Time evolution of the autocorrelation function in dynamical replica theory

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Abstract

Asynchronous dynamics given by the master equation in the Sherrington–Kirkpatrick (SK) spin-glass model is studied based on dynamical replica theory (DRT) with an extension to take into account the autocorrelation function. The dynamical behaviour of the system is approximately described by dynamical equations of the macroscopic quantities: magnetization, energy contributed by randomness and the autocorrelation function. The dynamical equations under the replica symmetry assumption are derived by introducing the subshell equipartitioning assumption and exploiting the replica method. The obtained dynamical equations are compared with Monte Carlo simulations, and it is demonstrated that the proposed formula describes well the time evolution of the autocorrelation function in some parameter regions. The study offers a reasonable description of the autocorrelation function in the SK spin-glass system.

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((Some figures may appear in colour only in the online journal)

1. Introduction

Random systems that possess frustration have many metastable states in their free energy landscape, which are expected to contribute to the system behaviour. Spin-glass systems are representative of such systems and their equilibrium properties have been extensively studied in a mean-field model, the Sherrington–Kirkpatrick (SK) model. Mean-field theory based on the replica method reveals the existence of a transition associated with replica symmetry breaking (RSB) that is interpreted with the aid of Thouless–Anderson–Palmer (TAP) theory as the exponential appearance of metastable states in the free energy landscape [1, 2]. A quantitative understanding of the metastable states is attempted by counting the number of solutions of the TAP equation or belief propagation equation [3–5].

Below a critical temperature, spin glasses in experiments show slow relaxation dynamics and an ageing effect in which the dynamical properties depend on the history even after a
long time period [6]. The dynamical behaviour and equilibrium properties are considered to be related, but the connection is not fully understood. It has been suggested that the dynamical behaviour of the SK model is qualitatively similar to spin glasses in experiments even though the realistic spin glasses have finite dimensions [7]. Therefore, clarifying the relaxation dynamics in the SK model provides insight into the connection between the equilibrium and dynamical properties, as well as into the dynamics of random systems. For this purpose, an analytical description of the relaxation dynamics in the SK model that can be compared with our knowledge of the equilibrium states would be of significant use.

Dynamical replica theory (DRT) provides a statistical-mechanical description of the dynamics in random systems [8]. In DRT, the time evolution of macroscopic quantities is derived according to the replica method. A characteristic feature of DRT is the introduction of an assumption called subshell equipartitioning in which the microscopic states in the subshell that are characterized by a certain macroscopic quantity value appear with the same probability. By introducing this assumption, one can obtain closed dynamical equations that describe the time evolution of the macroscopic quantities. The derivation of the dynamical equations and their DRT solutions are tractable even if the relaxation dynamics over a long time are considered, although an exact description of the dynamics is sacrificed. DRT offers a contrasting perspective compared to the generating functional method [9, 10] in which the dynamical equations are exact but difficult to solve over long time steps.

The physical implications of subshell equipartitioning and its influence on the accuracy of the theoretical prediction have been studied [11, 12]. The assumption removes the microscopic time correlation, and the system only depends on the past value of the macroscopic quantities that characterize the subshell. It is well known that the time correlation is significant in describing the dynamics of random systems. In fact, it has been demonstrated that analytical descriptions of the dynamics are improved by taking into account the time correlation in the Hopfield model [13, 14] and in the interference canceller of code-division-multiple-access multiuser detection [15]. However, DRT in its present form cannot deal with the time correlation, and hence in order to apply DRT to random systems and discuss the time correlation, an extension should be introduced.

A central quantity in the study of the dynamics of spin glasses is the spin autocorrelation function that describes the correlation between microscopic states at two time steps. It is experimentally and numerically observed with the conjugate response function as a quantity that indicates the slow relaxation dynamics [6] and its analytical description has been studied, in particular by Langevin dynamics of soft spins [16, 17]. In this paper, I study the sequential dynamics of the SK model with DRT while including the time evolution of the autocorrelation function. DRT has been studied mainly for two cases where the subshell is characterized by a magnetization and energy set contributed by randomness [8] or a conditional local field distribution [18]. I start with the simplest DRT [8] and introduce the autocorrelation function into the formulation. The evolution equation of the autocorrelation function is derived based on the replica method and its dependence on the phase is discussed.

This paper is organized as follows. In section 2, I explain the model settings and time evolution of the joint probability of microscopic states at two time steps, which is described by Glauber dynamics [19]. In section 3, macroscopic quantities, magnetization, energy contributed by randomness and the autocorrelation function are introduced to characterize the microscopic state space. The closed formula of the evolutional dynamics of the quantities is obtained following the procedures of DRT, and the expressions under the replica symmetry (RS) assumption are derived. In section 4, the obtained dynamical equations are numerically solved and the results are compared with Monte Carlo (MC) simulations. Finally, section 5 is devoted to conclusions and the outlook for further developments.
2. Model setup

The system being studied here is a mean-field spin-glass model, the SK model, which consists of $N$ Ising spins $S = \{S_i\}(i = 1, \ldots, N)$. The Hamiltonian is given by

$$H(S|J) = -\sum_{i<j} J_{ij} S_i S_j - h \sum_i S_i,$$

where $J_{ij}$ is the interaction between spins $S_i$ and $S_j$, and $h$ is the external field. The interactions $J = \{J_{ij}\}$ are symmetric ($J_{ij} = J_{ji}$) and assumed to be independent and identically distributed according to the Gaussian distribution with a mean of $J_0/N$ and a variance of $J^2/N$,

$$P_J(J_{ij}) = \sqrt{\frac{N}{2\pi J^2}} \exp\left[\frac{N}{2J^2} \left( J_{ij} - \frac{J_0}{N} \right)^2 \right]. \quad (i \neq j).$$

Here, I consider the conditional probability of a microscopic spin configuration $\sigma$ at $t_w + t$ under a given spin configuration $s$ at time $t_w$ and interaction $J$ as $p(\sigma; t_w + t|s; t_w, J)$. The time $t_w$ is called the waiting time. The two configurations $\sigma$ and $s$ are related to each other through $\sigma = s \circ d$, where $d = \{d_i\}(i = 1, \ldots, N)$ is a vector consisting of Ising variables, and $\circ$ denotes the Hadamard product, which is a product with respect to each component; the $i$th component of $s \circ d$ is $s_i d_i$. When $d_i = 1$, the $i$th spin configurations at time $t_w$ and those at time $t_w + t$ are the same, otherwise they differ. The time evolution of the microscopic conditional probability in the direction of $t$ is given by Glauber dynamics, which is an asynchronous update of spin configurations described by the master equation,

$$\frac{d}{dt} p(s \circ d; t_w + t|s; t_w, J) = \sum_{k=1}^N \left[ p(s \circ F_k d; t_w + t|s; t_w, J) w_k(s \circ F_k d|J) \right. \right.$$

$$\left. \left. - p(s \circ d; t_w + t|s; t_w, J) w_k(s \circ d|J) \right]\right], \quad (3)$$

where $F_k$ is an operator that flips the configuration of the $k$th Ising variable as $F_k s = [s_1, \ldots, -s_k, \ldots, s_N]$. The transition probability under a given $J$, $w_k(s|J)$, is given by

$$w_k(s|J) = \frac{1}{2} (1 - s_k \tanh(\beta h_k(s|J))), \quad (4)$$

where $h_k(s|J) = \sum_{j \neq k} J_{kj} s_j + h$ is the local field of the $k$th spin. The time evolution of the joint probability $p(\sigma; t_w + t, s; t_w, J)$ is obtained by multiplying both sides of (3) by the probability distribution of a microscopic state at time $t_w$, $p(s; t_w, J)$. The fixed point of (3) obtained at $t \rightarrow \infty$ corresponds to the equilibrium distribution

$$p(\sigma; t_w + t, s; t_w, J) \rightarrow \frac{1}{Z} p(s; t_w, J) \exp(-\beta H(\sigma|J)), \quad (5)$$

where $Z$ is the normalization constant.

