Dynamical age of the universe as a constraint on the parametrization of dark energy equation of state

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Abstract

The dynamical age of the universe depends upon the rate of the expansion of the universe, which explicitly involves the dark energy equation of state parameter $w(z)$. Consequently, the evolution of $w(z)$ has a direct imprint on the age of the universe. We have shown that the dynamical age of the universe as derived from CMB data can be used as an authentic criterion, being independent of the priors like the present value of the Hubble constant $H_0$ and the cosmological density parameter $\Omega_M^0$, to constrain the range of admissible values of $w$ for quiescence models and to test the physically viable parametrizations of the equation of state $w(z)$ in kinessence models. An upper bound on variation of dark energy density is derived and a relation between cosmological density parameters and the transition redshift is established.

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I. INTRODUCTION

The dark energy is now well established as a dominant constituent of the present day universe. Its existence is directly inferred from the acceleration in cosmic expansion as indicated by SNIa observation \([1, 2]\) and indirectly from CMB anisotropy measurements \([3]\) and Integrated Sachs-Wolfe effect \([4]\). But, the nature of dark energy still remains enigmatic whether it arises from the cosmological constant, quintessence or phantom fields with constant equation of state parameter or varying \(w(z)\). The most commonly used method to explore the dark energy models in the literature is to assume an adhoc equation of state \(w(z) \equiv p_X/\rho_X\) for dark energy and parametrize \(w(z)\) or dimensionless dark energy function \(f(z) \equiv \rho_X(z)/\rho^0_X\) by one, two or more free parameters and then constrain these parameters by fitting to the observed SNIa, CMB and LSS data. Using this method Wang and Tegmark \([5]\) have discussed four viable parametrizations and put tight constraints over \(f(z)\) in a rather model independent way. We have adopted a different approach in this paper to constrain \(w(z)\) using dynamical age of the universe as the criterion to test the physical viability of different parametrizations. First, we argue that the age of the universe known observationally from various independent methods is found to be reasonably consistent and it can be used as an effective constraint on the evolution of the dark energy equation of state \(w(z)\) because the dynamical age of the universe depends on the Hubble expansion rate, which essentially involves \(w(z)\). The possibility of using the age of the universe as a constraint has been discussed in recent papers \([3,6-12]\). The oldest globular clusters yield ages of about 12.5 Gyr with an uncertainty of 1.5 Gyr \([13]\). Assuming the genesis of dark energy from the cosmological constant \((w = -1)\), the SNIa data yield the product of age and Hubble constant \(H_0t_0 = 0.96 \pm 0.04\) \([14]\). Taking \(H_0 = 72\) Mpc\(^{-1}\) km s\(^{-1}\), it gives \(t_0 = 13.04 \pm 0.5\) Gyr. Padmanabhan \([15]\) has given the maximum likelihood for \(H_0t_0 = 0.94\) based on the analysis of SNIa observations. This can be consistent with \(\Omega = 1\) models only if \(\Omega^0_M \simeq 0.3\) and \(\Omega^0_X \simeq 0.7\). It is noteworthy that the SNe observations constrain the combination \(H_0t_0\) better than the individual parameters. With the implicit assumption of the cosmological constant \((w = -1)\), the WMAP data \([16]\) yields \(t_0 = 13.7 \pm 0.2\) Gyr. Relaxation of this constraint would lead to variable estimates of \(t_0\) depending upon the prior choice of \(H_0, \Omega^0_M\) and \(\Omega^0_X\). On the other hand, Knox et al \([17]\) have given a method for precise determination of the dynamical age of the universe from CMB anisotropy measurements. The best estimate of
the dynamical age of the universe, coming from CMB data, is $t_0 = 14.0 \pm 0.5$ Gyr. Although this method also presumes the cosmological constant, it is shown that the variation of the equation of state away from $w = -1$ at fixed $\theta_s$ (the angle subtended by the acoustic horizon on the last scattering surface) has very little effect on the age of the universe. Moreover, the CMB method of age determination does not involve the observational parameters $H_0$ and $\Omega^0_M$ and is therefore free from observational uncertainties involved in the measurement of these parameters. These diverse observations leading to mutually agreeable estimates of the age of the universe reinforce our confidence in using the dynamical age of the universe as an effective tool to constrain the equation of state and the dark energy density parameter $\Omega_X$. We would take $t_0 = 14.0 \pm 0.5$ Gyr (based on CMB anisotropy observations [17]) as the most reliable estimate of the present age of the universe since it is not sensitive to variation in the values of the Hubble constant $H_0$ and the fractional energy density $\Omega^0_M$ of the non-relativistic matter in the universe. We would use it as the standard criterion for the dynamical age of the universe for comparison with a variety of kinesence models [18] whose equation of state $w(z)$ is assumed to have different parametrizations as discussed in Sec. IV of the paper.

