Critical domain walls in the Ashkin–Teller model

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Abstract. We study the fractal properties of interfaces in the 2d Ashkin–Teller model. The fractal dimension of the symmetric interfaces is calculated along the critical line of the model in the interval between the Ising and the four-states Potts models. Using Schramm’s formula for crossing probabilities we show that such interfaces cannot be related to the simple SLE\textsubscript{κ}, except for the Ising point. The same calculation on non-symmetric interfaces is performed in the four-states Potts model: the fractal dimension is compatible with the result coming from Schramm’s formula, and we expect a simple SLE\textsubscript{κ} in this case.

Keywords: classical Monte Carlo simulations, fractal growth (theory), interfaces in random media (theory)

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1. Introduction

The study of critical systems has been of interest to physicists for at least five decades. In two dimensions many exact methods to study critical systems were invented by physicists and mathematicians, such as the Yang–Baxter equation [1], conformal field theories (CFT) [2], and recently Schramm–Loewner evolution (SLE) [3]. Most of the critical systems in two dimensions can be formulated in terms of fluctuating loops: for example, in the most familiar case, i.e. the Ising model, they simply correspond to domain walls between regions of opposite magnetization. The SLE is based on direct investigation of these critical loops and is based on probability techniques (for a review, see [4]). According to SLE all of the conformally invariant curves in two dimensions can be parametrized by the SLE drift $\kappa$, related to the fractal dimension of the curves via $D = 1 + \kappa/8$.

One of the most interesting statistical systems to study from the SLE point of view is the Ashkin–Teller (AT) model [5]. This model is interesting at least from two points of view: first, it has a rich phase diagram including a critical line, and, secondly, it has an interesting physical realization as selenium adsorbed on the Ni(100) surface [6]. Following the above motivations we investigate the critical loops in this model on the critical line.

The definition of the Ashkin–Teller model on an arbitrary graph is as follows: on each vertex $i$ of the graph lives a field $s_i$ with values $s_i = e^{-i(\pi/2)q_i}$, where $q_i = 0–3$. Then the partition function of the model is $Z = \sum \prod W(s_i, s_j)$, where

$$W(s_i, s_j) = (1 + x_1 s_i^* s_j + x_2 s_i^{*2} s_j^2 + x_1 s_i^{*3} s_j^3),$$

with the product over pairs of adjacent sites. The above partition function is apparently $\mathbb{Z}_4$ symmetric and for $x_1 = x_2$ reduces to the four-states Potts model. The model was solved exactly on the square lattice by mapping it to the six-vertex model and some of the critical exponents were found by mapping the model to the solid-on-solid (SOS) model ([7]–[9]; see also [10] for the phase diagram of the anisotropic case). The self-dual Ashkin–Teller model on the square lattice is described by the line $x_2 + 2x_1 = 1$; moreover, it is exactly solvable all along this line. Some special points along this line are in the universality class of well-known models: the point $x_1 = x_2 = \frac{1}{2}$ is in the universality class of the four-states Potts model. The model is critical for all points on the self-dual line with
$x_1 \geq \frac{1}{3}$. At the point $x_1 = \frac{1}{3} \sin(\pi/16)/\sin(3\pi/16)$, called the Fateev–Zamolodchikov (FZ) point, the model is fully integrable and can be described by $\mathbb{Z}_4$ parafermionic CFT [11]. By an easy change of variables, see (6), one can show that the above partition function corresponds to the Hamiltonian of two coupled Ising models which decouple for $x_2 = x_1^2$, so for $x_1 = \sqrt{2} - 1$ we have the Ising universality class. Since the critical properties of the fluctuating curves at the above special points on the critical line are known, one can use them to extrapolate the results to other points.

From the CFT point of view it is well known that the field theory describing the Ashkin–Teller model at the critical line is the $c = 1$ orbifold conformal field theory [12]–[14]. This CFT comes from compactifying the free bosonic field theory to a circle with radius $r$ and then requiring a $\mathbb{Z}_2$ symmetry for the bosonic fields (for a review of this conformal field theory see [15, 16]). To fix the notation take the free field theory with action $S = (1/4\pi) \int \partial \phi \partial \phi$ and then compactify the bosonic field on a circle of radius $r$.

