PERIODIC HIGGS BUNDLES OVER CURVES

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Abstract. In this article, we study periodic Higgs bundles and their applications. We obtain the following results: i). an elliptic curve has infinitely many primes of supersingular reduction if and only if any periodic Higgs bundle over it is a direct sum of torsion line bundles; ii). the uniformizing de Rham bundle attached to a generic projective hyperbolic curve is not one-periodic, and it is motivic iff it admits a modular embedding (e.g. Shimura curves, triangle curves); iii). there is an explicit Deuring-Eichler mass formula for the Newton jumping locus a Shimura curve of Hodge type. We propose the periodic Higgs conjecture, which would imply an arithmetic Simpson correspondence. The conjecture holds in rank one case.

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1. INTRODUCTION

This is a sequel to the work [KS20], in which we studied periodic parabolic de Rham bundles on curves. In this article, we study periodic Higgs bundles on curves. Let $C$ be a smooth projective curve over $\mathbb{C}$. Recall the following definition.

Definition 1.1 (Definition 3.13 of [KS20]). A parabolic Higgs bundle $(E, \theta)$ over $C$ is called periodic if the following properties hold:

(i) $(E, \theta)$ is graded;
(ii) the parabolic weight of $E$ is rational. Let $D$ be the support of its quasi-parabolic structure;


(iii) there exists a positive integer $f$ and a spreading-out
\[(C, \mathcal{D}, \mathcal{E}, \Theta) \to S,\]
where $S$ is an integral scheme of finite type over $\mathbb{Z}$, such that for all
gometric points $s \in S$, the reduction $(\mathcal{E}_s, \Theta_s)$ at $s$ is periodic of period
\[\leq f\] with respect to all $W_2(k(s))$-liftings $\tilde{s} \to S$.

This notion is independent of the choice of spreading-out.

**Definition 1.2.** A parabolic Higgs bundle $(E, \theta)$ on $C$ is called *motivic* if there
exists a non-empty open subset $U \subset C$, a smooth projective morphism $f : Y \to U$, and a non-negative integer $i \geq 0$ such that $(E, \theta)$ is isomorphic to a summand of
the canonical parabolic extension of
\[\text{Gr}_{\text{Fil}} H^i_{dR}(Y/U).\]

The notion of the canonical parabolic extension is essentially due to Iyer-Simpson. The main motivation for the notion of a periodic Higgs bundle is the fact that motivic Higgs bundles are periodic, which follows immediately from
[KS20, Theorem 1.3].

In this paper, we restrict ourselves to Higgs bundles over projective curves without parabolic or logarithmic structure. In this context, if a Higgs bundle $(E, \theta)$ is motivic, then there exists a smooth projective morphism $Y \to U$ such that $(E, \theta)|_U$ is a summand of $\text{Gr}_{\text{Fil}} H^i_{dR}(Y/U)$. The purpose of this paper is to study the arithmetic and geometric meaning of a Higgs bundle over $C$ being periodic.

It follows from [KS20, Proposition 3.15] that a periodic Higgs bundle over $C$ must be semistable of degree zero. Thus we start with the case of $C$ being a genus one curve.\footnote{There is no nontrivial semistable Higgs bundle of degree zero over $\mathbb{P}^1$.} To state our first discovery, we need the following definition.

**Definition 1.3.** A genus one curve $C/\mathbb{C}$ has infinitely many primes of supersingular reduction if for a spreading-out $C \to S$ of $C/\mathbb{C}$, where $S$ is of finite type over
$\text{Spec} (\mathbb{Z})$, there exist closed points $s$ of $S$ of arbitrarily large residue characteristic such that the reduction $C_s \to s$ is a supersingular genus one curve.

This notion is independent of the choice of spreading-out.

**Theorem (Theorem 2.1).** Let $C$ be a genus one curve over $\mathbb{C}$. Then the following
two statements are equivalent.

(i) There exists infinitely many primes of supersingular reduction of $C$.

(ii) Any periodic Higgs bundle over $C$ is a direct sum of torsion line bundles.

Again, we emphasize that property (i) refers to any spreading out of $C$.

**Remark 1.4.** If the defining field of $C$ is transcendental, then property (i) in
the above theorem holds because the number of supersingular elliptic curves over
$\mathbb{F}_p$ grows as $p \to \infty$. By a famous result of N. Elkies, it also holds true if $C$ is
defined either over a number field of odd degree over $\mathbb{Q}$ [Elk87] or a number field
that admits at least one real embedding [Elk89]. If one believes Conjecture 1.14

$^1$There is no nontrivial semistable Higgs bundle of degree zero over $\mathbb{P}^1$.\footnote{There is no nontrivial semistable Higgs bundle of degree zero over $\mathbb{P}^1$.}
below, then property (i) should hold for any elliptic curve over \( \mathbb{C} \)! Though the set of supersingular primes for \( C \) defined over (say) \( \mathbb{Q} \) is infinite, it is of Dirichlet density zero if \( C \) is non-CM \( [SB77] \).

We next turn to higher genus curves. Here is one natural question we want to address: can one understand the uniformization of a hyperbolic curve algebraically? G. Faltings \( [Fa83] \) gives a negative answer to the question; his result roughly says that one is unable to construct(characterize) the uniformizing flat connection \( (\nabla_{\text{unif}} \in \text{Fil}_{\text{unif}}) \) in the following diagram) in a purely algebraic manner. However, we note that every projective hyperbolic curve is indeed equipped with a related purely algebraic object: the uniformizing Higgs bundle \( (E_{\text{unif}}, \theta_{\text{unif}}) \), which is uniquely defined up to tensoring with a two-torsion line bundle.

\[
\begin{array}{ccc}
(V_{\text{unif}}, \nabla_{\text{unif}}, \text{Fil}_{\text{unif}}) & \xrightarrow{\text{GrFil}} & (E_{\text{unif}}, \theta_{\text{unif}})
\end{array}
\]

By the nonabelian Hodge theory over \( \mathbb{C} \), \( \nabla_{\text{unif}} \) can be obtained from \( (E_{\text{unif}}, \theta_{\text{unif}}) \) by solving the Higgs-Yang-Mills equation, which is transcendental in nature. Instead of considering the question only at the archimedean place, we spread out \( (E_{\text{unif}}, \theta_{\text{unif}}) \) and \( (V_{\text{unif}}, \nabla_{\text{unif}}, \text{Fil}_{\text{unif}}) \), and then study various mod \( p \) reductions for large \( p \)s. Specifically, we attach a canonical preperiodic de Rham-Higgs flow with initial term \( (V_{\text{unif}}, \nabla_{\text{unif}})_p \) as well as a canonical preperiodic Higgs-de Rham flow with initial term \( (E_{\text{unif}}, \theta_{\text{unif}})_p \). One asks about the possible relations between the algebraic structure of \( C \) and dynamical behaviors of these flows for varying \( p \). The following is perhaps the most basic question one could ask.

**Question 1.5.** When is the uniformizing de Rham bundle \( (V_{\text{unif}}, \nabla_{\text{unif}}, \text{Fil}_{\text{unif}}) \) over \( C \) one-periodic? When is the uniformizing Higgs bundle \( (E_{\text{unif}}, \theta_{\text{unif}}) \) one-periodic?

If every genus \( g \) algebraic curve over \( \mathbb{C} \) had one-periodic uniformizing de Rham bundle, then there would be an algebraic construction of the uniformizing flat connection of \( C \), as one may see from the following diagram:

\[
\begin{array}{ccc}
(V_{\text{unif}}, \nabla_{\text{unif}})_p & \xrightarrow{\text{GrFil}_{\text{unif},p}} & (E_{\text{unif}}, \theta_{\text{unif}})_p
\end{array}
\]

where \( C_p^{-1} \) is the inverse Cartier transform \( [OV] \) at the prime \( p \). In other words, if \( (V_{\text{unif}}, \nabla_{\text{unif}}) \) is one-periodic, then the following two statements hold.
• The uniformizing Higgs bundle \((E_{\text{unif}}, \theta_{\text{unif}})\) is one-periodic.
• For \(p \gg 0\), the inverse Cartier transforms \(C_p^{-1}\) of \((E_{\text{unif}}, \theta_{\text{unif}})_p\) “glue” to the uniformizing flat connection \((V_{\text{unif}}, \nabla_{\text{unif}})\).

As suggested by Faltings’ result, one does not expect this holds for a general curve. Call a smooth projective curve \(C/\mathbb{C}\) of genus \(g\) generic if the following holds: the associated moduli map \(\text{Spec}(\mathbb{C}) \to \mathcal{M}_g\) to the moduli stack of genus \(g\) curves over \(\text{Spec}(\mathbb{Z})\) is dominant. Then we have the following result.

**Theorem** (Theorem 3.2). Let \(C\) be a generic hyperbolic curve of genus \(g\). Then the uniformizing Higgs bundle \((E_{\text{unif}}, \theta_{\text{unif}})\) is not one-periodic. Consequently, away from a countable union of proper closed subsets of \(\mathcal{M}_g(\mathbb{C})\), neither the uniformizing de Rham bundle nor the uniformizing Higgs bundle is one-periodic.

One may speculate that \((E_{\text{unif}}, \theta_{\text{unif}})\) is not periodic for the generic curve. We indeed conjecture this to be the case, but proving this seems substantially more difficult than the one-periodic case. To investigate the situation of an arbitrary period, let us recall the following notion.

**Definition 1.6.** Let \(X/\mathbb{C}\) be a smooth algebraic curve and let \(W\) be a quaternionic Shimura variety uniformized by some product of upper half planes:

\[
\mathfrak{h}_1 \times \cdots \times \mathfrak{h}_r \to W.
\]

We say that a morphism \(X \to W\) is a modular embedding if there exists some index \(1 \leq j \leq r\) such that the induced map on universal covers \(\mathfrak{h} \to \mathfrak{h}_1 \times \cdots \times \mathfrak{h}_r\) projects to an isomorphism \(\mathfrak{h} \to \mathfrak{h}_j\).

We have then the following result, which is an elementary combination of results due to Corlette-Simpson [CS08] and the theory of modular embeddings.

**Theorem** (Proposition 3.6). Let \(X/\mathbb{C}\) be a smooth algebraic curve. Then the following are equivalent.

1. \(X\) admits a modular embedding;
2. \((V_{\text{unif}}, \nabla_{\text{unif}})\) is motivic;
3. \((E_{\text{unif}}, \theta_{\text{unif}})\) is motivic.

By [KS20, Theorem 1.3], the uniformizing de Rham bundle of a projective hyperbolic curve admitting a modular embedding is periodic.

**Remark 1.7.** If one assumes the de Rham version of Conjecture 1.11 below, then \((V_{\text{unif}}, \nabla_{\text{unif}}, \text{Fil}_{\text{unif}})\) is motivic if and only if it is periodic. Therefore, the hyperbolic curves \(C\) with periodic uniformizing de Rham bundles conjecturally belong to the narrow class: curves admitting modular embeddings. Note that Shimura curves give the first such examples. Recall that triangle curves are those smooth projective curves whose fundamental groups are commensurable to a triangle group. A result of [CW90] asserts that triangle curves admit also modular embeddings. A surprising conjecture due to Chudnovsky-Chudnovsky [CC90] even implies that Shimura curves and triangle curves are the only examples of smooth projective hyperbolic curves admitting modular embeddings. However, the conjecture remains largely open so far.\(^2\)

\(^2\)In the case of affine curves, there is a third type of known example: Teichmüller curves.
As remarked, the uniformizing Higgs bundle attached to a Shimura curve is periodic. We can refine its arithmetic meaning to obtain quantitative results. Indeed, using Deligne’s modèle étrange \[\text{[Del72, §6]},\] we are able to describe explicitly the canonical periodic Higgs de-Rham flow over the (good) reduction modulo \(p\) of a Shimura curve. As a direct application, it allows us to define the Hasse-Witt invariant attached to a Shimura curve with good reduction at \(p\). Combining this with the “one clump theorem” \[\text{[Kri18]}\] and the old result of \[\text{[SZZ]}\] for modèle étrange, we resolve \[\text{[SZZ, Conjecture 1.3]}\] for any Shimura curve of Hodge type. We elaborate on this point.

The classical Deuring-Eichler mass formula counts the number of supersingular \(j\)-invariants \[\text{[Deu41]}\] and may be deduced from the fact that the degree of the Hasse polynomial attached to the Legendre family of elliptic curves is \(\frac{p-1}{2}\). The formula may be regarded as the stacky mass of the supersingular locus of the modular curve with level one structure: let \(S\) be the set of isomorphism classes of supersingular elliptic curves over \(\mathbb{F}_p\). Then

\[
\sum_{[E] \in S} \frac{1}{|\text{Aut}(E)|} = \frac{p - 1}{24}.
\]

Since the stacky Euler characteristic of \(M_{1,1}(\mathbb{C})\) is equal to \(-\frac{1}{12}\), the right hand side of the above formula can be written as \((1 - p)^{\chi_{\text{top}}(M_{1,1}(\mathbb{C}))}\). As a result, if \(\mathcal{M}\) is the good reduction modulo \(p\) of a modular curve \(\mathcal{M}\), then the mass of the supersingular locus in \(\mathcal{M}\) is \((1 - p)^{\chi_{\text{top}}(\mathcal{M}(\mathbb{C}))}\). In this way, the mass formula for a modular curve at \(p\) is expressed as the product of two terms: the first is a linear in \(p\) and the second is purely topological. We will show this pattern persists for general Shimura curves of Hodge type.

Let \(F\) be a totally real field and \(D\) a quaternion algebra over \(F\) which splits at exactly one real place \(\tau\). Let \(G\) be the reductive \(\mathbb{Q}\)-group given as follows: \(G_{\mathbb{Q}} := \text{Res}_{F/\mathbb{Q}}D^\times\). Let \(M\) be an associated Shimura curve of Hodge type defined by a symplectic representation \(G_{\mathbb{Q}} \rightarrow \text{GSp}_{2n}\) together with a level structure. Let \(p\) be an odd prime of \(F\) such that \(p\) is unramified over \(\mathbb{Q}\) and does not divide the discriminant of \(D\). Set \(\mathbb{F}_q = k_p\), the residue field of \(F\) at \(p\). By a result of Kisin \[\text{[Kis10, Theorem 2.3.8]}\], there is a canonical smooth integral model \(\mathcal{M}\) of \(M\) over the ring of integers \(\mathcal{O}_{F_p}\), together with an abelian scheme over the integral model. Our result on Deuring-Eichler mass formula reads as follows.

**Theorem 1.8** (Corollary \[4.16\], Corollary \[4.20\]). Let \(\mathcal{M} \hookrightarrow \mathcal{A}_n\) be the good reduction of a Shimura curve as above.

(i) There are exactly two Newton strata appearing in \(\mathcal{M}(\mathbb{F}_p)\);

(ii) The Newton jumping locus \(N_p \subset \mathcal{M}(\mathbb{F}_p)\) is independent of symplectic representation of \(G_{\mathbb{Q}}\);

(iii) The mass formula reads:

\[
|N_p| = (1 - q)^{\chi_{\text{top}}(\mathcal{M}(\mathbb{C}))}.\]
Remark 1.9. There has been much recent work on the Ekedahl-Oort strata on the good reduction of Shimura varieties of Hodge type, in particular by Goldring-Koskivirta and Zhang [GK16, Zha14]. Their work yields another approach to parts of Theorem 1.8, but the techniques are rather different. We briefly discuss this circle of ideas.

Many generalized Hasse invariants for the good reduction of Hodge-type Shimura varieties have been recently constructed in [GK16] by constructing Hasse invariants on the stack of G-zips. These Hasse invariants cut out the (reduced) Ekedahl-Oort strata and are “automatically” Hecke-equivariant. The stack of G-zips is analogous to a period domain. Call their Hasse invariant $H^G_K$. Zhang proves that the open Ekedahl-Oort stratum is independent of the choice of symplectic embedding realizing $(G, X)$ as a Shimura datum of Hodge-type [Zha14, Theorem 0.1], which implies Theorem 1.8 (ii). Hence the dependence of $H^G_K$ on the choice of symplectic embedding is minimal and in particular the support of $\text{div}(H^G_K)$ is independent of the symplectic embedding. On the other hand, it is not clear that $\text{div}(H^G_K)$ is multiplicity-free. Our Hasse-Witt invariant is multiplicity-free by explicit calculation from [SZZ].

