UNIQUENESS OF SOLUTIONS FOR KELLER-SEGEL SYSTEM OF POROUS MEDIUM TYPE COUPLED TO FLUID EQUATIONS

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Abstract. We prove the uniqueness of Hölder continuous weak solutions via duality argument and vanishing viscosity method for the Keller-Segel system of porous medium type equations coupled to the Stokes system in dimensions three. An important step is the estimate of the Green function of parabolic equations with lower order terms of variable coefficients, which seems to be of independent interest.

1. Introduction

In this paper, we consider a mathematical model of the dynamics of swimming bacteria Bacillus subtilis in [20], where the movement of bacteria is formulated as a form of porous medium equation. More precisely, we are concerned with the following model

\begin{align*}
\partial_t \eta + v \cdot \nabla \eta - \Delta \eta^{1+\alpha} + \nabla \cdot (\chi(c)\eta^q \nabla c) &= 0 & \text{in } \mathbb{R}_T^3, \\
\partial_t c + v \cdot \nabla c - \Delta c + \kappa(c)\eta &= 0 & \text{in } \mathbb{R}_T^3, \\
\partial_t v - \Delta v + \nabla p + \eta \nabla \phi &= 0 & \text{in } \mathbb{R}_T^3, \\
\nabla \cdot v &= 0 & \text{in } \mathbb{R}_T^3, \\
\eta(0, x) &= \eta_0(x), & c(0, x) &= c_0(x), & v(0, x) &= v_0(x) & \text{in } \mathbb{R}^3,
\end{align*}

where \( \alpha > 0 \) and \( q \geq 1 \) are given constants, and \( \mathbb{R}_T^3 = (0, T) \times \mathbb{R}^3 \). Here \( \eta, c, v, \) and \( p \) indicate biological cell density, the oxygen concentration, the fluid velocity, and the pressure, respectively. The nonnegative functions \( k(c) \) and \( \chi(c) \) denote the oxygen consumption rate and the chemotactic sensitivity, which are assumed to be locally bounded functions of \( c \). Furthermore, the function \( \phi \) is a time-independent potential function which indicates, for example, the gravitational force or centrifugal force. The system was proposed by Tuval et al. in [20] (see also [5]) for the case that \( \alpha = 0, q = 1 \) and fluid equations are the Navier-Stokes system (see e.g. [2, 3, 4, 21, 22] for related mathematical results).

Very recently, in [7, Theorem 1.8-Theorem 1.10], the existence of weak solutions and Hölder continuous weak solutions are proved under certain assumptions on \((\alpha, q, \chi, \kappa)\) (compare to [8, 11, 19, 23]). It is, however, unknown whether or not such solutions are unique. Our main motivation is to show the uniqueness of the Hölder continuous weak solutions of (1.1), for which we take the following simplified model by taking \( \chi = 1, \kappa(c) = c \) and \( q = 1 \), since the system (1.1) is highly nonlinear and general \( \chi, \kappa \) and \( q > 1 \) seem to be beyond our analysis.

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(see Remark 2). So, we consider the following system of equations
\begin{align}
\partial_t \eta + v \cdot \nabla \eta - \Delta \eta^{1+\alpha} + \nabla \cdot (\eta \nabla c) &= 0 \quad \text{in } \mathbb{R}^3_T, \quad (1.2a) \\
\partial_t c + v \cdot \nabla c - \Delta c + c \eta &= 0 \quad \text{in } \mathbb{R}^3_T, \quad (1.2b) \\
\partial_t v - \Delta v + \nabla p + \eta \nabla \phi &= 0 \quad \text{in } \mathbb{R}^3_T, \quad (1.2c) \\
\nabla \cdot v &= 0 \quad \text{in } \mathbb{R}^3_T. \quad (1.2d)
\end{align}

\eta(0, x) = \eta_0(x), \quad c(0, x) = c_0(x), \quad v(0, x) = v_0(x) \quad \text{in } \mathbb{R}^3. \quad (1.2e)

Since the system (1.2) satisfies the assumptions in [7, Theorem 1.10], it is straightforward that there exist Hölder continuous weak solutions when \( \alpha > \frac{1}{8} \) and initial data are sufficiently regular. In this case, we can show that such solutions become unique. Our main result reads as follows:

**Theorem 1.1.** Let \( \alpha > \frac{1}{8} \) and \( (\eta_0, c_0, v_0) \) satisfy
\begin{align*}
\eta_0(1 + |x| + |\ln \eta_0|) &\in L^1(\mathbb{R}^3), \quad \eta_0 \in L^\infty(\mathbb{R}^3), \\
c_0 &\in L^\infty(\mathbb{R}^3) \cap H^1(\mathbb{R}^3) \cap W^{1,m}(\mathbb{R}^3), \quad v_0 \in L^2(\mathbb{R}^3) \cap W^{1,m}(\mathbb{R}^3) \quad \text{for any } m < \infty.
\end{align*}

Then, Hölder continuous weak solutions of the system (1.2) are unique.

We note that there are some known results regarding uniqueness of Hölder continuous weak solutions of Keller-Segel equations of the porous medium type (see [13, 15]).

\begin{align}
\partial_t \eta - \Delta \eta^{1+\alpha} + \nabla \cdot (\eta \nabla c) &= 0 \quad \text{in } \mathbb{R}^3_T, \quad (1.3a) \\
\partial_t c - \Delta c + c - \eta &= 0 \quad \text{in } \mathbb{R}^3_T, \quad (1.3b) \\
\eta(0, x) = \eta_0(x), \quad c(0, x) = c_0(x) \quad \text{in } \mathbb{R}^3. \quad (1.3c)
\end{align}

In principle, we apply the duality argument and the vanishing viscosity method used in [15]. However, the nonlinearity of the equation for \( c \), presence of the fluid equations as well as drift terms in the equations of \( \eta \) and \( c \), cause other kinds of difficulties, which do not seem to allow the techniques in [15] directly applicable. Nevertheless, we prove the uniqueness of Hölder continuous weak solutions of the system (1.2) via adapted methods of proofs in [15] with estimates of the Green function of parabolic equations.

In the following we briefly explain how our methods proceed. Let \( (\eta_1, c_1, v_1, p_1) \) and \( (\eta_2, c_2, v_2, p_2) \) be two Hölder continuous weak solutions of the system (1.2) and let
\[ \eta = \eta_1 - \eta_2, \quad c = c_1 - c_2, \quad v = v_1 - v_2, \quad p = p_1 - p_2. \]

We then see that \( (\eta, c, v, p) \) solves
\begin{align}
\partial_t \eta + v_1 \cdot \nabla \eta + v_2 \cdot \nabla \eta_2 - \Delta (\eta_1^{1+\alpha} - \eta_2^{1+\alpha}) + \nabla \cdot (\eta \nabla c_1 - \eta_2 \nabla c) &= 0 \quad \text{in } \mathbb{R}^3_T, \quad (1.4a) \\
\partial_t c + v_1 \cdot \nabla c + v_2 c_2 - \Delta c + c_1 \eta + v \cdot \nabla c_2 &= 0 \quad \text{in } \mathbb{R}^3_T, \quad (1.4b) \\
\partial_t v - \Delta v + \nabla p + \eta \nabla \phi &= 0 \quad \text{in } \mathbb{R}^3_T, \quad (1.4c) \\
\nabla \cdot v &= 0 \quad \text{in } \mathbb{R}^3_T. \quad (1.4d)
\end{align}

Next, we express \( c \) and \( v \) as integral forms involving \( \eta \) by using the representation formula via the Green functions for parabolic equations with lower order terms and Stokes system in three dimensions. We then substitute them to the equation of \( \eta \) to get an equation of the form \( (\mathcal{D} \eta, \Phi) = 0 \) for any appropriate test function \( \Phi \), where \( \mathcal{D} \) is a differential operator involving \( \eta_1, \eta_2, v_1, c_1, \phi \) for \( \eta \) and \( ((\cdot, \cdot)) \) is the pairing in space and time. Let \( \mathcal{D}^* \) be the
adjoint operator for \( D \) defined at (3.6) in section 3. Then \(( (D\eta, \Phi) ) = ( (\eta, D^*\Phi) ) = 0.\) If we can solve \( D^*\Phi = 0 \) for \( \Phi \), given any \( \Phi_0 \in X \) (which will specified below) it would follow that \( \eta \equiv 0 \), establishing uniqueness of solutions to the system (1.2).

When we prove Theorem 1.1, we formulate the dual problem in terms of a new function \( \zeta \) defined in (3.8), and it contains the term \( A^{1+\alpha} \Delta \zeta \), where \( A^{1+\alpha} \) is defined in (3.4). Since \( A^{1+\alpha} \) is degenerate, we add \( \delta \Delta \) to the equation of \( \zeta \). After solving the equation of \( \zeta^\delta \) for each \( \delta > 0 \), we show that

\[
\lim_{\delta \to 0} \delta \int_{\mathbb{R}^3} f(\tau, x) \Delta \zeta^\delta(\tau, x) dx d\tau = 0
\]

for all \( f \in L^2(\mathbb{R}^3) \).

One of main tools of proving Theorem 1.1 is some point-wise estimates of the Green function of a parabolic equation with lower order terms of variable coefficients, which seems to be of independent interest. More precisely, consider the equation of the form

\[
\begin{align*}
\partial_t f + a \cdot \nabla f + bf - \Delta f &= F \quad \text{in } \mathbb{R}_T^3, \\
f(0, x) &= f_0(x) \quad \text{in } \mathbb{R}^3,
\end{align*}
\]

(1.5)

where \( a : \mathbb{R}_T^3 \to \mathbb{R}^3 \) and \( b : \mathbb{R}_T^3 \to \mathbb{R} \) are given vector field and scalar function, respectively. We then have the following:

**Theorem 1.2.** Let \( a \in C^{\beta,\beta/2}(\mathbb{R}_T^3) \), \( b \in C^{\beta,\beta/2}(\mathbb{R}_T^3) \) for some \( 0 < \beta < 1 \), with \( \nabla \cdot a = 0 \) and \( b \geq 0 \). Then, a solution of the system (1.5) can be written as

\[
f(t, x) = \int_{\mathbb{R}^3} \Gamma(t, x, 0, y)f_0(y)dy + \int_0^t \int_{\mathbb{R}^3} \Gamma(t, x, s, y)F(s, y)dyds,
\]

where \( \Gamma(t, x, s, y) \) is the fundamental solution of a parabolic operator \( L \):

\[
L = \partial_t - \Delta + a \cdot \nabla + b.
\]

Moreover, \( \Gamma(t, x, s, y) \) satisfies the following pointwise bounds:

\[
|\Gamma(t, x, s, y)| \leq C_0(t-s)^{-\frac{3}{2}} \exp\left(-\frac{c|x-y|^2}{t-s}\right), \quad (0 \leq s < t \leq T),
\]

