Rigid current Lie algebras

Michel GOZE - Elisabeth REMM

Université de Haute Alsace, F.S.T.
4, rue des Frères Lumière - 68093 MULHOUSE - France

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Abstract

A current Lie algebra is contructed from a tensor product of a Lie algebra and a commutative associative algebra of dimension greater than 2. In this work we are interested in deformations of such algebras and in the problem of rigidity. In particular we prove that a current Lie algebra is rigid if it is isomorphic to a direct product \( g \times g \times \ldots \times g \) where \( g \) is a rigid Lie algebra.

1 Current Lie algebras

If \( g \) is a Lie algebra over a field \( K \) and \( A \) a \( K \)-associative commutative algebra, then \( g \otimes A \), provided with the bracket

\[ [X \otimes a, Y \otimes b] = [X, Y] \otimes ab \]

for every \( X, Y \in g \) and \( a, b \in A \) is a Lie algebra. If \( \text{dim}(A) = 1 \) such an algebra is isomorphic to \( g \). If \( \text{dim}(A) > 1 \) we will say that \( g \otimes A \) with the previous bracket is a current Lie algebra.

In [6] we have shown that if \( \mathcal{P} \) is a quadratic operad, there is an associated quadratic operad, noted \( \tilde{\mathcal{P}} \) such that the tensor product of a \( \mathcal{P} \)-algebra by a \( \tilde{\mathcal{P}} \)-algebra is a \( \mathcal{P} \)-algebra for the natural product. In particular, if the operad \( \mathcal{P} \) is \( \text{Lie} \), then \( \tilde{\text{Lie}} = \text{Lie}^! = \text{Com} \) and a \( \text{Com} \)-algebra is a commutative associative algebra. In this context we find again the notion of current Lie algebra.

In this work we study the deformations of a current Lie algebra and we show that a current Lie algebra is rigid if and only if it is isomorphic to \( g \times g \times \ldots \times g \) where \( g \) is a rigid Lie algebra. The notion of rigidity is related to the second group of the Chevalley cohomology. For the current Lie algebras, this group is not wellknown. Recently some relation between \( H^2(\mathcal{G} \otimes A, g \otimes A) \) and \( H^2(g, g) \) and \( H^2_H(A, A) \) are given in [7] but often when \( g \) is abelian. Let us note also that the scalar cohomology has been studied in [5].
2 Determination of rigid current Lie algebras

2.1 On the rigidity of Lie algebras

Let us remind briefly some properties of the variety of Lie algebras (for more details, see [1]). Let \( g \) be a \( n \)-dimensional \( \mathbb{K} \)-Lie algebra. Since the underlying vector space is isomorphic to \( \mathbb{K}^n \), there exists a one to one correspondence between the set of Lie brackets of \( n \)-dimensional Lie algebras and the skew-symmetric bilinear maps \( \mu : \mathbb{K}^n \times \mathbb{K}^n \to \mathbb{K}^n \) satisfying the Jacobi identity. We denote by \( \mu_g \) this bilinear map corresponding to \( g \). In this framework, we can identify \( g \) with the pair \((\mathbb{K}^n, \mu_g)\). Let us fix definitively a basis \( \{X_1, ..., X_n\} \) of \( \mathbb{K}^n \). The structure constants \( (C^k_{ij})_g \) of \( g \) are given by

\[
\mu_g(X_i, X_j) = \sum_{k=1}^{n} C^k_{ij} X_k
\]

and we can identify \( \mu_g \) with the \( N \)-uple \((C^k_{ij})_g \) with \( N = \frac{n^2(n-1)}{2} \). The Jacobi identity satisfied by \( \mu_g \) is equivalent to the polynomial system :

\[
\sum_{l=1}^{n} C^l_{ij} C^s_{lk} + C^l_{jk} C^s_{li} + C^l_{ki} C^s_{lj} = 0. \tag{1}
\]

