If the growth condition (2.4b) for \(\Psi_u\) holds uniformly in \(u \in U\), then more
general external forces \(f \in L^1(0, T; H) + L^p(0, T; V^*)\) can be considered.
2.3 Discussion of the assumptions

Apart from the remarks made above, we want to discuss certain Conditions more in detail or provide examples to the abstract setting.

As the name suggests, we consider in this case evolution equations of second order with nonlinear damping, i.e., where $\partial \Psi_{\alpha}(t)$ is nonlinear and in general multi-valued. As already mentioned in the literature review (Section 1.3), this has not been studied before.

Ad (2.Ψ). The Condition (2.Ψa) allows us to consider nonsmooth dissipation potentials and the assumption $\Psi_{\alpha}(0) = 0$ is not restrictive as the potential is uniquely determined up to a constant. The growth condition (2.Ψb) here is crucial to employ an integration by parts formula for the second derivative $u''$ proven in [EmT11], see Lemma 3.3 below. As we mentioned in Remark 2.1 ii), the liminf estimate in Condition (2.Ψc) is already implied by the MOSCO-convergence $\Psi_{\alpha} \xrightarrow{M} \Psi_{\alpha}$ for all sequences $u_{\alpha} \rightarrow u$. The Mosco convergence is related to the graph convergence of its subdifferential and stronger than the $\Gamma$-convergence. A prototypical example for such a dissipation potential which fulfill Condition (2.Ψa)-(2.Ψd) is

$$\Psi_{\alpha}(v) = \int_{\Omega} \left( g_{1}(\nabla u) \frac{1}{p} |\nabla v(x)|^p + g_{2}(\nabla u) |\nabla v(x)| \right) \, dx \quad \text{or}$$

$$\Psi_{\alpha}(v) = \int_{\Omega} \left( g_{1}(u) \frac{1}{p} |v(x)|^p + g_{2}(u) |v(x)| \right) \, dx$$

on $V = W^{1,p}_{0}(\Omega)^m$ or $V = L^p(\Omega)^m$ with $m \in \mathbb{N}$ and $p \in (1, + \infty)$, respectively, where $g_1, g_2 : \mathbb{R}^m \rightarrow \mathbb{R}$ are continuous function satisfying further conditions depending on the concrete form of the energy functional. See Chapter 4, where we discuss more general dissipation potentials. This type of dissipation potentials occur in rate-independent systems such as in plasticity [MiR15], in ferromagnetic hysteresis [Vis00, MRS13] or in Elastoviscoplasticity [RaR03], see Section 5.

Ad (2.E). The crucial assumption we make for the energy functional $\mathcal{E}_t = \mathcal{E}^1 + \mathcal{E}^2_t$ is that the leading part $\mathcal{E}^1$ is defined by a bounded, symmetric, and strongly positive bilinear form $b : U \times U \rightarrow \mathbb{R}$. As mentioned in Remark 2.2, the Conditions (2.Ed) and (2.Ee) make sure that we can control the time derivative of the energy functional and that we can pass to the limit in the energy-dissipation equality (2.5) in Section 3.4. The Condition (2.Em) is needed in order to obtain bounds for the subgradients of $\mathcal{E}_t$, which in turn is necessary to obtain a priori estimates for $u''$. The problem is that the bounds in Lemma 3.2 only gives a priori estimates for the sum of the subgradient of $\mathcal{E}_t$ and $u''$ which necessitates an independent bound. Condition (2.Eg) could be replaced by the more general condition that $\partial \mathcal{E}_t$ is a bounded operator.

A prototypical example is given by

$$\mathcal{E}_t(u) = \int_{\Omega} \left( \frac{1}{2} |u|^2 + W(u) \right) \, dx + \int_{\Omega} \nabla u : E \nabla u \, dx - \langle f(t), u \rangle_{U^* \times U}$$

on $U = H^1_{0}(\Omega)^m$ with $m \in \mathbb{N}$, where $W : \mathbb{R}^5 \rightarrow \mathbb{R}$ is a $\lambda$-convex and continuously differentiable function, e.g., $W(u) = (1 - u^2)^2$, $E : \mathbb{R}^m \rightarrow \mathbb{R}^m$ a uniformly positive definite and symmetric matrix, and $f \in C^1([0,T]; U^*)$. This type of example occurs very often in models for ferro-magnetism where the solution $u$ is the so-called magnetization, see...
Ad (2.B). The continuity condition (2.Ba) means that $B$ is a continuous perturbation of $\partial \mathcal{E}_t$. In practice, the term $B$ contains all non-variational and non-monotone contributions of lower order in terms of growth as well as spatial derivatives. This is reflected by Condition (2.Bb), where $B$ satisfies a growth condition in terms of the dissipation potential and its convex conjugate as well as the energy functional as well as the kinetic energy. In fact, the growth condition shows that the higher the order of the growths of $\Psi_u$ and $\mathcal{E}_t$ are, the more we can allow for the growth of the perturbation. Condition (2.Bb) ensures that we can control the perturbation in order to derive appropriate bounds. Both conditions can be generalized in a framework that instead of a point-wise continuity and a pointwise growth condition, a continuity on suitable Bochner spaces can be imposed as well as a growth condition on the level of time integrals. Furthermore, it would be sufficient to define the perturbation on the domain of the subdifferential of $\mathcal{E}_t$, see, e.g., [Aka11, Ö82], where this has been considered for evolution inclusions of first order. An example for the perturbation is given by

$$B(t, u, v) = \int_{\Omega} (\pm |u(x)|^p + |v(x)|^q) \, dx$$

on $W = L^q(\Omega)^m$ and $\bar{W} = L^q(\Omega)^m$ for appropriate $p, q \geq 1$. Obviously, $B$ is neither variational nor monotone.

### 2.4 Statement of the main result

Having discussed all assumptions, we are in a position to state the main result which again includes the notion of solution to (1.1).

**Theorem 2.4 (Main result).** Let the nonlinearly damped inertial system 

$$(U, V, W, \bar{W}, H, \mathcal{E}, \Psi, B, f)$$

be given and fulfill Assumptions (2.E), (2.Ψa)-(2.Ψc) as well as (2.B) and Assumption (2.f). Then, for every $u_0 \in U$ and $v_0 \in H$, there exists a solution to (1.1), i.e., there exist functions $u \in C_w([0, T]; U) \cap W^{1,\infty}(0, T; H) \cap W^{2,q}_{\text{loc}}(0, T; U^* + V^*)$ with $u - u_0 \in W^{1,q}(0, T; V)$ and $\eta \in L^q(0, T; V^*)$ satisfying the initial conditions $u(0) = u_0$ in $U$ and $u'(0) = v_0$ in $H$ such that

$$u''(t) + \eta(t) + D\mathcal{E}_t(t) + B(t, u(t), u'(t)) = f(t) \quad \text{in } U^* + V^*,
\eta(t) \in \partial \Psi_{u(t)}(u'(t)) \quad \text{in } V^*,$$

(2.4)

for almost every $t \in (0, T)$. Furthermore, the energy-dissipation balance

$$\frac{1}{2} |u'(t)|^2 + \mathcal{E}_t(u(t)) + \int_0^t \left( \Psi_{u(t)}(u'(r)) + \Psi_{u'(t)}'(S(r) - D\mathcal{E}_r(r)) - u''(r) \right) \, dr
= \frac{1}{2} |v_0|^2 + \mathcal{E}_0(u_0) + \int_0^t \partial_r \mathcal{E}_r(u(r)) \, dr + \int_0^t (S(r), u'(r))_{V^* \times V} \, dr$$

(2.5)

holds for almost every $t \in (0, T)$, where $S(r) := f(r) - B(r, u(r), u'(r))$, $r \in [0, T]$, and if $V \hookrightarrow U$, then (2.5) holds for all $t \in [0, T]$.

### 3 Proof of the main result

The proof of Theorem 2.4 consists of the following main steps:
1. We discretize the inclusion in time via a semi-implicit Euler scheme with time step \( \tau > 0 \) and show solvability of the discrete problem.

2. We define interpolations functions and show a priori estimates for them.

3. We show compactness of the interpolation functions in suitable spaces.

4. We pass to the limit with \( \tau \searrow 0 \) and show existence of solutions to the regularized problem.

In the following, each step will be carried out in a subsection.

3.1 Variational approximation scheme

The proof of Theorem 2.4 again relies on a semi-implicit time discretization scheme. Therefore, we will proceed in a similar way to the case in the previous section. The main difference and difficulty arises in identifying the (a priori) weak limits associated with the nonlinear terms \( D\mathcal{E} \) and \( \partial\Psi \). Again, for \( N \in \mathbb{N}\setminus\{0\} \), let

\[
I_\tau = \{ 0 = t_0 < t_1 < \cdots < t_n = n\tau < \cdots < t_N = T \}
\]

be an equidistant partition of the time interval \([0,T]\) with step size \( \tau := T/N \), where we again omit the dependence of the nodes from the partition on the step size. Discretizing inclusion (1.1) in a semi-implicit manner yields

\[
\frac{V^n_\tau - V^{n-1}_\tau}{\tau} + \partial\Psi_{U^{n-1}_\tau} (V^n_\tau) + D\mathcal{E}_\tau(U^n_\tau) + B(t_n, U^{n-1}_\tau; V^{n-1}_\tau) \ni f^n_\tau \quad \text{in} \quad U^*_\tau + V^*_\tau \quad (3.1)
\]

for \( n = 1, \ldots, N \) with \( V^n_\tau = \frac{U^n_\tau - U^{n-1}_\tau}{\tau} \). The value \( U^n_\tau \) is to be determined recursively from the variational approximation scheme

\[
U^0_\tau \in U \cap V \quad \text{and} \quad V^0_\tau \in V \quad \text{are given; whenever} \quad U^1_\tau, \ldots, U^{n-1}_\tau \in D \cap V \quad \text{are known,}
\]

\[
\begin{cases}
U^n_\tau \in J_{\tau, t_{n-1}} (U^{n-1}_\tau, U^{n-2}_\tau; B(t_n, U^{n-1}_\tau, V^{n-1}_\tau) - f^n_\tau)
\end{cases}
\]

(3.2)

for \( n = 1, \ldots, N \), where \( J_{\tau, t} (v, w; \eta) := \arg\min_{u \in U \cap V} \Phi(r, t, v, w, \eta; u) \) and \( U^{-1}_\tau = U^0_\tau - V^0_\tau \tau \) with

\[
\Phi(r, t, v, w, \eta; u) = \frac{1}{2r^2} |u - 2v + w|^2 + r\Psi_u \left( \frac{u - v}{r} \right) + \mathcal{E}_{t+r} (u) - \langle \zeta, u \rangle_{V^*_\tau \times V}
\]

for \( r \in \mathbb{R}^+, t \in [0,T) \) with \( r + t \in [0,T] \), \( u \in U \cap V, v \in V, w \in H \) and \( \zeta \in V^* \). The solvability of the discrete problem (3.2) and that every solution fulfills the Euler–Lagrange equation (3.1) is ensured by the following lemma.

Lemma 3.1. Let the nonlinearly damped inertial system \((U, V, W, H, \mathcal{E}, \Psi)\) be given and fulfill the Conditions (2.Ea)-(2.Ee), (2.Ee), (2.Ef) and (2.Pa)-(2.Pb). Furthermore, let \( r \in (0,T) \) and \( t \in [0,T) \) with \( r + t \leq T \) as well as \( v \in V, w \in H \) and \( \zeta \in V^* \). Then, the set \( J_{\tau, t} (v, w; \eta) \) is non-empty and single valued if \( r \leq \frac{1}{3\lambda} \), where \( \lambda \) is from (2.Ef). Furthermore, to every \( u \in J_{\tau, t} (v, w; \eta) \), there exists \( \eta \in \partial \Psi_u \left( \frac{u - v}{r} \right) \subset V^* \) such that

\[
\frac{u - 2v - w}{r^2} + \eta + D\mathcal{E}_t (u) + \zeta = 0 \quad \text{in} \quad U^*_\tau + V^*_\tau.
\]
Proof. The proof follows from the direct methods of the calculus of variations as well as Lemma A.1. Let $u \in U \cap V, v, w \in V, \zeta \in V^*$ and $r \in (0, t_0), t \in [0, T)$ with $r + t \leq T$ be given. First of all, the Fenchel–Young inequality and the boundedness of the energy from below yield

$$\Phi(r, t, v, w, \eta; u) = \frac{1}{2r^2} |u - 2v + w|^2 + \mathcal{E}_{t+r}(u) - \langle \zeta, u \rangle_{V^* \times V}$$

which implies on the one hand $\inf_{u \in U \cap V} \Phi(r, t, v, w, \eta; u) < -\infty$. On the other hand, we observe that

$$\inf_{u \in U \cap V} \Phi(r, t, v, w, \eta; u) \leq \frac{1}{2r^2} |u_0 - 2v + w|^2 + r\Psi_{u_0} \left( \frac{u_0 - v}{r} \right) + \mathcal{E}_{t+r}(u_0) - \langle \eta, u_0 \rangle_{V^* \times V}$$

for any $u_0 \in U \cap V$, so that $\inf_{u \in U \cap V} \Phi(r, t, v, w, \eta; u) < +\infty$ holds as well. It remains to show that the global minimum is achieved by an element of $D$. In order to show that, let $(u_n)_{n \in \mathbb{N}} \subset U \cap V$ be a minimizing sequence for $\Phi(r, t, v, w, \eta; \cdot)$. From (3.3), we deduce that $(u_n)_{n \in \mathbb{N}} \subset U \cap V$ is contained in a sublevel set of $\mathcal{E}_{t+r}(\cdot)$ and thus by Assumptions (2.Ec) and (2.Ψc) bounded in $U \cap V$. Hence, there exists a subsequence (not relabeled) which converges weakly in $U \cap V$ towards a limit $\bar{u} \in U \cap V$. By the weak lower semicontinuity of the mapping $u \mapsto \Phi(r, t, v, w, \eta; u)$, we have

$$\Phi(r, t, v, w, \eta; \bar{u}) \leq \liminf_{n \to \infty} \Phi(r, t, v, w, \eta; u_n) = \inf_{\bar{v} \in V} \Phi(r, t, v, w, \eta; \bar{v}),$$

and therefore $u \in J_{r,t}(v, w; \eta) \neq \emptyset$. \hfill \Box

Thus, Lemma A.1 ensures that minimizer of the mapping

$$u \mapsto \Phi(r, t, v, w, \eta; u_n)$$

fulfil the Euler–Lagrange equation (3.1) for a $\eta \in \partial_{U \cap V} \Psi_u \left( \frac{u-v}{\sqrt{r}} \right) \subset U^* + V^*$ where the subdifferential is taken on the space $U \cap V$ which can be realized by restricting the functional $\Psi_u$ to the space $U \cap V$. It remains to show that $\eta \in V^*$. Applying Lemma A.2, there holds

$$r\Psi_u \left( \frac{u-v}{r} \right) + r\Psi^*_u (\eta) = \left\langle \eta, \frac{u-v}{r} \right\rangle_{(U^* + V^*) \times (U \cap V)}$$

where $\Psi^*_u$ is the convex conjugate of $\Psi_u$ on $U \cap V$. Taking into account $f = f^{**}$ for any proper, convex and lower semicontinuous functional (see [EkT99, Proposition 4.1, p. 18]), it is easy to show that

$$\Psi^*_u (\xi) = \begin{cases} 
\Psi_u (\xi) & \text{if } \xi \in V^* \\
+\infty & \text{else}
\end{cases}$$

which in regard of (3.5) immediately shows $\eta \in V^*$. 

