ON THE INTERVALS OF A THIRD BETWEEN FAREY FRACTIONS

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Abstract. The spacing distribution between Farey points has drawn attention in recent years. It was found that the gaps $\gamma_{j+1} - \gamma_j$ between consecutive elements of the Farey sequence produce, as $Q \to \infty$, a limiting measure. Numerical computations suggest that for any $d \geq 2$, the gaps $\gamma_{j+d} - \gamma_j$ also produce a limiting measure whose support is distinguished by remarkable topological features. Here we prove the existence of the spacing distribution for $d = 2$ and characterize completely the corresponding support of the measure.

1. Introduction

Let $\mathcal{F}_Q = \{\gamma_1, \ldots, \gamma_N\}$ be the Farey sequence of order $Q$, which is defined to be the set of all subunitary irreducible fractions with denominators $\leq Q$, arranged in ascending order. For any interval $\mathcal{I} \subset [0,1]$, we write $\mathcal{F}_Q(\mathcal{I}) = \mathcal{F}_Q \cap \mathcal{I}$. The cardinality of $\mathcal{F}_Q(\mathcal{I})$ is well known to be $N_\mathcal{I}(Q) = 3|\mathcal{I}|Q^2/\pi^2 + O(Q \log Q)$. When $\mathcal{I} = [0,1]$ we write shortly $N(Q)$ instead of $N_{[0,1]}(Q)$. Since $\mathcal{F}_Q$ contains a large number of fractions obtained by a combined process of division, sieving and sorting of integers from $[1, Q]$, one would apriori expect little or even no special structure in the set of all differences between consecutive fractions (which we also call intervals of a second). Though, this expectation is not fulfilled. This is sustained from many points of view by a series of authors, such as Franel [4], Kanemitsu, Sita Rama Chandra Rao and Siva Rama Sarma [9], Hall and Tenenbaum [7, 8], Hall [5], Augustin, Boca and the authors [1], who have studied the set of gaps between consecutive Farey fractions. A regularity is expected also in the set of larger gaps $\gamma^{(d+1)} - \gamma'$, where $\gamma'$ runs over $\{\gamma_1, \ldots, \gamma_{N-d}\}$ and $d \geq 2$. (We use up-scripts, such as $\gamma', \gamma'', \gamma''', \ldots$ to write...
consecutive elements of $\mathfrak{F}_Q$.

It is our object to treat here the case $d = 2$, that is, the case of *intervals of a third*.

Geometrical representations of the set of pairs of neighbor intervals of fractions from $\mathfrak{F}_Q$ created for different values of $Q$ reveals sets of points whose density concentrates on different parts of the plane. The aesthetical qualities of the pictures catches attention immediately. For for any $d \geq 1$ they look like a swallow and the main topological distinctions are in the number of folds of the tail. Thus, when $d = 1$ (neighbor pairs of intervals of a second) the swallow has a one-fold tail (see [1]). When $d = 2$, the case treated in the present paper, the swallow has a two-fold tail (see Figure 1) and in Section 3 we have calculated explicitly the equations of the frontier. In the cases $d \geq 3$ the tail appears always to have a three-folded tail, but this is more complex and its characterization will appear in a separate paper.

Given $N$ real numbers $x_1 \leq x_2 \leq \cdots \leq x_N$ with mean spacing 1, we consider *the $h$-th level of intervals of a third probability* $\mu_{2,h}$ on $R_+^h$, defined, for $f \in C_c([0, \infty))$, by

$$\int_{[0, \infty)^h} f d\mu_{2,h} = \frac{1}{N-h-1} \sum_{j=1}^{N-h-1} f(x_{j+2} - x_j, x_{j+3} - x_{j+1}, \ldots, x_{j+h+1} - x_{j+h-1}).$$

In our case, we normalize $\mathfrak{F}_Q(I)$ to get the sequence $x_j = N(Q, I)\gamma_j/|I|$, $1 \leq j \leq N(Q, I)$ with mean spacing equal to one. Accordingly, we get the sequence $(\mu_{2,h}^{Q, I})_{Q \geq 1}$ of the $h$-th level of intervals of a third probabilities on $[0, \infty)^h$. We show that this sequence converges, as $Q \to \infty$, to a probability measure $\mu_{2,h}$, which is independent of $I$, and can be expressed explicitly.

