BOUNDARY SOLUTIONS OF THE
CLASSICAL YANG-BAXTER EQUATION

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Abstract. We define a new class of unitary solutions to the classical Yang-Baxter equation (CYBE). These “boundary solutions” are those which lie in the closure of the space of unitary solutions to the modified classical Yang-Baxter equation (MCYBE). Using the Belavin-Drinfel’d classification of the solutions to the MCYBE, we are able to exhibit new families of solutions to the CYBE. In particular, using the Cremmer-Gervais solution to the MCYBE, we explicitly construct for all $n \geq 3$ a boundary solution based on the maximal parabolic subalgebra of $\mathfrak{sl}(n)$ obtained by deleting the first negative root. We give some evidence for a generalization of this result pertaining to other maximal parabolic subalgebras whose omitted root is relatively prime to $n$. We also give examples of non-boundary solutions for the classical simple Lie algebras.

0. Introduction. Let $\mathfrak{g}$ be a simple Lie algebra, either over $\mathbb{C}$ or split over $\mathbb{R}$. To fix the terminology call an $r \in \mathfrak{g} \wedge \mathfrak{g}$ a (constant, “unitary”) solution to the classical Yang-Baxter equation (CYBE) if $\langle r, r \rangle := \langle r_{12}, r_{13} \rangle + \langle r_{12}, r_{23} \rangle + \langle r_{13}, r_{23} \rangle = 0$, and call it a solution to the modified CYBE (MCYBE) if instead $\langle r, r \rangle$ is a non-zero multiple of the unique, up to scalar multiple, invariant of $\wedge^3 \mathfrak{g}$. The simply connected algebraic group $G$ with Lie $(G) = \mathfrak{g}$ operates on both sets, which together with multiplication by non-zero scalars defines a natural concept of equivalence.

The solutions to the MCYBE have been constructively classified by Belavin and Drinfel’d and depend on certain “admissible triples” involving the root system of $\mathfrak{g}$, see [BD], or [CP, §3.1.A and §3.1.B]. These triples have the form $(\Pi_1, \Pi_2, T)$ where $\Pi_i$ is a proper subset of $\Pi$, the set of positive simple roots, and $T : \Pi_1 \to \Pi_2$ is a bijection satisfying certain properties (see Definition 2.1 for a precise formulation of these properties). These triples serve as the “discrete” parameter for the space of solutions to the MCYBE; there is also a “continuous” parameter: it is a certain $(d - \#\Pi_1)/2$-dimensional affine subspace of $\mathfrak{h} \wedge \mathfrak{h}$ where $d$ is the rank of $\mathfrak{g}$ and $\mathfrak{h}$ is a fixed Cartan subalgebra. When $\mathfrak{g} = \mathfrak{sl}(2)$ there is only the trivial (or empty)

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triple which has $\Pi_1 = \Pi_2 = \emptyset$ and there is a unique solution, $e_{12} \wedge e_{21}$, to the MCYBE lying in $\mathfrak{sl}(2) \wedge \mathfrak{sl}(2)$. For $\mathfrak{g} = \mathfrak{sl}(3)$, there are essentially two admissible triples, the trivial triple and the Cremmer-Gervais triple. The trivial triple determines the following family of solutions to the MCYBE:

$$e_{12} \wedge e_{21} + e_{13} \wedge e_{31} + e_{23} \wedge e_{32} + \lambda \cdot (e_{11} \wedge e_{22} - e_{11} \wedge e_{33} + e_{22} \wedge e_{33})$$

where $\lambda$ is an arbitrary scalar and is the continuous parameter for this family. In contrast, the continuous parameter for the Cremmer-Gervais triple is uniquely determined. The solution to the MCYBE associated to this triple is

$$e_{12} \wedge e_{21} + e_{13} \wedge e_{31} + e_{23} \wedge e_{32} + \frac{1}{3} (e_{11} \wedge e_{22} - e_{11} \wedge e_{33} + e_{22} \wedge e_{33}) + 2 e_{12} \wedge e_{32}. \quad (0.1)$$

This Cremmer-Gervais triple (and its associated solution to the MCYBE) have analogs in higher dimensions, (see our remarks preceding Theorem 2.6 and formula (5.8)). Actually, what we call the Cremmer-Gervais triple or Cremmer-Gervais solution to the MCYBE is never mentioned in [CG]. Instead, the authors there construct (a non-standard) quantum Yang-Baxter matrix. With a knowledge of the Belavin-Drinfel’d classification, it is easy to infer that the classical limit of their quantum Yang-Baxter matrix is, essentially, the Cremmer-Gervais solution mentioned above. The Cremmer-Gervais triple for $\mathfrak{sl}(n)$ is described in [GGS2] (see [GGS3] for a more detailed exposition); the case $n = 3$ can also be found in [FG1]. A derivation of the Belavin-Drinfel’d classification using homological and deformation-theoretic methods can be found in [FG1] and [FG2].

If $\mathfrak{g} = \mathfrak{sl}(n)$, there are triples such that $\Pi_1$ – which must omit at least one positive simple root – omits exactly one. This is possible only when $n \geq 3$. Similar considerations hold for other split simple $\mathfrak{g}$ when exactly two roots are omitted. Such triples determine a unique solution to the MCYBE since the continuous parameter has dimension zero. Identifying the positive simple roots of $\mathfrak{sl}(n)$ with $\{1, 2, \ldots, n - 1\}$ in their natural order, we will prove (see Theorem 2.6) that the index of the omitted root must be relatively prime to $n$ and $T$ is then unique: Denoting that index by $n - i$, the root omitted from $\Pi_2$ is then $i$ and for all $j$ between 1 and $n - 1$ except $n - i$, we have $T(j) = j + i \mod n$. The solution of the MCYBE constructed with such a triple will be called a generalized Cremmer-Gervais solution, the original Cremmer-Gervais case being that where $i = 1$ and $T(j) = j + 1$ for all $1 \leq j \leq n - 1$.

For solutions to the CYBE, by contrast, there is presently only a non-constructive description due to Stolin, see [S1], [S2], or [CP, §3.1.D]. A Lie subalgebra $\mathfrak{f}$ of $\mathfrak{g}$ is quasi-Frobenius if it has a non-degenerate 2-cocycle with coefficients in the ground field $k$; it is Frobenius
if it has a non-degenerate 2-coboundary, that is, if there is a linear map $F : \mathfrak{f} \to k$ such that $F([a, b])$ is a non-degenerate skew form on $\mathfrak{f} \wedge \mathfrak{f}$. A simple Lie algebra can not itself be Frobenius, so $\mathfrak{f}$ is necessarily a proper subalgebra. Every solution $r$ of the CYBE in $\mathfrak{g}$ has a largest subalgebra $\mathfrak{g}_0$, which we will call its carrier, on which it is non-degenerate. That subalgebra is necessarily quasi-Frobenius, the 2-cocycle being just the inverse of $r$. That is, if relative to some basis $b_1, \ldots, b_m$ of $\mathfrak{g}_0$ we have $r = \sum r_{ij} b_i \wedge b_j$ then $F(b_i, b_j) = (r^{-1})_{ij}$ is a non-degenerate 2-cocycle. Conversely, if $\mathfrak{f}$ is quasi-Frobenius with non-degenerate 2-cocycle $F$ then the inverse of $F$ is a solution $r$ to the CYBE whose carrier is $\mathfrak{f}$. Finally, $r$ and $r'$ with corresponding $(\mathfrak{f}, F)$ and $(\mathfrak{f}', F')$ are equivalent if there is an inner automorphism of $\mathfrak{g}$ carrying $\mathfrak{f}$ to $\mathfrak{f}'$ and carrying $F$ to a 2-cocycle of $\mathfrak{f}'$ cohomologous to $F'$. If $\mathfrak{f}$ equals $\mathfrak{f}'$ and is Frobenius then $r$ and $r'$ are equivalent, so solutions to the CYBE with Frobenius carriers can be classified by their carriers, but the problem remains difficult. The classification of quasi-Frobenius subalgebras is harder yet since, for example, all even-dimensional abelian subalgebras are quasi-Frobenius. Stolin has carefully used the preceding analysis to list (up to equivalence) all of the quasi-Frobenius subalgebras of $\mathfrak{sl}(2)$ and $\mathfrak{sl}(3)$, see [S2]. He gives the unique solution, $(e_{11} - e_{22}) \wedge e_{12}$, of the CYBE which lies in $\mathfrak{sl}(2) \wedge \mathfrak{sl}(2)$, but he does not explicitly exhibit any which lie in $\mathfrak{sl}(3) \wedge \mathfrak{sl}(3)$.

In view of this we consider a restricted problem. If $r$ is a solution to the MCYBE and we identify it with $\lambda \cdot r$ for any scalar $\lambda$, then we can view this equivalence class as a point inside the projective space $\mathbb{P}(\mathfrak{g} \wedge \mathfrak{g})$ of lines in $\mathfrak{g} \wedge \mathfrak{g}$. We will show in Theorem 4.1 that the set of solutions, $\mathcal{M}$, to the MCYBE is a quasi-projective variety, in other words $\mathcal{M}$ is an open subset of its closure. The boundary points of $\mathcal{M}$ must be solutions to the CYBE. We call these boundary solutions, and ask if it is possible to classify constructively at least these. We show in Theorem 4.3 that the boundary solutions form a proper subset of the solutions to the CYBE by explicitly constructing non-boundary solutions for for the classical simple Lie algebras.

