Variants on Andrica’s conjecture assuming the Riemann hypothesis

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Abstract:
It is (reasonably) well-known that the Riemann hypothesis is not sufficient to prove Andrica’s conjecture: \( \forall n \geq 1, p_{n+1} - \sqrt{p_n} \leq 1? \) But can one at least get tolerably close? I shall first show that with a logarithmic modification, (provided one assumes the Riemann hypothesis), one has

\[
\frac{\sqrt{p_{n+1}}}{\ln p_{n+1}} - \frac{\sqrt{p_n}}{\ln p_n} < 1; \quad (n \geq 1).
\]

Then, by considering more general \( m^{th} \) roots, I shall show that

\[
\sqrt[m]{p_{n+1}} - \sqrt[m]{p_n} < \frac{4e^{-1}}{m-2}; \quad (n \geq 1; \ m > 2).
\]

The key technical step in both these results is, (again assuming the Riemann hypothesis), to develop a fully explicit bound on prime gaps:

\[
g_n = p_{n+1} - p_n < 2\sqrt{p_n}\ln p_n; \quad (n \geq 1).
\]

In three appendices I will show how the numerical pre-factors in these bounds can be further tightened, present (slightly weaker but perhaps somewhat cleaner) “lim sup” forms of these results, and update the region on which Andrica’s conjecture is unconditionally verified.

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1 Introduction

The Riemann hypothesis continues to defeat all attempts to prove or disprove it [1–6]. On the other hand assuming the Riemann hypothesis provides a wealth of intriguing tentative suggestions regarding the distribution of the prime numbers [1–6]. However, even the Riemann hypothesis is insufficient to prove Andrica’s conjecture [7, 8]:

\[
\forall n \geq 1, \quad \sqrt{p_{n+1}} - \sqrt{p_n} \leq 1?
\]  

(1.1)

In the current article we shall seek to derive results as close to Andrica’s conjecture as possible. The basic tools we use are based on the behaviour of prime gaps under the Riemann hypothesis. I shall first develop the fully explicit bound:

**Theorem 1 (Prime gaps).** Assuming the Riemann hypothesis,

\[
g_n = p_{n+1} - p_n < 2\sqrt{p_n \ln p_n}.
\]  

(1.2)

(In one of the appendices I will refine the pre-factor 2 to replace it by \(1 + \frac{2375}{25746}\).)

The “close to Andrica” results I will prove below are these:

**Theorem 2 (Logarithmic modification of Andrica).** Assuming the Riemann hypothesis,

\[
\forall n \geq 1, \quad \frac{\sqrt{p_{n+1}}}{\ln p_{n+1}} - \frac{\sqrt{p_n}}{\ln p_n} \leq 1.
\]  

(1.3)

**Theorem 3 (Higher root modification of Andrica).** Assuming the Riemann hypothesis,

\[
\forall n \geq 1, \forall m > 2, \quad \sqrt[p_n+1]{p_n} - \sqrt[p_n]{p_n} \leq \frac{4 e^{-1}}{(m - 2)}.
\]  

(1.4)

In the appendices I will first tighten the numerical pre-factors in these bounds, and then present (slightly weaker but perhaps cleaner) “lim sup” forms of these results.

2 Known results on prime gaps assuming Riemann hypothesis

Two older theorems addressing this issue are “ineffective” (meaning one or more implicit constants are known to be finite but are otherwise undetermined):

**Theorem 4 (Cramer 1919 [9, 10]).** Assuming the Riemann hypothesis,

\[
g_n = p_{n+1} - p_n = O(\sqrt{p_n \ln(p_n)}).
\]  

(2.1)
(Unfortunately, this particular theorem only gives qualitative, but not quantitative, information.)

**Theorem 5** (Goldston 1982 [11]). Assuming the Riemann hypothesis,

\[ g_n = p_{n+1} - p_n \leq 4\sqrt{p_n} \ln(p_n); \quad n \text{ sufficiently large.} \quad (2.2) \]

(This particular theorem gives quantitative information about the size of prime gaps, but only qualitative information as to the domain of validity.) Two considerably more recent theorems are fully explicit:

**Theorem 6** (Ramare and Saouter 2003 [12]). Assuming the Riemann hypothesis, for \( x \geq 2 \) there exists a prime in the interval

\[ \left( x - \frac{8}{5} \sqrt{x} \ln x, x \right]. \quad (2.3) \]

**Theorem 7** (Dudek 2014 [13]). Assuming the Riemann hypothesis, for \( x \geq 2 \) there exists a prime in the interval

\[ \left( x - \frac{4}{\pi} \sqrt{x} \ln x, x \right]. \quad (2.4) \]

It is the recent 2014 result by Dudek that we shall modify for our purposes.

