From polygons to ultradiscrete Painlevé equations

Christopher M. Ormerod and Yasuhiko Yamada

Abstract. Starting with polygons arising from tropical genus one curves, the rays of those tropical curves are constrained in a way that defines a bounded polygon. When we relax this constraint, the resulting curves do not close, giving rise to a system of spiraling polygons. The piecewise linear transformations that preserve the forms of those rays form tropical rational presentations of groups of affine Weyl type. We present a selection of spiraling polygons with three to eleven sides with groups of piecewise linear transformations coinciding with the Bäcklund transformations and the evolution equations for the ultradiscrete Painlevé equations.

1. Introduction

A significant contribution to our understanding of the Painlevé equations, both discrete and continuous, has been their characterization in terms of their rational surfaces of initial conditions [35, 49]. These works related the symmetries of the Painlevé equations to Cremona isometries of rational surfaces [24, 29, 30], which are groups of affine Weyl type [7, 17, 25]. This gave a geometric interpretation to many works based on symmetries of the Painlevé equations [19, 20, 33]. In the autonomous limit, these equations are elliptic or QRT mappings [42, 45] and their associated surfaces of initial conditions are rational elliptic surfaces [6].

The ultradiscrete Painlevé equations are integrable second order mappings defined over the max-plus semifield [45]. Given a subtraction free discrete Painlevé equation, one may obtain an ultradiscrete Painlevé equation by applying the ultradiscretization process [55]. The ultradiscretization process famously related integrable difference equations with integrable cellular automata [52, 54, 55], hence, the process is thought to preserve integrability [21, 45]. The ultradiscrete Painlevé equations possess many of the same properties as the continuous and discrete Painlevé equations, albeit, in some tropical form, such as tropical Lax representations [15, 37], tropical singularity confinement [14, 38], symmetry groups of affine Weyl type [18, 19] and special solutions of rational and hypergeometric type [28, 36, 51]. The ultradiscrete QRT maps may also be obtained as autonomous limits of the ultradiscrete Painlevé equations [31, 41].

The ultradiscrete QRT maps preserve the level sets of tropical biquadratic functions [31, 41]. Since every non-degenerate level set of a tropical biquadratic is

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a genus one curve, another way of putting this is that the ultradiscrete QRT maps are automorphisms of tropical elliptic surfaces. Given the geometric interpretation of tropical singularity confinement \[38\], the positions of the rays in a pencil of any genus one curves play the same role as the positions of the base points in a pencil of genus one curves. In this way, there is an analogous constraint on the positions of the rays of a pencil of genus one curve, which when removed this constraint, gives rise to a system of spiraling polygons, as seen in Figure 1.1. This mimics the generalization of elliptic surfaces to surfaces of initial conditions for discrete Painlevé equations.

This article is concerned with groups of piecewise linear transformations of the plane which preserves the forms of the spiraling polygons. We specify a selection spiraling polygons with between three and eleven sides whose groups of transformations form representations of affine Weyl groups with types that coincide with those of the Bäcklund transformations for the multiplicative Painlevé equations \[49\]. A list of the correspondences is provided in Table 1. This provides a geometric interpretation for the group of Bäcklund transformations of the ultradiscrete QRT maps and ultradiscrete Painlevé equations.

This construction replicates the ultradiscretization of known subtraction-free affine Weyl representations of Kajiwara et. al. \[18\], however, our derivation does not use or require the ultradiscretization procedure. Finding generators for the representations is reduced to combinatorial properties of the underlying polygons. By considering genus one tropical plane quartic and sextic curves, we treat polygons with up to eleven sides. Tropical biquadratics also appeared in this context in the work of Rojas \[48\], Nobe \[31\] and Scully \[50\], as do a very small collection of the symmetries we list in \[48\].

We set out this paper as follows: we first briefly review a geometric setting for QRT mappings and the discrete Painlevé equations in \[2\] then we review the ultradiscretization procedure with some relevant tools from tropical geometry in \[8\]. We then present a description of a canonical classes of transformations that

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{A fibration of closed tropical curves (left) corresponds to ultradiscrete QRT maps. Breaking this closure condition results in spirals (right), which corresponds to ultradiscrete Painlevé equations.}
\end{figure}
| Sides | Polygon | Affine Weyl group | Painlevé equation |
|-------|---------|-------------------|------------------|
| 3     | Triangle| $A_0^{(1)}$       |                  |
| 4     | Quadrilateral | $A_1^{(1)}$, $A_1^{(1)} + D_8$ | u-P_I, u-P'_I |
| 5     | Pentagon | $(A_1 + A_1)^{(1)}$ | u-P_{III} |
| 6     | Hexagon | $(A_2 + A_1)^{(1)}$ | u-P_{III}/u-P_{IV} |
| 7     | Heptagon | $A_4^{(1)}$ | u-P_{V} |
| 8     | Octagon | $D_5^{(1)}$ | u-P_{V,II} |
| 9     | Enneagon | $E_6^{(1)}$ | u-P($A_2^{(1)}$) |
| 10    | Decagon | $E_7^{(1)}$ | u-P($A_1^{(1)}$) |
| 11    | Undecagon | $E_8^{(1)}$ | u-P($A_1^{(1)*}$) |

Table 1. A labelling of the various polygons and the affine Weyl groups of symmetries they possess.

2. The geometry of QRT and discrete Painlevé equations

The QRT mappings are integrable second order autonomous difference equations [42, 43]. They are Lax integrable, measure preserving and possess the singularity confinement property. The QRT maps may broadly be considered the discrete analogue of elliptic functions [56]. To construct a QRT map, one takes two linearly independent biquadratics, $h_0$ and $h_1$, and a generic point, $p = (x, y)$, which we associate with an element, $z = [z_0 : z_1] \in \mathbb{P}_1$, by the relation

$$z_0 h_0(x, y) + z_1 h_1(x, y) = 0.$$ (2.1)

If we let $h(x, y) = h_0(x, y)/h_1(x, y)$, then the QRT map, $\phi : (x, y) \rightarrow (\tilde{x}, \tilde{y})$, is defined by the condition that $\tilde{x}$ and $\tilde{y}$ are related by

$$h(x, y) = h(\tilde{x}, \tilde{y}),$$ (2.2a)

$$h(x, \tilde{y}) = h(\tilde{x}, \tilde{y}),$$ (2.2b)

where the trivial solution, $x = \tilde{x}$ and $y = \tilde{y}$ are discarded [42, 43]. In this way, the mapping is an endomorphism of the curve defined by (2.1) for each value of $z$.

If we take a point in the intersection of the curves $h_0(x, y) = 0$ and $h_1(x, y) = 0$, then $z_0$ and $z_1$ can be chosen arbitrarily, hence, the entire pencil of curves intersect these points. These points are called base-points and the number of base points for any pencil of biquadratics is 8, counting multiplicities. A case in which there are eight distinct base points in $\mathbb{R}^2$ is depicted in Figure 2.5. By blowing up these base points, possibly multiple times in the case of higher multiplicities, we obtain a surface with a fibration by smooth biquadratic curves (i.e., elliptic curves). Lifting the QRT map to this surface gives an automorphism of an elliptic surface [6, 56].

A classic example is the QRT-map defined by the invariant

$$(2.3) \quad h(x, y) = \frac{y}{a_3} + \frac{y}{a_4} + \frac{(a_1 + a_2)b_1b_2}{ya_1a_2} + \frac{(y + b_1)(y + b_2)}{xy} + \frac{x(y + b_3)(y + b_4)}{ya_3a_4},$$
where we require the condition

\[(2.4) \quad a_1a_2b_3b_4 = b_1b_2a_3a_4.\]

The mapping, \((x, y) \to (\tilde{x}, \tilde{y})\), is specified by relations

\begin{align*}
\tilde{x}x &= \frac{a_3a_4(\tilde{y} + b_1)(\tilde{y} + b_2)}{(\tilde{y} + b_3)(\tilde{y} + b_4)}, \\
\tilde{y}y &= \frac{b_3b_4(x + a_1)(x + a_2)}{(x + a_3)(x + a_4)}. 
\end{align*}

(2.5a) \hspace{1cm} (2.5b)

The base points of (2.5) lie on the lines \(x, y = 0, \infty\) in \(P^2\). The blow-up at these points, with (2.4) as a constraint, is an elliptic surface [6].

The discrete Painlevé equations are integrable second order difference equations that admit the continuous Painlevé equations as a continuum limit [44] and QRT maps in an autonomous limit. The discrete Painlevé equations and QRT mappings are integrable by many of the same criteria; Lax integrability [12, 39], vanishing algebraic entropy [3] and singularity confinement [44].

One way in which one could obtain a non-autonomous second order difference equation from a QRT mapping is by assuming the parameters vary in a manner that preserves the singularity confinement property [44]. Given the autonomous system defined by (2.5), we may deautonomize to the system

\begin{align*}
\tilde{y}y &= \frac{b_3b_4(x + a_1t)(x + a_2t)}{(x + a_3)(x + a_4)}, \\
\tilde{x}x &= \frac{a_3a_4(\tilde{y} + qb_1t)(\tilde{y} + qb_2t)}{(\tilde{y} + b_3)(\tilde{y} + b_4)}. 
\end{align*}

(2.6a) \hspace{1cm} (2.6b)

where \(x = x(t), y = y(t), \tilde{x} = x(qt)\) and \(\tilde{y} = y(qt)\). If we think of this as a difference equation for \(y = y_n\) and \(x = x_n\), with independent parameter, \(n\), this is equivalent to \(n\) appearing in an exponent as \(t = t_0q^n\). The parameter \(q \in \mathbb{C}\) is a constant defined by the relation

\[(2.7) \quad q = \frac{a_1a_2b_3b_4}{b_1b_2a_3a_4}.\]
This system was first derived as a connection preserving deformation \[12\]. While these are often thought of as nonlinear \(q\)-difference equations in \(t\), from the viewpoint of symmetries, it is more conducive to think of (2.6) as a mapping

\[
\phi : \left( \frac{a_1, a_2, a_3, a_4}{b_1, b_2, b_3, b_4} : x, y \right) \rightarrow \left( \frac{qa_1, qa_2, a_3, a_4}{qb_1, qb_2, b_3, b_4} : \tilde{x}, \tilde{y} \right),
\]

where \(\tilde{x}\) and \(\tilde{y}\) are related by (2.6) and we absorb \(t\) into the definitions of \(a_1, a_2, b_1\) and \(b_2\) (equivalent to setting \(t = 1\) in (2.6)). When we blow up the eight points, \(P = \{p_i\} \subset \mathbb{P}_1^2\), given by

\[
p_1 = (-a_1, 0), \quad p_2 = (-a_2, 0), \quad p_3 = (-a_3, \infty), \quad p_4 = (-a_4, \infty),
\]
\[
p_5 = (0, -b_1), \quad p_6 = (0, -b_2), \quad p_7 = (\infty, -b_3), \quad p_8 = (\infty, -b_4),
\]

the resulting surface, \(X_P\), has been called a generalized Halphen surface \[49\]. Lifting the mapping defined by (2.6) is not an automorphism of \(X_P\), but rather an isomorphism, \(\varphi : X_P \rightarrow \tilde{X_P}\), where \(\tilde{P}\) is the set of points defined by the image of (2.8). This map is bijective for the same reasons as for the QRT case. In the autonomous limit as \(q = 1\), (2.7) coincides with (2.4), \(P = \tilde{P}\) and \(\varphi\) is an isomorphism of elliptic surfaces that coincides with the lift of (2.5).

In the same way as (2.5), the blow-up points for (2.6) lie on the lines \(x, y = 0, \infty\), as shown in Figure 2.2. Projectively, in coordinates \([x_0 : x_1, y_0 : y_1]\), they lie on the decomposable curve defined by \(x_0x_1y_0y_1 = 0\).

