The enumeration of coverings of closed orientable Euclidean manifolds $G_3$ and $G_5$.

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Abstract
There are only 10 Euclidean forms, that is flat closed three dimensional manifolds: six are orientable and four are non-orientable. The aim of this paper is to describe all types of $n$-fold coverings over orientable Euclidean manifolds $G_3$ and $G_5$, and calculate the numbers of non-equivalent coverings of each type. We classify subgroups in the fundamental groups $\pi_1(G_3)$ and $\pi_1(G_5)$ up to isomorphism and calculate the numbers of conjugated classes of each type of subgroups for index $n$. The manifolds $G_3$ and $G_5$ are uniquely determined among the others orientable forms by their homology groups $H_1(G_3) = \mathbb{Z}_3 \times \mathbb{Z}$ and $H_1(G_5) = \mathbb{Z}$.

Key words: Euclidean form, platycosm, flat 3-manifold, non-equivalent coverings, crystallographic group.

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Introduction
Let $\mathcal{M}$ be a manifold with fundamental group $\Gamma = \pi_1(\mathcal{M})$. Two coverings

$$p_1 : \mathcal{M}_1 \rightarrow \mathcal{M} \text{ and } p_2 : \mathcal{M}_2 \rightarrow \mathcal{M}$$

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are said to be equivalent if there exists a homeomorphism \( h : \mathcal{M}_1 \to \mathcal{M}_2 \) such that \( p_1 = p_2 \circ h \). According to the general theory of covering spaces, any \( n \)-fold covering is uniquely determined by a subgroup of index \( n \) in the group \( \Gamma \). The equivalence classes of \( n \)-fold covering of \( \mathcal{M} \) are in one-to-one correspondence with the conjugacy classes of subgroups of index \( n \) in the fundamental group \( \pi_1(\mathcal{M}) \). See, for example, ([7], p. 67).

In such a way the following two natural problems arise. The first one is to calculate the number of subgroups of given finite index \( n \) in \( \pi_1(\mathcal{M}) \). The second problem is to find the number of conjugacy classes of subgroups of index \( n \) in \( \pi_1(\mathcal{M}) \).

The problem of enumeration for nonequivalent coverings over a Riemann surface with given branch type goes back to the paper [8] by Hurwitz, in which the number of coverings over the Riemann sphere with given number of simple (of order two) branching points was determined. Later, in [9], it has been proved that this number can be expressed in the terms of irreducible characters of symmetric groups. The Hurwitz problem was studied by many authors. A detailed survey of the related results is contained in ([13], [10]). For closed Riemann surfaces, this problem was completely solved in [16]. However, of most interest is the case of unramified coverings. Let \( s_\Gamma(n) \) denote the number of subgroups of index \( n \) in the group \( \Gamma \), and let \( c_\Gamma(n) \) be the number of conjugacy classes of such subgroups. According to what was said above, \( c_\Gamma(n) \) coincides with the number of nonequivalent \( n \)-fold coverings over a manifold \( \mathcal{M} \) with fundamental group \( \Gamma \). The numbers \( s_\Gamma(n) \) and \( c_\Gamma(n) \) for the fundamental group of a closed surface (orientable or not) were found in ([14], [15], [17]). In the paper [18], a general method for calculating the number \( c_\Gamma(n) \) of conjugacy classes of subgroups in an arbitrary finitely generated group \( \Gamma \) was given. Asymptotic formulas for \( s_\Gamma(n) \) in many important cases were obtained by T. W. Müller and his collaborators ([19], [20], [21]).

In the three-dimensional case, for a large class of Seifert fibrations, the value of \( s_\Gamma(n) \) was calculated in [11] and [12]. In our previous papers [1] and [2] the numbers \( s_\Gamma(n) \) and \( c_\Gamma(n) \) were determined for the fundamental group of non-orientable Euclidian manifolds \( \mathcal{B}_1 \) and \( \mathcal{B}_2 \) whose homologies are \( H_1(\mathcal{B}_1) = \mathbb{Z}_2 \oplus \mathbb{Z}^2 \) and \( H_1(\mathcal{B}_2) = \mathbb{Z}^2 \) and for the orientable Euclidian manifolds \( \mathcal{G}_2 \) and \( \mathcal{G}_1 \) with \( H_1(\mathcal{G}_2) = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z} \) and \( H_1(\mathcal{G}_1) = \mathbb{Z}_2 \oplus \mathbb{Z} \) respectively.

The aim of the present paper is to investigate \( n \)-fold coverings over orientable Euclidean three dimensional manifolds \( \mathcal{G}_3 \) and \( \mathcal{G}_5 \), with the first homologies \( H_1(\mathcal{G}_3) = \mathbb{Z}_3 \oplus \mathbb{Z} \) and \( H_1(\mathcal{G}_5) = \mathbb{Z} \). We classify subgroups of finite index in the fundamental groups of \( \pi_1(\mathcal{G}_3) \) and \( \pi_1(\mathcal{G}_5) \) up to isomorphism. Then we calculate the number of subgroups and the number of conjugacy classes of subgroups of each isomorphism type for a given index \( n \).

We note that numerical methods to solve these and similar problems for the three-dimensional crystallogical groups were developed by the Bilbao group [4]. The first homologies of all the three-dimensional crystallogical groups are determined in [22].

**Notations**

We use the following notations: \( s_{H,G}(n) \) is the number of subgroups of index \( n \) in the group \( G \), isomorphic to the group \( H \); \( c_{H,G}(n) \) is the number conjugacy classes of sub-
groups of index \( n \) in the group \( G \), isomorphic to the group \( H \).

Also we will need the following combinatorial functions:

\[
\sigma_0(n) = \sum_{k \mid n} 1, \quad \sigma_1(n) = \sum_{k \mid n} k, \quad \sigma_2(n) = \sum_{k \mid n} \sigma_1(k), \quad \omega(n) = \sum_{k \mid n} k \sigma_1(k),
\]

\[
\theta(n) = |\{(p, q) \mid p > 0, q \geq 0, p^2 - pq + q^2 = n\}|.
\]

In all cases we consider the function vanished if \( n \not\in \mathbb{N} \).

**Remark.** It can be shown that \( \theta(n) = \sum_{k \mid n} \left( \frac{k}{3} \right) \), where \( \left( \frac{k}{3} \right) \) is the Legendre symbol, see [6] p.112. This representation clarifies the analogy between the functions \( \sigma_1(n) \) and \( \theta(n) \), and makes the appearance of the latter one less amazing. However, this representation will not be used further.

## 1 Overview

The main goal of this paper is to prove the following results.

The first theorem provides the complete solution of the problem of enumeration of subgroups of a given finite index in \( \pi_1(\mathcal{G}_3) \).

**Theorem 1.** Every subgroup \( \Delta \) of finite index \( n \) in \( \pi_1(\mathcal{G}_3) \) is isomorphic to either \( \pi_1(\mathcal{G}_3) \) or \( \pi_1(\mathcal{G}_1) \cong \mathbb{Z}^3 \). The respective numbers of subgroups are

\[
(i) \quad s_{\pi_1(\mathcal{G}_3), \pi_1(\mathcal{G}_3)}(n) = \sum_{k \mid n} k \theta(k) - \sum_{k \mid \frac{n}{3}} k \theta(k),
\]

\[
(ii) \quad s_{\mathbb{Z}^3, \pi_1(\mathcal{G}_3)}(n) = \omega(n^3). 
\]

The next theorem provides the number of conjugacy classes of subgroups of index \( n \) in \( \pi_1(\mathcal{G}_3) \) for each isomorphism type. That is the number of non-equivalent \( n \)-fold covering \( \mathcal{G}_3 \), which have a prescribe fundamental group.

