Classical solution of a sigma model in curved background

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Abstract

We have solved equations of motion of a σ–model in curved background using the fact that the Poisson–Lie T-duality can transform them into the equations in the flat one. For finding solution of the flat model we have used transformation of coordinates that makes the metric constant. The T-duality transform was then explicitly performed.

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1 Introduction

Klimčík and Ševčik in their seminal work [1] described the conditions and procedure for transforming solutions of a σ–model to those of a dual one. Namely, let us assume that the σ–model is defined on a Lie group $G$ on which a covariant second order tensor field $F$ is given. The classical action of the σ–model then is

$$ S_F[\phi] = \int d^2x \partial_+ \phi^\mu F_{\mu\nu}(\phi) \partial_\nu \phi^\nu $$

(1)

where the functions $\phi^\mu : \mathbb{R}^2 \rightarrow \mathbb{R}$, $\mu = 1, 2, \ldots, \dim G$ are obtained by the composition $\phi^\mu = y^\mu \circ \phi$ of a map $\phi : \mathbb{R}^2 \rightarrow G$ and a coordinate map $y : U_g \rightarrow \mathbb{R}^n$, $n = \dim G$ of a neighborhood of an element $\phi(x_+, x_-) = g \in G$. If $F$ satisfies

$$ \mathcal{L}_{v_i}(F)_{\mu\nu} = F_{\mu\kappa} v^\kappa_j \tilde{F}^{jk}_i v^\lambda_\nu F_{\lambda\nu}, \quad i, \mu, \nu = 1, \ldots, \dim G $$

(2)
where \( v_i \) form a basis of left–invariant fields on \( G \) and \( f^i_{jk} \) are structure coefficients of a Lie group \( \tilde{G} \), \( \dim \tilde{G} = \dim G \), then there is a relation between solutions of the equations of motion for \( S_F \) and \( S_{\tilde{F}} \) where \( \tilde{F}_{\mu\nu} \) is a second order tensor field on \( \tilde{G} \).

The relation between the solutions \( \phi(x_+, x_-) \) of the model given by \( F \) and \( \tilde{\phi}(x_+, x_-) \) of the model given by \( \tilde{F} \) is given by two possible decompositions of elements \( d \) of Drinfel’d double

\[
d = g \cdot \tilde{h} = \tilde{g} \cdot h
\]

where \( g, h \in G, \tilde{g}, \tilde{h} \in \tilde{G} \).

The map \( \tilde{h} : \mathbb{R}^2 \to \tilde{G} \) that we need for this duality transform satisfies the equations

\[
((\partial_+ \tilde{h}), \tilde{h}^{-1})_j = -A_{+j} := -v_j^\lambda F_{\lambda\mu}(\phi) \partial_+ \phi^\mu
\]

\[
((\partial_- \tilde{h}), \tilde{h}^{-1})_j = -A_{-j} := \partial_- \phi^\lambda F_{\lambda\mu}(\phi) v_j^\nu
\]

Even though the equations (3–5) define the so called Poisson–Lie T–duality transformation their solution is usually very complicated to use them for finding the solutions. There are three steps in performing the transformation:

1. You must know a solution \( \phi(x_+, x_-) \) of the \( \sigma \)–model given by \( F \).
2. For the given \( \phi(x_+, x_-) \) you must find \( \tilde{h}(x_+, x_-) \) i.e. solve the system of PDE’s (4, 5).
3. For given \( d(x_+, x_-) = \phi(x_+, x_-).\tilde{h}(x_+, x_-) \in D \) you must find the decomposition \( d(x_+, x_-) = \phi(x_+, x_-).h(x_+, x_-) \) where \( \phi(x_+, x_-) \in \tilde{G}, h(x_+, x_-) \in G \).

