PyR@TE 2
A Python tool for computing RGEs at two-loop.

F. Lyonnet\textsuperscript{a}, I. Schienbein\textsuperscript{b}

\textsuperscript{a}Southern Methodist University, Dallas, TX 75275, USA
\textsuperscript{b}Laboratoire de Physique Subatomique et de Cosmologie, UJF Grenoble 1, CNRS/IN2P3, INPG, 53 Avenue des Martyrs, F-38026 Grenoble, France

Abstract
Renormalization group equations are an essential tool for the description of theories across different energy scales. Even though their expressions at two-loop for an arbitrary gauge field theory have been known for more than thirty years, deriving the full set of equations for a given model by hand is very challenging and prone to errors. To tackle this issue, we have introduced in [1] a Python tool called PyR@TE: Python Renormalization group equations @ Two-loop for Everyone. With PyR@TE, it is easy to implement a given Lagrangian and derive the complete set of two-loop RGEs for all the parameters of the theory. In this paper, we present the new version of this code, PyR@TE 2, which brings many new features and in particular it incorporates kinetic mixing when several U(1) gauge groups are involved. In addition, the group theory part has been greatly improved as we introduced a new Python module dubbed PyLie that deals with all the group theoretical aspects required for the calculation of the RGEs as well as providing very useful model building capabilities. This allows the use of any irreducible representation of the SU($n$), SO(2$n$) and SO(2$n$ + 1) groups. Furthermore, it is now possible to implement terms in the Lagrangian involving fields which can be contracted into gauge singlets in more than one way. As a byproduct, results for a popular model (SM+complex triplet) for which, to our knowledge, the complete set of two-loop RGEs has not been calculated before are presented in this paper. Finally, the two-loop RGEs for the anomalous dimension of the scalar and fermion fields have been implemented as well. It is now possible to export the coupled system of beta functions into a numerical C++ function, leading to a consequent speed up in solving them.

Keywords:  Renormalization group equations, quantum field theory, running coupling constants, model building, physics beyond the Standard Model

1. Introduction

The renormalization group equations (RGEs) for general non-supersymmetric gauge theories have been known at two-loop accuracy for more than 30 years [2, 3, 4, 5, 6, 7]. Due to their importance, these results have been thoroughly scrutinized in the literature and a complete re-evaluation has been performed in [8]. The general RGEs have been implemented in the Python program PyR@TE [1]\textsuperscript{2} which automatically

\textsuperscript{1}Corresponding author: florian.lyonnet@lpsc.in2p3.fr
\textsuperscript{2}The code and various tutorials are publicly available at http://pyrate.hepforge.org.

Preprint submitted to Computer Physics Communications August 26, 2016
generates the full two-loop RGEs for all the (dimensionless and dimensionful) parameters of a general gauge theory once the gauge group and particle content have been specified by the user via simple text files. All known typos in the original series of papers by Machacek and Vaughn have been taken into account (see, e.g., [8] and the appendix of Ref. [9]) and the code has been extensively validated against the literature. In addition, independently of PyR@TE, Mathematica routines were developed and cross-checked against PyR@TE; these routines are now part of SARAH 4 [10].

The code was designed to be easily usable and to provide the user with maximum flexibility for a variety of calculations. For example, one can choose to calculate the β-function of only one of the terms defined in the potential or even neglect a specific contribution to a particular RGE. These options are particularly useful for quick validations when developing a model, or to avoid calculating terms that are time consuming and that will be neglected in the end. The full list of options can be found in Tab. 1 and will be further described below. Once the RGEs have been calculated by PyR@TE the results can be exported to \LaTeX and Mathematica or stored in a Python data structure for further processing. In addition, it is possible to create a numerical Python function that can be used to solve the RGEs, e.g. using the provided Python tool box\footnote{Note that a Mathematica routine to solve the exported RGEs is also provided.}.

Since the first release of PyR@TE \cite{1} three years ago, the code has been improved and extended by several new features described in this article which warrants the release of a new version 2 of PyR@TE:

- PyR@TE 2 is now able to handle semi-simple gauge groups with more than one Abelian gauge factor including the mixing between the associated gauge kinetic fields. The corresponding two-loop β-functions for the gauge couplings were studied in \cite{11, 12} and the general two-loop β-functions for all other dimensionless parameters were derived in \cite{13}. Our implementation employs the method proposed in Ref. \cite{14} where also the two-loop β-functions for the dimensionful parameters can be found. We have taken into account all the corrections at two-loop and compared them to SARAH4 \cite{10} where possible.

- PyR@TE 1 supported SU(\(n\)) groups up to SU(6) as well as U(1) with the possibility to add higher SU(\(n\)) gauge groups upon request. For each one of these groups the information on the irreducible representations (irreps) with dimensions between \(n\) and \(n^2 - 1\) were stored in a database. For PyR@TE 2, a dedicated module, PyLie, was written to deal with all the theory aspects required for the calculation of the RGEs such that no database is required anymore. This allows us to deal with any irrep of the SU(\(n\)), SO(2\(n\)), and SO(2\(n\) + 1) Lie algebras.

- Another important new feature is that PyR@TE 2 allows fields to be contracted into gauge singlets in different ways. As a typical example, there are two possible quartic terms for a SU(2) triplet \(\Delta\),

\[
\text{Tr}\{\Delta^\dagger \Delta \Delta^\dagger \Delta\}, \text{Tr}\{\Delta^\dagger \Delta\} \text{Tr}\{\Delta^\dagger \Delta\},
\]

which can now be consistently treated in PyR@TE 2.

- In addition, the code has been extended in various ways: (i) the two-loop expressions for the anomalous dimensions of fermion and scalar fields \cite{8} are now available in the code, (ii) it is now possible to export the coupled system of beta functions into a numerical C++ function, (iii) finally, some minor bugs, collected since version 1.0.0 (currently version 1.1.9 online), have been fixed.

The rest of this paper is organized as follows: In Section 2 we briefly summarize how to install and use PyR@TE and discuss some of the new options available in PyR@TE 2. Section 3 introduces the new module
PyLie and the model building capabilities. Our implementation of the effects of the gauge kinetic mixing on
the two-loop RGEs of a general gauge field theory is described in Section 4, where we also present, as an
example, a discussion of the Standard Model supplemented by an additional U(1)$_{B-L}$ symmetry. In Section 5
we describe how to handle products of fields which can be contracted to gauge singlets in different ways, and
some additional new features of PyR@TE 2 are introduced in Section 6. Finally, in Section 7 we present
our conclusions. Results for the full set of two-loop $\beta$-functions for two popular models (SM+U(1)$_{B-L}$,
SM+complex triplet) have been relegated to the appendix.

2. First steps with PyR@TE

In this section we provide a brief summary of how to use PyR@TE and discuss some of the new
functionalities/options available in PyR@TE 2. For additional information we refer to [1].

PyR@TE 2 can be downloaded from the web page http://pyrate.hepforge.org where also a
manual and tutorials are provided. To install PyR@TE 2, simply open a shell and type:

1. cd $HOME
2. wget http://pyrate.hepforge.org/downloads/pyrate-2.0.0.tar.gz
3. tar xfvz pyrate-2.0.0.tar.gz
4. cd pyrate-2.0.0/

As an indication, we list the version of the various modules with which PyR@TE 2 was developed. However, the code is likely to work with slightly different versions of these modules with the exception of SymPy:

- Python $\geq$ 2.7.10$^4$ [15]
- NumPy $\geq$ 1.7.1 [16] and SciPy 0.12.0 [17]
- SymPy = 1.0.0 [18]
- IPython $\geq$ 4.0.1 [19]
- PyYAML $\geq$ 3.10 [20]

In order to run PyR@TE the user needs to provide a simple text file, the model file, which specifies the
various elements of the model under consideration such as the gauge group and the particle content.
Each of these model files contains sections such as Groups, Fermions, RealScalars, CplxScalar, Potential,
and so on. For instance, the SM gauge group structure is specified via

1. Groups: {'U1': U1, 'SU2L': SU2, 'SU3c': SU3}

in which a unique identifier must be used for each of the gauge group factors, (SU2L, SU3c, U1). Interac-
tions in the potential are specified in a very similar way where we distinguish QuarticTerms, Yukawas,
ScalarMasses, FermionMasses, TrilinearTerms. For instance, the Higgs quartic term of the SM
Lagrangian, $\mathcal{L} = \frac{\lambda}{2} |H|^4$, would be defined by

$^4$PyR@TE was developed with Python 2.7.10 but should work with more recent/older versions with the exception of Python 3
for which it has not been tested.
QuarticTerms: {
  'λ': {Fields : [H,H*,H,H*], Norm : 1/2}}
}

while the up-quark Yukawa term, \(-L = Y_u \bar{Q} H^c u_R + \text{h.c.}\), reads

Yukwas: {
  'Y_{u}': {Fields: [Qbar,H*,uR], Norm: 1},
}

in which \(H, H^*\) represent the Higgs field and its charge conjugate respectively, \(Q\) the adjoint left-handed quark doublet and \(u_R\) the right-handed quark singlet.

Examples of such model files are shipped with the code and can be found in the 'models' subdirectory of the PyR@TE installation.