The 'electromagnetic' conformal spectrum of the model is

$$X_{em} = \frac{e^2}{2r^2} + \frac{r^2 m^2}{2},$$

where $e$ and $m$ are the electric and magnetic charges respectively. The action is invariant under $\phi \rightarrow -\phi$. To get an orbifold CFT we should project all the operator content of the circle theory to the operators which respect this symmetry; however, this theory cannot be modular invariant without introducing some twisted operators which come from the discrete $\mathbb{Z}_2$ symmetry of the model. In the case of the $\mathbb{Z}_2$ orbifold CFT we have four operators. Two of them have conformal weights $\frac{1}{16}$, and we will call them $\sigma$, which is reminiscent of the spin operator in the Ising model. The other two have conformal weights $\frac{9}{16}$ and are called $\tau$. This theory is conformally invariant for all real values of $r$ and by changing the radius we can change continuously the critical exponents of the model except the twist operators which are invariant under a change of orbifold radius.

On the square lattice we have the following equality between the radius and the coupling of the Ashkin–Teller model on the critical line [12, 15]:

$$\sin \left( \frac{\pi r^2}{8} \right) = \frac{1}{2} \left( \frac{1}{x_1} - 1 \right)$$

(an analogous relation for practical calculations on the triangular lattice is given in the last section). A lattice-invariant way to write the above equation, which is also useful in numerical calculations, is

$$\frac{1}{\nu} = 2 - \frac{2}{r^2},$$

where $\nu$ is the thermal exponent. Some well-known points are the following: $r = 2$ is the four-states Potts model, $r = \sqrt{3}$ describes $\mathbb{Z}_4$ parafermionic CFT (see [11]), $r = \sqrt{2}$ is in the Ising universality class—indeed it is a pair of decoupled Ising models—and finally $r = 1$ is in the universality class of the $XY$ model.

There are different possibilities to define critical curves in the Ashkin–Teller model [17]–[21]. In the spin representation one can think about the domain walls between one spin and the other three, or the domain walls between two definite spins and the other two. It is obvious that at the Ising point the latter ones, if properly chosen, give the domain walls of the Ising model, so we will mainly focus on a 'symmetric' choice.
recovering standard Ising interfaces on one of the two underlying \( \mathbb{Z}_2 \) systems, except at the Potts point where we consider also the former choice. We should stress here that there is no direct connection between these interfaces and the domain walls of the orbifold Gaussian free field theory. The contour lines of Gaussian free field theory are related to a different kind of interfaces discussed in [21]. As was recently proven in [22], the SLE drift of the ‘symmetric’ choice at the Ising point is \( \kappa = 3 \). At the FZ point there is a prediction by Santachiara [23] stating that the critical curves are related to \( \kappa = \frac{10}{3} \). Finally, in the four-states Potts model the common belief is that the SLE drift is \( \kappa = 4 \). Except for the Ising case (for which there is now a mathematical proof) there is no definite argument showing the specific connections of the above predictions to particular interfaces. In other words we do not know exactly to which specific interfaces the above predictions, coming from scaling-limit arguments, should be associated. In this paper we examine the above predictions indirectly and we also systematically study the ‘symmetric’ choice of the domain walls. Our method is based on calculating the fractal dimension of the introduced interfaces and comparing the results with those coming from checking Schramm’s formula for the crossing probability. The most direct way to see the conformal invariance of the interfaces is to use the inverse Loewner equation, find the corresponding drift and compare it with the Brownian motion for a normal SLE\( \kappa \) and with more complicated stochastic processes for the SLE\( \kappa, \rho, \rho \). However, it is difficult to get results with high accuracy by working directly with the SLE equation: a numerically more accessible way is to rule out the possibility of having SLE\( \kappa \) by just calculating the crossing probability and checking it against Schramm’s formula.

