BRILL–NOETHER THEORY OF MAXIMALLY SYMMETRIC GRAPHS

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Abstract. We analyze the Brill–Noether theory of trivalent graphs and multigraphs having largest possible automorphism group in a fixed genus. For trivalent multigraphs with loops of genus at least 3, we show that there exists a graph with maximal automorphism group which is Brill–Noether special. We prove similar results for multigraphs without loops of genus at least 6, as well as simple graphs of genus at least 7. This analysis yields counterexamples, in any sufficiently large genus, to a conjecture of Caporaso.

1. Introduction

A graph $\Gamma$ of genus $g$ is said to be Brill–Noether general if for all positive integers $r$ and $d$ such that the number

$$\rho^r_d(g) = g - (r + 1)(g - d + r)$$

is negative, there exist no effective divisors on $\Gamma$ of degree $d$ and rank at least $r$. We begin by recalling the following conjecture of Caporaso [4, Conjecture 6.6(1)].

Conjecture 1. Assume $g \geq 2$. Let $\Gamma$ be a trivalent graph of genus $g$ with largest possible automorphism group (among trivalent graphs of genus $g$). Then $\Gamma$ is Brill–Noether general.

The conjecture is motivated by an analogy between the moduli space $\overline{M}_g$ of Deligne-Mumford stable curves and the moduli space $\mathcal{M}_g^{\text{trop}}$ of tropical curves. The latter is the space parametrizing genus $g$ metric graphs with integer vertex-weights. The divisor theory of algebraic curves is related to that of graphs by means of Baker’s Specialization Lemma, see [2].

For fixed $g$, there is a natural order reversing bijection between the set of strata of these two moduli spaces. The classical moduli space $\overline{M}_g$ is stratified by the combinatorial type of the dual graphs of stable curves, while $\mathcal{M}_g^{\text{trop}}$ is stratified by the underlying combinatorial type of the metric graphs. That is, given a combinatorial graph $\Gamma$, there is a cone in the moduli space $\mathcal{M}_g^{\text{trop}}$ parametrizing all metric graphs having $\Gamma$ as their underlying combinatorial graph. In particular, the top dimensional stratum on the classical side are the points of $\overline{M}_g$ corresponding to smooth curves. The dual graph of a smooth genus $g$ curve is a single vertex with weight $g$. On
the other hand, the top dimensional stratum on the tropical side correspond to trivalent metric graphs with no nonzero vertex weights. A general smooth curve has no automorphisms, so Caporaso proposes that its analog should be a trivalent graph with the greatest possible number of automorphisms. Since a general curve is Brill–Noether general, Caporaso conjectures that the trivalent graphs with largest number of automorphisms are also Brill–Noether general. Here we prove that in any sufficiently larger genus, there exists a trivalent graph (or multigraph) with largest possible automorphism group that is not Brill–Noether general.

In what follows, we will refer to trivalent graphs or multigraphs achieving largest possible automorphism group as **maximally symmetric**.

**Theorem 2.** Let $g \geq 3$. There exists a maximally symmetric genus $g$ trivalent multigraph (possibly with loops) that is not Brill–Noether general.

Trivalent multigraphs with loops are the dual graphs of stable curves in the zero stratum of $\mathcal{M}_g$. We may also analyze Caporaso’s conjecture for dual graphs of curves which have only smooth components. These are multigraphs without loops.

**Theorem 3.** Let $g \geq 6$. There exists a maximally symmetric genus $g$ trivalent multigraph with no loops that is not Brill–Noether general.

Finally, we analyze the Brill–Noether theory of maximally symmetric trivalent simple graphs. That is, graphs without loops or multiple edges.

**Theorem 4.** Let $g \geq 7$. There exists a maximally symmetric genus $g$ trivalent simple graph that is not Brill–Noether general.

Our methods are explicit and combinatorial. We rely on the analysis of maximally symmetric trivalent graphs pursued by van Opstall and Veliche in [7, 8]. In each sufficiently large genus, we exhibit a maximally symmetric graph together with a divisor of low degree and rank 1.

The preceding theorems are sharp. The maximally symmetric genus 2 multigraph with loops is Brill–Noether general. When $g \leq 5$, all genus $g$ maximally symmetric multigraphs without loops are Brill–Noether general. The same is true for maximally symmetric simple graphs when $g \leq 6$.

