On flat complete causal Lorentzian manifolds

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Abstract

We describe up to finite coverings causal flat affine complete Lorentzian manifolds such that the past and the future of any point are closed near this point. We say that these manifolds are strictly causal. In particular, we prove that their fundamental groups are virtually abelian. In dimension 4, there is only one, up to a scaling factor, strictly causal manifold which is not globally hyperbolic. For a generic point of this manifold, either the past or the future is not closed and contains a lightlike straight line.

1 Introduction

Let $M_n$ be Minkowski spacetime of dimension $n$ and $P_n$ be Poincare group of its affine automorphisms. We consider manifolds that can be realized as quotient spaces $M_n/\Gamma$, where $\Gamma \cong \pi_1(M)$ is a discrete subgroup of $P_n$ whose action on $M_n$ is free and proper. These manifolds may be defined in differential geometric terms as geodesically complete Lorentzian manifolds with vanishing torsion and curvature. They are characterized by the existence of an atlas of coordinate charts with coordinate transformations in $P_n$, and the completeness. The latter means that any affine mapping of a segment in $\mathbb{R}$ to $M$ extends affinely to $\mathbb{R}$. Also, these manifolds are complete affine manifolds with a compatible Lorentzian metric.

Complete affine manifolds are studied since 60s. The following question is known as Auslander’s conjecture: is it true that $\pi_1(M)$ for a compact flat complete affine $M$ is virtually solvable? It remains unanswered but in many cases has the affirmative answer (see [1], [4] for details and further references). If $M$ is not compact then $\pi_1(M)$ may be free non-Abelian according to the remarkable example by Margulis [7] that gives the negative answer to the question of Milnor [9]. In paper [6] by Fried, Auslander’s conjecture was proved for Lorentzian compact 4-manifolds; also, [6] contains a description of causal two ended quotient spaces $H/\Gamma$, where $H$ is a subgroup of $P_n$ that is simply transitive on $M_n$.

In what follows, we assume that manifolds are Lorentzian, flat and complete if the contrary is not stated explicitly. The Lorentzian metric defines in each tangent space the pair of closed convex round cones. Choosing one of them,
we get locally a cone field. It can be extended to the global cone field on \( M \) or on a two-sheet covering space of \( M \). If \( M \) admits no closed timelike curves then \( M \) is said to be causal. We describe up to finite coverings complete flat causal Lorentzian manifolds which satisfy the following additional condition: the past and the future of any point \( p \in M \) are closed near \( p \). We say that these manifolds are strictly causal. Generic causal manifolds of the paper [6] are not strictly causal. On the other hand, nontrivial strictly causal manifolds are never globally hyperbolic. Manifolds of the latter class are well understood (the paper [2] by Barbot contains many useful information on them, including a classification, without the assumption of the completeness). We give an explicit (parametric) description of strictly causal manifolds up to finite coverings. The simplest nontrivial example has dimension 4: \( M = M_4/\Gamma \), where \( \Gamma \cong \mathbb{Z} \). Its causal properties are somewhat surprising. The manifold \( M \) is the disjoint union

\[
M = M^+ \cup M^0 \cup M^-,
\]

where \( M^+ \) and \( M^- \) are open and \( M^0 \) is closed (moreover, \( M^0 \) is an affine hyperplane). For any \( p \in M^+ \), its future \( F_p \) is closed; the past \( P_p \) is not closed and contains lightlike straight lines. Furthermore, \( M \) admits an involutive automorphism that transposes \( M^+ \) and \( M^- \) (hence the future of a point in \( M^- \) is not closed). For \( p \in M^0 \), both \( F_p \) and \( P_p \) are closed and contains no lightlike line. Also, any strictly causal 4-manifold that is not globally hyperbolic is homothetic to the manifold of this example.

In contrast to most of the cases above, the fundamental group \( \pi_1(M) \) of a strictly causal manifold \( M \) is virtually abelian. Moreover, \( M \) can be finitely covered by the product of a torus and Euclidean space. A finite cover of \( M \) can be realized as a (topologically trivial) vector bundle over \( M_k/\Gamma \), where \( k \leq n \), \( M_k \) is embedded to \( M_n \) as an affine \( \Gamma \)-invariant subspace, \( \Gamma \) is unipotent in \( M_k \), and the holonomy group is defined by a bounded linear representation of \( \Gamma \) in the fibre. Thus the problem is reduced to the case of unipotent \( \Gamma \). Up to finite coverings, there are two types of these manifolds. The first, elliptic, consists of manifolds that admit a Riemannian metric such that the identical mapping is affine. In other words, linear parts of transformations in \( \Gamma \) keep some positive definite quadratic form; then \( \Gamma \) contains a finite index subgroup of translations by vectors in some spacelike vector subspace. For manifolds of the second (parabolic) type, \( \Gamma \) is a uniform lattice in a vector group \( T \) whose action in \( M_n \) is free and quadratic on \( T \).

Some results in this article overlap with the recent paper [2]).

2 Preliminaries and statement of results

Fixing the origin \( o \in M_n \), we identify \( M_n \) with the real vector space \( V \) equipped with a Lorentzian form \( \ell \) of the signature \((+, - , \ldots, -)\). The set \( \ell(v, v) \geq 0 \) is the union of two closed convex round cones in \( V \). Let \( C \) be one of them. The group \( \Gamma \) is assumed to act freely and properly in \( V \) by affine transformations whose linear
parts keep $t$ and $C$. We denote by $\kappa$ the quotient mapping $\mathcal{M}_n \rightarrow M = V/\Gamma$ and define the past $P_p$ and the future $F_p$ of $p \in M$ by

$$P_p = \kappa(v - C), \quad F_p = \kappa(v + C), \quad v \in \kappa^{-1}(p).$$

Clearly, $P_p$ and $F_p$ do not depend on the choice of $v$. On $M$, this defines the field of pointed convex cones $C_p = d_v \kappa(v + C), v \in V$. The manifold $M$ is said to be causal if $M$ admits no closed piecewise smooth timelike paths. A smooth path $\eta$ is called timelike if $\eta'(t) \in C_{\eta(t)}$ for all $t$; for lightlike paths, $\eta'(t) \in \partial C_{\eta(t)}$ (note that lightlike paths are timelike). The definition naturally extends to piecewise smooth paths. Clearly, any timelike curve in $M$ can be lifted to a timelike curve in $V$ and the projection of a timelike curve in $V$ is timelike. The following observation is useful: $M$ is causal if and only if

$$v \in V, \, \gamma \in \Gamma, \, \gamma(v) \in v + C \quad \implies \quad \gamma = e, \quad (1)$$

where $e$ denotes the identity of $\Gamma$. Indeed, if $\gamma \neq e$ then $\gamma(v) \neq v$ and the projection to $M$ of the segment with endpoints $v$ and $\gamma(v)$ is a nontrivial closed timelike curve. Conversely, we get a timelike curve $\tilde{\eta}$ in $V$ lifting a timelike curve $\eta$ in $M$; hence $\tilde{\eta}$ lies in $v + C$ if it starts at $v \in \kappa^{-1}(p), \, p \in M$. If $\eta$ is closed and nontrivial then its endpoint is $\gamma(v)$ for some $\gamma \in \Gamma \setminus \{e\}$.

We say that $M$ is strictly causal if $M$ is causal and for each $p \in M$ there exists a neighbourhood $U$ of $p$ such that $U \cap F_p$ and $U \cap P_p$ are closed in $U$. For $\gamma \in \Gamma$, set

$$\gamma(v) = \lambda(\gamma)v + \tau(\gamma), \quad \text{where} \quad \lambda(\gamma) \in O(\ell), \, \tau(\gamma) \in V; \quad (2)$$

$$G = \lambda(\Gamma).$$

Clearly, $\lambda : \Gamma \rightarrow O(\ell)$ is a homomorphism and for all $\gamma, \nu \in \Gamma$

$$\tau(\gamma \circ \nu) = \lambda(\gamma)\tau(\nu) + \tau(\gamma).$$

A linear subspace $X \subset V$ is called spacelike if $X \cap C = \{0\}$; $X$ is spacelike if and only if $\ell$ is negative definite on $X$. We say that $M, \Gamma$ and $G$ are unipotent if $G$ consists of unipotent linear transformations. The classification problem can be reduced to the unipotent case.

