Resampling-based Confidence Intervals for Model-free Robust Inference on Optimal Treatment Regimes

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Abstract
Recently, there has been growing interest in estimating optimal treatment regimes which are individualized decision rules that can achieve maximal average outcomes. This paper considers the problem of inference for optimal treatment regimes in the model-free setting, where the specification of an outcome regression model is not needed. Existing model-free estimators are usually not suitable for the purpose of inference because they either have non-standard asymptotic distributions, or are designed to achieve fisher-consistent classification performance. This paper first studies a smoothed robust estimator that directly targets estimating the parameters corresponding to the Bayes decision rule for estimating the optimal treatment regime. This estimator is shown to have an asymptotic normal distribution. Furthermore, it is proved that a resampling procedure provides asymptotically accurate inference for both the parameters indexing the optimal treatment regime and the optimal value function. A new algorithm is developed to calculate the proposed estimator with substantially improved speed and stability. Numerical results demonstrate the satisfactory performance of the new methods.

Keywords: Confidence interval; Individualized treatment rule; Inference; Optimal treatment regime; Weighted bootstrap.

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1 Introduction

Applications in clinical medicine, public policy, internet marketing and other scientific areas often involve seeking for an individualized treatment rule (or regime, policy) to maximize the potential benefit. For example, Gail and Simon [1985] and Zhang et al. [2012] observed that younger patients with primary operable breast cancer and lower PR levels are likely to benefit more from the treatment L-phenylalanine mustard and 5-fluorouracil (PF) rather than from PF plus tamoxifen (PFT). Several successful estimation strategies have been developed, including Q-learning [Watkins and Dayan, 1992, Murphy, 2005a, Chakraborty et al., 2010, Qian and Murphy, 2011, Song et al., 2015], A-learning [Robins et al., 2000, Murphy, 2003, 2005b, Moodie and Richardson, 2010, Shi et al., 2018], model-free methods [Robins et al., 2008, Orellana and Robins, 2010, Zhang et al., 2012, Zhao et al., 2012, 2015, Athey and Wager, 2017, Linn et al., 2017, Zhou et al., 2017, Zhu et al., 2017a, Wang et al., 2018, Qi et al., 2018, Lou et al., 2018], tree or list-based methods [Laber and Zhao, 2015, Cui et al., 2017, Zhu et al., 2017b, Zhang et al., 2018], targeted learning ensembles approach [Díaz et al., 2018], among others.

Although there exists a rich literature on estimation, the associated inference problem has not been studied until recently. In this setting, there are two separate but related inference targets: one is the parameter $\beta_0$ indexing the theoretically optimal treatment regime and the other is the theoretically optimal value function $V(\beta_0)$. The former inference problem aims to quantify the importance of different predictors on making an optimal treatment decision, while the latter constructs a confidence interval for the maximally achievable expected performance which can be used as a gold standard to evaluate alternative treatment regimes.

For Q-learning, several inference methods have been investigated: Laber et al. [2010] proposed a novel locally consistent adaptive confidence interval for $\beta_0$, Chakraborty et al. [2013] proposed a practically convenient adaptive $m$-out-of-$n$ bootstrap for inference on $\beta_0$, Chakraborty et al. [2014] introduced a double bootstrap approach for inference for $V(\beta_0)$, Song et al. [2015] considered inference for $\beta_0$ based on the asymptotic distribution theory for penalized Q-learning. Recently, Jeng et al. [2018] developed Lasso-based procedure for inference on $\beta_0$ in the A-learning framework. However, accurate inference based on Q-learning and A-learning needs reliable model specification. Luedtke and Van Der Laan [2016] developed interesting theory for inference for $V(\beta_0)$ under exceptional laws. Their approach requires to estimate the conditional treatment effect either based on a working model or in a completely nonparametric fashion.
Different from the aforementioned literature, we aim to develop a model-free approach for making inference for both $\beta_0$ and $V(\beta_0)$. This would be useful to alleviate the sensitivity of inference with respect to the underlying generative model, the specification of which is often challenging in real data analysis. Despite the recent progress in robust estimation for optimal treatment regimes, there are several obstacles to construct a confidence interval for $\beta_0$ in a model-free fashion using existing robust estimators, see Section 2.2 for detailed discussions. This paper first proposes a smoothed model-free estimator for the optimal treatment regime and introduce a proximal algorithm which substantially improves both the computational speed and the accuracy. We prove that the smoothed robust estimator has an asymptotic normal distribution and converges to $\beta_0$ with a rate that can be made arbitrarily close to $n^{-1/2}$. We rigorously justify the validity of a resampling approach for inference.

The remaining of the paper is organized as follows. Section 2 provides the notation, motivations and an introduction to the new method with a new algorithm. Section 3 carefully studies the statistical properties for estimation and inference. Section 4 reports the results from Monte Carlo simulations. Section 5 analyzes a clinical data set from the Childhood Adenotonsillectomy Trial (CHAT). Section 6 concludes with some discussions. The appendix gives the technical assumptions and presents several useful lemmas, while the detailed technical derivations are given in the supplemental file.

2 Proposed Methods

2.1 Problem Setup

Let $A$ be a binary variable (0 or 1) denoting the treatment. For each subject, we observe a vector of covariates $x \in \mathbb{R}^p$ and an outcome $Y \in \mathbb{R}$. Assuming without loss of generality that larger outcome is preferred. To evaluate the treatment effect, we adopt the potential or counterfactual outcome framework [Neyman, 1990, Rubin, 1978] for causal inference. Let $Y_{1}^*$ and $Y_0^*$ be the potential outcome had the subject received treatment 1 and 0, respectively. In reality, we observe either $Y_{1}^*$ or $Y_0^*$, but never both. It is assumed that the observed outcome is the potential outcome corresponding to the treatment the subject actually receives (consistency assumption in causal inference), that is $Y = Y_{1}^*A + Y_0^*(1 - A)$. Assume $A$ and $\{Y_0^*, Y_1^*\}$ are independent conditional on $x$, that is, no unmeasured confounding. In addition, we assume
that the stable unit treatment value assumption [Rubin, 1986] and the positivity assumption are both satisfied, where the former requires a subject’s outcome from receiving a treatment is not influenced by the treatment received by other subjects and the latter requires that \(0 < P(A = a|x) < 1, \forall x\), almost surely.

An individualized treatment rule or a treatment regime, denoted by \(d(x)\), is a mapping from the space of covariates to the set of treatment options \(\{0, 1\}\). Let \(Y^*(d)\) be the potential outcome had a subject with covariates \(x\) received the treatment assigned by \(d(x)\). We have

\[
Y^*(d) = Y^*_1 d(x) + Y^*_0 (1 - d(x)).
\]

Given a collection \(\mathcal{D}\) of treatment regimes, the optimal treatment regime \(\arg\max_{d \in \mathcal{D}} \mathbb{E}(Y^*(d))\) leads to the maximal average outcome if being implemented in the population.

In practice, it is often desirable to have an interpretable treatment regime. Here, we focus on the popular class of index rules, given by \(\mathcal{D} = \{I(x^T \beta > 0) : \beta \in \mathbb{B}\}\), where \(I(\cdot)\) is the indicator function and \(\mathbb{B}\) is a compact subset of \(\mathbb{R}^p\). For a given \(\beta \in \mathbb{B}\), we sometimes write the corresponding treatment regime \(I(x^T \beta > 0)\) as \(d_\beta(x)\) or \(d_\beta\) for simplicity. The value function \(V(\beta) = \mathbb{E}(Y^*(d_\beta))\) measures the effectiveness of the treatment regime \(d_\beta\). We are interested in estimating the parameter indexing the optimal rule

\[
\beta_0 = \arg\max_{\beta \in \mathbb{B}} V(\beta).
\]

For identifiability, we assume that there exists a covariate whose conditional distribution given the other covariates is absolutely continuous and its coefficient is normalized to have absolute value one. The existence of such a covariate is satisfied in many real applications. Without loss of generality (one can rearrange the labels of the predictors), we assume \(x_1\) is a predictor that satisfies the condition. We write \(\beta = (\beta_1, \beta^T) \in \mathbb{R}^p\). Correspondingly, we write \(x = (x_1, x^T)^T\). This identifiability condition has been popular used for index models. An alternative identifiability condition is to assume \(\beta_0\) has euclidean norm equal to one, which requires \(\beta_0\) to be a boundary point of a unit sphere and generally leads to more involved technical arguments [Zhu and Xue, 2006].
2.2 Challenges of inference based on existing robust estimators

It is known that the parameter indexing the optimal treatment regime $\beta_0$ corresponds to the parameter of the Bayes rule of a weighted classification problem. This is due to the important observation [Qian and Murphy, 2011, Zhang et al., 2012, Zhao et al., 2012] that the value function $V(\beta)$ can be equivalently expressed as

$$V(\beta) = \mathbb{E}\left[ \frac{Y}{\pi(A, x)} I\{A = d_\beta(x)\} \right],$$

where $\pi(A, x) = P(A = 1|x)$ is the propensity score of the treatment and is equal to 0.5 in a randomized trial. Expression (3) is the foundation for robust or policy-search estimators for optimal treatment regime, which aim to alleviate the practical difficulty of specifying a reliable generative regression model, which describes not only how the treatment influences the outcome but also how the treatment and the covariates interact.

A robust estimator can be obtained by directly maximizing an unbiased sample estimator of the expectation in (3), which was the approach in Zhang et al. [2012]. In a randomized trial, based on the observed data $\{(x_i, Y_i, A_i), i = 1, \ldots, n\}$, which are independent copies of $(x, Y, A)$, $V(\beta)$ can be consistently estimated by its sample analog

$$V_n(\beta) = \frac{2}{n} \sum_{i=1}^{n} \{A_i I(x_i^T \beta > 0) + (1 - A_i) I(x_i^T \beta \leq 0)\} Y_i.$$  

(4)

Leaving out the terms in $V_n(\beta)$ that do not depend on $\beta$, we can estimate $\beta_0$ by

$$\arg \max_{\beta \in \mathbb{R}} M_n(\beta) = \arg \max_{\beta \in \mathbb{R}} \frac{2}{n} \sum_{i=1}^{n} (2A_i - 1) I(x_i^T \beta > 0) Y_i.$$  

(5)

However, as revealed in Wang et al. [2018] such a direct estimator for the Bayes rule belongs to a class of nonstandard $M$ estimators. It converges at a cubic-root rate to a nonnormal limiting distribution that is characterized by the maximizer of a centered Gaussian process with a parabolic drift. The nonstandard asymptotics is a consequence of the so-called \textit{sharp-edge effect} [Kim and Pollard, 1990]. Inference based on this approach is challenging due to the nonstandard asymptotics as the naive bootstrap procedure is not consistent.

A fruitful line of research replaces the 0-1 loss in (4) by the the surrogate hinge loss [Zhao et al.,
2012, Zhou et al., 2017, Lou et al., 2018] or a logistic loss [Jiang et al., 2019]. This results in a useful approach which is not computationally convenient but also enjoys guaranteed generalization error bound. However, despite the successful predictive performance, it is challenging to base inference directly on the resulted estimator. As a cost of the surrogate loss, the resulted decision rule is Fisher consistent, that is the sign of the decision function matches that sign\(x^T\beta_0\).

Although it ensures correct optimal decision rule making, it is not guarantee that the parameters indexing the resulted decision rule are estimation consistent for \(\beta_0\), see Lin [2002].

### 2.3 Smoothed Model-free Inference for Optimal Treatment Regime

For clarity of presentation, we assume the data are collected in a randomized trial but the results can be extended to an observational study under relatively mild assumptions, see Section 6 for more discussions. To facilitate inference, we study an alternative estimator which can be considered as a compromise between the two robust estimation approaches described in Section 2.2. Instead of replacing the indicator function with the hinge loss function, we replace it with a smoothed approximation. Formally, we estimate \(\beta_0\) by

\[
\hat{\beta}_n = \arg\max_{\beta \in \mathcal{B}} \bar{M}_n(\beta) = \arg\max_{\beta \in \mathcal{B}} \frac{2}{n} \sum_{i=1}^{n} (2A_i - 1) K\left(\frac{x_i^T \beta}{h_n}\right)Y_i,
\]

where \(K(\cdot)\) is a smoothed approximation to the indicator function, and \(h_n\) is a sequence of smoothing parameter that goes to zero as \(n \to \infty\). The function \(K(\cdot)\) is required to satisfy some general regularity conditions given in Appendix. For example, the cumulative distribution function of standard normal distribution meets all the requirements.

The motivation for the above new estimator is three-fold. First, as \(h_n\) goes to zero at an appropriate rate, the parameter indexing the optimal treatment regime or the Bayes rule can be estimated at a rate arbitrarily close to \(n^{-1/2}\), see Section 3.1. Second, smoothing the indicator function circumvents the aforementioned nonstandard asymptotics and would lead to a feasible bootstrap inference procedure with theoretical guarantee, see Section 3.2. Third, it also alleviates the computational challenge due to nonsmoothness, see Section 2.4 for a new efficient algorithm. Goldberg et al. [2014] proposed SoftMax Q-learning approach to alleviate the nonsmoothness problem in Q-learning but have not explore the associated inference theory.

For inference, we apply a resampling technique called "weighted bootstrap" which assigns
Note that independent and identically distributed positive random weights to each observation. This resampling scheme was proposed in Rubin [1981]. Barbe and Bertail [1995] provided a comprehensive introduction, see also Ma and Kosorok [2005] and Cheng and Huang [2010] for recent interesting developments. The bootstrapped estimate of the smoothed robust estimator is defined as

$$
\hat{\beta}_n^* = \arg \max_{\beta \in B} \hat{M}_n^*(\beta) = \arg \max_{\beta \in B} n^{-1} \sum_{i=1}^{n} r_i (2A_i - 1) K \left( \frac{x_i^T \beta}{h_n} \right) Y_i,
$$

where $r_1, \ldots, r_n$ are random weights satisfying conditions given in Section 3.2. Following notation introduced earlier, let $\hat{\beta}_n^* = (\hat{\beta}_{n1}^*, \hat{\beta}_{n2}^*, \ldots, \hat{\beta}_{np}^*)^T$, where $|\hat{\beta}_{n1}^*| = 1$ and $\hat{\beta}_n^* = (\hat{\beta}_{n2}^*, \ldots, \hat{\beta}_{np}^*)^T$. For $j = 2, \ldots, p$, let $\xi_j^{*(\alpha/2)}$ and $\xi_j^{*(1-\alpha/2)}$ be the $(\alpha/2)$-th and $(1 - \alpha/2)$-th quantile of the bootstrap distribution of $(nh)^{1/2} (\hat{\beta}_n^* - \hat{\beta}_j)$, respectively, where $\alpha$ is a small positive number. We can estimate $\xi_j^{*(\alpha/2)}$ and $\xi_j^{*(1-\alpha/2)}$ from a large number of bootstrap samples. An asymptotic $100(1 - \alpha)$% bootstrap confidence interval for $\beta_{0j}$, $j = 2, \ldots, p$, is given by

$$
\{ \hat{\beta}_j - (nh)^{-1/2} \xi_j^{*(1-\alpha/2)}, \hat{\beta}_j - (nh)^{-1/2} \xi_j^{*(\alpha/2)} \}.
$$

Next, we consider inference for the optimal value. Define

$$
V_n^*(\beta) = \frac{2}{n} \sum_{i=1}^{n} r_i \{ A_i I(x_i^T \beta > 0) + (1 - A_i) I(x_i^T \beta \leq 0) \} Y_i.
$$

Note that $V_n^*(\beta)$ can be considered as a perturbed version of the $V_n$ defined in (4). Let $d_n^{*(\alpha/2)}$ and $d_n^{*(1-\alpha/2)}$ be the $(\alpha/2)$-th and $(1 - \alpha/2)$-th quantile of the bootstrap distribution of $n^{1/2} (V_n^*(\hat{\beta}_n) - V_n(\hat{\beta}_n))$, respectively. An asymptotic $100(1 - \alpha)$% bootstrap confidence interval for $V(\beta_0)$ is

$$
\{ V_n(\hat{\beta}_n) - n^{-1/2} d_n^{*(1-\alpha/2)}, V_n(\hat{\beta}_n) - n^{-1/2} d_n^{*(\alpha/2)} \}.
$$

### 2.4 A Proximal Algorithm

The smoothed robust estimator largely alleviates the computational challenge due to the non-smooth indicator function. However, the objective function is still a nonconvex function of the parameter. Such nonconvexity is inherent to robust estimation of optimal treatment regime [Qian and Murphy, 2011]. We employ a proximal gradient descent algorithm, originally proposed in Nesterov [2007], which applies to a large class of nonconvex problems. In our set-
ting, this algorithm substantially improves the computational speed and can accommodate high-dimensional covariates.

Consider an optimization problem with an objective function $\Phi(\beta)$. Nesterov [2007] assumes that $\Phi(\beta)$ has the decomposition $\Phi(\beta) = f(\beta) + \Psi(\beta)$, over a convex set $Q$, where $f$ is a differentiable function but not necessarily convex, and $\Psi$ is closed and convex on $Q$. In our setting, we take $-\tilde{M}_n(\beta)$ as the $f$ function, and set $\Psi(\beta) \equiv 0$. Following Nesterov [2007], we generate a sequence of iterates $\{\beta(t), t = 0, 1, 2, \ldots\}$ such that

$$
\beta(t) = \arg \min_{\beta \in \mathbb{B}} \left\{ -\tilde{M}_n(\beta^{(t-1)}) - \langle \nabla \tilde{M}_n(\beta^{(t-1)}), \beta - \beta^{(t-1)} \rangle + \alpha_t \|\beta - \beta^{(t-1)}\|^2 + \Psi(\beta) \right\}
$$

$$
= \arg \min_{\beta \in \mathbb{B}} \left\{ -\frac{2}{n} \sum_{i=1}^{n} (2A_i - 1)K'(\frac{x_i^T \beta^{(t-1)}}{h_n}) \frac{x_i^T (\beta - \beta^{(t-1)})}{h_n} Y_i + \alpha_t \|\beta - \beta^{(t-1)}\|^2 \right\},
$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product between two vectors. Observe that the above minimization problem has a closed-form solution

$$
\beta(t) = \beta^{(t-1)} + (n\alpha_t)^{-1} \sum_{i=1}^{n} (2A_i - 1)K'(\frac{x_i^T \beta^{(t-1)}}{h_n}) \frac{x_i Y_i}{h_n}.
$$

Hence the algorithm can be updated efficiently. The algorithm stops when the following criterion is met:

$$
\tilde{M}_n(\beta^{(t)}) < \tilde{M}_n(\beta^{(t-1)}) + \langle \nabla \tilde{M}_n(\beta^{(t-1)}), \beta^{(t)} - \beta^{(t-1)} \rangle - \alpha_t \|\beta^{(t)} - \beta^{(t-1)}\|^2,
$$

where $\alpha_t$ is a sequence of small positive numbers. To choose $\alpha_t$, inspired by Fan et al. [2018], we employ an expanding series, which ensures that the stepsize diminishes during the update process. Details for this algorithm is provided in the supplementary material.