For any $\gamma_i = a_i/q_i$ and $\gamma_j = a_j/q_j$ in $\mathfrak{F}_Q$, we set $\Delta(\gamma_i, \gamma_j) = \Delta(i, j) = -|\frac{a_i}{q_i} \frac{a_j}{q_j}|$. This is the numerator of the difference $\gamma_j - \gamma_i$. It is well known that $\Delta(\gamma', \gamma'') = -1$ for any consecutive elements of $\mathfrak{F}_Q$, and it turns out that this equality is responsible for the existence of the $h$-spacing distribution of the Farey sequence. Though, this relation is no longer true for larger intervals, but there is a convenient replacement. To see this, let us note that a Farey fraction can be uniquely determined by its two predecessors. Indeed, if
\( \frac{q'}{q} < \frac{q''}{q'} < \frac{q'''}{q''} \) are consecutive fractions of \( \mathcal{F}_Q \), we have \( a''' = ka'' - a' \) and \( q''' = kq'' - q' \), where \( k = \Delta(\gamma', \gamma''') = \left[ \frac{q' + Q}{q''} \right] \).

The basic idea of our procedure is to parametrize the set of \( h \)-tuples of intervals of a third in terms of just two variables that run over a completely described domain. The set of pairs of consecutive denominators of fractions in \( \mathcal{F}_Q \) are exactly the elements of

\[ \{(q', q''): 1 \leq q', q'' \leq Q, \ q' + q'' > Q \text{ and } (q', q'') = 1\} . \]

Since we are mainly interested in what happens when \( Q \to \infty \), we reduce the scale \( Q \) times, and consider the background triangle \( \mathcal{T} = \{(x, y): 0 < x \leq 1, \ x + y > 1\} \),
called the Farey triangle. We split it into a series of polygons as follows. Firstly, for each $(x, y) \in \mathbb{R}^2$, we set $L_0(x, y) = x$, $L_1(x, y) = y$, and then, for $i \geq 2$, we define recursively:

$$L_i(x, y) = \left[\frac{1 + L_{i-2}(x, y)}{L_{i-1}(x, y)}\right] L_{i-1}(x, y) - L_{i-2}(x, y).$$

Then, as in [3], we consider the map $k : \mathcal{T} \to (\mathbb{N}^*)^h$, $k(x, y) = (k_1(x, y), \ldots, k_h(x, y))$,

where $k_i(x, y) = \left[\frac{1 + L_{i-1}(x, y)}{L_i(x, y)}\right]$. The functions $k_i(x, y)$ are locally constant, and the subsets of $\mathcal{T}$ on which they are constant plays a special role. Thus, for any $k \in (\mathbb{N}^*)^h$, we get the convex polygon

$$\mathcal{T}_k = \{(x, y) \in \mathcal{T} : k(x, y) = k\}.$$

Notice that $\mathcal{T} = \bigcup_{k \in (\mathbb{N}^*)^h} \mathcal{T}_k$ and $\mathcal{T}_k \cap \mathcal{T}_{k'} = \emptyset$ whenever $k \neq k'$.

Next we consider the application $\Phi_{2,h} : \mathcal{T} \to (0, \infty)^h$ defined by

$$\Phi_{2,h}(x, y) = \frac{3}{\pi^2} \left( \frac{k_1(x, y)}{L_0(x, y)L_2(x, y)}, \frac{k_2(x, y)}{L_1(x, y)L_3(x, y)}, \ldots, \frac{k_h(x, y)}{L_{h-1}(x, y)L_{h+1}(x, y)} \right).$$

Our main result shows that, indeed, for $Q \to \infty$, the sequence $(\mu_{2,h}^{Q,T})_{Q \geq 1}$ converges to a measure and $\Phi_{2,h}(x, y)$ is the needed tool to describe its support.

**Theorem 1.** The sequence $(\mu_{2,h}^{Q,T})_{Q \geq 1}$ converges weakly to a probability measure $\mu_{2,h}$, which is independent of $\mathcal{I}$. The support $\mathcal{D}_{2,h}$ of $\mu_{2,h}$ is the closure of the range of $\Phi_{2,h}$, and

$$\mu_{2,h}(\mathcal{C}) = 2\text{Area}(\Phi_{2,h}^{-1}(\mathcal{C})).$$

for any parallelepiped $\mathcal{C} = \prod_{j=1}^h (\alpha_j, \beta_j) \subset (0, \infty)^h$.

In Table 1 from Section 3 we provide explicit formulae for all the pieces that form $\mathcal{D}_{2,2}$.

**2. The Existence of the Limiting Measure**

It is plain that in order to prove Theorem 1 it suffices to see the effect of $\mu_{2,h}$ on bounded parallelepipeds. For any $\mathcal{C} = \prod_{j=1}^h (\alpha_j, \beta_j) \subset (0, \infty)^h$, we define

$$\mu_{2,h}^{Q,T}(\mathcal{C}) := \frac{1}{N_{\mathcal{I}}(Q)} \cdot \# \left\{ \gamma_j \in \mathcal{S}_{\mathcal{I}}(Q) : \frac{\alpha_i|\mathcal{I}|}{N_{\mathcal{I}}(Q)} < \gamma_{j+i+1} - \gamma_{j+i-1} < \frac{\beta_i|\mathcal{I}|}{N_{\mathcal{I}}(Q)}, i = 1, \ldots, h \right\}.$$
We have to show that the sequence \( \{ \mu_{2,h}^{Q,\mathcal{I}} \} \) is convergent when \( Q \to \infty \) and the limit is independent of \( \mathcal{I} \). In the beginning we treat the case of the complete interval \( \mathcal{I} = [0, 1] \).

