Optimal Eavesdropping in Quantum Cryptography. II. 
Quantum Circuit

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Abstract

It is shown that the optimum strategy of the eavesdropper, as described in the preceding paper, can be expressed in terms of a quantum circuit in a way which makes it obvious why certain parameters take on particular values, and why obtaining information in one basis gives rise to noise in the conjugate basis.

1 Introduction

The preceding paper [1] discusses the maximum information which an eavesdropper, Eve, can obtain for a given error rate between Alice, who sends signals, and Bob, their legitimate recipient, in the context of the BB84 cryptographic scheme [2]. In the present paper we show that Eve’s optimum strategy, discussed in Sec. III of the preceding paper, can be embodied in a simple quantum circuit, of the type proposed for quantum computation [3], together with appropriate initial states of the two qubits which constitute Eve’s probe, and suitable final measurements.

The quantum circuit is extremely simple: it involves only two gates, of the controlled-not variety, although adding a third gate could be advantageous under some circumstances (discussed towards the end of Sec. 3). When it is analyzed using the same consistent history methods we used earlier [4] to simplify the final Fourier transform in Shor’s factorization algorithm [5], it is immediately obvious how the parameters in Eve’s initial state are related to the error rates, both errors in the transmission from Alice to Bob, and the errors which determine the mutual information between Alice and Eve.

The results of the preceding paper which are essential for understanding the present one are summarized in Sec. 2 below. The quantum circuit is described in analyzed in Sec. 3, and a brief summary is presented in Sec. 4.

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2 Errors and Information

In the BB84 scheme, Alice transmits a signal using a qubit described by a two-dimensional Hilbert space; for example, the polarization of a photon, or the spin of a spin half particle. The kets $|x\rangle$ and $|y\rangle$ form an orthonormal basis of this space, and

$$|u\rangle = (|x\rangle + |y\rangle)/\sqrt{2}, \quad |v\rangle = (|x\rangle - |y\rangle)/\sqrt{2},$$

form an alternative (conjugate) basis. Alice chooses one of these bases at random, and then transmits one of the basis vectors, also chosen at random. Her qubit, hereafter denoted by $a$, is intercepted by Eve and made to interact with a probe consisting of two qubits, which we shall call $e$ and $f$. After this, Eve sends $a$ on to Bob, who measures it in one of the two bases, again chosen at random. Eventually Alice announces publicly the basis she used for transmission of the signal, and in those cases in which Bob measured in the same basis (the other cases are of no interest, for Alice and Bob discard the results), Eve, who now knows the basis Alice employed, measures the qubits in her probe in order to estimate which signal Alice sent.

In Sec. II of the preceding paper it was shown that the average information (in the Shannon sense) $I_{xy}$ which Eve obtains about Alice’s signal when the latter uses the $xy$ basis is bounded by

$$I_{xy} \leq \frac{1}{2} \phi \left[ 2\sqrt{D_{uv}(1 - D_{uv})} \right],$$

where

$$\phi(z) = (1 + z) \ln(1 + z) + (1 - z) \ln(1 - z),$$

and $D_{uv}$ is the error rate produced by the interaction with Eve’s probe when Alice transmits and Bob measures in the $uv$ basis. Similarly, when Alice sends a signal in the $uv$ basis, the average information $I_{uv}$ which Eve can gain is bounded by a similar inequality

$$I_{uv} \leq \frac{1}{2} \phi \left[ 2\sqrt{D_{xy}(1 - D_{xy})} \right],$$

with $D_{xy}$ the error rate from Alice to Bob when they employ the $xy$ basis.

3 Quantum Circuit

The quantum circuit used in Eve’s optimum strategy is shown in Fig. 1. The three horizontal lines represent three qubits thought of as moving from left to right as time increases. The top line is Alice’s qubit $a$ on its way to Bob, while the two lower lines show qubits $e$ and $f$ representing Eve’s probe. These interact with Alice’s qubit through two controlled-not gates, labeled 1 and 2 and indicated by solid vertical lines. The dashed vertical lines indicate where the qubits are at two specific times which we will want to refer to later.

