Comparison and Analysis of Different Numerical Schemes in Sod’s One-dimensional Shock Tube Problems

Ning Ji*

1School of Aeronautic of Science and Engineering, Beihang University, Beijing, 100191, China

*Corresponding author’s e-mail: bhjining@126.com

Abstract. In this paper, we apply a variety of numerical methods to calculate and analyze the Sod’s one-dimensional shock tube problem. Different schemes show different resolution capabilities for rarefaction waves, shock waves, and contact discontinuities. Under different grid density, we further analyze the performance of various schemes, and carry out error analysis to calculate the numerical accuracy. In conclusion, the accuracy of different numerical schemes decreases at discontinuities, and NND scheme and WENO scheme can effectively eliminate the non-physical oscillation.

1. Introduction
The Sod’s one-dimensional shock tube problem is a classic problem in Riemann problem, that is, there is a virtual diaphragm in the shock tube to separate the fluid into two different states (or different density, velocity and pressure). At time t=0, the virtual diaphragm is suddenly withdrawn, and different forms of waves (or shock wave, rarefaction wave and contact discontinuity) will appear in the flow field, thus changing the distribution of the fluid’s state. The one-dimensional shock tube problem has a unique advantage in testing the stability and resolution of the numerical schemes. Its advantage lies in that the control equation has fewer dimensions and variables and contains various CFD discontinuities. The solution of these discontinuous problems has always been the difficulty and core problem in the development of CFD, which has special significance in numerical simulation. Riemann problem has its own exact solution, which can verify the precision of the numerical method[1]. In this paper, FVS scheme, FDS scheme, NND scheme and fifth-order WENO scheme are used to calculate and analyze one-dimensional shock tube problems and the performance of each scheme is compared.

2. Governing Equation
2.1. Shock Tube Problem
The calculation region is $x = (-5, 5)$, initial discontinuous distribution at $t = 0$:

$$
(u, \rho, p) = \begin{cases} 
(0, 1, 1) & x \in [-5, 0) \\
(0, 0.125, 0.1) & x \in [0, 5] 
\end{cases}
$$

(2.1)

where $x$ is the distance $m$; $t$ the time $s$; $\rho$ the density $kg \cdot m^{-3}$; $u$ the velocity $m \cdot s^{-1}$; $p$ the pressure $N \cdot m^{-2}$. This paper will present the results at $t = 2.0s$. 

Content from this work may be used under the terms of the Creative Commons Attribution 3.0 licence. Any further distribution of this work must maintain attribution to the author(s) and the title of the work, journal citation and DOI.

Published under licence by IOP Publishing Ltd
2.2. Discontinuities Distribution
At \( t = 0 \), the velocity inside the tube is zero, and the high-pressure zone is to the left of the diaphragm. When the diaphragm is removed, the initial pressure difference will propagate to the right as a shock wave, while an isentropic rarefaction wave will propagate to the left. The discontinuous distribution at \( t > 0 \) is shown, and the fluid in the tube is divided into five regions (fig. 2). Region 1 and region 5 are regions to which the wave has not yet propagated. Region 2 is isentropic rarefaction wave, and its internal state parameters are continuous and smooth[2][3]. Region 3 is the region after the expansion wave propagation, and 4 is the region after the shock wave propagation. The two regions are separated by a contact discontinuity. Only the density changes and the velocity and pressure are the same.

\[
\begin{align*}
(u, p, \rho) &= (0, 1, 1) \\
(u, p, \rho) &= (0, 0.125, 0.1)
\end{align*}
\]

Figure 1. Initial condition

Figure 2. Discontinuities distribution

2.3. One-dimensional Euler Equations
Consider the conserved form of Euler equations for one-dimensional unsteady flow without considering the heat source, heat conduction and volume forces:

\[
\begin{align*}
\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} &= 0 \\
\frac{\partial (\rho u)}{\partial t} + \frac{\partial (\rho u^2 + p)}{\partial x} &= 0 \\
\frac{\partial E}{\partial t} + \frac{\partial (u(E + p))}{\partial x} &= 0
\end{align*}
\]

where \( E \) is the energy per unit volume (length).