\[3.1 \text{ VARIATIONAL APPROXIMATION SCHEME} \]
3.2 Discrete Energy-Dissipation inequality and a priori estimates

In this section, we derive a priori estimates to the approximate solutions. Thus, let the initial values \( u_0 \in U \cap V \) and \( v_0 \in V \) as well as the time step \( \tau > 0 \) be given and fixed. As before, we will assume more general initial values in the main existence result and approximate by suitable sequences of values. Then, for given approximate values \( (U^0_\tau)_{n=0}^N \) with \( U^0_\tau := u_0 \) and \( V^0_\tau = v_0 \) obtained from the variational approximation scheme (3.2), we define the piecewise constant and linear interpolations The piecewise constant and linear interpolations are defined by

\[
\bar{U}_\tau(0) = \underline{U}_\tau(0) = \hat{U}_\tau(0) := U^0_\tau = u_0 \quad \text{and} \quad U_\tau(t) := \frac{t_n - t}{\tau} U^{n-1}_\tau + \frac{t - t_{n-1}}{\tau} U^n_\tau \quad \text{for } t \in [t_{n-1}, t_n), \quad \text{and} \quad U_\tau(T) = U^N_\tau, \ n = 1, \ldots, N,
\]

as well as

\[
\bar{V}_\tau(0) = \underline{V}_\tau(0) = \hat{V}_\tau(0) := V^0_\tau = v_0 \quad \text{and} \quad V_\tau(t) := \frac{t_n - t}{\tau} V^{n-1}_\tau + \frac{t - t_{n-1}}{\tau} V^n_\tau \quad \text{for } t \in [t_{n-1}, t_n), \quad \text{and} \quad V_\tau(T) = V^N_\tau, \ n = 1, \ldots, N,
\]

where \( V^n_\tau = \frac{U^n_\tau - U^{n-1}_\tau}{\tau} \) for \( n = 1, \ldots, N \). We note that \( \hat{U}'_\tau = \bar{V}_\tau \) in the weak sense. Furthermore, we define the function \( f_\tau : [0, T] \to H \) by

\[
f_\tau(t) = f_\tau^n = \frac{1}{\tau} \int_{t_{n-1}}^{t_n} f(\sigma) \, d\sigma \quad \text{for } t \in (t_{n-1}, t_n], \ n = 1, \ldots, N,
\]

\[
f_\tau(T) = f_\tau^N.
\]

Furthermore, by Lemma 3.1, there exists a sequence \( (\eta^n_\tau)_{n=1}^N \subset V^* \) of subgradients fulfilling \( \eta^n_\tau \in \partial \Psi_{U(n) \bar{U}(n)}(V_\tau^n) \), \( n = 1, \ldots, N \), such that

\[
\frac{V^n_\tau - V^{n-1}_\tau}{\tau} + \eta^n_\tau + D\Psi_{U(n)}(U^n_\tau) + B(t_n, U^{n-1}_\tau, V^{n-1}_\tau) = f^n_\tau \quad \text{in } U^* + V^*, \ n = 1, \ldots, N.
\]

Then, we define the measurable function \( \eta_\tau : [0, T] \to V^* \) by

\[
\eta_\tau(t) = \eta^n_\tau \quad \text{for } t \in (t_{n-1}, t_n], \ n = 1, \ldots, N, \quad \text{and} \quad \eta_\tau(T) = \eta^N_\tau. \quad \text{(3.9)}
\]

For notational convenience, we also introduce the piecewise constant functions \( \underline{t}_\tau : [0, T] \to [0, T] \) and \( \bar{t}_\tau : [0, T] \to [0, T] \) given by

\[
\underline{t}_\tau(0) = 0 \quad \text{and} \quad \underline{t}_\tau(t) := t_n \quad \text{for } t \in (t_{n-1}, t_n], \quad \text{and} \quad \underline{t}_\tau(T) = T \quad \text{and} \quad \underline{t}_\tau(t) := t_n \quad \text{for } t \in [t_{n-1}, t_n), \quad n = 1, \ldots, N.
\]

\[
\bar{t}_\tau(t) \to t \quad \text{and} \quad \bar{t}_\tau(t) \to t \quad \text{as } \tau \to 0.
\]

Having defined the interpolations, we are in the position to show the a priori estimates in the following lemma.
Lemma 3.2 (A priori estimates). Let the system \((U, V, W, H, \mathcal{E}, \Psi, B, f)\) be given and satisfy the Assumptions (3.E), (3.Y), (3.B) as well as Assumption (3.f). Furthermore, let \(U^*_\tau, \overline{U^*_\tau}, \overline{V^*_\tau}, V^*_\tau, \dot{V}^*_\tau, \eta^*_\tau\) and \(f^*_\tau\) be the interpolations associated with the given values \(u_0 \in U \cap V\) and \(v_0 \in V\) as well as the step size \(\tau > 0\). Then, the discrete energy-dissipation inequality

\[
\begin{align*}
\int_{t_0}^{t_1} \left( \psi^*_{\overline{U^*_\tau}(r)}(\nabla^*_\tau(r)) + \psi^*_{\overline{U^*_\tau}(r)}\left( S^*_\tau(r) - \dot{V}^*_\tau(r) - \text{DE}_{\tau(r)}(\overline{U^*_\tau(r)}) \right) \right) \, dr \\
+ \frac{1}{2} \left| \nabla^*_\tau(t) \right|^2 + \mathcal{E}_{\tau(t)}(\overline{U^*_\tau}(t)) \\
\leq \frac{1}{2} \left| \nabla^*_\tau(s) \right|^2 + \mathcal{E}_{\tau(s)}(\overline{U^*_\tau}(s)) + \int_{t_0}^{t_1} \partial_t \mathcal{E}_\tau(U^*_\tau(r)) \, dr + \int_{t_0}^{t_1} \langle S^*_\tau(r), \nabla^*_\tau(r) \rangle_{U^*_\tau \times U} \, dr \\
+ \tau \lambda \int_{t_0}^{t_1} \left| \nabla^*_\tau(r) \right|^2 \, dr
\end{align*}
\] (3.11)

holds for all \(0 < s < t \leq T\), where we have introduced the short-hand notation \(S^*_\tau(r) \coloneqq f^*_\tau(r) - B(\overline{U^*_\tau}(r), \overline{U^*_\tau}(r), \overline{V^*_\tau}(r)), r \in [0, T]\). Moreover, there exist positive constants \(M, \tau^* > 0\) such that the estimates

\[
\begin{align*}
\sup_{t \in [0, T]} \left| \overline{V^*_\tau}(t) \right| &\leq M, \quad \sup_{t \in [0, T]} |\mathcal{E}_\tau(\overline{U^*_\tau}(t))| \leq M, \quad \sup_{t \in [0, T]} |\partial_t \mathcal{E}_\tau(\overline{U^*_\tau}(t))| \leq M, \\
\int_0^T \left( \psi_{\overline{U^*_\tau}(r)}(\nabla^*_\tau(r)) + \psi_{\overline{U^*_\tau}(r)}\left( S^*_\tau(r) - \dot{V}^*_\tau(r) - \text{DE}_{\tau(r)}(\overline{U^*_\tau(r)}) \right) \right) \, dr &\leq M
\end{align*}
\] (3.12)

(3.13)

hold for all \(0 < \tau \leq \tau^*\). In particular, the families of functions

\[
\begin{align*}
(\overline{U^*_\tau})_{0 < \tau \leq \tau^*} &\subset L^\infty(0, T; U), \\
(\nabla^*_\tau)_{0 < \tau \leq \tau^*} &\subset L^2(0, T; V), \\
(\eta^*_\tau)_{0 < \tau \leq \tau^*} &\subset L^q(0, T; V^*), \\
(\dot{V}^*_\tau)_{0 < \tau \leq \tau^*} &\subset L^{\min\{q^*, 2\}}(0, T; U^* + V^*), \\
(B^*_\tau)_{0 < \tau \leq \tau^*} &\subset L^2(0, T; V^*), \\
(D\mathcal{E}^2_{\tau}(\overline{U^*_\tau}))_{0 < \tau \leq \tau^*} &\subset L^\infty(0, T; \overline{W}^*),
\end{align*}
\] (3.14a,b,c,d,e,f)

are uniformly bounded with respect to \(\tau\) in the respective spaces, where \(q^* > 0\) is the conjugate exponent to \(q > 1\) and \(\nu \in (0, 1)\) being from Assumption (2.Bb). Finally, there holds

\[
\begin{align*}
\sup_{t \in [0, T]} \left( \|U^*_\tau(t) - \overline{U^*_\tau}(t)\|_V + \|\dot{U}^*_\tau(t) - \overline{U^*_\tau}(t)\|_V \right) &\to 0 \\
\sup_{t \in [0, T]} \left( \|\nabla^*_\tau(t) - \dot{V}^*_\tau(t)\|_{U^* + V^*} + \|\nabla^*_\tau(t) - \overline{V^*_\tau}(t)\|_{U^* + V^*} \right) &\to 0
\end{align*}
\] (3.15)

as \(\tau \to 0\).

Proof. Let \((U^n_\tau)_{n=1}^N \subset U \cap V\) be the approximative values obtained from the variational approximation scheme (3.2) which satisfy by Lemma A.1 the EULER–LAGRANGE equation

\[
f^n_\tau - B(t_n, U^n_\tau - 1, V^{n-1}_\tau) - \frac{V^n_\tau - V^{n-1}_\tau}{\tau} - D\mathcal{E}_{t_n}(U^n_\tau) = \eta^n_\tau \in \partial \Psi_{U^{n-1}_\tau}(V^n_\tau)
\] (3.16)
3 PROOF OF THE MAIN RESULT

for all \( n = 1, \ldots, N \). According to Lemma A.2, inclusion (3.16) is equivalent to

\[
\Psi_{U^n}^{-1}(V^n) + \Psi_{U^n}^{-1} \left( f^n - B(t_n, U^{n-1}_\tau, V^{n-1}_\tau) - \frac{V^n - V^{n-1}_\tau}{\tau} \right) - \mathcal{E}_{t_n}(U^n_\tau)
\]

\[
= \left< f^n - B(t_n, U^{n-1}, V^{n-1}) - \frac{V^n - V^{n-1}_\tau}{\tau}, - \mathcal{E}_{t_n}(U^n_\tau), V^n \right>_{V^* \times V}, \quad n = 1, \ldots, N.
\]

(3.17)

Furthermore, the enhanced FRÉCHET subdifferentiability (2.Ef) yields

\[
- \left\langle \mathcal{E}_{t_n}(U^n_\tau), U^n - U^{n-1}_\tau \right\rangle_{(U^*_\tau + V^*_\tau) \times (U \tau V)} \leq \mathcal{E}_{t_n}(U^{n-1}_\tau) + \lambda |U^n_\tau - U^{n-1}_\tau|^2
\]

\[
= \mathcal{E}_{t_{n-1}}(U^{n-1}_\tau) - \mathcal{E}_{t_n}(U^n_\tau) + \lambda |U^n_\tau - U^{n-1}_\tau|^2 + \int_{t_{n-1}}^{t_n} \partial_r \mathcal{E}_r(U^{n-1}_\tau) \, dr
\]

(3.18)

for all \( n = 1, \ldots, N \). Then, plugging in the inequality (3.18) into (3.17) and making use of the identity

\[
(V^n - V^{n-1}, V^n) = \frac{1}{2} \left( |V^n|^2 - |V^{n-1}|^2 + |V^n - V^{n-1}|^2 \right) \quad n = 1, \ldots, N,
\]

(3.19)

as well as the fact that \( (w, v)_{V^* \times V} = (w, v) \) whenever \( v \in V \) and \( w \in H \), we obtain

\[
\frac{1}{2} |V^n|^2 + \mathcal{E}_{t_n}(U^n_\tau) + \tau \Psi_{U^n}^{-1}(V^n) \leq \mathcal{E}_{t_{n-1}}(U^{n-1}_\tau) + \int_{t_{n-1}}^{t_n} \partial_r \mathcal{E}_r(U^{n-1}_\tau) \, dr + \lambda \int_{t_{n-1}}^{t_n} |V^n|^2 \, dr + \tau \left< S^n_\tau, V^n \right>_{V^* \times V}
\]

for all \( n = 1, \ldots, N \), where \( S^n_\tau := f^n - B(t_n, U^{n-1}_\tau, V^{n-1}_\tau), n = 1, \ldots, N \). Summing up the inequalities over \( n \) yields (3.11). The estimates (3.12) and (3.13) are obtained by employing the discrete version of GRONWALL’s lemma and the following estimates: First, employing Condition (2.Bb) and the FENCHEL–YOUNG inequality, we obtain

\[
\tau \left< S^n_\tau, V^n \right> = - \left< B(t_n, U^{n-1}_\tau, V^{n-1}_\tau) + f^n_\tau, V^n \right>_{V^* \times V}
\]

\[
= - \left< B(t_n, U^{n-1}_\tau, V^{n-1}_\tau), V^n \right>_{V^* \times V} + \left< f^n_\tau, V^n \right>_{V^* \times V}
\]

\[
\leq c \tau \Psi_{U^{n-1}}(V^n) + \tau \beta \left( 1 + \mathcal{E}_{t_n}(U^{n-1}_\tau) + |V^n|^2 + \Psi_{U^{n-1}}(V^{n-1}_\tau) \right)
\]

\[
+ \tau \varepsilon \Psi_{U^{n-1}}(V^{n-1}_\tau) + \tau C
\]

for positive constants \( \varepsilon, C = C(\varepsilon, \beta) > 0 \) such that \( \varepsilon < \frac{1}{2c} \) and \( C = \frac{\beta \gamma}{\varepsilon} \). For \( n = 1 \), we choose \( r = 0 \) and for \( n \geq 2 \), we choose \( r = \tau \) and note that \( U^{n-1}_\tau - \tau V^{n-1}_\tau = U^{n-2}_\tau \). Second, using Condition (2.Ed), there holds

