Two Influence Maximization Games on Graphs Made Temporal∗

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February 22, 2023

Abstract

To address the dynamic nature of real-world networks, we generalize competitive diffusion games and Voronoi games from static to temporal graphs, where edges may appear or disappear over time. This establishes a new direction of studies in the area of graph games, motivated by applications such as influence spreading. As a first step, we investigate the existence of Nash equilibria in 2-player competitive diffusion and Voronoi games on different temporal graph classes. Even when restricting our studies to temporal trees and cycles, this turns out to be a challenging undertaking, revealing significant differences between the two games in the temporal setting. Notably, both games are equivalent on static trees and cycles. Our two main technical results are (algorithmic) proofs for the existence of Nash equilibria in 2-player competitive diffusion and temporal Voronoi games when the edges are restricted not to disappear over time.

Keywords: Temporal Graph Games, Competitive Diffusion Games, Voronoi Games, Nash Equilibria, Temporal Graph Classes

1 Introduction

As graph games help us to reason about a networked world, playing games on graphs is an intensively researched topic since decades. In this work, we focus on competitive games on undirected graphs. Here, some external parties influence a small subset of agents who then spread some information through the network. Typical application scenarios for these occur, for instance, when political parties want to gain influence in a social network or in the context of viral marketing.

Looking at two prominent, somewhat similar representatives, namely competitive diffusion games and Voronoi games, we put forward to model network dynamics more realistically. Specifically, while to the best of our knowledge almost all work on graph games focused on static graphs, we initiate the study of these two games on temporal graphs. Roughly speaking, in a temporal graph, the edge set may evolve over discrete time steps, while the vertex set remains unchanged.

∗An extended abstract of this article appeared in the proceedings of the Thirtieth International Joint Conference on Artificial Intelligence (IJCAI 2021), pages 45–51, 2021.
†Supported by DFG projects MaMu (NI 369/19) and ComSoc-MPMS (NI 369/22).
‡Supported by DFG project MATE (NI 369/17).
yielding time-ordered graph layers with different edge sets. Moving to the temporal setting has dramatic consequences. For example, while competitive diffusion games and Voronoi games are equivalent on static trees and cycles [26] and we understand well their properties in these simple but important special cases, they are no longer equivalent in the temporal case and their properties are much more challenging to analyze.

Our study takes a first step towards understanding both games on temporal paths, trees, and cycles. It turns out that these backbone structures of graphs already confront us with several technically challenging questions when looking for the existence (and computation) of Nash equilibria—one of the most fundamental game-theoretic concepts. We refer to the next section for all formal definitions and examples of the two (temporal) games. Intuitively speaking, in both games one can think of each player having a color and trying to color as many vertices as possible by her own color; the coloring process starts in a vertex freely chosen by each player and acts through the neighborhood relation of the graph. Herein, the distance of a vertex to the start vertices plays a central role. In both games, a player colors all vertices that are closest to her start vertex. Moreover, while in competitive diffusion games a vertex that is at the same distance to two start vertices of competing players can still get one of the two colors, this is not the case for Voronoi games.

Related work. Competitive diffusion games were introduced by Alon et al. [2]. Research on competitive diffusion games so far mainly focused on the existence of Nash equilibria on a variety of graph classes for different numbers of players [2, 8, 17, 23, 24, 25, 27]. Also, the (parameterized) computational complexity of deciding the existence of a Nash equilibrium has been studied [14, 18].

Voronoi games have been originally studied for a one-dimensional or two-dimensional continuous space [1, 4, 8, 9, 15]. There, it is typically assumed that players choose their initial sets of points sequentially and that a player wins the game if a certain fraction of all points is closest to her. Voronoi games on graphs have also been studied on different graph classes for various numbers of players [3, 10, 16, 20, 26, 28]. Again a focus lies on determining for which graphs a Nash equilibrium exists and how to compute one.

From a broader perspective, analyzing games played on graphs from a game-theoretic perspective is an intensively researched topic; recent examples include Schelling games [7, 12], b-matching games [19], or network creation games [11].

As mentioned before, we know of basically no work systematically studying graph games in a temporal setting. The only exception we are aware of is in the context of pursuit-evasion games: Erlebach and Spooner [13] studied the pursuit-evasion game of cops and robbers on some specific temporal graphs, namely so-called edge-periodic graphs. Also different from our studies, their focus was on computing winning strategies for the players. Morawietz et al. [22] and Morawietz and Wolf [21] extended this study, also answering an open question of Erlebach and Spooner [13].

Our contributions. We put forward the study of game-theoretic models on temporal graphs. We do so by generalizing two well-studied (static) graph games to temporal graphs, namely competitive diffusion games and Voronoi games. For the two resulting temporal graph games we analyze the (constructive) existence of Nash equilibria on different temporal graph classes, focusing on different types of temporal paths, trees, and cycles (see Table 1 for an overview of our results). We observe that, in contrast to the static case where both games are equivalent and a Nash equilibrium is guaranteed to exist [23], the games exhibit far more complex dynamics and a quite different behavior on temporal trees and cycles. Our main results are two involved proofs of guaranteed existence of Nash equilibria, namely in temporal diffusion games and
Table 1: Overview of our results. “✓” means that a Nash equilibrium is not guaranteed to exist. “✗” means that a Nash equilibrium always exists. Entries in parentheses are implied by other table cells. See Section 2 for formal definitions. Voronoi games and diffusion games on static trees and static cycles are guaranteed to admit a Nash equilibrium.

|                      | Temporally connected | Monotonically growing | Monotonically shrinking |
|----------------------|----------------------|-----------------------|------------------------|
| **Diffusion**        |                      |                       |                        |
| Temporal Paths       | ✓ (Theorem 3)        | ✓ (Theorem 4)         | ✓ (Theorem 5)          |
| Temporal Trees       | ✓ (Theorem 3)        | ✓ (Theorem 7)         | ✓ (Theorem 8)          |
| Temporal Cycles      | ✓ (Theorem 5)        | ✓ (Theorem 7)         | ✗ (Theorem 8)          |
| **Voronoi**          |                      |                       |                        |
| Temporal Paths       | ✗ (Theorem 82)       | ✓ (Corollary 81)      |                        |
| Temporal Trees       | ✗ (Theorem 83)       | ✓ (Theorem 84)        |                        |
| Temporal Cycles      | ✗ (Theorem 82)       | ✓ (Theorem 85)        | ✗ (Corollary 84)       |

temporal Voronoi games on so-called monotonically growing temporal cycles. On a high level, one conclusion from our work is that temporal Voronoi games seem to exhibit a more intricate behavior than temporal diffusion games.

**Organization of the paper.** In Section 2, we provide some background on temporal graphs and define temporal competitive diffusion and temporal Voronoi games. In Section 3, we analyze temporal competitive diffusion games. We first consider these games on temporal trees (Section 3.1) and afterwards on temporal cycles (Section 3.2). In Section 3.2.3, we present our technically most involved result for diffusion games, i.e., we show that every temporal competitive diffusion game on a monotonically growing temporal cycle admits a Nash equilibrium. After that, in Section 3.2, we turn to temporal Voronoi games. Here, the structure is a bit different. We first show negative results on monotonically shrinking paths and cycles (Section 4.1) and then on temporally connected trees and cycles (Section 4.2). Lastly, in Section 4.3, we study monotonically growing trees and in Section 4.4, we present our most complicated result showing that a Nash equilibrium exists in all temporal Voronoi games on monotonically growing cycles.

## 2 Preliminaries

For $a \leq b \in \mathbb{N}$, let $[a, b] := \{a, a+1, \ldots, b\}$, $[a, b] := \{a, a+1, \ldots, b-1\}$, and $]a, b[ := \{a+1, \ldots, b\}$. Further, let $[n] := [1, n]$ for $n \in \mathbb{N}$. For any proposition $P$, the Iverson bracket $[P]$ evaluates to 1 if $P$ is true and to 0 if $P$ is false.

### 2.1 Temporal graphs

A **temporal graph** $G = (V, (E_t)_{t=1}^{\infty})$ consists of a set of vertices $V$ and a sequence of edge sets $(E_t)_{t=1}^{\infty}$ with $E_t \subseteq \binom{V}{2}$. We call the graph $G_t = (V, E_t)$ the $t$-th layer of $G$ and $G_{\infty} = (V, E_\infty)$ with $E_\infty := \bigcup_{t=1}^{\infty} E_t$ the **underlying graph** of $G$.

If there is an integer $\tau(G)$ such that $E_t = E_\tau$ for all $t \geq \tau(G)$, then the minimum such integer $\tau(G)$ is called the **lifetime** of $G$. Intuitively, this means that the graph stops changing after time $\tau(G)$. Since the temporal games we consider (which will be defined in Section 2.2) always end after some finite amount of time (depending on the temporal graph), it does not
constitute a loss of generality to assume that all our temporal graphs have finite lifetime $\tau$. We will then simply omit specifying $E_t$ for $i > \tau$ (implying that $E_i = E_\tau$ for all $i > \tau$).

A temporal forest (resp. tree, path, cycle) is a temporal graph whose underlying graph is a forest (resp. path, cycle). In the case of paths, we will use the convention that the path runs from left to right and that its vertices are denoted 1, $\ldots$, $n$ in that order. Similarly, we will denote the vertices of a cycle by either 1, $\ldots$, $n$ or 0, $\ldots$, ($n$ − 1) in counterclockwise order.

For the definition of temporal Voronoi games, we require a notion of temporal distance between two vertices. In a temporal graph $G = (V, (E_t)_{t=1}^{\infty})$, we define a temporal walk from a vertex $v_0$ to a vertex $v_d$ as a sequence of tuples $\{(v_0, v_1), (v_1, v_2), t_2, \ldots, (v_{d-1}, v_d), t_d\}$ such that the following properties hold:

1. $t_i < t_{i+1}$ for all $i \in [d-1]$,
2. $\{v_{i-1}, v_i\} \in E_{t_i}$ for all $i \in [d]$.

We refer to $t_d$ as the arrival time of the temporal walk. Moreover, we call a temporal walk from $v_0$ to $v_d$ foremost if there is no temporal walk from $v_0$ to $v_d$ with a smaller arrival time. We now define the temporal distance $\text{td}(u, v)$ from $u$ to $v$ as the arrival time of a foremost walk from $u$ to $v$. If there is no such walk, then set $\text{td}(u, v) = \infty$. Notably, in contrast to the static case, temporal distances are not necessarily symmetric, that is, $\text{td}(u, v) \neq \text{td}(v, u)$ is possible. By convention, we set $\text{td}(v, v) = 0$ for any vertex $v$. For two vertices $u$ and $v$, we say that $u$ reaches $v$ until step $\ell$ if $\text{td}(u, v) \leq \ell$ and that $u$ reaches $v$ in step (or at time) $\ell$ if $\text{td}(u, v) = \ell$.

The set of all vertices reachable from $v$ until time $\ell$ in $G$ is denoted by $\Omega_{\ell}^{G}(v) := \{w \mid \text{td}(v, w) \leq \ell\}$ where the subscript $v$ is omitted if it is clear from context.

A temporal graph $G$ is temporally connected if $\text{td}(u, v) < \infty$ for all vertex pairs $u, v$. Further, we call $G$ monotonically growing if no edge disappears over time, i.e., $E_t \subseteq E_{t+1}$ for all $t$. Symmetrically, $G$ is monotonically shrinking if edges do not appear over time, i.e., $E_{t+1} \subseteq E_t$ for all $t$.

2.2 Games on temporal graphs

We focus on games with two players. Nevertheless, to highlight the nature of our definitions, we directly introduce both games for an arbitrary number of players. Since the games are somewhat similar, we start by making some general definitions for both temporal games before describing the specifics. For a temporal graph $G = (V, (E_t)_{t=1}^{\infty})$ and a number $k \in \mathbb{N}$ of players, $\text{Diff}(G, k)$ (resp. $\text{Vor}(G, k)$) denotes the $k$-player temporal diffusion game (temporal Voronoi game) on the temporal graph $G$, where each player has a distinct color in $[k]$. Moreover, we use the color 0 to which we refer as gray. The strategy space of each player $i \in [k]$ is the vertex set $V$, that is, each player $i$ selects a single vertex $p_i \in V$, which is then immediately colored by her color $i$. If two players pick the same vertex, then it is colored gray. A strategy profile of the game is a tuple $(p_1, \ldots, p_k) \in V^k$ containing the initially chosen vertex of each player. We also use the term position to refer to the vertex $p_i$ chosen by player $i$.

Now, for both games the strategy profile $(p_1, \ldots, p_k)$ determines an initial (partial) coloring of the graph at time $t = 0$. We use $U^t_i(p_1, \ldots, p_k)$ to denote the set of vertices having color $i$ at time $t$. Then $U^0_i(p_1, \ldots, p_k) = \{p_i\}$, unless some other player chose the same vertex, in which case $U^0_i(p_1, \ldots, p_k) = \emptyset$. In each of the two games, the coloring at time $t$ then determines the coloring at time $t+1$ by the specific rules described below.

Let further $u^t_i(p_1, \ldots, p_k) := |U^t_i(p_1, \ldots, p_k)|$ be the number of vertices having color $i$ at time $t$. As vertices will never change their color once they have been colored, the sequence

\footnotetext[1]{We consider foremost walks since earliest arrival seems more natural than other concepts such as fast or shortest in the context of influence spreading. Moreover, we require strict inequality between $t_i$ and $t_{i+1}$ since this follows the static case and the diffusion process more closely.}
If an uncolored vertex is simultaneously reached by two or more players, then it is colored gray. The displayed strategy profile is a Nash equilibrium in Vor($G_t$) for every player $i \in [k]$ plays a best response to the other players in the strategy profile ($p_1, \ldots, p_k$) if for all vertices $p' \in V$ it holds that $u_i(p_1, \ldots, p_{i-1}, p', p_{i+1}, \ldots, p_k) \leq u_i(p_1, \ldots, p_k)$. A strategy profile ($p_1, \ldots, p_k$) is a Nash equilibrium if every player $i \in [k]$ plays a best response to the other players. Any strategy profile of the form ($p, p, \ldots, p$) in which all players choose the same vertex is called trivial.

It remains to specify how the strategy profile ($p_1, \ldots, p_k$) determines the coloring of the vertices $V$ in the two games.

**Temporal diffusion games.** In a temporal diffusion game, the temporal graph $G$ is colored by the following propagation process over time. We call a vertex uncolored if no color has been assigned to it (so far). In step $t$, we consider the layer $G_t$. We color a so far uncolored vertex $v$ with color $i \in [k]$ if $v$ has at least one neighbor in $G_t$ that is colored with color $i \in [k]$ and no neighbor in $G_t$ that is colored with any other color $j \in [k] \setminus \{i\}$. Every uncolored vertex with at least two neighbors in $G_t$ colored by two different colors $i, j \in [k]$ is colored gray.

**Temporal Voronoi games.** In a temporal Voronoi game, at time $t$ a vertex $v$ is colored with color $i \in [k]$ if $p_i$ is the only position of a player that can reach $v$ until time $t$ by a temporal walk. (If an uncolored vertex is simultaneously reached by two or more players, then it is colored gray.) In effect, $v \in U_i(p_1, \ldots, p_k)$ if and only if $t \delta(p_i, v) < t \delta(p_j, v)$ holds for all $j \neq i$.

Note that we defined temporal diffusion games and temporal Voronoi games such that both temporal games played on a temporal graph $G$ with lifetime 1 are equivalent to the (non-temporal) game played on the static graph $G_1$. An example of a temporal diffusion game and a temporal Voronoi game is shown in Figure 1. Notably, in the displayed temporal diffusion game each player colors a superset of the vertices they color in the temporal Voronoi game. In fact, by definition of the two games, this holds for every temporal graph (as in the static case).

**Difference games.** We also define difference versions of both temporal diffusion and Voronoi games, as they will prove useful in analyzing the classical versions. We only define these for $k = 2$ players, since there is no single canonical generalization to more players. The key difference in
difference games is that the payoff of player 1 becomes $\Delta(p_1, p_2) := u_1(p_1, p_2) - u_2(p_1, p_2)$ and the payoff of player 2 $\Delta(p_2, p_1) = -\Delta(p_1, p_2)$. Note that this turns the game into a zero-sum game. Additionally, in these games, the second player is forbidden to select the same vertex as the first player — otherwise she would always be able to force a tie.

We denote the difference versions of temporal diffusion and Voronoi games by $\Delta\text{Diff}(G, 2)$ and $\Delta\text{Vor}(G, 2)$, respectively. We also set $\Delta^t(p_1, p_2) := u_1^t(p_1, p_2) - u_2^t(p_1, p_2)$.

It is interesting to note that, in two-player games where all vertices are eventually colored (possibly in gray), difference games are equivalent to classical games in which the players are each awarded half a point for gray vertices.

3 Temporal diffusion games

As already mentioned, we only consider the 2-player setting here and in the rest of the paper. We start with temporal trees and then move to temporal cycles.

3.1 Temporal trees

We first prove that a Nash equilibrium is in general not guaranteed to exist in temporal diffusion games, not even on temporal paths. Afterwards, we show that as soon as a temporal tree is temporally connected, every game admits a Nash equilibrium.

We start by showing that the temporal diffusion game on the temporal path depicted in Figure 1 (where every edge of the underlying graph only occurs in one layer) does not admit a Nash equilibrium. This is in contrast to the static case where a Nash equilibrium is guaranteed to exist on every path [23, Theorem 1].

**Theorem 1.** There is a temporal path $P$ such that there is no Nash equilibrium in $\text{Diff}(P, 2)$.

**Proof.** We prove the theorem by showing that for the temporal path $P = ([6], (E_1, \ldots, E_5))$ with lifetime 5 and $E_t = \{\{t, t + 1\}\}$ for $t \in [5]$ (see Figure 1), there is no Nash equilibrium in $\text{Diff}(P, 2)$. For the sake of contradiction, assume that $(p_1, p_2)$ with $p_1 < p_2$ is a Nash equilibrium in $\text{Diff}(P, 2)$. Player 1 colors all vertices in $[\max(1, p_1 - 1), p_2 - 1]$ and player 2 all vertices in $[p_2, n]$.

Clearly, $p_2 = p_1 + 1$ must hold, as otherwise player 2 can color additional vertices by moving to vertex $p_1 + 1$. Moreover, if $p_1 \geq 3$, then player 1 can additionally color vertex 1 by moving to vertex 1. Thus, $p_1 \in \{1, 2\}$, $p_2 = p_1 + 1$, and thereby $p_2 \in \{2, 3\}$. However, this implies that player 1 colors at most two vertices and can increase her payoff by moving to vertex 4, thereby coloring three vertices. \hfill $\square$

3.1.1 Temporally connected trees

If all vertices can pairwise reach each other (i.e., if the graph is temporally connected), then a Nash equilibrium for temporal diffusion games is guaranteed to exist on temporal trees.

