Weak generalized lifting property, Bruhat intervals and Coxeter matroids

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Coxeter matroids

- Generalization of Whitney’s (ordinary) matroids
- Introduced by I. Gelfand and V. Serganova in 1987
- Studied by many people such as V. Borovik, I. Gelfand, M. Goresky, R. MacPherson, V. Serganova, A. Vince, N. White, A. Zelevinsky...
- Lies at the intersection of Combinatorics, Algebra, Geometry, Optimization Theory

We want to tell you about:
**Theorem (Caselli-D’Adderio-M).** Bruhat intervals of finite Coxeter groups are Coxeter matroids

Main new tool in the proof of the theorem:
**Weak generalized lifting property**
(true for all finite and infinite Coxeter groups)
(W, S)  Coxeter system

• W Coxeter group
• S = \{s_1, \ldots, s_n\} Coxeter generators
• relations: \( s_i^2 = e \) (involutions)
  \( (s_i s_j)^{m_{ij}} = e \) \( m_{ij} \in \mathbb{N}_{\geq 2} \cup \infty \)

Finite Coxeter groups are Reflection Groups
W acting on a real vector space V

\( \ell(w) := \min\{k : w \text{ is a product of } k \text{ generators}\} \)  length

\( \Phi = \Phi^+ \sqcup \Phi^- \) (positive and negative) roots

T  reflections

\( T \xleftrightarrow{\sim} \Phi^+ \) bijection

\( t \mapsto \alpha_t \)
\textbf{Bruhat order} on $W$

it is the transitive closure of $u < v \iff \exists t \in T : \begin{cases} v = tu \\ \ell(v) = \ell(u) + 1 \end{cases}$

\textbf{Properties:}

\begin{itemize}
  \item the identity $e$ is the bottom element
  \item the poset is ranked by length function $\ell$
  \item there exists a top element iff $W$ is finite
\end{itemize}

\textbf{Hasse diagram:} upword edge from $u$ to $v$ iff $u < v$

We also label the edge with the positive root $\alpha_t$ corresponding to $t$
EXAMPLE: $S_3$

$(W, S = \{s_1, s_2\})$ relations: $s_1^2 = s_2^2 = (s_1s_2)^3 = e$

$W = \{e, s_1, s_2, s_1s_2, s_2s_1, s_1s_2s_1 = s_2s_1s_2\} \simeq$ symmetric group $S_3$
$W$ of type $A_3$: the symmetric group $S_4$
Let \((W, S)\) be finite, \(J \subseteq S\).

- \(W_J := \langle J \rangle\) parabolic subgroup generated by \(J\)
- \(W^J := \{w \in W : ws > w \ \forall s \in J\}\) minimal left coset representatives

There is a unique decomposition

\[
W \sim \leftrightarrow W^J \times W_J
\]

\[
w \mapsto w^J \cdot w_J
\]

\[
W^J \sim \leftrightarrow W/W_J
\] bijection

Fix \(p \in V\) s. t. \((p, \alpha_s)\) \(= 0\) if \(s \in J\)
\(< 0\) if \(s \notin J\)

\(\delta_p : W/W_J \rightarrow V \)
\(vW_J \mapsto v(p)\)

well-defined since \(W_J\) fixes \(p\)

Given \(\emptyset \neq \mathcal{M} \subseteq W/W_J\), define a polytope associated with \(\mathcal{M}\):

\[
\Delta_\mathcal{M}(p) = \text{convex hull of } \delta_p(\mathcal{M})
\]

shorthand: \(\Delta_\mathcal{M}(p) = \Delta_{\mathcal{M}}\)
If $W = S_n$, $J = \emptyset$, $\mathcal{M} = W$, then $\Delta_{\mathcal{M}}$ is the classical permutohedron.
w-Bruhat order on $W$:

\[ u \leq^w v \iff w^{-1}u \leq w^{-1}v \]

Note: $\leq^e = \leq$

w-Bruhat order on $W/W_J$:

**Theorem/Definition:** Every $A \in W/W_J$ has a $\min^w$ and a $\max^w$ w.r.t. $\leq^w$.

Let $A, B \in W/W_J$, $w \in W$. TFAE:

- $A \leq^w B$
- $\min^w A \leq^w \min^w B$
- $\max^w A \leq^w \max^w B$
- $a \leq^w b$ for some $a \in A$ and $b \in B$

$\emptyset \neq \mathcal{M} \subseteq W/W_J$ is a **Coxeter matroid for $W$ and $J$** if it satisfies the Maximality Property

- for all $w \in W$, there exists $A \in \mathcal{M}$ s. t. $B \leq^w A$ for all $B \in \mathcal{M}$
EXAMPLE: ORDINARY MATROIDS

- $(W, S)$ of type $A_{n-1}$
- $W \simeq S_n$ the symmetric group on $[n] := \{1, \ldots, n\}$
- $S = \{s_1, s_2, \ldots, s_{n-1}\}$ with $s_i = (i, i+1)$.

If $J = S \setminus \{s_k\}$ then

- $W_J \simeq S_k \times S_{n-k}$
- every $b \in W/W_J$ corresponds to a subset $B$ of $[n]$ of cardinality $k$

\[ W/W_J \overset{\sim}{\longleftrightarrow} \binom{n}{k} \] bijection

With these choices:

- $\{\text{Coxeter matroids for } W \text{ and } J\} = \{\text{ordinary matroids on } [n] \text{ of rank } k\}$

The $B$’s are the bases of the matroid
The theorem translates the definition of a Coxeter matroid into geometric terms.