The structure of the paper is as follows: in section 2 we fix the notations and give the outline of the numerical procedure; in section 3 we calculate the fractal dimension of the interfaces in the region between the Ising and the four-states Potts models; section 4 is a check of Schramm’s formula for the introduced interfaces; finally, in section 5 we summarize our findings and also make some comments about possible exact formulae for describing them.

### 2. Definitions and procedure

The numerical part of this work is conveniently expressed in the coupling space \( \beta, \alpha \), where

\[
Z = \sum_{\{\sigma, \tau\}} \prod_{\langle ij \rangle} e^{S_{ij}},
\]

\[
S_{ij} = \beta(\sigma_i \sigma_j + \tau_i \tau_j) + \alpha(\sigma_i \sigma_j \tau_i \tau_j).
\]

Here, \( \sigma \) and \( \tau \) are two Ising variables; the correspondence is \( s_i = e^{-i(\pi/4)} / \sqrt{2}(\sigma_i + i\tau_i) \), so that

\[
x_1 = \frac{e^{2\beta} - e^{-2\beta}}{e^{2\beta} + e^{-2\beta} + 2e^{-2\alpha}}, \quad x_2 = \frac{e^{2\beta} + e^{-2\beta} - 2e^{-2\alpha}}{e^{2\beta} + e^{-2\beta} + 2e^{-2\alpha}}.
\]

In this language the Ising model has \( \alpha = 0 \), and the Potts line is \( \alpha = \beta \); the numerical work was carried on in the triangular lattice because in this case an interface can be defined in a unique and natural way (the dual lattice possesses only three-link joints); in
this geometry, the critical line is described by \[24\]
\[e^{4\alpha}(e^{4\beta} - 1) = 2. \tag{8}\]

Two kinds of interfaces are considered in the following, called 12|34 and 1|234. Here, the numbers denote the four possible spin states \(q_i\), and the interface is realized by imposing that the spins on either half of the border can assume only some of them. In the two-Ising notation above, one can identify, modulo \(\mathbb{Z}_4\) transformations, \(1 = (\sigma = +, \tau = +), 2 = (+, -), 3 = (-, -)\) and \(4 = (-, +)\).

Most of this work deals with the 12|34 interface, which amounts to imposing the boundary condition \(\sigma = +1\) on one half of the system’s border and \(\sigma = -1\) on the other, while leaving \(\tau\) free. This allows for a fast cluster-based update strategy, which is an adaptation of the Swendsen–Wang prescription applied alternatively to the Fortuin–Kasteleyn clusters of the \(\sigma\) and \(\tau\) sublattices, thus helping to prevent the critical slowing down \[25\] (in practice, the fixed boundaries are represented by an additional layer of \(\sigma\) spins which participate in forming the clusters but are not allowed to be flipped). For the 1|234 interface, instead, we used a local Metropolis accept/reject algorithm: the interface was induced by restricting the pool of possible spins on the system boundary to \(\{1\}\) and \(\{2, 3, 4\}\) on the two halves. In this case the simulations were performed on graphics processing units, so as to exploit their huge parallelization capability, with the CUDA programming libraries.

The strategy to identify the fractal dimension \(D\) associated to a particular point \((\beta, \alpha)\) along the critical line of the AT model went as follows: for a variety of system sizes \(L \times L\) the associated interface length \(S(L)\) was measured over a large number of configurations; this function is expected to have the leading-order behaviour
\[S(L) = f_L(L) = aL^D. \tag{9}\]

Given a spin configuration, the associated interface length \(S\) is obtained by recolouring isolated sign-clusters to leave only two regions, and then counting the links connecting spins of opposite sign (much in the same way as described in [17], with the difference that instead of an actual recolouring we build the simply connected tree of neighbouring sign-clusters in order to track, among all opposite-colour interfaces, the desired one).