It is worth emphasizing that this algorithm can be easily adapted to the high-dimensional setting by taking $\Psi(\beta)$ as a regularization function, such as the $L_1$ penalty function.
3 Statistical Properties

3.1 Consistency and Asymptotic Normality of the Smoothed Estimator

To lay the foundation for inference, we first present the statistical properties of the smoothed robust estimator \( \hat{\beta}_n \) defined in (6). All the regularity conditions are summarized in Appendix A. Theorem 1 below shows that \( \hat{\beta}_n \) is consistent for the parameter indexing the optimal treatment regime. Comparing with the asymptotic normality result in Theorem 2, the consistency requires very mild conditions and serves as a precursor step for proving asymptotic normality. See Section S3 of the online supplementary material for the proofs of Theorem 1 and Theorem 2.

**Theorem 1.** Under (A1) - (A3) and assume \( K(\cdot) \) satisfies (K1), then \( \hat{\beta}_n = \beta_0 + o_p(1) \).

Recall that for identification, we write \( \beta_0 = (\beta_{01}, \beta_{02})^T \in \mathbb{R}^p \) where \( |\beta_{01}| = 1 \). Similarly, we write \( \hat{\beta}_n = (\hat{\beta}_{n1}, \hat{\beta}_{n2})^T \in \mathbb{R}^p \) where \( |\hat{\beta}_{n1}| = 1 \). With the above consistency result, we have \( P(\hat{\beta}_{n1} = \beta_{01}) \to 1 \) as \( n \to \infty \). In the following, we focus on studying the asymptotic distribution of \( \hat{\beta}_n \). To this end, we introduce some additional notations. Define \( S(z, \tilde{x}) = \mathbb{E}[Y^*_{\tilde{x}} - Y^*_0 | z, \tilde{x}] \), where \( z = \tilde{x}^T \beta_0 \). Note that there is a one-to-one transformation between \((z, \tilde{x})\) and \(x = (x_1, \tilde{x}^T)^T\). Hence, \( S(z, \tilde{x}) \) is a measure of the conditional treatment effect. Let \( S^{(1)}(0, \tilde{x}) \) denote the partial derivative of \( S(z, \tilde{x}) \) with respect to \( z \). Furthermore, we define

\[
D = a_1 \mathbb{E}\{\tilde{x}\tilde{x}^T f(0|\tilde{x})\mathbb{E}(Y_{1\tilde{x}}^2 + Y_{0\tilde{x}}^2 | z = 0, \tilde{x})\}, \quad (11)
\]

\[
Q = a_2 \mathbb{E}\{\tilde{x}\tilde{x}^T f(0|\tilde{x})S^{(1)}(0, \tilde{x})\}, \quad (12)
\]

where \( f(z|\tilde{x}) \) denotes the conditional probability density function of \( z \) given \( \tilde{x} \), \( a_1 = 2 \int \{K'(\nu)\}^2 d\nu \), and \( a_2 = \int \nu K''(\nu) d\nu \), with \( K'(\cdot) \) and \( K''(\cdot) \) denoting the first- and second-derivative of \( K(\cdot) \), respectively.

**Theorem 2.** Assume \( K(\cdot) \) satisfies (K1) - (K3) for some \( b \geq 2 \), \( h_n = o(n^{-1/(2b+1)\}) \) and \( n^{-1}h_n^{-4} = o(1) \). Then under (A1) - (A5),

1. \( \sqrt{n}h_n (\hat{\beta}_n - \beta_0) \to N(0, Q^{-1}DQ^{-1}) \) in distribution as \( n \to \infty \).

2. \( \sqrt{n}\{V_n(\hat{\beta}_n) - V(\beta_0)\} \to N(0, U) \) in distribution as \( n \to \infty \), where \( V_n(\cdot) \) is defined in (4) and \( U = \text{Var}\{Y^*(d_{\beta}_0)\} + \mathbb{E}\{(Y^*(d_{\beta}_0))^2\} \).
Remark 1. Theorem 2 implies that $\hat{\beta}_n$ achieves a convergence rate arbitrarily close to $n^{-\frac{b}{2b+1}}$. If we take $K(\cdot)$ to be the cumulative distribution function of the standard normal distribution, then (K1) – (K3) are satisfied with $b = 2$, and the corresponding smoothed estimator converges at a rate arbitrarily close to $n^{-2/5}$. With a carefully designed $K(\cdot)$ function which satisfied (K1) – (K3) with $b$ sufficiently large, the convergence rate can be further improved. For example, one such $K(\cdot)$ function satisfying (K1) - (K3) with $b = 4$ is given by $K(v) = \left[0.5 + \frac{105}{104} \left(\frac{v}{5} - \frac{5}{3} \left(\frac{v}{5}\right)^3 + \frac{7}{5} \left(\frac{v}{5}\right)^5 - \frac{3}{7} \left(\frac{v}{5}\right)^7\right)\right]I(-5 \leq v \leq 5) + I(v > 5)$. This function would lead to an $n^{-4/9}$ convergence rate and first appeared in Horowitz [1992], which dealt with smoothing estimator in a different setting. Our setting and proofs are very different. Especially, our proofs substantially simplified the traditional methods for handling a smoothed objective function.

Remark 2. The key components of the proofs are modern empirical process techniques. In particular, we introduce some recent empirical process results [Giné and Sang, 2010, Mason, 2012] on VC classes of functions that involve smoothing parameters, which were originally developed for uniform asymptotics with data-driven bandwidth selection and have not been applied to the types of problems considered here. These new techniques lead to simpler proof and are of independent interest. Our technical derivation for this and other results in the paper employ recent techniques developed by Giné and Sang [2010] and Mason [2012] for VC classes of functions that involve smoothing parameters, see Appendix A. Carefully handling function classes involving a smoothing parameter is nontrivial. The literature usually either impose a lower positive bound on $h$ to avoid the process to blow up or requires more involved computation on the entropy bound for such classes. In contrast, the new techniques are based on a geometric argument and avoid the usually intensive entropy computation.

3.2 Justification for Resampling-based Inference

Let $r_1, ..., r_n$ be a random sample from a distribution of a positive random variable with mean one and variance one. Assume the random weights $r_1, ..., r_n$ are independent of the data. Recall that

$$\hat{\beta}_n^* = \arg\max_{\beta \in \mathcal{B}} \hat{M}_n^*(\beta) = \arg\max_{\beta \in \mathcal{B}} n^{-1} \sum_{i=1}^n r_i (2A_i - 1)K\left(\frac{x_i^T\beta}{h_n}\right)Y_i.$$
Hence, two different sources of randomness contribute to the distribution of $\hat{\beta}_n^*$ in this setup: one due to the random data and the other due to the random weights.

We next provide a rigorous justification for the validity of the bootstrap procedures proposed in Section 2.3. We establish that the bootstrap distribution asymptotically imitates the distribution of the original estimator. Let $r = \{r_1, \ldots, r_n\}$ be the collection of the random bootstrap weights and $w = \{W_1, \ldots, W_n\}$ be the random sample of observations, where $W_i = (x_i, A_i, Y_i)$.

Given a sequence of random variables $R_n$, $n = 1, \ldots, n$, we write $R_n = o_{pr}(1)$ if for any $\epsilon > 0, \delta > 0$, we have $P_w(P_{r|w}(|R_n| > \epsilon) > \delta) \to 0$ as $n \to \infty$. In the bootstrap literature, $R_n$ is said to converge to zero in probability, conditional on the data.

**Theorem 3.** Under (A1) – (A3), (A6) and assume $K(\cdot)$ satisfies (K1), then

1. $\hat{\beta}_n^* = \hat{\beta}_n + o_{pr}(1)$;
2. $\sqrt{n}\{V_n^*(\hat{\beta}_n) - V_n(\hat{\beta}_n)\} = N(0, U) + o_{pr}(1)$.

Part (2) of Theorem 3 suggests that we can use the perturbed value function defined in (9) with the plugged-in estimator $\hat{\beta}_n$ to estimate the asymptotic variance of the estimated optimal value in Theorem 2. This establishes the asymptotic validity of the confidence interval in (10), which allows for inference for the value function. The validity of the confidence interval in (8) for $\beta_0$ is ensured by Theorem 4 below.

**Theorem 4.** Assume $K(\cdot)$ satisfies (K1) – (K3) for some $b \geq 2$, $h_n = o(n^{-1/(2b+1)})$, and $\log(n) = o(nh_n^4)$. Under (A1) - (A6), $\sqrt{n}h_n(\hat{\beta}_n^* - \hat{\beta}_n) = N(0, Q^{-1}DQ^{-1}) + o_{pr}(1)$.

**Remark 3.** The proofs of Theorems 3 and 4 are given in Section S3 of the online supplementary material. We make use of the recent results in which allow for using an unconditional argument to derive conditional results. The use of the unconditional argument can be particularly convenient to combine with the Donsker class properties.

To better understand the behavior of the proposed inference procedure, we also study the properties of the smoothed estimator and its bootstrapped version under a moving parameter or local asymptotic framework. See Section S1 of the online supplementary material.

## 4 Simulation Results

We generate random data from the model $Y = \exp(x^T\eta) + Ax^T\beta + \epsilon$, where $\epsilon \sim N(0, 1)$, $x = (x_0, x_1, x_2, x_3)^T = (x_0, \bar{x}^T)^T$, $x_0 = 1$ and $\bar{x}$ follows a 3-dimensional multivariate normal
distribution with mean zero and identity covariance matrix. We set \( \eta = (-1, -0.5, 0.5, -0.5)^T \), and consider two settings for \( \beta \). In setting 1, we have \( \beta = (-2, -2, 2, 2)^T \); while in setting 2 we have \( \beta = (-2, -2, 2, 0)^T \) with \( x_3 \) being an inactive variable for the optimal treatment regime. The optimal treatment regime is given by \( I(x^T \beta \leq 0) \). As discussed in Section 2.1, for identifiability, we adopt the normalization \( |\beta_1| = 1 \), corresponding to the coefficient of the continuous covariate \( x_1 \). Under this normalization, the population parameter indexing the optimal treatment regime is \( \beta^{opt} = (\beta_0^{opt}, \beta_1^{opt}, \beta_2^{opt}, \beta_3^{opt}) = (-1, -1, 1, 1) \) in setting 1, and \( (-1, -1, 1, 0) \) in setting 2.

We first study the finite sample performance of the smoothed robust estimator in Section 2.3. The smoothed robust estimator is computed using the proximal algorithm in Section 2.4, where we choose \( K(\cdot) \) to be the cumulative distribution function of standard normal distribution and set \( h_n = 0.9 n^{-0.2} \min \{ \text{std}(x_i^T \beta), \text{IQR}(x_i^T \beta)/1.34 \} \), as suggested in Silverman [1986], where “std” denotes the standard deviation function, and “IQR” denotes the interquartile range. We compare with the nonsmooth estimator in (5), which was computed using the genetic algorithm, using the “genoud” function in R package “rgenoud” [Mebane, Jr. and Sekhon, 2011], as suggested in Zhang et al. [2012]. We consider 1000 simulation runs and three different sample sizes \( n = 300, 500, 1000 \) in the simulation experiment. Table 1 reports the bias and standard deviation of the estimate for the parameters indexing the optimal treatment regime, the match ratio (percentage of times the estimated optimal treatment regime matches the true one), and the bias and standard deviation of the estimated optimal value.

The results in Table 1 demonstrates that the smoothed robust estimate has smaller bias and substantially smaller standard deviation comparing with the nonsmooth robust estimator, particular for the smaller sample size setting. It also leads to higher match ratio. In addition, the expected value functions with the true parameter \( \beta^{opt} \) and random policy are simulated via Monte Carlo simulation with \( 10^7 \) replicates; for Setting 1, the optimal value turns out to be 1.14, and the value function with random policy is -0.47; and for Setting 2, the true optimal value is 0.93, and the value function with random policy is -0.29. This implies that our smoothed estimator can also estimate the empirical function values more accurately, relative to the value of random treatment assignment. When taking the computation time into consideration, the non-smoothed estimator requires about 4 seconds for each run, while the smoothed estimator only needs 0.002 seconds. This suggests a substantial reduction in computational costs.

We next investigate the bootstrap confidence interval in Section 2.3. For each data set, we generate positive random weights from a distribution with mean one and variance one. Based
Table 1: Performance of smoothed estimators

| n     | Method     | $\beta_0^{opt}$ | $\beta_1^{opt}$ | $\beta_2^{opt}$ | $\beta_3^{opt}$ | Match Ratio | $V_n(\beta_n)$ |
|-------|------------|-----------------|-----------------|-----------------|-----------------|-------------|----------------|
| Setting 1 |
| 300 Smooth | -0.04 (0.29) | 0 (0) | 0.03 (0.28) | 0.03 (0.31) | 99.51% | -0.01 (0.16) |
| Non-smooth | -0.32 (2.37) | 0 (0) | 0.13 (1.60) | 0.30 (2.51) | 96.31% | 0.06 (0.16) |
| 500 Smooth | -0.02 (0.21) | 0 (0) | 0.00 (0.21) | 0.02 (0.21) | 99.66% | -0.01 (0.13) |
| Non-smooth | -0.15 (0.40) | 0 (0) | 0.05 (0.36) | 0.15 (0.44) | 98.08% | 0.05 (0.13) |
| 1000 Smooth | -0.01 (0.14) | 0 (0) | 0.00 (0.14) | 0.01 (0.14) | 99.90% | 0.00 (0.09) |
| Non-smooth | -0.08 (0.24) | 0 (0) | 0.01 (0.22) | 0.06 (0.26) | 98.90% | 0.04 (0.09) |
| Setting 2 |
| 300 Smooth | -0.05 (0.26) | 0 (0) | 0.04 (0.25) | 0.01 (0.17) | 99.23% | -0.01 (0.15) |
| Non-smooth | -0.33 (1.13) | 0.00 (0.06) | 0.14 (0.96) | 0.09 (0.40) | 95.08% | 1.30 (0.15) |
| 500 Smooth | -0.03 (0.20) | 0 (0) | 0.02 (0.18) | -0.00 (0.14) | 99.57% | 0.00 (0.12) |
| Non-smooth | -0.13 (0.31) | 0 (0) | 0.03 (0.32) | 0.06 (0.23) | 97.62% | 0.05 (0.12) |
| 1000 Smooth | -0.01 (0.13) | 0 (0) | 0.01 (0.13) | 0.00 (0.09) | 99.79% | 0.00 (0.09) |
| Non-smooth | -0.07 (0.21) | 0 (0) | 0.02 (0.21) | 0.04 (0.17) | 98.65% | 0.04 (0.09) |

on 100 bootstrap estimators, we construct 95% bootstrap confidence intervals for the parameters indexing the optimal treatment regime. Table 2 summarizes the empirical coverage rate and the average interval length based on 500 Monte Carlo data sets. Results in Table 2 confirm that the bootstrap confidence intervals have desirable coverage probabilities with reasonable lengths. As sample size increases, the length of the confidence interval decreases significantly. As for computation time, on average one bootstrap run takes less than 0.2 seconds.

Finally, we explore several nonregular settings, where the optimal treatment regimes may be non-unique, motivated by Laber et al. [2010]. In these cases, the parameter indexing the optimal treatment regime is not uniquely identifiable but inference for the optimal value may still be feasible. We focus here on the bootstrap confidence interval for the optimal value. In setting 3, the same data generative model as before is used with $\beta = (1, 2, 0.02, 0)^T$. For setting 4 and 5, $\beta = (-1, 1, 0, 0)^T$, however, the first random covariate $x_1$ is generated from the discrete uniform distribution on the set $\{-1, 0, 1, 2\}$ and $\{1, 2\}$, respectively, instead of the standard normal distribution. For completeness, the bootstrap confidence intervals for the optimal value in setting 1 and setting 2 are also studied.

Let $p$ denote the probability of generating a covariate vector $x$ such that $x^T\beta = 0$. This is a useful measure of the nonregularity of the model [Laber et al., 2010]. According to this measurement, setting 1 – 3 are regular (R) cases with $p = 0$; while setting 4 and 5 are nonregular.
Table 2: Performance of bootstrap confidence intervals for optimal treatment regimes

| n     | Setting 1 |           |           |           |
|-------|-----------|-----------|-----------|-----------|
|       | Coverage Rate | 92.6% | 100% | 93.2% | 91.0% |
|       | Average Length | 1.36 | 0 | 1.26 | 1.38 |
| 300   | Coverage Rate | 92.2% | 100% | 93.0% | 92.6% |
|       | Average Length | 0.81 | 0 | 0.79 | 0.84 |
| 500   | Coverage Rate | 92.6% | 100% | 94.0% | 93.4% |
|       | Average Length | 0.54 | 0 | 0.53 | 0.56 |
| 1000  | Coverage Rate | 93.4% | 100% | 92.6% | 95.8% |
|       | Average Length | 1.12 | 0 | 1.01 | 0.71 |
|       | Coverage Rate | 94.2% | 100% | 93.8% | 94.6% |
|       | Average Length | 0.75 | 0 | 0.72 | 0.51 |
|       | Coverage Rate | 94.0% | 100% | 93.0% | 95.4% |
|       | Average Length | 0.50 | 0 | 0.48 | 0.35 |

(NR) with $p = 0.25$ for setting 4 and $p = 0.5$ for setting 5.

