IN INVARIANTS UNDER DEFORMATION OF THE ACTIONS OF TOPOLOGICAL GROUPS

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Abstract. Let \( \varphi \) and \( \varphi' \) be two homotopic actions of the topological group \( G \) on the topological space \( X \). To an object \( A \) in the \( G \)-equivariant derived category \( D_\varphi(X) \) of \( X \) relative to the action \( \varphi \) we associate an object \( A' \) of category \( D_{\varphi'}(X) \), such that the corresponding \( G \)-equivariant compactly supported cohomologies \( H^*_G,c(X, A) \) and \( H^*_G,c(X, A') \) are isomorphic. When \( G \) is a Lie group and \( X \) is a subanalytic space, we prove that the \( G \)-equivariant cohomologies \( H^*_G(X, A) \) and \( H^*_G(X, A') \) are also isomorphic.

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1. Introduction

Palais and Stewart proved in [10] that the actions of a compact Lie group \( G \) on a closed smooth manifold are rigid; that is, if \( \psi \) and \( \psi' \) are homotopic \( G \)-actions on a closed manifold \( N \), then there exists a diffeomorphism \( h \) of \( N \), such that \( h \circ \psi \circ h^{-1} = \psi' \) (see also [4, page 191]). Thus, in particular, \( h \) defines a bijective correspondence between the fixed point sets of \( \psi \) and \( \psi' \).

The situation is different, when either the group \( G \) is non compact, or the space on which it acts is not a smooth manifold. A simple example of non rigidity of \( \mathbb{R} \)-actions is given in [11], where are defined two homotopic \( \mathbb{R} \)-actions on \( S^1 \), for which there is no a diffeomorphism of \( S^1 \) intertwining them. In what follows, we show another example of a non rigid action of a non compact Lie group on \( \mathbb{R}P^1 \).

Let \( G \) be the Borel subgroup of \( SL(2, \mathbb{R}) \) consisting of the matrices

\[
g = \begin{pmatrix} r & y \\ 0 & r^{-1} \end{pmatrix}, \quad r \in \mathbb{R}^\times, \ y \in \mathbb{R}.
\]

For \( s \in [0, 1] \), we define the Lie group homomorphism \( \psi^s : G \to \text{Diff}(\mathbb{R}P^1) \) putting

\[
\psi^s(g)([a : b]) = [ra + syb : r^{-1}b].
\]

\( \{ \psi^s \}_{s \in [0, 1]} \) is a homotopy between \( \psi^0 \) and \( \psi^1 \). The fixed point set of the action \( \psi^0 \) is \( \{ [0 : 1], [1 : 0] \} \), but the set of fixed points of \( \psi^1 \) is \( \{ [1 : 0] \} \). Thus, there is no a diffeomorphism of \( \mathbb{R}P^1 \) which intertwines \( \psi^0 \) and \( \psi^1 \).

The preceding examples show that the actions of topological groups do admit, in general, non trivial deformations. The purpose of this note is to determine objects which are invariants under deformations of the action of a topological group \( G \) on a topological space \( X \). The underlying general idea we will develop is the following: One can expect that homotopic actions on a topological space \( X \) give rise to homeomorphic homotopy quotients of \( X \). On the other hand, the equivariant

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cohomologies are essentially the cohomology of these quotients. Therefore, one can expect the existence of an isomorphism between the equivariant cohomologies of $X$ with coefficients in sheaves which are related through the homotopy.

**Statement of main results.** Given a topological space $Y$, we write $Sh(Y)$ for the abelian category of sheaves of $\mathbb{C}$-vector spaces on $Y$. By $D^b(Y)$ we denote the bounded derived category of $Sh(Y)$ $\mathbb{S}$.

Associated with the action of a topological group $G$ on the topological space $X$ there is the homotopy quotient $X_G := EG \times_G X$, and the bounded equivariant derived category $D^b_G(X)$ (see $\mathbb{2}$). Basically, an object $A$ of $D^b_G(X)$ is a triple $(A_X, \bar{A}, \alpha)$ consisting of an object $A_X$ of the derived category $D^b(X)$, an object $\bar{A} \in D^b(X_G)$ and $\alpha$ is an isomorphism of $D^b(EG \times X)$ between the inverse images of $A_X$ and $\bar{A}$ by the natural maps of this diagram

$$X \leftarrow EG \times X \rightarrow X_G.$$ 

The $G$-equivariant cohomology $H_G(X; A)$ of $X$ with values in $A$ is the cohomology of the homotopy quotient $X_G$ with coefficients in $A$; i.e., $H(X_G; \bar{A})$.

By $\mathcal{H}$ we will denote a subgroup of the homeomorphism group of $X$. Let $\varphi : G \rightarrow \mathcal{H}$ be a group homomorphism; that is, a $G$-action on $X$. From now on, $D^b_G(X)$ stands for the equivariant derived category associated with the $G$-action $\varphi$ on $X$. On the other hand, the homomorphism $\varphi$ induces a map $\Phi : BG \rightarrow B\mathcal{H}$ between the respective classifying spaces. We write $X(\varphi)$ for the quotient $\Phi^*(E\mathcal{H}) \times_{\mathcal{H}} X$, where $\Phi^*(E\mathcal{H})$ is the pullback by $\Phi$ of the universal $\mathcal{H}$-bundle $E\mathcal{H}$.

We will denote by $\mathcal{D}^b(X, \varphi)$ the category whose objects are triples $\mathcal{A} = (A_X, \bar{A}, \alpha)$, where $A_X \in D^b(X)$, $\bar{A} \in D^b(X(\varphi))$ and $\alpha$ an isomorphism between the inverse images of $A_X$ and $\bar{A}$ by the projections maps in

$$X \leftarrow \Phi^*(E\mathcal{H}) \times X \rightarrow X(\varphi).$$

Using the universal property of the pullback $\Phi^*(E\mathcal{H})$, we will construct a functor

$$\delta : \mathcal{D}^b(X, \varphi) \rightarrow D^b_{\varphi'}(X).$$

Roughly speaking, $\delta$ is a “forgetful” functor from the $\mathcal{H}$-equivariance to the weaker $G$-equivariance.

Let $\varphi, \varphi' : G \rightarrow \mathcal{H}$ be two $G$-actions on $X$ which are homotopic by means of a family of group homomorphisms from $G$ to $\mathcal{H}$. The homotopy gives rise to an equivalence $\xi^*$ between the categories $\mathcal{D}^b(X, \varphi')$ and $\mathcal{D}^b(X, \varphi)$.

The $G$-equivariant (relative to the action $\varphi$) cohomology of $X$ with compact supports and coefficients in $A \in D^b_{\varphi}(X)$ will be denoted by $H_{\varphi, c}(X; A)$. We will prove the following theorem.

**Theorem 1.** Let $G$ be a topological group and $\varphi, \varphi' : G \rightarrow \mathcal{H}$ be continuous group homomorphisms that are homotopic through a family of group homomorphisms from $G$ to $\mathcal{H}$. If $X$ and $G$ are locally compact, then the cohomologies $H_{\varphi, c}(X; A)$ and $H_{\varphi', c}(X; A')$ are isomorphic

$$H_{\varphi, c}(X; A) \simeq H_{\varphi', c}(X; A'),$$

where $A = \delta(A)$, $A' = \delta'(\xi^*)^{-1}(A)$ and $A$ is any object of $\mathcal{D}^b(X, \varphi)$.

