EXACT RECOVERY OF COMMUNITY DETECTION IN K-PARTITE GRAPH MODELS

ZHONGYANG LI

Abstract. We study the vertex classification problem on a graph in which the vertices are in $k (k \geq 2)$ different groups, or communities, and edges are only allowed between vertices in distinct groups. The observation is the weighted adjacency matrix, perturbed by a Gaussian Orthogonal Ensemble (GOE), or Gaussian Unitary Ensemble (GUE) matrix. Different from the standard symmetric $\mathbb{Z}_2$-synchronization in which there are 2 communities with equal number of vertices, we do not assume that the number of vertices in each group is the same. For the exact recovery of the maximum likelihood estimation (MLE) with various weighted adjacency matrix, we prove a sharp phase transition result with respect to the intensity $\sigma$ of the Gaussian perturbation. These weighted adjacency matrices may be considered as natural models for the electric network. Surprisingly, the threshold, or critical value, of $\sigma$ does not depend on whether or not the sample space for MLE is restricted to the classifications such that the number of vertices in each group is the same as the true value. In contrast to the well-studied $\mathbb{Z}_2$-synchronization, a new complex version of the semi-definite programming (SDP) is designed to efficiently implement the community detection problem when the number of communities $k$ is greater than 2, and a common region for $\sigma$ such that SDP exactly recovers the true classification is obtained, which is independent of $k$.

1. Introduction

Most graphs of interest display community structure, i.e., their vertices are organized into groups, called communities, clusters or modules. In some cases, edges are concentrated within groups. For example, vertices of a graph may represent scientists, edges join coauthors. Vertices representing scientists working on the same research topic, where collaborations are more frequent. Likewise, communities could represent proteins with similar function in protein-protein interaction networks, groups of friends in social networks, websites on similar topics on the web graph, and so on. In some other cases, edges may only be possible between vertices in distinct groups. For instance, in an electrical network, electrical current can only be observed where there is a difference in electrical potential. Identifying communities may offer insight on how the network is organized. It allows us to focus on regions having some degree of autonomy within the graph. It helps to classify the vertices, based on their role with respect to the communities they belong to. For instance we can distinguish vertices totally embedded within their clusters from vertices at the boundary of the clusters, which may act as brokers between the modules and, in that case, could play a major role both in holding the modules together and in the dynamics of spreading processes across the network.
Identifying different communities in the stochastic block model is a central topic in many fields of science and technology; see [1] for a summary. A lot of spectacular work has been done when the graph has two equal-sized communities, see, for example, [7, 8, 2] for an incomplete list. Community detection with two equal-sized communities has also been studied on hyper-graphs, see [6]. In this paper, we instead study the community detection on a graph in which there are $k$ ($k \geq 2$) distinct clusters, not necessarily equal-sized, and edges are only allowed between vertices in different communities. This corresponds to the well-known $k$-partite graph in graph theory. The observation is a weighted adjacency matrix, perturbed by a Gaussian Orthogonal Ensemble (GOE), or Gaussian Unitary Ensemble (GUE) matrix. These weighted adjacency matrices, as explained later, may be considered as natural models for the electric network. Given the observation, we prove a sharp phase transition result with respect to the intensity $\sigma$ of the Gaussian perturbation for the exact recovery of the maximum likelihood estimation (MLE). Interestingly, the threshold, or critical value, of $\sigma$ does not depend on whether or not we restrict the sample space for MLE to those classifications in which the number of vertices of each group, or community, is the same as the true value. These results are obtained with the help of analyzing the Gaussian distribution through various inequalities.

Semidefinite programming (SDP) is one of the most exciting developments in mathematical programming in the 1990s. SDP has applications in such diverse fields as traditional convex constrained optimization, control theory, and combinatorial optimization. A linear programming problem is one in which we wish to maximize or minimize a linear objective function of real variables over a polytope. In semidefinite programming, we instead use real-valued vectors and are allowed to take the dot product of vectors; non-negativity constraints on real variables in LP (linear programming) are replaced by semi-definiteness constraints on matrix variables in SDP (semidefinite programming). Because SDP is solvable via interior point methods, most of these applications can usually be solved very efficiently in practice as well as in theory.

It is well-known that the community detection problem with $k = 2$ equal-sized communities may be efficiently solved by a semi-definite programming algorithm; see, for instance, [4, 5]. When there are $k \geq 3$ different communities, we can design a “complex version” of the semi-definite programming for efficient recovery. The idea is to relax the constraint on the rank of the optimal solution, solve the optimization problem on a larger space of the semi-definite matrices, and then achieve efficient recovery. We also obtain an interval of $\sigma$ to guarantee the exact recovery of the SDP, by applying the celebrated result of the Tracy-Widom fluctuation of the maximal eigenvalue of the GOE matrix; see [9].

2. **Main Results.**

In this section, we shall state the main results proved in the paper. We first discuss the basic definition and notation of the $k$ partite graph, where $k$ is a positive integer whose value is at least 2.

A $k$-partite graph $G = (V,E)$ is a graph whose vertices can be colored by $k$ different colors such that any two vertices of the same color cannot be adjacent. Assume $V = [n] = \{1, 2, \ldots, n\}$.
\{1, 2, \ldots, n\} be a set of vertices. Let

\[ R_k = \{c_1, \ldots, c_k\} \subset \mathbb{R} \]

be a set consisting of \(k\) distinct real numbers, where \(k \geq 2\) is a positive integer.

Let \(y : [n] \to R_k\) be a mapping from the set of vertices to the set of colors, i.e., it assigns

\[ y^{-1}(c_i) = \{j \in [n] : y(j) = c_i\} \]

where \(n_i\)'s are positive integers satisfying

\[ \sum_{i=1}^{k} n_i = n. \tag{2.1} \]

and for \(1 \leq i \leq k\)

\[ y^{-1}(c_i) = \{j \in [n] : y(j) = c_i\}. \]

We consider the following cases:

1. \(n_1 \geq n_2 \geq \ldots \geq n_k\) are fixed and satisfy (2.1); let \(\Omega_{n_1, \ldots, n_k}\) be the set of all mappings defined by

\[ \Omega_{n_1, \ldots, n_k} = \{x : [n] \to R_k | \exists \eta \in \Sigma_k : x^{-1}(c_i) = n_i; |x^{-1}(c_1)| \geq |x^{-1}(c_2)| \geq \ldots \geq |x^{-1}(c_k)|\} \]

2. let \(\Omega\) be the set of all mappings defined by

\[ \Omega = \{x : [n] \to R_k\}. \]

2.1. Real Weighted Adjacency Matrix with Gaussian Perturbation. We first consider the community detection problem when the observation is a real weighted adjacency matrix with Gaussian perturbation.

**Theorem 2.1.** Let \(n_1 \geq \ldots \geq n_k\) are the numbers of vertices in the \(k\) different colors, respectively. For a mapping \(y \in \Omega_{n_1, \ldots, n_k}\), let \(G(y)\) be the \(n \times n\) square matrix whose entries are defined by

\[ G_{ij}(y) = y(i) - y(j); \tag{2.2} \]

where \(1 \leq i, j \leq n\). Let

\[ T = G(y) + \sigma W \tag{2.3} \]

where \(W\) is a random \(n \times n\) matrix with i.i.d. standard Gaussian entries. Let \(k\) be fixed and let \(n \to \infty\). Let \((v_1 = \frac{n_1}{n}, \ldots, v_k = \frac{n_k}{n})\) be fixed as \(n \to \infty\). We have

\[ \hat{y} = \arg\min_{x \in \Omega_{n_1, \ldots, n_k}} \|T - G(x)\|_{F}^2; \tag{2.4} \]

\[ \tilde{y} = \arg\min_{x \in \Omega} \|T - G(x)\|_{F}^2. \tag{2.5} \]
(1) Assume there exists $\delta > 0$, such that
\[
\sigma^2 < \frac{(1 - \delta) \min_{1 \leq i < j \leq k} (c_i - c_j)^2 n}{4 \log n}
\]
Then
\[
\lim_{n \to \infty} p(\hat{y}; \sigma) = 1; \quad \text{and} \quad \lim_{n \to \infty} p(\tilde{y}; \sigma) = 1
\]

(2) Assume there exists $\delta > 0$, such that
\[
\sigma^2 > \frac{(1 + \delta) \min_{1 \leq i < j \leq k} (c_i - c_j)^2 n}{4 \log n}
\]
then
\[
\lim_{n \to \infty} p(\hat{y}; \sigma) = 0; \quad \text{and} \quad \lim_{n \to \infty} p(\tilde{y}; \sigma) = 0.
\]

\[\text{Theorem 2.2.} \]
Let $n_1 \geq \ldots \geq n_k$ are the numbers of vertices in the $k$ different colors, respectively. Let $K(y)$ be the $n \times n$ square matrix whose entries are defined by
\[
K_{i,j}(y) = \begin{cases} 
1 & \text{if } y(i) \neq y(j) \\
0 & \text{if } y(i) = y(j)
\end{cases}
\]
where $1 \leq i, j \leq n$. Let
\[
R = K(y) + \sigma W
\]
where $W$ is a random $n \times n$ matrix with i.i.d. standard Gaussian entries as before. Let $k$ be fixed and let $n \to \infty$. Let $(v_1 = \frac{n_1}{n}, \ldots, v_k = \frac{n_k}{n})$ be fixed as $n \to \infty$. We have
Let
\[
\tilde{y} = \arg\min_{x \in \Omega} \|R - K(x)\|_F^2
\]
\[
\overline{y} = \arg\min_{x \in \Omega_{n_1, \ldots, n_k}} \|R - K(x)\|_F^2.
\]

(1) Assume there exists $\delta > 0$, such that
\[
\sigma^2 < \frac{(1 - \delta) (n_k + n_{k-1})}{4 \log n}
\]
Then
\[
\lim_{n \to \infty} p(\tilde{y}; \sigma) = 1; \quad \text{and} \quad \lim_{n \to \infty} p(\overline{y}; \sigma) = 1
\]

(2) Assume there exists $\delta > 0$, such that
\[
\sigma^2 > \frac{(1 + \delta) (n_k + n_{k-1})}{4 \log n}
\]
then
\[
\lim_{n \to \infty} p(\tilde{y}; \sigma) = 0; \quad \text{and} \quad \lim_{n \to \infty} p(\overline{y}; \sigma) = 0.
\]
In Theorems 2.1 and 2.2, we observe different weighted adjacency matrix for the $k$-partite graph, both of which are perturbed by a constant multiple, i.e. multiplied by a parameter $\sigma$, of a matrix with i.i.d. standard Gaussian entries. The weighted adjacency matrix in 2.1 can be considered as a natural model for an electrical network; where each one of the $n$ vertices has one of the $k$ distinct electric potentials. The weight of each (oriented) edge is the difference of electric potentials between the initial point and terminal point; which is proportional to the electric current on the edge. Given the observation, the goal is to identify the electric potential differences of any two vertices. We consider the probability of the exact recovery of the maximum likelihood estimate (MLE), and find a sharp threshold with respect to the parameter $\sigma$. In theorem 2.2, we observe the uniformly-weighted adjacency matrix for the undirected graph, perturbed by a noise which is a $\sigma$-multiple of a matrix with i.i.d. standard Gaussian entries. Given the observation, the goal is to determine whether two vertices have the same color or not. Again we find a sharp threshold for the probability of the exact recovery of the MLE with respect to the parameter $\sigma$. From these two theorems, we can see that we may either choose the sample space for the maximum likelihood estimate as all the possible assignments of $k$ potentials to $n$ vertices in Theorem 2.1 (all the possible classifications of $n$ vertices in $k$ distinct groups in Theorem 2.2), or choose the sample space for the maximum likelihood estimate to be restricted on all the classifications such that the number of vertices of each type coincides with that of the true value - either way we obtain the same threshold.

Theorem 2.1 is proved in Sections 3 and 4; Theorem 2.2 is proved in Sections 5 and 6. The proofs are based on various inequalities of Gaussian distributions.

2.2. Complex Unitary Matrix with Gaussian Perturbation. Now we consider the community detection problem with $k \geq 2$ different communities when the observation is a complex unitary matrix with Gaussian perturbation. Community detection problems with such an observation matrix may be efficiently recovered by the semi-definite programming.

Let $d_1, \ldots, d_k \in [0, 2\pi)$ be $k$ distinct real numbers. Let $i$ satisfy $i^2 = -1$ be the imaginary unit. Let $y : [n] \rightarrow \{e^{id_1}, \ldots, e^{id_k}\}$ be a mapping which assigns each vertex in $[n]$ a unique color represented by a complex number of modulus 1. Let $\Theta$ be the set consisting of all such mappings.

For a mapping $y \in \Theta$, let $P(y)$ be an $n \times n$ matrix whose entries are given by

$$P_{a,b}(y) = y(a)\overline{y(b)} = e^{\text{Log}[y(a)]-\text{Log}[y(b)]}$$

where $\overline{y(b)}$ is the complex conjugate of $y(b)$ and $\text{Log}[y(b)]$ is the principal branch of the complex logarithmic function.

For each $x \in \Theta$, if we consider $x$ as an $n \times 1$ vector given by

$$x = (x(1), \ldots, x(n))^t,$$

then

$$P(x) = xx^t;$$

which is a rank-1 positive semi-definite Hermitian matrix whose diagonal entries are 1.
Define
\[ U = P(y) + \sigma W_c \]
where \( W_c \) is the standard GUE random matrix. More precisely, \( W_c \) is a random Hermitian matrix whose diagonal entries are i.i.d. standard real Gaussian random variables \( (\mathcal{N}(0, 1)_{\mathbb{R}}) \), and upper triangular entries are i.i.d. standard complex Gaussian random variables \( (\mathcal{N}(0, 1)_{\mathbb{C}}) \).

**Assumption 2.3.**  
(1) The number of vertices in each color is the same. 
(2) \( e^{id_1}, \ldots, e^{id_k} \) are the \( k \)th root of unity. Without loss of generality, assume that \( d_l = \frac{2(l-1)\pi}{k} \), for \( l = 1, \ldots, k \).

Let \( \Theta_A \) be the subset of \( \Theta \) consisting of all the mappings satisfying the above two assumptions.

Given each observation \( U \), the goal is to determine the colors of all the \( n \) vertices. Let

\[ y^A = \arg\min_{x \in \Theta_A} \| U - P(x) \|_F^2 \]

Note that for any \( x \in \Theta \),
\[ \| P(x) \|_F^2 = \sum_{1 \leq a, b \leq n} x(a)x(b)x(b)x(a) = n^2, \]
which is independent of \( x \). Hence we have
\[ y^A = \arg\max_{x \in \Theta_A} \Re \langle U, P(x) \rangle \]
where \( \Re \) denotes the real part of a complex number, and \( \langle \cdot, \cdot \rangle \) denotes the inner product of two matrices defined by

\[ \langle M_1, M_2 \rangle = \sum_{i,j \in [n]} M_1(i,j)M_2(i,j). \]

where \( M_1, M_2 \in \mathbb{C}^{n \times n} \).

**Theorem 2.4.** Let \( k \) be fixed and let \( n \to \infty \). Under Assumption 2.3 we have 

(a) If there exists \( \delta > 0 \), such that

\[ \sigma^2 < \frac{(1 - \delta) \left[ n(1 - \cos \frac{2\pi}{k}) \right]}{2 \log n} \]

then
\[ \lim_{n \to \infty} p(y^A ; \sigma) = 1 \]

(b) If there exists \( \delta > 0 \), such that

\[ \sigma^2 > \frac{(1 + \delta) \left[ n(1 - \cos \frac{2\pi}{k}) \right]}{2 \log n} \]

then
\[ \lim_{n \to \infty} p(y^A ; \sigma) = 0. \]
Define

\[ V = P(y) + \sigma \text{diag}(y) W_s \text{diag}(\bar{y}) \]

where \( W_s \) is the standard GOE random matrix. More precisely, \( W_s \) is a random symmetric matrix whose diagonal entries and upper triangular entries are i.i.d. standard real Gaussian random variables \( \mathcal{N}(0,1)_{\mathbb{R}} \).