Once the interface has been obtained, and all sites have been identified as lying on the left or on the right of it, collecting data for the crossing probability is straightforward. What we measure is \(F(x)\), where \(0 \leq x \leq 1\) is a coordinate along a segment perpendicular to the interface and located halfway between the two opposite sides of the system, and \(F\) is the probability that the point at \(x\) lies above the interface. The theoretical Schramm prediction for this function—provided the interface is described by a SLE\(_\kappa\)—is formulated on the upper half-plane, with the interface connecting points 0 and \(+i\infty\); the original rectangular system is an approximation of the infinite strip with width 1 in the complex plane, where the point on the segment has complex coordinates \(z(0, x)\), which are subsequently mapped in a conformal way to the half-plane via \(w = e^{\pi z}\) (figure 1). In order to have a good approximation of the whole half-plane, then, it is necessary to work with elongated systems, that is, we examined aspect ratios \(\ell = L_y/L_x\) from 1 to 5. Plugging those transformation into the Schramm formula \[26\],
\[P_\kappa(w) = \frac{1}{2} + \frac{\Gamma(4/\kappa)t}{\sqrt{\pi\Gamma((8 - \kappa)/2\kappa)^2}}F_1\left(1, 4; 3; -t^2\right), \tag{10}\]
where \( _2F_1 \) is the hypergeometric function and \( t = \text{Re}(w)/\text{Im}(w) \), we get the following formula for practical applications:

\[
P_\kappa(x) = \frac{1}{2} - \frac{\Gamma(4/\kappa) \cos(\pi x)}{\sqrt{\pi} \Gamma((8 - \kappa)/2\kappa)} _2F_1 \left( \frac{1}{2}, \frac{3\kappa - 8}{2\kappa}; \frac{3}{2}; \cos^2(\pi x) \right).
\]  

(11)

3. Fractal dimension from the interface length

The subleading corrections to (9) are still not completely clear, and their direct numerical investigation is very hard, so we tried several possible functional forms: if the resulting values of \( D \) appear to agree to some extent, one can use this pool of results to assess the systematic errors involved in determining the fractal dimension. Beyond the leading-order formula quoted above, we considered a power-law correction

\[
f_{2p}(L) = a_1 L^D + a_2 L^w, \quad w < D
\]

(12)

(with \( w \) either left free or fixed to one), and a logarithmic subleading term

\[
f_{ln}(L) = a_1 L^D + c \log(L/L_0).
\]

(13)

In practice, we fit the measured \( S(L) \) to these functions in a range \( L_{\text{min}} \leq L \leq L_{\text{max}} \) by varying \( L_{\text{min}} \) looking for plateaux of stable parameters and acceptable \( \chi^2/\text{ndf} \). We also tried, at the Potts point, other functional forms inspired by the RG arguments in [27], but they had to be discarded in favour of the above.

For the 12|34 interface, we examined six points on the triangular-lattice critical line: the three exactly known Potts, FZ and Ising universality classes (4P, FZ and I respectively), plus three others points, labelled B, C and D. We also collected data for the 1|234 interface at the Potts point. In table 1, we characterize all points by reporting the correlation exponent \( \nu \), which is known as a function of the couplings [28].

For each point (and each choice of interface), we measured \( S(L) \) for various tens of values of system size up to \( L \sim 1000–2000 \). The statistics employed was approximately of half a million configurations for each \( L \) and each point in phase space.

We noticed that the introduction of the secondary term in the behaviour of \( S(L) \) makes the resulting fractal dimension generally higher than the value \( D^{(1)} \) coming from
Table 1. Couplings and correlation exponent for the six points that we considered on the critical line.

| Point name | $\beta$ | $\alpha$ | $\nu$ |
|------------|---------|---------|------|
| 4P log(2)  | $\frac{1}{4}$ | log(2)  | 2    |
| D         | 0.174007 | 0.1718484 | 0.678870 |
| FZ        | $\frac{1}{4} \log \left(1 + \frac{2}{\sqrt{3}}\right)$ | $\log 3$  | 3    |
| B         | 0.216942 | 0.0924748 | 0.823489 |
| C         | 0.242110 | 0.0505546 | 0.897473 |
| I         | $\log 3$ | 0       | 1    |

Table 2. Fractal dimensions obtained for each point from subleading-order and leading-order fits, as described in the text. Information on the data sets is also provided.

