Topological approach to the generalized $n$-centre problem

S. V. Bolotin and V. V. Kozlov

Abstract. This paper considers a natural Hamiltonian system with two degrees of freedom and Hamiltonian $H = \|p\|^2/2 + V(q)$. The configuration space $M$ is a closed surface (for non-compact $M$ certain conditions at infinity are required). It is well known that if the potential energy $V$ has $n > 2\chi(M)$ Newtonian singularities, then the system is not integrable and has positive topological entropy on the energy level $H = h > \sup V$. This result is generalized here to the case when the potential energy has several singular points $a_j$ of type $V(q) \sim -\text{dist}(q,a_j)^{-\alpha_j}$. Let $A_k = 2 - 2k^{-1}$, $k \in \mathbb{N}$, and let $n_k$ be the number of singular points with $A_k \leq \alpha_j < A_{k+1}$. It is proved that if

$$\sum_{2 \leq k \leq \infty} n_k A_k > 2\chi(M),$$

then the system has a compact chaotic invariant set of collision-free trajectories on any energy level $H = h > \sup V$. This result is purely topological: no analytical properties of the potential energy are used except the presence of singularities. The proofs are based on the generalized Levi-Civita regularization and elementary topology of coverings. As an example, the plane $n$-centre problem is considered.

Bibliography: 29 titles.

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1. Introduction

Let $M$ be a connected 2-dimensional manifold, the configuration space of a natural Hamiltonian system with two degrees of freedom. Passing to a double covering, we may assume that $M$ is oriented. The Hamiltonian $H$ on $T^*M$ is quadratic in the momentum:

$$H(q, p) = \frac{1}{2} \| p \|^2 + V(q), \quad p \in T^*_qM,$$

where $\| \cdot \|$ is a Riemannian metric on $M$. The corresponding Lagrangian is

$$L(q, \dot{q}) = \frac{1}{2} \| \dot{q} \|^2 - V(q).$$

Trajectories of the system satisfy the Newton equation

$$\frac{D\dot{q}}{dt} = -\nabla V(q),$$

where $D/dt$ is the covariant derivative and $\nabla$ the gradient vector.

We assume that the metric is smooth $^2$ on $M$, and the potential energy $V$ is smooth except at a finite set of singular points

$$\Delta = \{a_1, \ldots, a_n\} \subset M.$$

More precisely, $V$ is smooth on $M \setminus \Delta$ and in a small neighbourhood

$$B_j = B(a_j, \varepsilon) = \{q \in M : d(x, a_j) \leq \varepsilon\}$$

of $a_j$ it has the form

$$V(q) = -\frac{f_j(q)}{d(q, a_j)^{\alpha_j}} + U_j(q), \quad \alpha_j > 0, \quad m_j = f_j(a_j) > 0,$$

where the functions $f_j$ and $U_j$ are smooth on $B_j$. The distance $d(q, a_j)$ is measured in the Riemannian metric $\| \cdot \|$. The configuration space of the system is $\widehat{M} = M \setminus \Delta$, and the phase space is $T^*\widehat{M}$.

The orders $\alpha_j$ of the singularities are arbitrary positive numbers. The most physical are Newtonian singularities with $\alpha_j = 1$. Singularities with $\alpha_j \geq 2$ are usually called strong force singularities $^1$. As discovered already by Poincaré, for strong force singularities it is easy to prove the existence of periodic solutions by variational methods (see $^1[1]$, $^1[16]$). The reason is that trajectories colliding with such singularities have infinite Hamilton or Maupertuis action. The case of singularities with $0 < \alpha_j < 2$ is more difficult. We call a singularity weak if $0 < \alpha_j < 1$, and moderate if $1 < \alpha_j < 2$. Newtonian singularities with $\alpha_j = 1$ and singularities with $\alpha_j = 2$ are critical.

We call singularities with $\alpha_j = 2$ Jacobi singularities. Jacobi studied the $n$-body problem in Euclidean space with the potential of degree $-2$ and discovered a simple

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$^1$We use the same notation for the norms of a vector and a covector.

$^2$Smooth means of class at least $C^3$. 
behaviour of the moment of inertia of the system with respect to the centre of mass as a function of time.

When $\alpha_j > 2$ we say that the singularity is strong. In proving the existence of chaotic trajectories there is a considerable difference between strong and Jacobi singularities.

Therefore,

$$\Delta = \Delta_{\text{weak}} \cup \Delta_{\text{newt}} \cup \Delta_{\text{mod}} \cup \Delta_{\text{jac}} \cup \Delta_{\text{strong}}.$$  

A standard example is the generalized $n$-centre problem in $\mathbb{R}^2$:

$$H(q, p) = \frac{1}{2}|p|^2 + V(q), \quad V(q) = -\sum_{j=1}^{n} \frac{m_j}{|q - a_j|^{\alpha_j}} + U(q), \quad m_j > 0, \quad (1.6)$$

where $|\cdot|$ is the Euclidean metric and the function $U$ is smooth and bounded above on $\mathbb{R}^2$. Usually it is assumed that the orders $\alpha_j$ are equal, but we do not impose this assumption.

When all singularities are Newtonian ($\alpha_j = 1$) and $n \geq 3$, the $n$-centre problem has a chaotic invariant set on any energy level $H = h > \sup V$ for purely topological reasons (see [3], [21], [6]). For $n = 2$ the classical 2-centre problem with $\alpha_j = 1$ and $U = 0$ has a first integral which is quadratic in the momentum.

For strong singularities ($\alpha_j > 2$) the $n$-centre problem has a chaotic invariant set on the level $H = h > \sup V$ for $n \geq 2$, again for purely topological reasons. In the present paper we prove sufficient topological conditions for the chaotic behavior of the $n$-centre problem for arbitrary $\alpha_j > 0$, generalizing some results in [14], [28], and [29]. For recent references, see [14].

We consider trajectories of the system (1.1) with a fixed value of the total energy

$$H = h > \sup M V. \quad (1.7)$$

When $h < \sup_{M} V$, the motion of the system occurs in the domain of possible motion $D_h = \{ q \in M : V \leq h \}$. In this case our main results do not hold. Then only much weaker results can be proved (see Remark 2.3). In particular, our results do not hold for repelling singularities, since then $\sup V = +\infty$.

The problem of the integrability of the system (1.1) on the energy level (1.7) was discussed in [9]. One of the results there is the following theorem. Let $\chi(M)$ be the Euler characteristic of $M$.

**Theorem 1.1.** Let $M$ be a closed manifold. Suppose that there are only moderate and Newtonian singularities with $1 \leq \alpha_j < 2$. If

$$\sum_{j=1}^{n} \alpha_j > 2\chi(M), \quad (1.8)$$

then the Hamiltonian system has no conditional first integrals that are non-constant polynomials in the momentum with smooth coordinates (no Birkhoff conditional integrals [2]) on the energy level $\{ H = h \} \subset T^* M$ for $h > \sup M V$. 
If there are no singularities (Δ = ∅), then Theorem 1.1 follows from the result of [23]: a natural analytic system on a surface with genus greater than one cannot have non-constant analytic conditional integrals on energy levels \( h > \sup M V \). The proof in [23] also works for smooth systems and smooth conditional integrals which are polynomials in the momentum. When there are only Newtonian singularities, the condition (1.8) gives \( n = \# \Delta > 2\chi(M) \). In this case Theorem 1.1 was proved in [4].

Theorem 1.1 also holds for systems with gyroscopic (or magnetic) forces when the symplectic form is

\[
dp \wedge dq + \pi^* \omega, \tag{1.9}
\]

where \( \omega \) is a closed 2-form on \( M \) and \( \pi: T^*M \to M \) is the projection (see [9]). However, if the form \( \omega \) is non-exact, then under the condition (1.8) the system has no non-constant Birkhoff integrals, but it may have non-constant conditional integrals analytic in the momentum on an energy level (1.7), and hence it has no chaotic trajectories (see, for example, [5]). For example, it is sufficient to take for the Jacobi metric a metric with constant negative curvature and let \( \omega = c\Omega \), where \( \Omega \) is the area form and \( c \) is a large constant. Then all trajectories with energy \( h \) will be periodic, so the system has a non-constant analytic conditional integral. However, there are no non-constant conditional integrals that are polynomials in the momentum.

For natural systems the condition (1.8) probably implies the existence of chaotic trajectories, but this has not been proved in the general case.

Below we will give topological conditions slightly stronger than (1.8) which imply the existence of chaotic trajectories, and in particular the positiveness of the topological entropy. A simple corollary of the main theorem (see Theorem 2.1 below) is the following result.

**Theorem 1.2.** Let \( M \) be a closed manifold. If \( 1 \leq \alpha_j < 2 \) and

\[
n > 2\chi(M), \tag{1.10}
\]

then for \( h > \sup M V \) the system has a compact invariant set with positive topological entropy on the energy level \( \{H = h\} \subset T^*M \).

When all singularities are Newtonian, Theorem 1.2 was proved in [4] using the global Levi-Civita regularization [25] (see also [6]). Moderate singularities with \( 1 < \alpha_j < 2 \) are mostly non-regularizable. However, the idea of the proof of Theorem 1.2 is the same as in [4]. First we will prove the following.

**Proposition 1.1.** There exist a closed surface \( N \), a smooth Riemannian metric on \( N \), and a smooth \( K \)-sheeted covering \( \phi: N \to M \) branched over the singular set \( \Delta \) with the following properties.

(a) The covering has \( K = 2 \) sheets if \( n \) is even and \( K = 4 \) sheets if \( n \) is odd.
(b) The Euler characteristic of \( N \) is

\[
\chi(N) = K \left( \chi(M) - \frac{n}{2} \right) < 0.
\]

(c) The map \( \phi \) takes minimal geodesics on \( N \) to trajectories of the Hamiltonian system on the energy level \( \{H = h\} \) (with changed parametrization) which do
not collide with $\Delta$. Regularizable Newtonian singularities are an exception: some minimal geodesics may pass through the set $\phi^{-1}(\Delta_{\text{newt}})$, and they correspond to trajectories on $M$ reflecting from Newtonian singularities.

A geodesic on $N$ is said to be minimal if it minimizes the distance between any two points on its lift to the universal covering $\tilde{N} \to N$. Note that non-minimal geodesics on $N$ do not necessarily correspond to trajectories with energy $h$ of the Hamiltonian system.

In particular, any non-trivial homotopy class of closed curves in $N$ contains a minimal geodesic which is projected onto a periodic orbit with energy $h$ having no collisions with $\Delta_{\text{mod}}$.