Although, the present dark energy density $\rho^0_X = (4.8 \pm 1.2) \times 10^{-30}$ gm/cm$^3$ is precisely known [5] and the cosmological density parameter $\Omega^0_X = 0.73$ from WMAP data [16], we have no definite estimate of $\rho_X(z)$ and $\Omega_X(z)$ at small $z$ or large $z$ in the past. As such, we do not know whether the dark energy evolves with cosmic time or not and if it does then what is the mode of its evolution. However, it is possible to find $\Omega_X(z)$ precisely in terms of $w(z)$ at the point of transition $z_T$ from the decelerating phase to accelerating expansion phase by taking $q = 0$ in Eq. (8). We have discussed in our previous paper [19], how the transition red shift $z_T$ may be used to compute the age of the universe $t_m$ up to the transition epoch and also the dynamical age of the universe $t_0$ up to the present epoch. We have used this technique in Sec. III of this paper to investigate the range of admissible values of $w$ for quiessence model, which are compatible with the range of dynamical age of the universe. The same constraint is applied in Sec. IV to test the physically viable parametrizations of $w(z)$ assuming the slow roll down condition for the scalar field in kinesence models.
II. EXPANSION DYNAMICS OF THE DARK ENERGY

During the matter dominated era onwards, the contribution of CMB photons and neutrinos to the cosmic energy density is trivial and the major energy constituents are non-relativistic matter (baryonic matter and dark matter) and the dark energy. The Friedmann equations for a spatially flat ($k = 0$) universe are

\[
H^2 = \frac{8\pi G}{3} \left[ \rho_M + \rho_X \right] \\
= H_o^2 \left[ \Omega^o_M (1 + z)^3 + \Omega^o_X f(z) \right]
\]  

(1)

and

\[
\frac{\ddot{a}}{a} = -\frac{4\pi G}{3} \left[ \rho_M + \rho_X (1 + 3w) \right] \\
= -\frac{H^2}{2} \left[ \Omega_M + \Omega_X (1 + 3w) \right]
\]  

(2)

where $H(z)$ is the Hubble parameter. The energy density of the non-relativistic matter $\rho_M$ is given by

\[
\rho_M(z) = \rho^0_M (1 + z)^3
\]  

(3)

and the dark energy density $\rho_X$ is written as

\[
\rho_X(z) = \rho^0_X f(z)
\]

with

\[
f(z) = \exp \left[ 3 \int_0^z \frac{1 + w(z')}{1 + z'} dz' \right]
\]  

(4)

In particular for quiessence models ($w = constant$)

\[
f(z) = (1 + z)^{3(1+w)}
\]  

(5)

This condition holds good during ‘tracking’ wherein slow varying equation of state ($w \simeq const \tan t$) is a pre-requisite.

In case of the cosmological constant ($w = -1$), $f(z) = 1$ and $\rho_X = \rho_\Lambda = const \tan t$. In all other cases, the dark energy density $\rho_X$ evolves with the redshift both for varying and non-varying $w(z)$. Logarithmic differentiation of Eq. (4) yields

\[
w(z) = -1 + \frac{1 + z}{3} \frac{d\rho_X}{\rho_X} dz
\]  

(6)
Using Eq. (6), a suitable parametrization for the dark energy density can be assumed to find the \( w(z) \) which conforms to observational constraints. Using the cosmic energy density parametric relation \( \Omega_M + \Omega_X = 1 \), \( \frac{\omega_M}{\rho_X} = \frac{\Omega_M}{\Omega_X} \) and for a spatially flat universe, Eq. (6) may be written in the form

\[
w(z) = \frac{1 + z}{3} \frac{\Omega'_X}{\Omega_X (1 - \Omega_X)}
\] (7)

where

\[
\Omega'_X = \frac{d\Omega_X}{dz}
\]

It is a significant relation as it reveals how the dark energy density varies with the evolution of the equation of state parameter \( w(z) \). With the help of Eq. (7), the Friedmann Eq. (2) can be recast in the form

\[
2q - 1 = 3 w(z) \Omega_X = -\frac{d \ln \Omega_M}{d \ln (1 + z)}
\] (8)

