Standard dilations of $q$-commuting tuples

SANTANU DEY

October 24, 2003

Abstract

Here we study dilations of $q$-commuting tuples. In [BBD] the authors gave the correspondence between the two standard dilations of commuting tuples and here these results have been extended to $q$-commuting tuples. We are able to do this when $q$-coefficients `$q_{ij}$' are of modulus one. We introduce ‘maximal $q$-commuting subspace’ of a $n$-tuple of operators and ‘standard $q$-commuting dilation’. Our main result is that the maximal $q$-commuting subspace of the standard noncommuting dilation of $q$-commuting tuple is the ‘standard $q$-commuting dilation’. We also introduce $q$-commuting Fock space as the maximal $q$-commuting subspace of full Fock space and give a formula for projection operator onto this space. This formula for projection helps us in working with the completely positive maps arising in our study. The first version of the Main Theorem (Theorem 19) of the paper for normal tuples using some tricky norm estimates and then use it to prove the general version of this theorem.

KEY WORDS: Dilation, $q$-Commuting Tuples, Complete Positivity

MATHEMATICS SUBJECT CLASSIFICATION: 47A45, 47A20
1. Introduction

A generalization of contraction operator in multivariate operator theory is a contractive $n$-tuple which is defined as follows:

**Definition 1.** A $n$-tuple $T = (T_1, \ldots, T_n)$ of bounded operators on a Hilbert space $H$ such that $T_1 T_1^* + \cdots + T_n T_n^* \leq I$ is a contractive $n$-tuple, or a row contraction.

Along the lines of [BBD], we will study the dilation of a class of operator tuples defined as follows:

**Definition 2.** A $n$-tuple $T = (T_1, \ldots, T_n)$ is said to be $q$-commuting if $T_j T_i = q_{ij} T_i T_j$ for all $1 \leq i < j \leq n$, where $q_{ij}$ are complex numbers.

Such operator tuples appear often in Quantum Theory ([C] [M] [Pr]). Here we introduce ‘maximal $q$-commuting piece’ and using this and a particular representation of permutation group we give a definition for $q$-commuting Fock space when $q$-coefficients ‘$q_{ij}$’ are of modulus one. We have this condition for $q$-coefficient for almost all the results here. This $q$-commuting Fock space is different from the twisted Fock space of M. Bożejko and R. Speicher ([BS1]) or that of P. E. T. Jorgensen ([JSW]). In section 2 we give formula for the projection of full Fock space onto this space. We obtain a special tuple of $q$-commuting operators and show that it is unitarily equivalent to the tuple of shift operators of [BB]. We are able to show that the range of the operator $A$ defined in equation (2.4) gives an isometry onto the $q$-commuting Fock Space tensored with a Hilbert space when $T$ is a pure contractive tuple (this operator was used by Popescu and Arveson in [Po3], [Po4], [Ar2] and for $q$-commuting case by Bhat and Bhattacharyya in [BB]). Using this we are able to give a condition equivalent to the assertion of the Main Theorem to hold for $q$-commuting purely contractive tuple. In section 3 the proof of the particular case of Theorem 19 where $T$ is also $q$-spherical unitary (introduced in section 3) is more difficult than the version for commuting tuple and we had to carefully choose the terms and proceed in a way that ‘$q_{ij}$’ of the $q$-commuting tuples get absorbed or cancel out when we simplify the terms. Also unlike [BBD] we had to use an inequality related to completely positive map before getting the result through norm estimates. We are not able to generalize section 4 of [BBD]. In the last section here we calculate the distribution of $S_i + S_i^*$ with respect to the vacuum expectation and study some properties of the related operator spaces.

For operator tuples $(T_1, \ldots, T_n)$, we need to consider the products of the form $T_{\alpha_1} T_{\alpha_2} \cdots T_{\alpha_m}$, where each $\alpha_k \in \{1, 2, \ldots, n\}$. We would have the following a notation for such products. Let $\Lambda$ denote the set $\{1, 2, \ldots, n\}$ and $\Lambda^m$ denote the $m$-fold cartesian product of $\Lambda$ for $m \geq 1$. Given $\alpha = (\alpha_1, \ldots, \alpha_m)$ in $\Lambda^m$, $T^{\alpha}$ will mean the operator $T_{\alpha_1} T_{\alpha_2} \cdots T_{\alpha_m}$. Let $\hat{\Lambda}$ denote $\bigcup_{n=0}^{\infty} \Lambda^n$, where $\Lambda^0$ is just the set $\{0\}$ by convention and by $T^{\emptyset}$ we would mean the identity operator of the Hilbert space where $T_i$'s are acting.

Let $S_m$ denote the group of permutation on $m$ symbols $\{1, 2, \ldots, m\}$. For a $q$-commuting tuple $T = (T_1, \ldots, T_n)$, consider the product $T_{x_1} T_{x_2} \cdots T_{x_m}$ where $1 \leq x_i \leq n$. If we replace a consecutive pair say $T_{x_i} T_{x_{i+1}}$ of operators in the above product by $q_{x_{i+1}x_i} T_{x_{i+1}} T_{x_i}$ and do finite number of such operations with different choices of consecutive pairs of these operators appearing in the subsequent product of operators after each such operation, we will get a permutation $\sigma \in S_m$ such
that the final product of operators can be written as $kT_{x_{\sigma}^{-1}(1)}T_{x_{\sigma}^{-1}(2)}...T_{x_{\sigma}^{-1}(m)}$ for some $k \in \mathbb{C}$, i.e., $T_{x_{1}}T_{x_{2}}...T_{x_{m}} = kT_{x_{\sigma}^{-1}(1)}T_{x_{\sigma}^{-1}(2)}...T_{x_{\sigma}^{-1}(m)}$. For defining $q$-commuting tuple in definition 2 we needed the known fact that this $k$ depends only on $\sigma$ and $x_{i}$, and not on the different choice of above operations that give rise to the same final product of operators $T_{x_{\sigma}^{-1}(1)}T_{x_{\sigma}^{-1}(2)}...T_{x_{\sigma}^{-1}(m)}$. It also follows from the Proposition 5 in section 2.

**Definition 3.** Let $\mathcal{H}, \mathcal{L}$ be two Hilbert spaces such that $\mathcal{H}$ be a closed subspace of $\mathcal{L}$ and let $T, R$ are $n$-tuples of bounded operators on $\mathcal{H}, \mathcal{L}$ respectively. Then $R$ is called a dilation of $T$ if

$$R_{i}^{*}u = T_{i}^{*}u$$

for all $u \in \mathcal{H}, 1 \leq i \leq n$. In such a case $T$ is called a piece of $R$. If $T$ is a $q$-commuting tuple (i.e., $T_{j}T_{i} = q_{ij}T_{i}T_{j}$, for all $i, j$), then it is called a $q$-commuting piece of $R$. A dilation $R$ of $T$ is said to be a minimal dilation if $\text{span}\{R^{\alpha}h : \alpha \in \check{\Lambda}, h \in \mathcal{H}\} = \mathcal{L}$. And if $R$ is a tuple of $n$ isometries and is a minimal dilation of $T$, then it is called the minimal isometric dilation or the standard noncommuting dilation of $T$.

A presentation of the standard noncommuting dilation taken from [Po1] has been used here to proof the main Theorem. All Hilbert spaces that we consider will be complex and separable. For a subspace $\mathcal{H}$ of a Hilbert space, $P_{\mathcal{H}}$ will denote the orthogonal projection onto $\mathcal{H}$. Standard noncommuting dilation of $n$-tuple of bounded operators, is unique upto unitary equivalence (refer [Po1-4]). Extensive study of standard noncommuting dilation was carried out by Popescu. He generalized many one variable results to multivariable case. It is easy to see that if $R$ is a dilation of $T$ then

$$(1.1) \quad T^{\alpha}(\overline{T^{\beta}})^{*} = P_{\mathcal{H}}R^{\alpha}(R^{\beta})^{*}|_{\mathcal{H}},$$

and for any polynomials $p, q$ in $n$-noncommuting variables

$$p(T)(q(T))^{*} = P_{\mathcal{H}}p(R)(q(R))^{*}|_{\mathcal{H}}.$$
\( e_{\alpha_1} \otimes e_{\alpha_2} \otimes \cdots \otimes e_{\alpha_n} \) in the full Fock space \( \Gamma(\mathbb{C}^n) \) and \( e^0 \) will denote the vacuum vector \( \omega \). Then the (left) creation operators \( V_i \) on \( \Gamma(\mathbb{C}^n) \) are defined by

\[
V_i x = e_i \otimes x
\]

where \( 1 \leq i \leq n \) and \( x \in \Gamma(\mathbb{C}^n) \) (here \( e_i \otimes \omega \) is interpreted as \( e_i \)). It is obvious that the tuple \( \mathbf{V} = (V_1, \ldots, V_n) \) consists of isometries with orthogonal ranges and \( \sum V_i V_i^* = I - I_0 \), where \( I_0 \) is the projection onto the vacuum space. Let us define \( q \)-commuting Fock space as the subspace \( (\Gamma(\mathbb{C}^n))^q(\mathbf{V}) \) and let it be denoted by \( \Gamma_q(\mathbb{C}^n) \).

Let \( \mathcal{S} = (S_1, \ldots, S_n) \) be the tuple of operators on \( \Gamma_q(\mathbb{C}^n) \) where \( S_i \) is the compression of \( V_i \) to \( \Gamma_q(\mathbb{C}^n) \):

\[
S_i = P_{\Gamma_q(\mathbb{C}^n)} V_i |_{\Gamma_q(\mathbb{C}^n)}.
\]

Clearly each \( V_i^* \) leaves \( \Gamma_q(\mathbb{C}^n) \) invariant.

Then it is easy to see that \( \mathcal{S} \) satisfies \( \sum S_i S_i^* = I^q - I_0^q \) (where \( I^q, I_0^q \) are identity, projection onto vacuum space respectively in \( \Gamma_q(\mathbb{C}^n) \)). So \( \mathbf{V} \) and \( \mathcal{S} \) are contractive tuples, \( S_j S_i = q_{ij} S_i S_j \) for all \( 1 \leq i, j \leq n \), and \( S_i^* x = V_i^* x \) for \( x \in \Gamma_q(\mathbb{C}^n) \).

The following result gives a description for maximal \( q \)-commuting piece.

**Proposition 5.** Let \( R = (R_1, \ldots, R_n) \) be a \( n \)-tuple of bounded operators on a Hilbert space \( \mathcal{M} \), \( \mathcal{K}_{ij} = \overline{\text{span}} \{ R^\alpha_i (q_{ij} R_i R_j - R_j R_i) h : h \in \mathcal{M}, \alpha \in \Lambda \} \) for all \( 1 \leq i, j \leq n \), and \( \mathcal{K} = \overline{\text{span}} \{ \cup_{i,j=1}^{n} \mathcal{K}_{ij} \} \). Then \( \mathcal{M}^q(\mathbf{R}) = \mathcal{K}^\perp \) and \( \mathcal{M}^q(\mathbf{R}) = \{ h \in \mathcal{M} : (\mathcal{K}_{ij} R_j^* R_i^* - R_i^* R_j^*) (R^\alpha_i)^* h = 0, \forall 1 \leq i, j \leq n, \alpha \in \Lambda \} \).

The above Proposition can be easily proved using argument similar to the proof of Proposition 4 of [BBD].

