Discrete-Time Quantum Field Theory and the Deformed Super Virasoro Algebra

M. Chaichian
High Energy Physics Division, Department of Physical Sciences, University of Helsinki and
Helsinki Institute of Physics, P.O. Box 64, FIN-00014 Helsinki, Finland

P. Prešnajder
Department of Theoretical Physics, Comenius University
Mlynská dolina, SK-84248 Bratislava, Slovakia

We show that the deformations of Virasoro and super Virasoro algebra, constructed earlier on an abstract mathematical background, emerge after Wick rotation, with an exact treatment of discrete-time free field models on a circle. The deformation parameter is $\lambda = \tau / \rho$ is the ratio of the discrete-time scale $\tau$ and the radius $\rho$ of the compact space.

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I. INTRODUCTION

Over many years much attention has been paid to the Virasoro algebra and super Virasoro algebra which play an important role in conformal field theory and string theory (see, e.g., [1], [2]). The super Virasoro algebra is an infinite Lie superalgebra with even generators $B_n$ and $H_n$ (quadratic in bosonic and fermionic oscillators respectively), and supplemented by odd generators $G_r$ (linear in bosonic and fermionic oscillators). The index $n$ is integer, $r \in \mathbb{Z} + 1/2$ for Neveu-Schwarz sector, or $r \in \mathbb{Z}$ in Ramond sector (Sugawara construction).

The deformations of the super Virasoro algebra are related to deformations of this Sugawara construction: All even generators $B_n^k$, $H_n^k$ and odd the ones $G_r^k$ are quadratic expressions in deformed bosonic and fermionic oscillators. The second index $k$ guarantees that $\{B_n^k\}$ closes to the double indexed deformed Virasoro algebra [3]. The supersymmetric extension formed by $\{B_n^k, H_n^k, G_r^k\}$ was found in [4] (the smallest set of $k$ for which these algebras close, is $k = 1, 3, \ldots$). When the deformation is removed the $k$ dependence becomes trivial.

Such deformations have been intensively studied in [5-8] mainly in connection with a formal deformations in the conformal and/or string field theories. In [7-9] it was shown that the second index $k$ is closely related to the point-splitting of Virasoro currents and to the deformation of the conformal symmetry.

Our construction in [6] represents a particular realization of the bosonization of Zamolodchikov-Faddeev algebras, which proved to be a natural framework for the deformed Virasoro algebras [6] (for an overview see [10]). However, until now meaning of the deformation, i.e. the physical interpretation of the deformation parameter $\tau$ has not been clear. In this paper we connect the parameter $\tau$ of the deformed super Virasoro algebra with the discreteness of time in QFT.

In Sec. II, we describe first the free fermionic field on a circle subject to a standard continuous time evolution, in which framework the Virasoro algebra appears naturally. Then we show how the bosonic realization of the deformed Virasoro, proposed in [6], emerges within discrete-time dynamics formulated in [7-9].

In Sec. III, we describe first the free fermionic field on a circle within discrete-time approach modifying the approach proposed in [12]. We recover the fermionic realization of the deformed Virasoro algebra. Finally, the bosonic and fermionic realizations are extended to the deformed super Virasoro algebra.

II. SCALAR FIELD ON A CIRCLE

Real time model. The field action for a free massless real scalar field on the circle $S^1$ with radius $\rho$ is defined by

$$ S[\Phi] = \frac{1}{4\pi\rho} \int_{\mathbb{R} \times S^1} dt d\tau [ (\rho \partial_t \Phi)^2 - (\partial_\tau \Phi)^2 ] . \quad (1) $$

We can expand the field $\Phi(t, \varphi) = \Phi^*(t, \varphi)$ into the Fourier modes

$$ \Phi(t, \varphi) = \sum_{k > 0} \left[ c_k(t) e^{ik\varphi} + e^{-ik\varphi} c_k^*(t) \right] , \quad k \text{ – integer} . \quad (2) $$

