Infrared renormalization of two-loop integrals and the chiral expansion of the nucleon mass

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Abstract

We describe details of the renormalization of two-loop integrals relevant to the calculation of the nucleon mass in the framework of manifestly Lorentz-invariant chiral perturbation theory using infrared renormalization. It is shown that the renormalization can be performed while preserving all relevant symmetries, in particular chiral symmetry, and that renormalized diagrams respect the standard power counting rules. As an application we calculate the chiral expansion of the nucleon mass to order $\mathcal{O}(q^6)$.

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I. INTRODUCTION

Chiral perturbation theory (ChPT) [1, 2, 3] is the effective field theory of the strong interactions at low energies (for an introduction see, e.g., [4, 5, 6]). It relies on a perturbative expansion in terms of small parameters \( q/\Lambda \), where \( q \) denotes a quantity like the pion mass or external momenta that are small relative to the scale \( \Lambda \), which for ChPT is expected to be of the size 1 GeV. One of the essential ingredients of ChPT is a consistent power counting, which assigns a chiral order \( D \) to each Feynman diagram for the process in question and which predicts that diagrams of higher orders are suppressed. Assuming the coefficients of the perturbative expansion to be of natural size one would expect contributions at order \( D+1 \) to be suppressed by a factor \( q/\Lambda \) compared to contributions at order \( D \). For \( q \) of the order of the pion mass and \( \Lambda \approx 1 \) GeV, this corresponds to a correction of about 20%. In the mesonic sector of ChPT this rough estimate seems accurate, however, the situation is less clear for the baryonic sector. While for example the chiral expansion of the nucleon mass shows a good convergence behavior, the nucleon axial coupling \( g_A \) receives large contributions from higher-order terms [7]. Further examples include the electromagnetic form factors of the nucleon (see, e.g., [8, 9]), which only describe the data for very low values of momentum transfer. For some of these quantities higher-order contributions clearly play an important role. The description of the nucleon form factors can be improved by the inclusion of vector mesons as explicit degrees of freedom, which corresponds to the resummation of higher-order contributions [3, 9].

The convergence properties of baryon chiral perturbation theory (BChPT) are also of great importance for lattice QCD. Due to numerical costs, present lattice calculations still require pion masses larger than the physical one, and results obtained on the lattice have to be extrapolated to the physical point. ChPT as an expansion in the pion mass is the appropriate tool to perform such extrapolations, which again poses the question for which values of small parameters the ChPT expansion gives reliable predictions.

There are several renormalization schemes for manifestly Lorentz-invariant BChPT at the one-loop level that result in a consistent power counting while preserving all relevant symmetries [10, 11, 12, 13, 14, 15]. The most commonly used of these is the infrared (IR) regularization of Ref. [11]. All these renormalization schemes have in common that there is a relation between the chiral order and the loop expansion, so that the investigation of higher chiral orders requires the evaluation of multi-loop diagrams. In Ref. [16] a reformulated version of the IR regularization has been introduced that is also applicable to multi-loop diagrams [17]. Reference [18] contains a different generalization of IR regularization to two-loop diagrams.

In this paper we describe the renormalization procedure for two-loop integrals in manifestly Lorentz-invariant BChPT within the reformulated version of infrared regularization of Ref. [16]. The results of a calculation of the nucleon mass up to and including order \( \mathcal{O}(q^6) \) have been reported in Ref. [19]. To the best of our knowledge, this is the first complete two-loop BChPT calculation in a manifestly Lorentz-invariant framework. Here, we describe the details of the calculation. In particular we show that the renormalization procedure preserves all relevant symmetries and that renormalized two-loop diagrams obey the standard power counting rules. A calculation of the nucleon mass to order \( \mathcal{O}(q^5) \) was performed in Ref. [20] in the framework of HBChPT (see, e.g., [21, 22]), and Ref. [23] contains the leading non-analytic contributions to the axial-vector coupling \( g_A \) at two-loop order obtained from renormalization group techniques.
This paper is organized as follows. In Section II we review the main features of infrared renormalization at the one-loop level that are essential for the following. Section III contains a brief overview over the general aspects of the renormalization of two-loop diagrams. The infrared renormalization of products of one-loop integrals is described in Sec. IV while the discussion of genuine two-loop integrals follows in Sec. V. An application of these methods can be found in Sec. VI which is followed by a summary. Explicit expressions for the appearing integrals can be found in the appendix.

II. INFRARED REGULARIZATION OF ONE-LOOP INTEGRALS

The method of infrared regularization \[11\] was developed as a manifestly Lorentz-invariant renormalization scheme preserving all relevant symmetries. It results in diagrams obeying the standard power counting (see Sec. VI). Infrared regularization is based on dimensional regularization and the analytic properties of loop integrals, and in its original formulation is applicable to one-loop integrals containing pion and nucleon propagators in the one-nucleon sector of ChPT. In Ref. \[16\] a different formulation of infrared regularization was presented which reproduces the results of the original formulation up to arbitrary order. The advantage of the new formulation is that it can also be applied to multi-loop diagrams and diagrams containing additional degrees of freedom \[9, 16, 17, 24, 25\]. Since IR regularization can, in fact, be viewed as a renormalization scheme we also refer to it as infrared renormalization. We briefly describe those features of the renormalization of one-loop integrals which are important for the renormalization of two-loop integrals.

Denote a general one-loop integral containing pion and nucleon propagators by

\[ H_{\pi N\ldots}(q_1, \ldots, p_1, \ldots) = i \int \frac{d^n k}{(2\pi)^n} \frac{1}{a_1 \cdots a_m b_1 \cdots b_l}, \]  

where \( a_i = (k + q_i)^2 - M^2 + i0^+ \) and \( b_j = (k + p_j)^2 - m^2 + i0^+ \) are related to pion and nucleon propagators, respectively, and \( n = 4 + 2\epsilon \) is the number of space-time dimensions. Infrared renormalization consists of splitting the integral into an infrared singular part \( I_{\pi N\ldots} \) and an infrared regular part \( R_{\pi N\ldots} \),

\[ H_{\pi N\ldots} = I_{\pi N\ldots} + R_{\pi N\ldots}, \]  

or for short

\[ H = I + R. \]  

The advantage of splitting the original integral into two parts is that the infrared singular part \( I_{\pi N\ldots} \) satisfies the power counting, while \( R_{\pi N\ldots} \) contains terms that violate the power counting. In addition, the infrared singular and infrared regular parts differ in their analytic properties. For noninteger \( n \) the expansion of \( I_{\pi N\ldots} \) in small quantities results in only noninteger powers of these variables, while \( R_{\pi N\ldots} \) only contains analytic contributions. In the formulation of Ref. \[16\] one obtains the infrared regular part \( R_{\pi N\ldots} \) by reducing \( H_{\pi N\ldots} \) to an integral over Schwinger or Feynman parameters, expanding the resulting expression in small quantities such as pion masses or small momenta, and interchanging summation and integration.

As an example consider the integral

\[ H_{\pi N}(0, -p) = i \int \frac{d^n k}{(2\pi)^n} \frac{1}{[k^2 - M^2 + i0^+][((k - p)^2 - m^2 + i0^+].} \]  

3
To apply the reformulated version of IR renormalization we combine the two propagators using
\[ \frac{1}{ab} = \int_0^1 \frac{dz}{(1-z)a + zb}, \]
and perform the integration over the loop momentum \( k \), resulting in
\[ H_{\pi N}(0, -p) = -\frac{1}{(4\pi)^{n/2}} \Gamma(2 - n/2) \int_0^1 dz \left| C(z) \right|^{(n/2) - 2}, \tag{5} \]
where \( C(z) = m^2 z^2 - (p^2 - m^2)(1 - z)z + M^2(1 - z) - i0^+ \). Next, we expand \( \left| C(z) \right|^{(n/2) - 2} \) in \( p^2 - m^2 \) and \( M^2 \) and interchange summation and integration. This generates the chiral expansion of the infrared regular part \( R \), which is given by
\[ R = -\frac{m^{n-4}\Gamma(2 - n/2)}{(4\pi)^{n/2}(n - 3)} \left[ 1 - \frac{p^2 - m^2}{2m^2} + \frac{(n - 6)(p^2 - m^2)^2}{4m^4(n - 5)} + \frac{(n - 3)M^2}{2m^2(n - 5)} + \cdots \right], \tag{6} \]
and which coincides with the expansion of \( R \) given in Ref. [11].

Symmetries introduce relations among various Green functions of the theory, called Ward-Fradkin-Takahashi identities (Ward identities for short) [26, 27, 28]. Expressions containing the integrals \( H \) satisfy the Ward identities, \(^1\) since they are derived from an invariant Lagrangian and dimensional regularization does not violate the symmetries. Since \( I \), for noninteger \( n \), only contains nonanalytic terms, while \( R \) consists of analytic contributions only, each part has to satisfy the Ward identities separately in order for the sum \( H = I + R \) not to violate any symmetry.

Both the infrared regular and the infrared singular parts contain additional divergences not present in the original integral \( H \). Since these additional divergences do not appear in \( H \), they have to cancel in the sum of \( I + R = H \). This means that
\[ \frac{R^{\text{add}}}{\epsilon} = -\frac{I^{\text{add}}}{\epsilon}. \tag{7} \]

The \( \epsilon \) expansion of \( H \) is given by
\[ H = \frac{H^{UV}}{\epsilon} + H^{(0)} + O(\epsilon) = \frac{H^{UV}}{\epsilon} + \frac{I^{\text{add}}}{\epsilon} + \frac{R^{\text{add}}}{\epsilon} + \tilde{I} + \tilde{R}, \tag{8} \]
where \( H^{UV}/\epsilon \) denotes the ultraviolet divergence of \( H \), \( H^{(0)} \) refers to the terms independent of \( \epsilon \), and we have explicitly shown the additional divergences in the second line. In BChPT the renormalization can be performed in a two-step process. First, all divergences are absorbed, and then additional finite terms are subtracted. In the standard approach the divergences are absorbed using the \( \overline{\text{MS}} \) scheme. In this scheme one subtracts the quantity
\[ \frac{1}{32\pi^2} \left[ \frac{1}{\epsilon} - \ln(4\pi) + \gamma_E - 1 \right], \]
\(^1\) In the following we use the phrase that integrals satisfy the Ward identities, by which we mean that expressions containing these integrals satisfy the Ward identities.
where \( \gamma_E = -\Gamma'(1) \), and sets the appearing t’Hooft parameter \( \mu = m \), where \( m \) is the nucleon mass in the chiral limit. Here, in order to simplify the calculation, we apply minimal subtraction (MS) with a t’Hooft parameter \( \tilde{\mu} \), absorbing only terms proportional to \( \epsilon^{-1} \), and then set \( \tilde{\mu} = \frac{m}{(4\pi)^{3/2}} e^{\frac{3E-1}{2}} \). This is completely equivalent to the standard approach.

The infrared renormalized expression \( H^r \) of the integral \( H \) is defined as its finite infrared singular term,

\[
H^r = \tilde{I},
\]

which satisfies the power counting since all terms violating it are contained in \( R \). One of the fundamental properties used in the construction of the effective Lagrangian is the invariance under symmetries of the underlying theory. It is therefore of utmost importance that these symmetries are not violated at any step in the calculations. We now show that the definition of the renormalized integral \( H^r \) of Eq. (9) satisfies this requirement [11]. The original integral \( H \) is obtained from a chirally symmetric Lagrangian using dimensional regularization, which preserves all symmetries. Therefore expressions containing \( H \) satisfy the Ward identities; and in particular their \( \epsilon \) expansions satisfy the Ward identities order by order. As explained above, \( R \) satisfies the Ward identities separately from \( I \). This also means that the Ward identities are satisfied order by order in the \( \epsilon \) expansion of \( R \) and \( I \), respectively. Therefore the identification of the renormalized integral \( H^r \) as \( H^r = \tilde{I} \) does not violate any symmetry constraints. Since the sum of additional divergences cancels, the term which is subtracted from \( H \) is given by

\[
\tilde{R} = \frac{H_{UV}}{\epsilon} + \tilde{R}.
\]

With Eq. (7) and the definition of Eq. (10) we can write

\[
H = \tilde{I} + \tilde{R}.
\]

Within the framework of dimensional regularization, the dimensional counting analysis of Ref. [30] provides a method to obtain expansions of loop integrals in small parameters. This method is described in detail in Appendix A. Here we show how the infrared regular and infrared singular parts of the integral \( H \) are related to the different terms obtained from this method. Using dimensional counting, \( H \) is written as

\[
H = G_1 + G_2.
\]

For \( G_1 \) we simply expand the integrand in \( M \) and interchange summation and integration. \( G_2 \) is obtained by rescaling the integration variable \( k \rightarrow \frac{M}{m} k \) and then expanding the integrand with subsequent interchange of summation and integration. For \( p^2 = m^2 \) the method of obtaining \( G_1 \) is the same as the one used to determine the expansion of the infrared-regular part \( R \). It follows that

\[
G_1 = \sum_n R_n = R,
\]

while \( G_2 \) gives the chiral expansion of the infrared singular term \( I \),

\[
G_2 = \sum_n I_n,
\]

\footnote{Note that the notation used here differs from the one found in the Appendix to avoid confusion with terms in the \( \epsilon \) expansion of \( H \).}
where $R_n$ and $I_n$ are the terms in the chiral expansion of the infrared regular and infrared singular parts, respectively. It should be noted that the expansion of $I$ does not always converge in the entire low-energy region. For the integrals considered in the calculation of the nucleon mass, however, the expansion of $I$ converges. The identification of $G_1$ and $G_2$ with the infrared regular and infrared singular parts, respectively, is used below to show that the renormalization process in the two-loop sector does not violate the considered symmetries.

### III. General Features of the Renormalization of Two-Loop Integrals

We give a brief description of the general renormalization procedure for two-loop integrals before presenting details of the IR renormalization. The discussion follows Ref. [31].

At the two-loop level integrals not only contain overall UV divergences, but can also contain subdivergences for the case where one integration momentum is fixed while the other one goes to infinity. As an example consider the two-loop diagram of Fig. 1 (a). It contains one-loop subdiagrams, shown in Fig. 1 (b). The renormalization of subdiagrams requires vertices as shown in Fig. 1 (c), which are of order $\hbar^3$. At order $\hbar^2$ these vertices appear in so-called counterterm diagrams as the one shown in Fig. 1 (d). When the sum of the original diagram and the one-loop counterterm diagrams, Fig. 1 (a) and twice the contribution from Fig. 1 (d), respectively, is considered, the remaining divergence is local and can be absorbed by counterterms. In order to renormalize a two-loop diagram one has to take into account all corresponding one-loop counterterm diagrams.

In our calculation we encounter two general types of two-loop integrals. The first type can be directly written as the product of two one-loop integrals, while this decomposition is not possible for the second type.

### IV. Infrared Renormalization of Products of One-Loop Integrals

Consider the product of two one-loop integrals,

$$H = H_a H_b.$$  \hspace{1cm} (15)

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Footnote 3: Here, the power of $\hbar$ denotes the order in the loop expansion.
$H$ is a two-loop integral and the result of a dimensional counting analysis reads (see App. A2)

$$H = H^{(0,0)} + H^{(1,0)} + H^{(0,1)} + H^{(1,1)}$$

$$= F_1 + F_2 + F_3 + F_4,$$  \(16\)

where $F_1$, $F_2 + F_3$, and $F_4$ satisfy the Ward identities separately due to different analytic structures, i.e. different overall powers of $M$ in $n$ dimensions. Using Eq. (3), $H$ can also be expressed as

$$H = I_a I_b + I_a R_b + R_a I_b + R_a R_b,$$  \(17\)

where again $I_a I_b$, $I_a R_b + R_a I_b$, and $R_a R_b$ satisfy the Ward identities individually.

To renormalize the integral $H$ we need to add the contributions of (renormalized) counterterm integrals. The vertex used in the counterterm integral is determined by standard IR renormalization of a one-loop subintegral. In a one-loop calculation we do not have to consider terms proportional to $\epsilon$ for the subtraction terms since, at the end of the calculation, the limit $\epsilon \to 0$ is taken. At the two-loop level, however, the subtraction terms are multiplied with terms proportional to $\epsilon^{-1}$ from the second loop integration. Therefore the choice whether or not to include the terms proportional to $\epsilon$ in one-loop subtraction terms results in different finite contributions in the two-loop integrals. In addition to the UV divergences and the terms proportional to $\epsilon^0$ we choose the subtraction terms for one-loop integrals to contain all positive powers of $\epsilon$, \(4\)

$$\tilde{R} = \frac{H^{UV}}{\epsilon} + \tilde{R}^{(0)} + \epsilon \tilde{R}^{(1)} + \cdots.$$  \(18\)

This choice is crucial for the preservation of the relevant symmetries as is discussed in the following. $H$ contains two subintegrals, $H_a$ and $H_b$. The expressions for the unrenormalized counterterm integrals then read

$$- \tilde{R}_a H_b - \tilde{R}_b H_a,$$  \(19\)

which themselves need to be renormalized applying IR renormalization. The $H_i$ ($i = a, b$) are one-loop integrals from which we would subtract the term $\tilde{R}_i$ in a one-loop calculation, excluding the additional divergences. However, the term $\tilde{R}_j$ multiplying $H_i$ contains terms with positive powers of $\epsilon$, so that in the product of $\tilde{R}_j$ and $R_i$ we get finite terms from the additional divergences in $R_i$ (see Eq. (7)). These would not be removed if we chose the subtraction term to be $\tilde{R}_j R_i$. Instead we define the subtraction term for the product $\tilde{R}_j H_i$ to be

$$- \tilde{R}_j R_i + \frac{H^{UV} R^{\text{add}}_i}{\epsilon^2} + \tilde{R}_j^0 R^{\text{add}}_i,$$  \(20\)

i.e. we subtract all finite terms stemming from the additional divergences in $R_i$ but do not subtract the additional divergences themselves. This is analogous to the one-loop sector, where we do not subtract the additional divergences in the infrared regular part either (see Eqs. (8) and (10)).