Given each observation \( V \), the goal is to determine the colors of all the \( n \) vertices. We may consider the following optimization problem

\[
\begin{align*}
\text{max} \quad & \Re \langle V, X \rangle \\
\text{subject to} \quad & X_{ii} = 1 \\
\text{and} \quad & X \succeq 0
\end{align*}
\]

where \( X \succeq 0 \) means that \( X \) is a positive semi-definite Hermitian matrix.

Then we have the following theorem

\textbf{Theorem 2.5.} Let \( p(Y;\sigma) \) be the probability that the solution \( Y \) of \( (2.17) \) coincides with \( y \bar{y}^T \) where \( y \) is the true mapping for colors of vertices. If there exists a constant \( \delta > 0 \) independent of \( N \), such that \( \sigma < \frac{(1-\delta)\sqrt{n}}{\sqrt{2\log n}} \), then

\[
\lim_{n \to \infty} p(Y;\sigma) = 1
\]

In Theorem 2.5, it turns out that the bound \( \frac{(1-\delta)\sqrt{n}}{\sqrt{2\log n}} \) on \( \sigma \) to guarantee the exact recovery of the SDP is independent of \( k \) - the total number of communities. However, if we instead use the GUE matrix \( W_c \) instead of \( \text{diag}(y) W_s \text{diag}(\bar{y}) \) to represent the noise, Theorem 2.4 shows that threshold of \( \sigma \) to guarantee the exact recovery of MLE does depend on \( k \). This threshold is of order \( O \left( \frac{1}{n \log n} \right) \) when \( k \sim n \). Since the SDP is an algorithm obtained relaxation of constraints of the MLE, one may naturally expect a smaller common bound for \( \sigma \) to guarantee the exact recovery of the SDP for all \( k \), when the noise is represented by \( W_c \).

Theorem 2.4 is proved in Section 7.1; and Theorem 2.5 is proved in Section 7.2.

3. Proof of Theorem 2.1 when the number of vertices in each color is fixed

We first consider Case (1), and assume that \( n_1 \geq \ldots \geq n_k \) are the numbers of vertices in the \( k \) different colors, respectively. For a mapping \( y \in \Omega_{n_1,\ldots,n_k} \), let \( G(y), T \) be defined as in \( (2.2), (2.3) \), respectively.

Given a sample \( T \), the goal is to determine the color \( y \) of all the vertices. Let \( \hat{y} \) be defined by \( (2.4) \). Then

\[
\hat{y} = \arg\max_{x \in \Omega_{n_1,\ldots,n_k}} \langle G(x), T \rangle
\]

Let

\[
p(\hat{y},\sigma) = \Pr(\hat{y} = y)
\]
For \( x \in \Omega_{n_1, \ldots, n_k} \), define
\[
f(x) = \langle G(x), T \rangle
\]
Then
\[
p(y, \sigma) = \Pr \left( \max_{x \in \Omega_{n_1, \ldots, n_k} \setminus \{y\}} f(x) > f(y) \right)
\]
Note that
\[
f(x) - f(y) = \langle G(y), G(x) - G(y) \rangle + \sigma \langle W, G(x) - G(y) \rangle.
\]
The expression above shows that \( f(x) - f(y) \) is a Gaussian random variable with mean \( \langle G(y), G(x) - G(y) \rangle \) and variance \( \sigma^2 \|G(x) - G(y)\|^2_F \).

For \( i, j \in \{1, 2, \ldots, k\} \), let
\[
S_{i,j}(x, y) = \{1 \leq l \leq n : x(l) = c_i, y(l) = c_j\};
\]
i.e., \( S_{i,j}(x, y) \) consists of all the vertices which have color \( c_i \) in \( x \) and color \( c_j \) in \( y \).

Let \( t_{i,j}(x, y) = |S_{i,j}(x, y)| \). We may write \( t_{i,j} \) instead of \( t_{i,j}(x, y) \) when there is no confusion. We have
\[
\sum_{j \in \{1, 2, \ldots, k\}} t_{i,j} = n_i, \quad \forall i \in \{1, 2, \ldots, k\}
\]
\[
\sum_{i \in \{1, 2, \ldots, k\}} t_{i,j} = n_j, \quad \forall j \in \{1, 2, \ldots, k\}
\]
For each vertex \( l \in S_{i,j} \), the inner product of the row in \( G(x) \) corresponding to \( l \) and the row in \( G(y) \) corresponding to \( l \) is
\[
\sum_{u=1}^{k} \sum_{v=1}^{k} t_{u,v} (c_i - c_u)(c_j - c_v)
\]
Then
\[
\langle G(x), G(y) \rangle = \sum_{l=1}^{n} \langle G_l(x), G_l(y) \rangle
\]
\[
= \sum_{i,j,u,v \in \{1, 2, \ldots, k\}} t_{i,j} t_{u,v} (c_i - c_u)(c_j - c_v)
\]
\[
= 2 \left[ \sum_{i,j \in \{1, 2, \ldots, k\}} t_{i,j} c_i c_j \right] \left[ \sum_{u,v \in [k]} t_{u,v} \right] - 2 \left[ \sum_{i,j \in \{1, 2, \ldots, k\}} t_{i,j} c_i \right] \left[ \sum_{u,v \in [k]} t_{u,v} c_v \right]
\]
\[
= 2n \left[ \sum_{i,j \in \{1, 2, \ldots, k\}} t_{i,j} c_i c_j \right] - 2 \left[ \sum_{i \in \{1, 2, \ldots, k\}} n_i c_i \right]^2,
\]
where the last identity follows from (3.2) and (3.3).
Note that
\[ t_{i,j}(y, y) = \begin{cases} n_i & \text{if } i = j \\ 0 & \text{else} \end{cases} \]

Therefore
\[ \langle G(y), G(y) \rangle = 2n \sum_{i \in [k]} [n_i \cdot c_i^2] - 2 \left( \sum_{i \in [k]} n_i \cdot c_i \right)^2 \]

Let
\[ M(x, y) = -\mathbb{E}[f(x) - f(y)] = -(G(y), G(x) - G(y)) = 2n \sum_{i \in [k]} [n_i \cdot c_i^2] - 2 \left( \sum_{i,j \in [k]} t_{i,j} \cdot c_i \cdot c_j \right) \]

Then
\[ \text{Var}[f(x) - f(y)] = 2\sigma^2 M(x, y) \]

Then for \( x \in \Omega_{n_1, \ldots, n_k} \setminus \{ y \} \)
\[ \Pr(f(x) - f(y) > 0) = \Pr_{\xi \sim \mathcal{N}(0,1)} \left( \xi \geq \frac{\sqrt{M(x, y)}}{\sqrt{2\sigma}} \right) \]

Using the standard Gaussian tail bound \( \Pr_{\xi \sim \mathcal{N}(0,1)}(\xi > x) < e^{-\frac{x^2}{2}} \), we obtain
\[ \Pr(f(x) - f(y) > 0) \leq e^{-\frac{M(x, y)}{4\sigma^2}} \]

Then
\[ 1 - p(\hat{y}; \sigma) \leq \sum_{x \in \Omega_{n_1, \ldots, n_k} \setminus \{ y \}} \Pr(f(x) - f(y) \geq 0) \leq \sum_{x \in \Omega_{n_1, \ldots, n_k} \setminus \{ y \}} e^{-\frac{M(x, y)}{4\sigma^2}} \]

**Lemma 3.1.** Under the constraint (3.2) and (3.3), \( M(x, y) \) achieves its minimum when
\[ t_{i,i} = n_i \]
\[ t_{i,j} = 0, \quad \text{if } i \neq j. \]

and the minimal value of \( M(x, y) \) is 0.

**Proof.** Note that
\[ \sum_{i,j \in [k]} t_{i,j} \cdot c_i \cdot c_j \leq \sum_{i,j \in [k]} \frac{(c_i^2 + c_j^2)t_{i,j}}{2} = \sum_{i \in [k]} [n_i \cdot c_i^2]; \]
where the identity holds when \( c_i = c_j \) whenever \( t_{i,j} \neq 0 \). Then the lemma follows from the assumption that \( c_i \neq c_j \) whenever \( i \neq j \).

Let
\[
u_i = \sum_{j=1}^{k} u_{i,j} = \frac{n_i}{n}, \quad \forall 1 \leq i \leq k;
\]

Note that \( \sum_{i=1}^{k} \nu_i = 1 \). Let \( B \) be the set given by
\[
B = \left\{ (t_{1,1}, \ldots, t_{k,k}) \in \prod_{i,j \in [k]} [\min\{n_i, n_j\}] : \sum_{i=1}^{k} t_{i,j} = n_j, \sum_{j=1}^{k} t_{i,j} = n_i \right\}.
\]

Note also that for each \((t_{1,1}, \ldots, t_{k,k}) \in B\), the number of mappings \( x \in \Omega_{n_1, \ldots, n_k} \) with \( t_{i,j}(x,y) = t_{i,j} \) for all \( i, j \in [k] \) is no more than
\[
\frac{\prod_{l=1}^{k} [n_l!]}{\prod_{1 \leq i,j,k \leq [t_{i,j}]!]}.
\]

Then
\[
\sum_{x \in \Omega_{n_1, \ldots, n_k} \setminus \{y\}} e^{-\frac{M(x,y)}{4a^2}} \leq \sum_{(t_{1,1}, \ldots, t_{k,k}) \in B} \frac{\prod_{l=1}^{k} [n_l!]}{\prod_{1 \leq i,j,k \leq [t_{i,j}]!}} e^{-\frac{M(x,y)}{4a^2}}.
\]

By Stirling’s formula, we obtain that when each one of \( n_1, \ldots, n_k \) is large,

\[
(3.4) \quad \frac{\prod_{l=1}^{k} [n_l!]}{\prod_{1 \leq i,j,k \leq [t_{i,j}]!}} \sim \frac{\prod_{l=1}^{k} [n_l]}{(2n\pi)^{k/2} \prod_{1 \leq i,j,k \leq [t_{i,j}]}} \left[ \frac{\prod_{l=1}^{k} v_{i,j}^{n_l}}{\prod_{1 \leq i,j,k \leq [t_{i,j}]}} \right]^n
\]

Then we have
\[
\sum_{x \in \Omega_{n_1, \ldots, n_k} \setminus \{y\}} e^{-\frac{M(x,y)}{4a^2}} \leq I_1 + I_2
\]

where
\[
I_1 = \int_{D \setminus \{D \}} \frac{n^{(k-1)(k-2)/2} \prod_{1 \leq l < k} \sqrt{v_l} \left[ \frac{\prod_{l=1}^{k} v_{i,j}^{n_l}}{\prod_{1 \leq i,j,k \leq [t_{i,j}]}} \right]^n \ e^{-\frac{M(x,y)}{4a^2}}} dV
\]

and
\[
I_2 = \sum_{x \in \Omega_{n_1, \ldots, n_k} \setminus \{y\} : \left( t_{1,1} \ldots, t_{k,k} \right) \in D} e^{-\frac{M(x,y)}{4a^2}}.
\]

Here \( D \) is the domain given by
\[
D = \left\{ (u_{1,1}, \ldots, u_{k,k}) : u_{i,j} \geq 0, \text{ for } 1 \leq i, j \leq k; \sum_{j=1}^{k} u_{i,j} = v_i, \sum_{i=1}^{k} u_{i,j} = v_j \right\}.
\]
For a small positive number \( \epsilon > 0 \), let \( D_{\epsilon} \) be the domain given by
\[
D_{\epsilon} = \{(u_{1,1}, \ldots, u_{k,k}) \in D : u_{i,i} \geq v_i - \epsilon, \forall 1 \leq i \leq k\}
\]
Note that the dimension of \( D \) is \( k^2 - 2k + 1 \).

**Lemma 3.2.** Assume \((u_{1,1}, \ldots, u_{k,k}) \in D \setminus D_{\epsilon}\). Then
\[
M(x, y) \geq C_0 \epsilon n^2
\]
where \( C_0 > 0 \) is a constant given by
\[
C_0 = \min_{i,j \in [k], i \neq j} (c_i - c_j)^2
\]

**Proof.** When \((u_{1,1}, \ldots, u_{k,k}) \in D \setminus D_{\epsilon}\), we have there exists \(1 \leq i \leq k\),
\[
\sum_{j \in [k], j \neq i} u_{i,j} \geq \epsilon
\]
Note that
\[
M(x, y) = 2n \left\{ \sum_{i \in [k]} [n_i \cdot c_i^2] - \left[ \sum_{i,j \in [k]} t_{i,j} \cdot c_i \cdot c_j \right] \right\}
\]
\[
= n \sum_{i \in [k]} \sum_{j \in [k]} t_{i,j} (c_i - c_j)^2 \geq \epsilon C_0 n^2.
\]
Then the lemma follows. \(\square\)

**Proposition 3.3.** Let \( k \) be fixed and let \( n \to \infty \). Assume there exists \( \delta > 0 \), such that
\[
\sigma^2 < \frac{(1 - \delta)C_0 n}{4 \log n}
\]
where \( C_0 > 0 \) is a constant given by (3.5). Then for any fixed \((v_1, \ldots, v_k)\) satisfying \( v_i > 0 \) and \( \sum_{i=1}^k v_i = 1 \)
\[
\lim_{n \to \infty} p(\hat{y}; \sigma) = 1.
\]

**Proof.** First of all, for any fixed \( \epsilon > 0 \), if (4.6) holds, then \( \sigma \sim o(\sqrt{n}) \), by Lemma 3.2 and the fact that for any fixed \((v_1, \ldots, v_k)\) satisfying \( \sum_{i=1}^k v_i = 1 \),
\[
\prod_{i=1}^k v_i^{u_{i,i}} \leq k
\]
We obtain that \( \lim_{n \to \infty} I_1 = 0 \).