### 3 Prime gaps: Proof of Theorem 1

To modify Dudek’s 2014 theorem into the result we want, we shall argue as follows: For \( n \geq 2 \) we take \( x = p_{n+1} - 2 \), then

\[ p_n > (p_{n+1} - 2) - \frac{4}{\pi} \sqrt{p_{n+1} - 2} \ln(p_{n+1} - 2). \quad (3.1) \]

That is

\[ g_n < 2 + \frac{4}{\pi} \sqrt{p_{n+1} - 2} \ln(p_{n+1} - 2). \quad (3.2) \]

We now want to massage this into something a little more tractable. Note that on the domain of interest \( (p \geq 2) \) we have

\[ (\sqrt{p} \ln p)' = \frac{1}{2} \frac{\ln p + 2}{\sqrt{p}} > 0; \quad (\sqrt{p} \ln p)'' = \frac{1}{4} p^{-3/2} \ln p < 0. \quad (3.3) \]
So the function $\sqrt{p} \ln p$ is monotone and convex. Then certainly
\[ g_n < 2 + \frac{4}{\pi} \sqrt{p_{n+1}} \ln(p_{n+1}), \] (3.4)
and appealing to Bertrand's theorem ($p_{n+1} < 2p_n$)
\[ g_n < 2 + \frac{4}{\pi} \sqrt{2p_n} \ln(2p_n) = 2\sqrt{p_n} \ln p_n \left\{ \frac{2\sqrt{2}}{\pi} + \frac{2\sqrt{2}}{\pi} \ln 2 + \frac{1}{\sqrt{p_n} \ln p_n} \right\}. \] (3.5)
But the quantity in braces is monotone decreasing, and less than unity for $p_n \geq 755$, corresponding to $n \geq 134$. So we certainly have $g_n < 2\sqrt{p_n} \ln p_n$ for $n \geq 134$. A quick computer verification shows that this also holds for $n \in \{1, 2, 3, \ldots, 133, 134\}$ and so we have
\[ g_n < 2\sqrt{p_n} \ln p_n \quad \text{for} \quad n \geq 1. \] (3.6)
This is still by no means an optimal bound, but it is, (given the Riemann hypothesis and Dudek's 2014 theorem [13]), both extremely easy to establish and extremely easy to work with. Note this improves Goldston's 1982 bound [11] in two specific ways: The pre-factor 4 is reduced to 2, but more importantly the domain of validity is now extended to include all prime numbers. In one of the appendices I will further reduce the prefactor 2 to $1 + \frac{2375}{25746}$, but for current purposes the prefactor 2 is good enough.

4 Logarithmic modification of Andrica: Proof of Theorem 2

Now consider the function $\sqrt{p} / \ln p$ and note
\[ \left( \frac{\sqrt{p}}{\ln p} \right)' = \frac{\ln p - 2}{2\sqrt{p} \ln^2 p} > 0 \quad (p > e^2); \] (4.1)
\[ \left( \frac{\sqrt{p}}{\ln p} \right)'' = -\frac{\ln^2 p - 8}{4p^{3/2} \ln^3 p} < 0 \quad (p > e^{\sqrt{8}}). \] (4.2)
So $\sqrt{p} / \ln p$ is certainly monotone and convex for $p \geq 17$. Thence
\[ \frac{\sqrt{p_{n+1}}}{\ln p_{n+1}} - \frac{\sqrt{p_n}}{\ln p_n} < \frac{\ln p_n - 2}{2\sqrt{p_n} \ln^2 p_n} g_n < \frac{\ln p_n - 2}{\ln p_n} = 1 - \frac{2}{p_n} < 1; \quad (p_n \geq 17; n \geq 7). \] (4.3)
A quick verification shows that this also holds for $n \in \{1, 2, 3, 4, 5, 6\}$ and so we have
\[ \frac{\sqrt{p_{n+1}}}{\ln p_{n+1}} - \frac{\sqrt{p_n}}{\ln p_n} < 1; \quad (n \geq 1). \] (4.4)
This is again by no means an optimal bound, but it is, (given the Riemann hypothesis and our prime gap result), both easy to establish and easy to work with.
Higher-root modification of Andrica: Proof of Theorem 3