One more equation to close the system:

\[
E = \frac{p}{\gamma - 1} + \frac{1}{2} \rho u^2
\]

Denote:

\[
U = \begin{bmatrix} \rho \\ u \\ E \end{bmatrix}, \quad f(U) = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ u(E + p) \end{bmatrix}
\]

We get:

\[
\frac{\partial U}{\partial t} + \frac{\partial f(U)}{\partial x} = 0
\]

\[
A = \frac{\partial f}{\partial U} = \begin{bmatrix} 0 & 1 & 0 \\ \frac{(3 - \gamma)u^2}{2} & (3 - \gamma)u & \gamma - 1 \\ \frac{(\gamma - 2)u^3}{2} - \frac{ue^2}{\gamma - 1} & \frac{c^2}{\gamma - 1} + \frac{(3 - 2\gamma)u^2}{2} & \gamma u \end{bmatrix}
\]

where \( c \) is the acoustic velocity, which can be given as:

\[
c^2 = \frac{\gamma p}{\rho}
\]

The matrix \( A \) can be diagonalized:
\[ A = S^{-1}AS \]  

(2.10)

where \( A \) is the Diagonal matrix, \( S \) the transformational matrix.

The eigenvalues of matrix \( A \) are:

\[ \lambda_1 = u, \quad \lambda_2 = u - c, \quad \lambda_3 = u + c. \]  

(2.11)

3. Numerical Methods

3.1. Flux Vector Splitting

For the convection term in Euler equation, the information of the wave is transferred from the upstream to the downstream, so the information of the upstream is more important. The upwind scheme can reflect the physical characteristics of the equation in the propagation direction of wave and flow, which has the natural advantage of expressing the flow characteristics. On the contrary, the central scheme is symmetric, not directional, and requires the addition of artificial viscosity to suppress the shock \[4\].

Euler equation has the following characteristics:

\[ \frac{\partial f(U)}{\partial x} = A \frac{\partial U}{\partial t} \]  

(3.1)

So \( f(U) \) can do flux splitting. The fluxes are divided according to the positive and negative propagation velocity, and the upwind splitting is carried out respectively: forward difference is adopted in the positive direction, and backward difference is adopted in the negative direction \[5\]. Different FVS schemes can be constructed by different splitting methods, among which Steger-Warming and Van Leer schemes are the most representative.

\[ \frac{\partial U}{\partial t} + \frac{\partial f^+}{\partial x} + \frac{\partial f^-}{\partial x} = 0 \]  

(3.2)

3.1.1. Steger-Warming Scheme \[6\]

According to \( A = S^{-1}AS \), \( f \) can be splitted as:

\[ A = A^+ + A^-, \quad A^\pm = S^{-1}A^\pm S, \quad f^\pm = A^\pm U = S^{-1}A^\pm SU, \quad A^\pm = \text{diag}(\lambda_k^\pm), \quad k = 1, 2, 3 \]  

(3.3)

To make the flux splitting of the eigenvalue continuously differentiable at 0:

\[ \lambda_k^\pm = \frac{\lambda_k \pm \sqrt{\lambda_k^2 + \varepsilon^2}}{2} \]  

(3.4)

where \( \varepsilon \) is a very small number.

\[ f^\pm = \frac{\rho}{2\gamma} \begin{bmatrix} 2(\gamma - 1)\lambda_k^\pm + \lambda_k^\pm \left( u + c \right) + \lambda_k^\pm \left( u - c \right) \\left( \gamma - 1 \right)\lambda_k^\pm u^2 + \lambda_k^\pm \left( u - c \right)^2 + \lambda_k^\pm \left( u + c \right)^2 + f^* \end{bmatrix} \]  

