A NEW PARTIAL GEOMETRY \( pg(5, 5, 2) \)

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Abstract. We construct a new partial geometry with parameters \( pg(5, 5, 2) \), not isomorphic to the partial geometry of van Lint and Schrijver.

1. Introduction

A partial geometry \( pg(s, t, \alpha) \) is a partial linear space with lines of degree \( s + 1 \) and points of degree \( t + 1 \) such that for every non-incident point-line pair \( (P, \ell) \), there are exactly \( \alpha \) points on \( \ell \) collinear with \( P \). Consequently, the number of points is \( v = (s + 1)(st/\alpha + 1) \) and the point graph is strongly regular with parameters \( srg(v, s(t + 1), s - 1 + t(\alpha - 1), \alpha(t + 1)) \). The dual of a \( pg(s, t, \alpha) \) is a \( pg(t, s, \alpha) \), hence formulae for the number of lines and parameters of the line graph are obtained by exchanging \( s \) and \( t \). For results about partial geometries we refer to [4, 5, 15].

Van Lint and Schrijver [13] constructed a partial geometry with parameters \( pg(5, 5, 2) \). Another construction of the same partial geometry was given in [11]. This is one of only three known proper partial geometries with \( \alpha = 2 \), all of them sporadic examples [5]. It was denoted as a partial geometry of Type 6 in [15] and was characterized in [6] as the only \( pg(s, t, 2) \) with an abelian Singer group of rigid type (i.e. such that the stabilizer of every line is trivial). A stronger theorem along the same lines was proved in [10, Corollary 52]. Furthermore, in [7] the geometry of van Lint and Schrijver was listed as a counterexample to the Manickam–Miklós–Singhi conjecture for partial geometries.

In this note we construct another partial geometry with parameters \( pg(5, 5, 2) \), not isomorphic to the partial geometry of van Lint and Schrijver. The new partial geometry was discovered by prescribing automorphism groups and performing computer searches, with techniques similar to the ones used in [9] for quasi-symmetric designs. In 2000 Mathematics Subject Classification. 51E14.

Key words and phrases. partial geometry, strongly regular graph, MMS conjecture.

This work has been supported by the Croatian Science Foundation under the projects 6732 and 9752.
Section 2 we describe a computer-free construction of the new \( pg(5, 5, 2) \) by changing some lines of the geometry of van Lint and Schrijver. Properties of the new partial geometry are described in Section 3.

2. Construction of the new partial geometry

A partial geometry \( pg(5, 5, 2) \) has \( v = 81 \) points and as many lines. We shall denote the geometry of van Lint and Schrijver by \( G = (P, L) \), where the set of lines \( L \) consists of 6-element subsets of the set of points \( P \). Two constructions of \( G \) are given in [13], the first using cyclotomy in the finite field \( \mathbb{F}_{81} \), and the second using the dual code of the repetition code in \( \mathbb{F}_3^5 \). The first construction does not essentially use multiplication in \( \mathbb{F}_{81} \) and we describe it here in purely linear algebraic terms.

Let \( V \) be a four-dimensional vector space over \( \mathbb{F}_3 \) and \( \{e_1, e_2, e_3, e_4\} \) a basis. Define \( e_5 = -\sum_{i=1}^4 e_i \) and \( S = \{0, e_1, e_2, e_3, e_4, e_5\} \). Then \( L = \{x + S \mid x \in V\} \) is the set of lines of \( G = (V, L) \). In [13], the fifth roots of unity were used as the vectors \( e_1, \ldots, e_5 \), but the proof that this is a \( pg(5, 5, 2) \) only relies on the following observation. If a linear combination of the vectors in \( S \) is zero, then it must be of the form \( \alpha \cdot 0 + \beta \cdot (e_1 + \ldots + e_5) \). Clearly this holds starting from any basis.

There are \( \left[ \begin{array}{c} 4 \\ 3 \end{array} \right] = 40 \) three-dimensional subspaces of \( V \) and they intersect \( S \) in 1, 2, 3 or 4 vectors. Take a subspace \( N_0 \leq V \) such that \( |N_0 \cap S| = 3 \), e.g. \( N_0 = \langle e_1, e_2, e_3 - e_4 \rangle \), and denote its cosets by \( N_1 = e_3 + N_0 \) and \( N_2 = e_3 + e_4 + N_0 \). Note that \( |N_1 \cap S| = 3 \) and \( |N_2 \cap S| = 0 \), from which we get intersection sizes of \( N_0 \) with the lines of \( G \):

\[
|N_0 \cap (x + S)| = \begin{cases} 
3, & \text{for } x \in N_0 \cup N_2, \\
0, & \text{for } x \in N_1.
\end{cases}
\]

(A three-dimensional subspace \( N \leq V \) such that \( |N \cap S| = 2 \) is a 2-ovoid, i.e. intersects every line of \( G \) in exactly 2 points; see [14].)

