Behavioural equivalences
for coalgebras with unobservable moves

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Abstract

We introduce a general categorical framework for the definition of weak behavioural equivalences, building on and extending recent results in the field. This framework is based on parameterised saturation categories, i.e. categories whose hom-sets are endowed with complete orders and a suitable iteration operators; this structure allows us to provide the abstract definitions of various (weak) behavioural equivalence. We show that the Kleisli categories of many common monads are categories of this kind. On one hand, this allows us to instantiate the abstract definitions to a wide range of existing systems (weighted LTS, Segala systems, calculi with names, etc.), recovering the corresponding notions of weak behavioural equivalences; on the other, we can readily provide new weak behavioural equivalences for more complex behaviours, like those definable on presheaves, topological spaces, measurable spaces, etc.

1 Introduction

Since Aczel’s seminal work [2], the theory of coalgebras has been recognized as a good framework for the study of concurrent and reactive systems [30]: systems are represented as maps of the form \( X \to BX \) for some suitable behavioural functor \( B \). By changing the underlying category and the functor we can cover a wide range of cases, from traditional labelled transition systems to systems with I/O, quantitative aspects, probabilistic distribution, stochastic rates, and even systems with continuous state. Frameworks of this kind are very useful both from a theoretical and a practical point of view, since they prepare the ground for general results and tools which can be readily instantiated to various cases, and moreover they help us to discover connections and similarities between apparently different notions. In particular, Milner’s strong bisimilarity can be characterized by the final coalgebraic semantics and coalgebraic bisimulation; this has paved the way for the definition of strong bisimilarity for systems with peculiar computational aspects and many other important results (such as Turi and Plotkin’s bialgebraic approach to abstract GSOS [38]). More recently, Hasuo et al. [21] have showed that, when the functor \( B \) is of the form \( TF \) where \( T \) is a monad, the trace equivalence for systems of the form \( X \to TX \) can
be obtained by lifting $F$ to the Kleisli category of $T$. This has led to many results connecting formal languages, automata theory and coalgebraic semantics [9, 10, 33, 34].

These remarkable achievements have boosted many attempts to cover other equivalences from van Glabbeek’s spectrum [39]. However, when we come to behavioural equivalences for systems with unobservable (i.e., internal) moves, the situation is not as clear. The point is that what is “unobservable” depends on the system: in LTSs these are internal steps (i.e., $\tau$-transitions), but in systems with quantitative aspects, or dealing with resources, internal steps may still have observable effects. This has led to many definitions, often quite ad hoc. Some follow Milner’s “double arrow” construction (i.e., strong bisimulations of the system saturated under $\tau$-transitions), but in general this construction does not work; in particular for quantitative systems we cannot apply directly this schema, and many other solutions have been proposed; see e.g. [6, 7, 14, 19, 25, 36]. In non-deterministic probabilistic systems, for example, the counterpart of Milner’s weak bisimulation is Segala’s weak bisimulation [32], which differs from Baier-Hermann’s [5].

This situation points out the need for a general, uniform framework covering many weak behavioural equivalences at once. This is the problem we aim to address in this paper. Analysing previous work in this direction [11, 19, 25], a common trait we notice is that circular definitions of behavioural equivalences are turned into equations in a suitable domain of approximants; these equations are then solved taking advantage of some fixed point theory. Different equations and different domains yield different notions of weak bisimilarity. These observations lead us to introduce the notion of parameterised saturation categories. Basically, a PS-category is a category whose hom-sets are endowed with

- a join operator (in order to “merge” approximants);
- an iteration operator (for calculating the solutions of circular definitions as fixed points);
- a complete order (for finding the minimal fixed points).

Using this structure, in PS-categories we can define (and solve) the abstract equations corresponding to many kinds of weak observational equivalence. For example, we will show the abstract schemata corresponding to Milner’s and Baier-Hermann’s versions of weak bisimulations; hence, these two different bisimulations are applications of the same general framework. Then, we show that the Kleisli categories of many monads commonly used for defining behavioural functors are actually PS-categories; this allows us to port the definitions above to a wide range of behaviours in different categories (such as presheaf categories, topological spaces and measurable spaces).

It is interesting to point out that the notion of “unobservability” is embedded in the monads used in these construction. In fact, we show that unobservability can be considered as a separated computational effect, with its own computational monad which can be “plugged into” the behavioural functor, in a modular fashion. This allows us to decide which aspects have to be omitted by the behavioural equivalence just by choosing the suitable unobservation monad.

We assume the reader to be familiar with the theory of coalgebras and behavioural equivalences. We refer the interested reader to [3].

**Synopsis** In Section 2 we introduce parameterised saturation categories, and show that the Kleisli categories of many monads used for defining behavioural functors are PS-categories. Then, in Section 3 we describe how to embed the notion of unobservability in these monads, still keeping their Kleisli a PS-category. In Section 4 we show how an abstract definition of weak behavioural equivalence can be given in this theory of parameterised saturation; specific cases can be readily recovered by instantiating this construction to specific behavioural functors. Some conclusions and direction for further work are in Section 5. Omitted proofs can be found in Appendix A.
2 Saturation

In this section we introduce parameterised saturation categories, providing some sufficient conditions for a category to present this structure. Then, we illustrate how Kleisli categories of several monads of interest are actually parameterised saturation categories.

2.1 Parameterised saturation categories

Definition 1. Let \((V, \otimes, I)\) be a monoidal category. A category \(C\) is said to be enriched over \(V\) (or just \(V\)-enriched) if it has hom-objects in \(V\) and composition \((\cdot) : C(Y, Z) \otimes C(X, Y) \to C(X, Z)\) is a map in \(V\) for any \(X, Y, Z \in C\).

Let \((\text{Pos}^\vee, \times, 1)\) be the cartesian category of partial orders with binary joins as objects and monotonic maps as morphisms. Note that morphisms in \(\text{Pos}^\vee\) do not necessarily preserve joins.

Consider a \(\text{Pos}^\vee\)-enriched category \(C\). The enrichment in \(C\) guarantees that the composition is monotonic in each component i.e.

\[ f \leq f' \implies g \circ f \leq g \circ f' \land f \circ h \leq f' \circ h. \]

whereas joins are not necessarily preserved. In general however, the following inequalities \(f \circ (g \vee h) \geq f \circ g \vee f \circ h\) and \((g \vee h) \cdot i \geq g \circ i \vee h \circ i\) hold true.

Definition 2. A \(\text{Pos}^\vee\)-enriched category \(C\) is said to be a parameterised saturation category if for a span of morphisms \(X \xleftarrow{f} X \xrightarrow{g} Y\) in \(C\) there is \(f^* : X \to Y\) such that it is the least morphism satisfying:

\[ f^* = g \vee f^* \circ f. \]  

(1)

Let \((\omega\text{-Cpo}, \times, 1)\) be the cartesian category with partial orders admitting suprema of ascending \(\omega\)-chains as objects and maps preserving such suprema as morphisms. A category is \(\omega\text{-Cpo}\)-enriched whenever has \(\omega\)-complete partial orders has hom-objects and composition is monotone and continuous in both components. All of the examples considered in this paper form a special type of \(\omega\text{-Cpo}\)-enriched category i.e. their hom-objects admit also binary joins. For these the following proposition is of importance.

Proposition 1. If a category \(C\) is \(\omega\text{-Cpo}\)-enriched with hom-sets admitting binary joins then it is a parameterised saturation category.

Proof. Follows directly by the fact that for any \(f, g\) with suitable domain and codomain the assignment \(F_{f, g}(x) = g \vee x \circ f\) is monotonic with the least fixed point given by \(\bigvee_{n \in \omega} F_{f, g}^n(g)\).

For the sake of simplicity of notation put \(f^* \triangleq f_{id}^*\).

Proposition 2. Assume the following properties hold for any morphisms \(f, g, h\) with suitable domain and codomain:

- \(f \circ (g \vee h) = f \circ g \vee f \circ h\) (left distributivity),
- \(f \vee g \circ h \leq g \implies f \circ h^* \leq g\) ("star fixed point induction rule").

Then: \(f^*_g = g \circ f^*\).

Proof. We have \(g \circ f^* = g \circ (id \vee f^* \circ f) = g \vee g \circ f^* \circ f\). Hence, \(f^*_g \leq g \circ f^*\). To prove the converse note \(g \vee f^*_g \circ f \leq f^*_g\) which implies that \(g \circ f^* \leq f^*_g\).  

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As we will see further in the paper, many families of systems with internal moves form Pos\textsuperscript{2}-enriched categories. Moreover, least fixed points of the assignments \(x \mapsto g \lor x \circ f\) play a fundamental role in their weak behavioural equivalences. A careful study of Proposition 2 suggests that the family of parameterised fixed points \(\{f^*_g\}_{g \colon X \to Y}\) can in some cases be obtained by post-composing the least fixed point of \(x \mapsto id \lor x \circ f\) with \(g\). We have the following:

**Proposition 3.** If \(C\) is a \(\omega\)-Cpo-enriched category whose hom-sets admit binary joins satisfying left distributivity, then star fixed point induction rule holds.

Unfortunately, as we will see in the next subsection, left distributivity is often not satisfied by our main examples (e.g. \(Kl(F[0, \infty])\)). This justifies the need for a more general theory, like the one developed in this paper.

We would like to pinpoint some connections between PS-categories and the established notions in the literature. Fixed points and fixed point operators are becoming increasingly important for coalgebras with invisible steps; for instance, they are used in the theory of trace semantics \([10, 12, 34]\): the fixed point operator \((-)^{\dagger}\) which assigns to \(f : X \to X + Y\) the morphism \(f^{\dagger} : X \to Y\), and satisfies certain axioms, plays a crucial role in this setting. Although existence of \((-)^{\dagger}\) is often guaranteed by certain enrichment of hom-sets \([10, 34]\) one can consider this enrichment implicitly. Note that the situation is different with PS-categories. In this case, we explicitly require the hom-sets to be posets admitting arbitrary binary joins and ask certain fixed points to exist. It should be mentioned here, that among the notions known in the categorical fixed point literature one of the closest would be *iteration grove theories* (see e.g. \([8, 17]\)). In short, an iteration grove theory is a theory whose hom-sets admit an algebraic structure and which comes with the iteration operator \((-)^{\dagger}\). In this case, existence of \((-)^{\dagger}\) is equivalent to existence of a generalized star fixed point operator \([17]\), whose properties are similar to those of our \((\ast)^{\dagger}\). However, since the algebraic structure of a grove theory is assumed to satisfy left distributivity, this notion is too narrow for our purposes.

### 2.2 Examples of parameterised saturation categories

Here, we will list some important examples of monads whose Kleisli categories are parameterised saturation categories.

**Powerset monad** A standard example of a monad that fits our setting is the powerset monad \(\mathcal{P} : Set \to Set\), whose Kleisli category \(Kl(\mathcal{P})\) is isomorphic to \(Rel\) - the category of sets as objects, and binary relations as morphisms with relation composition as the morphism composition. It is easy to see that hom-sets of \(Kl(\mathcal{P})\) admit arbitrary joins which are preserved by the composition. Hence, this category is a parameterised saturation category.