The surface $N$ is a sphere with more than one handle. For example, if $M = S^2$ and $n = 2k$, then the Euler characteristic of $N$ is $2(2 - k)$ and the genus is $k - 1$, so that $N$ is a sphere with $k - 1$ handles.

To sketch the proof of Theorem 1.2, recall that the geodesic flow on a closed surface $N$ with genus greater than one has a compact chaotic invariant set. This was essentially known to Morse and Hedlund, who proved the existence of an infinite number of minimal heteroclinic geodesics. One can show [6] that these heteroclinics are topologically transverse. A rigorous proof of the positiveness of the topological entropy was given by Dinaburg [15]. In fact, the set of minimal geodesics also forms a compact chaotic invariant set with positive topological entropy (see [19]).

As mentioned above, the corresponding trajectories on $M$ may have regularizable collisions with $\Delta_{\text{newt}}$, but then there is another set on the level $\{H = h\}$ with positive topological entropy and no collisions.

We briefly recall what happens when there are collisions with regularizable Newtonian singularities (see also [4]). Suppose for simplicity that $n$ is even. There is a sheet-exchanging involution $\sigma : N \to N$, $\phi \circ \sigma = \phi$, preserving the Riemannian metric and such that the set of fixed points of $\sigma$ is $\phi^{-1}(\Delta)$. The global Levi-Civita regularization $\phi : N \to M$ completely removes the Newtonian singularities, so that the geodesics $\gamma$ on $N$ may pass freely through $\phi^{-1}(\Delta_{\text{newt}})$. If $\gamma(0) \in \phi^{-1}(\Delta_{\text{newt}})$, then the minimizer $\gamma$ will be $\sigma$-reversible:

$$\gamma(-t) = \sigma \gamma(t).$$

The corresponding trajectory $q(t) = \phi(\gamma(t))$ on $M$ will have a reflection from a Newtonian singularity:

$$q(0) \in \Delta_{\text{newt}} \text{ and } q(-t) = q(t).$$

However, only a small proportion of homotopy classes of closed curves on $N$ are $\sigma$-reversible. A minimizer in a non-reversible homotopy class will be projected onto a collision-free trajectory with energy $h$. Similarly, a large proportion of minimal (on the universal covering $\tilde{N}$) non-periodic geodesics will have no collisions with Newtonian singularities $\phi^{-1}(\Delta_{\text{newt}})$.

In the next section we formulate a generalization of Theorem 1.2.

2. Main results

Keeping in mind applications to the $n$-centre problem (1.6), we also consider systems with non-compact configuration space $M$. Then we have to impose certain
conditions at infinity. A simple way out is to assume that there is a compact geodesically convex domain $D$ in $M$ containing all singularities.

Let $D \subset M$ be a compact domain with smooth boundary $\partial D$. When $M$ is a closed manifold, we set $D = M$; then $\partial D$ is empty. Fix an energy level

$$H = h > \sup_D V. \quad (2.1)$$

By Maupertuis’ principle, the trajectories $\gamma: [a, b] \to M$ with energy $h$ are (up to a reparametrization) geodesics of the Jacobi metric

$$\|\dot{q}\|_h = g_h(q, \dot{q}) = \max_p \{\langle p, \dot{q} \rangle : H(q, p) = h\} = \sqrt{2(h - V(q))\|\dot{q}\|}, \quad (2.2)$$

that is, extremals of the Maupertuis action functional

$$J(\gamma) = \int_a^b \|\dot{\gamma}(t)\|_h \, dt.$$ 

Under the condition (2.1), the square of the Jacobi metric is a positive-definite Riemannian metric in the domain $\hat{D} = D \setminus \Delta$. Studying the Hamiltonian flow on the level $\{H = h\} \cap T^*\hat{D}$ is equivalent to studying the geodesic flow of the Riemannian metric $g^2_h$.

The boundary $\partial D$ is said to be geodesically convex for energy $h$ if it is geodesically convex with respect to the Jacobi metric $g_h$. Thus, for every trajectory $q(t)$ with energy $h$ such that $q(0) \in \partial D$ and $\dot{q}(0) \in T_{q(0)}(\partial D)$, there exists an $\varepsilon > 0$ such that

$$q(t) \notin D \setminus \partial D \quad \text{for} \quad -\varepsilon < t < \varepsilon.$$ 

For example, a domain bounded by a non-self-intersecting periodic trajectory with energy $h$ is geodesically convex.

Let $\nu$ be the inward unit normal vector to $\partial D$ with respect to the Riemannian metric $\|\cdot\|$, and let $\kappa$ be the geodesic curvature corresponding to the normal $\nu$. Thus, if $\tau$ is the unit tangent vector and $s$ the arc length along $\partial D$, then

$$\frac{D\tau}{ds} = \kappa \nu.$$

By (1.3) the boundary is geodesically convex if for the motion with energy $h$ along the boundary the normal force $F_{\text{norm}} = -\langle \nabla V, \nu \rangle$ is smaller than the normal acceleration

$$a_{\text{norm}} = \langle \frac{D\dot{q}}{dt}, \nu \rangle = \kappa \|\dot{q}\|^2 = 2\kappa(h - V).$$

Thus the domain $D$ is geodesically convex for energy $h$ if and only if

$$-\langle \nabla V(q), \nu(q) \rangle \leq 2\kappa(q)(h - V(q)), \quad q \in \partial D. \quad (2.3)$$

We divide the singularities into classes depending on their strength. Let

$$A_k = 2 - \frac{2}{k}, \quad k = 1, 2, 3, \ldots, \infty. \quad (2.4)$$
Then

\[ A_1 = 0, \quad A_2 = 1, \quad A_3 = \frac{4}{3}, \quad A_4 = \frac{3}{2}, \quad \ldots, \quad A_\infty = 2. \]

Singularities with \( \alpha_j = A_k \) are regularizable (Knauf [21]; see §6).

Let

\[ \Delta_{\text{reg}} = \{ a_j : \alpha_j = A_k \text{ for some } k = 2, 3, \ldots \}, \]

and let

\[ \Delta_k = \{ a_j : A_k \leq \alpha_j < A_{k+1} \}. \]

Hence

\[ \Delta_1 = \Delta_{\text{weak}}, \quad \Delta_\infty = \{ a_j : \alpha_j \geq 2 \} = \Delta_{\text{jac}} \cup \Delta_{\text{strong}}. \]

Let \( n_k = \#\Delta_k \) and

\[ A(\Delta) = \sum_{2 \leq k \leq \infty} n_k A_k = n_2 + \frac{4}{3} n_3 + \frac{3}{2} n_4 + \cdots + 2 n_\infty. \]

The next theorem is the main result of this paper.

**Theorem 2.1.** Let \( D \subset M \) be a compact connected domain containing all singularities, and let \( h > \sup_D V \). Suppose that the boundary \( \partial D \) is geodesically convex for energy \( h \). If

\[ A(\Delta) > 2\chi(D), \quad (2.5) \]

then:

1) there are infinitely many non-contractible collision-free periodic orbits with energy \( H = h \) in \( \bar{D} = D \setminus \Delta \);

2) the phase flow on the energy level \( \{ H = h \} \) has a compact chaotic invariant set with positive topological entropy.

When all singularities are Newtonian (then \( A(\Delta) = n \)), Theorem 2.1 is an old result (see, for example, [3], [4], [21]). Theorem 2.1 is also well known when all singularities are strong (then \( A(\Delta) = 2n \)): see [16] and [1].

When all singularities are moderate or Newtonian, that is, \( 1 \leq \alpha_j < 2 \), we have \( A(\Delta) \geq n \), so that for a closed surface \( M \) Theorem 2.1 implies Theorem 1.2. However, in this case Theorem 2.1 also gives a stronger statement.

When all singularities are weak with \( 0 < \alpha_j < 1 \), we have \( A(\Delta) = 0 \), and so the condition (2.5) is \( \chi(D) < 0 \), as if there were no singularities. Thus, weak singularities are ignored in Theorem 2.1.

Since \( A(\Delta) \leq \sum_{j=1}^n \alpha_j \), the condition (2.5) is slightly stronger than (1.8). The assertion of Theorem 2.1 was proved in [9] under the additional assumptions that \( D \) is homeomorphic to a plane domain and \( \Delta = \Delta_{\text{reg}} \), that is, all the singularities are regularizable [21]. Then \( A(\Delta) = \sum \alpha_j \), so the condition (2.5) coincides with (1.8).

One can partly describe symbolic dynamics in the chaotic set in Theorem 2.1.

**Theorem 2.2.** There exist a surface \( X \) with boundary, a \( K \)-sheeted smooth covering

\[ \phi : X \to D \setminus (\Delta_{\text{Jac}} \cup \Delta_{\text{strong}}) \]
branched over the set $\Delta_{\text{newt}} \cup \Delta_{\text{mod}}$ of Newtonian and moderate singularities, and a smooth complete\textsuperscript{3} Riemannian metric on $X$ such that:

(a) the boundary $\partial X$ is geodesically convex;

(b) the projections to $D$ of minimal geodesics on the universal covering of the surface $X$ are trajectories with energy $H = h$ having no collisions with the set $\Delta$ (except possibly with regularizable singularities in $\Delta_{\text{reg}}$);

(c) the Euler characteristic

$$
\chi(X) = K \left( \chi(D) - \frac{1}{2} A(\Delta) \right)
$$

is negative when (2.5) holds.

Theorem 2.1 follows from Theorem 2.2 and well-known properties of geodesic flows on closed surfaces (see [20]). Our surface $X$ has a geodesically convex boundary, but the required properties of the geodesic flow can be extended to this case (see [6], for example).

Remark 2.1. Using a formula obtained in [19], one can roughly estimate the topological entropy of the geodesic flow of the Jacobi metric:

$$
h_{\text{top}} \geq \sqrt{\frac{\pi (A(\Delta) - 2\chi(D))}{\text{vol}(D)}},
$$

where the volume is computed in the Jacobi metric.

Remark 2.2. Theorem 2.1 can be generalized to the case of systems with exact gyroscopic forces, when the gyroscopic 2-form $\omega$ in (1.9) is exact, that is, it is the differential of a 1-form $\langle w(q), dq \rangle$. By the change $p \to p - w(q)$ the symplectic form (1.9) can be replaced by the standard form $dp \wedge dq$, and the Hamiltonian is replaced by

$$
H(q, p) = \frac{1}{2} \| p - w(q) \|^2 + V(q).
$$

Then the condition (1.7) needs to be replaced by the condition

$$
h > \max_{q \in D} \left( V(q) + \frac{1}{2} \| w(q) \|^2 \right).
$$

Under this condition the Jacobi metric

$$
g_h(q, \dot{q}) = \| \dot{q} \|_h + \langle w(q), \dot{q} \rangle
$$

is a positive-definite Finsler metric on $D$. The definition of geodesic convexity also needs to be modified, since the Jacobi metric is irreversible. If the gyroscopic form is $\omega = u(q) \Omega$, where $\Omega$ is the area form on $M$, then we need to replace (2.3) by the condition

$$
-\langle \nabla V(q), \nu(q) \rangle \leq 2\kappa(q)(h - V(q)) - |u(q)| \sqrt{2(h - V(q))}, \quad q \in \partial D.
$$

However, for simplicity we consider only natural Hamiltonian systems. For systems with gyroscopic forces see [4].