The two diagrams in Fig. 1 represent the same quantum circuit, but show its action in two different bases: the $xy$ basis in (a) and the $uv$ basis in (b). By $xy$ basis we now mean the basis of the full eight-dimensional Hilbert space corresponding to qubits $a$, $e$, and $f$, with each qubit in either an $|x\rangle$ or a $|y\rangle$ state. Thus the basis vectors are of the form $|aef\rangle = |xyy\rangle, |xyx\rangle, \text{ and so forth.}$ Similarly, the $uv$ basis is constituted by vectors of the...
form $|ae\rangle = |uvw\rangle$, and so forth; each of the qubits is in either the state $|u\rangle$ or the state $|v\rangle$, where these are related to $|x\rangle$ and $|y\rangle$ through (1).

In Fig. 1(a), the $xy$ basis, gate 1 is a unitary transformation which when applied to qubits $a$ and $e$, with the left letter in each ket referring to $a$ and the right to $e$, yields

$$|xx\rangle \rightarrow |xx\rangle, \quad |xy\rangle \rightarrow |xy\rangle, \quad |yx\rangle \rightarrow |yx\rangle, \quad |yy\rangle \rightarrow |yy\rangle. \quad (5)$$

That is, if $a$ is in state $|y\rangle$, $e$ is flipped from $|x\rangle$ to $|y\rangle$ or vice versa, whereas if $a$ is in state $|x\rangle$, $e$ remains unchanged; in either case, $a$ retains its original value. Such a gate is called “controlled-not”, because flipping a bit corresponds to logical negation, and whether or not the target qubit $e$ is flipped depends on the state of the control qubit $a$. Note that $f$ is not involved in gate 1, so (5) can be extended to the eight basis vectors of the full Hilbert space by inserting a third letter in each of the kets to represent the state of $f$, the same letter on each side of the arrow: $|yx\rangle \rightarrow |yy\rangle$, $|yy\rangle \rightarrow |yy\rangle$, and so forth.

Gate 2 in Fig. 1(a) is another controlled-not operation, but now $f$ is the control qubit and $a$ the target qubit, whereas $e$ is not involved (as indicated by the absence of any symbol at the intersection of its line with the vertical line representing the gate). The action of gate 1 followed by gate 2 results in the following unitary transformation on the $xy$ basis vectors:

$$|xxx\rangle \rightarrow |xxx\rangle, \quad |xxy\rangle \rightarrow |xxy\rangle, \quad |xyx\rangle \rightarrow |xyx\rangle, \quad |xyy\rangle \rightarrow |xyy\rangle, \quad |yx\rangle \rightarrow |yx\rangle, \quad |yxx\rangle \rightarrow |yxx\rangle, \quad |yxy\rangle \rightarrow |yxy\rangle, \quad |yyy\rangle \rightarrow |yyy\rangle. \quad (6)$$

If instead of the $xy$ basis, the $uv$ basis, see (1), is employed for all three qubits, the same circuit, corresponding to the same unitary transformation (8), takes the form shown in Fig. 1(b). The reason is that if both qubits involved in a controlled-not gate are changed from the $xy$ to the $uv$ basis, the action of the gate can again be represented as a controlled-not, but with the control and target qubits interchanged, as the reader can easily verify using (1). Thus (8) is equivalent, again with qubit $a$ on the left and $e$ on the right, to

$$|uu\rangle \rightarrow |uu\rangle, \quad |uv\rangle \rightarrow |uv\rangle, \quad |vu\rangle \rightarrow |vu\rangle, \quad |vv\rangle \rightarrow |vv\rangle, \quad (7)$$

as can be checked by employing (1) with (8). The result in the $uv$ basis of the two gates acting in succession can be worked out either by combining (1) with (8) or, more simply, by employing (9) followed by the corresponding transformation for gate 2 in Fig. 1(b).

The following is then an optimum strategy for Eve. She prepares qubits $e$ and $f$ in initial states

$$|e_0\rangle = \sqrt{1 - \Delta_u}|x\rangle + \sqrt{\Delta_u}|y\rangle = \sqrt{1 - D_u}|u\rangle + \sqrt{D_u}|v\rangle, \quad |f_0\rangle = \sqrt{1 - D_u}|x\rangle + \sqrt{D_u}|y\rangle = \sqrt{1 - \Delta_u}|u\rangle + \sqrt{\Delta_u}|v\rangle, \quad (8)$$

where, with $w = uv$ or $xy$, $\Delta_w$ and $D_w$ are related through the formulas:

$$\Delta_w = \frac{1}{2} - \sqrt{D_w(1-D_w)}, \quad D_w = \frac{1}{2} - \sqrt{\Delta_w(1-\Delta_w)} \quad (9)$$

