A Generalization of Smillie’s Theorem on Strongly Cooperative Tridiagonal Systems

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Abstract

Smillie (1984) proved an interesting result on the stability of nonlinear, time-invariant, strongly cooperative, and tridiagonal dynamical systems. This result has found many applications in models from various fields including biology, ecology, and chemistry. Smith (1991) has extended Smillie’s result and proved entrainment in the case where the vector field is time-varying and periodic.

We use the theory of linear totally nonnegative differential systems developed by Schwarz (1970) to give a generalization of these two results. This is based on weakening the requirement for strong cooperativity to cooperativity, and adding an additional observability-type condition.

I. INTRODUCTION

For two vectors \( a, b \in \mathbb{R}^n \) let \( a \leq b \) denote that \( a_i \leq b_i \) for all \( i \). The system \( \dot{x} = f(x) \) is called cooperative if for any two initial conditions \( a \leq b \) the corresponding solutions satisfy \( x(t, a) \leq x(t, b) \) for all time \( t \geq 0 \). In other words, the solutions preserve the (partial) ordering \( \leq \) between the initial conditions. The system is strongly cooperative if for any two initial conditions \( a \leq b \), with \( a \neq b \), the corresponding solutions satisfy \( x(t, a) < x(t, b) \) for all time \( t > 0 \).

Monotone dynamical systems satisfy a similar condition but for a more general ordering \( \leq^K \) that is defined using a suitable cone \( K \subseteq \mathbb{R}^n \). Hirsch’s quasi-convergence theorem \cite{Hirsch} shows that monotonicity has far-reaching implications on the asymptotic behavior of the solutions. Roughly speaking, it implies that almost all bounded trajectories converge to an equilibrium. However, monotone systems can contain complicated dynamics such as chaotic invariant sets (although these cannot be attractors) \cite{Smith}.

Stronger results hold for cooperative systems with an additional structure. Let \( \mathbb{M}^+ \) denote the set of \( n \times n \) matrices that are tridiagonal, and with positive entries on the super- and sub-diagonals. In an interesting paper, Smillie \cite{Smillie} considered the time-invariant nonlinear cooperative system:

\[
\dot{x}(t) = f(x(t)),
\]

where \( x(t) \in \mathbb{R}^n \) and \( f : \mathbb{R}^n \to \mathbb{R}^n \) is \( n-1 \) times differentiable. He showed that if the Jacobian \( J(x) := \partial f(x) / \partial x \in \mathbb{M}^+ \) for all \( x \), then every trajectory of \( \dot{x} = f(x) \) either eventually leaves any compact set or converges to an equilibrium point.\(^1\) Smillie’s result has found many applications (see, e.g. \cite{Smith2}, \cite{Smith3}, \cite{Smith4}).

Smillie’s proof is quite interesting and is based on studying the number of sign variations in the vector \( z(t) := \dot{x}(t) \). Recall that for a vector \( y \in \mathbb{R}^n \) with no zero entries the number of sign variations in \( y \) is

\[
\sigma(y) := |\{ i \in \{1, \ldots, n-1\} : y_i y_{i+1} < 0 \}|.
\]

The domain of \( \sigma \) can be extended, via continuity, to the open set \( \mathcal{V} := \{ y \in \mathbb{R}^n : y_1 \neq 0, \ y_n \neq 0, \ \text{and if } y_i = 0 \text{ for some } i \in \{2, \ldots, n-1\} \text{ then } y_{i-1} y_{i+1} < 0 \} \). For example, for \( n = 3 \) the vector \( y := [2 \; \varepsilon \; -2]^T \in \mathcal{V} \) and \( \sigma(y) = 1 \) for all \( \varepsilon \in \mathbb{R} \).

\(^1\)Note that Fiedler and Gedeon \cite{Fiedler} have proved a similar result, but using a very different technique.

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To explain Smillie’s proof, let $z := \dot{x}$. Then (1) yields
\[ \dot{z}(t) = J(x(t))z(t). \] (3)
This linear time-varying system is sometimes referred to as the variational equation.

Smillie showed that if $z(\tau) \not\in V$ for some time $\tau$ then $z(\tau^-), z(\tau^+) \in V$ and
\[ \sigma(z(\tau^+)) < \sigma(z(\tau^-)). \]
In other words, $\sigma(z(t))$ is piecewise constant and whenever it changes its value it can only decrease. Since $\sigma$ takes values in $\{0, \ldots, n-1\}$, this implies that $z(t) \in V$ for all $t \geq 0$ except perhaps for up to $n-1$ discrete points. By the definition of $V$, we conclude that there exists a time $s$ such that $z_1(t) \neq 0$ (and $z_n(t) \neq 0$) for all $t \geq s$. Thus, $z_1(t)$ (and $z_n(t)$) is either eventually positive or eventually negative.

Smillie used this to show that for any $a$ in the state-space of (1) the omega limit set $\omega(a)$ includes no more than a single point. Hence, every trajectory either eventfully leaves any compact set or converges to an equilibrium.