**Corollary 6.** Suppose \( \mathbf{R}, \mathbf{T} \) are \( n \)-tuples of operators on two Hilbert spaces \( \mathcal{L}, \mathcal{M} \). Then the maximal \( q \)-commuting piece of \( (R_1 \oplus T_1, \ldots, R_n \oplus T_n) \) acting on \( \mathcal{L} \oplus \mathcal{M} \) is \( (R_1^q \oplus T_1^q, \ldots, R_n^q \oplus T_n^q) \) acting on \( \mathcal{L}^q \oplus \mathcal{M}^q \) and the maximal \( q \)-commuting piece of \( (R_1 \otimes I, \ldots, R_n \otimes I) \) acting on \( \mathcal{L} \otimes \mathcal{M} \) is \( (R_1^q \otimes I, \ldots, R_n^q \otimes I) \) acting on \( \mathcal{L}^q \otimes \mathcal{M} \).

**Proof:** Clear from Proposition 6. \( \square \)

**Proposition 7.** Let \( \mathbf{T}, \mathbf{R} \) are \( n \)-tuples of bounded operators on \( \mathcal{H}, \mathcal{L} \), with \( \mathcal{H} \subseteq \mathcal{L} \), such that \( \mathbf{R} \) is a dilation of \( \mathbf{T} \). Then \( \mathcal{H}^q(\mathbf{T}) = \mathcal{L}^q(\mathbf{R}) \cap \mathcal{H} \) and \( \mathbf{R}^q \) is a dilation of \( \mathbf{T}^q \).

**Proof:** This can be using arguments similar to proof of Proposition 7 of [BBD]. \( \square \)

2. A \( q \)-Commuting Fock Space

For a \( q \)-commuting \( n \)-tuple \( \mathbf{T} \) on a finite dimensional Hilbert space \( \mathcal{H} \) say of dimension \( m \), because of the relation

\[
\text{Spectrum}(T_i T_j) \cup \{0\} = \text{Spectrum}(T_j T_i) \cup \{0\} = \text{Spectrum}(q_{ij} T_i T_j) \cup \{0\},
\]

we get \( q_{ij} \) is either 0 or \( m^{th} \)-root of unity.

Here after whenever we deal with \( q \)-commuting tuples we would have another condition on the tuples that \( |q_{ij}| = 1 \) for \( 1 \leq i, j \leq n \). However Proposition 5, Proposition 6 and Corollary 7 does not need this assumption. Let \( \mathbf{T} = (T_1, \ldots, T_n) \) be a \( q \)-commuting tuple and consider the product \( T_{x_1} T_{x_2} \cdots T_{x_m} \) where \( 1 \leq x_i \leq n \). Let \( \sigma \in S_m \). As transpositions of the type
Let $\sigma^{-1}$ be $\tau_1 \ldots \tau_s$ where for each $1 \leq i \leq s$ there exist $k_i$ such that $1 \leq k_i \leq m - 1$ and $\tau_i$ is a transposition of the form $(k_i, k_i + 1)$. Let $\tilde{\sigma}_i = \tau_{i+1} \tau_i \ldots \tau_s$ for $1 \leq i \leq s - 1$ and $\tilde{\sigma}_s$ be the identity permutation. Let us define $y_i = x_{\sigma_i(k_i)}$ and $z_i = x_{\tilde{\sigma}_i(k_i+1)}$. If we substitute $T_{y_i}T_{z_i}$ by $q_{z_iy_i}T_{z_i}T_{y_i}$ corresponding to $\tau_s$, substitute $T_{y_{s-1}}T_{z_{s-1}}$ by $q_{z_{s-1}y_{s-1}}T_{z_{s-1}}T_{y_{s-1}}$ corresponding to $\tau_{s-1}$, and so on till we substitute the corresponding term for $\tau_1$, we would get $q_1^\sigma(x) \ldots q_s^\sigma(x)T_{x_{\sigma_1(1)}}T_{x_{\sigma_1(2)}} \ldots T_{x_{\sigma_1(m)}}$ where $q_i^\sigma(x) = q_{z_iy_i}$. That is $T_{x_1}T_{x_2} \cdots T_{x_m} = q_1^\sigma(x) \ldots q_s^\sigma(x)T_{x_{\sigma_1(1)}}T_{x_{\sigma_1(2)}} \ldots T_{x_{\sigma_1(m)}}$. Let $q^\sigma(x) = q_1^\sigma(x) \ldots q_s^\sigma(x)$ where $q_i^\sigma(x) = q_{z_iy_i}$.

**Proposition 8.** Let $\mathcal{T} = (T_1, \ldots, T_n)$ be a $q$-commuting tuple and consider the product $T_{x_1}T_{x_2} \cdots T_{x_m}$ where $1 \leq x_i \leq n$. Suppose $\sigma \in \mathcal{S}_m$ and $q^\sigma(x)$ be as defined above. Then

$$q^\sigma(x) = \prod q_{x_{\sigma^{-1}(i)}}x_{\sigma^{-1}(i)},$$

where product is over $\{(i, k) : 1 \leq i < k \leq m, \sigma^{-1}(i) > \sigma^{-1}(k)\}$. Moreover $q^\sigma(x)$ does not depend on the choice of $\sigma$.

**Proof:** We have

$$q^\sigma(x) = q_1^\sigma(x) \ldots q_s^\sigma(x)$$

where $q_i^\sigma(x) = q_{z_iy_i}$. For a pair $i, k$ such that $1 \leq i < k \leq m$ let $k' = \sigma^{-1}(k)$ and $i' = \sigma^{-1}(i)$. Let $\sigma = \tau_1 \cdots \tau_s$ and $\tilde{\sigma}_i$ be as defined above. If $i' > k'$ then there are odd number of transpositions $\tau_r$ for $1 \leq r \leq m$ such that they interchange the positions of $i'$ and $k'$ in the image of $\tilde{\sigma}_r$ when we consider the composition $\tau_r \tilde{\sigma}_r$. And for $1 \leq i < k \leq m$ if $i' < k'$ then there are even number of transpositions $\tau_r$ for $1 \leq r \leq m$ such that they interchange the positions of $i'$ and $k'$ in the image of $\tilde{\sigma}_r$ when we consider the composition $\tau_r \tilde{\sigma}_r$. For the first transposition in $\tau_r$ that interchanges $i'$ and $k'$, the corresponding factor in $q^\sigma(x)$ say $q_{x_{i'}}^\sigma(x)$ is $q_{z_{i'}x_{i'}}$, for the second transposition that interchanges $i'$ and $k'$, the corresponding factor is $q_{x_{i'}x_{k'}}$, for the third transposition that interchanges $i'$ and $k'$, the corresponding factor is $q_{x_{i'}x_{i'}}$, and so on. But $(q_{x_{i'}x_{k'}})^{-1} = q_{x_{k'}x_{i'}}$ and so

$$q^\sigma(x) = \prod q_{x_{\sigma^{-1}(i)}}x_{\sigma^{-1}(i)},$$

where product is over $\{(i, k) : 1 \leq i < k \leq m, \sigma^{-1}(i) > \sigma^{-1}(k)\}$. \(\square\)

Following similar arguments it is easy to see that if there exist $\sigma \in \mathcal{S}_m$ such that $(x_1, \ldots, x_n) = (x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(m)})$, then $q^\sigma(x) = 1$.

Let $U_{\sigma}^{m,q}$ be defined on $(\mathbb{C}^n)^{\otimes m}$ by

$$U_{\sigma}^{m,q}(e_{x_1} \otimes \ldots \otimes e_{x_m}) = q^\sigma(x)e_{x_{\sigma^{-1}(1)}} \otimes \ldots \otimes e_{x_{\sigma^{-1}(m)}}$$

on the standard basis vectors and extended linearly on $(\mathbb{C}^n)^{\otimes m}$. As $|q_{ij}| = 1$ for $1 \leq i, j \leq n$, $U_{\sigma}^{m}$ is unitary and $U_{\sigma}^{m}$ extends uniquely to a unitary operator on $(\mathbb{C}^n)^{\otimes m}$.

Let

$$(\mathbb{C}^n)^{\otimes m} = \{u \in (\mathbb{C}^n)^{\otimes m} : U_{\sigma}^{m,q}u = u \ \forall \sigma \in \mathcal{S}_m\}$$

and $(\mathbb{C}^n)^{\otimes m} = \mathbb{C}$

**Lemma 9.** The map defined from $\mathcal{S}_m$ to $B((\mathbb{C}^n)^{\otimes m})$ defined by $\sigma \rightarrow U_{\sigma}^{m,q}$ for all $\sigma \in \mathcal{S}_m$ is a representation.
**Proof:** Let $\otimes_{i=1}^{m} e_{x_i}, \otimes_{i=1}^{m} e_{y_i} \in (\mathbb{C}^n)^{\otimes m}$, $1 \leq x_i, y_i \leq n$. Suppose there exist $\sigma \in S_m$ such that $\otimes_{i=1}^{m} e_{y_i} = \otimes_{i=1}^{m} e_{x_{\sigma^{-1}(i)}}$. Then $\langle U_{\sigma}^{m,q}(\otimes_{i=1}^{m} e_{x_i}), \otimes_{i=1}^{m} e_{y_i} \rangle = q^2(x)$ and $\langle \otimes_{i=1}^{m} e_{x_i}, U_{\sigma^{-1}}^{m,q}(\otimes_{i=1}^{m} e_{y_i}) \rangle = \overline{q(\sigma^{-1})(y)}$. Also,

$$q^{(\sigma^{-1})}(y) = \prod q_{y_{\sigma(k)}} y_{\sigma(i)} = \prod q_{x_{\sigma^{-1}(i)}}$$

where the products are over $\{ (i, k) : 1 \leq i < k \leq m, \sigma(i) > \sigma(k) \}$. If we substitute $k = \sigma^{-1}(i')$ and $i = \sigma^{-1}(k')$ in the last term we get

$$q^{(\sigma^{-1})}(y) = \prod q_{x_{\sigma^{-1}(i')}} y_{\sigma^{-1}(k')} = \left( \prod q_{x_{\sigma^{-1}(k')}} y_{\sigma^{-1}(i')} \right)^{-1} = (q^\sigma(x))^{-1}$$

where the products are over $\{ (i', k') : 1 \leq i' < k' \leq m, \sigma^{-1}(i') > \sigma^{-1}(k') \}$. So

$$q^\sigma(x) = (q^{(\sigma^{-1})}(y))^{-1} = \overline{q^{(\sigma^{-1})}(y)}.$$ 

The last equality holds as $|q_{ij}| = 1$. This implies $\langle U_{\sigma}^{m,q}(\otimes_{i=1}^{m} e_{x_i}), \otimes_{i=1}^{m} e_{y_i} \rangle = \langle \otimes_{i=1}^{m} e_{x_i}, U_{\sigma^{-1}}^{m,q}(\otimes_{i=1}^{m} e_{y_i}) \rangle$. If there does not exist any $\sigma \in S_m$ such that $\otimes_{i=1}^{m} e_{y_i} = \otimes_{i=1}^{m} e_{x_{\sigma^{-1}(i)}}$ then

$$\langle U_{\sigma}^{m,q}(\otimes_{i=1}^{m} e_{x_i}), \otimes_{i=1}^{m} e_{y_i} \rangle = 0 = \langle \otimes_{i=1}^{m} e_{x_i}, U_{\sigma^{-1}}^{m,q}(\otimes_{i=1}^{m} e_{y_i}) \rangle$$

for all $\sigma' \in S_m$. So $(U_{\sigma}^{m,q})^* = U_{\sigma^{-1}}^{m,q}$ for $\sigma \in S_m$, when acting on the basis elements of $(\mathbb{C}^n)^{\otimes m}$, and hence is true for all elements $(\mathbb{C}^n)^{\otimes m}$.

