Obstructions, Extensions and Reductions. 
Some applications of Cohomology *

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Abstract

After introducing some cohomology classes as obstructions to orientation and spin structures etc., we explain some applications of cohomology to physical problems, in especial to reduced holonomy in $M$- and $F$-theories.

1 Orientation

For a topological space $X$, the important objects are the homology groups, $H_*(X,A)$, with coefficients $A$ generally in $\mathbb{Z}$, the integers. A bundle $\xi : E(M,F)$ is an extension $E$ with fiber $F$ (acted upon by a group $G$) over an space $M$, noted $\xi : F \to E \to M$ and it is itself a Čech cohomology element, $\xi \in \check{H}^1(M,G)$. The important objects here the characteristic cohomology classes $c(\xi) \in H^*(M,A)$.

Let $M$ be a manifold of dimension $n$. Consider a frame $e$ in a patch $U \subset M$, i.e. $n$ independent vector fields at any point in $U$. Two frames $e$, $e'$ in $U$ define a unique element $g$ of the general linear group $GL(n,R)$ by $e' = g \cdot e$, as $GL$ acts freely in $\{e\}$. An orientation in $M$ is a global class of frames, two frames $e$ (in $U$) and $e'$ (in $U'$) being in the same class if det $g > 0$ where $e' = g \cdot e$ in the overlap of two patches. A manifold is orientable if it is

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possible to give (globally) an orientation; choosing an orientation the manifold becomes oriented. So the questions are: first, when a manifold is orientable and second, if so, how many orientations are there. These questions are tailor-made for a cohomological answer.

This is about the easiest example translatable in simple cohomological terms by obstruction theory. Let \( \tau \) be the principal bundle of the tangent bundle to \( M \), so the total space \( B \) is the set of all frames over all points:

\[
\tau : GL(n, R) \to B \to M
\]

Matrices in \( GL^+ \) with \( \det > 0 \) have index two in \( GL \), hence are invariant, with \( Z_2 \) as quotient: we form therefore an associated bundle \( w_1 = w_1(\tau) \):

\[
\begin{array}{ccc}
GL^+ & \downarrow & \\
\tau : GL & \to B & \to M \\
& \downarrow \downarrow & \parallel \\
w_1 : Z_2 & \to B/2 & \to M
\end{array}
\]

Our space \( M \) is orientable if the structure group reduces to \( GL^+ \). The set of principal \( G \)-bundles over \( M \) is noted \( \hat{H}^1(M, G) \) and it is a cohomology set (in Čech cohomology) \( \ddagger \). Thus \( \tau \in \hat{H}^1(M, GL) \), and we have associated to \( \tau \) another bundle, name it \( \text{Det} \tau \equiv w_1(\tau) \in H^1(M, Z_2) \), called the first Stiefel-Whitney class of \( \tau \) (as a real vector bundle). The Čech cohomology set \( H^1 \) becomes a bona fide abelian group for \( G \) abelian, whence we suppress the \( \ddagger \), and the associated bundle, still presently principal, becomes a \( Z_2 \) cohomology class.

Now we have the induced exact cohomology sequence, i.e.

\[
H^0(M, Z_2) \to \hat{H}^1(M, GL^+) \to \hat{H}^1(M, GL) \to H^1(M, Z_2)
\]

\( \tau \) lives in the third group, and by exactness it has antecedent (i.e., \( M \) is orientable) if it goes to zero in the final group: the middle bundle above reduces if and only if the quotient splits: so we have the result:

\[
M \text{ is orientable if and only if the first Stiefel-Whitney class of } \tau, \text{ that is } w_1 \in H^1(M, Z_2), \text{ is zero.}
\]

In other words: \( M \) is orientable if \( \tau \) reduces its group \( GL \) to the connected subgroup \( GL^+ \). According to the fundamental exactness relation, orientability means section in the lower bundle, and as it is principal, the bundle is trivial, hence its class \( (w_1) \) is the zero of the cohomology group: the lower bundle in \( [2] \) splits.
Alternatively, $M$ is orientable if it has a volume form, which is the same as a global frame mod det $> 0$ transformations in overlapping patches.