To explain the basic idea underlying Smillie’s analysis of $\sigma(z(t))$, consider the case $n = 3$. Seeking a contradiction, assume that the sign pattern of $z(t)$ near some time $t = \tau$ is as follows:
\[
\begin{array}{ccc}
  t = \tau^- & t = \tau & t = \tau^+ \\
  z_1(t) & + & 0 & - \\
  z_2(t) & + & + & + \\
  z_3(t) & + & + & + \\
\end{array}
\]
Note that in this case $\sigma(t) := \sigma(z(t))$ increases from $\sigma(\tau^-) = 0$ to $\sigma(\tau^+) = 1$. However, using (3) and the structure of the Jacobian yields
\[
\dot{z}(\tau) = \begin{bmatrix}
  * & + & 0 \\
  + & * & + \\
  0 & + & * \\
\end{bmatrix}
\begin{bmatrix}
  0 \\
  + \\
  + \\
\end{bmatrix}
\]
where $+ \ [\ast]$ means a positive [arbitrary] value, and thus $\dot{z}_1(\tau) > 0$, and the case described in the table above cannot take place. Smillie’s analysis shows rigorously that when $\sigma(t)$ changes it can only decrease. This is based on direct analysis of the ODEs and is nontrivial due to the fact that if an entry $z_i(t)$ becomes zero at some time $t = \tau$ (thus perhaps leading to a change in $\sigma(t)$ near $\tau$) one must consider the possibility that higher-order derivatives of $z_i(t)$ are also zero at $t = \tau$.

Smith [22] has extended Smillie’s result to time-varying, $T$-periodic cooperative systems under similar assumptions on the structure of the Jacobian. He showed that every trajectory eventually leaves any compact set or converges to a periodic trajectory with the same period $T$. This entrainment property is important in many natural and artificial systems. For example, biological organisms are often exposed to periodic excitations like the 24h solar day, and the periodic cell division process. Proper functioning often requires entrainment to such excitations [17], [13]. Epidemics of infectious diseases often correlate with seasonal changes and the required interventions, such as pulse vaccination, may also need to be periodic [8]. Traffic flow is often controlled by periodically-varying traffic lights, and in this context entrainment means that the traffic flow converges to a periodic pattern with the same period as the traffic lights [12].

It has been recently shown [15] that the sign variation diminishing property (SVDP) of $\dot{z}(t)$ underlying these results follows as a special case from the seminal (yet largely forgotten) work of Schwarz [18] on totally positive differential systems (TPDSs). These are linear time-varying systems whose transition matrix is totally positive (TP) for all time. To explain this, recall that a matrix $A \in \mathbb{R}^{n \times n}$ is called totally nonnegative (TN) if all its minors are nonnegative, and totally positive (TP) if all its minors are positive. Such matrices have a rich and beautiful theory [4], [16]. In particular, they satisfy powerful SVDPs: multiplying a vector by a TN matrix cannot increase the number of sign variations in the vector.
Schwarz [18] studied the following problem. Fix a time interval \((a, b)\) with \(-\infty \leq a < b \leq \infty\). Let \(A(t)\) be a continuous matrix function on \((a, b)\), and consider the linear matrix differential equation \(\dot{\Phi} = A\Phi\), with \(\Phi(t_0) = I\). When will \(\Phi(t)\) be TN [TP] for every pair \((t_0, t)\) with \(a < t_0 < t < b\)? A system that satisfies this property is called a totally nonnegative [positive] differential system. Schwarz also described the implications of TNDS [TPDS] on the sign variations of the vector solution \(z(t)\) of \(\dot{z} = Az\).

As shown in [15], these results are closely related to the theorems of Smillie, Smith, and others. Indeed, the assumptions of Smillie on the Jacobian imply that (3) is TPDS. However, the work of Schwarz has been largely forgotten and its implications to the analysis of nonlinear dynamical systems have been overlooked.

Here we generalize one of the results of Schwarz on TNDSs, and then use this to generalize the results of Smillie and Smith. Roughly speaking, this generalization is based on weakening the requirement for a triadogonal Jacobian with positive entries on the super- and sub-diagonal to requiring a tridiagonal Jacobian with nonnegative entries on the super- and sub-diagonals, and adding a suitable observability-type condition.

The next section briefly reviews totally positive and totally nonnegative linear differential systems and their properties. Section III describes our main results, and the final section concludes.

II. PRELIMINARIES

We begin with reviewing definitions and results from the theory of totally nonnegative and totally positive matrices that will be used later on. We consider only square and real matrices, as this is the case that is relevant for our applications. For more information and proofs see the monographs [4], [16] and the survey paper [1]. Unfortunately, this field suffers from nonuniform terminology. We follow the more modern terminology used in [4].

**Definition 1.** A matrix \(A \in \mathbb{R}^{n \times n}\) is called totally nonnegative [totally positive] if the determinant of every square submatrix is nonnegative [positive].

In particular, if \(A\) is TN [TP] then every entry of \(A\) is nonnegative [positive].

