Imprints of an extended Chevallier-Polarski-Linder parametrization on the large scales

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In the present work we perform a systematic analysis of a new dark energy parametrization and its various corrections at first and higher orders around the presence epoch $z = 0$, where the first order correction of this dark energy parametrization is the well known Chevallier-Polarski-Linder model. We have considered up to the third order corrections of this parametrization and investigate the models at the level of background and perturbations. The models have been constrained using the latest astronomical datasets from a series of potential astronomical data, such as the cosmic microwave background observations, baryon acoustic oscillations measurements, recent Pantheon sample of the supernova type Ia and the Hubble parameter measurements. From the analyses we found that all parametrization favor the quintessential character of the dark energy equation of state where the phantom crossing is marginally allowed (within 68% CL). Probably the interesting outcome of the present work is that as long as we increase the higher order corrections, the parameter $w_x$ quantifying the dynamical nature of the dark energy parametrization becomes weak in magnitude and hence this eventually confers that the higher order corrections of the general dark energy parametrization are not much significant. Finally, we perform the Bayesian analysis using MCEvidence to quantify the statistical deviations of the parametrizations compared to the standard $\Lambda$CDM cosmology. The Bayesian analysis reports that $\Lambda$CDM is favored over all the DE parametrizations.

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1. INTRODUCTION

According to the theory of general relativity, one possible way to describe the recent observational evidences is to introduce the dark energy, a hypothetical fluid with large negative pressure [1]. However, apart from this negativity condition on the pressure of dark energy, no one knows what exactly this particular fluid is. The simplest explanation to the dark energy theory comes through the introduction of positive cosmological constant, $\Lambda$, which does not evolve with the time. But, the cosmological constant already suffers from two major problems, one which is recognized as the cosmological constant problem and the other is the cosmic coincidence problem. Thus, although as stated by a series of observational data, the $\Lambda$-cosmology is an elegant version to model the recent observational features of the universe, the problems associated with the above motivate us to think of the scenarios beyond the standard $\Lambda$-cosmology paradigm.

The simplest extension to $\Lambda$-cosmology is the $w_x$-cosmology in which $w_x$ is the dark energy equation-of-state quantified as the ratio of pressure to its density, mathematically which is $w_x = p_x/\rho_x$. One can identify that $p_x$ and $\rho_x$ are respectively the pressure and energy density of the dark energy fluid. The equation-of-state $w_x$ being $-1$ recovers the $\Lambda$-cosmology. In general one can assume $w_x (\neq -1)$ to be either time independent or dependent while the latter scenario is the most general one. Thus, in the present work we shall focus on the alternative cosmologies to the $\Lambda$-cosmology in which the dark energy equation-of-state is evolving with the expansion of the universe.

The parametrization of $w_x$ could be any function of the redshift $z$ or the scale factor $a(t)$ of the Friedmann-Lemaître-Robertson-Walker universe; note that, $1 + z = a_0/a(t)$, where $a_0$ is the present value of the scale factor in this universe. Thus, since $w_x \equiv w_x(z) \equiv w_x(a)$ could be any arbitrary function of the redshift or the scale factor, therefore, in principle this gives us a complete freedom to pick up any particular model of interest and test it with the observational data in order to see whether that model is able to correctly describe the evolution of the universe. In fact one can realize that the introduction of the dark energy equation-of-state is a reverse mechanism to probe the expansion history of the universe. Going back to literature, one can find that this particular area of cosmology has been investigated well both at the level of background and perturbations where various parametrizations for $w_x$ were introduced earlier [2,12] and later [13,22]. Precisely, the dark energy-parametrization with only a single free parameter, with two free parameters, with three free parameters and finally with more than three parameters have been rigorously studied by various investigators.

The aim of the present work is slightly different. Here, we are considering an exponential dark energy...
parametrization that in its first order approximation around \( z = 0 \) recovers the CPL parametrization, and further we allow its higher order corrections in order to investigate how such extended corrections affect the evolution of the universe both at the level of background and perturbations. More specifically, we consider up to the third order expansion of the exponential dark energy model. We remark that in general every analytic function for the equation-of-state parameter around the \( z = 0 \) describes the CPL parametrization in the first correction; however, while we want to assume a general Taylor expansion of an analytic function \( f(a) \) around \( a = 1 \), i.e. \( f(a) = \sum_{i=0}^\infty w_i (a-1)^i \), every new term which is introduced in the correction provides a new degree of freedom, a free parameter, in the model. Consequently, the models will have different degrees of freedom and they will not be in comparison. Hence, special relations amount the constants \( w_i \) should be considered, and for our analysis we assume that \( w_0 \) is free while \( w_j = \frac{w_{j+1}}{j+1} \), which \( j \neq 0 \), in which \( f(a) \) is now the exponential function. However, by this approach we will get a remarkable information on how the nonlinear terms in the parametrizations of the equation-of-state affect the viability of the model in higher-redshifts.

The work has been organized in the following way. In section 2.1 we introduce the models for \( w_x(z) \) and describe the general equations at the level of background and perturbations. In section 2.2 we describe the observational data and the statistical analysis that are used to constrain the models. After that in section 2.3 we describe the observational constraints extracted from the models using the astronomical data described in section 2.1. Then in section 2.4 we compute the evidences of the dark energy parametrizations through the MCEvidence. Finally, we close the work in section 2.5 with a brief summary of everything.