**Theorem 1.** A strictly causal flat complete Lorentzian manifold $M$ can be finitely covered by $V/\Gamma$, where $\Gamma$ is abelian and satisfies following conditions: there exists $\tilde{o} \in V$ and linear subspaces $V_0, V_1 \subseteq V$ such that $V = V_0 \oplus V_1$ and

1) the decomposition is $\ell$-orthogonal and $G$-invariant, $\ell$ is Lorentzian in $V_0$ and negative definite in $V_1$;

2) the affine subspace $U = \tilde{o} + V_0$ is $\Gamma$-invariant and $\Gamma$ is unipotent in it;

3) the mapping of the restriction to $U$ is injective on $\Gamma$ and $U/\Gamma$ is strictly causal.
Furthermore, the action of $\Gamma$ in $V_1$ is linear and can be defined by an arbitrary homomorphism $\alpha : \Gamma \to O(\ell|_{V_1})$.

In other words, the action of $\Gamma$ splits into the unipotent affine and bounded linear ones analogously to Euclidean case. The manifold $V/\Gamma$ is isomorphic to the total space of a topologically trivial vector bundle with the unipotent base $U/\Gamma$, the fibre $V_1$ and the holonomy representation $\alpha$. The decomposition is not unique if $G$ has nontrivial fixed points which are orthogonal to $\tau(\Gamma)$. We describe two types of $\Gamma$ that classify unipotent $M = V/\Gamma$ up to finite coverings.

(i) $T, L \subset V$ are linear subspaces such that $V = T \oplus L$, where the sum is orthogonal, $T$ is spacelike, $\dim L = 1$ and $L \cap \text{Int}(C) \neq \emptyset$. The group $\Gamma$ is a uniform lattice in $T$ that acts in $V$ by translations (i.e. $\tau(\gamma) = \gamma$ and $\lambda(\gamma)$ is identical in (2)).

(ii) Let $v_0, v_1 \in \partial C$ satisfy $\ell(v_0, v_1) = 1$ and set

$$L = \mathbb{R}v_0, \quad W = L^\perp, \quad N = W \cap v_1^+, \quad l_0(v) = \ell(v, v_0).$$

The hyperplane $W = N \oplus L$ is tangent to $\partial C$ at $v_0$. The form $\ell$ is nonpositive and degenerate in $W$, and nondegenerate and negative in $N$. Any $x \in N$ corresponds the following linear transformation of $V$:

$$\nu(x)v = v + l_0(v)x - \left(\ell(v, x) + \frac{1}{2}l_0(v)\ell(x, x)\right)v_0. \quad (3)$$

A straightforward calculation shows that $\nu$ is a homomorphism of the vector group $N$ to $SO(\ell)$ (in fact, $\nu$ identifies it with the factor $N$ in the Iwasawa decomposition $KAN$ for $SO(\ell)$). Note that for $w \in W$

$$\nu(x)w = w - \ell(w, x)v_0.$$

Further, let $T \subset N$ be a linear subspace and set $\tilde{T} = T + L$. We may identify $T$ with $\tilde{T}/L$. Let $\Gamma$ be a uniform lattice in $T$ and $a$ be an $\ell$-symmetric linear mapping

$$a : T \to N,$$

$$\ell(ax, y) = \ell(x, ay), \quad x, y \in T. \quad (4)$$

Any $a$ as above defines the affine action of $T$ in $V$ by setting

$$\lambda(x) = \nu(ax); \quad (5)$$

$$\tau(x) = x - \frac{1}{2}l_0(ax, x)v_0; \quad (6)$$

$$\gamma_x(v) = \lambda(x)v + \tau(x).$$

Here is a necessary and sufficient condition for this action to be free:

$$\ker(1 + ta) = 0 \quad \text{for all } t \in \mathbb{R}, \quad (7)$$

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where 1 is the identical mapping in \( T \). If \( a \) satisfies (4) then it admits the unique decomposition \( a = a' + a'' \), where
\[
\ell(ax, y) = \ell(a'x, y) = \ell(x, a'y), \quad x, y \in T,
\]
\( a' : T \to T \)
is the self-adjoint transformation of \( T \) corresponding to the symmetric bilinear form \( \ell(ax, y) \) and \( a'' \) is a linear mapping
\( a'' : T \to T \perp \cap N. \)

The condition (7) can be rewritten as follows:
\[
t \in \mathbb{R}, \quad a'x = tx \neq 0 \implies a''x \neq 0. \quad (8)
\]
Since \( a' \) is self-adjoint it has real eigenvalues and (8) implies that \( a'' \neq 0 \) if \( a' \neq 0 \); then \( T \neq W \). Moreover, (7) is true if and only if the action of \( \Gamma \) is free and proper (Lemma 19). Also, note that \( \Gamma \) consists of pure translations if \( a = 0 \); then it is of the type (i) in fact.

**Theorem 2.** If \( M \) is strictly causal and unipotent then it admits a finite covering by \( V/\Gamma \) with \( \Gamma \) as in (i) or (ii); in the latter case (7) holds. Conversely, if \( \Gamma \) is as in (i) or in (ii) with fulfilled (7) then \( V/\Gamma \) is strictly causal. In both cases, \( V/\Gamma \) is homeomorphic to the product of the torus \( T/\Gamma \) and the vector space \( V/T \).

The affine structure in \( V/\Gamma \) is not the product one if \( a \neq 0 \).

Here is the simplest example of a non-elliptic manifold of the type (ii) in the least possible dimension. The group \( \Gamma \) in it is a subgroup of some group of the paper [6]. For its generic point, either the past or the future is not closed and contains a lightlike straight line.

**Example 1.** Let \( \dim V = 4 \). Set
\[
\ell(v, v) = 2v_0v_3 - v_1^2 - v_2^2, \quad v = (v_0, v_1, v_2, v_3)^\top \in V = \mathbb{R}^4; \quad \lambda(n) = \begin{pmatrix} 1 & 0 & n & \frac{2}{n} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & n \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \tau(n) = \begin{pmatrix} 0 \\ n \\ 0 \\ 0 \end{pmatrix}, \quad n \in \mathbb{Z}; \quad (9)
\]
\[
\gamma_n(v) = \lambda(n)v + \tau(n). \quad (10)
\]
Then \( \Gamma = \{\gamma_n\}_{n \in \mathbb{Z}} \) is the cyclic infinite group; if \( e_0, e_1, e_2, e_3 \) is the standard basis of \( V \) then \( \Gamma = \mathbb{Z}e_1, l_0(v) = v_3, ae_1 = e_2 \). A calculation shows that
\[
\ell(\gamma_n(v) - v, \gamma_n(v) - v) = -n^2(1 + v_3^2). \quad (11)
\]
Hence \( \Gamma \) is causal and free; clearly, \( \Gamma \) is proper. For \( u = (u_0, u_1, u_2, u_3)^\top \),
\[
\ell(\gamma_n(v) - u, \gamma_n(v) - u) = a(v) + 2nb(v) - n^2(1 + u_3v_3), \quad (12)
\]
\[
b(v) = u_2v_3 - u_3v_2 + u_1 - v_1, \quad (13)
\]
\[
a(v) = \ell(v - u, v - u). \quad (14)
\]
Let $\kappa : V \to V/\Gamma$ be the quotient mapping and denote $M = V/\Gamma$, $p = \kappa(u)$. If $u_3 = 0$ then the coefficient at $n^2$ is constant and negative. Hence

$$\ell(\gamma_n(v) - u, \gamma_n(v) - u) < 0$$

for all sufficiently large $n$. Consequently, each orbit of $\Gamma$ has a finite number of common points with $C$ or $-C$. Moreover, for any $v \in V$ there exist a neighbourhood $U_v$ of $v$ and $n_v \in \mathbb{N}$ such that

$$\text{card}(\Gamma y \cap C) \leq n_v \quad \text{for all} \quad y \in U_v.$$ 

Therefore, $F_p$ and $P_p$ are closed in $M$ for all points $u$ in the hyperplane $u_3 = 0$.

Let $u_3 > 0$. Similar arguments show that the projection of the set

$$\left\{ v \in V : v_3 > -\frac{1}{u_3}, \ell(v,v) \geq 0 \right\}$$

to $M$ is locally closed. Hence $F_p$ is closed and $P_p \cap U_p$ is closed in $U_p$ for some neighbourhood $U_p$ of $p$ in $M$. According to (12), if

$$v_3 < -\frac{1}{u_3} \quad (14)$$

then $\ell(\gamma_n(v) - u, \gamma_n(v) - u) > 0$ for all sufficiently large $n$. Hence $\kappa(v) \in P_p$ for each $v$ that satisfies (14). Clearly, $\Gamma$ does not change the coordinate $v_3$; a computation with (13) shows what the same is true for $b(v)$ if $1 + v_3 u_3 = 0$.

