ON THE AUTOMORPHISM GROUPS OF MODELS IN $\mathbb{C}^2$

NINH VAN THU* AND MAI ANH DUC**

Abstract. In this note, we consider models in $\mathbb{C}^2$. The purpose of this note is twofold. We first show a characterization of models in $\mathbb{C}^2$ by their noncompact automorphism groups. Then we give an explicit description for automorphism groups of models in $\mathbb{C}^2$.

1. Introduction

For a domain $\Omega$ in the complex Euclidean space $\mathbb{C}^n$, the set of biholomorphic self-maps forms a group under the binary operation of composition of mappings, which is called automorphism group (Aut($\Omega$)). The topology on Aut($\Omega$) is that of uniform convergence on compact sets (i.e., the compact-open topology).

A boundary point $p \in \partial \Omega$ is called a boundary orbit accumulation point if there exist a sequence $\{f_j\} \subset \text{Aut}(\Omega)$ and a point $q \in \Omega$ such that $f_j(q) \to p$ as $j \to \infty$.

The classification of domains with noncompact automorphism groups is pertinent to the study of the geometry of the boundary at an orbit accumulation point. In this note, we consider a model

$$M_H = \{ (z_1, z_2) \in \mathbb{C}^2 : \Re z_2 + H(z_1) < 0 \},$$

where $H$ is a homogeneous subharmonic polynomial of degree $2m$ ($m \geq 1$) which contains no harmonic terms. It is a well-known result of F. Berteloot [7] that if $\Omega \subset \mathbb{C}^2$ is pseudoconvex, of D’Angelo finite type near a boundary orbit accumulation point, then $\Omega$ is biholomorphically equivalent to a model $M_H$. For the case $\Omega$ is strongly pseudoconvex, this result was proved by B. Wong [30] and J. P. Rosay [25]; indeed, the model is biholomorphically equivalent to the unit ball. These results motivate the following several concepts.

We first prove the following theorem.
Theorem 1. Let $\Omega$ be a domain in $\mathbb{C}^2$ and let $p \in \partial \Omega$. Suppose that $\Omega$ satisfies Condition $(M_H)$ at $p$ and there exist a sequence $\{f_n\} \subset \text{Aut}(\Omega)$ and $q \in \Omega$ such that $\{f_n(q)\}$ converges tangentially to order $\leq 2m$ ($= \deg(H)$) to $p$. Then $\Omega$ is biholomorphically equivalent to the model $M_H$.

Remark 1. Because of Condition $(M_H)$ at $p$, $\Omega$ is of finite type at $p$. Therefore, it is proved in [7] that $\Omega$ is biholomorphically equivalent to some model $M_{H'}$, where $H'$ is a subharmonic homogeneous polynomial. But we do not know the relationship between $H$ and $H'$. Theorem 1 tells us that $H'$ is exactly equal to $H$.

For a domain $\Omega$ in $\mathbb{C}^n$, the automorphism group is not easy to describe explicitly; besides, it is unknown in most cases. For instance, the automorphism groups of various domains are given in [10, 16, 20, 21, 23, 27, 28]. Recently, explicit forms of automorphism groups of certain domains have been obtained in [1, 8, 9].

The second part of this note is to describe automorphism groups of models in $\mathbb{C}^2$. If a model is symmetric, i.e. $H(z_1) = |z_1|^{2m}$, then it is biholomorphically equivalent to the Thullen domain $E_{1,m} = \{(z_1, z_2) \in \mathbb{C}^2 : |z_2|^2 + |z_1|^{2m} < 1\}$; the Aut($E_{1,m}$) is exactly the set of all biholomorphisms
\[(z_1, z_2) \mapsto \left(\frac{e^{i\theta_1} z_2 - a}{1 - \overline{a} z_2}, \frac{e^{i\theta_2} (1 - |a|^2)^{1/2m} z_1}{(1 - \overline{a} z_2)^{1/m}}\right)\]
for some $a \in \mathbb{C}$ with $|a| < 1$ and $\theta_1, \theta_2 \in \mathbb{R}$ (cf. [16, Example 9, p.20]). Let us denote by $\Omega_m = \{(z_1, z_2) \in \mathbb{C}^2 : \text{Re} z_2 + (\text{Re} z_1)^{2m} < 0\}$. All the other models, which are not biholomorphically equivalent to $E_{1,m}$ or $\Omega_m$, will be treated together, as the generic case. Let us denote $T^1_t, T^2_t, R_\theta, S_\lambda$ by the following automorphisms:
\begin{align*}
T^1_t & : (z_1, z_2) \mapsto (z_1 + it, z_2); \\
T^2_t & : (z_1, z_2) \mapsto (z_1, z_2 + it); \\
R_\theta & : (z_1, z_2) \mapsto (e^{i\theta} z_1, z_2); \\
S_\lambda & : (z_1, z_2) \mapsto (\lambda z_1, \lambda^{2m} z_2),
\end{align*}
where $t \in \mathbb{R}$, $\lambda > 0$, and $\exp(i\theta)$ is an $L$-root of unity (see Section 4).

With these notations, we obtain the following our second main result.

Theorem 2. If $m \geq 2$, then
\begin{enumerate}
\item[(i)] \text{Aut}(\Omega_m) is generated by \{T^1_t, T^2_t, R_\theta, S_\lambda \mid t \in \mathbb{R}, \lambda > 0\};
\item[(ii)] For any generic model $M_H$, \text{Aut}(M_H) is generated by \{T^2_t, R_\theta, S_\lambda \mid t \in \mathbb{R}, \lambda > 0$, and $\exp(i\theta)$ is an $L$-root of unity\}.
\end{enumerate}

Let $S(\Omega)$ denote the set of all boundary accumulation points for Aut($\Omega$). Then it follows from Theorem 2 that
\begin{enumerate}
\item[(i)] $S(E_{1,m}) = \{(e^{i\theta}, 0) \in \mathbb{C}^2 : \theta \in [0, 2\pi]\}$;
\item[(ii)] $S(\Omega_m) = \{(it, is) \in \mathbb{C}^2 : t, s \in \mathbb{R}\} \cup \{\infty\}$;
\item[(iii)] $S(M_H) = \{(it, 0) \in \mathbb{C}^2 : t \in \mathbb{R}\} \cup \{\infty\}$ for any generic model $M_H$.
\end{enumerate}

We remark that, for any model $M_H$ in $\mathbb{C}^2$, $S(M_H)$ is a smooth submanifold of $\partial M_H$. Moreover, the D’Angelo type is constant and maximal along $S(M_H)$. In addition, the behaviour of orbits in any model $M_H \subset \mathbb{C}^2$ is well-known. For instance, if there exist a point $q \in M_H$ and a sequence $\{f_n\} \subset \text{Aut}(M_H)$ such that
\(\{f_n(q)\}\) converges to some boundary accumulation point \(p \in S(M_H) \setminus \{\infty\}\), then it must converge tangentially to order \(\leq \deg(H)\) to \(p\). In the past twenty years, much attention has been given to the behaviour of orbits near an orbit accumulation point. We refer the reader to the articles \([17, 19, 18]\), and references therein for the development of related subjects.

