Determinism plus chance in random matrix theory

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ABSTRACT: We study Hamiltonians consisting of a deterministic term plus a random term. Using a diagrammatic approach and introducing the concept of “gluon connectedness,” we calculate the density of energy levels for a wide class of probability distributions governing the random term, thus generalizing a result obtained recently by Brézin, Hikami, and Zee. The method used here may be applied to a broad class of problems involving random matrices.

Some four decades ago, Wigner\textsuperscript{1} proposed studying the distribution of energy levels of a random Hamiltonian given by

\[ H = \varphi \]  

(1)

where \( \varphi \) is an \( N \) by \( N \) hermitean matrix taken from the distribution

\[ P(\varphi) = \frac{1}{Z}e^{-N\text{tr}V(\varphi)}. \]  

(2)

with \( Z \) fixed by \( \int d\varphi P(\varphi) = 1 \). This problem has been studied intensively by Dyson, Mehta, and others over the years.\textsuperscript{2,3,4} Two years ago, Brézin and Zee discovered that, remarkably, while the density of eigenvalues depends\textsuperscript{5} on \( V \), the correlation between the density of eigenvalues, when suitably scaled, is independent\textsuperscript{6} of \( V \). This universality has been clarified and extended by other authors,\textsuperscript{7,8,9} studied numerically,\textsuperscript{10} and furthermore, shown to hold even when the distribution (2) is generalized to a much broader class of distributions.\textsuperscript{11} We expect that the discussion to be given below will hold also for this broader class of distributions, but for the sake of simplicity we will not work this through here.
In recent work,\textsuperscript{12,13} Brézin and Zee have generalized this Wigner problem to the case of a Hamiltonian given by the sum of a deterministic term and a random term

\[ H = H_0 + \varphi \] (3)

Here $H_0$ is a diagonal matrix with diagonal elements $\epsilon_i$, $i = 1, 2, \ldots N$, and $\varphi$ a random matrix taken from the ensemble (2). For the Gaussian case, namely with $V(\varphi) = \frac{1}{2} \varphi^2$, Pastur\textsuperscript{14} has long ago determined the density of eigenvalues. The work described in Ref. 12 went beyond Pastur’s work in that the correlation function between the density of eigenvalues in the Gaussian case was also determined. More recently, in a work with Brézin and Hikami,\textsuperscript{13} we managed to determine the density of eigenvalues for $V(\varphi) = \frac{1}{2} \varphi^2 + g \varphi^4$ to all orders in $g$. The correlation function was also computed, but only to first order in $g$.

This problem of “determinism plus chance” may be regarded as a generic problem in physics, and as such represents a significant generalization of Wigner’s problem. For example, consider an electron moving in a magnetic field and scattering off impurities. We note that these “deterministic plus random” problems are considerably more difficult than the purely random problems defined in (1) and (2). A standard approach to solving the purely random problem involves diagonalizing the random matrix $\varphi$ and then use orthogonal polynomials to disentangle the resulting expression. Clearly, in (3) we cannot diagonalize $\varphi$ without un-diagonalizing $H_0$ and thus the orthogonal polynomial approach fails.

In this paper, we point out that the problem given in (3) is a special case of a broader class of problems involving the addition of random matrices. The deterministic Hamiltonian $H_0$ may in turn be replaced by a random Hamiltonian. Indeed, a deterministic matrix is but a special case of a random matrix. We will extend the work of Brezin, Hikami, and Zee\textsuperscript{13} and determine the density of eigenvalues for the Hamiltonian given in (3) for an arbitrary $V$.

Our work is inspired by recent advances in the mathematical literature involving the theory of non-commutative probability and operator algebra.\textsuperscript{15,16,17} A number of physicists have already brought these advances to the attention of the physics community.\textsuperscript{18,19,20} While our work is thus inspired, we will not be using the mathematical approach given in Ref. 15,16,17, but instead will be based on the diagrammatic approach developed in Ref. 12 and subsequent work.\textsuperscript{21,13}

Consider a Hamiltonian given by

\[ H = \varphi_1 + \varphi_2 \] (4)
with the matrices $\varphi_{1,2}$ taken from the probability distribution

$$P(\varphi_1, \varphi_2) = \frac{1}{Z} e^{-Ntr[V_1(\varphi_1) + V_2(\varphi_2)]} \equiv P_1(\varphi_1)P_2(\varphi_2). \quad (5)$$

Notice that the probability distribution factorizes. (This is known as “free” in the mathematical literature.) The problem defined in (3) represents a special case. Previously, with D’Anna and with Brézin we have studied the problem given in (4) but for the more difficult case in which $P(\varphi_1, \varphi_2)$ contains terms linking $\varphi_1$ and $\varphi_2$. Indeed, a detailed determination of the correlation function over all “distance scales” is a non-trivial problem even for Gaussian distributions. The discussion in this paper goes through precisely because $\varphi_1$ and $\varphi_2$ do not couple to each other in $P(\varphi_1, \varphi_2)$.