**Remark 5.** In [7, Proposition 6.5] it is proved that when the $g$ is $3 \cdot 2^m$, $3(2^m+1)$, $3(2^m+2^p)$, or $3(2^m+2^p+1)$ (where $p > m \geq 0$), there is a unique maximally symmetric multigraph with no loops. A direct application of the proof of this result also shows that when $g$ is $3 \cdot 2^m$, or $3(2^m + 2^p)$ there is a unique maximally symmetric multigraphs with loops. In other words, in these genera, the theorems above preclude the existence of maximally symmetric Brill–Noether general graphs.

The heuristic used by van Opstall and Veliche is that maximally symmetric trivalent graphs should be “as close to trees as possible”. As we will see in Section 3 this is achieved by attaching trees to a small graph, such as a
single vertex or a triangle, and placing appendages on the leaves of the trees to contribute genus. Our analysis suggests that graphs of this general structure are not Brill–Noether general for large genus. Together with Remark 5, this leads us to pose the following conjecture.

**Conjecture 6.** Let $\Gamma$ be a genus $g$ trivalent maximally symmetric graph or multigraph (with or without loops). If $g$ is sufficiently large, then $\Gamma$ is not Brill–Noether general.

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## 2. Background

### 2.1. Divisors on Graphs.

We now briefly recall the fundamentals of divisor theory on graphs and Brill–Noether theory. For further details, see [2, 3, 4].

A graph $\Gamma$ will mean a finite connected graph possibly with loops and multiple edges. The vertex set of $\Gamma$ will be denoted by $V(\Gamma)$ and edge set $E(\Gamma)$. We will be explicit when restricting our analysis to the cases of graphs without multiple edges or loops. The genus of $\Gamma$, denoted $g(\Gamma)$, is the first Betti number of $\Gamma$. Since $\Gamma$ is a connected graph, we have

$$g(\Gamma) = |E(\Gamma)| - |V(\Gamma)| + 1.$$  

A **divisor** is an element of the free abelian group on the vertices of $\Gamma$:

$$\text{Div}(\Gamma) := \left\{ \sum_{v \in V(\Gamma)} n_v v : n_v \in \mathbb{Z} \right\}.$$  

The **degree** of a divisor $D = \sum v n_v v$ is $\deg(D) = \sum v n_v$. The divisor $D$ is said to be **effective** if $n_v \geq 0$ for all $v$.

The **intersection pairing** on $\text{Div}(\Gamma)$ is obtained by extending the following formula bilinearly. For $v, w \in V(\Gamma)$,

$$\langle v, w \rangle := \begin{cases} \# of edges between v and w : v \neq w \\ -\deg(v) + 2\text{loop}(v) \end{cases} : v = w$$

where $\text{loop}(v)$ is the number of loops based at $v$ and $\deg(v)$ is the number of edges incident to $v$. Note that a vertex with a single loop and no other edges has degree 2.

As with the classical theory of divisors on algebraic curves, there is a natural notion of linear equivalence between divisors on graphs. For any $\mathbb{Z}$-valued function $f$ on $V(\Gamma)$, we may define an integer

$$\text{ord}_f(v) := \sum_{w \in V(\Gamma)} \langle w, v \rangle f(w).$$
Given such a function $f$ on $V(\Gamma)$, we define a divisor

$$\text{div}(f) := \sum_{v \in V(\Gamma)} \text{ord}_f(v)v.$$  

Divisors that are obtained in this fashion are called principal. We say that $D$ and $D'$ are linearly equivalent, and write $D \sim D'$ if $D - D'$ is principal. Given a divisor $D$ we denote by $|D|$ the set of all effective divisors that are linearly equivalent to $D$.

As with the divisor theory of algebraic curves, there is a natural notion of rank for divisors on graphs.

**Definition 7.** The rank of $D$, denoted $r(D)$, is the largest integer $r$ such that $|D - E|$ is nonempty for all effective divisors $E$ of degree $r$. Note that if $|D| = \emptyset$, then $r(D) = -1$.

### 2.2. Chip-Firing and Divisors.

It is sometimes helpful to visualize divisors as configurations of “chips” on the graph. For instance, the divisor $17v + 5w$ corresponds to 17 chips on vertex $v$ and 5 chips on vertex $w$. Given a vertex $v$, we say $v$ is fired if $\text{deg}(v)$ chips are moved from $v$, distributing one chip to each vertex adjacent to $v$. See for example Figure 1. Two divisors $D$ and $D'$ are linearly equivalent if $D'$ can be reached from $D$ by a finite sequence of chip firing moves. Chip firing on multigraphs is performed by first subdividing loop and multiple edges to obtain a simple graph, and then following the above process.