2. BASIC EQUATIONS AND THE MODELS

Considering a spatially flat Friedmann-Lemaitre-Robertson-Walker line element \( ds^2 = -dt^2 + a^2(t) \sum_{i=1}^{3} dx_i^2 \) (where \( a(t) \) is the expansion scale factor of the universe), in the context of the Einstein gravity, we assume that (i) matter is minimally coupled to gravity, (ii) there is no interaction between any two fluids under consideration and (iii) all the fluids satisfy barotropic equation of state, i.e., \( p_i = w_i \rho_i \), in which \( w_i \) being the barotropic state parameter for the \( i \)-th fluid having \( (\rho_i, p_i) \) as its energy density and pressure, respectively. Precisely, we consider that the total energy density of the universe is, \( \rho_{tot} = \rho_r + \rho_b + \rho_c + \rho_x \) and the total pressure thus becomes \( p_{tot} = p_r + p_b + p_c + p_x \). Here, the subscripts \( r, b, c \) and \( x \) respectively stands for radiation, baryons, cold dark matter and dark energy. Thus, the barotropic indices are, \( w_r = 1/3 \), \( w_b = w_c = 0 \) and we assume \( w_x \) to be dynamical. The Einstein’s field equations for the above FLRW universe can be written down as

\[
H^2 = \frac{8\pi G}{3} \rho_{tot},
\]

\[
\dot{H} = -4\pi G (\rho_{tot} + p_{tot}),
\]

in which an overhead dot represents the cosmic time differentiation and \( H \equiv \dot{a}/a \) is the Hubble rate of this universe. Now, using 1 and 2 (or alternatively the Bianchi’s identity), one can find the balance equation

\[
\dot{\rho}_{tot} + 3H (\rho_{tot} + p_{tot}) = 0.
\]

from which using the relation between pressure and energy density for the radiation, baryons, and cold (pressureless-) dark matter, one can find that \( \rho_r = \rho_r_0 a^{-4} \), \( \rho_m = \rho_b + \rho_c = (\rho_b_0 + \rho_c_0) a^{-3} \). Here, \( \rho_0 \) is the present value of \( \rho_i \). And finally, the evolution of the dark energy fluid can be given by,

\[
\rho_x = \frac{\rho_{x0}}{a_0} \left(a\right)^{-3} \exp \left(-3 \int_{a_0}^{a} \frac{w_x(a')}{{a'}} \, da' \right),
\]

where \( \rho_{x0} \) being the current value of \( \rho_x \) and \( a_0 \) is the present value of the scale factor that we set to be unity \( (a_0 = 1) \) without any loss of generality. We further note that the scale factor is related to the redshift that we shall frequently use hereafter via \( 1 + z = a_0/a = 1/a \). Thus, once the dark energy equation of state is prescribed, the evolution of the dark energy density can be found.

As we discussed above, we consider that the dark energy fluid follows a general parametrization in the following way:

\[
w_x(z) = (w_0 - w_a) + w_a \exp\left(\frac{z}{1+z}\right)
\]

where \( w_0 \) is the present value of the dark energy equation of state, that means, \( w_x(z = 0) = w_0 \) and \( w_a \) is another free parameter. The model 5 is very interesting by its construction since one can easily recognize that it could return a number of interesting parametrization that includes the classic Chevallier-Polarski-Linder parametrization \( w_x(z) = w_0 + w_0 z/(1 + z) \) if we take the first approximation of the exponential function in 6. We expand the exponential function of 6 upto its first, second and third order corrections leading to the following class of dark energy parametrization:
\[
\begin{align*}
  w_x(a) &= w_0 + w_a \frac{z}{1+z}, \\
  w_x(a) &= w_0 + w_a \left[ \frac{z}{1+z} + \frac{1}{2!} \left( \frac{z}{1+z} \right)^2 \right], \\
  w_x(a) &= w_0 + w_a \left[ \frac{z}{1+z} + \frac{1}{2!} \left( \frac{z}{1+z} \right)^2 + \frac{1}{3!} \left( \frac{z}{1+z} \right)^3 \right],
\end{align*}
\]

and for convenience we call the dark energy parametrization of equations (7), (8) and (9) as “Extension 1” (Ext1 in short), “Extension 2” (Ext2 in short) and “Extension 3” (Ext 3 in short), respectively.

At the end of this section, we would like to present the qualitative features of the present dark energy parametrizations in terms of the evolution of their equations of state and the deceleration parameters. In order to do so, we assume three different values of \( w_0 \), namely, \( w_0 = -0.95 \), \( w_0 = -1 \) and \( w_0 = -1.1 \) and in each case we consider various values of \( w_a \) to understand how the curves behave with the increasing of the \( w_a \) parameter. In Fig. 1 we show the evolution of the dark energy parameterizations (7), (8), (9) and (10) setting the present value of the dark energy equation of state at \( w_0 = 0.95 \) where we allow different values of \( w_a \) such as \( w_a = -0.3, -0.2, -0.1, 0, 0.1, 0.2, 0.3 \). The curve with \( w_a = 0 \) simply returns \( w = w_0 \) and this has been kept to compare with other curves having \( w_a \neq 0 \). From Fig. 4, we notice that for \( w_a < 0 \), the dark energy equation of state allows its phantom character which is much pronounced at high redshifts, while for \( w_a > 0 \), the reverse scenario is found. In a similar fashion, we show the evolution of the equations of states for the DE parametrizations for \( w_0 = -1 \) (Fig. 2) and \( w_0 = -1.1 \) (Fig. 3) where we have kept the same values of \( w_a \) as used in Fig. 1. From both the figures, namely, Fig. 2 and Fig. 3 we have similar observation to that of Fig. 1.

