On the Renormalization of Heavy Quark Effective Field Theory

WOLFGANG KILIAN  THOMAS MANNEL

Institut für Kernphysik
Technische Hochschule Darmstadt
Schlossgartenstr. 9, D–64289 Darmstadt
Germany

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Abstract

The construction of heavy quark effective field theory (HqEFT) is extended to arbitrary order in both expansion parameters \( \alpha_s \) and \( 1/m_q \). Matching conditions are discussed for the general case, and it is verified that this approach correctly reproduces the infrared behaviour of full QCD. Choosing a renormalization scheme in the full theory fixes the renormalization scheme in the effective theory except for the scale of the heavy quark field. Explicit formulae are given for the effective Lagrangian, and one-loop matching renormalization constants are computed for the operators of order \( 1/m \). Finally, the multiparticle sector of HqEFT is considered.
1 Introduction

The Heavy Quark Limit (HQL) has become a very useful tool for the description of systems involving one heavy quark. The main progress consists in the exploitation of additional symmetries occurring in the limit of infinite mass of heavy quarks. These additional symmetries allow model independent predictions for systems with heavy quarks, e.g., they yield model independent relations between form factors of weak transition matrix elements and also model independent absolute and relative normalizations of form factors at certain kinematic points.

The predictions obtained in the HQL receive corrections which are governed by two small parameters. The first type of corrections are the QCD short distance corrections which are obtained in a perturbation series in the parameter $\alpha_s(m)$, the strong coupling constant at the scale of the mass $m$ of the heavy quark. The second type are the recoil corrections governed by the small parameter $\Lambda_{\text{QCD}}/m$.

A convenient tool to deal with these corrections to the HQL is the so called Heavy Quark Effective Field Theory (HqEFT) which was originally formulated in \[2, 3, 4\]. The construction of any effective theory consists of two steps \[5\]. The first step is to identify the degrees of freedom which are irrelevant at the scales under consideration and to remove them. In the language of functional integrals these degrees of freedom are integrated out in the functional integral. Although generally the action of the full theory is the integral of a local Lagrangian, this first step will leave us with a nonlocal action functional. However, this nonlocality is connected to the large mass $M$ of the irrelevant degrees of freedom and an effective theory may be constructed, if the nonlocal action functional can be expanded into local terms where the expansion parameter is $1/M$.

This point of view has been elaborated in \[6\] for the case of HqEFT by identifying the massive degrees of freedom and by explicitly integrating them out from the functional integral of QCD. In this way the tree–level contributions in all orders of $1/m$ have been obtained. In other words, this approach yields the correct HqEFT at the scale of the heavy quark mass $m$, i.e., the tree–level matching of full QCD to the effective theory. However, computing the coefficients in the HqEFT Lagrangian to higher order in the perturbation expansion necessarily involves loop calculations using the Feynman rules of HqEFT. In the past these matching contributions usually have been obtained by evaluating corresponding diagrams in the full and effective theory separately, which is inconvenient because of spurious infrared (IR) divergences that cancel only in the final result.

The second problem with this approach is connected to the multiparticle states of HqEFT. In the HQL the numbers of heavy quarks and heavy antiquarks are separately conserved and the derivation given in \[6\] only deals with the one–particle sector of HqEFT. However, it has been noticed that in the two–particle sector some unusual features of HqEFT appear which are related to the fact that the naively calculated anomalous dimensions pick up an imaginary part which...
leads to phases in the Wilson coefficient functions. It has been shown later [7] that one may in fact define real anomalous dimensions by redefining the multiparticle states of HqEFT in an appropriate way.

The purpose of the present paper is to clarify these two points and to supplement the discussion given in [6]. We shall extend the work in [6] by stating the matching and renormalization conditions for the effective theory beyond tree level and thus provide a method to construct the effective theory to arbitrary order in the two expansion parameters. The ideas of [2] are extended in order to show that this theory indeed reproduces the results of full QCD in the given approximation. In Sec. 2 we fix our notation by showing at the level of Greens functions that the Lagrangian given in [6] correctly reproduces the QCD Greens functions at tree level. In Sec. 3 we include also loop effects and discuss the matching of HqEFT to full QCD. It is possible to summarize all corrections in an effective Lagrangian which takes a simple form, and thus to provide explicit formulae for the matching coefficients to all orders. As an application, in Sec. 4 we calculate the one–loop matching coefficients of the operators in the Lagrangian up to order $1/m$. Finally we address the problems of multiparticle states in HqEFT and conclude.

