A characterization of absorbing sets in coalition formation games*

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May 16, 2024

Abstract

Given a standard myopic dynamic process among coalition structures, an absorbing set is a minimal collection of such structures that is never left once entered through that process. Absorbing sets are an important solution concept in coalition formation games, but they have drawbacks: they can be large and hard to obtain. In this paper, we characterize an absorbing set in terms of a collection consisting of a small number of sets of coalitions that we refer to as a “reduced form” of a game. We apply our characterization to study convergence to stability in several economic environments.

JEL classification: C71, C78.

Keywords: Coalition formation, absorbing set, reduced form of a game, convergence to stability.

1 Introduction

Coalition formation games have become a central focus in a substantial body of literature addressing diverse social and economic issues. This includes topics such as the establishment of cartels, lobbying, customs unions, conflicts, the provision of public goods, and the formation of political parties (see Ray, 2007; Ray and Vohra, 2015). Coalition formation games encompass

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*We thank the Advisory Editor and two anonymous reviewers for their valuable comments. We also thank Elena Molis for her participation in the early stages of the paper, and Coralio Ballester, Jordi Massó, Alejandro Neme, Oscar Volij, and Peio Zuazo-Garin for their suggestions and comments. Inarra acknowledges financial support from the Spanish Ministry of Economy and Competitiveness (project PID2019-107539GB-I00), and from the Basque Government (project IT1367-19); and the hospitality of the Instituto de Matemática Aplicada San Luis, Argentina. Bonifacio and Neme acknowledge financial support from the UNSL through grants 032016 and 030320, from the Consejo Nacional de Investigaciones Científicas y Técnicas (CONICET) through grant PIP 112-200801-00655, and from Agencia Nacional de Promoción Científica y Tecnológica through grant PICT 2017-2355; and the hospitality of the University of the Basque Country (UPV/EHU), Spain, during their respective visits.

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matching models that have significantly improved the understanding and design of mechanisms for various applications, such as school choice and kidney exchange (see Roth, 2018, and references therein), one-sided problems like the roommate problem, as well as two-sided problems ranging from the classical marriage problem to many-to-one matching problems with peer effects and externalities.

A widely studied solution concept for coalition formation games is that of a stable coalition structure.\(^1\) A coalition structure is a partition of the set of agents into coalitions and a stable coalition structure is a coalition structure that "protects" its coalitions in the following way: Whenever some agents have an incentive to deviate to an external coalition (outside the coalition structure), another coalition in the structure impedes its formation because there is (at least) one agent that prefers that coalition to the external coalition. Note, however, that a coalition formation game may have no stable coalition structure.

In coalition formation games, there is a more general solution concept called absorbing set.\(^2\) An important feature of these sets is that they are known to exist for every coalition formation game. Consider the dynamic process where an unstable coalition structure undergoes adjustments when certain agents collectively decide to deviate and form a new coalition, with abandoned agents becoming singletons in the new structure.\(^3\) An absorbing set is a minimal collection of coalition structures that, once entered throughout this dynamic process, is never left. Furthermore, a stable coalition structure generates an absorbing set consisting of only that coalition structure. We call “trivial” to these absorbing sets. “Non-trivial” absorbing sets, in counter-position, have several coalition structures, none of which are stable.

In the case of roommate problems, coalition formation is well-understood. A roommate problem has either trivial absorbing sets or non-trivial absorbing sets (Inarra et al., 2013). The existence of non-trivial absorbing sets in these problems is equivalent to the existence of a set consisting of an odd number of coalitions presenting cyclical behavior, called an “odd ring” (Tan, 1991). Furthermore, in the trivial absorbing set case, starting from an arbitrary coalition structure a path to a stable coalition structure is guaranteed (Diamantoudi et al., 2004). However, none of these results hold in more general settings where trivial and non-trivial absorbing sets can coexist.

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\(^1\) In the literature on coalition formation games the papers by Banerjee et al. (2001); Bogomolnaia and Jackson (2002); Iehlé (2007) identify structures of preferences that guarantee the existence of stable coalition structures. Echenique and Yenmez (2007) develop an algorithm for matching markets with preferences over colleagues to determine the existence of stable matchings. Pycia (2012) and Gallo and Inarra (2018), in different contexts, study what sharing rules induce stable coalition formation games. Two other notable contributions to studying sharing rules and stable coalition structures are Barberà et al. (2015) and Herings et al. (2021).

\(^2\) This notion has been studied by several authors under different names and in different contexts. As far as we know, Schwartz (1970) was the first to introduce it for collective decision-making problems. See also Inarra et al. (2013); Jackson and Watts (2002); Olaizola and Valenciano (2014). The union of absorbing sets gives the “admissible set” (Kalai and Schmeidler, 1977). Recently, Demuynck et al. (2019) define the “myopic stable set” in a very general class of social environments and study its relation to other solution concepts.

\(^3\) This process corresponds to the \(\gamma\)-model of Hart and Kurz (1983). These authors argue that when a coalition is formed through the agreement of all its members, and subsequently some agents withdraw, the agreement dissolves, and the remaining agents become singletons. In our analysis, this assumption fits well, as our modeling may not encompass all possible coalitions, and in such instances, the agents who are abandoned may not be able to remain as a “permissible” coalition.
Absorbing sets in general settings have other drawbacks. On one hand, in order to compute an absorbing set, every coalition structure must be computed. On the other hand, in order to define them, a dynamic process among coalition structures has to be specified. To avoid such drawbacks, our goal is to characterize an absorbing set in terms of a collection consisting of a small number of sets of coalitions, which we call a “reduced form”. This reduced form condenses all the relevant information on what coalition structures will form in the absorbing set.

To illustrate the complexity of an absorbing set consider the game presented in Table 1.4

|   | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  |
|---|----|----|----|----|----|----|----|----|
| 12| 23 | 34 | 46 | 45 | 678| 78 | 78 |
| 13| 12 | 13 | 45 | 56 | 56 | 678| 678|
| 1 | 2  | 23 | 34 | 5  | 46 | 7  | 8  |
|   | 3  | 4  | 6  |    |    |    |    |

Table 1: An 8-agent game.

Let us take a look at the preferences in Table 1. Agent 1, for instance, prefers coalition 12 to coalition 13, and no other coalition (to which agent 1 belongs) is “permissible” for this agent. It can be checked that this game has only one absorbing set, depicted in Figure 1. This absorbing set is non-trivial and has fourteen coalition structures. Note that coalition 78 is present in every coalition structure of the absorbing set.

In Table 1, we can also observe that the set of coalitions \{13,12,23\} presents “cyclical behavior”: agent 1 prefers coalition 12 to coalition 13, agent 3 prefers coalition 13 to coalition 23, and agent 2 prefers coalition 23 to coalition 12. Observe that this cyclical behavior among coalitions induces a cyclical behavior among coalition structures. For instance, in Figure 1, \{12,3,46,5,78\} dominates \{13,2,46,5,78\}, \{13,2,46,5,78\} dominates \{1,23,46,5,78\}, and in turn \{1,23,46,5,78\} dominates \{12,3,46,5,78\}. The same occurs with the set of coalitions \{45,46,56\} and, for instance, with coalition structures \{12,3,45,6,78\}, \{12,3,46,5,78\}, and \{12,3,4,56,78\}. Notice that, although both sets of coalitions \{13,12,23\} and \{45,46,56\} present cyclical behavior, their influence on the absorbing set is different since:

(i) in each coalition structure of the absorbing set, there is one coalition of the set \{45,46,56\}, and

(ii) some coalition structures have no coalition of the set \{13,12,23\} present.

The only deviations from coalitions in the set \{45,46,56\} that agents can perform within the absorbing set involve only coalitions in that same set. To see this, notice that the only agent having incentive to deviate from any coalition structure of the absorbing set to an external coalition is agent 6 (to coalition 678). As coalition 78 is formed in each coalition structure of the absorbing set, and agent 7 prefers coalition 78 to coalition 678, such deviation will never be performed. So the cyclical behavior of the set of coalitions \{45,46,56\} is never disrupted. However, this is not true for the set \{13,12,23\}. For instance, agent 3 has the incentive to

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4In order to ease notation, throughout the paper we denote coalitions without curly brackets and commas, i.e., coalition \{6,7,8\} is simply written 678.
Figure 1: Non-trivial absorbing set of game.

deviate from coalition structure \{13, 2, 4, 56, 78\} to the external coalition 34, and there is no way to impede such deviation. When this happens, the cyclical behavior of the set of coalitions \{13, 12, 23\} is disrupted.

Now, consider the collection formed by the following three sets: \{45, 46, 56\}, \{78\}, and \{1, 2, 3\}. We claim that this collection condenses all the relevant information to reconstruct the absorbing set. As just analyzed, sets \{45, 46, 56\}, and \{78\} “protect” each other from external deviations, implying (i) and the fact that coalition 78 is present in every coalition structure of the absorbing set. Furthermore, some coalition structures in the absorbing set have all agents of the third set as singletons, implying (ii). Collections of this type, which we call “reduced forms”, are our proposal to characterize absorbing sets. They are composed by

- sets consisting of one non-singleton coalition that we call “fixed components”,
- sets consisting of coalitions presenting cyclical behavior that we call “generalized rings”, and
- a set consisting of singleton coalitions.

Fixed components and generalized rings in a reduced form protect each other in a stable way, whereas any fixed component or generalized ring formed by agents of the set of singletons is not protected by this collection. Our main finding is that each absorbing set generates a reduced form and, conversely, each reduced form generates an absorbing set. Thus, by knowing the
reduced form, any coalition structure of the absorbing set can be identified. These coalition structures include the coalition of each fixed component, some coalitions of the generalized rings, and some permissible coalitions that agents of the set of singletons belong to.

A reduced form with no generalized rings can be identified with a trivial absorbing set. Therefore, as a by-product of our main result, we find that a coalition formation game having a reduced form with no generalized ring has at least one stable coalition structure.

The essence of generalized ring and reduced form notions have already been identified in roommate problems. On the one hand, a generalized ring adapts the notion of an odd ring to general coalition formation games. On the other hand, the idea of reduced form has been studied in roommate problems under the name of “maximal stable partitions” (Inarra et al., 2013). An important difference between roommate problems and general coalition formation games is that the existence of a generalized ring itself does not imply either the non-existence of trivial absorbing sets or the existence of a reduced form to which that generalized ring belongs.

A pertinent question for coalition formation games with a stable coalition structure is whether agents always reach stability when left to interact on their own, or whether there is a need to introduce an arbitrator so that agents can achieve stability. We say that a game exhibits convergence to stability if starting from any coalition structure, there is a path towards a stable coalition structure throughout the domination relation. The reduced form of a game provides us with a tool for presenting a necessary and sufficient condition for general coalition formation games to guarantee that agents achieve stability on their own: If no reduced form of the game has generalized rings, convergence to stability is guaranteed.

As an application of this result on convergence to stability, we show that games satisfying the “common ranking property” (Farrell and Scotchmer, 1988) exhibit convergence to stability, but games satisfying the “weak top coalition property” (Banerjee et al., 2001) and the “ordinally balance property” (Bogomolnaia and Jackson, 2002) may lack convergence to stability.

We also analyze some economic environments in which coalitions produce an output to be divided among their members according to a pre-specified sharing rule. In such environments, the sharing rule naturally induces a game where each agent ranks the coalitions to which she belongs according to the payoffs that she gets. We focus on two types of sharing rules: Bargaining rules and rationing rules. We show that games induced by “pairwise aligned” bargaining rules (Pycia, 2012), which include the Nash bargaining rule (Nash, 1950), exhibit convergence to stability. A similar result is obtained in the context of rationing for “parametric” rules (see Young, 1987; Stovall, 2014), which include several of the most thoroughly-studied rules in the rationing literature.

The rest of the paper is organized as follows. Section 2 presents the preliminaries. Section 3 is divided into four subsections. Subsection 3.1 presents the definition of a generalized ring which is then used in Subsection 3.2 to define a reduced form. Subsection 3.3 characterizes absorbing sets in terms of reduced forms. In Subsection 3.4 we compare reduced forms to

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5 A typical coalition structure of the absorbing set of the previous example has the coalition of the fixed component \{78\}, a coalition of the generalized ring \{45,46,56\}, and permissible coalitions formed by the rest of the agents. For instance, \{1,2,3,45,6,78\} or \{12,34,56,78\}.