**Convex combinations monad(s)** The convex combinations monad was first introduced in its full generality by Jacobs in \([22]\) to study trace semantics for combined possibilistic and probabilistic systems. Independently, Brengos in \([11]\) and Goncharov and Pattison in \([19]\) have tweaked Jacobs’ construction slightly, so that the resulting monads are more suitable to model the so-called Segala systems and their weak bisimulations. Jacobs’ monad, Brengos’ monad and Goncharov-Pattison’s monad form Kleisli categories which are \(\omega\)-Cpo-enriched and whose hom-sets admit binary joins. Hence, their Kleisli are parametrized saturation categories. For the purposes of this paper we take the convex combinations monad \(CM : Set \to Set\) to be that considered in \([11, \S 8]\).
Countable generalized multiset monad  This monad was used in [25] to present weak bisimulations for weighted transition systems where weights are drawn from a semiring structure offering a modular approach to model several behavioural aspects as showed in [26].

Before describing the monad and its Kleisli category let us recall some preliminary definitions. A semiring $W = (W, +, 0, \cdot, 1)$ is said to be \textit{positively ordered} whenever $W$ admits a partial order $(W, \leq)$ such that 0 is the bottom element of this ordering and semiring operations are monotonic in both components i.e.: $x \leq y$ implies $x \cdot z \leq y \cdot z$ and $z \cdot x \leq z \cdot y$ for $\circ \in \{+, \cdot\}$ and $x, y, z \in W$. A semiring is positively ordered iff it is \textit{zerosumfree} i.e. $x + y = 0$ implies $x = y = 0$. In this case the natural order $x < y \iff \exists z. x + z = y$ is the weakest one rendering $W$ positively ordered.

A positively ordered semiring $W$ is said to be $\omega$-\textit{complete} if it has countable sums given as $\sum_{i \leq \omega} x_i = \sup\{ \sum_{j \in J} x_j \mid J \subset \omega \}$. It is called $\omega$-\textit{continuous} if suprema of ascending $\omega$-chains exist and are preserved by both operations i.e.: $y \circ \bigvee_i x_i = \bigvee_i y \circ x_i$ and $\bigvee_i x_i \circ z = \bigvee_i x_i \circ z$ for $\circ \in \{+, \cdot\}$ and $x, y, z \in W$.

Let $W$ be an $\omega$-complete semiring. Consider the \textbf{Set} endofunctor $\mathcal{F}_W$ given on any set $X$ and on any function $f : X \rightarrow Y$ as follows:

$$
\mathcal{F}_W X \triangleq \{ \varphi : X \rightarrow W \mid |\{x \mid \varphi(x) \neq 0\}| \leq \omega \} \quad \mathcal{F}_W f(\varphi)(y) \triangleq \sum_{\{x \mid f(x) = y\}} \varphi(x).
$$

This functor extends to the \textbf{countable generalized multiset monad} (a.k.a. $\omega$-\textit{complete semimodule monad}) whose multiplication $\mu$ and unit $\eta$ are given on their components by:

$$
\mu_X(\varphi)(x) \triangleq \sum_{\psi \in \mathcal{F}_W X} \varphi(\psi) \cdot \psi(x) \quad \eta_X(x)(x') \triangleq \begin{cases} 1 & \text{if } x = x' \\ 0 & \text{otherwise.} \end{cases}
$$

For any $f, g \in \text{KL}(\mathcal{F}_W)(X, Y)$ define:

$$
f \leq g \iff f(x)(y) \leq g(x)(y) \text{ for any } x, y \in X.
$$

\textbf{Proposition 4.} If the order in $W$ admits binary joins and $W$ is $\omega$-continuous then $\text{KL}(\mathcal{F}_W)$ is a $\omega$-\textit{Cpo-enriched category whose hom-set admit binary joins and hence a parameterised saturation category.}

\textbf{Proof.} Follows directly from $\omega$-continuity of $W$ and Proposition 1. \hfill \Box

The countable powerset monad $\mathcal{P}_\omega$ is $\mathcal{F}_B$ where $B$ denotes the boolean semiring $\{tt, ff\} \cup \{\vee, \wedge\}$. The monad of discrete probability distributions $\mathcal{D}$ is a submonad of $\mathcal{F}_{[0, \infty]}$ for the semiring of non-negative real numbers extended with the infinity (i.e. the free $\omega$-completion of the partial semiring $[0, 1]$).

\textbf{Probability distributions monad}  The Kleisli category for the (sub)distribution monad $\mathcal{D}_{\leq 1}$ is not a parameterised saturation category: despite being $\omega$-Cpo-enriched (see e.g. [21]), it lacks binary joins. However, this monad can be embedded into monads whose Kleisli supports saturation and therefore offers a context where this operation can be carried out also on fully-probabilistic systems. Examples are $\mathcal{C}_\omega$ and $\mathcal{F}_{[0, \infty]}$.

As we will better discuss in the last part of the paper, different embeddings yield different saturations and hence different weak behavioural equivalences. In fact, fully probabilistic systems can be seen as “deterministic” Segala systems but, in general, Baier and Hermann’s weak bisimulation and Segala’s weak bisimulation do not coincide.
Proof. Recall that $\mathcal{T}$ given by Proposition 5. If $\mathcal{C}$ be a small category and let $(T, \mu, \eta)$ be a monad on $\text{Set}$ such $\mathcal{Kl}(T)$ is a parameterised saturation category. We define the extension of $T$ to the presheaf category $[\mathcal{C}, \text{Set}]$ as the monad given by $T^C X \triangleq T \circ X$, $\mu^C_X \triangleq \mu X$, and $\eta^C_X \triangleq \eta X$. Coherence follows from Cat being a 2-category.

The category $\mathcal{Kl}(T^C)$ has presheaves on $\mathcal{C}$ as objects and natural transformations from $\mathcal{Nat}(-, T(-))$ as morphisms; hence the following isomorphism holds:

$$\mathcal{Kl}(T^C) \simeq [\mathcal{C}, \mathcal{Kl}(T)]$$

**Proposition 5.** If $\mathcal{Kl}(T)$ is enriched over $\text{Pos}$ or $\omega\text{-Cpo}$ so is $[\mathcal{C}, \mathcal{Kl}(T)]$.

**Proof.** Recall that $[\mathcal{C}, \mathcal{Kl}(T)](X, Y) = \mathcal{Nat}(X, TY)$. \hfill \qed

Let $I$ be a skeleton of the category of finite sets and injective functions. The presheaf category $[I, \text{Set}]$ is the context of several works on coalgebraic semantics for calculi with names. In particular, $\mathcal{P}$ is precisely the component expressing non-determinism in the behavioural functors capturing the late and early semantics of the $\pi$-calculus [18]. Enrichment extends pointwise to $\mathcal{Kl}(\mathcal{P})$ e.g. $f \leq g \iff \forall n \in I, f_n \leq g_n$, rendering the category $\mathcal{PS}$.

Another example of systems with names is offered by the Fusion calculus. Differently from $\mathcal{P}$, $\mathcal{F}$, a parameterised saturation category.

**Vietoris monad** Let us consider an example in a topological setting. Let $(X, \Sigma_X)$ be a compact Hausdorff space, and $KX$ the set of compact subsets of $X$. The Vietoris topology $\Sigma_{\mathcal{V}(X, \Sigma_X)}$ on $KX$ is described by the base consisting of sets of the following form:

$$\nabla\{U_1, \ldots, U_n\} = \{F \in KX | F \subseteq \bigcup_{i=1}^n U_i \text{ and } F \cap U_i \neq \emptyset \text{ for } 1 \leq i \leq n\}$$

where $n \in \mathbb{N}$ and each $U_i \in \Sigma_X$. This extends to an endofunctor $\mathcal{V}$ over $\mathcal{CHaus}$, the category of compact Hausdorff spaces and continuous function, whose action takes forward images $\mathcal{V}f(X') = f(X')$ for every continuous function $f : (X, \Sigma_X) \rightarrow (Y, \Sigma_Y)$ [41]. This functor has a central rôle in modal logics since Esakia’s seminal work on topological Kripke frames\footnote{Topological Kripke frames are usually defined on Stone spaces.} [16]. Moreover, it extends to a monad mimicking the lifting of $\mathcal{P}$ whose multiplication and unit are given, on each component $(X, \Sigma_X)$, as follows:

$$\mu_{(X, \Sigma_X)}(Y) = \bigcup Y \quad \eta_{(X, \Sigma_X)}(x) = \{x\}.$$  

The structure of sets of compact subsets extends to the Kleisli category for the Vietoris monad in the obvious way: arrows are ordered and joined in pointwise manner. Kleisli composition preserves this structure rendering $\mathcal{Kl}(\mathcal{V})$ enriched over $\omega\text{-Cpo}$ and, by existence of binary joins in $\mathcal{Kl}(\mathcal{V})$, a parameterised saturation category.

**Proposition 6.** The category $\mathcal{Kl}(\mathcal{V})$ is a parameterised saturation category.

**Proof.** By Proposition 1 we only have to show that $\mathcal{Kl}(\mathcal{V})$ is enriched over $\omega\text{-Cpo}$ and its hom-sets have binary joins. We conclude by Lemma 7 below. \hfill \qed
Lemma 7. The category $KL(V)$ is enriched over $Jsl$, the category of join semilattice and join preserving maps.

Proof. Joins in $V(X,\Sigma_X)$ (i.e., unions) are extended pointwise to $KL(V)$.

The Vietoris monad and its Kleisli category share some similarities with $P$, $CM$ and their Kleisli: beside being PS-categories, all of them present left distributivity and hence are a host for Proposition 2.

Measures monad Let $(X,\Sigma_X)$ be a measurable space and let $\Delta(X,\Sigma_X)$ be the set of all measures $\varphi : \Sigma_X \to [0,\infty]$ on $(X,\Sigma_X)$. For each measurable set $M \in \Sigma_X$ there is a canonical evaluation function $ev_M : \Delta(X,\Sigma_X) \to [0,\infty]$ s.t. $ev_M(\varphi) = \varphi(M)$; these evaluation maps allow us to endow $\Delta(X,\Sigma_X)$ with the smallest $\sigma$-algebra rendering each $ev_M$ measurable w.r.t. the Borel $\sigma$-algebra on $[0,\infty]$.\textmd{(i.e. the initial $\sigma$-algebra w.r.t. $\left\{ev_M \mid M \in \Sigma_X\right\}$).} This definition extends to an endofunctor $\Delta$ over $Meas$, the category of measurable spaces and measurable functions, acting on any $(X,\Sigma_X)$ and $f : (X,\Sigma_X) \to (Y,\Sigma_Y)$ as:

$$\Delta(X,\Sigma_X) \triangleq (\Delta(X,\Sigma_X),\Sigma_{\Delta(X,\Sigma_X)}) \quad \text{and} \quad \Delta f(\varphi) \triangleq \varphi \circ f^{-1}$$

Lemma 8. The functor $\Delta : Meas \to Meas$ extends to a monad $(\Delta,\mu,\eta)$ whose multiplication and unit are given, on each component $(X,\Sigma_X)$, as:

$$\mu_{(X,\Sigma_X)}(\varphi)(M) \triangleq \int_{\Delta(X,\Sigma_X)} \psi(M)\varphi(d\psi) = \int_{\Delta(X,\Sigma_X)} ev_M d\varphi$$

$$\eta_{(X,\Sigma_X)}(x)(M) \triangleq \delta_x(M) = \chi_M(x)$$

where $\chi_M : X \to \{0,1\}$ is the indicator function on $M \in \Sigma_X$.