We now show that this renormalization procedure for the counterterm integrals does not violate the Ward identities. We know that the subtraction terms $S$ for one-loop integrals do not violate the Ward identities and result in a modification of the coupling constants and

\(4\) In a calculation at two-loop order it is sufficient to include terms proportional to $\epsilon$. Higher powers of $\epsilon$ are required for the generalization to multi-loop diagrams.
fields in the Lagrangian. The counterterm integrals are then calculated with the help of this new Lagrangian which means that the term
\[- S H_i \] (21)
also respects all Ward identities. \( H_i \) is a one-loop integral and Eq. (21) can be written as
\[- S I_i - S R_i \] (22)
where \(- S I_i \) and \(- S R_i \) satisfy the Ward identities separately. In particular, the Ward identities are satisfied term by term in an expansion in \( \epsilon \) for \( S I_i \) and \( S R_i \), respectively. The expansion for \( S I_i \) is given by
\[
S I_i = \left( \frac{S^{\text{div}}}{\epsilon} + S^{\text{fin}} \right) \left( \frac{I^{\text{add}}_i}{\epsilon} + I^{\text{fin}}_i \right) = \frac{S^{\text{div}} I^{\text{add}}_i}{\epsilon^2} + \frac{1}{\epsilon} \left[ S^{\text{div}} I^{\text{fin}}_i + S^{\text{fin}} I^{\text{add}}_i \right] + \ldots . \quad (23)
\]
Suppose we choose the finite part of the counterterm to vanish,\(^5\)
\[ S^{\text{fin}} = 0. \]
In this case we can see that the term proportional to \( \epsilon^{-1} \) is given by
\[
\frac{1}{\epsilon} S^{\text{div}} I^{\text{fin}}_i . \quad (24)
\]
It has to satisfy the Ward identities since for this choice of \( S \) it is the only term proportional to \( \epsilon^{-1} \) in the \( \epsilon \) expansion of \( S I_i \). By changing the renormalization scheme to also include finite terms in the subtraction terms, the product in Eq. (24) does not change, but we obtain the more general expression of Eq. (23). Considering the term proportional to \( \epsilon^{-1} \) and keeping in mind that Eq. (24) respects the Ward identities we now see that
\[
\frac{1}{\epsilon} S^{\text{fin}} I^{\text{add}}_i \quad (25)
\]
satisfies the Ward identities separately. Since the additional divergences have to cancel in the sum of \( I_i \) and \( R_i \) it follows that \( I^{\text{add}}_i = - R^{\text{add}}_i \) and
\[
- \frac{1}{\epsilon} S^{\text{fin}} R^{\text{add}}_i \quad (26)
\]
does not violate any symmetry constraints. Using the fact that \( S R_i \) respects all symmetries and choosing the subtraction term \( S \) to be \( \tilde{R}_j \) (which only contains UV divergences),
\[ S = \tilde{R}_j, \quad S^{\text{div}} = H_j^{\text{UV}}, \quad S^{\text{fin}} = \tilde{R}_j^{(0)}, \]
it follows that
\[
- \frac{H_j^{\text{UV}} R^{\text{add}}_i}{\epsilon^2} - \tilde{R}_j^{(0)} R^{\text{add}}_i / \epsilon \quad (27)
\]
\(^5\) In baryonic ChPT this would result in terms violating the power counting. So far we are only concerned with the symmetries of the theory, which are conserved for \( S^{\text{fin}} = 0 \). The issue of power counting is addressed below.
satisfies the Ward identities and therefore also our prescription for the subtraction terms of the counterterm diagrams of Eq. (20) satisfies the Ward identities.

Using the above method the sum of the original expression and the renormalized counterterm integrals gives

\[ H_a H_b - \tilde{R}_a H_b + \tilde{R}_a R_b - \frac{H_{a}^{UV} R_{a}^{\text{add}}}{\epsilon^2} - \frac{R_{a}^{(0)} R_{a}^{\text{add}}}{\epsilon} - \frac{\tilde{R}_b H_a + \tilde{R}_b R_a - \frac{R_{a}^{\text{add}} H_{b}^{UV}}{\epsilon^2}}{\epsilon} \]

\[ = I_a I_b + I_a R_b + I_b R_a - \tilde{R}_a I_b - \frac{H_{a}^{UV} R_{b}^{\text{add}}}{\epsilon^2} - \frac{R_{a}^{(0)} R_{b}^{\text{add}}}{\epsilon} - \tilde{R}_b I_a \]

\[ = I_a I_b + I_a (R_b - \tilde{R}_b) + I_b (R_a - \tilde{R}_a) - \frac{H_{a}^{UV} R_{b}^{\text{add}}}{\epsilon^2} - \frac{R_{a}^{(0)} R_{b}^{\text{add}}}{\epsilon} - \tilde{R}_a \]

\[ - \frac{R_{b}^{(0)} R_{a}^{\text{add}}}{\epsilon} + R_a R_b. \]  

The difference between \( R_i \) and \( \tilde{R}_i \) is only given by the additional divergences \( R_i^{\text{add}} / \epsilon \), resulting in

\[ \left( \tilde{I}_a + \frac{I_a^{\text{add}}}{\epsilon} \right) \left( \tilde{I}_b + \frac{I_b^{\text{add}}}{\epsilon} \right) + \left( \tilde{I}_a + \frac{I_a^{\text{add}}}{\epsilon} \right) \frac{R_{b}^{\text{add}}}{\epsilon} + \left( \tilde{I}_b + \frac{I_b^{\text{add}}}{\epsilon} \right) \frac{R_{a}^{\text{add}}}{\epsilon} \]

\[ - \frac{H_{a}^{UV} R_{b}^{\text{add}}}{{\epsilon^2}} - \frac{R_{a}^{(0)} R_{b}^{\text{add}}}{\epsilon} - \tilde{R}_a \frac{R_{b}^{\text{add}}}{\epsilon} - \frac{R_{b}^{(0)} R_{a}^{\text{add}}}{\epsilon} + R_a R_b. \]  

Using \( I_i^{\text{add}} = - R_i^{\text{add}} \) we obtain

\[ \tilde{I}_a \tilde{I}_b - \frac{I_a^{\text{add}} I_b^{\text{add}}}{\epsilon^2} - \frac{H_{a}^{UV} R_{b}^{\text{add}}}{\epsilon^2} - \frac{R_{a}^{(0)} R_{b}^{\text{add}}}{\epsilon} - \tilde{R}_a \frac{R_{b}^{\text{add}}}{\epsilon} + R_a R_b + O(\epsilon). \]

Expanding \( R_a R_b \) in \( \epsilon \) and simplifying the resulting expression gives

\[ \tilde{I}_a \tilde{I}_b - \frac{I_a^{\text{add}} I_b^{\text{add}}}{\epsilon^2} + \frac{R_{a}^{\text{add}} R_{b}^{\text{add}}}{\epsilon^2} + \frac{H_{a}^{UV} H_{b}^{UV}}{\epsilon^2} + \frac{H_{a}^{UV} \tilde{R}_{b}^{(0)} + \tilde{R}_{a}^{(0)} H_{b}^{UV}}{\epsilon} + (R_a R_b)^{(0)} + O(\epsilon), \]

where \((R_a R_b)^{(0)}\) stands for the terms proportional to \( \epsilon^0 \) in the product \( R_a R_b \). Using again \( I_i^{\text{add}} = - R_i^{\text{add}} \) we see that all terms containing the additional divergences vanish,

\[ \tilde{I}_a \tilde{I}_b + \frac{H_{a}^{UV} H_{b}^{UV}}{\epsilon^2} + \frac{H_{a}^{UV} \tilde{R}_{b}^{(0)} + \tilde{R}_{a}^{(0)} H_{b}^{UV}}{\epsilon} + (R_a R_b)^{(0)} + O(\epsilon). \]

The term \( R_a R_b \) satisfies the Ward identities, in particular each term in the \( \epsilon \) expansion of \( R_a R_b \) does so individually. This means that we can subtract the finite part of \( R_a R_b \) by a counterterm. The terms proportional to \( \epsilon^{-2} \) and \( \epsilon^{-1} \) stem from the UV divergences in \( H_a \) and \( H_b \). These terms also satisfy the Ward identities individually and are absorbed in counterterms. As desired, the renormalized result for the product of two one-loop integrals including the counterterm integrals is then simply the product of the renormalized one-loop integrals,

\[ (H_a H_b) = \tilde{I}_a \tilde{I}_b. \]
Besides respecting all symmetries the renormalization prescription must also result in a proper power counting for renormalized integrals. The chiral order of a product of two integrals is the sum of the individual orders. For a one-loop integral the infrared singular part $\tilde{I}$ satisfies the power counting. Therefore the result of Eq. (33) also satisfies power counting.

V. INFRARED RENORMALIZATION OF TWO-LOOP INTEGRALS RELEVANT TO THE NUCLEON MASS CALCULATION

In this section we describe the renormalization procedure for two-loop integrals that do not directly factorize into the product of two one-loop integrals. We follow the general method presented in Ref. [17], but give more details. First we show how the proper renormalization of two-loop integrals and of the corresponding counterterm integrals preserves the underlying symmetries. Next we describe a simplified formalism to arrive at the same results while greatly reducing the calculational difficulties.

A. General method

Denote a general two-loop integral contributing to the nucleon mass by $H_2$,

$$H_2(a, b, c, d, e | n) = \int \int \frac{d^n k_1 d^n k_2}{(2\pi)^{2n}} \frac{1}{A^a B^b C^c D^d E^e},$$

where

$$A = k_1^2 - M^2 + i0^+,$$
$$B = k_2^2 - M^2 + i0^+,$$
$$C = k_1^2 + 2p \cdot k_1 + i0^+,$$
$$D = k_2^2 + 2p \cdot k_2 + i0^+,$$
$$E = k_1^2 + 2p \cdot k_1 + 2k_1 \cdot k_2 + 2p \cdot k_2 + k_2^2 + i0^+. \quad (35)$$

Using a dimensional counting analysis we can write $H_2(a, b, c, d, e | n)$ as

$$H_2 = F_1 + F_2 + F_3 + F_4. \quad (36)$$

$F_1$ is obtained by simply expanding the integrand in $M$ and interchanging summation and integration. For $F_2$ we rescale the first loop momentum $k_1$ by

$$k_1 \mapsto \frac{M}{m} k_1,$$

expand the resulting integrand in $M$, and interchange summation and integration. $F_3$ is obtained analogously to $F_2$, only that instead of $k_1$ the second loop momentum $k_2$ is rescaled,

$$k_2 \mapsto \frac{M}{m} k_2. \quad (38)$$

---

6 For brevity we employ the notation $H_2$ for $H_2(a, b, c, d, e | n)$ in the following discussion.
Finally $F_4$ is defined as the result from simultaneously rescaling both loop momenta,

$$k_1 \mapsto \frac{M}{m} k_1, \quad k_2 \mapsto \frac{M}{m} k_2,$$

and expanding the integrand with subsequent interchange of summation and integration. $F_1$, $F_2 + F_3$, and $F_4$ separately satisfy the Ward identities due to different overall factors of $M$. This is analogous to the one-loop sector, where the infrared singular and infrared regular parts separately satisfy the Ward identities, since the infrared singular part is nonanalytic in small quantities for noninteger $n$, while the infrared regular term is analytic. As in the one-loop case the interchange of summation and integration generates additional divergences not present in $H_2$ in each of the terms $F_1$, $F_2 + F_3$, and $F_4$. Again, these additional divergences cancel in the sum of all terms.

In addition to the two-loop integral we also need to determine the corresponding subintegals. To identify the first subintegral we consider the $k_1$ integration in $H_{\text{sub}}$, $H_{\text{sub}} = \int \frac{d^n k_1}{(2\pi)^n} \frac{1}{A^a C^b E^d}$. This is a one-loop integral which is renormalized using “standard” infrared renormalization. The infrared regular part $R_{\text{sub}}$ of this integral is obtained by expanding the integrand in $M$ and interchanging summation and integration. The only term in Eq. (40) depending on $M$ is $A$. Symbolically we write

$$R_{\text{sub}} = \sum \int \frac{d^n k_1}{(2\pi)^n} \frac{1}{A^a C^b E^d},$$

where underlined expressions are understood as an expansion in $M$. $R_{\text{sub}}$ contains additional divergences, and we define $\tilde{R}_{\text{sub}}$ as $R_{\text{sub}}$ without these divergences,

$$\tilde{R}_{\text{sub}} = R_{\text{sub}} - \frac{R_{\text{add sub}}}{\epsilon}.$$

As in the definition of Eq. (35), $\tilde{R}_{\text{sub}}$ again contains all terms of positive powers of $\epsilon$. Since $H_{\text{sub}}$ is a standard one-loop integral, $\tilde{R}_{\text{sub}}$ will satisfy the Ward identities and can be absorbed in counterterms of the Lagrangian.

Using these counterterms as a vertex we obtain a counterterm integral of the form

$$H_{\text{CT}} = -\int \frac{d^n k_2}{(2\pi)^n} \tilde{R}_{\text{sub}} \frac{1}{B^b D^d}.$$

$H_{\text{CT}}$ is generated by a Lagrangian that is consistent with the considered symmetries. Therefore, $H_{\text{CT}}$ satisfies the Ward identities. Inserting Eqs. (41) and (42) we rewrite $H_{\text{CT}}$ as

$$H_{\text{CT}} = -\int \frac{d^n k_2}{(2\pi)^n} \sum \int \frac{d^n k_1}{(2\pi)^n} \frac{1}{A^a B^b C^c D^d E^e} + \int \frac{d^n k_2}{(2\pi)^n} \frac{R_{\text{add sub}}}{\epsilon} \frac{1}{B^b D^d}.$$

Equation (44) still needs to be renormalized. After the $k_1$ integration has been performed, Eq. (44) is a one-loop integral and standard infrared renormalization can be used. To

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7 Note that for the integrals of interest here, the UV divergence is included in the infrared regular part $R$. 

11
obtain the infrared singular part $I_{CT1}$, we rescale $k_2 \mapsto \frac{M}{m} k_2$, expand in $M$, and interchange summation and integration. Symbolically we write

$$I_{CT1} = -\sum \int \frac{d^n k_2}{(2\pi)^n} \sum \int \frac{d^n k_1}{(2\pi)^n} \frac{1}{\Lambda^a B^b C^c D^d E^e} + \sum \int \frac{d^n k_2}{(2\pi)^n} \epsilon^{-1} R_{add}^{sub_1} \frac{1}{\frac{B^b D^d}{E^e}},$$

(45)

where double-underlined quantities are first rescaled and then expanded. Note that $R_{add_1}^{sub}$ can also depend on $k_2$ through the denominator $E$ in Eq. (40). Since $I_{CT1}$ is obtained from a one-loop integral that satisfies the Ward identities through the standard infrared renormalization process, it will itself satisfy the Ward identities. The infrared renormalized expression for the counterterm integral,

$$\tilde{I}_{CT1} = I_{CT1} - \frac{I_{CT1}^{add_1}}{\epsilon},$$

(46)

then also satisfies the Ward identities. Note that $I_{CT1}^{add_1}$ itself contains terms proportional to $\epsilon^{-1}$, since it stems from the one-loop counterterm for the subintegral, but we choose not to include any terms proportional to positive powers of $\epsilon$. This means that $\epsilon^{-1} I_{CT1}^{add_1}$ only contains terms proportional to $\epsilon^{-2}$ and $\epsilon^{-1}$. The expression for $\tilde{I}_{CT1}$ therefore does not contain any divergent terms stemming from additional divergences.

We now show how $\tilde{I}_{CT1}$ is related to the term $F_3$ of Eq. (36). As explained above, $F_3$ is obtained by rescaling $k_2$, expanding the resulting integrand and interchanging summation and integration. In the above notation this would correspond to

$$F_3 = \sum \int \int \frac{d^n k_1 d^n k_2}{(2\pi)^{2n}} \frac{1}{\Lambda^a B^b C^c D^d E^e}. $$

(47)

Comparing with the first term in Eq. (45) we see that the integrands in both cases are expanded in the same way. Therefore, when adding the counterterm diagram $\tilde{I}_{CT1}$ to $H_2$ it cancels parts of $F_3$. The difference between $\tilde{I}_{CT1}$ and $F_3$ is that in $F_3$ the terms stemming from the additional divergences $R_{add_1}^{sub_1}$ (including finite terms) as well as the additional divergences $I_{CT1}^{add_1}/\epsilon$ that are proportional to $\epsilon^{-2}$ and $\epsilon^{-1}$ are not subtracted. As pointed out above, the original integral $H_2$ only contains UV divergences, therefore the additional divergences cancel in the sum $F_1 + F_2 + F_3 + F_4$. Apart from these contributions, the terms remaining in the sum $\tilde{I}_{CT1} + F_3$ are the finite contributions stemming from the additional divergences in $R_{sub_1}$. Since in $F_3$ the variable $k_2$ is rescaled before expanding while the $k_1$ variable remains unchanged, $F_3$ can be considered as a sum of products of infrared singular and infrared regular terms, which we symbolically write as

$$F_3 = \sum R_1 I_2.$$

(48)

In this notation the remaining finite terms are $\sum R_1^{add} I_2^{(1)}$, where $\epsilon^{-1} R_1^{add}$ is the additional divergence of $R_1$ and $I_2^{(1)}$ is the part of $I_2$ proportional to $\epsilon$.

The second subdiagram can be calculated analogously, and is related to the term $F_2$ in Eq. (36).
Taking the above considerations into account we obtain for the sum of the original integral $H_2$ and the corresponding counterterm integrals

\[
H_2 + \tilde{I}_{CT1} + \tilde{I}_{CT2} = F_1 + F_2 + F_3 + F_4 + \tilde{I}_{CT1} + \tilde{I}_{CT2} \\
= \tilde{F}_1 + \tilde{F}_4 + \sum R_{1}^{add} I_2^{(1)} + \sum R_{2}^{add} I_1^{(1)} \\
= \tilde{F}_1 + \tilde{F}_4 - \sum I_1^{add} I_2^{(1)} - \sum I_2^{add} I_1^{(1)},
\]

where $\tilde{F}_1$ indicates that the additional divergences are excluded.

The expression in Eq. (49) satisfies the Ward identities since each term in the sum on the left side of the first line does so individually. $F_1$ separately satisfies the Ward identities, in particular this is the case for each term in its $\epsilon$ expansion. This means that we can subtract the finite part of $\tilde{F}_1$ by an overall counterterm without violating the symmetries. Since the remaining UV divergences also satisfy the Ward identities, absorbing them in an overall counterterm does not violate the symmetries. The result for the renormalized two-loop diagram is then

\[
H_r^2 = \tilde{F}_4 - \sum I_1^{add} I_2^{(1)} - \sum I_2^{add} I_1^{(1)}.
\]

Since all subtractions preserve the symmetries $H_r^2$ will satisfy the Ward identities.

So far we have subtracted pole parts in the epsilon expansion. Following [11] we choose to absorb the combination

\[
\frac{1}{(4\pi)^2} \left[ \frac{1}{n-4} - \frac{1}{2} (\log(4\pi) - \gamma_E + 1) \right]
\]

instead, which is achieved by simply replacing the original t’Hooft parameter $\tilde{\mu}$ by

\[
\tilde{\mu} \rightarrow \frac{\mu}{(4\pi)^{1/2}} e^{\frac{\gamma_E - 1}{2}}
\]

(see also App. [D]).

$F_4$ is obtained by rescaling both $k_1$ and $k_2$ and satisfies the power counting. Since the terms $I_i$ result from the rescaling of $k_i$, the product $I_1 I_2$ has the same analytic structure in $M$ as $F_4$, and therefore satisfies the power counting. This means that also the renormalized integral $H_r^2 = \tilde{F}_4 - \sum I_1^{add} I_2^{(1)} - \sum I_2^{add} I_1^{(1)}$ obeys the power counting.

\[\[\]

B. Simplified method

In the previous subsection we have established the concept of infrared renormalization of two-loop integrals. The procedure outlined above is quite involved when applied to actual calculations of physical processes. Therefore, we now describe a simpler method of obtaining the renormalized expression $H_r^2$ which, however, is only applicable to integrals with a single small scale. This is the case for the calculation of the nucleon mass, whereas e.g. the nucleon form factors contain the momentum transfer as an additional small quantity.

