Optimal Point-to-Point Codes in Interference Channels: An Incremental I-MMSE approach

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Abstract

A recent result of the authors shows a so-called I-MMSE-like relationship that, for the two-user Gaussian interference channel, an I-MMSE relationship holds in the limit, as $n \to \infty$, between the interference and the interfered-with receiver, assuming that the interfered-with transmission is an optimal point-to-point sequence (achieves the point-to-point capacity). This result was further used to provide a proof of the “missing corner points” of the two-user Gaussian interference channel. This paper provides an information theoretic proof of the above-mentioned I-MMSE-like relationship which follows the incremental channel approach, an approach which was used by Guo, Shamai and Verdú to provide an insightful proof of the original I-MMSE relationship for point-to-point channels. Finally, some additional applications of this result are shown for other multi-user settings: the Gaussian multiple-access channel with interference and specific K-user Gaussian Z-interference channel settings.

I. INTRODUCTION

A fundamental relationship between information theory and estimation theory has been revealed by Guo, Shamai and Verdú [1]. This relationship in its basic form regards the input and output of an additive Gaussian noise channel and relates the input-output mutual information to the minimum mean-square error (MMSE) when estimating the input from the output. This basic relationship holds for any arbitrary input distribution to the channel as long as the mutual information is finite [2]. Moreover it extends from the scalar channel to the vector channel and holds for any dimension $n$. This relationship, referred to as the I-MMSE relationship, has had many extensions and more importantly many applications. It has been shown to provide insightful proofs to known results in information theory and multi-user information theory and extend upon them to provide new observations (see [3] and [4] for a more general overview of this relationship).

In a recent result by the authors [5] the I-MMSE relationship has been used to examine the two-user Gaussian interference channel. More specifically, the work examined the Gaussian Z-interference channel assuming that the interfered-with user transmits at maximum rate (as if there is no interference). The work has shown that the rate of the interference must be limited as if it must also be reliably decoded by the interfered-with receiver while considering the interfered-with transmission as independent and identically distributed (i.i.d.) Gaussian noise. This
result resolved the “missing corner point” of the capacity region of the two-user Gaussian interference channel. The central result in [5] which allowed us to conclude the above is an I-MMSE like relationship between the interference and the interfered-with receiver, meaning that the same I-MMSE relationship holds when the input is the interference and the output is the interfered-with receiver. However, this relationship holds only in the limit, as \( n \to \infty \), and only at a limited range of SNRs. Given this relationship the conclusion regarding the two-user Gaussian interference channel follows directly. The proof of this I-MMSE like relationship given in [5] is an estimation theoretic proof. In this work we revisit this main result and show that a more elegant proof which follows one of the more insightful proofs of the I-MMSE relationship in [1], the incremental channel proof, can be established. This new proof provides additional support and insight into this result.

In parallel to [5] the problem of the “missing corner point” of the two-user Gaussian interference channel capacity region has been investigated and resolved also by Polyanskiy and Wu [6]. Although the conclusions are similar, meaning that the effect of an optimal point-to-point transmission on the interference in terms of information theoretic measures is as if an i.i.d. Gaussian input has been transmitted, the approach is quite different. We consider their work in more detail so as to emphasize the differences between the two methods of proof.

Finally, in this work we also consider some applications of this result to more elaborate channel models. We show that since the Gaussian multiple access channel (MAC) has a combined transmission which behaves as an optimal point-to-point sequence we can describe a subset of the capacity region of the MAC with interference. We also examine two operationally significant settings of the K-user Gaussian Z-interference channel.

The rest of this work is structured as follows: we begin in Section II with the model, the I-MMSE like relationship, which is the core of this work, and explain in detail why we refer to it as such. Section III is the core of the paper and provides the main steps of the incremental channel approach proof. In this section we first briefly review the original proof as given in [1] and only then detail the proof of the I-MMSE like relationship emphasizing the differences between the two proofs. As will be detailed the main differences require two important results given in Theorems 2 and 3 which will be discussed in the following sections. Moreover, Section III-C discusses the differences between the I-MMSE like approach to the proof of the “missing corner point” the proof of Polyanskiy and Wu [6]. Section IV provides the proof of Theorem 2 and Section V provides the proof of Theorem 3. The MAC with interference is considered in Section VI and the K-user Gaussian Z-interference settings are considered in Section VII. We conclude the paper with a short summary in Section VIII.

II. THE MODEL AND I-MMSE-LIKE RELATIONSHIP

Consider the following sequence of channel outputs parametrized by \( \gamma \):

\[
Y_n(\gamma) = \sqrt{\gamma \text{snr}_2} Z_n + \sqrt{\gamma \text{snr}_1} X_n + N_n
\]  

(1)

where \( N_n \) represents a standard additive Gaussian noise vector with independent components. \( X_n \) and \( Z_n \) are independent of each other (no cooperation between the transmitters) and independent of the additive Gaussian noise vector. We further assume, for simplicity and without loss of generality, that both inputs are zero-mean. The
subscript $n$ denotes that all vectors are length $n$ vectors. $\text{snr}_1$ and $\text{snr}_2$ are both non-negative scalar parameters and allow us to assume an average power constraint of 1 on $X_n$, without loss of generality, that is,

$$\frac{1}{n} \mathbb{E}\{\|x_n\|^2\} \leq 1$$

(2)

where $\| \cdot \|$ denotes the Euclidian norm. The parameter $a$ is also a non-negative scalar parameter, and is used here for consistency with the two-user Gaussian interference problem discussed in [5]. We assume that there is a sequence of point-to-point capacity achieving codebooks (i.e., that approach capacity, as $n \to \infty$, with vanishing probability of error). $X_n$ carries a message from the length $n$ codebook. Thus, when $n \to \infty$,

$$R_x = \frac{1}{2} \log (1 + \text{snr}_1),$$

(3)

where $R_x$ denotes the rate achieved in transmitting the message carried by $X_n$. The above sequence of channel outputs contains $Y_n(1)$ when $\gamma = 1$ which is depicted in Figure 1. As we consider only one output, it seems that the setting is more similar to the Gaussian MAC; however by setting requirements only on the reliable decoding of one of the two transmitted messages, this channel provides insights into the Gaussian two-user interference channel as shown in [5].

![Fig. 1. The Gaussian point-to-point channel with interference.](image)

In [5] the following result was obtained:

**Theorem 1:** [5, Theorem 4] For any independent random process over $\{X_n, Z_n\}_{n \geq 1}$, both component of which are of bounded variance, where $\{X_n\}_{n \geq 1}$ results in a “good” code sequence with reliable decoding from an output of an additive white Gaussian noise (AWGN) channel at $\text{snr}_1$, we have that

$$2 \frac{d}{d\gamma} I(Z; Y(\gamma)) = \begin{cases} \text{MMSE}\left(Z\sqrt{\frac{\gamma a \text{snr}_2}{1 + \gamma \text{snr}_1}} Z + N\right), & \gamma \in [0, 1) \\ \text{MMSE}\left(\sqrt{a \text{snr}_2} Z + \sqrt{\text{snr}_1} X|Y(\gamma)\right), & \gamma \geq 1 \end{cases}$$

(4)
The importance of this result in understanding the effect of maximal rate codes and obtaining the corner point of the two-user Gaussian interference channel is discussed in detail in [5] (which also provides a detailed introduction to the two-user Gaussian interference channel). The emphasis of the current paper is on providing an alternative proof of this result. Before doing so we wish to consider in more detail the meaning of this result. The Guo, Shamai and Verdú I-MMSE relationship [1] in its vector version states that, for any input \( X_n \) of arbitrary distribution (as long as the mutual information is finite [2]):

\[
\frac{d}{d\text{snr}} I \left( X_n; \sqrt{\text{snr}} H X_n + N_n \right) = \frac{1}{2} \text{Tr} \left( E_{X_n}(\text{snr}) \right)
\]  

where \( E_{X_n}(\gamma) \) is the MMSE matrix defined as follows:

\[
E_{X_n}(\gamma) = E \left\{ (X_n - E \{ X_n | \sqrt{\gamma} X_n + N_n \}) (X_n - E \{ X_n | \sqrt{\gamma} X_n + N_n \})^T \right\}.
\]  

We use the following notation that unifies the scalar and vector cases through the MMSE function:

\[
\text{MMSE}(X_n; \text{snr}) = \frac{1}{n} \text{Tr} \left( E_{X_n}(\text{snr}) \right).
\]  

Moreover, whenever we consider the normalized mutual information and MMSE function in the limit as \( n \to \infty \) we use the following notation:

\[
I(X; Y) = \lim_{n \to \infty} \frac{1}{n} I(X_n; Y_n)
\]

\[
\text{MMSE}(X; \text{snr}) = \lim_{n \to \infty} \text{MMSE}(X_n; \text{snr}).
\]  

We have given these definitions in order to emphasize the similarity between the result of Theorem 1 in the region of \( \gamma \in [0, 1) \), and the I-MMSE result, justifying referring to it as an I-MMSE-like relationship. The similarity can be shown using the chain rule of derivation. To observe this first note that (1) can be transformed as follows:

\[
\sqrt{\frac{1}{1 + \gamma \text{snr}_1}} Y_n(\gamma) = \sqrt{\frac{\gamma \text{snr}_2}{1 + \gamma \text{snr}_1}} Z_n + \sqrt{\frac{\gamma \text{snr}_1}{1 + \gamma \text{snr}_1}} X_n + \sqrt{\frac{1}{1 + \gamma \text{snr}_1}} N_n
\]

\[\hat{Y}_n(\gamma') = \sqrt{\gamma'} Z_n + Q_{\gamma,n}\]  

where we denoted \( \gamma' = \frac{\gamma \text{snr}_2}{1 + \gamma \text{snr}_1} \) and \( Q_{\gamma,n} \) is an additive noise (we discuss its distribution and variance in the sequel). Note that this noise changes with \( \gamma \) (or \( \gamma' \)); thus the notation. This one-to-one transformation has no effect on the mutual information, and so by the chain rule we have

\[
\frac{d}{d\gamma'} I \left( Z; \hat{Y}(\gamma') \right) = \frac{d}{d\gamma'} I \left( Z; \hat{Y}(\gamma) \right) \frac{d\gamma'}{d\gamma}.
\]  

Since

\[
\frac{d\gamma'}{d\gamma} = \frac{a \text{snr}_2}{(1 + \gamma \text{snr}_1)^2}
\]  

we have that Theorem 1 can be equivalently written (for \( \gamma \in [0, 1) \)) as

\[
\frac{d}{d\gamma} I \left( Z; \hat{Y}(\gamma) \right) = \frac{1}{2} \text{MMSE} \left( Z | \sqrt{\gamma} Z + N \right)
\]  

from which the similarity to the I-MMSE relationship is clearer.
Having shown this, it is important to note the differences. First and foremost, the I-MMSE relationship is given for any finite \( n \), whereas the relationship of Theorem 1 is given in the limit as \( n \to \infty \). This is a crucial difference since only in the limit does the transmission through \( X \) have attributes of a Gaussian i.i.d. input.