\[
\int_{t_{n-1}}^{t_n} \partial_r \mathcal{E}_r(U^{n-1}_\tau) \, dr \leq \int_{t_{n-1}}^{t_n} C_1 \mathcal{E}_r(U^{n-1}_\tau) \, dr \leq C_1 \int_{t_{n-1}}^{t_n} \mathcal{G}(U^{n-1}_\tau) \, dr.
\]
Inserting the obtained inequalities into (3.11) and summing up all inequalities from 1 to \( n \) we find a positive constant \( C > 0 \) such that
\[
\frac{1}{2} |V_{\tau}^n|^2 + \frac{1}{C_1} G(U_{\tau}^n) + \int_0^{t_n} \left( (1 - \alpha(\tau)) \Psi_{L_\varphi(r)}(\nabla r(r)) + \Psi^*_L(r) \left( S_r(r) - \nabla' r(r) - \xi(r) \right) \right) \, d r \\
\leq C \left( |v_0|^2 + \mathcal{E}_0(u_0) + T + \| f \|^2_{L^2(0,T;H)} + \Psi_{u_0}(v_0) \right) \\
+ C \int_0^{t_n} \left( |\nabla r(r)|^2 + G(\nabla r(r)) \right) \, d r ,
\]
where \( \alpha(\tau) := c + \tilde{c} + \tau \frac{\lambda}{\mu} < 1 \) for all \( \tau < \tau^* := \min \{ \frac{\mu}{\lambda}(1 - c - \tilde{c}), 1 \} \). In the step (3.20), we made use of the estimate for the interpolation \( f_\tau \)
\[
\| f_\tau \|^2_{L^2(0,T;H)} = \sum_{k=1}^n \tau|f_k|^2 \\
= \sum_{k=1}^n \frac{1}{\tau} \int_{\tau_{k-1}}^{\tau_k} |f(\sigma)|^2 d \sigma \\
\leq \sum_{k=1}^n \frac{1}{\tau} \int_{\tau_{k-1}}^{\tau_k} |f(\sigma)|^2 d \sigma = \int_0^{t_n} |f(\sigma)|^2 d \sigma \leq \| f \|^2_{L^2(0,T;H)}.
\]
Then, by the discrete version of GRONWALL’S lemma (see, e.g., [AGS08, Lemma 3.2.4, p. 68]), there exists a constant \( M > 0 \) such that (3.12) and (3.13) are satisfied. Now, we seek to show the bounds in (3.14). Due to the bound obtained in (3.12) and (3.13), the coercivity of \( \Psi \) and \( \Psi^* \) yield the boundedness of \( (\nabla r)_{0 < \tau \leq \tau^*} \subset L^2(0,T;V) \) and \( (\eta r)_{0 < \tau \leq \tau^*} = (S_r - \hat{V}'_r - D\mathcal{E}_0^2(\hat{U}'))_{0 < \tau \leq \tau^*} \subset L^q(0,T;V^*) \) uniformly in \( \tau < 1 \). The uniform boundedness of \( (B_\tau)_{0 < \tau \leq \tau^*} \subset L^2_n(0,T;V^*) \) follows from Assumptions (2.Bb) and (2.Ψb):
\[
\tilde{c}_M \int_0^T \| B_\tau(r) \|_{V^*} \, d r \leq \int_0^T \Psi_{\xi_\varphi(r)} \left( B(\nabla r(r), \nabla w(r), \nabla w(r)) \right) \, d r \\
\leq \int_0^T c \Psi_{\xi_\varphi(r)} \left( B(\nabla r(r), \nabla w(r), \nabla w(r)) \right) \, d r \\
\leq \int_0^T \left( C((1 + \mathcal{E}_0(r))(\nabla w(r)) \, d r + \| \nabla w(r) \|^2_{V^*} + \Psi_{\xi_\varphi(r)}(\nabla w(r)) \right) \, d r \\
\leq N
\]
for positive constants \( C, N > 0 \) independent of \( \tau \), where \( c \in (0,1) \) is from Assumption (2.Bb) and where we have used the fact that for all \( \zeta \in V^* \) the mapping \( r \mapsto r \Psi^*(\zeta/r) \) is monotonically decreasing on \( (0, +\infty) \) which follows from the convexity of \( \Psi^* \) and \( \Psi^*(0) = 0 \). Since \( (f_\tau)_{0 < \tau \leq \tau^*} \) is uniformly bounded in \( L^2(0,T;H) \), it follows that \( (\hat{V}'_\tau + \xi_\tau)_{0 < \tau \leq \tau^*} \) is uniformly bounded in \( L^{\min(q^*,2)}(0,T;V^*) \) as well. Finally, Assumption (2.Eg) implies a uniform bound for \( (D\mathcal{E}_0^2(\hat{U}'))_{0 < \tau \leq \tau^*} \) in \( L^\infty(0,T;W^*) \). Since all previous families of functions are bounded in the common space \( L^{\min(q^*,2)}(0,T;U^* + V^*) \), we deduce that \( (\hat{V}'_\tau)_{0 < \tau \leq \tau^*} \) is uniformly bounded in \( L^{q^*}(0,T;U^* + V^*) \) with respect to \( \tau \). Finally, the convergences (3.15) follow from the bounds of \( (\hat{V}'_\tau)_{0 < \tau \leq \tau^*} \subset L^{\min(q^*,2)}(0,T;U^* + V^*) \) and \( (\nabla r)_{0 < \tau \leq \tau^*} \subset L^q(0,T;V) \) and the estimates
\[
\| \nabla r(t) - \nabla r(t) \|_V \leq \| \nabla r(t) - \nabla r(t) \|_V = \int_{\tau(t)}^{\tau_1(t)} \| \nabla r(r) \|_V \, d r \quad \text{and}
\| \hat{V}'_\tau(t) - \hat{V}'(t) \|_{U^* + V^*} \leq \| \hat{V}'_\tau(t) - \hat{V}'(t) \|_{U^* + V^*} = \int_{\tau(t)}^{\tau_1(t)} \| \hat{V}'_\tau(r) \|_{U^* + V^*} \, d r
\]
for all \( t \in [0,T] \) which completes the proof.
3.3 Compactness

In this section, we prove the (weak) compactness of the approximate solutions in suitable Bochner spaces in order to pass to the limit in the weak formulation of the discrete inclusion (3.1) as the step size vanishes. After identifying all the weak limits, we will indeed obtain a solution to the Cauchy problem (1.1). The compactness result is given in the following lemma.

Lemma 3.3 (Compactness). Under the same assumptions of Lemma 3.2, let \( (\tau_n)_{n \in \mathbb{N}} \) be a vanishing sequence of step sizes and let \( u_0 \in U \cap V \) and \( v_0 \in V \). Then, there exists a subsequence, still denoted by \( (\tau_n)_{n \in \mathbb{N}} \), a pair of functions \((u, \eta)\) with

\[
  u \in C_w([0, T]; U) \cap W^{1,q}(0, T; V) \cap W^{1,\infty}(0, T; H) \cap W^{2,q^*}(0, T; U^* + V^*)
\]

and fulfilling the initial values \( u(0) = u_0 \) in \( U \) and \( u'(0) = v_0 \) in \( H \) such that the following convergences hold

\[
\begin{align*}
  U_{\tau_n}, \hat{U}_{\tau_n} &\rightharpoonup u \quad \text{in } L^\infty(0, T; U \cap V), \quad (3.23a) \\
  U_{\tau_n} &\rightharpoonup u \quad \text{in } L^r(0, T; \hat{W}) \quad \text{for any } r \geq 1, \quad (3.23b) \\
  V_{\tau_n}, \hat{V}_{\tau_n} &\rightharpoonup u' \quad \text{in } L^q(0, T; V) \cap L^\infty(0, T; H), \quad (3.23c) \\
  V_{\tau_n}, \hat{V}_{\tau_n} &\rightharpoonup u' \quad \text{in } L^q(0, T; W), \quad (3.23d) \\
  f_{\tau_n} &\rightharpoonup f \quad \text{in } L^2(0, T; H), \quad (3.23e)
\end{align*}
\]

\[
  B_{\tau_n} \rightharpoonup B(\cdot, u(\cdot), u'(\cdot)) \quad \text{in } L^q(0, T; V^*), \quad (3.23f)
\]

\[
\begin{align*}
  \hat{U}_{\tau_n}(t), \hat{V}_{\tau_n}(t), U_{\tau_n}(t) &\to u(t) \quad \text{in } U \quad \text{for all } t \in [0, T], \quad (3.23g) \\
  U_{\tau_n}(t) &\to u(t) \quad \text{in } V \quad \text{for all } t \in [0, T], \quad (3.23h) \\
  V_{\tau_n}(t), \hat{V}_{\tau_n}(t), \hat{U}_{\tau_n}(t) &\to u(t) \quad \text{in } \hat{W} \quad \text{for all } t \in [0, T], \quad (3.23i) \\
  \hat{V}_{\tau_n}(t), \hat{V}_{\tau_n}(t), \hat{U}_{\tau_n}(t) &\to u'(t) \quad \text{in } V \quad \text{for all } t \in [0, T], \quad (3.23j) \\
  \hat{V}_{\tau_n}(t), \hat{V}_{\tau_n}(t), \hat{U}_{\tau_n}(t) &\to u'(t) \quad \text{in } W \quad \text{for all } t \in [0, T], \quad (3.23k)
\end{align*}
\]

Furthermore, if the dissipation potential satisfies in addition Assumption (2.4d), then, there holds

\[
  \nabla_{\tau_n} \to u' \quad \text{in } L^{\max(2,q)}(0, T; U), \quad (3.24a) \\
  \hat{U}_{\tau_n} \to u \quad \text{in } C([0, T]; U). \quad (3.24b)
\]

Finally, the function \( u \) satisfies the inequality

\[
\frac{1}{2}|v_0|^2 + \frac{1}{2}|u'(t)|^2 + \mathcal{E}_0(u_0) - \mathcal{E}_1(u(t)) + \int_0^t \partial_r \mathcal{E}_r(u(r)) \, dr \\
\leq -\int_0^t (u''(r) + D \mathcal{E}_r(u(r)), u'(r))_{V^*, V}
\]

for almost every \( t \in (0, T) \).
3.3 COMPACTNESS

Proof. The convergences (3.23a), (3.23c), (3.23f), and (3.23g) follow (up to a subsequence) from the bounds shown in (3.14a) and Remark 2.2 i). We note that by standard arguments, we can indentify the weak limits in (3.23c) and (3.23f) to be \( u' \) and \( u'' \), respectively, denoting with \( u \) the weak limit in (3.23a). From the fact that \( L^\infty(0; T; X) \cap C_w([0, T]; Y) = C_w([0, T]; X) \) for two BANACH spaces \( X \) and \( Y \) with \( X \) being reflexive such that the continuous and dense embedding \( X \hookrightarrow Y \) holds, see, e.g., in LIONS & MAGENES [LiM68, Lemma 8.1, p. 275], we derive that for \( X = U \) and \( Y = H \), there holds \( u \in C_w([0, T]; U) \). The convergences (3.23b) and (3.23d) follow from the LIONS–AUBIN–DUBINSKIĬ lemma\(^1\) (see, e.g., [DrJ12, Theorem 1]). We proceed with proving the pointwise convergence (3.23k)-(3.23o). First, we note that, from \( \hat{V}_{\tau_n} \in W^{1,1}(0, T; U^* + V^*) \rightarrow C([0, T]; U^* + V^*) \) and (3.23f), there holds \( \hat{V}_{\tau_n} \rightarrow u'(t) \) in \( U^* + V^* \) as \( n \rightarrow \infty \) for all \( t \in [0, T] \). Since \( \hat{V}_{\tau_n}(t) \) is uniformly bounded in \( H \) for all \( t \in [0, T] \), it is (up to a subsequence) weakly convergent in \( H \) to \( u'(t) \). Since the weak limit is unique in \( U^* + V^* \), we obtain with the subsequence principle the convergence of the whole sequence. Together with the strong convergence in (3.15), this implies (3.23n). With the same argument, we can show the pointwise weak convergences (3.23k) and (3.23l) where in the latter convergence we use the fact that \( u_0 \in U \cap V \).

Since \( C([0, T]; H) \) is dense in \( L^2(0, T; H) \), there exists for every \( \varepsilon > 0 \) a function \( f^\varepsilon \in C([0, T]; H) \) such that \( \| f^\varepsilon - f \|_{L^2(0, T; H)} < \varepsilon / 3 \). We obtain

\[
\| f^\varepsilon_{\tau_n} - f \|_{L^2(0, T; H)} \leq \| f^\varepsilon_{\tau_n} - f^\varepsilon_{\tau_n} \|_{L^2(0, T; H)} + \| f^\varepsilon_{\tau_n} - f^\varepsilon \|_{L^2(0, T; H)} + \| f^\varepsilon - f \|_{L^2(0, T; H)} \\
\leq \| f - f^\varepsilon \|_{L^2(0, T; H)} + \| f^\varepsilon_{\tau_n} - f^\varepsilon \|_{L^2(0, T; H)} + \| f^\varepsilon - f \|_{L^2(0, T; H)} \\
\leq \varepsilon / 3 + \varepsilon / 3 + \varepsilon / 3 = \varepsilon
\]

for sufficiently small step sizes \( \tau_n \), where we also used the estimate (3.21) for the first term. The second term can be made smaller than \( \varepsilon / 3 \) for sufficiently small step sizes because of the uniform continuity of \( f^\varepsilon \). Now, we want to show the convergence of the perturbation in (3.23j). To do so, we denote by \( B(u)(t) = B(t, u(t), u'(t)) \), \( t \in [0, T] \), the associated NEMITSKĬ operator and recall that \( B_{\tau_n}(t) = B(\tau_{\tau_n}(t), U_{\tau_n}(t), V_{\tau_n}(t)) \). First, the pointwise convergences (3.23m) and (3.23o) together with the continuity condition (2.2a) implies

\[
\| B_{\tau_n}(t) - B(u)(t) \|_{V^*} \to 0 \quad \text{as } n \to \infty \text{ a.e. in } (0, T).
\]

By the growth condition (2.2b), we can show that \( B(u) \in L^2(0, T; V^*) \) in the same way as in (3.22). Hence, \( B_{\tau_n} - B(u) \in L^2(0, T; V^*) \) is uniformly bounded in \( n \in \mathbb{N} \) by a constant \( \bar{M} > 0 \). Then, by EGOROV’s theorem, for every \( \varepsilon > 0 \) there exists a subset \( E \subset [0, T] \) with measure \( \mu(E) < \varepsilon \) such that

\[
\lim_{n \to \infty} \sup_{t \in [0, T] \setminus E} \| B_{\tau_n}(t) - B(u)(t) \|_* = 0.
\]

Therefore, for every \( \varepsilon > 0 \) there exists an index \( N \in \mathbb{N} \) such that for all \( n \geq N \), there holds

\[
\| B_{\tau_n}(t) - B(u)(t) \|_* \leq \varepsilon \quad \text{for all } t \in [0, T] \setminus E.
\]

\(^1\)The LIONS–AUBIN–DUBINSKIĬ lemma is a particular version of the classical LIONS–AUBIN lemma.
Invoking the latter estimate, we obtain
\[
\|B_{\tau_n} - \mathcal{B}(u)\|_{L^p(0,T;V^*)} \\
\leq \left( \int_E \|B_{\tau_n}(t) - \mathcal{B}(u(t))\|_V^p \, dt \right)^{\frac{1}{p}} + \left( \int_{[0,T] \setminus E} \|B_{\tau_n}(t) - \mathcal{B}(u(t))\|_V^p \, dt \right)^{\frac{1}{p}} + \varepsilon T^{\frac{1}{p}} \\
\leq \mu(E)^{-\nu} \left( \int_E \|B_{\tau_n}(t) - \mathcal{B}(u(t))\|_V^p \, dt \right)^{\frac{1}{p}} \\
\leq \varepsilon^{-\nu} \|M\| + \varepsilon T^{\frac{1}{p}}
\]
and hence \((3.23j)\). Further, from the growth condition \((2.Eg)\), we obtain
\[
\|D\mathcal{E}_{\tau_n}^2(\mathcal{U}_{\tau_n}(t))\|_{V^*} \leq C_3(1 + \mathcal{E}^2(\mathcal{U}_{\tau_n}(t)) + \|\mathcal{U}_{\tau_n}(t)\|_{\tilde{V}})
\]
and in view of the a priori estimates \((3.12)\),
\[
\|D\mathcal{E}_{\tau_n}^2(\mathcal{U}_{\tau_n}(t))\|_{\tilde{V}^*} \leq C \quad \text{for all } t \in [0,T].
\]
Together with the convergence \((3.23m)\) and the continuity condition \((2.Ec)\), this leads to \((3.23h)\). The assertions \((3.24a)\) and \((3.24b)\) follow immediately from Assumption \((2.\Psi d)\) and
\[
\limsup_{n \to \infty} \int_0^T \|\nabla_{\tau_n}(r) - u'(r)\|_{V^*}^{\max\{p,2\}} \, dr \\
\leq \limsup_{n \to \infty} \int_0^T \langle \eta_n(r) - \eta(r), \nabla_{\tau_n}(r) - u'(r) \rangle_{V^* \times V} \, dr \leq 0
\]
and \(\eta(t) \in \partial \psi_{\Psi_u(t)}(u'(t))\) a.e. in \((0,T)\), which we will show in the proof of the main result. It remains to show the inequality \((3.25)\). The difficulty in proving the aforementioned inequality is that we are not allowed to split the duality pairing in the integral on the right-hand side and consider each integral separately. However, since \((3.25)\) is a slight modification of Lemma 6 in [EmT11], we follow their proof and regularize the function \(u'\) by its so-called STEKLOV average. For a function \(v \in L^p(0,T;X), p \geq 1\), defined on a BANACH space \(X\) and being extended by zero outside \([0,T]\), the STEKLOV average is, for sufficiently small \(h > 0\), given by
\[
S_h v(t) := \frac{1}{2h} \int_{t-h}^{t+h} v(r) \, dr.
\]
It is readily seen that \(S_h v \in L^p(0,T;X)\) and \(\|S_h v\|_{L^p(0,T;X)} \leq \|v\|_{L^p(0,T;X)}\). Furthermore, it can be shown by a regularization argument that \(S_h v \to v\) in \(L^p(0,T;X)\) as \(h \to 0\), see, e.g., [DiU77, Theorem 9, p. 49].

