DYNAMICAL SYSTEMS WITH FIRST- AND SECOND-CLASS CONSTRAINTS AND THE SECOND NOETHER THEOREM

S.A.Gogilidze\textsuperscript{1}, Yu.S.Surovtsev\textsuperscript{2}
\textsuperscript{1} Tbilisi State University, Tbilisi, University St.9, 380086 Republic of Georgia,
\textsuperscript{2} Bogoliubov Laboratory of Theoretical Physics, Joint Institute for Nuclear Research,
141 980 Dubna, Moscow Region, Russia

Abstract

Dynamical systems, described by Lagrangians with first- and second-class constraints, are investigated. In the Dirac approach to the generalized Hamiltonian formalism, the classification and separation of the first- and second-class constraints are presented with the help of passing to an equivalent canonical set of constraints. The general structure of second-class constraints is clarified. The method of constructing the generator of symmetry transformations in the second Noether theorem is given. It is proved that second-class constraints do not contribute to the transformation law of the local symmetry which entirely is entirely stipulated by all the first-class constraints.

1 Introduction

The role of the second Noether theorem as a basis of constrained theories is well known. Our objective is to construct a generator of local-symmetry transformations in this theorem for the general case of dynamical systems with first- and second-class constraints. A number of works is devoted to this matter beginning with the basic works by Dirac \cite{1}. For systems with only first-class constraints this problem was solved in our previous papers \cite{2}-\cite{4}. However, there have recently appeared papers \cite{5, 6} in which one even asserts that second-class constraints contribute also to a generator of gauge transformations which become global in the absence of first-class constraints \cite{5}. In the present work, we extend our scheme to theories also with second-class constraints and prove that these constraints do not contribute to the local-symmetry transformation law. For this aim, in the framework of the generalised Hamiltonian formalism (with no modifications) we first outline a separation scheme of constraints into the first- and second-class ones which is based on passing to an equivalent canonical set of constraints. Just the latter will be needed in what follows when deriving the local-symmetry transformations. Note that only in recent years there have appeared real schemes of such separation of constraints \cite{7, 8} proposed, however, for a modified generalised Hamiltonian formalism. Enough complete exposition of the structure of theories with second-class constraints is given in the works \cite{9, 10}.

2 Separation of First- and Second-Class Constraints

Let us consider a system with a number of degrees of freedom \( N \) described by a degenerate Lagrangian \( L(q, \dot{q}) \), where \( q = (q_1, \ldots, q_N) \) and \( \dot{q} = dq/dt = (\dot{q}_1, \ldots, \dot{q}_N) \) are generalized
coordinates and velocities, respectively. After passing to the Hamiltonian formalism, let $A$ primary constraints be obtained in the phase space $(q, p)$. Further, in accordance with the Dirac procedure of breeding constraints \[3\], let us have a complete and irreducible system of constraints $\phi^m_\alpha$, where $\alpha = 1, \cdots, A$ and $m_\alpha = 1, 2, \cdots, M_\alpha$ $(\sum_{\alpha=1}^A M_\alpha = M)$. Let
\[
\text{rank} \left\{ \phi^m_\alpha, \phi^m_\beta \right\} = 2R < M,
\]
which implies the presence of $2R$ constraints of second class $\Psi^m_\alpha$ and $M-2R$ constraints of first class $\Phi^m_\alpha$ \[3\]. The constraint sets $(\Phi, \Psi)$ and $\phi^m_\alpha$ are related with each other by the equivalence transformation. A possibility of constructing the set $(\Phi, \Psi)$ was indicated by Dirac. However, for practical aims (for example, to elucidate the role of second-class constraints in gauge transformations [2]) the knowledge of an explicit form of the set $(\Phi, \Psi)$ is needed. Pass to the $(\Phi, \Psi)$ set through several successive stages.

