Entropy in the NUT-Kerr-Newman Black Holes Due to an 
Arbitrary Spin Field

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Abstract: Membrane method is used to compute the entropy of the NUT-Kerr-Newman black holes. It is found that even though the Euler characteristic is greater than two, the Bekenstein-Hawking area law is still satisfied. The formula $S = \chi A/8$ relating the entropy and the Euler characteristic becomes inapplicable for non-extreme four dimensional NUT-Kerr-Newman black holes.

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I. Introduction

Ever since Bekenstein and Hawking discovered that the entropy of a black hole is proportional to its surface area in the 1970s, more efforts, have been devoted to studying the statistical origin of the black hole entropy. In 1985, 't Hooft proposed brick-wall method and studied the statistical mechanics of a free scalar field in the Schwarzschild black hole background[1]. The reason for the introduction of the 'brick wall' method is that the density of states approaching the horizon diverges. Thus in order to avoid the divergence in the entropy, a cut-off seems has to be introduced, which can be regarded as a 'brick wall'. Mathematically, this method requires the region of non-zero wave function is limited in $r_+ + \varepsilon$ and $L$, where $r_+$ is the radius of event horizon and $\varepsilon$, $L$ are ultraviolet cut-off and infrared cut-off respectively, and the relations: $\varepsilon \leq r_+, L \geq r_+$ should be satisfied. Even though it seems very arbitrary to introduce a cut-off in the brick wall method, it has been widely applied to scalar field and fermion field in various black hole background[2-11], where it showed that the leading term of the entropy for both bosons and fermions is proportional to the area of the event horizon.

Although the brick wall method achieved great success in the study of black hole entropy, when it is applied to non-spherical black holes such as Kerr, and Kerr-Newman black holes, the calculation seems rather complex[12,13]. In particular, the brick wall method need one has to adopt small approximation mass and remove the the term which is not proportional to area in the integration. Moreover, if one applies the brick wall method to Schwarzschild-de Sitter space-time[14]. Different from non-de Sitter space-time, there are two event horizons in Schwarzschild-de Sitter space-time. The two event horizons have different temperatures. Therefore the radiation between them is not in thermal equilibrium. It is clear that one should not use the brick wall method which bases on the thermal equilibrium. In other words, the region $[r_+ + \varepsilon, L]$ is not in thermal equilibrium. One should notice that former work conducted under the brick wall method tells us that the leading term of entropy comes from the contribution of the field very close to the horizon. Thus one can chose a thin membrane of quantum fields in the vicinity of each event horizon[15]. If the
distance from the membrane is \( \delta, \varepsilon \) and \( \delta \) should have the same order. That is to say we may assume the thickness of the membrane is also \( \varepsilon \). And then the fields in the membrane \([r_+ + \varepsilon, r_+ + 2\varepsilon]\) can be regarded as in locally thermal equilibrium. In fact, Hawking radiation also comes from the vacuum fluctuation in the vicinity of event horizon. Therefore we might regard the two event horizons as two independent thermal equilibrium systems and consider them respectively. This method is also applied to non-de Sitter space-time. We call this method 'membrane method'. The physical picture of this method is very clear.

On the other hand, the membrane method enables one free from dealing with the angular modes in the computation as if he/she can separate field equations into two parts: the radial equations and the angular equations. That is because just dealing with radial modes can give the Bekenstein-Hawking area law. If the worry is about the coefficient, then in the membrane method, as in the brick-wall method, one has to adjust the cut-off distance to get the usual coefficient, so missing out modes is not going to make matters worse.

However, an important issue should be pointed out here is the singularities of the metric of NUT charged spacetime, which is called Misner strings[16]. In order to avoid the singularities the time coordinate must be periodic. In the Euclidean section this forces a periodicity proportional to the NUT charge that must be matched to the usual periodicity requirement following from the elimination of conical singularities in the \((r, t)\) section. Thus the NUT charge and the rotation parameter must be analytically continued. Series of papers have worked on it [17-22]. But when involving rotation in spacetime, it is not clear that the vanishing of the metric function at the horizon yields the same physics its non-Wick rotated version. Fortunately, R.B.Mann managed to calculate the entropy of Kerr-NUT class spacetime by using a boundary counterterm prescription motivated by the AdS-CFT conjecture[23]. We will take the above issues into account in our present paper. Moreover, the geometrical properties of several gravitational instantons have been investigated in [24]. As it was speculated by some authors that the geometrical properties play an essential role in the explanation of intrinsic thermodynamics of black holes and entropy of a black hole can be expressed in the following formula: \( S = A\chi/8[24] \), in this paper, we will go further to
study the entropy of the NUT-Kerr-Newman black holes. Their entropy of spin fields is calculated by using the membrane method (The gravitational field doesn’t be taken into account because it seems difficult to separate the field equations). The results show that as the cut-off is properly chosen, the entropy in the black hole satisfies the Bekenstein-Hawking area law. What is more, it seems that the formula \( S = A \chi / 8 \) has its limitations: the author only discusses its application to four dimensional sphere-topology black holes.