**Theorem 2.** Let \( \mathcal{N} \to \mathcal{G}_3 \) be an \( n \)-fold covering over \( \mathcal{G}_3 \). If \( n \) is not divisible by 3 then \( \mathcal{N} \) is homeomorphic to \( \mathcal{G}_3 \). If \( n \) is divisible by 3 then \( \mathcal{N} \) is homeomorphic to either \( \mathcal{G}_3 \) or \( \mathcal{G}_1 \). The corresponding numbers of nonequivalent coverings are given by the following formulas:

\[
(i) \quad c_{\pi_1(\mathcal{G}_3), \pi_1(\mathcal{G}_3)}(n) = \sum_{k \mid n} \theta(k) + \sum_{k \mid \frac{n}{3}} \theta(k) - 2 \sum_{k \mid \frac{n}{9}} \theta(k)
\]

\[
(ii) \quad c_{\mathbb{Z}^3, \pi_1(\mathcal{G}_3)}(n) = \frac{1}{3} \left( \omega(n^3) + 2 \sum_{k \mid \frac{n}{3}} \theta(k) + 4 \sum_{k \mid \frac{n}{9}} \theta(k) \right).
\]
The next two theorems are analogues of Theorem 1 and Theorem 2 respectively for the manifold $G_5$.

**Theorem 3.** Every subgroup $\Delta$ of finite index $n$ in $\pi_1(G_5)$ is isomorphic to either $\pi_1(G_3)$ or $\pi_1(G_2)$ or $\pi_1(G_1) \cong \mathbb{Z}^3$. The respective numbers of subgroups are

(i) \[ s_{\pi_1(G_5), \pi_1(G_5)}(n) = \sum_{k|n, (n, 6)=1} k\theta(k) \]

(ii) \[ s_{\pi_1(G_3), \pi_1(G_3)}(n) = \sum_{k|\frac{n}{2}} k\theta(k) - \sum_{k|\frac{n}{6}} k\theta(k) \]

(iii) \[ s_{\pi_1(G_2), \pi_1(G_3)}(n) = \omega\left(\frac{n}{3}\right) - \omega\left(\frac{n}{6}\right) \]

(iv) \[ s_{\pi_1(G_3), \pi_1(G_3)}(n) = \omega\left(\frac{n}{6}\right). \]

**Theorem 4.** The numbers of $n$-fold covering over $G_5$ is given by the following formulas:

(i) \[ c_{\pi_1(G_5), \pi_1(G_5)}(n) = \sum_{k|n, (n, 6)=1} \theta(k) \]

(ii) \[ c_{\pi_1(G_3), \pi_1(G_3)}(n) = \sum_{k|\frac{n}{2}} \theta(k) - \sum_{k|\frac{n}{6}} \theta(k) \]

(iii) \[ c_{\pi_1(G_2), \pi_1(G_3)}(n) = \frac{1}{3} \left( \sigma_2\left(\frac{n}{3}\right) + 2\sigma_2\left(\frac{n}{6}\right) - 3\sigma_2\left(\frac{n}{12}\right) + 2 \sum_{k|\frac{n}{4}} \theta(k) - 2 \sum_{k|\frac{n}{6}} \theta(k) \right) \]

(iv) \[ c_{\mathbb{Z}^3, \pi_1(G_3)}(n) = \frac{1}{6} \left( \omega\left(\frac{n}{6}\right) + \sigma_2\left(\frac{n}{6}\right) + 3\sigma_2\left(\frac{n}{12}\right) + 4 \sum_{k|\frac{n}{4}} \theta(k) + 4 \sum_{k|\frac{n}{6}} \theta(k) \right) \]

In the Appendix we present the Dirichlet generating functions for the above sequences.

## 2 Preliminaries

Further we use the representations for the fundamental groups $\pi(G_3)$ and $\pi(G_5)$ given in $[25]$ and $[3]$.

\[ \pi_1(G_3) = \langle x, y, z : xyx^{-1}y^{-1} = 1, zxz^{-1} = y, zyz^{-1} = (xy)^{-1} \rangle. \]  
(2.1)

\[ \pi_1(G_5) = \langle \bar{x}, \bar{y}, \bar{z} : \bar{x}\bar{y}\bar{x}^{-1}\bar{y}^{-1} = 1, \bar{z}\bar{x}\bar{z}^{-1} = \bar{x}\bar{y}, \bar{z}\bar{y}\bar{z}^{-1} = \bar{x}^{-1} \rangle. \]  
(2.2)

We will widely use the following statements.
Proposition 1. Let $\Delta$ be a subgroup of finite index $n$ in $\mathbb{Z}^2$. Then $\Delta$ have a pair of generators of the form $(a,0)$ and $(\mu,b)$ where $a$ and $b$ are positive integers with $ab = n$ and $\mu$ is a nonnegative integer with $0 \leq \mu < a$. Furthermore, the set of subgroups $\Delta$ with $|\mathbb{Z}^2 : \Delta| = n$ bijectively corresponds to the set of pairs of generators of described form. The number of such subgroups $\Delta$ is $\sigma_1(n)$.

Let $\Delta$ be a subgroup of finite index $n$ in $\mathbb{Z}^3$. Then $\Delta$ have a set of three generators $(a,0,0)$, $(\mu,b,0)$ and $(\nu,\lambda,c)$ where $a$, $b$, $c$ are positive integers with $abc = n$, $\mu$, $\nu$ are integers with $0 \leq \mu, \nu < a$ and $\lambda$ is an integer with $0 \leq \lambda < b$. Furthermore, the set of subgroups $\Delta$ with $|\mathbb{Z}^3 : \Delta| = n$ bijectively corresponds to the set of triplets of generators of described form. The number of such subgroups $\Delta$ is $\omega(n)$.

Corollary 1. Given an integer $n$, by $S(n)$ denote the number of pairs $(H,\nu)$, where $H$ is a subgroup of index $n$ in $\mathbb{Z}^2$ and $\nu$ is a coset of $\mathbb{Z}^2/H$ with $2\nu = 0$ (we use the additive notation). Then

$$S(n) = \sigma_1(n) + 3\sigma_1\left(\frac{n}{2}\right).$$

Lemma 1. Let $H \leq G$ be an abelian group and its subgroup of finite index. Let $\phi : G \to G$ be an endomorphism of $G$, such that $\phi(H) \leq H$ and the index $|G : \phi(G)|$ is also finite. Then the cardinality of kernel of $\phi : G/H \to G/H$ equals to the index $|G : (H + \phi(G))|$. For the proofs, see Lemma 1 in [2].

Remark 1. Combining Lemma 1 and Corollary 1 we get the following observation. Given a subgroup $H \leq \mathbb{Z}^2$, the number of $\nu \in \mathbb{Z}^2/H$, such that $2\nu = 0$, is equal to $|\mathbb{Z}^2/((2,0),(0,2),H)|$. Indeed, taking $\mathbb{Z}^2$ as $G$, $H$ as $H$ and $\phi : g \to 2g$, $g \in \mathbb{Z}^2$ as $\phi$ one gets the desired equality. Since for each $H$ the numbers $|\{(\nu| \nu \in \mathbb{Z}^2/H, 2\nu = 0)\}|$ and $|\mathbb{Z}^2/((2,0),(0,2),H)|$ coincide, their sums taken over all subgroups $H$ also coincide, that is

$$S(n) = \sum_{H \leq \mathbb{Z}^2, |\mathbb{Z}^2/H| = n} |\{(\nu| \nu \in \mathbb{Z}^2/H, 2\nu = 0)\}| = \sum_{H \leq \mathbb{Z}^2, |\mathbb{Z}^2/H| = n} |\mathbb{Z}^2/((2,0),(0,2),H)|.$$ 

Definition 1. Consider the group $\mathbb{Z}^2$. By $\ell$ denote the automorphism $\ell : \mathbb{Z}^2 \to \mathbb{Z}^2$ given by $(x, y) \to (y, -x - y)$.