Defining \(K v(t) = \int_0^t v(r) \, dr\), we commence with calculating
\[
- \int_s^t \langle (S_h u')(r) + D\mathcal{E}_r(u_0 + (KS_h u')(r)), (S_h u')(r) \rangle_{V^* \times V} \, dr \\
- \int_s^t \langle (S_h u')(r) + E(u_0 + (KS_h u')(r)), D\mathcal{E}_r^2(u_0 + (KS_h u')(r)), (S_h u')(r) \rangle_{V^* \times V} \, dr \\
= \frac{1}{2}(S_h u')(s)^2 - \frac{1}{2}(S_h u')(t)^2 + \mathcal{E}^1(u_0 + (KS_h u')(s)) - \mathcal{E}^1(u_0 + (KS_h u')(t)) \\
+ \mathcal{E}^2_s(u_0 + (KS_h u')(s)) - \mathcal{E}^2_s(u_0 + (KS_h u')(t))
\]
3.4 Proof of Theorem 2.4

for all \(s, t \in [0, T]\) where we have applied the integration by parts formula, since the duality pairing can be split now due to the fact that \((S_h u')(t) = \frac{1}{2h} (\tilde{u}(t + h) - \tilde{u}(t - h))\), where \(\tilde{u}\) is a continuous extension of \(u\) outside \([0, T]\) which makes sense, since \(u \in L^\infty(0, T; U) \cap W^{1,1}(0, T; H) \subset C_w([0, T]; U)\) and therefore \(S_h u' \in L^2(0, T; U)\). However, we are not allowed to perform the limit passage after splitting up all the integrals, since the duality pairing in the limit would not be well defined because we only know that \(u'' + D\mathcal{E}_t(u) \in L^2(0, T; V^*)\). Nevertheless, since we have assumed \(V \hookrightarrow \hat{W}\), we can treat the term involving \(D\mathcal{E}_t^2 : \hat{W} \to \hat{W}^* \hookrightarrow V^*\) separately. First, taking into account

\[
u_0 + (KS_h u')(t) = u_0 + \frac{1}{2h} \int_{t-h}^{t+h} \tilde{u}(r) \, dr - \frac{1}{2} \int_{t-h}^{t+h} \tilde{u}(r) \, dr
\]

and that \(u \in C_w([0, T]; U) \subset C([0, T]; \hat{W})\) since \(U \hookrightarrow \hat{W}\), there holds

\[
\lim_{h \to 0} (u_0 + (KS_h u')) = u \quad \text{in} \quad C([0, T]; \hat{W}).
\]

Finally, by the continuity of \(\mathcal{E}_t^2\) and \(D\mathcal{E}_t^2\), the convergences (3.27) and \(S_h u' \to u'\) in \(L^q(0, T; V)\) as \(h \to 0\), there holds

\[
\begin{align*}
= - \int_s^t (D\mathcal{E}_r^2(u(r)), u'(r)) \, V^* \, dr \\
= \lim_{h \to 0} \int_s^t (D\mathcal{E}_r^2(u_0 + (KS_h u')(r)), (S_h u')(r)) \, V^* \, dr \\
= \lim_{h \to 0} \left( \mathcal{E}_s^2(u_0) - \mathcal{E}_t^2(u_0) \right) - \mathcal{E}_s^2(u_0) + \mathcal{E}_t^2(u_0) + (KS_h u')(s) - \mathcal{E}_s^2(u_0 + (KS_h u')(t)) \\
= \mathcal{E}_s^2(u_0) - \mathcal{E}_t^2(u_0) \tag{3.28}
\end{align*}
\]

for all \(s, t \in [0, T]\). Second, it has been shown in [EmT11, Lemma 6] that

\[
- \int_0^t (u''(r) + E(u(r)), u'(r)) \, V^* \, dr \\
\leq \frac{1}{2} |v_0|^2 - \frac{1}{2} |u'(t)|^2 + \mathcal{E}(u_0) - \mathcal{E}(u(t))
\]

for almost every \(t \in (0, T)\). The latter inequality together with (3.28) implies (3.25), which completes the proof.

\[\square\]

3.4 Proof of Theorem 2.4

Let \(u_0 \in U, v_0 \in H\) and \((\tau_n)_{n \in N}\) be a vanishing sequence of positive step sizes. Let \((u_0^k)_{k \in \mathbb{N}} \subset U \cap V\) and \((v_0^k)_{k \in \mathbb{N}} \subset V\) be such that \(u_0^k \to u_0\) in \(U\) and \(v_0^k \to v_0\) in \(H\) as \(k \to \infty\). We let \(k \in \mathbb{N}\) be fixed and we denote the interpolations associated with the initial data \(u_0^k\) and \(v_0^k\) again by (3.6)-(3.9). Henceforth, we suppress the dependence of the interpolations on \(k\) for simplicity. By the previous lemma, there exists a subsequence (relabeled as before) of the interpolations and limit functions \(u \in C_w([0, T]; U) \cap W^{1,\infty}(0, T; H) \cap W^{1,1}(0, T; V^*) \cap W^{1,\infty}(0, T; U^* + V^*)\) (notice that \(u_0^k \in U \cap V\) and \(u(0) = u_0^k\) in \(U\) and \(u'(0) = v_0^k\) in \(H\) such that the convergences (3.23) hold, where we again suppress the dependence of the limit functions on \(k\). First, we prove that the inclusion (2.4) holds. To do so, we note that the Euler–Lagrange equation (3.16) reads

\[
\begin{align*}
\tilde{V} \tau_n(t) + \eta_\tau_n(t) + D\mathcal{E}_\tau_n(t)(\tilde{U} \tau_n(t)) + S \tau_n(t) &= 0 \quad \text{in} \quad U^* + V^*, \\
\eta_n(t) &\in \partial_V \Psi_{\tau_n}(t)(\tilde{V} \tau_n(t)) \tag{3.29}
\end{align*}
\]
for all \( t \in (0, T) \), where \( S_{\tau_n}(t) = B(\overline{U}_{\tau_n}(t), \underline{V}_{\tau_n}(t), \overline{U}_{\tau_n}(t)) - f_{\tau_n}(t), t \in [0, T] \). Testing equation (3.29) with \( w \in L^{\text{max}(2,q)}(0, T; U \cap V) \), we obtain

\[
\int_0^T \langle \dot{V}_{\tau_n}(r) + \eta_{\tau_n}(r) + D\mathcal{E}_{\tau_n}(\overline{U}_{\tau_n}(s)) \rangle + S_{\tau_n}(r), w(r) \rangle_{U^* + V^*} \, dr = 0.
\]

Then, with the aid of the convergences (3.23), we are allowed to pass to the limit in the weak formulation obtaining

\[
\int_0^T \langle u''(r) + \eta(r) + D\mathcal{E}_u(u(s)) + B(t, u(r), u'(r)) - f(r), w(r) \rangle_{U^* + V^*} \, dr = 0
\]

for all \( w \in L^{\text{max}(2,q)}(0, T; U \cap V) \). Then, by a density argument and the fundamental lemma of calculus of variations, we deduce

\[
u'(t) + \eta(t) + D\mathcal{E}_u(u(t)) + B(t, u(t), u'(t)) = f(t) \quad \text{in } U^* + V^*
\]

for a.e. \( t \in (0, T) \). We shall identify the weak limit \( \eta \) as subgradient of the dissipation potential almost everywhere, i.e., \( \eta(t) \in \partial V_\Psi_a(t)(u(t)) \) for almost every \( t \in (0, T) \). For that purpose, we will employ Lemma A.4 with \( f_n(t, v) = \Psi_{U_{\tau_n}}'(v) \) and \( f(t, v) = \Psi_{u(t)}(v) \) for all \( v \in X = V \) and \( n \in \mathbb{N} \). Assumption (A.3) is already fulfilled by Condition (2.Ψc). Hence, it remains to show that

\[
\limsup_{n \to \infty} \int_0^T \langle \eta_n(t), \overline{V}_{\tau_n}(t) \rangle_{V^* \times V} \, dt \leq \int_0^T \langle \eta(t), u'(t) \rangle_{V^* \times V} \, dt.
\]

In order to show the latter lim sup estimate, we use the fact that \( \eta_{\tau_n} \) can be expressed through the remaining terms of the EULER–LAGRANGE equation (3.29). Therefore, we will split the integral on the left-hand side of (3.30) and note first that

\[
-\int_0^t \langle \dot{V}_{\tau_n}(r), \nabla_{\tau_n}(r) \rangle_{V^* \times V} \, dr
\]

\[
= -\int_0^t \langle \dot{V}_{\tau_n}(r), \nabla_{\tau_n}(r) \rangle_{V^* \times V} \, dr + \int_0^t \langle \dot{V}_{\tau_n}(r), \nabla_{\tau_n}(r) \rangle_{V^* \times V} \, dr
\]

\[
= \frac{1}{2} |v_0|^2 - \frac{1}{2} |\nabla_{\tau_n}(t)|^2 + \int_0^t \langle \dot{V}_{\tau_n}(r), \nabla_{\tau_n}(r) \rangle_{V^* \times V} \, dr
\]

\[
\leq \frac{1}{2} |v_0|^2 - \frac{1}{2} |\nabla_{\tau_n}(t)|^2,
\]

where we used the fundamental theorem of calculus for the absolutely continuous function \( t \mapsto \frac{1}{2} |\nabla_{\tau_n}(t)|^2 \) and that the estimate

\[
\int_0^t \langle \dot{V}_{\tau_n}(r), \nabla_{\tau_n}(r) \rangle_{V^* \times V} \, dr
\]

\[
= \sum_{i=1}^{m-1} \int_{t_{i-1}}^{t_i} \left( \frac{V^r_{\tau_n} - V^{i-1}_{\tau_n}}{\tau_n}, \frac{V^r_{\tau_n} - t_i - t_{i-1}}{\tau_n} - V^{i}_{\tau_n} \right) \, dr
\]

\[
+ \int_{t_{m-1}}^{t} \left( \frac{V^m_{\tau_n} - V^{m-1}_{\tau_n}}{\tau_n}, \frac{V^m_{\tau_n} - t_m - t_{m-1}}{\tau_n} - V^{m}_{\tau_n} \right) \, dr
\]

\[
= -\sum_{i=1}^{m-1} \int_{t_{i-1}}^{t_i} \left( \frac{V^r_{\tau_n} - V^{i-1}_{\tau_n}}{\tau_n}, \frac{V^i_{\tau_n} - V^{i-1}_{\tau_n}}{\tau_n} \frac{t_i - r}{\tau_n} \right) \, dr
\]

\[
- \int_{t_{m-1}}^{t} \left( \frac{V^m_{\tau_n} - V^{m-1}_{\tau_n}}{\tau_n}, \frac{V^m_{\tau_n} - V^{m-1}_{\tau_n}}{\tau_n} \frac{t_m - r}{\tau_n} \right) \, dr
\]

\[
= -\sum_{i=1}^{m-1} \int_{t_{i-1}}^{t_i} \frac{t_i - r}{\tau_n^2} |V^i_{\tau_n} - V^{i-1}_{\tau_n}|^2 \, dr - \int_{t_{m-1}}^{t} \frac{t_m - r}{\tau_n^2} |V^m_{\tau_n} - V^{m-1}_{\tau_n}|^2 \, dr \leq 0
\]
with \( t \in (t_{m-1}, t_m] \) for some \( m \in \{1, \ldots, N\} \).