So, let us consider the antisymmetric matrix $K^{11}$ with elements $K^{1\beta}_{\alpha \gamma} = \{\phi^1_\alpha, \phi^1_\beta\}$, and let
\[
\text{rank} \left\| K^{1\beta}_{\alpha \gamma} \right\| \Sigma \equiv A_1 = 2R_1 < A
\]
($\Sigma$ means this equality to hold on the primary constraint surface $\Sigma_1$), i.e. $A_1$ primary constraints exhibit their nature of second class already at this stage provided that subsequent stages of our procedure do not change the established properties. One can regard the principal minor of rank $A_1$, disposed in the left upper corner of the matrix $K^{11}$, to be nonzero. Write down
\[
\{\phi^1_\alpha, \phi^1_\beta\} = f_{\alpha \beta \gamma} \phi^1_\gamma + D_{\alpha \beta}, \quad \alpha, \beta, \gamma = 1, \cdots, A,
\]
where $D_{\alpha \beta} \Sigma \equiv F_{\alpha \beta}$. Among $F_{\alpha \beta}$ ($\alpha = 2, \cdots, A_1$) at least one element is nonzero in accordance with the supposition \[3\]. Let $F_{12} \neq 0$. Pass to a new set of constraints:
\[
^1\phi^1_{1} = \phi^1_{1}, \quad ^1\phi^1_{2} = \phi^1_{2}, \quad ^1\phi^1_{\alpha} = \phi^1_{\alpha} + ^1u_{\alpha 1} \phi^1_{1} + ^1u_{\alpha 2} \phi^1_{2}, \quad \alpha = 3, \cdots, A.
\]
The left superscripts indicate a stage of our procedure and will be omitted in the resultant expressions. Determine $^1u_{\alpha 1} = D_{2\alpha}/D_{12}$, $^1u_{\alpha 2} = -D_{1\alpha}/D_{12}$, $\alpha = 3, 4, \cdots, A$, to guarantee the fulfilment of the requirements: $\{^1\phi^1_{1}, ^1\phi^1_{\alpha}\} \Sigma \equiv 0$, $\{^1\phi^1_{2}, ^1\phi^1_{\alpha}\} \Sigma \equiv 0$, and to obtain: $^1D_{12} = -^1D_{21} \Sigma \equiv 1F_{12} = F_{12} \neq 0$, $^1D_{\alpha \beta} \Sigma \equiv F_{\alpha \beta} = 0$, $\alpha = 1, 2$, $\beta = 3, 4, \cdots, A$.

So, by means of the transformation $^1\phi^1_{\alpha} = ^1\Lambda_{\alpha \beta} \phi^1_\beta$, $\det ||^1\Lambda_{\alpha \beta}|| = 1$, we obtain
\[
^1K^{1\beta}_{\alpha \gamma} = \{^1\phi^1_{\alpha}, ^1\phi^1_{\beta}\} = ^1\Lambda_{\alpha \sigma} ^1\Lambda_{\beta \tau} K^{1\tau}_{\sigma \gamma} + O(\phi^1_{\alpha}),
\]
where $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $O$ are zero blocks, and $||^1\mathcal{F}_{\alpha \beta}||$ is $(A - 2) \times (A - 2)$-block which must be reduced to the quasidiagonal form at the next stages of the procedure.
This procedure must be iterated \( R_1 = A_1/2 \) times, and we shall obtain

\[
R_1 \mathbf{K}^{11} \equiv \begin{pmatrix}
F_{12} \cdot \mathbf{J} & \mathbf{O} & \ldots & \mathbf{O} \\
\mathbf{O} & F_{34} \cdot \mathbf{J} & \ldots & \mathbf{O} \\
\vdots & \vdots & \ddots & \vdots \\
\mathbf{O} & \mathbf{O} & \ldots & R_1^{-1} F_{A_1-1} A_1 \cdot \mathbf{J}
\end{pmatrix} \mathbf{O}.
\]  

(6)

The corresponding equivalent set of primary constraints is determined by the relation:

\[
R_1 \phi^1_{\alpha} = R_1 A_{\alpha\beta} R_1^{-1} A_{\beta\gamma} \cdots 1 A_{\sigma\tau} \phi^1_{\tau} = \Lambda_{\alpha\beta} \phi^1_{\beta} , \quad \det \mathbf{X} = 1.
\]

Denoting the second-class constraints by the letter \( \psi(\Psi) \), we now have the following set of primary constraints: \( [\psi^1_{a_1}]_{a_1=1} \), \( [\phi^1_{a_1}]_{a_1=1} \) with the properties

\[
\{ \psi^1_{a_1}, \psi^1_{b_1} \} \equiv \begin{cases} 
F_{a_1 b_1} 
eq 0, & a_1 = 2k + 1, \ b_1 = 2k + 2 \ (k = 0, 1, \ldots, A - 1) \\
0 & \text{and conversely,} \\
0 & \text{in other cases,}
\end{cases}
\]