Recall that a parabolic Higgs bundle over $\mathbb{C}_{\log}$ is said to be motivic if it is the grading of a motivic parabolic de Rham bundle (see [KS20, Definition 2.16]. [KS20, Theorem 1.3] (and its proof) asserts that any motivic parabolic de Rham bundle over $\mathbb{C}_{\log}$ is periodic. As a consequence, we have the following

Theorem 1.10. Let $C$ be a smooth projective curve over $\mathbb{C}$ and $D$ a reduced effective divisor of $C$. Then any motivic parabolic Higgs bundle over $\mathbb{C}_{\log}$ is periodic.

We make the following conjecture.

Conjecture 1.11 (Periodic Higgs conjecture). Let $C, D$ be as in Theorem 1.10. Then any periodic parabolic Higgs bundle over $\mathbb{C}_{\log}$ is motivic.

We may show the conjecture in rank one case.

Proposition 1.12. A rank one parabolic Higgs bundle $(L, \theta)$ is periodic if and only if its parabolic pullback becomes trivial under some cyclic cover whose branch divisor contains the quasi-parabolic structure of $L$. In particular, Conjecture 1.11 holds for rank one objects.

An important weaker version of Conjecture 1.11 is the following

Conjecture 1.13 (Arithmetic Simpson correspondence). The functor $\text{Gr}$ from the category of periodic parabolic de Rham bundles over $C$ to the category of periodic parabolic Higgs bundles over $C$ is an equivalence of semisimple categories over $\mathbb{C}$.

We would like to single out the semisimplicity in the above conjecture, which is a purely algebraic property about periodic objects.

Conjecture 1.14 (Semisimplicity conjecture). Let $C$ be a smooth projective algebraic curve over $\mathbb{C}$. Then any periodic parabolic Higgs/de Rham bundle over $C$ is semisimple.
In particular, a periodic parabolic Higgs bundle over \( C \) is conjectured to be \textit{polystable} (compare \cite[Proposition 3.15]{KS20}). To approach the conjectural essential surjectivity of the grading functor, we would like to emphasize that this is actually a conjectural \textit{compatibility} of the Simpson correspondence over \( C \) with the Ogus-Vologodsky correspondence in characteristic \( p \).

**Conjecture 1.15** (Compatibility conjecture). Let \((E, \theta)\) be a one-periodic parabolic Higgs bundle over \( C \). Let \((V, \nabla)\) be the corresponding parabolic connection under the Simpson correspondence over \( C \) \cite{Sim90a}. Then for any spreading-out of \((C, V, \nabla)\) defined over \( S \), an integral scheme of finite type over \( \mathbb{Z} \), there exists a closed subset \( Z \subset S \) with finite image in \( \text{Spec} \mathbb{Z} \) such that for all geometric points \( s \in S - Z \) and all \( W_2(k(s))\)-liftings \( \tilde{s} \subset S - Z \), the following relation holds

\[
(V, \nabla)_s \cong C_{s \tilde{s}}^{-1}(E, \theta)_s.
\]

More colloquially, the above conjecture says that if \((E, \theta)\) is a one-periodic, then the inverse Cartier transforms modulo \( p \) for all \( p \gg 0 \) “glue” to a global connection, and moreover, this is the connection furnished by the complex Simpson correspondence.

## 2. Periodic Higgs bundles on genus one curves

This section begins our arithmetic study of periodic Higgs bundle. In this section, we shall prove the following

**Theorem 2.1.** Let \( C \) be a genus one curve over \( \mathbb{C} \). Then the following two statements are equivalent:

(i) There are infinitely many supersingular primes in any spreading-out of \( C \).

(ii) Any periodic Higgs bundle over \( C \) is a direct sum of torsion line bundles.

Let \( C \) be a spreading-out of \( C \), which is a smooth projective scheme over \( S \), an integral scheme of finite type over \( \mathbb{Z} \). If the defining field of \( C \) is transcendental, then \( S \) has positive relative dimension over \( \mathbb{Z} \). Since the number of supersingular \( j \)-invariants increases with \( p \), it follows that for almost all prime number \( ps \), there exist primes \( s \in S \) over \( p \) such that \( C \) is of supersingular reduction at \( s \). By Proposition 1.12, one obtains the following classification result.

**Corollary 2.2.** Let \( C \) be a very general elliptic curve over \( \mathbb{C} \). Then the set of periodic Higgs bundles over \( C \) is just the set of direct sums of torsion line bundles.

One can regard the corollary as a small piece of evidence for the periodic Higgs conjecture. This is because the periodic Higgs conjecture predicts that each periodic Higgs bundle is polystable and therefore is simply a direct sum of line bundles over an elliptic curve.

Let us first collect several basic facts on Higgs bundles over an elliptic curve. Let \( C \) be an elliptic curve over an algebraically closed field \( k \) and \((E, \theta)\) a Higgs bundle over \( C \). We set \( N \) to be the unique (up to isomorphism) indecomposable vector bundle of rank 2 on \( C \) that is the extension of \( \mathcal{O}_C \) by \( \mathcal{O}_C \).

**Lemma 2.3.** Let \( C/k \) be an elliptic curve and let \((E, \theta)\) be a Higgs bundle on \( C \). Then \((E, \theta)\) is Higgs semistable if and only if \( E \) is bundle semistable.
Proof. The result is well known. For convenience of the reader, we include a proof. Note that the if-part is clear by definition. Assume now \((E, \theta)\) is Higgs semistable. Assume to the contrary that \(E\) is unstable. Let \(F \subset E\) be the maximal destabilizer which is of positive degree. Then the Higgs field induces the composite map

\[ F \to E \overset{\theta}{\to} E \otimes \Omega_{C/k} \cong E \to E/F, \]

which is zero for degree reason. Hence \(F\) is a \(\theta\)-invariant subsheaf. But \(\mu(F) > \mu(E)\), which is a contradiction. \(\square\)

A Higgs bundle is indecomposable if it cannot be written into direct sum of two proper Higgs subbundles. Clearly, any Higgs bundle is a direct sum of indecomposable Higgs bundles.

Lemma 2.4. Assume \(\text{char } k = 0\). An indecomposable vector bundle of rank \(r\) over \(C\) is of form \(S^{r-1}N \otimes L\), where \(N\) is the unique nontrivial extension of \(\mathcal{O}_C\) by \(\mathcal{O}_C\), and \(L\) is a line bundle.

Proof. This is [Ati57, Theorem 9], see also [Oda, p. 60]. \(\square\)

We present two proofs of the following lemma.

Lemma 2.5. Assume \(\text{char } k = 0\). Any indecomposable semistable graded Higgs bundle \((E, \theta)\) of degree zero is, up to tensoring with a line bundle of degree zero, a successive extension of \((\mathcal{O}_C, 0)\)s.

Proof. Up to a shift of indices, we may write \(E = \bigoplus_{0 \leq i \leq n} E^i \) with \(\theta : E^i \to E^{i-1}\) nonzero for each \(i > 0\). By Lemma 2.3, \(E^i\) is written into a direct sum

\[ E^i = \bigoplus_{j} L^{ij} \otimes S^{r_{ij}}N, \]

where \(L^{ij}\) is some line bundle. We claim that the \(L^{ij}\)s are all isomorphic to each other. First we observe the following simple fact: If there is a nonzero morphism \(L \otimes S^{r'}N \to L' \otimes S^{r''}N\), then \(\text{deg } L \leq \text{deg } L'\); furthermore, if \(\text{deg } L = \text{deg } L'\), then \(L \cong L'\). Note also that, if \(\theta(L^{ij} \otimes S^{r_{ij}}N) = 0\) for some \(ij\), then \(\text{deg } L^{ij} \leq 0\) by semistability. As \(\theta\) is nilpotent, it follows that \(\text{deg } L^{ij} \leq 0\) for each \(ij\). But since \(\text{deg } E = 0\), one must have \(\text{deg } L^{ij} = 0\) for each \(ij\). Assume \(E^n\) contains \(m\) indecomposable factors. We denote by \(S_{il}, 1 \leq l \leq m\) the subset of \(L^{ij}\)s in \(E^i\) with \(L^{ij} \cong L^{n_{il}}\). First, each \(L^{ij}\) belongs to some \(S_{il}\). This is because the direct sum of these indecomposable factors in \(E^i\) with \(L^{ij}\) in some \(S_{il}\), together with the induced Higgs field, forms a direct factor of \((E, \theta)\) which must be equal to \((E, \theta)\) since \((E, \theta)\) is indecomposable. Second, for any \(l_1 \neq l_2\), there exists some \(i\) such that \(S_{il_1} \cap S_{il_2} \neq \emptyset\). This is also clear, for otherwise the direct sum of those indecomposable factors in \(E^i\) with \(L^{ij}\) in \(S_{il}\) forms a proper direct factor of \((E, \theta)\), contradicting the indecomposability of \((E, \theta)\). Therefore \(L^{n_{il}}\)s are isomorphic to each other by the second point, and all \(L^{ij}\)s are isomorphic by the first point. The claim is proved. Therefore,

\[ (E, \theta) = (L, 0) \otimes (E', \theta'), \]

where \(L\) is a line bundle of degree zero and each indecomposable factor of \(E'\) is of form \(S'N\). To show that \((E', \theta')\) is a successive extension of \((\mathcal{O}_C, 0)\)s, we do by
induction on rank. Note that we shall not use the indecomposibility of \((E', \theta')\) in the following argument. So \(E' = \bigoplus_i E_i'\) with \(E_i' = \bigoplus_j S_{r_j} N\). Any \(S_{r_j} N\) contains a unique subline bundle \(O_C\) which gives rise to a Higgs sub line bundle \((O_C, 0)\) of \((E', \theta')\). Note that \((E', \theta')/(O_C, 0)\) is again graded and the bundle part is a direct sum of \(S^a N\). By induction, it is a successive extension of \((O_C, 0)s\). The lemma follows. □

Second proof of Lemma 2.5. We prove a slightly stronger statement: if \((E, \theta)\) is semistable Higgs bundle of degree 0 on \(C\), then up to tensoring with a line bundle of degree zero, \((E, \theta)\) is the successive extensions of \((O_C, 0)s\).

By the Simpson correspondence, it suffices to prove the following. Let \(\Pi := \mathbb{Z}^2\). Let \(V\) be an indecomposable finite dimensional complex representation of \(\Pi\) of dimension \(r\). Then there exists a character \(\chi\) of \(\Pi\) such that \(V \otimes \chi^{-1}\) is the iterated extension of trivial characters of \(\Pi\).

We prove this claim by induction. The base case is \(r = 1\), which is trivial. Now suppose we have proven the claim for rank \(r - 1\) and let \(V\) be an indecomposable finite dimensional representation of \(\Gamma\) of rank \(r\). As \(\Pi\) is abelian, \(V^{ss}\) is the direct sum of characters. Therefore there is a short exact sequence:

\[
0 \to W \to V \to \chi \to 0.
\]

By induction hypothesis, we may suppose that \(W\) is isomorphic to an iterated extension of a character \(\psi\). As \(V\) is supposed to be indecomposable, the associated extension class in \(\text{Ext}^1_{\Pi}(\chi, W)\) must be non-zero. On the other hand, this extension group is isomorphic to the topological cohomology group \(H^1(\mathbb{T}^2, \chi^{-1} \otimes W)\), where \(\mathbb{T}^2\) is the 2-torus because \(\mathbb{T}^2\) is a \(K(\Pi, 1)\). We claim this cohomology group vanishes if \(\chi \neq \psi\). Indeed, by the long exact sequence in cohomology, this vanishing reduces to the following fact: \(H^1(\mathbb{T}^2, \chi^{-1} \otimes \psi)\) vanishes if \(\chi \neq \psi\). Therefore \(\chi = \psi\), as desired. □

Lemma 2.6. Assume \(\text{char } k = 0\). Let \(E = (E, \theta)\) be a semistable graded Higgs bundle of degree zero. Consider the set \(S\) whose elements are direct sum of degree zero Higgs sub line bundles of \((E, \theta)\). Then \(S\) is nonempty and has a unique maximal element.

Proof. We write \(E = \bigoplus_i E_i\) into a direct sum of indecomposable graded Higgs bundles. Clearly each \(E_i\) is semistable of degree zero. By Lemma 2.5 each \(E_i\) contains a degree zero Higgs sub line bundle. Hence \(S\) is nonempty. Assume we have two maximal elements in \(S\), say \(L = \bigoplus_i L_i\) and \(M = \bigoplus_j M_j\). Consider the summation map

\[
\alpha : L \oplus M \to E.
\]

As both sides of \(\alpha\) are semistable of the same degree, \(\text{im } \alpha\) is also semistable of degree zero. As \(L \oplus M\) is polystable of degree zero, \(\text{im } \alpha\) is a direct sum of degree zero line bundles and has strictly larger rank than that of \(L\) which contradicts the maximality of \(L\). The lemma is proved. □

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3The argument we give will work more generally for \(\Pi = \mathbb{Z}^k\)
4R.K. learned this from Michael Gröchenig, see the Betti proof of [Gro18, Lemma 1.8].
Finally, we introduce the following notation. The (nilpotent) Higgs field $(\mathcal{O}_C^{\oplus 2}, id)$ refers to the matrix

$$
\text{id} := \begin{pmatrix}
0 & I_{\text{d} \mathcal{O}_C} \\
0 & 0
\end{pmatrix} : \mathcal{O}_C^{\oplus 2} \to \mathcal{O}_C^{\oplus 2} \otimes \Omega^1_C \cong \mathcal{O}_C^{\oplus 2}.
$$

If $\mathcal{C} \to S$ is a spreading-out of $C$, then this Higgs bundle also clearly spreads out over $\mathcal{C} \to S$.

**Lemma 2.7.** Let $C/\mathbb{C}$ be an elliptic curve. Then $C$ has infinitely many primes of supersingular reduction if and only if the Higgs bundles $(N,0)$ and $(\mathcal{O}_C^{\oplus 2}, id)$ are non-periodic.

**Proof.** Recall the notion of having infinitely many primes of supersingular reduction: let $C \to S = \text{Spec} \,(A)$ be a spreading-out of $C$, where $A \subset \mathbb{C}$ is a subring of finite type over $\mathbb{Z}$. Then $C \to S$ has infinitely many primes of supersingular reduction if and only there are points of $S$ of arbitrarily large residue characteristic such that the fiber $C_s$ is isomorphic to a supersingular elliptic curve.

Let us first show the if part. Let $s$ be a closed point of $S$. We claim that $(N,0)$ is non-periodic at $s$ if and only if $C_s$ is supersingular. Indeed, if $s$ is a supersingular prime, the bundle part of $C^{-1}(N_s,0)$ is $F_s^* N_s \cong \mathcal{O}_C^{\oplus 2}$, while its connection part is the canonical connection. So $\text{Gr} \circ C^{-1}(N_s,0) \cong (\mathcal{O}_C^{\oplus 2},0)$, which is non-isomorphic to $(N_s,0)$. A simple fact is that the trivial Higgs bundle $(\mathcal{O}_C^{\oplus 2},0)$ goes to the trivial Higgs bundle under the flow operator (with respect to any Hodge filtration). Therefore $(N_s,0)$ is preperiodic but not periodic. On the other hand, if $s$ is an ordinary prime, the bundle of $C^{-1}(N_s,0)$ is nothing but $N_s$ and therefore $\text{Gr}_{\text{Fil}^r} \circ C^{-1}(N_s,0) \cong (N_s,0)$, where $\text{Fil}^r$ stands for the trivial Hodge filtration. So $(N_s,0)$ is (one-)periodic at $s$. The claim is proved. Therefore, if $(N,0)$ is non-periodic, then $C$ must have infinitely many primes of supersingular reduction.