Some matrices with a special structure are known to be TN. We review two such examples. The first is important in proving the SVD of TN matrices. The second example is closely related to Smillie's results.

**Example 1.** Let \(E_{i,j} \in \mathbb{R}^{n \times n}\) denote the matrix with all entries zero, except for entry \((i, j)\) that is one. For \(p \in \mathbb{R}\) and \(i \in \{2, \ldots, n\}\), let

\[
L_i(p) := I + pE_{i,i-1}, \\
U_i(p) := I + pE_{i-1,i}.
\]

Matrices in this form are called elementary bidiagonal (EB) matrices. If the identity matrix \(I\) in (4) is replaced by a diagonal matrix \(D\) then the matrices are called generalized elementary bidiagonal (GEB). It is straightforward to see that EB matrices are TN when \(p \geq 0\), and that GEB matrices are TN when \(p \geq 0\) and the diagonal matrix \(D\) is componentwise nonnegative.

**Example 2.** Consider the tridiagonal matrix

\[
A = \begin{bmatrix}
a_1 & b_1 & 0 & \ldots & 0 \\
c_1 & a_2 & \ddots & \ldots & \vdots \\
0 & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & b_{n-1} & \vdots \\
0 & \ldots & \ldots & c_{n-1} & a_n
\end{bmatrix}
\]
where $b_i, c_i \geq 0$ for all $i$. In this case, the dominance condition
\[ a_i \geq b_i + c_{i-1} \quad \text{for all } i \in \{1, \ldots, n\}, \tag{6} \]
with $c_0 := 0$ and $b_n := 0$, guarantees that $A$ is TN [4, Ch. 0].

An important subclass of TN matrices that are “close” to TP matrices are the oscillatory matrices studied in the pioneering work of Gantmacher and Krein [7]. A matrix $A \in \mathbb{R}^{n \times n}$ is called oscillatory if $A$ is TN and there exists an integer $k > 0$ such that $A^k$ is TP. It is well-known that a TN matrix $A$ is oscillatory if and only if it is non-singular and irreducible [4, Ch. 2], and that in this case $A^{n-1}$ is TP.

**Example 3.** Consider the matrix $A = \begin{bmatrix} 1 & \varepsilon & 0 \\ \varepsilon & 1 & \varepsilon \\ 0 & \varepsilon & 1 \end{bmatrix}$, with $\varepsilon \in (0, 1/2)$. This matrix is non-singular (as $\det(A) = 1 - 2\varepsilon^2$), TN (by the result in Example [2]), and irreducible, so it is an oscillatory matrix.

Here $A^{n-1} = A^2 = \begin{bmatrix} 1 + \varepsilon^2 & 2\varepsilon & \varepsilon^2 \\ 2\varepsilon & 1 + 2\varepsilon^2 & 2\varepsilon \\ \varepsilon^2 & 2\varepsilon & 1 + \varepsilon^2 \end{bmatrix}$, and it is straightforward to verify that this matrix is indeed TP.

More generally, the matrix $A$ in (5) with $b_i, c_i > 0$ and the dominance condition (6) is TN and irreducible. If it is also non-singular then it is oscillatory.

An important property, that will be used throughout, is that the product of two TN [TP] matrices is a TN [TP] matrix. This follows immediately from Definition [1] and the Cauchy-Binet formula for the minors of the product of two matrices [11, Ch. 0].

When using TN matrices to study dynamical systems, it is important to bear in mind that in general coordinate transformations do not preserve TN. An important exception, however, is positive diagonal scaling: if $D$ is a diagonal matrix with positive entries on the diagonal then multiplying a matrix $A$ by $D$ either on the left or right changes the sign of no minor, so in particular $DAD^{-1}$ is TN [TP] if and only if $A$ is TN [TP].

**A. Sign variation diminishing property**

As noted above $\sigma(y)$ is not well-defined for all $y$. We recall two definitions of the number of sign variations that are well-defined for all $y \in \mathbb{R}^n$. Let $s^-(y)$ denote the number of sign variations in the vector $y$ after deleting all its zero entries, and let $s^+(y)$ denote the maximal possible number of sign variations in $y$ after each zero entry is replaced by either +1 or −1. Note that $s^-(y) \leq s^+(y)$ for all $y \in \mathbb{R}^n$.

For example, for $y = [2 \ 0 \ 1 \ -2 \ 0 \ 2.3]'$, $s^-(y) = 2$ and $s^+(y) = 4$. Let $\mathcal{W} := \{y \in \mathbb{R}^n : s^-(y) = s^+(y)\}$. Note that if $y \in \mathcal{W}$ then $y$ cannot have two adjacent zero coordinates. An immediate yet important observation is that $\mathcal{W} = \mathcal{V}$.