Let us now mention a few necessary definitions. Define the Green’s function

$$G(z) \equiv \left\langle \frac{1}{N}tr\frac{1}{z-H} \right\rangle = \int \int d\varphi_1 d\varphi_2 P(\varphi_1, \varphi_2) \frac{1}{N} tr\frac{1}{z-(\varphi_1 + \varphi_2)} \quad (6)$$

The density of eigenvalues is then given by

$$\rho(\mu) = \left\langle \frac{1}{N}tr\delta(\mu - H) \right\rangle = -\frac{1}{\pi} \text{Im}G(\mu + i\epsilon).$$

In this paper we focus on the density of eigenvalues, leaving the correlation for a future work. Note that the factors of $N$ are chosen in our definitions such that the interval over which $\rho(\mu)$ is non-zero is finite (i.e., of order $N^0$) in the large $N$ limit.

We may regard the distribution (5) as defining a $(0 + 0)$-dimensional field theory. The Feynman diagram expansion is then simply obtained by expanding $G(z)$ in inverse powers of $z$ and doing the integrals in (6). As explained in Ref. 12, it is useful to borrow the terminology of large $N$ quantum chromodynamics from the particle physics literature, and speak of quark and gluon lines. See figure (1) for a graphical representation. (It is of course not necessary to use this language, and readers not familiar with this language can simply think of the diagrams as representing the different terms one encounters in doing the integral in (6).) The quark propagator simply comes from the explicit factor of $z$ in (6) and is represented by a single line and given by $\frac{1}{z}$. The quadratic terms in $V_1(\varphi_1)$ and $V_2(\varphi_2)$ determine the gluon propagator, represented by double lines. Here, in a minor departure from large $N$ quantum chromodynamics we have two types of gluons, corresponding to $\varphi_1$ and $\varphi_2$. The gluon propagators are proportional to

$$\left\langle \varphi_{\alpha j}^i \varphi_{\beta l}^k \right\rangle \propto \delta_{\alpha \beta} \delta_{i j} \delta_{k l} \frac{1}{N} \quad (7)$$

The non-Gaussian terms in $V_1(\varphi_1)$ and $V_2(\varphi_2)$ describe the interaction between the gluons.

The reason that we can solve this problem is because, while the two types of gluons have arbitrarily complicated interactions among themselves, they do not interact with each
other. Note that while the gluons both interact with the quark, our problem is such that we do not have to include quark loops and thus the quark does not induce interaction between the two gluons. This is clear from the definition of our problem. Another way of saying this is to note that the Green’s function may be represented, by using the replica trick, as

\[ G(z) = \lim_{n \to 0} \int D\psi D\varphi P(\varphi) \psi_1^\dagger \psi_1 e^{-\sum_{\alpha=1}^n \psi_\alpha^\dagger (z-\varphi) \psi_\alpha} \]  

(8)

Note that in this language the \( \psi \)'s represent the quark fields and \( \varphi \) the gluon fields. The interaction between gluon and quarks are given by \( \psi_\alpha^\dagger \varphi \psi_\alpha \). (Color indices are suppressed here.) The interaction of the gluons with each other is determined by \( P(\varphi) \). Since internal quark loops are proportional to the number of replicas \( n \), they vanish in the \( n \to 0 \) limit.

Let us then calculate the Green’s function, which as usual can be written as (see figure 2)

\[ G(z) = \frac{1}{z - \Sigma(z)} \]  

(9)

in terms of the one-particle irreducible self energy \( \Sigma^i_j(z) = \delta^i_j \Sigma(z) \). The self-energy is then determined by the set of diagrams in figure (3) with the corresponding equation

\[ \Sigma(z) = \frac{1}{N} tr \varphi_1 >_{gc} + \frac{1}{N} tr \varphi_1^2 >_{gc} G(z) + \frac{1}{N} tr \varphi_1^3 >_{gc} G(z)^2 + \ldots + (1 \leftrightarrow 2) \]

\[ = \sum_{k=1}^\infty \frac{1}{N} tr \varphi_1^k >_{gc} G(z)^{k-1} + (1 \leftrightarrow 2) \]

\[ = \frac{1}{G} [\frac{1}{N} tr \frac{1}{1-\varphi_1 G} >_{gc} - 1] + (1 \leftrightarrow 2) \]

\[ = \frac{1}{G} [\frac{1}{G} G_{gc1} \frac{1}{G} - 1] + (1 \leftrightarrow 2) \]  

(10)

In order to write this equation, we have to invoke the factorization of \( P(\varphi_1, \varphi_2) \), which tells us that the two kinds of gluons do not interact, and the large \( N \) limit, which tells us that the two kinds of gluon lines cannot cross.

