The Analytical Solution for Consolidation of Vertical Drain with Vacuum Preloading Based on Equivalent Annular Drain

Dongyang Wan, Yuguo Zhang, Hanyue Yang, Mengmeng Duan, Weijie Zhang, Xiaojie Shi

School of Civil Engineering and Architecture, Zhongyuan University of Technology, Zhengzhou, Henan, 451191, China

*Corresponding author’s e-mail: 961439153@qq.com

Abstract. Based on the equivalent annular band-shaped drain, and considering the vacuum preloading and the variation of radial permeability coefficient of the smeared soil zone, the mathematical model is established. An analytic method is used to derive the general solution for the consolidation of vertical drain based on equivalent annular and arbitrary variation of load with time. Based on the general solutions, the detailed closed-form solutions are further derived for the cases of instantaneously loading and linear loading. Based on the closed solution, the calculation program is compiled and the consolidation curves of the vacuum preloading shaft foundation based on the annular equivalent are drawn. The results show that the equivalent annular consolidation rate is less than that given by the equivalent circumference. With the increase of the area reduction coefficient, the consolidation rate of the equivalent annular approaches to the consolidation rate of the equivalent circumference. With the decrease of the permeability coefficient of the vertical drain, the influence of the permeability coefficient on the consolidation rate is weakened. The dissipation rate of pore water under instantaneous load is greater than that of linear pore water dissipation.

1. Introduction
Vacuum preloading is an effective drainage consolidation method, and has broad prospects for development [1]. Therefore, a large number of scholars have conducted theoretical research on the consolidation of vertical drain under vacuum preloading. Tran et al. have established the equivalent plane strain model of vertical drain pipe on soft foundation, and given the analytical solution of consolidation based on the combined vacuum preloading of embankment [2]. Zhang Yiping and others established the space axial symmetry deformation model of vacuum preloading. By using displacement method and Hansbo’s sand well consolidation theory, the analytical solution of vacuum preloading consolidation of soft soil foundation was deduced [3]. Considering the influence of the original stress state of the soil, using the volumetric strain formula, the consolidation theory of the shaft foundation under vacuum preloading is analyzed by Fang Dechun and others[4], and the analytic solution of consolidation is obtained. In recent years, the vacuum preloading method has been widely applied in soft soil foundation treatment [5-10]. As a substitute for sand drains, plastic drainage boards have been greatly developed in accelerating consolidation of soft soil foundations [11-13]. Huang Chaoxuan and others assumed that the vertical drainage body was an elliptic cylinder, and the analytical solution of vertical drain consolidation was obtained [14] based on the elliptic cylinder coordinate system theory. Considering the clogging behavior of drainage plate, Chen Junsheng and others deduced the correction solution of sand well foundation consolidation under vacuum preloading,
and verified the correctness of consolidation theory by comparing with model test [15]. The above methods are all equivalent according to the circumference of drainage plate [16-18]. Lu Menmeng et al. put forward a new ring equivalent method. The circumference and cross-sectional area of the equivalent circular drainage are the same as those of the drainage plate, and the analytical solution of the consolidation of the drainage plate foundation under the ring equivalent is obtained [19]. Analytical solution of foundation consolidation based on annular equivalent preloading drainage plate has been studied by scholars, but the annular equivalent drainage plate foundation consolidation under vacuum preloading has not been studied.

Based on annular equivalent, the consolidation of plastic drainage slab foundation under vacuum preloading is studied analytically. And the analytical solution of the consolidation of plastic drainage slab foundation under vacuum preloading is deduced considering three modes of the variation of horizontal permeability coefficient of soil in the smeared area. Finally, the effect of annular equivalent is compared and analyzed.

2. Mathematical model

2.1 annular equivalent
The annular equivalence of plastic drainage boards is shown in Figure 1 [19].