Next we show the only-if part. It suffices to show that $(\mathcal{O}_C^{\oplus 2}, id)$ is non-periodic if $C$ has infinitely many supersingular primes. We claim that that $(\mathcal{O}_C^{\oplus 2}, id)$ is non-periodic at $s$ if and only if the bundle of $C^{-1}(\mathcal{O}_C^{\oplus 2}, id)$ is isomorphic to $N_s$, the unique non-split extension of $\mathcal{O}_C$ by $\mathcal{O}_C$. Let us assume that the bundle of $C^{-1}(\mathcal{O}_C^{\oplus 2}, id)$ is isomorphic to $\mathcal{O}_C^{\oplus 2}$. Since the connection of $C^{-1}(\mathcal{O}_C^{\oplus 2}, id)$ has nonzero $p$-curvature, there exists a sub line bundle which is isomorphic to $\mathcal{O}_C$ and not invariant under the connection. Taking it as the Hodge filtration of $C^{-1}(\mathcal{O}_C^{\oplus 2}, id)$, one obtains the periodicity for $(\mathcal{O}_C^{\oplus 2}, id)$ in this case. This shows the only-if part of the claim. It remains to show the if part. Now suppose that the bundle of $C^{-1}(\mathcal{O}_C^{\oplus 2}, id)$ is isomorphic to $N_s$. Hence $\text{Gr} \circ C^{-1}(\mathcal{O}_C^{\oplus 2}, id)$ is either $(N_s,0)$ or $(\mathcal{O}_C^{\oplus 2},0)$. For either case, it is easily seen that it can only go to one of these two possibilities under flow operator. In particular, it never flows back to $(\mathcal{O}_C^{\oplus 2}, id)$ and is hence non-periodic. It is well-known that for any $W_2$-lifting of a supersingular elliptic curve, the relative Frobenius does not lift. In other words, with respect to any $W_2$-lifting of $C_s$, the obstruction class in $H^1(C_s, F_{C_s}^* T_{C_s}) \cong H^1(C_s, \mathcal{O}_{C_s})$ of lifting Frobenius is nonzero. Finally, as the bundle part of $C^{-1}(\mathcal{O}_C^{\oplus 2}, id)$ is the exponential twisting of $F_{C_s}^* (\mathcal{O}_C^{\oplus 2}) \cong \mathcal{O}_C^{\oplus 2}$, it is
easy to see it is isomorphic to \( N_s \) by its very construction \cite{LSZ15}. This concludes the proof.

\[ \square \]

**Proposition 2.8.** Notation as Lemma 2.7. Suppose that \( C \) has infinitely many supersingular primes. Then any Higgs bundle which is a nontrivial extension of \((\mathcal{O}_C^{\oplus r}, 0)\) by \((\mathcal{O}_C^{\oplus s}, 0)\) with \( r, s \geq 1 \) is non-periodic.

**Proof.** Certainly we may assume \( E \) be graded (otherwise it cannot be periodic), so we write \( E \) into a direct sum of indecomposable graded Higgs bundles. As the bundle of \( E \) is an extension of \( \mathcal{O}_C^{\oplus r} \) by \( \mathcal{O}_C^{\oplus s} \), it follows that the bundle of \( E \) is a direct sum \( S^{\oplus n_s} \) by Lemma 2.5. We do induction on the rank of \( E \). The rank two case is nothing but Lemma 2.7. Assuming the truth for any such a rank \( n \) Higgs bundle, we consider the rank \( n + 1 \) case. By Lemma 2.5, we may find a Higgs subbundle \((\mathcal{O}_C, 0)\) of \( E \), so that we have a short exact sequence of graded Higgs bundles

\[
0 \to (\mathcal{O}_C, 0) \to E \xrightarrow{\pi} F \to 0.
\]

If \( F \) is a nontrivial extension of \((\mathcal{O}_C^{\oplus r'}, 0)\) by \((\mathcal{O}_C^{\oplus s'}, 0)\) for some \( r', s' \), then \( F \) is non-periodic by induction hypothesis. As \((\mathcal{O}_C, 0)\) is certainly periodic, \( E \) cannot be periodic: the category of periodic Higgs bundles is abelian by \cite[Lemma 3.10]{KS20}. So we may assume \( F \) is the trivial Higgs bundle \((\mathcal{O}_C^{\oplus n'}, 0)\) and the above exact sequence is non-split. We consider various projections \( p_i : F = \mathcal{O}_C^{\oplus n} \to \mathcal{O}_C^{\oplus n-1}, 0 \leq i \leq n \), where the \( i \)-th projection means omitting the \( i \)-th factor of \( F \). Set \( E_i \) to be the kernel of \( \pi \circ p_i \). We claim that at least one of \( E_i \), which is a rank two graded Higgs bundle, must be a nontrivial extension of \( \mathcal{O}_C \) by \( \mathcal{O}_C \). The claim implies the \( n + 1 \) case because of Lemma 2.7 and \cite[Lemma 3.10]{KS20}. To show the claim, we assume the contrary that each \( E_i \) is trivial and look at the summation map

\[
\alpha : \oplus_i E_i \to E
\]

It is easy to check the following properties of \( \alpha \): i) \( \ker \alpha \) is isomorphic to \( \mathcal{O}_C^{\oplus n} \); ii) the composite \( \pi \circ \alpha : \oplus_i E_i \to F \) is surjective. Hence \( \text{im } \alpha = \sum_i E_i \subset E \) gives rise to a splitting of \( \pi \). This is a contradiction.

\[ \square \]

We can now proceed to the proof of Theorem 2.1.

**Proof of Theorem 2.7.** The direction from (ii) to (i) is a consequence of Lemma 2.7. So it remains to show the direction from (i) to (ii). We prove it by induction on the rank of \( E \). The rank one case follows from Proposition 1.12. For the induction step, we use the existence of the unique maximal set of direct sum of degree zero Higgs sub line bundles \( S = \oplus_i L_i \subset E \), as proved in Lemma 2.6. If \( E \) is periodic, then \( S \) must be also periodic by uniqueness. By induction, \( S \) and the quotient bundle \( E/S \) are direct sum of torsion line bundles. We claim that the extension of \( E/S \) by \( S \) defined by \( E \) is trivial. Clearly, the claim completes the whole proof. As the (non)-triviality of the extension class will not change under isogeny, we may take a suitable isogeny \( n : C \to C \), such that \( n^* E \) becomes an extension of \((\mathcal{O}_C^{\oplus r}, 0)\) by \((\mathcal{O}_C^{\oplus s}, 0)\). If the extension class of \( n^* E \) was nontrivial, then Proposition 2.8 asserts that \( n^* E \) is non-periodic. Consequently, \( E \) cannot be periodic, which contradicts our assumption, so the extension must be trivial. \[ \square \]
3. Uniformization of hyperbolic curves

In this section, we explain arithmetic applications to the theory of (hyperbolic) uniformization of smooth projective curves of genus $\geq 2$. From the perspective of nonabelian Hodge theory, uniformization is given by the Hitchin-Simpson uniformization: let $C$ be a connected smooth projective curve of genus $\geq 2$. Choose and then fix a square root $K^{1/2}_C$ of the canonical bundle of $C$. Form the uniformizing Higgs bundle attached to $C$: $E_{unif} \cong K^{1/2}_C \oplus K^{-1/2}_C$ and $\theta_{unif}$ is given by the matrix:

$$\theta_{unif} := \begin{pmatrix} 0 & Id_{K^{1/2}_C} & 0 \\ 0 & 0 & 0 \end{pmatrix} : E_{unif} \to E_{unif} \otimes K_C \cong K^{3/2}_C \oplus K^{1/2}_C.$$

The complex local system corresponding to $(E_{unif}, \theta_{unif})$ under the Simpson correspondence is the complexification of the $\mathbb{R}$-PVHS coming from $\rho_{unif} : \pi^\text{top}_1(C) \to \text{SL}_2(\mathbb{R})$, the projectivization of which is exactly the uniformizing monodromy representation. Let $(V_{unif}, \nabla_{unif}, \text{Fil}_{unif})$ be the de Rham bundle over $C$ corresponding to $\rho_{unif}$ under the Riemann-Hilbert correspondence. One knows that $\text{Gr}_{\text{Fil}_{unif}}(V_{unif}, \nabla_{unif}) = (E_{unif}, \theta_{unif})$; Moreover, $V_{unif}$ is the unique nontrivial extension of $K^{-1/2}_C$ by $K^{1/2}_C$ and $\text{Fil}_{unif} \cong K^{1/2}_C$ is the maximal destabilizer of $V_{unif}$. G. Faltings attempted to single out $\nabla_{unif}$ among a complex affine space $\mathbb{A}$ of $3g - 3$ dimension using the sole condition that the corresponding local system is real. It turns out that it is not the case, as Faltings showed that such a set does not consist of only one point but probably a discrete subset of $\mathbb{A}$. On the other hand, S. Mochizuki developed a $p$-adic Teichmüller theory which is a $p$-adic analogue of the strategy of G. Faltings. Later, the work put a major part of Mochizuki’s theory into the framework of $p$-adic nonabelian Hodge theory. In this section, the basic question we are going to investigate is the following

**Question 3.1.** Notation as above. When is the uniformizing de Rham bundle $(V_{unif}, \nabla_{unif}, \text{Fil}_{unif})$ periodic? How about the uniformizing Higgs bundle $(E_{unif}, \theta_{unif})$?

As explained in the introduction, the motivation of the question is, like that of Faltings, an algebraic characterization/construction of $\nabla_{unif}$.

3.1. Non one-periodicity for generic curves. Let $C$ be a generic projective hyperbolic curve. That is, $C$ is a connected smooth projective curve of genus $g \geq 2$ over $\mathbb{C}$ such that for some spreading-out $\mathcal{C}$ of $C$, defined over $S$ over $\mathbb{Z}$, the associated moduli map $S \times \mathbb{F}_p \to \mathcal{M}_g \times \mathbb{F}_p$ is dominant for all $p \gg 0$. In this subsection, we show the following

**Theorem 3.2.** Let $C$ be a generic projective hyperbolic curve. Then neither $(V_{unif}, \nabla_{unif}, \text{Fil}_{unif})$ nor $(E_{unif}, \theta_{unif})$ over $C$ is one-periodic.

**Proof.** First observe that it suffices to show $(E_{unif}, \theta_{unif})$ is not one-periodic. The proof uses some basic results established in [LSYZ19a]. In our argument, we may always choose a large enough $p$. Let $s \in S$ be a closed point over $p$, such that the
reduction \((E_{\text{unif}}, \theta_{\text{unif}})_s\) over the smooth projective curve \(C_s\) is stable. We claim that if \((E_{\text{unif}}, \theta_{\text{unif}})_s\) is periodic at \(s\), then each term in the periodic flow must be stable. Indeed, any periodic Higgs bundle is semistable by Lemma [LSYZ19b, Theorem 1.5]. On the other hand, the penultimate de Rham term has a unique (up to shifting index) filtration whose associated graded is stable by [LSYZ19b, Lemma 7.1]. As inverse Cartier is an equivalence of categories, this means the penultimate Higgs term is stable. Iterating this procedure yields the result. In fact, by the uniqueness of the filtration, it follows that the filtration for a periodic flow at each step must be the Harder-Narasimhan filtration (up to a shift of index). Once there is no ambiguity in the choice of filtrations, the choice of \(W_2\)-liftings is the only factor in defining a Higgs-de Rham flow initializing \((E_{\text{unif}}, \theta_{\text{unif}})_s\).

In [LSYZ19a], the affine subspace \(A\) and the periodic cone \(K\) which are closed subvarieties of \(H^1(C_s, F^sK_{C_s}^{-1})\) are introduced. The basic property of the subset \(A \cap K \subset H^1(C_s, F^sK_{C_s}^{-1})\), as shown in the proof of [LSYZ19a, Proposition 3.6], is that its each element corresponds to an isomorphism class of \(W_2\)-liftings such that the maximal destabilizer of the corresponding inverse Cartier transform of \((E_{\text{unif}}, \theta_{\text{unif}})_s\) is of degree \(g - 1\). Therefore, the set \(A \cap K\) contains all possible \(W_2\)-liftings such that \((E_{\text{unif}}, \theta_{\text{unif}})_s\) can be one-periodic (in fact each element of \(A \cap K\) gives rise to a periodic flow of period at most two). It remains to show, under the generic condition, there always exists some \(W_2\)-lifting of \(C_s\), corresponding to a first order thickening \(\tilde{s} \subset S\), such that the isomorphism class is not in \(A \cap K\). For a given moduli point in \(M_g \times \overline{\mathbb{F}}_p\), it is hard to determine the size of the set \(A \cap K\) explicitly. However, it is determined in Proposition 4.7 [LSYZ19a] that for a totally degenerate curve, the set \(A \cap K\) consists of one unique point. Therefore, by a degeneration argument (see the proof of [LSYZ19a, Proposition 4.9]), for a general moduli point in \(M_g \times \overline{\mathbb{F}}_p\), the set \(A \cap K\) is a nonempty and finite. Now that the map \(S \times \overline{\mathbb{F}}_p \to M_g \times \overline{\mathbb{F}}_p\) is dominant, there exists an \(s \in S \times \overline{\mathbb{F}}_p\) such that the map is étale at \(s\) and its image in \(M_g \times \overline{\mathbb{F}}_p\) satisfies the previous property. As \(A \cap K\) is only a finite set and the tangent space of \(S\) at \(s\) is of positive dimension, one finds \(W_2\)-liftings \(\tilde{s}\) of \(s\) such that \((E_{\text{unif}}, \theta_{\text{unif}})_s\) cannot be periodic with respect to those liftings. Therefore for such \(s\), \((E_{\text{unif}}, \theta_{\text{unif}})_s\) is not periodic at this prime. As \(s\) can have arbitrary large characteristic, the bundle \((E_{\text{unif}}, \theta_{\text{unif}})\) over \(C\) is not periodic, as claimed. \(\square\)

Remark 3.3. By Conjecture [L4] the uniformizing de Rham/Higgs bundle over a generic bundle is not \(f\)-periodic for any \(f \in \mathbb{N}_{>0}\). Using the current technique, it seems difficult to show this. The work [Mo99] is perhaps related to this question.

3.2. Motivicity and modular embeddings. In this subsection, we summarize the theory of modular embeddings and use the work of Corlette-Simpson to prove Theorem 3.7 and Theorem 3.9. The theory of modular embeddings is due primarily to Cohen-Wolfart, Schmutz Schaller-Wolfart, and Kucharczyk [CW90, SM00, Kuc18]. We closely follow [Kuc18]. Let \(F\) be a totally real field of degree \(d\) over \(\mathbb{Q}\) and let \(D\) be a quaternion algebra over \(F\). We index the embeddings \(\tau_i: F \hookrightarrow \mathbb{R}\) so that only the places \(\tau_1, \ldots, \tau_r\) split \(B\). We further assume

\[\text{We call this type of argument the black-hole principle.}\]
that $D$ is split at at least one $\tau$. For every $1 \leq j \leq r$ we fix an isomorphism
\[ s_j : B \otimes_{\tau_j} \mathbb{R} \cong M_{2 \times 2}(\mathbb{R}) \].

Fix $\mathcal{O} \subset B$ an order and $\mathcal{O}^1$ be the invertible elements of reduced norm 1. By the above choices, $\mathcal{O}^1$ embeds as an arithmetic lattice inside of $(SL_2(\mathbb{R}))^r$. Moreover, any subgroup $\Gamma \subset (PGL_2(\mathbb{R}))^r$ commensurable to the image of $\mathcal{O}^1$ is also an arithmetic lattice.