Taken together with (12), this implies that the set

$$I_u = \left\{ v \in V : v_3 = -\frac{1}{u_3}, b(v) = 0, \ell(v,v) < 0 \right\}$$

is $\Gamma$-invariant. Hence $\kappa(I_u)$ has no common point with $P_p$. Clearly, $I_u \neq \emptyset$ and $\kappa(I_u) \subset \text{clos } P_p$. Thus $P_p$ is not closed. Moreover, $P_p$ contains lightlike lines parallel to $L = \mathbb{R}e_0$.

If $u_3 < 0$ then, repeating this with minor changes, we get that the past is closed but the future is not, and that the future contains lightlike lines; also, that $M$ is strictly causal.

Let $M^0, M^+, M^-$ be the three subsets of $M$ considered above, in the same order. Points in $M^0$ are distinguished by any of the following properties: 1) both the past and the future are closed; 2) neither the past nor the future contain lightlike lines. Thus $M$ is not homogeneous. The symmetry $(v_0, v_1, v_2, v_3) \to (-v_0, v_1, -v_2, -v_3)$ commutes with $\Gamma$. Hence it defines an involution $\iota$ in $M$; evidently, $\iota$ transposes $M^+$ and $M^-$. \hfill \Box

In dimension 4, which is of course of particular interest. Theorem 2 makes possible to find all the considered manifolds. We omit the elliptic case which is clear. We say that Lorentzian manifolds $(M_1, \ell_1)$ and $(M_2, \ell_2)$ are homothetic if there exists $t > 0$ such that $(M_1, t\ell_1)$ and $(M_2, \ell_2)$ are isometric.
Theorem 3. If a 4-dimensional strictly causal flat complete Lorentzian manifold is not elliptic then it is homothetic to the manifold of Example 1.

Theorems 1 and 2 contain the following assertion which we formulate separately as a proposition (in fact, it is the major point of the proof).

Proposition 1. The fundamental group $\pi_1(M)$ of a strictly causal flat complete Lorentzian manifold $M$ is virtually abelian.

We say that $M$ is elliptic if $\Gamma$ contains a finite index subgroup whose restriction to the affine space $U$ of Theorem 1 consists of translations; otherwise, $\Gamma$ and $M$ are said to be parabolic (the hyperbolic case is impossible, see Lemma 3). Thus elliptic manifolds correspond to (i) and parabolic to (ii) with $\alpha \neq 0$ (modulo Theorem 1). We formulate some properties which distinguish these classes. A noncompact manifold $M$ has two ends if for any compact set $K \subset M$ the number of unbounded components in $M \setminus K$ is not greater than 2 and is equal to 2 for some $K$. A submanifold of $M$ is called spacelike if the restriction of $-\ell$ to it is a Riemannian metric. A Cauchy hypersurface in a Lorentzian manifold is a smooth spacelike submanifold of codimension 1 which disconnects $M$ and such that each unextendible timelike (in particular, lightlike) curve intersects it by a single point. The existence of a Cauchy hypersurface is the main ingredient of the definition of a globally hyperbolic Lorentzian manifold (see [5], [3], [2]).

Proposition 2. Let $M$ be a strictly causal flat complete Lorentzian manifold. Each of following conditions implies that $M$ is elliptic:

1) $M$ is affinely equivalent to some Euclidean space form;
2) the past of any point contains no lightlike straight line;
3) the future of any point contains no lightlike straight line;
4) $M$ admits a Cauchy hypersurface;
5) $M$ has two ends.

Conversely, elliptic manifolds satisfy 1)–4).

Causal manifolds $M_4/\Gamma$ with two ends, where $\Gamma$ is a discrete subgroup of a simply transitive group of automorphisms of $M_4$, were described in [6]. Most of them are not strictly causal. The case 4) is a direct consequence of the paper [2]. We note that the past of some point in the Cauchy hypersurface contains lightlike lines if $M$ is not elliptic (this is a contradiction).

Let $\gamma \in \Gamma$, $g = \lambda(\gamma)$. We denote by $F_\gamma$ or $F_g$ the set of all fixed points of $g$,

$$V_\gamma = \{\lambda(\gamma)v - v : v \in V\},$$

Int ($X$) is the interior of a set $X$, span $X$ is its linear span, $L(V)$ denotes the space of all linear mappings $V \to V$. The projection of $C$ to $\mathbb{RP}^{n-1}$ may be identified with the closed unit ball $B_{n-1}$ in $\mathbb{R}^{n-1}$; the group $O(\ell)$ acts in it by
Moebius transformations. If $g \in O(\ell)$ has a fixed point in the open unit ball then $g$ is called elliptic; this is equivalent to

$$F_g \cap \text{Int}(C) \neq \emptyset.$$  \hspace{1cm} (15)

If $g$ is not elliptic then it has one or two fixed points in the unit sphere and is called parabolic or hyperbolic, respectively. If $g \in O(\ell)$ is hyperbolic and keeps $C$ then its eigenvectors corresponding to fixed points in the sphere belong to the boundary of $C$ and the eigenvalues are positive. If $g$ is not hyperbolic then it has no real eigenvalues different from $\pm 1$. Let

$$F = \bigcap_{g \in G} F_g$$

be the set of $G$-fixed points in $V$. If

$$F \cap \text{Int}(C) \neq \emptyset$$  \hspace{1cm} (16)

then we say that $G, \Gamma, M$ are elliptic (for $M$, this definition is equivalent to that was given above).

Vector spaces are always finite dimensional and real. We refer to [10] for the exposition of space forms of symmetric spaces.

3 Proof of results

In following lemmas, $M = V/\Gamma$ is assumed to be causal; the assumption that $M$ is strictly causal is always stated explicitly. Some of these lemmas are known and are proved for the sake of completeness. The condition (1) is evidently equivalent to

$$(V_\gamma + \tau(\gamma)) \cap C = \emptyset \text{ for all } \gamma \in \Gamma \setminus \{e\}. \hspace{1cm} (17)$$

This implies a necessary condition for (1):

$$V_\gamma \cap \text{Int}(C) = \emptyset, \quad \gamma \in \Gamma.$$  \hspace{1cm} (18)

To prove it, note that $v \in \text{Int}(C)$ implies $tv + \tau(\gamma) \in \text{Int}(C)$ for sufficiently large $t > 0$. The lemma below contains one more reformulation of (1).

**Lemma 1.** $M$ is causal if and only if

$$\ell(\gamma(v) - v, \gamma(v) - v) < 0 \text{ for all } v \in V, \gamma \in \Gamma \setminus \{e\}. \hspace{1cm} (19)$$

**Proof.** The inequality in (19) holds if and only if $\gamma(v) - v \notin C \cup (-C)$. If $\gamma(v) - v \in -C$ then $\gamma^{-1}(\gamma(v)) - \gamma(v) \in C$. Hence (17) is equivalent to (19). \qed

In the following lemma we collect some elementary facts concerning groups of linear transformations of $C$ (they hold for any pointed generating convex cone).
Lemma 2. The following assertions are equivalent:

1) \( G \) is elliptic;

2) \( G \) is bounded in \( \text{GL}(V) \) (i.e. its closure is compact);

3) all \( G \)-orbits in \( \text{Int}(C) \) are bounded;

4) there exists a bounded \( G \)-orbit in \( \text{Int}(C) \).

Furthermore, if \( G \) is abelian then it is elliptic if and only if each its element is elliptic.

Proof. If \( v \in \text{Int}(C) \) is \( G \)-fixed then the set \((C - v) \cap (v - C)\) is a bounded \( G \)-invariant neighbourhood of zero. Hence \( G \) is bounded and 2) follows from 1).

Implications 2) \( \Rightarrow \) 3) \( \Rightarrow \) 4) are obvious. Suppose that \( Gv \) is bounded for some \( v \in \text{Int}(C) \). Then each \( G \)-orbit in the linear span \( W \) of \( Gv \) is bounded. Hence there exists a bounded \( G \)-invariant neighborhood of zero in \( W \). Therefore, the restriction of \( G \) to \( W \) is bounded in \( \text{GL}(W) \). Clearly, the cone \( C \cap W \) is \( G \)-invariant. Using the averaging over the closure of \( G \) in \( \text{GL}(W) \), we get a \( G \)-fixed point in \( \text{Int}(C) \). Thus 4) implies 1).

If \( G \) is abelian and each \( g \in G \) is elliptic then the complexification of \( V \) decomposes into the direct sum of \( G \)-eigenspaces (note that each \( g \)-eigenspace is \( G \)-invariant and each \( g \in G \) is semisimple). Since \( g \)-eigenvalues are bounded, this implies that \( G \) is bounded. The converse is clear. \( \square \)

Lemma 3. \( G \) contains no hyperbolic element.