Inserting (2) into the action, we obtain

$$ S[\Phi] = \int_{\mathbb{R}} \frac{dt}{\rho} \sum_{k > 0} \left[ \rho^2 c_k^*(t) c_k(t) - k^2 c_k^*(t) c_k(t) \right] . \quad (3) $$
The canonically conjugate momentum to the modes \( c_k(t) \) and \( c_k^*(t) \) are \( \pi_k(t) = \rho \dot{c}_k^*(t) \) and \( \pi_k^*(t) = \rho \dot{c}_k(t) \), respectively. Solving the corresponding equations of motion we obtain the solution
\[
\Phi(t, \varphi) = \frac{i}{\sqrt{2}} \sum_{k \neq 0} \frac{1}{k} (a_k e^{-ik(t-\rho \varphi)/\rho} + b_k e^{-ik(t+\rho \varphi)/\rho}) .
\]
Here we have used the notation \( a_{-k} = a_k^* \) and \( b_{-k} = b_k^* \).

The terms with expansion coefficients \( a_k \) are interpreted as the right-movers on a closed bosonic string, whereas those solutions with \( b_k \) as the left-movers. They are independent, and we can treat them separately.

The equal-time canonical commutation relations
\[
[c_k(t), \pi_{k'}(t)] = i \hbar k' , k, k' > 0 , \quad \pi_k(t) = 0 , k, k' \neq 0 .
\]

Next we collect essential steps of the discrete-time approach proposed in [12]. In this approach the trajectory \( q(t) \) is replaced by a finite set of variables \( q_n \), \( n = 0, 1, \ldots, N \), interpreted as positions at the given discrete-times \( t_n : q_n = q(t_i + n \tau), \tau = (t_f - t_i)/N \). The action integral is replaced by the finite sum
\[
S_{\tau}[q] = \sum_{n=0}^{N-1} S(q_n, q_{n+1}) .
\]

The function \( S_n \equiv S(q_n, q_{n+1}) \), called the system function, specifies the dynamics of the system in question. The equations of motion have the form, \[12\]:
\[
\partial_{q_n} S_{\tau}[q] = \partial_{q_n} [S_{\tau}[q] + S_n] = 0 , \quad n = 1, \ldots, N-1 .
\]

A natural choice for \( S_n \) can be the time-slice Hamilton principal function \( S_n = S_c(q_n, q_{n+1}) \), i.e. the value of the action integral calculated for the classical path \( q(t) \) starting at the given point \( q_n = q(t_n) \) and terminating at \( q_{n+1} = q(t_{n+1}) \).

For systems quadratic in positions and momenta, like harmonic oscillator, the problem of quantization is reduced to the standard one. Namely, the discrete-time dynamical variables \( q_n \) and \( p_n \) should be replaced by operators satisfying the canonical "equal-time" commutations relations: \([q_n, p_n] = i\).

For such systems the momentum \( p_n \) conjugated to \( q_n \) can be defined as the product of particle mass \( \mu \) and the "discrete-time velocity":
\[
p_n = \frac{\mu}{2\tau} (q_{n+1} - q_{n-1}) , \quad \text{see e.g. [12]-[14]}. \quad \text{The canonical commutation relation then reads:}
\]
\[
[q_n, \frac{\mu}{2\tau} (q_{n+1} - q_{n-1})] = i .
\]

Let us now consider the harmonic oscillator described by the Lagrangian
\[
L(q, \dot{q}) = \frac{1}{2} \mu \dot{q}^2 - \frac{1}{2} \mu^2 \omega^2 q^2 .
\]

The corresponding Hamilton principal function in variables \( q_n = q(n \tau) \) is:
\[
S_c(q_n, q_{n+1}) = \frac{\mu \omega}{2 \sin \omega \tau} [(q_n^2 + q_{n+1}^2) \cos \omega \tau - 2q_n q_{n+1}].
\]

The solutions of equations of motion are
\[
q_n = \frac{i}{\omega} \sqrt{\frac{\omega \tau}{2 \sin \omega \tau}} \left( e^{-i \omega \tau} - a - e^{i \omega \tau} a^* \right) .
\]

The quantization can be now performed directly. Replacing \( a \) and \( a^* \) by annihilation and creation operators satisfying
\[
[a, a^*] = \frac{\sin \omega \tau}{\mu \tau} ,
\]

the canonical commutation relation \[14\] follows directly.

