Big Bang nucleosynthesis and baryogenesis in power-law $f(R)$ gravity: 
Revised constraints from the semianalytical approach

David Wenjie Tian
Faculty of Science, Memorial University, St. John’s, Newfoundland, A1C 5S7, Canada

Abstract
In this paper we investigate the primordial nucleosynthesis in $\mathcal{L} = e^{2-2\beta}R^\beta + 16\pi m_p^{-2}L_m$ gravity, where $e$ is a constant balancing the dimension of the field equation, and $1 < \beta < (4 + \sqrt{6})/5$ for the positivity of energy density and temperature. From the semianalytical approach, the influences of $\beta$ to the decoupling of neutrinos, the freeze-out temperature and concentration of nucleons, the opening of deuterium bottleneck, and the $^4$He abundance are all extensively analyzed; then $\beta$ is constrained to $1 < \beta < 1.05$ for $e = 1$ [1/s] and $1 < \beta < 1.001$ for $e = m_P$ (Planck mass). Supplementarily from the empirical approach, abundances of the lightest elements (D, $^4$He, $^7$Li) are computed by the model-independent best-fit formulae for nonstandard primordial nucleosynthesis, and we find the constraint $1 < \beta \leq 1.0505$ which corresponds to the extra number of neutrino species $0 < \Delta N_{\text{eff}} \leq 0.6365$; also, the $^7$Li abundance problem cannot be solved by $\mathcal{L} = e^{2-2\beta}R^\beta + 16\pi m_p^{-2}L_m$ gravity for this domains of $\beta$. Finally, the consistency with the mechanism of gravitational baryogenesis is estimated.

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1. Introduction

In the past few decades, the increasingly precise measurements for the cosmic abundances of the lightest elements have imposed stringent constraints to the thermal history of the very early Universe. The observed protium, deuterium (D) and $^4$He abundances prove to agree well with those predicted by the standard Big Bang nucleosynthesis (BBN) in general relativity (GR).

As is well known, any modification to the Hubble expansion rate and the time-temperature correspondence would affect the decoupling of neutrinos, the freeze-out of nucleons, the time elapsed to open the deuterium bottleneck, and the abundances of $^4$He along with the other light elements. To better meet the observations from the very early Universe, nonstandard BBN beyond the $SU(3)_c \times SU(2)_W \times U(1)_Y$ minimal standard model [1] or beyond the standard gravitational framework of GR have also received a lot of discussion, such as nonstandard BBN in scalar-tensor gravity [2, 3, 4, 5], Brans-Dicke gravity with a varying energy term related to the cosmic radiation background [6, 7], $f(R)$ gravity [8, 9, 10], $f(G)$ generalized Gauss-Bonnet gravity [11]. Nonstandard BBN in helps constrain these modified gravities from the properties of the very early Universe, which supplements the popular constraints from the accelerated expansion of the late-time Universe.

So far, nonstandard BBN in $f(R) \propto R^\beta$ gravity has been involved in Ref.[8] and studied in Refs.[9, 10]; however, these earlier investigations are not satisfactory. Ref.[8] only calculated the nonstandard...
decoupling temperature of nucleons; the BBN energy scale was inappropriately extended to $T \leq 100$ MeV, and the interconversion rate $\Gamma_{n-p}$ between neutrons and protons was incorrectly equated to the approximate rate at the high-energy domain $T \gg m_n-m_p \approx 1.2933$ MeV. Ref.[9] continued to investigate the primordial $^4$He synthesis in $f(R) \propto R^b$ gravity from a semianalytical approach; however, the BBN process after neutrinos’ decoupling was numerically calculated using the standard Hubble expansion of GR rather than the generalized Hubble rate in $f(R) \propto R^b$ gravity. Also, due to the inconsistent setups of the geometric conventions, the domain of $\beta$ was incorrectly set as $(4-\sqrt{6})/5 < \beta < 1$ in Refs.[8, 9], which had led to quite abnormal behaviors for $\beta \approx (4-\sqrt{6})/5$. Ref.[10] corrected the domain of $\beta$ into $1 < \beta < (4+\sqrt{6})/5$, and re-constrained the parameter $\beta$ by the abundances of both deuterium and $^4$He; however, the computations were carried out using the public BBN code, and the details regarding the influences of $\beta$ to the BBN process were not brought to light.

In this work, we aim to overcome the flaws in Refs.[9, 10], and reveal every detail of the BBN process in $f(R) \propto R^b$ gravity. This paper is analyzed as follows. Section 2 introduces the generalized Friedmann equations for the radiation-dominated Universe in generic $f(R)$ gravity. In Sec. 3, the power-law $f(R)$ gravity with the total Lagrangian density $\mathcal{L} = e^{2-2\beta}R^b + 16\pi m_p^2 \mathcal{L}_m$ is set up ($\epsilon$ being some constant balancing the dimensions of the field equation), with the nonstandard Hubble expansion and the generalized time-temperature relation derived. The decoupling of neutrinos is studied in Sec. 4, while the temperature and neutrons’ concentration at the nucleon freeze-out are computed in Sec. 5. In Sec. 6, the opening of the deuterium bottleneck and the primordial $^4$He abundance are found out, which exerts constraints to the parameter $\beta$ compared with the $^4$He abundance in astronomical measurement. The semianalytical discussion in Secs. 4–6 for $\mathcal{L} = e^{2-2\beta}R^b + 16\pi m_p^2 \mathcal{L}_m$ gravity is taken the GR limit $\beta \to 1$ in Sec. 7 to recover the standard BBN. Moreover, the primordial abundances of deuterium, $^4$He and $^7$Li are calculated in Sec. 8 from the empirical approach using the model-independent best-fit formulae, which supplements the results from the semianalytical approach. Finally, the consistency of $\mathcal{L} = e^{2-2\beta}R^b + 16\pi m_p^2 \mathcal{L}_m$ gravity with the gravitational baryogenesis is estimated in Sec. 9.

Throughout this paper, for the physical quantities involved in the thermal history of the early Universe, we use the natural unit system of particle physics which sets $c = \hbar = k_B = 1$ and is related with le système international d’unités by $1 \text{MeV} = 1.16 \times 10^{10}$ kelvin $= 1.78 \times 10^{-30}$ kg $= (1.97 \times 10^{-13}$ meters$)^{-1} = (6.58 \times 10^{-22}$ seconds$)^{-1}$. On the other hand, for the spacetime geometry, we adopt the conventions $\Gamma^\nu_{\beta\gamma} = \Gamma^\nu_{\gamma\beta} = \partial_\gamma \Gamma^\nu_{\delta\beta} - \partial_\beta \Gamma^\nu_{\delta\gamma} + \partial_\delta \Gamma^\nu_{\gamma\beta}$ and $R_{\mu\nu} = R^\alpha_{\mu\alpha\nu}$ with the metric signature $(-, +, +, +)$.