Our paper is organized as follows. In the next section, we discuss the singularities of the NUT-Kerr-Newman metric and make some mathematic provision. We separate equations in SecIII. Then we will go ahead, in SecIV, with the reduction of the radial part to the one-dimensional wave equation through a series of transformations, we calculate the entropy. In the last section, we present our conclusions.

II. NUT-Kerr-Newman metric

The NUT-Kerr-Newman space-times can be written in Boyer-Lindquist coordinates as [25](here \( \Lambda = 0 \))

\[
\begin{align*}
  ds^2 &= \frac{\Delta - a^2 \sin^2 \theta}{\rho^2} dt^2 + 2 \frac{(r^2 + a^2) a \sin^2 \theta - \Delta \left( a - \frac{(n - a \cos \theta)^2}{a} \right)}{\rho^2} dtd\phi \\
  -\frac{\rho^2}{\Delta} dr^2 - \rho^2 d\theta^2 - \frac{(r^2 + a^2)^2 \sin^2 \theta - \Delta \left( a - \frac{(n - a \cos \theta)^2}{a} \right)^2}{\rho^2} d\phi^2,
\end{align*}
\]

where \( \rho^2, \bar{\rho}, \Delta \), are defined by

\[
\begin{align*}
  \rho^2 &= \bar{\rho} \cdot \bar{\rho}^*, \bar{\rho} = r + i (a \cos \theta - n), \\
  \Delta &= r^2 + a^2 - n^2 + Q^2 - 2Mr,
\end{align*}
\]

where \( M, a, \) and \( n \), are the mass, angular momentum per unit mass, and the NUT parameter (\( n \) is also called gravitational magnetic type mass). This metric has a singularity at \( \theta = 0 \) and \( \theta = \pi \). The \( \cos \theta \) term in the metric means that a small loop around the axis does not shrink to zero length at \( \theta = 0 \) and at \( \theta = \pi \). This singularity can be regarded as the analogue of a Dirac string in electrodynamics, and is called the Misner string.

The Euclidean NUT-Kerr-Newman instanton is
\[ ds^2 = \frac{\Delta + a^2 \sin^2 \theta}{\rho^2} dt^2 + 2 \frac{(r^2 - a^2) \sin^2 \theta - \Delta \left(a - \frac{(n + a \cos \theta)^2}{a}\right)}{\rho^2} dtd\phi \\
+ \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + \frac{(r^2 - a^2)^2 \sin^2 \theta + \Delta \left(a - \frac{(n + a \cos \theta)^2}{a}\right)^2}{\rho^2} d\phi^2, \]

where

\[ \rho^2 = r^2 - (n + a \cos \theta)^2, \]
\[ \Delta = r^2 + a^2 - n^2 + Q^2 - 2Mr. \]

It was argued by Gibbons and Hawking that there are two kinds of basic gravitational instantons based on the dimension of the fixed point set of the continuous isometry group[26]. The first kind contains an isolated fixed point. The second type contains a 2-dimensional fixed point set is called a bolt. It also has been shown that entropy can be associated with a broader and qualitatively different gravitational system, which contains Misner strings[20]. Gravitational entropy arises whenever it is not possible to foliate a given spacetime in the Euclidean regime by a family of surfaces of constant time \( \tau \). Such breakdown in foliation can occur if the U(1) has a fixed point set of even co-dimension \( d_f < d - 2 \) (called a nut)[26]. If the fixed point set has co-dimension \( d-2 \) then the usual relationship between entropy and area holds. However, if there exists additional fixed point sets with lower co-dimensionality then the relationship between area and entropy is generalized. Following the idea of [23], we find the location of the nut is at \( r = a = r_n \) and the surface gravity is given by

\[ \kappa = \frac{(r_+ - r_-)}{2(r_+^2 - a^2)} \]

Here \( r_+ \) and \( r_- \) satisfies \( \Delta = (r - r_+)(r - r_-) = 0 \). The regularity in the \((r, \tau)\) section implies that \( \tau \) has period \( 2\pi/\kappa \). If we proceeding along this line, we can obtain the entropy of scalar field in Euclidean NUT-Kerr-Newman background that it is \( \pi(r_+^2 - a^2) \), which is in agreement with Ref.[23].