(7)

\[
\{ \psi^1_{a_1}, \phi^1_{b_1} \} \equiv 0, \quad \{ \phi^1_{a_1}, \phi^1_{b_1} \} \equiv 0.
\]

(8)

It is clear that the constraints \( \psi^1_{a_1} \) do not generate secondary constraints. Furthermore, with the help of the transformation \( \phi^m_{\alpha} = \phi^m_{\alpha} + C^m_{\alpha \beta_1} \psi^1_{\beta_1} \)

one can attain that

\[
\{
\psi^1_{a_1}, \phi^m_{\beta_1}\} \equiv 0, \quad m_\alpha = 2, \ldots, M_\alpha.
\]

Now let us turn to \( \phi^m_{\alpha}, \alpha_1 = 1, \ldots, A - A_1 \). Let rank \( \| \phi^1_{a_1}, \phi^2_{b_1} \| \) \( \equiv A_2 < A - A_1 \)

(\( \Sigma \) is the surface of all constraints). Further, one has to proceed by analogy with the preceding case with the differences that now the transformation of primary constraints will generate the transformation of secondary constraints according to Dirac’s scheme, and in this case we have

\[
\{ \phi^1_{a_1}, \phi^2_{b_1}\} \equiv \{ \phi^1_{a_1}, \phi^2_{b_1}\}.
\]

Therefore, \( \mathbf{K}^{12|\Sigma} \) will be given in the diagonal form with only nonvanishing diagonal elements. So, due to \( \{ \psi^2_{a_2}, \psi^2_{b_2}\} \equiv 0 \ (a_2, b_2 = 1, \ldots, A_2) \), we obtain, at this stage, \( A_2 \) one-linked chains of second-class constraints which are in involution on \( \Sigma \) with each other and with all other constraints.

Next one must consider the constraints \( \phi^m_{\alpha_2}, \alpha_2 = 1, \ldots, A - A_1 - A_2 \). Let rank \( \| \phi^1_{a_2}, \phi^3_{b_2} \| \) \( \equiv A_3 < A - A_1 - A_2 \).

With the help of the Jacobi identity we obtain \( \{ \phi^1_{a_2}, \phi^3_{b_2}\} \equiv \{ \phi^1_{b_2}, \phi^3_{a_2}\} \).

Proceeding by analogy with the previous case, we obtain the matrix \( \mathbf{K}^{13|\Sigma} \) in the quasidiagonal form with only nonvanishing elements along the principal diagonal \( F^1_{a_3 b_3} \neq 0 \) (where if \( a_3 = 2k + 1, \ b_3 = 2k + 2 \) and conversely; \( k = 0, 1, \ldots, A_3 - 2 \)).

Now we have the relations

\[
\{ \psi^2_{a_3}, \psi^3_{b_3}\} \equiv \{ \psi^3_{a_3}, \psi^3_{b_3}\} \equiv 0, \quad \{ \psi^2_{a_3}, \psi^2_{b_3}\} \equiv \{ \psi^1_{a_3}, \psi^3_{b_3}\} \equiv \{ \psi^3_{a_3}, \psi^1_{b_3}\}; \quad a_3, b_3 = 1, \ldots, A_3.
\]

Thus, two-linked doubled chains of second-class constraints are obtained. Constraints of such different formations are in involution on \( \Sigma \) with each other and with all other constraints since all the previously established properties are kept.
This procedure must be continued. We emphasize that every subsequent stage preserves the properties of transformed constraints, which were obtained at the preceding stages, and, therefore, the secondary, tertiary, etc. constraints do not mix themselves into primary constraints. If after carrying out certain \( n \)-th stage it is found that
\[
\text{rank} \left\{ \phi_{\alpha_n}^{m_n}, \phi_{\beta_n}^{m_n} \right\} = \sum_{i=1}^{\infty} A_i = 0, \quad \alpha_n, \beta_n = 1, \cdots, \]
these remaining constraints \( \phi_{\alpha_n}^{m_n} \) are all of first class.

So, the final set of constraints \( (\Phi, \Psi) \) is obtained from the initial one \( \phi_{\alpha}^{m} \) by the equivalence transformation.