Roughly speaking, $\Delta$ can be though as the “measurable equivalent” of $F_{[0,\infty]}$ and, likewise $F_{[0,\infty]}$ generalizes the probability distribution monad $D$, $\Delta$ generalizes the probability measure monad $\Delta_{=1}$ (a.k.a. Giry monad [13, 15, 28]).

Each measurable space $\Delta(X,\Sigma_X)$ presents a partial order induced by the pointwise extension of the natural order on $[0,\infty]$. For any two measures $\varphi,\psi \in \Delta(X,\Sigma_X)$ we have:

$$\varphi \leq \psi \iff \forall M \in \Sigma_X. \varphi(M) \leq \psi(M).$$

For any two measures $\varphi,\psi \in \Delta(X,\Sigma_X)$, their join is given on $M \in \Sigma_X$ as:

$$\varphi \lor \psi(M) \triangleq \sup\{\varphi(M_1) + \psi(M_2) \mid M_1 \in \Sigma_X, M_1 \subseteq M, M_2 = M \setminus M_1\}.$$  

For any increasing $\omega$-chain $(\varphi_i)_{i \in \mathbb{N}}$ in $\Delta(X,\Sigma_X)$, its supremum does exist and it is given on each $M \in \Sigma_X$ as:

$$\bigvee_{i \in \omega} \varphi_i(M) \triangleq \lim_{i \to \infty} \varphi_i(M) = \sup_i \varphi_i(M).$$

Each $\Delta(X,\Sigma_X)$ is an $\omega$-Cpo with binary joins. This structure extends pointwise to the hom-objects of $KL(\Delta)$ and is preserved by composition in $KL(\Delta)$.

Proposition 9. $KL(\Delta)$ is $\omega$-Cpo-enriched and its hom-sets admit binary joins, hence a parameterised saturation category.
3 The monadic structure of (un)observables

Originally [20, 34], coalgebras with silent actions were introduced in the context of coalgebraic trace semantics as coalgebras of the type $T(F + I\text{id})$ for a monad $T$ and an endofunctor $F$ on $\mathbb{C}$ (e.g. $\mathcal{P}((A + \{\tau\}) \times I\text{id})$). Intuitively, $T$, $F$, and $I\text{id}$ describe the “branching”, “observable”, and “unobservable” computational aspects respectively with the monad defining how moves are concatenated. For many systems modelled by coalgebras in $\mathbb{Set}$ observables are actually given by a set of labels (or alphabet) $A$. In this case:

$$T(F + I\text{id}) = T(A \times I\text{id} + I\text{id}) \cong T(A_{\tau} \times I\text{id}),$$

where $A_{\tau} \triangleq A + \{\tau\}$. However, this readily generalizes to more complex notions of observables and outside the $\mathbb{Set}$; e.g., the alphabet may be given as measurable space (cf. FlatCCS [4]) or as a presheaf (cf. $\pi$-calculus).

In this section we show how unobservable moves can be endowed with a suitable monadic structure. We build on the approach introduced in [11] where Brengos showed that given some mild assumptions on $T$ and $F$ we may either introduce a monadic structure on a lifting $F + I\text{id}$ of the functor $F + I\text{id}$ to the Kleisli category $\mathbb{Kl}(T)$ or embed the lifting $F + I\text{id}$ into the monad $F^\ast$, where $F^\ast$ denotes the free monad over the lifting $F + I\text{id}$ into the monad $\mathbb{Kl}(T)$ of $F$. The monadic structure of (un)observables on $\mathbb{Kl}(T)$ is composed with $T$ (since $\mathbb{Kl}(T)$ is monadic on $\mathbb{C}$) yielding a monadic structure on $T(F + I\text{id})$ and $TF^\ast$ respectively.

This result corroborates the view of unobservables as a computational effect, along the line of Moggi’s theory, and plays a fundamental rôle in our treatment of systems with unobservable moves: it allows us not to specify the invisible moves explicitly and lets a monadic structure of the behavioural functor to handle them internally. Instead of considering $T(F + I\text{id})$-coalgebras we consider $T'\text{-coalgebras}$ for a monad $T'$ on an arbitrary category $\mathbb{C}$ allowing us to combine the results of this section with parameterised saturation.

3.1 Liftings to and monads on $\mathbb{Kl}(T)$

The Kleisli category for a monad $(T, \mu, \eta)$ on $\mathbb{C}$ is monadic over $\mathbb{C}$:

$$\begin{array}{ccc}
\mathbb{C} & \xrightarrow{(-)^T} & \mathbb{Kl}(T) \\
\downarrow & \Downarrow & \downarrow \\
U_T & \Rightarrow & \mathbb{Kl}(T)
\end{array}$$

The left adjoint $(-)^T$ of the forgetful $U_T$ is the inclusion functor, defined on any object $X \in \mathbb{C}$ as the identity and on any morphism $f : X \to Y \in \mathbb{C}$ as $f^T = \eta_Y \circ f$.

We say that a functor $F : \mathbb{C} \to \mathbb{C}$ lifts to an endofunctor $\overline{F} : \mathbb{Kl}(T) \to \mathbb{Kl}(T)$ provided that the following diagram commutes:

$$\begin{array}{ccc}
\mathbb{Kl}(T) & \xrightarrow{F} & \mathbb{Kl}(T) \\
\downarrow (-)^T & & \downarrow (-)^T \\
\mathbb{C} & \xrightarrow{F} & \mathbb{C}
\end{array}$$

Given a functor $F : \mathbb{C} \to \mathbb{C}$ there is a one-to-one correspondence between its liftings $\overline{F} : \mathbb{Kl}(T) \to \mathbb{Kl}(T)$ and distributive laws $\lambda : FT \Rightarrow TF$ between the functor $F$ and the monad $T$ (see e.g. [23]
for a detailed definition and properties). Given a distributive law \( \lambda : FT \Rightarrow TF \) we define \( F : \mathcal{K}l(T) \to \mathcal{K}l(T) \) on any object \( X \) and any morphism \( f : X \to Y \) in \( \mathcal{K}l(T) \) as:
\[
FX \triangleq FX \quad Ff \triangleq \lambda_Y \circ Ff.
\]
Conversely, a lifting \( F : \mathcal{K}l(T) \to \mathcal{K}l(T) \) gives rise to a distributive law \( \lambda : FT \Rightarrow TF \) defined on each component \( \lambda_X \) as \( F(id_{TX}) \).

In [21] Hasuo et al. showed that *shapey functors* admit canonical distributive laws (hence liftings) if the monad is commutative\(^2\) (w.r.t. cartesian products) and \( \mathcal{C} \) has countable coproducts (hence, so does \( \mathcal{K}l(T) \) as the coproducts in \( \mathcal{K}l(T) \) come from the coproducts in \( \mathcal{C} \)). In loc. cit. a functor is said to be *shapey* whenever it is described by the grammar:
\[
F ::= \mathbb{I}d \mid A \mid F_1 \times F_2 \mid \prod_{i \in I} F_i
\]
where \( A \in \mathcal{C} \) and \( I \subseteq \mathbb{N} \). The canonical distributive law is defined by structural recursion. Note that a strong\(^2\) monad (i.e. not necessarily commutative) is sufficient to cope with the classic case of “labels with silent actions” i.e. \( A \times \mathbb{I}d + \mathbb{I}d \).

**Example 1.** Take \( \mathcal{C} = \mathcal{P} \) and \( T = \mathcal{P} \). The powerset monad is commutative and its double strength \( \text{dstr}_{X,Y} : \mathcal{P}X \times \mathcal{P}Y \to \mathcal{P}(X \times Y) \) is given by the formula:
\[
\text{dstr}_{X,Y}(X', Y') = X' \times Y'.
\]
Hence, the functor \( A \times \mathbb{I}d : \mathcal{C} \to \mathcal{C} \) lifts to \( A \times \mathbb{I}d : \mathcal{K}l(\mathcal{P}) \to \mathcal{K}l(\mathcal{P}) \) acting as \( A \times \mathbb{I}d \) on objects and as \( \text{dstr}(\eta_A \times \mathbb{I}d) \) on morphisms. In particular, for any object \( X \in \mathcal{K}l(\mathcal{P}) \) and any morphism \( f : X \to \mathcal{P}Y \) in \( \mathcal{K}l(\mathcal{P}) \) we have:
\[
(A \times \mathbb{I}d)X = A \times X \quad \text{and} \quad (A \times \mathbb{I}d)f(a, x) = \{(\sigma, y) \mid y \in f(x)\}.
\]
If the lifting of a functor presents a monad structure in \( \mathcal{K}l(T) \), say \( (S, \nu, \theta) \), we have the following two adjoint situations:
\[
\begin{array}{c}
\mathcal{C} \\
\downarrow \mathcal{U}_T \\
\mathcal{K}l(T)
\end{array}
\quad \quad \quad
\begin{array}{c}
\mathcal{K}l(T) \\
\downarrow \mathcal{U}_\Sigma \quad \quad \downarrow \mathcal{K}l(\mathcal{S})
\end{array}
\]

The adjoint situation defined by their composition endows \( TS : \mathcal{C} \to \mathcal{C} \) with a monad structure whose multiplication and unit are defined as:
\[
\mu_S \circ T\mu_S \circ T^2\nu \circ T\lambda_S \quad \text{and} \quad \theta.
\]
(2)

For any \( f : X \to Y \) and \( g : Y \to Z \) in \( \mathcal{K}l(TS) = \mathcal{K}l(\mathcal{S}) \) their composite \( g \cdot f \) is defined in \( \mathcal{C} \) as:
\[
\begin{array}{c}
X \\
\downarrow f \\
TSY \\
\downarrow TSG
\end{array}
\quad \quad \quad
\begin{array}{c}
(TS)^2Z \\
\downarrow T^2\lambda_S \\
T^2\nu \\
\downarrow T^3\lambda_S \\
T^2\mu_S \\
\downarrow T^2SZ \\
\downarrow \mu_S \\
TSZ
\end{array}
\]
(3)

\(^2\)A monad \((T, \mu, \eta)\) on a monoidal category is said to be *strong* (w.r.t. the monoidal structure) if it is equipped with a natural transformation \( \text{str} : \mathbb{I}d \otimes T \to T(\mathbb{I}d \otimes \mathbb{I}d) \), called *tensorial strength*, coherent with the structure of monads and monoidal categories. Likewise, a monad \((T, \mu, \eta)\) on a symmetric monoidal category is *commutative* whenever it is equipped with a *double tensorial strength* \( \text{dstr} : T \otimes T \to T(\mathbb{I}d \otimes \mathbb{I}d) \) and.
The adjoint situation in (2) allows us to compose several computational aspects and, when each layer is well-behaved w.r.t. the underlying enrichment, to define parameterised saturation in a uniform manner. Here, being well-behaved means that the functor of a monad $(\mathcal{S}, \nu, \theta)$ on the $\omega$-Cpo-enriched $\mathcal{Kl}(T)$ is locally continuous, i.e. it preserves suprema of ascending $\omega$-chains:

$$\mathcal{S}(\bigvee_{i<\omega} f_i) = \bigvee_{i<\omega} (\mathcal{S}f_i).$$

Note that it is not necessarily the case that $\mathcal{S}$ is a monad: we only assume that its lifting is.

**Theorem 10.** Assume that $\mathcal{Kl}(T)$ is $\omega$-Cpo-enriched category whose hom-objects admit binary joins. If $\mathcal{S}$ is locally continuous then $\mathcal{Kl}(\mathcal{S}) = \mathcal{Kl}(TS)$ is a parameterised saturation category.

