THE SYMPLECTIC PLACTIC MONOID, CRYSTALS AND MV CYCLES

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Abstract. We consider the case of the complex symplectic group, for which we give interpretations in terms of the affine Grassmannian of plactic relations and word reading associated to readable galleries, which we define. This leads us to associate an MV cycle to any readable gallery, and to prove that this association yields a morphism of crystals.

1. Introduction

This paper is part of a project which was started in [GL05] by Gausset and Littelmann, the aim of which is to establish an explicit relationship between the path model and the set of MV cycles used by Mirković and Vilonen for their Geometric Satake equivalence proven in [MV07].

1.1. Given a complex connected reductive linear algebraic group $G$, the choice of a Borel subgroup containing a fixed maximal torus allows one to consider the coweight lattice $X^\vee$, and in it a dominant integral coweight $\lambda$. Let $G = G(\mathbb{C}(t))/G(\mathbb{C}[[t]])$ be the affine Grassmannian associated to $G$. The set of coweights $X^\vee$ can be seen as a subset of $G$ and one may consider the $G(\mathbb{C}[[t]])$-orbit of $\lambda$ in $G$. The closure of this orbit, $X_\lambda$ has the structure of a complex algebraic variety which is usually singular. The Geometric Satake equivalence identifies the complex irreducible highest weight module $L(\lambda)$ of $G^\vee$ with the intersection cohomology of $X_\lambda$, a basis of which is given by the classes of certain subvarieties of $X_\lambda$ called MV cycles. The set of these subvarieties is denoted by $Z(\lambda)$. The Geometric Satake equivalence implies that the elements of $Z(\lambda)$ are in one to one correspondence with the vertices of the crystal $B(\lambda)$. In [BG01], Braverman and Gaitsgory endow the set $Z(\lambda)$ with a crystal structure and show the existence of a crystal isomorphism $\varphi : B(\lambda) \sim \rightarrow Z(\lambda)$.

1.2. In [GL05], Gaussent and Littelmann define a set $\Gamma(\gamma_\lambda)^{LS}$ of LS galleries, which are galleries in the affine building $J^a$ associated to $G$, and they endow this set with a crystal structure and an isomorphism of crystals $B(\lambda) \sim \rightarrow \Gamma(\gamma_\lambda)^{LS}$. They view the latter as a subset of the $T$-fixed points in a desingularization $\Sigma_{\gamma_\lambda} \rightarrow X_\lambda$. To each of these particular fixed points $\delta \in \Gamma(\gamma_\lambda)^{LS}$ corresponds a Bialynicki-Birula cell $C_\delta$. Gaussent and Littelmann show in [GL05] that the the closure $\pi(C_\delta)$ is an MV cycle, and Baumann and Gaussent show in [BG08] that the map...
is a crystal isomorphism with respect to the crystal structure on $Z(\lambda)$
given in [BG01] by Braverman and Gaitsgory. It is natural to ask whether
the closures $\pi(C_\delta)$ are still MV cycles for a more general choice of fixed
point $\delta$.

1.3. In [GL12] they consider one skeleton galleries, which are piecewise
linear paths in $X^\vee$. Such galleries can be interpreted in terms of Young
tableaux for types $A_n, B_n$ and $C_n$. For $G^\vee = \text{SL}_n(\mathbb{C})$, Gaussent, Littelmann
and Nguyen show in [GLN13] that for any fixed point $\delta \in \Sigma^T_{\gamma\lambda}$, the closure
$\pi(C_\delta)$ is in fact an MV cycle. They achieve this using combinatorics of
Young tableaux such as word reading and the well known Knuth relations,
and by relating them to the Chevalley relations for root subgroups which
hold in the affine Grassmannian $G$. In [Tor14] it is observed that word read-
ing is a crystal morphism, and this allows one to prove that in this case, the
map from all galleries to MV cycles is in fact a morphism of crystals.

It was conjectured in [GLN13] that generalizations of their results hold
for arbitrary complex semi-simple algebraic groups, in terms of the plactic
algebra defined by Littelmann in [Lit96]. It is with this in mind that we
formulate and state our results.

1.4. Results. We work with $G^\vee = \text{SP}_{2n}(\mathbb{C})$. We define a set $\Gamma(\gamma\lambda)^{\text{LS}} \supset \Gamma(\gamma\lambda)^R$ of readable galleries, which have an explicit formulation in terms of Young tableaux. There is a certain word reading described by Lecouvey in
[Lec02] which we show to be a crystal morphism when restricted to readable
galleries.

We obtain results similar to those obtained in [GLN13] concerning the
defining relations of the symplectic plactic monoid, described explicitly by
Lecouvey in [Lec02], as well as words of readable galleries. These results, to-
gether with the work of Gaussent-Littelmann [GL05], [GL12], and Baumann-
Gaussent [BG08] allow us to show in Theorem 7.1 that given $\delta \in \Gamma(\gamma\lambda)^R$,
there is an associated dominant coweight $\nu_\delta \leq \lambda$ such that:

1. $\pi(C_\delta)$ is an MV cycle in $X_{\nu_\delta}$.
2. The map

$$\Gamma(\gamma\lambda)^R \rightarrow \bigcup_{\gamma \in \Gamma(\gamma\lambda)^R} Z(\nu_\gamma)$$

$$\delta \mapsto \pi(C_\delta)$$
is a morphism of crystals. Moreover, this map induces an isomorphism when restricted to connected components. We then give some examples of galleries $\delta \in \Sigma^T_{\gamma, \lambda} - \Gamma(\gamma, \lambda)^R$ for which $\pi(C_{\delta})$ is not an MV cycle in $\mathcal{Z}(\nu_\delta)$.

1.5. This paper is organized as follows. In Section 2 we introduce our notation and recall several general facts about affine Grassmannians, MV cycles, galleries in the affine building, generalised Bott-Samelson varieties as varieties of galleries, and concrete descriptions of Bialynicki-Birula cells in them. From Section 3 on we work with $G^\vee = \text{SP}_{2n}(\mathbb{C})$. We introduce symplectic keys which correspond to the Young tableaux description of one skeleton galleries. We endow them with a crystal structure which coincides with the one given in [GL05], [Lit95], and define their words as in [Lec02]. In Section 4 we recall the definition of LS galleries and we define readable galleries, and we state Proposition 4.4 concerning a geometric interpretation of reading their words. In Section 5 we show that reading the word of a readable gallery is a morphism of crystals, and in Section 6 we define the symplectic plactic monoid and recall the description given by Lécouvey in [Lec02], as well as stating Proposition 6.5. In Section 7 we state and prove Theorem 7.1, and in Section 8 we provide a detailed example as well as proofs for Propositions 4.4 and 6.5. Finally, in Section 9, we provide some ‘degenerate’ examples as described in the previous paragraph.

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2. Preliminaries

2.1. Notation. Throughout this section, we consider $G$ to be a complex connected reductive algebraic group associated to a root datum $(X, X^\vee, \Phi, \Phi^\vee)$, with maximal torus $T \subset G$ having $X$ as its character group and Langlands dual $G^\vee$. We identify the Weyl group $W$ with the quotient $N_G(T)/T$, where $N_G(T)$ is the normaliser of $T$ in $G$. We fix opposite Borel subgroups $B^+ \supset T$ of $G$ corresponding to a choice of positive/negative roots $\Phi^\pm \subset \Phi$ and a set of simple roots $\Delta \subset \Phi^+$. With this choice comes also a a set of dominant weights $X^+$, respectively dominant coweights $X^\vee^+$, as well as a dominance order $\preceq$ on the sets $X, X^\vee$. Denote the non-degenerate pairing between $X$ and $X^\vee$ by $\langle -, - \rangle$. For a sum of positive roots $\lambda = \sum_{\alpha \in \Delta} n_\alpha \alpha; \ n_{\alpha \in \mathbb{N}}$ let $\text{ht}(\lambda) = \sum_{\alpha \in \Delta} n_\alpha$, and analogously for positive sums of elements in $\Delta^\vee$. 
To the Borel subgroups $B^\pm$ are attached their unipotent radicals $U^\pm$, each of which is generated by the set of one parameter subgroups $\{U_\alpha : \alpha \in \Phi^\pm\}$. We denote elements of such root subgroups by $U_\alpha(b)$ for $b \in \mathbb{C}$. For each cocharacter $\lambda \in X^\vee = \text{Hom}(\mathbb{C}^*, T)$ and each non-zero complex number $a \in \mathbb{C}^*$, denote by $a^\lambda$ the image $\lambda(a) \in T$. The following identities hold in $G$:

- For any $\lambda \in X^\vee$, $a \in \mathbb{C}^*$, $b \in \mathbb{C}$, $\alpha \in \Phi$,
  \[ a^\lambda U_\alpha(b) = U_\alpha(a^{(\alpha, \lambda)}b)a^\lambda \]  

- (Chevalley’s commutator formula) Given linearly independent roots $\alpha, \beta \in \Phi$, there exist numbers $c_{i,j,\alpha,\beta} \in \{\pm 1, \pm 2, \pm 3\}$ such that, for all $a, b \in \mathbb{C}$,
  \[ U_\beta(b)^{-1}U_\alpha(a)^{-1}U_\beta(b)U_\alpha(a) = \prod_{i,j \geq 0} U_\alpha^i U_\beta^j (i^ic_jb^j) \]  

where the product is taken over all pairs of positive integers $(i,j)$ such that $i\alpha + j\beta$ is a root, and is taken in order of increasing $i + j$.

### 2.2. Affine Grassmannians

Let $O = \mathbb{C}[[t]]$ denote the ring of complex formal power series and let $K = \mathbb{C}((t))$ denote its field of fractions, complex Laurent power series. For any $\mathbb{C}$-algebra $R$, denote by $G(R)$ the set of $R$-valued points. The set $G = G(K)/G(O)$ is called the **affine Grassmannian** associated to $G$. We will denote the class in $G$ of an element $g \in G(K)$ by $[g]$. A coweight $\lambda : \mathbb{C}^* \to T \subset G$ determines a point in $G$ and hence a class $[t^\lambda] \in G$. This map is injective, and we may therefore consider $X^\vee$ as a subset of $G$.

The study of $U^\pm(K)$ orbits on $G(K)$ gives rise to the Iwasawa decomposition

\[ G = \bigsqcup_{\lambda \in X^\vee} U^\pm(K)[t^\lambda], \]

while the study of $G(O)$ orbits results in the Cartan decomposition

\[ G = \bigsqcup_{\lambda \in X^\vee,+} G(O)[t^\lambda]. \]

Each $G(O)$-orbit has the structure of an algebraic variety induced from the pro-group structure of $G(O)$ and it is known that for $\lambda \in X^{\vee,+}$,

\[ \overline{G(O)[t^\lambda]} = \bigsqcup_{\mu \in X^{\vee,+}, \mu \leq \lambda} G(O)[t^\mu] \]

We call the closure $\overline{G(O)[t^\lambda]}$ a **generalised Schubert variety** and we denote it by $X_\lambda$. In 2.5, we will view certain resolutions of singularities of them in terms of galleries in the affine building associated to $G$. 
2.3. MV Cycles and Crystals. We denote the set of all irreducible components of a given topological space $Y$ by $\text{Irr}(Y)$. Let $\lambda, \mu \in X^\vee$. Then the intersection $U^+(K)[t^\mu] \cap G(O)[t^\lambda]$ is not empty only if $\mu - \lambda \in \Phi^\vee$ and if $\mu$ lies in the convex hull of $W\lambda$ in $X^\vee \otimes \mathbb{Z} \mathbb{R}$.

Let $\lambda \in X^\vee$ be antidominant and denote by $L(w_0 \lambda)$ the irreducible rational representation of $G^\vee$ with lowest weight $\lambda$. Mirković and Vilonen proved in [MV07] (Theorem 3.2 and Corollary 7.4) that the closure $U^+(K)[t^\mu] \cap G(O)[t^\lambda]$ is pure dimensional of dimension $ht(\mu - \lambda)$ and the number of irreducible components coincides with the dimension of the $\mu$-weight space of $L(w_0 \lambda)$.

We consider thus the following sets:

$$Z(\lambda)_\mu = \text{Irr}(U^+(K)[t^\mu] \cap G(O)[t^\lambda])$$

$$Z(\lambda) = \bigsqcup_{\mu \in X^\vee} Z(\lambda)_\mu.$$

The elements of these sets are called MV cycles. In [BG01], Section 3.3 Braverman and Gaitsgory have given the set $Z(\lambda)$ a crystal structure and have shown the existence of an isomorphism of crystals $B(\lambda) \rightarrow Z(\lambda)$.

We do not use the definition of this crystal structure, but we denote by $\tilde{f}_i$ (respectively $\tilde{e}_i$) the corresponding root operators.

2.4. Galleries in the Affine Building. Let $J^a$ be the affine building associated to $G$ and $K$. It is a union of simplicial complexes called apartments, each of which is isomorphic to the Coxeter complex of type the extended Dynkin diagram associated to $G$. The affine Grassmannian $G$ can be $G(K)$ equivariantly embedded into the building $J^a$, which also carries a $G(K)$ action. Denote by $\Phi^\text{aff}$ the set of (real) affine roots associated to $\Phi$, namely tuples $(\alpha, n) \in \Phi \times \mathbb{Z}$.

Now let $\mathbb{A} = X^\vee \otimes \mathbb{Z} \mathbb{R}$. For each $(\alpha, n) \in \Phi^\text{aff}$, consider

$$H_{\alpha, n} := \{ x \in \mathbb{A} : \langle \alpha, x \rangle = n \}$$

$$H^+_{\alpha, n} := \{ x \in \mathbb{A} : \langle \alpha, x \rangle \geq n \}$$

$$H^-_{\alpha, n} := \{ x \in \mathbb{A} : \langle \alpha, x \rangle \leq n \}$$

Denote by $W^a$ the affine Weyl group generated by all the affine reflections $s_{\alpha, n}$ with respect to the affine hyperplanes $H_{\alpha, n}$. We have an embedding $W \hookrightarrow W^a$ given by $s_{\alpha} \mapsto s_{\alpha, 0}$. The dominant Weyl chamber is the set

$$C^+ = \{ x \in \mathbb{A} : \langle \alpha, x \rangle > 0 \ \forall \alpha \in \Delta \}$$

and the fundamental alcove is in turn

$$\Delta^f = \{ x \in C^+ : \langle \alpha, x \rangle < 1 \ \forall \alpha \in \Delta \}.$$
the apartment of $J^a$ which contains all of the weights $X$ (coming from the embedding $G \hookrightarrow J^a$) with the Coxeter complex having as faces all possible intersections of hyperplanes and their associated closed positive and negative half spaces. This is also known as the **standard apartment**. For each face $F$ in the standard apartment, denote its stabiliser in $U^+(K)$ by $U_F$ and its stabiliser in $W^a$ by $W^a_F$.