**Theorem (Gelfand–Serganova).** Let $\emptyset \neq \mathcal{M} \subseteq W/W_J$. TFAE

- $\mathcal{M}$ is a Coxeter matroid
- for every edge of $\Delta_{\mathcal{M}}$, there exists a reflection of $W$ that flips that edge
- every edge of $\Delta_{\mathcal{M}}$ is parallel to a root in $\Phi$

- One of the most important tool of the theory
- Geometric interpretation of Coxeter matroids as polytopes with certain symmetry property
- Surprisingly simple (although cryptomorphic) definition of a Coxeter matroid
- This is why roots play a fundamental role.
$W = S_3 \quad J = \emptyset$
$M = \{ s_1, s_2, s_1s_2, s_2s_1 \}$ is not a Coxeter matroid.
\[ \mathcal{M} = \{s_1, s_1 s_2, s_2 s_1, s_1 s_2 s_1\} \] is a Coxeter matroid.
Theorem (Caselli-D’Adderio-M).
Let \((W, S)\) be a finite Coxeter system. For all \(J \subseteq S\) and all \(x, y \in W^J\) with \(x \leq y\), the parabolic Bruhat interval

\[
\{ z \in W^J : x \leq z \leq y \}
\]

is a Coxeter matroid.

In 2015, Kodama and Williams prove the theorem for \(W\) of type \(A\) and \(J = \emptyset\).
Let $\mathcal{M}$ be a Bruhat interval:

$$\mathcal{M} = [x, y] = \{ z \in W : x \leq z \leq y \}$$

$\Delta_\mathcal{M}$ is the **Bruhat interval polytope** corresponding to $\mathcal{M}$

To prove the theorem, we

- translate the problem into geometric terms using Gelfand–Serganova Theorem
- need to prove that the edges of $\Delta_\mathcal{M}$ are parallel to roots
- study actually all faces of $\Delta_\mathcal{M}$
- use several algebro-combinatorial tools in the theory of Coxeter groups
- use a new tool: the Weak Generalized Lifing Property
Classical Lifting Property (Verma). Let $u, v \in W$ with $u < v$ and $s \in S$. If $u \triangleleft su$ and $sv \triangleleft v$, then $su \leq v$ and $u \leq sv$.

Pros. Characterizes Bruhat order. Has many consequences: e.g. the interval is closed under multiplication by $s$.

Cons. For some $u, v \in W$, there are no such $s \in S$. 
**Generalized Lifting Property** For all $u, v \in W$ with $u < v$, there exists $t \in T$ s.t. $u \prec tu \leq v$ and $u \preceq tv \prec v$

**Pros.** Existence of such $t$. It holds for $W = S_n$ (Tsukerman–Williams ’15) and, more generally, for $W$ simply laced (Caselli–Sentinelli ’17)

**Cons.** It doesn’t hold for $W$ not simply laced (Caselli–Sentinelli ’17)
**Weak Generalized Lifting Property (C-D-M)** Given $u, v \in W$ with $u < v$, let $R_v = \{ \alpha_t \in \Phi^+ : u \leq tv < v \}$ and $R_u = \{ \beta_r \in \Phi^+ : u < rv \leq v \}$. Then $\text{Cone}(R_v) \cap \text{Cone}(R_u) \neq \{0\}$.

**Cons.** It is “weak”

**Pros.** It holds for all (finite and infinite) Coxeter systems
A LEMMA

**Lemma.** Let $F$ be a face of $\Delta_{[x,y]}(p)$. If $F$ contains $u(p)$ and $v(p)$ for some subinterval $[u,v]$, then there exists a complete chain $C$ from $u$ to $v$ such that $z(p) \in F$ for all $z \in C$.

By induction on $\ell(v) - \ell(u)$.

Let $f \in V^*$ be such that $f = c$ is the hyperplane containing $F$, and $f < c$ is the halfspace containing $\Delta_{[x,y]}(p) \setminus F$.

By the Weak Generalized Lifting Property

$$\sum_{i \in I} b_i \beta_{r_i} = \sum_{j \in J} a_j \alpha_{t_j} \neq 0$$

with $u \prec r_i u \leq v$, $u \leq t_j v < v$, and $a_j, b_i > 0$.
Recall
\[ r_i(u(p)) = u(p) + c_i \beta_{ri} \]
\[ t_j(v(p)) = v(p) - d_j \alpha_{tj} \]
with \( c_i, d_j > 0 \)

Since all points \( r_i(u(p)) \) and \( t_j(v(p)) \) belong to \( \Delta_{[x,y]}(p) \)

\[
\begin{align*}
 f(\beta_{ri}) &\leq 0 & f(\alpha_{tj}) &\geq 0 \\
\text{Thus } f(\sum b_i \beta_{ri}) = f(\sum a_j \alpha_{tj}) &= 0, \text{ and} \\
 f(\beta_{ri}) = f(\alpha_{tj}) &= 0
\end{align*}
\]

therefore, all points \( r_i u(p) \) and \( t_j v(p) \) lie in \( F \).

By the induction hypothesis, there is a complete chain \( C' \) from \( r_1 u \) to \( v \) such that \( z(p) \in F \) for all \( z \in C' \). Take the chain \( C = C' \cup \{u\} \).
GRAZIE!