3. Time evolution of macroscopic quantities

The macroscopic states are defined by the following macroscopic quantities:

$$m_0(s) = \frac{1}{N} \sum_{i=1}^N s_i, \quad r_0(s) = \frac{1}{N} \sum_{i<j} \left( J_{ij} - \frac{J_0}{N} \right) s_i s_j \quad (6)$$

$$m(\sigma) = \frac{1}{N} \sum_{i=1}^N \sigma_i, \quad r(\sigma) = \frac{1}{N} \sum_{i<j} \left( J_{ij} - \frac{J_0}{N} \right) \sigma_i \sigma_j \quad (7)$$

3
\[ c(d) = \frac{1}{N} \sum_{j=1}^{N} \sigma_j s_j = \frac{1}{N} \sum_{j=1}^{N} d_j, \]  

(8)

where \( m_0 \) and \( r_0 \) are the magnetization and the absolute value of the energy contributed by randomness (referred to as randomness energy, hereafter) at time \( t_w \), respectively, and \( m \) and \( r \) are the corresponding values at time \( t_w + t \). The quantities \( m_0 \) and \( r_0 \) do not depend on time \( t \). The quantity \( c \) is the autocorrelation function between time \( t_w + t \) and \( t_w \), and is a function of \( d \). The probability distribution of the macroscopic quantities is given by

\[
P_{t_w+t, t_w}(\Omega\{J\}) = \sum_{\sigma, s} p(\sigma; t_w + t, s; t_w\{J\}) \delta(m_0 - m(s)) \delta(r_0 - r(s)) \delta(m - m(\sigma))
\times \delta(r - r(\sigma)) \delta(c - c(d)),
\]

(9)

where \( \Omega = \{m_0, r_0, m, r, c\} \). There are three Ising variables \( \sigma, s \) and \( d \), but the relationship \( \sigma = s \circ d \) reduces the number of the independent variables to two. The double summation over the microstates of \( \sigma \) and \( s \) should be handled keeping in mind the relationship \( \sigma = s \circ d \). The time evolution of \( P_{t_w+t, t_w}(\Omega\{J\}) \) in the direction of \( t \) is derived by substituting (3) and (4) into (9) and employing the Taylor expansion with respect to 1/\( N \):

\[
\frac{d}{dt} P_{t_w+t, t_w}(\Omega\{J\})
= -\frac{\partial}{\partial m} P_{t_w+t, t_w}(\Omega\{J\}) \left( \frac{1}{N} \sum_{k=1}^{N} \left( \langle \tanh \beta(J_{0m} + J_{2k}(\sigma\{J\} + h)) \rangle_\Omega - m \right) \right)
- \frac{\partial}{\partial r} P_{t_w+t, t_w}(\Omega\{J\}) \left( \frac{1}{N} \sum_{k=1}^{N} \left( \langle z_k \tanh \beta(J_{0m} + J_{2k}(\sigma\{J\} + h)) \rangle_\Omega - 2r \right) \right)
- \frac{\partial}{\partial c} P_{t_w+t, t_w}(\Omega\{J\}) \left( \frac{1}{N} \sum_{k=1}^{N} \left( \langle z_k \tanh \beta(J_{0m} + J_{2k}(\sigma\{J\} + h)) \rangle_\Omega - c \right) \right),
\]

(10)

where \( z_k(\sigma\{J\}) = \sum_{i \neq k} (J_{ki} - J_{0}/N) \sigma_i \) is the local field contributed by randomness of the \( k \)th spin. A detailed explanation of the derivation of (10) is shown in appendix A. The notation \( \langle \cdots \rangle_\Omega \) represents the average within the subshell specified by the set of macroscopic quantities \( \Omega \) as

\[
\langle \langle \cdots \rangle_\Omega \rangle = \frac{\sum_{\sigma, s} \cdots p(\sigma; t_w + t, s; t_w\{J\}) W(\sigma, s|\Omega)}{\sum_{\sigma, s} p(\sigma; t_w + t, s; t_w\{J\}) W(\sigma, s|\Omega)}.
\]

(11)

where \( W(\sigma, s|\Omega) \equiv \delta(m_0 - m(s)) \delta(r - r(s)) \delta(m - m(\sigma)) \delta(r - r(\sigma)) \delta(c - c(d)) \). By regarding (10) as the total differential form with respect to \( m \), \( r \) and \( c \), their dynamical equations are given by

\[
\frac{d}{dt} m = \int dz \sum_{S = \pm 1} D(z; S, \Omega, J) \tanh(\beta(J_{0m} + J_{2z} + h)) = m
\]

(12)

\[
\frac{d}{dt} r = \int dz \sum_{S = \pm 1} D(z; S, \Omega, J) z \tanh(\beta(J_{0m} + J_{2z} + h)) = 2r
\]

(13)

\[
\frac{d}{dt} c = \int dz \sum_{S = \pm 1} D(z; S, \Omega, J) S \tanh(\beta(J_{0m} + J_{2z} + h)) = c,
\]

(14)
where a spin variable $S$ takes the value $\pm 1$ and $z$ is the effective noise caused by quenched randomness. The effective noise and the spin variable $S$ are distributed according to $D(z, S|\Omega, J)$, which is given by

$$D(z, S|\Omega, J) = \lim_{N \to \infty} \sum_{\sigma, A} \frac{1}{N^2} \sum_{k} \delta(z - z_k(\sigma, J)) \delta_{\mu, \nu} P_{\mu, \nu, A, B} (\Omega|J), \quad \text{eq. (15)}$$

where $\delta_{\mu, \nu}$ is the Kronecker delta.