2. Generalized Friedmann equations in $f(R)$ gravity

As a straightforward generalization of the Hilbert-Einstein action $S_{\text{HE}} = \int \sqrt{-g} \, d^4x \left( R + 16\pi m_p^{-2} \mathcal{L}_m \right)$, $f(R)$ gravity is given by the action

$$ S = \int d^4x \sqrt{-g} \left[ f(R, \epsilon) + 16\pi m_p^{-2} \mathcal{L}_m \right], \quad (1) $$

where $R$ is the Ricci scalar of the spacetime, and $\epsilon$ is some constant balancing the dimensions of the field equation. Also, $m_p$ refers to the Plank mass, which is related to Newton’s constant $G$ by $m_p := 1/\sqrt{G}$ and takes the value $m_p \approx 1.2209 \times 10^{22}$ MeV. Variation of Eq.(1) with respect to the inverse metric $\delta S/\delta g^{\mu\nu} = 0$ yields the field equation

$$ f_R R_{\mu\nu} - \frac{1}{2} f + \left( g_{\mu\nu} \Box - \nabla_\mu \nabla_\nu \right) f_R = 8\pi m_p^{-2} \mathcal{T}^{(m)}_{\mu\nu}, \quad (2) $$
where \( f_R := df(R, \varepsilon)/dR \), \( \Box \) denotes the covariant d’Alembertian \( \Box := g^{\alpha \beta} \nabla_\alpha \nabla_\beta \), and the stress-energy-momentum tensor \( T^{(m)}_{\mu \nu} \) of the physical content is defined by the matter Lagrangian density \( \mathcal{L}_m \) via 
\[
T^{(m)}_{\mu \nu} := -\frac{2}{\sqrt{-g}} \frac{\delta (\sqrt{-g} \mathcal{L}_m)}{\delta g^{\mu \nu}}.
\]
This paper considers the spatially flat, homogeneous and isotropic Universe, which, in the \((t, r, \theta, \varphi)\) comoving coordinates along the cosmic Hubble flow, is depicted by the Friedmann-Robertson-Walker (FRW) line element
\[
ds^2 = -dt^2 + a(t)^2 \left(dr^2 + r^2(d\theta^2 + \sin^2 \theta d\varphi^2)\right),
\](3)
where \( a(t) \) denotes the cosmic scale factor. Assume a perfect-fluid material content \( T^{(n)}_{\mu \nu} = \text{diag}[-\rho, P, P, P] \), with \( \rho \) and \( P \) being the energy density and pressure, respectively. Then Eq.(2) under the flat FRW metric yields the generalized Friedmann equations
\[
\frac{3}{a} \frac{\dot{a}}{a} f_R - \frac{1}{2} f - 3 \frac{\dot{a}}{a} f_{RRR} = -8\pi m_p^2 \rho,
\](4)
\[
\left( \frac{\dot{a}}{a} + 2 \frac{\ddot{a}}{a^2} \right) f_R - \frac{1}{2} f - f_{RRR} \dddot{R} - 3 \frac{\dot{a}}{a} f_{RRR} \dot{R} = 8\pi m_p^2 P,
\](5)
where overdot denotes the derivative with respect to the comoving time, \( f_{RR} := \frac{d^2 f(R, \varepsilon)/dR^2}{f} \), and \( f_{RRR} := \frac{d^3 f(R, \varepsilon)/dR^3}{f} \). Moreover, the equation of local energy-momentum conservation, \( \nabla^\mu T^{(m)}_{\mu \nu} = 0 \), gives rise to
\[
\dot{\rho} + 3 \frac{\dot{a}}{a} (\rho + P) = 0.
\](6)
When \( f(R, \varepsilon) = R \), one recovers Einstein’s equation \( \mathcal{R}_{\mu \nu} - \frac{1}{2} \mathcal{R} g_{\mu \nu} = 8\pi m_p^2 T^{(m)}_{\mu \nu} \) for GR, as well as the standard Friedmann equations \( 3 \frac{\dot{a}^2}{a^2} = -8\pi m_p^2 \rho \) and \( 3 \frac{\ddot{a}}{a} = -4\pi m_p^2 (\rho + P) \).

The very early (i.e. the first few minutes) Universe is absolutely radiation-dominated, with the equation of state \( \rho = 3P \). Thus Eq.(6) integrates and yields that the radiation density is related to the cosmic scale by
\[
\rho = \rho_0 a^{-4} \propto a^{-4}.
\](7)
\( \rho \) attributes to the energy densities of all relativistic species, which are exponentially greater than those of the nonrelativistic particles, and therefore \( \rho = \sum \rho_i (\text{boson}) + \frac{7}{8} \sum \rho_j (\text{fermion}) = \sum \frac{\pi^2}{30} g_i^{(b)} T_i^4 (\text{boson}) + \frac{7}{8} \sum \frac{\pi^2}{30} g_j^{(f)} T_j^4 (\text{fermion}) \), where \( \{g_i^{(b)}, g_j^{(f)}\} \) are the numbers of statistical degrees of freedom for relativistic bosons and fermions, respectively. More concisely, normalizing the temperatures of all relativistic species with respect to photons’ temperature \( T_\gamma \equiv T \), one has the generalized Stefan-Boltzmann law
\[
\rho = \frac{\pi^2}{30} g_* T^4 \quad \text{with} \quad g_* := \sum_{\text{boson}} g_i^{(b)} \left( \frac{T_i}{T} \right)^4 + \frac{7}{8} \sum_{\text{fermion}} g_j^{(f)} \left( \frac{T_j}{T} \right)^4,
\](8)
where, in thermodynamic equilibrium, \( T \) is the common temperature of all relativistic particles.

3. Power-law \( f(R) \) gravity

This paper works with the specific power-law nonlinear gravity
\[
S = \int d^4x \sqrt{-g} \left( e^{2\phi} R^8 + 16\pi m_p^2 \mathcal{L}_m \right),
\](9)
where $\beta = \text{constant} > 0$. Recall that for GR with $\beta = 1$, the first Friedmann equation $3\dot{a}^2/a^2 = -8\pi G m_{\odot}^2 \rho_0 a^{-4}$ yields the behavior $a = a_0 t^{1/2} \propto t^{1/2}$. Similarly, assume a power-law solution ansatz $a = a_0 t^\alpha$ with the index $\alpha = \text{constant} > 0$ – note that this ansatz proves valid for $\mathcal{L} = e^2 - 2\beta R^2 + 16\pi m_{\odot}^2 \mathcal{L}_m$ gravity, though invalid for generic $f(R)$ gravity. This way, the generalized first Friedmann equation (4) yield

$$\beta = 2\alpha, \quad H := \frac{\dot{a}}{a} = \frac{\beta}{2t},$$

(10)

and

$$\left[ \frac{12(\beta - 1)}{\beta} H^2 \right]^\beta \left( \frac{-5\beta^2 + 8\beta - 2}{\beta - 1} \right) = 32\pi e^{2\beta - 2} m_{\odot}^2 \rho,$$

(11)

where $H$ refers to the cosmic Hubble parameter. The weak, strong and dominant energy conditions for classical matter fields require the energy density $\rho$ to be positive definite, and as a consequence, the positivity of the left hand side of Eq.(11) limits $\beta$ to the domain

$$1 < \beta < \frac{4 + \sqrt{6}}{5} \approx 1.2899.$$

(12)

Note that the Ricci scalar for the flat FRW metric with $a = a_0 t^{\beta/2}$ reads

$$R = 6 \left( \frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} \right) = \frac{3\beta(\beta - 1)}{t^2},$$

(13)

so $R > 0$ and $R^2$ is always well defined in this domain.

Eqs.(8) and (11) imply that the expansion rate of the Universe is related to the radiation temperature by

$$H = \left( \frac{\beta}{12(\beta - 1)} \right)^{1/2} \left( \frac{(\beta - 1) g_*}{-5\beta^2 + 8\beta - 2} \right)^{1/2\beta} \left( \sqrt{\frac{32\pi^3 T^2}{30 m_{\odot}}} \right)^{1/\beta} e^{1/\beta}$$

$$= 0.2887 \times \sqrt{\frac{\beta}{\beta - 1}} \times \left( \frac{(\beta - 1) g_*}{-5\beta^2 + 8\beta - 2} \right)^{1/\beta} \left( 0.7164 \cdot T_{\text{MeV}}^2 \right)^{1/\beta} e^{1/\beta - 1} [s^{-1}],$$

(14)

where $T_{\text{MeV}}$ refers to the value (dimensionless) of temperature evaluated in the unit of MeV, $T = T_{\text{MeV}} \times [1 \text{ MeV}], \epsilon_s$ is the value of $\epsilon$ in the unit of $[1/s]$, and numerically $T^2/m_{\odot} = T_{\text{MeV}}^2/8.0276$ [s$^{-1}$].

