Efficient quantum circuits for perfect and controlled teleportation of n-qubit non-maximally entangled states of generalized Bell-type

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Abstract

An efficient and economical scheme is proposed for the perfect quantum teleportation of n-qubit non-maximally entangled state of generalized Bell-type. A Bell state is used as the quantum channel in the proposed scheme. It is also shown that the controlled teleportation of this n-qubit state can be achieved by using a GHZ state or a GHZ-like state as quantum channel. The proposed schemes are economical because for the perfect and controlled teleportation of n-qubit non-maximally entangled state of generalized Bell-type we only need a Bell state and a tripartite entangled state respectively. It is also established that there exists a family of 12 orthogonal tripartite GHZ-like states which can be used as quantum channel for controlled teleportation. The proposed protocols are critically compared with the existing protocols.

1 Introduction

The beauty of quantum circuit lies in the fact that it can perform certain tasks which are impossible in the classical world. For example, we can consider teleportation and super dense coding. The teleportation is a quantum task in which an unknown quantum state is transmitted from a sender (Alice) to a spatially separated receiver (Bob) via an entangled quantum channel and with the help of some classical communications. The original scheme was proposed by Bennet et al. in 1993 [1]. Since then large number of teleportation schemes and their applications have been reported [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16]. Some of these teleportation schemes are also experimentally realized by different groups [17, 18, 19, 20, 21, 22]. The initial proposals of teleportation were meant for perfect teleportation of an unknown qubit $\alpha|0\rangle + \beta|1\rangle$. By perfect teleportation we mean that the success rate of teleportation is unity. This requires a maximally entangled quantum channel. But immediately after the pioneering work of Bennet it was realized that teleportation is possible even when the quantum channels are non-maximally entangled. In that case the success rate of Bob will not be unity and the teleportation scheme would be called probabilistic.

In last two decades several authors have proposed protocols for transformation of more complex quantum states. For example, schemes for teleportation of EPR pair $(\alpha|00\rangle + \beta|11\rangle)$ [3], an arbitrary two qubit state $\alpha|00\rangle + \beta|01\rangle + c|10\rangle + d|11\rangle$ [4] and an arbitrary $n$-qubit state [10] are proposed in recent past. At the same time, possibility of many-party quantum teleportation has been studied by different groups and these studies lead to schemes for controlled teleportation (CT) or quantum information splitting (QIS). In these schemes, Alice shares prior entanglement with Bob and at least one Charlie (supervisor). Charlie is supervisor in the sense that he can control the channel between Alice and Bob. Bob can properly construct the state sent by Alice if Charlie cooperates (i.e. if Charlie send the value of his measurements to Bob). Such schemes for controlled teleportation of an unknown qubit are recently studied by Karlsson [8], Jaewoo [14], Zhang et al. [2] and Yang et al. [3] by using different tripartite entangled states as quantum channel. To be precise, Karlsson has used GHZ state, Zhang et al. [2] and Yang et al. [3] have used GHZ-like state and Joo [14] has used W state. Here we would like to note that in 1999, Hillery [23] had proposed a protocol for quantum secret sharing (QSS). Now if we consider that the quantum secret is a quantum state then a QSS scheme reduces to a scheme for quantum state sharing (QSTS) or quantum information splitting (QIS). We can visualize the situation as if a secret (which is a quantum state) is teleported to Charlie and Bob and one of them (Bob) can obtain the secret (the quantum state) if the other one (Charlie) collaborates. Now it is easy to realize that in our context, $CT = QIS = QSTS \subset QSS$. Consequently the proposed protocols may found specific applications in quantum secret communication schemes. In a CT protocol the measurements of Alice and Charlie are communicated to Bob via classical communication channels and Bob uses these information to choose the unitary operation to be applied by him. Consequently, any CT protocol may be reduced to usual teleportation scheme either by keeping the Charlie’s bit with Alice or by communicating the bit to Bob. But that would only make the quantum channel more complex. Further we wish to add that Werner [24] had shown that there exist a one to one correspondence between dense coding and teleportation when these schemes are assumed to be tight. As we have already mentioned different quantum channels may be used to teleport the same quantum state. So far the choice of quantum channel is a bit ad hoc and a trend of introducing more and more complex quantum channels have been observed in recent past. For example, Brown state [27], W state [14], etc. are recently introduced as quantum channel for teleportation. But with the present
experimental efficiency it is quite difficult to construct and maintain such multi-particle entangled states. Keeping these facts in mind we have tried to minimize the quantum channel cost. To do so we have generalized the existing ideas and have shown that the non-maximally entangled state of generalized Bell-like can be teleported (perfectly or in controlled manner) by using an optimal quantum channel. The paper also establishes that there exist a family of quantum states of GHZ-type which can be used for perfect and probabilistic controlled teleportation. The paper also produces some of the recently reported results as special cases of the more general results obtained here. For example, teleportation schemes proposed by Cola [13], An Ba [15], Zhang et al. [24] and Yang et al. [3] etc. can be obtained as special cases of the present work.

