Generalized General Relativistic Magnetohydrodynamic Equations for Plasmas of Active Galactic Nuclei in the Era of the Event Horizon Telescope

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Abstract

The generalized general relativistic magnetohydrodynamic (generalized GRMHD) equations have been used to study specific relativistic plasma phenomena, such as relativistic magnetic reconnection or wave propagation modified by nonideal MHD effects. However, the $\Theta$ term in the generalized Ohm’s law, which expresses the energy exchange between two fluids composining a plasma, has yet to be determined in these equations. In this paper, we determine the $\Theta$ term based on the generalized relativistic Ohm’s law itself. This provides closure of the generalized GRMHD equations, yielding a closed system of the equations of relativistic plasma. According to this system of equations, we reveal the characteristic scales of nonideal MHD phenomena and clarify the applicable condition of the ideal GRMHD equations. We evaluate the characteristic scales of the nonideal MHD phenomena in the M87$^*$ plasma using the Event Horizon Telescope observational data.

Unified Astronomy Thesaurus concepts: Black hole physics (159); Plasma physics (2089); General relativity (641); Magnetic fields (994); Active galactic nuclei (16); High energy astrophysics (739)

1. Introduction

The observation of the supermassive black hole shadow at the center of the giant elliptical galaxy M87 with impressive images by the Event Horizon Telescope (EHT) Collaboration (EHT Collaboration et al. 2019a) has brought us into a new era of black hole plasma physics. To extract detailed information concerning the dynamics of the plasma, the black hole’s gravitational field, and the black hole itself, it is necessary to develop fully general relativistic models of the accretion flow, associated winds and relativistic jets, and the emission properties of the plasmas. The most popular approach to modeling dynamic relativistic sources is known as the “ideal general relativistic magnetohydrodynamic (GRMHD) approximation.” Over the past few decades, a number of ideal GRMHD codes have been developed and applied to a large variety of astrophysical scenarios (Koide et al. 1998, 1999, 2000, 2002, 2006; Gammie et al. 2003; Koide 2003, 2004; McKinney 2006; Del Zanna et al. 2007; McKinney & Blandford 2009; McKinney et al. 2013; Radice & Rezzolla 2013). The EHT team also found that the images produced by GRMHD simulations with general relativistic ray-tracing calculations (EHT Collaboration et al. 2019b; Porth et al. 2019) are consistent with the asymmetric ring feature seen in the EHT data. Comparing the GRMHD simulations and the EHT images, the EHT Collaboration team concluded that the brightness asymmetry in the ring can be explained by the relativistic beaming of the emission from plasma rotating close to the speed of light around a black hole spinning clockwise (EHT Collaboration et al. 2019b).

In these ideal GRMHD simulations, both the finite electrically resistive effect and a number of plasma effects, such as the Hall effect, thermo-electromotive force, and electron inertia, are neglected. These neglected effects may modify the dynamics of the plasma and the magnetic field in processes like magnetic reconnection. Actually, Acciari et al. (2009) presented simultaneous radio and $\gamma$-ray observations of M87$^*$ and showed that radio knots were ejected from the core of the galaxy where the TeV $\gamma$-ray flare occurred. This knot may be recognized by the plasmoids formed by the magnetic reconnection like coronal-mass ejection from the Sun. Hirota et al. (2013, 2015) and Comisso & Asenjo (2014) showed that electron inertia causes collisionless magnetic reconnection.

To investigate the specific properties of relativistic plasmas with nonideal MHD effects, we must use generalized GRMHD, including a generalized relativistic version of Ohm’s law. The generalized GRMHD equations were introduced on the basis of the two-fluid approximation of plasma in the Kerr metric in a pioneering study by Khanna (1998). More generalized equations from the general relativistic Vlasov–Boltzmann equation in time-varying spacetime were formulated by Meier (2004). Koide (2008) introduced some peculiar quantity-average definitions to derive the generalized special relativistic MHD equations for pair plasma from the relativistic two-fluid equations without any additional approximations. Koide (2009) extended these generalized special relativistic MHD equations for pair plasma to any two-component plasma, including electron–ion (normal) plasma. The general relativistic version of generalized MHD for any kind of plasma, including pair and normal plasmas, was given by Koide (2010). Recently, nonideal MHD effects (e.g., related to relativistic magnetic reconnection and wave propagation) have been analyzed by a number of authors using the generalized relativistic MHD equations (Asenjo & Comisso 2015; Asenjo et al. 2015; Yang & Wang 2016, 2018; Kawazura 2017; Kawazura et al. 2017; Yang 2017, 2019a, 2019b, 2019c; Liu et al. 2018, 2019). The equations of the generalized relativistic MHD are identical to the relativistic equations of the two-fluid approximation with specific averages of physical variables. However, the $\Theta$ term in the generalized Ohm’s law was not determined in the previous papers (Koide 2008, 2009, 2010). This $\Theta$ term describes energy transport from the negatively charged plasma to the positively charged one. To determine the $\Theta$ term, Koide (2009) assumed that the relative velocity of the two fluids is not so large that the frictional force is proportional to
the relative velocity and obtained
\[
\Theta = \frac{\theta}{2e\rho_e} \times \left( \frac{\rho_e^2}{J_\mu} + J_\mu J_\nu \Delta \mu \left[ \frac{n^2 - \Delta \mu \mu \left( \frac{\rho_e^2}{2} \right)}{n^2 - \Delta \mu \mu \left( \frac{\rho_e^2}{2} \right)} \right] \right).
\]

where the definitions of variables \(e, \mu, \Delta \mu, n, J_\mu\), and \(\rho_e\) are shown in Section 2. Unfortunately, the \(\Theta\) term cannot be determined via Equation (1) because it contains the unknown parameter, \(\theta\) (0 ≤ θ ≤ 1).

In this paper, we determine the \(\Theta\) term in the generalized Ohm’s law using the covariant form of the generalized relativistic Ohm’s law itself. Previously, to determine the \(\Theta\) term, we had considered a number of approaches, including the relativistic Vlasov–Boltzmann equation and the collisional fluid two-fluid approximation; however, we finally found that such efforts are not necessary to determine the \(\Theta\) term. Using this term, we obtain an explicitly closed system of the generalized GRMHD equations.

Using the closed system of generalized GRMHD equations, we determine the characteristic scales at which nonideal MHD effects, such as the resistive electromotive force effect, the Hall effect, the thermo-electromotive force, and the current-carrier (electrons, in the normal plasma case) inertia effect, become significant in the generalized Ohm’s law. For this purpose, we introduce some plasma parameters from linear analysis of special relativistic plasma waves.

Using the parameters of plasmas obtained from observations by the EHT Collaboration, we can evaluate the characteristic scales of the nonideal MHD phenomena of the plasmas around M87*. The evaluated scales show that in global phenomena at the scale of the horizon radius, the additional terms of the Hall effect, thermo-electromotive force, and electron inertia, including the electric resistivity, are negligible. Thus, the ideal GRMHD is regarded as a good approximation of the global dynamics of the plasmas around M87*.

In Section 2, we review the generalized GRMHD equations based on the general relativistic two-fluid equations. We derive the \(\Theta\) term of the generalized Ohm’s law by itself. In Section 3, we briefly show the 3+1 formalism of the generalized GRMHD equations with the normal observer frame. In Section 4, we introduce some plasma parameters obtained by linear analysis of relativistic plasma waves and show the characteristic scales of the nonideal MHD phenomena of the relativistic plasmas. We evaluate the significance of the nonideal MHD terms of the generalized GRMHD equations in Section 5 using the plasma parameters introduced in Section 4 and reveal the characteristic scales of the nonideal MHD effects. We apply these characteristic scales to the plasma around the black hole of M87* using observational data from the EHT Collaboration (EHT Collaboration et al. 2019b) in the last part of Section 4. The final section presents a summary of this paper.

2. Generalized GRMHD Equations

2.1. Review of the Generalized GRMHD Equations

We review the generalized GRMHD equations based on the general relativistic two-fluid equations (Koide 2010). For simplicity, we assumed that the plasma is composed of two fluids, where one fluid consists of positively charged particles with mass \(m_+\) and electric charge \(e\) and the other fluid consists of negatively charged particles with mass \(m_-\) and electric charge \(-e\). We take no account of radiation cooling effect, plasma viscosity, and self-gravity in order to study the fundamentals of interaction between magnetic fields and resistive plasmas around the spinning black holes. We also assumed that the plasmas are heated only by ohmic heating and disregarded nuclear reactions, pair creation, and annihilation.

The spacetime, \((x^0, x^1, x^2, x^3) = (t, x^1, x^2, x^3)\), is characterized by a metric \(g_{\mu\nu}\), where a line element is given by \(ds^2 = g_{\mu\nu}dx^\mu dx^\nu\). Here we use units in which the speed of light, the dielectric constant, and the magnetic permeability in vacuum all are unity: \(c = 1, \epsilon_0 = 1, \mu_0 = 1\). When we consider a black hole with mass \(M_{\text{BH}}\), we use the unit system such as \(GM_{\text{BH}} = 1\), where \(G\) is the gravitational constant. The relativistic equations of the two fluids and the Maxwell equations are

\[
\nabla_\nu (n_\pm U_{\nu}^\pm) = 0, \quad \nabla_\nu (h_{\pm} U_\nu^\pm U_\mu^\pm) = -\nabla_\nu p_\pm \pm e n_\pm \mu_{\theta\mu} U_\nu^\pm F_{\sigma\nu} \pm R_\mu, \quad \nabla_\nu \ast F_{\mu\nu} = 0, \quad \nabla_\nu F_{\mu\nu} = J_\mu, \]

where variables with subscripts, plus/minus (±), are those of the fluid of positively/negatively charged particles, \(n_\pm\) is the proper particle number density, \(p_\pm\) is the proper pressure, \(h_\pm\) is the relativistic enthalpy density, \(U_\mu^\pm\) is the 4-velocity, \(\nabla_\nu\) is the covariant derivative, \(F_{\mu\nu}\) is the electromagnetic field tensor (\(A_\mu\) is the 4-vector potential), \(\ast F_{\mu\nu}\) is the dual tensor of \(F_{\mu\nu}\), \(R_\mu\) is the fractional 4-force density between the two fluids, and \(J_\mu\) is the 4-current density. We will often write a set of the components of the 4-vector using a bold italic font, e.g., \(U_\mu = (U_0^\pm, U_1^\pm, U_2^\pm, U_3^\pm)\), \(J = (J_0^\pm, J_1^\pm, J_2^\pm, J_3^\pm)\), \(R = (R_0^\pm, R_1^\pm, R_2^\pm)\). We further define the Lorentz factor \(\gamma_\pm = U_0^\pm / c\), the 3-velocity \(V_\pm = U_\nu^\pm / \gamma_\pm\), the electric field \(E_i = F_{0i}\), the magnetic flux density \(B_k = F_{ik}\) (\(e^{ijk}\) is the Levi–Civita tensor), and the electric charge density \(\rho_e = J_0^\pm\).