Lemma 2. A subgroup $H \leq \mathbb{Z}^2$ is preserved by $\ell$ if and only if $H$ is generated by a pair of elements of the form $(p,q), (-q,p-q)$. In this case $|\mathbb{Z}^2/H| = p^2 - pq + q^2$. For a given integer $n$ the number of invariant under $\ell$ subgroups $H$ of index $n$ in $\mathbb{Z}^2$ is $\theta(n)$.

Proof. Suppose $H$ is generated by elements $(p,q)$ and $(-q,p-q) = \ell((p,q))$. Then obviously $\ell(H) = H$. Also $|\mathbb{Z}^2/H| = p^2 - pq + q^2$, since $p^2 - pq + q^2$ is the number of integer points in a fundamental domain of $H$ on the plane.

Vice versa, suppose $\ell(H) = H$. Denote $d(x,y) = x^2 - xy + y^2$. Let $u = (p,q) \in H \setminus \{0\}$ be an element with the minimal value of $d(u)$. Consider the subgroup $H_1 = \langle u, \ell(u) \rangle \leq H$. Assume $H_1 \neq H$ and $v \in H \setminus H_1$. Since $H_1 = \langle u, \ell(u) \rangle$, the fundamental domain of $H_1$ is a parallelogram with vertices $0, u, \ell(u), u + \ell(u)$. That means that the plane splits
into the parallelograms of the form \( w, w + u, w + \ell(u), w + u + \ell(u), w \in H_1 \), each of them splits into two right triangles. One of this triangles contains \( v \). Note that the distance from a point inside a right triangle to one of its vertices is not greater then the side of this triangle. This contradicts the minimality of \( d(u) \), thus \( H_1 = H \).

To find the number of subgroups \( H \) note that the number of pairs \((p, q)\) with \( p^2 - pq + q^2 = n \), \( p > 0 \), \( q \geq 0 \) is \( \theta(n) \). As it was proven above, for each pair \((p, q)\) of the above type two pairs \((p, q)\) and \( \ell((p, q)) \) generate a subgroup \( H \) of the required type. Moreover \( d(p, q) \) takes the minimal value among \( d(v), v \in H \setminus \{0\} \). Suppose two different pairs \((p, q)\) and \((p', q')\) correspond the same subgroup \( H \). Then \((p - p', q - q') \in H \) and \( 0 < d(p - p', q - q') < d(p, q) \), which contradiction proves that there is a one-to-one correspondence between pairs \((p, q)\) and subgroups \( H \). \( \square \)

Before formulating the next corollary note that \( \ell(\nu), \nu \in \mathbb{Z}^2/H \) is well-defined if \( \ell(H) \leq H \).

**Corollary 2.** Let \( n \) be an integer. Consider the set of all subgroups \( H \) of \( \mathbb{Z}^2 \). Given an integer \( n \), by \( T(n) \) denote the number of pairs \((H, \nu)\), where \( H \) is a subgroup of index \( n \) in \( \mathbb{Z}^2 \) with \( \ell(H) = H \) and \( \nu \) is a coset of \( \mathbb{Z}^2/H \) with \( \ell(\nu) = \nu \). Then

\[
T(n) = \theta(n) + 2\theta\left(\frac{n}{3}\right).
\]

**Proof.** Consider subgroup \( H \leq \mathbb{Z}^2 \) with \(|\mathbb{Z}^2 : H| = n \) and \( \ell(H) = H \). Lemma 2 claims that \( H \) has a pair of generators \( (p, q), (-q, p - q) \), where \( p^2 - pq + q^2 = n \). Suppose \( \ell(\nu) = \nu \) holds for some coset \( \nu \in \mathbb{Z}^2/H \). Let \( (a, b) \in \mathbb{Z}^2 \) be a representative of coset \( \nu \). Then \( \nu = \ell(\nu) = (a + b, -a + 2b) \in i(p, q) + j(-q, p - q) \). That is \( (a, b) = i(2q, 3p) + j(-q, p - q) \) for some integer \( i, j \). Then modulo \( ((p, q), (-q, p)) \) there are only three different choices for pairs \((a, b)\) corresponding to \( i = j = 0 \), \( i = j = 1 \) and \( i = j = 2 \). The first pair is always integer, the latter two are integer if and only if \( p + q \equiv 0 \mod 3 \). Also, \( p + q \equiv 0 \mod 3 \) if and only if \( 3 \mid p^2 - pq + q^2 = n \). That is, for a fixed \( H \) there is one choice of \( \nu \) if \( 3 \nmid n \) and three choices if \( 3 \mid n \). By Lemma 2 the number of possible subgroups \( H \) is \( \theta(n) \). So \( R(n) = \theta(n) \) if \( 3 \nmid n \) and \( R(n) = 3\theta(n) \) if \( 3 \mid n \). Finally note \( \theta(\frac{n}{3}) = \theta(n) \) if \( 3 \mid n \) and \( \theta(\frac{n}{3}) = 0 \) otherwise. Then we have the required \( R(n) = \theta(n) + 2\theta\left(\frac{n}{3}\right) \). \( \square \)

**Remark 2.** Similar to Remark 1 get

\[
T(n) = \sum_{\ell(H) = H < \mathbb{Z}^2, |\mathbb{Z}^2/H| = n} |\{\nu|\nu \in \mathbb{Z}^2/H, \ell(\nu) = \nu\}| = \sum_{\ell(H) = H < \mathbb{Z}^2, |\mathbb{Z}^2/H| = n} |\mathbb{Z}^2/\langle(1, -1), (1, 2), H\rangle|.
\]

3 **On the covering of \( \mathcal{G}_3 \)**

3.1 **The structure of the group \( \pi_1(\mathcal{G}_3) \)**

The following proposition provides the canonical form of an element in \( \pi_1(\mathcal{G}_3) = \langle x, y, z : xyx^{-1}y^{-1} = 1, zxz^{-1} = y, zyz^{-1} = (xy)^{-1}\rangle \). The proof is similar to the proof of Proposition 2 in [2].
Proposition 2. (i) Each element of $\pi_1(G_3)$ can be represented in the canonical form $x^ay^bz^c$ for some integer $a, b, c$.

(ii) The product of two canonical forms is given by the formula

\[
x^ay^bz^c \cdot x^dy^ez^f = \begin{cases} 
  x^{a+d}y^{b+e}z^{c+f} & \text{if } c \equiv 0 \mod 3 \\
  x^{a-e}y^{b+d-e}z^{c+f} & \text{if } c \equiv 1 \mod 3 \\
  x^{a-d+e}y^{b-d}z^{c+f} & \text{if } c \equiv 2 \mod 3 
\end{cases}
\quad (3.3)
\]

(iii) The canonical epimorphism $\phi_{G_3} : \pi_1(G_3) \to \pi_1(G_3)/\langle x, y \rangle \cong \mathbb{Z}$, given by the formula $x^ay^bz^c \to c$ is well-defined.

(iv) The representation in the canonical form $g = x^ay^bz^c$ for each element $g \in \pi_1(G_3)$ is unique.

Routinely follows from the definition of the group.