In the case when $G$ is a Lie group and $X$ is a subanalytic space, we prove the existence of an isomorphism between the $G$-equivariant cohomologies $H_{\varphi}(X; A)$ and $H_{\varphi'}(X; A')$. More precisely,
Theorem 2. Let $\varphi, \varphi' : G \to \mathcal{H}$ be group actions on $X$, such that

1. $\varphi$ and $\varphi'$ are homotopic through a family of group homomorphisms from $G$ to $\mathcal{H}$,
2. $X$ is a subanalytic space,
3. $G$ is a Lie group with a finite number of connected components.

Then the cohomologies $H_\varphi(X; A)$ and $H_{\varphi'}(X; A')$ are isomorphic

$$H_{\varphi'}(X; A') \simeq H_\varphi(X; A),$$

where $A = \delta(A'), A' = \delta'((\xi^*)^{-1}(A))$ and $A$ is any object of $\mathcal{D}^b(X, \varphi)$.

In the case that the group $G$ is a compact torus $T$ and $X$ is a compactifiable space (see [5, Sections 3 and 6]), we will prove the following theorem.

Theorem 3. Let $X$ be a compactifiable space and $\varphi, \varphi' : T \to \mathcal{H}$ be continuous group homomorphisms homotopic through a family of group homomorphisms. Denoting by $F$ and $F'$ the fixed point sets of the corresponding actions on $X$, then

$$H(F; \mathbb{C}) \simeq H(F'; \mathbb{C}).$$

2. Equivariant derived categories

2.1. Homotopy quotients. In this subsection, we define the spaces $X(\varphi)(n)$, which are “approximations” to the space $X(\varphi)$ introduced in Section 1.

Let $X$ be a Hausdorff locally compact topological space and $G$ a topological group. As we said, $\mathcal{H}$ stands for a topological subgroup of the homeomorphism group of $X$, endowed with the compact open topology. By $\varphi : G \to \mathcal{H}$ we denote a continuous group homomorphism.

Given a positive integer $n$, let $EG(n) \to BG(n)$ be the approximation to the universal $G$-bundle given by the $n$-fold joint $G \times \cdots \times G$ in the Milnor’s construction (see [9, 7, page 53]). Following Husemoller, we write $\langle g, t \rangle$ for the elements of $EG(n)$, where $(g, t)$ is the sequence $(t_1, g_1, \ldots, t_n, g_n)$ satisfying the properties detailed in [7]. As it is well-known, the topological space $EG(n)$ is $\kappa(n)$-acyclic, where $\kappa(n)$ is a number that tends to infinity when $n$ grows.

It is easy to check in the Milnor’s construction that the group homomorphism $\varphi$ induces a continuous map $\phi_n : EG(n) \to EH(n)$, satisfying $\phi_n(eg) = \phi_n(e)\varphi(g)$, for all $e \in EG(n)$ and all $g \in G$. In fact, in the above notation the map $\phi_n$ is given by

$$\phi_n(\langle g, t \rangle) = \langle \varphi(g), t \rangle. \quad (2.1)$$

In turn, $\phi_n$ induces a continuous map $\Phi_n : BG(n) \to B\mathcal{H}(n)$, between the corresponding approximations to the classifying spaces. The map $\phi_n$ factors through $\Phi_n^* (EH(n))$, the pullback of $EH(n)$ by the map $\Phi_n$,

$$\begin{array}{ccc}
EG(n) & \xrightarrow{\tau_n} & \Phi_n^* (EH(n)) \\
\phi_n \downarrow & & \Phi_n \\
EH(n) & & \\
\end{array} \quad (2.2)$$

The $G$-action on $X$ determines an associated bundle to $EG(n)$ with fiber $X$, that will be denoted $X_{\varphi}(n) := EG(n) \times_\varphi X$. We have also the corresponding homotopy
quotient $X_{\mathcal{H}}(n) := EH(n) \times_{\mathcal{H}} X$. The map between the associated bundles $X_{\varphi}(n)$ and $X_{\mathcal{H}}(n)$ induced by $\phi_n$, will be denoted by $f_n$. The map $f_n$ factors through $X(\varphi)(n) := \Phi^*_n(EH(n)) \times_{\mathcal{H}} X$; $f_n = \nu_n \circ \rho_n$.

\[
\begin{array}{c}
X_{\mathcal{H}}(n) \xrightarrow{\rho_n} X(\varphi)(n) \\
| \downarrow f_n \quad | \downarrow \nu_n \\
X_{\mathcal{H}}(n)
\end{array}
\]

In this subsection, we assume that the value of $n$ has been fixed and we will omit the $n$’s in $BG(n)$, $EG(n)$, $X_{\varphi}(n)$, $X(\varphi)(n)$, $\phi_n$, $\rho_n$, $f_n$, etc.

Some objects and arrows we have introduced can be organized in the following commutative diagram in the category of topological spaces

\[
\begin{array}{ccc}
X & \xrightarrow{p} & EG \times X \\
\downarrow 1 & & \downarrow q \\
X & \xrightarrow{\tau \times 1} & X_{\varphi}
\end{array}
\]

\[
\begin{array}{ccc}
\Phi^*(EH) \times X & \xrightarrow{Q} & X(\varphi)
\end{array}
\]

where $p$ and $P$ are the corresponding projection maps and $q$ and $Q$ the quotient maps.

In the following commutative diagram are involved the bundles with fiber $X$ that we have mentioned

\[
\begin{array}{ccc}
X_{\varphi} & \xrightarrow{\rho} & X(\varphi) \\
\downarrow \pi & & \downarrow \nu \\
BG & \xrightarrow{\Phi} & BH
\end{array}
\]

\[
\begin{array}{ccc}
& \xrightarrow{f} & \quad & \quad & \quad & \quad \\
& \quad & X(\varphi) & \xrightarrow{\nu} & X_{\mathcal{H}} \\
BG & \xrightarrow{1} & BG & \xrightarrow{\Phi} & BH
\end{array}
\]

**Proposition 4.** The map $\rho : X_{\varphi} \to X(\varphi)$ is an isomorphism of bundles over the identity of $BG$.

**Proof.** The mapping $\rho$ is given by $\rho(q(e, x)) = Q(\phi(e), x)$, for $e \in BG$ and $x \in X$. Fixed an element $e^0$ in the fibre of $b$ in the fibration $EG \to BG$, then $\pi^{-1}(b) = \{q(e^0, x) \mid x \in X\}$ and $\pi^{-1}(b)$ can be parametrized by the points of $X$. An analogous parametrization of $\hat{\pi}^{-1}(b)$ is defined by choosing an element $e'$ in the fibre over $b$ in $\Phi^*(EH) \to BG$; in particular, we can choose $e' = \phi(e^0)$. With these choices the map $\pi^{-1}(b) \to \hat{\pi}^{-1}(b)$, restriction of $\rho$, is simply the map $x \mapsto x$. That is, $\rho$ induces homeomorphisms between $\pi^{-1}(b)$ and $\hat{\pi}^{-1}(b)$, for all $b \in BG$.

For $i \in \{1, \ldots, n\}$, let $t_i$ be the natural map $(g, t) \in EG \mapsto t_i \in [0, 1]$ (see [7]). The family

\[
\{V_i := \hat{\pi}(t_i^{-1}(0, 1))\}_{i=1,\ldots,n},
\]

$\hat{\pi}$ being the projection $EG \to BG$, is an open covering of $BG$.