Here the alphabetic index \((i, j, k)\) runs from 1 to 3.

To derive one-fluid equations of the plasma, we define the average and difference variables as

\[
\rho = m_+ n_+ \gamma_+ + m_- n_- \gamma_-, \quad n = \frac{\rho}{m}, \quad p = p_+ + p_- - p_\pm, \\
U_\mu = \frac{1}{\rho}(m_+ n_+ U_{\mu}^+ + m_- n_- U_{\mu}^-), \\
J_\mu = e(n_+ U_{\mu}^+ - n_- U_{\mu}^-),
\]

where \(\gamma_\pm\) is the Lorentz factor of the two fluids observed by the local center-of-mass frame of the plasma \(S'\) and \(m = m_+ + m_-\). Hereafter, a prime is used to denote the variables of the
According to Koide (1980, 1989, 2010), we introduce the scalar \( \Theta \) by

\[
\nabla \times \mathbf{F} = \frac{\eta}{n_e} \mathbf{J},
\]

where \( \mathbf{R} \) represents density of power (energy per unit time) transported from the negatively charged fluid to the positively charged fluid. Equations (16) and (17) yield

\[
\mathbf{R} = -\eta [\mathbf{J} - \rho' \mathbf{e} (1 + \Theta) \mathbf{U}],
\]

where \( \rho' \mathbf{e} \) is the charge density observed by the local center-of-mass frame of the two fluids \( S' \) and \( \rho' = -U_e \mathbf{J} \). Using the above variables, we have one-fluid equations from the two-fluid Equations (2) and (3),

\[
\nabla \rho(\mathbf{U}') = 0,
\]

\[
\nabla \left[ \mu \frac{\mathbf{h}_+}{\rho} \mathbf{J} + \frac{\mathbf{h}_+}{\rho} \mathbf{U}' + \frac{\Delta \mathbf{h}}{2\eta \mathbf{e}} (\mathbf{J} \mathbf{U}' + \mathbf{J} \mathbf{U}') \right] = -\nabla \mathbf{p} + \mathbf{J}' \mathbf{F}' \mathbf{v},
\]
2.2. Derivation of the $\Theta$ Term in the Generalized Ohm’s Law

We derive the $\Theta$ term in the generalized Ohm’s law (21) by Equation (21) itself. In the plasma rest frame $S'$, Equation (21) yields

$$\frac{\mu m}{n_e^2} \nabla_j q_{\mu j}^\nu = \frac{1}{2ne} \nabla_j \epsilon^\nu (\Delta_\mu \rho - \Delta \rho) + \left( U^\nu - \frac{\Delta_\mu}{n_e} J^\nu \right) F_{\mu j}^\nu - \eta [J^\nu - \rho_e'(1 + \Theta) U^\nu].$$

(26)

When we take $\mu = 0$ in the equations of the generalized Ohm’s law, we have

$$\frac{\mu m}{n_e^2} \nabla_j q_{\mu j}^\nu = \frac{\Delta_\mu}{n_e} J^\nu + \frac{1}{2ne} \partial_j (\Delta_\mu \rho - \Delta \rho) - \eta \rho_e' \Theta.$$  

(27)

Equation (27) yields Equation (23) with identities $J^\nu E_i = J^\nu U_i F_{\mu j}^\nu, \partial_j \delta^{\mu j} = U_j \partial_i$, and $\nabla_j q_{\mu j}^\nu = U^\nu \nabla_j q_{\mu j}^\nu$. The derivation of Equation (23) clearly shows that the first, second, and last terms on the right-hand side of Equation (23) vanish when the Hall effect, thermo-electromotive force, and current-carrier inertia are negligible, respectively. We express the form of the $\Theta$ term in several cases as follows:

1. Standard Ohm’s law: When the Hall effect, thermo-electromotive force, and current-carrier inertia are negligible, the $\Theta$ term vanishes and Equation (21) becomes the well-known standard relativistic Ohm’s law. Then we have

$$\eta \rho_e' \Theta = 0.$$  

(28)

2. A case of Hall term only: When the electric resistivity, thermo-electromotive force, and current-carrier inertia are negligible, the $\Theta$ term also vanishes because $E' \cdot J' = 0$, that is,

$$\eta \rho_e' \Theta = 0.$$  

(29)

3. Standard Ohm’s law with Hall term: When the thermo-electromotive force and current-carrier inertia are negligible, the $\Theta$ term is given by

$$\eta \rho_e' \Theta = \frac{\Delta_\mu}{n_e} U^\nu J^\nu F_{\mu j}^\nu.$$  

(30)

according to Equation (23).

4. A case of negligible current-carrier inertia: When the thermo-electromotive force is significant while energy-stress tensor of the 4-current density $q_{\mu j}^\nu$ is negligible, the $\Theta$ term includes the time derivative of plasma pressure. Then we have

$$\eta \rho_e' \Theta = \frac{\Delta_\mu}{n_e} U^\nu J^\nu F_{\mu j}^\nu - \epsilon^\nu \partial_j (\Delta_\mu \rho - \Delta \rho).$$

(31)

5. Generalized Ohm’s law with most general form: When the current-carrier inertia is significant, the energy-stress tensor of 4-current density $q_{\mu j}^\nu$ is not negligible and the $\Theta$ term becomes complex. The term $q_{\mu j}^\nu$ may be negligible when the Hall effect and thermo-electromotive force are not negligible, while the terms of the Hall effect and thermo-electromotive force are not negligible when the term $q_{\mu j}^\nu$ is not negligible as discussed in Section 5.

To reveal the physical meaning of the form of the $\Theta$ term given by Equation (23), we calculated the $\Theta$ term in the case of isothermal two fluids in charge neutrality as $p_{\pm} = n_0 T_0$, $n_\pm = n_0 \mp n_0$, $u_\pm^\nu = 0$, $u_\parallel^\nu = 0$, $\gamma_+ = \gamma_- = 1$, where $T_0$ is the temperature of the two fluids. We write $\Theta$ of this case by $\Theta_{iso}$. To keep the temperature of the two fluids equal, we have to distribute the same amount of the thermal energy released by the Joule heating to the two fluids. When the kinetic energy of the positively/negatively charged fluid is released with the power density $S_+$ and $S_-$, respectively, the energy density per unit time transported from the negatively charged fluid to the positively charged fluid is

$$R^\nu = \frac{1}{2} (S_+ + S_-) - \frac{m_-}{m_+} (S_+ + S_-) = \frac{1}{2} \Delta_\mu (S_+ + S_-).$$

(32)

As the Joule heating is given by $S_+ + S_- = E_i J_i^\nu$, we obtain

$$\eta \rho_e' \Theta_{iso} = R^\nu = \frac{\Delta_\mu}{2ne} E_i J_i^\nu = \frac{\Delta_\mu}{2} \frac{J^\nu}{n_e} F_{\mu j}^\nu.$$  

(33)

Equation (33) is half of $\eta \rho_e' \Theta$ in Equation (23). This means that the Hall effect with resistivity causes the temperature difference between the positively charged fluid and the negatively charged fluid. In the case of a normal plasma, we find $\eta \rho_e' \Theta > \eta \rho_e' \Theta_{iso}$ because $\Delta_\mu \approx 1, E' \cdot J' > 0$ for Joule heating. Then, Joule heating causes the ion fluid temperature to be higher than the electron temperature.

This also suggests that even in the resistive plasma the positively charged fluid and the negatively charged fluid do not exchange their thermal energy without the Hall effect, thermo-electromotive force, or current-carrier inertia effect.

3. 3+1 Formalism

We derive a 3+1 formalism of the equations of the “normal observer frame” in this paper. The line element of the displacement $dx^\mu$ in the spacetime is represented by

$$ds^2 = g_{\mu \nu} dx^\mu dx^\nu = - \alpha^2 dt^2 + \sum_{i,j} \gamma_{ij} (dx^i + \beta^i dt)(dx^j + \beta^j dt),$$

(34)

where $\gamma_{ij} = g_{ij}, g_{0i} = g_{0j}/\beta^j$, and $\alpha = \sqrt{-g_{00} - \sum_i g_{0i} \beta^j \beta^i}/2 = \sqrt{-g_{00} + \sum_i g_{0i} \beta^i \beta^i}/2$.

We introduce a local inertia frame called the “normal observer frame,” $(\tilde{t}, \tilde{x}^1, \tilde{x}^2, \tilde{x}^3)$ as

$$d\tilde{s}^2 = -d\tilde{t}^2 + \gamma_{ij} d\tilde{x}^i d\tilde{x}^j,$$  

(35)

where

$$d\tilde{t} = \alpha dt, \quad d\tilde{x}^i = dx^i + \beta^i dt.$$  

(36)

Here we have $g = \det(g_{\mu \nu}) = -\alpha^2 \gamma = -\alpha^2 \det(\gamma_{ij}).$ The 4-velocity of the normal observer frame is $N^{\mu} = (1/\alpha, -\beta^i/\alpha), N_{\mu} = (-\alpha, 0, 0, 0)$. Denoting these components observed by the normal observer frame with tildes and using

$^2$ When we write any contravariant vector by $a^\mu$, according to Equation (36), the contravariant vector in the normal observer frame, $a^\mu$, is given by $a^\mu = a^\mu\cdot \delta^a_\mu, a^\mu = a^\mu + \beta^a \delta^a_\mu = a^\mu - \alpha N^a a^\mu$. A covariant vector $a_i$ is $a_i = \frac{1}{2}(a_0 - \beta^0 a_i) = \frac{1}{2} a_i + \alpha N^a a_i, a_i = a_i$. In the normal observer frame $x^i$ we have $a^\mu = -a_0$ and $a^0 = \gamma a^i; \gamma_{ij} = \delta^i_j (\delta^i_j$ is the Kronecker delta)$.$
the equations in footnote 2, we have
\[
\tilde{\epsilon} \equiv U^0 = \alpha U^0, \quad \tilde{v}^i \equiv \frac{U^i}{U^0} = \frac{1}{\gamma} U^i - N_i U^0 \gamma^{-1}, \quad (37)
\]
\[
\epsilon + \gamma \rho \equiv T^{00} = \alpha^2 T^{00}, \quad \tilde{p}^i \equiv T^{0i} = \alpha T^{0i} - \alpha^2 N_i T^{00}, \quad (38)
\]
\[
T^{ij} \equiv T^{ij} - \alpha N^i T^{0j} - \alpha N^j T^{0i} + \alpha^2 T^{00}, \quad (39)
\]
\[
E_{i} \equiv F_{i0} = -F_{i0} = F_0 + \sum_j N_j F_{ij}, \quad \tilde{F}_{ij} \equiv F_{ij} = F_{ij} \quad (40)
\]
\[
\tilde{\rho}_c \equiv j^0 = \alpha j^0, \quad \tilde{J}^i = J^i - \alpha N^i j^0. \quad (41)
\]

The relationship between the variables measured in the normal observer frame is similar to that of ideal special relativistic MHD but not identical (Koide et al. 1996).