As examples, $RP^2$ is not orientable, but $RP^3$ is orientable, where $RP^n$ is the real $n$-dimensional projective space of rays in $R^{n+1}$. The reason is the antipodal map $(-1, ..., -1)$ in $S^n$ leading to $RP^n = S^n/Z_2$ is a rotation for $n$ odd, but a reflection for $n$ even; note $RP^n$ is not simply connected for any $n$. To have 1-cohomology in any ring the first cohomology group $H^1(M, Z)$ has to be $\neq 0$: simply connected spaces are orientable.

Notice the structure group $GL$ reduces always to the orthogonal group $O = O(n)$: any manifold is, in its definition, paracompact, and in any paracompact space there are partitions of unity, hence for a manifold a Riemann metric is always possible:

\[
\begin{array}{ccc}
O(n) & \downarrow & \\
\downarrow & & \\
GL(n, R) & \rightarrow & B \rightarrow M \\
\downarrow & & \\
R^{n(n+1)/2} & \rightarrow & E \rightarrow M \\
\end{array}
\] (4)

As the lower row fibre is contractible, the horizontal middle bundle lifts $O(n) \rightarrow B_0 \rightarrow M$, where $B_0$ is the set of orthogonal frames: any manifold is riemanizable. Note $O(n)$ is the maximal compact subgroup of $GL(n)$, and this is why the quotient is contractible. By contrast, not every manifold admits a Lorentzian metric: it needs to have a field of time-like vectors globally defined.

Hence the characteristic class $w_1 \in H^1(M, Z_2)$ is the obstruction to orientability: it measures if an orientation is possible in a manifold. The next question is: If the obstruction is zero, how many orientations are there? Again, the answer is written by (3): the elements in $H^1(M, GL^+)$ falling into $\tau$ are the coset labelled by $H_0(M, Z_2)$. In particular, if the manifold is connected, $H^0(M, A) = A$, and then the zeroth Betti number is $b_0 = 1$: hence, as then $H^0(M, Z_2) = 2$,

A connected orientable manifold has exactly two orientations.

2 Spin structure

The orthogonal group $O(n)$ is neither connected nor simply connected; 0-connectivity questions lead to the first Stiefel-Whitney class, 1-connectivity
to the second (sw) class. Suppose the manifold $M$ is orientable already and write the covering group $Spin(n) \to SO(n)$:

$$Z_2 \to Spin(n) \to SO(n).$$  \hfill (5)

For $n > 2$ this bundle is the universal covering bundle, as $\pi_1(SO(n)) = Z_2$, $n > 2$. For $n = 2$ the spin bundle still covers twice, but now $\pi_1(SO(2)) = Z$.

Endow now $M$ with a riemannian structure (there is no restriction on doing this, see above), and write

$$Z_2 = Z_2 \downarrow \downarrow \ Spin(n) \to B \to M \downarrow \downarrow \| \tau : SO(n) \to B \to M$$  \hfill (6)

We say that a manifold admits a spin structure (it is spinable) if the (rotation) tangent bundle lifts to a spin bundle. Again, there is a precise homological answer. From the exact sequence, and with $\tau$ living in third group

$$H^1(M, Z_2) \to \hat{H}^1(M, Spin(n)) \to \hat{H}^1(M, SO(n)) \to H^2(M, Z_2),$$  \hfill (7)

we see, as before, that if we call $w_2$ the image of $\tau$, there is an obstruction to spinability, called the second Stiefel-Whitney class,

$$w_2 = w_2(\tau) \in H^2(M, Z_2)$$  \hfill (8)

and a manifold is spinable if and only if $w_2(M) = 0$. For example, spheres and genus-$g$ surfaces are spinable. If the obstruction is zero, how many spin structures are there? Again, a simple look at (7) gives the answer: there are as many as $H^1(M, Z_2)$, which is a finite set, of course. For example, for an oriented surface of genus $g$, $H^1(\Sigma_g, Z_2) = Z_2^{2g}$, hence $\# = 2^{2g}$, as is well known in string theory; recall that the first Betti number $b_1(\Sigma_g) = 2g$.

