Marginally bound resonances of charged massive scalar fields in the background of a charged reflecting shell

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We study analytically the characteristic resonance spectrum of charged massive scalar fields linearly coupled to a spherically symmetric charged reflecting shell. In particular, we use analytical techniques in order to solve the Klein-Gordon wave equation for the composed charged-shell-charged-massive-scalar-field system. Interestingly, it is proved that the resonant oscillation frequencies of this composed physical system are determined by the characteristic zeroes of the confluent hyper-geometric function. Following this observation, we derive a remarkably compact analytical formula for the resonant oscillation frequencies which characterize the marginally-bound charged massive scalar field configurations. The analytically derived resonance spectrum is confirmed by numerical computations.

I. INTRODUCTION

Recent analytical explorations [1] of the coupled Einstein-scalar field equations have revealed the intriguing fact that spherically symmetric compact reflecting stars [2] cannot support regular matter configurations made of static self-gravitating neutral scalar fields in their exterior regions. The theorem proved in [1] has therefore revealed the interesting fact that these horizonless physical objects share the no-scalar-hair property with asymptotically flat black holes [3–6].

On the other hand, later studies [7] of the static sector of the Klein-Gordon wave equation for a scalar field of proper mass $\mu$ and charge coupling constant $q$ have explicitly demonstrated that charged compact reflecting objects have a much richer phenomenological structure. In particular, it has been proved in [7] that, for a compact reflecting shell (or equivalently, a compact reflecting ball) of electric charge $Q$, there exists a discrete set of shell radii, $\{R_n(\mu, qQ, l)\}_{n=0}^{\infty}$ [8], which can support static regular bound-state configurations made of charged massive scalar fields.

The analytical study presented in [7] has focused on the physical properties of the static sector of the composed charged-shell-charged-massive-scalar-field configurations. It is important to emphasize that the compact reflecting shell studied in [7] was assumed to have a negligible self-gravity [see Eq. (2) below]. This weak gravity assumption implies, in particular, that the spacetime outside the reflecting shell is well approximated by the flat space Minkowski metric [9].

The main goal of the present paper is to explore the physical properties of the composed charged-shell-charged-massive-scalar-field configurations [7]. In particular, we shall analyze the resonance oscillation spectrum $\{\omega_n(\mu, qQ, l, R)\}_{n=0}^{\infty}$ which characterizes the stationary bound-state configurations of the linearized charged massive scalar fields in the background of the charged reflecting shell. These bound-state (spatially localized) field configurations are characterized by proper resonant frequencies which are bounded from above by the relation

$$\omega^2_{\text{field}} < \mu^2.$$  \hspace{1cm} (1)

The inequality (1) guarantees that the stationary charged massive scalar field configurations are characterized by normalizable radial eigenfunctions which are spatially bounded to the central charged reflecting shell [see Eq. (8) below].

In order to explore the resonance spectrum of the composed charged-shell-charged-massive-scalar-field system, we shall study in this paper the physical and mathematical properties of the Klein-Gordon wave equation for stationary charged massive scalar fields linearly coupled to a charged compact reflecting shell. Interestingly, as we shall explicitly show below, the resonant oscillation frequencies which characterize the marginally-bound charged massive scalar field configurations can be determined analytically.

II. DESCRIPTION OF THE SYSTEM

We shall explore the dynamics of a charged massive scalar field $\Psi$ which is linearly coupled to a spherically symmetric charged reflecting shell. The central supporting shell is assumed to have a negligible self-gravity [11](2):

$$M, Q \ll R,$$ \hspace{1cm} (2)
where \( \{ M, Q, R \} \) are respectively the mass, electric charge, and proper radius of the central reflecting shell.