The generalized GRMHD equations except for the Ohm’s law (4), (19), and (20) are written as
\[
\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\nu} (\sqrt{-g} \rho U^\nu) = 0, \quad (42)
\]
\[
1 \frac{\partial}{\partial x^\nu} (\sqrt{-g} T^{\nu\rho}) + \Gamma_{\nu\rho}^{\gamma} T^{\gamma\nu} = 0, \quad (43)
\]
\[
\partial_\nu F_{\lambda \rho} + \partial_\rho F_{\nu \lambda} + \partial_\lambda F_{\nu \rho} = 0, \quad (44)
\]
\[
\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\nu} (\sqrt{-g} F^\nu) = -J^\nu, \quad (45)
\]
where we used the following relations: \( \nabla_{\mu} a^\nu = \partial_{\mu} a^\nu + \Gamma_{\mu\nu}^\rho a^\rho \) for any 4-vector \( a^\nu \), \( \Gamma_{\mu\nu}^\rho = \partial_{\mu} (\ln \sqrt{-g}) \), and \( F_{\nu \rho} = -F_{\rho \nu} \). With respect to the Ohm’s law (21), we have the following form:
\[
\frac{\mu m}{ne^2} \nabla_{\nu} q^{\nu \rho} = \frac{\mu m}{ne^2} \left[ \frac{1}{\sqrt{-g}} \partial_\nu (\sqrt{-g} q^{\nu \rho}) + \Gamma_{\nu\rho}^{\gamma \mu} q^{\gamma \mu} \right]
\]
\[
= \frac{1}{2ne^2} \nabla_\nu (\Delta \mu - \Delta \rho) + \left( U^\nu - \frac{\Delta \mu}{ne \rho} \right) F_{\nu \rho} - \eta [U^\nu - \rho \epsilon (1 + \Theta) U^\nu]. \quad (46)
\]

Using the normal observer variables, we obtain the following set of 3+1 formalism of the general GRMHD equations from Equations (42)–(45) and (46):
\[
\frac{\partial}{\partial t} (\gamma L \rho) = -\frac{1}{\sqrt{\gamma L}} \frac{\partial}{\partial x^\nu} [\gamma \gamma L \rho (v^\nu + N^\nu)], \quad (47)
\]
\[
\frac{\partial}{\partial t} P_i = -\frac{1}{\sqrt{\gamma L}} \frac{\partial}{\partial x^k} [\gamma \gamma L (T_{ik} + N^k P_i)] - \frac{\partial \alpha}{\partial x^i} \tilde{v}^i - \frac{\partial}{\partial x^k} (\alpha N^k) P_k + \alpha \frac{\partial \alpha}{\partial x^i} T_{jk}, \quad (48)
\]
\[
\frac{\partial}{\partial t} \tilde{\epsilon} = -\frac{1}{\sqrt{\gamma L}} \frac{\partial}{\partial x^\nu} [\gamma \gamma L (p^\nu + N^\nu \tilde{v})] - \frac{\partial \alpha}{\partial x^i} \tilde{v}^i - \left[ \gamma \tilde{v}^i \frac{\partial}{\partial x^j} (\alpha N^k) + \frac{1}{2} \alpha N^k \frac{\partial}{\partial x^i} N^\nu \right] T_{ji}, \quad (49)
\]
\[
\left( U^\nu - \frac{\Delta \mu}{ne} J^\nu \right) F_{i \nu} - \eta [J^\nu - \tilde{p}_c (1 + \Theta) U^\nu] + \frac{1}{2ne} \frac{\partial}{\partial x^i} (\Delta \mu - \Delta \rho)
\]
\[
= \frac{1}{\alpha ne^2} \left[ \frac{\partial}{\partial t} q^\nu + \frac{\partial}{\partial \gamma L} (\alpha \sqrt{\gamma L} (q^\nu + \gamma L N^\nu q^\nu)) \right] + \frac{\partial}{\partial x^i} (\alpha N^k) q^\nu_k - \alpha \frac{\partial \alpha}{\partial x^i} T_{ji}, \quad (50)
\]
\[
\eta \rho c \Theta = \Delta \mu J^\nu U^\nu F_{i \nu} + \left( U^\nu \frac{1}{\gamma L} \frac{\partial}{\partial \gamma L} + U^\nu \frac{\partial}{\partial x^\nu} (\Delta \mu - \Delta \rho) \right)
\]
\[
+ \frac{1}{ne^2} U^\nu \left[ \frac{1}{\gamma L} \frac{\partial}{\partial \gamma L} (\gamma L q^\nu - \frac{\partial \alpha}{\partial \gamma L} q^\nu) \right], \quad (51)
\]
\[
\frac{\partial}{\partial t} E^\nu + \alpha (J^\nu + \tilde{\rho}_c N^\nu) = \frac{\partial}{\partial x^\nu} [\alpha (B_k - \epsilon_{knu} N^\nu E^\nu)], \quad (53)
\]
\[
\frac{\partial}{\partial t} F^\nu = -\frac{\partial}{\partial \gamma L} (\gamma L B^\nu) = 0, \quad (54)
\]
\[
\frac{\partial}{\partial t} B^\nu = -\epsilon_{k\nu} \frac{\partial}{\partial x^\mu} (\alpha N^\mu B^\nu) = 0, \quad (55)
\]

Here we used formulae about the covariant derivative of a symmetric tensor (A13) and (A16) in the Appendix. With respect to Equation (51), we assumed \( \frac{\partial \gamma L}{\partial \gamma L} = 0 \) and used \( \nabla_{\mu} q^\nu = \frac{1}{\sqrt{\gamma L}} \frac{\partial}{\partial x^\nu} (\gamma L q^\nu) = \frac{1}{2} \frac{\partial}{\partial \gamma L} q^\nu + \frac{1}{2} \frac{\partial}{\partial x^\nu} q^\nu \), where \( \frac{\partial}{\partial \gamma L} = \frac{\partial}{\partial \gamma L} + N^\nu \frac{\partial}{\partial x^\nu} \) and \( \frac{\partial}{\partial \gamma L} = \frac{\partial}{\partial x^\nu} \). The terms with \( \alpha \) express the gravitation and the time lapse. The terms with \( N^\nu \) express the frame-dragging effect around a spinning black hole.

4. Characteristic Parameters of Relativistic Plasma

Introduced by Linear Analyses of Plasma Waves

We newly proposed a closed system of generalized GRMHD Equations (19)–(21) that are applicable not only for electron–ion (normal) but also for pair plasmas in the previous section. In this section, we introduce the characteristic parameters of relativistic plasma (\( \omega_p, \omega_c, u_A, c_p, \lambda_D \), etc.) using the linearized equations of these equations concerning various plasma waves and perturbations.

We investigate oscillations and waves propagating in a uniform, rest plasma and a uniform magnetic field in the flat spacetime (\( \alpha = 1 \) and \( N^\nu = 0 \)). For convenience, we use the 3-vector form like \( U = (U^0, U^1, U^2) \), \( J = (J^1, J^2, J^3) \), \( B = (B^1, B^2, B^3) \), \( E = (E^1, E^2, E^3) \), \( q = (q^0, q^1, q^2) \). In the flat spacetime, linearized equations of perturbations \( \tilde{\rho} = \rho - \tilde{\rho}, \tilde{p} = p - \tilde{p}, \tilde{h} = h - \tilde{h}, \tilde{U} = U, \tilde{B} = B - \tilde{B}, \) and \( \tilde{E} = E \) are derived by Equations (47)–(54) as
\[
\frac{\partial}{\partial t} \tilde{\rho} = -\tilde{\rho} \nabla \cdot \tilde{U}, \quad \tilde{h} \frac{\partial}{\partial t} \tilde{U} = -\nabla \tilde{\rho} + \tilde{J} \times \tilde{B}, \quad (56)
\]
Using enthalpy $H$, we consider the adiabatic EOS for single-component relativistic fluids, which are in thermal equilibrium, has been known and is given by

$$\frac{\rho}{\rho_{\pm}} = \frac{K_2(\rho_{\pm}/\bar{\rho}_{\pm})}{K_3(\rho_{\pm}/\bar{\rho}_{\pm})} = H_{\gamma}\left(\frac{\bar{\rho}_{\pm}}{\rho_{\pm}}\right)$$  \hfill (60)

(Chandrasekhar 1938; Synge 1957). Here $K_2$ and $K_3$ are the modified Bessel functions of the second kind of order two and three, respectively. When we consider the adiabatic one-component fluid in the rest frame, the adiabatic condition\(^5\) yields

$$\frac{\bar{n}_{\pm}}{n_{\pm}} = \frac{\bar{n}_{\pm} - \bar{p}_{\pm}}{\bar{\rho}_{\pm}}.$$  \hfill (61)