The goal of this paper is to present an example of a three–dimensional \( \sigma \)–model with nontrivial (i.e. curved) background for which all the three steps can be done so that the \( \sigma \)–model can be explicitly solved by this transformation. The tensor \( \tilde{F} \) of this model is

\[
\tilde{F}_{\mu\nu}(\tilde{y}) = \begin{pmatrix}
\frac{\tilde{y}_1}{\kappa^2 + U \kappa \tilde{y}_1} & -\frac{\tilde{y}_1}{\kappa^2 + U \kappa} & \frac{1}{\kappa} \\
\frac{\tilde{y}_1}{\kappa^2 + U \kappa} & \frac{\tilde{y}_1}{\kappa^2 + U \kappa} & 0 \\
\frac{\tilde{y}_1}{\kappa^2 + U \kappa} & -\frac{\tilde{y}_1}{\kappa^2 + U \kappa} & 0
\end{pmatrix}
\]

where \( U \) and \( \kappa \) are constants. The Gauss curvature of the metric \( \tilde{G}_{\mu\nu}(\tilde{y}) := (\tilde{F}_{\mu\nu}(\tilde{y}) + \tilde{F}_{\nu\mu}(\tilde{y}))/2 \) is

\[
R = \frac{7U^4}{8\kappa (\kappa^2 + U \tilde{y}_1)^2}
\]

so that for \( U \neq 0 \) we have a \( \sigma \)–model in a curved background (and with nontrivial torsion). The equations of motion have the form

\[
\partial_- \partial_+ \phi^\mu + \Gamma^\mu_{\nu\lambda} \partial_- \phi^\nu \partial_+ \phi^\lambda = 0
\]

where

\[
\Gamma^\mu_{\nu\lambda} := \frac{1}{2} \tilde{G}^{\mu\rho}(\tilde{F}_{\rho\lambda,\nu} + \tilde{F}_{\nu\rho,\lambda} - \tilde{F}_{\nu\lambda,\rho}).
\]
2 T-duality of the model

The reason why the above given model can be solved is that it is T-dual to a model with the flat background (Actually it was constructed in this way). It is easy to check that the tensor $\tilde{F}$ satisfies the equations dual to (2)

$$\mathcal{L}_{\tilde{v}}(\tilde{F})_{\mu \nu} = \tilde{F}_{\mu \kappa} v^\kappa_j f^j_{ik} \tilde{v}^k \tilde{F}_{\lambda \nu},$$

for vector fields on $\mathbb{R}^3$ that are left–invariant with respect to the Abelian group structure and $f^j_{ik}$ being structure constants of the Lie algebra given by

$$[T^1, T^2] = 0, \quad [T^2, T^3] = T^1, \quad [T^3, T^1] = 0.$$  (11)

It means that the equations of motion of the $\sigma$–model can be rewritten (see [1], [2]) as equations on the six–dimensional Drinfel’d double $D$ – connected Lie group whose Lie algebra $D$ admits a decomposition

$$D = \tilde{G} + G$$

into two subalgebras that are maximally isotropic with respect to a bilinear, symmetric, nondegenerate, ad–invariant form.

In this case, the subalgebras $\tilde{G}$ and $G$ are the three–dimensional Abelian and the second Bianchi algebra (11) so that the Poisson–Lie T–duality reduces to the nonabelian T-duality. The tensor field $\tilde{F}$ can be obtained as

$$\tilde{F}(\tilde{\phi}) = (E + \tilde{\pi}(\tilde{\phi}))^{-1}$$

where

$$E = \begin{pmatrix} 0 & U & \kappa \\ 0 & \kappa & 0 \\ \kappa & 0 & 0 \end{pmatrix}$$

and the matrix function $\tilde{\pi}(\tilde{\phi})$ follows from the adjoint representation of $\tilde{G}$ on $D$ (see e.g. [3]). Similarly, the tensor field $F$ of the dual $\sigma$–model can be obtained as

$$F(\phi) = e(\phi) E e(\phi)^t,$$  (14)

where the matrix $e(\phi)$ is the vielbein field on the group $G$ corresponding to the second Bianchi algebra (11) and $e(\phi)^t$ is its transpose. From this formula one gets

$$F_{\mu \nu}(\phi^\nu) = \begin{pmatrix} 0 & U & \kappa \\ 0 & \kappa & 0 \\ \kappa & 2 U \phi^2 & 2 \kappa \phi^2 \end{pmatrix}$$

The metric of this model is flat in the sense that its Riemann tensor vanishes.