In section 5, we present some families of boundary solutions to the CYBE lying in $\mathfrak{sl}(n) \wedge \mathfrak{sl}(n)$. The most intriguing of these are associated to maximal parabolic subalgebras of $\mathfrak{sl}(n)$. To every $i$ between 1 and $n - 1$ we can associate a maximal parabolic subalgebra $\mathfrak{p}_i$ of $\mathfrak{sl}(n)$, namely, that generated by the Cartan subalgebra, all positive root vectors, and all negative root vectors except $e_{i+1, i}$. It is known that $\mathfrak{p}_i$ is Frobenius if and only if $i$ and $n$ are relatively prime, see [E1]. Also $H^2(\mathfrak{p}_i) = 0$, see [F], and so a maximal parabolic subalgebra is Frobenius if and only if it is quasi-Frobenius. Thus $\mathfrak{p}_i$ is the carrier of a solution to the CYBE if and only if $(i, n) = 1$ in which case there is a unique solution (up to equivalence) in $\mathfrak{p}_i \wedge \mathfrak{p}_i$. We conjecture that in this case the solution is in fact a boundary solution and lies in the closure of a suitable orbit of the associated generalized Cremmer-Gervais solution to the
MCYBE; we show that this is indeed true for the “end” (maximal) parabolic subalgebras \( p_1 \) and \( p_{n-1} \), see Theorem 5.9 where we give the formula for this boundary solution. When \( n = 3 \), our formula gives

\[
\begin{pmatrix}
\frac{2}{3} e_{11} - \frac{1}{3} e_{22} - \frac{1}{3} e_{33} \\
\frac{1}{3} e_{11} + \frac{1}{3} e_{22} - \frac{2}{3} e_{33} \\
e_{23} + e_{13} \wedge e_{32}
\end{pmatrix}
\]

which clearly has carrier \( p_1 \), the subalgebra of \( \mathfrak{sl}(3) \) consisting of traceless matrices of the form

\[
\begin{pmatrix}
* & * & * \\
0 & * & * \\
0 & * & *
\end{pmatrix}.
\]

It seems remarkable that this solution to the CYBE can be constructed directly from (0.1), the associated Cremmer-Gervais solution to the MCYBE. We also show that the Cremmer-Gervais solution to the MCYBE and the boundary solution of the CYBE with carrier \( p_1 \) are part of a natural simple three-dimensional submodule of \( \mathfrak{sl}(n) \wedge \mathfrak{sl}(n) \) under the adjoint action of the principal three-dimensional subalgebra of \( \mathfrak{sl}(n) \).

We can prove our conjecture also for the case \( n = 5, i = 2 \) and show how these results might be extended to establish the general case, see Conjecture 6.1. Finally, we close the paper with a brief discussion about constructing solutions to the quantum Yang-Baxter equation from solutions to the CYBE.

Our main interest in solutions to the MCYBE and CYBE is their connection to Poisson geometry and quantizations. If \( r \in \mathfrak{g}_0 \wedge \mathfrak{g}_0 \) is a solution to the CYBE with carrier \( \mathfrak{g}_0 \), then there is a left invariant symplectic structure on the corresponding simply connected algebraic group \( G_0 \). This means that the coordinate ring \( \mathcal{O}(G_0) \) of \( G_0 \) is equipped with a left invariant skew bracket, \( \{\ ,\ \} \) (the “Poisson bracket”), which satisfies the Jacobi identity and has \( \{f, gh\}_r = \{f, g\}_r h + \{f, h\}_r g \) for all \( f, g, h \in \mathcal{O}(G) \). If \( r = \sum \lambda_{i j} a_i \wedge b_j \) then the associated Poisson bracket is given by \( \{f, g\}_r = \sum \lambda_{i j} (A_i(f) \cdot B_j(g) - B_j(f) \cdot A_i(g)) \) where \( A_i \) and \( B_j \) are the left-invariant vector fields on \( G \) corresponding to \( a_i \) and \( b_j \). Similarly, there is a right invariant symplectic structure, \( \{\ ,\ \}'_r \) on \( \mathcal{O}(G) \) given by \( \{f, g\}'_r = \sum \lambda_{i j} (A'_i(f) \cdot B'_j(g) - B'_j(f) \cdot A'_i(g)) \) where \( A'_i \) and \( B'_j \) are the right-invariant vector fields on \( G \) corresponding to \( a_i \) and \( b_j \).

If \( G \) is an algebraic group, then it is natural to consider Poisson-Lie structures on the Hopf algebra \( \mathcal{O}(G) \). A Poisson Lie structure is a Poisson bracket on \( \mathcal{O}(G) \) which is compatible with the comultiplication \( \Delta : \mathcal{O}(G) \to \mathcal{O}(G) \otimes \mathcal{O}(G) \). This means that \( \Delta(\{f, g\}) = \{\Delta f, \Delta g\} \) for all \( f, g \in \mathcal{O}(G) \). Drinfel’d proves in [D1] that all Poisson-Lie structures on \( \mathcal{O}(G) \) are of the form \( \{f, g\} = \{f, g\}_r - \{f, g\}'_r \) where \( r \in \mathfrak{g} \wedge \mathfrak{g} \) satisfies either the MCYBE or CYBE. This bracket, however, is neither left nor right invariant.
Although we will not discuss deformations in this paper, we would like to point out that Poisson (and Poisson-Lie) structures correspond to first order or infinitesimal deformations, which by the above remarks in turn correspond to solutions \( r \) of the CYBE and the MCYBE. Starting with such an \( r \), one would like, optimally, to have a way of constructing a deformation with \( r \) as its classical limit. For details of the existence of such deformations, see [BFGP], [D2] [EK1], and [EK2]. For some recent results on how these deformations might be constructed see [GGS3] and [GZ].

1. The classical Yang-Baxter equations. Suppose that \( \mathfrak{g} \) is a finite dimensional simple Lie algebra, over \( \mathbb{C} \) or split over \( \mathbb{R} \). Throughout this paper, we will simply use \( k \) to denote the ground field. All tensor products will be taken over \( k \), and if \( W \) is a \( k \)-vector space then we will view \( W \wedge W \) and \( W \wedge W \wedge W \) as subspaces of \( W \otimes W \) and \( W \otimes W \otimes W \) in the natural way. In particular, \( w_1 \wedge w_2 \) corresponds to \( \frac{1}{2}(w_1 \otimes w_2 - w_2 \otimes w_1) \). Let \( U\mathfrak{g} \) be the universal enveloping algebra of \( \mathfrak{g} \) and let \( G \) be a simply connected algebraic group with Lie(\( G \)) = \( \mathfrak{g} \). For \( r = \sum a_i \otimes b_i \in \mathfrak{g} \otimes \mathfrak{g} \) define \( \langle r, r \rangle \in U\mathfrak{g} \otimes U\mathfrak{g} \otimes U\mathfrak{g} \) to be

\[
[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}]
\]

where \( r_{12} = r \otimes 1, r_{23} = 1 \otimes r, r_{13} = \sum a_i \otimes 1 \otimes b_i \).

Definitions 1.1.

(1) An element \( r \in \mathfrak{g} \otimes \mathfrak{g} \) is a solution to the classical Yang-Baxter equation (CYBE) if \( \langle r, r \rangle = 0 \).

(2) An element \( r \in \mathfrak{g} \otimes \mathfrak{g} \) is a solution to the modified classical Yang-Baxter equation (MCYBE) if \( \langle r, r \rangle \) is non-zero and \( \mathfrak{g} \)-invariant.

(3) A solution to either the CYBE or MCYBE is unitary if \( r \) is skew-symmetric and non-unitary otherwise.

Actually, what we have just defined are constant solutions to the Yang-Baxter equations. There are related non-constant solutions to these equations, (those which depend on a spectral parameter), but our only concern here is with the constant solutions. We will also only be considering unitary solutions to the CYBE and MCYBE. As discussed in the introduction, it is these solutions which have important homological and geometrical meanings. A natural question therefore is to determine and describe, if possible, the set of all \( r \in \mathfrak{g} \wedge \mathfrak{g} \) which satisfy either the CYBE or the MCYBE. In our analysis of this question, it will be useful to identify \( r \in \mathfrak{g} \wedge \mathfrak{g} \) with \( \lambda \cdot r \) for any scalar \( \lambda \in k^\times \) and so if \( d = \dim_k (\mathfrak{g}) \), we can view \( r \) as an element of the \( \binom{d}{2} - 1 \) dimensional projective space \( \mathbb{P}(\mathfrak{g} \wedge \mathfrak{g}) \). Similarly we can view \( \langle r, r \rangle \), which necessarily lies in \( \mathfrak{g} \wedge \mathfrak{g} \wedge \mathfrak{g} \), as an element of the \( \binom{d}{3} - 1 \) dimensional projective \( \mathbb{P}(\mathfrak{g} \wedge \mathfrak{g} \wedge \mathfrak{g}) \).
Notation. Let $C$ and $M$ denote, respectively, the subsets of $\mathbb{P}(g \wedge g)$ consisting of solutions to the CYBE and MCYBE.

There is an important notion of equivalence of solutions to these equations which comes from the diagonal action of the group $G$ on $g \wedge g$. Specifically, if $g \in G$ and $r = \sum a_i \wedge b_i$ then we say that $r$ and $g \cdot r = \sum g a_i g^{-1} \wedge g b_i g^{-1}$ are equivalent. This notion of equivalence is well-defined since both $M$ and $C$ are invariant under under the action of $G$.