The dynamics of a scalar field \( \Psi \) of proper mass \( \mu \) and charge coupling constant \( q \) in the background of a charged shell is determined by the Klein-Gordon wave equation \([14–18]\)

\[
((\nabla^\nu - iqA^\nu)(\nabla_\nu - iqA_\nu) - \mu^2)\Psi = 0 ,
\]

(3)

where \( A_\nu = -\delta^\nu_0 Q/r \) is the electromagnetic potential of the central charged shell. Substituting into the wave equation \([3]\) the decomposed expression \([19]\)

\[
\Psi(t, r, \theta, \phi) = \int \sum_{lm} e^{im\phi} S_{lm}(\theta) R_{lm}(r; \omega)e^{-i\omega t} d\omega
\]

(4)

for the eigenfunction of the stationary charged massive scalar field, one finds that the radial scalar eigenfunction \( R_{lm}(r) \) is determined by the ordinary differential equation \([14–18]\)

\[
\frac{d}{dr} \left( r^2 \frac{dR_{lm}}{dr} \right) + \left[ (\omega r - qQ)^2 - (\mu r)^2 - K_l \right] R_{lm} = 0 ,
\]

(5)

where \( K_l = l(l+1) \) is the angular eigenvalue which characterizes the angular part \( S_{lm}(\theta) \) of the charged scalar eigenfunction \([20, 21]\). Note that the radial equation (5) is invariant under the symmetry transformation

\[
qQ \rightarrow -qQ \quad \text{with} \quad \omega \rightarrow -\omega .
\]

(6)

We shall henceforth assume without loss of generality that \([22]\)

\[
qQ > 0 .
\]

(7)

The ordinary radial differential equation (5), which determines the characteristic eigenfunctions \( \{ R_{lm}(r) \} \) of the stationary bound-state charged massive scalar field configurations, should be supplemented by the physically motivated asymptotic boundary condition

\[
\Psi(r \rightarrow \infty) \sim r^{-1+\kappa} e^{-\sqrt{\mu^2-\omega^2} r}
\]

(8)

at spatial infinity, where \( \kappa \equiv -qQ\omega/\sqrt{\mu^2-\omega^2} \) [see Eq. (16) below]. In the bounded frequency regime \( -\mu < \omega < \mu \) [see (11)], the large-\( r \) asymptotic behavior (8) corresponds to spatially localized (normalizable) configurations of the charged massive scalar fields which are bounded to the central charged shell \([23]\). In addition, the presence of the central reflecting shell dictates the inner radial boundary condition

\[
\Psi(r = R) = 0
\]

(9)

on the characteristic scalar eigenfunctions at the surface \( r = R \) of the charged reflecting shell.

The radial differential equation (5), supplemented by the boundary conditions (8) and (9), determines the \textit{discrete} resonance spectrum \( \{ \omega_n(\mu, qQ, l, R) \}^{n=\infty}_{n=0} \) which characterizes the composed charged-spherical-shell-charged-massive-scalar-field configurations. Interestingly, as we shall explicitly show in the next section, this ordinary differential equation is amenable to an \textit{analytical} treatment.

III. THE RESONANCE CONDITION OF THE STATIONARY CHARGED MASSIVE SCALAR FIELDS IN THE BACKGROUND OF THE CHARGED SPHERICAL SHELL

In the present section we shall use analytical techniques in order to derive a remarkably compact resonance condition [see Eq. (20) below] for the characteristic resonant oscillation frequencies \( \{ \omega_n(\mu, qQ, l, R) \}^{n=\infty}_{n=0} \) of the composed charged-spherical-shell-charged-massive-scalar-field system.

It proves useful to define the new radial eigenfunction

\[
\psi_{lm} = rR_{lm} ,
\]

(10)

in terms of which the radial equation (5) takes the characteristic form \([24]\)

\[
\frac{d^2\psi}{dr^2} + \left[ \left( \frac{\omega - qQ}{r} \right)^2 - \mu^2 - \frac{l(l+1)}{r^2} \right] \psi = 0
\]

(11)
of a Schrödinger-like ordinary differential equation. Interestingly, this equation can be expressed in the familiar form
of the Whittaker differential equation (see Eq. 13.1.31 of [20])

$$\frac{d^2 \psi}{dz^2} + \left[ -\frac{1}{4} - \frac{qQ\omega}{\sqrt{\mu^2 - \omega^2} z} + \frac{(qQ)^2 - l(l+1)}{z^2} \right] \psi = 0,$$

where we have used here the dimensionless radial coordinate

$$z = 2\sqrt{\mu^2 - \omega^2} r.$$  

The general solution of the Whittaker differential equation (12) can be expressed in terms of the confluent hyper-
geometric functions (see Eqs. 13.1.32 and 13.1.33 of [20]) [25]:

$$\psi(z) = e^{-\frac{1}{2}z} z^{\frac{1}{2} + \beta} \left[ A \cdot U\left(\frac{1}{2} + \beta - \kappa, 1 + 2\beta, z\right) + B \cdot M\left(\frac{1}{2} + \beta - \kappa, 1 + 2\beta, z\right) \right],$$

where \{A, B\} are normalization constants,

$$\beta \equiv \sqrt{(l + \frac{1}{2})^2 - (qQ)^2},$$

and

$$\kappa \equiv -\frac{qQ\omega}{\sqrt{\mu^2 - \omega^2}}.$$  

The asymptotic radial behavior of the scalar function (14) is given by (see Eqs. 13.1.4 and 13.1.8 of [20])

$$\psi(z \to \infty) = A \cdot z^\kappa e^{-\frac{1}{2}z} + B \cdot \frac{\Gamma(1 + 2\beta)}{\Gamma\left(\frac{1}{2} + \beta - \kappa\right)} z^{-\kappa} e^{\frac{1}{2}z}. $$

(17)

Taking cognizance of the physically motivated boundary condition (8) for the stationary normalizable bound-state
resonances of the composed charged-spherical-shell-charged-massive-scalar-field configurations, one concludes that the
coefficient of the exponentially exploding term in the asymptotic expression (17) should vanish:

$$B = 0.$$  

(18)

Thus, we conclude that the radial eigenfunction which characterizes the stationary bound-state resonances of the
charged massive scalar fields in the background of the central charged shell is given by the compact expression

$$\psi(z) = A \cdot e^{-\frac{1}{2}z} z^{\frac{1}{2} + \beta} U\left(\frac{1}{2} + \beta - \kappa, 1 + 2\beta, 2\sqrt{\mu^2 - \omega^2} r\right),$$

(19)

where \(U(a, b, z)\) is the confluent hypergeometric function of the second kind [20].

Taking cognizance of the inner boundary condition (9), which is dictated by the presence of the central reflecting
shell, together with the expression (19) for the radial scalar eigenfunction, one obtains the resonance condition

$$U\left(\frac{1}{2} + \sqrt{\ell^2 - \alpha^2} + \frac{\alpha\omega}{\sqrt{1 - \omega^2}}, 1 + 2\sqrt{\ell^2 - \alpha^2}, 2\gamma\sqrt{1 - \omega^2}\right) = 0$$

(20)

for the stationary composed charged-spherical-shell-charged-massive-scalar-field configurations, where we have used
here the dimensionless physical parameters

$$\alpha \equiv qQ ; \quad \ell \equiv l + \frac{1}{2} ; \quad \gamma \equiv \mu R ; \quad \omega \equiv \frac{\omega}{\mu}.$$  

(21)

The analytically derived resonance equation (20) determines the discrete family of resonant oscillation frequencies
\(\{\omega_n(\alpha, \gamma, \ell)\}_{n=0}^{\infty}\) which characterize the stationary bound-state configurations of the linearized charged massive
scalar fields in the background of the central charged reflecting shell.
IV. UPPER BOUND ON THE CHARACTERISTIC RESONANT FREQUENCIES OF THE CHARGED MASSIVE SCALAR FIELDS

In the present section we shall derive a remarkably compact upper bound on the resonant oscillation frequencies which characterize the composed charged-shell-charged-massive-scalar-field system. Substituting
\[ R = r^\delta \Phi \]  \hspace{1cm} (22)
into the radial differential equation (5), one obtains
\[ r^2 \frac{d^2 \Phi}{dr^2} + 2(\delta + 1) r \frac{d\Phi}{dr} + \left[ (\omega r - qQ)^2 - (\mu r)^2 - l(l+1) + \delta(\delta+1) \right] \Phi = 0 . \]  \hspace{1cm} (23)

The boundary conditions (8) and (9), together with Eq. (22), imply that the radial eigenfunction \( \Phi(r) \) which characterizes the stationary bound-state scalar configurations must have (at least) one extremum point in the interval \( r_{\text{ext}} \in (R, \infty) \).