6 Notice that, in the game of Table 1, generalized ring \{13,12,23\} does not belong to the unique reduced form.
“maximal stable partitions” in roommate problems. Section 4 is devoted to applying our characterization result to the study of convergence to stability in several economic environments. Section 5 contains some final remarks. Appendix A discusses the relation between two well-known concepts in the literature (rings in preferences and cycles of coalition structures) which is essential for the proof of the characterization result, presented in Appendix B.

2 Preliminaries

Let \( N = \{1, \ldots, n\} \) be a finite set of agents. A non-empty subset \( C \) of \( N \) is called a coalition. Each agent \( i \in N \) has a strict, transitive preference relation on the set of coalitions to which she belongs, denoted by \( \succ_i \). Given coalitions \( C \) and \( C' \), when agent \( i \in C \cap C' \) prefers coalition \( C \) to \( C' \) we write \( C \succ_i C' \). We say that \( C \) is (unanimously) preferred to \( C' \), and write \( C \succ C' \), if \( C \succ_i C' \) for each \( i \in C' \cap C \).

A set of agents \( N \) and a preference profile for such agents \( \succ_N = (\succ_i)_{i \in N} \) define a coalition formation game which is denoted by \((N, \succ_N)\). Let \( \mathcal{K} = \{C \subseteq 2^N : |C| \geq 1 \text{ and } C \succeq_i \{i\} \text{ for each } i \in N\} \) be the set of permissible coalitions of game \((N, \succ_N)\).

Let \( \Pi \) denote the set of partitions of \( N \) formed by permissible coalitions, which we call coalition structures. A generic element of \( \Pi \) is denoted by \( \pi \). For each \( \pi \in \Pi \) and each \( i \in N \), \( \pi(i) \) denotes the coalition in \( \pi \) that contains agent \( i \). Furthermore, given coalition structure \( \pi \) and a coalition \( C \in \mathcal{K} \setminus \pi \), we say that \( C \) blocks \( \pi \) if \( C \succ \pi(i) \) for each \( i \in C \).

The main solution concept for a coalition formation game is that of stability, namely, a coalition structure that is immune to the deviation of coalitions. A coalition structure \( \pi \in \Pi \) is stable if the existence of \( C \in \mathcal{K} \) and \( i \in C \) such that \( C \succ_i \pi(i) \) implies the existence of \( j \in C \) such that \( \pi(j) \succ_j C \). In other words, a stable coalition structure “protects” its coalitions from external deviations. Hereafter, a stable coalition structure is denoted by \( \pi^N \).

Another solution concept for a coalition formation game is that of an “absorbing set”, which can be described as follows: Given a dynamic process defined between coalition structures by means of a domination relation, an absorbing set is a minimal collection of coalition structures that, once entered throughout this dynamic process, is never left. In this paper, we adopt the standard (myopic) dynamic process in which a coalition structure is replaced by a new one where a coalition of better-off agents is formed, agents that are abandoned appear as singletons, and all other coalitions remain unchanged.

Let \((N, \succ_N)\) be a coalition formation game. The domination relation \( \gg \) over \( \Pi \) is defined as follows: \( \pi' \gg \pi \) if and only if there is \( C \in \mathcal{K} \) such that

(i) \( C \in \pi' \) and \( C \succ \pi(i) \) for each \( i \in C \),

(ii) for each \( C' \in \pi \) such that \( C' \cap C \neq \emptyset \), \( \pi'(j) = \{j\} \) for each \( j \in C' \setminus C \),

(iii) for each \( C' \in \pi \) such that \( C' \cap C = \emptyset \), \( C' \in \pi' \).

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7 Here, \( 2^N \) denotes the collection of non-empty subsets of \( N \).
8 The dynamic process just described is inspired by the \( \gamma \)-model by Hart and Kurz (1983). Other dynamic processes can be considered. For instance, the \( \delta \)-model in Hart and Kurz (1983). For roommate problems, Tamura (1993) considers a similar process but abandoned agents instead of remaining single join together in a new coalition.
Notice that \(\gg\) is a binary relation that is irreflexive, antisymmetric, and not necessarily transitive. To stress the role of coalition \(C\), \(\pi'\) is said to dominate \(\pi\) via \(C\), and \(\pi' \gg \pi\) via \(C\) is written. Condition (i) says that each agent \(i\) of coalition \(C\) improves in \(\pi'\) with respect to her position in \(\pi\). Condition (ii) says that the agents in coalitions from which one or more of them depart (to form \(C\)) become singleton sets in \(\pi'\). Condition (iii) says that coalitions that suffer no departures in \(\pi\) remain unchanged in \(\pi'\).

Given the domination relation \(\gg\) between coalition structures, let \(\gg^T\) be the transitive closure of \(\gg\). That is, given any two coalition structures \(\pi\) and \(\pi'\), \(\pi' \gg^T \pi\) if and only if there is a finite sequence of coalition structures \(\pi = \pi^0, \pi^1, \ldots, \pi^J = \pi'\) such that, for all \(j \in \{1, \ldots, J\}\), \(\pi^j \gg \pi^{j-1}\). Now, we are in a position to formalize the notion of absorbing set.

**Definition 1** A non-empty set of coalition structures \(\mathcal{A} \subseteq \Pi\) is an absorbing set whenever for each \(\pi \in \mathcal{A}\) and each \(\pi' \in \Pi \setminus \{\pi\}\),
\[
\pi' \gg^T \pi \text{ if and only if } \pi' \in \mathcal{A}.
\]

If \(|\mathcal{A}| \geq 3\), \(\mathcal{A}\) is said to be a non-trivial absorbing set. Otherwise, the absorbing set is trivial.

Notice that coalition structures in \(\mathcal{A}\) are symmetrically connected by the relation \(\gg^T\), and that no coalition structure in \(\mathcal{A}\) is dominated by a coalition structure outside the set.

**Remark 1** Facts on absorbing sets.

(i) There is always an absorbing set.

(ii) An absorbing set \(\mathcal{A}\) contains no stable coalition structure if and only if \(|\mathcal{A}| \geq 3\).

(iii) \(\pi^N\) is a stable coalition structure if and only if \(\{\pi^N\}\) is a trivial absorbing set.

(iv) For each non-stable coalition structure \(\pi \in \Pi\), there are an absorbing set \(\mathcal{A}\) and a coalition structure \(\pi' \in \mathcal{A}\) such that \(\pi' \gg^T \pi\).

Remark 1 (i) is implied by the finiteness of the set of coalition structures. Remark 1 (ii) is implied by the antisymmetry of \(\gg\). Remark 1 (iii) says that a stable coalition structure is, by definition, immune to the deviation of coalitions and is, therefore, a maximal element for the domination relation \(\gg\). Remark 1 (iv) says that from any non-stable coalition structure, there is a finite sequence of such structures that reaches a coalition structure in an absorbing set. This solution concept can be illustrated with the following example.

**Example 1** Consider the game given by the following table:

|   | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 | 12| 23| 34| 45| 15|
| 2 | 123| 123| 123| 345| 15|
| 3 | 15| 12| 23| 4 | 5 |
| 4 | 1 | 2 | 3 |   |   |

This game has two absorbing sets: a trivial absorbing set corresponding to the unique stable coalition structure \(\{123, 45\}\) and a non-trivial absorbing set depicted in Figure 2. 

\[\diamond\]
3 A characterization of absorbing sets

In this section, we present a characterization of an absorbing set throughout a collection of sets of coalitions that condenses all its relevant information, which we call “reduced form”. Before we state the characterization result, we present the key ingredients of that construct.

3.1 Generalized rings

As argued in the Introduction, there are sets of coalitions presenting cyclical behavior, and some of these sets induce the cyclical behavior of coalition structures in a non-trivial absorbing set. In this subsection, we formalize these sets that we call generalized rings.

For the game in Example 1, consider the collection of coalitions \( B = \{12, 15, 45, 34, 23\} \). This collection presents cyclical behavior among coalitions since \( 12 \succ 15 \succ 45 \succ 34 \succ 23 \succ 12 \). Furthermore, it induces cyclical behavior among coalition structures in the unique non-trivial absorbing set, because for each coalition in \( B \) there are two coalition structures of the non-trivial absorbing set such that one dominates the other via that coalition, for instance, \( \{12, 34, 5\} \) dominates \( \{12, 34, 5\} \) via 45. Note that collection \( \{123, 15, 45, 34\} \) also presents cyclical behavior among coalitions because \( 123 \succ 15 \succ 45 \succ 34 \succ 123 \). However, it does not induce cyclical behavior among coalition structures since there is no pair of coalition structures in the non-trivial absorbing set such that one dominates the other via 123.

Notice that the previous example suggests that we need to request an additional feature to a collection of coalitions presenting cyclical behavior among themselves in order to induce a cyclical behavior among coalition structures of a non-trivial absorbing set. A generalized ring is a collection of coalitions satisfying such an additional feature. Before presenting it formally, we define some notions that will be useful later.

Let \( B \subseteq \mathcal{K} \) be a collection of non-singleton coalitions. A set \( \mathcal{M} \subseteq B \) is maximal for \( B \) if:

1. \( C, C' \in \mathcal{M} \) implies \( C \cap C' = \emptyset \), and
2. for each \( C \in B \setminus \mathcal{M} \) there is \( C' \in \mathcal{M} \) such that \( C \cap C' \neq \emptyset \).

Condition (i) says that coalitions of a maximal set are pairwise disjoint. Condition (ii) establishes that the set cannot be enlarged. Denote by \( \mathcal{M}_B \) the collection of all maximal sets for \( B \).
Given a collection of non-singleton coalitions \( B \subseteq K \) and a coalition \( C \in K \setminus B \), we say that \( C \) breaks \( B \) if there is a maximal set \( M \in \mathcal{M}_B \) such that:

(i) there is a coalition \( C' \in M \) such that \( C \cap C' \neq \emptyset \), and

(ii) \( C \succ C'' \) for each \( C'' \in M \) such that \( C'' \cap C \neq \emptyset \).

Condition (i) says that there is always a coalition in \( M \) with agents in common with \( C \). Condition (ii) says that coalition \( C \) is preferred to each coalition in \( M \) that has agents in common with \( C \).

**Definition 2** Given a game \((N, \succ_N)\), a set of non-singleton coalitions \( B \subseteq K \) is a generalized ring if

(i) \( C \succ^T C' \) for each pair \( C, C' \in B \) with \( C \neq C' \),

(ii) for each maximal set \( M \in \mathcal{M}_B \) there is \( C \in B \setminus M \) such that \( C \) breaks \( M \).

Notice that \( |B| \geq 3 \). Condition (i) says that each coalition in the set is transitively preferred to any other coalition in the set. Condition (ii) says that for each set of disjoint coalitions of the generalized ring, there is another coalition of the generalized ring that breaks the set.

Consider the set \( \{15, 123, 34, 45\} \) in Example 1. This set fulfills Condition (i) but not Condition (ii) in Definition 2 because, for example, the maximal set \( \{15, 34\} \) cannot be broken by any other coalition in the set. Now, consider the set \( \{15, 12, 23, 34, 45\} \) in Example 1. This set fulfills both Conditions (i) and (ii) in Definition 2. It is easy to see that such a set consists of all the non-singleton coalitions that belong to a coalition structure of the non-trivial absorbing set depicted in Figure 2.

The following example illustrates a set of coalitions fulfilling Condition (ii) but not Condition (i) in Definition 2.

**Example 2** Consider the game given by the following table:

|   | 1   | 2   | 3   | 4   | 5   | 6   |
|---|-----|-----|-----|-----|-----|-----|
| 12| 23  | 13  | 45  | 56  | 46  |     |
| 123| 123 | 123 | 456 | 456 | 456 |     |
| 13 | 12  | 23  | 46  | 45  | 56  |     |
| 1  | 2   | 3   | 4   | 5   | 6   |     |

In this game there are two generalized rings: \( \{13, 12, 23\} \) and \( \{46, 45, 56\} \). Notice that their union fulfills Condition (ii) but not Condition (i). Therefore, such a union is not a generalized ring.

By delving deeper into the analysis we can distinguish two different types of generalized rings depending on whether each coalition structure of a non-trivial absorbing set includes a maximal set of the generalized ring or not. In Example 1, the first situation occurs. The maximal sets of the generalized ring \( \{12, 23, 34, 45, 15\} \) are \( \{12, 34\}, \{12, 45\}, \{23, 45\}, \{23, 15\}, \) and \( \{34, 15\} \). Each coalition structure of the non-trivial absorbing set includes one of these maximal sets. This is not always the case, though. To show this, we present the following example.