We then plot the evolution of the deceleration parameter for all the DE parametrizations, namely, (7), (8), (9) and (10). Here we have considered three fixed values of \( w_0 \), namely, \( w_0 = -0.95, -1, -1.1 \) but in each case we have assumed different values of \( w_a \) similar to what we have shown in Fig. 1, Fig. 2 and Fig. 3. Finally, we depict the evolution of the deceleration parameter in Fig. 4 (for \( w_0 = -0.95 \)), Fig. 5 (for \( w_0 = -1 \)) and Fig. 6 (for \( w_0 = -1.1 \)). The qualitative features of the three figures (Fig. 4, Fig. 5 and Fig. 6) representing the evolution of the deceleration parameters are same, as one can see that irrespective of the values of \( w_0 \), a fine transition from the past decelerating phase to the current accelerating one is observed, however, the impacts of \( w_a \) should be discussed. For that reason, we only consider Fig. 4 because the other figures lead to same conclusion. From Fig. 4 we see that for negative values of \( w_a \) the transition redshifts are shifting towards higher redshifts (although mild) while for positive values of \( w_a \), we see the reverse, that means the transition redshifts are shifting towards lower values of the redshift.

Overall, we find that the models at the level of background do not exhibit any deviations from one another. This is not surprising because the deviations between the cosmological models are usually reflected from their analysis at the level of perturbations. In what follows we shall consider the perturbation equations for all the DE parametrizations in this work.

We start with the following metric which is the perturbed form of the FLRW line element:

\[
\text{ds}^2 = a^2(\eta) \left[ -d\eta^2 + (\delta_{ij} + h_{ij})dx^i dx^j \right],
\]

Here, \( \eta \) denotes the conformal time; \( \delta_{ij}, h_{ij} \) are the unperturbed and the perturbative metric tensors, respectively. Now, considering the perturbed Einstein’s field equations, for a mode with wave-number \( k \) one can write down (13–45):

\[
\delta_i' = -(1 + w_i) \left( \theta_i + \frac{h^i}{2} \right) - 3H \left( \frac{\delta p_i}{\delta \rho_i} - w_i \right) \delta_i - 9H^2 \left( \frac{\delta p_i}{\delta \rho_i} - c_{a,i}^2 \right) \left( 1 + w_i \right) \frac{\theta_i}{k^2},
\]

\[
\theta_i' = -H \left( 1 - 3\frac{\delta p_i}{\delta \rho_i} \right) \theta_i + \frac{\delta p_i/\delta \rho_i}{1 + w_i} k^2 \delta_i - k^2 \sigma_i,
\]

where \( \delta_i = \delta \rho_i/\rho_i \) is the density perturbation for the \( i \)-th fluid; the prime associated to any quantity denotes the derivatives with respect to conformal time; \( H = a'/a \) is the conformal Hubble parameter; \( \theta_i \equiv i k^j v_j \) is the divergence of the \( i \)-th fluid velocity; \( h = h^j \) is the trace of the metric perturbations \( h_{ij} \); \( \sigma_i \) denotes the anisotropic stress related to the \( i \)-th fluid. Let us also note that \( c_{a,i}^2 = p_i/\rho_i \), is the adiabatic speed of sound of the \( i \)-th fluid which can also be written in terms of other physical quantities as \( c_{a,i}^2 = w_i - \frac{w_i'}{3H(1+w_i)} \), where we fix the sound speed \( c_s^2 = \delta p_i/\delta \rho_i \) to be unity. Finally, we also note that we have neglected the anisotropic stress from the system for simplicity.
3. OBSERVATIONAL DATA

For the convenience of the reader and for our presentation we provide the details of the observational data used to constrain the dynamical dark energy parametrization and also the methodology.

- Cosmic microwave background observations: the cosmic microwave background (CMB) observations are one of the powerful data to probe the nature of dark energy. Here we use the CMB from Planck 2015 [46, 47]. The high-$\ell$ temperature and polarization data as well as the low-$\ell$ temperature and polarization data from Planck 2015 (precisely the dataset: Planck TT, TE, EE + lowTEB) [46, 47] have been considered.

- Baryon acoustic oscillations: The baryon acoustic oscillations (BAO) data from different superovulation missions are used [49–51].

- Supernovae Type Ia: We also use latest released Pantheon sample [52] from the Supernovae Type Ia.

FIG. 1: We show the evolution of the dark energy parametrizations for different values of $w_a$ with a fixed value of $w_0 = -0.95$

FIG. 2: We show the evolution of the dark energy parametrizations for different values of $w_a$ with a fixed value of $w_0 = -1$
FIG. 3: We show the evolution of the dark energy parametrizations for different values of $w_a$ with a fixed value of $w_0 = -1.1$.

FIG. 4: The evolution of the deceleration parameter depicting a clear transition from the past decelerating phase to the current accelerating phase for all the dark energy models has been presented for different values of $w_a$ and with a fixed value of $w_0 = -0.95$. One can easily notice that as long as $w_a$ increase from its negative to positive values, the transition redshift shifts more closer to the present epoch.

- Hubble parameter measurements: Finally, we use the Hubble parameter measurements from the Cosmic Chronometers (CC) [53].

Now we come to the technical part of the statistical analysis. Thus, we have performed the fitting analysis using the modified version of cosmomc [54] [55], an efficient markov chain monte carlo package equipped with a convergence diagnostic given by the Gelman and Rubin statistics [56]. This package includes the support for the Planck 2015 likelihood code [47] (see http://cosmologist.info/cosmomc/). In Table 1 we have shown the flat priors on the model parameters that have been used during the observational analysis. Perhaps it might be important to mention here that in the present analysis we have used Planck 2015 likelihood [47] instead of Planck 2018 likelihood (although the cosmological parameters from Planck 2018 are already available [57]) because Planck 2018 likelihood code is not public yet. However, it will be worth to run the same codes that we
FIG. 5: The evolution of the deceleration parameter depicting a clear transition from the past decelerating phase to the current accelerating phase for all the dark energy models has been presented for different values of \( w \) and with a fixed value of \( w_0 = -1 \). Regarding the relation between \( w_a \) and the transition redshift \( a_t \), we have exactly similar relation to what we have observed in Fig. q-1.