Proof. Let \( \gamma \in \Gamma \) be hyperbolic, \( g = \lambda(\gamma) \). Then there exist \( v_1, v_2 \) such that

\[ gv_k = \mu_k v_k, \quad \mu_k \neq 1, \quad v_k \in \partial C, \quad k = 1, 2, \quad v_1 \neq v_2. \]

Their two-dimensional linear span \( W \) intersects \( \text{Int}(C) \). Since \( W \) is \( g \)-invariant and 1 is not the eigenvalue of \( g \) in \( W \) we have \( W \subseteq V_\gamma \), contradictory to (18). \( \square \)

Lemma 4. Let \( V \) be a real vector space and \( K \) be a subgroup of \( \text{GL}(V) \) such that the function \( \text{Tr} \) is bounded on \( K \). Then \( K \) is either bounded or reducible.

Proof. Set

\[ A = \text{span} K, \]
\[ B = \{ x \in L(V) : \sup_{h \in K} |\text{Tr} x h| < \infty \}, \]
\[ N = A^\perp = \{ x \in L(V) : \text{Tr} x h = 0 \quad \text{for all} \quad h \in K \}. \]

Due to the assumption of the lemma, \( K \subseteq B \). Hence \( A \subseteq B \). Clearly, \( N \subseteq B \) and \( A \) is an algebra. Set \( I = A \cap N \); standard arguments shows that \( I \) is a two-side ideal in \( A \).

If \( I = 0 \) then the bilinear form \( \text{Tr} xy \) is nondegenerate in \( A \). Therefore, any linear functional on \( A \) is bounded on \( K \). Thus \( K \) is bounded.

For any \( x \in I \) and positive integer \( n \), \( \text{Tr} x^n = 0 \); hence \( x \) is nilpotent, and Engel's theorem implies that the space \( Z = \{ v \in V : Iv = 0 \} \) is nontrivial. If \( I \neq 0 \) then \( Z \neq V \). Since \( Z \) is evidently \( A \)-invariant, \( K \) is reducible. \( \square \)
Lemma 5. Let $K$ be a subgroup of $O(\ell)$ that keeps $C$ and contains no hyperbolic element. Then $K$ has a fixed point in $C \setminus \{0\}$.

Proof. The assumption implies that eigenvalues of any $h \in K$ are contained in the unit circle. Hence $|\text{Tr} h| \leq \dim V$. If $K$ is irreducible then it is bounded by Lemma 4. Then its closure is compact and we get $K$-fixed points in $\text{Int}(C)$ by averaging.

Let $K$ be reducible. Then there exists a proper $K$-invariant space $W$. The invariant space $L = W \cap W^\perp$ is at most one dimensional since the Lorentzian form $\ell$ vanishes on it. If $\dim L = 1$ then all points of the line $L$ are $K$-fixed: otherwise, $K$ either contains an element that has positive eigenvalue $\mu \neq 1$ which is hyperbolic or does not keep $C$ (note that one of the two halflines in $L$ lie in $\partial C$). Thus the assertion is true if $L \neq 0$. If $L = 0$ then $V = W \oplus W^\perp$. Since $\ell$ is nondegenerate in each of these spaces, either one of them is one dimensional and intersects $\text{Int}(C)$ or the restriction of $\ell$ to one of them is Lorentzian. The first case is clear. Thus we may use the induction on $\dim V$ starting with the obvious case $\dim V = 1$. □

Lemma 6. If $G$ is elliptic then $M$ admits a flat Riemannian metric making it an Euclidean space form (i.e. a flat geodesically complete Riemannian manifold) such that the identical mapping is affine.

Proof. By definition, there exists $v_0 \in F \cap \text{Int}(C)$. The form

$$t\ell(v_0, v)^2 - \ell(v, v),$$

is $G$-invariant. Moreover, for sufficiently large $t > 0$ it is positive definite. Hence it induces the desired metric on $M = V/\Gamma$. □

In that follows, we assume that $G$ is not elliptic if the contrary is not stated explicitly. By Lemma 5, there exists

$$v_0 \in \partial C \cap F.$$  \hspace{1cm} (21)

For this vector $v_0$, we define $L, W, l_0$ as in (ii). By (15), $F \cap \text{Int}(C) = \emptyset$. Hence

$$L \subseteq F \subseteq W.$$  \hspace{1cm} (22)

Since $l_0$ is $G$-invariant,

$$l_0(\gamma(v) - v) = l_0(\tau(\gamma)),$$

in particular, the left part does not depend on $v$. The following lemma is obvious.

Lemma 7. The group $\Gamma$ keeps the family of hyperplanes in $V$ that are parallel to $W$; moreover, $G$ and the commutator group of $\Gamma$ keep each of them. The mapping

$$\alpha : \gamma \mapsto l_0(\tau(\gamma)).$$  \hspace{1cm} (23)

is a homomorphism $\Gamma \to \mathbb{R}$. □
The aim of subsequent lemmas is to prove that 
\[ \tau(\Gamma) \subset W. \] (24)

**Lemma 8.** If \( V_{\gamma} \supseteq L \) then \( \tau(\gamma) \in W. \)

**Proof.** Let \( v \in V \) be such that \( \lambda(\gamma)v - v = v_0. \) Set \( b = \tau(\gamma). \) For all \( t \in \mathbb{R}, \)
\[
\ell(\gamma(tv) - tv, \gamma(tv) - tv) = \ell(tv_0 + b, tv_0 + b) = 2t\ell_0(b) + \ell(b, b). \tag{25}
\]
If \( b \notin W \) then \( \ell_0(b) \neq 0. \) Hence \( \ell(\gamma(tv) - tv) > 0 \) for some \( t \in \mathbb{R} \) and \( M \) cannot be causal by Lemma 1. Thus \( b \in W. \)

Note that \( V_{\gamma} \perp v_0 \) since \( v_0 \) is \( G \)-fixed. Hence \( V_{\gamma} \subseteq W \) and \( V_{\gamma} \cap L = \{0\} \implies V_{\gamma} \cap L = \{0\}. \) (26)

**Lemma 9.** \( V_{\gamma} \cap L = \{0\} \) if and only if \( \gamma \) is elliptic.

**Proof.** Since \( V_{\gamma} \subseteq W \) and \( W^\perp = L, \) the assumption \( V_{\gamma} \cap L = \{0\} \) implies that \( \ell \) is negative definite on \( V_{\gamma}. \) Therefore, \( V = V_{\gamma} \oplus V_{\gamma}^\perp \)
and \( \lambda(\gamma) \) generates a bounded subgroup of \( \text{GL}(V_{\gamma}). \) Clearly, \( \lambda(\gamma) \) is identical in \( V_{\gamma}^\perp. \) Thus \( \lambda(\gamma) \) generates a bounded subgroup of \( \text{GL}(V) \) and is elliptic by Lemma 2. Conversely, if \( \gamma \) is elliptic then \( \lambda(\gamma) \) is semisimple according to the same Lemma. Since \( L \subseteq F_{\gamma}, \) this implies \( V_{\gamma} \cap L = \{0\}. \)

**Corollary 1.** \( V_{\gamma} \supseteq L \) if and only if \( \gamma \) is parabolic.

**Proof.** Since \( \dim L = 1, \) \( V_{\gamma} \supseteq L \) is equivalent to \( V_{\gamma} \cap L = \{0\}. \)

**Corollary 2.** If \( \tau(\gamma) \notin W \) then \( \gamma \) is elliptic.

**Proof.** Combine Lemma 8 and Corollary 1.

Set
\[
\Gamma' = \{ \gamma \in \Gamma : \tau(\gamma) \in W \}, \quad G' = \lambda(\Gamma').
\] (27)
By Lemma 7, \( \Gamma' \) and \( G' \) are normal in \( \Gamma, G, \) respectively.

**Lemma 10.** If \( G' \) is elliptic then \( G \) is elliptic.

**Proof.** Let \( F' \) be the set of all \( G' \)-fixed points in \( V. \) If \( G' \) is elliptic then
\[
F' \cap \text{Int}(C) \neq \emptyset. \tag{28}
\]
Since \( G' \) is normal, \( F' \) is \( G \)-invariant. The action of \( G \) in \( F' \) can be considered as the action of \( G/G', \) by Lemma 7, \( G/G' \) is abelian. Due to Corollary 2, each \( g \notin G' \) is elliptic. Taken together with (28), this satisfies the assumption of Lemma 2 for the group \( G/G', \) the space \( F', \) and the cone \( F' \cap C. \) Therefore, \( G \) has a fixed point in \( \text{Int}(C) \cap F' \subseteq \text{Int}(C). \) Hence \( G \) is elliptic.
According to Lemma 6 and Lemma 10, there are two levels of the problem: first, the case of non-elliptic $G$ under the additional assumption $\Gamma = \Gamma'$, and second, the covering $\mu : M' \to M$, where $M' = V/\Gamma'$. We show that the second is trivial (i.e. $\Gamma = \Gamma'$).