### 2.3. Brill–Noether Theory.

Let $C$ be a smooth projective algebraic curve. Brill–Noether theory of algebraic curves is concerned with the geometry of the scheme $W^r_d(C)$, parametrizing linear equivalence classes of divisors of degree $d$ moving in a linear system of rank at least $r$ on $C$. Brill–Noether theory of graphs seeks to develop an analogous combinatorial theory. One aspect of this is the existence of effective divisors of degree $d$ which have rank at least $r$ for positive integers $d$ and $r$.

Given a genus $g$ graph $\Gamma$ the Brill–Noether number, denoted $\rho^r_d(g)$, is defined to be

$$\rho^r_d(g) := g - (r + 1)(g - d + r).$$  

For an algebraic curve $C$, $\rho^r_d(g)$ is the naive expected dimension of $W^r_d(C)$. The classical Brill–Noether theorem states that for $\rho^r_d(g) < 0$, a general
curve of genus \( g \) has no divisors of degree \( d \) and rank at least \( r \). See [6] for more details. In [5], Cools, Draisma, Payne and Robeva prove a “tropical Brill–Noether theorem” which is then used to deduce the classical result. This tropical Brill–Noether theorem asserts the existence of a metric graph \( \Gamma_g \) in each genus \( g \) such that \( \Gamma_g \) has no divisors of degree \( d \) and rank at least \( r \) when \( \rho^r_d(g) \) is negative. With this in mind, a genus \( g \) graph \( \Gamma \) is said to be \textbf{Brill–Noether general} if for all \( r, d \geq 0 \) such that \( \rho^r_d(g) \) is negative, \( \Gamma \) has no divisors of degree \( d \) and rank at least \( r \). A graph is called \textbf{special} if it is not general.

3. Proof of Theorems

3.1. Maximally Symmetric Multigraphs with Loops. All graphs in this subsection are trivalent multigraphs, possibly with loop edges. These are precisely the dual graphs of stable curves in the zero-stratum of \( \overline{M}_g \). In [7], van Opstall and Veliche demonstrate explicit bounds for the automorphism group of a genus \( g \) trivalent multigraph with loops together with a graph \( C_g \) which achieves this bound. We now review the construction.

For \( n \geq 3 \) we construct trees \( T_n \) and the genus \( g \) graphs \( C_g \) using the following algorithms given in [7, Definition 3.1, 3.3].

\begin{definition}
For each \( n \geq 1 \), \( T_n \) is defined as follows:
\begin{itemize}
  \item Place \( n \) vertices in a row. Call this level one. Assume that level \( k \) is formed and has vertices labeled \( v_1, \ldots, v_m \). If \( m \) is even, form level \( k + 1 \) by adding vertices labeled as follows:
    \[ v_{\{1,2\}}, v_{\{3,4\}}, \ldots, v_{\{m-1,m\}}. \]
  \item If \( m \) is odd, form level \( k + 1 \) by adding vertices labeled as follows:
    \[ v_{\{1,2\}}, v_{\{3,4\}}, \ldots, v_{\{m-2,m-1\}}. \]
\end{itemize}

That is, level \( k + 1 \) contains precisely \( \left\lfloor \frac{m}{2} \right\rfloor \) vertices. Connect each vertex \( v_{\{r,r+1\}} \) in level \( k + 1 \) to the vertices \( v_r \) and \( v_{r+1} \) in level \( k \).
\begin{itemize}
  \item If there is an unpaired vertex in level \( k \) and one in a level smaller than \( k \), place a vertex in level \( k + 1 \) and connect these two vertices to it.
  \item Once level \( k \) is formed, if there are exactly two vertices left unpaired at any stage, connect them by an edge. If there are exactly three vertices left at any stage, add a new vertex and edges between this vertex and each of the three unpaired vertices.
\end{itemize}

Notice that the subscript \( n \) refers to the number of leaves in the tree.

\begin{definition}
For each \( g \geq 3 \), the graph \( C_g \) is defined as follows:
\begin{itemize}
  \item If \( g \neq 3 \cdot 2^m + 1 \), for \( m > 0 \) and \( g \neq 3(2^m + 2^p) \) for \( m > p > 0 \), form \( C_g \) from \( T_g \) by placing a loop on each leaf.
\end{itemize}
\end{definition}
The trees $T_5$ and $T_6$. In each case, the larger vertices correspond to the level one vertices.