FIG. 6: The evolution of the deceleration parameter depicting a clear transition from the past decelerating phase to the current accelerating phase for all the dark energy models has been presented for different values of \( w \) and with a fixed value of \( w_0 = -1.1 \). Similar to Fig. 4 and Fig. 5 we observe similar relation between \( w_a \) and the transition redshift.

use for the present models but with the new Planck 2018 likelihood which will enable us to understand any effective changes in the cosmological parameters and consequently more stringent constraints on them as well.
FIG. 7: 68% and 95% CL contour plots for various combinations of the model parameters of the general parametrization of $\theta$ (Gen) have been shown for different observational combinations. The figure also contains the one dimensional marginalized posterior distributions for the parameters shown in the two dimensional contour plots.

FIG. 8: 68% and 95% CL contour plots for various combinations of the model parameters of Ext1 of $\theta$ [the CPL parametrization] have been shown for different observational combinations. The figure also contains the one dimensional marginalized posterior distributions for the parameters shown in the two dimensional contour plots.
FIG. 9: 68% and 95% CL contour plots for various combinations of the model parameters of the Ext2 of (8) have been shown for different observational combinations. The figure also contains the one dimensional marginalized posterior distributions for the parameters shown in the two dimensional contour plots.

FIG. 10: 68% and 95% CL contour plots for various combinations of the model parameters of the Ext3 of (9) have been shown for different observational combinations. The figure also contains the one dimensional marginalized posterior distributions for the parameters shown in the two dimensional contour plots.
FIG. 11: We show the \((w_0, w_a)\) plane for the present dynamical dark energy parametrizations using different observational datasets. The left graph for the dataset CMB+BAO, the middle graph for the dataset CMB+BAO+Pantheon and the right graph stands for the observational dataset CMB+BAO+Pantheon+CC.

| Parameter       | Prior               |
|-----------------|---------------------|
| \(\Omega_m h^2\) | \([0.005, 0.1]\)    |
| \(\Omega_c h^2\) | \([0.01, 0.99]\)   |
| \(\tau\)        | \([0.01, 0.8]\)    |
| \(n_s\)         | \([0.5, 1.5]\)     |
| \(\log[10^{10} A_s]\) | \([2.4, 4]\) |
| \(1000_{MC}\)   | \([0.5, 10]\)      |
| \(w_0\)         | \([-2.0]\)         |
| \(w_a\)         | \([-3.3]\)         |

TABLE I: The table shows the flat priors on the model parameters used during the statistical analysis.

4. OBSERVATIONAL CONSTRAINTS AND THE ANALYSIS

In this section we describe the observational constraints on all the dark energy parametrization, namely the general parametrization of eqn. (6), Extension 1 or the CPL parametrization of eqn. (7), Extension 2 of eqn. (8) and extension 3 of eqn. (9) using various astronomical datasets summarized in section 3. In particular, we focus on the two key parameters of the dark energy parametrization, namely, \(w_0\) and \(w_a\) in order to investigate the qualitative changes in the parametrization as long as nonlinear terms are considered. In what follows we describe the observational constraints extracted from each dark energy scenario.

Let us first focus on the general dark energy parametrization given in equation (6). We have constrained this dark energy scenario using different observational combinations, the results of which are summarized in Table I. From Table I, one can see that the best constraints on the model parameters are achieved for the combinations CMB+BAO+Pantheon and CMB+BAO+Pantheon+CC since the addition of Pantheon and CC to the combination CMB+BAO significantly decrease the error bars on the model parameters. We find that for the general parametrization, the mean value of the dark energy equation of state at present, i.e., \(w_0\) is always in the quintessential regime while looking at the exact estimations on \(w_0\) for CMB+BAO+Pantheon \((w_0 = -0.963^{+0.062}_{-0.082} \text{ at } 68\% \text{ CL})\) and for CMB+BAO+Pantheon+CC \((w_0 = -0.933^{+0.072}_{-0.070} \text{ at } 68\% \text{ CL})\), it is also clear that \(w_0\) could cross the \(w_0 = -1\) boundary, but of course marginally. Additionally, we find that the remaining key parameter \(w_a\) may assume nonnull values, however, \(w_a = 0\) is allowed within 68\% CL of course. For a better understanding of all the parameters of this model, in Fig. 7 we have shown the one dimensional posterior distributions for some selected parameters of this model as well as the two dimensional contour plots for various combinations of the parameters. From Fig. 7 one can see that all the parameters shown in this figure are correlated with each other. Specifically, we find a strong correlation between \(w_0, w_a\) and \(H_0\).

We now consider the first extension of the general parametrization (6) that leads to the well known CPL parametrization of (7). The results of this parametrization are also extracted using the same observational datasets applied to the general DE parametrization which can be found from Table III. One can easily see that the conclusions on \(w_0\) and \(w_0\) remain same similar to what we have found in the general parametrization (6). So, effectively we see that the first approximation of the original parametrization (6) returns similar fit to the original parametrization (6). Similarly, for this parametrization we plot Fig. 8 containing the one dimensional marginalized posterior distributions as well as the two dimensional contour plots at 68\% and 95\% CL.

Then we move to the observational constraints of the next parametrization given in eqn. (8). The results for this parametrization are shown in Table IV and in Fig. 9 we have shown the graphical variations of the model parameters. We find that this parametrization behaves similarly
to the previous two parametrizations, that means concerning the free parameters \( w_0 \), \( w_a \), we have exactly similar conclusion as observed with model 6 and model 7.