(1.6)

where \( c = \frac{1}{4} - \epsilon \) for any \( 0 < \epsilon < \frac{1}{4} \) and \( C_0 = C_0(T, \|a\|_{L_\infty(\mathbb{R}_T^3)}, \|b\|_{L_\infty(\mathbb{R}_T^3)}, \epsilon) \). Moreover,

\[
\begin{align*}
|\nabla_x \Gamma(t, x, s, y)| &\leq C_1(t-s)^{-\frac{3+1}{2}} \exp\left(-\frac{c_1|x-y|^2}{t-s}\right), \quad (0 \leq s < t \leq T), \\
|\nabla_y \Gamma(t, x, s, y) + |\partial_t \Gamma(t, x, s, y)| &\leq C_2(t-s)^{-\frac{3+2}{2}} \exp\left(-c_2\frac{|x-y|^2}{t-s}\right), \quad (0 \leq s < t \leq T)
\end{align*}
\]

(1.7)

where \( c_i > 0 \) is an absolute constant and \( C_i = C_i(T, \|a\|_{C^{\beta,\beta/2}(\mathbb{R}_T^3)}, \|b\|_{C^{\beta,\beta/2}(\mathbb{R}_T^3)}) , i = 1, 2. \)

**Remark 1.** In fact, the assumption that \( b \geq 0 \) in Theorem 1.2 is not so essential. If we set \( v(x, t) = e^{-\mu t}u(x, t) \), then \( v \) satisfies \( L v = e^{-\mu t}(Lu - \mu u) \), or equivalently,

\[
\tilde{L} v := L v + \mu v = e^{-\mu t}L u.
\]

The assumption that \( b \in C^{\beta,\beta/2}(\mathbb{R}_T^3) \) particularly implies that \( b \) is bounded, and thus we can make \( b + \mu \geq 0 \) by choosing \( \mu \) large enough, for example, \( \mu = \|b\|_{L_\infty(\mathbb{R}_T^3)} \leq \|b\|_{C^{\beta,\beta/2}(\mathbb{R}_T^3)}. \)
Note that the new operator \( \tilde{L} \) satisfies the hypothesis of Theorem 1.2. Therefore, we can apply Theorem 1.2 to \( v \) and transfer the results back to \( u \).

This paper is organized as follows. In section 2 we recall some useful notations and known results. Section 3 is devoted to proving Theorem 1.1. In section 4, we discuss the fundamental solutions of some parabolic equations and provide the proof of Theorem 1.2.

2. Preliminaries

2.1. Notations. • All generic constants will be denoted by \( C \). We write \( C = C(p_1, p_2, \cdots) \) to mean a constant that depends on \( (p_1, p_2, \cdots) \). We follow the convention that such constants can vary from expression to expression and even between two occurrences within the same expression. \( f^\delta \) denotes the dependence of a function on a parameter \( \delta \).

• \( L^p \) and \( W^{k,p} \) are the usual Lebesgue spaces and the Sobolev spaces. \( L^p L^q \) denotes the Banach set of Bochner measurable functions \( f \) from \( (0, T) \) to \( L^q(\mathbb{R}^3) \) such that \( \| f(t) \|_{L^q} \in L^p(0, T) \).

• For \( 1 < s, p < \infty \) and \( 0 < T \leq \infty \), we define a function space

\[
P_T^{s,p}(\mathbb{R}^3) = \left\{ f \in \mathcal{D}'((0, T) \times \mathbb{R}^3) : \partial_t f \in L^p_T L^p(\mathbb{R}^3), f \in L^p_T W^{2,p}(\mathbb{R}^3) \right\}
\]

with the norm

\[
\| f \|_{P_T^{s,p}} = \| \partial_t f \|_{L^p_T L^p} + \| f \|_{L^p_T W^{2,p}}.
\]

We also use the real interpolation space for initial data (\cite{18}, Chapter 3)

\[
I_0^{s,p}(\mathbb{R}^3) = (L^p(\mathbb{R}^3), W^{2,p}(\mathbb{R}^3))_{1-s,p}.
\]

• We finally introduce the Hölder space \( C^{\beta,\gamma}(\mathbb{R}^3_T) \), \( 0 < \beta, \gamma < 1 \):

\[
C^{\beta,\gamma}(\mathbb{R}^3_T) = \left\{ f \in C(\mathbb{R}^3_T) : \| f \|_{C^{\beta,\gamma}(\mathbb{R}^3_T)} < \infty \right\},
\]

where

\[
\| f \|_{C^{\beta,\gamma}(\mathbb{R}^3_T)} = \sup_{(t,x) \in \mathbb{R}^3_T} |f(t, x)| + \sup_{(t,x),(s,y) \in \mathbb{R}^3_T, x \neq y, s \neq t} \frac{|f(t, x) - f(s, y)|}{|t - s|^{\gamma} + |x - y|^{\beta}}.
\]

In particular, we set \( C^{\beta}(\mathbb{R}^3_T) = C^{\beta,\beta}(\mathbb{R}^3_T) \).

2.2. Parabolic equations. To show uniqueness of solutions of the system \( (1.2) \), we use the vanishing viscosity method. This requires to solve the equation of the form

\[
\partial_t f - (\delta + V(t, x)) \Delta f + \mu f = g(t, x) \quad \text{in } \mathbb{R}^3_T
\]

\[
f(0, x) = f_0(x) \quad \text{in } \mathbb{R}^3.
\]

Lemma 2.1. \cite[Lemma 3.4]{15} Let \( 1 < s, p < \infty \) and \( 0 < T < \infty \). Suppose \( V \) satisfies the following conditions:

\[
V \in L^\infty_T (L^1 \cap L^\infty) \cap C^\beta(\mathbb{R}^3_T), \quad V(t, x) \geq 0
\]

for some \( 0 < \beta < 1 \). Then for every \( g \in L^s_T L^p \) and \( f_0 \in I_0^{s,p} \), there exists a unique solution \( f \) of \( (2.1) \) on \( (0, T) \). Moreover, there exists

\[
\mu_1 = \mu_1(\beta, \| V \|_{C^\beta}, \delta, s, p, T)
\]

such that

\[
\| \partial_t f \|_{L^s_T L^p} + \delta \| \nabla^2 f \|_{L^s_T L^p} + \mu \| f \|_{L^s_T L^p} \leq C \left( \| g \|_{L^s_T L^p} + \| f_0 \|_{I_0^{s,p}} \right)
\]
for all $\mu \geq \mu_1$, where $C = C(\beta, \|V\|_{C^3}, \delta, s, p, T)$ is a constant independent of $\mu \geq \mu_1$.

We next consider the Stokes system

\begin{align}
\partial_t v - \Delta v + \nabla p &= F \quad \text{in } \mathbb{R}^3, \\
\nabla \cdot v &= 0 \quad \text{in } \mathbb{R}^3, \\
v(0, x) &= v_0(x) \quad \text{in } \mathbb{R}^3.
\end{align}

(2.2a)

(2.2b)

(2.2c)

Let $\mathbb{P}$ be the Leray projection operator. Then, we can write $v$ as

$$v(t, x) = e^{t\Delta}v_0 + \int_0^t e^{(t-s)\Delta}\mathbb{P}F(s)ds,$$

where $e^{t\Delta}$ is the heat kernel. We note that $S(t) := e^{t\mathbb{P}}$ has the following $L^p$ estimates: for $1 \leq p \leq r \leq \infty$

$$\|S(t)f\|_{L^r} \leq C(p, r)t^{-\frac{3}{2}(\frac{1}{r} - \frac{1}{2})}\|f\|_{L^p},$$

(2.3)

$$\|\nabla S(t)f\|_{L^r} \leq C(p, r)t^{-\frac{3}{2}(\frac{1}{r} - \frac{1}{2})-\frac{1}{2}}\|f\|_{L^p}.$$

Without the operator $\mathbb{P}$, we have the bounds (2.3) from the heat kernel $e^{t\Delta}$ [15]. Since the operator $\mathbb{P}$ is a singular integral operator of degree 0, it is a bounded operator such that

$$\|\mathbb{P}f\|_{L^p} \leq C(p)\|f\|_{L^p}$$

and thus we have (2.3) as well.

We also consider the equation:

\begin{align}
\partial_t f + a \cdot \nabla f + bf - \Delta f &= 0 \quad \text{in } \mathbb{R}^3, \\
f(0, x) &= f_0(x) \quad \text{in } \mathbb{R}^3.
\end{align}

(2.4)

Under the assumptions on $a$ and $b$ in Theorem 1.2, (1.6) and (1.7) imply the following estimates: for $1 \leq p \leq r \leq \infty$

\begin{align}
\left\| \int_{\mathbb{R}^3} \Gamma(t, x, 0, y)f(y)dy \right\|_{L^r} &\leq C(p, r, a, b)t^{-\frac{3}{2}\left(\frac{1}{r} - \frac{1}{2}\right)}\|f\|_{L^p}, \\
\left\| \int_{\mathbb{R}^3} \nabla_y \Gamma(t, x, 0, y)f(y)dy \right\|_{L^r} &\leq C(p, r, a, b)t^{-\frac{3}{2}\left(\frac{1}{r} - \frac{1}{2}\right)-\frac{1}{2}}\|f\|_{L^p}.
\end{align}

(2.5)

3. Proof of Theorem 1.1

In order to show the uniqueness of Hölder continuous weak solutions, let $(\eta_1, c_1, v_1, p_1)$ and $(\eta_2, c_2, v_2, p_2)$ be two solutions of the system (1.2) and let

$$\eta = \eta_1 - \eta_2, \quad c = c_1 - c_2, \quad v = v_1 - v_2, \quad p = p_1 - p_2.$$

Then, $(\eta, c, v, p)$ satisfies the following equations in $\mathbb{R}^3$:

\begin{align}
\partial_t \eta + v_1 \cdot \nabla \eta + v \cdot \nabla \eta_2 - \Delta \left(\eta_1^{1+\alpha} - \eta_2^{1+\alpha}\right) + \nabla \cdot (\eta \nabla c_1 - \eta_2 \nabla c) &= 0 \quad \text{in } \mathbb{R}^3, \\
\partial_t c + v_1 \cdot \nabla c + \eta_2 c - \Delta c + c_1 \eta + v \cdot \nabla c_2 &= 0 \quad \text{in } \mathbb{R}^3, \\
\partial_t v - \Delta v + \nabla p + \eta \nabla \phi &= 0 \quad \text{in } \mathbb{R}^3, \\
\nabla \cdot v &= 0 \quad \text{in } \mathbb{R}^3, \\
\eta(0, x) = c(0, x) = v(0, x) &= 0 \quad \text{in } \mathbb{R}^3.
\end{align}

(3.1a)

(3.1b)

(3.1c)

(3.1d)

(3.1e)
We first express \( v \) in terms of \( \eta \):

\[
v(t, x) = - \int_0^t \int_{\mathbb{R}^3} S(t - s, x - y) (\eta \nabla \phi)(s, y) dy ds.
\] (3.2)

By Theorem 1.2, we also express \( c \) as

\[
c(t, x) = - \int_0^t \int_{\mathbb{R}^3} \Gamma(t, x, s, y) (c_1 \eta + v \cdot \nabla c_2)(s, y) dy ds,
\] (3.3)

where \( \Gamma \) is the fundamental solution in Theorem 1.2 with \( a \) and \( b \) replaced by \( v_1 \) and \( \eta_2 \). So, we can reformulate the dual system of (3.1) by the dual problem of \( \eta \). Once uniqueness of \( \eta \) is proved, uniqueness of \( c \) and \( v \) then comes automatically.