The second difference regards the fact that this relationship has been shown for \( \gamma \in [0, 1) \). The second line in (12) which gives a relationship for \( \gamma \geq 1 \) is an immediate consequence of the chain rule of mutual information and the I-MMSE relationship (see [5] for details) and holds in general for all \( \gamma \geq 0 \). Thus, one may think that this is only a matter of being able to extend the proof for \( \gamma \geq 1 \); however this is not the case. We show this by considering a specific distribution over \( Z_n \), namely \( Z_n \sim \mathcal{N}(0, I_n) \). Note that

\[
I(Z_n; X_n; Y_n(\gamma)) = I(Z_n; Y_n(\gamma)) + I(X_n; Y_n(\gamma)|Z_n) \\
= I(X_n; Y_n(\gamma)) + I(Z_n; Y_n(\gamma)|X_n) .
\]

Thus,

\[
I(Z_n; Y_n(\gamma)) = I(X_n; Y_n(\gamma)) + I(Z_n; Y_n(\gamma)|X_n) - I(X_n; Y_n(\gamma)|Z_n) .
\]

Since \( Z_n \) is Gaussian i.i.d. and \( X_n \) is taken from a “good” point-to-point code sequence designed for reliable communication at \( \text{snr}_1 \) we have that the exact behavior of \( I(X_n; Y_n(\gamma)) \) and \( I(X_n; Y_n(\gamma)|Z_n) \) is known for all \( \gamma \) in the limit as \( n \to \infty \). Moreover,

\[
\frac{1}{n} I(Z_n; Y_n(\gamma)|X_n) = \frac{1}{2} \log(1 + \gamma \text{snr}_2)
\]

for any \( n \) and any \( \gamma \). Thus, we have the exact behavior of \( \frac{1}{n} I(Z_n; Y_n(\gamma)) \) in the limit, as \( n \to \infty \):

\[
I(Z; Y(\gamma)) = \begin{cases} 
\frac{1}{2} \log \left( 1 + \frac{\gamma \text{snr}_2}{1 + \gamma \text{snr}_1} \right), & \gamma \in [0, 1) \\
\frac{1}{2} \log \left( 1 + \gamma (\text{snr}_2 + \text{snr}_1) \right), & \gamma \in \left[ 1, \frac{1}{1 - \text{asnr}_2} \right) \\
\frac{1}{2} \log (1 + \gamma \text{asnr}_2), & \gamma \geq \frac{1}{1 - \text{asnr}_2}
\end{cases}
\]

and also its derivative with respect to \( \gamma \):

\[
2 \frac{d}{d\gamma} I(Z; Y(\gamma)) = \begin{cases} 
\frac{1}{2} \frac{\text{asnr}_2}{(1 + \gamma \text{snr}_1)^2}, & \gamma \in [0, 1) \\
\frac{\text{asnr}_2 + \text{snr}_1}{1 + \gamma (\text{asnr}_2 + \text{snr}_1)}, & \gamma \in \left[ 1, \frac{1}{1 - \text{asnr}_2} \right) \\
\frac{\text{asnr}_2}{1 + \gamma \text{asnr}_2}, & \gamma \geq \frac{1}{1 - \text{asnr}_2}
\end{cases}
\]

It is now evident that an I-MMSE like relationship does not hold for \( \gamma \geq 1 \). Surely, once \( \gamma \) is large enough the transmission in \( X \) is reliably decoded (recall that we assume here that \( Z \) is i.i.d. Gaussian), and thus once \( \gamma \) is sufficiently large (\( \gamma \geq \frac{1}{1 - \text{asnr}_2} \)) the transmission can be removed and the behavior reduces to that of \( Z \) (i.i.d. Gaussian) over an AWGN channel but with standard additive noise and not of variance \( 1 + \gamma \text{snr}_1 \) as in the region of \( \gamma \in [0, 1) \). Moreover, even before this can be done, when \( \gamma \in \left[ 1, \frac{1}{1 - \text{asnr}_2} \right) \), the expression is no longer as in \( \gamma \in [0, 1) \) and the I-MMSE like relationship no longer holds.
III. THE INCREMENTAL CHANNEL APPROACH

This section provides the main result of this paper which is an alternative proof of the result in Theorem 1 using the incremental channel approach. The starting point of this section is the following channel model:

\[ Y_n(\gamma) = \sqrt{\gamma}Z_n + N_n \]  

where \( Z_n \) is the input to the channel of some arbitrary distribution, \( \gamma \) denotes the SNR and \( N_n \) denotes the additive noise of variance one. When \( N_n \) is i.i.d. Gaussian the above channel is the AWGN channel model for which the I-MMSE relationship holds [1]. We begin by recalling the main steps of the incremental channel proof of the I-MMSE relationship as was given in [1]. Note that (9) is also an instance of the above channel model, where the additive noise is \( Q_{\gamma,n} \) and has specific properties. The second part of this section will be dedicated to this specific setting and will show that we can follow the steps of the incremental channel proof to obtain the result in Theorem 1.

A. The I-MMSE Incremental Channel Proof

The incremental channel proof of the I-MMSE relationship in [1] has two main ingredients. First the authors of [1] observe that proving the I-MMSE relationship in its standard form (5) is equivalent to showing the following:

\[ \lim_{\delta \to 0} \left[ I(Z_n; Y_n(\text{snr} + \delta)) - I(Z_n; Y_n(\text{snr})) \right] = \frac{\delta}{2} E \{ (Z_n - E\{Z_n\})^2 \} + o(\delta). \]  

(19)

Second, the above formalism is reminiscent of the approximated behavior of the input-output mutual information at weak SNRs given in the following lemma, proved in [7, Lemma 5.2.1], [8, Theorem 4], implicitly in [9] and also in [1, Lemma 1, Appendix II],

Lemma 1 ([1], [7], [8]): As \( \delta \to 0 \), the input-output mutual information of the canonical Gaussian channel

\[ Y = \sqrt{\delta}Z + U \]  

where \( E\{Z^2\} < \infty \) and \( U \sim N(0,1) \) is independent of \( Z \), is given by

\[ I(Y; Z) = \frac{\delta}{2} E \{ (Z - E\{Z\})^2 \} + o(\delta). \]  

(21)

Thus, most steps in the proof are dedicated to showing that the mutual information difference in (19) is equivalent to a transmission over an AWGN channel on which the results of Lemma 1 can be applied. This is done by observing first that the above difference can be written as a conditional mutual information due to a Markov chain relationship. We will see that this step extends verbatim and does not depend on the i.i.d. Gaussian distribution of the additive noise in the channel. More precisely, defining

\[ Y_{n,1} = Z_n + \sigma_1 N_{1,n} \]

\[ Y_{n,2} = Y_{n,1} + \sigma_2 N_{2,n} \]  

(22)
where
\[
\sigma_1^2 = \frac{1}{1 + \text{snr}}
\]
\[
\sigma_2^2 = \frac{1}{\text{snr}}
\]
we have that
\[
I(Z_n; Y_n(\text{snr} + \delta)) - I(Z_n; Y_n(\text{snr})) = I(Z_n; Y_{n,1}) - I(Z_n; Y_{n,2}).
\]
(24)

Since \(Z_n - Y_{n,1} - Y_{n,2}\) form a Markov chain we have that the above difference is
\[
I(Z_n; Y_{n,1}) - I(Z_n; Y_{n,2}) = I(Z_n; \text{snr} Y_{n,2} + \delta Z_n + \sqrt{\delta} N_n | Y_{n,2})
\]
(25)

where
\[
N_n = \frac{1}{\sqrt{\delta}} (\delta \sigma_1 N_{n,1} - \text{snr} \sigma_2 N_{n,2}).
\]
(26)

It is clear to see that \(N_n\) is standard Gaussian independent of \(Z_n\); however the second important observation required here is to show that \(N_n\) is independent of \((Z_n, Y_{n,2})\). This is done by showing that \(N_n\) is uncorrelated with \(\sigma_1 N_{n,1} + \sigma_2 N_{n,2}\) the Gaussian noise of \(Y_{n,2}\). Since both are Gaussian, this leads to independence. Thus, the fact that the additive noise is Gaussian becomes crucial in the proof. Using this observation the above conditional mutual information can be written as
\[
I(Z_n; Y_{n,1}| Y_{n,2} = y_{n,2}) = I(Z_n; \sqrt{\delta} Z_n + N_n | Y_{n,2} = y_{n,2})
\]
(27)
meaning that the above is equivalent to an AWGN channel in which the input is distributed according to \(P_{Z_n|Y_{n,2}=y_{n,2}}\), and the SNR is \(\delta\). It remains to apply Lemma 1 and take the expectation with respect to \(Y_{n,2}\). Noting the definition of the MMSE function in (7) this concludes the I-MMSE relationship proof.

B. The Incremental Gaussian Interference Channel

We now consider the additive noise channel as given in (9), that is
\[
\tilde{Y}_n(\gamma') = \sqrt{\gamma'} Z_n + Q_{\gamma,n}.
\]
(28)
The first observation, parallel to the original I-MMSE proof, is that the I-MMSE like relationship in Theorem 1 is equivalent to the following:
\[
\left. \lim_{\delta \to 0} \lim_{n \to \infty} \frac{1}{n} \left[ I(Z_n; \tilde{Y}_n(\text{snr} + \delta)) - I(Z_n; \tilde{Y}_n(\text{snr})) \right] \right| = \delta \lim_{n \to \infty} \frac{1}{n} \text{Tr} \left( R_{Z_n|\tilde{Y}_n(\text{snr})} \right) + o(\delta). \tag{29}
\]
for all \(\text{snr}, \text{snr} + \delta \in \left[ 0, \frac{\text{snr}_2}{1 + \text{snr}_1} \right]\).

The second observation, parallel to the original I-MMSE proof, is that we require an approximation of the behavior of the mutual information at weak SNRs; however, since the additive noise in (28) is not i.i.d. Gaussian
and furthermore depends on \( \gamma \), understanding the exact extension of Lemma 1 needed here requires some analysis of this channel.

We thus target the following mutual information difference:

\[
I \left( Z_n; \tilde{Y}_n^{(\text{snr} + \delta)} \right) - I \left( Z_n; \tilde{Y}_n^{(\text{snr})} \right).
\] (30)

Simple arithmetic gives us the value of \( \gamma \) for these two values of \( \gamma' \), meaning \( \text{snr} + \delta \) and \( \text{snr} \),

\[
\gamma_{\text{snr}} = \frac{\text{snr}}{\alpha_{\text{snr}2} - \text{snr} \alpha_{\text{snr}1}}
\]

\[
\gamma_{\text{snr} + \delta} = \frac{\text{snr} + \delta}{\alpha_{\text{snr}2} - \text{snr} (\text{snr} + \delta)},
\] (31)

so as to substitute them in the definition of the additive noise \( Q_{\gamma,n} \). This gives us the following:

\[
\tilde{Y}_n^{(\text{snr})} = \sqrt{\text{snr}} Z_n + Q_{\gamma_{\text{snr}},n},
\]

\[
= \sqrt{\text{snr}} Z_n + \frac{\gamma_{\text{snr}} \text{snr}_1}{1 + \gamma_{\text{snr}} \text{snr}_1} X_n + \sqrt{\frac{1}{1 + \gamma_{\text{snr}} \text{snr}_1}} N_n
\]

\[
= \sqrt{\text{snr}} Z_n + \frac{\text{snr}_1}{\alpha_{\text{snr}2}} \alpha_{\text{snr}} X_n + \sqrt{\frac{1}{1 - \alpha_{\text{snr}} \text{snr}}} N_n
\]

\[
= \sqrt{\text{snr}} Z_n + \sqrt{\alpha_{\text{snr}}} X_n + \sqrt{\frac{1}{1 - \alpha_{\text{snr}}}} N_n
\] (32)

where we denoted \( \alpha \equiv \frac{\text{snr}_1}{\alpha_{\text{snr}2}} \). In the same manner we have

\[
\tilde{Y}_n^{(\text{snr} + \delta)} = \sqrt{\text{snr} + \delta} Z_n + \sqrt{\alpha (\text{snr} + \delta)} X_n + \sqrt{1 - \alpha (\text{snr} + \delta)} N_n.
\] (33)

This can also be written as follows:

\[
Y_{\text{snr} + \delta} = \sqrt{\frac{1}{\text{snr} + \delta}} \tilde{Y}_n^{(\text{snr} + \delta)} = Z_n + \sqrt{\frac{\alpha}{\text{snr} + \delta}} X_n + \sqrt{\frac{1}{\text{snr} + \delta} - \alpha} N_n
\]

\[
Y_{\text{snr}} = \sqrt{\frac{1}{\text{snr}}} \tilde{Y}_n^{(\text{snr})} = Z_n + \sqrt{\alpha} X_n + \sqrt{\frac{1}{\text{snr}} - \alpha} N_n.
\] (34)

Observe that, as expected, the distribution of the noise component changes as a function of \( \text{snr} \). As \( \text{snr} \) increases the additive noise has a smaller i.i.d. Gaussian component as compared with the \( X_n \) fraction of the additive noise.