Figure 2.2. The positions of the blow-up points for (2.5) and (2.6).

If we were to follow up the construction of the surface, one notices that if we were to interchange the blow-up points, we obtain a surface that is isomorphic. We notice that the blow-up co-ordinates, \((z_0^1 : z_1^1)\) and \((z_3^3 : z_1^3)\), for the points, \(p_1\) and \(p_3\) respectively, satisfy the relations

\[
z_1^1(x + a_1) = z_3^3 y, \quad z_3^3(x + a_3) = \frac{z_0^1}{y},
\]

then if we define the transformation \((x, y) \rightarrow (\hat{x}, \hat{y})\), by

\[
\hat{x} = x, \quad \hat{y} = \frac{y + a_3}{x + a_1},
\]

then the blow-up co-ordinates in \(\hat{x}\) and \(\hat{y}\) satisfy

\[
z_1^1(\hat{x} + a_3) = z_0^1 \hat{y}, \quad z_3^3(\hat{x} + a_1) = \frac{z_0^1}{\hat{y}}.
\]
This transformation also has a scaling effect on the positions of \(p_5\) and \(p_6\).

\[
\begin{pmatrix}
a_1, a_2, a_3, a_4 \\
b_1, b_2, b_3, b_4
\end{pmatrix} : x, y \rightarrow \begin{pmatrix}
a_3, a_2, a_1, a_4 \\
b_1 \frac{a_3}{a_1}, b_2 \frac{a_2}{a_1}, b_3, b_4
\end{pmatrix} : \hat{x}, \hat{y}
\]

Both the constraint, (2.5), and the variable \(p\), defined by (2.7), remain valid on the new surface, hence, this is an isomorphism surfaces.

Let \(\sigma_{ij}\) denote the transformation that interchanges blow-ups at \(p_i\) and \(p_j\), then we have a natural set of elements, \(w_0 = \sigma_{78}, w_1 = \sigma_{56}, w_2 = \sigma_{57}, w_3 = \sigma_{13}, w_4 = \sigma_{12}\) and \(w_5 = \sigma_{34}\), with two Dynkin diagram automorphisms, \(p_1\) and \(p_2\), form a representation of an affine Weyl group of type \(D_5^{(1)}\). Furthermore, as an infinite order isomorphism, both (2.5) and (2.6) may be represented as a product of these involutions as

\[
T = p_2 \circ w_2 \circ w_0 \circ w_1 \circ w_2 \circ p_1 \circ w_3 \circ w_5 \circ w_4 \circ w_3.
\]

In many cases, such birational representations were studied independently.

While we have been considering biquadratics over \(\mathbb{P}_1^2\), we may extend these arguments to plane curves in \(\mathbb{P}_2^2\) via the birational mapping, \(\pi : \mathbb{P}_1^2 \rightarrow \mathbb{P}_2^2\), defined by

\[
\pi : ([x_0 : x_1], [y_0 : y_1]) = [x_0 y_0 : x_1 y_0 : x_0 y_1],
\]

which is not defined when \(x_0 = y_0 = 0\) (corresponding to \((\infty, \infty)\)). The inverse, \(\pi^{-1}([u_0 : u_1 : u_2]) = ([u_0 : u_1], [u_0 : u_2])\), is not defined at \([0 : 0 : 1]\) and \([0 : 1 : 0]\). These maps are isomorphisms between the copies of \(\mathbb{C}^2\) defined by \(x_0 = y_0 = 1\) and \(u_0 = 1\) respectively (or more precisely, \(x_0, y_0\) and \(u_0\) are not 0). Any biquadratic curve,

\[
b(x, y) = \sum b_{i,j} x_0^i x_1^{2-i} y_0^j y_1^{2-j} = 0,
\]

not going through \((\infty, \infty)\) (i.e., \(b_{0,0} = 0\)), is mapped, via \(\pi\), to a cubic plane curve

\[
c(u) = \sum_{0 \leq i, j \leq 2, i+j > 0} c_{i,j} u_0^{i+j-1} u_1^{2-i} u_2^{2-j},
\]

which goes through \([0 : 0 : 1]\) and \([0 : 1 : 0]\). In this way, our two generating biquadratics, \(h_0\) and \(h_1\) from (2.1), map to two cubic planar curves which generally intersect at 9 points (also constrained). In this way, we can naturally pass from a pencil of biquadratics on \(\mathbb{P}_1^2\), which is resolved by blowing up eight points to a pencil of cubic plane curves, and a surface obtained by blowing up \(\mathbb{P}_2^2\) at nine points.

In passing from the QRT maps to discrete Painlevé equations via singularity confinement, where the base points are allowed to move, one obtains one of either three types of nonautonomous mappings: difference, \(q\)-difference or elliptic difference equations. The points can still lie in non-generic positions, but the additional constraint associated with the QRT mappings is relaxed. The positions and multiplicities of these nine points determine the symmetries of the surface and of the mapping. All the equations admitting ultradiscretization (or tropicalization) are special cases of \(q\)-difference equations, where all the parameters are assumed to be positive. The class of surfaces giving rise to \(q\)-difference equations was studied by Looijenga [24].

When the nine points are in any non-generic position and appear with different multiplicities, one can not interchange blow-up points in any ad-hoc manner. For example, in the case of (2.5), the points lie on four distinct lines with an intersection
form of type $A_3^{(1)}$, and the positions of those points are subject to the constraint \((2.4)\). In this way we obtain two root systems, one describing the symmetry group of the equation, the other describing the symmetry group of the surface. A degeneration diagram which lists the of symmetries of the surface and the mappings for the $q$-Painlevé equations is given in Figure 2.3.

By identifying the Picard lattices of isomorphic surfaces, we have an alternative interpretation of these mappings and their symmetries \([49]\). From the theory of rational surfaces (as blow-ups of the minimal surfaces $\Sigma_0 = \mathbb{P}_1$ or $\Sigma_1 = \mathbb{P}_2$), we have the isomorphism $\text{Pic}(X) = H^1(X, \mathcal{O}^*) \cong H_2(X, \mathbb{Z})$, with an endowed intersection form \([29, 30]\). The interchange of blow-up and blow-down structures \([1]\) preserves this intersection form and leaves the canonical class fixed \([24]\), so we may interpret these as reflections in $\text{Pic}(X)$. This defines a group of Cremona isometries, which are of affine Weyl type. The work of Sakai extended \([24]\) and realized the action of the translational Cremona isometries as discrete Painlevé equations \([49]\).

### 3. Tropicalization

Tropicalization can be thought of as the pointwise application of a nonarchimedean valuation to geometric structures. Tropicalization sends curves to lines, surfaces to polygons and more generally, smooth structures to piecewise linear ones \([4, 47]\). In the integrable community a non-analytic limit known as ultradiscretization was being used as a way of obtaining new and interesting piecewise linear integrable systems \([55]\). Relating tropicalization with ultradiscretization gives us a way of understanding the geometry of ultradiscrete systems \([38]\).

Let us first consider the ultradiscretization procedure as it was originally considered in \([55]\). Given a subtraction free rational function in a number of strictly positive variables, $f(x_1, \ldots, x_n)$, we introduce ultradiscrete variables, $X_1, \ldots, X_n$, related by $x_i = e^{X_i}/\epsilon$. The ultradiscretization of $f$ is obtained by the limit

\[
F(X_1, \ldots, X_n) := \lim_{\epsilon \to 0^+} \epsilon \ln f(x_1, \ldots, x_n).
\]

The subtraction free nature of the function is required so that we need not consider the logarithm of a negative number. Roughly speaking, the ultradiscretization procedure replaces variables and binary operations as follows:

\begin{align*}
(3.2a) & \quad x_1 x_2 \to X_1 + X_2, \\
(3.2b) & \quad x_1 + x_2 \to \max(X_1, X_2), \\
(3.2c) & \quad x_1/x_2 \to X_1 - X_2,
\end{align*}

where there is no (natural) replacement of subtraction.
Given a difference equation, such as (2.6), we may apply the ultradiscretization procedure to obtain a system known as u-PVI [45], given by

\begin{align}
X + \tilde{X} &= A_3 + A_4 + \max(Q + T + B_1, \tilde{Y}) + \max(Q + T + B_2, \tilde{Y}) \\
&\quad - \max(B_3, \tilde{Y}) - \max(B_4, \tilde{Y}), \\
Y + \tilde{Y} &= B_3 + B_4 + \max(A_1 + T, \tilde{X}) + \max(A_2 + T, X) \\
&\quad - \max(B_3, X) - \max(B_4, X),
\end{align}

where the variable $Q$ is specified by the relation

\[ Q = A_1 + A_2 - A_3 - A_4 - B_1 - B_2 + B_3 + B_4. \]

A special case of this system was shown to arise as an ultradiscrete connection preserving deformation [37]. In the same way as the original equation, we may think of this as a mapping

\[ \Phi : \left( A_1, A_2, A_3, A_4, B_1, B_2, B_3, B_4, X, Y \right) \rightarrow \left( Q + A_1, Q + A_2, A_3, A_4, Q + B_1, Q + B_2, B_3, B_4, \tilde{X}, \tilde{Y} \right). \]

If we take the autonomous case, when $Q = 0$, then the above ultradiscrete Painlevé equation becomes an ultradiscrete QRT mapping (i.e., the ultradiscretization of (2.5)), which was introduced in [41] and studied from a tropical geometric viewpoint by Nobe [31].

The ultradiscretization of (2.3) gives the following piecewise linear function

\[ H(X, Y) = \max(Y - A_3, Y - A_4, B_1 + B_2 \max(-A_1, -A_2) - Y, \max(Y, B_1) + \max(Y, B_2) - X - Y, X - Y + \max(Y, B_3) + \max(Y, B_4) - A_3 - A_4), \]

which is also an invariant of the ultradiscrete QRT map, i.e., $H(X, Y) = H(\tilde{X}, \tilde{Y})$ [31]. Furthermore, the evolution of the ultradiscrete QRT map defines a linear evolution on the Jacobian of the invariant, hence, the ultradiscrete QRT map may be expressed in terms of the addition law on a tropical elliptic curve [31, 59].

While we may be able to solve (2.2) in a subtraction free manner, given an invariant such as (2.3), solving $H(X, Y) = H(\tilde{X}, \tilde{Y})$ involves a max on both the left and right. Our approach is different in that we only consider transformations that preserve the structure of the curve defined by (3.4), which is a tropical curve. Any automorphism of this tropical curve can be expressed in terms of compositions of more fundamental operations. We need to consider these curves more carefully, hence, we will briefly review some tropical geometry [47].

The discrete dynamical system, (3.3), is most naturally defined over a tropical semifield [40], more precisely the max-plus semifield, which is the set $\mathbb{T} = \mathbb{R} \cup \{-\infty\}$, equipped with the binary operations

\begin{align}
X_1 \oplus X_2 &:= \max(X_1, X_2), \\
X_1 \otimes X_2 &:= X_1 + X_2,
\end{align}

which are known as tropical addition and tropical multiplication respectively. The element $-\infty$ is a tropical additive identity and 0 is the tropical multiplicative identity [40].

The geometry of objects over the tropical semifields is the subject of tropical geometry [47]. A tropical polynomial, $F \in \mathbb{T}[x_1, \ldots, x_n]$ defines a piecewise linear
function from $\mathbb{T}^n \to \mathbb{T}$, given by
\begin{equation}
F(X_1, \ldots, X_n) = \max_j (C_j + A_{j1}X_1 + \ldots + A_{jn}X_n),
\end{equation}
where $\{A_{i,j}\}$ are integers and $\{C_j\}$ are elements of $\mathbb{T}$. The tropical variety associated with $F \in \mathbb{T}[X_1, \ldots, X_n]$, denoted $\mathcal{V}(F)$, is defined to be
\begin{equation}
\mathcal{V}(F) = \{X = (X_1, \ldots, X_n) \in \mathbb{T}^n \text{ such that } F \text{ is not differentiable at } x\},
\end{equation}
which occurs precisely when one argument of the max-expression becomes dominant over another argument [47].