2. Solutions of the modified classical Yang-Baxter equation. We first discuss $M$, the space of solutions to the MCYBE. Remarkably, Belavin and Drinfel’d have given a constructive description of $M$ in [BD]. They show, in particular, that $M$ is a finite disjoint union of components each of which is determined by certain data associated with the root system of $g$. To describe their work, we first need to recall some important facts and notation pertaining to finite dimensional simple Lie algebras. Let $(\cdot, \cdot)$ be an invariant non-degenerate symmetric bilinear form on $g$, let $h \subset g$ be a fixed a Cartan subalgebra, and denote the root system by $\Phi$. Write $\Phi = \Phi^+ \cup \Phi^-$ where $\Phi^+$ and $\Phi^-$ are the positive and negative roots, respectively, and let $\Pi \subset \Phi^+$ be a basis of positive simple roots. There is then a basis $\{x_\tau | \tau \in \Phi^+\} \cup \{x_{-\tau} | \tau \in \Phi^+\} \cup \{h_\tau | \tau \in \Pi\}$ of $g$ for which $(x_\tau, x_{-\tau}) = 1$ for all $\tau \in \Phi^+$. This gives a triangular decomposition $g = n^- \oplus h \oplus n^+$ where $n^-$ and $n^+$ are the nilpotent subalgebras spanned by the negative and positive root vectors relative to the chosen Cartan subalgebra. The data which determine the components of $M$ are called, using the terminology of [BD], admissible triples.

Definition 2.1 [BD]. An admissible triple is a triple $T = (\Pi_1, \Pi_2, T)$ where $\Pi_1$ and $\Pi_2$ are subsets of $\Pi$, with $\#(\Pi_1) = \#(\Pi_2)$ and $T : \Pi_1 \to \Pi_2$ is a bijection with the properties that

1. $T$ preserves the Killing form, i.e., $(T(x_\pi), T(x_\rho)) = (x_\pi, x_\rho)$ for all $\pi, \rho \in \Pi_1$, and
2. for every $\pi \in \Pi_1$ there is a positive integer $m$ such that $T^m \pi \notin \Pi_1$.

The following theorem proved in [BD] (which we state in a manner convenient for our purposes), shows that the set of admissible triples gives a decomposition of $M$ into mutually disjoint components.

Theorem 2.2 ([BD]). Let $M \in \mathbb{P}(g \wedge g)$ be the set of solutions to the MCYBE. For every admissible triple $T$, there is a non-empty subset $M_T$ of $M$ such that the following hold:

1. $M = \bigcup M_T$ where $T$ runs through the set of all admissible triples.
2. If $T \neq T'$, then $M_T \cap M_{T'} = \emptyset$.

To each triple $T$ there is a certain affine subvariety $B_T \subset h \wedge h$ which determines the dimension of $M_T$. Its description, given in the next theorem, uses for every $\pi \in \Pi_1$ the
map $1 \otimes \pi : \mathfrak{h} \otimes \mathfrak{h} \to \mathfrak{h}$ defined by $(1 \otimes \pi)(h_\tau \otimes h_\rho) = \pi(h_\rho) h_\tau$. In addition to $\mathcal{B}_T$, there is a uniquely defined element $\alpha \in \mathfrak{n}_+ \wedge \mathfrak{n}_-$ determined by $T$ which is used to describe $\mathcal{M}_T$. Its definition uses the linear extension of $T$ to the set $\hat{\Pi}_1 \subset \Phi_+$ of positive roots which are sums of simple roots in $\Pi_1$. If $\pi \in \hat{\Pi}_1$ and $\varrho \in \Phi_+$, then define $\pi \prec \varrho$ if there is an $m > 0$ such that $T^m \pi = \varrho$.

**Theorem 2.3 ([BD]).** Let $T = (\Pi_1, \Pi_2, T)$ be an admissible triple and let $\mathcal{B}_T$ be the set of all $\beta \in \mathfrak{h} \wedge \mathfrak{h}$ for which

$$(1 \otimes (T(\pi) - \pi)) \beta = \frac{1}{2} (h_{T(\pi)} + h_\pi) \quad \text{for all} \quad \pi \in \Pi_1.$$

1. $\mathcal{B}_T$ is an affine variety of dimension $\binom{d}{2}$ where $d = \#(\Pi_1 - \Pi)$.
2. If

$$\gamma = \sum_{\pi \in \Pi_+} x_\pi \wedge x_{-\pi} \quad \text{and} \quad \alpha = 2 \sum_{\pi \prec \varrho} x_\pi \wedge x_{-\varrho},$$

then $\gamma + \beta + \alpha$ is a solution to the MCYBE for any $\beta \in \mathcal{B}_T$.
3. Every solution of the MCYBE is equivalent to a unique solution of the form $\gamma + \beta + \alpha$ associated to some triple $T$.

The previous two theorems thus give a complete constructive description of $\mathcal{M}$ up to equivalence; the triple $T$ is the “discrete” parameter and the variety $\mathcal{B}_T$ is the “continuous” parameter of of the component $\mathcal{M}_T$. These theorems do not, however, give a direct way of finding specific $\beta \in \mathcal{B}_T$. For some triples associated to $\mathfrak{sl}(n)$ we are able to completely describe $\mathcal{B}_T$, see Theorem 2.9. Details of our remarks about the Belavin-Drinfel’d classification can be found in either [BD] or [CP, §3.1.A and §3.1.B].

Before going on to discuss $C$, we will examine some important triples and their associated solutions to the MCYBE. First, let us recall the natural partial ordering of triples discussed in [GGS3, §6]. If $T = (\Pi_1, \Pi_2, T)$ and $T' = (\Pi'_1, \Pi'_2, T')$ are triples then set $T < T'$ if $T$ is the restriction of $T'$ to some subset $\Pi_1$ of $\Pi'_1$. The trivial triple, $\mathcal{T}_0$, is the smallest element in this ordering and has $\Pi_1 = \Pi_2 = \emptyset$. The $\alpha$-term of the corresponding solution is zero while the $\beta$-term can be arbitrarily chosen from $\mathfrak{h} \wedge \mathfrak{h}$ and so if $l$ is the rank of $\mathfrak{g}$, this determines, up to equivalence, an $\binom{l}{2} + 1$ dimensional family of solutions to the MCYBE. The dimension of this family of solutions is greater than the dimension of any family of solutions associated to a non-trivial triple.

The simplest example of a non-trivial triple occurs when $\mathfrak{g} = \mathfrak{sl}(3)$. The two simple roots of this Lie algebra may be identified with the set $\{1, 2\}$ and the associated triple has $\Pi_1 = \{1\}, \Pi_2 = \{2\}$, and $T(1) = 2$. Here, $\beta$ is unique since $\Pi_1$ omitted all but
one positive simple root. The corresponding solution \( \gamma + \beta + \alpha \) is the Cremmer-Gervais solution to the MCYBE, see (0.1). This triple has an analog in \( \mathfrak{sl}(n) \) for all \( n > 3 \). If we make the natural identification of \( \Pi \) with \( \{1, 2, \ldots, n-1\} \) and set \( \Pi_1 = \{1, 2, \ldots, n-2\} \) and \( \Pi_2 = \{2, 3, \ldots, n-1\} \), then the triple \( T_{CG} \) (notation explained below) determined by the rule \( T(i) = i + 1 \) is admissible. This triple is maximal in the given partial ordering because \( \Pi_1 \) omitted only one root. Moreover, the \( \beta \)-term for the associated solution to the MCYBE is uniquely determined and hence there is a unique solution \( r_{CG} \) (given in (5.8)) to the MCYBE associated to this triple. We call this the Cremmer-Gervais solution because the associated non-unitary solution to the CYBE is the classical limit of the quantum Yang-Baxter matrix exhibited in [CG]. In contrast with the standard quantum groups and their multiparameter versions, little is known about the Cremmer-Gervais quantum groups. See [H] for some recent progress in understanding their structure.

For \( \mathfrak{sl}(n) \), there are other triples for which, like the Cremmer-Gervais triple, \( \Pi_1 \) consists of all but one positive simple root of \( \mathfrak{sl}(n) \). For each of these, the \( \beta \)-term is uniquely determined and so, up to equivalence, they each correspond to a unique solution to the MCYBE. We call these the generalized Cremmer-Gervais triples. These are clearly maximal elements in the partial ordering of the triples mentioned earlier. If we again make the identification of \( \Pi \) with the set \( \{1, 2, \ldots, n-1\} \) then the conditions for building an admissible generalized Cremmer-Gervais triple translate into finding a bijection \( T : S_1 \rightarrow S_2 \) between subsets \( S_1 \) and \( S_2 \) of \( \{1, 2, \ldots, n-1\} \) such that:

(2.4) For every \( i \in S_1 \) there is an \( r \) such that \( T^r i \notin S_1 \), and
(2.5) \( T \) respects adjacency, i.e., if \( i, j \in S_1 \) then \( |i - j| = 1 \) implies \( |Ti - Tj| = 1 \), and conversely.

As the next theorem shows, the number of generalized Cremmer-Gervais triples depends on the factorization of \( n \).