Now let us consider \( I_2 \). We consider the following “smoothing” process for \( t_{i,j} \)’s: for \( i, j \in [k] \) and \( i \neq j \), choose a vertex in \( S_{i,i}(x, y) \), change its color in \( x \) from \( c_i \) to \( c_j \); choose a vertex in \( S_{j,j}(x, y) \), change its color in \( x \) from \( c_j \) to \( c_i \). Then the change in \( M(x, y) \) is
\[
[-n(t_{i,j} + t_{j,i}) + n(t_{i,j} + 1 + t_{j,i} + 1)](c_i - c_j)^2 = 2n(c_i - c_j)^2 \geq 2nC_0,
\]
where \( C_0 \) is given in (3.5).
We continue this process until the resulting \((u_{1,1}, \ldots, u_{k,k})\) is outside \( D_{\epsilon}\).
\[ I_2 \leq \sum_{l=1}^{\infty} n^{2l} e^{-\frac{\sqrt{2\pi} \sigma}{2l}} \]

It is straightforward to check that when \( \sigma \) satisfies (4.6), there exists sufficiently small \( \epsilon > 0 \), such that \( \lim_{n \to \infty} I_2 = 0 \). Then the proposition follows. \( \square \)

**Proposition 3.4.** Let \( k \) be fixed and let \( n \to \infty \). Assume there exists \( \delta > 0 \), such that

\[
\sigma^2 > \frac{(1 + \delta)C_0 n}{4 \log n}
\]

then

\[
\lim_{n \to \infty} p(\hat{y}; \sigma) = 0.
\]

**Proof.** Assume \( u, v \in [k] \), \( u \neq v \) such that

\[
(c_u - c_v)^2 = \min_{i,j \in [k], i \neq j} (c_i - c_j)^2 = C_0
\]

For \( y \in \Omega \), \( a, b \in \{1, 2, \ldots, n\} \) such that \( c_u = y(a) \neq y(b) = c_v \). Let \( y^{(ab)} \) be the coloring of vertices defined by

\[
y^{(ab)}(i) = \begin{cases} 
y(i) & \text{if } i \in \{1, 2, \ldots, n\} \setminus \{a, b\} \\
c_v & \text{if } i = a \\
c_u & \text{if } i = b
\end{cases}
\]

Then

\[
1 - p(\hat{y}; \sigma) \geq \Pr \left( \bigcup_{a,b \in [n], c_u = y(a) \neq y(b) = c_v} \left[ f(y^{(ab)}) - f(y) > 0 \right] \right),
\]

since any of the event \( [f(y^{(ab)}) - f(y) > 0] \) implies \( \hat{y} \neq y \). Recall that

\[
f(y^{(ab)}) - f(y) = \langle G(y), G^{(ab)} - G(y) \rangle + \sigma \langle W, G^{(ab)} - G(y) \rangle
\]

\[= -2n(c_u - c_v)^2 + \sigma \langle W, G^{(ab)} - G(y) \rangle.
\]

So \( 1 - p(\hat{y}; \sigma) \) is at least

\[
\Pr \left( \bigcup_{a,b \in [n], c_u = y(a) \neq y(b) = c_v} \left[ f(y^{(ab)}) - f(y) > 0 \right] \right) \geq \Pr \left( \max_{a,b \in [n], c_u = y(a) \neq y(b) = c_v} \sigma \langle W, G^{(ab)} - G(y) \rangle > 2nC_0 \right)
\]

For \( i \in \{u, v\} \), let \( H_i \subset y^{-1}(c_i) \) such that \( |H_i| = \frac{n}{\log n} = h \). Then

\[
1 - p(\hat{y}; \sigma) \geq \Pr \left( \max_{a \in H_u, b \in H_v} \sigma \langle W, G^{(ab)} - G(y) \rangle > 2nC_0 \right)
\]

Let \( (X, Y, Z) \) be a partition of \( [n]^2 \) defined by

\[
X = \{ \alpha = (\alpha_1, \alpha_2) \in [n]^2, \{\alpha_1, \alpha_2\} \cap [H_u \cup H_v] = \emptyset \}
\]

\[
Y = \{ \alpha = (\alpha_1, \alpha_2) \in [n]^2, |\{\alpha_1, \alpha_2\} \cap [H_u \cup H_v]| = 1 \}
\]

\[
Z = \{ \alpha = (\alpha_1, \alpha_2) \in [n]^2, |\{\alpha_1, \alpha_2\} \cap [H_u \cup H_v]| = 2 \}
\]
For $\eta \in \{\mathcal{X}, \mathcal{Y}, \mathcal{Z}\}$, define the $n \times n$ matrix $W_\eta$ from the entries of $W$ as follows

$$W_\eta(i, j) = \begin{cases} 0 & \text{if } (i, j) \notin \eta \\ W(i, j) & \text{if } (i, j) \in \eta \end{cases}$$

For each $a \in H_u$ and $b \in H_v$, let

$$\begin{align*}
\mathcal{X}_{ab} &= \langle W_{\mathcal{X}}, G(y^{(ab)}) - G(y) \rangle \\
\mathcal{Y}_{ab} &= \langle W_{\mathcal{Y}}, G(y^{(ab)}) - G(y) \rangle \\
\mathcal{Z}_{ab} &= \langle W_{\mathcal{Z}}, G(y^{(ab)}) - G(y) \rangle
\end{align*}$$

**Claim 3.5.** The followings are true:

1. $\mathcal{X}_{ab} = 0$ for $a \in H_u$ and $b \in H_v$.
2. For each $a \in H_u$ and $b \in H_v$, the variables $\mathcal{Y}_{ab}$ and $\mathcal{Z}_{ab}$ are independent.
3. Each $\mathcal{Y}_{ab}$ can be decomposed into $Y_a + Y_b$ where $\{Y_a\}_{a \in H_u} \cup \{Y_b\}_{b \in H_v}$ is a collection of i.i.d. Gaussian random variables.

**Proof.** It is straightforward to check (1). (2) holds because $\mathcal{Y} \cap \mathcal{Z} = \emptyset$.

For $s \in H_u \cup H_v$, let $\mathcal{Y}_s \subseteq \mathcal{Y}$ be defined by

$$\mathcal{Y}_s = \{\alpha = (\alpha_1, \alpha_2) \in \mathcal{Y} : \alpha_1 = s, \text{ or } \alpha_2 = s\}.$$  

Note that for $s_1, s_2 \in H_u \cup H_v$ and $s_1 \neq s_2$, $\mathcal{Y}_{s_1} \cap \mathcal{Y}_{s_2} = \emptyset$. Moreover, $\mathcal{Y} = \bigcup_{s \in H_u \cup H_v} \mathcal{Y}_s$.

Therefore

$$\mathcal{Y}_{ab} = \sum_{s \in H_u \cup H_v} \langle W_{\mathcal{Y}_s}, G(y^{(ab)}) - G(y) \rangle$$

Note also that $\langle W_{\mathcal{Y}_s}, G(y^{(ab)}) - G(y) \rangle = 0$, if $s \notin \{a, b\}$. Hence

$$\mathcal{Y}_{ab} = \sum_{\alpha \in \mathcal{Y}_a \cup \mathcal{Y}_b} [W(\alpha)] \cdot \{[G(y^{(ab)}) - G(y)](\alpha)\}$$

Note that for $\alpha \in \mathcal{Y}_a$,

$$[G(y^{(ab)}) - G(y)](\alpha) = \begin{cases} c_v - c_u & \text{if } \alpha_1 = a \\ c_u - c_v & \text{if } \alpha_2 = a. \end{cases}$$

So,

$$Y_a := \sum_{\alpha \in \mathcal{Y}_a} [W(\alpha)] \cdot \{[G(y^{(ab)}) - G(y)](\alpha)\}$$

$$= \left\{ \sum_{\alpha \in \mathcal{Y}_a : \alpha_1 = a} [W(\alpha)] - \sum_{\alpha \in \mathcal{Y}_a : \alpha_2 = a} [W(\alpha)] \right\} (c_v - c_u)$$

Similarly, define

$$Y_b := \left\{ \sum_{\alpha \in \mathcal{Y}_b : \alpha_2 = b} [W(\alpha)] - \sum_{\alpha \in \mathcal{Y}_b : \alpha_1 = b} [W(\alpha)] \right\} (c_v - c_u)$$
Then $Y_{ab} = Y_a + Y_b$ and $\{Y_s\}_{s \in H_u \cup H_v}$ is a collection of independent Gaussian random variables. Moreover, the variance of $Y_s$ is equal to $(2n - 4h)C_0$ independent of the choice of $s$. □

By the claim, we obtain

$$\langle W, G(y^{(ab)}) - G(y) \rangle = Y_a + Y_b + Z_{ab}$$

Moreover,

$$\max_{a \in H_u, b \in H_v} Y_a + Y_b + Z_{ab} \geq \max_{a \in H_u, b \in H_v} (Y_a + Y_b) - \max_{a \in H_u, b \in H_v} (-Z_{ab}) = \max_{a \in H_u} Y_a + \max_{b \in H_v} Y_b - \max_{a \in H_u, b \in H_v} (-Z_{ab})$$

Recall the following tail bound result on the maximum of Gaussian random variables:

**Lemma 3.6.** Let $G_1, \ldots, G_N$ be Gaussian random variables with variance 1. Let $\epsilon > 0$ be a constant independent of $N$. Then

$$\Pr \left( \max_{i=1, \ldots, N} G_i > (1 + \epsilon) \sqrt{2 \log N} \right) \leq N^{-\epsilon}$$

and moreover, if $G_i$’s are independent,

$$\Pr \left( \max_{i=1, \ldots, N} G_i < (1 - \epsilon) \sqrt{2 \log N} \right) \leq \exp(-N^{o(\epsilon)})$$

By the lemma we obtain

$$\max_{a \in H_u} Y_a \geq (1 - 0.01\epsilon) \sqrt{2 \log h \cdot 2C_0 (n - 2h)}$$

$$\max_{b \in H_v} Y_b \geq (1 - 0.01\epsilon) \sqrt{2 \log h \cdot 2C_0 (n - 2h)}$$

$$\max_{a \in H_u, b \in H_v} Z_{ab} \leq (1 + \epsilon) \sqrt{4 \log h \cdot \max \{\text{Var}(Z_{ab})\}}$$

with probability $1 - o_n(1).$ (Here $o_n(1) \rightarrow 0$ as $n \rightarrow \infty$.) Moreover,

$$\text{Var}Z_{ab} = \|G(y^{(ab)}) - G(y)\|^2 - \text{Var}(Y_a) - \text{Var}(Y_b)$$

$$= 4C_0 n - 4C_0 (n - 2h)$$

$$= 8C_0 h$$

which is $o(n).$ Hence

$$\max_{a \in H_1, b \in H_2} \langle W, G(y^{(ab)}) - G(y) \rangle \geq 2(1 - 0.01\epsilon - o(1)) \sqrt{2 \log n \cdot 2C_0 (n - 2h)}$$

$$\geq 4(1 - 0.01\epsilon - o(1)) \sqrt{C_0 n \log n}$$

with probability $1 - o_n(1).$ Since $\sigma^2 > \frac{(1 + \delta)C_0 n}{4 \log n}$, we have

$$\Pr \left( \max_{a, b \in \{1, 2, \ldots, n\}, y(a) \neq y(b)} \sigma \langle W, G(y^{(ab)}) - G(y) \rangle > 2C_0 (1 + \delta) n \right) \geq 1 - o_n(1)$$

Then the lemma follows. □
4. Proof of Theorem 2.1 when the number of vertices in each color is arbitrary

Now we consider Case (2), and assume that for each \( x \in \Omega \), \( n_1(x), \ldots, n_k(x) \) are arbitrary positive integers satisfying (2.1) and denoting the number of vertices in the colors \( c_1, \ldots, c_k \) under the mapping \( x \), respectively. For a mapping \( y \in \Omega \), let \( G(y) \) be defined as in (2.2); and let \( T \) be defined as in (2.3).

Given a sample \( T \), the goal is to determine the color \( y \) of all the vertices. Let \( \tilde{y} \) be defined by (2.5). Then

\[
\tilde{y} = \arg\min_{x \in \Omega} \|G(x)\|_F^2 - 2\langle G(x), T \rangle
\]

Let

\[
p(\tilde{y}, \sigma) = \Pr(\tilde{y} = y)
\]

For \( x \in \Omega \), define

\[
d(x) = \|G(x)\|_F^2 - 2\langle G(x), T \rangle
\]

Then

\[
p(\tilde{y}, \sigma) = \Pr \left( d(y) < \min_{x \in \Omega \setminus \{y\}} d(x) \right)
\]

Note that

\[
d(x) - d(y) = \|G(x)\|_F^2 - \|G(y)\|_F^2 - 2\langle G(y), G(x) - G(y) \rangle - 2\sigma \langle W, G(x) - G(y) \rangle.
\]

The expression above shows that \( d(x) - d(y) \) is a Gaussian random variable with mean \( \|G(x)\|_F^2 - \|G(y)\|_F^2 - 2\langle G(y), G(x) - G(y) \rangle \) and variance \( 4\sigma^2 \|G(x) - G(y)\|_F^2 \).

For \( i, j \in \{1, 2, \ldots, k\} \), let \( S_{i,j}(x, y) \) be defined as in (7.17), and let \( t_{i,j}(x, y) = |S_{i,j}(x, y)| \). Then

\[
(4.1) \quad \sum_{j \in \{1, 2, \ldots, k\}} t_{i,j}(x, y) = n_i(x), \quad \forall i \in \{1, 2, \ldots, k\}
\]

\[
(4.2) \quad \sum_{i \in \{1, 2, \ldots, k\}} t_{i,j}(x, y) = n_j(y), \quad \forall j \in \{1, 2, \ldots, k\}
\]

Note that

\[
\langle G(x), G(y) \rangle = \sum_{l=1}^{n} (G_l(x), G_l(y))
\]

\[
= \sum_{i,j,u,v \in \{1, 2, \ldots, k\}} t_{i,j} t_{u,v} (c_i - c_u) (c_j - c_v)
\]

\[
= 2 \left[ \sum_{i,j \in [k]} t_{i,j} \cdot c_i \cdot c_j \right] \left[ \sum_{u,v \in [k]} t_{u,v} \right] - 2 \left[ \sum_{i,j \in [k]} t_{i,j} \cdot c_i \right] \left[ \sum_{u,v \in [k]} t_{u,v} \cdot c_v \right]
\]

\[
= 2n \left[ \sum_{i,j \in [k]} t_{i,j} \cdot c_i \cdot c_j \right] - 2 \left[ \sum_{i \in [k]} n_i(x) \cdot c_i \right] \left[ \sum_{j \in [k]} n_j(y) \cdot c_j \right],
\]
where the last identity follows from (4.1) and (4.2).

In particular

$$\langle G(y), G(y) \rangle = 2n \sum_{i \in [k]} \left[ n_i(y) \cdot c_i^2 \right] - 2 \left[ \sum_{i \in [k]} n_i(y) \cdot c_i \right]^2$$

Let

$$Q(x, y) = \mathbb{E}[d(x) - d(y)] = \|G(x)\|^2_{F} + \|G(y)\|^2_{F} - 2\langle G(x), G(y) \rangle$$

$$= 2n \sum_{i, j \in [k]} t_{i, j}(x, y)(c_i - c_j)^2 - 2 \left[ \sum_{i \in [k]} n_i(x)c_i - \sum_{j \in [k]} n_j(y)c_j \right]^2$$

$$= \sum_{u, v, i, j \in [k]} t_{u, v}(x, y)t_{i, j}(x, y)(c_u - c_v - c_i + c_j)^2$$

Then

$$\text{Var}[d(x) - d(y)] = 4\sigma^2 Q(x, y)$$

Then for $x \in \Omega \setminus \{y\}$

$$\Pr(d(y) - d(x) > 0) = \Pr_{\xi \sim \mathcal{N}(0, 1)} \left( \xi \geq \frac{\sqrt{Q(x, y)}}{2\sigma} \right)$$

Using the standard Gaussian tail bound $\Pr_{\xi \sim \mathcal{N}(0, 1)}(\xi > x) < e^{-\frac{x^2}{2}}$, we obtain

$$\Pr(d(y) - d(x) > 0) \leq e^{-\frac{Q(x, y)}{8\sigma^2}}$$

Then

$$1 - p(\bar{y}; \sigma) \leq \sum_{x \in \Omega \setminus \{y\}} \Pr(d(y) - d(x) \geq 0)$$

$$= \sum_{x \in \Omega \setminus \{y\}} \Pr_{\xi \sim \mathcal{N}(0, 1)} \left( \xi \geq \frac{\sqrt{Q(x, y)}}{2\sigma} \right)$$

$$\leq \sum_{x \in \Omega \setminus \{y\}} e^{-\frac{Q(x, y)}{8\sigma^2}}$$

Lemma 4.1. Under the constraint (4.2), $Q(x, y)$ achieves its minimum when

$$t_{i, i} = n_i(y)$$

$$t_{i, j} = 0, \quad \text{if } i \neq j.$$ 

and the minimal value of $Q(x, y)$ is 0.