We continue with the term involving the derivative of the energy functional and start with the linear part:

\[
- \int_0^t \langle E\mathring{\partial}_{\tau_n}(r), \mathring{\nabla}_{\tau_n}(r) \rangle_{U^* \times U} \, dr
\]

\[
= - \int_0^t \langle E\mathring{\partial}_{\tau_n}(r), \nabla_{\tau_n}(r) \rangle_{U^* \times U} \, dr + \int_0^t \langle E\mathring{\partial}_{\tau_n}(r), \mathring{\nabla}_{\tau_n}(r) \rangle_{U^* \times U} \, dr
\]

\[
= \mathcal{E}^i(u_0) - \mathcal{E}^i(\mathring{U}_{\tau_n}(t)) + \int_0^t \langle \mathring{E}(\mathring{U}_{\tau_n}(r) - \mathring{U}_{\tau_n}(r)), \mathring{\nabla}_{\tau_n}(r) \rangle_{U^* \times U} \, dr
\]

\[
\leq \mathcal{E}^i(u_0) - \mathcal{E}^i(\mathring{U}_{\tau_n}(t)),
\]

where we used

\[
\int_0^t \langle \mathring{E}(\mathring{U}_{\tau_n}(r) - \mathring{U}_{\tau_n}(r)), \mathring{\nabla}_{\tau_n}(r) \rangle_{U^* \times U} \, dr \leq 0,
\]

which can be shown in the same way as above by using the strong positivity of \( E \). As for the nonlinear part, we obtain by employing the \( \lambda \)-convexity of \( \mathcal{E}^2_i \) that

\[
- \int_0^t \langle \mathcal{D}\mathcal{E}^2_{\tau_n}(t), \mathring{\nabla}_{\tau_n}(r) \rangle_{U^* \times U} \, dr
\]

\[
= - \sum_{i=1}^{m-1} \langle \mathcal{D}\mathcal{E}^2_{t_i}(U_n), U_i - U_{t_i} \rangle_{U^* \times U} - \frac{t - t_{m-1}}{\tau_n} \langle \mathcal{D}\mathcal{E}^2_{t_{m-1}}(U_{m-1}), U_{m-1} - U_{t_{m-1}} \rangle_{U^* \times U}
\]

\[
\leq - \sum_{i=1}^{m-1} \left( \mathcal{E}^2_{t_i}(U_i - U_{t_i}) - \mathcal{E}^2_{t_i}(U_{t_i}) - \lambda \left| U_i - U_{t_i} \right|^2 \right)
\]

\[
- \frac{t - t_{m-1}}{\tau_n} \left( \mathcal{E}^2_{t_{m-1}}(U_{m-1}) - \mathcal{E}^2_{t_{m-1}}(U_{t_{m-1}}) - \lambda \left| U_{m-1} - U_{t_{m-1}} \right|^2 \right)
\]

\[
= - \sum_{i=1}^{m-1} \left( \mathcal{E}^2_{t_i}(U_i - U_{t_i}) - \mathcal{E}^2_{t_i}(U_{t_i}) + \int_{t_{i-1}}^{t_i} \partial_r \mathcal{E}^2_r(U_{t_i}) \, dr + \lambda \tau_n \left| V_{t_i} \right|^2 \right)
\]

\[
+ \frac{t_m - t}{\tau_n} \left( \mathcal{E}^2_{t_{m-1}}(U_{m-1}) - \mathcal{E}^2_{t_{m-1}}(U_{t_{m-1}}) - \lambda \left| U_{m-1} - U_{t_{m-1}} \right|^2 \right)
\]

\[
= \mathcal{E}^2_0(u_0) - \mathcal{E}^2_{\tau_n(t)}(\mathring{U}_{\tau_n}(t)) + \int_0^{\tau_n(t)} \partial_r \mathcal{E}^2_r(\mathring{U}_{\tau_n}(r)) \, dr + I_n(t),
\]

where

\[
I_n(t) = \frac{t_m - t}{\tau_n} \left( \mathcal{E}^2_{t_{m-1}}(U_{m-1}) - \mathcal{E}^2_{t_{m-1}}(U_{t_{m-1}}) - \lambda \left| U_{m-1} - U_{t_{m-1}} \right|^2 \right) + \lambda \tau_n \int_0^{\tau_n(t)} \left| \mathring{\nabla}_{\tau_n}(r) \right|^2 \, dr.
\]

Now, we want to make use of the inequality (3.25). However, the aforementioned inequality only holds true for almost every \( t \in (0, T) \). Therefore, we take a sequence of increasing values \((\beta_i)_{i \in \mathbb{N}}, \beta_i \in (0, T)\) for all \( i \in \mathbb{N} \), converging to \( T \) for which (3.25) holds true. Then, choosing \( t = \beta_i \), we obtain with the convergences (3.23k), (3.23n), (3.23m), and (3.23d), the sequential weak lower semicontinuity of \( \mathcal{E}^1_i \) and \( | \cdot | \) and the continuity of \( \mathcal{E}^2_i \), the limes...
superior condition and growth condition (2.3d) on \( \partial_t \mathcal{E}_t^2 \) and FATOU’s Lemma that

\[
\limsup_{n \to \infty} -\int_0^{\beta_t} \langle \dot{V}_r(t), r \rangle + D\mathcal{E}_{\tau_n}(r(\overline{U}_{\tau_n}(r)), \nabla_{\tau_n}(r))_{V^* \times V} \, dr \\
\leq \limsup_{n \to \infty} \left( \frac{1}{2} |v_0|^2 - \frac{1}{2} |\dot{V}_r(\beta_t)|^2 + \mathcal{E}_0(u_0) - \mathcal{E}_1(\dot{U}_{\tau_n}(\beta_t)) - \mathcal{E}_{\tau_n}(\beta_t)(\nabla_{\tau_n}(\beta_t)) \right) \\
\quad + \int_0^{\beta_t} \partial_r \mathcal{E}_r(\nabla_{\tau_n}(r)) \, dr + I_n(t) \\
\leq \frac{1}{2} |v_0|^2 - \frac{1}{2} |u'(\beta_t)|^2 + \mathcal{E}_0(u_0) - \mathcal{E}_1(u(\beta_t)) + \int_0^{\beta_t} \partial_r \mathcal{E}_r(u(r)) \, dr.
\]

Since \( u \in C_{w}([0, T]; U) \) and \( u' \in L^\infty(0, T; H) \cap W^{1,1}(0, T; U^* + V^*) \subset C_{w}([0, T]; H) \), Lemma 3.3 then yields

\[
\frac{1}{2} |v_0|^2 - \frac{1}{2} |u'(\beta_t)|^2 + \mathcal{E}_0(u_0) - \mathcal{E}_1(u(\beta_t)) + \int_0^{\beta_t} \partial_r \mathcal{E}_r(u(r)) \, dr \\
\leq - \int_0^{\beta_t} (u''(r) + D\mathcal{E}_r(u(r)), u'(r))_{V^* \times V}.
\]

Then, in view of the convergences (3.23i) and (3.23j), the EULER–LAGRANGE equation (3.29), we obtain

\[
\limsup_{n \to \infty} \int_0^{\beta_t} \langle \eta_n(t), \nabla_{\tau_n}(t) \rangle_{V^* \times V} \, dt \\
= \limsup_{n \to \infty} \int_0^{\beta_t} \langle S_{\tau_n}(t) - \dot{V}_r(t) - D\mathcal{E}_{\tau_n}(t)(\overline{U}_{\tau_n}(t)), \nabla_{\tau_n}(t) \rangle_{V^* \times V} \, dt \\
\leq \int_0^{\beta_t} \langle f(t) - B(t, u(t), u'(t)) - u''(t) - D\mathcal{E}_r(u(t)), u'(t) \rangle_{V^* \times V} \, dt \\
= \int_0^{\beta_t} \langle \eta(t), u'(t) \rangle_{U^* \times U} \, dt.
\]

Together with Condition (2.3c) and Lemma A.4, this implies \( \eta(t) \in \partial_t \Psi_{u(t)}(u'(t)) \) for almost every \( t \in (0, \beta_t) \) for all \( l \in \mathbb{N} \). Letting \( l \to \infty \) leads to \( \eta(t) \in \partial_t \Psi_{u(t)}(u'(t)) \) for almost every \( t \in (0, T) \). This shows for each \( k \in \mathbb{N} \) the existence of a function \( u \) satisfying the inclusion (2.4), and the initial values \( u(0) = u_0^k \in U \cap V \) and \( u'(0) = v_0^k \in V \). Denote with \( (u_k)_{k \in \mathbb{N}} \) the sequence of solutions to the associated sequence of initial values, and with \( (\eta_k)_{k \in \mathbb{N}} \) the subgradients of \( \Psi_{u_k(t)}(u_k'(t)) \). In the last step, we want to show that there exists a limit function \( u \) which satisfies (2.4) and (2.5) as well as the initial values \( u(0) = u_0 \in U \) and \( u'(0) = v_0 \in H \). We recall that \( u_k^0 \to u_0 \) in \( U \) and \( v_k^0 \to v_0 \) in \( H \) as \( k \to \infty \). The next steps are the following:

1. We derive a priori estimates based on the energy-dissipation inequality (2.5),

2. We show compactness of the sequences \( (u_k)_{k \in \mathbb{N}} \) and \( (\eta_k)_{k \in \mathbb{N}} \) in appropriate spaces,

3. We pass to the limit in the inclusion 2.4 as \( k \to \infty \).

**Ad 1.** Let \( t \in [0, T] \) and \( \mathcal{N} \subset (0, T] \) a set of measure zero such that \( \mathcal{E}_{\tau_n}(\overline{U}_{\tau_n}(s)) \to \mathcal{E}_t(u(s)) \) and \( \nabla_{\tau_n}(s) \to u'(s) \) for each \( s \in [0, T] \setminus \mathcal{N} \). Then, employing the convergences
(3.23), we obtain
\[
\frac{1}{2}|u_k'(t)|^2 + \mathcal{E}_t(u_k(t)) + \int_0^t \left( \psi_{u_k(r)}(u_k'(r)) + \psi_{u_k(r)}^*(S_k(r) - D\mathcal{E}_r(u_k(r)) - u_k''(r)) \right) \, dr
\]
\[
\leq \liminf_{n \to \infty} \left( \frac{1}{2} |\nabla \tau_n(t)|^2 + \mathcal{E}_t(\tau_n(t)) \right)
\]
\[
+ \int_0^{\tau_n(t)} \left( \psi_{\tau_n(r)}(\nabla \tau_n(r)) + \psi_{\tau_n(r)}^*(S_{\tau_n(r)} - D\mathcal{E}_{\tau_n(r)}(\nabla \tau_n(r)) - \nabla \tau_n'(r)) \right) \, dr
\]
\[
\leq \limsup_{n \to \infty} \left( \frac{1}{2} |v_0|^2 + \mathcal{E}_0(u_0^k) + \int_0^{\tau_n(t)} \partial_r \mathcal{E}_r(U_{\tau_n(r)}) \, dr \right)
\]
\[
+ \int_0^{\tau_n(t)} \langle S_{\tau_n(r)}, \nabla \tau_n(r) \rangle V^* \, dr + \tau \lambda \int_0^{\tau_n(t)} \|\nabla \tau_n(r)\|^2 \, dr
\]
\[
= \frac{1}{2} |v_0|^2 + \mathcal{E}_0(u_0^k) + \int_0^t \partial_r \mathcal{E}_r(u_k(r)) \, dr + \int_0^t \langle S(r), u_k'(r) \rangle V^* \, dr
\]
for all \( t \in [0, T] \), where \( S_k(r) = f(r) - B(r, u_k(r), u_k'(r)) \). Again, taking into account Condition (2.Ed), (2.Bb), and (2.Bb), we obtain with the classical lemma of GRONWALL
\[
\frac{1}{2}|u_k'(t)|^2 + \mathcal{E}_t(u_k(t)) + \int_0^t \left( \psi_{u_k(r)}(u_k'(r)) + \psi_{u_k(r)}^*(S_k(r) - D\mathcal{E}_r(u_k(r)) - u_k''(r)) \right) \, dr \leq C_B.
\]
for all \( t \in [0, T] \) for a constant \( C_B > 0 \).

**Ad 2.** With the same reasoning as for the interpolations, we obtain the convergences
\[
\begin{align*}
\lim u_k & \to u \quad \text{in } L^\infty(0, T; U), \quad (3.31a) \\
n_k - u_0^k & \to u - u_0 \quad \text{in } L^\infty(0, T; V), \quad (3.31b) \\
u_k(t) & \to u(t) \quad \text{in } U \quad \text{for all } t \in [0, T], \quad (3.31c) \\
u_k(t) - u_0^k & \to u(t) - u_0 \quad \text{in } V \quad \text{for all } t \in [0, T], \quad (3.31d) \\
u_k & \to u \quad \text{in } L^r(0, T; \tilde{W}) \quad \text{for any } r \geq 1, \quad (3.31e) \\
u_k(t) & \to u(t) \quad \text{in } \tilde{W} \quad \text{for all } t \in [0, T], \quad (3.31f) \\
u_k'(t) & \to u' \quad \text{in } L^q(0, T; V) \cap L^\infty(0, T; H), \quad (3.31g) \\
u_k''(t) & \to u'' \quad \text{in } L^0(0, T; H) \quad \text{for all } p \geq 1, \quad (3.31h) \\
u_k'(t) & \to u'(t) \quad \text{in } H \quad \text{for all } t \in [0, T], \quad (3.31i) \\
\eta^k_{\tau_n} & \to \eta \quad \text{in } L^q(0, T; V^*), \quad (3.31j) \\
\mathcal{E}_t & \to \mathcal{E}_t \quad \text{in } L^2(0, T; U^*), \quad (3.31k) \\
\lim \mathcal{E}_t^2 & \to \mathcal{E}_t^2 \quad \text{in } L^r(0, T; U^*) \quad \text{for any } r \geq 1, \quad (3.31l) \\
u_k'' & \to u'' \quad \text{in } L^{\min(2,q)}(0, T; U^* + V^*), \quad (3.31m) \\
B(\cdot, u_k, u_k') & \to B(\cdot, u, u') \quad \text{in } L^r(0, T; V^*), \quad (3.31n)
\end{align*}
\]
and if \( \psi_u \) satisfies (2.\( \Psi \)d), then
\[
\begin{align*}
u_k' & \to u' \quad \text{in } L^{\max(2,q)}(0, T; U), \quad (3.31n) \\
u_k & \to u \quad \text{in } C([0, T]; U).
\end{align*}
\]

**Ad 3.** Therefore, \( u \in C_w([0, T]; U) \cap W^{1,\infty}([0, T]; H) \cap W^{2,q'}(0, T; U^* + V^*) \) with \( u - u_0 \in W^{1,q}(0, T; V) \) and \( \eta \in L^{q'}(0, T; V^*) \) satisfies the initial conditions \( u(0) = u_0 \) in \( U \) and
\[ u'(0) = v_0 \] in \( H \). Along the same lines as for the interpolations, we obtain with Condition (2.Ψc) and Lemma A.4 that \( u \) and \( \eta \) satisfy the inclusions (2.4). It remains to show the energy-dissipation balance (2.5). The inequality
\[
\frac{1}{2} |u'(t)|^2 + E_t(u(t)) + \int_0^t \left( \psi_{u(r)}(u'(r)) + \psi_{u(r)}^*(S(r) - D\mathcal{E}_r(u(r)) - u''(r)) \right) \, dr \\
\leq \frac{1}{2} |v_0|^2 + E_0(u_0) + \int_0^t \partial_r \mathcal{E}_r(u(r)) \, dr + \int_0^t \langle S(r), u'(r) \rangle_{V^* \times V} \, dr,
\]
for all \( t \in [0, T] \) with \( S(r) = f(r) - B(\xi, u(r), u'(r)) \) is obtained by passing to the limit as \( k \to \infty \) while taking into account the convergences (3.23). Then, employing again (3.25) and the Fenchel–Young inequality, we obtain
\[
\int_0^t \left( \psi_{u(r)}(u'(r)) + \psi_{u(r)}^*(S(r) - D\mathcal{E}_r(u(r)) - u''(r)) \right) \, dr \\
\leq \frac{1}{2} |v_0|^2 - \frac{1}{2} |u'(t)|^2 + E_0(u_0) - E_T(u(t)) + \int_0^t \partial_r \mathcal{E}_r(u(r)) \, dr \\
+ \int_0^t \langle S(r), u'(r) \rangle_{V^* \times V} \, dr \\
\leq \int_0^t \langle D\mathcal{E}_r(u(r)) - u''(r), u'(r) \rangle_{V^* \times V} \, dr + \int_0^t \langle S(r), u'(r) \rangle_{V^* \times V} \, dr \\
= \int_0^t \langle S(r) - D\mathcal{E}_r(u(r)) - u''(r), u'(r) \rangle_{V^* \times V} \, dr \\
\leq \int_0^t \left( \psi_{u(r)}(u'(r)) + \psi_{u(r)}^*(S(r) - D\mathcal{E}_r(u(r)) - u''(r)) \right) \, dr
\]
for almost every \( t \in (0, T) \). Now, if \( V \hookrightarrow U \), then the inequality (3.25) indeed holds as equality for all \( t \in [0, T] \) by the classical integration by parts formula. This shows (2.5), and hence the completion of the proof.