As examples of spin manifolds, $CP^{2n+1}$ is spinable, but $CP^{2n}$ is not. For example, $CP^1 = S^2$, no sw classes; as for $CP^2$, we have that $b_2 = 1$; recall also Euler number ($CP^n$) = $n + 1$.

There is an alternative characterization of spin structures due to Milnor [2], which avoids using Čech cohomology sets. From (6) we have the exact sequence (taking values in $Z_2$):

$$0 \to H^1(M, Z_2) \to H^1(B, Z_2) \to H^1(SO(n), Z_2) \to H^2(M, Z_2),$$  \hfill (9)
and accepting \( w_2 = 0 \) we see again the number of spin structures to be \( \#H^1(M, \mathbb{Z}_2) \), as \( \tilde{B} \) is the total space of the lifted bundle, and lives in the second group.

3 The first Chern class

The Det map \( O(n) \to O(n)/SO(n) = \mathbb{Z}_2 \) can be performed also in complex bundles, with structure group \( U \) instead of \( O \):

When does a complex bundle \( \eta \) reduce to the unimodular group? Write

\[
\begin{align*}
SU(n) & \xrightarrow{\eta} U(n) \to B \to M \\
c_1 & : U(1) \to B' \to M
\end{align*}
\]

The associated bundle \( \text{Det} \eta \in H^1(M, U(1)) \) determines the first Chern class of \( \eta \) by the resolution \( Z \to R \to U(1) = S^1 \), as

\[
c_1(\eta) \in H^2(M, Z) = H^1(M, U(1)).
\]  

\( c_1(\eta) = 0 \) is the condition for reduction to the \( SU \) subgroup, an important restriction in compactifying spaces in \( M \)-theory, see later.

For the general definition of Stiefel-Whitney (and Pontriagin) classes of real vector bundles, and for the Chern classes of complex vector bundles, the insuperable source is [3].

4 Euler class as Obstruction

A more sophisticated example is provided by the Euler class. Look for manifolds \( M \) with a global 1-frame, i.e. a global zeroless vector field: let \( M \) be orientable; from the coset \( S^{n-1} = SO(n)/SO(n-1) \)

\[
\begin{align*}
SO(n-1) & \xrightarrow{\tau} SO(n) \to B \to M \\
& \quad \downarrow \downarrow \parallel \\
S^{n-1} & \xrightarrow{\tau'} B'' \to M
\end{align*}
\]

A 1-frame exists if the (unit) sphere bundle has a section. The last bundle produces a map

\[
H^{n-1}(S^{n-1}, R) \to H^n(M, R)
\]  

5
The image of the fundamental class of \( H^{n-1}(S^{n-1}, Z) \) is the Euler class \( e \) of \( M \). The condition of reduction is clearly that \( e = 0 \). Here \( e[M] = \chi \), the Euler number. The result is the well-known condition for a manifold to admit a global 1-frame: zero Euler number. It is funny (and easy to understand) that the theorem has a positive side: you can compute the Euler number by counting the Windungzahl of the zeros of any vector field, the Poincare-Hopf theorem.

5 Nonabelian Group Extensions

(Cfr. ([4]), Ch. 7). Consider the group extension problem: given groups \( K \) and \( Q \), find \( G \) such that \( K \subset G \) normal and \( G/K = Q \).