Now, due to the infinite divisibility of the Gaussian distribution we can write the following:

\[
Y_{\text{snr} + \delta} = Z_n + \sqrt{\alpha} X_n + \sigma_1 N_{1,n}
\]

\[
Y_{\text{snr}} = Y_{\text{snr} + \delta} + \sigma_2 N_{2,n}
\] (35)

where \( N_{1,n} \) and \( N_{2,n} \) are independent standard random vectors and

\[
\sigma_1^2 = \frac{1}{\text{snr} + \delta} - \alpha
\]

\[
\sigma_1^2 + \sigma_2^2 = \frac{1}{\text{snr}} - \alpha.
\] (36)

From this it is clear that we have a Markov chain relationship \( Z_n - Y_{\text{snr} + \delta} - Y_{\text{snr}} \). As such the mutual information difference in (30) can also be written as

\[
I \left( Z_n; Y_{\text{snr} + \delta} \right) - I \left( Z_n; Y_{\text{snr}} \right) = I \left( Z_n; Y_{\text{snr} + \delta} | Y_{\text{snr}} \right)
\] (37)
without regard for the specific properties of $X_n$. Thus, this Markov chain relationship is quite general. By a similar linear combination as used in [1] we have that

$$(\text{snr} + \delta)Y_{\text{snr} + \delta} = \text{snr}Y_{\text{snr} + \delta} + \delta Y_{\text{snr} + \delta}$$

$$= \text{snr} (Y_{\text{snr}} - \sigma_2 N_{2,n}) + \delta \left( Z_n + \sqrt{\alpha} X_n + \sigma_1 N_{1,n} \right)$$

$$= \text{snr} Y_{\text{snr}} + \delta Z_n + \delta \sqrt{\alpha} X_n + \delta \sigma_1 N_{1,n} - \text{snr} \sigma_2 N_{2,n}$$

$$= \text{snr} Y_{\text{snr}} + \delta Z_n + \delta \sqrt{\alpha} X_n + \sqrt{\delta} \hat{N}_n$$

(38)

where

$$\hat{N}_n = \frac{1}{\sqrt{\delta}} \left( \delta \sigma_1 N_{1,n} - \text{snr} \sigma_2 N_{2,n} \right).$$

(39)

It is easy to observe that $\hat{N}_n$ is an i.i.d. Gaussian random vector, as it is a combination of two independent i.i.d. Gaussian vectors. Moreover, it is simple to show that its variance is $1 - \delta \alpha$ (see Appendix A). However, there is an additional component in the noise $\sqrt{\delta} X_n$ with covariance matrix $\delta \alpha R_{X_n}$, where $R_{X_n}$ is the covariance of $X_n$. To conclude, the conditional mutual information can be written as

$$I \left( Z_n; Y_{\text{snr} + \delta} \right) = I \left( Z_n; \frac{1}{\sqrt{\delta}} \text{snr} Y_{\text{snr}} + \sqrt{\delta} Z_n + \sqrt{\delta} \alpha X_n + \hat{N}_n | Y_{\text{snr}} \right)$$

$$= I \left( Z_n; \sqrt{\delta} Z_n + \sqrt{\delta} \alpha X_n + \hat{N}_n | Y_{\text{snr}} \right).$$

(40)

Given the above and following the original I-MMSE incremental channel approach proof we are missing two crucial observations. The first is the independence of the noise and the input signal conditioned on $Y_{\text{snr}}$ which makes every increment an additive noise channel. This result is given in the next theorem.

**Theorem 2:** Given the above model and definitions, we have

$$\lim_{n \to \infty} \frac{1}{n} I \left( Z_n; \sqrt{\delta} \alpha X_n + \hat{N}_n | Y_{\text{snr}} \right) = 0 \quad \text{(41)}$$

and

$$\lim_{n \to \infty} \frac{1}{n} I \left( Y_{\text{snr}}; \sqrt{\delta} \alpha X_n + \hat{N}_n \right) = 0 \quad \text{(42)}$$

for all $\text{snr}, \text{snr} + \delta \in \left[ 0, \frac{\text{snr}}{1 + \text{snr}} \right]$.

The second is an extension of the approximation of the mutual information at weak SNRs to both the infinite dimensional case as well as to the specific distribution of the additive noise ($\sqrt{\delta} \alpha X_n + \hat{N}_n$, where $X_n$ is taken from a “good” code sequence). This result is given in the next theorem.

**Theorem 3:** Assume a “good” code sequence $\{X_n\}_{n \geq 1}$ which attains capacity over the Gaussian point-to-point channel and independent Gaussian noise $\hat{N}$ of variance $1 - \delta \alpha$. For any signal $\{Z_n\}_{n \geq 1}$ of bounded variance which is asymptotically independent of $\sqrt{\delta} \alpha X_n + \hat{N}_n$, we have that for $\delta \to 0$

$$\lim_{n \to \infty} \frac{1}{n} I \left( Z_n; \sqrt{\delta} Z_n + \sqrt{\delta} \alpha X_n + \hat{N}_n \right) = \frac{\delta}{2} \lim_{n \to \infty} \frac{1}{n} \mathbb{E} \left\{ \| Z_n - \mathbb{E} \{ Z_n \} \|^2 \right\} + o(\delta). \quad \text{(43)}$$

The proofs of these results are given in Sections IV and V, respectively.
Given these results we may continue following the original incremental channel approach proof of the I-MMSE [1]. Due to Theorem 2 we may apply Theorem 3 on (40) and obtain
\[
\lim_{n \to \infty} \frac{1}{n} I \left( Z_n; \sqrt{\delta} Z_n + \sqrt{\delta} X_n + \hat{N}_n | Y_{snr} = y \right) = \frac{\delta}{2} \lim_{n \to \infty} \frac{1}{n} \text{Tr} \left( R_{Z_n | Y_{snr} = y} \right) + o(\delta).
\] (44)

Taking the expectation with respect to \( Y_{snr} \) we have that
\[
\lim_{n \to \infty} \frac{1}{n} I \left( Z_n; \sqrt{\delta} Z_n + \sqrt{\delta} X_n + \hat{N}_n | Y_{snr} \right) = \frac{\delta}{2} \lim_{n \to \infty} \frac{1}{n} \text{Tr} \left( E_{Z_n} (Y_{snr}) \right) + o(\delta)
\] (45)

which according to (37) means that
\[
\lim_{n \to \infty} \frac{1}{n} \left[ I (Z_n; Y_{snr} + \delta) - I (Z_n; Y_{snr}) \right] = \frac{\delta}{2} \lim_{n \to \infty} \frac{1}{n} \text{Tr} \left( E_{Z_n} (Y_{snr}) \right) + o(\delta)
\] (46)

for \( \delta \to 0 \). As this is an equivalent form of Theorem 1, which concludes the proof.

C. Discussion

Before proceeding to the proofs of Theorem 2 and 3 we wish to emphasize a few points. The above result provides a simple proof of the “missing corner point” of the two-user Gaussian interference channel capacity region (see [5] and [10]). In parallel to our approach Polyanskiy and Wu [6] followed a different approach which also resulted in an insightful proof to this problem. Although the conclusion that an optimal point-to-point code sequence has an i.i.d. Gaussian effect, in terms of information theoretic measures, on the additional transmission, is obtained by both methods, the approaches are quite different. In [6] the authors examined the difference between two differential entropies, that of the interfered-with output and the other when instead of the point-to-point optimal sequence we have i.i.d. Gaussian noise. They show, more generally, that under regularity conditions (which hold for a signal convolved with Gaussian noise) information theoretic measures are Lipschitz continuous with respect to the Wasserstein distance. Using Talagrans’ inequality the difference can be bounded by the divergence, and thus when the divergence tends to zero the difference between the two differential entropies also tends to zero. Given the observations by Han and Verdú in [11] and by Shamai and Verdú in [12] which examined the output distributions of “good” code sequences and showed that the divergence indeed tends to zero, and using the data-processing inequality, this work fills the missing step in the proof as presented by Costa [13] (see [14] for more details).

A central difference between this approach and the one presented here is that the fact that the interfered-with signal is an optimal point-to-point sequence is used in [6] only so as to conclude regarding the properties of the output distribution at the desired SNR (the actual output of the Gaussian interference channel, \( \gamma = 1 \) in our formalism). In the approach presented in this work the understanding of the behavior of optimal point-to-point sequences at every SNR is used to analyze the incremental effect of such a sequence, thus concluding with an I-MMSE-like relationship. To emphasize this difference, note that for this approach we require an extension of the results in [12] for every SNR and not only at the desired output SNR.

Another related recent result is due to Calmon, Polyanskiy and Wu [15, Lemma 1]. They show that when the MMSE behavior when estimating a signal from the output of an AWGN channel is close to that of the linear estimator the input distribution is almost Gaussian in terms of the Kolmogorov-Smirnov distance. This result is
then applied, using the I-MMSE relationship, on mutual information close to capacity. To conclude, although this result uses a different measure, the conclusion is similar - input distributions with mutual information or MMSE behavior that is close to that of the Gaussian input poses similar properties to those of the Gaussian distribution to some extent.

Before concluding this discussion we wish to briefly explain how the result of Theorem 1 resolves the “missing corner point” of the two-user Gaussian interference channel capacity region. This application is given in detail in [5]. Note first that Theorem 1 provides an expression for the interference-output mutual information which is that of a transmission through an AWGN channel where the interfered-with transmission effect is that of additional additive Gaussian noise. On the other hand we have a requirement of reliable communication of the interfered-with transmission, i.e., at the desired output this transmission can be fully decoded and removed (a one to one mapping is required). Thus, we have two descriptions of the interference-output mutual information that correspond to the transmission of the interference through an AWGN channel but with different SNRs. Such an equality can hold if only if the MMSE of the interference from the output at these SNRs is zero (due to the I-MMSE relationship). This conclusion allows us to directly maximize the multi-letter expression for the capacity of the two-user Gaussian interference channel, given by Ahlswede [16].

IV. SIGNAL AND NOISE INDEPENDENCE: PROOF OF THEOREM 2

The independence that we need is a conditional independence between $Z_n$ and $\hat{Q}_n$ given the output $Y_{\text{snr}}$ in the limit as $n \to \infty$, since this is the incremental channel. Unconditioned these two are certainly independent since $\hat{Q}_n$ is simply $X_n$ and the additive Gaussian noise $\hat{N}_n$. However, conditioned on $Y_{\text{snr}}$ things are not as clear.

More specifically, we prove an asymptotic independence. Since we consider an approximation of the mutual information at weak SNR, in the limit as $n \to \infty$, this asymptotic independence suffices for our purpose.

Proof of Theorem 2: Using the chain rule of mutual information we have

$$I(Z_n, Y_{\text{snr}}; \sqrt{\delta \alpha} X_n + \hat{N}_n) = I(Z_n; \sqrt{\delta \alpha} X_n + \hat{N}_n) + I(Y_{\text{snr}}; \sqrt{\delta \alpha} X_n + \hat{N}_n|Z_n)$$

$$= I(Y_{\text{snr}}; \sqrt{\delta \alpha} X_n + \hat{N}_n|Z_n)$$

(47)

where the second equality is due to the independence of $Z_n$ and $\hat{Q}_n$. Alternatively, we have

$$I(Z_n, Y_{\text{snr}}; \sqrt{\delta \alpha} X_n + \hat{N}_n) = I(Y_{\text{snr}}; \sqrt{\delta \alpha} X_n + \hat{N}_n) + I(Z_n; \sqrt{\delta \alpha} X_n + \hat{N}_n|Y_{\text{snr}}).$$

(48)

Putting the above two together we have that

$$I(Z_n; \sqrt{\delta \alpha} X_n + \hat{N}_n|Y_{\text{snr}}) = I(Y_{\text{snr}}; \sqrt{\delta \alpha} X_n + \hat{N}_n|Z_n) - I(Y_{\text{snr}}; \sqrt{\delta \alpha} X_n + \hat{N}_n)$$

$$\leq I(Y_{\text{snr}}; \sqrt{\delta \alpha} X_n + \hat{N}_n|Z_n)$$

$$= I(\frac{1}{\text{snr}} Y_{\text{snr}}; \sqrt{\alpha} X_n + \frac{1}{\sqrt{\delta}} \hat{N}_n|Z_n)$$

$$= I(\sqrt{\alpha} X_n + \sigma_1 N_{1,n} + \sigma_2 N_{2,n}; \sqrt{\alpha} X_n + \sigma_1 N_{1,n} - \frac{\text{snr}}{\sigma_2} \sigma_2 N_{2,n}).$$

(49)
Thus by showing the independence of $\sqrt{\alpha}X_n + \sigma_1 N_{1,n} + \sigma_2 N_{2,n}$ and $\sqrt{\alpha}X_n + \sigma_1 N_{1,n} - \frac{\text{snr}}{\delta} \sigma_2 N_{2,n}$ in the limit as $n \to \infty$ we also prove the conditional independence in the limit. For simplicity we denote the following

$$Y_1 \equiv \sqrt{\alpha}X_n + \sigma_1 N_{1,n} + \sigma_2 N_{2,n}$$

$$Y_2 \equiv \sqrt{\alpha}X_n + \sigma_1 N_{1,n} - \frac{\text{snr}}{\delta} \sigma_2 N_{2,n}$$

and we wish to examine the mutual information between them. Note that both depend on $n$ (removed to simplify notation). Moreover, both depend on $\text{snr}$ and $\delta$ (recall the definition of $\sigma_1$ and $\sigma_2$ (36)). In Appendix B we show that

$$I(Y_1; Y_2) = D(P_{y_{1,2},G} || P_{y_{1,2},G}) + D(P_{y_{1,2},G} || P_{y_{1,2},G}) - D(P_{y_{1,2}} || P_{y_{1,2}}) - D(P_{y_{1,2}} || P_{y_{1,2}})$$

(51)

where $P_{y_{1,2},G}$ is a Gaussian distribution with the same covariance as that of the true distribution over $(Y_1, Y_2)$.