Here, we turn to calculate the entropy of NUT-Kerr-Newman black holes by the improved brick wall method in curvature spacetime. The null-vectors of the Newman-Penrose formalism [27] we take.
\[ l^\mu = \left[ \frac{(r^2 + a^2)}{\Delta}, 1, 0, \frac{a}{\Delta} \right], \]
\[ n^\mu = \frac{1}{2\rho^2} \left[ (r^2 + a^2), -\Delta, 0, a \right], \]
\[ m^\mu = \frac{1}{\sqrt{2\rho}} \left[ \frac{i \left( a - \frac{(n - a \cos \theta)^2}{a} \right)}{\sin \theta}, 0, 1, \frac{i}{\sin \theta} \right]. \]  

(3)

We find that the non-vanishing spin-coefficients [27] are:

\[ \pi = \frac{iasin\theta}{\sqrt{2\rho^2}}; \mu = -\frac{\Delta}{2\rho^2 \rho^*}; \alpha = \pi - \beta^*; \beta = \frac{\cos \theta}{2\sqrt{2\rho^2 \sin \theta}}; \]
\[ \tau = \frac{-iasin\theta}{\sqrt{2\rho^2}}; \rho = -\frac{1}{\rho^*}; \gamma = \frac{1}{4\rho^2} \frac{d\Delta}{dr} + \mu. \]  

(4)

Assuming that the azimuthal and time dependence of our fields will be of the form \( e^{i(m\phi - \omega t)} \), we find that the directional derivatives are

\[ D = l^\mu \partial_\mu = D_0, \Delta = n^\mu \partial_\mu = -\frac{\Delta}{2\rho^2} D_0^+, \]
\[ \delta = m^\mu \partial_\mu = \frac{1}{\rho \sqrt{2}} L_0^+, \delta^* = m^*\mu \partial_\mu = \frac{1}{\rho^* \sqrt{2}} L_0 \]  

(5)

where \( D_n, D_n^+, L_n, L_n^+, K, H \) are defined by

\[ D_n = \partial_r + \frac{iK}{\Delta} + \frac{n}{\Delta} \frac{d\Delta}{dr}, \]
\[ D_n^+ = \partial_r - \frac{iK}{\Delta} + \frac{n}{\Delta} \frac{d\Delta}{dr}, \]
\[ L_n = \partial_\theta + H + ncot\theta, \]
\[ L_n^+ = \partial_\theta - H + ncot\theta, \]
\[ K = am - \omega \left( r^2 + a^2 \right), \]
\[ H = \frac{m}{\sin \theta} - \frac{a^2 - (n - a \cos \theta)^2}{a \sin \theta} \omega. \]  

(6)

Thus K and H have the relation

\[ K - aH \sin \theta = -\rho^2 \omega. \]  

(7)

These differential operators satisfy some identities
\begin{align*}
\Delta D_{n+1} &= D_n \Delta \quad \text{(8)} \\
\Delta D^+_{n+1} &= D^+_n \Delta. \quad \text{(9)} \\
(sin \theta) L_{n+1} &= L_n \sin \theta \quad \text{(10)} \\
(sin \theta) L^+_{n+1} &= L^+_n \sin \theta \quad \text{(11)} \\
(D + \frac{m}{\bar{\rho}^*}) \left( L + \frac{im \sin \theta}{\bar{\rho}^*} \right) &= \left( L + \frac{im \sin \theta}{\bar{\rho}^*} \right) \left( D + \frac{m}{\bar{\rho}^*} \right) \quad \text{(12)}
\end{align*}