In particular, the eigenfunction \( \Phi(r) \) is characterized by the relations
\[ \{ \frac{d\Phi}{dr} = 0 \text{ and } \Phi \cdot \frac{d^2\Phi}{dr^2} < 0 \} \text{ for } r = r_{\text{ext}} \]  \hspace{1cm} (25)
at this extremum point. Taking cognizance of Eqs. (23) and (25), one deduces that the composed stationary charged-shell-charged-massive-scalar-field configurations are characterized by the relation \( (\omega r_{\text{ext}} - qQ)^2 - (\mu r_{\text{ext}})^2 - l(l+1) + \delta(\delta+1) > 0, \) or equivalently
\[ \left( \omega - \frac{qQ}{r_{\text{ext}}} \right)^2 > \mu^2 + \frac{l(l+1) - \delta(\delta+1)}{r_{\text{ext}}^2} . \]  \hspace{1cm} (26)
The strongest lower bound on the expression \( |\omega - qQ/r_{\text{ext}}| \) can be obtained by maximizing the r.h.s of (26). In particular, the term \(-\delta(\delta+1)/r_{\text{ext}}^2\) is maximized for \( \delta = -1/2 \), in which case one finds from (26) the characteristic inequality \( |\omega - qQ/r_{\text{ext}}| > \sqrt{\mu^2 + \left( l + \frac{1}{2} \right)^2} / r_{\text{ext}}, \) which implies \[ \omega - \frac{qQ}{r_{\text{ext}}} < -\sqrt{\mu^2 + \left( l + \frac{1}{2} \right)^2} r_{\text{ext}} . \]  \hspace{1cm} (27)

Taking cognizance of Eqs. (11) and (27), one finds that the resonant oscillation frequencies which characterize the composed stationary charged-shell-charged-massive-scalar-field configurations are restricted to the regime \[ 0 < \omega + \mu < \frac{qQ}{R} . \]  \hspace{1cm} (28)
The characteristic inequalities (28) can be expressed in the dimensionless compact form [see Eq. (21)] \[ -1 < \varpi < -1 + \frac{\alpha}{\gamma} . \]  \hspace{1cm} (29)

V. THE CHARACTERISTIC RESONANCE SPECTRUM OF THE STATIONARY COMPOSED CHARGED-SHELL-CHARGED-MASSIVE-SCALAR-FIELD CONFIGURATIONS

The analytically derived equation (20) for the characteristic resonant oscillation frequencies of the stationary composed charged-shell-charged-massive-scalar-field configurations can easily be solved numerically. Interestingly, one finds that, for given values of the dimensionless physical parameters \( \{ \alpha, \gamma, \ell \} \) [see Eq. (21)], the composed physical system is characterized by a discrete spectrum
\[ \varpi_{\infty} < \cdots < \varpi_2 < \varpi_1 < \varpi_0 \equiv \varpi_{\text{max}} < -1 + \frac{\alpha}{\gamma} \]  \hspace{1cm} (30)
of resonant oscillation frequencies, where \( \varpi_{\infty} \to -1 \) [see Eq. (37) below].
TABLE I: Resonant oscillation frequencies of the composed charged-spherical-shell-charged-massive-scalar-field system with \( l = 0 \) and \( \gamma = 1 \). We display the largest resonant oscillation frequency \( \omega_{\text{max}}(\alpha) \) of the charged massive scalar fields for various values of the dimensionless charge coupling constant \( \alpha \) [see Eq. (21)]. It is found that the dimensionless resonant oscillation frequency \( \omega_{\text{max}}(\alpha) \) is a monotonically increasing function of the dimensionless physical parameter \( \alpha \). In accord with our analytical derivation, the resonant oscillation frequencies which characterize the composed physical system are bounded from above by the compact relation \( \omega_{\text{max}} < \alpha/\gamma - 1 \) [see Eq. (29)].

| Dimensionless shell radius \( \gamma \) | 1     | 3     | 5     | 7     | 9     | 11    |
|---------------------------------------|-------|-------|-------|-------|-------|-------|
| \( \omega_{\text{max}}(\gamma; l = 0, \alpha = 5) \) | 0.1612 | -0.1649 | -0.4510 | -0.5851 | -0.6641 | -0.7168 |
| \( \omega_{\text{max}}(\gamma; l = 1, \alpha = 5) \) | 0.0814 | -0.2005 | -0.4668 | -0.5941 | -0.6700 | -0.7209 |

