Unbounded quantum Fisher information in two-path interferometry with finite photon number

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Abstract

The minimum error of unbiased parameter estimation is quantified by the quantum Fisher information in accordance to the Cramér–Rao bound. We indicate that only superposed NOON states by simultaneous measurements can achieve the maximum quantum Fisher information with form ⟨\hat{N}²⟩ for a given photon number distribution by a POVM in linear two-path interferometer phase measurement. We present a series of specified superposed states with infinite quantum Fisher information but with finite average photon numbers. The advantage of this unbounded quantum Fisher information will be beneficial to many applications in quantum technology.

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(Some figures may appear in colour only in the online journal)

1. Introduction

Precise interferometric measurement plays a key role in many scientific and technological applications, such as quantum metrology, imaging, sensing and information processing [1]. The fundamental sensitivity bounds of phase measurement in the Mach–Zehnder interferometer (MZI), as shown in figure 1(a), are of broad interest for those areas. For N identical uncorrelated particles, the error of the phase measured in MZI on average decreases as 1/(\langle N \rangle)¹/² which is the shot-noise limit (SNL). In 1980s, it was pointed out that by using coherent light together with squeezed vacuum we could beat the SNL [2]. It is also shown that using NOON state [3] and quantum entanglement allows interferometric sensitivity that also surpasses this limit. Instead of SNL, the ultimate limit imposed by quantum mechanics is the Heisenberg limit (HL) with a generally accepted form 1/(\langle N \rangle) [4, 5]. Experiments exploring those topics have been performed in various systems with photons [6, 7], ions [8], cold-atoms [9] and Bose–Einstein condensates [10, 11]. However, it seems to be indicated in some works that the HL with form...
Figure 1. (a) Setup of Mach–Zehnder interferometer. $|\Psi\rangle_{in}$ is the input state. Both beam-splitters are 50 : 50. The unknown phase shift $\phi$ is to be estimated. A controllable phase $\theta$, in the other arm, is used in the experiment to achieve the best precision. $\hat{U}_{\phi-\theta}$ is the phase operator and $\hat{U}_{\phi-\theta}|\Psi\rangle$ is the intermediate state. Passing through the second beam-splitter is the output state $|\Psi\rangle_{out}$. We perform photon number counting measurements at two ports of the Mach–Zehnder interferometer to measure the output state, respectively. (b) Setup of modified Mach–Zehnder interferometer. The first beam-splitter is replaced by an entangled photon source. We assume that the entangled photon source can produce NOON states and superposed NOON states.

$1/\langle \hat{N} \rangle$ could be violated while the experiments are performed with a fluctuating number of particles [12, 13]. Then, HL for an unfixed number of particles has become a focus of great attention as well as contention in all the parameter estimation schemes.

In this paper, we investigate the quantum Fisher information (QFI) in two kinds of two-path interferometers for a fluctuating number of photons since the well-known Cramér–Rao bound (CRB) whose leading role played by QFI is often used to estimate sensitivity of phase measurement. To achieve the maximum QFI and the lowest CRB, we should use the optimal states, and implement the optimal measurements (optimal POVM). We investigate the measurement applied in two kinds of linear two-path interferometers, as shown in figure 1. Note that figure 1(a) is for a MZI, and figure 1(b) is a modified Mach–Zehnder interferometer (MMZI) where the first beam-splitter is replaced by an entangled photon source. We show that the optimal measurement scheme can be expressed by a group of compatible observables. We confirm that the maximum QFI for a definite photon number probability distribution, written as $\langle \hat{N}^2 \rangle$, is saturated with a superposition of NOON states in MMZI. In particular, we present a superposition of NOON states with a specified probability distribution and a finite average photon number. We find that QFI for this state can be infinite, i.e. an arbitrary
high phase sensitivity is obtained by Cramér–Rao inequality while it still has a finite average photon number. Apparently, this phase sensitivity does not violate the Heisenberg uncertainty relation, but seems to break the HL.

Nevertheless, CRB is only asymptotically tight for infinitely many trials of measurements under the unbiased estimate assumption [14, 15], and it includes no prior information of the phase probability distribution [16]. Therefore, HL can merely be violated under some very special situation, for example the phase is in a restricted neighborhood near zero as the case of distinguishing states, and the limit obtained by CRB is of little use when we consider the resources required for the prior information of the problem [17–19]. Similar theoretical results based on the quantum speed limit [20] are also obtained in [21]. Although CRB can sometimes grossly underestimate the achievable error, QFI and CRB are still of great use for many applications in quantum techniques such as parameter estimation for noisy systems [22, 23], limits of imaging [24], distinguishability of states [25] and quantum Zeno effect [26]. Related topics and recent developments can be found in [27–36].