III. Spin fields in NUT-Kerr-Newman space-time

The Maxwell equations in the Newman-Penrose formalism take on the forms\cite{27}

\begin{align*}
D \phi_1 - \delta^* \phi_0 &= (\pi - 2\alpha) \phi_0 + 2\bar{\rho} \phi_1 - \kappa \phi_2, \quad \text{(13)} \\
D \phi_2 - \delta^* \phi_1 &= -\lambda \phi_0 + 2\pi \phi_1 + (\bar{\rho} - 2\epsilon) \phi_2, \quad \text{(14)} \\
\delta \phi_1 - \Delta \phi_0 &= (\mu - 2\gamma) \phi_0 + 2\tau \phi_1 - \sigma \phi_2, \quad \text{(15)} \\
\delta \phi_1 - \Delta \phi_0 &= -\nu \phi_0 + 2\mu \phi_1 + (\tau - 2\beta) \phi_2. \quad \text{(16)}
\end{align*}

Using Eqs.(4) and (5), then making the transformations

\begin{align*}
\phi_0 &= \Phi_0, \quad \phi_1 = \frac{1}{\sqrt{2\bar{\rho}^*}} \Phi_1
\end{align*}

and

\begin{align*}
\phi_2 &= \frac{1}{(2\bar{\rho}^*)^2} \Phi_1,
\end{align*}

we find that Eqs.(13)-(16) become

\begin{align*}
\begin{bmatrix} D_0 + \frac{1}{\bar{\rho}^*} \end{bmatrix} \Phi_1 &= \begin{bmatrix} L_1 - \frac{im \sin \theta}{\bar{\rho}^*} \end{bmatrix} \Phi_0, \quad \text{(17)} \\
\begin{bmatrix} D_0 - \frac{1}{\bar{\rho}^*} \end{bmatrix} \Phi_2 &= \begin{bmatrix} L_0 + \frac{im \sin \theta}{\bar{\rho}^*} \end{bmatrix} \Phi_1, \quad \text{(18)} \\
\begin{bmatrix} L^+_0 + \frac{im \sin \theta}{\bar{\rho}^*} \end{bmatrix} \Phi_1 &= -\Delta \begin{bmatrix} D^+_1 - \frac{1}{\bar{\rho}^*} \end{bmatrix} \Phi_0, \quad \text{(19)} \\
\begin{bmatrix} L^+_1 + \frac{im \sin \theta}{\bar{\rho}^*} \end{bmatrix} \Phi_2 &= -\Delta \begin{bmatrix} D^+_0 + \frac{1}{\bar{\rho}^*} \end{bmatrix} \Phi_1. \quad \text{(20)}
\end{align*}

From Eqs.(18) and (20), \( \Phi_1 \) can be eliminated to give
Similarly, from Eqs.(19) and (21) there is
\[
\left( \Delta D_1^+ D_1^+ + L_0^+ L_1 - 2i\omega \bar{\rho} \right) \Phi_0 = 0 \tag{21}
\]

Assuming \( \Phi_0 = R_{+1}(r)S_{+1}(\theta) \) and \( \Phi_2 = R_{-1}(\theta)S_{-1}(\theta) \), we can separate the variables of Eqs.(22) and (23) to be
\[
\left( \Delta D_1^+ D_1^+ - 2ir\omega \right) R_{+1} = \lambda R_{+1}, \tag{23}
\]
\[
\left( \Delta D_0^+ D_0^+ + 2ir\omega \right) R_{-1} = \lambda R_{-1}, \tag{24}
\]
\[
\left[ L_0^+ L_1 + 2\omega acos\theta - n \right] S_{+1} = -\lambda S_{+1}, \tag{25}
\]
\[
\left[ L_0 L_1^+ - 2\omega acos\theta - n \right] S_{-1} = -\lambda S_{-1}, \tag{26}
\]

here \( \lambda \) is the separation constant. For the Dirac field, the wave equations for a massless Dirac particles are\[28\]
\[
(D + \varepsilon - \rho) F_1 + (\delta + \pi - \alpha) F_2 = 0,
\]
\[
\left( \Delta' - \mu - \gamma \right) F_2 + (\delta + \beta - \tau) F_1 = 0,
\]
\[
(D + \varepsilon^* - \rho^*) G_2 - (\delta + \pi^* - \alpha^*) G_1 = 0,
\]
\[
\left( \Delta' - \mu^* - \gamma^* \right) G_1 - (\delta + \beta^* - \tau^*) G_2 = 0, \tag{27}
\]

where \( F_1, F_2, G_1, G_2 \) are 4-component spinors, \( \alpha, \beta, \gamma, \varepsilon, \pi, \mu, \rho, \tau \) etc are Newman-Penrose symbols, and \( \alpha^*, \beta^* \) are the complex conjugates of \( \alpha, \beta \) etc.