An important observation to be made here is that when the two players pick vertices $p_1$, $p_2$, then player 1 will color (at least) all the connected components of $T_i - p_1$ except for the one containing $p_2$. Symmetrically, this also holds for player 2.

We can thus observe the following.

**Lemma 2.** Let $T$ be a temporally connected tree and let $(p_1, p_2)$ be any Nash equilibrium of $\text{Diff}(T, 2)$. Then $p_1$ and $p_2$ must be adjacent.


Figure 2: A monotonically shrinking temporal path $P$ for which neither in $\text{Diff}(P, 2)$ nor in $\text{Vor}(P, 2)$ a Nash equilibrium exists.

Proof. If $p_1, p_2$ are not adjacent, then let $x \notin \{p_1, p_2\}$ be any vertex on the path connecting $p_1$ and $p_2$. Without loss of generality, $x \notin U_1(p_1, p_2)$. Then $U_1(x, p_2) \supset U_1(p_1, p_2)$, i.e., player 1 can improve by moving to $x$.

Note that in the situation where both players start on adjacent vertices, the payoffs are identical to the static diffusion game played on $T_{\downarrow}$. From Lemma 2, it is then not hard to see that a Nash equilibrium must always exist and that it will always be located at the centroid of $T_{\downarrow}$ (this is a vertex $c$ minimizing the maximum size of any component of $T_{\downarrow} - c$).

Theorem 3. Let $T$ be a temporally connected temporal tree. Then $\text{Diff}(T, 2)$ contains a Nash equilibrium which can be computed in linear time.

The formal proof of Theorem 3 proceeds exactly as in the static case ([23, Thm. 4]), thus we omit it. Note that the centroid of a graph can be computed in linear time by a folklore dynamic programming algorithm.

We remark that Theorem 3 can also be extended to disjoint unions of temporally connected trees. We can then simply apply Theorem 3 to a largest tree. If the resulting payoff for either player is less than the size of the second-largest tree, then they will simply switch to that tree. In either case the resulting strategy profile is a Nash equilibrium.

3.1.2 Monotonically shrinking paths

We have seen above that a Nash equilibrium is guaranteed to exist on temporally connected and thereby also on monotonically growing temporal paths. It turns out that enforcing the opposite, that is, the graph is monotonically shrinking, does not guarantee the existence of a Nash equilibrium:

Theorem 4. There is a monotonically shrinking temporal path $P$ consisting of two layers which only differ in one edge such that there is no Nash equilibrium in $\text{Diff}(P, 2)$.

Proof. Let $P = ([8], (E_1, E_2))$ (of lifetime 2) with $E_1 = \{\{i, i + 1\} | i \in [7]\}$ and $E_2 = E_1 \setminus \{\{2, 3\}\}$ (see Figure 2). For the sake of contradiction, assume that $(p_1, p_2)$ is a Nash equilibrium in $\text{Diff}(P, 2)$. We distinguish two cases:

1) $p_1 \in [3]$ or $p_2 \in [3]$: Let $p_1 = i \in [3]$. Then, $p_2 = i + 1$, as this is the unique best response of player 1. Player 1 can improve by deviating to vertex $i + 2$.

2) $p_1 \in [4, 8]$ and $p_2 \in [4, 8]$: If $p_1 = i \in [4, 5]$, then $p_2 = i + 1$ is the unique best response of player 2 from the relevant interval $[4, 8]$. Player 1 can improve by choosing vertex 3. If $p_1 = 6$, then $p_2 = 3$ is the unique best response of player 2. If $p_1 = i \in [7, 8]$, then $p_2 = i - 1$ is the unique best response of player 2 from $[4, 8]$. Player 1 can improve by choosing vertex 3.

Intuitively, the reason why a disappearing edge is enough to prevent the existence of a Nash equilibrium on a temporal path is that players may want to play in the immediate surrounding of such a disappearing edge, in order to color some part of the temporal path that otherwise
remains uncolored (in Figure 2, these are the vertices \{1, 2\}). However, if the disappearing edge is not located around the center, then the player close to this edge is at risk of losing many vertices to the other player. Such possibly “lost” vertices do not appear in monotonically growing temporal graphs where edges are not allowed to disappear.

3.2 Temporal cycles

In this section, we prove that in contrast to paths, a Nash equilibrium may fail to exist on temporally connected cycles and monotonically shrinking cycles. However, enforcing that edges do not disappear over time is enough to guarantee the existence of a Nash equilibrium.

3.2.1 Temporally connected cycles

The guaranteed existence of a Nash equilibrium on temporally connected trees (Theorem 3) does not extend to temporally connected cycles despite the fact that still all vertices will ultimately be colored. This can be shown using the graph depicted in Figure 1 with an additional layer connecting all vertices to a cycle and a similar argument as in Theorem 1.

**Theorem 5.** There is a temporally connected temporal cycle \( C \) such that there is no Nash equilibrium in \( \text{Diff}(C, 2) \).

**Proof.** Let \( C = ([n], (E_1, \ldots, E_n)) \) be the temporal cycle of size \( n \geq 6 \) with \( E_t = \{\{t, t+1\}\} \) for \( t < n \) and \( E_n = \{\{n, 1\}\} \cup \{\{i, i+1\} \mid i \in [n-1]\} \) of lifetime \( n \). We prove the statement by showing that there is no Nash equilibrium on \( C \). Let \( (p_1, p_2) \) be a strategy profile in \( \text{Diff}(C, 2) \) with \( p_1 < p_2 \). Then, player 1 colors all vertices in \([p_1, p_2 - 1]\) and player 2 colors all vertices in \([p_2, n]\). If \( p_1 > 1 \), then player 1 colors vertex \( p_1 - 1 \). If \( p_1 > 2 \), then the vertices \([1, p_1 - 2]\) are distributed evenly between the players with one vertex being colored gray if the number of vertices in \([1, p_1 - 2]\) is odd. We consider two different cases.

1. First, assume that \( p_2 \neq p_1 + 1 \). By moving to vertex \( p_1 + 1 \) player 2 colors more vertices than before.
2. Second, assume that \( p_2 = p_1 + 1 \). If \( p_1 \geq 3 \), then at least one vertex in \([1, p_1]\) is colored gray or by player 2. Thus, player 1 can color additional vertices by moving to vertex 1. Otherwise, it holds that \( p_1 \leq 2 \). In this case, player 1 colors at most two vertices. Since we assumed that \( p_2 = p_1 + 1 \), it follows that \( p_2 \leq 3 \). By moving to vertex 4 player 1 colors all vertices in \([4, n]\). Since \( n \geq 6 \), these are at least three vertices, so that vertex 4 is better for player 1 than her current vertex.

3.2.2 Monotonically shrinking cycles

The example in Figure 2 can be modified to show that enforcing edges to only disappear over time is not enough to guarantee the existence of a Nash equilibrium on a temporal cycle (as in the case of temporal paths).

**Theorem 6.** There is a monotonically shrinking temporal cycle \( C \) with lifetime 2 such that there is no Nash equilibrium in \( \text{Diff}(C, 2) \).
Proof. We prove the theorem by considering a temporal cycle (see Figure 3) similar to the monotonically shrinking temporal path without Nash equilibrium used in Theorem 4.

Specifically, let $C = ([11], (E_1, E_2))$ (of lifetime 2) with $E_1 = \{\{i, i + 1\} \mid i \in [10]\} \cup \{\{11, 1\}\}$ and $E_2 = E_1 \setminus \{\{1, 2\}, \{3, 4\}, \{9, 10\}, \{10, 11\}, \{11, 1\}\}$. For the sake of contradiction, let $(p_1, p_2)$ be a Nash equilibrium in $\text{Diff}(C, 2)$.

1. $p_1 \in [2]$ or $p_2 \in [2]$: Let $p_1 = i \in [2]$. Then there exist two best responses of player 2, that is vertex 3 and vertex 10: If $p_2 = 3$, then player 1 can improve by deviating to vertex 4. If $p_2 = 10$, then player 1 can improve by moving to vertex 9.

2. $p_1 \in [3, 4]$ or $p_2 \in [3, 4]$: Let $p_1 = i \in [3, 4]$. Then, $p_2 = i + 1$ is the unique best response of player 2. Player 1 can improve by moving to vertex $i + 2$.

3. $p_1 = 10$ or $p_2 = 10$: Let $p_1 = 10$. Then, $p_2 = 9$ is the unique best response of player 2. Player 1 can improve by moving to vertex 8.

4. $p_1 = 11$ or $p_2 = 11$: Let $p_1 = 11$. Then, $p_2 = 3$ or $p_2 = 4$ are the best responses of player 2. In both cases, Player 1 can improve by moving to vertex 5.

5. $p_1 \in [5, 9]$ and $p_2 \in [5, 9]$: If $p_1 = i \in [5, 6]$, then $p_2 = i + 1$ is the unique best response of player 2 from $[5, 9]$. Player 1 can improve by choosing vertex 4. If $p_1 = 7$, then $p_2 = 4$ is the unique best response. If $p_1 \in [8, 9]$, then the best responses of player 2 from $[5, 9]$ is $p_2 = i - 1$. Player 1 can improve by choosing vertex 4.

\[\Box\]

3.2.3 Monotonically growing cycles

If we require that edges do not disappear over time, then a Nash equilibrium is again guaranteed to exist:

Theorem 7. On every monotonically growing temporal cycle $C = ([n], (E_i)_{i \in [\tau]})$ a Nash equilibrium in $\text{Diff}(C, 2)$ exists and can be found in $O(\tau \cdot n)$ time.

Proving Theorem 7 will be the goal of this subsection. We split the proof into three parts. In Part I, we start by first studying temporal difference diffusion games which turn out to be simpler. In Part II, we formally analyze the relationship between Nash equilibria in temporal difference diffusion games and Nash equilibria in temporal diffusion games. Generally speaking, we prove that for every monotonically growing temporal cycle $C$, at least one Nash equilibrium of those found in $\Delta\text{Diff}(C, 2)$ is also a Nash equilibrium in $\text{Diff}(C, 2)$. We put together all our results in Part III, where we describe an algorithm that constructs a Nash equilibrium for each monotonically growing temporal cycle.

The relevance of temporal difference diffusion games for finding Nash equilibria of temporal diffusion games can be motivated by the following observation.

Observation 8. For every non-trivial strategy profile $(p_1, p_2)$ and monotonically growing temporal cycle $C$, no vertex is uncolored and at most two vertices are colored gray in $\text{Diff}(C, 2)$. 

Figure 3: A monotonically shrinking temporal cycle $C$ for which no Nash equilibrium in $\text{Diff}(C, 2)$ exists.
Proof. Since $C$ is monotonically growing, layer $\tau$ of $C$ is a cycle. Consequently, the players can eventually spread their color on the cycle until they are stopped by a vertex that is already colored. Since each player starts from only one position, vertices colored by the different players “meet” at exactly two places. At each place, at most one vertex can be colored gray. Clearly, no uncolored vertices remain. $\square$

In the remainder of this subsection, we will assume that the given monotonically growing temporal cycle $C = ([n], (E_t)_{t \in \tau_C})$ has $n > 1$ vertices and lifetime $\tau > 1$ (otherwise insert an edgeless layer at the beginning). Moreover, we may assume by symmetry that the edge $\{n, 1\}$ only appears in layer $\tau$, i.e., $\{n, 1\} \notin E_{\tau-1}$. We denote by $\overline{F} = \overline{F}(C)$ the temporal linear forest (i.e., disjoint union of paths) obtained from $C$ by replacing layers $\tau, \tau + 1, \ldots$ by empty layers. Recall that $\Omega_C^\ell(v)$ is the set of vertices reachable from $v$ until time $\ell$ in $G$. Note that

$$\Omega_{\overline{F}}^\ell(v) = \Omega_{\overline{F}}^{\min(t,\tau-1)}(v) = \Omega_C^{\min(t,\tau-1)}(v)$$

holds for all vertices $v$ and times $t$. For the sake of brevity we will abbreviate $\Omega_C^{\tau-1}(v) = \Omega_{\overline{F}}^{\tau-1}(v)$ as $\Omega(v)$.

Recall that for two vertices $u, v \in V$ we say that $u$ reaches $v$ until step $\ell$ if $td(u, v) \leq \ell$.

Part I: Temporal difference diffusion games. We show how to find a Nash equilibrium in a temporal difference diffusion game on a monotonically growing temporal cycle.

We can simplify this problem by considering temporal difference diffusion games on $\overline{F}$ due to the following lemma.

Lemma 9. A strategy profile is a Nash equilibrium in $\DeltaDiff(C, 2)$ if and only if it is a Nash equilibrium in $\DeltaDiff(\overline{F}(C), 2)$.

Proof. Since each player starts spreading her color from one position and since layer $\tau$ of $C$ is a cycle, both players color the same number of vertices from step $\tau$ on. It follows that $\Delta_\tau^{-1}(u, v) = \Delta(u, v)$ for all $u, v \in [n]$ in $C$. Consequently, in $C$, a player can improve $\Delta(p_1, p_2)$ by changing her position if and only if she can improve $\Delta_\tau^{-1}(p_1, p_2)$ by changing her position. Clearly, $\Delta_\tau^{-1}$ in $C$ is exactly the payoff in $\DeltaDiff(\overline{F}(C), 2)$. $\square$

Note that when we use $\Delta_\tau^{-1}(u, v)$ in the following we will not always state explicitly the graph on which we play, as this will always be $\overline{F}(C)$ (and we anyway have that $\Delta_\tau^{-1}(u, v)$ in $C$ is equal to $\Delta_\tau^{-1}(u, v)$ in $\overline{F}(C)$).

Thus, in the following we focus on temporal difference diffusion games on $\overline{F}(C)$. In preparation, we prove two elementary observations about reachability in temporal forests.

Observation 10. Let $u, v, \alpha \in [n]$ be three vertices of $\overline{F}$ and $t \in \mathbb{N}$. If $td(u, \alpha) \leq t$ and $td(v, u) \leq t - |\alpha - u|$, then $td(v, \alpha) \leq t$.

Proof. If $td(u, \alpha) < td(v, \alpha)$, then every edge between $u$ and $\alpha$ must already be present when $\alpha$ is reached from $v$. Thus we then have $td(v, \alpha) \leq td(v, u) + |\alpha - u| \leq t - |\alpha - u| + |\alpha - u| = t$. $\square$

Observation 11. In $\overline{F}$ it holds for all vertices $u \leq w \leq v$ that $td(v, w) \leq td(u, v) + |v - w| - 1$.

Proof. All edges between $u$ and $v$ (and thus between $w$ and $v$) are present at time $td(u, v)$ and all later times. $\square$

It turns out that there exists a special type of vertices (called nice vertices) from which a Nash equilibrium in $\overline{F}$ can be easily constructed. These nice vertices are defined as follows. (Recall that a centroid of a path is a vertex located in the middle of it; if the path has an even number of middle vertices then both middle vertices are centroids.)
**Definition 12.** A vertex $v$ is called nice if it is a centroid of $\Omega(v)$ and maximizes the cardinality of that set.

For an example of a nice vertex, see Figure 4. We first show that at least one nice vertex exists in $\mathcal{F}$. For this, we use the following lemma.

**Lemma 13.** Let $v$ be any vertex and let $m$ be a central vertex of $\Omega(v)$. Then, $\Omega(m) \supseteq \Omega(v)$.

**Proof.** Let $\Omega(v) := [\alpha, \beta]$. We can assume that $v < m$ by symmetry. It suffices to show that $m$ reaches $\alpha$ and $\beta$ until time $\tau - 1$. In order to show this, for technical reasons, in the following, we work on the monotonically growing temporal linear forest $\mathcal{F}$ obtained from $\mathcal{F}$ by replacing the empty layers $\tau, \tau + 1, \ldots$ by layer $\tau - 1$ (note that $\mathcal{F}$ and $\mathcal{F}$ are identical in the first $\tau - 1$ layers and thus in particular $m$ reaches the same vertices until step $\tau - 1$ in both graphs). By Observation 11 in $\mathcal{F}$

$$\text{td}(m, v) \leq \text{td}(v, m) + |v - m| - 1. \quad (1)$$

Since $v$ reaches $\beta$ until time $\tau - 1$ and since $v$ has to pass $m$ in order to reach $\beta$, we conclude that $v$ reaches $m$ until time $\tau - 1 - |m - \beta|$. Applying this to Equation (1), we get

$$\text{td}(m, v) \leq \tau - 1 - |m - \beta| + |v - m| - 1. \quad (2)$$

Since $m$ is a central vertex of $[\alpha, \beta]$, it holds that $|m - \beta| + 1 \geq |m - \alpha|$. Applying this to Equation (2), we get

$$\text{td}(m, v) \leq \tau - |\alpha - m| + |v - m| - 1$$

$$= \tau - |\alpha - v| - 1.$$

Since $m$ reaches $v$ until time $\tau - |\alpha - v|$, we conclude by Observation 10 that $m$ reaches $\alpha$ until time $\tau - 1$. Also $m$ clearly reaches $\beta$ until time $\tau - 1$ since $v$ does so and $v < m \leq \beta$.

From Lemma 13, the guaranteed existence of a nice vertex directly follows:

**Lemma 14.** There exists at least one nice vertex in $\mathcal{F}$.

**Proof.** We construct a nice vertex as follows. Let

$$R := \max_{v \in [n]} |\Omega(v)|$$

be the maximum number of vertices reachable from any vertex in $\mathcal{F}$. Let $v \in [n]$ be a vertex that reaches $R$ vertices in $\mathcal{F}$ and let $m$ be a central vertex of $\Omega(v)$. By Lemma 13, $\Omega(m) \supseteq \Omega(v)$ and since the latter set is maximal, equality must hold. Consequently, $m$ is a nice vertex.

Intuitively, it seems to be a good strategy for a player to play on a nice vertex $v$, as she colors as many vertices as possible in $\Delta\text{Diff}(\mathcal{F}, 2)$ in the absence of a second player “stealing” vertices from her and the other player can “steal” at most half of these vertices (by playing adjacent to her). In fact, we will show that if a player plays on a nice vertex, she is guaranteed to color at least as many vertices as the other player in $\Delta\text{Diff}(\mathcal{F}, 2)$.

To prove this, we need the following lemma about how the sets of vertices reachable by the players determine their payoffs.