![Fig.1 Band-shaped drain and the equivalent annular drain](image)

The drainage board is transformed into circular drainage body according to the principle of equal circumference, i.e.

\[ r_w = \frac{w + b}{\pi} \]  

where \( r_w \) is the external radius of circular drainage; \( w \) and \( b \) are the width and thickness of drainage boards respectively.

At the same time, it is required that the cross-sectional area of the annular drain body is equal to the cross section area of the drainage plate. The area reduction coefficient \( \zeta \) is introduced here, and the area of the annular drainage body is equal to the product of the area of the outer circle and the area reduction coefficient, i.e.

\[ wb = \zeta \pi r_w^2 \]  

The formula of area reduction coefficient can be obtained by substituting Eq. (1) into Eq. (2).

\[ \zeta = \frac{\pi \eta}{(1 + \eta)^2} \]
where $\eta$ is the ratio of thickness to width of drain plate, $\eta = b/w$.

2.2 Computing model
Fig. 2 is a diagram for calculating the vertical drain of radial consolidation. The top surface of the shaft foundation is completely permeable, and the bottom surface is completely impervious. Among them: the upper part is sand cushion; $H$ is the thickness of soft soil foundation. $r$ and $z$ are radial and vertical coordinates respectively; $r_w$ is the radius of shaft; $r_s$ is the radius of smearing area; $r_e$ is the radius of the shaft area; $k_w$ is the permeability coefficient of shaft; $k_r(r)$ is the permeability coefficient of the soil in the smearing area; $k_h$ is the horizontal permeability coefficient of undisturbed soil; $m_v$ is the volume compression coefficient; $u_w$ is the pore pressure at any point in the shaft; $u_r$ is the pore pressure at any point in the soil; $\gamma_w$ is the severity of water; $-P(t)$ is vacuum negative pressure.

2.3 Basic assumption
The basic assumptions are as follows: (1) The soil mass is completely saturated; (2) Soil particles and water are incompressible, and the deformation of the soil is entirely caused by the discharge of pore water; (3) The use of plastic drainage board is considered radial seepage, and the seepage of water in the soil obeys Darcy law; (4) The condition of equal strain is established; (5) By introducing the area reduction factor, the continuous assumption of well circumferential flow is revised.

\[
\left[2\pi r \frac{k_r(r)}{\gamma_w} \frac{\partial u_r}{\partial r}\right]_{r_e} = \frac{\zeta \pi r^2}{\gamma_w} \frac{\partial^2 u_e}{\partial z^2}
\]  

(4)

(6)Vacuum degree remains unchanged along the depth. (7)Radial permeability coefficients of the smeared area exhibit three variation patterns, i.e. constant (pattern I), linear (pattern II) and parabolic (pattern III).

3. Governing equations and their boundary conditions
Only radial seepage is considered, the basic equation of radial seepage consolidation can be obtained.

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( k_r(r) \frac{\partial u_r}{\partial r}\right) = \frac{\partial e}{\partial t}
\]  

(5)

where $k_r(r) = k_h f(r)$ is the radial permeability coefficient of the soil, $f(r)$ is a function of the radial distance away from the shaft, depending on the variation pattern of the horizontal permeability coefficient of the surrounding soil.
Based on equal strain conditions
\[
\frac{\partial \varepsilon_v}{\partial t} = -m_r \frac{\partial u_r}{\partial t}
\]  
(6)

Average pore pressure
\[
\bar{u}_r = \frac{1}{\pi (r_c^2 - r_w^2)} \int_{r_w}^{r_c} 2\pi r u_r dr
\]  
(7)
where \( \varepsilon_v \) is the volume strain at any point in the affected zone (equal to the vertical strain); \( u_r \) is the average pore pressure at any depth in the affected zone.