### 3.1. Closed formula of the dynamical equations

Following the procedures of DRT, a closed formula of the dynamical equations of the macroscopic variables is obtained by assuming the realization of two properties: self-averaging and subshell equipartitioning. The self-averaging property guarantees that the spin-noise joint distribution, $D(z, S|\Omega, J)$, which depends on a realization of $J$, converges to a typical distribution $D(\cdot, \Omega) = [D(\cdot, S|\Omega, J)]_N$ in the limit $N \to \infty$, where $[\cdot]_N$ is the average over the randomness $J$ according to the probability distribution (2). Subshell equipartitioning ensures that the microscopic states within a subshell specified by the macroscopic quantities $\Omega = \{m_0, r_0, m, r, c\}$ appear with the same probability at each time step. As a consequence of these assumptions, the spin-noise distribution $D(z, S|\Omega)$ is replaced with $D(z, S|\Omega)$ as given by

$$D(z, S|\Omega) = \lim_{N \to \infty} \frac{\sum_{\sigma, A} \frac{1}{N^2} \sum_{k} \delta(z - z_k(\sigma, J)) \delta_{\mu, \nu} \mathcal{W}(\sigma, s|\Omega)}{\sum_{\sigma, A} \mathcal{W}(\sigma, s|\Omega)} . \quad \text{eq. (16)}$$

At equilibrium, the probability distribution of the microscopic states is given by the Boltzmann distribution with a Hamiltonian (1) that can be expressed in terms of $m$ and $r$, and hence the subshell equipartitioning property with $m$ and $r$ is realized. However, the property is not expected to hold in cases far from equilibrium, and it is crucial to apply the subshell equipartitioning assumption to non-equilibrium states for theoretical accuracy.

The average over the quenched randomness in (16) is calculated by introducing the replica expression as

$$D(z, S|\Omega) = \lim_{n \to 0} \lim_{N \to \infty} \sum_{\sigma^1, \ldots, \sigma^n} \sum_{|s|} \left[ \frac{1}{N} \sum_{k=1}^N \delta(z - z_k(\sigma^1)) \prod_{a=1}^n \mathcal{W}(\sigma^a, s^a|\Omega) \right] . \quad \text{eq. (17)}$$

The right-hand side of (17) is calculated for a positive integer $n$ and then analytically continued to a non-integer $n$ by taking the limit to 0. After the calculations explained in section A.2, the spin-noise distribution can be expressed as

$$D(z, S|\Omega) = \lim_{n \to 0} \lim_{N \to \infty} \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{1}{2} \left( z - \overline{z} \right)^2 \right) e^{-N \Phi - P} , \quad \text{eq. (18)}$$

where $P = \frac{1}{4} \sum_{a<b} (\rho^a \rho^b + \rho^b \rho^a + \rho_0 \rho_0^a + \rho_0 \rho_0^a), \overline{z} = \rho^a \rho^a + \rho_0 \rho_0^a + c, \Phi = \max_{\delta, \Omega} \left[ -\frac{1}{2} \sum_{a=1}^n (m^a \mu^a + m_0 \mu_0^a + r_0 \rho^a + r_0 \rho_0^a + c_0 v^a) + \frac{1}{4} \sum_{a=1}^n (\rho^a 2 + 2 c^2 \rho_0^a + \rho_0^2) - \frac{1}{2} \sum_{a<b} (\rho^a \rho^b + \rho_0^a \rho_0^b + \rho_0^a \rho_0^b + \rho_0^b \rho_0^b + \rho_0^b \rho_0^b + \rho_0^b \rho_0^b + \rho_0^b \rho_0^b + \rho_0^b \rho_0^b) + \log \sum_{\sigma, A} e^\Sigma \right] . \quad \text{eq. (19)}$$

$$\Sigma = \sum_{a} (\mu^a \sigma^a + \mu_0^a \sigma_0^a + \gamma^a \sigma^a \sigma^a) + \sum_{a<b} (\delta_{ab} \rho^a \rho^b \sigma^a \sigma^b + \delta_{0} \rho_0^a \rho_0^b \sigma_0^a \sigma_0^b + \delta_{1} \rho_0^a \rho_0^b \sigma_0^a \sigma_0^b + \delta_{0} \rho_0^a \rho_0^b \sigma_0^a \sigma_0^b + \delta_{1} \rho_0^a \rho_0^b \sigma_0^a \sigma_0^b + \delta_{0} \rho_0^a \rho_0^b \sigma_0^a \sigma_0^b) . \quad \text{eq. (20)}$$
and \((\mathcal{A})_2 = \sum_{\{q^a\}_{a=1}^n} \sum_{\{\rho^a\}_{a=1}^n} e^{2q^a} \sum_{\{\rho^a\}_{a=1}^n} e^{2\rho^a}\). The variables \(\hat{\Omega} = \{q^a_0, \rho^a_0\}, \{\rho^a\}, \{q^a\}\) and \(\mathcal{Q} = \{q^{ab}_0, \{q^{ab}\}\}(a, b = 1, \ldots, n, a \neq b)\) are the conjugates of \(\Omega = \{m_0, r_0, m, r, c\}\) and the overlap parameters, respectively. The notation extr\(\hat{\Omega}, \mathcal{Q}\)\(\mathcal{F}(\hat{\Omega}, \mathcal{Q})\) denotes the extremization of a function \(\mathcal{F}(\hat{\Omega}, \mathcal{Q})\) with respect to \(\hat{\Omega}\) and \(\mathcal{Q}\). At the extremum of (19), the variables are related to the physical quantities through

\[
m = \frac{1}{n} \sum_a \sum_{\sigma \in \mathbb{S}} \sigma^a e^{2\hat{q}}_a, \quad m_0 = \frac{1}{n} \sum_a \sum_{\sigma \in \mathbb{S}} \sigma^a e^{2\hat{\rho}}_a, \quad c = \frac{1}{n} \sum_a \sum_{\sigma \in \mathbb{S}} \sigma^a e^{2\hat{q}}_a e^{2\hat{\rho}}_a,
\]

(21)

\[
r = \frac{1}{2n} \sum_{ab} (q^{ab}_0 \rho^b_0 + q^{ab}_0 \rho^b_0), \quad r_0 = \frac{1}{2n} \sum_{ab} (q^{ab}_0 \rho^b_0 + q^{ab}_0 \rho^b_0)
\]

(22)

and

\[
q^{ab} = \sum_{\sigma \in \mathbb{S}} \sigma^a \sigma^b e^{2\hat{q}}, \quad q^{ab}_0 = \sum_{\sigma \in \mathbb{S}} \sigma^a \sigma^b e^{2\hat{\rho}}, \quad q^a = \sum_{\sigma \in \mathbb{S}} \sigma^a e^{2\hat{q}}.
\]

(23)

The distribution (18) is coincident with that of the original DRT when the extremization of (19) is handled under the condition \{\mu_0^a = \rho_0^a = \gamma^a\} = 0, where 0 is the \(n\)-dimensional vector whose components are all 0. The condition means that \(m_0, r_0\) and \(c\) do not characterize a subshell.