**Lemma 15.** Let $u, v \in [n]$ and $[a_u, b_u] := \Omega(u)$ and $[a_v, b_v] := \Omega(v)$. If $a_u < a_v \leq b_u < b_v$, then in $\Delta\text{Diff}(\mathcal{F}, 2)$ we have $U_1(u, v) = [a_u, [x] - 1]$ and $U_2(u, v) = [[x] + 1, b_v]$ where $x = (b_u + a_v)/2$.  

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Proof. Since \( a_v \leq b_u < b_v \), it follows that the edge \( \{b_u, b_v + 1\} \) appears no later than time \( \tau - 1 \).
Thus, we have \( td(u, b_u) = \tau - 1 \) and analogously \( td(v, a_v) = \tau - 1 \). Also, we clearly have
\( td(u, b_u - 1) \leq \tau - 2 \) and \( td(v, a_v + 1) \leq \tau - 2 \). If \( a_v = b_u \) or \( a_v + 1 = b_u \), then we are done.
Otherwise, \( a_v + 1 \leq b_u - 1 \) and \( a_v + 1 \) and \( b_u - 1 \) are the furthest vertices reached by \( v \) resp. \( u \) until time \( \tau - 2 \). In that case delete layer \( \tau - 1 \) from \( \mathcal{F} \) and recursively apply this lemma to see that
\( U_1^{-2}(u, v) = [a'_u, x'] - 1 \) and \( U_2^{-2}(u, v) = [x' + 1, a'_v] \) where \( a'_u \leq u, b'_v \geq v \), and \( x' = (b_u + a_v + 1)/2 = x \). From this the claim follows. \( \square \)

**Lemma 16.** Let \( u < v \in \{n\} \) such that neither of \( \Omega(u) \) and \( \Omega(v) \) contains the other. Then
\( \Delta^{\tau-1}(u, v) = |\Omega(u)| - |\Omega(v)| \).

Proof. If \( \Omega(u) \cap \Omega(v) = \emptyset \), then each player colors exactly the vertices the player reaches and the
statement follows. Thus, it remains to consider the case \( \Omega(u) \cap \Omega(v) \neq \emptyset \). Using the notation of Lemma \( \text{[15]} \),
this and \( u < v \) implies that \( a_u < a_v \leq b_u < b_v \). Thus, applying Lemma \( \text{[15]} \) we have that
\[
\Delta^{\tau-1}(u, v) = [x - 1] - a_u + 1 - (b_v - [x + 1] + 1) \\
= 2x - a_u - b_v \\
= b_u + a_v - a_u - b_v \\
= |\Omega(u)| - |\Omega(v)|. 
\]

Using Lemma \( \text{[16]} \) we are now ready to show that if a player plays on a nice vertex, then
the other player can color at most the same number of vertices.

**Lemma 17.** If \( u \) is a nice vertex, then \( \Delta^{\tau-1}(u, v) \geq 0 \) for all \( v \in \{n\} \).

Proof. If \( u = v \) then the result is obvious. If \( u \neq v \), first consider the case that that \( \Omega(v) \subseteq \Omega(u) \).
Since \( u \) is a central vertex of \( \Omega(u) \), player 1 colors at least half the vertices of \( \Omega(u) \). Thereby,
player 2 cannot color more vertices than player 1, thus \( \Delta^{\tau-1}(u, v) \geq 0 \).

Otherwise, as \( u \) is a nice vertex, \( \Omega(u) \) and \( \Omega(v) \) must be incomparable. In this case, it holds
that \( \Delta^{\tau-1}(u, v) = |\Omega(u)| - |\Omega(v)| \) by Lemma \( \text{[16]} \). Since \( u \) is a nice vertex, \( v \) cannot reach more
vertices than \( u \), that is \( \Delta^{\tau-1}(u, v) \geq 0 \). \( \square \)

In the following, we construct a Nash equilibrium using nice vertices. With Lemma \( \text{[17]} \), it
is easy to show that each strategy profile where both players play on different nice vertices is a
Nash equilibrium:

**Lemma 18.** If \( u \) and \( v \) are different nice vertices, then \( (u, v) \) is a Nash equilibrium in
\( \Delta^{\tau}(\mathcal{F}, 2) \).

Proof. Since \( u \) and \( v \) are both nice vertices, we conclude by Lemma \( \text{[17]} \) that \( \Delta^{\tau}(u, v) = 0 \).
Additionally, by Lemma \( \text{[17]} \) no player can improve her payoff in \( \Delta^{\tau}(\mathcal{F}, 2) \) by moving to a
different position. As a result, \( (u, v) \) is a Nash equilibrium. \( \square \)

Next, we show that if only one nice vertex exists, then the nice vertex and any vertex
adjacent to it form a Nash equilibrium (see Figure \( \text{[3]} \) for an example). In order to prove that
such a strategy profile is always a Nash equilibrium, we need the following lemma:

**Lemma 19.** Assume that \( m \in \{n\} \) is a central vertex of \( \{\alpha, \beta\} \subseteq \Omega(m) \) and that the number
of vertices in \( \{\alpha, \beta\} \) is odd and larger than 1. Then \( \Omega(m + 1) \supseteq [\alpha + 1, \beta] \).
Figure 4: A temporal difference diffusion game on $F$ where only one nice vertex exists. The nice vertex is vertex 5 which is the position of player 1. Player 2 colors only one vertex less than player 1 by playing directly next to her on vertex 6. This strategy profile is a Nash equilibrium.

Proof. Clearly $\Omega(m + 1) \supseteq [m + 1, \beta]$. As $m$ is the unique central vertex of $[\alpha, \beta]$, it holds that $|m + 1 - \beta| = |\alpha + 1 - m|$. Since $m$ reaches $\beta$ until time $\tau - 1$, it follows that $m$ reaches $m + 1$ (and thus also $m + 1$ reaches $m$) no later than at time

$$\tau - 1 - |m + 1 - \beta| = \tau - 1 - |\alpha + 1 - m|.$$ 

From this we can derive by Observation 10 that $m + 1$ reaches $\alpha + 1$ until time $\tau - 1$.

Using Lemma 19 we show that if player 1 plays on the unique nice vertex and player 2 plays next to player 1, then player 1 colors exactly one vertex more than player 2. Using this, we prove that the strategy profile is a Nash equilibrium.

**Lemma 20.** If there is a unique nice vertex $u$, then $(u, u + 1)$ is a Nash equilibrium in $\Delta\text{Diff}(F, 2)$.

Proof. Let $[\alpha_u, \beta_u] := \Omega(u)$. Then $R := |\Omega(u)|$ is the maximum number of vertices reachable from any vertex in $F$. Note that $R > 1$ since we assume $n > 1$.

Observe that $u < n$ (as $u$ is the unique nice vertex) and write $v := u + 1$. We start by computing $\Delta^{-1}(u, v)$. Since $u$ is the only nice vertex, by Lemma 19 $\Omega(v) \supseteq [\alpha_u + 1, \beta_u]$. We have $\beta_u + 1 \notin \Omega(v)$ or $v$ would be a nice vertex. As $R$ is odd, player 2 colors the $\frac{R - 1}{2}$ vertices in $[v, \beta_u]$ and player 1 colors the $\frac{R - 1}{2} + 1$ vertices in $[\alpha_u, u]$. That is, $\Delta^{-1}(u, v) = 1$.

Suppose now for the sake of contradiction that player 2 can improve her payoff by moving from $v$ to some vertex $v' \neq u$ with $\Delta^{-1}(u, v') < 1$. We consider two cases.

1. First, assume that $\Omega(u)$ and $\Omega(v')$ are incomparable. By Lemma 19 it holds that $\Delta^{-1}(u, v') = |\Omega(u)| - |\Omega(v')|$. Since there is only one nice vertex, it needs to hold that $|\Omega(v')| < R$. It follows that $\Delta^{-1}(u, v') \geq 1$ leading to a contradiction.

2. Otherwise, it holds that $\Omega(u) \subseteq \Omega(v')$ or $\Omega(v') \subseteq \Omega(u)$. Since $|\Omega(u)| = R$, it cannot hold that $\Omega(u) \subset \Omega(v')$. Consequently, $\Omega(v') \subseteq \Omega(u)$. Since $u$ is the central vertex of $\Omega(u)$, player 1 colors at least $\frac{R - 1}{2} + 1$ vertices and player 2 colors at most $\frac{R - 1}{2}$ vertices, that is, $\Delta^{-1}(u, v') \geq 1$, which is a contradiction.

For the sake of contradiction, assume that player 1 can improve her payoff by moving from $u$ to $u'$, that is, $\Delta^{-1}(u', v) > 1$. We consider two cases.

1. First, assume again that $\Omega(u')$ and $\Omega(v)$ are incomparable. By Lemma 19 it holds that $\Delta^{-1}(u', v) = |\Omega(u')| - |\Omega(v)|$. We showed before that $|\Omega(v)| \geq R - 1$. Since $R$ is the maximum number of vertices reachable from any vertex in $F$, it holds that $|\Omega(u')| \leq R$. Hence, $\Delta^{-1}(u', v) = |\Omega(u')| - |\Omega(v)| \leq 1$ leading to a contradiction.
2. Otherwise, it holds that $\Omega(u') \subseteq \Omega(v)$ or $\Omega(v) \subseteq \Omega(u')$. First, assume that $\Omega(u') \subseteq \Omega(v)$. Recall that we have argued above that $[\alpha_u + 1, \beta_u] \subseteq \Omega(v)$. Since $v = u + 1$, $v$ reaches $\frac{R-1}{2}$ vertices to the right of $v$ and at least $\frac{R-1}{2}$ vertices to the left of $v$ in $\mathcal{F}$. Thus, independent of whether $u'$ is left or right of $v$, player 2 colors at least $\frac{R-1}{2}$ vertices and, as $\Omega(u') \subseteq \Omega(v)$, player 1 colors at most $\frac{R-1}{2} + 1$ vertices. It follows that $\Delta^{\tau-1}(u', v) \leq 1$.

We now have everything at hand to find a Nash equilibrium in $\Delta\text{Diff}(\mathcal{F}, 2)$. By Lemma 9 this result can be directly extended to temporal difference diffusion games on monotonically growing temporal cycles.

**Lemma 21.** If $u$ and $v$ are different nice vertices, then $(u, v)$ is a Nash equilibrium in $\Delta\text{Diff}(\mathcal{C}, 2)$.

Otherwise, let $v$ be the unique nice vertex. Then, $(v, v+1)$ is a Nash equilibrium in $\Delta\text{Diff}(\mathcal{C}, 2)$.

**Proof.** By Lemmas 14, 18 and 20, the claims hold for $\Delta\text{Diff}(\mathcal{F}, 2)$. By Lemma 9 this result transfers to $\Delta\text{Diff}(\mathcal{C}, 2)$.

**Part II: From difference diffusion games to diffusion games.** In Part I, we described how to find Nash equilibria in temporal difference diffusion games on monotonically growing temporal cycles based on the notion of nice vertices. We now prove that, for each monotonically growing temporal cycle $\mathcal{C}$, at least one Nash equilibrium in $\Delta\text{Diff}(\mathcal{C}, 2)$ described in Lemma 21 is also a Nash equilibrium in $\text{Diff}(\mathcal{C}, 2)$.

To this end, we prove that in several cases a Nash equilibrium $(u, v)$ in $\Delta\text{Diff}(\mathcal{C}, 2)$ colors at most one vertex in gray. This directly implies that $(u, v)$ is also a Nash equilibrium in $\text{Diff}(\mathcal{C}, 2)$ as proven in the following lemma:

**Lemma 22.** If $(u, v)$ is a Nash equilibrium in $\Delta\text{Diff}(\mathcal{C}, 2)$ which results in at most one gray vertex, then $(u, v)$ is a Nash equilibrium in $\text{Diff}(\mathcal{C}, 2)$.

**Proof.** Without loss of generality and for the sake of contradiction, assume that player 1 gets a higher payoff by moving to vertex $u'$. Since $(u, v)$ is a Nash equilibrium in $\Delta\text{Diff}(\mathcal{C}, 2)$, the payoff of player 2 increases by at least the same amount. However, since all but at most one vertex is colored by one of the players for $(u, v)$ in $\text{Diff}(\mathcal{C}, 2)$, it is not possible that both players color an additional vertex.

We observe that if the two players play next to each other, then at most one vertex is colored gray. This yields two easy cases based on Lemma 21 in which a Nash equilibrium exists:

**Lemma 23.**

1. If $v$ is the unique nice vertex, then $(v, v+1)$ is a Nash equilibrium in $\text{Diff}(\mathcal{C}, 2)$.

2. Let $\Omega$ be a maximum-size set of vertices reachable from a vertex in $\overline{\mathcal{F}}(\mathcal{C})$. If $|\Omega|$ is even, then its two central vertices form a Nash equilibrium in $\text{Diff}(\mathcal{C}, 2)$.

**Proof.** In both cases the players play on adjacent vertices and the resulting strategy profile is a Nash equilibrium in $\Delta\text{Diff}(\mathcal{C}, 2)$ by Lemma 21. As the players start on adjacent vertices, there exists at most one gray vertex. Thus Lemma 22 implies the claim.
It remains to consider the case where there exist two nice vertices in $\mathcal{F}(\mathcal{C})$ and the size of the maximum set of vertices reachable by any vertex in $\mathcal{F}(\mathcal{C})$ is odd. In this case, there do not always exist two adjacent nice vertices. However, we can prove that in this case there either exists a pair of nice vertices such that at most one vertex is colored gray if the players select them or all pairs of nice vertices are Nash equilibria. To be able to make this distinction, we define the distance between two vertices as follows:

**Definition 24.** Let $u, v \in [n]$ with $u < v$. The distances between $u$ and $v$ are defined as $d_1(u, v) := v - u$ and $d_2(u, v) := n - (v - u)$.

We can show that we get two gray vertices on a monotonically growing temporal cycle $\mathcal{C}$ where the maximum number of vertices reachable from any vertex in $\mathcal{F}(\mathcal{C})$ is odd if and only if the distances between the positions of the players are even in both directions:

**Lemma 25.** Let $u, v$ be two distinct nice vertices, each reaching an odd number of vertices in $\mathcal{F}$. Then, the strategy profile $(u, v)$ in $\text{Diff}(\mathcal{C}, 2)$ results in two gray vertices if and only if both $d_1(u, v)$ and $d_2(u, v)$ are even.

**Proof.** Say $u < v$. We will show that one vertex of $[u, v]$ is colored gray if and only if $d_1(u, v)$ is odd. The same arguments can then be also applied to the complement part of $\mathcal{C}$.

By assumption, $|\Omega(u)| = |\Omega(v)|$ is odd and $u, v$ are the central vertices of $\Omega(u)$, $\Omega(v)$ respectively. In particular, the sets $\Omega_u := \Omega(u) \cap [u, v]$ and $\Omega'_v := \Omega(v) \cap [u, v]$ have the same size.

Now if $\Omega_u \cap \Omega'_v \neq \emptyset$, then the claim follows from Lemma 19 (with the possibly gray vertex being $x$). Otherwise, until time $\tau - 1$ the two players color exactly $\Omega_u$ resp. $\Omega'_v$. At each subsequent time step each player reaches exactly one new vertex from the set $[u, v] \setminus (\Omega(u) \cup \Omega(v))$. If $d_1(u, v)$ is odd, then this set has odd size, implying that eventually its central vertex will be colored gray. Otherwise, if $d_1(u, v)$ is even, then this set has even size so no vertex will be colored gray.

From Lemma 25 we can derive the following:

**Lemma 26.** Let $u, v$ be two distinct nice vertices, each reaching an odd number of vertices in $\mathcal{F}$. If $d_1(u, v)$ or $d_2(u, v)$ are odd, then $(u, v)$ is a Nash equilibrium in $\text{Diff}(\mathcal{C}, 2)$.

**Proof.** By Lemma 24 we can conclude that the strategy profile $(u, v)$ results in only at most one vertex being colored in gray in $\text{Diff}(\mathcal{C}, 2)$. Moreover, by Lemma 21 $(u, v)$ is a Nash equilibrium in $\Delta\text{Diff}(\mathcal{C}, 2)$. Applying Lemma 22 it follows that $(u, v)$ is a Nash equilibrium in $\text{Diff}(\mathcal{C}, 2)$.

By Lemmas 23 and 20 it only remains to consider the case where there exist two or more nice vertices, each reaching an odd number of vertices in $\mathcal{F}$ and where all pairs of nice vertices have even distance in both directions. We will prove that in this case, in fact, every pair of nice vertices is a Nash equilibrium:

**Lemma 27.** Let the nice vertices be $v_1, \ldots, v_z$ where $z \geq 2$. Assume that each of them reaches an odd number of vertices in $\mathcal{F}$ and that for all $i \neq j \in [z]$, the distances $d_1(v_i, v_j)$ and $d_2(v_i, v_j)$ are even. Then, $(v_i, v_j)$ is a Nash equilibrium in $\text{Diff}(\mathcal{C}, 2)$ for all $i \neq j \in [z]$.

**Proof.** Fix some pair of nice vertices $u, v$. For the sake of contradiction, assume that $(u, v)$ is not a Nash equilibrium in $\text{Diff}(\mathcal{C}, 2)$. Without loss of generality, assume that player 1 can increase her payoff by $x$ when moving to vertex $u'$.

First, observe that by Lemma 25 and Observation 8 the strategy profile $(u, v)$ results in exactly two vertices being colored in gray and no vertices being uncolored in $\text{Diff}(\mathcal{C}, 2)$.
Since \( u \) and \( v \) are both nice vertices and both players color the same number of vertices in \( \text{Diff}(C, 2) \) after step \( \tau - 1 \), we conclude by Lemma \[24\] that \((u, v)\) is a Nash equilibrium in \( \Delta \text{Diff}(C, 2) \) and by Lemma \[17\] that \( \Delta(u, v) = 0 \) in \( \text{Diff}(C, 2) \). As \((u, v)\) is a Nash equilibrium in \( \Delta \text{Diff}(C, 2) \), no player can deviate to color more vertices than the other player. Thus, for player 1 to be able to increase her payoff by \( x \) by moving to \( u' \), also the payoff for player 2 for \((u', v)\) has to increase by \( x \). This directly implies that \( x = 1 \), as only two vertices are colored gray for the strategy profile \((u, v)\). Moreover, this implies that \( \Delta(u', v) = \Delta(u, v) = 0 \) in \( \text{Diff}(C, 2) \) and that the strategy profile \((u', v)\) results in all vertices being colored with color 1 or 2 in \( \text{Diff}(C, 2) \).