TABLE II: Resonant oscillation frequencies of the composed charged-spherical-shell-charged-massive-scalar-field system with \( l = 1 \) and \( \alpha = 5 \). We present the largest resonant oscillation frequency \( \omega_{\text{max}}(\gamma) \) of the charged massive scalar fields for various values of the dimensionless mass-radius parameter \( \gamma \) [see Eq. (21)]. It is found that the dimensionless resonant oscillation frequency \( \omega_{\text{max}}(\gamma) \) is a monotonically decreasing function of the dimensionless physical parameter \( \gamma \). Note that the characteristic resonant frequencies of the composed physical system conform to the analytically derived upper bound \( \omega_{\text{max}} < \alpha/\gamma - 1 \) [see Eq. (29)].

Interestingly, as we shall now prove explicitly, the resonance condition (20), which determines the characteristic resonant oscillation frequencies \( \{\omega_n(\alpha, \gamma, \ell)\}_{n=0}^{\infty} \) of the charged massive scalar fields in the background of the central charged reflecting shell, can be solved\textit{ analytically} in the asymptotic regime

\[
\omega \to -1^+ \tag{31}
\]

of \textit{marginally-bound} charged massive scalar field configurations.

Using the asymptotic approximation (see Eq. 13.5.16 of [20])

\[
U(a, b, x) \simeq \Gamma\left(\frac{1}{2}b - a + \frac{1}{4}\right)\pi^{-\frac{3}{4}}e^{\frac{x}{2}}x^{-\frac{1}{2}b} \cos\left[\sqrt{2(b - 2a)}x + \pi(a - \frac{1}{2}b + \frac{1}{4})\right] \quad \text{for} \quad a \to -\infty \tag{32}
\]

of the confluent hypergeometric function, one can express the resonance condition (20) in the form

\[
\cos\left(\sqrt{-8\alpha\gamma\omega} + \frac{\alpha\omega\pi}{\sqrt{1 - \omega^2}} + \frac{1}{4}\pi\right) = 0 \; . \tag{33}
\]

Defining the dimensionless physical parameter

\[
x = \sqrt{1 - \omega^2} \; , \tag{34}
\]

one finds from (33)

\[
x = \frac{\alpha}{n - \frac{1}{4} + c} \; ; \quad n \in \mathbb{Z} \tag{35}
\]
in the \( x \ll 1 \) regime [see Eqs. (31) and (34)], where

\[
c \equiv \sqrt{8\alpha \gamma \pi},
\]

and the integer \( n \gg 1 \) is the resonance parameter of the stationary bound-state charged massive scalar field modes.

Taking cognizance of Eqs. (34) and (35), one obtains the remarkably compact analytical formula

\[
\omega = \left[ 1 - \frac{1}{2} \left( \frac{\alpha}{n - \frac{1}{4} + c} \right)^2 \right]; \quad n \gg 1
\]

(37)

for the characteristic resonant oscillation frequencies of the composed charged-shell-charged-massive-scalar-field system in the regime (31) of marginally-bound scalar configurations.

It is worth noting that the analytically derived resonance spectrum (37) implies, in accord with the numerical data presented in Table I, that the resonant oscillation frequencies which characterize the composed charged-shell-charged-massive-scalar-field configurations are a monotonically increasing function of the dimensionless charge coupling constant \( \alpha \). In addition, the analytically derived formula (37) implies, in accord with the numerical data presented in Table II, that the characteristic resonant oscillation frequencies of the charged massive scalar fields are a monotonically decreasing function of the dimensionless mass-radius parameter \( \gamma \).