Once the model file has been written, PyR@TE is simply run by entering the following command (in the directory where PyR@TE is installed) passing the name of the model file with the '-m' option. For example, for the Standard Model the command would be

```
python pyR@TE.py -m models/SM.model -a
```

Note that the -a flag specifies PyR@TE to calculate the RGEs for all the terms in the potential. By default (no flag specified), PyR@TE does not calculate anything. The most common flags are -gc (gauge couplings), -yu (Yukawas), -qc (quartic couplings), -v (verbose), -tl (calculate at two-loop). The code was designed to provide the user with maximum flexibility with regard to what to calculate. For example, one can select to calculate the \(β\)-function of only one of the terms defined in the potential or even to neglect a specific contribution to a particular RGE. Restricting the calculation to \(Y_{\{u}\}\) is easily done by using the -only/-onl flag

```
python pyR@TE.py -m models/SM.model -yu -onl ['Y_{u}']
```

As another example, the following command line would calculate the RGEs of the gauge couplings, -gc, and the quartic term, -qc, of the SM at one-loop neglecting the contribution \(A_{abcd}\), -Skip/-sk ['CAabcd'],

```
python pyR@TE.py -m SM.model -gc -qc -sk ['CAabcd']
```

These options are particularly useful for quick validations when developing a model or to avoid calculating terms that are time consuming and that will be neglected in the end. The full list of options can be found in Tab. 1. The new options in version two can be found at the bottom of the table starting with the Skip command. The -ipl, -sa, -fa, -gutn, -kin options will be discussed in Section 6.

3. Lie algebra calculations within PyR@TE 2: PyLie

Within PyR@TE, the Clebsch-Gordan Coefficients (CGCs) were obtained from a pre-generated database of which the content was fixed. In order to provide the user with more flexibility, a new Python module dubbed PyLie was developed to deal with all the group calculations. This includes the calculation of the

\(^5\)We refer the reader to [1] for the definition.

\(^6\)Even though several extensions of the original database have been produced to include additional irreps, e.g. 15 of \(SU(3)\).
Table 1: List of all options that can be used to control PyR@TE 2. The bottom part of the table, starting with the Skip option, lists the new options in PyR@TE 2.

| Option | Keyword | Default | Description |
|--------|---------|---------|-------------|
| --Settings/-f | - | - | Specify the name of a .settings file. |
| --Model/-m | Model | - | Specify the name of a .model file. |
| --verbose/-v | verbose | False | Set verbose mode. |
| --VerboseLevel/-vl | VerboselLevel | Critical | Set the verbose level: Info, Debug, Critical |
| --Gauge-Couplings/-gc | Gauge-Couplings | False | Calculate the gauge couplings RGEs. |
| --Quartic-Couplings/-qc | Quartic-Couplings | False | Calculate the quartic couplings RGEs. |
| --Yukawas/-yu | Yukawas | False | Calculate the Yukawa RGEs. |
| --ScalarMass/-sm | ScalarMass | False | Calculate the scalar mass RGEs. |
| --FermionMass/-fm | FermionMass | False | Calculate the fermion mass RGEs. |
| --Trilinear/-tr | Trilinear | False | Calculate the trilinear term RGEs. |
| --All-Contributions/-a | all-Contributions | False | Calculate all the RGEs. |
| --Two-Loop/-tl | Two-Loop | False | Calculate at two-loop order. |
| --Weyl/-w | Weyl | True | The particles are Weyl spinors. |
| --LogFile/-lg | LogFile | True | Produce a log file. |
| --LogLevel/-lv | LogLevel | Info | Set the log level: Info, Debug, Critical |
| --LatexFile/-tex | LatexFile | RGEsOutput.tex | Set the name of the LaTeX output file. |
| --LatexOutput/-texOut | LatexOutput | True | Produce a LaTeX output file. |
| --Results/-res | Results | ./results | Set the directory of the results. |
| --Pickle/-pkl | Pickle | False | Produce a pickle output file. |
| --PickleFile/-pf | PickleFile | RGEsOutput.pickle | Set the name of the pickle output file. |
| --TotxtMathematica/-tm | ToM | False | Produce an output to Mathematica. |
| --TotxtMathFile/-tmf | ToMF | RGEsOutput.txt | Set the name of the Mathematica output file. |
| --Export/-e | Export | False | Produce the numerical output. |
| --Export-File/-ef | ExportFile | BetaFunction.py | File in which the beta functions are written. |
| --Skip/-sk | Skip | - | Set the different terms to neglect in the calculation. |
| --Only/-onl | Only | - | Set the only terms to calculate (the others skipped). |
| --PyLie/-ipl | PyLie | False | Starts the interactive PyLie mode. |
| --ScalarAnomalous/-sa | ScalarAnomalous | False | Calculate the scalar anomalous dimensions RGEs. |
| --FermionAnomalous/-fa | FermionAnomalous | False | Calculate the fermion anomalous dimensions RGEs. |
| --SetGutNorm/-gutn | SetGutNorm | False | Gut normalization: g1 → √3/5g'. |
| --KinMix/-kin | KinMix | False | Ignore kinetic mixing terms. |

Quadratic Casimir, Dynkin index, matrix representations as well as the CGCs. Therefore, PyR@TE 2 is able to handle any irreducible representation of SU(n) as well as SO(2n) and SO(2n + 1). PyLie consists of three main classes that represent Cartan Matrices, the Lie Algebra and the permutation group $S_n$ and is principally a Python translation of the corresponding Mathematica methods of SUSYNO [21]. In this section, we describe how to use this module through PyR@TE in order to obtain the relevant information regarding the calculation of the RGEs as well as for model building.

---

7Note that Lie algebra related routines are also available in form of the Mathematica package LieART[22].
3.1. Interactive PyLie

To initiate PyR@TE in an interactive mode we use the flag -ipl:

```
python pyR@TE.py -ipl
```

This will start a prompt in which the user can request any group theory information by issuing the corresponding command:

- **Invariants** group irreps ⇒ CGCs of the product of the irreps of group,
- **Matrices** group irrep ⇒ matrix representation of the generators of irrep of group,
- **Casimir** group irrep ⇒ quadratic Casimir of irrep of group,
- **Dynkin** group irrep ⇒ Dynkin index of irrep of group,
- **DimR** group irrep ⇒ dimension of irrep of group,
- **GetDynk** group dim ⇒ returns the Dynkin label (of the highest weight) of the irrep of dimension dim\(^8\).
- **AdjR** group ⇒ returns the Dynkin label (of the highest weight) of the adjoint representation of group,
- **FondR** group ⇒ returns the Dynkin label (of the highest weight) of the fundamental representation of group,
- **DimAdj** group ⇒ returns the dimension of the adjoint representation of group.

The information provided via the interactive mode is identical to what PyR@TE uses to calculate the RGEs. Therefore, it is easy to work out the mapping between a given notation and PyR@TE. This is particularly important for the CGCs, obtained via Invariants since there might exist several CGCs for a given combination of fields.

3.2. Model building with PyLie

For convenience, we implemented a couple of SUSYNO methods that are useful for model building which supplement the basic functions used to calculate the RGEs:

- **RepsUpTo** group dim ⇒ returns all the irreps of group that have dimension lower or equal to dim. The output is a list with terms of the form (dimension, Dynkin label). As an example, we can check the representations of SU(3) with dimension lower than 21:

```
RepsUpTo SU3 21
```

```
>> [(1, [0, 0]), (3, [1, 0]), (3, [0, 1]), (6, [0, 2]), (6, [2, 0]),
   (8, [1, 1]), (10, [0, 3]), (10, [3, 0]), (15, [2, 1]), (15, [1, 2]),
   (15, [4, 0]), (15, [0, 4]), (21, [0, 5]), (21, [5, 0])]
```

which as expected contains the 1, 3, 6, 10, 15, 21 and their respective conjugated counterparts as well as the 15\(^7\).

\(^8\)If there is more than one irrep of dimension dim an error message is returned.
• **ReduceProduct** group irreps ⇒ calculates the Clebsch-Gordan decomposition of the tensor product of the irreps of group. The output is a list of terms of the form ((Dynkin label, multiplicity), dimension). For instance, the result for $15 \times 15 \times 15 \times 15$ reads:

```latex
1 \text{ReduceProduct SU3 }[[1,2],[2,1],[1,2],[2,1]]
2 \quad >> \left(([[0, 0], 14), 1), \left([[1, 1], 72), 8), \left([[3, 0], 59), 10), \right)
3 \left([[0, 3], 59), 10), \left([[2, 2], 121), 27), \left([[0, 6], 28), 28), \right)
4 \left([[6, 0], 28), 28), \left([[1, 4], 88), 35), \left([[4, 1], 88), 35), \right)
5 \left([[0, 9], 2), 55), \left([[9, 0], 2), 55), \left([[3, 3], 104), 64), \right)
6 \left([[7, 1], 18), 80), \left([[1, 7], 18), 80), \left([[3, 5], 60), 81), \right)
7 \left([[6, 3], 21), 154), \left([[8, 2], 4), 162), \left([[2, 8], 4), 162), \right)
8 \left([[5, 5], 12), 216), \left([[4, 7], 3), 260), \left([[7, 4], 3), 260), \right)
9 \left([[6, 6], 1), 343)]]
```

As can be seen, the decomposition contains 14 singlets.

• **PermutationOfSymmetryInvs** group irreps ⇒ returns all the information on the permutation symmetries of the invariants. Its output contains two parts: (i) It first lists the groups of indices that mix together. Let $k$ be the number of groups and $n_i$ ($i = 1, \ldots, k$) the number of indices in the $i$th group. (ii) This is followed by a list of the irreps of $S_{n_1} \times S_{n_2} \times \ldots \times S_{n_k}$ under which the invariants transform. Let’s exemplify this with the above product $15 \times 15 \times 15 \times 15$ which contains 14 invariants that can be decomposed as:

```latex
1 \text{PermutationSymmetryOfInvs SU3 }[[1,2],[2,1],[1,2],[2,1]]
2 \quad >> \left([[1, 3], [2, 4]),
3 \left([[1, 1], (1, 1)], 5), \left([[2), (2)], 5),
4 \left([[1, 1], (2)], 2), \left([[2), (1, 1)], 2])
```

The first item, $[[1, 3], [2, 4]$] tells us that the first and third fields are identical as well as the second and fourth i.e., $k = 2, n_1 = 2, n_2 = 2$. Then it shows that the 14 invariants decompose into 4 irreps of $S_2 \times S_2$: (a) The $(1, 1), (1, 1))$ with multiplicity 5 which is the one fully anti-symmetric under the exchange of either the 1st and 3rd or the 2nd and 4th indices. (b) The $(2), (2))$ again with multiplicity 5 which is fully symmetric under the exchange of the indices among the first (second) group respectively. (c) Finally, with multiplicity 2, the representations $(1, 1), (2))$ and $(2), (1, 1))$ with mixed symmetry properties.