For the aims of this work $\mathcal{S}$ will be a functor modelling internal moves i.e. with shape $F + \mathcal{I}d$. The following result states that these functors extends to monads on $\mathcal{Kl}(T)$ whenever the category has zero morphisms.

**Theorem 11** ([11]). Let $F$ be the lifting to $\mathcal{Kl}(T)$ of $F$. If $\mathcal{C}$ has binary coproducts and $\mathcal{Kl}(T)$ has zero morphisms\(^3\) then $F + \mathcal{I}d = F + \mathcal{I}d$ extends to a monad whose unit is $\text{inr} : \mathcal{I}d \to F + \mathcal{I}d$ and whose multiplication is

$$[\text{inl}, \text{id}][F(0 + \mathcal{I}d) + \text{id}] : (F + \mathcal{I}d)^2 \to F + \mathcal{I}d.$$

**Example 2.** Let $T = \mathcal{P}$ and let $A$ be an arbitrary set. By Example 1 the Set-endofunctor $A\tau \times \mathcal{I}d$ lifts to $\mathcal{Kl}(\mathcal{P})$. By Theorem 11 its lifting $A\tau \times \mathcal{I}d \cong A\tau \times \mathcal{I}d + \mathcal{I}d$ can be equipped with a monadic structure $(A\tau \times \mathcal{I}d, \nu, \theta)$ whose multiplication and monad unit are given on their $X$-components by:

$$\theta_X(x) = \{(\tau, x)\} \quad \text{and} \quad \nu_X(a, b, x) = \left\{ \begin{array}{ll} \{(a, x)\} & \text{if } a = \tau \\ \{(b, x)\} & \text{if } b = \tau \\ \emptyset & \text{otherwise.} \end{array} \right.$$  

Now, the functor $\mathcal{P}(A\tau \times \mathcal{I}d)$ carries a monadic structure which is a consequence of composing two adjunctions $\text{Set} \rightleftarrows \mathcal{Kl}(\mathcal{P}) \rightleftarrows \mathcal{Kl}(A\tau \times \mathcal{I}d)$ as described earlier in this subsection. The composition in $\mathcal{Kl}(\mathcal{P}(A\tau \times \mathcal{I}d))$ is given as follows. For $f : X \to \mathcal{P}(A\tau \times X)$ and $g : Y \to \mathcal{P}(A\tau \times Z)$ we have:

$$g \cdot f(x) = \{(a, z) \mid x \overset{a}{\to}_f y \overset{\tau}{\to}_g z \text{ or } x \overset{\tau}{\to}_f y \overset{a}{\to}_g z \text{ for some } y \in Y, a \in A\tau\} \quad (4)$$

See [11] for details.

Although the zero morphism assumption is met by all the examples proposed in this paper, this is not the only way to define a monad on $\mathcal{Kl}(T)$ from $F + \mathcal{I}d$ modelling the behaviour of internal moves. In fact, Brengos showed in [11] that if $F$ admits a free monad then this can be lifted (by structural recursion) to $\mathcal{Kl}(T)$. We refer the reader to [11] for more details.

Intuitively, Theorem 11 follows the line of how weak behavioural equivalences compose un-observable and observable moves. Monad law for unit reflects the fact that weak behavioural equivalences assume self-loops of unobservable actions. However, note that $T(F + \mathcal{I}d)$-coalgebras are not required to present them (i.e. that every coalgebra $\alpha$ simulates the unit $\text{inr} \leq \alpha$); this behaviour is a consequence of saturation.

---

\(^3\)We say that a category has **zero morphisms** if for any two objects $X, Y$ there is a morphism $0_X, Y$ which is an annihilator w.r.t. composition, i.e.: $f \circ 0 = 0 \circ g$ for any morphisms $f, g$ with suitable domain and codomain.
Likewise, monad law for multiplication generalizes the derivation rules
\[
\begin{align*}
  x \xrightarrow{\tau} y & \quad y \xrightarrow{\alpha} z & \quad x \xrightarrow{\alpha} y & \quad y \xrightarrow{\alpha} z \\
  x \xrightarrow{\tau} z & & x \xrightarrow{\alpha} z & & x \xrightarrow{\alpha} y & \quad y \xrightarrow{\alpha} z
\end{align*}
\]
describing transitivity of unobservable moves and left and right “absorption” of unobservable moves by observable ones. The use of zero morphisms to “kill” consecutive observables corresponds to the absence of rules for this case.

Remarkably, small tweaks in the monad defined by Theorem 11 allow us to deal with various interactions between observable and unobservable moves; e.g., we can cover the grupidal nature of reversible computations by considering a multiplication sketched by the following rules:
\[
\begin{align*}
  x \xrightarrow{\alpha} y & \quad y \xrightarrow{\alpha^{-1}} z & \quad x \xrightarrow{\alpha^{-1}} y & \quad y \xrightarrow{\alpha} z \\
  x \xrightarrow{\tau} z & & x \xrightarrow{\tau} z & & x \xrightarrow{\tau} z & \quad y \xrightarrow{\tau} z
\end{align*}
\]

3.2 Examples of systems with unobservables

**Labelled transition systems**  Labelled transition systems with silent actions [27,31] are modelled as coalgebras of the type \( P(A, \times \text{I}d) \) [30]. In Example 2 we have seen how the functor \( P(A, \times \text{I}d) \) can be equipped with a monadic structure which handles unobservable moves internally. We have the following.

**Proposition 12.** \( \text{Kl}(P(A, \times \text{I}d)) \) is a parameterised saturation category.

**Segala systems** Probabilistic systems [32], known in the coalgebraic literature as Segala systems, can be modelled as coalgebras of the type \( \mathcal{PD}(A, \times \text{I}d) \) [35]. However, due to a lack of a distributive law between monads \( P \) and \( D \) [40] and following Varacca’s idea [40] further studied by Jacobs in [22], Brengos in [11] proposes to consider these systems as \( \mathcal{CM}(A, \times \text{I}d) \)-coalgebras (see [11] for a discussion on consequences of this treatment). From now on, whenever we refer to “Segala systems” we refer to \( \mathcal{CM}(A, \times \text{I}d) \)-coalgebras. The functor \( \mathcal{CM}(A, \times \text{I}d) \) carries a monadic structure as described in [11]. Moreover, we have the following.

**Proposition 13.** \( \text{Kl}(\mathcal{CM}(A, \times \text{I}d)) \) is a parameterised saturation category.

**Weighted transition systems** The monad \( (F_W, \mu, \eta) \) is commutative and its double strength \( \text{dstr} : \mathcal{F}_W \times \mathcal{F}_W \Rightarrow \mathcal{F}_W(\text{I}d \times \text{I}d) \) is defined, on each \( X, Y \), as:
\[
\text{dstr}_{X,Y}(\varphi, \psi)(x, y) = \varphi(x)\psi(y)
\]
Therefore, shapely functors have canonical liftings to \( \text{Kl}(\mathcal{F}_W) \).

**Lemma 14.** Let \( W \) be \( \omega \)-continuous. Canonical liftings of shapely functors to \( \text{Kl}(\mathcal{F}_W) \) are locally continuous whenever \( \text{Kl}(\mathcal{F}_W) \) is \( \omega \)-Cpo-enriched.

By this lemma and zero morphisms in \( \text{Kl}(\mathcal{F}_W) \), we can apply the main results of this section to prove that Kleisli categories of functors like \( \mathcal{F}_W(\text{F} + \text{I}d) \) are parameterised saturation categories.

**Proposition 15.** Let \( F \) be a shapely endofunctor on \( \text{Set} \). If \( W \) is \( \omega \)-continuous and admits binary joins then \( \text{Kl}(\mathcal{F}_W(F + \text{I}d)) \) is a parameterised saturation category.
Consider the functor $F_W(A \times \mathbb{I}d) \cong F_W(A \times \mathbb{I}d + \mathbb{I}d)$ of Weighted LTS. The canonical lifting of $A \times \mathbb{I}d$ is defined, on every $f : X \to Y$ in $\text{Kl}(F_W)$, as:

$$(A \times \mathbb{I}d)f(a, x)(b, y) = \eta_\mathbb{I}(b) f(x)(y)$$

where $x \in X$, $y \in Y$, and $a, b \in A$. By Theorem 11 the functor $A \times \mathbb{I}d$ extends to a monad whose unit $\theta$ is defined as $\theta_X(x) = \mathbb{I}r(x) = \mathbb{I}o \circ \eta_{A, x} \tau (x)$ and whose multiplication $\nu$ id defined as

$$\nu_X(a, \tau, x)(b, y) = \eta_{A, x} \tau (a, x)(b, y)$$

where $X \in \text{Set}$, $x, y \in X$, and $a, b \in A$.

Composition in $\text{Kl}(A \times \mathbb{I}d)$ follows from (3). Let $f, g$ be two maps in $\text{Kl}(F_W(A \times \mathbb{I}d))$ with suitable domains and codomains, then their composite $g \cdot f$ is defined on each $x \in X$ as the weight function mapping

- each $(\tau, z) \in \{\tau\} \times Z$ to $\sum_{y \in Y} g(y)(\tau, z) \cdot f(x)(\tau, y)$
- each $(a, z) \in A \times Z$ to $\sum_{y \in Y} g(y)(a, z) \cdot f(x)(\tau, y) + \sum_{y \in Y} g(y)(\tau,(z) \cdot f(x)(a, y)$.

Remark 3. The above expressions turn out to be precisely those used in [19, 25] to define and compute weak bisimulations for weighted transition systems.

**Nominal systems**  
Consider the $[\mathbb{I}, \text{Set}]$-endofunctor

$$B_\pi \triangleq P^3(N \times N \times \mathbb{I}d + N \times \mathbb{I}d N + N \times \delta + \mathbb{I}d)$$

describing the late semantics for $\pi$-calculus [18]; here $\delta : [\mathbb{I}, \text{Set}] \to [\mathbb{I}, \text{Set}]$ is the *dynamic allocation* endofunctor, defined by $\delta X_n = X_{n+1}$; $N_n = n$ and $(P^3) X_n = P(X_n)$ is the extension of the powerset monad described in Section 2. Hence, for any presheaf $X$ and stage $n \in \mathbb{I}$ we have

$$B_\pi(X)_n = P(n \times n \times X_n + n \times (X_n)^n \times X_{n+1} + n \times X_{n+1} + X_n)$$

describing the behaviour for processes with $n$ free names: the four components of the coproduct describe output, input, bound input and internal synchronization (i.e. $\tau$ transitions), respectively.

Let us define $F = F_\mathbb{I} + F_{\mathbb{I}} + F_{\mathbb{I}}$, where $F_\mathbb{I} = N \times N \times \mathbb{I}d$, $F_{\mathbb{I}} = N \times \mathbb{I}d N$, $F_{\mathbb{I}} = N \times \delta$; then, $B_\pi = P^3(F + \mathbb{I}d)$. The lifting $F + \mathbb{I}d$ to $\text{Kl}(P^3)$ is given, for any $f : X \to Y \in \text{Kl}(P^3)$, by the coputuple:

- $F_{\mathbb{I}} f_n = 3\text{-str} \circ (\eta_\mathbb{I} \times \eta_\mathbb{I} \times f_n)$
- $F_{\mathbb{I}} f_n = (n + 2)\text{-str} \circ (\eta_\mathbb{I} \times (n \cdot f_n) \times f_{n+1})$
- $F_{\mathbb{I}} f_n = n \times f_{n+1}$
- $F_{\mathbb{I}} f_n = n \times f_{n+1}$
- $F_{\mathbb{I}} f_n = n \times f_{n+1}$
- $F_{\mathbb{I}} f_n = n \times f_{n+1}$

where $\cdot$ denotes the copower and $k\text{-str}$ stands for a suitable $k$-fold strength of $P$. Note that all but $F_{\mathbb{I}}$ are canonical (at each stage) in the sense of [21].