**Theorem 2.6.** Suppose that \( S_1 \) and \( S_2 \) are subsets of \( \{1, \ldots, n-1\} \) with \( \#S_1 = n-2 \) and that \( T : S_1 \rightarrow S_2 \) determines a generalized Cremmer-Gervais triple. Let the omitted element of \( S_1 \) be \( n-i \). Then \( i \) and \( n \) are relatively prime, the omitted element of \( S_2 \) is \( i \), and \( T \) sends every \( j \in S_1 \) to \( j + i \mod n \).

**Remark 2.7.** A consequence of the Theorem is that, modulo \( n \), \( S_1 = \{i, 2i, \ldots, (n-2)i\} \), \( S_2 = \{2i, 3i \ldots, (n-1)i\} \) and these sets have a natural order (this order, however, is not compatible with the order in \( \mathbb{N} \)).

**Proof of 2.6.** Consider first the case where \( i = 1 \), so \( S_1 = \{1, \ldots, n-2\} \). The matter is trivial for \( n = 3 \) so we may suppose \( n > 3 \). Since \( T \) preserves adjacency, we must have \( S_2 = \{2, \ldots, n-1\} \), else \( T \) would be a bijection of \( S_1 \) with itself and condition 2.4 would be
violated. The only question then is whether $T$ preserves order, i.e., sends every $j$ to $j+1$ (as we claim), or reverses order, sending $j$ to $n-j$. The latter is impossible, for $\{2, \ldots, n-2\}$ would then be a non-empty subsegment of $S_1$ carried onto itself by $T$, and this would violate condition 2.4. Exactly the same argument applies when $i = n-1$, so we may now suppose that $1 < i < n-1$. Removing $n-i$ then breaks the string of integers $1, \ldots, n-1$ into two non-empty segments, $S_1' = \{1, \ldots, n-i-1\}$ and $S_1'' = \{n-i+1, \ldots, n-1\}$. Their images must again be separated segments of $\{1, \ldots, n-1\}$, for otherwise some pair of non-adjacent elements of $S_1$ would be carried by $T$ to adjacent elements and condition 2.5 would be violated. Now the only way that $S_1'$ and $S_1''$ can be sent to non-adjacent and non-overlapping segments of $\{1, \ldots, n-1\}$ without either being sent into itself is if $S_1'$ is sent to a terminal segment and $S_1''$ to an initial segment of $\{1, \ldots, n-1\}$ i.e., that the image of $S_1'$ is $\{i+1, \ldots, n-1\}$ and that of $S_1''$ is $\{1, \ldots, i-1\}$. In particular, the element omitted from $S_2$ must be $i$. Moreover, we can not have $i = n-i$ or $T$ would carry $S_1$ onto itself.

Now the possibilities for $T$ are that it either preserves or reverses the order in $S_1'$, i.e., that if $j \in S_1'$ then either $T(j) = j+i$ or $T(j) = n-j$, and similarly for $S_1''$. One sees now that $i$ and $n$ must be relatively prime. For suppose that $(n, i) = m > 1$. Then in each of the four possible cases, $T$ carries a multiple of $m$ lying in $S_1$ to another multiple of $m$, and the complement of the set of multiples of $m$ is not empty. This complement is contained in $S_1$, so we would have a subset of $S_1$ carried bijectively onto itself by $T$, which is impossible. All that remains, therefore, is to show that $T$ preserves the order separately in $S_1'$ and in $S_1''$. Note that these segments are non-empty and of different lengths. We must show that it is impossible for $T$ to reverse the order either in $S_1'$ or in $S_1''$ or in both. The last case is trivial, for then $T1 = n-1$ and $T(n-1) = 1$, so condition 2.4 would be violated. The arguments for the other cases are similar, so we shall do only one explicitly. Suppose that $S_1' = \{1, \ldots, n-i-1\}$ is the longer segment, and that $T$ reverses its order, sending any $j$ with $1 \leq j \leq n-i-1$ to $n-j$. It follows that $i+1 \leq n-i-1$ so $\{i+1, \ldots, n-i-1\}$ is a non-empty subsegment of $S_1'$ which is carried by $T$ onto itself, which is impossible. \hfill \Box

Let $T_i$ denote the generalized Cremmer-Gervais triple described by Theorem 2.6. As stated earlier, $M_{T_i}$ contains, up to equivalence, a unique solution to the MCYBE because there is only a single $\beta \in h \setminus h$ which satisfies

$$\langle 1 \otimes (T(\pi) - \pi) \rangle \beta = \frac{1}{2}(h_{T(\pi)} + h_{\pi}) \quad \text{for all} \quad \pi \in \Pi_1. \quad (2.8)$$

The following explicitly determines this element. Since $g = \mathfrak{sl}(n)$ we make the identifications of $\Pi$ with $\{1, 2, \ldots, n-1\}$, of $x_j$ with $e_{j,j+1}$, and of $h_j$ with $e_{jj} - e_{j+1,j+1}$.

**Theorem 2.9.** Suppose that $i$ and $n$ are relatively prime and that $T_i$ is the generalized
Cremmer-Gervais triple. Let \( \beta = \sum_{l<j} b_{p,q} e_{pp} \wedge e_{qq} = (1/2) \sum_{p<q} b_{p,q}(e_{pp} \otimes e_{qq} - e_{qq} \otimes e_{pp}) \) be the unique solution to (2.8). If \( p - q = si \mod n \) then \( b_{p,q} = (1/n)(n - 2s) \).

**Proof.** Recall that for \( T_i \) we have \( \Pi_1 = \{ i, 2i, \ldots, (n-2)i \} \), \( \Pi_2 = \{ 2i, 3i, \ldots, (n-1)i \} \), and \( T : \Pi_1 \to \Pi_2 \) is given by \( T(ri) = (r+1)i \mod n \) (see Theorem 2.6). With the indicated identifications above, finding a \( \beta \in h \wedge h \) which satisfies (2.8) means we must have

\[
(1 \otimes (e_{(r+1)i,(r+1)i+1} - e_{ri,ri+1})) \beta = \frac{1}{2}(e_{(r+1)i,(r+1)i+1} - e_{(r+1)i+1,(r+1)i+1} + e_{ri,ri} - e_{ri+1,ri+1})
\]

for all \( ri \in \Pi_1 \). For \( p < q \), set \( b_{q,p} = -b_{p,q} \). Then an elementary computation shows that

\[
(1 \otimes e_{si,si+1}) \beta = \left( \sum_{j \neq si,si+1} (b_{j,si} - b_{j,si+1})e_{jj} \right) - b_{si,si+1}(e_{si,si} + e_{si+1,si+1}). \tag{2.11}
\]

The result now follows by combining (2.10) and (2.11) with the given formula for the coefficients \( b_{p,q} \). \( \square \)

Theorems 2.3, 2.6, and 2.9 thus provide an explicit description of all generalized Cremmer-Gervais triples and their corresponding solutions to the MCYBE.

### 3. Solutions of the classical Yang-Baxter equation and quasi-Frobenius Lie algebras.

The space \( \mathcal{C} \) of skew solutions to the CYBE is, in a sense, less understood than \( \mathcal{M} \). In fact, as Belavin and Drinfel’d remark, a constructive classification of the solutions to the CYBE is essentially intractable as it would, in particular, require knowledge of all abelian subalgebras \( a \subset g \) since any \( r \in a \wedge a \) has \( \langle r, r \rangle = 0 \). On the positive side, there is a homological interpretation of the unitary solutions to the CYBE, see [BD, Proposition 2.4]. In [S1] and [S2], Stolin used this connection to provide a non-constructive description of \( \mathcal{C} \) in terms of quasi-Frobenius Lie algebras.

**Definition 3.1.**

1. A Lie algebra \( \mathfrak{f} \) is **Frobenius** if there exists a linear functional \( f \in \mathfrak{f}^* \) such that the skew bilinear form \( [ , ]_f : f \wedge f \to k \) which sends \( x \wedge y \) to \( f([x,y]) \) is non-degenerate (here \( k \) is \( \mathbb{C} \) or \( \mathbb{R} \)). Equivalently, \( [ , ]_f \) is a non-degenerate two-coboundary for the Lie algebra cohomology of \( \mathfrak{f} \) with coefficients in \( k \).

2. A Lie algebra \( \mathfrak{f} \) is **quasi-Frobenius** if there exists a non-degenerate two-cocycle \( F : f \wedge f \to k \). (Note that a Frobenius Lie algebra is also quasi-Frobenius.)

**Remark 3.2.** A result of A. I. Ooms states that any finite dimensional Frobenius Lie algebra has a primitive enveloping algebra. The converse holds in case the Lie algebra is algebraic, see [O].

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The connection between quasi-Frobenius Lie algebras and the CYBE is that if $F_{ij}$ is the matrix of the form $F$ relative to some basis $x_1, \ldots, x_d$ of $\mathfrak{f}$ then $r = \sum (F^{-1})_{ij} x_i \wedge x_j \in \mathfrak{f} \wedge \mathfrak{f}$ is a skew solution to the CYBE. Conversely, if we start with a non-degenerate skew solution $\sum r_{ij} x_i \wedge x_j \in \mathfrak{f} \wedge \mathfrak{f}$ to the CYBE, then the map $F : \mathfrak{f} \wedge \mathfrak{f} \to k$ where $F(x_i, x_j) = (r^{-1})_{ij}$ is a non-degenerate two-cocycle and so $\mathfrak{f}$ is quasi-Frobenius. A simple Lie algebra $\mathfrak{g}$ can not itself be quasi-Frobenius and so any solution $r \in \mathfrak{g} \wedge \mathfrak{g}$ to the CYBE is necessarily degenerate. For any such $r$, however there always exists a largest subalgebra $\mathfrak{f} \subset \mathfrak{g}$ on which $r$ is non-degenerate, see [CP, Propositions 2.2.5 and 2.2.6]. That subalgebra is necessarily quasi-Frobenius. Thus, finding solutions to the CYBE in $\mathfrak{g} \wedge \mathfrak{g}$ is equivalent to finding pairs $(\mathfrak{f}, F)$ where $\mathfrak{f}$ is a subalgebra of $\mathfrak{g}$ and $F$ is a non-degenerate two-cocycle on $\mathfrak{f}$. We call $\mathfrak{f}$ the carrier of $r$.