A typical consequence of Theorem \([2]\) and the Berteloot’s result \([7]\) is as follows.

**Corollary 1.** Let \(Ω\) be a domain in \(\mathbb{C}^2\). Suppose that there exist a point \(q \in Ω\) and a sequence \(\{f_j\} \subset \text{Aut}(Ω)\) such that \(\{f_j(q)\}\) converges to \(p_\infty \in \partial Ω\). Assume that the boundary of \(Ω\) is smooth, pseudoconvex, and of D’Angelo finite type near \(p_\infty\) \((\tau(\partial Ω, p_\infty) = 2m)\). Then exactly one of the following alternatives holds:

(i) If \(\dim \text{Aut}(Ω) = 2\) then
\[\Omega \cong M_H = \{(z_1, z_2) \in \mathbb{C}^2 : \text{Re } z_2 + H(z_1) < 0\},\]
where \(M_H\) is a generic model in \(\mathbb{C}^2\) and \(\deg(H) = 2m\).

(ii) If \(\dim \text{Aut}(Ω) = 3\) then
\[\Omega \cong Ω_m = \{(z_1, z_2) \in \mathbb{C}^2 : \text{Re } z_2 + (\text{Re } z_1)^{2m} < 0\}.\]

(iii) If \(\dim \text{Aut}(Ω) = 4\) then
\[\Omega \cong E_{1,m} = \{(z_1, z_2) \in \mathbb{C}^2 : \text{Re } z_2 + |z_1|^{2m} < 0\} \cong \{(z_1, z_2) \in \mathbb{C}^2 : |z_2|^2 + |z_1|^{2m} < 0\}.\]

(iv) If \(\dim \text{Aut}(Ω) = 8\) then
\[\Omega \cong \mathbb{B}^2 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 < 1\}.\]

The dimensions 0, 1, 5, 6, 7 cannot occur with \(Ω\) as above.

For the case that \(\partial Ω\) is real analytic and of D’Angelo finite type near a boundary orbit accumulation point (without the hypothesis of pseudoconvexity), a similar result as the above corollary was obtained in \([29]\) by using a different method. In addition, it was shown in \([3]\) that a smoothly bounded \(Ω\) in \(\mathbb{C}^2\) with real analytic boundary and with noncompact automorphism group, must be biholomorphically equivalent to \(E_{1,m}\).

This paper is organized as follows. In Section 2, we review some basic notions needed later. In Section 3, we prove Theorem \([1]\). Finally, the proof of Theorem \([2]\) is given in Section 4.

2. Definitions and results

First of all, we recall the following definitions.

**Definition 1** (see \([11]\)). Let \(Ω \subset \mathbb{C}^n\) be a domain with \(C^\infty\)-smooth boundary and \(p \in \partial Ω\). Then the D’Angelo type \(τ(\partial Ω, p)\) of \(\partial Ω\) at \(p\) is defined as
\[τ(\partial Ω, p) := \sup_\gamma \frac{ν(\rho ∘ γ)}{ν(γ)},\]
where \(ρ\) is a defining function of \(Ω\) near \(p\), the supremum is taken over all germs of nonconstant holomorphic curves \(γ : (\mathbb{C}, 0) \to (\mathbb{C}^n, p)\). We say that \(p\) is a point of \textit{finite type} if \(τ(\partial Ω, p) < ∞\) and of \textit{infinite type} if otherwise.

**Definition 2.** Let \(X, Y\) be complex spaces and \(F \subset Hol(X, Y)\).

(i) A sequence \(\{f_j\} \subset F\) is \textit{compactly divergent} if for every compact set \(K \subset X\) and for every compact set \(L \subset Y\) there is a number \(j_0 = j_0(K, L)\) such that \(f_j(K) ∩ L = \emptyset\) for all \(j \geq j_0\).
The family $\mathcal{F}$ is said to be not compactly divergent if $\mathcal{F}$ contains no compactly divergent subsequences.

**Definition 3.** A complex space $X$ is called taut if for any family $\mathcal{F} \subset \text{Hol}(\Delta, X)$, there exists a subsequence $\{f_j\} \subset \mathcal{F}$ which is either convergent or compactly divergent, where $\Delta = \{z \in \mathbb{C} : |z| < 1\}$.

We recall the concept of Carathéodory kernel convergence of domains which is relevant to the discussion of scaling methods (see [14]). Note that the local Hausdorff convergence can replace the normal convergence in case the domains in consideration are convex.

**Definition 4** (Carathéodory Kernel Convergence). Let $\{\Omega_\nu\}$ be a sequence of domains in $\mathbb{C}^n$ such that $p \in \bigcap_{\nu=1}^\infty \Omega_\nu$. If $p$ is an interior point of $\bigcap_{\nu=1}^\infty \Omega_\nu$, the Carathéodory kernel $\hat{\Omega}$ at $p$ of the sequence $\{\Omega_\nu\}$, is defined to be the largest domain containing $p$ having the property that each compact subset of $\hat{\Omega}$ lies in all but a finite number of the domains $\Omega_\nu$. If $p$ is not an interior point of $\bigcap_{\nu=1}^\infty \Omega_\nu$, the Carathéodory kernel $\hat{\Omega}$ is $\{p\}$. The sequence $\{\Omega_\nu\}$ is said to converge to its kernel at $p$ if every subsequence of $\{\Omega_\nu\}$ has the same kernel at $p$.