We are led to introduce in (10) the notion of “gluon connectedness,” denoted by “gc” henceforth. The necessity for this notion is illustrated by the shaded blob describing the interaction of the gluons in figure (3d): it should not include the diagram shown in figure (4): this class of diagrams is already included in figure (3b). In other words, a gluon connected blob with \( k \) external gluon lines is such that it cannot be separated into two blobs, with \( k_1 \) gluon lines and \( k_2 \) gluon lines respectively, (with \( k_1 + k_2 = k \) of course). In the last line we have defined the “gluon connected Green’s function”

\[ G_{gc1}(z) = \frac{1}{N} tr \frac{1}{z - \varphi_1} >_{gc} \]  

(11)
and similarly $G_{gc2}(z)$. The operations implied in (11) are clearly allowed since $<\frac{1}{N} tr \cdot >_{gc}$ is a linear operation. Note also that we have not assumed that $V_\alpha$ is an even function of its argument. In particular, we include a possible tadpole term indicated by $<\frac{1}{N} tr \varphi_\alpha >_{gc}$ in (10).

We should emphasize that the shaded blobs include interactions between gluons to all orders. It is very complicated, if not hopeless, to calculate these blobs in terms of $V_1$ and $V_2$, but fortunately, as we will show below, we do not have to calculate them explicitly. In our previous papers, we regarded the cubic, quartic, and so on, terms in $V_\alpha$ as interactions and proceeded to calculate the Green’s function and correlation function in terms of the various coupling constants. We follow a different strategy here, and try to express the Green’s function $G(z)$ directly in terms of $G_1(z)$ and $G_2(z)$ where

$$G_\alpha(z) \equiv <\frac{1}{N} tr \frac{1}{z - \varphi_\alpha}>$$ (12)

for $\alpha = 1, 2$ are the Green’s functions for two separately and purely random problems. (The average in (12) is performed with the distribution $P_\alpha(\varphi_\alpha) = \frac{1}{Z_\alpha} e^{-NtrV_\alpha(\varphi_\alpha)}$ of course.) In this way, we attempt to bypass having to deal with $V_1$ and $V_2$ altogether.

To see how to do this, let us go back to the simpler problem defined by (1) and (2). Following the same diagrammatic analysis leading to (10) we find that the Green’s function $G(z)$ and self energy $\Sigma(z)$ for this simpler problem are related by

$$\Sigma(z) = \frac{1}{G} \left[ \frac{1}{G} G_{gc}(\frac{1}{G}) - 1 \right]$$ (13)

with, evidently,

$$G_{gc}(z) \equiv <\frac{1}{N} tr \frac{1}{z - \varphi}>_{gc}$$ (14)

Combining (14) and (9) we find

$$\frac{1}{G^2} G_{gc}(\frac{1}{G}) = z$$ (15)

It is convenient to define a “Blue’s function”

$$B(z) \equiv \frac{1}{z^2} G_{gc}(\frac{1}{z})$$ (16)

Thus, we learn that the Blue’s function is the functional inverse of the Green’s function

$$B(G(z)) = z$$ (17)

From the normalization of the probability distribution $P(\varphi)$ we obtain trivially the “sum rule” $G(z) \to \frac{1}{z}$ as $z \to \infty$, thus implying that the Blue’s function $B(z) \to \frac{1}{z}$ as $z \to 0$. 

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Let us now go back to the more involved problem defined by (4). First, we define for \( \alpha = 1, 2 \) two Blue’s functions \( B_\alpha \) as the functional inverse of \( G_\alpha \) respectively. We now see that (10), when combined with (9), says simply that

\[
 z + \frac{1}{G} = B_1(G) + B_2(G) \tag{18}
\]

Thus, the law of addition for the Blue’s function is given by

\[
 B_{1+2}(z) = B_1(z) + B_2(z) - \frac{1}{z} \tag{19}
\]

This equation tells us how to obtain the Blue’s function associated with \( \varphi_1 + \varphi_2 \) from the Blue’s functions associated with \( \varphi_1 \) and \( \varphi_2 \).