All the above equations are also separated by using Newman-Penrose formalism. The radial equations are given by
\[
\Delta^{\frac{1}{2}} D_0^+ \left( \Delta^{\frac{1}{2}} D_0 R_{-\frac{1}{2}} \right) = \lambda^2 R_{-\frac{1}{2}}, \tag{28}
\]
\[
\Delta^{\frac{1}{2}} D_0 \left( \Delta^{\frac{1}{2}} D_0^+ R_{\frac{1}{2}} \right) = \lambda^2 R_{\frac{1}{2}}, \tag{29}
\]
\[
L_{\frac{3}{2}} \left( L_{\frac{3}{2}} S_{\frac{1}{2}} \right) = -\lambda^2 S_{\frac{1}{2}}, \tag{30}
\]
\[
L_{\frac{1}{2}} \left( L_{\frac{1}{2}} S_{-\frac{1}{2}} \right) = -\lambda^2 S_{-\frac{1}{2}}, \tag{31}
\]
For scalar field, the separated equations can achieved directly from the Klein-Gordon equation

$$\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\mu} \left( \sqrt{-g} g^{\mu\nu} \frac{\partial \Phi}{\partial x^\nu} \right) = 0. \quad (32)$$

The radial equation is

$$\frac{\partial}{\partial r} \Delta \frac{\partial}{\partial r} R(r) + \frac{[r^2 + a^2 - am]^2}{\Delta} R(r) = \lambda^2 R(r). \quad (33)$$

We can rewrite the above equation with $D_n, D^+_n$

$$\Delta D_0 D^+_0 R_0 = \lambda^2 R_0. \quad (34)$$

The radial equations (23),(29)and (34) can be combined into

$$\Delta D_s D^+_s(2s-1) - 2s(2s-1)i\omega r \right] R_s = \lambda^2 R_s. \quad (35)$$

On the other hand, equations (24) and (28) can be written as

$$\left[ \Delta D^+_1 D_0 + 2s(2s-1)i\omega r \right] R_{-s} = \lambda^2 R_{-s}. \quad (36)$$

where $s$ is the spin number. It is clear that as $s = 0, s = \frac{1}{2}$ and $s = 1$, Eq.(35) corresponds to scalar, Dirac and Maxwell field respectively.

IV. Entropy

By using the Wentzel-Kramers-Brillouin approximation and substituting equations $R_s = e^{i\ell(r)}$ into Eqs.(35), the wave numbers are obtained as follows,

$$k_r^2 = K^2 \Delta^2 + \frac{2s(2s-1)}{\Delta} + \frac{s(2s-1)(s-1)}{\Delta} \left( \frac{d\Delta}{dr} \right)^2 - \frac{\lambda^2}{\Delta}. \quad (37)$$

Considering while $s = 0, \frac{1}{2}, 1$ the third term of the equation is zero, we remove it from our equation. Therefore we have

$$k_r = \frac{1}{\Delta} \sqrt{[(r^2 + a^2)\omega - am]^2 + 2\Delta s(2s-1) - \Delta \lambda^2}. \quad (38)$$

The horizon equation can be written as
\[ \Delta = (r - r_+) (r - r_-) = 0, \tag{39} \]

where \( r_+ \) and \( r_- \) are the radius of the black hole event horizon, the inner horizon respectively. We assume that the boson field is in the Hartle-Hawking vacuum state. The spectrum of Hawking radiation in NUT-Kerr-Newman space-time can be written as

\[ N_{\omega}^2 = \frac{1}{e^{\beta_+ (\omega - m \Omega_+) \pm 1}}, \tag{40} \]

where \( N_{\omega}, \beta_+, \omega \) and \( \Omega_+ \) are the radiation intensity, the inverse of Hawking temperature, the energy of particles, and the angular velocity of event horizon. Moreover, the angular velocity, the Hawking temperature of the black hole event horizon are defined by

\[ \Omega_+ = \lim_{r \to r_H} \frac{-g_{t\phi}}{g_{\phi\phi}} r_+^2 + a^2, \tag{41} \]
\[ T_+ = \frac{\kappa}{2\pi} = \frac{1}{\beta_+} = \frac{(r_+ - r_-)}{4\pi (r_+^2 + a^2)}, \tag{42} \]

The \( \beta_+ \) here is in agreement with that of [25] considering \( \Lambda = 0 \).