Radial boundary condition
\[
\begin{align*}
    r &= r_v, & \frac{\partial u_r}{\partial r} &= 0; \\
    r &= r_w, & u_r &= u_w.  
\end{align*}
\]  
(8) (9)

Using boundary conditions (8), we can integrate on both sides of Eq. (5)
\[
\frac{\partial u_r}{\partial r} = \frac{\gamma_r}{2k_h} \frac{\partial \varepsilon_v}{\partial t} \left( \frac{r_v^2}{f(r)} - \frac{r}{f(r)} \right)
\]  
(10)

The on both sides of the Eq. (10) is again integrated by the boundary condition (9)
\[
u_r = \frac{\gamma_r}{2k_h} \frac{\partial \varepsilon_v}{\partial t} (r_v^2 B_0 - C_0) + u_w
\]  
(11)
where
\[
B_h = \int_{r_v}^{r_w} \frac{1}{f(\xi)} d\xi, \quad C_h = \int_{r_v}^{r_w} \frac{\xi}{f(\xi)} d\xi
\]  
(12)

The Eq. (11) is substituted by Eq. (7)
\[
\bar{u}_r = \frac{\gamma_r r_v^2 F}{2k_h} \frac{\partial \varepsilon_v}{\partial t} + u_w
\]  
(13)
where
\[
F = \frac{2(c_r^2 B_0 - C_1)}{r_v^2 (r_c^2 - r_v^2)}
\]  
(14)
\[
B_1 = \int_{r_v}^{r_w} r B_0 dr, \quad C_1 = \int_{r_v}^{r_w} r C_0 dr
\]  
(15)

\( F \) corresponds to the three models of the horizontal permeability coefficient of soil in the smearing area. The specific expression is [20].

It can be obtained by Eq. (8) and \( r = r_w \)
\[
\left. \frac{\partial u_r}{\partial r} \right|_{r=r_w} = \frac{\gamma_r r_w}{2k_h} (n^2 - 1) \frac{\partial \varepsilon_v}{\partial t}
\]  
(16)
where \( n = r_v / r_w \).

The Eq. (16) is substituted by Eq. (4)
\[ \frac{\partial^2 u_w}{\partial z^2} = -\frac{\gamma_w}{\xi k_w} \left( n^2 - 1 \right) \frac{\partial e_w}{\partial t} \]  

(17)

By Eq. (6), Eq. (13) and Eq. (17) are obtained.

\[ \frac{\partial \bar{u}_t}{\partial t} = -\frac{2k_w E_e}{\gamma_o F_e} (\bar{u}_t - u_w) \]  

(18)

\[ \frac{\partial^2 u_w}{\partial z^2} = -\frac{8k_h \left( n^2 - 1 \right)}{\xi k_u d^2 F} (u_t - u_w) \]  

(19)

where \( E_s \) is the modulus of compressibility of soil.

The governing equations of \( u_w \) are obtained by means of Eq. (18) and Eq. (19)

\[ \frac{\partial^2 u_w}{\partial z^2} + \frac{2k_w E_e}{\gamma_w} \frac{\partial^2 u_w}{\partial z^2} - \frac{8k_h \left( n^2 - 1 \right)}{\xi k_u d^2 F} \frac{\partial u_w}{\partial t} = 0 \]  

(20)

Vertical boundary condition

\[ z = 0 \, , \, u_w = \bar{u}_t = -P(t) \]  

(21)

\[ z = H \, , \, \frac{\partial u_w}{\partial z} = \frac{\partial \bar{u}_t}{\partial z} = 0 \]  

(22)

At this point, the basic equations and boundary conditions for equal strain radial consolidation have been given.

4. Equation solution

Set the solution of equation (20) to be

\[ u_w = -P(t) + \sum_{m=0}^{\infty} T(t) \sin \left( \frac{M}{H} z \right) \]  

(23)

where \( M = \frac{(2m+1)\pi}{2}, \quad m = 0,1,2,3, \cdots \).

