SURFACES WITH PRYM-CANONICAL HYPERPLANE SECTIONS

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Abstract. In this paper, we will explicit some general properties regarding surfaces with Prym-canonical hyperplane sections and the geometric genus of their possible singularities. Moreover, we will construct new examples of this type of surfaces.

1. Introduction

Let \( g \geq 3 \). It is well-known that a Prym curve is a pair \((C, \alpha)\), where \( C \) is a smooth genus \( g = p_g(C) \) curve and \( \alpha \) is a non-zero 2\(-\)torsion point of \( \text{Pic}^0(C) \). In the following, we will consider the so called Prym-canonical map, that is the rational map

\[
\phi_{[\omega_C(\alpha)]} : C \dashrightarrow \mathbb{P}^{g-2}
\]

defined by \( [\omega_C(\alpha)] \). In general, the pair \((C, \omega_C(\alpha))\) is called Prym-canonical curve. The complete linear system \([\omega_C(\alpha)]\) is base point free unless \( C \) is hyperelliptic and \( \alpha \simeq \mathcal{O}_C(p - q) \), with \( p \) and \( q \) ramification points of the \( g_2^1 \). Moreover, it defines an embedding if and only if \( C \) does not have a \( g_4^1 \) such that \( \alpha \sim \mathcal{O}_C(a + b - x - y) \), where \( 2(a + b) \) and \( 2(x + y) \) are members of the \( g_4^1 \) (see [3], Lemma 2.1). If \( \phi_{[\omega_C(\alpha)]} \) is an embedding, we say that \( C \simeq \phi(C) \subset P^{g-2} \) is a Prym-canonical (embedded) curve. If \( g < 5 \), then the Prym-canonical map cannot be an embedding, as observed in [3], so we will work only with \( g \geq 5 \).

We say that a surface \( X \) has Prym-canonical hyperplane sections if it can be birationally realized in some projective space \( \mathbb{P}^{g-1} \), for \( g \geq 5 \), such that a general hyperplane section \( C \) of \( X \) is a smooth Prym-canonical (embedded) curve of genus \( g \).

In this paper we will analyze the complex projective surfaces with Prym-canonical hyperplane sections up to birational equivalence. In particular, Section 2 is devoted to studying the first properties regarding surfaces with Prym-canonical hyperplane sections. We will show that these surfaces can be birationally equivalent to ruled surfaces or to \( \mathbb{P}^2 \) or to Enriques surfaces. In any case, there is only one effective anticanonical divisor \( W' \) on \( X' \), the minimal resolution of the singularities of \( X \), and, if \( \pi : X' \rightarrow X \), then the anticanonical divisor of \( X' \) is contracted by \( \pi \) and every singularity \( x \in X \) such that \( \pi^{-1}(x) \) does not meet \( \text{supp}(W') \) is a rational double point. We will also show that surfaces with Prym-canonical hyperplane sections birationally equivalent to non-rational ruled surfaces over a base curve of genus \( q > 0 \) have non rational singularities on them such that the sum of the geometric genus of their singularities is \( q \).

At the best of our knowledge, the only known examples of surfaces with Prym-canonical hyperplane sections are the Enriques surfaces and a surface in \( \mathbb{P}^5 \) of degree 10 obtained as image of the blowing up of \( \mathbb{P}^2 \) in the 10 nodes of an irreducible rational plane curve of degree 6. In Section 3 we will construct new examples of these surfaces. In particular, we will construct four new examples, one birationally equivalent to an elliptic ruled surface, another birationally equivalent to a ruled...
surface over a base curve of genus $q \geq 3$, again another one birationally equivalent to a rational ruled surface and finally, we will construct a new example of surface with Prym-canonical hyperplane sections birationally equivalent to $\mathbb{P}^2$.

Acknowledgements. The results of this paper are contained in my PhD-thesis. I would like to express my deepest gratitude to my three advisors, Ciro Ciliberto, Concettina Galati and Andreas Leopold Knutsen, for their useful and indispensable advice and I would also like to acknowledge PhD-funding from the Department of Mathematics and Computer Science of the University of Calabria and funding from Research project “Families of curves: their moduli and their related varieties” (CUP E81—18000100005, P.I. Flaminio Flamini) in the framework of Mission Sustainability 2017 - Tor Vergata University of Rome.

2. Preliminary results

We recall that a surface $X \subseteq \mathbb{P}^{g-1}$ is a surface with Prym canonical hyperplane sections if its general hyperplane section $C$ is a Prym canonical embedded curve. We start with the following remarks.

Remarks 2.1.

- The generic hyperplane section of $X$ is irreducible and smooth, whence $X$ has at most isolated singularities.
- Let $\pi : X' \rightarrow X$ be the minimal resolution of singularities of $X$ and let $C' = \pi^*C$ be the inverse image of a general hyperplane section. For a general $C' \in |\pi^*C|$, we have that $C' \cong C$ and $\mathcal{O}_{C'}(C') \cong \mathcal{O}_C(1) \cong \omega_{C'}(\alpha)$, with $\alpha$ a non trivial two-torsion element of $\text{Pic}^0(C')$. By the adjunction formula and because $X'$ is smooth, we can say that $\alpha = -K_{X'}|_{C'}$, in particular $K_{X'} \cdot C' = 0$.

From now on, we will assume that $C$ is projectively normal with respect to its embedding in $\mathbb{P}^{g-2}$. For example, if $\phi_{\omega_C^1} : C \hookrightarrow \mathbb{P}^{g-2}$ is a Prym-canonical embedding and the Clifford index $\text{Cliff}(C) \geq 3$, then $C$ is projectively normal with respect to the given embedding (see Theorem 1, [10]). In general, there are also projectively normal curves $C \subseteq \mathbb{P}^{g-2}$ with $\text{Cliff}(C) < 3$.

We state some general properties regarding surfaces with Prym-canonical hyperplane sections.

Theorem 2.1. Let $X$ be a surface with Prym-canonical hyperplane section $C$ of genus $g \geq 5$ and let $\pi : X' \rightarrow X$ be the minimal resolution of its singularities. If $C$ is projectively normal with respect to its embedding in $\mathbb{P}^{g-2}$, then:

- $h^1(\mathcal{O}_X(n)) = 0$ and $h^2(\mathcal{O}_X(n)) = 0$ for any $n \geq 0$, in particular $h^1(\mathcal{O}_X) = 0$ and $h^2(\mathcal{O}_X) = 0$, whence $p_a(X) = 0$;
- $X$ is projectively normal;
- the Kodaira dimension $\kappa(X')$ equals to $-\infty$ or $0$;
- $\deg(X) = 2g - 2$.

Proof. By assumption, $C$ is a projectively normal curve in $\mathbb{P}^{g-2}$, thus

$$H^0(\mathcal{O}_{\mathbb{P}^{g-1}}(n)) \rightarrow H^0(\mathcal{O}_C(n)),$$

for any $n \geq 0$. As a consequence the map $H^0(\mathcal{O}_X(n)) \rightarrow H^0(\mathcal{O}_C(n))$ is surjective for any $n \geq 0$.

Let us consider the exact sequence

$$0 \rightarrow \mathcal{O}_X(n-1) \rightarrow \mathcal{O}_X(n) \rightarrow \mathcal{O}_C(n) \rightarrow 0.$$
Then, the second part of the long exact sequence associated with (1) is
\[ 0 \to H^1(\mathcal{O}_X(n-1)) \to H^1(\mathcal{O}_X(n)) \to H^1(\mathcal{O}_C(n)) \to \]
\[ \to H^2(\mathcal{O}_X(n-1)) \to H^2(\mathcal{O}_X(n)) \to 0. \]

By Serre’s Theorem, there is a sufficiently large \( n_0 \) such that \( h^1(\mathcal{O}_X(n)) = 0 \), for any \( n \geq n_0 \). From the exact sequence (2) and applying descending induction on \( n \), we obtain that \( h^1(\mathcal{O}_X(n)) = 0 \), for any \( n \geq 0 \).

It is clear that \( H^1(\mathcal{O}_C(n)) = H^1(\mathcal{O}_C(n(K_C + \alpha))) \). For \( n = 1 \), we have that \( h^1(\mathcal{O}_C(K_C + \alpha)) = 0 \). Moreover, because \( \deg(K_C + \alpha) = 2g - 2 \), then \( \deg(n(K_C + \alpha)) > 2g - 2 \) for \( n \geq 2 \), so \( h^1(\mathcal{O}_C(n(K_C + \alpha))) = 0 \) (see [11], Example IV.1.3.4). Again by Serre’s Theorem and applying descending induction on \( n \), from the long exact sequence (2) we can conclude that \( h^2(\mathcal{O}_X(n)) = 0 \), for any \( n \geq 0 \).

To prove that \( X \) is projectively normal, it is enough to show that \( X \) is normal and that the map \( H^0(\mathcal{O}_{\mathbb{P}^n-1}(n)) \to H^0(\mathcal{O}_X(n)) \) is surjective for any \( n \geq 0 \).

Let \( \eta : \tilde{X} \to X \) be the normalization of \( X \). We consider the following exact sequence on \( X \):
\[ 0 \to \mathcal{O}_X \to \eta_*\mathcal{O}_{\tilde{X}} \to F \to 0, \]
where \( \text{supp}(F) \subset \text{Sing}(X) \). Since \( X \) has isolated singularities, then \( F \cong H^0(F) \cong \oplus_{i=1}^s (\mathcal{O}_i/\mathcal{O}_i) \), where \( \mathcal{O}_i \) is the local ring of \( x_i \) on \( X \) and \((\eta_*\mathcal{O}_{\tilde{X}})_{x_i} = \mathcal{O}_i \) is the normalization of \( \mathcal{O}_i \) in the function field of \( X \), with \( \text{Sing}(\tilde{X}) = \{ x_1, ..., x_s \} \).

We know that \( H^0(\mathcal{O}_X) = k \) because \( X \) is irreducible. By the properties of pushforward and because \( \tilde{X} \) is still irreducible, it is obvious that \( H^0(\eta_*\mathcal{O}_{\tilde{X}}) \cong H^0(\mathcal{O}_{\tilde{X}}) \cong k \). Moreover \( h^1(\mathcal{O}_X) = 0 \) by the previous part of this Proposition. For the long exact sequence associated with (3), we have that \( h^0(F) = 0 \). By definition of \( F \), it is true that \( \mathcal{O}_i \cong \mathcal{O}_i \), for any \( i = 1, ..., s \). We conclude that \( X \) is normal.

The surjectivity of \( H^0(\mathcal{O}_{\mathbb{P}^n-1}(n)) \to H^0(\mathcal{O}_X(n)) \) is trivial for \( n = 0 \). Let us consider the following diagram, where \( H \) is a general hyperplane in \( \mathbb{P}^{n-1} \) and \( C = X \cap H \):
\[
\begin{array}{ccc}
0 & \longrightarrow & H^0(\mathcal{O}_{\mathbb{P}^n-1}(n-1)) \\
\downarrow r_1 & & \downarrow r_2 \\
0 & \longrightarrow & H^0(\mathcal{O}_X(n-1)) \\
\downarrow r_3 & & \downarrow r_4 \\
0 & \longrightarrow & H^0(\mathcal{O}_X(n)) & \longrightarrow & H^0(\mathcal{O}_C(n)) \\
\end{array}
\]

We observe that \( r_3 \) is surjective because \( C \) is projectively normal and \( r_1 \) is surjective by the inductive hypothesis. Then \( r_2 \) is also surjective and the claim is proved.

Consider the following exact sequence:
\[ 0 \to \mathcal{O}_X(-C' + mK_X') \to \mathcal{O}_X(mK_X') \to \mathcal{O}_C(mK_X') \to 0. \]

Since \( -K_X/C' \sim \alpha \), for \( \alpha \) a non-zero two torsion element of \( C' \), then we have that \( (-C' + mK_X') \cdot C' = -\alpha^2 = 2 - 2g < 0 \). Whence \( h^0(\mathcal{O}_X(-C' + mK_X')) = 0 \) otherwise, if this divisor was effective, it would be a fixed component of \( |C'| \) that is a linear system without base locus by definition. At the same time
\[ h^0(\mathcal{O}_C(mK_X')) = \begin{cases} 0 & \text{if } m \text{ odd} \\ 1 & \text{if } m \text{ even.} \end{cases} \]

Consequently the plurigenus \( P_m(X') := h^0(\mathcal{O}_X'(mK_X')) \leq h^0(\mathcal{O}_C'(mK_{X'})) \leq 1 \). Then the Kodaira dimension \( \kappa(X') = -\infty \) or 0.

- We have \( \deg(X) = \deg(C) = C^2 = \deg(K_C + \alpha) = 2g - 2 \).

\[ \square \]

Remarks 2.2.

1. Since the Kodaira dimension is a birational invariant for smooth varieties, if \( \kappa(X') = -\infty \), then the minimal model \( X'' \) of \( X' \) is a ruled surface or \( \mathbb{P}^2 \) (see [11], Theorem V.6.1).