We shall say that a sequence $\{\Omega_\nu\}$ of domains in $\mathbb{C}^n$ converges normally to $\hat{\Omega}$ (denoted by $\lim \Omega_\nu = \hat{\Omega}$) if there exists a point $p \in \bigcap_{\nu=1}^\infty \Omega_\nu$ such that $\{\Omega_\nu\}$ converges to its Carathéodory kernel $\hat{\Omega}$ at $p$.

Now we recall several results which will be used later on. The following proposition is a generalization of the theorem of Greene-Krantz [15] (cf. [13]).

**Proposition 1.** Let $\{A_j\}_{j=1}^\infty$ and $\{\Omega_j\}_{j=1}^\infty$ be sequences of domains in a complex manifold $M$ with $\lim A_j = A_0$ and $\lim \Omega_j = \Omega_0$ for some (uniquely determined) domains $A_0$, $\Omega_0$ in $M$. Suppose that $\{f_j : A_j \to \Omega_j\}$ is a sequence of biholomorphic maps. Suppose also that the sequence $\{f_j : A_j \to M\}$ converges uniformly on compact subsets of $A_0$ to a holomorphic map $F : A_0 \to M$ and the sequence $\{g_j := f_j^{-1} : \Omega_j \to M\}$ converges uniformly on compact subsets of $\Omega_0$ to a holomorphic map $G : \Omega_0 \to M$. Then one of the following two assertions holds.

(i) The sequence $\{f_j\}$ is compactly divergent, i.e., for each compact set $K \subset A_0$ and each compact set $L \subset \Omega_0$, there exists an integer $j_0$ such that $f_j(K) \cap L = \emptyset$ for $j \geq j_0$, or

(ii) There exists a subsequence $\{f_{j_k}\} \subset \{f_j\}$ such that the sequence $\{f_{j_k}\}$ converges uniformly on compact subsets of $A_0$ to a biholomorphic map $F : A_0 \to \Omega_0$.

In closing this section we recall the following lemma (see [12]).

**Lemma 1** (F. Berteloot). Let $\sigma_\infty$ be a subharmonic function of class $\mathcal{C}^2$ on $\mathbb{C}$ such that $\sigma_\infty(0) = 0$ and $\int_{\mathbb{C}} \partial \bar{\partial} \sigma_\infty = +\infty$. Let $\{\sigma_k\}$ be a sequence of subharmonic functions on $\mathbb{C}$ which converges uniformly on compact subsets of $\mathbb{C}$ to $\sigma_\infty$. Let $\Omega$ be any domain in a complex manifold of dimension $m$ ($m \geq 1$) and let $z_0$ be a fixed point in $\Omega$. Denote by $M_k$ the domain in $\mathbb{C}^2$ defined by

$$M_k = \{(z_1, z_2) \in \mathbb{C}^2 : \text{Re } z_2 + \sigma_k(z_1) < 0\}.$$
Then any sequence \( h_k \in \text{Hol}(\Omega, M_k) \) such that \( \{h_k(z_0), k \geq 1\} \in M_\infty \) admits a subsequence which converges uniformly on compact subsets of \( \Omega \) to an element of \( \text{Hol}(\Omega, M_\infty) \).

3. **Asymptotic behaviour of orbits in a model in \( \mathbb{C}^2 \)**

Let \( P \) be a subharmonic polynomial. Let us denote by \( M_P \) the model given by

\[
M_P = \{(z_1, z_2) \in \mathbb{C}^2 : \rho(z_1, z_2) := \text{Re } z_2 + P(z_1) < 0\}.
\]

Let \( \Omega \) be a domain in \( \mathbb{C}^2 \). Suppose that \( \partial \Omega \) is pseudoconvex, finite type, and smooth of class \( C^\infty \) near a boundary point \( p \in \partial \Omega \). In [7], F. Berteloot proved that if \( p \) is a boundary orbit accumulation point for \( \text{Aut}(\Omega) \), then \( \Omega \) is biholomorphically equivalent to a model \( M_H \), where \( H \) is a homogeneous subharmonic polynomial of degree \( 2m \) which contains no harmonic terms with \( \|H\| = 1 \). Here and in what follows, denote by \( \|P\| \) the maximum of absolute values of the coefficients of a polynomial \( P \). Let us denote by \( P_{2m} \) the space of real valued polynomials on \( \mathbb{C} \) with degree less than or equals to \( 2m \) and which do not contain any harmonic term and by

\[
\mathcal{H}_{2m} = \{ H \in P_{2m} \text{ such that } \text{deg}(H) = 2m \text{ and } H \text{ is homogeneous and subharmonic} \}.
\]

From now on, let \( H \in \mathcal{H}_{2m} \) be as in Theorem [7] Taking the risk of confusion we employ the notation

\[
H_j := \frac{\partial^j H}{\partial z_1^j}, \quad H_{j,q} := \frac{\partial^{j+q} H}{\partial z_1^j \partial \bar{z}_1^q}
\]

throughout the paper for all \( j, q \in \mathbb{N}^* \).

For each \( a = (a_1, a_2) \in \mathbb{C}^2 \), let us define

\[
H_a(w_1) = \frac{1}{\epsilon(a)} \sum_{j,q>0} H_{j,q}(a_1)(j+q)! \tau(a)^{j+q} w_1^j \bar{w}_1^q,
\]

where \( \epsilon(a) = |\text{Re } a_2 + H(a_1)| \) and \( \tau(a) \) is chosen so that \( \|H_a\| = 1 \). We note that \( \sqrt{\epsilon(a)} \lesssim \tau(a) \lesssim \epsilon(a)^{1/(2m)} \). Denote by \( \phi_a \) the holomorphic map

\[
\phi_a : \mathbb{C}^2 \to \mathbb{C}^2
\]

\[
z \mapsto w = \phi_a(z),
\]

given by

\[
\begin{align*}
w_2 &= \frac{1}{\epsilon(a)} \left[z_2 - a_2 - \epsilon(a) + 2 \sum_{j=1}^{2m} \frac{H_j(a_1)}{j!} (z_1 - a_1)^j \right] \\
w_1 &= \frac{1}{\tau(a)} [z_1 - a_1].
\end{align*}
\]

It is easy to check that \( \phi_a \) maps biholomorphically \( M_H \) onto \( M_{H_a} \) and \( \phi_a(a) = (0, -1) \). Now let us consider a domain \( \Omega \) in \( \mathbb{C}^2 \) satisfying Condition \((M_H)\) at a boundary point \( p \in \partial \Omega \). With no loss of generality, we can assume \( p = (0, 0) \) and

\[
\Omega \cap U = \{(z_1, z_2) \in U : \rho(z_1, z_2) = \text{Re } z_2 + H(z_1) < 0\}.
\]

Assume that there exist a sequence \( \{f_n\} \subset \text{Aut}(\Omega) \) and a point \( q \in M_H \) such that \( \eta_n := f_n(q) \to (0, 0) \) as \( n \to \infty \).
Remark 2. By Proposition 2.1 in [7], Ω is taut and after taking a subsequence we may assume that for each compact subset $K \subset \Delta$ there exists a positive integer $n_0$ such that $f_n(K) \subset \Omega \cap U$ for every $n \geq n_0$.