**Definition 3.4.** Let $\Gamma$ be as above, and let $\Delta \subset PGL_2(\mathbb{R})^+$ be a lattice, and let $1 \leq j \leq r$. A modular embedding for $\Delta$ with respect to $\tau_j$ is a pair $(\tilde{f}, \phi)$ where

(i) the map $\phi$ is a group homomorphism $\Delta \to \Gamma$;

(ii) the map $\tilde{f}$ is a holomorphic map $\mathfrak{h} \to \mathfrak{h}_1 \times \ldots \times \mathfrak{h}_r$ from the upper half plane to the product of upper half planes

such that $\tilde{f}$ is equivariant for the homomorphisms $\phi$ and the composition $\mathfrak{h} \to (\mathfrak{h})^r \to \mathfrak{h}_j$ is a biholomorphism.

That this definition is compatible with [Kuc18, Definition 1.1] follows from Proposition 4.1 and the proceeding remarks on p. 218 of loc. cit.

Let $X := \frac{[\mathfrak{h}/\Delta]}{}$ and $S := (\mathfrak{h})^r/\Gamma$; here $S$ is a (connected) quaternionic polydisk Shimura variety. Then the data of $(\tilde{f}, \phi)$ yields a holomorphic map
\[ f : X \to S \]

which is (uniquely) algebraizable by a fundamental theorem of Borel. We sometimes refer to $f$ as the modular embedding.

There is a dichotomy between when $B/F$ is (globally) split or not: $B$ is a division algebra if and only if $S$ is compact [Kuc18, p. 213]. When $B \cong M_{2 \times 2}(F)$, the resulting Shimura variety is called a Hilbert-Blumenthal modular variety.

Let $S$ be a quaternionic polydisk Shimura variety. Then for each place $\tau_j : F \hookrightarrow \mathbb{C}$ there is a tautological rank 2 local system $L_j$ on $S$ given by
\[ \pi_1(S) \cong \Gamma \hookrightarrow B \otimes_{\tau_j} \mathbb{C} \to M_{2 \times 2}(\mathbb{C}) \].

More precisely, for $1 \leq j \leq r$, $L_j$ is a local system with values in $SL_2(\mathbb{R})$ and for $r + 1 \leq j \leq d$, $L_j$ has values in $SU(2)$, thought of as the norm 1 units of $\mathbb{H}$.

**Lemma 3.5.** [Kucharczyk] Let $f : X \to S$ be a morphism from an algebraic curve to a quaternionic polydisk Shimura variety as above. Then the following are equivalent.

(i) $X$ admits a modular embedding.

(ii) There exists $1 \leq j \leq r$ such that $f^*L_j$ is Fuchsian on $X$.

More precisely, assume (ii). If $X$ is compact, $f$ will be a modular embedding. If $X$ is affine, then there exists a finite étale cover $g : X' \to X$ so that $g \circ f$ is modular.

**Proof.** First of all, if $f^*L_j$ is Fuchsian on $X$, then we claim that $1 \leq j \leq r$. Indeed, otherwise the Higgs field would be identically 0.

When $X$ is projective, this follows from the equivalence of (i) and (ii) in [Kuc18, Theorem 4.5]. When $X$ is affine it follows from the equivalence of (i) and (iii) in
[Kuc18, Theorem 4.9] together with Proposition 2.8 of *loc. cit.* Note that the finite cover is only needed to ensure that the monodromy is unipotent. Finally, if \( g \circ f : X' \to S \) is modular and the degree of \( g \) is \( a \), then we may construct a modular embedding \( X \to S \times \cdots \times S \) to the \( a \)-fold product of \( S \).

\[ \square \]

**Lemma 3.6.** Let \( X/\mathbb{C} \) admit a modular embedding. Then the (logarithmic) uniformizing flat connection \((V_{\text{unif}}, \nabla_{\text{unif}})\) is motivic and is hence periodic.

**Proof.** Let \( f : X \to S \) be a modular embedding. By choosing a strange model as in Appendix \([\text{A}]\), one may construct a non-canonical abelian scheme \( A_S \to S \) whose cohomology contains all of the tautological local systems as summands. Pulling back, we see that the uniformizing local system is a summand of the first singular cohomology of \( A_X \to X \). It follows that the uniformizing flat connection is a summand of \( \mathcal{H}_{\text{dR}}(A_X/X) \). By the main theorem of [KS20], the uniformizing flat connection is therefore periodic.

\[ \square \]

The following theorem follows from an elementary combination of results due to Simpson and Corlette-Simpson.

**Theorem 3.7.** (Simpson, Corlette-Simpson) Let \( X/\mathbb{C} \) be a smooth hyperbolic curve. Let \( A \subset \mathbb{C} \) be a ring of algebraic integers and let \( \rho : \pi_1^{\text{top}}(X) \to SL_2(A) \) be an \( A \)-local system such that

- for every embedding \( \sigma : A \hookrightarrow \mathbb{C} \), the induced integral representation \( \sigma \rho : \pi_1^{\text{top}}(X) \to SL_2(A) \to SL_2(\mathbb{C}) \) underlies a complex (polarized) variation of Hodge structure.

Then there exists a map \( f : X \to S \) to a quaternionic polydisk Shimura variety as above such that

- for one of the tautological local systems \( L_i \) on \( S \), \( f^* L_i \) is isomorphic to the local system attached to \( \rho \otimes \mathbb{C} \);
- all of the other local systems \( f^* L_j \) are Galois conjugate to \( f^* L_i \).

**Proof.** That the sum

\[ \bigoplus_{\sigma : A \to \mathbb{C}} (\sigma \rho)^{\otimes 2} \]

comes from a family of abelian varieties is stated as [CS08, Lemma 8.3] and is attributed to Simpson. We sketch the argument for completeness.

First we copy of an argument from [Sim92, p. 57-58]. Let \( L \) be the field \( \mathbb{Q}(\text{Tr}(\rho(\gamma)))_{\gamma \in \pi_1(X)} \) generated by the \( \rho \)-traces of elements of \( \pi_1(X) \). Now, each \( \sigma \rho \) is polarized, we see that \( \bar{\sigma} \rho = \sigma^* \rho \). In particular, \( \sigma^{-1} \bar{\sigma} \rho \cong \rho^* \). Therefore for \( \gamma \in \pi_1(X) \),

\[ \sigma^{-1} \bar{\sigma} \text{Tr} \rho(\gamma) = \text{Tr} \rho^*(\gamma) \]

\[ \text{this choice depends on some auxiliary data} \]
is independent of $\sigma$. Moreover, the traces of $\rho^*$ are all in $L$. Therefore $C := \sigma^{-1}\bar{\sigma}$ yields a well-defined field automorphism of $L$. Now for each complex embedding $\sigma : L \hookrightarrow \mathbb{C}$, $C$ is the restriction of complex conjugation. Therefore $L$ is either totally real (if $C$ is the identity) or a CM field, i.e., a totally imaginary (quadratic) extension of a totally real field.

A lemma due to M. Larsen [Sim92, Lemma 4.8] then implies that $A$ may be chosen to be the ring of integers in a purely imaginary quadratic extension of a totally real field, see also the beginning of [CS08, §10]. Now we claim that

$$\bigoplus_{\sigma : A \rightarrow \mathbb{C}} (\sigma \rho)^{2i}$$

underlies an effective weight 1 variation of Hodge structures. If $\sigma \rho$ has two different non-zero Hodge numbers, then by Griffiths transversality the two non-zero Hodge numbers must be next to each other for the variation to not be isotrivial. By twisting, we may arrange for the VHS on $\sigma \rho$ to be effective of weight 1. If $\sigma \rho$ is unitary, we add another copy and twist so that it has type $(0,1)$ and $(1,0)$. Now the arguments of [CS08, §10] apply. Specifically, Proposition 10.2, Lemmas 10.3 and 10.4, and finally Proposition 10.5 of loc. cit. all hold for $\rho$ (despite the fact that $\rho$ may well not be rigid!). In other words, the hypothesis that every Galois conjugate of $\rho$ underlies a PVHS is precisely what one needs to make the arguments of §10 of loc. cit. go through. Then, [CS08, Theorem 9.3] implies there is a map to $f : X \rightarrow S$ where $S$ is a “polydisk Shimura modular DM stack”. Here, polydisk implies that the universal cover is of the form

$$\mathfrak{h} \times \cdots \times \mathfrak{h} \rightarrow S,$$

which yields $\dim S$ complex rank 2 local systems $L_i$ on $S$. There exists an $i$ such that $f^*L_i$ is isomorphic to the local system attached to $\rho \otimes \mathbb{C}$. We may choose $S$ such that tautological rank 2 local systems are all Galois conjugate to our fixed $L_i$. Therefore, on the induced abelian scheme $\pi : A \rightarrow X$, we have the following decomposition:

$$R^1\pi_*\mathbb{C} \cong \bigoplus_{\sigma : A \rightarrow \mathbb{C}} (\sigma \rho)^{m_\sigma}.$$

Finally, by the discussion in Appendix [A], we see that the $S$ constructed by Corlette-Simpson exactly yields one of our quaternionic polydisk Shimura varieties.

Theorem 3.8. Let $X/\mathbb{C}$ be a smooth, algebraic curve. Let $L$ (resp. $(E, \theta)$) be a complex local system (resp. Higgs bundle) that is irreducible, has rank 2, and has trivial determinant. Suppose $L$ (resp. $(E, \theta)$) is geometric. Then $L$ (resp. $(E, \theta)$) comes from a family of abelian varieties on $X$.

Proof. By assumption, there is a non-empty $U \subset X$, a smooth projective morphism $\pi : Y_U \rightarrow U$, and an integer $i \geq 0$ so that $L$ is a subquotient of $R^i\pi_*\mathbb{C}$. By semi-simplicity, $L$ is in fact a summand of $R^i\pi_*\mathbb{C}$. It follows that $M$ may be conjugated into a representation $\pi_1(U) \rightarrow SL_2(A)$ where $A$ is a ring of algebraic integers. Then for every $\sigma \in \text{Hom}(A, \mathbb{C})$, $\sigma M|_U$ is also a summand of of $R^i\pi_*\mathbb{C}$.
and hence carries a PVHS by Deligne’s semisimplicity theorem. Apply Theorem 3.7 to obtain a map $f_U: U \to S$ to a polydisk Shimura DM stack such that there is a tautological local system $L_j$ on $S$ with $f_U^*L_j \cong M|_U$ and with all of the other $f_U^*L_k$ Galois conjugate to $M|_U$.

The local system $M|_U$ and all of its Galois conjugates extend to $X$. Using Corlette-Simpson’s model of $S$ (a.k.a. a strange model) and the Grothendieck-Néron-Ogg-Shafarevich criterion, $f_U$ map extends uniquely to a map $f: X \to S$.

The map $\pi_1(U) \to \pi_1(X)$ is surjective, so $f^*L_i \cong M$. □

Proposition 3.9. Let $X/C$ be a smooth algebraic curve. Then the following are equivalent.

(1) $X$ admits a modular embedding;
(2) $(V_{unif}, \nabla_{unif})$ is motivic;
(3) $(E_{unif}, \theta_{unif})$ is motivic.

Proof. This follows immediately from Theorem 3.8 and Lemma 3.6. □

4. Mass formula of Shimura curves

In this section, we propose a new approach to study Newton stratifications of Shimura varieties of Hodge type. Although our result is restricted only to the curve case, the non-abelian Hodge theoretical part of our method can be generalized to higher dimensional Shimura varieties in a quite straightforward manner. In particular, we expect the existence of the canonical periodic Higgs-de Rham flows over Shimura varieties as motivated by the theory of canonical $\mathbb{C}$-VHS of Calabi-Yau type over locally symmetric domains (see [SZ12]).

4.1. Canonical periodic Higgs-de Rham flows. In this subsection, we work on a detailed description of the canonical periodic Higgs-de Rham flows over Shimura curves. The relevant output from the canonical flow to the mass formula is the deduction of a Hasse-Witt map attached to a Shimura curve.

Let $M_0/k$ be a geometrically connected component of the good reduction of a Shimura curve as in Notation A.3 with $F$ a totally real field and $p$ a prime of $F$. Fix a choice $K_{M_0}^{1/2}$ of square root of the canonical bundle $K_{M_0}$. Recall the uniformizing Higgs bundle attached to $M_0$: $E_{unif} \cong K_{M_0}^{1/2} \oplus K_{M_0}^{-1/2}$ and $\theta_{unif}$ is given by the matrix:

$$\theta_{unif} := \begin{pmatrix} 0 & Id_{K_{M_0}^{1/2}} \\ 0 & 0 \end{pmatrix}: E_{unif} \to E_{unif} \otimes K_{M_0} \cong K_{M_0}^{3/2} \oplus K_{M_0}^{1/2}.$$

We will show that the Higgs bundle $(E_{unif}, \theta_{unif})$ is periodic, compute this period, and use this periodicity to construct the Hasse-Witt map attached to the curve $X$. Let $\mathcal{M}_0$ be the $W_2(k)$-scheme coming from the reduction of the global Shimura curve modulo $p^2$.

Theorem 4.1. Notation as above. Let $f = [F_p : \mathbb{Q}_p]$. Then, with respect to the $W_2$-lifting $\mathcal{M}_0$ of $M_0$, the Higgs bundle $(E_{unif}, \theta_{unif})$ is $f$-periodic. More precisely, we have the following diagram of a periodic Higgs-de Rham flow:
where for \(0 \leq i \leq f - 2\), the bundles \(F_{\mathcal{M}_0}^i H\) are stable (hence the Harder-Narasimhan filtration \(\text{Fil}_{\text{HN}}\) over them are trivial), and \(F_{\mathcal{M}_0}^{f-1} H\) is unstable (hence the Harder-Narasimhan filtration over it is nontrivial).

**Remark 4.2.** All of the terms in the periodic Higgs de-Rham flow are essentially determined by the curve \(\tilde{\mathcal{M}}_0\) over \(W_2\). Note that \(f\) is the first natural number such that the \(f\)th-iterated Frobenius pullback of \(H\) become unstable. Also, since the Higgs bundle \((E_{\text{unif}}, \theta_{\text{unif}})\) is Higgs stable, the isomorphism \(\psi\) of graded Higgs bundles is unique up to scalar. Furthermore, for another choice of \((E_{\text{unif}}, \theta_{\text{unif}})\), the diagram will differ by tensoring with a two-torsion line bundle.

**Proof.** The periodicity of \((E_{\text{unif}}, \theta_{\text{unif}})\) is an intrinsic property of the Shimura curve (over \(W_2\)). However, in order to show this property, we need to make use of a strange model as in §2 and [Car86, §2]. Choose an auxiliary imaginary quadratic extension \(\mathbb{Q}(\alpha)\) where \(p\) splits and set \(K = F(\alpha)\). One gets a new Shimura curve of PEL type from \(B := D \otimes_F K\) [Car86, 2.6]. In particular, there is an abelian scheme \(\mathcal{A}_0 \to \mathcal{M}_0\) over the integral model, which we may reduce modulo \(p\) to obtain \(\mathcal{A}_0 \to \mathcal{M}_0\). We have the following property by [Car86, 2.6.2 (b)]: there exists an order \(O \subset B\), maximal at \(p\), such that

\[
O \hookrightarrow \text{End}_{\mathcal{M}_0}(\mathcal{A}_0).
\]

We will use several results from [SZZ], specifically Corollary 4.2, Proposition 4.3, 4.4 and Theorem 7.3. Notice that neither the assumption \(p \geq 2g(\mathcal{M}_0)\) in Proposition 4.4 nor the assumption \(p \geq \max\{2g(\mathcal{M}_0), 2([F : \mathbb{Q}] + 1)\}\) in Theorem 7.3 of loc. cit are necessary. This is because of the explicit reconstruction of the inverse Cartier transform of Ogus-Vologodsky in terms of exponential twisting (see [LSZ15]). Let \(p\mathcal{O}_F = \prod_i p_i\) be the prime decomposition, with \(p = p_1\). Choose an isomorphism \(\mathbb{C} \to \overline{\mathbb{Q}}_p\) such that under the induced identification

\[
\text{Hom}(F, \overline{\mathbb{Q}}) = \prod_i \text{Hom}_{\overline{\mathbb{Q}}_p}(F_{p_i}, \overline{\mathbb{Q}}_p).
\]

\(\tau\) is sent to \(\text{Hom}_{\overline{\mathbb{Q}}_p}(F_{p_1}, \overline{\mathbb{Q}}_p)\). The graded Higgs bundle \((E, \theta)\) attached to the family \(f_0 : \mathcal{A}_0 \to \mathcal{M}_0\) over \(\mathbb{F}_p\) is one-periodic by [LSZ15, Proposition 4.1]. That is, there is a natural isomorphism of graded Higgs bundles

\[
\text{Gr}_{\text{Frod}} \circ C^{-1}(E, \theta) \cong (E, \theta).
\]
The Higgs bundle \((E, \theta)\) is irreducible as an \(O\)-module. Pick a subfield \(L \subseteq D\), containing \(F\), that splits \(D\) and leaves \(p\) inert. Set \(\mathcal{O}' := \mathcal{O} \cap L \otimes_F K \subseteq \text{End}_{\mathcal{M}_0}(\mathcal{A}_0)\). With regards to \(\mathcal{O}'\) the Higgs bundle decomposes into direct sum of rank two graded Higgs subbundles:

\[
(E, \theta) = \bigoplus_{\phi \in \text{Hom}(L, \mathbb{C})} (E_{\phi}, \theta_{\phi}) \oplus (E_{\bar{\phi}}, \theta_{\bar{\phi}}).
\]

by [SZZ, Proposition 4.1]. We explain the notation. Fix the diagram \(L \supseteq F \subseteq K\) and an embedding \(\mathbb{Q}(\alpha) \hookrightarrow \mathbb{C}\). Then we have a map \(\text{Hom}(L, \mathbb{C}) \hookrightarrow \text{Hom}(L \otimes_F K, \mathbb{C})\) given by sending \(\alpha\) to the chosen image in \(\mathbb{C}\). Given \(\phi \in \text{Hom}(L, \mathbb{C})\), identify it with the induced map \(\text{Hom}(L \otimes_F K, \mathbb{C})\); then \(\bar{\phi}\) is the conjugate of \(\phi\) with respect to \(\alpha\).