The Eq. (23) has already satisfied the boundary condition Eq. (21) and Eq. (22), and the average pore pressure is

\[ \bar{u}_t = -P(t) + \sum_{m=0}^{\infty} \left( 1 + \frac{\xi k_u d^2 F}{8k_h (n^2 - 1)} \frac{M^2}{H^2} \right) T(t) \sin \left( \frac{M}{H} z \right) \]  

(24)

The initial condition is

\[ t = 0 \, , \, \bar{u}_t = 0 \, . \]  

(25)

By Eq. (24) and Eq. (25), there are

\[ -P(0) + \sum_{m=0}^{\infty} \left( 1 + \frac{\xi k_u d^2 F}{8k_h (n^2 - 1)} \frac{M^2}{H^2} \right) T(0) \sin \left( \frac{M}{H} z \right) = 0 \]  

(26)

The two sides of Eq. (26) are multiplied by \( \sin \left( \frac{M}{H} z \right) \) and integral to \([0, H]\) on \( z \), and the orthogonality of trigonometric series can be obtained
\[
T_w(0) = \frac{2}{M} P(0) \frac{1}{1 + \frac{\xi k_s d_F^2 F M^2}{8k_h (n^2 - 1) H^2}}
\]

(27)

The Eq. (23) is substituted by Eq. (20), and it can be obtained after arranging
\[
\frac{\xi k_s d_F^2 F M^2}{8k_h (n^2 - 1) H^2} \sum_{n=0}^{\infty} T(t) \sin \left( \frac{M}{H} z \right) + \frac{\xi k_s E_s M^2}{Y_s (n^2 - 1) H^2} \sum_{n=0}^{\infty} T(t) \sin \left( \frac{M}{H} z \right) - \sum_{n=0}^{\infty} T'(t) \sin \left( \frac{M}{H} z \right) = -\frac{\partial P(t)}{\partial t}
\]

(28)

Similarly, both sides of Eq. (28) are multiplied by \( \sin \left( \frac{M}{H} z \right) \) and integral to \([0, H]\) on \( z \), and the orthogonality of trigonometric functions series can be obtained
\[
T_w'(t) + \beta T_w(t) = \frac{2}{M} \frac{\partial P(t)}{\partial t} \frac{1}{1 + \frac{\xi k_s d_F^2 F M^2}{8k_h (n^2 - 1) H^2}}
\]

(29)

where
\[
\beta = \frac{8c_h}{1 + \frac{8k_h (n^2 - 1) H^2}{\xi k_s d_F^2 F M^2}}
\]

(30)

The Eq. (28) is the first order linear differential equation of \( T_w(t) \), and the Eq. (27) is its definite solution condition. The general solution is
\[
T_w(t) = e^{-\beta t} \left[ \frac{2}{M} \frac{P(0)}{1 + \frac{\xi k_s d_F^2 F M^2}{8k_h (n^2 - 1) H^2}} + \int_{0}^{t} \frac{\partial P(\tau)}{\partial \tau} e^{\beta \tau} d\tau \right]
\]

(31)

The average pore pressure of soil can be obtained by substituting Eq. (31) into Eq. (24)
\[
\bar{P}_u = -P(t) + \sum_{n=0}^{\infty} \frac{2}{M} \left[ \frac{P(0)}{1 + \frac{\xi k_s d_F^2 F M^2}{8k_h (n^2 - 1) H^2}} + \int_{0}^{t} \frac{\partial P(\tau)}{\partial \tau} e^{\beta \tau} d\tau \right] \sin \left( \frac{M}{H} z \right)
\]

(32)

The total average consolidation degree of the foundation is
\[
U = \frac{1}{H} \int_{0}^{H} \bar{u}_d dz = \frac{P(t)}{P_u} \sum_{n=0}^{\infty} \frac{2}{M} \left[ \frac{P(0)}{1 + \frac{\xi k_s d_F^2 F M^2}{8k_h (n^2 - 1) H^2}} + \int_{0}^{t} \frac{\partial P(\tau)}{\partial \tau} e^{\beta \tau} d\tau \right] e^{-\beta t}
\]

(33)

where \( P_u \) is the maximum vacuum degree.