4. Boundary solutions to the classical Yang-Baxter equation. Even though the Belavin-Drinfel’d classification of solutions to the MCYBE and the homological interpretation of solutions of the CYBE have been known for some time now, there has been little attention paid to the interface between these equations. As we shall see next, certain solutions to the CYBE can be viewed as limiting cases of solutions to the MCYBE.

**Theorem 4.1.** Let $\mathcal{C}$ and $\mathcal{M}$ be the subsets of $\mathbb{P}(\mathfrak{g} \wedge \mathfrak{g})$ consisting of solutions to the CYBE and MCYBE, respectively, and denote their Zariski closures by $\overline{\mathcal{C}}$ and $\overline{\mathcal{M}}$.

1. $\mathcal{C} = \overline{\mathcal{C}}$, i.e. $\mathcal{C}$ is a (closed) variety.
2. $\mathcal{M}$ is a quasi-projective variety, i.e. it is an open subset of $\overline{\mathcal{M}}$.
3. Any point on the boundary of $\mathcal{M}$ lies in $\mathcal{C}$ and hence is a solution to the CYBE.

**Proof.** (1) Suppose $r = \sum_{p,q} r_{pq} x_p \wedge x_q$ and $\langle r, r \rangle = \sum_{i,j,k} c_{ijk} x_i \wedge x_j \wedge x_k$. It is easy to see that each $c_{ijk} = f_{ijk}(r)$ for some homogeneous quadratic polynomial function $f_{ijk} : \mathbb{P}(\mathfrak{g} \wedge \mathfrak{g}) \to k$ and so $\mathcal{C}$ is a (closed) variety.

(2) and (3) Consider the map which sends $r$ to $\langle r, r \rangle$; it is a well-defined map $\phi : (\mathbb{P}(\mathfrak{g} \wedge \mathfrak{g}) - \mathcal{C}) \to \mathbb{P}(\mathfrak{g} \wedge \mathfrak{g} \wedge \mathfrak{g})$. Since $\mathfrak{g}$ is simple, the space of invariants in $\mathfrak{g} \wedge \mathfrak{g} \wedge \mathfrak{g}$ consists of all non-zero multiples of some fixed non-zero invariant in $\mathfrak{g} \wedge \mathfrak{g} \wedge \mathfrak{g}$. As a subspace of $\mathbb{P}(\mathfrak{g} \wedge \mathfrak{g} \wedge \mathfrak{g})$, this is just a single point which we denote $\omega$. Therefore $\mathcal{M} = \phi^{-1}(\omega)$ and so $\mathcal{M}$ is a closed variety.
subset of \((\mathbb{P}(\mathfrak{g} \wedge \mathfrak{g}) - \mathcal{C})\). Hence every element of \(\overline{\mathcal{M}} - \mathcal{M}\) must lie in \(\mathcal{C}\). Finally, to prove that \(\mathcal{M}\) is open in \(\overline{\mathcal{M}}\), it suffices to show that \(\overline{\mathcal{M}} - \mathcal{M}\) is a closed subset of \(\overline{\mathcal{M}}\). But this is trivial since \((\overline{\mathcal{M}} - \mathcal{M}) = \mathcal{C} \cap \overline{\mathcal{M}}\) and both \(\mathcal{C}\) and \(\overline{\mathcal{M}}\) are closed. □

In light of the preceding theorem, we can focus on a more restricted class of solutions to the CYBE.

**Definition 4.2.** Let \(\mathcal{C}\) and \(\overline{\mathcal{M}}\) be defined as in Theorem 4.1. An element \(r \in \mathfrak{g} \wedge \mathfrak{g}\) is a **boundary solution** to the CYBE if \(r \in \mathcal{C} \cap \overline{\mathcal{M}}\), i.e. if \(r\) is in the closure of, but not contained in the space of solutions of the MCYBE.

It is not a priori clear that there exist non-boundary solutions, but using the Belavin-Drinfel’d classification of solutions to the MCYBE, we can at least show that for the classical simple Lie algebras, there do indeed exist non-boundary solutions of the CYBE.

**Proposition 4.3.** Let \(\mathfrak{g}\) be a classical simple Lie algebra and set \(W = \mathfrak{g} \cap X\) where \(X\) is the space spanned by all matrix units \(e_{ij}\) with \(i \leq \left[\frac{n}{2}\right]\) and \(j \geq \left[\frac{n}{2}\right]\). Then, if the rank of \(\mathfrak{g}\) is large, the generic element of \(W \wedge W\) is a non-boundary solution of the CYBE.

**Proof.** We only give the proof for \(\mathfrak{g} = \mathfrak{sl}(n)\) since it easily modifies to the other cases. Since \(W = X\) for \(\mathfrak{g} = \mathfrak{sl}(n)\), the dimension of \(W \wedge W\) is on the order of \(\frac{n^4}{32}\) for large \(n\). Moreover, since \(W\) is abelian, every element of \(W \wedge W\) satisfies the CYBE. Now the component of the solutions to the MCYBE of maximal dimension corresponds to the triple with \(\Pi_1 = \Pi_2 = \emptyset\). The associated solution to the MCYBE has \(\alpha = 0\) and \(\beta \in \mathfrak{h} \wedge \mathfrak{h}\) can be arbitrary. Now since \(\dim(\mathfrak{sl}(n)) = n^2 - 1\) and \(\dim(\mathfrak{h} \wedge \mathfrak{h}) = \binom{n-1}{2}\), this component has dimension at most \((n^2 - 1) + \frac{1}{2}(n-1)(n-2)\) which is on the order of \(\frac{3}{2}n^2\) for large \(n\). Hence the generic element of \(W \wedge W\) must be a non-boundary solution of the CYBE since its dimension exceeds \(\frac{3}{2}n^2\). □

5. **Construction of boundary solutions.** Although we presently can not classify, or even find all boundary solutions, we are able to exhibit several (previously unknown) families of solutions to the CYBE. We will do this by analyzing the action of the group \(G\) on certain solutions to the MCYBE obtained from Theorem 2.3. The most intriguing of these is found using the Cremmer-Gervais solution to the MCYBE and is associated to the maximal parabolic subalgebra of \(\mathfrak{sl}(n)\) obtained by deleting the first negative root. This family, and all others we have, can be found using the following elementary (but useful) result. In the following, we treat \(t\) as a formal variable and enlarge the coefficient ring to \(k[t]\).

**Proposition 5.1.** Suppose that \(r \in \mathfrak{g} \wedge \mathfrak{g}\) and \(r_t = (r + tr_1 + \cdots + t^mr_m) \in \mathfrak{g} \wedge \mathfrak{g}\) are solutions of the MCYBE with \(\langle r, r \rangle = \langle r_t, r_t \rangle\). Then \(r_m\) is a boundary solution to the CYBE.
Proof. Since \( \langle r_t, r_t \rangle = \langle r, r \rangle + t(\langle r_1, r \rangle + \langle r, r_1 \rangle) + \cdots + t^{2m} \langle r_m, r_m \rangle \) and \( \langle r, r \rangle = \langle r_t, r_t \rangle \), the coefficients of each of the terms with positive degree in \( t \) must vanish identically. In particular, we must have \( \langle r_m, r_m \rangle = 0 \) and so \( r_m \) satisfies the CYBE. Now dividing \( r_t \) by \( t^m \) we obtain \( (r/t^m) + (r_1/t^{m-1}) + \cdots + r_m \) which satisfies the MCYBE and, any polynomial function which vanishes on \( r_t/t^m \) must also vanish on \( r_m \) and so \( r_m \) is a boundary solution. □

An efficient way to construct boundary solutions of this sort is to consider the orbit of a fixed solution, \( r \), of the MCYBE under the action of \( \exp(ta) \in G \) with \( a \in g \) nilpotent. Since \( \langle r, r \rangle \) is invariant it follows that \( r \) and \( r_t = \exp(ta) \cdot r \) satisfy the hypotheses of the proposition. This idea was first considered in [GGS1, §15] and was used to produce the first family of boundary solutions to the CYBE.