Thus a Lie algebra is a point of \( \mathbb{K}^N \) whose coordinates \( (C^k_{ij})_g \) satisfy (1). So the set of \( n \)-dimensional Lie algebras on \( \mathbb{K} \) is identified with the algebraic variety \( L_n \) embedded into \( \mathbb{K}^N \) and defined by the system of polynomial equations (1). We will always denote by \( \mu \) a point of \( L_n \). The algebraic group \( GL(n, \mathbb{K}) \) acts on \( L_n \) by:

\[
(f, \mu) \in GL(n, \mathbb{K}) \times L_n \longrightarrow \mu_f \in L_n \tag{2}
\]

where \( \mu_f \) is given by \( \mu_f(X, Y) = f^{-1}(\mu(f(X), f(Y))) \) for every \( X, Y \in \mathbb{K}^n \). The orbit \( O(\mu) \) of \( \mu \) related to this action corresponds to the Lie algebras isomorphic to \( g = (\mu, \mathbb{K}^n) \). We provide the algebraic variety \( L_n \) with the Zariski topology.

Definition 1 The Lie algebra \( g = (\mu, \mathbb{K}^n) \) is rigid if the orbit \( O(\mu) \) is open in \( L_n \).

Let us suppose that \( \mathbb{K} \) is an algebraically closed field. A way of building rigid Lie algebras rests on the Nijenhuis Richardson Theorem: Let \( H^*(g, g) \) be the Chevalley cohomology of \( g \). If \( H^2(g, g) = 0 \) then \( g \) is rigid. Let us note that the converse is false, numerous examples are described in [1] (in fact a rigid Lie algebra whose cohomology \( H^2(g, g) \) is not trivial is such that the affine schema \( L_n \) given by the Jacobi ideal is not reduced to the point \( \mu \) defining \( g \).)

An intuitive way of defining the notion of rigidity is to consider a rigid algebra as indeterminate, that is any close algebra is isomorphic to it. A general definition of deformations was proposed in [3]. Let \( A \) be a commutative \( \mathbb{K} \)-algebra of valuation such that the residual field \( A/\mathfrak{m} \) is isomorphic to \( \mathbb{K} \) where \( \mathfrak{m} \) is the maximal ideal of \( A \). If \( g \) is a \( \mathbb{K} \)-Lie algebra then the tensor product \( g \otimes A \) is an \( A \)-algebra denoted by \( g_A \).

Definition 2 A deformation of \( g \) is an \( A \)-Lie algebra \( g'_A \) such that the underlying \( A \)-module is \( g_A \) and the brackets \([u, v]_{g'_A}\) and \([u, v]_{g_A}\) of \( g'_A \) and \( g_A \) satisfy

\[
[u, v]_{g'_A} - [u, v]_{g_A} \in g \otimes \mathfrak{m}.
\]
When \( A = \mathbb{C}[t] \) we find the classical notion of deformation given by Gerstenhaber. When \( A \) is the ring of limited elements in a Robinson nonarchimedean extension of \( \mathbb{C} \), we find the notion of perturbations \([4]\). If \( g'_A \) is a deformation of \( g \) then we have

\[
[u, v]_{g'_A} = [u, v]_{g_A} + \sum_{i=1}^{k} \epsilon_1 \epsilon_2 \cdots \epsilon_i \phi_i,
\]

where \( \epsilon_i \in \mathfrak{m} \) and \( \{\phi_1, \ldots, \phi_k\} \) a family of independant skewsymmetric bilinear maps on \( \mathbb{K}^n \). In particular \( \phi_1 \in Z^2(\mathfrak{g}, \mathfrak{g}) \) and if \( g'_A \) is isomorphic to \( g_A \) this map belongs to \( B^2(\mathfrak{g}, \mathfrak{g}) \). We deduce that the deformations of \( g \) are parametrized by \( H^2(\mathfrak{g}, \mathfrak{g}) \). In the following, we are going to determine the current Lie algebras which are rigid.

### 2.2 The manifold \( L_{(p,q)} \)

Let \( g = g_p \otimes A_q \) be a current Lie algebra where \( g_p \) is a \( p \)-dimensional Lie algebra and \( A_q \) a \( q \)-dimensional associative commutative algebra. We suppose that \( \mathbb{K} \) is an algebraically closed field of characteristic 0. Let \( \{X_1, \ldots, X_p\} \) be a basis of \( g_p \) and \( \{e_1, \ldots, e_p\} \) a basis of \( A_q \). If we denote by \( \{C_{ij}^k\} \) and \( \{D_{ab}^c\} \) the structure constants of \( g_p \) and \( A_q \) with regards to these bases, then the Lie bracket \( \mu_g = \mu_{g_p} \otimes \mu_{A_q} \) of \( g \) where \( \mu_{g_p} \) is the multiplication of \( g_p \) and \( \mu_{A_q} \) the multiplication of \( A_q \), satisfy:

\[
\mu_g(X_i \otimes e_a, X_j \otimes e_b) = \sum_{k,c} C_{ij}^k D_{ab}^c X_k \otimes e_c,
\]

and the structure constants of \( g \) with respect to the basis \( \{X_i \otimes e_a\}_{i=1,\ldots,p; a=1,\ldots,q} \) are \( \{C^k_{ij} D^c_{ab}\} \). The Jacobi relations so are written

\[
\sum_{l,r} C^l_{ij} C^s_{lk} D^r_{ab} + C^l_{jk} C^s_{li} D^r_{bc} D^t_{ra} + C^l_{ki} C^s_{lj} D^r_{ca} D^t_{rb} = 0
\]

\( \forall(s, t) \in \{1, \ldots, p\} \times \{1, \ldots, q\} \). These polynomial relations define a structure of algebraic variety denoted by \( L_{(p,q)} \) and embedded in the vector space whose coordinates are the structure constants \( \{C^k_{ij} D^c_{ab}\} \). It is a closed subvariety of \( L_{pq} \). Let \( G(p, q) \) be the algebraic group \( G(p, q) = GL(p) \otimes GL(q) \). This group acts naturally on \( L_{(p,q)} \) by

\[
(f \otimes g)(\mu_{g_p} \otimes \mu_{A_q})(X \otimes a, Y \otimes b) = f^{-1}(\mu_{g_p}(f(X), f(Y))) \otimes g^{-1}(\mu_{A_q}(g(a), g(b))).
\]

We denote by \( O_{p,q}(g_p \otimes A_q) \) the orbit in \( L_{(p,q)} \) of \( \mu_g \) corresponding to this action.

Thus there are two types of deformations:

- The deformations of \( g \) in the manifold \( L_{pq} \). These deformations are parametrized by the second Chevalley cohomology space \( H^2(\mathfrak{g}, \mathfrak{g}) \).

- The deformations of \( g \) in the manifold \( L_{(p,q)} \). They are parametrized by \( H^2_{\mathcal{H}}(\mathfrak{g}_p, \mathfrak{g}_p) \oplus H^2_{\mathcal{H}}(A_q, A_q) \) where \( H^2_{\mathcal{H}}(A_q, A_q) \) is the Harrison cohomology of the associative commutative algebra \( A_q \).

**Definition 3** The Lie algebra \( g_p \otimes A_q \) is rigid in \( L_{(p,q)} \) if the orbit \( O_{p,q}(\mu_g) \) is open (in the Zariski sense). It is rigid if the orbit \( O(\mu_g) \) related to the action of \( GL(pq) \) in \( L_{pq} \) is open.
Proposition 6 A current Lie algebra \( g = g_p \otimes \mathcal{A}_q \) is rigid in \( L_{(p,q)} \) if and only if \( g_p \) is rigid in \( L_p \) and \( \mathcal{A}_q \) is rigid in \( \text{Com}(q) \), the variety of \( q \)-dimensional associative commutative \( \mathbb{K} \)-algebras.

Example. \( p = 2, q = 2 \) \( (\mathbb{K} = \mathbb{C}) \) There is, up to isomorphisms, only one 2-dimensional rigid Lie algebra. It is defined by \([X_1, X_2] = X_2\). There is only one 2-dimensional associative commutative algebra. It is given by \( e_1^2 = e_1, e_2^2 = e_2 \) and corresponds to the semi-simple algebra \( A^2_2 = M_1(\mathbb{K}) \times M_1(\mathbb{K}) \) where \( M_n(\mathbb{K}) \) is the algebra of \( n \)-matrices on \( \mathbb{K} \).

The Lie algebra \( g_2 \otimes A^2_1 \) is rigid in \( L_{(2,2)} \). This algebra is isomorphic to \( g_2 \times g_2 \). It is also rigid in \( L_4 \).