(3.5)

\[ f^* = \frac{(3 - \gamma) \left( \lambda_k^+ + \lambda_k^- \right) c^2}{2(\gamma - 1)} \]  

(3.6)

3.1.2. Van Leer Scheme \[7\]

The flux vector is divided according to the local Mach number:

\[ \begin{cases} Ma \geq 1 & f^+ = f, \ f^- = 0 \\ Ma \leq -1 & f^+ = 0, \ f^- = f \end{cases} \]  

(3.7)

when \( |Ma| < 1 \),
where

\[
\mathbf{f} = \begin{bmatrix} \rho u \\ u(\rho u^2 + p) \\ u(E + p) \end{bmatrix}, \quad \mathbf{f}_i^\pm = \pm \rho c \left( \frac{M + 1}{2} \right)^2
\]

\[
(3.8)
\]

3.2. Flux Difference Splitting
Steger-Warming and Van Leer schemes both belong to flow vector splitting method, and the original variables are not decoupled, so the errors of each variable will be transferred to each other. In contrast to the FVS scheme, another development line of the upwind scheme is the FDS (flux difference splitting) scheme, the most famous of which is the Roe scheme[8]. The Roe scheme linearizes the equation by constructing the average growth rate matrix and decoupling the variables. The Roe scheme has high resolution for shock waves and contact discontinuities. However, when the eigenvalues of the Jacobi matrix of flux are small, the entropy condition is not satisfied, and non-physical numerical solutions are generated, so entropy correction is necessary.

For the region \((x_l, x_r)\), the average growth rate matrix can be constructed:

\[
\mathbf{\bar{A}} = \begin{bmatrix} 0 & 1 & 0 \\ -\frac{(3 - \gamma) \bar{u}^2}{2} & (3 - \gamma) \bar{u} & \gamma - 1 \\ \frac{(\gamma - 2) \bar{u}^3}{2} - \frac{\bar{u} \bar{c}^2}{\gamma - 1} - \frac{\bar{c}^2}{\gamma - 1} + \frac{(3 - 2\gamma) \bar{u}^2}{2} & \gamma \bar{u} \end{bmatrix}
\]

\[
(3.10)
\]

where \(\bar{u}\) is the average velocity, \(\bar{c}\) is the average local acoustic velocity:

\[
\bar{u} = \frac{\sqrt{\rho_l} u_l + \sqrt{\rho_r} u_r}{\sqrt{\rho_l} + \sqrt{\rho_r}}
\]

\[
H = \frac{\gamma \rho}{(\gamma - 1) \rho} + \frac{1}{2} u^2, \quad \bar{H} = \frac{\sqrt{\rho_l} H_l + \sqrt{\rho_r} H_r}{\sqrt{\rho_l} + \sqrt{\rho_r}}
\]

\[
\bar{c} = (\gamma - 1) \left( \bar{H} - \frac{\bar{u}^2}{2} \right)
\]

\[
(3.12)
\]

\[
(3.13)
\]

The matrix can be diagonalized:

\[
\mathbf{\bar{A}} = \mathbf{\bar{S}}^{-1} \mathbf{\bar{A}} \mathbf{\bar{S}}
\]

\[
(3.14)
\]

The flux in \((x_i, x_{i+1})\):

\[
f_{i+1/2} = \frac{1}{2} \left[ f(x_i) + f(x_{i+1}) \right] - \frac{1}{2} \mathbf{\bar{S}}^{-1} \mathbf{\bar{A}} \mathbf{\bar{S}} (U_{i+1} - U_i)
\]

\[
(3.15)
\]

\(|\mathbf{\bar{A}}| = \text{diag}(|\lambda_1|, |\lambda_2|, |\lambda_3|)\) may be very small, resulting in the discontinuous derivative in the calculation process and the numerical oscillation. Entropy correction is required:

\[
\lambda_i = \begin{cases} 
|\lambda_i| & |\lambda_i| > \varepsilon \\
\frac{\lambda_i^2 + \varepsilon^2}{2\varepsilon} & |\lambda_i| \leq \varepsilon
\end{cases}
\]

\[
(3.16)
\]
3.3. Elimination of Non-physical Oscillations for Numerical Solutions

In the numerical solution of the discontinuous problem, non-physical numerical oscillations occur near the discontinuity, except for the low accuracy scheme with large dissipation. To solve this problem, the method of adding limiter function and constructing essentially non-oscillatory (ENO) scheme are often used.