Instead of calculating the subintegrals of the original integral $H_2$, consider just the terms in $F_4$. $F_4$ itself is a sum of two-loop integrals. Each two-loop integral contains one-loop subintegrals, which correspond to performing only one loop integration while keeping the other one fixed. These subintegrals contain divergences, resulting in divergent as well as
finite contributions when the second loop integration is performed. In addition to the subintegral contributions, $F_4$ contains finite parts and additional divergences originating in the interchange of summation and integration when generating $F_4$. We can symbolically write $F_4$ as

$$F_4 = \bar{F}_4 + \frac{\bar{F}_{4,\text{add},2}}{\epsilon^2} + \frac{\bar{F}_{4,\text{add},1}}{\epsilon} + \frac{F_{4,\text{Sub},1,\text{div}}}{\epsilon} F_4^{k_1} + \frac{F_{4,\text{Sub},2,\text{div}}}{\epsilon} F_4^{k_2} + \frac{F_{4,\text{Sub},3,\text{div}}}{\epsilon} F_4^{k_3}. \quad (53)$$

Here, the finite parts of $F_4$ are denoted by $\bar{F}_4$ to distinguish them from $\bar{F}_4$ in Eq. (49). The bar notation is also used for the divergent terms $\bar{F}_{4,\text{add},2}$ and $\bar{F}_{4,\text{add},1}$ to show that these are not the complete divergent expressions for $F_4$, but only the additional divergences of order $\epsilon^{-2}$ and $\epsilon^{-1}$, respectively. The terms $\epsilon^{-1} F_{4,\text{Sub},1,\text{div}}$ denote the divergences of the subintegral with respect to the integration over $k_1$, while $F_4^{k_i}$ stands for the remaining second integration of the counterterm integral. Note that the divergent part of the first loop integration over $k_i$ in general depends on the second loop momentum $k_j$. This dependence is included in the expression $F_4^{k_j}$.

We now show how the different parts in Eq. (53) are related to expressions in $F_2$ and $F_3$ and then describe the simplified renormalization method. $F_4$ is obtained from the original integral $H$ by rescaling $k_1$ and $k_2$, expanding the resulting integrand in $M$ and interchanging summation and integration. For the denominators of Eq. (33) the rescaling results in

$$k_1^2 - M^2 + i0^+ \mapsto \left(\frac{M}{m}\right)^2 (k_1^2 - m^2 + i0^+),$$

$$k_2^2 - M^2 + i0^+ \mapsto \left(\frac{M}{m}\right)^2 (k_2^2 - m^2 + i0^+),$$

$$k_1^2 + 2p \cdot k_1 + i0^+ \mapsto \left(\frac{M}{m}\right) \left(\frac{M}{m} k_1^2 + 2p \cdot k_1 + i0^+\right),$$

$$k_2^2 + 2p \cdot k_2 + i0^+ \mapsto \left(\frac{M}{m}\right) \left(\frac{M}{m} k_2^2 + 2p \cdot k_2 + i0^+\right),$$

$$k_1^2 + 2p \cdot k_1 + 2k_1 \cdot k_2 + 2p \cdot k_2 + k_2^2 + i0^+ \mapsto \left(\frac{M}{m}\right) \left(\frac{M}{m} k_1^2 + 2p \cdot k_1 + 2k_1 \cdot k_2 + 2p \cdot k_2 + \frac{M}{m} k_2^2 + i0^+\right).$$

After the interchange of summation and integration one can perform the substitution $k_i \mapsto \frac{m}{M} k_i$ to bring the denominators $k_1^2 - m^2 + i0^+$ and $k_2^2 - m^2 + i0^+$ back into the form $A$ and $B$, respectively. The result can be interpreted as obtained from the original integral by leaving $A$ and $B$ unchanged and expanding $C$ in $k_1^2$, $D$ in $k_2^2$, and $E$ in $k_1^2 + 2k_1 \cdot k_2 + k_2^2$, respectively. Symbolically

$$F_4 \sim \sum \int \int \frac{d^n k_1 d^n k_2}{(2\pi)^{2n}} \frac{1}{[k_1^2 - M^2 + i0^+]^{a/2} [k_2^2 - M^2 + i0^+]^{b/2} [k_1^2 + 2p \cdot k_1 + i0^+]^{c/2}} \times \frac{1}{[k_2^2 + 2p \cdot k_2 + i0^+]^{d/2} [k_1^2 + 2p \cdot k_1 + 2k_1 \cdot k_2 + 2p \cdot k_2 + k_2^2 + i0^+]^{e/2}}, \quad (54)$$

where we have used the underlined notation to mark terms that we have expanded in.

The divergent parts of the $k_1$ subintegral stem from the integration region $k_1 \to \infty$. They can be generated by further expanding each term in $F_4$ in inverse powers of $k_1$. This
corresponds to an expansion in positive powers of $M$ for the first denominator and in positive powers of $2p \cdot k_2$ in the resulting last propagator,

$$
\frac{F_{4,\text{Sub},\text{div}}^{k_2}}{\epsilon} F_4^{k_2} \sim \sum \int \int \frac{d^n k_2 d^n k_1}{(2\pi)^{2n}} \frac{1}{[k_1^2 - M^2 + i0^+]^a[k_2^2 - M^2 + i0^+]^b} \times \frac{1}{[k_1^2 + 2p \cdot k_1 + i0^+]^c[k_2^2 + 2p \cdot k_2 + i0^+]^d} \times \frac{1}{[k_1^2 + 2p \cdot k_1 + 2k_1 \cdot k_2 + 2p \cdot k_2 + k_2^2 + i0^+]^e}. \tag{55}
$$

We see that the expression for $F_4^{k_2}$ is of the form

$$
F_4^{k_2} \sim \sum \int \frac{d^n k_2}{(2\pi)^n} \frac{f_{\mu\nu\lambda\ldots} k_2^\mu k_2^\nu k_2^\lambda \ldots}{k_2^2 - M^2 + i0^+]^b[2p \cdot k_2 + i0^+]^{d+i_1}}, \tag{56}
$$

where $f_{\mu\nu\lambda\ldots}$ denotes the coefficients that result from the expansion in Eq. (55).

Next we show that $F_4^{k_2}$ is related to terms in $F_3$. $F_3$ is generated from the original integral $H_2$ by rescaling $k_2$, expanding the resulting integrand, and interchanging summation and integration. After the substitution $k_2 \mapsto \frac{m}{M} k_2$ and using the above notation we write

$$
F_3 \sim \sum \int \int \frac{d^n k_1 d^n k_2}{(2\pi)^{2n}} \frac{1}{[k_1^2 - M^2 + i0^+]^a[k_2^2 - M^2 + i0^+]^b[k_2^2 + 2p \cdot k_1 + i0^+]^c} \times \frac{1}{[k_2^2 + 2p \cdot k_2 + i0^+]^d} \times \frac{1}{[k_1^2 + 2p \cdot k_1 + 2k_1 \cdot k_2 + 2p \cdot k_2 + k_2^2 + i0^+]^e} \times \frac{1}{[2p \cdot k_2 + i0^+]^{d+j_1}} \times \frac{1}{[k_2^2 - M^2 + i0^+]^b \times [k_2^2 + 2p \cdot k_1 + i0^+]^{1+c+j_2} [k_2^2 + 2p \cdot k_1 + i0^+]^{1+c+j_2}}. \tag{57}
$$

We see that $F_3$ is the sum of products of one-loop (tensorial) integrals. As explained above these products of one-loop integrals are in fact products of infrared singular and infrared regular parts of integrals (see Eq. (48)),

$$
F_3 = \sum R_1 I_2,
$$

and the expressions for $I_2$ are given by

$$
I_2 \sim \sum \int \frac{d^n k_2}{(2\pi)^n} \frac{k_2^\alpha k_2^\beta k_2^\gamma \ldots}{k_2^2 - M^2 + i0^+]^b[2p \cdot k_2 + i0^+]^{d+i_2}}. \tag{58}
$$

Considering the $k_2$ integrals of Eqs. (55) and (57) one sees that one has expanded in the same quantities. While the ordering of the expansions as well as the interchanges of summation and integration are different, the two expansions are equivalent. Therefore, comparing Eqs. (56) and (58), one finds that for each term in $F_4^{k_2}$ there is a corresponding term in $I_2$, or symbolically

$$
F_4^{k_2} = I_2. \tag{59}
$$
An analogous analysis for the second subintegral gives

\[ F_{4}^{k_1} = I_1. \] (60)

As a next step we show that the divergences of the \( F_4 \) subintegrals are related to the additional divergences of the integrals \( R_i \) in \( F_2 \) and \( F_3 \). From Eq. (55) we see that the divergent part of the \( k_1 \) subintegral is given by integrals of the type

\[
F_{\text{Sub}}^{1} \sim \sum \int \frac{d^n k_1}{(2\pi)^n} \frac{1}{[k_1^2 - M^2 + i0^+]^{a[k_1^2 + 2p \cdot k_1 + i0^+]^{c}}}
\times \frac{1}{[k_1^2 + 2p \cdot k_1 + 2k_1 \cdot k_2 + 2p \cdot k_2 + k_2^2 + i0^+]^{e}]
\sim \sum \int \frac{d^n k_1}{(2\pi)^n} \frac{k_1^\mu k_1^\nu \cdots}{[k_1^2 - M^2 + i0^+]^{a[2p \cdot k_1 + i0^+]^{c+e+l_2}}}. \] (61)

The infrared regular integrals \( R_1 \) in Eq. (57) read

\[
R_1 \sim \sum \int \frac{d^n k_1}{(2\pi)^n} \frac{k_1^\mu k_1^\nu \cdots}{[k_1^2 + i0^+]^{a+m_1[k_1^2 + 2p \cdot k_1 + i0^+]^{c+e+m_2}}}. \] (62)

\( F_{\text{Sub}}^{1} \) and \( R_1 \) can be interpreted as the infrared singular and infrared regular parts of the auxiliary integrals

\[
h \sim \sum \int \frac{d^n k_1}{(2\pi)^n} \frac{k_1^\mu k_1^\nu \cdots}{[k_1^2 - M^2 + i0^+]^{a[k_1^2 + 2p \cdot k_1 + i0^+]^{\beta}}}, \] (63)

respectively. Since \( h \) is a “standard” one-loop integral that is only UV divergent, the additional divergences in its IR regular part \( R_1 \) must cancel exactly with the divergences in its IR singular part \( F_{\text{Sub}}^{1,\text{div}} \). Therefore,

\[
\frac{F_{\text{Sub}}^{1,\text{div}}}{\epsilon} = -\frac{R_1^{\text{add}}}{\epsilon}, \] (64)

and, using \( R_1^{\text{add}} = -I_1^{\text{add}} \), it also follows that

\[
\frac{F_{\text{Sub}}^{1,\text{div}}}{\epsilon} = \frac{I_1^{\text{add}}}{\epsilon}. \] (65)

Analogously

\[
\frac{F_{\text{Sub}}^{2,\text{div}}}{\epsilon} = -\frac{R_2^{\text{add}}}{\epsilon} = \frac{I_2^{\text{add}}}{\epsilon}. \] (66)

Having established the relationship between the terms in \( F_4 \) and the terms in \( F_2 \) and \( F_3 \) we now describe the renormalization procedure. Our method consists of treating each two-loop integral contributing to \( F_4 \) as an independent integral. We then renormalize each two-loop integral in the \( \bar{\text{MS}} \) scheme, i.e. we

- determine the divergences in the subintegrals,
- use the divergences as vertices in one-loop counterterm integrals that are added to \( F_4 \),
- perform an additional overall subtraction by absorbing all remaining divergences in counterterms,

- replace \( \tilde{\mu} = \frac{\mu}{(4\pi)^{1/2}} e^{\frac{\pi E - 1}{2}} \) and set \( \mu = m \).

The divergences in the subintegrals are given by \( \epsilon^{-1} F_{4,\text{div}} \). The one-loop counterterm integrals using these divergences read

\[- \frac{F_{4,\text{div}}}{\epsilon} F_{4,k_1}^1 - \frac{F_{4,\text{div}}}{\epsilon} F_{4,k_2}^2. \tag{67}\]

According to Eqs. (59), (60), (65), and (66) this can be written as

\[- \frac{I_{2,\text{add}}}{\epsilon} I_1 - \frac{I_{1,\text{add}}}{\epsilon} I_2. \tag{68}\]

When added to \( F_4 \) we obtain

\[ F_4 - \frac{I_{2,\text{add}}}{\epsilon} I_1 - \frac{I_{1,\text{add}}}{\epsilon} I_2. \tag{69}\]

Using the notation of Subsec. [V.A] we write \( F_4 \) as the sum of the additional divergences and a remainder \( \tilde{F}_4 \),

\[ F_4 = \frac{F_{4,\text{add},2}}{\epsilon^2} + \frac{F_{4,\text{add},1}}{\epsilon} + \tilde{F}_4. \tag{70}\]

Note that the divergent terms \( F_{4,\text{add},i} \) are not the divergent expressions \( \tilde{F}_{4,\text{add},i} \) of Eq. (53).

Performing the \( \epsilon \) expansion for the integrals \( I_i \),

\[ I_i = \epsilon^{-1} I_{i,\text{add}} + I_i^{(0)} + \epsilon I_i^{(1)} + \cdots, \]

the sum of \( F_4 \) and the counterterm integrals is given by

\[ \frac{F_{4,\text{add},2}}{\epsilon^2} + \frac{F_{4,\text{add},1}}{\epsilon} - 2 \frac{I_{2,\text{add}}}{\epsilon^2} I_{1,\text{add}} - \frac{I_{1,\text{add}}}{\epsilon} I_2 - \frac{I_{2,\text{add}}}{\epsilon} I_{1,\text{add}} + \tilde{F}_4 - I_{1,\text{add}} I_{1,\text{add}} - I_{2,\text{add}} I_{1,\text{add}}. \tag{71}\]

We now show that the remaining divergences are analytic in \( M^2 \) and can therefore be absorbed by counterterms. Recall that the sum of all additional divergences has to vanish, since they are not present in the original integral,

\[ 0 = \frac{F_{1,\text{add},2}}{\epsilon^2} + \frac{F_{1,\text{add},1}}{\epsilon} + \frac{F_{2,\text{add},2}}{\epsilon^2} + \frac{F_{2,\text{add},1}}{\epsilon} + \frac{F_{3,\text{add},2}}{\epsilon^2} + \frac{F_{3,\text{add},1}}{\epsilon} + \frac{F_{4,\text{add},2}}{\epsilon^2} + \frac{F_{4,\text{add},1}}{\epsilon}. \tag{72}\]

As shown above \( F_2 \) and \( F_3 \) are the sums of products of one-loop integrals, so Eq. (72) can be rewritten as

\[ 0 = \frac{F_{1,\text{add},2}}{\epsilon^2} + \frac{F_{1,\text{add},1}}{\epsilon} + \frac{I_{1,\text{add}} R_{2,\text{add}}}{\epsilon^2} + \frac{I_{1,\text{add}} R_{1,\text{add}}}{\epsilon^2} + \frac{I_{2,\text{add}} R_{1,\text{add}}}{\epsilon^2} + \frac{I_{2,\text{add}} R_{1,\text{add}}}{\epsilon^2} + \frac{I_{2,\text{add}} R_{1,\text{add}}}{\epsilon}. \tag{73}\]
Making use of $I_i^{\text{add}} = -R_i^{\text{add}}$ the sum of all additional divergences takes the form

$$0 = \frac{F_1^{\text{add},2}}{\epsilon} + \frac{F_1^{\text{add},1}}{\epsilon} - \frac{F_1^{\text{add}}}{\epsilon} I_1^{(0)} - \frac{R_2^{\text{add}}}{\epsilon} I_2^{(0)} - \frac{R_1^{\text{add}}}{\epsilon} I_2^{(0)}$$

$$-2 \frac{I_1^{\text{add}} I_2^{\text{add}}}{\epsilon^2} - \frac{I_1^{\text{add}}}{\epsilon} I_2^{(0)} - \frac{I_2^{\text{add}}}{\epsilon} I_1^{(0)} - \frac{R_4^{\text{add},2}}{\epsilon^2} + \frac{F_4^{\text{add},1}}{\epsilon}.$$  \(\text{(74)}\)

All terms in $F_1$ for the two-loop integral as well as the infrared regular terms in one-loop integrals are analytic in $M^2$. Therefore the first line in Eq. \(\text{(74)}\) is analytic in $M^2$. Since the sum of all terms vanishes the second line also has to be analytic. This second line, however, comprises exactly the remaining divergences in Eq. \(\text{(71)}\), which are therefore analytic in $M^2$ and can be subtracted. After these divergences have been absorbed in counterterms, the resulting expression for the renormalized contribution of $F_4$ reads

$$F_4^r = \tilde{F}_4 - \sum I_1^{\text{add}} I_2^{(1)} - \sum I_2^{\text{add}} I_1^{(1)},$$  \(\text{(75)}\)

where we have explicitly shown the sums again. Comparing with Eq. \(\text{(50)}\) we see that our result exactly reproduces the expression for the renormalized original integral $H_2^r$.

\[C. \; \epsilon\text{-dependent factors}\]

For actual calculations it is often convenient to reduce tensorial integrals to scalar integrals before performing the dimensional counting analysis as well as the renormalization. The reduction of the tensorial integrals can result in $\epsilon$-dependent factors multiplying the scalar integrals. These change the form of the result of Eq. \(\text{(50)}\) since additional finite terms can appear. Let the $\epsilon$-dependent factor be given by

$$\phi(\epsilon) = \phi^{(0)} + \epsilon \phi^{(1)} + \epsilon^2 \phi^{(2)} + \cdots.$$  \(\text{(76)}\)

Consider performing the $k_1$ integration first. Suppose that from the result one can extract an $\epsilon$-dependent factor $\varphi_1(\epsilon)$, and the subsequently performed $k_2$ integration leads to another $\epsilon$-dependent factor, $\varphi_2(\epsilon)$, with

$$\phi(\epsilon) = \varphi_1(\epsilon) \cdot \varphi_2(\epsilon).$$  \(\text{(77)}\)

One can also perform the $k_2$ integration first, which leads to a different factor $\tilde{\varphi}_2(\epsilon)$, followed by the $k_1$ integration resulting in a factor $\tilde{\varphi}_1(\epsilon)$ with

$$\phi(\epsilon) = \tilde{\varphi}_2(\epsilon) \cdot \tilde{\varphi}_1(\epsilon).$$  \(\text{(78)}\)

The terms $\varphi_1(\epsilon) = \varphi^{(0)}_1 + \epsilon \varphi^{(1)}_1 + \epsilon^2 \varphi^{(2)}_1 + \cdots$ and $\tilde{\varphi}_2(\epsilon) = \tilde{\varphi}^{(0)}_2 + \epsilon \tilde{\varphi}^{(1)}_2 + \epsilon^2 \tilde{\varphi}^{(2)}_2 + \cdots$ can then directly be taken into account when determining the divergent contributions from subintegrals. The result $H_2^{r,\phi}$ for the renormalized integral $\phi(\epsilon) H_2$ reads

$$H_2^{r,\phi} = \tilde{F}_4^{\phi} - \varphi_1^{(0)} I_1^{\text{add}} \left( \tilde{\varphi}_2^{(2)} I_2^{\text{add}} + \tilde{\varphi}_2^{(1)} I_2^{(0)} + \varphi_2^{(0)} I_2^{(1)} \right) - \tilde{\varphi}_2^{(0)} I_2^{\text{add}} \left( \tilde{\varphi}_1^{(2)} I_1^{\text{add}} + \tilde{\varphi}_1^{(1)} I_1^{(0)} + \varphi_1^{(0)} I_1^{(1)} \right),$$  \(\text{(79)}\)

where $\tilde{F}_4^{\phi}$ denotes the finite terms in $\phi(\epsilon) F_4$, and $I_i^{(0)}$, $\varphi_1^{(0)}$ and $\tilde{\varphi}_2^{(0)}$ are the $\epsilon$-independent terms in $I_i$, $\varphi_1$ and $\tilde{\varphi}_2$, respectively. Our simplified method still holds provided the $\epsilon$-dependent factors are taken into account.
ASI 1 1 1 1

FIG. 2: Two-loop diagram contributing to the nucleon self-energy.