• **SnIrrepDim** irrep ⇒ returns the dimension of an $S_n$ irrep. This allows one to check for instance that indeed the $(1, 1)$ and $(2)$ of $S_2$ are of dim 1.

```latex
1 \text{SnIrrepDim }[1,1]
2 \quad >> 1
3 \text{SnIrrepDim }[2]
4 \quad >> 1
```

Finally, note that PyR@TE 2 still has a database which is being updated each time PyR@TE is run either to calculate some RGEs or querying results through PyLie. This allows us to greatly improve the execution time when large representations are used. The interactive mode is exited via the command `exit` or `quit`. 7
4. Kinetic Mixing at two-loop

Multiple U(1) group factors are present in many BSM theories and an appropriate treatment of the mixing of the gauge kinetic terms is indispensable for studying the scale dependence of these theories. Indeed, it has been shown that the effects of kinetic mixing (in the RGEs) can be important [23]. In this section, we summarize the modifications of the RGEs in general gauge theories following the approach in [14] and discuss how these modifications are implemented in PyR@TE 2. We illustrate the effects of kinetic mixing for the SM supplemented by an additional U(1)$_{B-L}$ symmetry.

4.1. Kinetic mixing in a general gauge field theory

The impact of kinetic mixing on the two-loop RGEs of the dimensionless parameters of general gauge field theories has been calculated in [13]. Recently, a new approach has been proposed and applied first to supersymmetric (SUSY) and then to non-SUSY theories [24, 14]. This approach is particularly well suited for implementation in a computer code as there is no need for new parameters compared to the case without kinetic mixing and the required set of modifications to be applied to the RGEs of Machacek and Vaughn is minimal. We now review the main features of this approach.

With $n$ U(1) group factors, the field strength tensors related to different U(1) can mix as expressed by the term in the Lagrangian

$$L_{\text{kin.}} \ni -\frac{1}{4} F_{\mu\nu}^T \xi F^{\mu\nu}. \quad (1)$$

$F_{\mu\nu}$ is a vector of $n$ tensor fields corresponding to the U(1) group factors. Correspondingly, $\xi$ is a $n \times n$ real symmetric matrix introducing $\frac{1}{2} n(n-1)$ extra dynamical parameters (in addition to the $n$ diagonal parameters which are present also if the kinetic mixing is neglected). The approach followed by [13] is to calculate the beta functions for each one of these new parameters as well as the modifications induced to the RGEs of the other parameters.

An equivalent approach [24, 14] is to trade these $\frac{1}{2} n(n-1)$ parameters with effective gauge couplings. These new gauge couplings populate the off diagonal terms of an extended gauge coupling matrix

$$G \equiv \tilde{G} \xi^{-1/2}, \quad (2)$$

with $\tilde{G}$ the original diagonal gauge coupling matrix. This is easily seen by performing a redefinition of the vector fields $V^\prime = \xi^{1/2} V$, and we refer the reader to [24, 14] for more details. The replacement rules are then derived by substituting the polynomials involving the gauge couplings by the adequate matrix structure.

There are a couple of key points/features that make this approach interesting: (i) there is no new parameter for which one needs to calculate the beta function, (ii) the list of replacement rules is short, and (iii) the rules are simpler to apply as they take a simple matrix form.

As an illustration, we show here the results for the replacement rules involving the scalar generators, $\Theta^a$. We denote $C_A(R)(S_A(R))$ the quadratic Casimir (Dynkin index) of the representation $R$ of the gauge group $A$ and $C(\mathcal{G})$ the quadratic Casimir of the adjoint representation of the group $\mathcal{G}$. Correcting a typo in Eq. (23) of

---

9 However, the gauge couplings must be promoted to a non-diagonal matrix of effective gauge couplings.
where SU(2) in the literature and in particular the stability of the electroweak vacuum has been analysed [25, 26, 27].

χ right-handed neutrinos and an extra complex scalar the baryon and lepton number, respectively. In addition to the SM particle spectrum we also consider three

$(1)$

4.3. SM-U

matrix multiplications.

specified by the user. Note however, that the non-diagonal entries of the gauge matrix defined in Eq. (2) are switched on when several U(1) group factors are present in the model file. To allow the user to ignore these

4.2. Kinetic Mixing in PyR@TE

components of an eigenstate of the U(1) gauge interactions, respectively, and zero otherwise.

Finally, in the above equation we made use of $W^R \equiv G^T Q^R$ with $Q^R$ the column vector of charges of a given field and the antisymmetric tensor \( \delta_{ab} \) equal to the imaginary unit \( i \) if 'a' and 'b' are the real and imaginary components of an eigenstate of the U(1) gauge interactions, respectively, and zero otherwise.

4.2. Kinetic Mixing in PyR@TE

We implemented the complete set of replacement rules [24] in PyR@TE 2 such that all the effects of the kinetic mixing are consistently taken into account at two-loop. These modifications are automatically switched on when several U(1) group factors are present in the model file. To allow the user to ignore these terms we implemented the "-KinMix/-kin" switch.

Once the RGEs have been computed, PyR@TE exports the results as usual into the different formats specified by the user. Note however, that the non-diagonal entries of the gauge matrix defined in Eq. (2) are denoted by \( g_{ij} \). The difference in speed between taking into account the kinetic mixing terms and neglecting them has been found to be minimal as for most of the terms the additional contributions are obtained by matrix multiplications.

4.3. SM-U(1)$_{B-L}$ example

As an example, we apply our implementation to the SM-U(1)$_{B-L}$ where the SM gauge group SU(3)$_c \times$ SU(2)$_L \times$ U(1)$_Y$ is supplemented by an additional U(1)$_{B-L}$ group factor with $B - L$ charge where $B$ and $L$ are the baryon and lepton number, respectively. In addition to the SM particle spectrum we also consider three right-handed neutrinos and an extra complex scalar $\chi \sim (1, 1, 0, -2)$. This setup has been extensively studied in the literature and in particular the stability of the electroweak vacuum has been analysed [25, 26, 27].

As explained above, we work with the full 2×2 gauge coupling matrix

\[
G \equiv \begin{pmatrix}
g_{YY} & g_{YB} \\
g_{BY} & g_{BB}
\end{pmatrix}.
\]
In this basis, the covariant derivative for a field \( \phi \) charged under the two Abelian group factors reads

\[
D_\mu = \partial_\mu - i\bar{Q}_\phi^\dagger G A_\mu ,
\]

with the column vector of vector bosons \( A_\mu = \left( \begin{array}{c} A_{\mu}^Y \\ A_{\mu}^Z \\ \end{array} \right) \).

Because the matrix \( \xi \) contains \( \frac{1}{2}n(n - 1) = 1 \) mixing parameter, by extension the matrix \( G \) only has three independent parameters. Therefore, it is usual to work in the triangular basis in which these degrees of freedom are apparent. The two basis are related by an orthogonal rotation \( G' = GO_R^T = \left( \begin{array}{c} g & \tilde{g} \\ 0 & g' \end{array} \right) \). Hence, to obtain the RGEs of \( (g, \tilde{g}, g') \) from \( (g_{YY}, g_{BY}, g_{BB}) \) one must

1. first compute the derivative of the product \( GO_R^T \), \( d(GO_R^T)/dt = \left( \begin{array}{cc} \beta_g & \beta_{\tilde{g}} \\ 0 & \beta_{g'} \end{array} \right) = (dG/dt)O_R^T + G(dO_R^T/dt) \),

2. substitute the derivatives of the couplings, \( dg_{YY}/dt \equiv \beta_{g_{YY}}, dg_{BY}/dt \equiv \beta_{g_{BY}}, dg_{BB}/dt \equiv \beta_{g_{BB}} \), by their expressions as given by PyR@TE and finally,

3. express the result in terms of \( g, \tilde{g}, g' \) using \( G = G' O_R \).

Recently, the two-loop RGEs in this basis have been obtained in [25] in the limit \( Y_e \approx 0 \) against which we validated our results. In addition, we provide the beta functions for the scalar mass terms. For the discussion we show only the one-loop expression for all the couplings other than the gauge couplings and relegate the two-loop expressions to Appendix A.