Finally, we focus on the last parametrization of this series, namely Extension 3 shown in eqn. 9. We have summarized the results in Table VII and in Fig. 10 we have shown the corresponding graphical variations of the model parameters. Looking at all the parameters, it is clear that except \( w_a \), the other parameters have similar constraints as already found in models 6 and 7. For the \( w_a \) parameter, we see that for the last two combinations (the CMB data alone provide higher values, similar to other models as well), the mean values of \( w_a \) as well as the error bars are significantly reduced compared to the previous models. In fact for the last combination that means for CMB+BAO+Pantheon+CC, the estimated value of \( w_a \) is very close to zero which means that the dynamical nature of the dark energy parametrization is very weak, however, the statistical estimation also offers its non-null value finding, \( w_a = -0.0941 + 0.089 + 0.158 \times 10^{-2} \) or the 68% CL, CMB+BAO+Pantheon+CC. The difference in the results can be found from Fig. 11 which clearly shows that for this parametrization, the \( w_a \) plane is much reduced compared to others.

Thus, based on the analyses presented above one can see that as long as we consider the higher order corrections in the generalized parametrization 6, the parameter \( w_a \) quantifying the dynamical nature of the dark energy parametrization becomes weak. For a detailed understanding, we refer to Table VII.

Now we investigate how the present dark energy parametrization, namely, the Ext1 of 7 using various observational datasets. We note that \( H_0 \) is in the units of km/Mpc/sec and \( \Omega_{m0} \) is the present value of \( \Omega_m = \Omega_c + \Omega_b + H_0 \) in the units of km/sec/Mpc.

### Table II: Observational constraints on various free parameters at 68% and 95% CL for the dynamical dark energy state parameter \( w(z) = (w_0 - w_a) + w_a \exp (\frac{z}{z_0}) \) (Gen) using different astronomical datasets have been presented. Let us note that \( \Omega_{m0} \) is the present value of \( \Omega_m = \Omega_c + \Omega_b + H_0 \).

| Parameters | CMB+BAO | CMB+BAO+Pantheon | CMB+BAO+Pantheon+CC |
|------------|---------|------------------|----------------------|
| \( \Omega_c^2 \) | 0.1191 + 0.0013 + 0.0267 \times 10^{-2} | 0.1191 + 0.0013 + 0.0267 \times 10^{-2} | 0.1191 + 0.0013 + 0.0267 \times 10^{-2} |
| \( \Omega_b^2 \) | 0.0222 + 0.00015 + 0.00029 | 0.0222 + 0.00015 + 0.00029 | 0.0222 + 0.00015 + 0.00029 |
| \( \Omega_{m0} \) | 0.340 + 0.032 + 0.051 | 0.340 + 0.032 + 0.051 | 0.340 + 0.032 + 0.051 |
| \( \sigma_8 \) | 0.801 + 0.026 + 0.056 | 0.801 + 0.026 + 0.056 | 0.801 + 0.026 + 0.056 |
| \( H_0 \) | 64.14 + 3.85 + 5.60 | 68.84 + 4.80 + 5.15 | 68.84 + 4.80 + 5.15 |

### Table III: Observational constraints on the dark energy parametrization, namely, the Ext1 of 7 using various observational datasets. We note that \( H_0 \) is in the units of km/Mpc/sec and \( \Omega_{m0} \) is the present value of \( \Omega_m = \Omega_c + \Omega_b + H_0 \).

1 We have considered \( w_0 = -0.95 \) since from the observational...
Table IV: Observational constraints on the dark energy parametrization, namely, the Ext2 of [9] using various observational datasets. We note that $H_0$ is in the units of km/Mpc/sec and $\Omega_{m0}$ is the present value of $\Omega_m = \Omega_b + \Omega_c$.

Table V: Observational constraints on the dark energy parametrization, namely, the Ext3 of [9] using various observational datasets. We note that $H_0$ is in the units of km/Mpc/sec and $\Omega_{m0}$ is the present value of $\Omega_m = \Omega_b + \Omega_c$.

Fig. 12] one can easily find that as long as $w_a$ increases, the higher order corrections of the generalized model [9], that means model in eqn. [9] gets differentiated from the remaining models. While on the other hand, for the negative values of $w_a$ (corresponding to the lower graphs of Fig. 12), we don’t find any kind of differences between the models. In a similar way, in Fig. 13 we have shown plots representing the matter power spectra for positive and negative values of $w_a$ with fixed $w_b = -0.95$. The upper graphs of Fig. 13 correspond to $w_a > 0$ while the lower graphs correspond to $w_a < 0$. From this figure, we have similar observation as already noticed in Fig. 12.

5. BAYESIAN EVIDENCE

A general and natural question that we will be looking for in this section is that, how the models are efficient compared to the standard CDM cosmology. Thus, we need a statistical comparison between all four dynamical DE parametrizations where the base model will be fixed as ΛCDM. This statistical comparison comes through the Bayesian evidence. Here we apply publicly available code MCEvidence [52, 59] to compute the evidences of the models. The use of MCEvidence is very easy since the code only needs the MCMC chains used to extract the free parameters of the DE parametrizations.

While dealing with Bayesian analysis we need the posterior probability of the model parameters (denoted by $\theta$), given a specific observational data ($x$) with any prior information for a model ($M$). Following Bayes theorem, one can write,

$$ p(\theta|x,M) = \frac{p(x|\theta,M) p(\theta|M)}{p(x|M)}, $$

(12)

where $p(x|\theta,M)$ is the likelihood as a function of $\theta$ and $\pi(\theta|M)$ refers to the prior information. Here, the quantity $p(x|M)$ appearing in the denominator of (12) is the Bayesian evidence that we actually need for the model

analyses of the models presented in this work the best constraint on $w_0$ is around that value.

