null-plane Quantum Universal $R$-matrix

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Abstract

A non-linear map is applied onto the (non-standard) null-plane deformation of (3+1) Poincaré algebra giving rise to a simpler form of this triangular quantization. A universal $R$-matrix for the null plane quantum algebra is then obtained from a universal $T$-matrix corresponding to a Hopf subalgebra. Finally, the associated Poincaré Poisson–Lie group is quantized by using the FRT approach.
1 Introduction

In a previous work a new triangular quantum deformation of the (3+1) Poincaré algebra was introduced [1]. This quantization was shown to deform in a consistent way both the kinematical and dynamical contents [2] of the null-plane Poincaré symmetry, obtaining deformed Hamiltonians, spin and position operators. The aim of this letter is to provide a quantum universal $R$-matrix for this quantum Poincaré algebra that completes this deformed model, including its corresponding quantum group.

We recall that this problem has been recently solved for the (2+1) case by means of a contraction method and taking as starting point the non-standard deformation of $so(2, 2)$ [3, 4]. As we shall see in the sequel, provided a non-linear change of basis inspired in the results given in [4] has been performed, the (3+1) universal $R$-matrix can be obtained by using a universal $T$-matrix technique [5, 6]. An interesting consequence of this procedure will be the factorized form of the universal $R$-matrix, that is given by ordered (usual) exponentials of the elements appearing within the corresponding classical $r$-matrix. This kind of factorized expressions appears in a natural way in connection with transfer matrices problems in quantum field theory where the $R$-matrix we obtain could be useful in order to construct new integrable examples.

Let us perform the following transformation on the null-plane generators $\{\bar{P}_+, \bar{P}_i, \bar{P}_e, \bar{F}_i, \bar{K}_3, \bar{J}_3\}$ ($i = 1, 2$) of the quantum Poincaré algebra $U_{\mathbb{P}}(3 + 1)$ [1]:

$$
P_+ = \bar{P}_+; \quad E_i = \bar{E}_i; \quad J_3 = \bar{J}_3; \quad i = 1, 2;
\quad P_+ = e^{zP_0} \bar{P}_-; \quad P_i = e^{zP_0} \bar{P}_i,
\quad F_1 = e^{zP_0} \left( \bar{F}_1 - z\bar{E}_1 \bar{P}_- - z\bar{J}_3 \bar{P}_2 \right),
\quad F_2 = e^{zP_0} \left( \bar{F}_2 - z\bar{E}_2 \bar{P}_- + z\bar{J}_3 \bar{P}_1 \right),
\quad K_3 = e^{zP_0} \left( \bar{K}_3 - z\bar{E}_1 \bar{P}_1 - z\bar{E}_2 \bar{P}_2 \right).
$$

(1.1)

After this change of basis, the resulting coproduct $\Delta$, non-vanishing commutation relations, counit $\epsilon$ and antipode $\gamma$ read

$$
\Delta(X) = 1 \otimes X + X \otimes 1, \quad \text{for} \quad X \in \{P_+, E_i, J_3\},
\Delta(Y) = 1 \otimes Y + Y \otimes e^{2zP_0}, \quad \text{for} \quad Y \in \{P_-, P_i\},
\Delta(F_1) = 1 \otimes F_1 + F_1 \otimes e^{2zP_0} - 2zP_- \otimes E_1 e^{2zP_0} - 2zP_2 \otimes J_3 e^{2zP_0},
\Delta(F_2) = 1 \otimes F_2 + F_2 \otimes e^{2zP_0} - 2zP_- \otimes E_2 e^{2zP_0} + 2zP_1 \otimes J_3 e^{2zP_0},
\Delta(K_3) = 1 \otimes K_3 + K_3 \otimes e^{2zP_0} - 2zP_1 \otimes E_1 e^{2zP_0} - 2zP_2 \otimes E_2 e^{2zP_0};
$$

(1.2)

$$
[K_3, P_+] = \frac{e^{2zP_0} - 1}{2z}, \quad [K_3, P_-] = -P_- - zP_1^2 - zP_2^2,
[K_3, E_i] = E_i e^{2zP_0}, \quad [K_3, F_i] = -F_i - 2zK_3 P_i,
[J_3, P_i] = -\varepsilon_{ij3} P_j, \quad [J_3, E_i] = -\varepsilon_{ij3} E_j, \quad [J_3, F_i] = -\varepsilon_{ij3} F_j,
[E_i, P_j] = \delta_{ij} e^{2zP_0} - 1, \quad [F_i, P_j] = \delta_{ij} (P_+ + zP_1^2 + zP_2^2),
$$