Table 3 summarizes the empirical coverage rate and average length for the 95% bootstrap confidence intervals for the optimal value functions based on 500 Monte Carlo data sets. The results demonstrate that the bootstrap confidence intervals for the optimal value have desirable coverage rates with reasonable interval lengths, even in the nonregular cases. For comparison, we also report the percentage of times these bootstrap confidence would cover the value function from a random policy. The percentage is really low, which implies that the proposed method performs much better than random assignment even in the nonregular cases.

5 A Real Data Example

We analyze a clinical data set from the Childhood Adenotonsillectomy Trial (CHAT). This is a randomized study designed to test whether early adenotonsillectomy (eAT, denoted as treatment 1) is helpful to improve neurocognitive functioning, behavior and quality of life for children with mild to moderate obstructive sleep apnea, compared with watchful waiting plus supportive care (WWSC, denoted as treatment 0), see Marcus et al. [2013]. In this trial, 464 children with mild to moderate obstructive sleep apnea syndrome, ages 5 to 9.9 years, were randomly assigned to eAT and WWSC. Some biochemical and neurocognitive test results were recorded before the
Table 3: Performance of bootstrap confidence intervals for value functions

| n    | Setting Type | 1    | 2    | 3    | 4    | 5    |
|------|--------------|------|------|------|------|------|
| 300  | Coverage Rate| 93.0%| 92.6%| 96.4%| 97.2%| 95.4%|
|      | Average Length| 0.67 | 0.61 | 0.78 | 0.40 | 0.41 |
|      | CR for random policy| 0%  | 0%  | 0%  | 0%  | 31.2%|
| 500  | Coverage Rate| 93.8%| 94.0%| 96.0%| 95.2%| 94.4%|
|      | Average Length| 0.52 | 0.47 | 0.62 | 0.31 | 0.31 |
|      | CR for random policy| 0%  | 0%  | 0%  | 0%  | 12.4%|
| 1000 | Coverage Rate| 93.6%| 95.4%| 97.0%| 96.0%| 96.0%|
|      | Average Length| 0.37 | 0.33 | 0.43 | 0.22 | 0.22 |
|      | CR for random policy| 0%  | 0%  | 0%  | 0%  | 0.8% |

We consider the baseline Apnea-Hypopnea Index (AHI), with a natural log-transformation as recommended by Marcus et al. [2013], as an explanatory variable. AHI is the number of apneas or hypopneas recorded during the study per hour of sleep. It is an important measurement of the quality of sleep and is commonly used by doctors to classify the severity of sleep apnea. Marcus et al. [2013] suggested that black children tend to experience different improvements with eAT comparing with children from other races. We hence include race (binary, 1=African American, 0 for others) as another covariate. For the outcome variable, to balance the benefits and adverse effects from eAT, we adopt a composite score. The composite score uses the ratio of the follow-up AHI and baseline AHI (both with natural log-transformations) as an effective measure of benefit. One the other hand, it takes into account the adverse events documented according to the CHAT study manual of procedures as penalty.

We estimate the optimal treatment regime in the class of treatment regimes $\mathcal{D} = \{I(\beta_0 + \beta_1 \text{AHI} + \beta_2 \text{race} > 0) : |\beta_1| = 1\}$. Table 4 summarizes the estimated coefficients and the 95% bootstrap confidence intervals. The confidence intervals suggest that the coefficients are all significantly different from 0. The analysis suggests that it is reasonable to assign WWSC to those children with milder symptoms (lower AHI). It also suggests that black children display more improvement in the AHI scale with eAT. The results are consistent with those observed empirically in Redline et al. [2011], Marcus et al. [2013] and Dean et al. [2016]. The average outcome with randomized treatment is 0.288. While the estimated average outcome corresponding to the estimated optimal treatment regime is 0.063, with a 95% bootstrap confidence interval
(−0.126, 0.260). This suggested a significant reduction of the composite outcome score when applying the optimal treatment regime.

Table 4: Real Data Analysis

| Variable | Smoothed Estimator | 95% CI     |
|----------|--------------------|-----------|
| Intercept| 0.56               | (0.34, 0.97) |
| AHI      | 1                  | [1, 1]    |
| race     | 0.39               | (0.22, 0.65) |

6 Discussions

We propose a model-free approach for making inference about the parameter indexing the optimal treatment regime and the optimal value when the optimal treatment regime is assumed to belong to a class of feasible decision rules.

Although we focus on a randomized trial, the methods and theory can be extended to observation studies under reasonable assumptions without much difficulty. Assume \( \pi(x) = P(A = 1|x) \) can be modeled as \( \pi(x, \xi) \) where \( \xi \) is a finite-dimensional parameter, such as based on logistic regression. Let \( \hat{\xi} \) be an estimate of \( \xi \). Under the popular assumption of no unmeasured confounding, a smoothed robust estimator for \( \beta_0 \) can be constructed as

\[
\arg \max_{\beta \in \mathcal{B}} \frac{1}{n} \sum_{i=1}^{n} \{ A_i K \left( \frac{x_i^T \beta}{h_{w_i}} \right) + (1 - A_i) (1 - K \left( \frac{x_i^T \beta}{h_{w_i}} \right)) \} Y_i.
\]

The bootstrap inference procedure can be implemented similarly as described in Section 2.3.

It has been observed that the optimal treatment regime may not be unique if there exists a subpopulation who responds similarly to the two treatment options. The complexity of non-regularity may arise, see the discussions in Robins [2004], Moodie and Richardson [2010], Laber et al. [2010] and Luedtke and Van Der Laan [2016]. Uniform inference under non-regularity or exceptional laws is an important and challenging problem. The simulation results show that our bootstrap confidence interval for the optimal value function displays a fair degree of robustness in the two examples where nonregularity occurs. As an example, in simulation setting 5, if \( x_1 = 1 \), then the subject responds the same to the two treatment options; while if \( x_1 = 2 \), the subject benefits from treatment 1. There are four decision rules of interest for this example. The optimal treat-
ment rule is nonunique as one may assign either treatment 0 (say no treatment or a standard, less expensive treatment) or treatment 1 to those subjects with \( x_1 = 1 \). A relative simple approach to breaking the nonuniqueness is to introduce a secondary criterion. For example, one may argue that under the principle of avoiding over-treatment, there exists a unique optimal decision rule of interest, in this case \( I(x_1 = 2) \), which would not assign treatment 1 when ambiguity exists in order to reduce costs and avoid potential risks. Based on the sample, this unique optimal treatment regime can be consistently estimated by selecting the decision rule that maximizes the sample average treatment effect while treating the smallest proportion of the population.

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R. Zhu, Y.-Q. Zhao, G. Chen, S. Ma, and H. Zhao. Greedy outcome weighted tree learning of optimal personalized treatment rules. *Biometrics*, 73(2):391–400, 2017b.
A Regularity Conditions and Useful Lemmas

We first state some regularity conditions, where (K1)–(K3) are assumptions imposed on K(·), while (A1)–(A6) are assumptions imposed on the data.

(K1) K(·) is twice differentiable, K(·), K′(·) and K″(·) all bounded variation on the real line. Furthermore, \( \lim_{\nu \to -\infty} K(\nu) = 0 \) and \( \lim_{\nu \to \infty} K(\nu) = 1 \); \( \int [K′(\nu)]^2 d\nu \) and \( \int [K″(\nu)]^2 d\nu \) are both finite.

(K2) For some integer \( b \geq 2 \), and any \( 1 \leq i \leq b \), \( \int |\nu^i K′(\nu)| d\nu < \infty \); \( \int_{-\infty}^{\infty} \nu^i K′(\nu) d\nu = 0 \) for \( 1 \leq i \leq b - 1 \) and \( \int_{-\infty}^{\infty} \nu^b K′(\nu) d\nu = d \neq 0 \).

(K3) For any integer \( i \) between 0 and \( b \), any \( \eta > 0 \), and any sequence \( \{h_n\} \) converging to 0, \( \lim_{n \to \infty} h_n^{i-b} \int_{|h_n\nu| > \eta} |\nu^i K′(\nu)| d\nu = 0 \), and \( \lim_{n \to \infty} h_n^{-1} \int_{|h_n\nu| > \eta} |K″(\nu)| d\nu = 0 \).

(A1) \( \mu(a, x) \) is bounded for almost all \( x \), and \( a = 0, 1 \); \( Y^*_a - \mu(a, x) \), \( a = 0, 1 \), has a sub-Gaussian distribution for almost every \( x \).

(A2) The support of the distribution of \( x \) is not contained in any proper linear subspace of \( \mathbb{R}^p \). For almost every \( \bar{x} \), the distribution of \( x_1 \) conditional on \( \bar{x} \) has everywhere a positive density. The components of \( \bar{x} \) are bounded by \( M_x \).

(A3) Let \( S(z, \bar{x}) = E\{Y^*_a - Y^*_0 | z, \bar{x}\} \), where \( z = x^T \beta_0 \). For almost every \( \bar{x} \), \( S(0, \bar{x}) = 0 \). And for every \( \epsilon > 0 \), \( \sup_{\|\beta - \beta_0\| > \epsilon} E\{I(x^T \beta > 0)S(z, \bar{x})f(z|\bar{x})\} < E\{I(x^T \beta_0 > 0)S(z, \bar{x})f(z|\bar{x})\} \).

(A4) Given any integer \( 0 \leq i \leq b - 1 \), for all \( z \) in a neighborhood of 0, \( f^{(i)}(z|\bar{x}) \) is a continuous function of \( z \) and satisfies \( |f^{(i)}(z|\bar{x})| < M_f \) for almost every \( \bar{x} \), where \( M_f > 0 \) is a constant.

(A5) Let \( S^{(i)}(0, \bar{x}) \), \( i = 0, 1, \ldots, b \), denote the \( i \)th partial derivative of \( S(z, \bar{x}) \) with respect to \( z \). For \( 0 \leq i \leq b \), for all \( z \) in a neighborhood of 0, \( S^{(i)}(z, \bar{x}) \) is a continuous function of \( z \) and satisfies \( |S^{(i)}(z, \bar{x})| < M_s \) for almost every \( \bar{x} \), where \( M_s > 0 \) is a constant. The matrices \( E\{\bar{x}\bar{x}^T f(0|\bar{x})S^{(1)}(0, \bar{x})\} \) and \(-E\{\bar{x}\bar{x}^T (\bar{x}^T \beta_0) f(0|\bar{x})S^{(1)}(0, \bar{x})\} \) are negative definite.

(A6) The random weights \( r_1, \ldots, r_n \) form a random sample from a distribution of a positive random variable with mean one and variance one. Assume that \( r_i - E(r_i) \) has a sub-Gaussian distribution, \( i = 1, \ldots, n \).
Remark 4. The bounded variation assumption on $K(\cdot)$, $K'(\cdot)$ and $K''(\cdot)$ are relatively weak (Chapter 6, Apostol et al. [1974]). This and other assumptions in (K1)-(K2) are satisfied if $K(\cdot)$ is taken to be the distribution function of standard normal distribution ($b = 2$) or the function in Remark 1 ($b = 4$). However, $K(\cdot)$ is not required to be a cumulative distribution function. The bounded variation assumption implies that $K(\cdot)$, $|K'(\cdot)|$ and $|K''(\cdot)|$ are uniformly bounded. Our assumptions on the data are also relatively mild. Condition (A1) imposes mild assumption on the tail distribution of $Y_{a}^* - \mu(a, x)$, $a = 0, 1$, and allows for both normal distribution and many other nonnormal distributions. Condition (A3) is a margin type condition to ensure identification of $\beta_0$.

Let

$$G = \{A(x^T \beta > 0)Y + (1 - A)I(x^T \beta \leq 0)Y : \beta \in B\},$$

$$G^* = \{(r - 1)\{A(x^T \beta > 0) + (1 - A)I(x^T \beta \leq 0)\}Y : \beta \in B\}.$$ 

It is easy to see $G$ and $G^*$ are both Donsker classes of functions. Next, we state a useful lemma concerning the Donsker properties of several other classes of functions that involve a smoothing parameter, as well as four technical lemmas that are useful for the proof of the main theorems and are proved based on the Donsker properties using empirical processes techniques. Their proofs can be found in the online supplementary material.

**Lemma A1.** Under (K1), (A1)-(A3), the following six classes of functions are Donsker classes.

$$\mathcal{F} = \{(2A - 1)K(x^T \beta/h)Y : \beta \in B, h \in (0, 1]\},$$

$$\mathcal{F}^* = \{r(2A - 1)K(x^T \beta/h)Y : \beta \in B, h \in (0, 1]\},$$

$$\mathcal{H} = \{(2A - 1)K'(\frac{x}{h} + \psi^T \bar{x})\hat{\bar{x}}Y : \psi \in \Psi, h \in (0, 1]\},$$

$$\mathcal{H}^* = \{r(2A - 1)K'(\frac{x^T \beta}{h})\hat{\bar{x}}Y : \beta \in B, h \in (0, 1]\},$$

$$\mathcal{Q} = \{(2A - 1)K''(\frac{x^T \beta}{h})\hat{\bar{x}}^2Y : \beta \in B, h \in (0, 1]\},$$

$$\mathcal{Q}^* = \{r(2A - 1)K''(\frac{x^T \beta}{h})\hat{\bar{x}}^2Y : \beta \in B, h \in (0, 1]\},$$

where $\Psi = \{\psi : \psi \in \mathbb{R}^{p-1}, ||\psi|| \leq \frac{n}{\sqrt{2(\ln n)}}\}$, with $|| \cdot ||$ denoting the $l_2$ norm.

**Lemma A2.** Let $G_i(x_i, \beta, h_n) = (2A_i - 1)K'(\frac{x_i^T \beta}{h_n})Y_i - E\{(2A_i - 1)I(x_i^T \beta > 0)Y_i\}$. Under Assumptions (A1)-(A3) and (K1), sup $|\beta| \in \mathbb{R}$$n^{-1} \sum_{i=1}^{n} G_i(x_i, \beta, h_n)| \xrightarrow{p} 0$.

**Lemma A3.** For any $\theta \in \mathbb{R}^{p-1}$, let $R_n(\theta) = \frac{x_n}{n^2} \sum_{i=1}^{n} (2A_i - 1)K'(\frac{x_i^T \beta}{h_n} + \theta^T \bar{x}_i)\hat{\bar{x}}_i Y_i$. let $\eta > 0$ be such that $S^{(1)}(z, \bar{x})$, $S^{(2)}(z, \bar{x})$, and $f^{(1)}(z|\bar{x})$ exist and are uniformly bounded for almost every $\bar{x}$
if $|z| \leq \eta$. Define $\Theta_n = \{\theta : \theta \in \mathbb{R}^{p-1}, h_n||\theta|| \leq \frac{\eta}{2\sqrt{p-1}M_s}\}$. Assume the conditions of Theorem 2 are satisfied, then (1) $\sup_{\theta \in \Theta_n} ||R_n(\theta) - ER_n(\theta)|| \xrightarrow{p} 0$. (2) There are finite numbers $\alpha_1$ and $\alpha_2$ such that for all $\theta \in \Theta_n$, $||ER_n(\theta) - Q\theta|| \leq o(1) + \alpha_1 h_n||\theta|| + \alpha_2 h_n||\theta||^2$ uniformly over $\theta \in \Theta_n$.

Lemma A4. Define $G^*_i(x_i, \beta, h_n) = (2A_i - 1)r_iK(\frac{x_i^T\beta}{h_n})Y_i - E\{r_i(2A_i - 1)I(x_i^T\beta > 0)Y_i\}$. Under Assumptions (A1)-(A3) and (K1), $\sup_{\beta \in B} \left| n^{-1} \sum_{i=1}^{n} G^*_i(x_i, \beta, h_n) \right| = o_{p_{rw}}(1)$, where $o_{p_{rw}}(1)$ denotes a random sequence that converges to zero in probability with respect to the joint distribution of $(r, w)$.

Lemma A5. Assume the conditions of Theorem 4 are satisfied, then $(nh_n)^{1/2}\{T^*_n(\hat{\beta}_n; h_n) - T^*_n(\beta_0; h_n)\} = o_{p_{rw}}(1)$, where $T^*_n(\beta, h_n)$ is defined as follows:

$$T^*_n(\beta; h_n) = \frac{\partial \widehat{M}^*_n(\beta, h_n)}{\partial \beta} = \frac{2}{n} \sum_{i=1}^{n} r_i(2A_i - 1)K(\frac{x_i^T\beta}{h_n}) \widehat{x}_i Y_i.$$

Supplementary material

The supplementary material is constructed as follows. In Section S1, we study the properties of the smoothed estimator and its bootstrapped version under a moving parameter or local asymptotic framework. Section S2 states a preliminary lemma which validates the idea of estimating the value function $V(\beta)$ by its sample analog. Section S3 includes all proofs of Theorem 1–4 in the paper. Section S4 and Section S5 present the proofs of technical lemmas mentioned in the paper appendix and Section S3, respectively. Section S6 provides the proofs of Theorem 5 and Theorem 6 shown in Section S1. Finally, Section S7 presents the pseudo codes for the proximal algorithm we proposed in Section 2.4 of the main paper.

S1 Moving Parameter Asymptotics

To better understand the behavior of the proposed inference procedure, we study the properties of the smoothed estimator and its bootstrapped version under a moving parameter or local asymptotic framework, as motivated by Laber et al. [2010].

Consider the following semiparametric model

$$Y = \mu(x) + x^T(\beta_0 + b_n s)xA + \epsilon,$$  \hspace{1cm} (S1)
Theorem 6. Assume \( \epsilon \) is a sub-Gaussian random error term with mean zero and variance \( \sigma^2 \). The local model (S1) perturbs \( \beta_0 = (\beta_{01}, \beta_0) \) (with \( |\beta_{01}| = 1 \)) by a small quantity \( b_n s \), with \( b_n \) being a sequence of real numbers that converges to zero as \( n \rightarrow \infty \) and \( s = (s_1, \tilde{s}) \) is a fixed \( p \)-dimensional vector. We write \( s = (s_1, \tilde{s}) \) and assume \( s_1 = 0 \) to avoid complications that are not relevant to the main results. When \( b_n = 0 \), the optimal treatment regime is given by \( I(x^T \beta_0 > 0) \).