From Equation (60), we find

$$\frac{\bar{p}_{\pm}}{\bar{p}_{\pm}} = \frac{H'_{\gamma}\left(\frac{\bar{\rho}_{\pm}}{\rho_{\pm}}\right)}{H'_{\gamma}\left(\frac{\bar{\rho}_{\pm}}{\rho_{\pm}}\right)} - 1 = \Gamma_a\left(\frac{\bar{p}_{\pm}}{\rho_{\pm}}\right).$$  \hfill (62)

In general, $\Gamma_a$ is not constant and is a function of $\bar{p}_{\pm}/\rho_{\pm}$. $\Gamma_a$ is called the effective adiabatic index.\(^4\) Using $\Gamma_{\nu}$, we obtain the EOS for the plasma (Koide 2010) as

$$h = n\left[H_{\gamma}\left(\frac{p + \Delta p}{2\rho_{\pm}}\right)\frac{m_{\pm}^2}{\rho_{\pm}} + H_{\gamma}\left(\frac{p - \Delta p}{2\rho_{\pm}}\right)\frac{m_{\pm}^2}{\rho_{\pm}}\right].$$  \hfill (63)

$$\Delta h = 2n^2\mu m \left[H_{\gamma}\left(\frac{p + \Delta p}{2\rho_{\pm}}\right)m_+ - H_{\gamma}\left(\frac{p - \Delta p}{2\rho_{\pm}}\right)m_-\right].$$  \hfill (64)

where

$$\rho_{\pm} = \frac{\rho^2 + \frac{m_{\pm}^2\rho}{e}}{U_{\gamma}J_{\gamma}} - \frac{\left(\frac{m_{\pm}^2}{e}\right)^{\frac{1}{2}}J_{\gamma}^{\frac{1}{2}}}{\frac{1}{\gamma}}.$$  \hfill (65)

When $\bar{p}_{\pm} \ll \bar{\rho}_{\pm}$ or $\bar{p}_{\pm} \gg \bar{\rho}_{\pm}$, $\Gamma_a$ is asymptotically constant ($\Gamma_a(\bar{p}_{\pm}/\bar{\rho}_{\pm}) \approx \Gamma_a(\bar{p}_{\pm}/\bar{\rho}_{\pm})$); thus, we have $1/\Gamma_a = \bar{\rho}_{\pm} = \bar{p}_{\pm}$.\(^5\) That is,

$$\frac{\bar{p}_{\pm}}{\rho_{\pm}} = \frac{\bar{p}_{\pm}}{\bar{p}_{\pm}} = \Gamma_a(\bar{p}_{\pm}, \bar{\rho}_{\pm}).$$  \hfill (66)

In general, $\frac{\bar{p}_{\pm}}{\rho_{\pm}} = \Gamma_a(\bar{p}_{\pm}, \bar{\rho}_{\pm})$ depends on both $\bar{p}_{\pm}$ and $\bar{\rho}_{\pm}$.

\(^{4}\) The polytropic index is given by $N_{\rho\rho} = (\Gamma_a - 1)^{-1}$.

\(^{5}\) When we consider a fluid element of particle number $N$, volume $V$, and enthalpy $H$, the first law of thermodynamics is $dH = dQ + Vdp$, where $dQ$ is the heat energy from the outside of the fluid and vanishes in the adiabatic case. Using $dH = H dV$, $N = nV$, $dQ = 0$, we easily obtain Equation (61).

4.1. Longitudinal Modes of Plasma Waves and Oscillations

First, we derive a dispersion relation of longitudinal oscillation modes ($\hat{U}(\mathbf{k}, \mathbf{J})$) in an unmagnetized, rest plasma with uniform, finite pressure $\bar{p}$. For simplicity, we assume that the temperatures of the two fluids are the same: $T = T_+ = T_-$. Using Equation (5) and the zeroth component of Equation (8), we have $\gamma \eta_{\pm} = (\gamma \eta_{\pm} \approx m_{\pm}^2 \rho_{\pm}/e)/m$ when $\gamma = \gamma_+ \approx \gamma_-$. Using these equations, we have

$$\frac{\partial \eta}{\partial t} = -\frac{2}{m} \nabla \cdot (\rho \vec{U}) + \frac{\mu}{e} (\nabla \cdot \vec{J}).$$  \hfill (67)

In the present nonrelativistic case, $\gamma = 1$, we have

$$\frac{\partial \eta}{\partial t} = -\frac{2}{m} \nabla \cdot (\rho \vec{U}) + \frac{\mu}{e} (\nabla \cdot \vec{J}).$$  \hfill (68)

If $\Gamma_a(\bar{p}_{\pm}/\bar{\rho}_{\pm}) = \Gamma_a(\bar{p}_{\pm}/\bar{\rho}_{\pm}) \equiv \Gamma_a$ is uniform and constant, we obtain

$$\frac{\partial \bar{p}}{\partial t} = -2\Gamma_a \vec{T} \nabla \cdot (\bar{p} \vec{U}) - \frac{2\Gamma_a \bar{\rho}}{2e} \nabla \cdot (\bar{p} \vec{J}).$$  \hfill (69)

$$\frac{\partial \bar{\rho}}{\partial t} = -\frac{\Gamma_a \bar{T}}{e} \nabla \cdot \vec{J}.$$  \hfill (70)

We have the linearized equations

$$\vec{\bar{U}} = -\nabla \bar{\rho},$$  \hfill (71)

$$\frac{\bar{p}}{(\bar{\rho} \bar{e})^2} \frac{\partial \vec{J}}{\partial t} = -\frac{1}{2\bar{\rho} \bar{e}} \nabla \left(\Gamma_a \bar{p} \left(\bar{p} - \bar{\rho} \bar{\rho}\right)\right) - \frac{\gamma \eta_{\pm}}{2\bar{\rho} \bar{e}} \nabla \left(\Gamma_a \bar{p} \left(\bar{p} - \bar{\rho} \bar{\rho}\right)\right) + \vec{\bar{F}},$$  \hfill (72)

$$\vec{\bar{J}} + \frac{\partial}{\partial t} \vec{\bar{F}} = \vec{0}.$$  \hfill (73)

These equations yield

$$\frac{\mu \bar{h}}{(\bar{\rho} \bar{e})^2} \frac{\partial^2 \vec{J}}{\partial t^2} = -\frac{\Delta \mu \bar{p} \bar{F}}{\bar{\rho}} \nabla \cdot \left(\nabla \vec{U} \cdot \vec{J}\right)$$  \hfill (67)

$$+ \frac{\Gamma_a \bar{T}}{2\bar{\rho} \bar{e}} \left(1 + (\Delta \mu_{\pm}^2)\nabla \cdot \left(\Gamma_a \bar{F} \vec{J}\right) - \vec{J}, \right. \hfill (74)

$$\frac{\mu \bar{h}}{(\bar{\rho} \bar{e})^2} \frac{\partial^2 \vec{U}}{\partial t^2} = 2\frac{\Gamma_a \bar{T} \bar{p}}{m} \nabla \left(\Gamma_a \bar{J} \left(\nabla \cdot \vec{U}\right) \right) - \frac{\Delta \mu \Gamma_a \bar{T}}{\bar{e}} \nabla \left(\Gamma_a \bar{J} \left(\nabla \cdot \vec{J}\right)\right).$$  \hfill (75)

When $\vec{U}$ vanishes, Equation (74) yields

$$\frac{\mu \bar{h}}{(\bar{\rho} \bar{e})^2} \omega_p^2 \vec{J} = \frac{\Gamma_a \bar{T}}{2\bar{\rho} \bar{e}} \left(1 + (\Delta \mu_{\pm}^2)k \vec{J}\right).$$  \hfill (76)

Using the condition $\vec{J} \left| \vec{k} \right|$ in the longitudinal mode, we have the dispersion relation

$$\omega^2 = \frac{1 - 2\mu}{\mu} c_s^2 k^2 + \omega_p^2,$$  \hfill (77)

where $c_s = \sqrt{\frac{\mu}{\bar{\rho}}} = \sqrt{\frac{\mu}{\bar{p} \bar{e}}}$ is the sound speed and $\omega_p = \sqrt{\frac{\mu}{\bar{\rho} \bar{e}}}$ is the plasma frequency. When we take $c_s = c_s^0 = \sqrt{\bar{p} / \bar{e}}$, $k = i \sqrt{2}/\lambda_D$, and we have the “extended Debye length,”

$$\lambda_D = \sqrt{\frac{1 - 2\mu}{\mu} c_s^0} \omega_p^0 = 2 \frac{1 - 2\mu}{\mu} \frac{\lambda_D}{\omega_p^0} \frac{\bar{p}}{\bar{\rho} \bar{e}}.$$  \hfill (78)
which expresses the characteristic length of shielding of the electric field around an electric charge. Using the extended Debye length, the dispersion relation (77) is written by

$$\omega^2 = \omega_p^2 \left(1 + \frac{\mu}{2} \chi^2 k^2 \right).$$

(79)

With respect to the plasma oscillation mode, because $\vec{J} || \vec{k}$ and $\vec{U} || \vec{k}$ in the longitudinal modes, Equations (74) and (75) yield

$$-\omega^2 \frac{\mu \bar{e}}{n \bar{e}} \vec{J}_i = \frac{\Delta \mu \Gamma_s^2}{e} k^2 \vec{U}_i - \frac{\Gamma_s^2}{2ne^2} (1 + \Delta \mu^2) k^2 \vec{J}_i - \vec{J}_i,$$

$$-\omega^2 \bar{e} \vec{U}_i = -\frac{2\Gamma_s^2}{m} \bar{e} k^2 \vec{J}_i + \frac{\Delta \mu \Gamma_s^2}{e} k^2 \vec{J}_i,$$

(80)

where $\vec{J}_i \equiv (J \cdot k)/k$ and $\vec{U}_i \equiv (U \cdot k)/k$, $k = 0$. Then, we get the following dispersion relation:

$$\left[ \omega^2 - \frac{c_s^2}{2\mu} (1 + \Delta \mu^2) k^2 - \omega_p^2 \right] \left( \omega^2 - 2c_s^2 k^2 \right) = \frac{\Delta \mu^2}{\mu} (c_s k)^4.$$

(82)

In the case of an electron–ion (normal) plasma ($\Delta \mu \approx 1$, $\mu = m_e/m \ll 1$, $\omega \gg c_s /\sqrt{\mu}$), the dispersion relation becomes

$$\omega^2 = \frac{c_s^2}{2\mu} k^2 + \omega_p^2.$$

(83)

This expression shows the dispersion relation of the plasma oscillation for the plasma with finite pressure. In the case of $\omega^2 \ll \omega_p^2/\mu$, we have the dispersion relation of sound waves

$$\omega^2 = 2c_s^2 k^2.$$

(84)

In the case of a pair plasma ($\Delta \mu = 0$, $\mu = 1/4$), we have two modes

$$\omega^2 = \omega_p^2 + 2c_s^2 k^2,$$

(85)

and

$$\omega^2 = 2c_s^2 k^2.$$

(86)

The former is the dispersion relation of plasma oscillation, and the latter is that of the sound wave.