2 HqEFT at Tree Level

We shortly review the derivation of the HqEFT at tree level as it has been given in [6]. The heavy quark fields occurring in the full QCD Lagrangian
\[ \mathcal{L} = \bar{\psi}(i\slashed{D} - m)\psi + \bar{\eta}\psi + \bar{\psi}\eta \] (1)
are rewritten using the projections on upper and lower components with respect to the reference frame given by the velocity vector $v$
\[ P_v^\pm = \frac{1 \pm \gamma^0}{2} \] (2)
as
\begin{align*}
P_v^+\psi(x) &= e^{-imv\cdot x}h_v(x), \\
P_v^-\psi(x) &= e^{-imv\cdot x}H_v(x).
\end{align*} (3) (4)

With the corresponding parameterization of the heavy quark sources
\begin{align*}
\rho_v(x) &= P_v^+ e^{imv\cdot x}\eta(x), \\
R_v(x) &= P_v^- e^{imv\cdot x}\eta(x)
\end{align*} (5) (6)
and after integration over the $H_v$ fields in the functional integral, the tree–level Lagrangian becomes a nonlocal expression
\[ \mathcal{L}_v^{(0)} = \bar{h}_v i(v \cdot D) h_v + (\bar{h}_v iD_v^\dagger + \bar{R}) \frac{1}{2m + i(v \cdot D) - i\epsilon (iD_v^\dagger h_v + R_v)} \]
\[ + \bar{\rho}_v h_v + \bar{h}_v \rho_v, \] (7)
where the transverse derivative is given by
\[ D^\perp_v = \partial^\perp - \gamma(v \cdot D). \] (8)

The propagator of the heavy quark field can be read off as
\[ S^+_v(k) = \frac{iP^+_v}{v \cdot k + i\epsilon} \] (9)

and contains only forward propagation in time.

The nonlocal effective Lagrangian may be expanded in orders of \(1/m\) as
\[ L^{(0)}(0) = \bar{h}_v i(v \cdot D)h_v + (\bar{h}_v i\partial + \bar{R}_v) \sum_{n=0}^{\infty} \left( -\frac{iv \cdot D}{2m} \right)^n i(i\partial h_v + R_v) \]
\[ + \tilde{\rho}_v h_v + \tilde{h}_v \rho_v, \] (10)
where the backward propagation is lost if the series is truncated at any finite order in \(1/m\). In these expressions we have retained the sources \(R_v\) of the lower component fields. They may be dropped if only Greens functions sandwiched between \(P^+_v\) projectors are to be calculated, which is true if the external states are eigenstates of the lowest order HQEFT Lagrangian. However, if the external state contains a \(P^-_v\) projection, differentiations with respect to \(R_v\) are necessary.

It may be verified that the Lagrangian \((10)\) correctly reproduces the tree level Greens functions of the full QCD Lagrangian to arbitrary order in the \(1/m\) expansion. To see this, we rewrite the geometric series in \((10)\) as
\[ -\frac{i}{2m} P^-_v \sum_{n=0}^{\infty} \left( -\frac{iv \cdot D}{2m} \right)^n P^-_v = S^-_v + S^-_v(igA)S^-_v + S^-_v(igA)S^-_v(igA)S^-_v + \ldots, \] (11)
where the expression \(S^-_v\) which becomes in momentum space
\[ S^-_v(k) = -iP^-_v \frac{1}{2m} \sum_{n=0}^{\infty} \left( -\frac{v \cdot k}{2m} \right)^n = \frac{-iP^-_v}{2m + v \cdot k} \] (12)
may be interpreted as the propagator of the lower component field. To any finite order in the \(1/m\) expansion this is a local expression and therefore it is part of the interaction terms in the effective Lagrangian.

Using the Feynman rules as derived from \((10)\), with the upper component part of the Lagrangian in the form
\[ \bar{h}_v i(v \cdot D)h_v = \bar{h}_v P^+_v i(v \cdot \partial)P^+_v h_v + \bar{h}_v P^+_v (gA)P^+_v h_v, \] (13)
where we have made explicit the projection operators, we may collect all terms that can occur in a Feynman diagram between two insertions of the gluon field \(gA\). If both vertices contain \(P^+_v\) projectors, they sum up to
\[ G^{++}_v = S^+_v + S^+_v(ikS^-_v i\kappa)S^+_v + S^+_v(ikS^-_v i\kappa)S^+_v(ikS^-_v i\kappa)S^+_v + \ldots \]
\[ = iP^+_v \frac{1 + v \cdot k/2m}{v \cdot k + k^2/2m}. \] (14)
Similarly, if one insertion contains a $P_v^-$ projection, we have

$$G_v^{+-} = G_v^{++} i \frac{k/2m}{P_v} P_v^-,$$

whereas between two $P_v^-$ projectors there can be an additional term from the geometric series (11)

$$G_v^{--} = S_v^- i \frac{k/2m}{P_v} P_v^-,$$

In the sum of the four possible contributions the projection operators disappear

$$G_v^{++} + G_v^{+-} + G_v^{-+} + G_v^{--} = i \frac{m \delta + k + m}{(mv + k)^2 - m^2},$$

and the propagator of the full theory is recovered. The same is true if one or both of the gluon field insertions are replaced by external sources. Note that in the latter case the term quadratic in $R_v$ becomes important.