We distinguish two cases both leading to a contradiction to the observations from above.

1. First, assume that \( \Omega(u') \subseteq \Omega(v) \) or \( \Omega(v) \subseteq \Omega(u') \). Since \( v \) is nice, this can only mean that \( \Omega(u') \subseteq \Omega(v) \). As player 2 plays on the only central vertex of \( \Omega(v) \) and since both players color the same number of vertices in \( \text{Diff}(C, 2) \) after step \( \tau - 1 \), it follows that player 2 colors more vertices than player 1 in \( \text{Diff}(C, 2) \). This contradicts that \( \Delta(u', v) = 0 \) in \( \text{Diff}(C, 2) \).

2. Otherwise, \( \Omega(u') \) and \( \Omega(v) \) are incomparable. By Lemma \[16\] and since both players color the same number of vertices in \( \text{Diff}(C, 2) \) after step \( \tau - 1 \), it holds that \( 0 = \Delta(u', v) = |\Omega(u')| - |\Omega(v)| \) in \( \text{Diff}(C, 2) \). Let \( m \) be the central vertex of \( \Omega(u') \). By Lemma \[13\] \( \Omega(m) = \Omega(u') \). Thus, by Lemma \[16\] it holds that \( \Delta^{\tau - 1}(m, v) = |\Omega(u')| - |\Omega(v)| = 0 \).

As the edge \( \{1, n\} \) appears only in layer \( \tau \), at most one vertex can be colored gray in the first \( \tau - 1 \) steps. As at most one vertex can be colored gray after \( \tau - 1 \) steps, as \( \Delta^{\tau - 1}(u', v) = \Delta^{\tau - 1}(m, v) = 0 \), and as for both \((u', v)\) and \((m, v)\) exactly the vertices in \( \Omega(u') \cup \Omega(v) \) are colored after \( \tau - 1 \) steps, it follows that the coloring of the vertices after \( \tau - 1 \) steps for \((m, v)\) and for \((u', v)\) in \( \text{Diff}(C, 2) \) needs to be the same. This also directly implies that the coloring of all vertices for \((m, v)\) and for \((u', v)\) at the end of \( \text{Diff}(C, 2) \) needs to be the same. Since the strategy profile \((u', v)\) results in no vertices being colored gray, the same holds for the strategy profile \((m, v)\). However, \( m \) and \( v \) are both nice vertices. By our initial assumption, both \( d_1(m, v) \) and \( d_2(m, v) \) are even. Thus, by Lemma \[25\] it follows that the strategy profile \((m, v)\) results in two vertices being colored gray, which contradicts our previous observation.

Thus, \((u, v)\) is a Nash equilibrium.

Since we have exhausted all cases, we showed that there is a Nash equilibrium in every temporal diffusion game on a monotonically growing temporal cycle. In the next part, we summarize our result in an algorithm.

**Part III: The algorithm.** We claim that Algorithm \[1\] returns a Nash equilibrium for every temporal diffusion game on a monotonically growing temporal cycle \( C = ([n], (E_i)_{i \in [\tau]}) \) in \( \mathcal{O}(n \cdot \tau) \) time. We show the correctness of Algorithm \[1\] in Lemma \[28\] and prove its running time in Lemma \[29\].

**Lemma 28.** Algorithm \[1\] returns a Nash equilibrium for every temporal diffusion game on a monotonically growing temporal cycle \( C \).

**Proof.** Note first that by Lemma \[24\] every strategy profile returned by Algorithm \[1\] is a Nash equilibrium in \( \Delta \text{Diff}(C, 2) \). In the following, we iterate over all cases of Algorithm \[1\] and argue that Algorithm \[1\] always returns a Nash equilibrium in \( \text{Diff}(C, 2) \).

In Case \[1\] and Case \[2\] Lemma \[23\] applies. Otherwise, a strategy profile returned by Algorithm \[1\] falls under Case \[2(a)\]. Consequently, there are at least two nice vertices.
**Algorithm 1** Compute a Nash equilibrium on a monotonically growing cycle.

**Input:** Monotonically growing temporal cycle $C = ([n], (E_i)_{i \in [\tau]})$.

**Output:** A Nash equilibrium in Diff($C$, 2).

$R :=$ maximum number of vertices that are reachable from any vertex in $\mathcal{F}(C)$.

1. If $R$ is even, then return the two central vertices of a size-$R$ vertex set that is reachable from some vertex in $\mathcal{F}(C)$.

2. If $R$ is odd, then let $v_1, v_2, \ldots, v_z$ be the nice vertices.
   
   (a) If $z \geq 2$:
      i. If there is a pair $v_i \neq v_j$ such that $d_1(v_i, v_j)$ or $d_2(v_i, v_j)$ is odd, then return $(v_i, v_j)$.
      ii. Otherwise, return $(v_1, v_2)$.
   
   (b) Otherwise, return $(v_1, v_1 + 1)$.

The players play on nice vertices with odd distance in at least one direction. Thus, Lemma 26 applies. Considering Case 2(a)ii, all pairs of nice vertices have even distance in both directions. Thus, Lemma 27 applies.

**Lemma 29.** Let $C = ([n], (E_i)_{i \in [\tau]})$ be a monotonically growing temporal cycle of size $n$. Algorithm 1 on $C$ runs in $O(n \cdot \tau)$ time.

**Proof.** First, we need to compute $R$. To do so, we compute for each vertex $v$ the set $\Omega(v)$ of vertices reached in $\mathcal{F}(C)$. This can be done in $O(\tau)$ time for each vertex. Subsequently, we compute the central vertices of each distinct set of vertices of size $R$ reachable by some vertex, which can each be done in constant time.

Additional computation is only needed in Case 2a where we know that $R$ is odd and $z \geq 2$. If the number $n$ of vertices in $C$ is odd, then, for each $u, v \in [n]$, either $d_1(u, v)$ or $d_2(u, v)$ is odd. Consequently, if $n$ is odd, then $(v_1, v_2)$ is a solution for Case 2(a)ii. Otherwise, $n$ is even. It follows that for all $v_i, v_j$ with $i \neq j \in [z]$, $d_1(v_i, v_j)$ is even if and only if $d_2(v_i, v_j)$ is even. Thus, it is enough to consider the distances between two nice vertices in one direction. Without loss of generality, assume that $v_1 < v_i$ with $i \in [2, z]$. We compute all distances $d_1(v_1, v_i)$. If $d_1(v_1, v_i)$ is odd, then we found a solution. Otherwise, $d_1(v_1, v_i)$ is even for all $i \in [2, z]$. However, this directly implies that $d_1(v_j, v_i)$ is even for all $i \neq j \in [2, z]$. Thus, all nice vertices have even distance in both directions, so that Case 2(a)ii holds. We summarize that for Case 2(a)ii and Case 2(a)iii we only have to compute the distances between $v_1$ and all other nice vertices $v_i$ with $i \in [2, z]$. This can be done in $O(n)$ time.

Altogether, Algorithm 1 runs in $O(\tau \cdot n)$ time.

Theorem 7 now directly follows from Lemma 28 and Lemma 29.

## 4 Temporal Voronoi games

In this section, we study temporal Voronoi games. In contrast to temporal diffusion games, here, the color of a vertex $v$ is determined solely by the temporal distances from the players’ positions to $v$.
4.1 Monotonically shrinking paths and cycles

In Section 2, we observed that temporal diffusion games and temporal Voronoi games might already differ on a simple temporal path. In contrast to this, both games are equivalent on monotonically shrinking forests and cycles, as no foremost walk ever needs to wait at any vertex in these graphs.

**Lemma 30.** Let \( G = (V, (E_i)_{i \in [\tau]}) \) be a monotonically shrinking forest or cycle and let \( p_1, p_2 \in V \). For the strategy profile \((p_1, p_2)\), the final coloring of the vertices is the same in \( \text{Diff}(G, 2) \) and in \( \text{Vor}(G, 2) \).

**Proof.** As already noted in Section 2, a vertex \( v \) colored with color \( i \in [2] \) in \( \text{Vor}(G, 2) \) is colored the same in \( \text{Diff}(G, 2) \), as \( p_i \) reaching \( v \) first implies that \( p_i \) also reaches every vertex on a foremost walk from \( p_i \) to \( v \) first.

To see that on monotonically shrinking forest and cycles the converse also holds, assume that \( v \) gets colored with color \( 1 \) in \( \text{Diff}(G, 2) \). Note that there is exactly one temporal walk from \( p_1 \) to \( v \) which does not use vertex \( p_2 \) and no vertex repeatedly. Since no foremost walk ever needs to wait in \( G \) (as no new edges appear over time), this temporal walk must have fewer edges than any temporal walk from \( p_2 \) to \( v \). For the same reason, the walk from \( p_1 \) to \( v \) consists of exactly \( td(p_1, v) \) edges. Thus, \( td(p_1, v) < td(p_2, v) \). □

In particular, using Lemma 30 we can transfer Theorem 4 and Theorem 6 to temporal Voronoi games:

**Corollary 31.** There is a monotonically shrinking temporal path \( P \) and a monotonically shrinking temporal cycle \( C \) both consisting of two layers such that there is no Nash equilibrium in \( \text{Vor}(P, 2) \) and no Nash equilibrium in \( \text{Vor}(C, 2) \).

4.2 Temporally connected paths and cycles

In contrast to temporal diffusion games, a Nash equilibrium in a temporal Voronoi game on a temporally connected path may fail to exist. In fact, the underlying dynamics of temporal Voronoi games on temporally connected paths might be quite intriguing and far more complex than for temporal diffusion games (as highlighted in the next subsection).

**Theorem 32.** There is a temporally connected path \( P \) and a temporally connected cycle \( C \) such that there is no Nash equilibrium in \( \text{Vor}(P, 2) \) and no Nash equilibrium in \( \text{Vor}(C, 2) \).

**Proof.** Let \( P = ([8], (E_1, E_2)) \) be the temporal path from Figure 2. By Lemma 30 and the proof of Theorem 4, there is no Nash equilibrium in \( \text{Vor}(P, 2) \). Note that for all sufficiently large \( N \), modifying layer \( N \) of \( P \) does not affect the dynamics of the Voronoi game: At that point in time, either every vertex has already been reached by some player, or both players have reached the same set of vertices, therefore any vertices left uncolored can only become gray. In particular, we may replace layer \( N \) and all subsequent layers by a complete path or cycle, which proves the claim. □

4.3 Monotonically growing trees

We now turn to monotonically growing temporal trees and prove that every temporal Voronoi game on such a tree admits a Nash equilibrium. In the next Section 4.4, we reuse some ideas presented here to also prove a result about temporal Voronoi games on monotonically growing cycles. Specifically, this subsection is devoted to proving the following theorem:
Theorem 33. On every monotonically growing temporal tree $T$ a Nash equilibrium in $\text{Vor}(T, 2)$ exists and can be found in $O(n^2)$ time.

In order to prove Theorem 33, we first analyze the best responses of a player to a given strategy of the other player. For this, we introduce the concept of boundaries.

We call an edge $B = \{b, b'\} \in E(G)$ a boundary of a vertex $v$ if a player positioned at $v$ can cross $B$ at the earliest possible time, i.e., if $\min\{td(v, b), td(v, b')\} < \tau_B$. To better distinguish them from vertices, we will use capital letters for boundaries.

Let $B$ be the set of all boundaries of some vertex $v$. Then the connected components of $G_\downarrow - B$ are called the boundary components of $v$. For any vertex $w$, we define $j_v(w)$ to be the boundary component of $v$ that contains $w$. Note that $j_v(v) = \{v\}$. The following observation is the basis for our study of Nash equilibria. Note that it does not only hold for monotonically growing trees, but all temporal graphs. Compare also Figure 5.

Lemma 34. Let $v, w$ be any two vertices of a temporal graph $G$. Then $U_1(w, v) \subseteq j_v(w)$.

Proof. To reach any vertex $x \notin j_v(w)$, the player positioned at $w$ must cross some boundary of $v$. Since the player at $v$ can cross that boundary at the same time or earlier, $w$ cannot reach any vertices beyond that boundary before $v$ does.

Subsequently, we always assume that the graph $T$ is a monotonically growing tree $T = (V, (E_i)_{i \in [\tau]})$. For two vertices $v, w$ we will use the notation $[v, w]$ to refer to the unique (static) path from $v$ to $w$ in $T_\downarrow$. We say that a vertex $x$ or an edge $e$ is between $v$ and $w$ if it is part of $[v, w]$.

Complementary to Lemma 34, we now find that boundaries are the only way for a player to “catch up” with another player. An important special case of the following lemma is the situation where $x = x'$. (For an example, consider Figure 5 with $v = p_1$ and $x = x' = p_2$.)

Lemma 35. Let $v, w$ be two vertices of $T$ and $x \in [v, w]$. Assume $x'$ to be any vertex with $td(x', x) < td(v, x)$. Then $td(v, w) \geq td(x', w)$, and equality holds if and only if $[x, w]$ contains a boundary of $v$. 

Figure 5: Example Voronoi game on a monotonically growing temporal path graph. On the bottom, the boundary components of $p_1 = 5$ (upper row, blue) and $p_2 = 7$ (lower row, red) are indicated. As guaranteed by Lemma 34, the sets of vertices colored by the players satisfy $\{p_1\} = U_1(p_1, p_2) \subseteq j_{p_1}(p_1) = \{p_1, 6\}$ and $[p_2, 12] = U_1(p_2, p_1) \subseteq j_{p_2}(p_2) = [6, 12]$. 




Proof. It is clear that \( \text{td}(v, w) \geq \text{td}(x', w) \) since any temporal path from \( v \) to \( w \) must pass \( x \), and \( x' \) reaches \( x \) before \( v \) does.

Furthermore, if the path \([x, w]\) contains a boundary \( B \) of \( v \), then let \( \text{td}(v, B) \) be the time \( v \) reaches either endpoint of \( B \). Since \( B \) is a boundary of \( v \), \( \text{td}(v, B) < \tau_B \). Then \( \text{td}(x', x) < \text{td}(v, x) \leq \text{td}(v, B) < \tau_B \), therefore \( v, x \), and \( x' \) must all be on the same side of \( B \) (with \( w \) on the other side). Thus a temporal path from \( x' \) to \( w \) must cross \( B \) before reaching \( w \). Since \( B \) is a boundary of \( v \), this implies that \( x' \) cannot reach \( w \) earlier than \( v \) does.

Conversely, assume that \( \text{td}(v, w) = \text{td}(x', w) \). Then let \( w' \) be the first vertex on \([x, w]\) with \( \text{td}(v, w') = \text{td}(x', w') \). Since \( \text{td}(v, x) > \text{td}(x', x) \), we have \( w' \neq x \). So there is a vertex \( y \) of \([x, w]\) right before \( w' \). Since \( \text{td}(x', y) \leq \text{td}(v, y) - 1 \leq \text{td}(v, w') - 2 = \text{td}(x', w') - 2 \), a foremost path from \( x' \) to \( w' \) must “wait” at least one time step before crossing \([y, w']\). This proves that that path crosses \([y, w']\) at the earliest possible time, \( \tau_{[y, w']} \). This makes \([y, w']\) a boundary of \( x' \) and thus also of \( v \), since \( \text{td}(v, w') = \text{td}(x', w') \) implies that \( v \) and \( x' \) cross \([y, w']\) simultaneously. \( \square \)

With \( T_v \) being a tree, every boundary splits it in two connected components, about which we can make the following observation.

Lemma 36. Let \( v \) be a vertex of \( T \) and \( B \) a boundary of \( v \). Let \( C \) and \( C' \) be the two connected components of \( T_v \setminus \{B\} \), with \( v \in C \). Then \( \text{td}(v, w) \leq \text{td}(w', w) \) for any vertices \( w \in C \) and \( w' \in C' \).

Proof. Let \( B = \{b, b'\} \) with \( b \in C \) and \( b' \in C' \). Then \( b \in [w', w] \) and \( B \in [w', b] \). Since \( B \) is a boundary of \( v \), we have \( \text{td}(v, b) < \tau_B \leq \text{td}(w', b) \). Thus, \( \text{td}(v, w) \leq \text{td}(w', w) \) by Lemma 35. \( \square \)

For any boundary component \( C \) of a vertex \( v \), we call the vertex \( x \in C \) that minimizes \( \text{td}(v, x) \) the \emph{entry vertex} of \( C \) and the last edge of \([v, x]\) the \emph{entry boundary} of \( C \) (note that this edge is a boundary of \( v \)).

Lemma 37. Let \( v \) be any vertex, \( C \) be any boundary component of \( v \) with entry vertex \( x \). Then for any vertex \( w \), \( U_1(w, v) = C \) if and only if \( x \in U_1(w, v) \).

Proof. The forward direction is immediate since \( x \in C \). For the reverse, note that by Lemma 34 we must have \( U_1(w, v) \subseteq j_x(v) = C \). So it only remains to prove that \( U_1(w, v) \supseteq C \). So let \( y \in C \) be any vertex. By definition of \( C \), \([x, y]\) does not contain a boundary of \( v \). Thus we may apply Lemma 35 to get \( \text{td}(v, y) > \text{td}(w, y) \). \( \square \)

From Lemma 37 we can derive the following about best responses.

Lemma 38. Let \( x \) be a vertex of \( T \) and \( y \) a best response to \( x \). Then \( U_1(y, x) = j_x(y) \) and all edges leaving \( j_x(y) \) are boundaries of \( y \).

Proof. By Lemma 34 it holds that \( U_1(y, x) \subseteq j_x(y) \). Let \( b \) be the entry vertex of \( j_x(y) \), then \( U_1(b, x) = j_x(y) \) by Lemma 37. Thus \( U_1(y, x) = j_x(y) \), or \( b \) would have been a better response to \( x \) than \( y \). Since \( \text{td}(y, v) < \text{td}(x, v) \) for all \( v \in j_x(y) \), all the edges leaving \( C \) (which must be boundaries of \( x \) by definition) are boundaries of \( y \), too. \( \square \)

Looking back at Figure 5 we can thus deduce that \( p_2 \) was not a best response to \( p_1 \). Instead, the red player should have played at 6.

Now we can already show the following powerful structural result.

