L-invariants of Hilbert modular forms

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Abstract

In this paper we show that under certain condition the Fontaine–Mazur L-invariant for a Hilbert eigenform coincides with its Teitelbaum type L-invariant, and thus prove a conjecture of Chida, Mok and Park.

Introduction

In the remarkable paper [17] Greenberg and Stevens proved a formula for the derivative at \( s = 1 \) of the \( p \)-adic \( L \)-function of an elliptic curve \( E \) over \( \mathbb{Q} \) (i.e. a modular form of weight 2) when \( p \) is a prime of split multiplicative reduction, which is the exceptional zero conjecture proposed by Mazur, Tate, and Teitelbaum [21]. An important quantity in this formula is the \( L \)-invariant, namely \( \mathcal{L}(E) = \log_p(q_E)/v_p(q_E) \) where \( q_E \in \mathbb{P}^\times \) is the Tate period for \( E \).

For higher weight modular forms \( f \), since [21], a number of different candidates for the \( L \)-invariant \( L(f) \) have been proposed. These include:

1. Fontaine-Mazur’s \( L \)-invariant \( L_{FM} \) using \( p \)-adic Hodge theory,
2. Teitelbaum’s \( L \)-invariant \( L_T \) built by the theory of \( p \)-adic uniformization of Shimura curves,
3. an invariant \( L_C \) by Coleman’s theory of \( p \)-adic integration on modular curves,
4. an invariant \( L_{DO} \) due to Darmon and Orton using “modular-form valued distributions”,
5. Breuil’s \( L \)-invariant \( L_B \) by \( p \)-adic Langlands theory.

Now, all of these invariants are known to be equal [2, 4, 10, 19]. The readers are invited to consult Colmez’s paper [11] for some historical account.

The exceptional zero conjecture for higher weight modular forms has been proved by Steven using \( L_C \) [34], by Kato–Kurihara–Tsuji using \( L_{FM} \) (unpublished), by Orton using \( L_{DO} \) [25], by Emerton using \( L_B \) [14] and by Bertolini–Darmon–Iovita using \( L_T \) [2].

In [23] Mok addressed special cases of the exceptional zero conjecture in the setting of Hilbert modular forms. In [7] Chida, Mok and Park introduced the Teitelbaum type \( L \)-invariant for Hilbert modular forms, and conjectured that Teitelbaum type \( L \)-invariant coincides with the Fontaine-Mazur \( L \)-invariant. We state this conjecture precisely below.

Fix a prime number \( p \). Let \( F \) be a totally real field, \( g = [F : \mathbb{Q}] \) and \( \mathfrak{p} \) a prime ideal of \( F \) above \( p \). Let \( f_{\infty} \) be a Hilbert eigen newform with even weight \((k_1, \ldots, k_g, w)\) and level divided exactly by \( \mathfrak{p} \) (i.e. not by \( p^2 \)). Here “even weight” means that \( k_1, \ldots, k_g, w \) are all even.

On one hand, by Carayol’s result [6] we can attach to \( f_{\infty} \) a \( p \)-adic representation of \( G_{\mathbb{Q}} = \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \). This Galois representation (restricted to \( G_{\mathbb{Q}_p} \)) is semistable and thus we can attach to it a Fontaine-Mazur \( L \)-invariant \( L_{FM}(f_{\infty}) \).
On the other hand, Chida, Mok and Park attached to an automorphic form $f$ on a totally definite quaternion algebra over $F$ (of the same weight $(k_1, \cdots, k_g, w)$) a Teitelbaum type $L$-invariant $L_T(f)$ under the following assumption

$$\text{(CMP)} \quad f \text{ is new at } p \text{ and } U_p f = N_p^{w/2} f.$$  

Both $L_{FM}(f_\infty)$ and $L_T(f)$ are vector valued. See Section 1.2 and Section 8.2 for their precise definitions.

**Conjecture 0.1.** If $f_\infty$ and $f$ are associated to each other by the Jacquet-Langlands correspondence, then $L_{FM}(f_\infty) = L_T(f)$.

Our main result is the following

**Theorem 0.2.** (=Theorem 9.4) Assume that $F$ satisfies the following condition:

there is no place other than $p$ above $p$.

Let $f_\infty$ and $f$ be as above. Then $L_{FM}(f_\infty) = L_T(f)$.

We sketch the proof of Theorem 0.2. Our method is similar to that in [19]. The Galois representation attached to $f_\infty$ comes from the étale cohomology $H^1_{et}$ of some local system on a Shimura curve. The technical part of our paper is the computation of the filtered $\varphi_q$-isocrystal attached to this local system. On one hand, Coleman and Iovita [10] provided a precise description of the monodromy operator, which is helpful for computing Fontaine-Mazur’s $L$-invariant. On the other hand, the Teitelbaum type $L$-invariant is closely related to the de Rham cohomology of the filtered $\varphi_q$-isocrystal by Coleman integration and Schneider integration. Our precise description of the filtered $\varphi_q$-isocrystal allows us to compute Fontaine-Mazur’s $L$-invariant and the Teitelbaum type $L$-invariant. Finally, analyzing the relation among the monodromy operator, Coleman integration and Schneider integration finishes the proof.

When $F$ has more than one place (say $r$ places) above $p$, our method of computing filtered $\varphi_q$-isocrystals is not valid. To make it work, one might have to consider the Shimura variety studied by Rapoport and Zink [27, Chapter 6] (which is of dimension $r$) instead of the Shimura curve. Coleman and Iovita’s result is valid only for curves, and so cannot be applied directly. We plan to address this problem in a future work.

Our paper is organized as follows. Fontaine-Mazur’s $L$-invariant is introduced in Section 1. Coleman and Iovita’s result is recall in Section 2. Section 3 is devoted to compute the filtered $\varphi_q$-isocrystal attached to the universal special formal module. We introduce various Shimura curves, and study their $p$-adic uniformizations following Rapoort and Zink respectively in Section 4 and Section 5. In Section 6 we use the result in Section 3 to determine the filtered $\varphi_q$-isocrystals attached to various local systems on Shimura curves. In Section 7 we recall the theory of de Rham cohomology of certain local systems, and in Section 8 we recall Chida, Mok and Park’s construction of Teitelbaum type $L$-invariant. Finally in Section 9 we combine results in Section 2, Section 6 and Section 7 to prove our main theorem.

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Notations

For two \( \mathbb{Q} \)-algebras \( A \) and \( B \), write \( A \otimes B \) for \( A \otimes_{\mathbb{Q}} B \). For a ring \( R \) let \( R^\times \) denote the multiplicative group of invertible elements in \( R \). For a linear algebraic group over \( \mathbb{Q} \) we will identify it with its \( \mathbb{Q} \)-valued points.

Let \( F \) be a totally real number field, \( g = [F : \mathbb{Q}] \). Let \( p \) be a fixed prime. Suppose that \( p \) is inertia in \( F \), i.e. there exists exactly one place of \( F \) above \( p \), denoted by \( p \). If \( q \) is a power of \( p \), we use \( v_p(q) \) to denote \( \log_p q \).

Let \( \mathbb{A}_f \) denote \( \mathbb{Q} \otimes \hat{\mathbb{Z}} \) and let \( \mathbb{A}_f^p \) denote \( \mathbb{Q} \otimes \bigotimes_{\ell \neq p} \mathbb{Z}_\ell \). Similarly for any number field \( E \) let \( \mathbb{A}_{E,f} \) denote \( E \otimes \hat{\mathbb{Z}} \), the group of finite adèles of \( E \).

Fix an algebraic closure of \( F_p \), denoted by \( \overline{F}_p \), and let \( \mathbb{C}_p \) be the completion of \( \overline{F}_p \) with respect to the \( p \)-adic topology. By this way we have fixed an embedding \( F_p \hookrightarrow \mathbb{C}_p \). The Galois group \( G_{F_p} = \text{Gal}(\overline{F}_p/F_p) \) can be naturally identified with the group of continuous \( F_p \)-automorphisms of \( \mathbb{C}_p \).

1 Fontaine-Mazur invariant

1.1 Monodromy modules and Fontaine-Mazur \( L \)-invariant

Let \( F_{p0} \) be the maximal absolutely unramified subfield of \( F_p \). Let \( q \) be the cardinal number of the residue field of \( F_p \).

Let \( \mathcal{B}_{\text{cris}}, \mathcal{B}_{\text{st}} \) and \( \mathcal{B}_{\text{dR}} \) be Fontaine’s period rings [16]. As is well known, there are operators \( \varphi \) and \( N \) on \( \mathcal{B}_{\text{st}} \), and a descending \( \mathbb{Z} \)-filtration on \( \mathcal{B}_{\text{dR}} \); \( \mathcal{B}_{\text{cris}} \) is a \( \varphi \)-stable subring of \( \mathcal{B}_{\text{st}} \) and \( N \) vanishes on \( \mathcal{B}_{\text{cris}} \). Put \( \mathcal{B}_{\text{st}},F_p := \mathcal{B}_{\text{st}} \otimes_{\mathcal{B}_{\text{dR}}} F_p \); \( \mathcal{B}_{\text{st}},F_p \) can be considered as a subring of \( \mathcal{B}_{\text{dR}} \). We extend the operators \( \varphi_{q} = \varphi^{F_p(q)} \) and \( N \) \( F_p \)-linearly to \( \mathcal{B}_{\text{st}},F_p \).

Let \( K \) be either a finite unramified extension of \( F_p \) or the completion of the maximal unramified extension of \( F_p \) in \( \mathbb{C}_p \). Write \( G_{K} \) for the group of continuous automorphisms of \( \mathbb{C}_p \) fixing elements of \( K \). By our assumption on \( K \) we have

\[
(B_{\text{cris},F_p})^{G_{K}} = (B_{\text{st},F_p})^{G_{K}} = (B_{\text{dR}})^{G_{K}} = K.
\]

Let \( L \) be a finite extension of \( \mathbb{Q}_p \). For an \( L \)-linear representation \( V \) of \( G_{K} \), we put

\[
D_{\text{st},F_p}(V) := (V \otimes_{\mathbb{Q}_p} \mathcal{B}_{\text{st},F_p})^{G_{K}}.
\]

This is a finite rank \( L \otimes \mathbb{Q}_p \)-module. If \( V \) is semistable, then \( D_{\text{st},F_p}(V) \) is a filtered \((\varphi_q,N)\)-module: the \((\varphi_q,N)\)-module structure is induced from the operators \( \varphi_{q} = 1_V \otimes \varphi_q \) and \( N = 1_V \otimes N \) on \( V \otimes_{\mathbb{Q}_p} \mathcal{B}_{\text{st},F_p} \); the filtration comes from that on \( V \otimes_{\mathbb{Q}_p} \mathcal{B}_{\text{dR}} \). Note that \( \varphi_q \) and \( N \) are \( L \otimes \mathbb{Q}_p \)-linear.

If \( L \) splits \( F_p \), then \( L \otimes \mathbb{Q}_p K \) is isomorphic to \( \bigoplus_{\sigma} L \otimes_{\sigma,F_p} K \), where \( \sigma \) runs through all embeddings of \( F_p \) into \( L \). Here the subscript \( \sigma \) under \( \otimes \) indicates that \( F_p \) is considered as a subfield of \( L \) via \( \sigma \). Let \( \mathcal{E}_{\sigma} \) be the unity of the subring \( L \otimes_{\sigma,F_p} K \).

We shall need the notion of monodromy modules. This notion is introduced in [20]. However we will use the slightly different definition given in [19].

Let \( T \) be a finite-dimensional commutative semisimple \( \mathbb{Q}_p \)-algebra. A \( T \)-object \( D \) in the category of filtered \((\varphi_q,N)\)-modules, is called a 2-dimensional monodromy \( T \)-module, if the following hold:

- \( D \) is a free \( T_{F_p} \)-module of rank 2 (\( T_{F_p} = T \otimes_{\mathbb{Q}_p} F_p \)),

\[
\text{3}
\]
• the sequence $D \xrightarrow{N} D \xrightarrow{N} D$ is exact.
• there exists an integer $j_0$ such that $\text{Fil}^{j_0} D$ is a free $T_{F_p}$-submodule of rank 1 and $\text{Fil}^{j_0} D \cap \ker(N) = 0$.

**Lemma 1.1.** ([19, Lemma 2.3]) If $D$ is a monodromy $T$-module, then there exists a decomposition $D = D^{(1)} \oplus D^{(2)}$ where $D^{(1)}$ and $D^{(2)}$ are $\varphi_{q}$-stable free rank one $T_{F_p}$-submodules such that $N : D \to D$ induces an isomorphism $N|_{D^{(2)}} : D^{(2)} \xrightarrow{\sim} D^{(3)}$.

Let $D$ be a monodromy $T$-module and let $j_0$ be as above. The Fontaine-Mazur $L$-invariant of $D$ is defined to be the unique element in $T_{F_p}$, denoted as $L_{FM}(D)$, such that $x - L_{FM}(D) N(x) \in \text{Fil}^{j_0} D$ for every $x \in D^{(2)}$.

What we are interested in is the case when $T$ is an $L$-algebra, where $L$ is a field splitting $F_p$. Note that we have a decomposition of $T_{F_p}$:

$$ T_{F_p} \xrightarrow{\sim} \bigoplus_{\sigma} T_{\sigma}, \quad t \otimes a \mapsto (\sigma(a)t)_{\sigma}, \quad (1.1) $$

where $\sigma$ runs through all embeddings of $F_p$ in $L$. The index $\sigma$ in $T_{\sigma}$ indicates that $T$ is considered as an $F_p$-algebra via $\sigma$. Then we have a decomposition of $D$ by $D \simeq \bigoplus_{\sigma} D_{\sigma}$, where $D_{\sigma} = e_{\sigma} \cdot D$. Each $D_{\sigma}$ is stable under $\varphi_{q}$ and $N$. Note that, for every $j$, $\text{Fil}^{j} D$ is a $T_{F_p}$-submodule. Thus the filtration on $D$ restricts to a filtration on $D_{\sigma}$ for each $\sigma$, and satisfies $\text{Fil}^{j} D = \bigoplus_{\sigma} \text{Fil}^{j} D_{\sigma}$ for all $j \in \mathbb{Z}$.

Using the decomposition (1.1) we may write $L_{FM}(D)$ in the form $(L_{FM,\sigma}(D))_{\sigma}$. It is easy to see that $L_{FM,\sigma}(D)$ is the unique element in $T$ such that $x - L_{FM,\sigma}(D) N(x) \in \text{Fil}^{j_0} D_{\sigma}$ for every $x \in D^{(2)}_{\sigma}$. We also call $(L_{FM,\sigma}(D))_{\sigma}$, a vector with values in $T$, the Fontaine-Mazur $L$-invariant of $D$.

### 1.2 Fontaine-Mazur $L$-invariant for Hilbert modular forms

Let $\{\tau_1, \cdots, \tau_n\}$ be the set of real embeddings $F \hookrightarrow \mathbb{R}$. Fix a multiweight $k = (k_1, \cdots, k_g, w) \in \mathbb{N}^{g+1}$ satisfying $k_i \geq 2$ and $k_i \equiv w \mod 2$.

Let $\pi = \otimes_{v} \pi_v$ be a cuspidal automorphic representation of $GL(2, \mathbb{A}_F)$ such that for each infinite place $\tau_v$, the $\tau_v$-component $\pi_{\tau_v}$ is a holomorphic discrete series representation $D_{k_v}$. Let $n$ be the level of $\pi$.

Carayol [6] attached to such an automorphic representation (under a further condition) an $\ell$-adic Galois representation, which will be recalled below.

Let $L$ be a sufficiently large number field of finite degree over $\mathbb{Q}$ such that the finite part $\pi_{\infty} = \otimes_{p|\infty} \pi_p$ of $\pi$ admits an $L$-structure $\pi_{\infty}^{L}$. The fixed part $(\pi_{\infty}^{L})^{K(\pi)}$ is of dimension 1 and generated by an eigenform $f_{\infty}$. In this case we write $f_{\infty}$ for $\pi$.

The local Langlands correspondence associates to every irreducible admissible representation $\pi$ of $GL(2, F_v)$ defined over $L$ a 2-dimensional $L$-rational Frobenius-semisimple representation $\sigma(\pi)$ of the Weil-Deligne group $WD(F_v/F_p)$. Let $\hat{\sigma}(\pi)$ denote the dual of $\sigma(\pi)$.

For an $\ell$-adic representation $\rho$ of $\text{Gal}(\overline{F}/F)$, let $\rho_p$ denote its restriction to $\text{Gal}(\overline{F}/F_p)$, $\rho_p$ the Weil-Deligne representation attached to $\rho_p$ and $\rho_{F_{\text{ss}}}$ the Frobenius-semisimplification of $\rho_p$.

**Theorem 1.2.** [6] Let $f_{\infty}$ be an eigenform of multiweight $k$ satisfying the following condition:
If \( g = [F : \mathbb{Q}] \) is even, then there exists a finite place \( q \) such that the \( q \)-factor \( \pi_{f_{\infty,q}} \) lies in the discrete series.

Then for any prime number \( \ell \) and a finite place \( \lambda \) of \( L \) above \( \ell \), there exists a \( \lambda \)-adic representation \( \rho = \rho_{f_{\infty}} : \text{Gal}(\overline{F}/F) \to \text{GL}_{L_{\lambda}}(V_{f_{\infty},\lambda}) \) satisfying the following property:

For any finite place \( p \nmid \ell \) there is an isomorphism

\[
\rho_{f_{\infty},\lambda,p}^F \simeq \sigma(\pi_{f_{\infty},p}) \otimes L_{\lambda}
\]

of representations of the Weil-Deligne group \( WD(F_p/F_p) \).

Saito [28] showed that when \( p \mid \ell, \rho_{f_{\infty},\lambda,p} \) is potentially semistable.

Now we assume that \( \ell = p \), \( p \) is the prime ideal of \( F \) above \( p \), and \( L \) contains \( F \). Let \( \varPsi \) be a prime ideal of \( L \) above \( p \).