3.3.1. NND (Non-Oscillatory Non-Free-Parameter Dissipation Scheme)

In the 1980s, Zhang[9] analyzed the reason why the difference numerical solution would oscillate near the shock wave, and modified the flux by adding the Min mod limiter[10]. Different difference schemes were adopted before and after the shock wave, which met certain relation in the third-order dispersion term, and could effectively eliminate the non-physical oscillation. For the second-order precision scheme, the main term in the corrected equation was the third-order dispersion term. In other words, the dispersion error is greater than the dissipation error. The positive dispersion scheme will generate oscillations on the right side of the discontinuous (wave front). The negative dispersion scheme produces oscillations on the left side of the break (wave rear).

| Table 1 Front and rear dispersion characteristics of shock waves in NND scheme |
|------------------|------------------|------------------|------------------|------------------|
| Wave | Second order central difference $\gamma_3 < 0$ | Wave | Second order upwind difference $\gamma_3 > 0$ |
| Wave front | Wave rear | Wave front | Wave rear |
| $\frac{U^n_j - U^{n+1}_j}{\Delta t} + \frac{f^n_{j+\frac{1}{2}} - f^n_{j-\frac{1}{2}}}{\Delta x} = 0$ | $\frac{f^n_{j+\frac{1}{2}} - f^n_{j-\frac{1}{2}}}{\Delta x} = 0$ |

The flux in $(x_i, x_{i+1})$:

$$f^n_{i+\frac{1}{2}} = f^n_{i+\frac{1}{2}} + f^n_{i+\frac{1}{2}}$$

$$f^n_{i+\frac{1}{2}} = f^n_{i+\frac{1}{2}} + \frac{1}{2} \text{Min mod}(f^n_{i+1} - f^n_{i+2}, f^n_{i+1} - f^n_{i+3})$$

$$f^n_{i+\frac{1}{2}} = f^n_{i+\frac{1}{2}} - \frac{1}{2} \text{Min mod}(f^n_{i+1} - f^n_{i+2}, f^n_{i+2} - f^n_{i+1})$$

where

$$\text{Min mod}(x, y) = \frac{\text{sgn}(x) + \text{sgn}(y)}{2} \min(|x|, |y|)$$

3.3.2. WENO (Weighted Essentially Non-oscillatory Scheme)

WENO scheme is a high-precision shock wave capture method. The basis points are divided into multiple templates, and each template independently approximates the point derivative to obtain multiple differences. Then the weighted average of multiple difference results is made according to the smoothness of each template. The smoother the area is, the greater the weight will be. This paper uses the fifth-order WENO scheme of Jiang and Shu[11]. For the positive flux, take the weighted average of three templates of $(x_{i-2}, x_{i-1}, x_i), (x_{i-1}, x_i, x_{i+1}), (x_{i+1}, x_{i+2}, x_{i+3})$:

$$f^{i+1}_{i+\frac{1}{2}} = \omega_1 f^{i+1}_{i+\frac{1}{2}} + \omega_2 f^{i+1}_{i+\frac{1}{2}} + \omega_3 f^{i+1}_{i+\frac{1}{2}}$$