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\(^9\)Here \( \succ^T \) denotes the transitive closure of relation \( \succ \). That is, \( C' \succ^T C \) if and only if there is a finite sequence of coalitions \( C = C_0, C_1, \ldots, C_j = C' \) such that, for all \( j \in \{1, \ldots, J\} \), \( C_j \succ C_{j-1} \).
Example 3  Consider the game given by the following table:

|   | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  |
|---|----|----|----|----|----|----|----|----|
| 1 | 12 | 23 | 356| 145| 356| 678| 78 | 678|
| 145| 12 | 23 | 46 | 145| 46 | 678| 78 | 78 |
| 1  | 2  | 3  | 4  | 5  | 356| 7  | 8  | 6 |

The unique generalized ring is \{145, 12, 23, 356, 46\} and its maximal sets are \{145, 23\}, \{12, 356\}, \{12, 46\}, and \{23, 46\}. Notice that coalition 356 breaks the maximal set \{145, 23\} and has a non-empty intersection with two coalitions: 145 and 23. This means that coalition structures \{1, 2, 356, 4, 78\}, \{1, 2, 3, 46, 5, 78\}, \{145, 2, 3, 6, 78\}, \{12, 3, 4, 5, 6, 78\}, and \{1, 23, 4, 5, 6, 78\} belong to the non-trivial absorbing set depicted in Figure 3. Note that these coalition structures do not include any maximal set of the generalized ring.

We now formally define the two types of generalized rings.

Definition 3  A generalized ring \(\mathcal{B}\) is **compact** if for each \(\mathcal{M} \in \mathcal{M}_\mathcal{B}\) and each \(C \in \mathcal{B}\) such that \(C\) breaks \(\mathcal{M}\), there is a unique \(C' \in \mathcal{M}\) such that \(C \cap C' \neq \emptyset\). Otherwise, we say that the generalized ring is **non-compact**.
Note that in a non-compact generalized ring there is $M \in \mathcal{M}_B$, a coalition $C \in B$ such that $C$ breaks $M$, and at least two coalitions in $M$ that intersect $C$. Observe that the generalized ring $\{12, 23, 34, 45, 15\}$ in Example 1 is compact. In contrast, the generalized ring $\{145, 12, 23, 356, 46\}$ in Example 3 is non-compact.

### 3.2 The reduced form of a coalition formation game

We have shown a close connection between non-trivial absorbing sets and generalized rings. One or more generalized rings generate a non-trivial absorbing set. However, not all generalized rings can generate a non-trivial absorbing set. The concept of reduced form of a coalition formation game enables us to distinguish between those generalized rings that generate a non-trivial absorbing set and those that do not. The ingredients of a reduced form are generalized rings, sets consisting of one non-singleton coalition (called “fixed components”), and a set of singletons.

Consider again the game in Table 1 presented in the Introduction. Recall that this game has a unique non-trivial absorbing set. Furthermore, there are two generalized rings, $\{13, 12, 23\}$ and $\{45, 46, 56\}$, and:

- Each coalition structure of the absorbing set contains coalition 78.
- There are coalition structures of the absorbing set that contain no coalition of the generalized ring $\{13, 12, 23\}$, for instance, coalition structure $\{1, 2, 3, 45, 6, 78\}$.
- Each coalition structure of the absorbing set contains a coalition of the generalized ring $\{45, 46, 56\}$.

Notice that, from any coalition structure of the absorbing set, agent 6 has the incentive to deviate to external coalition 678. As coalition 78 belongs to each coalition structure of the absorbing set, 78 “impedes coalition 678 from being formed”. Generalized rings can also play this role. The formal definition of this notion is now presented. Given a set of coalitions $\mathcal{D}$, denote by $N(\mathcal{D})$ the set of agents that belong to (at least) one coalition in $\mathcal{D}$, that is, $N(\mathcal{D}) \equiv \bigcup_{C \in \mathcal{D}} C$.

**Definition 4** Given a generalized ring or a fixed component $B$ and a coalition $C \in \mathcal{K} \setminus B$ such that $N(\mathcal{B}) \cap C \neq \emptyset$, we say that $B$ **impedes coalition $C$ to be formed** if:

(i) $B$ is a compact generalized ring and for each $M \in \mathcal{M}_B$ there is $C' \in B$ with $C' \cap C \neq \emptyset$ and an agent $i \in C' \cap C$ such that $C' \succ_i C$.

(ii) $B$ is a non-compact generalized ring or a fixed component and for each $C' \in B$ there is an agent $i \in C' \cap C$ such that $C' \succ_i C$.\(^{10}\)

Condition (i) states that each maximal set of a compact generalized ring contains a coalition with an agent that prefers that coalition to the external coalition. Condition (ii) states that for each coalition of a non-compact generalized ring or for the coalition of each fixed component, an agent prefers that coalition(s) to the external coalition. Notice that, for compact generalized rings, maximal sets are the objects that impede the formation of external coalitions.

\(^{10}\)Notice that if $B$ is a fixed component, there is only one such $C'$.
because only the maximal sets of a compact generalized ring appear in all the coalition structures of an associated absorbing set (see Figure 2). For non-compact generalized rings, however, their coalition(s) are the objects that impede the formation of external coalitions because there are coalition structures of an associated absorbing set that only have one coalition of the non-compact generalized ring. For fixed components, the situation is similar to non-compact generalized rings because they consist of only one coalition.

Our proposal of reduced form for the non-trivial absorbing set of the game presented in Table 1 contains the generalized ring \( \{45, 46, 56\} \), the fixed component \( \{78\} \), and the set of singletons \( \{1, 2, 3\} \). Notice that the generalized ring \( \{45, 46, 56\} \) and the fixed component \( \{78\} \) of this reduced form “protect” each other from external blocking coalitions. However, they do not “protect” the generalized ring \( \{13, 12, 23\} \), and for that reason, agents 1, 2, 3 belong to the set of singletons of the reduced form. Next, we formalize the notion of protection. A collection (of sets of coalitions) \( \mathcal{R} \) is complete if it consists of generalized rings, fixed components, or a set of singletons that fulfills (i) \( \bigcup_{D \in \mathcal{R}} N(D) = N \), together with (ii) \( N(D) \cap N(D') = \emptyset \) for each pair \( D, D' \in \mathcal{R} \).

**Definition 5** Let \( \mathcal{R} \) be a complete collection. A generalized ring or a fixed component \( B \) is said to be protected by \( \mathcal{R} \) if for each coalition \( C \) that breaks \( B \) there is \( B' \in \mathcal{R} \) such that \( B' \) impedes \( C \) to be formed.\(^{11}\)

Now, we are in a position to formally define the notion of a reduced form of a game.

**Definition 6** Let \((N, \succ_N)\) be a coalition formation game. A reduced form of \((N, \succ_N)\) is a complete collection \( \mathcal{R} \) that may include generalized rings, fixed components, or a set of singletons (denoted by \( S \)) and satisfies the following:

(i) each generalized ring and each fixed component in \( \mathcal{R} \) is protected by \( \mathcal{R} \).

(ii) each generalized ring and each fixed component not in \( \mathcal{R} \) generated by agents in \( S \) is not protected by \( \mathcal{R} \).

Condition (i) states that the reduced form protects itself from external deviations. Condition (ii) is a maximality condition: it states that if a generalized ring or a fixed component is added to the collection, the resulting collection no longer protects itself from external deviations.

In Example 1, the game has two reduced forms: \( \{\{15, 12, 23, 34, 45\}\} \) and \( \{\{123\}, \{45\}\} \). The first reduced form contains only one generalized ring, while the second contains only two fixed components. In Example 2, the game has four reduced forms: \( \{\{123\}, \{456\}\}, \{\{13, 12, 23\}, \{456\}\}, \{\{123\}, \{46, 45, 56\}\}, \text{ and } \{\{13, 12, 23\}, \{46, 45, 56\}\} \). The game presented in Table 1 in the Introduction has only one reduced form, consisting of the generalized ring \( \{45, 46, 56\} \), the fixed component \( \{78\} \), and the set of singletons \( \{1, 2, 3\} \).

\(^{11}\)Notice that \( B \) need not be included in \( \mathcal{R} \).
3.3 The characterization result

In this subsection, we relate reduced forms of a game with absorbing sets of such a game. Recall that, by Remark 1, a trivial absorbing set consists of a collection that has as its unique element a stable coalition structure. The following remark shows how a trivial absorbing set can be identified with a reduced form that has only fixed components and (possibly) a set of singletons.

Remark 2 Suppose there is a reduced form that has no generalized ring. In that case, the coalition formation game has a trivial absorbing set, and by Remark 1 (iii) it has a stable coalition structure.

Even though a reduced form and an absorbing set are different concepts, we show they are related. We characterize absorbing sets in terms of reduced forms. Formally,

Theorem 1 For each coalition formation game, each absorbing set generates a reduced form and, conversely, each reduced form generates an absorbing set.

The proof of Theorem 1 is relegated to Appendix B. Essential to this proof is the relation between rings in preferences on one hand and cycles of coalition structures on the other. This relation is analyzed in Appendix A.

The game presented in Table 1 in the Introduction has only one absorbing set, with fourteen coalition structures. Such absorbing set is associated to reduced form \(\{\{1,2,3\},\{45,46,56\},\{78\}\}\). The game presented in Example 1 has two absorbing sets, one trivial and one non-trivial with five coalition structures. They are associated with reduced forms \(\{\{123\},\{45\}\}\) and \(\{\{15,12,23,34,45\}\}\), respectively. The game presented in Example 2 has four absorbing sets: one trivial, associated with reduced form \(\{\{123\},\{45\}\}\); two non-trivial with three coalition structures each, associated with reduced forms \(\{\{13,12,23\},\{456\}\}\) and \(\{\{123\},\{46,45,56\}\}\), respectively; and one non-trivial with nine coalition structures, associated with reduced form \(\{\{13,12,23\},\{46,45,56\}\}\). Notice that no intrinsic property of a generalized ring generates a non-trivial absorbing set. What is important is whether a generalized ring belongs to a reduced form. But, in a game, a generalized ring can belong to some reduced forms (and not to others), or even to none. For instance, in the example of Table 1, the generalized ring \(\{13,12,23\}\) does not belong to the unique reduced form; whereas in Example 2 the generalized ring \(\{46,45,56\}\) belongs to only two of the four different reduced forms.

3.4 Reduced forms in roommate problems

An important class of coalition formation games is the class of roommate problems, introduced by Gale and Shapley (1962). In these games, each agent has preferences over all coalitions of cardinality two to which she belongs. As is known, a roommate problem may not admit stable coalition structures. In this section we show that the notion of reduced form has already been identified in roommate problems under the name of “maximal stable partitions” by Inarra et al. (2013). To do this, we compare the notion of reduced form with the notion of “stable partition” (Tan, 1991) in roommate problems. We show that every reduced form is a stable partition, but only “maximal stable partitions” are reduced forms.
Tan (1991) proves that a roommate problem has no stable coalition structures if and only if there is a “stable partition” with an odd ring. In our terminology, a complete collection \( \mathcal{P} \) is a stable partition if, whenever coalition \( C \) breaks a fixed component or a ring \( B \in \mathcal{P} \), there is another fixed component or ring \( B' \in \mathcal{P} \) such that \( C \setminus N(B) \subseteq N(B') \) and \( R \succ C \) for each \( R \in B' \) with \( R \cap C \neq \emptyset \). This can be illustrated with the following example:

Example 4 (Example 2 in Inarra et al., 2013) Consider the game given by this table:

|    | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | a  |
|----|----|----|----|----|----|----|----|----|----|----|
| 12 | 23 | 13 | 47 | 58 | 69 | 57 | 68 | 49 |  a |
| 13 | 12 | 23 | 48 | 59 | 67 | 67 | 48 | 59 |   |
| 14 | 24 | 34 | 49 | 57 | 68 | 17 | 58 | 69 |   |
| 15 | 25 | 35 | 45 | 46 | 47 | 78 | 79 |   |   |
| 16 | 26 | 36 | 46 | 56 |  6 | 79 | 89 | 89 |   |
| 17 | 27 | 37 | 14 |  5 | 78 | 8  | 9  |   |   |
| 18 | 28 | 38 | 24 |  7 |    |    |    |    |    |
| 19 | 29 | 39 | 34 |    |    |    |    |    |    |
| 1  | 2  | 3  | 4  |    |    |    |    |    |    |

This example has three stable partitions: \( \{\{12, 23, 13\}, \{48\}, \{57\}, \{a\}\} \), \( \{\{12, 23, 13\}, \{49\}, \{57\}, \{68\}, \{a\}\} \), and \( \{\{12, 23, 13\}, \{47\}, \{58\}, \{69\}, \{a\}\} \). The first two are also reduced forms. The third one is a complete collection but not a reduced form because fixed component \( \{47\} \) is not protected by the collection: Coalition 17 breaks fixed component \( \{47\} \) and the collection does not impedes the formation of coalition 17. Thus, for the roommate problem, the notion of stable partition is weaker than the notion of reduced form.