Since $\|H_{\eta_n}\| = 1$, passing to a subsequence if necessary, we can assume that $\lim H_{\eta_n} = H_\infty$, where $H_\infty \in \mathcal{P}_{2m}$ and $\|H_\infty\| = 1$.

**Proposition 2.** Ω is biholomorphically equivalent to $M_{H_\infty}$.

**Proof.** Let $\psi_n := \phi_{\eta_n} \circ f_n$ for each $n \in \mathbb{N}^*$ and consider the following sequence of biholomorphisms

$$\psi_n : f_n^{-1}(\Omega \cap U) \rightarrow M_{H_{\eta_n}} \quad \eta \mapsto (0, -1).$$

By Lemma 1 and by Remark 2, after taking a subsequence we may assume that $\{\psi_n\}$ converges uniformly on any compact subsets of $\Omega$ to a holomorphic map $g : \Omega \rightarrow M_{H_\infty}$. In the other hand, since $\Omega$ is taut we can assume that $\{\psi_n^{-1}\}$ converges also uniformly on any compact subset of $M_{H_\infty}$ to a holomorphic map $\tilde{g} : M_{H_\infty} \rightarrow M_H$. Therefore it follows from Proposition [H] that $g$ is biholomorphic, and hence $\Omega$ is biholomorphically equivalent to $M_{H_\infty}$.

□

**Remark 3.** dist($\eta_n, \partial M_H$) $\approx \epsilon_n := |\rho(\eta_n)|$.

**Remark 4.** i) Let $\{\eta_n\}$ be a sequence in $M_H$ which converges tangentially to order $2m$ to $(0, 0)$. Set $\epsilon_n := |\rho(\eta_n)| \approx |\eta_n|^{2m}$. Then we have

$$|\Re \eta_n| = |\epsilon_n + H(\eta_n)| \lesssim |\eta_n|^{2m}.$$

ii) Suppose that $\{\eta_n\}$ is a sequence in $M_H$ which converges tangentially to order $< 2m$ to $(0, 0)$. Then we have $|\eta_n|^{2m} = o(\epsilon_n)$ and we thus obtain the following estimate

$$|\Re \eta_n| = |\epsilon_n + H(\eta_n)| \approx |\epsilon_n|.$$

**Lemma 2.** If $\{\eta_n\} \subset M_H$ converges tangentially to order $2m$ to $(0, 0)$, then $\deg(H_\infty) = 2m$ and moreover $M_{H_\infty}$ is biholomorphically equivalent to $M_H$.

**Proof.** Since $\{\eta_n\}$ converges tangentially to order $2m$ to $(0, 0)$, it follows that $|\eta_n|^{2m} \approx \epsilon_n \approx d(\eta_n, \partial \Omega)$. Let $a_{j, q}(\eta_n) := \frac{H_{j, q}(\eta_n) \tau(\eta_n)^{j+q}}{(j+q)! \epsilon_n}$ for each $j, q > 0$ with $j + q \leq 2m$. Then we have the following estimate

$$|a_{j, q}(\eta_n)| \lesssim \frac{|\eta_n|^{2m-j-q} \tau(\eta_n)^{j+q}}{(j+q)! \epsilon_n} \lesssim \left( \frac{\tau(\eta_n)}{|\eta_n|} \right)^{j+q}.$$

Since $\|H_{\eta_n}\| = 1$, we have $\tau(\eta_n) \gtrsim |\eta_n|^{1/(2m)}$, and therefore $\tau(\eta_n) \approx |\eta_n|^{1/(2m)}$. This implies that $\deg(H_\infty) = 2m$. Without loss of generality we can assume that $\lim \frac{\tau(\eta_n)}{|\eta_n|^{1/(2m)}} = \alpha$ and $\lim \frac{\tau(\eta_n)}{\epsilon_n^{1/(2m)}} = \beta$. We note that

$$a_{j, q}(\eta_n) = \frac{H_{j, q}(\eta_n) \tau(\eta_n)^{j+q}}{(j+q)! \epsilon_n} = \frac{1}{(j+q)!} \left( \frac{\tau(\eta_n)}{|\eta_n|^{1/(2m)}} \right)^{j+q} H_{j, q} \left( \frac{\eta_n}{\epsilon_n^{1/(2m)}} \right)$$
for any $j, q > 0$. Then we obtain $\lim a_{j, q}(\eta_n) = \frac{1}{(j+q)!} \beta^{j+q} H_{j, q}(\alpha) w^j H^q \bar{w}$ for each $j, q > 0$; hence

$$H_\infty(w_1) = \sum_{j, q > 0} \frac{1}{(j+q)!} \beta^{j+q} H_{j, q}(\alpha) w^j H^q \bar{w}$$

$$= H(\alpha + \beta w_1) - H(\alpha) - 2\Re \sum_{j=1}^{2m} \frac{H_j(\alpha)}{j!} (\beta w_1)^j.$$ 

So, the holomorphic map given by

$$\begin{cases}
  t_2 = w_2 - H(\alpha) - 2 \sum_{j=1}^{2m} \frac{H_j(\alpha)}{j!} (\beta w_1)^j \\
  t_1 = \alpha + \beta w_1
\end{cases}$$

is biholomorphic from $M_{H_\infty}$ onto $M_H$.  \(\square\)