3 Solution of the curved model

In the following subsections we are going to perform the above given steps of the duality transform between solutions of equations of motion for $S_F$ and $S_{\tilde{F}}$. 

3
3.1 Solution of the flat model

Even though we know that the model given by the tensor (15) is on the flat background it is not easy to find the functions $\phi^\mu(x_+, x_-)$ that solve the equation of motion given by the action $S_F[\phi]$ because the Christoffel symbols are not zero in spite of the fact that the metric is flat. To solve the equation of motion we must express $\phi^\mu$ in terms of coordinates $\xi$ for which the metric become constant. This was done in [4] for even more general forms of flat metrics.

Transformation of coordinates

$$
\phi^1 = \xi_1 - 2 \xi_2 \Omega - \frac{8 \Omega^3}{3} + \frac{U}{4 \kappa}(\xi_2 + 2 \Omega^2)^2
$$
$$
\phi^2 = \xi_2 + 2 \Omega^2
$$
$$
\phi^3 = 2 \Omega - \frac{U}{2 \kappa}(\xi_2 + 2 \Omega^2)
$$

where $\Omega = \xi_3/2 + \xi_2 U/(4 \kappa)$ transform the metric obtained as the symmetric part of (15) to constant

$$
G'(\xi) = \begin{pmatrix}
0 & U/2 & \kappa \\
U/2 & \kappa & 0 \\
\kappa & 0 & 0
\end{pmatrix},
$$

and equations of motion transform to the wave equations so that

$$
\xi_j(x_+, x_-) = W_j(x_+) + Y_j(x_-)
$$

with arbitrary $W_j$ and $Y_j$. Functions $\phi^\mu(x_+, x_-)$ that solve the equations of motion for $S_F[\phi]$ then follow from (16) and (17).

This finishes the first step in obtaining the solution of the $\sigma$–model in the curved background by the duality transform. The second step in the duality transform requires solving the system (15).

3.2 Solution of the system (15)

The coordinates $\tilde{h}_\nu$ in the Abelian group $\tilde{G}$ can be chosen so that the left–hand sides of the equations (15) are just $\partial_\pm \tilde{h}_\nu$. The right–hand sides are

$$
A_+ = \begin{pmatrix}
U \partial_+ \phi^2 + \kappa \partial_+ \phi^3 \\
-\kappa \phi^3 \partial_+ \phi^3 + \kappa \partial_+ \phi^2 - U \phi^3 \partial_+ \phi^2 \\
\kappa \partial_+ \phi^1 + U \phi^2 \partial_+ \phi^2 + 2 \kappa \phi^2 \partial_+ \phi^3
\end{pmatrix}
$$

and

$$
A_- = \begin{pmatrix}
-\kappa \partial_- \phi^3 \\
-U \partial_- \phi^1 - \kappa \partial_- \phi^2 - U \phi^2 \partial_- \phi^3 + \kappa \phi^3 \partial_- \phi^3 \\
-\kappa \partial_- \phi^1 - 2 \kappa \phi^2 \partial_- \phi^3
\end{pmatrix}
$$