**Theorem 1.3.** (=Theorem 9.2) Let \( f_{\infty} \) be as in Theorem 1.2, \( \ell = p \) and \( \lambda = \varPsi \). If \( f_{\infty} \) is new at \( p \) (when \( [F : \mathbb{Q}] \) is odd, we demand that \( f_{\infty} \) is new at another prime ideal), then \( \rho_{f_{\infty},\varPsi,p} \) is a semistable (non-crystalline) representation of \( \text{Gal}(\overline{F}/F_p) \), and the filtered \( (\varphi_q,N) \)-module \( D_{st,F_p}(\rho_{f_{\infty},\varPsi,p}) \) is a monodromy \( L_{\varPsi} \)-module.

**Remark 1.4.** The conditions in Theorem 1.2 and Theorem 1.3 are used to ensure that via the Jacquet-Langlands correspondence \( f_{\infty} \) corresponds to a modular form on the Shimura curve \( M \) associated to a quaternion algebra \( B \) that splits at exactly one real place; in Theorem 1.3 the quaternion algebra \( B \) is demanded to be ramified at \( p \). See Section 4.1 for the construction of \( M \).

Thus \( D_{st,F_p}(\rho_{f_{\infty},\varPsi,p}) \) is associated with the Fontaine-Mazur \( L \)-invariant. We define the *Fontaine-Mazur \( L \)-invariant* of \( f_{\infty} \), denoted by \( L_{FM}(f_{\infty}) \), to be that of \( D_{st,F_p}(\rho_{f_{\infty},\varPsi,p}) \).

## 2 Local systems and the associated filtered \( \varphi_q \)-isocrystals

Let \( X \) be a \( p \)-adic formal \( \mathcal{O}_{F_p} \)-scheme. Suppose that \( X \) is analytically smooth over \( \mathcal{O}_{F_p} \), i.e. the generic fiber \( X^{\text{an}} \) of \( X \) is smooth.

The filtered convergent \( \varphi \)-isocrystals attached to local systems are studied in [15, 10]. It is more convenience for us to compute the filtered convergent \( \varphi_q \)-isocrystals attached to the local systems that we will be interested in. From now on, we will ignore “convergent” in the notion.

Filtered \( \varphi_q \)-isocrystal is a natural analogue of filtered \( \varphi \)-isocrystal. To define it one needs the notion of \( F_p \)-enlargement. An \( F_p \)-enlargement of \( X \) is a pair \((T, x_T)\) consisting of a flat formal \( \mathcal{O}_{F_p} \)-scheme \( T \) and a morphism of formal \( \mathcal{O}_{F_p} \)-scheme \( x_T : T_0 \to X \), where \( T_0 \) is the reduced closed subscheme of \( T \) defined by the ideal \( \pi \mathcal{O}_T \).

An isocrystal \( \mathcal{E} \) on \( X \) consists of the following data:
- for every \( F_p \)-enlargement \((T, x_T)\) a coherent \( \mathcal{O}_T \otimes_{\mathcal{O}_{F_p}} F_p \)-module \( \mathcal{E}_T \),
- for every morphism of \( F_p \)-enlargements \( g : (T', x_{T'}) \to (T, x_T) \) an isomorphism of \( \mathcal{O}_{T'} \otimes_{\mathcal{O}_{F_p}} F_p \)-modules \( \theta_g : g^*(\mathcal{E}_T) \to \mathcal{E}_{T'} \).

The collection of isomorphisms \( \{\theta_g\} \) is required to satisfy certain cocycle condition. If \( T \) is an \( F_p \)-enlargement of \( X \), then \( \mathcal{E}_T \) may be interpreted as a coherent sheaf \( \mathcal{E}_T^{an} \) on the rigid space \( T^{an} \).

As \( X \) is analytically smooth over \( \mathcal{O}_{F_p} \), there is a natural integrable connection

\[
\nabla_X : E_X^{an} \to E_X^{an} \otimes \Omega_{X^{an}}^1.
\]
Note that an isocrystal on $X$ depends only on $X_0$. Let $\varphi_q$ denote the absolute $q$-Frobenius of $X_0$. A $\varphi_q$-isocrystal on $X$ is an isocrystal $\mathcal{E}$ on $X$ together with an isomorphism of isocrystals $\varphi_q : \varphi_q^* \mathcal{E} \to \mathcal{E}$. A filtered $\varphi_q$-isocrystal on $X$ is a $\varphi_q$-isocrystal $\mathcal{E}$ with a descending $\mathbb{Z}$-filtration on $E_X^\an$.

The following well known result compares the de Rham cohomology of a filtered $\varphi_q$-isocrystal and the étale cohomology of the $\mathbb{Q}_p$-local system associated to it.

**Proposition 2.1.** [15, Theorem 3.2] Suppose that $X$ is a semistable proper curve over $\mathcal{O}_{F_p}$. Let $\mathcal{E}$ be a filtered $\varphi_q$-isocrystal over $X$ and $\mathcal{F}$ be a $\mathbb{Q}_p$-local system over $X_{\overline{\mathbb{F}}_p}$ that are attached to each other. Then the Galois representation $\mathcal{H}_{\mathcal{E}}^1(X_{\overline{\mathbb{F}}_p}, \mathcal{E})$ of $G_{F_p}$ is semistable, and the filtered $(\varphi_q, N)$-module $D_{\mathcal{E}} \mathcal{F}_p(\mathcal{H}_{\mathcal{E}}^1(X_{\overline{\mathbb{F}}_p}, \mathcal{E}))$ is isomorphic to $H_{\mathcal{E}}^1(X^\an, \mathcal{E})$.

Coleman and Iovita [10] gave a precise description of the monodromy $N$ on $H_{\mathcal{E}}^1(X^\an, \mathcal{E})$.

Now let $X$ be a connected, smooth and proper curve over $\mathbb{F}_q$. For any $e \in E(X)$ there is a natural residue map $\text{Res}_e : H^1_{\mathcal{E}}(X^\an, \mathcal{E}) \to H^0_{\mathcal{E}}(X^\an, \mathcal{E})$ [10, Section 4.1]. These residue maps induce a map

$$\bigoplus_{e \in E(X)} \text{Res}_e : \left( \bigoplus_{e \in E(X)} H^1_{\mathcal{E}}(X^\an, \mathcal{E}) \right)^+ \to \left( \bigoplus_{e \in E(X)} H^0_{\mathcal{E}}(X^\an, \mathcal{E}) \right) .$$

**Proposition 2.2.** [10, Theorem 2.6, Remark 2.7] The monodromy operator $N$ on $H^1_{\mathcal{E}}(X^\an, \mathcal{E})$ coincides with the composition

$$\iota \circ \left( \bigoplus_{e \in E(X)} \text{Res}_e \right) \circ \left( H^1_{\mathcal{E}}(X^\an, \mathcal{E}) \to \left( \bigoplus_{e \in E(X)} H^1_{\mathcal{E}}(X^\an, \mathcal{E}) \right)^+ \right)$$

where $H^1_{\mathcal{E}}(X^\an, \mathcal{E}) \to \left( \bigoplus_{e \in E(X)} H^1_{\mathcal{E}}(X^\an, \mathcal{E}) \right)^+$ is the restriction map.
3 The universal special formal module

3.1 Special formal modules and Drinfeld’s moduli theorem

Let $B_p$ be the quaternion algebra over $F_p$ with invariant $1/2$. So $B_p$ is isomorphic to $F_p^{(2)}[\Pi]$; $\Pi^2 = \pi$ and $\Pi a = \tilde{a}\Pi$ for all $a \in F_p^{(2)}$. Here, $\pi$ is a fixed uniformizer of $F_p$, $F_p^{(2)}$ is the unramified extension of $F_p$ of degree 2, and $a \mapsto \tilde{a}$ denotes the nontrivial $F_p$-automorphism of $F_p^{(2)}$.

Let $\mathcal{O}_{B_p}$ be the ring of integers in $B_p$. Let $F_{p0}$ be the maximal absolutely unramified subfield of $F_p$, $k$ the residue field of $F_p$, and $F_p^{(2)}$ the unramified extension of $F_{p0}$ of degree 2.

Let $\mathfrak{O}^{ur}$ denote the maximal unramified extension of $\mathfrak{O}_{F_p}$, $\hat{\mathfrak{O}}^{ur}$ its $\pi$-adic completion. Fix an algebraic closure $\bar{k}$ of $k$. We identify $\hat{\mathfrak{O}}^{ur}/\pi\hat{\mathfrak{O}}^{ur}$ with $\bar{k}$. Then $W(\bar{k}) \otimes_{\mathfrak{O}^{ur}} \mathfrak{O}_{F_p} \cong \hat{\mathfrak{O}}^{ur}$. Let $\hat{\mathfrak{O}}^{ur}_p$ be the fractional field of $\hat{\mathfrak{O}}^{ur}$.

We use the notion of special formal $\mathcal{O}_{B_p}$-module in [13].

First we fix a special formal $\mathcal{O}_{B_p}$-module over $\bar{k}$, $\Phi$, as in [27, (3.54)]. Let $\iota$ denote the natural embedding of $F_{p0}$ into $W(\bar{k})[1/p]$. Then all embeddings of $F_{p0}$ into $W(\bar{k})[1/p]$ are $\varphi^j \circ \iota$ ($0 \leq j \leq v_p(q) - 1$). We have the decomposition

$$\mathcal{O}_{B_p} \otimes_{Z_p} W(\bar{k}) = \prod_{j=0}^{v_p(q)-1} \mathcal{O}_{B_p} \otimes_{\mathfrak{O}^{ur}_p} \varphi^j \circ \iota W(\bar{k}).$$

Let $u \in \mathcal{O}_{B_p} \otimes_{Z_p} W(\bar{k})$ be the element whose $\varphi^j \circ \iota$-component with respect to this decomposition is

$$u_{\varphi^j \circ \iota} = \begin{cases} \Pi \otimes 1 & \text{if } j = 0, \\ 1 \otimes 1 & \text{if } j = 1, \ldots, v_p(q) - 1. \end{cases}$$

Let $\overline{F}$ be the $1 \otimes \varphi$-semilinear operator on $\mathcal{O}_{B_p} \otimes_{Z_p} W(\bar{k})$ defined by

$$\overline{F}x = (1 \otimes \varphi)x \cdot u, \quad x \in \mathcal{O}_{B_p} \otimes_{Z_p} W(\bar{k}).$$

Let $\overline{V}$ be the $1 \otimes \varphi^{-1}$-semilinear operator on $\mathcal{O}_{B_p} \otimes_{Z_p} W(\bar{k})$ such that $\overline{F} \overline{V} = p$. Then

$$(\mathcal{O}_{B_p} \otimes_{Z_p} W(\bar{k}), \overline{V}, \overline{F})$$

is a Dieudonné module over $W(\bar{k})$ with an action of $\mathcal{O}_{B_p}$ by the left multiplication. Let $\Phi$ be the special formal $\mathcal{O}_{B_p}$-module over $\bar{k}$ whose contravariant Dieudonné crystal is $(\mathcal{O}_{B_p} \otimes_{Z_p} W(\bar{k}), \overline{V}, \overline{F})$.

Let $\iota_0$ and $\iota_1$ be the extensions of $\iota$ to $F_p^{(2)}$. Then

$$\varphi^j \iota_0, \varphi^j \iota_1 \quad (0 \leq j \leq v_p(q) - 1)$$

[1] The Dieudonné crystal in [27, (3.54)] is the covariant Dieudonné crystal of $\Phi$. The duality between our contravariant Dieudonné crystal and the covariant Dieudonné crystal is induced by the trace map

$$< \cdot, \cdot >: \mathcal{O}_{B_p} \times \mathcal{O}_{B_p} \to Z_p, (x, y) \mapsto \operatorname{tr}_{F_p/F_p} \left( \delta_{F_p/Q_p}^{-1} \operatorname{tr}_{B_p/F_p} (xy^t) \right),$$

where $\delta_{F_p/Q_p}$ is the different of $F_p$ over $Q_p$, $\operatorname{tr}_{B_p/F_p}$ is the reduced trace map, and $y \mapsto y^t$ is the involution of $B_p$ such that $H^t = H$ and $a^t = \bar{a}$ if $a \in F_p^{(2)}$. Then we have $< b \cdot x, y > = < x, b^t \cdot y >$ for any $b \in \mathcal{O}_{B_p}$.

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are all embeddings of $F_{p_0}^{(2)}$ into $W(\bar{k})[1/p]$. We have

$$\mathcal{O}_{B_p} \otimes_{\mathbb{Z}_p} W(\bar{k}) = \prod_{j=0}^{v_p(q)-1} \mathcal{O}_{B_p} \otimes_{\mathcal{O}_{B_p}^{(2)}, \varphi_j} W(\bar{k}) \times \prod_{j=0}^{v_p(q)-1} \mathcal{O}_{B_p} \otimes_{\mathcal{O}_{B_p}^{(2)}, \varphi_j} W(\bar{k}),$$

where $\mathcal{O}_{B_p}$ is considered as an $\mathcal{O}_{B_p^{(2)}}$-module by the left multiplication. Let $X$ be the element of $\mathcal{O}_{B_p} \otimes_{\mathbb{Z}_p} W(\bar{k})$ whose $\varphi_j \circ \iota_0$-component ($0 \leq j \leq v_p(q) - 1$) is $1 \otimes 1$ and whose $\varphi_j \circ \iota_1$-component ($0 \leq j \leq v_p(q) - 1$) is $\Pi \otimes 1$. Similarly, let $Y$ be the element whose $\varphi_j \circ \iota_0$-component ($0 \leq j \leq v_p(q) - 1$) is $\Pi \otimes 1$ and whose $\varphi_j \circ \iota_1$-component ($0 \leq j \leq v_p(q) - 1$) is $\pi \otimes 1$. Then $\{X, Y\}$ is a basis of $\mathcal{O}_{B_p} \otimes_{\mathbb{Z}_p} W(\bar{k})$ over $\mathcal{O}_{B_p^{(2)}} \otimes_{\mathbb{Z}_p} W(\bar{k})$.

Note that $\text{GL}(2, F_p) = (\text{End}_{\mathcal{O}_{B_p}}^{\phi} \Phi)^\times$ [27, Lemma 3.60]. We normalize the isomorphism such that the action on the $\varphi$-module

$$(\mathcal{O}_{B_p} \otimes_{\mathbb{Z}_p} W(\bar{k}), \bar{F})[1/p] = (B_p \otimes_{\mathbb{Q}_p} W(\bar{k})[1/p], \bar{F})$$

is given by $\begin{bmatrix} a & b \\ c & d \end{bmatrix} X = (a \otimes 1) X + (c \otimes 1) Y$ and $\begin{bmatrix} a & b \\ c & d \end{bmatrix} Y = (b \otimes 1) X + (d \otimes 1) Y$.

Let $\bar{D}_0$ denote the $\varphi$-module $$(B_p \otimes_{\mathbb{Q}_p} \bar{F}_p^{ur}, \bar{F}^{v_p(q)})$$

coming from the $\varphi$-module $(\mathcal{O}_{B_p} \otimes_{\mathbb{Z}_p} W(\bar{k}), \bar{F})[1/p]$.

We describe Drinfeld’s moduli problem. Let $\text{Nilp}$ be the category of $\bar{\mathcal{O}}^{ur}$-algebras on which $\pi$ is nilpotent. For any $A \in \text{Nilp}$, let $\psi$ be the homomorphism $\hat{k} \rightarrow A/\pi A$; let $\text{SFM}(A)$ be the set of pairs $(G, \rho)$ where $G$ is a special formal $\mathcal{O}_{B_p}$-module over $A$ and $\rho : \Phi_{A/\pi A} = \psi_* \Phi \rightarrow G$ is a quasi-isogeny of height zero.

We state a part of Drinfeld’s theorem [13] as follows.

Let $\mathcal{H}$ be the Drinfeld upper half plane over $F_p$, i.e. the rigid analytic $F_p$-variety whose $\mathbb{C}_p$-points are $\mathbb{C}_p - F_p$.

**Theorem 3.1.** The functor $\text{SFM}$ is represented by a formal scheme $\tilde{\mathcal{H}} \otimes \bar{\mathcal{O}}^{ur}$ over $\bar{\mathcal{O}}^{ur}$ whose generic fiber is $\tilde{\mathcal{H}}_{\bar{\mathcal{O}}^{ur}} = \mathcal{H} \otimes \bar{F}_p^{ur}$.

Let $\mathcal{G}$ be the universal special formal $\mathcal{O}_{B_p}$-module over $\tilde{\mathcal{H}} \otimes \bar{\mathcal{O}}^{ur}$. There is an action of $\text{GL}(2, F_p)$ on $\mathcal{G}$ (see [3, Chapter II (9.2)]): The group $\text{GL}(2, F_p)$ acts on the functor $\text{SFM}$ by $g : (\psi : G, \rho) = (\psi \circ \text{Frob}^{-n} ; G, \rho \circ \psi_*(g^{-1} \circ \text{Frob}^n))$ if $v_p(\det g) = n$. Here, $v_p$ is the valuation of $\mathbb{C}_p$ normalized such that $v_p(\pi) = 1$.

### 3.2 The filtered $\varphi_q$-isocrystal attached to the universal special formal module

It is rather difficult to describe $\mathcal{G}$ precisely. However, we can determine the associated (contravariant) filtered $\varphi_q$-isocrystal $\mathcal{M}$.

In the following, we write $\mathcal{O}_{\mathcal{H}, \bar{F}_p^{ur}}$ for $\mathcal{O}_{\mathcal{H} \otimes \bar{F}_p^{ur}}$ and $\Omega_{\mathcal{H}, \bar{F}_p^{ur}}$ for the differential sheaf $\Omega_{\mathcal{H} \otimes \bar{F}_p^{ur}}$.