Following Inarra et al. (2013), we use the term maximal stable partition to refer to those stable partitions with the maximal set of satisfied agents, i.e. agents with no incentive to change partners. The following result can then be established:

Proposition 1 For each roommate problem, each maximal stable partition generates a reduced form and, conversely, each reduced form generates a maximal stable partition.

Proof. Theorem 1 in Inarra et al. (2013) proves that there is a bijection between maximal stable partitions and absorbing sets. Our Theorem 1 states that there is a bijection between absorbing sets and reduced forms. Therefore, the result follows straightforwardly.

4 Application: Convergence to stability

In this section, we study an application of our characterization result. A relevant question for coalition formation games with a stable coalition structure (henceforth, stable coalition formation games) is whether agents always reach stability when left to interact on their own, or

\[\text{For a formal definition of “ring” see Appendix A.}\]
whether there is a need to introduce an arbitrator so that agents can achieve stability. There are coalition formation games, such as two-sided matching models (one-to-one and many-to-one) and one-sided matching models (roommate problems), in which no arbitrator is needed to reach stability (see Roth and Vandewalle, 1990; Chung, 2000; Klaus and Klijn, 2005; Kojima and Ünver, 2008; Eriksson and Häggström, 2008; Diamantoudi et al., 2004, for more details). However, this is not always the case in more general coalition formation games.

The reduced form of a game provides us with a tool for presenting a necessary and sufficient condition for general coalition formation games to guarantee that agents achieve stability on their own. We say that a stable coalition formation game \((N, \succ_N)\) exhibits convergence to stability if for each non-stable coalition structure \(\pi \in \Pi\) there is a stable coalition structure \(\pi^N \in \Pi\) such that \(\pi^N \succ^T \pi\). Given a stable coalition formation game, it is clear that if no reduced form has a generalized ring, then the game exhibits convergence to stability. This is because each absorbing set then is trivial. However, if there is a reduced form with a generalized ring, then the game does not exhibit convergence to stability. Formally,

**Proposition 2** A stable coalition formation game exhibits convergence to stability if and only if none of its reduced forms has a generalized ring.

**Proof.** Let \((N, \succ_N)\) be a stable coalition formation game.

\((\Rightarrow)\) Assume that \((N, \succ_N)\) has a reduced form with a generalized ring. By Proposition 4 in Appendix B, the reduced form induces a non-trivial absorbing set \(A\). Let \(\pi^N\) be a stable coalition structure, so \(\pi^N \in \Pi \setminus A\). Thus, by Definition 1 there is no \(\pi \in A\) such that \(\pi^N \succ^T \pi\). Therefore, \((N, \succ_N)\) does not exhibit convergence to stability.

\((\Leftarrow)\) Assume that \((N, \succ_N)\) has every reduced form with no generalized rings. By Remark 2, this means that each reduced form can be identified with a stable coalition structure, so the game has only trivial absorbing sets. By Remark 1 (iii) and (iv), for each non-stable coalition structure \(\pi \in \Pi\) there is a stable coalition structure \(\pi^N \in \Pi\) such that \(\pi^N \succ^T \pi\). \(\square\)

Recall that, for the particular case of a roommate problem, there are either trivial or non-trivial absorbing sets (see Diamantoudi et al., 2004; Inarra et al., 2013, for more details). Thus, if a roommate problem has a reduced form with no generalized rings, then all reduced forms of the problem have no generalized rings. Therefore, the following result holds:

**Corollary 1** A stable roommate problem always exhibits convergence to stability.

Several domain restrictions have been studied throughout the literature to guarantee stability for general coalition formation games. Some of them are the “common ranking property” (Farrell and Scotchmer, 1988), the “weak top coalition property” (Banerjee et al., 2001), the “ordinally balance property” (Bogomolnaia and Jackson, 2002), and the “pairwise alignment property” (Pycia, 2012).

A coalition formation game satisfies the **common ranking property** if there is an ordering \(\succ\) over \(K\) such that, for each \(i \in N\) and each \(C, C' \in K_i\), we have \(C \succ_i C'\) if and only if \(C \succ C'\). It is straightforward to see that a game satisfying this property has no generalized ring, so the following result holds.

---

13For TU-games, the similar question of accessibility to the core is studied in Kóczy and Lauwers (2004).
Corollary 2 A coalition formation game satisfying the common ranking property exhibits convergence to stability.

The following example presents two games, one satisfying the ordinally balanced property and the other satisfying the weak top coalition property. Although these classes of games impose some degree of commonality on agents’ preferences guaranteeing stability, they may lack convergence to stability.

Example 5 (see Bogomolnaia and Jackson, 2002, Section 4). Consider the following two coalition formation games:

|   | 1   | 2   | 3   |
|---|-----|-----|-----|
|   | 12  | 23  | 13  |
| **123** | **123** | **123** |
| 13  | 12  | 23  |
| 1   | 2   | 3   |

|   | 1   | 2   | 3   |
|---|-----|-----|-----|
| 123 | 23  | 13  |
| 12  | 12  | **123** |
| 13  | **123** | 23  |
| 1   | 2   | 3   |

The game in the first table is ordinally balanced and the one in the second table satisfies the weak top coalition property. In both cases, there are two reduced forms: \{\{123\}\} with no generalized ring, guaranteeing stability; and \{\{13,12,23\}\} with a generalized ring, implying lack of convergence by Proposition 2.

In the remaining of this section, we apply Proposition 2 to analyze convergence to stability in games induced by bargaining solutions and rationing rules.

4.1 Coalition formation games and bargaining solutions

Pycia (2012) presents a model in which there is a set of agents, each endowed with a utility function, who form coalitions that produce outputs to be distributed among its members. He shows that under a rich domain of preferences, and some restrictions on coalitions, there is a stable coalition structure for each preference profile if and only if agents’ preferences satisfy pairwise alignment. Agents’ preferences are pairwise aligned if any two agents rank coalitions that contain both of them in the same way. Formally, a game \((N,\succsim_N)\) satisfies the pairwise aligned property if for all \(C,C' \in \mathcal{K}\) and all \(i,j \in C \cap C'\) it holds that \(C \succ_i C'\) if and only if \(C \succ_j C'\).

Given a set of agents \(N\) and a set of coalitions \(\mathcal{K} \subseteq 2^N\setminus\{\emptyset\}\), a coalitional bargaining problem is a tuple \((U_N,y(C)_{C \in \mathcal{K}})\) where \(U_N = (U_i)_{i \in N}\) is a vector of utility functions \(U_i : \mathbb{R}_+ \rightarrow \mathbb{R}_+\) and, for each \(C \in \mathcal{K}\), \(y(C)\) is the output produced by coalition \(C\). When agent \(i \in C\) gets the

---

14We say that a collection of coalitions \(B \subseteq N\) is balanced if there is a vector of positive weights \(\lambda_S\), such that for each agent \(i \in N\), \(\sum_{S \in B : i \in S} \lambda_S = 1\) (see Bondareva, 1963; Shapley, 1967). A coalition formation game satisfies the ordinally balanced property if for each balanced collection of coalitions \(B\) there is a coalition structure \(\pi \in \Pi\) such that for each \(i \in N\) there is \(S \in B\) with \(i \in S\) such that \(\pi(i) \succ_i S\).

15A coalition \(W \subseteq G \subseteq N\), is a weak top coalition of \(G\) if it has an ordered coalition structure \((S_1, ..., S_{\ell})\) such that (i) any agent in \(S_1\) weakly prefers \(W\) to any subset of \(G\) and (ii) for any \(k > 1\), any agent in \(S_k\) needs cooperation of at least one agent in \(\cup_{m<k} S_m\) in order to form a strictly better coalition than \(W\) (Banerjee et al., 2001). A game satisfies the weak top coalition property if each coalition has a weak top coalition.
share $x$ of output $y(C)$ her utility gives her $U_i(x)$. Given $C \in K$, the bargaining problem for $C$ is $(U_C, y(C))$ where $U_C = (U_i)_{i \in C}$ is the utility vector of agents in $C$ and $y(C)$ is the output of coalition $C$. An allocation for the bargaining problem for $C$, is a vector $x = (x_i)_{i \in C} \in \mathbb{R}_+^C$ such that $\sum_{i \in C} x_i = y(C)$. A bargaining rule is a mapping that associates an allocation with each bargaining problem.

Given a coalitional bargaining problem $(U_N, y(C)_{C \in K})$, a bargaining rule $F$ induces a coalition formation game $(N, \succ_N)$ in the following way: for each $i \in N$ and each pair $C, C' \in K$ with $i \in C \cap C'$, if $F_i(U_C, y(C)) > F_i(U_{C'}, y(C'))$ then $C \succ_i C'$. Note that for the game to be well-defined, no pair of bargaining problems should allocate the same amount to agent $i$. A bargaining rule is pairwise aligned if the coalition formation game induced is pairwise aligned for each bargaining problem.

**Corollary 3** Any coalition formation game induced by a pairwise aligned bargaining rule exhibits convergence to stability.

**Proof.** Let $(N, \succ_N)$ be a coalition formation game induced by a pairwise aligned bargaining rule. Pycia (2012) guarantees that $(N, \succ_N)$ is a stable coalition formation game with no generalized rings. By Proposition 2, $(N, \succ_N)$ exhibits convergence to stability. □

Given $C \in K$, the Nash bargaining rule (Nash, 1950) for problem $(U_C, y(C))$ is determined by solving:

$$
\max_{x_i \geq 0} \prod_{i \in C} U_i(x) \text{ subject to } \sum_{i \in C} x_i = y(C).
$$

The Nash bargaining rule is included in the class of pairwise aligned bargaining rules (see Pycia, 2012, p.331) and therefore guarantees convergence to stability. Another important bargaining solution is Kalai-Smorodinsky’s (Kalai and Smorodinsky, 1975). Given $C \in K$, the Kalai-Smorodinsky bargaining rule for problem $(U_C, y(C))$ is determined by solving:

$$
\frac{U_i(x_i)}{U_i(y(C))} = \frac{U_j(x_j)}{U_j(y(C))} \text{ for all } i, j \in C \text{ subject to } \sum_{i \in C} x_i = y(C).
$$

Not all games induced by the Kalai-Smorodinsky bargaining rule are stable. However, even if one considers only stable coalition formation games induced by the Kalai-Smorodinsky solution, it is found that they may lack convergence to stability. We show this in the following example.