We assume, for convenience, that both quantities are between 0 and 1/2, so that (8) defines one quadrant of a circle of radius 1/2 centered at $(1/2, 1/2)$ in the $(D_w, \Delta_w)$ plane. It is easy to check that the second equality in each line in (8) is consistent with (1) and (9).
After the initial preparation, qubits \( e \) and \( f \) interact with Alice’s qubit \( a \) in the quantum circuit of Fig. 1, and Eve allows \( a \) to go on to Bob while storing \( e \) and \( f \) until Alice announces the basis in which the signal was transmitted. Then Eve measures both qubits \( e \) and \( f \) in the basis (\( xy \) or \( uv \)) announced by Alice.

One can check that this is an optimum strategy by applying the unitary transformation (6) to the initial state
\[
|a⟩ \otimes |e_0⟩ \otimes |f_0⟩,
\]
expressed as a linear combination of the \( xy \) basis states, to obtain \(|X⟩ \) if \(|a⟩ = |x⟩ \) and \(|Y⟩ \) if \(|a⟩ = |y⟩ \), in the notation of the preceding paper, and \(|U⟩ \) and \(|V⟩ \) by means of:
\[
|U⟩ = (|X⟩ + |Y⟩)/\sqrt{2}, \quad |V⟩ = (|X⟩ − |Y⟩)/\sqrt{2}.
\]

Then using the projectors \( \{E_λ\} \) and \( \{F_λ\} \) defined in Sec. III of the preceding paper, one can verify that the conditions given there (in Sec. II) for saturating the bounds (2) and (11) are satisfied.

However, we shall use an alternative approach, which yields more insight into the choice of coefficients in (8): we shall calculate the error rates and mutual information directly from the quantum circuit in Fig. 1 and verify that (2) and (11) are satisfied as equalities. Let us begin with the situation in which Alice announces, and Eve measures in, the \( xy \) basis, which can best be understood using Fig. 1(a). First consider the case in which \( D_{xy} \) and \( Δ_{uv} \)—note that Eve can choose them independently—are both equal to zero, so that both \( e \) and \( f \) are initially in the state \(|x⟩, (8)\). Then gate 1 simply copies qubit \( a, |x⟩ \) or \(|y⟩ \), to qubit \( e \), so that by measuring \( e \) in the \( xy \) basis, Eve knows precisely which signal Alice sent. Furthermore, since \( f \) is in the state \(|x⟩ \), qubit \( a \) remains unchanged on its way from Alice to Bob, so Eve’s intervention causes no error.

If \( Δ_{uv} = 0 \) but \( D_{xy} \) is positive, Eve will again be able to determine which signal Alice sent by measuring \( e \). However, measuring \( f \) will yield \(|y⟩ \) with probability \( D_{xy} \) and \(|x⟩ \) with probability \( 1 − D_{xy} \). Since \( f \) is the control qubit for gate 2, if it is in state \(|y⟩ \), an error will be produced in the transmission from Alice to Bob, while if it is in state \(|x⟩ \), there will be no error. Hence Eve’s measurement of \( f \), while it tells her nothing about which signal Alice sent, shows her whether or not, in this particular case, Bob’s measurement yielded the same or the opposite result from what Alice transmitted.

In the preceding paragraph we used a process of retrodiction, in which we inferred the prior state of qubit \( f \) from Eve’s measurement. This can lead to quantum paradoxes when it is not used in the proper way, but in the present context it can be justified, just as in [4], by using an appropriate framework or family of consistent histories [3]. However, rather than employing retrodiction for qubit \( e \) as well, it is more straightforward to adopt at the outset an appropriate consistent family, which we shall call the \( xy \) framework, based upon all three qubits being in either an \(|x⟩ \) or a \(|y⟩ \) state at a time \( t_1 \) shortly after \( t_0 \), when (3) applies, but before qubits \( a \) and \( e \) reach the first gate, see Fig. 1, and at all later times as well [6]. In this framework, \( e \) at time \( t_1 \) is in the state \(|y⟩ \) with probability \( Δ_{uv} \), and in \(|x⟩ \) with probability \( 1 − Δ_{uv} \), the absolute squares of the corresponding coefficients in (3), while for \( f \) these probabilities are \( D_{xy} \) and \( 1 − D_{xy} \). As long as we are using the \( xy \) framework, these probabilities can be thought of in the same way as in a classical stochastic theory [3].