2. If \( \kappa(X') = 0 \), then \( X' \) is a minimal Enriques surface. Let us show this.

Let us suppose that \( X' \) is not minimal. Then there is a \((-1\text{)}\)-curve \( E' \) on \( X' \). Because \( \kappa(X') = 0 \), let \( m > 0 \) be such that \( |mK_{X'}| \) contains only one effective divisor \( D' \). It is obvious that \( E' \) is a component of \( D' \). Now \( \mathcal{O}_C(D') \cong \mathcal{O}_C'(mK_{X'}) \cong \mathcal{O}_{C'} \) because \( -K_{X'}|_{C'} \) is a non-zero two torsion element and \( m \) is even as seen in the previous Proposition. Therefore \( D' \) and consequently \( E' \) are contracted to a point on \( X \) by \( \pi \), contradicting the minimality of the resolution \( \pi \).

It is true that \( 12K_{X'} \sim 0 \) because \( X' \) is minimal and \( \kappa(X') = 0 \) (see [11], Theorem V.6.3). By the classification of minimal surfaces, there is a smallest \( m \geq 1 \) such that \( mK_{X'} \sim 0 \) and the possibilities are \( m = \{1, 2, 3, 4, 6\} \) (see [6] and [4]). If \( m = 1 \), then \( K_{X'} \sim 0 \), whence \( \mathcal{O}_C(C') \cong \mathcal{O}_C'(K_{C'} - K_{X'}) \cong \mathcal{O}_{C'}(K_{C'}) \). This is not possible because \( C' \) is a Prym-canonical curve. So we exclude the cases in which \( X' \) is a K3 surface or an abelian surface.

If \( X' \) was a hyperelliptic surface, it would not contain curves with negative self-intersection, then we would have \( X = X' \) smooth by Mumford’s Theorem (see [14], Chapter 1). By definition, a hyperelliptic surface is irregular, contradicting the first point of Proposition 2.1. In conclusion \( X' \) is a minimal Enriques surface.

We want to determine the possible singularities on \( X \) surface with Prym-canonical hyperplane sections. The following Proposition determines the geometric genus of the singularities that occur on \( X \).

**Proposition 2.3.** With the same assumptions as before, if \( \text{Sing}(X) = \{x_1, ..., x_s\} \) is the locus of the singular points of \( X \), then:

- if \( X \) is birationally equivalent to an Enriques surface or \( \mathbb{P}^2 \), then \( X \) can only contain rational points as singularities;
- if \( X \) is birationally equivalent to a ruled surface \( X'' \) over a base curve of genus \( q \geq 0 \), then \( \sum_{i=1}^{s} p_g(x_i) = q \), where \( p_g(x_i) \) is the geometric genus of the singular point \( x_i \).

**Proof.** These results can be obtained using the following exact sequence, which one gets from the Leray spectral sequence for the sheaf \( \mathcal{O}_{X'} \) and the morphism \( \pi \) (see [9], pag. 462):

\[ 0 \rightarrow H^1(\mathcal{O}_X) \rightarrow H^1(\mathcal{O}_{X'}) \rightarrow H^0(R^1\pi_*\mathcal{O}_X) \rightarrow H^2(\mathcal{O}_X) \rightarrow ... \]

We know that \( h^1(\mathcal{O}_X) = h^2(\mathcal{O}_X) = 0 \) by Theorem 2.1 so

\[ \sum_{i=1}^{s} p_g(x_i) = h^0(R^1\pi_*\mathcal{O}_{X'}) = h^1(\mathcal{O}_{X'}), \]

where \( p_g(x_i) \) is the geometric genus of the singular point \( x_i \).
• If $X$ is birationally equivalent to an Enriques surface, then $X'$ is a minimal Enriques surface by Remark 2.2. So $h^1(\mathcal{O}_{X'}) = 0$ by definition and $\sum_{i=1}^s p_g(x_i) = 0$, whence $X$ can only contain rational singularities.

If $X$ is birationally equivalent to $\mathbb{P}^2$, then it is clear that $h^1(\mathcal{O}_{X'}) = 0$ and $\sum_{i=1}^s p_g(x_i) = 0$. This proves the first part of this Proposition.

• If $X$ is birationally equivalent to a ruled surface $X''$ over a base curve of genus $g \geq 0$, then $h^1(\mathcal{O}_{X''}) = q(X'') = q$. Consequently $q(X') = q(X'') = q$.

So $\sum_{i=1}^s p_g(x_i) = h^1(\mathcal{O}_{X'}) = q$.

\[ \square \]

The following results give other information regarding the singularities that occur on $X$. First of all, we sketch the proof of a preliminary lemma.

**Lemma 2.4.** Let $X$ be a surface with Prym-canonical hyperplane section $C$ and let $\pi : X' \to X$ be the minimal resolution of singularities of $X$. Then $\pi_*(2K_{X'}) \sim 0$.

**Proof.** Let $C_m \in \lfloor mC \rfloor$ be smooth satisfying $C_m \cap \text{Sing}(X) = \emptyset$. We put $C'_m = \pi^*(C_m)$. We have that $-2K_{X'} \cdot C'_m \sim -2mK_{X'} \cdot C' \sim 0$ in the Chow Ring $A(X')$. Since rational equivalence and linear equivalence coincide on a curve, then $-2K_{X'}|_{C'_m} \sim 0$, for any $m \geq 1$.

We also observe that $\pi_*(2K_{X'})|_{C'_m} \sim 0$. Indeed, since $\pi$ is an isomorphism in a neighborhood of $C'_m$, then $\pi_*\mathcal{O}_{C'_m} \cong \mathcal{O}_{C_m}$ and $\mathcal{O}_{C'_m} \cong \pi^*(\mathcal{O}_{C_m})$. Using the projection formula (see [11], Exe II.5.1) and the previous results, we obtain that $\mathcal{O}_{C'_m} \cong \pi_*\mathcal{O}_{C'_m} \cong \pi_*\mathcal{O}_{2K_{X'} \odot \mathcal{O}_{C'_m}} = \pi_*\mathcal{O}_{2K_{X'} \odot \pi^*(\mathcal{O}_{C_m})} \cong \pi_*\mathcal{O}_{2K_{X'}} \odot \mathcal{O}_{C_m}$.

By a known result of Zariski ([3], Theorem 4), if $\pi_*(2K_{X'})|_{C_m} \sim 0$ for any $m$, then there exists a divisor $D$ such that $D - \pi_*\mathcal{O}_{2K_{X'}}$ and $D|_{C_m} \sim 0$. Since $C_m$ is very ample on $X$, then $D \sim 0$. So $\pi_*(2K_{X'}) \sim 0$. This proves the lemma.

\[ \square \]

**Theorem 2.2.** The dimension $\dim (-2K_{X'}) = 0$, in particular, if $W'$ is the effective anticanonical divisor on $X'$, then either $W' \sim 0$ or $\text{supp}(W') = \pi^{-1}(x_1, \ldots, x_r)$ for certain singularities $x_i \in X$, for $i = 1, \ldots, r$.

**Proof.** Since $\pi_*(2K_{X'}) \sim 0$ by the previous Lemma, then either $2K_{X'} \sim 0$ or there is a bicanonical divisor $2K_{X'}$ on $X'$ with support in $\pi^{-1}(\text{Sing}(X))$. In the latter case, let $2K_{X'} = \sum m_i F_i - \sum n_j G_j$ be the decomposition in reduced and irreducible components, with $m_i, n_j \in \mathbb{N}_{>0}$ and $F_i \neq G_j$, for all $i, j$. Let $F = \sum m_i F_i$ and $G = \sum n_j G_j$.

Suppose $F \neq 0$. By Mumford’s Theorem (see [14], Chapter 1), we have that $F_i^2 < 0$ for any $i$ and, because the intersection form on $\pi^{-1}(\text{Sing}(X))$ is negative definite, also $F^2 < 0$. So there is an $i_0$ such that $F \cdot F_{i_0} < 0$. Up to renaming the index, we suppose that $i_0 = 1$. It is obvious that $F_1 \cdot G \geq 0$. Since $F_1$ is an irreducible component, then

$$0 \leq p_a(F_1) = 1 + \frac{1}{2} F_1 \cdot (F_1 + K_{X'}) = 1 + \frac{1}{2} F_1^2 + \frac{1}{4} F_1 \cdot F - \frac{1}{4} F_1 \cdot G.$$

The only possibility is $F_1^2 = -1$. Thus $F_1$ is a $(-1)$-curve, contradicting the minimality of $\pi$. Hence $F = 0$.

Thus either $2K_{X'} \sim 0$ or there are effective anticanonical divisors with support in $\pi^{-1}(\text{Sing}(X))$. Then $\dim (-2K_{X'}) = 0$.

Let $| -2K_{X'}| = \{W'\}$. To conclude the proof, we have only to show that, if $x \in \text{Sing}(X)$ is such that $\pi^{-1}(x)$ meets $\text{supp}(W')$, then $\pi^{-1}(x)$ does not contain curves which are not part of $\text{supp}(W')$. 


Suppose that there is an irreducible curve $E \subset \pi^{-1}(x)$ which is not part of $\text{supp}(W')$. Since $X$ is normal by Theorem 2.1 Point 2, then $\pi^{-1}(x)$ is connected, so we can assume that $E$ intersects $W'$ and $E \cdot W' > 0$. Then

$$0 \leq p_a(E) = 1 + \frac{1}{2} E^2 + \frac{1}{2} E \cdot K_X = 1 + \frac{1}{2} E^2 - \frac{1}{4} E \cdot W'.$$

Again by Mumford’s Theorem, we have $E^2 < 0$. So the only possible case for which the previous inequality is valid is: $E \cdot W' = 2$, $E^2 = -1$ and $p_a(E) = 0$. This contradicts the minimality of $\pi$. □

Remark 2.5. By the previous Theorem, we observe that, if $X$ is smooth, then $X = X'$ and $W' \sim 0$. Since $p_a(X) = p_a(X) = 0$ by Theorem 2.1 Point 1, then $X$ is an Enriques surface by [11], Theorem V.6.3.

Lemma 2.6. If $W'$ is the unique effective antibicanonical divisor on $X'$, then:

a singularity $x \in X$ such that $\pi^{-1}(x)$ does not meet $\text{supp}(W')$ is a rational double point.

Proof. Let $x \in X$ be a singularity such that $\pi^{-1}(x)$ does not meet $\text{supp}(W')$. Let $T$ be an irreducible component of the connected component $\pi^{-1}(x)$, then $T \cdot W' = 0$. So

$$0 \leq p_a(T) = 1 + \frac{1}{2} T^2 + \frac{1}{2} T \cdot K_X = 1 + \frac{1}{2} T^2 - \frac{1}{4} T \cdot W' = 1 + \frac{1}{2} T^2.$$

By Mumford’s Theorem $T^2 < 0$, so the only possible case for which the inequality above is valid is: $T^2 = -2$ and $p_a(T) = 0$. Then all the irreducible components of $\pi^{-1}(x)$ are smooth rational curves with self-intersection $-2$. We can call these curves $E_i$, for $i = 1, \ldots, n$.

We can prove that $x$ must be a rational singularity using [13], Proposition-Definition 2.1. Let $Z_0 = \sum_{i=1}^{n} a_i \cdot E_i$, for $a_i \geq 0$ not all zero, the fundamental cycle associated with $x$. First of all, $p_a(Z_0) = 1 + \frac{1}{2} Z_0^2 + \frac{1}{2} Z_0 \cdot K_X$. Now $Z_0 \cdot K_X = -\frac{1}{2} Z_0 \cdot W' = -\frac{1}{2} \sum_{i=1}^{n} a_i E_i \cdot W' = 0$, while $Z_0^2 < 0$ since $Z_0$ is contracted by $\pi$. So $p_a(Z_0) < 1$. By [16], Lemma 1.1, we have that $p_a(E_i) \leq p_a(Z_0)$ for every $E_i$ contained in $Z_0$. Since $p_a(E_i) = 0$ as computed before, then the only possible case is $p_a(Z_0) = 0$, so $x$ is a rational singularity.

In conclusion, $x \in X$ is a rational double point. Indeed, by the adjunction formula, the self-intersection $Z_0^2 = 2p_a(Z_0) - 2 - Z_0 \cdot K_X = 2p_a(Z_0) - 2 = -2$. □

Remark 2.7. We have already seen that, if $X$ is birationally equivalent to an Enriques surface, then $X$ can only contain rational singularities. Moreover, by the previous Lemma, we conclude that it can only contain rational double points as singularities.

3. Examples

In this chapter, we will construct examples of surfaces with Prym-canonical hyperplane sections.

3.1. Surfaces with Prym-canonical hyperplane sections birationally equivalent to ruled surfaces. Let us fix some notation about minimal smooth ruled surfaces in which we will follow [11], Chapter V.2.