**Remark 3.4.** If we take a closer look at the proof, we notice that we are allowed to consider the case \( b \equiv 0 \) if the compact embedding \( V \hookrightarrow \widetilde{W} \) holds. This ensures that we pass to the limit in the nonlinear term \( D\mathcal{E}_2 \).

**Remark 3.5.** The proof of Theorem 2.4 reveals that one can consider dissipation potentials that depend on a parameter \( \varepsilon \). In this case, the Condition (2.Ψa) is assumed to hold for every \( \varepsilon \geq 0 \) while Condition (2.Ψb) holds uniformly in \( \varepsilon \geq 0 \). Condition (2.Ψc) can either be replaced with the Mosco-convergence \( \psi_{u_n}^0 \xrightarrow{M} \psi_u^0 \) for every sequence \( u_n \rightharpoonup u \) as \( \varepsilon \downarrow 0 \), or with a more general liminf estimate (A.3).

**Remark 3.6.** We can also allow a nonsmooth energy functionals \( \mathcal{E}^2 \) by regularizing the energy functional using the \( p \)-Moreau–Yosida regularization and then passing to the limit as the regularization parameter vanishes. In that case we have to impose convexity, a growth condition as well as time independence of \( \mathcal{E}^2 \). We refer to [Bac20] where this case has been executed in the case where the principal part of the operator acting on the solution is linear. Since this can be done in the exact same way as in [Bac20], this will not be considered here.

## 4 Applications

In this section, we want to apply the abstract result developed in the previous sections to a concrete examples. First, we discuss in detail some physically motivated example to illustrate the strength of the theory.
5 Visco-elasto-plastic model for martensitic phase transformation in shape-memory alloys

In this example, we consider equations which describe a solid-solid phase transition in shape-memory alloys driven by stored-energy and a dissipation mechanism. As critically discussed in [RaR03], a commonly used model describing this phenomena is for the isothermal case given by

\[ \rho \partial_t u + \nu (-1)^n \Delta^n \partial_t u - \nabla \cdot (\sigma(\nabla u)) + \mu (-1)^m \Delta^m u = f, \]

where \( f \in L^2(0,T;H^{-1,2}(\Omega)^d) \), \( m, n \in \mathbb{N} \) and \( \mu, \nu \geq 0 \) are non-negative real values. Here, \( \rho \geq 0 \) denotes the density of the body, \( u : \Omega \times [0,T] \to \mathbb{R}^d \) the displacement of the body, which is related to the deformation \( y \) by \( u(x,\cdot) = y(x,\cdot) - x \) on a reference body configuration \( \Omega \), and \( \sigma : \mathbb{R}^{d \times d} \to \mathbb{R}^{d \times d} \) the Piola–Kirchhoff stress tensor depending on the gradient \( \nabla u \). The stress \( \sigma \) is, in general, not monotone and for hyperelastic materials given by the derivative of a potential \( \varphi : \mathbb{R}^{d \times d} \to \mathbb{R} \) describing the specific stored energy, i.e., \( \sigma = \varphi' \) in turn is not quasiconvex. The contribution of \( \mu (-1)^m \Delta^m u \) in the equations models a capillarity-like behaviour of the solid and \( \nu (-1)^n \Delta^n \partial_t u \) describes a higher order viscosity. According to the authors, experiments show that the hysteretic phenomena in shape memory alloys are rate independent. That implies that the equation (5.1) does not model plasticity effects appropriately. The authors in [RaR03] suggest to incorporate a correction term into the equations which describes plasticity effects of the body. More precisely, they introduce a dissipation function \( \lambda \) that is nonnegative and homogeneous of degree one that captures this hysteretic response. The governing equations are then given by

\[ \rho \partial_t u + \nu (-1)^n \Delta^n \partial_t u - \nabla \cdot (\sigma_p + \sigma(\nabla u)) + \mu (-1)^m \Delta^m u = f, \]

\[ \sigma_p \in \text{Sgn} (\lambda'(\nabla u(x))) : \nabla \partial_t u(x)) \lambda'(\nabla u(x)), \]

where \( \text{Sgn} : \mathbb{R} \to \mathbb{R} \) is here the multi-valued and one-dimensional sign function, \( \sigma_p : \mathbb{R}^{d \times d} \to \mathbb{R}^{d \times d} \) is the plastic stress, and \( \lambda : \mathbb{R}^{d \times d} \to \mathbb{R} \) is a so-called phase indicator and thus indicates the status of the phase of \( \nabla u \). For a more physical discussion of the model, we refer to [PIR02, RaR03, AGR03] and the references therein where the model has been studied extensively by ROUBIČEK and coauthors. In [PIR02], the authors showed the existence of weak solutions for the case \( n = m = 2 \), \( \alpha > 0 \), and \( \beta > 0 \). In [RaR03], the authors show the existence of very weak solution for the critical cases \( n = 0, \nu, \mu \geq 0 \) and \( m \geq 3 \) which can not be tackled in our framework due to the growth condition on \( \Psi \) that requires \( p > 1 \), see Condition (2.Ψc).

With the theory developed here, we show the existence of solutions for the cases \( \nu, \mu > 0 \) and \( n \geq 1, m \geq 2 \) which is not know in the literature under the assumptions on \( \phi \) presented here. We note that our theory also allows the case \( \mu = 0 \) since \( \nu > 0 \) creates enough regularity, see Remark 3.4. For the sake of simplicity, we supplement the equation (5.1) with homogeneous DIRICHLET & NEUMANN boundary conditions. Non-homogeneous can be considered in a standard way, see [PIR02]. Before we state the main result, we set a system of equations and define the associated functionals, operators and

\[ \text{See, e.g., [Rou13, Remark 6.5, p. 175].} \]
spaces. We consider the initial-boundary value problem

\[
\begin{aligned}
(\text{P1}) & \\
\begin{cases}
\rho \partial_t u + \nu (-1)^n \Delta^n \partial_t u - \nabla \cdot (\sigma_p + \sigma(\nabla u)) + \mu (-1)^m \Delta^m u = f & \text{in } \Omega_T, \\
\sigma_p(x, t) \in Sgn(\lambda' \langle \nabla u(x, t) \rangle : \nabla \partial_t u(x, t)) \lambda'(\nabla u(x, t)) & \text{a.e. in } \Omega_T, \\
u(x, 0) = u_0(x) & \text{on } \Omega, \\
u'(x, 0) = v_0(x) & \text{on } \Omega, \\
\frac{\partial \nu}{\partial n}(x, t) = 0 & \text{on } \partial \Omega \times [0, T], k = 0, \ldots, \max\{m, n\} - 1,
\end{cases}
\end{aligned}
\]

where \(Sgn : \mathbb{R} \rightarrow \mathbb{R}\) is here the multi-valued and one-dimensional sign function, \(\sigma_p : \mathbb{R}^{d \times d} \rightarrow \mathbb{R}^{d \times d}\) is the plastic stress, and \(\lambda : \mathbb{R}^{d \times d} \rightarrow \mathbb{R}\) is a so-called phase indicator and thus indicates the phase status of \(\nabla u\). Moreover, \(\rho : \mathbb{R}^d \rightarrow [0, \infty)\) is a measurable function satisfying \(\rho \geq \rho(x) \geq \rho > 0\) for a.e. \(x \in \Omega\). We want to show the existence of a weak solution to (P3) for any initial data \(u_0 \in H^m_0(\Omega)^d\) and \(v_0 \in L^2(\Omega)^d\) and external forces \(f \in L^2(0, T; H^{-\max\{m, n\}}(\Omega)^d)\) in the following sense: there exists a function \(u \in C_w(0, T; H^1(\Omega)^d) \cap W^{1, \infty}(0, T; L^2(\Omega)^d) \cap H^2(0, T; H^{-m}(\Omega)^d + H^{-n}(\Omega)^d)\) with \(u - u_0 \in H^1(0, T; H^2(\Omega)^d)\) and \(\sigma_p \in L^2(0, T; H^{-\max\{m, n\}}(\Omega)^d)\) satisfying the initial conditions \(u(0) = u_0, v(0) = v_0\), the integral equation

\[
\int_0^T \left( \langle \rho u'', v \rangle + \int_\Omega (\nu \nabla^2 \partial_t u : \nabla^2 v + (\sigma_p + \sigma(\nabla u)) : \nabla v + \mu \nabla^m u : \nabla^m v) \, dx \right) \, dt,
\]

\[
= \int_0^T \langle f, v \rangle \, dt \quad \text{for all } v \in L^2(0, T; H^0_{\max\{m, n\}}(\Omega)^d),
\]

(5.2)

such that \(\sigma_p(x, t) \in Sgn(\lambda' \langle \nabla u(x, t) \rangle : \nabla \partial_t u(x, t)) \lambda'(\nabla u(x, t))\) a.e. in \(\Omega_T\) and the energy-dissipation balance

\[
\frac{1}{2} \|\rho u'(t)\|_{L^2(\Omega)^d}^2 + \frac{\mu}{2} \|u(t)\|_{H^2(\Omega)^d}^2 + \int_\Omega \varphi(\nabla u(t)) \, dx + \int_{\Omega} \text{Var}(\lambda(\nabla u)) \, dx + \int_0^t \frac{\nu}{2} \|u'(r)\|_{H^2(\Omega)^d}^2 \, dr
\]
\[
+ \int_0^t \psi(t)(f(r) - \nabla \cdot (\sigma_p + \sigma(\nabla u(r))) + \mu (-1)^m \Delta^m u(r) - u''(r)) \, dr
\]
\[
= \frac{1}{2} \|\rho v_0\|_{L^2(\Omega)^d}^2 + \frac{\mu}{2} \|u_0\|_{H^2(\Omega)^d}^2 + \int_\Omega \varphi(\nabla u_0) \, dx + \int_0^t \langle f(t), u'(r) \rangle_{V^* \times V} \, dr,
\]

(5.3)

for almost all \(t \in (0, T)\) and if \(n \geq m\), then for all \(t \in (0, T)\), where \(\psi_u\) denotes the convex conjugate of \(\psi_u\) defined below. Applying [EkT99, Corollary 3.5, p. 335] and [AtB86, Theorem 1.1, pp. 126], one can show that

\[
\psi_u^*(\xi) = \min_{\eta \in H^m_0(\Omega)^d} \left\{ \min_{\varphi \in L^2(\Omega)^d} \int_\Omega f^*(x, \varphi(x)) \, dx + \frac{1}{2\nu} \|\xi - \eta\|_{H^2(\Omega)^d}^2 \right\},
\]

where \(f^*\) is the real-valued convex conjugate of \(f(x, p) = \lambda'(\nabla u(x)) : p\) with respect to \(p \in \mathbb{R}^{m \times d}\). Moreover, \(\text{Var}(\lambda)\) denotes the total variation of \(\lambda\) over \([0, t]\) and \(\langle \cdot, \cdot \rangle\) denotes the duality pairing between \(H^m_0(\Omega)^d\) and its dual space \(H^{-\max\{m, n\}}(\Omega)^d\), where \(H^k(\Omega)^d\) is the SOBOLEV space of all measurable functions whose weak derivative exist up to the order \(k \in \mathbb{N}\) and are square-integrable, and the traces of all derivatives up to the order \(k - 1\) vanish on the boundary \(\partial \Omega\). It is readily seen that these spaces equipped with the inner product \((v, w)_{H^k_0 \times H^{-k}} = \int_\Omega \nabla^k v : \nabla^k w \, dx\) form a HILBERT space, where \(\nabla\) is the FROBENIUS inner product. It is well known that by a classical density argument and the POINCARÉ–FRÉDRIECHS inequality, the norm induced by this inner
product and is equivalent to the standard norm. The (half)norm of $H^k_0(\Omega)^d$ is denoted by $|\cdot|_{k,2} = |\cdot|_{H^k_0(\Omega)}$ and with $|\cdot|_{-n,2}$, we denote the dual norm. Now, since the stored energy $\varphi$ was not supposed to satisfy any convexity assumption, we have in general two possibilities of approaching this problem. On the one hand, we can treat the stress $\sigma$ as strongly continuous perturbation of the capillarity if $\sigma$ has at most linear growth. On the other hand, if we assume the stress satisfies an Andrews–Ball type condition allowing any polynomial growth for $\sigma$, we can treat the stored energy $\varphi$ as part of the energy functional.

Now, we suppose that $\sigma$ fulfills, aside from certain growth and continuity conditions, a potential, and that $\sigma$ satisfies an Andrews–Ball type condition which was originally introduced by ANDREWS & BALL to show global existence of solutions for the one-dimensional equations in viscoelastodynamics, i.e., when $\nu > 0$, $n = 1$ and $\mu = 0$, see [And80, AnB82]. The existence of weak solutions to the aforementioned case in arbitrary dimensions has already been studied in a more general abstract setting in [EmŠ13] by making the crucial assumption that the operator $B + \lambda A$ is monotone for some $\lambda > 0$, which in practice generalizes the Andrews–Ball condition. The latter condition states that $\sigma$ is monotone in the large, i.e., there exists a positive value $R > 0$ such that

$$
(\sigma(F) - \sigma(\tilde{F})) : (F - \tilde{F}) > 0 \quad \text{for all } F, \tilde{F} \in \mathbb{R}^{d \times d} \text{ with } |F - \tilde{F}| \geq R,
$$

where $|\cdot|$ is the norm induced by the FROBENIUS inner product $:\cdot$. We will impose the more general assumption of the convexity of $\varphi + \frac{\lambda}{2}|\cdot|^2$ which is in this smooth setting equivalent to the monotonicity of $\sigma + \lambda \id$. However, if $m \in \mathbb{N}$ is sufficiently large so that we can again treat the stored energy $\varphi$ as strongly continuous perturbation, then the previous condition is redundant. Therefore, we will not explicitly focus on this case. Having said that, the exact conditions which we impose on the stress $\sigma$ and the phase indicator $\lambda$ are the following:

(5.2a) There exists a continuously differentiable function $\varphi : \mathbb{R}^{d \times d} \to \mathbb{R}$ such that $\sigma = \varphi'$.

(5.2b) There exist positive constants $c^1_\sigma, C^1_\sigma > 0$ and $p > 1$ such that

$$
|\sigma(F)| \leq C^1_\sigma(1 + |F|^{p-1}) \quad \text{and} \quad c^1_\sigma|F|^p - C^1_\sigma \leq |\varphi(F)| \leq C^1_\sigma(1 + |F|^p) \quad \text{for all } F \in \mathbb{R}^{d \times d}.
$$

(5.2c) There exists a positive number $\lambda > 0$ such that $\varphi + \frac{\lambda}{2}|\cdot|^2$ is convex.

(5.2d) There holds $\lambda \in C^2(\mathbb{R}^{m \times d})$ such that $\lambda'$ is bounded.