Let us now apply this results to the free scalar field on a circle with radius \( \rho \). Putting \( c_k(t) = (1/\sqrt{2})[c_k^R(t) + i c_k^I(t)] \), the formula \[13\] represents an action for two infinite sets of harmonic oscillators \( c_k^R = \text{Re} c_k \) and \( c_k^I = \text{Im} c_k \) with frequencies \( \omega = k/\rho \), \( k \)-positive integer. The corresponding time-slice principal Hamilton function is
\[
S_c(c_n^k, c_{n+1}^k) = \sum_{k>0} \frac{k}{2 \sin k \lambda} [(c_n^k c_{n+1}^k + c_n^k c_{n+1}^k) \cos k \lambda - e^{k \lambda} c_n^k c_{n+1}^k] .
\]

Here \( \lambda = \tau/\rho \) is the natural dimensionless parameter induced by the discreteness of the time.

The quantum discrete-time version is obtained straightforwardly, by repeating for any oscillator the steps which led from the Lagrangian \[11\] to the solutions \[12\] given in terms of annihilation and creation operators satisfying \[14\]. Performing this procedure, we obtain the discrete-time fields on a circle
\[
\Phi_n(\varphi) = \frac{i}{\sqrt{2}} \sum_{k \neq 0} \frac{\lambda}{\sin k \lambda} (a_k e^{-ik(n \lambda - \varphi)} + b_k e^{-ik(n \lambda + \varphi)}) ,
\]

with annihilation and creation operators satisfying the deformed commutation relations
\[
[a_k, a_{k'}^*] = [b_k, b_{k'}^*] = \frac{\sin k \lambda}{\lambda} \delta_{k+k',0} .
\]
\[ [a_k, b_{k'}] = 0, \quad k, k' \neq 0. \tag{17} \]

Here, \( a_{-k} = a_k^* \), \( b_{-k} = b_k^* \). The operators \( a_k, k > 0 \), can be interpreted as annihilation operators, provided \( \sin(k\lambda) > 0 \). This is guaranteed if the admissible values of \( k \) are restricted to \( 0 < k < \pi \).

**Euclidean time model.** In order to analyze the Euclidean case, we have to substitute the time \( t \) by \( -it \); as a result the the Euclidean action is defined by

\[ S[\Phi] = \frac{1}{4\pi \rho} \int_{\mathbb{R} \times S^1} dt d\varphi (\rho \partial_t \Phi)^2 + (\partial_\varphi \Phi)^2, \tag{18} \]

with the field

\[ \Phi(t, \varphi) = \sum_{k > 0} \left[ c_k(t) e^{ik\varphi} + e^{-ik\varphi} c_k^*(-t) \right], \tag{19} \]

satisfying the Euclidean reality condition, \( \Phi^d(t, \varphi) = \Phi(t, \varphi) \). The solution corresponding Euler-Lagrange equations in terms of the variables \( z = e^{(t+i\varphi)/\rho} \) and \( \bar{z} = e^{(t-i\varphi)/\rho} \), reads

\[ \Phi(z, \bar{z}) = i \sqrt{2} \sum_{k \neq 0} \frac{1}{k} (a_k z^{-k} + b_k \bar{z}^{-k}) \equiv \Phi(z) + \Phi(\bar{z}). \tag{20} \]

The canonical equal-time commutation relations are satisfied provided that \( a_k \) and \( b_k \), \( k \neq 0 \), satisfy commutation relations (17).

The Euclidean discrete-time version is obtained by analytic continuation, (12): the discrete-time step \( \tau \) is replaced by \( -i\tau \). Performing this, the Euclidean Hamilton principal function is:

\[ S_c(c_n^k, c_{n+1}^k) = \sum_{k \neq 0} \frac{k}{2 \sinh k\lambda} \left[ (e^{nk}\cosh k\lambda + e^{-nk}\cosh k\lambda) \right. \]

\[ -\left. e^{nk}\cosh k\lambda - e^{-nk}\cosh k\lambda \right] . \tag{21} \]