VI. NUMERICAL CONFIRMATION

In the present section we shall verify the validity of the analytically derived formula (37) for the discrete spectrum of resonant oscillation frequencies which characterize the stationary composed charged-shell-charged-massive-scalar-field configurations. In Table III we present the resonant frequencies \( \omega_{\text{analytical}}(n; \alpha, l, \gamma) \) of the charged fields as obtained from the analytically derived formula (37) in the regime (31) of marginally-bound scalar configurations. We also present the corresponding resonant frequencies \( \omega_{\text{numerical}}(n; \alpha, l, \gamma) \) of the charged fields as obtained from a direct numerical solution of the resonance condition (20) which characterizes the composed charged-shell-charged-massive-scalar-field system. We find a remarkably good agreement [34] between the approximated resonant oscillation frequencies of the marginally-bound charged massive scalar fields [as calculated from the analytically derived formula (37)] and the corresponding exact resonant oscillation frequencies of the charged fields [as computed numerically from the resonance condition (20)].

| Formula       | \( \omega(n = 3) \)  | \( \omega(n = 4) \)  | \( \omega(n = 5) \)  | \( \omega(n = 6) \)  | \( \omega(n = 7) \)  | \( \omega(n = 8) \)  | \( \omega(n = 9) \)  | \( \omega(n = 10) \) |
|---------------|-----------------------|-----------------------|-----------------------|-----------------------|-----------------------|-----------------------|-----------------------|-----------------------|
| Analytical [Eq. (37)] | -0.98404  | -0.98851  | -0.99134  | -0.99323  | -0.99457  | -0.99555  | -0.99628  | -0.99685  |
| Numerical [Eq. (20)] | -0.98354  | -0.98827  | -0.99120  | -0.99315  | -0.99451  | -0.99551  | -0.99625  | -0.99683  |

TABLE III: Resonant oscillation frequencies of the composed charged-spherical-shell-charged-massive-scalar-field system with \( \alpha = 1 \), \( l = 0 \), and \( \gamma = 10 \). We display the resonant oscillation frequencies \( \omega_{\text{analytical}}(n; \alpha, l, \gamma) \) of the charged massive scalar fields as obtained from the analytically derived compact formula (37). We also display the corresponding resonant oscillation frequencies \( \omega_{\text{numerical}}(n; \alpha, l, \gamma) \) of the charged fields as obtained from a direct numerical solution of the resonance condition (20) which characterizes the composed stationary charged-shell-charged-massive-scalar-field configurations. One finds a remarkably good agreement [34] between the approximated resonant oscillation frequencies of the marginally-bound charged massive scalar fields [as calculated from the analytically derived formula (37)] and the corresponding exact resonant oscillation frequencies of the charged fields [as computed numerically from the resonance equation (20)].

VII. SUMMARY AND DISCUSSION

Recent analytical studies [1, 7] of the static sector of the Klein-Gordon wave equation have explicitly proved that spherically symmetric charged compact reflecting objects can support static bound-state matter configurations made of charged massive scalar fields. In the present paper we have focused on the non-static sector of the composed charged-shell-charged-field system. In particular, we have explored the physical properties of the stationary bound-state charged-shell-charged-massive-scalar-field configurations.

Our main goal was to determine the resonance oscillation spectrum \( \{ \omega_{\mu}(\mu, qQ, l, R) \}_{n=0}^{\infty} \) which characterizes the Klein-Gordon wave equation for charged massive scalar fields linearly coupled to the central charged reflecting shell. The main physical results derived in the present paper are as follows:
(1) We have proved that the resonant oscillation frequencies which characterize the stationary bound-state configurations of the charged massive scalar fields in the background of the charged reflecting shell are restricted to the regime [see Eqs. (1) and (28)]
\[-\mu < \omega < \min(\mu, \frac{qQ}{R} - \mu). \tag{38}\]

(2) It was shown that, for given values of the dimensionless physical parameters \{qQ, \mu R, l\}, there exists a discrete spectrum of resonant oscillation frequencies \{\omega_n(qQ, \mu R, l)\}_{n=0}^{\infty} which characterize the composed charged-shell-charged-massive-scalar-field system. Interestingly, we have proved that the resonant frequencies of this composed physical system are determined by the characteristic zeroes of the confluent hypergeometric function [see Eq. (20)]. In particular, it has been explicitly shown that the characteristic resonance spectrum of the composed charged-shell-charged-field system is determined by the analytically derived resonance equation [see Eqs. (20) and (21)]
\[U\left(\frac{1}{2} + \sqrt{\left(l + 1/2\right)^2 - (\mu q Q)^2} + q Q \omega/\sqrt{\mu^2 - \omega^2}, 1 + 2 \sqrt{\left(l + 1/2\right)^2 - (\mu q Q)^2}; 2 \sqrt{\mu^2 - \omega^2} R\right) = 0. \tag{39}\]

(3) It was found that the characteristic resonant frequency \(\omega_{\text{max}}\) [see Eq. (21)] of the composed charged-shell-charged-massive-scalar-field system is a monotonically increasing function of the dimensionless charge coupling parameter \(qQ\) and a monotonically decreasing function of the dimensionless mass-radius parameter \(\mu R\).