Next, we investigate the bulk compressional wave of the magnetized plasma. When $\vec{k} || \vec{B}$ and $\vec{U} || \vec{B}$, the dispersion relation is the same as that of the nonmagnetized plasma wave. Then, we investigate the case of $\vec{k} \perp \vec{B}$, $\vec{k} \perp \vec{B}$, $\vec{k} || \vec{U}$, and $\rho_e = 0$. Because $\rho_e = 0$, we have $\Delta p = 0$. In the case of $k \ll \omega_p$ and $\omega \ll \omega_c$, the left-hand side and the first term and the Hall term on the right-hand side of Ohm’s law (57) are negligible. Using the linearized Equations (56)–(59), we have the dispersion relation

$$\omega^2 = \nu_f^2 k^2,$$

(87)

where $\nu_f^2 = \frac{v_f^2 + \beta_f^2}{1 + \beta_f^2}$ is the 3-velocity of the fast wave. It is also noted that $\nu_f < 1$.

4.2. Transverse Wave Propagating along the Magnetic Field

We investigate transverse waves propagating through the ideal MHD plasma along the magnetic field lines,

$$\vec{B} || \vec{k}, \vec{E}, \vec{B}, \vec{U} || \vec{k}, \eta = 0.$$

(88)

We assume that any perturbation $\vec{A}$ is proportional to $\exp(ik \cdot r - i\omega t)$. The linearized equations become

$$-i\omega \vec{U} = \vec{J} \times \vec{B},$$

$$-i\omega \frac{\mu \bar{e}}{n \bar{e}} \vec{J} = (\bar{e} \vec{U} - \Delta \mu \vec{J}) \times \vec{B} + \bar{e} \vec{E},$$

$$ik \times \vec{E} = 0, ik \cdot \vec{B} = 0,$$

$$-i\omega \vec{B} = -ik \times \vec{E}, \vec{J} = i\omega \vec{E} = \vec{J} \times \vec{B}.$$

(90)

(91)

(92)

From these linearized equations, we have

$$-\omega^2 \frac{\mu \bar{e}}{n \bar{e}} \sqrt{k} \mathbf{J} = \frac{\bar{e}}{h} \left( \omega^2 - k^2 \right) \mathbf{J}^\ast \bar{B}_0^2$$

$$+ \Delta \mu \omega^2 \left( \omega^2 - k^2 \right) (\bar{B}_0 \mathbf{J}^\ast \bar{B}_0^2 - \bar{e} \omega^2 \mathbf{J}^\ast \bar{B}_0^2).$$

(93)

Then, we obtain the dispersion relation of the transverse modes,

$$\left( \omega^2 - k^2 \right) \left( \frac{\mu \bar{e}}{n \bar{e}} \omega^2 - \frac{\Delta \mu \bar{B}_0}{\omega_p} \omega - \frac{\bar{B}_0^2}{h} \right) - \omega^2 = 0,$$

(94)

that is,

$$\left( \omega^2 - k^2 \right) \left( \frac{1}{\omega_p^2} \omega^2 - \frac{\Delta \mu \omega_c}{\omega_p^2} \omega - \frac{\omega_c^2}{h} \right) - \omega^2 = 0,$$

(95)

(96)

where $\omega_c = \sqrt{\frac{eB/m}{\rho_e}}$ is the 4-Alfvén velocity and $\omega_c = \frac{eB_0 \rho_e}{\mu_0 m h}$. Note that $\omega_c$ corresponds to the cyclotron frequency, which we call the “extended cyclotron frequency.” If we set the pressure to be zero, $\omega_c$ reduces to the cyclotron frequency of the charged particle with mass $m$ and charge $e$ in the magnetic field $B$.

$\omega_c = eB/m$. In general, we have the relation between the plasma parameters,

$$\frac{\omega_c}{\omega_p} = \frac{\mu_0}{\sqrt{\mu}},$$

(97)

When we consider the limit $\omega \gg \omega_p, \omega_c$, the dispersion relation (96) yields

$$\omega = \pm \sqrt{k^2 + \omega_p^2 + \frac{\Omega_{FR}^2}{2}} = \omega_{\pm},$$

(98)

where

$$\Omega_{FR} = \omega_+ - \omega_- = \frac{\Delta \mu \omega_c}{\omega_p^2} \omega_p^2 - \left( \omega_\pm \omega_c \right)^2 \approx \frac{\Delta \mu \omega_c}{k^2} \omega_p^2.$$
presents the angular velocity of electromagnetic wave polarity of Faraday rotation.

5. Estimation of Nonideal MHD Terms of Generalized GRMHD Equations

Here we summarize a complete system of the generalized GRMHD equations (Equations (19)–(21) and (23)) derived from the general relativistic two-fluid equations as

\[
\nabla \cdot (\rho U^{\nu}) = 0, \quad (100)
\]

\[
\nabla \left[ \frac{\hbar U^{\nu} U^{\mu} + \frac{\mu B}{(ne)^2} J^{\mu} J^{\nu} + \Delta h}{2ne} (U^{\mu} J^{\nu} + J^{\mu} U^{\nu}) \right] = -\nabla \rho \cdot p + \lambda \cdot F_{\nu}, \quad (101)
\]

\[
U^{\nu} F_{\nu} = \eta [J^{\rho} - \rho' (1 + \Theta) U^{\rho}] + \frac{\Delta \mu}{ne} J^{\rho} F_{\rho} - \frac{1}{2ne} \nabla \left( \Delta \mu p - \Delta p \right) + \frac{\mu m}{e} \nabla q_{\mu}, \quad (102)
\]

\[
\eta \rho' \Theta = -\frac{\Delta \mu}{ne} J^{\rho} F_{\rho} + \frac{1}{2ne} U^{\sigma} \partial_\sigma (\Delta \mu p - \Delta p) - \frac{\mu m}{e} U^{\sigma} \nabla q_{\sigma}, \quad (103)
\]

where \(q^{\mu\nu} = \frac{n_i}{nm} (U^{\mu} J^{\nu} + J^{\mu} U^{\nu}) - \frac{\Delta \mu}{n_i} J^{\mu} J^{\nu} + \frac{\Delta h}{m} e^{\mu\nu} U^{\mu} U^{\nu} \) is a tensor of the electric current. It is noted that Equations (102) and (103) are not independent because the latter comes from the former.

We evaluate the significance of the nonideal MHD terms in the generalized GRMHD equations in plasma. We introduce the “primary (primitive)” parameters of the plasmas as follows. In the SI units, the primary plasma parameters are written as

\[
\omega_p^{\text{prm}} = \sqrt{\frac{ne^2}{\mu_0 c_0}}, \quad \omega_e^{\text{prm}} = \frac{eB}{\mu_0}, \quad \nu_A^{\text{prm}} = \sqrt{\frac{\Gamma_{\nu}}{\rho}}, \quad (104)
\]

where \(c_0\) and \(\mu_0\) are the permittivity and permeability of vacuum, respectively. Using the primary plasma parameters, we write the plasma parameters introduced in this paper as

\[
\omega_p = \omega_p^{\text{prm}} \sqrt{r_{\text{rel}}}, \quad \omega_e = \omega_e^{\text{prm}} f_{\text{rel}}, \quad \nu_A = \nu_A^{\text{prm}} \sqrt{f_{\text{rel}}}, \quad (105)
\]

where \(f_{\text{rel}} = \rho/h \leq 1\) is the relativistic factor of the internal energy of the plasma. The primary plasma parameters are calculated as

\[
\omega_p^{\text{prm}} = 5.64 \times 10^6 \left( \frac{n}{10^4 \text{ cm}^{-3}} \right)^{1/2} \left( \frac{\mu \text{m}}{m_e} \right)^{-1/2} \text{ s}^{-1}, \quad (106)
\]

\[
\omega_e^{\text{prm}} = 1.76 \times 10^7 \left( \frac{B}{1 \text{ G}} \right) \left( \frac{\mu \text{m}}{m_e} \right)^{-1} \text{ s}^{-1}, \quad (107)
\]

\[
\nu_A^{\text{prm}} = 2.18 \times 10^9 \left( \frac{B}{1 \text{ G}} \right) \left( \frac{n}{10^4 \text{ cm}^{-3}} \right)^{-1/2} \left( \frac{m}{m_1} \right)^{-1} \text{ cm s}^{-1}, \quad (108)
\]

\[
c_s^{\text{prm}} = 1.05 \times 10^6 \left( \frac{k_B T}{10^{10} \text{ K}} \right)^{1/2} \left( \frac{m}{m_e} \right)^{-1/2} \left( \frac{\Gamma_{\nu}}{4/3} \right)^{1/2} \text{ cm s}^{-1}. \quad (109)
\]

Using the plasma parameters, we estimate the significance of the nonideal MHD terms in the generalized GRMHD equations: the terms of resistive electromotive force, Hall effect, thermo-electromotive force, and current-carrier inertia in the generalized Ohm’s law (102) and the energy stress of electric 4-current density in Equation (101). The left-hand side of Equation (102) is written as \(U^\nu F_{\mu} = \gamma E_i + \epsilon_{\mu\nu} B^\nu B^\mu\). We compare the terms on the right-hand side of Equation (102) with the term \(\epsilon_{\mu\nu} B^\nu B^\mu\). We take the 4-Alfvén velocity \(\nu_A\) as the characteristic value of 4-velocity of the plasma, \(U = \sqrt{U_{\mu} U^\mu}\).