### 3 Local Symmetry Transformations

A group of phase-space coordinate transformations, which map each solution of the Hamiltonian equations of motion into the solution of the same equations, will be called a symmetry transformation. Under these transformations the action functional is quasi-invariant within a surface term. So, we shall consider the action
\[
S = \int_{t_1}^{t_2} dt \ (p \ \dot{q} - H_T)
\]
where
\[
H_T = H' + u_{\alpha} \Phi_{\alpha}^1, \quad H' = H_c + \sum_{i=1}^{\infty} (K_{1_i})^{-1}_{a_i} (\Psi_{a_i}^i, H_c) \Psi_{a_i}^1,
\]
\( H_c \) is the canonical Hamiltonian, \( u_{\alpha} \) are the Lagrange multipliers. We shall require a quasi-invariance of action (9) with respect to transformations:
\[
\begin{align*}
q_i' &= q_i + \delta q_i, \quad \delta q_i = \{q_i, G\}, \\
p_i' &= p_i + \delta p_i, \quad \delta p_i = \{p_i, G\}.
\end{align*}
\]
The generator \( G \) will be looked for in the form
\[
G = \varepsilon_{\alpha}^{m} \Phi_{\alpha}^{m} + \eta_{a_i}^{m_i} \Psi_{a_i}^{m_i}.
\]
So, under transformations (11) we have
\[
\delta S = \int_{t_1}^{t_2} dt (\delta p \ \dot{q} + p \ \delta \dot{q} - H_T) = \int_{t_1}^{t_2} dt \left[ \frac{d}{dt} (p \ \frac{\partial G}{\partial p} - G) - \frac{\partial G}{\partial t} - \{G, H_T\} \right].
\]
From (13) we see: in order that transformations (11) were the symmetry ones, it is necessary that
\[
\frac{\partial G}{\partial t} + \{G, H_T\} \equiv 0.
\]
The fact that the last equality must be realized on the primary-constraint surface \( \Sigma_1 \), can easily be interpreted if one remembers that \( \Sigma_1 \) is the whole \( (q, \dot{q}) \)-space image in the phase space. Since under the operation of the local-symmetry transformation group the configuration space is being mapped into itself in a one-to-one manner, the one-to-one mapping of \( \Sigma_1 \) into itself corresponds to this in the phase space. Therefore, at looking for
the generator $G$, it is natural to require also $\Sigma_1$ to be conserved under transformations (11), i.e. the requirement (14) must be supplemented by the demands

$$\{G, \Psi^1_{\alpha_i}\} \Sigma_1 \equiv 0, \quad \{G, \Phi^1_{\alpha}\} \Sigma_1 \equiv 0.$$  \hfill (15)

Further, we shall use the following Poisson brackets:

$$\{\Phi^{m_\alpha}_{\alpha}, H'\} = g^{m_\alpha m_\beta}_{\alpha} \Phi^{m_\beta}_{\beta}, \quad m_\beta = 1, \cdots, m_\alpha + 1,$$  \hfill (16)

$$\{\Psi^{m_\alpha}_{\alpha}, H'\} = g^{m_\alpha m_\alpha}_{\alpha} \Phi^{m_\alpha}_{\alpha} + \sum_{k=1}^{n} h^{m_\alpha m_\beta}_{a_i b_k} \Psi^{m_\beta}_{b_k}, \quad m_{\beta} = m_\alpha + 1,$$  \hfill (17)

$$\{\Phi^{m_\alpha}_{\alpha}, \Phi^{m_\beta}_{\beta}\} = f^{m_\alpha m_\gamma m_\gamma}_{\alpha \beta \gamma} \Phi^{m_\gamma}_{\gamma}.$$  \hfill (18)

The general properties of the structure functions can be extracted from the consideration of the previous section.

So, from eqs. (14) and (12) with taking account of (16)-(18) we write down

$$\left(\dot{g}^{m_\alpha m_\beta}_{\alpha \beta} + \sum_{i=1}^{n} \eta^{a_i}_{a_i} \dot{g}^{m_\alpha m_\alpha}_{a_i a_i} \right) \Phi^{m_\alpha}_{\alpha} + \sum_{i=1}^{n} \left(\eta^{a_i}_{a_i} + m_{\beta}^{b_k} h^{m_\gamma m_\alpha}_{b_k a_i} \right) \Psi^{m_\alpha}_{a_i} + u_\alpha \{G, \Phi^1_{\alpha}\} \Sigma_1 \equiv 0.$$  \hfill (19)

Taking into consideration (15), we have $u_\alpha \{G, \Phi^1_{\alpha}\} \Sigma_1 \equiv 0$.