2.1. The case \( \mathcal{I} = [0, 1] \). In the following we write shortly \( \mu_{2,h}^{Q} \) instead of \( \mu_{2,h}^{Q,[0,1]} \).

With the notations from the Introduction, we see that \( \gamma_{j+i-1} - \gamma_{j+i+1} = k_{j+i}/q_{j+i}q_{j+i+1} \). Then \( \mu_{2,h}^{Q} \) can be written as

\[
\mu_{2,h}^{Q}(C) = \frac{1}{N(Q)} \cdot \# \left\{ \gamma_j \in \mathcal{F}(Q) : \frac{N(Q)}{\beta_i} < \frac{q_{j+i+1}q_{j+i-1}}{k_{j+i}} < \frac{N(Q)}{\alpha_i}, \ i = 1, \ldots, h \right\}. \tag{1}
\]

Knowing that \( q_{j+i} = QL_i(q_j/Q, q_{j+1}/Q) \), we consider the set

\[
\Omega^Q(C) = \left\{ (x, y) \in QT : \frac{N(Q)}{Q^2\beta_i} < \frac{L_{i-1}(\frac{x}{Q}, \frac{y}{Q})L_{i+1}(\frac{x}{Q}, \frac{y}{Q})}{k_i(\frac{x}{Q}, \frac{y}{Q})} < \frac{N(Q)}{Q^2\alpha_i}, \ i = 1, \ldots, h \right\}. \tag{2}
\]

Since neighbor denominators in \( \mathcal{F}_Q \) are always coprime, relation \( \text{(1)} \) turns into

\[
\mu_{2,h}^{Q}(C) = \frac{1}{N(Q)} \cdot \# \left\{ (x, y) \in \Omega^Q(C) \cap \mathbb{N}^2 : \gcd(x, y) = 1 \right\}.
\]

Next, we select the points with coprime coordinates using Möbius summation (cf. \cite[Lemma 2]{1}), and we find that

\[
\mu_{2,h}^{Q}(C) = \frac{1}{N(Q)} \left( \frac{6}{\pi^2} \text{Area} \ (\Omega^Q(C)) + O\left( \text{length} \ (\partial \Omega^Q(C)) \log Q \right) \right). \tag{3}
\]

Splitting \( \mathcal{T} \) into the series of polygons \( T_k \), we see that the error term in \( \text{(3)} \) is \( O(Q \log Q) \).

In the main term, we replace \( \Omega^Q(C) \) by the bounded set \( \Omega(C) = \Omega^Q(C)/Q \). These yield

\[
\mu_{2,h}^{Q}(C) = \frac{6Q^2}{\pi^2 N(Q)} \text{Area} \ (\Omega(C)) + O \left( \frac{\log Q}{Q} \right). \tag{4}
\]

It remains to replace in \( \text{(1)} \) the set \( \Omega(C) \) by a set as in \( \text{(2)} \), but with bounds independent of \( Q \) in the corresponding inequalities. This set is

\[
\mathcal{D}(C) := \left\{ (x, y) \in QT : \frac{3}{\pi^2 \beta_i} < \frac{L_{i-1}(x,y)L_{i+1}(x,y)}{k_i(x,y)} < \frac{3}{\pi^2 \alpha_i}, \ i = 1, \ldots, h \right\}. \tag{5}
\]

Notice that \( \mathcal{D}(C) \) is exactly \( \Phi_{2,h}^{-1}(C) \). The replacement does not change the error term because, via \( N(Q) = 3Q^2/\pi^2 + O(Q \log Q) \), we have:

\[
\max_{1 \leq i \leq h} \left\{ \left| \frac{N(Q)}{\alpha_i Q^2} - \frac{3}{\pi^2 \alpha_i} \right|, \left| \frac{N(Q)}{\beta_i Q^2} - \frac{3}{\pi^2 \beta_i} \right| \right\} = O \left( \frac{\log Q}{Q} \right), \tag{6}
\]

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which implies
\[
\text{Area} \left( \Omega(C) \triangle \Omega(C) \right) = O_c \left( \frac{\log Q}{Q} \right).
\] (7)

Therefore, by (5) and (7), we get
\[
\mu_{2,h}^Q(C) = 2 \text{Area} \left( \Omega(C) \right) + O_c \left( \frac{\log Q}{Q} \right).
\] (8)

In particular, this gives \( \mu_{2,h}(C) = \lim_{Q \to \infty} \mu_{2,h}^Q(C) = 2 \text{Area} \left( \Omega(C) \right) \), concluding the proof of the theorem when \( I = [0, 1] \).