---

2See [35] for some information about $\mathcal{G}$ and [38] for a higher rank analogue.
As is observed in [15] and [27], except for the filtration, the \( \varphi_q \)-isocrystal \( \mathcal{M} \) is constant. So it is naturally isomorphic to the \( \varphi_q \)-isocrystal

\[
\tilde{D}_0 \otimes \tilde{F}_p^u \mathcal{O}_{\mathcal{H}, \tilde{F}_p^u}
\]

with the \( q \)-Frobenius being \( F^{u_p(2)}(q) \otimes \varphi_q \mathcal{H}_{\tilde{F}_p^u} \) and the connection being

\[
1 \otimes \text{d} : \tilde{D}_0 \otimes \tilde{F}_p^u \mathcal{O}_{\mathcal{H}, \tilde{F}_p^u} \to \tilde{D}_0 \otimes \tilde{F}_p^u \Omega_{\mathcal{H}, \tilde{F}_p^u}.
\]

Next we determine the filtration on \( \tilde{D}_0 \otimes \tilde{F}_p^u \mathcal{O}_{\mathcal{H}, \tilde{F}_p^u} \). For any \( F_p \)-algebras \( K \) and \( L, L \otimes_{Q_p} K \) is isomorphic to \( L \otimes_{F_p} K \) via the homomorphism \( L \otimes_{Q_p} K \to L \otimes_{F_p} K \), \( \ell \otimes a \mapsto \ell \otimes a \). If \( L \) is a field extension of \( F_p \) containing all embeddings of \( F_p \), then \( L \otimes_{Q_p} K = \bigoplus_{\tau: F_p \to L} L \otimes_{F_p} K \) and \( (L \otimes_{Q_p} K)_{\text{non}} \) corresponds to the non-canonical embeddings. We apply this to \( L = F_p \) and \( K = \hat{F}_p^{ur} \) as an \( F_p \otimes_{Q_p} \hat{F}_p^{ur} \)-module. Then \( \tilde{D}_0 \) splits into two parts: one is the canonical part which corresponds to the natural embedding \( \text{id} : F_p \hookrightarrow F_p \), and the other is the non-canonical part. Correspondingly, \( \tilde{D}_0 \otimes \hat{F}_p^{ur} \mathcal{O}_{\mathcal{H}, \hat{F}_p^{ur}} \) splits into two parts, the canonical part \( B_p \otimes_{F_p} \mathcal{O}_{\mathcal{H}, \hat{F}_p^{ur}} \) and the non-canonical part. Since \( F_p \) acts on the Lie algebra of any special formal \( \mathcal{O}_{B_p} \)-module through the natural embedding, the filtration on the non-canonical part is trivial.

The filtration on the canonical part is closely related to the period morphism [15, 27]. Let us recall the definition of the period morphism [38, Section 2.2]. We will use the notations in [38].

Let \( M(\Phi) \) be the Cartier module of \( \Phi \), a \( \mathbb{Z}/2\mathbb{Z} \)-graded module. The \( \mathbb{Z}/2\mathbb{Z} \)-grading depends on the choice of \( F_p \)-embedding of \( F_p^{(2)} \) into \( \hat{F}_p^{ur} \). We choose the one, \( i_0 \), that restricts to \( i_0 \), and denote the other \( F_p \)-embedding by \( i_1 \). We fix a graded V-basis \( \{ 0^0, 0^1 \} \) of \( M(\Phi) \) such that \( V g^0 = \Pi g^0 \) and \( V g^1 = \Pi g^1 \). Then \( \{ 0^0, 0^1, V g^0, V g^1 \} \) is a basis of \( M(\Phi)[1/p] \) over \( \hat{F}_p^{ur} \); \( F_p^{(2)} \subset B_p \) acts on \( \hat{F}_p^{ur} 0^0 \oplus \hat{F}_p^{ur} 0^1 \) by \( i_0 \), and acts on \( \hat{F}_p^{ur} V g^0 \oplus \hat{F}_p^{ur} V g^1 \) by \( i_1 \). See [13] for the definition of Cartier module and the meaning of graded V-basis. See [27, (3.55)] for the relation between \( M(\Phi) \) and the covariant Dieudonné module attached to \( \Phi \). In loc. cit. Cartier module is called \( \tau \)-Weyl(L)-crystal.

Let \( R \) be any \( \pi \)-adically complete \( \hat{F}_p^{ur} \)-algebra. Drinfeld constructed for each \( (\psi; G, \rho) \in \text{SFM}(R) \) a quadruple \( (\eta, T, u, \rho) \). Let \( M = M(G) \) be the Cartier module of \( G, N(M) \) the auxiliary module that is the quotient of \( M \oplus M \) by the submodule generated by elements of the form \( (\nu x, -\Pi x) \) and \( \beta_M \) the quotient map \( M \oplus M \to M(\Phi) \). For \( (x_0, x_1) \in M \oplus M \), we write \( ((x_0, x_1)) \) for \( \beta_M(x_0, x_1) \). Then we have a map \( \varphi_M : N(M) \to N(M) \). See [38, Definition 4] for its definition. Put

\[
\eta_M := N(M)_{\varphi_M}, \quad T_M := M/V M;
\]

both \( \eta_M \) and \( T_M \) are \( \mathbb{Z}/2\mathbb{Z} \)-graded. Let \( u_M : \eta_M \to T_M \) be the \( \mathcal{O}_{F_p} \)-linear map of degree 0 that is the composition of the inclusion \( \eta_M \hookrightarrow N(M) \) and the map

\[
N(M) \to M/V M, \quad ((x_0, x_1)) \mapsto x_0 \mod VM.
\]

Then \( \eta_M(\Phi) \) is a free \( \mathcal{O}_{F_p} \)-module of rank 4 with a basis

\[
\{ ((0^0, 0)), (0^1, 0), (V g^0, 0), (V g^1, 0) \}.
\]
where \((g^0, 0), ((V^g)^1, 0)\) are in degree 0, and \(((g^1, 0), ((Vg^0, 0))\) are in degree 1. The quasi-isogeny \(\rho : \psi_*\Phi \to G_{R/p^R}\) induces an isomorphism

\[ \rho : \eta_{\mathfrak{M}(\Phi)}^0 \otimes \mathcal{E}_{F_p} F_p \xrightarrow{\sim} \eta_{\mathfrak{M}(G)}^0 \otimes \mathcal{E}_{F_p} F_p. \]

Then the period of \((G, \rho)\) is defined by

\[ z(G, \rho) = \frac{u_M' \circ \rho((Vg^1, 0))}{u_M' \circ \rho((g^0, 0))}, \quad (3.1) \]

where \(u_M'\) is the map \(\eta_{\mathfrak{M}(G)}^0 \otimes \mathcal{E}_{F_p} F_p \to T^0_{\mathfrak{M}} \otimes_R R[1/p]\) induced by \(u_M\).

Note that considered as a \(\varphi\)-module, \(M(\Phi)[1/p]\) is the dual of \(B_p \otimes_{F_p} \hat{F}_p^\text{ur}\), the canonical part of \(\hat{D}_0\). Let \(\{v_0, v_1, v_2, v_3\}\) be the basis of \(B_p \otimes_{F_p} \hat{F}_p^\text{ur}\) over \(\hat{F}_p^\text{ur}\) dual to \(\{\pi g^1, g^0, Vg^0, Vg^1\}\). Then

\[
\begin{align*}
\text{Fil}^0 B_p \otimes_{F_p} \mathcal{O}_{\mathcal{H}, \hat{F}_p^\text{ur}} &= B_p \otimes_{F_p} \mathcal{O}_{\mathcal{H}, \hat{F}_p^\text{ur}} \\
\text{Fil}^1 B_p \otimes_{F_p} \mathcal{O}_{\mathcal{H}, \hat{F}_p^\text{ur}} &= \text{the } \mathcal{O}_{\mathcal{H}, \hat{F}_p^\text{ur}}\text{-submodule generated by } \\
& \hat{F}_p^\text{ur} \cdot (v_1 + zv_3) \oplus \hat{F}_p^\text{ur} \cdot (zv_0 + v_2) \\
\text{Fil}^2 B_p \otimes_{F_p} \mathcal{O}_{\mathcal{H}, \hat{F}_p^\text{ur}} &= 0.
\end{align*}
\]

Here \(z\) is the canonical coordinate on \(\mathcal{H}_{\hat{F}_p^\text{ur}}\).

We decompose \(B_p \otimes_{F_p} \hat{F}_p^\text{ur}\) into two direct summands:

\[ B_p \otimes_{F_p} \hat{F}_p^\text{ur} = B_p \otimes_{F_p^{(2)}} \hat{F}_p^\text{ur} \oplus B_p \otimes_{F_p^{(2)}, \tau} \hat{F}_p^\text{ur}, \]

where \(B_p\) is considered as an \(F_p^{(2)}\)-module by left multiplication. Let \(e_0\) and \(e_1\) denote the projection to the first summand and that to the second, respectively. We may choose \(g^0, g^1\) such that \(v_0 = e_0 X, v_1 = e_1 Y, v_2 = e_0 Y, v_3 = e_1 X\). Thus

\[
\begin{align*}
\text{Fil}^0 B_p \otimes_{F_p} \mathcal{O}_{\mathcal{H}, \hat{F}_p^\text{ur}} &= B_p \otimes_{F_p} \mathcal{O}_{\mathcal{H}, \hat{F}_p^\text{ur}} \\
\text{Fil}^1 B_p \otimes_{F_p} \mathcal{O}_{\mathcal{H}, \hat{F}_p^\text{ur}} &= \text{the } F_p^{(2)}\otimes_{F_p} \mathcal{O}_{\mathcal{H}, \hat{F}_p^\text{ur}}\text{-submodule generated by } zX + Y \\
\text{Fil}^2 B_p \otimes_{F_p} \mathcal{O}_{\mathcal{H}, \hat{F}_p^\text{ur}} &= 0.
\end{align*}
\]

Finally we note that the induced action of \(\text{GL}(2, F_p)\) on \(\mathcal{H}\) is given by \([\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}] z = \frac{az + b}{cz + d}\).

**4 Shimura curves**

Fix a real place \(\tau_1\) of \(F\). Let \(B\) be a quaternion algebra over \(F\) that splits at \(\tau_1\) and is ramified at other real places \(\{\tau_2, \ldots, \tau_g\}\) and \(p\).
4.1 Shimura curves $M$, $M'$ and $M''$

We will use three Shimura curves studied by Carayol [5] and recall their constructions below (see also [28]).

Let $G$ be the reductive algebraic group over $\mathbb{Q}$ such that $G(R) = (B \otimes R)^{\times}$ for any $\mathbb{Q}$-algebra $R$. Let $Z$ be the center of $G$; it is isomorphic to $T = \text{Res}_{F/\mathbb{Q}}G_m$. Let $\nu : G \rightarrow T$ be the morphism obtained from the reduced norm $\text{Nrd}_{B/F}$ of $B$. The kernel of $\nu$ is $G^{\text{der}}$, the derived group of $G$, and thus we have a short exact sequence of algebraic groups

$$1 \longrightarrow G^{\text{der}} \longrightarrow G \overset{\nu}{\longrightarrow} T \longrightarrow 1.$$ 

Let $X$ be the $G(\mathbb{R})$-conjugacy class of the homomorphism

$$h : \quad \mathbb{C}^\times \rightarrow G(\mathbb{R}) = \text{GL}_2(\mathbb{R}) \times \mathbb{H}^\times \times \cdots \times \mathbb{H}^\times$$

$$z = x + \sqrt{-1}y \mapsto \left( \begin{bmatrix} x & y \\ -y & x \end{bmatrix}^{-1}, 1, \ldots, 1 \right),$$

where $\mathbb{H}$ is the Hamilton quaternion algebra. The conjugacy class $X$ is naturally identified with the union of upper and lower half planes. Let $M = M(G, X) = (M_U(G, X)_U$ be the canonical model of the Shimura variety attached to the Shimura pair $(G, X)$; the canonical model is defined over $F$, the reflex field of $(G, X)$. There is a natural right action of $G(\mathbb{A}_f)$ on $M(G, X)$. Here and in what follows, by abuse of terminology we call a projective system of varieties simply a variety.

Take an imaginary quadratic field $E_0 = \mathbb{Q}(\sqrt{-a})$ (a a square-free positive integer) such that $p$ splits in $E_0$. Put $E = FE_0$ and $D = B \otimes_F E = B \otimes_{\mathbb{Q}} E_0$. We fix a square root $\rho$ of $-a$ in $\mathbb{C}$. Then the prolonging of $\tau_i$ to $E$ by $x + y\sqrt{-a} \mapsto \tau_i(x) + \tau_i(y)\rho$ (resp. $x + y\sqrt{-a} \mapsto \tau_i(x) - \tau_i(y)\rho$) is denoted by $\tau_i$ (resp. $\overline{\tau}_i$).

Let $T_E$ be the torus $\text{Res}_{E/\mathbb{Q}}G_m$, $T_E^1$ the subtorus of $T_E$ such that $T_E^1(\mathbb{Q}) = \{ z \in E : z\overline{z} = 1 \}$. We consider the amalgamate product $G'' = G \times_Z T_E$, and the morphism $\tilde{\nu} : G'' = G \times_Z T_E \rightarrow T'' = T \times T_E^1$ defined by $(g, z) \mapsto (\nu(g)z\overline{z}, z\overline{z})$. Consider the subtorus $T'' = \mathbb{G}_m \times T_E^1$ of $T''$, and let $G'$ be the inverse image of $T'$ by the map $\nu''$. The restriction of $\nu''$ to $G'$ is denoted by $\nu'$. Both the derived group of $G'$ and that of $G''$ are identified with $G^{\text{der}}$, and we have two short exact sequences of algebraic groups

$$1 \longrightarrow G^{\text{der}} \longrightarrow G' \overset{\nu'}{\longrightarrow} T' \longrightarrow 1$$

and

$$1 \longrightarrow G^{\text{der}} \longrightarrow G'' \overset{\nu''}{\longrightarrow} T'' \longrightarrow 1.$$ 

The complex embeddings $\tau_1, \ldots, \tau_9$ of $E$ identify $G''(\mathbb{R})$ with $\text{GL}_2(\mathbb{R}) \cdot \mathbb{C}^\times \times \mathbb{H}^\times \cdot \mathbb{C}^\times \times \cdots \times \mathbb{H}^\times \cdot \mathbb{C}^\times$. We consider the $G'(\mathbb{R})$-conjugacy class $X'$ (resp. $G''(\mathbb{R})$-conjugacy class $X''$) of the homomorphism

$$h' : \quad \mathbb{C}^\times \rightarrow G'(\mathbb{R}) \subset G''(\mathbb{R}) = \text{GL}_2(\mathbb{R}) \cdot \mathbb{C}^\times \times \mathbb{H}^\times \cdot \mathbb{C}^\times \times \cdots \times \mathbb{H}^\times \cdot \mathbb{C}^\times$$

$$z = x + \sqrt{-1}y \mapsto \left( \begin{bmatrix} x & y \\ -y & x \end{bmatrix}^{-1} \otimes 1, 1 \otimes z^{-1}, \ldots, 1 \otimes z^{-1} \right).$$

Let $M' = M(G', X')$ and $M'' = M(G'', X'')$ be the canonical models of the Shimura varieties defined over their reflex field $E$. There are natural right actions of $G'(\mathbb{A}_f)$ and $G''(\mathbb{A}_f)$ on $M'$ and $M''$, respectively.
Put $T_{E_0} = \text{Res}_{\mathbb{Q}}^{E_0} G_m$. Using the complex embeddings $\tau_1, \ldots, \tau_g$ of $E$, we identify $T_E(\mathbb{R})$ with $\mathbb{C}^\times \times \cdots \times \mathbb{C}^\times$; similarly via the embedding $x + y\sqrt{-a} \to x + y\rho$ we identify $T_{E_0}(\mathbb{R})$ with $\mathbb{C}^\times$.

Consider the homomorphisms

\[
\begin{align*}
\tilde{h}_E &: \mathbb{C}^\times \to T_E(\mathbb{R}) = \mathbb{C}^\times \times \cdots \times \mathbb{C}^\times, \quad z \mapsto (z^{-1}, 1, \cdots, 1), \\
\tilde{h}_{E_0} &: \mathbb{C}^\times \to T_{E_0}(\mathbb{R}) = \mathbb{C}^\times, \quad z \mapsto z^{-1}.
\end{align*}
\]

Let $N_E = M(T_E, h_E)$ and $N_{E_0} = M(T_{E_0}, h_{E_0})$ be the canonical models attached to the pairs $(T_E, h_E)$ and $(T_{E_0}, h_{E_0})$ respectively. Then $N_E$ is defined over $E$, and $N_{E_0}$ is defined over $E_0$.

Consider the homomorphism $\alpha : G \times T_E \to G''$ of algebraic groups inducing

\[B^\times \times E^\times \to G''(\mathbb{Q}) \subset (B \otimes_\mathbb{Q} E)^\times, \quad (b, e) \mapsto b \otimes N_{E_0}(e)e^{-1}\]

on $\mathbb{Q}$-valued points, and the homomorphism $\beta : G \times T_E \to T_{E_0}$ inducing

\[N_{E/E_0} \circ \text{pr}_2 : B^\times \times E^\times \to E_0^\times\]

on $\mathbb{Q}$-valued points. Here, $N_{E/E_0}$ denotes the norm map $E^\times \to E_0^\times$. Since $h' = \alpha \circ (h \times h_E)$ and $h_{E_0} = N_{E/E_0} \circ h_E$, $\alpha$ and $\beta$ induce morphisms of Shimura varieties $M \times N \to M''$ and $M \times N_E \to N_{E_0}$ again denoted by $\alpha$ and $\beta$ respectively. We have the following diagram

\[
\begin{array}{c}
M \xrightarrow{\text{pr}_1} M \times N_E \xrightarrow{\alpha} M'' \leftarrow M' \\
\downarrow \beta \\
N_{E_0}.
\end{array}
\]

### 4.2 Connected components of $M$, $M \times N_E$, $M'$ and $M''$

We write $\tilde{G}$ for $G \times T_E$ and write $\tilde{M}$ for $M \times N_E$. For $\sharp = \emptyset, \prime, \prime''$, as $B$ is ramified at $p$, there exists a unique maximal compact open subgroup $U^\sharp_{p,0}$ of $G^\sharp(\mathbb{Q}_p)$. Then $U'_{p,0} = U''_{p,0} \cap G''(\mathbb{Q}_p)$ and $U^\prime_{p,0} = \alpha(U_{p,0})$.