Lemma 39. Let \( x \) be any vertex of \( T \) and \( y \) a best response to \( x \). Then \( U_1(y, x) = j_x(y) \) and \( U_1(x, y) = j_y(x) \) are disjoint sets, connected by an edge. Furthermore, \( V \setminus (j_x(y) \cup j_y(x)) \) can be partitioned into sets which are boundary components of both, \( x \) and \( y \).
Let \( B = \{ b, b' \} \) be the entry boundary of \( j_x(y) \) with \( b' \in j_x(y) \) being the entry vertex (see also Figure 6). By Lemma 38, \( U_1(y, x) = j_x(y) \) and \( B \) is a boundary of \( y \), too. Note that \([x, b] \subseteq U_1(x, y)\). In particular, \( y \) cannot have any boundaries in \([x, b]\). Thus, \( b \) is the entry vertex of \( j_y(x) \) and \( U_1(x, y) = j_y(b) = j_y(x) \) by Lemma 37.

It remains to show that \( j_x(v) = j_y(v) \) for any \( v \in V \setminus (j_x(y) \cup j_y(x)) \). Say without loss of generality that \( v \) is on the same side of \( B \) as \( x \) and let \( B' \) be the entry boundary of \( j_y(v) \). By the above, \( x \) and \( y \) are on the same side of \( B' \), and by Lemma 38, \( B' \) is also a boundary of \( y \) (due to \( B \) being a boundary of \( x \)). Thus, \( td(x, w) = td(y, w) \) for all \( w \in j_y(v) \). Therefore \( j_y(v) \) must also be a boundary component of \( x \).

Note that the fact that \( j_x(y) \) and \( j_y(x) \) are connected by an edge in Lemma 39 implies that for any \( v \in V \setminus (j_x(y) \cup j_y(x)) \) the two (identical) boundary components \( j_x(v) \) and \( j_y(v) \) also have the same entry boundary.

Since \((v, w)\) forms a Nash equilibrium if and only if \( v \) and \( w \) are mutual best responses, our strategy of proving Theorem 38 will be to consider a best response dynamic where both players alternatingly pick a best response to the other’s previous choice. We will then show that this process must eventually reach a Nash equilibrium. For this, we require the following lemma.

**Lemma 40.** Let \( x \) be any vertex of \( T \), \( y \) a best response to \( x \), and \( z \) a best response to \( y \). Let \( B \) be the entry boundary of \( j_x(y) \). If \( x \) and \( z \) are on the same side of \( B \), then \((x, y)\) or \((y, z)\) is a Nash equilibrium.

**Proof.** By Lemma 39, \( B \) is a shared boundary of \( x \) and \( y \). Also \( U_1(x, y) = j_y(x) \), so if \( z \in j_y(x) \) then \( U_1(z, y) \subseteq j_y(z) = j_y(x) = U_1(y, x) \), which would make \( x \) a best response to \( y \), proving the claim.

So assume now \( z \notin j_y(x) \), i.e., \( j_y(z) \neq j_y(x) \). By Lemma 39, it holds that \( j_x(z) = j_y(z) \) and both of these have the same entry boundary, \( B' \) say (see Figure 4). Let \( C, C' \) be the two connected components of \( T_i - \{ B' \} \), with \( z \in C' \) and \( \{ x, y \} \subseteq C \). Furthermore, by Lemma 39, every boundary of \( z \) in \( C \) is also a boundary of \( x \) and \( y \). Also, \( B \) is clearly no boundary of \( z \) as \( B \) is between \( y \) and \( B' \). Thus, \( j_z(y) \supseteq j_x(y) \cup j_y(x) \).

By Lemma 38, \( B' \) is a boundary of \( z \). Thus, by Lemma 39, all boundaries of \( x \) in \( C' \) are also boundaries of \( z \). So if some \( y' \in C' \) is a best response to \( z \), then \( j_z(y') \subseteq j_x(y') \). But then we must have \(|j_z(y')| \leq |j_x(y')| \leq |j_x(z)| < |j_x(y) \cup j_y(x)| \leq |j_x(y)|\) by choice of \( y \). This contradicts \( y' \) being a best response to \( z \).

So any best response \( y' \) to \( z \) must necessarily lie in \( C \). We may assume \( y' \notin j_x(y) \supseteq j_x(y) \cup j_y(x) \), or \( y \) would already be a best response to \( z \), making \((y, z)\) a Nash equilibrium. Let \( B'' \) be the entry boundary of \( j_x(y') \). Since \( j_x(y) \) touches \( B' \), both, \( x \) and \( z \) are on the same side of \( B'' \). Also, \( B'' \) is a boundary of \( x \) by Lemma 39. Therefore \( j_x(y') = j_x(y') \). Consequently,
Figure 7: Illustration of the proof of Lemma 40.

\[ |j_z(y')| = |j_x(y')| \leq |j_z(y)| < |j_z(y) \cup j_y(x)| \leq |j_z(y)| \] by choice of \( y \), thus contradicting the choice of \( y' \). □

We can now prove the existence of a Nash equilibrium in \( \text{Vor}(T) \).

**Lemma 41.** Let \( x_0 \) be any vertex of \( T \). Define \( x_{i+1} \) iteratively to be the entry vertex of a largest boundary component of \( x_i \). Then there is some index \( j < |V| \) such that \( (x_j, x_{j+1}) \) forms a Nash equilibrium.

**Proof.** By Lemma 34 and Lemma 37, each \( x_{i+1} \) is a best response to \( x_i \).

For any \( i \), let \( B_i \) be the entry boundary of \( j_x(x_{i+1}) \) and \( C_i, C_i' \), the two connected components of \( T_i - \{B_i\} \), with \( x_i \in C_i \).

By Lemma 40 if \( x_{i+2} \in C_i \), then we have found a Nash equilibrium. Thus, we may assume that \( x_{i+2} \in C_i' \) for all \( i < |V| \). Therefore, also \( B_{i+1} \subseteq C_i' \) and \( C_{i+1} \subseteq C_i' \). Since the sequence \( C_0' \supset C_1' \supset C_2' \supset \ldots \) must terminate after at most \( |V| \) steps, this concludes the proof. □

From Lemma 41, we can easily deduce our main result Theorem 33.

**Proof of Theorem 33.** Compute a sequence \( x_0, x_1, \ldots \) as in Lemma 41 until we encounter a Nash equilibrium. Observe that computing the reach times from any given vertex \( v_i \) to all other vertices takes \( \mathcal{O}(|V|) \) time, since we assume that the first appearance of any edge can be found in constant time. Consequently, we can also determine all boundaries of \( v_i \) and thus compute \( v_{i+1} \) in \( \mathcal{O}(|V|) \) time. By Lemma 41 we will find a Nash equilibrium after at most \( |V| \) iterations, and we can easily test for this by comparing \( |j_x(x_{i-1})| \) and \( |j_x(x_{i+1})| \). Thus, we need \( \mathcal{O}(|V|^2) \) time overall.

We close this section by remarking that the existence of Nash equilibria extends to monotonically growing forests as well. To see this, simply apply the argument of Lemma 41 to a connected component of maximum size. If at any point the best response to \( x_i \) is not \( x_{i+1} \) but some vertex \( y \) located in a different connected component, then \( (x_i, y) \) already forms a Nash equilibrium.

### 4.4 Monotonically growing cycles

In the previous subsection we have seen that Nash equilibria always exist in temporal Voronoi games on monotonically growing trees. Now we extend this result also to monotonically growing cycles. However, the proof becomes more involved here.

In this subsection, let \( C = ([0, n - 1], (E_i)_{i \in [n]}) \) be a monotonically growing temporal cycle graph (note that it will be convenient here to number the vertices starting with 0). We split the proof into two parts. In the first part, in Section 4.4.1, we make some structural observations that allow us to classify all Nash equilibria in temporal difference Voronoi games. Subsequently, these structural results also help us to find Nash equilibria in (normal) temporal Voronoi games in Section 4.4.2.
As was the case for temporal diffusion games, we will first look into temporal difference Voronoi games, which will help us to subsequently find Nash equilibria for Voronoi games.

4.4.1 Temporal difference Voronoi games

As was the case for temporal diffusion games, we will first look into temporal difference Voronoi games, which will help us to subsequently find Nash equilibria for Voronoi games.

For vertices, we use chains of inequalities like \( a < b < c \) to express that when starting from vertex \( a \) and moving in positive direction on \( C \) (that is, in increasing order of vertices), then vertex \( b \) is encountered before \( c \). We denote subintervals of the cycle by \([a, b] := \{v \in V \mid a \leq v \leq b\}\), thus \([a, b] \cup [b, a] = V\). For any \( x \in \mathbb{Z} \), we use \( x \equiv \) to denote the unique \( x \in V = [0, n - 1] \) with \( x \equiv x \pmod{n} \). Note that \([a, b] = b - a + 1\).

The definition of a boundary carries over from Section 4.3. However, we will now distinguish between left and and right boundaries. An edge \( B = \{b, b + 1\} \) is a right boundary of a vertex \( v \) if \( \text{td}(v, b) < \tau_B \) and a left boundary if \( \text{td}(v, b + 1) < \tau_B \). Note that there might be one boundary that is both a left and a right boundary, i.e., where \( v \) reaches both \( b \) and \( b + 1 \) before \( \tau_{(x,x+1)} \).

Clearly, every boundary component \( C \) of \( v \) is now adjacent to exactly two boundaries of \( v \). If both of these are left (resp. right) boundaries, then we call \( C \) a left (resp. right) boundary component. Otherwise, if \( C \) is not the trivial boundary component \( \{v\} \), then we call \( C \) the (unique) outer boundary component of \( v \), denoted \( O_v \). If \( v \) has no such boundary component, then we set \( O_v = \emptyset \).

All boundary components except \( O_v \) are called inner boundary components. See Figure 8 for an example. Note that \( w \in O_v \) is intuitively equivalent to the fact that \( v \) will never “catch up” with \( w \) from behind (see also Figure 9). Due to this, we observe the following. (Recall that \( \Omega^t(v) \) is the set of all vertices reachable from \( v \) until time \( t \)).

**Lemma 42.** If \( v \) is a vertex of \( C \) and \( w \in O_v \), then for all \( t \)

\[
\Omega^t(v) \supseteq \Omega^t(w) \iff \Omega^t(v) = V.
\]

**Proof.** Assume \( \Omega^t(v) \supseteq \Omega^t(w) \). By monotonicity, it suffices to consider the minimal such \( t \), thus we may assume that there is \( x \in \Omega^{t-1}(w) \backslash \Omega^{t-1}(v) \). Since \( x \in \Omega^t(v) \), there must be a neighbor \( x' \) of \( x \) with \( x' \in \Omega^{t-1}(v) \) and \( \tau_{(x,x')} \leq t \). Say without loss of generality \( x' = x - 1 \). If \( x + 1 \in \Omega^t(w) \subseteq \Omega^t(v) \), then \( \Omega^t(v) = V \), since \( v \) cannot have reached \( x + 1 \) via \( x \) yet. Otherwise \( (x + 1 \notin \Omega^t(w)) \), as \( x \in \Omega^t(w) \), we must have \( \tau_{(x,x+1)} > t \), making this edge a right boundary of \( v \) with \( v < w < \{x, x + 1\} \), which would contradict \( w \in O_v \).

The reverse implication is trivial. \( \square \)
Figure 9: Example instance with $\Omega^6(v)$ (blue, inner arc) and $\Omega^6(w)$ (red, outer arc) indicated. Note how $\Omega^6(v) \supseteq \Omega^6(w)$ since $v$ has “caught up” with $w$ in positive (counterclockwise) direction. By Lemma 42 we can thus conclude that $w \notin O_v$. (In fact, $O_v = \{v-1\}$.)

Boundaries keep their importance also on cycles, since Lemma 34 also applies to cycles. We also observe the following:

**Lemma 43.** Let $u < v \leq w$ be vertices of $C$. If $\{w, w+1\}$ is a right boundary of $u$, then it is also a right boundary of $v$.

**Proof.** Because $\{w, w+1\}$ is a right boundary of $u$, $td(u, w) < \tau_{\{w, w+1\}}$. Thus, as $u < v \leq w$, we have $td(v, w) \leq td(u, w) < \tau_{\{w, w+1\}}$, implying that $w$ is also a right boundary of $v$.

Analogously, it holds for $w \leq u < v$ that if $\{w-1, w\}$ is a left boundary of $v$, then it is also a left boundary of $u$.

Lastly, we will also make use of the following fact.

**Lemma 44.** Let $u, v$ be vertices of $C$. If $B$ is a common right boundary of $u$ and $v$ with $u < v < B$, then every edge $C$ with $B < C < u$ is a right boundary of $u$ if and only if it is a right boundary of $v$.

**Proof.** Let $B = \{b, b+1\}$ and let $C = \{c, c+1\}$ be a right boundary of $u$ or $v$. This implies that $B$ is not a left boundary of $u$ or $v$ and thus $td(u, b+1) = td(v, b+1) = \tau_{\{b, b+1\}}$ and further that $td(u, c) = td(v, c)$, which establishes the claim.

As indicated by the following lemma, playing in the outer component of your opponent is often a good idea:

**Lemma 45.** Let $v, w$ be vertices of $C$. If $v \notin O_w$ and $w \in O_v$, then there is some time $t$ with $\Omega^t(v) \subset \Omega^t(w)$.

**Proof.** Let $j_w(v) = [B, B']$ be a left boundary component of $w$ without loss of generality. Note that $B$ is also a left boundary of $v$ by Lemma 43. Set $B = \{b-1, b\}$ and $t := td(w, b)$. Then $\Omega^t(v) \subseteq \Omega^t(w)$.

Assume for contradiction that $\Omega^t(v) = \Omega^t(w)$. Thus, by Lemma 32, $\Omega^t(w) = \Omega^t(v) = V$. In particular $td(v, b-1) \leq t$, therefore $B$ is also a right boundary of $v$. Thus, $O_v = [B, B] = \emptyset$, but $w \in O_v$.

If both players play in their opponent’s outer components, then each of them is guaranteed to win all their respective inner components.

**Lemma 46.** Let $v, w$ be vertices of $C$. If $v \in O_w$ and $w \in O_v$, then $U_1(v, w) \supseteq V \setminus O_v$. 24
Proof. Recall that \( V \setminus O_v \) is the union of all inner boundary components of \( v \). Let \( I = [B, B'] \) be any inner boundary component of \( v \), say without loss of generality a right one. Then \( w \in O_v \subseteq [B', v] \). Since \( B' \) is a boundary of \( v \) and \( v \in O_w \), we have \( td(v, u) < td(w, u) \) for all \( u \in [v, B'] \supseteq I \). \qed

We call an edge \( \{x, x+1\} \) a left (resp. right) blocker for the vertex \( v \) if \( td(v, x+1) < \tau(x, x+1) - 1 \) (resp. \( td(v, x) < \tau(x, x+1) - 1 \)), i.e., if a temporal path from \( v \) to \( x \) needs to wait at least one time step at \( x + 1 \). Clearly, blockers are a special case of boundaries.

To keep track of the situation even after a player has reached all vertices of \( C \), using the following step count functions is sometimes preferable to just using \( \Omega \). For any \( t \geq 0, v \in [n] \), the number of steps that one can move from \( v \) to the left until time \( t \) is

\[
\omega^t_<(v) := \begin{cases} 0 & \text{if } t \leq 0, \\ \omega^{t-1}_<(v) & \text{if } 0 < t < \tau(v - \omega^{t-1}_<(v) - 1, v - \omega^{t-1}_<(v)) \\ \omega^{t-1}_<(v) + 1 & \text{otherwise.} \end{cases}
\]

Symmetrically, the number of possible steps to the right is

\[
\omega^t_>(v) := \begin{cases} 0 & \text{if } t \leq 0, \\ \omega^{t-1}_>(v) & \text{if } 0 < t < \tau(v + \omega^{t-1}_>(v) - 1, v + \omega^{t-1}_>(v) + 1) \\ \omega^{t-1}_>(v) + 1 & \text{otherwise.} \end{cases}
\]

Note that we do not stop once \( v \) has reached \( v \) again, i.e., \( \omega^t_<(v) \) can be larger than \( n \). Note further that the second case in the above definitions only occurs if the respective edge is a blocker of \( v \). Also, setting \( \omega^t(v) := 1 + \omega^t_<(v) + \omega^t_>(v) \), the following is true for all \( t \geq 0 \) (both cases may hold simultaneously):

\[
\omega^t(v) = \begin{cases} |\Omega^t(v)| & \text{if } \Omega^t(v) \neq V \text{ or there is an edge } e \text{ with } \tau_e > t, \\ \omega^{t-1}(v) + 2 & \text{if there is no edge } e \text{ with } \tau_e > t. \end{cases}
\] (3)

Using these notions, we can give the following easy characterization of boundaries.

**Lemma 47.** Two distinct vertices \( v, w \) of \( C \) have a common right boundary \( B = \{b, b+1\} \) with \( v < w < B \) if and only if

\[
\omega^t_>(v) = \omega^t_>(w) + w - v
\]

holds for some time \( t \). In that case, equality holds for all \( t \geq td(v, b) \).

Analogously, \( v \) and \( w \) have a common left boundary \( B = \{b-1, b\} \) with \( B < w < v \) if and only if

\[
\omega^t_<(v) = \omega^t_<(w) + v - w
\]

holds for some time \( t \). In that case, equality holds for all \( t \geq td(v, b) \).

**Proof.** We only prove the first half, as the second half follows by symmetry.

If \( B \) is a right boundary of \( v \) and \( w \) with \( v < w < B \), then the player starting at \( w \) will be forced to wait at \( b \) until time \( \tau_B - 1 \geq td(v, b) \). Thus, eventually

\[
v + \omega^{td(v, b)}_>(v) = b = w + \omega^{td(v, b)}_>(w).
\]

It is further clear that \( v + \omega^t_<(v) = w + \omega^t_>(w) \) must then also hold for all \( t > td(v, b) \).
For the converse, assume now \( t \) to be chosen minimally such that

\[
\omega_{\leq}^{t-1}(v) < \omega_{\leq}^{t-1}(w) + w - v \leq \omega_{\leq}^{t}(w) + w - v = \omega_{\leq}^{t}(v).
\]

Since \( \omega_{\leq}^{t}(v) \leq \omega_{\leq}^{t-1}(v) + 1 \), we must then have \( \omega_{\leq}^{t}(w) = \omega_{\leq}^{t-1}(w) \). Therefore, the edge \( \{w + \omega_{\leq}^{t}(w), w + \omega_{\leq}^{t}(w) + 1\} \) is a (right) blocker for \( w \) and thus also a (right) boundary of \( v \), as \( \omega_{\leq}^{t}(w) + w - v = \omega_{\leq}^{t}(v) \).

We specifically note the following consequence of Lemma 47.