### 3.2. Replica symmetry

I introduce the RS assumption here for the conjugate and overlap parameters as

\[
\mu^a = \mu, \quad \mu_0^a = \mu_0, \quad \gamma^a = \gamma.
\]

(24)

\[
q^{ab} = q, \quad q^{ab}_0 = q_0, \quad q^{ab}_0 = q_0.
\]

(25)

\[
\rho^a = \rho, \quad \rho^a_0 = \rho_0.
\]

(26)

With these assumptions, the macroscopic quantities given by (21) are transformed to

\[
m = \int Dq^a D\rho^a \tan X + \tanh X_0 \tan Y
\]

(27)

\[
m_0 = \int Dq^a_0 D\rho^a_0 \tan X + \tanh X_0 \tan Y
\]

(28)

\[
c = \int Dq^a D\rho^a \tan Y + \tanh X_0 \tan X \tan Y
\]

(29)

where

\[
X = \mu + \rho \mu \sqrt{q}
\]

(30)

\[
X_0 = \mu_0 + \rho_0 \mu_0 \sqrt{q}_0 + \rho_0 \mu_0 \sqrt{q} - \frac{q^2}{q}
\]

(31)

\[
Y = \gamma - q \rho \mu_0.
\]

(32)

\(J. \text{Phys. A: Math. Theor.} \textbf{46} \text{ (2013) 165001} \) A. Sakata
The conjugates \( \rho \) and \( \rho_0 \) are derived by solving (22) as
\[
\rho = \frac{2\{r_0(c^2 - q_0^2) - r(1 - q_0^2)\}}{(c^2 - q_0^2)^2 - (1 - q_0^2)(1 - q^2)},
\]
\[
\rho_0 = \frac{2\{r(c^2 - q_0^2) - r_0(1 - q_0^2)\}}{(c^2 - q_0^2)^2 - (1 - q_0^2)(1 - q^2)},
\]
and the overlaps are transformed to
\[
q = \int DuDV \left( \frac{\tanh X + \tanh X_0 \tanh Y}{1 + \tanh X_0 \tanh X \tanh Y} \right)^2
\]
\[
q_0 = \int DuDV \left( \frac{\tanh X_0 + \tanh X \tanh Y}{1 + \tanh X_0 \tanh X \tanh Y} \right)^2
\]
\[
q_1 = \int DuDV \frac{\tanh X_0 + \tanh X \tanh Y}{1 + \tanh X_0 \tanh X \tanh Y}.
\]
Derivations of (27)–(29) and (35)–(37) are given in section A.3. The spin-noise distribution under the RS assumption is given by a function of the RS quantities as
\[
D(\mathbb{S}, \Omega) = \int \frac{DuDV}{8\sqrt{2\pi}}
\]
\[
\times \left\{ (1 + S) \frac{(1 + \tanh X_1)(1 + \tanh X_0)(1 + \tanh Y)}{1 + \tanh X_1 \tanh X_0 \tanh Y} e^{-\frac{i}{2}(-\mathbb{S}, \Omega)^2} + (1 - S) \frac{(1 + \tanh X_2)(1 - \tanh X_0)(1 - \tanh Y)}{1 + \tanh X_2 \tanh X_0 \tanh Y} e^{-\frac{i}{2}(\mathbb{S}, \Omega)^2} + (1 + S) \frac{(1 - \tanh X_3)(1 + \tanh X_0)(1 - \tanh Y)}{1 + \tanh X_3 \tanh X_0 \tanh Y} e^{-\frac{i}{2}(\mathbb{S}, \Omega)^2} + (1 - S) \frac{(1 - \tanh X_4)(1 - \tanh X_0)(1 + \tanh Y)}{1 + \tanh X_4 \tanh X_0 \tanh Y} e^{-\frac{i}{2}(\mathbb{S}, \Omega)^2} \right\},
\]
which is a mixture of four Gaussian distributions. The means of the four Gaussian distributions are given by
\[
\begin{align*}
\bar{z}_1 &= \rho(1 - q) + \rho_0(c - q_0), \\
\bar{z}_2 &= \rho(1 - q) - \rho_0(c - q_0), \\
\bar{z}_3 &= -\rho(1 - q) - \rho_0(c - q_0), \\
\bar{z}_4 &= -\rho(1 - q) + \rho_0(c - q_0),
\end{align*}
\]
and the functions \( X_i \) and \( X_0^i \) for \( i = 1, \ldots, 4 \) are given by
\[
X_i = \mu + \rho \sqrt{q} \left\{ \sqrt{1 - qu + \sqrt{q}(z - \bar{z}_i)} \right\},
\]
\[
X_0^i = \mu_0 + \rho_0 \left\{ \frac{q_0}{\sqrt{q}} \left( \sqrt{1 - qu + \sqrt{q}(z - \bar{z}_i)} \right) + \sqrt{q_0 - \frac{q_0^2}{q}} \right\}.
\]

3.3. Fixed point of the macroscopic equations

The dynamical equations reach the fixed point, for which \( dm/dt = 0, dr/dt = 0 \), and \( dc/dt = 0 \), after a long time evolution of \( t \to \infty \). At the fixed point denoted by \( m^*, r^* \) and \( c^* \), the following relationships hold:
\[
m^* = \int \frac{Dz}{m \to 0} \lim_{m \to 0} \frac{\beta(J_0 m + J(z + \bar{c}) + h)}{z}
\]
To do so, the equations of the variables \( \Delta_1 \) correspond to the time evolution of \( m \) and overlap parameters, and noise distributions, shown in appendices B and C, respectively, recursively based on (35)–(37) under fixed conjugates. The time dependence of the conjugates \( \Delta_1 \) evolutional dynamics are treated discretely in the numerical simulation with a unit step of \( t_0 = 1 \). The changes over time, \( \Delta m = m(t + \Delta t) - m(t) \), \( \Delta r = r(t + \Delta t) - r(t) \) and \( \Delta c = c(t + \Delta t) - c(t) \), are obtained by calculating the distribution \( D(z, S|\Omega) \) at each time step. To do so, the equations of the variables \( (\mu, \mu_0, \gamma, q, q_0, q_1, r_0) \), (27)–(29) and (35)–(37), should be solved under the given values of the macroscopic quantities \( \Omega(t) = \{m_0, r_0, m(t), r(t), c(t)\} \). The conjugates \( (\mu, \mu_0, \gamma) \) and overlaps \( (q, q_0, q_1) \) are solved alternately until (27)–(29) and (35)–(37) are satisfied simultaneously; the conjugates are implicitly solved by the Newton method based on (27)–(29) under fixed values of the overlaps, and the overlaps are solved recursively based on (35)–(37) under fixed conjugates. The time dependence of the conjugates and overlap parameters, and noise distributions, shown in appendices B and C, respectively, correspond to the time evolution of \( m, r \) and \( c \) shown in this section.