In section 2 we have proposed a new scheme for teleportation of an unknown n-qubit non-maximally entangled state of the form \(|\psi\rangle = \alpha |x\rangle + \beta |\bar{x}\rangle\) by using a Bell state. In section 3 we have shown that the CT of the same state is possible if we use a GHZ state as quantum channel. In section 4 we have shown that the CT scheme described in section 3 may also be achieved by using any member of a family of 12 orthogonal GHZ-like states as quantum channel. Some of the existing results are also obtained as special cases of this protocol. Finally section 5 is dedicated to conclusions.

2 Protocol for teleportation of an unknown n-qubit non-maximally entangled state of the form \(|\psi\rangle = \alpha |x\rangle + \beta |\bar{x}\rangle\) using Bell state as quantum channel

Suppose we wish to teleport an n-qubit non-maximally entangled state of the form

\[
|\psi\rangle^\pm = \alpha |x\rangle \pm \beta |\bar{x}\rangle
\]

(1)

where \(\alpha^2 + \beta^2 = 1\), \(x\) varies from 0 to \(2^n - 1\) and \(\bar{x} = 1^{\otimes n} \oplus x\) in modulo 2 arithmetic. This state will reduce to generalized Bell state (GBS) [25] for \(\alpha = \beta = \frac{1}{\sqrt{2}}\). Usual Bell state and GHZ state are special cases of GBS for \(n = 2\) and \(n = 3\) respectively. Consequently a successful teleportation scheme for the above state will be able to teleport GHZ state, Bell state in general and EPR states in particular. Since \(\frac{1}{\sqrt{2}} (|x\rangle \pm |\bar{x}\rangle)\) is called generalized Bell state [23], we may call \(\alpha |x\rangle + \beta |\bar{x}\rangle\) as generalized Bell-type state. Now, without loss of generality we may consider the unknown generalized Bell-type state to be teleported as

\[
|\psi\rangle_{unknown} = \alpha |x\rangle + \beta |\bar{x}\rangle = \alpha |x_1x_2..x_n\rangle + \beta |\bar{x}_1\bar{x}_2..\bar{x}_n\rangle
\]

(2)

where \(x_i \in \{0, 1\}\). Alice may teleport this unknown state to Bob in following five steps:

Step 1: Let Alice and Bob share a Bell state of the form

\[
|\psi\rangle_{channel} = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle)_{AB}.
\]

(3)

The first qubit of this maximally entangled state is kept with Alice and the second qubit is sent to Bob. Thus this particular Bell state [3] constitutes our quantum channel. The proposed protocol is valid for any Bell state. Now the input state of the circuit (i.e. the state of the system after Alice receives the unknown n-qubit state to be teleported) is

\[
|\Psi_{in}\rangle = |\psi\rangle_{unknown} \otimes |\psi\rangle_{channel} = \frac{1}{\sqrt{2}} (\alpha |x_1x_2x_3..x_n00\rangle + \alpha |x_1x_2x_3..x_n11\rangle + \beta |\bar{x}_1\bar{x}_2\bar{x}_3..\bar{x}_n00\rangle + \beta |\bar{x}_1\bar{x}_2\bar{x}_3..\bar{x}_n11\rangle)_{A_1A_2..A_nAB}
\]

(4)

Here the first \((n + 1)\) qubits are with Alice and the last qubit is with Bob. Step 2: Alice performs \(n\) Cnot operations on her qubits (as shown in Fig. 1) by using the first qubit (\(A_1\)) as the control qubit and the next \(n\) qubits as target qubits.
Then Alice measures her entangled qubits by applying a measurement. This operation transforms the input state of the system (4) to

\[
|\Psi\rangle_1 = \frac{1}{\sqrt{2}} [\alpha |x_1 (x_1 + x_2) (x_1 + x_3) ... (x_1 + x_n) x_10 \rangle + \beta |\bar{x}_1 (x_1 + x_2) (x_1 + x_3) ... (x_1 + x_n) x_10 \rangle]_{A_1 A_2 ... A_n AB}
\]

Here we may use the identities

\[
(a \oplus b) = (\bar{a} \oplus \bar{b}) \\
(a \oplus 0) = a \\
(a \oplus 1) = \bar{a}
\]

where \(a, b \in \{0, 1\}\), to obtain

\[
|\Psi\rangle_1 = \frac{1}{\sqrt{2}} [\alpha |x_1 (x_1 + x_2) (x_1 + x_3) ... (x_1 + x_n) x_10 \rangle + \alpha |x_1 (x_1 + x_2) (x_1 + x_3) ... (x_1 + x_n) \bar{x}_11 \rangle + \beta |\bar{x}_1 (x_1 + x_2) (x_1 + x_3) ... (x_1 + x_n) \bar{x}_10 \rangle + \beta |\bar{x}_1 (x_1 + x_2) (x_1 + x_3) ... (x_1 + x_n) \bar{x}_11 \rangle]_{A_1 A_2 ... A_n AB}
\]