This paper is organized as follows. In section 2, we demonstrate the optimal measurement scheme by simultaneous measurements in linear two-path interferometers. In section 3, we obtain the maximum QFI in the optimal measurement scheme. In section 4, we discuss the measurement scheme written as one operator. In section 5, we present a superposition of NOON states with infinite QFI and finite average photon number. Superpositions of NOON states in MMZI and dual Fock states in MZI are compared in section 6. Finally, a conclusion is given in section 7.

2. Simultaneous measurements

Two-path optical interferometers are widely studied to estimate a completely unknown optical phase $\phi$ from the photon numbers observed in two ports of the output. Here we express the quantum mechanics of two-path interferometers in terms of the spin-$N/2$ algebra of the Schwinger representation, as shown in [37],

\begin{align*}
\hat{J}_1 &= (\hat{a}^\dagger \hat{b} + \hat{b}^\dagger \hat{a})/2, \\
\hat{J}_2 &= (\hat{a}^\dagger \hat{b} - \hat{b}^\dagger \hat{a})/2i, \\
\hat{J}_3 &= (\hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b})/2,
\end{align*}

(1)

where $\hat{a}$ and $\hat{b}$ are the annihilation operators of the two paths. Eigenstates of $\hat{J}_3$ can be defined as usual by Fock space representation as,

$$|j, m\rangle = |j + m\rangle_a |j - m\rangle_b.$$  

(2)

The phase shift of $\phi$ between the arms of the interferometer can be expressed by a unitary transformation $\hat{U}_\phi = \exp(-i\phi\hat{J}_3)$. The unitary transformation of a beam-splitter is given by $\hat{B} = \exp(i\pi\hat{J}_1/2)$.

Then, we consider that a linear phase estimation scheme in two-path interferometers can be divided into three parts: state preparation, phase transformation and measurement [4, 38, 39], see figure 2. The procedure in [40] is a special example in which phase transformation repeats $n$ times during each measurement.

In this section, our discussion is focused on the measurement part. We perform photon number counting measurements at two ports of linear two-path interferometers to measure the output state $|\Psi\rangle_{out}$, respectively, written as $\hat{n}_a$ and $\hat{n}_b$. It is obvious that $\hat{n}_a$ and $\hat{n}_b$ are compatible observables, $[\hat{n}_a, \hat{n}_b] = 0$. Thus, measurements $\hat{n}_a$ and $\hat{n}_b$ can be performed simultaneously. This pair of measurements cannot be expressed as $\hat{n}_a \otimes \hat{n}_b$ because we obtain two independent
results that can be written as a column matrix, $(n_a, n_b)^T$, from simultaneous measurements of any two-mode Fock state,
\[
\begin{pmatrix}
\hat{n}_a \\
\hat{n}_b
\end{pmatrix} = \sum_j \sum_{m=-j}^{j} C_{j,m} \begin{pmatrix} j+m \end{pmatrix}_a |j-m\rangle_b = \sum_{m=-j}^{j} \sum_{m=-j}^{j} C_{j,m} \begin{pmatrix} j+m \end{pmatrix}_a |j-m\rangle_b,
\]
where $\sum_j \sum_{m=-j}^{j} |C_{j,m}|^2 = 1$. However, the operator $\hat{n}_a \otimes \hat{n}_b$ only obtains a result $n_a \times n_b$, and is useless to describe a pair of simultaneous measurements. Here we define simultaneous measurements as using several compatible observables to perform a united measurement for a state at the same time. In fact, it is sufficient and necessary for them to be measurable consecutively, such that the joint probability of the results does not depend on the order of the measurement [1]. Moreover, we show these pairs of simultaneous measurements are equivalent,
\[
\begin{pmatrix}
\hat{n}_a \\
\hat{n}_b
\end{pmatrix} \Leftrightarrow \begin{pmatrix}
\hat{n}_b \\
\hat{n}_a
\end{pmatrix} \Leftrightarrow \begin{pmatrix}
\hat{N} \\
\hat{\Delta}
\end{pmatrix} \Leftrightarrow \begin{pmatrix}
\hat{\Delta} \\
\hat{N}
\end{pmatrix},
\]
where $\hat{N} = \hat{n}_a + \hat{n}_b$ is total output photon number and $\hat{\Delta} = \hat{n}_b - \hat{n}_a$ is the difference between the two ports.