Condition (5.2c) is in fact equivalent to the following Andrews–Ball type condition:

$$
(\sigma(F) - \sigma(\tilde{F})) : (F - \tilde{F}) \geq -\lambda |F - \tilde{F}|^2 \quad \text{for all } F, \tilde{F} \in \mathbb{R}^{d \times d}. \quad (5.4)
$$

This follows from the convexity and Gâteaux differentiability of $\varphi + \frac{\lambda}{2}|\cdot|^2$ and the parallelogram identity of $|\cdot|$. The Andrews–Ball condition in turn necessitates (5.4) if $\sigma$ is in addition locally Lipschitz continuous, see [EmŠ13]. Typically, $\varphi$ is a polynomial of order less than or equal to 4 and is a multi-well potential, e.g., $\varphi(e) = (e^2 - 1)^2$ which is covered in our framework. As we mentioned before, this condition on the energy functional is more general than the ones considered in [RaR03] with the assumption that $\sigma \in C^2$ with $\sigma''$ bounded or in [PIR02] with the assumption that $\sigma = \sigma_1 + \sigma_2 \in C^1$ and
\( \sigma_1 \) convex and \( \sigma_2 \) bounded. In particular, the case \( \phi(e) = (e^2 - 1)^2 = e^4 + 1 - 2e^2 \) is not included in the aforementioned works.

The evident choice of the spaces are \( U = H^n_0(\Omega)^d, V = W = H^n_0(\Omega)^d, W = W_0^{1,p}(\Omega)^d, \) and \( H \) as before. Further, we assume again \( f \in L^2(0,T; H^{-\max\{m,n\}}(\Omega)^d) \). We choose \( p > 1 \) such that

\[
H^n_0(\Omega)^d \hookrightarrow W_0^{1,p}(\Omega)^d \hookrightarrow L^2(\Omega)^d. \tag{5.5}
\]

This, enables us to view the stored energy \( E_1 \) defined below as a strongly continuous perturbation of \( E_2 \). For example, we obtain for \( m = 2 \) in dimension \( d = 2 \) all values \( p > 1 \) and in dimension \( d = 3 \) the range \( 6/5 \leq p \leq 6 \). Then, the dissipation potential \( \Psi \) and the energy functional \( E \) are given by

\[
\Psi_u(v) = \int_\Omega \left( \frac{\nu}{2} |\nabla v(x)|^2 + |\lambda'(\nabla u(x)) : \nabla v(x)| \right) \, dx = \Psi^1(v) + \Psi^2_u(v)
\]

and

\[
E(u) = \int_\Omega \left( \varphi(\nabla u(x)) + \frac{\mu}{2} |\nabla u(x)|^2 \right) \, dx = E^1(u) + E^2(u),
\]

respectively, and therefore, \( B \equiv 0 \). Here, we assume \( n \geq 1, \nu, \mu > 0 \). We start by verifying the assumptions on the dissipation potential \( \Psi_u \).

**Assumptions on \( \Psi_u \):** The dissipation potential \( \Psi_u \) obviously complies with Condition (2.\( \Psi_a \)) as it is convex and finite everywhere on \( V \). Since \( \lambda' \) is supposed to be bounded, the growth condition (2.1) in Condition (2.\( \Psi_b \)) is easily verified by the Poincaré–Friedrichs inequality. In order to proof Condition (2.\( \Psi_d \)), we proof that for any sequence \( u_n \to u \) as \( n \to \infty \) with \( \sup_{n \in \mathbb{N}, t \in [0,T]} E_t(u_n) < +\infty \), there holds \( \Psi_u \to \Psi_u \) in the sense of MOSCO convergence, see Remark 2.1 ii). As mentioned in the same remark, from [Ste08, Lemma 4.1], we infer that \( \Psi_u \to \Psi_u \) implies Condition (2.\( \Psi_d \)). In order to show (2.2), we distinguish the cases \( n = 1 \) and \( n \geq 2 \).

**Ad \( n = 1 \):** Let \( v_n \to v \) and \( u_n \to u \) as \( n \to \infty \) with \( \sup_{n \in \mathbb{N}, t \in [0,T]} E_t(u_n) < +\infty \). By the compact embedding (5.5), there exists a subsequence (labeled as before) and a function \( g \in W^{1,p}(\Omega)^d \) such that

\[
\nabla u_n(x) \to \nabla u(x) \quad \text{for a.e.} \ x \in \Omega, \tag{5.6}
\]

\[
|\nabla u_n(x)| \leq g(x) \quad \text{for a.e.} \ x \in \Omega. \tag{5.7}
\]

We define

\[
f(x, z, \xi) = |\lambda'(z)\xi| + |\xi|^2
\]

and note that since \( \lambda' \) is continuous, \( f \) satisfies the assumptions of Theorem 2 in [EkT99] which implies

\[
\Psi_u(v) = \int_\Omega \left( |\nabla v(x)|^2 + |\lambda'(\nabla u(x)) : \nabla v(x)| \right) \, dx \tag{5.8}
\]

\[
\leq \liminf_{n \to \infty} \int_\Omega \left( |\nabla u_n(x)|^2 + |\lambda'(\nabla u_n(x)) : \nabla v_n(x)| \right) \, dx = \Psi_u(v_n). \tag{5.9}
\]
From the fact, that $\lambda''$ is bounded, it follows that $\lambda'$ has at most linear growth. Hence, by the dominated convergence theorem, we obtain that for the constant sequence $\tilde{v}_n = v$ for all $n \in \mathbb{N}$, there holds

\[
\Psi_u(v) = \int_{\Omega} \left( |\nabla v(x)|^2 + |\lambda'(\nabla u(x)) : \nabla v(x)| \right) \, dx
\]

\[
= \lim_{n \to \infty} \int_{\Omega} \left( |\nabla v(x)|^2 + |\lambda'(\nabla u_n(x)) : \nabla v(x)| \right) \, dx = \Psi_u(v_n)
\]

(5.10)

(5.11)

from which the MOSCO convergence (2.2) follows.

**Ad $n \geq 2$:** Let $v_n \to v$ in $V$ and $u_n \to u$ in $U$ as above. Since $n \geq 2$, by the compact embedding $H^0_0(\Omega)^d \hookrightarrow H^1_0(\Omega)^d$, there holds $v_n \to v$ in $H^1_0(\Omega)^d$. Then, taking again (5.6) into account, we obtain (5.8) and (5.10) with the dominated convergence theorem, whence Condition (2.2).

**Assumptions on $E$:** We need to verify the Conditions (2.Ed)- (2.Eg) for $E$. From the assumptions (5.2a)-(5.2d) and the fact that $E$ is time-independent, Conditions (2.Ed)- (2.Ef) are easily verified. From Assumption (5.2c) and the parallelogram identity of the norm of $H^m_0(\Omega)^d$, it follows that

\[
E(\vartheta u + (1 - \vartheta)v) \leq \vartheta E(u) + (1 - \vartheta)E(v) + \vartheta(1 - \vartheta) \left( \lambda |u - v|^2_{H^1_0(\Omega)^d} - \mu |u - v|^2_{H^m_0(\Omega)^d} \right)
\]

for all $t \in [0, T], \vartheta \in [0, 1]$ and $u, v \in U$. Employing the Gagliardo–Nirenberg inequality (see, e.g., [Nir66], [Fri69] or [Zei90a, Section 21.19]), there exists constants $c_1, c_2 > 0$ such that

\[
|u - v|_{H^1_0(\Omega)^d}^2 \leq c_1 |u - v|_{H^m_0(\Omega)^d}^{2/m} |u - v|_{L^2(\Omega)^d}^{2(m-1)/m} + c_2 |u - v|_{L^2(\Omega)^d}^2
\]

\[
\leq \varepsilon |u - v|_{H^m_0(\Omega)^d}^2 + C(\varepsilon) |u - v|_{L^2(\Omega)^d}^2 + c_2 |u - v|_{L^2(\Omega)^d}^2
\]

where we employed Young’s inequality in the last step for $\varepsilon > 0$ and a constant $C(\varepsilon) > 0$. Choosing $\varepsilon > 0$ sufficiently small (e.g. $\varepsilon < \mu$), we obtain the $\lambda$-convexity of $E$ for a $\Lambda := C(\varepsilon) + c_2$. Thus, the energy functional $E$ fulfills Condition (2.Ef). Further, Assumption (5.2a)-(5.2d) imply that $E^1$ and $E_2$ are Fréchet differentiable on $U$ and $\bar{U}$, respectively, with the derivatives is given by

\[
\langle D\mathcal{E}_2(u), v \rangle_{U^* \times U} = \mu \int_{\Omega} \nabla^m u(x) \cdot \nabla^m v(x) \, dx
\]

\[
\langle D\mathcal{E}_1(u), v \rangle_{\bar{U}^* \times \bar{U}} = \int_{\Omega} \sigma(\nabla u(x)) : \nabla v(x) \, dx
\]

Consequently, by the subdifferential calculus, the subdifferentials are single-valued with $\partial_u \mathcal{E}_1(u) = \{D\mathcal{E}_1(u)\}$ and $\partial_{\bar{U}} \mathcal{E}_2(u) = \{D\mathcal{E}_2(u)\}$. Hence, $\xi_1 \in \partial_u \mathcal{E}_1(u)$ and $\xi_2 \in \partial_{\bar{U}} \mathcal{E}_2(u)$ if and only if $\xi_1 = \mu(-1)^m \Delta^m u \in U^*$ and $\xi_2 = -\nabla \cdot \sigma(\nabla u) \in \bar{U}$, respectively. Then,
Condition (2.Eg) follows from the following estimate:
\[
\langle \nabla^2 u, v \rangle_{W^1 \times \tilde{W}} = \int_{\Omega} \sigma(\nabla u) : \nabla v \, dx \\
\leq \left( \int_{\Omega} |\sigma(\nabla u)|^{p/(p-1)} \, dx \right)^{(p-1)/p} \left( \int_{\Omega} |\nabla v|^p \, dx \right)^{1/p} \\
\leq \left( C_2 \int_{\Omega} (1 + |\nabla u|^p) \, dx \right)^{1/p} \left( \int_{\Omega} \varphi(\nabla u) \, dx \right)^{p/(p-1)} \\
\leq C \left( 1 + \int_{\Omega} \varphi(\nabla u) \, dx \right) \|v\|_{\tilde{W}},
\]
where have employed Hölder’s and Young’s inequality as well as the growth condition (5.2b), and where \( C > 0 \) denotes a generic constant. This shows Condition (2.Eg). Along the same lines, we obtain the inequality
\[
\langle \nabla^2 (u - v), w \rangle_{W^1 \times \tilde{W}} = \int_{\Omega} \sigma(\nabla u) : \nabla w \, dx \\
\leq \left( \int_{\Omega} |\sigma(\nabla u) - \sigma(\nabla v)|^{p/(p-1)} \, dx \right)^{(p-1)/p} \left( \int_{\Omega} \varphi(\nabla v) \, dx \right)^{p/(p-1)} \\
\leq C \left( 1 + \int_{\Omega} \varphi(\nabla u) \, dx \right) \|v\|_{\tilde{W}}
\]
and therefore
\[
\|\nabla^2 u - \nabla^2 v\|_{\tilde{W}} \leq \left( \int_{\Omega} |\sigma(\nabla u) - \sigma(\nabla v)|^{p/(p-1)} \, dx \right)^{(p-1)/p}.
\]
Then, Condition (2.Ec) follows from the dominated convergence theorem. Since we verified all conditions of Theorem 2.4, there exists for every \( u_0 \in U \) and \( v_0 \in H \), a weak solution \( u \in C_w([0, T]; H^m) \cap W^{1,\infty}(0, T; L^2(\Omega)^d) \cap H^2(0, T; H^{-m}(\Omega)^d) \cap H^{-n}(\Omega)^d \) with \( u - u_0 \in H^1(0, T; H^m(\Omega)^d) \) and \( \sigma_p \in L^2(0, T; H^{-n}(\Omega)^d) \) satisfying the integral equation (5.2). Noting that there holds
\[
\text{Var} \lambda(\nabla u) = \int_0^t \left| \frac{\partial}{\partial r} \lambda(\nabla u(r)) \right| \, dr = \int_0^t \left| \lambda'(\nabla u(r)) : \nabla \partial_t u(r) \right| \, dr = \int_0^t \Psi_2^2(u'(r)) \, dr,
\]
we infer the energy-dissipation balance (5.3).

**Remark 5.1.** We note that this result has not been shown before in the literature and that there are no abstract results that can address this type of problem.

### 5.1 Differential Inclusion II

The following example is a nonlinearly damped inertial system and can for smooth dissipation potentials be interpreted as a viscous regularization of the KLEIN–GORDON equation. The equations supplemented with initial and boundary conditions read

(P2) \[
\begin{align*}
\partial_t u - \nabla \cdot p - \Delta u + b(u) &= f \quad \text{in } \Omega_T, \\
p(x, t) \in &\partial_u \psi(x, u(x, t), \nabla \partial_t u(x, t)) \quad \text{a.e. in } \Omega_T, \\
u(x, 0) &= u_0(x) \quad \text{on } \Omega, \\
u'(x, 0) &= v_0(x) \quad \text{on } \Omega, \\
u(x, t) &= 0 \quad \text{on } \partial \Omega \times [0, T], \\
\frac{\partial u}{\partial \nu}(x, t) &= 0 \quad \text{on } \partial \Omega \times [0, T].
\end{align*}
\]
If $\psi = 0$ and $b(u) = \gamma u$ for a constant $\gamma > 0$, then the equation in (P4) reduces to the classical KLEIN–GORDON equation, which is a relativistic wave equation with applications in relativistic quantum mechanics that is related to the SCHRODINGER equation.

We make the following assumptions on the functions $\psi$ and $b$. For simplicity, we choose $d = 1$ and note that the case $d \geq 2$ can be (under stronger assumptions) be treated in a similar way.

(5.a) The function $\psi : \Omega \times \mathbb{R} \times \mathbb{R} \rightarrow [0, +\infty)$ is a CARATHÉODORY function such that $\psi(x, y, \cdot)$ is a proper, lower semicontinuous, and convex, and $\psi(y, y, 0) = 0$ for almost every $x \in \Omega$ and all $y \in \mathbb{R}$.

(5.b) There exists a real number $q > 1$ and positive constants $c_\psi, C_\psi > 0$ such that

$$c_\psi^q (|z|^q - 1) \leq \psi(x, y, z) \leq C_\psi (1 + |z|^q)$$

for a.e. $x \in \Omega$ and all $z \in \mathbb{R}^m$, $y \in \mathbb{R}$, $|y| \leq R$.

(5.c) The function $b : \Omega \rightarrow \mathbb{R}$ is a continuous function and there exist a real number $p > 1$ and a constant $C_b > 0$ such that

$$|b(u)| \leq C_b(|u|^{p-1} + 1) \quad \text{for all } u \in \mathbb{R}.$$ 

Accordingly, the function spaces are given by $V = W^{1,q}_0(\Omega)$, $U = H^1_0(\Omega)$, $\overline{W} = L^{\max(p, 2)}(\Omega)$ and $H = L^2(\Omega)$. Then, we identify the dissipation potential $\psi : V \rightarrow \mathbb{R}$ and the energy functional $\mathcal{E} : U \rightarrow [0, +\infty)$ as

$$\psi_u(v) = \int_\Omega \psi(x, u(x), \nabla v(x)) \, dx \quad \text{and} \quad \mathcal{E}(u) = \frac{1}{2} \int_\Omega |\nabla u(x)|^2 \, dx,$$

respectively. The perturbation $B : \overline{W} \rightarrow V^*$ is given by

$$\langle B(u), w \rangle_{\overline{W}^* \times \overline{W}} = \int_\Omega b(u(x)) w(x) \, dx.$$ 

We note that the conjugate functional $\psi_u^*$ can, in general, not be expressed as an integral functional over $\Omega$, since it is defined on $W^{-1,q'}(\Omega)$.

