Exponential confidence region based on the projection density estimate.

Recursivity of these estimations.

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Abstract

We investigate the famous Tchentzov’s projection density statistical estimation in order to deduce the exponential decreasing tail of distribution for the natural normalized deviation.

We modify these estimations assuming the square integrability of estimated function, to make it recursive form, which is more convenient for applications, however they have at the same speed of convergence as the for the classical ones in the composite Hilbert space norm.

Key words and phrases.

Projection statistical density estimation, measure and measurable space, Riesz - Fisher theorem, integrability and square integrability, probability space, random series and variables, sample, orthogonal function and series, Hilbert space and composite norm, orthonormal complete system, order, convergence, adaptivity, Fourier coefficients, sample, random variable (r.v.), density, consistency, ordinary and adaptive projection estimation, slowly varying function, recursivity, error, observations, optimal rule, triangle inequality.
1 Introduction.

Let \( (X = \{x\}, \mathcal{B}, \mu) \) be measurable space equipped with sigma finite and separable measure \( \mu \). Separability implies that the metric space \( \mathcal{B} \) relative the distance function

\[
D(A_1, A_2) := \mu(A_1 \setminus A_2) + \mu(A_2 \setminus A_1)
\]

is separable.

Let also \( \xi_i, i = 1, 2, \ldots, \xi := \xi_1 \) be a sample, i.e. a sequence of independent identical distributed random variables (r.v.) defined on certain probability space \( (\Omega = \{\omega\}, \mathcal{V}, \mathbb{P}) \) with expectation \( \mathbb{E} \) and variance \( \text{Var} \), having at the same non-negative integrable and in addition square integrable unknown density function \( f = f(x), x \in X \) relative the measure \( \mu \):

\[
\mathbb{P}(\xi_i \in A) = \int_A f(x) \mu(dx), \ A \in \mathcal{B},
\]

such that

\[
||f||^2(L_2(X, \mu)) = ||f||^2 = ||f||_2^2 := \int_X f^2(x) \mu(dx) < \infty, \quad (1)
\]

and as ordinary

\[
f(x) \geq 0; \quad \int_X f(x) \mu(dx) = ||f||_1 = 1.
\]

Our goal in this report is building of the exponential decreasing confidence region in the Hilbert space norm for the unknown density as well as to offer and investigate the recursive projection statistical estimate of the density function consistent in the mixed Hilbert space norm, having however at the same speed of convergence as the classical optimal projection estimates.

Let us clarify briefly the difference between ordinary and recursive estimations on the example of classical kernel density estimates. The classical kernel Parzen - Rosenblatt estimate \( f[PR]_n(x) \) of \( f(x), x \in \mathbb{R}^d, d = 1, 2, \ldots \) has a form

\[
f[PR]_n(x) := \frac{1}{n h^d(n)} \sum_{i=1}^{n} V \left( \frac{x - \xi_i}{h(n)} \right),
\]

where \( h(n) \to 0, n \to \infty \). In contradiction, the so - called recursive Wolverton - Wagner’s estimate \( f[WW]_n(x) \) is defined as follows

\[
f[WW]_n(x) := n^{-1} \sum_{i=1}^{n} \frac{1}{h(i)^d} V \left( \frac{x - \xi_i}{h(i)} \right).
\]
The second estimate has at the same asymptotical properties as the first one when \( n \to \infty \) but it obey’s the recursive form

\[
f_{WW}(x) = \frac{n - 1}{n} f_{WW}(x) + \frac{1}{h^d(n)} V \left( \frac{x - \xi_n}{h(n)} \right), \quad n \geq 2,
\]

therefore it is more convenient for the practical using.

## 2 Main result. Non-adaptive projection estimations.

We will start from the classical projection Tchentzov’s density estimations, see the classical works [26], [27]. The more modern adaptive estimations will be discuss briefly further.

Let \( \{\phi_k\} = \{\phi_k(x)\}, \ x \in X \) be certain complete orthonormal system functions in the (separable) Hilbert space \( L_2 = L_2(X, \mu) \):

\[
(\phi_k, \phi_l) = \int_X \phi_k(x) \phi_l(x) \mu(dx) = \delta_{kl}, \quad k, l = 1, 2, \ldots ,
\]

where \( \delta_{kl} \) is the Kroneker’s symbol.