As an example consider the diagram of Fig. 2. Ignoring constant factors, one can show that in a calculation up to order $\mathcal{O}(q^6)$ the nucleon mass only receives contributions from

$$\gamma^\mu (\not{p} - m) \gamma^\alpha (\not{p} + m) \gamma^\nu (\not{p} - m) \gamma^\beta \int \frac{d^{n+2}k_1 d^{n+2}k_2}{(2\pi)^{2n+4}} \frac{g^{\alpha\beta} g^{\mu\nu}}{ABCDE}, \quad (80)$$

where the denominators are given in Eq. (35). One would also obtain the expression of Eq. (80) if one considered a diagram with fictitious particles as shown in Fig. 3 (a), with Feynman rules given by

$$k^\mu g^{\mu\nu} \frac{1}{k^2 - M^2 + i0^+}, \quad \frac{\not{p} - m}{k^2 + 2p \cdot k + i0^+}, \quad \frac{\not{p} + m}{k^2 + 2p \cdot k + i0^+}, \quad \gamma_\mu, \quad \gamma_\mu.$$

The subintegral corresponding to performing the $k_1$ integration first is shown in Fig. 3 (b). With the Feynman rules above it is proportional to

$$(n - 3) (4m^2 \gamma_\nu - 4mp_\nu), \quad (81)$$

so that we can identify $\varphi_1 (\epsilon) = n - 3 = 1 + 2\epsilon$. The subsequent $k_2$ integration corresponds to the diagram of Fig. 3 (c), where the diamond-shaped vertex is given by the result of the $k_1$ integration. One finds that the term proportional to $p_\nu$ only contributes to higher orders and can be ignored. The remaining expression is proportional to

$$(n - 3) \gamma_\mu (\not{p} - m) \gamma_\nu g^{\mu\nu} = -2m(n - 3)(n - 1), \quad (82)$$

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and therefore $\varphi_2(\epsilon) = n - 1 = 3 + 2\epsilon$. On the other hand, considering the $k_2$ integration first leads to an analogous analysis with the results $\varphi_2(\epsilon) = n - 3 = 1 + 2\epsilon = \varphi_1(\epsilon)$ and $\tilde{\varphi}_1(\epsilon) = n - 1 = 3 + 2\epsilon = \varphi_2(\epsilon)$.

In the cases where one cannot identify the individual contributions to $\phi(\epsilon)$ from the integrations of $k_1$ and $k_2$, respectively (this happens for example for tensor integrals of the type $k_1^\mu k_2^\nu$), one has to perform the dimensional counting analysis before reducing the tensor integrals.

VI. APPLICATION: NUCLEON MASS TO ORDER $\mathcal{O}(q^6)$

As an application we consider the nucleon mass up to and including order $\mathcal{O}(q^6)$. The result for the chiral expansion obtained from this calculation has been published in Ref. [19]. Here, we present more details of the calculations.

A. Lagrangian and power counting

The effective Lagrangian is given by the sum of a purely mesonic and a one-nucleon part,

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_2 + \mathcal{L}_4 + \cdots + \mathcal{L}_{\pi N}^{(1)} + \mathcal{L}_{\pi N}^{(2)} + \mathcal{L}_{\pi N}^{(3)} + \mathcal{L}_{\pi N}^{(4)} + \mathcal{L}_{\pi N}^{(5)} + \mathcal{L}_{\pi N}^{(6)} + \cdots$$

The purely mesonic Lagrangian at order $\mathcal{O}(q^2)$ is given in Ref. [2]. Reference [32] contains the mesonic Lagrangian at order $\mathcal{O}(q^4)$ as well as the lowest-order nucleonic Lagrangian. We use the conventions of Ref. [11] for the Lagrangian at order $\mathcal{O}(q^2)$ and of Ref. [33] for the Lagrangians at order $\mathcal{O}(q^3)$ and $\mathcal{O}(q^4)$. While the complete Lagrangians at order $\mathcal{O}(q^5)$ and $\mathcal{O}(q^6)$ have not yet been constructed, up to the order we are considering, vertices from these two Lagrangians only appear as contact terms. The light quark masses are proportional to the square of the pion mass, and only analytic expressions containing the quark masses appear in the effective Lagrangian. Therefore the nucleon mass does not receive any contributions from the Lagrangian at order $\mathcal{O}(q^5)$ in our calculation. The contributions from the Lagrangian at order $\mathcal{O}(q^6)$ are of the form $\tilde{g}_1 M^6$, where $\tilde{g}_1$ denotes a linear combination of low-energy coupling constants (LECs) from $\mathcal{L}_{\pi N}^{(6)}$.

---

8 Here we consider perfect isospin symmetry.
The bare Lagrangians are decomposed into renormalized and counterterm parts. Here we only show explicit results obtained from the renormalized Lagrangians, i.e., all appearing coupling constants are renormalized coupling constants. The renormalization procedure can then be viewed as simply replacing loop integrals by their infrared singular parts.

We use the following standard power counting [35,36]: Each loop integration in $n$ dimensions is counted as $q^n$, a pion propagator as $q^{-2}$, a nucleon propagator as $q^{-1}$ and vertices derived from $L_i$ and $L_{ij}^{(j)}$ as $q^i$ and $q^j$, respectively.

B. Inclusion of contact interaction insertions

To simplify the calculation we include the self-energy contributions from contact term diagrams in the nucleon propagator [11]. The advantage of this choice is that all self-energy diagrams with contact interaction insertions in the propagator are summed up automatically.

In terms of the nucleon mass in the chiral limit $m$ the full nucleon propagator can be written as

$$S_N(p) = \frac{1}{p - m - \Sigma(p, m) + i0^+}, \quad (84)$$

where $-i\Sigma(p, m)$ is the sum of all one-particle irreducible self-energy diagrams. The physical nucleon mass $m_N$ is determined by the solution to the equation

$$S_N^{-1}|_{p = m_N} = [p - m - \Sigma(p, m)]|_{p = m_N} = 0. \quad (85)$$

The self-energy receives contributions from contact terms as well as from loop diagrams,

$$\Sigma(p, m) = \Sigma_c + \Sigma_{\text{loop}}(p, m). \quad (86)$$

Due to the form of the BChPT Lagrangian used here, $\Sigma_c$ for the nucleon is independent of $p$. Inserting Eq. (86) into Eq. (85) one finds

$$[p - m - \Sigma_c - \Sigma_{\text{loop}}(p, m)]|_{p = m_N} = 0. \quad (87)$$

In a loop expansion Eq. (87) has the perturbative solution

$$m_N = m + \Sigma_c + \mathcal{O}(\hbar). \quad (88)$$

In the above the propagator which is used in the calculation of the self-energy diagrams has been chosen to be

$$S_N(p) = \frac{1}{p - m + i0^+}. \quad (89)$$

However, one can also choose this propagator to be

$$\tilde{S}_N(p) = \frac{1}{p - m - \Sigma_c + i0^+}. \quad (90)$$

This corresponds to including in the free Lagrangian those terms bilinear in $\bar{\Psi}, \Psi$ which generate the contact term diagrams in the self-energy contribution. The advantage of this choice is that all self-energy diagrams with contact interaction insertions in the propagator
are summed up automatically. With this choice of the propagator the self-energy is now given by the sum of loop diagrams only, i.e.

$$\Sigma(\not{p}, m) \rightarrow \tilde{\Sigma}_{\text{loop}}(\not{p}, \tilde{m}),$$  \hspace{1cm} (91)$$

where

$$\tilde{m} = m + \Sigma_c.$$  \hspace{1cm} (92)$$

As an additional benefit, when working to two-loop accuracy, one can set $p = \tilde{m}$ in the expression of two-loop diagrams, since corrections are at least of order $O(\hbar^3)$. To obtain $m_N$ one has to solve the equation

$$\tilde{S}_N^{-1}(m_N) = [\not{p} - \tilde{m} - \tilde{\Sigma}_{\text{loop}}(\not{p}, \tilde{m})]_{p = m_N} = 0.$$  \hspace{1cm} (93)$$

Inserting the loop expansion for $\tilde{\Sigma}_{\text{loop}}(\not{p}, \tilde{m})$,

$$\tilde{\Sigma}_{\text{loop}}(\not{p}, \tilde{m}) = \hbar\Sigma^{(1)}_{\text{loop}}(\not{p}, \tilde{m}) + \hbar^2\Sigma^{(2)}_{\text{loop}}(\not{p}, \tilde{m}) + \cdots,$$  \hspace{1cm} (94)$$

using the ansatz

$$m_N = \tilde{m} + \hbar\Delta m_1 + \hbar^2\Delta m_2 + \cdots$$  \hspace{1cm} (95)$$

and expanding around $\tilde{m}$ we obtain up to the two-loop level

$$0 = \tilde{m} + \hbar\Delta m_1 + \hbar^2\Delta m_2 - \tilde{m} - \hbar\Sigma^{(1)}_{\text{loop}}(\tilde{m} + \hbar\Delta m_1, \tilde{m}) - \hbar^2\Sigma^{(2)}_{\text{loop}}(\tilde{m}, \tilde{m})$$
$$= \hbar \left[ \Delta m_1 - \tilde{\Sigma}^{(1)}_{\text{loop}}(\tilde{m}, \tilde{m}) \right] + \hbar^2 \left[ \Delta m_2 - \Delta m_1 \tilde{\Sigma}^{(1)'}_{\text{loop}}(\tilde{m}, \tilde{m}) - \tilde{\Sigma}^{(2)}_{\text{loop}}(\tilde{m}, \tilde{m}) \right],$$  \hspace{1cm} (96)$$

where $\tilde{\Sigma}^{(1)'}_{\text{loop}}(\not{p}, \tilde{m})$ denotes the derivative of $\tilde{\Sigma}^{(1)}_{\text{loop}}(\not{p}, \tilde{m})$ with respect to $p$. The solutions for $\Delta m_1$ and $\Delta m_2$ are given by

$$\Delta m_1 = \tilde{\Sigma}^{(1)}_{\text{loop}}(\tilde{m}, \tilde{m}),$$  \hspace{1cm} (97)$$

$$\Delta m_2 = \tilde{\Sigma}^{(1)}_{\text{loop}}(\tilde{m}, \tilde{m})\tilde{\Sigma}^{(1)'}_{\text{loop}}(\tilde{m}, \tilde{m}) + \tilde{\Sigma}^{(2)}_{\text{loop}}(\tilde{m}, \tilde{m}).$$  \hspace{1cm} (98)$$

To obtain the nucleon mass up to chiral order $O(q^6)$ one needs to determine $\Sigma_c$, $\Delta m_1$ and $\Delta m_2$ up to that order. In the following we will not directly evaluate $\Delta m_2$ as given in Eq. (95) since we are only interested in the combination $\hbar\Delta m_1 + \hbar^2\Delta m_2$ with $\hbar = 1$. Instead, as indicated in the first line of Eq. (96), we will use the result for $\Delta m_1$ to determine $\tilde{\Sigma}^{(1)}_{\text{loop}}(\tilde{m} + \hbar\Delta m_1, \tilde{m})$ directly (see Eq. (103) below).

In principle, the nucleon propagator is a $2 \times 2$-matrix in isospin space. For arbitrary values of the up and down quark masses the propagator is a diagonal matrix; since however in this work the isospin-symmetric case $m_u = m_d$ is considered, the masses of proton and neutron are identical and the propagator is proportional to the unit matrix.

C. Contact terms

The contributions to the nucleon mass from contact interactions are given by

$$\delta m_c = -4c_1 M^2 - (16\epsilon_{38} + 2\epsilon_{115} + 2\epsilon_{116}) M^4 + \hat{g}_1 M^6$$
$$= -4c_1 M^2 - \hat{\epsilon}_1 M^4 + \hat{g}_1 M^6,$$  \hspace{1cm} (99)$$
FIG. 4: One-loop diagrams contributing to the nucleon self-energy up to order $O(q^6)$.

where $M^2$ is the lowest-order expression for the square of the pion mass. We use the notation

$$\hat{e}_1 = 16e_{38} + 2e_{115} + 2e_{116}$$

and $\hat{g}_1$ denotes a linear combination of LECs from the Lagrangian at order $O(q^6)$.

D. One-loop diagrams

The one-loop diagrams contributing to the nucleon mass up to order $O(q^6)$ are shown in Fig. (4). Diagrams (a) and (d) are of order $O(q^3)$ and $O(q^4)$, respectively, and have been determined in Ref. [11]. Diagrams (b) and (c) are of order $O(q^5)$, while the power counting gives $D = 6$ for diagrams (e) and (f).

Using dimensional regularization the unrenormalized results for the one-loop diagrams
up to order $\mathcal{O}(q^6)$ read

\[
\Sigma_{1(a)} = -\frac{3g_A^2}{4F^2} \left[ (\phi + \tilde{\phi})H_N + (\phi + \tilde{\phi})M^2H_{\pi N}(p^2) + (p^2 - \tilde{m}^2)\phi H^{(p)}_{\pi N}(p^2) \right],
\]

\[
\Sigma_{1(b)} = -\frac{3g_A}{F^2} (2d_{16} - d_{18})M^2 \left[ (\phi + \tilde{\phi})H_N + (\phi + \tilde{\phi})M^2H_{\pi N}(p^2) + (p^2 - \tilde{m}^2)\phi H^{(p)}_{\pi N}(p^2) \right],
\]

\[
\Sigma_{1(c)} = -\frac{3g_A^2}{2F^2} (\phi + \tilde{\phi})M^2 \left\{ l_3M^2 \left[ H_{\pi\pi} - (p^2 - \tilde{m}^2)H_{\pi\pi N}(p^2) - \phi(\phi + \tilde{\phi})H^{(p)}_{\pi\pi N}(p^2) \right] - l_4 \left[ H_{\pi} - (p^2 - \tilde{m}^2)H_{\pi N}(p^2) - \phi(\phi + \tilde{\phi})H^{(p)}_{\pi N} \right] \right\},
\]

\[
\Sigma_{1(d)} = \frac{3}{F^2} \left( (2c_1 - c_3)M^2H_{\pi} - c_2 \frac{p^2}{m^2} H^{(00)}_{\pi} \right),
\]

\[
\Sigma_{1(e)} = -\frac{12}{F^2} \left\{ [2(e_{14} + e_{19}) - e_{36} - 4e_{38}] M^4H_{\pi} + 2 [e_{15} + e_{20} + e_{35}] \frac{p^2}{m^2} M^2H^{(00)}_{\pi} + 6e_{16} \frac{p^4}{m^4} H^{(0000)}_{\pi} \right\},
\]

\[
\Sigma_{1(f)} = \frac{6}{F^4} \left\{ 2c_1 \left[ l_3M^2H_{\pi\pi} - l_4H_{\pi} \right] - c_2 \frac{p^2}{m^2} \left[ (l_3 - l_4)H_{\pi} + l_3M^2H_{\pi\pi} \right] - c_3 \left[ (l_3 - l_4)H_{\pi} - l_3M^2H_{\pi\pi} \right] \right\}.
\]

The integrals $H_{\pi}, H_{\pi N}(p^2), \ldots$ are given in App. B. Various combinations of fourth-order baryonic LECs appear through the vertex in diagram (e). To simplify the notation we use $\hat{\epsilon}_1$ as defined in Eq. (100) and

\[
\hat{\epsilon}_2 = 2e_{14} + 2e_{19} - e_{36} - 4e_{38},
\]

\[
\hat{\epsilon}_3 = e_{15} + e_{20} + e_{35}
\]

for these combinations in the following.

To determine the contribution of these diagrams to the nucleon mass we evaluate the expressions of Eq. (101) at $p^2 = m_N^2$ between on-shell spinors. To the order we are working, we can use $m_N = \tilde{m} + \hbar\Delta m_1$ and thus obtain $\tilde{\Sigma}_{\text{loop}}^{(1)}(\tilde{m} + \hbar\Delta m_1, \tilde{m})$ in Eq. (96). Further, we renormalize the one-loop integrals by replacing them with the corresponding infrared singular parts. The infrared renormalized expressions for the mass contributions, denoted
by a superscript \( r \), up to order \( M^6 \) are given by

\[
\delta m_{1(a)}^r = -\frac{3g_A^2}{32\pi F^2} M^3 - \frac{3g_A^2}{64\pi^2 F^2 m} \left[ 2 \ln \frac{M}{\mu} + 1 \right] M^4 \\
+ \frac{3g_A^2}{1024\pi^3 F^4 m^2} \left[ 4\pi^2 F^2 + 3g_A^2 m^2 + 9g_A^2 m^2 \ln \frac{M}{\mu} \right] M^5 \\
- \frac{g_A^2}{2048\pi^4 F^4 m^3} \left[ 27\pi^2 g_A^2 m^2 + 384\pi^2 c_1 F^2 m - 16\pi^2 F^2 - 9m^2(g_A^2 - c_2 m) \right] \\
+ 3m \left[ -15g_A^2 m + 16c_1(3m^2 + 16\pi^2 F^2) + 3m^2(c_2 - 8c_3) \right] \ln \frac{M}{\mu} \\
- 54m^2 \left( g_A^2 - 8c_1 m + c_2 m + 4c_3 m \right) \ln^2 \frac{M}{\mu} \right] M^6,
\]

\[
\delta m_{1(b)}^r = -\frac{3g_A}{8\pi F^2} \left( 2d_{16} - d_{18} \right) M^5 - \frac{3g_A}{16\pi^2 F^2 m} \left( 2d_{16} - d_{18} \right) \left[ 2 \ln \frac{M}{\mu} + 1 \right] M^6,
\]

\[
\delta m_{1(c)}^r = -\frac{3g_A^2}{32\pi F^4} \left( 3l_3 - 2l_4 \right) M^5 - \frac{3g_A^2}{32\pi^2 F^4 m} \left[ 3l_3 - l_4 + 2(2l_3 - l_4) \ln \frac{M}{\mu} \right] M^6,
\]

\[
\delta m_{1(d)}^r = \frac{3}{128\pi^2 F^2} \left[ c_2 + \ln \frac{M}{\mu} \left( 32c_1 - 4c_2 - 16c_3 \right) \right] M^4 + \frac{3c_1 c_2}{16\pi^2 F^2 m} \left[ 4 \ln \frac{M}{\mu} + 1 \right] M^6,
\]

\[
\delta m_{1(e)}^r = \frac{M^6}{32\pi^2 F^2} \left[ 6\tilde{e}_3 + 5e_{16} \right] - \frac{3M^6}{8\pi^2 F^2} \ln \frac{M}{\mu} \left[ 4\tilde{e}_2 + 2\tilde{e}_3 + e_{16} \right],
\]

\[
\delta m_{1(f)}^r = \frac{3}{64\pi^2 F^4} \left[ 16c_1 l_3 - c_2 l_4 - 8c_3 l_3 \right] M^6 + \frac{3}{16\pi^2 F^4} \left[ 8c_1 (l_3 - l_4) \right. \\
\left. -(c_2 + 4c_3)(2l_3 - l_4) \right] M^6 \ln \frac{M}{\mu}.
\]

The scale dependence of the renormalized low-energy constants is governed by

\[
l_{i,0} = l_i(\mu) + \gamma_i \lambda, \quad d_{i,0} = d_i(\mu) + \frac{\delta_i}{F^2} \lambda, \quad \epsilon_{i,0} = \epsilon_i(\mu) + \frac{\epsilon_i}{m F^2} \lambda,
\]

where the subscript \( 0 \) denotes bare quantities and

\[
\lambda = \frac{\mu^{n-4}}{16\pi^2} \left\{ \frac{1}{n-4} - \frac{1}{2} \left[ \ln(4\pi) + \Gamma'(1) + 1 \right] \right\}.
\]

The coefficients \( \gamma_i \) are given by [2]

\[
\gamma_3 = -\frac{1}{2}, \quad \gamma_4 = 2,
\]

while the \( \delta_i \) can be taken from Ref. [33] \(^\text{9}\)

\[
\delta_{16} = \frac{1}{2} g_A + g_A^3, \quad \delta_{18} = 0.
\]

\(^{9}\) Note that the numbering of terms in the Lagrangian used here differs from Ref. [33].
The LECs $\hat{e}_2, \hat{e}_3, e_{16}$ appear through the vertex shown in Fig. 4(e), which also gives the contact term contribution at order $O(q^4)$ to $\pi N$ scattering as analyzed in Ref. [37]. We can therefore relate the combinations of LECs used here to the ones in Ref. [37] (here denoted by a superscript $BL$), resulting in

\[ \hat{e}_1 = -e_1^{BL}, \quad \hat{e}_2 = \frac{1}{8} e_3^{BL}, \quad \hat{e}_3 = \frac{1}{16} e_4^{BL}, \quad e_{16} = \frac{1}{16} e_6^{BL}. \]

Using the expressions for the renormalized couplings given in App. E of Ref. [37] we find

\[ \hat{\varepsilon}_1 = -\frac{3}{2} \frac{g_A}{A} + \frac{3}{2} (8c_1 - c_2 - 4c_3)m, \]
\[ \hat{\varepsilon}_2 = -\frac{1}{8} \left( 1 + 3g_A^2 + \frac{22}{3} g_A^4 + 8c_1m + c_2m - 4c_3m \right), \]
\[ \hat{\varepsilon}_3 = \frac{1}{16} \left( 10 + 12g_A^2 + \frac{52}{3} g_A^4 + 8c_2m \right), \]
\[ \varepsilon_{16} = -\frac{1}{16} \left( 12 + 8g_A^2 + 8g_A^4 \right). \]

(107)

E. Two-loop diagrams

The two-loop diagrams relevant for a calculation of the nucleon self-energy up to order $O(q^6)$ are shown in Fig. 5. According to the power counting there are further diagrams at the given order. An example would be diagram 5(c) with one first-order vertex replaced by a second-order one. As a result of our calculation we find that these diagrams give vanishing contributions to the nucleon mass up to the order we are considering.