Consider a random sample \( \{(x_i, A_i, Y_i), i = 1, ..., n\} \) from (S1). We estimate \( \beta_0 \) by the smooth robust estimator introduced in Section 2.3, that is, \( \hat{\beta}_n = \arg \max_{\beta \in \mathbb{R}^p} n^{-1} \sum_{i=1}^{n} (2A_i - 1) K(\frac{x_i^T \beta}{h_n}) Y_i \). Correspondingly, the confidence interval is constructed using the formula in (8) based on the bootstrapped estimator \( \hat{\beta}^*_n = \arg \max_{\beta \in \mathbb{R}^p} n^{-1} \sum_{i=1}^{n} r_i (2A_i - 1) K(\frac{x_i^T \beta}{h_n}) Y_i \). That is, we study the behavior of the procedures proposed earlier which are constructed in a model-free fashion when the underlying data are generated by (S1). To study the local asymptotics, define

\[
D_0 = 2a_1 E\{\tilde{x}x^T f(0|\tilde{x}) (E[\mu^2(x)|z = 0, \tilde{x}] + \sigma^2)\} \quad \text{and} \quad Q_0 = a_2 E\{\tilde{x}x^T f(0|\tilde{x})\},
\]

where \( a_i \) \((i = 1, 2)\) is defined in Section 3.1. As before, write \( \hat{\beta}_n = (\hat{\beta}_{n1}, \hat{\beta}_n)^T \in \mathbb{R}^p \) and \( \hat{\beta}^*_n = (\hat{\beta}^*_{n1}, \hat{\beta}^*_n)^T \).

The following two theorems show that asymptotic normality holds for \( \hat{\beta}_n \) and \( \hat{\beta}^*_n \) for \( b_n \) chosen at appropriate rate. If the sequence \( b_n \) goes to zero faster that \( (nh_n)^{-1/2} \), the smoothed estimator is asymptotically unbiased and the bootstrap confidence interval for \( \beta_0 \) is asymptotically accurate. The proofs of these results can be found in Section S6.

**Theorem 5.** Assume \( K(\cdot) \) satisfies (K1) - (K3) for some \( b \geq 2 \), \( h_n = o(n^{-1/(2b+1)}) \) and \( n^{-1} h_n^{-4} = o(1) \). If \( b_n = (nh_n)^{-1/2} \), then under (A1), (A2), (A4),

\[
\sqrt{nh_n}(\hat{\beta}_n - \beta_0) \rightarrow N\left( -a_2^{-1} \tilde{s}, Q_0^{-1} D_0 Q_0^{-1} \right)
\]

in distribution as \( n \rightarrow \infty \).

**Theorem 6.** Assume \( K(\cdot) \) satisfies (K1) - (K3), for some \( b \geq 2 \), \( h_n = o(n^{-1/(2b+1)}) \), and \( \log(n) = o(n h_n^{-4}) \). If \( b_n = (nh_n)^{-1/2} \), then under (A1), (A2), (A4), (A6),

\[
\sqrt{nh_n}(\hat{\beta}^*_n - \beta_n) = N\left( -a_2^{-1} \tilde{s}, Q_0^{-1} D_0 Q_0^{-1} + o_p(1) \right).
\]
S2 A Preliminary Lemma

Lemma 1. \( E\{[I(A = 1)I(x^T\beta > 0) + I(A = 0)I(x^T\beta \leq 0)]Y]\} = \frac{1}{2}V(\beta). \)

Proof of Lemma 1: By the iterative expectation formula,

\[
E\{[I(A = 1)I(x^T\beta > 0) + I(A = 0)I(x^T\beta \leq 0)]Y]\}
= E_{A,x}\{[I(A = 1)I(x^T\beta > 0) + I(A = 0)I(x^T\beta \leq 0)]E(Y|A, x)\}
= E_{A,x}\{I(A = 1)I(x^T\beta > 0)\mu(1, x) + I(A = 0)I(x^T\beta \leq 0)\mu(0, x)\}
= \frac{1}{2}E_x\{I(x^T\beta > 0)\mu(1, x) + I(x^T\beta \leq 0)\mu(0, x)\} = \frac{1}{2}V(\beta).
\]

\[\square\]

S3 Proof of Theorems 1–4

Proof of Theorem 1: We observe that \( \hat{\beta}_n \) maximizes \( \widehat{M}_n(\beta, h_n) \) over \( \beta \in \mathbb{B} \). Lemma A2 implies that \( \sup_{\beta \in \mathbb{B}} |\widehat{M}_n(\beta, h_n) - M(\beta)| \to 0 \) in probability as \( n \to \infty \), where \( M(\beta) = E\{(2A_i - 1)I(x_i^T\beta > 0)Y_i\} \). Condition (A3) implies that for every \( \epsilon > 0 \), \( \sup_{\|\beta - \beta_0\| > \epsilon} M(\beta) < M(\beta_0) \). Hence, \( \hat{\beta}_n \) is consistent by Theorem 5.7 in van der Vaart [2000].

The asymptotic distribution of \( \hat{\beta}_n \) depends critically on the properties of the gradient and the Hessian matrix of the objective function, \( \widehat{M}_n(\beta, h_n) \). Define

\[
T_n(\beta; h_n) = \frac{\partial \widehat{M}_n(\beta, h_n)}{\partial \beta} = \frac{2}{n} \sum_{i=1}^{n} (2A_i - 1)K'(\frac{x_i^T\beta}{h_n}) \tilde{x}_i Y_i, \quad (S2)
\]

\[
Q_n(\beta; h_n) = \frac{\partial^2 \widehat{M}_n(\beta, h_n)}{\partial \beta R} = \frac{2}{n} \sum_{i=1}^{n} (2A_i - 1)K''(\frac{x_i^T\beta}{h_n}) \tilde{x}_i \tilde{x}_i^T Y_i, \quad (S3)
\]

Lemmas 2 and 3 below establish useful properties of \( T_n(\beta; h_n) \) and \( Q_n(\beta; h_n) \), respectively. The proofs of these two lemmas are given in Section S5.
Proof of Theorem 2: By Taylor expansion, we have

\[ \beta_n \text{ where } \lim_{n \to \infty} E\{(nh_n)^{1/2}T_n(\beta_0; h_n)\} = 0, \quad \text{and} \quad \lim_{n \to \infty} \text{Var}\{(nh_n)^{1/2}T_n(\beta_0; h_n)\} = D. \]

Lemma 3. Let \( \beta_n^r \) be any value between \( \hat{\beta}_n \) and \( \beta_0 \). Assume the conditions of Theorem 2 are satisfied, then \( Q_n(\beta_n^r; h_n) \xrightarrow{p} Q \), where \( Q \) is defined in (12).

Let \( V''(\cdot) \) be the Hessian matrix of \( V(\cdot) \) with respect to \( \tilde{\beta} \), i.e., \( \frac{\partial^2 V(\beta)}{\partial \beta \partial \beta^T} \). Lemmas 4 below describes the continuity of \( V''(\cdot) \). The proof is given in Section S5.

Lemma 4. Let \( \beta_n^r \) be any value between \( \hat{\beta}_n \) and \( \beta_0 \). Assume the conditions of Theorem 2 are satisfied, then \( V''(\beta_n^r) \xrightarrow{p} I_V \), where \( I_V = -E\{S^{(1)}(0, \tilde{x}) f(0|\tilde{x}) \tilde{x} \tilde{x}^T (\tilde{x}^T \tilde{\beta}_0)\} \).

Proof of Theorem 2: By Taylor expansion, we have

\[ T_n(\hat{\beta}_n; h_n) = T_n(\beta_0; h_n) + Q_n(\beta_n^r; h_n)(\hat{\beta}_n - \beta_0), \]

where \( \beta_n^r \) is between \( \hat{\beta}_n \) and \( \beta_0 \). The definition of \( \hat{\beta}_n \) implies that \( T_n(\hat{\beta}_n; h_n) = 0 \), where \( 0 \) denotes a \( (p - 1) \)-dimensional vector of zeroes. Lemma 3 indicates that

\[ \hat{\beta}_n - \beta_0 = (-Q + o_p(1))^{-1} T_n(\beta_0; h_n). \]

To prove the theorem, it suffices to verify \( (nh_n)^{1/2}T_n(\beta_0; h_n) \xrightarrow{d} N(0, D) \). It is known from Lemma B1 that \( E(nh_n)^{1/2}\{T_n(\beta_0; h_n)\} \to 0 \). It is sufficient to prove that \( (nh_n)^{1/2} \gamma^T \{T_n(\beta_0; h_n) - ET_n(\beta_0; h_n)\} \) is asymptotically \( N(0, \gamma^T D \gamma) \) for any constant vector \( \gamma \in \mathbb{R}^{p-1} \) such that \( ||\gamma|| = 1 \). Let

\[ q_i = 2(2A_i - 1)(nh_n)^{1/2}K'(\tilde{x}_i^T \beta_0) \frac{\gamma^T \tilde{x}_i}{h_n} Y_i. \]

It follows the proof of Lemma 2 that \( \lim_{n \to \infty} \text{Eq}_i = 0 \), and \( \lim_{n \to \infty} \text{Eq}_i^2/n = \gamma^T D \gamma \).

To apply Lyapunov central limit theorem, we will verify that

\[ \lim_{n \to \infty} \left( s_n^4 \right)^{-1} \sum_{i=1}^{n} \text{E}\{(q_i - \text{Eq}_i)^4\} = 0, \quad (S4) \]

where \( \lim_{n \to \infty} (n^{-2}s_n^2) = \lim_{n \to \infty} \sum_{i=1}^{n} \text{Var}(n^{-1}q_i) = \gamma^T D \gamma \). We observe that the left-side of \( (S4) \) is bounded from above (up to a positive constant) by \( \lim_{n \to \infty} n^{-3}\text{E}(q_i^4) + \lim_{n \to \infty} n^{-3}(\text{Eq}_i)^4 = I_1 + I_2 \). As
(E_{q_i})^4 \rightarrow 0$, we have $I_2 = o(1)$. To evaluate $I_1$, note that

$$n^{-3}E(q_i^4) = 16(nh_n^2)^{-1}E\{K'(x_i^T \beta \overline{h}_n)^{4}(\gamma^T \tilde{x}_i)^{4}Y_i^{4}\}.$$ 

Since $Y$ has sub-Gaussian tail, then for any integer $k \geq 1$, $E|Y|^k$ is finite. So with the boundedness of $K(\cdot)$ and $\tilde{x}$, $E\{K'(x_i^T \beta)\overline{h}_n)^{4}(\gamma^T \tilde{x}_i)^{4}Y_i^{4}\}$ is finite. Then $n^{-1}h_n^{-2} = o(1)$ implies $I_1 = o(1)$. Therefore, the Lyapunov condition is satisfied. This proves (1).

To prove (2), we observe that

$$\sqrt{n}(V_n(\tilde{\beta}_n) - V(\beta_0)) = \sqrt{n}\{V_n(\beta_0) - V(\beta_0)\} + \sqrt{n}\{V_n(\tilde{\beta}_n) - V_n(\beta_0) - V(\tilde{\beta}_n) + V(\beta_0)\}$$

$$+ \sqrt{n}\{V(\tilde{\beta}_n) - V(\beta_0)\}$$

$$= I_1 + I_2 + I_3,$$

where the definition of $I_i$ ($i = 1, 2, 3$) is clear from the context. Similarly as the proof for part (2) of Theorem 1 we have $I_1 \overset{d}{\rightarrow} N(0, U)$ and $I_2 = o_p(1)$. Note that with the consistency result in Theorem 1, we have $P(\tilde{\beta}_{n1} = \beta_{01}) \rightarrow 1$ as $n \rightarrow \infty$. By Taylor expansion, we have

$$I_3 = \sqrt{n}(\tilde{\beta}_n - \tilde{\beta}_0)^T V'(\beta_0) + \sqrt{n}(\tilde{\beta}_n - \tilde{\beta}_0)^T V''(\beta^*) (\tilde{\beta}_n - \beta_0)/2 + o_p(1),$$

where $V'(\cdot)$ and $V''(\cdot)$ denote the gradient vector and Hessian matrix of $V(\cdot)$ with respect to $\tilde{\beta}$, respectively; $\beta^*$ is between $\tilde{\beta}_0$ and $\tilde{\beta}_n$. As $\beta_0$ is the maximizer of $V(\cdot)$, we have $V'(-\beta_0) = 0$. Let $\lambda_{max}(\cdot)$ be the eigenvalue with the greatest absolute value. The second term is upper bounded by $|\lambda_{max}(V''(\beta^*))|\sqrt{n}|\tilde{\beta}_n - \tilde{\beta}_0|^2/2$, which is of order $O_p(n^{-1/2}h^{-1}) = o_p(1)$ by Lemma 4, Assumption (A5) and the first part of the theorem on the convergence rate. This proves (2). 

In the rest of this appendix, we will prove the theory for bootstrap based inference. As described in Section 3.2, given a sequence of random variables $R_n, n = 1, \ldots, n$, we write $R_n = o_{p_r}(1)$ if for any $\epsilon > 0, \delta > 0$, we have $P_{w}(P_{r|w}(|R_n| > \epsilon) > \delta) \rightarrow 0$ as $n \rightarrow \infty$. Furthermore, $o_{pw}(1)$ denotes a random sequence that converges to zero in probability with respect to the joint distribution of $(r, w)$; and $o_{pw}(1)$ denotes a random sequence that converges to zero in probability with respect to the distribution of $r$ only. By Lemma 3 of Cheng and Huang [2010], if $R_n = o_{pw}(1)$, then $R_n = o_{p_r}(1)$. In particular, if $R_n$ depends only on the data $w$ but not on the random weights $r$ and if $R_n = o_{pw}(1)$, then it is easy to see $R_n = o_{pw}(1)$, and hence it is $o_{p_r}(1)$. In
this part of proof, we will include subscripts in the probability and expectation to clarify which probability distribution is used in the calculation.

**Proof of Theorem 3:** By definition, \( \hat{\beta}_n^* \) maximizes \( \tilde{M}_n^*(\beta, h_n) \) over \( \beta \in \mathbb{B} \). First, by combining Lemma A2 and Lemma A4 and recognizing that \( E_n \{ (2A_i - 1) I(x_i^T \beta > 0) Y_i \} = E_n E_{\tau |w} \{ r_i (2A_i - 1) I(x_i^T \beta > 0) Y_i \} \), we have \( \sup_{\beta \in \mathbb{B}} | \tilde{M}_n^*(\beta, h_n) - \tilde{M}_n(\beta, h_n) | = o_{prw}(1) \). By Lemma 3 of Cheng and Huang [2010], \( \sup_{\beta \in \mathbb{B}} | \tilde{M}_n^*(\beta, h_n) - \tilde{M}_n(\beta, h_n) | = o_{prw}(1) \). By Theorem 5.7 in van der Vaart [2000], to prove the theorem, it is sufficient to show that for any \( \epsilon > 0 \),

\[
\lim_{n \to \infty} P_w \left( \sup_{||\beta - \beta_n|| > \epsilon} \{ \tilde{M}_n(\beta, h_n) - \tilde{M}_n(\hat{\beta}_n, h_n) \} < 0 \right) = 1. \quad (S5)
\]

By Lemma A2, \( \tilde{M}_n(\beta, h_n) - \tilde{M}_n(\hat{\beta}_n, h_n) = M(\beta) - M(\hat{\beta}_n) + o_{prw}(1) \). Furthermore, the consistency of \( \hat{\beta}_n \) implies that for all sufficiently large \( n \), any \( \beta \) that satisfies \( ||\beta - \hat{\beta}_n|| > \epsilon \) would also satisfy \( ||\beta - \beta_0|| \geq \epsilon/2 \). Condition (A3) implies that \( \sup_{||\beta - \beta_0|| > \epsilon/2} M(\beta) < M(\beta_0) \). Hence, \( (S5) \) holds. This proves (1).

To prove (2), we observe that \( \sqrt{n}(V_n^*(\beta) - V_n(\beta)) = n^{-1/2} \sum_{i=1}^{n} (r_i - 1) \{ (A_i I(x_i^T \beta > 0) + (1 - A_i) I(x_i^T \beta \leq 0)) \} Y_i \), which has mean zero. The Donsker property of the function class \( G^* \) and the fact \( \hat{\beta}_n = \beta_0 + o_{prw}(1) \) implies that

\[
\sqrt{n} \left[ \{ V_n^*(\hat{\beta}_n) - V_n(\hat{\beta}_n) \} - \{ V_n^*(\beta_0) - V_n(\beta_0) \} \right] = o_{prw}(1), \quad (S6)
\]

by Lemma 19.24 of van der Vaart [2000]. By assumption (A6) and the classical central limit theorem, \( \sqrt{n}(V_n^*(\beta_0) - V_n(\beta_0)) = N(0, U) + o_{prw}(1) \). Hence, \( \sqrt{n} \{ V_n^*(\hat{\beta}_n) - V_n(\hat{\beta}_n) \} = N(0, U) + o_{prw}(1) \). Lemma 3 in Cheng and Huang [2010] implies (2) holds.

To prove Theorem 4, we define the following gradient function and Hessian matrix corresponding to the randomly weighted objective function

\[
T_n^*(\beta; h_n) = \frac{\partial \tilde{M}_n^*(\beta, h_n)}{\partial \beta} = \frac{2}{n} \sum_{i=1}^{n} r_i (2A_i - 1) K'(\frac{x_i^T \beta}{h_n}) \frac{\hat{x}_i Y_i}{h_n}, \quad (S7)
\]

\[
Q_n^*(\beta; h_n) = \frac{\partial^2 \tilde{M}_n^*(\beta, h_n)}{\partial \beta \partial \beta^T} = \frac{2}{n} \sum_{i=1}^{n} r_i (2A_i - 1) K''(\frac{x_i^T \beta}{h_n}) \frac{\hat{x}_i x_i^T Y_i}{h_n^2}. \quad (S8)
\]
Lemma 5 below characterizes the asymptotic property of the Hessian matrix. Its proof is given in the supplementary material.