Then, in view of the functional independence of constraints, in order to carry out the equality (19), one must demand the coefficients of the constraints $\Phi^{m_\alpha}_{\alpha}$ and $\Psi^{m_\alpha}_{a_i}$ to vanish.

Before analyzing these conditions to satisfy the equality (19), let us consider in detail the conditions of the $\Sigma_1$ conservation (15) starting from the former. Its realization would mean

$$\sum_{k=1}^{n} \eta^{m_{\beta}}_{b_k} \{\Psi^{m_{\beta}}_{b_k}, \Psi^1_{a_i}\} \Sigma_1 \equiv 0.$$  \hfill (20)

In this equality for each value of $a_i$ in the double sum over $k$ and over $b_k$ the only nonvanishing Poisson brackets are those at $b_k = a_i, \ M_{b_k} = i$; therefore

$$\eta^{a_i}_{a_i} = 0 \quad \text{for} \quad i = 1, \cdots, n.$$  \hfill (20)

Now considering the requirement for the coefficients of the constraints $\Psi^{m_\alpha}_{a_i}$ in eq. (19) to vanish

$$\eta^{m_\alpha}_{a_i} + \sum_{k=1}^{n} \eta^{b_k}_{b_k} h^{m_{\beta} m_\alpha}_{b_k a_i} \Sigma_1 \equiv 0.$$  \hfill (21)

we obtain with taking account of (20) that this system of algebraic equations for unknowns $\eta^{m_{\beta}}_{b_k}$ has only a trivial solution $\eta^{m_{\beta}}_{b_k} = 0$ since

$$\det \| h^{i-k-1}_{i-k} \| \neq 0 \quad (\text{and} \ \det \| g^{M_{\alpha-k-1} M_{\alpha-k}}_{\alpha} \| \neq 0),$$

that is easily proved as a consequence of functional independence of all constraints with the help of the method by contradiction [3].

So, it is proved that the second-class constraints do not contribute to a generator of local-symmetry transformations.

Returning to the second condition of the $\Sigma_1$ conservation (13) under transformations (11), we see that it will be fulfilled if

$$\{\Phi^1_{\alpha}, \Phi^{m_\beta}_{\beta}\} = f^{m_\beta}_{\alpha} \Phi^1_{\alpha} \Phi^{m_\beta}_{\beta}.  \hfill (22)$$
This relation means a quasi-algebra of the special form where first-class primary constraints make an ideal of quasi-algebra formed by all first-class constraints.

To determine the multipliers $\varepsilon^{ma}_\alpha$ in the generator (12), we have only a requirement of vanishing coefficients of constraints $\Phi^{ma}_\alpha$ in (19) 

$$\dot{\varepsilon}^{ma}_\alpha + \varepsilon^{m\beta}_\alpha g^{\beta m\alpha}_\beta = 0, \quad m_\beta = m_\alpha - 1, \ldots, M_\alpha.$$  

In system of equations (23) the number of unknowns exceeds the number of equations by the number

$$F = A - \sum_{i=1}^{n} A_i$$

of the first-class primary constraints; therefore, the system (23) can be solved to within $F$ arbitrary functions. Introducing arbitrary functions $\varepsilon^{M_\alpha}_\alpha (\alpha = 1, \ldots, F)$ and inserting them into equations (23), we obtain a system of algebraic equations, that always have a solution [4]. Then, the generator of local-symmetry transformations takes the form:

$$G = B^{ma_\beta}_\alpha m^{\alpha\beta}_\alpha, \quad m_\beta = m_\alpha, \ldots, M_\alpha,$$  

where

$$\varepsilon^{(M_\alpha-m_\beta)}_\beta \equiv (d^{M_\alpha-m_\beta} dt^{M_\alpha-m_\beta}) \varepsilon^{(M_\alpha-m_\beta)}_\beta (t), \quad B^{ma_\beta}_\alpha$$

are, generally speaking, functions of $q$ and $p$ and their derivatives up to the order $M_\alpha - m_\alpha - 1$. The obtained generator (24) satisfies the group property

$$\{ G_1, G_2 \} = G_3$$

where the transformation $G_3$ is realized by carrying out two successive transformations $G_1$ and $G_2$. The amount of group parameters $\varepsilon^{(t)}_\alpha$, which determine a rank of a quasigroup of these transformations, equals the number of primary constraints of first class. As can be seen from formula (24), the transformation law may include both arbitrary functions $\varepsilon^{(t)}_\alpha (t)$ and their derivatives up to and including the order $M_\alpha - 1$; the highest derivatives $\varepsilon^{(M_\alpha-1)}_\alpha$ should always be present.