Step 3: Alice applies a Hadamard operation [8] on her first qubit \(|x_1\rangle\) to transform the state \(|\Psi\rangle_1\) to

\[
|\Psi\rangle_2 = \frac{1}{\sqrt{2}} [\alpha (-1)^{x_1} |x_1 (x_1 + x_2) (x_1 + x_3) ... (x_1 + x_n) x_10 \rangle + \alpha |x_1 (x_1 + x_2) (x_1 + x_3) ... (x_1 + x_n) \bar{x}_11 \rangle + \beta (-1)^{x_1} |\bar{x}_1 (x_1 + x_2) (x_1 + x_3) ... (x_1 + x_n) \bar{x}_10 \rangle + \beta (-1)^{x_1} |\bar{x}_1 (x_1 + x_2) (x_1 + x_3) ... (x_1 + x_n) \bar{x}_11 \rangle]_{A_1 A_2 ... A_n AB}
\]

By using \((x_1 + x_n) = e_{n-1}\) we can obtain

\[
|\Psi\rangle_2 = \frac{1}{2} [\alpha (-1)^{x_1} |xe_1 e_2 ... e_{n-1} x_10 \rangle + \alpha |\bar{x}_1 e_1 e_2 ... e_{n-1} \bar{x}_10 \rangle + \alpha (-1)^{x_1} |xe_1 e_2 ... e_{n-1} \bar{x}_11 \rangle + \alpha |\bar{x}_1 e_1 e_2 ... e_{n-1} \bar{x}_11 \rangle + \beta (-1)^{x_1} |\bar{x}_1 e_1 e_2 ... e_{n-1} \bar{x}_10 \rangle + \beta |\bar{x}_1 e_1 e_2 ... e_{n-1} \bar{x}_11 \rangle + \beta (-1)^{x_1} |xe_1 e_2 ... e_{n-1} \bar{x}_11 \rangle + \beta |xe_1 e_2 ... e_{n-1} x_11 \rangle]_{A_1 A_2 ... A_n AB}
\]

Here we can easily observe that Alice’s first and last qubits \((A_1\text{ and }A)\) are entangled with the Bob’s qubit \((B)\) and the remaining \((n - 1)\) qubits of Alice (i.e. \(A_2\text{ and }...\text{A}_n\)) are separable.

Step 4: Alice measures all her qubits in computational basis and classically communicate the results to Bob. For simplicity of visualization we may consider this measurement as two step process in which Alice measures the separable qubits first by applying a measurement \(M_{n-1} = M_{A_2 A_3 ... A_n}\) in computational basis. This measurement reduces the state to

\[
|\Psi\rangle_3 = \frac{1}{2} [( -1)^{x_1} |x_1 x_1\rangle_{A_1} (\alpha |0\rangle + (-1)^{x_1} \beta |1\rangle)_B + |\bar{x}_1 x_1\rangle_{A_1} (\alpha |0\rangle + ( -1)^{x_1} \beta |1\rangle)_B + (-1)^{x_1} |x_1 \bar{x}_1\rangle_{A_1} (\alpha |1\rangle + ( -1)^{x_1} \beta |0\rangle)_B + |\bar{x}_1 \bar{x}_1\rangle_{A_1} (\alpha |1\rangle + ( -1)^{x_1} \beta |0\rangle)_B]_{A_1 A_2 ... A_n AB}
\]

Then Alice measures her entangled qubits by applying a measurement \(M_{A_1 A}\) . We will show in the next step that Bob can reconstruct the initial state \([8]\) by using the values of \(e_{1...e_{n-1}}\) provided he has \(\alpha |x_1\rangle + \beta |\bar{x}_1\rangle\).

Step 5: Now it is easy to observe from \([10]\) that Bob may construct \(\alpha |x_1\rangle + \beta |\bar{x}_1\rangle\) by using the result of \(M_{A_1 A}\). This is so because from \([10]\) we observe that if the measurement on Alice’s first qubit yield 0 then we do not need to change the relative phase of Bob’s state. On the other hand if the measurement yield 1 then Bob has to apply a phase flip operation \(\text{“Z”}\) to get the correct relative phase. Again if the second measurement yield zero then the bit values are proper but if the second measurement yield 1 then Bob has to apply a bit flip operation \(\text{“X”}\) to obtain \(\alpha |x_1\rangle + \beta |\bar{x}_1\rangle\). To be more precise, according to the outcome of \(M_{A_1 A}\), Bob applies suitable gates as shown in Table 1 to his qubit [8]. After constructing \(\alpha |x_1\rangle + \beta |\bar{x}_1\rangle\), Bob creates \((n - 1)\) ancilla qubits \(|B_1 B_2 ... B_{n-1}\rangle\) in accordance to the outcome of \(M_{n-1}\)

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1 A Hadamard operation is defined by

\[H|a\rangle = \frac{(-1)^{a} |a\rangle + |\bar{a}\rangle}{\sqrt{2}}\] where \(a \in \{0, 1\}\)

2 Global phase is ignored here.
The input state of this circuit for CT is shown in Fig. 3. Controlled teleportation of an unknown Gorbachev’s circuit [6] for the same purpose requires a GHZ channel, 16 gates and quantum cost of the circuit is 10. Thus the special case of classical communication can be replaced by non-local unitary operation we can convert the generalized circuit shown in Fig. 1 to a circuit for measurement less teleportation of a 2-qubit non-maximally entangled quantum state as shown in Fig 2. This is done only for comparison purpose. For teleporting a 2-qubit non-maximally entangled quantum state we need a Bell state to quantum channel, 9 gates and quantum cost [26] of our circuit is 7 where as Gorbachev’s circuit [6] for the same purpose requires a GHZ channel, 16 gates and quantum cost of the circuit is 10. Thus the special case of our circuit (shown in Fig. 2) has lesser gate count, lesser quantum cost and lesser quantum channel cost as compared to Gorbachev’s circuit [6] for the same purpose.