Now, let \(\sigma\) denote the absolute Frobenius of the base field \(\overline{F}_p\), and let \(*\) denote the involution of \(L/F\). Since the inverse Cartier transform \(C^{-1}\) is \(\sigma\)-linear and \(\text{Gr}_{\text{Frod}}\) is linear, the flow operator \(\text{Gr}_{\text{Frod}} \circ C^{-1}\) transforms the \(\phi\)-factor to the \(\sigma(\phi)\)-factor. The \(\sigma\)-orbit of a \(\phi\) with the property \(\phi |_F = \tau\) reads

\[
\text{Hom}_{\mathbb{Q}_p}(L_{\overline{p}}, \mathbb{Q}_p) = \{\phi_1, \ldots, \phi_f, \bar{\phi}_1, \ldots, \bar{\phi}_f\}.
\]

It follows from [SZZ, Proposition 4.3] that

\[
(\text{Gr}_{\text{Frod}} \circ C^{-1})^f(E_{\phi_1}, \theta_{\phi_1}) = (E_{\phi_1^*}, \theta_{\phi_1^*}) \cong (E_{\phi_1}, \theta_{\phi_1}).
\]

By the argument of [SZZ, Proposition 4.4], it follows that \((E_{\phi_1}, \theta_{\phi_1})\) is a maximal Higgs field and therefore differs from our chosen uniformizing Higgs bundle by a two-torsion line bundle. Therefore \(((\text{unif}_{\phi}, \theta_{\text{unif}}))\) initiates an \(f\)-periodic flow.

Finally, we describe the terms of this flow intrinsically in terms of \(\mathcal{M}_0\). By the analysis of the Higgs subbundles in [SZZ, Proposition 4.1], one knows that the Higgs fields of

\[
(\text{Gr}_{\text{Frod}} \circ C^{-1})^i(E_{\phi_1}, \theta_{\phi_1}), 1 \leq i \leq f - 1
\]

are zero. By [SZZ, Proposition 6.6 (ii)], \(F^iH, 0 \leq i \leq f - 2\) is stable, and \(F^{*f-1}H\) becomes unstable. By [SZZ, Corollary 7.4], the Hodge filtration on \((F^{*f-1}H, \nabla_{\text{can}})\) is nothing but the Harder-Narasimhan filtration.

\[
\Box
\]

**Construction of the Hasse-Witt map:** Set \(L = K^{1/2}_{\mathcal{M}_0}\). Applying the functor \(C^{-1}\) to the Higgs subbundle \((L^{-1}, 0) \subset (\text{unif}_{\phi}, \theta_{\text{unif}})\), we obtain \((F_{\mathcal{M}_0}L^{-1}, \nabla_{\text{can}}) \subset (H, \nabla)\). In particular, \(F_{\mathcal{M}_0}L^{-1} \subset H\) and consequently \(F_{\mathcal{M}_0}^{*f}L^{-1} \subset F_{\mathcal{M}_0}^{*f-1}H\). Since

\[
\text{Gr}_{\text{Fil}_H}(F_{\mathcal{M}_0}^{*f-1}H, \nabla_{\text{can}}) \cong (E_{\text{unif}}, \theta_{\text{unif}})),
\]

it follows that

\[
0 \to L \to F_{\mathcal{M}_0}^{*f-1}H \to L^{-1} \to 0
\]

is a short exact sequence. The Hasse-Witt map is defined to be the composite

\[
\text{HW}(\mathcal{M}_0) : F_{\mathcal{M}_0}^{*f}L^{-1} \to F_{\mathcal{M}_0}^{*f-1}H \to L^{-1}.
\]
We may also regard $\text{HW}(\mathcal{M}_0)$ as a global section of the line bundle $F_{\mathcal{M}_0}^* L \otimes L^{-1} \cong L^{p-1}$. Since $p$ is odd, the bundle $L^{p-1}$ is independent of the choice $K_{\mathcal{M}_0}^{1/2}$ and by Remark 4.2 the section $\text{HW}(\mathcal{M}_0) \in \Gamma(X, L^{p-1})$ is also independent of this choice. Note also that $p$ being odd means that $\text{HW}(\mathcal{M}_0)$ is a pluricanonical differential form; in other words, it is a modular form modulo $p$. We shall call this natural section the Hasse-Witt invariant of $\mathcal{M}_0$.

**Remark 4.3.** Given any smooth curve whose (logarithmic) uniformizing Higgs bundle $(E_{\text{unif}}, \theta_{\text{unif}})$ is periodic, we can construct a Hasse-Witt invariant. We do not know what properties to expect in general from these invariants.

4.2. **One-clump theorem.** The second ingredient in our approach to the mass formula is the so-called “one clump theorem” [Kri18], whose origin is traced back to the work [Mo98] of S. Mochizuki. Its usage here stems from the observation that the Newton jumping locus is invariant under a Hecke correspondence. This phenomenon has been abstracted into the notion of clump of an étale correspondence without a core. The “one clump theorem” then implies the Newton jumping locus (under any symplectic embedding) is precisely the zero-locus of the Hasse-Witt map established as above. The goal of this subsection is to collect (and modify) several results in [Kri18] that we will use for the proof of the mass formula.

**Definition 4.4.** Let $X/k$, $Y/k$, and $Z/k$ be curves. A morphism of curves (over $k$) is a morphism of $k$-schemes that is $X \rightarrow Y$ finite, generically separable, and dominant. A correspondence of curves over $k$ is a diagram

$$
\begin{array}{ccc}
Z & \rightarrow & Y \\
\downarrow f & & \downarrow g \\
X & \leftarrow & Y
\end{array}
$$

where $f$ and $g$ are morphisms of curves.

**Definition 4.5.** A correspondence of geometrically connected curves $X \leftarrow Z \rightarrow Y$ over $k$ has no core if

$$
k \cong k(X) \cap_{k(Z)} k(Y)
$$

**Remark 4.6.** By [Kri18, Prop. 4.2], the property of “having a core” is unchanged under arbitrary field extension $L/k$.

**Definition 4.7.** Let $X \leftarrow Z \rightarrow Y$ be a correspondence of curves over $k$. A clump is a finite set $S \subset Z(\overline{k})$ such that $f^{-1}(f(S)) = g^{-1}(g(S)) = S$. A clump $S$ is étale if $f$ and $g$ are étale at all points $s \in S$.

**Theorem 4.8.** [Kri18, Theorem 9.6] Let $X \leftarrow Z \rightarrow Y$ be a correspondence of geometrically connected curves over $k$ without a core. Then there is at most one étale clump.

Theorem 4.8 is a main input in proving Theorem 1.8. We will briefly introduce the main concepts that are relevant to the “one clump theorem”.
Definition 4.9. Let

\[
\begin{array}{ccc}
Z & \overset{f}{\longrightarrow} & X \\
\downarrow & & \downarrow \\
Y & \overset{g}{\longleftarrow} & Z
\end{array}
\]

be a correspondence of geometrically connected curves over \(k\). An invariant line bundle \(\mathcal{L}\) is a triple \((\mathcal{L}_X, \mathcal{L}_Y, \phi)\) where \(\mathcal{L}_X\) is a line bundle on \(X\), \(\mathcal{L}_Y\) is a line bundle on \(Y\), and \(\phi: f^* \mathcal{L}_X \rightarrow g^* \mathcal{L}_Y\) is an isomorphism of line bundles on \(Z\). An invariant section of \(\mathcal{L}\) is a pair \((s_X, s_Y)\) \(\in H^0(X, \mathcal{L}_X) \oplus H^0(Y, \mathcal{L}_Y)\) such that \(f^* s_X = \phi^* g^* s_Y\). The space of invariant sections is denoted by \(H^0(\mathcal{L})\) and its dimension is denoted by \(h^0(\mathcal{L})\).

Then the one clump theorem essentially follows from the following theorem, which is [Kri18, Proposition 8.2, Corollary 8.10]:

Theorem 4.10. Let \(X \leftarrow Z \rightarrow Y\) be a correspondence of curves over \(k\) without a core. Let \(\mathcal{L}\) be an invariant line bundle. Then \(h^0(\mathcal{L}) \leq 1\). If \(\mathcal{L}\) and \(\mathcal{M}\) are invariant line bundles of positive degree, then there exist positive integers \(m, n\) such that \(\mathcal{L} \otimes^m \cong \mathcal{M} \otimes^n\).

The utility of this concept is the following: the Hasse-Witt map \(HW(\mathcal{M}_0)\) yields an invariant pluricanonical differential form on the good reduction of a Hecke correspondence of Shimura curves. We will need several auxiliary results about étale correspondences without a core.

Lemma 4.11. Let \(X \leftarrow Z \rightarrow Y\) be an étale correspondence of geometrically connected curves over \(k\), where \(Z\) is hyperbolic. Then the following are equivalent:

1. \(X \leftarrow Z \rightarrow Y\) has a core
2. There exists a finite extension \(l/k\), a geometrically connected curve \(W/l\), and a finite étale map \(W \rightarrow Z\) such that the induced maps \(W \rightarrow X_l\) and \(W \rightarrow Y_l\) are Galois.

Proof. [Kri18, Lemma 4.2] implies that \(X \leftarrow Z \rightarrow Y\) has a core if and only if there exists a finite field extension \(l/k\) and a finite, generically separable map \(W' \rightarrow Z_l\) such that the induced maps to \(X_l\) and to \(Y_l\) are Galois. Here, “Galois” means on the level of function fields. In particular, if such \(W\) exists as in (1), then the correspondence has a core. On the other hand, if there is a core, then there is a finite extension \(l/k\) and a curve \(W' \rightarrow Z_l\) that is finite Galois over \(X_l\) and \(Y_l\) (in the sense of function fields), again by Lemma 4.2 of loc. cit. Given any such morphism, there is a unique factorization \(W' \rightarrow W \rightarrow Z_l\) where the first map is a ramified covering of curves and the second arrow a maximal finite étale subcover. The uniqueness implies that \(W \rightarrow X_l\) and \(W \rightarrow Y_l\) are Galois, as desired.

Lemma 4.12. Let \(S = \text{Spec}(R)\) be the spectrum of a discrete valuation ring with generic point \(\eta\) and special point \(s\). Let \(\mathcal{X}, \mathcal{Y},\) and \(\mathcal{Z}\) be smooth, proper curves
over $S$ whose generic fibers are all geometrically connected and have genus at least 2. Let

$$
\begin{array}{c}
Z \\
\downarrow f \\
\downarrow g \\
\rightarrow X \\
\rightarrow Y
\end{array}
$$

be a finite étale correspondence of schemes. Then the base-changed correspondence of curves over $\eta$ has a core if and only if the correspondence of curves over $s$ has a core.

**Proof.** This is essentially contained in [Kri18, Lemma 4.10, 4.14], but we present a slightly different proof of the “not having a core specializes” direction. First of all, having a core is invariant under field extension; therefore, we may assume that $s$ is the spectrum of an algebraically closed field. Suppose $X_s \leftarrow Z_s \rightarrow Y_s$ has a core. Then there exists a connected curve $W_s$ over $s$, equipped with a map $W_s \rightarrow Z_s$, such that under the induced maps $W_s$ is finite Galois over $X_s$ and $Y_s$. As finite étale covers canonically lift over finite-order thickenings of the base, $W_s$ canonically deforms to a formal scheme $\tilde{W}$ over $Spf(R)$. As $W_s$ is a smooth projective curve, the formal scheme $\tilde{W}$ canonically algebraizes to a smooth projective curve $\mathcal{W}$ over $S$ by Grothendieck’s formal existence theorem. Therefore by Lemma 4.11, the correspondence of curves over $\eta$ has a core.

For the other direction, that “having a core specializes” see [Kri18, Lemma 4.14].

\[ \square \]

### 4.3. Proof

We construct certain Hecke correspondences for Shimura curves. Then using Theorem 4.8 and work of Wortmann on the $\mu$-ordinary locus of Hodge-type Shimura varieties, we conclude that there are exactly two Newton strata. In fact, our techniques show there are exactly two central leaves.

**Definition 4.13.** Let $(G, X)$ denote a Shimura curve datum, given by $F$ and $D$. Let $K \subset G(\mathbb{A}_f)$ be a sufficiently small open compact subgroup such that the associated Shimura curve $Sh_K(G, X)$ is a scheme. Let $l$ be primes of $\mathbb{Q}$ and $\lambda \mid l$ be a prime of $F$ such that

- $l$ is unramified in $F$
- The prime $\lambda$ has inertial degree 1 over $l$
- $D$ is split at $\lambda$
- $K$ is hyperspecial at $l$

We define the correspondence of curves, $T_\lambda$, as the Hecke correspondence associated to $g \in G(\mathbb{A}_f)$ given by $(1, \ldots, 1, g_\lambda, 1, \ldots)$ where $g_\lambda$ is the matrix $\begin{pmatrix} l & 0 \\ 0 & 1 \end{pmatrix}$ under a choice of isomorphism $D \otimes F_\lambda \cong M_{2 \times 2}(\mathbb{Q}_l)$.

**Lemma 4.14.** Let $(G, X)$ be a Shimura curve datum and let $K \subset \mathbb{A}_f$ be a sufficiently small open compact subgroup so that $Sh_K(G, X)$ is a scheme. Then
(1) $T_\lambda$ is a finite étale correspondence of compact complex curves that descends to $F$, the reflex field of $(G,X)$, over which the canonical model of $M$ is defined.

(2) When $T_\lambda$ is restricted to geometrically connected components, it is an étale correspondence of curves without a core of bidegree $(l + 1, l + 1)$.

(3) If $K$ is hyperspecial at $p$, and $l \neq p$, then $T_\lambda$ has good reduction modulo $p$. When restricted to any geometrically connected component modulo $p$, it yields an étale correspondence of curves without a core.

Proof. By construction, $T_\lambda$ is a finite étale correspondence of (possibly disconnected) compact Riemann surfaces. We first prove that the correspondence, when restricted to (complex) connected components, has no core.