Several remarks: (i) Though the condition (22) means a constraint quasi-algebra of a special form, it is fulfilled in most of the physically interesting theories. However, in the available literature there are examples of Lagrangians where this condition (22) does not hold, e.g., Polyakov’s string and other model Lagrangians [4]. In the case of theories with only first-class constraints we have shown [4] that there always exist equivalent sets of constraints, for which the condition (22) holds valid, and given a method of passing to a set like these. This proof and method are valid also for theories with first- and second-class constraints. (ii) The corresponding transformations of local symmetry in the Lagrangian formalism are determined in the following way:

$$\delta q_i (t) = \{ q_i (t), G \} |_{p= \Omega^{q_i}_\omega}, \quad \delta \dot{q}_i (t) = \frac{d}{dt} \delta q_i (t).$$  

(iii) If time derivatives of $q$ and $p$ are present in $B^{ma_\beta}_\alpha$, the Poisson brackets in (11) are not determined. This problem was solved in our previous work [4]. It is shown that local-symmetry transformations are canonical in the extended (by Ostrogradsky) phase space using the formalism of theories with higher derivatives [11, 12].

Examples: 1. Consider the Lagrangian

$$L = (\dot{q}_1 + \dot{q}_2)q_3 + \frac{1}{2} \dot{q}_3^2 - \frac{1}{2} q_2^2.$$  

6
In the phase space we obtain two primary constraints \( \phi_1 = p_1 - q_3 \), \( \phi_2 = p_2 - q_3 \) and the total Hamiltonian:
\[
H_T = \frac{1}{2}(p_3^2 + q_2^2) + u_1\phi_1 + u_2\phi_2.
\]

From the self-consistency conditions of the theory we derive two secondary constraints \( \phi_1^2 = p_3 \), \( \phi_2^2 = p_3 - q_2 \) and the Lagrangian multipliers \( u_1 = u_2 = 0 \). Let us calculate the matrix
\[
W = \|K^{m_\alpha m_\beta}\| = \|\{\phi_\alpha^{m_\alpha}, \phi_\beta^{m_\beta}\}\|:
\]
\[
\begin{pmatrix}
0 & 0 & -1 & -1 \\
0 & 0 & -1 & -2 \\
1 & 1 & 0 & 0 \\
1 & 2 & 0 & 0
\end{pmatrix}.
\]

We see that \( \text{rank} W = 4 \), i.e. all constraints are of second class; therefore, \( W \) have a quasidiagonal (antisymmetric) form. Performing our procedure, we pass to the equivalent canonical set of constraints \( \Psi_a^{m_\alpha} \):
\[
\begin{pmatrix}
\Psi_1^1 \\
\Psi_1^2 \\
\Psi_2^1 \\
\Psi_2^2
\end{pmatrix}
= \begin{pmatrix}
1 & 0 & 0 & 0 \\
1 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 1 & -1
\end{pmatrix}
\begin{pmatrix}
\phi_1^1 \\
\phi_2^1 \\
\phi_1^2 \\
\phi_2^2
\end{pmatrix}
= \begin{pmatrix}
p_1 - q_3 \\
p_1 - p_2 \\
p_3 \\
-p_2 - q_2
\end{pmatrix}.
\]

For the last set of constraints the quasidiagonal form of \( W' \) will have a canonical structure:
\[
W' = \begin{pmatrix}
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix}.
\]

For quasi-invariance of the action with respect to transformations \( \Xi \) with the generator
\[
G = \eta_1^1 \Psi_1^1 + \eta_2^1 \Psi_1^2 + \eta_1^2 \Psi_2^1 + \eta_2^2 \Psi_2^2,
\]

it is necessary to realize the condition \( \{G, \Psi_a^1\}_{\Sigma_1} = 0 \), \( a = 1, 2 \).