Notation. By $\Gamma$ denote the subgroup of $\pi_1(G_3)$ generated by $x, y$.

Our goal is to introduce some easy invariants, similar to those of Proposition 1.

Definition 2. Suppose all elements of $\pi_1(G_3)$ are represented in the canonical form. Let $\Delta$ be a subgroup of finite index $n$ in $\pi_1(G_3)$. Put $H(\Delta) = \Delta \cap \Gamma$. By $a(\Delta)$ denote the minimal positive exponent of $z$ among all the elements of $\Delta$. Choose an element $Z_\Delta$ with such exponent of $z$, represented in the form $Z_\Delta = hz^{a(\Delta)}$, where $h \in \Gamma$. By $\nu(\Delta) = hH(\Delta)$ denote the coset in the coset decomposition $\Gamma/H(\Delta)$.

Note that the invariants $a(\Delta), H(\Delta)$ and $\nu(\Delta)$ are well-defined. In particular the latter one does not depends on a choice of $Z_\Delta$. Also $a(\Delta)[\Gamma : H(\Delta)] = [\pi_1(G_2) : \Delta]$.

Definition 3. A 3-plet $(a, H, \nu)$ is called $n$-essential if the following conditions holds:

(i) $a$ is a positive divisor of $n$,

(ii) $H$ is a subgroup of index $n/a$ in $\Gamma$ with $H \lhd \pi_1(G_3)$ if $3 \nmid a$,

(iii) $\nu$ is an element of $\Gamma/H$.

The next proposition show that the introduced invariants are sufficient to enumerate the subgroups of finite index.

Proposition 3. There is a bijection between the set of $n$-essential 3-plets $(a, H, \nu)$ and the set of subgroups $\Delta$ of index $n$ in $\pi_1(G_3)$, such that $(a, H, \nu) = (a(\Delta), H(\Delta), \nu(\Delta))$, given by the correspondence $\Delta \leftrightarrow (a(\Delta), H(\Delta), \nu(\Delta))$. Moreover, $\Delta \cong \mathbb{Z}^3$ if $3 \mid a(\Delta)$ and $\Delta \cong \pi_1(G_3)$ otherwise.

The next few lemmas are auxiliary statements needed for the proof of Proposition 3.

Lemma 3. If $3 \nmid a(\Delta)$ then $H(\Delta) \lhd \pi_1(G_3)$. 

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Proof. Recall $Z_\Delta = hz^{a(\Delta)} \in \Delta$. Then $H(\Delta)^{Z_\Delta} = H(\Delta)$. Since $H(\Delta)^x = H(\Delta)^y = H(\Delta)^3 = H(\Delta)$, the former fact yields $H(\Delta)^g = H(\Delta), \ g \in \pi_1(G_3)$. □

Lemma 4. For arbitrary $n$-essential 3-plet $(a, H, \nu)$ there exists a subgroup $\Delta$ in the group $\pi_1(G_3)$ such that $(a, H, \nu) = (a(\Delta), H(\Delta), \nu(\Delta))$.

Proof. Take an $n$-essential 3-plet $(a, H, \nu)$. In case $3 \nmid a$ consider the following construction

$$\Delta = \{hz^{(3l+1)a(\Delta)} H|l \in \mathbb{Z}\} \cup \{hz^{a(\Delta)}hz^{-a(\Delta)}z^{(3l+2)a(\Delta)} H|l \in \mathbb{Z}\} \cup \{z^{3a(\Delta)} H|l \in \mathbb{Z}\}.$$ 

One can check that $\Delta$ is a subgroup of index $n$ in $\pi_1(G_3)$ and $(a(\Delta), H(\Delta), \nu(\Delta)) = (a, H, \nu)$.

Similarly, in case $3 | a$ the set

$$\Delta = \{h^l z^{a(\Delta)} H|l \in \mathbb{Z}\}$$

is the required subgroup. □

Proof of Proposition 3. Consider the family of subgroups $\Delta$ of index $n$ in $\pi_1(G_3)$, and the family of $n$-essential 3-plets. The definition of notions $a(\Delta)$, $H(\Delta)$ and $\nu(\Delta)$ together with Lemma 3 provide the correspondence from subgroups to $n$-essential 3-plets. Since the above invariants are well-defined, each subgroup $\Delta$ corresponds to only one 3-plet. By virtue of Lemma 4 each 3-plet corresponds to some subgroup, also different 3-plets correspond to different 3-plets. The bijection part is proven.

If $3 \nmid a(\Delta)$ Lemma 3 implies that $\Delta$ is a subgroup of $\langle x, y, z^3 \rangle$. Thus $\Delta$ is a subgroup of finite index in $\mathbb{Z}^3$, hence $\Delta$ is isomorphic to $\mathbb{Z}^3$ itself.

Consider case $a(\Delta) \equiv 1 \mod 3$. Since $H(\Delta)$ is a subgroup of finite index in $\langle x, y \rangle \cong \mathbb{Z}^2$, we have $H(\Delta) \cong \mathbb{Z}^2$. By Lemmas 2 and 3 there is a pair of elements, that generates $H(\Delta)$, that have the form form $(x^p y^q, x^{-q} y^{p+q})$. Let $h$ be an arbitrary element in the coset $\nu(\Delta)$. Put $X = x^p y^q$, $Y = x^{-q} y^{p+q}$ and $Z = hz^{a(\Delta)}$. Direct verification shows that the relations $XYX^{-1}Y^{-1} = 1$, $ZXZ^{-1} = Y$, and $ZYZ^{-1} = (XY)^{-1}$ holds. Further we call this relations the proper relations of the subgroup $\Delta$. Thus the map $x \rightarrow X$, $y \rightarrow Y$, $z \rightarrow Z$ can be extended to an epimorphism $\pi_1(G_3) \rightarrow \Delta$. To prove that this epimorphism is really an isomorphism we need to show that each relation in $\Delta$ is a corollary of proper relations. We call a relation, that is not a corollary of proper relations an improper relation.

Assume the contrary, i.e. there are some improper relations in $\Delta$. Since in $\Delta$ the proper relations holds, each element can be represented in the canonical form, given by Proposition 2 in terms of $X, Y, Z$, by using just the proper relations. That is each element $g$ can be represented as

$$g = X^a Y^b Z^c.$$ 

If there is an improper relation then there is an equality

$$X^a Y^b Z^c = X^{a'} Y^{b'} Z^{c'}, \quad (3.4)$$ 

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Consider these two cases separately.

Proceed to the proof of Theorem 1. Proposition 3 claims that each subgroup \( G \) of finite index in \( \pi_1(G_3) \) isomorphic to \( \pi_1(G_3) \) or \( \mathbb{Z}^3 \), depending upon whether \( a(\Delta) \) is a multiple of 3. Consider these two cases separately.

**Case (i).** Let \( \Delta \) be a subgroup of \( \pi_1(G_3) \) isomorphic to \( \pi_1(G_3) \). To find the number of such subgroups, by Proposition 3 we need to calculate the cardinality of the set of \( n \)-essential 3-plets with \( 3 \nmid a \).