The three template fluxes are constructed as follows:

$$f^{i+1}_{i+\frac{1}{2}} = \frac{1}{3} f_{i+1} - \frac{7}{6} f_{i+2} + \frac{11}{6} f_i$$

$$f^{i+1}_{i+\frac{1}{2}} = -\frac{1}{6} f_{i-1} + \frac{5}{6} f_i + \frac{1}{3} f_{i+1}$$
where

\[ IS_1 = \frac{1}{4} (f_{i-2} - 4f_{i-1} + 3f_i)^2 + \frac{13}{12} \left( f_{i-2} - 2f_{i-1} + f_i \right)^2 \]

\[ IS_2 = \frac{1}{4} (f_{i-1} - f_{i+1})^2 + \frac{13}{12} (f_{i-1} - 2f_i + f_{i+1})^2 \]

\[ IS_3 = \frac{1}{4} (3f_i - 4f_{i+1} + f_{i+2})^2 + \frac{13}{12} (f_i - 2f_{i+1} + f_{i+2})^2 \]

For negative flux, the three templates are: \( \{x_{i+2}, x_{i+1}, x_i\}, \{x_{i+1}, x_i, x_{i-1}\}, \{x_i, x_{i-1}, x_{i-2}\} \).

4. Numerical Results

4.1. Comparison of Different Schemes of Flux Splitting Methods
Figure 3. Results of different schemes and partial enlargement of density distribution

The distribution of density, pressure and velocity at $t = 2.0s$ are shown respectively (fig. 3). In the specific calculation process, the spatial grid is divided into 1000 ($\Delta x = 0.01$). By comparing the results of different calculation schemes with the exact solution, several schemes adopted are basically consistent with the exact solution. To better see the differences in several schemes, figure 3 shows a partial magnification of the density distribution. First-order Steger-Warming and Van Leer schemes, especially at discontinuities, have large dissipation. The second-order scheme can capture the discontinuities more accurately than the first-order scheme, but there are obvious non-physical oscillations near the discontinuities, especially at the shock wave. Roe scheme is a first-order scheme, which is basically similar to FVS scheme and has a large dissipation. To be more precise, Van Leer scheme and Roe scheme are closer to the exact solution near rarefaction wave than first-order Steger-Warming scheme. These numerical schemes have higher resolution for shock waves than for contact discontinuities.

4.2. Elimination of Non-physical Oscillations
From the previous calculation, it can be concluded that if the low precision scheme with large dissipation is not used in the numerical solution of the discontinuous function, Gibbs phenomenon will occur, that is, non-physical oscillation will occur near the discontinuity. The distribution of density, pressure and velocity at are shown respectively (fig. 4). In the specific calculation process, the spatial grid is divided into 2000. After adding NND scheme and fifth-order WENO scheme to Steger-Warming scheme, compared with second-order Steger-Warming scheme, non-physical shocks were basically eliminated. The capturing accuracy of NND scheme in discontinuity is similar to that of Steger-Warming in the second order, and it has a good suppression of non-physical oscillation near discontinuity. Compared with other schemes, Steger-Warming scheme with the addition of fifth-order WENO scheme modification has a much higher accuracy, and the calculated results are very close to the exact solution.

4.3. Grid Density
In the previous calculation process, the grid points were set at 1000 and 2000 respectively, so there was little difference in the calculation results of each scheme, making it difficult to judge the advantages and disadvantages of each scheme. In this section, the number of grid points is reduced to 50.
Figure 5. Results of sparse grid

With the decrease of the number of grids, the precision of numerical schemes decreases obviously. However, all kinds of computing schemes can still simulate the five regions in the flow field well, and the sensitivity to grid is relatively low, which shows the good performance of the upwind schemes.