On $V_i$ we consider the local section $\chi_i$ of $EG$ defined by

$$
\chi_i([g, t]) = [g, t]g_i^{-1} = (t_1g_1g_i^{-1}, \ldots, t_ng_ng_i^{-1}),
$$

where $[g, t] = \pi((g, t))$. The equality $[g, t] = [g', t']$ implies the existence of an element $b \in G$ such that $g_j' = g_jb$, for all $j$ such that $t_j \neq 0$. If $[g, t] = [g', t'] \in V_i$, the following holds:

\[
\begin{array}{c}
\chi_i([g, t]) = [g, t]g_i^{-1} = (t_1g_1g_i^{-1}, \ldots, t_ng_ng_i^{-1}) \\
\chi_i([g', t']) = [g', t']g_i^{-1} = (t_1g_1g_i^{-1}, \ldots, t_ng_ng_i^{-1})
\end{array}
\]
then \( b = g_i^{-1}g_i' \); thus, \( g_jg_i^{-1} = g_j'g_i'^{-1} \), and the section \( \chi_i \) is well defined. On \( V_i \cap V_j \), 
\( \chi_j = \chi_i \cdot m_{ij} \), with \( m_{ij}([g, t]) = g_jg_i^{-1} \).

Analogously, \( \{\tilde{V}_i\}_i \), with
\[
\tilde{V}_i = \{([h, t] | t_i \neq 0) \in BH, 
\]
is an open covering of \( BH \). On each \( \tilde{V}_i \) one can define the corresponding section \( \tilde{\chi}_i \) of \( E \). The respective transition functions are \( \tilde{m}_{ij}([h, t]) = h_ih_j^{-1} \).

Since the map \( \Phi : BG \to BH \) is defined by \( \Phi([g, t]) = [\varphi(g), t] \), the inverse image \( \Phi^{-1}(\tilde{V}_i) = V_i \), and the transition functions of \( \Phi^*(E^H) \) associated with the above trivializations are the maps
\[
[g, t] \in V_i \cap V_j \mapsto \varphi(g_i)\varphi(g_j^{-1}) \in H.
\]

On other hand, \( X_\varphi \) is the bundle associated to \( EG \) through the representation \( \varphi \) of \( G \) in the homeomorphisms of \( X \). Therefore, its transition functions are the functions \( [g, t] \mapsto \varphi(g_i)\varphi(g_j^{-1}) \). Thus, the bundles \( X_\varphi \) and \( X(\varphi) \) are isomorphic, since they have homeomorphic fibers and the same transition functions with respect to the families of local sections \( \{\chi_i\}_i \) and \( \{\tilde{\chi}_i\}_i \).

\[\square\]

**Corollary 5.** If \( \mathcal{H} \) is a subgroup of \( H \) such that \( \varphi \) can be expressed as the composition \( i \circ \hat{\varphi} \), with \( i : \mathcal{H} \hookrightarrow H \), then \( X(\varphi) \) and \( X(\hat{\varphi}) \) are isomorphic bundles.

If \( X \) is a locally compact topological space and \( G \) is a locally compact topological group, then the spaces \( X(\varphi) \simeq X_\varphi \) and \( BG \) are locally compact, as well, and we can consider functors direct image with compact support between the categories of sheaves on these spaces.

By Proposition 4, the left square in (2.3) is obviously cartesian. Then we have the “base change” formulas given in the following proposition.

**Proposition 6.** Let assume that

1. \( X \) is a locally compact topological space,
2. \( G \) is locally compact topological group.

Then there is a canonical isomorphism between the functors \( R\pi_!\rho^* \) and \( R\tilde{\pi}_! \)

\[
R\pi_!\rho^* \simeq R\tilde{\pi}_! : D^b(X(\varphi)) \to D^b(BG).
\]

And a canonical isomorphism between the functors

\[
R\pi_*\rho^* \simeq R\tilde{\pi}_* : D^b(X(\varphi)) \to D^b(BG).
\]

2.2. **The space** \( X(\varphi) \). Next, we define the space \( X(\varphi) \) as the direct limit of the \( X(\varphi)(n) \)'s. We insert the label \( n \) in the notations. Let us consider the filtrations associated with the group \( G \)

\[
\cdots \subset EG(n) \subset EG(n+1) \subset \cdots \quad \text{and} \quad \cdots \subset BG(n) \subset BG(n+1) \subset \cdots
\]

and the ones corresponding to the group \( H \). If \( \varphi : G \to H \) is a \( G \)-action on \( X \), the maps

\[
\Phi_n : BG(n) \to BH(n) \quad \text{and} \quad \tau_n : EG(n) \to \Phi_n^*(E^H(n))
\]
are compatible with the filtrations. That is, for \( n < m \) we denote

\[
j_{mn} : EG(n) \hookrightarrow EG(m), \quad \text{and} \quad i_{mn} : \Phi_n^*(E^H(n)) \hookrightarrow \Phi_m^*(E^H(m)),
\]
then $\tau_m j_{mn} = i_{mn} \tau_n$. Similar relations are valid for the maps $\Phi_n$ and the inclusions $BG(n) \rightarrow BG(m)$. On the other hand, $j_{mn}$ induces a map $\psi_{mn} : X_\varphi(n) \rightarrow X_\varphi(m)$, and in turn $i_{mn}$ induces $\bar{\psi}_{mn} : X(\varphi)(n) \rightarrow X(\varphi)(m)$. In summary, the following diagrams are commutative

\[
\begin{array}{ccc}
\Phi_n^*(E\mathcal{H}(n)) & \xrightarrow{i_{mn}} & \Phi_n^*(E\mathcal{H}(m)) \\
\tau_n \uparrow & & \tau_m \uparrow \\
EG(n) & \xrightarrow{j_{mn}} & EG(m)
\end{array}
\quad
\begin{array}{ccc}
X(\varphi)(n) & \xrightarrow{\bar{\psi}_{mn}} & X(\varphi)(m) \\
\rho_\varphi \uparrow & & \rho_\varphi \uparrow \\
X_\varphi(n) & \xrightarrow{\psi_{mn}} & X_\varphi(m)
\end{array}
\]

For later references, we copy left square of (2.3) showing the dependence of $n$ of its elements.

\[(sq(n)) \quad X_\varphi(n) \xrightarrow{\rho_n} X(\varphi)(n) \quad \bar{\pi}_n \downarrow \quad \pi_n \quad BG(n) \xrightarrow{1} BG(n)\] 

The maps $\psi_{mn}, \bar{\psi}_{mn}$ and the inclusion $BG(n) \subset BG(m)$ connect the vertices of diagram $(sq(n))$ with the corresponding vertices of $(sq(m))$, and the resulting cube is a commutative diagram. Taking the inductive limit, we obtain a homeomorphism

\[\rho : X_\varphi := \lim X_\varphi(n) \longrightarrow X(\varphi) := \lim X(\varphi)(n),\]

which is an isomorphism over the identity between the bundles $X_\varphi \xrightarrow{\pi} BG$ and $X(\varphi) \xrightarrow{\bar{\pi}} BG$.

2.3. The category $\mathcal{D}^b(X, \varphi)$. In this subsection, we assume over the space $X$ and the group $G$ the hypotheses of Proposition 6. Then, as we said, the spaces $X_\varphi(n)$ and $X(\varphi)(n)$ are also locally compact.