For each \( 3 \nmid a \) there are \( \theta(\frac{a}{3}) \) subgroups \( H \) in \( \Gamma \) such that \( |\Gamma : H| = \frac{n}{a} \) and \( H \triangleleft \pi_1(G_3) \). Also there are \( \frac{n}{a} \) different choices of a coset \( \nu \). Thus, for each \( 3 \nmid a \) the number of \( n \)-essential 3-plets is \( \frac{n}{a} \theta(\frac{a}{3}) \). So, the total number of subgroups is given by

\[
s_{\pi_1(G_3), \pi_1(G_3)}(n) = \sum_{\mathbf{a} | n, 3 \nmid a} \frac{n}{a} \theta(\frac{n}{a}) = \sum_{\mathbf{a} | n} \frac{n}{a} \theta(\frac{n}{a}) - \sum_{\mathbf{a} | n} \frac{n}{3a} \theta(\frac{n}{3a}) = \sum_{k | n} k \theta(k) - \sum_{k | \frac{n}{3}} k \theta(k).
\]

**Case (ii).** Similarly to the previous case, we get the formula

\[
s_{\mathbb{Z}^3, \pi_1(G_3)}(n) = \sum_{3 \nmid a} \frac{n}{3a} \sigma_1(\frac{n}{3a}) = \omega(\frac{n}{3}).
\]

### 3.2 The proof of Theorem 1

Proceed to the proof of Theorem 1. Proposition 3 claims that each subgroup \( \Delta \) of finite index in \( \pi_1(G_3) \) or \( \mathbb{Z}^3 \), depending upon whether \( a(\Delta) \) is a multiple of 3. Consider these two cases separately.

**Case (i).** Let \( \Delta \) be a subgroup of \( \pi_1(G_3) \) isomorphic to \( \pi_1(G_3) \). To find the number of such subgroups, by Proposition 3 we need to calculate the cardinality of the set of \( n \)-essential 3-plets with \( 3 \nmid a \).

For each \( 3 \nmid a \) there are \( \theta(\frac{a}{3}) \) subgroups \( H \) in \( \Gamma \) such that \( |\Gamma : H| = \frac{n}{a} \) and \( H \triangleleft \pi_1(G_3) \). Also there are \( \frac{n}{a} \) different choices of a coset \( \nu \). Thus, for each \( 3 \nmid a \) the number of \( n \)-essential 3-plets is \( \frac{n}{a} \theta(\frac{a}{3}) \). So, the total number of subgroups is given by

\[
s_{\pi_1(G_3), \pi_1(G_3)}(n) = \sum_{\mathbf{a} | n, 3 \nmid a} \frac{n}{a} \theta(\frac{n}{a}) = \sum_{\mathbf{a} | n} \frac{n}{a} \theta(\frac{n}{a}) - \sum_{\mathbf{a} | n} \frac{n}{3a} \theta(\frac{n}{3a}) = \sum_{k | n} k \theta(k) - \sum_{k | \frac{n}{3}} k \theta(k).
\]

**Case (ii).** Similarly to the previous case, we get the formula

\[
s_{\mathbb{Z}^3, \pi_1(G_3)}(n) = \sum_{3 \nmid a} \frac{n}{3a} \sigma_1(\frac{n}{3a}) = \omega(\frac{n}{3}).
\]

### 3.3 The proof of Theorem 2

#### 3.3.1 Overall scheme of the proof

The proof of both cases follows the general scheme that we describe first. Recall that a subgroup \( G \) of finite index in \( \pi_1(G_3) \) has one of the following isomorphism types: \( \mathbb{Z}^3 \) or \( \pi_1(G_3) \). We use the standard notation \([g_1, g_2] = g_1 g_2 g_1^{-1} g_2^{-1}, g_1, g_2 \in \pi_1(G_3)\). Also, given subgroups \( G_1, G_2 \leq \pi_1(G_3) \) by \([G_1, G_2]\) we denote the subgroup, generated by the elements \([g_1, g_2], g_1 \in G_1, g_2 \in G_2\). Fix an isomorphism type of a subgroup. Further \( \Delta \) will always denote a subgroup of this isomorphism type of index \( n \in \pi_1(G_3) \). In each case we point a normal subgroup of finite index \( \Lambda \leq \pi_1(G_3) \) such that two conditions are met:
1° for any $\lambda \in \Lambda$ and any $\Delta$ holds $Ad_{\lambda}(H(\Delta)) = H(\Delta)$,

2° $[\Lambda, Z(\Delta)] = [H(\Lambda), Z(\Delta)]$, where $Z(\Delta)$ is given by Definition 2.

Call an intermediate conjugacy class $\Delta^\Lambda$ of $\Delta$ the set of subgroups $\Delta^\lambda$, $\lambda \in \Lambda$. Denote the number of such classes by $c^\Lambda_{G, \pi_1(G_3)}$, where $G$ isomorphic to one of $\mathbb{Z}^3$ or $\pi_1(G_3)$.

We propose an algorithm to uniformly calculate $c^\Lambda_{G, \pi_1(G_3)}$. Given $\Delta$, a subgroup $\Delta' \in \Delta^\Lambda$ have the following invariants: $a(\Delta') = a(\Delta)$, $H(\Delta') = H(\Delta)$ and $\nu(\Delta') \in \nu(\Delta)/[\Lambda, Z(\Delta)]$. Keep in mind that $[\Lambda, Z(\Delta)] \leq \Gamma$, since $\Gamma$ is normal in $\pi_1(G_3)$ and $\pi_1(G_3)/\Gamma$ is abelian. Thus for a fixed pair $(a, H)$ there are $|\Gamma : \langle [\Lambda, Z(\Delta)], H \rangle|$ partial conjugacy classes $\Delta^\Lambda$, each corresponding to the pair $(a, H)$. This let us to enumerate partial conjugacy classes.

The factor-group $\pi_1(G_3)/\Lambda$ acts by conjugation on partial conjugacy classes. An orbit of partial conjugacy classes form a conjugacy class, thus we use the Burnside’s lemma to calculate the number of conjugacy classes. To do this, we introduce one more definition.

**Definition 4.** Given $u \in \pi_1(G_3)/\Lambda$, by $B(u)$ denote the number of partial conjugacy classes, preserved by the conjugation with $u$: $B(u) = |\{ \Delta^\Lambda | (\Delta^\Lambda)^u = \Delta^\Lambda \}|$. In particular, $B(1)$ is the number of partial conjugacy classes.

Now we are done with the general scheme and proceed to its realization in specific cases.

### 3.3.2 Case (i)

Put $\Lambda = \pi_1(G_3)$. Proposition 3 claims $H(\Delta) \triangleleft \pi_1(G_3)$ in case $\Delta \cong \pi_1(G_3)$, thus (1°) holds. Recall that $Z(\Delta) = h_{x^a(\Delta)}$. Direct calculation through (3.3) shows that $[\Lambda, Z(\Delta)] = (xy^{-1}, xy^2) = [H(\Lambda), Z(\Delta)]$.