\textsuperscript{3}Therefore, the corresponding metric space is complete. Equivalently, geodesics exist until they exit $X$ across $\partial X$. 

Remark 2.3. As already mentioned, our results do not work for $h < \sup V$. Then only much weaker results can be proved. Suppose that the domain of possible motion

$$D = D_h = \{q \in M : V(q) \leq h\}$$

is compact and $\nabla V \neq 0$ on the boundary $\partial D$. The Jacobi metric vanishes on $\partial D$. Theorem 2.2 also holds in this case, but now the metric in $X \setminus \partial X$ is, of course, incomplete. However, the results in [8] imply that the number of minimizing geodesic arcs starting and ending on the boundary $\partial X$ is at least $\text{rank } H_1(X, \partial X, \mathbb{Z})$.

These geodesics are projected by the covering $\phi : X \to D$ onto reversible periodic orbits (brake orbits, or librations) with energy $h$ which have no collisions with $\Delta$, except possibly at regularizable singularities in $\Delta_{\text{reg}}$. If there are no regularizable singularities, $\Delta_{\text{reg}} = \emptyset$, then there will be no collisions, so there exist at least $\text{rank } H_1(X, \partial X, \mathbb{Z})$ collision-free reversible periodic orbits with energy $h$. In contrast to the case $h > \sup V$, we obtain only a finite number of periodic orbits and there is no hope of obtaining chaotic trajectories (at least, using the elementary topological methods of this paper).

Remark 2.4. Theorems 2.1 and 2.2 are proved in §8. In §4 we show that without loss of generality we may assume in the proof that there are no strong singularities. In the main part of the paper we also assume that there are no Jacobi singularities. In the presence of Jacobi singularities the proof of the first part of Theorem 2.1 does not change, but the proof of the second part requires additional arguments. The case of non-empty $\Delta_{\text{jac}}$ is discussed in §9.

3. Examples

(i) $M = \mathbb{T}^2$ is a torus. Since $\chi(\mathbb{T}^2) = 0$, by Theorem 2.1 the existence of $n \geq 1$ singularities with $\alpha_j \geq 1$ implies chaotic behaviour on energy levels $h > \sup V$. For Newtonian or strong singularities this is well known (see [4], [21], [22], for example). In [24] it is proved that in the presence of singularities with $1/2 < \alpha_j < 2$ on $\mathbb{T}^2$ there are no first integrals which are polynomials in the momentum and are integrable with respect to the coordinates in the whole phase space.\footnote{In [24] systems with singularities on $\mathbb{T}^d$ were also studied.} But such non-integrability in general does not imply chaotic behaviour of the system. We do not know if the existence of a weak singularity on $\mathbb{T}^2$ always implies that the topological entropy is positive.

(ii) $M = S^2$ is a sphere. Then $2\chi(S^2) = 4$. If all singularities are strong, then $A(\Delta) = 2n$, so Theorem 2.1 works for $n \geq 3$. A system on a sphere with two strong singularities may be integrable.

Indeed, take a metric of revolution on a sphere, place the singularities at antipodal points, and take a potential of revolution depending only on the distance to these points. Then the angular momentum about the axis of revolution will be a first integral.

If there are $n$ singularities with $\alpha_j \geq 3/2$, then we have $A(\Delta) \geq 3n/2$, so Theorem 2.1 works for $n \geq 3$. If there are $n$ singularities with $\alpha_j \geq 3/4$, then Theorem 2.1 works for $n \geq 4$. If there are $n$ Newtonian singularities, then $A(\Delta) = n$, so Theorem 2.1 works for $n \geq 5$. For $n = 4$ Newtonian singularities the system may
have a first integral quadratic in the momentum on an energy level \( H = h > \sup V \) (see [5]).

Indeed, take an arbitrary Riemannian metric \( \| \cdot \| \) on \( S^2 \) and a set \( \Delta \subset S^2 \) with \( \#\Delta = 4 \). The metric defines a conformal structure on \( S^2 \). There exists a holomorphic differential \( \omega \) of degree 2 that has simple poles at each point in \( \Delta \). Then \( |\omega| \) is a Riemannian metric on \( S^2 \setminus \Delta \) conformally equivalent to \( \| \cdot \| \). Define the kinetic energy by \( T = \| \dot{q} \|^2/2 \) and define the potential energy so that \( h - V = |\omega|/T \). Then \( V \) is a function on \( S^2 \setminus \Delta \) which has four Newtonian singularities. The Jacobi metric \( g_h^2 \) is a constant multiple of the flat metric \( |\omega| \), so its geodesic flow is integrable: it has the first integral \( \Re \omega \), which is quadratic in the momentum.

If there are three Newtonian singularities and a fourth stronger one with \( \alpha_4 \geq 4/3 \), then we obtain that \( A(\Delta) \geq 3 + 4/3 > 4 \), so there is a chaotic invariant set for energies \( h > \sup V \).

(iii) The generalized \( n \)-centre problem. Since \( M = \mathbb{R}^2 \) is non-compact, to apply Theorem 2.1 to the Hamiltonian (1.6) we need to find a compact set \( D \subset \mathbb{R}^2 \) containing all singularities which is geodesically convex for energy \( h > \sup V \). We can try to take for \( D \) a disk \( |q| \leq R \) with a sufficiently large radius \( R \). This works under a convexity assumption (compare with (2.3)):

\[
\langle \nabla V(q), q \rangle \leq 2(h - V(q)), \quad |q| = R.
\] (3.1)

For example, (3.1) holds for large \( R \) if \( h > \sup V \) and

\[
\limsup_{|q| \to +\infty} \langle \nabla U(q), q \rangle \leq 0.
\]

Then for large \( R \) the disk \( D = B(0, R) \) is geodesically convex for energy \( h > \sup V \).

**Theorem 3.1.** If \( A(\Delta) > 2 \), then the generalized \( n \)-centre problem in \( \mathbb{R}^2 \) has a compact chaotic invariant set and an infinite number of minimizing (on a suitable branched covering) periodic collision-free trajectories on the energy level \( H = h > \sup V \).

Trajectories in the chaotic set are projections of minimal geodesics of a smooth Riemannian metric on a geodesically convex compact \( K \)-sheeted covering \( X \) of the domain \( D \) with Euler characteristic

\[
\chi(X) = K \left( 1 - \frac{1}{2} A(\Delta) \right) < 0.
\]

If the orders of all the singularities with \( 1 \leq \alpha_j < 2 \) belong to a single interval \( A_k \leq \alpha_j < A_{k+1} \), then the covering has degree \( K = k \) and the surface \( X \) is homeomorphic to the Riemann surface of the function

\[
z \mapsto \sqrt[k]{(z - a_1) \cdots (z - a_n)}.
\]

In general, we can set

\[
K = \prod_{2 \leq k < \infty, \ n_k \neq 0} k,
\]

which is, of course, not optimal.
We will show that Theorem 3.1 holds without the convexity condition (3.1), but then the choice of a geodesically convex compact domain $D \subset \mathbb{R}^2$ needs to be different. The proof of Theorem 3.1 without the convexity condition is given in §5.

For example, if there are two singularities, one Newtonian and the other with $\alpha_2 \geq 4/3$, then $A(\Delta) \geq 1 + 4/3 > 2$, so Theorem 3.1 applies. Theorem 3.1 also gives chaotic behaviour if there are three or more singularities of at least Newtonian strength.

In a recent paper [14] it was proved that for the classical $n$-centre problem with $U = 0$ and only moderate singularities, the minimizers of the Maupertuis action functional in certain admissible homotopy classes of closed curves in $\mathbb{R}^2 \setminus \Delta_{\text{mod}}$ do not have collisions with singularities. We prove a slightly more general statement in §5. The results in [14] imply chaotic behaviour for the $n$-centre problem with $n \geq 4$ moderate singularities. The assumptions of Theorem 3.1 are considerably weaker.

**Remark 3.1.** The case $U = 0$ is relatively simple because then the Jacobi metric has negative Gaussian curvature (see [21]), and then Theorem 3.1 can be improved. For example, it also holds when $h = \sup V = 0$. Indeed, in that case

$$2(-V(q)) - \langle \nabla V(q), q \rangle = \sum \frac{m_j}{|q - a_j|^{\alpha_j}}(2 - \alpha_j + O(|q|^{-1})).$$

Hence, if there is at least one singularity with $\alpha_j < 2$, then for large $R$ the boundary of the disk $B(0, R)$ is geodesically convex for energy $h = 0$.

**Remark 3.2.** The assumptions of Theorem 3.1 are purely topological: no analytical properties of the potential energy are involved except the presence of singularities. Of course, under additional analytical assumptions much stronger results can also be proved for the $n$-centre problem in $\mathbb{R}^d$. For example, for Newtonian singularities and large energy $h \to +\infty$ (or, equivalently, for small masses $m_j$) Aubry’s anti-integrable limit technique can be used to describe the symbolic dynamics of chaotic orbits (see, for example, [21], [22], [10], [12], [7]). But this requires $n \geq 4$ singularities not lying on one line. Theorem 2.1 works for $n \geq 3$ and all energies $h > \sup V$.

**Remark 3.3.** We conjecture that Theorem 3.1 also holds for the $n$-centre problem in $\mathbb{R}^3$. If all singularities are Newtonian, then this was proved in [11] using the KS-regularization and deep results of Gromov and Paternain (see [27]). The elementary methods of the present paper will not work for the spatial problem since they make essential use of the fact that $\mathbb{R}^2 \setminus \Delta$ is not simply connected.

In the rest of the paper we prove Theorems 1.2 and 2.1, and also remove the convexity assumption in Theorem 3.1.