As we are studying the connected components, we work with the derived group $G^{der}$. Write the correspondence of geometrically connected curves as $M \xleftarrow{\ell} Z \xrightarrow{g} M$. By the assumptions on $l$ and $\lambda$, $G^{der}(\mathbb{Z}_l) \cong SL_2(\mathbb{Z}_l) \times H$, where $H$ is an $l$-adic group. Then the correspondence $T_\lambda$ “comes from” the pair of inclusions of $l$-adic groups

$\Gamma_0(l) \quad \Gamma_0(l) \quad \Gamma_0(l)$

Here, $\Gamma_0(l)$ is the subgroup of $SL_2(\mathbb{Z}_l)$ that reduces to the standard Borel mod $l$, the left hand inclusion is the standard inclusion, and the right hand inclusion is given by conjugation by the element $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

By Lemma 4.11 to prove that $M \xleftarrow{\ell} Z \xrightarrow{g} M$ has no core, it suffices to prove that there is no finite étale cover $h : W \to Z$ such that $f \circ h$ and $g \circ h$ are both Galois. This is equivalent to proving that there is no finite index subgroup $\Gamma' \subset \Gamma_0(l)$ that is normal under both of the above embeddings into $SL_2(\mathbb{Z}_l)$, a question in pure group theory.

Fix $V = \mathbb{Q}_l^\oplus 2$ and consider the set $\mathcal{L}$ of lattices (a.k.a. free rank 2 $\mathbb{Z}_l$-submodules) up to homothety in $V$. This discrete set can be enchanced into a tree, the Bruhat-Tits tree $\mathcal{T}$, as follows. Given two lattices $L_1$ and $L_2$, we join their representatives in $\mathcal{L}$ by an edge if $L_1$ is a sublattice of index $l$ in $L_2$. (This is symmetric because the elements of $\mathcal{L}$ are lattices up to homothety.) It is well-known that $\mathcal{T}$ is an $l + 1$-regular tree [SB77, Theorem 1, p. 98-101]. The group $SL_2(\mathbb{Q}_l)$ acts on $\mathcal{T}$ and the stabilizer of a vertex is isomorphic to $SL_2(\mathbb{Z}_l)$. Moreover, the stabilizer of an edge $e$ between $v$ and $w$ is $\Gamma_0(l)$ [SB77, p. 106-107], and in fact $\text{Stab}(v) \leftarrow \text{Stab}(e) \rightarrow \text{Stab}(w)$ is isomorphic to $SL_2(\mathbb{Z}_l) \leftarrow \Gamma_0(l) \rightarrow SL_2(\mathbb{Z}_l)$ (see [SB77, Corollaire 1, p. 110] and its following comments).

Now we claim that there is no finite index subgroup $\Gamma'$ of $\Gamma_0(l)$ that is normal in both copies of $SL_2(\mathbb{Z}_l)$. Suppose that $\Gamma'$ is a normal subgroup of $\text{Stab}(v) \cong SL_2(\mathbb{Z}_l)$. Let $\mathcal{T}_{\Gamma'}$ be the subgraph of $\mathcal{T}$ fixed by $\Gamma'$. The ends emanating from $v$ are in bijective correspondence with $\mathbb{P}V$ in an $SL_2(\mathbb{Z}_l)$-equivariant way [SB77, P. 101]: hence $SL_2(\mathbb{Z}_l)$ acts transitively on the ends emanating from $v$. As
\( \Gamma' \subset SL_2(\mathbb{Z}_l) \) is normal and of finite index, the subgraph \( \mathcal{T}_{\Gamma'} \) is a ball around \( v \) of some finite radius. On the other hand, there is no finite subgraph of \( \mathcal{T} \) that is a ball around both \( v \) and \( w \); therefore there is no finite index subgroup \( \Gamma' \subset \Gamma_0(l) \) that is normal under both of the embeddings into \( SL_2(\mathbb{Z}_l) \). Translating back to algebraic geometry, there is no finite étale cover of \( \mathbb{Z} \) that is Galois over both copies of \( M \). By Lemma 4.11 this finite étale correspondence of curves over \( \mathbb{C} \) has no core. The (bi)degree is just the index of a Borel in \( SL_2(\mathbb{F}_l) \); by considering the action on \( \mathbb{P}^1_{\mathbb{F}_l} \), this is \( l + 1 \).

By the discussion on canonical models in Section A, \( T_\lambda \) descends to the reflex field of \( (G, X) \) because the construction of canonical models is Hecke-equivariant. When \( K \) is hyperspecial at \( p \) and \( l \neq p \), then by the discussion on integral canonical models, for each prime \( p \mid p \), the Hecke correspondence has a model over \( \mathcal{O}_{\mathbb{F}_p} \). We may take an unramified extension to pick out the geometrically connected components by [Kis10, 2.2.4]; then the special fibers are also geometrically connected by Zariski’s principle of connectedness. By applying Lemma 4.12 we see that the reduction modulo \( p \) of the Hecke correspondence of geometrically connected components has no core.

For each geometric point \( x \in \mathcal{M}(\overline{\mathbb{F}}_p) \), we may consider the associated abelian variety \( U_x \). It follows from a theorem of Grothendieck-Katz that the Newton polygon jumps on a Zariski closed subset of \( \mathcal{M}(\overline{\mathbb{F}}_p) \). For each geometrically connected component \( \mathcal{M}_i \) of \( \mathcal{M} \), we have an abelian scheme \( U_i \to \mathcal{M}_i \) over \( \overline{\mathbb{F}}_p \), these are all isomorphic. Therefore there are only finitely many points of \( \mathcal{M}(\overline{\mathbb{F}}_p) \) where the Newton polygon jumps.

**Definition 4.15.** Notation as in A.4 Then the Newton jumping locus \( \mathcal{N}_p \) is the (finite) set of geometric points of \( x \in \mathcal{M}(\overline{\mathbb{F}}_p) \) such that \( U_x \) has a Newton polygon which is higher than the generic Newton polygon.

**Corollary 4.16.** Notations as in A.4. Then the Newton jumping locus \( \mathcal{N}_p \) consists of only one Newton polygon. The set \( \mathcal{N}_p \subset \mathcal{M}(\overline{\mathbb{F}}_p) \) is independent of the rational symplectic representation realizing \( (G, X) \) as a Hodge-type Shimura curve.

**Proof.** We first argue that \( \mathcal{N}_p \) intersects every geometrically connected component non-trivially. By [Wort13, Theorem 1.1], the \( \mu \)-ordinary locus of \( \mathcal{M} \) is open and dense. Any pair of points \( x \in \mathcal{M}(\overline{\mathbb{F}}_p) \setminus \mathcal{N}_p \) are Newton generic and hence \( \mu \)-ordinary. Then [Wort13, Theorems 1.2] implies thats \( U_x[p^\infty] \) and \( U_y[p^\infty] \) are isomorphic as \( p \)-divisible groups over \( \overline{\mathbb{F}}_p \). If \( \mathcal{N}_p \) were empty, then in particular \( \mathcal{M} \) would live entirely in a fixed Ekedahl-Oort strata of \( \mathscr{A}_{n,1,K'} \). On the other hand, such EO strata are quasi-affine by [Oort01, Theorem 1.2]. As each \( \mathcal{M}_i \) is proper and connected, this proves that the map is constant, which is a contradiction. In particular, the Newton polygon jumps on every (geometrically) connected component. Now, as the \( U_i \to \mathcal{M}_i \) are all isomorphic over \( \overline{\mathbb{F}}_p \), the special Newton polygons are the same on the different geometrically connected components. Note that it still, à priori, depends on the choice of rational symplectic embedding.
Pick a $\lambda$ in $F$, prime to $p$, such that the conditions in Definition 4.13 are met. Then $T_\lambda$ restricts to an étale correspondence without a core over $\overline{\mathbb{F}}_p$ by Lemma 4.14.

Recall that $s^*U$ is isogenous to $t^*U$ as abelian schemes over $X$ by Lemma A.3. Therefore if $N_p$ consisted of more than one type of Newton polygon, then there would be at least two clumps. On the other hand, Theorem 4.8 implies that there is at most one clump.

Now, the set $N_p \subset \mathcal{M}(\mathbb{F}_p)$ does not depend on the choice of symplectic representation: indeed, the Hecke correspondence $\mathcal{M} \leftarrow X \rightarrow \mathcal{M}$ is independent of the choice of symplectic representation by [Kis10, 2.3.8 (ii)] and $N_p$ is the unique clump of this Hecke correspondence.

Remark 4.17. In fact, $s^*U$ is separably isogenous to $t^*U$ by the construction of the Hecke correspondence $T_\lambda$; therefore, for $x, y \in N_p$, the $p$-divisible groups $\mathcal{U}_x[p^\infty]$ and $\mathcal{U}_y[p^\infty]$ are isomorphic as $p$-divisible groups over $\overline{\mathbb{F}}_p$ by Theorem 4.8.

Proposition 4.18. Notation as above. Then the Hasse-Witt map $HW(\mathcal{M}_0)$ has the following properties:

(i) $HW(\mathcal{M}_0) \neq 0$;
(ii) $\text{div}(HW(\mathcal{M}_0))$ is multiplicity free;
(iii) The Newton jumping locus of $\mathcal{U}_0 \rightarrow \mathcal{M}_0$ is $\text{div}(HW(\mathcal{M}_0))$

Proof. Part (i) follows from [SZZ, Remark 3.4] by Honda-Tate theory. Part (ii) is given by [SZZ, Proposition 5.3] with an explicit calculation using displays. (Both computations rely on a fixed strange model.) Finally, consider the Hecke correspondence $T_\lambda$:

\[ s \quad \mathcal{Z}_0 \quad t \]
\[ s \quad \mathcal{M}_0 \quad t \]

By Lemma 4.14 $T_\lambda$ is an étale correspondence without a core. Note that

\[ s^*(E_{unif}, \theta) \cong t^*(E_{unif}, \theta) \]

because both $s$ and $t$ are finite étale; moreover, this pulled-back Higgs bundle is uniformizing on $Z$. As $(E_{unif}, \theta)$ is $f$-periodic, so is the pulled-back Higgs bundle. From the construction of the Hasse-Witt invariant, it follows that $s^*(HW(\mathcal{M}_0))$ and $t^*(HW(\mathcal{M}_0))$ are both proportional to $HW(Z_0)$.

Therefore the Hasse-Witt invariant yields an non-zero invariant, pluricanonical differential form on $T_\lambda$. By taking the zero-locus, we obtain that $\text{div}(HW)$,
considered as a finite set of points, is a clump. As the Newton jumping locus
is non-empty (see the proof of Corollary 4.16), it also yields a clump on $T_\lambda$. By
Theorem 4.8 these two clumps must coincide; hence $\mathcal{N}_p = \text{div}(HW(\mathcal{M}_0))$ as
desired. □

**Remark 4.19.** There is another “intrinsic” characterization of $\mathcal{N}_p$. By [Car86,
§1.4], there exists a “universal” Barsotti-Tate group on the limit $\hat{M} \otimes F_p$ of a
certain (prime-to-$p$) tower of Shimura curves. More precisely, there exists a height
2, dimension 1 divisible $\mathcal{O}_p$-module on $\hat{M} \otimes F_p$. Using strange models, Carayol
shows that this extends to the canonical integral model and exists on the limit $\hat{M}$
modulo $p$ [Car86, §6.4]. After Drinfeld, there are two types of such objects over
an algebraically closed field of characteristic $p$: “ordinary” and “supersingular”
[Car86, §6.7]. In particular, given a geometric point $x$ of $\mathcal{M}$, it is in $\mathcal{N}_p$ if and
only if the corresponding $\mathcal{O}_p$-module over $\hat{M}_x$ is supersingular.

The mass formula immediately follows.

**Corollary 4.20.** Notation as in Theorem 1.8. Then the following mass formula
holds:

$$|\mathcal{N}_p| = (p^f - 1)(g - 1),$$

where $f = [F_p : \mathbb{Q}_p]$ and $g$ is the genus of the curve $X$.

**Proof.** The Newton jumping locus in each geometrically connected component has
the same cardinality. Hence, we reduce to a geometrically connected component.
By Proposition 4.18 this is the degree of

$$HW(\mathcal{M}_0) \in H^0(\mathcal{M}_0, K_{\mathcal{M}_0}^{\otimes p^f - 1}).$$

This degree is $(p^f - 1)(g - 1)$, as desired. □

**Proof of Theorem 1.8.** Combine Corollary 4.16 and Corollary 4.20. □

## 5. Periodic Higgs Conjecture

Let $C$ be a smooth projective curve over $\mathbb{C}$. We introduce several categories of
parabolic objects over $C$. Let $\text{DR}^{\text{par}}(C)$ be the category of parabolic de Rham
bundles over $C$. Let $V_i = (V_i, \nabla_i, Fil_i), i = 1, 2$ be two objects. Nonzero morphism
$V_1 \to V_2$ may be defined only when the support of the quasi-parabolic structure
of $V_1$ is contained in that of $V_2$, and it is a parabolic morphism which is parallel
and respects the filtration. Let $\text{HG}^{\text{par}}(C)$ be the category of parabolic graded
Higgs bundles over $C$. Again, nonzero morphisms of two objects may be defined
only when the condition on the support of quasi-parabolic structure is satisfied,
and a morphism means a parabolic morphism respecting the Higgs field and the
grading structure. By [KS20, Lemma 2.14], there is an obvious associated graded
functor:

$$\text{Gr} : \text{DR}^{\text{par}}(C) \to \text{HG}^{\text{par}}(C).$$
We shall introduce subcategories which sit in the four corners of the following commutative diagram, whose definition we recall below:

\[
\begin{align*}
\text{MDR}^{\text{Par}}(\mathcal{C}) & \longrightarrow \text{PDR}^{\text{Par}}(\mathcal{C}) \\
\text{MHG}^{\text{Par}}(\mathcal{C}) & \longrightarrow \text{PHG}^{\text{Par}}(\mathcal{C}).
\end{align*}
\]

The categories \( \text{PDR}^{\text{Par}}(\mathcal{C}) \) and \( \text{PHG}^{\text{Par}}(\mathcal{C}) \) are the categories of periodic parabolic de Rham bundles over \( \mathcal{C} \) and periodic parabolic Higgs bundles over \( \mathcal{C} \) respectively. Objects and morphisms of these two categories are defined in [KS20, Definition 3.13].

In this paragraph, we extensively use [KS20, Theorem 1.3] (and its proof). The symbol \( \text{MDR}^{\text{Par}}(\mathcal{C}) \) denotes the full subcategory of motivic parabolic de Rham bundles over \( \mathcal{C} \). Its objects are motivic parabolic de Rham bundles, which are direct summands of parabolic Gauß-Manin systems over \( \mathcal{C} \). By [KS20, Lemma 4.1], any motivic parabolic de Rham bundle over \( \mathcal{C} \) is completely reducible: it decomposes into a direct sum of irreducible parabolic de Rham bundles. So \( \text{MDR}^{\text{Par}}(\mathcal{C}) \) is a semisimple category over \( \mathcal{C} \). The symbol \( \text{MHG}^{\text{Par}}(\mathcal{C}) \) denotes the subcategory of motivic parabolic Higgs bundles over \( \mathcal{C} \). Any motivic parabolic Higgs bundle over \( \mathcal{C} \) is a direct sum of parabolic polystable Higgs bundles of degree zero, so the category is also semisimple over \( \mathcal{C} \). Clearly, \( \text{MDR}^{\text{Par}}(\mathcal{C}) \) and \( \text{PDR}^{\text{Par}}(\mathcal{C}) \) (resp. \( \text{MHG}^{\text{Par}}(\mathcal{C}) \) and \( \text{PHG}^{\text{Par}}(\mathcal{C}) \)) are subcategories of \( \text{DR}^{\text{Par}}(\mathcal{C}) \) (resp. \( \text{HG}^{\text{Par}}(\mathcal{C}) \)).