3 Controlled teleportation of an unknown $n$-qubit non-maximally entangled state of the form $\alpha |x\rangle + \beta |\bar{x}\rangle$ using a GHZ state as quantum channel.

It is already mentioned in the introduction that the CT is equivalent to QIS and QSTS. In this multiparty teleportation scheme Alice sends a secret message to a party say Bob in cooperation with another party Charlie. Bob can create the secret message with the information obtained from Alice and Charlie. In our case the message is generalized scheme Alice sends a secret message to a party say Bob in cooperation with another party Charlie. Bob can create the message generalized as control qubit and rest $(n-1)$ qubits as target qubits. and retrieve the unknown state [2]. As the measurement and classical communication can be replaced by non-local unitary operation we can convert the generalized circuit shown in Fig. 1 to a circuit for measurement less teleportation of a 2-qubit non-maximally entangled quantum state as shown in Fig 2. This is done only for comparison purpose. For teleporting a 2-qubit non-maximally entangled quantum state we need a Bell state as quantum channel, 9 gates and quantum cost [26] of our circuit is 7 where as Gorbachev’s circuit [6] for the same purpose requires a GHZ channel, 16 gates and quantum cost of the circuit is 10. Thus the special case of our circuit (shown in Fig. 2) has lesser gate count, lesser quantum cost and lesser quantum channel cost as compared to Gorbachev’s circuit [6] for the same purpose.

Let Alice, Bob and Charlie share a GHZ state as quantum channel

$$|\psi\rangle_{ABC} = \frac{1}{\sqrt{2}} (|000\rangle + |111\rangle)_{ABC} \quad (11)$$

The first qubit of the circuit is kept with Alice and the second and third qubits are sent to Bob and Charlie respectively. The input state of this circuit for CT is shown in Fig. 3.

$$|\Psi\rangle = |\psi\rangle_{unknown} \otimes |\psi\rangle_{channel} = \alpha |x_1 x_2 x_3 \ldots x_n 000\rangle + \alpha |x_1 x_2 x_3 \ldots x_n 111\rangle + \beta |x_1 x_2 x_3 \ldots x_n 000\rangle + \beta |x_1 x_2 x_3 \ldots x_n 111\rangle |A_1 A_2 \ldots A_n ABC$$ \quad (12)

After application of $(n-1)$ Cnots, the input state of the system transforms to

$$|\Psi\rangle_1 = |\alpha x_1 (x_1 \oplus x_2) \ldots (x_1 \oplus x_n) 000\rangle + |\alpha x_1 (x_1 \oplus x_2) \ldots (x_1 \oplus x_n) (x_1 \oplus 1) 111\rangle + |\beta x_1 (x_1 \oplus x_2) \ldots (x_1 \oplus x_n) 000\rangle + |\beta x_1 (x_1 \oplus x_2) \ldots (x_1 \oplus x_n) 111\rangle |A_1 A_2 \ldots A_n ABC$$ \quad (13)

Upto this point the CT scheme is similar to the usual teleportation scheme described in last section but after this point Charlie applies the Hadamard transformation on her qubit. This operation of Charlie transforms the state of the system to

| $M_{A_1 A_2}$ | Bob’s state | Gates | Bob’s state after production of Ancilla | Bob’s final state |
|----------------|-------------|------|-----------------------------------|------------------|
| [00] $\alpha |0\rangle + \beta |1\rangle$ | I | $\alpha (0e_1e_2\ldots e_{n-1}) + \beta |1e_1e_2\ldots e_{n-1}\rangle$ | $\alpha |x_1\ldots x_n\rangle + \beta |x_1\ldots x_n\rangle$ |
| [01] $\alpha |1\rangle + \beta |0\rangle$ | X | $\alpha (0e_1e_2\ldots e_{n-1}) + \beta |1e_1e_2\ldots e_{n-1}\rangle$ | $\alpha |x_1\ldots x_n\rangle + \beta |x_1\ldots x_n\rangle$ |
| [10] $\alpha |0\rangle - \beta |1\rangle$ | Z | $\alpha (0e_1e_2\ldots e_{n-1}) - \beta |1e_1e_2\ldots e_{n-1}\rangle$ | $\alpha |x_1\ldots x_n\rangle + \beta |x_1\ldots x_n\rangle$ |
| [11] $\alpha |1\rangle - \beta |0\rangle$ | iY | $\alpha (0e_1e_2\ldots e_{n-1}) - \beta |1e_1e_2\ldots e_{n-1}\rangle$ | $\alpha |x_1\ldots x_n\rangle + \beta |x_1\ldots x_n\rangle$ |