We again employ dimensional regularization. The unrenormalized expressions for the...
FIG. 5: Two-loop diagrams contributing to the nucleon self-energy up to order $\mathcal{O}(q^6)$. 
mass contributions of the diagrams of Fig. 5 up to order $\mathcal{O}(q^6)$ can be reduced to

$$
\delta m_{2(a)} = -\frac{6g^4_A}{F^4} \pi^2 m^3(n-1)(n-3)H_2(1,1,1,1|n+2),
$$

$$
\delta m_{2(b)} = \frac{9g^4_A}{F^4} m\pi^2(n-1) \left\{ 2m^2H_2(1,1,1,0,2|n+2) - M^2H_2(1,2,1,0,1|n+2) \right. \\
- 4m^2M^2 \left[ H_2(1,2,2,0,1|n+2) + H_2(1,1,2,0,2|n+2) \right] \\
- 32\pi^2 m^2 \left[ 2H_2(1,2,1,0,3|n+4) + H_2(1,2,2,0,2|n+4) \right] \left. \right\},
$$

$$
\delta m_{2(c)} = \frac{1}{2} \frac{3}{F^4} m \left\{ \frac{1}{2} H_2(1,1,0,0,0|n) - M^2H_2(1,1,0,0,1|n) - H_2(1,0,0,0,1|n) \\
+ 8\pi^2(n-1)H_2(1,1,0,0,2|n+2) - 16\pi^2(M^2 - 2m^2)H_2(1,2,0,0,2|n+2) \right\},
$$

$$
\delta m_{2(d)} = \frac{24g^2_A}{F^4} \left[ 32\pi^2 m^3(n-1) \left[ 2H_2(1,2,1,0,3|n+4) + H_2(1,2,2,0,2|n+4) \right] \\
+ \pi^2 m M^2 \left[ (n-1)H_2(1,2,1,0,1|n+2) + H_2(1,2,2,0,2|n+2) \right] \right\},
$$

$$
\delta m_{2(e)} = -\frac{48g^2_A}{F^4} \pi^2 m^2(n-1) \left\{ c_3 \left[ H_2(1,1,0,0,2|n+2) - 64\pi^2 m^2H_2(2,1,1,0,3|n+4) \right] \\
+ c_4 \left[ 2H_2(1,1,0,0,2|n+2) - 2M^2H_2(1,2,1,0,1|n+2) - 64\pi^2 m^2H_2(2,1,1,0,3|n+4) \right] \right\},
$$

$$
\delta m_{2(f)} = \frac{3g^2_A}{2F^4} m M^4 H_N(1,1|n) H_N(1,1|n),
$$

$$
\delta m_{2(g)} = \frac{24g^2_A}{F^4} \pi^2 m^2(n-1) \left[ (n-2)c_4 - c_3 \right] H_N(1,1|n+2) H_N(1,1|n+2),
$$

$$
\delta m_{2(h)} = \frac{18g^2_A}{F^4} \pi m^2 M^2(n-1) \left[ \frac{c_2}{n} + c_3 - 2c_1 \right] H_N(1,0|n) H_N(1,2|n+2) \\
- \frac{72g^2_A}{F^4} \pi^2 m^2 M^2(n-1) \left[ \frac{c_2}{n} + c_3 - 2c_1 \right] H_N(2,0|n+2) H_N(1,2|n+2),
$$

$$
\delta m_{2(i)} = \frac{-g^2_A}{F^4} m M^2 H_N(1,0|n) H_N(1,1|n),
$$

$$
\delta m_{2(j)} = \frac{1}{8} \frac{4M^2}{F^4} \left[ 5c_1 - 4\frac{c_2}{n} - 4c_3 \right] H_N(1,0|n) H_N(1,0|n),
$$

$$
\delta m_{2(k)} = \frac{1}{4} \frac{2}{F^4} \left\{ 3 \left[ \frac{c_2}{n} + c_3 - 2c_1 \right] M^4 H_N(1,0|n) H_N(2,0|n) \\
+ \left[ 7\frac{c_2}{n} + 7c_3 - 8c_1 \right] M^2 H_N(1,0|n) H_N(1,0|n) \right\},
$$

$$
\delta m_{2(l)} = \frac{g^2_A}{4F^4} m H_N(1,0|n) \left[ 4H_N(0,1|n) + 7M^2 H_N(1,1|n) + 3M^4 H_N(2,1|n) \right]. \tag{108}
$$

The integrals $H_N(a,b|n)$ and $H_2(a,b,c,d,e|n)$ are defined in App. C Here we have expressed tensor integrals in terms of scalar integrals in higher dimensions where convenient (see App. C and also Ref. 38).

After performing the infrared renormalization as described in Sec. V the contributions to
the nucleon mass up to order $\mathcal{O}(q^6)$ read

\[
\begin{align*}
\delta m_{2(a)}^r &= -\frac{g_A^4}{512\pi^3 F^4} \left[ 3M^5 \left( 1 + \ln \frac{M}{\mu} \right) - \frac{M^6}{48\pi m} \left( 5 + 36\pi^2 + 48 \ln \frac{M}{\mu} \right) \right], \\
\delta m_{2(b)}^r &= \frac{g_A^4}{1024\pi^3} M^5 \left( 1 + 3 \ln \frac{M}{\mu} \right) - \frac{27}{4096\pi^4 m} M^6 \left( 1 + 6 \ln \frac{M}{\mu} + 4 \ln^2 \frac{M}{\mu} \right), \\
\delta m_{2(c)}^r &= \frac{g_A^4}{M^6} \left[ \pi^2 \ln^2 \frac{M}{\mu} \right], \\
\delta m_{2(d)}^r &= -\frac{g_A^2}{1536\pi^3 F^4 m} M^6 \left[ 1 + 9\pi^2 - 6 \ln \frac{M}{\mu} \right], \\
\delta m_{2(e)}^r &= \frac{g_A^2}{128\pi^2 F^4} M^6 [c_3 - 2c_4], \\
\delta m_{2(f)}^r &= -\frac{3g_A^2}{512\pi^2 F^4 m} M^6, \\
\delta m_{2(g)}^r &= \frac{g_A^2}{128\pi^2 F^4} [c_3 - 2c_4] M^6, \\
\delta m_{2(h)}^r &= -\frac{9g_A^2}{256\pi^4 F^4} M^6 \left[ (c_3 - 2c_1) \left( \ln \frac{M}{\mu} + 3 \ln^2 \frac{M}{\mu} \right) + \frac{c_2}{16} \left( -1 + \ln \frac{M}{\mu} \right) + 12 \ln^2 \frac{M}{\mu} \right], \\
\delta m_{2(i)}^r &= \frac{g_A^2}{128\pi^3 F^4} \ln \frac{M}{\mu} M^5 + \frac{g_A^2}{256\pi^4 F^4 m} \left( \ln \frac{M}{\mu} + 2 \ln^2 \frac{M}{\mu} \right) M^6, \\
\delta m_{2(j)}^r &= -\frac{M^6}{128\pi^4 F^4} \left[ (5c_1 - c_2 - 4c_3) \ln^2 \frac{M}{\mu} + \frac{c_2}{4} \ln \frac{M}{\mu} \right], \\
\delta m_{2(k)}^r &= \frac{M^6}{512\pi^4 F^4} \left[ (12c_1 + c_2 - 6c_3) \ln \frac{M}{\mu} + 2(28c_1 - 5c_2 - 20c_3) \ln^2 \frac{M}{\mu} \right], \\
\delta m_{2(l)}^r &= -\frac{17g_A^2}{1024\pi^3 F^4} \ln \frac{M}{\mu} M^5 - \frac{g_A^2}{1024\pi^4 F^4 m} \left( 13 \ln \frac{M}{\mu} + 20 \ln^2 \frac{M}{\mu} \right) M^6.
\end{align*}
\]

**F. Results and discussion**

Combining the contributions from the contact interactions with the one- and two-loop results we obtain for the nucleon mass up to order $\mathcal{O}(q^6)$

\[
m_N = m + k_1 M^2 + k_2 M^3 + k_3 M^4 \ln \frac{M}{\mu} + k_4 M^4 + k_5 M^5 \ln \frac{M}{\mu} + k_6 M^5 + k_7 M^6 \ln^2 \frac{M}{\mu} + k_8 M^6 \ln \frac{M}{\mu} + k_9 M^6.
\]
The coefficients $k_i$ are given by

$$k_1 = -4c_1,$$
$$k_2 = -\frac{3g^2_A}{32\pi F^2},$$
$$k_3 = -\frac{3}{32\pi^2 F^2 m} (g^2_A - 8c_1 m + c_2 m + 4c_3 m),$$
$$k_4 = -\hat{c}_1 - \frac{3}{128\pi^2 F^2 m} (2g^2_A - c_2 m),$$
$$k_5 = \frac{3g^2_A}{1024\pi^3 F^4} (16g^2_A - 3),$$
$$k_6 = \frac{3g^2_A}{256\pi^3 F^4} \left[ g^2_A + \frac{\pi^2 F^2}{m^2} - 8\pi^2 (3l_3 - 2l_4) - \frac{32\pi^2 F^2}{g_A} (2d_{16} - d_{18}) \right],$$
$$k_7 = -\frac{3}{256\pi^4 F^4 m} \left[ g^2_A - 6c_1 m + c_2 m + 4c_3 m \right],$$
$$k_8 = -\frac{g^4_A}{64\pi^4 F^4 m} - \frac{3g^2_A}{1024\pi^4 F^4 m^2} \left[ 384\pi^2 F^2 c_1 + 5m + 192\pi^2 m (2l_3 - l_4) \right]$$
$$- \frac{3g^2_A}{8\pi^2 F^2 m} \left[ 2d_{16} - d_{18} \right] + \frac{3}{256\pi^4 F^4} \left[ 2c_1 - c_3 \right] + \frac{3}{8\pi^2 F^2 m} \left[ 2c_1 c_2 - 4\hat{c}_2 m - 2\hat{c}_3 m - c_{16} m \right]$$
$$+ \frac{3}{16\pi^2 F^4} \left[ 8c_1 (l_3 - l_4) - (c_2 + 4c_3) (2l_3 - l_4) \right],$$
$$k_9 = \hat{g}_1 - \frac{g^4_A}{24576\pi^4 F^4 m} \left( 49 + 288\pi^2 \right) - \frac{3g^2_A}{16\pi^2 F^2 m} (2d_{16} - d_{18})$$
$$- \frac{g^2_A}{1536\pi^4 F^4 m^2} \left[ m^2 (1 + 18\pi^2) - 12\pi^2 F^2 + 144\pi^2 m^2 (3l_3 - l_4) + 288\pi^2 F^2 m c_1 - 24\pi^2 m^2 (c_3 - 2c_4) \right] + \frac{3}{64\pi^2 F^4} \left[ 8(2c_1 - c_3) l_3 - c_2 l_4 \right]$$
$$+ \frac{1}{2048\pi^4 F^4 m} \left[ 1 - 384\pi^2 F^2 c_1 c_2 + 384\pi^2 F^2 m \hat{e}_3 + 320\pi^2 F^2 m e_{16} \right].$$

In general, the expressions of the coefficients in the chiral expansion of a physical quantity differ in various renormalization schemes, since analytic contributions can be absorbed by redefining LECs. However, this is not possible for the leading nonanalytic terms, which therefore have to agree in all renormalization schemes. Comparing our result with the HBChPT calculation of Ref. [20], we see that the expressions for the coefficients $k_2$, $k_3$, and $k_5$ agree as expected. At order $\mathcal{O}(q^6)$ also the coefficient $k_7$ has to be the same in all renormalization schemes. Note that, while $k_6 M^5$ and $k_8 M^6 \ln \frac{M}{\mu}$ are nonanalytic in the quark masses, the algebraic form of the coefficients $k_6$ and $k_8$ are renormalization scheme dependent. This is due to the different treatment of one-loop diagrams in different renormalization schemes. The counterterms for one-loop subdiagrams depend on the renormalization scheme and produce nonanalytic terms proportional to $M^5$ and $M^6 \ln \frac{M}{\mu}$ when used as vertices in counterterm diagrams. We find that our result for $k_6$ coincides with the HBChPT calculation of Ref. [20] except for a term proportional to $d_{28}$, which, however, does not have a finite contribution for manifestly Lorentz-invariant renormalization schemes [33]. Therefore, at order $\mathcal{O}(q^5)$ the chiral expansion of the IR renormalized result reproduces the HBChPT result.

The result for the nucleon mass should be scale-independent at each order in the chiral
expansion, and showing this scale independence serves as a check of our results. The terms up to and including order $O(M^4)$ have been discussed previously \cite{37}. Using the expressions for the scale dependence of the renormalized couplings of Eqs. (104), (105) and (106) we see that the contribution at order $O(M^5)$ is independent of $\mu$ as required. At order $O(M^6)$, the scale dependence of the LEC $\hat{g}_1$ is not known which prevents a complete analysis at this order. However, $\hat{g}_1$ does not contribute to terms proportional to $\ln M \ln \mu$ since it only appears in the analytic expression at $O(M^6)$. We can therefore analyze the terms proportional to $\ln M \ln \mu$ which must vanish if our result is to be scale-independent. This is the case, which can be shown using the expressions of Eqs. (104)-(107).

The numerical contributions from higher-order terms cannot be calculated so far, since most expressions in Eq. (111) contain LECs which are not reliably known in IR renormalization. In order to get an estimate of these contributions we consider several terms for which the LECs have previously been determined. The coefficient $k_5$ is free of higher-order LECs and is given in terms of the axial-vector coupling constant $g_A$ and the pion decay constant $F$. While the values for both $g_A$ and $F$ should be taken in the chiral limit, we evaluate $k_5$ using the physical values $g_A = 1.2695(29)$ \cite{40} and $F = 92.42(26)$ MeV. Setting $\mu = m_N$, $m_N = (m_p + m_n)/2 = 938.92$ MeV, and $M = M_{\pi^+} = 139.57$ MeV we obtain $k_5 M^5 \ln(M/m_N) = -4.8 \text{ MeV}$. This amounts to approximately 31% of the leading nonanalytic contribution at one-loop order, $k_2 M^3$. The mesonic LECs appearing in $k_6$ can be found in Ref. \cite{2} and are given by $l_5(m_N) = 1.4 \times 10^{-3}$ and $l_4(m_N) = 3.7 \times 10^{-3}$ at the scale $\mu = m_N$. The parameter $d_{18}$ can be related to the Goldberger-Treiman discrepancy \cite{37} and is given by $d_{18} = -0.80 \text{ GeV}^{-2}$. The LEC $d_{16}$, however, is not as reliably determined. In order to estimate the magnitude of the contribution stemming from $k_6$ we use the central value from the reaction $\pi N \rightarrow \pi\pi N$, $d_{16}(m_N) = -1.93 \text{ GeV}^{-2}$ \cite{41,42}. It should be noted that the calculation of Ref. \cite{41} was performed in HBChPT, and employing the obtained value for $d_{16}$ in an infrared renormalized expression therefore only gives an estimate of the size of the corresponding term. The resulting contribution is $k_6 M^5 = 3.7 \text{ MeV}$ and cancels large parts of the nonanalytic term $k_5 M^5 \ln(M/m_N)$. In Ref. \cite{43} the parameter $d_{16}$ has been determined by a fit to lattice data. At the scale $\mu = m_N$ it is given by $d_{16}(m_N) = 4.11 \text{ GeV}^{-2}$, which does not agree with the result from the reaction $\pi N \rightarrow \pi\pi N$. With this value of $d_{16}$ we find $k_6 M^5 = -7.6 \text{ MeV}$. The LECs appearing in $k_7$ have been determined in Ref. \cite{37}, and we obtain $k_7 M^6 \ln^2(M/m_N) = 0.3 \text{ MeV}$.

The terms $k_8$ and $k_9$ contain LECs from the fourth order Lagrangian $\mathcal{L}^{(4)}_{\pi N}$ which have not been determined. We try to get a very rough estimate of the size of these contributions by assuming that all these LECs as well as $\hat{g}_1$ are of natural size, that means $e_i \sim 1 \text{ GeV}^{-3}$ and $\hat{g}_1 \sim 1 \text{ GeV}^{-5}$. We choose the $d_{16}$ value from $\pi N \rightarrow \pi\pi N$ and use the above values for the other LECs. Setting all appearing $e_i = 0 \text{ GeV}^{-3}$ gives a contribution $k_8 M^6 \ln(M/m_N) \approx 10^{-2} \text{ MeV}$. The choice $e_i = 5 \text{ GeV}^{-3}$ results in $k_8 M^6 \ln(M/m_N) \approx 0.9 \text{ MeV}$, while $e_i = -5 \text{ GeV}^{-3}$ gives $k_8 M^6 \ln(M/m_N) \approx -0.9 \text{ MeV}$. A similar analysis for the term $k_9 M^6$ gives $k_9 M^6 \approx -2.8 \text{ MeV}$ for all $e_i = 0 \text{ GeV}^{-3}$ and $\hat{g}_1 = 0 \text{ GeV}^{-5}$, while setting $e_i = 5 \text{ GeV}^{-3}$, $\hat{g}_1 = 5 \text{ GeV}^{-5}$ and $e_i = -5 \text{ GeV}^{-3}$, $\hat{g}_1 = -5 \text{ GeV}^{-5}$ results in $k_9 M^6 \approx -2.5 \text{ MeV}$. The numbers obtained here are only very rough estimates. Choosing $e_{14} = e_{15} = 5 \text{ GeV}^{-3}$ and $e_{16} = e_{19} = e_{20} = e_{35} = e_{36} = e_{38} = 1 \text{ GeV}^{-3}$, $\hat{g}_1 = 1 \text{ GeV}^{-5}$ leads to large cancelations between the terms $k_8 M^6 \ln(M/m_N)$ and $k_9 M^6$, resulting in a complete contribution at order $O(M^6)$ of about 0.3 MeV. As a check we also use the value of $d_{16}$ as obtained in Ref. \cite{43}, which results in contributions from $k_8$ that are about a factor 10 larger, while the dependence of $k_9$ on $d_{16}$
FIG. 6: Pion mass dependence of the term $k_5 M^5 \ln(M/m_N)$ (solid line) for $M < 400 \text{MeV}$. For comparison also the term $k_2 M^3$ (dashed line) is shown.

is much less pronounced. Clearly a more reliable determination of the higher-order LECs is desirable.