Next we require the following results:

Lemma 2: For $\text{snr}, \delta \in [0, \frac{\text{snr}^2}{1+\text{snr}^2})$

$$\lim_{n \to \infty} \frac{1}{n} D(P_{y_1} || P_{y_1, G}) = 0$$

$$\lim_{n \to \infty} \frac{1}{n} D(P_{y_2} || P_{y_2, G}) = 0.$$  

(52)

Proof: The proof relies on the results of Han and Verdú [11] and Shamai and Verdú [12] and is given in Appendix C.

Lemma 3: For $\text{snr}, \text{snr} + \delta \in [0, \frac{\text{snr}^2}{1+\text{snr}^2})$

$$\lim_{n \to \infty} \frac{1}{n} D(P_{y_{1,2}} || P_{y_{1,2}, G}) = 0.$$  

(53)

Meaning that for these values of $\text{snr}$ they are asymptotically jointly Gaussian.

Proof: The proof is given in Appendix D.

The third claim regards “good”, point-to-point capacity achieving code sequences.

Lemma 4: For any “good”, point-to-point capacity achieving, code sequence $\{X_n\}_{n \geq 1}$ which complies with the power constraint we have that

$$\lim_{n \to \infty} \lambda_i(R_{X_n}) = 1$$  

(54)

where $R_{X_n}$ denotes its sequence of covariance matrices and $\lambda_i(\cdot)$ denotes the $i^{th}$ eigenvalue function.

Proof: The proof is given in Appendix E.
We now calculate the correlation matrix between $Y_1$ and $Y_2$:

$$B \equiv \mathbf{E}\{Y_2 Y_1^T\} = \mathbf{E}\{\sqrt{\alpha}X + \delta \sigma_1 N_1 - \text{snr}\sigma_2 N_2\} (\sqrt{\alpha}X + \sigma_1 N_1 + \sigma_2 N_2)^T\}
= \alpha \mathbf{R}_{X_n} + \mathbf{E}\{\delta \sigma_1 N_1 - \text{snr}\sigma_2 N_2\}(\sigma_1 N_1 + \sigma_2 N_2)^T\}
= \alpha \mathbf{R}_{X_n} + \sigma_1^2 \mathbf{I} - \frac{\text{snr}}{\delta} \sigma_2^2 \mathbf{I}
= \alpha \mathbf{R}_{X_n} + \left(\frac{1}{\text{snr} + \delta} - \alpha\right) \mathbf{I} - \frac{\text{snr}}{\delta} \mathbf{I}
= \frac{1}{\text{snr} + \delta} \mathbf{I} - \alpha \mathbf{I} - \frac{\text{snr}}{\delta} \mathbf{I}
= \alpha \left(\mathbf{R}_{X_n} - \mathbf{I}\right). \quad (55)$$

Its eigenvalues are then given by

$$\alpha \lambda I \left(\mathbf{R}_{X_n} - \mathbf{I}\right) = \alpha \left(\lambda I \left(\mathbf{R}_{X_n} - \mathbf{I}\right) - 1\right). \quad (56)$$

Since $X_n$ is a “good” code sequence, using Lemma 4 we have that the eigenvalues of the correlation matrix approach zero in the limit, as $n \to \infty$. Note also that the matrix is Hermitian since

$$B = \alpha \left(\mathbf{R}_{X_n} - \mathbf{I}\right) = B^T. \quad (57)$$

Finally, the fourth claim is

**Lemma 5:** Assume $P_{G_0} \sim \mathcal{N}(0, \Sigma_0)$ and $P_{G_1} \sim \mathcal{N}(0, \Sigma_0)$ of dimension $n$ such that

$$\Sigma_0 = \begin{pmatrix} A & B^T \\ B & C \end{pmatrix}, \quad \Sigma_1 = \begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix} \quad (58)$$

and both $A$ and $C$ are non-singular. We have that

$$D(P_{G_0}||P_{G_1}) = -\frac{1}{2} \ln \prod_{i=1}^n (1 - \lambda_i(C^{-1}BA^{-1}B^T))$$

$$= -\frac{1}{2} \sum_{i=1}^n \ln (1 - \lambda_i(C^{-1}BA^{-1}B^T)). \quad (59)$$

**Proof:** The proof is given in Appendix F. \hfill \blacksquare

Using the above lemma when $P_{G_0}$ is the Gaussian distribution $P_{y_1, G; y_2, G}$ and $P_{G_1}$ is the Gaussian distribution $P_{y_1, G; y_2, G}$ thus complying with the assumption in the lemma (the non-singularity of $A$ and $C$ is easily verified) where the matrix $B$ is the correlation matrix $E\{Y_2 Y_1^T\}$, we have an expression for the Kullback-Leibler (KL) divergence. Moreover, since $B$ is a Hermitian matrix we can use Schur Decomposition and write it as $U \Lambda_B U^{-1}$, where $U$ is a unitary matrix and $\Lambda_B$ is a diagonal matrix which contains the eigenvalues of $B$ on its diagonal (thus in the limit converges to the zero matrix). Using similarity we have that

$$\lambda_i(\Lambda_B U^{-1} A^{-1} U \Lambda_B) = \lambda_i(\Lambda_B A^{-1} B U^{-1}). \quad (60)$$
Thus, we can conclude that in the limit
\[
\lim_{n \to \infty} \lambda_i(BA^{-1}B^T) = 0.
\] (61)

Moreover, note that both $C^{-1}$ and $BA^{-1}B^T$ are positive semi-definite matrices, thus we can bound from both above and below any eigenvalue of the product using results from majorization theory [17][Equation 2.0.3]
\[
\max_{i+j=t+n} \lambda_i(C^{-1})\lambda_j(BA^{-1}B^T) \leq \lambda_i(C^{-1}BA^{-1}B^T) \leq \min_{i+j=t+1} \lambda_i(C^{-1})\lambda_j(BA^{-1}B^T).
\] (62)

Thus, in the limit since we have shown that the eigenvalues of $BA^{-1}B^T$ converge to zero we can conclude that also the eigenvalues of $C^{-1}BA^{-1}B^T$ go to zero
\[
\lim_{n \to \infty} \lambda_i(C^{-1}BA^{-1}B^T) = 0.
\] (63)

and due to the result of Lemma 5
\[
\lim_{n \to \infty} D(P_{G_0} \| P_{G_1}) = 0.
\] (64)

Putting everything together - the above result, Lemma 2 and Lemma 3 in (113) (normalized) - and taking $n \to \infty$ we have
\[
\lim_{n \to \infty} I(Y_1; Y_2) = \\
\lim_{n \to \infty} \frac{1}{n} \left[ D(P_{Y_1,Y_2} \| P_{Y_1,G,Y_2,G}) + D(P_{Y_1,G,Y_2,G} \| P_{Y_1,G}P_{Y_2,G}) - D(P_{Y_1} \| P_{Y_1,G}) - D(P_{Y_2} \| P_{Y_2,G}) \right] = 0.
\] (65)

Finally, from (49) and the non negativity of the mutual information we can also conclude that
\[
\lim_{n \to \infty} \frac{1}{n} I\left(Y_{\text{snr}}; Q_n \right) = 0.
\] (66)

This concludes the proof.  

V. APPROXIMATION AT WEAK SNR: PROOF OF THEOREM 3

As shown in Section III the key to the incremental channel approach proof in [1] was to reduce the proof of the relationship for all SNRs to the case of vanishing SNR, a domain that capitalizes on the result given in Lemma 1 (proved in [7, Lemma 5.2.1], [8, Theorem 4], implicitly in [9] and also in [1, Appendix II]). In Section III we have shown that the incremental channel approach proof can be extended and the proof of the I-MMSE like relationship, for $\text{snr} \in \left[0, \frac{\text{snr}_g}{1+\text{snr}_g}\right)$, can be reduced to the case of vanishing SNRs; however the approximation at weak SNRs must be extended to that given in Theorem 3. The most obvious difference between this extension and the result in Lemma 1 is that it is given in the limit, as blocklength $n$ goes to infinity. The reason is two-fold, first because this is the regime of interest when considering capacity of the a given channel; second, in this regime we can put to use the fact that the sequence $\{X_n\}_{n \geq 1}$ is a “good” code sequence for the AWGN channel. This leads to the second difference, which is that the additive noise is constructed from a combination of a Gaussian i.i.d. sequence of variance $1 - \delta \alpha$ and a “good” code sequence of variance $\delta \alpha$. 

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Before we proceed to the proof of Theorem 3 we wish to note that the signal \( \{Z_n\}_{n \geq 1} \) in Theorem 3 is any arbitrary signal of bounded variance. Moreover, we do not assume zero mean. This is important as this result is used on the conditional version \( Z_n | Y_{\text{snr}} = y \).

**Proof of Theorem 3:** Following the proof [7, Lemma 5.2.1] we provide upper and lower bounds on the mutual information in the limit, as \( n \to \infty \). We begin with the upper bound:

\[
I \left( Z; \sqrt{\delta} Z + \sqrt{\delta} \alpha X + \hat{N} \right) = h(\sqrt{\delta} Z + \sqrt{\delta} \alpha X + \hat{N}) - h(\sqrt{\delta} \alpha X + \hat{N})
\]

\[
\leq \lim_{n \to \infty} -\frac{1}{2n} \log \left( (2\pi e)^n |\delta R_{Z_n} + \delta \alpha R_{X_n} + (1 - \delta \alpha) I_n| \right) - \frac{1}{2} \log (2\pi e)
\]

\[
= \lim_{n \to \infty} -\frac{1}{2n} \log \prod_{i=1}^{n} (\lambda_i(\delta R_{Z_n} + \delta \alpha R_{X_n}) + (1 - \delta \alpha))
\]

\[
= \lim_{n \to \infty} -\frac{1}{2n} \log \sum_{i=1}^{n} \lambda_i(\delta R_{Z_n} + \delta \alpha R_{X_n}) + (1 - \delta \alpha)
\]

\[
b \leq \lim_{n \to \infty} \frac{1}{2n} \log \left( \frac{1}{n} \text{Tr}(\delta R_{Z_n} + \delta \alpha R_{X_n}) + (1 - \delta \alpha) \right)
\]

\[
= \lim_{n \to \infty} \frac{1}{2n} \log \left( \frac{1}{n} \text{Tr}(\delta R_{Z_n}) + \frac{1}{n} \text{Tr}(\delta \alpha R_{X_n}) + (1 - \delta \alpha) \right)
\]

\[
c \leq \lim_{n \to \infty} \frac{1}{2n} \log \left( \frac{\delta}{n} \text{Tr}(R_{Z_n}) + \delta \alpha + (1 - \delta \alpha) \right)
\]

\[
= \lim_{n \to \infty} \frac{1}{2n} \log \left( \frac{\delta}{n} \text{Tr}(R_{Z_n}) + 1 \right)
\]

\[
d \leq \lim_{n \to \infty} \frac{\delta}{2n} \text{Tr}(R_{Z_n})
\]

\[
= \frac{\delta}{2} \lim_{n \to \infty} \frac{1}{n} \text{E} \{ \|Z_n - E \{Z_n\}\|^2 \} \tag{67}
\]

where in inequality \( a \) we use the maximum differential entropy result and the fact that \( \{X_n\}_{n \geq 1} \) is a “good” code sequence. Inequality \( b \) is due to the concavity of the log function and Jensen’s inequality. Inequality \( c \) is due to the power constraint (2) and the monotonicity of the log function. The last inequality is due to the inequality \( \log(1 + x) \leq x \) for all non-negative \( x \).