Moreover, as time elapses after the Big Bang, the space expands and the Universe cools. Eq.(14) along with $H = \beta/(2t)$ lead to the time-temperature relation

$$t = \sqrt{3\beta(\beta - 1)} \left( \frac{1}{(\beta - 1) g_*} \right)^{1/\beta} \left( \frac{30 m_{\odot}}{32\pi^3 T^2} \right)^{1/\beta} e^{1/\beta - 1}$$

$$= \sqrt{3\beta(\beta - 1)} \left( \frac{1.3959}{T_{\text{MeV}}^2} \right)^{1/\beta} e^{1/\beta - 1} [s].$$

(15)

Eqs.(14) and (15) play important roles in studying the primordial nucleosynthesis and the baryogenesis. For the calculations in the subsequent sections, we will utilize two choices of $\epsilon$ to balance the dimensions:

(a) $\epsilon = 1 [s^{-1}]$. This choice can best respect and preserve existent investigations in mathematical relativity for the $f(R)$ class of modified gravity, which have been analyzed for $\mathcal{L} = f(R) + 16\pi m_{\odot}^2 \mathcal{L}_m$
without caring the physical dimensions. Supplementarily, we have \( \epsilon = 6.58 \times 10^{-22} \) MeV.

(b) \( \epsilon = m_p \). The advantage of this choice is there is no need to employ new parameters outside the mathematical expression \( \mathcal{L} = f(R) + 16\pi m_p^2 \mathcal{L}_m \). Supplementarily, \( m_{ps} := m_p [1 / s] \approx 0.1854 \times 10^{44} \text{[s}^{-1}] \). However, the disadvantage of this choice is also apparent. For example, Eq.(9) with \( \epsilon = m_p \) is mathematically and gravitationally equivalent to

\[
S = \int d^4x \sqrt{-g} \left( R^\beta + 16\pi m_p^2 \mathcal{L}_m \right) = \int d^4x \sqrt{-g} \left( R^\beta + 16\pi G^{2-\beta} \mathcal{L}_m \right),
\]

and thus hence, the deviation between \( f(R) = R \) and \( f(R) = R^\beta \) would indicate a departure of the matter-gravity coupling strength from Newton’s constant \( G \) to \( G^{2-\beta} \); as a consequence, to constrain the parameter \( \beta \) for \( L = R^\beta / m_p^{2-2} + 16\pi m_p^{-2} \mathcal{L}_m \), one just need to check the measurement of Newton’s constant rather than recalculate the testable gravitational processes.

4. Chemical and thermal equilibrium of nucleons, and weak freeze-out of neutrinos

According to the \( SU(3)_c \times SU(2)_W \times U(1)_Y \) minimal standard model, primordial nucleosynthesis happens after the temperature drops below \( T = 10 \) MeV, when all mesons have decayed into nucleons. At \( T \approx 10 \) MeV, photons are in thermal equilibrium with neutrons and protons, which are interconverted by the two-body reactions

\[
n + \nu_e \rightleftharpoons p + e^- , \quad n + e^+ \rightleftharpoons p + \bar{\nu}_e ,
\]

as well as the neutron decay/fusion

\[
n \rightleftharpoons p + e^- + \bar{\nu}_e .
\]

When the nuclear reaction rate \( \Gamma(n \rightleftharpoons p) \) is faster than the Hubble expansion rate, the interconversions Eqs.(17) and (18) are fast enough to maintain neutrons and protons in thermal equilibrium.

Introduce the following dimensionless quantity for the number concentration of neutrons among all baryons,

\[
X_n = \frac{n_n}{n_n + n_p},
\]

and thus before the opening of the deuterium bottleneck, the proton concentration is \( X_p = 1 - X_n = \frac{n_p}{n_n + n_p} \).

Regard neutrons and protons as the two energy states of nucleons, and the approximated Maxwell-Boltzmann energy distribution function yields

\[
\frac{X_n^{eq}}{X_p^{eq}} = \frac{n_n}{n_p} = \exp \left( -\frac{Q}{T} + \frac{\mu_e - \mu_{\nu_e}}{T} \right) \approx \exp \left( -\frac{Q}{T} \right),
\]

or

\[
X_n^{eq} = \frac{1}{1 + \exp \left( \frac{Q}{T} \right)},
\]

where \( Q := m_n - m_p = 1.2933 \) MeV denotes the neutron-proton mass difference (with \( m_n = 939.5654 \) MeV, \( m_p = 938.2721 \) MeV), and we have applied the standard-model assumption \( \mu_{\nu_e} = 0 \) and the fact that \( \mu_e \ll T \) for \( T \gtrsim 0.03 \) MeV. Eq.(21) implies that \( X_n^{eq} \rightarrow 1/2 = X_p^{eq} \) for \( T \gg 1.2933 \) MeV, and \( X_n^{eq} \) gradually decreases as the temperature drops, while nucleons remain in weak-interaction equilibrium until neutrinos decouple.
Neutrinos are in equilibrium with photons, nucleons and electrons via weak interactions and elastic scattering. The interaction rate is

\[
\Gamma_{\nu e} \simeq 1.3 G_F^2 T^5 \simeq 0.2688 T_{\text{MeV}}^5 \text{[s}^{-1}],
\]

where \( G_F \) is Fermi’s constant in beta decay and generic weak interactions, and \( G_F = 1.1664 \times 10^{-11} \text{MeV}^{-2} \). Neutrinos decouple when \( \Gamma_{\nu e} = H \), and according to Eqs. (14) and (22), the weak freeze-out temperature
\( T_\nu^f \) is the solution to

\[
T_{\text{MeV}}^{5-2/\beta} = 1.0741 \times \sqrt{\frac{\beta}{\beta - 1}} \times \left[ 0.7164 \times \sqrt{\frac{(\beta - 1) g_*}{-5\beta^2 + 8\beta - 2}} \right]^{1/\beta} e_1^{1-1/\beta} .
\]  
(23)

Fig.1 and Fig.2 illustrate the dependence of \( T_\nu^f \) on \( \beta \) for two different choices of \( \epsilon \), and some typical values of \( T_\nu^f \) have been collected in Tables 1 and 2. Note that in the calculation of \( T_\nu^f \), we have used \( g_* = g_\ast(T : 1 \sim 10 \text{ MeV}), g_b = 2 \) (photon) and \( g_f = 2 \times 2(e^\pm) + 2 \times 3.046 \) (neutrino) = 10.092, and thus the effective number of degree of freedom \( g_* = g_b + \frac{7}{8} g_f = 10.8305 \), with all these relativistic species in thermal equilibrium at the same temperature. That is to say, the effective number of species for light neutrinos is set to be \( N_{\text{eff}} = 3.046 \) rather than \( N_{\text{eff}} = 3 \); this correction attributes to the fact that the neutrino decoupling is actually a thermal process of finite time rather than an instantaneous event [21].

5. Temperature and concentration at nucleon freeze-out

5.1. Temperature for freeze-out of nucleons

After the weak freeze-out of neutrinos, the neutron concentration deviates from the equilibrium value in Eq.(21), and the evolution of \( X_n \) satisfies

\[
\frac{dX_n}{dt} = -\Gamma_{n \rightarrow p} X_n + \Gamma_{p \rightarrow n} (1 - X_n) = -\Gamma_{n \rightarrow p} \left(1 + e^{-\frac{Q}{T}}\right)(X_n - X_{n}^{\text{eq}}) ,
\]  
(24)

where \( \Gamma_{n \rightarrow p}/\Gamma_{p \rightarrow n} \) denotes the reaction rate to convert neutrons/protons into protons/neutrons. When nucleons and leptons are carried apart by the Hubble expansion faster than their collisions, the reactions in Eqs.(17) and (18) cease and \( X_n \) freezes out.