Recall first the relations, for \( Z_H \): center of \( H \)

\[
\begin{align*}
H/Z_H & \equiv \operatorname{Int} H \quad \text{and} \quad \operatorname{Aut} H/\operatorname{Int} H \equiv \operatorname{Out} H
\end{align*}
\]

(14) for any group \( H \). If there is a solution \( G/K = Q \) to our problem, write

\[
\begin{array}{ccc}
Z_K & \downarrow \\
\downarrow & & \downarrow \\
K & G & Q \\
\downarrow & \downarrow & \downarrow \\
\operatorname{Int} K & \operatorname{Aut} K & \operatorname{Out} K
\end{array}
\]

(15) where you construct the last two vertical arrows. So any extension determines a morphism \( \alpha : Q \to \operatorname{Out} K \). Inverse question is: given \( \alpha \in \operatorname{Hom}(Q, \operatorname{Out} K) \), are there extensions? How many? Note first, given \( \alpha \) there is an extension

\[
\begin{array}{ccc}
\alpha : & \operatorname{Int} K & \to X & \to Q \\
\downarrow & \downarrow & \downarrow \\
\operatorname{Int} K & \to \operatorname{Aut} K & \to \operatorname{Out} K
\end{array}
\]

(16)

That is

\[
\begin{array}{ccc}
Z_K & \to X & \to Q \\
\to \downarrow & \to \downarrow \\
K & \to X & \to Q
\end{array}
\]

(17)

We need to lift the horizontal sequence to have extensions, and we see that there is an obstruction in the exact cohomology sequence,

\[
H^*(Q, Z_K) \to H^*(Q, K) \to H^*(Q, \operatorname{Int} K) \to H^{*+1}(Q, Z_K)
\]

(18)
So $\alpha$ produces extensions iff the image of $\alpha$ in the last group is zero: this is the obstruction. Now if we add our knowledge that the abelian extensions are given by some $H^2$, the obstructions lies in $H^3(Q, Z_K)$, and if it is zero, the number of extensions is $H^2(Q, Z_K)$; and the obstruction lies in the third group:

$$w(\alpha) \in H^3(Q, Z_K);$$

all this is very similar to the spin or orientation problems.

## 6 Structure of Lie groups

An unexpected problem where an obstruction is necessary appears in the existence of simple Lie groups. Consider the next simplest group, $SU(3)$. In the natural 3 representation, the group leaves the unit sphere invariant, with $SU(2)$ as little group:

$$SU(2) = S^3 \rightarrow SU(3) \rightarrow S^5 \subset R^6 = C^3$$

and regard this as a bundle extension; bundles over $n$-spheres are classified by $\pi_{n-1}(G)$, where $G$ is the structure group. So here, as

$$\pi_4(S^3) = Z_2$$

we have just two solutions, the direct product (which cannot be the group $SU(3)$, because $S^5$ is not parallellizable), and the other, necessarily $SU(3)$: the existence of non-trivial bundles, here (in this case) for nontrivial homotopy classes are crucial for the existence of Lie groups. Incidentally, the map $S^4 \rightarrow S^3$ generating $SU(3)$ is easy to describe: it is the suspension of the second Hopf bundle. (I thank D. Freed for this remark):

$$\beta : S^1 \rightarrow S^3 \rightarrow S^2$$

$$\Sigma \downarrow \quad \Sigma \downarrow$$

$$S^4 \rightarrow S^3$$

For all Lie groups besides the “atom in the category”, $SU(2) = S^3$ the same obstructions obtain; we leave the details. It would be nice to invert the question: to deduce the simple Lie groups from nontrivial extensions... Incidentally, the Hopf $\beta$ bundle is the second on the series of higher homotopy groups of spheres $\pi_{4n-1}(S^{2n}) = Z+...$, related to the Hopf invariant and to the nonexistence of division algebras besides R, C, H and O [5]. For expressions of simple compact Lie groups as finite twisted products of odd spheres, see [6].
7 Special Holonomy manifolds

Since the advent of $M$-Theory (1995; Townsend, Witten, Polchinski [7]) the problem of compactification of extra dimensions, from 10 to 4 in one extreme to 12 down to 2 in the other, is becoming more and more acute. If one is a true believer in $M$ (or $F$) theory (as I tend to be), this problem is perhaps the central one in physics. The arguments for extra dimensions are overwhelming, and so are the reasons why we live in four large dimensions. In a nutshell, geometric description of nongravitational forces requires extra dimensions, while interactions transmitted via massless particles do not make physical sense outside four (i.e. the $1/r$ potential law).