The problematic direction is the lower bound. As done in [7, Lemma 5.2.1] we rely on [18, Theorem 2] where a multidimensional, continuous alphabet, memoryless channel with weak SNR and a peak power constraint was considered. Thus, in order to use this result a truncation argument was used in [7, Lemma 5.2.1]. We follow the same approach; however there are a few significant differences in our proof as compared to the proof in [7, Lemma 5.2.1]. First of all we consider length-\( n \) random vectors (where as [7, Lemma 5.2.1] considered a scalar input signal). Second, we are interested in the regime of \( n \to \infty \). Third, the additive noise is not additive white Gaussian noise, and thus some delicacy is required when using [18, Theorem 2]. We will emphasize these differences throughout the proof and see their effect.

We begin by following the proof of [7, Lemma 5.2.1] where a truncation argument was used. Since we consider
length-$n$ random vectors we assume a per-component peak limited input. Let $\kappa > 0$ be arbitrary (large). Let

$$Z_\kappa^n = [Z_\kappa^1, \cdots, Z_\kappa^n]^T$$

$$Z_i^\kappa = \begin{cases} Z_i, & |Z_i| < \kappa \\ \kappa, & \text{otherwise} \end{cases}$$

$$S_\kappa^n = \begin{cases} 1, & \forall i \in [1,n], |Z_i| < \kappa \\ 0, & \text{otherwise} \end{cases}$$

(68)

Note that such a restriction is also guaranteed to be peak limited

$$\|Z_\kappa^n\| < \sqrt{n\kappa}$$

(69)

which will allow us to apply the result of [18, Theorem 2]. We also denote

$$Y_\kappa^n = \sqrt{\delta}Z_\kappa^n + \sqrt{\delta\alpha}X_n + \hat{N}_n$$

(70)

where

$$\hat{Q}_n = \sqrt{\delta\alpha}X + \hat{N}.$$  

(71)

We now turn to consider the following mutual information quantity:

$$I(Z_\kappa^n; Y_\kappa^n) = I(Z_\kappa^n; \sqrt{\delta}Z_\kappa^n + \hat{Q}_n).$$

(72)

We claim the following:

**Lemma 6:** Assuming inputs of bounded variances, when $\delta \to 0$ we have

$$I(Z_\kappa^n; Y_\kappa^n) \geq \frac{\delta'}{2\alpha} \text{Tr}(R_{Z_\kappa^n}) - \frac{\delta'^2}{2} \max_{i \in [1,n]} \lambda_i(R_{X_n}) \text{Tr}(R_{Z_\kappa^n}) + o(\delta)$$

(73)

where $\delta' = \frac{\delta\alpha}{1 + \delta\alpha}$.

**Proof:** The proof relies on [18, Theorem 2] and is given in Appendix G.

We first define $\hat{\delta} = \frac{\delta}{1 + \delta\alpha}$ and note that

$$o(\delta) = o(\delta') = o(\hat{\delta}).$$

(74)

This is due to the fact that for any $f(\cdot) \in o(\delta)$ we have that

$$\lim_{\delta \to 0} \frac{f(\delta)}{\delta} = 0$$

$$\lim_{\delta \to 0} \frac{f(\delta)(1 + \delta\alpha)}{\alpha\delta} = \lim_{\delta \to 0} \frac{f(\delta)}{\alpha\delta} + \lim_{\delta \to 0} f(\delta) = 0$$

(75)

and similarly for $\hat{\delta}$. Moreover, note that both $\delta'$ and $\hat{\delta}$ are monotonically non-decreasing functions of $\delta$, and thus as $\delta \to 0$ also $\delta' \to 0$ and $\hat{\delta} \to 0$.

It is straightforward to observe that the second term in (73) belongs to $o(\delta')$ since

$$\lim_{\delta' \to 0} \frac{\delta'^2}{2} \max_{i \in [1,n]} \lambda_i(R_{X_n}) \text{Tr}(R_{Z_\kappa^n}) = 0.$$  

(76)
The first term in (73) can be written as
\[
\frac{\delta}{2} \text{Tr} \left( R Z_n^\kappa \right) + \left( \frac{\delta}{2} - \delta \right) \text{Tr} \left( R Z_n^\kappa \right) = \frac{\delta}{2} \text{Tr} \left( R Z_n^\kappa \right) - \frac{1}{2} \frac{\alpha \delta^2}{1 + \alpha \delta} \text{Tr} \left( R Z_n^\kappa \right) \tag{77}
\]
where the second term belongs to \( o(\delta) \):
\[
\lim_{\delta \to 0} - \frac{1}{2} \frac{\alpha \delta}{1 + \alpha \delta} \text{Tr} \left( R Z_n^\kappa \right) = 0. \tag{78}
\]
Thus, (73) can be written as follows:
\[
I \left( Z_n^\kappa; Y_n^\kappa \right) \geq \frac{\delta}{2} \text{Tr} \left( R Z_n^\kappa \right) + o(\delta). \tag{79}
\]
As shown in [7, Equation (83)] due to the data processing inequality and the chain rule for mutual information we can conclude that
\[
I \left( Z_n; Y_n \right) \geq I \left( Z_n^\kappa; Y_n | S^\kappa = 1 \right) P(S^\kappa = 1). \tag{80}
\]
Note that the last equality in [7, Equation (83)] becomes an inequality in our case due to the definition of \( S^\kappa \) in which for \( S^\kappa = 0 \) we do not have a deterministic input to the channel and thus the mutual information in this case is not zero; however we can simply disregard this term and obtain a lower bound. More importantly, note that the above inequality holds also once we take \( n \to \infty \). Note that this does not remove the restriction on the input signal which is still per-component limited. Now consider
\[
\lim_{n \to \infty} \frac{1}{n} I \left( Z_n^\kappa; Y_n | S^\kappa = 1 \right) \overset{a}{=} \lim_{n \to \infty} \frac{1}{n} I \left( Z_n^\kappa; Y_n^\kappa | S^\kappa = 1 \right)
\]
\[\overset{\text{a}}{=} \lim_{n \to \infty} \frac{1}{n} \left[ I \left( Z_n^\kappa; Y_n^\kappa | S^\kappa = 1 \right) + \frac{\delta}{2} \text{Tr} \left( R Z_n^\kappa | S^\kappa = 1 \right) + o(\delta) \right]. \tag{81}
\]
where in transition \( a \) we use the assumption that in the limit as \( n \to \infty \) we have independence between the noise and the input signal, and since \( S^\kappa \) is merely a function of the input signal we also have independence between the noise and \( S^\kappa \). We can now apply Lemma 6 to this quantity and conclude that
\[
\lim_{n \to \infty} \frac{1}{n} I \left( Z_n^\kappa; Y_n | S^\kappa = 1 \right) \geq \lim_{n \to \infty} \frac{1}{n} \left[ \frac{\delta}{2} \text{Tr} \left( R Z_n^\kappa | S^\kappa = 1 \right) + o(\delta) \right]. \tag{82}
\]
It remains only to take \( \kappa \to \infty \) in which case
\[
\lim_{\kappa \to \infty} P(S^\kappa = 1) = 1 \tag{83}
\]
and
\[
\lim_{\kappa \to \infty} \lim_{n \to \infty} \frac{1}{n} \text{Tr} \left( R Z_n^\kappa | S^\kappa = 1 \right) = \frac{1}{n} \text{Tr} \left( R Z_n \right). \tag{84}
\]
Thus, we obtain the following lower bound
\[
\lim_{n \to \infty} \frac{1}{n} I \left( Z_n; Y_n \right) \geq \lim_{n \to \infty} \frac{1}{n} \left[ \frac{\delta}{2} \text{Tr} \left( R Z_n \right) + o(\delta) \right]. \tag{85}
\]
This concludes the proof. 

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VI. THE GAUSSIAN MAC WITH INTERFERENCE

Consider a Gaussian MAC channel:

\[ Y_n = \sqrt{\text{snr}_1} X_{1,n} + \sqrt{\text{snr}_2} X_{2,n} + N_n \]  

(86)

where \( X_{1,n} \) carries the message of transmitter 1, denoted as \( W_1 \), and \( X_{2,n} \) carries the message of transmitter 2, denoted as \( W_2 \). The two transmissions are independent (no cooperation between the transmitters). The additive noise, \( N_n \), is assumed to be standard Gaussian and independent of both messages. We further have an average power constraint on the transmitted codewords of 1.

The capacity region of the above channel is well known [19]:

\[ R_1 \leq \frac{1}{2} \log(1 + \text{snr}_1) \]  

(87)

\[ R_2 \leq \frac{1}{2} \log(1 + \text{snr}_2) \]  

(88)

\[ R_1 + R_2 \leq \frac{1}{2} \log(1 + \text{snr}_1 + \text{snr}_2) \]  

(89)

and can be obtained by either Gaussian codebook sequences and time-sharing, or Gaussian superposition code sequences, in which case no synchronization is required [20].

A simple observation on capacity achieving code sequences is that

\[ I(X_1, X_2; Y) = I(\sqrt{\text{snr}_1} X_1 + \sqrt{\text{snr}_2} X_2; Y) \]

\[ = \frac{1}{2} \log(1 + \text{snr}_1 + \text{snr}_2) \]  

(90)

meaning the combined codeword is an optimal point-to-point codeword, and thus we have that

\[ \text{MMSE}(\sqrt{\text{snr}_1} X_1 + \sqrt{\text{snr}_2} X_2; \gamma) = \begin{cases} \frac{\text{snr}_1 + \text{snr}_2}{1 + \gamma(\text{snr}_1 + \text{snr}_2)}, & \gamma \in [0, 1) \\ 0, & \gamma \geq 1 \end{cases} \]  

(91)

Consider now an additional interfering transmitter, i.e.,

\[ Y_n = \sqrt{\text{snr}_1} X_{1,n} + \sqrt{\text{snr}_2} X_{2,n} + \sqrt{\text{asnr}_2} Z_n + N_n \]  

(92)

where all signals are as defined above and the additional independent signal \( Z_n \) is an interfering signal which is not required to be reliably decoded by the receiver. This setting is depicted in Figure 2. For this setting we can claim the following:

Theorem 4: Given the above Gaussian MAC with interference we have that the interference must comply with the following property:

\[ \text{MMSE}(Z; \gamma) = 0, \quad \forall \gamma \geq \frac{a\text{snr}_2}{1 + \text{snr}_1 + \text{snr}_2}, \]

(93)

regardless of the strength of the interference.
Proof: This is a direct consequence of the fact that $\sqrt{\text{snr}_1}X_{1,n} + \sqrt{\text{snr}_2}X_{2,n}$ is a codeword from a “good” code sequence for the AWGN channel. Hence, the I-MMSE like relationship of Theorem 1 applies on any additional interference through the channel. So, on the one hand we have, in the limit as $n \to \infty$,

$$I(Z; Y) = I\left(Z; \sqrt{\frac{a\text{snr}_2}{1 + \text{snr}_1 + \text{snr}_2}}Z + N\right)$$

(94)

where $N$ is standard additive Gaussian noise. On the other hand in the limit, as $n \to \infty$ we have reliable decoding of $X_1$ and $X_2$ and therefore also of their linear combination $\sqrt{\text{snr}_1}X_1 + \sqrt{\text{snr}_2}X_2$. Thus,

$$I(Z; Y) = I(Z; Y, \sqrt{\text{snr}_1}X_1 + \sqrt{\text{snr}_2}X_2)$$

$$= I(Z; \sqrt{a\text{snr}_2}Z + N).$$

(95)

From the above we can conclude that

$$I\left(Z; \sqrt{\frac{a\text{snr}_2}{1 + \text{snr}_1 + \text{snr}_2}}Z + N\right) = I(Z; \sqrt{a\text{snr}_2}Z + N)$$

(96)

and thus,

$$\int_{\frac{a\text{snr}_2}{1 + \text{snr}_1 + \text{snr}_2}}^{\text{snr}_2} \text{MMSE}(Z; \gamma) d\gamma = 0$$

(97)
meaning,

$$\text{MMSE}(Z; \gamma) = 0, \quad \forall \gamma \geq \frac{asnr_z}{1 + \text{snr}_1 + \text{snr}_2}. \quad (98)$$

This concludes the proof.