Finally, the dynamical age of the universe is given by

\[
t_0 = \int_0^\infty \frac{dz}{(1 + z) H(z)} = H_0^{-1} \int_0^\infty \frac{dz}{(1 + z) \left[ \Omega'_M (1 + z)^3 + \Omega'_X f(z) \right]^{1/2}}
\] (9)

The impact of the evolution of dark energy on the dynamical age of the universe can be clearly seen from the fact that for a given functional form of \( f(z) \), the contribution of \( \rho_X \) to \( H(z) \) in Eq. (1) goes on decreasing with more negative values of \( w(z) \). Consequently, the expansion age of the universe given by Eq. (9) increases with evolution of \( w(z) \) to more negative values. At the same time, the cosmic expansion accelerates according to Eq. (2) for more negative values of \( w(z) \). Thus, the expansion dynamics of the universe revolves around the equation of state parameter \( w(z) \).

### III. QUIESENCE MODELS

Quiessence models \[18\] of dark energy \( (w = \text{constant}) \) find wide application in tracker field theory \[21,22,23\] wherein slow roll down condition of scalar fields demands \( w \simeq \text{constant} \). Melchiorri et al. \[24\] have combined constraints from CMB observations (including latest WMAP data), large scale structure, luminosity measurement of SN type Ia and Hubble Space
Telescope measurements to find the bounds of dark energy equation of state parameter to
be $-1.38 < w < -0.82$. On the basis of WMAP data for the dynamical age of the universe,
Johri [19] has shown that $w$ lies in a thin strip around $w = -1$, viz

$$-1.18 < w < -0.93$$

(10)

taking $H_0^{-1} = 13.65$ Gyr.

At the transition epoch ($q = 0$), Eq. (8) simplifies to the differential equation

$$\frac{dw}{dx} = -\frac{w(1 + 3w)}{1 + x}$$

(11)

where $x = z_T$ is the redshift at the transition from deceleration to accelerating expansion
phase corresponding to equation of state parameter $w$. A particular integral of Eq. (11) yields

$$1 + z_T = \left[-(1 + 3w)\frac{\Omega_X^0}{\Omega_M^0}\right]^{-1/3w}$$

(12)

In fact, Eq. (12) follows directly from Eq. (2) for quiescence models. For a prior choice of
the ratio $\frac{\Omega_X^0}{\Omega_M^0}$, it is interesting to plot the variation of $z_T$ versus $w$ according to the Eq. (12)
as shown in our previous paper [19]. Eq. (11) gives the gradient of the $z_T \sim w$ curve. This
equation can be alternatively written as

$$\frac{1}{\Omega_M(z_T)} \frac{d\Omega_M}{dz} \bigg|_{z = z_T} = \frac{1}{1 + z_T}$$

(13)

and

$$\Omega_X(z_T) = 1 + (1 + z_T) \frac{d\Omega_X}{dz} \bigg|_{z = z_T}$$

(14)

In Table 1, we have given values of transition redshift $z_T$ for various quiescence models with
dark energy equation of state parameter $w$, the dark energy density parameter $\Omega_X(z_T)$ and
corresponding age of the universe $t_0$ taking $H_0^{-1} = 13.77$ Gyr. We have also plotted the
variation of $z_T$ and $t_0$ with respect to $w$ in Fig. 1. It is noticed that there exists an inverse
correlation between the variation of $z_T$ and $t_0$ with respect to $w$ in quiescence models of dark
energy. Jimenez et al. [9] have detected a similar age-redshift correlation between the ages
of the oldest galaxies and their redshifts.
IV. KINESSENCE MODELS

In case of kinessence models with varying equation of state \( w = w(z) \), there are various competing dark energy models with different parametrizations of \( w(z) \) such as one index parametrizations by Gong et al. [26], two index parametrizations by Huterer et al. [27], Weller et al. [28], Chevallier et al. [29], Linder [30], Jassal et al. [32], Upadhye et al. [33] as well as Wetterich [34] and four index parametrizations by Hannestad et al. [36] and Lee [37]. The question is how well the dark energy equation of state and the cosmological density parameter \( \Omega_X \) in these models behave with increasing redshift \( z \). We have discussed the critical evaluation of some of these models in our earlier work [25]. In the following section, we examine the cosmic age predictions of these models given in Table 2 and compare them with the observational estimates of the dynamical age of the universe \( t_0 = 14.0 \pm 0.5 \) Gyr derived from CMB anisotropy measurements [17].