In this paper, we consider both the normal plasma (ion–electron plasma) and the pair plasma (positron–electron plasma). In the case of the normal plasma, we have \(\mu = m_e\) and \(n = \rho/m \approx n_{i_1} \approx n_i \approx n_e\). In the pair plasma case, we have \(\mu = m_i/2\) and \(n = \rho/m \approx (1/2) (n_e n_i^{1/2} + n_i n_e^{1/2}) \approx n_e\). Here we use the condition of \(m_i/m_e = 1386 \gg 1\), the charge neutrality \((n_e \approx n_i)\), and \(n_{i_1} \approx n_i \approx n_{i_1} \approx 1\). In both cases, we approximately have \(n_e \approx n_i\). Then, we have

\[
\eta = \frac{m_e \nu_{i_1}}{2n_e^2} = \frac{\mu \nu_A}{\nu_{i_1}} \left( \frac{\nu_{i_1}}{\nu_A} \right) \approx \frac{\mu \nu_A}{\nu_{i_1}} \frac{m_e}{2\nu_{i_1}}, \quad (110)
\]

where \(\nu_{i_1}\) is the collision frequency between the + and – particles and \(\nu_{i_1}\) is the collision rate between the electrons and ions (Miyamoto 1987, 1989),

\[
\nu_{i_1} = \frac{n_i e^4 \Lambda}{25.8 \pi^{1/2} \mu_0 \mu_e^{1/2} \nu_{i_1}^{3/2}} \left( \frac{T_e}{10^9 \text{ K}} \right)^{-3/2} \left( \frac{n_e}{10^4 \text{ cm}^{-3}} \right)[\text{ s}^{-1}], \quad (111)
\]

where \(\Lambda\) is the Coulomb logarithm (\(\Lambda \approx 20\)).

We evaluate the significance of the nonideal MHD terms of the generalized relativistic Ohm’s law as follows. Here we use \(J \sim B/\mu_0 L, U \sim u_A, \tau = L/U,\) and the relation

\[
\frac{\omega_e}{\omega_p} = \frac{\nu_A}{\sqrt{\tau}}. \quad (112)
\]

1. The electric resistivity term:

\[
\mathcal{I}_i = \frac{\eta J}{UB} = \frac{\eta}{\mu_0 U L} = \frac{1}{\tau} = \frac{\eta}{\mu_0 u_A L} = \frac{1}{2} \frac{\mu m_e \nu_A}{\omega_e} \frac{1}{\mu \omega_c \tau} = f_i \frac{1}{2 m_e \omega_e \nu_A}, \quad (113)
\]

where \(f_i = \frac{1}{2 m_e \omega_e \nu_A} \). Here, in the normal or pair plasma, we have \(\frac{\nu_{i_1}}{\nu_{i_1}} < 1\). Furthermore, in a thin plasma, like a plasma around a supermassive black hole, we have \(\nu_{i_1} / \nu_{i_1} \ll 1\). Then, we usually use \(f_i < 1\).
2. The Hall term:
\[
I_H = \frac{\Delta \mu_B}{\mu B} = \frac{\Delta \mu B}{\mu \nu e_A L} = \frac{\Delta \mu}{\mu \nu e_A L} = \frac{1}{\mu \omega_c}.
\]
(114)
where \( f_H = \Delta \mu \). In the normal plasma, we have \( \Delta \mu = 1-2m_e/2m_i < 1 \). In the pair plasma, we have \( \Delta \mu = 0 \). We also have \( f_H < 1 \).

3. Thermo-electromotive force term:
\[
I_{\text{th}} = \frac{1}{\mu B} \frac{\beta_B}{\beta_p} \frac{2 \mu_0}{\nu e_A B L} = \frac{1}{\mu B} \frac{1}{f_h} = \frac{1}{\mu \omega_c}.
\]
(115)
where \( \beta_B = p/(B^2/2\mu_0) \) is the plasma beta and \( f_h = \beta_h/2 \). In the magnetically dominated plasma, we have \( f_h \lesssim 2 \). Then, we also have \( f_h \lesssim 1 \).

4. Current-carrier inertia term:
\[
I_{\text{cci}} = \frac{\mu h^4}{\nu e_A (\mu_0 L)^2} = \frac{\mu h^4}{h (\mu_0 L)^2} = \left( \frac{f_{\text{cci}}}{\mu \omega_c} \right)^2.
\]
(116)
where \( f_{\text{cci}} = h^4/h = \sqrt{1 - \Delta \mu \Delta h/h} \). When the plasma temperature is not relativistic, we have \( f_{\text{cci}} \sim 1 \). However, when the plasma temperature is relativistic, \( f_{\text{cci}} \) becomes \( \sim 1/\sqrt{\mu} \), maximally.

We also evaluate the significance of the nonideal MHD term of the momentum Equation (101). The nonideal MHD term of Equation (101) is the second and third terms in the brackets on the left-hand side of Equation (101), which is the energy-stress tensor due to the current-carrier inertia.

5. Energy-stress tensor due to the current-carrier inertia on the momentum Equation (101):
\[
T_{\text{cci}} = \frac{\mu h^4}{\nu e_A (\mu_0 L)^2} = \frac{\mu h^4}{h (\mu_0 L)^2} = \frac{1}{\mu \omega_c}.
\]
(117)

When we use \( U = L/\tau \sim u_A \) and Equation (122), we have
\[
T_{\text{cci}} = \left( \frac{f_{\text{cci}}}{\mu \omega_c} \right)^2 = T_{\text{Ohm}}.
\]
(118)
We found that the nonideal MHD conditions of the current-carrier inertia term of the generalized Ohm’s law and the generalized momentum equation are identical. Then, we use the characteristic scale of the nonideal MHD effect due to current-carrier inertia of the generalized Ohm’s law.

According to the above conclusion, we conclude that the nonideal MHD effects are the resistive electromagnetic force, Hall effect, thermo-electromotive force, and current-carrier inertia. With respect to the three former effects, putting aside the details of the factors \( f_r, f_{\text{Ohm}}, f_{\text{Ohm}} \leq 1 \), we have the characteristic timescale
\[
\tau_{\text{Ohm}} \equiv \frac{1}{\mu \omega_c}.
\]
With respect to the current-carrier inertia terms, putting aside the detail of the factor \( f_{\text{cci}} \), we have the characteristic timescale
\[
\tau_{\text{cci}} \equiv \frac{1}{\sqrt{\mu} \omega_c}.
\]
(120)
It is noted that \( \tau_{\text{Ohm}} \) gives a more severe condition of the nonideal MHD effects than \( \tau_{\text{Ohm}} \) because of \( \tau_{\text{Ohm}} \leq \tau_{\text{Ohm}} \).

The characteristic length scales of the nonideal MHD effects are
\[
L_{\text{Ohm}} = u_A \tau_{\text{Ohm}} = \frac{u_A}{\mu \omega_c} = \frac{c}{\sqrt{\mu} \omega_p} = \frac{1}{\sqrt{\mu}} l_s,
\]
(121)
\[
L_{\text{cci}} = u_A \tau_{\text{cci}} = \frac{u_A}{\mu \omega_c} = \frac{c}{\omega_p} = l_s,
\]
(122)
where \( l_s = c/\omega_p \) is the plasma skin depth. Here we also have \( L_{\text{Ohm}} \leq \sqrt{\mu} L_{\text{Ohm}} < L_{\text{Ohm}} \), and \( L_{\text{Ohm}} \) gives a more severe condition of the nonideal MHD effects than \( L_{\text{Ohm}} \). Then, to confirm that nonideal MHD terms are all negligible, we just investigate whether both \( \tau_{\text{Ohm}} \) and \( L_{\text{Ohm}} \) are much smaller than the timescales and length scales of the plasma phenomena \( \tau \) and \( L \), \( \tau \gg \tau_{\text{Ohm}} \), \( L \gg L_{\text{Ohm}} \) because the factors \( f_r, f_{\text{Ohm}}, f_{\text{Ohm}} \) are usually equal to or less than unity. When \( \tau_{\text{Ohm}} \gg \tau \) or \( L_{\text{Ohm}} \gg L \), it is possible that \( \tau_{\text{Ohm}} \ll \tau \) or \( L_{\text{Ohm}} \ll L \) for normal plasma because \( \sqrt{\mu} = 1/\sqrt{1836} = 1/42.8 \) for normal plasma. In such a case, we neglect the current-carrier inertia terms of the generalized Ohm’s law and momentum equations, while the resistive term, Hall effect, and thermo-electromotive force are significant. Otherwise, that is, \( \tau \lesssim \tau_{\text{Ohm}} \) and \( L \lesssim L_{\text{Ohm}} \), all of the nonideal MHD effects are significant if the factors are nearly equal to unity \( f_r, f_{\text{Ohm}}, f_{\text{Ohm}} \sim 1 \).

It is noted that the finite resistivity may cause the magnetic reconnection and drastic phenomena even if the term is much smaller than the term \( U \times B \). Then, we have to take into account the electric resistive term even if \( S_A \) is much larger than unity. Furthermore, it is noted that current-carrier inertia also may cause the magnetic reconnection (Hirota et al. 2013, 2015).

For estimation of the variables of plasmas around these black holes, we have to give the black hole mass \( M_{\text{BH}} \), plasma density \( \rho \), and magnetic field \( B \) of the plasma. We had no direct observation that determines the variables around any black holes before the observation of the EHT Collaboration (EHT Collaboration et al. 2019a, 2019b, 2021a). Now, we employ the data set of \( M_{\text{BH}} \), accretion rate \( M \), \( \rho \), temperature \( T \), and \( B \) in the accretion disks of M87” from the EHT Collaboration results.

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5 The more precise characteristic timescales of resistive, Hall, thermo-electromotive force, and current-carrier inertia effects are given by \( \tau_r = f_r \tau_{\text{Ohm}} \), \( \tau_H = f_H \tau_{\text{Ohm}} \), \( \tau_{\text{Ohm}} = \tau_{\text{Ohm}} \), respectively.