The functor $F + \mathbb{I}d$ extends to a monad: the unit is $\text{inr} \cdot \text{inr}$ and maps everything into the unobservable (fourth) component of the coproduct; the multiplication, at each stage, concatenates unobservable transitions discarding pairs of observables by means of suitable zero morphisms as expected. In particular, it is defined as follows:

$$\nu_X(\tau, (a, b, x)) = \{(a, b, x)\}$$
$$\nu_X(\tau, (a, \tau, x)) = \{(a, \tau)\}$$
$$\nu_X(\tau, (a, \tau, x)) = \{(a, \tau)\}$$
$$\nu_X(\tau, y) = \emptyset$$

$$\nu_X(a, \tau, x) = \{(a, \bar{a}, x)\}$$
$$\nu_X(a, \tau, x) = \{(a, \bar{a}, x)\}$$
$$\nu_X(a, \tau, x) = \{(a, \bar{a}, x)\}$$

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where \(a, b\) are names at \(n\)-th stage, \(\overline{x, (\tau, x)}\) are \((n + 1)\)-tuples and \(y\) covers the cases left out.

Composition in \(\mathcal{Kl}(F + Id)\) follows directly by (3). In particular, for any two compatible morphisms \(f : X \rightarrow Y\) and \(g : Y \rightarrow Z\) their composite \((g \cdot f)\) maps each \(x \in X_n\) to the set resulting from the union of the following sets:

\[
\begin{align*}
\{(a, b, z) & \mid (a, b, y) \in f_n(x) \land (\tau, z) \in g_n(y)\} \\
\{& (a, (z_1, \ldots, z_{n+1})) \mid (a, (y_1, \ldots, y_{n+1})) \in f_n(x) \land (\tau, z_i) \in g_n(y_i)\} \\
\land & (\tau, z_{n+1}) \in g_{n+1}(y_{n+1}) \\
\{& (a, (z_1, \ldots, z_{n+1})) \mid (\tau, y) \in f_n(x) \land (a, (z_1, \ldots, z_{n+1})) \in g_n(y)\} \\
\{& (a, z) \mid (a, y) \in f_n(x) \land (\tau, z) \in g_n(y)\} \\
\{& (\tau, z) \mid (\tau, y) \in f_n(x) \land (\tau, z) \in g_n(y)\}
\end{align*}
\]

where \(a, b\) are names at \(n\)-th stage. Using a more evocative notation, composition is characterized by the following derivation rules:

\[
\begin{align*}
x \overset{\tau}{\rightarrow} f_n \rightarrow y \quad & y \overset{a(l)}{\rightarrow} g_n \rightarrow z \\
x \overset{a(l)}{\rightarrow} (g \cdot f)_n \rightarrow z \\
x \overset{a(l)}{\rightarrow} (g \cdot f)_n \rightarrow z \\
x \overset{a(l)}{\rightarrow} (g \cdot f)_n \rightarrow z \\
x \overset{a(l)}{\rightarrow} (g \cdot f)_n \rightarrow z \\
x \overset{a(l)}{\rightarrow} (g \cdot f)_n \rightarrow z \\
x \overset{a(l)}{\rightarrow} (g \cdot f)_n \rightarrow z \\
x \overset{a(l)}{\rightarrow} (g \cdot f)_n \rightarrow z \\
\end{align*}
\]

The Kleisli category for the monad \(\mathcal{P}(F + Id)\) is a parameterised saturation category since the lifting \(F + Id\) is locally continuous.

**Proposition 16.** The category \(\mathcal{Kl}(\mathcal{P}(F + Id))\) is a PS-category.

The above construction can be easily adapted to many other behaviours with unobservables and local resources, or in general modelled in some presheaf category.

**Topological Kripke frames** The Vietoris monad on \(\text{cHaus}\) is commutative. Its double strength is defined, on each component, as:

\[
dstr_{(X, \Sigma_X), (Y, \Sigma_Y)}(X', Y') = X' \times Y'.
\]

Therefore, shapely functors lift canonically to \(\mathcal{Kl}(\mathcal{V})\).

**Lemma 17.** Canonical liftings of shapely functors are locally continuous.

Continuity and existence of zero morphisms allow us to apply the main results of this section to ensure that Kleisli categories of functors like \(\mathcal{V}(F + Id)\) are parameterised saturation categories.

**Proposition 18.** Let \(F\) be a shapely endofunctor on \(\text{cHaus}\). The category \(\mathcal{Kl}(\mathcal{V}(F + Id))\) is a parameterised saturation category.

In the following we instantiate the result on a CCS-like behaviour where labels are drawn from a complete Hausdorff space \((A, \Sigma_A)\) plus a distinguished silent action \(\tau\). Henceforth, let
\[
F = (A, \Sigma_A) \times I. \] The canonical lifting of \( F + I \) on \( Kl(\mathcal{V}) \) is defined, on any continuous map \( f : (X, \Sigma_X) \to (Y, \Sigma_Y), \) as:

\[
F \times I = id(A, \Sigma_A) \times f.
\]

This functor extends to a monad \((F \times I, \nu, \theta)\) as in Theorem 10; its unit and multiplication are defined, on each complete Hausdorff space \((X, \Sigma_X),\) as:

\[
\begin{align*}
\nu_{(X, \Sigma_X)}(a, \tau, x) &= \{(a, x)\} \\
\nu_{(X, \Sigma_X)}(\tau, a, x) &= \{(a, x)\} \\
\theta_{(X, \Sigma_X)}(x) &= \{(\tau, x)\} \\
\nu_{(X, \Sigma_X)}(a, b, x) &= \emptyset.
\end{align*}
\]

Composition in \( Kl(F \times I) \) follows by (4) and, for any two morphisms \( f \) and \( g \) with suitable domain and codomain, the composite \( g \cdot f \) is:

\[
(g \cdot f)(x) = \{(a, z) \mid (a, y) \in f(x) \land (\tau, z) \in g(y) \text{ or } (\tau, y) \in f(x) \land (a, z) \in g(y)\}.
\]

**Measure systems** The measure monad \((\Delta, \mu, \eta)\) on \textbf{Meas} is commutative; its double strength \(dstr : \Delta \times \Delta \to \Delta(I \times I)\) is defined on each component as:

\[
dstr_{(X, \Sigma_X), (Y, \Sigma_Y)}(\varphi, \psi)(M \times N) = \varphi(M)\psi(N).
\]

Therefore, shapely functors have canonical liftings to \( Kl(\Delta).\)

**Lemma 19.** Canonical liftings of shapely functors are locally continuous.

Continuity and existence of zero morphisms allow us to apply the main results of this section to ensure that Kleisli categories of functors like \(\Delta(F + I)\) are parameterised saturation categories.

**Proposition 20.** Let \( F \) be a shapely endofunctor on \textbf{Meas}. The category \( Kl(\Delta(F + I)) \) is a parameterised saturation category.

In the following we instantiate the result on “measurable LTS” like those described e.g. by FlatCCS [4]: a calculus for CCS-like processes living in the real plane. To this end, let \( F = (A, \Sigma_A) \times I. \) The canonical lifting of \( F + I \) is defined, on every morphism \( f : (X, \Sigma_X) \to (Y, \Sigma_Y) \in Kl(\Delta), \) as:

\[
(F + I)f(a, x)(M \times N) = \chi_M(a)f(x)(N).
\]

Unit and multiplication of the canonical monad \((F + I, \nu, \theta)\) are defined, on each measurable space \((X, \Sigma_X)\) by \(\theta_{(X, \Sigma_X)}(x) = \text{inr}(x) = \text{inr} \circ \delta_{(\tau, x)}\) and as

\[
\begin{align*}
\nu_{(X, \Sigma_X)}(a, \tau, x)(M) &= \chi_M(a, x) \\
\nu_{(X, \Sigma_X)}(\tau, a, x)(M) &= \chi_M(a, x) \\
\nu_{(X, \Sigma_X)}(\tau, \tau, x)(M) &= \chi_M(\tau, x) \\
\nu_{(X, \Sigma_X)}(a, b, x)(M) &= 0
\end{align*}
\]

respectively. Composition in \( Kl(F + I) \) follows directly by (3). In particular, for any two morphisms \( f : (X, \Sigma_X) \to (Y, \Sigma_Y) \) and \( g : (Y, \Sigma_Y) \to (Z, \Sigma_Z) \) the composite \((g \cdot f)\) is defined as follows:

\[
(g \cdot f)(x)(M \times N) = \int_{\Delta S(Z, \Sigma_Z)} \psi(M) f(x) \circ (\mu_{S(Z, \Sigma_Z)} \circ \Delta_{(Z, \Sigma_Z)} \circ Sg)^{-1}(\psi)
\]

where \( S = F + I. \)
4 Saturation and behavioural equivalences

In this section we present weak behavioural equivalences via parameterised saturation as a way to compare $T$-coalgebras with unobservables. We insist in the use of the term behavioural instead of bisimulation since our approach is based on kernel bisimulations (a.k.a. cocongruences) [37] and emphasize the role of unobservables which, thanks to the modularity of our approach, are not limited to the usual case of $\tau$-transitions. Henceforth we assume $\mathcal{Kl}(T)$ to be a parameterised saturation category where observables are given a monadic structure handling the unobservable effects as described in Section 3.

4.1 (Weak) behavioural equivalences via parameterised saturation

Many notions of weak bisimulations present some degree of circularity. In fact, how unobservable transitions are combined into one may depend on the equivalence classes induced by the bisimulation itself. For instance, according to Baier and Hermann [5], a weak bisimulation for fully probabilistic systems compares states depending on their probability of reaching (i.e. the minimal execution paths to) each equivalence class of the bisimulation itself. Likewise, Aceto’s resource bisimulation for weighted systems [1] considers maximal executions paths ending inside a given class. This dependence is captured by saturation being parameterised on the cospan underlying a kernel bisimulation, as in the next general definition.

**Definition 3** (PS-behavioural equivalence). Let $\alpha$ and $\beta$ be two $T$-coalgebras. A span $(p,q)$ is a parametrically saturated behavioural equivalence if and only if, it is the pullback of an epic sink $(i,j)$ making the diagram below commute.

\[
\begin{array}{ccc}
X & \xrightarrow{i} & C \\
\downarrow{\alpha} & & \downarrow{\beta} \\
T_X & \xrightarrow{T_i} & T_C \\
\downarrow{T_j} & & \downarrow{T_Y} \\
\end{array}
\]

Notice that $\gamma$ s.t. $\alpha^* \gamma = \gamma \circ i$ may not exist in general. Reworded, not every “quotient” on the state space of a system is stable w.r.t. the saturated system.

**Definition 4** (Homomorphism preservation). Parameterised saturation for $T$-coalgebras is said to preserve homomorphisms iff $\alpha^* \gamma = \gamma \circ f$ for any coalgebra homomorphism $f : \alpha \rightarrow \beta$.