We have been assumed above that the density function is square integrable: \( f(\cdot) \in L_2; \) therefore in has a form (theorem of Riesz - Fisher)

\[
f(x) = \sum_{k=1}^{\infty} c_k \phi_k(x),
\]

empirical Fourier coefficients, so that

\[
\lim_{N \to \infty} \rho(N) = 0, \quad \rho(N) = \rho[f](N) \overset{def}{=} \sum_{k=N+1}^{\infty} c_k^2.
\]

As long as

\[
c_k = \int_X \phi_k(x) f(x) \mu(dx) = E \phi_k(\xi),
\]

the unbiased and consistent as \( n \to \infty \) with probability one as well as in the mean square sense statistical estimate of the Fourier coefficients \( c_k(n) \) has a form

\[
c_k(n) \overset{def}{=} n^{-1} \sum_{i=1}^{n} \phi_k(\xi_i).
\]

The classical so-called projection estimate of \( f(x) \) offered by N.N.Tchentzov [26], [27] has a form
\[ f_{N,n}(x) \overset{\text{def}}{=} \sum_{k=1}^{N} c_k(n) \phi_k(x), \quad N = 2, 3, 4, \ldots, \quad (6) \]

the number \( N \) is called order of this estimation.

The quantity of this approximation may be measured by means of the mixed (anisotropic) square Hilbert norm

\[ \Delta(N, n) = \Delta[f_{N,n}, f] \overset{\text{def}}{=} E \|f_{N,n} - f\|^2. \quad (7) \]

**Lemma 2.1.** Suppose that the orthonormal system \( \{\phi_k(\cdot)\} \) is uniformly bounded:

\[ M \overset{\text{def}}{=} \sup_{k=1,2,\ldots} \sup_{x \in X} \phi_k^2(x) < \infty. \quad (8) \]

Then \( \Delta(N, n) \leq A(N, n) \), where

\[ A(N, n) \overset{\text{def}}{=} \frac{MN}{n} + \rho(N). \quad (9) \]

**Proof.** By virtue of orthonormality

\[
\|f_{N,n} - f\|^2 = \sum_{k=1}^{N} (c_k(n) - c_k)^2 + \sum_{k=N+1}^{\infty} c_k^2 = \sum_{k=1}^{N} (c_k(n) - c_k)^2 + \rho(N).
\]

Further,

\[ E(c_k(n) - c_k)^2 = n^{-1} \text{Var}\{ \phi_k(\xi) \} \leq n^{-1} E[\phi_k(\xi)]^2 =
\]

\[ n^{-1} \int_X \phi_k^2(x) f(x) \mu(dx) \leq M n^{-1}; \]

following

\[ E\|f_{N,n} - f\|^2 \leq MN/n + \rho(N) = A(N, n), \quad (10) \]

Q.E.D.

Introduce the following important optimal, more exactly, pseudo - optimal, variable (number of summands)

\[ N_0 = N_0(n) = N_0[n, M, \rho] \overset{\text{def}}{=} \arg\min_{N=2,3,\ldots,N} A(N, n), \quad (11) \]

and correspondingly the optimal rate of convergence of offered projection estimate is equal to
\[ A^*(n) \stackrel{\text{def}}{=} \min_N A(N, n) = A(N_0, n). \] (12)

**Example 2.1.** Let \( c_k^2 \asymp C_1/k^{1+\gamma} \), \( C_j, \gamma = \text{const} \in (0, \infty) \). Then

\[ \rho(N) \asymp C_2N^{-\gamma}, \quad N_0(n) \asymp C_3N^{-\gamma} \]

and finally

\[ A^*(n) \asymp C_4n^{-\gamma/(1+\gamma)}, \quad n \geq 1. \]

**Example 2.2.** Let now \( c_k^2 \asymp C_5 \exp(-C_6k) \), \( k = 1, 2, \ldots \); then

\[ \rho(N) \asymp C_7 \exp(-C_8N), \quad N_0(n) \asymp C_9 \ln n, \quad n \geq 2, \]

and finally in this case

\[ A^*(n) \asymp C_{10} \ln n/n, \quad n \geq 2. \]

Ibragimov I.A. and Hasminskii R.Z. in [16] proved that this speed of convergence is optimal in the mixed Hilbert norm \( L_2(\Omega \otimes X) \) sense on the widely classes of estimated functions \( f \).