2.3 Structure of rigid current Lie algebras

Recall that if \( g \) is a finite dimensional rigid Lie algebra it admits a decomposition \( g = s \oplus t \oplus n \) where \( t \oplus n \) is the radical of \( g \), \( t \) is a maximal abelian subalgebra whose adjoint operators \( ad \) \( X \), \( X \in t \) are semi-simple and \( n \) is the nilradical. If \( g = g_p \otimes \mathcal{A}_q \) is rigid then \( g_p \) is rigid in \( L_p \). If \( g_p \) is solvable then \( g \) too and we have

\[ g_p = t_p \oplus n_p \quad \text{and} \quad g = t \oplus n. \]

Since \( n_p \otimes \mathcal{A}_q \) is a nilpotent ideal of \( g \), \( n_p \otimes \mathcal{A}_q \subset n \).

Lemma 5 If \( g = g_p \otimes \mathcal{A}_q \) is rigid, then \( \mathcal{A}_q \) has a non zero idempotent.

Proof. If \( \mathcal{A}_q \) is a nilalgebra then \( g \) is nilpotent. In fact if \( X \in g_p \) and \( a \in \mathcal{A}_q \) we have
\[
[ad(X \otimes a)]^m = (ad X)^m \otimes (L_a)^m \quad \text{where} \quad L_a : \mathcal{A}_q \to \mathcal{A}_q \text{ is the left multiplication by } a. 
\]
Since \( \mathcal{A}_q \) is a nilalgebra, every element is nilpotent and there exits \( m_0 \) such that \( (L_a)^{m_0} = 0 \).

Thus \( ad(X \otimes a) \) is a nilpotent operator for any \( X \) and \( a \). This implies that \( g \) is nilpotent. Let \( f \) be a derivation of \( g_p \). Then \( f \otimes Id \) is a derivation of \( g \). Since \( g_p \) is rigid, we can find an inner non trivial derivation \( ad X \) which is diagonal. In this case \( ad X \otimes Id \) is a non trivial diagonal derivation of \( g \). By hypothesis \( g \) is rigid. But any rigid nilpotent Lie algebra is characteristically nilpotent, that is, every derivation is nilpotent. We have a contradiction and \( \mathcal{A}_q \) can not be a nilalgebra. Since it is finite dimensional, it admits a non zero idempotent.

Proposition 6 If \( g = g_p \otimes \mathcal{A}_q \) is rigid then \( \mathcal{A}_q \) is an associative commutative rigid unitary algebra in \( \text{Com}(q) \).

Proof. Let \( e \) be in \( \mathcal{A}_q \) and satisfying \( e^2 = e \). The associated Pierce decomposition

\[ \mathcal{A}_q = \mathcal{A}_q^{00} \oplus \mathcal{A}_q^{10} \oplus \mathcal{A}_q^{01} \oplus \mathcal{A}_q^{11} \]

where

\[ \mathcal{A}_q^{ij} = \{ x \in \mathcal{A}_q \text{ such that } e \cdot x = ix, x \cdot e = jx \} \]

reduce to \( \mathcal{A}_q = \mathcal{A}_q^{11} \oplus \mathcal{A}_q^{00} \) because \( \mathcal{A}_q \) is commutative and we have \( \mathcal{A}_q^{11} \cdot \mathcal{A}_q^{00} = \{0\} \). Thus \( \mathcal{A}_q \) is a direct sum of two commutative algebras. Since \( \mathcal{A}_q \) is rigid, the algebras \( \mathcal{A}_q^{ij} \) and
\( \mathcal{A}_q^{00} \) are also rigid. The subalgebra \( \mathcal{A}_q^{11} \) is unitary (e is the unit element). From the previous lemma \( \mathcal{A}_q^{00} \) has an idempotent and admits a decomposition

\[
\mathcal{A}_q^{00} = \mathcal{A}_q^{0011} \oplus \mathcal{A}_q^{0000}
\]

with \( \mathcal{A}_q^{0011} \neq \{0\} \). By induction we deduce that

\[
\mathcal{A}_q = \mathcal{A}_q^1 \oplus \ldots \oplus \mathcal{A}_q^p
\]

with \( \mathcal{A}_q^i \) with unit \( e_i \) and \( \{e_1, \ldots, e_p\} \) is a system of pairwise orthogonal idempotents. Then \( e_1 + \ldots + e_p \) is a unit of \( \mathcal{A}_q \).