**Remark 5.1.** We remind our reader that the category \( \text{PDR}^{\text{Par}}(\mathcal{C}) \) is in fact a subcategory of \( \text{PDR} \), which is introduced in [KS20, §5]. They are à priori not the same category. However, the periodic de Rham conjecture loc. cit. predicts that they should be naturally equivalent. We briefly remind the reader of the difference. Loosely, \( \text{PDR} \) is the filtered colimit of the categories \( \text{PDR}(U) \) as \( U \subset \mathcal{C} \) ranges through non-empty Zariski opens. On the other hand, \( \text{PDR}^{\text{Par}}(\mathcal{C}) \) is the filtered colimit of \( \text{PDR}^{\text{Par}}(\mathcal{C}_{\log}) \) as we vary the “divisor at ∞”, i.e., \( D \). In particular, the category \( \text{PDR} \) potentially has more morphisms than the category \( \text{PDR}^{\text{Par}}(\mathcal{C}) \).

The vertical arrows of the above diagram are given by \( \text{Gr} \), the functor taking to a de Rham bundle to its associated graded Higgs bundle.

**Lemma 5.2.** The following properties hold:

(i) The functor \( \text{Gr} : \text{PDR}^{\text{Par}}(\mathcal{C}) \to \text{PHG}^{\text{Par}}(\mathcal{C}) \) is faithful;

(ii) The functor \( \text{Gr} : \text{MDR}^{\text{Par}}(\mathcal{C}) \to \text{MHG}^{\text{Par}}(\mathcal{C}) \) is an equivalence of semisimple categories over \( \mathcal{C} \).

**Proof.** The functor \( \text{Gr} \) is additive and \( \mathbb{C} \)-linear. (i) follows from the periodicity (see the proof of [KS20, Proposition 5.2]). As for (ii), one notes that by definition, \( \text{Gr} \) is essentially surjective on motivic objects. By [KS20, Lemma 4.4 (iii)],

---

8Motivic parabolic de Rham bundles are periodic and motivic parabolic Higgs bundles are periodic.
Gr maps simple objects to simple objects. Therefore, Gr is an equivalence of categories.

We propose the following

**Conjecture 5.3 (Periodic Higgs conjecture).** Let \( C \) be a smooth projective curve over \( \mathbb{C} \). Then the inclusion functor

\[
\iota : \text{MHG}^{\text{par}}(C) \to \text{PHG}^{\text{par}}(C)
\]

is essentially surjective, i.e., it is an equivalence of categories.

We proceed to the proof of Proposition 1.12, which confirms Conjecture 5.3 in rank one case.

**Proof.** Let \((L, \theta)\) be a parabolic graded Higgs line bundle over \( C \). Assume its parabolic pullback via a cyclic cover \( \pi : C' \to C \) is trivial, where the branch locus of \( \pi \) contains the support of the quasi-parabolic structure of \( L \). Let \( G \) be the automorphism group of \( \pi \). First of all, the Higgs field \( \theta \) must be zero, because its usual pullback coincides the parabolic pullback away from the branch locus and is hence zero by assumption. Next, we claim that \((L, 0)\) is a direct summand of the associated grading to \( H_{dR}^0(U'|U), \nabla_{GM} \). By the Biswas-Iyer-Simpson correspondence ([KS20, Proposition 2.11]) and the extension lemma ([KS20, Lemma 2.15]), the parabolic pushforward of \((O_{U'}, d)\) is a direct summand of \( \text{Gr}(H_{dR}) \). So by Theorem 1.10, \((L, \theta)\) is periodic.

Conversely, we are given a periodic parabolic Higgs line bundle \((L, \theta)\) over \( C \). By periodicity, \( \theta = 0 \). By [KS20, Lemma 3.11], one finds a cyclic cover \( \pi : C' \to C \) whose branch divisor contains \( D \) such that the parabolic pullback \( \pi^*(L, 0) \) is a periodic Higgs line bundle over \( C' \). Note that a Higgs line bundle \((E, 0)\) being periodic is equivalent to saying that \( E_p \) is torsion of order divisible by \( p^f - 1 \) for some fixed \( f \) and for almost all primes \( p \). Let \( C \) be a spreading-out of \( C \), which is defined over an affine scheme \( S \) absolutely of finite type. As deg \( E = 0 \), \( E \) defines an element \([E]\) in \( \text{Pic}^0(C)(k(S)) \). For any point \( v \in S(\bar{Q}) \), the specialization of \([E]\) at \( v \), which is a point of \( \text{Pic}^0(C)(k(v)) \), must be torsion by a deep result of R. Pink [Pink, Theorem 5.3]. By the main theorem of [Ma89], due to D. Masser, \([E]\) is a torsion point of \( \text{Pic}^0(C)(k(S)) \). Therefore \( E \) is a torsion line bundle over \( C \). It follows that there exists some unramified cover \( \pi' : C'' \to C' \) such that the pullback of \( \pi^*L \) via \( \pi'' \) becomes trivial. That means, there is a nonempty Zariski open subset \( U \subset C \) such that \( L|_U \) is étale trivializable. The corresponding rank one local system to the degree zero parabolic line bundle \( L \) under the Simpson correspondence therefore has finite image. So there exists a cyclic cover \( \tilde{\pi} : \tilde{C} \to C \) such that the parabolic pullback of \( L \) via \( \tilde{\pi} \) becomes trivial. This concludes the proof. \( \square \)
Now suppose $\iota$ is surjective on Higgs objects (i.e., suppose that Conjecture 5.3 is true). Then Lemma 5.2 implies that it must be also surjective on de Rham objects. That is, the inclusion functor induces the following identity

$$\text{MDR}^\text{par}(C) = \text{PDR}^\text{par}(C).$$

By the commutativity of the above diagram and by [KS20, Proposition 4.2, Lemma 4.3], we would obtain a subsidiary conjecture of Conjecture 5.3.

**Conjecture 5.4** (Arithmetic Simpson correspondence). Let $C$ be a smooth projective curve over $\mathbb{C}$. The grading functor

$$\text{Gr} : \text{PDR}^\text{par}(C) \to \text{PHG}^\text{par}(C)$$

is an equivalence of semisimple categories.

To conclude the article, we explain how Conjecture 1.15 may follow from the periodic Higgs conjecture, at least in the one-periodic case.

**Lemma 5.5.** Let $C$ be a smooth projective curve over $\mathbb{C}$, let $D \subset C$ be a reduced divisor, and let $(E, \theta) \in \text{MHG}^\text{par}(C)_{\log}$ be a motivic parabolic Higgs bundle that is assumed to be one-periodic. Then the Simpson correspondence is compatible with the Ogus-Vologodsky correspondence.

**Proof.** By the functoriality of the Simpson correspondence, the commutativity of the inverse Cartier transform with parabolic pullback ([KS20, Lemma 3.8]) and the Biswas-Iyer-Simpson correspondence ([KS20, Proposition 2.11]), we may assume that $(E, \theta)$ has trivial parabolic structure. The periodic Higgs conjecture implies that $(E, \theta)$ would be a direct summand of a logarithmic Kodaira-Spencer system $\text{Gr}_F_{\text{hod}, s}^\text{F}(\text{Hod}, \nabla_{\text{GM}})$, associated to a quasi-semistable family $f$ over $C$. We may take a spreading-out of $f$, defined over some integral scheme $S$ absolutely of finite type. For all closed points $s \in S$ of sufficiently large residue characteristic and any $W_2(k(s))$-lifting $\tilde{s} \subset S$, one has two filtrations on $C_{s < \tilde{s}}(E, \theta)_{s}$ in question: one comes from the assumption that $(E, \theta)$ is one-periodic which is denoted by $\text{Fil}_s$; the second comes from the following fact-by [KS20, Proposition 4.5], there is an isomorphism

$$\psi_s : (\text{Hod}, \nabla_{\text{GM}})_{s} \cong C_{s < \tilde{s}}^{s} \text{Gr}_{\text{hod}, s} \text{Gr}_F_{\text{hod}, s}^\text{F}(\text{Hod}, \nabla_{\text{GM}})_{s},$$

and therefore $\text{Gr}_F_{\text{hod}, s}$ induces a filtration via $\psi_s$ on the subspace

$$C_{s < \tilde{s}}^{-1}(E, \theta)_{s} \subset C_{s < \tilde{s}}^{s} \text{Gr}_{\text{hod}, s}^\text{F}(\text{Hod}, \nabla_{\text{GM}})_{s},$$

which we denote also by $\text{Gr}_F_{\text{hod}, s}$. Note that both $\text{Gr}_{\text{hod}, s}^\text{F}(E, \theta)_{s}$ and $\text{Gr}_{\text{Fil}, s}^\text{F}(E, \theta)_{s}$ are polystable. By [Lan14, Corollary 5.6], there is an isomorphism of logarithmic Higgs bundles:

$$\text{Gr}_{\text{hod}, s}^\text{F}(E, \theta)_{s} \cong \text{Gr}_{\text{Fil}, s}^\text{F}(E, \theta)_{s},$$

and therefore by one-periodicity of $(E, \theta)$ an isomorphism of logarithmic Higgs bundles

$$\text{Gr}_{\text{hod}, s}^\text{F}(E, \theta)_{s} \cong (E, \theta)_{s}.$$
By [KS20, Lemma 4.3], the functor $Gr_{\text{F} \text{ hod}}$ establishes a bijection from the set of isomorphism classes of irreducible de Rham factors of $(H_{dR}, \nabla^G_{\text{GM}})$ to the set of isomorphism classes of graded stable factors of $Gr_{\text{F} \text{ hod}}(H_{dR}, \nabla^G_{\text{GM}})$. Clearly, the equivalence by a shift of indices on the filtrations of the de Rham side corresponds to the equivalence by a shift of indices on the grading structures of the Higgs side. It follows that $Gr_{\text{F} \text{ hod}}$ induces a bijection from the set of isomorphism classes of irreducible logarithmic flat subbundles of $(H_{dR}, \nabla_{\text{GM}})$ to the set of isomorphism classes of stable logarithmic Higgs subbundles of $Gr_{\text{F} \text{ hod}}(H_{dR}, \nabla^G_{\text{GM}})$. This property is preserved for reductions modulo all closed points $s \in S$ with sufficiently large residue characteristic. Now let $(V, \nabla)$ be the corresponding connection to $(E, \theta)$ under the Simpson correspondence over $\mathbb{C}$. It must be isomorphic to a direct summand of $(H_{dR}, \nabla^G_{\text{GM}})$, since the Simpson correspondence for $(H_{dR}, \nabla^G_{\text{GM}})$ (and its direct summands) is realized by the functor $Gr_{\text{F} \text{ hod}}$. By the injectivity of the functor $Gr_{\text{F} \text{ hod}, s}$ by the above discussion, it follows that there is an isomorphism of logarithmic flat bundles

$$(V, \nabla)_s \cong C^1_{X_s}(E, \theta)_s$$

as desired. \hfill \Box

Appendix A. Quaternionic Shimura varieties and Deligne’s modècle étrange

In this section, we collect necessary results on Shimura varieties, integral models, strange models, and Hecke correspondences. Our main sources are Kisin [Kis10] and Carayol [Car86] for the good reduction of Shimura curves (first due to Morita [Mor81]). We explicate the important fact that the quaternionic polydisk Shimura varieties occurring in [Kuc18] are finite étale covers of the “polydisk Shimura modular DM stacks” of [CS08, §9] using strange models.

Let $F$ be a totally real field of degree $d$ over $\mathbb{Q}$ and let $D$ be a quaternion algebra over $F$ that is split at $r \geq 1$ real places of $F$. Labelling the places $\tau_1, \ldots, \tau_d$, we suppose the first $r$ split $D$ and we fix splittings $s_i \cong D \otimes_{F, \tau_i} \mathbb{R}$ for $1 \leq i \leq r$. We describe two perspectives on “quaternionic polydisk Shimura varieties”:

**Connected quaternionic polydisk Shimura varieties.** Pick a maximal order $O_D \subset D$. Via the splittings we may embed

$$\Gamma := O_D^{N=1} \hookrightarrow ((PGL_2(\mathbb{R}))^r)$$

where $N = 1$ signifies “norm=1”. Given these choices, the quotient

$$X_{\Gamma} := [\Gamma \backslash (\mathfrak{h})^r]$$

is isomorphic to the analytification of a connected complex Deligne-Mumford stack. A connected quaternionic polydisk *arithmetic* or *Shimura-arithmetic* variety is a finite étale cover of such a $X_{\Gamma}$; i.e., given a torsion-free finite index subgroup $\Gamma' \subset \Gamma$, the quotient $\Gamma' \backslash (\mathfrak{h})^r$ is a connected quaternionic polydisk Shimura variety.
Quaternional polydisk Shimura varieties. Consider $G := \text{Res}_{F/Q}(D^*)$ as an algebraic group over $\mathbb{Q}$. We get $G_\mathbb{R}$-conjugacy class $X$ of cocharacters

$$S \to G_\mathbb{R} \cong GL_2(\mathbb{R})^r \times U(2)^{d-r}$$

by sending an element $g = x + \sqrt{-1}y \in S(\mathbb{R}) = \mathbb{C}^*$ to

$$\left( \begin{array}{cc} x & y \\ -y & x \end{array} \right), \ldots, \left( \begin{array}{cc} x & y \\ -y & x \end{array} \right), 1, \ldots, 1).$$

Then $(G, X)$ is a Shimura datum. We have the explicit description: $X = (P_1(\mathbb{C}) \setminus P_1(\mathbb{R}))^r$.

Let $A_f$ denote the finite adèles over $\mathbb{Q}$. We obtain an action of $G(\mathbb{Q})$ on $X$ via the embedding $G(\mathbb{Q}) \hookrightarrow G(\mathbb{R})$. Then the quaternionic polydisk Shimura variety associated to $(G, X)$ is the following quotient:

$$\text{Sh}(G, X) := G(\mathbb{Q}) \backslash G(\mathbb{A}_f) \times X$$

This is the inverse limit of adèle double quotients

$$\text{Sh}_K(G, X) := [G(\mathbb{Q}) \backslash G(\mathbb{A}_f) \times X/K]$$

where $K$ runs through sufficiently small (i.e., neat) open compact subgroups of $G(\mathbb{A}_f)$ [Milne, Theorem 5.28]. In the general formulation, potential disconnectedness is built in: $\text{Sh}_K(G, X)$ is a finite union of connected Shimura varieties. If $T := \text{Res}_{F/Q} G_m$, then there is a map $\nu: T \to G_m$ given by “reduced norm” (with kernel $G_1$, the derived group of $G$).

If $D/F$ is not globally split and $d > 1$, the Shimura variety $\text{Sh}(G, X)$ is not naturally a moduli space of abelian varieties. However, by using Deligne’s modèles étranges, we can construct (non-canonically) a different Shimura variety with isomorphic geometrically connected components that does underlie a “modular” PEL-type family of abelian varieties. We follow [Car86, §2] closely.

Choose an auxiliary imaginary quadratic extension $\mathbb{Q}(\alpha)$ and set $E := F(\alpha)$ with involution $z \mapsto \bar{z}$. Following Carayol, we always consider $E$ as a fixed subfield of $\mathbb{C}$. Let $B := D \otimes_F E$. Let $T_E := \text{Res}_{E/Q} G_m$ and let $U_E \subset T_E$ be the subgroup determined by $z\bar{z} = 1$. Let $G' := G \ast_{Z(G)} T_E$ be the amalgamated product (a.k.a. coproduct).

Consider the homomorphism $T_E \to T \times U_E$ given as $z \mapsto (z\bar{z}, z/\bar{z})$. This agrees on $Z(G) \cong T$ with the homomorphism $G \to T \times U_E$ given as $g \mapsto (\nu(g), 1)$. Hence there is an induced homomorphism

$$G' \overset{\nu'}{\to} T \times U_E$$

defined by $(g, z) \mapsto (\nu(g)z\bar{z}, z/\bar{z})$. Consider the subtorus $T' := G_m \times U_E \subset T \times U_E$ and let $G' := (\nu')^{-1}(T')$. Then the derived groups of $G$ and $G'$ are isomorphic. We now follow [Car86, 2.13] to construct $X'$ so that $(G', X')$ is a Shimura datum.