To write the dual equation of \( \eta \), let

\[
A^{\alpha + 1}(t) := A^{\alpha + 1}(\eta_1(t, x), \eta_2(t, x)) = \frac{\eta_1^{\alpha + 1}(t, x) - \eta_2^{\alpha + 1}(t, x)}{\eta_1(t, x) - \eta_2(t, x)}.
\] (3.4)

We multiply (3.1a) by \( \Phi \in C_c^\infty(\mathbb{R}^3_T) \). Then, via the integration by parts, we have

\[
\int_{\mathbb{R}^3} \eta(t, x)\Phi(t, x) dx + \int_0^t \int_{\mathbb{R}^3} \eta(t, x)\delta \Delta \Phi(\tau, x) dx d\tau
= \int_0^t \int_{\mathbb{R}^3} \eta(\tau, x)\partial_t \Phi(\tau, x) dx d\tau + \int_0^t \int_{\mathbb{R}^3} \eta(\tau, x) \left(\delta \Delta \Phi(\tau) + A^{\alpha + 1}(\tau) \Delta \Phi(\tau, x)\right) dx d\tau
+ \int_0^t \int_{\mathbb{R}^3} \eta(\tau, x) (v \cdot \nabla \Phi)(\tau, x) dx d\tau + \int_0^t \int_{\mathbb{R}^3} \eta(\tau, x) (\nabla c_1 \cdot \nabla \Phi)(\tau, x) dx d\tau
+ \int_0^t \int_{\mathbb{R}^3} \eta_2(\tau, x) (v \cdot \nabla \Phi)(\tau, x) dx d\tau + \int_0^t \int_{\mathbb{R}^3} (\eta 2 \nabla c)(\tau, x) \nabla \Phi(\tau, x) dx d\tau
= N_1 + N_2 + N_3 + N_4 + N_5 + N_6,
\]

where we add \( \delta \Delta \) because \( A^{\alpha + 1}(\tau) \) is degenerate. We note that \( N_1, N_2, N_3 \) and \( N_4 \) are already given in the dual form. By (3.2),

\[
N_5 = - \int_0^t \int_{\mathbb{R}^3} \eta(\tau, x) \nabla \phi(x) \cdot \left( \int_\tau^t \int_{\mathbb{R}^3} S(\hat{\tau} - \tau, x - y) (\eta_2(\hat{\tau}, y) \nabla \Phi(\hat{\tau}, y)) dy d\hat{\tau} \right) dx d\tau.
\]

Also, by (3.3) we rewrite \( N_6 \) as

\[
N_6 = \int_0^t \int_{\mathbb{R}^3} \eta(\tau, x) c_1(\tau, x) \left[ \int_\tau^t \int_{\mathbb{R}^3} \nabla y \Gamma(\hat{\tau}, y, \tau, x) \cdot (\eta_2(\hat{\tau}, y) \nabla \Phi(\hat{\tau}, y)) dy d\hat{\tau} \right] dx d\tau
+ \int_0^t \int_{\mathbb{R}^3} v(\tau, x) \cdot \nabla c_2(\tau, x) \left[ \int_\tau^t \int_{\mathbb{R}^3} \nabla y \Gamma(\hat{\tau}, y, \tau, x) \cdot (\eta_2(\hat{\tau}, y) \nabla \Phi(\hat{\tau}, y)) dy d\hat{\tau} \right] dx d\tau
= N_{61} + N_{62}.
\]

Using (3.2), we proceed further to rewrite \( N_{62} \) as

\[
N_{62} = - \int_0^t \int_{\mathbb{R}^3} \eta(\tau, x) \nabla \phi(x) \cdot \left\{ \int_\tau^t \int_{\mathbb{R}^3} S(\hat{\tau} - \tau, x - y) \nabla c_2(\hat{\tau}, y) \right\}
\times \left[ \int_\tau^t \int_{\mathbb{R}^3} \nabla z \Gamma(\rho, z, \hat{\tau}, y) \cdot (\eta_2(\rho, z) \nabla \Phi(\rho, z)) dz d\rho \right] dy d\hat{\tau} dx d\tau.
\]
Collecting all terms together, we obtain
\[
\int_{\mathbb{R}^3} \eta(t,x) \Phi(t,x) dx + \delta \int_{\mathbb{R}^3} \eta(\tau,x) \Delta \Phi(\tau,x) dx d\tau = \int_{\mathbb{R}^3} \eta(\tau,x) \mathcal{D}^*(\tau,x) dx d\tau,  
\]  
(3.5)
where
\[
\mathcal{D}^*(\tau,x) = \partial_{\tau} \Phi + \delta \Delta \Phi + A^{\alpha+1} \Delta \Phi + v_1 \cdot \nabla \Phi + \nabla c_1 \cdot \nabla \Phi 
- \nabla \phi(x) \cdot \int_{\tau}^{t} \int_{\mathbb{R}^3} S(\tilde{\tau} - \tau,x-y) \eta_2(\tilde{\tau},y) \nabla \Phi(\tilde{\tau},y) dy d\tilde{\tau} 
+ c_1(\tau,x) \int_{\tau}^{t} \int_{\mathbb{R}^3} \nabla_y \Gamma(\tilde{\tau},y,\tau,x) \cdot (\eta_2(\tilde{\tau},y) \nabla \Phi(\tilde{\tau},y)) dy d\tilde{\tau} 
- \nabla \phi \cdot \int_{\tau}^{t} \int_{\mathbb{R}^3} S(\tilde{\tau} - \tau,x-y) \nabla c_2(\tilde{\tau},y) 
\times \left[ \int_{\tau}^{\tilde{\tau}} \int_{\mathbb{R}^3} \nabla_z \Gamma(\rho,z,\tilde{\tau},y) (\eta_2(\rho,z) \nabla \Phi(\rho,z)) dz d\rho \right] dy d\tilde{\tau}. 
\]  
(3.6)

At this point, we could finish the proof of Theorem 1.1 if we could show the followings.
• Find a solution \( \Phi^\delta \) solving \( \mathcal{D}^*(\tau,x) = 0 \) for a.e. \( (\tau,x) \in \mathbb{R}_t^3 \) for each \( \delta > 0 \).
• \( \lim_{\delta \to 0} \int_{\tau}^{t} \int_{\mathbb{R}^3} \eta(\tau,x) \Delta \Phi^\delta(\tau,x) dx d\tau = 0 \).

Let \( \lim_{\delta \to 0} \Phi^\delta = \Phi \) in \( \mathcal{P}_T^{s,p} \). Then, (3.5) implies that
\[
\int_{\mathbb{R}^3} \eta(t,x) \Phi(t,x) dx = 0.  
\]  
(3.7)
But, in order to say \( \eta \equiv 0 \) from (3.7), we should remove the time dependency in \( \Phi(t,x) \). To do this, let \( \zeta^\delta(t - \tau,x) = \Phi^\delta(\tau,x) \). Then, \( \Phi^\delta(t,x) = \zeta^\delta(0,x) \). Let
\[
\theta = t - \tau, \quad \tilde{\theta} = t - \tilde{\tau}, \quad \rho = t - \sigma. 
\]
Then, \( \mathcal{D}^*(\tau,x) = 0 \) is equivalent to solve the following equation for each \( \delta > 0 \)
\[
\partial_{\theta} \zeta^\delta - (\delta \Delta + A^{\alpha+1}(t-\theta)) \zeta^\delta - v_1(t-\theta) \cdot \nabla \zeta^\delta - \nabla c_1(t-\theta) \cdot \nabla \zeta^\delta 
+ \nabla \phi \int_{0}^{\theta} \int_{\mathbb{R}^3} S(\theta - \tilde{\theta},x-y) \eta_2(t - \tilde{\theta},y) \nabla \zeta^\delta(\tilde{\theta},y) dy d\tilde{\theta} 
- c_1(t-\tau) \int_{0}^{\theta} \int_{\mathbb{R}^3} \nabla_y \Gamma(\tilde{\theta},y,\theta,\tau,x) \cdot (\eta_2(t - \tilde{\theta},y) \nabla \zeta^\delta(\tilde{\theta},y)) dy d\tilde{\theta} 
+ \nabla \phi \cdot \int_{0}^{\theta} \int_{\mathbb{R}^3} S(\theta - \tilde{\theta},x-y) \nabla c_2(t - \tilde{\theta},y) \mathcal{T}(\hat{\theta},t,y,\zeta^\delta) dy d\tilde{\theta} = 0, 
\]  
(3.8)
where
\[
\mathcal{T}(\hat{\theta},t,y,\zeta^\delta) = \int_{0}^{\hat{\theta}} \int_{\mathbb{R}^3} \nabla_z \Gamma(t - \sigma,z,t - \hat{\theta},y) \eta_2(t - \sigma,z) \nabla \zeta^\delta(\sigma,z) dz d\sigma.  
\]  
(3.9)

**Lemma 3.1.** Let \( \alpha > \frac{1}{2}, \ 2 < s,p < \infty \) and \( 0 < t < T \). For any \( \zeta_0 \in \mathcal{I}^{s,p}_T \), there exists a unique solution \( \zeta^\delta \in \mathcal{P}_T^{s,p} \) of (3.8) for each \( \delta > 0 \). Moreover, for a.e. \( 0 < t < T \) and for all
\[ \chi \in L^2_t L^2, \]
\[ \lim_{\delta \to 0} \int_0^t \int_{\mathbb{R}^3} \chi(\tau, x) \Delta \zeta^\delta(\tau, x) dx d\tau = 0. \]

If Lemma 3.1 can be proved, the proof of Theorem 1.1 can be finalized. More precisely, by the dual problem (3.8), \( D^*(\tau, x) = 0 \) for a.e. \((\tau, x) \in \mathbb{R}^3_T\). So,
\[ \int_{\mathbb{R}^3} \eta(t, x) \zeta^\delta(x) dx + \delta \int_{\mathbb{R}^3} \eta(\tau, x) \Delta \Phi^\delta(\tau, x) dx d\tau = 0. \]
Moreover, the second part of Lemma 3.1, with \( \lim_{\delta \to 0} \zeta^\delta = \zeta_0 \), implies that
\[ \int_{\mathbb{R}^3} \eta(t, x) \zeta_0(x) dx = 0 \]
for a.e. \( t \in (0, T) \). Since \( \zeta_0 \in \mathcal{I}_0^{s,p} \) is arbitrary, we conclude that \( \eta \equiv 0 \).