**Lemma 5.** Let $\beta^{sr}_n$ be a variable between $\beta^{s}_n$ and $\beta_n$. Assume the conditions of Theorem 4 are satisfied, then $Q_n^*(\beta^{sr}_n; h_n) = Q + o_p(1).

**Proof of Theorem 4:** By Taylor expansion, $T_n^*(\beta^{s}_n; h_n) = T_n^*(\beta_n; h_n) + Q_n^*(\beta^{sr}_n; h_n)(\beta^{s}_n - \beta_n)$, where $\beta^{sr}_n$ is between $\beta^{s}_n$ and $\beta_n$. By the definition of $\beta^{s}_n$, we have $T_n^*(\beta^{s}_n; h_n) = 0$. By Lemma B4, we have

$$\beta^{s}_n - \beta_n = - (Q + o_p(1))^{-1} T_n^*(\beta_n; h_n).$$

It remains to show $(nh_n)^{1/2} T_n^*(\beta_n; h_n) = N(0, D) + o_p(1)$. By Lemma A5, we only need to show $(nh_n)^{1/2} T_n^*(\beta_0; h_n) = N(0, D) + o_p(1)$. Observe that

$$E_{r|w}\{(nh_n)^{1/2} T_n^*(\beta_0; h_n)\} = (nh_n)^{1/2} T_n(\beta_0; h_n),$$

$$\text{Var}_{r|w}\{(nh_n)^{1/2} T_n^*(\beta_0; h_n)\} = 4 \sum_{i=1}^n \left\{ K'(x_i^T \beta_0 \frac{\partial}{\partial h_n}) \right\}^2 \frac{x_i x_i^T}{h_n} Y_i^2.$$

Lemma 2 implies that

$$\lim_{n \to \infty} E_{w} E_{r|w}\{(nh_n)^{1/2} T_n^*(\beta_0; h_n)\} = 0, \quad \text{and} \quad \lim_{n \to \infty} E_{w}\text{Var}_{r|w}\{(nh_n)^{1/2} T_n^*(\beta_0; h_n)\} = D.$$

It suffices to prove that for any constant vector $\gamma \in \mathbb{R}^{p-1}$ such that $||\gamma|| = 1$,

$$(nh_n)^{1/2} \gamma^T \left\{ T_n^*(\beta_0; h_n) - ET_n^*(\beta_0; h_n) \right\} = N(0, \gamma^T D \gamma) + o_p(1).$$

Define $q_i^{s} = 2 r_i(2A_i - 1)(nh_n)^{1/2} K'(x_i^T \beta_0 \frac{\partial}{\partial h_n}) \frac{Y_i}{h_n}$, where $E_{r|w} q_i^{s} = q_i$, and $E_{r|w}(q_i^{s^2}) = 2q_i^2$, for $q_i$ defined in the proof of Theorem 2. To check the Lyapunov condition, it suffices to prove that

$$(s_n^*)^{-1} \sum_{i=1}^n E_{r|w}\{(q_i^{s} - E_{r|w}q_i^{s})^4\} \xrightarrow{a.s.} 0,$$

where $s_n^* = \sum_{i=1}^n \text{Var}_{r|w}(q_i^{s})$. Similarly as Theorem 2, the Lyapunov condition holds if

$$(s_n^*)^{-1} \sum_{i=1}^n E_{r|w}(q_i^{s^4}) \xrightarrow{a.s.} 0, \quad \text{and} \quad (s_n^*)^{-1} \sum_{i=1}^n (E_{r|w}q_i^{s^4}) \xrightarrow{a.s.} 0.$$
Since $r$ and $Y$ both have sub-Gaussian tails, we know that for any integer $k \geq 1$, $E|r|^k$ and $E|Y|^k$ are finite. Then it is easy to compute that

$$s_n^2 = 4n^2h_n^{-1}I_1, \quad \sum_{i=1}^n E_{r|w}(q_i^4) = 16n^3h_n^{-2}E(r^4)I_2, \quad \sum_{i=1}^n (E_{r|w}(q_i^4))^4 = 16n^3h_n^{-2}I_2,$$

where

$$I_1 = n^{-1} \sum_{i=1}^n \left\{ K' \left( \frac{x^T_i \beta_0}{h_n} \right)^2 (\gamma^T \tilde{x}_i)^2 Y_i^2 \right\}, \quad I_2 = n^{-1} \sum_{i=1}^n \left\{ K' \left( \frac{x^T_i \beta_0}{h_n} \right)^4 (\gamma^T \tilde{x}_i)^4 Y_i^4 \right\}.$$

According to (K1) and (A1)-(A2), we know that $I_1 \overset{a.s.}{\longrightarrow} E_w I_1$ and $I_2 \overset{a.s.}{\longrightarrow} E_w I_2$. With the continuous mapping theorem, it is easy to conclude that $I_1^{-2}I_2 \overset{a.s.}{\longrightarrow} (E_w I_1)^{-2}E_w I_2$. We therefore have

$$(s_n^4)^{-1} \sum_{i=1}^n E_{r|w}(q_i^4) = n^{-1}E(r^4)I_1^{-2}I_2 \overset{a.s.}{\longrightarrow} 0, \quad (s_n^4)^{-1} \sum_{i=1}^n (E_{r|w}(q_i^4))^4 = n^{-1}I_1^{-2}I_2 \overset{a.s.}{\longrightarrow} 0.$$

This verifies the Lyapunov condition and finishes the proof.  

\[ \Box \]

**S4 Proof of Auxiliary Results in Appendix A**

**Proof of Lemma A1:** We give below the proof for $\mathcal{F}$. Proofs for the other classes of functions are similar. Since $K(\cdot)$ is continuous, and has bounded variation on the real line, by Jordan’s Theorem in Section 6.3 in Royden and Fitzpatrick [2010], there exist bounded, nondecreasing, right continuous functions $K_1$ and $K_2$ on $\mathbb{R}$ such that $K = K_1 - K_2$. Let $\mathcal{F}_1 = \{(2A - 1)K_1(\frac{x^T \beta}{h})Y : \beta \in \mathbb{B}, h \in (0, 1]\}$, and $\mathcal{F}_2 = \{(2A - 1)K_2(\frac{x^T \beta}{h})Y : \beta \in \mathbb{B}, h \in (0, 1]\}$. Furthermore, let $\mathcal{F}_{10} = \{K_1(\frac{x^T \beta}{h}) : \beta \in \mathbb{B} \subset \mathbb{R}^p, h \in (0, 1]\}$. We will first prove $\mathcal{F}_{10}$ is a VC class by similar techniques as in Giné and Sang [2010] and Mason [2012]. It is sufficient to show the collection of all subgraphs $S_0 = \left\{ \{(x, t) : K_1(\frac{x^T \beta}{h}) < t \} : K_1(\frac{x^T \beta}{h}) \in \mathcal{F}_{10} \right\}$ forms a VC class of sets in $\mathcal{X} \times \mathbb{R}$.

Since $K_1(\cdot)$ is a bounded, nondecreasing function, assume $\lim_{x \to -\infty} K_1(x) = m_1$ and $\lim_{x \to \infty} K_1(x) = m_2$. Note that

$$\left\{ (x, t) : K_1(\frac{x^T \beta}{h}) < t \right\} = \left\{ (x, t) : -x^T \beta + hK_1^{-1}(t) > 0 \right\},$$

where $K_1^{-1}(t) = -\infty$ if $t \leq m_1$, is $K_1^{-1}(t)$ for $m_1 < t \leq m_2$ and is $\infty$ if $t > m_2$. Let $\psi_{\beta,h}(x, t) = -x^T \beta + hK_1^{-1}(t)$, $S_1 = \{(x, t) : x \in \mathcal{X}, t \in (m_1, m_2]\}$ and $S_2 = \{(x, t) : x \in \mathcal{X}, t > m_2\}$. Then
for any $\beta \in \mathbb{B} \subset \mathbb{R}^p$, $h \in (0, 1]$.

\[ \{ (x, t) : K_1 \left( \frac{x^T \beta}{h} \right) < t \} = \{ (x, t) : \psi_{\beta, h}(x, t) I((x, t) \in S_1) > 0 \} \cup S_2. \]

Note that $\psi_{\beta, h}(x, t)$ is in a finite dimensional space of functions when restricted to $S_1$. This implies the collection $\{ (x, t) : \psi_{\beta, h}(x, t) I((x, t) \in S_1) > 0 \}$ is a VC subgraph class (Lemma 2.6.15, van der Vaart and Wellner [1996]). $\{S_2\}$ is obviously VC. Hence, $S_0$ is also VC, and hence Donsker. As $(2A - 1)Y$ is square integrable and does not depend on $(\beta, h)$, $F_1$ is a Donsker class with a square integrable envelope (Theorem 2.10.6, van der Vaart and Wellner [1996]). Similarly, $F_2$ is also a Donsker class. Then by the Donsker presentation property, $F$ is Donsker.

\[ \square \]

**Proof of Lemma A2:** The Donsker property of $F$ implies that as $n \to \infty$,

\[ \sup_{\beta \in \mathbb{B}} \left| 2n^{-1} \sum_{i=1}^{n} \left\{ (2A_i - 1)K \left( \frac{x_i^T \beta}{h_n} \right) Y_i - E[(2A_i - 1)K \left( \frac{x_i^T \beta}{h_n} \right) Y_i] \right\} \right| \to 0. \]

It is sufficient to show

\[ \sup_{\beta \in \mathbb{B}} \left| E \left\{ 2(2A_i - 1)[K \left( \frac{x_i^T \beta}{h_n} \right) - I(x_i^T \beta > 0)] Y_i \right\} \right| \to 0. \]

Note that

\[ E \left\{ 2(2A_i - 1)[K \left( \frac{x_i^T \beta}{h_n} \right) - I(x_i^T \beta > 0)] Y_i \right\} = E \left\{ \left[ K \left( \frac{x_i^T \beta}{h_n} \right) - I(x_i^T \beta > 0) \right] (Y_i^* \{(1) - Y_i^* (0)\}) \right\} = E \left\{ \left[ K \left( \frac{x_i^T \beta}{h_n} \right) - I(x_i^T \beta > 0) \right] (\mu(1, x_i) - \mu(0, x_i)) \right\}. \]

According to (A1), we know that $\mu(a, x)$ is bounded for almost all $x$, and $a = 0, 1$.

\[ \sup_{\beta \in \mathbb{B}} \left| E \left\{ 2(2A_i - 1)[K \left( \frac{x_i^T \beta}{h_n} \right) - I(x_i^T \beta > 0)] Y_i \right\} \right| \leq \sup_{\beta \in \mathbb{B}} c E \left| K \left( \frac{x_i^T \beta}{h_n} \right) - I(x_i^T \beta > 0) \right|. \]
for some positive constant $c$. For any positive constant $\tau$,

$$\sup_{\beta \in \mathbb{B}} \left| E \{ 2(2A_i - 1) \left[ K \left( \frac{x_i^T \beta}{h_n} \right) - I(x_i^T \beta > 0) \right] Y_i \} \right| \leq I_1 + I_2,$$

where

$$I_1 = \sup_{\beta \in \mathbb{B}} c E \left| \left[ K \left( \frac{x_i^T \beta}{h_n} \right) - I(x_i^T \beta > 0) \right] I(|x_i^T \beta| \geq \tau) \right|,$$

$$I_2 = \sup_{\beta \in \mathbb{B}} c E \left| \left[ K \left( \frac{x_i^T \beta}{h_n} \right) - I(x_i^T \beta > 0) \right] I(|x_i^T \beta| < \tau) \right|.$$

By the property of $K(\cdot)$, $\forall \tau > 0$, the expectation can be made arbitrary small, uniformly in $\beta$, for all $n \geq n_0$, where $n_0$ is a positive integer. So $I_1 \to 0$. On the other hand,

$$I_2 \leq c \sup_{\beta \in \mathbb{B}} P \left( |x_i^T \beta| < \tau \right) = c \sup_{\beta \in \mathbb{B}} P \left( -\tau - \tilde{x}_i^T \beta < x_1 < \tau + \tilde{x}_i^T \beta \right) \leq c' \tau.$$

As $\tau$ is an arbitrary positive constant, $I_2 \to 0$. This proves the lemma. \hfill \square

**Proof of Lemma A3:** (1) Let $k_{ni}(\psi) = (2A_i - 1)K' \left( \frac{x_i + \psi^T \tilde{x}_i}{h} \right) \tilde{x}_i Y_i$. It suffices to show that

$$\sup_{\psi \in \Psi} \left\| (nh_n^2)^{-1} \sum_{i=1}^n \left\{ k_{ni}(\psi) - E k_{ni}(\psi) \right\} \right\|_2 \to 0,$$

where $\Psi = \{ \psi : \psi \in \mathbb{R}^{p-1}, \| \psi \| \leq \frac{\eta}{2 \sqrt{p-1} M_x} \}$.

The Donsker property of $\mathcal{H}$ implies that

$$\sup_{\psi \in \Psi} \sup_{h \in (0,1]} \left\| n^{-1} \sum_{i=1}^n \left\{ k_{ni}(\psi) - E k_{ni}(\psi) \right\} \right\| = O_p(n^{-1/2}).$$

Then since $h_n \to 0$ and $nh_n^4 \to \infty$, we can derive that

$$\sup_{\psi \in \Psi} \left\| (nh_n^2)^{-1} \sum_{i=1}^n \left\{ k_{ni}(\psi) - E k_{ni}(\psi) \right\} \right\| \leq h_n^{-2} \sup_{\psi \in \Psi} \sup_{h \in [0,1]} \left\| n^{-1} \sum_{i=1}^n \left\{ k_{ni}(\psi) - E k_{ni}(\psi) \right\} \right\|$$

$$\leq O_p(n^{-1/2}h_n^{-2}) = o_p(1).$$
(2) $E[R_n(\theta)] = I_{n1} + I_{n2}$, where

$$I_{n1} = \frac{1}{h_n^2} \int_{|z| \leq \eta} K'\left(\frac{z}{h_n} + \theta^T \bar{x}\right) \bar{x}S(z, \bar{x}) f(z|\bar{x}) dz dP(\bar{x}),$$

and

$$I_{n2} = \frac{1}{h_n^2} \int_{|z| > \eta} K'\left(\frac{z}{h_n} + \theta^T \bar{x}\right) \bar{x}S(z, \bar{x}) f(z|\bar{x}) dz dP(\bar{x}).$$

From (A4) and (A5), we can say that for some $M > 0$,

$$||I_{n2}|| \leq \frac{M}{h_n^2} \int_{|z| > \eta} K'\left(\frac{z}{h_n} + \theta^T \bar{x}\right) dz dP(\bar{x}).$$

Let $\zeta = z/h_n + \theta^T \bar{x}$. Since $h_n ||\theta|| \leq \frac{\eta}{2\sqrt{p-1}M_x}$ and $||\bar{x}|| \leq \sqrt{p-1}M_x$ by (A3), then $|z| > \eta$ implies that

$$|\zeta| > \frac{\eta}{2h_n}, \quad \text{and} \quad ||I_{n2}|| \leq \frac{M}{h_n} \int_{h_n|\zeta| > \eta/2} K'(\zeta) d\zeta.$$

And from (K3), it converges to 0 as $n \to \infty$. Therefore,

$$\lim_{n \to \infty} \sup_{\theta \in \Theta_n} ||I_{n2}|| = 0.$$

When $|z| \leq \eta$, then we have:

$$S(z, \bar{x}) f(z|\bar{x}) = S^{(1)}(0, \bar{x}) f(0|\bar{x}) z + [S^{(1)}(0, \bar{x}) f^{(1)}(\epsilon_2|\bar{x}) + S^{(1)}(\epsilon_1, \bar{x}) f^{(1)}(0|\bar{x})] z^2,$$

where $\epsilon_1$ and $\epsilon_2$ are between 0 and $z$. So $I_{n1} = J_{n1} + J_{n2}$, where

$$J_{n1} = \frac{1}{h_n^2} \int_{|z| \leq \eta} K'\left(\frac{z}{h_n} + \theta^T \bar{x}\right) \bar{x}z S^{(1)}(0, \bar{x}) f(0|\bar{x}) dz dP(\bar{x})$$

$$= \int_{|\zeta - \theta^T \bar{x}| \leq \eta/h_n} K'(\zeta) S^{(1)}(0, \bar{x}) f(0|\bar{x}) \bar{x}(\zeta - \bar{x}^T \theta) d\zeta dP(\bar{x}),$$

And from (K3), it converges to 0 as $n \to \infty$. Therefore,

$$\lim_{n \to \infty} \sup_{\theta \in \Theta_n} ||I_{n2}|| = 0.$$
and

\[
J_{n2} = \frac{1}{h_n^2} \int_{|z| > \eta} K' \left( \frac{\hat{z}}{h_n} + \theta^T \hat{x}\right) S^{(1)}(0, \hat{x}) f^{(1)}(\epsilon_2 | \hat{x}) + S^{(1)}(\epsilon_1, \hat{x}) f^{(1)}(0 | \hat{x}) \right] \hat{z}^2 d\hat{z} d\mathcal{P}(\hat{x})
\]

\[
= h_n \int_{\zeta - \theta^T \hat{x} > \eta/h_n} K' \left( \zeta \right) \hat{x} \left[ S^{(1)}(0, \hat{x}) f^{(1)}(\epsilon_2 | \hat{x}) + S^{(1)}(\epsilon_1, \hat{x}) f^{(1)}(0 | \hat{x}) \right] \left( \zeta - \hat{x}^T \theta \right)^2 d\zeta d\mathcal{P}(\hat{x}).
\]

Since \( \int \zeta K' (\zeta) d\zeta = 0 \) by (K2), and \( |h_n \theta^T \hat{x}| \leq \eta/2 \),

\[
\left| \int_{\zeta - \theta^T \hat{x} \leq \eta/h_n} \zeta K' (\zeta) d\zeta \right| \leq \int_{\zeta \leq \eta/2h_n} |\zeta| K' (\zeta) d\zeta.
\]

By (K2), \( \int_{|\zeta| \leq \eta/2h_n} |\zeta K' (\zeta)| d\zeta \) is bounded uniformly over \( n \) and \( \theta \in \Theta_n \), and converges to 0.