Now consider the function $\sqrt[p]{p} = p^{1/m}$ and restrict attention to $p > 0$ and $m > 1$. Then

$$
(p^{1/m})' = \frac{1}{m} p^{1/m-1} > 0; \quad (p^{1/m})'' = -\frac{m-1}{m^2} p^{1/m-2} < 0; \quad (5.1)
$$

Thus the function $\sqrt[p]{p} = p^{1/m}$ is monotone and convex, so we have

$$
\sqrt[p]{p_{n+1}} - \sqrt[p]{p_n} < \frac{1}{m} p_n^{1/m-1} g_n < \frac{2}{m} p_n^{1/m-1/2} \ln(p_n); \quad (n \geq 1). \quad (5.2)
$$

If $m \leq 2$ this result is true but not particularly interesting. For $m > 2$ the function $p^{1/m-1/2} \ln p$ rises from zero (at $p = 1$) to a maximum, and subsequently dies back to zero asymptotically as $p \to \infty$. The maximum occurs at $p_{\text{critical}} = \exp(2m/(m-2))$ where the function takes on the value $(p^{1/m-1/2} \ln p)_{\text{max}} = 2me^{-2/(m-2)}$, thereby implying

$$
\sqrt[p]{p_{n+1}} - \sqrt[p]{p_n} < \frac{4e^{-1}}{m-2}; \quad (n \geq 1; m > 2). \quad (5.3)
$$

This is the result we were seeking to establish. This is again by no means an optimal bound, but it is, (given the Riemann hypothesis and our prime gap result), both easy to establish and easy to work with.

Note that for $m = 2$, the situation relevant to the standard Andrica conjecture, we merely have

$$
\sqrt{p_{n+1}} - \sqrt{p_n} < \ln(p_n); \quad (n \geq 1). \quad (5.4)
$$

For $m = 2$ this is not enough to conclude anything useful regarding the standard Andrica conjecture.

In contrast for $m = 3$ and $m = 4$ we certainly have:

**Corollary 1** (Cube root modification of Andrica). *Assuming the Riemann hypothesis,*

$$
\forall n \geq 1, \quad \sqrt[p]{p_{n+1}} - \sqrt[p]{p_n} \leq \frac{3}{2}. \quad (5.5)
$$

**Corollary 2** (Fourth root modification of Andrica). *Assuming the Riemann hypothesis,*

$$
\forall n \geq 1, \quad \sqrt[p]{p_{n+1}} - \sqrt[p]{p_n} \leq \frac{3}{4}. \quad (5.6)
$$
6 Discussion

While the Riemann hypothesis provides (among very many other things) a nice explicit bound on prime gaps, it is still not quite sufficient to prove Andrica’s conjecture — though as seen above, one can get reasonably close. There are a number of places where the argument might be tightened: First the $4/\pi$ factor in Dudek’s 2014 theorem might be improved, see the appendices, second the pre-factor 2 in our prime gap bound can be somewhat reduced — the presentation above was designed to be simple and direct, not optimal.

Of course the really big improvement would be if any of these results could be made unconditional. While the numerical evidence certainly suggests this, a proof seems impossible with current techniques.

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A Some tighter bounds

Consider this slightly more recent and tighter theorem, which is a little trickier to adapt to current purposes:

**Theorem 8** (Dudek–Grenie–Molteni 2015 [14]). Assume the Riemann hypothesis. Set $c := \frac{1}{2} + \frac{2}{\ln x}$. Then for any $x \geq 2$ there is a prime in the interval

$$\left( x - c\sqrt{x} \ln x, x + c\sqrt{x} \ln x \right). \quad (A.1)$$

To turn this into an explicit bound of the form we are seeking, define $\tilde{p}_n = \frac{p_n + p_{n+1}}{2}$. Then $p_n = \tilde{p}_n - g_n/2$ and $p_{n+1} = \tilde{p}_n + g_n/2$ are symmetrically placed, and are the closest primes to $\tilde{p}_n$. Now take $x = \tilde{p}_n$ and apply the theorem above. One deduces

$$g_n < 2c\sqrt{\tilde{p}_n} \ln \tilde{p}_n = \sqrt{p_n} (\ln \tilde{p}_n + 4); \quad (n \geq 1). \quad (A.2)$$