(1.3)
\[ [E_i, F_j] = \delta_{ij} K_3 + \varepsilon_{ij3} J_3 e^{2zP_+}, \quad [P_+, F_i] = -P_i, \]
\[ [F_1, F_2] = 2z(P_1 F_2 - P_2 F_1), \quad [P_-, E_i] = -P_i; \quad (1.3) \]
\[ \epsilon(X) = 0; \quad \text{for} \; X \in \{P_\pm, P_i, E_i, F_i, K_3, J_3\}; \quad (1.4) \]
\[ \gamma(X) = -X \quad \text{for} \; X \in \{P_+, E_i, J_3\}, \]
\[ \gamma(Y) = -Y e^{-2zP_+} \quad \text{for} \; X \in \{P_-, P_i\}, \]
\[ \gamma(F_1) = -(F_1 + 2zP_- E_1 + 2zP_2 J_3) e^{-2zP_+}, \]
\[ \gamma(F_2) = -(F_2 + 2zP_- E_2 - 2zP_1 J_3) e^{-2zP_+}, \]
\[ \gamma(K_3) = -(K_3 + 2zP_1 E_1 + 2zP_2 E_2) e^{-2zP_+}. \quad (1.5) \]

Note that both the coproduct and commutation relations are now much simpler when compared to the original ones [1]; in particular, the quantum component \( W^q_+ \) of the Pauli-Lubanski operator has no contribution in (1.3). In general, the map (1.1) can be used to reproduce in this new basis the physically relevant operators introduced in [1]. For instance, the deformed square of the mass \( M^2_\parallel \) is now
\[ M^2_\parallel = P_+ - \frac{1 - e^{-2zP_+}}{z} - P_1^2 e^{-2zP_+} - P_2^2 e^{-2zP_+}, \quad (1.6) \]
and it induces a deformed null-plane evolution governed by a \( q \)-Schrödinger equation that has been studied for the (2+1) case in [4].

The key of our construction of the universal \( R \)-matrix is to focus on the six generators appearing in the classical \( r \)-matrix underlying this quantum deformation
\[ r = 2(K_3 \wedge P_+ + E_1 \wedge P_1 + E_2 \wedge P_2), \quad (1.7) \]
since they close a Hopf subalgebra \( U_\pm g \) after quantization. The universal \( T \)-matrix for this Hopf subalgebra can be computed, and this canonical element will give rise to a (factorized) universal \( R \)-matrix for \( U_\pm g \) in a straightforward way. The important point is that this \( R \)-matrix can be shown to be a universal \( R \)-matrix for the whole null-plane quantum Poincaré algebra. As a first application of this result, the null-plane quantum Poincaré group will be obtained.

### 2 Universal \( T \)-matrix for \( U_\pm g \)

The universal \( T \)-matrix of a Hopf algebra, considered for the first time by Fronsdal and Galindo [3, 4], is the Hopf algebra dual form
\[ T = \sum_{\mu} X^\mu \otimes p_\mu, \quad (2.1) \]
where \( \{X^\mu\} \) is a basis for the Hopf algebra and \( \{p_\mu\} \) its dual: \( \langle p_\nu, X^\mu \rangle = \delta_\nu^\mu \). Let us remark that, in spite of the presence of a particular basis in (2.1), the \( T \)-matrix is by definition basis-independent.
We are interested in the Hopf algebra dual form for \( U_\mathbb{C} ^g \) with generators \( \{ P_+, P, E_i, K_3 \} \) \( (i = 1, 2) \). Let us choose the basis \( X^{abcdef} = E_i^a E_j^b P_+^c K_3^d P_e^f \). Its dual basis will be given by the monomials \( p_{lmnpqr} \) such that

\[
\langle p_{lmnpqr}, X^{abcdef} \rangle = \delta_i^a \delta_j^b \delta_k^c \delta_l^d \delta_m^e \delta_n^f.
\]  

(2.2)