3.1. Proof of Lemma 3.1. To solve the equation (3.8), we first introduce the exponential factor:
\[ \zeta^\delta(\theta, x) = e^{\mu \theta} \psi^\delta(\theta, x), \]
where \( \mu \) is a constant to be determined when we apply Lemma 2.1 later. Then, \( \psi^\delta \) satisfies
\[ \partial_\theta \psi^\delta - (\delta \Delta + A^{\alpha+1}(t-\theta)) \psi^\delta + \mu \psi^\delta - v_1(t-\theta) \cdot \nabla \psi^\delta - \nabla c_1(t-\theta) \cdot \nabla \psi^\delta \]
\[ + \nabla \phi \int_0^\theta \int_{\mathbb{R}^3} S(\theta - \hat{\theta}, x-y) \eta_2(t-\hat{\theta}, y) e^{\mu(\hat{\theta} - \theta)} \nabla \psi^\delta(\hat{\theta}, y) dy d\hat{\theta} \]
\[ - c_1(t-\tau) \int_0^\theta \int_{\mathbb{R}^3} \nabla \Gamma(t-\hat{\theta}, y, t-\theta, x) \cdot \left( \eta_2(t-\hat{\theta}, y) e^{\mu(\hat{\theta} - \theta)} \nabla \psi^\delta(\hat{\theta}, y) \right) dy d\hat{\theta} \]
\[ + \nabla \phi \cdot \int_0^\theta \int_{\mathbb{R}^3} S(\theta - \hat{\theta}, x-y) \nabla c_2(t-\hat{\theta}, y) T(\hat{\theta}, t, y, \psi^\delta) dy d\hat{\theta} = 0, \]
where \( T(\hat{\theta}, t, y, \psi^\delta) \) is defined by (3.9) with \( \zeta^\delta \mapsto e^{\mu(\sigma-\theta)} \psi^\delta \). Since the exponential factors \( e^{\mu(\hat{\theta} - \theta)} \) and \( e^{\mu(\sigma-\theta)} \) are less than 1, they do not affect the arguments below. Hence, we consider the following equation, with the same notation \( \psi^\delta \),
\[ \partial_\theta \psi^\delta - (\delta \Delta + A^{\alpha+1}(t-\theta)) \psi^\delta + \mu \psi^\delta - v_1(t-\theta) \cdot \nabla \psi^\delta - \nabla c_1(t-\theta) \cdot \nabla \psi^\delta \]
\[ + \nabla \phi \int_0^\theta \int_{\mathbb{R}^3} S(\theta - \hat{\theta}, x-y) \eta_2(t-\hat{\theta}, y) \nabla \psi^\delta(\hat{\theta}, y) dy d\hat{\theta} \]
\[ - c_1(t-\tau) \int_0^\theta \int_{\mathbb{R}^3} \nabla \Gamma(t-\hat{\theta}, y, t-\theta, x) \cdot \left( \eta_2(t-\hat{\theta}, y) \nabla \psi^\delta(\hat{\theta}, y) \right) dy d\hat{\theta} \]
\[ + \nabla \phi \cdot \int_0^\theta \int_{\mathbb{R}^3} S(\theta - \hat{\theta}, x-y) \nabla c_2(t-\hat{\theta}, y) T(\hat{\theta}, t, y, \psi^\delta) dy d\hat{\theta} = 0. \]
\[ (3.10) \]
So, instead of proving Lemma 3.1, we prove the following equivalent lemma.

Lemma 3.2. Let \( \alpha > \frac{1}{2} \), \( 2 < s, p < \infty \) and \( 0 < t < T \). For any \( \psi_0 \in \mathcal{I}_0^{s,p} \), there exists a unique solution \( \psi^\delta \in \mathcal{P}^{s,p}_T \) of (3.10) for each \( \delta > 0 \). Moreover, for a.e. \( 0 < t < T \) and for all \( \chi \in L^2_t L^2 \),
\[ \lim_{\delta \to 0} \int_0^t \int_{\mathbb{R}^3} \chi(\tau, x) \Delta \psi^\delta(\tau, x) dx d\tau = 0. \]
The proof of Lemma 3.2 consists of two parts.

3.1.1. **Unique solvability of (3.10).** We construct a unique solution to the equation (3.10) via the iteration argument. Let \( \psi_1 = e^{\tau G} \psi_0 \) and and for \( k \geq 2 \),

\[
\partial_\theta \psi_k^\delta - (\delta + A^{\alpha+1}(t - \theta)) \Delta \psi_k^\delta + \mu \psi_k^\delta = G_{k-1},
\]

where we choose \( G_1 \) by putting \( \psi_1 = e^{\tau G} \psi_0 \) in place of \( \psi_{k-1}^\delta \) below and

\[
G_{k-1} = v_1(t - \theta) \cdot \nabla \psi_{k-1}^\delta + \nabla c_1(t - \theta) \cdot \nabla \psi_{k-1}^\delta \\
- \nabla \phi \int_0^\theta \int_{\mathbb{R}^3} S(\theta - \tilde{\theta}, x - y) \eta_2(t - \tilde{\theta}, y) \nabla \psi_{k-1}^\delta(\tilde{\theta}, y) dyd\tilde{\theta} \\
+ c_1(t - \tau) \int_0^\theta \int_{\mathbb{R}^3} \nabla y \Gamma(t - \tilde{\theta}, y, t - \theta, x) \cdot \left( \eta_2(t - \tilde{\theta}, y) \nabla \psi_{k-1}^\delta(\tilde{\theta}, y) \right) dyd\tilde{\theta} \\
- \nabla \phi \cdot \int_0^\theta \int_{\mathbb{R}^3} S(\theta - \tilde{\theta}, x - y) \nabla c_2(t - \tilde{\theta}, y) T(\tilde{\theta}, y, \psi_{k-1}^\delta) dyd\tilde{\theta} \\
= F_1 + F_2 + F_3 + F_4 + F_5.
\]

We now estimate \( G_{k-1} \) in \( L_T^p L^p \). First,

\[
\|F_1\|_{L_T^p L^p} + \|F_2\|_{L_T^p L^p} \leq \left( \|v_1\|_{L_T^\infty L^\infty} + \|\nabla c_1\|_{L_T^\infty L^\infty} \right) \|\nabla \psi_{k-1}^\delta\|_{L_T^p L^p}.
\]

Using (2.3), we also have

\[
\|F_3\|_{L_T^p L^p} \leq C(s, p)(1 + T)^2 \|\nabla \phi\|_{L^\infty} \|\eta_2\|_{L_T^\infty L^\infty} \|\nabla \psi_{k-1}^\delta\|_{L_T^p L^p}.
\]

By (2.5),

\[
\|F_4\|_{L_T^p L^p} \leq C_1 \|c_1\|_{L_T^\infty L^\infty} \|\eta_1\|_{L_T^\infty L^\infty} \int_0^\theta \frac{1}{\sqrt{T - \theta}} \|\nabla \psi_{k-1}^\delta(\tilde{\theta})\|_{L^p} d\tilde{\theta}
\]

and hence we have, with \( s > 2 \),

\[
\|F_4\|_{L_T^p L^p} \leq C_1(1 + T)^2 \|c_1\|_{L_T^\infty L^\infty} \|\eta_2\|_{L_T^\infty L^\infty} \|\nabla \psi_{k-1}^\delta\|_{L_T^p L^p},
\]

where \( C_1 \) is the constant defined in Theorem 1.2.

Similarly,

\[
\|F_5\|_{L_T^p L^p} \leq C_1(1 + T)^4 \|\nabla \phi\|_{L^\infty} \|\nabla c_2\|_{L_T^\infty L^\infty} \|\eta_2\|_{L_T^\infty L^\infty} \|\nabla \psi_{k-1}^\delta\|_{L_T^p L^p}.
\]

By Lemma 2.1,

\[
\|\partial_\theta \psi_k^\delta\|_{L_T^p L^p} + \delta \|\nabla^2 \psi_k^\delta\|_{L_T^p L^p} + \mu \|\psi_k^\delta\|_{L_T^p L^p} \leq C_* \|\psi_0\|_{L^p} + C_{ss} \|\nabla \psi_{k-1}^\delta\|_{L_T^p L^p},
\]

where

\[
\mu \geq \mu_1(\beta, \|\eta_1\|_{C^\beta}, \|\eta_2\|_{C^\beta}, \delta, s, p, T), \quad C_* = C(\beta, \|\eta_1\|_{C^\beta(B^2)}, \|\eta_2\|_{C^\beta(B^2)}, \delta, s, p, T),
\]

\[
C_{ss} = C \left( C_*, \|\nabla \phi\|_{L^\infty}, \|v_1\|_{L_T^\infty L^\infty}, \|c_1\|_{L_T^\infty W^{1, \infty}}, \|\nabla c_2\|_{L_T^\infty L^\infty}, \|\eta_2\|_{L_T^\infty L^\infty}, \|\eta_1\|_{L_T^\infty L^\infty}, T \right).
\]

Using the following interpolation

\[
\|\nabla \psi_{k-1}^\delta\|_{L^p} \leq \epsilon \|\nabla^2 \psi_{k-1}^\delta\|_{L^p} + \frac{4}{\epsilon} \|\psi_{k-1}^\delta\|_{L^p}, \quad \epsilon = \frac{\delta}{2C_{ss}},
\]

we conclude that the solution is unique.

\[
\text{UNIQUENESS OF SOLUTIONS FOR KELLER-SEGEL-FLUID MODEL}
\]
we have
\[
\|\partial_t \psi_k^\delta\|_{L^2_tL^p_x} + \delta \|\nabla^2 \psi_k^\delta\|_{L^2_tL^p_x} + \mu \|\psi_k^\delta\|_{L^2_tL^p_x} \leq C_\ast \|\psi_0\| + \frac{\delta}{2} \|\nabla^2 \psi_{k-1}^\delta\|_{L^2_tL^p_x} + \frac{8C_{2\ast}^2}{\delta} \|\psi_{k-1}^\delta\|_{L^2_tL^p_x}.
\]
Let
\[
\mu \geq \max \left\{ \mu_1, \frac{8C_{2\ast}^2}{\delta} \right\}.
\]
Then, we finally obtain
\[
\|\partial_t \psi_k^\delta\|_{L^2_tL^p_x} + \delta \|\nabla^2 \psi_k^\delta\|_{L^2_tL^p_x} + \mu \|\psi_k^\delta\|_{L^2_tL^p_x} \\
\leq C_\ast \|\psi_0\|_{L^2_0L^p} + \frac{1}{2} \left( \|\partial_t \psi_{k-1}^\delta\|_{L^2_tL^p_x} + \delta \|\nabla^2 \psi_{k-1}^\delta\|_{L^2_tL^p_x} + \mu \|\psi_{k-1}^\delta\|_{L^2_tL^p_x} \right).
\]
(3.12)
Let
\[
M(T) = \|\psi_0\|_{L^2_0L^p} \sup_{0 \leq t \leq T} C_\ast.
\]
Then, (3.12) implies that the sequence \(\{\psi_k\}\) is uniformly in \(P^{s,p}_T\), bounded above by \(2M(T)\). Moreover, using the same argument, we can derive that
\[
\|\partial_t (\psi_{k+1}^\delta - \psi_k^\delta)\|_{L^2_tL^p_x} + \delta \|\nabla^2 (\psi_{k+1}^\delta - \psi_k^\delta)\|_{L^2_tL^p_x} + \mu \|\psi_{k+1}^\delta - \psi_k^\delta\|_{L^2_tL^p_x} \\
\leq \frac{1}{2} \left( \|\partial_t (\psi_{k}^\delta - \psi_{k-1}^\delta)\|_{L^2_tL^p_x} + \delta \|\nabla^2 (\psi_{k}^\delta - \psi_{k-1}^\delta)\|_{L^2_tL^p_x} + \mu \|\psi_{k}^\delta - \psi_{k-1}^\delta\|_{L^2_tL^p_x} \right).
\]
(3.13)
Hence \(\{\psi_k^\delta\}\) is a Cauchy sequence in \(P^{s,p}_T\). Therefore, there exists a unique \(\psi^\delta \in P^{s,p}_T\) such that
\[
\lim_{k \to \infty} \|\psi_k^\delta - \psi^\delta\|_{P^{s,p}_T} = 0.
\]
By taking \(k \to \infty\) to (3.11), we obtain a unique solution \(\psi^\delta\) of (3.10) for each \(\delta > 0\).