Let $A$ be a TN EB matrix. Pick $x \in \mathbb{R}^n$, and let $y := Ax$. Then there exists at most one index $i$ such that $\sgn(y_i) \neq \sgn(x_i)$, and either $y_i = x_i + px_{i-1}$ or $y_i = x_i + px_{i+1}$, and since $p \geq 0$ the sign can change only in the “direction” of $x_{i-1}$ or $x_{i+1}$. In either case, neither $s^-$ or $s^+$ may increase. We conclude that if $A$ is TN EB then $s^-(Ax) \leq s^-(x)$, and
\[ s^+(Ax) \leq s^+(x). \tag{7} \]

A similar argument shows that if $A$ is TN GEB then $s^-(Ax) \leq s^-(x)$. However, (7) does not hold in general for a TN GEB matrix $A$. For example, $A = 0$ is TN GEB and clearly $s^+(Ax) = s^+(0) = n - 1$ may be larger than $s^+(x)$.

This SVDP can be extended to all TN matrices using a fundamental decomposition result that states that any TN matrix can be expressed as a product of TN GEB matrices [4, Ch. 2].
Theorem 1. \cite{Ch. 4} If $A \in \mathbb{R}^{n \times n}$ is TN then
\begin{equation}
    s^{-}(Ax) \leq s^{-}(x), \text{ for all } x \in \mathbb{R}^{n}.
\end{equation}

If $A$ is TN and nonsingular then
\begin{equation}
    s^{+}(Ax) \leq s^{+}(x), \text{ for all } x \in \mathbb{R}^{n}.
\end{equation}

If $A$ is TP then
\begin{equation}
    s^{+}(Ax) \leq s^{-}(x), \text{ for all } x \in \mathbb{R}^{n} \setminus \{0\}.
\end{equation}

If $A$ is TN and nonsingular then $s^{+}(Ax) \leq s^{-}(x)$ holds if either $x$ has no zero entries or $Ax$ has no zero entries.

At this point we can already reinterpret Simillie’s result using the SVDP of TP matrices. To explain this, consider for simplicity the system $\dot{z} = Jz$, with $J$ a constant matrix and $J \in \mathbb{M}^{+}$. Then clearly there exists $\varepsilon > 0$ sufficiently small such that the matrix $\exp(J\varepsilon) = I + \varepsilon J + o(\varepsilon)$ is nonsingular, TN (by the result in Example \cite{2}), and irreducible for all $\varepsilon \in (0, \varepsilon)$. Thus, $(\exp(J\varepsilon))^{n-1}$ is TP, implying that $\exp(J\varepsilon)$ is in fact TP for all $\varepsilon > 0$ sufficiently small. Since $z(\varepsilon) = \exp(J\varepsilon)z(0)$, we conclude from Theorem \cite{1} that $s^{+}(z(\varepsilon)) \leq s^{-}(z(0))$ for all $z(0) \neq 0$.

This suggests the following question: when is the transition matrix associated with $\dot{x} = Ax$ TN [or TP]? This is exactly the question addressed by Schwarz in \cite{18}. He introduced the following definitions.

Definition 2. Consider the matrix differential equation $\dot{\Phi} = A\Phi$, $\Phi(t_0) = I$, where $A(t)$ is a continuous matrix on the time interval $t \in (a, b)$. The system is called a totally nonnegative differential system (TNDS) if $\Phi(t)$ is TN for any pair $(t_0, t)$ with $a < t_0 \leq t < b$. It is called a totally positive differential system (TPDS) if $\Phi(t)$ is TP for any pair $(t_0, t)$ with $a < t_0 < t < b$.

For the case where $A(t)$ is continuous in $t$, Schwarz derived a necessary and sufficient condition for a system to be TNDS or TPDS. Let $\mathbb{M} \subset \mathbb{R}^{n \times n}$ denote the set of tridiagonal matrices with nonnegative entries on the sub- and super-diagonals.

Theorem 2. \cite{18} Consider the matrix differential system $\dot{\Phi} = A\Phi$, $\Phi(t_0) = I$, where $A(t)$ is a continuous matrix for $t \in (a, b)$. The system is TNDS iff $A(t) \in \mathbb{M}$ for all $t \in (a, b)$. It is TPDS iff $A(t) \in \mathbb{M}$ for all $t \in (a, b)$, and every entry on the super- and sub-diagonals of $A(t)$ is not zero on a time interval.

Schwarz also analyzed the implications of TNDS or TPDS to the sign variations in the vector solution of $\dot{z}(t) = A(t)z(t)$.

The next section describes our main results. These are based on weakening the requirement $J(t) \in \mathbb{M}^{+}$ for all $t$ to $J(t) \in \mathbb{M}$ for all $t$ and adding an observability-type condition.