A **gallery** is a sequence of faces in the affine building $J^a$

\[
\gamma = (V_0 = 0, E_0, V_1, \ldots, E_k, V_{k+1})
\]

such that:

- For each $i \in \{1, \ldots, k\}$, $V_i \subset E_i \supset V_{i+1}$.
- Each face labelled $V_i$ has dimension zero (is a vertex) and each face labelled $E_i$ has dimension one (is an edge).
- The last vertex $V_k$ is a **special vertex**, which means that its stabiliser in the affine Weyl group $W^a$ is isomorphic to the finite Weyl group $W$ associated to $G$.

If, in addition, each face in the sequence belongs to the standard apartment, then $\gamma$ is called a **combinatorial gallery**. In this case, the second condition is equivalent to requiring the vertex to be a coweight $V_{k+1} \in X^\vee$.

**Remark 1.** The galleries we defined are actually called one-skeleton galleries in the literature, however since in this framework we only speak of the latter, we need only say gallery.

**Remark 2.** By (1), given $\lambda \in X^\vee$, the root subgroup $U_{(\alpha, n)}$ stabilises the point $[t^\lambda] \in G \hookrightarrow J^a$ if and only if $\lambda \in H^-_{\alpha, n}$.

### 2.5. Bott-Samelson varieties.

Given a combinatorial gallery $\gamma$ as above, there is an associated gallery

\[
\gamma^f := (V_0^f = 0, E_0^f, V_1^f, \ldots, V_{k+1}^f)
\]

with each $V_i$, respectively $E_i$ contained in the fundamental alcove and elements $w_i \in W^a_{V_i^f}/W^a_{E_i^f}$ such that $w_0 \cdots w_j(V_j) = V_j^f$. If two galleries have the same associated gallery we say that the two galleries have the same **type**.

Consider the associated sequence of stabilisers of $\gamma^f$ in $G(K)$:

\[
(G(\mathcal{O}), P_{E_0^f}, P_{V_1^f}, \ldots, P_{V_{k+1}^f}).
\]

Note that $G(\mathcal{O}) \supset P_{E_i^f} \subset P_{V_i^f}$ for all $i \in \{1, \ldots, k\}$.

**Definition 2.1.** The **Bott Samelson variety** of type $\gamma^f$ is the quotient of

\[
G(\mathcal{O}) \times P_{V_1^f} \times \cdots \times P_{V_{k+1}^f}
\]
by the following action of $P_{E_0} \times \cdots \times P_{E_k}$:

$$(p_0, p_1, \cdots, p_k) \cdot (q_0, \cdots, q_k) := (q_0 p_0, p_0^{-1} q_1 p_1, \cdots, p_k^{-1} q_k p_k)$$

We will denote it by $\Sigma_\gamma$.

**Remark 3.** The pro-group structure of the groups $P_{V_i}, P_{E_i}$ assures that $\Sigma_\gamma$ is in fact a variety, and it is also smooth. It may be identified with the set of all galleries in the building of the same type as $\gamma^f$, as well as with the set of all sequences of parahoric subgroups of $G(\mathcal{K})$

$$(G(\mathcal{O}), Q_0, P_1, \cdots, P_{k+1})$$

where $Q_i$ is $G(\mathcal{K})$-conjugated to $P_{E_i}$ and $P_i$ to $P_{V_i}$. See [GL12], Proposition 2. When considered as a set of galleries, the $T$-fixed points of $\Sigma_\gamma$ are precisely the combinatorial galleries with same type as $\gamma^f$. We will denote the set of combinatorial galleries of the same type as a given gallery $\delta$ by $\Gamma(\delta)$.

Let $\omega \in \mathbb{A}$ be a fundamental coweight. We define a combinatorial gallery which starts at 0 and ends at $\omega$. Let $V_1, \cdots, V_k$ be the vertices in the standard apartment which lie on the open line segment joining 0 and $\omega$, numbered such that $V_{i+1}$ lies on the open line segment joining $V_i$ and $\omega$. Note that $k \in \{0, 1, 2\}$. Let further $E_i$ denote the face contained in $\mathbb{A}$ which contains the vertices $V_i$ and $V_{i+1}$. The gallery

$$\gamma_\omega := (0 = V_0, E_0, V_1, E_1, \cdots, E_k, V_{k+1} = \omega)$$

is called a fundamental gallery.

Now let $\lambda \in X^+$ be a dominant integral coweight, and let $\gamma_\lambda$ be a gallery with endpoint the coweight $\lambda$ and such that it is a concatenation of fundamental galleries, where concatenation of two combinatorial galleries $\gamma_1 \ast \gamma_2$ is defined by translating the second one to the endpoint of the first one. Then, if $\lambda^f$ is the point in the orbit $W^a \lambda$ contained in the fundamental alcove, the map

$$\begin{align*}
\Sigma_{\gamma_\lambda} & \xrightarrow{\pi} X_\lambda \\
[g_0, \cdots, g_r] & \mapsto g_0 \cdots g_r \lambda^f
\end{align*}$$

(5)

is a resolution of singularities of the generalised Schubert variety $X_\lambda$.

**Remark 4.** The fact that the above map is in fact a resolution of singularities is due to the fact that such a gallery is minimal. This resembles the condition for usual Bott Samelson varieties associated to a reduced expression, and is made precise in [GL12], sections 4.3 and 5.2.
2.6. The Retraction at Infinity, cells and positive crossings. Let $\eta: C^* \to T$ be a generic one parameter subgroup such that for all $u \in U^+(K)$,

$$\lim_{s \to 0} \eta(s) u \eta(s)^{-1} = 1 \quad (6)$$

Let $\gamma^f$ be a combinatorial gallery contained in the fundamental alcove and $\gamma \in \Sigma_{\gamma^f}$ a $T$-fixed point. Consider the associated Bialynicki-Birula cell

$$C_\gamma = \{ x \in \Sigma_{\gamma^f} : \lim_{s \to 0} \eta(s)x = \gamma \}.$$

We can write the associated Bott-Samelson variety as a disjoint union of these cells:

$$\Sigma_{\gamma^f} = \bigsqcup_{\delta \in \Sigma_{\gamma^f}} C_\delta.$$

It is clear by our choice of $\eta$ in (6) that $U^+(K)\delta \subset C_\delta$ for $\delta$ as above. In fact,

$$C_\gamma = U^+(K)\gamma. \quad (7)$$

The second inclusion may be checked directly using relations such as 1 or 2, or it may be understood in terms of the so called retraction at infinity

$$r_\infty : \mathcal{J}^a \to \mathbb{A}.$$

The fibres of this retraction are precisely the $U^+(K)$-orbits in $\mathcal{J}^a$. In symbols, for any face $F$ in the standard apartment $\mathbb{A}$,

$$r_\infty^{-1}(F) = U^+(K)F.$$

For more information and proofs of these statements see [Ron09], Chapter 9 and [GL05], Section 3.4. This retraction induces a map

$$r_{\gamma^f} : \Sigma_{\gamma^f} \to \Gamma(\gamma^f)$$

defined by applying $r_\infty$ to each of the faces of a given gallery. Its fibres are $U^+(K)$-orbits, see [GL05] Proposition 5. We may hence write

$$\Sigma_{\gamma^f} = \bigsqcup_{\delta \in \Sigma_{\gamma^f}} r_{\gamma^f}^{-1}(\delta),$$

thus obtaining the equality (7).

The latter description of the cells $C_\gamma$ allows a very explicit combinatorial description. To obtain it, one needs to calculate the stabiliser of $\gamma$ in $U^+(K)$. The latter is generated by the root subgroups $U(\alpha,n)$ for $(\alpha,n) \in \Phi^+ \times \mathbb{Z}$. Any
face $F \subset A$ is stabilised by such a coroot subgroup precisely when $F \subset H_{\alpha,n}$. One may then parametrize the cells $U^+(\mathcal{K})$ explicitly as follows.

Let $\gamma$ be a combinatorial gallery denoted as above. The subgroup $U^+(\mathcal{K})$ is generated by the root subgroups $U_{(\alpha,n)}$ for $\alpha$ a positive root, $n \in \mathbb{Z}$. Now, for each $i \in \{1, \ldots, r\}$, consider the product $U_{V_i}$ of all the root subgroups in $U^+(\mathcal{K})$ which fix $V_i$ but not $F_i$ (in any fixed order). More precisely,

$$U_{V_i} := \prod_{(\alpha,n) \in \Phi_i} U_{(\alpha,n)},$$

where

$$\Phi_i = \{ (\alpha, n) \in \Phi^\text{aff} : V_i \subset H_{(\alpha,n)} \text{ and } E_i \notin H_{(\alpha,n)} \}.$$ 

Note that $U_{V_i}/U_{E_i} \cong U_i$. Now write $\gamma$ in terms of Definition 2.1 as

$$\gamma = [\gamma_0, \ldots, \gamma_r].$$

In [GL12], Proposition 6, Gaussent and Littelmann give the following description of the cell $C_\gamma$.

**Theorem 2.2.** The map

$$U_{V_0} \times U_{V_1} \times \cdots \times U_{V_{r-1}} \to \Sigma_{\gamma^t}$$

$$(u_0, \ldots, u_r) \mapsto [u_0\gamma_0, \gamma_0^{-1}u_1\gamma_0\gamma_1, \ldots, \gamma_0\cdots\gamma_{r-1}^{-1}u_r\gamma_0\cdots\gamma_r]$$

is injective and has image $C_\gamma$.

We illustrate the theorem with an example, and refer the reader to [GL12] for a full proof.

**Example 2.3.** If $\gamma = (V_0, E_0, V_1)$, then

$$U^+(\mathcal{K})\gamma \cong U^+(\mathcal{K})/U_{E_0} \cong U_{V_0}/U_{E_0} \cong U_{V_0}$$

The following statement may be deduced from the fact that the cells $C_\gamma$ are $U^+(\mathcal{K})$-stable.

**Corollary 2.4.**

$$\pi(C_{\gamma^t}) = U_{V_0} \cdots U_{V_{r-1}}[t^{\mu_r}] = U_{V_0} \cdots U_{V_r}[t^{\mu_r}]$$

3. **Crystals, Symplectic Keys and Combinatorial Galleries**

From this point on, our coroot datum $(X, \Phi, X^\vee, \Phi^\vee)$ is given as follows.
We consider $\mathbb{R}^n$ with canonical basis $\{\epsilon_1, \ldots, \epsilon_n\}$ and standard inner product $\langle -, - \rangle$, i.e. $\langle \epsilon_i, \epsilon_j \rangle = \delta_{ij}$. Let

$$
\Phi := \{\pm \epsilon_i, \epsilon_i \pm \epsilon_j\}_{i,j \in \{1, \ldots, n\}}
$$

$$
\Phi^\vee := \{\tilde{\alpha} := \frac{2\alpha}{(\alpha, \alpha)}\}
$$

$$
X := \{v \in \mathbb{R}^n : \langle v, \alpha \rangle \in \mathbb{Z}\}
$$

$$
X^\vee := \{v \in \mathbb{R}^n : \langle \alpha, v \rangle \in \mathbb{Z}\}
$$

We choose a basis $\Delta \subset \Phi$ given by

$$
\Delta = \{\epsilon_i - \epsilon_{i+1}, \epsilon_n : i \in \{1, \ldots, n-1\}\},
$$

and so

$$
\Delta^\vee = \{\epsilon_i - \epsilon_{i+1}, 2\epsilon_n : i \in \{1, \ldots, n-1\}\}
$$

is a basis for $\Phi^\vee$. Then $X$ (respectively $X^\vee$) has a $\mathbb{Z}$-basis given by $\{\omega_i\}_{i \in \{1, \ldots, n\}}$ (respectively $\{\tilde{\omega}_i\}_{i \in \{1, \ldots, n\}}$), where

$$
\omega_i = \epsilon_1 + \cdots + \epsilon_i \quad 1 \leq i \leq n-1
$$

$$
\omega_n = \frac{1}{2}(\epsilon_1 + \cdots + \epsilon_n)
$$

$$
\tilde{\omega}_1 = \epsilon_1 + \cdots + \epsilon_i \quad 1 < i \leq n.
$$

Then $G = SO(2n+1, \mathbb{C})$ and $G^\vee = Sp(2n, \mathbb{C})$.

### 3.1. Symplectic keys

Consider the ordered alphabet $C_n = \{1 < 2 < \cdots < n-1 < n < \overline{n} < \cdots < \overline{1}\}$. We denote by $W_{C_n}$ the word monoid on $C_n$.

**Definition 3.1.** A symplectic block is either a single box filled in with some letter of $C_n$ or a filling with letters of $C_n$ of an arrangement of boxes consisting of only two columns of the same length strictly larger than 1, with entries strictly increasing in columns and such that

- No letter of $C_n$ appears both barred and unbarred in the same column.
- One column may be obtained from the other (up to re-ordering) by replacing one of its entries $a \in C_n$ by $\overline{a}$ an even number of times.

A symplectic key is a concatenation of symplectic blocks.

**Example 3.2.** The following is a symplectic key:

```
1 2
2 1
```

The following is not:

```
1 2 3
2 1
```

A symplectic key will also be denoted by writing down the columns in list form, each column being a subset of elements in $C_n$ satisfying the appropriate formulation of the first condition in Definition 3.1, and reading the columns
from right to left. For example, the first symplectic key above may also be denoted by

\[ \left( \{1\}, \{1, \overline{3}\}, \{2, \overline{1}\} \right) \]

The type of a symplectic key is the list of column sizes read from right to left, once for each block. The symplectic key in the above example has type \((1, 2)\).