The solution of equations of motion for the field \( \Phi_n(\varphi) \equiv \Phi(z, \bar{z}) \) possesses in the variables \( z = e^{n\lambda+iv} \) and \( \bar{z} = e^{n\lambda-iv} \) the mode expansion

\[ \Phi(z, \bar{z}) = i \sqrt{2} \sum_{k \neq 0} \frac{1}{k} (a_k z^{-k} + b_k \bar{z}^{-k}) \equiv \Phi(z) + \Phi(\bar{z}), \tag{22} \]

where \( [k]_- = (1/\lambda) \sinh k \). The Euclidean reality condition \( \Phi^d_n(\varphi) = \Phi^*_n(\varphi) = \Phi_n(\varphi) \) is satisfied provided \( a_{-k} = a_k^* \) and \( b_{-k} = b_k^* \). The canonical commutation relations among fields \( \Phi(z, \bar{z}) \) and field momenta

\[ \Pi(z, \bar{z}) = i \sqrt{2} \sum_{k \neq 0} (a_k z^{-k} + b_k \bar{z}^{-k}). \tag{23} \]

induce the following commutation relations for the oscillator pairs:

\[ [a_k, b_{k'}] = 0, [a_k, a_{k'}] = [b_k, b_{k'}] = [k]_- \delta_{k+k',0}. \tag{24} \]

Since \( [k]_- \neq 0 \) for any \( k > 0 \), there is no restriction on allowed range of \( k \). This is different from the real discrete-time case.

The deformed Virasoro algebra generators can be expressed as contour integrals (over circle in the complex with a given radius \( r = e^{i\lambda} \), \( \lambda \)-positive integer):

\[ B_n^k = \oint \frac{dz}{2\pi i} z^n :\Pi(e^{k\lambda/2}z)\Pi(e^{-k\lambda/2}z) : \]

\[ = \frac{1}{2} \sum_j [k(\frac{n}{2} - j)]_+ a_j a_{n-j} + , \tag{25} \]

where \( \Pi(z) = \frac{1}{2\pi} \sum a_k z^{-k} \) is the holomorphic part of the field momentum \( \Pi(z, \bar{z}) \). The anti-holomorphic part \( \Pi(\bar{z}) = \frac{1}{2\pi} \sum b_k \bar{z}^{-k} \) gives rise to another set of Virasoro generators expressed in terms of \( b_k \).

### III. SUPERSYMMETRIC EXTENSION

**Fermionic oscillator.** Let us consider the continuous time fermionic Lagrangian

\[ L = \frac{i}{2} \bar{\xi}^* \dot{\xi} - \frac{i}{2} \bar{\xi} \dot{\bar{\xi}} - \omega \bar{\xi}^* \xi , \tag{26} \]

depending on complex Grassmann variables \( \xi \) and \( \xi^* \) and a parameter \( \omega \) either positive or negative. Since \( L \) is linear in velocities, we can specify only two of the four values \( \xi_i, \xi^*_i \) and \( \xi_f, \xi^*_f \) at initial and final times.

The field action which for fixed \( \xi^*_f \) and \( \xi_f \) leads to equations of motion, is

\[ S[\xi^*, \xi] = \int dt \left[ \frac{i}{2} \xi^* \dot{\xi} - \frac{i}{2} \bar{\xi} \dot{\bar{\xi}} - \omega \bar{\xi}^* \xi \right] + \frac{i}{2} \xi^*_f \xi_f + \frac{i}{2} \xi_f \xi_f . \tag{27} \]

Inserting here the solutions \( \xi = e^{-i\omega(t-t_i)} \xi_i, \xi^* = e^{-i\omega(t-t_i)} \xi^*_i \) of equations of motion, we obtain the principal Hamilton function \( S_c(\xi^*_i, \xi_i) = ie^{-i\omega(t-t_i)} \xi^*_i \xi_i \). Alternatively, for fixed \( \xi^*_i \) and \( \xi_f \) the proper field action

\[ S[\xi^*, \xi] = \int dt \left[ \frac{i}{2} \xi^* \dot{\xi} - \frac{i}{2} \bar{\xi} \dot{\bar{\xi}} - \omega \bar{\xi}^* \xi \right] - \frac{i}{2} \xi_f \xi_f - \frac{i}{2} \xi^*_f \xi^*_f . \tag{28} \]

leads to the principal Hamilton function \( S_c(\xi^*_f, \xi_f) = ie^{i\omega(t-t_f-t)} \xi^*_f \xi^*_f \).