2.2. The short interval case. Suppose now that \( I \subset [0, 1] \) is fixed. In order to impose the condition that only the fractions from \( I \) are involved in the calculations, we employ the fundamental property of neighbor fractions in \( \mathcal{F}_Q \). This says that if \( \gamma' = a'/q' \) and \( \gamma'' = a''/q'' \) are consecutive then \( a''q' - a'q'' = 1 \). Consequently, \( a'' \equiv (q')^{-1} \pmod{q''} \), and this allows us to write the fraction \( a''/q'' \) in terms of \( q' \) and \( q'' \). Thus
\[
a''/q'' \in I \iff (q')^{-1} \pmod{q''} \in q''I.
\]

This time we have to estimate
\[
\mu_{2,h}^{Q,I}(C) = \frac{1}{N_I(Q)} \cdot \# \left\{ (q', q'') \in Q^2 : \frac{N_I(Q)}{|I|Q^{2\beta_i}} < \frac{L_{i-1}(\frac{q'}{Q}, \frac{q''}{Q})L_{i+1}(\frac{q'}{Q}, \frac{q''}{Q})}{k_i(\frac{q'}{Q}, \frac{q''}{Q})} < \frac{N_I(Q)}{|I|Q^{2\alpha_i}}, \right. \\
\left. \text{for } i = 1, \ldots, h; \quad (q')^{-1} \pmod{q''} \in q''I \right\}.
\] (9)

We may write (9) as
\[
\mu_{2,h}^{Q,I}(C) = \frac{1}{N_I(Q)} \sum_{q=1}^{Q} N_q(\mathcal{J}_c^Q(q), qI),
\] (10)

where
\[
N_q(\mathcal{J}_1, \mathcal{J}_2) = \# \{ (m, n) \in \mathcal{J}_1 \times \mathcal{J}_2 : mn \equiv 1 \pmod{q} \},
\]

for any \( \mathcal{J}_1, \mathcal{J}_2 \subset [0, Q - 1] \) and
\[
\mathcal{J}_c^Q(q) = \left\{ x \in (Q - q, Q] : \frac{N_I(Q)}{|I|Q^{2\beta_i}} < \frac{L_{i-1}(\frac{q'}{Q}, \frac{q''}{Q})L_{i+1}(\frac{q'}{Q}, \frac{q''}{Q})}{k_i(\frac{q'}{Q}, \frac{q''}{Q})} < \frac{N_I(Q)}{|I|Q^{2\alpha_i}}, \right. \\
\left. \text{for } i = 1, \ldots, h \right\}.
\]

For the best available technique to estimate the size of \( N_q(\mathcal{J}_1, \mathcal{J}_2) \) one requires bounds for Kloosterman sums (cf. [2]). This is done when \( \mathcal{J}_1 \) and \( \mathcal{J}_2 \) are intervals, but it may be easily extended for finite unions of subintervals of \( [0, q - 1] \) (as the set \( \mathcal{J}_c^Q(q) \) is), even with
the same same formula. For our needs here, it suffices a version with a slightly weaker term:

\[ N_q(\mathcal{J}_c^Q(q), qI) = \frac{\varphi(q)|\mathcal{J}_c^Q(q)|}{q} \cdot |I| + O_{c,\varepsilon}(q^{1/2+\varepsilon}). \]

(11)

Inserting (11) into (10), we get

\[ \mu_{2,h}(C) = \frac{|I|}{N_I(Q)} \sum_{q=1}^{Q} \frac{\varphi(q)|\mathcal{J}_c^Q(q)|}{q} + O_{c,\varepsilon}(Q^{-1/2+\varepsilon}). \]

(12)

To calculate the sum in (12), we employ the Euler-MacLaurin formula, noticing the fact that \(|\mathcal{J}_c^Q(q)|\), as a function of \(q\), is piecewise continuous differentiable on \([0, 1]\). We obtain

\[ \sum_{q=1}^{Q} \frac{\varphi(q)|\mathcal{J}_c^Q(q)|}{q} = \frac{1}{\zeta(2)} \int_{1}^{Q} |\mathcal{J}_c^Q(q)| \, dq + O\left( \left( \sup_{1 \leq q \leq Q} |\mathcal{J}_c^Q(q)| + \int_{1}^{Q} \frac{\partial}{\partial q} |\mathcal{J}_c^Q(q)| \, dq \right) \log Q \right). \]

(13)

The size of the error term is estimated observing, firstly, that \(|\mathcal{J}_c^Q(q)| \leq Q\). Secondly, by the definition of \(\mathcal{J}_c^Q(q)\) it follows that there exists a partition of \([1, Q]\) in finitely many intervals with the property that the cardinality of \(\mathcal{J}_c^Q(q)\) is monotonic on each of them. Therefore

\[ \int_{1}^{Q} \frac{\partial}{\partial q} |\mathcal{J}_c^Q(q)| \, dq = O_c(Q). \]

