Berglund-Hübsch Mirror Symmetry via Vertex Algebras

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Received: 1 November 2011 / Accepted: 10 November 2012
Published online: 9 April 2013 – © Springer-Verlag Berlin Heidelberg 2013

Abstract: We give a vertex algebra proof of the Berglund-Hübsch duality of nondegenerate invertible potentials. We suggest a way to unify it with the Batyrev-Borisov duality of reflexive Gorenstein cones.

1. Introduction

Ever since its been first discovered in the early 1990’s, mirror symmetry served as an inspiration to algebraic and symplectic geometers. The original physical motivation behind it centers around the notion of $N = (2, 2)$ superconformal field theory, which is a very rich and only partially axiomatized structure. There are physical methods of assigning such theories to various combinatorial and/or geometric data. In certain instances, the theories one obtains from two different sets of data are isomorphic via a very special mirror involution, which leads to a deep connection between the two sets of data. The earliest example was the prediction of the (virtual) number of rational curves of given degree on a generic quintic threefold [CaOGP].

Most classical treatments of mirror symmetry revolve around the calculation of the so-called $A$ and $B$ chiral rings which are particular substructures of $N = (2, 2)$ superconformal field theory. A mirror set of data is characterized by the property that the two rings are interchanged. The $A$ ring of one set of data is supposed to be isomorphic to the $B$ ring of the mirror set of data and vice versa. In many cases these two rings can be constructed mathematically from the initial data, even while the whole theory can only be constructed physically. For example, one can start with a Calabi-Yau manifold $X$ with a complexified Kähler class $[w]$. Then the $A$ ring is the quantum cohomology of $X$ with the parameters specialized to $[w]$. The $B$ ring is the cohomology of the exterior algebra of the tangent bundle of $X$.

In the examples of interest, the $A$ and $B$ rings of the theory come with a double grading and with a natural vector space isomorphism between them. If $\hat{c} > 0$ is the central charge of the theory (equal to the dimension of $X$ in the above example), then
this isomorphism sends a \((p, q)\)-graded piece of the \(A\) ring to the \((\hat{c} - p, q)\) graded piece of the \(B\) ring. Of course, this isomorphism is not compatible with the product structures.

Mirror symmetry construction of Batyrev \cite{Ba2} and its generalization to Calabi-Yau complete intersections in toric varieties known as Batyrev-Borisov construction is one the best understood settings of mirror symmetry. In contrast, the Berglund-Hübsch construction of \cite{BeH} has unfortunately received too little attention, despite its appealing simplicity. Recently, papers of Krawitz \cite{Kra} and Chiodo-Ruan \cite{ChR} partially corrected this injustice by proving the conjectural mirror symmetry of Berglund-Hübsch at the level of double graded dimensions, and by connecting the Landau-Ginzburg version of the theory to the geometric notion of orbifold cohomology. Importantly, Krawitz was able to define the dual potential and group of Berglund-Hübsch construction in full generality, beyond the original examples of Berglund and Hübsch. This paper hopes to further advance the understanding of Berglund-Hübsch-Krawitz duality and to also sketch a path that will likely lead to a combined setting that includes both Batyrev-Borisov and Berglund-Hübsch constructions as special cases. Specifically, we find that the vertex algebra approach to mirror symmetry, originally developed for Batyrev-Borisov setting in \cite{Bo2} and \cite{Bo3} can be applied with some modifications to the Berglund-Hübsch setting. It allows us to reprove the result of \cite{Kra} and moreover to show the ring isomorphism between the \(A\) ring of a Berglund-Hübsch potential and the \(B\) ring of the dual potential (with the appropriate choices of orbifoldizations). In the process we are lead to a natural combinatorial setup for the unification of the two constructions.

The vertex algebra approach to mirror symmetry aims to construct a larger algebraic structure which contains the \(A\) and \(B\) rings of the theory as two subspaces, with the induced structure of supercommutative double graded rings. The mirror set of data gives the same larger structure, up to a natural involution that has the effect of switching \(A\) and \(B\) rings. Specifically, this larger algebraic structure is known as vertex (in some treatments chiral) algebra with \(N = 2\) structure, and the \(A\) and \(B\) rings are the chiral rings of this algebra, in the sense of \cite{LeVW}. From the physics viewpoint this vertex algebra is the state space of the half-twisted theory. We recall the definition of these algebras in Sect. 4. The aforementioned isomorphism of \(A\) and \(B\) rings of the same theory comes from a natural physical construction called spectral flow. The original calculations of \cite{Bo2} have been heavily inspired by the chiral de Rham complex construction of Malikov, Schechtman and Vaintrob, see \cite{MSV}. However, the setting of this paper is more combinatorial, since the underlying geometry is often not clear.

The structure of the paper is as follows. In Sect. 2 we recall the definitions of Berglund-Hübsch potentials, following \cite{Kra}, and then rephrase the construction in the terms that are similar to those of Batyrev-Borisov duality. Specifically, we encode a pair of dual potentials and groups \((W, G)\) and \((W^\vee, G^\vee)\) by a pair of dual lattices \(M\) and \(N\) and collections of elements \(\Delta\) and \(\Delta^\vee\) in these lattices. The pairings of elements of \(\Delta\) and \(\Delta^\vee\) encode the matrix of degrees that occurs in the definition of Berglund-Hübsch potentials. The elements of \(\Delta\) and the elements of \(\Delta^\vee\) generate cones \(K_M\) and \(K_N\) respectively. While \(K_M\) and \(K_N\) are not quite dual to each other (as would be the case in Batyrev-Borisov setting) they are close to being dual, because of the nondegeneracy of the potentials.

In Sect. 3 we give a novel description of \(A\) and \(B\) rings of Berglund-Hübsch construction by what essentially amounts to a switch to logarithmic coordinates. We describe the