From here we obtain \( \eta_1^2 = \eta_2^2 = 0 \). Next from (21) we establish \( \eta_1^1 = \eta_2^1 = 0 \), i.e. the second-class constraints of the system \( \Psi_a^{m_\alpha} \) do not generate transformations of either local symmetry or global one.

2. Now consider the well-known case of spinor electrodynamics. In the phase space \( (A_\mu, \psi, \bar{\psi}) \) and \( \pi_\mu, p_\psi, p_{\bar{\psi}} \) are the generalized coordinates and momenta, respectively; \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \) we have the canonical Hamiltonian
\[
H_\psi = \int d^3x \left[ \frac{1}{4} F_{ij} F^{ij} + \frac{1}{2} \pi^i \pi^i + \pi_i \partial_i A_0 + ip_\psi A_0 \psi + \bar{\psi} \gamma_i (\partial_i - ieA_i) \psi + m \bar{\psi} \psi \right].
\]

and three primary constraints: \( \phi_1^1 = \pi_0 \), \( \phi_2^1 = p_\psi - i\bar{\psi} \gamma_0 \), \( \phi_3^1 = p_{\bar{\psi}} \). Among the conditions of the constraint conservation in time \( \dot{\psi}^i_1 = 0 \) \( (i = 1, 2, 3) \) two last ones serve for determining the Lagrangian multipliers. From the first condition we obtain one secondary constraint \( \phi_1^2 = \partial_i \pi^i - ie p_\psi \). Calculating the matrix
\[
W = \delta(x - x') \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & i\gamma_0 & -iep_\psi \\
0 & i\gamma_0 & 0 & -e\gamma_0 \psi \\
iep_\psi & e\gamma_0 \psi & 0 & 0
\end{pmatrix},
\]
we see that \( \text{rank} \mathbf{W} = 2 \); therefore, two constraints are of second class and the other two are of first class. Now implementing our procedure, we shall pass to the canonical set of constraints by the equivalence transformation

\[
\begin{pmatrix}
\Psi_1^1 \\
\Psi_2^1 \\
\Phi_1^1 \\
\Phi_2^1 \\
\end{pmatrix}
= \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
ie\psi & -i\bar{\psi} & 0 & 1 \\
\end{pmatrix}
\begin{pmatrix}
\phi_1^1 \\
\phi_2^1 \\
\phi_3^1 \\
\phi_4^1 \\
\end{pmatrix}
= \begin{pmatrix}
p\psi - i\bar{\psi}\gamma_0 \\
p_{\psi} \\
0 \\
\partial_i\pi^i - ie(p_{\psi}\psi + \bar{\psi}p_{\psi}) \\
\end{pmatrix}
\]

where the constraints are already separated into the ones of first and second class since now the matrix \( \mathbf{W} \) has the form

\[
\mathbf{W}' = \delta(x - x') \begin{pmatrix}
0 & i\gamma_0 & 0 & 0 \\
i\gamma_0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{pmatrix}.
\]

Further, we look for the generator \( G \) in the form

\[
G = \int d^3x [\eta_1^1 \Psi_1^1 + \eta_2^1 \Psi_2^1 + \varepsilon_1^1 \Phi_1^1 + \varepsilon_2^1 \Phi_2^1].
\]

If the first condition (13) of conservation of \( \Sigma_1 \) under the transformations (11) we derive \( \eta_1^1 = \eta_2^1 = 0 \), i.e. the constraints of second class do not contribute to \( G \). The second condition (13) is realized because \( \{\Phi_1^1, \Phi_2^1\} = 0 \). Eq.(23) accepts the form:

\[
\varepsilon_1^2 - \varepsilon_1^1 = 0, \quad \text{i.e.} \quad \varepsilon_1^1 = \varepsilon \quad \text{where} \quad \varepsilon \equiv \varepsilon_1^2.
\]

Therefore, we have

\[
G = \int d^3x \{\varepsilon \pi_0 + \varepsilon [\partial_i\pi^i - ie(p_{\psi}\psi + \bar{\psi}p_{\psi})]\},
\]

from which it is easily to obtain the gauge transformations in the phase space and the well-known transformation rule: \( \delta A_\mu = \partial_\mu \varepsilon, \quad \delta \psi = ie\varepsilon\psi, \quad \delta \bar{\psi} = -ie\varepsilon\bar{\psi} \).

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