On the other hand, as shown in Fig.1 and Fig.2, \( T_\nu^f \) is positively related to \( \beta \), and \( T_\nu^f \) is always minimized in the GR limit \( \beta \rightarrow 1 \), with \( \min(T_\nu^f) = \lim_{\beta \rightarrow 1} T_\nu^f = 1.3630 \text{ MeV} \); moreover, Eq.(15) indicates \( t \propto T^{-2/\beta} \), thus by setting \( T = \min(T_\nu^f) = 1.3630 \) MeV in Eq.(15) one gets the upper limit of \( t_\nu^f \) for any \( 1 < \beta < (4 + \sqrt{6})/5 \), which is depicted in Fig.3 and Fig.4 with \( t_\nu^f := \max(t_\nu^f) \). Thus, the mean time of neutron decay \( \tau_n = 880.0 \pm 0.9 \) [s] [19] is always far greater than the time elapsed from big bang to neutrino freeze-out. Furthermore, as will be shown in Tables 1 and 2, the time \( t_\nu^f \) by nucleons’ freeze-out also happens within the first few seconds and is still far less than \( \tau_n \). Hence, the rate of three-body reaction in Eq.(18) is negligible by the stage of free-out.

The combined reaction rate \( \Gamma_{n \rightarrow p} \) for the two-body reactions in Eq.(17) is [16]

\[
\Gamma_{n \rightarrow p} = \frac{255}{\tau_n} \left(\frac{T}{Q}\right)^5 \left[\left(\frac{Q}{T}\right)^2 + 6\left(\frac{Q}{T}\right) + 12\right] \text{ [s}^{-1}\text{]} .
\]  
(25)

Introduce the dimensionless variable

\[
x := \frac{Q}{T} ,
\]  
(26)

and then Eq.(25) becomes \( \Gamma_{n \rightarrow p} = \frac{255}{\tau_n} x^{-5}(x^2 + 6x + 12) \text{ [s}^{-1}\text{]} \). Also, the Hubble parameter can be recast into
\[ H(x) = 0.2887 \times \frac{\beta}{\beta - 1} \times \left( \sqrt{\frac{(\beta - 1) g_s}{-5\beta^2 + 8\beta - 2}} \right)^{1/\beta} \left( \frac{1.1983}{x^2} \right)^{1/\beta} e_x^{-1/\beta} \text{ [s}^{-1}] \]  

Figure 3: \( \tau_{v}^{f} = \text{max} \tau_{v}^{i} \) (in [sec]) for \( \varepsilon = 1 \text{ sec}^{-1} = 6.58 \times 10^{-22} \text{ MeV}, T=1.3630 \text{ MeV} \)

Figure 4: \( \tau_{v}^{f} = \text{max} \tau_{v}^{i} \) (in [sec]) for \( \varepsilon = m_P = 1.221 \times 10^{22} \text{ MeV}, T=1.3630 \text{ MeV} \)

where \( H(Q) := H(T = Q) = H(x = 1) \), and \( H(Q) \) is a constant carrying the parameter \( \beta \). If there were
no decay of neutrons, the $X_n$ would freeze out when $\Gamma_{n\to p}(x) = H(x)$,

$$\frac{255 \, x^2 + 6x + 12}{x^5} = \frac{2.887}{\sqrt{\beta - 1}} \times \left( \frac{\sqrt{(\beta - 1) \, g_s}}{-5\beta^2 + 8\beta - 2} \right)^{1/\beta} \left( 1.1983 / x^2 \right)^{1/\beta} \left( e^{-1} \right)^{1/\beta}.$$  \hspace{1cm} (28)

so the freeze-out temperature $T_{f\text{MeV}} = \frac{1.2993}{x_f}$ can be found out by solving $x_f$ from

$$\frac{x^2 + 6x + 12}{x^{5-2/\beta}} = 0.9963 \frac{\sqrt{\beta}}{\beta - 1} \times \left( \frac{\sqrt{(\beta - 1) \, g_s}}{-5\beta^2 + 8\beta - 2} \right)^{1/\beta} \left( 1.1983 \right)^{1/\beta} \left( e^{-1} \right)^{1/\beta}.$$  \hspace{1cm} (29)

An exact solution to Eq.(29) is difficult to work out, so it is numerically solved for a series of $\beta$ in the domain $1 < \beta < (4 + \sqrt{6})/5$, as shown in Tables 1 and 2.

### 5.2. Freeze-out concentration of neutrons

To figure out the concentration $X_n$ of neutrons at the freeze-out temperature $T_f$, rewrite Eq.(24) into

$$\frac{dX_n}{dt} = \frac{dX_n}{dx} \frac{dx}{dT} \frac{dT}{dt} = -\frac{dX_n}{dx} \cdot \dot{x} \cdot \frac{\dot{T}}{T} = -\Gamma_{n\to p} (1 + e^{-x}) \left( X_n - X_n^{eq} \right).$$  \hspace{1cm} (30)

Eqs.(8) and (14) imply that

$$T = \left( \frac{30}{\pi^2 g_s^\rho} \right)^{1/4} = \left( \frac{30 \, e^{-\beta} \, m_P^2 \, (-5\beta^2 + 8\beta - 2)}{32 \pi^3 g_s^\rho} \right)^{1/4} \left( 3 \beta (\beta - 1) \right)^{1/4} \left( t^{-\beta/2} \right),$$  \hspace{1cm} (31)

and thus $\dot{T} / T = -\beta/(2t) = -H(t) = -H(x) = -H(Q) x^{-2/\beta}$, which recasts Eq.(30) into

$$\frac{dX_n}{dx} = -\Gamma_{n\to p} \frac{x^{\beta - 1}}{H(Q)} (1 + e^{-x}) \left( X_n - X_n^{eq} \right).$$  \hspace{1cm} (32)

Define a new function $F(x) := X_n - X_n^{eq}$ to describe the departure of $X_n$ from the ideal equilibrium concentration, and transform $dX_n/dx$ into the evolution equation for $dF(x)/dx$:

$$\frac{dF(x)}{dx} + \Gamma_{n\to p} \frac{x^{\beta - 1}}{H(Q)} (1 + e^{-x}) F(x) = \frac{e^x}{(1 + e^x)^2}.$$  \hspace{1cm} (33)

Its general solution is $F(x) = \tilde{F}(x) E(x)$, where

$$\tilde{F}(x) = \exp \left[ - \int^x \Gamma_{n\to p} \frac{y^{\beta - 1}}{H(Q)} (1 + e^{-y}) \, dy \right],$$  \hspace{1cm} (34)

and $E(x)$ satisfies

$$\frac{dE(x)}{dx} = \frac{1}{\tilde{F}(x)} \frac{e^x}{(1 + e^x)^2}.$$  \hspace{1cm} (35)
Integrating $\tilde{F}(x)E(x)$, we obtain

$$F(x) = \int_{x}^{\infty} d\tilde{x} \frac{e^\tilde{x}}{(1 + e^\tilde{x})^2} \exp \left[ - \int_{\tilde{x}}^{\infty} \frac{\Gamma_{n\to p} y^{\frac{3}{2} - 1}}{H(Q)} (1 + e^{-y}) \, dy \right], \quad (36)$$

and the reverse of $F(x) = X_n - X_n^{\text{eq}}$ leads to

$$X_n = X_n^{\text{eq}} + \int_{x}^{\infty} d\tilde{x} \frac{e^\tilde{x}}{(1 + e^\tilde{x})^2} \exp \left[ - \int_{\tilde{x}}^{\infty} \frac{\Gamma_{n\to p} y^{\frac{3}{2} - 1}}{H(Q)} (1 + e^{-y}) \, dy \right]. \quad (37)$$