**Theorem 7** Let \( \mathfrak{g}_p \) be a rigid Lie algebra with solvable non nilpotent radical such that \( Z(\mathfrak{g}) = \{0\} \). Then \( \mathfrak{g} = \mathfrak{g}_p \otimes \mathcal{A}_q \) is rigid if and only if \( \mathcal{A}_q = M_1^0 \) is given by

\[
e_i^2 = e_i, \quad i = 1, \ldots, q \quad \text{and} \quad e_i \cdot e_j = 0 \quad \text{if} \quad i \neq j.
\]

**Proof.** Since \( \mathcal{A}_q \) is unitary, the radical of \( \mathfrak{g} \) solvable and non nilpotent. Moreover \( Z(\mathfrak{g}_p) = \{0\} \) implies that \( Z(\mathfrak{g}) = \{0\} \). In fact if \( U = \sum_{j,a} \alpha_{ja}X_j \otimes x_a \) is in the center of \( \mathfrak{g} \), then \( [U, X \otimes 1] = 0 \) for each \( X \in \mathfrak{g}_p \). Thus

\[
\sum \alpha_{ja}[X_j, X] \otimes x_a = 0.
\]

We have \( [\sum_j \alpha_{ja}X_j, X] = 0 \) for each \( a \) and \( X \). So \( \sum_j \alpha_{ja}X_j \in Z(\mathfrak{g}_p) \) for any \( a \). Therefore \( \alpha_{ja} = 0 \) for any \( a \) and \( U = 0 \).

Consequently \( \mathfrak{g} \) is a rigid Lie algebra with trivial center whose radical is non nilpotent. This implies that all derivations are inner. Let \( f \) be a non trivial derivation of \( \mathcal{A}_q \). Since \( \mathcal{A}_q \) is commutative, it is necessarily external. Then \( Id \otimes f \) is a derivation of \( \mathfrak{g} \) and satisfies \( (Id \otimes f)(X \otimes 1) = X \otimes f(1) = 0 \) because \( f(1 \cdot 1) = 2f(1) = f(1) = 0 \). Suppose that \( Id \otimes f \in Int(\mathfrak{g}) \), that is \( Id \otimes f = ad(\sum \alpha_{ij}X_i \otimes x_j) \). Thus \( (Id \otimes f)(X \otimes 1) = \sum \alpha_{ij}[X_i, X] \otimes x_j = 0 \) which implies \( \sum \alpha_{ij}[X_i, X] = 0 \) for any \( j \) and \( X \). So \( \sum \alpha_{ij}X_i \in Z(\mathfrak{g}_p) \) for any \( j \). Since the center is trivial, then \( \sum \alpha_{ij}X_j = 0 \) for any \( j \) and \( Id \otimes f \notin Int(\mathfrak{g}) \). There is a contradiction. Therefore \( \mathcal{A}_q \) is such that any external derivation is trivial. We deduce that \( \mathcal{A}_q = M_1^0 \).

### 3 Cohomology and deformations

a) The Chevalley cohomology of current Lie algebras was computed in [7] for the degrees 1 and 2. It is shown that the algebra of derivation satisfies

\[
\text{Der}(\mathfrak{g} \simeq \mathfrak{g}_p \otimes \mathcal{A}_q) = \text{Der}(\mathfrak{g}_p) \otimes \mathcal{A}_q \oplus \text{Hom}(\mathfrak{g}_p/[\mathfrak{g}_p, \mathfrak{g}_p], Z(\mathfrak{g}_p)) \oplus \frac{\text{End}(\mathcal{A}_q)}{\mathcal{A}_q + \text{Der}\mathcal{A}_q}.
\]

More precisely, let \( f = f_1 \otimes f_2 \) be a derivation of \( \mathfrak{g} \). Denote \( \mu_1 \) the Lie product of \( \mathfrak{g} \) and \( \mu_2 \) the product of \( \mathcal{A}_q \). Then \( f \) is a derivation of \( \mathfrak{g} \) if and only if

\[
\mu_1(f_1(X), Y) \otimes \mu_2(f_2(a), b) + \mu_1(X, f_1(Y)) \otimes \mu_2(f_2(b), a) - f_1(\mu_1(X, Y)) \otimes f_2(\mu_2(a, b)) = 0
\]
for any $X,Y \in g_1$ and $a,b \in A_q$. If $A_q$ is unitary, by considering $a = b = 1$, we obtain