**Example 6** Consider a risk-averse firm $f$ and a risk-neutral firm $g$ that can employ either one or two risk-averse workers $w_1, w_2$ whose utilities are given by

$$
U_f(x) = x^{1/4}, U_g(x) = x, U_{w_1}(x) = x^{1/6}, U_{w_2}(x) = x^{1/2}.
$$

The following table provides the coalitions and the allocation given by the Nash and the Kalai-Smorodinsky (K-S) bargaining solutions for different levels of outputs:

---

16 We normalize all bargaining problems so that the disagreement point is equal to the origin.

17 Pycia (2012) shows that each pairwise aligned bargaining rule induces a stable coalition formation game (Corollary 1 in Pycia (2012)) with a rich domain of preferences. Lemmata 3 and 4 in Pycia (2012) state that a coalition formation game with rich domain and pairwise aligned preferences has no “$n$-cycles in preferences”. The non-existence of “$n$-cycles in preferences” in his setting implies the non-existence of generalized rings in our setting.
| Coalitions | f w₁ w₂ | g w₁ w₂ | f w₁ | f w₂ | g w₁ | g w₂ |
|------------|---------|---------|------|------|------|------|
| Outputs    | 43      | 83      | 20   | 37   | 1    | 1    |
| Nash       | (11.7, 7.8, 23.5) | (49.8, 8.3, 24.9) | (12.8) | (12.3, 24.7) | (0.8, 0.2) | (0.7, 0.3) |
| K-S        | (12.7, 6.9, 23.4) | (49.6, 3.8, 29.6) | (11.4, 8.6) | (14.1, 22.9) | (0.8, 0.2) | (0.6, 0.4) |

The coalition formation game induced by Nash bargaining is:

\[
\begin{array}{cccc}
  f & g & w_1 & w_2 \\
  f w_2 & g w_1 & w_1 & w_2 \\
  f w_1 & g & f w_1 & f w_2 \\
  f w_1 w_2 & g & f w_1 w_2 & f w_1 w_2 \\
  f & g & g w_1 & g w_2 \\
\end{array}
\]

Observe that the unique reduced form is \{ {f}, {g w₁ w₂} \} composed of a singleton and a fixed component implying not only stability but also convergence to stability. The coalition formation game induced by Kalai-Smorodinsky bargaining is:

\[
\begin{array}{cccc}
  f & g & w_1 & w_2 \\
  f w_2 & g w_1 w_2 & g w_1 & w_2 \\
  f w_1 & g w_1 & f w_1 w_2 & f w_1 w_2 \\
  f w_1 w_2 & g w_2 & f w_1 w_2 & f w_1 w_2 \\
  f & g & g w_1 & g w_2 \\
\end{array}
\]

Observe that there are two reduced forms: \{ {f w₁ w₂}, {g} \} and \{ {f w₁, f w₂, g w₁ w₂} \}. The first one is composed only of a fixed component and a singleton, guaranteeing stability. In contrast, the second one is composed of a generalized ring, implying lack of convergence to stability. ♦

### 4.2 Coalition formation games and rationing rules

In the model considered by Gallo and Inarra (2018), there is a set of agents with claims and each coalition of agents produces an output which is insufficient to meet the claims of its members. Formally, given set of agents \( N \) and a set of coalitions \( K \subseteq 2^N \setminus \{\emptyset\} \), a coalitional rationing problem is a tuple \((d_N, y(C))_{C \in K}\) where \( d_N = (d_i)_{i \in N} \in \mathbb{R}^N_+ \) is a claims vector, \( y(C) \in \mathbb{R}_+ \) is the output of coalition \( C \) and \( \sum_{i \in C} d_i \geq y(C) \) for each \( C \in K \). Given \( C \in K \), the rationing problem for \( C \) is \((d_C, y(C))\) where \( d_C = (d_i)_{i \in C} \) is the claims’ vector of agents in \( C \) and \( y(C) \) is the output of coalition \( C \). An allocation for the rationing problem \((d_C, y(C))\) is a vector \( x = (x_i)_{i \in C} \in \mathbb{R}^C_+ \) such that \( \sum_{i \in C} x_i = y(C) \). A rationing rule is a mapping that associates an allocation with each rationing problem.

Given a coalitional rationing problem \((d_N, y(C))_{C \in K}\), a rationing rule \( F \) induces a coalition formation game \((N, K, \succ\, N)\) in the following way: for each \( i \in N \) and each pair \( C, C' \in K \)
with \( i \in C \cap C' \), if \( F_i(d_C, y(C)) > F_i(d_{C'}, y(C')) \) then \( C \succ_i C' \). Note that for the game to be well-defined, no pair of rationing problems should allocate the same amount to agent \( i \).

One of the most important classes of rules for rationing problems is the class of parametric rules (see Young, 1987; Stovall, 2014). The proportional, constrained equal awards, constrained equal losses, and the Talmud and reverse Talmud rules are symmetric parametric rules while the sequential priority rule is an asymmetric parametric rule.

Let \( f \) be a collection of functions \( \{f_i\}_{i \in N} \) where each \( f_i : \mathbb{R}_+ \times [a, b] \rightarrow \mathbb{R}_+ \) is continuous and weakly increasing in \( \lambda \), \( \lambda \in [a, b] \), \( -\infty \leq a < b \leq \infty \) and for each \( i \in N \) and \( d_i \in \mathbb{R}_+ \), \( f_i(d_i, a) = 0 \) and \( f_i(d_i, b) = d_i \). Given \( f \), a parametric (rationing) rule \( F \) is defined as follows. For each problem \( (d, y) \) and each \( i \in N \),

\[
F_i(d, y) = f_i(d_i, \lambda) \quad \text{where } \lambda \text{ is chosen so that } \sum_{i \in N} f_i(d_i, \lambda) = y. \tag{19}
\]

**Corollary 4** Any coalition formation game induced by a parametric rule exhibits convergence to stability.

**Proof.** Let \((N, \succ_N)\) be a coalition formation game induced by a parametric rule. Gallo and Inarra (2018) guarantee that \((N, \succ_N)\) is a stable coalition formation game with no generalized rings.\(^{20}\) By Proposition 2, \((N, \succ_N)\) exhibits convergence to stability. \(\square\)

The random arrival rule (O’Neill, 1982) fails to guarantee stability. Moreover, focusing only on stable coalition formation games induced by the random arrival rule, we find that they may lack convergence to stability. The following example illustrates the different behavior of the proportional rule and the random arrival rule when inducing coalition formation games.\(^{21}\)

**Example 7** Assume that there is a call to finance research projects and that several researchers are ready to submit a project. Each researcher has an aspiration, which depends on her CV, regarding the compensation she believes she deserves. Let \( N = \{1, 2, 3, 4, 5, 6, 7, 8, 9\} \) be the set of researchers with the following aspirations:

\[
c_1 = c_2 = c_5 = c_7 = c_8 = c_9 = 50, \quad c_3 = c_4 = c_6 = 10.
\]

Researchers can form various teams but participate in only one. Funding depends on the quality of the project, which in turn depends on the team composition, and there is not enough money to meet

---

\(^{18}\)When the rule is symmetric, \( f_i \) is the same for all agents.

\(^{19}\)In the literature, \( f \) is said to be a parametric representation of \( F \).

\(^{20}\)Gallo and Inarra (2018) show that each parametric rationing rule induces a stable coalition formation game with no “rings in preferences” (Proposition 1 in Gallo and Inarra, 2018). The non-existence of “rings in preferences” in their setting implies the non-existence of generalized rings in our setting.

\(^{21}\)For each \( C \in \mathcal{K} \), each \((d_C, y(C))\), and each \( i \in C \),

**Proportional rule, Prop:**

\[
\text{Prop}_i(d_C, y(C)) = \frac{d_i}{\sum_{j \in C} d_j} \cdot y(C).
\]

**Random arrival rule, RA:**

\[
\text{RA}_i(d_C, y(C)) = \frac{1}{|C|!} \left( \sum_{\sigma \in \mathcal{O}^C} \min \left\{ d_i, \max \left\{ y(C) - \sum_{j \in C, j \neq i} d_j, 0 \right\} \right\} \right),
\]

where \( \mathcal{O}^C \) denote the class of strict orders on \( C \), with generic element \( \prec \).
the aspirations of all possible teams. Assume that the money assigned to each potential team is distributed according to the random arrival and proportional rules. The table below shows the coalitions, the outputs, and the distribution of the outputs obtained from these two rules.

| Coalitions | Outputs | RA      | Prop    |
|------------|---------|---------|---------|
| \{15\}     | 34      | (17,17) | (17,17) |
| \{45\}     | 20      | (5,15)  | \left(\frac{10}{3}, \frac{50}{3}\right) |
| \{123\}    | 53      | \left(\frac{73}{3}, \frac{73}{3}, \frac{13}{3}\right) | \left(\frac{265}{11}, \frac{265}{11}, \frac{53}{11}\right) |
| \{34\}     | 9       | \left(\frac{9}{7}, \frac{9}{7}\right) | \left(\frac{9}{7}, \frac{9}{7}, \frac{15}{7}\right) |
| \{68\}     | 9       | (17,17) | (17,17) |
| \{78\}     | 34      | (13, \frac{73}{3}, \frac{23}{3}) | \left(\frac{53}{11}, \frac{265}{11}, \frac{265}{11}\right) |
| \{679\}    | 53      | (15,5)  | \left(\frac{50}{7}, \frac{10}{3}\right) |
| \{26\}     | 20      | (15,17) | \left(\frac{10}{7}, \frac{50}{3}\right) |

The coalition formation game induced by random arrival rationing is:

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|
| 123 | 123 | 34 | 45 | 15 | 26 | 679 | 78 | 679 |
| 15 | 26 | 123 | 34 | 45 | 68 | 78 | 68 | 9 |
| 1 | 2 | 3 | 4 | 5 | 679 | 7 | 8 | 6 |

Observe that there are two reduced forms: \{\{15\}, \{26\}, \{34\}, \{78\}, \{9\}\} and \{\{123\}, \{45\}, \{679, 68, 78\}\}. The first one is composed of fixed components and a singleton, guaranteeing stability. In contrast, the second one is composed of two fixed components and a generalized ring, implying a lack of convergence to stability. The coalition formation game induced by proportional rationing is:

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|
| 123 | 123 | 123 | 34 | 15 | 26 | 679 | 78 | 679 |
| 15 | 26 | 34 | 45 | 45 | 679 | 78 | 68 | 9 |
| 1 | 2 | 3 | 4 | 5 | 68 | 7 | 8 | 6 |

The only reduced form is \{\{123\}, \{45\}, \{679\}, \{8\}\}, with no generalized ring.

5 Final remarks

Our paper contributes to the literature on coalition formation by characterizing absorbing sets in terms of reduced forms of the game. We show that a reduced form condenses all the relevant information that can be extracted from an absorbing set without having to compute it. Furthermore, the reduced form of a game has the advantage over absorbing sets that the domination relation between coalition structures need not be considered.

Given a reduced form, we can reconstruct its associated absorbing set. Each coalition structure of this set contains:

(i) a maximal set of each compact generalized ring,
(ii) a subset of a maximal set of each non-compact generalized ring,\textsuperscript{22} and
(iii) all the fixed components.

Agents gathered in the set of singletons of the reduced form appear in some coalition structures of the absorbing set forming non-singleton coalitions and in others as singletons.

In this paper, we also use the notion of reduced form to shed light on the problem of convergence to stability in coalition formation games for some economic environments.

We study stable coalition formation games satisfying properties such as common ranking, ordinary balance, weak top coalition, and pairwise alignment. Furthermore, stable coalition formation games induced by bargaining solutions and sharing rules are also considered.

Finally, our approach opens up several interesting research directions, including the following: The paper relies on a dynamic process between coalition structures which is consistent with the standard blocking definition in that all members of the blocking coalition become strictly better off, and assumes that abandoned agents appear as singletons in the newly formed coalition structure. However, another possibility is for the abandoned agents to get together as in, for instance, the $\delta$-model of Hart and Kurz (1983) or the marriage model of Tamura (1993). How our notion of reduced form adapts to this new dynamic is an open question.

Another interesting direction to explore is considering indifferences in preference relations. We think such an adaptation is not straightforward but could be accomplished by carefully adjusting several of the definitions involved in our analysis. We leave this interesting extension for further research.

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\textsuperscript{22}There are coalition structures containing only one coalition of a non-compact generalized ring and others containing a maximal set of the coalitions in that generalized ring.
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A Relation between rings and cycles

This appendix studies the relationship between two well-known concepts in the literature: Cycles of coalition structures and rings of coalitions. We show that each cycle induces a ring and each ring induces a cycle. This result links non-trivial absorbing sets with generalized rings in the following way: Each non-trivial absorbing set consists of one or more cycles, each cycle induces a ring, and a generalized ring can be constructed by merging all overlapping rings.
A cycle of coalition structures is an ordered set of coalition structures that shows cyclical behavior. That is, for each pair of consecutive coalition structures of the ordered set, the successor coalition structure dominates its predecessor.\(^{23}\) Formally,

**Definition 7** An ordered set of coalition structures \((\pi_1, \ldots, \pi_J) \subseteq \Pi\), with \(J \geq 3\), is a cycle if \(\pi_{j+1} \gg \pi_j\) for \(j = 1, \ldots, J\) subscript modulo \(J\).

A ring of coalitions is an ordered set of non-singleton coalitions that behaves cyclically, i.e., for each pair of consecutive coalitions\(^{24}\) of the ordered set the successor coalition is preferred to its predecessor (see Diamantoudi et al., 2004; Inarra et al., 2013; Tan, 1991, among others). Formally,

**Definition 8** An ordered set of non-singleton coalitions \((R_1, \ldots, R_J) \subseteq K\), with \(J \geq 3\), is a ring if \(R_{j+1} \succ R_j\) for \(j = 1, \ldots, J\) subscript modulo \(J\).