3.1.2. Vanishing viscosity of (3.10). We now use the vanishing viscosity limit to (3.10). We multiply the equation (3.10) by \(-\Delta \psi^\delta\) and integrate it over \(\mathbb{R}^3\). Then, for \(0 < \theta < t\)
\[
\frac{1}{2} \frac{d}{d\theta} \|\nabla \psi^\delta(\theta)\|_{L^2}^2 + \mu \|\nabla \psi^\delta\|_{L^2}^2 + \int (\delta + A^{\alpha+1}(t - \theta)) |\Delta \psi^\delta|^2 dx \\
= \int_{\mathbb{R}^3} (v_1(t - \theta) \cdot \nabla \psi^\delta) \Delta \psi^\delta dx + \int (\nabla c_1(t - \theta) \cdot \nabla \psi^\delta) \Delta \psi^\delta dx \\
- \int_{\mathbb{R}^3} (\nabla \phi \cdot \int_0^\theta \int_{\mathbb{R}^3} S(\theta - \theta, x - y) \eta_2(t - \theta, y) \nabla \psi^\delta(\theta, y) dy d\theta) \Delta \psi^\delta dx \\
+ \int_{\mathbb{R}^3} (c_1(t - \tau) \int_0^\theta \int_{\mathbb{R}^3} \nabla_y \Gamma(t - \theta, y, t - \theta, x) \cdot (\eta_2(t - \theta, y) \nabla \psi^\delta(\theta, y)) dy d\theta) \Delta \psi^\delta dx \\
- \int_{\mathbb{R}^3} (\nabla \phi \cdot \int_0^\theta \int_{\mathbb{R}^3} S(\theta - \theta, x - y) \nabla c_2(t - \theta, y) \mathcal{T}(\theta, t, y, \psi^\delta) dy d\theta) \Delta \psi^\delta dx \\
= I + II + III + IV + V.
\]
(3.14)

- By the divergence free condition of \(v_1\), we have
\[
I \leq \|\nabla v_1(t - \theta)\|_{L^\infty} \|\nabla \psi^\delta\|_{L^2}^2.
\]
• By the integration by parts,
\[
II \leq \left( \| \nabla^2 c_1(t - \theta) \|_{L^\infty} + \frac{1}{2} \| \Delta c_1(t - \theta) \|_{L^\infty} \right) \| \nabla \psi^\delta \|_{L^2}^2.
\]
• Also, by the integration by parts
\[
III \leq \| \phi \|_{W^{2,\infty}} \| \nabla \psi^\delta \|_{L^2}^2 + \left\| \int_0^\theta \int_{\mathbb{R}^3} S(\theta - \hat{\theta}, x - y) \eta_2(t - \hat{\theta}, y) \nabla \psi^\delta(\hat{\theta}, y) dyd\hat{\theta} \right\|_{H^1}^2.
\]
Let
\[
U(\theta, x) = \int_0^\theta \int_{\mathbb{R}^3} S(\theta - \hat{\theta}, x - y) \eta_2(t - \hat{\theta}, y) \nabla \psi^\delta(\hat{\theta}, y) dyd\hat{\theta}.
\]
Then, \( U \) satisfies the following system:
\[
\partial_\theta U - \Delta U + \nabla p_1 = \eta_2(t - \theta) \nabla \psi^\delta \quad \text{in} \, \mathbb{R}^3_T,
\]
\[
\nabla \cdot U = 0 \quad \text{in} \, \mathbb{R}^3_T,
\]
\[
U(0, x) = 0 \quad \text{in} \, \mathbb{R}^3.
\]
for some scalar function \( p_1 \). Thus,
\[
\frac{1}{2} \frac{d}{d\theta} \| U \|_{L^2}^2 + \| \nabla U \|_{L^2}^2 \leq \| U \|_{L^6} \| \eta_2(t - \theta) \|_{L^3} \| \nabla \psi^\delta \|_{L^2}^2
\]
\[
\leq \frac{1}{2} \| \nabla U \|_{L^2}^2 + C \| \eta_2(t - \theta) \|_{L^3} \| \nabla \psi^\delta \|_{L^2}^2
\]
and hence
\[
\frac{d}{d\theta} \| U \|_{L^2}^2 + \| \nabla U \|_{L^2}^2 \leq C \| \eta_2(t - \theta) \|_{L^3} \| \nabla \psi^\delta \|_{L^2}^2.
\]
This implies that
\[
\int_0^\theta III \, d\hat{\theta} \leq \left( \| \phi \|_{W^{2,\infty}} + C(1 + t^2) \| \eta_2 \|_{L^\infty} \right) \int_0^\theta \left\| \nabla \psi^\delta(\hat{\theta}) \right\|_{L^2}^2 d\hat{\theta}.
\]
• In a similar way to III, we estimate IV. First,
\[
IV \leq c_1(t - \theta) \| \nabla \psi^\delta \|_{L^2}^2
\]
\[
+ \left\| \int_0^\theta \int_{\mathbb{R}^3} \nabla \Gamma(t - \hat{\theta}, y, t - \theta, x) \cdot \left( \eta_2(t - \hat{\theta}, y) \nabla \psi^\delta(\hat{\theta}, y) \right) dyd\hat{\theta} \right\|_{H^1}^2.
\]
Let
\[
U(\theta, x) = \int_0^\theta \int_{\mathbb{R}^3} \nabla \Gamma(t - \hat{\theta}, y, t - \theta, x) \cdot \left( \eta_2(t - \hat{\theta}, y) \nabla \psi^\delta(\hat{\theta}, y) \right) dyd\hat{\theta}.
\]
After integration by parts, \( U \) satisfies the following equation
\[
\partial_\theta U + v_1 \cdot \nabla U + \eta_2 U - \Delta U = \nabla \cdot \left( \eta_2(t - \theta) \nabla \psi^\delta \right) \quad \text{in} \, \mathbb{R}^3_T,
\]
\[
U(0, x) = 0 \quad \text{in} \, \mathbb{R}^3.
\]
Then, using the divergence-free condition of \( v_1 \) and the sign condition of \( \eta_2 \),
\[
\frac{1}{2} \frac{d}{d\theta} \| U \|_{L^2}^2 + \| \nabla U \|_{L^2}^2 \leq \| \nabla U \|_{L^2} \| \eta_2(t - \theta) \|_{L^\infty} \| \nabla \psi^\delta \|_{L^2}
\]
\[
\leq \frac{1}{2} \| \nabla U \|_{L^2}^2 + C \| \eta_2(t - \theta) \|_{L^\infty} \| \nabla \psi^\delta \|_{L^2}^2.
\]
and hence
\[ \|U\|_{L^6_T L^2}^2 + \|\nabla U\|_{L^6_T L^2}^2 \leq C \|\eta_2\|_{L^\infty_T L^\infty}^2 \|\nabla \psi^\delta\|_{L^6_T L^2}^2. \]

So, we obtain
\[ \int_0^\tau V d\theta \leq \left( \|c_1\|_{L^\infty_T W^{1,\infty}}^2 + (1 + t^2) \|\eta_2\|_{L^\infty_T L^\infty}^2 \right) \int_0^\theta \|\nabla \psi^\delta(\theta)\|_{L^2}^2 d\theta. \]

- We finally estimate V:
\[ V \leq \|\phi\|_{W^{2,\infty}}^2 \|\nabla \psi^\delta\|_{L^2}^2 + \left\| \int_0^\theta \int_{\mathbb{R}^3} S(\theta - \hat{\theta}, x - y) \nabla c_2(t - \hat{\theta}, y) \mathcal{T}(\hat{\theta}, t, y, \psi^\delta) dy d\hat{\theta} \right\|_{H^1}^2. \]

Let
\[ U(\theta, x) = \int_0^\theta \int_{\mathbb{R}^3} S(\theta - \hat{\theta}, x - y) \nabla c_2(t - \hat{\theta}, y) \mathcal{T}(\hat{\theta}, t, y, \psi^\delta) dy d\hat{\theta}. \]

Then, U satisfies the following system:
\[ \partial_\theta U - \Delta U + \nabla p_2 = \nabla c_2(t - \theta) \mathcal{T}(\theta, t, y, \psi^\delta) \quad \text{in} \ \mathbb{R}^3_T, \]
\[ \nabla \cdot U = 0 \quad \text{in} \ \mathbb{R}^3_T, \]
\[ U(0, x) = 0 \quad \text{in} \ \mathbb{R}^3 \]

for some scalar function p_2. Then,
\[ \frac{1}{2} \frac{d}{d\theta} \|U\|_{L^2}^2 + \|\nabla U\|_{L^2}^2 \leq \|U\|_{L^6} \|\nabla c_2(t - \theta)\|_{L^3} \|\mathcal{T}(\theta, t, y, \psi^\delta)\|_{L^2} \]
\[ \leq \frac{1}{2} \|\nabla U\|_{L^2}^2 + 2 \|\nabla c_2(t - \theta)\|_{L^3}^2 \|\mathcal{T}(\theta, t, y, \psi^\delta)\|_{L^2}^2 \]
\[ \leq \frac{1}{2} \|\nabla U\|_{L^2}^2 + C \|\nabla c_2(t - \theta)\|_{L^3}^2 \|\eta_2\|_{L^\infty_T L^\infty}^2 \int_0^\theta \|\nabla \psi^\delta(\theta)\|_{L^2}^2 d\theta, \]

which implies that
\[ \|U\|_{L^6_T L^2}^2 + \|\nabla U\|_{L^6_T L^2}^2 \leq C \theta^2 \|\nabla c_2\|_{L^\infty_T L^3}^2 \|\eta_2\|_{L^\infty_T L^\infty}^2 \|\nabla \psi^\delta\|_{L^6_T L^2}^2. \]

Therefore, we obtain
\[ \int_0^\theta V d\theta \leq \left( \|\phi\|_{W^{2,\infty}}^2 + C(1 + t^2) \|\nabla c_2\|_{L^\infty_T L^3}^2 \|\eta_2\|_{L^\infty_T L^\infty}^2 \right) \int_0^\theta \|\nabla \psi^\delta(\theta)\|_{L^2}^2 d\theta. \]