Then, we show that the simultaneous measurements in two-path interferometers can be described by two equivalent POVMs,
\[
\hat{E}(N, \Delta) \Leftrightarrow \hat{E}'(n_a, n_b) = \begin{pmatrix}
N \Delta \\
N - \Delta
\end{pmatrix}_a \begin{pmatrix}
N + \Delta \\
N - \Delta
\end{pmatrix}_b = \begin{pmatrix}
N + \Delta \\
N - \Delta
\end{pmatrix}_b \begin{pmatrix}
N + \Delta \\
N - \Delta
\end{pmatrix}_a.
\]
\[
\hat{E}(n_a, n_b) \quad \text{and} \quad \hat{E}(N, \Delta) \quad \text{are complete in the sense that,}
\]
\[
\sum_{n_a, n_b} \hat{E}(n_a, n_b) = \sum_{N} \sum_{\Delta} \hat{E}(N, \Delta) = \hat{I},
\]
where $\hat{I}$ is a unit operator. In subsequent sections, we use the equivalent simultaneous measurements $\hat{E}(N, \Delta)$ for simplicity.

### 3. Fisher information and Cramér–Rao bound

To estimate the parameter $\phi$, a POVM $\{\hat{E}(\xi)\}$ is performed on the output state, and $\phi$ is inferred from the measurement results $\xi$. The final measurement results are related to Fisher
information (FI) which takes the form,

$$F_{E(ξ)}[Ψ_φ] = \sum_ξ P(ξ|φ) \left( \frac{\partial \ln P(ξ|φ)}{\partial φ} \right)^2,$$

(7)

where $|Ψ_φ⟩ = \hat{U}_φ|Ψ⟩$ and likelihood function $P(ξ|φ) = ⟨Ψ_φ|\hat{E}(ξ)|Ψ_φ⟩$ is the probability of measurement results $ξ$ given that the true system phase is $φ$. CRB places a limit on the mean of the square of the unbiased phase error via the Cramér–Rao inequality,

$$Δφ \geq 1 / \sqrt{F_{E(ξ)}[Ψ_φ]}.$$

(8)

We note that FI is additive, and for $M$ repeated trials of the same measurement, the FI is $M \times F_{E(ξ)}[Ψ_φ]$, which leads to CRB on the phase uncertainty, $Δφ \geq 1/(M \times F_{E(ξ)}[Ψ_φ])^{1/2}$. QFI for state $|Ψ_φ⟩$ is the maximum,

$$F_Q[Ψ_φ] := \max_{|E(ξ)⟩} F_{E(ξ)}[Ψ_φ],$$

(9)

which is saturated by a particular POVM [25, 41]. In the following discussion, we show that the measurement with form $\hat{E}(N, Δ)$ in equation (5) saturates the QFI when state $|Ψ⟩$ is a superposed NOON state.

When total photon number fluctuates, we consider a pure state generated from the entangled photon source in figure 1(b),

$$|Ψ⟩ = \sum_N \sqrt{P(N)|ψ_N⟩},$$

(10)

where $P(N)$ refers to the probability distribution of total photon number $N$ and $|ψ_N⟩ = \sum_{m=-N/2}^{N/2} \sqrt{P(m)}|n⟩$ when the phase is $φ$, we write the probability of any output result $(N, Δ)$ as $P(N, Δ|φ) = \langle Ψ|\hat{E}(N, Δ)|Ψ⟩_{out}$, in which $|Ψ⟩_{out} = \hat{B}\hat{U}_φ|Ψ⟩$ is the output state. For the output result $N = n_0 + n_0$, the probability is $P(N|φ) = \sum_{Δ=-N}^{N} \langle Ψ|\hat{E}(N, Δ)|Ψ⟩_{out}$. The conditional probability is $P(N|φ) = P(N, Δ|φ)/P(N|φ)$, which is $[\hat{B}, \hat{N}] = [\hat{U}_φ, \hat{N}] = 0$, the probability distribution of photon number does not change after beam-splitter or phase, $P(N|φ) = P(N)$ and $\partial P(N|φ)/\partial φ = 0$. Hence, FI can be written as,

$$F_{E(N, Δ)}[Ψ] = \sum_N P(N) F_{E(ξ)}[Ψ_N].$$

(11)