Next let $\sigma \in S_m$ be equal to $\sigma_1 \sigma_2$ for some $\sigma_1, \sigma_2 \in S_m$. We would show that $U_{\sigma}^{m,q} = U_{\sigma_1}^{m,q} U_{\sigma_2}^{m,q}$. Let $e_x = e_{x_1} \otimes \ldots \otimes e_{x_m}$ where $x_j \in \{1, \ldots, n\}$ for $1 \leq j \leq m$. Let $\sigma_1^{-1} = \tau_1 \ldots \tau_r$ and $\sigma_2^{-1} = \tau_{r+1} \ldots \tau_s$ where for each $1 \leq i \leq s$, there exist $k_i$ such that $1 \leq k_i \leq m - 1$ and $\tau_i$ is a transposition of the form $(k_i, k_i + 1)$.

$$U_{\sigma_1}^{m,q} U_{\sigma_2}^{m,q}(e_{x_1} \otimes \ldots \otimes e_{x_m}) = U_{\sigma_1}^{m,q}(q^{\sigma_2}(x)e_{x_{\sigma_2^{-1}(1)}} \otimes \ldots \otimes e_{x_{\sigma_2^{-1}(m)}}) = q^{\sigma_1}(z)q^{\sigma_2}(x)e_{x_{\sigma_1^{-1}(1)}} \otimes \ldots \otimes e_{x_{\sigma_1^{-1}(m)}}$$

where $e_z = e_{z_1} \otimes \ldots \otimes e_{z_m}$, i.e., $z_i = x_{\sigma_2^{-1}(i)}$. But as $\sigma = \tau_1 \ldots \tau_r \tau_{r+1} \ldots \tau_s$ it is easy to see that $q^{\sigma}(x) = q^{\sigma_1}(z)q^{\sigma_2}(x)$ using the definition of $q^{2}(x)$. So we get

$$U_{\sigma_1}^{m,q} U_{\sigma_2}^{m,q}(e_{x_1} \otimes \ldots \otimes e_{x_m}) = q^{\sigma}(x)e_{x_{\sigma^{-1}(1)}} \otimes \ldots \otimes e_{x_{\sigma^{-1}(m)}} = U_{\sigma}^{m,q}(e_{x_1} \otimes \ldots \otimes e_{x_m}).$$

And hence $U_{\sigma_1}^{m,q} U_{\sigma_2}^{m,q} = U_{\sigma}^{m,q}$. 

Now if we use $Y_{\sigma}^{m,q}$ also to denote a operator in $\Gamma(\mathbb{C}^n)$ which acts as $U_{\sigma}^{m,q}$ on $(\mathbb{C}^n)^{\otimes m}$ and $I$ on the orthogonal, we get a representation of $S_m$ on $B(\Gamma(\mathbb{C}^n))$. In the next Lemma and Proposition we derive a formula for the projection operator onto the $q$-commuting Fock space.

**Lemma 10.** Let $P_m$ be a operator on $(\mathbb{C}^n)^{\otimes m}$ defined by

$$P_m = \frac{1}{m!} \sum_{\sigma \in S_m} U_{\sigma}^{m,q}.$$ 

Then $P_m$ is a projection of $(\mathbb{C}^n)^{\otimes m}$ onto $(\mathbb{C}^n)^{\otimes m}$.

**Proof:** First we see that

$$P_m^* = \frac{1}{m!} \sum_{\sigma \in S_m} (U_{\sigma}^{m,q})^* = \frac{1}{m!} \sum_{\sigma \in S_m} U_{\sigma^{-1}}^{m,q} = P_m.$$
Consider a permutation \( \sigma' \in S_m \).
\[
P_m U_{\sigma'}^{m,q} = \frac{1}{m!} \sum_{\sigma \in S_m} U_{\sigma}^{m,q} = \frac{1}{m!} \sum_{\sigma \in S_m} U^{m,q}_{\sigma} = P_m.
\]
Similarly \( U_{\sigma'}^{m,q} P_m = P_m \). So \( P_m^2 = P_m \) and hence \( P_m \) is a projection.

**Proposition 11.** \( \oplus_{m=0}^{\infty} (\mathbb{C}^n)^{\otimes m} = \Gamma_q (\mathbb{C}^n) \)

**Proof:** Let \( Q = \oplus_{m=0}^{\infty} P_m \) be the projection of \( \Gamma (\mathbb{C}^n) \) onto \( \oplus_{m=0}^{\infty} (\mathbb{C}^n)^{\otimes m} \) where \( P_m \) is a defined in Lemma 9. For transposition \( (1, 2) \), let us define \( U_{(1, 2)}^{\alpha} \) as \( \oplus_{m=0}^{\infty} U_{(1, 2)}^{m,q} \) where \( U_{(1, 2)}^{0,q} = I \) and \( U_{(1, 2)}^{1,q} = I \).

Let \( \otimes_{i=1}^{k} e_i \in (\mathbb{C}^n)^{\otimes k}, 1 \leq x_i \leq n \). Then
\[
U_{(1, 2)}^{m,q} V_i \otimes_{j=1}^{m} e_{x_j} = U_{(1, 2)}^{m,q} \{ e_j \otimes e_i \otimes (\otimes_{i=1}^{k} e_{x_i}) \} = q_{ij} e_i \otimes e_j \otimes (\otimes_{i=1}^{k} e_{x_i}) = q_{ij} V_i \otimes_{j=1}^{m} e_{x_j}.
\]

Next we would show that \( \oplus_{m=0}^{\infty} (\mathbb{C}^n)^{\otimes m} \) is left invariant by \( V_i^* \). Let \( \otimes_{j=1}^{m} e_{x_j} \in (\mathbb{C}^n)^{\otimes m}, 1 \leq x_j \leq n \). Then \( V_i^* \{ P_m (\otimes_{j=1}^{m} e_{x_j}) \} \) is zero if none of \( x_j \) is equal to \( i \). Otherwise \( V_i^* \{ P_m (\otimes_{j=1}^{m} e_{x_j}) \} \) is some non-zero element belonging to \( \oplus_{m=0}^{\infty} (\mathbb{C}^n)^{\otimes (m-1)} \) because of the following. Let \( x_j = i \) iff \( j \in \{ i_1, ..., i_p \} \), and let \( A_k \) be the set of all \( \sigma \in S_m \) such that \( \sigma^{-1} \) sends 1 to \( i_k, 1 \leq k \leq p \), then each element of \( A_k \) is a composition \( \tau \sigma' \) where \( \tau \) is the transposition \( (1, i_k) \) and a permutation \( \sigma' \) for which \( (\sigma')^{-1} \) keeps 1 fixed and permutes rest of the \( m-1 \) symbols. As \( V_i \) are isometries with orthogonal ranges,
\[
V_i^* \{ P_m (\otimes_{j=1}^{m} e_{x_j}) \} = V_i^* \{ \frac{1}{m!} \sum_{\sigma \in S_m} U_{\sigma}^{m,q} (\otimes_{j=1}^{m} e_{x_j}) \} = \frac{1}{m!} \sum_{k=1}^{p} V_i^* \{ \sum_{\sigma \in A_k} U_{\sigma}^{m,q} (\otimes_{j=1}^{m} e_{x_j}) \}
\]
\[
= \frac{1}{m!} \sum_{k=1}^{p} V_i^* \{ \sum_{\sigma' \in A_k} U_{\tau}^{m,q} U_{\sigma'}^{m,q} (\otimes_{j=1}^{m} e_{x_j}) \}
\]
\[
= \sum_{k=1}^{p} a_k(x) P_{m-1} (\otimes_{j=1}^{m} e_{x_1} \otimes \cdots \hat{e}_{x_{i_k}} \otimes \cdots \otimes e_{x_m})
\]
where \( a_k(x) \) are constants and \( \hat{e}_{x_{i_k}} \) denotes the term \( e_{x_1} \otimes \cdots \otimes e_{x_{p-1}} \otimes e_{x_{p+1}} \otimes \cdots \otimes e_{x_m} \). This shows that \( \oplus_{m=0}^{\infty} (\mathbb{C}^n)^{\otimes m} \) is left invariant by \( V_i^* \).

Using these and the results of Lemma 9 we have the following. Taking \( R_i = Q V_i Q \) for \( \alpha \in \Lambda^m \) we get
\[
R_i R_j R_{\alpha} \omega = Q V_i V_j V_{\alpha} \omega = Q U_{(1, 2)}^{m+2,q} q_{ji} V_j V_j V_{\alpha} \omega = q_{ji} Q V_j V_j V_{\alpha} \omega = q_{ji} (R_j)(R_i) R_{\alpha} \omega.
\]
So \( (Q V_i Q, ..., Q V_n Q) \) is a \( q \)-commuting piece of \( V \). To show maximality we make use of Proposition 6. Suppose \( x \in \Gamma (\mathbb{C}^n) \) and \( \langle x, V_{\alpha} (q_{ji} V_j V_j - V_j V_j) y \rangle = 0 \) for all \( \alpha \in \Lambda, 1 \leq i, j \leq n \) and \( y \in \Gamma (\mathbb{C}^n) \). Suppose \( x_m \) is the \( m \)-particle component of \( x \), i.e., \( x = \oplus_{m=0}^{\infty} x_m \) with \( x_m \in (\mathbb{C}^n)^{\otimes m} \) for \( m \geq 0 \). For \( m \geq 2 \) and any permutation \( \sigma \) of \( \{ 1, 2, ..., m \} \) we need to show that the unitary \( U_{\sigma}^{m,q} : (\mathbb{C}^n)^{\otimes m} \rightarrow (\mathbb{C}^n)^{\otimes m} \), defined by equation (2.1) leaves \( x_m \) fixed. Since \( S_m \) is generated by the set of transpositions \( \{ (1, 2), ..., (m-1, m) \} \) it is enough to verify \( U_{\sigma}^{m,q} (x_m) = x_m \) for permutations \( \sigma \) of the form \( (i, i + 1) \). So fix \( m \) and \( i \) with \( m \geq 2 \) and \( 1 \leq i \leq (m - 1) \). We have
\[
\langle \oplus_{p=0}^{x_m}, V_{\alpha}^\beta (q_{ki} V_k V_i - V_i V_k) V_{\alpha}^\beta \omega \rangle = 0,
\]
(2.3)
for every $\beta \in \tilde{\Lambda}$, $1 \leq k, l \leq n$. This implies that
\[
\langle x_m, e^\alpha \otimes (q_{kl} e_k \otimes e_t - e_t \otimes e_k) \otimes e^\beta \rangle = 0
\]
for any $\alpha \in \Lambda^{i-1}, \beta \in \Lambda^{m-i-1}$. So if
\[
x_m = \sum a(s, t, \alpha, \beta)e^\alpha \otimes e_s \otimes e_t \otimes e^\beta
\]
where the sum is over $\alpha \in \Lambda^{i-1}, \beta \in \Lambda^{m-i-1}$ and $1 \leq s, t \leq n$, and $a(s, t, \alpha, \beta)$ are constants, then for fixed $\alpha$ and $\beta$ it follows from equation (2.3) that $q_{kl} a(k, l, \alpha, \beta) = a(l, k, \alpha, \beta)$ or $q_{kl} a(k, l, \alpha, \beta) = a(l, k, \alpha, \beta)$.