We now define a new incidence structure \( G' = (V, L') \) by changing the lines of \( G \) disjoint from \( N_0 \), and retaining the 3-secants of \( N_0 \). Take \( S' = \{0, e'_1, e'_2, e'_3, e'_4, e'_5\} \) for \( e'_1 = -e_1 + e_3 \), \( e'_2 = -e_1 + e_3 - e_4 \), \( e'_3 = -e_2 + e_4 \), \( e'_4 = -e_2 - e_3 + e_4 \) and define

\[
L' = \{x + S' \mid x \in N_1\} \cup \{x + S \mid x \in N_0 \cup N_2\}.
\]

The set \( S' \) is of the same form as \( S \), i.e. \( \{e'_1, e'_2, e'_3, e'_4\} \) is a basis of \( V \) and \( e_5 = -\sum_{i=1}^4 e'_i \). It has the same intersection pattern with the cosets of \( N_0 \), namely \( |N_0 \cap S'| = |N_1 \cap S'| = 3 \), \( |N_2 \cap S'| = 0 \). From this and the observation \( \Delta(S) \cap \Delta(S') \cap N_0 = \emptyset \) it follows that \( G' \) is a partial linear space with points and lines of degree 6. Here \( \Delta(S) = \ldots \)
\{x - y \mid x, y \in S, x \neq y\} is the set of differences of \(S\). To prove that \(G'\) is a partial geometry with \(\alpha = 2\), it suffices to check that the point graph \(\Gamma_1(G')\) is strongly regular with parameters \(\text{srg}(81, 30, 9, 12)\) by the Lemma from [2]. Because \(\Delta(S) \cap (N_1 \cup N_2) = \Delta(S') \cap (N_1 \cup N_2)\) holds, collinearity in \(G\) and \(G'\) differs only for points \(x, y \in N_1\) and \(x, y \in N_2\). In all other cases the common neighbours of \(x\) and \(y\) are the same as in \(\Gamma_1(G)\), which we know is a \(\text{srg}(81, 30, 9, 12)\). If \(x, y \in N_1\) are collinear, one can check that they have three common neighbours, and the arguments are the same for \(x, y \in N_2\). Finally, to prove that the partial geometries \(G\) and \(G'\) are not isomorphic, take two collinear points \(x = 0\) and \(y = e_1\). The set of points collinear with both \(x\) and \(y\) is \(A \cup B \cup \{z\}\) for \(A = \{e_2, e_3, e_4, e_5\}, B = \{e_1 - e_2, e_1 - e_3, e_1 - e_4, -e_1 + e_2 + e_3 + e_4\}, \) and \(z = -e_1\) (in both geometries). In [1] it was noted that the collinearity graph of this subconfiguration of \(G\) always consists of two copies of the complete graph \(K_4\) and an isolated vertex. Indeed, pairs of points in \(A\) and pairs of points in \(B\) are collinear and there are no other collinearities between points of \(A \cup B \cup \{z\}\) in \(G\). In \(G'\) we lose mutual collinearity of \(e_1 - e_3, e_1 - e_4, \) and \(-e_1 + e_2 + e_3 + e_4\) and gain no further collinearities, so \(B\) is a star graph.

3. Properties of the new partial geometry

In [1], the full automorphism group of \(G\) was determined as \(\text{Aut}(G) = \mathbb{F}_3^4 \rtimes S_6\) (semidirect product with normal subgroup \(\mathbb{F}_3^4\)) of order 58,320. The new geometry \(G'\) clearly has \(N_0 = \mathbb{F}_3^3\) as automorphism group. Using nauty [11] and GAP [8], we found that the full automorphism group is \(\text{Aut}(G') = \mathbb{F}_3^3 \rtimes G\) of order 972, where \(G \leq S_6\) is a subgroup of order 36 isomorphic to \(\mathbb{F}_3^3 \rtimes \mathbb{Z}_4\). The group \(\text{Aut}(G')\) is not transitive and does not have Singer subgroups. The point orbits of \(\text{Aut}(G')\) are \(N_0\) and \(N_1 \cup N_2\), and the line orbits are \(\{x + S' \mid x \in N_1\}\) and \(\{x + S \mid x \in N_0 \cup N_2\}\).

The point graphs \(\Gamma_1(G)\) and \(\Gamma_1(G')\) are strongly regular with parameters \(\text{srg}(81, 30, 9, 12)\) and are not isomorphic. The full automorphism group \(\text{Aut}\Gamma_1(G)\) is twice as large as \(\text{Aut}(G)\) [11], while the full automorphism groups \(\text{Aut}\Gamma_1(G')\) and \(\text{Aut}(G')\) coincide. Using Cliquer [12], we found that the graphs \(\Gamma_1(G)\) and \(\Gamma_1(G')\) are faithfully geometric, meaning that they only support their respective partial geometries up
to isomorphism. The graph $\Gamma_1(\mathcal{G})$ has 162 cliques of size 6 and the graph $\Gamma_1(\mathcal{G}')$ has 108 such cliques.

The two geometries are self-dual, hence their line graphs $\Gamma_2(\mathcal{G})$ and $\Gamma_2(\mathcal{G}')$ are isomorphic to the point graphs. Of the 162 cliques of size 6 in $\Gamma_2(\mathcal{G})$, 81 correspond to stars (sets of 6 lines through a single point). The remaining 81 cliques correspond to negative lines, i.e. sets of the form $-(x+S)$, $x \in V$. For every negative line there are 6 lines of $\mathcal{G}$ intersecting it in precisely one point and they form a clique in $\Gamma_2(\mathcal{G})$. Negative lines were used in [7] to show that $\mathcal{G}$ does not have the strict MMS star property. There are 27 cliques of size 6 in the line graph $\Gamma_2(\mathcal{G}')$ that are not stars. They can be used in the same way to show that the new partial geometry $\mathcal{G}'$ is also a counterexample to the MMS conjecture.

Note added in proof

After submitting the paper, the author learned that the new partial geometry was also independently discovered in [3] by a different method.

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