**Definition 3.3.** The word of a symplectic key \(K\) is the symplectic key of type \((1, \ldots, 1)\) obtained from \(K\) as follows:

- Write \(K\) as a concatenation of symplectic blocks \(B_i\):
  \[ K = B_1 \ast \cdots \ast B_r \]
  (8)
  where concatenation \(*\) of blocks is carried out starting with \(B_1\) from right to left, so that it coincides with concatenation of the corresponding galleries.

- For each of the blocks \(B_i\), let \(P_i = \{p_1, \ldots, p_s\}\) be the set of unbarred letters appearing in the right column of \(B_i\) and let \(N_i = \{n_1, \ldots, n_k\}\) be the set of barred letters appearing in the left column of \(B_i\), both written in increasing order. Define
  \[ w_i := p_1 \cdots p_s \overline{n_k} \cdots \overline{n_1} \]

- The word of \(K\) is then
  \[ w(K) := w_1 \cdots w_r \]

**Remark 5.** We consider all combinatorial galleries of type \((1, \ldots, 1)\) as elements of the word monoid \(W_{\mathcal{C}_n}\) via the bijection

\[ \{\text{Combinatorial galleries of type } (1, \ldots, 1)\} \rightarrow W_{\mathcal{C}_n} \]

\[ \{\{a_1\}, \ldots, \{a_r\}\} \mapsto a_1 \cdots a_r \]

and we use both notations freely.

**Example 3.4.** Let

\[ B_1 = \begin{array}{c}
1
\end{array}, \quad B_2 = \begin{array}{c}
\frac{1}{2} \frac{2}{1}
\end{array} \]

and

\[ K = B_1 \ast B_2 = \begin{array}{c}
1 2 1
\frac{2}{1} \frac{1}{2}
\end{array} \]

Then \(w_1 = 1, w_2 = 22\), and \(w(K) = 122\).
3.2. Crystal structure on symplectic keys. For \( i \in \{1, \ldots, n\} \) we introduce crystal operators \( e_i, f_i \) on the set of symplectic blocks endowing the latter with a crystal structure. We refer to [BG01], Section 1, for basic facts that we use concerning crystals. For a detailed account see [Kas95].

Let \( K \) be a symplectic key expressed as a concatenation of symplectic blocks \( B_i \), as in (8). We first tag some of the blocks \( B_i \) by \( \sigma \in \{+,-,+-,-+\} \). Whenever it doesn’t contain one of the letters \( \{i, i+1, \bar{i}, \bar{i}+1\} \) for \( i \neq n \), respectively one of \( \{n, \bar{n}\} \) the block is left untagged. If the block consists of one single box, then

- If its filling is \( i \) or \( i + 1 \), tag the block with \((+)\).
- If it is \( \bar{i} \) or \( i + 1 \), tag it with \((-)\).

Now assume that the block \( B_i \) consists of more than one box. Note that by the definition of a symplectic block, if a letter \( j \) appears in one of its columns, then either \( j \) or \( \bar{j} \) must appear in the other one. If either \( i \) or \( i + 1 \) appear in a column, tag the column with \((+)\), and if either \( \bar{i} \) or \( i + 1 \) appear, tag it with a \((-)\). In the other cases, leave the block untagged, unless an element of \( \{i, i+1\} \) appears in both columns, in which case the complete block is tagged with \((+)\), or one of \( \{\bar{i}, i + 1\} \), in which case the block is tagged with a \((-)\). Now ignore all untagged blocks of \( K \) to get a sub-key, and then ignore all \((-+)\), recursively producing sub-keys until a sub-key with tagging of the form

\[(+)^r(-)^s\]

is obtained.

To apply \( f_i \) (respectively \( e_i \)), change the block or column corresponding to the rightmost \((+)\) (respectively leftmost \((-)\)) tag by replacing \( i \) by \( i + 1 \), \( i + 1 \) by \( \bar{i} \) (respectively \( i + 1 \) by \( \bar{i} \) by \( i + 1 \)) and, if \( i = n \), \( n \) by \( \bar{n} \) (respectively \( \bar{n} \) by \( n \)).

Example 3.5. Let \( n = 2 \). Then,

\[
\begin{array}{c|c|c}
1 & 1 & 2 \\
2 & 2 & 2 \\
\end{array}
\rightarrow
\begin{array}{c|c|c}
1 & 1 & 1 \\
2 & 2 & 1 \\
\end{array}
\rightarrow
\begin{array}{c|c|c}
1 & 1 & 2 \\
2 & 1 & 2 \\
\end{array}
\]

3.3. The combinatorial gallery associated to a symplectic key. Let \( K \) be a symplectic key, block notation as above. For each \( k \in \{1, \ldots, n\} \) define \( \varepsilon_k := -\varepsilon_k \). For \( l \in \{1, \ldots, r\} \), write

\[B_l = (\{i_1, \ldots, i_{r_l}\}, \{j_1, \ldots, j_{r_l}\})\]
Let
\[ t_1 = \frac{1}{2} \sum_{s \in \{i_1, \ldots, i_r\}} \varepsilon_s \]
\[ t_2 = \frac{1}{2} \sum_{s \in \{j_1, \ldots, j_r\}} \varepsilon_s \]

Now define the galleries
\[ \gamma^1_k := 0 \in E_1 = \{ at_1 : a \in [0, 1] \} \rightarrow t_1 \]
\[ \gamma^2_k := 0 \in E_2 = \{ at_2 : a \in [0, 1] \} \rightarrow t_2 \]

and define the combinatorial gallery \( \gamma_k := \gamma^1_k \ast \gamma^2_k \) as the concatenation of the previously defined galleries.

**Remark 6.** Combinatorial galleries describe piecewise linear paths in \( \Lambda \) starting at the origin and ending at a coweight. The crystal structure we have defined here for combinatorial galleries is a combinatorial version of the one defined in [Lit95], Section 1 for paths.

### 4. Lakshmibai Seshadri and readable Keys

Lakshmibai Seshadri (LS) galleries were introduced by Gaussent and Littelmann in [GL05], Section 5 as combinatorial galleries of maximal dimension. They show ([GL05], Theorem C) that if \( \lambda \in X^{v,+} \) is a dominant integral coweight, \( \gamma_\lambda \) is the unique LS-gallery with endpoint \( \lambda \), \( (\Sigma_{\gamma_\lambda}, \pi) \) the associated Bott-Samelson variety as in (5), and \( \delta \in \Gamma(\gamma_\lambda) \) is a \( T \)-fixed point which is also an LS gallery, then \( \pi(C_\delta) \) is an MV-cycle in \( Z(\lambda) \). Moreover, Baumann and Gaussent show in [BG08], Theorem 5.8 that the map
\[ \Gamma(\gamma_\lambda)^{LS} \rightarrow Z(\lambda) \]
\[ \delta \mapsto \pi(C_\delta) \]

is an isomorphism of crystals. In this section we introduce certain symplectic keys called LS blocks and LS tableaux. The galleries associated to the latter as in the previous section are the LS galleries \( \Gamma(\gamma_\lambda)^{LS} \) introduced in [GL05]. We define the set \( \Gamma(\gamma_\lambda)^R \supset \Gamma(\gamma_\lambda)^{LS} \) of readable galleries as those combinatorial galleries which correspond to concatenations of LS blocks and certain galleries contained in the fundamental chamber of weight zero.

#### 4.1. Lakshmibai Seshadri Keys

For a subset \( X \subseteq C_n \), we denote the corresponding subset of barred elements by \( \overline{X} := \{ \overline{x} : x \in X \} \).
Definition 4.1. A symplectic block $K$ is called a Lakshmibai Seshadri block, or LS block, if there exist positive integers $k, r, s$ such that $2k + r + s \leq n$, and disjoint sets of positive integers

$$
A = \{a_i : 1 \leq i \leq r, a_1 < \cdots < a_r\}
$$

$$
B = \{b_i : 1 \leq i \leq s, b_1 < \cdots < b_s\}
$$

$$
Z = \{z_i : 1 \leq i \leq k, z_1 < \cdots < z_k\}
$$

$$
T = \{t_i : 1 \leq i \leq r, t_1 < \cdots < t_k\}
$$

such that

$$
K = (\{Z, T, A, B\}, \{T, Z, A, B\})
$$

and such that the elements of $T$ are uniquely characterised by the properties

$$
t_k = \max\{t \in C : t < z_k, t \notin Z \cup A \cup B\}
$$

$$
t_{j-1} = \max\{t \in C : t < \min(z_{j-1}, t_j), t \notin Z \cup A \cup B\} \text{ for } j < k.
$$

An LS key is any concatenation of LS blocks, and an LS tableau is an LS key such that its type is a list of weakly decreasing integers and such that the entries are weakly increasing from left to right. Galleries associated to LS Tableau are called LS galleries, and are denoted by $\Gamma^{LS}$. If the type of a gallery $\gamma$ is fixed, they are donoted by $\Gamma(\gamma)^{LS}$. We do not specify specific notation for LS keys since we will use them to construct readable galleries next.

Example 4.2. The symplectic block

$$
\begin{array}{c}
1 & 2 \\
2 & T
\end{array}
$$

is an LS tableau, with $A = B = \emptyset, Z = \{2\}, T = \{1\}$. The symplectic block

$$
\begin{array}{c}
1 & Z \\
2 & T
\end{array}
$$

is not LS, and note that its entries are even increasing in rows.

Remark 7. The above definition is a reformulation of Proposition 18, iii) of [GL12].

Remark 8. For $G' = \text{SL}_n(\mathbb{C})$, LS tableau are semi-standard Young tableaux, see [GL12], Section 8.4.

Remark 9. If $K$ is an LS tableau, then $f_i(K)$ is again an LS tableau, see [GL05], Corollary 2.

Remark 10. Let $\lambda = a_1 \omega_1 + \cdots + a_n \omega_n \in X^{\vee,+}$ be a dominant integral coweight, where for $1 \leq i \leq n$, we denote the i-th fundamental coweight by $\omega_i$. In their paper [KN94], Theorem 4.5.1, Kashiwara and Nakashima show that there is
4.2. Readable keys.

**Definition 4.3.** A symplectic block $B$ is called a **readable block** if it is LS or of the form

$$B = (\{1, \ldots, k\}, \{\bar{k}, \ldots, \bar{1}\}).$$

A **readable key** is a concatenation of readable blocks. Galleries associated to readable keys are called **readable galleries**, and are denoted by $\Gamma^R$. If the type of a gallery $\gamma$ is fixed, they are denoted by $\Gamma(\gamma)^R$.

We have the following result about words of readable galleries, which we prove in Section 8. We will use it in Section 7.1. It is in this sense that such galleries are called **readable**.

**Proposition 4.4.** Let $B$ be a readable block. Let $\gamma, \nu$ be two readable galleries, and $(\Sigma, \pi), (\Sigma', \pi')$ the resolutions corresponding to $\gamma \ast B \ast \nu$, respectively $\gamma \ast w(B) \ast \nu$, as in (5). Then

$$\pi(C_{\gamma \ast B \ast \nu}) = \pi'(C_{\gamma \ast w(B) \ast \nu}).$$

5. Word Reading is a Crystal Morphism

This section is the ‘symplectic’ version of [Tor14]. It will be useful in Section 7.1. We show the following statement.

**Proposition 5.1.** The map

$$\Gamma^R \xrightarrow{w} \mathcal{W}_{\mathcal{C}_n}$$

$$K \mapsto w(K)$$

is a crystal morphism.

**Example 5.2.** Let $n = 2$ and $B$ be the block

$$\begin{array}{c}
1 & 2 \\
\hline
2 & 1
\end{array}$$

Then,

$$w(B) = 2 \bar{2}$$

$$f_1(B) = \begin{array}{c}
\bar{2} & 2 \\
\hline
\bar{1} & \bar{1}
\end{array}$$

$$w(f_1(B)) = f_1(w(B)) = 2 \bar{1}$$
Proof of Proposition 5.1. Let $B$ be a readable block. Note that if it is labelled by $(-)$ then its word is labelled by $(-)$, and if two blocks $B_1, B_2$ are labelled by $(+)$ and $(-)$ respectively, then the word $w(B_1 * B_2)$ is in turn labelled by $(-)$. If the block $B$ is not labelled, then its word $w(B)$ is either labelled by a $(-)$ or not at all.

It is therefore enough to show that $f_i(w(B)) = w(f_i(B))$ for any $i \in \{1, \ldots, n\}$ and any readable block $B$. If $B$ is not LS, then it is clearly killed by $f_i$ and its word is of the form

$$w(B) = 12\ldots k\bar{k}\ldots\bar{2}1$$

which is also killed by $f_i$. If $B$ is LS, let $A, B, Z, T$ the subsets of $C_n$ as in Definition 4.1. Now, if $j \notin \{x, x-1 : x \in B \cup A \cup Z \cup T\}$, then by definition $f_j(B) = 0 = f_j(w(B))$. We write the full proof in the case that $j \in T \cup A$, the other cases being very similar, switching the barred and non-barred letters in the arguments below.

Claim 1. For $i \in \{1, \ldots, k\}$, $w(f_{t_i}(B)) = f_{t_i}(w(B))$.

First note that by conditions (9), $i = n$ is not possible, nor is $t_i + 1 \notin Z \cup A \cup B \cup T$ since then $t_i < t_i + 1 < t_i + 1 < z_i$ and $t_i + 1 < t_i + 1$ so $t_i$ wouldn’t satisfy (9).

Case. For some $j \in \{1, \ldots, k\}$, $t_i + 1 = t_j$.

Since the $t_i$ all appear either barred or unbarred in any one column, and since they don’t appear at all in the word, then $f_{t_i}$ annihilates both $B$ and $w(B)$.

Case. For some $j \in \{1, \ldots, k\}$, $t_i + 1 = z_j$.