4.4. Error Analysis

In the previous calculation process, the Roe scheme has the same accuracy as the first-order scheme of FVS method, the NND scheme has the same accuracy as the second-order scheme of FVS method, and the fifth-order WENO scheme has the highest accuracy. In this section, the errors of each scheme are further calculated to demonstrate the performance of each scheme. In the calculation process, the first-order explicit scheme is used for time step calculation, so for FVS first-order scheme and Roe scheme, \( \Delta t \sim \Delta x \); For FVS second order scheme and NND scheme, \( \Delta t \sim \Delta x^2 \). For the fifth order WENO scheme, \( \Delta t \sim \Delta x^5 \). The error is expressed as the deviation of the numerical solution from the exact solution.

| N   | \( L_1 \) error | \( L_1 \) order | \( L_2 \) error | \( L_2 \) order |
|-----|-----------------|----------------|----------------|----------------|
| 250 | 2.490E-02       | -              | 3.900E-03      | -              |
| 500 | 1.650E-02       | 0.59           | 2.400E-03      | 0.70           |
| 1000| 1.070E-02       | 0.62           | 1.400E-03      | 0.78           |
| 2000| 6.300E-03       | 0.76           | 6.775E-04      | 1.05           |
| 4000| 3.600E-03       | 0.81           | 3.401E-04      | 0.99           |

| N   | \( L_1 \) error | \( L_1 \) order | \( L_2 \) error | \( L_2 \) order |
|-----|-----------------|----------------|----------------|----------------|
| 250 | 2.040E-02       | -              | 3.600E-03      | -              |
| 500 | 1.360E-02       | 0.58           | 2.300E-03      | 0.65           |
| 1000| 8.600E-03       | 0.66           | 1.300E-03      | 0.82           |
N is the number of grid points. $L_1$ error and $L_2$ error are calculated. The Roe scheme is a first-order scheme, and its error is similar to the first-order Van Leer scheme, while the first-order Steger-Warming scheme has a larger error, which is consistent with the results shown in the previous figure, indicating its poor ability to capture rarefaction waves.

### Table 3. Accuracy for second-order schemes

| Scheme                       | N     | $L_1$ error | $L_1$ order | $L_1^*$ error | $L_1^*$ order | $L_2$ error | $L_2$ order |
|------------------------------|-------|-------------|-------------|---------------|---------------|-------------|-------------|
| Second-order Steger-Warming  | 500   | 6.700E-03   | -           | 2.621E-05     | -             | 1.900E-03   | -           |
|                              | 1000  | 4.300E-03   | 0.64        | 6.303E-06     | 2.06          | 1.200E-03   | 0.66        |
|                              | 2000  | 2.100E-03   | 1.03        | 1.426E-06     | 2.14          | 5.394E-04   | 1.15        |
|                              | 4000  | 1.200E-03   | 0.81        | 2.188E-07     | 2.70          | 2.739E-04   | 0.98        |
| Second-order Van Leer        | 500   | 6.500E-03   | -           | 2.519E-05     | -             | 2.000E-03   | -           |
|                              | 1000  | 4.100E-03   | 0.66        | 6.587E-06     | 1.94          | 1.200E-03   | 0.74        |
|                              | 2000  | 2.100E-03   | 0.96        | 1.459E-06     | 2.17          | 5.356E-04   | 1.16        |
|                              | 4000  | 1.100E-03   | 0.93        | 2.228E-07     | 2.71          | 2.641E-04   | 1.02        |
| NND Steger-Warming           | 500   | 5.900E-03   | -           | 3.827E-05     | -             | 1.700E-03   | -           |
|                              | 1000  | 3.400E-03   | 0.8         | 1.105E-05     | 1.79          | 1.100E-03   | 0.63        |
|                              | 2000  | 1.600E-03   | 1.09        | 3.146E-06     | 1.81          | 3.765E-04   | 1.55        |
|                              | 4000  | 7.736E-04   | 1.06        | 8.772E-07     | 1.84          | 1.758E-04   | 1.1         |

$N$ is the number of grid points. $L_1$ error, $L_1^*$ error and $L_2$ error are calculated. $L_1^*$ covers $2.5 \leq x \leq 3$(smooth region). Due to discontinuities, the overall accuracy of the second-order scheme of FVS method is still only about one order, but in the smooth region, the second-order scheme of FVS method can guarantee second-order accuracy. On the whole, NND scheme is better than the second-order scheme of FVS method, but in the smooth region, the accuracy is not as good as FVS schemes.