| Point name | $L$ values | $L$ maximum | $D$       | $D^{(1)}$ |
|------------|------------|-------------|-----------|-----------|
| 4P | 25 | 768 | 1.4330(40) | 1.4325(25) |
| 4P | 36 | 1400 | 1.4805(10) | 1.4745(15) |
| D  | 41 | 1000 | 1.4900(100) | 1.4720(20) |
| FZ | 115 | 2400 | 1.4320(50) | 1.4215(15) |
| B  | 44 | 2200 | 1.3950(100) | 1.3915(20) |
| C  | 38 | 1600 | 1.3870(90) | 1.3745(15) |
| I  | 79 | 2400 | 1.3751(24) | 1.3714(8) |

We have observed that, within the accuracy permitted by the data (interface lengths are not self-averaging, thus making it very difficult to get very small errors in large systems), the exponent $w$ for the power-law secondary term is never far from one, which motivated the attempt to fit with $w$ fixed to one.

A possible cause of bias in the fractal dimension could come from the influence of the fixed borders. That is, on systems with aspect ratio of one and forced boundaries, the value of $S(L)$ is still influenced in a noticeable way by the particular way in which the boundary conditions are imposed. To quantify this effect, we have carried out two sets of simulations with the same statistics and system sizes at the Ising point: the first set (‘jagged’) had the layer of fixed spins on the boundaries arranged in a way compliant with the regular geometry of the triangular-lattice system (as in all the other simulations we performed), while in the other (‘non-jagged’) every bordering site had links to exactly two fixed spins regardless of the underlying geometry (figure 3). The resulting fractal dimensions (obtained with a leading-order fit) differ by about two standard deviations, signalling that results from square systems are noticeably influenced by the borders.
Figure 2. Values of $D$ and $D^{(1)}$ from the fits described in the text to the $S(L)$ data for the Ising point. Error bars for the subleading-term fits are omitted for clarity, however they never exceeded 0.008. The leading-order fit gives the exact known answer only for $L_{\text{min}}$ beyond 1000, while the other functional forms indicate a rather mutually consistent result somewhat earlier.

Figure 3. In the ‘jagged’ way of imposing boundary conditions (left), some active boundary spins (blue) connect to one fixed site (red) and some others to three. In the ‘non-jagged’ scheme (right), every bordering active site touches exactly two fixed spins.

4. Crossing probability and SLE$_\kappa$

The analysis of the crossing probability was carried on at the same points where we studied $S(L)$, including the 1|234 case for the Potts point. We considered systems with aspect ratio ranging from $\ell = L_y/L_x = 1$ to 5, and transverse side up to $L_x = 300$; we collected data from about $1-6 \times 10^5$ configurations.

The basic idea, not dissimilar from the approach found, e.g., in [17, 29], is to identify the optimal $\kappa$, that is, the value for which (10) best describes the measured data, by minimizing the sum of the squared deviations. We added, however, key features to this approach, in an attempt to keep systematic errors—coming from the fact that the system
Critical domain walls in the Ashkin–Teller model

Figure 4. Best $\kappa$ as a function of the cutoff for the $12|34$ crossing probability at the Ising point for a variety of systems of size $L_x \times L_y$. The data start to stay constant for aspect ratios of at least three.

is discrete, finite and with limited aspect ratio—under control: first, we exclude from our analysis a variable halo around the endpoints, by performing the minimization in the domain $\epsilon < x < 1 - \epsilon$; this gives the optimum as a function of the cutoff, $\kappa(\epsilon)$; secondly, and most important, we allow for some ‘shrinking’ of the curve: this is motivated by the fact that at the endpoints of the $x$ range the fixed boundaries squeeze, to an unknown extent, the area where the interface can actually live undisturbed. The analysis is then carried on after the transformation

$$x \rightarrow x' = \frac{1}{2} + r(x - \frac{1}{2}),$$

and the shrinking factor $r$ is minimized together with the drift $\kappa$ (both found as functions of $\epsilon$; finally, we look for stable plateaux in $\epsilon$, at cutoffs as small as possible). On general grounds, we expect the optimal solution $(\kappa, r)$ to have $r \lesssim 1$. We test this analysis at the Ising point, where it is known that the $12|34$ interface is described by a SLE$_\kappa$ with $\kappa = 3$, in accordance with our $S(L)$ investigation. The result, for sufficiently elongated systems, is $\kappa = 3.002(3)$ and $r = 0.98$ (figure 4), to be compared with the outcome of the basic analysis on the same data, $\kappa = 3.08$ (the same value found in [17]).