Note that the following inner-product is also referred in [BB] in Definition (1.1) in general case. Now let $\mathcal{P}$ be the vector space of all polynomials in $q$-commuting variables $z_1, \ldots, z_n$ that is $z_j z_i = q_{ij} z_i z_j$. Any multi-index $\underline{k}$ is a ordered $n$-tuple of non-negative integers $(k_1, \ldots, k_n)$. We shall write $k_1 + \ldots + k_n$ as $|\underline{k}|$. The special multi-index which has 0 in all positions except the $i$th one, where it has 1, is denoted by $\underline{e}_i$. For any non-zero multi-index $\underline{k}$ the monomial $z_1^{k_1} \ldots z_n^{k_n}$ will be denoted by $z^{\underline{k}}$ and for the multi-index $\underline{k} = (0, \ldots, 0)$, let $z^0$ be the complex number 1. Let us have the following inner product with it. Declare $z^{\underline{k}}$ and $z^0$ orthogonal if $\underline{k}$ is not the same as $\underline{l}$ as ordered multi-indices. Let
\[
\|z^{\underline{k}}\|^2 = \frac{k_1! \cdots k_n!}{|\underline{k}|!}.
\]
Note that the following inner product is also referred in [BB] in Definition (1.1) in general case. Now define $\mathcal{H}'$ to be the closure of $\mathcal{P}$ with respect to this inner product. Define a tuple $\underline{S}' = (S'_1, \ldots, S'_n)$ where each $S'_i$ is defined for $f \in \mathcal{P}$ by
\[
S'_i f(z_1, \ldots, z_n) = z_i f(z_1, \ldots, z_n)
\]
and $S_i$ is linearly extended to $\mathcal{H}'$. In the case of our standard $q$-commuting $n$-tuple $\underline{S}$ of operators on $\Gamma_q(\mathbb{C}^n)$, when $\underline{k} = (k_1, \ldots, k_n)$ let $\underline{S}^\underline{k} = S_1^{k_1} \cdots S_n^{k_n}$ and when $\underline{k} = (0, \ldots, 0)$ let $\underline{S}^\underline{k} = 1$.

Using (2.2) and the fact that $V_i$’s are isometries with orthogonal ranges for $\underline{k} = (k_1, \ldots, k_n), |\underline{k}| = m$ we get
\[
\|\underline{S}^\underline{k}\omega\|^2 = \langle P_m V_{\underline{k}}^\omega V_{\underline{k}}^* \omega, V_{\underline{k}}^\omega \rangle = \langle \frac{1}{|\underline{k}|!} \sum_{\sigma \in S_m} U_{\sigma}^m, Q^m V_{\underline{k}}^\omega, V_{\underline{k}}^\omega \rangle = \frac{k_1! \cdots k_n!}{|\underline{k}|!}.
\]

If we denote $\underline{V}^\underline{k} \omega$ by $e_{x_1} \otimes \cdots \otimes e_{x_m}, 1 \leq x_i \leq n$, then to get the last term of the above equation we used the fact that there are $k_1! \cdots k_n!$ permutations $\sigma \in S_m$ such that $e_{x_1} \otimes \cdots \otimes e_{x_m} = e_{x_{\sigma^{-1}(1)}} \otimes \cdots \otimes e_{x_{\sigma^{-1}(m)}}$. Next we show that the above tuples $\underline{S}'$ and $\underline{S}$ are unitarily equivalent.

**Proposition 12.** Let $\underline{S}' = (S'_1, \ldots, S'_n)$ be the operator tuples on $\mathcal{H}'$ as introduced above and let $\underline{S} = (S_1, \ldots, S_n)$ be the standard $q$-commuting tuple of operators on $\Gamma_q(\mathbb{C}^n)$. Then there exist unitary $U : \mathcal{H}' \to \mathcal{H}$ such that $US'_i = S_i U$ for $1 \leq i \leq n$. 
Proof: Define \( U : \mathcal{P} \to \Gamma_q(\mathbb{C}^n) \) as
\[
U \left( \sum_{|k| \leq s} b_k z^k \right) = \sum_{|k| \leq s} b_k S^k \omega
\]
where \( b_k z^k \in \mathcal{P}, b_k \) are constants. As \( \|z^k\| = \|S^k \omega\| \) we have
\[
\| \sum_{|k| \leq s} b_k z^k \| = \sum_{|k| \leq s} |b_k|^2 \|z^k\|^2 = \sum_{|k| \leq s} |b_k|^2 \|S^k \omega\|^2 = \| \sum_{|k| \leq s} b_k S^k \omega \|^2.
\]
So we can extend it linearly to \( \mathcal{H}' \) and \( U \) is a unitary.
\[
US'_i \left( \sum_{|k| \leq s} b_k z^k \right) = U(z_i \sum_{|k| \leq s} b_k z^k) = q_i^{k_{i-1}} \cdots q_i^{k_{i}} U \left( \sum_{|k| \leq s} b_k z^{k+i} \right) = q_i^{k_{i-1}} \cdots q_i^{k_{i}} \sum_{|k| \leq s} b_k S^k \omega = S_i \left( \sum_{|k| \leq s} b_k S^k \omega \right) = S_i U \left( \sum_{|k| \leq s} b_k z^k \right),
\]
i. e., \( US'_i = S_i U \) for \( 1 \leq i \leq n \). \( \square \)

For any complex number \( z \), the \( z \)-commutator of two operators \( A, B \) is defined as:
\[
[A, B]_z = AB - zBA.
\]

The following Lemma holds for \( S \) as \( S' \) and \( S \) are unitarily equivalent and the same properties have been proved for \( S' \) in [BB].

**Lemma 13.** (1) Each monomial \( S_k \omega \) is an eigenvector for \( S_i^* S_i - I \), so that it is a diagonal operator on the standard basis. In fact,
\[
\sum_{i=1}^{n} S_i^* S_i \left( \frac{S_k \omega}{\|S_k \omega\|^2} \right) = \left( \sum_{i=1}^{n} \frac{\|S_k \omega\|^2}{\|S_k \omega\|^2} \right) S_k \omega.
\]
Also \( \sum S_i^* S_i - I \) is compact.
(2) The commutator \([S_i^*, S_i]\) is as follows:
\[
[S_i^*, S_i] S_k \omega = \left( \frac{\|S_k \omega\|^2}{\|S_k \omega\|^2} - \frac{\|S_k \omega\|^2}{\|S_k \omega\|^2} \right) S_k \omega, \text{ when } k_i \neq 0.
\]
If \( k_i = 0 \), then \( [S_i^*, S_i] S_k \omega = S_i^* S_k \omega = \frac{\|S_k \omega\|^2}{\|S_k \omega\|^2} S_k \omega. \)
(3) \([S_i^*, S_j] \) is compact for all \( 1 \leq i, j \leq n \).

The map \( U^{m,q} : S_m \to \Gamma(\mathbb{C}^n) \) given by
\[
U^{m,q}(\sigma) = U^{m,q}_\sigma
\]
gives the representation of \( S_m \) on \( \Gamma(\mathbb{C}^n) \). It is easy to see that for all \( q = (q_{ij})_{n \times n}, |q_{ij}| = 1 \), the representations are isomorphic or similar by checking the characters of the representations. They
have same characters. But for the representations of permutation groups it follows that they are unitarily equivalent representations. So there exist unitary \( W^q : \Gamma(\mathbb{C}^n) \to \Gamma(\mathbb{C}^n) \) such that
\[
(2.4) \quad W^q P_{\Gamma_S(\mathbb{C}^n)} = P_{\Gamma_S(\mathbb{C}^n)} W^q.
\]
This \( W^q \) is not unique as for \( k \in \mathbb{C} \) such that \( |k| = 1 \), the operator \( kW^q \) is also a unitary which satisfy equation (2.4). We will give one such \( W^q \) explicitly.

For \( m \in \mathbb{N} \), \( y_i \in \Lambda \) define \( W^{m,q} \) over \( (\mathbb{C}^m)^\otimes^m \) as
\[
W^{m,q}(e_{y_1} \otimes \ldots \otimes e_{y_m}) = q^{\sigma^{-1}}(x)e_{y_1} \otimes \ldots \otimes e_{y_m}.
\]
where \( x = (x_1, \ldots, x_m) \) is the tuple got by rearranging \( (y_1, \ldots, y_m) \) in nondecreasing order and \( \sigma \in S_m \) such that \( y_i = x_{\sigma(i)} \). From Proposition 8 it’s clear that \( q^{\sigma^{-1}}(x) \) does not depend upon the choice of \( \sigma \). And
\[
W^{m,q} P_{\Gamma_S(\mathbb{C}^n)}(e_{y_1} \otimes \ldots \otimes e_{y_m}) = W^{m,q}\left( \frac{1}{m!} \sum_{\tau \in S_m} e_{y_{\tau^{-1}(1)}} \otimes \ldots \otimes e_{y_{\tau^{-1}(m)}} \right)
\]
\[
= \frac{1}{m!} \sum_{\tau \in S_m} q^{(\tau^{-1})^{-1}}(x)e_{y_{\tau^{-1}(1)}} \otimes \ldots \otimes e_{y_{\tau^{-1}(m)}}
\]
\[
= \frac{1}{m!} \sum_{\tau \in S_m} q^{\sigma^{-1}}(x)e_{y_{\tau^{-1}(1)}} \otimes \ldots \otimes e_{y_{\tau^{-1}(m)}}
\]
\[
= \frac{1}{m!} \sum_{\tau \in S_m} q^\sigma(x_{\sigma(1)}, \ldots, x_{\sigma(m)}) q^{\sigma^{-1}}(x)e_{y_{\tau^{-1}(1)}} \otimes \ldots \otimes e_{y_{\tau^{-1}(m)}}
\]
\[
= P_{\Gamma_S(\mathbb{C}^n)} q^{\sigma^{-1}}(x)e_{y_1} \otimes \ldots \otimes e_{y_m}
\]
\[
= P_{\Gamma_S(\mathbb{C}^n)} W^{m,q}(e_{y_1} \otimes \ldots \otimes e_{y_m}).
\]
So, \( W^{m,q} P_{\Gamma_S(\mathbb{C}^n)} = P_{\Gamma_S(\mathbb{C}^n)} W^{m,q} \) and \( W^q = \bigoplus_{m=0}^\infty W^{m,q} \) gives the required unitary which satisfy equation (2.4)(here \( W^{0,q} = I \)). Also note that for \( \Gamma^m_q(\mathbb{C}^m) \) and \( \Gamma_q^m(\mathbb{C}^m) \) we have unitary \( W^q(W^q)^* \) such that
\[
W^q(W^q)^* P_{\Gamma_q(\mathbb{C}^n)} = P_{\Gamma_q(\mathbb{C}^n)} W^q(W^q)^*.
\]

3. Dilation of \( q \)-Commuting Tuples and the Main Theorem

**Definition 14.** Let \( T = (T_1, \ldots, T_n) \) be a contractive tuple on a Hilbert space \( \mathcal{H} \). The operator \( \Delta_T = [I - (T_1T_1^* + \cdots + T_nT_n^*)]^\frac{1}{2} \) is called the defect operator of \( T \) and the subspace \( \Delta_T(\mathcal{H}) \) is called the defect space of \( T \). The tuple \( T \) is said to be pure if \( \sum_{\alpha \in \Lambda^m} T^{\alpha} (T^{\alpha})^* \) converges to zero in strong operator topology as \( m \) tends to infinity.