- Collecting all terms together, we have
\[ \|\nabla \psi^\delta(t)\|_{L^2}^2 \leq \mu \|\nabla \psi^\delta\|_{L^2_T L^2}^2 \int_0^t \int (\delta + A^{\alpha+1}(t - \tau)) \|\Delta \psi^\delta\|_{L^2_x}^2 dx d\tau \]
\[ \leq \|\nabla \psi_0\|_{L^2}^2 + \int_0^t \mathcal{J}(t) \|\nabla \psi^\delta(\tau)\|_{L^2}^2 d\tau, \]

where
\[ \mathcal{J}(t) = \|\phi\|_{W^{2,\infty}}^2 + \|\nabla v_1\|_{L^\infty_T L^\infty} + \|\nabla^2 c_1\|_{L^\infty_T L^\infty} + \|\Delta c_1\|_{L^\infty_T L^\infty} \]
\[ + C(1 + t^2) \|\eta_2\|_{L^\infty_T L^\infty}^2 + \|\nabla c_1\|_{L^\infty_T L^\infty}^2 + C(1 + t^2) \|\nabla c_2\|_{L^\infty_T L^3}^2 \|\eta_2\|_{L^\infty_T L^\infty}^2. \]
We note that Hölder regularities in [7] are enough to control $J(t)$. Hence, we derive the following a prior estimate by Gronwall’s inequality

$$\sup_{0 \leq t \leq T} \| \nabla \psi_\delta(t) \|^2_{L^2} \leq \| \nabla \psi_0 \|^2_{L^2} e^{2T J(T)}.$$ 

Therefore,

$$\sup_{0 \leq t \leq T} \| \nabla \psi_\delta(t) \|^2_{L^2} \leq C(T, J(T)) \| \nabla \psi_0 \|^2_{L^2}$$

and thus

$$\delta \int_0^T |\Delta \psi_\delta|^2 dx d\tau \leq C(T, J(T)) \| \nabla \psi_0 \|^2_{L^2}.$$ 

This implies that

$$\lim_{\delta \to 0} \delta \int_0^t \int_{\mathbb{R}^3} \chi(\tau, x) \Delta \psi_\delta(\tau, x) dx d\tau = 0$$

for a.e. $0 < t < T$ and for all $\chi \in L^2_{t}L^2$. This completes the proof of Lemma 3.2.

**Remark 2.** The vanishing viscosity argument does not work with $q > 1$ of the system (1.1). Indeed, when we estimate the right-hand side of (3.14), we need to perform the integration by parts to move one derivative in $\Delta \zeta$ to the other terms in $II$. If $q > 1$, $A^{1+\alpha}$ appears in $II$ and it requires that $\eta \in W^{1,\infty}(\mathbb{R}^3_T)$, which is beyond the regularity in [7]. Developing new methods dealing with (1.1) for the case $q > 1$ and possibly other equations, having solutions but no uniqueness, will be some of our next study.

### 4. Fundamental solution of a parabolic equation

We here present the proof of Theorem 1.2. In fact, Theorem 1.2 is a corollary of a more general statement, Proposition 4.1. In this section, we shall denote by $\mathcal{L}$ a parabolic operator on $\mathbb{R}^n_T = (0, T) \times \mathbb{R}^n (n \geq 1)$ defined by

$$\mathcal{L} u = u_t - \text{div}(A \nabla u) + a \cdot \nabla u + bu.$$ 

The coefficients $A = A(t, x)$ is an $n \times n$ (not necessarily symmetric) matrix with entries $a^{ij}(t, x)$ that is uniformly parabolic and bounded. More precisely, we assume

$$\lambda |\xi|^2 \leq A(t, x)\xi \cdot \xi, \quad |A(t, x)\xi \cdot \eta| \leq \Lambda |\xi||\eta|, \quad \forall \xi, \eta \in \mathbb{R}^n, \quad \forall (t, x) \in \mathbb{R}^n_T$$

for some positive constants $\lambda$ and $\Lambda$. We assume that $a = (a^1, \ldots, a^n)$ is a divergence-free vector field. Then the adjoint operator $\mathcal{L}^* = \mathcal{L}^*$ of $\mathcal{L}$ is given by

$$\mathcal{L}^* u = -u_t - \text{div}(A^T \nabla u) - a \cdot \nabla u + bu.$$ 

We define the parabolic distance between the points $X = (t, x)$ and $Y = (s, y)$ in $\mathbb{R}^{n+1}$ as

$$|X - Y|_p := \max(\sqrt{|t - s|}, |x - y|).$$

We use the following notations for basic cylinders in $\mathbb{R}^{n+1}$:

$$Q_r^-(X) = (t - r^2, t) \times B_r(x),$$

$$Q_r^+(X) = (t, t + r^2) \times B_r(x),$$

$$Q_r(X) = (t - r^2, t + r^2) \times B_r(x).$$
**Proposition 4.1.** Let $a = (a^1, \ldots, a^n)$ be a divergence-free vector field and $b$ be a nonnegative function on $\mathbb{R}^n_T := (0, T) \times \mathbb{R}^n$. Assume that $a$ and $b$ are bounded on $\mathbb{R}^n_T$. Then, there exists a unique fundamental solution $\Gamma(t, x, s, y)$ of $\mathcal{L}$ on $\mathbb{R}^n_T$ which satisfies the following pointwise bound:

$$|\Gamma(t, x, s, y)| \leq C(t - s)^{-\frac{n}{2}} \exp \left( -c \frac{|x - y|^2}{t - s} \right), \quad (0 < s < t < T), \quad (4.2)$$

where $c = \frac{1}{4\Lambda} - \epsilon$ for any $0 < \epsilon < \frac{1}{4\Lambda}$ and $C = C(n, \lambda, \Lambda, T, \|a\|_{L^\infty(\mathbb{R}^n_T)}, \|b\|_{L^\infty(\mathbb{R}^n_T)}, \epsilon)$.

**Proof of Proposition 4.1.** To prove this proposition, we closely follow methods used in [6]. We recall some notations introduced there. For $U \subset \mathbb{R}^{n+1}$ and $I(U)$ for the set of all points $(t_0, x)$ in $U$ and $I(U)$ for the set of all $t$ such that $U(t)$ is nonempty. We denote

$$\|u\|_{U}^2 = \|\nabla u\|_{L^2(U)}^2 + \operatorname{ess sup}_{t \in I(U)} \|u(\cdot, t)\|_{L^2(U(t))}^2.$$  

We ask reader to consult [6] for definition of functions spaces such as $\dot{V}^{1,0}_2(Q)$. We first note that if $u \in \dot{V}^{1,0}_2(\mathbb{R}^n_T)$ is the weak solution of the problem

$$\begin{cases}
\mathcal{L}u = f & \text{in } (t_0, t_1) \times \mathbb{R}^n, \\
u = \psi & \text{on } \{t_0\} \times \mathbb{R}^n,
\end{cases}$$

then $u$ satisfies the energy inequality

$$\sup_{t_0 \leq t \leq t_1} \int_{\mathbb{R}^n} |u(t, x)|^2 \, dx + \lambda \int_{t_0}^{t_1} \int_{\mathbb{R}^n} |\nabla u(t, x)|^2 \, dx \, dt \leq \int_{\mathbb{R}^n} |\psi(x)|^2 \, dx + C \left( \int_{t_0}^{t_1} \int_{\mathbb{R}^n} |f(t, x)|^{\frac{2(n+2)}{n+4}} \, dx \, dt \right)^{\frac{n+4}{n+2}}, \quad (4.3)$$

where $C = C(n, \lambda, \Lambda)$. Here, we essentially use the assumption that $\nabla \cdot a = 0$ and $b \geq 0$. A similar statement is true for an adjoint problem.

We construct an approximate fundamental solution as follows. For $Y = (s, y) \in \mathbb{R}^n_T$ denote

$$d_Y := \sqrt{\max(s, T - s)}$$

so that $X \in \mathbb{R}^n_T$ when $|X - Y|_p < d_Y$. For $0 < \epsilon < d_Y$, let $v_\epsilon \in \dot{V}^{1,0}_2(\mathbb{R}^n_T)$ be a unique weak solution of the problem

$$\begin{cases}
\mathcal{L}u = \frac{1}{|Q^\epsilon_Y|}Q^\epsilon_Y(\mathcal{L}) & \text{in } Q^\epsilon_Y(\mathcal{L}), \\
u(0, \cdot) = 0.
\end{cases} \quad (4.4)$$

With the aid of the energy inequality (4.3), the unique solvability of the problem (4.4) in $\dot{V}^{1,0}_2(\mathbb{R}^n_T)$ follows from the Galerkin method described in [14, §III.5]. Observe that the energy inequality (4.3) implies

$$\|v_\epsilon\|_{\mathbb{R}^n_T} \leq Ce^{-\frac{s}{\epsilon}}. \quad (4.5)$$

Next, for a given $F \in C_c^\infty(\mathbb{R}^n_T)$, let $u \in \dot{V}^{1,0}_2(\mathbb{R}^n_T)$ be the weak solution of the *backward problem*

$$\begin{cases}
\mathcal{L}^*u = F & \text{in } (0, T) \times \mathbb{R}^n, \\
u(T, \cdot) = 0.
\end{cases} \quad (4.6)$$

Then, we have the identity

$$\int_{\mathbb{R}^n_T} v_\epsilon F \, dx \, dt = \int_{Q^\epsilon_Y(\mathcal{L})} u \, dx \, dt. \quad (4.7)$$
If we assume that $F$ is supported in $Q^+_R(X_0) \subset \mathbb{R}^n_T$, then an inequality similar to (4.3) yields
\[
\|u\|_{\mathbb{R}^n_T} \leq C\|F\|_{L^{2(n+2)/(n+4)}(Q^+_R(X_0))},
\] (4.8)

By the well-known embedding theorem (see e.g., [14, §II.3]), we have
\[
\|u\|_{L^{2(n+2)/n}(\mathbb{R}^n_T)} \leq C(n)\|u\|_{\mathbb{R}^n_T}.
\]

By combining the above two inequalities and using Hölder’s inequality, we get
\[
\|u\|_{L^2(Q^+_R(X_0))} \leq C(n, \lambda, \Lambda)R\|F\|_{L^{2(n+2)/(n+4)}(Q^+_R(X_0))}.
\] (4.9)

**Lemma 4.1.** Let $u$ be a weak solution of
\[
\mathcal{L}u = F \text{ in } Q^-_R(X_0),
\]
where $Q^-_R(X_0) \subset \mathbb{R}^n_T$. Then $u$ is locally Hölder continuous in $Q^-_R(X_0)$. In particular, $u$ is locally bounded in $Q^-_R(X_0)$ and for any $p > 0$, we have the estimate
\[
\sup_{Q^-_R(X_0)} |u| \leq C \left( \int_{Q^-_R(X_0)} |u|^p \, dx \, dt \right)^{\frac{1}{p}} + CR^2\|F\|_{L^\infty(Q^-_R(X_0))},
\]
where $C = C(n, p, \lambda, \Lambda, \|a\|_{L^\infty(\mathbb{R}^n_T)}, \|b\|_{L^\infty(\mathbb{R}^n_T)})$. A similar statement is true for a weak solution of
\[
\mathcal{L}^*u = F \text{ in } Q^+_R(X_0),
\]
where $Q^+_R(X_0) \subset \mathbb{R}^n_T$.