To avoid the use of non locally compact topological spaces in the definition of $\mathcal{D}^b(X, \varphi)$, we will construct this category as a “direct limit” of a sequence of categories, following a procedure similar to the one employed for the definition of the $G$-equivariant category $D^b_G(X)$ in [2, page 27].

The category $\mathcal{D}^b(X, \varphi)$. Let $\mathcal{A}_n := (A_X, \tilde{A}_n, \alpha_n)$ be a triple consisting of

\[A_X \in D^b(X), \quad \tilde{A}_n \in D^b(X(\varphi)(n))\]

and $\alpha_n$ an isomorphism in $D^b(\Phi_n^*(E\mathcal{H}(n)) \times X)$ from $P_n^*(A_X)$ to $Q_n^*(\tilde{A}_n)$ (see diagram (cd(n))). That is, with the notation introduced in [2, page 17], the triple $\mathcal{A}_n$ defines an object of the category $D^b_H(X, \Phi_n^*(E\mathcal{H}(n)) \times X)$.

Let $\mathcal{A} := \{\mathcal{A}_n\}_{n=1,2,...}$ be a sequence of triples, such that each of which satisfies the properties stated above. Furthermore, we assume that the following compatibility conditions hold

\[(2.8) \quad \bar{\psi}_{mn}^*(\tilde{A}_m) = \tilde{A}_n, \quad (i_{mn} \times 1)^*(\alpha_m) = \alpha_n,\]

for $n < m$.

The above sequence $\mathcal{A}$ does not defines an object of the category $D^b_H(X)$, since the limit of the $\Phi_n^*(E\mathcal{H}(n))$’s is not contractible. However, we can construct a new category, $\mathcal{D}^b(X, \varphi)$, whose objects are the $\mathcal{A}$’s.
Given two objects \( \mathcal{A} = \{(A_X, \bar{A}_n, \alpha_n)\}_n \) and \( \mathcal{B} = \{(B_X, \bar{B}_n, \beta_n)\}_n \) in the category \( \mathcal{D}^b(X, \varphi) \), a morphism \( \sigma : \mathcal{A} \to \mathcal{B} \) in \( \mathcal{D}^b(X, \varphi) \) is a sequence of pairs \( \{(\sigma_X, \bar{\sigma}_n)\}_n \), with \( \sigma_X : A_X \to B_X \) and \( \bar{\sigma}_n : \bar{A}_n \to \bar{B}_n \), such that

\[
\beta_n \circ P_n^*(\sigma_X) = Q_n^*(\bar{\sigma}_n) \circ \alpha_n,
\]

and for \( n < m \) they satisfy the compatibility condition

\[
(2.9) \quad \tilde{\psi}_{mn}^*(\bar{\sigma}_m) = \bar{\sigma}_n.
\]

The functor \( \delta \). On the other hand, \( \mathcal{A}_n \) gives rise to the triple \( A_n := (A_X, \bar{A}_n, a_n) \), where

\[
(2.10) \quad A_X = A_X \in D^b(X), \quad \bar{A}_n = p_n^*(\bar{A}_n) \in D^b(X, \varphi(n)),
\]

and

\[
(2.11) \quad a_n := (\tau_n \times 1)^*(\alpha_n)
\]

is an isomorphism in \( D^b(EG(n) \times X) \) between \( p_n^*(A_X) \to q_n^*(\bar{A}) \) (see \( \text{cd}(n) \)). In other words, \( A_n \) is an object of \( D^b(X, \text{EG}(n) \times X) \), when we consider in \( X \) the \( G \)-action defined by \( \varphi \).

Given the object \( \mathcal{A} = \{\mathcal{A}_n\} \), each \( \mathcal{A}_n \) gives rise to a triple \( A_n = (A_X, \bar{A}_n, a_n) \). Since \( \text{cd}(n) \) and \( \text{cd}(m) \) together with the maps between them determined by \( j_{mn}, l_{mn}, \psi_{mn} \) and \( \tilde{\psi}_{mn} \) form a commutative cuboid, then from (2.8), it follows

\[
(2.12) \quad \psi_{mn}^*(\bar{A}_m) = \bar{A}_n, \quad (j_{mn} \times 1)^*(a_m) = a_n.
\]

As the \( G \)-space \( \text{EG}(n) \times X \) defines a resolution \( \kappa(n) \)-acyclic of \( X \), the sequence \( A = \{A_n\}_n \) can be considered as an object of the equivariant derived category \( D^b_{\varphi}(X) \) [2, page 27].

On the other hand, given \( \sigma : \mathcal{A} \to \mathcal{B} \), we define \( s_X := \sigma_X \) and \( \bar{s}_n := \rho_n^*(\bar{\sigma}_n) \). From (2.11) together with the commutativity of \( (\text{cd}(n)) \), it follows \( q_n^*(\bar{s}_n) \circ a_n = b_n \circ p_n^*(s_X) \). By (2.9),

\[
\psi_{mn}^*(\bar{s}_m) = \bar{s}_n.
\]

Thus, the family \( s = \{(s_X, \bar{s}_n)\}_n \) is a morphism in \( D^b_{\varphi}(X) \) from \( A \) to \( B \). So, we have the functor

\[
(2.13) \quad \delta : \mathcal{D}^b(X, \varphi) \to D^b_{\varphi}(X) ; \quad \mathcal{A} \mapsto A, \quad \sigma \mapsto s.
\]

Examples. With the constant sheaves \( \mathbb{C}_X \) and \( \mathbb{C}_X(\varphi(n)) \), we can construct the triple

\[
C_n := (\mathbb{C}_X, \mathbb{C}_X(\varphi(n)), \text{Id}) \in D^b_{\text{H}}(X, \Phi_n^*(\text{EH}(n)) \times X),
\]

since the constant sheaf on a space \( Z \) is defined through the inverse image by \( \mu : Z \to \text{pt} \), the constant map to a point. Then, the object \( \delta(C) \in D^b_{\varphi}(X) \) is defined by the sequence \( \{C_n\} \), with

\[
C_n := (\mathbb{C}_X, \mathbb{C}_X(\varphi(n)), \text{Id}) \in D^b_{\varphi}(X, \text{EG}(n) \times X),
\]

again by the above mentioned property of the constant sheaf.

Let us assume that \( X \) is a topological space with finite cohomological dimension. The dualizing object \( D_X \) on \( X \) is defined as the inverse image with compact support of \( \mathbb{C}_{\text{pt}} \) by the constant map to a point. Since \( Q^* \mu^! = \mu^! \) and \( P^* \mu^! = \mu^! \), we can construct the following object of the category \( D^b_{\text{H}}(X, \Phi_n^*(\text{EH}(n)) \times X) \)

\[
(2.14) \quad \mathcal{D}_n = (D_X, D_X(\varphi(n)), \text{Id}) \in D^b_{\text{H}}(X, \Phi_n^*(\text{EH}(n)) \times X),
\]
which gives rise to the object $\delta(D) \in D^b_{\varphi}(X)$.