Another equivalent algebraic characterization of tropical varieties relies on nonarchimedean valuations. Every non-zero algebraic function, $f \in \mathbb{C}(t)$, admits a representation as a Puiseux series,
\begin{equation}
f(t) = c_1t^{q_1} + c_2t^{q_2} + \ldots,
\end{equation}
where $c_1 \neq 0$ and $\{q_i\}$ are rational and ordered such that $q_i < q_{i+1}$. The function, $\nu : \mathbb{C}(t) \to \mathbb{T}$, given by
\begin{equation}
\nu(f) = -q_1,
\end{equation}
is a nonarchimedean valuation. This may be extended to an algebraically and topologically closed field with a valuation ring of $\mathbb{R}$, which we simply denote $\mathbb{K} = \overline{\mathbb{C}(t)}$ [20]. If $I \subset \mathbb{K}[x_1^{\pm1}, \ldots, x_n^{\pm1}]$ is an ideal, then we define $V(I) \subset \mathbb{K}^n$ as
\begin{equation}
V(I) = \{(x_1, \ldots, x_n) : f(x_1, \ldots, x_n) = 0 \text{ for all } f \in I\}.
\end{equation}
The tropical variety associated with $I$ is the topological closure of the point-wise application of $\nu$ to $V(I)$, i.e., $\mathcal{V}(I) = \nu(V(I)) \subset \mathbb{T}^n$. For every tropical variety $\mathcal{V}(F)$, there exists a function, $f$, such that $\mathcal{V}(F) = \nu(V(f))$, hence, we may define a tropical variety in terms of either piecewise linear functions or ideals of $\mathbb{K}[x_1^{\pm1}, \ldots, x_n^{\pm1}]$. The equivalence of the set of points of non-differentiability and the image of the valuations is outlined in [47]. Each tropical curve is a collection vertices, finite line segments, called edges, and a collection of semi-infinite line segments, called rays.

In the same way as affine $n$-space may be considered to be embedded in projective space, we may naturally consider $\mathbb{T}^n$ as being embedded in tropical projective space. Define the equivalence relation, $\sim$, on $\mathbb{T}^{n+1}$ so that
\[
V \sim U \text{ if and only if } V = U + \lambda(1, 1, \ldots, 1),
\]
for some $\lambda$, then tropical projective $n$-space is the set
\[
TP^n = \mathbb{T}^n / \sim.
\]
A tropical function of the form (3.6) is said to be homogeneous if there exists a $d$ such that for every $j$
\[
\sum_i A_{j,i} = d.
\]
The set of non-differentiable points of a tropically homogeneous polynomial defines a tropical projective variety.

Given a rational function in a number of variables, $f(x_1, \ldots, x_n)$, we can lift the function up to the field of algebraic functions by letting $x_i = t^{X_i}$ for some $X_i$, then the ultradiscretization procedure is known to coincide with
\begin{equation}
F(X_1, \ldots, X_n) = \nu(f(x_1, \ldots, x_n)),
\end{equation}
where $\nu : \mathbb{C}(t) \to \mathbb{T}$, given by
\begin{equation}
\nu(f) = -q_1,
\end{equation}
is a nonarchimedean valuation. This may be extended to an algebraically and topologically closed field with a valuation ring of $\mathbb{R}$, which we simply denote $\mathbb{K} = \overline{\mathbb{C}(t)}$ [20]. If $I \subset \mathbb{K}[x_1^{\pm1}, \ldots, x_n^{\pm1}]$ is an ideal, then we define $V(I) \subset \mathbb{K}^n$ as
\begin{equation}
V(I) = \{(x_1, \ldots, x_n) : f(x_1, \ldots, x_n) = 0 \text{ for all } f \in I\}.
\end{equation}
The tropical variety associated with $I$ is the topological closure of the point-wise application of $\nu$ to $V(I)$, i.e., $\mathcal{V}(I) = \nu(V(I)) \subset \mathbb{T}^n$. For every tropical variety $\mathcal{V}(F)$, there exists a function, $f$, such that $\mathcal{V}(F) = \nu(V(f))$, hence, we may define a tropical variety in terms of either piecewise linear functions or ideals of $\mathbb{K}[x_1^{\pm1}, \ldots, x_n^{\pm1}]$. The equivalence of the set of points of non-differentiability and the image of the valuations is outlined in [47]. Each tropical curve is a collection vertices, finite line segments, called edges, and a collection of semi-infinite line segments, called rays.

In the same way as affine $n$-space may be considered to be embedded in projective space, we may naturally consider $\mathbb{T}^n$ as being embedded in tropical projective space. Define the equivalence relation, $\sim$, on $\mathbb{T}^{n+1}$ so that
\[
V \sim U \text{ if and only if } V = U + \lambda(1, 1, \ldots, 1),
\]
for some $\lambda$, then tropical projective $n$-space is the set
\[
TP^n = \mathbb{T}^n / \sim.
\]
A tropical function of the form (3.6) is said to be homogeneous if there exists a $d$ such that for every $j$
\[
\sum_i A_{j,i} = d.
\]
The set of non-differentiable points of a tropically homogeneous polynomial defines a tropical projective variety.

Given a rational function in a number of variables, $f(x_1, \ldots, x_n)$, we can lift the function up to the field of algebraic functions by letting $x_i = t^{X_i}$ for some $X_i$, then the ultradiscretization procedure is known to coincide with
\begin{equation}
F(X_1, \ldots, X_n) = \nu(f(x_1, \ldots, x_n)),
\end{equation}
where $\nu : \mathbb{C}(t) \to \mathbb{T}$, given by
\begin{equation}
\nu(f) = -q_1,
for all subtraction free functions [36, 38]. The above extension, given by (3.10), is one of a number of ways to incorporate a version of subtraction into the ultradiscretization process [11, 22, 23, 34].

The most immediate consequence from the viewpoint of the geometry is that singularities of a map manifest themselves as points of non-differentiability [4, 38, 47]. This interpretation was also present in the work of Joshi and Lafortune who elucidated what the analogue of singularity confinement should be [14].

One of the characteristic features of the QRT map is that the invariant curves all intersect at the base points. From looking at the invariant curves of (2.5), depicted in Figure 1.1, this feature is not apparent in the tropical setting. When we consider the extension of the ultradiscretization via (3.10), another way of looking at the invariant is the tropical variety generated by the ideal

$$I_{H_0} = \langle h(x, y) - t^{H_0} \rangle,$$

in $\mathbb{K}[x, y]$. The tropical variety associated with this ideal is the set

$$(3.11) \quad \mathcal{V}(I_{H_0}) := \nu(V(I_{H_0})).$$

For each $x = t^X$, where $X \in \mathbb{Q}$, the equation

$$h(t^X, y) - t^{H_0} = 0$$

is quadratic in $y$, and as $\mathbb{K}$ is algebraically close, we have two algebraic solutions, $y_1$ and $y_2$ over $\mathbb{K}$. That is for each $X$, we obtain values $Y_1 = \nu(y_1)$ and $Y_2 = \nu(y_2)$ in $\mathbb{T}$, which form infinite rays (also called tentacles in [31, 59]). These form points of $\mathcal{V}(I_{H_0})$ that do not appear in the level set of $H(X, Y)$. Notice that each of the rays intersect on the lines at $X = \pm \infty$ and $Y = \pm \infty$, and positions of the rays give us define where on that line they intersect. The inclusion of the rays to the level sets, as seen in Figure 3.1, makes them smooth tropical curves in the sense of [47].

Figure 3.1. A tropical biquadratic with the rays labeled in red.

We may extend these tropical biquadratics to $\mathbb{TP}_1^2$ by using homogeneous coordinates $X = [X_0 : X_1]$ and $Y = [Y_0 : Y_1]$. The maps $\pi$ and $\pi^{-1}$ possess tropical analogues, $\Pi : \mathbb{TP}_1^2 \to \mathbb{TP}_2$ and $\Pi^{-1} : \mathbb{TP}_2 \to \mathbb{TP}_1^2$, given by

$$\Pi : ([X_0, X_1], [Y_0, Y_1]) \to [X_0 + Y_0 : X_1 + Y_0 : X_0 + Y_1],$$

$$\Pi^{-1} : [U_0 : U_1 : U_2] \to ([U_0 : U_1], [U_0 : U_2]).$$

These are isomorphisms between the copies of $\mathbb{T}^2$ specified by $X_0 = Y_0 = 0$ and $U_0 = 0$ respectively. The mapping $\Pi$ is not defined when $X_0 = Y_0 = -\infty$ and the
The inverse is not defined at \([-∞ : 0 : −∞]\) and \([−∞ : −∞ : 0]\). The image of any tropical biquadratic, which is the level set of a tropical biquadratic function

\[ H(X, Y) = \max_{i,j=0,1,2} \left( B_{i,j} + iX_0 + (2-i)X_1 + jY_0 + (2-j)Y_1 \right) , \]

in which \(A_{0,0} = −∞\), under the mapping \(Π\) is a tropical cubic plane curve specified by the level set

\[ H(U) = \max_{0≤i,j≤2, i+j>0} \left( C_{i,j} + (i+j-1)U_0 + (2-i)U_1 + (2-j)U_2 \right) . \]

While the rays over \(\mathbb{T}P^2\) are mapped to rays under \(Π\), the more general cubic plane curve is an enneagon. If one considers the enneagon as the image of the variety over \(K[x, y, z]\), one recovers nine rays counting multiplicities. The case of nine distinct rays is depicted in Figure 3.2. In this way, the information we have on rays in \(\mathbb{P}^2\) applies equally well to the rays in \(\mathbb{T}P^2\).

Figure 3.2. A tropical cubic plane curve with rays labeled in red.

As the rays define the positions of the vertices of each polygon, they will play an important role in the description of the symmetries. In Figure 3.1 and Figure 3.2, all the rays are asymptotic to one of three forms:

\[ L_i : X - A_i = 0, \quad L_j : Y - A_j = 0, \quad L_k : Y - X - A_k = 0. \]

Since the rays in Figure 3.1 and Figure 3.2 are part of every variety of the form (3.11), this is equivalent to each variety intersecting in \(\mathbb{T}P^2\) at points

\[ [A_i : −∞ : 0], \quad [−∞ : A_j : 0], \quad [0 : −∞ : A_k], \]

respectively. For the level set to close, there is a constraint on the positions of the rays, which when relaxed gives a spiral diagram. For smooth biquadratics, we obtain spiralling octagons (see Figure 1.1). In the smooth cubic case we obtain spiralling enneagons (see Figure 5.8). Degenerations arise when two rays to coincide, corresponding to setting some of the coefficients in \(H(X, Y)\) to \(−∞\).

This construction may be generalized to tropical genus one curves, which allows us to consider decagons and undecagons as level sets of tropical quartic and tropical sextic plane curves respectively. In these cases, one finds twelve and thirteen rays, counting multiplicities (when rays coincide). The decagon used will be a tropical quartic with four rays of order one of the form \(L_i : X - A_i\), four rays of order one of the form \(L_j : Y - A_j\) and two rays of order two of the form \(L_k : Y - X - A_k = 0\). This
would be the ultradiscretization of a curve of degree four with eight singularities of
order one and two of order two, which gives a genus of one curve by the degree-genus
formula,

\[ g = \frac{(d - 1)(d - 2)}{2} - \sum_k r_k(r_k - 1), \]

where \( d \) is the degree of the curve and the \( r_i \) is the order of the \( k \)-th singularity.
In a similar way, our undecagon is a the ultradiscretization of a genus one curve of
degree six curve with six rays of order one, two of order three and three of order
two. This formula remains valid for tropical varieties \([9]\).