- If $g = 3 \cdot 2^m + 1$, then form $C_g$ by following the construction for $T_{g-1}$, but at the last step join the last three vertices in a triangle, then placing a loop on each leaf.
- If $g = 3(2^m + 2^p)$ for $m > p \geq 0$, attach a binary tree of $m$ leaves and a binary tree of $p$ leaves to the leaves of a binary tree with two leaves. Arrange three copies of this configuration around a 3-star, and place a loop on each leaf. This is $C_g$.

The following proposition is a direct consequence of Proposition 3.4 and Main Theorem of [7].

**Proposition 10.** Let $g \geq 3$. The graph $C_g$ is a maximally symmetric genus $g$ trivalent multigraph with loops.

The hyperelliptic involution on $C_4$ and $C_5$ flips each loop. $C_7$ is not hyperelliptic

**Definition 11.** A graph $\Gamma$ is called hyperelliptic if it has a divisor of rank 1 and degree 2.

Similarly, an algebraic curve is hyperelliptic if it has a divisor of rank 1 and degree 2. An algebraic curve is hyperelliptic if and only if it admits an involution $\iota$ whose quotient is a genus 0 curve. In similar spirit, Baker and Norine show in [3] Theorem 5.2 that $\Gamma$ is hyperelliptic if and only if there exists an involution $\iota : \Gamma \rightarrow \Gamma$, such that $\Gamma/\iota$ is a tree. If $\iota$ exists, it is necessarily unique, and we refer to it as the hyperelliptic involution.
The divisor of rank 1 and degree 2 is unique up to linear equivalence, and is referred to as the **hyperelliptic divisor**.

For \( g \geq 3 \), hyperelliptic graphs are not Brill–Noether general, since \( \rho_1^1(g) \) is negative. We will use this fact repeatedly in the forthcoming proofs.

**Proof of Theorem 2.** We see immediately from the construction above that if \( g \neq 3 \cdot 2^m + 1 \), the graph \( C_g \) is is formed by attaching loops to the leaves of a tree. The automorphism flipping each loop is a hyperelliptic involution. Hence, \( C_g \) is hyperelliptic and consequently not Brill–Noether general for \( g \geq 3 \).

Let \( g = 3 \cdot 2^m + 1 \). Then \( C_g \) is formed from a triangle by attaching to each vertex (via an edge) hyperelliptic subgraphs of genus \( 2^m \), as in Definition 9. Consider the divisor \( D = 3v \) where \( v \) is a vertex of the central triangle. \( D \) has degree 3 and rank 1.

We now show that \( D \) has rank at least 1. We may assume that \( D \) is zero away from the vertices of the central triangle by using a sequence of chip firing moves to move any nonzero weight onto the triangle. This is a consequence of the following standard argument: suppose \( e \) is a bridge edge with vertices \( v \) and \( v' \). Then firing every vertex in the connected component of \( \Gamma \setminus e \) that contains \( v \) moves \( v \) to \( v' \), so \( v \sim v' \). It follows that any two vertices connected by a bridge edge are equivalent. The problem is now reduced the case of a divisor \( 3w - w' \) where \( w, w' \) are vertices of the central triangle. Since \( \Gamma \) is trivalent, firing \( w \) yields an effective divisor. Since \( \rho_1^1(g) < 0 \), \( C_g \) is not Brill–Noether general. \( \square \)

**Remark 12.** Let \( D \) be a special divisor on a graph \( \Gamma \). We may ask if \( D \) is the specialization of a divisor on an algebraic curve in the sense of Baker [2]. Loosely speaking, we ask whether \( \Gamma \) is Brill–Noether special for algebraic reasons rather than combinatorial ones. When \( g \neq 3 \cdot 2^m + 1 \), the hyperelliptic divisor is not the specialization of such a divisor on an algebraic curve. This follows from the fact that the hyperelliptic involution on this graph cannot be lifted to a hyperelliptic involution on an algebraic curve. See [1, Corollary 9.17].

### 3.2. Maximally Symmetric Multigraphs without Loops

All graphs in this subsection are trivalent with no loop edges, but may have multiple edges. We first describe graphs \( C'_g \) for all genera, following [7, Definition 3.5]. By a **cone** we mean a triangle with one edge doubled. Note that a cone has genus 2.