The Lagrangian (26) is singular and gives rise to first class constraints \( \eta(t) + \frac{i}{2} \xi^* = 0 \) and \( \eta^*(t) - \frac{i}{2} \xi = 0 \) among fermionic coordinates \( \xi = \xi(t), \xi^* = \xi^*(t) \) and corresponding momenta \( \eta = \eta(t), \eta^* = \eta^*(t) \). They lead to Dirac brackets

\[ \{\xi(t), \xi(t)\} = \{\xi^*(t), \xi^*(t)\} = 0, \quad \{\xi(t), \xi^*(t)\} = 1 \]

which after quantization are replaced by anticommutation relations

\[ [\xi(t), \xi(t)] = [\xi^*(t), \xi^*(t)] = 0, \quad [\xi(t), \xi^*(t)] = 1. \tag{29} \]
In the discrete-time case as dynamical variables we take \( \xi_n \equiv \xi(n\tau) \) for \( n \)-odd, and \( \xi_{n'} \equiv \xi^*(n'\tau) \) for \( n' \)-even (or equivalently, with the role of \( n \) even and odd interchanged). For the time-slice system function we can take either \( S_c(\xi_{n'+1}, \xi_n) + S_c(\xi_{n'-1}, \xi_n) \) or \( S_c(\xi_{n'}, \xi_{n'+1}) + S_c(\xi_{n'}, \xi_{n'+1}) \). Both choices give the same expression
\[
S_\tau = i \sum_n (\xi_{n+1}^* \xi_n e^{-i\omega \tau} - \xi_{n-1}^* \xi_n e^{i\omega \tau})
\]
\[
= i \sum_{n'} (\xi_{n'}^* \xi_{n'-1} e^{-i\omega \tau} - \xi_{n'}^* \xi_{n'+1} e^{i\omega \tau}) .
\] (30)
The corresponding equations of motion
\[
\xi_{n'+1} e^{i\omega \tau} = \xi_{n'-1} e^{-i\omega \tau} ; \quad \xi_{n+1}^* e^{-i\omega \tau} = \xi_{n-1}^* e^{i\omega \tau} ,
\] have the solution
\[
\xi_n = \frac{1}{\cos \omega \tau} e^{-i\omega \tau} b , \quad \xi_{n'} = \frac{1}{\cos \omega \tau} e^{i\omega \tau} b^* .
\] (31)
In the discrete-time case we replace \( \{ , \} \) by anticommutators
\[
[\xi_n, \xi_n] = [\xi_{n'}, \xi_{n'}] = 0 ,
\]
\[
[\xi_n, \frac{1}{2}(\xi_{n+1}^* + \xi_{n-1}^*)] = 1 , \quad \text{or} \quad [\xi_{n'}, \frac{1}{2}(\xi_{n'+1}^* + \xi_{n'-1}^*)] = 1 .
\] (32)
Here we have taken into account that \( \xi_n \) is affiliated with \( n \)-odd, whereas \( \xi_{n'} \) with \( n' \)-even, i.e. \( \xi_{n'} \) is a link variable with respect to \( \xi_n \), and vice versa. The second line in \( \{ , \} \) indicates both alternative choices (in the first (second) anticommutator we replaced \( \xi_{n'} \) (\( \xi_n \)) by the nearest neighbor average). Inserting here solutions \( \{ , \} \) we find that the anticommutation relations \( \{ , \} \) are satisfied, for both choices, provided that \( b \) and \( b^* \) satisfy anticommutation relations
\[
[b, b] = [b^*, b^*] = 0 , \quad [b, b^*] = \cos \omega \tau .
\] (33)
The left-hand-side of the last anticommutator represents a positive operator. Therefore, we require \( \cos \omega \tau > 0 \), i.e. \( -\pi/2 < \omega \tau < +\pi/2 \). For \( \omega > 0 \) we interpret \( b \) as an annihilation operator and \( b^* \) as a creation one, for \( \omega < 0 \) their interpretation is reversed.

The Euclidean version is obtained by the replacement \( \tau \rightarrow -i\tau \). Repeating all the steps leading to \( \{ , \} \), we obtain a fermionic oscillator pair satisfying the anticommutation relations
\[
[b, b] = [b^*, b^*] = 0 , \quad [b, b^*] = \cosh \omega \tau .
\] (34)
In this case there is no restriction on admissible values of \( \omega \tau \).