### 4. Convexity properties

First we prove simple convexity properties of small neighbourhoods of singularities. Let $a_j \in \Delta$ be a singularity of order $\alpha_j$ and let $B_j = B(a_j, \varepsilon)$ be the small disk (1.4).
Lemma 4.1. For $0 < \alpha_j < 2$ and sufficiently small $\varepsilon > 0$ the disk $B_j$ is geodesically convex in the Jacobi metric. For $\alpha_j > 2$ and small $\varepsilon > 0$ the boundary circle $S_j = \partial B_j$ is geodesically concave in the Jacobi metric, that is, the complement $D \setminus B_j$ is geodesically convex.

This property was already known to the founders of celestial mechanics (it follows from the Lagrange–Jacobi identity). However, we give a proof for completeness. For any $x \in S_j$, let $\nu(x)$ be the inward unit normal vector and $\kappa(x)$ the curvature (with respect to the metric $\| \cdot \|$). Then by (1.5),

\[
\kappa(x) = \varepsilon^{-1} + o(\varepsilon^{-1}),
\]

\[
V(x) = -m_j \varepsilon^{-\alpha_j} + o(\varepsilon^{-\alpha_j}),
\]

\[
\nabla V(x) = -\alpha_j m_j \varepsilon^{-\alpha_j - 1} \nu(x) + o(\varepsilon^{-\alpha_j - 1}).
\]

Hence

\[
2(h - V(x))\kappa(x) = 2m_j \varepsilon^{-\alpha_j - 1} + o(\varepsilon^{-\alpha_j - 1}),
\]

\[
-\langle \nu, \nabla V(x) \rangle = \alpha_j m_j \varepsilon^{-\alpha_j - 1} + o(\varepsilon^{-\alpha_j - 1}).
\]

The conclusion now follows from (2.3).

For critical Jacobi singularities with $\alpha_j = 2$ Lemma 4.1 does not work. Such singularities have to be treated separately.

Corollary 4.1. In the proof of Theorem 2.1 one can assume without loss of generality that there are no strong singularities.

Proof. By Lemma 4.1, for a strong singularity $a_j$ with $\alpha_j > 2$, the disk $B_j = B(a_j, \varepsilon)$ has a concave boundary $\partial B_j$. Hence the complement

\[
D' = \overline{D \setminus \bigcup_{\alpha_j > 2} B_j}
\]

is a geodesically convex compact domain containing the set

\[
\Delta' = \Delta \setminus \Delta_{\text{strong}} = \{a_j \in \Delta : \alpha_j \leq 2\}
\]

of non-strong singularities. We have

\[
A(\Delta') = \sum_{2 \leq k < \infty} n_k A_k = A(\Delta) - 2n_\infty,
\]

\[
\chi(D') = \chi(D) - n_\infty.
\]

Consequently,

\[
A(\Delta) - 2\chi(D) = A(\Delta') - 2\chi(D').
\]

Replacing $D$ by $D'$ and $\Delta$ by $\Delta'$ in Theorem 2.1, we obtain a system without strong singularities that satisfies the conditions of Theorem 2.1. □
5. Jacobi metric space

Trajectories with energy $h$ are geodesics of the Jacobi metric (2.2), that is, extremals of the Maupertuis–Jacobi action functional $J$.

Remark 5.1. In many recent papers (see, for example, [14], [28], [29]) a different Maupertuis action functional is used:

$$I(\gamma) = \left( \int_a^b ||\dot{\gamma}(t)||^2 \, dt \right) \left( \int_a^b [h - V(\gamma(t))] \, dt \right).$$

Then

$$I(\gamma) \geq \frac{J(\gamma)^2}{2},$$

with equality if and only if the energy is constant along $\gamma$. The functional $I$ has the advantage of being differentiable on a suitable Sobolev space. Using this functional makes sense when looking for minimax geodesics. However, if (as in the present paper) only minimal geodesics are being studied, then the classical Maupertuis functional $J$ in the Jacobi form is more convenient, since we can replace functional analysis by simple metric geometry.

Because the length $J(\gamma)$ of a curve is independent of the parametrization, we identify curves which differ by an orientation-preserving reparametrization and we parametrize all curves by the interval $[0, 1]$. The Jacobi metric makes $\hat{D} = D \setminus \Delta$ a metric space with the distance

$$\rho(x, y) = \inf \{J(\gamma) : \gamma \in C^1([0, 1], \hat{D}), \gamma(0) = x, \gamma(1) = y\}.$$

If $a_j$ is a strong singularity with $\alpha_j > 2$, then for $x \neq a_j$ we have

$$\rho(x, y) \sim d(y, a_j)^{1-\alpha_j/2} \to +\infty$$

as $y \to a_j$. For Jacobi singularities with $\alpha_j = 2$,

$$\rho(x, y) \sim -\log d(y, a_j) \to +\infty.$$

For singularities with $\alpha_j < 2$, there exists the limit

$$\rho(x, a_j) = \lim_{y \to a_j} \rho(x, y) < +\infty.$$

Thus the Jacobi metric is complete at Jacobi singularities and strong singularities. This was discovered by Poincaré and first used by Gordon [16]. Hence, the completion of the metric space $(\hat{D}, \rho)$ is the complete metric space $(D \setminus (\Delta_{jac} \cup \Delta_{strong}), \rho)$ with the standard topology.

From now on we assume that there are no Jacobi singularities with $\alpha_j = 2$. The critical case when some $\alpha_j = 2$ is treated separately in $\S$ 9. We have already removed strong singularities, so that $0 < \alpha_j < 2$ for all $j$: we have only weak, moderate, and Newtonian singularities. Then the completion of the metric space $(\hat{D}, \rho)$ is the compact metric space $(D, \rho)$.
Now we can define the length $J(\gamma) \in [0, +\infty]$ of any curve $\gamma \in C^0([0, 1], D)$ in a standard way:

$$J(\gamma) = \sup \left\{ \sum_{i=1}^{k} \rho(\gamma(t_{i-1}), \gamma(t_i)) : 0 = t_0 < t_1 < \cdots < t_k = 1 \right\}.$$  

It is natural to consider $J$ on the set

$$\{ \gamma \in C^0([0, 1], D) : J(\gamma) < \infty \}$$

of rectifiable curves such that if $\gamma$ is not a point curve (that is, $J(\gamma) \neq 0$), then $\gamma(t) \neq \text{const}$ on any interval. Since curves differing by an orientation-preserving reparametrization are identified, every element of the corresponding quotient space $\tilde{C}([0, 1], D)$ is uniquely represented by a curve $\gamma : [0, 1] \to D$ with parameterization proportional to the arc length:

$$J(\gamma|_{[0, s]}) = sJ(\gamma).$$

Then $\tilde{C}([0, 1], D)$ is embedded in $C^0([0, 1], D)$ and carries the $C^0$-topology.

It is well known that the length functional $J$ is lower semicontinuous and that for any $c > 0$ the set $\{ \gamma \in \tilde{C}([0, 1], D) : J(\gamma) \leq c \}$ is compact. Then $(D, \rho)$ is a compact length space [13]: for any points $x, y \in D$ there is a minimizing curve $\gamma$ such that

$$\gamma(0) = x, \quad \gamma(1) = y, \quad \text{and} \quad J(\gamma) = \rho(x, y).$$

By convexity, if $x, y \in D \setminus \partial D$, then the minimizer will not touch $\partial D$.

Let $\Gamma \subset C^0([0, 1], \hat{D})$ be a homotopy class of curves joining two points $x, y \in \hat{D}$. The compactness property implies the following lemma.

**Lemma 5.1.** The functional $J$ has a minimum on the closure $\Gamma$ of $\Gamma$ in the space $C^0([0, 1], D)$. Any minimizer $\gamma$ is a trajectory with energy $h$ or a chain of trajectories $\gamma = \gamma_1 \cdots \gamma_k$, where the curves $\gamma_i$ join pairs of singular points in $\Delta$.

The same holds if $\Gamma$ is a non-trivial homotopy class of closed curves in $\hat{D}$, that is, a path-connected component of $C^0(S^1, \hat{D})$, where $S^1 = [0, 1]/\{0, 1\}$. We call a homotopy class of closed curves in $\hat{D}$ non-trivial if it does not contain contractible loops or loops $\gamma : S^1 \to B_j$ contained in small neighbourhoods $B_j$ of singularities $a_j$. For a trivial class $\Gamma$, the minimum of $J$ in the closure of $\Gamma$ is attained at a point curve $\gamma \equiv a_j$.

In order to prove the existence of collision-free periodic trajectories with energy $h$ we need to show that the minimizer $\gamma$ does not pass through the singular set $\Delta$.

Let $a_j \in \Delta$. Take a sufficiently small $\varepsilon > 0$ and let $B_j = B(a_j, \varepsilon)$ be the closed disk in Lemma 4.1 and $S_j$ the corresponding circle. Since for $0 < \alpha_j < 2$ the disk $B_j$ is geodesically convex in the Jacobi metric, a minimizer joining a pair of points $x, y \in B_j$ stays in $B_j$. In particular,

$$\inf \{ J(\gamma) : \gamma \in C^0([0, 1], B_j), \gamma(0) = x, \quad \gamma(1) = y \} = \rho(x, y).$$

The next lemma means that the metric $\rho$ has a cone singularity at $a_j$, with total angle less than $2\pi$.  

Lemma 5.2. Let $0 < \alpha_j < 2$. Then there exists a $\lambda \in (0, 1)$ such that for sufficiently small $\varepsilon > 0$ and any points $x, y \in S_j$

$$\rho(x, y) < \lambda(\rho(x, a_j) + \rho(a_j, y)). \quad (5.1)$$

We will prove Lemma 5.2 in §6. By (5.1),

$$\rho(x, y) \leq \rho(x, a_j) + \rho(a_j, y) - 2r, \quad r = (1 - \lambda)\rho(S_j, a_j).$$

If a curve $\gamma$ joining points $x, y \in S_j$ enters the ball

$$\mathcal{B}_j = \mathcal{B}(a_j, \delta) = \{x : \rho(x, a_j) \leq \delta\} \subset B_j, \quad 0 < \delta < r,$$

then

$$J(\gamma) \geq \rho(x, a_j) + \rho(a_j, y) - 2\delta > \rho(x, y) + \mu, \quad \mu = 2r - 2\delta.$$

Thus, if

$$J(\gamma) \leq \rho(x, y) + \mu,$$

then $\gamma$ does not enter the ball $\mathcal{B}_j$.

Corollary 5.1. Let $\pi: \tilde{D} \to D$ be the universal covering, let $\tilde{\Delta} = \pi^{-1}(\Delta)$, let $\tilde{\rho}$ be the corresponding distance on $\tilde{D}$, and let $\tilde{J}$ be the length functional. If $\gamma$ is a minimizer of $\tilde{J}$ joining a pair of points in $\tilde{D} \setminus \tilde{\Delta}$, then $\gamma$ does not pass though $\tilde{\Delta}$.