4. piecewise linear transformations of polygons and spirals

Cremona transformations of the plane, and their subgroups, are a topic of
classical and modern interest \([8, 25, 17]\). The classical result of Noether \([32]\)
(see also \([8]\)) states that Cremona transformations are generated the quadratic
transformations, the simplest being the standard Cremona transformation

\[ \tau : [x : y : z] \rightarrow [yz : xz : xy], \]

which may be interpreted as the blow-up of the points \([1 : 0 : 0], [0 : 1 : 0] \text{ and} [0 : 0 : 1] \)combined with a blow-down on the co-ordinate lines given by \( xyz = 0 \). In a similar vein, our aim is to specify a generating set of tropical Cremona
transformations from which all the other transformations may be obtained. We
want to then specify subgroups of these that preserve a given spiral diagram.

To specify any spiral diagram, we begin with a parameterization of the asymp-
totic form of the rays,

\[ \mathcal{X} = \{ L_i \text{ where } L_i : a_iX + b_iY + c_i = 0, \text{ and } a_i, b_i, c_i \in \mathbb{Z} \}. \]

The shape of the spirals are determined by the invariants obtained in the au-
tonomous limit. We seek a group of transformations that preserve the forms of
these rays, more specifically, we seek transformations, \( \sigma \), such that

1. \( \sigma \) is a bijection of the plane.
2. For every ray, \( L_j \), there is a ray, \( L_i \), such that \( \sigma : L_i = \tilde{L}_j \), where \( \tilde{L}_j \)
differs only by some translation.

These may be thought of as tropical Cremona isometries, as these conditions replic-
cate conditions that require the canonical class and intersection form of the surface
be fixed.

Since the Cremona isometries are products of the interchange of blow-up and
blow-down structures \([1]\), and the positions of these blow-up points are encoded in
the positions of the rays, it is sufficient to consider the shearing transformations that
create and smooth out polygons whose vertices lie along these rays. Analogous
to the results of Noether \([8]\), we propose the following two generators:

\[ \iota_A : (X, Y) \rightarrow (X, Y + \max(0, X - A)), \]
\[ \Xi : (X, Y) \rightarrow (aX + cY, bX + dY), \]

where \( |ad - bc| = 1 \) and \( A \in \mathbb{T} \). The action of \( \iota_A \) can be seen as an analogous to
the interchange of blow-ups in the following way: if the vertices of the level sets of
a polygon trace out the rays, then \( \iota_A \) can smooth out all the vertices along a ray
asymptotic to, \( L : X = A \), while simultaneously creating a kink along all the level
sets along a ray of the same form, but in the opposite direction. This means that if all the rays intersected at a point \( P = (A, -\infty) \), the transformed polygon has rays that intersect at \( (A, \infty) \), or vice versa.

Let us use \( \iota_A \) to interchange rays that are of the same form in asymptotically opposite directions. Suppose we have two rays, \( L_i \) and \( L_j \), which satisfy

\[
L_i : X - A = 0 \quad \text{and} \quad L_j : X - B = 0,
\]

as \( Y \to -\infty \) and \( Y \to \infty \) respectively. In the simplest case, these rays are order one, in that the change in derivative is just one, in which case the transformation

\[
\sigma = \iota_B^{-1} \circ \iota_A : (X, Y) \to (X, Y + \max(0, X - A) - \max(0, X - B)),
\]

has the effect of creating a ray along the line \( X - A \) as \( Y \to \infty \) and smooting out a set of kinks along \( L_i \), and conservely doing the same for \( L_j \). If we think of the surface as being parameterized by \( A \) and \( B \), then this action has the effect of swapping \( A \) and \( B \). The overall shape of the resulting polygon does not change by this transformation and the action is an isomorphism of polygons. The action of \( \iota_A \) and \( \sigma \) on the plane is depicted in Figure 4.1 and the action on the level set of the form in Figure 3.1 is depicted in Figure 4.2.

\[ \text{Figure 4.1. Assuming } B < A, \text{ the effect of } \iota_B \text{ is depicted on the left, and } \sigma = \iota_A^{-1} \circ \iota_B \text{ on the right.} \]

Let us now consider how to swap rays given by

\[
L_i : X - A = 0 \quad \text{and} \quad L_j : Y - B = 0,
\]

as \( Y \to -\infty \) and \( X \to \infty \) respectively. To describe this transformation, let us consider the transformation, \( \rho : \mathbb{T}^2 \to \mathbb{T}^2 \), given by

\[
\rho : (X, Y) \to (X - \max(0, Y), X - \max(0, -Y)),
\]

whose inverse is given by

\[
\rho^{-1} : (X, Y) \to (\max(X, Y), Y - X).
\]

This transformation can be expressed as a composition of transformations of the form (4.2) and (4.3) as

\[
\rho : (X, Y) \xrightarrow{\Xi} (Y, X) \xrightarrow{\iota_A} (Y, X - \max(0, Y)) \xrightarrow{\Xi} (X - \max(0, Y), Y) \xrightarrow{\Xi} (X - \max(0, Y), Y + X - \max(0, Y)) = (X - \max(0, Y), X - \max(0, -Y)).
\]
Figure 4.2. A depiction of action of $\sigma$, described above on an octagon with rays $L_1$ and $L_2$. The blue octagon is the preimage and the green octagon is the image.

Roughly speaking, this sends every straight line of the form $X - A = 0$ to one that is bent 90 degree along the line $Y = X$. The conjugation of $\iota_A$ by $\rho$, which we label $\eta_A = \rho \circ \iota_A \circ \rho^{-1}$, is given by the expression

$$
\eta_A(X, Y) = (X - \max(0, Y - A), Y + \max(A, X, Y) - \max(A, Y)).
$$

It should be clear that this has the same effect as $\iota_A$ below the line $Y = X$, however, the effect of $\iota_A$ around $Y = \infty$ now occurs at $X = -\infty$. We may now state that the transformation swapping $L_i$ and $L_j$ is given by

$$
\sigma = \eta_B \circ \eta_A^{-1},
$$

whose max-plus expression may be simplified to

$$
\sigma(X, Y) = (B + X + \max(A, X, Y) - \max(A + B, B + X, A + Y),
A + Y + \max(B, X, Y) - \max(A + B, B + X, A + Y)),
$$

or equivalently, this is the tropical projective transformation

$$
\sigma([X : Y : Z]) = [B + X + \max(A + Z, X, Y) : A + Y + \max(B + Z, X, Y) : Z + \max(A + B + Z, B + X, A + Y)].
$$

The effect of $\eta_B$ is shown in Figure 4.3 and the effect on a cubic plane curve with these rays is depicted in Figure 4.4.

Lastly, we include a set of transformations simply permute the roles of two rays that are of the same type. For example, if we have a tentacle, described by $L_i : X - A = 0$ as $Y \to -\infty$ and another, described by $L_j : X - B = 0$ as $Y \to -\infty$, then one transformation simply swaps the roles of $A$ and $B$, which swaps $L_i$ and $L_j$. In this case, $\sigma$ acts the identity map on the plane and a simple transformation of the parameter space. This transformation can always be applied when there are two rays of the same form.

Each of these transformations is an isomorphism between either a collection of polygons or some polygon spiral diagram to another polygon spiral diagram of the same form, we can now specify that each of the ultradiscrete QRT mappings and ultradiscrete Painlevé equations are elements of the group of transformations that
preserve either the polygon or polygonal spiral diagrams, hence, may be expressed in terms of the simple transformations above. To do this, we consider (2.5) and (2.6) by parameterizing the rays as follows:

\[L_1 : X - A_1 = 0, \quad L_2 : X - A_2 = 0, \quad L_3 : X - A_3 = 0, \quad L_4 : X - A_4 = 0,\]
\[L_5 : Y - B_1 = 0, \quad L_6 : Y - B_2 = 0, \quad L_7 : Y - B_3 = 0, \quad L_8 : Y - B_4 = 0,\]

where \(L_1\) and \(L_2\) extend downwards, \(L_3\) and \(L_4\) extend upwards, \(L_5\) and \(L_6\) extend to the left and \(L_7\) and \(L_8\) extend to the right. We now have a group of type \(W(D_5^{(1)}) = \langle s_0, \ldots, s_5 \rangle\) where

\[s_0 = \sigma_{78}, \quad s_1 = \sigma_{56}, \quad s_2 = \sigma_{57},\]
\[s_3 = \sigma_{13}, \quad s_4 = \sigma_{12}, \quad s_5 = \sigma_{34},\]

with two Dynkin diagram automorphisms, \(p_1\) and \(p_2\), which are reflections through the line \(Y = (B_3 + B_4)/2\) and \(X = (A_3 + A_4)/2\) respectively. We can now write the ultradiscrete QRT mapping and the ultradiscrete Painlevé equation as the composition

\[(4.6) \quad T = p_2 \circ s_2 \circ s_0 \circ s_1 \circ s_2 \circ s_1 \circ p_1 \circ s_3 \circ s_5 \circ s_4 \circ s_3.\]
To show that each step is an isomorphism of spiral diagrams, we have depicted the nontrivial steps in $T$ on a typical spiral in Figure 4.5.

![Spiral Diagrams](image)

**Figure 4.5.** Starting with a single spiral, we show each significant step in the sequence (4.6). In blue, we show the previous result, in green is the result of the transformations. In the last step, we also show the original spiral (in red).

We can present the nontrivial actions of these transformations as

- $s_2 : X \rightarrow X + \max(Y, B_3) - \max(Y, B_1)$,
- $s_2 : A_1 \rightarrow A_1 + B_3 - B_1$,
- $s_2 : A_2 \rightarrow A_2 + B_3 - B_1$,
- $s_3 : Y \rightarrow Y + \max(X, A_3) - \max(X, A_1)$,
- $s_3 : B_1 \rightarrow B_1 + A_3 - A_1$,
- $s_3 : B_2 \rightarrow B_2 + A_3 - A_1$,
- $p_1 : Y \rightarrow B_3 + B_4 - Y$,
- $p_2 : X \rightarrow A_3 + A_4 - X$.

The composition in (4.6) gives (3.3).

**Remark 4.1.** The above constitutes the action on a tropical biquadratic that does not satisfy the requirement that the image under $\Pi$ is a tropical cubic plane curve. A cubic plane curve may be obtained by applying $\iota_{A_4}$, which has the effect of removing the ray given by $L_4$ and adding a ray given by the same formula, but pointing downward instead of upwards. Up to translational invariance, this is equivalent to the polygon considered in subsection 5.7. In particular, their groups are of the same affine Weyl type.

An aspect of defining the group of transformations for a polygon or spiral diagram that we have not introduced in the above example is that we may always
remove two parameters by taking into account uniqueness of a group of transformations up to translational equivalence. We can take this into account by insisting that two tentacles, of different asymptotic forms, pass through the origin. This means that we will often compose one of the above types of transformation with a translation so that any ray which is supposed to pass through the origin does so after the transformation. This fixes a representation based on which rays we choose to pass through the origin.

5. Tropical representations of affine Weyl groups

While the task of finding a subtraction free versions of the Cremona transformations in Sakai list was completed in \cite{18}, what we wish to present is a different perspective. The derivation of the following list of affine Weyl representations will be slightly different due to the manner in which they were derived. We will also provide some of the geometric motivation behind our choices of generators. To this end, we shall display a spiral diagram and a nontrivial translation for each of the cases in Table 1. When the Newton polygon is known, this will also accompany the spiral diagram on the right.