Next, we present an algorithm that constructs a ring of coalitions from a cycle of coalition structures. Let \(C = (\pi_1, \ldots, \pi_J)\) be a cycle of coalition structures, let \(C_j\) denote the coalition that is formed in \(\pi_j\), i.e. \(\pi_j \gg \pi_{j-1}\) via \(C_j\), and consider the ordered set \(C = (C_1, \ldots, C_J)\). To construct a ring, proceed as follows:

---

**Algorithm:**

**Step 1** Set \(R_1\) as any coalition in \(C\).

**Step \(t\)** Set

\[
R_t \equiv \min_{r \geq 1}\{C_j + r \mid C_j = R_{t-1} \text{ and } C_j \cap C_{j+r} \neq \emptyset \text{ with } j + r \mod J\}.
\]

IF \(R_t = R_s\) for \(s < t\),

THEN set \((R_{s+1}, \ldots, R_t)\), and STOP.

ELSE continue to Step \(t + 1\).

---

Notice that in each step of the algorithm, a different coalition of \(C\) is selected except in the last step, where one of the previously selected coalitions is singled out. Therefore, the algorithm stops in at most \(J + 1\) steps (recall that \(J = |C|\)).

The following proposition establishes that the ordered set \((R_{s+1}, \ldots, R_t)\), where the above algorithm determines \(s\), is a ring. Conversely, for each ring, a cycle of coalition structures can be constructed. This result is crucial for the proof of the main finding of the paper, Theorem 1, presented in Appendix B.

**Proposition 3** A coalition formation game has a ring of coalitions if and only if it has a cycle of coalition structures.

\(^{23}\)Here, given an ordered set of coalition structures with \(k\) elements, two coalition structures are consecutive if their indices differ by one (modulo \(k\)).

\(^{24}\)Here, given an ordered set of coalitions with \(k\) elements, two coalitions are consecutive if their indices differ by one (modulo \(k\)). Note that our concept of consecutive coalition is not related with the notion presented in Greenberg and Weber (1986).

\(^{25}\)There are other ways of defining cyclicity between coalitions (see Pycia, 2012; Inal, 2015, for more details).
Proof. \((\Leftarrow)\) Let \(C\) be a cycle of coalition structures. The application of the above algorithm results in the ordered set \((R_{1+1}, \ldots, R_\ell)\). To simplify notation, we rename the elements of the ordered set and write \((R_1, \ldots, R_\ell) = (\tilde{R}_{s+1}, \ldots, \tilde{R}_1)\). We claim that the ordered set \((R_1, \ldots, R_\ell)\) thus constructed is a ring, i.e., for each \(R_{j+1}\) and \(R_j\) in the ordered set, \(R_{j+1} \succ R_j\) and \(\ell \geq 3\). Take any coalition \(R_j\). Coalition \(R_{j+1}\) (modulo \(\ell\)) is the closest coalition that has a non-empty intersection with \(R_j\) (following the modular order of the coalition structures in cycle \(C\)), so all the coalition structures between the one in which \(R_j\) breaks and the one in which \(R_{j+1}\) breaks contain coalition \(R_j\). Let \(\pi\) and \(\pi'\) be the two consecutive coalition structures in \(C\) such that \(\pi' \gg \pi\) via \(R_{j+1}\). \(R_{j+1}\) is the breaking coalition, so \(R_{j+1}\) belongs to \(\pi'\). Furthermore, since \(R_j\) belongs to \(\pi\) and \(R_{j+1} \cap R_j \neq \emptyset\), by definition of the domination relation \(\gg\), \(R_{j+1} \succ R_j\). Furthermore, \(\ell \geq 3\). This holds for the following two facts: (i) there are at least two coalitions in the ordered set because all the coalitions that break in a cycle are also broken; (ii) if there are only two coalitions, say \(R_1\) and \(R_2\), then there is an agent \(i \in R_1 \cap R_2\) such that \(R_1 \succ_i R_2 \succ_i R_1\), which by transitivity implies \(R_1 \succ_i R_j\), which is a contradiction. Therefore, \((R_1, \ldots, R_\ell)\) is a ring.

\((\Rightarrow)\) Let \((R_1, \ldots, R_\ell)\) be a ring in coalition formation game \((N, \succ_N)\). Define the collection of coalition structures \(C = (\pi_1, \ldots, \pi_\ell)\) where \(\pi_j\) is as follows:

\[
\pi_j(i) = \begin{cases} 
R_i & \text{for } i \in R_j \\
\{i\} & \text{otherwise.}
\end{cases}
\]

By Definition 8, \(\pi_j \gg \pi_{j-1}\) via \(R_j\) for each \(j = 1, \ldots, \ell\) (subscript modulo \(\ell\)). Therefore, \(C\) is a cycle. \(\Box\)

Next, we illustrate the above result with an example.

Example 1 (Continued) Consider the ring \((15,12,23,34,45)\) in Example 1. The collection \(C = (\{12,34,5\}, \{12,3,45\}, \{1,23,45\}, \{15,23,4\}, \{15,2,34\})\) is a cycle of coalition structures. Starting from \(\{12,34,5\}\), the set of blocking coalitions between coalition structures is \(C = (45,23,15,34,12)\). Assume that Step 1 of the above algorithm selects coalition 45. The following steps select coalitions 15, 12, 23 and 34, respectively. The algorithm ends when coalition 45 is reached again and ring \((45,15,12,23,34)\) is obtained. \(\Box\)

Given a non-trivial absorbing set \(A\), by Definition 1, there is a collection of cycles forming \(A\). For each cycle, by Proposition 3, there is a collection of rings. We say that each one of these rings is derived from \(A\). Thus, by merging all the overlapping rings derived from \(A\), we can construct all generalized rings that belong to the reduced form associated with \(A\). In the following lemmata we formalize this construction.

Lemma 1 Let \(A\) be a non-trivial absorbing set and consider a maximal collection of overlapping rings derived from \(A\). If \(B\) is the set of all coalitions that belong to such rings, then \(B\) is a generalized ring.

Proof. Let \(A\) be a non-trivial absorbing set and consider a maximal collection of rings derived from \(A\) that overlaps. Let \(B\) be the set of all coalitions that belong to those rings. We now show that \(B\) fulfills Conditions (i) and (ii) of Definition 2. By construction of the overlapping rings, Condition (i) is fulfilled straightforwardly. To see Condition (ii), assume that there are a coalition \(\tilde{C} \in B\) and a coalition structure \(\tilde{\pi} \in A\) such that \(\tilde{C} \in \tilde{\pi}\). There are two cases to consider:
1. There is \( M \in \mathcal{M}_B \) such that \( M \subseteq \tilde{\pi} \). Thus, by construction of \( B \), there is \( C' \in B \) with \( C' \succ \tilde{C} \) such that \( C' \) breaks \( \tilde{\pi} \) and, therefore, \( C' \) breaks \( M \).

2. For each \( M \in \mathcal{M}_B \), \( M \not\subseteq \tilde{\pi} \). Take the set of coalitions \( B \cap \tilde{\pi} \). By definition of maximal set, there is \( \tilde{M} \in \mathcal{M}_B \) such that \( B \cap \tilde{\pi} \subseteq \tilde{M} \). Note that the agents in \( N(B) \setminus N(B \cap \tilde{\pi}) \) are singletons in \( \tilde{\pi} \). Thus, there is a \( \tilde{\pi} \) such that \( \tilde{\pi} \subseteq \tilde{\pi} \) and \( \tilde{\pi} \gg^T \tilde{\pi} \) by forming each coalition in \( \tilde{M} \setminus \tilde{\pi} \) with agents of \( N(B) \) that are singletons in \( \tilde{\pi} \). Now, by Case 1, there is a coalition \( C' \) that breaks \( \tilde{M} \).

Since Conditions (i) and (ii) of Definition 2 are fulfilled, \( B \) is a generalized ring. □

In this way, for a given non-trivial absorbing set, we can construct a collection of generalized rings. Formally,

**Lemma 2** A non-trivial absorbing set induces a collection of generalized rings.

**Proof.** Let \( A \) be a non-trivial absorbing set. Notice that, given any two different coalition structures in \( A \), by Definition 1 there is a cycle of coalition structures in \( A \) that includes those coalition structures. \( A \) can, therefore, be seen as the union of all such cycles. Thus, by Proposition 3, for each cycle of coalition structures in \( A \) there is a ring. Thus, by Lemma 1, all generalized rings induced by \( A \) are constructed. □

**B Proof of Theorem 1**

This appendix proves that each absorbing set can be identified with a reduced form. To prove that each reduced form \( \mathcal{R} \) generates an absorbing set, we first define a special class of coalition structures constructed from \( \mathcal{R} \), that we call \( \mathcal{R} \) - coalition structures (Definition 9). We show that each \( \mathcal{R} \) - coalition structure transitively dominates: (i) all other \( \mathcal{R} \) - coalition structures (Lemma 3), and (ii) all coalition structures that contain all non-singleton coalitions of that \( \mathcal{R} \) - coalition structure (Lemma 5). We also show that if a coalition structure transitively dominates a \( \mathcal{R} \) - coalition structure, then the converse also follows (Lemma 6).

Each \( \mathcal{R} \) - coalition structure, in turn, defines a set: The set that includes it together with all coalition structures that transitively dominate it. By the previous results, we show that this set only depends on \( \mathcal{R} \) (and not on the particular \( \mathcal{R} \)-coalition structure chosen to construct it). This set is proven to be an absorbing set (Proposition 4).

To show that each absorbing set generates a reduced form, first we observe that an absorbing set is formed by overlapping cycles of coalition structures. From the collection of all these cycles, we can construct all generalized rings derived from such absorbing set (Lemma 2). Furthermore, we identify the fixed components by considering the coalitions that appear in all coalition structures of that absorbing set. Thus, we construct a collection of generalized rings, fixed components, and a set of singletons consisting of the remaining agents. This collection turns out to be a reduced form (Theorem 1).

Now we start by formalizing the definition of a \( \mathcal{R} \) - coalition structure.

**Definition 9** Let \( (N, \succ_N) \) be a coalition formation game and \( \mathcal{R} \) a reduced form of that game. An \( \mathcal{R} \) – coalition structure is a coalition structure \( \pi_{\mathcal{R}} \) such that:
(i) for each compact generalized ring or each fixed component $B \in \mathcal{R}$, $\pi R \cap B = M$ for some $M \in \mathcal{M}_B$, and $\pi R(i) = \{i\}$ for each $i \in N(B) \setminus N(M)$.\(^{26}\)

(ii) for each non-compact generalized ring $B \in \mathcal{R}$, $\pi R \cap B = C$ for some $C \in B$, and $\pi R(i) = \{i\}$ for each $i \in N(B) \setminus C$.

(iii) for each $i \in S$, $\pi R(i) = \{i\}$.

Condition (i) says that for each compact generalized ring or each fixed component of $\mathcal{R}$, an $\mathcal{R}$–coalition structure includes:

- a maximal set of that generalized ring,
- the agents of that generalized ring not involved in that maximal set as singletons, and
- the coalition of each fixed component.

Condition (ii) says that for each non-compact generalized ring of $\mathcal{R}$, an $\mathcal{R}$–coalition structure includes:

- only one coalition of that generalized ring, and
- the agents of that generalized ring not involved in the coalition as singletons.

Condition (iii) says that each agent in $S$ is a singleton in an $\mathcal{R}$–coalition structure.

**Lemma 3** Let $\mathcal{R}$ be a reduced form and let $\pi R$ and $\pi'_R$ be two different $\mathcal{R}$–coalition structures. Then $\pi R \gg_T \pi'_R$.

**Proof.** Let $\mathcal{R}$ be a reduced form and let $\pi R$ and $\pi'_R$ be two different $\mathcal{R}$–coalition structures. By Definition 2, there are a sequence of coalitions $C_1, \ldots, C_J$ and a sequence of coalition structures $\pi_0, \ldots, \pi_J$ such that:

(i) $\pi_0 = \pi'_R$ and $\pi_J = \pi R$,

(ii) $C_j \in B$ for some $B \in \mathcal{R}$ and $C_j \cap C \neq \emptyset$ for some $C \in K \cap \pi_{j-1}$ for $j = 1, \ldots, J$,

(iii) $\pi_j \gg \pi_{j-1}$ via $C_j$ for each $j = 1, \ldots, J$.