A. One index parametrizations

1. Gong-Zhang 1st parametrization

The one index dark energy equation of state parameter \( w(z) \) is given by Gong et al. [26] as

\[
w(z) = \frac{w_0}{1 + z}
\]  

(15)

The best fit values to SN Ia ‘gold set’, SDSS and WMAP data are \( w_0 = -1.1 \) and \( \Omega_M^0 = 0.25 \). Hence, the parameters favor dark energy of phantom origin. Combining Eqs. (15) and (14), the dark energy density is given by

\[
\rho_X(z) = \rho_X^0 (1 + z)^3 \exp\left[\frac{3w_0z}{1 + z}\right]
\]  

(16)

and the dark energy parameter \( \Omega_X(z) \) can be written as

\[
\Omega_X(z) = \left[1 + \frac{\Omega_M^0}{\Omega_X^0} \exp\left(-\frac{3w_0z}{1 + z}\right)\right]^{-1}
\]  

(17)

We calculate the transition redshift \( z_T = 0.56 \) for this model by taking \( q = 0 \) in Eq.(8) and inserting for \( w \) and \( (\Omega_X)_T \) (the best fit values of the parameters \( w_0 \) and \( \Omega_M^0 \) ) from Eqs. (15) and (17) respectively. Knowing the transition redshift \( z_T \), the total dynamical age of the universe (worked out using the technique of [19]) turns out to be 13.33 Gyr. Since it lies
outside the observational range given by Knox et al \[17\], this parametrization is found to be incompatible with the age constraint. Following the same procedure, we have found the dynamical age of the universe for various parametrizations discussed in this section and have tested their physical viability on the basis of the age constraint. The results are summarized in Table 2.

2. Gong-Zhang 2nd parametrization

The second one index dark energy equation of state parameter $w(z)$ is given by \[26\]

$$w(z) = \frac{w_0}{1 + z} \exp \left( \frac{z}{1 + z} \right)$$

(18)

The best fit values to SN Ia ‘gold set’, SDSS and WMAP data are $w_0 = -0.97$ and $\Omega_M^0 = 0.28$ and the parameters favor dark energy of quintessence origin. Inserting for $w(z)$ from Eq. (18) into Eq. (4), one gets

$$\rho_X = \rho_X^0 (1 + z)^3 e^{3w_0 \frac{z}{1 + z} - 3w_0}$$

(19)

and the dark energy parameter $\Omega_X(z)$ turns out to be

$$\Omega_X(z) = \left[ 1 + \frac{\Omega_M^0}{\Omega_X^0} \exp \left( 3w_0 - 3w_0 e^{\frac{z}{1 + z}} \right) \right]^{-1}$$

(20)

For this parametrization, the transition red shift $z_T = 0.39$, we get $(\Omega_X)_T = 0.383$. The age of the universe $t_0 = 13.30$ Gyr.

B. Two index parametrizations

1. Linear-redshift parametrization

The dark energy equation of state parameter $w(z)$ is given by Huterer et al. \[27\] and Weller et al. \[28\] as

$$w(z) = w_o + w'z, \quad w' = \left( \frac{dw}{dz} \right)_{z=0}$$

(21)

It has been used by Riess et al. \[2\] for probing SN Ia observations at $z < 1$. The best fit values to SN Ia ‘gold set’ data \[31\] are $w_o = -1.40$, $w' = 1.67$ and $\Omega_M^0 = 0.30$. Hence, the parameters favor dark energy of phantom origin. Inserting Eqs. (21) into Eq.(4), one gets

$$\rho_X(z) = \rho_X^0 (1 + z)^{3(1+w_0-w')} \exp(3w'z)$$

(22)

and

$$\Omega_X(z) = \left[ 1 + \frac{\Omega_M^0}{\Omega_X^0} (1 + z)^{-3(w_0-w')} \exp(-3w'z) \right]^{-1}$$

(23)
In this case, the transition red shift \( z_T = 0.39 \), the corresponding dynamical age of the universe is 10.98 Gyr which lies beyond the observational estimates.