5.1. Triangles. At the bottom of the hierarchy of multiplicative surfaces in \cite{49} is the system with a symmetry of the dihedral group of order 6, which admits the presentation

$$D_6 = \langle p_1, p_2 | p_1^3 = p_2^2 = (p_2p_1)^2 = 1 \rangle.$$ 

This is the group permuting the three rays in figure 5.1. The rays may be parameterized by the equations

$$L_1 : Y - X - A = 0, \quad L_2 : 2X + Y - B = 0, \quad L_3 : X + 2Y - C = 0.$$ 

By exploiting scaling (i.e., $X \to X + \lambda$ and $Y \to Y + \mu$), we can reduce this to the case where we fix $B = C = 0$.

![Figure 5.1. The spiral diagram for the system with $A_0^{(1)}$-symmetry.](image)

In this way, let $p_1$ be the three cycle on the lines, given as $p_1 : (L_1, L_2, L_3) \to (L_3, L_1, L_2)$. Similarly, $p_2$ is a two cycle, which swaps $L_2$ and $L_3$. These are
explicitly given by the piecewise linear transformations
\[p_1 : X \rightarrow -X - Y - \frac{A}{3}, \quad p_2 : Y \rightarrow X + \frac{2A}{3}, \quad p_2 : A \rightarrow A,\]
\[p_2 : X \rightarrow Y, \quad p_2 : X \rightarrow Y, \quad p_2 : A \rightarrow -A.\]

As a very degenerate case, the transformations are simple given by (up to translations) a subgroup of actions of the type \((4,3)\).

It is natural to see that \(Q = A\). The limit which gives a fibration by tropical biquadratics is the limit as \(A = Q = 0\), with the polygons arising as level sets of \(H(X,Y) = \max(-X - Y, X,Y)\).

As \(D_6\) contains no elements of infinite order, there is no difference equation associated with this group.

5.2. Rectangles. We consider a quadrilateral that gives an affine Weyl group of type \(W(A_1^{(1)})\) with an additional \(D_8\) symmetry. In the same way as above, we may exploit scaling so that the red lines on the left of Figure 5.2 are fixed in a way that they pass through the origin, giving
\[L_1 : Y + X = 0, \quad L_3 : Y - X - A = 0,\]
\[L_2 : Y - X = 0, \quad L_4 : Y + X - B = 0.\]

This is depicted in figure 5.2

![Figure 5.2. The spiral diagram for the system with affine Weyl symmetry \(A_1^{(1)}\).](image)

The symmetry group for this system is the semidirect product of \(D_8 = \langle p_1, p_2 \rangle\) and \(W \left( A_1^{(1)} \right) = \langle s_0, s_1 \rangle\). A presentation is given by
\[D_8 \rtimes W \left( A_1^{(1)} \right) = \langle p_1, p_2, s_0, s_1 : p_1^4 = p_2^2 = (p_1 p_2)^2 = s_0^2 = s_1^2 = s_0 p_2 s_1 p_2 = 1 \rangle,\]
where the action of \(D_8\) is specified up to translation by a clockwise rotation of the four defining lines, \(p_1\), whereas \(p_2\) swaps \(L_3\) and \(L_4\). We write these transformations as
\[p_1 : X \rightarrow Y + \frac{B}{2}, \quad p_1 : Y \rightarrow -X - \frac{B}{2}, \quad p_1 : A \rightarrow B, \quad p_1 : B \rightarrow A,\]
\[p_2 : X \rightarrow X, \quad p_2 : Y \rightarrow -Y, \quad p_2 : A \rightarrow -B, \quad p_2 : B \rightarrow -A.\]
Let $s_0$ be the conjugation of the transformation depicted in Figure 4.1 with the piecewise linear transformation that makes $L_2$ and $L_3$ parallel to the $y$-axis (and $L_1$ and $L_4$ to the $y$-axis). The $s_1$ may be obtained in a similar manner with $L_1$ and $L_4$, giving

$$s_0 : X \to X + \max(0, Y - X + A) - \max(0, Y - X) - \frac{A}{2},$$

$$s_0 : Y \to Y + \max(0, Y - X + A) - \max(0, Y - X) + \frac{A}{2},$$

and $s_1 = p_2 \circ s_0 \circ p_2$, which we write as

$$s_1 : X \to X + \max(0, -X - Y - B) - \max(0, -X - Y) + \frac{B}{2},$$

$$s_1 : Y \to Y + \max(0, -X - Y) - \max(0, -X - Y - B) + \frac{B}{2},$$

$$s_1 : A \to -2B, \quad s_1 : B \to -B.$$

We find that $Q = A - B$ by tracing around the spiral. When $A = B$, we obtain the invariant

$$H(X, Y) = \max(-X, -Y, X - A).$$

For the element $T = s_1 \circ s_0$, we resort to co-ordinates $U$ and $V$, where $X = (U + V)/2$ and $Y = (U - V)/2$. The dynamical system in these variables is

$$\begin{align*}
\hat{U} - U &= B + 2\max(A, A + V) - 2\max(0, V), \\
\hat{V} - V &= 3A + 2\max(0, \hat{U}) - 2\max(B, 2A + \hat{U}), \\
T : A &\to A + 2Q, \quad T : A \to A - 2Q,
\end{align*}$$

where $\hat{U} = T(U)$ and $\hat{V} = T(V)$.

### 5.3. Quadralaterals

We have another quadralateral that does not possess the symmetry of a rectangle. We break the dihedral symmetry, fixing the parameterization of the four rays in the following manner:

$$\begin{align*}
L_1 : Y + X &= 0, & L_3 : Y + 2X - A &= 0, \\
L_2 : Y &= 0, & L_4 : Y - X - B &= 0,
\end{align*}$$

as depicted in Figure 5.3.

We write this group as

$$W(A^{(1)}_4) = (s_0, s_1 | (s_0)^2 = (s_1)^2),$$

where $s_1 \circ s_0$ is the element of infinite order. The first transformation, $s_0$, swaps the roles of $L_1$ with $L_2$ and $L_3$ with $L_4$, which we write as

$$\begin{align*}
s_0 : Y &\to -Y - X, & s_0 : A &\to -B, \\
s_0 : X &\to X, & s_0 : B &\to -A.
\end{align*}$$

The other involution, $s_1$, is a reflection around $X = B/2$ above $Y = 0$ (so that $L_4$ is sent to $L_1$) and a skewed reflection below $Y = 0$, given by $X \to -Y - X - B/2$, which simplifies to the following tropically rational transformation

$$\begin{align*}
s_1 : X &\to \max(0, -Y) - X - B & s_1 : A &\to -2B - A, \\
s_1 : Y &\to Y & s_1 : B &\to B.
\end{align*}$$
This is simply the conjugation of $\iota_0$ with a swap of $X$ and $Y$. We find $Q = A + B$ by tracing around one spiral. When $A = -B$, we obtain the invariant
$$H(X,Y) = \max(-X-Y,-X,Y,X-A).$$

The composition, $T = s_1 \circ s_0$, gives the evolution equations

(5.3a) \quad \tilde{X} + X = \max(0,Y+X) + A,

(5.3b) \quad \tilde{Y} + Y = -X,

(5.3c) \quad T : A \to A + Q, \quad T : B \to B - Q.

This element, $T$, is the generator for $\mathbb{Z}$ in the decomposition of $W(A_1^{(1)}) \cong \mathbb{Z} \times \mathfrak{S}_2$ in [18, 49]. Alternatively, we could write this system as a second order difference equation in $W = -Y$, where the resulting system becomes
$$W + 2\tilde{W} + \tilde{\tilde{W}} = \max(0,\tilde{W}) + A,$$
which coincides with a more standard version of an ultradiscrete version of the first Painlevé equation [45].

5.4. Pentagons. This case is associated with $u$-$P_{II}$. To preserve much of the structure of the two previous cases, we have parameterize the five rays as follows:

$L_1 : Y + X - A = 0,$ \quad $L_4 : Y + X - B = 0,$

$L_2 : Y = 0,$ \quad $L_5 : Y - X - C = 0,$

$L_3 : X = 0,$

as depicted in Figure 5.4.

A presentation of the group of transformations is
$$A_1^{(1)} \times A_1^{(1)} = \langle s_0, s_1, w_0, w_1 : s_1^2 = w_1^2 = 1 \rangle$$
where $s_0$ is a reflection around the line $Y = X$, and $s_1$ is the same the action of $s_1$ in the previous section in that above the line $Y = 0$, we have a reflection, and
below the line, we skew the plane. The generators are written
\[ s_0 : X \rightarrow Y, \quad p_0 : A \rightarrow B, \quad p_0 : C \rightarrow -C, \]
\[ s_0 : Y \rightarrow X, \quad p_0 : B \rightarrow A, \]
\[ s_1 : X \rightarrow \max(0, -Y) - X + B, \quad p_1 : A \rightarrow B + C, \quad p_1 : A_2 \rightarrow A - B, \]
\[ s_1 : Y \rightarrow Y, \quad p_1 : B \rightarrow B. \]
The other part of the group, generated by \( w_0 \) and \( w_1 \), can be generated by actions that swap \( L_1 \) and \( L_4 \) via a transformation that sheers between the lines \( L_1 \) and \( L_4 \), hence, swapping them, written as
\[ w_0 : X \rightarrow X + \max(0, X + Y - A) - \max(0, X + Y - B), \]
\[ w_0 : Y \rightarrow Y + \max(0, X + Y - B) - \max(0, X + Y - A), \]
\[ w_0 : A \rightarrow B, \quad w_0 : B \rightarrow A, \quad w_0 : C \rightarrow C + 2A - 2B, \]
and, \( w_1 \), is a piecewise linear sheering transformation swapping \( L_2 \) and \( L_3 \), which we write as
\[ w_1 : X \rightarrow X + C + \max(0, X, Y - C) - \max(0, X, Y), \]
\[ w_1 : X \rightarrow Y + \max(C, X, Y - C) - \max(0, X, Y), \]
\[ w_1 : A_0 \rightarrow A + C, \quad w_1 : B \rightarrow B + C, \quad s_1 : C \rightarrow -C. \]
Tracing around the figure gives us
\[ Q = B + C - A. \]
When \( C = A - B \), we obtain the invariant
\[ H(X, Y) = \max(-X - Y, -X, -Y, X - B, Y - A). \]
One simple translation is the composition, \( T = (s_0 \circ s_1)^2 \), which can be written as
\[ (5.4a) \quad \tilde{X} + X = \max(0, -Y) + B, \]
\[ (5.4b) \quad \tilde{Y} + Y = \max(0, -\tilde{X}) + B + C, \]
\[ (5.4c) \quad T : A \rightarrow A + Q, \quad T : B \rightarrow B + Q, \]
where the other obvious translation, \((w_0 \circ w_1)\), commutes with \(T\). This system is called \(u\)-P_{II}. Exact solutions of (5.4) were studied in \([28]\).

5.5. Hexagons. The tropical representation for \(W(A_2^{(1)} + A_1^{(1)})\) was one of the first to be written down \([19]\). There are a number of equivalent ways of obtaining a hexagon as a cubic plane curve, we choose one in which the rays replicates the representation of \([19]\). The particular choice is given by

\[
L_1 : Y = 0, \\
L_2 : Y - X - B_1 = 0, \\
L_3 : X = 0, \\
L_4 : Y - B_2 = 0, \\
L_5 : Y - X + A_0 - B_1 = 0, \\
L_6 : X + A_1 = 0,
\]

as depicted in Figure 5.5.