With two scalar fields, the effective potential can be written in the following form [25]

\[
V = \lambda_1 H^\dagger H H^\dagger H + \lambda_2 \chi^\dagger \chi \chi^\dagger \chi + \lambda_3 (H^\dagger H)(\chi^\dagger \chi) + \mu H^\dagger H + \mu_3 \chi^\dagger \chi ,
\]

and additional Yukawa terms have to be added to the Yukawa sector of the SM Lagrangian

\[
-\mathcal{L}_{Yuk} = -\mathcal{L}_{SM} + Y_{e} L H^\dagger v_R + Y_N \bar{N}_R^c v_R \chi + h.c. .
\]

Note that this model implements a type-I seesaw scenario for the three light neutrinos [25] with a Majorana mass term, \( M_{\nu_R}^2 v_R \), dynamically generated by the vacuum expectation value of the \( \chi \) field. For the sake of comparison with results in the literature we present results in which we neglect \( Y_{e} \) and \( Y_{\nu} \) and retain only the Yukawa coupling of the top quark, \( Y_t \). However, we keep the full dependence on \( Y_e \). Finally, we assume the Yukawa couplings of the heavy neutrinos to be universal, i.e., \( Y_{iN} = \delta^{ij} Y_N \). However, the full results are easily obtained running PyR@TE 2. In addition, a detailed study of the numerical impact of the kinetic mixing in this model is presented in [28]. Note that all the results for an arbitrary coupling \( x \) are given in the form

\[
\beta_x = \frac{1}{(4\pi)^2} \beta_x^{(1)} + \frac{1}{(4\pi)^2} \beta_x^{(2)} .
\]

Gauge Couplings

At one-loop the beta functions for the non-Abelian gauge couplings \((g_{2}(SU(2)), g_{3}(SU(3)))\) do not get modified by kinetic mixing and read

\[
\beta^{(1)}_{g_{2}} \equiv \frac{d}{dt} g_{2} = -\frac{19}{6} g_{2}^{3} ,
\]

\[
\beta^{(1)}_{g_{3}} \equiv \frac{d}{dt} g_{3} = -7 g_{3}^{3} .
\]
The Abelian part involves the three couplings $g', g, \tilde{g}$ and their products

$$\beta_s^{(1)} = \frac{d}{dt} g = \frac{41 g^3}{6}, \quad \beta_s^{(2)} = \frac{d}{dt} g = \frac{41 g^2 g'}{3} + \frac{32}{3} g^2 g' + 12 g (g')^2 + \frac{41 g^3}{6} + \frac{32 g^2 g'}{3}, \quad \beta_{g'}^{(1)} = \frac{d}{dt} g' = \frac{32}{3} g g' + \frac{41}{6} g^2 g' + 12 g^3.$$

At two-loop, the non-Abelian beta functions get an additional contribution coming from Eq. (16) of [14] leading to

$$\beta_{s_2}^{(2)} = 4 g_s^3 g' + \frac{3}{2} g_s^3 g' + \frac{3}{2} g_s^2 g' + 4 g_s^2 (g')^2 - \frac{3}{2} g_s^2 Y_t^2 + \frac{35 g_s^5}{6} + 12 g_s^3 g', \quad \beta_{s_3}^{(2)} = \frac{4}{3} g_s^3 g' + \frac{11}{6} g_s^2 g' + \frac{11}{6} g_s^2 g' + \frac{4}{3} g_s^3 (g')^2 - 2 g_s^3 Y_t^2 - 26 g_s^3 + \frac{9}{2} g_s^3 g', \quad \beta_{g'}^{(2)} = \frac{1}{18} g' \left( 328 g g' + 199 g^2 + 199 g^2 + 184 (g')^2 + 81 g_s^2 + 264 g_s^2 - 51 Y_t^2 - 9 Y_t^2 \right) + 4 g_s^2 g' \left( 328 g_s^2 + 224 (g')^2 + 54 g_s^2 + 48 g_s^2 - 15 Y_t^2 - 9 Y_t^2 \right) + 2 g' \left( 199 g_s^4 - 51 g_s^2 Y_t^2 - 9 g_s^2 Y_t^2 - 12 (g')^2 Y_t^2 - 36 (g')^2 Y_t^2 - 36 Y_t^2 (g')^2 + 800 (g')^4 + 64 g_s^2 (g')^2 + 27 g_s^2 (3 g_s^2 + 4 (g')^2) + 24 g_s^2 \left( 11 g_s^2 + 4 (g')^2 \right) \right) + 199 g_s^5 + 4 g_s^2 g' \left( 82 g_s^2 + 112 (g')^2 + 54 g_s^2 + 48 g_s^2 - 15 Y_t^2 - 9 Y_t^2 \right),$$

where, after detailed comparison, we now find complete agreement with [25] for the terms involving $Y_t$ and $Y_N^t$. Note that our results also agree with the ones from SARAH [10].

**Yukawa couplings**

The one-loop beta functions for the Yukawa couplings are given by

$$\beta_{Y_i}^{(1)} = \frac{1}{12} Y_i \left( -20 g g' - 17 g^2 - 17 g^2 - 8 (g')^2 - 27 g_s^2 + 96 g_s^2 + 54 Y_t^2 + 12 \text{Tr} \left( Y_i Y_i \right) \right), \quad \beta_{Y_N}^{(1)} = Y_N \left( -6 (g')^2 + 10 Y_N^2 + Y_N Y_T + Y_N Y_T \right),$$

$$\beta_{Y_T}^{(1)} = \frac{1}{4} Y_T \left( -20 g g' - 3 g^2 - 3 g^2 - 24 (g')^2 - 9 g_s^2 + 8 Y_N^2 + 12 Y_T^2 + 4 \text{Tr} \left( Y_T Y_T \right) \right) \frac{3}{2} Y_T Y_T Y_T.$$

Our expressions in Eq. (26), Eq. (27) and Eq. (28) agree with the results found in [25, 26][11] and SARAH.

---

[10] Note that the authors of [25] updated their original result accordingly in a new version of their article.

[11] The calculation of [26] is at one-loop and we therefore perform the comparison with their results in this limit.
**Quartic couplings**

Our SM-U(1)$_{B-L}$ model has three quartic couplings for which we find the following beta functions:

\[
\beta_{\lambda_1}^{(1)} = \frac{3}{4} g^3 \left( g^2 + g_1^2 - 4 \lambda_1 \right) + \frac{3g^4}{8} + \frac{3g^4}{8} - 3g^2 \lambda_1 + \frac{3}{4} g_2^2 \left( g^2 - 12 \lambda_1 \right) + \frac{9g_2^4}{8} + 24\lambda_1^2 + \lambda_3^2 + 12\lambda_1 Y_N^2 - 6Y_i^2 + 4\lambda_1 \text{Tr} \left( Y_i Y_f \right) - 2\text{Tr} \left( Y_i Y_f Y_i^c Y_i^c \right), \\
\beta_{\lambda_2}^{(1)} = 24\lambda_2 \left( Y_N^2 - 2 \left( g' \right)^2 \right) + 96 \left( g' \right)^4 + 20\lambda_2^2 + 2\lambda_3^2 - 48Y_N^4, \\
\beta_{\lambda_3}^{(1)} = \frac{1}{2} \left( 24g^2 \left( g' \right)^2 + \lambda_3 \left( -3g^2 - 3g^2 - 48 \left( g' \right)^2 - 9g_2^2 + 24\lambda_1 + 16\lambda_2 + 12Y^2 + 24Y_N^2 \right) \right) + 8\lambda_3^2 + \text{Tr} \left( Y_f Y_f^c \right) \left( 4\lambda_3 - 32Y_N^2 \right). 
\]

Again we have compared our results with the literature where possible. In the limit of no kinetic mixing, our results agree with SARAH. Taking the kinetic mixing into account, there are some differences in the terms of order $O(g^4)$ involving at least one Abelian coupling. This is easily traced back to the fact that Eq. (4) has not been implemented in SARAH. We find perfect agreement with [25] limiting our results to their theoretical assumptions stated above.

**Scalar mass terms**

Finally, we provide our results for the scalar mass beta functions at one-loop:

\[
\beta_{\mu_1}^{(1)} = \frac{1}{2} \left( -3\mu g^2 - 3g^2 \mu - 9g_2^2 \mu + 4\lambda_3 \mu_1 + 24\lambda_1 \mu_1 + 12\mu Y^2 + 4\mu \text{Tr} \left( Y_i Y_i^c \right) \right), \\
\beta_{\mu_2}^{(1)} = 4 \left( \mu_2 \left( -6 \left( g' \right)^2 + 2\lambda_2 + 3Y_N^2 \right) + \lambda_3 \mu \right). 
\]

The two-loop expressions for the Yukawa couplings, the quartic couplings, and the scalar mass terms can be found in Appendix A. It is important to note that perfect agreement with [25] was obtained for the provided parameters. In addition to these parameters, we also provide the RGEs for $Y_f$, $\mu$ and $\mu_2$.

### 5. Multiple Invariants

This section describes another addition to the code made recently [29] and merged to PyR@TE 2: the possibility to deal with multiple gauge singlets. The main motivation is to be able to automatically calculate the RGEs in situations where products of fields are present that allow for multiple ways to contract them into gauge singlets (see Section 3.2). Indeed, as already mentioned above, each term in the Lagrangian is specified in the model file by a set of fields. This makes it impossible to distinguish multiple singlets that could result from the contraction of the same set of fields. In PyR@TE 1, the chosen combination was the one in which the quartic term can be factorized in

\[
\frac{(a \otimes b) \otimes (c \otimes d)}{1} 
\]

where $a$, $b$, $c$, $d$ are the representations of the scalar fields involved. In general, assuming a set of $k$ fields $\phi_i$, where each $\phi_i$ belongs to a $D_i$-dimensional irrep of an SU(n) gauge group, we will denote the Clebsch-Gordan coefficients that give the contraction of indices to an invariant combination as $C$, i.e.

\[
C_{i_1i_2...i_k} \phi_{i_1} \phi_{i_2} ... \phi_{i_k}, 
\]
does not transform under SU(n). Here, the \( i_x (x = 1, \ldots, k) \) are the charge indices with respect to the gauge group. This section describes how to implement terms that have two or more such invariants \((C^1, C^2, \ldots)\) in PyR@TE 2.