We then proceeded to apply this technique to the other points: the shrinking factor was around 0.96–0.98 for all $12|34$ interfaces, while the $1|234$ at the Potts point yielded the puzzling result of $r \sim 1.025$; anyway, we identified a stable value of $\kappa$ for all points considered. In all cases, we start finding plateaux and mutually consistent values only at aspect ratios of $\ell = 3$ or more. The values of $\kappa$ can be compared with the relation between drift and interface fractal dimension from SLE, $D = 1 + \kappa/8$, and with the theoretical prediction at the exactly known points, see table 3 and figure 5.

The $12|34$ interface gives a drift compatible with the fractal dimension only at the Ising point, suggesting that as one moves away from Ising the interface is no longer described by a simple SLE$_\kappa$; moreover, at the Potts point we observe compatibility for the $1|234$ interface, but not in agreement with the theoretical expectation $\kappa = 4$. 

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5. Discussion and conclusions

We studied the particular interfaces 12|34 in the Ashkin–Teller model. The fractal dimensions were calculated all along the critical line in the interval between the Ising model and the four-states Potts model. We also checked Schramm’s formula to verify whether our interfaces are SLE_κ or not. Our numerical calculation shows that the interface 12|34 is not SLE_κ except at the Ising point. This may be expected from lattice arguments: our interfaces do not exhibit the domain Markov property, at the lattice level, except at the Ising point; consequently, the property does not hold at the continuum level as well. However, this argument is not always true: it is possible to have no domain Markov property at the lattice level—as seems to be the case for the disordered systems—but to recover it in the continuum limit [30]. Our numerical calculation shows that
most probably this phenomenon does not occur in our system. That means even if the interfaces are conformally invariant we cannot expect a match between the results coming from the fractal dimension calculations and Schramm’s formula. Although our numerical calculation shows that the interfaces are not SLE$_\kappa$ they can still be conformally invariant and related to some generalizations of SLE$_\kappa$, such as SLE$_{\kappa,p,p}$. One can use the above argument also for the 1/234 interfaces with the difference that, in this case, we have the domain Markov property just at the four-states Potts point. Our numerical calculation shows that the results coming from calculating the fractal dimension and Schramm’s formula agree; however, the result is not compatible with SLE$_4$. The mismatch can be related to the expected very slow convergence to the infinite-volume regime, with the appearance of logarithmic corrections (see, for example, [31]–[33]), however our results do not show a significant trend towards $\kappa = 4$ as the system is enlarged. At the FZ point, our calculation shows that, although the numerical outcome for the fractal dimension is almost compatible with the prediction in [23], there is still a discrepancy that did not seem to vanish even when simulating to very huge system sizes.

Accepting the conjectures at the Potts point $D = \frac{7}{2}$ and the FZ point $D = \frac{17}{12}$, and considering the exact result at the Ising point $D = \frac{11}{8}$, one can put forward an Ansatz about the fractal dimension of the 1234 interfaces as follows: since the critical exponents in the Ashkin–Teller model are related to the compactification radius of the orbifold with a rational function, it is plausible to argue that the relation between $D$ and the compactification radius is rational as well. Using the three known points one can derive the following equation:

$$D = \frac{7}{8} + \frac{r^2}{8} + \frac{1}{2r^2},$$

(15)

where $r$ is the compactification radius of the associated bosonic theory. The number $\frac{7}{8}$ appearing there could be a signal of the important role of the operator $\tau$ in calculating the fractal dimension of our interfaces. By combining (4) with the relations, valid for the Ashkin–Teller model [28],

$$\nu(\lambda) = \frac{2\pi - 2\lambda}{3\pi - 4\lambda}, \quad \lambda(\beta) = 3\arcsin\frac{\sqrt{3} - 1/\tanh 2\beta}{2}$$