![Figure 5.5. The spiral diagram for the discrete Painlevé equation with \(A_2^{(1)} + A_1^{(1)}\) symmetry.](image)

The affine Weyl group is given by

\[
W(A_2^{(1)} + A_1^{(1)}) = \langle s_0, s_1, s_2, r_0, r_1 : s_i^2 = r_i^2 = (s_{i+1})^3 \rangle.
\]

We have a natural \(A_2^{(1)}\) group acting on the pairs of lines opposite to each other, in particular, if we denote the piecewise linear transformation that shears the space between two lines (as in Figure 4.1), \(L_i\) and \(L_j\), by \(\sigma_{i,j}\), then we let \(s_0 = \sigma_{2,5}\), \(s_1 = \sigma_{3,6}\) and \(s_2 = \sigma_{1,4}\). The action of these elements may be written as

\[
\begin{align*}
    s_0 &: X \rightarrow X + \max(0, A_1 + X), \\
s_0 &: Y \rightarrow Y + \max(0, A_1 + X) - \max(0, Y), \\
s_1 &: X \rightarrow A_1 + X, \\
s_1 &: Y \rightarrow Y + \max(0, A_1 + X) - \max(0, X), \\
s_2 &: X \rightarrow X + \max(A_2, Y) - A_2 - \max(0, Y), \\
s_2 &: Y \rightarrow Y - A_2.
\end{align*}
\]

where the action on the parameters is

\[
\begin{align*}
    s_i &: A_i \rightarrow -A_i, \\
s_i &: A_j \rightarrow A_j - 2A_i.
\end{align*}
\]
The action of the $W(A_1^{(1)}) = \langle r_0, r_1 \rangle$ component is as follows:

\[
\begin{align*}
    r_0 : X &\to X + \max(X, X + Y - A_2, Y - A_1 - A_2) \\
    r_0 : Y &\to Y + \max(X, X + Y - A_2, Y - A_1 - A_2) \\
    r_1 : X &\to X + \max(X + B_1, Y, B_1) - \max(X, Y, 0) \\
    r_1 : Y &\to Y + \max(X, Y - B_1, -B_1) - \max(X, Y, 0).
\end{align*}
\]

The Dynkin diagram automorphisms comprise of a reflection around $L_2$, swapping $L_1$ with $L_3$ and $L_4$ with $L_6$, and hence, are given by

\[
\begin{align*}
p_1 : A_{0,1,2} &\to A_{1,2,0} & p_1 : B_{0,1} &\to B_{1,0}, \\
p_1 : X &\to Y - A_2, & p_2 : X &\to A_0 - B_1 + Y - X, \\
p_2 : A_{0,1,2} &\to -A_{0,2,1} & p_1 : B_{0,1} &\to B_{1,0}, \\
p_2 : X &\to A_2 - Y, & p_2 : Y &\to -A_1 - X,
\end{align*}
\]

which coincides with the original representation [19]. These come with the additional relations

\[p_1^3 = p_2^2 = p_1^{-1} \circ s_{i+1} \circ p_1 \circ s_i = p_2 \circ r_{i+1} \circ p_1 \circ r_i = 1.\]

We find the value of $Q$ is

\[Q = A_0 + A_1 + A_2 = B_0 + B_1,\]

which, in the autonomous limit $Q = 0$, gives the invariants

\[
\begin{align*}
    H(X, Y) = \max(-A_1 - B_1 - X, X, A_2 - Y, A_2 + X - Y, \\
    Y - B_1, Y - A_1 - B_1 - X).
\end{align*}
\]

We have two distinct evolution equations corresponding to different lattice directions, given by $T_1 = p_1 \circ s_2 \circ s_1$

\[
\begin{align*}
    (5.5a) \quad &\tilde{X} - Y = \max(Q + X, Y + B_0) - \max(X + A_1 + A_2, Y + B_0), \\
    (5.5b) \quad &\tilde{Y} - Y + X = A_0 + A_2 - B_1 + \max(\tilde{X}, 0) - \max(X, A_0 + A_2), \\
    (5.5c) \quad &T_1 : A_0 \to A_0 + Q, \quad T_1 : A_1 \to A_1 - Q,
\end{align*}
\]

which corresponds to a version of $u$-P$_{III}$ [19], and $T_2 = p_2 \circ r_0$

\[
\begin{align*}
    (5.5d) \quad &\tilde{X} - Y = A_2 + B_1 + \max(0, X, Y) - \max(0, Y, X + B_1), \\
    (5.5e) \quad &\tilde{X} - Y = B_1 - A_1 - A_2 + \max(0, X, Y) - \max(B_1, Y, X + B_1), \\
    (5.5f) \quad &T_2 : B_0 \to B_0 - Q, \quad T_2 : B_1 \to B_1 + Q.
\end{align*}
\]

which corresponds to a version $u$-P$_{IV}$ [19].

\textbf{5.6. Septagons.} This case is associated with $u$-P$_{IV}$ [45]. There are seven rays, specified as follows:

\[
\begin{align*}
    L_1 : Y = 0, & \quad L_4 : X = 0, \\
    L_2 : Y - B_0 = 0, & \quad L_5 : X - B_3 = 0, \\
    L_3 : Y + B_1 = 0, & \quad L_6 : X + B_4 = 0, \\
    L_7 : Y - X + B_1 + B_2 + B_3 + B_4 = 0, & \quad L_8 : X + B_3 - B_1 + B_2 + B_4 = 0.
\end{align*}
\]
which we depict in Figure 5.6.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{spiral_diagram.png}
\caption{The spiral diagram for the case of the ultradiscrete Painlevé equation with $A_4^{(1)}$ symmetry}
\end{figure}

The Weyl group is

$$W\left(A_4^{(1)}\right) = \langle s_0, \ldots, s_4 \rangle,$$

whose presentation follows from the Dynkin diagram below.

\begin{center}
\begin{tikzpicture}
\node (1) at (0,0) {1};
\node (2) at (1,0) {2};
\node (3) at (1,1) {3};
\node (4) at (0,1) {4};
\draw (1) -- (2);
\draw (2) -- (3);
\draw (3) -- (4);
\draw (4) -- (1);
\end{tikzpicture}
\end{center}

From the Dynkin diagram, the action of $s_i$ on $t_i$ is specified by

\begin{equation}
(5.6) \quad s_i : B_j = \begin{cases} 
-B_i & \text{if } i = j, \\
B_j + B_i & \text{if } i \neq j \text{ and node } i \text{ is adjacent to node } j, \\
B_j & \text{otherwise}.
\end{cases}
\end{equation}

From this point, we will choose the forms of rays so that that the action of $s_i$ on the parameters determined by the Dynkin diagram in this way.

The first action is one that interchanges $L_1$ and $L_2$ by the piecewise linear shearing transformations

$$s_0 : X \rightarrow X + \max(Y, B_0) - B_0 - \max(0, Y), \quad s_0 : Y \rightarrow Y - B_0.$$

The second transformation is a simple translation,

$$s_1 : Y \rightarrow Y + B_1,$$

which has the effect of moving $L_3$ to $L_1$, hence, redefining $L_1$ and $L_3$. The transformation $s_2$ has the form

$$s_2 : X \rightarrow X + \max(B_2, B_2 + X, Y) - \max(0, X, Y), \quad s_2 : Y \rightarrow Y + \max(0, B_2 + X, Y) - B_2 - \max(0, X, Y),$$

while $s_3$ simply is a translation in $X$ that redefines $L_4$ and is given by

$$s_3 : X \rightarrow X - B_3.$$
The last transformation is similar to $s_0$, but applied to the lines $L_4$ and $L_5$,

$$s_4 : X \to B_4 + X, \quad s_4 : Y \to \max(-X, B_4) - Y,$$

The Dynkin diagram automorphisms are generated by a rotation of the nodes

$$p_1 : B_i \to B_{i+1},$$
$$p_1 : X \to \max(0, X, Y) - X - Y,$$
$$p_1 : Y \to \max(0, X) - Y,$$

and a reflection

$$p_2 : B_0, 1, 2, 3, 4 \to -B_2, 1, 0, 4, 3,$$
$$p_2 : Y \to \max(0, X) - Y,$$

Tracing around the parameters provides a variable

$$Q = B_0 + B_1 + B_2 + B_3 + B_4.$$

In the autonomous limit, when $Q = 0$, the spiral diagram degenerates to a foliation by tropical cubic plane curves, specified by the level sets of

$$H(X, Y) = \max(Y, \max(0, B_1) - B_1 - B_4 - X, Y - B_3 - X, \max(0, B_3) - B_1 - B_3 - B_4 - Y, -B_1 - B_4 - X - Y, X - B_1 - B_3 - B_4 - Y).$$

The translation is expressed as the composition

$$T = s_4 \circ s_3 \circ s_2 \circ s_1 \circ p_1,$$

which corresponds to the evolution equation

$$(5.7a) \quad \tilde{X} + X = B_3 + \max(0, \tilde{Y}) + \max(0, \tilde{Y} + B_1) - \max(B_0, Q + \tilde{Y}),$$
$$(5.7b) \quad \tilde{Y} + Y = -B_1 - B_3 + \max(0, X) + \max(A_3, X) - \max(0, X + B_4),$$
$$(5.7c) \quad T : A_0 \to A_0 - Q, \quad T : A_4 \to A_4 + Q,$$

which is known as the ultradiscrete version of the fifth Painlevé equation [45].

5.7. Octagons. The biquadratic invariants obtained in the autonomous limit of $q$-P$_{VI}$ are not naturally mapped to cubic plane curves. We do have a eight sided tropical cubic plane curves, given by Figure 5.7, that are equivalent to these biquadratic invariants under a transformation. The rays of these cubic plane curves are parameterized by

$$L_1 : X = 0, \quad L_5 : Y = 0,$$
$$L_2 : X - B_2 = 0, \quad L_6 : Y + B_5 = 0,$$
$$L_3 : X - B_1 - B_2 = 0, \quad L_7 : Y - X - B_3 = 0,$$
$$L_4 : X + B_0 = 0, \quad L_8 : Y - X - B_3 - B_4 = 0.$$

The Weyl group is $W\left( D_5^{(1)} \right) = \langle s_0, \ldots, s_5 \rangle$. The Dynkin diagram is
Figure 5.7. The spiral diagram for the case of the ultradiscrete Painlevé equation with $D_5^{(1)}$ symmetry.