A typical example is that of a complex triplet of SU(2). Writing the triplet as a two-by-two matrix \( \Delta \), two quartic invariants can be formed which are usually written in the literature as

\[
\text{Tr}(\Delta^\dagger \Delta^\dagger \Delta^\dagger \Delta), \quad \text{Tr}(\Delta^\dagger \Delta)\text{Tr}(\Delta^\dagger \Delta). \quad (38)
\]

When there are several invariant contractions in a product of representations, \( \{C^1, C^2, \ldots, C^m\} \), any linear combination of the \( C_i \) is also an invariant. More specifically, we can trade the set \( \{C^1, C^2, \ldots, C^m\} \) for \( \{\sum_{j=1}^m A_{ij} C_j\}\), where \( A_{ij} \) is an arbitrary non-singular \( m \times m \) matrix. Even if one requires a specific normalization for the CGCs, one is left with an immense number of possibilities. It is therefore reasonable to use this freedom and write the invariants in an advantageous form. Assuming we have the product of \( k \) identical fields, the invariants will be in representations of \( S_k \) which one would like to be irreducible.

Following [21] the strategy of PyLie is to segregate the invariants in terms of irreps of \( S_k \).

As an example, it is common knowledge that the invariant of the product of two doublets of SU(2) is in the anti-symmetric irrep of \( S_2 \). Going forward, let us now consider the product \( 3 \otimes 3 \otimes 3 \otimes 3 \) of four representations of SU(2). In the general case where all four fields are different there exist three different invariants in \( S_2 \times S_2 \) that are segregated in the following way:

(i) two are in the \( (\{2\}, \{2\}) \) of \( S_2 \times S_2 \), i.e. they are completely symmetric under the exchange of the first two fields or the last two,

(ii) and one is in the \( (\{1\}, \{1\}) \) which is anti-symmetric under the exchange of the first two or last two fields.

To distinguish the various invariants in the model file we introduce a new keyword, CGCs.

5.1. The CGCs keyword

The user has the possibility to specify which one of the CGCs to use for each of the Lagrangian terms via a new keyword, CGCs. As an example, assuming that the invariant defined by Eq. (38) corresponds to the second CGC returned by PyR@TE, \( C^2 \), one would specify this term via

```python
QuarticTerms: {
    \'\lambda\': {Fields : [T*,T*,T*,T], Norm : 1, CGCs: {SU2L: [2]}}
}
```

Note that the argument of the CGCs keyword must be a list to accommodate the possibility that Eq. (38) is given by a linear combination of the two invariants of PyR@TE. An example is given below in Section 5.2. If the CGCs keyword is not specified then PyR@TE 2 returns the first one.

5.2. A toy model: SM+complex triplet

As an example, let us consider the SM extended by a triplet of SU(2) \( L \) of complex scalars, \( T \equiv (\Delta^+\Delta^+\Delta^0) \). With the two scalar fields \( H \sim (2, 1/2) \) and \( T \sim (3, 1) \) (transformed into a \( 2 \times 2 \) matrix \( \Delta = \sigma_iT_i \) where \( \sigma_{1,2,3} \) are the Pauli matrices) we write the following potential

\[
V = \lambda_1 H^\dagger HH^\dagger H + \lambda_2 \text{Tr}(\Delta^\dagger \Delta)H^\dagger H + \lambda_3 H^\dagger \Delta \Delta^\dagger H + \lambda_{\Delta_1} \text{Tr}(\Delta^\dagger \Delta)\text{Tr}(\Delta^\dagger \Delta) + \lambda_{\Delta_2} \text{Tr}(\Delta^\dagger \Delta^\dagger \Delta). \quad (40)
\]

As explained in Section 3, one obtains the list of CGCs implemented in PyR@TE for the contraction of four arbitrary triplet fields of SU(2) \( (a, b, c, d) \) by using PyLie. For instance,
Invariants SU2 $[[2,True],[2],[2,True],[2]]$

leads to three invariants$^{12}$, $C^{1,2,3}$. Once the substitutions $a, c \rightarrow T^i$ and $b, d \rightarrow T$ have been performed, $C^1$ vanishes and one is left with two invariants. Working out the details of the matching between these invariants and the ones in Eq. (38), Eq. (39) one arrives at the following relations for the CGCs of the $\lambda_{\Delta_1}$ and $\lambda_{\Delta_2}$ term, denoted $C_{\lambda_{\Delta_1}}$ and $C_{\lambda_{\Delta_2}}$, respectively:

$$C_{\lambda_{\Delta_1}} \rightarrow \frac{3}{7} C^2 + \frac{1}{14} C^3, \quad C_{\lambda_{\Delta_2}} \rightarrow \frac{1}{2} C^2.$$  \hspace{1cm} (41)

Similar relations are obtained for the other CGCs and then inserted in the model file via the keyword CGCs. The complete scalar potential entry of the model file then reads

```
QuarticTerms: {
    'lambda_1': {Fields: [H*,H*,H*,H], Norm: 1, CGCs: {SU2L: [1]}},
    'lambda_2': {Fields: [T*,T,H*,H], Norm: 1, CGCs: {SU2L: [1]}},
    'lambda_3': {Fields: [[H*,T,T*,H], [H*,T,T*,H]], Norm: [1/2, 1/sqrt(2)],
                  CGCs: {SU2L: [1,2]}},
    'lambda_{\Delta_1}': {Fields: [[T*,T,T*,T], [T*,T,T*,T]], Norm: [3/7, 1/14],
                          CGCs: {SU2L: [2,3]}},
    'lambda_{\Delta_2}': {Fields: [T*,T,T*,T], Norm: [1/2],
                        CGCs: {SU2L: [2]}}
}
```

Note that line 6 of the above snippet exemplifies how to enter terms that are linear combinations of PyR@TE CGCs. The complete set of RGEs is given in Appendix B.

6. Additional new features

In addition to the above described new capabilities, PyR@TE 2 benefits from all the modifications made to the code since the first release. Furthermore, we implemented the anomalous dimensions at two-loop for the scalar and fermion fields given in [8]. Finally, the numerical output has been optimized and it is now possible to solve the RGEs directly in C++ leading to a significant improvement in speed.

6.1. Scalar and Fermion Fields anomalous dimension at two-loop

The two-loop RGEs for the scalar and fermion field anomalous dimensions are expressed in terms of the quadratic Casimir operator $C_2$ and the Dynkin index $S_2$ of the gauge group which can be related to the generators $t^A$ for fermions and $\Theta^A$ for scalars

$$C_2^{ab}(S) = \Theta^A_{ac} \Theta^A_{cb}, \quad S_2(S)\delta_{AB} = \text{Tr}(\Theta^A \Theta^B),$$

$$C_2^{ab}(F) = t^A_{ac} t^A_{cb}, \quad S_2(F)\delta_{AB} = \text{Tr}(t^A t^B).$$ \hspace{1cm} (42, 43)

We also use the Dynkin index summed over all the states present in the model, $\bar{S}_{2,k}$ as defined in [1]. With these definitions, the anomalous dimensions for a single gauge group $G_k(g)$ read at two-loop [8]

$$\gamma^{i}_{ab} = \gamma^{S,i}_{ab} + \gamma^{s,HI}_{ab}, \quad \gamma^{F}_{ij} = \gamma^{F,J}_{ij} + \gamma^{F,HI}_{ij}.$$ \hspace{1cm} (44)

$^{12}$Note that $'[2]'$ is the Dynkin label of the 3 of SU(2).
Three new switches are also making their appearance in PyR@TE 2: (i) -Skip/sk, (ii) -Only/onl and (iii) -SetGutNorm/-gutn. As mentioned in Section 2, the user now has the possibility to neglect...
Table 2: Time to solve the coupled system of $\beta$-functions at 393 different points in log ($t$) for the SM + complex triplet at one- and two-loop using the two different numerical routines.

|       | NumPy | C++  |
|-------|-------|------|
| one-loop | 431   | 48.8 |
| two-loop | 2230  | 58.1 |

some of the terms in the calculation of the RGEs using (i). This is particularly useful when developing a new model or when one wants to neglect tiny terms at the generation level and speed up the calculation of the RGEs. The list of terms that can be neglected this way is based on the grouping usually done in the literature, e.g. [8]. For instance, the one-loop $\beta$-function for the quartic terms read

$$\beta^I_{abcd} = \Lambda^2_{abcd} - 8\kappa H_{abcd} + 2\kappa \Lambda^Y_{abcd} - 3g^2 \Lambda^S_{abcd} + 3g^4 A_{abcd}.$$  

Neglecting the terms proportional to the gauge couplings $\Lambda^S_{abcd}$, $A_{abcd}$ is achieved by passing the flag

```
-sk ['CLSabcd','CAabcd']
```

Note the capital C in front of the name of the invariant. Note also that all the possible terms are being printed on the screen when running the calculation with the verbose mode -v.

The -only/-onl option has a slightly different scope. It allows the user to calculate the RGEs for only some of the terms defined in the model file. The rest of the Lagrangian is still taken into account for the calculation. The syntax is very similar and one just passes the name of the couplings to include, e.g.

```
-onl ['\lambda_1','\lambda_2']
```

Finally, we introduced the possibility to replace the U(1) coupling ($g_1$) by the coupling $g'$ which has the standard SU(5) normalization, i.e., $g_1 = \sqrt{3/5}g'$. All the RGEs are then expressed in terms of $g'$ instead of $g_1$. This is enabled by passing -SetGutNorm/-gutn.

### 6.3. Efficient numerical solving of the RGEs

PyR@TE has also been extended by adding a numerical C++ output. The set of coupled differential equations is now exported to C++ in addition to the already existing Mathematica and Python outputs. The C++ code can then be compiled via the provided Makefile into a shared library and solved in Python using the routines which have already been provided for this purpose. This leads to a drastic improvement in speed for solving the RGEs. Tab. 2 reports the different times of execution at one- and two-loop. These numbers are obtained for the SM+complex triplet model detailed above and solved for 393 points in energy scale between $M_Z$ and $10^{19}$ GeV. As one can see, the C++ routine is a factor of almost 9 faster than the corresponding NumPy one. At two-loop, the difference is much bigger reaching a factor of 38. However, the time of execution grows linearly for both routines with the number of points in log ($t$) required.