The distribution of particles corresponding to Eq.(40) is given by

\[ a_l = \frac{\omega_l}{e^{\beta (\omega - \omega_0)} \pm 1}, \tag{43} \]

where \( a_l \) is the number of particles in the \( l \) energy level, \( \omega_l \) is the degeneracy of \( l \) energy level; \( \beta \) and \( \omega_0 \) denote \( \beta_+ \) and \( \Omega_+ \). It is clear that Eq.(43) will be meaningless if \( \omega < \omega_0 \), because \( a_l \) will be negative for bosons in such case. Thus this demands \( \omega \) in Eq.(40) satisfies \( \omega - m \Omega_+ > 0 \) for boson fields. However, just as it was pointed out by R. B. Mann in Ref.[23] that the Euclidean time variable should be taken to be periodic, so the Lorentzian time variable be periodic with period \( 8\pi n \). Therefore solutions to the wave equation must also be periodic, and so \( \omega \) in Eq.(33) must be \( C/(4n) \), where \( C \) is an integer. And also this demands in the following calculation the sum of \( \omega \) should be replaced by the sum of \( C \). Considering Eq.(40) we set \( E = C/(4n) - m \Omega_+ \). Further calculation will show that this does not change our results if we take \( E \) as the system energy. Thus the wave numbers refer to the horizon can be written as
\[ k_r = \frac{1}{\Delta} \sqrt{(r^2 + a^2)^2 (E + m\Omega_+ - m\Omega)^2 + 2\Delta s(s - 1) - \Delta^2}, \quad (44) \]

where \( \Omega = \frac{a}{r^2 + a^2} \). The free energy at temperature \( T_+ \) of the boson system is given by

\[ \beta_+ f = - \sum_E \ln \left( 1 \pm e^{-\beta_+ E} \right), \quad (45) \]

where + corresponds to fermion field and – corresponds to boson field.

According to semi-classical quantum theory, there is

\[ \sum_E \rightarrow \int_0^\infty dE g(E), \]

where \( g(E) = \omega \frac{d^dE}{dE} \) is the states density. \( \omega \) is the degeneracy of the fields (for scalar field and neutrino field, \( \omega = 1 \); for Maxwell field, \( \omega = 2 \)). The states number is

\[ \Gamma(E) = \sum_{m,\lambda} n_r(E, \lambda, m) = \int dm \int d\lambda \frac{1}{\pi} \int k_r(E, \lambda) dr. \quad (46) \]

The free energy can be calculated as follows

\[ -\beta_+ f = \pm \int_0^\infty dE g(E) \ln \left( 1 \pm e^{-\beta_+ E} \right), \]

\[ = \pm \beta_+ \int_0^\infty dE \omega' \frac{\Gamma(E)}{e^{\beta_+ E} \pm 1} \]

\[ = \frac{\beta_+ \omega'}{\pi} \int_0^{r_+ + 2\varepsilon} dE \int_{r_+ + \varepsilon}^{\lambda_{max}} d\lambda \int_{-\lambda}^{\lambda} dm \frac{1}{\Delta} \left( e^{\beta_+ E} \pm 1 \right)^{-1} \]

\[ \sqrt{(r^2 + a^2)^2 (E + m\Omega_+ - m\Omega)^2 + 2\Delta s(s - 1) - \Delta^2}, \quad (47) \]

where the separation constant \( \lambda \) is the angular quantum number which corresponds to \( l \) in the spherical space-time case. In this case, the range of magnetic quantum number \( m \) is \(-\lambda \leq m \leq \lambda\). \( \lambda_{max} \) corresponds to the fact that \( k_r \geq 0 \) (while \( k_r = 0, \lambda \) reach its maximum). Considering fermions field and bosons field, the results can be written as

\[ f_f = -\frac{7}{180} \cdot \frac{\pi^3}{\beta^4} \cdot \frac{(r_+^2 + a^2)^3}{(r_+ - r_-)^2} \cdot \frac{\omega'}{\eta^2}, \quad \text{(fermions field)} \quad (48) \]

\[ f_b = -\frac{2}{45} \cdot \frac{\pi^3}{\beta^4} \cdot \frac{(r_+^2 + a^2)^3}{(r_+ - r_-)^2} \cdot \frac{\omega'}{\eta^2}, \quad \text{(bosons field)} \quad (49) \]