5. Special solution

Figure 3 is the two common vacuum preloading distribution map. The above general solution is used to give a detailed solution.
6. Analysis of consolidation behavior

Based on the solution given, the influencing factors of foundation consolidation are calculated and analyzed for a kind of plastic drainage board with a width of 100 mm and a thickness of 4 mm, and the results are compared with the perimeter equivalence.
Fig. 4 is three models next week looks and so on, but the length and width of the different drainage boards using perimeter equivalent and annular equivalent of the linear vacuum load consolidation degree comparison. The equivalent radius is calculated according to Eq. (1), but the girth is equivalent to the area reduction. The annular equivalent will reduce the area according to the thickness-width ratio. If the aspect ratio is smaller, the reduction factor is smaller. According to Fig. 4, it can be seen that the degree of consolidation considering area reduction is less than that without area reduction in the three modes. With the increase of area reduction coefficient, the annular equivalent degree of consolidation approaches the circumference equivalent degree of consolidation.

$(a)$

$(b)$
Fig. 4 Influence of area reduction factor on consolidation degree curve (linear load)

Figure 5 is a comparison of the consolidation degree of the soil in the three smear areas when the drainage plate is subjected to different permeability coefficients under transient vacuum loading. It can be seen from the graph that the consolidation rate is the largest when the horizontal permeability coefficient of soil changes parabola, the second is linear change, and the smallest when the constant remains unchanged. With the decrease of permeability coefficient of drainage plate, the difference of consolidation rates calculated by different models is decreased.

Fig. 5 Comparison between the results of soil consolidation with three variation patterns of horizontal permeability of smeared soil (linear load)

Figure 6 is a comparison of the consolidation degree of the drainage plate's permeability coefficient when the instantaneous vacuum load changes the horizontal permeability coefficient of the soil in the smear area. It can be seen that with the smaller and smaller permeability coefficient of the drainage plate, the consolidation rate of the foundation becomes slower and slower.
Fig. 6 Influence of permeability coefficient of band-shaped drain on degree of consolidation (pattern III, instantly loading)

Fig. 7 shows the time-dependent dissipation rate of pore water under instantaneous and linear vacuum loads with different annular equivalent area reduction coefficients. It can be seen that the dissipation rate of pore water under the linear load is slower than that under instantaneous load. With the increase of reduction coefficient of annular equivalent area, the dissipation rate of pore water in foundation becomes larger and larger.

**7. Conclusion**

Based on the annular equivalent method of drainage plate, the analytical solution of consolidation of vertical drain is derived, and the results are compared and analyzed. The following conclusions are drawn:

(1) The degree of consolidation obtained by considering circumference equivalence is greater than that obtained by ring equivalent. With the increase of the area reduction factor, the equivalent consolidation rate of the ring approaches the equivalent consolidation rate of girth.

(2) It is assumed that the consolidation rate of the shaft foundation is the highest when the horizontal permeability coefficient of the soil in the smeared area varies in a parabolic way, followed
by a linear one, and is the smallest when the constant remains unchanged. With the decrease of permeability coefficient of drainage plate, the difference of consolidation rates calculated by the three models also decreases. If the permeability coefficient of drainage plate is smaller, the consolidation rate of shaft foundation is slower.

(3) The dissipation rate of pore water under linear load is slower than that under instantaneous load. With the increase of reduction coefficient of annular equivalent area, the dissipation rate of pore water in foundation becomes larger and larger.