The procedure for determining the Green’s function and hence the density of eigenvalues of the problem defined by (1) and (2) is then as follows: given the Green’s functions \( G_1 \) and \( G_2 \), determine the corresponding Blue’s functions \( B_1 \) and \( B_2 \), calculate \( B_{1+2} \) according to (19), then determine the functional inverse of \( B_{1+2} \) to find the desired Green’s function \( G(z) \).

Let us remark briefly on the connection to the mathematical literature. Voiculescu\(^{15}\) has introduced the “\( R \)-transform.” It turns out that the \( R \) function discussed by mathematicians is simply related to \( B \) by

\[
 B(z) = \frac{1}{z} + R(z) \]

In fact, we see that the Dyson-Schwinger equation (9) when combined with (17) gives simple

\[
 B(G(z)) = \frac{1}{G(z)} + \Sigma(z) \tag{20}
\]

Thus, the \( R \) function of the mathematicians is nothing but the self-energy \( \Sigma \) of the physicists expressed in terms of different arguments:

\[
 R(G(z)) = \Sigma(z) \tag{20}
\]

Let us now proceed by building up from a few simple examples. In the most trivial case, \( \varphi \) is not random at all, but fixed to be a constant \( c \) times the unit matrix. Then from the Green’s function \( G(z) = \frac{1}{z-c} \) we find the Blue’s function \( B(z) = c + \frac{1}{z} \). For a slightly less trivial example, let \( \varphi \) be a diagonal matrix with matrix elements given by \( \epsilon_i \) with \( i = 1, ..., N \). Then the corresponding Blue’s function is determined by

\[
 \frac{1}{N} \sum_i \frac{1}{B(z) - \epsilon_i} = z \tag{21}
\]

Next, let \( P(\varphi) \) be Gaussian (that is, \( V(\varphi) = \text{tr} \frac{1}{2} \varphi^2 \)). Then as is well known (see for example Ref. 12), the Green’s function is determined by

\[
 z = G(z) + \frac{1}{G(z)} \tag{22}
\]
Substituting $z \to B$, we obtain immediately that

$$B(z) = z + \frac{1}{z}$$

(23)

Now we are ready to do our first non-trivial problem. Consider the problem defined in (3). Since we know from (21) and (23) the Blue’s functions corresponding to the two terms in the Hamiltonian, we learn immediately from (19) the Blue’s function for $H$:

$$B_{1+2}(z) = B_1(z) + z + \frac{1}{z} - \frac{1}{z} = B_1(z) + z$$

(24)

with $B_1$ determined by (21) with the substitution $B \to B_1$. The desired Green’s function $G(z)$ is now determined by solving for the functional inverse

$$B_{1+2}(G) = z$$

(25)

or equivalently, upon using (24)

$$B_1(G) = z - G$$

(26)

Let us now evaluate the two sides of this equation with the function $G_1(\cdot)$. We obtain immediately

$$G(z) = G_1(z - G(z))$$

(27)

precisely the classic result of Pastur which was obtained diagrammatically in Ref. 12.

After these simple exercises, we can now immediately go on and solve the general version of the problem defined in (3): find the density of energy levels of a Hamiltonian given by $H = H_0 + \varphi$ with $\varphi$ drawn from the general distribution (2). With a slight shift in notation, let us call the Blue’s function associated with $H_0$ and with $\varphi$ respectively $B_0$ and $B_2$. Then the Blue’s function associated with $H$ is given by

$$B(z) = B_0(z) + B_2(z) - \frac{1}{z}$$

Substituting $z \to G(z)$ (where $G(z)$ is the unknown Green’s function associated with $H$), we find immediately that

$$B_0(G) = z + \frac{1}{G} - B_2(G)$$

(28)

Anticipating the next step, we define

$$\Sigma(z) = B_2(G(z)) - \frac{1}{G(z)}$$

(29)

Let us now evaluate both sides of (28) with the Green’s function $G_0(\cdot)$ associated with $H_0$. We find instantly that

$$G(z) = G_0(z - \Sigma(z)) = \frac{1}{N} \sum_i \frac{1}{z - \epsilon_i - \Sigma(z)}$$

(30)
Let us repeat this trick: rewrite (29) as
\[ B_2(G(z)) = \Sigma(z) + \frac{1}{G(z)} \]
and evaluate both sides with the function \( G_2(\cdot) \). We obtain
\[ G(z) = \frac{1}{G(z)} \]
\[ G(z) = G_2(\Sigma(z) + \frac{1}{G(z)}) \] (31)

These two equations, (30) and (31), allow us to determine the two unknown functions \( G(z) \) and \( \Sigma(z) \), provided we know the Green's function \( G_2(z) \). But what is the Green's function \( G_2(z) \)? It is just the Green's function associated with the random matrix \( \varphi \) drawn from the general distribution (2). But this was obtained by Brézin et al\(^5\) almost twenty years ago. These authors told us that (for \( V(z) \) an even polynomial for the sake of notational simplicity)
\[ G_2(z) = \frac{1}{2}[V'(z) - P(z)\sqrt{z^2 - a^2}] \] (32)
Here \( V'(z) \equiv \frac{dV}{dz} \), \( P(z) \) is a polynomial, and \( a \) determines the endpoints of the spectrum of eigenvalues. The quantities \( P(z) \) and \( a \) are determined\(^26\) by the “sum rule” that \( G_2 \to \frac{1}{z} \) as \( z \to \infty \).