We can express duality in an explicit way by means of two structure tensors \( E \) and \( F \) that give, respectively, the product and the coproduct in the Hopf algebra. For our purposes it suffices to consider the latter, so we have

\[
\Delta(X^{abcdef}) := F^{abcdef}_{ijklmn;pqrs} X^{ijklmn} \otimes X^{pqrs} \quad (2.3)
\]

\[
p_{ijklmn} p_{pqrs} = F^{abcdef}_{ijklmn;pqrs} p^{abcdef} . \quad (2.4)
\]

(2.4)

In order to compute the \( T \)-matrix we only need to know very few selected components of this tensor. This fact has been already used in [7], where all the essential reasonings needed to prove the following statements can be found. From (2.4) and taking into account that \( p_{000000} = 1 \) we get

\[
F^{abcdef}_{ijklmn;pqrs} = \delta_i^a \delta_j^b \delta_k^c \delta_l^d \delta_m^e \delta_n^f.
\]

(2.5)

\[
F^{abcdef}_{ijklmn;000000} = \delta_i^a \delta_j^b \delta_k^c \delta_l^d \delta_m^e \delta_n^f.
\]

\[
F^{abcdef}_{ijklmn;000000} = \delta_i^a \delta_j^b \delta_k^c \delta_l^d \delta_m^e \delta_n^f.
\]

(2.6)

And by computing the coproducts of some specific monomials \( X^{abcdef} \) and comparing them to (2.3), it can be checked that

\[
F^{abcdef}_{000000;000000} = \delta_i^a \delta_j^b \delta_k^c \delta_l^d \delta_m^e \delta_n^f.
\]

\[
F^{abcdef}_{000000;000000} = \delta_i^a \delta_j^b \delta_k^c \delta_l^d \delta_m^e \delta_n^f.
\]

(2.7)

\[
F^{abcdef}_{000000;000000} = \delta_i^a \delta_j^b \delta_k^c \delta_l^d \delta_m^e \delta_n^f.
\]

\[
F^{abcdef}_{000000;000000} = \delta_i^a \delta_j^b \delta_k^c \delta_l^d \delta_m^e \delta_n^f.
\]

(2.8)

\[
F^{abcdef}_{000000;000000} = \delta_i^a \delta_j^b \delta_k^c \delta_l^d \delta_m^e \delta_n^f.
\]

(2.9)

\[
\mathcal{T} = e^{E_2 \otimes \hat{e}_2} e^{E_1 \otimes \hat{e}_1} e^{P_+ \otimes \hat{a}_+} e^{K_3 \otimes \hat{a}_3} e^{P_1 \otimes \hat{a}_1} e^{P_2 \otimes \hat{a}_2}.
\]  

(2.10)
3 Universal $R$-matrix

A universal $R$-matrix for $U_zg$ can be easily deduced from the $T$-matrix (2.9) provided there exists an algebra homomorphism and coalgebra antihomomorphism $\Phi$ between the quantum algebra $U_zg$ and its associated dual Hopf algebra (quantum group) $Fun_z(G)$ [8]. If this condition is fulfilled, the element

$$R = (\text{id} \otimes \Phi)T,$$  \hspace{1cm} (3.1)

with $\Phi$ acting on the generators of $Fun_z(G)$, is a solution of the quantum Yang–Baxter equation and, moreover,

$$R\Delta(X)R^{-1} = \sigma \circ \Delta(X), \quad \text{with} \quad \sigma(a \otimes b) = b \otimes a. \hspace{1cm} (3.2)$$

In our case, let us compute the defining relations for $Fun_z(G)$. The elements in (2.7) are precisely dual to the generators of $U_zg$, so they can be taken as generators for $Fun_z(G)$. The main tool in order to derive the commutation rules among them is again the structure tensor $F$. From (2.4) we have

$$[p_{ijklmn}, p_{pqrs tu}] = (F^{abcdef}_{ijklmn, pqrs tu} - F^{abcdef}_{pqrs tu, ijkln})p_{abcdef} \hspace{1cm} (3.3)$$

for two arbitrary elements in $Fun_z(G)$. Let us recall that we can obtain information about the components of the tensor $F$ from the coproduct in the quantum algebra, exactly as we did in the previous section. In particular, the relevant terms in order to compute $[x, y]$, where $x$ and $y$ are two of the generators considered above, are $X \otimes Y$ and $Y \otimes X$ ($X$ and $Y$ being their respective dual elements) appearing in the coproduct of elements of $U_zg$ [4]. This, together with a careful preservation of the order throughout the computations allow us to derive the following relations