**Lemma 3.** If $\{\eta_n\} \subset M_H$ converges tangentially to order $< 2m$ to $(0, 0)$, then $H_\infty = H$.

**Proof.** It is easy to see that $\tau(\eta_n) \lesssim \epsilon_n^{1/(2m)}$. On the other hand, since $|\eta_n|^{2m} = o(\epsilon_n)$, we have for $j, q \in \mathbb{N}$ with $j, q > 0, j + q < 2m$ that

$$|a_{j, q}(\eta_n)| \lesssim \frac{|\eta_n|^{2m-j-q} \epsilon_n^{(j+q)/(2m)}}{(j+q)! \epsilon_n} \lesssim \left( \frac{|\eta_n|^{2m}}{\epsilon_n} \right)^{2m-j-q}.$$ 

Therefore $\lim a_{j, q}(\eta_n) = 0$ for any $j, q > 0$ with $j + q < 2m$, and thus $H_\infty = H$. Hence, the proof is complete. \(\square\)

**Proof of Theorem 4.** Let $\Omega$ and $\{f_n\}$ be a domain and a sequence, respectively, as in Theorem 4. Then, after a change of coordinates, we can assume that $p = (0, 0)$ and $\Omega \cap U = \{(z_1, z_2) \in U : \rho(z_1, z_2) = \Re z_2 + H(z_1) < 0\}$. Moreover, we may also assume that $\eta_n := f_n(q) \in U \cap M_H$ for all $n \in \mathbb{N}^*$. Therefore, it follows from Proposition 3, Lemma 2 and Lemma 4 that $\Omega$ is biholomorphically equivalent to $M_H$, which finishes the proof. \(\square\)

In the case that $\{\eta_n\}$ converges tangentially to order $> 2m$ to $(0, 0)$, we obtain the following proposition.

**Proposition 3.** Let $\{\eta_n\} \subset M_H$ be a sequence which converges tangentially to order $> 2m$ to $(0, 0)$. If there exist $j, q > 0$ with $j + q < 2m$ such that

$$\frac{\partial^{j+q} H}{\partial z^j \partial \bar{z}^q}(\eta_n) \approx |\eta_n|^{2m-j-q},$$

then $\tau(\eta_n) = o(\epsilon_n^{1/(2m)})$, and thus $\deg(H_\infty) < 2m$. 

\textbf{Proof.} Suppose otherwise that \( \tau(\eta_n) \approx \epsilon_n^{1/(2m)} \). Then since \( \epsilon_n = o(|\eta_n|^{2m}) \), one gets

\[
|a_j, q(\eta_n)| \approx \frac{|\eta_n|^{2m-j-q \epsilon_n^{(j+q)/(2m)}}}{(j+q)! \epsilon_n} \approx \left( \frac{|\eta_n|^{2m}}{\epsilon_n} \right)^{2m-j-q}.
\]

This implies that

\[
\lim_{n \to \infty} a_j, q(\eta_n) = +\infty,
\]

which is a contradiction. Thus, the proof is complete. \( \square \)

**Example 1.** Let \( E_{1,2} := \{(z_1, z_2) \in \mathbb{C}^2 : \Re z_2 + |z_1|^4 < 0\} \). Then the sequence \( \{(1/\sqrt{n}, -2/n)\} \) converges tangentially to order 4 to \((0, 0)\). But the sequence \( \{(1/\sqrt{n}, -1/n - 1/n^2)\} \) converges tangentially to order 8 to \((0, 0)\).

Let \( \rho(z_1, z_2) = \Re z_2 + |z_1|^4 \) and let \( \eta_n = (1/\sqrt{n}, -1/n - 1/n^2) \) for every \( n \in \mathbb{N} \).

We see that \( \rho(\eta_n) = -1/n - 1/n^2 + 1/n = -1/n^2 \approx -\text{dist}(\eta_n, \partial E_{1,2}) \). Set \( \epsilon_n = |\rho(\eta_n)| = 1/n^2 \). Then

\[
\rho(z_1, z_2) = \Re(z_2) + \left[ \frac{1}{\sqrt{n}} + z_1 - \frac{1}{\sqrt{n}} \right]^4
\]

\[
= \Re(z_2) + \frac{1}{n} \left[ \Re(z_1 - \frac{1}{\sqrt{n}}) \right]^2 + \frac{4}{\sqrt{n}} |z_1 - \frac{1}{\sqrt{n}}|^2 \Re(z_1 - \frac{1}{\sqrt{n}})
\]

\[
= \Re(z_2) + \frac{1}{n} \left[ \Re(z_1 - \frac{1}{\sqrt{n}}) \right]^2 + \frac{2}{n} |z_1 - \frac{1}{\sqrt{n}}|^2
\]

\[
= \Re(z_2) + \frac{4}{\sqrt{n}} \Re(z_1 - \frac{1}{\sqrt{n}}) + \frac{2}{\sqrt{n}} |z_1 - \frac{1}{\sqrt{n}}|^2.
\]

A direct calculation shows that \( \tau_n := \tau(\eta_n) = \frac{1}{\sqrt{n}^2} \) for all \( n = 1, 2, \ldots \) and thus the automorphism \( \phi_{\eta_n} \) by

\[
\phi_{\eta_n}^{-1}(w_1, w_2) = \left( \frac{1}{\sqrt{n}} + \tau_n w_1, \epsilon_n w_2 - \frac{1}{n} - \frac{4}{\sqrt{n} \sqrt{n}} \tau_n w_1 - \frac{2}{\sqrt{n}} \tau_n^2 w_1^2 \right);
\]

\[
\epsilon_n^{-1} \rho \circ \phi_{\eta_n}^{-1}(w_1, w_2) = \epsilon_n^{-1} \rho \left( \frac{1}{\sqrt{n}} + \tau_n w_1, \epsilon_n w_2 - \frac{1}{n} - \frac{4}{\sqrt{n} \sqrt{n}} \tau_n w_1 - \frac{2}{\sqrt{n}} \tau_n^2 w_1^2 \right)
\]

\[
= \Re(w_2) + \frac{1}{16n} |w_1|^4 + |w_1|^2 + \frac{1}{2 \sqrt{n}} |w_1|^2 \Re(w_1).
\]

We now show that there do not exist a sequence \( \{f_n\} \subset \text{Aut}(E_{1,2}) \) and \( a \in E_{1,2} \) such that \( \eta_n = f_n(a) \to (0, 0) \) in \( \partial E_{1,2} \) as \( n \to \infty \). Indeed, suppose that there exist such a sequence \( \{f_n\} \) and such a point \( a \in E_{1,2} \). Then by Proposition 2 \( E_{1,2} \) is biholomorphically equivalent to the following domain \( D := \{(w_1, w_2) \in \mathbb{C}^2 : \Re w_2 + |w_1|^2 < 0\} \simeq \mathbb{E}^2 \). It is impossible.