6 The more precise characteristic length scales of resistive, Hall, thermo-electromotive force, and current-carrier inertia effects are given by \( L_r = f_r L_{\text{Ohm}} \), \( L_H = f_H L_{\text{Ohm}} \), \( L_{\text{Ohm}} = f_{\text{Ohm}} L_{\text{Ohm}} \), and \( L_{\text{Ohm}} = f_{\text{Ohm}} L_{\text{Ohm}} \), respectively.
Using the EHT observation data, we check the characteristic scales of nonideal MHD effects $\tau_{c,\text{Ohm}} = 1/(\mu_\text{Ohm})$ and $L_{\text{c}} = l_s/\sqrt{\mu}$ on the plasma in M87*. Here we assume that the plasma observed by EHT is normal plasma ($\Delta \mu = 0$ and $\mu = 1/1836$). EHT Collaboration et al. (2019b) reported that the observation of M87* is explained by a simple, spherical, one-zone model for the source as

$$n_e = 2.9 \times 10^{14} \left( \frac{r}{5r_g} \right)^{-1.3} \beta_p^{0.62} \left( \frac{T_i}{3T_e} \right)^{-0.47} \text{[cm}^{-3}] , \quad (123)$$

$$B = 4.9 \left( \frac{r}{5r_g} \right)^{-0.63} \beta_p^{-0.19} \left( \frac{T_i}{3T_e} \right)^{0.14} [\text{G}] , \quad (124)$$

$$T_i = 0.202 \times 10^{12} \left( \frac{r}{5r_g} \right) [\text{K}] . \quad (125)$$

First, we evaluate the plasma parameters at $r = 5r_g$ in the model, for example. At $r = 5r_g$, we have $f = \mu = 1$, $\omega_\text{c} \approx \omega_{\text{prm}} = 8.6 \times 10^5 \text{s}^{-1}$, $u_\text{a} \approx v_\text{prm} = 2.8 \times 10^6 \text{cm} \text{s}^{-1}$, $\nu_\text{p} \approx \nu_{\text{prm}} = 9.6 \times 10^5 \text{s}^{-1}$, $c_\text{s} \approx c_{\text{prm}} = 4.9 \times 10^6 \text{cm} \text{s}^{-1}$, and $\nu_\text{rel} = 1.2 \times 10^{-12} \text{s}^{-1}$. Then, we found characteristic scales of the nonideal MHD phenomena as $\tau_{c,\text{Ohm}} = 1/(\mu_\text{Ohm}) = 5.0 \times 10^{-3} \text{s}$ and $L_{\text{c,Ohm}} = 1.3 \times 10^3 \text{m}$. Considering the spatial dependence on characteristic scales, $\tau_{c,\text{Ohm}} = 1/(\mu_\text{Ohm}) \propto B_{\text{rel}}^{-1} \propto v_\text{rel}^{0.14} (T_i/T_e)^{0.14} L_{\text{c,Ohm}} = c/\sqrt{\mu} \omega_\text{p} \propto n^{-1/3} r_{\text{rel}}^{1/2} \propto v_\text{rel}^{0.65} \beta_p^{-0.31} (T_i/T_e)^{0.24} r_{\text{rel}}^{-1/2}$, we have

$$\tau_{c,\text{Ohm}} = 5.0 \times 10^{-3} \left( \frac{r}{5r_g} \right)^{0.06} \beta_p^{0.11} \left( \frac{T_i}{T_e} \right)^{0.14} [\text{s}] , \quad (126)$$

$$L_{\text{c,Ohm}} = c/\sqrt{\mu} \omega_\text{p} = 1.3 \times 10^3 \left( \frac{r}{5r_g} \right)^{0.65} \beta_p^{0.24} \left( \frac{T_i}{T_e} \right)^{0.24} r_{\text{rel}}^{-1/2} [\text{m}] . \quad (127)$$

These scales become larger as the outer region becomes farther from the black hole.

Incidentally, we note that the extended Debye length of the plasma is $\lambda_\text{D} = 1/c_{\text{prm}}^2 \sim (1 - 2\mu)^{1/2} \beta_p^{1/2} \nu_\text{a}^{1/2} \mu_{\text{prm}} \sim 8.4 \text{ cm}$, where we calculate $\beta_p \sim 0.84$ at $r = 5r_g$. The particle number in the Debye sphere of a charged particle, $N_\text{D} = 4\pi^2 \lambda_\text{D}^3 n = 7 \times 10^9$, is much greater than unity, and the plasma has a collective property as a plasma.

As an example of the minimum of the characteristic scales of phenomena of plasmas ($L$ and $\tau$) around the black holes, we consider the current sheet that causes the magnetic reconnection in the accretion disk around the black hole. The minimum scales of the magnetic reconnection are roughly estimated by the thickness of the current sheet $L_{\text{CS}}$, which is calculated by the minimum scale of the magnetorotational instability (MRI), $L_{\text{CS}} \sim \lambda_{\text{MRI}} = 4 \left( \frac{c}{\sqrt{\mu} \omega_\text{p}} \right) / n^2$ (see Chapter 8 in Shibata et al. 1999 or Chapter 4 in Tajima & Shibata 2002). Here $\Omega = \sqrt{GM_{\text{BH}}/r^3} = c/\sqrt{2} r_s(r/r_s)^{3/2}$ is the angular velocity of the disk and $v_\text{a} = \sqrt{B^2/\mu_0} r_s$ is the Alfvén velocity; $\Omega \approx \sqrt{GM_{\text{BH}}/r^3} > c/v_\text{a}$. The Alfvén transit time of the current sheet is given by $\tau_s = L_{\text{CS}}/v_\text{a} = 8/\sqrt{3} r_s(r/r_s)^{3/2}$, where $r_s = r_\text{g}/c$ is the Schwarzschild transit time. From the M87* observation by EHT, we have $M_{\text{BH}} = 6.5 \times 10^9 M_\odot$, $r_s = 1.9 \times 10^{15} \text{ cm}$, and $\tau_\text{s} = 6.3 \times 10^5 \text{ s}$. The thickness of the current sheet is calculated as

$$L_{\text{CS}} = \frac{8}{\sqrt{3}} \left( \frac{r}{r_\text{g}} \right)^{3/2} \frac{v_\text{a}}{c} r_s = 7.5 \times 10^{15} \left( \frac{r}{5r_g} \right)^{1.52} [\text{cm}] . \quad (128)$$

The values of the spatial and temporal scales, $L_{\text{CS}}$ and $\tau_s$, are much larger than the critical variables $\tau_{c,\text{Ohm}} = \tau_{c,\text{ci}}/\sqrt{\mu}$ and $L_{\text{c,Ohm}} = L_{\text{c,ci}}/\sqrt{\mu}$, respectively. This suggests the validity of the resistive GRMHD equations in the phenomena in the reconnection regions around the black holes.

### 6. Summary

In this paper, we determined the $\Theta$ term of the generalized relativistic Ohm’s law, which had not been determined in our previous works (Koide 2008, 2009, 2010). We have now obtained an explicitly closed system of generalized GRMHD equations and evaluated the terms of the nonideal MHD effects in these equations (Equations (101) and (102)). There are two main characteristic scales of the nonideal MHD effects with respect to time and length. These scales come from the generalized Ohm’s law (102) and are given by $\tau_{c,\text{Ohm}} = 1/(\mu_\text{Ohm})$ and $L_{\text{c,Ohm}} = c/\sqrt{\mu} \omega_\text{p}$. In more detail, the additional characteristic timescales and length scales come from current-carryer inertia both in the generalized Ohm’s law and in the momentum equation and are given by $\tau_{c,\text{ci}} = \sqrt{\mu} \tau_{c,\text{Ohm}}$ and $L_{\text{c,ci}} = \sqrt{\mu} L_{\text{c,Ohm}}$, which are smaller by a factor $\sqrt{\mu} \leq 1/2$ than $\tau_{c,\text{Ohm}}$ and $L_{\text{c,Ohm}}$, respectively. We evaluated the additional terms of the generalized relativistic Ohm’s law with the plasma parameters ($T_i$, $n_e$, and $B$) obtained by EHT observations of M87* and found that the additional terms of resistive electromotive force, the Hall effect, thermo-electromotive force, and the current-carryer (electron) inertia effect are negligible compared to the $U \times B$ term of the generalized relativistic Ohm’s law for global-scale phenomena around the black hole, whose characteristic length scale is given by $L_{\text{CS}} \sim r_\text{s}$.

While the resistive term is negligible in the global phenomena around the black hole, the magnetic reconnection has been suggested to occur frequently in black hole magnetospheres by a number of ideal GRMHD simulations (e.g., Koide et al. 2000, 2006; McKinney 2006). However, it should be emphasized that magnetic reconnection in the ideal GRMHD simulations is caused by numerical resistivity; this resistivity often results in a fatal error for the numerical results and should be avoided. To perform GRMHD simulations of magnetic reconnection around a black hole without numerical resistivity, we must develop a highly accurate resistive GRMHD code. Recently, Inda-Koide et al. (2019) performed resistive GRMHD simulations of the magnetic reconnection around a black hole with a simple magnetic configuration and relativity small Reynolds number ($S_M \sim 10^5$). This work was the first resistive GRMHD simulation of magnetic reconnection around a black hole; however, the Reynolds number was not sufficiently large and the magnetic configuration was too simple to apply the results to astrophysical objects. To perform simulations with a suitably high magnetic Reynolds number and a magnetic configuration appropriate for the astrophysical situation, we require the advanced numerical technique of an implicit method with high accuracy (Bucciantini & Del Zanna 2013; Tomei et al. 2020).
On the other hand, explosive magnetic reconnection in a collisionless plasma has been proposed actually using the generalized Ohm’s law (Hirotá et al. 2013, 2015). The current-carrier inertia (electron inertia) term plays an important role in the explosive magnetic reconnection model. Comisso & Asenjo (2014) found similar magnetic reconnection with the generalized relativistic MHD equations given by Koide (2009). The normalized reconnection rate was given by

$$\frac{v_{\text{in}}}{c} \sim \sqrt{\frac{1}{S} + \frac{1}{4f L}}.$$  

(129)

where $v_{\text{in}}$ is the inflow velocity of plasma toward the reconnection region. The first term in the square root on the right-hand side represents the magnetic reconnection rate of the Sweet–Parker reconnection model, in which electric resistivity causes the reconnection. The second term in the square root represents the rate of the reconnection due to current-carrier (electron) inertia. The ratio of the first and second terms for the relativistic magnetic reconnection ($u_a \sim c$) is calculated as

$$\frac{1}{4f L} = \frac{\mu_0 u_a l_s}{4 \pi \sigma} = \frac{\mu_0 u_a c}{4 \pi \sigma} \frac{2 \omega_p}{m_e} \frac{\mu m}{\omega_p} m_e c \frac{\omega_p}{c} \frac{\mu m}{\omega_p} m_e c \approx 1.$$  

(130)

The magnetic reconnection due to the effect of current-carrier inertia would be significant compared to resistive magnetic reconnection. To confirm this explosive reconnection, a numerical simulation of the generalized GRMHD is required.