Condition (24) and $G$-invariance of $L$ define the affine action of $\Gamma$ in the quotient space $W/L$ which may be identified with $N$. Since $W = L^\perp$, the form $\ell$ induces a negative definite $G$-invariant form on $W/L$ which coincides with its restriction to $N$. Let $\phi : V \to V/L$ and $\kappa : V \to M = V/\Gamma$ be the quotient mappings. Then $\phi(W) = N$.

**Lemma 11.** Let (24) be true. If $M$ is causal then the action of $\Gamma$ in $N = W/L$ is free; if $M$ is strictly causal then it is discontinuous (i.e. each orbit is discrete).

**Proof.** It follows from (1) that the projection $\phi : W \to N$ is one-to-one on each $\Gamma$-orbit. This proves the first assertion.

Let $u, w \in W$ and $\{\gamma_n\}$ be a sequence in $\Gamma$ such that

$$\lim_{n \to \infty} \phi(\gamma_n(w)) = \phi(u).$$

Then $\gamma_n(w) = u + w_n + t_nv_0$, where $w_n \to 0$ in $W$ as $n \to \infty$ and $t_n \in \mathbb{R}$. Since $\gamma_n(w + tv_0) = \gamma_n(w) + tv_0$ for all $t \in \mathbb{R}$, this implies that $\kappa(w + tv_0)$ lies in the closure of $\kappa(u + L)$ in $M$ for each $t \in \mathbb{R}$. Also, it follows that for any neighbourhood $U$ of $p = \kappa(u)$ in $M$ and some $s \in \mathbb{R}$ the inclusion $\kappa(w + sv_0) \in U$ holds. Let $U$ be such that $U \cap F_p$ and $U \cap P_p$ are closed in $U$. Set $L^+ = \{tv_0 : t \geq 0\}$; clearly, $\kappa(w + sv_0)$ belongs to the closure of at least one of the sets $\kappa(u + L^+)$ and $\kappa(u - L^+)$. On the other hand, $\kappa(u - L^+) \cap U$ and $\kappa(u + L^+) \cap U$ are closed in $U$. Indeed, due to (24), $\kappa(W)$ is closed in $M$ and

$$\kappa(u + L^+) = F_p \cap \kappa(W), \quad \kappa(u - L^+) = P_p \cap \kappa(W)$$

(recall that $F_p = \kappa(u + C)$, $P_p = \kappa(u - C)$). Therefore, $\kappa(w + sv_0) \in \kappa(u + L)$; this means that $\phi(\gamma(w)) = \phi(u)$ for some $\gamma \in \Gamma$. Hence each orbit of $\Gamma$ in $W/L$ is closed. Consequently, all orbits are discrete. \qed

**Corollary 3.** If (24) is true and $M$ is strictly causal then $N/\Gamma$ is an Euclidean space form; in particular, $\Gamma$ is virtually abelian. \qed

**Lemma 12.** If $\Gamma$ is not elliptic then any abelian subgroup $\widetilde{\Gamma} \subseteq \Gamma$ of finite index contains a parabolic element.

**Proof.** If $\widetilde{\Gamma}$ consists of elliptic elements then $\widetilde{G} = \lambda(\widetilde{\Gamma})$ is elliptic by Lemma 2. Then $\widetilde{G}$ has a fixed point $w_0 \in \text{Int}(C)$. The orbit $Gw_0$ is finite; the sum of vectors in it belongs to $\text{Int}(C)$ and is $G$-fixed. \qed

Any affine action in $\mathbb{R}^m$ has the natural extension to linear one in $\mathbb{R}^{m+1}$. Precisely, the mapping $x \to ax + b$ can be realized as the restriction to the hyperplane $x_{m+1} = 1$ of the linear mapping $\tilde{a}$ that is equal to $a$ on $\mathbb{R}^m$ (embedded to $\mathbb{R}^{m+1}$ as the hyperplane $x_{m+1} = 0$) and satisfies

$$\tilde{a}e_{m+1} = e_{m+1} + b, \quad e_{m+1} = (0, \ldots, 0, 1).$$

(29)
Thus we may consider $\Gamma$ as a linear group in $\tilde{V} = V \oplus \mathbb{R}$.

**Lemma 13.** Let $\Gamma$ be an abelian group of affine transformations in $V$. Then there exist $\tilde{o} \in V$ and linear subspaces $V_0, V_1 \subseteq V$ such that

$$V = V_0 \oplus V_1,$$

the affine subspace $U = \tilde{o} + V_0$ is $\Gamma$-invariant, $\Gamma$ is unipotent in $U$, and $\Gamma$ is linear in $V_1$.

**Proof.** Let $\tilde{V} = V \oplus \mathbb{R}$ be as above and $\pi$ be the projection to the first component along the second one. Since $\Gamma$ is abelian, it admits the triangular realization in some linear base of $\tilde{V}$. Diagonal elements are characters of $\Gamma$ and $\tilde{V}$ decomposes over them. Let $\tilde{V}_0$ be the component of the trivial character. Then $\tilde{V} = \tilde{V}_0 \oplus V_1$, (31)

where $V_1$ is the sum of all other components. Clearly, $V$ is $\Gamma$-invariant (we identify $V$ with $V \oplus 0$). According to (29), the representation of $\Gamma$ in $\tilde{V}/V$ is trivial. Hence

$$V_1 \subseteq V$$

and $(V \oplus 1) \cap \tilde{V}_0 \neq \emptyset$. The set $U = \pi((V \oplus 1) \cap \tilde{V}_0)$ is an affine subspace of $V$. Since $\tilde{V}_0$ and $V \oplus 1$ are $\Gamma$-invariant, $U$ is invariant with respect to the affine action of $\Gamma$ in $V$. Pick any $\tilde{o} \in U$ and set $V_0 = \tilde{V}_0 \cap V$. Then $\tilde{o} \oplus 1 \in \tilde{V}$, $\tilde{V}_0 = \tilde{o} \oplus 1 + V_0$, $U = \tilde{o} + V_0$ and (30) follows from (31) and (32). \qed

We need a lemma that reduces the virtually abelian case to the abelian one.

**Lemma 14.** Let a group $\Gamma \subset \text{Aff}(V)$ be virtually abelian. Then there exists a finite index abelian subgroup $\tilde{\Gamma} \subseteq \Gamma$ such that $\alpha \in \text{Aff}(V)$, $\alpha^{-1}\Gamma\alpha = \Gamma \implies \alpha^{-1}\tilde{\Gamma}\alpha = \tilde{\Gamma}$.

**Proof.** Let $\Gamma'$ be an abelian subgroup of finite index in $\Gamma$, and let $\tilde{\Gamma}'$ be its algebraic closure. Then $\tilde{\Gamma} = \Gamma\tilde{\Gamma}'$ is closed being a finite union of closed sets; hence $\tilde{\Gamma}$ is the algebraic closure of $\Gamma$ (in particular, $\tilde{\Gamma}$ is a group). Let $\tilde{\Gamma}_0$ be the identity component of $\tilde{\Gamma}$. The group $\tilde{\Gamma} = \Gamma \cap \tilde{\Gamma}_0$ evidently satisfies the lemma. \qed

Any affine transformation $\gamma$ is the composition of its linear part $\lambda(\gamma)$ and the translation part $v \rightarrow v + \tau(\gamma)$. They commute if and only if $\tau(\gamma)$ is $\lambda(\gamma)$-fixed:

$$\lambda(\gamma)\tau(\gamma) = \tau(\gamma).$$

(33)

Of course, the decomposition and (33) depend on the choice of origin: if it is removed to $v \in V$ then $\tau(\gamma)$ is replaced by $\gamma(v) - v = \lambda(\gamma)v - v + \tau(\gamma)$. For the linearization of $\gamma$ in $V \oplus \mathbb{R}$, the existence of the origin satisfying (33) is equivalent to $\ker(1 - \gamma) \not\subseteq V \oplus 0$. Groups $\Gamma$ of (ii) do not have this property.
Lemma 15. Suppose that (24) is not true, $\Gamma'$ is an abelian subgroup of $\Gamma$, let $V_0, V_1, \bar{o}$ be as in Lemma 13 for $\Gamma'$ and remove origin to $\bar{o}$. Then (33) holds for any parabolic $\gamma \in \Gamma'$.