(4) We have explicitly proved that the resonant oscillation frequencies which characterize the stationary bound-state charged massive scalar fields can be determined analytically from the resonance condition \(\omega/\mu \to -1^+\) [see Eq. (31)] of marginally-bound scalar configurations. In particular, we have used analytical techniques in order to derive the remarkably compact analytical formula [see Eqs. (21), (35), and (37)]
\[\frac{\omega}{\mu} = -\left[1 - \frac{1}{2} \left(\frac{qQ}{n + \frac{1}{2} + \sqrt{8q Q \mu R/\pi}}\right)^2\right] ; \ n \gg 1. \tag{40}\]

for the resonant oscillation frequencies which characterize the marginally-bound stationary charged massive scalar field configurations.

(5) Finally, we have explicitly shown that the analytically derived formula (40) for the characteristic resonance oscillation spectrum of the marginally-bound charged-shell-charged-massive-scalar-field configurations agrees with direct numerical computations of the corresponding resonant oscillation frequencies.

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Taking cognizance of (1) and (7), one realizes that the alternative inequality
\[ \omega \neq qQ/r_{\text{ext}} \]
for brevity, we shall henceforth omit the angular harmonic indices
It is worth noting that the data presented in Table III reveals the interesting fact that the agreement between the exact
Note, in particular, that in the regime
It is worth emphasizing that the analysis presented in \cite{7} has focused on the physical properties of the
\[ qQ/\mu \ll qQ/r_{\text{ext}} \]
Note that, in the regime
It is worth emphasizing that the monotonic dependence of the resonant frequency \( \omega_{\text{res}} \) on \( \mu \) is an important feature of
both the spherically symmetric \( l = 0 \) field configurations and the non-spherically symmetric \( l = 1 \) field configurations.
It is worth noting that the data presented in Table \ref{table:results} reveals the interesting fact that the agreement between the exact
resonant oscillation frequencies \( \omega_{\text{res}} \) on \( \gamma \) is an important characteristic of both the spherically symmetric \( l = 0 \) field configurations and the non-spherically symmetric \( l = 1 \) field configurations.
It is worth noting that the data presented in Table \ref{table:results} reveals the interesting fact that the agreement between the exact
resonant oscillation frequencies \( \omega_{\text{res}} \) on \( \alpha \) is an important feature of both the spherically symmetric \( l = 0 \) field configurations and the non-spherically symmetric \( l = 1 \) field configurations.