Here we want to show, via simple examples, that compactification with extra conditions (like preserving $N = 1$ Supersymmetry) can be easily stated in cohomological terms, as reductions of the structure/holonomy group of different bundles.

Consider the “old” problem of compactifying the Heterotic String living in $10D$ down to $4D$. The tangent bundle of the compactifying manifold $K_6$ is

$$\tau : O(6) \to B \to M = K_6$$

Now we want $K_6$ to be a manifold orientable (to integrate), spin (to describe fermions) and with a (covariant) constant spinor field (to preserve $N = 1$ Susy in order to “understand” the scale of the Higgs mass together with the existence of chiral fermions). In terms of reduction:

$O(6)$ reduces to $SO(6)$; $SO(6)$ lifts to $Spin(6) = SU(4)$. $SU(4)$ reduces to $SU(3)$, which lies in $U(3):

$$SU(3)$$

$$\downarrow$$

$$\tilde{\tau} : U(3) \to B \to M$$

$$det \downarrow \quad \downarrow \quad \parallel$$

$$U(1) \to B' \to M$$

Now $R/Z = S^1$ induces (see above) $\det (\tilde{\tau} = c_1(\tau(M)))$, the first Chern class; hence $M$ is a complex manifold with $SU(3)$ holonomy, with the first Chern class $= 0$, and it can be seen that this implies the trace of the curvature zero; it is a Ricci-flat riemannian manifold: Calabi-Yau manifolds. The search for those manifolds was a prolific industry led by Phil Candelas in Austin in 1985-92 [8].

Manifolds with tangent structure groups less than maximal are therefore crucial for $M$-theory; let us see more examples.
8 Compactification in $M$-theory

Please notice, first, that the reduced holonomy problem is not the same as reducing the structure group, but in practice both are present together; the link is of course the two holonomy theorems:

(1) The structure group can be reduced to the holonomy group, the Ambrose-Singer theorem.

(2) The Lie algebra of the holonomy group is generated by the curvature of the connection producing the holonomy in the first place, the curvature theorem.

For a generic riemannian manifold, the possibility of isometry groups and reduced holonomy groups are antagonic: a generic manifold $M$ has no isometries, and maximal holonomy (e.g. $SO(\dim M)$ if $M$ orientable). Viceversa, special holonomy manifolds have no isometries in general (what poses a problem for the existence of gauge groups down in 4D by the KK mechanism, see later), and a very symmetric space, in fact maximally symmetric, like even-dim spheres, has irreducible holonomy $SO(n)$.

M. Berger (1955) classified holonomy groups, and came up with several series (like $O(2n) \supset U(n), O(4n) \supset Sp(n)$ etc.), and just two (in fact, three; one, corresponding to $Spin(9) \subset SO(16)$ was already known as a symmetric space) special cases:

$$Spin(7) \subset SO(8) \quad \text{and} \quad G_2 \subset SO(7) \quad (25)$$

Both turned out essential in $M$ and $F$ theories. Both come, of course, from the beautiful irreducible representations provided by the Clifford algebra. Notice this is irreducible holonomy, in the sense that the irrep of the subgroup has the same dim as that of the group, namely 8 and 7 bzw.

To demystify those cases it is enough to ask for those representations of the spin groups $Spin(n)$ which act trans in the unit sphere; and the answer is, besides the low dimensional cases in which there are repetitions (like $Spin(6) = SU(4)$), only two more: the 16 irrep of $Spin(9)$ and the 8 of $Spin(7)$. The first case gives only the Moufang plane $OP^2 = F_4/Spin(9)$. The other is very interesting:

The 8 irrep of $Spin(7)$ allows for the embedding $Spin(7) \subset SO(8)$. Now $SO(8)$ preserves a quadratic form. What else does $Spin(7)$ maintain? What manifolds have $Spin(7)$ holonomy? The construction of these manifolds starting by D. Joyce around 1995 [10] has been a great achievement. We shall comment on these constructions later, but let us finish first with the $G_2$ case.