If $U^\sharp$ is a subgroup of $G^\sharp(\mathbb{A})$ of the form $U^\sharp_{p,0}U_{\mathbb{Z},p}$ where $U_{\mathbb{Z},p}$ is a compact open subgroup of $G^\sharp(\mathbb{A}_f)$, we will write $M^\sharp_{0, U_{\mathbb{Z},p}}$ for $M^\sharp_{U_{\mathbb{Z},p}}$. Let $M^\sharp_{0}$ denote the projective system $(M^\sharp_{0, U_{\mathbb{Z},p}})_{U_{\mathbb{Z},p}}$; this projective system has a natural right action of $G^\sharp(\mathbb{A}_f)$.

**Lemma 4.1.** (a) For any sufficiently small $U^\sharp_{\mathbb{Z},p}$, each geometrically connected component of $M^\sharp_{0, U^\sharp_{\mathbb{Z},p}}$ is defined over a field that is unramified at all places above $p$.

(b) Let $\tilde{U}_p$ be a sufficiently small compact open subgroup of $\tilde{G}(\mathbb{A}_f)$. Then the morphism

\[
\tilde{M}_{0, \tilde{U}_p} \to M''_{0, \alpha(\tilde{U}_p)}
\]

induced by $\alpha$ is an isomorphism onto its image when restricted to every geometrically connected component.
Proof. When \( U^\sharp:p \) is sufficiently small, \( M^\sharp_{0,U^\sharp,p} \) is smooth. Let \( \pi_0(M^\sharp_{0,U^\sharp,p}) \) denote the group of geometrically irreducible components of \( M^\sharp_{0,U^\sharp,p} \) over \( \overline{\mathbb{Q}} \) (which must be connected since \( M^\sharp_{0,U^\sharp,p} \) is smooth). Then \( \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \) acts on \( \pi_0(M^\sharp_{0,U^\sharp,p}) \). This action is explicitly described by Deligne [12, Theorem 2.6.3], from which we deduce (a).

As \( \alpha \) induces an isomorphism from the derived group of \( \tilde{G} \) to that of \( G'' \), by [12, Remark 2.1.16] or [22, Proposition II.2.7] we obtain (b).

4.3 Modular interpolation of \( M' \)

Let \( \ell \mapsto \tilde{\ell} \) be the involution on \( D = B \otimes_{\mathbb{Q}} E_0 \) that is the product of the canonical involution on \( B \) and the complex conjugate on \( E_0 \). Choose an invertible symmetric element \( \delta \in D \) (\( \delta = \delta' \)). Then we have another involution \( \ell \mapsto T^* := \delta^{-1} \ell \delta \) on \( D \).

Let \( V \) denote \( D \) considered as a left \( D \)-module. Let \( \psi \) be the non-degenerate alternating form on \( V \) defined by \( \psi(x,y) = \text{Tr}_{E/\mathbb{Q}}(\sqrt{-} \cdot a \text{Trd}_{D/E}(x \delta y^*)) \), where \( \text{Tr}_{E/\mathbb{Q}} \) is the trace map and \( \text{Trd}_{D/E} \) is the reduced trace map. For \( \ell \in D \) put

\[
t(\ell) = \text{tr}(\ell; V_{\mathbb{C}}/\text{Fil}^0 V_{\mathbb{C}})
\]

where \( \text{Fil}^n \) is the Hodge structure defined by \( h' \). We have

\[
t(\ell) = (\tau_1 + \bar{\tau}_1 + 2\tau_2 + \cdots + 2\tau_g)(\text{tr}_{D/E}(\ell))
\]

for \( \ell \in D \). The subfield of \( \mathbb{C} \) generated by \( t(\ell), \ell \in D \), is exactly \( E \).

Choose an order \( \mathcal{O}_D \) of \( D \), \( T \) the corresponding lattice in \( V \). With a suitable choice of \( \delta \), we may assume that \( \mathcal{O}_D \) is stable by the involution \( \ell \mapsto \ell^* \) and that \( \psi \) takes integer values on \( T \). Put \( \mathcal{O}_\tilde{T} := \mathcal{O}_D \otimes \mathbb{Z} \) and \( \tilde{\mathcal{T}} := T \otimes \mathbb{Z} \).

In Section 5 when we consider the \( p \)-adic uniformization of the Shimura curves, we need to make the following assumption.

Assumption 4.2. We assume that \( \delta \) is chosen such that \( \tilde{T} \) is stable by \( U'_{p,0} \).

If \( U' \) is a sufficiently small compact open subgroup of \( G'({\mathbb{A}}_f) \) keeping \( \tilde{T} \), then \( M^\sharp_{U'} \) represents the following functor \( M_{U'} \) [28, Section 5]:

For any \( E \)-algebra \( R \), \( M_{U'}(R) \) is the set of isomorphism classes of quadruples \((A, \iota, \theta, \kappa)\) where

- \( A \) is an isomorphism class of abelian schemes over \( R \) with an endomorphism \( \iota: \mathcal{O}_D \to \text{End}(A) \) such that \( \text{tr}(\iota(\ell), \text{Lie}A) = t(\ell) \) for all \( \ell \in \mathcal{O}_D \).
- \( \theta \) is a polarization \( A \to \hat{A} \) whose associated Rosati involution sends \( \iota(\ell) \) to \( \iota(\ell^*) \).
- \( \kappa \) is a \( U' \)-orbit of \( \mathcal{O}_D \otimes \hat{\mathbb{Z}} \)-linear isomorphisms \( \tilde{T}(A) := \prod_{\ell} \text{Tr}(A) \to \tilde{T} \) such that there exists a \( \hat{\mathbb{Z}} \)-linear isomorphism \( \kappa': \tilde{T}(1) \to \hat{\mathbb{Z}} \) making the diagram

\[
\begin{array}{c}
\tilde{T}(A) \times \tilde{T}(A) \xrightarrow{(1, \theta^*)} \tilde{T}(A) \times \tilde{T}(\hat{A}) \xrightarrow{\kappa \times \kappa} \hat{\mathbb{Z}}(1) \\
\end{array}
\]

\[
\begin{array}{c}
\tilde{T} \times \tilde{T} \xrightarrow{\psi \otimes \hat{\mathbb{Z}}} \hat{\mathbb{Z}} \\
\end{array}
\]

commutative.

Let \( A_{U'} \) be the universal \( \mathcal{O}_D \)-abelian scheme over \( M^\sharp_{U'} \).
5 $p$-adic Uniformizations of Shimura curves

5.1 Preliminaries

We provide two simple facts, which will be useful later.

(i) Let $X$ be a scheme with a discrete action of a group $C$ on the right hand side, and let $Z$ be a group that contains $C$ as a normal subgroup of finite index. Fix a set of representatives $\{g_i\}_{i \in C \setminus Z}$ of $C \setminus Z$ in $Z$. We define a scheme $X \ast_C Z$ with a right action of $Z$ below. As a scheme $X \ast_C Z$ is $\bigsqcup_{C \setminus Z} X^{(g_i)}$, where $X^{(g_i)}$ is a copy of $X$. For any $g \in Z$ and $x^{(g_i)} \in X^{(g_i)}$, if $g_i g = h g_k$ with $h \in C$, then $x^{(g_i)} \cdot g = (x \cdot h)^{(g_k)}$. It is easy to show that up to isomorphism $X \ast_C Z$ and the right action of $Z$ are independent of the choice of $\{g_i\}_{i \in C \setminus Z}$.

(ii) Let $X_1$ and $X_2$ be two schemes whose connected components are all geometrically connected. Suppose that each of $X_1$ and $X_2$ has an action of an abelian group $Z$; $Z$ acts freely on the set of components of $X_1$ (resp. $X_2$). Let $C$ be a closed subgroup of $Z$. Then the $Z$-actions on $X_1$ and $X_2$ induce $Z/C$-actions on $X_1/C$ and $X_2/C$.

Lemma 5.1. If there exists a $Z/C$-equivariant isomorphism $\gamma : X_1/C \to X_2/C$, then there exists a $Z$-equivariant isomorphism $\tilde{\gamma} : X_1 \to X_2$ such that the following diagram

\[
\begin{array}{ccc}
X_1 & \xrightarrow{\tilde{\gamma}} & X_2 \\
\downarrow{\pi_1} & & \downarrow{\pi_2} \\
X_1/C & \xrightarrow{\gamma} & X_2/C
\end{array}
\]

is commutative, where $\pi_1$ and $\pi_2$ are the natural projections.

Proof. We identify $X_1/C$ with $X_2/C$ by $\gamma$, and write $Y$ for it. The condition on $Z$-actions implies that the action of $Z/C$ on the set of connected components of $Y$ is free and that the morphism $\pi_1$ (resp. $\pi_2$) maps each connected component of $X_1$ (resp. $X_2$) isomorphically to its image.

We choose a set of representatives $\{Y_i\}_{i \in I}$ of the $Z/C$-orbits of components of $Y$. Then $\{gY_i : \tilde{g} \in Z/C, i \in I\}$ are all different connected components of $Y$. For each $i \in I$ we choose a connected component $\tilde{Y}_i^{(1)}$ (resp. $\tilde{Y}_i^{(2)}$) of $X_1$ (resp. $X_2$) that is a lifting of $Y_i$. Then $\{g\tilde{Y}_i^{(1)} : g \in Z, i \in I\}$ (resp. $\{g\tilde{Y}_i^{(2)} : g \in Z, i \in I\}$) are all different connected components of $X_1$ (resp. $X_2$).

As $\pi_1|_{\tilde{Y}_i^{(1)}} : \tilde{Y}_i^{(1)} \to Y_i$ and $\pi_2|_{\tilde{Y}_i^{(2)}} : \tilde{Y}_i^{(2)} \to Y_i$ are isomorphisms, there exists an isomorphism $\tilde{\gamma}_i : \tilde{Y}_i^{(1)} \to \tilde{Y}_i^{(2)}$ such that $\pi_1|_{\tilde{Y}_i^{(1)}} = \pi_2|_{\tilde{Y}_i^{(2)}} \circ \tilde{\gamma}_i$. We define the morphism $\tilde{\gamma} : X^{(1)} \to X^{(2)}$ as follows: $\tilde{\gamma}$ maps $g\tilde{Y}_i^{(1)}$ to $g\tilde{Y}_i^{(2)}$, and $\tilde{\gamma}|_{g\tilde{Y}_i^{(1)}} = g \circ \tilde{\gamma}_i \circ g^{-1}$. Then $\tilde{\gamma}$ is a $Z$-equivariant isomorphism and $\pi_1 = \pi_2 \circ \tilde{\gamma}$. \hfill $\square$

5.2 Some Notations

Fix an isomorphism $C \cong \mathbb{C}_p$. Combining the isomorphism $C \cong \mathbb{C}_p$ and the inclusion $E_0 \hookrightarrow \mathbb{C}$, $x + y \sqrt{-a} \to x + y p$, we obtain an inclusion $E_0 \hookrightarrow \mathbb{Q}_p$ and $E \hookrightarrow F_p$. Thus $D \otimes \mathbb{Q}_p$ is isomorphic to $B_p \oplus B_p$.

Note that $G(\mathbb{Q}_p)$ is isomorphic to $B_p^\times$, $G'(\mathbb{Q}_p)$ is isomorphic to the subgroup

\[\{(a, b) : a, b \in B_p^\times, ab \in \mathbb{Q}_p^\times\}\]
of $B_p^\times \times B_p^\times$, and $G''(\mathbb{Q}_p)$ is isomorphic to

$$\{(a, b) : a, b \in B_p^\times, \bar{a}b \in F_p^\times\},$$

where $a \mapsto \bar{a}$ is the canonical involution on $B$. Note that $T_E(\mathbb{Q}_p)$ is isomorphic to $F_p^\times \times F_p^\times$, and $T_{E_0}(\mathbb{Q}_p)$ is isomorphic to $\mathbb{Q}_p^\times \times \mathbb{Q}_p^\times$. We normalize these isomorphisms such that $G'(\mathbb{Q}_p) \hookrightarrow G''(\mathbb{Q}_p)$ becomes the natural inclusion

$$\{(a, b) : a, b \in B_p^\times, \bar{a}b \in F_p^\times\} \hookrightarrow \{(a, b) : a, b \in B_p^\times, \bar{a}b \in F_p^\times\},$$

$\alpha : G(\mathbb{Q}_p) \times T_E(\mathbb{Q}_p) \to G''(\mathbb{Q}_p)$ becomes

$$B_p^\times \times (F_p^\times \times F_p^\times) \to \{(a, b) : a, b \in B_p^\times, \bar{a}b \in F_p^\times\},$$

$$(a, (t_1, t_2)) \mapsto (a \frac{N_{F_p/\mathbb{Q}_p}(t_1)}{t_1}, a \frac{N_{F_p/\mathbb{Q}_p}(t_2)}{t_2}),$$

and $\beta : G(\mathbb{Q}_p) \times T_E(\mathbb{Q}_p) \to T_{E_0}(\mathbb{Q}_p)$ becomes

$$B_p^\times \times (F_p^\times \times F_p^\times) \to \mathbb{Q}_p^\times \times \mathbb{Q}_p^\times,$$

$$(a, (t_1, t_2)) \mapsto (N_{F_p/\mathbb{Q}_p}(t_1), N_{F_p/\mathbb{Q}_p}(t_2)).$$

Let $B$ be the quaternion algebra over $F$ such that

$$\text{inv}_v(B) = \begin{cases} \text{inv}_v(B) & \text{if } v \neq \tau_1, p, \\ \frac{1}{2} & \text{if } v = \tau_1, \\ 0 & \text{if } v = p. \end{cases}$$

With $B$ instead of $B$ we can define analogues of $G$, $G'$ and $G''$, denoted by $\tilde{G}$, $\tilde{G}'$ and $\tilde{G}''$ respectively. For $\tilde{z} = \emptyset$, $''$ we have $\tilde{G}''(\mathbb{A}_f^p) = G''(\mathbb{A}_f^p)$; $\tilde{G}(\mathbb{Q}_p)$ is isomorphic to $\text{GL}(2, F_p)$; $\tilde{G}'(\mathbb{Q}_p)$ is isomorphic to $\text{GL}(2, F_p) \times \text{GL}(2, F_p)$, and $\tilde{G}''(\mathbb{Q}_p)$ is isomorphic to

$$\{([a_1 b_1], [a_2 b_2]) : a_i, b_i, c_i, d_i \in F_p, \begin{bmatrix} d_1 & -b_1 \\ c_1 & a_1 \end{bmatrix}, \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \in \mathbb{Q}_p^\times\}$$

of $\text{GL}(2, F_p) \times \text{GL}(2, F_p)$, and $\tilde{G}''(\mathbb{Q}_p)$ is isomorphic to

$$\{([a_1 b_1], [a_2 b_2]) : a_i, b_i, c_i, d_i \in F_p, \begin{bmatrix} d_1 & -b_1 \\ c_1 & a_1 \end{bmatrix}, \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \in F_p^\times\}.$$

If $\tilde{z} = 0$, let $\tilde{G}(\mathbb{Q}_p)$ act on $\mathcal{H}_{\mathbb{Q}_p}$ as in Section 3. If $\tilde{z} = \tilde{z}$, let $\tilde{G} = \tilde{G} \times T_E$ act on $\mathcal{H}_{\mathbb{Q}_p}$ by the projection to the first factor. If $\tilde{z} = \tilde{z}'$ or $''$, let $\tilde{G}''(\mathbb{Q}_p)$ act on $\mathcal{H}_{\mathbb{Q}_p}$ by the first factor. Let $\tilde{G}^2(\mathbb{Q}_p)$ act on $\mathcal{H}_{\mathbb{Q}_p}$ via its embedding into $G''(\mathbb{Q}_p)$.

The center of $\tilde{G}^2$, $Z(\tilde{G}^2)$, is naturally isomorphic to the center of $G^2$, $Z(G^2)$; we denote both of them by $Z^2$.

### 5.3 The $p$-adic uniformizations

Let $\tilde{z}$ be either $\tilde{z}$, $'$ or $''$. For any compact open subgroup $U^{\tilde{z}, p}$ of $G^2(\mathbb{A}_f^p)$, let $X_{U^{\tilde{z}, p}}^2$ denote $M_{U^{\tilde{z}, p}}^2 \times \text{Spec}(F_p)$, $\text{Spec}(\mathbb{F}_p)$.
Proposition 5.2. Suppose that Assumption 4.2 holds.

(a) Assume that $\mathfrak{z} = \mathfrak{z}'$ or $\mathfrak{z}''$. For any sufficiently small compact open subgroup $U^{\mathfrak{z}, \mathfrak{p}}$ of $G^{\mathfrak{z}}(\mathbb{A}^F)$, writing $U^\mathfrak{z} = U^{\mathfrak{z}, \mathfrak{p}}_0 U^{\mathfrak{z}, \mathfrak{p}}$, we have a $Z^{\mathfrak{z}}(\mathbb{Q}) \backslash Z^{\mathfrak{z}}(\mathbb{A}_f) / (Z^{\mathfrak{z}}(\mathbb{A}_f) \cap U^\mathfrak{z})$-equivariant isomorphism

$$X^\mathfrak{z}_{U^\mathfrak{z}, \mathfrak{p}} \cong \bar{G}^{\mathfrak{z}}(\mathbb{Q}) \backslash (\mathcal{H}_{\bar{F}_p^{\mathfrak{z}}} \times G^{\mathfrak{z}}(\mathbb{A}_f) / U^\mathfrak{z}).$$

(5.1)

Here, $\bar{G}^{\mathfrak{z}}(\mathbb{Q})$ acts on $\mathcal{H}_{\bar{F}_p^{\mathfrak{z}}}$ as mentioned above and acts on $G^{\mathfrak{z}}(\mathbb{A}^F) / U^{\mathfrak{z}, \mathfrak{p}}$ by the embedding $\bar{G}^{\mathfrak{z}}(\mathbb{Q}) \hookrightarrow \bar{G}^{\mathfrak{z}}(\mathbb{A}^F)$; in the case of $\mathfrak{z} = \mathfrak{z}'$ or $\mathfrak{z}''$, if $g \in \bar{G}^{\mathfrak{z}}(\mathbb{Q})$ satisfies $g_p = (a, b)$ with $a, b \in \text{GL}(2, F_p)$, then $g$ acts on $G^{\mathfrak{z}}(\mathbb{Q}_p) / U^{\mathfrak{z}, \mathfrak{p}}_0$ via the left multiplication by $(\Pi^a p^{\nu_\mathfrak{z}}(\text{det} \mathfrak{g}), \Pi^b p^{\nu_\mathfrak{z}}(\text{det} \mathfrak{g}))$; while, in the case of $\mathfrak{z} = \mathfrak{z}'$, $\bar{g} = (g, t) \in \bar{G}(\mathbb{Q})$ ($g \in G(\mathbb{Q}), t \in T_E(\mathbb{Q})$) acts on $\bar{G}(\mathbb{Q}_p) / U^{\mathfrak{z}, \mathfrak{p}}_0$ via the left multiplication by $(\Pi^a p^{\nu_\mathfrak{z}}(\text{det} g_p), t_p)$; the group $Z^{\mathfrak{z}}(\mathbb{Q}) \backslash Z^{\mathfrak{z}}(\mathbb{A}_f) / (Z^{\mathfrak{z}}(\mathbb{A}_f) \cap U^\mathfrak{z})$ acts on the right hand side of (5.1) by right multiplications on $\bar{G}^{\mathfrak{z}}(\mathbb{A}_f)$.