Proof. The lemma follows from the expression (4.3). □
Let
\[ u_{i,j}(x, y) = \frac{t_{i,j}(x, y)}{n}, \forall 1 \leq i, j \leq k; \tag{4.4} \]
\[ v_i(y) = \sum_{j=1}^{n} u_{j,i}(x, y) = \frac{n_i(y)}{n}, \forall 1 \leq i \leq k. \tag{4.5} \]

Note that \( \sum_{i=1}^{k} v_i(y) = 1 \). Let \( B \) be the set given by
\[
B = \left\{ (t_{1,1}, t_{2,1}, \ldots, t_{k,k}) \in \prod_{j \in [k]} [n_{j}(y)]^k : \sum_{i=1}^{k} t_{i,j}(x, y) = n_{j}(y) \right\}.
\]

Note also that for each \((t_{1,1}, t_{2,1}, \ldots, t_{k,k}) \in B\), the number of mappings \( x \in \Omega \) with \( t_{i,j}(x, y) = t_{i,j} \) for all \( i, j \in [k] \) is no more than
\[
\frac{\prod_{l=1}^{k} [n_{l}(y)]!}{\prod_{1 \leq i, j \leq k} [t_{i,j}(x, y)]!}.
\]

By Stirling’s formula as in (3.4), we obtain that
\[
\sum_{x \in \Omega_{n_{1}, \ldots, n_{k}} \setminus \{y\}} e^{-\frac{Q(x,y)}{8\sigma^2}} \leq J_1 + J_2
\]
where
\[
J_1 = \int_{\mathcal{D} \setminus \mathcal{D}_{\epsilon}} \frac{n^{\frac{k^2-k}{2}}}{(2\pi)^{\frac{k^2-k}{2}}} \frac{1}{\prod_{1 \leq i, j \leq k} v_{i,j}} \exp \left[ -\frac{\prod_{l=1}^{k} v_{l}(y)^{v_{l}(y)}}{\prod_{1 \leq i, j \leq k} (u_{i,j})^{u_{i,j}}} \right] n^{\frac{Q(x,y)}{8\sigma^2}} dV
\]
and
\[
J_2 = \sum_{x \in \Omega \setminus \{y\}} \sum_{(\frac{u_{1,1}}{n}, \ldots, \frac{t_{k,k}}{n}) \in \mathcal{D}_{\epsilon}} e^{-\frac{Q(x,y)}{8\sigma^2}}.
\]

Here \( \mathcal{D} \) is the domain given by
\[
\mathcal{D} = \left\{ (u_{1,1}, \ldots, u_{k,k}) : u_{i,i} \geq 0, \text{ for } 1 \leq i, j \leq k; \sum_{i=1}^{k} u_{i,j} = v_{j}(y), \right\}.
\]

Note that the dimension of \( \mathcal{D} \) is \( k^2 - k \).

For a small positive number \( \epsilon > 0 \), let \( \mathcal{D}_{\epsilon} \) be the domain given by
\[
\mathcal{D}_{\epsilon} = \{(u_{1,1}, \ldots, u_{k,k}) \in \mathcal{D} : u_{i,i} \geq v_{i}(y) - \epsilon, \forall 1 \leq i \leq k\}
\]

**Lemma 4.2.** Assume \((u_{1,1}, \ldots, u_{k,k}) \in \mathcal{D} \setminus \mathcal{D}_{\epsilon}\). Then if \( \epsilon > 0 \) is small enough,
\[
Q(x, y) \geq 4C_{0}\epsilon^{2}n^{2}
\]
where \( C_{0} > 0 \) is a constant given by (3.5).
Proof. When \((u_{1,1}, \ldots, u_{k,k}) \in D \setminus D_\epsilon\), we have there exists \(1 \leq i \leq k\),
\[
\sum_{j \in [k], j \neq i} u_{j,i} \geq \epsilon
\]
The following cases might occur

(1) there exists \(l \in [k]\), such that
\[
u_{l,l} > v_l(y) - \epsilon
\]
Then
\[
Q(x, y) = \sum_{u, v, i, j \in [k]} t_{u,v}(x, y) t_{i,j}(x, y) (c_u - c_v - c_i + c_j)^2
\]
\[
\geq t_{l,l}(x, y) \sum_{j \in [k], j \neq i} t_{j,i}(x, y) (c_i - c_j)^2
\]
\[
\geq (v_l(y) - \epsilon) \epsilon C_0 n^2
\]

(2) for all the \(p \in [k]\), we have
\[
\sum_{q \in [k], q \neq p} u_{q,p} \geq \epsilon
\]
Without loss of generality, assume that \(c_1 > c_2 > \ldots > c_k\). Then
\[
Q(x, y) = \sum_{u, v, i, j \in [k]} t_{u,v}(x, y) t_{i,j}(x, y) (c_u - c_v - c_i + c_j)^2
\]
\[
\geq \sum_{i \in [k], i \neq 1} \sum_{j \in [k], j \neq k} t_{i,1}(x, y) t_{j,k}(x, y) (c_i - c_1 - c_j + c_k)^2
\]
\[
\geq 4C_0 \epsilon^2 n^2
\]
When \(\epsilon\) is sufficiently small, the lemma follows. \qed

**Proposition 4.3.** Let \(k\) be fixed and let \(n \to \infty\). Assume there exists \(\delta > 0\), such that
\[
(4.6) \quad \sigma^2 < \frac{(1 - \delta) C_0 n}{4 \log n}
\]
where \(C_0 > 0\) is a constant given by (3.5). Then for any fixed \((v_1(y), \ldots, v_k(y))\) satisfying \(v_i(y) > 0\) and \(\sum_{i=1}^k v_i(y) = 1\)
\[
\lim_{n \to \infty} p(\tilde{y}; \sigma) = 1.
\]

**Proof.** First of all, for any fixed \(\epsilon > 0\), if (4.6) holds, then \(\sigma \sim o(\sqrt{n})\), by Lemma 4.2 We obtain that \(\lim_{n \to \infty} J_1 = 0\).

Now let us consider \(J_2\). Recall that
\[
Q(x, y) = 2n \sum_{i, j \in [k]} t_{i,j}(x, y) (c_i - c_j)^2 - 2 \left[ \sum_{i, j \in [k]} t_{i,j}(x, y) (c_i - c_j) \right]^2
\]
Note also that when \((u_{1,1}, \ldots, u_{k,k}) \in \mathcal{D}_\epsilon\), we obtain
\[
\left| \sum_{i,j \in [k]} t_{u,v}(x, y)(c_u - c_v) + (c_j - c_i) \right| = \sum_{i \in [k]} (n_i(x) - n_i(y)) c_i \leq k^2 \epsilon n \max_{1 \leq i \leq k} |c_i|.
\]

We consider the following “smoothing” process for \(t_{i,j}\)’s: for \(i, j \in [k]\) and \(i \neq j\), choose a vertex in \(S_{i,j}(x, y)\), change its color in \(x\) from \(c_i\) to \(c_j\). Then the change in \(Q(x, y)\) is
\[
2n(c_i - c_j)^2 + 2 \left[ \sum_{u,v \in [k]} t_{u,v}(x, y)(c_u - c_v) \right]^2 - 2 \left[ \sum_{i,j \in [k]} t_{u,v}(x, y)(c_u - c_v) + (c_j - c_i) \right]^2,
\]
\[
= (2n - 2)(c_i - c_j)^2 + 4(c_i - c_j) \left[ \sum_{u,v \in [k]} t_{u,v}(x, y)(c_u - c_v) \right] \geq (2n - 2)C_0 - C_1 \epsilon n
\]
where \(C_0\) is given in (3.5), and \(C_1 > 0\) is a constant given by
\[
C_1 := 4k^2 \max_{i \in [k]} |c_i| \max_{p,q \in [k]} |c_p - c_q|
\]

We continue this process until the resulting \((u_{1,1}, \ldots, u_{k,k})\) is outside \(\mathcal{D}_\epsilon\).
\[
J_2 \leq \sum_{l=1}^{\infty} n^l e^{-\frac{(2n-2)C_0 - C_1 \epsilon n}{8\sigma^2}}
\]
It is straightforward to check that when \(\sigma\) satisfies (4.6), there exists sufficiently small \(\epsilon > 0\), such that \(\lim_{n \to \infty} I_2 = 0\). Then the proposition follows. \(\square\)

**Proposition 4.4.** Let \(k\) be fixed and let \(n \to \infty\). Assume there exists \(\delta > 0\), such that
\[
\sigma^2 > \frac{(1 + \delta)C_0 n}{4 \log n}
\]
then
\[
\lim_{n \to \infty} p(\hat{y}; \sigma) = 0.
\]

**Proof.** Assume \(u, v \in [k], u \neq v\) such that
\[
(c_u - c_v)^2 = \min_{i,j \in [k], i \neq j} (c_i - c_j)^2 = C_0
\]
For \(y \in \Omega, a \in [n]\) such that \(c_a = y(a)\). Let \(y^{(a)}\) be the coloring of vertices defined by
\[
y^{(ab)}(i) = \begin{cases} y(i) & \text{if } i \in [n] \setminus \{a\} \\ c_v & \text{if } i = a \end{cases}
\]
Then
\[
1 - p(\hat{y}; \sigma) \geq \Pr \left( \cup_{a \in [n], c_a = y(a)} d(y^{(a)}) - d(y) < 0 \right),
\]
since any of the event \([d(y^{(a)}) - d(y) < 0]\) implies \(\tilde{y} \neq y\). Recall that

\[
d(y^{(a)}) - d(y) = \|G(y^{(a)}) - G(y)\|^2_Z - 2\sigma \langle W, G(y^{(a)}) - G(y) \rangle
\]

\[
= (2n - 2)(c_u - c_v)^2 + 4(c_u - c_v) \left[ \sum_{i,j \in [k]} t_{i,j}(y^{(a)}, y)(c_i - c_j) \right] - 2\sigma \langle W, G(y^{(ab)}) - G(y) \rangle.
\]

So \(1 - p(\tilde{y}; \sigma)\) is at least

\[
Pr \left( \cup_{a,b \in [n], c_u = y(a)} [d(y^{(a)}) - d(y) < 0] \right) \geq Pr \left( \max_{a,b \in [n], c_u = y(a)} \sigma (W, G(y^{(ab)}) - G(y)) > (n - 3)C_0 \right)
\]

Let \(H_u \subset y^{-1}(c_u)\) such that \(|H_u| = \frac{n}{\log^2 n} = h\). Then

\[
1 - p(\tilde{y}; \sigma) \geq Pr \left( \max_{a \in H_u} \sigma (W, G(y^{(a)}) - G(y)) > (n - 3)C_0 \right)
\]

Let \((\mathcal{X}, \mathcal{Y}, \mathcal{Z})\) be a partition of \([n] \times [n]\) defined by

\[
\mathcal{X} = \{ \alpha = (\alpha_1, \alpha_2) \in [n]^2, \{\alpha_1, \alpha_2\} \cap H_u = \emptyset \}
\]

\[
\mathcal{Y} = \{ \alpha = (\alpha_1, \alpha_2) \in [n]^2, |\{\alpha_1, \alpha_2\} \cap H_u| = 1 \}
\]

\[
\mathcal{Z} = \{ \alpha = (\alpha_1, \alpha_2) \in [n]^2, |\{\alpha_1, \alpha_2\} \cap H_u| = 2 \}
\]

For \(\eta \in \{\mathcal{X}, \mathcal{Y}, \mathcal{Z}\}\), define the \(n \times n\) matrix \(W_\eta\) from the entries of \(W\) as follows

\[
W_\eta(i, j) = \begin{cases} 0 & \text{if } (i, j) \notin \eta \\ W(i, j) & \text{if } (i, j) \in \eta \end{cases}
\]

For each \(a \in H_u\), let

\[
\mathcal{X}^{(a)} = \langle W_\mathcal{X}, G(y^{(a)}) - G(y) \rangle
\]

\[
\mathcal{Y}^{(a)} = \langle W_\mathcal{Y}, G(y^{(a)}) - G(y) \rangle
\]

\[
\mathcal{Z}^{(a)} = \langle W_\mathcal{Z}, G(y^{(a)}) - G(y) \rangle
\]

**Claim 4.5.** The followings are true:

1. \(\mathcal{X}^{(a)} = 0\) for \(a \in H_u\).
2. For each \(a \in H_u\), the variables \(\mathcal{Y}^{(a)}\) and \(\mathcal{Z}^{(a)}\) are independent.
3. \(\{\mathcal{Y}^{(a)}\}_{a \in H_u}\) is a collection of i.i.d. Gaussian random variables.

**Proof.** It is straightforward to check (1), (2) holds because \(\mathcal{Y} \cap \mathcal{Z} = \emptyset\).

For \(s \in H_u\), let \(\mathcal{Y}_s \subseteq \mathcal{Y}^{(a)}\) be defined by

\[
\mathcal{Y}_s = \{ \alpha = (\alpha_1, \alpha_2) \in \mathcal{Y} : \alpha_1 = s, \text{ or } \alpha_2 = s \}.
\]

Note that for \(s_1, s_2 \in H_u\) and \(s_1 \neq s_2\), \(\mathcal{Y}_{s_1} \cap \mathcal{Y}_{s_2} = \emptyset\). Moreover, \(\mathcal{Y} = \cup_{s \in H_u} \mathcal{Y}_s\). Therefore

\[
\mathcal{Y}^{(a)} = \sum_{s \in H_u} \langle W_{\mathcal{Y}_s}, G(y^{(a)}) - G(y) \rangle
\]
Note also that $\langle W, G(y^{(a)}) - G(y) \rangle = 0$, if $s \neq a$. Hence

$$\mathcal{Y}^{(a)} = \sum_{a \in \mathcal{Y}_a} [W(\alpha)] \cdot \{[G(y^{(a)}) - G(y)](\alpha)\}$$

Note that for $\alpha \in \mathcal{Y}_a$,

$$[G(y^{(a)}) - G(y)](\alpha) = \begin{cases} c_v - c_u & \text{if } \alpha_1 = a \\ c_u - c_v & \text{if } \alpha_2 = a. \end{cases}$$

So,

$$\mathcal{Y}^{(a)} = \sum_{a \in \mathcal{Y}_a} [W(\alpha)] \cdot \{[G(y^{(ab)}) - G(y)](\alpha)\}$$

$$= \left\{ \sum_{a \in \mathcal{Y}_a; \alpha_1 = a} [W(\alpha)] - \sum_{a \in \mathcal{Y}_a; \alpha_2 = a} [W(\alpha)] \right\} (c_v - c_u)$$

and $\{\mathcal{Y}^{(a)}\}_{a \in H_u}$ is a collection of independent Gaussian random variables. Moreover, the variance of $\mathcal{Y}^{(a)}$ is equal to $(2n - 2h)C_0$ independent of the choice of $s$. □

By the claim, we obtain

$$\langle W, G(y^{(a)}) - G(y) \rangle = \mathcal{Y}^{(a)} + \mathcal{Z}^{(a)}$$

Moreover,

$$\max_{a \in H_u} \mathcal{Y}^{(a)} + \mathcal{Z}^{(a)} \geq \max_{a \in H_u} \left[ \mathcal{Y}^{(a)} \right] - \max_{a \in H_u} \left[ \mathcal{Z}^{(a)} \right]$$

By the Lemma 3.6 we obtain

$$\max_{a \in H_u} \mathcal{Y}^{(a)} \geq (1 - 0.01 \epsilon) \sqrt{2 \log h \cdot 2C_0 (n - h)}$$

$$\max_{a \in H_u} \mathcal{Z}^{(a)} \leq (1 + \epsilon) \sqrt{2 \log h \cdot \max \text{Var}(Z^{(a)})}$$

with probability $1 - o_n(1)$. (Here $o_n(1) \to 0$ as $n \to \infty$.) Moreover,

$$\text{Var} \mathcal{Z}^{(a)} = \|G(y^{(a)}) - G(y)\|_F^2 - \text{Var}(\mathcal{Y}^{(a)})$$

$$= (2n - 6)C_0 - 2C_0 (n - h)$$

$$= C_0(2h - 6)$$

which is $o(n)$. Hence

$$\max_{a \in H_1, b \in H_2} \langle W, G(y^{(a)}) - G(y) \rangle \geq (1 - 0.01 \epsilon - o(1)) \sqrt{2 \log n \cdot 2C_0(n - h)}$$

$$\geq 2(1 - 0.01 \epsilon - o(1)) \sqrt{C_0 n \log n}$$

with probability $1 - o_n(1)$. Since $\sigma^2 > \left(\frac{1+\delta}{4 \log n}\right)C_0n$, we have

$$\Pr \left( \max_{a,b \in \{1,2,...,n\}, y(a) \neq y(b)} \sigma \langle W, G(y^{(a)}) - G(y) \rangle > C_0(1 + \delta)n \right) \geq 1 - o_n(1)$$

Then the lemma follows. □
5. Proof of Theorem 2.2 When the number of vertices in each color is arbitrary

Now we consider Case (2), and assume that for each \( x \in \Omega \), \( n_1(x) \geq \ldots \geq n_k(x) \) are arbitrary positive integers satisfying (2.1) and denoting the number of vertices in each color under the mapping \( x \), respectively. Let \( K(y) \) be defined as in (2.8), and \( R \) be defined as in (2.9).