In general, in a (real or complex) vector space $V$, the set $X$ of 2-forms and/or quadratic forms are “open” in the sense of the orbit space $X/GL(V)$.
But $p$-forms dimension grows, of course, like $n!/p!(n-p)! > n^2 = \dim GL(V)$. The exceptions occur naturally in low dimensions: A three-form in 6, 7 and 8 dimensions and self-dual 4-form in 8 dimensions.

In particular, the group leaving a regular 3-form invariant in 7-dim space is $G_2$. (Note how the dimensions match: $49 = 7 \times 7 = 35$ (dim 3-forms) + 14 (dim $G_2$)). Which 3-form? Bilinear forms produce scalar products, but trilinear forms produce an internal law $X \times X \to X$, so one guesses that in $R^8$ there is a product! Of course, there is: octonion multiplication. The appearance of octonions just reinforces the idea that $M$-theory, as a *unique* theory, has to include octonions, which are *unique* structures in mathematics, and responsible for most of their exceptional objects [12]. To repeat: the reason of the appearance of octonions in $M$-theory is this: in the 11 to 4 version, the compact manifold has to have $G_2$ holonomy because of supersymmetry; this group preserves a 3-form, which corresponds to the fully antisymmetric (alternative) octonion multiplication 3-form. No wonder, $G_2$ is the automorphism group of octonions.

In the $F$-theory version, the $12 \to 4$ descent implies $Spin(7)$ holonomy; but this group can be seen as unit-octonion “group” $S^7$ stabilized by $G_2$. In both cases of $11 = (1, 10)$ dimensions and $12 = (2, 10)$ it is remarkable that supersymmetry unveils octonion structures!

### 9 Structure Diagrams

It is time to express in diagrams what we are saying. First, there is the structure diagram for $G_2$: define it as the little (=isotropy) group for the *trans* action of $Spin(7)$ in the 7-sphere [5]:

\[
\begin{array}{ccc}
SU(3) & \to & SU(4) = Spin(6) \to S^7 \\
\downarrow & & \downarrow \\
G_2 & \to & Spin(7) \to S^7 \\
\downarrow & & \downarrow \\
S^6 & = & S^6
\end{array}
\]  (26)

So $G_2$ operates in the 6-sphere of unit imaginary octonions; but $G_2$ is also a subgroup of $SO(7)$, witness the $1$ *irrep*: the *torsion* diagram explains this:

\[
\begin{array}{ccc}
Z_2 & = & Z_2 \\
\downarrow & & \downarrow \\
G_2 & \to & Spin(7) \to S^7 \\
\downarrow & & \downarrow \\
G_2 & \to & SO(7) \to RP^7
\end{array}
\]  (27)
and the mixed diagram for $G_2$

$$
\begin{align*}
SU(2) & = SU(2) \\
\downarrow & \downarrow \\
SU(3) & \to \quad G_2 \quad \to \quad S^6 \\
\downarrow & \quad \downarrow \\
S^5 & \to \quad V_{11} \quad \to \quad S^6
\end{align*}
$$

(28)

where $V_{11}$ is a Stiefel manifold, generating the 2-torsion in $G_2$, in the odd sphere structure [13].

$$
G_2 = S^3 \times S^{11}
$$

(29)

Finally, we exhibit the richness of the $S^7$ sphere of unit octonions in the following special holonomy diagram

$$
\begin{align*}
Sp(1) & \subset SU(3) \subset G_2 \subset SO(7) \\
\downarrow & \downarrow & \downarrow \\
Spin(5) & \subset Spin(6) \subset Spin(7) \subset SO(8) = Spin(8)/\mathbb{Z}_2 \\
\downarrow & \quad \downarrow \\
S^7 & = S^7 = S^7 = S^7 \\
Sp(H) & \quad U(C) \quad Oct(O) \quad O(R)
\end{align*}
$$

(30)

10 Structure of Special Holonomy manifolds

We return to a question mentioned above. In the old Kaluza-Klein approach, where $M_D \to M_4$, one gets gauge forces in the lower space from the isometries of the compactifying manifold; for example, the $U(1)$ for the electromagnetic field in the original $5 \to 4$ reduction of Kaluza (1919). But now, where extra dimensions are there to stay, the argument does not work anymore! Special holonomy manifolds have, generically, no isometries; so if we are to rely on simple gravity in higher spaces, how do we get gauge forces in our mundane 4D space?