$X_n$ satisfies the initial condition $X_n(t \to 0) = X_n(T \gg Q) = X_n(x \to 0) = X_n^{\text{eq}} = 1/(1 + e^x)$. Without the decay of neutrons, $X_n$ would eventually freeze out after the decoupling of neutrinos; effectively setting $x = \infty$ in Eq. (37), we obtain the freeze-out concentration $X_n^{\infty} = X_n(x = \infty)$:

$$X_n^{\infty} = \int_{0}^{\infty} d\tilde{x} \frac{e^\tilde{x}}{(1 + e^\tilde{x})^2} \exp \left[ - \int_{\tilde{x}}^{\infty} \frac{\Gamma_{n\to p} y^{\frac{3}{2} - 1}}{H(Q)} (1 + e^{-y}) \, dy \right] = \int_{0}^{\infty} d\tilde{x} \frac{e^\tilde{x}}{(1 + e^\tilde{x})^2} \exp \left[ - \frac{255}{H(Q) \tau_n} \int_{\tilde{x}}^{\infty} \left( \frac{\gamma^2 + 16\gamma + 12}{\gamma^{\frac{3}{2}} - \gamma} \right) (1 + e^{-y}) \, dy \right], \quad (38)$$

where $X_n^{\infty}(x \to \infty) = 0$. Since an exact analytical result for $X_n^{\infty}$ is difficult (if not impossible) to find out, $X_n^{\infty}$ have been numerically integrated for different $\beta$, as shown in Tables 1 and 2.

6. Opening of deuterium bottleneck and helium synthesis

The number densities of neutrons, protons and deuterium (D), which are nonrelativistic particles at the energy scale $T < 10$ MeV, are separately

$$n_n = 2 \left( \frac{m_n T}{2 \pi} \right)^{3/2} e^{\frac{\mu_n - m_n}{T}}, \quad n_p = 2 \left( \frac{m_p T}{2 \pi} \right)^{3/2} e^{\frac{\mu_p - m_p}{T}}, \quad n_D = 3 \left( \frac{m_D T}{2 \pi} \right)^{3/2} e^{\frac{\mu_D - m_D}{T}}, \quad (39)$$

so the equilibrium of chemical potentials $\mu_D = \mu_n + \mu_p$ yields

$$X_D := 2 \frac{n_D}{n_n + n_p} = \frac{3}{2} \frac{n_n n_p}{n_n + n_p} \left( \frac{2 \pi m_D T}{m_n m_p} \right)^{3/2} e^{(\mu_n + \mu_p - m_D)/T} \quad (40)$$

$$= \frac{3}{2} X_n X_p n_b \left( \frac{2 \pi m_D}{T m_n m_p} \right)^{3/2} e^{B_D/T},$$

where $n_b = n_n + n_p$, and $B_D = m_n + m_p - m_D \approx 2.2246$ MeV refers to the deuteron binding energy (with $m_D = 1875.6129$ MeV, $m_n = 939.5654$ MeV, and $m_p = 938.2721$ MeV). Moreover, $n_p$ is related to the photon number density by

$$n_p = \frac{g_{sS}(T)}{g_{sS}(T_0)} \times \eta_{10} \times 10^{-10} \times n_\gamma = \eta_{10} \times 10^{-10} \times \frac{2 \xi(3)}{\pi^2} T^3 = 0.2346 \times 10^{-10} \eta_{10} T^3, \quad (41)$$

where $g_{sS}(T) = g_{sS}(T_0)$ after the electron-positron annihilation, and $\eta_{10} := 10^{10} \times n_b/n_\gamma$ describes the photon-to-baryon ratio $n_b/n_\gamma$ for the net baryons left after baryogenesis. Substituting Eq. (41) into
Table 1: $\varepsilon = 1$ [1/s] or $\varepsilon_s = 1$ for $H(Q)$

| $\beta$ | $T_\gamma^\nu$ [MeV] | $t_\gamma^\nu$ [s] | $X_\gamma^\nu$ | $T_\nu^\nu$ [MeV] | $t_\nu^\nu$ [s] | $H(Q)$ [1/s] | $X_n^\infty$ | $t_{BBN}$ [s] | $Y_p = 2X_n^{BBN}$ |
|---------|----------------------|--------------------|----------------|-------------------|----------------|----------------|---------------|-----------------|------------------|
| 1.289   | 2.4628755            | 0.026463           | 0.3716563      | 1.5171792         | 0.056119       | 8.965756       | 0.302760      | 8.638302        | 0.59952454       |
| 1.25    | 1.6377907            | 0.197342           | 0.3122414      | 0.9402799         | 0.479537       | 2.170797       | 0.218063      | 40.17368        | 0.41689076       |
| 1.2     | 1.5188824            | 0.276162           | 0.2991222      | 0.8472790         | 0.730566       | 1.662142       | 0.197121      | 61.87626        | 0.36777676       |
| 1.15    | 1.4636230            | 0.318533           | 0.2924263      | 0.7968552         | 0.917006       | 1.455879       | 0.183997      | 84.66697        | 0.33457403       |
| 1.1     | 1.4282857            | 0.344287           | 0.2879234      | 0.7585614         | 1.087931       | 1.333847       | 0.172927      | 112.8207        | 0.30461469       |
| 1.05    | 1.3997179            | 0.363572           | 0.2841493      | 0.7230330         | 1.279480       | 1.242265       | 0.161821      | 151.0530        | 0.27298984       |
| 1.01    | 1.3736317            | 0.384215           | 0.2805937      | 0.6909348         | 1.498067       | 1.166680       | 0.151417      | 195.2714        | 0.24297087       |
| $1 + 10^{-3}$ | 1.3646073        | 0.393550           | 0.2793385      | 0.6814495         | 1.575961       | 1.142595       | 0.155550      | 208.7735        | 0.24575649       |
| $1 + 10^{-4}$ | 1.3632380        | 0.395175           | 0.2791469      | 0.6801943         | 1.587108       | 1.139048       | 0.148162      | 210.3996        | 0.23372999       |
| $1 + 10^{-5}$ | 1.3630539        | 0.395406           | 0.2791211      | 0.6800382         | 1.588536       | 1.138576       | 0.147985      | 210.5848        | 0.23393413       |
| $1 + 10^{-6}$ | 1.3630308        | 0.395436           | 0.2791179      | 0.6800196         | 1.588709       | 1.138517       | 0.147967      | 210.6055        | 0.2336818        |
| $1 + 10^{-7}$ | 1.3630281        | 0.395440           | 0.2791175      | 0.6800174         | 1.588730       | 1.138510       | 0.147965      | 210.6078        | 0.23364422       |
| $1 + 10^{-8}$ | 1.3630277        | 0.395440           | 0.2791175      | 0.6800171         | 1.588732       | 1.138509       | 0.147965      | 210.6080        | 0.2336436        |
| $\beta \rightarrow 1^+$ | 1.362986         | 0.395458           | 0.2791116      | 0.6799823         | 1.588868       | 1.138509       | 0.147965      | 210.6044        | 0.2336534        |
Table 2: $\epsilon = m_\mu$ or $\epsilon_\tau = m_\mu = 0.1854 \times 10^4$ for $H(Q)$