2. Chevallier-Polarski-Linder parametrization

The dark energy equation of state parameter \( w(z) \) is given as \([29, 30]\)

\[
 w(z) = w_0 + \frac{w_1 z}{1 + z} \tag{24}
\]

The best fit values of \( w_0, w_1 \) and \( \Omega_m^0 \) to SN Ia ‘gold set’ data \([31, 35]\) are \(-1.6, 3.3\) and \(0.30\) respectively and the parameters suggest that the dark energy is of phantom origin. The dark energy density is given by

\[
 \rho_X(z) = \rho_X^0 (1 + z)^{3(1 + w_0 + w_1)} \exp(-\frac{3w_1 z}{1 + z}) \tag{25}
\]

and the dark energy parameter

\[
 \Omega_X(z) = \left[1 + \frac{\Omega_M^0}{\Omega_X^0} (1 + z)^{-3(w_0 + w_1)} \exp(\frac{3w_1 z}{1 + z})\right]^{-1} \tag{26}
\]

The transition red shift \( z_T = 0.35 \) and the calculated dynamical age of the universe \( t_0 = 11.23 \) Gyr lies beyond the observational estimates.

3. Jassal-Bagla-Padmanabhan parametrization

Jassal et al. have parametrized \( w(z) \) as \([32]\)

\[
 w(z) = w_0 + \frac{w_1 z}{(1 + z)^2} \tag{27}
\]

The best fit to SN Ia ‘gold set’ data are \( w_0 = -1.9, w_1 = 6.6 \) and \( \Omega_M^0 = 0.30 \). One has \( w(z) = w_0 = -1.9 \) at \( z = 0 \). The parameters suggest that the dark energy is of phantom origin. Combining Eq. \( (27) \) and Eq. \( (4) \) one obtains

\[
 \rho_X(z) = \rho_X^0 (1 + z)^{3(1 + w_0)} \exp\left[\frac{3w_1 z^2}{2(1 + z)^2}\right] \tag{28}
\]

and

\[
 \Omega_X(z) = \left[1 + \frac{\Omega_M^0}{\Omega_X^0} (1 + z)^{-3w_0} \exp\left\{-\frac{3w_1 z^2}{2(1 + z)^2}\right\}\right]^{-1} \tag{29}
\]

The transition redshift \( z_T = 0.3 \) and the corresponding age of the universe turns out to be 12.94 Gyr for the parametrization of Jassal et al. \([32]\).

4. Upadhye-Ishak-Steinhardt parametrization
Upadhye et al. have parametrized $w(z)$ as

$$w(z) = \begin{cases} 
  w_0 + w_1 z & \text{if } z < 1 \\
  w_0 + w_1 & \text{if } z \geq 1
\end{cases} \tag{30}$$

The best fit values of parameters for SN Ia ‘gold set’, galaxy power spectrum and CMB power spectrum data are $w_0 = -1.38$, $w_1 = 1.2$ and $\Omega_M^0 = 0.31$. Hence, the parameters suggest that the dark energy has phantom origin. The dark energy density is given by

$$\rho_x(z) = \rho_X^0 (1 + z)^{3(1+w_0-w_1)} \exp(3w_1 z) \quad \text{if } z < 1$$

$$= \rho_X^0 (1 + z)^{3(1+w_0+w_1)} \quad \text{if } z \geq 1 \tag{31}$$

and the dark energy density parameter is written as

$$\Omega_X(z) = \begin{cases} 
  \left[ 1 + \frac{\Omega_M^0}{\Omega_X^0} (1 + z)^{-3(w_0-w_1)} \exp(-3w_1 z) \right]^{-1} & \text{if } z < 1 \\
  \left[ 1 + \frac{\Omega_M^0}{\Omega_X^0} (1 + z)^{-3(w_0+w_1)} e^{-3w_1(1-2\ln 2)} \right]^{-1} & \text{if } z \geq 1
\end{cases} \tag{32}$$

The transition red-shift $z_T$ turns out to be 0.44 and corresponding dynamical age of the universe turns out to be 12.87 Gyr.

### 5. Wetterich Parametrization

Wetterich has parametrized the dark energy equation of state $w(z)$ as

$$w(z) = \frac{w_0}{1 + b \ln (1 + z)^2} \tag{33}$$

The best fit values to SN Ia ‘gold set’ data are $w_0 = -2.5$, $b = 4.0$ and $\Omega_M^0 = 0.3$. One has $w(z) = w_0 = -2.5$ at $z = 0$. The parameters suggest that the dark energy is of phantom origin. Inserting Eq.(33) for $w(z)$ in Eq. (11), one gets

$$\rho_X(z) = \rho_X^0 (1 + z)^{3 + \frac{3w_0}{1 + b \ln (1 + z)}} \tag{34}$$

and

$$\Omega_X(z) = \left[ 1 + \frac{\Omega_M^0}{\Omega_X^0} (1 + z)^{-\frac{3w_0}{1 + b \ln (1 + z)}} \right]^{-1} \tag{35}$$