If $X''$ is a minimal smooth ruled surface and $p : X'' \to \Gamma$ is the natural map on the base curve $\Gamma$ of genus $q \geq 0$, then $X'' = \mathbb{P}_1(\mathcal{E})$, where $\mathcal{E}$ is a normalized locally free sheaf of rank 2 on $\Gamma$. 
Let \( \wedge^2(\mathcal{E}) = \mathcal{O}_\Gamma(D) \), for \( D \in \text{Div}(\Gamma) \). The integer \( e = -\deg(D) \) is an invariant of \( X'' \).

Let \( C_0 \) be a section of \( p \) such that \( \mathcal{O}_{X''}(C_0) = \mathcal{O}_{X''}(1) \). Then \( C_0^2 = -e \). Moreover, we recall that the canonical divisor \( K_{X''} \sim -2C_0 + (K_\Gamma + D) \cdot f \), where \( (K_\Gamma + D) \cdot f \) denotes the divisor \( p^*(K_\Gamma + D) \) by abuse of notation, with \( K_\Gamma + D \) a divisor on \( \Gamma \).

Finally, if \( \mathcal{E} \) is decomposable, i.e. \( \mathcal{E} = \mathcal{O}_\Gamma \oplus \mathcal{O}_\Gamma(D) \), then we will denote by \( C_1 \) a fixed section of \( p \) disjoint from \( C_0 \). Thus \( C_1 \sim C_0 - D \cdot f \).

3.1.1. **The minimal model is a non-rational ruled surface.** We start recalling a simple example of surface with Prym-canonical hyperplane sections.

Let \( X'' = \mathbb{P}_\Gamma(\mathcal{O}_\Gamma \oplus \mathcal{O}_\Gamma(D)) \) be a minimal ruled surface over a base curve \( \Gamma \) of genus \( q \geq 5 \), with \( D \sim -K_\Gamma + \alpha \), where \( \alpha \) is a non-trivial two torsion divisor and \( K_\Gamma \) is the canonical divisor of \( \Gamma \), and let \( L'' = |C_1| \) be a linear system on \( X'' \). By [3], Lemma 2.1, if \( \Gamma \) is non-hyperelliptic and it does not admit \( g_3^1 \), then a general hyperplane section of \( X'' \) is Prym-canonical embedded. The images of fibres of \( X'' \) by \( i_{L''} \) are lines since \( C'' \cdot f = C_1 \cdot f = 1 \). It is not difficult to prove that \( X \) has only one singularity, so the map \( i_{L''} : X'' \rightarrow \mathbb{P}^{q-1} \) defined by the linear system \( L'' \) is such that \( X = i_{L''}(X'') \) is a cone on a Prym-canonical embedded curve and if a general hyperplane section \( C \) of \( X \) is projectively normal, then the geometric genus of the only singularity is \( q \) (see Proposition 2.3).

**Remark 3.1.** Let us consider \( L'' \subseteq |aC_0 + \Delta \cdot f| \), for \( a \geq 2 \) and \( \Delta \in \text{Pic}(\Gamma) \). If \( X'' = \mathbb{P}_\Gamma(\mathcal{O}_\Gamma \oplus \mathcal{O}_\Gamma(D)) \) is a minimal ruled surface over a base curve \( \Gamma \) of genus \( q \geq 5 \), then the images of fibres of \( X'' \) by the map \( i_{L''} \) associated with \( L'' \) are not lines since \( C'' \cdot f = (aC_0 + \Delta \cdot f) \cdot f = a > 1 \), for \( C'' \) a general curve in \( L'' \). Furthermore there is not another family of rational curves on \( X'' \) mapped into lines by \( i_{L''} \) because the genus of the base curve \( \Gamma \) is \( q > 0 \). Indeed, by the Riemann-Hurwitz formula (see [11], Corollary IV.2.4), there is not a curve of genus 0 (not equal to a fibre) on a ruled surface over a base curve of genus \( q > 0 \). Hence we never obtain \( X = i_{L''}(X'') \) as a cone.

Now we focus our attention on surfaces with Prym-canonical hyperplane sections birationally equivalent to ruled surfaces \( X'' \) over a non-hyperelliptic base curve of genus \( q \geq 3 \).

**Proposition/Example 3.1.** Let \( X'' = \mathbb{P}_\Gamma(\mathcal{O}_\Gamma \oplus \mathcal{O}_\Gamma(D)) \) be a minimal ruled surface over a non-hyperelliptic smooth base curve \( \Gamma \) of genus \( q \geq 3 \), for \( D \in \text{Div}(\Gamma) \). Let \( L'' = |aC_1| \) be a linear system with \( a \geq 2 \). If \( D \sim -K_\Gamma + \alpha \), for \( \alpha \) a non-zero two-torsion divisor, then the image \( X = i_{L''}(X'') \subseteq \mathbb{P}^{q-2}(q-1) \) of the morphism associated with \( L'' \) has Prym-canonical hyperplane sections and only one singularity. In particular, if a general hyperplane section \( C \) of \( X \) is projectively normal, then the geometric genus of the only singularity \( x \) is \( p_g(x) = q \).

**Proof.** It is easy to prove that the linear system \( L'' = |aC_1| \) is base-point free using [8], Proposition 36, for any \( a \in \mathbb{N}_{\geq 2} \). Moreover, since \( (C'')^2 > 0 \), for \( C'' \) a general element of \( L'' \), then, by Bertini’s Theorem, \( C'' \) is smooth and irreducible.

**CLAIM 1:** We have that \( -K_{X''} \) is not effective, while \( -2K_{X''} \) is.

**Proof.** We know that \( h^0(\mathcal{O}_{X''}(-2K_{X''})) = h^0(\mathcal{O}_{X''}(2C_0)) = h^0(\mathcal{O}_{\Gamma} + h^0(\mathcal{O}_\Gamma(D))) + h^0(\mathcal{O}_{\Gamma}(2D)) + h^0(\mathcal{O}_{\Gamma}(3D)) + h^0(\mathcal{O}_{\Gamma}(4D)) \) by [8], Lemma 35. Because \( \deg(D) = 2 - 2q < 0 \), then \( h^0(\mathcal{O}_{X''}(-2K_{X''})) = 1 \) and \( -2K_{X''} \) is effective.

On the other hand, we have that \( h^0(\mathcal{O}_{X''}(-K_{X''})) = h^0(\mathcal{O}_{X''}(2C_0 - (K_\Gamma + D) \cdot f)) = h^0(\mathcal{O}_{\Gamma}(-K_\Gamma - D)) + h^0(\mathcal{O}_{\Gamma}(-K_\Gamma)) + h^0(\mathcal{O}_{\Gamma}(-K_\Gamma + D)) \) by [8], Lemma
35. Since $\deg(D - K_T) = 4 - 4q < 0$ and $\deg(-K_T) < 0$, then $h^0(-K_{X''}) = h^0(\mathcal{O}_T(-K_T - D)) = h^0(\mathcal{O}_T(-\alpha)) = 0$, so $-K_{X''}$ is not effective. □

CLAIM 2: We prove that $\mathcal{O}_{C''}(-K_{X''}) \not\cong \mathcal{O}_{C''}$, while $\mathcal{O}_{C''}(-2K_{X''}) \cong \mathcal{O}_{C''}$, where $C'' \in L''$ is a general curve.

Proof. Since $-2K_{X''}$ is effective as seen in Claim 1 and $C'' \cdot (-2K_{X''}) = (aC_0 - aD \cdot f \cdot (4C_0) = 4aC_0^2 - 4a\deg(D) = 0$, then $\mathcal{O}_{C''}(-2K_{X''}) \cong \mathcal{O}_{C''}$.

Clearly also $C'' \cdot (-K_{X''}) = 0$. Since $h^0(\mathcal{O}_{X''}(-K_{X''})) = 0$ as seen in Claim 1, if we prove that $h^1(\mathcal{O}_{X''}(-K_{X''} - C'')) = 0$, then, from the exact sequence $0 \to \mathcal{O}_{X''}(-K_{X''} - C'') \to \mathcal{O}_{X''}(-K_{X''}) \to \mathcal{O}_{C''}(-K_{X''}) \to 0$,

we have that $h^0(\mathcal{O}_{C''}(-K_{X''})) = 0$, which implies that $\mathcal{O}_{C''}(-K_{X''}) \not\cong \mathcal{O}_{C''}$.

Thus, by Serre Duality, we have that $h^1(\mathcal{O}_{X''}(-K_{X''} - C'')) = h^1(\mathcal{O}_{X''}(2K_{X''} + C''))$. If we prove that $K_{X''} + C''$ is ample, then, by the Kodaira vanishing Theorem (see [11], Remark III.7.15), we have that $h^1(\mathcal{O}_{X''}(2K_{X''} + C'')) = 0$ and the claim is proved.

By [11], Proposition V.2.20, if $a > 2$, then $K_{X''} + C''$ is ample and the claim is satisfied, instead, if $a = 2$, we have that $K_{X''} + C''$ is not ample. About that, let us suppose that $\mathcal{O}_{C''}(-K_{X''}) \cong \mathcal{O}_{C''}$. Then the image of $X''$ by $i_{L''}$ is a surface with canonical hyperplane sections. By [7], Corollary 5.4, $X''$ contains only one effective anticanonical divisor. This contradicts Claim 1, then this claim is also satisfied for $a = 2$. □

We know that $h^0(\mathcal{O}_T) = 1$. Using the Riemann-Roch Theorem, we also have that $h^0(\mathcal{O}_T(-D)) = h^0(\mathcal{O}_T(\alpha)) + 2q - 2 + 1 + q = (2q - 2) + 1 + q$ and $h^0(\mathcal{O}_T(-mD)) = m(2q - 2) + 1 - q$, for any $m \in \mathbb{N} > 1$. Hence, using [3], Lemma 35, we obtain that $h^0(\mathcal{O}_{X''}(L'')) = (1 + \ldots + a)(2q - 2) + a(1 - q) + 1 = a(a + 1)(q - 1) + a(1 - q) + 1 = a^2(q - 1) + 1.$

So $X = i_{L''}(X'')$ is contained in $\mathbb{P}^{a^2(q - 1)}$, for $a^2(q - 1) \geq 4 \cdot 2 = 8$.

CLAIM 3: It remains to show that $L''$ defines a birational morphism $i_{L''}$, in particular an isomorphism outside $C_0$. So $i_{L''}|_{C''}$ is a Prym-canonical embedding, for $C'' \in L''$ a general divisor.

Proof. We can prove that $L''$ defines a birational map, in particular an isomorphism outside $C_0$, using [3], Theorem 38. Indeed, this happens if $-aD$ is very ample and if $| - aD + D|, | - aD + (a - 1)D| = | - D|$ and $| - aD + aD|$ are base-point free.

The last case is trivial. By [11], Corollary IV.3.2, since $\deg(-aD) = a(2q - 2) \geq 4q - 4 = 2q + (2q - 4) \geq 2q + 1$, then $-aD$ is very ample. Again by [11], Corollary IV.3.2, if $a \geq 3$, since $\deg(-aD + D) = a(2q - 2) + (2 - 2q) = (a - 1)(2q - 2) \geq 4q - 4 = 2q + (2q - 4) \geq 2q$, then $| - aD + D|$ is base-point free.

It remains to show that $| - D|$ is base-point free. Since $D \sim -K_T + \alpha$, if $| - D|$ has base points, then $\Gamma$ must be hyperelliptic (see [3], Lemma 2.1). This contradicts our assumptions. So $L''$ defines an isomorphism outside $C_0$, called $i_{L''}$.

A general $C'' \sim aC_1$ in $L''$ is disjoint from $C_0$ by definition, then

$i_{L''}|_{C''} : C'' \to \mathbb{P}^{a^2(q - 1) - 1}$

is an embedding. By the adjunction formula, we have that

$L''|_{C''} \cong K_{C''} - K_{X''}|_{C''}$
but in Claim 2 we have already proved that $-K_{X''} \sim 4C_0$ is a non trivial two-torsion divisor, so $i_{L''}|_{C''}$ is a Prym-canonical embedding.

We observe that, since $i_{L''}|_{C''} : C'' \to \mathbb{P}^{g-2}$ by definition of Prym-canonical map, then $g = g(C'') = a^2(q - 1) + 1$. The image $x \in X$ of $-2K_{X''} \sim 4C_0$ by $i_{L''}$ is a singular point. There are not other possible singularities because $L''$ is an isomorphism outside $C_0$. We have found examples of surfaces in $\mathbb{P}^{a^2(q-1)}$ with Prym-canonical hyperplane sections birationally equivalent to non-rational ruled surfaces, for $a \geq 2$ and $q \geq 3$.

If a general hyperplane section $C' \subset X$ is projectively normal, then, by Proposition [2,3] the singularity $x$ has geometric genus equal to $g$.

We can construct an example of surface with Prym-canonical hyperplane sections birationally equivalent to an elliptic ruled surface $X''$. 