Example 5.2 (see [GGS1] §15). Suppose that \( g = \mathfrak{sl}(n) \) and \( r = \gamma = \sum_{i<j} e_{ij} \wedge e_{ji} \). Note that \( \gamma \) lies in the component \( \mathcal{M}_{T_\emptyset} \) of \( \mathcal{M} \), corresponding to the trivial triple. An elementary computation shows that

\[
\exp(-te_{1n}) \cdot \gamma = \gamma + t \left( (e_{11} - e_{nn}) \wedge e_{1n} + 2 \sum_{i=2}^{n-1} e_{1i} \wedge e_{in} \right)
\]

(2.2.1)

and so by Proposition 5.1, the coefficient of the linear term, \((e_{11} - e_{nn}) \wedge e_{1n} + 2 \sum_{i=2}^{n-1} e_{1i} \wedge e_{in}\) is a boundary solution to the CYBE, (in [GGS1] this solution was denoted \( \gamma_\infty \), but this notation no longer seems suitable since, as we shall see, there are many other boundary solutions.) The carrier of this solution is the Lie algebra, \( \mathcal{H} \), consisting of matrices of the form

\[
\begin{pmatrix}
1 & * & \ldots & * & * \\
0 & 0 & \ldots & 0 & * \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 0 & * \\
0 & 0 & \ldots & 0 & -1
\end{pmatrix}
\]

and so is a one-dimensional extension of a Heisenberg algebra. (Arbitrary scalars can be substituted into those entries marked with \( * \).) Since \( \mathcal{H} \) is the carrier Lie algebra of a solution to the CYBE, it follows that \( \mathcal{H} \) is quasi-Frobenius. In this case though, it is easy to see that \( \mathcal{H} \) is actually Frobenius. The skew bilinear form \( [ , ]_{e_{1n}^*} \) associated to the linear functional \( e_{1n}^* : \mathcal{H} \to k \) is non-degenerate. In fact, the solution to the CYBE obtained from the inverse of \( [ , ]_{e_{1n}^*} \) is the boundary solution of Example 5.2. If we take \( n = 2 \) in that example, we obtain \((e_{11} - e_{22}) \wedge e_{12}\), the unique solution, up to equivalence, of the CYBE which lies in \( \mathfrak{sl}(2) \wedge \mathfrak{sl}(2) \).
We showed in [GGS3, §7] that the boundary solution of Example 5.2 was actually part of a larger family of solutions to the CYBE. At that time however, we were unable to determine whether elements of this family were boundary solutions. We show next that this larger family does indeed consist of boundary solutions.

Example 5.3 (see [GGS3, §7]). Let \( \mathfrak{g} = \mathfrak{sl}(n) \) and set \( a = \lambda_1 e_{1n} + \lambda_2 e_{2,n-2} + \cdots + \lambda_d e_{d,n-d+1} \) where \( d = \left\lfloor \frac{n}{2} \right\rfloor \) and \( \lambda_i \in k^\times \). Then

\[
\exp(-ta) \cdot \gamma = \gamma + t \left( \sum_{p=1}^{d} \left( \lambda_p^{-1}(e_{pp} - e_{n-p+1,n-p+1}) \wedge e_{p,n-p+1} + 2 \sum_{i=p+1}^{n-p} e_{pi} \wedge e_{i,n-p+1} \right) \right)
\]

and so the coefficient of the linear term is a boundary solution to the CYBE. The carrier for this solution is the Lie algebra \( \mathcal{H}' \) of matrices of the form

\[
\begin{pmatrix}
d_1 & * & \cdots & * \\
0 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & * \\
0 & \cdots & 0 & d_n
\end{pmatrix}
\]

where the diagonal entries \( d_1, \ldots, d_n \) are defined as follows: \( d_i = 1 \) for \( i \leq \frac{n}{2} \), \( d_i = -1 \) for \( i \geq \frac{n}{2} + 1 \), and \( d_i = 0 \) if \( n = 2i + 1 \). This Lie algebra is also Frobenius. The skew form associated to the linear functional \( e_{1n}^* + \cdots e_{d,n-d+1}^* : \mathcal{H}' \to k \) is non-degenerate and its inverse is the boundary solution constructed above.

Remark 5.4. A slightly more general version of the preceding two examples can be obtained by replacing \( \gamma \) with \( \gamma + \beta \) with \( \beta \in \mathfrak{h} \wedge \mathfrak{h} \) arbitrary; recall that this is the the generic element of \( \mathcal{M}_{T_0} \). In a qualitative sense though, the resulting boundary solutions are essentially identical to those with \( \beta = 0 \). The only difference is that the carrier Lie algebras have more general diagonal entries than just \( \pm 1 \) or 0.

As stated earlier, the most interesting boundary solutions we have so far are related to maximal parabolic subalgebras of \( \mathfrak{sl}(n) \). Henceforth we will assume that \( \mathfrak{g} = \mathfrak{sl}(n) \) and once again we will make the natural identification of \( \{1, \ldots, n-1\} \) with a fixed basis \( \Pi \) of positive simple roots. To each subset non-empty subset \( \Omega \in \Pi \), there is an associated parabolic subalgebra, \( \mathfrak{p}_\Omega \), of \( \mathfrak{sl}(n) \) which is generated by the Cartan subalgebra \( \mathfrak{h} \), all positive simple root vectors, and all negative simple root vectors except those of the form \( x_{-\pi} \) with \( \pi \in \Omega \). It is easy to see that \( \mathfrak{h} \oplus \mathfrak{n}_+ \subset \mathfrak{p}_\Omega \) for every \( \Omega \). Our primary focus will be on maximal parabolic subalgebras, those with \( \#\Omega = 1 \). The maximal parabolic subalgebra corresponding to \( \Omega = \{i\} \) will be denoted \( \mathfrak{p}_i \). In this case, the only missing negative root
vector is $e_{i+1,i}$. The special cases $p_1$ and $p_{n-1}$ will be called the end (maximal) parabolic subalgebras, they consist of matrices of the form

$$p_1 = \begin{pmatrix} * & * & \ldots & * \\ 0 & * & \ldots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & * & \ldots & * \end{pmatrix}$$

and

$$p_{n-1} = \begin{pmatrix} * & * & \ldots & * \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \ldots & * \\ 0 & \ldots & 0 & * \end{pmatrix}$$

where the only restriction on the entries marked $*$ is that the trace is zero. The end parabolic subalgebras have have the greatest dimension among all the $p_i$. Note that $p_i \cong p_{n-i}$ since they map to each other under the automorphism of $\mathfrak{sl}(n)$ obtained the non-trivial automorphism of its Dynkin diagram.

The next result settles the question of the existence (and non-existence) of solutions to the CYBE lying in $p_i \wedge p_i$.

**Theorem 5.5.** Let $p_i$ be the maximal parabolic subalgebra of $\mathfrak{sl}(n)$ obtained by deleting the negative simple root $e_{i+1,i}$.

1. $p_i$ is Frobenius if and only if $i$ and $n$ are relatively prime.
2. $H^2(p_i) = 0$ for all $i$.

**Proof.** (1) This is a special case of the theorem in [E1] which classifies Frobenius Lie algebras of the form $R + N$, where $R$ is a reductive subalgebra and $N$ is a unipotent radical which is either a simple $R$-module or abelian.

(2) This is an immediate consequence of the Hochschild-Serre spectral sequence, see [F]. The result can also be obtained directly by considering the decomposition of $p_i$ into its reductive and unipotent components. □

**Corollary 5.6.** A maximal parabolic subalgebra $\mathfrak{sl}(n)$ is quasi-Frobenius if and only if it is Frobenius.

**Proof.** Combine Theorem 5.5 with Definition 3.1. □

The preceding two results imply that $p_i$ is a carrier of a solution to the CYBE if and only if $i$ and $n$ are relatively prime and, in this case, the solution is uniquely determined up to equivalence. These results however only establish the existence of these solutions. If $i$ and $n$ are relatively prime, it seems natural to ask whether the solution to the CYBE with carrier $p_i$ is a boundary solution and, if so, is it possible to explicitly construct it from an appropriate solution to the MCYBE? We think that this is indeed the case. Recall that the other instance where we needed $i$ and $n$ to be relatively prime was in Theorem 2.6 which classified and constructed all possible generalized Cremmer-Gervais triples, $\mathcal{T}_i$, for $\mathfrak{sl}(n)$. 

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Each of these triples determines a unique (up to equivalence) solution, $r_i$, to the MCYBE. Now the deleted root from the subset $\Pi_2$ of $\Pi$ associated to $T_i$ is $i$. Thus the maximal parabolic subalgebra $p_i$ is, in a sense, canonically related to both the unique solution to the CYBE (since it is Frobenius) and the generalized Cremmer-Gervais triple $T_i$. This prompts the following:

**Conjecture 5.7.** Suppose that $i$ and $n$ are relatively prime and let $r_i$ be the unique solution to the MCYBE associated to the generalized Cremmer-Gervais triple $T_i$. We conjecture that the unique solution of the CYBE with carrier $p_i$ is, in a sense, canonically related to both the unique solution to the CYBE (since it is Frobenius) and the generalized Cremmer-Gervais triple $T_i$. This prompts the following:

We are able to verify this conjecture when $i = 1$, (and hence also for $i = n - 1$) and also for the triple $T_2$ associated to $\mathfrak{sl}(5)$. Our analysis of these cases suggests a procedure which we believe works in general, see Conjecture 6.1.

We begin with $i = 1$. This case corresponds to the Cremmer-Gervais triple $T_{CG}$. The explicit form for the associated solution, $r_{CG}$, to the MCYBE is

$$\left( \sum_{i<j} e_{ij} \wedge e_{ji} \right) + \left( \frac{1}{n} \sum_{i<j} (n + 2(i - j))e_{ii} \wedge e_{jj} \right) + 2 \left( \sum_{i<j} \sum_{m=1}^{j-i-1} e_{i,j-m}\wedge e_{j,i+m} \right). \quad (5.8)$$

**Theorem 5.9.** Suppose that $r_{CG} \in \mathfrak{sl}(n) \wedge \mathfrak{sl}(n)$ is the Cremmer-Gervais solution to the MCYBE.