**Corollary 48.** Let \( v, w \) be two vertices of \( C \). Then \( w \in O_v \) if and only if

\[
\begin{align*}
\omega_{\leq}^{t}(v) &< \omega_{\leq}^{t}(w) + w - v \\
\omega_{\leq}^{t}(v) &< \omega_{\leq}^{t}(w) + v - w
\end{align*}
\]

hold for all \( t \geq 0 \).

**Proof.** Clearly both inequalities hold for \( t = 0 \) (unless \( v = w \) in which case we are done). By Lemma 47 neither of the two can hold with equality for any value of \( t \). Combining this with the fact that both sides of the inequalities can only increase by 1 during each time step yields the claim.

The following technical lemma will be the key to solving temporal difference Voronoi games. It’s precondition can be paraphrased as “none of the two players has caught up with their opponent yet in neither direction”.

**Lemma 49.** Let \( v, w \) be two vertices of \( C \) and \( t \geq 0 \). If

\[
\begin{align*}
\frac{v - \omega_{\geq}^{t-1}(v)}{v + \omega_{\geq}^{t-1}(v)} &\neq \frac{w - \omega_{\geq}^{t-1}(w)}{w + \omega_{\geq}^{t-1}(w)} \\
\frac{v - \omega_{\geq}^{t-1}(v)}{v + \omega_{\geq}^{t-1}(v)} &\neq \frac{w - \omega_{\geq}^{t-1}(w)}{w + \omega_{\geq}^{t-1}(w)}
\end{align*}
\]

then

\[
\Delta^t(v, w) = u_1^t(v, w) - u_1^t(w, v) = \omega_1^t(v) - \omega_1^t(w).
\]

**Proof.** Note first that the assumption of the lemma must also hold for all \( t' < t \) (by Lemma 47 if we ever had equality then this would propagate to all subsequent times).

We now use induction over \( t \). Clearly, the claim holds for \( t = 0 \).

Observe that

\[
\begin{align*}
u_1^t(v, w) &= u_1^{t-1}(v, w) \\
&\quad + (\omega_{\leq}^{t-1}(v) - \omega_{\leq}^{t-1}(v)) \cdot [v - \omega_{\leq}^{t}(v) \notin \Omega^t(w)] \\
&\quad + (\omega_{\leq}^{t-1}(v) - \omega_{\leq}^{t-1}(v)) \cdot [v + \omega_{\leq}^{t}(v) \notin \Omega^t(w)].
\end{align*}
\]

Let now \( t_{vw} \) be the first time that \( v \) and \( w \) “meet” inside \([v, w]\), i.e., the smallest value for which \( \omega_{\geq}^{t}(v) + \omega_{\leq}^{t}(w) \geq v - w \). Let analogously \( t_{vw} \) be minimal with \( \omega_{\geq}^{t}(v) + \omega_{\leq}^{t}(w) \geq v - w \).

Assume now that \( \omega_{\geq}^{t}(v) > \omega_{\leq}^{t-1}(v) \). Then we claim that

\[
v + \omega_{\leq}^{t-1}(v) \in \Omega^t(w) \iff t \geq t_{vw}.
\]

The reason for this is that our assumption from the lemma implies that \( \omega_{\leq}^{t-1}(v) < v - w + \omega_{\leq}^{t-1}(v) \), therefore \( w \) cannot, at time \( t - 1 \), have reached \( v + \omega_{\leq}^{t-1}(v) \) by going in positive direction.
Consequently, \( w \) cannot reach \( v + \omega_{<}^{-1}(v) + 1 = v + \omega_{<}^{-1}(v) \) until time \( t \) by going in positive direction. This establishes that \( v + \omega_{>}^{-1}(v) \in \Omega^t(w) \) implies \( t \geq t_{vw} \). Moreover, if \( t \geq t_{vw} \), then by the definition of \( t_{vw} \) we have that \( v + \omega_{>}^{-1}(v) \in \Omega^t(w) \).

For symmetrical reasons, if \( \omega_{<}^{-1}(v) > \omega_{<}^{-1}(v) \), then

\[
v - \omega_{<}^{-1}(v) \in \Omega^t(w) \iff t \geq t_{vw}.
\]

Together, these two claims allow us to rewrite (1) as follows:

\[
u_1^t(v, u) = u_1^{-1}(v, w) + (\omega_{<}^{-1}(v) - \omega_{<}^{-1}(v)) \cdot [t < t_{vw}] + (\omega_{>}^{-1}(v) - \omega_{>}^{-1}(v)) \cdot [t < t_{vw}]
\]

(5)

Our assumptions from the lemma directly imply that \( \omega_{<}^{-1}(v) < w - v + \omega_{>}^{-1}(w) \) and \( \omega_{<}^{-1}(v) < v - w + \omega_{<}^{-1}(w) \). This and the definitions of \( t_{vw} \) and \( t_{vw} \) give us the following two implications (since any edge that has already been crossed by the player starting at \( w \) must also be available to the player starting at \( v \)).

\[
t \geq t_{vw} \implies \omega_{<}^{-1}(v) = \omega_{<}^{-1}(v) + 1
\]

(6)

\[
t \geq t_{vw} \implies \omega_{>}^{-1}(v) = \omega_{>}^{-1}(v) + 1
\]

(7)

By symmetry, (5)–(7) also hold with \( v \) and \( w \) swapped. From (5), we obtain

\[
u_1^t(v, u) - u_1^{-1}(v, w) = u_1^{-1}(v, w) - u_1^{-1}(v, w) + (\omega_{<}^{-1}(v) - \omega_{<}^{-1}(v)) \cdot [t < t_{vw}] + (\omega_{>}^{-1}(v) - \omega_{>}^{-1}(v)) \cdot [t < t_{vw}]
\]

\[
= \omega_{<}^{-1}(v) + \omega_{<}^{-1}(v) - \omega_{<}^{-1}(v) - \omega_{<}^{-1}(v) + (*)
\]

\[
= (\omega_{<}^{-1}(v) - \omega_{<}^{-1}(v)) \cdot [t \geq t_{vw}] + (\omega_{>}^{-1}(v) - \omega_{<}^{-1}(v)) \cdot [t \geq t_{vw}]
\]

\[
+ (\omega_{<}^{-1}(v) - \omega_{<}^{-1}(v)) \cdot [t < t_{vw}] + (\omega_{>}^{-1}(v) - \omega_{<}^{-1}(v)) \cdot [t < t_{vw}]
\]

(\*)

and, using (6) and (7),

\[
= (\omega_{<}^{-1}(v) - \omega_{<}^{-1}(v)) \cdot [t \geq t_{vw}]
\]

\[
+ (\omega_{>}^{-1}(v) - \omega_{<}^{-1}(v)) \cdot [t \geq t_{vw}]
\]

\[
+ (\omega_{<}^{-1}(v) - \omega_{<}^{-1}(v)) \cdot [t < t_{vw}]
\]

(\*)

\[
= \omega_{<}^{-1}(v) - \omega_{<}^{-1}(v) + \omega_{>}^{-1}(v) - \omega_{<}^{-1}(v)
\]

\[
= \omega_{<}^{-1}(v) - \omega_{<}^{-1}(v)
\]

(\*)

With Lemma (4) at hand, we can now easily determine the winner of a temporal difference Voronoi game if both players play in their opponent’s outer boundary component.

**Theorem 50.** Let \( v \) and \( w \) be vertices of \( C \) with \( v \in O_w \) and \( w \in O_v \). Then, for all \( t \geq 0 \),

\[
u_1^t(v, w) - u_1^t(w, v) = \omega^t(v) - \omega^t(w).
\]

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Theorem 51. Let \( v \) and \( w \) be vertices of \( C \) with \( w \not\in O_v \). Let \( \bar{t} = \bar{t}(v, w) \) and assume that \( \Omega^f(v) \supseteq \Omega^f(w) \). Then
\[
\begin{align*}
u_1^f(v, w) - u_1^f(w, v) &= \omega^f(v) - \omega^f(w) \quad \forall t \leq \bar{t}, \quad (i) \\
u_1^f(v, w) - u_1^f(w, v) &\geq \omega^f(v) - \omega^f(w) \geq 0 \quad \forall t \geq \bar{t}. \quad (ii)
\end{align*}
\]

Proof. The proof of \((i)\) is just a direct application of Lemma 49.

Assume without loss of generality that \( j_i(v) \) is a right boundary component. Then \( v \) has a right boundary \( B = \{b, b+1\} \) with \( v < w < B \). Observe that \( t \leq td(v, b) < \tau_B \) (as \( \Omega^f(v, b) \supseteq \Omega^f(v, b)(w) \)) and thus \( \omega^f(v) - \omega^f(w) \geq 0 \) follows from the definition \((3)\) of \( \omega \). Since, \( \Omega^f(v) \supseteq \Omega^f(w) \) for all \( t \geq \bar{t} \), we can further conclude from \((3)\) that \( \omega^f(v) - \omega^f(w) \geq 0 \) also for all \( t \geq \bar{t} \).

So it remains to show the first inequality of \((i)\). For this we also use induction, starting at \( t = \bar{t} \), for which the validity follows from \((i)\). Then (for \( t \geq \bar{t} \)) we have that \( \omega^f(v) - \omega^f(w) \) is a nonincreasing function of \( t \) and \( \omega^f(v) - \omega^f(w) \) is a constant function of \( t \). Therefore, as \( t \geq \bar{t} \) and thus \( \Omega^f(v) \supseteq \Omega^f(w) \), using the induction hypothesis, we get
\[
\begin{align*}
u_1^f(v, w) - u_1^f(w, v) &= u_1^f(v, w) - u_1^f(w, v) \\
&= u_1^f(v, w) - u_1^f(w, v) \\
&\geq u_1^f(v, w) - u_1^f(w, v) \\
&\geq \omega^f(v) - \omega^f(w) \\
&= \omega^f(v) - \omega^f(w).
\end{align*}
\]

Theorems \(50\) and \(51\) already show that a vertex \( v \) is a good choice to play on if \( \omega^f(v) \) is large for large values of \( t \). We make this more precise in the following theorem. For this, we first need to introduce the colexicographic order: For two monotone sequences \( (a^j)_i \), \( (b^j)_i \) of numbers in \( [n] \), we write
\[
(a^j)_i > (b^j)_i \iff \exists i : a^j > b^j \land \forall j > i : a^j \geq b^j.
\]

Note that, due to the monotonicity and boundedness of the sequences, this order is total.

We will use the shorthand notation \( v \succ w \) for two vertices \( v, w \) to denote that
\[
(\omega^f(v))_{t=\bar{t}(v, w)} \succ (\omega^f(w))_{t=\bar{t}(v, w)}
\]
which, by \((3)\), is equivalent to
\[
(\Omega^f(v))_{t=\bar{t}(v, w)} \succ (\Omega^f(w))_{t=\bar{t}(v, w)}.
\]

Further, we write \( v \sim w \) if
\[
(\omega^f(v))_{t=\bar{t}(v, w)} = (\omega^f(w))_{t=\bar{t}(v, w)}.
\]

Proof. The proof of \((i)\) is just a direct application of Lemma 49.

Assume without loss of generality that \( j_i(v) \) is a right boundary component. Then \( v \) has a right boundary \( B = \{b, b+1\} \) with \( v < w < B \). Observe that \( t \leq td(v, b) < \tau_B \) (as \( \Omega^f(v, b)(v) \supseteq \Omega^f(v, b)(w) \)) and thus \( \omega^f(v) - \omega^f(w) \geq 0 \) follows from the definition \((3)\) of \( \omega \). Since, \( \Omega^f(v) \supseteq \Omega^f(w) \) for all \( t \geq \bar{t} \), we can further conclude from \((3)\) that \( \omega^f(v) - \omega^f(w) \geq 0 \) also for all \( t \geq \bar{t} \).

So it remains to show the first inequality of \((i)\). For this we also use induction, starting at \( t = \bar{t} \), for which the validity follows from \((i)\). Then (for \( t \geq \bar{t} \)) we have that \( \omega^f(v) - \omega^f(w) \) is a nonincreasing function of \( t \) and \( \omega^f(v) - \omega^f(w) \) is a constant function of \( t \). Therefore, as \( t \geq \bar{t} \) and thus \( \Omega^f(v) \supseteq \Omega^f(w) \), using the induction hypothesis, we get
\[
\begin{align*}
u_1^f(v, w) - u_1^f(w, v) &= u_1^f(v, w) - u_1^f(w, v) \\
&= u_1^f(v, w) - u_1^f(w, v) \\
&\geq u_1^f(v, w) - u_1^f(w, v) \\
&\geq \omega^f(v) - \omega^f(w) \\
&= \omega^f(v) - \omega^f(w).
\end{align*}
\]

Theorems \(50\) and \(51\) already show that a vertex \( v \) is a good choice to play on if \( \omega^f(v) \) is large for large values of \( t \). We make this more precise in the following theorem. For this, we first need to introduce the colexicographic order: For two monotone sequences \( (a^j)_i \), \( (b^j)_i \) of numbers in \( [n] \), we write
\[
(a^j)_i > (b^j)_i \iff \exists i : a^j > b^j \land \forall j > i : a^j \geq b^j.
\]

Note that, due to the monotonicity and boundedness of the sequences, this order is total.

We will use the shorthand notation \( v \succ w \) for two vertices \( v, w \) to denote that
\[
(\omega^f(v))_{t=\bar{t}(v, w)} \succ (\omega^f(w))_{t=\bar{t}(v, w)}
\]
which, by \((3)\), is equivalent to
\[
(\Omega^f(v))_{t=\bar{t}(v, w)} \succ (\Omega^f(w))_{t=\bar{t}(v, w)}.
\]

Further, we write \( v \sim w \) if
\[
(\omega^f(v))_{t=\bar{t}(v, w)} = (\omega^f(w))_{t=\bar{t}(v, w)}.
\]
Theorem 52. Let $\omega$ and $w$. If $\omega \leq w$, so the claim follows by Theorem 50.

Using this newly introduced notation, we can now summarize our findings on temporal difference Voronoi games as follows.

**Theorem 52.** Let $v$, $w$ be two vertices in $C$ and $t = \bar{t}(v, w)$. Then,

$$u_1(v, w) \geq u_1(w, v) \iff \omega^t(v) \geq \omega^t(w) \iff v \succeq w.$$  

**Proof.** The second equivalence is due to the fact that $\Omega^t(v)$ and $\Omega^t(w)$ are inclusion-wise comparable for all $t \geq \bar{t}$. So it remains to prove the first equivalence.

Assume $\omega^t(v) \geq \omega^t(w)$, i.e. $\Omega^t(v) \supseteq \Omega^t(w)$ by choice of $\bar{t}$ and $\Omega$.

If $w \in O_v$, then $\Omega^t(v) = V$ by Lemma 42. This already tells us that all edges appear at or before time $t$ (we use $O_v \neq \emptyset$). In particular we have for all $t \geq \bar{t}$ that $\omega^t(v) = \omega^t(v) + 2$ and $\omega^t(w) = \omega^t(w) + 2$ and thus $\omega^t(v) - \omega^t(w) = \omega^t(v) - \omega^t(w)$. Furthermore $v \in O_w$ by Lemma 45 so the claim follows by Theorem 50.

If $w \notin O_v$, then we can directly apply Theorem 51 (i). The other implication, assume (by symmetry) $\omega^t(v) > \omega^t(w)$ and thus $\Omega^t(v) \supset \Omega^t(w)$. If $w \in O_v$, then the proof works exactly as above. If $w \notin O_v$, then by Theorem 51, $u_1^t(v, w) > u_1^t(v, w)$ and $u_1^t(v, w) \geq u_1^t(v, w)$ for all $t \geq \bar{t}$, this proves the claim.

Note that Theorem 52 also holds if all inequalities are replaced by their strict versions, since this is tantamount to simply negating and mirroring all three equivalent statements.

We call a vertex $v$ paramount if $v \succeq w$ for all other vertices $w$. Note that the existence of at least one paramount vertex $v$ is guaranteed by simply picking $v$ to maximize $(\omega^t(v))_{t=0}^\infty$ with respect to the colexicographic order.

We can now classify all Nash equilibria of temporal difference Voronoi games.

**Corollary 53.** A pair of vertices $(v, w)$ forms a Nash equilibrium in $\Delta\text{Vor}(C)$ if and only if $v$, $w$ are both paramount.

**Proof.** If $v$ and $w$ are paramount, then for all vertices $x$ by Theorem 52 $\Delta(v, x) \geq 0$ and $\Delta(w, x) \geq 0$. In particular, $\Delta(v, w) = -\Delta(w, v) = 0$ and neither player has a better response.

Assume now conversely that $(v, w)$ is a Nash equilibrium. If either vertex, say $v$, is not paramount, then there exists a vertex $w'$ with $w' > v$, thus $\Delta(w', v) > 0$ by Theorem 52. Then also $\Delta(w, v) > 0$ since $w$ is a best response to $v$. Since $\Delta(w, w) = 0 > \Delta(v, w)$, $v$ cannot be a best response to $w$.

Figure 10: In this monotonically growing cycle, $x \sim y$ and $x \sim z$, but $y \succ z$.

Since the colexicographic order is total, we have $v < w$ if and only if $v \not\succ w$. We remark that $\sim$ is not transitive and thus not an equivalence relation (see e.g. Figure 10).

Using this newly introduced notation, we can now summarize our findings on temporal difference Voronoi games as follows.
We conclude this subsection by sketching an algorithm to compute all paramount vertices.

**Theorem 54.** All paramount vertices of \( C \) can be computed in \( O(n^2) \) time.

**Proof.** For any vertex \( v \), we can compute a compact representation of the (formally infinite) vectors \( (\omega^<_v(t))_{t=1}^\infty \) and \( (\omega^>_v(t))_{t=1}^\infty \) in \( O(n) \) time, by making note of the points in time \( t \) at which the value of \( \omega^<_v(t) \) resp. \( \omega^>_v(t) \) increases. Knowing \( \omega^<_v(t) \), we need not store any further entries once we reach \( \omega^<_v(t) + \omega^>_v(t) \geq n \), as then clearly \( t \geq \max\{\tau_e \mid e \in E(C)\} \).

Having computed these two vectors for all vertices, we can subsequently test whether any given vertex \( v \) is paramount in \( O(n) \) time as follows.

Set \( \ell \leftarrow v-1 \) and \( r \leftarrow v+1 \). Starting at \( t = 1 \), iterate over all times \( t \) for which \( \omega^<_v(t) \geq \omega^<_v(t-1) \) or \( \omega^>_v(t) > \omega^>_v(t-1) \), i.e., all times where \( v \) is not stuck between two boundaries. Our algorithm will maintain the invariant that \( x \not\preceq v \) holds for all vertices \( x \) with \( \ell < x < r \).