### Table 4. Accuracy for WENO scheme

| WENO Steger-Warming | N     | $L_1$ error | $L_1$ order | $L_1^*$ error | $L_1^*$ order | $L_2$ error | $L_2$ order |
|---------------------|-------|-------------|-------------|---------------|---------------|-------------|-------------|
| 800                 | 2.900E-03 | -           | 6.50E-05    | -             | 4.69E-2      | -           | 1.10E-03   |
| 1000                | 2.50E-03  | 0.66        | 1.74E-05    | 5.91          | 4.10E-2      | 0.60        | 9.42E-04   | 0.69        |
| 1250                | 1.70E-03  | 1.73        | 6.33E-06    | 4.50          | 2.46E-2      | 2.28        | 5.32E-04   | 2.56        |
\[ L^*_t \text{ covers } 2.5 \leq x \leq 3 \text{ (smooth region)} \]

\[ L^{**}_t \text{ covers } 3.4 \leq x \leq 3.6 \text{ (shock wave)} \]

In general, the Jiang and Shu's fifth-order WENO scheme has less than third-order accuracy, close to fifth-order accuracy in the smooth region, and about 2.5-order accuracy in the discontinuous region.

5. Conclusion

For the problem of Sod's one-dimensional shock tube, a variety of numerical schemes were used for calculation and analysis. The discontinuities make the overall accuracy of all the higher-order computing schemes decrease significantly, among which the overall accuracy of second-order schemes such as NND is reduced to the first order. However, in the smooth region, these schemes show excellent performance with high precision.

Reference

[1] Li W.X. (2003) Shock Wave. In: Yanbin Wu. (Eds.), One-Dimensional Nonsteady Flow and Shock Waves. National Defense Industry Press, Beijing. pp. 295-308
[2] Chorin A.J., Marsden J.E. (2000) Gas Flow in One Dimension. In: June M. (Eds.), A Mathematic Introduction to Fluid Mechanics. Springer-Verlag, New York. pp. 103-145
[3] Zhang .M.Y., Jing S.R., Li G.J. (2006) One-Dimensional Flow of Ideal Compressible Fluid. In: Xinqi L.(Eds.), Higher Engineering Fluid Mechanics. Xi'an Jiaotong University Press, Xi'an. pp. 335-361
[4] Yan C. Euler Equation and Hyperbolic Equation. In: Shujian S.(Eds.), Computational Fluid Dynamics Methods and Applications. Beihang University Press, Beijing, pp. 62-78
[5] Ren Y. X., Chen H.X. (2006) Numerical method for compressible flow. In: Yanqing S. (Eds.), Fundamentals of Computational Fluid Dynamics. Tsinghua University Press, Beijing. pp. 145-177
[6] Steger J.L., Warming R.F. (1981) Flux vector splitting of the inviscid gas dynamic equations with application to finite-difference methods. Journal of Computational Physics, 40(2):263-293.
[7] Van L. B. (1982) Flux vector splitting for Euler equations. Lecture Notes in Physics, 23(2):447-478.
[8] Roe P. L. (1981) Approximate Riemann Solvers, Parameters Vectors and Difference Schemes. Journal of Computational Physics, 135(2):250-258.
[9] Chu Y.Z., Lu L.S., Ma Z.K., Zhang H.X. (1983) Numerical Solution of Supersonic Two-dimensional Turbulent Separation Flow. Journal of Aerodynamics, 01:42-47.
[10] Zhang H. X. (1988) Non-Oscillatory Non-Free-Parameter Dissipation Scheme. Journal of Aerodynamics, 6: 143-165.
[11] Jiang G.S., Wu C.C. (1999) A High-Order WENO Finite Difference Scheme for the Equations of Ideal Magnetohydrodynamics. Journal of Computational Physics, 150(2):561-594.