Abusing notation, we have complex embeddings $\tau_1, \ldots, \tau_d: E \to \mathbb{C}$, which yield an isomorphism
Let $h_E: \mathbb{S} \to (T_E)_\mathbb{R}$ be the map $z \mapsto (1, \ldots, 1, z, \ldots, z)$ where the first $r$ entries are “$1$”. Then the projection $h \times h_E: \mathbb{S} \to (G \times_T T_E)_\mathbb{R}$ factors through a map $h': \mathbb{S} \to G'_\mathbb{R}$. The $G'_\mathbb{R}$ conjugacy class of $h'$ is denoted by $X'$. In particular, we obtain a Shimura datum $(G', X')$.

We now explain how to realize this Shimura datum as being of PEL type [Car86, §2.2]. Denote by $b \mapsto \overline{b}$ the involution on $B$ which is tensor product of the natural involution on $D/F$ and conjugation on $E$. Let $\delta \in B$ be a symmetric element: $\delta = \overline{\delta}$. Then we may construct a new involution on $B$:

$$b \mapsto b^* := \delta^{-1} \overline{b} \delta.$$  

Let $V$ be the vector space underlying $B$. Choose $0 \neq \alpha \in E$ that is imaginary: $\overline{\alpha} = -\alpha$. Then we have the following form on $V$ (where $\text{tr}_{B/E}$ denotes the reduced trace):

$$\psi(v, w) := \text{Tr}_{E/\mathbb{Q}}(\alpha \text{tr}_{B/E}(v \delta w^*),$$

which is symplectic and non-degenerate. One checks that $G'$ is isomorphic to the group of $B$-linear symplectic similitudes $GSP(V, \psi)$. In particular, for any small open compact $K' \subset G'(\mathbb{A}_f)$, the Shimura variety $Sh_{K'}(G', X')$ carries a “universal” family of abelian varieties of dimension $4d$ with multiplication some order $\mathcal{O} \subset B$.

Suppose we had made our choice of $\alpha$ such that:

1. $E = F(\alpha) \subset D$ and
2. $E$ splits $D$.

Then $B \cong M_{2 \times 2}(E)$. In particular, after projecting by an idempotent, we obtain a family of $2d$-dimensional abelian varieties over $Sh_{K'}(G', X')$ that have multiplication by some order in $E$ such that the polarization is compatible with the $E$-action. In the language of [CS08, §9], this is (a finite ´etale cover of) a polydisk Shimura modular stack. Conversely, given the data of a totally real field $F$ together with a totally imaginary quadratic extension $E$: there exists a quaternion algebra $D/F$ such that $E \subset D$ and $E$ splits $D$. Therefore the moduli spaces of Corlette-Simpson yield, on the level of connected components, precisely what we have constructed above.

**Definition A.1.** Let $F$ be a totally real number field and let $D$ be a non-split quaternion algebra over $F$ that is split at exactly one real place. The Shimura curve datum associated to $F$ and $D$ is the quaternionic Shimura datum $(G, X)$ described above.

We now discuss Hecke correspondences. Let $(G, X)$ be a Shimura datum and $g \in G(\mathbb{A}_f)$. Then, $K_g := gKg^{-1} \cap K \subset G(\mathbb{A}_f)$ is also an open compact subgroup.

$$T_E(\mathbb{R}) = (E \otimes R)^* \cong (\mathbb{C}^*)^d.$$
The Hecke correspondence $T_g$ associated to $g$ is

\[
\begin{array}{ccc}
Sh_{K_g}(G,X) & \xrightarrow{s} & Sh_K(G,X) \\
& t & \downarrow \\
Sh_{K}(G,X) & \rightarrow & Sh_K(G,X)
\end{array}
\]

Here the first map is given by the natural inclusion $K_g \subset K$ and the second map is given by the inclusion $K_g \subset g\Gamma g^{-1} \cong K$ where the second identification comes from conjugation by $g$. This is a finite étale correspondence of smooth, quasi-projective, not-necessarily connected algebraic varieties, defined over $E$: see [Milne, Ch. 12-14], and especially 13.6 of loc. cit for the Hecke-equivariance property.

**Remark A.2.** Let $(G,X)$ be a Shimura curve datum, given by $F$ and $D$. Then the reflex field $E(G,X)$ is isomorphic to $F$ [Car86, 1.1.1].

For the Siegel Shimura datum $(GSP_{2n}, S^\pm)$, Hecke correspondences have a pleasant property. First we set some notation: if $K \subset GSP_{2n}(\mathbb{A}_f)$ is an open compact subgroup, we write $\mathcal{A}_{n,1,K}$ for the Shimura variety $Sh_K(GSP_{2n}, S^\pm)$; intuitively, it is the moduli space of principally polarized abelian varieties with K-level structure. Let $g \in GSP_{2n}(\mathbb{A}_f)$. Consider the Hecke correspondence

\[
\begin{array}{ccc}
\mathcal{A}_{n,1,K} & \xrightarrow{s} & \mathcal{A}_{n,1,K} \\
& t & \downarrow \\
\mathcal{A}_{n,1,K} & \rightarrow & \mathcal{A}_{n,1,K}
\end{array}
\]

Let $\mathcal{U} \rightarrow \mathcal{A}_{n,1,K}$ be the universal family of abelian varieties over $\mathcal{A}_{n,1,K}$. Then $s^* \mathcal{U}$ is isogenous to $t^* \mathcal{U}$.

Let $(G,X)$ denote a Shimura curve datum, let $p$ be a prime of $\mathbb{Q}$, and let $\mathfrak{p} | p$ be a prime of $F$. Suppose $K \subset G(\mathbb{A}_f)$ is hyperspecial at $p$ and neat. Then it follows from [Car86, §6.1] that $Sh_K(G,X)$ has a canonical integral model $\mathcal{I}_K(G,X)$ over $\mathcal{O}_{E,\mathfrak{p}}$. Now let $(G,X) \hookrightarrow (GSP_{2n}, S^\pm)$ realize $(G,X)$ as being of Hodge type. Then by [Kis10, 2.1.2] there exists $K' \subset GSP_{2n}(\mathbb{A}_f)$, hyperspecial at $p$, and an induced map of integral canonical models:

\[
\mathcal{I}_K(G,X) \rightarrow \mathcal{A}_{n,1,K'}
\]

which are smooth schemes over $\mathcal{O}_{E,\mathfrak{p}}$. In particular, from this data we obtain a “universal” abelian scheme over $\mathcal{I}_K(G,X)$.

**Corollary A.3.** Let $(G,X) \hookrightarrow (GSP_{2n}, S^\pm)$ be a Hodge-type Shimura datum with reflex field $E$. Let $p$ be a rational prime and let $\mathfrak{p} | p$ be a prime of $E$ over $p$. Let $K = K_p K^p \subset G(\mathbb{A}_f)$ be a sufficiently small open compact subgroup with $K_p \subset G(\mathbb{Q}_p)$ hyperspecial. Let $g \in G(\mathbb{A}_f^\mathfrak{p})$ and consider the Hecke correspondence of integral models over $\mathcal{O}_{E,\mathfrak{p}}$. 
Let $\mathcal{U}_G \to \mathcal{J}_K(G, X)$ be the family of abelian schemes, induced from a morphism $\mathcal{J}_K(G, X) \hookrightarrow \mathcal{A}_{n,1,K'}$ as above. Then $s^* \mathcal{U}_G$ is isogenous to $t^* \mathcal{U}_G$.

Proof. Denote by $g' \in GSp_{2n}(A_f)$ the image of $g$ under the embedding of Shimura data as in the hypothesis of the corollary. Then the Hecke correspondence for $(G, X)$ maps to a Hecke correspondence for $(GSp_{2n}, S^\pm)$:

Denote again by $\mathcal{U} \to \mathcal{A}_{n,1,K'}$ a universal family of abelian varieties over $\mathcal{A}_{n,1,K'}$. Then the fact that $s^* \mathcal{U}$ is isogenous to $t^* \mathcal{U}$ implies that $s^* \mathcal{U}_G$ is isogenous to $t^* \mathcal{U}_G$ as desired.

\[\square\]

**Notation A.4.** Let $(G, X)$ be a Shimura curve datum, let $K \subset G(A_f)$ be an open compact subgroup that is hyperspecial at $p$ and let $p|p$ be a prime of $F$.

- Let $M$ be the Shimura curve $Sh_K(G, X)$ over $F$ (or $\mathbb{C}$).
- Let $\mathcal{M}$ be the canonical integral model $\mathcal{J}_K(G, X)$ over $\mathcal{O}_F$.
- Let $\mathcal{M} := \mathcal{M} \otimes \mathbb{F}_p$ and $\mathcal{M}_0$ be a geometrically connected component of $\mathcal{M}$.

Given a rational symplectic representation $G \to GSp_{2n}$ realizing $Sh(G, X)$ as of Hodge-type, pick $K' \subset GSp_{2n}(A_f)$ such that we obtain $\mathcal{M} \hookrightarrow \mathcal{A}_{n,1,K'}$.

- Denote by $\mathcal{U} \to \mathcal{M}$ the induced abelian scheme over $\mathcal{M}$
- Denote by $\mathcal{U} \to \mathcal{M}$ the reduction modulo $p$ of $\mathcal{U} \to \mathcal{M}$.

**References**

[Ati57] M. Atiyah, Vector bundles over an elliptic curve, Proc. Lond. Math. Soc., III. Ser. 7, 414-472 (1957).

[Car86] P. Carayol, Sur la mauvaise réduction des courbes de Shimura, Comp. Math. 59 (1986), no. 2, 151-230.

[CC90] D. Chudnovsky, G. Chudnovsky, Computer algebra in the service of mathematical physics and number theory, Computers in mathematics, Lecture Notes in Pure and Appl. Math. 125, 109-232, 1990.

[CW90] P. Cohen, J. Wolfart, Modular embeddings for some nonarithmetic Fuchsian groups, Acta Arith. 56 (1990), no. 2, 93-110.

[CS08] K. Corlette, C. Simpson, On the classification of rank-two representations of quasiprojective fundamental groups, Compos. Math. 144 (2008), no. 5, 1271-1331.
[Del71] Deligne, P.: Théorème de Hodge II, Publ. Math. IHES, tome 40, 5-57, 1971.
[Del72] P. Deligne, Travaux de Shimura, Séminaire. Bourbaki, Lecture Notes in Mathematics, Vol. 244, Springer, Berlin, 1972, pp. 123-165.
[Deu41] M. Deuring, Die Typen der Multiplikatorenringe elliptischer Funktionenkoerper, Abh. Math. Sem. Hansischen Univ. 14 (1941), 19-272.
[Elk87] N. Elkies, The existence of infinitely many supersingular primes for every elliptic curve over $\mathbb{Q}$, Invent. math. 89, 561-567 (1987).
[Elk89] N. Elkies, Supersingular primes for elliptic curves over real number fields, Compos. Math. 72, No. 2, 165-172 (1989).
[Fa83] G. Faltings, Real projective structures on Riemann surfaces, Compositio Mathematicae, tome 48, no 2 (1983), p. 223-269.
[GK16] W. Goldring, J. Koskivirta, Strata Hasse invariants, Hecke algebras, and Galois representations, 2016.
[Gro18] M. Gröchenig, Epsilon Factors, notes for a course at FU Berlin, Spring 2018, http://individual.utoronto.ca/groechenig/epsilon.pdf (2020).
[Kis10] M. Kisin, Integral models for Shimura varieties of Abelian type, J. Amer. Math. Soc 23:4 (2010), 967-1012.
[Kri18] R. Krishnamoorthy, Correspondences without a core, Algebra & Number Theory 12:5 (2018) 1173-1214.
[KS20] R. Krishnamoorthy, M. Sheng, Periodic de Rham bundles over curves, 2020.
[Kuc18] R. Kucharczyk, Modular embeddings and automorphic Higgs bundles, Algebr. Geom. 5 (2018), no. 2, 200-238.
[LSZ15] G. Lan, M. Sheng, K. Zuo, Nonabelian Hodge theory via exponential twisting, Math. Res. Lett. 22 (2015), no. 3, 859-879.
[LSYZ19a] G. Lan, M. Sheng, Y. Yang, K. Zuo, Uniformization of p-adic curves via Higgs-de Rham flows, J. Reine Angew. Math. 747 (2019), 63-108.
[LSYZ19b] G. Lan, M. Sheng, K. Zuo, Semistable Higgs bundles, periodic Higgs bundles, and representations of the algebraic fundamental group, J. Eur. Math. Soc. (2019)
[Lan14] A. Langer, Semistable modules over Lie algebroids in positive characteristic, Documenta Mathematicae 19 (2014) 561-592.
[Lan19] A. Langer, Nearby cycles and semipositivity in positive characteristic, arXiv: 1902.05745 (2019).
[LS] M. Li, M. Sheng, Characterization of Beauville’s algebraic numbers via Hodge theory, arxiv: 2009.14529, (2020).
[Ma89] D. Masser, Specializations of finitely generated subgroups of Abelian varieties, Transactions of the American Mathematical Society, Volume 311, Number 1, 413-424, 1989.
[Milne] J. Milne, Introduction to Shimura Varieties, available at http://www.jmilne.org/math/xnotes/svi.pdf, version of September 16, 2017.
[Mo96] S. Mochizuki, A theory of ordinary $p$-adic curves, Publ. Res. Inst. Math. Sci. 32 (1996), no. 6, 957-1152.
[Mo98] S. Mochizuki, Correspondences on hyperbolic curves, J. pure and applied algebra, 131(3), 227-244, 1998.

[Mo99] S. Mochizuki, Foundations of $p$-adic Teichmüller Theory, AMS/IP Studies in Advanced Mathematics, Volume 11, 1999.

[Mor81] Y. Morita, Reduction modulo $\mathfrak{p}$ of Shimura curves, Hokkaido Mathematical Journal 10.2 (1981): 209-238.

[Oda] T. Oda, Vector bundles on an elliptic curve, Nagoya Mathematical Journal, Volume 43 September 1971 , pp. 41-72.

[Oort01] F. Oort, A stratification of a moduli space of abelian varieties, Moduli of abelian varieties. Birkhäuser, Basel(2001), 345-416.

[OV] A. Ogus, V. Vologodsky, Nonabelian Hodge theory in characteristic $p$, Publ. Math. Inst. Hautes études Sci. 106 (2007), 1-138.

[Pin] P. Pink, On the order of the reduction of a point on an abelian variety, Math. Ann. 330 (2004), no. 2, 275-291.

[SM00] P. Schaller, J. Wolfart, Semi-arithmetic Fuchsian groups and modular embeddings, J. London Math. Soc. (2) 61 (2000), no. 1, 13-24.

[SB77] J.-P. Serre, H. Bass, Arbres, amalgames, $SL_2$, Soc. math. France (1977).

[SZZ] M. Sheng, J. Zhang, K. Zuo, Higgs bundles over good reduction of Shimura curves associated with quaternion division algebra, J. reine angew. Math. 671 (2012), 223-248.

[SZ12] M. Sheng, K. Zuo, Polarized variation of Hodge structure of Calabi-Yau type and characteristic subvarieties over bounded symmetric domains, Math. Ann. 348, No. 1 (2012), 211-236.

[SZ16] M. Sheng, K. Zuo, Deuring’s mass formula for a Mumford family, Trans. Amer. Math. Soc. 368 (2016), no. 1, 169-207.

[Sim90a] C. Simpson, Harmonic bundles on noncompact curves, J. Amer. Math. Soc. 3 (1990), no. 3, 713-770.

[Sim92] C. Simpson, Higgs bundles and local systems, Inst. Hautes Études Sci. Publ. Math. No. 75 (1992), 5-95.

[VZ] E. Viehweg, K. Zuo, A characterization of certain Shimura curves in the moduli stack of abelian varieties, J. Differential Geom. 66 (2004), no. 2, 233-287.

[Wort13] D. Wortmann, The $\mu$-ordinary locus for Shimura varieties of Hodge type, arXiv preprint arXiv: 1310.6444(2013).

[Zha14] C. Zhang, Remarks on Ekedahl-Oort stratifications, arXiv: 1401.6632 (2014).

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