**Proof.** See [14, §III.8 and §III.10].

By utilizing (4.9) and the above lemma, we get
\[
\sup_{Q^+_R(X_0)} |u| \leq CR^2\|F\|_{L^\infty(Q^+_R(X_0))}.
\] (4.10)

If $Q^-_\epsilon(Y) \subset Q^+_R/2(X_0)$, then by (4.7) and (4.10), we obtain
\[
\left| \int_{Q^+_R(X_0)} \nu_\epsilon F \right| \leq \int_{Q^-_\epsilon(Y)} |u| \leq CR^2\|F\|_{L^\infty(Q^+_R(X_0))}.
\]

Therefore, by duality, it follows that we have
\[
\|\nu_\epsilon\|_{L^1(Q^+_R(X_0))} \leq CR^2.
\] (4.11)

We define the **averaged fundamental solution** $\Gamma^\epsilon(X, Y) = \Gamma^\epsilon(t, x, s, y)$ for $\mathcal{L}$ by setting
\[
\Gamma^\epsilon(t, Y) = \nu_\epsilon.
\]

**Lemma 4.2.** Let $X = (t, x)$, $Y = (s, y) \in \mathbb{R}^n_T$ and assume $0 < |X - Y|_p < \frac{1}{6}d_Y$. Then
\[
|\Gamma^\epsilon(X, Y)| \leq C |X - Y|^{-n}, \quad \forall \epsilon < \frac{1}{3} |X - Y|_p,
\] (4.12)

where $C = C(n, \lambda, \Lambda, \|a\|_{L^\infty(\mathbb{R}^n_T)}, \|b\|_{L^\infty(\mathbb{R}^n_T)})$. 

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Proof. Denote \( d = |X - Y|_p \), and let \( r = \frac{1}{3}d \), \( X_0 = (y, s - 4d^2) \), and \( R = 6d \). It is easy to see that for \( \epsilon < \frac{1}{3}d \), we have
\[
Q_\epsilon^{-}(Y) \subset Q_{R/2}^+(X_0), \quad Q_\epsilon^{-}(X) \subset Q_{R}^+(X_0).
\]
and also that \( v_\epsilon \) satisfies \( \mathcal{L} v_\epsilon = 0 \) in \( Q_\epsilon^{-}(X) \). Then, by Lemma 4.1, we have
\[
\|v_\epsilon\|_{L^\infty(Q_{R/2}^{-}(X))} \leq \frac{C}{r^{n+2}}\|v_\epsilon\|_{L^1(Q_\epsilon^{-}(X))}.
\]
Therefore, by (4.11), we have \( |v_\epsilon(X)| \leq Cr^{-n} \), which implies (4.12).

For \( \epsilon < r < R < \frac{1}{6}d_Y \), let \( \eta : \mathbb{R}^{n+1} \to \mathbb{R} \) be a smooth function such that
\[
0 \leq \eta \leq 1, \quad \eta \equiv 0 \text{ on } Q_r(Y), \quad \eta \equiv 1 \text{ on } Q_R(Y), \quad |\nabla \eta|^2 + |\nabla^2 \eta| + |\eta| \leq \frac{12}{(R-r)^2}. \tag{4.13}
\]
Recall that \( v_\epsilon \) satisfies (4.4). By testing with \( \eta^2 v_\epsilon \) and using assumption \( \nabla \cdot a = 0 \), we have
\[
0 = \int_{\mathbb{R}^n} \frac{1}{2} (\eta^2 v_\epsilon^2)_t - \int_{\mathbb{R}^n} \eta \eta_t v_\epsilon^2 + \int_{\mathbb{R}^n} \eta^2 A \nabla v_\epsilon \cdot \nabla v_\epsilon
+ \int_{\mathbb{R}^n} 2\eta A \nabla v_\epsilon \cdot \nabla \eta v_\epsilon - \int_{\mathbb{R}^n} a \cdot \nabla \eta \eta v_\epsilon^2 + \int_{\mathbb{R}^n} \eta^2 b v_\epsilon^2.
\]
Then by using (4.1) and \( b \geq 0 \), we get
\[
\int_{\mathbb{R}^n} \frac{1}{2} (\eta^2 v_\epsilon^2)_t + \lambda \int_{\mathbb{R}^n} \eta^2 |\nabla v_\epsilon|^2 \leq \int_{\mathbb{R}^n} \eta \eta_t |v_\epsilon|^2 + 2\Lambda \int_{\mathbb{R}^n} \eta |\nabla v_\epsilon| |\nabla \eta| |v_\epsilon| + \int_{\mathbb{R}^n} \eta |\nabla \eta| |a||v_\epsilon|^2.
\]
By using Young’s inequality and integrating over \( t \), we get
\[
\sup_{0 \leq t \leq T} \frac{1}{2} \int_{\mathbb{R}^n} \eta^2 v_\epsilon^2 + \lambda \int_{\mathbb{R}^n} \eta^2 |\nabla v_\epsilon|^2 \leq \int_{\mathbb{R}^n} \left\{ |\eta_t| + \frac{2\Lambda^2}{\lambda} |\nabla \eta|^2 + |a||\nabla \eta| \right\} v_\epsilon^2.
\]
Therefore, by (4.13) and noting that \( R - r < \sqrt{T} \) and so
\[
|\nabla \eta| \leq \sqrt{12/(R - r)} = \sqrt{12(R - r)/(R - r)^2} \leq \sqrt{12T/(R - r)^2},
\]
w\( e \) have
\[
\sup_{0 \leq t \leq T} \int_{\mathbb{R}^n} \eta^2 v_\epsilon^2 + \int_{\mathbb{R}^n} \eta^2 |\nabla v_\epsilon|^2 \leq \frac{C}{(R-r)^2} \int_{Q_R(Y) \setminus Q_T(Y)} v_\epsilon^2, \tag{4.14}
\]
where \( C = C(n, \lambda, \Lambda, T, \|a\|_{L^\infty(\mathbb{R}^n)}) \).

Then by setting \( r = \frac{1}{2}R \) in (4.14) and applying Lemma 4.2, we obtain
\[
\|\Gamma^\epsilon(\cdot, Y)\|_{L^2_{\mathbb{P}} \setminus Q_R(Y)}^2 \leq CR^{-2} \int_{\{R/2 < |X - Y|_p < R\}} |X - Y|_p^{-2n} dX \leq CR^{-n},
\]
whenever \( \epsilon < \frac{1}{6}R \) and \( R < \frac{1}{6}d_Y \). On the other hand, in the case when \( \epsilon \geq \frac{1}{6}R \), (4.5) yields
\[
\|\Gamma^\epsilon(\cdot, Y)\|_{L^2_{\mathbb{P}} \setminus Q_R(Y)}^2 \leq \|\Gamma^\epsilon(\cdot, Y)\|_{L^2_{\mathbb{P}}}^2 \leq C\epsilon^{-n} \leq CR^{-n}.
\]
Therefore, we have for all \( \epsilon < d_Y \) and \( R < \frac{1}{6}d_Y \)
\[
\|\Gamma^\epsilon(\cdot, Y)\|_{L^2_{\mathbb{P}} \setminus Q_R(Y)}^2 \leq CR^{-n}. \tag{4.15}
\]
We note that the estimate (4.15) corresponds to [6, (4.6)]. With Lemma 4.1 at hand, we can repeat the same argument as in [6] to construct the fundamental solution
\[
\Gamma(X, Y) = \Gamma(t, x, s, y)
\]
for $\mathcal{L}$ in $\mathbb{R}^n_T$. See Section 4.2 – 4.3 of [6] for details.

Next, we show that $\Gamma(x, t, y, s)$ satisfies the Gaussian bound (4.2). We modify an argument in [12], which is in turn based on Davies [9] and Fabes-Stroock [10]. Let $\psi$ be a bounded Lipschitz function on $\mathbb{R}^n$ satisfying $|\nabla \psi| \leq \gamma$ a.e. for some $\gamma > 0$ to be chosen later. Fix $s \in (0, T)$. For $s < t < T$, we define an operator $P^\psi_{s \rightarrow t}$ on $L^2(\mathbb{R}^n)$ as follows. For a given $f \in L^2(\mathbb{R}^n)$, let $u \in \dot{V}^1_2((s, T) \times \mathbb{R}^n)$ be the weak solution of the problem

$$
\begin{cases}
\mathcal{L} u = 0 \\
\int u(s, \cdot) = e^{-\psi} f.
\end{cases}
$$

Then, we define

$$
P^\psi_{s \rightarrow t} f(x) := e^{\psi(x)} u(t, x)
$$

so that

$$
P^\psi_{s \rightarrow t} f(x) = e^{\psi(x)} \int_{\mathbb{R}^n} \Gamma(t, x, s, y)e^{-\psi(y)} f(y) dy.
$$

Let us denote

$$
I(t) := \int_{\mathbb{R}^n} e^{2\psi(x)} u(t, x)^2 dx.
$$

Then, $I'(t)$ satisfies for a.e. $t \in (s, T)$ that

$$
I'(t) = -2 \int_{\mathbb{R}^n} A\nabla u \cdot \nabla (e^{2\psi} u) + e^{2\psi} u a \cdot \nabla u + e^{2\psi} bu^2 dx
\leq -2 \int_{\mathbb{R}^n} e^{2\psi} A\nabla u \cdot \nabla u dx - 4 \int_{\mathbb{R}^n} e^{2\psi} u A\nabla u \cdot \nabla \psi dx - 2 \int_{\mathbb{R}^n} a \cdot \nabla u e^{2\psi} u dx,
$$

where we used that $b \geq 0$.