**Remark.** If $\bar{\mathcal{H}}$ is a subgroup of $\mathcal{H}$ containing the image of $\varphi$, as in Corollary $\mathbb{V}$ the inclusion $E\bar{\mathcal{H}} \subset E\mathcal{H}$ induces a factorization of $\rho_n : X_{\varphi}(n) \rightarrow X(\varphi)(n)$ through $X(\bar{\varphi})(n)$, which in turn gives rise to the following factorization of the functor $\delta$

$$D^b_{\varphi}(X) \xrightarrow{\delta} D^b(X, \bar{\varphi}) \xrightarrow{\delta} D^b(X, \varphi).$$

According to Corollary $\mathbb{V}$ the spaces $X(\varphi)$ and $X(\bar{\varphi})$ are homeomorphic, but the categories $D^b(X, \varphi)$ and $D^b(X, \bar{\varphi})$ are not necessarily equivalent.

## 3. Deformation of group actions

Before to consider deformations of $G$-group actions on the space $X$, we will treat with a slighter general situation.

Given $\varphi, \varphi' : G \rightarrow \mathcal{H}$ two $G$-actions not necessarily homotopic. Let us assume that for each $n$ there are bundle isomorphisms over the identity of $BG(n)$

$$\omega_n : \Phi^*_n(E\mathcal{H}(n)) \rightarrow \Phi'^*_n(E\mathcal{H}(n)) \text{ and } \vartheta_n : X(\varphi)(n) \rightarrow X(\varphi')(n),$$

satisfying the compatibility conditions

$$\vartheta_n \circ Q_n = Q'_n \circ (\omega_n \times 1), \quad i_{mn}' \circ \omega_n = \omega_m \circ i_{mn}, \quad \tilde{\psi}_{mn}' \circ \vartheta_n = \vartheta_m \circ \tilde{\psi}_{mn}, \text{ for } n < m.$$  

For the sake of brevity, we say that the family $\vartheta = \{\vartheta_n\}$ is a $(\varphi, \varphi')$-cohomological morphism. (The reason for the adjective cohomological will be apparent below).

That is, for each $n$, we have the following commutative diagram of fiber bundles on $BG(n)$, that in turn is consistent with the one corresponding to the number $m$.

$$\xymatrix{ EG(n) \times X \ar[r]^{q_n} \ar[d]_{\tau_n \times 1} & X_{\varphi}(n) \ar[d]_{\rho_n} \ar[r]^{Q'_n} & X(\varphi')(n) \ar[d]_{\vartheta_n} \\ \Phi^*_n(E\mathcal{H}(n)) \times X \ar[r]^{Q_n} \ar[dr]_{\omega_n \times 1} & X(\varphi) \ar[dr]_{\vartheta_n} & X(\varphi'_n) \ar[dr]_{\vartheta_n} & .}$$

**Proposition 7.** If the space $X$ and the group $G$ are locally compact and $\vartheta$ is a $(\varphi, \varphi')$-cohomological morphism, then there exist the following canonical isomorphisms of functors

$$R(\bar{\pi}_{n}) : D^b(X(\varphi')(n)) \rightarrow D^b(BG(n))$$

$$R(\bar{\pi}'_{n}) : D^b(X(\varphi')(n)) \rightarrow D^b(BG(n)).$$
Proof. Since \( \vartheta_n \) is a homeomorphism, the diagram

\[
\begin{array}{ccc}
X(\varphi)(n) & \xrightarrow{\vartheta_n} & X(\varphi')(n) \\
\downarrow{\pi_n} & & \downarrow{\pi'_n} \\
BG(n) & \xrightarrow{1} & BG(n).
\end{array}
\]

is a cartesian square, obviously. Formula (2.4b) in [6] applied to this square gives the first isomorphism. On the other hand, the square consisting of the maps \( \vartheta_{n,-1} \), \( \bar{\pi}_n \), \( 1 \) and \( \bar{\pi}'_n \) is also cartesian. Applying (2.5a) of [6] to those cartesian squares, one obtains the second isomorphism.

Taking the inductive limits as \( n \) goes to \( \infty \) in (3.3), one prove that the diagram obtained from (3.3) deleting the label \( n \) is also commutative.

**Proposition 8.** The families \( \omega \) and \( \vartheta \) define an isomorphism \( \vartheta^* \) from the category \( D^b(X, \varphi') \) to \( D^b(X, \varphi) \).

**Proof.** The inverse image functors induced by the family of homeomorphisms \( \omega_n \) and \( \vartheta_n \) determine the isomorphism. More precisely, the functor \( (\vartheta^{-1})^*: D^b(X, \varphi) \to D^b(X, \varphi') \) defined by

\[
(\vartheta^{-1})^*(A) = \{(A_X, (\vartheta^{-1})^*(\bar{A}_n), (\omega_n^{-1} \times 1)^*(\alpha_n))\}_n
\]

gives the isomorphism between the categories. \qed

**Proposition 9.** Under the hypotheses of Proposition 7, given \( A \in D^b(X, \varphi) \), let \( \{(A_X, \bar{A}_n, a_n)\} = \delta(A) \in D^b_\varphi(X) \) and \( \{(A_X, \bar{A}_n', a'_n)\} = \delta'(\vartheta^{-1})^*(A) \in D^b_\varphi(X) \), then

- The objects \( R(\pi_n): (\bar{A}_n) \) and \( R(\pi'_n): (\bar{A}'_n) \) of \( D^b(BG(n)) \) are canonically isomorphic.
- \( R(\pi_n)_*(\bar{A}_n) \simeq R(\pi'_n)_*(\bar{A}'_n) \) canonically isomorphic.

**Proof.** From Proposition 6 together with (2.10), it follows

\[
R(\pi_n)_!(\bar{A}_n) = R(\pi_n)_!(\bar{A}_n) \simeq R(\bar{\pi}_n)_!(\bar{A}_n),
\]

where the isomorphism is canonical.

Analogously, \( R(\pi'_n)_!(\bar{A}'_n) \simeq R(\bar{\pi}'_n)_!(\bar{A}'_n) \). Using Proposition 6 we conclude

\[
R(\pi_n)_!(\bar{A}_n) \simeq R(\pi'_n)_!(\bar{A}'_n).
\]

Similarly, from Proposition 6 and Proposition 8 it follows our second claim. \qed

As above, we denote by \( \mu \) the constant map \( \mu : X \to pt \). Then we have the following commutative diagram

\[
\begin{array}{ccc}
X & \xleftarrow{p_n} & EG(n) \times X \xrightarrow{q_n} X_\varphi(n) \\
\mu \downarrow & & \varepsilon \downarrow \mu_G \\
pt & \xleftarrow{u} & EG(n) \xrightarrow{\bar{\pi}} BG(n) = pt_G(n),
\end{array}
\]

where \( \mu_G \) is the map induced between the homotopy quotients; that is, \( \mu_G = \pi_n \) (see \( sq(n) \)).
Let \( A = \{(A_X, \tilde{A}_n, a_n)\} \) be an object of \( D^b(X) \), then formula (2.4b) of [6] applied to the squares of (3.3) gives the isomorphisms
\[
(3.5) \quad u^* R\mu(A_X) \simeq R\mu_n^* (A_X), \quad \text{and} \quad R\tilde{\epsilon} q_n^* (\tilde{A}_n) \simeq \tilde{\pi}^* R(\tilde{\pi}_n)(\tilde{A}_n).
\]

The object \( R\mu(A) \in D^b_G(pt) \) is defined by the family of triples
\[
\{(R\mu(A_X), R(\pi_n)(\tilde{A}_n), c_n)\},
\]
where \( c_n \) is given by the composition of the canonical isomorphisms (3.5) with
\[
R\tilde{\epsilon}(a_n) : R\mu_n^* (A_X) \to R\tilde{\epsilon} q_n^* (\tilde{A}_n).
\]

Thus, we have defined a functor \( D^b_G(pt) \to D^b_G(pt) \), which will be denoted by \( R\mu \).