**Definition 13.** For each \( g \geq 3 \), the graph \( C'_g \) is defined as follows:

- If \( g = 3 \cdot 2^m \), \( m > 0 \), \( C'_g \) is formed three binary trees with \( 2^m - 1 \) leaves connected in a 3-star. A cone is placed on each leaf.

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1 A bridge edge of a connected graph \( \Gamma \) is an edge \( e \) such that if \( e \) is deleted, \( \Gamma \) becomes disconnected.
• If $g = 3 \cdot 2^m + 1$, $m > 0$, $C'_g$ is formed from $C'_{g-1}$ by expanding the central vertex to a triangle.
• If $g = 3 \cdot 2^m + 2$, $C'_g$ is formed from $C'_{g-2}$ by expanding the central vertex to a $K_{2,3}$.
• If $g = 3 \cdot (2^m + 1)$, $m > 1$, $C'_g$ is formed from $C'_{g-3}$ by adding double edges in the middle of the edges of the star.
• If $g = 3(2^m + 2^p)$, $m > p + 1$ and $p > 0$ attach two binary trees with $2^{p-1}$ and $2^{m-1}$ leaves at their roots, and arrange three of these at the ends of a star, placing a cone on each leaf.
• If $g = 3(2^m + 2^p)$, $m > p + 1$ and $p > 0$, insert double edges to the previous case in the edges emanating from the star.
• Otherwise if $g$ is even, form $C_{g/2}$ as in Section 3.1, replacing loops by cones.
• If $g$ is odd, then the last step of the construction of $T_{\lfloor g/2 \rfloor}$ (the tree in Section 3.1) ends by connecting 2 vertices by an edge or 3 by a star. In the former case, add a double edge in the middle of this edge. In the latter case, the subgraphs on one of the 3 branches of the star is not isomorphic to the other two. Add a double edge on this branch. This is $C'_g$.

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{multigraph_C_6_graph_C_4}
\caption{While the graph on the left is hyperelliptic, the graph on the right is not.}
\end{figure}

The following proposition is a direct consequence of Proposition 3.6 and Main Theorem of [7].

**Proposition 14.** Let $g \geq 10$. The graph $C'_g$ is a maximally symmetric genus $g$ trivalent multigraph without loops.

### 3.2.1. Low Genus Cases.
We first analyze Caporaso’s conjecture for multigraphs without loops in genus smaller than 10. The following results are obtained by explicitly determining the divisor classes of prescribed rank and degree on a given graph using E. Robeva’s program.\(^3\)

\(^2\)Recall that $K_{2,3}$ the complete bipartite graph, has 2 trivalent vertices and 3 bivalent vertices. The binary trees are attached to these bivalent vertices.

\(^3\)This program efficiently computes the number of degree $d$ rank $r$ divisors on a given simple graph. It can be found at [http://math.berkeley.edu/~erobeva/chip-firing.html](http://math.berkeley.edu/~erobeva/chip-firing.html)
In each case the hyperelliptic involution interchanges the vertices \( v \) and \( w \), while fixing all other vertices.

We denote by \( \tilde{C}_g \), the genus \( g \) “loop of loops” graph. Namely, the graph obtained from the \( 2(g - 1) \)-gon by doubling every other edge.

- In genus 3 the unique maximally symmetric multigraph is in fact simple. It is the tetrahedron, and is Brill–Noether general.
- In genus 4 the unique maximally symmetric multigraph is also simple. It is the complete bipartite graph \( K_{3,3} \), and is Brill–Noether general.
- In genus 5, the unique maximally symmetric multigraph is \( \tilde{C}_5 \). This graph is Brill–Noether general.
- In genus 6 and 8, the unique maximally symmetric multigraphs are \( C'_6 \) and \( C'_8 \) respectively. As we will see, these graphs are hyperelliptic and hence special.
- In genus 7, the unique maximally symmetric multigraph is \( \tilde{C}_7 \). This graph has an effective divisor of degree 4 and rank 1, and hence is special.
- In genus 9, the maximally symmetric multigraphs are \( C'_9 \) and \( \tilde{C}_9 \). These graphs are both Brill–Noether special. The former is hyperelliptic, and the latter has a divisor of degree 5 and rank 1.