When \( \sum T_i T_i^* = I \), we have \( \sum_{\alpha \in \Lambda^m} T^{\alpha} (T^{\alpha})^* = I \) for all \( m \) and hence \( T \) is not pure. Let \( \bar{T} \) be a pure contractive tuple on \( \mathcal{H} \). Take \( \bar{\mathcal{H}} = \Gamma(\mathbb{C}^n) \otimes \Delta_T(\mathcal{H}) \), and define an operator \( A : \mathcal{H} \to \bar{\mathcal{H}} \) by
\[
(3.1) \quad Ah = \sum_{\alpha} e^{\alpha} \otimes \Delta_T(T^{\alpha})^* h,
\]
where the sum is taken over all \( \alpha \in \bar{\Lambda} \) (this operator was used by Popescu and Arveson in [Po3], [Po4], [Ar2] and for \( q \)-commuting case by Bhat and Bhattacharyya in [BB]). \( A \) is an isometry and
we have $T^\alpha = A^*(V^\alpha \otimes I)A$ for all $\alpha \in \tilde{\Lambda}$ (see [Po4]). Also the tuple $\tilde{V} = (V_1 \otimes I, \ldots, V_n \otimes I)$ of operators on $\tilde{H}$ is a realization of the minimal noncommuting dilation of $T$.

Let $C^*(\tilde{V})$, and $C^*(\tilde{S})$ be unital $C^*$-algebras generated by tuples $\tilde{V}$ and $\tilde{S}$ (defined in the Introduction) on Fock spaces $\Gamma(\mathbb{C}^n)$ and $\Gamma_q(\mathbb{C}^n)$ respectively. For any $\alpha, \beta \in \tilde{\Lambda}$, $\tilde{V}^\alpha(I - \sum V_i V_i^*)$ is the rank one operator $x \mapsto \langle e^\beta, x e^\alpha \rangle$, formed by basis vectors $e^\alpha, e^\beta$ and so $C^*(\tilde{V})$ contains all compact operators. Similarly we see that $C^*(\tilde{S})$ also contains all compact operators of $\Gamma_q(\mathbb{C}^n)$. As $V_i^*V_j = \delta_{ij}I$, it is easy to see that $C^*(\tilde{V}) = \overline{\text{span}} \{ V^\alpha V^\beta : \alpha, \beta \in \tilde{\Lambda} \}$. As $q_{ij}$-commutators $[S_i^*, S_j]^a_{nij}$ are compact for all $i, j$, we can also get $C^*(\tilde{S}) = \overline{\text{span}} \{ S^\alpha S^\beta : \alpha, \beta \in \tilde{\Lambda} \}$.

Consider a contractive tuple $T$ on a Hilbert space $\mathcal{H}$. For $0 < r < 1$ the tuple $rT = (rT_1, \ldots, rT_n)$ is clearly a pure contraction. So by equation (2.4) we have an isometry $A_r : \mathcal{H} \to \Gamma(\mathbb{C}^n) \otimes \Delta_r(\mathcal{H})$ defined by

$$A_r h = \sum \alpha c^\alpha \Delta_r ((rT)^\alpha) h, \quad h \in \mathcal{H},$$

where $\Delta_r = (I - r^2 \sum T_i T_i^*)^1$. So for every $0 < r < 1$ we have a completely positive map $\psi_r : C^*(\tilde{V}) \to \mathcal{B}(\mathcal{H})$ defined by $\psi_r(X) = A_r^*(X \otimes I)A_r$, $X \in C^*(\tilde{V})$. By taking limit as $r$ increases to 1 (See [Po1-4] for details), we get a unital completely positive map $\psi$ from $C^*(\tilde{V})$ to $\mathcal{B}(\mathcal{H})$ (Popescu’s Poisson transform) satisfying

$$\psi(V^\alpha V^\beta) = T^\alpha(T^\beta)^*$$

for $\alpha, \beta \in \tilde{\Lambda}$.

As $C^*(\tilde{V}) = \overline{\text{span}} \{ V^\alpha V^\beta : \alpha, \beta \in \tilde{\Lambda} \}$, $\psi$ is the unique such completely positive map. Let the minimal Stinespring dilation of $\psi$ be unital $*$-homomorphism $\pi : C^*(\tilde{V}) \to \mathcal{B}(\tilde{H})$ where $\tilde{H}$ is a a Hilbert space containing $\mathcal{H}$, and

$$\psi(X) = P_{\mathcal{H}} \pi(X)|_{\mathcal{H}} \forall X \in C^*(\tilde{V}),$$

and $\overline{\text{span}} \{ \pi(X)h : X \in C^*(\tilde{V}), h \in \mathcal{H} \} = \tilde{H}$. Let $\tilde{V} = (\tilde{V}_1, \ldots, \tilde{V}_n)$ where $\tilde{V}_i = \pi(V_i)$ and so $\tilde{V}$ is the unique standard noncommuting dilation of $T$ and clearly $(\tilde{V}_i)^*$ leaves $\mathcal{H}$ invariant. If $T$ is $q$-commuting, by considering $C^*(\tilde{S})$ instead of $C^*(\tilde{V})$, and restricting $A_r$ in the range to $\Gamma_q(\mathbb{C}^n)$, and taking limits as $r$ increases to 1 as before we would get the unique unital completely positive map $\phi : C^*(\tilde{S}) \to \mathcal{B}(\mathcal{H})$, (also see [BB]) satisfying

$$\phi(S^\alpha S^\beta) = T^\alpha(T^\beta)^*$$

for $\alpha, \beta \in \tilde{\Lambda}$.

**Definition 15.** Let $T$ be a $q$-commuting tuple. Then we have a unique unital completely positive map $\phi : C^*(\tilde{S}) \to \mathcal{B}(\tilde{H})$ satisfying equation (3.2). Consider the minimal Stinespring dilation of $\phi$. Here we have a Hilbert space $\mathcal{H}_1$ containing $\mathcal{H}$ and a unital $*$-homomorphism $\pi_1 : C^*(\tilde{S}) \to \mathcal{B}(\mathcal{H}_1)$, such that

$$\phi(X) = P_{\mathcal{H}_1} \pi_1(X)|_{\mathcal{H}} \forall X \in C^*(\tilde{S}),$$

and $\overline{\text{span}} \{ \pi_1(X)h : X \in C^*(\tilde{S}), h \in \mathcal{H} \} = \mathcal{H}_1$. Let $\tilde{S}_i = \pi_1(S_i)$ and $\tilde{S} = (\tilde{S}_1, \ldots, \tilde{S}_n)$. Then $\tilde{S}$ is called the standard $q$-commuting dilation of $T$.

Standard $q$-commuting dilation is also unique up to unitary equivalence as minimal Stinespring dilation is unique up to unitary equivalence.
Lemma 16. Suppose $\mathcal{T} = (T_1, \ldots, T_n)$ is a $q$-commuting tuple on a Hilbert space $\mathcal{H}$ and let $A$ be the operator introduced in Equation (3.1). Then there exist a Hilbert space $\mathcal{K}$ such that $(S_1 \otimes I_\mathcal{K}, \ldots, S_n \otimes I_\mathcal{K})$ is a dilation of $\mathcal{T}$ and $\dim(\mathcal{K}) = \text{rank}(\Delta_\mathcal{T})$.

Proof: For a given $k = (k_1, \ldots, k_n)$ such that $|k| = m$ let us denote $e_{k_1}^1 \otimes \cdots \otimes e_{k_n}^n$ by $e_x$, $1 \leq x \leq n$ in the following calculation.

$$A(h) = \sum_{m=0}^{\infty} \sum_{\sigma \in S_m} e_{x_1} \cdots e_{x_m} \otimes \Delta_\mathcal{T}(T_{x_1} \cdots T_{x_m})h$$

So the range of $A$ is contained in $\tilde{\mathcal{H}}_q = \Gamma_q(\mathbb{C}^n) \otimes \Delta_\mathcal{T}(\mathcal{H})$. In other words now $\mathcal{H}$ can be considered as a subspace of $\tilde{\mathcal{H}}_q$. Moreover, $\tilde{\mathcal{S}} = (S_1 \otimes I, \ldots, S_n \otimes I)$, as a tuple of operators in $\tilde{\mathcal{H}}_q$ is the standard $q$-commuting dilation of $(T_1, \ldots, T_n)$. More abstractly we can get a Hilbert space $\mathcal{K}$ such that $\mathcal{H}$ can be isometrically embedded in $\Gamma_q(\mathbb{C}^n) \otimes \mathcal{K}$ and $(S_1 \otimes I_\mathcal{K}, \ldots, S_n \otimes I_\mathcal{K})$ is a dilation of $\mathcal{T}$ and $\sum_{\mathcal{S} \in \mathcal{S}}(S_1 \otimes I_\mathcal{K})h \in \mathcal{H}$, $\alpha \in \tilde{\mathcal{A}} = \Gamma_q(\mathbb{C}^n) \otimes \mathcal{K}$. There is a unique such dilation and up to unitary equivalence and $\dim(\mathcal{K}) = \text{rank}(\Delta_\mathcal{T})$.

Theorem 17. Let $\mathcal{T}$ be a pure contractive tuple on a Hilbert space $\mathcal{H}$.

1. Then the maximal $q$-commuting piece $\tilde{\mathcal{V}}_q$ of the standard noncommuting dilation $\tilde{\mathcal{V}}$ of $\mathcal{T}$ is a realization of the standard $q$-commuting dilation of $\mathcal{T}^q$ if and only if $\Delta_\mathcal{T}(\mathcal{H}) = \Delta_\mathcal{T}(\mathcal{H}^q(\mathcal{T}))$.

And if $\Delta_\mathcal{T}(\mathcal{H}) = \Delta_\mathcal{T}(\mathcal{H}^q(\mathcal{T}))$ then $\text{rank}(\Delta_\mathcal{T}) = \text{rank}(\Delta_\mathcal{T}^q) = \text{rank}(\tilde{\mathcal{V}}_q) = \text{rank}(\tilde{\mathcal{V}}) = \text{rank}(\Delta_\mathcal{T})$.

2. Let the standard noncommuting dilation of $\mathcal{T}$ be $\tilde{\mathcal{V}}$. If $\text{rank}(\Delta_\mathcal{T})$ and $\text{rank}(\Delta_\mathcal{T}^q)$ are finite and equal then $\tilde{\mathcal{V}}_q$ is a realization of the standard $q$-commuting dilation of $\mathcal{T}^q$.

Proof: The proof is similar to the proofs of that of Theorem 10 and Remark 11 of [BBD].

If the ranks of both $\Delta_\mathcal{T}$ and $\Delta_\mathcal{T}^q$ are infinite then we can not ensure that $\Delta_\mathcal{T}(\mathcal{H}) = \Delta_\mathcal{T}(\mathcal{H}^q(\mathcal{T}))$ and hence can not ensure the converse of the last Theorem, as seen by the following example. For any $n \geq 2$ consider the Hilbert space $\mathcal{H}_0 = \Gamma_q(\mathbb{C}^n) \otimes \mathcal{M}$ where $\mathcal{M}$ is of infinite dimension and let $\mathcal{R} = (S_1 \otimes I, \ldots, S_n \otimes I)$ be a $q$-commuting pure contractive $n$-tuple. Infact one can take any $\mathcal{R}$ to be any $q$-commuting pure $n$-tuple on some Hilbert space $\mathcal{H}_0$ with $\Delta_\mathcal{R}(\mathcal{H}_0)$ of infinite dimensional.