**Lemma 2.2.**

\[ c_{N_0}^2 \geq \frac{M}{n}; \quad c_{N_0+1}^2 \leq \frac{M}{n}, \quad n, N_0 \geq 2. \] (13)

**Proof.** Both the relations in (13) follows immediately from the obvious inequalities

\[ A(N_0 - 1, n) \leq A(N_0, n) \leq A(N_0 + 1, n), \quad N_0 = N_0(n). \] (14)

**Briefly, by definition of notation:**

\[ c_{N_0(n)}^2 \stackrel{\text{def}}{=} \frac{M}{n}, \quad n \to \infty. \]

Inversely, let the estimations

\[ c_L^2 \geq \frac{M}{n}; \quad c_{L+1}^2 \leq \frac{M}{n}, \quad n \geq 2. \] (15)

for some positive integer value \( L = L(n), \quad n \geq 2 \) holds true. Question: is the value \( L = L(n) \) actually the optimal numbers \( N_0 : \ L(n) = N_0(n) \)?

**Lemma 2.3.** Let the integer valued number \( L = L(n), \quad n \geq 2 \) be a minimal value for which the relation (15) holds true. Let also the numerical non-negative
sequence of square of coefficients $c_k^2$ be monotonically non-increasing for sufficiently greatest values $k$:

$$\exists k_0 = 2, 3, \ldots \; \forall k \geq k_0 \Rightarrow c_{k+1}^2 \leq c_k^2.$$  \hspace{1cm} (16)

Then $L(n) = N_0(n) = \text{argmin}_N A(N, n)$.

**Proof.** Let us ground first of all the existence and non-triviality of the variable $N_0 = N_0(n)$. Note that $\lim_{N \to \infty} A(N, n) = \infty$, therefore $N_0(n) < \infty$, $n \geq 2$. Further, denote by $\lfloor m \rfloor = \text{Ent}(m)$ the integer part the number $m$. We have

$$A([\sqrt{n}], \lfloor \sqrt{n} \rfloor) \leq C \frac{1}{\sqrt{n}} + \rho[f]([\sqrt{n}]) \to 0, \; n \to \infty,$$

as long as $f \in L_2(X, \mu)$. Following

$$\lim_{n \to \infty} A^*(n) = \lim_{n \to \infty} A(N_0(n), n) = 0.$$

Ultimately, let the estimations (15) is valid for some positive integer value $L = L(n)$, $n \geq 2$. Let also $l = 1, 2, \ldots$ be arbitrary positive integer number. We must ground the inequality

$$\rho(L) + \frac{ML}{n} \leq \rho(L + l) + \frac{M(N + l)}{n},$$

which is in turn quite equivalent to the next one

$$\sum_{j=1}^{l} c_{L+j}^2 \leq l \frac{M}{n}, \; l = 1, 2, \ldots;$$  \hspace{1cm} (18)

which follows immediately from the conditions of Lemma 2.3, because $c_{L+j}^2 \leq c_L^2$, $j = 1, 2, \ldots$.

The case of the negative values $l$ may be considered quite analogously. Namely, one has

$$\rho(L) + \frac{ML}{n} \leq \rho(L - l) + \frac{M(L - l)}{n}, \; l = 1, 2, \ldots, L - 1;$$

which is in turn quite equivalent to the next inequality

$$\sum_{j=1}^{l} c_{L-j}^2 \geq l \frac{M}{n}, \; l = 1, 2, \ldots, L - 1.$$  \hspace{1cm} (20)

**Remark 2.1.** Note that the assertion of Lemma 2.3 remains true only under both the relations (18) and (20).

**Remark 2.2.** Let an integer non-random value $K = K(n), n \geq 2$, $K(n) = 2, 3, \ldots, n - 1$ be such that there exists an universal constant $\Gamma = \text{const} \in [1, \infty)$ for which
∀n, N ∈ [2, n − 1] ⇒ A(K(n), n) ≤ Γ A(N, n).

Then obviously

\[ E\| f(K(n), n)(\cdot) - f(\cdot) \|^2 \leq \Gamma \cdot A^*(n) - \]

quasi-optimality.

**Remark 2.3.** As long as in the practice the values A(N, n) are unknown, one can use instead of him its statistical consistent estimations (adaptivity). Namely, define the variables

\[ \tau_n(N) \overset{def}{=} 2N \sum_{k=N+1}^{2N} c^2_k(n), \ n, N \geq 2, \ N \leq n/2. \]

It is proved in [4], see also [20] - [24] that as \( n \to \infty \) with probability one

\[ \tau_n(N) \asymp A(N, n), \]

and following

\[ \min_{N \in [2, n/2]} \tau_n(N) \sim A^*(n), \ \arg\min_{N \in [2, n/2]} \tau_n(N) \sim N_0(n), \]

also almost everywhere.