and for the solution $\phi^\mu(x_+, x_-)$ found in the previous section they become rather extensive expressions in $W(x_+)$ and $Y(x_-)$. Nevertheless, the equations (15) can be
and by repeated application of this formula we get
implies it can be easily done. We can use the Baker–Campbell–Hausdorff formula that now
To solve it might be rather complicated in general but in this case when the only
yield an equation for \( \tilde{h}_3(x_+ , x_-) \) and the two possible decompositions
As both \( G \) as \( \tilde{G} \) are solvable (even nilpotent) we can write all group elements as
solved and the general solution is
\[
\begin{align*}
\tilde{h}_1(x_+, x_-) &= \kappa (Y_3(x_-) - W_3(x_+)) - U W_2(x_+) - U \Omega^2, \\
\tilde{h}_2(x_+, x_-) &= \kappa (Y_2(x_-) - W_2(x_+)) + U Y_1(x_-) + \frac{U}{2} \beta(x_+, x_-) + \\
&\quad \frac{U}{2} (W_2(x_+) Y_3(x_-) - W_3(x_+) Y_2(x_-)) + \frac{2U}{3} \Omega^3 - \\
&\quad \frac{U^2}{2 \kappa} \left( \frac{1}{2} (W_2(x_+) + Y_2(x_-)) + \Omega^2 \right)^2, \\
\tilde{h}_3(x_+, x_-) &= \kappa (Y_1(x_-) - W_1(x_+)) + \kappa (W_2(x_+) Y_3(x_-) - W_3(x_+) Y_2(x_-)) + \\
&\quad C + \kappa \beta(x_+, x_-) - U \left( \frac{1}{2} (W_2(x_+) + Y_2(x_-)) + \Omega^2 \right)^2
\end{align*}
\]
where \( C \) is a constant,
\[
\Omega = \frac{1}{2} (W_3(x_+) + Y_3(x_-)) + \frac{U}{4 \kappa} (W_2(x_+) + Y_2(x_-))
\]
and the function \( \beta \) solves
\[
\begin{align*}
\partial_+ \beta &= W_3'(x_+) W_3(x_+) - W_3'(x_+) W_2(x_+), \\
\partial_- \beta &= Y_2(x_-) Y_3'(x_-) - Y_3(x_-) Y_2'(x_-).
\end{align*}
\]

### 3.3 Dual decomposition of elements of the Drinfel’d double

The final step in the dual transformation follows from the possibility of rewriting
\[
d(x_+, x_-) = \phi(x_+, x_-) \tilde{h}(x_+, x_-)
\]
as \( \tilde{\phi}(x_+, x_-) h(x_+, x_-) \) where \( \phi(x_+, x_-) \), \( h(x_+, x_-) \in G \), and \( \tilde{\phi}(x_+, x_-) \), \( \tilde{h}(x_+, x_-) \in \tilde{G} \). As both \( G \) and \( \tilde{G} \) are solvable (even nilpotent) we can write all group elements as
product of elements of one–parametric subgroups and the two possible decompositions yield an equation for \( \tilde{\phi}_\mu \) and \( \kappa \) in terms of \( \tilde{h}_\lambda \) and \( \phi^\rho \)
\[
e^{\phi^1 T_1} e^{\phi^2 T_2} e^{3 \phi^3 T_3} e^{\tilde{h}_1 \tilde{T}_1} e^{\tilde{h}_2 \tilde{T}_2} e^{\tilde{h}_3 \tilde{T}_3} = e^{\tilde{\phi}^1 \tilde{T}_1} e^{\tilde{\phi}^2 \tilde{T}_2} e^{\tilde{\phi}^3 \tilde{T}_3} e^{h^1 T_1} e^{h^2 T_2} e^{h^3 T_3}.
\]
To solve it might be rather complicated in general but in this case when the only nonzero Lie products are
\[
[T_2, T_3] = T_1, \ [T_2, \tilde{T}_1] = -\tilde{T}_3, \ [T_3, \tilde{T}_1] = \tilde{T}_2
\]
it can be easily done. We can use the Baker–Campbell–Hausdorff formula that now implies
\[
e^A e^B = e^{B e^A[A,B]}
\]
and by repeated application of this formula we get
\[
\tilde{\phi}_1 = \tilde{h}_1, \ \tilde{\phi}_2 = \tilde{h}_2 + \tilde{h}_1 \phi_3, \ \tilde{\phi}_3 = \tilde{h}_3 - \tilde{h}_1 \phi_2
\]
and $h^\nu = \phi^\nu$, $\nu = 1, 2, 3$. Inserting (16), (17) and (20) into (23) we get the solution of
the equations of motion for the $\sigma$–model given by the action $\tilde{S}$, where $\tilde{F}$ is given by (6).