In this paper, we used a simple 1D model (Equations (123)–(125)) based on the EHT observation of M87* (EHT Collaboration et al. 2019b) to evaluate the significance of the nonideal MHD effect in the surrounding plasma. Using this simple model, we concluded that the ideal GRMHD approximation works well for the global phenomena of plasma around M87*. However, the plasma around M87* is actually complex, since the region is composed of the torus, the accretion disk, the outflow (wind), and the jet. The 1D model is too simplistic to grasp the detailed plasma behavior around the black hole. We must improve the significant evaluation of the nonideal MHD effect based on a forthcoming, more actual model taken from EHT observations of not only M87* but also Sgr A*. It is worth continuing to check the significance of the nonideal MHD effect, because it would change the plasma dynamics drastically from the results of ideal GRMHD simulations.

Numerical simulation of the generalized GRMHD is necessary to confirm and reveal the specific phenomena caused by nonideal MHD effects. The EOSs (74) and (75) with respect to $h$ and $\Delta h$ in Koide (2009) provide closure to the generalized GRMHD Equations (100)–(103). Such numerical calculation is possible in principle, although it becomes drastically difficult compared to the ideal GRMHD simulations. This is because we have to treat the displacement current $\partial E/\partial t$ in Ampère’s law and the inertia of the current density $\sigma (\mu m/(ne^2)) \delta ([h^2/(mn)]) J/\partial t$ in the generalized Ohm’s law explicitly. In the ideal GRMHD calculations, the former is implicitly accounted for and the latter can be neglected entirely. Furthermore, we have to consider the zeroth component of Ohm’s law to calculate the enthalpy density difference, $\Delta h$, of relativistically hot plasmas around the black hole. Thus, appropriate simplifications of the generalized GRMHD equations, especially of Ohm’s law, are required for adequate numerical study. For this simplification, the characteristic scales of the nonideal MHD phenomena will provide a basic guide. The adjusted closed system of the generalized GRMHD equations will play a significant role in forthcoming numerical simulations of magnetized plasmas around the black hole in the new era with the EHT observations.

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**Appendix**

**Derivation of a 3+1 Formalism of Divergence of Symmetric Tensor**

We derive a 3+1 formalism of divergence of symmetric tensor, $\nabla_{\nu}T^{\mu\nu} = F^{\mu}$. The 4-acceleration of the normal frame is given by $a_{\mu} = N_{\mu}N^{\nu}$, and we have

$$a_{\mu} = (-\alpha N^k (\ln \alpha)_k, (\ln \alpha)_k), a^\mu = (0, \gamma^k (\ln \alpha)_k).$$  

(A1)

where the subscript “*” denotes the partial derivative $\partial/\partial x^k$. Using the projection tensor to the time constant hypersurface $\mathcal{D}^{\alpha\beta} = \delta^{\alpha\beta} + N^\alpha N^\beta$, we define the extrinsic curvature tensor by $K_{ij} \equiv -T^\mu_{ij} N_{\mu} N_{\nu} = -N_{ij}$, where the subscript “;” denotes the covariant derivative $\nabla_i$. In the stationary spacetime ($\gamma_{ij} = 0$), we have

$$K_{ij} = -\frac{1}{2\alpha} [\gamma_{ij} + \nabla_i (\ln N) + \nabla_j (\ln N)]$$  

$$= -\frac{1}{2\alpha} [\gamma_{ij} (\ln N)^k_j + \gamma_{jk} (\ln N)^k_j + \gamma_{ij} (\ln N)^k].$$  

(A2)

Using the normal vector of the hypersurface of time constant $N^\mu$ and projection tensor toward the hypersurface $\mathcal{D}^{\mu\nu}$, we separate the 4-vector $F^\mu$ into temporal and spatial components:

$$\tilde{F}^i \equiv -F^0 N_\mu = -T^\mu_{ij} N_{\mu},$$  

(A3)

$$\tilde{F}_i \equiv F^\mu N_{\mu} = T^\mu_{i\nu} P_{\mu\nu}.$$  

(A4)

Note that $\tilde{F}^i$ and $\tilde{F}_i$ reproduce $F^\mu$ as

$$F^\mu = \tilde{F} \mathcal{N}^\mu + \tilde{F}^i.$$  

(A5)

Equation (A3) yields a scalar-like equation such as the energy conservation law, and Equation (A4) yields a 3-vector conservation equation such as the momentum conservation law. Similarly, when we separate $T_{\mu\nu}$ into

$$\tilde{u} = T_{\mu\nu} N_\mu N_\nu,$$  

(A6)

$$\tilde{S}_i = -T_{\mu\nu} P_{i\nu} N_\mu,$$  

(A7)

$$\tilde{T}_{\mu\nu} = T_{\mu\nu} P_{\rho\sigma} P_{\rho\sigma}.$$  

(A8)

Here we found

$$T_{\mu\nu} = \tilde{u} N_{\mu} N_{\nu} + S^\mu N_\nu + N^\mu \tilde{S} + \tilde{T}^\mu \nu,$$  

(A9)

and $\tilde{u} = \tilde{T}^0 S_i = \tilde{T}_i = \tilde{T}^i = \tilde{F}^0 = \tilde{F}_i$. Equation (A3) is written by

$$\tilde{F}^i = -(T_{\mu\nu} N_\mu)_i + T_{\mu\nu} N_{\mu\nu}$$  

$$= -\frac{1}{\sqrt{-g}} (\sqrt{-g} T_{\mu\nu} N_\nu)_\mu + T_{\mu\nu} N_{\mu\nu}.$$  

(A10)
Using \( a_{\mu} = N_{\mu \sigma} \nu^\sigma \), \( K_{\nu} = -N_{\nu \rho} \sqrt{-g} = \alpha \sqrt{\gamma} \), and Equation (A9), we have
\[
\mathcal{F}^{\dagger} = \frac{1}{\alpha \sqrt{\gamma}} \left( \sqrt{\gamma} \mathcal{F} \right)^{\dagger} + \frac{1}{\alpha \sqrt{\gamma}} \left[ \alpha \sqrt{\gamma} (S^k + N^k \tilde{u}) \right]_{\nu} + (\ln \alpha)_{\nu} S^k - K_{\nu} \tilde{J}^k. \tag{A11}
\]

Multiplying by \( \alpha \), we obtain
\[
\alpha \mathcal{F}^{\dagger} = \frac{1}{\alpha \sqrt{\gamma}} \frac{\partial}{\partial t} \left( \sqrt{\gamma} \tilde{u} \right) + \frac{1}{\alpha \sqrt{\gamma}} \frac{\partial}{\partial x^k} \left[ \alpha \sqrt{\gamma} (S^k + N^k \tilde{u}) \right] + \tilde{u} N_{\nu}^{\kappa} \frac{\partial^\nu}{\partial x^\nu} \left[ \frac{1}{2} \alpha N^k \frac{\partial N_{\nu}^{\kappa}}{\partial x^\nu} \right] \tilde{J}^k. \tag{A12}
\]

When we use \( \alpha K_{\nu} \tilde{J}^k = \left[ \gamma_{jk} \frac{\partial}{\partial x^k} (\alpha N^k) + \frac{1}{2} \alpha N^k \frac{\partial \gamma_{jk}}{\partial x^k} \right] \tilde{J}^k \), we get
\[
\alpha \mathcal{F}^{\dagger} = \frac{\partial}{\partial t} \tilde{u} + \frac{1}{\alpha \sqrt{\gamma}} \frac{\partial}{\partial x^k} \left[ \alpha \sqrt{\gamma} (S^k + N^k \tilde{u}) \right] + \frac{\partial}{\partial x^k} \left[ \alpha \sqrt{\gamma} (S^k + N^k \tilde{u}) \right] \tilde{J}^k. \tag{A13}
\]

With respect to Equation (A4), we have
\[
\mathcal{F}^{\dagger} = T_{\nu \sigma}^\nu \left( \sqrt{\gamma} \right) \mathcal{F}^{\dagger} = \frac{1}{\gamma \sqrt{\gamma}} \left( \sqrt{\gamma} \mathcal{F} \right)^{\dagger} - \frac{1}{2} g_{\alpha \beta} T^{\alpha \beta}. \tag{A14}
\]

Using \( a_{\mu} = N_{\mu \sigma} \nu^\sigma \), \( \sqrt{-g} = \alpha \sqrt{\gamma} \), and Equation (A9), we get
\[
\mathcal{F}^{\dagger} = \frac{1}{\alpha \sqrt{\gamma}} \left( \sqrt{\gamma} \tilde{S}_k \right)_{\nu} + \frac{1}{\alpha \sqrt{\gamma}} \left[ \alpha \sqrt{\gamma} (\tilde{S}_k + N^k \tilde{u}) \right]_{\nu} + \tilde{u} \left( \ln \alpha \right)_{\nu} \tilde{S}_k - \frac{1}{2} \gamma_{\alpha \beta} \tilde{J}^\alpha \tilde{J}^\beta. \tag{A15}
\]

Multiplying by \( \alpha \), we obtain
\[
\alpha \mathcal{F}^{\dagger} = \frac{\partial}{\partial t} \tilde{S}_k + \frac{1}{\alpha \sqrt{\gamma}} \frac{\partial}{\partial x^k} \left[ \alpha \sqrt{\gamma} (\tilde{S}_k + N^k \tilde{u}) \right] + \frac{\partial}{\partial x^k} \tilde{u} \left( \ln \alpha \right)_{\nu} \tilde{S}_k - \frac{1}{2} \gamma_{\alpha \beta} \tilde{J}^\alpha \tilde{J}^\beta. \tag{A16}
\]

Furthermore, using a 3-covariant derivative, which is given by \( \nabla_k A^k, \) we have
\[
\alpha \mathcal{F}^{\dagger} = \frac{\partial}{\partial t} \tilde{S}_k + \left( \nabla_k (\alpha (\tilde{S}_k + N^k \tilde{u})) \right) + \tilde{u} \frac{\partial}{\partial x^k} \left( \alpha (\tilde{S}_k + N^k \tilde{u}) \right) - \frac{1}{2} \gamma_{\alpha \beta} \tilde{J}^\alpha \tilde{J}^\beta. \tag{A17}
\]