In this case $f_{t_i}(B)$ has the same first column as $B$, and the second column is obtained from $B$ by replacing $t_i$ by $z_j$ and $\bar{z}_j$ by $\bar{t}_i$. Also note that since $z_{j-1} < z_j < z_{j+1}$ then necessarily $z_{j-1} < t_i < z_{j+1}$ because $z_{j-1} = t_i$ is impossible. Analogously $t_{i-1} < z_j < t_{i+1}$, so the order of the entries in the new LS block $f_{t_i}(B)$ isn’t changed. The word $w(f_{t_i}(B))$ is obtained from $w(B)$ precisely by replacing $\bar{z}_j$ by $\bar{t}_i$, which is precisely $f_{t_i}(w(B))$.

Case. For some $j \in \{1, \ldots, r\}$, $t_i + 1 = a_j$.

In this case $f_{t_i}(B) = 0$, and also $f_{t_i}(w(B)) = 0$ since no $t_i's$ appear at all in the word, and the $a_j's$ always appear unbarred.

Case. For some $j \in \{1, \ldots, s\}$, $t_i + 1 = b_j$.

In this case $f_{t_i}(B)$ is obtained from $B$ replacing $t_i$ with $b_j$ and $\bar{b}_j$ with $\bar{t}_i$ in the left most column of $B$. As before, the order of the new entries does not change. Hence, the word $w(f_{t_i}(B))$ is obtained from $w(B)$ by replacing $\bar{b}_j$ with $\bar{t}_i$ which is precisely $f_{t_i}(w(B))$.

Claim 2. For $i \in \{1, \ldots, k\}$, $w(f_{a_i}(B)) = f_{a_i}(w(B))$. 

If \( a_i = n \), then \( f_{a_i}(B) \) is obtained from \( B \) by replacing, in both columns, \( n \) with \( \overline{n} \). On the other hand, \( f_{a_i}(w(B)) \) is obtained from \( w(B) \) by replacing \( n \) with \( \overline{n} \), and the ordering of the alphabet means that the word \( w(f_{a_i}(w(B))) \) indeed coincides with this.

**Case.** For some \( j \in \{1, \ldots, k\} \), \( a_i + 1 = t_j \).

The LS block \( f_{a_i}(B) \) is obtained from \( B \) by replacing, in the first column, \( a_i \) with \( t_j \) and \( t_j \) with \( \overline{a_i} \). The new word \( f_{a_i}(w(B)) \) is obtained from \( w(B) \) precisely by replacing \( a_i \) with \( t_j \).

**Case.** For some \( j \in \{1, \ldots, k\} \), \( a_i + 1 = z_j \).

The LS block \( f_{a_i}(B) \) is obtained from \( B \) by replacing, in the second column, \( a_i \) with \( z_j \) and \( z_j \) with \( \overline{a_i} \), while the word \( f_{a_i}(w(B)) \) is obtained from \( w(B) \) by replacing \( z_j \) with \( a_i \).

**Case.** For some \( j \in \{1, \ldots, r\} \), \( a_i + 1 = a_j \).

In this case \( f_{a_i}(B) = f_{a_i}(w(B)) = 0 \).

**Case.** For some \( j \in \{1, \ldots, s\} \), \( a_i + 1 = b_j \).

The LS block \( f_{a_i}(B) \) is obtained from \( B \) by replacing \( a_i \) with \( b_j \) and \( b_j \) with \( \overline{a_i} \) in the first column, while \( f_{a_i}(w(B)) \) is obtained from \( w(B) \) by replacing \( a_i \) with \( b_j \).

\[ \square \]

### 6. The Symplectic Plactic Monoid

**6.1. Equivalence of readable keys.** We introduce equivalence of readable galleries, as in [Lit96]. This equivalence may be restricted to the set of words \( W_{C_n} \) defined above. We call the associated quotient monoid the *symplectic plactic monoid*, then recall the explicit relations describing it which Lecouvey has given in [Lec02]. These contain the known *Knuth relations* for type A. It is worth pointing out that having such an explicit description of these relations is crucial for the purposes of this paper.

**Definition 6.1.** Two readable keys \( \gamma \) and \( \nu \) are **equivalent** if there exist readable galleries \( \gamma^+, \nu^+ \subset C^+ \) and integers \( i_1, \ldots, i_r \) such that

- The galleries corresponding to \( \gamma^+ \), respectively \( \nu^+ \) have the same endpoint \( \lambda \in X^+ \).

  \[ f_{i_1} \cdots f_{i_r} \gamma^+ = \gamma \]

  \[ f_{i_1} \cdots f_{i_r} \nu^+ = \nu \]

This relation will be denoted by \( \gamma \sim \nu \).

Proposition 6.2 below assures us that condition in Definition 6.1 is always satisfied for readable galleries.
Proposition 6.2. For a readable gallery $\gamma$, there exist indices $i_1, \ldots, i_r$ and a readable gallery $\gamma^+$ with its faces contained in the fundamental chamber, such that $f_{i_1} \cdots f_{i_r} \gamma^+ = \gamma$.

Proof. Let $V_n$ be the vector representation of $\text{SP}_{2n}(C)$. Then the crystal of words $W_{C_n}$ is isomorphic to the crystal associated to $T(V_n) = \bigoplus_{l \in \mathbb{Z}_{\geq 0}} V_n^{\otimes l}$, see for example [Lec02] section 2.1. Therefore Proposition 6.2 holds if $\gamma$ is a word, i.e. a gallery of type $(1, \cdots, 1)$. Since word reading is a morphism of crystals by Proposition 5.1, we conclude then that Proposition 6.2 holds for any readable gallery $\gamma$. □

Remark 11. This equivalence coincides with the one defined in [Lit96], Section 2 for paths in $X^+$.

6.2. The symplectic plactic monoid. We call the quotient monoid

$$\mathcal{P}_n := W_{C_n} / \sim.$$ 

the **symplectic plactic monoid**. In [Lec02], Lecouvey gave the following explicit description of $\mathcal{P}_n$:

Theorem 6.3. The symplectic plactic monoid is isomorphic to the quotient of the word monoid $W_{C_n}$ by the congruence generated by the following relations

(R1) 

\begin{align*}
yxz & \equiv yzx \quad x \leq y < z \\
xzy & \equiv zxy \quad x < y \leq z
\end{align*}

(R2) 

\begin{align*}
yx & - 1 x - 1 \equiv yx \bar{x} \\
\bar{x} x - 1 y & \equiv \bar{x} xy \\
1 < x \leq n, x \leq y \leq \bar{x}.
\end{align*}

(R3) 

\begin{align*}
a_1 \cdots a_r z \bar{b}_s \cdots b_1 & \equiv a_1 \cdots a_r \bar{b}_s \cdots \bar{b}_1
\end{align*}

for $a_i, b_i \in \{1, \cdots, n\}$ for all $i \in \{1, \cdots, \max\{s, r\}\}$, such that

\begin{align*}
a_1 < \cdots < a_r, \\
b_1 < \cdots < b_s,
\end{align*}

and such that the left hand side of the above expression is not the word of an LS block.

Example 6.4.

\begin{align*}
12\bar{2} & \sim 1\bar{1} \sim \emptyset \\
112 & \sim 121
\end{align*}
Remark 12. Relations (R1) are the Knuth relations in type A, while relation (R3) may be understood as the general relation which specialises to $1 \sim \emptyset$. For example the gallery $1 \sim$ has the origin as endpoint and is contained in the fundamental chamber.

6.3. Image closures and plactic relations. The following geometric interpretation of the plactic relations will be proved in Section 8.

Proposition 6.5. Let $w_1, w_2$ be two words. Then

$$w_1 \sim w_2 \implies \pi(C_{w_1}) = \pi'(C_{w_2})$$

where $(\Sigma_{w_1}, \pi)$ and $(\Sigma_{w_2}, \pi')$ are the corresponding Bott-Samelson resolutions described in 5.

7. Readable Galleries and MV Cycles

We state the principal result in this paper. The proof relies on Proposition 4.4 and Proposition 6.5, together with Proposition 5.1, the results of Gaussent-Littelmann ([GL05], Theorem C) and Baumann-Gaussent([BG08], Theorem 5.8).

Theorem 7.1. Let $\lambda \in X^+$ be a dominant integral coweight, and $\gamma_\lambda$ a readable gallery contained in $C^+$ with endpoint $\lambda$. Let $\delta \in \Gamma(\gamma_\lambda)^R$, and $(\Sigma_{\gamma_\lambda}, \pi)$ the corresponding Bott-Samelson resolution as in (5). Let $\nu_\delta$ be the endpoint of the associated gallery $\delta^+$ contained in the fundamental chamber as determined in Proposition 6.2. Then

(1) $\pi(C_\delta)$ is an MV cycle in $Z(\nu_\delta)$.

(2) The map

$$\Gamma(\gamma_\lambda)^R \to \bigcup_{\gamma \in \Gamma(\gamma_\lambda)^R} Z(\nu_\gamma)$$

$$\delta \mapsto \pi(C_\delta)$$

is a morphism of crystals.

Proof. Let $\lambda \in X^{\vee,+}, \gamma_\lambda, (\Sigma_{\gamma_\lambda}, \pi)$ and $\delta \in \Gamma(\gamma_\lambda)^R$ as in the statement of the Theorem. By Proposition 6.2, we know that there exists a gallery $\delta^+ \in \Gamma(\gamma_\lambda)^R$ (with endpoint $\nu_\delta$) contained in the fundamental chamber, and integers $i_1, \cdots, i_r$ such that

$$f_{i_1} \cdots f_{i_r}(\delta^+) = \delta$$

Let $\gamma^+$ be an LS gallery with endpoint $\nu_\delta$, and consider

$$\gamma := f_{i_1} \cdots f_{i_r} (\gamma^+).$$

Then, by Definition, $\delta \sim \gamma$, and hence, by Proposition 5.1, $w(\delta) \sim w(\gamma)$. Let $(\Sigma_{w(\gamma)}, \pi'), (\Sigma_{w(\delta)}, \pi'')$ as in (5). By Proposition 6.5,

$$\pi'(C_{w(\gamma)}) = \pi''(C_{w(\delta)})$$
Hence, by Proposition 4.4,
\[
\pi(C_\gamma) = \pi'(C_{w(\gamma)}) = \pi''(C_{w(\delta)}) = \pi(C_\delta).
\]
Since \(\gamma^+\) is LS, by Remark 9 so is \(\gamma\), and hence \(\pi(C_\gamma)\) is an MV cycle in \(\mathcal{Z}(\nu_\delta)\) as shown by Gaussent and Littelmann in \([\text{GL05}], \text{Theorem C}\). We conclude that then \(\pi(C_\delta)\) is an MV cycle in \(\mathcal{Z}(\nu_\delta)\).

Now let \(1 \leq i \leq n\). Since \(\delta \sim \gamma\) it follows that \(f_i(\delta) \sim f_i(\gamma)\), and so, since \(\pi(C_{f_i(\gamma)}) = \tilde{f}_i(\pi(C_\gamma))\) by \([\text{BG08}], \text{Theorem 5.8}\), we have:
\[
\pi(C_{f_i(\delta)}) = \pi(C_{f_i(\gamma)}) = \tilde{f}_i(\pi(C_\gamma)) = \tilde{f}_i(\pi(C_\delta)).
\]

\[
\square
\]

8. Counting Positive Crossings

In this section we provide proofs of Propositions 4.4 and 6.5. We first provide simple examples which exhibit the essence of the general calculations carried out.

8.1. Examples. Let \(n = 2\). Consider the symplectic keys
\[
K_1 = \begin{bmatrix} 1 & 1 & T \\ 2 & 2 \\ T & T \end{bmatrix}
\]
\[
K_2 = \begin{bmatrix} 2 & 1 & 2 \\ 2 & T & 1 \end{bmatrix}
\]

and their words
\[
w(K_1) = 112
\]
\[
w(K_2) = 222
\]

Their corresponding galleries all have endpoint \(\varepsilon_2\). The gallery corresponding to \(K_1\) is a T-fixed point in the Bott-Samelson variety \((\Sigma(\gamma_{\omega_1} * \gamma_{\omega_2}), \pi)\), the one corresponding to \(K_2\) is a T-fixed point in \((\Sigma(\gamma_{\omega_2} * \gamma_{\omega_1}), \pi')\), and the ones corresponding to the words are T-fixed points in \((\Sigma(\gamma_{\omega_1} * \gamma_{\omega_1} \gamma_{\omega_1}), \pi'')\). Also note that \(\gamma_{\omega_1} \gamma_{\omega_2} \gamma_{\omega_2} \gamma_{\omega_1}\) since they are both contained in the fundamental chamber, have the same endpoint \(\omega_1 + \omega_2\) and
\[
f_1f_2f_1(\gamma_{\omega_1} \gamma_{\omega_2}) = K_1
\]
\[
f_1f_2f_1(\gamma_{\omega_2} \gamma_{\omega_1}) = K_2
\]
so that \(K_1 \sim K_2\). Notice that in fact, by relation R2 in Theorem 6.3
\[
w(K_1) \sim w(K_2).
\]

with \(y = x = 2\).
We know that the image

\textbf{Claim.} $\pi(C_{K_1}) = \pi''(C_{w(K_1)})$

\textbf{Proof.} Points in $\pi(C_{w(K_1)})$ are products

$$U_{(\varepsilon_1,-1)}(a)U_{(\varepsilon_1+\varepsilon_2,-1)}(b)U_{(\varepsilon_1-\varepsilon_2,-1)}(c)U_{(\varepsilon_2,0)}(d)[t^{\varepsilon_2}]$$

$$= U_{(\varepsilon_1,-1)}(a + ec)U_{(\varepsilon_1+\varepsilon_2,-1)}(b + ec^2)U_{(\varepsilon_2,0)}(c)U_{(\varepsilon_1+\varepsilon_2,0)}(d)[t^{\varepsilon_2}]$$

with $a, b, c, d, e \in \mathbb{C}$.