Because qubit \( f \) at \( t_1 \), and thus at all later times until it is measured, is in state \(|y⟩ \) with probability \( D_{xy} \), this is also the probability of an error if Alice is transmitting to Bob in the
xy mode, as noted earlier. Next let us consider qubit e. The results of Eve’s measurement, which is the value e has when it leaves gate 1, will coincide with the initial value of a only if at $t_1$ e is in the state $|x\rangle$. Otherwise the measured value will be the reverse of what Alice transmits. Thus if one thinks of gate 1 as part of a communication channel from Alice (qubit a) to Eve (qubit e), it is a noisy channel with a probability $\Delta_{uv}$ that a bit will be flipped. The mutual information $I_{xy}$ associated with such a channel is easily computed, assuming that Alice transmits $|x\rangle$ or $|y\rangle$ with equal probability, and is given by the right side of (2).

To understand what happens when Alice sends a signal, and Eve makes her measurements, in the uv basis, we use Fig. 1(b) and an alternative consistent family, which we call the uv framework, in which all three qubits are in state $|u\rangle$ or $|v\rangle$ at $t_1$ and all later times. Then in this framework, $e$ is in state $|v\rangle$ with probability $D_{uv}$, and $|u\rangle$ with probability $1 - D_{uv}$, at $t_1$; see (3). For $f$ the corresponding probabilities are $\Delta_{xy}$ and $1 - \Delta_{xy}$. As $e$ is now the control qubit for gate 1, Fig. 1(b), it is at once obvious that when $e$ is in state $|v\rangle$, there will be an error in the transmission from Alice to Bob when they use the $uv$ basis. Thus the error rate in this basis is $D_{uv}$, the probability that $e$ is in state $|v\rangle$ at $t_1$, and hence at later times as well. Also by measuring $e$ (in the $uv$ basis) Alice can determine whether or not such an error has occurred. However, measuring $e$ tells her nothing about whether Alice sent a $|u\rangle$ or a $|v\rangle$. To obtain this information, she must measure qubit $f$. The task is a bit more complicated than in the case of the $xy$ basis considered earlier, because qubit $a$ may have been flipped through its interaction with $e$ before it is copied to $f$. However, since she also measures $e$, Eve can easily correct for this effect. Consequently, the noise in the channel between Alice (qubit $a$) and Eve (qubit $f$ corrected by $e$) is determined by the uncertainty in the value of $f$ at $t_1$; a bit passing from Alice to Eve through this channel will be flipped with probability $\Delta_{xy}$, the probability that $f$ is in state $|v\rangle$ at $t_1$. Again, the mutual information $I_{uv}$ for such a channel is easily computed, and is given by the right side of (4).

Consequently, one sees that the error rates produced by Eve’s employing the initial states in (3) are, indeed, $D_{xy}$ and $D_{uv}$ in the $xy$ and $uv$ bases, respectively, whereas the appropriate mutual information in each case saturates the corresponding bound, (2) or (3). This shows that Eve’s strategy employing the gates in Fig. 1 is, indeed, optimal. Furthermore, one can understand how Eve faces a trade-off between gaining information when Alice sends in one mode, and creating errors when Alice uses the other mode. If Alice only employed the $xy$ mode, Eve would, of course, set both $D_{xy}$ and $\Delta_{uv}$ equal to zero, as this would cause no errors in the transmission from Alice to Bob, and produce a perfect copy of Alice’s signal in qubit $e$. However, $\Delta_{uv} = 0$ is equivalent, (3), to $D_{uv} = 1/2$, so that obtaining the maximum possible information about the $xy$ transmission produces a large number of errors in the $uv$ mode. Similarly, setting $D_{xy} = 0$, while it produces no errors when Alice transmits in the $xy$ mode, has the consequence the $\Delta_{uv} = 1/2$, which means that Eve can extract no information whatever when Alice transmits in the $uv$ mode.