**Example 3.2.** Let $\Gamma$ be an elliptic curve and let $X'' = \mathbb{P}(\mathcal{O}_\Gamma \oplus \mathcal{O}_\Gamma(D))$ be a minimal ruled surface with base curve $\Gamma$, for $D \in \text{Div}(\Gamma)$. If $Q_i$, for $i = 1, 2, 3$, is a general point on $\Gamma$ and $\alpha, \beta \in \Gamma$ are two points such that $\alpha - \beta$ is a two torsion element of $\Gamma$, then we assume that $D = -Q_1 - Q_2 - Q_3 - \alpha + \beta$. So $e = -\deg(D) = 3$.

We consider the linear system $|3C_1| = |3C_0 + 3(Q_1 + Q_2 + Q_3 + \alpha - \beta) \cdot f|$. It is easy to prove that this linear system is base-point free using [3], Proposition 36, so, by Bertini’s Theorem, its general element $L$ is smooth and, since $L^2 = 9C_1^2 = 9e > 0$, it is also irreducible.

We call $f_i := Q_i \cdot f$, for $i = 1, 2, 3$. For any fibre $f$, we have that $\mathcal{L} \cdot f = 3$, so we can fix the 9 points
\[
Z := \{x_{1,1}, x_{1,2}, x_{1,3}, x_{2,1}, x_{2,2}, x_{2,3}, x_{3,1}, x_{3,2}, x_{3,3}\}
\]
of intersection between $\mathcal{L}$ and $f_i$, for $i = 1, 2, 3$. Since $3C_1$ is disjoint from $C_0$, we can assume that $Z \cap C_0 = \emptyset$. So we can consider the linear system $L'' \subset |3C_1|$ on $X''$ with $Z$ as base locus. In particular, we suppose that every curve $C'' \in L''$ simply passes through the 9 points, so $\mathcal{L}$ is an element of $L''$. By [3], Lemma 35, we have that $h^0(\mathcal{O}_{X''}(3C_0 + 3(Q_1 + Q_2 + Q_3 + \alpha - \beta) \cdot f)) = 19$, so $\dim L'' \geq 18 - 9 = 9$.

Since $\mathcal{L} \subset L''$ and smoothness is an open condition, then the general element $C''$ of $L''$ is smooth.

We know that $-K_{X''} \sim 2C_0 - D \cdot f$ and $h^0(\mathcal{O}_{X''}(-K_{X''})) = 4$ by [3], Lemma 35. We can show that $h^0(\mathcal{O}_{X''}(-K_{X''} - \mathcal{I}_Z)) = 0$. Indeed, if we suppose that there is an effective divisor $T \subset [-K_{X''} - \mathcal{I}_Z]$, then $T \cdot C_0 = 2C_0^2 - \deg(D) = 2(-3) + 3 < 0$, so $C_0$ is a fixed component of $T$. Moreover $T \cdot f = 2$ but $T$ contains 3 points of the fibres $f_i$, so it also contains $f_1, f_2, f_3$. Then
\[
| - K_{X''} - \mathcal{I}_Z | = | 2C_0 - D \cdot f - \mathcal{I}_Z | = C_0 + f_1 + f_2 + f_3 + | C_0 + (\alpha - \beta) \cdot f |,
\]
so
\[
\dim | - K_{X''} - \mathcal{I}_Z | = \dim | C_0 + (\alpha - \beta) \cdot f |.
\]

By [3], Lemma 35, we have that $h^0(\mathcal{O}_{X''}(C_0 + (\alpha - \beta) \cdot f)) = h^0(\mathcal{O}_{\Gamma}(\alpha - \beta) + h^0(\mathcal{O}_{\Gamma}(\alpha - \beta - Q_1 - Q_2 - Q_3 - \alpha + \beta)) = 0$. Hence $T$ effective does not exist.

It is not difficult to prove that $h^0(\mathcal{O}_{X''}(-2K_{X''})) = 10$ and, since $-2K_{X''} \sim 4C_0 + 2(Q_1 + Q_2 + Q_3) \cdot f$ and $4C_0 + 2(Q_1 + Q_2 + Q_3) \cdot f$ contains $Z$ with multiplicity 2, then also $h^0(\mathcal{O}_{X''}(-2K_{X''}) \cap \mathcal{I}_Z) > 0$.

Let $\phi : X' \to X''$ be the blowing up of $X''$ along the 9 points defining $Z$. Let $E_{i,j} \subset X'$ be the exceptional divisor of $x_{i,j}$, for $i, j \in \{1, 2, 3\}$, and let $f_i$ be the strict
transform of $f_i$, for $i = 1, 2, 3$. With abuse of notation, we call $C_0 := \phi^*(C_0)$ (we remark that $\phi^*(C_0)$ is the strict transform of $C_0$ since $Z \cap C_0 = \emptyset$), $\phi^*(\alpha \cdot f) := f_\alpha$ and $\phi^*(\beta \cdot f) := f_\beta$. Let $L'$ be such that $L'' = \phi \cdot L'$. Then the strict transform $C' \in L'$ of a general $C'' \in L''$ is of the form

$$C' = \phi^*(C'') - E_{1,1} - ... - E_{3,3} \sim 3C_0 + 3 \sum_{i=1}^{3} \tilde{f}_i + 3f_\alpha - 3f_\beta +$$

$$+ 2(E_{1,1} + ... + E_{3,3}).$$

Instead, using [11], Proposition V.3.3, we obtain that

$$-K_{X'} = \phi^*(-K_{X''}) - E_{1,1} - ... - E_{3,3} \sim 2C_0 + 3 \sum_{i=1}^{3} \tilde{f}_i + f_\alpha - f_\beta$$

and

$$-2K_{X'} \sim 4C_0 + 2\tilde{f}_1 + 2\tilde{f}_2 + 2\tilde{f}_3.$$ 

It is clear that $h^0(\mathcal{O}_{X'}(-K_{X'})) = h^0(\mathcal{O}_{X''}(-K_{X''} - \mathcal{I}_Z)) = 0$ while $h^0(\mathcal{O}_{X'}(-2K_{X'})) = h^0(\mathcal{O}_{X''}(-2K_{X''} - \mathcal{I}_Z)) > 0$.

Step by step, we can prove that the general hyperplane section $C'$ of $X'$ is a Prym-canonical embedded curve.

CLAIM 1: We have that $\mathcal{O}_{C'}(-K_{X'}) \not\cong \mathcal{O}_{C'}$ and $\mathcal{O}_{C'}(-2K_{X'}) \cong \mathcal{O}_{C'}$, where $C' \in L'$ is a general curve. In particular, $-K_{X'}|_{C'}$ is a non-zero two torsion divisor.

Proof. The intersection

$$C' \cdot (-K_{X'}) = 6C_0^2 + 6 \sum_{i=1}^{3} C_0 \cdot \tilde{f}_i + 3 \sum_{i=1}^{3} (C_0 \cdot \tilde{f}_1 + \tilde{f}_1^2) + 2[\tilde{f}_1 \cdot (E_{1,1} + E_{1,2} + E_{1,3}) + \tilde{f}_2 \cdot (E_{2,1} + E_{2,2} + E_{2,3}) + \tilde{f}_3 \cdot (E_{3,1} + E_{3,2} + E_{3,3})] = 6(-3) + 6(3) + 3(3-9) + 2(3+3) + 0$$

and clearly also $C' \cdot (-2K_{X'}) = 0$.

Since $-2K_{X'}$ is effective, then the anticanonical divisor of $X'$ is contracted by $i_{L'}$, in particular $\mathcal{O}_{C'}(-2K_{X'}) \cong \mathcal{O}_{C'}$.

On the contrary, we have that $h^0(\mathcal{O}_{X'}(-K_{X'})) = 0$, so $-K_{X'}$ is not effective. As seen in Claim 2 of Proposition 3.3, if we proved that $h^1(\mathcal{O}_{X'}(-K_{X'} - C')) = 0$, then $\mathcal{O}_{C'}(-K_{X'}) \not\cong \mathcal{O}_{C'}$.

Thus, by Serre Duality, it is clear that $h^1(\mathcal{O}_{X'}(-K_{X'} - C')) = h^1(\mathcal{O}_{X'}(2K_{X'} + C'))$.

If we prove that $K_{X'} + C'$ is big and nef, then, by the Kawamata-Viehweg vanishing Theorem (see [12] and [17]), the first cohomology $h^1(\mathcal{O}_{X'}(2K_{X'} + C')) = 0$.

In our case, since $2(\alpha - \beta) \sim 0$, then $2f_\alpha - 2f_\beta \sim 0$ and the divisor

$$K_{X'} + C' \sim C_0 + 2\tilde{f}_1 + 2\tilde{f}_2 + 2\tilde{f}_3 + 2E_{1,1} + ... + 2E_{3,3}.$$ 

Since $K_{X'} + C'$ is written as sum of irreducible and effective curves, then, to prove that $K_{X'} + C'$ is nef, it is enough to prove that $(K_{X'} + C') \cdot \delta \geq 0$, for any its irreducible component $\delta$. With some simple computations we obtain that this is true and because we have strictly positive intersections between $K_{X'} + C'$ and its components, then $K_{X'} + C'$ is also big. So the claim is satisfied. □
It is not difficult to compute that $C'^2 = 18$, so, by the adjunction formula, the genus $g(C') = 1 + \frac{1}{2}(C'^2 + K_{X'}) = 10$. We also observe that $C'$ is smooth because it is the strict transform of a general element $C''$ of $L''$, that is smooth. Since $-K_{X'}|_{C'}$ is a non-zero two torsion divisor as seen in Claim 1, we have that $L'|_{C'} = [K_{C'} - K_{X'}|_{C'}]$ defines a Prym-canonical map \[ \phi_{L'|_{C'}} : C' \rightarrow \mathbb{P}^8. \]

**CLAIM 2:** The rational map $\phi_{L'|_{C'}} : C' \rightarrow \mathbb{P}^8$ is an embedding, for any general curve $C' \in L'$.

**Proof.** First of all, we know that, if $L'|_{C'}$ has base points, then $C'$ is hyperelliptic by [3], Lemma 2.1. Moreover, since $C' \cdot \tilde{f} = 3$, where $\tilde{f}$ is the pullback of a general fibre $f$ of $X''$, then $C''$ is also a covering $3 : 1$ of the elliptic curve $\Gamma$. This is not possible by Castelnuovo-Severi inequality otherwise we would have $10 = g(C') \leq 2 \cdot 0 + 3 \cdot 1 + 1 \cdot 2 = 5$ (see [1]). Thus $L'|_{C'}$ is base-point free.

Thanks to [3], Corollary 2.2, we know that, if $C'$ is not bielliptic, then $L'|_{C'}$ is an embedding. Because $C'$ is a triple cover of $\Gamma$ as observed before, then $C''$ cannot be bielliptic again by Castelnuovo-Severi inequality otherwise we would have $10 \leq 2 \cdot 1 + 3 \cdot 1 + 1 \cdot 2 = 7$. Thus the claim is proved.

At this point, since $L'|_{C'}$ is base-point free, it is clear that $L'$ is also base-point free. Since the restriction $L'|_{C'}$ defines an embedding for each generic curve $C' \in L'$, then $\phi_{L'}$ is a birational map, generically 1 : 1.

Before we have showed that $\dim(L'') = \dim(L') \geq 9$. From the exact sequence \[ 0 \rightarrow \mathcal{O}_{X'}(C' - C') \rightarrow \mathcal{O}_{X'}(C') \rightarrow \mathcal{O}_{C'}(C') \rightarrow 0, \] we conclude that $h^0(\mathcal{O}_{X'}(C')) \leq 10$ since $\mathcal{O}_{X'}(C' - C') \cong \mathcal{O}_{X'}$ and $h^0(\mathcal{O}_{C'}(C')) = 9$. So we have that $h^0(\mathcal{O}_{X'}(C')) = 10$.

Then $X'$ has hyperplane sections that are Prym-canonical embedded and, in particular, we have found a new surface $X = \phi_{L'}(X') \subset \mathbb{P}^9$ with Prym-canonical hyperplane sections. Since the anticanonical divisor of $X'$ is connected, then its image $x \in X$ by $\phi_{L'}$ is a singular point. There are other possible rational double singularities on $X$ whose exceptional divisors on $X'$ do not intersect $-2K_{X'}$.

If a general hyperplane section $C' \subset X$ is projectively normal, then, by Proposition 2.3 the geometric genus $p_g(x)$ is equal to 1.

**Remark 3.3.** We can compute how many moduli the couple $(X'', L'')$ of previous example depends on.

The choice of the elliptic curve $\Gamma$ depends on one parameter. In addition we fix a divisor $D = -Q_1 - Q_2 - Q_3 - \alpha + \beta$ of degree $-3$, where $Q_i$ is a general point on $\Gamma$, for $i = 1, 2, 3$, and $\alpha - \beta$ is a non-zero two torsion element of $\Gamma$.

We know that there are only three non-zero two torsion points on $\Gamma$. Instead we observe that $|Q_1 + Q_2 + Q_3|$ is a linear system of dimension 2, so the choice of $\mathcal{O}_\Gamma(D)$ depends on $3 - 2 = 1$ parameter.