For $z_T = 0.26$, the calculated age of the universe $t_0 = 12.52$ Gyr in comparison to the age $14.0 \pm 0.5$ Gyr derived from CMB anisotropies by Knox et al. [17].
C. Four index parametrizations

1. Hannestad-Mörtsell parametrization

Let us now consider Hannestad parametrization \[36\] which involves 4-parameters.

\[
\begin{align*}
    w(z) &= w_0 w_1 \frac{a^p + a_s^p}{w_1 a^p + w_0 a_s^p} \\
       &= \frac{1 + (1+z)^p}{w_o^{-1} + w_1^{-1}(1+z)^p}
\end{align*}
\]

where \(w_0\) and \(w_1\) are the asymptotic values of \(w(z)\) in the distant future \((1 + z \to 0)\) and in the distant past \((z \to \infty)\) respectively. The \(a_s\) and \(p\) are the scale factor at the change over and the duration of the change over in \(w\) respectively. Taking the best fit values for the combined CMB, LSS and SN Ia data \[36\], \(w_0 = -1.8\), \(w_1 = -0.4\), \(q = 3.41\) and \(a_s = 0.50\) with a prior \(\Omega_M^0 = 0.38\), one has \(w(z = 0) = -1.38\) at \(z = 0\). These parameters suggest that the dark energy is of phantom origin. Further, the dark energy density \(\rho_X(z)\) is given by

\[
\rho_X(z) = \rho_X^0 (1 + z)^{3(w_1 + w_0 a_s^p)} \times \left[ \frac{(w_1 + w_0 a_s^p) (1 + z)^p}{w_1 + w_0 a_s^p (1 + z)^p} \right]^{-\frac{3(w_0 - w_1)}{p}}
\]

and the dark energy parameter is written as

\[
\Omega_X(z) = \left[ 1 + \frac{\Omega_M^0}{\Omega_X^0} (1 + z)^{-3w_1} \right] \left\{ \frac{w_1 + w_0 a_s^p (1 + z)^p}{w_1 + w_0 a_s^p (1 + z)^p} \right\}^{-\frac{3(w_0 - w_1)}{p}}
\]

At transition redshift \(z_T = 0.39\), the dark energy parameter \(\Omega_X(z_T) = 0.333\). The age of the universe \(t_0 = 12.52\) Gyr in comparison to \(t_0 = 14.0 \pm 0.5\) Gyr derived by Knox et al \[17\] from CMB observations.

2. Lee parametrization

The dark energy equation of state parameter \(w(z)\) is parametrized by Lee as \[37\]

\[
w(z) = \frac{w_0 \exp(px) + \exp(px_c)}{\exp(px) + \exp(px_c)}
\]

where

\[
x = \ln a = -\ln(1 + z)
\]

and the symbols \(w_r\) and \(x_c\) have the usual meaning as in \[37\]. The parameters \(w_r\) is chosen to be \(1/3\) using the tracking condition and \(w_0\) is taken as \(-3\). The other parameters are obtained by analyzing the separation of CMB peaks and the time variation of the fine
structure constant. For $\Omega^0_M = 0.27$ and $x_c = -2.64$, the range of $p$ is taken as $1.5 \leq p \leq 3.9$. At $z = 0$, one has $w(z = 0) = -0.832$. Hence, the parameters suggest that the dark energy is of quintessence origin. Inserting Eq. (39) into Eq. (4), one gets

$$\rho_X(z) = \rho^0_X (1 + z)^3 (a + a_{eq}) \left( \frac{a^p + a^p_{c1}}{1 + a^p_{c1}} \right)^\frac{4}{p}$$

The dark energy parameter $\Omega_X(z)$ is given by

$$\Omega_X(z) = \left[ 1 + \frac{\Omega^0_M (a + a_{eq})}{\Omega^0_X} \left( \frac{a^p + a^p_{c1}}{1 + a^p_{c1}} \right)^{-4/p} \right]^{-1}$$

For $p = 1.5$ and $3.9$, the dark energy parameter $\Omega_X$ is $0.353$ and $0.333$ at the transition redshift $z_T = 0.74 - 0.76$ respectively. In comparison to the observational estimate of the age derived by Knox et al. [17] $t_0 = 14.0 \pm 0.5$ Gyr, the age of the universe for this parametrization turns out to be $13.57$ and $13.67$ Gyr for $p = 1.5$ and $3.9$ respectively.