This means, firstly, that (2°) holds, and secondly that $|\Gamma : \langle [\Lambda, Z(\Delta)], H \rangle|$ equals 1 if $3 \nmid n$ and equals 3 if $3 \mid n$. For a fixed $a(\Delta)$ the number of subgroups $H(\Delta)$ is $\theta(\frac{n}{a(\Delta)})$. Keep in mind that $\theta(\frac{k}{3}) = \theta(k)$ if $\frac{k}{3}$ is integer, and $\theta(\frac{k}{3}) = 0$ otherwise. So the function $k \mapsto \begin{cases} \theta(k) & \text{if } 3 \nmid k \\ 3\theta(k) & \text{if } 3 \mid k \end{cases}$ is given by $\theta(k) + 2\theta(\frac{k}{3})$. Applying this and summing achieved number of pairs $(H, \nu)$ over all values of $a$ one gets:

$$c^\Lambda_{\pi_1(G_3), \pi_1(G_3)}(n) = \sum_{a \mid n} \theta\left(\frac{n}{a}\right) + 2\theta\left(\frac{n}{3a}\right) = \sum_{a \mid n} \theta\left(\frac{n}{a}\right) + \sum_{a \mid \frac{n}{3}} \theta\left(\frac{n}{3a}\right) - 2\sum_{a \mid \frac{n}{9}} \theta\left(\frac{n}{9a}\right) =$$

$$\sum_{k \mid n} \theta(k) + \sum_{k \mid \frac{n}{3}} \theta(k) - 2\sum_{k \mid \frac{n}{9}} \theta(k).$$

Since $\Lambda = \pi_1(G_3)$, the Burnside’s lemma is not needed to conclude $c^\Lambda_{\pi_1(G_3), \pi_1(G_3)}(n) = c^\Lambda_{\pi_1(G_3), \pi_1(G_3)}(n)$.
3.3.3 Case (ii)

Put \( \Lambda = \langle x, y, z^3 \rangle \). Since \( \Lambda \cong \mathbb{Z}^3 \) and \( \Delta \leq \Lambda \) for any \( \Delta \), conditions (1°) and (2°) hold. Also each partial conjugacy class consists of one subgroup, i.e. \( c_{\mathbb{Z}^3, \pi_1(G_3)}(n) = s_{\mathbb{Z}^3, \pi_1(G_3)}(n) = \omega(\frac{n}{3}) \).

The factor \( \pi_1(G_3)/\Lambda \) consists of three elements, represented by 1, \( z \), and \( z^2 \) respectively.

The condition \( \Delta^2 = \Delta \) is equivalent to \( (H(\Delta))^z = H(\Delta) \) and \( (\nu(\Delta))^z = \nu(\Delta) \) met simultaneously. Corollary 1 provides the number of such pairs \( (H, \nu) \) for a given value of \( a \). Summing over all the possible values (recall that \( 3 \mid a \)) one gets \( B(z) = B(z^2) = \sum_k \frac{\theta(k)}{k^2} + 2 \sum_{k|3} \theta(k) \).

By making use of Burnside’s lemma obtain

\[
c_{\mathbb{Z}^3, \pi_1(G_3)}(n) = \frac{1}{3} \left( \omega\left(\frac{n}{3}\right) + 2 \sum_{k|3} \theta(k) + 4 \sum_{k|3} \theta(k) \right).
\]

4 On the coverings of \( G_5 \)

4.1 The structure of the group \( \pi_1(G_5) \)

Recall that the fundamental group of \( G_5 \) is given by: \( \pi_1(G_5) = \langle x, \tilde{y}, \tilde{z} : \tilde{x}\tilde{y}\tilde{x}^{-1}\tilde{y}^{-1} = 1, \tilde{z}\tilde{x}\tilde{z}^{-1} = \tilde{x}\tilde{y}, \tilde{z}\tilde{y}\tilde{z}^{-1} = \tilde{x}^{-1} \rangle \). The following proposition provides the canonical form of an element in \( \pi_1(G_5) \).

**Proposition 4.** (i) Each element of \( \pi_1(G_5) \) can be represented in the canonical form \( \tilde{x}^a\tilde{y}^b\tilde{z}^c \) for some integer \( a, b, c \).

(ii) The product of two canonical forms is given by the formula

\[
\tilde{x}^a\tilde{y}^b\tilde{z}^c \cdot \tilde{x}^d\tilde{y}^e\tilde{z}^f = \begin{cases} 
\tilde{x}^{a+d}\tilde{y}^{b+e}\tilde{z}^{c+f} & \text{if } c \equiv 0 \mod 6 \\
\tilde{x}^{a+d-e}\tilde{y}^{b+d}\tilde{z}^{c+f} & \text{if } c \equiv 1 \mod 6 \\
\tilde{x}^{a-e}\tilde{y}^{b+d-e}\tilde{z}^{c+f} & \text{if } c \equiv 2 \mod 6 \\
\tilde{x}^{a-d}\tilde{y}^{b-e}\tilde{z}^{c+f} & \text{if } c \equiv 3 \mod 6 \\
\tilde{x}^{a-d+e}\tilde{y}^{b-d}\tilde{z}^{c+f} & \text{if } c \equiv 4 \mod 6 \\
\tilde{x}^{a+e}\tilde{y}^{b-d+e}\tilde{z}^{c+f} & \text{if } c \equiv 5 \mod 6 
\end{cases} \quad (4.6)
\]

(iii) The canonical epimorphism \( \phi_{G_5} : \pi_1(G_5) \to \pi_1(G_5)/\langle \tilde{x}, \tilde{y} \rangle \cong \mathbb{Z} \), given by the formula \( \tilde{x}^a\tilde{y}^b\tilde{z}^c \to c \) is well-defined.

(iv) The representation in the canonical form \( g = \tilde{x}^a\tilde{y}^b\tilde{z}^c \) for each element \( g \in \pi_1(G_5) \) is unique.

Routinely follows from the definition of the group.

**Notation.** Let \( \Gamma = \langle \tilde{x}, \tilde{y} \rangle \) be the subgroup of \( \pi_1(G_5) \) generated by \( \tilde{x}, \tilde{y} \).
Our goal is to introduce some easy invariants, similar to those in Proposition 3. That will let us to enumerate the subgroups.

**Definition 6.** Suppose all elements of $\pi_1(\mathcal{G}_5)$ are represented in the canonical form. Let $\Delta$ be a subgroup of finite index $n$ in $\pi_1(\mathcal{G}_5)$. Put $H(\Delta) = \Delta \cap \Gamma$. By $a(\Delta)$ denote the minimal positive exponent of $z$ among all the elements $x^a y^b z^c \in \Delta$. Choose an element $Z(\Delta)$ with such exponent of $z$, represented in the form $Z(\Delta) = h z^{a(\Delta)}$, where $h \in \Gamma$. By $\nu(\Delta) = h H(\Delta)$ denote the coset in $\Gamma/H(\Delta)$ containing $h$.

**Definition 6.** A 3-plet $(a, H, \nu)$ is called n-essential if the following conditions hold:

(i) $a$ is a positive divisor of $n$,

(ii) $H$ is a subgroup of index $n/a$ in $\Gamma$ with $H \triangleleft \pi_1(\mathcal{G}_5)$ if $3 \nmid a$,

(iii) $\nu$ is an element of $\Gamma/H$.

**Proposition 5.** There is a bijection between the set of n-essential 3-plets $(a, H, \nu)$ and the set of subgroups $\Delta$ of index $n$ in $\pi_1(\mathcal{G}_5)$, given by the correspondence $\Delta \leftrightarrow (a, H, \nu(\Delta))$. Moreover, $\Delta \cong \mathbb{Z}_3$ if $(a, 6) = 6$, $\Delta \cong \pi_1(\mathcal{G}_2)$ if $(a, 6) = 3$, $\Delta \cong \pi_1(\mathcal{G}_3)$ if $(a, 6) = 2$ and $\Delta \cong \pi_1(\mathcal{G}_5)$ if $(a, 6) = 1$.

**Proof.** The proof of Proposition 5 is similar to the proof of Proposition 3.

4.2 The proof of Theorem 3

In case (i) the argument similar to the proof of Theorem 1 leads to the formula:

$$s_{\pi_1(\mathcal{G}_5), \pi_1(\mathcal{G}_5)}(n) = \sum_{a \mid n, (a, 6) = 1} \frac{n}{a} \theta\left(\frac{n}{a}\right) = \sum_{k \mid n, \left(\frac{a}{k}, 6\right) = 1} k \theta(k).$$

The last equality is obtained by applying the inclusion–exclusion principle. The cases (ii), (iii) and (iv) can be treated in the similar way.