Table 1: Quantum gates applied by Bob according to different measurement outcomes of Alice.
and respectively. From (16) and (17) one can easily see that if Alice measures $A_1$ using Bell analyzers $M_{BM}$ while Charlie measures his qubit by a measurement $M_C$ in computational basis then the state of Bob reduces to a one qubit state which can be transformed into the state $\alpha |x_1\rangle + \beta |\bar{x}_1\rangle$ by applying an unitary transformation $U_B$ as shown in Table 2. The choice of the particular unitary operation depends on the outcome of the measurement of Alice ($M_{BM}$) and that of Charlie ($M_C$). Therefore, it is required that the Alice and Charlie sent their measurement outcomes $M_{BM}$ and $M_C$ to Bob using classical channels and after receiving them, Bob chooses the unitary operations as per the Table 2. According to measurement $M_{n-1}$ Bob will create $n-1$ ancillas $|B_1B_2B_{n-1}\rangle$, such that $|B_1B_2B_{n-1}\rangle = |e_1,e_{n-1}\rangle$. Finally Bob performs $n-1$ Cnot operations as shown in Fig. 3 to reconstruct the unknown state $|\psi\rangle_{unknown}$.
Recently it is reported by Zhang et al. \cite{GHZ} that CT of an unknown qubit is possible if Alice, Bob and Charlie share a GHZ-like tripartite maximally entangled state of the form $|\phi\rangle = \frac{1}{\sqrt{2}} (|000\rangle + |111\rangle + |101\rangle + |011\rangle)$. Almost in the same time Yang et al. have independently shown that the same task may be achieved by using $|\phi\rangle = \frac{1}{\sqrt{2}} (|001\rangle + |100\rangle + |010\rangle + |111\rangle)$.

Here we have shown that these states are not unique and there exists an underlying symmetry. To be precise, not only CT of an unknown qubit but CT of an $n$-qubit non-maximally entangled state of the form $|\psi\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle)$ is possible if we use quantum channel of the form

$$|\psi\rangle = \frac{1}{\sqrt{2}} \left( |0\rangle + e^{i\phi} |1\rangle \right)$$

where $i, j \in \{0, 1, 2, 3\}, i \neq j$ and $\psi_{i,j}$ are Bell states usually denoted as

$$\psi_i = |\psi_{i0}\rangle = |\psi_i^+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}, \quad \psi_j = |\psi_{j1}\rangle = |\psi_j^-\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}.$$

For the calculational simplicity we may use a more compact notation in which a Bell state is written as

$$\psi_i = \frac{|0,y\rangle + e^{i\phi} |1,x\rangle}{\sqrt{2}}$$

where $x, y$ are two digit binary representation of $i$ and $j$ respectively for example for $\psi_2 = |\psi_{10}\rangle, x = 1$ and $y = 0$. Now we can use this compact notation \cite{Yang} to describe the state of the quantum channel in general as

$$\frac{1}{\sqrt{2}} \left( \psi_i |0\rangle + \psi_j |1\rangle \right)_{ABC} = \frac{1}{\sqrt{2}} \left( \psi_{x,y} |0\rangle + e^{i\phi} \psi_{x',y'} |1\rangle \right)_{ABC} = \frac{1}{\sqrt{2}} \left( |0y0\rangle + (-1)^z |1y\rangle + |0y'1\rangle + (-1)^z |1y'\rangle \right)_{ABC}.$$

The subscripts $ABC$ used in \cite{Yang} indicates that the first qubit of the channel is kept with Alice and the second and third qubits are sent to Bob and Charlie respectively. The quantum circuit for generation of these states is given in Fig. 4. Now we may perform CT of an $n$-qubit state of the form \cite{Yang} by using \cite{Yang} as quantum channel. The circuit designed for this purpose is shown in Fig. 5.

Here the unknown state to be teleported is \cite{Yang} and the quantum channel to be used for the purpose is \cite{Yang}. Consequently the input state of the system is

$$|\Psi\rangle_1 = |\psi\rangle_{\text{unknown}} \otimes |\psi\rangle_{\text{channel}}$$

$$= \left( \left[ \alpha |a_1, a_2, ..., a_n\rangle + \beta |d_1, d_2, ..., d_n\rangle \right] \otimes \frac{1}{\sqrt{2}} \left( |0y0\rangle + (-1)^z |1y\rangle + |0y'1\rangle + (-1)^z |1y'\rangle \right) \right)_{A_1, A_2, ..., A_n, ABC}.$$

Now Alice performs $n$-CNOT operations on her qubits (this step is same as step 2 of Section 2) in accordance with Fig. 5. This operation transforms the input state $|\Psi\rangle_1$ into

\[\text{Figure 4: A circuit to create GHZ-like states $|u\rangle|v\rangle|w\rangle$ are the input bits to the circuit and the output is a GHZ-like state.}\]

\[\text{Figure 5: CT of $\alpha |x\rangle + \beta |\bar{x}\rangle$ using GHZ-like quantum state as quantum channel.}\]
The present proposals are already experimentally achieved \cite{17, 18, 19, 20, 21, 22}. We may claim that our protocols which extended for CT of much more general and complex state of the form (2) are economical compared to most of these proposals because our quantum channel consumes only one Bell state. Since we have used only a Bell state for perfect quantum teleportation, Yang et al. \cite{2} did special cases of the present work. But unfortunately there are some mistakes in the work of Yang et al. \cite{3}. Present work provides a generalized perception of the quantum channel and provide more options for experimentalist to experimentally realize the quantum information splitting. Furthermore the earlier protocols \cite{2, 3} were designed for CT of an unknown qubit \( |0\rangle \) but here it is extended for CT of much more general and complex state of the form \( |\psi\rangle = |0\rangle + \beta |1\rangle \).

