On-shell and off-shell improvement for Ginsparg-Wilson fermions

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We discuss the improvement of bilinear fermionic operators for Ginsparg-Wilson fermions. We present explicit formulae for improved Green’s functions, which apply both on-shell and off-shell.

1. Introduction

A fermion action that fulfills the Ginsparg-Wilson condition realizes chiral symmetry in an unconventional way that allows it to avoid the Nielsen-Ninomiya theorem [1,2]. This chiral symmetry automatically ensures that the hadron masses of the theory are free of $O(a)$ discretization errors.

However, if we are interested in going beyond this and calculating matrix elements (for example structure functions, decay constants etc.) we also need to know how to improve fermion operators. When we measure hadronic matrix elements, it is enough if the operators are improved for on-shell quantities. To do non-perturbative renormalisation we may want to measure off-shell Green’s functions with a virtuality large enough that we can reasonably compare with continuum perturbation theory, so improvement of off-shell Green’s functions is useful too.

2. Ginsparg-Wilson fermions

The continuum Dirac operator, $\hat{D}$, anticommutes with $\gamma_5$, but we know from the Nielsen-Ninomiya theorem that a lattice Dirac operator cannot do this without having problems such as doubling or non-locality. Ginsparg-Wilson fermions are defined by the anti-commutation relation

$$D_{GW} \gamma_5 + \gamma_5 D_{GW} = a D_{GW} \gamma_5 D_{GW}. \quad (1)$$

The standard way of writing down massive Ginsparg-Wilson fermions is to use the fermion matrix

$$M \equiv \begin{pmatrix} 1 & a\gamma_5 m_0 \\ a\gamma_5 m_0 & 1 \end{pmatrix} D_{GW} + m_0. \quad (2)$$

$M$ is local and always invertible.

Following [3], we can define the associated matrix

$$K_{GW} \equiv \left(1 - \frac{a}{2}m_0\right)^{-1} D_{GW}. \quad (3)$$

The Ginsparg-Wilson condition implies that

$$K_{GW} \gamma_5 + \gamma_5 K_{GW} = 0. \quad (4)$$

Thus $K_{GW}$ has the same chiral properties as the continuum Dirac operator. Also, the eigenvalues of $K_{GW}$ are imaginary, like the eigenvalues of the continuum $\hat{D}$. The fermion propagator we really want to find is the propagator calculated with $K_{GW}$.

3. Fermion propagator

The unimproved fermion propagator is obtained simply by inverting $M$.

$$S \equiv \langle \psi \bar{\psi} \rangle = \frac{1}{a^4} \langle M^{-1} \rangle. \quad (5)$$
However this propagator has unwanted contact terms which are of $O(a)$.

We would prefer the improved ($\ast$) propagator defined from the matrix $K_{GW}$:

$$S_\ast = \frac{1}{a^4} \left( \frac{1}{K_{GW} + m_0} \right).$$

(6)

Since $K_{GW}$ has the same chiral properties as the continuum Dirac operator, the improved propagator $S_\ast$ is automatically free from all $O(a)$ discretisation errors. However, we need a way to calculate $S_\ast$ directly from $M$, without having to construct the problematic $K_{GW}$. Substitute eq. (3), the definition of $K_{GW}$, into the above equation, and a little algebra gives:

$$S_\ast(x,y) = \frac{1}{1 + am_0b_\psi} \left( S(x,y) - \frac{a}{2} \lambda_\psi \delta(x-y) \right),$$

(7)

with improvement coefficients

$$b_\psi = -\frac{1}{2} \quad \text{and} \quad \lambda_\psi = 1. \quad (8)$$

Note that we never have to explicitly construct $K_{GW}$, the only matrix we invert is $M$, so the formula is applicable even when $D_{GW}$ has eigenvalues at $2/a$. The chiral violation in $S$ is concentrated in a $\delta$ function contact term at the origin, so it only gives a problem when we are interested in off-shell quantities.

To illustrate the difference between the unimproved and improved propagators we will show how improvement works in the case of the free fermion theory. Starting from the massless Wilson fermion matrix $D_W$, Neuberger [6] introduces the matrix $A$, defined by

$$A \equiv 1 - aD_W. \quad (9)$$

It can then be shown that the operator

$$D_N \equiv \frac{1}{a} \left( 1 - \frac{A}{\sqrt{A^\dagger A}} \right) \quad (10)$$

satisfies the Ginsparg-Wilson condition.