III. MAIN RESULTS

Our first result provides a bound on the number of isolated zeros of $z_1(t)$ and $z_n(t)$ in the system $\dot{z} = Az$ that is TNDS (but not necessarily TPDS). From here on we assume a more general case than in Schwarz \cite{18}, namely, that
\begin{equation}
    A : (a, b) \rightarrow \mathbb{R}^{n \times n}
\end{equation}
is a matrix of locally (essentially) bounded measurable functions.\footnote{We slightly modify the original definitions in \cite{18} to make them compatible with more modern terminology.}

Recall that \cite{11} implies that
\begin{equation}
    \dot{\Phi} = A\Phi, \quad \Phi(t_0) = I,
\end{equation}

admits a unique, locally absolutely continuous, invertible solution for all $t \in (a, b)$ (see, e.g., \cite{24}, Appendix C).
Theorem 3. Consider the time-varying linear system:

$$\dot{z}(t) = A(t)z(t), \quad (13)$$

with $A(t)$ satisfying $\Box$ and suppose that

$$\Phi(t, t_0) \text{ is TN for all } a < t_0 \leq t < b. \quad (14)$$

Let $p_i$ denote the number of isolated zeros of $z_i(t)$ on $(a, b)$. Then $\max\{p_1, p_n\} \leq n - 1.$

**Proof of Thm.\,3.** We prove the assertion for $z_1(t)$ (the proof for $z_n(t)$ is very similar). Since $\Phi(t)$ is TN and invertible for all $t \geq t_0$, Thm.\,1 implies that $s^+(z(t))$ is non-increasing with $t$. Schwarz \cite{18} has shown that if $\dot{z} = Az$ is a TNDS on $(a, b)$ and there exist times $r, q$ with $a < r < q < b$ such that $z_1(r) = 0$ and $z_1(q) \neq 0$ then $s^+(z(q)) < s^+(z(r))$. Since $s^+$ takes values in $\{0, 1, \ldots, n - 1\}$, we conclude that $z_1(t)$ cannot have more than $n - 1$ isolated zeros on $(a, b).$ \hfill $\square$

Note that Thm.\,3 only bounds the number of isolated zeros of $z_1(t)$ and $z_n(t)$. In particular, it does not rule out the possibility that $z_1(t)$ or $z_n(t)$ are zero on a time interval.

**Example 4.** Consider the case $n = 2$ and the constant matrix $A = \begin{bmatrix} a_{11} & 0 \\ 1 & a_{22} \end{bmatrix}$. Then $A \in \mathbb{M}$, so (13) is TNDS. The solution of $\dot{\Phi} = A\Phi$, $\Phi(0) = I$, is

$$\exp(At) = \begin{bmatrix} \exp(a_{11}t) & 0 \\ p(t) & \exp(a_{22}t) \end{bmatrix},$$

where $p(t) := \frac{\exp(a_{11}t) - \exp(a_{22}t)}{a_{11} - a_{22}}$ if $a_{11} \neq a_{22}$, and $p(t) := t\exp(a_{11}t)$, otherwise. Note that in both cases $p(t) > 0$ for all $t > 0$, so $\exp(At)$ is TN for all $t \geq 0$. From this we conclude that the solution of $\dot{z} = Az$ is $z(t) = \begin{bmatrix} 0 \\ \exp(a_{22}t)z_2(0) \end{bmatrix}$ if $z_1(0) = 0$, and $z(t) = \begin{bmatrix} \exp(a_{11}t)z_1(0) \\ p(t)z_1(0) + \exp(a_{22}t)z_2(0) \end{bmatrix}$, if $z_1(0) \neq 0$.

In the first case, $z_1(t)$ is zero for all $t \geq 0$ and $z_2(t)$ is either zero for all $t$ or has no zeros. In the second case, $z_1(t)$ has no zeros, and $z_2(t)$ has no more than a single isolated zero. \hfill $\square$

A. Applications to stability analysis

We are now ready to give a generalization of Smillie’s Theorem. In fact, we provide a generalization to a result of Smith on the stability of tridiagonal cooperative systems with a time-varying periodic vector field, and then specialize to the time-invariant case.

Consider the nonlinear time-varying dynamical system

$$\dot{x}(t) = f(t, x(t)), \quad (15)$$

whose trajectories evolve on an invariant set $\Omega \subset \mathbb{R}^n$, that is, for any $x_0 \in \Omega$ and any $t_0 \geq 0$ a unique solution $x(t, t_0, x_0)$ exists and satisfies $x(t, t_0, x_0) \in \Omega$ for all $t \geq t_0$. From here on we take $t_0 = 0$. We assume that the state-space $\Omega$ is convex and compact. We also assume that the Jacobian $J(t, x) = \frac{\partial}{\partial x} f(t, x)$ exists for all $t \geq 0$ and $x \in \Omega$.

We consider the case where $f$ is $T$-periodic, that is,

$$f(t, z) = f(t + T, z), \quad \text{for all } t \geq 0, \ z \in \Omega.$$

Note that in the particular case where $f$ is time-invariant this property holds for all $T$. We make two assumptions.

**Assumption 1.** For any $t \geq 0$ and any line $\gamma : [0, 1] \to \Omega$ the matrix

$$A(t) := \int_0^1 J(t, \gamma(r)) \, dr \quad (16)$$
is well-defined, locally (essentially) bounded, measurable, and satisfies the conditions for TNDS, i.e. \( A(t) \in \mathbb{M} \) for almost all \( t \in (a, b) \).

Note that since \( f \) is \( T \)-periodic so is \( A(t) \).

**Assumption 2.** For any solution of \( \dot{z}(t) = A(t)z(t) \) that is not the trivial solution either \( z_1(t) \) or \( z_n(t) \) has only isolated zeros.