So

\[
\lim_{n \to \infty} \sup_{\theta \in \Theta_n} \left| \int_{\zeta - \theta^T \hat{x} \leq \eta/h_n} \zeta K' (\zeta) S^{(1)}(0, \hat{x}) f(0 | \hat{x}) \hat{x} d\zeta d\mathcal{P}(\hat{x}) \right| = 0.
\]

In addition,

\[
\left| \hat{\theta}^T \hat{x} - \theta^T \hat{x} \right| \int_{\zeta - \theta^T \hat{x} \leq \eta/h_n} K' (\zeta) d\zeta \leq |h_n \theta^T \hat{x}| h_n^{-1} \int_{\zeta \leq \eta/h_n} |K' (\zeta)| d\zeta \leq \frac{\eta}{2h_n} \int_{\zeta \leq \eta/(2h_n)} |K' (\zeta)| d\zeta.
\]

Similarly, we also have:

\[
\lim_{n \to \infty} \sup_{\theta \in \Theta_n} \left| \int_{\zeta - \theta^T \hat{x} \leq \eta/h_n} \zeta K' (\zeta) S^{(1)}(0, \hat{x}) f(0 | \hat{x}) \theta^T \hat{x} \hat{x}^T d\zeta d\mathcal{P}(\hat{x}) - \theta^T Q \right| = 0.
\]

Then for \( J_{n2} \), there is some finite \( M > 0 \), and \( \alpha_1, \alpha_2 \) such that:

\[
\left\| J_{n2} \right\| \leq M h_n \int_{|\zeta - \theta^T \hat{x}| > \eta/h_n} |K' (\zeta)| (\zeta - \theta^T \hat{x})^2 d\zeta d\mathcal{P}(\hat{x}) \leq o(1) + \alpha_1 h_n \| \theta \| + \alpha_2 h_n \| \theta \|^2.
\]

In conclusion, \( \left\| E R_n (\theta) - Q \theta \right\| \leq o(1) + \alpha_1 h_n \| \theta \| + \alpha_2 h_n \| \theta \|^2. \)

\( \square \)

**Proof of Lemma A4:** The Donsker property of \( \mathcal{F}^* \) implies that as \( n \to \infty \),

\[
\sup_{\beta \in \mathbb{B}} \left| \frac{1}{n} \sum_{i=1}^{n} \left\{ (2A_i - 1) r_i K \left( \frac{x_i^T \beta}{h_n} \right) Y_i - E_{\theta} E_{r_i | u} \left[ (2A_i - 1) r_i K \left( \frac{x_i^T \beta}{h_n} \right) Y_i \right] \right\} \right| = o_{prw}(1).
\]

36
It is sufficient to show
\[
\sup_{\beta \in \mathbb{B}} \left| E_w E_r |w \left\{ 2(2A_i - 1) r_i \left[ K\left( \frac{x_i^T \beta}{h_n} \right) - I(x_i^T \beta > 0) \right] Y_i \right\} \right|
\]
\[
= \sup_{\beta \in \mathbb{B}} \left| E_w \left\{ 2(2A_i - 1) \left[ K\left( \frac{x_i^T \beta}{h_n} \right) - I(x_i^T \beta > 0) \right] Y_i \right\} \right| \to 0,
\]
which is verified in Lemma A2. Hence the lemma is proved.

**Proof of Lemma A5:** It follows from Lemma 3 of Cheng and Huang [2010] that it is sufficient to prove
\[
\sup_{||\beta - \beta_0|| \leq C(nh_n)^{-1/2}} \sqrt{n h_n} \left| T_n^* (\beta; h_n) - T_n^* (\beta_0; h_n) \right| = o_{pr_w}(1).
\]
According to Lemma 9.14 in Kosorok [2010], \( \mathcal{H}^* \) is a bounded uniform entropy integral (BUEI) class, and the proof of Lemma 9.13 implies that \( \forall 0 < \epsilon < 1 \), the \( \epsilon \)-covering number of \( \mathcal{H}^* \) satisfies \( N(\epsilon ||F||, \mathcal{H}^*, L(P)) \leq (\frac{4}{\epsilon})^v \), for some positive constants \( A \) and \( v \), and an envelop \( F \). Consider the stochastic process
\[
f \mapsto n^{-1/2} \sum_{i=1}^{n} f(W_i, r_i), \quad f \in \mathcal{H}^*, \quad W_i = (A_i, x_i, Y_i).
\]
Given \( Y, x, r \) all have sub-Gaussian distributions, \((f - \mathbb{P}f)\) is a separable sub-Gaussian process.
Since \( K''(\cdot) \) is bounded, we can derive by the Lipschitz property of \( K'(\cdot) \) that
\[
||\beta - \beta_0|| \leq C(nh_n)^{-1/2} \implies ||f - f_0|| \leq C'(nh_n^3)^{-1/2}.
\]
By the property of the increments for the separable sub-Gaussian process (Corollary 2.2.8 in van der Vaart and Wellner [1996]),
\[
E_w E_r |w \left\{ \sup_{||f - f'|| \leq C'(nh_n^3)^{-1/2}} \left\| n^{-1/2} \sum_{i=1}^{n} [f(W_i, r_i) - E_w E_r |w f] - n^{-1/2} \sum_{i=1}^{n} [f'(W_i, r_i) - E_w E_r |w f'] \right\| \right\}
\]
\[
\leq D \int_0^{C'(nh_n^3)^{-1/2}} \sqrt{\log(A/\epsilon)^v} d\epsilon \leq D'(nh_n^3)^{-1/2} \sqrt{\frac{1}{2} \log (nh_n^3)},
\]

37
for some positive constants $D$ and $D'$. Then by Markov inequality, for any $\delta > 0$,

\[
P_{r,w} \left( \sup_{\|\beta - \beta_0\| \leq C'(nh_n)^{-1/2}} \sqrt{nh_n} \left| T_n^* (\beta; h_n) - T_n^* (\beta_0; h_n) \right| > \delta \right) 
\leq P_{r,w} \left( \sup_{\|f - f_0\| \leq C'(nh_n)^{-1/2}} h_n^{-1/2} \left| n^{-1/2} \left( \sum_{i=1}^{n} f(W_i, r_i) - \sum_{i=1}^{n} f_0(W_i, r_i) \right) \right| > \delta \right) 
\leq (\delta h_n^{1/2})^{-1} E_{w} E_{r|w} \left\{ \sup_{\|f - f_0\| \leq C'(nh_n)^{-1/2}} n^{-1/2} \left| \sum_{i=1}^{n} \left[ f(W_i, r_i) - E_{w|w} f \right] - \sum_{i=1}^{n} \left[ f_0(W_i, r_i) - E_{w|w} f_0 \right] \right| \right\} 
\leq (\delta h_n^{1/2})^{-1} E_{w} E_{r|w} \left\{ \sup_{\|f - f_0\| \leq C'(nh_n)^{-1/2}} n^{-1/2} \left| E_{w|w} f - f_0 \right| \right\} 
\leq D' \delta^{-1} (nh_n^4)^{-1/2} \sqrt{\frac{1}{2} \log (nh_n^3)} + C' \delta^{-1} (nh_n^2)^{-1} \to 0,
\]

given $\log(n)/nh_n^4 = o_p(1)$, where $P_{r,w}(\cdot)$ denotes probability with respect to the joint distribution of $(r, w)$. The conclusion follows as $\delta > 0$ is arbitrary. \hfill \Box

S5 \hspace{1em} Proof of Auxiliary Results in Section S3

**Proof of Lemma 2:** (1) Let $\zeta = z/h_n$, then by (A1), we have

\[
E \{ h_n^{-b} T_n(\beta_0; h_n) \} = h_n^{-b} E \left\{ (2A - 1) K' \left( \frac{x^{T} \beta_0}{h_n} \right) \frac{\tilde{x}}{h_n} Y \right\} 
= h_n^{-b} E \left\{ K' \left( \frac{x^{T} \beta_0}{h_n} \right) \frac{\tilde{x}}{h_n} (Y_1^* - Y_0^*) \right\} 
= h_n^{-b} E \left\{ K' \left( \frac{z}{h_n} \right) \frac{\tilde{x}}{h_n} S(z, \tilde{x}) \right\} 
= h_n^{-b} \int K'(\zeta) \tilde{x} S(h_n \zeta, \tilde{x}) f(h_n \zeta | \tilde{x}) d\zeta dP(\tilde{x}).
\]

Under (A3), $S(0, \tilde{x}) = 0$ for almost every $\tilde{x}$, so the Taylor series expansions for $S(h_n \zeta, \tilde{x})$
and \( f(h_n \zeta | \bar{x}) \) can be written as

\[
S(h_n \zeta, \bar{x}) = \sum_{i=1}^{b-1} S^{(i)}(0, \bar{x}) \frac{(h_n \zeta)^i}{i!} + S^{(b)}(\xi_b, \bar{x}) \frac{(h_n \zeta)^b}{b!},
\]

\[
f(h_n \zeta | \bar{x}) = \sum_{j=0}^{b-i} f^{(j)}(0|\bar{x}) \frac{(h_n \zeta)^j}{j!} + f^{(b-i)}(\xi_i|\bar{x}) \frac{(h_n \zeta)^{b-i}}{(b-i)!},
\]

for \( i = 1, \ldots, b-1 \), where \( \xi_1, \ldots, \xi_b \) are scalars with values between 0 and \( h_n \zeta \). Combining these two expansions yields

\[
S(h_n \zeta, \bar{x}) f(h_n \zeta | \bar{x}) = S^{(b)}(\xi_b, \bar{x}) \frac{(h_n \zeta)^b}{b!} f(h_n \zeta | \bar{x}) + \sum_{i=1}^{b-1} S^{(i)}(0, \bar{x}) f^{(b-i)}(\xi_i|\bar{x}) \frac{(h_n \zeta)^b}{i!(b-i)!} \\
+ \sum_{i=1}^{b-1} \sum_{j=0}^{b-i} S^{(i)}(0, \bar{x}) f^{(j)}(0|\bar{x}) \frac{(h_n \zeta)^{i+j}}{i!j!}.
\]

(S9)

So we have

\[
E \{ h_n^{-b} T_n(\beta_0; h_n) \} = \int \zeta^b K'(\zeta) \bar{x} \left\{ S^{(b)}(\xi_b, \bar{x}) \frac{f(h_n \zeta | \bar{x})}{b!} + \sum_{i=1}^{b-1} S^{(i)}(0, \bar{x}) \frac{f^{(b-i)}(\xi_i|\bar{x})}{i!(b-i)!} \right\} d\zeta dP(\bar{x}) \\
+ \sum_{i=1}^{b-1} \sum_{j=0}^{b-i-1} \int h_n^{i+j} \zeta^{i+j} K'(\zeta) \bar{x} S^{(i)}(0, \bar{x}) \frac{f^{(j)}(0|\bar{x})}{i!j!} d\zeta dP(\bar{x})
\]

\[
= I_1 + I_2 + I_3,
\]

where for some \( \eta > 0 \),

\[
I_1 = \int \zeta^b K'(\zeta) \bar{x} \left\{ S^{(b)}(\xi_b, \bar{x}) \frac{f(h_n \zeta | \bar{x})}{b!} + \sum_{i=1}^{b-1} S^{(i)}(0, \bar{x}) \frac{f^{(b-i)}(\xi_i|\bar{x})}{i!(b-i)!} \right\} d\zeta dP(\bar{x}),
\]

\[
I_2 = \sum_{i=1}^{b-1} \sum_{j=0}^{b-i-1} \int_{|h_n \zeta| \leq \eta} h_n^{i+j} \zeta^{i+j} K'(\zeta) \bar{x} S^{(i)}(0, \bar{x}) \frac{f^{(j)}(0|\bar{x})}{i!j!} d\zeta dP(\bar{x}),
\]

\[
I_3 = \sum_{i=1}^{b-1} \sum_{j=0}^{b-i-1} \int_{|h_n \zeta| > \eta} h_n^{i+j} \zeta^{i+j} K'(\zeta) \bar{x} S^{(i)}(0, \bar{x}) \frac{f^{(j)}(0|\bar{x})}{i!j!} d\zeta dP(\bar{x}).
\]

Then from (K2), (A4)-(A5), and Lebesgue’s dominated convergence theorem, we have \( I_1 \to \)}
$H$, where $H = a_H \sum_{i=1}^{b+1} \frac{1}{2 \ell^{(b+1)} \nu} E \{ \tilde{x} S^{(1)} \} f^{(b+1)}(0|\tilde{x}) \} \) with $a_H = \int \nu^b K(\nu) d\nu$. Similarly, let $\eta \to 0$, we have $I_2 \to 0$ and $I_3 \to 0$. Therefore, $\lim_{n \to \infty} E \{(nh_n)^{1/2} T_n(\beta_0; h_n)\} = 0$, since $nh_n^{2b+1} = o(1)$.

(2) Let $t_i = 2(2A_i - 1) K'(\frac{x_i^T \beta_0}{h_n}) \frac{x_i}{h_n} Y_i$, then we have:

$$\text{Var}\{(nh_n)^{1/2} T_n\} = h_n \left( E(t_i^T - ET_n ET_n^T) \right)$$

$$= 4E \left[ K'(\frac{x_i^T \beta_0}{h_n}) \right] \frac{2 \tilde{x}_i \tilde{x}_i^T}{h_n} Y_i^2 - h_n ET_n ET_n^T$$

$$= 2E \left\{ K'(\frac{z}{h_n}) \frac{2 \tilde{x}_i \tilde{x}_i^T}{h_n} E \left( Y_1^* + Y_0^* | z, \tilde{x} \right) \right\} - h_n ET_n ET_n^T$$

$$= 2 \int K'(\frac{z}{h_n}) \frac{2 \tilde{x}_i \tilde{x}_i^T}{h_n} E \left( Y_1^* + Y_0^* | z, \tilde{x} \right) f(z|\tilde{x}) dzdP(\tilde{x}) - h_n ET_n ET_n^T$$

$$= 2 \int K'(\zeta) \frac{2 \tilde{x}_i \tilde{x}_i^T}{h_n} E \left( Y_1^* + Y_0^* | h_n \zeta, \tilde{x} \right) f(h_n \zeta|\tilde{x}) d\zeta dP(\tilde{x}) - h_n ET_n ET_n^T.$$

Since $h_n \to 0$, by (A1)-(A2) and the dominated convergence theorem, we have

$$\lim_{n \to \infty} \left\{ \text{Var}(nh_n)^{1/2} T_n \right\}$$

$$= \lim_{n \to \infty} \left\{ 2 \int K'(\zeta) \frac{2 \tilde{x}_i \tilde{x}_i^T}{h_n} E \left( Y_1^* + Y_0^* | h_n \zeta, \tilde{x} \right) f(h_n \zeta|\tilde{x}) d\zeta dP(\tilde{x}) - h_n ET_n ET_n^T \right\}$$

$$= 2 \int K'(\zeta)^2 d\zeta \int \tilde{x} \tilde{x}^T E \left( Y_1^* + Y_0^* | 0, \tilde{x} \right) f(0|\tilde{x}) dP(\tilde{x}) - 0 \ast HH^T$$

$$= a_1 \int \tilde{x} \tilde{x}^T f(0|\tilde{x}) E \left( Y_1^* + Y_0^* | 0, \tilde{x} \right) dP(\tilde{x}) = D.$$

This finishes the proof. \qed

**Proof of Lemma 3:** It suffices to prove the following two results:

(a) let $\theta_n = (\hat{\beta}_n - \beta_0)/h_n$, then $\theta_n = o_p(1)$;

(b) let $\{\beta_n\} = \{\hat{\beta}_n\}^T$ be any sequence in $\mathcal{B}$ such that $(\beta_n - \beta_0)/h_n \to 0$ as $n \to \infty$, then $Q_n(\beta_n; h_n) \overset{p}{\to} Q$.

To prove (a), we first note that $h_n \theta_n \overset{p}{\to} 0$ by Theorem 1. Lemma A3 then implies $R_n(\theta_n) \overset{p}{\to} 0$, and there exist some constants $\alpha_1$ and $\alpha_2$ such that $||Q \theta_n|| \leq o_p(1) + \alpha_1 h_n ||\theta_n|| + \alpha_2 h_n ||\theta_n||^2$. Since $Q$ is negative definite, we have inf $\frac{||Q \theta_n||}{||\theta_n||} = |\omega_{min}| > 0$, where $\omega_{min}$ is the eigenvalue of $Q$. 

40
with the smallest absolute value. It indicates that

\[ 0 < |\omega_{\min}| < \frac{||Q\theta_n||}{||\theta_n||} \leq o_p(\frac{||\theta_n||}{||\theta_n||}) + \alpha_1 h_n + \alpha_2 h_n ||\theta_n||. \]

Since \( h_n \to 0 \) and \( h_n ||\theta_n|| \to 0 \), if \( ||\theta_n|| = o_p(1) \) does not hold, then the right hand side of the above inequality would degenerate to \( o_p(1) \), which contradicts with the fact that it should be larger than \( |\omega_{\min}| > 0 \). Consequently, we have \( ||\theta_n|| = o_p(1) \).

To prove (b), let \( q_{ni}(\beta) = (2A_i - 1)K''\left(\frac{x^T \beta}{h_n}\right) \tilde{x}_i \tilde{x}_i^T Y_i \). It suffices to show that

\[ \sup_{||\beta - \beta_0|| \leq \epsilon h_n} \left\| n^{-1} \sum_{i=1}^{n} (h_n^{-2} q_{ni}(\beta) - Q) \right\| = o_p(1), \]

for arbitrary positive \( \epsilon \).