Given a sample \( R \), again we want to determine the color \( y \) of all the vertices. Let \( \tilde{y} \) be defined as in (2.11).

Again for \( i, j \in [k] \), let \( S_{i,j}(x,y) \) be defined as in (7.17), and \( t_{i,j}(x,y) = |S_{i,j}(x,y)| \). Then

\[
\sum_{i \in [k]} t_{i,j}(x,y) = n_j(y); \quad \sum_{j \in [k]} t_{i,j}(x,y) = n_i(x); \quad \sum_{i,j} t_{i,j}(x,y) = n
\]

and

\[
\langle K(x), K(y) \rangle = \sum_{i,j \in [k]} t_{i,j}(x,y) [n - n_i(x) - n_j(y) + t_{i,j}(x,y)]
\]

\[
= n^2 - \sum_{i \in [k]} [n_i(x)]^2 - \sum_{j \in [k]} [n_j(y)]^2 + \sum_{i,j \in [k]} [t_{i,j}(x,y)]^2
\]

In particular,

\[
\langle K(y), K(y) \rangle = n^2 - \sum_{j \in [k]} [n_j(y)]^2
\]

Hence we obtain

\[
\tilde{y} = \arg\max_{x \in \Omega} \left[ \sum_{j=1}^{k} n_j^2(x) + 2\langle K(x), R \rangle \right]
\]

Define

\[
g(x) = \sum_{j=1}^{k} n_j^2(x) + 2\langle K(x), R \rangle;
\]

then

\[
g(x) - g(y) = 2\sigma \langle K(x) - K(y), W \rangle + 2 \sum_{i,j \in [k]} [t_{i,j}(x,y)]^2 - \sum_{i=1}^{k} [n_i(x)]^2 - \sum_{j=1}^{k} [n_j(y)]^2
\]

Note that \( g(x) - g(y) \) is a Gaussian random variable with mean \( 2 \sum_{i,j \in [k]} [t_{i,j}(x,y)]^2 - \sum_{i=1}^{k} [n_i(x)]^2 - \sum_{j=1}^{k} [n_j(y)]^2 \) and variance \( 4\sigma^2 \| K(x) - K(y) \|_F^2 \).

Let

\[
L(x,y) = \sum_{i=1}^{k} [n_i(x)]^2 + \sum_{j=1}^{k} [n_j(y)]^2 - 2 \sum_{i,j \in [k]} [t_{i,j}(x,y)]^2
\]

Then

\[
\Pr(g(x) - g(y) > 0) = \Pr_{\xi \sim N(0,1)} \left( \xi > \frac{\sqrt{L(x,y)}}{2\sigma} \right) \leq e^{-\frac{L(x,y)}{8\sigma^2}}
\]
For \( y \in \Omega \), let \( C(y) \) consist of all the \( x \in \Omega_{n_1(y), \ldots, n_k(y)} \) such that \( x \) can be obtained from \( y \) by a permutation of colors with the same number of vertices. More precisely, \( x \in C(y) \subset \Omega \) if and only if the following condition holds

- for \( 1 \leq i, j \leq n \), \( y(i) = y(j) \) if and only if \( x(i) = x(j) \).

We define an equivalence relation on \( \Omega \) as follows: we say \( x, z \in \Omega \) are equivalent if and only if \( x \in C(z) \). Let \( \Omega \) be the set of all the equivalence classes in \( \Omega \). We have the following elementary lemma:

**Lemma 5.1.** If \( x, z \in \Omega \) are equivalent, then

\[
\sum_{i, j \in \{1, 2, \ldots, k\}} [t_{i, j}(x, y)]^2 = \sum_{i, j \in \{1, 2, \ldots, k\}} [t_{i, j}(z, y)]^2
\]

*Proof.* By definition if \( x, z \in \Omega \) are equivalent, then there exists a permutation \( \sigma \) of \( \{1, 2, \ldots, k\} \), such that for \( 1 \leq l \leq n \), we have

\[
z(l) = \sigma(x(l))
\]

and for \( 1 \leq i \leq k \),

\[
|x^{-1}(i)| = |z^{-1}(\sigma(i))|
\]

Then we have

\[
t_{i, j}(x, y) = t_{\sigma(i), j}(z, y)
\]

by summing over all the \( i, j \)’s in \( \{1, 2, \ldots, k\} \), and using the fact that \( \sigma \) is a permutation of \( \{1, 2, \ldots, k\} \), we obtain the lemma. \( \square \)

**Lemma 5.2.** If \( x \) and \( z \) are equivalent elements in \( \Omega \), then for any chosen sample \( W \), we have

\[
g(x) = g(z).
\]

*Proof.* By definition we have

\[
g(x) = 2\langle K(x), K(y) \rangle + 2\sigma(K(x), W) + \sum_{j=1}^{k} n_j(x)^2
\]

Recall that \( K_{i, j}(x) = 1 \) if and only if \( x(i) \neq x(j) \); if \( x(i) = x(j) \), \( K_{i, j}(x) = 0 \). Since \( x \) and \( z \) are equivalent \( x(i) \neq x(j) \) if and only if \( z(i) \neq z(j) \), therefore

\[
K(x) = K(z).
\]

Moreover, if \( x \) and \( z \) are equivalent, then \( x \in \Omega_{n_1(z), \ldots, n_k(z)} \); in particular this implies

\[
\sum_{j=1}^{k} [n_j(x)]^2 = \sum_{j=1}^{k} [n_j(z)]^2
\]

Then the lemma follows. \( \square \)
Let

\[ p(\tilde{y}, \sigma) = \Pr(\tilde{y} = y) \]

Then

\[ p(\tilde{y}, \sigma) = \Pr \left( g(y) > \max_{x \in \Omega \setminus C(y)} g(x) \right); \]

hence

\[ 1 - p(\tilde{y}, \sigma) \leq \sum_{x \in \Omega \setminus C(y)} \Pr(\tilde{f}(x) - \tilde{f}(y) \geq 0) \leq \sum_{x \in \Omega \setminus C(y)} e^{-\frac{L(x, y)}{8\sigma^2}}. \]

**Lemma 5.3.** For any \( x, y \in \Omega \), \( L(x, y) \geq 0 \); where the equality holds if and only if \( x \in C(y) \).

**Proof.** By (5.1) and (5.2) we have

\[ L(x, y) = \begin{cases} 2 \left[ \sum_{i \in [k]} \sum_{1 \leq j_1 < j_2 \leq k} t_{i, j_1}(x, y) t_{i, j_2}(x, y) + \sum_{j \in [k]} \sum_{1 \leq i_1 < i_2 \leq k} t_{i_1, j}(x, y) t_{i_2, j}(x, y) \right] \geq 0 \]

Obviously \( L(x, y) = 0 \) if \( x \in C(y) \). We only need to show that if \( L(x, y) = 0 \), then \( x \in C(y) \).

Note that if \( L(x, y) = 0 \), then

\[ t_{i, j_1}(x, y) t_{i, j_2}(x, y) = 0, \ \forall i \in [k], \ 1 \leq j_1 < j_2 \leq k; \text{ and} \]
\[ t_{i_1, j}(x, y) t_{i_2, j}(x, y) = 0, \ \forall j \in [k], \ 1 \leq i_1 < i_2 \leq k \]

Then for any fixed \( i \in [k] \), there exists exactly one \( j \in [k] \), such that \( t_{i, j} \neq 0 \); and for each fixed \( j \in [k] \), there exists exactly one \( i \in [k] \), such that \( t_{i, j} \neq 0 \). Then the lemma follows. \( \Box \)

We have the following theorem

**Theorem 5.4.** Let \( k \) be fixed and let \( n \to \infty \).

(1) If there exists \( \delta > 0 \), such that

\[ \sigma^2 < \frac{(1 - \delta)(n_k(y) + n_{k-1}(y))}{4\log n} \]

then

\[ \lim_{n \to \infty} p(\tilde{y}; \sigma) = 1 \]

(2) If there exists \( \delta > 0 \), such that

\[ \sigma^2 > \frac{(1 + \delta)(n_k(y) + n_{k-1}(y))}{4\log n} \]

then

\[ \lim_{n \to \infty} p(\tilde{y}; \sigma) = 0. \]
Proof. We first prove Part (1). Let $\tilde{B}$ be the set given by

$$\tilde{B} = \left\{(t_{1,1}, t_{1,2}, \ldots, t_{k,k}) \in \left(\prod_{i \in [k]} [n_i]\right)^k : \sum_{i=1}^{k} t_{i,j} = n_j \right\}.$$  

Note that for each $(t_{1,1}, t_{1,2}, \ldots, t_{k,k}) \in \tilde{B}$, the number of $C(x) \in \Omega$ such that for all $1 \leq i, j \leq k$, $t_{i,j}(x, y) = t_{i,j}$ is no more than

$$\frac{\prod_{i=1}^{k} [n_i]!}{\prod_{1 \leq i,j \leq k} [t_{i,j}]!}.$$

Then we have

$$\sum_{x \in \Omega \setminus C(y)} e^{-\frac{L(x,y)}{8\sigma^2}} \leq I_3 + I_4$$

where

$$I_3 = \int_{\tilde{D} \setminus \tilde{D}_\varepsilon} \frac{n^{\frac{k^2-k}{2}} \prod_{1 \leq i,j \leq k} \sqrt{v_i(y)} \prod_{1 \leq i,j \leq n} \sqrt{u_{i,j}}}{\prod_{1 \leq i,j \leq k} \left(\prod_{l=1}^{k} v_l(y) v_l(y)\right)^{\frac{n}{8\sigma^2}} \prod_{1 \leq i,j \leq k} (u_{i,j})^{u_{i,j}}} \prod_{1 \leq i,j \leq k} \left(\prod_{l=1}^{k} v_l(y) v_l(y)\right)^{n} e^{-\frac{L(x,y)}{8\sigma^2}} dV$$

and

$$I_4 = \sum_{C(x) \in \Omega, \left(\frac{t_{1,1}}{n}, \ldots, \frac{t_{k,k}}{n}\right) \in \tilde{D}_\varepsilon, C(x) \neq C(y)} e^{-\frac{L(x,y)}{8\sigma^2}}.$$

Here for $1 \leq i, j \leq k$, $u_{i,j}$ and $v_i(y)$ are given as in (4.4) and (4.5) $\tilde{D}$ is the $(k^2 - k)$-dimensional domain given by

$$\tilde{D} \left(\sum_{u_{i,j} \geq 0} \text{for } 1 \leq i, j \leq k; \sum_{i=1}^{k} u_{i,j} = v_j(y) \right).$$

For a small positive number $\varepsilon > 0$, let $\tilde{D}_\varepsilon$ be the domain given by

$$\tilde{D}_\varepsilon = \left\{(u_{1,1}, \ldots, u_{k,k}) : \forall i \in [k], \exists j \in [k] \text{ s.t. } u_{j,i} \geq v_i(y) - \varepsilon \text{ and } |v_j(x) - v_i(y)| < \varepsilon \right\}$$

**Lemma 5.5.** When $(u_{1,1}, \ldots, u_{k,k}) \in \tilde{D} \setminus \tilde{D}_\varepsilon$, by (5.3) we have

$$L(x, y) \geq \frac{\varepsilon}{k^2} n^2$$

Proof. By (5.3), we obtain

$$L(x, y) \geq \sum_{j \in [k]} \max_{i \in [k]} t_{i,j}(x, y) \left[n_j(y) - \max_{i \in [k]} t_{i,j}(x, y)\right]$$

$$\geq \sum_{j \in [k]} \frac{n_j(y)}{k} \varepsilon n_j(y) \geq \varepsilon \left(\sum_{j \in [k]} n_j(y)\right)^2 \frac{\varepsilon n^2}{k^2} = \frac{\varepsilon n^2}{k^2}$$
For any fixed $\epsilon > 0$, if (5.4) holds, then $\sigma \sim o(\sqrt{n})$, therefore $\lim_{n \to \infty} I_1 = 0$. 

Now let us consider $I_2$. For $i \in [k]$, assume

$$u_{w(i),i} = \max_{j \in [k]} u_{w,j,i},$$

where $w(i) \in [k]$. When $(u_{1,1}, \ldots, u_{k,k}) \in \mathcal{D}_C$, then $u_{w(i),i} > v_i(y) - \epsilon$ and $v_{w(i)}(x) = v_i(y)$.

We consider the following “smoothing” process for $t_i$'s: choose a vertex in $S_{w(i),i}(x,y)$, change its color in $x$ from $c_{w(i)}$ to $c_s$: where $s \in [k]$ an $s \neq j$. Then the change in $L(x,y)$ is

$$[n_{w(i)}(x) - 1]^2 + [n_s(x) + 1]^2 - [n_{w(i)}(x)]^2 - [n_s(x)]^2 - 2(t_{w(i),i} - 1)^2 - 2(t_s,i + 1)^2 + 2t_{w(i),i}^2 + 2t_{s,i}^2$$

$$= -2n_{w(i)}(x) + 2n_s(x) + 4(t_{w(i),i} - t_{s,i}) - 2 \geq 2[n_k(y) + n_{k-1}(y) - 3\epsilon n],$$

when $n$ is sufficiently large. We continue this process until the resulting $(u_{1,1}, \ldots, u_{k,k})$ is outside $\mathcal{D}_C$.

$$I_2 \leq \sum_{l=1}^{\infty} k^l n^l e - \frac{n_k(y) + n_{k-1}(y) - 3\epsilon n}{4\sigma^2}$$

It is straightforward to check that when $\sigma$ satisfies (4.7), there exists sufficiently small $\epsilon > 0$, such that $\lim_{n \to \infty} I_2 = 0$. Then Part (1) the theorem follows.