Weak behavioural equivalences are complete with respect to strong ones, under the mild assumption of homomorphism preservation.

**Theorem 21** (Strong bisimulations). If parameterised saturation preserves homomorphisms then every kernel bisimulation is also a weak one.

**Proof.** Let $(p,q) : X \leftrightarrow S \rightarrow Y$ be the kernel bisimulation for two coalgebras $\alpha$ and $\beta$ and let $(i,j) : \alpha \rightarrow \gamma \leftarrow \beta$ the associated cospan. By homomorphism preservation the saturation of this cospan yield a pair of commuting triangles as in (5) and therefore $(p,q)$ is a weak bisimulation. \qed

**Proposition 22.** Let $\zeta$ be the final $T$-coalgebra. For any $T$-coalgebra $\alpha$, let $\text{beh}$ be the coinductive extension of $\alpha$. Then we have: $\alpha^* \text{beh} = \zeta^* \circ \text{beh}$.  

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The homomorphism preservation hypotheses holds for a wide range of settings. In particular, it is met by any $Kl(T)$ whose joins arise from morphism codomains (i.e. from $T$) as compositions with product universal maps and join maps i.e. whenever $(f ∨ g) = (− ∨ −) ∘ (f, g)$ for $(− ∨ −) ∈ C(TX × TX, TX)$. Reworded, whenever binary joins are pointwise.

**Lemma 23.** If $(η ∨ h) ∘ f = f ∨ h ∘ f$ then homomorphisms are preserved by parameterised saturation.

**Lemma 24.** If binary joins are given by composing in $C$ product universal maps with join ones then they right distribute over composition in $C$.

**Double arrow** If composition in $Kl(T)$ is well-behaved with respect to binary joins as stated by Proposition 2 then Definition 3 is precisely a kernel bisimulation (i.e. a cocongruence) over the saturated systems where saturation is actually carried out by the star fix-point operator. In fact, by Proposition 2 and $Kl(T)$ being monadic over the underlying category $C$ we have that

$$α^*_i = i^* ∙ α^* = Ti ∙ α^*$$

rendering the middle triangles in (5) a cospan $(i, j)$ from $α^*$ and $β^*$ to the quotient system $γ$ i.e. a cocongruence.

Therefore, Definition 3 is adequate with respect to Milner’s well-known double arrow construction [27] which defines weak bisimulations as strong ones for systems obtained via the reflexive and transitive closure of $τ$-transitions i.e. systems saturated w.r.t. the star operator as proved in [11,12]. Moreover, the definition coincides with Aczel-Mendler’s coalgebraic bisimulation [2,37] for the saturated systems if $T$ preserves weak pullbacks.

**Proposition 25.** Under the assumptions of Proposition 2 every weak behavioural equivalence via parameterised saturation is a kernel bisimulation for the (star) saturated systems i.e. is adequate w.r.t. Milner’s double arrow construction.

The assumptions of Proposition 2 hold for Kleisli categories of the monads $P(A_τ × Id)$ and $CM(A_τ × Id)$. As a counter example, assumptions of Proposition 2 are not met by $Kl(F_{[0,∞]}(A_τ × Id))$. One can still apply Milner’s double arrow construction and then consider strong bisimulations on the $τ$-closure of fully probabilistic systems, but the equivalences so defined do not coincide with Baier and Hermann’s weak bisimulation.

### 4.2 Examples of weak behavioural equivalences

**Labelled transition systems** As mentioned above, weak bisimulation for LTSs is given via the double arrow construction. By Proposition 25 and [11] we have the following:

**Proposition 26.** For $P(A_τ × Id)$-coalgebras, PS-behavioural equivalence corresponds to Milner’s weak bisimulation for labelled transition systems [27,31].

**Segala systems** Weak bisimulation for $CM(A_τ × Id)$-coalgebras studied in [11] is defined via the double arrow construction and hence adequacy follows by Proposition 25 and [11]:

**Proposition 27.** For Segala systems $α$ and $β$, PS-behavioural equivalence between them corresponds to kernel bisimulation between $α^*$ and $β^*$. 
Weighted transition systems  Weak bisimulation for weighted transition systems was independently studied in [19, 25], covering Baier and Hermann’s weak bisimulation among others. Both works approach the problem by means of recursive equations describing how unobservable transitions are composed. This yields a saturated system akin to the linear equation systems in [5]. These equations depend on the state space partition induced by the weak bisimulation relation under definition.

In our settings, unobservables are hidden inside the arrow composition in $Kl(F_W(A_{τ} \times \mathbb{I}_{d}))$ then, it is easy to see that parameterised saturation instantiates to the aforementioned recursive equations. In fact, for any $α : X \rightarrow F_W(A_{τ} \times X)$ and $h : X \rightarrow Y$ the equation $β = h^τ \lor β \circ α$ defining parameterised saturation expands into the equation

$$β(x)(τ, y) = h^τ(x)(τ, y) ∨ \sum_{z \in X} α(x)(τ, z) \cdot β(z)(τ, y)$$

$$β(x)(a, y) = h^τ(x)(a, y) ∨ \sum_{z \in X} α(x)(τ, z) \cdot β(z)(a, y) + \sum_{z \in X} α(x)(a, z) \cdot β(z)(τ, y)$$

(in the semiring underlying $W$) where $x \in X$, $y \in Y$, and $a \in A$.

Both works (with minor distinctions) define weak bisimulations as kernels for morphisms being the least solution to the above equation(s).

Proposition 28. For WLTSs, PS-behavioural equivalence corresponds to weak bisimulation.

Therefore adequacy of PS-behavioural equivalences w.r.t. WLTS and systems subsumed by them follows directly from the results in [19, 25].

Corollary 29. The following correspondences hold:

(a) For fully probabilistic systems, PS-behavioural equivalence corresponds to (Baier-Hermann’s) weak bisimulation.

(b) For systems weighted over $(\mathbb{N} \cup \{\infty\}, \max, +)$, PS-behavioural equivalence corresponds to resource bisimulation.

Proof. (a) By Proposition 28 and [19, 25]. (b) By Proposition 28 and [19].

Weak (late) behavioural equivalence for the π-calculus  Since Proposition 2 holds true in this setting, Definition 3 instantiates to kernel bisimulation over saturated systems. Saturation instantiates to assignment mapping each late transition system to the least one closed under the following rules.

Here the single and double arrows denote the LTS for the given coalgebra and the saturated one respectively at each stage $n \in \mathbb{N}$, $a, b \in \mathbb{N}$, and every process is at stage $n$ except for $x_{n+1}, y_{n+1}$ which are at stage $n + 1$. 

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**Topological Kripke frames** Let $\alpha$ be a coalgebra for the functor $\mathcal{V}((A, \Sigma_A) \times Id + Id)$ on $\text{cHaus}$. The forgetful functor $U : \text{cHaus} \rightarrow \text{Set}$ extends to a forgetful

$$U : \text{CoAlg}(\mathcal{V}((A, \Sigma_A) \times Id + Id)) \rightarrow \text{CoAlg}(\mathcal{P}(A \times Id + Id)).$$

**Proposition 30.** If $(R, \Sigma_R)$ is a PS-behavioural equivalence for $\alpha$, then $R$ is a PS-behavioural equivalence (hence a weak bisimulation for the LTS described by) $U\alpha$.

**Proof.** Since both $\mathcal{Kl}(\mathcal{V}((A, \Sigma_A) \times Id + Id))$ and $\mathcal{Kl}(\mathcal{P}(A \times Id + Id))$ are a host for Proposition 2 (i.e., $\alpha^* = i \circ \alpha^*$) it suffices to prove that $U\alpha^* = (U\alpha)^*$ where $U$ is extended to $\mathcal{Kl}(\mathcal{V}((A, \Sigma_A) \times Id + Id))$ in the obvious way. Recall that $\alpha^* = \bigvee_{n < \omega} \alpha^n$, the thesis follows by $U$ being continuous in the sense that, for any ascending $\omega$-chain $(f_i)_{i \in \mathbb{N}}$, $U \bigvee_{i < \omega} f_i = \bigvee_{i < \omega} Uf_i$. We conclude by noticing that $U$ forgets the topology of each space and that suprema of ascending $\omega$-chains in both cases are pointwise suprema of ascending $\omega$-chains of compact subsets i.e. countable unions. □

**Measure systems** For simplicity’s sake, let us assume that there are no visible labels, i.e. $F + Id = Id$. Arrows in $\mathcal{Kl}(\Delta)$ can be seen as “measurable relations” (akin to stochastic relations [14]). Then, for any $\alpha : (X, \Sigma) \rightarrow (\Delta(X, \Sigma), \alpha^*)$ exactly the transitive and reflexive closure of the relation as expected. On the other hand, $\alpha^*_h$ is the analogue of Baier and Hermann’s weak bisimulation for continuous state systems.

5 Conclusions and future work

In this paper we have introduced a general theory for the definition of behavioural equivalences for coalgebras with unobservable moves. This framework is based on the notion of “parameterised saturation categories”, which support the definitions of various abstract behavioural equivalences. Remarkably, the Kleisli categories of many monads used in concurrency theory to define behavioural functors, are parameterised saturation categories: as we have showed, these include the powerset monad, the convex combinations monad, the generalized multiset (i.e., weighting) monad, the Vietoris monad, etc. Using this theory, we have provided a general abstract definition of weak behavioural equivalence; we have showed that this notion covers several “weak bisimilarities” defined in literature (for weighted LTS, Segala systems, calculi with names, etc.), just by choosing the corresponding behavioural functor. Notably, this theory applies also to categories different from $\text{Set}$ (e.g., we have considered presheaves and complete Hausdorff spaces).

The benefits of this framework theory are manifold. First, it provides a general, uniform way for defining behavioural equivalences in presence of unobservable moves; this allows us to readily obtain new (weak) behavioural equivalences, even in categories like $\text{Set}^+$ or $\text{Meas}$. Secondly, it is “normative”, in the sense that we can classify behavioural equivalences by looking at the form of their defining equations (e.g., we can tell “double arrow” constructions from Baier-Hermann’s). Finally, this theory singles out the computational aspect of “unobservability” from the definition of system behaviours, thus allowing for a modular development of the theory. In fact, although in this paper we have focused on weak behavioural equivalences, the theory of parameterised saturation categories should allow us to cover a wide spectrum of behavioural equivalences (e.g., delay, dagger, . . . ), just by changing the monadic structure of (un)observable actions and/or the form of the circular parameterised saturation. For instance, in equation (1) we used the join operator $\lor$, but the theory holds for any binary operation monotonic on both its components.

Another interesting elaboration on this theory is to consider Eilenberg-Moore categories of behavioural monads, instead of Kleisli. We expect to be able to capture trace equivalences, along the lines of [21].
Likely, the previous work closest to ours is [19]. In that work a $\text{dCpo}_1$-enrichment is required, which restricts its applicability; in fact, several computational monads do not respect this enrichment. Moreover, the development is done in $\text{Set}$ only. In this paper we have showed that a $\omega$-$\text{Cpo}$-enrichment with binary joins suffices, thus covering a wider range of categories (e.g. presheaf categories, $\text{Meas}$, $\text{Top}$, ... ) and behaviours.