This means that the FI of the state with a fluctuating photon number equals the probability summation of each FI of the superposed state which has a fixed photon number. As the situation that the photon number is fixed, FI of $|ψ_N⟩$ is $F_Q[|ψ_N⟩] = N^2$ if and only if $|ψ_N⟩$ is a NOON state, and FI of any other state is less than $N^2$. Therefore, we have,

$$F_{E(N, Δ)}[Ψ] ≤ \max_{|Ψ⟩} F_Q[|Ψ⟩] = (N^2),$$

(12)

where the equality can only be achieved when $|Ψ⟩$ is a superposition of NOON states. Moreover this is QFI, and $\{\hat{E}(N, Δ)\}$ is the optimal POVM. According to the Cramér–Rao inequality, we obtain CRB as

$$Δφ \geq 1/(\hat{N}^2)^{1/2}.$$

(13)

Therefore, we find the optimal measurement and optimal states for the maximum QFI in linear two-path interferometers. We remark that we can calculate FI by using either intermediate state $\hat{U}_φ|Ψ⟩$ or final output $|Ψ⟩_{out}$ in figure 1. It makes no difference to the result when POVM $\{\hat{E}(ξ)\}$ is optimal, as pointed out in [38].

Here we emphasize that CRB is asymptotically tight for unbiased estimation with infinite trials of measurements, and it does not provide a rigorous basis for HL when considering the
prior probability distribution of overall phase shift. Even so, QFI is still an extremely important quantity in quantum physics. When the system interacts with the environment, such as quantum noise and photon losses, QFI changes and the precision of phase sensitivity is affected. It is shown that QFI flow, written as \( \partial_t F_Q[|\Psi\rangle] \), directly characterizes the non-Markovianity of the quantum dynamics of open systems [42]. Therefore, QFI can also be evaluated by the phase sensitivity in the experiment to measure the non-Markovianity of quantum open systems. Moreover, an environment-assisted precision measurement is also available [43].

4. Measurement as one observable estimator

If a phase shift \( \phi \) is measured by the outcomes of an observable estimator \( \hat{A} \) in linear two-path interferometers, the estimator should be expressed as a binary function of \( \hat{n}_a \) and \( \hat{n}_b \),

\[
\hat{A} = f(\hat{n}_a, \hat{n}_b),
\]

which is the case considered in [44–46]. Because simultaneous measurements are complete for the two-mode Fock space, and are sufficient for phase \( \phi \), we obtain,

\[
F_{\hat{A}}[|\Psi_\phi\rangle] \leq F_{\hat{E}}[|\Psi_\phi\rangle],
\]

where the equality holds if the measurement \( \hat{A} \) of the state \( |\Psi_\phi\rangle \) is a sufficient statistic for underlying parameter \( \phi \) [47]; for example, an observable \( \hat{A} = \hat{n}_a + \sqrt{2}\hat{n}_b \) is sufficient for QFI for any state in the two-mode Fock space.

Note that one observable estimator that is sufficient for QFI when considering a fixed photon number may be insufficient when the photon number is fluctuating. For example, if the total photon number is fixed and known to be \( N \), momentum operator \( \hat{J}_3 \) [48] is sufficient to achieve the maximum QFI when state \( |\Psi\rangle \) generated from the entangled photon source in MMZI is a NOON state; but we confirm that \( \hat{J}_3 \) is not the optimal measurement for QFI when total photon number fluctuates.

We should also note that some operators, for example, the parity operator \( \hat{\Pi}_a = \exp(i\pi\hat{n}_a) \) [49, 12], are still useful in quantum metrology for highly precise parameter estimation, although they cannot saturate QFI for states with a fluctuating photon number. Moreover, when deriving the appropriate form of HL, it is reasonable to consider the situation that the parameter is estimated from the results of quantum measurement which is a sufficient statistic for the parameter.