\[
|\Psi\rangle_2 = (\alpha |a_1 \rangle \otimes a_2 \) \ldots (a_1 \otimes a_n) + \beta (|\bar{a}_1 \rangle \otimes \bar{a}_2 \) \ldots (|\bar{a}_1 \rangle \otimes |\bar{a}_n\rangle) \otimes \frac{1}{\sqrt{2}} \left( (|0\rangle |0\rangle + (-1)^z |1\rangle |0\rangle) + |0\rangle |y\rangle + (-1)^z |1\rangle |y\rangle) \right)_{A_1 A_2 \ldots A_n} ABC \\
= (\alpha |a_1 e_1 \ldots e_{n-1}\rangle + \beta |\bar{a}_1 e_1 \ldots e_{n-1}\rangle) \otimes \frac{1}{\sqrt{2}} \left( (|0\rangle |0\rangle + (-1)^z |1\rangle |0\rangle) + |0\rangle |y\rangle + (-1)^z |1\rangle |y\rangle) \right)_{A_1 A_2 \ldots A_n} ABC
\]

Alice measures the separable qubits \((A_2, A_3, \ldots, A_n)\) in computational basis and this correspond to the measurement \(M_{n-1}\). This measurement reduces the state \( |\Psi\rangle_2 \) to \(|S\rangle_{A_1 AB} |0\rangle_C + |T\rangle_{A_1 AB} |1\rangle_C \) where \(|S\rangle_{A_1 AB} \) and \(|T\rangle_{A_1 AB} \) can be written as

\[
|S\rangle_{A_1 AB} = \frac{1}{\sqrt{2}} \left( (\alpha |a_1 0\rangle + (-1)^z \alpha |a_1 1\rangle + \beta |\bar{a}_1 0\rangle + (-1)^z \beta |\bar{a}_1 1\rangle) \right)_{A_1 AB} \\
= \frac{1}{\sqrt{2}} \left( (\alpha |0\rangle + (-1)^z \alpha |1\rangle) \otimes (\alpha |y\rangle + \beta |\bar{y}\rangle) \right)_{A_1 A} \\
+ \frac{1}{\sqrt{2}} \left( (\alpha |0\rangle + (-1)^z \alpha |1\rangle) \otimes (\alpha |y\rangle + \beta |\bar{y}\rangle) \right)_{A_1 A} \otimes (\alpha |y\rangle - \beta |\bar{y}\rangle)
\]

and

\[
|T\rangle_{A_1 AB} = \frac{1}{\sqrt{2}} \left( (\alpha |a_1 0\rangle + (-1)^z \alpha |a_1 1\rangle + \beta |a_0 0\rangle + (-1)^z \beta |a_0 1\rangle) \right)_{A_1 AB} \\
= \frac{1}{\sqrt{2}} \left( (\alpha |a_0 0\rangle + (-1)^z \alpha |a_0 1\rangle) \otimes (\alpha |y\rangle + \beta |\bar{y}\rangle) \right)_{A_1 A} \\
+ \frac{1}{\sqrt{2}} \left( (\alpha |a_0 0\rangle + (-1)^z \alpha |a_0 1\rangle) \otimes (\alpha |y\rangle + \beta |\bar{y}\rangle) \right)_{A_1 A} \otimes (\alpha |y\rangle - \beta |\bar{y}\rangle)
\]

respectively. From \cite{21}, \cite{22} and \cite{24} one can easily see that if Alice measures the qubits 1 and 2 using Bell analyzers \(M_{BM}\) and Charlie measures his qubit by a measurement \(M_C\) in computational basis then the state of Bob reduces to a one qubit state which can be transformed into the unknown qubit state by applying an unitary transformation. The choice of the particular unitary operation depends on the outcome of the measurement of Alice \((M_{BM})\) and that of Charlie \((M_C)\). Therefore, when Alice and Charlie sent the outcomes of their measurements \((M_{BM} \text{ and } M_C)\) to Bob using classical channels then Bob can choose appropriate unitary operations as per the Table 3 to construct \(\alpha |x_1\rangle + \beta |\bar{x}_1\rangle\). After that Bob creates \(n-1\) ancillas \(|B_1 B_2 \ldots B_{n-1}\rangle\), such that \(|B_1 B_2 \ldots B_{n-1}\rangle = |e_1 \ldots e_{n-1}\rangle\) and follow the last part of the protocol described in section 2 to reconstruct the unknown state with the help of \(n-1\) Cnot operations as shown in Fig. 4.

From Table 3, it is clear that the every member of a family of quantum channels described by \(\frac{|\psi_0\rangle + |\psi_1\rangle}{\sqrt{2}}\) can be used for multiparty quantum teleportation. Yang et al. \cite{3} and Zhang et al. \cite{2} did special cases of the present work. But unfortunately there are some mistakes in the work of Yang et al. \cite{3}. Present work provides a generalized perception of the quantum channel and provide more options for experimentalist to experimentally realize the quantum information splitting. Further the earlier protocols \cite{2, 3} were designed for CT of an unknown qubit \(\alpha |0\rangle + \beta |1\rangle\) only but here it is extended for CT of much more general and complex state of the form \(\alpha |\psi\rangle\).