**Fermionic field on a circle.** The real time free fermionic field on a circle in Neveu-Schwarz sector satisfies antiperiodic boundary conditions \( \Psi(t, \varphi + 2\pi) = -\Psi(t, \varphi) \) and \( \Psi^*(t, \varphi + 2\pi) = -\Psi^*(t, \varphi) \). Such a field can be expanded as
\[
\Psi(t, \varphi) = \sum_r \xi_r(t) e^{ir\varphi} ; \quad \Psi^*(t, \varphi) = \sum_r \xi^*_r(t) e^{-ir\varphi} .
\] (35)
Here \( r \) is half-integer, \( \xi_r = \xi_r(t) \) and \( \xi^*_r = \xi^*_r(t) \) are anticommuting variables. Below, we consider only the field action on a circle with radius \( \rho \) for left-movers:
\[
S[\Psi^*, \Psi] = \frac{1}{2\pi \rho} \int_{D_\rho^1} dtd\varphi \left[ \frac{i}{\rho} (\Psi^* \bar{\Psi} - \bar{\Psi} \Psi) + i\bar{\Psi} \partial_\varphi \Psi \right] \pm \frac{i}{4\pi} \int_{S^1} d\varphi [\Psi^* \bar{\Psi} + \bar{\Psi} \Psi] .
\] (36)
The action for right-movers is obtained by replacement \( +i\partial_\varphi \rightarrow -i\partial_\varphi \). This system contains first class constraints which lead to the Dirac brackets:
\[
\{ \Psi(t, \varphi), \Psi^*(t', \varphi') \}_D = \{ \Psi^*(t, \varphi), \Psi(t', \varphi') \}_D = 0 ,
\]
\[
\{ \Psi(t, \varphi), \Psi^*(t', \varphi') \}_D = 2\pi \delta(\varphi - \varphi') .
\] (37)

The +/- signs in \( \{ , \} \) refer to the cases with the following fixed values of final and initial fields:
\[+ \text{ sign : } \Psi_i(\varphi) = \sum_r \xi_{ir} e^{ir\varphi} ; \quad \Psi^*_i(\varphi) = \sum_r \xi^*_{ir} e^{-ir\varphi} ,\]
\[\quad \Psi_f(\varphi) = \sum_r \xi_{fr} e^{ir\varphi} ; \quad \Psi^*_f(\varphi) = \sum_r \xi^*_{fr} e^{-ir\varphi} .\] (38)

Inserting the expansions \( \{ , \} \) into \( \{ , \} \), we obtain the action
\[
S[\Psi^*, \Psi] = \int dt \sum_r \left[ \frac{1}{2} \xi_r^*(t) \dot{\xi}_r(t) - \frac{i}{2} \xi_r^*(t) \dot{\xi}_r(t) \right]
\]
\[
- \frac{r}{\rho} \xi_r^*(t) \dot{\xi}_r(t) \pm \frac{i}{2} \sum_r [\xi^*_r \partial_t \xi_r + \xi_r \partial_t \xi^*_r] ,
\] (40)

describing the system of independent fermionic oscillators with frequencies \( \omega = r/\rho \).

For \( \{ , \} \) the solution of equations of motion
\[
\dot{\xi}_r(t) = e^{-ir(t-t_i)/\rho} \xi_{ir} , \quad \dot{\xi}_r^*(t) = e^{-ir(t-t_i)/\rho} \xi_{fr} ,
\] gives the Hamilton principal function
\[
S_c(\Psi_f, \Psi_i) = \int dt \sum_r e^{-ir(t-t_i)/\rho} \xi^*_r \dot{\xi}_r + \frac{i}{2} \sum_r e^{-ir(t-t_i)/\rho} \xi^*_r \xi_r .
\] (41)
Similarly, for \( \{ , \} \) the solution
\[
\dot{\xi}_r(t) = e^{ir(t-t_i)/\rho} \xi_{fr} , \quad \dot{\xi}_r^*(t) = e^{ir(t-t_i)/\rho} \xi_{ir} ,
\]
induces the Hamilton principal function
\[ S_\tau(\Psi', \Psi_f) = i \sum_r e^{ir(t_r - t_i)/\rho} \xi_f r \xi_{ir}^r . \] (42)