Eve’s strategy as described above requires that she store both qubits $e$ and $f$ while waiting for Alice to announce the basis used in sending the signal. Since the “storage costs” of preserving the qubits against decoherence, should there be a long delay, could be high, it is worth noting, as was pointed out in the preceding paper, that only one qubit needs to be preserved if Eve just wants to estimate which signal Alice sent, and is not interested in keeping track of whether an error occurred in the transmission from Alice to Bob. From
Fig. 1(a) it is evident that in the $xy$ case, qubit $f$ could be discarded after it emerges from gate 2, since Eve only uses $e$ to gain information about Alice’s signal. However, this would not work for the $uv$ basis, where the information of interest is contained in the correlation between the two qubits. There are, nonetheless, two obvious strategies available to Eve. She can measure qubit $f$ in the $uv$ basis immediately after it emerges from gate 2, and record the value in her notebook, while preserving qubit $e$ for later analysis. If Alice later announces that she used the $xy$ basis, Eve ignores the record in her notebook, and measures $e$ in the $xy$ basis. If, on the other hand, the basis turns out to be $uv$, Eve measures $e$ in that basis and uses it to correct the $f$ value she measured earlier. An alternative approach is to add a third gate to the circuit in Fig. 1, a controlled-not in which $e$ is the control and $f$ the target in the $xy$ basis (and, of course, the reverse in the $uv$ basis). After it passes through this third gate, Eve discards qubit $f$ and retains qubit $e$ for later measurement in whichever basis is appropriate; this is equivalent to the approach employed in the preceding paper.

The use of the quantum circuit does not solve the problem of whether Eve’s optimum strategy is essentially unique. One can show that interchanging (a) and (b) in Fig. 1, that is, employing the circuit in (a) for the $uv$ basis and that in (b) for the $xy$ basis, is equivalent to the original scheme preceded and followed by unitary transformations on the qubits of Eve’s probe, so that it is not different in any essential way. However, this obviously does not settle the question of uniqueness.

4 Conclusion

We have shown that Eve’s optimum strategy can be represented by a simple quantum circuit involving two controlled-not gates, along with the preparation of the two qubits of her probe in suitable initial states, and their later measurement in the same basis announced by Alice. In this circuit, the function of each qubit of the probe is clearly distinguished. For example, in the $xy$ basis, qubit $e$ is employed for extracting information about the signal Alice sends, while $f$ creates errors in the transmission from Alice to Bob. While these errors can be reduced to zero by Eve’s choice of a suitable initial state for $f$, this choice makes it impossible for her to obtain any information when Alice transmits in the $uv$ basis, for which Eve must extract the information using $f$.

While the use of the quantum circuit provides one with a certain amount of insight into eavesdropping strategies, it does not by itself provide a proof that the strategy is optimal, for which one needs the bounds derived in the preceding paper, nor does it show that there is a unique optimal strategy.

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References
[1] C. A. Fuchs et. al, Phys. Rev. A (preceding paper).

[2] C. H. Bennett and G. Brassard, in Proceedings of the IEEE International Conference on Computer, Systems, and Signal Processing, Bangalore, India (IEEE, New York, 1984), pp. 175–179.

[3] See, for example, D. P. DiVincenzo, Science 270, 255 (1995).

[4] R. B. Griffiths and C. S. Niu, Phys. Rev. Lett. 76, 3228 (1996).

[5] P. W. Shor, in Proceedings of the 35th Annual Symposium on Foundations of Computer Science, Santa Fe, 1994, edited by S. Goldwasser (IEEE Computer Society Press, Los Alamitos, California, 1994), p. 124.

[6] R. B. Griffiths, Phys. Rev. A 54, 2759 (1996), which also gives references to earlier work.

[7] To be more precise, at $t_0$ we employ a decomposition of the identity which includes projectors onto the states (10) with $a$ set equal to $x$ and $y$, and at all later times (except those at which two of the bits are actually interacting inside one of the gate, and which are not included in the histories) a decomposition of the identity onto the eight $xy$ basis states of the three bits.

[8] The qualifications are the same as in [7], except that $x$ and $y$ are replaced by $u$ and $v$. Also note that the $uv$ framework is incompatible with the $xy$ framework considered previously, so they cannot both be applied simultaneously to the same system. For further remarks on incompatible frameworks, see [6].

Figure 1: Quantum circuit representing the interaction of Eve’s probe, qubits $e$ and $f$, with Alice’s qubit $a$, in (a) the $xy$ basis and (b) the $uv$ basis.