5. A superposition of NOON states with arbitrary high phase sensitivity and finite average photon number

As we have already shown, a superposition of NOON states will always saturate the maximum QFI in equation (12). Here we present an interesting example of such a state which has an infinite QFI when the average photon number is finite. We consider a superposition of NOON states as follows:

\[
|\Psi(x)\rangle = \frac{1}{\sqrt{\xi(x)}} \sum_{N=1}^{\infty} \frac{1}{\sqrt{N^x}} |N\rangle_a |0\rangle_b + |0\rangle_a |N\rangle_b, \quad \xi(x) = \sum_{N=1}^{\infty} 1/N^x \quad \text{is Riemann zeta function,}
\]

where \( x \in (1, +\infty) \), \( \xi(x) = \sum_{N=1}^{\infty} 1/N^x \quad \text{is Riemann zeta function, and here it is the normalization factor. The probability of each NOON state, } (|N\rangle_a |0\rangle_b + |0\rangle_a |N\rangle_b)/\sqrt{2}, \quad \text{is } 1/(\xi(x) \times N^x) \quad \text{for photon number } N. \quad \text{When } x \leq 2, \quad \text{the average of photon number is infinite.} \]
We next consider the case $x = 3$, the photon number on average can be calculated as,

$$\langle \hat{N} \rangle = \frac{1}{\xi(3)} \sum_{N=1}^{\infty} \frac{N}{N^3} = \frac{\xi(2)}{\xi(3)} \approx 1.369. \quad (17)$$

This means that a superposed state, $|\Psi(3)\rangle$ in equation (16), has a finite average photon number, while the average of squared photon numbers, $\langle \hat{\hat{N}}^2 \rangle$, can be infinite, as calculated in the following:

$$\langle \hat{\hat{N}}^2 \rangle = \frac{1}{\xi(3)} \sum_{N=1}^{\infty} \frac{N^2}{N^3} = \frac{\xi(1)}{\xi(3)} \rightarrow \infty, \quad (18)$$

where we have used the fact $\xi(1) \rightarrow \infty$. For the general case, both $\langle \hat{N} \rangle$ and $\langle \hat{\hat{N}}^2 \rangle^{1/2}$ are dependent on $x$, the relation between the photon number on average and CRB of phase estimation for state equation (16), $\Delta \phi = 1/\langle \hat{\hat{N}}^2 \rangle^{1/2}$, is shown in figure 3(a). We can see from figure 3(a) that CRB reaches zero when $\langle \hat{N} \rangle$ approaches about 1.369, and as $\langle \hat{N} \rangle$ increases, CRB remains zero.

Note that in [4, 50], the following state was proposed:

$$|\Phi\rangle_{SW} = A \sum_{m=0}^{M} \frac{1}{m+1} |m\rangle, \quad (M \gg 1, A \simeq \sqrt{6/\pi^2}). \quad (19)$$

It is shown that QFI of this state can be arbitrarily high when the average photon number is infinite, while in our case as equation (16), the average photon number can be finite.

6. Superpositions of NOON states and dual Fock states

Dual Fock state, $|N\rangle_a|N\rangle_b$, is closely related to NOON state and is also widely used in the optical interferometric quantum measurement. For example, when input is a dual Fock state $|1\rangle_a|1\rangle_b$ in MZI, the intermediate state is a NOON state with photon number $N = 2$, see [6]. We next compare QFI of superposition of dual Fock states with that of NOON states in two examples. We emphasize that NOON states and dual Fock states are compared in different interferometers. NOON states are generated by the entangled photon source in figure 1(b), while dual Fock states are the input states of MZI in figure 1(a). The reason why we consider dual Fock states is that generating dual Fock states is easier than generating NOON states when photon number grows high.

The first example: we consider the input state as the two-mode squeezed vacuum (TMSV), which is a superposition of dual Fock states, $|\Psi\rangle_{in} = \sum_{N=0}^{\infty} p_N(|\hat{N}\rangle)|N\rangle_a|N\rangle_b$, where $p_N(|\hat{N}\rangle) = (1 - t_{\delta}^2)^{N/2}$ with $t_{\delta} = 1/(1 + 2/\langle \hat{N} \rangle)$, see [12]. When a dual Fock state $|\Psi\rangle_{in} = |N\rangle_a|N\rangle_b$ passes through the first beam-splitter, the intermediate state is,

$$|\Psi\rangle = \sum_{k=0}^{N} C_k \sqrt{\frac{(2N-2k)! (2k)!}{N! 2^{N+1/2}}} [(-1)^N |2N - 2k\rangle_a |2k\rangle_b + |2k\rangle_a |2N - 2k\rangle_b]. \quad (20)$$