Now we prove Part (2) of the theorem. For $y \in \Omega$, $a \in \{1, 2, \ldots, n\}$ such that $c_{k-1} = y(a)$. Let $y^{(a)}$ be defined by

$$y^{(a)}(i) = \begin{cases} y(i) & \text{if } i \neq a \\ c_k & \text{if } i = a. \end{cases}$$

Then

$$1 - p(\bar{y}; \sigma) \geq \Pr \left( \cup_{a \in [n]} [y^{-1}(c_{k-1})] [g(y^{(a)}) - g(y) > 0] \right)$$

Since any of the event $[g(y^{(a)}) - g(y) > 0]$ implies $\bar{y} \neq y$. Recall that

$$\text{(5.6)} \quad \langle \mathbf{K}(y), \mathbf{K}(y^{(a)}) - \mathbf{K}(y) \rangle = \sum_{i,j \in [k]} [t_{i,j}(y^{(a)}, y)]^2 - \sum_{i \in [k]} [n_i(y^{(a)})]^2 = -2n_k(y)$$

Then

$$g(y^{(a)}) - g(y) = \sum_{i \in [k]} [n_i(y^{(a)})]^2 - \sum_{i \in [k]} [n_i(y)]^2 + 2\langle \mathbf{K}(y), \mathbf{K}(y^{(a)}) - \mathbf{K}(y) \rangle + 2\sigma \langle \mathbf{W}, \mathbf{K}(y^{(a)}) - \mathbf{K}(y) \rangle$$

$$= -2n_{k-1}(y) - 2n_k(y) + 2 + 2\sigma \langle \mathbf{W}, \mathbf{K}(y^{(a)}) - \mathbf{K}(y) \rangle.$$ 

Let $H_{k-1} \subset y^{-1}(c_{k-1})$, such that $|H_{k-1}| = \frac{n}{\log^2 n} = h$. Then $1 - p(\bar{y}; \sigma)$ is at least

$$\Pr \left( \cup_{a \in [n]} [y^{-1}(c_{k-1})] [g(y^{(a)}) - g(y) > 0] \right)$$

$$\geq \Pr \left( \max_{a \in [n]} [y^{-1}(c_{k-1})] \sigma\langle \mathbf{W}, \mathbf{K}(y^{(a)}) - \mathbf{K}(y) \rangle > n_k(y) + n_{k-1}(y) - 1 \right)$$

$$\geq \Pr \left( \max_{a \in H_{k-1}} \sigma\langle \mathbf{W}, \mathbf{K}(y^{(a)}) - \mathbf{K}(y) \rangle > n_k(y) + n_{k-1}(y) - 1 \right)$$
Let \((X, Y, Z)\) be a partition of \([n]^2\) defined by
\[
X = \{\alpha = (\alpha_1, \alpha_2) \in [n]^2; \{\alpha_1, \alpha_2\} \cap [H_{k-1}] = \emptyset\}
\]
\[
Y = \{\alpha = (\alpha_1, \alpha_2) \in [n]^2; |\{\alpha_1, \alpha_2\} \cap [H_{k-1}]| = 1\}
\]
\[
Z = \{\alpha = (\alpha_1, \alpha_2) \in [n]^2; |\{\alpha_1, \alpha_2\} \cap [H_{k-1}]| = 2\}
\]

For \(\eta \in \{X, Y, Z\}\), define the \(n \times n\) matrix \(W_\eta\) from the entries of \(W\) as follows
\[
W_\eta(i, j) = \begin{cases} 
0 & \text{if } (i, j) \notin \eta \\
W(i, j) & \text{if } (i, j) \in \eta
\end{cases}
\]

For each \(a \in H_{k-1}\), let
\[
X_a = \langle W_X, G(y^{(a)}) - G(y) \rangle
\]
\[
Y_a = \langle W_Y, G(y^{(a)}) - G(y) \rangle
\]
\[
Z_a = \langle W_Z, G(y^{(a)}) - G(y) \rangle
\]

**Claim 5.6.** The followings are true:

1. \(X_a = 0\) for \(a \in H_{k-1}\).
2. For each \(a \in H_{k-1}\), the variables \(Y_a\) and \(Z_a\) are independent.

**Proof.** It is straightforward to check (1). (2) holds because \(Y \cap Z = \emptyset\). \(\square\)

For \(s \in H_{k-1}\), let \(Y^s \subseteq Y\) be defined by
\[
Y^s = \{\alpha = (\alpha_1, \alpha_2) \in Y: \alpha_1 = s, \text{ or } \alpha_2 = s\}.
\]

Note that for \(s_1, s_2 \in H_{k-1}\) and \(s_1 \neq s_2\), \(Y^{s_1} \cap Y^{s_2} = \emptyset\). Moreover, \(Y = \bigcup_{s \in H_2} Y^s\). Therefore
\[
Y_a = \sum_{s \in H_{k-1}} \langle W_{Y^s}, K(y^{(a)}) - K(y) \rangle
\]

Note also that \(\langle W_{Y^s}, K(y^{(a)}) - K(y) \rangle = 0\), if \(s \neq a\). Hence
\[
Y_a = \sum_{\alpha \in Y^a} [W(\alpha)] \cdot \{[K(y^{(a)}) - K(y)](\alpha)\}
\]

Note that for \(\alpha \in Y^a\),
\[
[K(y^{(a)}) - K(y)](\alpha) = \begin{cases} 
-1 & \text{if } \{\alpha_1, \alpha_2\} \cap [y^{-1}(c_k)] = 1 \\
1 & \text{if } \{\alpha_1, \alpha_2\} \cap [y^{-1}(c_k-1)] = 2 \\
0 & \text{else}
\end{cases}
\]

So,
\[
\sum_{\alpha \in Y^a} [W(\alpha)] \cdot \{[K(y^{(a)}) - K(y)](\alpha)\}
\] = \sum_{\alpha \in Y^a: \{\alpha_1, \alpha_2\} \cap [y^{-1}(c_k)] = 1} [W(\alpha)] - \sum_{\alpha \in Y^a: \{\alpha_1, \alpha_2\} \cap [y^{-1}(c_k-1)] = 2} [W(\alpha)]
\]

\(\{Y^a\}_{s \in H_{k-1}}\) is a collection of independent Gaussian random variables. Moreover, the variance of \(Y_a\) is equal to \(2(n_k(y) + n_{k-1}(y) - h)\) independent of the choice of \(s\).

By the claim, we obtain
\[
\langle W, K(y^{(a)}) - K(y) \rangle = Y_a + Z_a
\]
Moreover,
\[
\max_{a \in H_{k-1}} Y_a + Z_a \geq \max_{a \in H_{k-1}} Y_a - \max_{a \in H_{k-1}} (-Z_a)
\]

By Lemma 3.6 about the tail bound result of the maximum of Gaussian random variables, we obtain
\[
\max_{a \in H_{k-1}} Y_a \geq (1 - 0.01\epsilon) \sqrt{2\log h \cdot 2(n_k(y) + n_{k-1}(y) - h)}
\]
\[
\max_{a \in H_{k-1}} Z_a \leq (1 + \epsilon) \sqrt{2\log h \cdot \max_{a \in H_{k-1}} \text{Var}(Z_a)}
\]
with probability \(1 - o_n(1)\). (Here \(o_n(1) \to 0\) as \(n \to \infty\).) Note that
\[
\langle K(y^{(a)}), K(y^{(a)} - K(y)) \rangle = - \sum_{i,j \in [k]} [t_{i,j}(y, y^{(a)})]^2 + \sum_{i \in [k]} [n_i(y)]^2
\]
\[
= 2n_{k-1}(y) - 2
\]
Moreover, by (5.6) and (5.7)
\[
\text{Var}Z_a = \|K(y^{(a)}) - K(y)\|^2_F - \text{Var}(Y_a)
\]
\[
= 2[n_{k-1}(y) + n_k(y)] - 2(2n_{k-1}(y) + n_k(y) - h)
\]
\[
= 2h - 2
\]
which is \(o(n)\). Hence
\[
\max_{a \in H_{k-1}} \langle W, K(y^{(a)}) - K(y) \rangle \geq (1 - 0.01\epsilon - o(1)) \sqrt{2\log n \cdot 2(n_k(y) + n_{k-1}(y))}
\]
\[
\geq 2(1 - 0.01\epsilon - o(1)) \sqrt{n_k(y) + n_{k-1}(y)} \log n
\]
with probability \(1 - o_n(1)\). Since \(\sigma^2 > \frac{(1+\delta)[n_k(y) + n_{k-1}(y)]}{4\log n}\), we have
\[
\Pr \left( \max_{a,b \in \{1, \ldots, n\}, y(a) \neq y(b)} \sigma \langle W, K(y^{(a)}) - K(y) \rangle > (1 + \delta)[n_k(y) + n_{k-1}(y)] \right) \geq 1 - o_n(1)
\]
Then the lemma follows.
\[
\square
\]

6. Fixed Number of Vertices in Each Color and Adjacency Matrix

We again consider Case (1). Let \(y \in \Omega_{n_1, \ldots, n_k}\). Let \(K(y)\) be defined as in (2.8), and let \(R\) be defined as in (2.9). For given sample \(R\), let \(\overline{y}\) be defined as in (2.11).

We have the following theorem

**Theorem 6.1.** Let \(k\) be fixed and let \(n \to \infty\).

(1) If there exists \(\delta > 0\), such that
\[
\sigma^2 < \frac{(1 - \delta)[n_{k-1} + n_k]}{4 \log n}
\]
then
\[
\lim_{n \to \infty} p(\overline{y}; \sigma) = 1
\]
(2) If there exists $\delta > 0$, such that
\[
\sigma^2 > \frac{(1 + \delta)[n_{k-1} + n_k]}{4 \log n}
\]
then
\[
\lim_{n \to \infty} p(\mathbb{F}; \sigma) = 0.
\]

Proof. The theorem can be proved in a similar way as the proof of Theorem 5.4. \qed

7. Complex Unitary Matrix with Gaussian Perturbation

Now we consider the community detection problem when the observation is a complex unitary matrix plus a multiple of a GUE or GOE matrix. In the former case, we prove a threshold with respect the intensity $\sigma$ of the GUE perturbation for the exact recovery of the MLE. In the latter case, we develop a “complex version” of SDP algorithm for efficient recovery, and explicitly prove the region of the intensity of the GOE perturbation for the exact recovery of the SDP.

7.1. GUE perturbation. In this section, we prove Theorem 2.4.

For $x \in \Theta_A$, define
\[
r(x) = \Re \langle U, P(x) \rangle
\]
Note that
\[
r(x) - r(y) = \Re \{\langle P(y), P(x) - P(y) \rangle + \sigma \langle W_c, P(x) - P(y) \rangle\}
\]
which is a real Gaussian random variable with mean $\Re \{\langle P(y), P(x) - P(y) \rangle\}$, and variance $4\sigma^2 \sum_{i<j} \{1 - \Re \{x(i)x(j)y(i)y(j)\}\}$. Moreover,
\[
\Re \langle P(x), P(y) \rangle = n + 2 \sum_{i<j} \Re \{x(i)x(j)y(i)y(j)\}
\]
Hence
\[
\Re \{\langle P(y), P(x) - P(y) \rangle\} = -2 \sum_{i<j} \Re \{x(i)x(j)y(i)y(j)\}
\]
Let
\[
J(x, y) = -\mathbb{E}[r(x) - r(y)] = 2 \sum_{i<j} \Re \{x(i)x(j)y(i)y(j)\}
\]
Then for $x \in \Theta_A$
\[
\Pr(r(x) - r(y) > 0) = \Pr_{\xi \in \mathcal{N}(0,1)} \left( \frac{\sqrt{J(x, y)}}{\sqrt{2\sigma}} \right) \leq e^{-\frac{f(x, y)}{4\sigma}}.
\]

Lemma 7.1. For any $x, y \in \Theta_A$, $J(x, y) \geq 0$; $J(x, y) = 0$ if and only if there exists a fixed angle $\alpha$, such that $e^{i\alpha}x = y$. 

Proof. First of all, $J(x, y) \geq 0$ follows from the fact that $|\overline{x(i)x(j)y(i)y(j)}| = 1$. Moreover, $J(x, y) = 0$ if and only if for any $1 \leq i < j \leq n$, $\overline{x(i)x(j)y(i)y(j)} = 1$. Then the lemma follows. \hfill \Box

We define an equivalence class on $\Theta_A$ as follows. We say $x, y \in \Theta_A$ are equivalent if there exists a fixed angle $\alpha$, such that $e^{i\alpha}x = y$. For each $y \in \Theta$, let $C(y)$ be the equivalence class containing $y$. Then

**Lemma 7.2.** If $x, y \in \Theta_A$ are equivalent, then $r(x) = r(y)$.

Proof. Note that if $x \in C(y)$, then $P(x) = P(y)$. Then the lemma follows from (7.1). \hfill \Box

By Lemmas 7.1 and 7.3, we obtain

$$p(y^A; \sigma) = \Pr \left[ r(y^A) > \max_{x \in \Theta_A, x \notin C(y)} r(x) \right]$$

Hence

$$1 - p(y^A; \sigma) \leq \sum_{C(x) \subseteq [\Theta \setminus C(y)]} e^{-\frac{J(x, y)}{4\sigma^2}}$$

For $i, j \in \{1, 2, \ldots, k\}$, let

$$S_{i,j}(x, y) = \{1 \leq l \leq n : x(l) = e^{ik}, y(l) = e^{id} \};$$

i.e., $S_{i,j}(x, y)$ consists of all the vertices which have color $c_i$ in $x$ and color $c_j$ in $y$.

Let $t_{i,j}(x, y) = |S_{i,j}(x, y)|$. Again $t_{i,j}(x, y)$ satisfies (4.1) and (4.2). From (7.5), we obtain

$$J(x, y) = \sum_{i,j,p,q \in [k]} t_{i,j}(x, y) t_{p,q}(x, y) [1 - \cos(-d_i + d_p + d_j - d_q)]$$

$$= \sum_{i,j,p,q \in [k]} t_{i,j}(x, y) t_{p,q}(x, y) \left[ 1 - \cos \left( \frac{2\pi(p + j - q - i)}{k} \right) \right]$$

When $p, q, i, j \in [k], p + j - q - i \in [-2k, 2k - 2]$; hence $\cos \left( \frac{2\pi(p + j - q - i)}{k} \right) = 1$ if and only if $p + j - q - i \in \{-k, 0, k\}$. Therefore,

$$J(x, y) = \sum_{i,j,p,q \in [k], p + j - q - i \notin \{-k, 0, k\}} t_{i,j}(x, y) t_{p,q}(x, y) \left[ 1 - \cos \left( \frac{2\pi(p + j - q - i)}{k} \right) \right]$$

For $1 \leq i, j \leq k$, let $u_{i,j}(x, y)$ and $v_i(y)$ be defined as in (4.4) and (4.5). By Assumption 2.3,

$$n_1(x) = n_2(x) = \ldots = n_k(x) = n_1(y) = n_2(y) = \ldots = n_k(y) = \frac{n}{k}$$

Let $\hat{D}$ be a domain defined by

$$\hat{D} = \left\{ (u_{1,1}, \ldots, u_{k,k}) \in \left[ 0, \frac{1}{k} \right]^k : \sum_{j \in [k]} u_{i,j} = \frac{1}{k}, \sum_{i \in [k]} u_{i,j} = \frac{1}{k} \right\}$$
Note that the dimension of $\hat{D}$ is $(k - 1)^2$. For a fixed $\epsilon > 0$, let

$$\mathcal{D}_\epsilon = \left\{ (u_{1,1}, \ldots, u_{k,k}) \in \hat{D} : \sum_{i,j,p,q \in [k], p+j-q \in \{-k,0,k\}} u_{i,j}u_{p,q} \leq \epsilon \right\}$$