Moreover, every automorphism of $\Gamma$ lifts to an automorphism of $X''$ that means that, if $\phi : \Gamma \rightarrow \Gamma$ is an automorphism, then $X'' = \mathbb{P}_{\Gamma}(\mathcal{O}_\Gamma \oplus \mathcal{O}_\Gamma(D)) \cong \mathbb{P}_{\Gamma}(\mathcal{O}_\Gamma \oplus \mathcal{O}_\Gamma(\phi^*(D)))$. The group $\operatorname{Aut}(\Gamma)$ has dimension 1.

To construct the surface with Prym-canonical hyperplane sections of the previous example, we also fix a linear system $L'' \subset |3C_1| = |3C_0 - 3D \cdot f|$ with 9 simple base
The linear system $|3C_1|$ depends on the parameters fixed before. Instead the 9 simple base points are the points of intersection between a general element $L \in |3C_1|$ and the three fibres $f_i := Q_i \cdot f_i$ for $i = 1, 2, 3$. The choice of the effective divisor in a linear system of the type $|Q_1 + Q_2 + Q_3|$ that defines the three fibres $f_1, f_2, f_3$ depends on 2 parameters. In addition, as seen in the previous example, the nine points $\{ x_{1,1}, x_{1,2}, x_{1,3}, x_{2,1}, x_{2,2}, x_{2,3}, x_{3,1}, x_{3,2}, x_{3,3} \}$ impose independent conditions on the linear system $L''$, so they depend on 9 parameters.

The choice of the pair $(X'', L'')$ depends on $1 + 1 - 1 + 2 + 9 = 12$ parameters.

We know that $\dim(\text{Aut}(\mathbb{P}^3)) = 15$. We can consider $X''$ as the blowing up of the vertex of the cone $C_{X''}$ on a plane cubic of $\mathbb{P}^3$. If $C_T$ is the base curve of $C_{X''}$, there are $\infty^8$ plane cubics isomorphic to $C_T$. Since we can choose the vertex among all the possible points of $\mathbb{P}^3$ obtaining always isomorphic cones, then there are $\infty^{(8+3)} = \infty^{11}$ isomorphic cones of the type of $C_{X''}$ in $\mathbb{P}^3$.

Thus there are $\infty^4$ automorphism of $\mathbb{P}^3$ that fix $X''$ so, in conclusion, the couple $(X'', L'')$ depends on $12 - 4 = 8$ parameters.

Since the surface constructed in the previous example depends on 8 moduli while a general Enriques surface depends on 10 moduli, then the generic Enriques surface can degenerate to one of these surfaces since they depend on less parameters.

3.1.2. The minimal model is a rational ruled surface. We construct an example of surface with Prym-canonical hyperplane sections birationally equivalent to a rational ruled surface.

**Example 3.4.** Let $\Gamma$ be a rational smooth curve and let $X'' = \mathbb{P}_T(\mathcal{O}_T \oplus \mathcal{O}_T(D))$ be a minimal ruled surface with base curve $\Gamma$, for $D \in \text{Div}(\Gamma)$. We assume that $e = 4$, so $\deg(D) = -4$. Hence $X''$ is a Hirzebruch surface $F_4$.

We know that $-K_{X''} \sim 2C_0 - (K_T + D) \cdot f$, where $\deg(-K_T - D) = 2 + 4 = 6$. We put

$$-K_{X''} = 2C_0 + 2 \sum_{i=1}^{3} F_i,$$

where $F_1, F_2$ and $F_3$ are distinct and fixed fibres. We also set

$$W'' = 4C_0 + 3F + 3 \sum_{i=1}^{3} F_i \in | -2K_{X''}|,$$

where $F$ is a generic fibre distinct from $F_i$, for $i = 1, 2, 3$.

We consider the linear system $|4C_1| = |4C_0 - 4D \cdot f|$ on $X''$. Every element in $|4C_1|$ intersects every fibre of $X''$ in 4 points since $4C_1 \cdot f = 4$. In addition, we know that $h^0(\mathcal{O}_{X''}(4C_1)) = 45$ by [S], Lemma 35.

CLAIM 1: There is a smooth curve $L \in |4C_1|$ such that $L$ is tangent to $F, F_1, F_2$ and $F_3$ respectively in two points.

**Proof.** We can assume that $X'' = F_4$ is the blowing up of the vertex of a cone in $\mathbb{P}^5$ on a rational normal curve of $\mathbb{P}^4$. It is clear that $C_0$ is the exceptional divisor associated with the vertex.

Let $E = F + F_1 + F_2 + F_3$ be the curve intersection between the cone and a hyperplane $H_0$ of $\mathbb{P}^5$ passing through the vertex of the cone. With abuse of notation, the total transform of $E$ on $X''$ is $E = C_0 + F + F_1 + F_2 + F_3$. 

It is obvious that the linear systems $|2C_1|$ and $|3C_1|$ are base-point free. Then they respectively contain a general quadric $Q$ and a general cubic $C$, that are smooth by Bertini’s Theorem.

It is clear that $2Q$ intersects every fibre in two points with multiplicity 2. We call \( \{x_j, x_{1,j}, x_{2,j}, x_{3,j}\} \), for \( j = 1, 2 \), the intersections points between $2Q$ and $F + F_1 + F_2 + F_3$.

Let us consider a pencil $\mathcal{P}$ generated by $2Q$ and $E + C$. By Bertini’s Theorem, its curves may have singular points only on the base locus of the pencil. At this point we observe that $2Q \cdot C \sim 2(2C_1) \cdot (3C_1) = 12 \cdot 4 = 48$ since $C_1$ is a rational normal curve of degree 4. These 48 points are base points for the pencil, different from \( \{x_j, x_{1,j}, x_{2,j}, x_{3,j}\} \), with \( j = 1, 2 \), for the generality of $C$. Now $E + C$ has only 16 singular points since $E$ has only 4 singular points on $C_0$, while $C$ is smooth and disjoint from $C_0$ for its generality and $E \cdot C \sim (C_0 + 4F) \cdot 3C_1 = 12$. Since $Q$ is also disjoint from $C_0$ and it is general, then these 16 points are different from the 48 base points. Then $E + C$ is smooth in the 48 base points. The same is true for $2Q$. Hence also a general divisor $\mathcal{L}$ in the pencil $\mathcal{P}$ is smooth in the 48 base points.

Instead $2Q_{|E} = \{x_1, x_2, x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2}, x_{3,1}, x_{3,2}\}$. These are other 8 base points for $\mathcal{P}$. Since $2Q$ passes through \( \{x_j, x_{1,j}, x_{2,j}, x_{3,j}\} \), for \( j = 1, 2 \), with multiplicity 2 and since $E + C$ simply passes through the eight points ($E$ contains the fibres $F, F_1, F_2, F_3$ and $C$ does not contain these 8 points), then a general curve $\mathcal{L}$ is smooth in these 8 points and, in particular, it is tangent to $F, F_1, F_2, F_3$ in \( \{x_1, x_2, x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2}, x_{3,1}, x_{3,2}\} \).

Finally, we observe that $2Q \sim 2(2C_1) = 4C_1$ and similarly, we have $C \sim 3C_1$ and $E \sim C_0 + 4F$, so $E + C \sim 4C_1$. Then we have found a smooth curve $\mathcal{L} \in |4C_1|$ tangent to $F$ in $x_1$ and $x_2$ and tangent to $F_i$ in $x_{i,1}$ and $x_{i,2}$, for \( i = 1, 2, 3 \). \( \square \)

In the following figure, we analyze what happens blowing up all the intersection points between $\mathcal{L}$ and $W''$, also infinitely near. We observe that, since $\mathcal{L} \sim 4C_1$ and $4C_1$ is disjoint from $C_0$, then $\mathcal{L}$ does not intersect $C_0$. With abuse of notation, we will call the strict transforms of $C_0$, $F_i$ and $\mathcal{L}$ with the same names.

In the figure, we only focus on $F_1$, it is the same for $F_2, F_3$ and $F$.

- **STEP 1** We blow up the intersection points $x_{i,1}$ and $x_{i,2}$ on $X''$;
- **STEP 2** In $X'' := Bl_{x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2}, x_{3,1}, x_{3,2}}(X''')$, the curve $\mathcal{L}$ simply passes through the infinitely near base points $y_{i,1}$ and $y_{i,2}$, for $i = 1, 2, 3$. We also blow up these six points;
- **STEP 3** Again $\mathcal{L}$ intersects the exceptional divisors $E_{i,1}$ and $E_{i,2}$ respectively in $z_{i,1}$ and $z_{i,2}$ on $X''' := Bl_{y_{i,1}, y_{i,2}, z_{i,1}, z_{i,2}}(X''')$, for $i = 1, 2, 3$. We obtain $Y = Bl_{z_{1,1}, z_{1,2}}, z_{2,1}, z_{2,2}, z_{3,1}, z_{3,2}}(X''')$ blowing up these other six points.
With the same techniques as before, we also blow up \( \{x_1, x_2, y_1, y_2, z_1, z_2\} \in \mathcal{L} \cap F \) (as seen in the previous figure, they are infinitely near points). We define

\[
X' := \text{Bl}_{z_1, z_2} \left( \text{Bl}_{y_1, y_2} \left( \text{Bl}_{x_1, x_2}(Y) \right) \right).
\]

In \( X' \), there are no intersection points between \( \mathcal{L} \) and \(-2K_{X'}\).

After observing how the blowing up works, we consider \( L'' \subset |4C_1| \) as the linear system of the curves of \( |4C_1| \) simply passing through

\[
Z := \{x_1, x_2, y_1, y_2, z_1, z_2, x_{i,1}, x_{i,2}, y_{i,1}, y_{i,2}, z_{i,1}, z_{i,2}\}, \text{ for } i = 1, 2, 3.
\]

Then \( \dim L'' \geq 44 - 6 \cdot 4 = 20 \). It is clear that \( \mathcal{L} \) is an element of \( L'' \) and since smoothness is an open condition, then the general element \( C'' \) of \( L'' \) is smooth.

With the same notation as before, we can obtain that

\[
-K_Y = 2C_0 + 3 \sum_{i=1}^3 (2F_i + J_{i,1} + J_{i,2} + 2E_{i,1} + 2E_{i,2} + B_{i,1} + B_{i,2})
\]

while

\[
W_1'' = 4C_0 + 3F + 3 \sum_{i=1}^3 (3F_i + J_{i,1} + J_{i,2} + 2E_{i,1} + 2E_{i,2}) \in | -2K_Y |.
\]
At this point, we use the fact that, if $M$ is an effective divisor and $N_i$ are irreducible divisors, if $M \cdot N_1 < 0$, $(M - N_1) \cdot N_2 < 0$, and so on, the $\sum N_i$ is a partial fixed part of $|M|$. Thus, inductively, one can verify that all $2C_0 + \sum_{i=1}^{3}(2F_i + J_{i,1} + J_{i,2} + 2E_{i,1} + 2E_{i,2} + B_{i,1} + B_{i,2})$ is a fixed component of $|-K_X|$, so this is the only one effective curve in its linear system. In addition, the part $4C_0 + \sum_{i=1}^{3}(3F_i + J_{i,1} + J_{i,2} + 2E_{i,1} + 2E_{i,2})$ is the fixed part of $|-2K_Y|$ while its variable part is $3F$.

Similarly to before, we can compute that

$$-K_X' \sim 2C_0 + \sum_{i=1}^{3}(2F_i + J_{i,1} + J_{i,2} + 2E_{i,1} + 2E_{i,2} + B_{i,1} + B_{i,2}) +$$

$$-J_1 - J_2 - 2E_1 - 2E_2 - 3B_1 - 3B_2.$$ 

This is clearly not effective. Instead

$$W' = 4C_0 + 3F + J_1 + J_2 + 2E_1 + 2E_2 + \sum_{i=1}^{3}(3F_i + J_{i,1} + J_{i,2} + 2E_{i,1} + 2E_{i,2}) \in |-2K_X'|$$

is effective. In addition, all $4C_0 + 3F + J_1 + J_2 + 2E_1 + 2E_2 + \sum_{i=1}^{3}3F_i + J_{i,1} + J_{i,2} + 2E_{i,1} + 2E_{i,2}$ is a fixed component of $|-2K_X'|$ and consequently it is the only one effective divisor in its linear system.

Since all the divisors on $\Gamma$ of the same degree are linearly equivalent, then we observe that $-4D \cdot f \sim 4F + 4 \sum_{i=1}^{3}F_i$, so we can assume that

$$C'' \sim 4C_0 + 4F + 4 \sum_{i=1}^{3}F_i,$$

where $C''$ is a general element in $L''$. Then, its strict transform $C'$ on $X'$ is linearly equivalent to

$$C' \sim 4C_0 + 4F + \sum_{i=1}^{2}(3J_j + 6E_j + 5B_j) +$$

$$+ \sum_{i=1}^{3}(4F_i + 3J_{i,1} + 3J_{i,2} + 6E_{i,1} + 6E_{i,2} + 5B_{i,1} + 5B_{i,2}).$$

If $\phi : X' \rightarrow X''$ is the blowing up of $X''$ along the points of $Z$, let $L'$ be such that $L'' = \phi^*L'$, with $C'$ a general element.