### 3 Exponential confidence region based on the projection estimate.

Let us return to the arbitrary projection estimate (10). Define the (non-negative) variable

\[ \Delta = \Delta[f]_{N,n} \overset{def}{=} \frac{n}{M N} \left[ \| f_{N,n}(\cdot) - f(\cdot) \|^2_2 - \rho(N) \right], \]

and its tail function

\[ T[\Delta](t) \overset{def}{=} P(\Delta > t), \ t \geq 1, \]

which may be used by the building of the confidence region for the unknown density function \( f(\cdot) \) in the \( L_2(X) \) sense. Here \( 2 \leq N \leq n - 2, \ n \geq 5; \) the value \( N \) may be optimal or not.

**Theorem 3.1.** We conclude under formulated conditions

\[ T[\Delta](t) \leq \exp \left[ -t/ (e M) \right], \ t \geq e M. \]
Proof. We will use the known theory of Grand Lebesgue Spaces (GLS) of random variables having the exponential decreasing tail of distributions, see e.g. [1], [2], [3], [5], [6], [7], [17], [18], [19], [20] etc. Concrete, let \( \psi = \psi(p) \), \( 1 \leq p < b = \text{const} \leq \infty \) be measurable strictly positive: \( \inf_p \psi(p) > 0 \) function. By definition, the Banach rearrangement invariant Grand Lebesgue Space (GLS) \( G_\psi \) consists on all the random variables (r.v.) \( \{\xi\} \) having a finite norm

\[
\|\xi\|_{G_\psi} \overset{\text{def}}{=} \sup_{p \in (1,b)} \left\{ \frac{\|\xi\|_{L_{p,\Omega}}}{\psi(p)} \right\} < \infty, \tag{24}
\]

where as ordinary

\[
\|\xi\|_{L_{p,\Omega}} = \|\xi\|_p := \left[ \mathbb{E}|\xi|^p \right]^{1/p}, \quad 1 \leq p < \infty,
\]

\[
\|\xi\|_\infty = \text{vraisup}_{\omega \in \Omega} |\xi(\omega)|.
\]

The finiteness of some GLS norm \( \|\xi\|_{G_\psi} \) for the r.v. \( \xi \) is closely related with its tail behavior

\[
T[\xi](t) \overset{\text{def}}{=} \mathbb{P}(|\xi| > t), \quad t \geq e.
\]

Indeed, assume for definiteness that \( \|\xi\|_{G_\psi} = 1 \); then

\[
T[\xi](t) \leq \exp \left( - \sup_{p \in (1,b)} \left( p \ln t - p \ln \psi(p) \right) \right), \quad t \geq e; \tag{25}
\]

and inverse conclusion is true under simple natural conditions: if (25) there holds, then

\[
\xi \in G_\psi, \quad \Leftrightarrow \exists K = K(\psi) < \infty, \quad \|\xi\|_{G_\psi} \leq K(\psi).
\]

Example. Set for certain \( m = \text{const} \in (0, \infty) \)

\[
\psi_m(p) := p^{1/m}, \quad 1 \leq p < \infty.
\]

The GLS estimate

\[
\|\xi\|_{G_{\psi_m}} = \sup_{p \geq 1} \left[ \frac{\|\xi\|_{L_{p,\Omega}}}{\psi_m(p)} \right] < \infty \tag{26}
\]

is completely equivalent to the following tail inequality

\[
\exists \gamma = \gamma(m) \in (0, \infty) \Rightarrow T[\xi](t) \leq \exp \left( - \gamma(m) \ t^m \right), \quad t \geq 0. \tag{27}
\]

The case \( m = 2 \) correspondent to the famous subgaussian random variables. Standard notation \([17]\):

\[
\|\xi\|_{\text{Sub}} \overset{\text{def}}{=} \|\xi\|_{G_{\psi_2}} = \sup_{p \geq 1} \left[ \frac{\|\xi\|_p}{\sqrt{p}} \right].
\]

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It is known that $||\xi||_{\text{Sub}} \leq ||\xi||_{\infty}$ and if the r.v. - s $\{\eta_i\}, i = 1, 2, \ldots, n$ are subgaussian, independent, centered and identical distributed, then

$$\sup_n \left[ n^{-1/2} \left| \sum_{i=1}^{n} \eta_i \right|_{\text{Sub}} \right] = ||\eta||_{\text{Sub}}. \quad (28)$$