Then we have

$$\sum_{(x,y) \subseteq \Theta \setminus C(y)} e^{-\frac{J(x,y)}{4\sigma^2}} \leq \hat{I}_1 + \hat{I}_2$$

where

$$\hat{I}_1 = \int_{\mathcal{D} \setminus \mathcal{D}_\epsilon} \frac{n^{(k-1)(k-2)}}{(2\pi)^{k-1}} \prod_{1 \leq l \leq k} \sqrt{v_l(y)} \left[ \prod_{1 \leq i,j \leq n} \sqrt{u_{i,j}} \right]^{n} e^{-\frac{J(x,y)}{4\sigma^2}} dV$$

and

$$\hat{I}_2 = \sum_{(x,y) \subseteq \Theta \setminus C(y)} e^{-\frac{J(x,y)}{4\sigma^2}} \cdot \sum_{(t_{1,1}, \ldots, t_{k,k}) \in \mathcal{D}_\epsilon} \frac{1}{n}.$$

**Lemma 7.3.** For any fixed $\epsilon > 0$,

$$\lim_{n \to \infty} \hat{I}_1 = 0.$$

**Proof.** From the definition of the domains $\hat{D}$ and $\mathcal{D}_\epsilon$, as well as the expression (7.5), we obtain that when $\epsilon > 0$,

$$J(x,y) \geq \epsilon n^2 \left[ 1 - \cos \left( \frac{2\pi}{k} \right) \right].$$

Then the lemma follows. \(\square\)

From (7.5), we obtain

$$J(x,y) = \sum_{t_{1,1}} \sum_{i,j,p,q \in [k], \ p+j-q \equiv k \mod k} t_{i,j}(x,y)t_{p,q}(x,y) \left[ 1 - \cos \left( \frac{2u\pi}{k} \right) \right]$$

**Lemma 7.4.** For each $x \in \Theta_A$ satisfying $\left( \frac{t_{1,1}(x,y)}{n}, \ldots, \frac{t_{k,k}(x,y)}{n} \right) \in \mathcal{D}_\epsilon$, there exists $y' \in C(y)$ such that

$$\sum_{i,j \in [k], i \neq j} u_{i,j}(x,y') < \delta(\epsilon) := \frac{2(k-1)\epsilon}{1 + \sqrt{1 - 4\epsilon}}$$

**Proof.** From the definition (7.6) and the fact that $\sum_{i,j \in [k]} u_{i,j}(x,y) = 1$, we obtain

$$\sum_{i,j,p,q \in [k], p+j-q \in \{-k,0,k\}} u_{i,j}(x,y)u_{p,q}(x,y) \geq 1 - \epsilon;$$

which is the same as the following inequality

$$\sum_{l=0}^{k-1} \left[ \sum_{[i,j \in [k], (i-j) \mod k = l]} u_{i,j}(x,y) \right]^2 \geq 1 - \epsilon$$
Let $a = \frac{2}{1 + \sqrt{1 - \epsilon}}$. Again using the fact that $\sum_{i,j \in \mathbb{Z}} u_{i,j}(x, y) = 1$; if there exists $l_1, l_2 \in \{0, \ldots, k - 1\}$ and $l_1 \neq l_2$ and

$$
\min \left\{ \sum_{[i,j] \mod k = l_1} u_{i,j}(x, y), \sum_{[i,j] \mod k = l_2} u_{i,j}(x, y) \right\} \geq a;
$$

then

$$
\sum_{l=0}^{k-1} \left[ \sum_{[i,j] \mod k = l} u_{i,j}(x, y) \right]^2 \leq a^2 + (1 - a)^2 = 1 - 2\epsilon
$$

which is a contradiction to (7.9). Hence there exists at most one $l \in \{0, 1, \ldots, k - 1\}$, such that

$$
\sum_{[i,j] \mod k = l} u_{i,j}(x, y) \geq a
$$

Therefore

$$
\sum_{[i,j] \mod k = l} u_{i,j}(x, y) \geq 1 - (k - 1)a = 1 - \delta(\epsilon)
$$

Let $y' = e^{2\pi i \frac{l}{k}} y$, then the lemma follows. \hfill \Box

For given $x \in \Theta_A$, an elementary move of $x$ is obtained by swapping the colors of two vertices $u, v \in [n]$ satisfying

$$
(x(u) - x(v)) \mod k \in \{\pm l\},
$$

where $l \in [k - 1]$. We can obtain any $x \in \Theta_A$ by finitely many elementary moves starting from $y$.

For each $x \in \Theta_A$ satisfying \left(\frac{t_{1,1}(x,y)}{n}, \ldots, \frac{t_{k,k}(x,y)}{n}\right) \in \mathcal{D}_\epsilon$, let $y' \in C(y)$ such that (7.1) holds. Let $x' \in \Theta_A$ be obtained from $x$ by an elementary move of swapping the colors of two vertices $u$ and $v$ such that

- $u$ and $v$ satisfy (7.10); and
- $y'(u) = x(u)$, $y'(v) = x(v)$

Then

$$
t_{x(u),x(u)}(x, y') - 1 = t_{x(u),x(u)}(x', y')
$$

$$
t_{x(v),x(u)}(x, y') + 1 = t_{x(v),x(u)}(x', y')
$$

$$
t_{x(v),x(v)}(x, y') - 1 = t_{x(v),x(v)}(x', y')
$$

$$
t_{x(u),x(v)}(x, y') + 1 = t_{x(u),x(v)}(x', y')
$$

Let

$$
\beta_{u,v} := (x(u), x(u)), (x(u), x(v)), (x(v), x(u)), (x(v), x(v)).
$$

If $(i, j) \notin \beta_{u,v}$, we have $t_{i,j}(x, y) = t_{i,j}(x', y)$. Hence we have

$$
J(x, y) - J(x', y) = J(x, y') - J(x', y') = B_1 + B_2 + B_3
$$
where

\[
B_1 = 2 \left[ t_{x(u), x(u)}(x, y') + t_{x(v), x(v)}(x, y')(t_{x(u), x(v)}(x, y') + t_{x(u), x(v)}(x, y')) - (t_{x(u), x(u)}(x', y') + t_{x(v), x(v)}(x', y'))(t_{x(u), x(v)}(x', y') + t_{x(u), x(v)}(x', y')) \right] 1 - \cos \left( \frac{2l\pi}{k} \right)
\]

\[
= 4[t_{x(u), x(u)}(x', y') + t_{x(v), x(v)}(x', y') - 1] 1 - \cos \left( \frac{2l\pi}{k} \right)
\]

\[
= -\frac{8n}{k} \left[ 1 - \cos \left( \frac{2l\pi}{k} \right) \right] + O(\epsilon n) + O(1);
\]

\[
B_2 = 2[t_{x(v), x(u)}(x, y') t_{x(u), x(v)}(x, y') - t_{x(u), x(u)}(x', y') t_{x(u), x(v)}(x', y')] 1 - \cos \left( \frac{4l\pi}{k} \right)
\]

\[
= O(\epsilon n) + O(1);
\]

\[
B_3 = 2 \sum_{(i, j) \in \beta_{u, v}, (p, q) \notin \beta_{u, v}} [t_{i,j}(x, y') t_{p, q}(x, y') - t_{i,j}(x', y') t_{p, q}(x', y')] 1 - \cos \left( \frac{2\pi(p + j - q - i)}{k} \right)
\]

\[
= 2[t_{x(u), x(v)}(x, y') + t_{x(v), x(u)}(x, y') - t_{x(u), x(v)}(x', y') - t_{x(v), x(u)}(x', y')]
\]

\[
= \left( \sum_{i \notin \{x(u), x(v)\}} t_{i,j}(x, y) \right) \left[ 1 - \cos \left( \frac{2l\pi}{k} \right) \right] + O(\epsilon n) + O(1)
\]

\[
= -\frac{4n(k - 2)}{k} \left[ 1 - \cos \left( \frac{2l\pi}{k} \right) \right] + O(\epsilon n) + O(1)
\]

Therefore,

\[
J(x', y) - J(x, y) \geq 4n \left( 1 - \cos \frac{2l\pi}{k} \right) - O(\epsilon n) - O(1)
\]

\[
\geq 4n \left( 1 - \cos \frac{2\pi}{k} \right) - O(\epsilon n) - O(1)
\]

**Proof of Theorem 2.4.** We first prove Part (a). By Lemma 7.4, for each \( x \in \Theta_A \) satisfying \( \left( \frac{l_{1,1}(x,y)}{n}, \ldots, \frac{l_{k,k}(x,y)}{n} \right) \in \hat{D}_t \), let \( y' \in C(y) \), such that holds. Moreover, \( x \) may be obtained from \( y' \) by finitely many times of fundamental moves of the following type: choose two vertices \( u, v \in [n] \) such that they have colors \( e^{\frac{2\pi l_1}{k}} \) and \( y(v) = e^{\frac{2\pi (l + 1)}{k}} \), for some \( l \in [k] \), respectively; then exchange the colors of \( u \) and \( v \). We may continue this process until the resulting mapping does not satisfy \( \left( \frac{l_{1,1}(x,y)}{n}, \ldots, \frac{l_{k,k}(x,y)}{n} \right) \in \hat{D}_t \) any more. Then

\[
\hat{I}_2 \leq \sum_{l=1}^{\infty} k n 2l^2 e^{-\frac{n l \left[ 1 - \cos \left( \frac{\pi}{k} \right) \right]}{\pi^2}}
\]

If (2.16) holds, then \( \lim_{n \to \infty} \hat{I}_2 = 0 \), then Part (a) of the lemma follows from (7.3), (7.7) and Lemma 7.3.
Now we prove Part (b). For \( y \in \Theta_A, a, b \in [n] \) such that \( 1 = y(a) \neq y(b) = e^{2\pi i} \). Let \( y^{(ab)} \) be the coloring of vertices defined by

\[
y^{(ab)}(i) = \begin{cases} 
y(i) & \text{if } i \in \{1, 2, \ldots, n\} \setminus \{a, b\} \\
e^{2\pi i} & \text{if } i = a \\
1 & \text{if } i = b
\end{cases}
\]

Then

\[
1 - p(y^A; \sigma) \geq \Pr \left( \bigcup_{a, b \in [n], 1 = y(a) \neq y(b) = e^{2\pi i}} [r(y^{(ab)}) - r(y) > 0] \right),
\]

since any of the event \([r(y^{(ab)}) - r(y) > 0]\) implies \( y^A \neq y \). Recall that

\[
r(y^{(ab)}) - r(y) = \langle P(y), P(y^{(ab)}) - P(y) \rangle + \sigma \langle W_c, P(y^{(ab)}) - P(y) \rangle
\]

\[
= -4(n-2) \left(1 - \cos \frac{2\pi}{k}\right) - 2 \left(1 - \cos \frac{4\pi}{k}\right) + \sigma \langle W_c, P(y^{(ab)}) - P(y) \rangle.
\]

Let

\[
E(n) := 4(n-2) \left(1 - \cos \frac{2\pi}{k}\right) + 2 \left(1 - \cos \frac{4\pi}{k}\right)
\]

For \( l \in \{0, 1\} \), let \( H_l \subset y^{-1} \left(e^{2\pi i l/k}\right) \) such that \( |H_l| = \frac{n}{\log^2 n} = h \). Then

\[
1 - p(y^A; \sigma) \geq \Pr \left( \max_{a \in H_0, b \in H_1} \sigma \langle W, P(y^{(ab)}) - P(y) \rangle > E(n) \right)
\]

Let \((\mathcal{X}, \mathcal{Y}, \mathcal{Z})\) be a partition of \([n]^2\) defined by

\[
\mathcal{X} = \{\alpha = (\alpha_1, \alpha_2) \in [n]^2, \{\alpha_1, \alpha_2\} \cap [H_0 \cup H_1] = \emptyset\}
\]

\[
\mathcal{Y} = \{\alpha = (\alpha_1, \alpha_2) \in [n]^2, \{\alpha_1, \alpha_2\} \cap [H_0 \cup H_1] = \{\alpha_1, \alpha_2\}\}
\]

\[
\mathcal{Z} = \{\alpha = (\alpha_1, \alpha_2) \in [n]^2, \{\alpha_1, \alpha_2\} \cap [H_0 \cup H_1] = \{\alpha_1, \alpha_2\}\}
\]

For \( \eta \in \{\mathcal{X}, \mathcal{Y}, \mathcal{Z}\} \), define the \( n \times n \) matrix \( W_{\eta} \) from the entries of \( W \) as follows

\[
W_{\eta,c}(i, j) = \begin{cases} 
0 & \text{if } (i, j) \notin \eta \\
W_c(i, j) & \text{if } (i, j) \in \eta
\end{cases}
\]

For each \( a \in H_0 \) and \( b \in H_1 \), let

\[
\mathcal{X}_{ab} = \langle W_{\mathcal{X},c}, P(y^{(ab)}) - P(y) \rangle
\]

\[
\mathcal{Y}_{ab} = \langle W_{\mathcal{Y},c}, P(y^{(ab)}) - P(y) \rangle
\]

\[
\mathcal{Z}_{ab} = \langle W_{\mathcal{Z},c}, P(y^{(ab)}) - P(y) \rangle
\]

**Claim 7.5.** The followings are true:

1. \( \mathcal{X}_{ab} = 0 \) for \( a \in H_0 \) and \( b \in H_1 \).
2. For each \( a \in H_0 \) and \( b \in H_1 \), the variables \( \mathcal{Y}_{ab} \) and \( \mathcal{Z}_{ab} \) are independent.
3. Each \( \mathcal{Y}_{ab} \) can be decomposed into \( Y_a + Y_b \) where \( \{Y_a\}_{a \in H_0} \cup \{Y_b\}_{b \in H_1} \) is a collection of i.i.d. Gaussian random variables.
Proof. It is straightforward to check (1). (2) holds because \( \mathcal{Y} \cap Z = \emptyset \).

For \( s \in H_0 \cup H_1 \), let \( \mathcal{Y}_s \subseteq \mathcal{Y} \) be defined by

\[
\mathcal{Y}_s = \{ \alpha = (\alpha_1, \alpha_2) \in \mathcal{Y} : \alpha_1 = s, \text{ or } \alpha_2 = s \}.
\]

Note that for \( s_1, s_2 \in H_0 \cup H_1 \) and \( s_1 \neq s_2 \), \( \mathcal{Y}_{s_1} \cap \mathcal{Y}_{s_2} = \emptyset \). Moreover, \( \mathcal{Y} = \bigcup_{s \in H_0 \cup H_1} \mathcal{Y}_s \).