By using $\nabla \cdot a = 0$, we find that

$$
0 = \int_{\mathbb{R}^n} a \cdot \nabla (e^{\psi} u) e^{\psi} u dx = \int_{\mathbb{R}^n} a \cdot \nabla u e^{2\psi} u dx + \int_{\mathbb{R}^n} a \cdot \nabla \psi e^{2\psi} u^2 dx.
$$

For the rest of proof, we set

$$
\kappa := \|a\|_{L^\infty(\mathbb{R}^n_T)}.
$$

By using $|\nabla \psi| \leq \gamma$, we then obtain

$$
I'(t) \leq -2\lambda \int_{\mathbb{R}^n} e^{2\psi} |\nabla u|^2 dx + 4\Lambda \int_{\mathbb{R}^n} e^{\psi} |u| e^{\psi} |\nabla u| dx + 2\gamma \kappa \int_{\mathbb{R}^n} e^{2\psi} u^2 dx
\leq (2\lambda^2 \lambda^{-1}\gamma^2 + 2\kappa\gamma) \int_{\mathbb{R}^n} e^{2\psi} u^2 dx = 2(\lambda^2 \lambda^{-1}\gamma^2 + \kappa\gamma) I(t).
$$

The initial condition $u(s, \cdot) = e^{-\psi} f$ yields

$$
I(t) \leq e^{2(\lambda^2 \lambda^{-1}\gamma^2 + \kappa\gamma)(t-s)} \|f\|^2_{L^2(\mathbb{R}^n)}.
$$

Since $I(t) = \|P^\psi_{s \rightarrow t} f\|^2_{L^2(\mathbb{R}^n)}$, we have derived

$$
\|P^\psi_{s \rightarrow t} f\|_{L^2(\mathbb{R}^n)} \leq e^{(\lambda^2 \lambda^{-1}\gamma^2 + \kappa\gamma)(t-s)} \|f\|_{L^2(\mathbb{R}^n)}.
$$

(4.16)
By Lemma 4.1, we estimate 
\[ e^{-2\psi(x)}|P_{s\to t}f(x)|^2 = |u(x,t)|^2 \]
\[ \leq \frac{C}{(t-s)^{\frac{n+2}{2}}} \int_s^t \int_{B_{\sqrt{t-\tau}(x)}} |u(y,\tau)|^2 \, dy \, d\tau \]
\[ \leq \frac{C}{(t-s)^{\frac{n+2}{2}}} \int_s^t \int_{B_{\sqrt{t-\tau}(x)}} e^{-2\psi(y)}|P_{s\to t}f(y)|^2 \, dy \, d\tau. \]
Hence, by using (4.16) we find 
\[ |P_{s\to t}f(x)|^2 \leq C(t-s)^{-\frac{n+2}{2}} \int_s^t \int_{B_{\sqrt{t-\tau}(x)}} e^{2\psi(x)-2\psi(y)}|P_{s\to t}f(y)|^2 \, dy \, d\tau \]
\[ \leq C(t-s)^{-\frac{n+2}{2}} \int_s^t \int_{B_{\sqrt{t-\tau}(x)}} e^{2\gamma\sqrt{t-\tau}}|P_{s\to \tau}f(y)|^2 \, dy \, d\tau \]
\[ \leq C(t-s)^{-\frac{n+2}{2}} e^{2\gamma\sqrt{t-s}} \int_s^t \int_{B_{\sqrt{t-\tau}(x)}} e^{2(\lambda^2\lambda^{-1}\gamma^2+\kappa\gamma)(\tau-s)} \|f\|_{L^2(\mathbb{R}^n)}^2 \, d\tau \]
\[ \leq C(t-s)^{-\frac{n+2}{2}} e^{2\gamma\sqrt{t-s}+2(\lambda^2\lambda^{-1}\gamma^2+\kappa\gamma)(t-s)} \|f\|_{L^2(\mathbb{R}^n)}^2. \]
We have thus derived the following \( L^2 \to L^\infty \) estimate for \( P_{s\to t}^\psi \):
\[ \|P_{s\to t}^\psi f\|_{L^\infty(\mathbb{R}^n)} \leq C(t-s)^{-\frac{n+2}{2}} e^{\gamma\sqrt{t-s}+(\lambda^2\lambda^{-1}\gamma^2+\kappa\gamma)(t-s)} \|f\|_{L^2(\mathbb{R}^n)}. \] \hfill (4.17)
We also define the operator \( Q^\psi_{t\to s} \) on \( L^2(\mathbb{R}^n) \) by
\[ Q^\psi_{t\to s}g(y) = e^{-\psi(y)}v(y,s), \]
where \( v \) is the weak solution of the backward problem
\[
\begin{cases}
  \mathcal{L}^* u = 0 \\
  u(t,\cdot) = e^\psi g.
\end{cases}
\]
Then, by a similar calculation, we get 
\[ \|Q^\psi_{t\to s}g\|_{L^\infty(\mathbb{R}^n)} \leq C(t-s)^{-\frac{n+2}{2}} e^{\gamma\sqrt{t-s}+(\lambda^2\lambda^{-1}\gamma^2+\kappa\gamma)(t-s)} \|g\|_{L^2(\mathbb{R}^n)}. \]
Since (see [6, Section 5.1])
\[ \int_{\mathbb{R}^n} (P_{s\to t}^\psi f) \, g \, dx = \int_{\mathbb{R}^n} f \, (Q^\psi_{t\to s}g) \, dx, \]
by duality, for any \( f \in C_c^\infty(\mathbb{R}^n) \), we have 
\[ \|P_{s\to t}^\psi f\|_{L^2(\mathbb{R}^n)} \leq C(t-s)^{-\frac{n+2}{2}} e^{\gamma\sqrt{t-s}+(\lambda^2\lambda^{-1}\gamma^2+\kappa\gamma)(t-s)} \|f\|_{L^1(\mathbb{R}^n)} \]. \hfill (4.18)
Now, set \( r = \frac{s+t}{2} \) and observe that by the uniqueness, we have 
\[ P_{s\to t}^\psi f = P_{r\to t}^\psi (P_{s\to r}^\psi f), \quad \forall f \in C_c^\infty(\mathbb{R}^n). \]
Then, by noting that 
\[ t-r = r-s = \frac{1}{2}(t-s), \]
we obtain from (4.17) and (4.18) that for any \( f \in C_c^\infty(\mathbb{R}^n) \), we have 
\[ \|P_{s\to t}^\psi f\|_{L^\infty(\mathbb{R}^n)} \leq C(t-s)^{-\frac{n+2}{2}} e^{\gamma\sqrt{2(t-s)}+(\lambda^2\lambda^{-1}\gamma^2+\kappa\gamma)(t-s)} \|f\|_{L^1(\mathbb{R}^n)}. \]
For fixed $x, y \in \mathbb{R}^n$ with $x \neq y$, the above estimate imply, by duality, that
\[
e^{\psi(x) - \psi(y)} |\Gamma(t, x, y, s)| \leq C(t - s)^{-\frac{n}{2}} e^{\gamma \sqrt{2(t-s)}} e^{\gamma \sqrt{(\Lambda^2 - 1) \gamma^2 + \alpha t}} e^{\sqrt{2(t-s)}}.
\]

We now set
\[
\psi(z) = \gamma \min(|z - y|, |x - y|) \quad \text{and} \quad \gamma = \frac{\lambda}{2\Lambda^2} \frac{|x - y|}{t - s}.
\]

It should be clear that $\psi$ is a bounded Lipschitz function satisfying $|\nabla \psi| \leq \gamma$ a.e.,
\[
\psi(x) = \gamma |x - y| = \frac{\lambda}{2\Lambda^2} \frac{|x - y|^2}{t - s}, \quad \text{and} \quad \psi(y) = 0.
\]

Therefore, (4.19) yields
\[
|\Gamma(t, x, s, y)| \leq C(t - s)^{-\frac{n}{2}} \exp \left\{ \frac{\lambda}{\sqrt{2\Lambda^2}} \frac{|x - y|}{\sqrt{t - s}} + \frac{\lambda}{4\Lambda^2} \frac{|x - y|^2}{t - s} + \frac{\kappa \lambda}{2\Lambda^2} |x - y| - \frac{\lambda}{2\Lambda^2} \frac{|x - y|^2}{t - s} \right\}
\[
\leq C(t - s)^{-\frac{n}{2}} \exp \left\{ \left( \frac{\lambda}{\sqrt{2\Lambda^2} + \frac{\kappa \lambda}{2\Lambda^2}} \right) \frac{|x - y|}{\sqrt{t - s}} - \frac{\lambda}{4\Lambda^2} \frac{|x - y|^2}{t - s} \right\}.
\]

Note that for any $\epsilon \in (0, \lambda/4\Lambda^2)$, there exists a number $N = N(\lambda, \Lambda, T, \kappa, \epsilon)$ such that
\[
e^{\left( \frac{\lambda}{\sqrt{2\Lambda^2} + \frac{\kappa \lambda}{2\Lambda^2}} \right) r - \frac{\lambda}{4\Lambda^2} |r|^2} \leq Ne^{-\left( \frac{\lambda}{4\Lambda^2} - \epsilon \right) |r|^2}, \quad \forall r \geq 0,
\]
and recall that $\kappa = \|a\|_{L^\infty(\mathbb{R}^n)}$. Therefore, we obtain the Gaussian bound (4.2).

**Proof of Theorem 1.2.** To prove Theorem 1.2, we begin with the following lemma.

**Lemma 4.3.** Assume the hypothesis of Theorem 1.2 holds. Let $u$ be a solution of
\[
\mathcal{L}u = 0 \quad \text{in} \quad Q^-_T(X_0),
\]
where $Q^-_T(X_0) \subset \mathbb{R}^3$. Then, $\nabla u, u_t, \nabla^2 u$ are locally Hölder continuous in $Q^-_T(X_0)$. In particular, we have the following estimate:
\[
r \|\nabla u\|_{L^\infty(Q^-_{r/2}(X_0))} + r^2 \|\nabla^2 u\|_{L^\infty(Q^-_{r/2}(X_0))} + r^2 \|u_t\|_{L^\infty(Q^-_{r/2}(X_0))} \leq C \|u\|_{L^\infty(Q^-_{r/2}(X_0))},
\]
where $C = C(\|a\|_{C^{0,\beta/2}(\mathbb{R}^n)}, \|b\|_{C^{0,\alpha/2}(\mathbb{R}^n)}).

**Proof.** It is a consequence of the standard Schauder theory.

We apply Proposition 4.1 to the operator $\mathcal{L}$ and construct the fundamental solution $\Gamma(t, x, s, y)$. Note that we have $\lambda = \Lambda = 1$ in this case so that we have $\frac{\lambda}{4\Lambda^2} = \frac{1}{4}$, and thus (4.2) follows.

Next, to prove (1.7), let $r := \frac{1}{2} \sqrt{t - s}$ and note that $u(\cdot) := \Gamma(\cdot, Y)$ satisfies
\[
\mathcal{L}u = 0 \quad \text{in} \quad Q^-_r = Q^-_r(X).
\]

Then, by Lemma 4.3, we have
\[
\sqrt{t - s} |\nabla \Gamma(t, x, s, y)| + (t - s) |\nabla^2 \Gamma(t, x, s, y)| + (t - s) |\partial_t \Gamma(t, x, s, y)| \leq C \|u\|_{L^\infty(Q^-_r(X)).
\]

Note that for $Z = (\tau, z) \in Q^-_r(X)$, we have
\[
\sqrt{t - s} \leq \tau - s \leq t - s \quad \text{and} \quad |x - y| - \frac{1}{2} \sqrt{t - s} \leq |z - y| \leq |x - y| + \frac{1}{2} \sqrt{t - s}.
\]
Therefore, by (4.2), we have
\[ \| u \|_{L^\infty(Q_r^c(x))} \leq C(t - s)^{-\frac{3}{2}} e^{-c' \frac{|x - y|^2}{2(t - s)}}, \] (4.21)
where \( c' > 0 \) is an absolute constant and \( C = C(T, \| a \|_{C^{\beta, \beta/2}(\mathbb{R}^3)}, \| b \|_{C^{\beta, \beta/2}(\mathbb{R}^3)}). \) By (4.20) and (4.21), we obtain the estimate (1.7), which complete the proof of Theorem 1.2.

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