This implies the following weak version of Theorem 2.1.

Proposition 5.1. Suppose that all the singularities satisfy $0 < \alpha_j < 2$. Every non-trivial homotopy class of closed curves in $D$ contains a minimal closed geodesic having no collisions with $\Delta$. If $\chi(D) < 0$, then the geodesic flow of the Jacobi metric has a compact chaotic invariant set of minimal (on the universal covering of $D$) collision-free geodesics with positive topological entropy.

When there are no singularities ($\Delta = \emptyset$) and no boundary ($D = M$ is a closed surface of genus $\geq 2$), this is an old result essentially known to Morse [26] and Hedlund [18] (except for the rigorous definition of topological entropy, of course). They proved the existence of many minimal heteroclinic geodesics which, as we now know, implies the positiveness of the topological entropy. Katok proved (see [19]) that the set of minimizing geodesics on the universal covering $\tilde{M}$, after projection onto $M$, gives a compact invariant set for the geodesic flow on $M$ with positive topological entropy. This property can be extended to the case of surfaces with geodesically convex boundary (see [6], for example).

Indeed, suppose for simplicity that $\partial D$ consists of one closed curve. Take a minimal closed geodesic $\gamma$ in the homotopy class of the boundary $\partial D$; $\gamma$ bounds a domain $U \subset D$. Gluing two copies of $U$ along $\gamma$, we obtain a closed surface $N$ without boundary. Since the boundary curve has zero curvature, one can show that the Riemannian metric on $U$ defines a $C^2$ Riemannian metric on $N$ which is invariant under the involution $\sigma: N \to N$ interchanging the copies of $U$. Known results about minimal geodesics on closed surfaces can now be applied (see [19]).

If there are only weak singularities with $0 < \alpha_j < 1$, then Proposition 5.1 coincides with Theorem 2.1. In fact, we will deduce Theorem 2.1 from Proposition 5.1 by using successive Levi-Civita type regularizations.
Proportion 5.1. Let us modify the Jacobi metric $g_h$ in every ball $\mathcal{B}_j$, replacing it by a smooth Riemannian metric $g' \geq g_h$ on $D$ such that $g' = g_h$ on $D \setminus \bigcup \mathcal{B}_j$. Then every non-trivial homotopy class of closed curves in $D$ contains a minimal geodesic and, as explained above, there is a compact chaotic invariant set of minimizing (in the universal covering $\tilde{D} \to D$) geodesics of the metric $g'$. By Lemma 5.2 such geodesics cannot pass through the balls $\mathcal{B}_j$, so they are geodesics of the Jacobi metric $g_h$. □

For moderate singularities with $1 < \alpha_j < 2$ a stronger version of Lemma 5.2 is true: the total angle at the vertex $a_j$ of the cone is less than $\pi$. Let $\phi: B'_j \to B_j$ be a double covering branched over $a_j$ (the same as in the Levi-Civita regularization; see the next section). Thus, $a'_j = \phi^{-1}(a_j)$ is a single point, and $\phi^{-1}(x)$ consists of two points if $x \neq a_j$. We lift the metric $\rho$ to a metric $\rho'$ on $B'_j$.

Lemma 5.3. Let $1 < \alpha_j < 2$. Then there exists a $\lambda \in (0,1)$ such that for sufficiently small $\varepsilon > 0$ and any $x,y \in S'_j = \partial B'_j$, $\rho'(x,y) < \lambda(\rho'(x,a'_j) + \rho'(a'_j, y))$.

As in Lemma 5.2, there exist $\delta, \mu > 0$ such that any curve $\gamma$ joining $x,y \in S'_j$ and satisfying $J'(\gamma) \leq \rho'(x,y) + \mu$ does not enter the ball $\mathcal{B}'_j = \{x: \rho'(x,a'_j) \leq \delta\} \subset B'_j$.

Corollary 5.2. Let $1 < \alpha_j < 2$. Then a minimizer joining any points $x,y \in S'_j$ does not pass through $\mathcal{B}'_j$.

Note that for Newtonian singularities this is not true: if $x \neq y$ are points on $S'_j$ with $\phi(x) = \phi(y)$, then $\rho'(x,y) = \rho'(x,a'_j) + \rho'(a'_j, y)$.

Recall that $\Delta_{\text{mod}}$ is the set of moderate singularities with $1 < \alpha_j < 2$, and $\Delta_{\text{newt}}$ is the set of Newtonian singularities. Suppose that $\Sigma = \Delta_{\text{mod}} \cup \Delta_{\text{newt}} \neq \emptyset$ (otherwise Theorem 2.1 is reduced to Proposition 5.1). The following proposition is an elementary property of Riemann surfaces. It was used in [4] to deal with Newtonian singularities (see also [21] for $D = \mathbb{T}^2$).

Proposition 5.2. There exists a compact surface $D'$ with boundary and a $K$-sheeted ($K = 2$ or $K = 4$) smooth covering $\phi: D' \to D$ branched over $\Sigma$ such that each point in $\Sigma' = \phi^{-1}(\Sigma)$ has multiplicity 2. Thus, $\# \phi^{-1}(a_j) = \frac{K}{2}$ for $a_j \in \Sigma$ and $\# \phi^{-1}(x) = K$ for $x \in D \setminus \Sigma$.

We recall the proof of Proposition 5.2 below in the proof of Theorem 1.2 in §7.

Corollary 5.3. Let $\pi: \tilde{D} \to D'$ be the universal covering of $D'$, let $\Phi = \phi \circ \pi: \tilde{D} \to D$, and let $\tilde{\Delta} = \Phi^{-1}(\Delta)$. Then a minimizer joining a pair of points in $\tilde{D} \setminus \tilde{\Delta}$ does not pass through $\tilde{\Delta}$, except possibly for regularizable Newtonian singularities in $\Phi^{-1}(\Delta_{\text{newt}})$.
Lemma 5.3 will be proved in the next section using the generalized Levi-Civita regularization. It admits the following reformulation.

For given \(x, y \in S_j\) there exist two simple homotopy classes \(\Gamma_\pm(x, y)\) of curves in \(C^0([0, 1], B_j \setminus \{a_j\})\) that join \(x\) and \(y\) and pass on different sides of \(a_j\). Let
\[
\rho_\pm(x, y) = \inf\{J(\gamma) : \gamma \in \Gamma_\pm(x, y)\}.
\]
Then of course \(\rho(x, y) = \min\{\rho_+(x, y), \rho_-(x, y)\}\).

**Lemma 5.4.** Let \(1 < \alpha_j < 2\). Then there exists a \(\lambda \in (0, 1)\) such that for sufficiently small \(\varepsilon > 0\) and any \(x, y \in S_j\),
\[
\max\{\rho_+(x, y), \rho_-(x, y)\} < \lambda (\rho(x, a_j) + \rho(a_j, y)).
\]

As in Lemma 5.2, this implies that there exist \(\delta, \mu > 0\) such that any curve \(\gamma \in \Gamma_\pm(x, y)\) such that
\[
J(\gamma) \leq \rho_\pm(x, y) + \mu
\]
does not enter the ball \(B_j\).

As an application, we give a simple proof of the main result of [14]. Following [14], we call a homotopy class \(\Gamma\) of closed curves in \(C^0(S^1, D \setminus \Sigma)\) admissible if its representative with a minimal number of self-intersections has no simple subloops bounding a disk containing a single singularity \(a_j \in \Sigma\). This condition does not depend on the choice of a representative (see [17]).

**Corollary 5.4.** Let \(\Gamma\) be an admissible homotopy class of closed curves in \(C^0(S^1, D \setminus \Sigma)\). Then any minimizer \(\gamma\) of \(J\) in the closure \(\overline{\Gamma}\) in \(C^0(S^1, D)\) has no collisions with moderate singularities in \(\Delta_{\text{mod}}\). If \(\gamma\) has a collision with \(\Delta_{\text{newt}}\) at \(t = \tau\), then it is reversible:
\[
\gamma(\tau + t) = \gamma(\tau - t).
\]

Indeed, suppose that \(\gamma(\tau) = a_j \in \Delta_{\text{mod}}\). There exists a minimizing sequence \(\gamma_k \to \gamma\) with a minimal number of self-intersections (see [17]). Then \(\gamma_k\) enters the ball \(B_j\) for large \(k\), so there is a segment \(C\) of \(\gamma_k\) joining points \(x, y \in S_j\) in \(B_j\) and crossing \(B_j\). If the class \(\Gamma\) is admissible, then \(C\) does not contain a subloop going around \(a_j\), so \(C \in \Gamma_+(x, y)\) or \(C \in \Gamma_-(x, y)\). Then \(J(C) > \rho_\pm(x, y) + \mu\) by Lemma 5.4, so that \(J(\gamma) > \inf \Gamma J + \mu\), which yields a contradiction for large \(k\). \(\square\)

Now we use Lemma 5.4 to prove Theorem 3.1 without the convexity assumption.

**Proof of Theorem 3.1.** Take a simple closed curve in \(\mathbb{R}^2\) encircling all singularities and minimize the length functional \(J\) on the homotopy class \(\Gamma\) of this curve in \(\mathbb{R}^2 \setminus (\Delta \setminus \Delta_{\text{weak}})\). Since
\[
g_h(q, \dot{q}) \geq c|\dot{q}|, \quad 0 < c < \sqrt{2(h - \sup V)},
\]
by Lemma 5.1, the minimum will be achieved on a closed curve \(\gamma\) in the closure of the class \(\Gamma\) in \(C^0(S^1, \mathbb{R}^2)\). The curve \(\gamma\) has no self-intersections. Hence by Lemma 5.4, \(\gamma\) cannot pass through the singular set \(\Delta_{\text{mod}}\). If \(\gamma\) passes through \(\Delta_{\text{newt}}\), then \(\gamma\) will be a segment joining two Newtonian singularities and passed through twice in opposite directions. Then there are no other non-weak singularities. But this is impossible, since \(A(\Delta) > 2\). Consequently, \(\gamma\) does not pass through Newtonian singularities either, and therefore it bounds a compact domain \(D \subset \mathbb{R}^2\) with geodesically convex boundary and containing all non-weak singularities. Theorem 3.1 now follows from Theorem 2.1. \(\square\)
6. Levi-Civita regularization

Levi-Civita regularization is the main tool of this paper.
In a neighbourhood of any point of $M$ there exist local coordinates $q_1, q_2$ such that the Riemannian metric defining the kinetic energy has a conformal form:

$$\|\dot{q}\|^2 = g(q_1, q_2)(\dot{q}_1^2 + \dot{q}_2^2) = g(z)|\dot{z}|^2,$$

where $z = q_1 + iq_2 \in \mathbb{C}$. Passing to an orienting double covering, we may assume that $M$ is oriented. Then conformal coordinates endow $M$ with the structure of a Riemann surface.