Suppose $P_k = (p_{ij}^k)_{n \times n}$ for $1 \leq k \leq n$ are $n \times n$ matrices with complex entries such that

- if $i = k$, $j = k + 1$
- otherwise

$$p_{ij}^k = \begin{cases} t_k & \text{if } i = k, j = k + 1 \\ 0 & \text{otherwise} \end{cases} \quad \text{for } 1 \leq k < n$$

$$p_{ij}^n = \begin{cases} t_n & \text{if } i = n, j = 1 \\ 0 & \text{otherwise} \end{cases}$$
where $t_k$'s are complex numbers satisfying $0 < |t_k| < 1$. Let $\mathcal{H} = \mathcal{H}_0 \oplus \mathbb{C}^n$. Take $\mathcal{T} = (T_1, \ldots, T_n)$ where $T_k$ for $1 \leq k \leq n$ be operators on $\mathcal{H}$ defined by

$$T_k = \begin{bmatrix} R_k & P_k \end{bmatrix}$$

for $1 \leq k \leq n$.

So $\mathcal{T}$ is a pure contractive tuple, the maximal $q$-commuting piece of $\mathcal{T}$ is $R$ and $\mathcal{H}^q(\mathcal{T}) = \mathcal{H}_0$ (by Corollary 7). Here rank $(\Delta_{\mathcal{T}}) = \text{rank } (\Delta_{\mathcal{T}}) = \infty$ but $\Delta_{\mathcal{T}}(\mathcal{H}) = \Delta_{R}(\mathcal{H}_0) \oplus \mathbb{C}^n$. But the converse of Theorem 18 holds when rank of $\Delta_{\mathcal{T}}$ is finite.

Consider the case when $\mathcal{T}$ is a $q$-commuting tuple on Hilbert space $\mathcal{H}$ satisfying $\sum T_i T_i^* = I$. As $C^*(\mathcal{S})$ contains the ideal of all compact operators by standard $C^*$-algebra theory we have a direct sum decomposition of $\pi_1$ as follows. Take $\mathcal{H}_1 = \mathcal{H}_{1C} \oplus \mathcal{H}_{1N}$ where $\mathcal{H}_{1C} = \text{span}\{\pi_1(X)h : h \in \mathcal{H}, X \in C^*(\mathcal{S}) \text{ and } X \text{ is compact}\}$ and $\mathcal{H}_{1N}$ is the orthogonal complement of it. Clearly $\mathcal{H}_{1C}$ is a reducing subspace for $\pi_1$. Therefore $\pi_1 = \pi_{1C} \oplus \pi_{1N}$ where $\pi_{1C}(X) = P_{\mathcal{H}_{1C}} \pi_1(X) P_{\mathcal{H}_{1C}}$, and $\pi_{1N}(X) = P_{\mathcal{H}_{1N}} \pi_1(X) P_{\mathcal{H}_{1N}}$. Also $\pi_{1C}(X)$ is just the identity representation with some multiplicity. In fact $\mathcal{H}_{1C}$ can be written as $\mathcal{H}_{1C} = \Gamma_q(\mathbb{C}^n) \otimes \Delta_{\mathcal{T}}(\mathcal{H})$ (see Theorem 4.5 of [BB]) and $\pi_{1N}(X) = 0$ for compact $X$. But $\Delta_{\mathcal{T}}(\mathcal{H}) = 0$ and commutators $[S_i^*, S_i]$ are compact. So if we take $W_1 = \pi_{1N}(S_i)$, $W_2 = (W_1, \ldots, W_n)$ is a tuple of normal operators. It follows that the standard $q$-commuting dilation of $\mathcal{T}$ is a tuple of normal operators.

**Definition 18.** A $q$-commuting $n$-tuple $\mathcal{T} = (T_1, \ldots, T_n)$ of operators on a Hilbert space $\mathcal{H}$ is called a $q$-spherical unitary if each $T_i$ is normal and $T_1 T_1^* + \cdots + T_n T_n^* = I$.

If $\mathcal{H}$ is a finite dimensional Hilbert space and $\mathcal{T}$ is a $q$-commuting tuple on $\mathcal{H}$ satisfying $\sum T_i T_i^* = I$, then $\mathcal{T}$ a spherical unitary because each $T_i$ would be subnormal and all finite dimensional subnormal operators are normal (see [Ha]).

**Theorem 19.** (Main Theorem) Let $\mathcal{T}$ is a $q$-commuting contractive tuple on a Hilbert space $\mathcal{H}$. Then the maximal $q$-commuting piece of the standard noncommuting dilation of $\mathcal{T}$ is a realization of the standard $q$-commuting dilation of $\mathcal{T}$.

**Proof of the Theorem 19:** Let $\tilde{\mathcal{S}}$ denote the standard $q$-commuting dilation of $\mathcal{T}$ on a Hilbert space $\mathcal{H}_1$ and we follow the notations as in section 2. As $\mathcal{S}$ is also a contractive tuple, we have a unique unital completely positive map $\eta : C^*(\mathcal{V}) \to C^*(\mathcal{S})$, satisfying

$$\eta(\mathcal{V}^\alpha(\mathcal{V}^\beta)^*) = \mathcal{S}^\alpha(\mathcal{S}^\beta)^* \alpha, \beta \in \tilde{\Lambda}.$$ 

It is easy to see that $\psi = \phi \circ \eta$. Let unital $*$-homomorphism $\pi_2 : C^*(\mathcal{V}) \to \mathcal{B}(\mathcal{H}_2)$ for some Hilbert space $\mathcal{H}_2$ containing $\mathcal{H}_1$, be the minimal Stinespring dilation of the map $\eta(T) : C^*(\mathcal{V}) \to \mathcal{B}(\mathcal{H}_1)$ such that $\eta(T)(X) = P_{\mathcal{H}_1} \pi_2(X)|_{\mathcal{H}_1}$, $\forall X \in C^*(\mathcal{V})$, and $\text{span } \{\pi_2(X)h : X \in C^*(\mathcal{V}) \text{, } h \in \mathcal{H}_1\} = \mathcal{H}_2$. We get the following commuting diagram.
where all the down arrows are compression maps, horizontal arrows are unital completely positive maps and diagonal arrows are unital \(*\)-homomorphisms. Let \( \hat{V} = (\hat{V}_1, \ldots, \hat{V}_n) \) where \( \hat{V}_i = \pi_2(V_i) \).

We would show that \( \hat{V} \) is the standard noncommuting dilation of \( T \). We have this result if we can show that \( \pi_2 \) is a minimal dilation of \( \psi = \phi \circ \eta \) as minimal Stinespring dilation is unique up to unitary equivalence. For this first we show that \( \tilde{S} = (\pi_1(S_1), \ldots, \pi_1(S_n)) \) is the maximal \( q \)-commuting piece of \( \hat{V} \).

First we consider a particular case when \( T \) is a \( q \)-spherical unitary on a Hilbert space \( \mathcal{H} \). In this case we would show that standard commuting dilation and the maximal \( q \)-commuting piece of the standard noncommuting dilation of \( T \) is itself. We have this result if we can show that \( \pi_2 \) is a minimal dilation of \( \psi = \phi \circ \eta \).

For convenience, at some places we would identify \( \mathcal{H} \oplus \cdots \oplus \mathcal{H} \) with \( \mathbb{C}^n \otimes \mathcal{H} \) so that \((h_1, \ldots, h_n) = \sum_{i=1}^n e_i \otimes h_i \). Then

\[
D(h_1, \ldots, h_n) = D(\sum_{i=1}^n e_i \otimes h_i) = \sum_{i=1}^n e_i \otimes (h_i - \sum_{j=1}^n T_i^* T_j h_j)
\]

and the standard noncommuting dilation \( \tilde{V}_i \)

\[\tilde{V}_i(h \oplus \sum_{\alpha \in \Lambda} e^\alpha \otimes d_\alpha) = T_i h \oplus D(e_i \otimes h) \oplus e_i \otimes (\sum_{\alpha \in \Lambda} e^\alpha \otimes d_\alpha)\]

for \( h \in \mathcal{H}, d_\alpha \in \mathcal{D} \) for \( \alpha \in \tilde{\Lambda} \), and \( 1 \leq i \leq n \) (\( \mathbb{C}^n \omega \otimes \mathcal{D} \) has been identified with \( \mathcal{D} \)).
We have
\[ T_i^*T_i = T_i^*T_i \] and \( T_jT_i = q_{ij}T_iT_j \forall 1 \leq i, j \leq n. \)

Also by Fuglede-Putnam Theorem ([Ha] [Pu])
\[ T_j^*T_i = q_{ij}T_j^*T_i^* \] and \( T_j^*T_i^* = q_{ij}T_i^*T_j^* \forall 1 \leq i, j \leq n. \)

As \( \sum T_i^*T_i = I \), by direct computation \( D^2 \) is seen to be a projection. So, \( D = D^2 \). Note that \( q_{ij}q_{ij} = 1 \), i.e., \( q_{ij} = q_{ji} \). Then we get
\[
(3.5) \quad D(h_1, \ldots, h_n) = \sum_{i,j=1}^{n} e_i \otimes T_j(T_j^*h_i - q_{ij}T_i^*h_j) = \sum_{i,j=1}^{n} e_i \otimes T_j(h_{ij})
\]
where \( h_{ij} = T_j^*h_i - q_{ij}T_i^*h_j = T_j^*h_i - q_{ij}T_i^*h_j \) for \( 1 \leq i, j \leq n \). Note that \( h_{ii} = 0 \) and \( h_{ji} = -q_{ij}h_{ij} \).

As clearly \( \mathcal{H} \subseteq \mathcal{H}^q(V) \), lets begin with \( y \in \mathcal{H}^q \cap \mathcal{H}^q(V) \). We wish to show that \( y = 0 \). Decompose \( y \) as \( y = 0 \oplus \sum_{\alpha \in \Lambda} e^\alpha \otimes y_\alpha \), with \( y_\alpha \in \mathcal{D} \). We assume \( y \neq 0 \) and arrive at a contradiction. If for some \( \alpha, y_\alpha \neq 0 \), then \( \langle \omega \otimes y_\alpha, (V^\alpha)^*y \rangle = \langle e^\alpha \otimes y_\alpha, y \rangle = \langle y_\alpha, y_\alpha \rangle \neq 0 \). Since \( (V^\alpha)^*y \in \mathcal{H}^q(V) \), we can assume \( \|y_\alpha\| = 1 \) without loss of generality. Taking \( \bar{y}_m = \sum_{\alpha \in \Lambda^m} e^\alpha \otimes y_\alpha \), we get \( y = 0 \oplus \oplus_{m \geq 0} \bar{y}_m \). \( D \) being a projection its range is closed and as \( y_0 \in \mathcal{D} \), there exist some \( (h_1, \ldots, h_n) \) such that \( y_0 = D(h_1, \ldots, h_n) \). Let \( \bar{x}_0 = y_0 \), \( \bar{x}_1 = \sum_{i,j=1}^{n} e_i \otimes D(e_j \otimes h_{ij}) \), and for \( m \geq 1 \),
\[
\bar{x}_m = \sum_{i_1, \ldots, i_{m-1}, i, j=1}^{n} e_{i_1} \otimes \cdots \otimes e_{i_{m-1}} \otimes e_i \otimes D(e_j \otimes (\prod_{1 \leq r \leq s \leq m-1} q_{i_r,i_s})(\prod_{k=1}^{m-1} q_{i_k,i_{k+1}})T_{i_1}^* \cdots T_{i_{m-1}}^*h_{ij}).
\]
So \( \bar{x}_m \in (\mathbb{C}^n)^{\otimes m} \otimes \mathcal{D} \) for all \( m \in \mathbb{N} \). As \( T \) is \( q \)-commuting \( n \)-tuple and \( D \) is a projection, we have
\[
\sum_{1 \leq i < j \leq n} (q_{ij}V_iV_j - \tilde{V}_i\tilde{V}_j)q_{ji}h_{ij} = \sum_{1 \leq i < j \leq n} (q_{ij}T_iT_j - T_jT_i)q_{ji}h_{ij}
\]
\[
+ \sum_{1 \leq i < j \leq n} D(e_i \otimes T_jh_{ij} - q_{ij}e_j \otimes T_ih_{ij})
\]
\[
+ \sum_{1 \leq i < j \leq n} (e_i \otimes D(e_j \otimes h_{ij}) - q_{ij}e_j \otimes D(e_i \otimes h_{ij}))
\]
\[
= 0 + D(\sum_{i,j=1}^{n} e_i \otimes T_jh_{ij}) + \sum_{i,j=1}^{n} e_i \otimes D(e_j \otimes h_{ij})
\]
\[
= D^2(h_1, \ldots, h_n) + \sum_{i,j=1}^{n} e_i \otimes D(e_j \otimes h_{ij})
\]
\[
= \bar{x}_0 + \bar{x}_1.
\]
So by Proposition 6, $(y, \tilde{x}_0 + \tilde{x}_1) = 0$. Next let $m \geq 2$.