$$[\hat{k}_3, \hat{a}_+] = 2z(e^{\hat{k}_3} - 1), \quad [\hat{a}_i, \hat{a}_+] = 2z\hat{a}_i e^{\hat{k}_3}, \quad [\hat{e}_i, \hat{a}_j] = 2z\delta_{ij}(e^{\hat{k}_3} - 1), \hspace{1cm} (3.4)$$

and to show that the remaining commutators vanish. In order to find the coproduct of $Fun_z(G)$ we consider a $5 \times 5$ matrix representation of $U_zg$:

$$D(P_+) = \frac{1}{2}(e_{10} + e_{40}), \quad D(P_1) = e_{20}, \quad D(P_2) = e_{30}, \quad D(K_3) = e_{14} + e_{41},$$

$$D(E_1) = \frac{1}{2}(e_{12} + e_{21} - e_{24} + e_{42}), \quad D(E_2) = \frac{1}{2}(e_{13} + e_{31} - e_{34} + e_{43}), \hspace{1cm} (3.5)$$

where $e_{ij}$ is the matrix with a single 1 entry at row $i$, column $j$, and zeros at the remaining entries (note that this representation is a classical one). By applying the universal $T$-matrix (2.9) we obtain an element of the quantum group $Fun_z(G)$:

$$D(T) = \exp\{D(E_2)\hat{e}_2\} \exp\{D(E_1)\hat{e}_1\} \exp\{D(P_+)\hat{a}_+\}$$
$$\times \exp\{D(K_3)\hat{k}_3\} \exp\{D(P_1)\hat{a}_1\} \exp\{D(P_2)\hat{a}_2\}$$

5
of the classicalization of the (2+1) result given in [4], and can be seen as an ordered exponentiation of Poincaré algebra (1.2)–(1.5). This

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
\frac{1}{2}(\hat{a}_+ + \hat{e}_1 \hat{a}_1 + \hat{e}_2 \hat{a}_2) & \cosh(\hat{k}_3) + f & \frac{1}{2} \hat{e}_1 & \frac{1}{2} \hat{e}_2 & \sinh(\hat{k}_3) - f \\
\hat{a}_1 & \frac{1}{2} \hat{e}_1 e^{-\hat{k}_3} & 1 & 0 & -\frac{1}{2} \hat{e}_1 e^{-\hat{k}_3} \\
\hat{a}_2 & \frac{1}{2} \hat{e}_2 e^{-\hat{k}_3} & 0 & 1 & -\frac{1}{2} \hat{e}_2 e^{-\hat{k}_3} \\
\frac{1}{2}(\hat{a}_+ + \hat{e}_1 \hat{a}_1 + \hat{e}_2 \hat{a}_2) & \sinh(\hat{k}_3) + f & \frac{1}{2} \hat{e}_1 & \frac{1}{2} \hat{e}_2 & \cosh(\hat{k}_3) - f
\end{pmatrix}
\]

(3.6)

where

\[
f = \frac{1}{8}(\hat{e}_1^2 + \hat{e}_2^2) e^{-\hat{k}_3}.
\]

Hence, the coproduct of Fun$_z(G)$ can be derived from the multiplication of two quantum matrices $D(T) \otimes D(T)$:

\[
\Delta(\hat{k}_3) = 1 \otimes \hat{k}_3 + \hat{k}_3 \otimes 1, \quad \Delta(\hat{a}_i) = 1 \otimes \hat{a}_i + \hat{a}_i \otimes 1, \\
\Delta(\hat{e}_i) = e^{\hat{k}_3} \otimes \hat{e}_i + \hat{e}_i \otimes 1, \\
\Delta(\hat{a}_+) = e^{\hat{k}_3} \otimes \hat{a}_+ + \hat{a}_+ \otimes 1 - \hat{a}_1 e^{\hat{k}_3} \otimes \hat{e}_1 - \hat{a}_2 e^{\hat{k}_3} \otimes \hat{e}_2.
\]

(3.8)