The answer is spectacular, and I do not think it has been assimilated wholly by the scientific community: the special holonomy manifolds have a rich homology, and the non-trivial cycles (that is, the uncontractible spheres) can act as sources for gauge fields, following the pattern of singularities, $A - D - E$ classification, and Mac-Kay correspondence! [14], [15]. In a way, this is the generalization of the fact that the open string sustains gauge groups in its boundary, the singular points. Or, that compactification of M-theory in a segment necessitates two $E_8$ groups in the border, the Horawa-Witten mechanism.
We do not want to pursue a line of research which seems to be incomplete as yet. However, we cannot resist to consider the case of the perhaps simplest special (special but not exceptional) holonomy, the case of $K3$ (see an early example in [16]), which appears when the $11 \to 7$ or $10 \to 6$ descents; it is representative of the many sophisticated constructions of Joyce and others. So let us construct $K3$ [17].

1) Start with $\mathbb{R}^4$, divide by a lattice $L$ to generate a 4-Torus

$$T^4 = \mathbb{R}^4/L = \mathbb{R}^4/(\mathbb{Z} + \mathbb{Z} + \mathbb{Z} + \mathbb{Z})$$

(31)

2) Now apply a discrete group $\Gamma$ with a non-free action, for example the “parity” operation generating $\mathbb{Z}_2$:

$$\theta_i \to -\theta_i$$

(32)

for the four angles labelling $T^4$. Call $X = T^4/\Gamma$. The space $X$ is an orbifold, that is, a manifold with some special (singular) points, those fixed by $\Gamma$ (here there are $2^4 = 16$ points).

3) The 4-Torus is a complex surface, the $X$ space is also a complex surface, and as a complex manifold, there is a perfectly standard procedure (starting in the 19th century by Italian mathematicians!) to remove (blow-up) the singularities, trading them, in our case, by 2-spheres (complex projective lines, really). The resulting true, bona fide smooth manifold is called $K3$ in the literature (for Kummer, Kähler and Kodaira, amen for coincidence with the Himalaya peaks; the godfather seems to be A. Weil [17]).

The reader can think of converting the cone $x^2 + y^2 = z^2$ on the one-sheeted hyperboloid $x^2 + y^2 - z^2 = 1$ as a simple blow-up of the conic singularity, trading it by a circle.

It is useful to pursue the change in topology: the Betti numbers are $(1, 0, 0, 0, 0)$ for $\mathbb{R}^4$, $(1, 4, 6, 4, 1)$ for $T^4$, $(1, 0, 6, 0, 1)$ for $X$ and $(1, 0, 22, 0, 1)$ for $K3$.

Notice the change after the blow-up: each of the 16 singular points fattens to become a “hollow” 2-sphere, hence $b_2$ increases from 6 to 22; notice also the increase in “curvature” from $T^4$, which is flat: already in $X$ there is “point” curvature.

From the point of view of string theory, the point of introducing $K3$ is that string theory $IIA$ or $IIB$ dualizes with the heterotic string in this curious way, in six dimensions [19]:

$$II/K3 \approx \text{Het}/T^4$$

(33)
In other words, $K3$ “generates” whatever remains in $6D$ of the 496-dim gauge group extant in $10D$! We do not enter into details, as they are well known, albeit not well understood.

11 Acknowledgements

This is a slightly worked out edition of the talk I gave in Salamanca. It is meant to recall the past happy times forty years ago when Adolfo and myself were together in Barcelona learning the rudiments of cohomology (mainly from Prof. Juan Sancho Guimeira), with applications to projective representations and group extensions.

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