\[
\begin{align*}
    r^* &= \frac{1}{2} \int D_z \lim_{n \to 0} (z + \bar{z}) \tanh \beta (J_0 m + J (z + \bar{z}) + h) \Omega_1 \\
    c^* &= \int D_z \lim_{n \to 0} (s^1 \tanh (J_0 m + J (z + \bar{z}) + h)) \Omega_1,
\end{align*}
\]

where \( \int D_z = (2\pi)^{-1/2} \int_{-\infty}^{\infty} dz \exp(-z^2/2) \). Equations (41)–(43) are obtained by substituting \( D(z, S|\Omega) \) into (12)–(14). When the relationships \( \mu = \beta (J_0 m^* + h), \gamma = \rho \rho c^*, \rho = \beta J \) and \( c^* = q_i \) are satisfied under the RS assumption, the RS fixed-point equations (41)–(43) are transformed as

\[
m^* = \int Du \tanh \beta (J_0 m^* + h + J \sqrt{q_0} u)
\]

\[
r^* = \frac{\beta J}{2} (1 - q^2)
\]

\[
c^* = \int Du Du \tanh \beta (J_0 m^* + h + J \sqrt{q_0} u) \tanh \left( \mu_0 + \rho_0 \left( \sqrt{\frac{\rho}{q_0}} u + \sqrt{q_0 - \frac{q_0}{q_1}} v \right) \right).
\]

In this case, the overlaps are given by

\[
q_0 = \int Du \tanh^2 (\mu_0 + \rho_0 \sqrt{q_0} u)
\]

\[
q = \int Du \tanh^2 \beta (J_0 m^* + h + J \sqrt{q_0} u),
\]

and the equation to determine the conjugate \( \mu_0 \) is given by

\[
m_0 = \int Du \tanh (\mu_0 + \rho_0 \sqrt{q_0} u).
\]

The conjugates of the energy \( r \) and \( r_0 \) are given by \( \rho = 2r^*/(1 - q^2) \) and \( \rho_0 = 2r_0/(1 - q_0^2) \), respectively, and the former relationship indicates the consistency of \( \rho = \beta J \). Equations (44), (45) and (48) correspond to the magnetization, randomness energy and overlap in the SK model under the RS assumption, respectively, and hence the RS solution of the SK model is recovered as a solution of the fixed-point equations, as with the original DRT.

4. Results

The evolutional dynamics of \( m, r \) and \( c \), given by (21), are solved numerically. The continuous evolutional dynamics are treated discretely in the numerical simulation with a unit step of \( \Delta t = 0.01 \). The changes over time, \( \Delta m = m(t + \Delta t) - m(t), \Delta r = r(t + \Delta t) - r(t) \) and \( \Delta c = c(t + \Delta t) - c(t) \), are obtained by calculating the distribution \( D(z, S|\Omega) \) at each time step. To do so, the equations of the variables \( (\mu, \mu_0, \gamma, q, q_0, q_1) \), (27)–(29) and (35)–(37), should be solved under the given values of the macroscopic quantities \( \Omega(t) = \{m_0, r_0, m(t), r(t), c(t)\} \). The conjugates \( (\mu, \mu_0, \gamma) \) and overlaps \( (q, q_0, q_1) \) are solved alternately until (27)–(29) and (35)–(37) are satisfied simultaneously; the conjugates are implicitly solved by the Newton method based on (27)–(29) under fixed values of the overlaps, and the overlaps are solved recursively based on (35)–(37) under fixed conjugates. The time dependence of the conjugates and overlap parameters, and noise distributions, shown in appendices B and C, respectively, correspond to the time evolution of \( m, r \) and \( c \) shown in this section.
The results of the dynamical equations are compared with an MC simulation for $N = 4096$. The MC simulation settings are as follows. The initial spin configuration is randomly generated according to the distribution

$$P_S(S_i) = \frac{1 + m_0}{2} \delta_{S_i, 1} + \frac{1 - m_0}{2} \delta_{S_i, -1}, \quad (50)$$

and the interaction $J$ is generated according to the distribution (2). The MC dynamics are averaged over 1000 realizations of $J$ and 100 initial configurations of $S$ for each $J$. A set of randomly generated spin configurations and interactions generally provides $r(S) \sim O(N^{-1})$, which is sufficiently small to be regarded as zero at $N = 4096$. Therefore, the average over the initial configuration in the MC simulation corresponds to uniform sampling from the subshell characterized by $m_0$ and $r_0 = 0$, and the overlap $q_0$ at time $t_w = 0$ is given by $m_0^2$. In the MC simulation, a randomly chosen $k$th spin is flipped with a probability $w_k$ given by (4) every $1/N$ step to match the timescale to that of the DRT.

To clarify the effects of involving the dynamical equation of $c$ into the DRT, the dynamics of $m$ and $r$ are also compared with that of the original DRT [8], in which the subshell is characterized by $m$ and $r$ only. The original DRT and that including the autocorrelation function are denoted by $(m, r)$-DRT and $(m, r, c)$-DRT, respectively. The results in three parameter regions corresponding to the paramagnetic, ferromagnetic and spin-glass phases are shown here.

4.1. Time evolution of the magnetization and randomness energy

The time evolution of $m$ and $-r$ are shown in figures 1, 2 and 3 for $J_0/J = 1.0$ and $T/J = 2.0$ (paramagnetic phase), $J_0/J = 1.5$ and $T/J = 1.0$ (ferromagnetic phase), and $J_0/J = 0.0$ and $T/J = 0.5$ (spin-glass phase), respectively. The initial condition at $t_w = 0$ is set to $m_0 = 0.5$ and $r_0 = 0$.

In the paramagnetic phase (figure 1), the time evolution of $m$ and $r$ in the MC simulation is well described by the $(m, r, c)$-DRT proposed here as well as the original $(m, r)$-DRT. The time evolution does not change significantly by including the autocorrelation function into the DRT formula in the paramagnetic phase.

In the ferromagnetic phase (figure 2), the time evolution of $m$ predicted by the $(m, r, c)$-DRT is improved from that of the original $(m, r)$-DRT. A discrepancy between the MC
Figure 2. Time evolution of (a) $m$ and (b) $-r$ at $T/J = 1.0$ and $J_0/J = 1.5$, whose equilibrium state corresponds to a ferromagnetic phase. The lines are the same as those in figure 1.

Figure 3. Time evolution of (a) $m$ and (b) $-r$ at $T/J = 0.5$ and $J_0/J = 0.0$, whose equilibrium state corresponds to a spin-glass phase. The lines are the same as those in figure 1. The short time behaviour of the magnetization is shown in the inset of (a).

simulation and the DRT results appears in early time steps, but it is confirmed that the fixed point of the DRT is coincident with the MC results as $t \to \infty$. The time evolution of $r$ described by the $(m, r, c)$-DRT is almost coincident with that described by the original $(m, r)$-DRT. At sufficiently large $t$, the DRT predicts the correct value of $r$, but the decay to the fixed point at around $2 \lesssim t \lesssim 8$ is faster than for the MC result.

Figure 3 shows the time evolution of $m$ and $r$ in the spin-glass phase. The short time behaviour of $m$ in the $(m, r, c)$-DRT is slightly improved from that in the $(m, r)$-DRT, as shown in the inset of figure 3(a). The fixed point of $m$ in both DRTs is the same, i.e. the RS solution of the SK model. The behaviour of $r$ in both DRTs is similar, but the long time behaviour in the MC simulation is not correctly predicted by either DRT.