**Theorem 4.** If Assumptions 1 and 2 hold then every solution of (15) converges to a periodic trajectory with period \( T \).

**Remark 1.** Thm. 4 is a generalization of a result of Smith [22] who derived the same result under stronger conditions, namely, that \( A(t) \) satisfies the necessary conditions for TPDS (so in particular Assumption 1 holds). TPDS also means that the solution of \( \dot{z} = Az \) satisfies \( z(t) \in \mathcal{V} \) for all \( t \geq 0 \) except perhaps for up to \( n - 1 \) discrete time points [18]. By the definition of \( \mathcal{V} \), this means that Assumption 2 also holds. Thus, the result of Smith is a special case of Thm. 4.

If the vector field has the form \( f(t, x) = f(x, u) \), with \( u(t) \) \( T \)-periodic, one may view \( u \) as a periodic input that excites the system. Thm. 4 then implies that the system entrains to the excitation, as every solution also converges to a periodic solution with the same period as the excitation.

We will prove Thm. 4 in the case where all the zeros of \( z_1(t) \) are isolated (the proof in the case where the zeros of \( z_n(t) \) are isolated is very similar). The proof is similar to the proof in Smith [22] albeit under our weaker assumptions, but we include it for the sake of completeness. We require the following result.

**Lemma 1.** Pick \( a, b \in \Omega \), with \( a \neq b \), and consider the solutions \( x(t, a), x(t, b) \) of (15). Then there exists a time \( s \geq 0 \) such that for all \( t \geq s \) either \( x_1(t, a) > x_1(t, b) \) or \( x_1(t, a) < x_1(t, b) \).

**Proof of Lemma 1.** Let
\[
\gamma(t, r) := rx(t, a) + (1 - r)x(t, b), \quad r \in [0, 1],
\]
denote the line between the two solutions at time \( t \). Since \( \Omega \) is convex, \( \gamma(t, r) \in \Omega \) for all \( t \geq 0 \) and all \( r \in [0, 1] \). Let \( z(t) := x(t, a) - x(t, b) \). Then
\[
\dot{z}(t) = f(t, x(t, a)) - f(t, x(t, b))
\]
\[
= \int_0^1 \frac{d}{dr} f(t, \gamma(r)) dr
\]
\[
= A(t)z(t),
\]
with \( A(t) \) defined in (16). By Assumption 1 this is a TNDS. Hence, according to Thm. 3 \( z_1(t) \) has no more than \( n - 1 \) isolated zeros. Combining this with Assumption 2 which states that \( z_1(t) \) has only isolated zeros, we conclude that there exists \( s \geq 0 \) such that \( z_1(t) \neq 0 \) for all \( t \geq s \) and this completes the proof.

We can now prove Thm. 4. If for some \( a \in \Omega \) the solution \( x(t, a) \) is \( T \)-periodic then there is nothing to prove. Thus, suppose that for some \( a \in \Omega \) the solution \( x(t, a) \) of (15) is not \( T \)-periodic. Then the \( T \)-periodicity of the vector field implies that \( x(t+T, a) \) is another solution of (15) that is different from \( x(t, a) \). Lemma 1 implies that there exists an integer \( m \geq 0 \) such that \( x_1(kT, a) - x_1((k+1)T, a) \neq 0 \) for all \( k \geq m \). Without loss of generality, assume that
\[
x_1(kT, a) - x_1((k+1)T, a) > 0 \quad \text{for all} \quad k \geq m.
\]

Define the Poincaré map \( P_T : \Omega \to \Omega \) by \( P_T(y) := x(T, y) \). Then \( P_T \) is continuous, and for any integer \( k \geq 1 \) the \( k \)-times composition of \( P_T \) satisfies \( P_T^k(y) = x(kT, y) \). The omega limit set \( \omega_T : \Omega \to \Omega \) is defined by \( \omega_T(y) := \{ z \in \Omega : \text{there exists a sequence } n_1, n_2, \ldots \text{ with } n_k \to \infty \text{ and } \lim_{k \to \infty} P_T^{n_k}(y) = \}

z}. It is well-known that $\omega_T(y) \neq \emptyset$, $x(kT, y) \rightarrow \omega_T(y)$, and $\omega_T(y)$ is invariant under $P_T$, that is, $P_T(\omega_T(y)) = \omega_T(y)$. In particular, if $\omega_T(y) = \{q\}$ then $P_T(q) = q$, that is, the solution emanating from $q$ is $T$-periodic.