Let us choose conformal coordinates in a neighbourhood of a singular point $a_j \in \Delta$ so that it corresponds to $z = 0$. After a conformal change of variables, we may assume without loss of generality that in the metric (6.1)

$$g(z) = 1 + O(|z|^2).$$

Then for the distance in the metric $\| \cdot \|$ we have

$$d(q, a_j) = |z|(1 + O(|z|^2)).$$

By (1.5), the potential energy in a neighbourhood of the point $a_j$ takes the form

$$V = -\frac{f(z)}{|z|\alpha}(1 + O(|z|^2)) + U(z), \quad \alpha = \alpha_j, \quad f(0) = m_j > 0. \quad (6.2)$$

Following the idea of the Levi-Civita regularization [25], let us make a change of variables

$$z = \phi_\beta(w) = w^\beta, \quad w \in B(0, \varepsilon) = \{w \in \mathbb{C}: |w| \leq \varepsilon\}, \quad (6.3)$$

where the real number $\beta > 1$ is to be determined. For integers $\beta = k$ the change

$$\phi_k: B(0, \varepsilon) \to B(0, \varepsilon^k)$$

is a smooth $k$-sheeted covering branched over $w = 0$. For non-integer $\beta$ the map $\phi = \phi_\beta$ is correctly defined by the formula

$$\phi(w) = r^\beta e^{i\beta \theta}$$

in the domain

$$\{w = re^{i\theta}: 0 \leq r \leq \varepsilon, \ 0 \leq \theta < 2\pi\}.$$

Instead of making the change of variables (6.3) in the Hamiltonian, we will make the change in the Jacobi metric

$$g_h^2 = 2(h - V(z))g(z)|\dot{z}|^2$$

corresponding to the energy level $H = h > \sup V$. By (6.2), in the conformal coordinates near the singular point $a_j$ the Jacobi metric is

$$g_h^2 = 2g(z)\left(\frac{f(z)}{|z|\alpha}(1 + O(|z|^2)) + h - U(z)\right)|\dot{z}|^2. \quad (6.4)$$
After the transformation $\phi$, the metric $\bar{g} = \phi^* g_h$ takes the form
\[
\bar{g}^2 = 2\beta^2 g(w^\beta)\left(\frac{f(w^\beta)}{|w|^\alpha}(1 + O(|w|^{2\beta})) + (h - U(w^\beta))|w|^{2(\beta-1)}\right)|\dot{w}|^2,
\]
where
\[
\bar{\alpha} = \alpha \beta - 2(\beta - 1) = 2 - \beta(2 - \alpha).
\]

Let us choose $\beta$ so that $\bar{\alpha} = 0$. Then
\[
\alpha = 2 - \frac{2}{\beta}, \quad \beta = \frac{2}{2 - \bar{\alpha}}.
\]

If $0 < \alpha < 2$, then $1 < \beta < \infty$ (and vice versa). The Jacobi metric takes the form
\[
\bar{g}^2 = 2\beta^2 g(w^\beta)\left[f(w^\beta)(1 + O(|w|^{2\beta})) + (h - U(w^\beta))|w|^{2(\beta-1)}\right]|\dot{w}|^2.
\]

If $\beta = k$ is an integer, then $\alpha = A_k$ (see (2.4)), and the metric $\bar{g}^2$ is smooth and positive-definite on the disk $B(0, \varepsilon)$. Then $\phi = \phi_k$ is the generalized Levi-Civita regularization of degree $k$ introduced by Knauf [21]. For Newtonian singularities with $\alpha = 1$ and $\beta = 2$ we have the classical Levi-Civita regularization $z = w^2$.

The next lemma is a direct consequence of (6.5) and (6.6).

**Lemma 6.1.** Let $A_k < \alpha_j < A_{k+1}$. Then the generalized Levi-Civita transformation of degree $k$ centred at $a_j$ transforms the singularity $a_j$ into a weak singularity of order
\[
\bar{\alpha}_j = 2 - k(2 - \alpha_j).
\]

Singularities with $\alpha_j = A_k$ disappear: the metric $\bar{g}$ is smooth at the corresponding point. Jacobi singularities are transformed into Jacobi singularities with
\[
\bar{\alpha}_j = \alpha_j = 2.
\]

**Remark 6.1.** For Jacobi singularities, the substitution $z = e^w$ is more useful than the power-function substitution. However, we will not use it in this paper.

We are interested in the case of non-integer $\beta$, when regularization is impossible. Let $W$ be the cone obtained from the sector
\[
\left\{ w = re^{i\theta} : 0 \leq r \leq \varepsilon, \quad 0 \leq \theta \leq \frac{2\pi}{\beta} \right\}
\]
by identifying $\theta = 0$ with $\theta = 2\pi/\beta$. The total angle at the vertex $w = 0$ of the cone is then $2\pi/\beta < 2\pi$. The map
\[
\phi : W \to B(0, \varepsilon^\beta)
\]
is a homeomorphism, and is a diffeomorphism away from the vertex. The metric is smooth on the cone $W$ except at the vertex $w = 0$, and is nearly Euclidean for $\beta > 1$:
\[
\bar{g} = (c + O(|w|^{2(\beta-1)}))|\dot{w}|, \quad c = \beta \sqrt{2f(0)g(0)}.
\]

Let $L(\gamma)$ be the length of a curve $\gamma$ on $W$ in the Euclidean metric $|\dot{w}|$, and let $J(\gamma)$ be its length in the Jacobi metric $\bar{g}$. Then
\[
|J(\gamma) - cL(\gamma)| \leq C\varepsilon^{2(\beta-1)}L(\gamma).
\]
Proof of Lemma 5.2. Since $\beta > 1$, the angle at the vertex $w = 0$ of the cone $W$ is $2\pi/\beta < 2\pi$. Hence for any points $x, y \in \partial W$ the Euclidean distance $d(x, y)$ on $W$ does not exceed the length $2\varepsilon \sin(\pi/(2\beta))$ of a chord of the circle $|w| = \varepsilon$ that subtends the angle $\theta = \pi/\beta < \pi$. Consequently,

$$d(x, y) \leq 2\varepsilon \sin \frac{\pi}{2\beta} < 2\varepsilon = d(x, 0) + d(0, y). \quad (6.10)$$

Take $\lambda \in (\sin(\pi/(2\beta)), 1)$. Then

$$d(x, y) \leq \lambda (d(x, 0) + d(0, y)). \quad (6.11)$$

By (6.9), for small $\varepsilon > 0$ a similar estimate holds for the metric $\rho$.

Proof of Lemma 5.3. If $1 < \alpha < 2$, then $\beta > 2$. Let $W'$ be the cone obtained from the sector

$$\left\{ w = re^{i\theta} : 0 \leq r \leq \varepsilon, \ 0 \leq \theta \leq \frac{4\pi}{\beta} \right\}$$

by identifying $\theta = 0$ with $\theta = 4\pi/\beta < 2\pi$. Then we have a double covering $\phi: W' \to B(0, \varepsilon^\beta)$ which is smooth except at the vertex. The angle at the vertex of $W'$ is less than $2\pi$. Thus, an estimate similar to (6.10) for the Euclidean distance between points $x, y \in W'$ gives us (6.11) with $\lambda \in (\sin(\pi/\beta), 1)$. □

For Newtonian singularities, the angle at the vertex of $W'$ will be $2\pi$, so the proof fails. The singularity then completely disappears after the Levi-Civita regularization, so minimizing trajectories in $W'$ may pass through the singularity.

7. Proof of Theorem 1.2

We follow [4], where, however, only Newtonian singularities were studied.

Lemma 7.1. Let $\Lambda = \{a_1, \ldots, a_k\} \subset D$ be a finite set. Suppose that $k = \#\Lambda$ is even or $D$ is homeomorphic to a domain in the plane. Then there exists a smooth double covering $\phi: D' \to D$ branched over $\Lambda$ such that near each point $q \in \phi^{-1}(\Lambda)$ the map $\phi$ is the classical Levi-Civita regularization.

Proof. First suppose that $D$ is conformally equivalent to a domain in the complex plane. Let $Z$ be the hyperelliptic Riemann surface

$$Z = \left\{ (z, w) \in \mathbb{C}^2 : w^2 = \prod_{j=1}^{k} (z - a_j) \right\}. \quad (7.1)$$

Then $Z$ is a smooth surface, and the projection $\pi: Z \to \mathbb{C}$, $\pi(z, w) = z$, is a branched covering which is the classical Levi-Civita transformation $z = a_j + c_jw^2 + \cdots$ locally near $a_j' = (a_j, 0) = \pi^{-1}(a_j)$. We set

$$D' = \pi^{-1}(D) \subset Z \quad \text{and} \quad \phi = \pi|_{D'}.$$ 

If $D$ is not homeomorphic to a plane domain, then take a domain $U \subset D \setminus \partial D$ conformally equivalent to a disk in $\mathbb{C}$ and such that $\Lambda \subset U$. Define a surface $U' = \pi^{-1}(U) \subset Z$ and a branched double covering $\phi: U' \to U$ as above with $D$. 
replaced by $U$. Next we extend $\phi: U' \to U$ to a double covering $\phi: D' \to D$ as follows.

The boundary $\partial U$ is a closed curve. If $k = \# \Sigma$ is even, then $\partial U' = \phi^{-1}(\partial U)$ consists of two components, the closed curves $S_1$ and $S_2$, and $\phi: S_i \to \partial U$ is a diffeomorphism (for odd $k$ the boundary $\partial U'$ consists of a single closed curve and $\phi: \partial U' \to \partial U$ is a double covering). We attach two copies of $D \setminus U$ to $U'$ along the boundary circles $S_1$ and $S_2$ and obtain a smooth surface $D'$ and a smooth double covering $\phi: D' \to D$ branched over $\Lambda$. $\square$

We have already removed all strong singularities and have assumed (for now) that there are no Jacobi singularities. Let $\Sigma = \Delta_{\text{newt}} \cup \Delta_{\text{mod}}$. We prove Theorem 1.2 in the case when $k = \# \Sigma$ is even or $D$ is a plane domain. Let $\Lambda = \Sigma$ in Lemma 7.1. We lift the Jacobi metric $g_h$ on $D$ to a Riemannian metric on $D'$ with the singular set $\Delta' = \phi^{-1}(\Delta)$. For this metric Newtonian singularities disappear by Levi-Civita’s classical result. Weak singularities remain weak, but their number will double. By Lemma 6.1, moderate singularities with $1 < \alpha_j < 3/2$ become weak singularities of order $\alpha_j' = 2\alpha_j - 2 \in (0,1)$, while singularities with $3/2 < \alpha_j < 2$ become moderate. Singularities with $\alpha_j = 3/2$ become Newtonian with $\alpha_j' = 1$.