\[
\begin{align*}
\sum_{i_1, \ldots, i_{m-1}=1}^n \tilde{V}_{i_1} \cdots \tilde{V}_{i_{m-1}} \{ & \sum_{i, j=1}^n (q_{ij} \tilde{V}_i \tilde{V}_j - \tilde{V}_j \tilde{V}_i) (\prod_{1 \leq r < s \leq m-1} q_{riv_is}) (\prod_{k=1}^{m-2} q_{ij}) (T_{i_1}^* T_{i_2}^* \cdots T_{i_{m-2}}^* h_{i_{m-1}}) \\
= & \sum_{i_1, \ldots, i_{m-1}=1}^n e_{i_1} \otimes \cdots \otimes e_{i_{m-1}} \otimes \sum_{i, j=1}^n D((\prod_{1 \leq r < s \leq m-1} q_{riv_is})(\prod_{k=1}^{m-2} q_{ij}) (q_{ij} e_i \otimes T_j T_{i_1}^* \cdots T_{i_{m-2}}^* h_{i_{m-1}}) - e_j \otimes D(e_i \otimes T_i^* T_{i_1}^* \cdots T_{i_{m-2}}^* h_{i_{m-1}})) \\
- & \sum_{i_1, \ldots, i_{m-1}=1}^n e_{i_1} \otimes \cdots \otimes e_{i_{m-1}} \otimes \sum_{i, j=1}^n D(e_j \otimes (\prod_{1 \leq r < s \leq m-1} q_{riv_is})(\prod_{k=1}^{m-2} q_{ij}) (e_i \otimes T_i^* \cdots T_{i_{m-2}}^* h_{i_{m-1}})) \\
= & - \sum_{i_1, \ldots, i_{m-1}=1}^n e_{i_1} \otimes \cdots \otimes e_{i_{m-1}} \otimes \sum_{i, j=1}^n e_i \otimes D(e_j \otimes (\prod_{1 \leq r < s \leq m} q_{i_r iv_is})(\prod_{k=1}^{m-2} q_{ij}) (T_{i_1}^* \cdots T_{i_{m-1}}^* h_{i_{m-1}}) \\
\otimes (\prod_{1 \leq r < s \leq m-1} q_{riv_is}) (\prod_{k=1}^{m-2} q_{ij}) (T_{i_1}^* T_{i_2}^* \cdots T_{i_{m-2}}^* T_j^* h_{i_{m-1}} - q_{i_{m-1} j i} T_j T_{i_1}^* \cdots T_{i_{m-2}}^* h_{i_{m-1}}) \\
- & (\prod_{1 \leq r < s \leq m-2} q_{riv_is}) (\prod_{k=1}^{m-2} q_{ij}) (T_j T_{i_1}^* \cdots T_{i_{m-2}}^* T_j h_{i_{m-1}} - q_{i_{m-1} j i} T_j T_{i_1}^* \cdots T_{i_{m-2}}^* T_j h_{i_{m-1}})) \}
\end{align*}
\]

(in the term above, $i$ and $j$ have been interchanged in the last summation)

\[
\begin{align*}
= & - \sum_{i_1, \ldots, i_{m-2}, i, j=1}^n e_{i_1} \otimes \cdots \otimes e_{i_{m-2}} \otimes e_i \otimes (\prod_{1 \leq r < s \leq m-2} q_{riv_is})(\prod_{k=1}^{m-2} q_{ij}) (D(e_j \otimes T_i^* \cdots T_{i_{m-2}}^* h_{i_{m-1}}) \\
+ & \sum_{i_1, \ldots, i_{m-1}, i, j=1}^n e_{i_1} \otimes \cdots \otimes e_{i_{m-1}} \otimes e_i \otimes (\prod_{1 \leq r < s \leq m-1} q_{riv_is})(\prod_{k=1}^{m-1} q_{ij}) (D(e_j \otimes T_i^* \cdots T_{i_{m-1}}^* h_{i_{m-1}}) \\
= & -\tilde{x}_{m-1} + \tilde{x}_m.
\end{align*}
\]
Hence by proposition 6, $\langle y, \bar{x}_{m-1} - \bar{x}_m \rangle = 0$. Further we compute $\|\bar{x}_m\|$ for all $m \in \mathbb{N}$.

\[
\|\bar{x}_m\|^2 = \sum_{i_1, \ldots, i_{m-1}, i, j=1}^n e_{i_1} \otimes \cdots \otimes e_{i_{m-1}} \otimes e_i \otimes D(e_j \otimes \left( \prod_{1 \leq r < s \leq m-1} q_{r,i}\right) \left( \prod_{k=1}^{m-1} \left( q_{i_k,i_{k'}} \right)^* T_{i_1}^* \cdots T_{i_{m-1}}^* h_{i_{ij}} \right),
\]

\[
\sum_{i_1, \ldots, i_{m-1}, i', j'=1}^n e_{i_1'} \otimes \cdots \otimes e_{i'_{m-1}} \otimes e_{i'} \otimes D(e_{j'} \otimes \left( \prod_{1 \leq r' < s' \leq m-1} q_{r',i}\right) \left( \prod_{k'=1}^{m-1} \left( q_{i_{k'},i_{k'}} \right)^* T_{i_1}^* \cdots T_{i_{m-1}}^* h_{i'_{i'j'}} \right)
\]

\[
\sum_{i_1, \ldots, i_{m-1}, i'=1}^n D(e_{j'} \otimes \left( \prod_{1 \leq r' < s' \leq m-1} q_{r',i}\right) \left( \prod_{k'=1}^{m-1} \left( q_{i_{k'},i_{k'}} \right)^* T_{i_1}^* \cdots T_{i_{m-1}}^* h_{i'_{i'j'}} \right)
\]

\[
\sum_{i_1, \ldots, i_{m-1}, i', j'=1}^n \left( \prod_{k'=1}^{m-1} \left( q_{i_{k'},i_{k'}} \right)^* T_{i_1}^* \cdots T_{i_{m-1}}^* h_{i'_{i'j'}} \right)
\]

Let $\tau : B(\mathcal{H}) \to B(\mathcal{H})$ be defined by $\tau(X) = \sum_{i=1}^n T_i X T_i^*$ for all $X \in B(\mathcal{H})$, and let $\tilde{\tau}^m : M_n(B(\mathcal{H})) \to M_n(B(\mathcal{H}))$ be defined by $\tilde{\tau}^m(X) = (\tau^m(X_{ij}))_{n \times n}$ for all $X = (X_{ij})_{n \times n} \in M_n(B(\mathcal{H}))$. As $\tau$ is a completely positive map, $\tilde{\tau}^m$ is also a completely positive map.

So we have $\tilde{\tau}^m(D) \leq I$ and
\[ \| \tilde{x}_m \|^2 = \sum_{r=1}^{n} \langle \tilde{x}^m(D)(h_{r1} \ldots h_{rn}), (h_{r1} \ldots h_{rn}) \rangle \]
\[ \leq \sum_{r=1}^{n} \langle (h_{r1} \ldots h_{rn}), (h_{r1} \ldots h_{rn}) \rangle \]
\[ = \sum_{r,i} \langle h_{ri}, h_{ri} \rangle = \sum_{i,r=1}^{n} \langle T^*_i h_r - q_i T^*_i h_i, T^*_i h_r - q_i T^*_i h_i \rangle \]
\[ = \sum_{i,r=1}^{n} \{ (T^*_i T_i h_r - T^*_i T_i h_i, h_r) - (T^*_i T_i h_r - T^*_i T_i h_i, h_i) \} \]
\[ = \sum_{r=1}^{n} \langle h_r - \sum_{i=1}^{n} T^*_i T_i h_i, h_r \rangle - \sum_{i=1}^{n} \langle \sum_{r=1}^{n} T^*_i T_i h_r - h_i, h_i \rangle \]
\[ = 2 \sum_{r=1}^{n} \langle h_r - \sum_{i=1}^{n} T^*_i T_i h_i, h_r \rangle = 2 \langle D(h_1, \ldots, h_n), (h_1, \ldots, h_n) \rangle = 2 \| \tilde{x}_0 \|^2 = 2. \]

As \( \langle y, \tilde{x}_0 + \tilde{x}_1 \rangle = 0 \) and \( \langle y, \tilde{x}_{m-1} + \tilde{x}_m \rangle = 0 \) for \( m+1 \in \mathbb{N} \), we get \( \langle y, \tilde{x}_0 + \tilde{x}_m \rangle = 0 \) for \( m \in \mathbb{N} \). So \( 1 = \langle \tilde{y}_0, \tilde{y}_0 \rangle = \langle \tilde{y}_0, \tilde{x}_0 \rangle = -\langle \tilde{y}_m, \tilde{x}_m \rangle \). By Cauchy-Schwarz inequality, \( 1 \leq \| \tilde{y}_m \| \| \tilde{x}_m \| \), which implies \( \frac{1}{\sqrt{2}} \leq \| \tilde{y}_m \| \) for \( m \in \mathbb{N} \). This is a contradiction as \( y = 0 \oplus \oplus_{m \geq 0} \tilde{y}_m \) is in the Hilbert space \( \mathcal{H} \). This proves the particular case.