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\bibitem{7} S. Hod, Phys. Lett. B \textbf{763}, 275 (2016).
\bibitem{12} Here the integer \( l \) is the spherical harmonic index which characterizes the charged massive scalar field configurations \cite[see Eq. (1)]{12} below.
\bibitem{10} In particular, the spacetime outside the compact reflecting shell studied in \cite{10} was assumed to be characterized by the flat space metric
\[ ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \]
\bibitem{11} It is worth emphasizing that the analysis presented in \cite{7} has focused on the physical properties of the composed charged-shell-charged-massive-scalar-field system. On the other hand, in the present paper we shall analyze the physical properties of the composed charged-shell-charged-massive-scalar-field system for stationary (rather than static) bound-state field configurations.
\bibitem{13} We shall use natural units in which \( G = c = \hbar = k_B = 1 \).
\bibitem{22} Note that the weak gravity assumption \cite{22} implies that the spacetime metric outside the reflecting shell is well approximated by the flat space line element
\[ ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \]
\bibitem{23} It is worth noting that the physical parameters \( \{ \mu, q \} \) which characterize the bound-state charge massive scalar field stand respectively for \( \{ \mu/h, q/h \} \). These characteristic physical parameters of the stationary field configuration therefore have the dimensions of (length)^{-1}.
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\bibitem{20} M. Abramowitz and I. A. Stegun, \textit{Handbook of Mathematical Functions} (Dover Publications, New York, 1970). \bibitem{21} A. Ronveaux, \textit{Heun’s differential equations} (Oxford University Press, Oxford, UK, 1995). \bibitem{23} The symmetry transformation \cite{23} implies, in particular, that if \( \omega \) is a resonant frequency of the composed charged-shell-charged-massive-scalar-field system for some particular positive value of the dimensionless physical parameter \( qQ \), then \( -\omega \) is a resonant oscillation frequency of the composed charged-shell-charged-massive-scalar-field system with the corresponding negative value \( -qQ \) of the dimensionless charge coupling constant.
\bibitem{29} Note, in particular, that in the regime \( \omega^2 < \mu^2 \) \cite[see Eq. (1)]{29}, the radial asymptotic behavior \cite{30} corresponds to normalizable bound-state scalar configurations of \textit{finite} mass.
\bibitem{31} For brevity, we shall henceforth omit the angular harmonic indices \( \{ l, m \} \) of the stationary scalar field mode.
\bibitem{32} Here \( M(a, b, z) \) and \( U(a, b, z) \) are respectively the confluent hypergeometric function of the first kind and the confluent hypergeometric function of the second kind \cite{32}.
\bibitem{33} Taking cognizance of \cite{11} and \cite{7}, one realizes that the alternative inequality \( \omega - qQ/r_{\text{ext}} > \sqrt{\mu^2 + (l + \frac{1}{2})^2}/r_{\text{ext}} \) cannot be satisfied.
\bibitem{34} Here we have used the inequality \( qQ/r_{\text{ext}} < qQ/R \) for \( R_{\text{ext}} > R \) \cite[see Eq. (24)]{34}. In addition, we have used here the inequality
\[ -\sqrt{\mu^2 + (l + \frac{1}{2})^2}/r_{\text{ext}} < -\mu. \]
\bibitem{35} Note that, in the regime \( \mu > qQ/2R \), the analytically derived upper bound \( \omega < -\mu + qQ/R \) \cite[see Eq. (25)]{35} on the resonant oscillation frequencies of the charged massive scalar fields is stronger than the upper bound \( \omega < \mu \) \cite[see Eq. (1)]{11} which characterizes the stationary bound-state massive scalar field configurations.
\bibitem{36} Note that the Schwinger quantum phenomenon \cite{36} associated with the pair-production mechanism of oppositely charged particles in the electric field of the central charged shell is exponentially suppressed in the regime \( E_{\text{shell}} < E_c \) \cite{37, 38}, where \( E_{\text{shell}} = qQ/R^2 \) and \( E_c = \mu^2/q \) are respectively the electric field at the surface of the central charged shell and the critical electric field for the pair-production mechanism of oppositely charged particles \cite{37, 38}. Thus, this quantum effect is exponentially suppressed in the \( \alpha \ll \gamma^2 \) regime \cite[see Eq. (26)]{39}.
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\bibitem{41} It is worth emphasizing that the monotonic dependence of the resonant frequency \( \omega_{\text{res}}^{\max} \) on \( \alpha \) is an important feature of both the spherically symmetric \( l = 0 \) field configurations and the non-spherically symmetric \( l = 1 \) field configurations.
\bibitem{42} It is worth noting that the monotonic dependence of the resonant frequency \( \omega_{\text{res}}^{\max} \) on \( \gamma \) is an important characteristic of both the spherically symmetric \( l = 0 \) field configurations and the non-spherically symmetric \( l = 1 \) field configurations.
\bibitem{43} It is worth noting that the data presented in Table \ref{table:results} reveals the interesting fact that the agreement between the exact resonant oscillation frequencies \cite[as obtained from a direct numerical solution of the resonance condition (20)]{40} and the compact analytical formula \cite{44} is quite good even in the regime \( n = O(1) \). This observation is quite surprising since the analytically derived formula \cite{44} is the characteristic resonant oscillation frequencies of the composed charged-shell-charged-massive-scalar-field system is formally valid in the \( n \gg 1 \) regime or equivalently, in the regime \( x \ll 1 \), see Eqs.
(31), (34), and (35).