4. AUTOMORPHISM GROUP OF A MODEL IN \( \mathbb{C}^2 \)

In this section, we consider a model

\[
M_H := \{(z_1, z_2) \in \mathbb{C}^2 : \Re z_2 + H(z_1) < 0\},
\]
where
\[
H(z_1) = \sum_{j=1}^{2m-1} a_{2m-j} z_1^{2m-j} = a_m |z_1|^{2m} + 2 \sum_{j=1}^{m-1} |z_1|^{2j} \text{Re}(a_j z_1^{2m-2j}) \tag{1}
\]
is a nonzero real valued homogeneous polynomial of degree 2m, with \(a_j \in \mathbb{C}\) and \(a_j = \overline{a}_{2m-j}\). We will give the explicit description of \(\text{Aut}(M_H)\).

The D'Angelo type of \(\partial M_H\) is given by the following.

**Lemma 4.** \(\tau(\partial M_H, (\alpha, -H(\alpha) + it)) = m_\alpha\) for all \(\alpha \in \mathbb{C}\) and for all \(t \in \mathbb{R}\), where

\[
m_\alpha = \min\{j + q \mid j, q > 0, \frac{\partial^{j+q} H(\alpha)}{\partial z_1^j \partial \bar{z}_1^q} \neq 0\}.
\]

**Proof.** By the following change of variables

\[
\begin{cases}
    w_2 = z_2 + H(\alpha) + 2 \sum_{j=1}^{2m} \frac{H_j(\alpha)}{j!} (z_1 - \alpha)^j \\
    w_1 = z_1 - \alpha,
\end{cases}
\]

the defining function for \(M_H\) is now given by

\[
\rho(w_1, w_2) = \text{Re} \ w_2 + \sum_{j,q>0} \frac{H_{j,q}(\alpha)}{j!} w_1^j w_2^q.
\]

By a computation, we get \(\tau(\partial M_H, (\alpha, -H(\alpha) + it)) = m_\alpha\), and thus the proof is complete. \(\Box\)

Let \(P_k(\partial M_H)\) the set of all points in \(\partial M_H\) of D'Angelo type \(k\) (\(k\) is either a positive integer or infinity). Let us denote by \(\Gamma := \{(z_1, -H(z_1) + it) \mid t \in \mathbb{R}, z_1 \in \mathbb{C} \text{ with Re}(e^{it}z_1) = 0\}\) if \(H(z_1) = a \left[2 \text{Re}(e^{it}z_1)^{2m} - 2 \text{Re}(e^{it}z_1)^{2m}\right]\) for some \(a \in \mathbb{R}\) and for some \(\nu \in [0, 2\pi]\) and by \(\Gamma := \{(0, it) \mid t \in \mathbb{R}\}\) if otherwise.

**Lemma 5.** If \(m \geq 2\), then \(P_{2m}(\partial M_H) = \Gamma\) and \(\tau(\partial M_H, p) < 2m\) for all \(p \in \partial M_H \setminus \Gamma\).

**Proof.** It is not hard to show that \(\Gamma \subset P_{2m}(\partial M_H)\). Now let \(p = (\alpha, -H(\alpha) + it)\) (\(\alpha \neq 0\)) be any boundary point in \(\partial M_H \setminus \Gamma\). By Lemma 4 we see that \(\tau(\partial M_H, p) = m_\alpha \leq 2m\). We will prove that \(\tau(\partial M_H, p) < 2m\). Indeed, suppose that, on the contrary, \(\tau(\partial M_H, p) = m_\alpha = 2m\). This implies that \(H_{j,q}(\alpha) = 0\) for all \(j, q > 0\) and \(j + q < 2m\) and thus \(H_{1,1}(\alpha + z_1) = H_{1,1}(z_1)\) for all \(z_1 \in \mathbb{C}\). Let \(f(x,y) := H_{1,1}(x + iy)\) for all \(z_1 = x + iy \in \mathbb{C}\). By a change of affine coordinates in \(\mathbb{C}\), we may assume that \(\alpha = (1,0)\) and thus \(f(x + 1, y) = f(x, y)\) for all \((x, y) \in \mathbb{R}^2\). Hence, for each \(y \in \mathbb{R}\) \(f(x,y)\) is a periodic polynomial in \(x\), and thus \(f(x,y)\) does not depend on \(x\), i.e., \(f(x, y) = \beta y^{2m-2}\) for some \(\beta \in \mathbb{R}\).

Therefore by the above, we conclude that \(H_{1,1}(z_1) = \beta (\text{Re}(e^{it}z_1))^{2m-2}\) for some \(\beta \in \mathbb{R}^*\) and for some \(\nu \in [0, 2\pi]\) and \(\alpha\) satisfies \(\text{Re}(e^{it}\alpha) = 0\). Since \(H\) is a homogeneous polynomial of degree \(2m\) without harmonic terms, it is easy to show that \(H(z_1) = a \left[2 \text{Re}(e^{it}z_1)^{2m} - 2 \text{Re}(e^{it}z_1)^{2m}\right]\) for some \(a \in \mathbb{R}^*\) and \((\alpha, -H(\alpha) + it) \in \Gamma\), which is impossible. Thus the proof is complete. \(\Box\)

We recall the following lemma, proved by F. Berteloot (see [7]), which is the main ingredient in the proof of Theorem 2.
Lemma 6 (F. Berteloot). Let $Q \in \mathcal{P}_{2m}$ and $H \in \mathcal{H}_{2m}$. Suppose that $\psi : M_H \to M_Q$ is a biholomorphism. Then there exist $t_0 \in \mathbb{R}$ and $z_0 \in \partial M_Q$ such that $\psi$ and $\psi^{-1}$ extend to be holomorphic in neighborhoods of $(0, it_0)$ and $z_0$, respectively. Moreover, the homogeneous part of higher degree in $Q$ is equal to $\lambda H(e^{i\nu}z)$ for some $\lambda > 0$ and $\nu \in [0, 2\pi)$.