$$[\mu_1(f_1(X),Y) + \mu_1(X,f_1(Y)) - f_1(\mu_1(X,Y))] \otimes f_2(1) = 0$$

and $f_1$ is a derivation of $g_1$ as soon as $f_2(1) \neq 0$. Let us take $a = b$. The above identity reduce to:

$$f_1(\mu_1(X,Y)) \otimes (\mu_2(f_2(a),a) - f_2(a^2)) = 0.$$  

Thus, either $f_1$ satisfies $f_1(\mu_1(g_p,g_p)) = 0$, or $f_2$ satisfies $\mu_2(f_2(a),a) = f_2(a^2)$. This last identity becomes

$$\mu_2(f(a),b) + \mu_2(a,f(b)) = 2f_2(a,b)$$

by linearisation. Concerning the class of rigid Lie algebras that we consider, that is $\{a,b\}$, (see [7] for notations). But the second space was just computed when $g_p$ is abelian.

In the general case, the first space of cohomology is given in [7]:

$$H^1(g,g) \simeq H^1(g_p,g_p) \otimes A_q \oplus Hom(g_p,g_p) \otimes Der(A_q) \oplus Hom(g_p/[g_p,g_p], Z(g_p)) \otimes \frac{Hom(A_q,A_q)}{A_q + Der(A_q)}.$$

In this case, this reduce to

$$H^1(g,g) \simeq H^1(g_p,g_p) \otimes A_q \oplus Hom(g_p,g_p) \otimes Der(A_q).$$

If $g_p$ is rigid with non nilpotent nilradical, any derivation is inner. This implies $H^1(g_p,g_p) = 0$ and $H^1(g,g) = Hom(g_p,g_p) \otimes Der(A_q)$. So $H^1(g,g) = 0 \Leftrightarrow Der(A_q) = 0$. We find again the result.

**Proposition 8** Let $g_p$ be a rigid Lie algebra with a non nilpotent radical and a center reduced to zero. Then $g = g_p \otimes A_q$ is rigid if and only if $Der(A_q) = 0$.

b) A Chevalley 2-cochain $\varphi$ of $g = g_p \otimes A_q$ decomposes as

$$\varphi = \psi_1 \otimes \varphi_2 + \varphi_3 \otimes \psi_4$$

with $\psi_1 \in C^2(g_p,g_p)$, $\varphi_2 \in S^2(g_p,g_p)$ and $\varphi_3 \in S^2(g_p,g_p)$, $\psi_4 \in C^2(A_p,A_p)$, where $C^2(g_p,g_p)$ denotes the space of Chevalley 2-cochains of $g_p$, $S^2(g_p,g_p)$ the space of symmetric bilinear applications with values in $g_p$, $C^2(A_p,A_p)$ the space of 2-cochains of the Harrison cohomology of $A_q$. We deduce using this decomposition that $H^2(g,g) = (H^2)^' \oplus (H^2)^{''}$. The first space is compute in ([7], proposition 3.1). We find

$$(H^2)^' = H^2(g_p,g_p) \otimes A_q \oplus B(g_p,g_p) \otimes \frac{H^2_q(A_q,A_q)}{P_+ (A_q,A_q)} \oplus T(g_p,g_p) \otimes \frac{A(A_q,A_q)}{P_+ (A_q,A_q)}$$

(see [7] for notations). But the second space was just computed when $g_p$ is abelian.
Let $\mu_1 \otimes \mu_2 + \epsilon(\psi_1 \otimes \varphi_2 + \varphi_3 \otimes \psi_4)$ be an infinitesimal deformation of $\mu_1 \otimes \mu_2$. The linear part of the Jacobi identity gives the expression of a 2-cocycle of Chevalley cohomology of $\mu_1 \otimes \mu_2$. We find:

$$\delta_{\mu_1 \otimes \mu_2}(\psi_1 \otimes \varphi_2 + \varphi_3 \otimes \psi_4) = \Sigma \mu_1(\psi_1(X_1, X_2), X_3) \otimes \mu_2(\varphi_2(a_1, a_2), a_3) + \Sigma \mu_1(\varphi_3(X_1, X_2), X_3) \otimes \mu_2(\psi_4(a_1, a_2), a_3) + \Sigma \psi_1(\mu_1(X_1, X_2), X_3) \otimes \varphi_2(\mu_2(a_1, a_2), a_3) + \Sigma \varphi_3(\mu_1(X_1, X_2), X_3) \otimes \psi_4(\mu_2(a_1, a_2), a_3)$$

$$= 0$$

for any $X_1, X_2, X_3 \in \mathfrak{g}_p$ and $a_1, a_2, a_3 \in \mathcal{A}_q$ and the sum is taken on the cyclic permutations of $(1, 2, 3)$. We deduce