(14)

Then, forgathering (13), (14), (6) in (12) and using again the estimate \(N_I(Q) = 3|I|Q^2/\pi^2 + O(Q \log Q)\), we obtain

\[ \mu_{2,h}(C) = \frac{6|I|}{\pi^2 N_I(Q)} \int_{1}^{Q} |\mathcal{J}_c^Q(q)| \, dq + O_{c,\varepsilon}(Q^{-1/2+\varepsilon}) \]

\[ = 2 \text{Area } (\mathcal{D}(Q)) + O_{c,\varepsilon}(Q^{-1/2+\varepsilon}). \]

This concludes the proof of the theorem.

3. The Support of the Limiting Measure

For \(h = 1\), we have \(\mathcal{D}_{2,1} = [6/\pi^2, \infty)\). For \(h \geq 2\), by Theorem [11] it follows that \(\mathcal{D}_{2,h}\) is a countable union of hyper-surfaces in \([6/\pi^2, \infty)^h\).
The support $D_{2,h}$ has some striking features. Let us see them in the case $h = 2$. We write $k = (k, l)$ and observe that

$$T_{k,l} = \left\{ (x, y) \in T_k : \frac{1 + (l + 1)x}{k(l + 1) - 1} < y \leq \frac{1 + lx}{kl - 1} \right\}.$$

Roughly speaking, by definition we find that $T_k$ corresponds to the set of 3-tuples $(\gamma', \gamma'', \gamma''')$ of consecutive elements of $\mathcal{F}_Q$ with the property that $\Delta(\gamma', \gamma''') = k$. Similarly, $T_{k,l}$ corresponds to the set of 4-tuples $(\gamma', \gamma'', \gamma''', \gamma iv)$ of consecutive elements of $\mathcal{F}_Q$ with the property that $\Delta(\gamma', \gamma''') = k$ and $\Delta(\gamma'', \gamma iv) = l$. We remark that $T_{1,1} = \emptyset$, and also $T_{k,l} = \emptyset$ whenever both $k$ and $l$ are $\geq 2$, except when $(k, l) \in \{(2, 2); (2, 3); (2, 4); (3, 2); (4, 2)\}$. Notice that the symmetry of the Farey sequence of order $Q$ with respect to $1/2$ produces a sort of balance between the polygons $T_{k,l}$ and $T_{l,k}$.

Then the support $D_{2,h}$ is the closure of the image of the function $\Phi_{2,2}$, which can be written as

$$\Phi_{2,2}(x, y) = \frac{3}{\pi^2} \left( \frac{k}{xz}, \frac{l}{yt} \right),$$

in which $z = x - ky$, $t = y - lt$, for $(x, y) \in T_{k,l}$. A tedious, but elementary, computation allows us to find precisely the boundaries of $\Phi_{2,2}(T_{k,l})$. The image obtained is shown in Figure 1 and the equations are listed in Table 1. All the functions that produce the equations of the boundaries of $\Phi_{2,2}(T_{k,l})$ are either of the form $\frac{3}{\pi^2} \cdot \frac{et}{a + bt + ct \sqrt{t(t-d)}}$, with $t$ in a certain interval that might be unbounded, or the symmetric with respect to $x = y$ of such a curve. Here $a, b, c, d, e$ are integers.

We conclude by making a few remarks. Firstly, we mention that $\Phi_{2,2}$ has a symmetrization effect, namely, it makes $\Phi_{2,2}(T_{n,m})$ and to $\Phi_{2,2}(T_{m,n})$ to be symmetric with respect to the first diagonal $y = x$, for any $m, n \geq 1$. The diamond$^1$ $\Phi_{2,2}(T_{2,2})$ is the single nonempty domain $\Phi_{2,2}(T_{k,l})$ that has $y = x$ as axis of symmetry. The top of the beak of the swallow $D_{2,h}$ has coordinates $(6/\pi^2, 6/\pi^2)$. The asymptotes of the wings are $y = 6/\pi^2$ and $x = 6/\pi^2$. The highest density is on a region situated in the neck, where many components of the swallow overlap partially or completely.

$^1$Remark that the edges of $\Phi_{2,2}(T_{2,2})$ are close to being, but are not exactly straight lines. The same applies for the edges of the diamonds in the tail.
Table 1 below lists all the equations of the boundaries of $\Phi_{k,l}(t)$. In the head of the table $MN$ represents an edge of $T_{k,l}$ (listed in counterclockwise order, starting either from the East or from the North side) and $g_{MN}(t)$ is a parametrization of $\Phi_{k,l}(MN)$.