To prove the theorem we need to show that $\omega_T(a)$ is a singleton. Assume that this is not the case. Then there exist $p, q \in \omega_T(a)$ with $p \neq q$. By the definition of $\omega_T(a)$, there exist integer sequences $n_k \rightarrow \infty$ and $m_k \rightarrow \infty$ such that $\lim_{k \rightarrow \infty} x(n_k T, a) = p$ and $\lim_{k \rightarrow \infty} x(m_k T, a) = q$. Combining this with the monotonicity condition (17) yields $p_1 = q_1$. We conclude that all points in $\omega_T(a)$ have the same first coordinate. Now consider the solutions emanating from $p$ and from $q$ at time zero, that is, $x(t, p)$ and $x(t, q)$. We know that there exists an integer $m \geq 0$ such that, say,

$$x_1(kT, p) - x_1(kT, q) > 0 \text{ for all } k \geq m.$$ 

But since $p, q \in \omega_T(a)$, $x(kT, p), x(kT, q) \in \omega_T(a)$ for all $k$, and thus we already know that they have the same first coordinate, that is, $x_1(kT, p) = x_1(kT, q)$. This contradiction completes the proof of Thm. 4 \hfill $\blacksquare$

The time-invariant nonlinear dynamical system:

$$\dot{x}(t) = f(x(t))$$

is $T$ periodic for all $T > 0$, so Thm. 4 yields the following result.

**Corollary 1.** Suppose that the solutions of (13) evolve on an invariant compact and convex set $\Omega \subset \mathbb{R}^n$, that the matrix $J(x) := \frac{\partial f(x)}{\partial x} \in \mathbb{M}$ for all $x \in \Omega$, and that Assumption 2 holds. Then for every $x_0 \in \Omega$ the solution $x(t, x_0)$ converges to an equilibrium point.

This is a generalization of Smillie’s theorem [20]. Indeed, Smillie assumed that $J(x) \in \mathbb{M}^+$ for all $x \in \Omega$. Note that $J(x) \in \mathbb{M}$ means that $J(x)$ is tridiagonal and Metzler, so (15) is a tridiagonal cooperative system in the sense of Hirsch (see [23]).

To establish that Assumption 2 indeed holds one can use an observability-type test. Indeed, consider the general time-varying, nonlinear, cooperative, tridiagonal dynamical system:

$$\begin{align*}
\dot{x}_1 &= f_1(t, x_1, x_2), \\
\dot{x}_2 &= f_2(t, x_1, x_2, x_3), \\
&\vdots \\
\dot{x}_{n-1} &= f_{n-1}(t, x_{n-2}, x_{n-1}, x_n), \\
\dot{x}_n &= f_n(t, x_{n-1}, x_n).
\end{align*}$$

Then $z := \dot{x}$ satisfies

$$\dot{z} = J(t, x(t))z,$$

with

$$J = \begin{bmatrix}
\frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & 0 & 0 & \ldots & 0 \\
\frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} & 0 & \ldots & 0 \\
0 & 0 & \frac{\partial f_{n-1}}{\partial x_{n-2}} & \frac{\partial f_{n-1}}{\partial x_{n-1}} & \frac{\partial f_{n-1}}{\partial x_n} & \frac{\partial f_{n-1}}{\partial x_n} \\
0 & 0 & \ldots & 0 & \frac{\partial f_{n-1}}{\partial x_{n-2}} & \frac{\partial f_{n-1}}{\partial x_{n-1}} & \frac{\partial f_{n-1}}{\partial x_n} & \frac{\partial f_{n-1}}{\partial x_n}
\end{bmatrix}.$$ 

Suppose that

$$\frac{\partial f_i}{\partial x_{i+1}}(t, z) > 0 \text{ for all } t \geq 0, z \in \Omega, i = 1, \ldots, n - 1. \quad (20)$$

In other words, the entries on the super-diagonal of $J$ are positive.
Suppose that \( z_1(t) = 0 \) on a time interval \( I \in (a, b) \). Then
\[
0 = \dot{z}_1 = \frac{\partial f_1}{\partial x_2} z_2,
\]
and using (20) implies that \( z_2(t) = 0 \) on \( I \). Thus,
\[
0 = \dot{z}_2 = \frac{\partial f_2}{\partial x_3} z_3.
\]
Proceeding in this manner, we conclude that \( z_1(t) \) is zero on \( I \) only if \( z(t) \) is the trivial solution, i.e. (20) implies that Assumption 2 indeed holds.

Note that unlike in the results by Smillie and Smith we do not require any entry on the sub-diagonal of \( J \) to be positive. This demonstrates the fact that our results are more general, as if the entries on the sub-diagonal are only nonnegative then (19) is TNDS but not necessarily TPDS.

More generally, to check whether Assumption 2 holds, we can associate with (19) an output \( y := z_1 \), i.e. \( y = c' z \), with \( c := [1 \ 0 \ 0 \ldots \ 0]' \). Clearly, if \( z_1(t) \) is zero on a time interval \( I \) and \( z(t) \) is not the trivial solution then (19) is not observable. Thus, observability of (19) on any time interval implies that Assumption 2 holds. One can then apply well-known results guaranteeing the observability of time-varying linear systems (see, e.g., [19], [24, Chapter 6]) to establish that Assumption 2 holds. Of course, a similar approach can be used to establish that \( z_n(t) \) is not zero on a time interval.