**Proposition 9** If $\mathcal{A}_q$ is unitary then $\psi_1 \in Z^2(\mathfrak{g}_p, \mathfrak{g}_p)$ as soon as $\varphi_2(1, 1) \neq 0$.

If $X_1 = X_2 = X_3$, the above identity reduce to:

$$\mu_1(\varphi_3(X, X), X) \otimes \Sigma \mu_2(\psi_4(a_1, a_2), a_3) = 0.$$ 

**Proposition 10** If there exits $X \in \mathfrak{g}_p$ such that $\mu_1(\varphi_3(X, X), X) \neq 0$ then

$$\mu_2 \bullet \psi_4 = 0$$

with

$$\mu_2 \bullet \psi_4(a_1, a_2, a_3) = \Sigma \mu_2(\psi_4(a_1, a_2), a_3).$$

Note that $\psi$ is a 2-cocyle for the Harrison cohomology of $\mu_2$ if $\mu_2 \bullet \psi_4 = \psi_4 \bullet \mu_2$.

Suppose that $\mathfrak{g}$ is rigid solvable with trivial center. Then $\mathcal{A}_q$ is unitay and $\psi_1 \in Z^2(\mathfrak{g}_p, \mathfrak{g}_p)$ as soon as $\varphi_2(1, 1) \neq 0$.

### 4 Application : associative commutative real rigid algebras

#### 4.1 Real rigid Lie algebras

The study of the rigid real Lie algebras was lately initiated in [2]. Let us point out the principal results. An external torus of derivations of $\mathfrak{n}$ is an abelian subalgebra $\mathfrak{t}$ of $\text{Der}(\mathfrak{n})$, the Lie algebra of derivations of $\mathfrak{n}$, such as the elements are semi-simple. This means that complex derivations $f \otimes \text{Id} \in \mathfrak{t} \otimes \mathbb{C}$ are simultaneously diagonalizable. If $\mathfrak{t}$ is a maximal (for the inclusion) external torus of $\mathfrak{n}$ then $\mathfrak{t} \otimes \mathbb{C}$ is a maximal Malcev torus of $\mathfrak{n} \otimes \mathbb{C}$. As all the maximal torus of $\mathfrak{n} \otimes \mathbb{C}$ are conjugated with respect to $\text{Aut}(\mathfrak{n} \otimes \mathbb{C})$, their dimensions are equal. It is the same for the maximal torus $\mathfrak{t}$ of $\mathfrak{n}$. This dimension is called the rank of $\mathfrak{n}$. But contrary to the complex case, all the torus are not conjugated with respect to the group of automorphisms.

**Definition 11** Let $\mathfrak{n}$ be a finite dimensional real nilpotent Lie algebra. We call toroidal index of $\mathfrak{n}$ the number of conjugaison classes of maximal external torus with respect to the group of aurtomorphisms $\text{Aut}_R(\mathfrak{n})$ of $\mathfrak{n}$. 