Although $D_N$ satisfies the Ginsparg-Wilson condition exactly, it is not yet off-shell improved. If we expand for small momenta we find

$$D_N(p) = i\not{\psi} + \frac{1}{2} a p^2 + O(a^2 p^3), \quad (11)$$

which is no improvement over the Wilson propagator. However, if we calculate the improved propagator of eq. (7) we find that it has errors of $O(a^2)$. This is illustrated in Fig. 1, where we compare the trace of the Wilson action propagator, the unimproved Ginsparg-Wilson propagator and the improved propagator with the continuum result.

4. Improving bilinear operators

In this section we present the main conclusions from [7]. Let us consider improving the Green’s function for a flavour non-singlet operator of the form $\bar{\psi} \Gamma \tau D_\alpha \cdots D_\omega \psi$, where $\Gamma$ is a matrix in the Clifford algebra, $\tau$ a flavour matrix and $D_\alpha$ a covariant derivative.

As in the case of the propagator, we know that the Green’s function $G^{O}_\ast$

$$G^{O}_\ast = \frac{1}{a^4} \left( \frac{1}{K_{GW} + m_0} \right) \frac{1}{K_{GW} + m_0} \quad (12)$$

is free of $O(a)$ effects, but we want to express it in terms of the well-behaved matrix $M$. One
Figure 2. Green’s function improvement for the local vector current $\bar{\psi} \gamma_\mu \tau \psi$.

possibility is

$$G^O = \frac{1}{a^4} \frac{1}{1 + a m_0 b \psi} \left\langle M^{-1} \tilde{O} M^{-1} \right\rangle$$

where

$$\tilde{O} = (1 + a m_0 b \psi)(1 - \frac{a}{2} D_{GW}) O (1 - \frac{a}{2} D_{GW})$$

This is not the most general expression for $G^O$. We can use the identity

$$D_{GW} M^{-1} = \frac{1}{1 - a m_0 / 2} (\pm 0M^{-1} + \delta_{xy})$$

to get the alternative expression

$$G^O = \frac{1}{1 + a m_0 b \psi} \left[ G_o - \frac{a}{2} \lambda O C^O + \frac{a^2 \eta O}{4} \frac{1}{a^4} \left\langle O \right\rangle \right]$$

where

$$G_o \equiv \frac{1}{a^4} \left\langle M^{-1} O_s M^{-1} \right\rangle$$

$$O_s \equiv O + a c_0 m_0 O - \frac{a}{2} c_1 (D_{GW} O + O D_{GW})$$

$$+ \frac{a^2}{4} c_2 D_{GW} O D_{GW}$$

$$C^O \equiv \frac{1}{a^4} \left\langle OM^{-1} \right\rangle + \frac{1}{a^4} \left\langle M^{-1} O \right\rangle.$$  \hspace{1cm} (17)

$C^O$ involves just a single propagator, so it is a contact term which only contributes when the operator overlaps with the source or sink, and $\left\langle O / a^4 \right\rangle$, with no propagators, is a “double contact term” occurring when source, sink and operator all overlap.

Two improvement coefficients are free, for example $c_1$ and $c_2$, the others are then determined:

$$c_0 = \frac{1/2 - c_1}{1 - a m_0 / 2} - \frac{a m_0 c_2}{4(1 - a m_0 / 2)^2}$$

$$\lambda O = \frac{1 - c_1}{1 - a m_0 / 2} - \frac{a m_0 c_2}{2(1 - a m_0 / 2)^2}$$

$$\eta O = \frac{1 - c_2 - a m_0 / 2}{(1 - a m_0 / 2)^2}.$$  \hspace{1cm} (18)

The improvement formula is illustrated in Fig. 2. Again we see that the unimproved Ginsparg-Wilson Green’s function $\left\langle M^{-1} O M^{-1} \right\rangle / a^4$ is no better than the Wilson action Green’s function. It is only by using the improved operator that we can remove all $O(a)$ discretisation errors.

5. Conclusions

For Ginsparg-Wilson fermions we can construct off-shell improved Green’s functions by adding irrelevant operators, and we can write down the improvement coefficients in closed form.

The nature of the bosonic sector makes no difference, the results are the same for Abelian and non-Abelian gauge theories.

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