Table 1: The edges of $D_{2,2}$.

| $k, l$ | $MN$ | $\frac{\pi^2}{3} g_{MN}(t)$ | the domain of $t$ |
|-------|------|-----------------|-----------------|
| 1, 2  | $(\frac{1}{3}, 1); (0, 1)$ | $\frac{2t}{\sqrt{(t-4)}}$ | $\frac{9}{2} \leq t < \infty$ |
| 1, 2  | $(0, 1); (\frac{1}{4}, \frac{4}{3})$ | $\frac{16t}{-12+3t+5\sqrt{(t-8)}}$ | $\frac{25}{3} \leq t \leq \infty$ |
| 1, 2  | $(\frac{1}{5}, \frac{4}{5}); (\frac{1}{3}, 1)$ | $\frac{16t}{-12-3t+5\sqrt{(t+8)}}$ | $\frac{9}{2} \leq t \leq \frac{25}{3}$ |
| 1, 3  | $(\frac{1}{2}, 1); (\frac{1}{3}, 1)$ | $\frac{6t}{t+3\sqrt{(t-4)}}$ | $4 \leq t \leq \frac{9}{2}$ |
| 1, 3  | $(\frac{1}{3}, 1); (\frac{1}{4}, \frac{4}{5})$ | $\frac{12t}{-t+3\sqrt{(t+8)}}$ | $\frac{9}{2} \leq t \leq \frac{25}{3}$ |
| 1, 3  | $(\frac{1}{4}, \frac{4}{5}); (\frac{1}{5}, \frac{3}{4})$ | $\frac{24t}{-20+7t+9\sqrt{(t-8)}}$ | $8 \leq t \leq \frac{25}{3}$ |
| 1, 3  | $(\frac{1}{4}, \frac{3}{5}); (\frac{2}{5}, \frac{5}{7})$ | $\frac{24t}{-20+7t-9\sqrt{(t-8)}}$ | $8 \leq t < \frac{49}{6}$ |
| 1, 3  | $(\frac{2}{7}, \frac{5}{7}); (\frac{1}{2}, 1)$ | $\frac{54t}{-24+7t+11\sqrt{(t+12)}}$ | $4 \leq t < \frac{49}{6}$ |
| 1, 4  | $(\frac{3}{5}, 1); (\frac{1}{2}, 1)$ | $\frac{4t}{t-2\sqrt{(t-4)}}$ | $\frac{25}{6} \leq t \leq 4$ |
| 1, 4  | $(\frac{1}{2}, 1); (\frac{2}{7}, \frac{5}{7})$ | $\frac{12t}{-t+2\sqrt{(t+12)}}$ | $4 \leq t < \frac{49}{6}$ |
| 1, 4  | $(\frac{2}{7}, \frac{5}{7}); (\frac{1}{3}, \frac{2}{5})$ | $\frac{32t}{-28+11t-13\sqrt{(t-8)}}$ | $\frac{49}{6} \leq t \leq 9$ |
| 1, 4  | $(\frac{1}{3}, \frac{2}{5}); (\frac{3}{5}, 1)$ | $\frac{128t}{-40-13t+19\sqrt{(t+16)}}$ | $\frac{25}{6} \leq t \leq 9$ |
| 1, $l \geq 5$ | $(\frac{l-1}{l+1}, 1); (\frac{l-2}{l}, 1)$ | $\frac{2t}{(l-2)t-l\sqrt{(t-4)}}$ | $\frac{t^2}{2(t-2)} \leq t \leq \frac{(l+1)^2}{2(l-1)}$ |
| 1, $l \geq 5$ | $(\frac{l-2}{l}, 1); (\frac{l-3}{l+1}, \frac{l-1}{l+1})$ | $\frac{2(l-1)t}{(2-l)t+4\sqrt{(t+4l-4)}}$ | $\frac{t^2}{2(l-2)} \leq t \leq \frac{(l+1)^2}{2(l-3)}$ |
| 1, $l \geq 7$ | $(\frac{l-3}{l+1}, \frac{l-1}{l+1}); (\frac{l-2}{l+2}, \frac{l}{l+2})$ | $\frac{8t}{4+l+3(t-5)+l+3\sqrt{(t-8)}}$ | $\frac{(l+1)^2}{2(l-3)} \leq t \leq \frac{(l+2)^2}{2(l-2)}$ |
| 1, $l = 5, 6$ | $(\frac{l-3}{l+1}, \frac{l-1}{l+1}); (\frac{l-2}{l+2}, \frac{l}{l+2})$ | $\frac{8t}{4+l+3(t-5)+l+3\sqrt{(t-8)}}$ | $\frac{(l+1)^2}{2(l-3)} \leq t \leq \frac{(l+2)^2}{2(l-2)}$ |
| 1, $l \geq 5$ | $(\frac{l-2}{l+2}, \frac{l}{l+2}); (\frac{l-1}{l+1}, 1)$ | $\frac{4l^2}{((l-2)t+l+3\sqrt{(t+4l)})((l-1)+(l+1)\sqrt{(t+4l)})}$ | $\frac{(l+2)^2}{2(l-1)} \leq t \leq \frac{(l+2)^2}{2(l-2)}$ |