| $\beta$ | $T^\nu_\tau$ [MeV] | $t^\nu_\tau$ [s] | $X_n^\nu$ | $T^\nu_\tau$ [MeV] | $t^\nu_\tau$ [s] | $H(Q)$ [1/s] | $X_n^\infty$ | $t_{BBN}$ [s] | $Y_p = 2X_n^{BBN}$ |
|---------|---------------------|------------------|-----------|---------------------|------------------|-------------|-----------|-------------|------------------|
| 1.289   | 1601.768            | 2.2743$\times 10^{-16}$ | 0.499798  | 1106.8080          | 4.036$\times 10^{-16}$ | 4.503$\times 10^{10}$ | 0.49968   | 1.720$\times 10^{-9}$ | 0.999360        |
| 1.25    | 574.7002            | 3.7094$\times 10^{-14}$ | 0.499437  | 394.90361          | 6.761$\times 10^{-14}$ | 9.778$\times 10^{8}$ | 0.499098  | 8.919$\times 10^{-8}$ | 0.998196        |
| 1.2     | 221.2736            | 4.2086$\times 10^{-12}$ | 0.498539  | 150.79103          | 7.975$\times 10^{-12}$ | 2.704$\times 10^{7}$ | 0.497615  | 3.803$\times 10^{-6}$ | 0.995230        |
| 1.15    | 78.73242            | 7.0718$\times 10^{-10}$ | 0.495893  | 53.07006           | 1.404$\times 10^{-9}$ | 6.409$\times 10^{5}$ | 0.493163  | 1.923$\times 10^{-4}$ | 0.986326        |
| 1.1     | 24.60633            | 2.2686$\times 10^{-7}$  | 0.486863  | 16.28405           | 4.805$\times 10^{-7}$ | 11443.81   | 0.477623  | 0.013150         | 0.955232        |
| 1.05    | 6.481921            | 0.00017           | 0.450284  | 4.086189           | 0.00041           | 142.758    | 0.413515  | 1.31445          | 0.825796        |
| 1.01    | 1.904289            | 0.07503           | 0.336451  | 1.036945           | 0.25003           | 3.12859    | 0.226164  | 72.8185          | 0.416405        |
| $1 + 10^{-3}$ | 1.410608          | 0.33343           | 0.285602  | 0.711275           | 1.30963           | 1.26217    | 0.163013  | 188.995          | 0.263015        |
| $1 + 10^{-4}$ | 1.367772          | 0.38867           | 0.279780  | 0.683132           | 1.55789           | 1.15045    | 0.148929  | 208.314           | 0.235073        |
| $1 + 10^{-5}$ | 1.363507          | 0.39475           | 0.279185  | 0.680332           | 1.58559           | 1.13971    | 0.148062  | 210.375           | 0.233158        |
| $1 + 10^{-6}$ | 1.363076          | 0.39537           | 0.279124  | 0.680049           | 1.58841           | 1.13863    | 0.147974  | 210.585           | 0.232964        |
| $1 + 10^{-7}$ | 1.363033          | 0.39543           | 0.279118  | 0.680020           | 1.58870           | 1.13852    | 0.147966  | 210.606           | 0.232946        |
| $1 + 10^{-8}$ | 1.363028          | 0.39544           | 0.279117  | 0.680017           | 1.58873           | 1.13851    | 0.147965  | 210.608           | 0.232944        |
| $\beta \rightarrow 1^+$ | 1.362986          | 0.39546           | 0.279112  | 0.679982           | 1.58887           | 1.13851    | 0.147965  | 210.604           | 0.232943        |
Eq. (40), one has the deuterium concentration
\[
X_D = 10^{-10} \times \frac{3\zeta(3)}{\pi^2} \eta_{10} X_n X_p \left( \frac{2\pi}{T} \frac{m_D}{m_n m_P} \right)^{3/2} \exp^{B_{D}/T} T^3
\]
\[
\approx 5.6474 \times 10^{-14} \times \eta_{10} X_n X_p e^{B_{D}/T} T^{3/2}.
\]

Note that the value of \( \eta_{10} \) can be determined through
\[
\eta_{10} = \frac{n_b}{n_\gamma} = 10^{10} \times \frac{\rho_{\text{crit}} \Omega_b}{m_P n_\gamma} \approx 274 \Omega_b h^2 = 6.0472 \pm 0.0740,
\]

where \( h \) denotes the Hubble constant in the unit of 100 km·s\(^{-1}\)·Mpc\(^{-1}\), and we have adopted the latest Plank data \( \Omega_b h^2 = 0.02207 \pm 0.00027 \). At \( T_{\text{BBN}} \approx 0.079 \text{ MeV} \), the \( X_D \) peaks and \( X_n \) drops below the concentration predicted by beta decay. The deuterium bottleneck has broken and the remaining free neutrons are quickly fused into \(^4\text{He}\) through the sequence of reactions [12]
\[
\begin{align*}
n + p &\rightarrow D, \\
D + n &\rightarrow ^3\text{H} + p \rightarrow ^4\text{He}, \\
D + p &\rightarrow ^3\text{He} + n \rightarrow ^4\text{He}.
\end{align*}
\]

Following the time-temperature relation Eq. (15) with \( T_{\text{MeV}} = 0.079 \) and \( g_* = 3.383538 \), nucleosynthesis occurs at
\[
t_{\text{BBN}} = \sqrt{3\beta(\beta - 1)} \left( 121.5947 \sqrt{-\frac{5\beta^2 + 8\beta - 2}{\beta - 1}} \right)^{1/\beta} e^{\frac{1}{\beta - 1}} \text{[s]}.
\]

Here \( g_* \approx 3.3835 \); this because around the temperature \( T_{\text{BBN}} \ll m_e \approx 0.5110 \text{ MeV} \) after the electron-positron annihilation, only photons and neutrinos remain as relativistic species with \( T_{\nu}/T_{\gamma} = (4/11)^{1/3} \) (this ratio is independent of the number of neutrino species), hence \( g_*(T \leq m_e) = 2 + \frac{7}{8} \times 3.046 \times 2 \times \left( \frac{4}{11} \right)^{4/3} \approx 3.3835 \). Hence, the neutron concentration at BBN is
\[
X_n^{\text{BBN}} = X_n^{\infty} \exp \left( \frac{t_f - t_{\text{BBN}}}{\tau_n} \right),
\]

and the primordial \(^4\text{He}\) abundance is \( Y_p \approx 2X_n^{\text{BBN}} \). For different values of \( \beta \), \( t_{\text{BBN}} \), \( X_n^{\text{BBN}} \) and \( Y_p \) have been numerically calculated, and the results have been collected in Tables 1 and 2.

7. GR limit

In the limit \( \beta \rightarrow 1 \), the gravitational framework reduces from \( \mathcal{L} = e^{-2\phi} R^8 + 16\pi m_p^2 \mathcal{L}_m \) gravity to GR. In this GR limit, one has
\[
\lim_{\beta \rightarrow 1} \sqrt{\frac{\beta}{\beta - 1}} \times \left( \sqrt{-\frac{\beta - 1}{-5\beta^2 + 8\beta - 2}} \right)^{1/\beta} = 1,
\]

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and

\[
\lim_{\beta \to 1} \sqrt{\beta}(\beta - 1) \left( \frac{-5\beta^2 + 8\beta - 2}{\beta - 1} \right) = 1 = \lim_{\beta \to 1} \left[ \beta(\beta - 1) \right]^{\beta} \left( \frac{-5\beta^2 + 8\beta - 2}{\beta - 1} \right),
\]

so from Eqs. (14), (15) and (31) one recovers the standard Hubble expansion [note: by “standard” we mean the standard big bang cosmology of GR]

\[
\mathcal{H} = \frac{1}{2t} = \left( \frac{8\pi^3}{90g_*} \right)^{1/2} \frac{T^2}{m_p} \simeq 1.6602 \sqrt{g_*} \frac{T^2}{m_p} \simeq 0.2068 \sqrt{g_*} T_{\text{MeV}}^2 \text{[s}^{-1}] .
\]

as well as the the standard time-temperature relation

\[
t = \sqrt{\frac{90}{32\pi^3}} \frac{g_*}{T^2} \frac{m_p}{\sqrt{\mathcal{g}_*} T_{\text{MeV}}^2} \simeq \frac{2.4177}{\sqrt{\mathcal{g}_*} T_{\text{MeV}}} \text{[s]} \text{ or } t T_{\text{MeV}}^2 \simeq \frac{2.4177}{\sqrt{\mathcal{g}_*}}.
\]