Therefore, $\pi R \gg_T \pi'_R$. \(\square\)

The domination relation between coalition structures when a non-compact generalized ring is involved has a particular feature: When a maximal set of a non-compact generalized ring is included in a coalition structure, there is another coalition structure that contains only one coalition of that generalized ring that transitively dominates only one coalition in the maximal set. For instance, consider Example 3, which only has one non-compact generalized ring. In this example $\{145, 23, 6, 78\} \gg \{1, 23, 46, 5, 78\}$ via 145, $\{1, 2, 356, 4, 78\} \gg \{145, 23, 6, 78\}$ via 356, and $\{1, 2, 3, 46, 5, 78\} \gg \{1, 2, 356, 4, 78\}$ via 46. Therefore, $\{1, 2, 3, 46, 5, 78\} \gg_T \{1, 23, 46, 5, 78\}$. This “disintegrating” behavior of maximal sets is always present when the generalized rings are non-compact. The following lemma deals with this fact and is used to prove Lemma 5.

---

\(^{26}\)Given a maximal set $M$, denote by $N(M)$ the set of agents that belong to (at least) one coalition in $M$, that is, $N(M) = \bigcup_{C \in M} C$. 

---
Lemma 4. Let \( B \) be a non-compact generalized ring, \( \mathcal{M} \in \mathcal{M}_B \), and let \( \pi \in \Pi \) be such that \( \mathcal{M} \subseteq \pi \). If \( C \) is any coalition in \( B \), and \( \pi^* \in \Pi \) is such that

\[
\pi^*(i) = \begin{cases} 
C & \text{for } i \in C \\
\pi(i) & \text{for } i \in N \setminus N(B) \\
\{i\} & \text{otherwise}
\end{cases}
\]

then \( \pi^* \gg T \pi \).

Proof. Since \( B \) is non-compact, there are \( \tilde{C} \in B \) and \( \tilde{\mathcal{M}} \in \mathcal{M}_B \) such that \( \tilde{C} \) has non-empty intersection with at least two coalitions of \( \tilde{\mathcal{M}} \). Let \( \tilde{\pi} \in \Pi \) be such that

\[
\tilde{\pi}(i) = \begin{cases} 
\tilde{\mathcal{M}}(i) & \text{for } i \in N(\tilde{\mathcal{M}}) \\
\pi(i) & \text{for } i \in N \setminus N(B) \\
\{i\} & \text{otherwise}
\end{cases}
\]

Thus, by Definition 2, we have that \( \tilde{\pi} \gg T \pi \).

Assume that \( \tilde{C} \) has non-empty intersection with each coalition in \( \tilde{\mathcal{M}} \). Let \( \pi' \in \Pi \) be such that

\[
\pi'(i) = \begin{cases} 
\tilde{C} & \text{for } i \in \tilde{C} \\
\pi(i) & \text{for } i \in N \setminus N(B) \\
\{i\} & \text{otherwise}
\end{cases}
\]

Thus, \( \pi' \gg \tilde{\pi} \) via \( \tilde{C} \). By Definition 2, \( \pi^* \gg T \pi' \). Therefore, \( \pi^* \gg T \pi' \gg \tilde{\pi} \gg T \pi \) implying \( \pi^* \gg T \pi \), and the proof is complete.

Assume now that \( \tilde{C} \) has a non-empty intersection with \( k \) coalitions of \( \tilde{\mathcal{M}} \) with \( k < |\tilde{\mathcal{M}}| \). Let \( \mathcal{E} \subseteq \tilde{\mathcal{M}} \) be such that each coalition in \( \mathcal{E} \) is disjoint with \( \tilde{C} \). Thus, \( |\mathcal{E}| \geq 1 \). Let \( \pi_{\mathcal{E}} \in \Pi \) be such that

\[
\pi_{\mathcal{E}}(i) = \begin{cases} 
\tilde{C} & \text{for } i \in \tilde{C} \\
\mathcal{E}(i) & \text{for } i \in N(\mathcal{E}) \\
\pi(i) & \text{for } i \in N \setminus N(B) \\
\{i\} & \text{otherwise}
\end{cases}
\]

Thus, \( \pi_{\mathcal{E}} \gg \tilde{\pi} \) via \( \tilde{C} \). Now, it is possible to construct a sequence \( C_1, \ldots, C_m \) of coalitions of \( B \), a sequence \( \mathcal{E}_0, \ldots, \mathcal{E}_m \) of subsets of the maximal sets of \( B \) such that \( \mathcal{E}_0 = \mathcal{E} \cup \{\tilde{C}\} \) and \( \mathcal{E}_m \subseteq \tilde{\mathcal{M}} \) with \( \tilde{C} \) having a non-empty intersection with at least two coalitions in \( \mathcal{E}_m \), and a sequence \( \pi_{\mathcal{E}} = \pi_{\mathcal{E}_0}, \ldots, \pi_{\mathcal{E}_m} \) of coalition structures fulfilling the following conditions for each \( \ell = 1, \ldots, m \):

(i) \( C_\ell \in \mathcal{E}_\ell \subseteq \pi_{\ell} \),
(ii) \( \pi_{\ell} \gg \pi_{\ell-1} \) via \( C_\ell \),
(iii) \( |\mathcal{E}_\ell| = |\mathcal{E}| + 1 \).

Thus, \( \pi_{\mathcal{E}_m} \gg T \pi_{\mathcal{E}} \). Let \( \mathcal{E}' \subseteq \mathcal{E}_m \) be such that each coalition in \( \mathcal{E} \) is disjoint with \( \tilde{C} \). Let \( \tilde{\pi} \) be such that

\[
\tilde{\pi}(i) = \begin{cases} 
\tilde{C} & \text{for } i \in \tilde{C} \\
\mathcal{E}'(i) & \text{for } i \in N(\mathcal{E}') \\
\pi(i) & \text{for } i \in N \setminus N(B) \\
\{i\} & \text{otherwise}
\end{cases}
\]
Thus, \( \hat{\pi} \gg \pi_m \) via \( \hat{C} \). If \( \mathcal{E}' = \emptyset \), then \( \hat{\pi} = \pi' \). By Definition 2, \( \pi^* \gg^T \pi' \), so \( \pi^* \gg^T \pi' \gg \hat{\pi} \gg^T \pi \) implying \( \pi^* \gg^T \pi \), and the proof is complete. If \( \mathcal{E}' \neq \emptyset \), then \( \pi_{\mathcal{E}'} \) is defined similarly to \( \pi_{\mathcal{E}} \) and the same reasoning used when \( \pi_{\mathcal{E}} \) was defined can be repeated. Eventually, we reach a coalition structure \( \hat{\pi} \) such that the only coalitions in common with \( \hat{\mathcal{M}} \) are those that have non-empty intersection with \( \hat{C} \). Then, such coalition structure is dominated by \( \pi' \) via \( \hat{C} \). Thus, \( \pi' \gg \hat{\pi} \gg^T \pi_{\mathcal{E}} \gg \hat{\pi} \gg^T \pi \). By Definition 2, \( \pi^* \gg^T \pi' \). Therefore, \( \pi^* \gg^T \pi \). \( \square \)

**Lemma 5** Let \( \tilde{\mathcal{R}} \) be a reduced form and let \( \pi_{\tilde{\mathcal{R}}} \) be an \( \tilde{\mathcal{R}} \)–coalition structure. If \( \pi \in \Pi \) is such that each non-singleton coalition in \( \pi_{\tilde{\mathcal{R}}} \) belongs to \( \pi \), then \( \pi = \pi_{\tilde{\mathcal{R}}} \) or \( \pi \gg^T \pi \).

**Proof.** Let \( \tilde{\mathcal{R}} \) be a reduced form and let \( \pi_{\tilde{\mathcal{R}}} \) be an \( \tilde{\mathcal{R}} \)–coalition structure. Let be \( \mathcal{M} \) the set of all non-singleton coalitions in \( \pi_{\tilde{\mathcal{R}}} \) and consider \( \pi \in \Pi \) such that \( \mathcal{M} \subseteq \pi \). There are two cases to consider:

1. Each non-singleton coalition \( C \in \pi \setminus \mathcal{M} \) belongs to a generalized ring or fixed component of \( \tilde{\mathcal{R}} \). If \( \tilde{\mathcal{R}} \) only includes generalized rings of type 1 or fixed components then, by Definition 9, \( \pi = \pi_{\tilde{\mathcal{R}}} \) and the proof is complete. Assume that there is only one non-compact generalized ring, say \( \mathcal{B} \), in \( \tilde{\mathcal{R}} \). Thus, by Lemma 4, there are \( C' \in \mathcal{B} \) and \( \pi^* \in \Pi \) with

\[
\pi^*(i) = \begin{cases} 
C' & \text{for } i \in C' \\
\pi(i) & \text{for } i \in N \setminus N(\mathcal{B}) \\
\{i\} & \text{otherwise}
\end{cases}
\]

such that \( \pi^* \gg^T \pi \). By Definition 9, \( \pi^* \) is an \( \tilde{\mathcal{R}} \)–coalition structure. Thus, by Lemma 3, \( \pi_{\tilde{\mathcal{R}}} \gg^T \pi^* \) and the proof is complete. Assume now that there is more than one non-compact generalized ring. Applying Lemma 4 for each of these generalized rings, the proof follows similarly.

2. There is a non-singleton coalition \( C \in \pi \setminus \mathcal{M} \) such that \( C \) does not belong to any generalized ring or any fixed component of \( \tilde{\mathcal{R}} \). Thus, either \( C \) belongs to a generalized ring or it defines a fixed component (formed by agents in \( S \)), that we denote by \( \tilde{\mathcal{B}} \). In either case, by Definition 6, \( \tilde{\mathcal{B}} \) is not protected by \( \tilde{\mathcal{R}} \). Next define \( \pi^* \in \Pi \) as follows. If there is no non-compact generalized ring in \( \tilde{\mathcal{R}} \), \( \pi^* = \pi \). If there are non-compact generalized rings in \( \tilde{\mathcal{R}} \), using Lemma 4 repeatedly we can construct \( \pi^* \in \Pi \) such that it contains only one coalition of each non-compact generalized ring, \( \pi^*(i) = \pi(i) \) for each agent \( i \) not in any non-compact generalized ring, all remaining agents are singletons in \( \pi^* \), and \( \pi^* \gg^T \pi \).

Given a coalition structure \( \pi \in \Pi \), let \( |\pi|_\mathcal{K} \) denote the number of coalitions \( C \in \pi \) with \( |C| > 1 \).

**Claim:** there is a coalition structure \( \tilde{\pi} \) with set of non-singleton coalitions \( \mathcal{M}'' \) such that \( \mathcal{M} \subseteq \mathcal{M}'' \), \( \tilde{\pi} \gg^T \pi^* \), and \( |\tilde{\pi}|_\mathcal{K} < |\pi^*|_\mathcal{K} \).

To prove the Claim, note that in \( \pi^* \) there is a non-singleton coalition with a non-empty intersection with agents in \( N(\tilde{\mathcal{B}}) \), say \( C_0 \). First, assume that \( C_0 \subseteq N(\tilde{\mathcal{B}}) \). Thus, \( C_0 \) defines a fixed component or belongs to a generalized ring, and Definition 6 (ii) implies the existence of a coalition \( C_1 \) that breaks the fixed component or generalized ring to which \( C_0 \) belongs and nothing impedes \( C_1 \) from being formed. If there is no fixed component and
no generalized ring $B \in \mathcal{R}$ such that $C_1 \cap N(B) \neq \emptyset$, then $C_1$ defines a fixed component or belongs to a generalized ring formed by agents in $N(\tilde{B})$. In either case, Definition 6 implies the existence of a coalition $C_2$ that breaks the fixed component or the generalized ring to which $C_1$ belongs and nothing impedes $C_2$ from being formed. If there is neither a fixed component nor a generalized ring $B \in \mathcal{R}$ such that $C_2 \cap N(B) \neq \emptyset$, repeat the previous argument. Continuing this reasoning, it is possible to construct a sequence of coalitions $C_0, C_1, \ldots, C_m$ and a sequence of coalition structures $\pi_0, \pi_1, \ldots, \pi_{m-1}$ with $\pi_0 = \pi^*$ fulfilling the following conditions:

(i) $C_\ell \succ C_\ell-1$ for each $\ell = 1, \ldots, m$;
(ii) $C_\ell-1 \subseteq N(\tilde{B})$ and $\pi_\ell \succ \pi_{\ell-1}$ via $C_\ell$ for each $\ell = 1, \ldots, m - 1$; and
(iii) there is a fixed component or a generalized ring $B' \in \tilde{\mathcal{R}}$ such that $C_m \cap N(B') \neq \emptyset$.