To summarize, by taking into account the time evolution of the autocorrelation function, the short time behaviour of the magnetization is improved in the ferromagnetic and spin-glass phases, but there is little effect on the randomness energy in any phase.

4.2. Time evolution of the autocorrelation function

The time evolution of the autocorrelation function in the $(m, r, c)$-DRT at waiting time $t_w$ is compared to the MC simulation. The autocorrelation $c(t + t_w, t_w)$ for $t_w > 0$ is calculated
increases, the relaxation of the autocorrelation appears to be slow. The converged value of $J$. Phys. A: Math. Theor.

the paramagnetic and spin-glass phases irrespective of the value of $t$ as

dynamics (12) and (13) with certain initial conditions $m_0$ and $r_0$. The overlap at time $t_w$, $q(t_w)$, is determined by (35) for given values of $m(t_w)$ and $r(t_w)$. Therefore, the non-equilibrium state at time $t_w$ is assumed to be specified by three macroscopic quantities: $m(t_w)$, $r(t_w)$ and $q(t_w)$.

The time invariance of the autocorrelation function at sufficiently large $t_w$ is confirmed over the whole parameter region as $c(t) = \lim_{t_w \to \infty} c(t_w + t, t_w)$. When one sets $r_0 = 0$, a solution of the fixed-point equations is given by $\rho_0 = \gamma = 0$, as discussed in section 3.3, and in this case the fixed point of the autocorrelation corresponds to $c^* = q_t = m_0 \times m^*$. Therefore, as $t \to \infty$, if this fixed-point solution is stable, then the autocorrelation decreases to zero in the paramagnetic and spin-glass phases irrespective of the value of $m_0$. On the other hand, it converges to a finite value given by $m_0 m^*$ in the ferromagnetic phase.

Figure 4 shows the time evolutions of $c(t_w + t, t_w)$ at $J_0/J = 1.0$ and $T/J = 2.0$, in which the equilibrium state corresponds to the paramagnetic phase. Both of the results for $t_w = 0$ and $t_w = 10$ show good agreement with the results of the MC simulation. The autocorrelation decreases to zero as time increases for any $t_w$. In the paramagnetic phase, the solution $c^* = q_t = 0$ is a stable solution of the fixed-point equations.

In figure 5, the time evolutions of $c(t_w + t, t_w)$ for $t_w = 0$ and $t_w = 10$ are shown for the ferromagnetic phase ($J_0/J = 1.5$ and $T/J = 1.0$). The dynamics of the autocorrelation function are well described by the $(m, r, c)$-DRT for both $t_w$ values. As the waiting time $t_w$ increases, the relaxation of the autocorrelation appears to be slow. The converged value of $c(t_w + t, t_w)$ corresponds to $m(t_w) \times m(t_w + t)$ for both $t_w$ times. In the ferromagnetic phase, $c^* = q_t = m_0 m^*$ is a stable solution of the fixed-point equations, where $m^* \sim 0.56$ in this parameter region.

In figure 6, the time evolutions of the autocorrelation function in the spin-glass phase, $J_0/J = 0$ and $T/J = 0.5$ are shown for $t_w = 0$ and $t_w = 10$. At $t_w = 0$, the short time behaviour of the autocorrelation $c$ is well described by the $(m, r, c)$-DRT, but it deviates from the results of the MC simulation as $t$ increases; $(m, r, c)$-DRT describes a faster relaxation of $c$ than the actual relaxation. The fixed point of the autocorrelation function for $t_w = 0$ in the spin-glass phase is $c^* = 0$, as in the paramagnetic phase. As $t_w$ increases, the discrepancy between the autocorrelation function in the $(m, r, c)$-DRT and that in the MC simulation widens even for short time steps. For $t_w > 0$, the relaxation dynamics of $c$ predicted by $(m, r, c)$-DRT is slower than the actual dynamics, in contrast to the result at $t_w = 0$. The faster relaxation...
at $t_w = 0$ makes the initial condition $m_0 = m(t_w)$ and $r_0 = r(t_w)$ for $t_w > 0$ smaller and larger than the actual values, respectively. This discrepancy in the initial conditions provides a slow relaxation of $c(t_w + t, t_w)$ at $t_w = 10$. By comparing the results in the paramagnetic and spin-glass phases, it is found that the $(m, r, c)$-DRT can describe the slower relaxation of the autocorrelation function in the spin-glass phase better than in the paramagnetic phase.

To characterize the relaxation of the autocorrelation function, it is convenient to introduce the integral relaxation time defined by

$$\tau_{\text{int}}(t_w) = \int_0^t \! \! \! d' C(t_w + t', t_w).$$

The tendency of $\tau_{\text{int}}(t_w)$ as $t \to \infty$ is numerically checked by changing the integration range $t$. Figure 7 shows the temperature dependence of $\tau_{\text{int}}(t_w)$ at $J_0/J = 0$ and $h = 0$ for $t_w = 100$ with integration ranges $t = t_w$, $t = 5t_w$, and $t = 10t_w$. The spin-glass transition temperature in this parameter region is $T_c/J = 1$, as indicated by the vertical line in figure 7. The integral relaxation time in the paramagnetic phase ($T/J > 1$) does not depend on the integration range $t$, but it increases in the spin-glass phase ($T/J < 1$) as $t$ increases. This means that $\tau_{\text{int}}$ diverges as $t \to \infty$ in the spin-glass phase. Around the spin-glass transition temperature, $\tau_{\text{int}}$ exhibits a power law behaviour with respect to $T/J - T_c/J$ as shown in the inset of figure 7. In the DRT formulation, $\tau_{\text{int}}$ goes to infinity at $T_c$ for $\tau_{\text{int}}(t_w) = 1.4(T/J - T_c/J)^{-0.6}$ at sufficiently
Figure 7. The temperature dependence of the integral relaxation time $\tau_{\text{int}}$ at $J_0/J = 0$ and $h = 0$ for $t_w = 100$. The vertical line at $T_c/J = 1.0$ indicates the spin-glass transition temperature. The inset shows the dependence of $\tau_{\text{int}}$ on $(T − T_c)/J$ with a fitting function denoted by the dashed line.

large $t_w$ and $t$, which indicates that the $(m, r, c)$-DRT can detect the critical slowing down. However, the exponent of 0.6 is smaller than the expected value of 2.0 [20]. The proposed $(m, r, c)$-DRT qualitatively describes slow relaxation in the spin-glass phase and the critical behaviour around the spin-glass transition temperature, but further development is necessary for a more accurate description.

5. Summary and discussion

In this paper, I have examined the relaxation dynamics in the SK model using DRT and including the autocorrelation function. The joint probability of the microscopic states at times $t_w$ and $t_w + t$ was introduced, whose time evolution is described by Glauber dynamics. The microscopic states at both times are characterized by macroscopic quantities: the magnetization and energy contributed by randomness at time $t_w + t$ denoted by $m$ and $r$ and those at time $t_w$ denoted by $m_0$ and $r_0$, as well as the autocorrelation function $c$. Following the DRT procedures, closed dynamical equations were derived based on the self-averaging and subshell equipartitioning assumptions. The dynamical equations are governed by three overlap parameters $q$, $q_0$, $q_t$ and the conjugates $\mu$, $\mu_0$, and $\gamma$ under the RS assumption.