First, let \( \tilde{\beta}_n = \beta_0 + h_n \tilde{\theta}_n \) with \( \tilde{\theta}_n \to 0 \) as \( n \to \infty \). Now we have

\[
E\{h_n^{-2}q_{ni}(\beta_n)\} = E\left\{2(2A_i - 1)K''\left(\frac{x^T \beta_n}{h_n}\right) \tilde{x}_i \tilde{x}_i^T Y_i \right\} \\
= E\left\{K''\left(\frac{x^T \beta_0}{h_n} + \tilde{\theta}_n h_n \tilde{x}_i \right) \tilde{x}_i \tilde{x}_i^T S(z, \tilde{x}) \right\} \\
= \int K''\left(\frac{z}{h_n} + \tilde{\theta}_n h_n \tilde{x}_i \right) \tilde{x}_i \tilde{x}_i^T S(z, \tilde{x}) f(z|\tilde{x}) d\tilde{x} dP(\tilde{x}).
\]

By Taylor expansion and (A3), there exists a \( 0 < \epsilon < 1 \) such that \( S(z, \tilde{x}) = zS^{(1)}(\epsilon z, \tilde{x}) \). We have

\[
E\{h_n^{-2}q_{ni}(\beta_n)\} = \int K''\left(\frac{z}{h_n} + \tilde{\theta}_n h_n \tilde{x}_i \right) \tilde{x}_i \tilde{x}_i^T zS^{(1)}(\epsilon z, \tilde{x}) f(z|\tilde{x}) d\tilde{x} dP(\tilde{x}) \\
= \int (\zeta - \tilde{\theta}_n h_n \tilde{x}_i) K''(\zeta) \tilde{x}_i \tilde{x}_i^T S^{(1)}(\epsilon h_n(\zeta - \tilde{\theta}_n h_n \tilde{x}_i), \tilde{x}_i) f(h_n(\zeta - \tilde{\theta}_n h_n \tilde{x}_i)|\tilde{x}_i) d\zeta dP(\tilde{x}).
\]

Then by (K1), (A2), (A4)-(A5), and the dominated convergence theorem,

\[
\lim_{n \to \infty} E\{h_n^{-2}q_{ni}(\beta_n)\} = \lim_{n \to \infty} \int (\zeta - \tilde{\theta}_n h_n \tilde{x}_i) K''(\zeta) \tilde{x}_i \tilde{x}_i^T S^{(1)}(\epsilon h_n(\zeta - \tilde{\theta}_n h_n \tilde{x}_i), \tilde{x}_i) f(h_n(\zeta - \tilde{\theta}_n h_n \tilde{x}_i)|\tilde{x}_i) d\zeta dP(\tilde{x}) \\
= \lim_{n \to \infty} \int \zeta K''(\zeta) \tilde{x}_i \tilde{x}_i^T S^{(1)}(0, \tilde{x}_i) f(0|\tilde{x}_i) d\zeta dP(\tilde{x}) \\
= a_2 \int \tilde{x}_i \tilde{x}_i^T S^{(1)}(0, \tilde{x}_i) f(0|\tilde{x}_i) dP(\tilde{x}) = Q. \]
The Donsker property of $Q$ implies that for arbitrary $\epsilon > 0$,

$$\sup_{||\beta-\beta_0|| \leq \epsilon h_n} \left\| n^{-1} \sum_{i=1}^{n} \left\{ q_{ni}(\beta) - E q_{ni}(\beta) \right\} \right\| = O_p(n^{-1/2}).$$

Then since $h_n = o(n^{-1/(2b+1)})$ and $(nh_n^4)^{-1} = o(1)$, we can derive that

$$\sup_{||\beta-\beta_0|| \leq \epsilon h_n} \left\| n^{-1} \sum_{i=1}^{n} \left\{ h_n^{-2} q_{ni}(\beta) - Q \right\} \right\| \leq \sup_{||\beta-\beta_0|| \leq \epsilon h_n} \left\| (nh_n^2)^{-1} \sum_{i=1}^{n} \left\{ q_{ni}(\beta) - E q_{ni}(\beta) \right\} \right\| + o(1)$$

$$\leq h_n^{-2} \sup_{||\beta-\beta_0|| \leq \epsilon h_n} \left\| n^{-1} \sum_{i=1}^{n} \left\{ q_{ni}(\beta) - E q_{ni}(\beta) \right\} \right\|$$

$$\leq O_p(n^{-1/2}h_n^{-2}) = o_p(1).$$

\[ \square \]

**Proof of Lemma 4:** With the consistency result in Theorem 1, we have $P(\hat{\beta}_{n1} = \beta_{01}) \to 1$ as $n \to \infty$. Hence for any $\beta = (\beta_1, \beta^T)^T$ between $\beta_0$ and $\hat{\beta}_n$, note that $I(x^T \beta > 0) = I(z + \bar{x}^T(\bar{\beta} - \bar{\beta}_0) > 0)$. Then for $V(\beta) = E_x \{ \mu(1, x) I(x^T \beta > 0) + \mu(0, x) I(x^T \beta \leq 0) \}$, we have

$$V(\beta) = E_x \left\{ \mu(1, x) I(z + \bar{x}^T(\bar{\beta} - \bar{\beta}_0) > 0) + \mu(0, x) I(z + \bar{x}^T(\bar{\beta} - \bar{\beta}_0) \leq 0) \right\}$$

$$= \int \left\{ \int_{-\infty}^{\infty} E_x \{ \mu(1, x)|z, \bar{x}\} f(z|\bar{x}) dz + \int_{-\infty}^{\infty} E_x \{ \mu(0, x)|z, \bar{x}\} f(z|\bar{x}) dz \right\} dP(\bar{x}).$$

Let $\tilde{\delta} = \bar{\beta} - \tilde{\beta}_0$, then $\tilde{\delta} \overset{p}{\to} 0$ for $\beta$ between $\beta_0$ and $\hat{\beta}_n$, according to Theorem 1. Note that

$$V'(\beta) = \frac{\partial V(\beta)}{\partial \beta} = \int S(-\bar{x}^T \tilde{\delta}, \bar{x}) f(-\bar{x}^T \tilde{\delta}|\bar{x}) \bar{x} \bar{x}^T (\bar{\beta}_0 + \tilde{\delta}) dP(\bar{x}),$$

$$V''(\beta) = \frac{\partial V'(\beta)}{\partial \beta} = \int S(-\bar{x}^T \tilde{\delta}, \bar{x}) \left\{ f(-\bar{x}^T \tilde{\delta}|\bar{x}) - (\bar{x}^T \tilde{\beta}_0) f^{(1)}(-\bar{x}^T \tilde{\delta}|\bar{x}) \right\} \bar{x} \bar{x}^T dP(\bar{x})$$

$$- \int (\bar{x}^T \tilde{\delta}) \left\{ S^{(1)}(-\bar{x}^T \tilde{\delta}, \bar{x}) f(-\bar{x}^T \tilde{\delta}|\bar{x}) + S(-\bar{x}^T \tilde{\delta}, \bar{x}) f^{(1)}(-\bar{x}^T \tilde{\delta}|\bar{x}) \right\} \bar{x} \bar{x}^T dP(\bar{x})$$

$$- \int (\bar{x}^T \tilde{\beta}_0) S^{(1)}(-\bar{x}^T \tilde{\delta}, \bar{x}) f(-\bar{x}^T \tilde{\delta}|\bar{x}) \bar{x} \bar{x}^T dP(\bar{x})$$

$$= I_1 + I_2 + I_3,$$

42
where the definition of $I_i$ ($i = 1, 2, 3$) is clear from the context. By Taylor expansion, there exists some constant $0 < r_1 < 1$ such that

$$I_1 = \int - (\tilde{x}^T \delta) S^{(1)} \left( - r_1 \tilde{x}^T \delta, \tilde{x} \right) \left\{ f \left( - \tilde{x}^T \delta | \tilde{x} \right) ( - \tilde{x}^T \tilde{\beta}_0) f^{(1)} \left( - \tilde{x}^T \tilde{\beta}_0 | \tilde{x} \right) \right\} \tilde{x} \tilde{x}^T dP(\tilde{x}).$$

By (A2), (A4)-(A5), we know that the components of $\tilde{x}$, $S^{(i)}(z, \tilde{x})$ and $f^{(i)}(z|\tilde{x})$, $i = 0, 1$, are bounded for almost every $\tilde{x}$. Then for any $\tilde{\delta} \xrightarrow{p} 0$, it is easy to conclude $I_1 \xrightarrow{p} 0$ and $I_2 \xrightarrow{p} 0$. To evaluate $I_2$, note that for some constant $0 < r_2 < 1$,

$$I_2 = I_V + \int (\tilde{\delta}^T \tilde{x} \tilde{\beta}_0) \left\{ S^{(2)} \left( - r_2 \tilde{x}^T \delta, \tilde{x} \right) f \left( - \tilde{x}^T \delta | \tilde{x} \right) \right. + S^{(1)} \left( - \tilde{x}^T \delta, \tilde{x} \right) f^{(1)} \left( - r_2 \tilde{x}^T \delta | \tilde{x} \right) \right\} \tilde{x} \tilde{x}^T dP(\tilde{x}).$$

With the boundedness of the components of $\tilde{x}$, $S^{(1)}(z, \tilde{x})$, $S^{(2)}(z, \tilde{x})$, $f(z|\tilde{x})$ and $f^{(1)}(z|\tilde{x})$ from (A2), (A4)-(A5), we also have $I_2 \xrightarrow{p} I_V$ as $\tilde{\delta} \xrightarrow{p} 0$, where $I_V$ is negative definite by (A5). This finishes the proof. 

Recall from Section 3.2 of the main paper that $r = \{r_1, \ldots, r_n\}$ denotes the collection of the random bootstrap weights and $w = \{W_1, \ldots, W_n\}$ denotes the random sample of observations, where $W_i = (x_i, A_i, Y_i)$. Given a sequence of random variables $R_n$, $n = 1, \ldots, n$, we write $R_n = o_{p_r}(1)$ if for any $\epsilon > 0, \delta > 0$, we have $P_{w}(P_{r|w}(|R_n| > \epsilon) > \delta) \to 0$ as $n \to \infty$. In the bootstrap literature, $R_n$ is said to converge to zero in probability, conditional on the data. Let $E_{r|w}$ and $\text{Var}_{r|w}$ denote the conditional expectation and the conditional variance according to the distribution of $r$ given $x$. Furthermore, $o_{p_{r|w}}(1)$ denotes a random sequence that converges to zero in probability with respect to the joint distribution of $(r, w)$, and $o_{p_{r|w}}(1)$ denotes a random sequence that converges to zero in probability with respect to the distribution of $r$ only. By Lemma 3 of Cheng and Huang [2010], if $R_n = o_{p_{r|w}}(1)$, then $R_n = o_{p_r}(1)$. In particular, if $R_n$ depends only on the data $w$ but not on the random weights $r$ and if $R_n = o_{p_{w}}(1)$, then it is easy to see $R_n = o_{p_{r|w}}(1)$, and hence it is $o_{p_r}(1)$. In this part of proof, we will include subscripts in the probability and expectation to clarify which probability distribution is used in the calculation.

**Proof of Lemma 5:** It suffices to prove

(a) let $\hat{\theta}_n^* = (\hat{\beta}_n^* - \tilde{\beta}_n)/h_n$, we have $\hat{\theta}_n^* = o_{p_r}(1)$;
(b) let \( \{\beta_n\} = \{(\beta_{n1}, \beta_{n2})^T\} \) be any sequence in \( \mathbb{B} \) such that \( (\beta_n - \hat{\beta}_n)/h_n \to 0 \) as \( n \to \infty \), then \( Q_n^*(\beta_n; h_n) = Q + o_p(1) \).

To prove (a), for any \( \theta \in \mathbb{R}^{p-1} \), define \( R_n^*(\theta) = \frac{2}{n h_n} \sum_{i=1}^n r_i (2A_i - 1) K' \left( \frac{x_i}{h_n} + \theta^T \tilde{x}_i \right) \tilde{x}_i Y_i \). We observe

\[
\left\| E_{r|w} R_n^*(\theta) - Q \theta \right\| \leq \left\| E_{r|w} R_n^*(\theta) - E_w R_n(\theta) \right\| + \left\| E_w R_n(\theta) - Q \theta \right\| \\
= \left\| R_n(\theta) - E_w R_n(\theta) \right\| + \left\| E_w R_n(\theta) - Q \theta \right\|.
\]

By Lemma A3,

\[
\sup_{\theta \in \Theta_n} \left\| R_n(\theta) - E_w R_n(\theta) \right\| = o_p(1),
\]

\[
\left\| E_w R_n(\theta) - Q \theta \right\| \leq o(1) + \alpha_1 h_n ||\theta|| + \alpha_2 h_n ||\theta||^2,
\]

uniformly over \( \theta \in \Theta_n \) for some finite \( \alpha_1 \) and \( \alpha_2 \). Hence

\[
\left\| E_{r|w} R_n^*(\theta) - Q \theta \right\| \leq o(1) + \alpha_1 h_n ||\theta|| + \alpha_2 h_n ||\theta||^2,
\]

uniformly over \( \theta \in \Theta_n \). By Theorem 3, \( h_n \theta_n^* = o_p(1) \). So \( R_n^*(\theta_n^*) = o_p(1) \). So we have

\[
\left\| Q \theta_n^* \right\| \leq o(1) + \alpha_1 h_n ||\theta_n^*|| + \alpha_2 h_n ||\theta_n^*||^2.
\]

Then similarly to the proof of Lemma B2, we can show that \( \theta_n^* = o_p(1) \).

To prove (b), let \( q_{ni}^*(\beta) = (2A_i - 1) r K'' \left( \frac{x_i^T \beta}{h_n} \right) \tilde{x}_i \tilde{x}_i^T Y_i \). It suffices to show that

\[
\sup_{||\beta - \hat{\beta}_n|| \leq \epsilon h_n} \left| n^{-1} \sum_{i=1}^n \{ h_n^{-2} q_{ni}^*(\beta) - Q \} \right| = o_p(1),
\]

for arbitrary positive \( \epsilon \).

Let \( \hat{\beta}_n = \tilde{\beta}_n + h_n \hat{\beta}_n^* \) with \( \hat{\beta}_n^* \to 0 \). Consequently, \( \lim_{n \to \infty} E_w E_{r|w} \{ h_n^{-2} q_{ni}^*(\beta_n^*) \} = Q \).

The Donsker property of \( Q^* \) implies that for arbitrary \( \epsilon > 0 \),

\[
\sup_{||\beta - \hat{\beta}_n|| \leq \epsilon h_n} \sup_{h \in (0, 1]} \left| n^{-1} \sum_{i=1}^n \{ q_{ni}^*(\beta) - E_w E_{r|w} q_{ni}^*(\beta) \} \right| = O_p(n^{-1/2}).
\]
Then since $h_n = o(n^{-1/(2^b + 1)})$ and $(nh_n^2)^{-1} = o(1)$, we can derive that

$$
\sup_{||\beta - \beta_n|| \leq \epsilon h_n} n^{-1} \left\| \sum_{i=1}^{n} \left( h_n^{-2} q_{ni}^*(\beta) - Q \right) \right\| \leq \sup_{||\beta - \beta_n|| \leq \epsilon h_n} (nh_n^2)^{-1} \left\| \sum_{i=1}^{n} \left( q_{ni}^*(\beta) - E_{w_i} E_{r_i} q_{ni}^*(\beta) \right) \right\| + o(1)
$$

$$
\leq h_n^{-2} \sup_{||\beta - \beta_n|| \leq \epsilon h_n} \sup_{b \in [0, 1]} \left\| n^{-1} \sum_{i=1}^{n} \left( q_{ni}^*(\beta) - E_{w_i} E_{r_i} q_{ni}^*(\beta) \right) \right\|
$$

$$
\leq O_{p_{rw}}(n^{-1/2}h_n^{-2}) = o_{p_{rw}}(1).
$$

\[\square\]

S6 Proof of Results in Section S1

Let $Y$ be the response generated from the local model (S1). We can write $Y = \tilde{Y} + b_n \tilde{Y}$, where $\tilde{Y} = \mu(x) + Ax^T \beta_0 + \epsilon$ and $\tilde{Y} = A\tilde{x}^T \tilde{\beta}$. Note that $\tilde{Y}$ satisfies all the assumptions about the outcome variable in (A1), (A3) and (A5). It follows that all the preceding lemmas and theorems still hold if regarding $\tilde{Y}$ as the observed response $Y$. In addition, since $(2A - 1)\tilde{Y}$ is square integrable and does not depend on $(\beta, h)$, it implies that all classes listed in Lemma A1 are still VC classes with $\tilde{Y}$ as their responses. In the following proof, we use “$\tilde{\sim}$” to denote corresponding notation when we replace $Y$ with $\tilde{Y}$. For example, we define

$$
\tilde{M}_n(\beta, h_n) = 2n^{-1} \sum_{i=1}^{n} (2A_i - 1)I(x_i^T \beta > 0)\tilde{Y}.
$$

First we will prove the consistency of the smoothed estimator given the observed data \{$(x_i, A_i, Y_i)$, $i = 1, ..., n$\} from (S1).

**Lemma 6.** Under (A1), (A2) and assume $K(\cdot)$ satisfies (K1), if $b_n = o(1)$, then $\hat{\beta}_n = \beta_0 + o_p(1)$.

**Proof of Lemma 6:** We observe that $\hat{\beta}_n$ maximizes $\tilde{M}_n(\beta, h_n)$ over $\beta \in B$, and $\beta_0$ maximizes $\tilde{M}(\beta)$. Note that

$$
\sup_{\beta \in B} |\tilde{M}_n(\beta, h_n) - \tilde{M}(\beta)| \leq \sup_{\beta \in B} \left| 2b_n n^{-1} \sum_{i=1}^{n} (2A_i - 1)K \left( \frac{x_i^T \beta}{h_n} \right) \bar{Y}_i \right| + \sup_{\beta \in B} |\tilde{M}_n(\beta, h_n) - \tilde{M}(\beta)|.
$$
Lemma A2 implies that \( \sup_{\beta \in B} |\tilde{M}_n(\beta, h_n) - \tilde{M}(\beta)| = o_p(1) \). In addition, it is obvious that
\[
\sup_{\beta \in B} \left| \frac{b_n}{n} \sum_{i=1}^{n} (2A_i - 1) K \left( \frac{x_i^T \beta}{h_n} \right) \tilde{Y}_i \right| \leq \sup_{\beta \in B} \left| \frac{b_n}{n} \sum_{i=1}^{n} \left[ (2A_i - 1) K \left( \frac{x_i^T \beta}{h_n} \right) \tilde{Y}_i - E \left\{ K \left( \frac{x_i^T \beta}{h_n} \right) \tilde{x}_i \tilde{s} \right\} \right] \right| 
+ \sup_{\beta \in B} \left| \frac{b_n}{n} E \left\{ K \left( \frac{x_i^T \beta}{h_n} \right) \tilde{x}_i \tilde{s} \right\} \right|.
\]

The Donsker property of \( F \) ensures the first term converges to 0 in probability if \( b_n = o(\sqrt{n}) \). By the boundedness of \( K(\cdot) \) and \( x \), the second term also goes to 0 as \( b_n = o(1) \). So \( \sup_{\beta \in B} |\tilde{M}_n(\beta, h_n) - \tilde{M}(\beta)| = o_p(1) \) can be concluded.