Equating \( \mathcal{H} \) to the neutrino reaction rate \( \Gamma_{\nu} \) in Eq. (22), i.e. \( 0.2688 T_{\text{MeV}}^5 = 0.2068 \sqrt{g_*} T_{\text{MeV}}^2 \), one can find that neutrinos decouple at \( T = 1.3630 \text{ MeV} \) and \( t = 0.3955 \text{ [s]} \). Furthermore, equating \( \mathcal{H} \) to the combined two-body reaction rate \( \Gamma_{n \rightarrow p} \) in Eq. (25),

\[
\mathcal{H}(x) = \frac{\mathcal{H}(Q)}{x^2} = \frac{255 x^2 + 6x + 12}{x^5} ,
\]

where \( \mathcal{H}(Q) = 0.3459 \sqrt{g_*} \), it turns out that nucleons freeze out at \( x = 1.9020 \), \( T^* = 0.6800 \text{ MeV} \), and \( t_n^* = 1.5889 \text{ [s]} \). According to Eq. (37) with \( \beta \to 1 \), the neutron concentration after the weak freeze-out of neutrinos is determined by

\[
X_n(x) = X_n^\text{eq} + \int_x^\infty \frac{d\tilde{x}}{(1 + e^{\tilde{x}})^2} \exp \left[ - \int_x^{\tilde{x}} \frac{\Gamma_{n \rightarrow p} y}{\mathcal{H}(Q)} (1 + e^{-y}) dy \right] = X_n^\text{eq} + \int_x^\infty \frac{d\tilde{x}}{(1 + e^{\tilde{x}})^2} \exp \left[ - \frac{255}{\mathcal{H}(Q) \tau_n} \int_x^{\tilde{x}} y^{-4} (y^2 + 16y + 12) (1 + e^{-y}) dy \right] ,
\]

and thus in the absence of neutron decay \( X_n \) would freeze out to the concentration \( X_n(x \to \infty) \)

\[
X_n^\infty = \int_0^\infty \frac{d\tilde{x}}{(1 + e^{\tilde{x}})^2} \exp \left[ - \frac{255}{\mathcal{H}(Q) \tau_n} \frac{x^2 + 3x + 4 + e^{-\tilde{x}}(x + 4)}{\tilde{x}^3} \right] = 0.1480.
\]

Nucleosynthesis begins at \( T \approx 0.079 \text{ MeV} \), which corresponds to \( t_{\text{BBN}} = 210.6045 \text{ [s]} \). Hence, the neutron concentration at BBN is

\[
X_n^{\text{BBN}} = X_n^\infty \exp \left( \frac{t_{\text{BBN}} - t_n^*}{\tau_n} \right) = 0.1167 .
\]

and the primordial helium abundance is

\[
Y_p \approx 2X_n^{\text{BBN}} = 0.2334 .
\]

These numerical results are also collected in Tables 1 and 2 in the bottom row.
8. Empirical constraints from D and $^4$He abundances

So far we have calculated the primordial nucleosynthesis in $\mathcal{L} = e^{2\beta}R^8 + 16\pi m_p^2 \mathcal{L}_m$ gravity and GR from the semianalytical approach. We have seen that primordial synthesis and abundances of the lightest elements (D, $^4$He, and also $^3$H, $^3$He, $^7$Li) rely on the baryon-to-photon ratio $\eta_{10} = 10^{10} n_b/n_\gamma$ and the expansion rate $H$ of the Universe. In addition to the semianalytical approach, the abundances can be also be estimated in an empirical way at high accuracy [22, 23]. For nonstandard expansion $H = \frac{\beta}{\Pi}$ in $\mathcal{L} = e^{2\beta}R^8 + 16\pi m_p^2 \mathcal{L}_m$ gravity that deviates from the standard expansion $H = \frac{1}{\Pi}$ in GR, employ the nonstandard-expansion parameter

$$S := \frac{H}{\mathcal{H}} \Rightarrow S = \beta.$$  

(56)

It has been found that, for the priors $4 \leq \eta_{10} \leq 8$ and $0.85 \leq S \leq 1.15$, the primordial deuterium and $^4$He abundances satisfy the best-fit formulae

$$y_D := 10^2 \times \frac{D}{\mathcal{H}} = 46.5 \times (1 \pm 0.03) \times \left[ \eta_{10} - 6(S - 1) \right]^{-1.6}$$  

(57)

and

$$Y_p = (0.2386 \pm 0.0006) + 2 \times 10^{-4} \times (\tau - 885.7) + \frac{\eta_{10}}{625} \times \frac{S - 1}{6.25},$$  

(58)

the reverse of which respectively yield

$$S = \frac{\eta_{10}}{6} - \frac{1}{6} \left[ \frac{46.5 \times (1 \pm 0.03)}{y_D} \right]^{1/1.6} + 1$$  

(59)

and

$$S = 6.25 \times \left[ Y_p - (0.2386 \pm 0.0006) + 2 \times 10^{-4} \times (885.7 - \tau) \right] - \frac{\eta_{10}}{100} + 1.$$  

(60)

Recall that $\eta_{10} = 6.0472 \pm 0.0740$ for $\Omega_B h^2 = 0.02207 \pm 0.00027$, $\tau = 880.0 \pm 0.9$ [s], and according to the recommended values from the Particle Data Group [19], we have

$$y_D = 2.53 \pm 0.04, \quad Y_p = 0.2465 \pm 0.0097.$$  

(61)

Thus, Eqs.(59) and (60) lead to

$$S = 0.9797 \pm 0.0708 \quad \text{or} \quad 0.9089 \leq S = \beta \leq 1.0505 \quad \text{(deuterium)},$$  

(62)

$$S = 0.9960 \pm 0.1036 \quad \text{or} \quad 0.8925 \leq S = \beta \leq 1.0996 \quad \text{($^4$He)}.$$  

(63)

Here for the errors of mutually independent quantities in $\{x_i + \Delta x_i, x_j + \Delta x_j\} \mapsto y + \Delta y$, we have applied the propagation rules that $\Delta y = \sqrt{(\Delta x_i)^2 + (\Delta x_j)^2}$ for $y = x_i \pm x_j$, $\frac{\Delta y}{y} = \sqrt{(\frac{\Delta x_i}{x_i})^2 + (\frac{\Delta x_j}{x_j})^2}$ for $y = x_i x_j$ or $y = x_i/x_j(i \neq j)$, and $\Delta y = \sqrt{n} x^{n-1} \Delta x$ for $y = x^n$.

Combining Eq.(62) with Eq.(63), we find $0.9089 \leq S = \beta \leq 1.0505$; taking into account the positive energy density/temperature condition $1 < \beta < (4 + \sqrt{6})/5$ in Eq.(12), we further obtain $1 < S = \beta$. 