The points in $\pi(C_{K_1})$ are in turn products

$$U_{(\varepsilon_1,-1)}(a')U_{(\varepsilon_1+\varepsilon_2,-1)}(b')U_{(\varepsilon_2,0)}(c')U_{(\varepsilon_1+\varepsilon_2,0)}(d')[t^{\varepsilon_2}]$$

with $a', b', c', d' \in \mathbb{C}$. The latter equality holds by (2) and because $U_{(\varepsilon_1-\varepsilon_2,-1)}(e)$ stabilizes $[t^{\varepsilon_2}]$ by (1). Hence, setting the open conditions $c', a, a', a + ec \neq 0$ one finds a common dense subset of both $\pi(w(C_{K_1}))$ and $\pi(C_{K_1})$.\hfill $\square$

That $\pi(C_{K_2}) = \pi''(C_{w(K_2)})$ is much easier to check (this is a coincidence). It remains to show

\textbf{Claim.} $\pi''(C_{w(K_1)}) = \pi''(C_{w(K_2)})$

\textbf{Proof.} We write points of $\pi''(C_{w(K_2)})$ as products

$$U_{(\varepsilon_1,0)}(a)U_{(\varepsilon_1+\varepsilon_2,0)}(b)U_{(\varepsilon_1-\varepsilon_2,0)}(c)U_{(\varepsilon_2,0)}(d)[t^{\varepsilon_2}]$$

$$= U_{(\varepsilon_1+\varepsilon_2,0)}(b)U_{(\varepsilon_1,0)}(ea)U_{(\varepsilon_1+\varepsilon_2,-1)}(a^2c)U_{(\varepsilon_1-\varepsilon_2,-1)}(e)U_{(\varepsilon_2,0)}(a)U_{(\varepsilon_2,0)}(c)U_{(\varepsilon_1+\varepsilon_2,0)}(d)[t^{\varepsilon_2}]$$

$$= U_{(\varepsilon_1,-1)}(ea)U_{(\varepsilon_1+\varepsilon_2,-1)}(a^2c)U_{(\varepsilon_1-\varepsilon_2,-1)}(e)U_{(\varepsilon_2,0)}(a + c)U_{(\varepsilon_1+\varepsilon_2,0)}(d + b)[t^{\varepsilon_2}]$$

We conclude then that demanding the corresponding open conditions as in the proof of the previous Claim, we obtain the description of a common dense subset of $\pi''(C_{w(K_2)})$ and of $\pi''(C_{w(K_1)})$.\hfill $\square$

8.2. Truncated Images and Tails. Before proceeding to the proofs of Propositions 4.4 and 6.5 we consider the r-truncated image (see below) of a combinatorial gallery $\gamma$ at a special vertex $V_r = \mu_r$ and show in Proposition 8.1 that it is $U_{\mu_r}$-stable.

Let $\gamma$ be a combinatorial gallery as in (3) with endpoint the weight $V_k = \mu_k$ and let $r \leq k$ such that $V_r$ is a special vertex which we denote by $\mu_r \in X$. We know that the image $\pi(C_\gamma)$ is, by Corollary 2.4, stable under $U_0$.

\textbf{Proposition 8.1.} The r-\textit{truncated image} of $\gamma$

$$Z^{\geq r} := U_{V_r}U_{V_{r+1}}U_{V_k}[t^{\mu_k}]$$

is $U_{\mu_r}$-stable.
Proof. By Remark 2, we know that
\[ t_{\mu_r} U_0 t^{-1}_{\mu_r} = U_{\mu_r}. \]
On the other hand, we may also consider the \textit{r-truncated gallery} \( \gamma^{2r} \), which is the combinatorial gallery obtained from
\[ (V_r, E_r, V_{r+1}, \ldots, E_{k-1}, V_k) \]
by translating it to the origin. This gallery has endpoint \( \mu_k - \mu_r \) and is in turn a T-fixed point of a Bott-Samelson variety \((\Sigma, \pi')\). Since we know that the image \( \pi'(C_{\gamma^{2r}}) \) is \( U_0 \)-stable, we conclude that the r-truncated image \( Z^{2r} \) is indeed \( U_{\mu_r} \)-stable.

\begin{remark}
This Proposition is proven for \( SL_2(C) \) in [GLN13], Proposition 3. The proof we have provided is exactly the same, except for the restriction of only being able to truncate at special vertices.
\end{remark}

\section{Proof of Proposition 4.4.}

\textbf{Proof of Proposition 4.4.} We write the proof for \( \gamma = \emptyset \), an arbitrary choice of gamma changes nothing in the arguments for this case, but would introduce more notation. Speaking of which, we introduce the notation we do need. Assume that \( K \) is an LS block, and let \( A = \{a_1, \ldots, a_r\} \), \( Z = \{z_1, \ldots, z_k\} \), \( T = \{t_1, \ldots, t_k\} \), \( B = \{b_1, \ldots, b_s\} \) as in Definition 4.1. Given a set \( X \subseteq \mathbb{N} \) of natural numbers and a natural number \( r \leq n \), we will use the sets \( X^r = \{x \in X : x < n\} \) (respectively \( \lessdot, \rangle \)), and \( I_r \) will denote the set \( \{1, \ldots, r\} \). We know that \( a_1 < \cdots < a_r, b_1 < \cdots < b_s, z_1 < \cdots < z_k, t_1 < \cdots < t_k \), and in total they are ordered in some way. The following Claim allows us to write the image of \( C_{w(K)} \) down in a convenient way, regardless of the total order.

\begin{claim}
There is a dense subset of \( \pi(C_{w(K)}) \) which belongs to the set
\[ \bigcup_{a_r} U_{a_r} U_{z_1} \cdots U_{z_k} U_{b_s} \cdots U_{b_1} U_{z_k} \cdots U_{z_1} Z^{2s+r+2k+1} \]
where, for \( d \in Z \cup A \),
\[ U_d = U_{(\varepsilon_d, 0)} \prod_{l \in Z \cup A} U_{(\varepsilon_l \varepsilon_l, 0)} \prod_{l \in (Z \cup A) < d} U_{(\varepsilon_l \varepsilon_l, 0)} \prod_{l \in (Z \cup A) > d} U_{(\varepsilon_l \varepsilon_l, 0)} \]
and
\[ U_{b_l} = \prod_{l \in Z \cup A} U_{(\varepsilon_l \varepsilon_l, 0)} \prod_{l \in Z \cup A} U_{(\varepsilon_l \varepsilon_l, 1)} \]
\[ U_{z_l} = \prod_{l \in Z \cup A} U_{(\varepsilon_l \varepsilon_l, -1)} \prod_{l \in Z \cup A} U_{(\varepsilon_l \varepsilon_l, 0)} \]
\end{claim}

\textbf{Proof of Claim 3.} Assume \( z_i < a_j \). This means that \( z_i \) appears before \( a_j \) in \( w(K) \). The term belonging to \( U_{(\varepsilon_l, 0)} \) commutes with all the terms in \( U_{a_i} \), and we have, for \( l \neq a_i, z_i \):
\[ U_{(\varepsilon_l, a_i, 0)}(s) U_{(\varepsilon_l + \varepsilon_l, n)}(t) = U_{(\varepsilon_l + \varepsilon_l, n)}(s) U_{(\varepsilon_l, a_i, 0)}(t) U_{(\varepsilon_l, 0)}(s) \]
\[ U_{(\varepsilon_l, a_j, 0)}(s) U_{(\varepsilon_l + \varepsilon_l, n)}(t) = U_{(\varepsilon_l + \varepsilon_l, n)}(s) U_{(\varepsilon_l, a_j, 0)}(t) U_{(\varepsilon_l, 0)}(s) \]
Now, \( n \) is either 0 or 1. If it is 1, then \( l \in (\mathbb{Z} \cup A)^{<a_j} \). There are two possibilities for this, either \( z_i < l < a_j \) or \( l < z_i < a_j \). The first case is the only one relevant to the first of the equalities above, otherwise \( \varepsilon z_i - \varepsilon l \) is not a positive root. Since \( z_i \in (\mathbb{Z} \cup A)^{<l} \), \( U_{(\varepsilon z_i + \varepsilon l, 1)}(st) \) may be commuted back to \( U_{\mu_t} \) and proceed by induction. Likewise \( U_{(\varepsilon a_j + \varepsilon z_i, 1)}(st) \) already appears in \( U_{a_j} \). In the second case, \( l \in (\mathbb{Z} \cup A)^{<z_i} \) already appears in \( U_{a_j} \), and is handled below. If \( n = 0 \), then the only possibility left is \( z_i < a_j < l \). In the first of the above equalities, the term \( U_{(\varepsilon a_j + \varepsilon z_i, 0)}(st) \) commutes with \( U_{(\varepsilon z_i + \varepsilon l, 0)}(t) \) and also appears in \( U_{a_j} \). Again, these terms will be handled below. In the second of the above equalities, again \( U_{(\varepsilon a_j + \varepsilon z_i, 0)}(st) \) commutes with \( U_{(\varepsilon a_j + \varepsilon l, 0)}(t) \).

Now we proceed as above with the terms of \( U_{a_j} \) corresponding to the roots of the form \((\varepsilon_i + \varepsilon_j, n)\). First assume \( l \not\in (\mathbb{Z} \cup A)^{<z_i} \). We have, for \( l > a_j \):

\[
U_{(\varepsilon z_i + \varepsilon l, 0)}(s)U_{(\varepsilon a_j - \varepsilon z_i, 0)}(t) = U_{(\varepsilon a_j - \varepsilon l, 0)}(t)U_{(\varepsilon z_i + \varepsilon a_j, 0)}(st)U_{(\varepsilon z_i + \varepsilon l, 0)}(s)
\]

The term \( U_{(\varepsilon z_i + \varepsilon a_j, 0)}(st) \) in this case already appears in \( U_{a_j} \). Otherwise, \( U_{(\varepsilon z_i + \varepsilon a_j, 0)}(s) \) commutes with all elements belonging to \( U_{(\varepsilon z_i + \varepsilon l, 0)} \) and to \( U_{(\varepsilon a_j, 0)} \).

If \( l \in (\mathbb{Z} \cup A)^{<z_i} \), then terms of the form \( U_{(\varepsilon z_i + \varepsilon l, 0)} \) commute with all of the terms appearing in \( U_{a_j} \). Hence there is a dense open subset of \( U_{w(K)} \) which consists of products of non zero elements of the following form

\[
U_{a_1} \cdots U_{a_s} U_{z_1} \cdots U_{z_k} Z^{2r+k+1}
\]

Now we handle the barred terms appearing in the word \( w(K) \).

First assume \( z_i > b_j, l \not\in b_j \). This means that \( \bar{z}_i \) appears before \( \bar{b}_j \) in \( w(K) \). Then if \( l \in (A \cup Z)^{<z_i} \), all elements of \( U_{(\varepsilon l - \varepsilon z_i, 0)} \) commute with all elements in \( U_{b_j} \). Otherwise, for \( l \not\in (A \cup Z)^{<b_j} \),

\[
U_{(\varepsilon b_j - \varepsilon z_i, -1)}(s)U_{(\varepsilon l - \varepsilon b_j, 0)}(t) = U_{(\varepsilon l - \varepsilon b_j, 0)}(t)U_{(\varepsilon l - \varepsilon z_i, -1)}(st)U_{(\varepsilon b_j - \varepsilon z_i, -1)}(s)
\]

and for \( l \in (A \cup Z)^{<b_j} \)

\[
U_{(\varepsilon b_j - \varepsilon z_i, -1)}(s)U_{(\varepsilon l - \varepsilon b_j, 1)}(t) = U_{(\varepsilon l - \varepsilon b_j, 1)}(t)U_{(\varepsilon l - \varepsilon z_i, 0)}(st)U_{(\varepsilon b_j - \varepsilon z_i, -1)}(s).
\]

In both cases, the terms \( U_{(\varepsilon l - \varepsilon z_i, -1)}(st) \) respectively \( U_{(\varepsilon l - \varepsilon z_i, 0)}(st) \) belong to \( U_{a_j} \). We conclude that there is indeed a dense subset of \( \pi(C_w(K)) \) which consists of products of non-trivial elements which belong to the set

\[
U_{a_1} \cdots U_{a_r} U_{z_1} \cdots U_{z_k} \frac{U_{\bar{b}_j}}{U_{\bar{b}_j}} \cdots U_{\bar{b}_l} \frac{U_{\bar{z}_i}}{U_{\bar{z}_i}} \cdots U_{\bar{z}_m} Z^{2s+r+2k+1}
\]

Now we proceed to the proof of Proposition 4.4. An element of \( \pi(C_K) \) belongs to the set
We choose to write
\[ \mathcal{U}_{V_0} \bigcup \mathcal{U}_{V_1} \mathbb{Z}^2 \]

where
\[
\begin{align*}
\mathcal{U}_{K,a_i} &= \mathcal{U}((\varepsilon_{a_1},0) \prod_{l \in l_{i}^{a_i} \cup \mathbb{Z} \cup A} \mathcal{U}(\varepsilon_{a_1} - \varepsilon_{l},0) \prod_{l \in (\mathbb{Z} \cup A)^{a_1} \cup B \cup \mathcal{T}} \mathcal{U}(\varepsilon_{a_1} + \varepsilon_{l},0)) \\
\mathcal{U}_{K,z_i} &= \mathcal{U}((\varepsilon_{z_1},0) \prod_{l \in l_{i}^{z_i} \cup \mathbb{Z} \cup A} \mathcal{U}(\varepsilon_{z_1} - \varepsilon_{l},0) \prod_{l \in (\mathbb{Z} \cup A)^{z_1} \cup B \cup \mathcal{T}} \mathcal{U}(\varepsilon_{z_1} + \varepsilon_{l},0)) \\
\mathcal{U}_{K,B,T} &= \prod_{s,t \in \mathbb{Z} \cup A \cup B \cup T, s \in I_{a_i}^{z_i}, t \in I_{b_i}^{z_i}} \mathcal{U}(\varepsilon_{a_i} + \varepsilon_{b_i},0) \\
\mathcal{U}_{K,A} &= \prod_{i,j \in I_{b_i}^{z_i}} \mathcal{U}(\varepsilon_{a_i} + \varepsilon_{b_i},0) \\
\end{align*}
\]