Proof. According to [2], there exists a holomorphic function $\phi$ on $M_Q$ which is continuous on $\overline{M_Q}$ such that $|\phi| < 1$ for $z \in M_Q$ and tends to 1 at infinity. Let $\psi : M_H \to M_Q$ be a biholomorphism. We claim that there exists $t_0 \in \mathbb{R}$ such that $\lim_{z \to 0^-} \inf |\psi(0^+, x + it_0)| < +\infty$. Indeed, if this would not be the case, the function $\phi \circ \psi$ would be equal to 1 on the half plane $\{(z_1, z_2) \in \mathbb{C}^2 : \text{Re } z_2 < 0, z_1 = 0\}$ and this is impossible since $|\phi| < 1$ for $|z| \gg 1$. Therefore, we may assume that there exists a sequence $x_k < 0$ such that $\lim x_k = 0$ and $\lim \psi(0, x_k + it_0) = z_0 \in \partial M_Q$.

It is proved in [5] that under these circumstances $\psi$ extends homeomorphically to $\partial M_H$ on some neighbourhood of $(0, it_0)$. Then the result of Bell (see [4]) shows that this extension is actually diffeomorphic. Moreover, it follows from [5, Theorem 3] (see also [12, 20]) $\psi$ and $\psi^{-1}$ extend to be holomorphic in neighborhoods of $(0, it_0)$ and $z_0$, respectively. Therefore, the conclusion follows easily. □

Now we recall two basic integer valued invariants used in the normal form construction in [22]. Let $l = m_0 < m_1 < \cdots < m_p \leq m$ be indices in [11] for which $m_{i+1} \neq 0$. Denote by $L$ the greatest common divisor of $2m - 2m_0, 2m - 2m_1, \ldots, 2m - 2m_p$. If $l = m$, then $H(z_1) = a_m|z_1|^{2m} (a_m > 0)$ and it is known that $M_H$ is biholomorphically equivalent to the domain

$$E_{1, m} = \{(z_1, z_2) \in \mathbb{C}^2 : |z_2|^2 + |z_1|^{2m} < 1\}.$$

The automorphism group of $E_{1, m}$ is well-known (see [16, Example 9, p. 20]). So, in what follows we only consider the case $l < m$. Moreover, we consider the model $\Omega_m = \{(z_1, z_2) \in \mathbb{C} : \text{Re } z_2 + (\text{Re } z_1)^{2m} < 0\}$ and others which are not biholomorphically equivalent to it.

Remark 5. If $H(z_1) = a \left[2\text{Re}(e^{i\nu}z_1)^{2m} - 2\text{Re}(e^{i\nu}z_1)^{2m}\right]$ for some $a > 0$ and for some $\nu \in [0, 2\pi)$, then $M_H \simeq \Omega_m$ and $L = 2$. Indeed, $L = 2$ is obvious. Now let us denote by $\Phi : \mathbb{C}^2 \to \mathbb{C}^2$ the biholomorphism defined by $w_2 = z_2 - 2a(e^{i\nu}z_1)^{2m}; w_1 = 2a^{1/2m}e^{i\nu}z_1$. Then it is easy to check that $\Omega_m = \Phi(M_H)$. Hence, the assertion follows.

Lemma 7. $H(\exp(i\theta)z_1) = H(z_1)$ for all $z_1 \in \mathbb{C}$ if and only if $\exp(i\theta)$ is an L-root of unity.

Proof. We have

$$H(\exp(i\theta)z_1) = a_m|z_1|^{2m} + 2 \sum_{j=0}^{p} \left(|z_1|^{2j}\text{Re}\left\{a_{m_j}\exp(i(2m - 2m_j)\theta)z_1^{2m-2m_j}\right\}\right)$$

for all $z_1 \in \mathbb{C}$. Hence, we conclude that $H(\exp(i\theta)z_1) = H(z_1)$ for all $z_1 \in \mathbb{C}$ if and only if $\exp(i(2m - 2m_j)\theta) = 1$ for every $j = 0, \ldots, p$, which proves the assertion. □
Proof of Theorem 3. For $t \in \mathbb{R}$, $\lambda > 0$, and any $L$-root of unity $\exp(it\theta)$, consider the mappings
\[
T^1_t : (z_1, z_2) \mapsto (z_1 + it, z_2);
\]
\[
T^2_t : (z_1, z_2) \mapsto (z_1, z_2 + it);
\]
\[
R_\theta : (z_1, z_2) \mapsto (e^{i\theta}z_1, z_2);
\]
\[
S_\lambda : (z_1, z_2) \mapsto (\lambda z_1, \lambda^{2m}z_2).
\]
It is easy to check that $T^2_t, R_\theta, S_\lambda$ are in $\text{Aut}(M_H)$ and moreover $T^1_t \in \text{Aut}(M_H)$ if $H(z_1) = (\text{Re } z_1)^{2m}$ for all $z_1 \in \mathbb{C}$. Now let $f = (f_1, f_2)$ be any biholomorphism of $M_H$. It follows from Lemma 4 that there exist $p \in \Gamma$ and $q \in \Gamma$ such that $f$ and $f^{-1}$ extend to be holomorphic in neighborhoods of $p$ and $q$, respectively, and $f(p) = q$. Replacing $f$ by its composition with reasonable translations $T^2_t, T^1_t$, we may assume that $p = q = (0,0)$, and there exist neighborhoods $U_1$ and $U_2$ of $(0,0)$ such that $U_2 \cap \partial M_H = f(U_1 \cap \partial M_H)$, and $f$ and $f^{-1}$ are holomorphic in $U_1$ and $U_2$, respectively. Moreover, $f$ is a local CR diffeomorphism between $U_1 \cap \partial M_H$ and $U_2 \cap \partial M_H$.

Let us denote by $\mathcal{H} = \{ z \in \mathbb{C} : \text{Re } z < 0 \}$. We now define $g_1(z_2) := f_1(0, z_2)$ and $g_2(z_2) := f_2(0, z_2)$ for all $z_2 \in \mathcal{H}$. It follows from Lemma 4 that $f(U_1 \cap \Gamma) = U_2 \cap \Gamma$. Consequently, $g_1(it) = 0$ for all $-\epsilon_0 < t < \epsilon_0$ with $\epsilon_0 > 0$ small enough. By the Schwarz Reflection Principle and the Identity Theorem, we have $g_1(z_2) = 0$ for all $z_2 \in \mathcal{H}$. This also implies that $\text{Re } f_2(0, z_2) < 0$, and thus $g_2 \in \text{Aut}(\mathcal{H})$. Since $g_2(0) = 0$, it is known that $g_2(z_2) = \frac{\alpha z_2}{1 + i\beta z_2}$ for some $\alpha \in \mathbb{R}^*$ and $\beta \in \mathbb{R}$.