Proof. Let $\gamma \in \Gamma'$ be parabolic and set $W_0 = W \cap V_0$, $g = \lambda(\gamma)$, $b = \tau(\gamma)$. Note that $b \in W_0$

by Corollary 2 and Lemma 13. Since $g$ is orthogonal with respect to the form $\ell$ in $N = W/L$, it is semisimple in $N$. Hence $g$ is identical in $W_0/L$. Thus there exists a linear functional $l$ on $W_0$ such that

$$ gu = u + l(u)v_0 $$

for all $u \in W_0$. Clearly, $l(v_0) = 0$. If $l(b) = 0$ then $gb = b$ and (33) is true. Therefore, it is sufficient to prove that

$$ \ell(\gamma^n \circ \nu(o), \gamma^n \circ \nu(o)) > 0 $$

for some $\nu \in \Gamma$ and $n \in \mathbb{Z}$ assuming that (24) is false and

$$ l(b) \neq 0. $$

Also, we may assume that $\Gamma'$ is unipotent since $V_0$ is $\Gamma'$-invariant. Let $w \in W_0$. A calculation shows that

$$ \gamma^n(w) = w + nb + \left( nl(w) + \frac{n(n-1)}{2} l(b) \right) v_0. $$

If $v_1 \in V_0 \setminus W_0$ then $\ell(v_0, v_1) \neq 0$ since $\ell$ is nondegenerate in $V_0$. Replacing $v_1$ by $t_1v_1 + t_0v_0$ for suitable $t_1, t_0 \in \mathbb{R}$, we may assume

$$ \ell(v_1, v_1) = 0, \quad \ell(v_1, v_0) = 1. $$

If $gv_1 = v_1$ then $v_0 + v_1$ is a $g$-fixed point in Int($C$) but $g$ is supposed to be parabolic. Hence

$$ u_0 = gv_1 - v_1 \neq 0. $$

Clearly, $u_0 \in W$; we claim that

$$ u_0 \in W \setminus L. $$

If $u_0 \in L$ then $u_0 = gu_0 = g^{-1}u_0$. Since $v_1$ is lightlike,

$$ \ell(gv_1, v_1) = \ell(u_0, v_1) = \ell(g^{-1}u_0, v_1) = -\ell(g^{-1}v_1, v_1) = -\ell(v_1, gv_1). $$

Consequently, $\ell(u_0, v_1) = 0$ if $u_0 \in L$; by (38), $u_0 = 0$ but this contradicts to (39). This proves (40). Since $u_0 \in W$, $gu_0 - u_0 \in L$. Hence $gu_0 - u_0 = u_0 - g^{-1}u_0$; using (38) we get

$$ \ell(gu_0 - u_0, v_1) = \ell(g^2v_1 - 2gv_1 + v_1, v_1) = \ell(gv_1 - 2v_1 + g^{-1}v_1, v_1) = \ell(gv_1, v_1) + \ell(g^{-1}v_1, v_1) = 2\ell(gv_1, v_1) = -\ell(gv_1 - v_1, gv_1 - v_1) = -\ell(u_0, u_0). $$

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Due to (40), this implies \( \ell(gu_0 - u_0, v_1) \neq 0 \). Therefore, \( gu_0 - u_0 \neq 0 \) and

\[
g^n v_1 = v_1 + nu_0 + \frac{n(n-1)}{2} pv_0, \quad pv_0 = gu_0 - u_0. \tag{41}
\]

According to (38), (40) and the calculation above,

\[
p = \ell(gu_0 - u_0, v_1) = -\ell(u_0, u_0) \neq 0. \tag{42}
\]

Let \( v \in V_0, v = tv_1 + w \), where

\[
t = \ell(v, v_0)
\]

and \( w \in W_0 \). By (37) and (41),

\[
\gamma^n(v) = \gamma^n(tv_1 + w) = tg^n v_1 + \gamma^n(w) = tv_1 + w + n(tu_0 + b + l(w)v_0) + \frac{n(n-1)}{2} (tp + l(b))v_0.
\]

Relations

\[
\ell(u_0, v_0) = \ell(b, v_0) = \ell(v_1, v_1) = \ell(v_0, v_0) = \ell(w, v_0) = 0, \quad \ell(v_1, v_0) = 1,
\]

imply

\[
\ell(\gamma^n(v), \gamma^n(v)) = n^2(t(tp + l(b)) + \ell(tu_0 + b, tu_0 + b)) + O(n) = n^2(l(b) + \ell(b, b) + 2t\ell(u_0, b)) + O(n), \tag{43}
\]

where the latter equality holds due to (42). By (34) and (38), for all \( u \in W_0 \)

\[
l(u) = \ell(l(u)v_0, v_1) = \ell(gu - u, v_1) = \ell(u, g^{-1}v_1 - v_1) = -\ell(u, g^{-1}u_0).
\]

Since \( gu_0 - u_0 = sv_0 \) for some \( s \in \mathbb{R} \) and \( v_0 \) is \( g \)-fixed, we have \( g^{-1}u_0 = u_0 - sv_0 \). Therefore,

\[
l(u) = -\ell(u, u_0). \tag{44}
\]

It follows from (36) that the coefficient at \( n^2 \) is positive for sufficiently large \( t \) of the same sign as \( -l(b) \). Lemma 7 and the assumption \( \tau(\Gamma) \not\subset W \) imply the existence of \( \nu \in \Gamma \) such that this property is true for \( t = \ell(\tau(\nu), v_0) \). Then (35) holds for all sufficiently large \( n \in \mathbb{Z} \).

The linear part \( g \) of \( \gamma \) keeps paraboloids \( \partial C \cap (W + x) \) whose axes are collinear to \( v_0 \). The calculation above shows that the asymptotic behavior of powers \( \gamma^n \) is determined by \( \lambda(\gamma) \).

**Lemma 16.** Let \( \Gamma \) be abelian and unipotent. Suppose that (33) holds for each \( \gamma \in \Gamma \). Then

\[
\lambda(\gamma)\tau(\nu) = \tau(\nu) \tag{45}
\]

for all \( \gamma, \nu \in \Gamma \). Moreover, the mapping \( \gamma \to \tau(\gamma) \) is an isomorphism of \( \Gamma \) onto a uniform lattice in the linear span \( T \) of \( \tau(\Gamma) \).
Proof. Applying (33) to the composition \( \gamma \circ \nu = \nu \circ \gamma \), we get
\[
\lambda(\gamma)\lambda(\nu)(\lambda(\gamma)\tau(\nu) + \tau(\gamma)) = \lambda(\nu)\tau(\gamma) + \tau(\nu).
\]
Since \( \lambda(\gamma) \) and \( \lambda(\nu) \) commute and (33) holds for \( \gamma \) and \( \nu \), the left side is equal to \( \lambda(\gamma)^2\tau(\nu) + \lambda(\nu)\tau(\gamma) \). Therefore, \( \lambda(\gamma)^2\tau(\nu) = \tau(\nu) \). Since \( \lambda(\gamma) \) is unipotent, \( \lambda(\gamma) \) and \( \lambda(\gamma)^2 \) have identical sets of fixed points. This proves (45). The composition law for affine mappings and (45) imply
\[
\tau(\gamma \circ \nu) = \tau(\gamma) + \tau(\nu).
\]
Since \( \lambda(\gamma) \) is unipotent, \( \lambda(\gamma) \) and \( \lambda(\gamma)^2 \) have identical sets of fixed points. This proves (45). The composition law for affine mappings and (45) imply
\[
\tau(\gamma \circ \nu) = \tau(\gamma) + \tau(\nu).
\]
Since \( \lambda(\gamma) \) and \( \lambda(\nu) \) commute and (33) holds for \( \gamma \) and \( \nu \), the left side is equal to \( \lambda(\gamma)^2\tau(\nu) + \lambda(\nu)\tau(\gamma) \). Therefore, \( \lambda(\gamma)^2\tau(\nu) = \tau(\nu) \). Since \( \lambda(\gamma) \) is unipotent, \( \lambda(\gamma) \) and \( \lambda(\gamma)^2 \) have identical sets of fixed points. This proves (45). The composition law for affine mappings and (45) imply
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\[
\tau(\gamma \circ \nu) = \tau(\gamma) + \tau(\nu).
\]
Since \( \lambda(\gamma) \) is unipotent, \( \lambda(\gamma) \) and \( \lambda(\gamma)^2 \) have identical sets of fixed points. This proves (45). The composition law for affine mappings and (45) imply
\[
\tau(\gamma \circ \nu) = \tau(\gamma) + \tau(\nu).
\]
then \( \lambda(\gamma) = 1 \). Hence (33) is true for all \( \gamma \in \Gamma'' \). Applying consequently Lemma 16 and Lemma 17, we get

\[
\lambda(\gamma)b = b
\]

for all \( \gamma \in \Gamma'' \). Since \( b \notin W \), the two dimensional plane passing through \( b \) and \( v_0 \) intersects \( \text{Int}(C) \). Thus \( \gamma \) is elliptic. Hence \( \Gamma'' \) consists of elliptic elements; by Lemma 2, \( \Gamma'' \) is elliptic. Then \( \Gamma' \) and \( \Gamma \) are elliptic by Lemma 12 and Lemma 10, respectively.

\[\Box\]

**Proof of Proposition 1.** If \( \Gamma \) is elliptic then the assertion holds due to Lemma 6. If \( \Gamma \) is not elliptic then (24) is true for \( \Gamma \) by Proposition 3. Hence \( \Gamma \) is virtually abelian by Corollary 3.