Commuting the elements in the product which belong to \( \mathcal{U}_{K,B,T} \) forward, and applying
\[
\begin{align*}
\mathcal{U}((\varepsilon_{a_i} - \varepsilon_{l},0))(s) & \cdot \mathcal{U}((\varepsilon_{b_i} - \varepsilon_{l},-1))(t) = \mathcal{U}((\varepsilon_{a_i} - \varepsilon_{l},0))(s) \cdot \mathcal{U}((\varepsilon_{b_i} - \varepsilon_{l},-1))(t) \\
\mathcal{U}((\varepsilon_{a_i} - \varepsilon_{l},0))(s) & \cdot \mathcal{U}((\varepsilon_{b_i} - \varepsilon_{l},-1))(t) = \mathcal{U}((\varepsilon_{a_i} - \varepsilon_{l},0))(s) \cdot \mathcal{U}((\varepsilon_{b_i} - \varepsilon_{l},-1))(t)
\end{align*}
\]
for \( s, t \in \mathbb{C}^* \) we find a dense open subset of \( \pi(C_K) \) which is a subset of
\[ \mathcal{U}_{K,a_i} \cdot \mathcal{U}_{K,a_i} \bigcup \mathcal{U}_{K,z_i} \cdot \mathcal{U}_{K,z_i} \bigcup \mathcal{U}_{K,b_i} \cdot \mathcal{U}_{K,b_i} \bigcup \mathcal{U}_{K,T} \cdot \mathcal{U}_{K,A} \mathbb{Z}^{2 \varepsilon + 2k}, \quad (11) \]
where
\[
\begin{align*}
\mathcal{U}_{K,b_i} &= \prod_{l \in \mathbb{Z} \cup A \cup B \cup T, l \in I_{b_i}^{z_i}} \mathcal{U}(\varepsilon_{-b_i},0) \prod_{j \in I_{b_i}^{z_i}} \mathcal{U}(\varepsilon_{a_j} - \varepsilon_{b_i},0) \\
\mathcal{U}_{K,z_i} &= \prod_{l \in \mathbb{Z} \cup A \cup B, l \in I_{z_i}^{b_i}} \mathcal{U}(\varepsilon_{z_i},-1) \prod_{j \in I_{b_i}^{z_i}} \mathcal{U}(\varepsilon_{a_j} - \varepsilon_{z_i},0) \\
\mathcal{U}_{K,T} &= \prod_{s,t \in \mathbb{Z} \cup A \cup B, s \in I_{b_i}^{z_i}} \mathcal{U}(\varepsilon_{a_i} + \varepsilon_{b_i} \varepsilon_{z_i},-1)
\end{align*}
\]

Now, for \( a_i, z_i \in \Gamma^{a_j} \), elements in (11) which belong to \( \prod_{l \in (\mathbb{Z} \cup A)^{a_j}} \mathcal{U}(\varepsilon_{a_i} + \varepsilon_{l},0) \subset \mathcal{U}_{K,a_j} \) may be commuted backwards or forwards respectively and so we have a dense subset of \( \pi(C_K) \) of the form
\[ \bigcup'_{K,a_1} \cdots \bigcup'_{K,a_t} \bigcup_{K,z_1} \cdots \bigcup_{K,z_k} \bigcup'_{K,\bar{b}_1} \bigcup_{K,\bar{z}_1} \bigcup'_{K,\bar{b}_2} \bigcup_{K,\bar{z}_2} \cdots \bigcup'_{K,\bar{b}_k} \bigcup_{K,\bar{z}_k} \bigcup'_{K,T} \bigcup_{K,A} \mathbb{Z}^{2r+2k+1} \]

(12)

where

\[ \bigcup'_{K,a_j} = U_{(\varepsilon_{a_1},0)} \prod_{l \in I^a_n} U_{(\varepsilon_{a_l}-\varepsilon_l,0)} \prod_{l \in (Z \cup A)^{\varepsilon_1,B \cup T}} U_{(\varepsilon_{a_l}+\varepsilon_l,0)} \]

Now we go back to the image \( \pi(C_w(K)) \). We want a dense open subset of the latter which may be written as most recently above in (12). Let \( z_j, z_s \in I^{a_1}_n, j \in I^{s_1}_n \). Then, for \( a, b, c \in \mathbb{C}^* \),

\[ U_{(\varepsilon_{a_1}+\varepsilon_{z_j},1)}(a)U_{(\varepsilon_{z_s}+\varepsilon_{a_1},0)}(b)U_{(\varepsilon_{a_1}+\varepsilon_{z_j},1)}(a)U_{(\varepsilon_{a_1}+\varepsilon_{z_j},1)}(a)U_{(\varepsilon_{z_s}+\varepsilon_{a_1},0)}(c)U_{(\varepsilon_{a_1}+\varepsilon_{z_j},1)}(a)U_{(\varepsilon_{z_s}+\varepsilon_{a_1},0)}(c)U_{(\varepsilon_{a_1}+\varepsilon_{z_j},1)}(a) \]

Thus, the elements in the expression for \( \pi(C_w(K)) \) given by Claim 3 which belong to \( U_{(\varepsilon_{a_1}+\varepsilon_{z_j},1)} \), (which stabilize \( \nu \)) and similarly those belonging to \( U_{(\varepsilon_{z_s}+\varepsilon_{a_1},1)} \) (which also stabilize \( \nu \)) and \( U_{(\varepsilon_{a_1}+\varepsilon_{z_j},1)} \) may be commuted until the end in such a way that one gets an open subset of \( \pi(C_w(K)) \) with points non trivial products of the form

\[ \bigcup''_{K,a_1} \cdots \bigcup''_{K,a_t} \bigcup''_{K,z_1} \cdots \bigcup''_{K,z_k} \bigcup''_{B,T} \bigcup''_{K,\bar{b}_1} \bigcup''_{K,\bar{z}_1} \bigcup''_{K,\bar{b}_2} \bigcup''_{K,\bar{z}_2} \cdots \bigcup''_{K,\bar{b}_k} \bigcup''_{K,\bar{z}_k} \bigcup''_{K,T} \bigcup''_{K,A} \mathbb{Z}^{2k+r+s+1} \]

(13)

where

\[ \bigcup''_{K,a_i} = U_{(\varepsilon_{a_1},0)} \prod_{l \in I^a_n} U_{(\varepsilon_{a_l}-\varepsilon_l,0)} \prod_{l \in (Z \cup A)^{\varepsilon_1,B \cup T}} U_{(\varepsilon_{a_l}+\varepsilon_l,0)} \]

\[ \bigcup''_{K,z_i} = U_{(\varepsilon_{z_s},0)} \prod_{l \in I^{s_1}_n} U_{(\varepsilon_{z_l}-\varepsilon_l,0)} \prod_{l \in (Z \cup A)^{\varepsilon_z,B \cup T}} U_{(\varepsilon_{z_l}+\varepsilon_l,0)} \]

\[ \bigcup''_{B,T} = \prod_{l \in T \cup B} U_{(\varepsilon_{a_1}+\varepsilon_l,0)} \prod_{l \in T \cup B} U_{(\varepsilon_{z_s}+\varepsilon_l,0)} \]

\[ \bigcup''_{K,\bar{b}_i} = \prod_{l \in (Z \cup A)} U_{(\varepsilon_{\bar{b}_i}-\varepsilon_l,0)} \prod_{l \in (Z \cup A)} U_{(\varepsilon_{\bar{b}_i}+\varepsilon_l,0)} \]

\[ \bigcup''_{K,\bar{z}_i} = \prod_{l \in (Z \cup A)} U_{(\varepsilon_{\bar{z}_i}-\varepsilon_l,0)} \prod_{l \in (Z \cup A)} U_{(\varepsilon_{\bar{z}_i}+\varepsilon_l,0)} \]

Now, in the product \( \bigcup''_{B,T} \), elements belonging to \( U_{(\varepsilon_{a_1}+\varepsilon_{z_s},0)} \) commute with all elements in the product to their right, and so can be written at the end of the expression without changing anything. Now, let \( a, b \in \mathbb{C}^*, l, s \in \mathbb{Z} \).
The subgroups $U_{(\varepsilon_{a_i}+\varepsilon_{b_j},0)}$ can be commuted to the right where they appear already. The terms $U_{(\varepsilon_{a_i}+\varepsilon_{l},0)}(ab)$ and $U_{(\varepsilon_{a_i}+\varepsilon_{s},1)}(ac)$ can be commuted to the very beginning respectively to the end of the expression. The subgroups $\prod_{l \in \Gamma} U_{(\varepsilon_{a_i}+\varepsilon_{l},0)}$ stabilize the endpoint of the gallery $\nu$ and can, without producing any new terms, be commuted until the end of the product above. In a similar fashion, the subgroups $U_{(\varepsilon_{a_i}+\varepsilon_{a_j},0)}$, $U_{(\varepsilon_{a_i}+\varepsilon_{x},0)}$, $U_{(\varepsilon_{a_i}+\varepsilon_{y},0)}$ stabilize $\nu$ and can be omitted up to an open subset. The elements in $U_{(\varepsilon_{a_i},0)}$ can be commuted to the right where they appear already.

The subgroups $U_{(\varepsilon_{x}+\varepsilon_{t},0)}$ and $U_{(\varepsilon_{x}+\varepsilon_{b_j},0)}$ both stabilize $\nu$, and for $a, b, c, d \in \mathbb{C}^*$, $l \in I_{n}$, $l \notin Z \cup A$, $s \in I_{b_j}$ we get the relations:

$$U_{(\varepsilon_{x}+\varepsilon_{l},0)}(a)U_{(\varepsilon_{l}+\varepsilon_{x},0)}(b) = U_{(\varepsilon_{l}+\varepsilon_{x},0)}(ab)U_{(\varepsilon_{l}+\varepsilon_{x},0)}(a)$$

$$U_{(\varepsilon_{x}+\varepsilon_{l},0)}(c)U_{(\varepsilon_{a_i}+\varepsilon_{w},0)}(d) = U_{(\varepsilon_{a_i}+\varepsilon_{w},0)}(c)U_{(\varepsilon_{a_i}+\varepsilon_{w},0)}(d)U_{(\varepsilon_{a_i}+\varepsilon_{l},0)}(a)$$

And, for $l \notin Z \cup A$, $s \in Z \cup A$

$$U_{(\varepsilon_{x}+\varepsilon_{b_j},0)}(a)U_{(\varepsilon_{l}+\varepsilon_{x},0)}(b) = U_{(\varepsilon_{l}+\varepsilon_{x},0)}(ab)U_{(\varepsilon_{l}+\varepsilon_{x},0)}(a)$$

$$U_{(\varepsilon_{x}+\varepsilon_{b_j},0)}(c)U_{(\varepsilon_{a_i}+\varepsilon_{w},0)}(d) = U_{(\varepsilon_{a_i}+\varepsilon_{w},0)}(c)U_{(\varepsilon_{a_i}+\varepsilon_{w},0)}(d)U_{(\varepsilon_{a_i}+\varepsilon_{b_j},0)}(a)$$

Finally, note that the subgroups $U_{(\varepsilon_{a_i}+\varepsilon_{b_j},0)}$, $U_{(\varepsilon_{a_i}+\varepsilon_{b_j},0)}$, $U_{(\varepsilon_{b_i}+\varepsilon_{t},0)}$, $U_{(\varepsilon_{b_i}+\varepsilon_{t},0)}$ stabilize $\nu$, elements of $U_{(\varepsilon_{a_i}+\varepsilon_{b_j},0)}$ and $U_{(\varepsilon_{a_i}+\varepsilon_{b_j},0)}$ commute with all the terms in front of them, and that the following hold for $a, b \in \mathbb{C}^*$:

$$U_{(\varepsilon_{b_i}+\varepsilon_{b_j},0)}(a)U_{(\varepsilon_{b_j}+\varepsilon_{b_j},0)}(b) = U_{(\varepsilon_{b_i}+\varepsilon_{b_j},0)}(ab)U_{(\varepsilon_{b_i}+\varepsilon_{b_j},0)}(a)$$

$$U_{(\varepsilon_{a_i}+\varepsilon_{b_j},1)}(a)U_{(\varepsilon_{b_j}+\varepsilon_{b_j},1)}(b) = U_{(\varepsilon_{a_i}+\varepsilon_{b_j},1)}(ab)U_{(\varepsilon_{a_i}+\varepsilon_{b_j},1)}(a)$$

Hence, by computations similar or identical to previous ones we obtain a dense subset of $\pi(C_{w(K)})$ whose points are non trivial products belonging to the set (12).

Hence, by (12), there is a dense open subset of both $\pi(C_{w(K)})$ and of $\pi(C_{K})$.

If $K$ is not an LS block the Proposition may be directly checked to hold, since we may describe its word very precisely. □

8.4. Proof of Proposition 6.5.

Proof of Proposition 6.5. We verify (R1), (R2) and (R3) from Theorem 6.3 and make use of Proposition 8.1.
Claim 4. (R1) Let $x \leq y < z, w_1 = yxz, w_2 = yzx$. Then

$$\pi(C_{w_1}) = \pi(C_{w_2})$$

Proof of Claim 4.