The same conclusion has been used in [5] to obtain the corner points of the Gaussian two-user interference channel. We may obtain similar results for this setting, for example consider the weak interference case, meaning that $Z$ is required to be reliably decoded at a receiver

$$Y_{z,n} = \sqrt{\text{snr}_z} Z_n + N_n \quad (99)$$

and $a \in [0, 1)$. In this case we can obtain the following set of capacity boundary points by simply optimizing $I(Z; Y_z)$ and taking into account the property in Theorem 4.

**Corollary 1:** Given the above Gaussian MAC with weak interference the following rates are on the boundary of the capacity region:

$$(R_1, R_2, R_z) = \left( \frac{1}{2} \log (1 + \beta \text{snr}_1), \frac{1}{2} \log \left( 1 + \frac{(1 - \beta)\text{snr}_1 + \text{snr}_2}{1 + \beta \text{snr}_1} \right), \frac{1}{2} \log \left( 1 + \frac{asnr_z}{1 + \text{snr}_1 + \text{snr}_2} \right) \right) \quad (100)$$

for any $\beta \in [0, 1]$.

### VII. APPLICATIONS TO THE $K$-USER GAUSSIAN INTERFERENCE CHANNEL

In this section we consider two applications of the I-MMSE like property to the $K$-user Gaussian interference channel that have, to our understanding, practical meaning. In order to simplify the exposition we limit these settings to three transmitter-receiver pairs, however the extension to any $K$ users is straightforward.

**A. The Intermediate Transmitter**

Consider the following three-user Z-interference channel:

$$Y_{1,n} = \sqrt{\text{snr}_1} X_{1,n} + \sqrt{a_2 \text{snr}_2} X_{2,n} + N_{1,n}$$

$$Y_{2,n} = \sqrt{\text{snr}_2} X_{2,n} + \sqrt{a_3 \text{snr}_3} X_{3,n} + N_{2,n}$$

$$Y_{3,n} = \sqrt{\text{snr}_3} X_{3,n} + N_{3,n} \quad (101)$$

where $N_{i,n}$ are independent standard additive Gaussian noise vectors. The signals transmitted are $X_{i,n}$ and are independent of each other (no cooperation between the transmitters) and the Gaussian noise. This setting is depicted in Figure 3. Now, consider the setting in which the signal $X_{2,n}$ is an intermediate transmission added to a working setting in which the two other transmissions, $X_1$ and $X_3$, employ “good” codebook sequences and attain reliable decoding at the respective receivers. Given these assumptions the corner points of the two-user Gaussian Z-interference channel provide us with the maximum rate for $X_2$ that allows the system to continue operating with the same codebooks (same rates), for $X_1$ and $X_3$. 

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Corollary 2: The transmission of the intermediate node is limited to

\[ R_2 \leq \min \left\{ \frac{1}{2} \log \left( 1 + \frac{a_2 \text{snr}_2}{1 + \text{snr}_1} \right), \frac{1}{2} \log \left( 1 + \frac{\text{snr}_2}{1 + a_3 \text{snr}_3} \right) \right\}. \] (102)

Another update to the system that is required is that of the decoding scheme at \( Y_1 \) to a scheme that first decodes and removes the interfering transmission. The operation of \( Y_2 \) is similar: first reliably decode and remove the transmission from \( X_3 \) and then reliably decode the additional transmission, \( X_2 \).

B. Interference from Proportional Distances

The strength of the interference is often a function of the distance between the original transmission and the receiver with which it interferes. Let us consider the following specific setting in which this function is of a multiplicative form:

\[
Y_{1,n} = \sqrt{\text{snr}_1}X_{1,n} + \sqrt{a \text{snr}_2}X_{2,n} + \sqrt{a^2 \text{snr}_3}X_{3,n} + N_{1,n}
\]

\[
Y_{2,n} = \sqrt{\text{snr}_2}X_{2,n} + \sqrt{a \text{snr}_3}X_{3,n} + N_{2,n}
\]

\[
Y_{3,n} = \sqrt{\text{snr}_3}X_{3,n} + N_{3,n}.
\] (103)

In other words we have a cascade of three transmitter-receiver pairs, in which the strength of their interference decays by some \( a \in [0, 1) \) and is only one directional. This setting is depicted in Figure 4.
For such a setting we can again use the conclusions of Theorem 1, that in order to obtain reliable communication of a “good” code sequence we must reliably decode the interference, and claim the following:

Lemma 7: For the above model, the following rates are on the boundary of the capacity region:

\[(R_1, R_2, R_3) = \left( \frac{1}{2} \log(1 + \text{snr}_1), \frac{1}{2} \log \left(1 + \frac{\alpha \text{snr}_2}{1 + \text{snr}_1}\right), \frac{1}{2} \log \left(1 + \frac{(1 - \beta)\alpha \text{snr}_2 + \alpha^2 \text{snr}_3}{1 + \text{snr}_1 + \beta \alpha \text{snr}_2}\right) \right) \]  

(104)

for any \(\beta \in [0, 1]\).

Proof: From Theorem 1 we can conclude that if we have reliable decoding of \(X_1\) we also have that

\[\text{MMSE}(\sqrt{\alpha \text{snr}_2}X_2 + \sqrt{\alpha^2 \text{snr}_3}X_3; \gamma) = 0, \quad \forall \gamma \geq \frac{1}{1 + \text{snr}_1}\]  

(105)

meaning that we also have reliable decoding of the combined interference at \(Y_1\). Using [21, Equation (19)] we can rewrite the above condition as follows:

\[\frac{1}{\sqrt{\alpha}} \text{MMSE}(\sqrt{\text{snr}_2}X_2 + \sqrt{\alpha^2 \text{snr}_3}X_3; \gamma') = 0, \quad \forall \gamma' \geq \frac{1}{1 + \text{snr}_1}\]  

(106)

The advantage of the above model with its multiplicative decay of interference is that \(Y_2\) is a better receiver than \(Y_1\) in terms of this combined signal, that is we also have the reliable decoding of this combined interference at
This is the main conclusion which transforms $Y_2$ from a receiver that simply requires the reliable decoding of $X_2$ to a MAC receiver that requires the reliable decoding of $(X_2, X_3)$. For such a receiver we have a constraint on the sum-rate:

$$R_2 + R_3 \leq I(X_2, X_3; Y_2) \leq \frac{1}{2} \log \left( 1 + \frac{\alpha \text{snr}_2 + \alpha^2 \text{snr}_3}{1 + \text{snr}_1} \right)$$

(107)

where the second inequality is due to the conclusion in (106). Moreover, we can also conclude individual constraints on $R_2$ and $R_3$

$$R_2 \leq \frac{1}{2} \log \left( 1 + \frac{\alpha \text{snr}_2}{1 + \text{snr}_1} \right)$$

(108)

$$R_3 \leq \frac{1}{2} \log \left( 1 + \frac{\alpha^2 \text{snr}_3}{1 + \text{snr}_1} \right)$$

simply by using a genie-aided approach in which $Y_1$ is provided with either $X_3$ or $X_2$. Finally, note that there are no additional constraints on $R_3$ due to the requirement of reliable decoding at $Y_3$ since this is a stronger receiver than $Y_2$ in terms of the transmission of $X_3$. Putting (107) and (108) together we have that for any $\beta \in [0, 1]$

$$(R_2, R_3) \leq \left( \frac{1}{2} \log \left( 1 + \frac{\beta \text{snr}_2}{1 + \text{snr}_1} \right), \frac{1}{2} \log \left( 1 + \frac{(1 - \beta) \text{snr}_2 + \alpha^2 \text{snr}_3}{1 + \text{snr}_1 + \beta \text{snr}_2} \right) \right).$$

(109)

These rates are obtained by a “good” point-to-point code sequence for $X_1$ designed for the AWGN channel with SNR of $\text{snr}_1$ and a “good” Gaussian MAC code sequence for $(X_2, X_3)$ designed for reliable decoding at a receiver

$$\hat{Y} = \sqrt{\frac{\alpha \text{snr}_2}{1 + \text{snr}_1}} X_2 + \sqrt{\frac{\alpha^2 \text{snr}_3}{1 + \text{snr}_1}} X_3 + N.$$  

(110)

Such a code sequence will attain the desired rates and also comply with the requirement in (105) allowing the reliable decoding of $X_1$ at $Y_1$. Note that in this scheme $Y_1$ can reliably decode all three transmissions; however this is not necessarily the case for all optimal schemes. This concludes the proof.

\section*{VIII. Summary}

In this paper we have provided an alternative proof of the I-MMSE like relationship shown in [5, Theorem 4]. As opposed to the Guo, Shamai and Verdú I-MMSE relationship [1], which holds over the AWGN channel for any SNR, for any arbitrary input distribution and for any dimension $n$, the I-MMSE like relationship is limited in SNR and holds only in the limit as $n \to \infty$. However, it extends the I-MMSE relationship to an additive noise channel with a noise that is an output of a “good” code sequence transmitted through an AWGN channel. The original proof of this result relied on the I-MMSE relationship [1] and followed estimation theory arguments. The proof proposed here is an information theoretic proof and follows the incremental channel approach, which is most insightful. In this proof we see precisely how all the conditions are required for this relationship to hold. Specifically, we see that the asymptotic independence holds only for the relevant values of SNRs. For these values the extension of the truncation argument used in [7, Lemma 5.2.1] is possible and allows us to approximate the mutual information at weak SNRs. In is important to note that everything holds only once $n \to \infty$ where we can build on the properties of “good” code sequences for which the behavior tends to that of additive Gaussian noise.
The I-MMSE like relationship builds on the properties of a “good” code sequence specified for every SNR. In this sense this approach differs from the one taken by Polyanskiy and Wu [6] which builds on the properties of such codes at the output SNR, following observations regarding the output distribution of such codes as given in [11] and [12]. This difference raises many open questions, most notably: are there cross insights between these different approaches, and can this approach be extended beyond “good” code sequences, providing insight into the behavior of the full capacity region of the interference channel, which so far is mainly characterized via bounds.

**APPENDIX**

A. The Covariance of $\hat{N}_n$

Using the definitions in (36) we can calculate the covariance of $\hat{N}_n$ ($N_{1,n}$ and $N_{2,n}$ are i.i.d. Gaussian zero-mean random vectors so we need to calculate only the variance)

\[
\text{Var}(\hat{N}) = \delta \sigma_1^2 + \frac{\text{snr}}{\delta} \sigma_2^2
\]

\[
= \delta \left( \frac{1}{\text{snr} + \delta} - \alpha \right) + \frac{\text{snr}}{\delta} \left( \frac{1}{\text{snr}} - \alpha - \sigma_1^2 \right)
\]

\[
= \delta \left( \frac{1}{\text{snr} + \delta} - \alpha \right) + \frac{\text{snr}}{\delta} \left( \frac{1}{\text{snr}} - \frac{1}{\text{snr} + \delta} \right)
\]

\[
= \delta \left( \frac{1}{\text{snr} + \delta} - \alpha \right) + \frac{\text{snr}}{\delta} \frac{\delta}{\text{snr}(\text{snr} + \delta)}
\]

\[
= \delta \frac{\text{snr}}{\text{snr} + \delta} + \frac{\text{snr}}{\text{snr} + \delta} - \delta \alpha
\]

\[
= 1 - \delta \alpha. \tag{111}
\]