The time evolution of $m$ in the $(m, r, c)$-DRT is improved in the ferromagnetic and spin-glass phases compared with $(m, r)$-DRT, but that of $r$ does not change significantly in any phase. In the paramagnetic and ferromagnetic phases, the time evolution of the autocorrelation function $c(t_w + t, t_w)$ was well described by the proposed framework even at finite $t_w$. In the spin-glass phase, the short time behaviour of $c(t_w + t, t_w)$ at $t_w = 0$ was well described by the $(m, r, c)$-DRT, but the theoretical prediction deviated from the MC simulation as the waiting time $t_w$ and the difference between the two times $t$ increased. The temperature dependence of $c(t_w + t, t_w)$ was characterized by the integral relaxation time $\tau_{\text{int}}$, and it was found that slow relaxation in the spin-glass phase and the critical slowing down are qualitatively described by the current $(m, r, c)$-DRT formulation.

To improve the results in the spin-glass phase, the validity of the assumptions, RS and subshell equipartitioning with some macroscopic quantities should be discussed. A promising method for improvement is to introduce a higher order macroscopic quantity to characterize the subshell as in [18] and RSB [1, 2]. It is expected that the fixed point of the macroscopic
equations in DRT with a noise distribution calculated under the \( k \)-step RSB assumption corresponds to the equilibrium state under the \( k \)-step RSB assumption. However, there is no guarantee that the relaxation dynamics described by DRT are coincident with the actual relaxation dynamics even when the full-step RSB is assumed. An analysis of a system that includes a one-step RSB phase will assist in understanding how the RSB effect modifies the relaxation dynamics in DRT.

Another straightforward development is to take into account multitime correlations. One can obtain the dynamical equations of the multitime correlation function by starting with a joint probability distribution of the microscopic states at several time steps. In this case, the analytical method that is employed to derive the dynamical equations corresponds to the replica method for a multi-replicated system consisting of spin systems at each time step, which is called real-replica analysis and is employed to understand the RSB picture in the equilibrium state [21].

DRT has been developed here to include the dynamical equation of the autocorrelation function defined by the microscopic states at two specific times. From this study, the ability and prospects of DRT for describing the dynamics of random systems have been shown. A modification of the DRT proposed in this paper will provide an analytical description of the experimentally and numerically employed protocols for glassy systems. For example, it is expected that the fluctuation–dissipation relation can be checked in the present formulation by comparing the time evolutions of the susceptibility and the autocorrelation function [22]. The current theory is applicable to a \( p \)-body interaction system; in this case, the relationship between bifurcation in dynamical equations and static phase transition associated with RSB should be discussed [23].

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**Appendix A. Derivation of equations**

**A.1. Derivation of (10)**

The time evolution of the probability distribution with respect to macroscopic quantities \( \Omega \) is given by

\[
\frac{d}{dt} P_{t_0+t_1} (\Omega | J) = \sum_{\sigma,s} \frac{d}{dt} p(\sigma; t_0 + t, s; t_1 | J) W(\sigma, s | \Omega) \\
= \sum_{\sigma,s} \left\{ \sum_{k=1}^N (p(\sigma \circ F_k d; t_0 + t, s; t_1) w_k (s \circ F_k d) W(\sigma, s | \Omega) + p(\sigma \circ d; t_0 + t, s; t_1) w_k (s \circ d)) W(\sigma, s | \Omega) \right\}. 
\] 

(A.1)

The first term is invariant against the replacement \( F_k d \rightarrow d \) as

\[
\sum_{\sigma,s} p(\sigma \circ F_k d; t_0 + t, s; t_1) w_k (s \circ F_k d) W(s \circ d, s | \Omega) \\
\rightarrow \sum_{\sigma,s} p(\sigma \circ d; t_0 + t, s; t_1) w_k (s \circ d) W(s \circ F_k d, s | \Omega).
\] 

(A.2)
The probability mass function $\mathcal{W}(s \circ F_d, s|\Omega)$ is expanded as follows:

$$\mathcal{W}(s \circ F_d, s|\Omega) = \mathcal{W}(\sigma, s|\Omega) + \frac{2dz}{N} \left[ s \frac{\partial}{\partial m} + \frac{sk}{\sqrt{N}} \sum_{j=1}^{N} \left( J_{ij} - J_0 \right) s_j \right] \mathcal{W}(\sigma, s|\Omega) + O(N^{-2}).$$

(A.3)

By combining equations (A.1)–(A.3), we obtain (10).

A.2. Derivation of (18)

I introduce the integral expressions of the six delta functions in (17) (five delta functions are contained in $\mathcal{W}(\sigma, s|\Omega)$). After averaging over the quenched randomness and applying the saddle point method, the spin-noise distribution is given by

$$D(z, S|\Omega) = \lim_{n \to 0} \lim_{N \to \infty} \frac{1}{\sqrt{2\pi}} \int Dx e^{\Phi - \frac{1}{2} \sum_{\sigma, s} \delta_{\sigma, s} \exp[ix(z - \Xi)]},$$

(A.4)

where

$$\Phi' = \max_{\Xi \in \Omega} \left[ -\sum_{a=1}^{n} \left( m\mu^a + m_{0}\mu_{0}^a + r\rho_{0}^a + r_{0}\rho_{0}^a + c\gamma^a \right) + \frac{1}{4} \sum_{a} \left( \rho^2 + 2c^2 \rho^{2} \rho_{0}^2 + \rho_{0}^2 \right) \right] + \frac{1}{2} \sum_{a<b} \left( \rho_{0}^a \rho_{0}^b + \rho_{0}^a q_{0}^b \rho_{0}^b + \rho_{0}^a q_{0}^b \rho_{0}^b + \rho_{0}^a q_{0}^b \rho_{0}^b \right) + \log \sum_{s} \exp[\Xi] \right],$$

(A.5)

$$\Xi' = \sum_{a} \left( \mu \sigma^a + \mu_{0}\sigma_{0}^a + \gamma \sigma_{0}^a \sigma_{0}^a \right) + \sum_{a<b} \left( \rho_{0}^a \rho_{0}^b \rho_{0}^b + \rho_{0}^a q_{0}^b \rho_{0}^b + \rho_{0}^a q_{0}^b \rho_{0}^b + \rho_{0}^a q_{0}^b \rho_{0}^b \right) + \log \sum_{s} \exp[\Xi] \right],$$

(A.6)

The overlaps $Q = \{q_{ab}, q_{0}^a, \{q_{0}^a\} \}$ are introduced for the summation over the spin variables. The variables $\tilde{Q} = \{\tilde{q}_{ab}, \{\tilde{q}_{0}^a\} \}$ are the conjugates of $\{q_{ab}\}$ and $\{q_{0}^a\}$, respectively. At the extremum of $\Phi'$, the conjugates of the overlaps are given by

$$\tilde{q}_{ab} = \rho^{a} \rho^{b} \tilde{q}_{ab}, \tilde{q}_{0}^a = \rho_{0}^a \rho_{0}^b \tilde{q}_{0}^a, \tilde{q}_{0}^a = \rho_{0}^a \rho_{0}^b \tilde{q}_{0}^a.$$

(A.7)

respectively. By using the relationship (A.7) and integrating over $x$ in (A.4), the spin-noise distribution (18) is derived.