Case. First assume that all letters are unbarred and distinct as well as mutually distinct to each others barred versions. We have

$$\pi(C_{w_1}) = \mathcal{U}_y \mathcal{U}_x \mathcal{U}_z [\ell^{\varepsilon_x + \varepsilon_y + \varepsilon_z}]$$

where

$$\mathcal{U}_y = U_{(\varepsilon_y,0)} \prod_{l \in \Gamma^*_n} U_{(\varepsilon_y - \varepsilon_l,0)} U_{(\varepsilon_y + \varepsilon_l,0)}$$

$$\mathcal{U}_x = U_{(\varepsilon_x,0)} \prod_{l \in \Gamma^*_n \neq y} U_{(\varepsilon_x - \varepsilon_l,0)} U_{(\varepsilon_x - \varepsilon_y - 1,0)} \prod_{l \neq y} U_{(\varepsilon_x + \varepsilon_l,0)} U_{(\varepsilon_x + \varepsilon_y,1)}$$

$$\mathcal{U}_z = U_{(\varepsilon_z,0)} \prod_{l \in \Gamma^*_n \neq y} U_{(\varepsilon_z - \varepsilon_l,0)} \prod_{l \neq y,x} U_{(\varepsilon_z + \varepsilon_l,0)} U_{(\varepsilon_z + \varepsilon_y,1)} U_{(\varepsilon_z + \varepsilon_z,1)}$$

for $a, b \in \mathbb{C}^*, l \in \Gamma^*_n k \neq x, y, k \in \Gamma^*_n$:

$$U_{(\varepsilon_x - \varepsilon_y,0)}(a) U_{(\varepsilon_y + \varepsilon_z,1)}(b) = U_{(\varepsilon_x + \varepsilon_z,1)}(ab) U_{(\varepsilon_x - \varepsilon_y,0)}(a)$$

$$U_{(\varepsilon_x - \varepsilon_y,0)}(a) U_{(\varepsilon_y + \varepsilon_z,1)}(b) = U_{(\varepsilon_y + \varepsilon_z,1)}(b) U_{(\varepsilon_x + \varepsilon_y,1)}(ab) U_{(\varepsilon_x - \varepsilon_z,0)}(a)$$

$$U_{(\varepsilon_x - \varepsilon_z,0)}(a) U_{(\varepsilon_x + \varepsilon_z,0)}(b) = U_{(\varepsilon_x + \varepsilon_z,0)}(ab) U_{(\varepsilon_x + \varepsilon_y,1)}(ab) U_{(\varepsilon_x - \varepsilon_z,0)}(a)$$

$$U_{(\varepsilon_x - \varepsilon_z,0)}(a) U_{(\varepsilon_x + \varepsilon_z,0)}(b) = U_{(\varepsilon_x + \varepsilon_z,0)}(ab) U_{(\varepsilon_x + \varepsilon_y,1)}(ab) U_{(\varepsilon_x - \varepsilon_z,0)}(a)$$

$$U_{(\varepsilon_x - \varepsilon_z,0)}(a) U_{(\varepsilon_x + \varepsilon_z,0)}(b) = U_{(\varepsilon_x - \varepsilon_z,0)}(ab) U_{(\varepsilon_x + \varepsilon_y,1)}(ab) U_{(\varepsilon_x - \varepsilon_z,0)}(a)$$

$$U_{(\varepsilon_x - \varepsilon_z,0)}(a) U_{(\varepsilon_x + \varepsilon_z,0)}(b) = U_{(\varepsilon_x - \varepsilon_z,0)}(ab) U_{(\varepsilon_x + \varepsilon_y,1)}(ab) U_{(\varepsilon_x - \varepsilon_z,0)}(a)$$

Hence, the terms in $\mathcal{U}_z$ may be commuted through the terms in $\mathcal{U}_x$ to produce a dense subset of $\pi(C_{w_1})$ which coincides with the dense subset of $\pi(C_{w_2})$ given by non zero parameters in all the terms involved. The same calculation yields $\pi(C_{w_2}) = \pi(C_{w_1})$.

Case. Now assume $x, y, z$ are still all different, mutually distinct to each others barred versions and one of them is barred. Then $z = \overline{b}$ for some $b \leq n$, and assume $x < y < b$.

Then:

$$\pi(C_{w_1}) = \mathcal{U}_y \mathcal{U}_x \mathcal{U}_z [\ell^{\varepsilon_x + \varepsilon_y + \varepsilon_b}]$$
where
\[
\mathcal{U}_y = \bigcup_{\varepsilon_y,0} \prod_{l \in I_0^y} \mathcal{U}_{(\varepsilon_y-\varepsilon_l,0)} \prod_{l \in I_0^y} \mathcal{U}_{(\varepsilon_y+\varepsilon_l,0)} \\
\mathcal{U}_x = \bigcup_{\varepsilon_x,0} \prod_{l \in I_0^x, l \neq y} \mathcal{U}_{(\varepsilon_x-\varepsilon_l,0)} \prod_{l \in I_0^x, l \neq y} \mathcal{U}_{(\varepsilon_x+\varepsilon_l,0)} \prod_{l \in I_0^x, l \neq y} \mathcal{U}_{(\varepsilon_x+\varepsilon_l,1)} \\
\mathcal{U}_b = \prod_{l \in b, l \neq x, y} \mathcal{U}_{(\varepsilon_l-\varepsilon_b,0)} \mathcal{U}_{(\varepsilon_l-\varepsilon_b,1)} \mathcal{U}_{(\varepsilon_x-\varepsilon_b,1)}
\]

To obtain a dense subset of the latter which is also a dense subset of \(\pi(C_{w_2})\), we operate the terms in the product \(\mathcal{U}_b\) back until just after \(\mathcal{U}_y\). The relevant relations which appear are:

\[
\mathcal{U}_{(\varepsilon_x+\varepsilon_b,0)}(a) \mathcal{U}_{(\varepsilon_x-\varepsilon_b,1)}(b) = \mathcal{U}_{(\varepsilon_x-\varepsilon_b,1)}(b) \mathcal{U}_{(\varepsilon_x+\varepsilon_b,0)}(a) \\
\mathcal{U}_{(\varepsilon_x-\varepsilon_y-1)}(a) \mathcal{U}_{(\varepsilon_y-\varepsilon_b,1)}(b) = \mathcal{U}_{(\varepsilon_x-\varepsilon_b,0)}(ab) \mathcal{U}_{(\varepsilon_x+\varepsilon_b,1)}(b) \mathcal{U}_{(\varepsilon_x+\varepsilon_y-1)}(a)
\]

Hence, \(\pi(C_{w_1}) = \pi(C_{w_2})\), and, again, the same computation yields \(\pi(C_{\overline{w_1}}) = \pi(C_{\overline{w_2}})\).

If \(x < b < y\), then in \(\mathcal{U}_b\), the term \(\mathcal{U}_{(\varepsilon_y-\varepsilon_b,1)}\) does not appear, so there are less terms to commute, and if \(b < x < y\), then \(\mathcal{U}_x\) and \(\mathcal{U}_b\) commute!

**Case.** All letters distinct, mutually distinct to each others barred versions, two of them barred.

In this case, we have \(x < y < z\), with \(y = \overline{a}, z = \overline{b}, b < a\). If \(x < b < a\),

\[
\pi(C_{w_1}) = \mathcal{U}_b \mathcal{U}_x \mathcal{U}_{\{l \neq \overline{a}, \overline{b}\}}
\]

where

\[
\mathcal{U}_b = \prod_{l \neq \overline{a}, \overline{b}} \mathcal{U}_{(l-\varepsilon_b,0)} \\
\mathcal{U}_x = \bigcup_{\varepsilon_x,0} \prod_{l \in I_0^x, l \neq a} \mathcal{U}_{(\varepsilon_x-\varepsilon_l,0)} \prod_{l \in I_0^x, l \neq a} \mathcal{U}_{(\varepsilon_x+\varepsilon_l,0)} \prod_{l \in I_0^x, l \neq a} \mathcal{U}_{(\varepsilon_x+\varepsilon_l,1)} \\
\mathcal{U}_a = \prod_{l \in a, l \neq x} \mathcal{U}_{(l-\varepsilon_a,0)} \mathcal{U}_{(l-\varepsilon_a,1)}
\]

We operate the terms \(\prod_{l \neq \overline{a}, \overline{b}} \mathcal{U}_{(l-\varepsilon_b,0)}\) back until just before \(\mathcal{U}_b\). The only relation which shows up is

\[
\mathcal{U}_{(\varepsilon_x+\varepsilon_b,0)}(a) \mathcal{U}_{(\varepsilon_x-\varepsilon_b,0)}(b) = \mathcal{U}_{(\varepsilon_x-\varepsilon_b,0)}(b) \mathcal{U}_{(\varepsilon_x+\varepsilon_b,0)}(ab) \mathcal{U}_{(\varepsilon_x+\varepsilon_b,1)}(a)
\]

Hence

\[
\overline{\pi(C_{w_1})} = \pi(C_{w_2})
\]

If \(b < x < a\), the term \(\mathcal{U}_{(\varepsilon_x-\varepsilon_b,1)}\) doesn’t appear in \(\mathcal{U}_b\), and if \(b < a < x\), no terms in \(\mathcal{U}_{(\varepsilon_x-\varepsilon_a,0)}\) appear at all, but the relevant computation is in any
case the same as above. Hence we get the above equality in all cases, and repeating the same relations one also gets

\[ \pi(C_{xxy}) = \pi(C_{yx}) \]

**Case.** All letters distinct and all barred.

We have \( x = \overline{c}, y = \overline{b}, z = \overline{a} \), with \( a < b < c \). Then

\[ \pi(C_{w_1}) = U_{\overline{y}} U_{\overline{x}} U_{\overline{y}}[t^{-(\varepsilon_a + \varepsilon_b + \varepsilon_c)}] \]

where

\[ U_{\overline{y}} = \prod_{l < b} U(\varepsilon_y, 0) \]
\[ U_{\overline{x}} = \prod_{l < c, l \neq b} U(\varepsilon_x - \varepsilon_c, 0)U(\varepsilon_b - \varepsilon_c, -1) \]
\[ U_{\overline{y}} = \prod_{l < a} U(\varepsilon_1 - \varepsilon_a, 0) \]

The only relevant relation which appears when commuting \( U_{\overline{a}} \) back until just after \( U_{\overline{b}} \) is

\[ U(\varepsilon_a - \varepsilon_c, 0)(s)U(\varepsilon_1 - \varepsilon_a, 0)(t) = U(\varepsilon_1 - \varepsilon_c, 0)(st)U(\varepsilon_1 - \varepsilon_a, 0)(t)U(\varepsilon_a - \varepsilon_c, 0)(s) \]

Which as before allows to construct open subsets such that

\[ \pi(C_{w_1}) = \pi(C_{w_2}) \]
\[ \pi(C_{wy}) = \pi(C_{wz}) \]

The second of which is a consequence, as in the other cases, of the same calculation.

**Case.** One of the letters appears barred and unbarred.

The only possibility for this to happen is that \( x < y \leq n \) and \( z = \overline{y} \). We have

\[ \pi(C_{w_1}) = U_y U_x U_{\overline{y}}[t^{\varepsilon_x}] \]

with

\[ U_y = U(\varepsilon_y, 0) \prod_{l \in \Gamma_n^y} U(\varepsilon_y - \varepsilon_l, 0) \prod_{l \in \Gamma_n} U(\varepsilon_y + \varepsilon_l, 0) \]
\[ U_x = U(\varepsilon_x, 0) \prod_{l \in \Gamma_n} U(\varepsilon_x - \varepsilon_l, 0)U(\varepsilon_x - \varepsilon_y, -1) \prod_{l \in \Gamma_n} U(\varepsilon_x + \varepsilon_l, 0)U(\varepsilon_x + \varepsilon_y, 1) \]
\[ U_{\overline{y}} = \prod_{l < y, l \in \overline{x}} U(\varepsilon_1 - \varepsilon_y, 0)U(\varepsilon_x - \varepsilon_y, -1) \]

In operating the terms \( \prod_{l \in \Gamma_n} U(\varepsilon_1 - \varepsilon_y, 0) \) back until after \( U_y \), the following relation is the only one appearing:
\[ U_{(\varepsilon x + \varepsilon y, 1)}(a) U_{(\varepsilon l - \varepsilon y, 0)}(b) = U_{(\varepsilon l - \varepsilon y, 0)}(b) U_{(\varepsilon x + \varepsilon y, 1)}(a) U_{(\varepsilon l + \varepsilon x, 1)}(ab) \]

Hence, arguing as before, the Claim is proven in this case as well.

**Case.** Two letters are equal, both unbarred:

a) \( y \leq y < z \)

b) \( y < y \leq z \)

In a), we have

\[
\pi(C_{u_1}) = \mathbb{U}_y \mathbb{U}_y' \mathbb{U}_z \left[ t^{2\varepsilon_y + \varepsilon_z} \right]
\]

where

\[
\mathbb{U}_y = U_{(\varepsilon y, 0)} \prod_{l \leq \varepsilon y} U_{(\varepsilon y - \varepsilon l, 0)} U_{(\varepsilon y + \varepsilon l, 0)}
\]

\[
\mathbb{U}_y' = U_{(\varepsilon y, 1)} \prod_{l \leq \varepsilon y} U_{(\varepsilon y - \varepsilon l, 1)} U_{(\varepsilon y + \varepsilon l, 1)}
\]

\[
\mathbb{U}_z = U_{(\varepsilon z, 0)} \prod_{l \leq \varepsilon z} U_{(\varepsilon z - \varepsilon l, 0)} U_{(\varepsilon y + \varepsilon z, 2)}
\]

In commuting the terms in \( U_{(\varepsilon x, 0)} \prod_{l \leq \varepsilon y} U_{(\varepsilon x - \varepsilon l, 0)} \prod_{l \leq \varepsilon y} U_{(\varepsilon z, 0)} \prod_{l \leq \varepsilon y} U_{(\varepsilon y + \varepsilon z, 2)} \) of \( \mathbb{U}_z \), we get the following relations:

\[
U_{(\varepsilon y - \varepsilon z, 1)}(a) U_{(\varepsilon z - \varepsilon l, 0)}(b) = U_{(\varepsilon y - \varepsilon z, 1)}(ab) U_{(\varepsilon y - \varepsilon z, 1)}(a)
\]

\[
U_{(\varepsilon y - \varepsilon z, 1)}(a) U_{(\varepsilon z, 0)}(b) = U_{(\varepsilon y - \varepsilon z, 1)}(ab) U_{(\varepsilon y - \varepsilon z, 1)}(b)
\]

Thus we get a dense open subset of \( \pi(C_{u_1}) \) which allows to establish the Claim in this case, and the same calculation gives \( \mathbb{U}_y \).

**Case.** Two letters are equal, one is barred:

a) \( y \leq y < z = \overline{b} \)

b) \( y < y \leq z = \overline{b} \)

Assuming first that \( y < b \), we get

\[
\pi(C_{u_1}) = \mathbb{U}_y \mathbb{U}_y' \mathbb{U}_y \left[ t^{2\varepsilon_y - \varepsilon_b} \right]
\]

where \( \mathbb{U}_y, \mathbb{U}_y' \) are as in the previous case and

\[
\mathbb{U}_b = \prod_{l \leq y, l < b} U_{(\varepsilon l - \varepsilon b, 0)} U_{(\varepsilon y - \varepsilon b, 2)}
\]

Again, we must commute back the terms \( \prod_{l \leq y, l < b} U_{(\varepsilon l - \varepsilon b, 0)} \) until just after \( \mathbb{U}_y \).