QFI of each superposed state is $(2N - 4k)^2$. Hence, QFI of the intermediate state equation (20) can be calculated to be $2N^2 + 2N$. Then QFI of the TMSV is $\sum_{N=0}^{\infty} p_N(|\hat{N}\rangle)(2N^2 + 2N) = \langle \hat{\hat{N}}^2 \rangle + 2\langle \hat{N} \rangle$. In comparison, for a superposition of NOON states with the same probability distribution as TMSV, $|\Psi\rangle = \sum_{N=0}^{\infty} \sqrt{p_N(|\hat{N}\rangle)(|2N\rangle_a|0\rangle_b + |0\rangle_a|2N\rangle_b)} / \sqrt{2}$, the QFI is $\sum_{N=0}^{\infty} p_N(|\hat{N}\rangle)4N^2 = 2\langle \hat{N}^2 \rangle + 2\langle \hat{N} \rangle$. This state saturates the maximum QFI for this probability distribution of total photon number. CRBs of these two states are shown in figure 3(a).
Figure 3. (a) Phase uncertainty $\Delta \phi$ against average photon number $\langle \hat{N} \rangle$. Green dash-dot line is for the SNL $1/\langle \hat{N} \rangle^{1/2}$, black dash-dot line is for conventional HL $1/\langle \hat{N} \rangle$, purple dash-dot line denotes CRB of TMSV dual Fock state $1/(\langle \hat{N}^2 \rangle + 2\langle \hat{N} \rangle)^{1/2}$, blue dash line indicates the minimum CRB in TMSV states $1/12\langle \hat{N}^2 \rangle + 2\langle \hat{N} \rangle^{1/2}$ and red solid line is for CRB of superposed NOON states in equation (16). When $\langle \hat{N} \rangle \geq 1.369$, the state in equation (16) obtains an infinite QFI and a zero CRB. (b) Phase uncertainty $\Delta \phi$ against average photon number $\langle \hat{N} \rangle$. Red solid line is for the CRB obtained by the superposition of NOON states similar to equation (16), and black solid line is for the superposition of dual Fock states. When $\langle \hat{N} \rangle < 2.737$, QFI for NOON state is larger than for dual Fock state, while when $\langle \hat{N} \rangle \geq 2.737$, both QFIs are infinite.

Second example: similar to the state in equation (16), we consider a superposition of dual Fock states with the same probability distribution of the photon number, $|\Psi(x)\rangle_{\text{dualFock}} = \sum_{N=1}^{\infty} |N\rangle_a |N\rangle_b / \sqrt{\xi(x)} \propto N^x$. When $x = 3$, the average photon number is $\langle \hat{N} \rangle = \frac{2\xi(2)}{\xi(3)} \approx 2.737$, and QFI can be calculated as $F_Q(\phi) = \sum_{N=1}^{\infty} \left[ \frac{2\xi(1)}{\xi(3)} + \frac{2\xi(2)}{\xi(3)} \right] \rightarrow \infty$. Subsequently, it has a zero CRB similar to the state in equation (16). The results of these two states are plotted in
figure 3(b). Note that for probability distribution $1/\zeta(x) \times N^x$, the photon number of NOON state is $2N$ in comparison with that of dual Fock state. We can see that when $\langle \hat{N} \rangle < 2.737$, QFI for NOON state case is larger than the case for dual Fock state; when $\langle \hat{N} \rangle \geq 2.737$, CRBs of both states are zeroes.

Generally, for any pure intermediate state written as equation (10), the form of superposed state $|\psi_N\rangle$ and the distribution of photon number $P(N)$ are both important to achieve a high QFI in linear two-path interferometers, which can be regarded as the effects of entanglement as well as superposition. A similar discussion can also be found in [13]. Another interesting result is that the states with infinite QFI might not have any Zeno dynamics since the quantum Zeno time scale expressed in terms of QFI, written as $\tau_{QZ} = 2/\sqrt{mF_Q}$ [26], approaches zero.

7. Conclusion and discussion

We propose that the maximum QFI of states with a fluctuating or fixed number of particles takes the form $\langle \hat{N}^2 \rangle$ in a two-path interferometer. Our result shows that the superposed NOON states by simultaneous measurements achieve this optimal QFI in MMZI. We also present a specified superposition of NOON states with probability distribution related to Riemann zeta function. This state has an infinite QFI but with a finite average photon number via linear two-path interferometer measurements. We also compare the case of dual Fock state in MZI with that of NOON state in MMZI. Our work presents that the advantage of this unbounded quantum Fisher information can be obtained by only a few superposed terms, which will be beneficial to the future development of quantum technology.

We assume that the entangled photon source in figure 1(b) could produce superposed NOON states. One area for future research is to prepare superposed NOON states when some achievements have been made for the formation of high-NOON states [51, 52].

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