(b) The isomorphisms in (a) can be chosen such that, for either $\mathfrak{z} = \mathfrak{z}'$ and $\mathfrak{z} = \mathfrak{z}''$, or $\mathfrak{z} = \mathfrak{z}'$ and $\mathfrak{z} = \mathfrak{z}''$, we have a commutative diagram

$$
\begin{array}{ccc}
X^\mathfrak{z}_{U^\mathfrak{z}, \mathfrak{p}} & \longrightarrow & \bar{G}^{\mathfrak{z}}(\mathbb{Q}) \backslash (\mathcal{H}_{\bar{F}_p^{\mathfrak{z}}} \times G^{\mathfrak{z}}(\mathbb{A}_f) / U^\mathfrak{z}) \\
\downarrow & & \downarrow \\
X^\mathfrak{z}_{U^\mathfrak{z}, \mathfrak{p}} & \longrightarrow & \bar{G}^{\mathfrak{z}}(\mathbb{Q}) \backslash (\mathcal{H}_{\bar{F}_p^{\mathfrak{z}}} \times G^{\mathfrak{z}}(\mathbb{A}_f) / U^\mathfrak{z})
\end{array}
$$

-compatible with the $Z^{\mathfrak{z}}(\mathbb{Q}) \backslash Z^{\mathfrak{z}}(\mathbb{A}_f) / (Z^{\mathfrak{z}}(\mathbb{A}_f) \cap U^\mathfrak{z})$-actions on the upper and the $Z^{\mathfrak{z}}(\mathbb{Q}) \backslash Z^{\mathfrak{z}}(\mathbb{A}_f) / (Z^{\mathfrak{z}}(\mathbb{A}_f) \cap U^\mathfrak{z})$-actions on the lower, where the left vertical arrow is induced from the morphism $M^1 \to M^2$, and the right vertical arrow is induced by the identity morphism $\mathcal{H}_{\bar{F}_p^{\mathfrak{z}}} \to \mathcal{H}_{\bar{F}_p^{\mathfrak{z}}}$ and the homomorphism $\alpha : \bar{G} = G \times T_E \to G''$ or the inclusion $G' \hookrightarrow G''$. Here, in the case of $\mathfrak{z} = \mathfrak{z}'$ and $\mathfrak{z} = \mathfrak{z}''$, $U^\mathfrak{z} = \alpha(U^\mathfrak{z})$; in the case of $\mathfrak{z} = \mathfrak{z}'$ and $\mathfrak{z} = \mathfrak{z}''$, $U^\mathfrak{z} = U^\mathfrak{z} \cap G'(\mathbb{A}_f)$.

The conclusions of Proposition 5.2 especially (a) are well known [27, 37]. However, the author has no reference for (b), so we provide some detail of the proof.

**Proof.** Assertion (a) in the case of $\mathfrak{z} = \mathfrak{z}'$ comes from [27, Theorem 6.50].

For the case of $\mathfrak{z} = \mathfrak{z}'$ and $\mathfrak{z} = \mathfrak{z}''$ we put

$$C = Z^{\mathfrak{z}}(\mathbb{Q}) \backslash Z^{\mathfrak{z}}(\mathbb{A}_f) / (Z^{\mathfrak{z}}(\mathbb{A}_f) \cap U^\mathfrak{z})$$

and

$$Z = Z^{\mathfrak{z}''}(\mathbb{Q}) \backslash Z^{\mathfrak{z}''}(\mathbb{A}_f) / (Z^{\mathfrak{z}''}(\mathbb{A}_f) \cap U^{\mathfrak{z}''}).$$

Then $X^\mathfrak{z}_{U^\mathfrak{z}, \mathfrak{p}}$ is $Z$-equivariantly isomorphic to $X^\mathfrak{z}_{U^\mathfrak{z}, \mathfrak{p}} \ast_C Z$, and $\bar{G}''(\mathbb{Q}) \backslash (\mathcal{H}_{\bar{F}_p^{\mathfrak{z}}} \times G''(\mathbb{A}_f) / U^{\mathfrak{z}''})$ is $Z$-equivariantly isomorphic to $\left( \bar{G}'(\mathbb{Q}) \backslash (\mathcal{H}_{\bar{F}_p^{\mathfrak{z}}} \times G'(\mathbb{A}_f) / U'') \right) \ast_C Z$. So (a) in the case of $\mathfrak{z} = \mathfrak{z}'$ and (b) in the case of $\mathfrak{z} = \mathfrak{z}'$, $\mathfrak{z} = \mathfrak{z}''$ follow.

Now we consider the rest cases. Let $H$ be the kernel of the homomorphism $\alpha : \bar{G} = G \times T_E \to G''$. Put

$$C = H(\mathbb{Q}) \backslash H(\mathbb{A}_f) / (H(\mathbb{A}_f) \cap \bar{U}), \quad Z = \bar{Z}(\mathbb{Q}) \backslash \bar{Z}(\mathbb{A}_f) / (\bar{Z}(\mathbb{A}_f) \cap \bar{U}).$$
Put \( X_1 = \widetilde{X}_{U_p} \) and \( X_2 = \widetilde{G}(\mathbb{Q})/(\mathcal{H}_{F_p^0} \times \widetilde{G}(\mathfrak{a}_f))/\overline{U}_{p,0}\overline{U}_p \). By Lemma 4.1 (a), all connected components of \( X_1 \) are geometrically connected; it is obvious that all connected components of \( X_2 \) are geometrically connected. Thus \( Z \) acts freely on the set of components of \( X_1 \) (resp. \( X_2 \)). Furthermore \( X_1/C \) is isomorphic to \( X''_{\alpha(\mathfrak{a}_f)} \) and \( X_2/C \) is isomorphic to \( G''(\mathbb{Q})/(\mathcal{H}_{F_p^0} \times G(\mathfrak{a}_f)/U_{p,0}\alpha(\overline{U}_p)) \). We have already proved that \( X_1/C \) is \( Z/C \)-equivariantly isomorphic to \( X_2/C \). Applying Lemma 5.1 we obtain (a) in the case of \( z = '-' \) and (b) in the case of \( z = '0' \).

**Remark 5.3.** By [37] the similar conclusion of Proposition 5.2 (a) holds for the case of \( z = 0 \). We use \( X_{U_p} \) to denote \( \widetilde{G}(\mathbb{Q})/(\mathcal{H}_{F_p^0} \times G(\mathfrak{a}_f)/U_{p,0}U_p) \), where the action of \( \widetilde{G}(\mathbb{Q}) \) on \( \mathcal{H}_{F_p^0} \times G(\mathfrak{a}_f)/U_{p,0}U_p \) is defined similarly.

### 6 Local systems and the associated filtered \( \varphi_q \)-isocrystals on Shimura Curves

#### 6.1 Local systems on Shimura curves

We choose a number field \( L \) splitting \( F \) and \( B \). We identify \( \{ \tau_i : F \to L \} \) with \( I = \{ \tau_i : F \to \mathbb{C} \} \) by the inclusion \( L \to \mathbb{C} \). Fix an isomorphism \( L \otimes \mathbb{Q} B = M(2, L)^I \). Then we have a natural inclusion \( G(\mathbb{Q}) \to GL(2, L)^I \).

For a multiweight \( k = (k_1, \ldots, k_g, w) \) with \( k_1 \equiv \cdots \equiv k_g \equiv w \mod 2 \) and \( k_1 \geq 2, \cdots, k_g \geq 2, \) we define the morphism \( \rho^{(k)} : G \to GL(n, L) (n = \prod_{i=1}^g (k_i - 1)) \) to be the product \( \otimes_{i \in I} [\text{Sym}^{k_i - 2} \otimes \text{det}^{(w-k_i)/2}] \circ p_{\rho_i} \). Here \( p_{\rho_i} \) denotes the contragradient representation of the \( i \)-th projection \( pr_i : GL(2, L)^I \to GL(2, L) \). The algebraic group defined by \( G^{c} \) in [22, Chapter III] is the quotient of \( G \) by \( \ker( N_{F/\mathbb{Q}} : F^\times \to \mathbb{Q}^\times ) \). As the restriction of \( \rho^{(k)} \) to the center \( F^\times \) is the scalar multiplication by \( N_{F/\mathbb{Q}}(\cdot) \), \( \rho^{(k)} \) factors through \( G^{c} \), so we can attach to the representation \( \rho^{(k)} \) a \( G(\mathfrak{a}_f) \)-equivariant smooth \( L_{\varphi} \)-sheaf \( \mathcal{F}(k) \) on \( M \).

Let \( p_2 : G''_{E_0} \to G_{E_0} \) be the map induced by the second projection on \( (D \otimes \mathbb{Q} E_0)^\times = D^\times \times D^\times \) corresponding to the conjugate \( E_0 \to E_0 \). As the algebraic representation \( \rho^{(k)} = \rho^{(k)} \circ p_2 \) factors through \( G^{c} \), we can attach to it a \( G''(\mathfrak{a}_f) \)-equivariant smooth \( L_{\varphi} \)-sheaf \( \mathcal{F}'(k) \) on \( M'' \). Let \( \mathcal{F}'(k) \) be the restriction of \( \mathcal{F}(k) \) to \( M' \).

We define a character \( \chi : T_0 \to \mathbb{G}_m \) such that on \( \mathbb{C} \)-valued point \( \tilde{\chi} \) is the inverse of the second projection \( T_0C = \mathbb{C}^\times \times \mathbb{C}^\times \to \mathbb{C}^\times \). Let \( \mathcal{F}(\tilde{\chi}) \) be the \( L_{\varphi} \)-sheaf attached to the representation \( \tilde{\chi} \). By [28] one has the following \( G(\mathfrak{a}_f) \times T(\mathfrak{a}_f) \)-equivariant isomorphism of \( L_{\varphi} \)-sheaves

\[
pr_1^* \mathcal{F}(k) \simeq \alpha^* \mathcal{F}'(k) \otimes \beta^* \mathcal{F}(\tilde{\chi}^{-1})^{(g-1)(w-2)}
\]

on \( M \times N \), where \( pr_1 \) is the projection \( M \times N \to M \).

Note that \( L \otimes \mathbb{Q} D \simeq (M_2(L) \times M_2(L))^I \). For each \( i \in I \), the first component \( M_2(L) \) corresponds to the embedding \( E_0 \subset L \subset \mathbb{C} \) and the second \( M_2(L) \) to its conjugate. Let \( \mathcal{F}' \) be the local system \( R^1 g_* L_{\varphi} \) where \( g : A \to M' \) is the universal \( \mathcal{O}_D \)-abelian scheme; it is a sheaf of \( L \otimes \mathbb{Q} D \)-modules. For each \( i \in I \), let \( e_i \in L \otimes \mathbb{Q} D \) be the idempotent whose \( (2, i) \)-th component is a rank one idempotent e.g. \( \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \) and the other components are zero. Let \( \mathcal{F}_i' \) denote the \( e_i \)-part \( e_i \cdot R^1 g_* L_{\varphi} \). Note that \( \mathcal{F}_i' \) does not depend on the choice of the rank one idempotent. By [28] we have an isomorphism of...
For \( k = (k, v) = (k_1, \ldots, k_g, v_1, \ldots, v_g) \), let \( \mathcal{F}'(k, v) \) be the local system \( \bigotimes_{i \in I} \left( \text{Sym}^{k_i-2} \mathcal{F}'_i \otimes (\det \mathcal{F}'_i)^{(v-w-k_i)/2} \right) \).

We can define more local systems on \( M' \). For \( (k, v) = (k_1, \cdots, k_g, v_1, \cdots, v_g) \), let \( \mathcal{F}'(k, v) \) be the local system \( \bigotimes_{i \in I} \left( \text{Sym}^{k_i-2} \mathcal{F}'_i \otimes (\det \mathcal{F}'_i)^{v_i} \right) \).

### 6.2 Filtered \( \varphi_q \)-isocrystals associated to the local systems

We use \( \tilde{k} \) uniformly to denote \( (k, v) = (k_1, \cdots, k_g, v_1, \cdots, v_g) \) (resp. \( k = (k_1, \cdots, k_g, w) \)) in the case of \( \tilde{\xi} = \emptyset \) (resp. \( \tilde{\xi} = \emptyset', \emptyset'' \)).

We shall need the filtered \( \varphi_q \)-isocrystal attached to \( \mathcal{F}(\tilde{k}) \). However, we do not know how to compute it. Instead, we compute that attached to \( \text{pr}_*^\dagger \mathcal{F}(\tilde{k}) \). As a middle step, we determine the filtered \( \varphi_q \)-isocrystals associated to \( \mathcal{F}(\tilde{k}) \) and \( \mathcal{F}'(\tilde{k}) \).

For any integers \( k \) and \( v \) with \( k \geq 2 \), and any inclusion \( \sigma : F_p \to L_{Q_p} \), let \( V_\sigma(k, v) \) be the space of homogeneous polynomials in two variables \( X_\sigma \) and \( Y_\sigma \) of degree \( k - 2 \) with coefficients in \( L_{Q_p} \); let \( \text{GL}(2, F_p) \) act on \( V_\sigma(k, v) \) by

\[
[a \ b] P(X_\sigma, Y_\sigma) = \sigma(ad - bc)^v P(\sigma(a)X_\sigma + \sigma(c)Y_\sigma, \sigma(b)X_\sigma + \sigma(d)Y_\sigma).
\]

For \( (k, v) = (k_1, \cdots, k_g, v_1, \cdots, v_g) \) we put

\[
V(k, v) = \bigotimes_{\sigma \in I} V_\sigma(k_\sigma, v_\sigma)
\]

where the tensor product is taken over \( L_{Q_p} \).

Let \( \tilde{G}^g \) (\( \tilde{\xi} = \emptyset', \emptyset'' \)) be the groups defined in Section 5.2. For \( \tilde{\xi} = \emptyset \), via the projection \( \tilde{G}^g(Q_{p}) \to \text{GL}(2, F_p) \), \( V(k) \) becomes a \( \tilde{G}^g(Q_{p}) \)-module. For \( \tilde{\xi} = \emptyset', \emptyset'' \), via the projection of \( \tilde{G}^g(Q_{p}) \subset \text{GL}(2, F_p) \times \text{GL}(2, F_p) \) to the second factor, \( V(k) \) becomes a \( \tilde{G}^g(Q_{p}) \)-module. In each case via the inclusion \( \tilde{G}^g(Q) \hookrightarrow \tilde{G}^g(Q_p) \), \( V(k) \) becomes a \( \tilde{G}^g(Q) \)-module. Using the \( p \)-adic uniformization of \( X^g = X^g_{U_{3,p}} \), we attach to this \( \tilde{G}^g(Q) \)-module a local system \( \mathcal{V}^g(\tilde{k}) \) on \( X^g \).

Let \( \varphi_{q, k, v} \) be the operator on \( V(k, v) \)

\[
\bigotimes_{\sigma} P_\sigma(X_\sigma, Y_\sigma) \mapsto \prod_{\sigma} \sigma(-\pi)^{v_\sigma} \cdot \bigotimes_{\sigma} P_\sigma(Y_\sigma, \sigma(\pi)X_\sigma).
\]

For \( k = (k_1, \cdots, k_g, w) \) we put

\[
V(k) = V(k_1, \cdots, k_g; (w - k_1)/2, \cdots, (w - k_g)/2)
\]

and

\[
\varphi_{q, k} = \varphi_{q, (k_1, \cdots, k_g; (w - k_1)/2, \cdots, (w - k_g)/2)}.
\]

Let \( \mathcal{F}^g(\tilde{k}) \) be the filtered \( \varphi_q \)-isocrystal \( \mathcal{V}^g(\tilde{k}) \otimes_{Q_p} \mathcal{O}_{X^g} \) on \( X^g \) with the \( q \)-Frobenius \( \varphi_{q, k} \otimes \varphi_q, \mathcal{O}_{X^g} \) and the connection \( 1 \otimes d : \mathcal{V}^g(\tilde{k}) \otimes_{Q_p} \mathcal{O}_{X^g} \to \mathcal{V}^g(\tilde{k}) \otimes_{Q_p} \Omega^1_{X^g} \); the filtration on

\[
\mathcal{V}^g(\tilde{k}) \otimes_{Q_p} \mathcal{O}_{X^g} = \bigoplus_{\tau : F_p \to L_{Q_p}} \mathcal{V}^g(\tilde{k}) \otimes_{\tau, F_p} \mathcal{O}_{X^g}
\]

(6.3)
is given by
\[
\text{Fil}^{j+\nu_r}(\mathcal{V}(\hat{k}) \otimes \tau.F_p \Theta_{\mathcal{X}})
\begin{cases}
\mathcal{V}(\hat{k}) \otimes \tau.F_p \Theta_{\mathcal{X}} & \text{if } j \leq 0, \\
\text{the } \Theta_{\mathcal{X}}\text{-submodule locally generated by polynomials} & \text{if } 1 \leq j \leq k_r - 2 \\
0 & \text{if } j \geq k_r - 1
\end{cases}
\]
with the convention that \(\nu_r = \frac{w-k_r}{2}\) in the case of \(\hat{k} = (k_1, \ldots, k_g, w)\), where \(z\) is the canonical coordinate on \(\mathcal{H}_{\bar{p}}\).