By the Riemann–Hurwitz formula,

$$\chi(D') = 2\chi(D) - k < 0.$$ 

In this case the assertion of Theorem 1.2 now follows from Proposition 5.1.

**Proof of Proposition 5.2.** For even $k = \# \Sigma$ Proposition 5.2 follows from Lemma 7.1. If $k$ is odd, then we first take a covering of the surface $D$, making $k$ even. If $D$ is not simply connected, then there exists a double covering $\psi: \tilde{D} \to D$ with connected $\tilde{D}$. Let $\Sigma = \psi^{-1}(\Sigma)$. Then $\# \Sigma = 2k$ is even. Now we can construct a double covering $\phi: D' \to \tilde{D}$ branched over $\Sigma$ as in Lemma 7.1. Then $\Phi = \psi \circ \phi: D' \to D$ is a branched 4-sheeted covering with

$$\chi(D') = 2\chi(\tilde{D}) - 2k = 2(2\chi(D) - k) < 0.$$ 

Near each point in $\Sigma' = \Phi^{-1}(\Sigma)$ the map $\Phi$ is the Levi-Civita transformation.

Suppose now that $D$ is simply connected and $k$ is odd. Since the case when $D$ is a disk was already covered in Lemma 7.1, we may assume that $D = S^2$. By the hypothesis of Theorem 1.2,

$$k = 2m + 1 > 2\chi(S^2) = 4,$$

so $m \geq 2$.

Let $\Sigma = \{a_1, \ldots, a_{2m+1}\}$. By Lemma 7.1, there exist a closed surface $N$ and a double covering $\psi: N \to S^2$ branched over the set $\Lambda = \{a_1, \ldots, a_{2m}\}$. On $N$ the orders $\alpha_j$ of these singularities will be replaced by $\alpha_j' = 2\alpha_j - 2$, so the singularities $a_j' = \psi^{-1}(a_j)$ with $j = 1, \ldots, 2m$ may disappear, become weak, or remain moderate. The last singularity $a_{2m+1}$ will be replaced by the set $\psi^{-1}(a_{2m+1}) = \{b_1, b_2\}$ of two moderate or Newtonian singularities. We have

$$\chi(N) = 2\chi(S^2) - 2m \leq 0,$$
so that the genus of the surface \( N \) is \( \geq 1 \) and there are at least two moderate or Newtonian singularities. This case has already been studied. Using Lemma 7.1 with \( \Lambda = \{ b_1, b_2 \} \), we finally obtain a 4-sheeted covering \( \phi: X \to S^2 \) branched over \( \Sigma \) such that each point \( q \in \phi^{-1}(\Sigma) \) has multiplicity 2. Near \( q \) the map \( \phi \) is the Levi-Civita regularization. 

Let \( \phi: D' \to D \) be the covering in Proposition 5.2. We have \( \#\phi^{-1}(\Sigma) = kK/2 \). By the Riemann–Hurwitz formula,

\[
\chi(D') = K\chi(D) - \frac{1}{2}kK < 0.
\]
Thus we have reduced Theorem 1.2 to Proposition 5.1 also when \( k \) is odd.

8. Proof of Theorem 2.1

The proof of Theorem 2.1 is similar to the proof of Theorem 1.2, but instead of the Levi-Civita regularization we use a sequence of generalized Levi-Civita regularizations. We have already removed all the strong singularities and have assumed that there are no Jacobi singularities. Let

\[
\Sigma = \Delta_{\text{mod}} \cup \Delta_{\text{newt}} = \bigcup_{i=1}^{m} \Delta_{k_i}, \quad \Delta_{k_i} \neq \emptyset, \quad k_i \geq 2.
\]

First we prove Theorem 2.1 when \( D \) is conformally equivalent to a domain in the complex plane. If \( \Sigma = \Delta_k \) for a single \( k \), then we can define the regularizing covering as in (7.1):

\[
Z = \left\{ (z, w) \in \mathbb{C}^2: w^k = \prod_{a_j \in \Delta_k} (z - a_j) \right\}. \tag{8.1}
\]

Then \( \pi: Z \to \mathbb{C} \) is a \( k \)-sheeted covering which is the generalized Levi-Civita regularization of degree \( k \) at each point \( a'_j = (a_j, 0) \).

In the general case let

\[
Z = \left\{ (z, w_1, \ldots, w_m) \in \mathbb{C}^{m+1}: w_i^{k_i} = \prod_{a_j \in \Delta_{k_i}} (z - a_j), \; i = 1, \ldots, m \right\}. \tag{8.2}
\]

It is easy to see that if the points \( a_j \) are distinct, then for \( (z, w_1, \ldots, w_m) \in Z \) the rank of the Jacobi matrix

\[
\frac{\partial (\Phi_1, \ldots, \Phi_m)}{\partial (z, w_1, \ldots, w_m)}, \quad \Phi_i = w_i^{k_i} - \prod_{a_j \in \Delta_{k_i}} (z - a_j),
\]
is \( m \). Consequently, \( Z \) is a smooth complex curve in \( \mathbb{C}^{m+1} \). The projection

\[
\pi: Z \to \mathbb{C}, \quad \phi(z, w_1, \ldots, w_m) = z,
\]
is a covering over \( \mathbb{C} \setminus \Sigma \) with the number of sheets

\[
\#\pi^{-1}(z) = \prod_{i=1}^{m} k_i = K, \quad z \notin \Sigma.
\]
The covering is branched over $\Sigma$: for $a_j \in \Delta_{k_i}$,
\[
\#\pi^{-1}(a_j) = k_1 \cdots \hat{k}_i \cdots k_m = \frac{K}{k_i}.
\]
The branching index of a point $q \in \pi^{-1}(a_j)$, $a_j \in \Delta_{k_i}$, equals $\nu(q) = k_i$. In a neighbourhood of $q$ the covering $\pi$ is the generalized Levi-Civita transformation (6.3) of degree $k_i$:
\[
z = a_j + c_j w_i^{k_i} + \cdots, \quad c_j^{-1} = \prod_{a_i \in \Delta_{k_i}, i \neq j} (a_j - a_i).
\]
Let $X = \pi^{-1}(D)$ and $\phi = \pi|_X$. We lift the Jacobi metric $g_h$ on $D \setminus \Delta$ to a Riemannian metric $\phi^*g_h$ on $X \setminus \phi^{-1}(\Delta)$. The boundary $\partial X = \phi^{-1}(\partial D)$ is geodesically convex. By Lemma 6.1 the singularity $a_j \in \Delta_{k_i}$ will be replaced by $K/k_i$ singularities $q \in \phi^{-1}(a_j)$ of order $\alpha_j' < \alpha_j$. Therefore, the singularity disappears if $\alpha_j = A_{k_i}$, and becomes weak if $A_{k_i} < \alpha_j < A_{k_i+1}$.
Let $n_{k_i} = \#\Delta_{k_i}$. By the Riemann–Hurwitz formula,
\[
\chi(X) = K\chi(D) - \sum_{i=1}^{m} \sum_{q \in \pi^{-1}(\Delta_{k_i})} (\nu(q) - 1)
\]
\[
= K\chi(D) - \sum_{i=1}^{m} n_{k_i} \frac{K}{k_i} (k_i - 1)
\]
\[
= K\chi(D) - \frac{K}{2} \sum_{i=1}^{m} n_{k_i} A_{k_i} = K\left(\chi(D) - \frac{1}{2} A(\Delta)\right).
\]
If (2.5) holds, then $\chi(X) < 0$, so Proposition 5.1 works. This proves Theorem 2.1 in the case when $D$ is homeomorphic to a plane domain.

If $D$ is not a plane domain, then we proceed as in the proof of Theorem 1.2. Take a domain $U \subset D$ containing $\Delta$ and conformally equivalent to a disk $B(0, 1) = \{z: |z| \leq 1\}$ in the complex plane. Define the covering $\pi: Z \rightarrow \mathbb{C}$ as in (8.2) and let $Y = \pi^{-1}(U)$ as above, with $D$ replaced by $U$. It is easy to see that the boundary $\partial Y$ is homeomorphic to the curve
\[
S = \{(z, w_1, \ldots, w_m) \in \mathbb{C}^{m+1}: |z| = 1, \ w_i^{k_i} = z^{n_{k_i}}, \ i = 1, \ldots, m\}.
\]
Indeed, we can assume that $U = B(0, 1)$. Since $\Sigma \subset U$, for $R \geq 1$ the topology of the surface $Y_R = \pi^{-1}(B(0, R))$ does not depend on $R$. But for large $R$ the boundary $\partial Y_R$ is given by the equations $w_i^{k_i} = z^{n_{k_i}}(1 + O(R^{-1}))$.

If all the ratios $n_{k_i}/k_i = d_i$ are integers, then $S$ consists of $K = \prod_{i=1}^{m} k_i$ closed curves:
\[
S = \bigcup_{\sigma} S_\sigma, \quad S_\sigma = \{(z, w_1, \ldots, w_m): |z| = 1, \ w_i = \sigma_i z^{d_i}, \ i = 1, \ldots, m\},
\]
where $\sigma_i$ are roots of unity of degree $k_i$ ($\sigma_i^{k_i} = 1$) and $\sigma = (\sigma_1, \ldots, \sigma_m)$. The projection $\pi: S_\sigma \rightarrow \partial U$ is a diffeomorphism. Hence we can attach to $Y$ a copy of $D \setminus U$ along each circle $S_\sigma$ and obtain a smooth surface $X$ and a branched $K$-sheeted covering $\phi: X \rightarrow D$ as above. Therefore, we have proved the following.
Lemma 8.1. Suppose that $D$ is a plane domain or that all the ratios $n_{k_i}/k_i$ are integers. Then there exists a $(K = \prod_{i=1}^{m} k_i)$-sheeted branched covering $\phi: X \rightarrow D$ such that near every point $q \in \phi^{-1}(\Delta_{k_i})$ the map $\phi$ is the generalized Levi-Civita regularization of degree $k_i$. The Euler characteristic of $X$ is given by

$$\chi(X) = K \left( \chi(D) - \frac{1}{2} A(\Delta) \right).$$

Of course, the hypothesis of Lemma 8.1 is very restrictive, but the general case reduces to the one under consideration. If $D$ is not simply connected, then we first take a non-branched covering $\psi: \tilde{D} \rightarrow D$ of degree $K = \prod_{i=1}^{m} k_i$. Then $\Sigma$ is replaced by $\tilde{\Sigma} = \psi^{-1}(\Sigma)$ and $n_{k_i}$ is replaced by $\tilde{n}_{k_i} = Kn_{k_i}$. Lemma 8.1 is applicable with $D$ and $\Sigma$ replaced by $\tilde{D}$ and $\tilde{\Sigma}$. The composition of the coverings $\phi: X \rightarrow \tilde{D}$ and $\psi: \tilde{D} \rightarrow D$ gives the desired branched covering $\Phi: X \rightarrow D$ of degree $K^2$.