5 Conclusions

In our protocol we have used maximally entangled states as quantum channel. It is straightforward exercise to extend these protocols to the situations in which non-maximally entangled states of the similar form are used as quantum channel. In that case the success probability of the teleportation will not be unity. Thus the teleportation schemes would become probabilistic. For example, if we use \((\alpha |\psi_1\rangle |0\rangle + \beta |\psi_2\rangle |1\rangle)\) as the teleportation channel in place of GHZ-like state in Section 4 or \(\alpha |00\rangle + \beta |11\rangle\) in place of \(\frac{|00\rangle + |11\rangle}{\sqrt{2}}\) in Section 2 then we will obtain a probabilistic teleportation scheme. Of course the nature of unitary operations conducted by Bob will be different. Similarly the existing schemes for probabilistic teleportation can be easily converted to the scheme for perfect teleportation. There are several proposals for perfect and probabilistic teleportation \cite{10, 11, 16, 19}. Our scheme is economical compared to most of these proposals because our quantum channel consumes only one Bell state. Since we have used only a Bell state for perfect quantum teleportation of \(n\)-qubit Bell state and a GHZ for controlled teleportation of \(n\)-qubit Bell state. It is obvious that the minimum amount of non-local quantum resource is used here as quantum channel. The same amount of resource are used by An Ba \cite{15} in their excellent proposal of teleporting \(\alpha |0\rangle^\otimes n + \beta |1\rangle^\otimes n\) using an EPR state. Our proposal is more general as it can teleport \(\alpha |x\rangle + \beta |\bar{x}\rangle\) in general. Further An Ba’s protocol may be obtained as a special case of our protocol. As in An Ba case \(e_1 e_2 \ldots e_{n-1}\) will always be \(|0\rangle^\otimes n-1\). We don’t need the corresponding Cnot gates in encoding circuit in Fig. 1. We don’t even need to measure or classically communicate them. Consequently, Bob always prepares his ancillas in state \(|B_1 B_2 \ldots B_{n-1}\rangle = |0\rangle^\otimes n-1\). The present protocol is not only more generalized compared to An Ba protocol \cite{15} but it is much more simpler too. Further our protocol always require only two measurement and two classical communications to teleport \(\alpha |0\rangle^\otimes n + \beta |1\rangle^\otimes n\) but the An Ba protocol always require three measurements and sometime three classical communications too. Since the quantum gates, the quantum channels and the measurements used in the present proposals are already experimentally achieved \cite{17, 18, 19, 20, 21, 22}. We may claim that our protocols which
| Quantum Channel | $M_{BM}$ | $M_C$ | $U$ | $M_C$ | $U_B$ |
|-----------------|----------|-------|-----|-------|-------|
| $\frac{1}{\sqrt{2}} (\psi^+ |0\rangle + |\psi^- |1\rangle)$ | $\phi^+_{A1,A}$ | $|0\rangle C$ | $1$ | $|1\rangle C$ | $Z$ |
| $\frac{1}{\sqrt{2}} (\psi^- |0\rangle + |\phi^+ |1\rangle)$ | $\phi_{A1,A}$ | $|0\rangle C$ | $Z$ | $|1\rangle C$ | $1$ |
| $\frac{1}{\sqrt{2}} (\psi^- |0\rangle + |\phi^- |1\rangle)$ | $\phi^+_{A1,A}$ | $|0\rangle C$ | $X$ | $|1\rangle C$ | $XZ$ |
| $\frac{1}{\sqrt{2}} (\psi^- |0\rangle + |\phi^- |1\rangle)$ | $\phi^-_{A1,A}$ | $|0\rangle C$ | $X$ | $|1\rangle C$ | $XZ$ |
| $\frac{1}{\sqrt{2}} (\psi^- |0\rangle + |\phi^- |1\rangle)$ | $\phi^-_{A1,A}$ | $|0\rangle C$ | $Z$ | $|1\rangle C$ | $1$ |
| $\frac{1}{\sqrt{2}} (\psi^- |0\rangle + |\phi^- |1\rangle)$ | $\phi_{A1,A}$ | $|0\rangle C$ | $X$ | $|1\rangle C$ | $XZ$ |
| $\frac{1}{\sqrt{2}} (\phi^+ |0\rangle + |\phi^- |1\rangle)$ | $\phi^+_{A1,A}$ | $|0\rangle C$ | $Z$ | $|1\rangle C$ | $XZ$ |
| $\frac{1}{\sqrt{2}} (\phi^- |0\rangle + |\phi^+ |1\rangle)$ | $\phi_{A1,A}$ | $|0\rangle C$ | $X$ | $|1\rangle C$ | $1$ |
| $\frac{1}{\sqrt{2}} (\phi^+ |0\rangle + |\phi^- |1\rangle)$ | $\phi^+_{A1,A}$ | $|0\rangle C$ | $X$ | $|1\rangle C$ | $XZ$ |
| $\frac{1}{\sqrt{2}} (\phi^- |0\rangle + |\phi^+ |1\rangle)$ | $\phi_{A1,A}$ | $|0\rangle C$ | $Z$ | $|1\rangle C$ | $1$ |
| $\frac{1}{\sqrt{2}} (\phi^+ |0\rangle + |\phi^- |1\rangle)$ | $\phi^+_{A1,A}$ | $|0\rangle C$ | $X$ | $|1\rangle C$ | $XZ$ |
| $\frac{1}{\sqrt{2}} (\phi^- |0\rangle + |\phi^+ |1\rangle)$ | $\phi_{A1,A}$ | $|0\rangle C$ | $Z$ | $|1\rangle C$ | $1$ |