Thus, in summary, for any distribution defined by \( V(\varphi) \), we can determine the Green’s function \( G(z) \) and hence the density of eigenvalues by solving (30) and (31). It is clearly convenient to define \( \sigma(z) \equiv \Sigma(z) + \frac{1}{G(z)} \). We can then simplify (31) slightly to\(^27\)
\[ P^2(\sigma)(\sigma^2 - a^2) = (V'(\sigma) - 2G)^2 \] (33)
Thus, we can use (33) to determine \( \sigma \), and hence \( \Sigma \), in terms of \( G \). Plugging this into (30) then gives us an equation for \( G \).

As mentioned earlier, Brézin, Hikami, and Zee\(^13\) recently used the equation of motion method and a detailed diagrammatic analysis to determine the Green’s function for the problem in (3) with the distribution defined by \( V(\varphi) = \frac{1}{2}\varphi^2 + g\varphi^4 \). It is straightforward, although slightly tedious, to verify that for this simple case, (33) reduces to equation (4.15) in Ref. 13. The analysis given here is considerably simpler.

In practice, with an arbitrary set of \( \epsilon_i \), (30) can only be solved numerically. One relatively simple example involves taking half of the \( \epsilon_i \)’s to be equal to \( +\epsilon \) and the other half to be equal to \( -\epsilon \). Thus, before the introduction of the random term \( \varphi \) into the Hamiltonian, the spectrum consists of two levels, which may for example represent the two lowest Landau levels, corresponding to spin up and spin down, in a spin-dependent quantum Hall fluid.\(^28,29,30\). The randomness will then broaden the two levels.
Clearly, by repeating the discussion given here, we can add an arbitrary number of random Hamiltonians together. The procedure is defined by (19). The deterministic plus random Hamiltonian studied here is just a special case. It will be interesting to see if this work can be generalized to a study of the universal correlation function discussed in Ref. 6,12,13.

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FIGURE CAPTIONS

Fig 1. Feynman rules: (a) quark propagator, (b) gluon propagator, (c) quark gluon vertex, (d) gluon interaction, illustrated here with a $g\varphi^4$ vertex.

Fig 2. Quark propagator and one-particle-irreducible self energy.

Fig 3. Quark self energy: the gluons shown explicitly are all of type 1. There are of course also type 2 gluons inside the quark propagator $G$.

Fig 4. A class of diagrams not included in (3d).

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25. Using dimensional analysis we see immediately that for $V(\phi) = \frac{1}{2} m^2 \phi^2$ the Blue’s function $B(z) = \frac{z}{m^2} + \frac{1}{z}$. Then (24) implies the usual Gaussian law of addition $m_{1+2}^{-2} = m_1^{-2} + m_2^{-2}$.

26. For the sake of completeness, let us record (and for the sake of simplicity, with $V$ an even function) that for $V(\phi) = \sum_{k=1}^{p} \frac{1}{2k} g_k \phi^{2k}$ we have

$$P(z) = \frac{1}{2} \sum_{k=1}^{p} g_k \sum_{n=0}^{k-1} \frac{(2n)!}{(n!)^2} \left(\frac{a^2}{4}\right)^n \lambda^{2k-2n-2}$$

and

$$\frac{1}{2} \sum_{k=1}^{p} g_k \frac{(2k)!}{(k!)^2} \left(\frac{a^2}{4}\right)^k = 1$$

In particular, for $V = \frac{1}{2} \phi^2 + \frac{g}{4} \phi^4$ we have

$$G(z) = \frac{1}{2} \left[ z + g z^3 - (1 + \frac{a^2 g}{2} + g z^2) \sqrt{z^2 - a^2} \right]$$

and $a^2 = \frac{2g}{3g} (\sqrt{1 + 12g} - 1) = 4(1 - 3g + 18g^2 + ....)$. 

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27. It is perhaps worth noting that $B_2$ satisfies an equation similar to (33), namely

$$P^2(B_2)(B_2^2 - a^2) = (V'(B_2) - 2z)^2$$

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