Chiral expansions like Eq. (110) play an important role in the extrapolation of lattice QCD results to physical quark masses, and the nucleon mass is an example that has been studied in detail (see, e.g., Refs. [43, 44, 45, 46, 47]). In Ref. [44] such an extrapolation was performed for the nucleon mass up to order $O(q^4)$, while Ref. [43] includes an analysis of the fifth-order terms. It was shown, as had also been argued in Ref. [42], that the terms at order $O(q^5)$ play an important role in the chiral extrapolation. As an illustration we consider the leading nonanalytic term at this order, $k_5 M^5 \ln(M/m_N)$. Its dependence on the pion mass is shown in Fig. 6 for pion masses below 400 MeV, which is considered a region where chiral extrapolations are valid (see, e.g., Refs. [48, 49]). We see that already at $M \approx 360 \text{MeV}$ the term $k_5 M^5 \ln(M/m_N)$ becomes as large as the leading nonanalytic term at one-loop order, $k_2 M^3$, indicating the importance of the fifth-order terms at unphysical pion masses. Since the contribution at order $O(M^6)$ depends on a number of unknown LECs, we do not attempt to perform a chiral extrapolation up to this order here, but restrict the discussion on the pion mass dependence of the term $k_5 M^5 \ln^2(M/m_N)$. Figure 7 shows this dependence for pion masses below 400 MeV. No errors are given for the LECs $c_1$, $c_2$, and $c_3$ in Ref. [37]. For an estimate we have assumed the relative errors of these LECs and of $g_A$ to be 20%, and the corresponding error for $k_7 M^6 \ln^2(M/m_N)$ is shown in Fig. 7. For comparison we also show the nonanalytic term at fourth order, $k_3 M^4 \ln(M/m_N)$. As expected, and in contrast to the fifth-order term, the two-loop term $k_7 M^6 \ln^2(M/m_N)$ is smaller than the one-loop contribution $k_3 M^4 \ln(M/m_N)$ in the considered pion mass region. Note that the relative difference in the pion mass dependence between $k_5 M^5 \ln(M/m_N)$ and $k_2 M^3$, as well as $k_7 M^6 \ln^2(M/m_N)$ and $k_3 M^4 \ln(M/m_N)$ is proportional to a factor $M^2 \ln(M/m_N)$, and that for the physical pion mass the differences in the two cases are comparable on an absolute scale. We also show the pion mass dependence of the terms $k_7 M^6 \ln^2(M/m_N)$ and $k_3 M^4 \ln(M/m_N)$ up to $M \approx 700 \text{MeV}$, which, however, is beyond the domain that is considered suitable for the application of Eq. (110). Again the sixth-order term remains much smaller than the fourth-order one, also at higher pion masses. However, the above considerations are not reliable predictions for the behavior of the complete two-loop contributions at unphysical quark masses. This is because here only one of the terms
FIG. 7: Pion mass dependence of the term $k_7 M^6 \ln^2(M/m_N)$ (solid line) for $M < 400\,\text{MeV}$. The shaded band corresponds to relative errors of 20% in the LECs. For comparison also the term $k_3 M^4 \ln(M/m_N)$ (dashed line) is shown.

FIG. 8: Pion mass dependence of the term $k_7 M^6 \ln^2(M/m_N)$ (solid line) for $M < 700\,\text{MeV}$. The shaded band corresponds to relative errors of 20% in the LECs. For comparison also the term $k_3 M^4 \ln(M/m_N)$ (dashed line) is shown.

at order $O(q^6)$ is considered, and the contribution of the analytic term proportional to $M^6$ can be considerably larger than $k_7 M^6 \ln^2(M/m_N)$ depending on the values of the unknown LECs.

G. Nucleon $\sigma$ term

The Feynman-Hellmann theorem \cite{50, 51} relates the nucleon mass to the value of the nucleon scalar form factor at zero momentum transfer, the so-called $\sigma$ term (see, e.g., \cite{52, 53}),

$$\sigma(q^2 = 0) = M^2 \frac{\partial m_N}{\partial M^2}.$$  \hspace{1cm} (112)
Applying the Feynman-Hellmann theorem to Eq. (110), the chiral expansion of \( \sigma(0) \) is given by

\[
\sigma(0) = k_1 M^2 + \frac{3}{2} k_2 M^3 + 2k_3 M^4 \ln \frac{M}{\mu} + \left( \frac{k_3}{2} + 2k_4 \right) M^4 + \frac{5}{2} k_5 M^5 \ln \frac{M}{\mu} + \frac{1}{2} (k_5 + 5k_6) M^5 + 3k_7 M^6 \ln^2 \frac{M}{\mu} + (k_7 + 3k_8) M^6 \ln \frac{M}{\mu} + \left( \frac{k_8}{2} + 3k_9 \right) M^6.
\] (113)

The first four terms have already been determined in Ref. [11]. To estimate the contributions of the terms of order \( \mathcal{O}(M^5) \) we use the same values for the LECs as above, in particular the value of \( d_{16} \) as extracted from the reaction \( \pi N \rightarrow \pi \pi N \). The combined contributions at order \( \mathcal{O}(M^5) \) are

\[
\frac{5}{2} k_5 M^5 \ln \frac{M}{\mu} + \frac{1}{2} (k_5 + 5k_6) M^5 \approx -0.2 \text{ MeV}.
\] (114)

Due to the dependence of the order \( \mathcal{O}(M^6) \) nucleon mass contribution on the specific values of the unknown LECs \( e_i \), we do not attempt to evaluate the terms at order \( \mathcal{O}(M^6) \) in Eq. (113).

**VII. SUMMARY**

We have shown details of how to consistently renormalize two-loop diagrams in manifestly Lorentz-invariant BChPT within the framework of infrared regularization. The renormalization procedure preserves all relevant symmetries such that renormalized expressions fulfill the relevant Ward identities. Renormalized diagrams also obey the standard power counting. We have presented a simplified method of renormalizing diagrams with one small scale, which relies on dimensional analysis. Integrals of this kind appear, e.g., in the calculation of the nucleon mass or the axial-vector coupling constant. In this method integrals derived from the original expressions are renormalized using the \( \tilde{\text{MS}} \) scheme, which simplifies the calculations considerably. As an application we have calculated the nucleon mass up to and including order \( \mathcal{O}(q^6) \). For physical values of the pion mass, the numerical estimate of the two-loop contributions is reasonably small. For example, the estimate of the \( \mathcal{O}(q^5) \) term of the nucleon \( \sigma \) term is \(-0.2 \text{ MeV} \). However, when considering the nucleon mass as a function of the pion mass, we have seen that already at a pion mass of 360 MeV the nonanalytic contribution at order \( \mathcal{O}(q^5) \) may become as large as the nonanalytic \( \mathcal{O}(q^3) \) contribution. From this one cannot conclude that the chiral expansion breaks down at this value of the pion mass since the analytic terms at \( \mathcal{O}(q^5) \) might cancel parts of the nonanalytic term. One should, however, take special care when performing chiral extrapolations beyond this value.

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APPENDIX A: DIMENSIONAL COUNTING ANALYSIS

Analytic expressions for two-loop integrals, especially when two mass scales such as the pion mass \( M \) and the nucleon mass in the chiral limit \( m \) appear in the same integral, can be extremely difficult to obtain. Since we are interested in the chiral expansion of the considered integrals in the present work, we do not have to find a closed-form solution to the appearing integrals, but can use a method called dimensional counting analysis \([30]\) for the evaluation of integrals. A closely related way of calculating loop integrals is the so-called “strategy of regions” \([54]\). Here we present an illustration of dimensional counting for one- and two-loop integrals.

### 1. One-loop integrals

The advantage of the dimensional counting analysis for one-loop integrals lies in its applicability to dimensionally regulated integrals containing several different masses. Consider integrals with two different mass scales, \( M \) and \( m \), where \( M < m \), and a possible external momentum \( p \) with \( p^2 \approx m^2 \). Dimensional counting provides a method to reproduce the expansion of the integral for small values of \( M / p^2 - m^2 \). To that end one rescales the loop momentum \( k \mapsto \frac{M^{\alpha_i}}{\tilde{k}} \), where \( \alpha_i \) is a non-negative real number. After extracting an overall factor of \( M \) one expands the integrand in positive powers of \( M \) and interchanges summation and integration. The sum of all possible rescalings with subsequent expansions with nontrivial coefficients then reproduces the expansion of the result of the original integral.

To be specific, consider the integral

\[
H_{\pi N}(p^2) = \frac{i}{(2\pi)^n} \int \frac{d^nk}{(k^2 - M^2 + i0^+)((k+p)^2 - m^2 + i0^+)}.
\]

(A1)

It can be evaluated analytically and the result is given in App. [B]. After rescaling one obtains

\[
H_{\pi N}(p^2) \mapsto \frac{i}{(2\pi)^n} \int \frac{M^{\alpha_i} d^n\tilde{k}}{[k^2M^{2\alpha_i} - M^2 + i0^+][(k^2 + 2p \cdot \tilde{k}M^{\alpha_i} + p^2 - m^2 + i0^+)].
\]

(A2)

No overall factor of \( M \) can be extracted from the second propagator, which is therefore expanded in positive powers of \( M \). As a result the integration variable \( \tilde{k} \) only appears in positive powers in the expanded expression of this propagator. If \( 0 < \alpha_i < 1 \) one can extract the factor \( M^{-2\alpha_i} \) from the first propagator, which takes the form

\[
\frac{1}{k^2 - M^{2-2\alpha_i} + i0^+}.
\]

(A3)

Expanding in positive powers of \( M \) and interchanging summation and integration one obtains integrals of the type

\[
\int d^n\tilde{k} \frac{1}{(k^2 + i0^+)^j}.
\]

(A4)

Combined with the expansion of the second propagator the resulting coefficients in the expansion in \( M \) are integrals of the type

\[
\int d^n\tilde{k} \frac{\tilde{k}^m}{(k^2 + i0^+)^j},
\]

(A5)
which vanish in dimensional regularization. For the case $1 < \alpha_i$ the first propagator in 
Eq. \((A2)\) can be rewritten as

$$\frac{1}{M^2} \frac{1}{(\tilde{k}^2 M^{2\alpha_i-2} - 1 + i0^+)}.$$ \hspace{1cm} (A6)

Expanding in $M$ and combining with the expansion of the second propagator one obtains integrals of the type

$$\int d^n \tilde{k} \tilde{k}^i,$$ \hspace{1cm} (A7)

which, again, vanish in dimensional regularization. The only contributions to $H_{\pi N}(p^2)$ can therefore stem from $\alpha_i = 0$ and $\alpha_i = 1$. For $\alpha_i = 0$ one obtains

$$H_{\pi N}^{(0)}(p^2) = \frac{i}{(2\pi)^n} \sum_{i=0}^{\infty} (M^2)^i \int \frac{d^n k}{[k^2 + i0^+]^{1+i}[((k + p)^2 - m^2 + i0^+]},$$ \hspace{1cm} (A8)

while the expression for $\alpha_i = 1$ reads

$$H_{\pi N}^{(1)}(p^2) = \frac{i}{(2\pi)^n} \sum_{i=0}^{\infty} (-1)^i \frac{M^{n-2+i}}{(p^2 - m^2)^{1+i}} \int \frac{d^n k (\tilde{k}^2 M + 2p \cdot \tilde{k})^i}{[k^2 - 1 + i0^+]}. \hspace{1cm} (A9)$$

The expansion of $H_{\pi N}(p^2)$ is then given by

$$H_{\pi N}(p^2) = H_{\pi N}^{(0)}(p^2) + H_{\pi N}^{(1)}(p^2), \hspace{1cm} (A10)$$

which correctly reproduces the result of App. \[B\]?

2. Two-loop integrals

While one of the advantages of the dimensional counting method lies in its applicability to integrals containing several mass scales, a difficulty arises for the calculation of the nucleon mass. Since integrals have to be evaluated on-mass-shell, the two small scales $M$ and $p^2 - m^2$ are not independent of each other and are comparable in size. Therefore an expansion in $\frac{M}{p^2 - m^2}$ does not converge. By the choice of the nucleon propagator mass to include all contact interaction contributions, the terms $p^2 - m^2$ in the propagator can be neglected in two-loop integrals since they are of higher order in the loop expansion. The two-loop integrals contributing to the nucleon mass are therefore reduced to integrals with only one small mass scale, for which an expansion in $M$ can be obtained.

For the extension of the dimensional counting method to two-loop integrals

$$H_2(a, b, c, d, e|n) = \int \int \frac{d^n k_1 d^n k_2}{(2\pi)^{2n}} \frac{1}{[k_1^2 - M^2 + i0^+]^{a}[k_2^2 - M^2 + i0^+]^{b}}$$

$$\times \frac{1}{[k_1^2 + 2p \cdot k_1 + i0^+]^{c}[k_2^2 + 2p \cdot k_2 + i0^+]^{d}} [((k_1 + k_2)^2 + 2p \cdot k_1 + 2p \cdot k_2 + i0^+)^e],$$ \hspace{1cm} (A11)
one has to consider all possible combinations of rescaling the integration variables \( k_1 \mapsto M^{\alpha_i} \tilde{k}_1, \ k_2 \mapsto M^{\beta_i} \tilde{k}_2 \). The expansion of the two-loop integral is then given by

\[
H_2(a, b, c, d, e|n) = \sum_{\alpha_i, \beta_j} M^{\varphi(\alpha_i, \beta_j)} h^{(\alpha_i, \beta_j)}(a, b, c, d, e|n)
\]

\[
= \sum_{\alpha_i, \beta_j} H^{(\alpha_i, \beta_j)}(a, b, c, d, e|n),
\]

where \( \varphi(\alpha_i, \beta_j) \) is the overall power of \( M \) extracted for each rescaling, the functions \( h^{(\alpha_i, \beta_j)}(a, b, c, d, e|n) \) are the expressions for the integrated expansions, and we have defined \( H^{(\alpha_i, \beta_j)}(a, b, c, d, e|n) = M^{\varphi(\alpha_i, \beta_j)} h^{(\alpha_i, \beta_j)}(a, b, c, d, e|n) \) to simplify the notation. Following the discussion of the one-loop sector one sees that the only combinations \( (\alpha_i, \beta_j) \) that give non-vanishing contributions are \((0, 0), (1, 0), (0, 1) \) and \((1, 1)\), so that a two-loop integral is given by

\[
H_2(a, b, c, d, e|n) = H^{(0,0)}(a, b, c, d, e|n) + H^{(1,0)}(a, b, c, d, e|n) + H^{(0,1)}(a, b, c, d, e|n)
+ H^{(1,1)}(a, b, c, d, e|n).
\]

(A13)

To shorten the notation and to avoid confusion with other superscripts used for further expansions in this paper the contributions corresponding to \( H^{(0,0)}, H^{(1,0)}, H^{(0,1)}, \) and \( H^{(1,1)} \) are also denoted by \( F_1, F_2, F_3 \) and \( F_4 \), respectively.

From a technical point of view it is convenient to consider the rescaling \( k_1 \mapsto (M/m)^{\alpha_i} k_1, \ k_2 \mapsto (M/m)^{\beta_i} k_2 \), since then the integration variables \( \tilde{k} \) have dimension of momenta. This also facilitates the evaluation of certain loop integrals appearing in the calculation of the nucleon mass.

As an example consider the integral \( H_2(1, 1, 1, 1, 1|n) \). For \((0, 0)\) the resulting integrals read

\[
H^{(0,0)}(1, 1, 1, 1, 1|n) = \sum_{i,j} M^{2i+2j} \int \int \frac{d^n \tilde{k}_1 d^n \tilde{k}_2}{(2\pi)^{2n}} \frac{1}{[k_1^2 + i0^+]^{1+i}[k_2^2 + i0^+]^{1+j} [k_1^2 + 2p \cdot k_1 + i0^+][k_2^2 + 2p \cdot k_2 + i0^+].}
\]

(A14)

While still a two-loop integral that does not directly factorize into the product of one-loop integrals, the vanishing of the mass scale \( M \) simplifies the evaluation of the integral. The rescaling of only \( k_1 \) leads to

\[
H^{(1,0)}(1, 1, 1, 1, 1|n) = \sum_{i,j,l} (-1)^{j+l} M^{n-3+2i+j+l} n^{3-n-j-l} \int \int \frac{d^n \tilde{k}_1 d^n \tilde{k}_2}{(2\pi)^{2n}} \frac{1}{[k_1^2 - m^2 + i0^+]}
\times \frac{(\tilde{k}_1^2)^i (M \tilde{k}_2^2 + 2p \cdot \tilde{k}_1 + 2\tilde{k}_1 \cdot k_2)^l}{[k_2^2 + i0^+]^{1+i} [2p \cdot k_1 + i0^+]^{1+j} [k_2^2 + 2p \cdot k_2 + i0^+]^{2+l}},
\]

(A15)

while the expression for \( H^{(0,1)}(1, 1, 1, 1, 1|n) \) can be obtained by substituting \( \tilde{k}_1 \mapsto \tilde{k}_2 \) and \( k_2 \mapsto k_1 \) in Eq. (A15). One sees that the integrals of Eq. (A15) can be reduced to the
product of tensorial one-loop integrals, which is a considerable simplification compared to the original integral. The last contribution stems from $\alpha_i = 1, \beta_i = 1$ and reads

$$H^{(1,1)}(1, 1, 1, 1 | n) = \sum_{i,j,l} (-1)^{i+j+l} \left( \frac{M}{m} \right)^{2n-7+i+j+l} \int \frac{d^n k_1 d^n k_2}{(2\pi)^{2n}} \left[ \frac{1}{k_1^2 - m^2 + i0^+} \right]$$

$$\times \frac{(k_1^2)^i (k_2^2)^j (\tilde{k}_1^2 + 2\tilde{k}_1 \cdot \tilde{k}_2 + \tilde{k}_2^2)^l}{[k_2^2 - m^2 + i0^+] [2p \cdot \tilde{k}_1 + i0^+] [2p \cdot k_2 + i0^+] [2p \cdot \tilde{k}_1 + 2p \cdot \tilde{k}_2 + i0^+]^{1+l}}$$

where the integration can be reduced to the evaluation of a set of basis integrals (see App. D). The sum of all four contributions reproduces the $M$ expansion of the integral $H_2(1, 1, 1, 1 | n)$,

$$H_2(1, 1, 1, 1 | n) = H^{(0,0)}(1, 1, 1, 1 | n) + H^{(1,0)}(1, 1, 1, 1 | n) + H^{(0,1)}(1, 1, 1, 1 | n) + H^{(1,1)}(1, 1, 1, 1 | n).$$