Using arguments similar to that of Theorem 13 of [BBD], the proof of the general case (that is when \( T_i \) is not necessarily normal) and the proof of \( \tilde{V} \) is the standard noncommuting dilation of \( T \), both follows. \( \square \)

4. Distribution of \( S_i + S_i^* \) and Related Operator Spaces

Let \( \mathcal{R} \) be the von Neumann algebra generated by \( G_i = S_i + S_i^* \) for all \( 1 \leq i \leq n \). We are interested in calculating the moments of \( S_i + S_i^* \) with respect to the vacuum state and inferring about the distribution. The vacuum expectation is given by \( \epsilon(T) = \langle \omega, T \omega \rangle \) where \( T \in \mathcal{R} \). So,

\[ \epsilon((S_i + S_i^*)^n) = \langle \omega, (S_i + S_i^*)^n \omega \rangle = \left\{ \begin{array}{ll}
C_n = 0_{\frac{n}{n+1}(\frac{1}{2})} & \text{if } n \text{ is odd} \\
C_n = 1_{\frac{n+1}{n+2}(\frac{1}{2})} & \text{otherwise}
\end{array} \right. \]

where \( C_n \) the catalan number (refer [Com]). This shows that \( S_i + S_i^* \) has semicircular distribution. Further this vacuum expectation is not tracial on \( \mathcal{R} \) for \( n \geq 2 \) as

\[ \epsilon(G_2G_2G_1G_1) = \langle \omega, (S_2 + S_2^*)(S_2 + S_2^*)(S_1 + S_1^*)(S_1 + S_1^*) \omega \rangle = \omega, S_2 S_2^* S_1 S_1 + S_2 S_2^* S_1 S_1 \omega = 1 \]
\[ \epsilon(G_2G_1G_1G_2) = \langle \omega, (S_2 + S_2^*)(S_1 + S_1^*)(S_2 + S_2^*) \omega \rangle = \langle \omega, S_2 S_2^* S_1 S_2 + S_2 S_1 S_1 S_2 \omega \rangle = \frac{1}{2} \]
We would now investigate using arguments of theory of operator spaces introduced by Effros and Ruan [ER]. Here we follow the ideas of [BS2] and [HP]. Operator spaces which are Hilbert spaces are called Hilbertian operator spaces. For some Hilbert space \( \tilde{\mathcal{H}} \) and \( a_i \in B(\tilde{\mathcal{H}}), 1 \leq i \leq n \) define

\[
\|(a_1, \cdots, a_n)\|_{\text{max}} = \max\left(\sum_{i=1}^{n} a_i a_i^*, \sum_{i=1}^{n} a_i^* a_i^{1/2}\right).
\]

Let us denote the operator space

\[
\begin{pmatrix}
  r_1 & 0 & \cdots & 0 \\
  \vdots & \ddots & \vdots & \vdots \\
  0 & \cdots & r_n & 0 \\
  r_n & 0 & \cdots & 0
\end{pmatrix} \oplus \begin{pmatrix}
  r_1 & \cdots & r_n \\
  0 & \cdots & 0 \\
  \vdots & \ddots & \vdots \\
  0 & \cdots & 0
\end{pmatrix}
\]

by \( E_n \). Let \( \{e_{ij} : 1 \leq i, j \leq n\} \) denote the standard basis of \( M_n \) and \( \delta_i = e_{i1} \oplus e_{1i} \). Then one has

\[
\left\| \sum_{i=1}^{n} a_i \otimes \delta_i \right\|_{B(\tilde{\mathcal{H}}) \otimes M_n} = \|(a_1, \cdots, a_n)\|_{\text{max}}.
\]

**Theorem 20.** The operator space generated by \( G_i, 1 \leq i \leq n \) is completely isomorphic to \( E_n \).

**Proof:** It's enough to show that for \( a_i \in B(\tilde{\mathcal{H}}), 1 \leq i \leq n \) we have

\[
\|(a_1, \cdots, a_n)\|_{\text{max}} \leq \sum_{i=1}^{n} a_i \otimes G_i \|_{\tilde{\mathcal{H}} \otimes \Gamma_4(\mathbb{C}^n)} \leq 2\|(a_1, \cdots, a_n)\|_{\text{max}}
\]

\[
\left\| \sum_{i=1}^{n} a_i \otimes S_i^* \right\|_{\tilde{\mathcal{H}} \otimes \Gamma_4(\mathbb{C}^n)} = \left\| \sum_{i=1}^{n} (a_i \otimes 1)(1 \otimes S_i^* \right) \|_{\tilde{\mathcal{H}} \otimes \Gamma_4(\mathbb{C}^n)}
\]

\[
\leq \left\| \sum_{i=1}^{n} a_i a_i^* \|_{\tilde{\mathcal{H}}} \right\| \sum_{i=1}^{n} S_i S_i^* \|_{\Gamma_4(\mathbb{C}^n)} \leq \left\| \sum_{i=1}^{n} a_i a_i^* \|_{\Gamma_4(\mathbb{C}^n)} \right\| \sum_{i=1}^{n} S_i S_i^* \|_{\Gamma_4(\mathbb{C}^n)}
\]

Similarly

\[
\left\| \sum_{i=1}^{n} a_i \otimes S_i \|_{\tilde{\mathcal{H}} \otimes \Gamma_4(\mathbb{C}^n)} = \left\| \sum_{i=1}^{n} (1 \otimes S_i)(a_i \otimes 1) \|_{\tilde{\mathcal{H}} \otimes \Gamma_4(\mathbb{C}^n)}
\]

\[
\leq \left\| \sum_{i=1}^{n} a_i^* a_i \|_{\Gamma_4(\mathbb{C}^n)} \right\| \sum_{i=1}^{n} S_i S_i^* \|_{\Gamma_4(\mathbb{C}^n)}
\]

So

\[
\left\| \sum_{i=1}^{n} a_i \otimes G_i \right\|_{\tilde{\mathcal{H}} \otimes \Gamma_4(\mathbb{C}^n)} \leq 2\|(a_1, \cdots, a_n)\|_{\text{max}}
\]
Let $\mathcal{S}$ denote the set of all states on $B(\tilde{\mathcal{H}})$. Now using the fact that $\epsilon(G_i G_j) = \langle \omega, S_i S_j \omega \rangle = \delta_{ij}$ we get

$$\| \sum_{i=1}^n a_i \otimes G_i \|_{\tilde{\mathcal{H}} \otimes \Gamma_q(C^n)}^2 \geq \sup_{\tau \in \mathcal{S}} (\tau \otimes \epsilon)[\left( \sum_{i=1}^n a_i \otimes G_i \right)^* \sum_{j=1}^n a_j \otimes G_j]$$

$$= \sup_{\tau \in \mathcal{S}} \tau \left( \sum_{i=1}^n a_i^* a_i \right) = \left\| \sum_{i=1}^n a_i^* a_i \right\|$$

Using similar arguments

$$\| \sum_{i=1}^n a_i \otimes G_i \|_{\tilde{\mathcal{H}} \otimes \Gamma_q(C^n)}^2 \geq \left\| \sum_{i=1}^n a_i a_i^* \right\|.$$

\[ \square \]

**Acknowledgements:** The author is supported by a research fellowship from the Indian Statistical Institute. The author is thankful to B. V. Rajarama Bhat and Tirthankar Bhattacharyya for many helpful discussions.

**References**

[A] J. Agler: The Arveson extension theorem and coanalytic models, Integral Equations and Operator Theory, 5 (1982), 608-631. MR 84g:47011.

[AP1] Arias, A.; Popescu, G.: Noncommutative interpolation and Poisson transforms, Israel J. Math., 115(2000), 205-234. MR 2001i:47021.

[Ar1] Arveson, W. B.: *An Invitation to $C^*$-algebras*, Graduate Texts in Mathematics, No. 39, Springer-Verlag, New York-Heidelberg (1976). MR 58#23621.

[Ar2] Arveson, W. B.: Subalgebras of $C^*$-algebras III, Multivariable operator theory, Acta Math., 181(1998), no. 2, 159-228. MR 2000e:47013.

[At1] Athavale, A.: On the intertwining of joint isometries, J. Operator Theory, 23(1990), 339-350. MR 91i:47029.

[At2] Athavale, A.: Model theory on the unit ball in $C^m$, J. Operator Theory, 27(1992), 347-358. MR 94i:47011.

[BB] Bhat, B. V. Rajarama; Bhattacharyya, T.: A model theory for $q$-commuting contractive tuples, J. Operator Theory, 47 (2002), 97-116.

[BBD] Bhat, B. V. Rajarama; Bhattacharyya, T.; Dey, Santanu : Standard noncommuting and commuting dilations of commuting tuples, to appear in Trans. Amer. Math. Soc.

[B] Bhattacharyya, T.: Dilation of contractive tuples: a survey; to appear in *Survey of Analysis and Operator Theory*, Proceedings of CMA, Volume 40.

[BS1] Bożejko, M., Speicher, R.: An example of a generalized Brownian motion, Commun. Math. Phys. 137 (1991), no. 3, 519-531 MR 92m:46096.

[BS2] Bożejko, M., Speicher, R.: Completely positive maps on Coxeter groups, deformed commutation relations, and operator spaces, Math. Ann. 300 (1994), no. 1, 97-120 MR 95g:46105.
[Bu] Bunce, J. W.: Models for \( n \)-tuples of noncommuting operators, J. Funct. Anal., 57 (1984), 21-30. MR 85k:47019.

[Con] Connes, A.: Noncommutative Geometry, Academic Press, (1994) MR 95j:46063.

[Com] Comtet, L.: Advanced combinatorics, D. Reidel Publishing Co., (1974) MR 57 #124.

[Cu] Cuntz, J.: Simple \( C^* \)-algebras generated by isometries, Commun. Math. Phys., 57 (1977), 173-185. MR 57#7189.

[Da] Davis, C.: Some dilation and representation theorems. Proceedings of the Second International Symposium in West Africa on Functional Analysis and its Applications (Kumasi, 1979), 159-182. MR 84e:47012.

[DKS] Davidson K. R.; Kribs, D. W.; Shpigel, M.E.: Isometric dilations of non-commuting finite rank \( n \)-tuples, Canad. J. Math., 53 (2001) 506-545. MR 2002f:47010.

[ER] Effros, E. G.; Ruan, Z. J.: Operator spaces, London Mathematical Society Monographs. New Series, 23, (2000) MR 2002a:46082.

[Fr] Frazho, A. E.: Models for noncommuting operators, J. Funct. Anal., 48 (1982), 1-11. MR 84h:47010.

[Ha] Halmos, P. R.: A Hilbert Space Problem Book, Second Edition, Graduate Texts in Mathematics 19, (Springer-Verlag, New York-Berlin 1982) MR 84e:47001.

[HP] Haagerup, U.; Pisier, G.: Bounded linear operators between \( C^* \)-algebras, Duke Math. J., 71 (1993), no. 3, 889-925. 94k:46112.

[JSW] Jorgensen, P. E. T.; Schmitt, L. M.; Werner, R. F.: \( q \)-canonical commutation relations and stability of the Cuntz algebra, Pacific J. Math. 165 (1994), no. 1, 131-151. MR 95g:46116.

[M] Majid, S.: Foundations of Quantum Group Theory, Cambridge University Press, (1995) MR 97g:17016.

[Pa] Parrott, S.: Unitary dilations for commuting contractions, Pacific J. Math. 34 (1970), 481-490. MR 42#3607.

[Po1] Popescu, G.: Isometric dilations for infinite sequences of noncommuting operators, Trans. Amer. Math. Soc., 316 (1989), 523-536. MR 90c:47006.

[Po2] Popescu, G.: Characteristic functions for infinite sequences of noncommuting operators, J. Operator Theory, 22 (1989), 51-71. MR 91m:47012.

[Po3] Popescu, G.: Poisson transforms on some \( C^* \)-algebras generated by isometries. J. Funct. Anal. 161 (1999), 27-61. MR 2000m:46117.

[Po4] Popescu, G.: Curvature invariant for Hilbert modules over free semigroup algebras, Advances in Mathematics, 158 (2001), 264-309. MR 2002b:46097.

[Pr] Prugovecki, E.: Quantum Mechanics in Hilbert Space, Academic Press, Second Edition (1981) MR 84k:81005.

[Pu] Putnam, C. R.: Commutation properties of Hilbert space Operators and Related Topics, Springer-Verlag (1967). MR 36#707.