1. Let $x = \frac{1}{2}[(n - 1)e_{12} + (n - 2)e_{23} + \cdots + 1 \cdot e_{n-1,n}] \in \mathfrak{sl}(n)$ and set

$$d_p = \frac{n-p}{n}(e_{11} + e_{22} + \cdots e_{pp}) - \frac{p}{n}(e_{p+1,p+1} + e_{p+2,p+2} + \cdots e_{nn}).$$

Then $[x, [x, r]] = 0$ and so $\exp(-t x) \cdot r_{CG} = r_{CG} + t [x, r]$. Thus

$$[x, r] = b_{CG} = \left( \sum_{p=1}^{n-1} d_p \wedge e_{p,p+1} \right) + \sum_{i<j} \sum_{m=1}^{j-i-1} e_{i,j-m+1} \wedge e_{j,i+m}$$

is a boundary solution to the CYBE.

2. The carrier for $b_{CG}$ is $p_1$, the maximal parabolic subalgebra of $\mathfrak{sl}(n)$ obtained by deleting the first negative root.

3. The inverse of $b_{CG}$ is the skew (non-degenerate) bilinear form on $p_1$ associated the the linear functional $f = e_{12}^* + e_{23}^* + \cdots e_{n-1,n}^*: p_1 \to k$. That is, $b_{CG}^{-1}(p, q) = f([p, q])$ for all $p, q \in p_1$. 

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The theorem presents several intriguing questions. First, notice that there is a striking similarity between the individual terms of the last summand of $r_{CG}$ and $b_{CG}$. For every term $e_{i,j-m} \wedge e_{j,i+m}$ of the last summand of $r_{CG}$, there is a similar term of the last summand of $b_{CG}$, namely $e_{i,j-m+1} \wedge e_{j,i+m}$. The only difference is the +1 in the second subscript of the first wedge factor. Is this related to the bijection $T$ from Cremmer-Gervais triple $T_{CG}$? Recall that in this case for every $i \in \Pi_1$ we had $T(i) = i + 1$.

Theorem 5.9 also provides an interesting connection between $r_{CG}$, $b_{CG}$, and the principal three-dimensional subalgebra of $\mathfrak{sl}(n)$. Recall that this subalgebra (which is isomorphic to $\mathfrak{sl}(2)$) has generators

$E = (n - 1)e_{12} + (n - 2)e_{23} + \cdots + 1 \cdot e_{n-1,n},$

$F = 1 \cdot e_{21} + 2 \cdot e_{23} + \cdots + (n - 1)e_{n,n-1},$ and

$H = (n - 1)e_{11} + (n - 3)e_{22} + \cdots + (3 - n)e_{n-1,n-1} + (1 - n)e_{nn}.$

This copy of $\mathfrak{sl}(2)$ naturally acts on $\mathfrak{sl}(n) \wedge \mathfrak{sl}(n)$ via the adjoint action. In terms of this action, the first part of Theorem 5.9 says that $[E, r_{CG}] = -2b_{CG}$ and $[E, b_{CG}] = 0$. We can relate this to the $\mathfrak{sl}(2)$-module structure on $\mathfrak{sl}(n) \wedge \mathfrak{sl}(n)$ by means of the automorphism $\sigma : \mathfrak{sl}(n) \rightarrow \mathfrak{sl}(n)$ which sends $e_{ij}$ to $e_{n+1-i,n+1-j}$. It is easy to see that $\sigma^2$ is the identity, $\sigma(E) = F$, $\sigma(F) = E$, and $\sigma(r_{CG}) = -r_{CG}$. Therefore it follows that $[F, [F, \sigma(b_{CG})]] = 0$ and so $\sigma(b_{CG}) = 2[F, r_{CG}]$ is also a boundary solution. Moreover, the three-dimensional submodule of $\mathfrak{sl}(n) \wedge \mathfrak{sl}(n)$ generated by $r_{CG}$ under the action of the principal three-dimensional subalgebra of $\mathfrak{sl}(n)$ is simple since $\mathfrak{sl}(n)$ decomposes into a direct sum of simple submodules of dimensions 3, 5, 7, ..., $(2n - 1)$ under this action and hence $\mathfrak{sl}(n) \wedge \mathfrak{sl}(n)$ contains no invariants (one-dimensional submodules). But the only possible decomposition of the submodule generated by $r_{CG}$ would be into a simple three-dimensional submodule and a direct sum of invariants; there being none of the latter, the module must be simple. For this simple module, the highest and lowest weight vectors, $b_{CG}$ and $\sigma(b_{CG})$ are boundary solutions to the CYBE while the weight zero vector, $r_{CG}$, is a solution to the MCYBE.

Let us now consider the triple $T_2$ for $\mathfrak{sl}(5)$ and its associated solution $r_2$ to the MCYBE.
Explicitly, \( r_2 = \gamma + \beta + \alpha \) where

\[
\gamma = e_{12} \wedge e_{21} + e_{13} \wedge e_{31} + e_{14} \wedge e_{41} + e_{15} \wedge e_{51} + e_{23} \wedge e_{32} \\
+ e_{24} \wedge e_{42} + e_{25} \wedge e_{52} + e_{34} \wedge e_{43} + e_{35} \wedge e_{53} + e_{45} \wedge e_{54},
\]

\[
\beta = \frac{1}{5}\left(-e_{11} \wedge e_{22} - e_{22} \wedge e_{33} - e_{33} \wedge e_{44} - e_{44} \wedge e_{55} + 3e_{11} \wedge e_{33} \\
+ 3e_{22} \wedge e_{44} + 3e_{33} \wedge e_{55} - 3e_{11} \wedge e_{44} - 3e_{22} \wedge e_{55} + e_{11} \wedge e_{55}\right), \quad \text{and}
\]

\[
\alpha = 2(e_{23} \wedge e_{54} + e_{23} \wedge e_{21} + e_{23} \wedge e_{43} + e_{45} \wedge e_{21} + e_{45} \wedge e_{43} + e_{12} \wedge e_{43} + e_{13} \wedge e_{53}).
\]

To show that the solution to the CYBE with carrier \( p_2 \) lies in the boundary of an orbit of \( r_2 \) we again make use of a three-dimensional subalgebra of \( \mathfrak{sl}(5) \), but this one is not the principal one and the submodule of \( \mathfrak{sl}(5) \wedge \mathfrak{sl}(5) \) generated by \( r_2 \) will not be simple.

Set \( X = 2e_{13} + e_{24} + e_{35}, \ Y = e_{31} + e_{42} + 2e_{53} \) and \( H = [X,Y] = 2e_{11} + e_{22} - e_{44} - 2e_{55} \). Then we have \( [H,X] = 2X, \ [H,Y] = -2Y \), and so the subalgebra of \( \mathfrak{sl}(5) \) spanned by \( H, X \) and \( Y \) is isomorphic to \( \mathfrak{sl}(2) \). Let \( \sigma : \mathfrak{sl}(5) \rightarrow \mathfrak{sl}(5) \) once again denote the automorphism sending \( e_{i,j} \) to \( e_{n+1-i,n+1-j} \). Note that \( \sigma(X) = Y, \ \sigma(H) = H, \) and \( \sigma(r_2) = -r_2 \). Just as in the proof of Theorem 5.9, it is an easy computation to show that we have \( \exp(-tX) \cdot r_2 = r_2 + t[X,r_2] \) where

\[
[X,r_2] = \frac{1}{5}\left\{2(2e_{11} + 2e_{22} - 3e_{33} + 2e_{44} - 3e_{55}) \wedge e_{13} + (e_{11} + e_{22} + e_{33} - 4e_{44} + e_{55}) \wedge e_{24} \\
+ (e_{11} + e_{22} + e_{33} + e_{44} - 4e_{55}) \wedge e_{35} \right\} \\
+ \left\{(e_{14} \wedge e_{43} + e_{12} \wedge e_{23} + e_{25} \wedge e_{54} + e_{12} \wedge e_{45} + e_{15} \wedge e_{53} + e_{34} \wedge e_{45}) \right\}
\]

and so \( [X,r_2] \) is a boundary solution to the CYBE. However, its carrier is not \( p_2 \), which has dimension 18, but a 16-dimensional subalgebra. This is probably related to the fact that the present \( [X,r] \) is already in the boundary of the orbit of a “smaller” solution \( r_2' \) to the MCYBE.

Under the action of the current \( \mathfrak{sl}(2) \), we have that \( \mathfrak{sl}(5) \) decomposes into a direct sum of modules of dimensions 5, 4, 4, 3, 3, 2, 2, and 1, so \( \mathfrak{sl}(n) \wedge \mathfrak{sl}(n) \) has two invariants (coming from the exterior products of the 4 dimensional and 2 dimensional submodules with themselves). Setting \( \xi = e_{23} + e_{45} \) and \( \eta = \sigma(\xi) = e_{21} + e_{43} \), we can write \( \alpha = \alpha_0 + \alpha_1 \) where \( \alpha_0 = 2\xi \wedge \eta \) and \( \alpha_1 = 2(e_{23} \wedge e_{54} + e_{12} \wedge e_{43} + e_{13} \wedge e_{53}) \). Then, with the present \( X, Y, H \), one can check
the following relations: \([X, \xi] = 0\), \([X, \eta] = -\xi\), and so \([X, \alpha_0] = 0\). Similar relations for \(Y\) are easily obtained by applying \(\sigma\). Thus \(\alpha_0\) is invariant for the action of the current copy of \(\mathfrak{sl}(2)\) and so if \(r'_2 = r_2 - \alpha_0\) then \([X, r_2] = [X, r'_2]\) and \([X, r_2]\) is in the boundary of the orbit of \(r'_2\). It follows that the module generated by \(r'_2\) is 3-dimensional and simple, and the module generated by \(r_2\) is the direct sum of this module and the invariant \(\alpha_0\). Now one can check that \(r'_2\) is in fact one of the Belavin–Drinfel’d solutions to the MCYBE, namely that obtained by taking \(\Pi_1 = 1, 2, \Pi_2 = 3, 4\), with \(T(1) = 3, T(2) = 4\). Note that this triple is obtained from \(T_2\) by dropping the unique map, \(T(4) = 1\), which takes a root higher in the usual order to a lower one.