Let us return to the theorem 3.1. Denote

$$\delta_k(n) := n^{1/2}(c_k(n) - c_k) = n^{-1/2} \sum_{i=1}^{n} (\phi_k(\xi_i) - c_k);$$

therefore

$$n^{1/2} ||c_k(n) - c_k||_{\text{Sub}} \leq \sqrt{M}.$$  

We entail from here

$$||c_k(n) - c_k||_{\text{Sub}} \leq \frac{M}{n}. $$

One can apply the triangle inequality for the $G_{\psi_1}$ norm

$$|| \sum_{k=1}^{N} (c_k(n) - c_k)^2 ||_{G_{\psi_1}} \leq \frac{M N}{n}. \quad (29)$$

Denote

$$\Theta = \Theta[N, n] := \frac{n}{M N} \sum_{k=1}^{N} (c_k(n) - c_k)^2;$$

then $||\Theta||_{G_{\psi_1}} \leq 1$ and hence

$$T[\Theta](t) \leq \exp(-t/e), \; t \geq e.$$  

The proposition (39) follows immediately from the last estimate after simple calculations.

4 Recursions.

We intent in this section to offer a recursive modification of the projection density estimation having at the same up to finite multiplicative constant speed of convergence as classical ones.

Notice that the number of summands $N(n)$ is integer positive number, which is monotonically non-increasing relative the number $n$. Therefore it is reasonable
to suppose that it satisfied for all the sufficiently greatest values \( n \), say for \( n \geq 10 \), the following recursion

\[
N(n + 1) = N(n) + 1 \tag{30}
\]
or

\[
N(n + 1) = N(n). \tag{31}
\]

The first case (30) take place iff

\[
c_{N(n)}^2 > \frac{M}{n}, \tag{32}
\]

the second one take place when

\[
c_{N(n)}^2 \leq \frac{M}{n}. \tag{33}
\]

So, let us introduce the following *recursively defined* sequence of positive non-decreasing integer numbers \( Y = \{Y(n)\} \) such that \( Y(1) = 1 \) and consequently

\[
c_{Y(n)}^2 > \frac{M}{n} \Rightarrow Y(n + 1) := Y(n) + 1, \tag{34}
\]

and otherwise

\[
c_{Y(n)}^2 \leq \frac{M}{n} \Rightarrow Y(n + 1) := Y(n). \tag{35}
\]

**Theorem 4.1.** We retain all the conditions and notations of the theorem 3.1. Define the following centered and normalized deviation variable

\[
\Delta(Y) = \Delta[f]_{Y,N,n} \overset{\text{def}}{=} \frac{n}{MY(n)} \left[ || f_{Y(n),n} (\cdot) - f(\cdot)||^2_2 - \rho(Y(n)) \right], \tag{36}
\]

and its tail function

\[
T[\Delta(Y)](t) \overset{\text{def}}{=} P(\Delta(Y) > t), \ t \geq 1, \tag{37}
\]

Here as above \( 2 \leq Y(n) \leq n - 2, \ n \geq 5. \)

Assume in addition that

\[
Q = Q[Y] \overset{\text{def}}{=} \sup_n \left\{ \frac{A(Y(n),n)}{A^*(n)} \right\} < \infty. \tag{38}
\]

We conclude under formulated conditions by virtue of the proposition of remark 2.2
$T[\Delta(Y)](t) \leq \exp\left[-t/\left(e \ M \ Q\right)\right], \ t \geq e \ M \ Q. \quad (39)$

**Remark 4.1.** The condition $Q(Y) < \infty$ in (38) is satisfied for instance when

$$\exists C_1, C_2 \in (0, \infty), \ c_k^2 \asymp k^{-1-C_1} L(k), \ k \geq 2,$$

where $L = L(z), \ z \in [2, \infty)$ is non-negative continuous slowly varying at infinity function; as well as when

$$\exists C_1, C_2 \in (0, \infty), \ c_k^2 \asymp \exp\left(-C_1 k^{C_2} L(k)\right), \ k \geq 2.$$

5 Concluding remarks.

It is interest in our opinion to deduce the recursive projection density (and regression, spectral density) function estimates for the so-called adaptive estimates, having however the optimal rate of convergence, which does not dependent on the unknown, in general case, class of smoothness for estimating function. See for example [4], [21], [22].

**Acknowledgement.** The first author has been partially supported by the Gruppo Nazionale per l’Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM) and by Università degli Studi di Napoli Parthenope through the project “sostegno alla Ricerca individuale”.

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