IV. Conclusions

Entrainment is an important asymptotic property of dynamical systems and has many applications. In this paper we considered two interesting results derived by Smillie [20] and Smith [22], that guarantee stability and entrainment for nonlinear cooperative tridiagonal dynamical systems. Building upon the theory of TNDSs developed by Schwarz [18], we were able to generalize these stability and entrainment results under a weaker condition, namely that the Jacobian is tridiagonal but may have nonnegative (rather than positive) entries on its super- and sub-diagonals, along with a suitable observability-type condition.

As a topic for further research, we believe that combining the TNDS framework with an additional observability-type condition may be used to generalize other results that are based on using the number of sign variations in the vector of derivatives as a discrete-valued Lyapunov function (see, e.g., [6], [21]).

References

[1] T. Ando, “Totally positive matrices,” Linear Algebra Appl., vol. 90, pp. 165–219, 1987.
[2] L. O. Chua and T. Roska, “Stability of a class of nonreciprocal cellular neural networks,” IEEE Trans. Circuits and Systems, vol. 37, no. 12, pp. 1520–1527, 1990.
[3] P. Donnell, S. A. Baigent, and M. Banaji, “Monotone dynamics of two cells dynamically coupled by a voltage-dependent gap junction,” J. Theoretical Biology, vol. 261, no. 1, pp. 120–125, 2009.
[4] S. M. Fallat and C. R. Johnson, Totally Nonnegative Matrices. Princeton, NJ: Princeton University Press, 2011.
[5] B. Fiedler and T. Gedeon, “A Lyapunov function for tridiagonal competitive-cooperative systems,” SIAM J. Math. Anal., vol. 30, pp. 469–478, 1999.
[6] G. Fusco and W. M. Oliva, “Transversality between invariant manifolds of periodic orbits for a class of monotone dynamical systems,” J. Dyn. Differ. Equ., vol. 2, no. 1, pp. 1–17, 1990.
[7] F. R. Gantmacher and M. G. Krein, Oscillation Matrices and Kernels and Small Vibrations of Mechanical Systems. Providence, RI: American Mathematical Society, 2002, translation based on the 1941 Russian original.
[8] N. C. Grassly and C. Fraser, “Seasonal infectious disease epidemiology,” Proc. Royal Society B: Biological Sciences, vol. 273, p. 25412550, 2006.
[9] M. W. Hirsch, “Systems of differential equations that are competitive or cooperative II: Convergence almost everywhere,” SIAM J. Math. Anal., vol. 16, no. 3, pp. 423–439, 1985.
[10] ———, “Convergent activation dynamics in continuous time networks,” Neural Networks, vol. 2, no. 5, pp. 331–349, 1989.
[11] R. A. Horn and C. R. Johnson, Matrix Analysis, 2nd ed. Cambridge University Press, 2013.
[12] M. Margaliot, L. Grüne, and T. Kriecherbauer, “Entrainment in the master equation,” ArXiv e-prints, 2017. [Online]. Available: http://adsabs.harvard.edu/abs/2017arXiv171007321M
[13] M. Margaliot, E. D. Sontag, and T. Tuller, “Entrainment to periodic initiation and transition rates in a computational model for gene translation,” PLoS ONE, vol. 9, no. 5, p. e96039, 2014.


[14] M. Margaliot and T. Tuller, “Stability analysis of the ribosome flow model,” IEEE/ACM Trans. Comput. Biol. Bioinf., vol. 9, pp. 1545–1552, 2012.

[15] M. Margaliot and E. D. Sontag, “Revisiting totally positive differential systems: A tutorial and new results,” 2018, submitted. [Online]. Available: https://arxiv.org/abs/1802.09590

[16] A. Pinkus, Totally Positive Matrices. Cambridge, UK: Cambridge University Press, 2010.

[17] G. Russo, M. di Bernardo, and E. D. Sontag, “Global entrainment of transcriptional systems to periodic inputs,” PLOS Computational Biology, vol. 6, p. e1000739, 2010.

[18] B. Schwarz, “Totally positive differential systems,” Pacific J. Math., vol. 32, no. 1, pp. 203–229, 1970.

[19] L. M. Silverman and H. E. Meadows, “Controllability and observability in time-variable linear systems,” SIAM J. Control, vol. 5, no. 1, pp. 64–73, 1967.

[20] J. Smillie, “Competitive and cooperative tridiagonal systems of differential equations,” SIAM J. Mathematical Analysis, vol. 15, pp. 530–534, 1984.

[21] H. L. Smith, “A discrete Lyapunov function for a class of linear differential equations,” Pacific J. Math., vol. 144, no. 2, pp. 345–360, 1990.

[22] ——, “Periodic tridiagonal competitive and cooperative systems of differential equations,” SIAM J. Math. Anal., vol. 22, no. 4, pp. 1102–1109, 1991.

[23] ——, Monotone Dynamical Systems: An Introduction to the Theory of Competitive and Cooperative Systems, ser. Mathematical Surveys and Monographs. Providence, RI: Amer. Math. Soc., 1995, vol. 41.

[24] E. D. Sontag, Mathematical Control Theory: Deterministic Finite Dimensional Systems, 2nd ed. New York: Springer, 1998.