From Proposition 9 it follows the following proposition.

**Proposition 10.** If the group \( G \) and the space \( X \) are locally compact and \( \vartheta \) is a \((\varphi, \varphi')\)-cohomological morphism, then the isomorphism \( \vartheta^* : \mathfrak{D}^b(X, \varphi') \to \mathfrak{D}^b(X, \varphi) \) makes commutative, up to canonical isomorphism, the following diagram
\[
\begin{array}{ccc}
D^b_G(pt) & \xrightarrow{R\mu'} & D^b_G(pt) \\
\downarrow{R\mu} & & \downarrow{R\mu} \\
D^b_{\varphi'}(X) & \xrightarrow{\delta'} & D^b_{\varphi}(X) \\
\downarrow{\delta} & & \downarrow{\delta} \\
\mathfrak{D}^b(X, \varphi') & \xrightarrow{\vartheta^*} & \mathfrak{D}^b(X, \varphi),
\end{array}
\]

where \( R\mu \) and \( R\mu' \) are the functors induced by the constant map \( \mu : X \to pt \).

As we said, we denote by \( H_{\varphi,c}(X; A) \) the equivariant (with respect to the action \( \varphi \)) cohomology of \( X \) with compact supports and coefficients in \( A \in D^b_{\varphi}(X) \). By definition \( H_{\varphi,c}(X; A) = H(R\mu(A)) \) [2, page 115]. A direct consequence of above Proposition is the following corollary.

**Corollary 11.** Under the hypotheses of Proposition 10, if \( A \in \mathfrak{D}^b(X, \varphi) \), then
\[
H_{\varphi,c}(X; \delta(A)) \simeq H_{\varphi,c}(X; \delta(\vartheta^{-1})^*(A)).
\]

To introduce the functor direct image in the context of equivariant categories are necessary additional hypotheses on the topological space \( X \) and on the group \( G \), because the formula of change basis for the functor direct image holds only when the change is smooth [2, page 10].

We will assume that \( X \) is a subanalytic space. With respect to \( G \), we suppose that it is a group such that the spaces \( EG(n) \) are homeomorphic to finite dimensional manifolds. This property is guaranteed if \( G \) is a Lie group with a finite number of connected components (see [2] pages 34-35).

As in the case of the direct image with compact supports, we consider diagram (3.3) but assuming that \( EG(n) \) is a smooth manifold. Given \( A = \{(A_X, \tilde{A}_n, a_n)\} \in D^b_G(X) \), since the space \( EG(n) \) is smooth we have the following isomorphisms (see [2] page 13, [3] page 38)
\[
u^* R\mu_*(A_X) \simeq R\epsilon_* \mu_n^* (A_X), \quad \text{and} \quad R\tilde{\epsilon} q_n^* (\tilde{A}_n) \simeq \tilde{\pi}^* R(\tilde{\pi}_n)(\tilde{A}_n).
\]

The object \( R\mu_*(A) \in D^b_G(pt) \) is defined by the sequence of triples
\[
\{(R\mu_*(A_X), R\pi_*(\tilde{A}_n), e_n)\},
\]
where $e_n$ is given by the composition of the the above isomorphisms with $$R\epsilon_*(a_n) : R\epsilon_*p_n^*(A_X) \rightarrow R\epsilon_*q_n^*(\hat{A}_n).$$

In this way, we have defined a functor $R\mu_* : D_G^b(X) \rightarrow D_G^b(pt)$.

From Proposition 11, one deduces the following proposition.

**Proposition 12.** If $\vartheta$ is a $(\varphi, \varphi')$-cohomological morphism and

1. $X$ is a subanalytic space,
2. $G$ is a Lie group with a finite number of connected components.

Then for any $A \in \mathcal{D}'(X, \varphi)$, the objects $R\mu'_*(\vartheta^{-1})^*(A)$ and $R\mu_*(\vartheta(A))$ of $D_G^b(pt)$ are canonically isomorphic

$$R\mu'_*(\vartheta^{-1})^*(A) \simeq R\mu_*(\vartheta(A)),$$

where $R\mu_*$ and $R\mu'_*$ are the functors induced by the constant map $\mu : X \rightarrow pt$.

### 3.1. Homotopic $G$-actions

Let $\{\varphi^s : G \rightarrow \mathcal{H} \mid s \in [0, 1]\}$ be a homotopy consisting of continuous group homomorphisms. We put $\varphi := \varphi^0$, $\varphi' := \varphi^1$ and we write $\varphi \sim_n \varphi'$.

The maps $\phi_n, \phi'_n : EG(n) \rightarrow E\mathcal{H}(n)$ are also homotopic by means of the family

$$\{\langle g, t \rangle \mapsto \langle \varphi^s(g), t \rangle \}_{s \in [0, 1]}.$$  \hspace{1cm} (3.7)

In the same way, the maps $\Phi_n, \Phi'_n : BG(n) \rightarrow B\mathcal{H}(n)$ are homotopic through the homotopy $F_n$ defined by

$$BG(n) \times [0, 1] \rightarrow B\mathcal{H}(n) : (\langle g, t \rangle, s) \mapsto \langle \varphi^s(g), t \rangle =: \Phi^s_n(\langle g, t \rangle),$$

where $[g, t]$ is the image of $\langle g, t \rangle$ by the projection to $BG(n)$.

Since $\Phi_n$ and $\Phi'_n$ are homotopic, the $\mathcal{H}$-principal bundles $X(\varphi)(n)$ and $X(\varphi')(n)$ are isomorphic. In the following proposition, we recall the construction of this isomorphism using the fold joint structure of $EG(n)$. Some stuff of this construction will be used for proving Proposition 13.

**Proposition 13.** Let $\varphi, \varphi' : G \rightarrow \mathcal{H}$ be continuous group homomorphisms that are homotopic through a family of group homomorphisms. Then, for each positive integer $n$, there exist bundle isomorphisms over the identity of $BG(n)$

$$\lambda_n : \Phi^*_n(E\mathcal{H}(n)) \rightarrow \Phi'^*_n(E\mathcal{H}(n))$$

and $\xi_n : X(\varphi)(n) \rightarrow X(\varphi')(n)$, which satisfy $\xi_n \circ Q_n = Q'_n \circ (\lambda_n \times 1)$.

**Proof.** Fixed the integer $n$, we omit in this proof the label $n$ in our notations; so we write $EG$ for $EG(n)$, $BG$ for $BG(n)$, $\Phi$ for $\Phi_n$, etc.

The fibre $\mathcal{H}$-bundles on $BG \times I$, $F^*(E\mathcal{H})$ and $\hat{\pi}^*(\Phi^*(E\mathcal{H}))$ (where $\hat{\pi}$ is the projection of $BG \times I$ on the first factor) are isomorphic on $BG \times \{1\}$.

According to the proof of Theorem 9.6 of [7] (see [7] page 50), to extend the isomorphism to $BG \times I$, are sufficient a covering $\{V_i\}$ of $BG$, trivializations of $F^*(E\mathcal{H})$ over the $V_i$'s and a family of maps $w_i : BG \rightarrow I$, with $w_i^{-1}(0, 1) \subset V_i$.