Now consider the function $\sqrt{p} (\ln p + 4)$ on the domain $p \geq 2$, then

$$\left( \sqrt{p} (\ln p + 4) \right)' = \frac{1}{2} \frac{\ln p + 6}{\sqrt{p}} > 0; \quad \left( \sqrt{p} (\ln p + 4) \right)'' = -\frac{1}{4} p^{-3/2} (\ln p + 4) < 0. \quad (A.3)$$
So the function $\sqrt{p}(\ln p + 4)$ is monotone and convex. Then certainly

$$g_n < \sqrt{p_n}(\ln p_n + 4) + \frac{1}{2} \ln p_n + 6 \frac{g_n}{\sqrt{p_n}}.$$

(A.4)

Therefore

$$g_n \left\{ 1 - \frac{1}{4} \ln p_n + 6 \frac{1}{\sqrt{p_n}} \right\} < \sqrt{p_n}(\ln p_n + 4).$$

(A.5)

Thence for $p_n \geq 5$, that is $n \geq 3$, we have

$$g_n < \sqrt{p_n \ln p_n \times Q_n}; \quad Q_n = \left\{ \frac{1}{1 - \frac{1}{4} \ln p_n + 6 \frac{1}{\sqrt{p_n}}} \right\}.$$

(A.6)

The “quality factor” $Q_n$ in braces tends to monotonically to 1 from above as $p \to \infty$. Choose the specific prime $p_{77}^* = 678798899657777797 > 6.787 \times 10^{18}$, (this is the prime associated with the 77th maximal prime gap [15]), then

$$Q_{77}^* < 1 + \frac{2375}{25746}.$$

(A.7)

So we certainly have

$$g_n < \sqrt{p_n \ln p_n \times \left\{ 1 + \frac{2375}{25746} \right\}}; \quad (p_n \geq 678798899657777797).$$

(A.8)

On the other hand, one can also establish

$$g_n < \sqrt{p_n \ln p_n} ; \quad (5 \leq p_n \leq 678798899657777797).$$

(A.9)

To do this, let the triplet $(i, g^*_i, p^*_i)$ denote the $i^{th}$ maximal prime gap; of width $g^*_i$, starting at the prime $p^*_i$. (77 such maximal prime gaps are currently known [15], up to $g^*_{77} = 1510$ and $p^*_{77}$ as presented above.) Then certainly

$$\forall p_n \in [p^*_i, p^*_j] \quad \text{we have} \quad \frac{g_n}{\sqrt{p_n \ln p_n}} < q_{i,j} := \frac{g^*_j}{\sqrt{p^*_i \ln p^*_i}}.$$ 

(A.10)

Checking the known maximal prime gaps [15] establishes the claim. (We do not need to check each maximal prime gap individually, we just need to choose suitable $i$ and $j$ such that $q_{i,j} < 1$. For instance: $q_{4.5}$, $q_{5.7}$, $q_{7.20}$, and $q_{20.77}$ are all less than unity.

The remaining primes from 5 to $p^*_4 = 23$ are easily checked by hand.) Combining these results we now have:

**Theorem 9** (Prime gaps — tightened bound). **Assuming the Riemann hypothesis**

$$\forall n \geq 1, \quad g_n < \sqrt{p_n \ln(p_n)} \times \left\{ 1 + \frac{2375}{25746} \right\}.$$ 

(A.11)
This is certainly a tighter bound, but somewhat uglier than before. This bound will certainly be improved as additional maximal prime gaps are identified. (One might hope to ultimately replace the $1 + \frac{2375}{25746}$ by 1, but this would seem to require a somewhat different approach.) Using this tightened bound we see:

**Theorem 10** (Modified Andrica — tighter bound). *Assuming the Riemann hypothesis,*

$$\forall n \geq 1, \quad \sqrt[p_{n+1}]{p_{n+1}} - \sqrt[p_n]{p_n} \leq \frac{1}{2} \left\{ 1 + \frac{2375}{25746} \right\}.$$  \hspace{1cm} (A.12)

$$\forall n \geq 1, \forall m > 2, \quad \sqrt[p_{n+1}]{p_{n+1}} - \sqrt[p_n]{p_n} \leq \frac{2 e^{-1}}{(m - 2)} \left\{ 1 + \frac{2375}{25746} \right\}.$$  \hspace{1cm} (A.13)

Additionally we have the tightened corollaries:

**Corollary 3** (Modified Andrica — tighter bound). *Assuming the Riemann hypothesis,*

$$\forall n \geq 1: \quad \sqrt[p_{n+1}]{p_{n+1}} - \sqrt[p_n]{p_n} \leq \frac{41}{50}; \quad \sqrt[p_{n+1}]{p_{n+1}} - \sqrt[p_n]{p_n} \leq \frac{41}{100}.$$  \hspace{1cm} (A.14)