4.3 The proof of Theorem 4

The proof uses the overall scheme form section 3.3.1 so we just proceed to its realization in specific cases.

4.3.1 Case (i)

Put $\Lambda = \pi_1(\mathcal{G}_5)$. Proposition 3 claims $H(\Delta) \triangleleft \pi_1(\mathcal{G}_5)$ in case $\Delta \cong \pi_1(\mathcal{G}_5)$, thus $(1^*)$ holds. Direct calculation using (4.6) in case $a(\Delta) \equiv 1 \mod 6$ gives $[\bar{x}, Z(\Delta)] = \bar{y}^{-1}$ and $[\bar{y}, Z(\Delta)] = \bar{x}^{-1}$. In case $c(\Delta) \equiv 5 \mod 6$ we respectively get $[\bar{x}, Z(\Delta)] = \bar{x}\bar{y}$ and $[\bar{y}, Z(\Delta)] = \bar{x}^{-1}$. So in both cases $[\Gamma, Z(\Delta)] = \Gamma$. Firstly this means that $(2^*)$ holds. Secondly, conjugacy classes of subgroups $\Delta$ are in one-to-one correspondence with pairs $(a, H)$. Summing the number of choices of $H$ over all the possible values of $a$ get

$$c_{\pi_1(\mathcal{G}_5), \pi_1(\mathcal{G}_5)}(n) = \sum_{a \mid n, (a, 6) = 1} \theta\left(\frac{n}{a}\right) = \sum_{k \mid n} \theta(k) - \sum_{k \mid \frac{n}{2}} \theta(k) - \sum_{k \mid \frac{n}{3}} \theta(k) + \sum_{k \mid \frac{n}{6}} \theta(k).$$
4.3.2 Case (ii)

Put $\Lambda = \langle \bar{x}, \bar{y}, \bar{z}^2 \rangle$. Proposition 1 claims $H(\Delta) \triangleleft \pi_1(\mathcal{G}_5)$ in case $\Delta \cong \pi_1(\mathcal{G}_5)$, thus (1°) holds. Recall that $Z(\Delta) = h\bar{z}^a(\Delta)$. Direct calculation through (4.6) shows that 

$$[\Lambda, Z(\Delta)] = \langle \bar{x}\bar{y}^{-1}, \bar{x}\bar{y}^2 \rangle = [H(\Lambda), Z(\Delta)].$$

This means, firstly, that (2°) holds, and secondly that $|\Gamma : \langle [\Lambda, Z(\Delta)], H \rangle |$ equals 1 if $3 \nmid n$ and equals 3 if $3 | n$.

The factor $\pi_1(\mathcal{G}_5)/\Lambda$ consists of two elements, represented by 1 and $\bar{x}^3$ respectively. The conjugation with these elements preserves $(a, H)$, thus in case $3 \nmid n$ the partial conjugacy classes coincide with conjugacy classes, and there is only one conjugacy class for a fixed pair $(a, H)$. In case $3 | n$ for a fixed pair $(a, H)$ there are 3 partial conjugacy classes: namely $\Delta_0^\Lambda$, $\Delta_1^\Lambda$ and $\Delta_2^\Lambda$, where $\Delta_0 \leftrightarrow (a, H, 1)$, $\Delta_1 \leftrightarrow (a, H, \bar{y})$ and $\Delta_2 \leftrightarrow (a, H, \bar{y}^2)$. Note that the conjugation with $\bar{x}^3$ swaps the partial conjugacy classes $\Delta_1^\Lambda$ and $\Delta_2^\Lambda$. Thus for a fixed pair $(a, H)$ there are two conjugacy classes: $\Delta_0^{\pi_1(\mathcal{G}_5)} = \Delta_0^\Lambda$ and $\Delta_1^{\pi_1(\mathcal{G}_5)} = \Delta_2^\Lambda$.

Keep in mind that $\theta(\frac{n}{2}, \bar{y}) = \theta(n)$ if $\frac{n}{2}$ is integer, and $\theta(\frac{n}{2}, \bar{y}) = 0$ otherwise. Applying this and summing achieved number of conjugacy classes over all possible values of $a$ one gets:

$$c_{\pi_1(\mathcal{G}_5), \pi_1(\mathcal{G}_5)}(n) = \sum_{a \mid \frac{n}{2}, a \mid \frac{n}{6}, 3 \mid a} \theta(\frac{n}{2a}) + 2 \sum_{a \mid \frac{n}{6}, 3 \mid a} \theta(\frac{n}{6a}) = \sum_{a \mid \frac{n}{2}} \theta(k) - \sum_{a \mid \frac{n}{18}} \theta(k) = \sum_{k \mid \frac{n}{2}} \theta(k) - \sum_{k \mid \frac{n}{18}} \theta(k).$$

4.3.3 Case (iii)

Put $\Lambda = \langle \bar{x}, \bar{y}, \bar{z}^3 \rangle$. $Ad_{\bar{x}}$ and $Ad_{\bar{y}}$ are the identity transformation on $\Gamma$. $Ad_{\bar{z}}$ is given by $g \to g^{-1}$, $g \in \Gamma$. That is $Ad_{\bar{x}}$, $Ad_{\bar{y}}$ and $Ad_{\bar{z}}$ preserves $H(\Delta)$, i.e. (1°) holds.

Further $[H(\Lambda), Z(\Delta)] = \langle \bar{x}^2, \bar{y}^2 \rangle$. Recall the notation $Z(\Delta) = h\bar{z}^a(\Delta)$. Then $[\bar{z}^3, Z(\Delta)] = h^{-2} \in \langle \bar{x}^2, \bar{y}^2 \rangle$, so (2°) holds.

Applying Remark 1 and Corollary 1 and summing over all possible values of $a$ find

$$c_{\pi_1(\mathcal{G}_5), \pi_1(\mathcal{G}_5)}(n) = \sum_{a \mid n, 3 \mid a, 6 \mid a} \left( \sigma_1(\frac{n}{a}) + 3\sigma_1(\frac{n}{2a}) \right) = \sigma_2(\frac{n}{3}) + 2\sigma_2(\frac{n}{6}) - 3\sigma_2(\frac{n}{12}).$$

The factor $\pi_1(\mathcal{G}_5)/\Lambda$ consists of three elements, represented by 1, $\bar{x}^2$ and $\bar{x}^4$ respectively. Obviously the numbers of partial conjugacy classes, preserved by $Ad_{\bar{x}^2}$ and $Ad_{\bar{x}^4}$ coincide: $B(\bar{x}^2) = B(\bar{x}^4)$. To find them note the following. A partial conjugacy class $\Delta^\Lambda$ is preserved by $Ad_{\bar{x}^2}$ iff the hollowing conditions are met simultaneously:

(*) $(H(\Delta))^2 = H(\Delta)$

(**) $(\nu(\Delta))^2 = \bar{x}^{-2}$ must belong to the same conjugacy class in $\Gamma / \langle \bar{x}^2, \bar{y}^2, H \rangle$ as $\nu(\Delta)$. 

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By Lemma \( \text{2} \) the condition (*) implies that \( H(\Delta) \) have a pair of generators \((\bar{x}^p \bar{y}^q, \bar{x}^{-q} \bar{y}^{-q})\). The matrix \( \begin{pmatrix} p & q \\ -q & p - q \end{pmatrix} \) modulo 2 have the rank 0 or 2, never 1. So \( \Gamma / \langle \bar{x}^2, \bar{y}^2, \bar{x}^p \bar{y}^q, \bar{x}^{-q} \bar{y}^{-q} \rangle \) is either trivial or isomorphic to \( \mathbb{Z}_2^2 \). In the first case the condition (***) holds for the sole element, in the second case the condition (****) holds only for the coset of 0, and does not holds for three other cosets.