**Lemma 6.1.** When \(k_1 = \cdots = k_{i-1} = k_{i+1} = \cdots = k_g = 2, k_i = 3, \) and \(v_1 = \cdots = v_g = 0\), the filtered \(\varphi_q\)-isocrystal attached to \(\mathcal{F}'(k, v)\) is isomorphic to \(\mathcal{F}'(k, v)\).

**Proof.** Let \(\bar{e}_i \in L \otimes \mathbb{Q} D\) be the idempotent whose \((2, i)\)-th component is \(\begin{bmatrix}1 & 0 \\ 0 & 1\end{bmatrix}\) and the other components are zero. Let \(A\) be the universal \(\mathcal{O}_D\)-abelian scheme over \(M'\), \(\hat{A}\) the formal module on \(X'\) attached to \(A\). Note that \(\bar{e}_i(\mathfrak{o}_{L_p} \otimes_{\mathcal{O}_p} \hat{A})\) is just the pullback of \(\mathfrak{o}_{L_p} \otimes_{\pi, \mathcal{O}_p} \mathcal{G}\) by the projection \(X'_{U_p} \to \mathcal{G}(Q) \cap U''_{p,0})/\mathcal{H}_{\bar{p}}\) [27, 6.43], where \(\mathcal{G}\) is the universal special formal \(\mathcal{O}_{B_p}\)-module (forgetting the information of \(\rho\) in Drinfeld’s moduli problem).

As \(L_p\) splits \(B_p, L_p\), contains all embeddings of \(F_p^{(2)}\). The embedding \(\tau_i : F_p \to L_p\) extends in two ways to \(F_p^{(2)}\) denoted respectively by \(\tau_{i,0}\) and \(\tau_{i,1}\). Then
\[
\mathfrak{o}_{L_p} \otimes \tau_{i,0, \mathcal{O}_{F_p}} B_p = \mathfrak{o}_{L_p} \otimes \tau_{i,0, \mathcal{O}_{F_p}} \otimes B_p \oplus \mathfrak{o}_{L_p} \otimes \tau_{i,1, \mathcal{O}_{F_p}} \otimes B_p.
\]
We decompose \(\mathfrak{o}_{L_p} \otimes \tau_{i,0, \mathcal{O}_{F_p}} \mathcal{G}\) into the sum of two direct summands according to the action of \(\mathfrak{o}_{F_p^{(2)}} \subset \mathfrak{o}_{B_p}\): \(\mathfrak{o}_{F_p^{(2)}}\) acts by \(\tau_{i,0}\) on the first direct summand, and acts by \(\tau_{i,1}\) on the second. Without loss of generality we may assume that \(e_i\) in the definition of \(\mathcal{F}'\) (see Section 6.1) is chosen such that \(e_i\) is the projection onto the first direct summand. So \(e_i(\mathfrak{o}_{L_p} \otimes_{\mathcal{O}_p} \hat{A})\) is just the pullback of \(\mathfrak{o}_{L_p} \otimes_{\pi, \mathcal{O}_p} \mathfrak{G}\) by the projection \(X'_U \to (\mathcal{G}(Q) \cap U''_{p,0})/\mathcal{H}_{\bar{p}}\). Now the statement of our lemma follows from the discussion in Section 3.2. \(\Box\)

**Proposition 6.2.** The filtered \(\varphi_q\)-isocrystal attached to \(\mathcal{F}'(k, v)\) is isomorphic to \(\mathcal{F}'(k, v)\).

**Proof.** Let \(\mathcal{F}'\) denote the filtered \(\varphi_q\)-isocrystal attached to \(\mathcal{F}'\). By (6.4) the filtered \(\varphi_q\)-isocrystal attached to \(\mathcal{F}'(k, v)\) is isomorphic to
\[
\bigotimes_{i \in I} \left(\text{Sym}_{k_i - 2}(\mathcal{F}'(\det(\mathcal{F}'))^{(w-k_i)/2})\right).
\]

By Lemma 6.1 a simple computation implies our conclusion. \(\Box\)

**Corollary 6.3.** The filtered \(\varphi_q\)-isocrystal attached to \(\mathcal{F}'(k)\) is isomorphic to \(\mathcal{F}'(k)\).

**Proof.** This follows from Proposition 6.2 and [28, Lemma 6.1]. \(\Box\)

**Lemma 6.4.** The filtered \(\varphi_q\)-isocrystal associated to the local system \(\mathcal{F}(\tilde{\chi})\) over \((N_{E_0,0})_{\bar{p}}\) is \(\mathcal{F}(\tilde{\chi}) \otimes \Theta_{(N_{E_0,0})_{\bar{p}}}\) with the \(q\)-Frobenius being \(1 \otimes \varphi_q(N_{E_0,0})_{\bar{p}}\) and the filtration being trivial.
Lemma 6.5. The filtered $\varphi_q$-isocrystal attached to $\text{pr}_1^* F(k)$ is $\text{pr}_1^* \mathcal{F}(k)$.

Proof. By (6.1) the filtered $\varphi_q$-isocrystal attached to $\text{pr}_1^* F(k)$ is the tensor product of the filtered $\varphi_q$-isocrystal attached to $\alpha^* \mathcal{F}'(k)$ and that attached to $\beta^* \mathcal{F}(\chi^{-1}(q^{-1}(w-2)))$. Our conclusion follows from Proposition 5.2 (b) (in the case of $z = -$ and $z =''$), Corollary 6.3 and Lemma 6.4.

It is rather possible that the filtered $\varphi_q$-isocrystal attached to $\mathcal{F}(k)$ is $\mathcal{F}(k)$. But the author does not know how to descent the conclusion of Corollary 6.5 to $X_{U^+}$.

7 The de Rham cohomology

7.1 Covering filtration and Hodge filtration for de Rham cohomology

We fix an arithmetic Schottky group $\Gamma$ that is cocompact in $\text{PGL}(2, F_p)$. Then $\Gamma$ acts freely on $\mathcal{H}$, and the quotient $X_\Gamma = \Gamma \backslash \mathcal{H}$ is the rigid analytic space associated with a proper smooth curve over $F_p$. Here we write $\mathcal{H}$ for $\mathcal{H}_{F_p}$.

We recall the theory of de Rham cohomology of local systems over $X_\Gamma$ [30, 31, 32, 33].

We denote by $\hat{\mathcal{H}}$ the canonical formal model of $\mathcal{H}$. The curve $X_\Gamma$ has a canonical semistable module $X_\Gamma = \Gamma \backslash \hat{\mathcal{H}}$; the special fiber $X_{\Gamma, s}$ of $X_\Gamma$ is isomorphic to $\Gamma \backslash \hat{\mathcal{H}}_s$.

The graph $\text{Gr}(X_{\Gamma, s})$ is closely related to the Bruhat-Tits tree $\mathcal{T}$ for $\text{PGL}(2, F_p)$. The group $\Gamma$ acts freely on the tree $\mathcal{T}$. Let $\mathcal{T}_\Gamma$ denote the quotient tree. The set of connected components of the special fiber $X_{\Gamma, s}$ is in one-to-one correspondence to the set $V(\mathcal{T}_\Gamma)$ of vertices of $\mathcal{T}_\Gamma$, each component being isomorphic to the projective line over $k(= \text{the residue field of } F_p)$. Write $\{P_1^v\}_{v \in V(\mathcal{T}_\Gamma)}$ for the set of components of $X_{\Gamma, s}$. The singular points of $X_{\Gamma, s}$ are ordinary $k$-rational double singular points; they correspond to (unoriented) edges of $\mathcal{T}_\Gamma$. Two components $P_u^v$ and $P_v^v$ intersect if and only if $u$ and $v$ are adjacent; in this case, they intersect at a singular point. For simplicity we will
use the edge $e$ joining $u$ and $v$ to denote this singular point. There is a reduction map from $X_{T, s}^{\Gamma}$ to $X_{T, s}$. For a closed subset $U$ of $X_{T, s}$ let $|U|$ denote the tube of $U$ in $X_{T, s}^{\Gamma}$. Then $\{ |P_{e}^{1}\}_{e \in V(T_{s})}$ is an admissible covering of $X_{T, s}^{\Gamma}$. Clearly $|P_{\partial(e)}^{1}| = |e|$.

Let $L$ be a field that splits $F_{p}$. Fix an embedding $\tau : F_{p} \rightarrow L$. Let $V$ be an $L[\Gamma]$-module that comes from an algebraic representation of $\text{PGL}(2, F_{p})$ of the form $V(k)$ with $k = (k_{1}, \cdots, k_{2} : 2)$. We will regard $V$ as an $F_{p}$-vector space by $\tau$. Let $\mathcal{Y}$ be the hypercohomology of the complex $\mathcal{Y} \otimes_{\tau, F_{p}} \Omega^{1}_{X_{T}}$.

We consider the Mayer-Vietorus exact sequence attached to $H_{\text{dR}, \Gamma}(X_{T}, \mathcal{Y})$ with respect to the admissible covering $\{ |P_{e}^{1}\}_{e \in V(T_{s})}$. As a result we obtain an injective map

$$\iota : \left( \bigoplus_{e \in E(T_{s})} H_{\text{dR}, \Gamma}(\{ e \}, \mathcal{Y}) \right)^{-} / \left( \bigoplus_{v \in V(T_{s})} H_{\text{dR}, \Gamma}(\{ |P_{e}^{1}\}, \mathcal{Y}) \right) \hookrightarrow H_{\text{dR}, \Gamma}(X_{T, s}, \mathcal{Y}).$$

As $|P_{e}^{1}|$ and $|e|$ are quasi-Stein, a simple computation shows that $H_{\text{dR}, \Gamma}(\{ |P_{e}^{1}\}, \mathcal{Y})$ and $H_{\text{dR}, \Gamma}(\{ e \}, \mathcal{Y})$ are isomorphic to $V$. Let $H^{0}(V)$ be the space of $V$-valued functions on $V(T)$, $C^{1}(V)$ the space of $V$-valued functions on $E(T)$ such that $f(e) = -f(\bar{e})$. Let $\Gamma$ act on $C^{1}(V)$ as $f \mapsto \gamma \circ f \circ \gamma^{-1}$. Then we have a $\Gamma$-equivariant short exact sequence

$$0 \rightarrow V \xrightarrow{\partial} C^{0}(V) \xrightarrow{\partial} C^{1}(V) \rightarrow 0 \quad (7.1)$$

where $\partial(f)(e) = f(o(e)) - f(t(e))$. Observe that

$$\bigoplus_{v \in V(T_{s})} H_{\text{dR}, \Gamma}(\{ |P_{e}^{1}\}, \mathcal{Y}) \cong C^{0}(V)^{\Gamma},$$

$$\left( \bigoplus_{e \in E(T_{s})} H_{\text{dR}, \Gamma}(\{ e \}, \mathcal{Y}) \right)^{-} \cong C^{1}(V)^{\Gamma}$$

and the map

$$\bigoplus_{v \in V(T_{s})} H_{\text{dR}, \Gamma}(\{ |P_{e}^{1}\}, \mathcal{Y}) \rightarrow \left( \bigoplus_{e \in E(T_{s})} H_{\text{dR}, \Gamma}(\{ e \}, \mathcal{Y}) \right)^{-}$$

coincides with $\partial$. Thus

$$\left( \bigoplus_{e \in E(T_{s})} H_{\text{dR}, \Gamma}(\{ e \}, \mathcal{Y}) \right)^{-} / \bigoplus_{v \in V(T_{s})} H_{\text{dR}, \Gamma}(\{ |P_{e}^{1}\}, \mathcal{Y})$$

is isomorphic to $C^{1}(V)^{\Gamma} / \partial C^{0}(V)^{\Gamma}$. From (7.1) we get the injective map

$$\delta : C^{1}(V)^{\Gamma} / \partial C^{0}(V)^{\Gamma} \hookrightarrow H^{1}(\Gamma, V).$$

Let $C^{1}_{\text{har}}(V)$ be the space of harmonic forms

$$C^{1}_{\text{har}}(V) := \{ f : \text{Edge}(T) \rightarrow V | f(e) = -f(\bar{e}), \forall v, \sum_{t(e) = v} f(e) = 0 \}.$$

Fixing some $v \in V(T)$, let $\epsilon$ be the map $C^{1}_{\text{har}}(V)^{\Gamma} \rightarrow H^{1}(\Gamma, V)$ [7, (2.26)] defined by

$$c \mapsto (\gamma \mapsto \sum_{e : \gamma \circ e = v} c(e)),$$

$$\epsilon$$
where the sum runs over the edges joining \( v \) and \( \gamma v \); \( \epsilon \) does not depend on the choice of \( v \). By [7, Appendix A] \( \epsilon \) is minus the composition

\[
C^1_{\text{har}}(V)^\Gamma \to C^1(V)^\Gamma / \partial C^0(V)^\Gamma \overset{\delta}{\to} H^1(\Gamma, V),
\]

and is an isomorphism. Combining this with the injectivity of \( \delta \) we obtain that both the natural map \( C^1_{\text{har}}(V)^\Gamma \to C^1(V)^\Gamma / \partial C^0(V)^\Gamma \) and \( \delta \) are isomorphisms. Below, we will identify \( C^1_{\text{har}}(V)^\Gamma \) with \( C^1(V)^\Gamma / \partial C^0(V)^\Gamma \).

By [33] we have

\[
H^1_{\text{dR},\tau}(X_\Gamma, \mathcal{V}) \cong \{ \text{\( V \)-valued differentials of second kind on } X_\Gamma \} / \{ df \mid f \text{ a } \( V \)-valued meromorphic function on } X_\Gamma \},
\]

(7.3)

In loc. cit, de Shalit only considered a special case, but his argument is valued for our general case. If \( \omega \) is a \( \Gamma \)-invariant \( V \)-valued differential of the second kind on \( \mathcal{H} \), let \( F_\omega \) be a primitive of it [32], which is defined by Coleman’s integral [8]. \(^3\) Let \( P \) be the map

\[
P : H^1_{\text{dR},\tau}(X_\Gamma, \mathcal{V}) \to H^1(\Gamma, V), \quad \omega \mapsto (\gamma \mapsto \gamma(F_\omega) - F_\omega).
\]

Note that \( P \circ \iota \) coincides with \( \delta \). Thus \( P \) splits the inclusion \( \iota \circ \delta^{-1} : H^1(\Gamma, V) \to H^1_{\text{dR},\tau}(X_\Gamma, \mathcal{V}) \).

Let \( I \) be the map

\[
I : H^1_{\text{dR},\tau}(X_\Gamma, \mathcal{V}) \to C^1_{\text{har}}(V)^\Gamma, \omega \mapsto (e \mapsto \text{Res}_e(\omega)).
\]

Now, we suppose that \( \Gamma \) is of the form in [7, Appendix A]. We do not describe it precisely, but only point out that \( \Gamma_{\iota,0} \) in Section 7.2 is of this form.

**Proposition 7.1.** We have an exact sequence called the covering filtration exact sequence

\[
0 \to H^1(\Gamma, V) \overset{\iota \circ \delta^{-1}}{\to} H^1_{\text{dR},\tau}(X_\Gamma, \mathcal{V}) \overset{I}{\to} C^1_{\text{har}}(V)^\Gamma \to 0.
\]

**Proof.** What we need to prove is that the map

\[
H^1_{\text{dR},\tau}(X_\Gamma, \mathcal{V}) \to H^1(\Gamma, V) \oplus C^1_{\text{har}}(V)^\Gamma, \omega \mapsto (P(\omega), I(\omega))
\]

is an isomorphism. When \( V \) is the trivial module, this is already proved in [33]. So we assume that \( V \) is not the trivial module. First we prove the injectivity of the above map. For this we only need to repeat the argument in [33, Theorem 1.6]. Let \( \omega \) be a \( \Gamma \)-invariant differential form of second kind on \( \mathcal{H} \) such that \( P([\omega]) \) is in the class of \( \omega \) in \( H^1_{\text{dR},\tau}(X_\Gamma, \mathcal{V}) \). Let \( F_\omega \) be a primitive of \( \omega \). As \( I(\omega) = 0 \), the residues of \( \omega \) vanish, and thus \( F_\omega \) is meromorphic. As \( P(\omega) = 0 \), we may adjust \( F_\omega \) by a constant so that it is \( \Gamma \)-invariant. By (7.3) we have \( [\omega] = 0 \). To show the surjectivity we only need to compare the dimensions. By [7, Appendix A] we have

\[
\dim_{F_p} C^1_{\text{har}}(V)^\Gamma = \dim_{F_p} H^1(\Gamma, V)
\]

and

\[
\dim_{F_p} H^1(\Gamma, V) = \dim_{F_p} H^1(\Gamma, V^*)
\]

\(^3\)Precisely we choose a branch of Coleman’s integral.
where $V^* = \text{Hom}_{F_p}(V, F_p)$ is the dual $F_p[\Gamma]$-module. By [30, Theorem 1] we have
\[ \dim_{F_p} H^1_{\text{dR}, \tau}(X_{\Gamma}, \mathcal{V}) = \dim_{F_p} H^1(\Gamma, V) + \dim_{F_p} H^1(\Gamma, V^*). \]
Hence
\[ \dim_{F_p} H^1_{\text{dR}, \tau}(X_{\Gamma}, \mathcal{V}) = \dim_{F_p} C^1_{\text{bar}}(V)^\Gamma + \dim_{F_p} H^1(\Gamma, V), \]
as desired.}

We have also a Hodge filtration exact sequence
\[ 0 \rightarrow H^0(X_{\Gamma}, \mathcal{V} \otimes_{F_p, \tau} \Omega^1_{X_{\Gamma}}) \rightarrow H^1_{\text{dR}, \tau}(X_{\Gamma}, \mathcal{V}) \rightarrow H^1(X_{\Gamma}, \mathcal{V} \otimes_{F_p, \tau} \mathcal{O}_{X_{\Gamma}}) \rightarrow 0. \]
This exact sequence and the covering filtration exact sequence fit into the commutative diagram
\[ \begin{array}{c}
\begin{array}{c}
0 \\
H^1(\Gamma, V) \\
\downarrow \text{loc}^{-1}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
0 \\
H^0(X_{\Gamma}, \mathcal{V} \otimes_{F_p, \tau} \Omega^1_{X_{\Gamma}}) \\
\downarrow \approx \\
H^1_{\text{dR}, \tau}(X_{\Gamma}, \mathcal{V}) \\
\downarrow \approx \\
H^1(X_{\Gamma}, \mathcal{V} \otimes_{F_p, \tau} \mathcal{O}_{X_{\Gamma}}) \\
\downarrow I
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
C^1_{\text{bar}}(V)^\Gamma \\
\downarrow \\
0
\end{array}
\end{array} \]
The diagonal arrows are isomorphisms. Indeed, by [32, Section 3] the south-west arrow is an isomorphism; one easily deduces from this that the north-east arrow is also an isomorphism. In particular, we have a Hodge-like decomposition
\[ H^1_{\text{dR}, \tau}(X_{\Gamma}, \mathcal{V}) = H^0(X_{\Gamma}, \mathcal{V} \otimes_{F_p, \tau} \Omega^1_{X_{\Gamma}}) \bigoplus I \circ \delta^{-1}(H^1(\Gamma, V)). \tag{7.4} \]

### 7.2 De Rham cohomology of $\mathcal{F}(k)$

Let $B$ be as in Section 5.2. Write $\hat{B}^X := (\hat{B} \otimes \mathbb{A}_f)^X$ and $\hat{B}^{p, X} := (\hat{B} \otimes \mathbb{A}_f^p)^X$. Let $U$ be a compact open subgroup of $G(\mathbb{A}_f)$ that is of the form $U_{p,0}U^p$. We identify $U^p = \prod_{i \neq p} U_i$ with a subgroup of $\hat{B}^{p, X}$.