By recalling the coproduct and commutation rules of $U_z g$ given in (1.2) and (1.3), together with the expressions (3.8) and (3.4), we get the map we were looking for:

\[
\Phi(\hat{a}_+) = -2z K_3, \quad \Phi(\hat{a}_i) = -2z E_i, \quad \Phi(\hat{k}_3) = 2z P_+, \quad \Phi(\hat{e}_i) = 2z P_i.
\]

(3.9)

Hence, the universal $R$-matrix for $U_z g$ is:

\[
\mathcal{R} = \exp\{2z E_2 \otimes P_2\} \exp\{2z E_1 \otimes P_1\} \exp\{-2z P_+ \otimes K_3\} \\
\times \exp\{2z K_3 \otimes P_+\} \exp\{-2z P_1 \otimes E_1\} \exp\{-2z P_2 \otimes E_2\}.
\]

(3.10)

The $T$-matrix construction ensures that the element (3.10) is a solution of the quantum Yang–Baxter equation and that (3.2) is fulfilled for all the generators of the Hopf subalgebra. Furthermore, this condition is also true for the four remaining generators ($P_-, F_1, F_2$ and $J_3$). This fact can be proved by computing $\mathcal{R} \Delta \mathcal{R}^{-1}$ for each of them, with $\mathcal{R}$ written in the form $\mathcal{R} = e^{A_1} e^{A_2}$ where

\[
A_1 = 2z(E_1 \otimes P_1 + E_2 \otimes P_2 - P_+ \otimes K_3), \quad A_2 = -2z(P_1 \otimes E_1 + P_2 \otimes E_2 - K_3 \otimes P_+).
\]

(3.11)

Therefore, we conclude that (3.10) is a universal $R$-matrix for the null-plane deformation of Poincaré algebra (1.2)–(1.3). This $R$-matrix is just the natural generalization of the (2+1) result given in [4], and can be seen as an ordered exponentiation of the classical $r$-matrix (1.7).

4 Null-plane quantum Poincaré group

The following generic element of the quantum Poincaré group Fun$_z(P(3 + 1))$:

\[
D(\mathcal{P}) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
\frac{\hat{x}^+ + \hat{x}^-}{2} & \Lambda_0^0 & \Lambda_1^0 & \Lambda_2^0 & \Lambda_3^0 \\
\hat{x}^1 & \Lambda_0^1 & \Lambda_1^1 & \Lambda_2^1 & \Lambda_3^1 \\
\hat{x}^2 & \Lambda_0^2 & \Lambda_1^2 & \Lambda_2^2 & \Lambda_3^2 \\
\frac{\hat{x}^+ - \hat{x}^-}{2} & \Lambda_0^3 & \Lambda_1^3 & \Lambda_2^3 & \Lambda_3^3
\end{pmatrix},
\]

(4.1)
can be regarded as the non-commutative analogue of a null-plane Poincaré group element obtained by means of a $5 \times 5$ matrix representation of the quantum Poincaré algebra given by the six matrices $(3.3)$ together with

$$D(P_{\pm}) = e_{10} - e_{40}, \quad D(F_1) = e_{12} + e_{24} - e_{42},$$
$$D(J_3) = e_{23} - e_{32}, \quad D(F_2) = e_{13} + e_{31} + e_{43} - e_{43}, \quad (4.2)$$

where the exponentials of the translation generators $P_{\pm}, P_i$ have been located at the left. The quantum Lorentz coordinates $\hat{\Lambda}^\mu$ satisfy the pseudo-orthogonality condition:

$$\hat{\Lambda}^\mu \hat{\Lambda}^\rho \eta^{\mu\rho} = \eta^{\mu\rho}, \quad (\eta^{\mu\rho}) = \text{diag} (1, -1, -1, -1). \quad (4.3)$$

The coproduct of $Fun_\mathbb{Z}(P(3 + 1))$ is provided by $D(\mathcal{P}) \otimes D(\mathcal{P})$. The counit and antipode come from relations $\epsilon(D(\mathcal{P})) = I$ ($I$ is the $5 \times 5$ identity matrix) and $\gamma(D(\mathcal{P})) = D(\mathcal{P})^{-1}$. On the other hand, the associated commutation relations of the quantum Poincaré group can be deduced via the FRT approach $[9]$. The universal $R$-matrix $(3.11)$ written in the representation $(3.3)$ reduces to

$$D(\mathcal{R}) = I \otimes I + 2z( D(K_3) \wedge D(P_+) + D(E_1) \wedge D(P_1) + D(E_2) \wedge D(P_2)). \quad (4.4)$$