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A Omitted proofs

Proof of Proposition 3. Let $F(x) = \id \lor x \circ h$. We know that $h^* = \bigvee_{n \in \omega} F^n(id)$. Assume $f \lor g \circ h \leq g$. We will now inductively prove that $f \circ F^n(id) \leq g$. Indeed, the inequality holds for $n = 1$ as $f \circ (id \lor id \circ h) = f \circ (id \lor h) = f \lor f \circ h \leq f \lor g \circ h \leq g$. Now assume it holds for $n > 1$ and consider
\[ f \circ F^{n+1}(id) = f \circ (id \lor F^n(id) \circ h) = f \lor f \circ F^n(id) \circ h \leq f \lor g \circ h \leq g. \]
Hence, $f \circ h^* = f \circ \bigvee_n F^n(id) = \bigvee_n f \circ F^n(id) \leq g$. \hfill \Box

Proof of Lemma 8. Recall that any measurable function $f$ from a measurable space $(X, \Sigma_X)$ to $[0, \infty]$ there is a monotonic increasing sequence of non-negative simple functions $(f_n)_{n \in \mathbb{N}}$ such that $f = \lim_{n \to \infty} f_n$. In particular, such a sequence can be obtained by defining each $f_n$ as $\sum_{i=1}^{\lfloor 2^n \rfloor} \chi_{N_{n,i}}$ where $N_{n,i}$ is the $\Sigma_X$-measureable $\{x \in X \mid f(x) \in \left[\frac{i-1}{2^n}, \frac{i}{2^n}\right)\}$ for $i < 2^n$ and $\{x \in X \mid n \leq f(x)\}$ for $i = 2^n^n$.
For any $(X, \Sigma_X)$, $\varphi \in \Delta(X, \Sigma_X)$ and $M \in \Sigma_X$ we have that:
\[
(\mu_{(X,\Sigma_X)} \circ \eta_{(X,\Sigma_X)})(\varphi)(M) = \int_{\Delta(X,\Sigma_X)} e^{v_M} d\eta_{(X,\Sigma_X)}(\varphi) = \sup \left\{ \int_{\Delta(X,\Sigma_X)} f d\eta_{(X,\Sigma_X)}(\varphi) \bigg| 0 \leq f \leq e^{v_M}, f \text{ is simple} \right\} = \lim_{n \to \infty} \sum_{i=1}^{\lfloor 2^n \rfloor} \int_{\Delta(X,\Sigma_X)} \chi_{N_{n,i}}(\varphi) = \varphi(M)
\]
where each $N_{n,i}$ is the $i$-th measurable set of the $n$-th simple function defining $e^{v_M}$ as above. Likewise:
\[
(\mu_{(X,\Sigma_X)} \circ \Delta \eta_{(X,\Sigma_X)})(\varphi)(M) = \int_{\Delta(X,\Sigma_X)} e^{v_M} d\Delta \eta_{(X,\Sigma_X)}(\varphi) = \lim_{n \to \infty} \sum_{i=1}^{\lfloor 2^n \rfloor} \int_{\Delta(X,\Sigma_X)} \chi_{N_{n,i}} \varphi^{-1}(\eta_{(X,\Sigma_X)}(N_{n,i})) = \varphi(M)
\]
This completes the proof of coherence for the monoidal unit.

For any measurable function $f : X \to [0, \infty]$ assume that:
\[ \int_{(X,\Sigma_X)} f d\mu_{(X,\Sigma_X)}(\varphi) = \int_{\Delta(X,\Sigma_X)} \left( \int_{(X,\Sigma_X)} f d\rho \right) \varphi(d\rho). \tag{6} \]

For any measurable space $(X, \Sigma_X)$, measure $\varphi \in \Delta(X, \Sigma_X)$ and measurable set $M \in \Sigma_X$, coherence for the monoidal multiplication instantiates in:
\[
(\mu_{(X,\Sigma_X)} \circ \mu_{\Delta(X,\Sigma_X)})(\varphi)(M) = \int_{\Delta(X,\Sigma_X)} e^{v_M} d\mu_{\Delta(X,\Sigma_X)}(\varphi) = \int_{\Delta^2(X,\Sigma_X)} e^{v_M} d\rho \varphi(d\rho) = \int_{\Delta(X,\Sigma_X)} e^{v_M} \circ \mu_{(X,\Sigma_X)} d\varphi = \int_{\Delta^2(X,\Sigma_X)} e^{v_M} \circ \mu_{(X,\Sigma_X)}^{-1} d\varphi = (\mu_{(X,\Sigma_X)} \circ \Delta \mu_{(X,\Sigma_X)})(\varphi)(M)
\]

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where (7) follows from (6) and (8) from the change of variables theorem.

Finally, we prove (6). The argument follows Doberkat’s proof of the same equality in the case of bounded measurable functions and the Giry monad (cf. [15, Lem 3.2.2]). If \( f \) is the indicator functor \( \chi_M : X \to [0, \infty] \) (for \( M \in \Sigma_X \)) then (6) is precisely \( \mu_{(X, \Sigma_X)} \). By linearity of integral of simple functions (6) holds for any non-negative simple function. Since every non-negative measurable function is the limit of some monotone increasing sequence of non-negative simple functions, (6) holds, by monotone convergence theorem, on any non-negative measurable function completing the proof. \( \Box \)

**Proof of Proposition 9.** The proof is organized as follows: we show that (a) each \( \Delta(X, \Sigma_X) \) has binary joins and is an \( \omega \)-Cpo; (b) each hom-set of \( \mathcal{KL}(\Delta) \) has binary joins and is an \( \omega \)-Cpo; (c) composition is continuous in both components. Therefore \( \mathcal{KL}(\Delta) \) is \( \omega \)-Cpo-enriched, has binary joins, and hence (by Proposition 1) a PS-category.

The join of any \( \varphi, \psi \in \Delta(X, \Sigma_X) \) is given, on \( M \in \Sigma_X \), by:

\[
(\varphi \lor \psi)(M) \triangleq \sup\{\varphi(M_1) + \psi(M_2) \mid M_1 \in \Sigma_X, M_1 \subseteq M, M_2 = M \setminus M_1\}.
\]

In order to verify this claim first we need to show that \( \varphi \lor \psi \) is a measure. Indeed, \( (\varphi \lor \psi)(\emptyset) = 0 \). Consider a countable disjoint family of measurable sets \( M_i \in \Sigma_X \) and let \( M = \bigcup_i M_i \). Assume \( \varphi \lor \psi)(M) < \infty \) and \( \varphi \lor \psi)(M_i) < \infty \). In this case, for any \( \varepsilon > 0 \) there are measurable, disjoint sets \( N_1, N_2 \) s.t. \( M = N_1 \cup N_2 \) and

\[
\varphi \lor \psi(M) - \varepsilon < \varphi(N_1) + \psi(N_2) = \varphi\left(\bigcup_i M_i \cap N_1\right) + \psi\left(\bigcup_i M_i \cap N_2\right) = \sum_i \varphi(M_i \cap N_1) + \psi(M_i \cap N_2) \leq \sum_i (\varphi \lor \psi)(M_i).
\]

The above is true for any \( \varepsilon > 0 \) and hence \( \varphi \lor \psi\left(\bigcup_i M_i\right) \leq \sum_i (\varphi \lor \psi)(M_i) \). To see the inverse inequality is true consider arbitrary \( \varepsilon \) and note that for any \( M_i \) there are measurable, disjoint sets \( N_1^i, N_2^i \) such that \( M_i = N_1^i \cup N_2^i \) we have:

\[
(\varphi \lor \psi)(M_i) - \frac{\varepsilon}{2^i} \leq \varphi(N_1^i) + \psi(N_2^i).
\]

Hence,

\[
\sum_i (\varphi \lor \psi)(M_i) - \varepsilon \leq \sum_i \varphi(N_1^i) + \psi(N_2^i) = \varphi\left(\bigcup_i N_1^i\right) + \psi\left(\bigcup_i N_2^i\right) \leq \varphi \lor \psi(M).
\]

Hence, \( \sum_i (\varphi \lor \psi)(M_i) \leq (\varphi \lor \psi)(M) \). Let \( (\varphi \lor \psi)(M) = \infty \); then for any \( r > 0 \) there are disjoint and measurable sets \( N_1, N_2 \) such that \( M = N_1 \cup N_2 \) and

\[
r \leq \varphi(N_1) + \psi(N_2) = \sum_i \varphi(M_i \cap N_1) + \psi(M_i \cap N_2) \leq \sum_i (\varphi \lor \psi)(M_i).
\]

Hence, \( \sum_i (\varphi \lor \psi)(M_i) = \infty = \varphi \lor \psi(M) \). Similarly we verify the case for \( (\varphi \lor \psi)(M_i) = \infty \) for some \( i \). This proves \( \sigma \)-additivity of \( \psi \lor \varphi \). In order to complete the proof we will show that \( \psi \lor \varphi \) is indeed the supremum of \( \psi \) and \( \varphi \). Consider any measure \( \rho : \Sigma \to [0, \infty] \) such that \( \varphi, \psi \leq \rho \).

For any measurable \( M \) and any measurable and disjoint \( N_1, N_2 \) such that \( M = N_1 \cup N_2 \) we have:

\[
\rho(M) = \rho(N_1 \cup N_2) = \rho(N_1) + \rho(N_2) \geq \varphi(N_1) + \psi(N_2).
\]

Hence, \( \rho(M) \geq (\varphi \lor \psi)(M) \). Therefore, the hom-sets of \( \mathcal{KL}(\Delta) \) admit arbitrary binary joins, which follows by the fact that the order on these hom-sets is the pointwise extension of the order on measures.

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Any increasing \( \omega \)-chain \( (\varphi_i)_{i \in \mathbb{N}} \) in \( \Delta(X, \Sigma_X) \) has a supremum and it is given, on each \( M \in \Sigma_X \), as: \( \bigvee_{i<\omega} \varphi_i(M) \triangleq \lim_{i \to \infty} \varphi_i(M) = \sup \{ \varphi_i(M) \} \). We need to verify that \( \bigvee_{i<\omega} \varphi_i : \Sigma_X \to [0, \infty] \) is indeed a measure. We have \( \bigvee_{i<\omega} \varphi_i(\emptyset) = 0 \) and for a countable family of measurable sets \( M_n \in \Sigma_X \), the following is true:

\[
\bigvee_{i<\omega} \varphi_i \left( \bigcup_n M_n \right) = \sup_i \left\{ \varphi_i \left( \bigcup_n M_n \right) \right\} = \sup_i \left\{ \sum_n \varphi_i(M_n) \right\} = \\
\sup_i \left( \sum_n \varphi_i(M_k) \right) = \sup_i \left( \sum_{k=1}^n \varphi_i(M_k) \right) = \sup_i \left( \sum_{k=1}^n \varphi_i(M_k) \right) = \\
\sum_i \bigvee_{i<\omega} \varphi_i(M_k).
\]

This proves that \( \bigvee_{i<\omega} \varphi_i \) is a measure on \( (X, \Sigma_X) \).