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Reference
[1] Zhang YJ, Zhang YJ. (2000)Current Situation and Tendency of Improving Soft Foundation by Vacuum Preloading. WORLD GEOLOGY, 4: 375-378.
[2] Tuan A T, Toshiyuki M. (2008)Equivalent plane strain modeling of vertical drains in soft ground under embankment combined with vacuum preloading. Computers and Geotechnics, 35: 655-672.
[3] Zhang YP, Yan L, Yu YN, et al. (2011)Solutions for settlement and consolidation of soft ground with vacuum preloading. Rock and Soil Mechanics, 32: 149-154.
[4] Fang DC, Li YJ, Guo B. (2015)Consolidation analysis of vertical drain under vacuum preloading considering the influence of original stress state of soil. Journal of Highway and Transportation Research and Development, 11: 59-61.
[5] Han WJ, Liu SY, Zhang DW, et al. (2013) Nonlinear consolidation of subsoil by vacuum preloading based on bi-logarithmic coordinate compression model. Journal of Southeast University (Natural Science Edition), 43: 967-972.
[6] Bao SF, Zhou Q, Chen PS, et al. (2015) Consolidation analysis for sand drains foundations with non-uniform distribution of negative pressure boundary condition. Port & Waterway Engineering, 3: 12-20.
[7] Li FF, Xie KH, Deng YB. (2015) Analytical solution for consolidation by vertical drains with exponential flow under vacuum preloading. Journal of Southeast University (Natural Science Edition), 46: 1075-1081.
[8] Cao YP, Ding JW, Ma ZH, et al. (2016) Axisymmetric large-strain consolidation model for dredged clays with high water content under vacuum preloading. Journal of Southeast University (Natural Science Edition), 46: 860-865.
[9] Dong ZL, Chen PS, Mo HH, et al. (2010) Effects of permeability coefficients on consolidation of soft clay under vacuum preloading. Rock and Soil Mechanics, 31: 1452-1456.
[10] Zhou WH, Thomas ML, Zhao LS, et al. (2017)Analytical solutions to the axisymmetric consolidation of a multi-layer soil system under surcharge combined with vacuum preloading. Geotextiles and Geomembranes, 45: 487-498.
[11] Gao HQ, Mo HH. (2016)Theoretical study of reinforced depth of plastic vertical drain considering the water flux. Chinese Journal of Rock Mechanics and Engineering, 35: 3337-3342.
[12] Rohan W, Buddhima I, F.ASCE. (2006) Vertical drain consolidation with parabolic distribution of permeability in smear zone. Journal of Geotechnical and Geoenvironmental Engineering, 7: 937-941.
[13] P.V. Long, L.V. Nguyen, D.T. Bergado, et al. (2015) Performance of PVD improved soft ground using vacuum consolidation methods with and without airtight membrane. Geotextiles and Geomembranes, 43: 473-483.
[14] Huang CX, Deng YB, Hu GJ. (2017) Consolidation theory for prefabricated vertical drain assuming elliptic cross section and nonlinear well resistance. Chinese Journal of Rock Mechanics and Engineering, 36: 725-735.

[15] Chen JS, Mo HH, Li JH. (2017) Amendments consolidation theory of vertical sand drains considering drainage board clogging in vacuum preloading. Port & Waterway Engineering, 3: 150-156+168.

[16] Huang CX, Wang ZZ, Fang YL. (2017) Analytical solution of vacuum preloading foundation considering air leakage and nonlinear well resistance. Rock and Soil Mechanics, 38: 2574-2582.

[17] Gao GY, Nie CX, Zhang HQ, et al. (2017) Radial consolidation solution of plastic wick drain combined vacuum preloading. Journal of Tongji University (Natural Science), 45: 1290-1297.

[18] Jiang JA, Chen HY, Chen Y, et al. (2016) Analytical solutions to drainage consolidation considering vacuum loss in prefabricated vertical drain. Chinese Journal of Geotechnical Engineering, 38: 404-418.

[19] Lu MM, Zhang Q, Jing HW, et al. (2018) Consolidation of band-shaped drain based on equivalent annular drain. Chinese Journal of Rock Mechanics and Engineering, 37: 513-520.

[20] Xie KH, Lu MM, Liu GB. (2009) Equal strain consolidation for stone columns reinforced foundation. International Journal for Numerical and Analytical Methods in Geomechanics, 33: 1721-1735.