Step by step, we can prove that a general hyperplane section $C'$ of $X'$ is a Prym-canonical embedded curve.

CLAIM 2: We have that $\mathcal{O}_{C'}(-K_{X'}) \not\cong \mathcal{O}_{C'}$ while $\mathcal{O}_{C'}(-2K_{X'}) \cong \mathcal{O}_{C'}$. In particular, $-K_{X'}|_{C'}$ is a non-zero two torsion divisor.

Proof. It is easy to compute that $C' \cdot W' = 0$. Since $W'$ is effective, then it is contracted by the map defined by $L'$, in particular $\mathcal{O}_{C'}(-2K_{X'}) \cong \mathcal{O}_{C'}$.

It is clear that also $C' \cdot (-K_{X'}) = 0$ but this time we have that $h^0(\mathcal{O}_{X'}(-K_{X'})) = 0$. As seen in Claim 2 of Proposition 3.1, it is sufficient to show that $h^1(\mathcal{O}_{X'}(-K_{X'} - C')) = 0$ to prove that $\mathcal{O}_{C'}(-K_{X'}) \not\cong \mathcal{O}_{C'}$.

Using Serre Duality and the Kawamata-Viehweg vanishing Theorem (see [12] and [17]), if we prove that $K_{X'} + C'$ is big and nef, then the claim is satisfied. Now

$$K_{X'} + C' = 2C_0 + \sum_{i=1}^{3}(2F_i + 2J_{i,1} + 2J_{i,2} + 4E_{i,1} + 4E_{i,2} + 4B_{i,1} + 4B_{i,2}) +$$

$$+ 8F + 4J_1 + 4J_2 + 8E_1 + 8E_2 + 8B_1 + 8B_2.$$
Since $K_{X'} + C'$ is written as sum of irreducible and effective curves, then, to prove that $K_{X'} + C'$ is nef, it is enough to prove that $(K_{X'} + C') \cdot \delta \geq 0$, for any its irreducible component $\delta$. It is possible to compute that this is true. Because we have strictly positive intersections between $K_{X'} + C'$ and its components, then $K_{X'} + C'$ is also big. So the claim is satisfied. □

It is not difficult to compute that $C'^2 = 40$, so, by the adjunction formula, the genus $g(C') = 1 + \frac{1}{2}(C'^2) = 21$. We also observe that $C'$ is smooth because it is the strict transform of a general element $C''$ of $L''$, that is smooth. Since $-K_{X'}|_{C'}$ is a non-zero two torsion divisor as seen in Claim 2, we have that $L'_{|C'} = |K_{C'} - K_{X'}|_{C'}$ defines a Prym-canonical map

$$\phi_{L'_{|C'}} : C' \dashrightarrow \mathbb{P}^{19}.$$ 

We have already observed that $\dim(L'') = \dim(L') \geq 20$. From the exact sequence

$$0 \rightarrow \mathcal{O}_X(C' - C') \rightarrow \mathcal{O}_X(C') \rightarrow \mathcal{O}_{C'}(C') \rightarrow 0,$$

we conclude that $h^0(\mathcal{O}_X(C')) \leq 20$ since $\mathcal{O}_X(C' - C') \cong \mathcal{O}_X$, and $h^0(\mathcal{O}_{C'}(C')) = 19$. So we have that $h^0(\mathcal{O}_X(C')) = 21$ and $\phi_{L'}(X') \subseteq \mathbb{P}^{20}$.

CLAIM 3 : The rational map $\phi_{L'_{|C'}} : C' \dashrightarrow \mathbb{P}^{19}$ is an embedding, for any general curve $C' \in L'$.

Proof. First of all, we know that, if the Prym-canonical system $L'_{|C'}$ has base points, then $C'$ is hyperelliptic by [3], Lemma 2.1.

Let us suppose that $C'$ is hyperelliptic. Since we know that $C' \in X'$ is isomorphic to $C'' \sim 4C_1$ on $X''$, then $C''$ is also hyperelliptic.

The self-intersection $C''^2 = (4C_1)^2 = 64$. Furthermore $C''$ is nef since $C'' \sim 4C_0 + 4F + 4 \sum_{i=1}^{3} F_i$ and $(4C_0 + 4F + 4 \sum_{i=1}^{3} F_i) \cdot C_0 = 0$ and $(4C_0 + 4F + 4 \sum_{i=1}^{3} F_i) \cdot F = (4C_0 + 4F + 4 \sum_{i=1}^{3} F_i) \cdot F_i = 4$. Moreover, by [11], Proposition IV.5.2, we have that $|K_{C''}|$ is not very ample, precisely it does not separate any pair of points $p$ and $q$ such that $p + q$ is a member of the $g^1_2$ on $C''$. By the adjunction formula, we also have that $|K_{X''} + C''|$ does not separate such $p$ and $q$.

By [15], Theorem 1, there exists an effective divisor $E$ on $X''$ passing through $p$ and $q$ such that $C'' \cdot E < 4$. Since $C'' \sim 4C_1$, then $C'' \cdot E = 0$. This is not possible and thus $E$ cannot exist. We exclude the case $C''$ hyperelliptic and hence $L'_{|C'}$ is base-point free.

Furthermore, we can prove that $L'_{|C'}$ defines a birational map. Indeed, if this did not happen, we would have $C'$ bielliptic and the image of $X'$ via the map associated with $L'$ would be a surface in $\mathbb{P}^{20}$ with elliptic sections (see [3], Corollary 2.2). Since $20 > 9$, then the surface image in $\mathbb{P}^{20}$ could not be a Del Pezzo surface but it would be an elliptic cone. Anywhere $X'$ is a rational surface, so it cannot cover an elliptic cone. Then $L'_{|C'}$ defines a birational map.

More precisely, we can also show that $L'_{|C'}$ defines an embedding, for any general $C' \in L'$. By [3], Lemma 2.1, we know that $L'_{|C'}$ does not separate $p$ and $q$ (possibly infinitely near) if and only if $C'$ has a $g^1_2$ and $-K_{X'}|_{C'} \sim \mathcal{O}_{C'}(p + q - x - y)$, where $2(p + q)$ and $2(x + y)$ are members of the $g^1_4$.

We know that $C' \cong C''$ and $C'' \sim 4C_1$ has a $g^1_4$ defined by the fibres of the ruled surface $X''$. This is the only one. Indeed, if $C''$ had two $g^1_4$, then there would be a map $\psi : C'' \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$. If $\psi$ was a birational map, the image curve would be of the type $(4, 4)$ on $\mathbb{P}^1 \times \mathbb{P}^1$. Then its geometric genus would be at most $(4 - 1)(4 - 1) = 9$. Since $C'$ has genus 21, this case is excluded. Thus $\psi$ would be a map $2 : 1$ on a curve $D$. The image curve $D$ would be a curve of type $(2, 2)$ on $\mathbb{P}^1 \times \mathbb{P}^1$, so its
geometric genus would be \( g(D) \leq 1 \). Since \( C' \) is non-hyperelliptic as seen before, then \( g(D) = 1 \) and \( C' \) is bielliptic. Then \( C'' \) admits a singular correspondence. By Corollary 2.2 of [5], the map determined by the linear system \( |C'| \) is not birational, in particular it is \( 2:1 \) on a surface with elliptic sections. We have already excluded this possibility, so \( C'' \) has only one \( g_1^1 \).

It is clear that \( C'' \cap F \sim C'' \cap F_1 \sim C'' \cap F_2 \sim C'' \cap F_3 \) and we know that \( C'' \) is tangent in two points to this four fibres. So we have four pairs of points \( (p,q) \in F \), \( (x,y) \in F_1 \), \( (z,w) \in F_2 \) and \( (a,b) \in F_3 \) such that \( 2(p + q) \sim 2(x + y) \) and so on for all the possible cases. Since \( C'' \) is the strict transform of \( C'' \), it has the same characteristics of \( C'' \) and, after the blowing up, the four pairs of points that satisfy this property are the intersection points between \( C' \) and \( B_{i,j} \) and \( C' \) and \( B_j \), for \( i = 1,2,3 \) and \( j = 1,2 \). Now, with abuse of notation and using the expression of \( -K_X \), seen before, we have that

\[
-K_{X'}|_{C'} = \sum_{i=1}^{3} (B_{i,1} + B_{i,2})|_{C'} - (3B_1 + 3B_2)|_{C'} = x + y + w + z + a + b - 3p - 3q \sim x + y - w - z + a + b - p - q.
\]

At this point, we observe that

\[
x + y - w - z + a + b - p - q \not\sim x + y - p - q
\]

otherwise, if \( a + b - w - z \sim 0 \), then \( C' \) would have a \( g_1^2 \). Hence \( L'|_{C'} \) separate each pair of points and it defines an embedding.

At this point, since \( L'|_{C'} \) is base-point free, it is clear that \( L' \) is also base-point free. Since the restriction \( L'|_{C'} \) defines an embedding for each generic curve \( C' \in L' \), then \( \phi_{L'} \) is a birational map, generically \( 1:1 \).

Then \( X' \) has hyperplane sections that are Prym-canonical embedded. In particular, \( \phi_{L'}(X') \) is a surface with Prym-canonical hyperplane sections.

We have found a new surface \( X = \phi_{L'}(X') \subset \mathbb{P}^{20} \) with Prym-canonical hyperplane sections. Since \( W' \) is connected, then the image \( x \in X \) of \( W' \) is a rational singular point (see Proposition 2.3). There are other possible rational double singularities on \( X \) whose exceptional divisors on \( X' \) do not intersect \(-2K_{X'}\).

3.2 More surfaces with Prym-canonical hyperplane sections birationally equivalent to \( \mathbb{P}^2 \). We construct a new example of such surface whose minimal model is \( X'' = \mathbb{P}^2 \).

**Example 3.5.** Let \( X'' = \mathbb{P}^2 \) be such that \(-2K_{X''} \) is an irreducible sextic with 10 nodes \( \{x_1, ..., x_{10}\} \). Let \( L'' \) be a linear system of curves of degree 18 with base points \( \{x_1, ..., x_{10}\} \in X'' \) of multiplicity respectively \( r_i = 4 \), for \( i = 1,2,3 \), and \( r_i = 6 \), for \( i = 4, ..., 10 \). Let \( X' = Bl_{\{x_1, ..., x_{10}\}}(\mathbb{P}^2) \) be the blowing up of \( X'' \) along the base points of \( L'' \) and let \( L' \) be the strict transform of \( L'' \). We observe that the anticanonical divisor \(-K_{X'} \) is not effective.

Let

\[
C' \sim 18l - 4 \sum_{i=1}^{3} E_i - 6 \sum_{i=4}^{10} E_i
\]

be a general curve in \( L' \), where \( E_i \) is the exceptional divisor associated with \( x_i \), for \( i = 1, ..., 10 \). It is obvious that

\[
\deg(C'|_{C'}) = 18^2 - 3 \cdot 16 - 7 \cdot 36 = 24.
\]
We have that $h^0(\mathcal{O}_{X'}(C')) \geq \binom{20}{2} - 4 \frac{45}{2} - 7 \frac{67}{2} = 13$, so $\phi_L'(X') = X \subset \mathbb{P}^r$, for $r \geq 12$.

Since $-2K' \sim J = 6l - 2 \sum_{i=1}^{10} E_i$ is effective and $C'(-2K') = 0$ by construction, then $\mathcal{O}_{C'}(-2K_{C'}) \cong \mathcal{O}_{C'}$. So $L'$ contracts $J$ in a single point since $J$ is irreducible. Moreover, since $J$ is also rational, then $\phi_L'(J)$ is a rational singularity of multiplicity 4 because the fundamental cycle $Z_0 = J$ is such that $Z_0^2 = J^2 = -4$.