### Higher Genus Cases: Proof of Theorem

If \( g \neq 3 \cdot 2^m + 1 \), \( C'_g \) is hyperelliptic. The hyperelliptic involution reflects all cones, and interchanges the edges on all double edges. If the graph contains a \( K_{2,3} \) subgraph, the hyperelliptic involution is as depicted in Figure 5.

If \( g = 3 \cdot 2^m + 1 \), \( C'_g \) has a divisor of degree 3 and rank 1. The argument is nearly identical to that used for multigraphs with loops in the previous section. The relevant divisor in this case is \( 3v \), where \( v \) the top vertex of any cone.

### Maximally Symmetric Simple Graphs
All graphs in this subsection are simple and trivalent. We begin with an analysis of the conjecture for low genus.
3.3.1. Low Genus Cases. The following are the maximally symmetric trivalent graphs in each genus from [8] Remark 4.3:

- In genus 3, the unique maximally symmetric graph is the tetrahedron. It is Brill–Noether general.
- In genus 4, the unique maximally symmetric graph is $K_{3,3}$, the complete bipartite graph. It is Brill–Noether general.
- In genus 5, the unique maximally symmetric graph is the cube. It is Brill–Noether general.
- In genus 6, the unique maximally symmetric graph is the Petersen graph. It is Brill–Noether general.
- In genus 7, the unique maximally symmetric graph is depicted in Figure 6. The divisor depicted in this figure has degree 4 and rank 1, and the graph is hence special.
- In genus 8, the unique maximally symmetric graph is the Heawood graph. This graph has an effective divisor of degree 7 and rank 2 and is hence special.

![Figure 6](image)

**Figure 6.** The maximally symmetric simple graph of genus 7. The chip configuration depicted corresponds to a special divisor of degree 4 and rank 1.

We note that in genus 8, rank 1 divisors do not violate Caporaso’s conjecture. However, the divisor $7v$ for any vertex $v$ is a rank 2 special divisor.

3.3.2. Higher Genus Cases. In this section, we prove that for genus at least 9, the maximally symmetric trivalent graph is not Brill–Noether general. We extensively use the analysis of maximally symmetric trivalent graphs in [8]. The proof demands certain technical lemmas regarding the equivalence of divisors on certain types graphs.

By a **pinched tetrahedron** or a **pinched $K_{3,3}$** we mean the graphs depicted in Figure 7. These are obtained from the standard tetrahedron or $K_{3,3}$ by adding a new vertex to subdivide an edge.

**Definition 15** ([8] Definition 4.1). Let $A_m$ denote the graph formed by attaching a pinched tetrahedron (at its unique bivalent vertex) to each leaf of a binary tree having $2^m$ leaves. Similarly, let $B_m$ (for $m \geq 2$) denote the

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4In the $g = 7$ case, the optimal graph was produced via an exhaustive search of all trivalent graphs in this genus using data from Gordon Royle available at [http://school.maths.uwa.edu.au/~gordon/data.html](http://school.maths.uwa.edu.au/~gordon/data.html)
graph formed by attaching a pinched \( K_{3,3} \) to the leaves of binary tree having \( 2^m - 2 \) leaves. Observe that \( A_m \) has genus \( 3 \cdot 2^m \) and \( B_m \) has genus \( 2^m \).

For \( \Gamma = A_m \) or \( B_m \), we let \( T \) denote the binary subtree of \( \Gamma \) described above, and \( K \) a pinched \( K_{3,3} \) or tetrahedron. We abuse notation slightly, allowing \( K \) to be ambiguous in denoting these subgraphs. \( T \) is attached to each \( K \) at precisely one valence 1 vertex of \( T \).

**Lemma 16.** Let \( \Gamma = A_m \) or \( B_m \) be as above. Let \( D \) and \( D' \) be divisors on \( \Gamma \) of the form \( D = nv \) and \( D' = nw \), \( n \in \mathbb{Z} \) and \( v, w \in V(T) \). Then \( D \sim D' \).

**Proof.** The proof of this result is similar to that of Theorem 2. In particular, it suffices to show that \( v \sim v' \), where \( v \) and \( v' \) are vertices connected by a bridge edge and \( v' \) is closer to the root of \( T \) than \( v \). Firing each vertex in \( K \) as well as those vertices in \( T \) between \( K \) and \( v \) (including \( v \)) results in the cancelation of each chip except the one that arrives at \( v' \), proving the claim.

A simple application of the previous lemma yields the following.