Thus the direct analog of Theorem 5.9 does not produce the solution to the CYBE with carrier \(p_2\). However, we can successfully modify the proceeding to indeed find this solution. Let \(H_1 = \left(\frac{1}{5}\right)(-4e_{11} + 6e_{22} - 4e_{33} + 6e_{44} - 4e_{55})\). Then \(\exp(\eta) \cdot r_2 = r_2 + H_1 \wedge \eta\) is a solution to the MCYBE equivalent to \(r_2\). Now

\[
\exp(tX + \eta) \cdot r_2 = r_2 + H_1 \wedge \eta + t \{[X, r_2] - (3/2)H_1 \wedge \xi + \eta \wedge \xi\}.
\]

The coefficient of \(t\) must therefore be a boundary solution and one can check that its carrier is the maximal parabolic subalgebra

\[
p_2 = \begin{pmatrix}
* & * & * & * & *
0 & 0 & * & * & *
0 & 0 & * & * & *
0 & 0 & * & * & *
0 & 0 & * & * & *
\end{pmatrix}
\]

\[\begin{pmatrix}
* & * & * & * & *
* & * & * & * & *
0 & 0 & * & * & *
0 & 0 & * & * & *
0 & 0 & * & * & *
\end{pmatrix}
\]

6. Closing remarks. We close with two conjectures – one about how the foregoing analysis may extend to prove Conjecture 5.7 in all cases and the other about constructing quantum Yang-Baxter matrices from the solutions to the CYBE.

Let us consider the \(i\)th maximal parabolic subalgebra of \(\mathfrak{sl}(n)\) where \(i\) is relatively prime to \(n\) and hence defines a Belavin–Drinfel’d solution, \(r_i\), to the MCYBE in which the sole root omitted from \(\Pi_1\) is \(n - i\). As mentioned in Remark 2.7, the integers \(1, \ldots, n - 1\) have an order defined by the corresponding root mapping \(T\). That is, write them in the order \(i, 2i, \ldots, n - i\), always reducing modulo \(n\). So, for example, if \(n = 12\) and \(i = 5\) then the order is 5, 10, 3, 8, 1, 6, 11, 4, 9, 2, 7. Now define a string, \(s\), to be a maximal subsequence in which the integers appear in their natural order. In this case, we have the following strings: \(\{5, 10\}, \{3, 8\}, \{1, 6, 11\}, \{4, 9\}, \{2, 7\}\), which we will denote by \(s_1, \ldots, s_5\), respectively, and in the general case by \(s_1, \ldots, s_i\) since the number of strings is always equal to \(i\). Some will have length \([n/i]\) (greatest integer contained in \(n/i\)) and some will have length one
greater. Now for each string $s_1, \ldots, s_{i-1}$, i.e., for all except the last, define $e(s) = \sum e_{j,j+1}$ where the sum runs over the integers $j$ in the string. In the example we have $e(s_1) = e(5, 10) = e_5 + e_{10, 11}$, but $e(2, 7)$ is not defined. For every string $s_2, \ldots, s_i$, i.e. all but the first, define $e'(s) = \sum e_{j+1,j}$ where the sum again is over the integers in the string. Thus $e'(s_5) = e'(2, 7) = e_3 + e_8 + e_7$, while $e'(5, 10)$ is not defined. Notice that if, as before, $\sigma$ is the automorphism of $\mathfrak{sl}(n)$ sending $e_{jk}$ to $e_{n+1-j, n+1-k}$, then $\sigma$ sends the set of $e(s)$ which are defined to the set of $e'(s)$ which are defined.

Let $L_+$ now be the Lie subalgebra of $\mathfrak{sl}(n)$ generated by $e(s_1), \ldots, e(s_{i-1})$, let $L_- = \sigma(L_+)$ be that generated by $e'(s_2), \ldots, e'(s_i)$, and $L$ be the Lie subalgebra generated by both together. The intersection of $L$ with the set of diagonal matrices will be denoted $L_0$. In the case where $i = 1$ we have $L = 0$, since there is then but one string and hence no $e(s)$ and no $e'(s)$ are defined. The algebra $L$ is never semisimple. When $i = 2$ it is of dimension 2 and abelian, but there is a non-trivial semisimple part whenever $i > 2$. (In the case $n = 5, i = 2$, the unique $e(s) = e(2, 4)$ was denoted by $\xi$ and the unique $e'(s) = e(1, 3)$ by $\eta$.) The radical $\mathcal{R}$ of $L$ is the direct sum of its part lying in $L_+$, which we denote by $\mathcal{R}_+$, and of its part in $L_-$, denoted $\mathcal{R}_-$. One can readily compute that in the example $n = 11, i = 5$, the semisimple part is a direct sum of two algebras, one isomorphic to $\mathfrak{sl}(2)$ and the other to $\mathfrak{sl}(3)$, its total dimension being 11, while the radical has dimension 12. Finally, we define a subalgebra isomorphic to $\mathfrak{sl}(2)$ as follows: set $X = \sum_{j=1}^{n-1}[(n-j)/i]e_{j,j+1}, \quad Y = \sigma X,$ and $H = [X, Y].$

**Conjecture 6.1.**

1. $[X, L_0] = [X, L_+] = 0$, while $[X, L_-] = L_+$ and $[X, \mathcal{R}_-] = \mathcal{R}_+$.
2. $\mathcal{R}$ is abelian. (One obtains the corresponding relations involving $Y$ by applying $\sigma$.) These assertions can be easily verified in the example and probably are not too difficult in general. The more difficult part is:
3. Let $r_i$ be the Belavin-Drinfel’d solution to the MCYBE associated to the triple $\mathcal{T}_i$ and suppose $z$ is a generic element of $\mathcal{R}_-$. Then

$$\exp(t X + z) \cdot r_i = r_i' + t \omega$$

where $r_i'$ is a solution to the MCYBE which is equivalent to $r_i$ and $\omega$ is a boundary solution whose carrier is precisely the maximal parabolic subalgebra $\mathfrak{p}_i$. (This is essentially what we have shown for $n = 5, p = 2$.)

The Lie algebra $L$ arises naturally by considering the terms which disappear from the $\alpha$-part of $r_i$ when one drops from the bijection $T$ of $\mathcal{T}_i$ the mappings carrying a root higher in the usual order to a lower one. But the significance of its decomposition is not clear, and
what is most mysterious is the representation of $\mathfrak{sl}(2)$ associated with each $i$ which seems to play an essential role.

We do not know in general which non-maximal parabolic subalgebras of $\mathfrak{sl}(n)$ are Frobenius or quasi-Frobenius, but if the conjecture is true it may suggest ways to determine when a parabolic is the carrier of a boundary solution to the CYBE (and hence at least quasi-Frobenius).

Finally, it is natural to consider the problem of explicitly finding a quantum Yang-Baxter matrix associated to a solution of the CYBE. Little is known in general about this problem. For many of the boundary solutions though the following result applies.

**Theorem 6.2** [GGS1]. Suppose that $b \in \mathfrak{sl}(n) \wedge \mathfrak{sl}(n)$ is a solution to the CYBE and that $b^3 = 0$ when viewed as an $n^2 \times n^2$ matrix under the Kronecker product. Then $B = \exp(t \ b) = 1 + t \ b + (t^2/2) b^2$ is a solution to the quantum Yang-Baxter equation, that is, $B_{12}B_{13}B_{23} = B_{23}B_{13}B_{12}$.

The boundary solution of Example 5.2 has cube zero and it seems likely that many others do as well. This is true of the Cremmer-Gervais boundary solution $b_{CG}$ for $n = 3$ (formula (0.2)). In this case we have that $1 + t \ b_{CG} + (t^2/2) b_{CG}^2 =$

\[
\begin{pmatrix}
1 & t/3 & t^2/36 & -t/3 & t^2/18 & 0 & t^2/36 & 0 & 0 \\
0 & 1 & t/6 & 0 & t/6 & t^2/36 & 0 & t^2/36 & 0 \\
0 & 0 & 1 & 0 & 0 & t/6 & 0 & t/2 & -t^2/18 \\
0 & 0 & 0 & 1 & -t/6 & t^2/36 & -t/6 & t^2/36 & 0 \\
0 & 0 & 0 & 0 & 1 & t/6 & 0 & -t/6 & t^2/18 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & t/3 \\
0 & 0 & 0 & 0 & 0 & -t/2 & 1 & -t/6 & -t^2/18 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -t/3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

is a solution to the quantum Yang-Baxter equation.

**Conjecture 6.3.** For all $n > 3$, the Cremmer-Gervais boundary solution $b_{CG}$ (Theorem 5.9.1) has cube zero.
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