**Claim 1:** The dimension $\dim(L') = 12$, so $\phi_L'(X') = X \subseteq \mathbb{P}^{12}$.

**Proof.** We can consider the following exact sequence, already tensored with $\mathcal{O}_{X'}(C')$:

\[(4)\quad 0 \to \mathcal{O}_{X'}(C' - J) \to \mathcal{O}_{X'}(C') \to \mathcal{O}_J(C') \to 0.
\]

Since $\mathcal{O}_{C'}(-2K_{C'}) \cong \mathcal{O}_{C'}$ and $J \in |-2K_{C'}|$, then $\mathcal{O}_J(C') \cong \mathcal{O}_J \cong \mathcal{O}_P$ since $J$ is rational. Thus we can rewrite (4) as

\[(5)\quad 0 \to \mathcal{O}_{X'}(12l - 2 \sum_{i=1}^{3} E_i - \sum_{i=4}^{10} E_i) \to \mathcal{O}_{X'}(18l - 4 \sum_{i=1}^{3} E_i - \sum_{i=4}^{10} E_i) \to \mathcal{O}_P \to 0.
\]

Similarly we obtain that

\[0 \to \mathcal{O}_{X'}(6l - 2 \sum_{i=4}^{10} E_i) \to \mathcal{O}_{X'}(12l - 2 \sum_{i=1}^{3} E_i - \sum_{i=4}^{10} E_i) \to \mathcal{O}_J(12l - 2 \sum_{i=1}^{3} E_i - \sum_{i=4}^{10} E_i) \to 0.
\]

\[(6)\]

It is possible to choose a quintuple of points among the 10 nodes \{x_1, ..., x_{10}\} of $J$ such that three of these points are not aligned, then an irreducible conic passing through this quintuple of points exists. Up to renaming the nodes of $J$, we suppose that a conic passing through \{x_4, ..., x_{8}\} exists. So let us consider the following exact sequences:

\[(7)\]

\[0 \to \mathcal{O}_{X'}(4l - \sum_{i=4}^{8} E_i - \sum_{i=9}^{10} E_i) \to \mathcal{O}_{X'}(6l - 2 \sum_{i=4}^{10} E_i - \sum_{i=9}^{10} E_i) \to \mathcal{O}_{L_{x_4} - E_{x_{10}}(4l - \sum_{i=4}^{8} E_i - \sum_{i=9}^{10} E_i)} \to 0;
\]

\[0 \to \mathcal{O}_{X'}(3l - \sum_{i=4}^{10} E_i) \to \mathcal{O}_{X'}(4l - \sum_{i=4}^{8} E_i - \sum_{i=9}^{10} E_i) \to \mathcal{O}_{L_{x_4} - E_{x_{10}}(4l - \sum_{i=4}^{8} E_i - \sum_{i=9}^{10} E_i)} \to 0.
\]

\[(8)\]

It obvious that $h^0(\mathcal{O}_{X'}(3l - \sum_{i=4}^{8} E_i) = \binom{8}{3} - 7 = 3$. Since it is an effective divisor on $X'$ and it has the expected dimension, then $h^1(\mathcal{O}_{X'}(3l - \sum_{i=4}^{10} E_i)) = 0$.

Because $l - E_9 - E_{10}$ is rational and \((l - E_9 - E_{10}) \cdot (4l - \sum_{i=4}^{8} E_i - \sum_{i=9}^{10} E_i) = 0\),

then $h^0(\mathcal{O}_{L_{x_4} - E_{x_{10}}(4l - \sum_{i=4}^{8} E_i - \sum_{i=9}^{10} E_i)} = 1$ and $h^1(\mathcal{O}_{L_{x_4} - E_{x_{10}}(4l - \sum_{i=4}^{8} E_i - \sum_{i=9}^{10} E_i)} = 0$.

From the exact sequence (8), we conclude that $h^0(\mathcal{O}_{X'}(4l - \sum_{i=4}^{8} E_i - \sum_{i=9}^{10} E_i)) = 3 + 1 = 4$ and $h^1(\mathcal{O}_{X'}(4l - \sum_{i=4}^{8} E_i - \sum_{i=9}^{10} E_i)) = 0$. 


Using the Riemann-Roch Theorem, since \( J \) and \( 2l - \sum_{i=4}^{10} E_i \) are rational, we obtain that
\[
h^0(\mathcal{O}_X(12l - 2 \sum_{i=1}^{3} E_i - 4 \sum_{i=4}^{10} E_i)) = 4 + 1 = 5
\]
and
\[
h^0(\mathcal{O}_{2l-\sum_{i=4}^{10} E_i}(6l - 2 \sum_{i=4}^{10} E_i)) = 2 + 1 = 3.
\]
From the exact sequence (7), we conclude that \( h^0(\mathcal{O}_X(6l - 2 \sum_{i=4}^{10} E_i)) = 7 \) and \( h^1(\mathcal{O}_X(6l - 2 \sum_{i=4}^{10} E_i)) = 0 \). Again, from the exact sequence (6), we have that \( h^0(\mathcal{O}_X(12l - 2 \sum_{i=1}^{3} E_i - 4 \sum_{i=4}^{10} E_i)) = 12 \) and \( h^1(\mathcal{O}_X(12l - 2 \sum_{i=1}^{3} E_i - 4 \sum_{i=4}^{10} E_i)) = 0 \). Finally, from the exact sequence (5), we obtain that
\[
h^0(\mathcal{O}_X(18l - 4 \sum_{i=1}^{3} E_i - 6 \sum_{i=4}^{10} E_i)) = 12 + 1 = 13.
\]
Then the claim is proved.

Step by step we want to show that a general hyperplane section of \( X' \) is a Prym-canonical embedded curve.

**CLAIM 2:** We prove that \( \mathcal{O}_{C'}(-K_{X'}) \not\cong \mathcal{O}_{C'} \).

**Proof.** If we show that \( h^1(\mathcal{O}_{X'}(-K_{X'} - C')) = 0 \), then, using the long exact sequence associated with
\[
0 \to \mathcal{O}_{X'}(-K_{X'} - C') \to \mathcal{O}_{X'}(-K_{X'}) \to \mathcal{O}_{C'}(-K_{X'}) \to 0
\]
and observing that \(-K_{X'}\) is not effective, we have that \( h^0(\mathcal{O}_{C'}(-K_{X'})) = 0 \), thus \( \mathcal{O}_{C'}(-K_{X'}) \not\cong \mathcal{O}_{C'} \).

Since
\[
-K_{X'} - C' \sim -15l + 3 \sum_{i=1}^{3} E_i + 5 \sum_{i=4}^{10} E_i,
\]
then it is not effective and \( h^0(\mathcal{O}_{X'}(-K_{X'} - C')) = 0 \). By Serre Duality, we have that \( h^2(\mathcal{O}_{X'}(-K_{X'} - C')) = h^0(\mathcal{O}_{X'}(2K_{X'} + C')) = h^0(\mathcal{O}_{X'}(12l - 2 \sum_{i=1}^{3} E_i - 4 \sum_{i=4}^{10} E_i)) = 12 \) as proved in Claim 1.

Using the Riemann-Roch Theorem, we conclude that \(-h^1(\mathcal{O}_{X'}(-K_{X'} - C')) + h^2(\mathcal{O}_{X'}(-K_{X'} - C')) = h^1(\mathcal{O}_{X'}(-K_{X'} - C')) + 12 = \frac{1}{2}(-15l + 3 \sum_{i=1}^{3} E_i + 5 \sum_{i=4}^{10} E_i)((-12l+2 \sum_{i=1}^{3} E_i + 4 \sum_{i=4}^{10} E_i) + 1 - 0) = 12 \), so \( h^1(\mathcal{O}_{X'}(-K_{X'} - C')) = 0 \) and the claim is proved. □

**CLAIM 3:** There are irreducible curves of degree 18 with exactly 3 quadruple points and 7 points of multiplicity six in the ten nodes of \( J \).

**Proof.** We observe that curves of the type \( J + D \), with \( D \in \{12l - 2 \sum_{i=1}^{3} E_i - 4 \sum_{i=4}^{10} E_i\} \) and \( J \) fixed part, are contained in \( L' = \{18l - 4 \sum_{i=1}^{3} E_i - 6 \sum_{i=4}^{10} E_i\} \). As proved in Claim 1, we have that \( \dim \{18l - 4 \sum_{i=1}^{3} E_i - 6 \sum_{i=4}^{10} E_i\} = 12 \) while \( \dim \{12l - 2 \sum_{i=1}^{3} E_i - 4 \sum_{i=4}^{10} E_i\} = 11 \), so the reducible curves \( J + D \) do not fill up all the linear system of the curves of degree 18. As consequence of Bertini’s Theorem (see [2], pag. 1), the generic curve of \( L' \) is irreducible (indeed the curves of the linear system \( L' \) with fixed part \( J \) define a sublinear system and moreover the sublinear system is not composed by a pencil, even more so the linear system \( L' \)).
Also curves of the type $2J + F$, with $F \in |6l - 2\sum_{i=4}^{10} E_i|$ and $2J$ fixed part, are contained in $L'$. Since these special curves of $L'$ have exactly quadruple points in three of the 10 nodes of $J$ and points of multiplicity 6 in seven of the 10 nodes of $J$, then the generic curves of the linear system $L'$ have the same property. Thus irreducible curves of degree 18 with exactly quadruple points in three of the 10 nodes of $J$ and points of multiplicity 6 in the remaining nodes of $J$ exist. □

Therefore the arithmetic genus, that is equal to the geometric genus of $C$, is

$$g(C') = \frac{17 \cdot 16}{2} - 3 \cdot 4 \cdot 3 - 7 \cdot 6 \cdot 5 = 13$$

by the Plücker Formula.

It remais to show that $L'$ defines an embedding outside the contracted curve $J$.

**CLAIM 4:** The linear system $L'$ is base-point free.

**Proof.** Let $\mathcal{X} = B_{x_{11}}(X')$, where $x_{11}$ is a point of a general $C' \in L'$. If $\mathcal{Y} = |18l - 4\sum_{i=1}^{10} E_i - 6 \sum_{i=4}^{10} E_i - E_{11}|$, for $E_{11}$ the exceptional divisor associated with $x_{11}$, then $L'$ is base point free if and only if

$$\dim(\mathcal{Y}) = \dim(L') - 1,$$

for any point $x_{11} \in C'$, for a general $C' \in L'$.

We have already proved that $\dim(L') = 12$, instead $\dim(\mathcal{Y}) \geq \binom{10}{2} - 3 \cdot 4 \cdot 3 - 7 \cdot 6 \cdot 5 - 1 - 1 = 11$. We observe that, since $x_{11} \in C'$ and $C'$ and $J$ are disjoint by assumptions, then $\mathcal{O}_{\mathcal{Y}}(\mathcal{Y}) \cong \mathcal{O}_{\mathcal{Y}^{'}}$, where $\mathcal{Y}'$ is a general curve in $\mathcal{Y}$ and $J$ is the strict transform of $J$ on $\mathcal{X}'$.

Similarly to the exact sequences (5), (6), we have the following:

(9)

$$0 \to \mathcal{O}_{\mathcal{X}}(12l - 2\sum_{i=1}^{3} E_i - 4\sum_{i=4}^{10} E_i - E_{11}) \to \mathcal{O}_{\mathcal{X}}(18l - 4\sum_{i=1}^{10} E_i - 6 \sum_{i=4}^{10} E_i - E_{11}) \to \mathcal{O}_{\mathcal{Y}} \to 0$$

$$\to \mathcal{O}_{\mathcal{X}}(6l - 2\sum_{i=4}^{10} E_i - E_{11}) \to \mathcal{O}_{\mathcal{X}}(12l - 2\sum_{i=1}^{3} E_i - 4 \sum_{i=4}^{10} E_i - E_{11}) \to$$

$$\to \mathcal{O}_{\mathcal{X}}(12l - 2\sum_{i=1}^{3} E_i - 4 \sum_{i=4}^{10} E_i - E_{11}) \to 0.$$ (10)

As in Claim 1, we suppose that an irreducible conic passing through $\{x_4, \ldots, x_8\}$ exists.

- If $x_{11} \in 2l - \sum_{i=4}^{8} E_i$, we can consider the exact sequence

  $$0 \to \mathcal{O}_{\mathcal{X}}(4l - 8\sum_{i=4}^{8} E_i - 2 \sum_{i=9}^{10} E_i) \to \mathcal{O}_{\mathcal{X}}(6l - 2\sum_{i=4}^{10} E_i - E_{11}) \to$$

  $$\to \mathcal{O}_{2l - \sum_{i=4}^{8} E_i - E_{11}}(6l - 2\sum_{i=4}^{10} E_i - E_{11}) \to 0.$$ (11)

From the exact sequence (5), we know that $h^0(\mathcal{O}_{\mathcal{X}}(4l - 8\sum_{i=4}^{8} E_i - 2 \sum_{i=9}^{10} E_i)) = 4$ and $h^1(\mathcal{O}_{\mathcal{X}}(4l - 8\sum_{i=4}^{8} E_i - 2 \sum_{i=9}^{10} E_i)) = 0$.