In the discrete-time approach, the field modes are described by fermionic variables \( \xi_n^r \equiv \xi^r(n\tau) \) for n-odd, and \( \xi_{n'}^r \equiv \xi^r*(n'\tau) \) for n'-even. Motivated by (37) and (38) we take the time-slice system function in the form
\[ S_\tau = \sum_r \sum_n \left( \xi_{n+1}^r \xi_n^r e^{-ir\lambda} - \xi_n^r \xi_{n-1}^r e^{ir\lambda} \right) \]
\[ = \sum_r \sum_{n'} \left( \xi_{n'}^r \xi_{n'-1}^r e^{-ir\lambda} - \xi_{n'}^r \xi_{n'+1}^r e^{ir\lambda} \right) . \] (43)

After quantization the discrete time analogs of Dirac brackets (34) are replaced by the anticommutation relations
\[ [\Psi_n(\varphi), \Psi_{n'}(\varphi') = [\Psi^*_n(\varphi), \Psi^*_{n'}(\varphi')] = 0 , \]
\[ [\Psi_n(\varphi), \frac{1}{2}(\Psi_{n+1}(\varphi) + \Psi_{n-1}(\varphi')) = 2\delta(\varphi - \varphi') . \] (44)

Here, we have taken into account the fact that \( \Psi_n(\varphi) \) is affiliated with n-odd, and \( \Psi^*_n(\varphi) \) with n'-even, and again we have replaced \( \Psi^*_n(\varphi) \) by the nearest neighbor average.

Expanding the fields \( \Psi_n(\varphi) \) and \( \Psi^*_n(\varphi) \) into solutions of equations of motion
\[ \xi_n^r = \frac{1}{\cos r\lambda} e^{-ir\lambda} b_r \] \( \xi_n^r = \frac{1}{\cos r\lambda} e^{ir\lambda} b_r \] \( \xi_{n'}^r = \frac{1}{\cos r\lambda} e^{in'\lambda} b^*_r \]
\( \xi_{n'}^r = \frac{1}{\cos r\lambda} e^{in'\lambda} b^*_r \), \( r > 0 , \)
\( \xi_n^r = \frac{1}{\cos r\lambda} e^{-in\lambda} b_r \] \( \xi_n^r = \frac{1}{\cos r\lambda} e^{-in\lambda} b_r \), \( r < 0 , \)
\( \xi_{n'}^r = \frac{1}{\cos r\lambda} e^{-in'\lambda} b^*_r \] \( \xi_{n'}^r = \frac{1}{\cos r\lambda} e^{-in'\lambda} b^*_r \) it can be easily seen that the anticommutation relations (44) are satisfied provided \( b_r, b^*_r, b^*_r \) and \( b^*_r \) satisfy the anticommutation relations
\[ [b_r, b^*_r] = 0 , [b_r, b^*_r] = [b^*_r, b^*_r] = [b_r, b^*_r] = \cos(\lambda r) \delta_{r+r',0} . \] (46)

Here we put \( b_{r-r} = b^*_r \) and \( b_{r-r} = b^*_r \). For a given \( \lambda \) the admissible values of \( r \) are specified by the inequality \( -\frac{1}{\lambda} \pi < r < \frac{1}{\lambda} \pi \).

In the Euclidean discrete-time case the anticommutation relations (46) are replaced by
\[ [b_r, b^*_r] = 0 , [b_r, b^*_r] = [b^*_r, b^*_r] = [b_r, b^*_r] = [b^*_r, b^*_r] = [r] + \delta_{r+r',0} , \] (47)
where \( [r] = \cosh r\lambda \) (there is no restriction on admissible values of \( r \)). Thus, we obtain two independent sets of fermionic oscillators both satisfying exactly the anticommutation relations required for the deformation in question (3). Any of them can be used for a fermionic realization of deformed Virasoro algebra and its supersymmetric extension.