(i) Covering of $BG$. For $i \in \{1, \ldots, n\}$, let $t_i$ be the natural map $\langle g, t \rangle \in EG \mapsto t_i \in [0, 1]$ (see [7] page 53). The family

$$\{V_i := \hat{\pi}(t_i^{-1}(0, 1))\}_{i=1,\ldots,n},$$

$\hat{\pi}$ being the projection $EG \rightarrow BG$, is an open covering of $BG$.

(ii) The functions $w_i$. The maps $w_i : [g, t] \in BG \mapsto t_i \in [0, 1]$ satisfy the required property.
(iii) Trivializations of $F^*(EH)$. Next, we construct a family $\{\eta_i\}_{i = 1, \ldots, n}$ of local sections of the $H$-bundle $F^*(EH)$, where the domain of $\eta_i$ is $V_i \times I$. We put

$$\eta_i : V_i \times I \to F^*(EH) : ([g, t], s) \mapsto (\varphi^s(g), t)\varphi^s(g_i^{-1}).$$

More explicitly,

$$\langle \varphi^s(g), t \rangle \varphi^s(g_i^{-1}) := \langle t_1 \varphi^s(g_1)\varphi^s(g_i^{-1}), \ldots, t_i 1, \ldots, t_n \varphi^s(g_n)\varphi^s(g_i^{-1}) \rangle.$$

The equality $[g, t] = [g', t']$ implies the existence of an element $b \in G$ such that $g'_j = g_j b$, for all $j$ such that $t_j \neq 0$. In particular, if $[g, t] = [g', t'] \in V_i$, then $b = g_i^{-1} g'_i$; thus, $\varphi^s(g_j)\varphi^s(g_i^{-1}) = \varphi^s(g'_j)\varphi^s(g_i^{-1})$, and the section $\eta_i$ is well defined.

With these ingredients, and following the construction given in [7], one can define an isomorphism $\lambda$ of $H$-bundles over the identity between $\Phi^*(EH)$ and $\Phi^*(EH)$.

The isomorphism $\lambda$ of principal bundles induces another one between the associated bundles $X(\varphi)$ and $X(\varphi')$,

$$\xi : X(\varphi) \to X(\varphi'),$$

satisfying $\xi \circ Q = Q' \circ (\lambda \times 1)$.

\[
\square
\]

3.2. Compatibility of the $\xi_n$’s with each other. The homotopy

$$(g, s) \in G \times [0, 1] \mapsto \varphi^s(g) \in H$$

between $\varphi$ and $\varphi'$ consisting of group homomorphisms induces the homotopy $F_n$ between $\Phi_n$ and $\Phi'_n$ given (3.8). As the spaces $BG(n)$’s and the $BH(n)$’s are fold joints, the homotopies $F_n$ and $F_m$ are compatible with the inclusions $BG(n) \hookrightarrow BG(m)$ and $BH(n) \hookrightarrow BH(m)$. One has the following facts:

(1) For $n < m$, the diagram

$$
\begin{array}{ccc}
F^*_n(EH(n)) & \xrightarrow{(t_1, \ldots, t_n)} & F^*_m(EH(m)) \\
| & & | \\
BG(n) \times [0, 1] & \xrightarrow{(t_1, \ldots, t_n)} & BG(m) \times [0, 1]
\end{array}
$$

is commutative.

(2) The same property holds for the bundles

$$\hat{\pi}_n^*(\Phi_n^*EH(n)) \text{ and } \hat{\pi}_m^*(\Phi'_m^*EH(m)),$$

where $\hat{\pi}_n$ is the projection $BG(n) \times [0, 1] \to BG(n)$.

(3) We denote by $\hat{\pi}_{n+1}$ the quotient map $EG(n + 1) \to BG(n + 1)$, and for $j = 1, \ldots, n + 1$ we put $V_j(n + 1) := \hat{\pi}_{n+1}(t_j^{-1}(0, 1))$. Then, for $i = 1, \ldots, n$, $V_i(n + 1) \cap BG(n) = V_i$, where $V_i := \hat{\pi}_n(t_i^{-1}(0, 1))$, as in the proof of Proposition [13]

(4) Analogously, for $j = 1, \ldots, n + 1$ one defines the maps

$$w_j(n + 1) : BG(n + 1) \to [0, 1] : [g, t] \mapsto t_j.$$

Obviously, if $i = 1, \ldots, n$ the restriction of $w_i(n + 1)$ to $BG(n)$ coincides with the map $w_i$ defined in the proof of Proposition [13].
For \( j = 1, \ldots, n+1 \) we put
\[
\eta_j(n+1) : V_j(n+1) \times I \to F_{n+1}^*(E\mathcal{H}(n+1)),
\]
\[
([g, t], s) \mapsto \langle \phi^s(g), t \rangle \phi^s(g)^{-1}.
\]
For \( 1 \leq i \leq n \), the restriction of \( \eta_i(n+1) \) to \( V_i \times I \) coincides with the map \( \eta_i \) defined in (3.10).

By these compatibilities, the homeomorphism
\[
\lambda_{n+1} : \Phi^\ast (E\mathcal{H}(n+1)) \to \Phi^\ast (E\mathcal{H}(n+1)),
\]
constructed following the process reminded in Proposition 13 restricted to \( \Phi(n) \) coincides with \( \lambda_n \).

Thus, the sequences of homeomorphisms \( \{ \lambda_n \}_n \) and \( \{ \xi_n \}_n \) constructed in this way are consistent with the inclusions. We have the following proposition.

**Proposition 14.** Let \( \varphi, \varphi' : G \to \mathcal{H} \) be continuous group homomorphisms that are homotopic through a family of group homomorphisms, then for each \( n \) there exist homeomorphisms
\[
\lambda_n : \Phi^\ast (E\mathcal{H}(n)) \to \Phi^\ast (E\mathcal{H}(n)), \quad \text{and} \quad \xi_n : X(\varphi)(n) \to X(\varphi')(n),
\]
satisfying
\[
i'_{mn} \circ \lambda_n = \lambda_m \circ i_{mn}, \quad \bar{\psi}_{mn} \circ \xi_n = \xi_m \circ \bar{\psi}_{mn},
\]
for \( n < m \).

From Proposition 14 together with Proposition 13 it follows the corollary.

**Corollary 15.** The family \( \xi = \{ \xi_n \} \) is a \( (\varphi, \varphi') \)-cohomological morphism.

Thus, one has the corresponding functor \( \xi^\ast \), mentioned in Section 1 that gives an equivalence between the categories \( \mathcal{D}^h(X, \varphi') \) and \( \mathcal{D}^h(X, \varphi) \).

**Proof of Theorem 1.** The theorem is an immediate consequence of Corollaries 15 and 11.

As a corollary of Theorem 1 we deduce the following result, which can also be proved directly without making use of the derived categories.

**Corollary 16.** If \( \varphi \) and \( \varphi' \) are group homomorphisms as in Theorem 1 then
\[
H_{\varphi,c}(X; \mathbb{C}) \simeq H_{\varphi',c}(X; \mathbb{C}).
\]

**Proof.** The result can be proved applying Theorem 1 to the object \( \mathcal{C} \in \mathcal{D}^h(X, \varphi) \) defined in Subsection 2.3.

**Proposition 17.** Let \( G \) be a topological group and \( \varphi, \varphi' : G \to \mathcal{H} \) such that \( \varphi \sim_h \varphi' \).