Now we are going to prove that $f$ is biholomorphic between neighborhoods of the origin. To do this, it suffices to show that $J_f(0,0) \neq 0$ (a similar proof shows that $J_{f^{-1}}(0,0) \neq 0$). To derive a contradiction, we suppose that $J_f(0,0) = 0$. By the above we can write
\[
f(z_1, z_2) = (z_1a(z_1, z_2), g_2(z_2) + z_1b(z_1, z_2)),
\]
where $a$ and $b$ are holomorphic functions defined on neighborhoods of $(0,0)$, respectively. By shrinking $U_1$ if necessary, we can assume that $a, b$ are defined on $U_1$.

Take derivative of $f$ at points $(0, z_2)$ we have
\[
df(z_1, z_2) = \begin{pmatrix}
a(z_1, z_2) \\
b(z_1, z_2)
\end{pmatrix}
\begin{pmatrix}
z_1a(z_1, z_2) \\
g_2'(z_2) + z_1b(z_1, z_2)
\end{pmatrix}.
\]
Therefore we obtain $J_f(0, z_2) = a(0, z_2)g_2'(z_2)$ for every $z_2$ small enough. We note that $J_f(0, z_2) \neq 0$ for all $z_2 \in \mathcal{H}$, $g_2'(0) = \alpha \neq 0$, and $J_f(0,0) = 0$. This implies that $a'(z_1, z_2) = O(|z|)$. Since $f(z_1, z_2) \in M_H \cap U_2$ for all $(z_1, z_2) \in M_H \cap U_1$,
\[
\text{Re}(g_2(z_2) + z_1b(z_1, z_2)) + H(z_1a(z_1, z_2)) \leq 0
\]
for all $(z_1, z_2) \in M_H \cap U_1$. Because of the invariance of $M_H$ under any map $S_t(t > 0)$, one gets
\[
\text{Re}(g_2(t^{2m}z_2) + t^{2m}z_1b(tz_1, t^{2m}z_2)) + H(tz_1a(tz_1, t^{2m}z_2)) \leq 0
\]
for every $(z_1, z_2) \in M_H \cap U_1$ and for every $t \in (0,1)$.
Expand the function $b$ into the Taylor series at the origin so that

\[ b(z_1, z_2) = \sum_{j,k=0}^{\infty} b_{j,k} z_1^j z_2^k, \]

where $b_{j,k} \in \mathbb{C}$ for all $j, k \in \mathbb{N}$. Hence the equation (2) can be re-written as

\[
\rho \circ f(tz_1, t^{2m}z_2) = \text{Re} \left( \frac{t^{2m}z_2}{1 + i\beta t^{2m}z_2} + tz_1 \sum_{j,k=0}^{\infty} b_{j,k}(tz_1)^j (t^{2m}z_2)^k \right)
+ H(tz_1a(tz_1, t^{2m}z_2)) \leq 0
\]  

(3)

for every $(z_1, z_2) \in \overline{M_H} \cap U_1$ and for every $t \in (0, 1)$.

Now let us denote by $j_0 = \min \{ j \mid b_{j,0} \neq 0 \}$ if $b(z_1, 0) \neq 0$ and $j_0 = +\infty$ if otherwise. We divide the argument into three cases as follows.

**Case 1.** $0 \leq j_0 \leq 2m - 2$. Note that we can choose $\delta_0 > 0$ and $\epsilon_0 > 0$ such that $H(z_1) < \epsilon_0$ for all $|z_1| < \delta_0$. Since $(-\epsilon_0, z_1) \in U \cap M_H$ for all $|z_1| < \delta_0$, taking $\lim_{t \to 0^+} \rho \circ f(tz_1, t^{2m}\epsilon_0)$ we obtain $\text{Re}(b_{j_0,0}z_1^{j_0+1}) \leq 0$ for all $|z_1| < \delta_0$, which leads to a contradiction.

**Case 2.** $j_0 = 2m - 1$. It follows from (3) that

\[
\lim_{t \to 0^+} \frac{1}{t^{2m}} \rho \circ f(tz_1, t^{2m}z_2) = \text{Re}(\alpha z_2 + b_{2m-1,0}z_1^{2m}) = 0
\]

for all $(z_1, z_2) \in U_1$ with $\text{Re} z_2 + H(z_1) = 0$. This implies that $H(z_1) = \text{Re}(\frac{b_{2m-1,0}}{\alpha} z_1^{2m})$ for all $|z_1| < \delta_0$ with $\delta_0 > 0$ small enough. It is absurd since $H$ contains no harmonic terms.

**Case 3.** $j_0 > 2m - 1$. Fix a point $(z_1, z_2) \in U_1 \cap \partial M_H$ with $\text{Re}(z_2) \neq 0$. From (3) one has

\[
\lim_{t \to 0^+} \frac{1}{t^{2m}} \rho \circ f(tz_1, t^{2m}z_2) = \text{Re}(\alpha z_2) = 0,
\]

which is impossible.

Altogether, we conclude that $f$ is a local biholomorphism between neighborhoods $U_1$ and $U_2$ of the origin satisfying $f(U_1 \cap \partial M_H) = U_2 \cap \partial M_H$. Therefore by Corollary 5.3, p. 909 and the Identity Theorem, we have

\[ f(z_1, z_2) = (\lambda e^{i\theta} z_1, \lambda^{2m} z_2) \]

for all $(z_1, z_2) \in M_H$, where $e^{i\theta}$ is an $L$-root of unity. Thus $f = S_\lambda \circ R_\theta$, and hence the proof is complete. □

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*Center for Geometry and its Applications, Pohang University of Science and Technology, Pohang 790-784, The Republic of Korea - and - Department of Mathematics, Vietnam National University at Hanoi, 334 Nguyen Trai, Thanh Xuan, Hanoi, Vietnam
E-mail address: thunv@vnu.edu.vn, thunv@postech.ac.kr
**Department of Mathematics, Hanoi National University of Education, 136 Xuan Thuy str., Hanoi, Vietnam

E-mail address: ducphuongma@gmail.com