At each point in $\phi^{-1}(\Delta_{k_i})$ the covering is the generalized Levi-Civita regularization of degree $k_i$. Hence, all singularities in $\Delta_{\text{reg}}$ are now regularized, and the remaining singularities are now weak by Lemma 6.1. The assertion of Theorem 2.1 follows from Proposition 5.1.

It remains to consider the case when $D$ is simply connected. Then we can assume that $D = S^2$. As in Theorem 1.2, this case is more difficult. The desired covering $\phi: X \rightarrow S^2$ will be different for different types of singularities. In general the smaller $A(\Delta)$ is, the more subtle the construction is.

For example, let us take the smallest possible $A(\Delta) > 4$ satisfying the conditions of Theorem 1.1. Then three singularities $a_1$, $a_2$, and $a_3$ are Newtonian, and the fourth has order $\alpha_4 = 4/3$, so $A(\Delta) = 4\frac{1}{3}$. Take the Newtonian singularities $a_1$ and $a_2$ and perform the global Levi-Civita regularization in Lemma 7.1 branched over the set $\Lambda = \{a_1, a_2\}$. The Euler characteristic of the new configuration space will be

$$2\chi(S^2) - 2 = 2,$$

so it is still a sphere. The singularities $a_1$ and $a_2$ will disappear, $a_3$ will be replaced by two Newtonian singularities $b_1$ and $b_2$, and $a_4$ will be replaced by two singularities of order $4/3$.

Repeating the regularization in Lemma 7.1 with $\Lambda = \{b_1, b_2\}$, we will get rid of all the Newtonian singularities, but there will remain four singularities of order $4/3$: $c_1$, $c_2$, $c_3$, $c_4$. The configuration space is still $S^2$. Next we perform the regularization $\phi: X \rightarrow S^2$ of degree $K = 3$ in Lemma 8.1 branched over $c_1$, $c_2$, and $c_3$. The Euler characteristic of the regularized surface $X$ will be

$$\chi(X) = 3\chi(S^2) - \frac{3}{2} \cdot 3 \cdot \frac{4}{3} = 0,$$

so that $X = \mathbb{T}^2$. The singularities $c_1$, $c_2$, and $c_3$ are now regularized, but there are three singularities of order $4/3$ in $\phi^{-1}(c_4)$. A further regularization of degree 3 branched over these singularities will produce a configuration space $Y$ with negative Euler characteristic. The composition of all these coverings will be a branched

\footnote{Recall that $D$ is oriented, so it is a sphere with handles and holes.}
covering \( \Phi: Y \to S^2 \) of degree \( K = 36 \). All singularities are now regularized and \( \chi(Y) < 0 \).

Other types of singularities on \( S^2 \) are treated in a similar way. For example, if there are three singularities \( a_1, a_2, \) and \( a_3 \) with \( 1 \leq \alpha_j < 4/3 \) and \( a_4 \) with \( 4/3 \leq \alpha_4 < 3/2 \), then we take the same covering \( \Phi: Y \to S^2 \). On \( Y \) all singularities become weak, so Proposition 5.1 applies.

Theorems 2.1 and 2.2 are now proved under the assumption that there are no Jacobi singularities.

9. Jacobi singularities

Until this point we have assumed that there are no Jacobi singularities. Suppose that we have already taken care of Newtonian and moderate singularities by the generalized Levi-Civita regularization as described in the previous section, and we have obtained a geodesically convex domain \( D \) containing only weak and Jacobi singularities. Then under the assumptions of Theorem 2.1,

\[
\chi(D \setminus \Delta_{\text{jac}}) = \chi(D) - n < 0, \quad n = \#\Delta_{\text{jac}} \geq 1.
\]

For a Jacobi singularity \( a_j \) with \( \alpha_j = 2 \), we have

\[
\rho(x, a_j) = +\infty \quad \text{for } x \neq a_j,
\]

as for strong singularities. Hence the Jacobi metric on \( D \setminus \Delta_{\text{jac}} \) is complete, but the boundaries of the balls \( B_j \) have no concavity property as for strong singularities. However, any non-trivial homotopy class of closed curves in \( D \setminus \Delta_{\text{jac}} \) contains a minimizing closed curve. The only exceptions are trivial homotopy classes of small closed loops \( \gamma \) in \( B_j \) going \( k \) times around a Jacobi singularity \( a_j \). The length \( J(\gamma) \) of such a loop is bounded: it is close to \( 2\pi k \sqrt{2m_j} \). For trivial such classes, the infimum of \( J \) may be attained on a trivial curve \( \gamma \equiv a_j \).

A curve \( \gamma \) in a non-trivial homotopy class \( \Gamma \subset C^0(S^1, D \setminus \Delta_{\text{jac}}) \) cannot be pulled close to a Jacobi singularity \( a_j \) without increasing the length \( J(\gamma) \) to \( +\infty \). Consequently, for \( J(\gamma) \leq C \) the curve \( \gamma \) stays in a compact subdomain of \( D \setminus \Delta_{\text{jac}} \), so \( J \) has a minimum on \( \Gamma \). This argument is due to Poincaré and Gordon [16].

Thus, we still obtain the existence of an infinite number of minimal periodic orbits corresponding to non-trivial homotopy classes in \( C^0(S^1, D \setminus \Delta_{\text{jac}}) \), and therefore the first assertion of Theorem 2.1 is also proved in the presence of Jacobi singularities.

The proof of the second assertion is a little harder. To obtain a compact invariant set with positive topological entropy a different argument is needed, since we do not have concavity of the boundaries of neighbourhoods of Jacobi singularities. Obtaining chaotic trajectories as limits of minimal periodic geodesics when homotopy classes of closed curves in \( D \setminus \Delta_{\text{jac}} \) become more and more complicated will not work, since minimal closed geodesics may come closer and closer to a Jacobi singularity while making more and more revolutions around it. In the limit we will not obtain a compact invariant set. However, the proof will work if we first surround pairs of Jacobi singularities by simple closed geodesics. This requires the existence of a sufficiently large even number of Jacobi singularities. We can always achieve
this, since the order of a Jacobi singularity does not change under the Levi-Civita regularization: see Lemma 6.1. We consider several cases.

If \( D \) is not simply connected, then \( \chi(D) \leq 0 \). Take a double covering \( \tilde{D} \rightarrow D \), replacing \( D \) with a new domain \( \tilde{D} \) with the Euler characteristic \( \chi(\tilde{D}) = 2\chi(D) \). Now there are \( 2n \) Jacobi singularities \( b_1, \ldots, b_{2n} \). We take simple closed curves bounding disks containing only pairs of Jacobi singularities \( b_i, b_{n+i} \), and we take minimizing curves \( \gamma_i \) in the corresponding homotopy classes \( \Gamma_i \), using, for example, the parabolic flow. The minimizer geodesics exist since curves \( \gamma \in \Gamma_i \) with bounded length \( J(\gamma) \leq C \) cannot be pulled close to \( \Delta_{\text{Jac}} \): they stay outside small neighbourhoods of the singularities. The minimizers \( \gamma_i \) are simple curves without self-intersections, and \( \gamma_i \cap \gamma_j = \emptyset \) for \( i \neq j \). Hence \( \gamma_i \) bounds a disk \( D_i \) containing the points \( a_1, b_1, b_{n+i} \), and \( D_i \cap D_j = \emptyset \) for \( i \neq j \). We replace \( \tilde{D} \) by a geodesically convex domain \( D' = \tilde{D} \setminus \bigcup D_i \) with the Euler characteristic

\[
\chi(D') = 2\chi(D) - n < 0.
\]

Now Proposition 5.1 works.

If \( D \) is simply connected, then it is a sphere or a disk. Suppose first that \( D \) is a disk with \( \chi(D) = 1 \); then \( n \geq 2 \). Perform the Levi-Civita regularization in Lemma 7.1 for the Jacobi singularities \( \Lambda = \{a_1, a_2\} \). The domain \( D \) will be replaced by \( D' \) with Euler characteristic

\[
\chi(D') = 2\chi(D) - 2 = 0,
\]

so \( D' \) is a cylinder \( S^1 \times [0,1] \). By Lemma 6.1, the singularities \( a_1 \) and \( a_2 \) will be replaced by a pair of Jacobi singularities \( b_1 \) and \( b_2 \). Take a minimizing closed geodesic around \( b_1 \) and \( b_2 \) bounding a disk \( U \) and delete \( U \) from \( D \). We again obtain a geodesically convex domain \( D \setminus U \) with Euler characteristic

\[
\chi(D \setminus U) = \chi(D) - 1 = -1.
\]

Finally, let \( D = S^2 \); then \( n \geq 3 \). If \( n = 2k \geq 6 \), then as above we can remove \( k \) disks containing pairs of Jacobi singularities and obtain a domain \( D' \) with Euler characteristic \( \chi(D') = 2 - k < 0 \). If \( n \) is odd or \( n < 6 \), we perform the Levi-Civita regularization in Lemma 7.1 for the Jacobi singularities \( a_1 \) and \( a_2 \). Then the configuration space will be replaced by a closed surface \( N \) such that

\[
\chi(N) = 2\chi(S^2) - 2 = 2,
\]

so \( N = S^2 \). But now there are at least \( 2n - 2 \geq 4 \) Jacobi singularities \( b_1, \ldots, b_{2n-2} \), two of which are obtained from \( a_1 \) and \( a_2 \) while the other \( 2(n-2) \) are obtained from the remaining \( n-2 \) singularities. Using Lemma 7.1 with the four singularities \( b_1, b_2, b_3, \) and \( b_4 \), we obtain the new configuration space \( T^2 \) and

\[
4 + 2(2n - 6) = 4n - 8 \geq 4
\]

Jacobi singularities. This case has already been considered.

Theorem 2.1 is now completely proved, also in the presence of Jacobi singularities.
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