2 See github.com/yabebalFantaye/MCEvidence.
TABLE VI: For a clear understanding and comparison of the dark energy key parameters \((w_0, w_a)\) extracted from all the parametrizations, we show their estimations at 68% CL. Here, CB = CMB+BAO, CBP = CMB+BAO+Pantheon, CBPC = CMB+BAO+Pantheon+CC.

| Datasets | Parameters | Gen | Ext1 (CPL) | Ext2 | Ext3 |
|---------|------------|-----|------------|------|------|
| CB      | \((-0.537\pm0.442, -1.154\pm0.894)\) | \((-0.524\pm0.374, -1.403\pm0.731)\) | \((-0.616\pm0.428, -1.006\pm0.909)\) | \((-0.581\pm0.348, -0.371\pm0.274)\) |
| CBP     | \((-0.963\pm0.060, -0.231\pm0.291)\) | \((-0.947\pm0.076, -0.306\pm0.367)\) | \((-0.949\pm0.074, -0.296\pm0.277)\) | \((-0.933\pm0.065, -0.116\pm0.085)\) |
| CBPC    | \((-0.933\pm0.071, -0.33\pm0.288)\) | \((-0.950\pm0.075, -0.291\pm0.338)\) | \((-0.946\pm0.072, -0.302\pm0.310)\) | \((-0.951\pm0.075, -0.075\pm0.089)\) |

FIG. 12: Cosmic microwave background spectra for the present dynamical dark energy parameterizations have been shown for different values of the \(w_a\) parameter with fixed \(w_0 = -0.95\). The upper left, middle, and right graphs stands for \(w_a = 0.1, 0.2\) and \(0.3\), respectively while the lower left, middle and right graphs stands respectively for \(w_a = -0.1, -0.2\) and \(-0.3\). From the graphs, one can notice that difference between the models is observed only for the positive values of \(w_a\), namely, \(w_a = 0.2\) and \(w_a = 0.3\), while for negative values of \(w_a\) upto \(w_a \leq 0.3\), we do not find any kind of differences between the models.

| \(\ln B_{ij}\) | Evidence for model \(M_i\) |
|-----------------|-----------------------------|
| \(0 \leq \ln B_{ij} < 1\) | Weak |
| \(1 \leq \ln B_{ij} < 3\) | Definite/Positive |
| \(3 \leq \ln B_{ij} < 5\) | Strong |
| \(\ln B_{ij} \geq 5\) | Very strong |

\[
p(M_j|x) = \frac{\pi(M_j) p(x|M_j)}{\pi(M_j) p(x|M_j)} = \frac{\pi(M_j)}{\pi(M_j)} B_{ij},
\]

in which \(B_{ij} = \frac{p(x|M_j)}{p(x|M_j)}\), is the Bayes factor of the model \(M_i\) relative to \(M_j\). And based on the values of \(B_{ij}\) (alternatively, in \(B_{ij}\)) we quantify the observational support of the underlying model \(M_i\) relative to \(M_j\). The quantification is done through the widely accepted Jeffreys scales [60] (see Table VII). We also note that the negative values of \(\ln B_{ij}\) indicate that the reference model \((M_j)\) is preferred over the underlying model \((M_i)\).

In Table VIII we have shown the values of \(\ln B_{ij}\) computed for all DE parametrizations considering all the datasets. We find that the values of \(\ln B_{ij}\) are all negative indicating that \(\Lambda\)CDM is always preferred and this is true for all the observational datasets.
The dark energy, a hypothetical fluid in Einstein gravity is the main concern of this work. This dark energy, as examined by many investigators since the year 1998, could be anything obeying only one condition that the pressure of the fluid should be negative. Thereafter, a cluster of dark energy models have been introduced and confronted with the observational data, see \(^1\) to get an overview of the models.

Among them an interesting construction of the dark energy models comes through the equation of state of dark energy, \(w_x = p_x/\rho_x\) which in principle is the function of the underlying cosmological time parameter, usually the function of the redshift. Technically, there is no such restriction to pick up any specific functional form for \(w_x\), however, the viability of the model is only tested through the observational data and its effects on the large scale structure of the universe indeed. According to the investigations performed in the last couple of years, the Chevallier-Polarski-Linder parametrization is a feasible and well functioning dark energy parametrization with the observational data. The present work is motivated in the same direction whilst we have investigated something different as follows.

We have introduced a new dark energy parametrization \(^3\) having a novel feature. The model recovers the well known CPL parametrization in its first order Taylor
series expansion around \( z = 0 \). Thus, the model actually presents a generalized version of the CPL parametrization. Since the model is a nonlinear generalized version of the CPL model, thus, a natural inquiry one may ask for is, how its higher order corrections are important for the expansion history of the universe, and moreover, how the higher order corrections could affect the evolution of the universe at the level of background and perturbations. In order to investigate these issues, we have considered the generalized model \( 6 \) together with its first, second and third order Taylor approximations around the present cosmic epoch \( z = 0 \), given in equations \( 7, 8 \) and \( 9 \). Since the original model \( 6 \) contains only two free parameters \( w_0 \) (current value of the dark energy equation of state) and \( w_a \) (parameter quantifying the dynamical nature of the DE), thus its extensions contain the same free parameters. We then constrain all the models using a class of astronomical data, such as CMB, BAO, Pantheon from SNIa and the Hubble parameter measurements (summarized in section 3).

The observational constraints are summarized in Table I (for eqn. (6)), Table II (for eqn. (7)), Table III (for eqn. (8)) and Table IV (for eqn. (9)) and the graphical variations of the model parameters are also shown in Fig. 7, Fig. 8, Fig. 9 and Fig. 10, respectively for the general, Ext1, Ext2, and Ext3. From the analyses, it is clear that up to Ext2, the cosmological parameters assume similar constraints while the third order correction (Ext3 of eqn. (9)) presents slightly different results compared to others (see Fig. 11). Precisely, for the third order correction (eqn. (9)), we find that the parameter \( w_0 \) quantifying the dynamical nature of the model is very weak (\( \equiv w_0 \sim 0 \)) at least for the last two combinations (i.e., CMB+BAO+Pantheon and CMB+BAO+Pantheon+CC). We mention that for the CMB+BAO combination, the error bars of the parameters for all the models are significantly large compared to others (hence weakly constrained) and moreover, one can notice that for this particular dataset the parameters \( w_0 \) and \( w_a \) are strongly correlated to each other (as seen from Fig. 11). Such a disparity has also been reflected from the analysis at large scale structure. Looking at Fig. 12 and Fig. 13 one can see that as soon as we allow the higher order corrections, the model (Ext3) is distinguished from the others and the dynamical nature of the model becomes insignificant.