If \( X \) and \( G \) are locally compact, and \( X \) has finite cohomological dimension, then
\[
H_{\varphi',c}(X; D_X) \simeq H_{\varphi,c}(X; D_X).
\]

**Proof.** It is a consequence of Theorem 1 applied to the object \( \mathcal{D} \) defined in (2.14).

**Proof of Theorem 2.** The equivariant cohomology of \( X \), with respect to the \( G \)-action \( \varphi \), with coefficients in \( A \in D^b_c(X) \) is \( H_\varphi(X; A) := H(R\mu_* A) \). Thus, Proposition 12 and Corollary 15 imply Theorem 2.
From Theorem [2] we deduce.

**Corollary 18.** Under the hypotheses of Theorem [2], the equivariant cohomologies $H_\varphi(X; \mathbb{C})$ and $H_{\varphi'}(X; \mathbb{C})$ are isomorphic.

**Proof.** Theorem [2] applied to the object $C \in D^b(X, \varphi)$ gives the corollary. □

### 4. Localization

In this section, we assume that $G$ is the torus $T = U(1)^n$ which acts on a $T$-compactifiable space $X$. This hypothesis guarantees that there are finitely many different types of $T$-orbits, although $X$ is not necessarily compact.

The $T$-equivariant cohomology with complex coefficients of a point $H_T(\text{pt}; \mathbb{C}) = H(BT; \mathbb{C})$ can be identified to the algebra $\mathbb{C}[t_C]$, of polynomials on the complexification $t_C$ of the Lie algebra of $T$. In this identification the generators of the polynomial algebra are considered with even degree [1, page 3].

Let $\Sigma$ denote the multiplicative subset of $\mathbb{C}[t_C]$ consisting of the non-zero polynomials, and by $F$ we denote the fixed point set of the $T$-action. For $A \in D^b_T(X)$, from the localization theorem [5, Sect. 6] one deduces that the restriction map defines an isomorphism

$$H_T(X; A)_\Sigma \to H_T(F; A)_\Sigma$$

between the corresponding localization modules.

As $T$ acts trivially on $F$, $ET \times_T F = BT \times F$. Thus, for the particular case of complex coefficients

$$H_T(F; \mathbb{C}) \simeq H(BT; \mathbb{C}) \otimes \mathbb{C} H(F; \mathbb{C}).$$

Hence,

$$H_T(X; \mathbb{C})_\Sigma \simeq \mathbb{C}(t_C^*) \otimes \mathbb{C} H(F; \mathbb{C}),$$

where $\mathbb{C}(t_C^*)$ is the field of rational functions on $t_C$. In other words, $H_T(X; \mathbb{C})_\Sigma$ is the result of the extension of scalars in $H(F; \mathbb{C})$ from $\mathbb{C}$ to $\mathbb{C}(t_C^*)$.

**Proof of Theorem [3]** The isomorphism is a consequence of Corollary [18] together with (4.2) and the fact that the cohomologies are $\mathbb{C}$-vector spaces. □

**Corollary 19.** Under the hypotheses of Theorem [3],

$$\bigoplus_{S \in \mathcal{C}} H(S; \mathbb{C}) \simeq \bigoplus_{S' \in \mathcal{C}'} H(S'; \mathbb{C}),$$

where $\mathcal{C}$ and $\mathcal{C}'$ are the sets of connected components of $F$ and $F'$, respectively. In particular, if $F$ and $F'$ are finite sets, then

$$\#F = \#F'.$$

**Corollary 20.** If the hypotheses of Theorem [3] are satisfied, then

$$H_\varphi(F; \mathbb{C}) \simeq H_{\varphi'}(F'; \mathbb{C}).$$
Proof. From Theorem 3 together with (11), it follows
\[ H_\varphi(F; \mathbb{C}) \simeq H(BT; \mathbb{C}) \otimes_\mathbb{C} H(F; \mathbb{C}) \simeq H(BT; \mathbb{C}) \otimes_\mathbb{C} H(F'; \mathbb{C}) \simeq H_\varphi(F'; \mathbb{C}). \]

Let \( \varphi \) be a \( T \)-action on \( X \) and \( F \) the set of fixed points. The kernel of the restriction map \( H_\varphi(X; \mathbb{C}) \to H_\varphi(F; \mathbb{C}) \) is a torsion \( H(BT; \mathbb{C}) \)-submodule. We can consider the \( H(BT; \mathbb{C}) \)-submodule \( M \) of \( H_\varphi(F; \mathbb{C}) \) consisting of those elements which admit an extension to equivariant cohomology classes on \( X \). We will show that under certain hypotheses \( M \) is isomorphic to \( H_\varphi(X; \mathbb{C}) \).

Since \( BT \) is a simply connected space, the \( E_2 \) page of the Leray-Serre cohomology spectral sequence of the fibration \( X \to ET \times_T X \to BT \) is
\[ E_2^{p,q} = H^p(BT; \mathbb{C}) \otimes_\mathbb{C} H^q(X; \mathbb{C}). \]

If \( H^\text{odd}(X; \mathbb{C}) = 0 \), as the cohomology of \( BT \) vanishes in the odd degrees, the differential \( d_r : E_r^{p,q} \to E_r^{p+r,q-r+1} \) is zero; so, the spectral sequence collapses at the step \( E_2 \). In the terminology of [5], the \( T \)-space \( X \) is equivariantly formal. Thus,
\[ H_\varphi(X; \mathbb{C}) = H(ET \times_T X; \mathbb{C}) = H(BT; \mathbb{C}) \otimes_\mathbb{C} H(X; \mathbb{C}). \]

That is, \( H_\varphi(X; \mathbb{C}) \) is a free finite generated \( \mathbb{C}[t^*_C] \)-module. Therefore, \( H_\varphi(X; \mathbb{C}) \) can be identified with a submodule of \( H_\varphi(F; \mathbb{C}) \) (see [5, page 42]); that is, \( M \) is isomorphic to \( H_\varphi(X; \mathbb{C}) \). Furthermore, since the \( T \)-action on \( F \) is trivial, \( H_\varphi(F; \mathbb{C}) \) is a free \( \mathbb{C}[t^*_C] \)-module. Thus, we have proved the proposition.

**Proposition 21.** If \( H^\text{odd}(X; \mathbb{C}) = 0 \), then \( M \) is a free \( \mathbb{C}[t^*_C] \)-submodule of \( H_\varphi(F; \mathbb{C}) \) isomorphic to \( H_\varphi(X; \mathbb{C}) \).

When \( T = U(1) \), \( \mathbb{C}[t^*_C] \) is a principal entire ring, and the free finite generated submodule \( M \) of the \( \mathbb{C}[t^*_C] \)-module \( H_\varphi(F; \mathbb{C}) \) has associated the corresponding elementary divisors.

**Proposition 22.** Let \( \varphi \) and \( \varphi' \) be homotopic \( U(1) \)-actions on a compactifiable space \( X \), satisfying the hypotheses of Theorem 3. If \( H^\text{odd}(X; \mathbb{C}) = 0 \), then the elementary divisors of \( M \) and \( M' \) as submodules of \( H_\varphi(F; \mathbb{C}) \) and \( H_\varphi(F'; \mathbb{C}) \) (resp.) coincide.

**Proof.** From Corollary 18, Proposition 21 and Corollary 20 we deduce the proposition.

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