Since this element fulfills the property $(3.2)$ we can apply the prescription

$$D(\mathcal{R})D(\mathcal{P})_1D(\mathcal{P})_2 = D(\mathcal{P})_2D(\mathcal{P})_1D(\mathcal{R}), \quad (4.5)$$

where $D(\mathcal{P})_1 = D(\mathcal{P}) \otimes I$ and $D(\mathcal{P})_2 = I \otimes D(\mathcal{P})$, thus obtaining the commutation rules of $Fun_\mathbb{Z}(P(3 + 1))$:

$$[\hat{x}^+, \hat{x}^i] = -2z \hat{x}^i, \quad [\hat{x}^+, \hat{x}^-] = -2z \hat{x}^-, \quad [\hat{x}^i, \hat{x}^-] = 0, \quad [\hat{x}^1, \hat{x}^2] = 0,$$
$$[\hat{\Lambda}^\mu, \hat{\Lambda}^\rho] = 0, \quad \nu, \mu, \rho, \sigma = 0, 1, 2, 3;$$

$$[\hat{\Lambda}^\mu, \hat{x}^+] = -2z \delta_{\mu 0}(\hat{\Lambda}^3_\mu - \delta_{\nu 0} + \delta_{\nu 3}) - 2z \delta_{\mu 3}(\hat{\Lambda}^0_\nu + \delta_{\nu 0} - \delta_{\nu 3}) + z(\hat{\Lambda}^0_\mu + \hat{\Lambda}^3_\mu)(\hat{\Lambda}^0_\nu + \hat{\Lambda}^3_\nu),$$

$$[\hat{\Lambda}^\mu, \hat{x}^-] = \frac{1}{2} z \delta_{\mu 0}(-\delta_{\nu 0} + \delta_{\nu 3}) + \frac{1}{2} z \delta_{\mu 3}(-\delta_{\nu 0} + \delta_{\nu 3}) + \frac{1}{2} z (\hat{\Lambda}^0_\mu + \hat{\Lambda}^3_\mu)(\hat{\Lambda}^0_\nu - \hat{\Lambda}^3_\nu),$$

$$[\hat{\Lambda}^\mu, \hat{x}^1] = z \delta_{\mu 2} \hat{\Lambda}^1_\nu + z \delta_{\mu 1}(-\hat{\Lambda}^0_\nu + \hat{\Lambda}^3_\nu + \delta_{\nu 0} - \delta_{\nu 3}) + z \hat{\Lambda}^1_\nu(\hat{\Lambda}^0_\nu + \hat{\Lambda}^3_\nu - 1),$$

$$[\hat{\Lambda}^\mu, \hat{x}^2] = z \delta_{\mu 1} \hat{\Lambda}^2_\nu + z \delta_{\mu 2}(-\hat{\Lambda}^0_\nu + \hat{\Lambda}^3_\nu + \delta_{\nu 0} - \delta_{\nu 3}) + z \hat{\Lambda}^2_\nu(\hat{\Lambda}^0_\nu + \hat{\Lambda}^3_\nu - 1). \quad (4.6)$$

The commutators among the quantum coordinates $\hat{x}^\pm, \hat{x}^i$ can be interpreted as the null-plane quantum Poincaré plane. It is also worth mentioning that the universal $R$-matrix can be used to construct a $q$-differential calculus $[10]$ on the null-plane quantum Poincaré group; in general, non-standard deformations exhibit interesting properties in this context $[11]$. It can be checked these commutators are a Weyl quantization of the Poisson brackets of the coordinate functions $\{x^\pm, x^i, \Lambda^\mu \}$ on the classical Poincaré group, which can be obtained by means of the Poisson bivector

$$\{ D(\mathcal{P}) \hat{\otimes} D(\mathcal{P}) \} = [r, D(\mathcal{P}) \hat{\otimes} D(\mathcal{P})], \quad (4.7)$$
writing the classical $r$-matrix (1.7) in terms of the matrix representation (3.3) and applying the corresponding pseudo-orthogonality relations (4.3). This was the method used in [12, 13] to construct the $\kappa$-Poincaré group (recall that, in this case, only the (2+1) universal $R$-matrix has been found in [14]).

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