Write $\hat{B} = \bigsqcup_{i=1}^h \tilde{B}^x_{x_i}U^p$. For $i = 1, \cdots, h$, put
\[ \tilde{\Gamma}_i = \{ \gamma \in \tilde{B}^x : \gamma \in (x_i)_t U_t(x_i)^{-1} \text{ for } t \neq p \}. \]
Then $X_{U^p}$ is isomorphic to
\[ \hat{B}^x \backslash (\mathcal{H} \times G(\mathbb{Q}_p))/U_{p,0} \times \hat{B}^{p, X} / U^p \cong \bigsqcup_{i=1}^h \tilde{\Gamma}_i \backslash (\mathcal{H} \times \mathbb{Z}). \]
Here we identify \( \mathbb{Z} \) with \( G(\mathbb{Q}_p)/U_{p,0} \). Note that \( \Gamma_i \) acts transitively on \( G(\mathbb{Q}_p)/U_{p,0} \).

Note that, for every point in \( \mathbb{Z} = G(\mathbb{Q}_p)/U_{p,0} \) it is fixed by \( \gamma \in \mathcal{B}^\infty \) if and only if \( \gamma_p \) is in \( \text{GL}(2, \mathcal{O}_{F_p}) \). Put

\[
\Gamma_{i,0} = \{ \gamma \in \Gamma_i : |\det(\gamma_p)|_p = 1 \} = \{ \gamma \in \mathcal{B}^\infty : \gamma \in (x_i)/U_i(x_i)^{-1} \text{ for } i \neq p \text{ and } |\det(\gamma_p)|_p = 1 \}.
\]

Let \( \Gamma_{i,0} \) be the image of \( \Gamma_{i,0} \) in \( \text{PGL}(2, F_p) \). Then we have an isomorphism

\[
X_{U^p} \cong \bigsqcup_{i=1}^h \Gamma_{i,0} \backslash \mathcal{H}.
\] (7.5)

Applying the constructions in Section 7.1 to each part \( \Gamma_{i,0} \backslash \mathcal{H} \) of \( X_{U^p} \), we obtain operators \( \iota, P \) and \( I \).

8 Automorphic Forms on totally definite quaternion algebras and Teitelbaum type \( L \)-invariants

In this section we recall Chida, Mok and Park’s definition of Teitelbaum type \( L \)-invariant [7].

8.1 Automorphic Forms on totally definite quaternion algebras

We recall the theory of automorphic forms on totally definite quaternion algebras.

Let \( \mathcal{B} \) be as in Section 5, which is a totally definite quaternion algebra over \( F \). Let \( \Sigma = \prod_i \Sigma_i \) be a compact open subgroup of \( \mathcal{B}^\infty \).

Let \( \chi_{F,cyc} : \mathbb{A}_F^\times / F^\times \rightarrow \mathbb{Z}_p^\times \) be the Hecke character obtained by composing the cyclotomic character \( \chi_{\mathbb{Q},cyc} : \mathbb{A}_\mathbb{Q}^\times / \mathbb{Q}^\times \rightarrow \mathbb{Z}_p^\times \) and the norm map from \( \mathbb{A}_F^\times \) to \( \mathbb{A}_\mathbb{Q}^\times \).

**Definition 8.1.** An automorphic form on \( \mathcal{B}^\times \), of weight \( k = (k_1, \cdots, k_g, w) \) and level \( \Sigma \), is a function \( f : \mathcal{B}^\times \rightarrow V(k) \) that satisfies

\[
f(\gamma z u b v) = \chi_{F,cyc}^2(z) \chi_{F,cyc}(u_p^{-1}) \cdot f(b)
\]

for all \( \gamma \in \mathcal{B}^\times \), \( u \in \Sigma \), \( b \in \mathbb{B}^\times \), and \( z \in \mathbb{F}^\times \). Denote by \( S^B_k(\Sigma) \) the space of such forms. Remark that our \( S^B_k(\Sigma) \) coincides with \( S^B_{k',v}(\Sigma) \) for \( k' = (k_1 - 2, \cdots, k_g - 2) \) and \( v = (w - k_1, \cdots, w - k_g) \) in [7].

Observe that a form \( f \) of level \( \Sigma \) is determined by its values on the finite set \( \mathcal{B}^\times \backslash \mathcal{B}^\times / \Sigma \). As in Section 7.2 write \( \mathcal{B}^\times = \bigsqcup_{i=1}^h \mathcal{B}^\times x_i \text{GL}(2, F_p) \Sigma_i \); for \( i = 1, \cdots, h \), put

\[
\Gamma_i = \{ \gamma \in \mathcal{B}^\times : \gamma_i \in (x_i)/(\Sigma_i(x_i)^{-1} \text{ for } i \neq p \}.
\]

Then we have a bijection

\[
\bigsqcup_{i=1}^h \Gamma_i \backslash \text{GL}(2, F_p)/\Sigma_p \cong \mathcal{B}^\times \backslash \mathcal{B}^\times / \Sigma.
\]
The class of \( g \) in \( \widetilde{\Gamma}_i / \text{GL}(2,F_p) / \Sigma_p \) corresponds to the class of \( x_{i,p}g \) in \( \widetilde{B}/\widetilde{B}/\Sigma \), where \( g \) is the element of \( \widetilde{B} \) that is equal to \( g \) at the place \( p \), and equal to the identity at each other place. Using this we can attach to an automorphic form \( f \) of weight \( k \) and level \( \Sigma \) an \( h \)-tuple of functions \( (f_1, \cdots , f_h) \) on \( \text{GL}(2,F_p) \) with values in \( V(k) \) defined by \( f_i(g) = f(x_{i,p}g) \). The function \( f_i \) satisfies
\[
f_i(\gamma g uz) = \chi_{F,iyc}^2(z)u^{-1}.f_i(g)
\]
for all \( \gamma \in \widetilde{\Gamma}_i, g \in \text{GL}(2,F_p), u \in \Sigma_p \) and \( z \in F_p^\times \).

For each prime \( l \) of \( F \) such that \( B \) splits at \( l \), \( l \neq p \), and \( \Sigma \) is maximal, one define a Hecke operator \( T_l \) on \( S^B_k(\Sigma) \) as follows. Fix an isomorphism \( \iota_l : B_l \rightarrow M_2(F_l) \) such that \( \Sigma \) becomes identified with \( \text{GL}_2(\sigma_{F_l}) \). Let \( \pi_l \) be a uniformizer of \( \sigma_{F_l} \). Given a double coset decomposition
\[
\text{GL}_2(\sigma_{F_l}) = \bigsqcup b_i \text{GL}_2(\sigma_{F_l})
\]
we define the Hecke operator \( T_l \) on \( S^B_k(\Sigma) \) by
\[
(T_l f)(b) = \sum_i f(bb_i).
\]
We define \( U_p \) similarly. Let \( T_{1} \) be the Hecke algebra generated by \( U_p \) and these \( T_l \).

Denote by \( \sigma_{F_l}^p \) the ring of \( p \)-integers of \( F \) and \( (\sigma_{F_l}^p)^\times \) the group of \( p \)-units of \( F \). We have \( \widetilde{\Gamma}_i \cap F^\times = (\sigma_{F_l}^p)^\times \). For \( i = 1, \cdots , h \), put \( \Gamma_i = \widetilde{\Gamma}_i/(\sigma_{F_l}^p)^\times \). Consider the following twisted action of \( \Gamma \) on \( V(k) \):
\[
\gamma \ast v = |\text{Nrd}_{B/F}\gamma|^{\frac{1}{p-2}} \gamma_p \ast v.
\]
Then \( (\sigma_{F_l}^p)^\times \) is trivial on \( V(k) \), so we may consider \( V(k) \) as a \( \Gamma \)-module via the above twisted action.

8.2 \textbf{Teitelbaum type} \textbf{L}-\textbf{invariants}

Chida, Mok and Park \cite{7} defined Teitelbaum type \textbf{L}-\textbf{invariant} for automorphic forms \( f \in S^B_k(\Sigma) \) satisfying the condition \textbf{(CMP)} given in the introduction:
\[
f \text{ is new at } p \text{ and } U_p f = \mathcal{N} p^{w/2} f.
\]
We recall their construction below.

We attach to each \( f_i \) a \( \Gamma_i \)-invariant \( V(k) \)-valued cocycle \( c_{f_i} \), where \( \Gamma_i \) acts on \( V(k) \) via \( \ast \). For \( e = (s,t) \in E(T) \), represent \( s \) and \( t \) by lattices \( L_s \) and \( L_t \) such that \( L_s \) contains \( L_t \) with index \( \mathcal{N} p \).

Let \( g_e \in \text{GL}(2,F_p) \) be such that \( g_e(\sigma_{F_p}^2) = L_s \). Then we define \( c_{f_i}(e) := |\text{det}(g)|^{\frac{1}{p-2}} g_e \ast f_i(g_e) \).

If \( f \) satisfies \textbf{(CMP)}, then \( c_{f_i} \) is in \( C^*_h(V(k) / \Gamma_i) \) \cite[Proposition 2.7]{7}. Thus we obtain a vector of harmonic cocycles \( c_T = (c_{f_1}, \cdots , c_{f_h}) \).

For each \( c \in C^*_h(V(k) / \Gamma_i) \), we define \( \kappa^{\text{sch}}_c \) to be the following function on \( \Gamma_i \) with values in \( V(k) \): fixing some \( v \in V(T) \), for each \( \gamma \in \Gamma \), we put
\[
\kappa^{\text{sch}}_c(\gamma) := \sum_{c: v \rightarrow \gamma v} c(e)
\]
25
where \( e \) runs over the edges in the geodesic joining \( v \) and \( \gamma v \). As \( c \) is \( \Gamma_i \)-invariant, \( \kappa_c^{\text{sch}} \) is a 1-cocycle. Furthermore the class of \( \kappa_c^{\text{sch}} \) in \( H^1(\Gamma_i, V(k)) \) is independent of the choice of \( v \). Hence we obtain a map

\[
\kappa_c^{\text{sch}} : \bigoplus_{i=1}^h C^1_{\text{har}}(V(k))^{\Gamma_i} \to \bigoplus_{i=1}^h H^1(\Gamma_i, V(k)).
\]

By [7, Proposition 2.9] \( \kappa_c^{\text{sch}} \) is an isomorphism.

For each \( \sigma : F_p \to L_{\mathfrak{F}} \), let \( L_{\mathfrak{F},\sigma}(k,v) \) be the dual of \( V_{\sigma}(k,v) \) with the right action of \( \text{GL}(2,F_p) \): if \( g \in \text{GL}(2,F_p) \), \( P' \in L_{\mathfrak{F},\sigma}(k,v) \) and \( P \in V_{\sigma}(k,v) \), then \( (P', g \cdot P) = (P'|_\sigma, P) \). We realize \( L_{\mathfrak{F},\sigma}(k,v) \) by the same space as \( V_{\sigma}(k,v) \), with the right \( \text{GL}(2,F_p) \)-action

\[
\langle X^j Y^{k-2-j}, X^j Y^{k-2-j'} \rangle = \begin{cases} 1 & \text{if } j = j' \\ 0 & \text{if } j \neq j' \end{cases}
\]

and the right \( \text{GL}(2,F_p) \)-action

\[
P|_{\begin{bmatrix} a & b \\ c & d \end{bmatrix}} = P(\sigma(a)X_\sigma + \sigma(b)Y_\sigma, \sigma(c)X_\sigma + \sigma(d)X_\sigma).
\]

Put

\[
L_{\mathfrak{F}}(k)^\tau := \bigotimes_{\sigma \neq \tau} L_{\mathfrak{F},\sigma}(k_{\sigma}, (w - k_{\sigma})/2)
\]

with the right action of \( \text{GL}(2,F_p) \), where the tensor product is taken over \( L_{\mathfrak{F}} \).

For each harmonic cocycle \( c \in C^1_{\text{har}}(V(k))^{\Gamma_i} \), the method of Amice-Velu and Vishik allows one to define the \( V(k)^\tau \)-valued rigid analytic distribution \( \mu_c^\tau \) on \( P^1(F_p) \) such that the value of \( \int_{U_\mathfrak{F}} t^j \mu_c^\tau(t) \in V(k)^\tau \) (0 \( \leq j \leq k_\tau - 2 \)) satisfies

\[
\langle Q, \int_{U_\mathfrak{F}} t^j \mu_c^\tau(t) \rangle = \frac{(X^j Y^{k_\tau-2-j} \otimes Q, c(e))}{(k_\tau - 2)}
\]

for each \( Q \in L_{\mathfrak{F}}(k)^\tau \).  \(^4\)

Using \( \mu_c^\tau \) we obtain a \( V(k)^\tau \)-valued rigid analytic function \( g_c^\tau \), precisely a global section of \( V(k)^\tau \otimes_{\tau,F_p} \mathcal{O}_{\mathcal{H},F_{\mathfrak{F}}^\tau} \) by

\[
g_c^\tau(z) = \int_{P^1(F_p)} \frac{1}{z-t} \mu_c^\tau(t)
\]

for \( z \in \mathcal{H}_{F_{\mathfrak{F}}^\tau} \). The function \( g_c^\tau \) satisfies the transformation property: for \( \gamma \in \Gamma_i \), let \( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \) be the image of \( \gamma \) in \( B_p \cong \text{GL}(2,F_p) \); then

\[
g_c^\tau(\gamma \cdot z) = |\text{Nrd}_{B/F}(\gamma)|^{\frac{k_\tau - 2}{2}} (\text{Nrd}_{B/F}(\gamma))^{\frac{-2 - b}{2}} (cz + d)^{k_\tau} \gamma \cdot g_c^\tau(z).
\]

Consider \( V(k) \) as an \( F_p \)-module via \( \tau : F_p \hookrightarrow L_{\mathfrak{F}} \). We define the \( V(k) \)-valued cocycle \( \lambda^\tau_c \) as follows. Fix a point \( z_0 \in \mathcal{H} \). For each \( \gamma \in \Gamma_i \) the value \( \lambda^\tau_c(\gamma)(z) \) is given by the formula: for \( Q \in L_{\mathfrak{F}}(k)^\tau \),

\[
\langle X^j Y^{k_\tau-2-j} \otimes Q, \lambda^\tau_c(\gamma)(z) \rangle = \left( k_\tau - 2 \right) \int_{z_0}^{z_0} z^j g_c^\tau(z)dz.
\]

\(^4\)There are minors in the definitions of \( \mu_c^\tau \) and \( \lambda^\tau_c \) in [7]. See [36] Definition 6 and the paragraph before Proposition 9.
\[0 \leq j \leq k_r - 2,\] where the integral is the branch of Coleman’s integral chosen in Section 7. Then \( \lambda^c_r \) is a 1-cocycle on \( \Gamma_i \), and the class of \( \lambda^c_r \) in \( H^1(\Gamma_i, V(k)) \), denoted by \([\lambda^c_r]\), is independent of the choice of \( z_0 \). This defines a map

\[
\kappa^\text{col,}\tau : \bigoplus_{i=1}^h \mathcal{C}^1_{\text{har}}(T, V(k))^\Gamma_i \rightarrow \bigoplus_{i=1}^h H^1(\Gamma_i, V(k)), \quad (c_i)_i \mapsto ([\lambda^c_r])_i.
\]

As \( \kappa^\text{sch} \) is an isomorphism, for each \( \tau \) there exists a unique \( \ell_{\tau} \in L_{\mathbb{Q}} \) such that

\[
\kappa^\text{col,}\tau(c_{\ell}) = \ell_{\tau} \kappa^\text{sch}(c_{\ell}).
\]

The Teitelbaum type \( L \)-invariant of \( f \), denoted by \( \mathcal{L}_T(f) \), is defined to be the vector \( (\ell_{\tau})_{\tau} \) [7, Section 3.2]. We also write \( \mathcal{L}_{T,\tau}(f) \) for \( \ell_{\tau} \).

### 9 Comparing \( L \)-invariants

Let \( B, \tilde{B}, G \) and \( \tilde{G} \) be as before. Let \( n^- \) be the conductor of \( \tilde{B} \). By our assumption on \( \tilde{B} \), \( \mathfrak{p} \nmid n^- \) and the conductor of \( B \) is \( \mathfrak{p} n^- \). Let \( n^+ \) be an ideal of \( \mathcal{O}_F \) that is prime to \( \mathfrak{p} n^- \), and put \( n := \mathfrak{p} n^+ n^- \).