\textbf{APPENDIX B: INTEGRALS AT THE ONE-LOOP LEVEL}

Using dimensional regularization the one-loop integrals are defined as

$$H_\pi = i \int \frac{d^n k}{(2\pi)^n} \left[ \frac{1}{k^2 - M^2 + i0^+} \right],$$

$$g^{\mu\nu} H_\pi^{(00)} = i \int \frac{d^n k}{(2\pi)^n} \left[ \frac{k_\mu k_\nu}{k^2 - M^2 + i0^+} \right],$$

$$(g^{\mu\nu} g^{\rho\sigma} + g^{\mu\rho} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\rho}) H_\pi^{(0000)} = i \int \frac{d^n k}{(2\pi)^n} \left[ \frac{k_\mu k_\nu k_\rho k_\sigma}{k^2 - M^2 + i0^+} \right],$$

$$H_{\pi\pi} = i \int \frac{d^n k}{(2\pi)^n} \left[ \frac{1}{k^2 - M^2 + i0^+} \right],$$

$$g^{\mu\nu} H_{\pi\pi}^{(00)} = i \int \frac{d^n k}{(2\pi)^n} \left[ \frac{k_\mu k_\nu}{k^2 - M^2 + i0^+} \right],$$

$$H_N = i \int \frac{d^n k}{(2\pi)^n} \left[ \frac{1}{k^2 - m^2 + i0^+} \right],$$

$$H_{\pi N}(p^2) = i \int \frac{d^n k}{(2\pi)^n} \left[ \frac{1}{[k^2 - M^2 + i0^+][(k + p)^2 - m^2 + i0^+]} \right],$$

$$p^\mu H_{\pi N}^{(p\mu)}(p^2) = i \int \frac{d^n k}{(2\pi)^n} \left[ \frac{k^\mu}{[k^2 - M^2 + i0^+][(k + p)^2 - m^2 + i0^+]} \right],$$

$$H_{\pi\pi N}(p^2) = i \int \frac{d^n k}{(2\pi)^n} \left[ \frac{1}{[k^2 - M^2 + i0^+][2(k + p)^2 - m^2 + i0^+]} \right],$$

$$p^\mu H_{\pi\pi N}^{(p\mu)}(p^2) = i \int \frac{d^n k}{(2\pi)^n} \left[ \frac{k^\mu}{[k^2 - M^2 + i0^+][2(k + p)^2 - m^2 + i0^+]} \right].$$
The tensorial loop integrals can be reduced to scalar ones \cite{5,6} and we obtain

\[
H_\pi^{(00)} = \frac{M^2}{n} H_\pi,
\]

\[
H_\pi^{(00)} = \frac{1}{n} \left[ H_\pi + M^2 H_\pi \right],
\]

\[
H_\pi^{(0000)} = \frac{M^4}{n(n+2)} H_\pi,
\]

\[
H_\pi^{(0)}(p^2) = \frac{1}{2p^2} \left[ H_\pi - H_N - (p^2 - m^2 + M^2) H_{\pi N}(p^2) \right],
\]

\[
H_\pi^{(p)}(p^2) = \frac{1}{2p^2} \left[ H_\pi - H_{\pi N} - (p^2 - m^2 + M^2) H_{\pi \pi N}(p^2) \right].
\]

Defining

\[
\lambda = \frac{\mu^{n-4}}{16\pi^2} \left\{ \frac{1}{n-4} - \frac{1}{2} \ln(4\pi) - \gamma_E + 1 \right\},
\]

where \(\gamma_E = -\Gamma'(1)\) is Euler’s constant, and

\[
\Omega = \frac{p^2 - m^2 - M^2}{2mM},
\]

the scalar loop integrals are given by \cite{15}

\[
H_\pi = 2M^2 \lambda + \frac{M^2}{8\pi^2} \ln \frac{M}{\mu},
\]

\[
H_\pi^{(0)}(p^2) = 2\lambda + \frac{1}{16\pi^2} \left[ 1 + 2 \ln \frac{M}{\mu} \right],
\]

\[
H_N = 2m^2 \lambda + \frac{m^2}{8\pi^2} \ln \frac{m}{\mu},
\]

\[
H_{\pi \pi N}(p^2) = 2\lambda + \frac{1}{16\pi^2} \left[ -1 + \frac{p^2 - m^2 + M^2}{p^2} \ln \frac{M}{\mu} + \frac{2mM}{p^2} F(\Omega) \right],
\]

where

\[
F(\Omega) = \begin{cases} 
\sqrt{\Omega^2 - 1} \ln (-\Omega - \sqrt{\Omega^2 - 1}), & \Omega \leq -1, \\
\sqrt{1 - \Omega^2} \arccos(-\Omega), & -1 \leq \Omega \leq 1, \\
\sqrt{\Omega^2 - 1} \ln (\Omega + \sqrt{\Omega^2 - 1}) - i\pi \sqrt{\Omega^2 - 1}, & 1 \leq \Omega.
\end{cases}
\]

The integral \(H_{\pi \pi N}\) can be obtained from \(H_{\pi N}(p^2)\) by differentiating with respect to \(M^2\).

**APPENDIX C: INTEGRALS AT THE TWO-LOOP LEVEL**

We define the one-loop integral

\[
H_{\pi N}(a, b|n) = \int \frac{d^n k}{(2\pi)^n} \frac{1}{[k^2 - M^2 + i0^+]^a [k^2 + 2p \cdot k + i0^+]^b}.
\]
Note that we have not included a factor $i$ in the definition, since the one-loop integrals $H_{\pi N}(a, b|n)$ always appear in the product $H_{\pi N}(a_1, b_1|n_1)H_{\pi N}(a_2, b_2|n_2)$.

The two-loop integrals $H_2(a, b, c, d, e|n)$ are defined as

$$H_2(a, b, c, d, e|n) = \iiint \frac{d^n k_1 d^n k_2}{(2\pi)^{2n}} \frac{1}{A^a B^b C^c D^d E^e},$$

where

$$A = k_1^2 - M^2 + i0^+, \quad B = k_2^2 - M^2 + i0^+, \quad C = k_1^2 + 2p \cdot k_1 + i0^+, \quad D = k_2^2 + 2p \cdot k_2 + i0^+, \quad E = k_1^2 + 2p \cdot k_1 + 2k_1 \cdot k_2 + 2p \cdot k_2 + k_2^2 + i0^+.$$ 

The product of two one-loop integrals with the same space-time dimension $n$ can then be written as

$$H_{\pi N}(a_1, b_1|n)H_{\pi N}(a_2, b_2|n) = H_2(a_1, a_2, b_1, b_2, 0|n).$$

We do not attempt to evaluate the integrals $H_2(a, b, c, d, e|n)$ here. Instead the expressions for the integrals relevant to the nucleon mass including the corresponding counterterm contributions are given in App. [D]

Tensorial integrals have been reduced to scalar ones in the same dimension using methods similar to the one-loop integrals, or the following relations to scalar integrals in higher dimensions have been used:

$$H_2^\mu(a, b, c, d, e|n) = \iiint \frac{d^n k_1 d^n k_2}{(2\pi)^{2n}} \frac{k_1^\mu}{A^a B^b C^c D^d E^e}$$

$$= -16\pi^2 p^\mu \left[ b c H_2(a, b + 1, c + 1, d, e|n + 2) + c d H_2(a, b, c + 1, d + 1, e|n + 2) + c e H_2(a, b, c + 1, d + 1|n + 2) + b e H_2(a, b + 1, c, d, e + 1|n + 2) \right],$$

$$H_2^{\mu\nu}(a, b, c, d, e|n) = \iiint \frac{d^n k_1 d^n k_2}{(2\pi)^{2n}} \frac{k_1^\mu k_2^\nu}{A^a B^b C^c D^d E^e}$$

$$= \frac{(4\pi)^2}{2} g^{\mu\nu} \left[ b H_2(a, b + 1, c, d, e|n + 2) + d H_2(a, b, c, d + 1, e|n + 2) + e H_2(a, b, c, d, e + 1|n + 2) \right] + \mathcal{O}(p),$$

$$H_2^{\mu\nu\lambda}(a, b, c, d, e|n) = \iiint \frac{d^n k_1 d^n k_2}{(2\pi)^{2n}} \frac{k_1^\mu k_2^\nu k_1^\lambda}{A^a B^b C^c D^d E^e}$$

$$= -\frac{(4\pi)^4}{2} \left[ g^{\mu\nu} p^\lambda + g^{\mu\lambda} p^\nu + g^{\nu\lambda} p^\mu \right] \left[ b (b + 1) c H_2(a, b + 2, c + 1, d, e|n + 4) + b (b + 1) e H_2(a, b + 2, c, d, e + 1|n + 4) \right] + \mathcal{O}(p^3),$$

$$H_2^{\mu,\nu}(a, b, c, d, e|n) = \iiint \frac{d^n k_1 d^n k_2}{(2\pi)^{2n}} \frac{k_1^\mu k_2^{\nu}}{A^a B^b C^c D^d E^e}$$

$$= -\frac{(4\pi)^2}{2} g^{\mu\nu} e H_2(a, b, c, d, e + 1|n + 2) + \mathcal{O}(p),$$

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\[ H_2^{\mu\alpha\beta}(a, b, c, d, e|n) = \int \int \frac{d^n k_1 d^n k_2}{(2\pi)^{2n}} \frac{k_1^\alpha k_2^\beta}{A^a B^b C^c D^d E^e} \]

\[ = -\frac{(4\pi)^2}{2} g^{\alpha\beta} p^\mu [c H_2(a, b, c + 1, d, e|n + 2) + e H_2(a, b, c, d, e + 1|n + 2)] \]

\[ + \frac{(4\pi)^4}{2} \left[ g^{\alpha\beta} p^\mu + g^{\mu\alpha} p^\beta + g^{\mu\beta} p^\alpha \right] [a e (e + 1) H_2(a + 1, b, c, d, e + 2|n + 4) \]

\[ + a d e H_2(a + 1, b, c, d, e + 1|n + 4) \]

\[ + c d e H_2(a, b, c + 1, d + 1, e + 1|n + 4) \]

\[ + d e (e + 1) H_2(a, b, c, d + 1, e + 2|n + 4)] + \mathcal{O}(p^3), \]

\[ H_2^{\alpha\beta,\mu\nu}(a, b, c, d, e|n) = \int \int \frac{d^n k_1 d^n k_2}{(2\pi)^{2n}} \frac{k_1^\alpha k_2^\beta k_1^\mu k_2^\nu}{A^a B^b C^c D^d E^e} \]

\[ = \frac{(4\pi)^4}{4} \left[ g^{\alpha\beta} g^{\mu\nu} + g^{\alpha\mu} g^{\beta\nu} + g^{\alpha\nu} g^{\beta\mu} \right] e (e + 1) H_2(a, b, c, d, e + 2|n + 4) \]

\[ + \frac{(4\pi)^2}{4} g^{\alpha\beta} g^{\mu\nu} H_2(a, b, c, d, e|n + 2) + \mathcal{O}(p). \]

Here, \( \mathcal{O}(p) \) stands for terms proportional to \( p^\rho \), where \( \rho \) denotes the Lorentz index corresponding to the integral under consideration. In our calculation of the nucleon mass these terms appear in combination with expressions like \( (\not p - m) \gamma_\rho (\not p + m) \), resulting in higher-order contributions that are not considered.

**APPENDIX D: EVALUATION OF TWO-LOOP INTEGRALS**

As seen in Sec. [V] the calculation of the two-loop integrals relevant to the nucleon mass reduces to the evaluation of the \( F_4 \) part of the respective integrals. The \( F_4 \) parts are sums of tensor integrals, which can be reduced to scalar integrals [50] of the form

\[ H_2^{(1,1)}(a, b, c, d, e|n) = \frac{1}{(2\pi)^{2n}} \frac{1}{[k_1^2 - m^2 + i0^+][k_2^2 - m^2 + i0^+][2p \cdot k_1 + i0^+]^c} \]

\[ \times \frac{1}{[2p \cdot k_2 + i0^+]^d[2p \cdot k_1 + 2p \cdot k_2 + i0^+]^e}, \]  

(D1)

where the superscript \( (1, 1) \) indicates that these integrals have been obtained after rescaling both integration variables (see App. [A]) and \( a, b, c, d, e \) are integers. Depending on the values of the exponents \( c, d, \) and \( e \), one can evaluate \( H_2^{(1,1)}(a, b, c, d, e|n) \) with the help of several basic integrals.

1. \( e = 0 \)

If the exponent \( e \) vanishes, the integral can be written as the product of one-loop integrals,

\[ H_2^{(1,1)}(a, b, c, d, 0|n) = H_1^{(1)}(a, c|n) H_1^{(1)}(b, d|n), \]  

(D2)
where
\[ H_1^{(1)}(a, b|n) = \int \frac{d^n k}{(2\pi)^n} \frac{1}{[k^2 - m^2 + i0^+]^a[2p \cdot k + i0^+]^b} \]
\[ = \frac{i^{1-2a-2b}}{2^n (4\pi)^{n/2}} \frac{\Gamma[\frac{1}{2}]\Gamma[a + \frac{b}{2} - \frac{n}{2}]}{\Gamma[a] \Gamma[\frac{b+1}{2}]} (m^2)^{n/2-a-b/2} (p^2)^{-b/2}. \] (D3)

2. \( c = d = 0, c \neq 0 \)

If \( c = d = 0 \), the expression for the integral \( H_2(a, b, 0, 0, e|n) \) reads
\[ H_2^{(1,1)}(a, b, 0, 0, e|n) \]
\[ = \int \int \frac{d^n k_1 d^n k_2}{(2\pi)^{2n}} \frac{1}{[k_1^2 - m^2 + i0^+]^a[k_2^2 - m^2 + i0^+]^b[2p \cdot k_1 + 2p \cdot k_2 + i0^+]^c} \]
\[ = \frac{\Gamma[\alpha + \frac{\beta}{2} - \frac{n}{2}]\Gamma[\alpha + \frac{\beta}{2} - \frac{n}{2}]\Gamma[\alpha + \frac{\beta}{2} - \frac{n}{2}]\Gamma[\alpha + \frac{\beta}{2} - \frac{n}{2}]}{\Gamma[\alpha] \Gamma[\beta] \Gamma[\epsilon] \Gamma[\alpha + b + e - n]} \] (D4)

3. \( d = 0, c \neq 0, e \neq 0 \) and \( c = 0, d \neq 0, e \neq 0 \)

For vanishing \( d \) with non-vanishing \( c \) and \( e \) we consider the integral \( H_2^{(1,1)}(a, b, c, 0, e|n) \) for \( p^2 = m_N^2 \).
\[ H_2^{(1,1)}(a, b, c, d, e|n) = \int \int \frac{d^n k_1 d^n k_2}{(2\pi)^{2n}} \frac{1}{[k_1^2 - m^2 + i0^+]^a[k_2^2 - m^2 + i0^+]^b[2p \cdot k_1 + i0^+]^c} \]
\[ \times \frac{1}{[2p \cdot k_2 + i0^+]^d[2p \cdot k_1 + 2p \cdot k_2 + i0^+]^e} \] \( p^2 = m_N^2 \). (D5)

Note that the mass terms \( m \) in the first two propagators stem from the rescaling of the loop momenta, while we have to consider \( p^2 = m_N^2 \) when evaluating the nucleon mass. In the calculations performed in this work the difference between \( p^2 = m^2 \) and \( p^2 = m_N^2 \) in these integrals is of higher order.

The result for \( H_2^{(1,1)}(a, b, c, 0, e|n) \) is given by the sum
\[ H_2^{(1,1)}(a, b, c, 0, e|n) = \sum_{l=0}^{c-1} \left( \begin{array}{l} c-1 \end{array} \right) (-1)^l Z_{(c+1-l)/2}(a, b, c, 0, e|n), \] (D6)

where
\[ Z_\alpha(a, b, c, 0, e|n) = \frac{\Gamma[\alpha + 1]}{\Gamma[\alpha + 2]} \frac{\Gamma[\alpha + \frac{1}{2}]}{\Gamma[\alpha + \frac{1}{2}]} \frac{\Gamma[\alpha + \frac{1}{2}]}{\Gamma[\alpha + \frac{1}{2}]} \frac{\Gamma[\alpha + \frac{1}{2}]}{\Gamma[\alpha + \frac{1}{2}]} \frac{\Gamma[\alpha + \frac{1}{2}]}{\Gamma[\alpha + \frac{1}{2}]} \frac{\Gamma[\alpha + \frac{1}{2}]}{\Gamma[\alpha + \frac{1}{2}]} \]
\[ \times 3F_2 \left( \begin{array}{l} 1, c/2 + e/2, a + c/2 + e/2 - n/2 \\ 0, 0, 0 \\ 1, a + b + c + e - n \end{array} \right) \] (D7)

and \( 3F_2 \left( \begin{array}{l} a, b, c \\ d, e \end{array} \right) \) is a hypergeometric function. The case \( c = 0, d \neq 0, e \neq 0 \) is obtained by replacing \( c \) with \( d \) and interchanging \( a \) and \( b \) in Eq. (D7).
4. \( c \neq 0, d \neq 0, e \neq 0 \)

For the case that none of the exponents \( c, d, e \) vanishes, it is convenient to perform an expansion into partial fractions,

\[
\frac{1}{[2p \cdot k_2 + i0^+] [2p \cdot k_1 + 2p \cdot k_2 + i0^+]} = \frac{1}{[2p \cdot k_1 + i0^+] [2p \cdot k_2 + i0^+]} - \frac{1}{[2p \cdot k_1 + i0^+] [2p \cdot k_1 + 2p \cdot k_2 + i0^+]},
\]

(D8)

until one obtains a sum of integrals of the form \( H_2^{1,1}(a, b, \tilde{c}, 0, \tilde{e}) \) and \( H_2^{1,1}(a, b, \tilde{c}, d, 0) \), which are evaluated as described above.

5. **Subtraction terms**

In addition to the integrals given above the evaluation of the subintegrals for the \( F_4 \) terms requires the integrals

\[
H_1^{1,1}(a, b; \omega|n) = \int \frac{d^nk}{(2\pi)^n [k^2 - m^2 + i0^+]^a [2p \cdot k + \omega + i0^+]^b},
\]

(D9)

where \( \omega = 2p \cdot q \) with \( q \) the second loop momentum. The integral \( H_1^{1,1}(a, b; \omega|n) \) is given by the sum

\[
H_1^{1,1}(a, b; \omega|n) = \frac{i^{-1-2a-2b} m^n - 2a - 2b}{(4\pi)^{n/2}} \sum_{l=0}^{\infty} \frac{\Gamma[b + \frac{l}{2}] \Gamma[a + b - \frac{n}{2} + \frac{l}{2}]}{2\Gamma[a] \Gamma[b] \Gamma[l + 1]} \left( \frac{\omega}{m^2} \right)^l \left( \frac{m^2}{p^2} \right)^{b/2 + l/2}.
\]

(D10)

The sum contains an infinite number of terms. However, when performing the second loop integration over \( q \) in the considered counterterm integrals, increasing orders of \( \omega = 2p \cdot q \) contribute to increasing chiral orders. Therefore only a finite number of terms in Eq. (D10) is needed in the calculation of the nucleon mass.