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Example. The toroidal index of the real abelian Lie algebra $a_n$ of dimension $n$ is equal to $\lfloor n/2 \rfloor + 1$ where $[p]$ is the integer part of the rational number $p$. In fact, let $\{X_1, \ldots, X_n\}$ be a basis of $a_n$. Let us denote by $f_i$ the derivation defined by $f_i(X_j) = \delta_i^j X_j$ and by $f_{1,2p}$ the derivation given by
\[
\begin{align*}
f_{1,2p}(X_{2p-1}) &= X_{2p}, \\
f_{1,2p}(X_{2p}) &= X_{2p-1}.
\end{align*}
\]
Up a conjugation the maximal exterior torus are the subalgebras of $gl(n, \mathbb{R})$ generated by
\[
\begin{align*}
t_1 &= \mathbb{R}\{f_1, \ldots, f_n\} \\
t_2 &= \mathbb{R}\{f_1, f_1 + f_2, f_3, \ldots, f_n\} \\
t_3 &= \mathbb{R}\{f_1, f_1 + f_2, f_1, f_3, f_4, f_5, \ldots, f_n\} \\
&\vdots \\
t_n &= \mathbb{R}\{f_1, f_1 + f_2, f_1, f_3, f_4, \ldots, f_1, f_{n-1}, f_{n-1} + f_n\}
\end{align*}
\]
if $n$ is even, if not the last relation is replaced by
\[
t_n = \mathbb{R}\{f_1, f_1 + f_2, f_1, f_3, f_4, \ldots, f_1, f_{n-2} + f_{n-1}, f_n\}.
\]

4.2 Real rigid associative commutative algebras

Let $t_2$ be the real nonabelian 2-dimensional Lie algebra. There exists a basis $\{X_1, X_2\}$ with regard to which the bracket is given by $[X_1, X_2] = X_2$. Let $A_n$ be a $n$-dimensional real rigid associative commutative algebra. Its complexified is isomorphic to $M(1)^n_2$. Thus the real current Lie algebra $g = t_2 \otimes A_n$ is rigid. We deduce that its complexified is rigid and isomorphic to $t_2^n$. These remarks allow to write the following decomposition:
\[
g = t_2 \otimes A_n = t_n \oplus a_n
\]
where $a_n$ is the $n$-dimensional abelian Lie algebra. We can deduce from this the structure of $A_n$. In fact, if $\{Y_1, \ldots, Y_n\}$ is a basis of $t_n$ corresponding to the derivations $f_1, f_1 + f_2, \ldots, f_1, f_{2s} + f_{2s+1}, \ldots, f_n$ described in the previous section, the the Lie bracket of $g$ satisfies
\[
\begin{align*}
[Y_1, X_1] &= -X_2, \quad [Y_1, X_2] = X_1 \\
[Y_2, X_1] &= X_1, \quad [Y_2, X_2] = X_2 \\
&\quad \vdots \\
[Y_{2s-1}, X_{2s-1}] &= X_2s, \quad [Y_{2s-1}, X_{2s}] = X_2s \\
[Y_{2s}, X_{2s-1}] &= X_2s-1, \quad [Y_{2s}, X_{2s}] = X_2s \\
[Y_i, X_i] &= X_i, \quad i = 2s + 1, \ldots, n.
\end{align*}
\]
Let $\{e_1, \ldots, e_n\}$ be a basis of $A_n$ such that the isomorphism between $t_2 \otimes A_n$ and $t_n \oplus a_n$ is given by $U_1 \otimes e_i = Y_i$ and $X_{2i} = U_2 \otimes e_{2i-1}$, $X_{2i-1} = U_2 \otimes e_{2i}$ for $i = 1, \ldots, s$ and $X_j = U_2 \otimes e_j$ for $j = 2s + 1, \ldots, n$. The rigid associative algebra $A_n$ is thus defined by
\[
\begin{align*}
e_{2i-1}^2 &= e_{2i-1}, & i &= 1, \ldots, s \\
e_{2i-1}e_{2i} &= e_{2i}e_{2i-1} = e_{2i}, & i &= 1, \ldots, s \\
e_{2i}^2 &= -e_{2i-1}, & i &= 1, \ldots, s \\
e_j^2 &= e_j, & j &= 2s + 1, \ldots, n.
\end{align*}
\]
**Proposition 12** Let $A_n$ be a $n$-dimensional real rigid associative algebra. There exists an integer $s$, $1 \leq s \leq n$ and a basis $\{e_1, ..., e_n\}$ of $A_n$ such that the multiplication of $A_n$ is given by

$$
\begin{align*}
    e_{2i-1}^2 &= e_{2i-1}, & i &= 1, \ldots, s \\
    e_{2i-1}e_{2i} &= e_{2i}e_{2i-1} = e_{2i}, & i &= 1, \ldots, s \\
    e_{2i}^2 &= -e_{2i-1}, & i &= 1, \ldots, s \\
    e_j^2 &= e_j, & j &= 2s + 1, \ldots, n.
\end{align*}
$$

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