**Lemma 17.** Let \( v \) be the unique vertex shared by \( K \) and \( T \). Suppose \( D = 4v \) and \( E = -w \) for some vertex \( w \in K \). Then \( D + E \sim D' \geq 0 \).

For certain \( g \), maximally symmetric graphs can be constructed by attaching copies of the \( A_m \) and \( B_m \) to cycle graphs. In light of this, we have the following lemma.

**Lemma 18.** Let \( \Gamma \) be defined by attaching a copy of \( A_m \) or \( B_m \) to each vertex \( v_1, \ldots, v_n \) in a cycle graph \( G_n \) of length \( n \). Let \( D = k \cdot nv_i \) and \( D' = k \cdot nv_j \), \( k \in \mathbb{Z} \). Then \( D \sim D' \).

**Proof.** It suffices to consider \( k = 1 \). By Lemma 16, chips on the attached \( A_m \) or \( B_m \) can be ignored.

We will show that \( nv_1 \sim nv_n \) from which the result immediately follows. To this end, let \( D_i = jv_1 - (j - 1)v_{i+1} \). Then if \( D_{i+1} = (j + 1)v_{i+1} - jv_{i+2} \), it follows that by firing the vertex \( v_{i+1} \) \( j \) times, \( D_i \sim D_{i+1} \).

Now, write \( n(v_1) = v_1 + (n - 1)v_1 \). Apply the above with \( i = 1 = j \) to \( v_1 + 0 \cdot v_2 \) successively to obtain

\[
D \sim (n - 1)v_1 + 2v_2 - v_3) \sim (n - 1)v_1 + 3v_3 - 2v_4) \sim \cdots \sim D',
\]
completing the proof.

Repeated applications of the previous lemmas yield the following two useful facts which we will use in the main proof.

**Lemma 19.** Let $G$ be defined by attaching three copies of $A_m$ or $B_m$ to the degree 2 vertices of $K_{2,3}$. For any $n \in \mathbb{Z}$ and choice $v, w$ of degree 2 vertices in $K_{2,3}$, the divisors $D = (2n)v$ and $D' = (2n)w$ are equivalent.

**Lemma 20.** Let $G$ be a graph defined by joining together two triangles via the identification of an edge $(w_1, w_2)$. Say $v_1$ and $v_2$ are the other vertices of $G$. Then, if $D = (2n)v_1$ and $D' = (2n)v_2$, $D \sim D'$.

We are now ready to prove Theorem 4. In [8], van Opstall and Veliche prove sharp numerical bounds for the size of the automorphism group of a genus $g$ trivalent graph. The authors show that these bounds are sharp by explicitly constructing a maximally symmetric simple graph $C''_g$. These graphs are constructed in similar fashion to the cases of multigraphs. We now show that this graph $C''_g$ is special for sufficiently large genus. For an explicit construction of the graphs $C''_g$ we refer the reader to the aforementioned paper.

**Proof of Theorem 4.** In each case, components $A_m$ or $B_m$ occur in $C''_g$, thus, the tree $T$ appears as well.

**Case 1.** Suppose that $C''_g$ consists of some collection of $A_m$’s or $B_m$’s joined at a common root, a common edge or path, by a $K_{2,3}$, by a square, or by two triangles that share a common edge. Define $D = 4v$ for any vertex $v \in T$. $D$ has rank at least one. This is shown by Lemmas 16 and 17, together with one of Lemmas 18, 19 or 20 depending on whether a square, $K_{2,3}$ or two triangles appear in the graph. Furthermore, $\rho_1(g) < 0$ when $g > 6$ as is the case here, whence the result follows.

**Case 2.** Suppose that $C''_g$ consists of some collection of $A_m$’s or $B_m$’s joined to the vertices of a triangle or to a path in which a triangle appears at one end. Let $v$ and $w$ be two distinct vertices of the triangle. Define $D = 3v + 2w$. Lemmas 16 to 18 guarantee that $D$ has rank at least 1. Furthermore, $\rho_1(g) < 0$ when $g > 8$ as is the case here, whence the result follows.

**Case 3.** Suppose that $C''_g$ consists of some collection of $A_m$’s or $B_m$’s joined to the vertices of a pentagon. Define $D = 5v$ for some vertex $v \in T$. Then, Lemmas 16 to 18 guarantee that $D$ has rank at least 1. Furthermore, $\rho_1(g) < 0$ when $g > 8$ as is the case here, whence the result follows.

Together these exhaust all possible configurations for $C''_g$, and the result follows. 

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