### B Some lim sup results

Instead of carefully bounding $Q_n$ we can use the fact that $\lim_{n \to \infty} Q_n = 1$ from above to deduce (still assuming the Riemann hypothesis):

**Theorem 11** (Lim sup bounds).

$$\limsup_{n \to \infty} \frac{g_n}{\sqrt[p_n]{p_n} \ln p_n} \leq 1.$$  \hspace{1cm} (B.1)

$$\limsup_{n \to \infty} \left( \sqrt[p_{n+1}]{p_{n+1}} - \sqrt[p_n]{p_n} \right) \leq \frac{1}{2}.$$  \hspace{1cm} (B.2)

$$\limsup_{n \to \infty} \left( \sqrt[p_{n+1}]{p_{n+1}} - \sqrt[p_n]{p_n} \right) \leq \frac{2 e^{-1}}{(m - 2)}; \quad (m > 2).$$  \hspace{1cm} (B.3)

$$\limsup_{n \to \infty} \left( \sqrt[p_{n+1}]{p_{n+1}} - \sqrt[p_n]{p_n} \right) \leq \frac{2}{e}.$$  \hspace{1cm} (B.4)

$$\limsup_{n \to \infty} \left( \sqrt[p_{n+1}]{p_{n+1}} - \sqrt[p_n]{p_n} \right) \leq \frac{1}{e}.$$  \hspace{1cm} (B.5)

It may sometimes be preferable to work with these lim sup bounds.

In contrast, lim inf results are, (due to Zhang’s theorem), now essentially trivial.
Theorem 12 (Zhang [16]).

\[ \liminf_{n \to \infty} g_n = C < \infty. \]  \hfill (B.6)

We only need to know that \( \liminf_{n \to \infty} g_n \) is bounded \( C < \infty \), we do not need to know any specific value of \( C \). Going back to the proof of our key theorems, the boundedness of \( \liminf_{n \to \infty} g_n \), together with the fact that the relevant quantities are non-negative, is enough to deduce:

Theorem 13 (Lim inf bounds).

\[ \liminf_{n \to \infty} \frac{g_n}{\sqrt{p_n} \ln p_n} = 0. \]  \hfill (B.7)

\[ \liminf_{n \to \infty} \left( \frac{\sqrt{p_{n+1}}}{\ln p_{n+1}} - \frac{\sqrt{p_n}}{\ln p_n} \right) = 0. \]  \hfill (B.8)

\[ \liminf_{n \to \infty} \left( \sqrt{\frac{p_{n+1}}{p_n}} - \sqrt{\frac{p_n}{p_{n+1}}} \right) = 0; \quad (m > 1). \]  \hfill (B.9)

C Unconditional result for the standard Andrica conjecture

Andrica’s conjecture can be rearranged to be equivalent to

\[ g_n \leq 2\sqrt{p_n} + 1. \]  \hfill (C.1)

In an unpublished note (see comments in [5, 8]) Imran Ghory rephrased this in terms of maximal prime gaps. Using the \((i, g_i^*, p_i^*)\) triplet notation for prime gaps introduced above, Imran Ghory observed the equivalent of

\[ \forall p_n \in [p_i^*, p_j^*] \quad g_n \leq g_j^*; \quad 2\sqrt{p_n} + 1 \geq 2\sqrt{p_i^*} + 1. \]  \hfill (C.2)

That is, Andrica’s conjecture certainly holds on the interval \( p_n \in [p_i^*, p_j^*] \) if one has

\[ g_j^* \leq 2\sqrt{p_i^*} + 1; \quad \text{that is} \quad \left( \frac{g_j^*}{2} - 1 \right)^2 < p_i^*. \]  \hfill (C.3)

But this is easily checked to hold on the intervals \([p_{26}^*, p_{77}^*], [p_{12}^*, p_{26}^*], [p_7^*, p_{12}^*], [p_5^*, p_{77}^*], [p_4^*, p_5^*], [p_3^*, p_4^*], [p_2^*, p_3^*], \text{ and } [p_1^*, p_2^*] \). Thus Andrica’s conjecture certainly holds up to the 77th maximal prime gap, \( p_{77}^* = 678798899965777797 > 6.787 \times 10^{18} \). This bound will certainly be improved as additional maximal prime gaps are identified.
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The key step seems to have been an unpublished observation by Imran Ghory.

See also reference [5], and the discussion in appendix C where the verification range is slightly extended to the 77th maximal prime gap, approximately $6.787 \times 10^{18}$.

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