Binary joins of measures are extended to hom-objects of \( KL(\Delta) \) in a pointwise manner: \( (f \vee g)(x) \triangleq f(x) \vee g(x) \) for any \( f \) and \( g \) in \( KL(\Delta)((X, \Sigma_X), (Y, \Sigma_Y)) \) and \( x \in X \). In order to verify the claim we have to show that the \( f \vee g \) is measurable. Since \( \Sigma_{\Delta(Y, \Sigma_Y)} \) is the initial \( \sigma \)-algebra w.r.t. \( \{ev_M \mid M \in \Sigma_X\} \) it suffices to show that \( ev_M \circ (f \vee g) \) is measurable for any \( M \in \Sigma_X \). The join \( (f \vee g) \) is pointwise and defined as the supremum of measurable functions:

\[
(f \vee g)(x)(M) = \sup \left\{ (ev_{M_1} \circ f)(x) + (ev_{M_2} \circ g)(x) \mid M_1 \in \Sigma_X, M_1 \subseteq M, M_2 = M \setminus M_1 \right\}
\]

Hence, the claim follows by showing that the above supremum can be always reformulated into one over a countable set. This set is formed by a suitable family of measurable functions \( ev_{M_1} \circ f + ev_{M_2} \circ g \) parameterised by a countable sequence of suitable pairs \( (M_1, M_2) \). In fact, the supremum is taken over a subset of \([0, \infty]\) and hence can always be define as the limit of some non-decreasing \( \omega \)-chain of its elements.

Suprema for ascending \( \omega \)-chains of measurable functions are defined by pointwise extension of suprema for ascending \( \omega \)-chains of measures i.e. for any ascending \( \omega \)-chain \( (f_i)_{i \in \mathbb{N}} \) in \( KL(\Delta)((X, \Sigma_X), (Y, \Sigma_Y)) \) and \( x \in X \):

\[
\left( \bigvee_{i<\omega} f_i \right)(x) \triangleq \bigvee_{i<\omega} f_i(x) = \sup \{ f_i(x) \}.
\]

Measurability follows by \( \Sigma_{\Delta(X, \Sigma_X)} \) being the initial \( \sigma \)-algebra w.r.t. \( \{ev_M \mid M \in \Sigma_X\} \) and by suprema being given pointwise:

\[
\left( \bigvee_{i<\omega} f_i \right)(x)(M) = \sup \{ f_i(x)(M) \} = \lim_{i \to \infty} f_i(x)(M).
\]

We then show that composition is continuous in both components (i.e. suprema are preserved). Let \( (f_i)_{i \in \mathbb{N}} \) be an ascending \( \omega \)-chain in \( KL(\Delta) \) as above and let \( h : (W, \Sigma_W) \to (X, \Sigma_X) \) and \( k : (Y, \Sigma_Y) \to (Z, \Sigma_Z) \) be arrows in \( KL(\Delta) \). Composition is left continuous: \( \bigvee f_i \circ h = (\bigvee f_i) \circ h \). In fact, the following holds:

\[
(\mu_{Z, \Sigma_Z} \circ \Delta(\bigvee f_i) \circ h)(x)(M) = \int_{\Delta(X, \Sigma_X)} ev_M dh(x) \circ (\bigvee f_i)^{-1} = \\
\int_{\Delta(Y, \Sigma_Y)} ev_M \circ \bigvee f_i dh(x) \circ (\bigvee f_i)^{-1} = \\
\int_{\Delta(Y, \Sigma_Y)} ev_M \circ \bigvee f_i dh(x) = (\bigvee ev_{M \circ f_i} \circ h)(x)(M)
\]

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where (10) follows from the monotonic convergence theorem and (9) from
\[(ev_M \circ \bigvee f_i) (x) = (\bigvee f_i)(x)(M) = \sup \{ f_i(x)(M) \} = (\bigvee ev_M \circ f_i)(x).\]
Likewise \( g \cdot f_i = g \cdot \bigvee f_i \) completing the proof. \( \square \)

**Proof of Theorem 10.** The partial order on \( Kl(S)(X,Y) \) is imposed by the partial order on \( Kl(T)(X,SY) \). Hence, it is clear that the order on hom-sets in \( Kl(S) \) admits binary joins as the order on hom-sets in \( Kl(T) \). Now take an ascending chain \( f_0 \leq f_2 \leq \ldots \) in \( Kl(S)(X,Y) \) and morphisms \( g,h \) in \( Kl(S) \) with suitable domain and codomain. Let \( \bullet \) and \( \cdot \) denote the compositions in \( Kl(S) \) and \( Kl(T) \) respectively. We have:
\[
\begin{align*}
g \bullet \bigvee f_n & = \nu \cdot Sg \cdot \bigvee f_n = \bigvee \nu \cdot Sg \cdot f_n = \bigvee g \cdot f_n, \\
\bigvee f_n \cdot h & = \nu \cdot S \bigvee f_n \cdot h = \bigvee \nu \cdot S f_n \cdot h = \bigvee f_n \cdot h.
\end{align*}
\]
This proves that \( Kl(S) \) is \( \omega \)-Cpo-enriched. By Proposition 1 we conclude that it is a parameterised saturation category. \( \square \)

**Proof of Proposition 12.** It follows by Theorem 10. Indeed, \( Kl(P) \equiv Rel \) is an \( \omega \)-Cpo-enriched category with binary joins. Since the powerset monad \( P \) is commutative, the shapely functor \( S = A_r \times Id \) lifts to \( A_r \times Id : Kl(P) \rightarrow Kl(P) \). Moreover, \( A_r \times Id \) can be equipped with a monadic structure as described in Example 2. Since \( A_r \times Id \) preserves arbitrary joins in \( Kl(P) \) [11] it is locally continuous. This proves that \( Kl(A_r \times Id) = Kl(P(A_r \times Id)) \) is \( \omega \)-Cpo-enriched. Additionally, it is easy to see that hom-sets in \( Kl(P(A_r \times Id)) \) admit binary joins. This proves the assertion. \( \square \)

**Proof of Proposition 13.** The proof of this theorem follows the lines of the proof of Theorem 12 for \( CM \) instead of \( P \). Indeed, by [11] the category \( Kl(CM) \) is \( \omega \)-Cpo-enriched and admits binary joins. The functor \( A_r \times Id \) lifts to an endofunctor \( A_r \times Id \) on \( Kl(CM) \) which can be equipped with a monadic structure following the guidelines of [11]. The functor \( A_r \times Id \) is locally continuous [11]. By Theorem 10 the category \( Kl(A_r \times Id) = Kl(CM(A_r \times Id)) \) is a parameterised saturation category. \( \square \)

**Proof of Lemma 14.** The proof is carried out by structural recursion akin to Hasuo’s proof of the lemma for \( P \) and \( D \) (cf. [21, Lem. 2.6]). Because the ordering is pointwise it suffices to show that \( d \) and \( \pi \) are continuous map between CPOs. Both follow by \( W \) being continuous and suprema being pointwise. \( \square \)

**Proof of Proposition 15.** Zero morphisms are defined pointwise by means of the constantly zero weight function—i.e. the empty (mult)iset. By Theorem 11 \( F + Id \) is a monad on \( Kl(FW) \) and by Lemma 14 its endofunctor is locally continuous. We conclude by Proposition 4 and Theorem 10. \( \square \)

**Proof of Proposition 16.** By Theorem 10 we only need to prove that \( F + Id \) (where \( F = F_1 + F_2 + F_3 = N \times N \times Id + N \times Id^N + N \times \delta \) is locally continuous. The proof is quite straightforward since \( F \) is shapely except for \( Id^N \) and \( \delta \) with the latter being a special case of the former. Recall that suprema of ascending \( \omega \)-chains in \( Kl(P^i) \) are given at each stage \( n \in \mathbb{N} \) as join in \( Kl(P) \). We conclude by noticing that at stage \( n \) \( Id^N \) is shapely:
\[
\begin{align*}
\bigvee_{i<\omega} f_{i,n} = \left( n \cdot \bigvee_{i<\omega} f_{i,n} \right) \times \bigvee_{i<\omega} f_{i,n+1} = \bigvee_{i<\omega} (n \cdot f_{i,n}) \times f_{i,n+1} = \bigvee_{i<\omega} Id^N f_{i,n}.
\end{align*}
\]
for any and ascending ω-chain \((f_i)_{i \in \mathbb{N}}\).

**Proof of Lemma 17.** The proof is carried out by structural recursion akin to Hasuo’s proof of [21, Lem. 2.6]. Because the ordering is pointwise it suffices to show that \(\text{dstr}\) and coprojections are continuous map between CPOs.

Let \((X_i)_{i \in \mathbb{N}}\) and \((Y_i)_{i \in \mathbb{N}}\) be two ascending ω-chains in \(\mathcal{V}(X, \Sigma_X)\) and \(\mathcal{V}(Y, \Sigma_Y)\) respectively. Continuity of \(\text{dstr}_{(X, \Sigma_X), (Y, \Sigma_Y)}(X', Y')\) follows by \(\bigvee_{i<\omega} X_i \times Y_i = (\bigvee_{i<\omega} X_i) \times (\bigvee_{i<\omega} Y_i)\) and by definition of cartesian product of CPOs. Continuity of \(\text{Tin}_i\) follows directly by definition of coproducts in \(\text{cHaus}\).

**Proof of Proposition 18.** Zero morphisms map every element of their domain to the emptyset of their codomain. By Theorem 11 \(\overline{F} + Id\) is a monad on \(\mathcal{Kl}(\mathcal{V})\) and by Lemma 17 its endofunctor is locally continuous. We conclude by Proposition 6 and Theorem 10.

**Proof of Lemma 19.** The proof is carried out by structural recursion akin to Hasuo’s proof of [21, Lem. 2.6]. Because the ordering is pointwise it suffices to show that \(\text{dstr}\) and coprojections are continuous map between CPOs.

Let \((\varphi_i)_{i \in \mathbb{N}}\) and \((\psi_i)_{i \in \mathbb{N}}\) be two ascending ω-chains in \(\Delta(X, \Sigma_X)\) and \(\Delta(Y, \Sigma_Y)\) respectively. Continuity of \(\text{dstr}_{(X, \Sigma_X), (Y, \Sigma_Y)}(\varphi, \psi)\) follows by \(\bigvee_{i<\omega} \varphi_i(M) \psi_i(N) = (\bigvee_{i<\omega} \varphi_i(M)) (\bigvee_{i<\omega} \psi_i(N))\) and by definition of cartesian product of CPOs. Continuity of \(\text{Din}_i\) follows directly by definition of coproducts in \(\text{Meas}\).

**Proof of Proposition 20.** Zero morphisms are defined pointwise by means of the constantly zero measure. By Theorem 11 \(\overline{F} + Id\) is a monad on \(\mathcal{Kl}(\Delta)\) and by Lemma 19 its endofunctor is locally continuous. We conclude by Proposition 9 and Theorem 10.

**Proof of Lemma 23.** Let \(f : \alpha \to \beta\) be a \(T\)-coalgebra map. Consider the equation

\[
\beta^* \circ f = (\eta \vee \beta^* \cdot h) \circ f = \eta_X \circ f \vee (\beta^* \cdot \beta \circ f) = f^X \vee (\beta^* \cdot Tf \circ \alpha) = f^X \vee (\beta^* \circ f \cdot \alpha)
\]

where \(\cdot\) and \(\circ\) denote composition in \(\mathcal{Kl}(T)\) and \(\text{C}\) respectively. Replacing \(x\) for \(\beta^* \circ f\) we get exactly \(x = f^X \vee x \cdot \alpha\).

**Proof of Lemma 24.** Follows by \((- \vee -) \circ (f, g) \circ h = (- \vee -) \circ (f \circ h, g \circ h)\) which holds by the universal property of products.