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| $k, l$ | $MN$ | \( \pi_{3}^{2} g_{MN}(t) \) | the domain of \( t \) |
|-------|------|----------------|----------------|
| 2, 1  | (1, 1); \( \frac{1}{3}, \frac{2}{3} \) | \( \frac{4t^{2}}{(t+2)(t-2)} \) | \( 2 \leq t \leq 6 \) |
| 2, 1  | \( \frac{1}{3}, \frac{2}{3} \); \( \frac{2}{5}, \frac{3}{5} \) | \( \frac{9t}{-12+4t-5\sqrt{t(t-6)}} \) | \( 6 \leq t \leq \frac{25}{4} \) |
| 2, 1  | \( \frac{2}{5}, \frac{3}{5} \); (1, 1) | \( \frac{9t}{-12-4t+5\sqrt{t(t+6)}} \) | \( 2 \leq t \leq \frac{25}{4} \) |
| 2, 2  | (1, \( \frac{1}{3} \)); (1, 1) | \( \frac{-8t^{2}}{(t+2)(t-6)} \) | \( 2 \leq t \leq \frac{10}{3} \) |
| 2, 2  | (1, 1); \( \frac{2}{5}, \frac{3}{5} \) | \( \frac{-6t}{t+2\sqrt{t(t+6)}} \) | \( 2 \leq t \leq \frac{25}{4} \) |
| 2, 2  | \( \frac{2}{5}, \frac{3}{5} \); \( \frac{1}{2}, \frac{1}{2} \) | \( \frac{18t}{-30+13t-14\sqrt{t(t-6)}} \) | \( \frac{25}{4} \leq t \leq 8 \) |
| 2, 2  | \( \frac{1}{2}, \frac{1}{2} \); (1, \( \frac{1}{3} \)) | \( \frac{50t}{-30-11t+14\sqrt{t(t+10)}} \) | \( \frac{10}{3} \leq t \leq 8 \) |
| 2, 3  | (1, \( \frac{1}{3} \)); (1, \( \frac{1}{3} \)) | \( \frac{-12t^{2}}{(t+2)(t-10)} \) | \( \frac{10}{3} \leq t \leq \frac{14}{3} \) |
| 2, 3  | \( \frac{1}{2}, \frac{1}{2} \); \( \frac{1}{2}, \frac{1}{2} \) | \( \frac{-30t}{4t+6\sqrt{t(t+10)}} \) | \( \frac{10}{3} \leq t \leq 8 \) |
| 2, 3  | \( \frac{1}{2}, \frac{1}{2} \); (1, \( \frac{1}{3} \)) | \( \frac{27t}{24-2t+7\sqrt{t(t-6)}} \) | \( 8 \leq t \leq \frac{25}{4} \) |
| 2, 3  | \( \frac{4}{5}, \frac{3}{5} \); (1, \( \frac{1}{3} \)) | \( \frac{147t}{-56-22t+27\sqrt{t(t+14)}} \) | \( \frac{14}{3} \leq t \leq \frac{25}{4} \) |
| 2, 4  | (1, \( \frac{1}{3} \)); (1, \( \frac{1}{3} \)) | \( \frac{-16t^{2}}{(t+2)(t-14)} \) | \( \frac{14}{3} \leq t \leq 6 \) |
| 2, 4  | (1, \( \frac{1}{3} \)); \( \frac{4}{5}, \frac{3}{5} \) | \( \frac{28t}{-3t+4\sqrt{t(t+14)}} \) | \( \frac{14}{3} \leq t \leq \frac{25}{4} \) |
| 2, 4  | \( \frac{4}{5}, \frac{3}{5} \); (1, \( \frac{2}{3} \)) | \( \frac{36t}{30-t+8\sqrt{t(t-6)}} \) | \( 6 \leq t \leq \frac{25}{4} \) |
| 3, 1  | (1, \( \frac{1}{3} \)); (1, \( \frac{1}{3} \)) | \( \frac{-9t^{2}}{(t+5)(t-6)} \) | \( 3 \leq t \leq \frac{15}{4} \) |
| 3, 1  | (1, \( \frac{1}{3} \)); \( \frac{1}{2}, \frac{1}{2} \) | \( \frac{9t^{2}}{(t+3)(2t-3)} \) | \( 3 \leq t \leq 6 \) |
| 3, 1  | \( \frac{1}{2}, \frac{1}{2} \); (1, \( \frac{1}{3} \)) | \( \frac{32t}{-72+3t-11\sqrt{3t(t-16)}} \) | \( 6 \leq t \leq \frac{147}{20} \) |
| 3, 1  | \( \frac{4}{5}, \frac{3}{5} \); (1, \( \frac{1}{3} \)) | \( \frac{50t}{-60-23t+9\sqrt{3t(t+20)}} \) | \( \frac{15}{4} \leq t \leq \frac{147}{20} \) |
| 3, 2  | (1, \( \frac{1}{3} \)); (1, \( \frac{1}{3} \)) | \( \frac{-14t^{2}}{(t+3)(t-15)} \) | \( \frac{15}{4} \leq t \leq 6 \) |
| 3, 2  | (1, \( \frac{1}{3} \)); \( \frac{4}{5}, \frac{3}{5} \) | \( \frac{10t}{-2t+\sqrt{t(9t+60)}} \) | \( \frac{15}{4} \leq t \leq \frac{147}{20} \) |
| 3, 2  | \( \frac{4}{5}, \frac{3}{5} \); \( \frac{3}{5}, \frac{2}{5} \) | \( \frac{64t}{-168+79t-27\sqrt{3t(t-16)}} \) | \( \frac{147}{20} \leq t \leq \frac{25}{3} \) |
| 3, 2  | \( \frac{3}{5}, \frac{2}{5} \); (1, \( \frac{1}{2} \)) | \( \frac{64t}{-72-11t+7\sqrt{3t(t-16)}} \) | \( 6 \leq t \leq \frac{25}{3} \) |