**Proof of Theorem 1.** Due to Proposition 1, we may assume that \( \Gamma \) is abelian. Then Lemma 13 implies the existence of the decomposition; moreover, 2) is an easy consequence of the construction (recall that \( V_1 \) is the sum of all nontrivial components and \( V_0 \) is the unipotent component of \( G \) in \( V \)). Further, 3) holds since the action of \( \Gamma \) is free. The decomposition is evidently \( \ell \)-orthogonal and \( G \)-invariant. By Lemma 5, \( C \) contains a nontrivial fixed point \( v_0 \). Since \( \dim v_0^\perp \cap C \leq 1 \), \( V_1 \) is spacelike. Hence \( \ell \) is negative definite in it. This implies that \( \ell \) is Lorentzian in \( V_0 = V_1^\perp \). Remaining assertions are clear.

\[\Box\]

**Lemma 18.** Let \( \Gamma \) be unipotent. Suppose that isomorphic embeds \( \Gamma \) to the vector group \( W/L \) as a uniform lattice in some its linear subspace. Then there exists \( v_1 \in \partial C \) and subspaces \( T \subseteq N \subseteq W \) such that the action of \( \Gamma \) in \( V \) is subject to formulas of (ii).

**Proof.** Note that any choice of \( v_1 \in \partial C \) (we shall refine it later) such that \( \ell(v_0, v_1) = 1 \) identifies \( W/L \) with the space \( N = W \cap v_1^\perp \), the restriction of \( \phi \) to \( W \) with the orthogonal projection \( \pi \) onto \( N \), and \( T \) with the linear span of \( \phi(\tau(\Gamma)) \). We may assume that \( \Gamma \) is embedded to \( T \) by \( \phi \circ \tau \); then \( \phi \circ \tau(x) = x \) for all \( x \in \Gamma \). Since \( \lambda(x) \) is \( \ell \)-orthogonal and unipotent in \( W/L \) for any \( x \in \Gamma \), \( \lambda(x) = 1 \) in \( W/L \). Thus

\[
\begin{align*}
\tau(x) &= x + \xi(x)v_0, \\
\lambda(x)w &= w + \eta(x, w)v_0,
\end{align*}
\]

where \( \xi \) and \( \eta \) are real valued functions, \( x \in \Gamma \) and \( w \in W \). Clearly, \( \eta(x, w) \) is linear on \( w \in W \). Since \( v_0 \) is \( G \)-fixed and

\[
\tau(x + y) = \lambda(x)\tau(y) + \tau(x) = \lambda(y)\tau(x) + \tau(y),
\]

we get

\[
\begin{align*}
\lambda(x)\tau(y) &= \lambda(x)(y + \xi(y)v_0) = y + (\eta(x, y) + \xi(y))v_0, \\
\tau(x + y) - x - y &= (\eta(x, y) + \xi(x) + \xi(y))v_0 = \xi(x + y)v_0.
\end{align*}
\]
Combining equalities above, we get formulas of (ii) for $\lambda$ true for $w$ for all $x \in T$. Let $z \in T$ be such that $\zeta(x) = \ell(z, x)$ for all $x \in T$. Replacing $T$ by the space $\{x + \zeta(x)v_0 : x \in T\}$, $v_1$ by a suitable combination of $v_1 - z$ and $v_0$, and redefining $N$, we may assume that $\zeta = 0$. For $x \in \Gamma$, set

$$ax = \pi(\lambda(x)v_1),$$

where $\pi$ is the orthogonal projection onto $N$. Then

$$\lambda(x)v_1 = \sigma(x)v_1 + ax + \omega(x)v_0,$$

where $\sigma, \omega$ are functions on $\Gamma$. Since $ax \in N$, $v_1, v_0 \in \partial C$, $\ell(v_1, v_0) = 1$, $\lambda(x) \in \text{SO}(\ell)$, and $v_0$ is $G$-fixed,

$$\sigma(x) = \ell(\lambda(x)v_1, v_0) = \ell(v_1, \lambda(x)^{-1}v_0) = 1; \quad 0 = \ell(\lambda(x)v_1, \lambda(x)v_1) = 2\omega(x) + \ell(ax, ax).$$

If $w \in W$ then $w - \lambda(x)w \in L$. Hence $\pi(\lambda(x)w) = \pi(w)$; in particular, this is true for $w = v_1 - \lambda(y)v_1$ and we get

$$a(x + y) = \pi(\lambda(x + y)v_1) = \pi(\lambda(x)v_1) + \pi(\lambda(x)(\lambda(y)v_1 - v_1)) = ax + ay,$$

$$ao = \pi(v_1) = o$$

for all $x, y \in \Gamma$. Therefore, $a$ extends to a linear operator $T \to N$. Clearly, $\lambda(x)^{-1} = \lambda(-x)$. Due to (38) and (47), if $x \in \Gamma$ and $w \in N$ then

$$\eta(x, w) = \ell(v_1, \lambda(x)w) = \ell(\lambda(-x)v_1, w) = -\ell(ax, w).$$

Since $\lambda(x)v_0 = v_0$, (47) extends the derived formula to the case $w \in W$ by setting $\eta(x, v_0) = 0$. For any $v \in V$ we have $v = l_0(v)v_1 + w$, where $w \in W$. Combining equalities above, we get formulas of (ii) for $\lambda$ and $\tau$:

$$\lambda(x)v = v + l_0(v)ax - (\ell(ax, v) + \frac{1}{2} l_0(v)\ell(ax, ax))v_0, \quad (48)$$

$$\tau(x) = x - \frac{1}{4} \ell(ax, x)v_0, \quad (49)$$

where $v \in V$, $x \in \Gamma$. Since $\eta$ is symmetric, $a$ is $\ell$-symmetric in $T$. \hfill $\square$

**Lemma 19.** Each of following conditions is equivalent to (7):

1) the action (ii) of $T$ in $V$ is free;

2) the action (ii) of $\Gamma$ in $V$ is free and proper.

**Proof.** The action of $T$ in $V$ is not free if and only if

$$\gamma_x(v) - v = (x + l_0(v)ax) - \left( \ell(v, ax) + \frac{1}{2} l_0(v)\ell(ax, ax) + \frac{1}{2} \ell(ax, x) \right) v_0 = 0$$

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for some $x \in T \setminus \{0\}, v \in V$. This equality is equivalent to the system
\[
\begin{aligned}
x + l_0(v)ax &= 0; \\
\ell(v, ax) &= 0.
\end{aligned}
\]
The first equation has a solutions $x, v$ such that $x \neq 0$ if and only if (7) is not true. If $\dim V > 1$ then vector $v$ can be removed without changing $l_0(v)$ to satisfy the second equation (note that $ax \not\in L \setminus \{0\}$). Since the assertion is clear for $\dim V = 1$, (7) is the same as 1.

Hyperplanes $X_s = \{v \in V : l_0(v) = s\}$ are $T$-invariant; the space $L$ is parallel to each of them. Hence $T$ acts in hyperplanes $\phi(X_s)$ in $V/L$ by translations $\nu \to \nu + x + sax$. Therefore, the action of the uniform lattice $\Gamma \subseteq T$ in all $\phi(X_s)$ is free and proper if and only if $T$ acts freely. If $\Gamma$ is free and proper in all these hyperplanes then it is free and proper in $V/L$ since the fundamental parallelepiped for lattices in $\phi(X_s)$ depends on $s$ continuously. Then the same is true for $V$. If $\Gamma$ is not free in $V/L$ then it is not free in $V$ due to the calculation above. A minor modification of these arguments shows that the action of $\Gamma$ is not proper if $T$ is not free in $V/L$ but $\Gamma$ is free (then there exists $v \in V$ such that $\kappa(v + L)$ is an irrational winding in some torus in $V/\Gamma$).

\begin{proof}[Proof of Theorem 2] If $\Gamma$ is elliptic then (i) holds according to Lemma 6 and known results on Euclidean space forms (see [10, Chapter 2]). Note that $\Gamma$ acts in the spacelike space $L^\perp$.