(16)

(the second one is specialized to the triangular lattice), we find the following expression, useful for checking the above Ansatz:

$$r^2(\beta) = 4 - \frac{12}{\pi} \arcsin\left(\frac{\sqrt{3} - 1/\tanh(2\beta)}{2}\right).$$

(17)

In figure 5 we have compared this prediction with the results coming from the numerical calculation. The formula roughly describes the numerical data, but with no perfect compatibility. An important feature of this formula is the prediction of a minimum for $D$ at the Ising point, that was recently confirmed numerically in [20]. In addition, the above Ansatz predicts the correct value $D = \frac{3}{2}$ for the XY point. Interestingly, one can map the model at the XY point into the $O(n = 2)$ model on the honeycomb lattice. The argument goes as follows. At the XY point of the triangular lattice, the weight associated to neighbouring sites with different spins, i.e. $W(\tau_i \neq \tau_j, \sigma_i \neq \sigma_j)$, is zero. This can happen just at the XY point of the triangular Ashkin–Teller model because there we
have $1 - 2x_1 + x_2 = 0$. Renormalizing the weight of the neighbouring interactions with like spins to one, i.e. $W(\tau_i = \tau_j, \sigma_i = \sigma_j) = 1$, gives $W(\tau_i \neq \tau_j, \sigma_i = \sigma_j) = W(\tau_i = \tau_j, \sigma_i \neq \sigma_j) = 1/\sqrt{2}$. If we draw a line on the honeycomb-lattice link dual to the $(i, j)$ link of the original lattice for all pairs of unlike spins $\sigma_i \neq \sigma_j$, we will get a loop model with the following partition function:

$$Z = \sum_C \left( \frac{1}{\sqrt{2}} \right)^b 2^d,$$  \hspace{1cm} (18)

where $b$ is the number of bonds, $d$ is the number of loops and the sum is over all loop configurations $C$. The above formula is the partition function of the $O(n = 2)$ model at the critical point [9]. It was conjectured by many authors, see for example [4], that these interfaces are described by SLE$_4$. This result is even stronger than the prediction of our formula because it also claims that the interfaces are conformal and related to the simple SLE$_4$. The very interesting property of the $XY$ point is that most of the interfaces that can be defined [20], including $1|234$, exhibit the domain Markov property; thus we conjecture that they are all SLE$_4$.

Another very interesting phenomenon is that for the Ashkin–Teller model we expect different continuum limits for the same boundary conditions on different underlying lattices. The simple way to see this is to consider the end point of the physical portion of the critical line on the square lattice, which is $(x_1, x_2) = (1/2, 0)$ with $\nu = 2$. Here we expect the same behaviour as the $XY$ point of the triangular lattice, i.e. the weight associated to neighbouring sites with different spins is zero and so we expect to have the domain Markov property. This means that if the interfaces were conformal we would expect simple SLE$_\kappa$. This is not necessarily true on the triangular lattice with $\nu = 2$ because at the lattice level we do not have the domain Markov property. The conclusion is that the same boundary conditions on the different lattices for the statistical models with some internal symmetries could have different continuum limits.

Finally, we would like to make some observations on the numerical pitfalls in identifying fractal dimensions correctly. The triangular lattice has the advantage of allowing for an unambiguous interface definition, i.e. with no need to define a ‘tie-break’ rule to deal with four-line junctions as is the case for the square lattice; on the other hand, it seems that this geometry somewhat enhances the finite-size effects, requiring the simulation of very large systems to achieve a stable value for $D$ (as opposed, for instance, to the rather stable results from moderate sizes for the square lattice [20]). Moreover, even though we could not apply the predictions in [27] for the subleading terms in $S(L)$ (because they are formulated in a different setting from ours), there are some indications that fitting numerical data simply to the leading-order behaviour does not give reliable fractal dimensions. Another potential concern is our choice—widely common in the literature—of working with square aspect ratios $\ell = 1$: as shown by the test of jagged versus non-jagged boundary conditions, a proper determination of $D$ should reasonably be conducted on systems fulfilling to some extent the requirement $L_x \gg L_y$.

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