B. Equation (51)

\[
I(Y_1; Y_2) = D(P_{y_1y_2} || P_{y_1} P_{y_2}) = \int \ln \frac{P_{y_1y_2}}{P_{y_1} P_{y_2}} P_{y_1} P_{y_2}
\]

\[
= \int \ln \frac{P_{y_1y_2} P_{y_1,c} P_{y_2,c}}{P_{y_1} P_{y_2}} P_{y_1} P_{y_2}
\]

\[
= \int \ln \left( \frac{P_{y_1y_2}}{P_{y_1,c} P_{y_2,c}} \right) P_{y_1} P_{y_2} + \int \ln \frac{P_{y_1,c}}{P_{y_1}} P_{y_2} P_{y_1,c} P_{y_2,c}
\]

\[
= D(P_{y_1y_2} || P_{y_1,c} P_{y_2,c}) + \int \ln \frac{P_{y_1,c}}{P_{y_1}} P_{y_2} P_{y_1,c} + \int \ln \frac{P_{y_2,c}}{P_{y_2}} P_{y_1} P_{y_2}
\]

\[
= D(P_{y_1y_2} || P_{y_1,c} P_{y_2,c}) - D(P_{y_1} || P_{y_1,c}) - D(P_{y_2} || P_{y_2,c}). \tag{112}
\]
Let us define a Gaussian distribution which has the same covariance as $P_{y_1,y_2}$ and denote it as $P_{y_1,G,y_2,G}$. Thus we can rewrite the above as

$$I(Y_1;Y_2) = \int \ln \frac{P_{y_1,y_2}}{P_{y_1,G,y_2,G}} P_{y_1,y_2} - D(P_{y_1} \parallel P_{y_1,G}) - D(P_{y_2} \parallel P_{y_2,G})$$

$$= \int \ln \frac{P_{y_1,y_2}}{P_{y_1,G,y_2,G}} P_{y_1,y_2} - D(P_{y_1} \parallel P_{y_1,G}) - D(P_{y_2} \parallel P_{y_2,G})$$

$$= \int \ln \frac{P_{y_1,y_2}}{P_{y_1,G,y_2,G}} P_{y_1,y_2} + \int \ln \frac{P_{y_1,G,y_2,G}}{P_{y_1,G} P_{y_2,G}} P_{y_1,y_2} - D(P_{y_1} \parallel P_{y_1,G}) - D(P_{y_2} \parallel P_{y_2,G})$$

$$= \int \ln \frac{P_{y_1,y_2}}{P_{y_1,G,y_2,G}} P_{y_1,y_2} + \int \ln \frac{P_{y_1,G,y_2,G}}{P_{y_1,G} P_{y_2,G}} P_{y_1,G,y_2,G} - D(P_{y_1} \parallel P_{y_1,G}) - D(P_{y_2} \parallel P_{y_2,G})$$

$$= D(P_{y_1,y_2} \parallel P_{y_1,G,y_2,G}) + D(P_{y_1,G,y_2,G} \parallel P_{y_1,G} P_{y_2,G}) - D(P_{y_1} \parallel P_{y_1,G}) - D(P_{y_2} \parallel P_{y_2,G})$$

(113)

where transition $a$ is due to the fact that the distributions in the ln are both zero-mean Gaussian and thus this is equivalent to the second moments which are identical whether calculated according to $P_{y_1,G,y_2,G}$ or to $P_{y_1,y_2}$.

C. Proof of Lemma 2

Proof: This claim is a direct extension of the result of Han and Verdú [11, Theorem 15] for finite alphabet inputs and its extension to continuous inputs given in [12, Theorem 2]. As shown in [12] the convergence of the empirical output distribution in normalized divergence follows directly from [12, Theorem 1] and [12, Lemma 1]. On the other hand [12, Theorem 1] has been extended in [22] (see also [23, Theorem 1]) where it was shown that a “good”, capacity achieving, code sequence follows the maximum mutual information for all SNRs up to the SNR of reliable communication. Thus, the assumption in [12, Lemma 1, equation (15)] is fulfilled by such code sequences, and the conclusion in [12, Theorem 2] can be extended to the output distribution at any SNR up to the SNR of reliable communication.

It remains only to examine the exact SNRs for which this holds. Recall the two outputs considered here as defined in (50):

$$Y_1 \equiv \sqrt{\alpha} X_n + \sigma_1 N_{1,n} + \sigma_2 N_{2,n}$$

$$Y_2 \equiv \sqrt{\alpha} X_n + \sigma_1 N_{1,n} - \frac{\text{snr}}{\delta^2} \sigma_2 N_{2,n}.$$  

(114)

Considering first $Y_1$ we have that the SNR is

$$\frac{\alpha}{\sigma_1^2 + \sigma_2^2} = \frac{\alpha \text{snr}}{1 - \alpha \text{snr}}$$

$$= \frac{\text{snr}_1 \text{snr}}{\alpha \text{snr}_2 - \text{snr}_1 \text{snr}}.$$  

(115)

Since $X_n$ is taken from a “good” code sequence designed for SNR of snr$_1$ we have that

$$\frac{\text{snr}_1 \text{snr}}{\alpha \text{snr}_2 - \text{snr}_1 \text{snr}} \in [0, \text{snr}_1)$$  

(116)

results in the required mutual information behavior for the output $Y_1$. This is equivalent to

$$\text{snr} \in \left[0, \frac{\alpha \text{snr}_2}{1 + \text{snr}_1}\right].$$  

(117)
Similarly, for $Y_2$ we have that the SNR is
\[
\frac{\alpha}{\sigma^2_1 + \frac{\text{snr}}{\delta} \sigma^2_2} = \frac{\alpha}{\frac{1}{\text{snr} + \delta} - \alpha + \frac{\text{snr}^2}{\delta^2} \left( \frac{1}{\text{snr}} - \frac{1}{\text{snr} + \delta} \right)} = \frac{\frac{\alpha}{\text{snr} + \delta} - \alpha + \frac{\text{snr}}{\delta} \sigma^2_2}{\frac{\alpha}{\text{snr} + \delta} - \frac{\alpha}{\delta}}.
\]
Note that the above equation does not contain $\text{snr}$; however it still contains $\delta$. The condition that this SNR will fall within the region $[0, \text{snr}_1]$ in which we have the desired mutual information behavior reduces to
\[
\delta \leq \frac{\text{asnr}_2}{1 + \text{snr}_1}.
\]
This conclude the proof.

D. Proof of Lemma 3

Proof: We need to prove that $P_{y_1,Y_2}$ approaches a Gaussian distribution for all relevant values of $\text{snr}$. In order to prove this we show that every linear combination of the two approaches a Gaussian random vector. We examine the distribution of $\beta Y_1 + \delta Y_2$ for every value of $\beta$. Since we know that the distribution of $Y_1$ approaches a Gaussian distribution [11], this will suffice. We have
\[
\beta Y_1 + \delta Y_2 = \beta \left( \sqrt{\alpha} X_n + \sigma_1 N_{1,n} + \sigma_2 N_{2,n} \right) + \delta \sqrt{\alpha} X_n + \delta \sigma_1 N_{1,n} - \text{snr} \sigma_2 N_{2,n}
\]
\[= \sqrt{\alpha} (\beta + \delta) X_n + \sigma_1 N_{1,n} (\beta + \delta) + \sigma_2 N_{2,n} (\beta - \text{snr}).
\]
This is the output of the transmission of $X_n$ through an AWGN channel. Thus, as shown in the proof of Lemma 2 the output distribution is approximately Gaussian when the SNR is $[0, \text{snr}_1]$. Denoting the noise variance by $\sigma^2$ we thus need to have $\frac{\alpha(\beta + \delta)^2}{\sigma^2} \in [0, \text{snr}_1]$ for all values of $\beta$, or in other words
\[
\sigma^2 \geq \frac{\alpha}{\text{snr}_1} (\beta + \delta)^2
\]
\[
\sigma^2_1 (\beta + \delta)^2 + \sigma^2_2 (\beta - \text{snr})^2 \geq \frac{\alpha}{\text{snr}_1} (\beta + \delta)^2
\]
\[
\frac{1 - \alpha (\text{snr} + \delta)}{\text{snr} + \delta} (\beta + \delta)^2 + \frac{\delta}{\text{snr} (\text{snr} + \delta)} (\beta - \text{snr})^2 \geq \frac{\alpha}{\text{snr}_1} (\beta + \delta)^2
\]
\[
\text{snr} \left( 1 - \left( 1 + \frac{1}{\text{snr}_1} \right) \alpha (\text{snr} + \delta) \right) (\beta + \delta)^2 + \delta (\beta - \text{snr})^2 \geq \frac{\alpha}{\text{snr}_1} (\beta + \delta)^2 \text{snr} (\text{snr} + \delta)
\]
\[
\text{snr} \left( 1 - \left( 1 + \frac{1}{\text{snr}_1} \right) \alpha (\text{snr} + \delta) \right) (\beta^2 + 2 \beta \delta + \delta^2) + \delta (\beta^2 - 2 \beta \text{snr} + \text{snr}^2) \geq 0.
\]
This results in a parabolic equation of the following form:
\[
\beta^2 \left[ \text{snr} \left( 1 - \left( 1 + \frac{1}{\text{snr}_1} \right) \alpha (\text{snr} + \delta) \right) + \delta \right] + 2 \beta \left[ \delta \text{snr} \left( 1 - \left( 1 + \frac{1}{\text{snr}_1} \right) \alpha (\text{snr} + \delta) \right) - \delta \text{snr} \right] + \delta^2 \text{snr} \left( 1 - \left( 1 + \frac{1}{\text{snr}_1} \right) \alpha (\text{snr} + \delta) \right) + \delta \text{snr}^2 \geq 0.
\]
Applying some algebra on the above we obtain the following:

\[
\beta^2 (\text{snr} + \delta) \left( 1 - \left( 1 + \frac{1}{\text{snr}} \right) \alpha \text{snr} \right) - 2 \beta \left( 1 + \frac{1}{\text{snr}} \right) \delta \text{snr} (\text{snr} + \delta) \left( 1 - \left( 1 + \frac{1}{\text{snr}} \right) \delta \alpha \right) \geq 0
\]

\[
\beta^2 \left( 1 - \left( 1 + \frac{1}{\text{snr}} \right) \alpha \text{snr} \right) - 2 \beta \left( 1 + \frac{1}{\text{snr}} \right) \delta \alpha \text{snr} + \delta \text{snr} \left( 1 - \left( 1 + \frac{1}{\text{snr}} \right) \delta \alpha \right) \geq 0.
\]

(123)

For this inequality to hold for all values of \( \beta \) we need

- A positive coefficient of \( \beta^2 \).
- No real valued roots.

The first condition results in

\[
1 - \left( 1 + \frac{1}{\text{snr}} \right) \alpha \text{snr} \geq 0.
\]

(124)

The second condition results in

\[
\Delta = \left( -2 \left( 1 + \frac{1}{\text{snr}} \right) \delta \alpha \text{snr} \right)^2 - 4 \left( 1 - \left( 1 + \frac{1}{\text{snr}} \right) \alpha \text{snr} \right) \delta \text{snr} \left( 1 - \left( 1 + \frac{1}{\text{snr}} \right) \delta \alpha \right) \leq 0
\]

\[
4 \delta \text{snr} \left( 1 + \frac{1}{\text{snr}} \right) \alpha (\text{snr} + \delta) - 1 \leq 0
\]

(125)

Recalling that \( \alpha = \frac{\text{snr}}{\text{snr}_2} \), the above two conditions result in

\[
\text{snr} \leq \frac{1}{\left( 1 + \frac{1}{\text{snr}} \right) \alpha} = \frac{\text{snr}_2}{1 + \text{snr}_1}
\]

\[
\text{snr} + \delta \leq \frac{1}{\left( 1 + \frac{1}{\text{snr}} \right) \alpha} = \frac{\text{snr}_2}{1 + \text{snr}_1}.
\]

(126)

This means that we only require that \( \text{snr} + \delta \leq \frac{\text{snr}_2}{1 + \text{snr}_1} \) to have a “smiling” parabola with no real roots, that is, positive for all values of \( \beta \). In such a case using the extension to Han and Verdú [11] we can conclude that the combined signal is approximately Gaussian. Thus, the joint distribution of \( Y_1 \) and \( Y_2 \) is approximately Gaussian.