We analogously specify the generators as we did before, where we denote the generators (in terms of $\sigma_{i,j}$ which swaps $L_i$ with $L_j$),

\[
\begin{align*}
    s_0 &= \sigma_{1,4}, \\
    s_1 &= \sigma_{3,4}, \\
    s_2 &= \sigma_{1,3}, \\
    s_3 &= \sigma_{5,7}, \\
    s_4 &= \sigma_{7,8}, \\
    s_5 &= \sigma_{5,6}.
\end{align*}
\]

These generators may be written

\[
\begin{align*}
    s_0 &: X \to X + B_0, \\
    s_0 &: Y \to Y + \max(0, X + B_0) - \max(0, X), \\
    s_2 &: X \to X - B_2, \\
    s_5 &: Y \to Y + B_5, \\
    s_3 &: X \to X + \max(B_3 + \max(0, X), Y) - \max(0, X, Y), \\
    s_3 &: Y \to X + \max(0, X + B_3, Y) - \max(0, X, Y) - B_3,
\end{align*}
\]

and the Dynkin diagram automorphisms, $p_1$ and $p_2$, are

\[
\begin{align*}
    p_1 &: B_{0,1,2,3,4,5} \to -B_{5,4,3,2,1,0}, \\
    p_1 &: X \to \max(0, X) - Y, \\
    p_1 &: Y \to \max(0, X, Y) - X - Y, \\
    p_2 &: B_{0,1,2,3,4,5} \to -B_{0,1,2,3,5,4} \\
    p_2 &: X \to -X, \\
    p_2 &: Y \to Y - X - B_3,
\end{align*}
\]

Tracing around the particular values gives us the variable

\[
Q = B_0 + B_1 + 2B_2 + 2B_3 + B_4 + B_5
\]

In particular, when $Q = 0$, we obtain the invariant

\[
H(X, Y) = \max(\max(0, -B_3) - X, Y - X, \max(0, -B_1, -B_1 - B_2) - B_5 - Y, \\
B_0 + Y, \max(0, -B_1, -B_1 - B_2) + X - B_2 - B_5 - Y, -b_5 - X - Y, \\
B_0 + B_3 + \max(0, B_4) + X, 2X - Y - B_1 - 2B_2 - B_5).
\]

The usual translation that is associated with the dynamics of $u$-PVI and the symmetry QRT equation is the action of

\[
T = p_2 \circ p_1 \circ p_2 \circ s_1 \circ s_2 \circ s_3 \circ s_5 \circ s_4 \circ s_3 \circ s_2 \circ s_1.
\]
To express the evolution of this system, we invert the transformation that expressed the invariant as a cubic by introducing the coordinate

\[ Z = \max(0, X) + \max(0, X - B_2) - \max(0, X + B_0) - Y, \]

which means the evolution is expressed as

\[
\begin{align*}
X + \bar{X} &= B_2 - B_5 + \max(0, Z) - \max(0, Z + B_0 + B_2 + B_3) \\
&\quad + \max(B_5, Z) - \max(0, Z + B_0 + B_2 + B_3 + B_4), \\
\end{align*}
\]

\[
\begin{align*}
Z + \bar{Z} &= B_5 - B_2 + \max(0, \bar{X}) - \max(0, \bar{X} + Q + B_0) \\
&\quad + \max(B_2, \bar{X}) - \max(0, \bar{X} + Q - B_1 - B_2),
\end{align*}
\]

which is equivalent to (3.3) above.

### 5.8. Enneagons

It is at this point we go beyond the QRT maps defined by biquadratic cases [31, 48]. We exploit the translational freedom to parameterize the rays so that \( L_1 \) and \( L_4 \) coincide with the \( y \)-axis and \( x \)-axis respectively. The remaining lines are given by

\[
\begin{align*}
L_1 : X &= 0, \\
L_2 : X - B_2 &= 0, \\
L_3 : X - B_1 - B_2 &= 0, \\
L_4 : Y &= 0, \\
L_5 : Y + B_6 &= 0, \\
L_6 : Y + B_0 + B_6 &= 0, \\
L_7 : Y - X + B_3 &= 0, \\
L_8 : Y - X + B_3 + B_4 &= 0, \\
L_9 : Y - X + B_3 + B_4 + B_5 &= 0.
\end{align*}
\]

The relevant diagram is an irregular enneagon, given in Figure 5.8.

![Figure 5.8](image)

**Figure 5.8.** The spiral diagram for the case of the ultradiscrete Painlevé equation with \( E_6^{(1)} \) symmetry

The Weyl group is \( W\left(E_6^{(1)}\right) = \langle s_0, \ldots, s_6 \rangle \). The presentation, and action on the parameters, is specified by the Dynkin diagram below.

![Dynkin Diagram](image)
We may now parameterize the affine Weyl group actions by first specifying the generators that have a little effect on $X$ and $Y$, by letting $s_0 = \sigma_{56}$, $s_1 = \sigma_{23}$, $s_2 = \sigma_{12}$, $s_4 = \sigma_{78}$, $s_5 = \sigma_8$ and $s_6 = \sigma_{45}$. In these cases, the effect on $X$ and $Y$ are trivial, except for $s_2$ and $s_6$, which have the effect

$$s_2 : X \rightarrow X - B_2, \quad s_6 : Y \rightarrow Y + B_6.$$  

The action of $s_3$ is

$$s_3 : X \rightarrow X + \max(B_3, B_3 + X, Y) - \max(0, X, Y),$$

$$s_3 : Y \rightarrow Y - B_3 + \max(0, B_3 + X, Y) - \max(0, X, Y).$$

The Weyl group is of type $W(E_6^{(1)}) = \langle s_0, \ldots, s_6 \rangle$ extended by the group of automorphisms of the Dynkin diagram

$$p_1 : B_{0,1,2,3,4,5,6} \rightarrow -B_{5,1,2,3,6,0,4},$$

$$p_1 : X \rightarrow -X, \quad p_2 : Y \rightarrow Y - X - B_3,$$

$$p_2 : B_{0,1,2,3,4,5,6} \rightarrow -B_{1,0,6,3,4,5,2},$$

$$p_2 : X \rightarrow Y, \quad p_2 : Y \rightarrow X.$$  

Tracing around the enneagon, we obtain the variable

$$Q = B_0 + B_1 + 2B_2 + 3B_3 + 2B_4 + B_5 + B_6.$$  

In the autonomous limit, when $Q = 0$, this spiral diagram degenerates to give a fibration by cubic plane curves, which may be expressed as the tropical curves that arise as the level sets of

$$H(X,Y) = \max(2X - Y - B_1 - B_2, \max(0, -B_2, -B_1 - B_2) + X - Y - B_2),$$

$$\max(0, -B_5, -B_4 - B_5) + X - B_1 - 2B_2 - B_3, \max(0, -B_2, -B_1 - B_2) - Y, Y + \max(0, -B_1, -B_4 - B_5) - B_1 - 2B_2 - 2B_3 - B_4, 2Y + B_0 + 2B_6 - X, Y + B_6 + \max(0, B_0, B_0 + B_6) - X, \max(0, B_6, B_0 + B_6) - X, -X - Y).$$

The translation that is associated with the dynamics of the discrete Painlevé equation in this case is given by

$$T = p_1 \circ p_2 \circ s_1 \circ s_2 \circ s_3 \circ s_4 \circ s_6 \circ s_0 \circ s_3 \circ s_2 \circ s_1 \circ s_6 \circ s_3 \circ s_2 \circ s_4 \circ s_3 \circ s_6 \circ s_0.$$  

Though the evolution equation is very complicated, the action can be evaluated quite easily by the geometric method in [16]. We consider a translation where only the two rays move, say $L_i$ and $L_j$. Let $T_i$ move to a point, $T(L_i)$, in which $T(L_i)$ and the other rays define a closed curve. Any point in $\mathbb{TP}_2$, say $P = (X,Y)$, is now on some closed polygon, $C$. We send $P$ to $T(P)$, satisfying

$$(5.9) \quad T(P) + T(L_i) = P + L_j,$$

where $L_i$ on $C$ is interpreted in terms of the intersection with the ray and the curve and the addition is in accordance with the group law on $C$ (see [59]). The position of $T(L_j)$ then satisfies

$$L_i + L_j = T(L_i) + T(L_j),$$

on $C$. We have illustrated this in Figure 5.9.
5.9. Decagons. The rational surface associated with $A_1^{(1)}$ symmetry was obtained by blowing up three points on a line and six on a quadratic curve \[49\]. The most general tropical cubic plane curve is a nonagon, hence, to describe the decagon spirals, we need to consider spiral degenerations of quartic plane curves with some double rays. This gives us 10 rays, which we parameterize as follows:

- $L_1 : X + B_4 = 0$
- $L_2 : X + B_4 + B_5 = 0$
- $L_3 : X + B_4 + B_5 + B_6 = 0$
- $L_4 : X + B_4 + B_5 + B_6 + B_7 = 0$
- $L_5 : Y = 0$
- $L_6 : Y - B_3 = 0$
- $L_7 : Y - B_3 - B_2 = 0$
- $L_8 : Y - B_3 - B_2 - B_1 = 0$
- $L_9 : Y - X - B_0 = 0$
- $L_{10} : Y - X = 0$

The resulting system of spiraling polygons is depicted in Figure 5.10.

The resulting system has a group of Bäcklund transformations of type $W(E_7^{(1)}) = \langle s_0, \ldots, s_7 \rangle$. This groups Dynkin diagram is below.

\[1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5 \rightarrow 6 \rightarrow 7\]
The reflections are given by $s_0 = \sigma_{9,10}$, $s_1 = \sigma_{78}$, $s_2 = \sigma_{67}$, $s_3 = \sigma_{56}$, $s_5 = \sigma_{12}$, $s_6 = \sigma_{23}$ and $s_7 = \sigma_{34}$, hence, the nontrivial actions are given by

$$
\begin{align*}
&\ s_0 : X \rightarrow X + B_0, \quad s_3 : Y \rightarrow Y + B_3, \quad s_3 : X \rightarrow X + B_3, \\
&\ s_4 : X \rightarrow X - B_4 + \max(B_4, X, Y) - \max(B_4, X, B_4 + Y), \\
&\ s_4 : Y \rightarrow Y + \max(0, X, Y) - \max(B_4, X, B_4 + Y).
\end{align*}
$$

We also have a single Dynkin diagram automorphism. This Dynkin diagram automorphism has the effect of sending $X$ to $-X$, and $Y$ to $-Y$, which swaps all the eight first order rays, however, it also has the effect of reflecting the two rays, $L_9$ and $L_{10}$, in opposite direction, hence, we compose this a transformation of the form of $\iota_A$ along $L_9$ and $L_{10}$, giving

$$
\begin{align*}
p_1 : X &\rightarrow Y + B_4 - \max(Y, X) - \max(Y, B_0 + X), \\
p_1 : Y &\rightarrow X + B_0 + B_4 - \max(Y, X) - \max(Y, B_0 + X), \\
p_1 : B_{0,1,2,3,4,5,6,7} &\rightarrow B_{0,7,6,5,4,3,2,1},
\end{align*}
$$

Tracing around the spirals reveals that

$$Q = 2B_0 + B_1 + 2B_2 + 3B_3 + 4B_4 + 3B_5 + 2B_6 + B_7,$$

which in the autonomous limit, when $Q = 0$, gives a closing of spirals to give the following invariant

$$H(X, Y) = \max(0, Y + \mu_1, 2Y + \mu_2, X + \mu_5, 2X + \mu_6),$$

$$\max(Y + \mu_3, X - B_0 + \mu_7) + \max(Y, X + B_0) + \max(X, Y),$$

$$2\max(Y, X + B_0) + 2\max(X, Y) + \mu_4) - X - Y,$$

where the values of $\mu_i$ are defined by

$$\begin{align*}
\max(0, X) + \max(0, B_3 + X) + \max(0, B_3 + B_2 + X) \\
+ \max(0, B_3 + B_2 + B_1 + X)
= \max(0, \mu_1 + X, \mu_2 + 2X, \mu_3 + 3X, \mu_4 + 4X), \\
\max(0, X - B_4) + \max(0, X - B_4 - B_3) + \max(0, X - B_4 - B_5 - B_6) \\
+ \max(0, X - B_4 - B_5 - B_6 - B_7)
= \max(0, \mu_5 + X, \mu_6 + 2X, \mu_7 + 3X, \mu_8 + 4X).
\end{align*}$$

A translation associated with the ultradiscrete Painlevé equation is given by

$$T = p_1 \circ s_1 \circ s_2 \circ s_3 \circ s_4 \circ s_0 \circ s_5 \circ s_4 \circ s_3 \circ s_2 \circ s_1 \circ s_6 \circ s_5 \circ s_4 \circ s_0 \circ s_3 \circ s_2 \circ s_1,$$

whose action on $T^2$ is too complicated to be written here. However, the evolution moves just two rays, and hence, the action of $T$ may be described by the theory of [16], where the evolution takes the form

$$(5.10) \quad T(P) + T(L_i) = P + L_j,$$

where the enneagons of (5.9) are replaced with dodecagons.
5.10. Undecagon. The blow-up points in the original classification on \( \mathbb{P}_2 \) lie on a single nodal cubic. This configuration is birationally equivalent (by a series of blow-ups and blow-downs) to a configuration in \( \mathbb{P}_2 \) in which there are three order two singularities on the line at \( y = 0 \), two order three singularities on the line \( x = 0 \) and six order one singularities on the line \( z = 0 \).