The only relations which appear are, for \( y < l < b, a, b \in \mathbb{C}^* \),

\[
U_{(\varepsilon y - \varepsilon l, 1)}(a) U_{(\varepsilon l - \varepsilon b, 0)}(b) = U_{(\varepsilon l - \varepsilon b, 0)}(b) U_{(\varepsilon y - \varepsilon b, 1)}(ab) U_{(\varepsilon y - \varepsilon l, 1)}(a)
\]

Note that if \( b < y \), then \( \mathbb{U}_y' \) and \( \mathbb{U}_y' \) commute. This proves the Claim in situation a), but the same calculation yields the Claim in case b).
Case. Two letters are equal, one is barred:

a) \( \overline{b} = y \leq y < z = \overline{a} \)
b) \( \overline{b} = y < z \leq z = \overline{a} \)

We again compute \( a) \), the computation for \( b) \) being exactly the same. We have

\[
\pi(C_{w_1}) = U_b U'_b U_\pi [t^{-(\epsilon_a + 2\epsilon_b)}]
\]

where

\[
U_b = \prod_{l < b} U_{(\epsilon_l - \epsilon_b, 0)}
\]
\[
U'_b = \prod_{l < b} U_{(\epsilon_l - \epsilon_b, 1)}
\]
\[
U_\pi = \prod_{l < a} U_{(\epsilon_l - \epsilon_a, 0)}
\]

Since \( U_\pi, U'_b \) commute with each other we are done, as it is the same case when dealing with \( b) \). \( \square \)

Claim 5. (R2) Let \( 1 < x \leq n, x \leq y \leq \overline{\pi} \). Consider also

\[
w_1 = yx - 1x - 1
\]
\[
w_2 = yx\overline{\pi}
\]
\[
w_3 = x - 1x - 1y
\]
\[
w_4 = x\overline{\pi}y.
\]

Then

a) \( \overline{\pi(C_{w_1})} = \overline{\pi(C_{w_2})} \)
b) \( \overline{\pi(C_{w_3})} = \overline{\pi(C_{w_4})} \)

Proof of Claim 5.

Case. Assume that \( y \) is unbarred, and that it is strictly larger than \( x \). We have

\[
\pi(C_{w_1}) = U_y U_x U_\pi [t^{\epsilon_y}]
\]
\[
\pi(C_{w_2}) = U_y U_{x-1} U_{x-1} [t^{\epsilon_y}]
\]
where

\[ U_y = \prod_{l \leq y} U_{(\varepsilon_l, 0)} \prod_{l \leq y} U_{(\varepsilon_y - \varepsilon_l, 0)} \prod_{l \leq y} U_{(\varepsilon_y + \varepsilon_l, 0)} \]

\[ U_x = \prod_{l \leq x} U_{(\varepsilon_x - \varepsilon_l, 0)} U_{(\varepsilon_x - \varepsilon_y, -1)} \prod_{l \leq y} U_{(\varepsilon_x + \varepsilon_l, 0)} U_{(\varepsilon_x + \varepsilon_y, 1)} \]

\[ U_{\mathbb{P}} = \prod_{l \leq x} U_{(\varepsilon_l - \varepsilon_x - 1, -1)} \]

\[ U_{x-1} = \prod_{l \leq x} U_{(\varepsilon_l - \varepsilon_x - 1, -1)} \]

\[ U_{x-1} = \prod_{l \leq x} U_{(\varepsilon_{x-1} - \varepsilon_l, -1)} \prod_{l \leq y} U_{(\varepsilon_{x-1} - \varepsilon_y, -2)} \prod_{l \leq y} U_{(\varepsilon_{x-1} + \varepsilon_l, -1)} U_{(\varepsilon_{x-1} + \varepsilon_y, 0)} \]

Note that any element from \( U_x \) leaves the tail \( \prod_{l \leq x} U_{x-1} \) stable, and likewise any element in \( \prod_{l < x} \) stabilises \( U_x U_{\mathbb{P}} \). We show that

\[ U_{\mathbb{P}} \subset \prod_{l \leq x} U_{x-1} \text{ and } U_{x-1} \subset U_x U_{\mathbb{P}}. \]

Then we get

\[ \pi(C_{\mathbb{U}_y}) = U_y U_x U_{\mathbb{P}} \left[ t^{\varepsilon_y} \right] \subset U_y U_{x-1} [t^{\varepsilon_y}] \subset U_y U_x U_{\mathbb{P}} [t^{\varepsilon_y}]; \]

Establishing the Claim in this case. To see \( U_{\mathbb{P}} \subset \prod_{l \leq x} U_{x-1} \), observe that all terms in \( U_{\mathbb{P}} \) stabilise the final weight \( \nu \), and that, for \( l < x - 1 < k \), the following relation holds for \( a, b \in \mathbb{C}^* \):

\[ U_{(\varepsilon_l, \varepsilon_{k-1})}(a) U_{(\varepsilon_{k-1}, -1)}(b) = U_{(\varepsilon_l, \varepsilon_k, -1)}(ab) U_{(\varepsilon_{k-1}, -1)}(b) U_{(\varepsilon_l - \varepsilon_{x-1}, 0)}(a) \]

To see \( U_{x-1} \subset U_x U_{\mathbb{P}} \), we have the following relations for \( a, b \in \mathbb{C}^* \).

\[ U_{(\varepsilon_x, 0)}(a) U_{(\varepsilon_{x-1} - \varepsilon_x, -1)}(b) = \]

\[ U_{(\varepsilon_{x-1}, -1)}(ab) U_{(\varepsilon_x + \varepsilon_{x-1}, -1)}(a^2 b) U_{(\varepsilon_{x-1} - \varepsilon_x, -1)}(b) U_{(\varepsilon_x, 0)}(a) \]

\[ U_{(\varepsilon_{x-1} - \varepsilon_y, -1)}(a) U_{(\varepsilon_{x-1} - \varepsilon_x, -1)}(b) = \]

\[ U_{(\varepsilon_{x-1} - \varepsilon_y, -2)}(ab) U_{(\varepsilon_{x-1} - \varepsilon_x, -1)}(b) U_{(\varepsilon_{x-1} - \varepsilon_y, -1)}(a) \]

\[ U_{(\varepsilon_x + \varepsilon_y, 1)}(a) U_{(\varepsilon_{x-1} - \varepsilon_x, -1)}(b) = \]

\[ U_{(\varepsilon_y + \varepsilon_{x-1}, 0)}(ab) U_{(\varepsilon_{x-1} - \varepsilon_x, -1)}(b) U_{(\varepsilon_x + \varepsilon_y, 1)}(a) \]

For \( l > x - 1, l \neq y \):

\[ U_{(\varepsilon_{x-1} - \varepsilon_x, -1)}(b) = U_{(\varepsilon_{x-1} - \varepsilon_x, -1)}(ab) U_{(\varepsilon_{x-1} - \varepsilon_x, -1)}(b) U_{(\varepsilon_x - \varepsilon_l, 0)}(a), \]

and for \( l \neq y \):

\[ U_{(\varepsilon_x + \varepsilon_l, 0)}(a) U_{(\varepsilon_x - \varepsilon_x, -1)}(b) = U_{(\varepsilon_x - \varepsilon_l, -1)}(ab) U_{(\varepsilon_x - \varepsilon_x, -1)}(b) U_{(\varepsilon_x + \varepsilon_l, 0)}(a) \]

Hence a) holds, and the same computation means that b) holds as well.

**Case.** Assume \( y = 7 \).
Once more, computations for a) and for b) are the same ones, we do only a). In this case we have
\[
\pi(C_{w_1}) = U_{x}^{\pi} \mathcal{U}_{x} [t^{-\varepsilon_b}]
\]
\[
\pi(C_{w_2}) = U_{x}^{\pi} \mathcal{U}_{x-1} [t^{-\varepsilon_b}]
\]
where
\[
U_{x-1} = \prod_{l \in I_{x-1}^{+}, l \neq b} U(\varepsilon_{x-1} - \varepsilon_{l}, 0) \prod_{l \in I_{x-1}^{+}, l \neq b} U(\varepsilon_{x-1} + \varepsilon_{l}, 0) U(\varepsilon_{x-1} + \varepsilon_{b}, -2)
\]
\[
U_{x} = U(\varepsilon_{x}, 0) \prod_{l \in I_{x}^{+}, l \neq b} U(\varepsilon_{x} - \varepsilon_{l}, 0) U(\varepsilon_{x} - \varepsilon_{b}, 1) \prod_{l \in I_{x}^{+}, l \neq b} U(\varepsilon_{x} + \varepsilon_{l}, 0) U(\varepsilon_{x} + \varepsilon_{b}, -1)
\]
\[
U_{\pi} = \prod_{l \in I_{x}} U(\varepsilon_{l} - \varepsilon_{x}, -1)
\]
It is now clear that one need only perform the same computations as in the previous case to conclude the Claim.

The last case is also similar:

**Case.** Assume now \( x = y. \)

Then we have
\[
\pi(C_{w_1}) = U_{x} U_{x}^{\pi} \mathcal{U}_{x} [t^{\varepsilon_x}]
\]
\[
\pi(C_{w_2}) = U_{x} U_{x-1}^{\pi} \mathcal{U}_{x-1} [t^{\varepsilon_x}]
\]
where
\[
U_{x}^{\prime} = U(\varepsilon_{x}, 1) \prod_{l \in I_{x}^{+}} U(\varepsilon_{x} - \varepsilon_{l}, 1) \prod_{l \in I_{x}^{+}} U(\varepsilon_{x} + \varepsilon_{l}, 1)
\]
\[
U_{\pi}^{\prime} = \prod_{l \in I_{x}} U(\varepsilon_{l} - \varepsilon_{x}, -2)
\]
\[
U_{x-1}^{\pi} = \prod_{l \in I_{x-1}} U(\varepsilon_{l} - \varepsilon_{x}, -1)
\]
\[
U_{x-1} = U(\varepsilon_{x-1}, -1) \prod_{l \in I_{x-1}^{+}} U(\varepsilon_{x-1} - \varepsilon_{l}, -1) \prod_{l \in I_{x-1}^{+}} U(\varepsilon_{x-1} + \varepsilon_{l}, -2) \prod_{l \in I_{x}} U(\varepsilon_{x-1} + \varepsilon_{l}, -1) U(\varepsilon_{x-1} + \varepsilon_{x}, 0)
\]
As in the previous two cases, we show that
\[
U_{\pi}^{\prime} \subset U_{x}^{\pi} \mathcal{U}_{x-1} \quad \text{and} \quad U_{x-1} \subset U_{x}^{\prime} U_{\pi}
\]
To claim the first condition, we only need the following relation for \( a, b \in \mathbb{C}^{*}, l < x - 1: \)
\[
U(\varepsilon_{l} - \varepsilon_{x-1}, 0) U(\varepsilon_{x-1} - \varepsilon_{x}, -2) (b) = U(\varepsilon_{l} - \varepsilon_{x}, -2) (ab) U(\varepsilon_{x-1} - \varepsilon_{x}, -2) (b) U(\varepsilon_{l} - \varepsilon_{x-1}, 0) (a)
\]
The relations needed to show the second contention are exactly the same ones as for the previous two cases. Hence we conclude that the Claim holds.
\[\square\]
Claim 6. (R3) Let \( w \) be a word that is not the word of an LS block, and such that it has the form
\[
w_1 = a_1 \cdots a_r z b^*_s \cdots b_1, \quad \text{and let} \quad w_2 = a_1 \cdots a_r b^*_s \cdots b_1
\]
with \( a_1 < \cdots < a_r < b_s > \cdots > b_1 \).

Then
\[
\pi(C_{w_1}) = \pi(C_{w_2}).
\]

Proof of Claim 6. We have
\[
\pi(C_{w_1}) = U_{a_1} \cdots U_{a_r} U_{z} U_{\tau} U_{b^*_s} \cdots U_{b^*_1}[\nu^0]
\]
where
\[
U_z = U_{(\varepsilon_1, 0)} \prod_{i \in I^s} U_{(\varepsilon_i - \varepsilon_l, 0)} \prod_{l \in A} U_{(\varepsilon_l + \varepsilon_l, 0)} \prod_{a_i \in A} U_{(\varepsilon_l + \varepsilon_a_i, 1)}
\]
\[
U_{\tau} = \prod_{a_i \in A} U_{(\varepsilon_a_i, -\varepsilon_z, 0)}
\]
and \( \nu = \sum a_i - \sum b_j \). The terms in \( U_z \) all stabilise the final weight and commute with all \( U_{b^*_j} \), while the terms of \( U_{\tau} \) all appear in \( U_{a_i} \) and commute with all terms of \( U_{a_l} \) for \( l > i \). This concludes the proof of the Claim with the usual arguments. \( \square \)

9. Non-Examples for non-readable galleries

Let \( n = 2 \) and \( \lambda = \varepsilon_1 + \varepsilon_2 \), and \( (\Sigma_{\gamma}, \pi) \) the corresponding Bott-Samelson as in (5). Let \( \gamma \) be the gallery corresponding to the block
\[
\begin{array}{c}
1 \\
2
\end{array}
\begin{array}{c}
\gamma \\
\tau
\end{array}
\]

Then points in \( \pi(C_{\gamma}) \) are of the form
\[
U_{(\varepsilon_1 + \varepsilon_2, -1)}(b)[\nu^0]
\]
for \( b \in \mathbb{C} \), hence form an affine set of dimension 1. However, elements in \( \mathcal{Z}(\lambda) \) have dimension \( \text{ht}(\lambda) = \text{ht}(\varepsilon_1 - \varepsilon_2 + 2\varepsilon_2) = 2 \), so \( \pi(C_{\gamma}) \) cannot be an MV cycle in \( \mathcal{Z}(\lambda) \). This example can be used to construct several more non examples by concatenating \( \gamma \) with itself any number of times.
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