An example of a simple solution dependent on both $x_+$ and $x_-$ is

\begin{align*}
\tilde{\phi}_1(x_+, x_-) &= -U \sin t \cos x, \\
\tilde{\phi}_2(x_+, x_-) &= -2\kappa \cos t \sin x + \frac{U^2}{2\kappa} \sin^2 t \cos^2 x, \\
\tilde{\phi}_3(x_+, x_-) &= U \sin^2 t \cos^2 x
\end{align*}

(24)

where $t = (x_+ + x_-)/2$, $x = (x_+ - x_-)/2$ or more generally

\begin{align*}
\tilde{\phi}_1(x_+, x_-) &= -\frac{U}{2} (Y_1(x_-) + W_2(x_+)), \\
\tilde{\phi}_2(x_+, x_-) &= \kappa (Y_2(x_-) - W_2(x_+)) + \frac{U^2}{8\kappa} (Y_2(x_-) + W_2(x_+))^2, \\
\tilde{\phi}_3(x_+, x_-) &= \frac{U}{4} (Y_2(x_-) + W_2(x_+))^2
\end{align*}

(25)

obtained from (23) for

$$Y_1 = W_1 = 0, \ Y_3 = \frac{U}{2\kappa} Y_2, \ W_3 = -\frac{U}{2\kappa} W_2, \ W_2, Y_2 \text{ arbitrary.}$$

Another very simple solution is

$$\tilde{\phi}_1(x_+, x_-) = 0, \ \tilde{\phi}_2(x_+, x_-) = U Y_1(x_-), \ \tilde{\phi}_3(x_+, x_-) = \kappa (Y_1(x_-) - W_1(x_+)).$$

(26)

obtained for $Y_2 = W_2 = Y_3 = W_3 = 0$, $W_1, Y_1$ arbitrary.

4 Conclusions

We have explicitly solved the equations of motion of the three–dimensional $\sigma$–model in
the curved background (6) by the Poisson–Lie T-duality transformation. The solution
$\tilde{\phi}(x_+, x_-)$ is given by composition of the formulas (23), (20), (16) and (17). Even
though the transformation is known for more than ten years it is for the first time
when it was used, to the best knowledge of the author, for finding an explicit solution.
The reason may be that performing the three steps of the transformation mentioned
in the Introduction may be rather difficult in general.

To solve the equations of motion we have used the fact that we know several $\sigma$–models in
the curved background that can be transformed to the flat ones (see [3]). We were also able to find the transformation of group coordinates of the flat model to those
for which the metric is constant. In the latter coordinates solution of the flat $\sigma$–model
reduces to the solution of the wave equation. Performability of the next two steps of
the Poisson–Lie T–duality transformation depends critically on the complexity of the
structure of Drinfel’d double where the $\sigma$–models live. In our case one of the subgroup
of the decomposition of the Drinfel’d double was Abelian and the other one nilpotent.
Because of that the systems of equations (4) and (5) separate and the formula (22) for
solution of (21) can be used. More complicated cases are under investigation now.
Let us note that in [5] we have tried to solve the equations of motion for $\sigma$–models on the solvable groups with curved backgrounds by the Inverse scattering method. It turned out that for Lax pairs linear in currents $g^{-1}\partial_\mu g$ the $\sigma$–models solvable by the Inverse scattering method must have constant Christoffel symbols which is not the case of (6).

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