The construction of \( \tilde{Y} \) implies that for every \( \tau > 0 \),
\[
\sup_{||\beta - \beta_0|| > \tau} \tilde{M}(\beta) - \tilde{M}(\beta_0) = \sup_{\beta \in B} 2E \{ (2A_i - 1) \tilde{Y}_i [I(x_i^T \beta > 0) - I(x_i^T \beta_0 > 0)] \}
= \sup_{||\beta - \beta_0|| > \tau} E \{ x_i^T \beta_0 [I(x_i^T \beta > 0) - I(x_i^T \beta_0 > 0)] \} < 0.
\]

Hence, \( \hat{\beta}_n = \beta_0 + o_p(1) \) is derived from Theorem 5.7 in van der Vaart (2000).

**Proof of Theorem 5:** The proof of Theorem 2 implies that it suffices to verify:
(a) \( (nh_n)^{1/2} T_n(\beta_0; h_n) \overset{d}{\rightarrow} N(a_2^{-1} Q_0 \tilde{s}, D_0) \);
(b) \( Q_n(\beta^*_n; h_n) = Q_0 + o_p(1) \) for any \( \beta^*_n \) is between \( \hat{\beta}_n \) and \( \beta_0 \).

To prove (a), note that Lemma 2 indicates that
\[
E(nh_n)^{1/2} \{ T_n(\beta_0; h_n) \} \rightarrow a_2^{-1} Q_0 \tilde{s}, \quad \text{and} \quad \text{Var}(nh_n)^{1/2} \{ T_n(\beta_0; h_n) \} \rightarrow D_0.
\]

It is sufficient to prove that \( (nh_n)^{1/2} \gamma^T \{ T_n(\beta_0; h_n) - ET_n(\beta_0; h_n) \} \) is asymptotically \( N(\tilde{0}, \gamma^T D_0 \gamma) \) for any fixed vector \( \gamma \in \mathbb{R}^{p-1} \) such that \( ||\gamma|| = 1 \). Define
\[
q_i = 2(2A_i - 1)(nh_n)^{1/2} K' \left( \frac{x_i^T \beta_0}{h_n} \right) \frac{\gamma^T \tilde{x}_i \tilde{Y}_i}{h_n}
= \tilde{q}_i + 2(2A_i - 1) K' \left( \frac{x_i^T \beta_0}{h_n} \right) \frac{\gamma^T \tilde{x}_i \tilde{Y}_i}{h_n}
= \tilde{q}_i + \tilde{\tilde{q}}_i
\]




46
for \( \tilde{q}_i \) defined as in the proof of Theorem 2. With Lyapunov central limit theorem, we will verify

\[
\lim_{n \to \infty} \left( s_n \right)^{-4} \sum_{i=1}^{n} E \{(q_i - E_qi)^4 \} = 0, \tag{S10}
\]

where \( \lim_{n \to \infty} (n^{-1}s_n)^2 = \lim_{n \to \infty} \sum_{i=1}^{n} \text{Var}(n^{-1}q_i) = \gamma^T D_0 \gamma \). The fact that \( \lim_{n \to \infty} n^{-3} E \{(\tilde{q}_i - E_{\tilde{q}_i})^4 \} = 0 \) implies that the left-side of (S10) is bounded from above (up to a positive constant) by

\[
\lim_{n \to \infty} n^{-3} E(\tilde{q}_i^4) + \lim_{n \to \infty} n^{-3} (E_{\tilde{q}_i})^4 = \lim_{n \to \infty} (n^3 h_n^4)^{-1} 8E \left\{ K' \left( \frac{x_i \beta_0}{h_n} \right)^4 (\gamma^T \tilde{x}_i \tilde{\gamma}^T )^4 \right\}
\]

\[
+ \lim_{n \to \infty} \left( n^3 h_n^4 \right)^{-1} \left\{ E \left[ K' \left( \frac{x_i \beta_0}{h_n} \right) (\gamma^T \tilde{x}_i \tilde{\gamma}^T ) \right] \right\}^4.
\]

With the boundedness of \( K'(\cdot) \) and \( \tilde{x} \), \( (n^3 h_n^4)^{-1} = o(1) \) implies the Lyapunov condition is satisfied, and (a) follows. To prove (b), note that

\[
\sup_{\beta \in \mathbb{B}} ||Q_n(\beta; h_n) - Q_n(\beta; h_n)|| = \sup_{\beta \in \mathbb{B}} \left\{ \frac{2b_n}{n} \sum_{i=1}^{n} (2A_i - 1) K'' \left( \frac{x_i \beta}{h_n} \right) \tilde{x}_i \tilde{x}_i^T \tilde{Y}_i \right\}
\]

\[
\leq O_p \left( (n^2 h_n^5)^{-1/2} \right) + O(b_n h_n^{-1}) \sup_{\beta \in \mathbb{B}} \left\{ E \left[ K'' \left( \frac{x_i \beta}{h_n} \right) \tilde{x}_i \tilde{x}_i^T \tilde{Y}_i \tilde{Y}_i^T \right] \right\}
\]

\[
= O_p \left( (n^2 h_n^5)^{-1/2} \right) + O \left( (n h_n^3)^{-1/2} \right) = o_p(1),
\]

since \( Q \) is a VC class. Then it suffices to show that \( \theta_n = (\hat{\beta}_n - \beta_0)/h_n = o_p(1) \). Consider \( R_n(\theta) \) defined as in Lemma A3. The Donsker properties of \( H \) imply that

\[
\sup_{\theta \in \Theta_n} ||R_n(\theta) - \tilde{R}_n(\theta)|| \leq O_p \left( (n^2 h_n^5)^{-1/2} \right) + O(b_n h_n^{-1}) \sup_{\theta \in \Theta_n} \left\{ E \left[ K' \left( \frac{z_i}{h_n} + \tilde{\theta} \right) \tilde{x}_i \tilde{x}_i^T \tilde{Y}_i \right] \right\}
\]

\[
= O_p \left( (n^2 h_n^5)^{-1/2} \right) + O \left( (n h_n^3)^{-1/2} \right) = o_p(1),
\]

where \( \Theta_n \) is defined in Lemma A3. Combined with Lemma A3, it implies that \( \sup_{\theta \in \Theta_n} ||R_n(\theta) - Q_0\theta|| \leq o(1) + \alpha_1 h_n ||\theta|| + \alpha_2 h_n ||\theta||^2 \). By the definition of \( \theta_n \), we know that \( h_n \theta_n \nrightarrow 0 \), and \( R_n(\theta_n) \nrightarrow 0 \). Then from the proof of Lemma 3, (b) can be concluded. \( \square \)

For the asymptotic distribution for bootstrap estimators with the moving parameter framework, we first prove its consistency.

47
Lemma 7. Under (A1), (A2), (A6) and assume $K(\cdot)$ satisfies (K1), if $b_n = o(1)$, then $\hat{\beta}_n = \hat{\beta}_n + o_p(1)$.

Proof of Lemma 7: By definition, $\hat{\beta}_n$ maximizes $\tilde{M}_n^*(\beta, h_n)$ over $\beta \in \mathbb{B}$. First, given $\mathcal{F}^{*\text{new}}$ is a VC class, the Donsker property and Lemma A2 jointly indicate that

$$
\sup_{\beta \in \mathbb{B}} |\tilde{M}_n^*(\beta, h_n) - \tilde{M}_n(\beta, h_n)| \leq \sup_{\beta \in \mathbb{B}} \left| \frac{2b_n}{n} \sum_{i=1}^{n} (r_i - 1)(2A_i - 1)K \left( \frac{x_i^T \beta}{h_n} \right) \tilde{Y}_i \right|
$$

$$+ \sup_{\beta \in \mathbb{B}} |\tilde{M}_n(\beta, h_n) - \tilde{M}_n(\beta, h_n)| = o_{pr}(1).
$$

By Lemma 3 of Cheng & Huang (2010), $\sup_{\beta \in \mathbb{B}} |\tilde{M}_n^*(\beta, h_n) - \tilde{M}_n(\beta, h_n)| = o_p(1)$. By Theorem 5.7 in van der Vaart (2000), to prove the theorem, it is sufficient to show that for any $\epsilon > 0$,

$$
\lim_{n \to \infty} P_w \left( \sup_{||\beta - \hat{\beta}_n|| > \epsilon} \{ \tilde{M}_n(\beta, h_n) - \tilde{M}_n(\hat{\beta}_n, h_n) \} < 0 \right) = 1. \tag{S11}
$$

Note that Lemma A2 and the consistency of $\hat{\beta}_n$ implies that

$$
\sup_{||\beta - \hat{\beta}_n|| > \epsilon} \{ \tilde{M}_n(\beta, h_n) - \tilde{M}_n(\hat{\beta}_n, h_n) \} \leq \sup_{||\beta - \hat{\beta}_n|| > \epsilon} \left\{ \tilde{M}_n(\beta, h_n) - \tilde{M}_n(\beta_0, h_n) \right\} + \left\{ \tilde{M}_n(\beta_0, h_n) - \tilde{M}_n(\hat{\beta}_n, h_n) \right\}
$$

$$+ \sup_{||\beta - \hat{\beta}_n|| > \epsilon} \left| \frac{2b_n}{n} \sum_{i=1}^{n} (2A_i - 1)K \left( \frac{x_i^T \beta}{h_n} \right) - K \left( \frac{x_i^T \hat{\beta}_n}{h_n} \right) \right| \tilde{Y}_i
$$

$$= \sup_{||\beta - \hat{\beta}_n|| > \epsilon} \left\{ \tilde{M}(\beta) - \tilde{M}(\beta_0) \right\} + o_p(1).
$$

Furthermore, the consistency of $\hat{\beta}_n$ implies that for all sufficiently large $n$, any $\beta$ that satisfies $||\beta - \hat{\beta}_n|| > \tau$ would also satisfy $||\beta - \beta_0|| \geq \tau/2$. Hence, Lemma 6 implies (S11) holds.

Proof of Theorem 6: The proofs of Theorem 4 indicate that it is sufficient to verify:

(a) $Q_n^*(\beta_n^*; h_n) = Q_0 + o_p(1)$ for any $\beta_n^*$ is between $\hat{\beta}_n$ and $\hat{\beta}_n$;

(b) $(nh_n)^{1/2}T_n^*(\hat{\beta}_n; h_n) = N(a_2^{-1}Q_0^{1/2}, D_0) + o_p(1)$. 

48
To prove (a), the fact that $\mathcal{Q}^*$ is a VC class implies that

$$
\sup_{\theta \in \Theta} \left\| Q_n^{*}(\theta; h_n) - \bar{Q}_n^{*}(\theta; h_n) \right\| = \sup_{\theta \in \Theta} \left\| \frac{2b_n}{n} \sum_{i=1}^{n} r_i (2A_i - 1) K''(\frac{x_i T \beta}{h_n}) \frac{\tilde{x}_i \tilde{x}_iT}{h_n^{2}} \tilde{Y}_i \right\|
$$

$$
\leq O_p((n^2 h_n^5)^{-1/2}) + O(b_n h_n^{-1}) \sup_{\theta \in \Theta} \left\| E_w \left\{ K''(\frac{x_i T \beta}{h_n}) \frac{\tilde{x}_i \tilde{x}_iT}{h_n} \tilde{x}_i \tilde{S} \right\} \right\|
$$

$$
= O_p((n^2 h_n^5)^{-1/2}) + O((nh_n^3)^{-1/2}) = o_{p_r}(1).
$$

It suffices to show that $\theta_n^* = (\hat{\beta}_n^* - \beta_0)/h_n = o_{p_r}(1)$. For $R_n^*(\theta)$ defined in the proof of Lemma 5, the fact that $\mathcal{H}^*$ is a VC class indicates that

$$
\sup_{\theta \in \Theta_n} \left\| R_n^*(\theta) - \tilde{R}_n^*(\theta) \right\| \leq O_p((n^2 h_n^5)^{-1/2}) + O(b_n h_n^{-1}) \sup_{\theta \in \Theta_n} \left\| E_w \left\{ K'(\frac{\tilde{z}_i}{h_n} + \theta^T \tilde{x}_i) \frac{\tilde{z}_i \tilde{S}}{h_n} \right\} \right\|
$$

$$
= O_p((n^2 h_n^5)^{-1/2}) + O((nh_n^3)^{-1/2}) = o_{p_r}(1),
$$

where $\Theta_n$ is defined in Lemma A3. Combined with Lemma B4, it implies that $\sup_{\theta \in \Theta_n} \left\| R_n^*(\theta) - Q_0 \theta \right\| \leq o(1) + \alpha_1 h_n \| \theta \| + \alpha_2 h_n \| \theta \|^2$. By the definition of $\theta_n^*$, we know that $h_n \theta_n^* = o_{p_r}(1)$, and $R_n^*(\theta_n^*) = o_{p_r}(1)$. Then from the proof of Lemma B4, (a) can be concluded. To prove (b), the proof of Lemma A5 and the VC class $\mathcal{H}^*$ implies that $\lim_{n \to \infty} \left\{ (nh_n)^{1/2} \right\} T_n^*(\beta_0; h_n) = \mathbb{E} \left\{ T_n^*(\beta_0; h_n) \right\} = 0$. Then observe that

$$
\lim_{n \to \infty} E_w \left\{ \left( (nh_n)^{1/2} T_n^*(\beta_0; h_n) - \mathbb{E} \left\{ T_n^*(\beta_0; h_n) \right\} \right) \right\} = a_2^{-1} Q_0 \tilde{z},
$$

$$
\lim_{n \to \infty} E_w \left\{ \text{Var}_w \left\{ (nh_n)^{1/2} T_n^*(\beta_0; h_n) \right\} \right\} = D_0.
$$

It is sufficient to prove that $\lim_{n \to \infty} \left\{ (nh_n)^{1/2} \gamma^T \left\{ T_n^*(\beta_0; h_n) - \mathbb{E} T_n^*(\beta_0; h_n) \right\} \right\} = \mathcal{N}(0, \gamma^T D_0 \gamma) + o_{p_r}(1)$ for any fixed vector $\gamma \in \mathbb{R}^{p-1}$ such that $\| \gamma \| = 1$. Define

$$
q_i^* = 2r_i (2A_i - 1) (nh_n)^{1/2} K' \left( \frac{x_i T \beta_0}{h_n} \right) \gamma^T \tilde{x}_i Y_i
$$

$$
= \tilde{q}_i^* + 2r_i (2A_i - 1) K' \left( \frac{x_i T \beta_0}{h_n} \right) \gamma^T \tilde{x}_i Y_i = \tilde{q}_i^* + q_i^*,
$$

49
for $q^*_i$ defined as in the proof of Theorem 4. To check the Lyapunov condition, it suffices to prove

$$
\lim_{n \to \infty} (s^*_n)^{-4} \sum_{i=1}^{n} E\{(q^*_i - E q^*_i)^4\} = 0,
$$

where $(s^*_n)^2 = \sum_{i=1}^{n} \text{Var}_w(q^*_i)$. Similarly as the proof of Theorem 5, the Lyapunov condition holds if $(s^*_n)^{-4} \sum_{i=1}^{n} E\{|w(q^*_i - E q^*_i)|^4\} \xrightarrow{a.s.} 0$, and $(s^*_n)^{-4} \sum_{i=1}^{n} (E\{|w q^*_i|^2\})^4 \xrightarrow{a.s.} 0$.

Since $r$ is sub-Gaussian, then $E|r|^k$ is finite for any positive integer $k$. Hence with bounded $K(\cdot)$ and $\bar{x}$ and fixed $s$, the strong law of large numbers and the continuous mapping theorem imply that

$$
(s^*_n)^{-4} \sum_{i=1}^{n} E\{|w(q^*_i - E q^*_i)|^4\} \xrightarrow{a.s.} \lim_{n \to \infty} E\{|w(s^*_n)|^2\}^{-2} \lim_{n \to \infty} E\{|w(q^*_i - E q^*_i)|^4\} = 0,
$$

$$(s^*_n)^{-4} \sum_{i=1}^{n} (E\{|w q^*_i|^2\})^4 \xrightarrow{a.s.} \lim_{n \to \infty} E\{|w(s^*_n)|^2\}^{-2} \lim_{n \to \infty} E\{|w q^*_i|^4\} = 0.
$$

This verifies the Lyapunov condition and (b) follows.

\[\square\]

### S7 Pseudo Codes for the Proximal Algorithm

**Algorithm 1** Proximal $(\beta^{(0)}, \alpha_0, \gamma)$

1: Set $t = 0$.
2: Set $\text{diff} = 0$.
3: while $\text{diff} \geq 0$ do
4: \hspace{1em} $t \leftarrow t + 1$.
5: \hspace{1em} $\alpha_t \leftarrow \gamma \alpha_{t-1}$.
6: \hspace{1em} $\delta_t \leftarrow (n \alpha_t)^{-1} \sum_{i=1}^{n} (2A_i - 1) K'\left(\frac{x_i^T \beta^{(t-1)}}{h_n}\right) \frac{x_i}{h_n} Y_i$.
7: \hspace{1em} $\beta^{(t)} \leftarrow \beta^{(t-1)} + \delta_t$.
8: \hspace{1em} $\text{diff} \leftarrow 2 \sum_{i=1}^{n} (2A_i - 1) Y_i \left\{ K\left(\frac{x_i^T \beta^{(t)}}{h_n}\right) - K\left(\frac{x_i^T \beta^{(t-1)}}{h_n}\right) - h_n^{-1} x_i^T \delta_t K'\left(\frac{x_i^T \beta^{(t-1)}}{h_n}\right) \right\} + \alpha_t ||\delta_t||^2$.
9: end while
10: Output $\beta^{(t)}$.

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