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where $\varepsilon$ is the ultraviolet regulator, which satisfies $0 < \varepsilon \ll r_+$. This manifests that the integral over the quantum number $m$ does not diverge, therefore we need not to regularize the $m$ integral. On the other hand, the membrane model illustrates that the black hole entropy mainly comes from the vicinity of event horizon. Thus we have taken into account the following equation in the integration with respect to $m$,

$$\lim_{r \to r_+} \Omega = \Omega_+.$$  (50)

We also used the median theorem in the integration with respect to $r$, hence $\varepsilon < \eta < 2\varepsilon$. Eq.(50) is also used in the integration with respect to $r$.

We are now ready to obtain the entropy due to arbitrary spin field of the NUT-Kerr-Newman black hole from the standard formula

$$S = \beta^2 \frac{\partial F}{\partial \beta}.$$  (51)

As to fermion field, one componential entropy can be written as

$$S_{1f} = \frac{7}{45} \cdot \frac{\pi^3}{\beta^3} \cdot \frac{(r_+^2 + a^2)^3 \omega'}{(r_+ - r_-)^2} \cdot \frac{\varepsilon}{n^2}.$$  (52)

There are four components of the wave function refer to fermion fields. Therefore the whole black hole entropy is given by

$$S_f = 4S_{1f} = \frac{28}{45} \cdot \frac{\pi^3}{\beta^3} \cdot \frac{(r_+^2 + a^2)^3 \omega'}{(r_+ - r_-)^2} \cdot \frac{\varepsilon}{n^2}.$$  (53)

Similarly, the entropy of boson fields can be obtained as

$$S_b = \frac{8}{45} \cdot \frac{\pi^3}{\beta^3} \cdot \frac{(r_+^2 + a^2)^3 \omega'}{(r_+ - r_-)^2} \cdot \frac{\varepsilon}{n^2}.$$  (54)

Eq.(51) is in same form as that in [18] We choose the cut-off as $\frac{1}{\varepsilon} = 90\beta$. Here $\varepsilon$ and $\eta$ in Eq.(47) and Eq.(48) are of the same order. Therefore $\frac{\varepsilon}{\eta^2} \sim \frac{1}{\varepsilon} = 90\beta$, then the entropy in Eq.(47) and Eq.(48) satisfies the area law

$$S_f = \frac{7}{8} \cdot 4\pi (r_+^2 + a^2) \omega' = \frac{7}{8} A_+ \omega',$$  (55)

$$S_b = \frac{1}{4} \cdot 4\pi (r_+^2 + a^2) \omega' = \frac{1}{4} A_+ \omega',$$  (56)
where $A_+$ is the area of black hole event horizon. The results in Eq.(55) is in agreement with our results in Ref.[29] considering $\Lambda = 0$.

V. Discussion and Conclusion

We have discussed the issue that arises singularities in NUT-Kerr-Newman spacetime and have studied the entropy due to arbitrary spin field in the NUT-Kerr black holes whose Euler’s characteristic is over two. Our results is in agreement with the results in [23]. Since the cut-off was properly chosen, the NUT-Kerr-Newman black hole entropy is identified with the Bekenstein-Hawking area law. As the topology of the NUT-Kerr-Newman black hole is special, and its Euler’s characteristic is greater than two, we can see that the formula $S = \frac{1}{8} \chi A$ is not applicable to our case. Therefore, it means that this formula in [24] has its limitations: the equation does not apply to high dimensional black holes ($\chi (s^{2\kappa+1}) = 0$), even does not apply to four dimensional black holes. The NUT-Kerr-Newman black hole is such an example. Therefore the relations between a black hole topology and its entropy need further investigation. In addition, we can see from the results that the electromagnetic, Dirac and scalar field entropies of the following black holes: Schwarzschild black hole, Reissner-Nordsrtöm black hole, Kerr black hole and NUT-Kerr black hole are embodied as special cases of the NUT-Kerr-Newman black hole entropy.

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