Therefore

\[
\mathcal{Y}_{ab} = \sum_{s \in H_0 \cup H_1} \langle \mathbf{W}_{\mathcal{Y}_s, c}, \mathbf{P}(y^{(ab)}) - \mathbf{P}(y) \rangle
\]

Note also that \( \langle \mathbf{W}_{\mathcal{Y}_s, c}, \mathbf{P}(y^{(ab)}) - \mathbf{P}(y) \rangle = 0 \), if \( s \notin \{a, b\} \). Hence

\[
\mathcal{Y}_{ab} = \sum_{\alpha \in \mathcal{Y}_a \cup \mathcal{Y}_b} [\mathbf{W}_c(\alpha)] \cdot \{[\mathbf{P}(y^{(ab)}) - \mathbf{P}(y)](\alpha)\}
\]

Note that for \( \alpha \in \mathcal{Y}_a \),

\[
[\mathbf{P}(y^{(ab)}) - \mathbf{P}(y)](\alpha) = \begin{cases} (e^{\frac{2\pi \alpha_1}{k}} - 1)y(\alpha_2) & \text{if } \alpha_1 = a \\ y(\alpha_1)(e^{-\frac{2\pi \alpha_1}{k}} - 1) & \text{if } \alpha_2 = a. \end{cases}
\]

So,

\[
Y_a := \sum_{\alpha \in \mathcal{Y}_a} [\mathbf{W}_c(\alpha)] \cdot \{[\mathbf{P}(y^{(ab)}) - \mathbf{P}(y)](\alpha)\}
\]

\[
= \left\{ \sum_{\alpha \in \mathcal{Y}_a, \alpha_1 = a} y(\alpha_2)\mathbf{W}_c(\alpha) + \sum_{\alpha \in \mathcal{Y}_a, \alpha_2 = a} y(\alpha_1)\mathbf{W}_c(\alpha) \right\} \left( \cos \frac{2\pi}{k} - 1 \right)
\]

\[
+ i \left\{ \sum_{\alpha \in \mathcal{Y}_a, \alpha_1 = a} y(\alpha_2)\mathbf{W}_c(\alpha) - \sum_{\alpha \in \mathcal{Y}_a, \alpha_2 = a} y(\alpha_1)\mathbf{W}_c(\alpha) \right\} \sin \frac{2\pi}{k}
\]

Similarly, define

\[
Y_b := \left\{ \sum_{\alpha \in \mathcal{Y}_b, \alpha_2 = b} y(\alpha_2)\mathbf{W}_c(\alpha) - \sum_{\alpha \in \mathcal{Y}_b, \alpha_1 = b} y(\alpha_1)\mathbf{W}_c(\alpha) \right\} \left( 1 - \cos \frac{2\pi}{k} \right)
\]

\[
- i \left\{ \sum_{\alpha \in \mathcal{Y}_b, \alpha_1 = b} y(\alpha_2)\mathbf{W}_c(\alpha) - \sum_{\alpha \in \mathcal{Y}_b, \alpha_2 = b} y(\alpha_1)\mathbf{W}_c(\alpha) \right\} \sin \frac{2\pi}{k}
\]

Then \( \mathcal{Y}_{ab} = Y_a + Y_b \) and \( \{Y_s\}_{s \in H_0 \cup H_1} \) is a collection of independent Gaussian random variables. Moreover, the variance of \( Y_s \) is equal to \( 8(n - 2h)(1 - \cos \frac{2\pi}{k}) \) independent of the choice of \( s \).

By the claim, we obtain

\[
\langle \mathbf{W}_c, \mathbf{P}(y^{(ab)}) - \mathbf{P}(y) \rangle = Y_a + Y_b + Z_{ab}
\]
Moreover,
\[
\max_{a \in H_0, b \in H_1} Y_a + Y_b + Z_{ab} \geq \max_{a \in H_0, b \in H_1} (Y_a + Y_b) - \max_{a \in H_0, b \in H_1} (-Z_{ab}) = \max_{a \in H_0} Y_a + \max_{b \in H_1} Y_b - \max_{a \in H_0, b \in H_1} (-Z_{ab})
\]

By Lemma 3.6 we obtain
\[
\max_{a \in H_0} Y_a \geq (1 - 0.01\epsilon) \sqrt{2 \log h \cdot 8(n - 2h)} \left(1 - \cos \frac{2\pi}{k}\right)
\]
\[
\max_{b \in H_1} Y_b \geq (1 - 0.01\epsilon) \sqrt{2 \log h \cdot 8(n - 2h)} \left(1 - \cos \frac{2\pi}{k}\right)
\]
\[
\max_{a \in H_0, b \in H_1} Z_{ab} \leq (1 + \epsilon) \sqrt{4 \log h \cdot \text{Var}(Z_{ab})}
\]
with probability \(1 - o_n(1)\). (Here \(o_n(1) \to 0\) as \(n \to \infty\).) Moreover,
\[
\text{Var} Z_{ab} \leq 32h^2
\]
which is \(o(n)\). Hence
\[
\max_{a \in H_1, b \in H_2} \langle W_c, P(y^{(ab)}) - P(y) \rangle \geq 2(1 - 0.01\epsilon - o(1)) \sqrt{2 \log n \cdot 8(n - 2h)} \left(1 - \cos \frac{2\pi}{k}\right)
\]
\[
\geq 8(1 - 0.01\epsilon - o(1)) \sqrt{1 - \cos \frac{2\pi}{k}} n \log n
\]
with probability \(1 - o_n(1)\). Since \(\sigma^2 > \frac{(1+\delta)[n(1-\cos \frac{2\pi}{k})]}{2 \log n}\), we have
\[
\text{Pr} \left(\max_{a \in H_0, b \in H_1} \sigma(W, G(y^{(ab)}) - G(y)) > E(n)\right) \geq 1 - o_n(1)
\]
Then the lemma follows.

7.2. Algorithm: Complex Semi-Definite Programming and GOE perturbation.
In this section, we prove Theorem 2.5.

Assume
\[
V = V_1 + iV_2; \quad X = X_1 + iX_2
\]
where \(U_1\) and \(X_1\) are \(n \times n\) real symmetric matrices, and \(U_2\) and \(X_2\) are \(n \times n\) real anti-symmetric matrices.

Let
\[
\tilde{X} = \begin{pmatrix} X_1 & -X_2 \\ X_2 & X_1 \end{pmatrix}, \quad \tilde{U} = \begin{pmatrix} V_1 & V_2 \\ -V_2 & V_1 \end{pmatrix},
\]
then the complex optimization problem (2.17) is equivalent to the following real optimization problem

\[ \text{max} \langle \tilde{V}, \tilde{X} \rangle \]

subject to

\[ \tilde{X}_{ii} = 1, \text{ for } 1 \leq i \leq 2n \]
\[ \tilde{X}_{p,q} = \tilde{X}_{p+n,q+n}, \text{ for } 1 \leq p \leq n, 1 \leq q \leq n \]
\[ \tilde{X}_{p,q+n} = -\tilde{X}_{p+n,q}, \text{ for } 1 \leq p \leq n, 1 \leq q \leq n. \]

and

\[ \tilde{X} \succeq 0 \]

The dual program of (7.12) is

\[ \text{min} \ \text{tr}(Z) \]

subject to

\[ Z - \tilde{V} + \begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix} + \begin{pmatrix} 0 & B \\ B & 0 \end{pmatrix} \succeq 0 \]

\[ Z \text{ is diagonal} \]

By complementary slackness

\[ X = X_1 + iX_2 = \mathbf{y}^t \mathbf{y} = (\mathbf{y}_1 + i\mathbf{y}_2)(\mathbf{y}_1 - i\mathbf{y}_2)^t \]

is the unique optimum solution of (2.17) if and only if there exists a dual feasible solution \((Z, A, B)\), such that

\[ \langle Z - \tilde{V} + \begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix} + \begin{pmatrix} 0 & B \\ B & 0 \end{pmatrix}, \begin{pmatrix} \mathbf{y}_1 \mathbf{y}_1^t + \mathbf{y}_2 \mathbf{y}_2^t & \mathbf{y}_1 \mathbf{y}_2^t - \mathbf{y}_2 \mathbf{y}_1^t \\ \mathbf{y}_2 \mathbf{y}_1^t - \mathbf{y}_1 \mathbf{y}_2^t & \mathbf{y}_1 \mathbf{y}_1^t + \mathbf{y}_2 \mathbf{y}_2^t \end{pmatrix} \rangle = 0; \]

which is equivalent to

\[ \langle Z - \tilde{V} + \lambda_1 \begin{pmatrix} 11^t & 0 \\ 0 & 11^t \end{pmatrix}, \begin{pmatrix} \mathbf{y}_1 \mathbf{y}_1^t + \mathbf{y}_2 \mathbf{y}_2^t & \mathbf{y}_1 \mathbf{y}_2^t - \mathbf{y}_2 \mathbf{y}_1^t \\ \mathbf{y}_2 \mathbf{y}_1^t - \mathbf{y}_1 \mathbf{y}_2^t & \mathbf{y}_1 \mathbf{y}_1^t + \mathbf{y}_2 \mathbf{y}_2^t \end{pmatrix} \rangle = 0. \]

Assume

\[ Z = \begin{pmatrix} Z_1 & 0 \\ 0 & Z_2 \end{pmatrix}, \]

where \(Z_1, Z_2\) are \(n \times n\) real diagonal matrices. Then (7.14) is equivalent to

\[ \Re [\mathbf{y}^t(Z_1 + Z_2 - 2\mathbf{V})\mathbf{y}] = 0 \]

Note that

\[ \Im [\mathbf{y}^t(Z_1 + Z_2 - 2\mathbf{V})\mathbf{y}] = \langle Z_1 + Z_2 - 2\mathbf{V}, \mathbf{y}_2 \mathbf{y}_1^t - \mathbf{y}_1 \mathbf{y}_2^t \rangle - \langle 2\mathbf{V}, \mathbf{y}_1 \mathbf{y}_1^t + \mathbf{y}_2 \mathbf{y}_2^t \rangle \]

which is identically zero since each term on the right hand side of (7.16) is the inner product of a symmetric matrix and an anti-symmetric matrix. Moreover, if the minimizer of (7.13) is unique, then as a Hermitian matrix, the second smallest eigenvalue of

\[ S := Z_1 + Z_2 - 2\mathbf{V} \]

is strictly positive.
From (7.15) and the fact that $S$ is positive semi-definite, we obtain $S\mathbf{y} = 0$. Hence

$$Z_1(i,i) + Z_2(i,i) = 2\sum_{j=1}^{n} \overline{y(i)}V(i,j)y(j);$$

and therefore,

(7.16) \hspace{1cm} S(i,i) = 2\sum_{j\in[n], j\neq i} \overline{y(i)}V(i,j)y(j) - 2V(i,i)

For $i \neq j$,

(7.17) \hspace{1cm} S(i,j) = -2V(i,j)

For a Hermitian matrix $M$, we define the Laplacian $\Delta(M)$ of $M$ by

$$\Delta(M) := \text{diag}(M\mathbf{1}) - M$$

where $\mathbf{1}$ is the column vector all of whose entries are 1. Then from (7.16), (7.17), explicit computations show that

$$S = 2\text{diag}(\mathbf{y})[\Delta(\text{diag}(\mathbf{y})V\text{diag}(\mathbf{y}))]\text{diag}(\mathbf{y})$$

Moreover,

$$\text{diag}(\mathbf{y})\text{Udiag}(\mathbf{y}) = \text{diag}(\mathbf{y})[\mathbf{y}\mathbf{y}' + \sigma\text{diag}(\mathbf{y})W_s\text{diag}(\mathbf{y})]\text{diag}(\mathbf{y})$$

$$= 1\mathbf{1}' + \sigma W_s$$

Therefore

(7.18) \hspace{1cm} \Delta[\text{diag}(\mathbf{y})\text{Udiag}(\mathbf{y})] = n\left(I_{n\times n} - \frac{1}{n}1\mathbf{1}'\right) + \sigma\Delta[W_s].

The matrix $n\left(I_{n\times n} - \frac{1}{n}1\mathbf{1}'\right)$ has rank $(n - 1)$ and two distinct eigenvalues: 0 and $n$. The eigenvalue 0 has multiplicity 1 and $n$ has multiplicity $(n - 1)$. Note that 0 is also an eigenvalue of the matrix $\Delta[W_s]$, and the $n$-dimensional vector $1$ is an eigenvector with respect to the eigenvalue 0 for both the matrix $n\left(I_{n\times n} - \frac{1}{n}1\mathbf{1}'\right)$ and the matrix $\Delta[W_s]$. Therefore the matrix (7.18) is positive definite if

(7.19) \hspace{1cm} \sigma\|\Delta[W_s]\| \leq n,

where $\| \cdot \|$ is the spectral norm of a matrix defined to be the largest modulus of its eigenvalues. We have, by the triangle inequality,

(7.20) \hspace{1cm} \|\Delta W_s\| \leq \max_{i\in[n]} \left| \sum_{j=1}^{n} W_s(i,j) \right| + \|W_s\|.

The matrix $W_s$ is a standard GOE matrix. Recall the following proposition about the largest eigenvalue of the standard GOE matrix.
Proposition 7.6. \textit{(Tracy-Widom \cite{9})} Let $\lambda_{\text{max}}^{(n)}$ be the largest eigenvalue of an $n \times n$ GOE matrix, then when $n$ is large,

$$\lambda_{\text{max}}^{(n)} \sim \sqrt{2n} + \frac{n^{-\frac{1}{6}} \xi_1}{\sqrt{2}}$$

where $\xi_1$ is a random variable independent of $n$ with the GOE Tracy-Widom distribution.

By Proposition 7.6, for any fixed $\delta > 0$,

$$\lim_{n \to \infty} P(\|W_s\| \geq (1 + \delta)\sqrt{2n}) = 0.$$ \hfill (7.21)

Now we consider the distribution of $\max_{i \in [n]} \left| \sum_{j=1}^{n} W_s(i, j) \right|$. Since we require that $W_s$ is a symmetric matrix, the identically distributed Gaussian random variables $\left\{ \sum_{j=1}^{n} W_s(i, j) \right\}_{i \in [n]}$ are no longer independent. In our case the random vector $\left( \sum_{j=1}^{n} W_s(1, j), \ldots, \sum_{j=1}^{n} W_s(n, j) \right)$ is a Gaussian random vector with mean 0 and covariance matrix given by

$$\Sigma = \begin{pmatrix}
    n & 1 & \ldots & 1 \\
    1 & n & \ldots & 1 \\
    \vdots & \vdots & \ddots & \vdots \\
    1 & \ldots & 1 & n
\end{pmatrix}$$

We have the following proposition

Proposition 7.7. \textit{(Theorem 2.2 of \cite{3})} Consider a triangular array of normal random variables $\xi_{n,i}$, $i = 1, 2, \ldots$, and $n = 1, 2, \ldots$, such that for each $n$, $\{\xi_{n,i}, i \geq 0\}$ is a stationary normal sequence. Assume $\xi_{n,i} \sim N(0, 1)$. Let $\rho_{n,j} := E(\xi_{n,i}, \xi_{n,i+j})$ and assuming that

$$(1 - \rho_{n,j}) \log n \to \delta_j \in (0, \infty], \text{ for all } j \geq 1 \text{ as } n \to \infty$$

Assume that there exist positive integers $l_n$ satisfying $l_n = o(n)$ and for which

$$\lim_{n \to \infty} \sup_{j \geq n} |\rho_{n,j}| \log n = 0.$$  

and

$$\lim_{m \to \infty} \lim_{n \to \infty} \sup_{j=m} \sum_{j=m}^{l_n} n \frac{1-\rho_{n,j}}{1+\rho_{n,j}} \frac{\rho_{n,j}}{(\log n)^{\frac{1}{2}}} = 0$$

Then

$$\lim_{n \to \infty} \Pr(\max_{1 \leq i \leq n} \xi_{n,i} \leq u_n(x)) = \exp[-\theta \exp(-x)]$$

where

$$u_n(x) = \frac{x}{a_n} + b_n;$$

and

$$a_n = \sqrt{2 \log n}$$
$$b_n = \sqrt{2 \log n} + \frac{\log \log n + \log 4\pi}{2\sqrt{2 \log n}}$$

and $\theta \in [0, 1]$ is a constant. In particular $\theta = 1$ if $\delta_j = \infty$ for all $j \geq 1$. 

By Proposition 7.7,

$$\max_{i \in [n]} \left| \sum_{j=1}^{n} W_s(i, j) \right| \geq (1 - \delta) \sqrt{2 \log n \left[ \max_{i \in [n]} \left[ \text{Var} \left( \sum_{j \in [n]} W_s(i, j) \right) \right] \right]}$$

$$= (1 - \delta) \sqrt{2n \log n}$$

with probability $1 - o_n(1)$. Moreover, by Lemma 3.6, for any fixed $\epsilon > 0$

$$\max_{i \in [n]} \left| \sum_{j=1}^{n} W_s(i, j) \right| \leq (1 + \epsilon) \sqrt{2n \log n}$$

with probability $1 - o(1)$.

Then Theorem 2.5 follows from (7.19), (7.20), (7.21) and (7.22).

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