For any such time \( t \), compare \( \Omega^t(\ell) \) and \( \Omega^t(r) \). If \( \Omega^t(\ell) \supset \Omega^t(v) \), then \( v \) is clearly not paramount and we can abort. If conversely \( \Omega^t(\ell) \subset \Omega^t(v) \), then we must have \( \ell \lec v \) since we did not abort at any previous step. Thus, we may then decrement \( \ell \) by 1 and repeat the comparison.

Symmetrically, compare also \( \Omega^t(r) \) and \( \Omega^t(v) \), possibly incrementing \( r \).

During the above, if \( \ell \) and \( r \) ever pass each other (i.e., if \( r+1 = \ell \) ), then we can immediately conclude that \( v \) is paramount as we have compared \( v \) to all other vertices.

If this does not happen, then we must reach a point where \( \Omega^t(v) \) is inclusion-wise incomparable to both, \( \Omega^t(\ell) \) and \( \Omega^t(r) \). Then observe that \( \Omega^t(v) \) must also be incomparable to \( \Omega^t(x) \) for all vertices \( x \) with \( r < x < \ell \). We then continue with the next value of \( t \).

Note that once we reach \( \omega^<_v(t) + \omega^>_v(t) \geq n \), we will iterate at most \(|n/2|\) more times — afterwards every vertex has reached every other vertex. Thus we perform at most \( O(n) \) loop iterations. Since we also update \( \ell \) and \( r \) at most \( n \) times overall, our algorithm runs in \( O(n) \) time. \( \Box \)

### 4.4.2 Temporal Voronoi games

Moving on to temporal Voronoi games, paramount vertices remain a reasonable choice for the players. However, the situation becomes slightly more intricate. Call a vertex \( w \) a paramount response to another vertex \( v \), if \( w \) is a best response to \( v \) and furthermore \( w \succeq w' \) for all other best responses \( w' \). Then in most cases a paramount vertex and a paramount response will form a Nash equilibrium; but this does not hold universally, as seen e.g. in Figure 11.

Our goal in this section will be to prove the following theorem, which states that we will reach a Nash equilibrium by iterating the above at most two times.
Theorem 55. Let $x$ be a paramount vertex, $y$ a paramount response to $x$, and $z$ a paramount response to $y$. Then $(x, y)$ or $(y, z)$ is a Nash equilibrium of Vor($C$).

From Theorem 55 the main result of this subsection directly follows:

Corollary 56. Every instance of Vor($C$) has a Nash equilibrium which can be found in $O(n^2)$ time.

Proof. We use Theorem 55. As seen in the previous subsection, a paramount vertex $x$ always exists and we can find it in $O(n^2)$ time. Since we can determine all best responses to any given vertex in $O(n^2)$ time, computing $y$ and $z$ works similarly.

The proof of Theorem 55 is split into three parts, Lemmas 58, 59 and 60. For the first and easiest part, we only need the following lemma.

Lemma 57. Let $x$ be any vertex of $C$ and $y$ a best response to $x$. If $j_x(y)$ is an inner boundary component, then the corresponding boundaries of $x$ are also boundaries of $y$.

Proof. Let $j_x(y) = [b, b']$ be a right boundary component of $x$ without loss of generality. Note that $U_1(y, x) \subseteq j_x(y)$ (Lemma 51) and, since $y$ is a best response to $x$, also $U_1(y, x) \supseteq U_1(b, x) = j_x(y)$. In particular, $td(y, b) < td(x, b) = \tau_{b-1}(b)$ and $td(y, b') < td(x, b') < \tau_{b'}(b' + 1)$. \hfill \Box

We can now give the first of three parts of the proof of Theorem 55 in the form of the following lemma. It covers all cases where $z \notin O_x$.

Lemma 58. Let $x$ be a paramount vertex of $C$, $y$ a best response to $x$, and $z$ a best response to $y$ with $z \notin O_x$. Then $(x, y)$ is a Nash equilibrium of Vor($C$).

Proof. It suffices to show that $u_1(z, y) \leq u_1(x, y)$. Let $j_x(z) = [a, b]$ be a left boundary component of $x$ without loss of generality. We distinguish three cases.

If $y \in j_x(z)$, then Lemma 57 applies to $y$ (since $j_x(z) = j_x(y)$ is an inner component) and yields $j_y(z) \subseteq j_x(z)$. Otherwise, if $y \notin j_x(z)$ and additionally $td(x, b) \geq td(y, b)$, we have $b < y < x$, as $[a-1, a]$ is a left boundary of $x$ and $y \notin [a, b]$. By Lemma 43 we get that $[b, b+1]$ is a left boundary of $y$, implying $j_y(z) = j_x(z)$ (recall Lemma 41). So we have in both of these cases that $U_1(z, y) \subseteq j_y(z) \subseteq j_x(z)$ (by Lemma 44). Also, $|j_x(z)| = u_1(b, x) \leq u_1(y, x)$ where the inequality holds because $y$ is a best response to $x$. Furthermore, $u_1(y, x) \leq u_1(x, y)$ since $x$ is paramount (Theorem 22). Together this gives $u_1(z, y) \leq u_1(b, x) \leq u_1(y, x) \leq u_1(x, y)$.

It remains to consider the case that $y \notin j_x(z)$ and $td(x, b) < td(y, b)$. Since $x$ has no blockers in $[a, b]$ and $td(x, a) < \tau_{b-1}(a)$, we then also have $td(x, a) < td(y, a)$ and thus $j_x(z) \subseteq U_1(x, y)$. Further, note that $U_1(z, y) \subseteq U_1(z, x) \cup U_1(x, y)$. Since $U_1(z, x) \subseteq j_x(z)$ (Lemma 44), this yields $U_1(z, y) \subseteq j_x(z) \subseteq U_1(x, y)$, which proves the claim. \hfill \Box

For the next part of the proof of Theorem 55 we first need to briefly investigate how a section of $C$ (i.e., a temporal path graph) is split between the two players if neither of them has any blockers in the area. To this end, assume for simplicity that the temporal path graph $P$ is obtained from $C$ by deleting the edge $\{n-1, 0\}$.

Lemma 59. Let $x < v \leq w < y$ be vertices of $P$. If $x$ and $y$ have no blockers in $[v, w]$, then in Vor($P$)

\[ U_1(x, y) \cap [v, w] = [v, \min\{\lceil z \rceil - 1, w\}] \text{ and} \]
\[ U_1(y, x) \cap [v, w] = [\max\{\lfloor z \rfloor + 1, v\}, w], \]

where

\[ z = \frac{v + w + td(y, w) - td(x, v)}{2}. \]

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Proof. It is easy to verify that, unless one of \( x, y \) reaches all of \([v, w]\) before the other, the two will meet each other exactly at point \( z \) as above, where \( z \) might be either an integer (i.e., a vertex) or have fractional part \( 0.5 \) (i.e., an edge). If \( z \) is an integer, then the corresponding vertex is colored gray, and \([z] - 1 = z - 1 \in U_1(x, y)\) and \([z] + 1 = z + 1 \in U_1(y, x)\). Otherwise, we have \([z] - 1 = [z] \in U_1(x, y)\) and \([z] + 1 = [z] \in U_1(y, x)\). This proves the claim. \( \Box \)

From Lemma 60, we can directly conclude the following, in which \( \text{med}(a, b, c) \) denotes the median of the three numbers \( a, b, c \).

**Lemma 60.** Let \( x < v \leq w < y \) be vertices of \( \mathcal{P} \). Define

\[
M := \text{med} \left( 0, \left\lfloor \frac{w - v + \text{td}(y, w) - \text{td}(x, v)}{2} \right\rfloor, w - v + 1 \right).
\]

Then the following holds for \( \text{Vor}(\mathcal{P}) \):

(i) If \( x \) has no blockers in \([v, w]\), then \( |U_1(x, y) \cap [v, w]| \geq M \).

(ii) If \( y \) has no blockers in \([v, w]\), then \( |U_1(x, y) \cap [v, w]| \leq M \).

We can now apply Lemma 60 to an outer boundary component on the cycle as follows.

**Lemma 61.** Let \( x, a, \) and \( b \) be vertices of \( \mathcal{C} \), \( O_x = [a, b] \) and suppose that \( \text{td}(x, a) \leq \text{td}(x, b) \). Then

\[
u_1(a, x) \geq \min \left\{ \left\lfloor \frac{b - a + 1 + \text{td}(x, b) - \text{td}(x, a)}{2} \right\rfloor, b - a + 1 \right\}.
\]

Proof. If \( a = b \), then the statement is easy to verify. Otherwise, \( \{a, a + 1\} \) is no boundary of \( x \), therefore \( \text{td}(a, a + 1) = \tau(a, a + 1) \leq \text{td}(x, a) \) and we may assume equality without loss of generality since we are interested in a lower bound for \( u_1(a, x) \). Then, as \( x \) has no boundary in \([a, b], a \) has no blockers in \([a + 1, b]\). Thus, by applying Lemma 60 (i), we obtain

\[
u_1(a, x) = 1 + |U_1(a, x) \cap [a + 1, b]| \\
\geq 1 + \text{med} \left( 0, \left\lfloor \frac{b - (a + 1) + \text{td}(x, b) - \text{td}(a, a + 1)}{2} \right\rfloor, b - a \right) \\
= \min \left\{ \left\lfloor \frac{b - a + 1 + \text{td}(x, b) - \text{td}(x, a)}{2} \right\rfloor, b - a + 1 \right\},
\]

where the first equality holds by Lemma 59 as \( \mathcal{J}_x(a) = [a, b] \) and the last equality holds as we have assumed that \( \text{td}(a, a + 1) = \text{td}(x, a) \) and \( \text{td}(x, a) \leq \text{td}(x, b) \).

We also record the following easy observation.

**Lemma 62.** Let \( v \) be a vertex of \( \mathcal{C} \) with left boundary \( A \) and right boundary \( B \). If a vertex \( w \notin[A, B] \) has no boundaries inside \([A, B]\), then \( U_1(v, w) \supseteq [A, B] \).

Proof. Suppose a vertex \( x \in [A, B] \) had \( \text{td}(w, x) \leq \text{td}(v, x) \). Say without loss of generality \( x \in [A, v] \). Since \( \tau_A > \text{td}(v, x) \), \( w \) must first reach \( x \) via \( v \). By Lemma 59, this contradicts \( w \) not having any boundaries in \([A, v]\) (we can treat \( \mathcal{C} \) as a path by cutting at \( A \)).

Using the results above, we can now prove the second puzzle piece of Theorem 55.

---

3We do not consider the boundaries \( A \) and \( B \) to be inside \([A, B]\).
Lemma 63. Let $x$ be a paramount vertex of $C$, $y$ a best response to $x$, and $z$ a best response to $y$ and suppose that $y \notin O_x$ and $z \in O_x$. Then $(x, y)$ is a Nash equilibrium of $\text{Vor}(C)$.

Proof. It suffices to show $u_1(x, y) \geq u_1(z, y)$. Without loss of generality $y_1(y)$ is a right boundary component of $x$. Let $B_1$ be the rightmost boundary $B$ of $x$ with $x < B < y$ (note that $B_1$ is one of the two boundaries enclosing $y_1(y)$). Let further $B_2$ be the rightmost (right) boundary of $x$ and $B_3$ the leftmost (left) boundary of $x$, that is, $[B_2, B_3] = O_x$. See Figure 12 for an illustration. We set $t_1 := \tau_{B_1}$.

First consider the case that $x \notin O_y$. Then $y$ has a left boundary $B$ with $B < x < y$. By Lemma [53] $B$ is also a left boundary of $x$, therefore $B_3 < B < x$. Thus, $B_3$ is also a left boundary of $y$ by Lemma [54]. Since $B_2$, too, is a common right boundary of $x$ and $y$ by Lemma [53] we then have $\text{td}(x, v) = \text{td}(y, v)$ for all $v \in [B_2, B_3]$, and in particular $O_y = O_x = [B_2, B_3]$. From this and Lemma [54] we get that $U_1(z, y) = U_1(z, x)$. Thus, $u_1(x, y) \leq u_1(z, y) = u_1(z, x) \leq u_1(y, x)$ by the choice of $y$ and $z$. Since $x$ is paramount, all of these inequalities must be equalities (by Theorem [52]), and the claim is proven.

In the second case, we have $x \in O_y$. Let $\xi$ be the number of vertices in $[B_3, B_1]$, $\eta$ the number of vertices in $[B_1, B_2]$, and $\zeta$ the number of vertices in $O_x = [B_2, B_3]$ (compare Figure 12). By Lemma [57] $B_1$ is a left boundary of $y$, and, since $x \in O_y$, $B_1$ is the leftmost left boundary of $y$. (Note that there cannot be a left boundary of $y$ between $B_1$ and $x$, since this would contradict $B_1$ being a right boundary of $x$.) Therefore, since $B_3$ is a left boundary of $x$ but not of $y$, $t_3 < t_1 + \xi$. Also, by Lemma [58] $B_2$ is a right boundary of $y$ and by Lemma [54] the rightmost right boundary of $y$. Thus, $O_y = [B_2, B_1]$.

Further, we have

$$t_2 \geq t_1 + \eta \geq t_1 + u_1(y, x) \tag{8}$$

where the first inequality holds as $B_1$ and $B_2$ are boundaries of $x$ and the second inequality holds by Lemma [54]. Note that $|t_2 - t_3| < \zeta$ since $O_x \neq \emptyset$. Moreover, using that $O_y = [B_2, B_1]$, applying Lemma [54] yields $[B_3, B_1] \subseteq U_1(x, y)$. It remains to find out how many vertices from $[B_2, B_3]$ are colored by $x$ (where we already know that the rest will be colored gray). To this end, we use Lemma [53] (mirrored), since $x$ clearly has no blockers in $O_x = [B_2, B_3]$ and since we can ignore the fact that $y$ may reach vertices via $B_3$ (as $x \in O_y$ will have reached them first). This gives us

$$u_1(x, y) \geq \xi + \text{med} \left(0, \left\lfloor \frac{\zeta - 1 + t_2 - t_3}{2} \right\rfloor, \zeta \right) = \xi + \left\lfloor \frac{\zeta - 1 + t_2 - t_3}{2} \right\rfloor. \tag{9}$$
Let $B_3 := [b_3 - 1, b_3]$. Again using Lemma 61, since $U_1(z, y) \subseteq [B_2, B_1]$ (both ends being boundaries of $y$) and as we have already observed above that $y$ has no boundaries and thus in particular no blockers in $[B_3, B_1]$, we get

$$u_1(z, y) \leq \zeta + \text{med} \left( 0, \left\lceil \frac{\zeta - 1 + t_1 - \text{td}(z, b_3)}{2} \right\rceil, \xi \right) \leq \zeta + \text{med} \left( 0, \left\lceil \frac{\zeta - 1 + t_1 - t_3}{2} \right\rceil, \xi \right) \leq \zeta + \left\lceil \frac{\zeta - 1 + t_1 - t_3}{2} \right\rceil,$$

(10)

where we used for the second inequality that $\text{td}(z, b_3) \geq t_3$. Finally, by Lemma 61 and as $y$ is a best response to $x$, it follows

$$u_1(y, x) \geq \min \left\{ \left\lceil \frac{\zeta + |t_2 - t_3|}{2} \right\rceil, \zeta \right\} \geq \left\lceil \frac{\zeta + |t_2 - t_3|}{2} \right\rceil.$$

(11)

Using the above, we then derive by (10) and (11)

$$2(u_1(x, y) - u_1(z, y)) \geq 2\zeta + 2 \left\lceil \frac{\zeta - 1 + t_2 - t_3}{2} \right\rceil - 2\zeta - 2 \left\lceil \frac{\zeta - 1 + t_1 - t_3}{2} \right\rceil \\
= 2\zeta - 2\zeta + \zeta - 1 + t_2 - t_3 - \zeta + 1 - t_1 + t_3 - 1 \\
\geq t_3 + t_2 - 2t_1 - \zeta \quad \quad \quad \quad \quad \quad \quad \quad \quad \text{since } t_3 < t_1 + \xi \\
\geq t_3 + t_2 - 2t_1 - 2u_1(y, x) + |t_2 - t_3| \quad \quad \quad \quad \quad \quad \quad \quad \quad \text{by (11)} \\
\geq t_3 - t_2 + |t_2 - t_3| \quad \quad \quad \quad \quad \quad \quad \quad \quad \text{by (8)} \\
\geq 0$$

Thus, $u_1(x, y) \geq u_1(z, y)$. \qed

The following easy result will help us in the last part of the proof of Theorem 55.

**Lemma 64.** If $v \in O_w$ and $w \in O_v$ are two vertices of $C$, then the pair of positions $(v, w)$ produces at most two gray vertices in $\text{Vor}(C)$.

**Proof.** By Lemma 12 $\Omega'(v) \subseteq \Omega'(w)$ or $\Omega'(w) \subseteq \Omega'(v)$ for some $t$ implies that $\Omega'(v) = \Omega'(w) = V$. Thus, any gray vertex must be reached from the two players from opposite sides, so there can be at most one gray vertex in each of the two components in which $v$ and $w$ divide $C$. \qed

We need one last ingredient before we can solve the final piece of the puzzle:

**Lemma 65.** Let $x$ be a paramount vertex of $C$, and $y$ a paramount response to $x$ with $y \sim x$. Then $y$ is paramount.

**Proof.** Suppose for contradiction that there was a vertex $y' \succ x$. Since $x \sim y$, clearly $y' \neq x$. Assume without loss of generality that $y < y' < x$. Note that we must have $\tilde{i}(y, y') < \tilde{i}(x, y)$, for otherwise $y' \succ x$ would follow from the definition of $\succ$, contradicting the fact that $x$ is paramount. Therefore $\Omega'(y, y')(y') \neq V$, so $y \notin O_{y'}$ by Lemma 12. Thus, $\gamma_{y'}(y)$ is a left boundary interval. Let $B$ be its left boundary, i.e., the rightmost left boundary of $y'$ with $B < y < y'$.

As $\tilde{i}(y, y') < \tilde{i}(x, y)$, vertex $x$ has no boundaries inside $[B, y']$, thus $[B, y'] \subseteq U_1(y', x)$ by Lemma 62. Then it is easy to see that $U_1(y, x) \subseteq U_1(y', x)$, i.e., $y'$ is a best response to $x$. Since $y'$ is paramount, this contradicts the choice of $y$ as paramount response to $x$. \qed
Now, we can finally prove the remaining case of Theorem 55.