The tropical analogue requires we have a configuration of two, three and six rays, which, by exploiting the translational invariance, may be parameterized by

\[
\begin{align*}
L_1 & : X = 0, & L_7 : Y - X - B_3 - B_4 = 0, \\
L_2 & : X - B_0 = 0, & L_8 : Y - X - B_3 - B_4 - B_5 = 0, \\
L_3 & : Y = 0, & L_9 : Y - X - B_3 - B_4 - B_5 - B_6 = 0, \\
L_4 & : Y + B_2 = 0, & L_{10} : Y - X - B_3 - B_4 - B_5 - B_6 - B_7 = 0, \\
L_5 & : Y + B_1 + B_2 = 0, & L_{11} : Y - X - B_3 - B_4 - B_5 - B_6 - B_7 - B_8 = 0, \\
L_6 & : Y - X - B_3 = 0.
\end{align*}
\]

Such a configuration is depicted in Figure 5.11.

![Figure 5.11. A spiral diagram for the \( E^{(1)}_8 \) case.](image-url)

The top case of the multiplicative type Painlevé equations of [49] is one that has a symmetry group of type \( W(E^{(1)}_8) = \langle s_0, \ldots, s_8 \rangle \). The Dynkin diagram is below.

```
```

There are no Dynkin diagram automorphisms, and we may specify the elements \( s_0 = \sigma_{03}, s_1 = \sigma_{45}, s_2 = \sigma_{12}, s_4 = \sigma_{67}, s_5 = \sigma_{78}, s_6 = \sigma_{89}, s_7 = \sigma_{9,10} \) and
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By tracing around the figure, we find that

\[ s_8 = \sigma_{10,11}. \]

These actions and \( s_3 \) have the effect

\[
\begin{align*}
    s_0 &: X \to X - B_0, \\
    s_2 &: Y \to Y + B_2, \\
    s_3 &: X \to X + \max(B_3, B_3 + X, Y) - \max(0, X, Y), \\
    s_3 &: Y \to Y - B_3 + \max(0, B_3 + X, Y) - \max(0, X, Y).
\end{align*}
\]

By tracing around the figure, we find that

\[ Q = 3B_0 + 2B_1 + 4B_2 + 6B_3 + 5B_4 + 4B_5 + 3B_6 + 2B_7 + B_8. \]

In the autonomous limit, this becomes a foliation of tropical sextic curves, specified by the level sets of

\[ H(X, Y) = \max\{iX + jY + c_{i,j} \mid 0 \leq i, 0 \leq j, i + j \leq 6\} - 2X - 3Y, \]

where

\[
\begin{align*}
    c_{0,0} &= 2\lambda_3, \quad c_{0,2} = 2\lambda_2, \quad c_{0,4} = 2\lambda_1, \\
    c_{1,1} &= \max(2\kappa_2 + \mu_5, \kappa_1 - \kappa_2 + \lambda_2 + \lambda_3), \\
    c_{1,2} &= \max(\kappa_1 - \kappa_2 + 2\lambda_3, 2\kappa_2 + \mu_5 + \lambda_2 - \lambda_3, \mu_1 + \lambda_3), \\
    c_{1,3} &= \max(\mu_1 + \lambda_2, \kappa_1 - \kappa_2 + \lambda_1 + \lambda_2, 2\kappa_2 + \mu_5 + \lambda_1 - \lambda_3), \\
    c_{1,4} &= \max(\mu_1 + \lambda_1, \kappa_1 - \kappa_2 + \lambda_2, 2\kappa_2 + \mu_5 - \lambda_3), \\
    c_{1,5} &= \mu_1, \\
    c_{2,1} &= \max(\kappa_1 + \kappa_2 + \mu_5, 2\kappa_1 - 2\kappa_2 + \lambda_2 + \lambda_3), \\
    c_{2,2} &= \max(\kappa_2 + \mu_4, 2\kappa_2 - \kappa_2, \kappa_1 + \kappa_2 + \mu_5 + \lambda_2 - \lambda_3, \kappa_1 - \kappa_2 + \mu_1 + \lambda_3, \\
    &\quad 2\kappa_1 - 2\kappa_2 + \lambda_3), \\
    c_{2,4} &= \mu_2, c_{3,0} = 3\kappa_1 - 3\kappa_2 + 2\lambda_3, c_{3,1} = \max(2\kappa_1 + \mu_5, \kappa_1 - 2\kappa_2 + \lambda_2 + \lambda_3), \\
    c_{3,2} &= \max(\kappa_1 + \mu_4, \kappa_2 + \mu_5 + \lambda_2 - \lambda_3, \kappa_1 + \mu_1 + \lambda_3, \kappa_1 - 2\kappa_2 + \lambda_1 + \lambda_3), \\
    c_{3,3} &= \mu_3, c_{4,1} = \max(\kappa_1 + \mu_5, -2\kappa_2 + \lambda_2 + \lambda_3), \\
    c_{4,2} &= \mu_4, \quad c_{5,1} = \mu_5, \quad c_{6,0} = \mu_6, \quad c_{0,1} = c_{0,3} = c_{2,3} = -\infty, \\
    \max(0, X + \kappa_1, 2X + \kappa_2) &= \max(0, X) + \max(0, X + B_0), \\
    \max(0, X + \lambda_1, 2X + \lambda_2, 3X + \lambda_3) &= \max(0, X) + \max(0, X - B_2) \\
    + \max(0, X - B_1 - B_2), \max(0, X + \mu_1, \cdots, 6X + \mu_6) &= \max(0, X + B_3) \\
    + \max(0, X + B_3 + B_4) + \cdots + \max(0, X + B_3 + B_4 + \cdots + B_8). \\
\end{align*}
\]

The translation is the composition

\[ T = s_3 \circ s_2 \circ s_4 \circ s_3 \circ s_1 \circ s_2 \circ s_5 \circ s_4 \circ s_3 \circ s_6 \circ s_5 \circ s_4 \circ s_0 \circ s_3 \circ s_2 \circ s_1 \]
\[ \circ s_7 \circ s_6 \circ s_5 \circ s_4 \circ s_3 \circ s_2 \circ s_8 \circ s_7 \circ s_6 \circ s_5 \circ s_4 \circ s_3 \circ s_0 \circ s_3 \circ s_4 \]
\[ \circ s_5 \circ s_6 \circ s_7 \circ s_8 \circ s_2 \circ s_3 \circ s_4 \circ s_5 \circ s_6 \circ s_7 \circ s_1 \circ s_2 \circ s_3 \circ s_0 \circ s_4 \]
\[ \circ s_5 \circ s_6 \circ s_3 \circ s_4 \circ s_5 \circ s_2 \circ s_1 \circ s_3 \circ s_4 \circ s_3 \circ s_4 \circ s_4 \circ s_0. \]

Once again, the evolution is too complicated in its tropical form to give here. However, the geometric interpretation is that the evolution is defined as

\[ (5.11) \quad T(P) + T(L_i) = P + L_j, \]

where the relevant polygons are undecagons.
6. Discussion of Dodecagons, Triskaidecagons and higher

We wish to briefly discuss some of the difficulties extending the above arguments to more than eleven sides. We have two constructions that we believed were related; tropical mappings of the plane arising from polygons with greater than 11 sides and tropical birational representations of the Weyl group $W(T_{p,q,r})$ constructed in [58], whose Dynkin diagram is given in Figure 6.1.

![Figure 6.1. A Dynkin diagram of type $T_{p,q,r}$.](image)

Each one of the polygons we have considered so far arise from tropical genus one curves. If we go to a higher number of sides, a simple combinatorial argument based on (3.12) shows us that the higher sided polygons must come from higher genus cases. The simplest example is the autonomous system defined on a pencil of tropical quartics such that there are four distinct rays of order one in each direction. Suppose we parameterize these by

- $L_1 : X = 0$,
- $L_2 : X - B_0 = 0$,
- $L_3 : X - B_0 - B_1 = 0$,
- $L_4 : X - B_0 - B_1 - B_2 = 0$,
- $L_5 : Y = 0$,
- $L_6 : Y + B_7 = 0$,
- $L_7 : Y + B_7 + B_8 = 0$,
- $L_8 : Y + B_7 + B_8 + B_9 = 0$,
- $L_9 : Y - X + B_3 = 0$,
- $L_{10} : Y - X + B_3 + B_4 = 0$,
- $L_{11} : Y - X + B_3 + B_4 + B_5 = 0$,
- $L_{12} : Y - X + B_3 + B_4 + B_5 + B_6 = 0$,

as labelled in Figure 6.2.

The diagram in Figure 6.2 has been obtained by following a path in which the rays have been fixed, and follow the level curves of a biquartic invariant where three of the parameters, the coefficients of $X + Y$, $2X + Y$ and $X + 2Y$, have been set to $-\infty$. Tracing around the diagram, we find that $Q$ is given by

$$Q = 3B_0 + 2B_1 + B_2 + 4B_3 + 3B_4 + 2B_5 + B_6 + 3B_7 + 2B_8 + B_9.$$ Define a translation, $T$, by letting $T(L_i)$ move to a point that defines a pencil of closed polygons. For any point $P$, we have a unique closed curve intersecting with $P$. As any closed curve is also convex, then any tropical line meet $C$ at three points, forming the basis for the desired group law. This means it still makes some sense to compute $T(P)$ by

$$T(P) + T(L_i) = P + L_j,$$

which is a discrete dynamical system with an autonomous limit admitting an invariant. To obtain a higher number of sides (with the constraint that all rays of the same form are of the same order), we may realize 13-sided polygons as specializations of tropical curves of degree twelve and 14-sided polygons as specializations...
of tropical curves of degree six. A rudimentary search reveals $n$-agons for all $n$ up to 30 sides.

The problem is that in the autonomous limit a naive extension to $W(T_{4,4,4})$ using canonical permutations and $s_3$ in the $E_8^{(1)}$ case does not preserve all quartic plane curves constructed. The action of this generator generally gives a curve of degree five, hence, it is not $s_3$ invariant. It seems likely that this fails to preserve all the required degenerate curves when the closed piecewise linear curve become small. It seems that the dynamics we describe may have an interpretation in terms of the addition on some tropical hyper-elliptic curve, for example \[10\] where certain tropical dynamics was studied by using the tropical addition formulae on the spectral curve of hyper-elliptic type.

7. Conclusion

What has been presented is a way of naturally obtaining a group of transformations that preserve the structure of a spiral diagram. It is possible to extend this to cases that do not arise as ultradiscrete Painlevé equations, however the invariants seem more elusive. The autonomous limits do not necessarily result in foliations of genus one curves. Where the role of the addition law on cubic plane curves in the Painlevé equations and QRT mappings is central \[16\], perhaps similar integrable systems could be based on the addition laws for hyperelliptic curves, which are in general, much more complicated \[5, 10\].

Another possible direction is to explore the tropical Cremona transformations more thoroughly. Interesting tropical versions of del Pezzo surfaces have emerged with $W(E_6)$ and $W(E_7)$ symmetry during the write-up of this paper \[46\]. A homological approach that follows \[29, 30, 7, 24\] more closely would also be of interest.
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Department of Mathematics, California Institute of Technology, 1200 E California Blvd, Pasadena, CA, 91125, USA.

Department of Mathematics, Kobe University, Rokko, 657–8501, Japan.