This concludes the proof.

\[\blacksquare\]

E. Proof of Lemma 4

Proof: Our basic assumption is that the code sequence \( \{X_n\}_{n \geq 1} \) is point-to-point capacity achieving, meaning that if we consider the following AWGN channel output:

\[
Y_n^c = \sqrt{\text{snr}} X_n + N_n
\]

(127)

the input-output mutual information will follow

\[
\lim_{n \to \infty} \frac{1}{n} I(X_n; Y_n^c) = \frac{1}{2} \log(1 + \text{snr}).
\]

(128)
Moreover, we are restricted to code sequences that comply with the following constraint:

\[
\frac{1}{n} \| x_n \|^2 \leq 1, \quad \forall x_n \in C_n,
\]  

(129)

meaning also that

\[
\frac{1}{n} \text{Tr} (R_{X_n}) = E \left\{ \frac{1}{n} \| X_n \|^2 \right\} \leq 1.
\]  

(130)

The input-output mutual information can be upper bounded for all \( n \geq 1 \) as follows:

\[
I (X_n; Y_n^c) \leq \frac{1}{2} \log |I_n + \text{snr} R_{X_n}| \\
= \frac{1}{2} \log \prod_{i=1}^{n} (1 + \text{snr} \lambda_i(R_{X_n})) \\
= \frac{1}{2} \sum_{i=1}^{n} \log (1 + \text{snr} \lambda_i(R_{X_n})).
\]  

(131)

Taking \( n \to \infty \) we have that

\[
\lim_{n \to \infty} I (X_n; Y_n^c) \leq \lim_{n \to \infty} \frac{1}{2} \sum_{i=1}^{n} \log (1 + \text{snr} \lambda_i(R_{X_n})).
\]  

(132)

The optimization problem

\[
\max_{n \to \infty} \frac{1}{2} \sum_{i=1}^{n} \log (1 + \text{snr} \lambda_i(R_{X_n})) \\
\text{s.t.} \quad \frac{1}{n} \text{Tr} (R_{X_n}) \leq 1
\]  

(133)

is a strictly concave problem, and thus the optimal solution is unique. This is due to the strict concavity of the log function. The solution is

\[
\max_{n \to \infty} \frac{1}{2} \sum_{i=1}^{n} \log (1 + \text{snr} \lambda_i(R_{X_n})) = \frac{1}{2} \log (1 + \text{snr})
\]  

(134)

and is obtained when \( \lambda_i = 1 \). As this optimal solution is approached in the limit we can write the following: for all \( \epsilon > 0 \) there exists an \( N_\epsilon \) such that

\[
\left| \frac{1}{n} \sum_{i=1}^{n} \frac{1}{2} \log (1 + \text{snr} \lambda_i(R_{X_n})) - \frac{1}{2} \log (1 + \text{snr}) \right| \leq \epsilon, \quad \forall n \geq N_\epsilon.
\]  

(135)

Since the optimization problem is strictly concave, approaching the optimal solution means that we are in the vicinity of the optimizing point (which is unique), \( i.e., \)

\[
|\lambda_i(R_{X_n}) - 1| \leq \epsilon', \quad \forall n \geq N_\epsilon.
\]  

(136)

This concludes the proof.
F. Proof of Lemma 5

Proof: Assume \( P_{G_0} \sim \mathcal{N}(0, \Sigma_0) \) and \( P_{G_1} \sim \mathcal{N}(0, \Sigma_0) \) are of dimension \( n \). Then

\[
D(P_{G_0} \mid \mid P_{G_1}) = \frac{1}{2} \left( \text{Tr}(\Sigma_1^{-1}\Sigma_0) - n + \ln \left( \frac{|\Sigma_1|}{|\Sigma_0|} \right) \right). \tag{137}
\]

We further assume that

\[
\Sigma_0 = \begin{pmatrix} A & B^T \\ B & C \end{pmatrix},
\]

\[
\Sigma_1 = \begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix}
\]

(138)

where \( A \) and \( C \) are non-singular. Given these assumptions we have that

\[
\Sigma_1^{-1} = \begin{pmatrix} A^{-1} & 0 \\ 0 & C^{-1} \end{pmatrix}
\]

(139)

and thus

\[
\text{Tr}(\Sigma_1^{-1}\Sigma_0) = n.
\]

(140)

Thus, the problem reduces to

\[
D(P_{G_0} \mid \mid P_{G_1}) = \frac{1}{2} \ln \left( \frac{|\Sigma_1|}{|\Sigma_0|} \right)
\]

\[
= \frac{1}{2} \ln |AC|
\]

\[
= \frac{1}{2} \ln \left| AC \left( I_n - C^{-1}BA^{-1}B^T \right) \right|
\]

\[
= \frac{1}{2} \ln \left| I_n - C^{-1}BA^{-1}B^T \right|
\]

\[
= \frac{1}{2} \ln \left| \left( I_n - C^{-1}BA^{-1}B^T \right)^{-1} \right|
\]

\[
= \frac{1}{2} \ln |(M)^{-1}| \tag{141}
\]

where in transition a we used the following definition

\[
M = I_n - C^{-1}BA^{-1}B^T.
\]

(142)

The determinant of \( M \) can be written as

\[
|M| = |I_n - C^{-1}BA^{-1}B^T|
\]

\[
= \prod_{i=1}^{n} \left( 1 - \lambda_i \left( C^{-1}BA^{-1}B^T \right) \right). \tag{143}
\]

This concludes the proof.
G. Proof of Lemma 6

Proof: We begin with a straightforward application of [18, Theorem 2] where we notice that the summation can be written as a trace function and also note that the Fisher information matrix (denoted by \( J \)) considered is that of the additive noise \( \tilde{Q}_n \) as also noted in [24, Corollary 2]. Thus we have that when \( \delta \to 0 \) (assuming bounded variances)

\[
I (Z_n^\kappa, Y_n^\kappa) = I \left( \sqrt{\delta} Z_n^\kappa, Y_n^\kappa \right) \\
= \frac{\delta}{2} \text{Tr} \left( J_{\tilde{Q}_n} R_{Z_n}^\kappa \right) + o \left( \delta \text{Tr} \left( R_{Z_n}^\kappa \right) \right) \\
= \frac{\delta}{2} \text{Tr} \left( J_{\tilde{Q}_n} R_{Z_n}^\kappa \right) + o (\delta).
\]

Now we put to use the specific structure of \( \tilde{Q}_n \):

\[
J_{\tilde{Q}_n} = J \sqrt{\delta \alpha} X_n + \tilde{N}_n
\]

\[
= \frac{a}{1 + \delta \alpha} J \sqrt{\frac{\delta \alpha}{1 + \delta \alpha}} X_n + \tilde{N}_n
\]

\[
= \frac{b}{1 + \delta \alpha} \left( I - \frac{\delta \alpha}{1 + \delta \alpha} E_{X_n} \left( \sqrt{\frac{\delta \alpha}{1 - \delta \alpha}} X_n + \tilde{N}_n \right) \right)
\]

(145)

where in transition a we have used a well-known property of the Fisher information (see for example [25, Equation (9)]) where \( \tilde{N}_n \) denotes a standard additive Gaussian noise. In transition b we use [1, Equation (57)] which is given specifically for the standard additive Gaussian noise channel, and \( E_{X_n} \left( \sqrt{\frac{\delta \alpha}{1 + \delta \alpha}} X_n + \tilde{N}_n \right) \) denotes the MMSE matrix when estimating \( X_n \) from the channel output \( \sqrt{\frac{\delta \alpha}{1 + \delta \alpha}} X_n + \tilde{N}_n \). We use this in (144) and obtain the following:

\[
I (Z_n^\kappa, Y_n^\kappa) = \frac{1}{2} \frac{\delta}{1 + \delta \alpha} \text{Tr} (R_{Z_n}^\kappa) - \frac{1}{2} \left( \frac{\delta}{1 + \delta \alpha} \right)^2 \alpha \text{Tr} \left( E_{X_n} \left( \sqrt{\frac{\delta \alpha}{1 - \delta \alpha}} X_n + \tilde{N}_n \right) R_{Z_n}^\kappa \right) + o (\delta)
\]

\[
= \frac{1}{2} \frac{\delta'}{\alpha} \text{Tr} (R_{Z_n}^\kappa) - \frac{1}{2} \delta'^2 \text{Tr} \left( E_{X_n} (\sqrt{\delta'} X_n + \tilde{N}_n) R_{Z_n}^\kappa \right) + o (\delta)
\]

(146)

where in the second transition we have used the definition \( \delta' = \frac{\delta \alpha}{1 + \delta \alpha} \). In order to complete the proof it remains to show only that

\[
\text{Tr} \left( E_{X_n} (\sqrt{\delta'} X_n + \tilde{N}_n) R_{Z_n}^\kappa \right) \leq \max_{i \in [1, n]} \lambda_{i} (R_{X_n}) \text{Tr} (R_{Z_n}^\kappa)
\]

\[
\sum_{j=1}^{n} \lambda_{j} (E_{X_n} (\sqrt{\delta'} X_n + \tilde{N}_n) R_{Z_n}^\kappa) \leq \max_{i \in [1, n]} \lambda_{i} (R_{X_n}) \sum_{j=1}^{n} \lambda_{j} (R_{Z_n}^\kappa)
\]

(147)

and thus we have a lower bound. In order to show this first require the following inequality [17, Equation 2.0.3]:

\[
\max_{i+j=t+n} \{ \lambda_i(A) \lambda_j(B) \} \leq \lambda_t(AB) \leq \min_{i+j=t+1} \{ \lambda_i(A) \lambda_j(B) \}.
\]

(148)

This inequality can be loosened as follows:

\[
\min_{i} \{ \lambda_i(A) \} \lambda_t(B) \leq \lambda_t(AB) \leq \max_{i} \{ \lambda_i(A) \} \lambda_j(B), \ \forall j
\]

(149)
where in the upper bound any choice of $j$ has a corresponding $i$ that results in $i + j = t + 1$ (all matrices are $n \times n$ and positive semi-definite). In the lower bound the situation is a bit different; however the choice of $j = t$ and $i = n$ is always possible, thus providing an immediate lower bound. We then take a lower bound on that by taking a lower bound on $i$ which might result in an eigenvalue lower than that at $i = n$. Taking the upper bound with $j = t$ we have that

$$
\sum_{j=1}^{n} \lambda_j \left( E_{X_n}(\sqrt{\delta'} X_n + \tilde{N}_n) R_{Z_n} \right) \leq \sum_{j=1}^{n} \max_{i \in [1,n]} \lambda_i \left( E_{X_n}(\sqrt{\delta'} X_n + \tilde{N}_n) \right) \lambda_j \left( R_{Z_n} \right)
$$

(150)

For the next step we require the following claim:

**Lemma 8:** For all $j \in [1,n]$ we have the following upper bound on the eigenvalues of the MMSE matrix:

$$
\lambda_j \left( E_{X_n}(\sqrt{\delta'} X_n + \tilde{N}_n) \right) \leq \max_{i \in [1,n]} \lambda_i \left( R_{X_n} \right).
$$

(151)

**Proof:** The proof is given in Appendix H.

Using the above we have that

$$
\sum_{j=1}^{n} \lambda_j \left( E_{X_n}(\sqrt{\delta'} X_n + \tilde{N}_n) R_{Z_n} \right) \leq \max_{i \in [1,n]} \lambda_i \left( E_{X_n}(\sqrt{\delta'} X_n + \tilde{N}_n) \right) \sum_{j=1}^{n} \lambda_j \left( R_{Z_n} \right)
$$

(152)

$$
\leq \max_{i \in [1,n]} \lambda_i \left( R_{X_n} \right) \sum_{j=1}^{n} \lambda_j \left( R_{Z_n} \right)
$$

(153)

thus, concluding the proof.

**H. Proof of Lemma 8**

**Proof:** The MMSE matrix of estimating $X_n$ from the AWGN channel output $\sqrt{\delta'} X_n + \tilde{N}_n$ is upper bounded in the positive semidefinite sense by the covariance matrix of $X_n$

$$
E_{X_n}(\sqrt{\delta'} X_n + \tilde{N}_n) \preceq R_{X_n}.
$$

(154)

Denote by $U$ the unitary matrix that diagonalizes $R_{X_n}$ and $\Lambda_{X_n}$ as its diagonal form. Thus, we have

$$
\Lambda_{X_n} - UE_{X_n}(\sqrt{\delta'} X_n + \tilde{N}_n)U^T \succeq 0.
$$

(155)

Also note that

$$
\Lambda_{X_n} \preceq \max_i \lambda_i \left( R_{X_n} \right) I_n.
$$

(156)

Putting the two together we have that

$$
\max_i \lambda_i \left( R_{X_n} \right) I_n - UE_{X_n}(\sqrt{\delta'} X_n + \tilde{N}_n)U^T \succeq 0.
$$

(157)

Since the eigenvalues of the above matrix are non-negative (a positive semi-definite matrix) we can conclude that for all $j \in [0,n]$

$$
\max_i \lambda_i \left( R_{X_n} \right) - \lambda_j \left( UE_{X_n}(\sqrt{\delta'} X_n + \tilde{N}_n)U^T \right) \geq 0
$$

(158)

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and since

$$\lambda_j \left( U \mathbf{E} X_n \left( \sqrt{\delta} \mathbf{X}_n + \tilde{N}_n \right) \right) = \lambda_j \left( \mathbf{E} X_n \left( \sqrt{\delta} \mathbf{X}_n + \tilde{N}_n \right) \right)$$  \hspace{1cm} (158)

due to similarity, we can conclude the proof.

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