**Lemma 66.** Let \( x \) be a paramount vertex of \( C \), \( y \) a paramount response to \( x \), and \( z \) a paramount response to \( y \), and suppose that \( y, z \in O_z \). Then \((x, y)\) or \((y, z)\) is a Nash equilibrium of \( \text{Vor}(C) \).

**Proof.** Without loss of generality, assume that \( x < y < z \). Further, we have \( x \in O_y \cap O_z \) by Lemma 45 and Theorem 52, as \( x \) is paramount. We prove the statement via case distinction.

**Case 1: \( z \notin O_y \)** Then \( j_y(z) \) is an inner boundary component of \( y \) and \( U_1(z, y) \subseteq j_y(z) \subseteq V \setminus O_y \subseteq U_1(y, x) \) by Lemmas 44 and 46. Since \( x \) is paramount, we have \( u_1(y, x) \leq u_1(x, y) \) (Theorem 52). Together, this gives \( u_1(z, y) \leq u_1(x, y) \). Thus, as \( z \) is a best response to \( y \), \( x \) is a best response to \( y \) and \( (x, y) \) is a Nash equilibrium.

**Case 2: \( z \in O_y \) and \( y \notin O_z \)** Then \( j_z(y) \) is an inner boundary component of \( z \) and \( U_1(y, z) \subseteq j_z(y) \subseteq V \setminus O_z \subseteq U_1(z, x) \) by Lemmas 44 and 46. Therefore,

\[
U_1(y, z) \subseteq U_1(y, z) \cup U_1(z, x) \subseteq j_z(y) \cup U_1(z, x) \subseteq U_1(z, x),
\]

which means that \( z \) is a best response to \( x \). Also, by Lemma 45 there is a time \( t \) with \( \Omega^t(z) \supset \Omega^t(y) \). By Theorem 52 this contradicts \( y \) being a paramount response to \( x \).

**Case 3: \( z \in O_y \) and \( y \in O_z \)** Define \( \Delta_+ := U_1(z, y) \setminus U_1(x, y) \) and \( \Delta_- := U_1(x, y) \setminus U_1(z, y) \). We assume \( |\Delta_+| > |\Delta_-| \) since otherwise \((x, y)\) is a Nash equilibrium and we are done.

Let \( t_{yx} \) be the first time for which \( \omega_{t_{yx}}^y(y) + \omega_{t_{yx}}^x(x) \geq x - y \), that is, the first time that \( x \) and \( y \) reach a common vertex in \([y, x]\). Let also \( t_{yz} \) be the first time for which \( \omega_{t_{yz}}^y(y) + \omega_{t_{yz}}^x(z) \geq z - y \). Note that as \( x < y < z \), we have \( t_{yz} \leq t_{yx} \).

Then we claim that \( t_{yz} < t_{yx} \). Otherwise, if \( t' := t_{yz} = t_{yx} \), then we would have

\[
x - y - \omega_{t'}^x(x) = \omega_{t'}^y(y) = z - y - \omega_{t'}^x(z)
\]

and thus

\[
\omega_{t'}^y(x) = \omega_{t'}^y(z) + x - y - z - y = \omega_{t'}^x(z) + x - z
\]

in contradiction to \( z \in O_z \) (Corollary 48).

Since \( z \in O_y \) and \( y \in O_z \), Lemma 43 implies that

\[
\forall t \geq t_{yz} : \quad \omega_{t}^y(z) = \omega_{t}^y(z) + 1 \quad \text{and} \quad \omega_{t}^x(y) = \omega_{t}^x(y) + 1
\]

and similarly, as \( y \in O_z \)

\[
\forall t \geq t_{yz} : \quad \omega_{t}^y(x) = \omega_{t}^y(x) + 1.
\]

Let \( \delta_+ := t_{yx} - t_{yz} > 0 \) (where the inequality holds as we have shown above that \( t_{yz} < t_{yx} \)) and define \( \lambda_{yz}, \lambda_{yz} \in \{0, 1\} \) as follows:

\[
\omega_{t_{yx}}^y(y) + \omega_{t_{yx}}^x(x) = x - y + \lambda_{yz},
\]

\[
\omega_{t_{yz}}^y(y) + \omega_{t_{yz}}^x(z) = z - y + \lambda_{yz}.
\]

Observe that \( \lambda_{yz} = 0 \) if and only if a player at position \( y \) and a player at \( x \) simultaneously reach some vertex in \([y, x]\), i.e., if some vertex in \([y, x]\) is colored gray. An analogous statement holds for \( \lambda_{yz} \).
Then, for all $t \geq t_{yx} > t_{yz}$, it holds
\[
\omega_t^l(x) - \omega_t^l(z) = \omega_t^{\rho z}(x) - \omega_t^{\rho y}(z) \quad \text{by (12), (13)}
\]
\[
= x - y + \lambda_{yx} - \omega_t^{\rho y}(y) - \omega_t^{\rho y}(z)
\]
\[
= x - y + \lambda_{yx} - \omega_t^{\rho y}(y) - \omega_t^{\rho y}(z) - 2\delta_+ \quad \text{by (12)}
\]
\[
= x - y + \lambda_{yx} - z - y - \lambda_{yz} - 2\delta_+
\]
\[
= x - z + \lambda_{yx} - 2\delta_+
\]
\[
= x - z - 2|\Delta^+| - \lambda_{xy} + \lambda_{yz}.
\]

In the above equation we used the fact that $|\Delta^+| = \delta_+ - \lambda_{yx} + \lambda_{yz}$, which is due to $y$ reaching exactly one vertex from $\Delta^+$ during each time $t' \in [t_{yz} + 1 - \lambda_{yz}, t_{yx} - \lambda_{yx}]$ (see also Figure 13).

Analogously, let $t_{zy}$ be the first time where $\omega_t^{\rho y}(z) + \omega_t^{\rho y}(y) \geq y - z$ and let $t_{xy}$ be the first time where $\omega_t^{\rho y}(x) + \omega_t^{\rho y}(y) \geq y - x$. Let $\delta_- := t_{zy} - t_{xy} > 0$ and define $\lambda_{xy}, \lambda_{xy}$ as
\[
\omega_t^{\rho y}(z) + \omega_t^{\rho y}(y) = y - z + \lambda_{xy},
\]
\[
\omega_t^{\rho y}(x) + \omega_t^{\rho y}(y) = y - x + \lambda_{xy}.
\]

Analogously to above, for all $t \geq t_{xy}$, it holds
\[
\omega_t^l(x) - \omega_t^l(z) = \omega_t^{\rho y}(x) - \omega_t^{\rho y}(z)
\]
\[
= \omega_t^{\rho y}(x) + \omega_t^{\rho y}(y) - y - z - \lambda_{xy}
\]
\[
= \omega_t^{\rho y}(x) + \omega_t^{\rho y}(y) + 2\delta_+ - y - z - \lambda_{xy}
\]
\[
= y - x + \lambda_{xy} + 2\delta_+ - y - z - \lambda_{xy}
\]
\[
= -x - z + \lambda_{xy} + 2\delta_+ - \lambda_{xy}
\]
\[
= -x - z + 2|\Delta^-| - \lambda_{xy} + \lambda_{yz}.
\]
Since \( x \) is paramount, we now obtain from Theorem 52 that, for sufficiently large \( t \), it holds
\[
0 \leq \omega^t(x) + \omega^t(z) - (\omega^t(z) + \omega^t(z)) \\
= 2(|\Delta_-| - |\Delta_+| + \lambda_{yz} + \lambda_{zy} - \lambda_{xy} - \lambda_{yx}) \leq 0,
\]
from which we conclude that
\[
\omega^t(x) = \omega^t(z), \quad |\Delta_+| = |\Delta_-| + 1, \quad \lambda_{yz} = \lambda_{zy} = 1, \quad \text{and} \quad \lambda_{xy} = \lambda_{yx} = 0.
\]

Furthermore,
\[
u_1(y, x) \leq \nu_1(x, y) = n - 2 - \nu_1(y, x) \leq n - 2 - \nu_1(z, x) \leq \nu_1(x, z) = u_1(z, x),
\]
where the first inequality is due to \( x \) being paramount, the first equality is by Lemma 64 and (16), the second inequality is due to the fact that \( y \) is a best response to \( x \), the third inequality is again by Lemma 64, and the last equality is by Theorem 50 and (14).

Since also \( \nu_1(z, x) \leq \nu_1(y, x) \) by choice of \( y \), all of the above inequalities are in fact equalities. In particular, \( \nu_1(x, y) = \nu_1(y, x) = (n - 2)/2 \). We can thus deduce by Lemma 65 that \( y \) is paramount. Therefore also \( u_1(y, z) \geq u_1(z, y) \).

We claim that \( z \) is paramount, too. Otherwise, we must have \( u_1(z, y) > \nu_1(x, y) \) by choice of \( z \). Furthermore \( u_1(y, z) > u_1(z, y) \) or we could deduce the claim from Lemma 65. Thus we get \( u_1(y, z) > u_1(z, y) > \nu_1(x, y) = n/2 - 1 \), i.e., \( u_1(y, z) + u_1(z, y) \geq n + 1 \), which is clearly impossible.

Due to this, \((y, z)\) is a Nash equilibrium in \( \Delta\text{Vor}(C, 2) \) by Corollary 53 (as well as \((x, y)\) and \((x, z)\)). As there are no gray vertices (by (15)), this implies that \((y, z)\) is also a Nash equilibrium in \( \text{Vor}(C) \): Any opportunity for one player to improve would have to come with a loss for the other player, which would contradict \((y, z)\) being a Nash equilibrium in \( \Delta\text{Vor}(C, 2) \).

As Lemmas 58, 63 and 66 together cover all possible cases, we have now successfully proved Theorem 55. Note that in the only case where \((x, y)\) did not form a Nash equilibrium (case 3 of Lemma 66), we showed \( y \) to be paramount. Thus there always exists some paramount vertex that is part of a Nash equilibrium.

5 Conclusion

Our work is meant to initiate further systematic studies of (not only competitive) games on (classes of) temporal graphs. There is a wealth of unexplored research directions to pursue.

Extending our work on temporal trees and cycles, there are also many more special temporal graphs to study such as temporal grids. Another direction is to consider variations of temporal diffusion and Voronoi games. For example, as already partly studied in difference diffusion/Voronoi games, the payoff could be defined as the difference of the number of vertices colored by the players. Related to this, “splitting” gray vertices between players is another possibility and was already studied on static games. Finally, for Voronoi games, there are several different temporal distance notions to consider. For example, one may study non-strict walks (i.e., walks that traverse multiple edges in a single time step). Note that these are trivial on monotonically shrinking temporal graphs. We conjecture that Theorem 33 still holds for such non-strict Voronoi games — in fact, we know of no instance of any monotonically growing temporal graph without a Nash equilibrium.
References

[1] Hee-Kap Ahn, Siu-Wing Cheng, Otfried Cheong, Mordecai J. Golin, and René van Oostrum. Competitive facility location: the Voronoi game. *Theoretical Computer Science*, 310(1-3): 457–467, 2004. doi:10.1016/j.tcs.2003.09.004

[2] Noga Alon, Michal Feldman, Ariel D. Procaccia, and Moshe Tennenholtz. A note on competitive diffusion through social networks. *Information Processing Letters*, 110(6):221–225, 2010. doi:10.1016/J.IPL.2009.12.009

[3] Sayan Bandyapadhyay, Aritra Banik, Sandip Das, and Hirak Sarkar. Voronoi game on graphs. *Theoretical Computer Science*, 562:270–282, 2015. doi:10.1016/J.TCS.2014.10.003

[4] Aritra Banik, Bhaswar B. Bhattacharya, and Sandip Das. Optimal strategies for the one-round discrete Voronoi game on a line. *Journal of Combinatorial Optimization*, 26(4): 655–669, 2013. doi:10.1007/s10878-011-9447-6

[5] Matthias Bentert, Anne-Sophie Himmel, André Nichterlein, and Rolf Niedermeier. Efficient computation of optimal temporal walks under waiting-time constraints. *Applied Network Science*, 5(1):73, 2020. doi:10.1007/s41109-020-00311-0

[6] Laurent Bulteau, Vincent Froese, and Nimrod Talmon. Multi-player diffusion games on graph classes. *Internet Mathematics*, 12(6):363–380, 2016. doi:10.1080/15427951.2016.1197167

[7] Ankit Chauhan, Pascal Lenzner, and Louise Molitor. Schelling segregation with strategic agents. In *Proceedings of the 11th International Symposium on Algorithmic Game Theory (SAGT)*, pages 137–149, 2018, arXiv:1806.08713 doi:10.1007/978-3-319-99660-8_13

[8] Otfried Cheong, Sariel Har-Peled, Nathan Linial, and Jiří Matoušek. The one-round Voronoi game. *Discrete & Computational Geometry*, 31(1):125–138, 2004. doi:10.1007/s00454-003-2951-4

[9] Mark de Berg, Sándor Kisfaludi-Bak, and Mehran Mehr. On one-round discrete Voronoi games. In *Proceedings of the 30th International Symposium on Algorithms and Computation (ISAAC)*, pages 37:1–37:17, 2019. doi:10.4230/LIPIcs.ISAAC.2019.37

[10] Christoph Dürr and Nguyen Kim Thang. Nash equilibria in Voronoi games on graphs. In *Proceedings of the 15th Annual European Symposium on Algorithms (ESA)*, pages 17–28, 2007, arXiv:cs/0702054 doi:10.1007/978-3-540-75520-3_4

[11] Hagen Echzell, Tobias Friedrich, Pascal Lenzner, and Anna Melnichenko. Flow-based network creation games. In *Proceedings of the 29th International Joint Conference on Artificial Intelligence (IJCAI)*, pages 139–145, 2020. doi:10.24963/IJCAI.2020/20

[12] Edith Elkind, Jiaru Gan, Ayumi Igarashi, Warut Suksompong, and Alexandros A. Voudouris. Schelling games on graphs. In *Proceedings of the 28th International Joint Conference on Artificial Intelligence (IJCAI)*, pages 266–272, 2019. doi:10.24963/IJCAI.2019/38
[13] Thomas Erlebach and Jakob T. Spooner. A game of cops and robbers on graphs with periodic edge-connectivity. In Proceedings of the 46th International Conference on Current Trends in Theory and Practice of Informatics (SOFSEM), pages 64–75, 2020. arXiv:1908.06828 doi:10.1007/978-3-030-38919-2_6

[14] Seyed Rasoul Etesami and Tamer Başar. Complexity of equilibrium in competitive diffusion games on social networks. Automatica, 68:100–110, 2016. doi:10.1016/J.AUTOMATICA.2016.01.063

[15] Sándor P. Fekete and Henk Meijer. The one-round Voronoi game replayed. Computational Geometry, 30(2):81–94, 2005. doi:10.1016/j.comgeo.2004.05.005

[16] Rainer Feldmann, Marios Mavronicolas, and Burkhard Monien. Nash equilibria for Voronoi games on transitive graphs. In Proceedings of the 5th International Workshop on Internet and Network Economics (WINE), pages 280–291, 2009. doi:10.1007/978-3-642-10841-9_26

[17] Naoka Fukuzono, Tesshu Hanaka, Hironori Kiya, Hirotaka Ono, and Ryogo Yamaguchi. Two-player competitive diffusion game: Graph classes and the existence of a Nash equilibrium. In Proceedings of the 46th International Conference on Current Trends in Theory and Practice of Informatics (SOFSEM), pages 627–635, 2020. doi:10.1007/978-3-030-38919-2_52

[18] Takehiro Ito, Yota Otachi, Toshiki Saitoh, Hisayuki Satoh, Akira Suzuki, Kei Uchizawa, Ryuhei Uehara, Katsuhisa Yamanaka, and Xiao Zhou. Competitive diffusion on weighted graphs. In Proceedings of the 14th Workshop on Algorithms and Data Structures (WADS), pages 422–433, 2015. doi:10.1007/978-3-319-21840-3_35

[19] Soh Kumabe and Takanori Maehara. Convexity of b-matching games. In Proceedings of the 29th International Joint Conference on Artificial Intelligence (IJCAI), pages 261–267, 2020. doi:10.24963/IJCAI.2020/37

[20] Marios Mavronicolas, Burkhard Monien, Vicky G. Papadopoulou, and Florian Schoppmann. Voronoi games on cycle graphs. In Proceedings of the 33rd International Symposium on Mathematical Foundations of Computer Science (MFCS), pages 503–514, 2008. doi:10.1007/978-3-540-85238-4_41

[21] Nils Morawietz and Petra Wolf. A timecop’s chase around the table. In Proceedings of the 46th International Symposium on Mathematical Foundations of Computer Science (MFCS), pages 77:1–77:18, 2021. doi:10.4230/LIPIcs.MFCS.2021.77

[22] Nils Morawietz, Carolin Rehs, and Mathias Weller. A timecop’s work is harder than you think. In Proceedings of the 45th International Symposium on Mathematical Foundations of Computer Science (MFCS), pages 71:1–71:14, 2020. doi:10.4230/LIPIcs.MFCS.2020.71

[23] Elham Roshanbin. The competitive diffusion game in classes of graphs. In Proceedings of the 10th International Conference on Algorithmic Applications in Management (AAIM), pages 275–287, 2014. doi:10.1007/978-3-319-07956-1_25

[24] Lucy Small and Oliver Mason. Nash equilibria for competitive information diffusion on trees. Information Processing Letters, 113(7):217–219, 2013. doi:10.1016/J.IPL.2013.01.011
[25] Yuki Sukenari, Kunihito Hoki, Satoshi Takahashi, and Masakazu Muramatsu. Pure Nash equilibria of competitive diffusion process on toroidal grid graphs. *Discrete Applied Mathematics*, 215:31–40, 2016. doi:10.1016/j.dam.2016.07.021

[26] Xiaoming Sun, Yuan Sun, Zhiyu Xia, and Jialin Zhang. The one-round multi-player discrete Voronoi game on grids and trees. *Theoretical Computer Science*, 838:143–159, 2020. doi:10.1016/j.tcs.2020.06.028

[27] Reiko Takehara, Masahiro Hachimori, and Maiko Shigeno. A comment on pure-strategy Nash equilibria in competitive diffusion games. *Information Processing Letters*, 112(3):59–60, 2012. doi:10.1016/j.ipl.2011.10.015

[28] Sachio Teramoto, Erik D. Demaine, and Ryuhei Uehara. The Voronoi game on graphs and its complexity. *Journal of Graph Algorithms and Applications*, 15(4):485–501, 2011. doi:10.7155/jgaa.00235