Note that as in PyR@TE, an example file called SolveRGEsCpp.py is produced with each run and illustrates how to solve the system of $\beta$-functions using the C++ routines. The C++ routines rely on the Armadillo [30] library for the matrix computation that must be installed independently of PyR@TE in order to use this functionality.
7. Conclusion

Renormalization group equations are a key ingredient to extrapolate theories to different energy scales. Three years ago, we released a Python code, PyR@TE, which automatically generates the full two-loop RGEs for all the dimensionless and dimensionful parameters of a general gauge theory. In this article, we have presented the new features implemented in PyR@TE 2. Most importantly, PyR@TE 2 now supports the kinetic mixing when several U(1) gauge groups are involved. In addition, a new Python module dubbed PyLie has been introduced that deals with all the group theoretical aspects required for the calculation of the RGEs and provides several functions useful for model building. In particular, any irreducible representation of the SU(n), SO(2n) and SO(2n + 1) groups is now supported. Furthermore, it is now possible to handle combinations of fields which can be contracted into gauge singlets in multiple ways. Finally, the two-loop RGEs for the anomalous dimensions of the scalar and fermion fields have been implemented as well. It is now possible to export the coupled system of beta functions into a numerical C++ function, leading to a consequent speed up in solving them.

Acknowledgments

We are grateful to Florian Staub, Renato Fonseca, Kristjan Kannike, Helena Kolešová and Fred Olness for many useful discussions. F.L. would like to thank Florian Staub for helping validating the implementation of the kinetic mixing. F.L. is also grateful to Luigi Delle Rose for providing insights on the implementation of the kinetic mixing and for the help in resolving early discrepancies in the U(1)$_{B-L}$ model.
Appendix A. SM-U(1)$_{B-L}$

Appendix A.1. Yukawa couplings beta function

\[ \beta_{Y_{\nu}}^{(2)} = -\frac{1}{48} Y_N \left( \frac{1}{4} (156g' + 93g^2 - 36Tr(Y_{\nu}Y_{\nu}^\dagger) + 93g^2 + 192(g')^2 + 135g_2^2\right) \]

\[ +192\lambda_1 - 8Y_N^2 - 108Y_1^2 + 2(Y_N (2g' g'(30Tr(Y_{\nu}Y_{\nu}^\dagger) + 252g^2 + 1012(g')^2) + 27g_2^2 + 144Y_1^2 + 50Y_2^2 + 8g' (15Tr(Y_{\nu}Y_{\nu}^\dagger) + 70g^2 + 1598(g')^2 - 54g_2^2 + 85Y_2^2) + 504g^3g' + 35g_2^2 + 15g^2Tr(Y_{\nu}Y_{\nu}^\dagger) + 120(g')^2 Tr(Y_{\nu}Y_{\nu}^\dagger) + 9g_2^2 (5Tr(Y_{\nu}Y_{\nu}^\dagger) \]

\[ -6g^2 + 18(g')^2 + 15Y_1^2 - 72Y_N Tr(Y_{\nu}Y_{\nu}^\dagger) - 54Tr(Y_{\nu}Y_{\nu}^\dagger Y_{\nu}Y_{\nu}^\dagger) + 35g_4^2 + 85g_2^2 Y_2^2 + 960(g')^2 Y_2^2 + 40(g')^2 Y_2^2 + 1560(g')^2 + 374g^2 (g')^2 + 480g_3^2 Y_2^2 - 138g_2^2 + 144Y_2^2 \]

\[ +12Y_2^2 - 192\lambda_3 Y_N^2 - 480Y_1^2 - 162Y_2^2 + 168Y_N^2 (Y_{\nu}Y_{\nu}^\dagger Y_{\nu}Y_{\nu}^\dagger) + 36 (Y_{\nu}Y_{\nu}^\dagger Y_{\nu}Y_{\nu}^\dagger) \] (A.3)
Appendix A.2. Scalar mass beta function

\[ \beta^{(2)}_{\mu} = \frac{1}{48} \tilde{g}^2 \left[ 60\mu \text{Tr} \left[ Y_{\nu} Y_{\nu}^\dagger \right] + 1105 g^2 \mu + 960 (g')^2 \mu_\chi + 816 \mu (g')^2 + 63 \tilde{g}_2 \mu + 1152 \lambda_1 \mu + 340 \mu Y_i^2 \right] \\
+ \frac{5}{3} \mu g g' \left[ 3 \text{Tr} \left[ Y_{\nu} Y_{\nu}^\dagger \right] + 8 g^2 + 5 Y_i^2 \right] + \frac{3}{16} g_2^2 \mu \left[ 20 \text{Tr} \left[ Y_{\nu} Y_{\nu}^\dagger \right] + 7 g^2 + 384 \lambda_1 + 60 Y_i^2 \right] \\
+ \frac{5}{4} g^2 \mu \text{Tr} \left[ Y_{\nu} Y_{\nu}^\dagger \right] + 10 \mu (g')^2 \text{Tr} \left[ Y_{\nu} Y_{\nu}^\dagger \right] - 13 \mu Y_N^2 \text{Tr} \left[ Y_{\nu}^\dagger Y_{\nu} \right] - 24 \lambda_1 \mu \text{Tr} \left[ Y_{\nu} Y_{\nu}^\dagger \right] \\
+ \frac{40}{3} \mu g^3 g' - 9 \mu \text{Tr} \left[ Y_{\nu} Y_{\nu}^\dagger \right] + \frac{1105 g^4 \mu}{96} + 24 g^2 \lambda_1 \mu + \frac{85}{12} g^2 \mu Y_i^2 + 64 \lambda_3 (g')^2 \mu_\chi \\
+ \frac{10}{3} \mu (g')^2 Y_i^2 - \frac{145 g_2^4 \mu}{16} + 40 \tilde{g}_2^2 \mu_\chi - 60 \lambda_3 \mu - 24 \lambda_3 \mu Y_N^2 - 72 \lambda_1 \mu Y_i^2 \\
- \frac{27 \mu Y_i^4}{2} + \frac{1105 \mu g^3}{96} \right] \tag{A.4} \]

\[ \beta^{(2)}_{\mu_\chi} = 88 \lambda_3 \mu \left[ \tilde{g}^2 - \text{Tr} \left[ Y_{\nu} Y_{\nu}^\dagger \right] + g^2 + 3 g_2^2 - 3 Y_i^2 \right] + \frac{640}{3} \tilde{g} (g')^3 \mu_\chi + \frac{422}{3} \tilde{g}^2 (g')^2 \mu_\chi \\
+ 40 \mu g^2 (g')^2 - 12 \mu_\chi Y_N^2 \left[ \text{Tr} \left[ Y_{\nu}^\dagger Y_{\nu} \right] - 5 (g')^2 + 8 \lambda_2 \right] - 24 \mu_\chi \mu_\chi \text{Tr} \left[ Y_N Y_N^\dagger Y_N Y_N^\dagger \right] \\
+ 256 \lambda_2 (g')^2 \mu_\chi + 648 (g')^4 \mu_\chi - 40 \lambda_2^2 \mu_\chi - 2 \lambda_3^2 \left( \mu_\chi + 4 \mu \right) \tag{A.5} \]
Appendix A.3. Quartic couplings beta function

We present here the complete list of the two-loops terms for the beta functions of the quartic terms for the SM-U(1)$_{B-L}$. Note that all the replacement rules of [14] have been implemented.

\[
\beta_{\lambda_i}^{(2)} = -\frac{379g^6}{48} - \frac{19}{4}Y_i^2g^4 + \frac{629\lambda_1g^4}{24} - \frac{1}{4}\text{Tr}[Y_iY_i^\dagger]g^4 - \frac{8}{3}Y_i^4g^2 + 36\lambda_1^2g^2 + \frac{85}{6}Y_i^2 \\
\lambda_1g^2 + \frac{5}{2}\lambda_1\text{Tr}[Y_iY_i^\dagger]g^2 + \frac{305g^4}{16} + 30Y_i^6 - 32g_2^2Y_i^4 \\
-312\lambda_1^3 - 4\lambda_1^3 - 144Y_i^2\lambda_1^2 - 12Y_i^2\lambda_1^3 - 10\lambda_1\lambda_1^2 - \frac{8}{3}Y_i^4(g')^2 \\
+32\lambda_1^3(g')^2 + \frac{20}{3}Y_i^2\lambda_1(g')^2 + 20\lambda_1\text{Tr}[Y_iY_i^\dagger](g')^2 \\
-8\text{Tr}[Y_iY_i^\daggerY_iY_i^\dagger](g')^2 - 3Y_i^4\lambda_1 + 80g_3^2Y_i^2\lambda_1 \\
-48\lambda_1^3\text{Tr}[Y_iY_i^\dagger] - \frac{1}{48}g_2^4\left(289g^2 + 108Y_i^2 + 436\lambda_1 + 36\text{Tr}[Y_iY_i^\dagger]\right) \\
-\frac{1}{48}g_2^4\left(559g^4 + 24\text{Tr}[Y_iY_i^\dagger]\right)g^2 \\
-5184\lambda_1^4 - 72Y_i^2\left(7g^2 + 15\lambda_1\right) - 36\lambda_1\left(13g^2 + 10\text{Tr}[Y_iY_i^\dagger]\right) - 12Y_i^2\lambda_1\text{Tr}[Y_iY_i^\dagger] \\
-\lambda_1\text{Tr}[Y_iY_i^\daggerY_iY_i^\dagger] + 8Y_i^2\text{Tr}[Y_iY_i^\daggerY_iY_i^\dagger] + 8Y_i^2\text{Tr}[Y_iY_i^\daggerY_iY_i^\dagger] \\
+10\text{Tr}[Y_iY_i^\daggerY_iY_i^\dagger]Y_iY_i^\dagger - \frac{32}{3}g^6g' \\
-\frac{2}{3}g^3\left(32g^2 + 16g_2^2 + 15\lambda_1^2 - 40\lambda_1 + 9\text{Tr}[Y_iY_i^\dagger]\right)g' - \frac{2}{3}g\left(16g^4 - 40\lambda_1g^2 + 9\text{Tr}[Y_iY_i^\dagger]\right)g^2 \\
+10Y_i^4 + 5Y_i^2\left(3g^2 - 5\lambda_1\right) - 15\lambda_1\text{Tr}[Y_iY_i^\dagger] + g_2^4\left(16g^2 - 9Y_i^2 + 9\text{Tr}[Y_iY_i^\dagger]\right) \\
+6\text{Tr}[Y_iY_i^\daggerY_iY_i^\dagger]g' - \frac{1}{48}g^4\left(1137g^4 + 559g_2^2 + 228Y_i^2 + 624(g')^2 \right) - 1258\lambda_1 \\
+12\text{Tr}[Y_iY_i^\dagger] - \frac{1}{48}g^4\left(1137g^4 + 624(g')^2g^2 - 2516\lambda_1g^2 + 24\text{Tr}[Y_iY_i^\dagger]\right)g^2 \\
+289g_4^2 + 128Y_i^4 - 1728\lambda_1^2 - 1632\lambda_1(g')^2 - 960\lambda_3(g')^2 + 576\text{Tr}[Y_iY_i^\dagger](g')^2 \\
-120\lambda_1\text{Tr}[Y_iY_i^\dagger] + 8Y_i^2\left(57g^2 + 24(g')^2 \right) - 85\lambda_1 \\
+2g_2^4\left(559g^2 - 252Y_i^2 + 312(g')^2 - 234\lambda_1 + 12\text{Tr}[Y_iY_i^\dagger]\right) = -\frac{379g^6}{48}
\]
Here we list the full set of RGEs for the SM:

\[
\beta_{\alpha_2}^{(2)} = \frac{-1}{3} \left( \frac{713}{3} (g')^2 g^4 - \frac{1024}{3} (g')^4 g^2 - \frac{713}{3} g^2 (g')^2 g^2 - 45 \tilde{g}^2 (g')^2 g^2 - 76 \tilde{g}^2 g^2 + 120 \lambda_1 (g')^2 g^2 + 80 \lambda_2 (g')^2 g^2 \right) \\
-4 \text{Tr} \left[ Y_{\nu} Y_{\nu} Y_{\nu} Y_{\nu} \right] (g')^2 g^2 + \text{Tr} \left[ Y_{N} Y_{N} Y_{N} Y_{N} \right] (g')^2 g^2 - \frac{512}{3} g^2 (g')^3 g - 160 \tilde{g}^2 (g')^3 g \\
-96 \text{Tr} \left[ Y_{\nu} Y_{\nu} \right] (g')^3 g - 64 \tilde{g}^2 (g')^4 - 192 \text{Tr} \left[ Y_{N} Y_{N} \right] (g')^4 - 11 \lambda_3^2 + 6 g^2 \text{Tr} \left[ Y_{N} Y_{N} Y_{N} Y_{N} \right] \\
3 \tilde{g}^2 \text{Tr} \left[ Y_{N} Y_{N} Y_{N} + \tilde{g}^2 + 3 g^2 \right] 2 - 12 Y_{N}^2 - 12 g^2 - 72 \lambda_1 - 48 \lambda_2 - 4 \lambda_3 \left[ Y_{N} Y_{N} \right] \\
\frac{1}{48} \lambda_3 \left( 557 g^4 + 340 \tilde{g}^2 g^2 + 1152 \lambda_1^2 g^2 + 60 \text{Tr} \left[ Y_{\nu} Y_{\nu} \right] g^2 + 557 \tilde{g}^4 - 435 \tilde{g}_I^4 - 648 Y_{\nu}^4 + 32256 (g')^4 \right) \\
+1920 \tilde{g}_I Y_{\nu}^2 - 2880 \tilde{g}_I^2 + 160 \lambda_2 (g')^2 + 12288 \lambda_2 (g')^2 + 480 \text{Tr} \left[ Y_{\nu} Y_{\nu} \right] (g')^2 \\
-3456 \lambda_4^2 - 4608 \lambda_5^2 - 1152 \lambda_2 \text{Tr} \left[ Y_{\nu} Y_{\nu} \right] + 18 \tilde{g}^2 (5 g^2 + 30 Y_{\nu}^2 + 192 \lambda_1 + 10 \text{Tr} \left[ Y_{\nu} Y_{\nu} \right] \\
288 Y_{\nu} \text{Tr} \left[ Y_{\nu} Y_{\nu} \right] - 1152 \text{Tr} \left[ Y_{N} Y_{N} Y_{N} Y_{N} \right] + 192 \text{Tr} \left[ Y_{N} Y_{N} Y_{N} Y_{N} \right] + 192 \text{Tr} \left[ Y_{N} Y_{N} Y_{N} Y_{N} \right] \\
-216 \text{Tr} \left[ Y_{\nu} Y_{\nu} Y_{\nu} Y_{\nu} \right] + 640 \lambda_3 (g' + g)^2 + 80 \tilde{g}^2 (g' - 2 Y_{\nu}^2 + 128 (g')^2 + 3 \text{Tr} \left[ Y_{\nu} Y_{\nu} \right] \\
+2 \tilde{g}^2 (557 g^2 + 45 \tilde{g}_I^2 + 170 Y_{\nu}^2 + 3976 (g')^2 + 576 \lambda_1 + 30 \text{Tr} \left[ Y_{\nu} Y_{\nu} \right] ) \\
\lambda_2 \left[ \text{Tr} \left[ Y_{\nu} Y_{\nu} \right] (g^2 + \tilde{g}^2 + 3 g^2 + 16 \tilde{g}^2 \right) + 8 \left( 6 \tilde{g}^2 (g')^2 + 4 \text{Tr} \left[ Y_{\nu} Y_{\nu} Y_{\nu} Y_{\nu} \right] + 5 \text{Tr} \left[ Y_{\nu} Y_{\nu} Y_{\nu} Y_{\nu} \right] \right] \right)
\]

\[
\beta_{\alpha_2}^{(2)} = \frac{1}{3} \left( -4 \right) \left( 320 \tilde{g}_I g' + 211 \tilde{g}_I^2 + 1584 (g')^2 \right) - 12 \text{Tr} \left[ Y_{N} Y_{N} Y_{N} Y_{N} \right] + 15 \lambda_3^2 - 30 \lambda_3 \tilde{g}^2 (g')^2 \\
+2048 \tilde{g} (g')^5 + 1336 \tilde{g}^2 (g')^4 - 3 \lambda_3^2 \tilde{g}^2 + 18 Y_{\nu}^2 \left( \text{Tr} \left[ Y_{\nu} Y_{\nu} \right] \right) - 5 (g')^2 - 32 (g')^4 \\
+10 \lambda_2^2 - 48 (g')^2 \text{Tr} \left[ Y_{N} Y_{N} Y_{N} Y_{N} \right] - 48 \text{Tr} \left[ Y_{N} Y_{N} Y_{N} Y_{N} \right] \\
-96 \text{Tr} \left[ Y_{N} Y_{N} Y_{N} Y_{N} Y_{N} Y_{N} \right] + 3 \lambda_2^2 \left[ Y_{\nu} Y_{\nu} \right] - 3 \lambda_2 \lambda_2^2 - 336 \lambda_2 \tilde{g}^2 (g')^2 + 5376 (g')^6 - 9 \tilde{g}_I^2 \lambda_2 \\
+180 \lambda_3^2 + 6 \lambda_3 + 9 \lambda_3 \tilde{g}^2 \right)
\]

Appendix B. RGEs for the SM+complex triplet

Here we list the full set of RGEs for the SM+complex triplet model including the two invariants coming from the contraction of four triplet fields defined by the potential in Eq. (40).

Appendix B.1. Gauge Couplings

\[
\frac{dg}{dt} = \beta_s \left[ \beta_s^I + \beta_s^H \right]
\]
Appendix B.1.1. Evolution of $g_2$

\[ (4\pi)^2 \beta_{g_2} |_{I} = -\frac{3}{2} g_2^3 \]
\[ (4\pi)^4 \beta_{g_2} |_{II} = +\frac{41}{4} g_1^2 g_3^3 + \frac{147}{4} g_2^5 + 12 g_2^3 g_3^2 \]

Appendix B.1.2. Evolution of $g_1$

\[ (4\pi)^2 \beta_{g_1} |_{I} = +\frac{53}{6} g_1^3 \]
\[ (4\pi)^4 \beta_{g_1} |_{II} = +\frac{857}{36} g_1^5 + \frac{123}{4} g_1^2 g_2^3 + \frac{44}{3} g_1^3 g_3^2 \]

Appendix B.1.3. Evolution of $g_3$

\[ (4\pi)^2 \beta_{g_3} |_{I} = -7 g_3^3 \]
\[ (4\pi)^4 \beta_{g_3} |_{II} = +\frac{11}{6} g_1^2 g_3^3 + \frac{9}{2} g_1^2 g_2^3 g_3 - 26 g_3^5 \]

Appendix B.2. Quartic Coupling

\[ \frac{d\lambda}{dt} = \beta_{\lambda I} + \beta_{\lambda II} \]