V. BOUNDS ON VARIATION OF DARK ENERGY

In the case of kinessence models with equation of state $w = w(z)$, $\rho_X$ and $\Omega_X$ are expressible in terms of $f(z)$ which depends upon the choice of parametrization of $w(z)$. So, it is not possible to predict the variation of $\Omega_X(z)$ precisely in the past without knowing the realistic form of $f(z)$. Various parametrizations as discussed in the Sec. 4.1 and 4.2 give only the tentative variation of $\rho_X$ and $\Omega_X$ with the redshift. However, we can put an upper bound on the variation of $\Omega_X(z)$ in the quintessence models. According to the Table 1, the dynamical age of the universe given by Knox et al. [17] constrains $w(z)$ to lie within the range $-1.60 \leq w(z) \leq -0.82$ which is consistent with WMAP upper bound $w < -0.8$ [16] and satisfies the dark energy condition $w(z) < -2/3$. Applying this condition to the integral in Eq. (4) leads to an upper bound on the dimensionless dark energy function $f(z)$ given by

$$f(z) < (1 + z)$$

Combining $\rho_X(z) \leq \rho^0_X (1 + z)$ with $\rho_M(z) = \rho^0_M (1 + z)^3$ we get the upper bound on $\rho_X$ and $\Omega_X$

$$\frac{\rho_X}{\rho_M} \leq \frac{\rho^0_X}{\rho^0_M} (1 + z)^{-2}$$
\[
\Omega_X^{-1} \geq 1 + \frac{\Omega_0^M}{\Omega_X^0} (1 + z)^2
\] (45)

As shown in the Fig. 2, to check the validity of Eq. (45), let us choose a quintessence model with \( w = -0.93 \) =constant. The corresponding \( \Omega_X = 0.358 \) from Table 1 at the transition redshift \( z_T = 0.760 \) whereas Eq. (45) yields the upper bound \( \Omega_X \leq 0.46 \) for \( z_T = 0.760, \Omega_M^0 = 0.27 \) and \( \Omega_X^0 = 0.73 \). Eq. (45) can also be applied, in principle, to dark energy models with varying equation of state parameter \( w(z) \) using well-behaved parametrizations to yield the upper bound of the fractional dark energy density \( \Omega_X \).

VI. CONCLUSIONS

A new approach to the exploration of dark energy parameters is discussed based on the impact of the evolution of dark energy equation of state \( w(z) \) on the dynamical age of the universe. The dynamical age depends upon the Hubble expansion of the universe of which dark energy is presently a dominant constituent. Therefore, the age of the universe carries the signature of the dark energy and can be used as an effective constraint to check the physical viability of the various quiessence and kininessence models as discussed in this paper. Unlike the conventional procedure of assuming a parametrization for \( w(z) \) and constraining it by fitting it to the observational data, we have calculated the dynamical age of the universe assuming different parametrizations and tested the physical viability of these parametrizations with the cosmic age constraint derived from the CMB data [17]. Direct comparison of the theoretically calculated age of the universe (Table 1 and 2) with the observational range of the cosmic age as laid down by CMB anisotropy measurements reveals that Lee parametrizations [37] satisfy both the astrophysical [25] and the cosmic age constraint [17]. Of course, the quintessence models in the range \(-1 < w < -0.82\), the cosmological constant and a wide class of phantom models also satisfy the cosmic age constraint.

Recently Jassal, Bagla and Padmanabhan [38] have carried out a detailed analysis of constraints on cosmological parameters from different observations with particular reference to equation of state parameter \( w(z) \). According to their analysis, the SN Ia observations alone favor phantom models \( (w \ll -1) \) with large \( \Omega_M \) whereas the WMAP observations favor models with \( w \sim -1 \) if dark energy perturbations are included. In the absence of any definite evidence of variation of dark energy density with the redshift, the cosmological
constant still remains a favorite candidate for dark energy as it remains compatible with astrophysical observations and satisfies the cosmic age constraint as laid down in this paper as well as the recently published WMAP three years data estimating $t_0 = 13.73^{+0.13}_{-0.17} \text{Gyr}$ and $H_0 = 73.4^{+2.8}_{-3.8} \text{Mpc}^{-1} \text{km s}^{-1}$.