A.3. Derivation of macroscopic variables under RS assumption

To obtain expressions (27)–(29) and (35)–(37), the function $f \equiv \sum_{\sigma, s} \varepsilon^s$ should be calculated. Under the RS assumption, it is given by

$$f = \sum_{\sigma, s} \exp \left[ -\frac{\rho_{0}^2 q_{ab}}{2} + \sum_{a} \left( \mu \sigma^a + \mu_{0}\sigma_{0}^a + (\gamma - q_{b} \rho_{0}^a) \sigma_{0}^a \sigma_{0}^a \right) \right]$$

$$+ \frac{\rho_{0}^2 q_{ab}}{2} \left( \sum_{a} \sigma^a \right)^2 + \frac{\rho_{0}^2 q_{ab}}{2} \left( \sum_{a} \sigma^a \right)^2 + q_{b} \rho_{0} \sum_{a} \sigma_{0}^a \sigma_{0}^a$$

$$= \int D\nu \int D\sigma \sum_{\sigma, s} \exp \left[ -\frac{\rho_{0}^2 q_{ab}}{2} + (\mu + \rho \sqrt{q_{ab}}) \sigma_{0}^a \right]$$

15

J. Phys. A: Math. Theor. 46 (2013) 165001
\[ + \left\{ \mu_0 + \rho_0 \left( \sqrt{\frac{q_0^2}{q}} \mu + \sqrt{\frac{q_0 - q_0}{q}} \right) \right\} s' + \{ y - q(t) \rho(t) \sigma} \right\} = \int D u D v [4 \cosh X_0 \cosh X \cosh Y + \sinh X_0 \sinh X \sinh Y]^n. \]  

(A.8)

where \( q_0 \geq q^2 \) is satisfied. The macroscopic quantities are derived as

\[ m = \lim_{n \to 0} \frac{1}{n f} \frac{\partial f}{\partial \mu}, \quad m_0 = \lim_{n \to 0} \frac{1}{n f} \frac{\partial f}{\partial \mu_0}, \quad c = \lim_{n \to 0} \frac{1}{n f} \frac{\partial f}{\partial \gamma} \]  

(A.9)

\[ q = \lim_{n \to 0} \frac{1}{n(n-1)} \left( \frac{1}{n} \frac{\partial^2 f}{\partial \mu^2} - n \right) \]  

(A.10)

\[ q_0 = \lim_{n \to 0} \frac{1}{n(n-1)} \left( \frac{1}{n} \frac{\partial^2 f}{\partial \mu_0^2} - n \right) \]  

(A.11)

\[ q_r = \lim_{n \to 0} \frac{1}{n(n-1)} \left( \frac{1}{n} \frac{\partial^2 f}{\partial \mu_0 \partial \mu} - nc \right). \]  

(A.12)

Appendix B. Overlap and conjugate parameters

To obtain the time evolution of the macroscopic variables, the overlaps \( q_0, q, q_r \) and conjugates \( \mu_0, \mu, \gamma \) should be solved at each time step. Figure B1 and B2 show the time dependence of the conjugates and overlap parameters, respectively, at \( t_w = 0 \). The initial condition is given by \( m_0 = 0.5 \) and \( r_0 = 0 \), and the corresponding time evolutions of the macroscopic quantities

![Figure B1](image1)

Figure B1. Time dependence of the conjugate parameters at (a) \( T/J = 2.0 \) and \( J_0/J = 1.0 \) (paramagnetic phase), (b) \( T/J = 1.0 \) and \( J_0/J = 1.5 \) (ferromagnetic phase), and (c) \( T/J = 0.5 \) and \( J_0/J = 0.0 \) (spin-glass phase) for \( t_w = 0 \).

![Figure B2](image2)

Figure B2. Time dependence of the overlap parameters at (a) \( T/J = 2.0 \) and \( J_0/J = 1.0 \) (paramagnetic phase), (b) \( T/J = 1.0 \) and \( J_0/J = 1.5 \) (ferromagnetic phase), and (c) \( T/J = 0.5 \) and \( J_0/J = 0.0 \) (spin-glass phase) for \( t_w = 0 \).
are shown in section 4. As shown in figure B1, the conjugate $\gamma$ decreases to 0 with increasing time. As $t \to \infty$, the conjugates $\mu_0$ and $\mu$ converge to the values given by $\tanh^{-1} m_0$ and $\beta (J_0 m^* + h)$, respectively. The causality that $q_0$ takes a constant value (in this case given by $m_0$) is achieved at all time steps by appropriately controlling the conjugates as shown in figure B2. The overlap between times $t_w$ and $t_w + t$, $q_t$ decreases to zero in the paramagnetic and spin-glass phases and converges to $m_0 \times m(t)$ in the ferromagnetic phase.

Appendix C. Noise distribution

The time evolutions of the noise distributions $D(z|\Omega(t_w, t)) = \sum_{S=\pm 1} D(z, S\Omega(t_w, t))$ for $t_w = 0$ are shown in figure C1 for (a) $T/J = 2$ and $J_0/J = 1$ (paramagnetic phase), (b) $T/J = 1.0$ and $J_0/J = 1.5$ (ferromagnetic phase), and (c) $T/J = 0.5$ and $J_0/J = 0$ (spin-glass phase), where $\Omega(t_w, t)$ denotes the set of macroscopic quantities $[m(t_w + t), r(t_w + t), m_0(t_w), r_0(t_w), c(t_w + t, t_w)]$. The initial conditions are given by $m_0 = 0.5$ and $r_0 = 0$, and the noise distribution at $t_w = t = 0$ is given by the Gaussian distribution with mean 0 and variance 1. For comparison, the noise distributions calculated by the MC simulation for $N = 4096$ are also shown in figure C1. In the paramagnetic and ferromagnetic phases, the time evolution of the noise distribution is well described by $(m, r, c)$-DRT, but in spin-glass phase, the noise distribution in $(m, r, c)$-DRT shows a faster convergence to a symmetric and double-peak distribution than the MC simulation. The result of the noise

![Figure C1](image-url). Noise distributions at $t_w = 0$ and $t = 0, 5, 10$ for (a) $T/J = 2.0$ and $J_0/J = 1.0$ (paramagnetic phase), (b) $T/J = 1.0$ and $J_0/J = 1.5$ (ferromagnetic phase), and (c) $T/J = 0.5$ and $J_0/J = 0.0$ (spin-glass phase). The results of the MC simulation are denoted by dashed lines.
distribution in a spin-glass phase is consistent with the faster relaxation of the macroscopic variables shown in section 4.

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