Since $h^0(\mathcal{O}_{2l - \sum_{i=4}^{8} E_i - E_{11}}(6l - 2\sum_{i=4}^{10} E_i - E_{11})) = 2$ by Riemann-Roch’s Theorem, then $h^0(\mathcal{O}_{\mathcal{X}}(6l - 2\sum_{i=4}^{10} E_i - E_{11})) = 6$ and $h^1(\mathcal{O}_{\mathcal{X}}(6l - 2\sum_{i=4}^{10} E_i - E_{11})) = 0$ from the exact sequence (11).
• If $x_{11} \notin 2l - \sum_{i=4}^{8} E_i$, then we consider the following

$$0 \to \mathcal{O}_X(4l - \sum_{i=4}^{8} E_i - 2E_9 - 2E_{10} - E_{11}) \to \mathcal{O}_X(6l - 2\sum_{i=4}^{10} E_i - E_{11}) \to$$

(12)

$$\to \mathcal{O}_{2l - \sum_{i=4}^{8} E_i}(6l - 2\sum_{i=4}^{10} E_i - E_{11}) \to 0.$$ 

♦ If $x_{11} \in l - E_9 - E_{10}$, we consider

$$0 \to \mathcal{O}_X(3l - \sum_{i=4}^{10} E_i) \to \mathcal{O}_X(4l - \sum_{i=4}^{8} E_i - 2\sum_{i=9}^{10} E_i - E_{11}) \to$$

(13)

$$\to \mathcal{O}_{l-E_9-E_{10}-E_{11}}(4l - \sum_{i=4}^{8} E_i - 2\sum_{i=9}^{10} E_i - E_{11}) \to 0.$$ 

It is obvious that $h^0(\mathcal{O}_X(3l - \sum_{i=4}^{10} E_i)) = 3$ and $h^0(\mathcal{O}_{l-E_9-E_{10}-E_{11}}(4l - \sum_{i=4}^{8} E_i - 2\sum_{i=9}^{10} E_i - E_{11})) = 0$. From the exact sequence (13), we obtain that $h^0(\mathcal{O}_X(4l - \sum_{i=4}^{8} E_i - 2\sum_{i=9}^{10} E_i - E_{11})) = 3$.

♦ If $x_{11} \notin l - E_9 - E_{10}$, we can consider the following exact sequences

$$0 \to \mathcal{O}_X(3l - \sum_{i=4}^{11} E_i) \to \mathcal{O}_X(4l - \sum_{i=4}^{8} E_i - 2\sum_{i=9}^{10} E_i - E_{11}) \to$$

(14)

$$\to \mathcal{O}_{l-E_9-E_{10}}(4l - \sum_{i=4}^{8} E_i - 2\sum_{i=9}^{10} E_i - E_{11}) \to 0;$$

$$0 \to \mathcal{O}_X(l - \sum_{i=9}^{11} E_i) \to \mathcal{O}_X(3l - \sum_{i=4}^{11} E_i) \to$$

(15)

$$\to \mathcal{O}_{2l-\sum_{i=4}^{8} E_i}(3l - \sum_{i=4}^{11} E_i) \to 0.$$ 

By assumption we have that $h^0(\mathcal{O}_X(l - \sum_{i=9}^{11} E_i)) = 0$. Because $h^0(\mathcal{O}_{2l-\sum_{i=4}^{8} E_i}(3l - \sum_{i=4}^{11} E_i)) = 2$, then, from the exact sequence (15) we conclude that $h^0(\mathcal{O}_X(3l - \sum_{i=4}^{11} E_i)) \leq 2$. Since $h^0(\mathcal{O}_X(3l - \sum_{i=4}^{11} E_i)) \geq \binom{6}{2} - 8 = 3$, then equality holds.

Moreover $h^0(\mathcal{O}_X(4l - \sum_{i=4}^{8} E_i - 2\sum_{i=9}^{10} E_i - E_{11})) \geq \binom{6}{2} - 6 - 3 - 3 = 3$. Since $h^0(\mathcal{O}_{l-E_9-E_{10}}(4l - \sum_{i=4}^{8} E_i - 2\sum_{i=9}^{10} E_i - E_{11})) = 1$, then, from the exact sequence (14), we have that $h^0(\mathcal{O}_X(4l - \sum_{i=4}^{8} E_i - 2\sum_{i=9}^{10} E_i - E_{11})) \leq 3$, so equality holds.
In both previous cases, we have found \( h^0(\mathcal{O}_{X'}(4l - \sum_{i=4}^{8} E_i - 2 \sum_{i=9}^{10} E_i - E_{11})) = 3 \). Since it is the expected dimension, then \( h^1(\mathcal{O}_{X'}(4l - \sum_{i=4}^{8} E_i - 2 \sum_{i=9}^{10} E_i - E_{11})) = 0 \).

Using the Riemann-Roch Theorem, we have that \( h^0(\mathcal{O}_{X'}(6l - 2 \sum_{i=4}^{10} E_i - E_{11})) = 3 \). So we obtain that \( h^0(\mathcal{O}_{X'}(6l - 2 \sum_{i=4}^{10} E_i - E_{11})) = 6 \) and \( h^1(\mathcal{O}_{X'}(6l - 2 \sum_{i=4}^{10} E_i - E_{11})) = 0 \) from the exact sequence \( \text{[12]} \).

In all two cases we have that \( h^0(\mathcal{O}_{X'}(6l - 2 \sum_{i=4}^{10} E_i - E_{11})) = 6 \).

With the same techniques as before, from the exact sequence \( \text{[10]} \), we have that \( h^0(\mathcal{O}_{X'}(12l - 2 \sum_{i=3}^{3} E_i - 4 \sum_{i=4}^{10} E_i - E_{11})) = 11 \) and \( h^1(\mathcal{O}_{X'}(12l - 2 \sum_{i=3}^{3} E_i - 4 \sum_{i=4}^{10} E_i - E_{11})) = 0 \) and finally, from the exact sequence \( \text{[9]} \), we obtain that \( h^0(\mathcal{O}_{X'}(18l - 4 \sum_{i=1}^{3} E_i - 6 \sum_{i=4}^{10} E_i - E_{11})) = 12 \). Then we can conclude that \( L' \) is base-point free.

By Bertini’s Theorem, since \( L' \) is base-point free, then the generic \( C' \in L' \) is smooth.

**Claim 5:** The linear system \( L' \) defines an embedding outside the contracted curve \( J \).

**Proof.** It is sufficient to show that \( \dim L' = \dim(L') - 2 \), where either \( L' = \sum_{i=1}^{10} E_i - 6 \sum_{i=4}^{10} E_i - E_{11} - E_{12} \), for \( E_{11} \) and \( E_{12} \) the exceptional divisors associated with any two distinct points \( x_{11} \) and \( x_{12} \) not belonging to \( J \), or \( L' = \sum_{i=1}^{10} E_i - 6 \sum_{i=4}^{10} E_i - E_{11} - 2E_{12} \), for \( E_{11} \) and \( E_{12} \) the exceptional divisors associated with any two points \( x_{11} \) and \( x_{12} \) infinitely near not belonging to \( J \).

We have already proved that \( \dim(L') = 12 \). Moreover we have that \( \dim(L') \geq \binom{20}{2} - 3\cdot\frac{4}{2} - 7\cdot\frac{7}{2} - 2 - 1 = 10 \). If \( C' \) is a general curve in \( L' \) and \( J \) is the strict transform of \( J \) on \( X' \), then \( C' \cdot J = 0 \) since we choose \( x_{11} \) and \( x_{12} \) not belonging to \( J \).

We will show that \( L' \) defines an embedding outside the contracted curve \( J \) assuming \( x_{11} \) and \( x_{12} \) distinct. The proof is similar for \( x_{11} \) and \( x_{12} \) infinitely near.

We can consider the following exact sequences:

\[
0 \to \mathcal{O}_{X'}(12l - 2 \sum_{i=1}^{3} E_i - 4 \sum_{i=4}^{10} E_i - E_{11} - E_{12}) \to \\
\to \mathcal{O}_{X'}(18l - 4 \sum_{i=1}^{3} E_i - 6 \sum_{i=4}^{10} E_i - E_{11} - E_{12}) \to \mathcal{O}_{p_1} \to 0;
\]

\[
0 \to \mathcal{O}_{X'}(6l - 2 \sum_{i=4}^{10} E_i - E_{11} - E_{12}) \to \mathcal{O}_{X'}(12l - 2 \sum_{i=1}^{3} E_i - 4 \sum_{i=4}^{10} E_i - E_{11} - E_{12}) \to \\
\to \mathcal{O}_{Z'}(12l - 2 \sum_{i=1}^{3} E_i - 4 \sum_{i=4}^{10} E_i - E_{11} - E_{12}) \to 0
\]

\[
0 \to \mathcal{O}_{X'}(2 \sum_{i=1}^{3} E_i - E_{11} - E_{12}) \to \mathcal{O}_{X'}(6l - 2 \sum_{i=4}^{10} E_i - E_{11} - E_{12}) \to \\
\to \mathcal{O}_{X'}(6l - 2 \sum_{i=4}^{10} E_i - E_{11} - E_{12}) \to 0
\]
We have that \( \dim h^0(\mathcal{O}_\mathcal{X}(2 \sum_{i=4}^{3} E_i - E_{11} - E_{12})) = 0 \). Again, by Serre Duality, we have that \( h^0(\mathcal{O}_\mathcal{X}(2 \sum_{i=4}^{3} E_i - E_{11} - E_{12})) = 0 \). Using the Riemann-Roch Theorem, we obtain that \( -h^0(2 \sum_{i=4}^{3} E_i - E_{11} - E_{12}) = \frac{1}{2}(2 \sum_{i=4}^{3} E_i - E_{11} - E_{12}) \cdot (3l + \sum_{i=4}^{3} E_i - \sum_{i=4}^{10} E_i - 2E_{11} - 2E_{12}) + 1 = -4 \).

Since \( \mathcal{J}(6l - 2 \sum_{i=4}^{10} E_i - E_{11} - E_{12}) = 8 \), then \( h^0(\mathcal{O}_\mathcal{X}(6l - 2 \sum_{i=4}^{10} E_i - E_{11} - E_{12})) = 9 \). Moreover \( h^0(\mathcal{O}_\mathcal{X}(6l - 2 \sum_{i=4}^{10} E_i - E_{11} - E_{12})) \geq (\frac{6}{7}) - 7 \cdot (3 - 2) = 5 \). To show that equality holds, it is sufficient to prove that \( h^1(\mathcal{O}_\mathcal{X}(6l - 2 \sum_{i=4}^{10} E_i - E_{11} - E_{12})) = 0 \).

We have that \( \dim [6l - 2 \sum_{i=4}^{10} E_i - E_{11} - E_{12}] \geq 5 \) while \( \dim [3l - \sum_{i=4}^{10} E_i] = (\frac{6}{7}) - 7 = 5 \cdot \frac{4}{7} - 7 = 3 \), so the reducible curves of the type \( (3l - \sum_{i=4}^{10} E_i) + F \) do not fill up all the linear system of the curves of degree 6. As consequence of Bertini’s Theorem (see [2], pag. 1), the generic curve \( D \) of \( [6l - 2 \sum_{i=4}^{10} E_i - E_{11} - E_{12}] \) is irreducible (indeed the curves of the linear system \( [6l - 2 \sum_{i=4}^{10} E_i - E_{11} - E_{12}] \) with fixed part \( 3l - \sum_{i=4}^{10} E_i \) define a sublinear system and moreover the sublinear system is not composed by a pencil, even more so the linear system \( [6l - 2 \sum_{i=4}^{10} E_i - E_{11} - E_{12}] \).

Let us consider the exact sequence

\[
0 \to \mathcal{O}_\mathcal{X} \to \mathcal{O}_\mathcal{X}(D) \to \mathcal{O}_D(D) \to 0.
\]

Since \( D^2 = 36 - 28 - 2 = 6 \) and \( p_a(D) = \frac{5}{2} \cdot \frac{7}{2} - 7 = 3 \), then \( h^1(\mathcal{O}_D(D)) = 0 \) (see [11], Example IV.1.3.4). Because \( h^1(\mathcal{O}_\mathcal{X}) = 0 \) by definition, then \( h^1(\mathcal{O}_\mathcal{X}(6l - 2 \sum_{i=4}^{10} E_i - E_{11} - E_{12})) = 0 \) from the exact sequence \((19)\). Consequently \( h^0(\mathcal{O}_\mathcal{X}(6l - 2 \sum_{i=4}^{10} E_i - E_{11} - E_{12})) = 5 \) from the exact sequence \((18)\).

Since \( \mathcal{J} \) is rational, then \( h^0(\mathcal{O}_\mathcal{X}(12l - 2 \sum_{i=1}^{3} E_i - 4 \sum_{i=4}^{10} E_i - E_{11} - E_{12})) = 5 \), so, from the exact sequence \((16)\), we have that \( h^0(\mathcal{O}_\mathcal{X}(12l - 2 \sum_{i=1}^{3} E_i - 4 \sum_{i=4}^{10} E_i - E_{11} - E_{12})) = 10 \) and \( h^1(\mathcal{O}_\mathcal{X}(12l - 2 \sum_{i=1}^{3} E_i - 4 \sum_{i=4}^{10} E_i - E_{11} - E_{12})) = 0 \).

Finally, from the exact sequence \((17)\), we obtain that \( h^0(\mathcal{O}_\mathcal{X}(18l - 4 \sum_{i=1}^{3} E_i - 6 \sum_{i=4}^{10} E_i - E_{11} - E_{12})) = 11 \). The claim is proved.

We have found a new example of rational surface \( X \subset \mathbb{P}^{12} \) of degree \( \deg(X) = C^{2}/\deg(C) = 24 \) with Prym-canonical hyperplane sections and only one singularity, a quartic rational singularity.

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