So the conjugacy classes \( \Delta^\Lambda \) with \( \Delta^z^2 \in \Delta^\Lambda \) are in one-to-one correspondence with the normal subgroups \( H(\Delta) \triangleleft \pi_1(G_5) \). Applying Lemma \( \text{2} \) and summing over all the possible values of \( a \) yields \( B(\bar{x}^2) = B(\bar{x}^4) = \sum_{k \mid n}^\Lambda \theta(k) - \sum_{k \mid n}^\nu \theta(k) \). Substituting to Burnside’s lemma obtain

\[
c_{\pi_1(G_2), \pi_1(G_5)}(n) = \frac{1}{3} \left( \sigma_2 \frac{n}{3} + 2 \sigma_2 \frac{n}{6} - 3 \sigma_2 \frac{n}{12} + 2 \sum_{k \mid \frac{n}{3}} \theta(k) - 2 \sum_{k \mid \frac{n}{6}} \theta(k) \right).
\]

### 4.3.4 Case (iv)

Put \( \Lambda = \langle \bar{x}, \bar{y}, \bar{z}^6 \rangle \). Since \( \Lambda \cong \mathbb{Z}^3 \) and \( \Delta \leq \Lambda \) for any \( \Delta \), conditions (1') and (2') hold. Also each partial conjugacy class consists of one subgroup, i.e. \( c_{\mathbb{Z}^3, \pi_1(G_5)}(n)^\Lambda = s_{\mathbb{Z}^3, \pi_1(G_5)}(n) = \omega(\frac{n}{6}) \).

The factor \( \pi_1(G_5) / \Lambda \) consists of six elements, represented by 1, \( \bar{x} \), \( \bar{x}^2 \), \( \bar{x}^3 \), \( \bar{x}^4 \) and \( \bar{x}^5 \) respectively.

The condition \( \Delta^\bar{x}^3 = \Delta \) is equivalent to \( 2\nu(\Delta) = 0 \). Corollary \( \text{2} \) provides the number of such pairs \((H, \nu)\) for a given value of \( a \), summing over all possible values (recall that \( 6 \mid a \)) one gets \( B(\bar{x}^2) = \sigma_2 \frac{n}{3} + 3 \sigma_2 \frac{n}{12} \).

The condition \( \Delta^\bar{x}^2 = \Delta \) is equivalent to \((H(\Delta))^{\bar{x}^2} = H(\Delta) \) and \((\nu(\Delta))^{\bar{x}^2} = \nu(\Delta) \) met simultaneously. Corollary \( \text{3} \) provides the number of such pairs \((H, \nu)\) for a given value of \( a \). Summing over all the possible values (recall that \( 6 \mid a \)) one gets \( B(\bar{x}^2) = B(\bar{x}^4) = \sum_{k \mid \frac{n}{3}} \theta(k) + 2 \sum_{k \mid \frac{n}{6}} \theta(k) \).

Finally, \( \Delta^\bar{x} = \Delta \) implies \((H(\Delta))^\bar{x} = H(\Delta) \) and \((\nu(\Delta))^\bar{x} = (\nu(\Delta))^\bar{x} = \nu(\Delta) \). The latter two equalities imply \( \nu(\Delta) = 0 \), so the unique subgroup \( \Delta \) correspond to a \( H(\Delta) \). Summing over all the possible values of \( a \) yields \( B(\bar{x}) = B(\bar{x}^5) = \sum_{k \mid \frac{n}{6}} \theta(k) \).

Substituting to Burnside’s lemma obtain

\[
c_{\mathbb{Z}^3, \pi_1(G_5)}(n) = \frac{1}{6} \left( \omega \frac{n}{6} + \sigma_2 \frac{n}{6} + 3 \sigma_2 \frac{n}{12} + 4 \sum_{k \mid \frac{n}{3}} \theta(k) + 4 \sum_{k \mid \frac{n}{6}} \theta(k) \right).
\]

### 5 Appendix

Given a sequence \( \{f(n)\} \) \( n \geq 1 \), a formal power series

\[
\hat{f}(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}
\]
is called Dirichlet generating function for $\{f(n)\}_{n=1}^{\infty}$, see for example, \[23\]. For the way to reconstruct the sequence $f(n)$ by $\hat{f}(s)$ see Perron's formula (for example \[24\]).

Here we present the Dirichlet generating functions for the calculated sequences $s_{H,G}(n)$ and $c_{H,G}(n)$. Since theorems 1–4 provides he explicit formulas, the remaining can is done by direct calculations which we omit here.

**Notations.** By $\zeta(s)$ we denote the Riemann zeta function. Define sequence $\{\chi(n)\}_{n=1}^{\infty}$ by $\chi(n) = \frac{1}{\sqrt{3}}(-\frac{1}{2} + \frac{\sqrt{3}}{2}i)^n - (-\frac{1}{2} - \frac{\sqrt{3}}{2}i)^n$ or equivalently $\chi(n) = \begin{cases} 1 & \text{if } n \equiv 1 \mod 3 \\ -1 & \text{if } n \equiv 2 \mod 3 \end{cases}$.

For the sake of brevity denote $\vartheta(s) = \tilde{\chi}(s)$. Note that $\vartheta(s)$ is the Dirichlet L-series for the multiplicative character $\chi(n)$.

| $G$ | $\mathcal{G}_3$ | $\mathcal{G}_5$ |
|-----|----------------|----------------|
| $\mathcal{G}_3^G$ | $3^{-s} \zeta(s) \zeta(s - 1) \zeta(s - 2)$ | $6^{-s} \zeta(s) \zeta(s - 1) \zeta(s - 2)$ |
| $\mathcal{G}_5^G$ | $3^{-s-1} \zeta(s) \left( \zeta(s - 1) \zeta(s - 2) + 2(1 + 2 \cdot 3^{-s}) \zeta(s) \vartheta(s) \right)$ | $6^{-s-1} \zeta(s) \left( \zeta(s - 1) \zeta(s - 2) + (1 + 3 \cdot 2^{-s}) \zeta(s) \zeta(s - 1) + 4(1 + 3^{-s}) \zeta(s) \vartheta(s) \right)$ |
| $\mathcal{G}_2^G$ | $3^{-s} (1 - 2^{-s}) \zeta(s) \zeta(s - 1) \zeta(s - 2)$ | $3^{-s} (1 - 2^{-s}) \zeta(s)^2 \left( (1 + 3 \cdot 2^{-s}) \zeta(s - 1) + 2 \vartheta(s) \right)$ |
| $\mathcal{G}_3^G$ | $(1 - 3^{-s}) \zeta(s - 1)^2 \vartheta(s - 1)$ | $2^{-s} (1 - 3^{-s}) \zeta(s - 1)^2 \vartheta(s - 1)$ |
| $\mathcal{G}_5^G$ | $(1 - 3^{-s}) (1 + 2 \cdot 3^{-s}) \zeta(s)^2 \vartheta(s)$ | $2^{-s} (1 - 3^{-s}) (1 + 3^{-s}) \zeta(s)^2 \vartheta(s)$ |
| $\mathcal{G}_5^G$ | $(1 - 2^{-s}) (1 - 3^{-s}) \zeta(s) \zeta(s - 1)^2 \vartheta(s - 1)$ | $(1 - 2^{-s}) (1 - 3^{-s}) \zeta(s)^2 \vartheta(s)$ |

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