6. **Results for \( H_2^{1,1}(a, b, c, d, e|n) \) and counterterm integrals**

The results for the \( H_2^{1,1} \) parts of the two-loop integrals contributing to the nucleon mass evaluated on-mass-shell are given by

\[
\bar{\mu}^{8-2n} H_2^{1,1}(1, 1, 0, 0, 1|n) = -\frac{1}{\epsilon^2} \frac{3M^4}{1024\pi^4 m^2} - \frac{1}{\epsilon} \frac{M^4}{1024\pi^4 m^2} \left[ 1 + 12 \ln \frac{M}{\mu} \right] - \frac{M^4}{2048\pi^4 m^2} \left[ \pi^2 + 10 \right.
\]

\[+ 8 \ln \frac{M}{\mu} + 48 \ln^2 \frac{M}{\mu} \right],
\]

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\[
\hat{\mu}^{8-2n} H_2^{(1,1)}(1, 1, 0, 0, 2|n + 2) \\
= \frac{1}{\epsilon^2} \frac{M^6}{24576\pi^6 m^2} - \frac{1}{\epsilon} \frac{M^6}{36864\pi^6 m^2} \left[ 1 - 6 \ln \frac{M}{\mu} + \frac{M^6}{442368\pi^6 m^2} \right] + 26 - 48 \ln \frac{M}{\mu} + 144 \ln^2 \frac{M}{\mu},
\]

\[
\hat{\mu}^{8-2n} H_2^{(1,1)}(1, 2, 0, 0, 2|n + 2) \\
= \frac{1}{\epsilon^2} \frac{M^4}{16384\pi^6 m^2} + \frac{1}{\epsilon} \frac{M^4}{4096\pi^6 m^2} \ln \frac{M}{\mu} + \frac{M^4}{98304\pi^6 m^2} \left[ \pi^2 + 6 + 48 \ln^2 \frac{M}{\mu} \right],
\]

\[
\hat{\mu}^{8-2n} H_2^{(1,1)}(1, 1, 1, 0, 2|n + 2) \\
= -\frac{1}{\epsilon^2} \frac{M^6}{98304\pi^6 m^2} + \frac{1}{\epsilon} \left\{ \frac{M^5}{12288\pi^6 m^3} + \frac{M^6}{147456\pi^6 m^4} \left[ 1 - 6 \ln \frac{M}{\mu} \right] \right\} \\
+ \frac{M^5}{36864\pi^6 m^3} \left[ 6 \ln(2) - 5 + 12 \ln \frac{M}{\mu} \right] - \frac{M^6}{1769472\pi^6 m^4} \left[ 75\pi^2 + 26 \right] \\
- 48 \ln \frac{M}{\mu} + 144 \ln^2 \frac{M}{\mu},
\]

\[
\hat{\mu}^{8-2n} H_2^{(1,1)}(1, 1, 2, 0, 2|n + 2) \\
= -\frac{1}{\epsilon^2} \frac{5M^4}{98304\pi^6 m^4} - \frac{1}{\epsilon} \frac{M^4}{147456\pi^6 m^4} \left[ 1 + 30 \ln \frac{M}{\mu} \right] - \frac{M^4}{1769472\pi^6 m^4} \left[ 87\pi^2 \right] \\
+ 82 + 48 \ln \frac{M}{\mu} - 720 \ln^2 \frac{M}{\mu},
\]

\[
\hat{\mu}^{8-2n} H_2^{(1,1)}(1, 2, 1, 0, 1|n + 2) \\
= \frac{1}{\epsilon^2} \frac{M^4}{49152\pi^6 m^2} - \frac{1}{\epsilon} \frac{M^4}{73728\pi^6 m^2} \left[ 1 - 6 \ln \frac{M}{\mu} \right] - \frac{M^4}{884736\pi^6 m^2} \left[ 69\pi^2 - 26 \right] \\
+ 48 \ln \frac{M}{\mu} - 144 \ln^2 \frac{M}{\mu},
\]

\[
\hat{\mu}^{8-2n} H_2^{(1,1)}(1, 2, 2, 0, 1|n + 2) \\
= \frac{1}{\epsilon^2} \frac{11M^4}{98304\pi^6 m^4} - \frac{1}{\epsilon} \left[ \frac{M^3}{12288\pi^5 m^3} - \frac{M^4}{73728\pi^6 m^4} \left( 5 + 33 \ln \frac{M}{\mu} \right) \right] \\
+ \frac{M^3}{18432\pi^5 m^3} \left[ 1 - \ln 8 - 6 \ln \frac{M}{\mu} \right] + \frac{M^4}{1769472\pi^6 m^4} \left[ 190 + 105\pi^2 \right] \\
+ 480 \ln \frac{M}{\mu} + 1584 \ln^2 \frac{M}{\mu},
\]

\[
2\hat{\mu}^{8-2n} H_2^{(1,1)}(1, 2, 1, 0, 3|n + 4) + \hat{\mu}^{8-2n} H_2^{(1,1)}(12202|n + 4) \\
= -\frac{1}{\epsilon^2} \frac{M^6}{786432\pi^8 m^8} + \frac{1}{\epsilon} \frac{M^6}{1179648\pi^8 m^8} \left[ 1 - 6 \ln \frac{M}{\mu} \right] - \frac{M^6}{14155776\pi^8 m^8} \left[ 3\pi^2 \right] \\
+ 26 - 48 \ln \frac{M}{\mu} + 144 \ln^2 \frac{M}{\mu},
\]

\[
\hat{\mu}^{8-2n} H_2^{(1,1)}(2, 1, 1, 0, 3|n + 4)
\]
\[
\bar{\mu}^{8-2n}H^{(1)}_{CT_1}(1, 1, 1, 1|n + 2) = \frac{1}{\epsilon^2} \frac{M^6}{1572864\pi^8 m^4} - \frac{1}{\epsilon} \frac{M^5}{235926} \left[ 1 - 6 \ln \frac{M}{\mu} \right] + \frac{M^6}{28311552\pi^8 m^4} \left[ 27\pi^2 + 26 - 48 \ln \frac{M}{\mu} + 144 \ln^2 \frac{M}{\mu} \right],
\]

where \(\bar{\mu}\) is the 't Hooft parameter and we have used \(\bar{\mu} = \mu e^{2n-\frac{1}{2}}\). In the above notation the \(\overline{\text{MS}}\) scheme of ChPT corresponds to subtracting all terms proportional to \(\epsilon^{-1}\) and setting \(\mu = m\).

The corresponding counterterm integrals \(H^{(1)}_{CT_1}(a, b, c, d, e)\) and \(H^{(1)}_{CT_2}(a, b, c, d, e)\) are given by

\[
\bar{\mu}^{8-2n}H^{(1)}_{CT_1}(1, 1, 0, 0, 0|n) = -\frac{1}{\epsilon^2} \frac{3M^4}{1024\pi^4 m^2} - \frac{1}{\epsilon} \frac{M^4}{2048\pi^4 m^2} \left[ 1 + 12 \ln \frac{M}{\mu} \right] - \frac{M^4}{4096\pi^4 m^2} \left[ \pi^2 + 5 + 4 \ln \frac{M}{\mu} \right] + 24 \ln^2 \frac{M}{\mu}.
\]

\[
\bar{\mu}^{8-2n}H^{(1)}_{CT_2}(1, 1, 0, 0, 0|n) = \frac{1}{\epsilon^2} \frac{M^4}{24576\pi^6 m^2} - \frac{1}{\epsilon} \frac{M^4}{73728\pi^6 m^2} \left[ 1 - 6 \ln \frac{M}{\mu} \right] + \frac{M^4}{884736\pi^6 m^2} \left[ 3\pi^2 + 22 - 24 \ln \frac{M}{\mu} + 72 \ln^2 \frac{M}{\mu} \right],
\]

\[
\bar{\mu}^{8-2n}H^{(1)}_{CT_2}(1, 1, 0, 0, 2|n + 2) = \frac{1}{\epsilon^2} \frac{M^4}{16384\pi^6 m^2} + \frac{1}{\epsilon} \frac{M^4}{32768\pi^6 m^2} \left[ 1 + 4 \ln \frac{M}{\mu} \right] + \frac{M^4}{196608\pi^6 m^2} \left[ \pi^2 + 3 + 12 \ln \frac{M}{\mu} \right] + 24 \ln^2 \frac{M}{\mu}.
\]

\[
\bar{\mu}^{8-2n}H^{(1)}_{CT_2}(1, 2, 0, 0, 2|n + 2) = \frac{1}{\epsilon^2} \frac{M^4}{16384\pi^6 m^2} - \frac{1}{\epsilon} \frac{M^4}{32768\pi^6 m^2} \left[ 1 - 4 \ln \frac{M}{\mu} \right] + \frac{M^4}{196608\pi^6 m^2} \left[ \pi^2 + 9 - 12 \ln \frac{M}{\mu} \right] + 24 \ln^2 \frac{M}{\mu}.
\]

\[
\bar{\mu}^{8-2n}H^{(1)}_{CT_2}(1, 1, 1, 0, 2|n + 2) = \frac{1}{\epsilon^2} \frac{M^6}{32768\pi^6 m^4} - \frac{1}{\epsilon} \frac{M^6}{589824\pi^6 m^4} \left[ 11 - 36 \ln \frac{M}{\mu} \right] + \frac{M^6}{3538944\pi^6 m^4} \left[ 9\pi^2 + 91 \right]
\]
\[
\mu^{8-2n} H_{\text{CT1}}^{(1,1)} (1, 1, 1, 0, 0, 2|n + 2) \]
\[
= -\frac{1}{\epsilon^2} 98304 \pi^6 m^4 \left[ 1 + 20 \ln \frac{M}{\mu} \right] + \frac{M^4}{589824 \pi^6 m^4} \left[ 5\pi^2 + 27 \right] + 360 \ln^2 \frac{M}{\mu},
\]
\[
\mu^{8-2n} H_{\text{CT2}}^{(1,1)} (1, 1, 0, 0|n + 2) \]
\[
= -\frac{1}{\epsilon^2} 16384 \pi^6 m^4 \left[ 1 - 4 \ln \frac{M}{\mu} \right] - \frac{M^4}{196608 \pi^6 m^4} \left[ \pi^2 + 9 \right] - 12 \ln \frac{M}{\mu} + 24 \ln^2 \frac{M}{\mu},
\]
\[
\mu^{8-2n} H_{\text{CT1}}^{(1,1)} (1, 2, 2, 0, 1|n + 2) \]
\[
= -\frac{1}{\epsilon^2} 98304 \pi^6 m^4 \left[ 1 + 28 \ln \frac{M}{\mu} \right] + \frac{M^4}{1179648 \pi^6 m^4} \left[ 7\pi^2 + 39 \right] + 12 \ln \frac{M}{\mu} + 168 \ln^2 \frac{M}{\mu},
\]
\[
\mu^{8-2n} H_{\text{CT2}}^{(1,1)} (1, 2, 2, 0, 1|n + 2) \]
\[
= -\frac{1}{\epsilon^2} 32768 \pi^6 m^4 \left[ 5 - 6 \ln 2 + 6 \ln \frac{M}{\mu} \right] + \frac{M^4}{393216 \pi^6 m^4} \left[ 5\pi^2 + 39 - 36 \ln \frac{M}{\mu} \right] + 120 \ln^2 \frac{M}{\mu},
\]
\[
2\mu^{8-2n} H_{\text{CT1}}^{(1,1)} (1, 2, 1, 0, 3|n + 4) + \mu^{8-2n} H_{\text{CT1}}^{(1,1)} (12202|n + 4) \]
\[
= -\frac{1}{\epsilon^2} 786432 \pi^8 m^4 \left[ 1 - 6 \ln \frac{M}{\mu} \right] - \frac{M^6}{28311552 \pi^8 m^4} \left[ 3\pi^2 + 22 \right] - 24 \ln \frac{M}{\mu} + 72 \ln^2 \frac{M}{\mu},
\]
\[
= 2\mu^{8-2n} H_{\text{CT2}}^{(1,1)} (1, 2, 1, 0, 3|n + 4) + \mu^{8-2n} H_{\text{CT2}}^{(1,1)} (12202|n + 4),
\]
\[
\mu^{8-2n} H_{\text{CT1}}^{(1,1)} (2, 1, 1, 0, 3|n + 4)
\]
\[\begin{align*}
&= -\frac{1}{\epsilon^2} \frac{M^6}{4718592 \pi^8 m^4} + \frac{1}{\epsilon} \frac{M^6}{28311552 \pi^8 m^4} \left[ 5 - 12 \ln \frac{M}{\mu} \right] - \frac{M^6}{169869312 \pi^8 m^4} \left[ 3\pi^2 \right] \\
&+ \frac{37 - 60 \ln \frac{M}{\mu} + 72 \ln^2 \frac{M}{\mu}}{2} ,
\end{align*}\]

\[\begin{align*}
\tilde{\mu}^{8-2n} H^{(1,1)}_{CT2}(2, 1, 1, 0, 3|n + 4) \\
&= \frac{1}{\epsilon^2} \frac{7M^6}{4718592 \pi^8 m^4} - \frac{1}{\epsilon} \frac{M^6}{28311552 \pi^8 m^4} \left[ 17 - 84 \ln \frac{M}{\mu} \right] + \frac{M^6}{169869312 \pi^8 m^4} \left[ 21 \pi^2 \right] \\
&+ 169 - 204 \ln \frac{M}{\mu} + 504 \ln^2 \frac{M}{\mu} ,
\end{align*}\]

\[\begin{align*}
\tilde{\mu}^{8-2n} H^{(1,1)}_{CT1}(1, 1, 1, 1|n + 2) \\
&= \frac{1}{\epsilon^2} \left[ \frac{M^5}{12288 \pi^5 m^5} + \frac{M^6}{36864 \pi^6 m^6} \right] - \frac{1}{\epsilon} \frac{M^6}{884736 \pi^6 m^6} \left[ 11 - 48 \ln \frac{M}{\mu} \right] \\
&- \frac{M^5}{36864 \pi^5 m^5} \left[ 5 - 6 \ln(2) - 6 \ln \frac{M}{\mu} \right] \\
&= \tilde{\mu}^{8-2n} H^{(1,1)}_{CT2}(1, 1, 1, 1|n + 2).
\end{align*}\]

The results for the products of one-loop integrals read

\[\begin{align*}
\tilde{\mu}^{8-2n} H^{(1,1)}_{2}(1, 1, 0, 0|n) \\
&= -\frac{1}{\epsilon^2} \frac{M^4}{256 \pi^4} - \frac{1}{\epsilon} \frac{M^4}{64 \pi^4} \ln \frac{M}{\mu} - \frac{M^4}{1536 \pi^4} \left[ \pi^2 + 6 + 48 \ln^2 \frac{M}{\mu} \right] ,
\end{align*}\]

\[\begin{align*}
\tilde{\mu}^{8-2n} H^{(1,1)}_{2}(1, 2, 0, 0|n) \\
&= -\frac{1}{\epsilon^2} \frac{M^2}{256 \pi^4} - \frac{1}{\epsilon} \frac{M^2}{256 \pi^4} \left[ 1 + \ln \frac{M}{\mu} \right] - \frac{M^2}{1536 \pi^4} \left[ \pi^2 + 6 + 24 \ln \frac{M}{\mu} + 48 \ln^2 \frac{M}{\mu} \right] ,
\end{align*}\]

\[\begin{align*}
\tilde{\mu}^{8-2n} H^{(1,1)}_{2}(1, 1, 1, 0|n) \\
&= -\frac{1}{\epsilon^2} \frac{M^4}{512 \pi^4 m^2} - \frac{1}{\epsilon} \left[ \frac{M^3}{256 \pi^3 m} + \frac{M^4}{512 \pi^4 m^2} \left( 1 + 4 \ln \frac{M}{\mu} \right) \right] + \frac{M^3}{256 \pi^3 m} \left[ 1 - 2 \ln(2) \right] \\
&- 4 \ln \frac{M}{\mu} - \frac{M^2}{3072 \pi^4 m^2} \left[ \pi^2 + 6 + 24 \ln \frac{M}{\mu} + 48 \ln^2 \frac{M}{\mu} \right] ,
\end{align*}\]

\[\begin{align*}
\tilde{\mu}^{8-2n} H^{(1,1)}_{2}(1, 2, 0, 1|n) \\
&= -\frac{1}{\epsilon^2} \frac{M^2}{512 \pi^4 m^2} - \frac{1}{\epsilon} \left[ \frac{M^3}{256 \pi^3 m} + \frac{M^2}{512 \pi^4 m^2} \left( 1 + 2 \ln \frac{M}{\mu} \right) \right] - \frac{M^3}{512 \pi^3 m} \left[ 1 + 2 \ln(2) \right] \\
&+ 4 \ln \frac{M}{\mu} - \frac{M^2}{3072 \pi^4 m^2} \left[ \pi^2 + 12 + 48 \ln \frac{M}{\mu} + 48 \ln^2 \frac{M}{\mu} \right] ,
\end{align*}\]

\[\begin{align*}
\tilde{\mu}^{8-2n} H^{(1,1)}_{2}(2, 1, 0, 2|n + 2) \\
&= \frac{1}{\epsilon^2} \frac{M^4}{8192 \pi^6 m^2} + \frac{1}{\epsilon} \frac{M^4}{2048 \pi^6 m^2} \ln \frac{M}{\mu} + \frac{M^4}{49152 \pi^6 m^2} \left[ \pi^2 + 6 + 48 \ln^2 \frac{M}{\mu} \right] ,
\end{align*}\]

\[\begin{align*}
\tilde{\mu}^{8-2n} H^{(1,1)}_{2}(1, 1, 1, 0|n) \\
&= -\frac{M^2}{256 \pi^2 m^2} .
\end{align*}\]
\[ \tilde{\mu}^{8-2n} H_{CT_1}^{(1)}(1, 1, 0, 0, 0|n) \]
\[ = \frac{1}{e^2} \frac{M^4}{256\pi^4} - \frac{1}{e} \frac{M^4}{128\pi^4} \ln \frac{M}{\mu} - \frac{M^4}{3072\pi^4} \ln \frac{M}{\mu} + \frac{M^4}{3072\pi^4} \left[ \pi^2 + 6 + 24 \ln^2 \frac{M}{\mu} \right] \]

The integral \( H_{CT_2}^{(1)}(1, 1, 0, 0|n) \) can be written as \( H_{CT_2}^{(1)}(1, 1, 0, 0|n) \) by the substitution \( k_2 \leftrightarrow k_2 + k_1 \), and

\[ H_{CT_2}^{(1)}(1, 0, 0, 1|n) = H_1^{(1)}(1, 0|n) H_2^{(1)}(0, 1|n) = 0, \]

since the infrared singular part \( H_1^{(1)}(01|n) \) of \( H_1(01|n) = -iN_1 \) vanishes.

The counterterm integrals corresponding to the products of one-loop integrals read

\[ \tilde{\mu}^{8-2n} H_{CT_1}^{(1)}(1, 1, 0, 0, 0|n) \]
\[ = \frac{1}{e^2} \frac{M^4}{256\pi^4} - \frac{1}{e} \frac{M^4}{128\pi^4} \ln \frac{M}{\mu} - \frac{M^4}{3072\pi^4} \ln \frac{M}{\mu} + \frac{M^4}{3072\pi^4} \left[ \pi^2 + 6 + 24 \ln^2 \frac{M}{\mu} \right] \]

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\[ \tilde{\mu}^{8-2n} H_{CT_2}^{(1,1)} (2, 1, 0, 2, 0 \mid n + 2), \]
\[ \tilde{\mu}^{8-2n} H_{CT}^{(1,1)} (1, 1, 1, 1, 0 \mid n) \]
\[ = 0 \]
\[ = \tilde{\mu}^{8-2n} H_{CT_2}^{(1,1)} (1, 1, 1, 1, 0 \mid n), \]
\[ \tilde{\mu}^{8-2n} H_{CT}^{(1,1)} (1, 1, 1, 1, 0 \mid n + 2) \]
\[ = 0 \]
\[ = \tilde{\mu}^{8-2n} H_{CT_2}^{(1,1)} (1, 1, 1, 1, 0 \mid n + 2). \]

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