Obviously, $\mathcal{E}$ satisfies all Conditions 2.2. In view of the compact embedding $H^1_0(\Omega) \hookrightarrow C(\overline{\Omega})$ and FATOU’s lemma, it is readily that $\psi_u$ satisfies Conditions (2.2a) and (2.2b). In order to verify Condition (2.2c), we show that for every sequence $u_n \rightharpoonup u$ in $U$ with $\sup_{n \in \mathbb{N}} \mathcal{E}(u_n) < +\infty$, there holds $\psi_{u_n} \rightharpoonup \psi_u$ in $\mathcal{M}$. As we mentioned in Remark 2.1 ii), the MOSCO-convergence $\psi_{u_n} \rightharpoonup \psi_u$ implies Condition (2.2c). The liminf estimate in the MOSCO-convergence follows from [Iof77, Theorem 3]. The limsup estimate is trivially fulfilled by choosing, for each $v \in V$, the constant sequence $v_n = v, n \in \mathbb{N}$, and the dominated convergence theorem.

If we assume $p \in (1, 2]$, and $f \in L^2(0, T; H)$, it is easy to check in the same way as in the previous examples that Conditions (2.2a), (2.2b), and (2.2b) are also fulfilled. Therefore, Theorem 2.4 ensures that for every initial values $v_0 \in H$ and $u_0 \in U$, the existence of a solution $u \in C_w([0, T]; U) \cap W^{1, \infty}(0, T; H) \cap W^{2, q'}(0, T; U^* + V^*)$ with $u - u_0 \in W^{1, q}(0, T; V)$ to (P4) satisfying the integral equation

$$\int_0^T \left( u'' v \right)_{(U^* + V^*) \times (U \cap V)} + \int_\Omega p \cdot \nabla v + b(u)v \, dx \, dt = \int_0^T \int_\Omega f v \, dx \, dt.$$
for all \( v \in L^{\min(2,q)}(0,T;U^* + V^*) \) with \( p(x,t) \in \partial_v \psi(x,u(x,t),\nabla \partial_t u(x,t)) \) a.e. in \( \Omega_T \), and the energy-dissipation balance
\[
\frac{1}{2} \| u'(t) \|_{L^2(\Omega)}^2 + \frac{1}{2} \| u(t) \|_{H^1_0(\Omega)}^2 + \int_0^T \left( \Psi_{u,t}(u'(r)) + \Psi^*_u(f(r) - u''(r) - \Delta u(r)) \right) dr
\]
\[
= \frac{1}{2} \| v_0 \|_{L^2(\Omega)}^2 + \frac{1}{2} \| u_0 \|_{H^1_0(\Omega)}^2 + \int_0^T \langle f(r), u'(r) \rangle_{L^2(\Omega) \times L^2(\Omega)} dr
\]
holds for almost every \( t \in (0,T) \) if \( q \in (1,2) \) and for all \( t \in (0,T) \) if \( q \geq 2 \).

5.2 Differential inclusion III

In the final example, we consider a nonlinearly damped inertial system which can not be treated with the known abstract results. The differential inclusion supplemented with initial and boundary conditions is given by
\[
\left\{ \begin{aligned}
\partial_t u + |\partial_t u|^{q-2} \partial_t u + p - \nabla \cdot (E \nabla u) + W'(u) &= f \quad \text{in } \Omega_T, \\
 p(x,t) &\in \text{Sgn} (\partial_t u(x,t)) \quad \text{a.e. in } \Omega_T, \\
u(x,0) &= u_0(x) \quad \text{on } \Omega, \\
u'(x,0) &= v_0(x) \quad \text{on } \Omega, \\
u(x,t) &= 0 \quad \text{on } \partial \Omega \times [0,T],
\end{aligned} \right.
\]

where \( q \geq 2 \), \( W : \mathbb{R}^3 \rightarrow \mathbb{R} \) is a \( \lambda \)-convex and continuously differentiable function, \( E : \mathbb{R}^m \rightarrow \mathbb{R}^m \) a uniformly positive definite and symmetric matrix, and \( f \in C^1([0,T];H^{-1}(\Omega)) \). As mentioned before, \( W(u) = (1 - u^2)^2 \) can be chosen to be a doubly well potential. We set \( U = H^1_0(\Omega) \), \( V = L^2(\Omega) \), and \( H = L^2(\Omega) \). The dissipation potential \( \Psi : V \rightarrow \mathbb{R} \) and the energy functional \( E : U \rightarrow [0, +\infty] \) are given by
\[
\Psi(u) = \int_{\Omega} \left( \frac{1}{q} |u(x)|^q + |v(x)| \right) dx \quad \text{and}
\]
\[
E_t(u) = \int_{\Omega} \left( \frac{1}{2} \nabla u(x) : E(x) \nabla u(x) + W(u(x)) \right) dx - \langle f(t), u \rangle_{U^* \times U},
\]
respectively. Consequently, \( B = 0 \) and \( E^2_t = 0 \). It is easily verified that all the assumptions are fulfilled, so that Theorem 2.4 ensures for any initial values \( v_0 \in H \) and \( u_0 \in U \) the existence of a solution \( u \in C_w([0,T];U) \cap W^{1,\infty}(0,T;H) \cap W^{2,q}(0,T;U^* + V^*) \) with \( u - u_0 \in W^{1,q}(0,T;V) \) to (P5) fulfilling the integral equation
\[
\int_0^T \left( \langle u''v \rangle_{(U^* + V^*) \times (U \cap V)} + \int_{\Omega} |\partial_t u|^{q-2} \partial_t uv + pv + \nabla u \cdot \nabla v \right) dx dt = \int_0^T \int_{\Omega} f v dx dt
\]
for all \( v \in L^{(2,q)}(0,T;U^* + V^*) \) with \( p(t,x) \in \text{Sgn}(u(x,t)) \) a.e. in \( \Omega_T \), and the energy-dissipation balance (2.5) holds for almost every \( t \in (0,T) \).

A Appendix

A.1 Subdifferential calculus

In this section, we want to collect some results from subdifferential calculus. Let \((X,\| \cdot \|)\) be a separable and reflexive BANACH space and denote with \((X^*,\| \cdot \|_*)\) its topological
dual space. Unlike the differential operator, the subdifferential operator is, in general, not linear. The following well known result shows under which assumptions the variational sum rule holds.

Lemma A.1 (Variational sum rule). 1) Let \( f_1 : X \to (-\infty, +\infty] \) and \( f_2 : X \to (-\infty, +\infty] \) be subdifferentiable and Fréchet differentiable at \( u \in D(\partial G) \cap D(\partial G) \neq \emptyset \), respectively. Then, there holds

\[
\partial (f_1 + f_2)(u) = \partial f_1(u) + Df_2(u),
\]

where \( Df_2 \) denotes the Fréchet derivative of \( f_2 \).

2) Let \( f_1 : X \to (-\infty, +\infty] \) and \( f_2 : X \to (-\infty, +\infty] \) be proper, lower semicontinuous and convex, and if there is a point \( \bar{u} \in \text{dom}(f_1) \cap \text{dom}(f_2) \) where \( f_2 \) is continuous, we have

\[
\partial (f_1 + f_2)(v) = \partial f_1(v) + \partial f_2(v) \quad \text{for all } v \in X. \tag{A.1}
\]

If \( f_2 \) is, in addition Gâteaux differentiable on \( V \), there holds \( \partial f_2(v) = \{ D_G f_2(v) \} \) and we have

\[
\partial (f_1 + f_2)(v) = \partial f_1(v) + D_G f_2(v) \quad \text{for all } v \in X,
\]

where \( D_G f_2 \) denotes the Gâteaux derivative of \( f_2 \).

Proof. The proof of assertion 1) follows immediately from the definition of a subdifferential and the proof of assertion 2) follows from Proposition 5.3. on p. 23 and Proposition 5.6 on p. 26 in [EkT99].

The next lemma establishes a deep connection between the subgradient of a function and its convex conjugate \( f^*(\xi) := \sup_{u \in V} \{ \langle \xi, u \rangle - f(u) \} \), \( \xi \in X^* \).

Lemma A.2. Let \( V \) be a Banach space and let \( f : V \to (-\infty, +\infty] \) be a proper, lower semicontinuous and convex functional and let \( f^* : V^* \to (-\infty, +\infty] \) be the convex conjugate of \( f \). Then for all \( (u, \xi) \in V \times V^* \), the following assertions are equivalent:

i) \( \xi \in \partial f(u) \) in \( V^* \);

ii) \( u \in \partial f^*(\xi) \) in \( V \);

iii) \( \langle \xi, u \rangle = f(u) + f^*(\xi) \) in \( \mathbb{R} \).

Proof. Proposition 5.1 and Corollary 5.2 on pp. 21 in [EkT99].

We denote with \( \mathcal{L}_{(0,T)} \) the Lebesgue \( \sigma \)-algebra of the interval \( [0, T] \) and with \( \mathcal{B}(X) \) the Borel \( \sigma \)-algebra of \( X \). A functional \( f : [0, T] \times X \to (-\infty, +\infty] \) is called normal integrand if it is \( \mathcal{L}_{(0,T)} \otimes \mathcal{B}(X) \)-measurable on \( [0, T] \times X \) and for a.e. \( t \in (0, T) \) the mapping \( v \mapsto f(t, v) \) is lower semicontinuous on \( X \). Note that if \( f \) is a normal integrand, then by the Pettis theorem, see, e.g., Diestel & Uhl [DiU77, Theorem 2, p. 42], the mapping \( t \mapsto f(t, v(t)) \) is Lebesgue measurable for any Bochner measurable functional \( v : [0, T] \to X \).
Theorem A.3. Let $X$ be a separable and reflexive Banach space and $f : [0, T] \times X \to (-\infty, +\infty]$ be a normal integrand such that $f(t, \cdot) : X \to (-\infty, +\infty]$ is for a.e. $t \in (0, T)$ a proper, lower semicontinuous and convex functional. Denote with $\alpha, \alpha^*$ that there exists constants $\xi$ then

$$f(t, v) + \alpha\|v\| + \beta \geq 0$$

for a.e. $t \in [0, T]$ and all $v \in X$.

and

$$f^*(t, \xi) + \alpha^*\|\xi\| + \beta^* \geq 0$$

for a.e. $t \in [0, T]$ and all $\xi \in X^*$.

Then following assertions hold

i) The functional $f^* : [0, T] \times X^* \to (-\infty, +\infty]$ is a normal integrand, and if $F$ is proper, then the conjugate functional $F^* : L^p(0, T; X^*) \to \mathbb{R}$ is proper, lower semicontinuous and convex, and is given by the integral functional

$$F^*(\xi) = \begin{cases} 
\int_0^T f^*(t, \xi(t)) \, dt & \text{if } f^*(\cdot, \xi(\cdot)) \in L^1(0, T), \\
+\infty & \text{otherwise.}
\end{cases}$$

ii) The functional $F$ is lower semicontinuous and convex on $L^p(0, T; X)$, and there holds $F(v) > -\infty$ for all $v \in L^p(0, T; X)$.

iii) Let $F$ be proper, and let $v \in \text{dom}(F)$ and $\xi \in L^p(0, T; X^*)$. Then, $\xi \in \partial F(v) \subset L^p(0, T; X^*)$ if and only if $\xi(t) \in \partial f(t, v(t)) \subset X^*$ for a.e. $t \in (0, T)$.

Proof. Assertions i) and ii) follow from Kenmochi [Ken75] and Rockafellar [Roc71, Proposition 2 & Theorem 2] as well as [EkT99, Proposition 4.1 & Corollary 4.1, p. 18], respectively. Assertion iii) follows from i), ii), Lemma A.2, and the fact that

$$\int_0^T (f(t, v(t)) + f^*(t, \xi(t)) - \langle \xi(t), v(t) \rangle) = 0 \quad (A.2)$$

if and only if

$$f(t, v(t)) + f^*(t, \xi(t)) - \langle \xi(t), v(t) \rangle = 0 \quad \text{a.e. in } (0, T),$$

which in turn follows from the fact that the integrand in (A.2) is by the Fenchel–Young inequality always non-negative.

In the next result, we show the

Lemma A.4. Let the functionals $f, f_n : [0, T] \times X \to (-\infty, +\infty]$ be given and fulfill the assumptions of Theorem A.3, and let $p \in (1, +\infty)$. Furthermore, let $(v_n)_{n \in \mathbb{N}} \subset L^p(0, T; X)$ and $(\xi_n)_{n \in \mathbb{N}} \subset L^p(0, T; X^*)$ with $\xi_n \in \partial F_n(v_n)$ such that $v_n \rightharpoonup v$ in $L^p(0, T; X)$ and $\xi_n \rightharpoonup \xi$ in $L^p(0, T; X^*)$ as $n \to \infty$ where $F_n$ is the integral functional associated to $f_n$. If

$$\int_0^T (f(t, v(t)) + f^*(t, \xi(t))) \, dt \leq \liminf_{n \to \infty} \int_0^T (f_n(t, v_n(t)) + f_n^*(t, \xi_n(t))) \, dt \quad (A.3)$$

and there holds

$$\limsup_{n \to \infty} \int_0^T \langle \xi_n(t) - \xi(t), v_n(t) - v(t) \rangle \, dt \leq 0, \quad (A.4)$$

then $\xi(t) \in \partial f(t, v(t))$ a.e. in $(0, T)$ and

$$\int_0^T (f(t, v(t)) + f^*(t, \xi(t))) \, dt = \lim_{n \to \infty} \int_0^T (f_n(t, v_n(t)) + f_n^*(t, \xi_n(t))) \, dt$$
Proof. By the Legendre–Fenchel inequality and Assumptions A.3 and A.4, we find
\[
\int_0^T \langle \xi(t), v(t) \rangle_{X^* \times X} \, dt \leq \int_0^T (f(t, v(t)) + f^*(t, \xi(t))) \, dt
\]
\[
\leq \liminf_{n \to \infty} \int_0^T (f_n(t, v_n(t)) + f_n^*(t, \xi_n(t))) \, dt
\]
\[
\leq \limsup_{n \to \infty} \int_0^T (f_n(t, v_n(t)) + f_n^*(t, \xi_n(t))) \, dt
\]
\[
= \limsup_{n \to \infty} \int_0^T \langle \xi_n(t), v_n(t) \rangle_{X^* \times X} \, dt
\]
\[
= \int_0^T \langle \xi(t), v(t) \rangle_{X^* \times X} \, dt.
\]
By Theorem A.3 and Lemma A.2, it follows that \( \xi(t) \in \partial f(t, v(t)) \subset X^* \) for a.e. \( t \in (0, T) \). \( \square \)

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