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| $k, l$ | $MN$ | $\frac{x^2}{3} g_{MN}(t)$ | the domain of $t$ |
|-------|-------|--------------------------|-----------------|
| 4, 1  | $(1, \frac{2}{5}); (1, \frac{1}{2})$ | $\frac{-16t^2}{(t+4)(t-12)}$ | $4 \leq t \leq \frac{28}{5}$ |
| 4, 1  | $(1, \frac{1}{2}); \left(\frac{3}{5}, \frac{2}{5}\right)$ | $\frac{16t^2}{(t+4)(3t-4)}$ | $4 \leq t \leq \frac{20}{3}$ |
| 4, 1  | $\left(\frac{2}{5}, \frac{2}{3}\right); \left(\frac{2}{3}, \frac{1}{3}\right)$ | $\frac{25t}{-80+37t-38\sqrt{t(t-5)}}$ | $9 \leq t \leq \frac{20}{3}$ |
| 4, 1  | $\left(\frac{2}{5}, \frac{1}{3}\right); (1, \frac{3}{7})$ | $\frac{49t}{-56-23t+26\sqrt{t(t+7)}}$ | $\frac{28}{5} \leq t \leq 9$ |
| 4, 2  | $(1, \frac{2}{5}); (1, \frac{2}{3})$ | $\frac{-32t^2}{(t+4)(t-28)}$ | $\frac{28}{5} \leq t \leq \frac{20}{3}$ |
| 4, 2  | $(1, \frac{2}{3}); (\frac{2}{3}, \frac{1}{3})$ | $\frac{14t}{-3t+4\sqrt{t(t+7)}}$ | $\frac{28}{5} \leq t \leq 9$ |
| 4, 2  | $\left(\frac{2}{5}, \frac{1}{3}\right); (1, \frac{2}{7})$ | $\frac{50t}{60-9t+16\sqrt{t(t-5)}}$ | $\frac{20}{3} \leq t \leq 9$ |
| $k \geq 5, 1$ | $(1, \frac{2}{k+1}); (1, \frac{2}{k})$ | $\frac{-k^2t^2}{(t-k^2+k)(t+k)}$ | $k \leq t \leq \frac{k(k+1)}{k-1}$ |
| $k \geq 5, 1$ | $(1, \frac{2}{k}); \left(\frac{k-1}{k+1}, \frac{2}{k+1}\right)$ | $\frac{k^2t^2}{(t+k)(k-1)(t-k)}$ | $k \leq t \leq \frac{k(k+1)}{k-1}$ |
| $k \geq 5, 1$ | $(\frac{k-1}{k+1}, \frac{2}{k+1}); \left(\frac{k}{k+2}, \frac{2}{k+2}\right)$ | $\frac{4(k+1)^2t^2}{(k+2)t-k\sqrt{kt(kt-4k-4)}}$ | $\frac{k(k+1)}{k-1} \leq t \leq \frac{(k+2)^2}{k}$ |
| $k \geq 5, 1$ | $(\frac{k}{k+2}, \frac{2}{k+2}); (1, \frac{2}{k+1})$ | $\frac{2(k+1)^2t^2}{(t-k\sqrt{kt(kt-4k-4)})} \frac{(k^2-2)t-k\sqrt{kt(kt-4k-4)}}{(-k^2+2)t+k\sqrt{kt(kt-4k-4)}}$ | $\frac{k(k+1)}{k-1} \leq t \leq \frac{(k+2)^2}{k}$ |

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