Finally, we perform the Bayesian analysis using the MCEvidence and compared the models with respect to the standard ΛCDM reference scenario. Our analysis reveals that ΛCDM is indeed favored over all the dynamical DE models. This is an expected result because the parameters space for all the dynamical DE models are of eight dimensional while the ΛCDM has only six parameters.

Last but not least, we would like to comment that the model (6), so far we are aware of the literature, is a new one in the field of dark energy which naturally recovers CPL parametrization in its first order approximation and sounds good with the Bayesian evidence. Therefore, a number of investigations can be performed in various contexts. We hope to address some of them in near future.

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[1] E. J. Copeland, M. Sami and S. Tsujikawa, *Dynamics of dark energy*, Int. J. Mod. Phys. D 15, 1753 (2006).
[2] M. Chevallier and D. Polarski, *Accelerating universes with scaling dark matter*, Int. J. Mod. Phys. D 10, 213 (2001).
[3] E. V. Linder, *Exploring the expansion history of the universe*, Phys. Rev. Lett. 90, 091301 (2003).
[4] A. R. Cooray and D. Huterer, *Gravitational lensing as a probe of quintessence*, Astrophys. J. 513, L95 (1999).
[5] G. Efstathiou, *Constraining the equation of state of the universe from distant type Ia supernovae and cosmic microwave background anisotropies*, Mon. Not. Roy. Astron. Soc. 310, 842 (1999).
[6] P. Astier, *Can luminosity distance measurements probe the equation of state of dark energy*, Phys. Lett. B 500, 8 (2001).
[7] J. Weller and A. Albrecht, *Future supernovae observations as a probe of dark energy*, Phys. Rev. D 65, 103512 (2002).
[8] C. Wetterich, *Phenomenological parameterization of quintessence*, Phys. Lett. B 594, 17 (2004).
[9] S. Hannestad and E. Mortsell, *Cosmological constraints on the dark energy equation of state and its evolution*, JCAP 0409, 001 (2004).
[10] H. K. Jassal, J. S. Bagla and T. Padmanabhan, *Observational constraints on low redshift evolution of dark energy: How consistent are different observations?*, Phys. Rev. D 72, 103503 (2005).
[11] Y. g. Gong and Y. Z. Zhang, *Probing the curvature and dark energy*, Phys. Rev. D 72, 043518 (2005).
[12] B. Feng, M. Li, Y. S. Piao and X. Zhang, *Oscillating quintom and the recurrent universe*, Phys. Lett. B 634, 101 (2006).
[13] S. Nojiri and S. D. Odintsov, *The Oscillating dark energy: Future singularity and coincidence problem*, Phys. Lett. B 637, 139 (2006).
Constraints on oscillating dark energy models, Phys. Lett. B 659, 14 (2008).

E. M. Barboza, Jr. and J. S. Alcaniz, A parametric model for dark energy, Phys. Lett. B 666, 415 (2008).

E. N. Saridakis, Theoretical Limits on the Equation-of-State Parameter of Phantom Cosmology, Phys. Lett. B 676, 7 (2009).

R. Lazkoz, V. Salzano and I. Sendra, Oscillations in the dark energy EoS: new MCMC lessons, Phys. Lett. B 694, 198 (2010).

J. Z. Ma and X. Zhang, Probing the dynamics of dark energy with novel parametrizations, Phys. Lett. B 699, 233 (2011).

H. Li and X. Zhang, Probing the dynamics of dark energy with divergence-free parametrizations: A global fit study, Phys. Lett. B 703, 119 (2011).

L. Feng and T. Lu, A new equation of state for dark energy model, JCAP 1111, 034 (2011).

I. Sendra and R. Lazkoz, SN and BAO constraints on (new) polynomial dark energy parametrizations: current results and forecasts, Mon. Not. Roy. Astron. Soc. 422, 776 (2012).

C. J. Feng, X. Y. Shen, P. Li and X. Z. Li, A New Class of Parametrization for Dark Energy without Divergence, JCAP 1209, 023 (2012).

E. Di Valentino, A. Melchiorri and J. Silk, Reconciling Planck with the local value of $H_0$ in extended parameter space, Phys. Lett. B 761, 242 (2016).

G. B. Zhao et al., Dynamical dark energy in light of the latest observations, Nat. Astron. 1, 627 (2017).

E. Di Valentino, A. Melchiorri, E. V. Linder and J. Silk, Constraining Dark Energy Dynamics in Extended Parameter Space, Phys. Rev. D 96, 023523 (2017).

E. Di Valentino, Crack in the cosmological paradigm, Nat. Astron. 1, 569 (2017).

W. Yang, R. C. Nunes, S. Pan and D. F. Mota, Effects of neutrino mass hierarchies on dynamical dark energy models, Phys. Rev. D 95, 103522 (2017).

M. Rezaei, M. Malekjani, S. Basilakos, A. Mehrabi and D. F. Mota, Constraints to Dark Energy Using PADE Parameterizations, Astrophys. J. 843, no. 1, 65 (2017).

R. J. F. Marcondes and S. Pan, Cosmic chronometers constraints on some fast-varying dark energy equations of state [arXiv:1711.06157 [astro-ph.CO]].

W. Yang, S. Pan and A. Paliathanasis, Latest cosmological constraints on some nonlinear parametric dark energy models, Mon. Not. Roy. Astron. Soc. 475, 2605 (2018).