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Table 1: Age of the universe with constant dark energy parameter $w$ and $H_0^{-1} = 13.77$ Gyr.

| $w$  | $z_T$ | $(\Omega_M)_T$ | $t_m$ | $t_0H_0$ | $t_0$ (Gyr) |
|------|-------|----------------|-------|-----------|-------------|
| -0.66| 0.636 | 0.495          | 0.613 | 1.128     | 15.53       |
| -0.70| 0.680 | 0.524          | 0.589 | 1.106     | 15.22       |
| -0.75| 0.718 | 0.556          | 0.570 | 1.082     | 14.91       |
| -0.80| 0.741 | 0.583          | 0.558 | 1.064     | 14.65       |
| -0.85| 0.754 | 0.608          | 0.552 | 1.049     | 14.44       |
| -0.90| 0.759 | 0.629          | 0.550 | 1.036     | 14.27       |
| -0.93| 0.760 | 0.642          | 0.550 | 1.030     | 14.18       |
| -0.95| 0.759 | 0.649          | 0.550 | 1.026     | 14.13       |
| -1.00| 0.755 | 0.667          | 0.552 | 1.017     | 14.01       |
| -1.02| 0.753 | 0.673          | 0.553 | 1.014     | 13.97       |
| -1.05| 0.749 | 0.683          | 0.555 | 1.010     | 13.91       |
| -1.10| 0.740 | 0.697          | 0.559 | 1.005     | 13.84       |
| -1.15| 0.730 | 0.710          | 0.564 | 1.000     | 13.77       |
| -1.18| 0.723 | 0.718          | 0.567 | 0.998     | 13.74       |
| -1.20| 0.719 | 0.722          | 0.569 | 0.996     | 13.72       |
| -1.35| 0.684 | 0.753          | 0.587 | 0.989     | 13.62       |
| -1.50| 0.648 | 0.778          | 0.607 | 0.986     | 13.57       |
| -1.60| 0.625 | 0.792          | 0.620 | 0.985     | 13.56       |
| -1.70| 0.603 | 0.804          | 0.632 | 0.985     | 13.57       |
| -1.80| 0.582 | 0.815          | 0.645 | 0.986     | 13.58       |
| -1.90| 0.562 | 0.825          | 0.657 | 0.987     | 13.60       |
| -2.00| 0.543 | 0.833          | 0.669 | 0.989     | 13.62       |
| -2.10| 0.526 | 0.841          | 0.681 | 0.991     | 13.65       |
| -2.20| 0.509 | 0.849          | 0.692 | 0.994     | 13.68       |
| -2.30| 0.494 | 0.855          | 0.703 | 0.996     | 13.72       |
| -2.40| 0.479 | 0.861          | 0.713 | 0.999     | 13.75       |
| -2.50| 0.465 | 0.867          | 0.723 | 1.001     | 13.79       |
Table 2: The age of the universe in parametric models of dark energy.

| Models                        | Reference | $z_T$ | $(\Omega x)_T$ | $t_0H_0$ | $t_0$ (Gyr) |
|-------------------------------|-----------|-------|----------------|-----------|-------------|
| Gong-Zhang 1st                | [26]      | 0.56  | 0.479          | 0.968     | 13.33       |
| Gong-Zhang 2nd                | [26]      | 0.39  | 0.383          | 0.966     | 13.30       |
| Linear red shift              | [27, 28]  | 0.39  | 0.443          | 0.797     | 10.98       |
| Chevallier-Polarski-Linder    | [29, 30]  | 0.35  | 0.451          | 0.816     | 11.23       |
| Jassal-Bagala-Padmanabhan     | [32]      | 0.30  | 0.467          | 0.939     | 12.94       |
| Upadhye-Ishak-Steinhardt      | [33]      | 0.44  | 0.392          | 0.934     | 12.87       |
| Wetterich                     | [34]      | 0.26  | 0.488          | 0.909     | 12.52       |
| Hannestad-Mörtsell            | [36]      | 0.39  | 0.333          | 0.909     | 12.52       |
| Lee ($p = 1.5$)               | [37]      | 0.74  | 0.353          | 0.985     | 13.57       |
| Lee ($p = 3.9$)               | [37]      | 0.76  | 0.333          | 0.993     | 13.67       |
Fig. 1: Transition Redshift $z_T$ and the dynamical age of the universe $t_0$ in quiescence models. The solid and dashed lines represent the variation of $z_T$ and $t$ with respect to $w$. The estimate of dynamical age of the universe by Knox et al. is shown by vertical lines.
Fig. 2: Upper Bound on the variation of $\Omega_x$ in the quintessence models.