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1.0505. Since $S$ is related to the extra number of effective neutrino species by

$$S := \frac{H}{\mathcal{H}} = \left(1 + \frac{7}{43} \Delta N_{\nu}\right)^{1/2} \quad \Rightarrow \quad \Delta N_{\nu} = \frac{43}{7} (\beta^2 - 1),$$

(64)

thus for $1 < S = \beta \leq 1.0505$, $\Delta N_{\nu}^{\text{eff}} := N_{\nu}^{\text{eff}} - 3$ is constrained by

$$0 < \Delta N_{\nu}^{\text{eff}} \leq 0.6365.$$  

(65)

Note that the theoretically predicted primordial abundance for $^7\text{Li}$ is found to respect the best-fit formula

$$y_{\text{Li}} := 10^{10} \frac{\text{Li}}{\text{H}} = \frac{(1 \pm 0.1)}{8.5} \times \left[\eta_{10} - 3(S - 1)\right]^2,$$

(66)

which, for the domain $1 < S = \beta \leq 1.0505$, gives rise to

$$y_{\text{Li}} = 4.0892 \pm 0.0012 (\beta = 1.0505) \quad \text{to} \quad 4.3022 \pm 0.0012 (\beta = 1).$$

(67)

Hence,

$$4.0880 \leq y_{\text{Li}} < 4.3034,$$

(68)

which is much greater than the observed abundance $y_{\text{Li}} = 1.6 \pm 0.3$ [19]. This indicates that the lithium problem remains unsolved in $\mathcal{L} = e^{2-2\beta R^\beta} + 16\pi m_p^{-2} \mathcal{L}_m$ gravity.

9. Consistency with gravitational baryogenesis

We just investigated the primordial nucleosynthesis in $\mathcal{L} = e^{2-2\beta R^\beta} + 16\pi m_p^{-2} \mathcal{L}_m$ gravity from the semianalytical and the empirical approaches. The nucleons building the lightest nuclei come from the net baryons left after baryogenesis, and in this section we will quickly check the consistency of $\mathcal{L} = e^{2-2\beta R^\beta} + 16\pi m_p^{-2} \mathcal{L}_m$ gravity with the baryon-antibaryon asymmetry using the framework of gravitational baryogenesis [25], which, compared with traditional Sakharov-type mechanisms, dynamically produces the required baryon asymmetry for an expanding Universe by violating the combined symmetry of charge conjugation, parity transformation and time reversal (CPT) while being in thermal equilibrium. In this mechanism, the dominance of baryons over antibaryons attribute to the coupling between the gradient of the Ricci curvature scalar $R$ and some current $J^\mu_B$ leading to net baryon-lepton charges:

$$\int d^4x \sqrt{-g} \left(\partial_\mu R\right) \frac{J^\mu_B}{M_*^2} = \int d^4x \sqrt{-g} \frac{\dot{R} (n_B - n_{\bar{B}})}{M_*^2},$$

(69)

where $M_*$ refers to the cutoff scale of the effective theory, and is estimated to take the value of the reduced Plank mass $M_* \approx m_p / \sqrt{8\pi}$.

The baryon asymmetry can be depicted by the dimensionless baryon-to-entropy ratio $n_B/s$ of the radiation-dominated Universe, with

$$n_B \approx \frac{1}{6} g_b \mu_B T^2 \quad \text{and} \quad s = \frac{2\pi^2}{45} g_{*s} T^3,$$

(70)

where $g_b = 28 = 2$ (photon) + $2 \times 8$ (gluon) + $3 \times 3$ ($W^\pm$, $Z^0$) + 1 (Higgs) for $T > m_{\text{top quark}} \approx 1.733 \times 10^5$ MeV, and $\mu_B := -\dot{R}/M_*^2$ acts as the effective chemical potential. Also, $g_{*s}$ denotes the entropic effective
Figure 5: $n_B/s$ for $\epsilon = 1 \text{ sec}^{-1} = 6.58 \times 10^{-22} \text{ MeV}$

Figure 6: $n_B/s$ for $\epsilon = 1 \text{ sec}^{-1} = 6.58 \times 10^{-22} \text{ MeV}$
number of degree of freedom, and is defined like \( g_s \) by

\[
g_{ss} := \sum_{boson} g^{(b)}_i \left( \frac{T_i}{T} \right)^3 + \frac{7}{8} \sum_{fermion} g^{(f)}_j \left( \frac{T_j}{T} \right)^3; \tag{71}
\]

one has \( g_{ss} = g_s \) at the baryogenesis era when all standard-model particles are relativistic and in equilibrium. \( g_f = 2 \times 3 \) (neutrino) + 2 \times 6 (charged lepton) + 12 \times 6 (quark) = 90, and \( g_{ss} = g_s = g_b + \frac{7}{8} g_f = 106.75 \). In \( \mathcal{L} = \epsilon^{2-2\beta} R^2 + 16\pi m_p^{-2} \mathcal{L}_m \) gravity for which \( \partial_\mu R \) or \( \dot{R} \) is nontrivial, Eqs.(15) and (70) lead to

\[
\frac{n_B}{s} = \frac{15}{4\pi^2} \frac{g_b}{g_{ss} M_d^2 T_d} \frac{\dot{R}}{T_d} = \frac{45}{2\pi^2} \frac{g_b \beta(\beta - 1)}{g_{ss}^3 M_d^4 T_d^2}
\]

\[
= \frac{5 \sqrt{3} \sqrt{3}}{2\pi^2} \frac{g_b}{g_{ss}} \frac{1}{\sqrt{\beta(\beta - 1)}} \left( \sqrt{\frac{(\beta - 1) g_s}{-5\beta^2 + 8\beta - 2}} \right)^{3/\beta} \left( \sqrt{\frac{32\pi^3 T_d^2}{30 e m_P^2}} \right)^{3/\beta} (e^3)^{3/\beta} M_d^2 T_d^2
\]

where \( T_d \approx 3.3 \times 10^{19} \) MeV is the upper bound on the tensor-mode fluctuations at the inflationary scale [26].

Following the observational value \( \Omega_b h^2 = 0.02207 \pm 0.00027 \) [20], we have the net-baryon-to-entropy ratio \( n_B/s = \frac{n_B}{n_T} = 3.8920 \times 10^{-9} \Omega_b h^2 = (8.5897 \pm 0.1051) \times 10^{-11} \), which remains constant during the expansion of the early Universe and imposes a constraint to \( n_B/s \). For \( \epsilon = \sqrt{3} \) rad/s, Eq.(72) respects this constraint for all \( 1 < \beta < (4 + \sqrt{6})/5 \), as shown in Figure 5, with minor violation for \( 1.001426 \) rad/s, as magnified in Fig. 6; however, this minor violation can be easily removed.

\[
\text{Figure 7: } n_B/s \text{ for } \epsilon = m_p
\]
by a fluctuation of $M_\star$ and $T_d$. For $\varepsilon = m_P$, this constraints is satisfied for $1 < \beta < 1.04255$, as shown in Fig. 7.

10. Conclusions

In this paper, we have investigated the nonstandard BBN in $\mathcal{L} = \varepsilon^2 - 2\beta R^2 + 16\pi m_P^2 \mathcal{L}_m$ gravity. The main results, compared with the standard BBN or the GR limit in Sec. 7, include Eq.(14) for the nonstandard Hubble expansion, Eq.(15) for the generalized time-temperature correspondence, Eq.(23) for the neutrino decoupling temperature $T_\nu$, Eq.(29) for the freeze-out temperature $T_{n_f}$ of nucleons, Eq.(37) for the out-of-equilibrium concentration $X_{n_0}^\infty$. From the data points in Tables 1 and 2, we have shown that every step of BBN is considerably $\beta$-dependent when running over the entire domain $1 < \beta < (4 + \sqrt{6})/5$.

In the semianalytical approach, $\beta$ is constrained to $1 < \beta < 1.05$ for $\varepsilon = 1 [1/s]$ and $1 < \beta < 1.001$ for $\varepsilon = m_P$. In the empirical approach, we have found $1 < \beta \leq 1.0505$ which corresponds to an extra number of neutrino species by $0 < \Delta N_{\nu_\alpha}^\text{eff} \leq 0.6365$. In theory, it might be possible for modified gravities to severely rescale the thermal history of the early Universe without changing the state of the current Universe. This requires the joint investigations of BBN, the cosmic radiation background, and the structure formation, and we will look into the possibility of such strongly modified gravities in our prospective studies.

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