Table 4: Controlled teleportation of $\alpha|x\rangle + \beta|x\rangle$ is possible by using quantum channels of the form $\frac{(\psi^+ |0\rangle + |\psi^- |1\rangle)}{\sqrt{2}}$. Yang et al. [??] have used $\frac{1}{\sqrt{2}} (\psi^+ |1\rangle + |\phi^+ |0\rangle)$ as their quantum channel and Zhang et al. [??] have used $\frac{1}{\sqrt{2}} (\psi^+ |0\rangle + |\phi^+ |1\rangle)$ as their quantum channel for controlled teleportation of $\alpha |0\rangle + \beta |1\rangle$. $U_B$ denotes the unitary operation to be applied by Bob after receiving the outcome of $M_{A1,A}$ and $M_C$. 


involve minimum non-local quantum resource is experimentally achievable. The biggest advantage of the proposed circuits lies in the fact that maintaining a multi-partite entangled quantum channel is costly. In our case the channel is optimal as far as the number of qubits are concerned. We have proposed two schemes for CT. One is by using GHZ state as quantum channel; and other is by using GHZ-like states as quantum channel. It is needless to say that the works in [2] and [3] are just special cases of our proposal. Further we would like to note that An Ba generalized the work of Cola [13] and consequently the work of Cola is a special case of An Ba and since An Ba protocol is special case of our protocol so Cola [13] protocol is also a special case of our protocol. Further an improved version of Gorbachev protocol for teleportation of 2-qubit entangled state is also obtained as a special case here. Another advantage of the present work lies in the fact that unitary operations to be performed by Bob is always one qubit operation. In many of the existing protocols Bob needs to implement more complex quantum gates. We end this paper with a justified expectation that the present proposals and their variants will find applications in dense coding, measurement less quantum error correction and secured quantum communication.

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References

[1] C. H. Bennet et al., Phys. Rev. Lett. 70 (1993) 1895.
[2] Q. Y. Zhang et al., Int. J. Theo. Phys. 48 (2009) 3331.
[3] K. Yang et al., Int. J. Theo. Phys. 48 (2009) 516.
[4] T. Gao, Z. Wang and F. Yan, Chinese Phys. Lett 20 (2003) 2094.
[5] B. S. Shi, Y. K. Jiang and G. C. Guo, Phys. Lett. A 268 (2000) 161.
[6] V. N. Gorbachev and A. I. Trubiko, J. Exp. Theo. Phys. 91 (2000) 894.
[7] R. F. Werner, J. Phys. A 34 (2001) 7081.
[8] A. Karlsson and M. Bourennane, Phys. Rev. A 58 (1998) 4394.
[9] W. L. Li, C. F. Li and G. C. Guo, Phys. Rev. A 61 (2000) 034301.
[10] H. Lu and G. C. Guo, Phys. Lett. A 276 (2000) 209.
[11] B. S. Shi and A. Tomita, Phys. Lett. A 296 (2002) 161.
[12] L. Vaidman, Phys. Rev. A 49 (1994) 1473.
[13] M. M. Cola and M. G. A. Paris, Phys. Lett. A 337 (2005) 10.
[14] J. Jaewoo and Y. J. Park, New J. Phys. 5 (2003) 136.
[15] N. B. An, Adv. Studies Theor. Phys. 1 (2007) 489.
[16] M. Cao and S. Zhu, Commun. Theor. Phys. 43 (2005) 69.
[17] D. Bouwmeester et al., Nature 390 (1997) 575.
[18] M. A. Nielsen, E. Knill and R. Laflamme, Nature 396 (1998) 52.
[19] A. Furusawa et al., Science 282 (1998) 706.
[20] Z. Zhao et al., Nature 430 (2004) 54.
[21] M. Riebe et al., Nature 429 (2004) 734.
[22] M. D. Barret et al., Nature 429 (2004) 737.
[23] M. Hillery, V. Buzek and A. Bertaiume, Phys. Rev. A 59 (1999) 1829.
[24] R. F. Werner, J. Phys. A: Math. Gen. 34 (2001) 7081.
[25] M. Gupta, A. Pathak, R. Srikanth and P. Panigrahi, AIP Conference Proceedings 864 (2006) 197.
[26] A. Banerjee and A. Pathak, Appl. Math. Info. Sc. (2010) in press.
[27] S. Muralidharan and P. K. Panigrahi , Phys.Rev.A 77 (2008) 032321.