The deformed Virasoro algebra generators are usually expressed in terms of an auxiliary Euclidean real field \( \psi(z) \) depending, for right-movers, on the variable \( z = e^{n\lambda+\rho} \). This field can be determined as follows. We restrict ourselves to modes with \( r > 0 \). The Euclidean field solutions then possess the expansions (into functions (48) with \( \lambda \) replaced \( -\lambda \))
\[ \Psi_n(\varphi) = \sum_{r>0} \frac{1}{\cos r\lambda} e^{-ir\lambda+ir\rho} b_r, \ n - \text{odd} , \]
\[ \Psi^*_n(\varphi) = \sum_{r>0} \frac{1}{\cos r\lambda} e^{ir\rho-ir\lambda} b^*_r, \ n' - \text{even} . \] (48)

In order to construct \( \psi(z) \) we take the proper discrete-time nearest neighbor combinations:
\[ \frac{1}{2}(\Psi_n(\varphi + \lambda) + \Psi_n(\varphi - \lambda)) + \frac{1}{2}(\Psi^*_{n+1}(\varphi) + \Psi^*_{n-1}(\varphi)) \]
\[ = \sum_{r>0} e^{in\lambda-r\varphi} b_r + \sum_{r>0} e^{in\lambda+r\varphi} b^*_r , \ n - \text{odd} . \]
\[ \frac{1}{2}(\Psi^*_n(\varphi + \lambda) + \Psi^*_n(\varphi - \lambda)) + \frac{1}{2}(\Psi^*_{n+1}(\varphi) + \Psi^*_{n-1}(\varphi)) \]
\[ = \sum_{r>0} e^{i(n'\lambda-r\varphi)} b^*_r + \sum_{r>0} e^{-i(n'\lambda-r\varphi)} b^*_r \] (50)

Equations (49) and (50) define the real Euclidean field
\[ \psi(z) = \sum_r b_r z^{-r} , \ z = e^{n\lambda+\rho} , \ n - \text{integer} . \] (51)

Repeating the same procedure for modes with \( r < 0 \), we obtain another auxiliary field \( \tilde{\psi}(\tilde{z}) \) in terms of \( \tilde{b}_r \).

The deformed Virasoro algebra generators in the fermionic realization are given in terms of \( \psi(z) \) as follows:
\[ H^k_n = \int \frac{dz}{2\pi i} \sum_r \frac{1}{2\lambda} \psi(e^{k\lambda/2}z) \psi(e^{-k\lambda/2}z) : \]
\[ = \frac{1}{2} \sum_r [k(n' - r)] : b_r \tilde{b}_{n-r} : . \] (52)

They satisfy, up to central term, the same commutation relations as \( B^k_n \) given in terms of \( \Pi(z) \) in (25). Both \( B^k_n \) and \( H^k_n \) span the even part of the deformed super Virasoro algebra, whereas the odd generators are given in terms of \( \psi(z) \) and \( \Pi(z) \):
\[ G^k_r = \int \frac{dz}{2\pi i} \sum z^r \psi(e^{k\lambda/2}z) \Pi(e^{-k\lambda/2}z) \]
\[
\sum_j e^{-k(\xi - j)\lambda} a_j b_{r-j}.
\]

The graded commutator relations generated by \(\{B_n^k, H_n^k, G_n^k\}\) can be found, e.g. in [9] (our definition of \([k]_\pm\) differs from that given there by factor \(\kappa = \sqrt{(1/\lambda) \sinh \lambda}\), \([k]_+\) is the same; consequently, the bosonic oscillators \(a_k\) used above are multiplied by constant \(\kappa\) with respect to those used in [9], the fermionic oscillators \(b_r\) are unchanged).

**IV. CONCLUDING REMARKS**

The deformed Virasoro algebra \(\mathfrak{g}\) and its supersymmetric extension \(\mathfrak{h}\) were suggested earlier on purely formal mathematical grounds. The field theoretical origin of the deformed (super) Virasoro algebras, formulated above in the framework of the Euclidean discrete-time QFT, can serve for a better (physical) motivation and understanding of its role in all related constructions (q-strings, q-vertex operators and Zamolodchikov-Faddeev algebras).

In this context it would be of great interest to extend our model to the deformed Kac-Moody algebras, see e.g. [15] and refs therein. It is plausible that this can be achieved along the same lines as in the Virasoro algebra case: (i) The Kac-Moody algebras can be defined as free field current algebras on a circle; (ii) Their deformations are realized in terms of various sets of deformed bosonic and/or fermionic oscillators. As other possible application could serve discrete-time integrable models, see [16].

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