Note that, by Conditions (i) and (ii) above, $C_m \cap N(\tilde{B}) \neq \emptyset$. Also, the existence of a fixed component or a generalized ring $B'$ in (iii) is ensured by the finiteness of the number of coalitions of the game and by Definition 6. Given that $B' \in \tilde{\mathcal{R}}$ is a generalized ring and does not impede $C_m$ from being formed, there is a maximal set of $B'$ (when $B'$ is of type 1) or a coalition in $B'$ (when $B'$ is non-compact) that has empty intersection with $C_m$. Use the term $E$ for that maximal set (when $B'$ is of type 1) or for the singleton that includes that coalition (when $B'$ is non-compact). By Definition 2, there is a coalition structure $\pi'$ such that $\pi' \gg T \pi_{m-1}$ with $\pi'(i) = \pi_{m-1}(i)$ for each $i \in N \setminus N(B')$, and either $\pi'(i) = \{i\}$ or $\pi'(i) \subseteq E$ for each $i \in N(B')$. Thus, there is $m$ such that $\pi_m \gg \pi'$ via $C_m$. Given that $B'$ is a generalized ring, $C_m \cap N(B') \neq \emptyset$, and $B'$ does not impede $C_m$ from being formed, there are $E'$ (a maximal set when $B'$ is of type 1 or a singleton that includes a coalition when $B'$ is non-compact), a coalition $R_0 \in E'$ such that $R_0$ breaks $E$ and $R_0 \supset C_m$, and a coalition structure $\pi''$ such that $\pi'' \gg \pi_m$ via $R_0$.28 Given that $E'$ may not be included in $\pi^*$, and $B'$ is a generalized ring, there are a sequence of coalitions $R_1, \ldots, R_s$ in $B'$, and a sequence of coalition structures $\tilde{\pi}_0, \tilde{\pi}_1, \ldots, \tilde{\pi}_s$ with $\tilde{\pi}_0 = \pi''$ fulfilling the following conditions for each $\ell = 1, \ldots, s$:

(i) $R_\ell \supset R_{\ell-1}$,
(ii) $\tilde{\pi}_\ell \gg \tilde{\pi}_{\ell-1}$ via $R_\ell$, and
(iii) $\tilde{\pi}_s(i) = \pi^*(i)$ for each $i \in N(B')$.

Let $\tilde{\pi} = \tilde{\pi}_s$. Note that, since $C_m \in \pi_m, R_0 \supset C_m$, $R_0$ breaks $E$, and $E \subseteq \pi_m$, it follows that $|\pi_m|_K > |\tilde{\pi}|_K$. Also, by construction of the sequence $\pi_0, \ldots, \pi_m$, $|\pi^*|_K \geq |\pi_m|_K$. Therefore, $|\pi'|_K > |\tilde{\pi}|_K$. Moreover, by Condition (iii) verified by the sequence $\tilde{\pi}_0, \tilde{\pi}_1, \ldots, \tilde{\pi}_s$, we have that $M \subseteq M''$. Thus, $\tilde{\pi}$ fulfills the conditions of the Claim and the claim holds when $C_0 \subseteq \tilde{B}$. Second, assume that $C_0 \subseteq \left( N \setminus N(\tilde{B}) \right) \neq \emptyset$. Thus, $C_0$ can be considered as the coalition $C_m$ in the previous reasoning, and the proof follows similarly. This completes the proof of the Claim.

Now, we conclude the proof of Lemma 5. First note that, by the Claim, a coalition structure $\tilde{\pi}$ is obtained such that $\tilde{\pi} \gg T \pi^*, M \subseteq M''$, and $|\tilde{\pi}|_K < |\pi^*|_K$. If $\tilde{\pi} = \pi \varnothing$, then

27 Otherwise, since $B'$ does not impede $C_m$ from being formed, $C_m$ would break $B'$. This contradicts Definition 6.

28 If $E$ is the unique set such that $C_m \cap N(E) = \emptyset$, the existence of $R_0$ such that $R_0 \supset C_m$ is guaranteed. If it is not unique, it can be selected such that $\pi'' \gg \pi_m$ via $R_0$. 

30
the proof is complete. Otherwise, there is \( C' \in M'' \setminus M \) such that \( C' \) does not belong to any fixed component or generalized ring of \( R \). By applying the Claim to coalition structure \( \tilde{\pi} \) we obtain a new coalition structure \( \tilde{\pi}' \) such that \( \tilde{\pi}' \gg^T \tilde{\pi}, M \subseteq M'' \) and \(|\tilde{\pi}'|_K < |\tilde{\pi}|_K < |\pi'|_K|_K|_K\). If \( \tilde{\pi} = \pi_R \), then the proof is completed. Otherwise, continue applying the Claim until coalition structure \( \pi_R \) is obtained.

\[ \square \]

**Lemma 6** Let \( R \) be a reduced form, let \( \pi_R \) be an \( R \)-coalition structure and let \( \pi \in \Pi \). If \( \pi \gg^T \pi_R \), then \( \pi_R \gg^T \pi \).

**Proof.** Let \( R \) be a reduced form, let \( \pi_R \) be an \( R \)-coalition structure and let \( \pi \in \Pi \) such that \( \pi \gg^T \pi_R \). By Definition 6, there is an \( R \)-coalition structure \( \pi' \) with set of non-singleton coalitions \( M' \) such that \( M' \subseteq \pi \). Thus, by Lemma 5, \( \pi' = \pi \) or \( \pi_R \gg^T \pi \). Then, by Lemma 3, \( \pi_R \gg^T \pi' \) and, therefore, \( \pi_R \gg^T \pi \).

\[ \square \]

Given a reduced form \( R \) and an \( R \)-coalition structure \( \pi_R \), the set generated by \( \pi_R \), denoted by \( A_R \), is the set formed by \( \pi_R \) together with all the coalition structures that transitively dominate it. Formally,

\[ A_R \equiv \{ \pi_R \} \cup \{ \pi \in \Pi : \pi \gg^T \pi_R \}. \]

The following result states that the set \( A_R \) is actually an absorbing set.

**Proposition 4** If \( R \) is a reduced form, then \( A_R \) is an absorbing set. Furthermore, if there is a (compact or non-compact) generalized ring in \( R \), \( A_R \) is a non-trivial absorbing set.

**Proof.** Let \( R = \{ B_1, \ldots, B_L, S \} \) be a reduced form. First, assume that \( B_1 = \{ C_1 \} \) for each \( \ell = 1, \ldots, L \). Thus, \( \pi_R \) is a stable coalition structure. If not, there is a coalition that blocks the coalition structure \( \pi_R \). Since \( R \) is a reduced form, such a blocking coalition must be formed by agents in \( S \), and it must belong to a fixed component or to a generalized ring, say \( B' \), that is not protected by \( R \). Therefore, there is another blocking coalition formed by agents in \( S \) that blocks \( B' \). Repeating this reasoning and by the finiteness of the set of permissible coalitions, we eventually reach a fixed component or a generalized ring formed by agents in \( S \) which is not blocked, generating a contradiction since \( R \) is a reduced form. Hence, \( \pi_R \) is stable and, by Remark 1 (i), \( A_R = \{ \pi_R \} \) is an absorbing set. Second, assume there is \( B \in R \) such that \(|B| \geq 3 \). Thus, \( B \) is a generalized ring of \( R \). By Definition 2, Definition 6, and Definition 9, there are at least three different \( R \)-coalition structures. Thus, by Lemma 3, \(|A_R| \geq 3 \). Let \( \pi \in A_R \) and consider \( \pi' \in \Pi \setminus \{ \pi \} \) such that \( \pi' \gg^T \pi \). As \( \pi \in A_R \), it follows that \( \pi \gg^T \pi_R \). Therefore, by transitivity of \( \gg^T \), \( \pi' \gg^T \pi_R \) and \( \pi' \in A_R \). Next, let \( \pi \) and \( \pi' \) be different coalition structures of \( A_R \). By definition of \( A_R \), it follows that \( \pi \gg^T \pi_R \) and \( \pi' \gg^T \pi_R \). By Lemma 6, it follows that \( \pi_R \gg^T \pi \). Thus, \( \pi' \gg^T \pi_R \) together with \( \pi_R \gg^T \pi \) and the transitivity of \( \gg^T \) imply \( \pi' \gg^T \pi \). This proves that \( A_R \) satisfies the conditions of Definition 1. Therefore, \( A_R \) is a non-trivial absorbing set.

\[ \square \]

Recall that, by the definition of a compact generalized ring, each coalition that breaks a maximal set has a non-empty intersection with only one coalition of that maximal set. Thus,
when a reduced form includes a compact generalized ring, each coalition structure of the absorbing set generated includes a maximal set of that generalized ring. By contrast, when a reduced form includes a non-compact generalized ring there are coalition structures in the generated absorbing set that contain only one coalition of such generalized ring (and the remaining agents involved in the said generalized ring appear as singletons in those coalition structures).

Note that, by Lemma 3, the absorbing set $A_{R}$ depends only on the reduced form $R$, and not on the specific $R$–coalition structure selected to construct it.

Finally, we are in a position to prove the main result of the paper.

Proof of Theorem 1. To prove that there is a bijection between reduced forms and absorbing sets for each coalition formation game, we show that $A$ is an absorbing set if and only if $A = A_{R}$ for a reduced form $R$. The proof ($\iff$) follows from Proposition 4. To prove ($\implies$), let $A$ be an absorbing set. If $|A| = 1$, by Remark 1 (ii), the unique element of $A$ is a stable coalition structure. Thus, by Remark 2, a reduced form can be induced. If $|A| > 1$, the reduced form $R$ is constructed as follows. First, by Lemma 2 of Appendix A, each of the generalized rings involved in $A$ can be identified. Let $B$ be the collection of all those generalized rings. Next, consider the subcollection

$$B^{*} = \{B \in R : \text{for each } \pi \in A \text{ there is } C \in B \text{ such that } C \in \pi\}$$

and include each generalized ring of $B^{*}$ as an element of $R$. Second, let $F = \{C \in K : C \in \pi \text{ for each } \pi \in A\}$ and for each $C \in F$ include $\{C\}$ as an element of $R$. Now, it remains to be shown that $R$ is a reduced form. To do this, take any $B \in R$. The goal is to confirm that $B$ is protected by $R$. Assume otherwise. Thus, there is $C \in K$ that breaks $B$ and no other element of $R$ impedes $C$ from being formed. This implies that there are $\pi, \pi' \in A$ and $C' \in B$ such that $\pi' \gg \pi$ via $C$, $C' \in \pi$ and $C \succ C'$. By the definition of absorbing set it also follows that $\pi \gg^{T} \pi'$. Thus, there is a cycle $C$ of coalition structures that contains $\pi$ and $\pi'$. By Proposition 3 in Appendix A, cycle $C$ induces a ring of coalitions that contains both coalitions $C'$ and $C$. Therefore, $C$ belongs to generalized ring $B$, which is absurd since $C$ breaks $B$. Hence, $B$ is protected by $R$ and Condition (i) of Definition 6 holds. To see that Condition (ii) of Definition 6 holds, let $B$ be a generalized ring or a fixed component formed by agents in $S$. We need to show that $B$ is not protected by $R$. Assume otherwise. There are two cases to consider:

1. No coalition breaks $B$. Thus, by construction of $R$, $B$ belongs either to $R^{*}$ or to $F$, contradicting the requirement that $B$ is formed by agents in $S$.

2. There is a coalition $C$ that breaks $B$ and $B$ is protected by $R$. Let $\pi_{R}$ be a $R$–coalition structure. Thus $\pi_{R}(i) = \{i\}$ for each $i \in N(B)$ (since $N(B) \subseteq S$). Since $C$ breaks $B$ and $B$ is protected by $R$, and the fact that Condition (i) holds, $C \subseteq N(B)$. Thus, there is $\pi \in A$ such that $\pi \gg \pi_{R}$ via $C$. Therefore, $B$ belongs either to $R^{*}$ or to $F$, contradicting the requirement that $B$ is formed by agents in $S$.

Therefore, $R$ is a reduced form.