For any prime ideal \( l \) of \( \mathcal{O}_F \), put

\[
\tilde{R}_l := \begin{cases} 
\text{an maximal compact open subgroup of } \tilde{B}^\times \quad &\text{if } l \text{ is prime to } n, \\
\text{the maximal compact open subgroup of } \tilde{B}^\times \quad &\text{if } l \text{ divides } n^- , \\
1 + \text{ an Eichler order of } \tilde{B} \text{ of level } l^{\text{val}(n^-)} &\text{if } l \text{ divides } \mathfrak{p} n^+.
\end{cases}
\]

Let \( \tilde{\Sigma} = \Sigma(n^+, n^-) \) be the level \( \prod_l \tilde{R}_l \). We write \( S^B_k(n^+, n^-) \) for \( S^B_k(\Sigma(n^+, n^-)) \). Similarly we define \( \Sigma = \Sigma(n^+, \mathfrak{p} n^-) \), a compact open subgroup of \( G(\mathbb{A}_f) \). Let \( S^B_k(n^+, \mathfrak{p} n^-) \) be the space of modular forms on the Shimura curve \( M \) of weight \( k \) and level \( \Sigma \).

Let \( k = (k_1, \cdots, k_g, \omega) \) be a multiweight such that \( k_1 \equiv \cdots k_g \equiv \omega \mod 2 \) and \( k_1, \cdots, k_g \) are all even and larger than 2. Let \( f_\infty \) be a (Hilbert) eigen cusp newform of weight \( k \) and level \( n \) that is new at \( \mathfrak{p} n^- \). Let \( f \in S^B_k(n^+, n^-) \) (resp. \( f_B \in S^B_k(n^+, \mathfrak{p} n^-) \)) be an eigen newform corresponding to \( f_\infty \) by the Jacquet-Langlands correspondence; \( f \) (resp. \( f_B \)) is unique up to scalars.

We further assume that \( f \) satisfies (CMP), so that we can attach to \( f \) the Teitelbaum type \( L \)-invariant \( \mathcal{L}_T(f) \). We define \( \mathcal{L}_T(f_\infty) \) to be \( \mathcal{L}_T(f) \). The goal of this section is to compare \( \mathcal{L}_{F,L}(f_\infty) \) and \( \mathcal{L}_T(f_\infty) \).

Let \( L \) be a (sufficiently large) finite extension of \( F \) that splits \( B \) and contains all Hecke eigenvalues acting on \( f_\infty \). Let \( \lambda \) be an arbitrary place of \( L \).

**Lemma 9.1.** [28, Lemma 3.1] There is an isomorphism

\[
H^1_{\text{et}}(M, F(k)_\lambda) \simeq \bigoplus_{\lambda'} \pi^{\infty}_{\lambda'}(L(f')) \otimes L(f') \bigotimes_{\lambda' \mid \lambda} (\rho_{f', \lambda'})
\]

of representations of \( G(\mathbb{A}_f) \times \text{Gal}(\overline{F}/F) \) over \( L_\lambda \). Here \( f' \) runs through the conjugacy classes over \( L \), up to scalars, of eigen newforms of multiweight \( k \) that are new at primes dividing \( \mathfrak{p} n^- \). The extension of \( L \) generated by the Hecke eigenvalues acting on \( f' \) is denoted by \( L(f') \), and \( \lambda' \) runs through places of \( L(f') \) above \( \lambda \).
By the strong multiplicity one theorem (cf. [26]) there exists a primitive idempotent $e_f \in T_S$ such that $e_f \mathbb{T}_S = L e_f$ and $e_f \cdot S_{0}^\Sigma (\mathbb{P}^n, n^{-}) = 1$. Lemma 9.1 tells us that $e_f \cdot H^1_{et}(\mathcal{M}^{(\mathcal{F}^d)}_{\Sigma}, \mathcal{F}(k), \lambda^\Sigma)$ is exactly $\rho_{f_B, \lambda}$, the $\lambda$-adic representation of $\text{Gal}(\overline{\mathbb{F}}_p/F)$ attached to $f_B$. By Carayol’s construction of $\rho_{f_{\infty, \lambda}}$ [6] $\rho_{f_{\infty, \lambda}}$ coincides with $\rho_{f_{B, \lambda}}$.

Now we take $\lambda$ to be a place above $\mathfrak{p}$, denoted by $\mathfrak{q}$.

Recall that in Section 7.2 and Section 8 we associate to $\Sigma$ the groups $\tilde{\Gamma}_{i, 0}, \tilde{\Gamma}_i, \Gamma_0, \Gamma_i$ ($i = 1, \ldots, h$). By (7.5) $X_\Sigma$ is isomorphic to $\prod_{i} X_{\Gamma_{i, 0}}$, where $X_{\Gamma_{i, 0}} = \Gamma_{i, 0} \setminus \overline{\mathfrak{F}_{\mathfrak{p}}^\Sigma}$.

**Theorem 9.2.** Let $f_B$ be as above. Then $\rho_{f_B, \mathfrak{q}, p}$ is a semistable (non-crystalline) representation of $\text{Gal}(\overline{\mathbb{F}}_p/F)$, and the filtered $(\varphi_q, N)$-module $D_{st, F_p}(\rho_{f_B, \mathfrak{q}, p})$ is a monodromy $L_{\mathfrak{q}, p}$-module.

**Proof.** To show that $\rho_{f_B, \mathfrak{q}, p}$ is semistable, we only need to prove that $H^1_{et}(\mathcal{X}_\Sigma, \mathcal{F}(k))$ is semistable, since $\rho_{f_B, \mathfrak{q}, p}$ is a subrepresentation of $H^1_{et}(\mathcal{X}_\Sigma, \mathcal{F}(k))$. But this follows from Proposition 2.1 and the fact that $X_\Sigma$ is semistable.

Next we prove that $D_{st, F_p}(\rho_{f_B, \mathfrak{q}, p})$ is a monodromy $L_{\mathfrak{q}, p}$-module. We only need to consider $D_{st, \overline{\mathbb{F}}_{\mathfrak{p}}}(\rho_{f_B, \mathfrak{q}, p})$ instead.

Twisting $f_B$ by a central character we may assume that $w = 2$.

Being a Shimura variety, $N_E$ is a family of varieties. But in the following we will denote $E_N$ to denote any one in this family that corresponds to a level subgroup whose $p$-factor is $\mathcal{O}_E^\times$. By the proof of Lemma 6.4, any geometric point of $(N_E)_{\overline{\mathbb{F}}_{\mathfrak{p}}}$ is defined over $\overline{\mathbb{F}}_{\mathfrak{p}}$. In other words, $(N_E)_{\overline{\mathbb{F}}_{\mathfrak{p}}}$ is the product of several copies of $\text{Spec}(\overline{\mathbb{F}}_{\mathfrak{p}})$.

Let $\text{pr}_1$ be the projection $X_\Sigma \times (N_E)_{\overline{\mathbb{F}}_{\mathfrak{p}}} \to X_\Sigma$. By Corollary 6.5 the filtered $\varphi_q$-isocrystal attached to $\text{pr}_1^* \mathcal{F}(k)$ is $\text{pr}_1^* \mathcal{F}(k)$. Note that

$$H^1_{et}(\mathcal{X}_\Sigma \times (N_E)_{\overline{\mathbb{F}}_{\mathfrak{p}}}, \text{pr}_1^* \mathcal{F}(k)) = H^0_{et}(\mathcal{N}_E_{\overline{\mathbb{F}}_{\mathfrak{p}}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} H^1_{et}(\mathcal{X}_\Sigma_{\overline{\mathbb{F}}_{\mathfrak{p}}}, \mathcal{F}(k)).$$

The $\text{Gal}(\overline{\mathbb{F}}_p/F_{\mathfrak{p}})$-representation $H^0_{et}(\mathcal{N}_E_{\overline{\mathbb{F}}_{\mathfrak{p}}}, \mathbb{Q}_p)$ is crystalline and the associated filtered $\varphi_q$-module is $H^0_{\text{dR}}(\mathcal{N}_E_{\overline{\mathbb{F}}_{\mathfrak{p}}}, \mathbb{Q}_p)$ with trivial filtration. Let $H^0$ denote this filtered $\varphi_q$-module for simplicity. As a consequence, we have an isomorphism of filtered $(\varphi_q, N)$-modules

$$H^0 \otimes_{\mathbb{Q}_p} D_{st, \overline{\mathbb{F}}_{\mathfrak{p}}}(H^1_{et}(\mathcal{X}_\Sigma_{\overline{\mathbb{F}}_{\mathfrak{p}}}, \mathcal{F}(k))) = H^0 \otimes_{\mathbb{Q}_p} H^1_{\text{dR}}(X_\Sigma, \mathcal{F}(k)).$$

Using the decomposition (6.3), for each embedding $\tau : F_p \hookrightarrow L_\mathfrak{q}$ we put

$$H^1_{\text{dR}, \tau}(X_\Sigma, \mathcal{F}(k)) := \text{H}^1(X_\Sigma, V(k) \otimes_{\tau, F_p} \mathcal{O}_{X_\Sigma}^\times).$$

In Section 8.2 we attached to $\mathfrak{f} = (f_1, \cdots, f_h)$ an $h$-tuple $g^\tau = (g^1_\tau, \cdots, g^h_\tau)$. Let $M_\tau(\mathfrak{f})$ denote the $L_{\mathfrak{q}}$-subspace of $\bigoplus_{\tau} H^1_{\text{dR}, \tau}(X_{\Gamma_{i, 0}}, \mathcal{F}(k))$ generated by the element

$$\omega_\mathfrak{f}^\tau = (g^i_\tau(\tau z)(z \tau X + Y_\tau)^{k_\tau-2}dz)_{1 \leq i \leq h}.$$ (Note that, when $w = 2$, the twisted action $\ast$ in Section 8 coincides with the original action.)

Therefore, we have

$$e_\mathfrak{f} \cdot \text{Fil}^{w+2}_0 H^1_{\text{dR}, \tau}(X_\Sigma, \mathcal{F}(k)) \supseteq M_\tau(\mathfrak{f}).$$

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Consider the pairing $<\cdot,\cdot>$ on $V(k)$ defined by

$$<Q_1,Q_2> = \text{the coefficient of } \prod_\sigma (X_\sigma Y_\sigma)^{k_\sigma - 2} \text{ in } Q_1Q_2.$$  

It is perfect and induces a perfect pairing on $H^1_{dR,\tau}(X_\Sigma, F(k))$. With respect to this pairing $\Fil^{k_\tau + 1} H^1_{dR,\tau}(X_\Sigma, F(k))$ is orthogonal to $\Fil^{k_\tau - 2} H^1_{dR,\tau}(X_\Sigma, F(k))$. As $e_\tau \cdot H^1_{dR,\tau}(X_\Sigma, F(k))$ is of rank 2 over $L_\Psi$, we obtain

$$e_\tau \cdot \Fil^{k_\tau + 1} H^1_{dR,\tau}(X_\Sigma, F(k)) = e_\tau \cdot \Fil^{k_\tau - 2} H^1_{dR,\tau}(X_\Sigma, F(k)) = M_\tau(f).$$

Thus

$$H^0 \otimes_{Q_\tau} \rho_\tau^* M_\tau(f) = H^0 \otimes_{Q_\tau} \Fil^{\min \{ k_\tau, -1 \}} D_{st,F_p} (\rho_{f_\tau,\varphi_\tau}) \tau.$$  

Let $\iota^\tau$ and $I^\tau$ be the operators attached to the sheaf $\mathcal{F}(k)$ over $X_\Sigma$ (see Section 7.2). Here, the superscript $\tau$ is used to emphasize the embedding $\tau : F_p \hookrightarrow L_\Psi$. Proposition 2.2 tells us that the monodromy $N$ on $H^1_{dR,\tau}(X_\Sigma, F(k))$ coincides with $\iota^\tau \circ I^\tau$. By Proposition 7.1 the kernel of $N$ is

$$\iota^\tau \circ \delta^{-1} \left( \bigoplus_i H^1(T_{i,0}, V(k)) \right).$$

So, by (7.4) the restriction of $N$ to $M_\tau(f)$ is injective. Hence,

$$\ker(N) \cap \Fil^{\min \{ k_\tau, -1 \}} D_{st,F_p} (\rho_{f_\tau,\varphi_\tau}) \tau = 0,$$

as desired.  

Let $P^\tau$ be the operator attached to $\mathcal{F}(k)$ (see Section 7.2).

**Lemma 9.3.** Let $\omega^\tau_\rho$ be as in the proof of Theorem 9.2. Then

$$P^\tau(\omega^\tau_\rho) = \kappa^{\text{col},\tau}(cf), \quad I^\tau(\omega^\tau_\rho) = cf.$$  

**Proof.** The first formula comes from the definitions.

The proof of the second formula is similar to that of [36, Theorem 3]. Let $\mu^\tau_i$ ($i = 1, \ldots, h$) be the rigid analytic distributions on $P^1(F_p)$ coming from $\rho_\tau$ (see Section 8). Recall that

$$g^\tau_i(z) = \int_{P^1(F_p)} \frac{1}{z-t} \mu^\tau_i(t).$$

For each edge $e$ of $\mathcal{T}$ let $B(e)$ be the affinoid open disc in $P^1(\mathbb{C}_p)$ that corresponds to $e$. Assume that $B(e)$ meets the limits set $P^1(F_p)$ in a compact open subset $U(e)$. Put

$$g^\tau_{i,e}(z) = \int_{U(e)} \frac{1}{z-t} \mu^\tau_i(t).$$

Let $a(e)$ be a point in $U(e)$. Expanding $\frac{1}{z-t}$ at $a(e)$ we obtain that

$$g^\tau_{i,e}(z) = \sum_{n=0}^{+\infty} \frac{1}{n+1} (z-a(e))^n \int_{U(e)} (t-a(e))^n \mu^\tau_i(t),$$

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and thus \( g_{i,e}^\tau(z) \) converges on the complement of \( B(e) \). Note that \( g_{i,e}^\tau - g_{i,e}^- \) is analytic on \( B(e) \). So, we have

\[
I^\tau(g_{i,e}^\tau(zX_\tau + Y_\tau)^{k_\tau - 2}dz)(e) = \text{Res}_e\left(g_{i,e}^\tau(zX_\tau + Y_\tau)^{k_\tau - 2}dz\right) = \text{Res}_e\left(g_{i,e}^-\left(zX_\tau + Y_\tau\right)^{k_\tau - 2}dz\right)
\]

\[
= \text{Res}_e\left(\int_{U(e)} \frac{(zX_\tau + Y_\tau)^{k_\tau - 2}}{z - t} \mu_i^\tau(t)\right) = \int_{U(e)} (tX_\tau + Y_\tau)^{k_\tau - 2} \mu_i^\tau(t) = c_i^\tau(e),
\]

where the fourth equality follows from the fact that \( \text{Res}_e \) commutes with \( \int_{U(e)} \cdot \mu_i^\tau(t) \).

\[\square\]

**Theorem 9.4.** Let \( f_\infty \) be as above. Then \( L_{FM}(f_\infty) = L_T(f_\infty) \).

**Proof.** Twisting \( f_\infty \) by a central character we may assume that \( w = 2 \).

Put \( D_\tau = H^0 \otimes_{\mathbb{Q}_p} eT_{dR,\tau}(X_{\Sigma}, \mathcal{F}(k)) \). Note that \( N = \iota^\tau \circ I^\tau \). As the kernel of \( N \) coincides with the image of \( \iota^\tau \circ \delta^{-1} \) and \( P^\tau \) splits \( \iota^\tau \circ \delta^{-1} \), we have \( D_\tau = \ker(N) \oplus \ker(P^\tau) \). Write \( \omega_\tau^x = x + y \) according to this decomposition. Then

\[\iota^\tau \circ \delta^{-1} \circ P^\tau(\omega_\tau^x) = x.\]  

(9.2)

By the proof of Theorem 9.2, \( y \) is non-zero and so \( N(y) \neq 0 \).

By Lemma 9.3 and the definition of Teitelbaum type \( L \)-invariant, \( L_{T,\tau}(f_\infty) \) is characterized by the property

\[\left(P^\tau - L_{T,\tau}(f_\infty)\iota \circ I^\tau\right)\omega_\tau^x = 0,\]  

(9.3)

where \( \iota \) is the map defined by (7.2) which coincides with \( \kappa^{sch} \). As \( \delta^{-1} \circ \iota = -\text{id} \) and \( \iota^\tau \circ I^\tau = N \), we have

\[\iota^\tau \circ \delta^{-1} \circ \iota \circ I^\tau(\omega_\tau^x) = -N(\omega_\tau^x).\]  

(9.4)

By (9.2), (9.3) and (9.4) we get

\[L_{T,\tau}(f_\infty)N(\omega_\tau^x) + x = 0.\]  

(9.5)

By the definition of Fontaine-Mazur \( L \)-invariant, \( L_{FM,\tau}(f_\infty) \) is characterized by the property

\[y - L_{FM,\tau}(f_\infty)N(y) \in H^0 \otimes_{\mathbb{Q}_p} \text{Fil}^{\frac{n + \min_y(x)}{2} - 2} H_{dR,\tau}^1 (X_{\Sigma}, \mathcal{F}(k)).\]  

(9.6)

Combining (9.5) and (9.6) we obtain

\[
\begin{align*}
(L_{FM,\tau}(f_\infty) - L_{T,\tau}(f_\infty))N(y) &= L_{FM,\tau}(f_\infty)N(y) - L_{T,\tau}(f_\infty)N(\omega_\tau^x) \\
&\in \omega_\tau^x + H^0 \otimes_{\mathbb{Q}_p} \text{Fil}^{\frac{n + \min_y(x)}{2} - 2} H_{dR,\tau}^1 (X_{\Sigma}, \mathcal{F}(k)) \\
&= H^0 \otimes_{\mathbb{Q}_p} \text{Fil}^{\frac{n + \min_y(x)}{2} - 2} H_{dR,\tau}^1 (X_{\Sigma}, \mathcal{F}(k)).
\end{align*}
\]

But \( N(y) \) is in \( \ker(N) \) and is non-zero, and by Theorem 9.2

\[
\ker(N) \cap H^0 \otimes_{\mathbb{Q}_p} \text{Fil}^{\frac{n + \min_y(x)}{2} - 2} H_{dR,\tau}^1 (X_{\Sigma}, \mathcal{F}(k)) = 0.
\]

Therefore

\[L_{FM,\tau}(f_\infty) - L_{T,\tau}(f_\infty) = 0,
\]

as wanted.  

\[\square\]
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