M. Jaber and A. de la Macorra, Probing a Steep EoS for Dark Energy with latest observations, Astropart. Phys. 97, 130 (2018).

S. Pan, E. N. Saridakis and W. Yang, Observational Constraints on Oscillating Dark-Energy Parametrizations, Phys. Rev. D 98, no. 6, 063510 (2018).

S. Vagnozzi, S. Dhawan, M. Gerbino, K. Freese, A. Goo-bar and O. Mena, Constraints on the sum of the neutrino masses in dynamical dark energy models with $w(z) \geq -1$ are tighter than those obtained in $\Lambda$CDM, Phys. Rev. D 98, no. 8, 083501 (2018).

X. D. Li et al., Cosmological Constraints from the Redshift Dependence of the Alcock-Paczynski Effect: Dynamical Dark Energy, Astrophys. J. 856, no. 2, 88 (2018).

G. Panotopoulos and . Rincn, Growth index and statefinder diagnostic of Oscillating Dark Energy, Phys. Rev. D 97, no. 10, 103509 (2018).

L. G. Jaime, M. Jaber and C. Escamilla-Rivera, New parametrized equation of state for dark energy surveys, Phys. Rev. D 98, no. 8, 083530 (2018).

W. Yang, S. Pan, E. Di Valentino, E. N. Saridakis and S. Chakraborty, Observational constraints on one-parameter dynamical-dark-energy parametrizations and the $H_0$ tension, [arXiv:1810.05141 [astro-ph.CO]].

W. Yang, S. Pan, E. Di Valentino and E. N. Saridakis, Observational constraints on dynamical dark energy with pivoting redshift, [arXiv:1811.06932 [astro-ph.CO]].

F. Pace, C. Schindl, D. F. Mota and A. Del Popolo, Halo collapse: virtualization by shear and rotation in dynamical dark-energy models, [arXiv:1811.12105 [astro-ph.CO]].

M. Du, W. Yang, L. Xu, S. Pan and D. F. Mota, Future Constraints on Dynamical Dark-Energy using Gravitational-Wave Standard Sirens, [arXiv:1812.01440 [astro-ph.CO]].

D. Tamayo and J. A. Vazquez, Fourier series expansion of the dark energy equation of state, [arXiv:1901.08679 [astro-ph.CO]].

V. F. Mukhanov, H. A. Feldman and R. H. Brandenberger, Theory of cosmological perturbations, Phys. Rept. 215, 203 (1992).

C. P. Ma and E. Bertschinger, Cosmological perturbation theory in the synchronous and conformal Newtonian gauges, Astrophys. J. 455, 7 (1995).

K. A. Malik and D. Wands, Cosmological perturbations, Phys. Rept. 475, 1 (2009).

R. Adam et al. [Planck Collaboration], Planck 2015 results. I. Overview of products and scientific results, Astron. Astrophys. 594, A1 (2016).

N. Aghanim et al. [Planck Collaboration], Planck 2015 results. XI. CMB power spectra, likelihoods, and robustness of parameters, Astron. Astrophys. 594, A11 (2016).

M. Betoule et al. [SDSS Collaboration], Improved cosmological constraints from a joint analysis of the SDSS-II and SNLS supernova samples, Astron. Astrophys. 568, A22 (2014).

F. Beutler et al., The 6dF Galaxy Survey: Baryon Acoustic Oscillations and the Local Hubble Constant, Mon. Not. Roy. Astron. Soc. 416, 3017 (2011).

A. J. Ross, L. Samushia, C. Howlett, W. J. Percival, A. Burden and M. Manera, The clustering of the SDSS DR7 main Galaxy sample $\hat{a}E1$. A $4\%$ measurement of the Hubble parameter at $z = 0.15$, Mon. Not. Roy. Astron. Soc. 449, no. 1, 835 (2015).

H. Gil-Marín et al., The clustering of galaxies in the SDSS-III Baryon Oscillation Spectroscopic Survey: BAO measurement from the LOS-dependent power spectrum of DR12 BOSS galaxies, Mon. Not. Roy. Astron. Soc. 460, no. 4, 4210 (2016).

D. M. Scônic et al., The Complete Light-curve Sample of Spectroscopically Confirmed SNe Ia from Pan-STARRS1 and Cosmological Constraints from the Combined Pantheon Sample, Astrophys. J. 859, no. 2, 101 (2018).

M. Moresco et al., A 6% measurement of the Hubble parameter at $z \sim 0.45$: direct evidence of the epoch of cosmic re-acceleration, JCAP 1605, no. 05, 014 (2016).

A. Lewis and S. Bridle, Cosmological parameters from
CMB and other data: A Monte Carlo approach, Phys. Rev. D 66, 103511 (2002).

[55] A. Lewis, A. Challinor and A. Lasenby, Efficient computation of CMB anisotropies in closed FRW models, Astrophys. J. 538, 473 (2000).

[56] A. Gelman and D. Rubin, Inference from iterative simulation using multiple sequences, Statistical Science 7, 457 (1992).

[57] N. Aghanim et al. [Planck Collaboration], Planck 2018 results. VI. Cosmological parameters, [arXiv:1807.06209] [astro-ph.CO].

[58] A. Heavens, Y. Fantaye, A. Mootoovaloo, H. Eggers, Z. Hosenie, S. Kroon and E. Sellentin, Marginal Likelihoods from Monte Carlo Markov Chains, [arXiv:1704.03472] [stat.CO].

[59] A. Heavens, Y. Fantaye, E. Sellentin, H. Eggers, Z. Hosenie, S. Kroon and A. Mootoovaloo, No evidence for extensions to the standard cosmological model, Phys. Rev. Lett. 119, no. 10, 101301 (2017).

[60] R. E. Kass and A. E. Raftery, Bayes Factors, J. Am. Statist. Assoc. 90, no.430, 773 (1995).