Appendix B.2.1. Evolution of $\lambda_1$

\[ (4\pi)^2 \beta_{\lambda_1} |_{I} = +24 \lambda_1^2 + \frac{5}{4} \lambda_2^2 + 3 \lambda_3 \lambda_2 + 3 \lambda_2^2 + \frac{9}{8} g_2^4 \]
\[ -3 \lambda_1 g_1^2 - 9 \lambda_1 g_2^2 + \frac{3}{8} g_1^4 + \frac{3}{4} g_1^2 g_2^2 \]

22
Appendix B.2.2. Evolution of $\lambda_3$

\[(4\pi)^4 \beta_{(1)}|_{II} = -312\lambda_1^3 + 15\lambda_2g_1^4 + 30\lambda_2g_2^4 - \frac{511}{48}g_1^6 + \frac{821}{24}\lambda_1g_1^4 + 10\lambda_3g_1^2 - \frac{691}{48}g_1^4g_2^2 + \frac{201}{16}g_2^6 + 17\lambda_3g_2^2 - 10\lambda_3g_1^2g_2^2 + 24\lambda_2g_1^2 - 30\lambda_1\lambda_3\lambda_2 - 30\lambda_1\lambda_3^2 + 24\lambda_3\lambda_2g_1^2 + 108\lambda_1^2g_2^2 - \frac{29}{2}\lambda_1\lambda_3^2 - 12\lambda_2^2 - 18\lambda_3\lambda_2^2 + 15\lambda_3g_2^4 - \frac{131}{16}g_1^2g_2^4 - 19\lambda_3\lambda_2^2 + \frac{15}{2}\lambda_3g_1^4 + 48\lambda_2^2g_2^2 + 36\lambda_1^2g_1^2 - \frac{13}{2}\lambda_1^4 + 48\lambda_3\lambda_2g_2^2 + \frac{39}{4}\lambda_1g_1^2g_2^2 + \frac{75}{8}\lambda_1g_2^4\]

\[(4\pi)^4 \beta_{(1)}|_{II} = +8\lambda_3\lambda_2 - \frac{15}{2}\lambda_1g_1^2 - \frac{33}{2}\lambda_3g_2^2 - 12g_1^2g_2^2 + 4\lambda_1\lambda_3 + 4\lambda_3^2 + 8\lambda_3\lambda_2 + 4\lambda_3\lambda_1\]
Appendix B.2.3. Evolution of $\lambda_2$

\[
(4\pi)^2\beta_{13}^I = + 12\lambda_2\lambda_2 + 16\lambda_2\lambda_1 - \frac{15}{2}\lambda_2 g_1^2 \\
+ 4\lambda_1\lambda_3 + 2\lambda_3\lambda_2 - \frac{33}{2}\lambda_2 g_2^2 + 3g_1^4 + 6g_1^2 g_2^2 \\
+ 12\lambda_1\lambda_2 + \lambda_3^2 + 6\lambda_3\lambda_1 + 4\lambda_2^2 + 6g_4^2
\]

\[
(4\pi)^4\beta_{13}|_{II} = + 60\lambda_1 g_2^4 - 72\lambda_2\lambda_2^2 - 96\lambda_2\lambda_1 \\
- 5\lambda_1\lambda_3 + 12\lambda_3\lambda_2 g_2^2 + 40\lambda_2 g_1^2 g_2^2 \\
+ 20\lambda_1 g_1^2 g_2^2 - 16\lambda_3\lambda_2 g_1^2 + 256\lambda_2\lambda_1 g_2^2 \\
- 16\lambda_1^2\lambda_3 - 42\lambda_3 g_1^2 g_2^2 + \frac{1235}{16}\lambda_2 g_2^4 \\
- 32\lambda_1\lambda_3\lambda_2 + 72\lambda_1\lambda_2 g_2^2 - 70\lambda_2^2\lambda_2 \\
+ 48\lambda_3\lambda_2 g_1^2 + \frac{185}{4}\lambda_3 g_4^4 + 20\lambda_1 g_1^2 g_2^2 \\
- 16\lambda_3\lambda_2^2 + 56\lambda_3\lambda_2 g_2^2 - 118g_1^2 g_2^2 \\
- 13\lambda_1^2 + 5\lambda_2^2 g_1^2 + 11\lambda_2^2 g_2^2 \\
+ 36\lambda_1\lambda_3 g_2^2 - \frac{11}{2}\lambda_3^2 + \frac{11}{4}\lambda_2^2 g_2^2 \\
- 18\lambda_1\lambda_3^2 - 16\lambda_3\lambda_2\lambda_1 - 48\lambda_3\lambda_2\lambda_1 \\
+ 30\lambda_1 g_1^4 + 108\lambda_3\lambda_1 g_2^2 - \frac{1157}{12}g_1^4 \\
- 60\lambda_1^2\lambda_2 - 24\lambda_1^2\lambda_2 + 192\lambda_2 g_1^2 g_2^2 \\
+ \frac{125}{4}\lambda_3 g_4^4 + 96\lambda_2 g_1^2 g_2^2 - 80\lambda_2\lambda_2^2 \\
- 16\lambda_2\lambda_2 g_2^2 + 30\lambda_2 g_1^4 - \frac{1877}{12}g_1^2 g_2^2 \\
+ 24\lambda_1 g_1^2 g_2^2 - 120\lambda_2\lambda_2 g_2^2 + \frac{59}{7}g_2^4 \\
+ 8\lambda_1\lambda_3 g_1^2 - 24\lambda_3\lambda_2^2 + \frac{207}{8}\lambda_2 g_1^2 g_2^2
\]
Appendix B.2.4. Evolution of $\lambda_{A1}$

$$
(4\pi)^2 \beta_{\lambda_{A1}} \bigg|_{I} = + 2\lambda_{A1} + 24\lambda_{A2}\lambda_{A1} - 12\lambda_{A1}g_{1}^2 - 12g_{1}^2 g_{2}^2
+ 6\lambda_{A2}^2 + 2\lambda_{2}^2 + 28\lambda_{A1} - 24\lambda_{A1}g_{2}^2 + 6g_{1}^4 + 15g_{2}^4
$$

$$
(4\pi)^4 \beta_{\lambda_{A1}} \bigg|_{II} = + 12\lambda_{A1}g_{2}^2 + 24\lambda_{A2}g_{1}^2 + 168\lambda_{A2}g_{2}^4
+ 3\lambda_{A2}^2 g_{2}^2 + 20\lambda_{A2}g_{2}^4 + 352\lambda_{A1}g_{2}^6
- 3\lambda_{A1}^2 + 5\lambda_{A1}g_{1}^4 + \frac{466}{3} g_{1}^4 g_{2}^2
- 284\lambda_{A2}^2 \lambda_{A1} - 20\lambda_{A2}^4 + \frac{841}{3} \lambda_{A1}g_{1}^4
- 96\lambda_{A2}^3 + 120\lambda_{A2}g_{1}^2 - 56\lambda_{A2}g_{1}^2 g_{2}^2 + 266\lambda_{A1}g_{2}^4
+ 4\lambda_{A2}^2 g_{1}^2 + 192\lambda_{A2}\lambda_{A1}g_{1}^2 - 8g_{1}^2
- 6\lambda_{A2}^2 + 10\lambda_{A1}g_{1}^2 + 120\lambda_{A2}g_{1}^4
- 12\lambda_{A2}g_{2}^2 + 10\lambda_{A2}g_{2}^4 + 10\lambda_{A2}g_{2}^4 - \frac{646}{3} g_{1}^6
- 20\lambda_{A2}^2 \lambda_{A1} + 176\lambda_{A2}g_{1}^2 + 86g_{1}^2 g_{2}^2
- 528\lambda_{A2}^2 \lambda_{A1} + 4\lambda_{A2}g_{1}^2 - 384\lambda_{A1}^3
- 4\lambda_{A2}^2 \lambda_{A2} - 144\lambda_{A2}g_{1}^2 g_{2}^2 + 12\lambda_{A2}g_{2}^2
- \lambda_{A2}^3 + 384\lambda_{A2} \lambda_{A1}g_{2}^2 - 209g_{2}^4
$$

Appendix B.2.5. Evolution of $\lambda_{A2}$

$$
(4\pi)^2 \beta_{\lambda_{A2}} \bigg|_{I} = + 18\lambda_{A2}^2 + 24\lambda_{A2}\lambda_{A1} - 12\lambda_{A2}g_{1}^2 + 24g_{1}^2 g_{2}^2 - 6g_{2}^4
+ \lambda_{A2}^3 - 24\lambda_{A2}g_{2}^2
$$

$$
(4\pi)^4 \beta_{\lambda_{A2}} \bigg|_{II} = - 7\lambda_{A2}^2 - 20\lambda_{A2} \lambda_{A1} + 96\lambda_{A2} \lambda_{A1}g_{1}^2
- 4\lambda_{A2}^3 + 2\lambda_{A2}g_{1}^4 + \frac{361}{3} \lambda_{A2}g_{1}^4 + 360\lambda_{A2}g_{1}^2 g_{2}^2
- 8\lambda_{A2}^2 \lambda_{A1} - 672\lambda_{A2} \lambda_{A1} + 144\lambda_{A2}^2 g_{2}^2 + 170g_{2}^4
- 20\lambda_{A2} \lambda_{A2} + 192\lambda_{A2} \lambda_{A1}g_{2}^2 - 14\lambda_{A2}g_{2}^4
- 8\lambda_{A2}^3 - 20\lambda_{A2}g_{1}^2 g_{2}^2 - 412g_{1}^2 g_{2}^2
- 228\lambda_{A2}^3 + 192\lambda_{A2} \lambda_{A1} g_{2}^2 - \frac{1652}{3} g_{1}^4 g_{2}^2
+ 120\lambda_{A2}^2 g_{1}^2 - 448\lambda_{A2} \lambda_{A1} g_{2}^2 - 48\lambda_{A1}g_{2}^4
$$