Let $\Gamma$ be non-elliptic. Then $G$ has a nontrivial fixed point $v_0 \in \partial C$ by Lemma 5, the space $W = v_0^\perp$ is $G$-invariant and tangent to $\partial C$ at $v_0$, the representation of $G$ in $V/W$ is trivial. Moreover, there is no $G$-fixed point in $V \setminus W$ (otherwise, $G$ is elliptic since each subspace that contains $v_0$ and is not contained in $W$ intersects $\text{Int}(C)$). Pick any $v_1 \in \partial C$ such that $\ell(v_0, v_1) = 1$ and set $L = \mathbb{R}v_0, N = v_1^\perp \cap W$ (thus $W, L, N$ are defined as in (ii) and (38) holds). By Proposition 3, $\tau(\Gamma) \subseteq W$. This makes possible to apply Lemma 11. According to it, $\Gamma$ acts in $W/L$ freely and discontinuously. Let $\phi$ be the quotient mapping. Since $-\ell$ induces Euclidean structure in $W/L$, we may assume that $\phi \circ \tau$ isomorphically embeds $\Gamma$ to the vector group $W/L$ (replacing $\Gamma$ by its finite index subgroup if necessary) as an uniform lattice in some linear subspace of $W/L$. Due to Lemma 18, the action of $\Gamma$ is subject to formulas of (ii) (the vector $v_1$ may be removed); Lemma 19 implies (7).

Conversely, let $\Gamma$ be as in (ii) and (7) be true (the case (i) is clear). Using relations $G \subseteq \text{SO}(\ell), \ell(v_0, x) = \ell(v_0, ax) = 0$, by a straightforward calculation we get for any $v, u \in V$ and $x \in \Gamma$:
\[
\ell(\gamma_x(u) - v, \gamma_x(u) - v) = \ell(x + l_0(v)ax, x + l_0(u)ax + \ldots, \tag{50}
\]
where dots denote summands that are linear or constant on $x$. It follows from definitions that $\ell$ is negative definite on $N$. Due to (7), for each $v \in V$ the quadratic form $\ell(x + l_0(v)ax, x + l_0(v)ax)$ is negative definite on $T$; if $u$ is sufficiently close to $v$ then the same is true for the form
\[
q_{u,v}(x) = \ell(x + l_0(v)ax, x + l_0(u)ax).
\]
Therefore, there exist a neighbourhood $U$ of $v$ in $V$ and $n_v \in \mathbb{N}$ such that for all $u \in U$ the number of $x$ satisfying the inequality

$$\ell(\gamma_x(u) - v, \gamma_x(u) - v) \geq 0$$

is less than $n_v$. Hence $\kappa((v+C)\cap U)$ and $\kappa((v-C)\cap U)$ are closed in $\kappa(U) \subset V/\Gamma$. Thus, $V/\Gamma$ is strictly causal.

Let $S$ be the union of affine subspaces

$$S_t = ((1 + ta)T)^\perp \cap X_t, \quad t \in \mathbb{R},$$

where $X_t = \{ v \in V : l_0(v) = t \}$. Due to (7), $S$ is homeomorphic to the vector space $V/T$. Let $v \in V_2$ and denote by $u$ the orthogonal projection of $v$ to $(1 + ta)T$ (the definition is correct since $\ell$ is nondegenerate in $N$). There exists the unique $x \in T$ such that $u + x +tax = 0$. Since $v_0, v_1 \perp N$ and $\lambda(x) \in SO(\ell)$, by (48) and (49) this implies that each $T$-orbit in $V$ has precisely one common point with $S$. Therefore, $V$ can be realized as the trivial vector bundle over $S$ with the fibre $T$. Thus $M$ is homeomorphic to $V/T \times T/\Gamma$.

**Proof of Theorem 3.** Let $M = V/\Gamma$ be a 4-manifold of the type (ii). Suppose that $a \neq 0$. Due to (7), this implies $N \neq T$. Hence $\dim T = 1$, $\dim N = 2$, and $\Gamma \cong \mathbb{Z}$. In the coordinate system whose origin is removed to a point $\tilde{o} \in V$ the translation part of an affine transformation $\gamma(v) = \lambda(v) + \tau(v)$ is replaced by

$$\hat{\tau}(\gamma) = \gamma(\tilde{o}) - \tilde{o} = \tau(\gamma) + \lambda(\gamma)\tilde{o} - \tilde{o}$$

and $\lambda(\gamma)$ does not change. A simple calculation with (3), (5), (6) shows that there is a point $\tilde{o} \in \mathbb{R}v_1$ such that $\hat{\tau}(x)$ contains no summands that are quadratic on $x \in \Gamma$ (note that $\dim T = 1$, $\ell(ax, ax) \neq 0$ if $x \neq 0$, and that only coefficients at $v_0$ have quadratic summands). In the proof of Theorem 2 we did not specify the position of origin. Hence we get the same formulas (with other $a$ and $v_1$ in general) removing it to $\tilde{o}$. Then $\tau(x)$ is linear on $x$ and (6) implies that $\ell(ax, x) = 0$ and $\tau(x) = x$. Let $e_1 \in T$ generate $\Gamma$. Multiplying $\ell$ by a suitable positive number, we may assume $\ell(e_1, e_1) = -1$. There exists the unique $t > 0$ such that

$$\ell(\lambda(e_1)(tv_1) - tv_1, \lambda(e_1)(tv_1) - tv_1) = t^2\ell(\lambda e_1, \lambda e_1) = -1.$$

Set $e_3 = tv_1, e_0 = v_0/t, e_2 = ta e_1$. In this base we get formulas of Example 1 (note that $a$ must be replaced by $ta$). Hence $M$ is homothetic to the manifold of this example.

Let $M' = V/\Gamma'$ satisfy assumptions of the theorem. Replacing it by a homothetic manifold, we may assume that $M'$ it is finitely covered by the manifold $M = V/\Gamma$ of Example 1. Then $\Gamma'$ is a subgroup of finite index in $\Gamma$. By Proposition 3, Lemma 11 and Corollary 3, $\Gamma'$ acts freely and properly in Euclidean 2-plane $W/\mathcal{L}$ as a group of isometric transformations. If $\Gamma'$ acts by translations then we may apply Lemma 18. Then we have the setting above. Note that $\dim T = 1$ and $\Gamma' \cong \mathbb{Z}$ in this case. Any other isometric transformation that $\Gamma'$
Due to (50), for \( P \) implies that \( M \) where \( u \in V \). The set is invariant with respect to the translations along \( W \) along \( L \) have two ends.

\( M \) suppose that \( \kappa \) is elliptic if 2) is true. Similar arguments prove 3).

Let \( \Gamma \) be of the type (ii) with \( a \neq 0 \). \( M \) be finitely covered by \( V/\Gamma \) and suppose that \( M \) admits the Cauchy hypersurface \( S \). Then for any \( v \in V \) the line \( \kappa(v+L) \) has a common point with \( S \). Hence we may assume that \( p = \kappa(v) \in S \) and \( l_0(v) > c \), where \( c \) is as in (51). Then \( P_p \) contains all lines \( \kappa(u+L) \), where \( u \in V \) satisfies \( l_0(u) < -c \). These lines cannot intersect \( S \) since \( S \) separates \( P_p \) and \( F_p \) being a Cauchy hypersurface, and we get a contradiction. Thus 4) implies that \( M \) is elliptic.

Since codim \( T \geq 2 \) in (ii) and \( M \) is homeomorphic to \( V/T \times T/\Gamma \), it cannot have two ends.

Let \( M \) be elliptic. Then \( G = \lambda(\Gamma) \) is finite and keeps \( C \). Hence there exists \( G \)-fixed vector \( v_0 \in \text{Int}(C) \) and \( \Gamma \)-invariant spacelike hyperplane \( H = v_0^\perp \) in \( V \). Evidently, \( \kappa(H) \) is a Cauchy hypersurface. Hence \( M \) satisfies 4). The function \( \ell(v_0, v) \) is \( \Gamma \)-invariant and strictly increasing along all lightlike lines; moreover, it is not bounded on each of them. Thus 2) and 3) are true if \( M \) is elliptic. \( \square \)

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