The $1/m$ expansion of a tree–level Greens function is unique. We have seen that the full propagator, and thus any Greens function, is correctly reproduced by the effective theory. Truncating the series in the effective Lagrangian at a given order $1/m^k$, we get an approximation to the full Greens function which differs by terms of order $1/m^{k+1}$:

$$G^{(0)}_{hl} (\lambda, m) - \tilde{G}^{(0)}_{hl} (\lambda, 1/m) = O \left( g^{h+l-2} \lambda^\delta (\lambda/m)^{k+1} \right),$$

where $\delta$ is the mass dimension of the Greens function, and the momenta of the $h$ external heavy and $l$ external massless lines are parameterized by

$$P_i = mv + \lambda x_i,$$
$$p_i = \lambda y_i,$$

so that the parameter $\lambda$ measures the deviation of the external momenta from mass shell.

Although these facts might seem trivial for the tree–level case, we note that some parts of the full theory propagator shrink to a point and become part of the interaction vertices in the effective theory. Thus, the one–particle irreducible (1PI) Greens functions of the full theory do not correspond to the 1PI functions in the effective theory. This fact has to be accounted for in the matching of higher dimensional operators at order $\alpha_s$ or higher.
3 Matching and Renormalization Conditions

If we want a statement like (18) to hold at higher orders in the loop expansion, we have to include in the effective Lagrangian all possible local operators up to a given mass dimension and heavy–quark number that cannot be excluded by symmetry arguments. Furthermore, the operators that are already present in the tree–level Lagrangian will be modified by multiplicative renormalizations which in general are ultraviolet (UV) divergent. However, no IR infinities can occur if the calculations are organized appropriately, so one can dispense of all IR regulators in the matching calculations.

In comparing the one loop Greens functions of the full and effective theories we first consider the Greens functions which contain no external heavy quark. Diagrams without any internal heavy quark line are identical in both theories. On the other hand, diagrams with heavy quark loops have no counterpart in the effective theory. This is true for any diagram even if two heavy quarks with different velocities are present: The heavy quark propagator in coordinate space

\[ S^+(v)(x) = \theta(x_0) \delta^3(\vec{x} - \vec{v}x_0/v_0) \]

(20)
describes a particle moving along a classical straight world line in the forward light cone. If two heavy propagators are connected at one point (the origin) in coordinate space, the loop cannot be closed at any other point, so the loop integral vanishes. To be precise, because at the origin two distributions are multiplied, it is equal to some undefined constant which can be got rid of by a simple renormalization. In contrast to light particles, a heavy particle loop cannot introduce any nonlocal interaction.

Thus, one can calculate the coefficients of operators without heavy quarks such as

\[ \frac{1}{(2m)^2} (D_\mu G_{\mu\rho})(D_\nu G^{\nu\rho}), \quad \frac{1}{(2m)^2} G_\mu^\nu G_\rho^\nu G_\rho^\mu. \]

(21)

by computing the corresponding loop diagrams in the full theory. This is just the result we would have obtained if we had integrated out the heavy quark completely. At one loop the result is given by the determinant of the heavy fermions in the functional integral \[ 3 \].

For Greens functions with external heavy quarks we need additional counterterms. These are to be chosen in such a way that the condition analogous to (18) for \( n \) loop Greens functions with \( h \) external heavy–quark and \( l \) external gluon lines\[ 3 \]

\[ \Delta_{hl}^{(n)}(\lambda, m) = G_{hl}^{(n)}(\lambda, m) - \bar{G}_{hl}^{(n)}(\lambda, 1/m) = O \left( g^{h+l-2} \alpha_s^n \lambda^\delta(\lambda/m)^{k+1} \ln^n(\lambda/m) \right), \]

(22)

\[ ^1 \text{Throughout this paper we ignore the ghost fields, and do not write out the pure gluon and gauge fixing terms in the Lagrangian. The ghost fields can be treated in the matching in the same way as the gluon fields. To order } 1/m^0 \text{ the question of BRS invariance has been addressed in } 3. \text{ A complete discussion to all orders is beyond the scope of this paper.} \]
can be satisfied for \( h \leq h_0 \), if we know that it is already true in lower order of the loop expansion. For the lowest order \((1/m^0)\) Greens functions with \( h_0 = 2 \) this has been shown in \([2]\). Extending the ideas developed there we shall show in the following paragraphs how the matching can be accomplished to arbitrary order in the \(1/m\) expansion.

For \( \delta + k \geq 0 \), (22) is equivalent to the requirement that the first \( \delta + k \) derivatives of \( \hat{\Delta}_{hl}^{(n)}(\lambda, m) \) with respect to external momenta vanish for \( \lambda \to 0 \). If the quantity \( \hat{\Delta}_{hl}^{(n)}(\lambda, m) \), which is defined in the same way as \( \Delta_{hl}^{(n)}(\lambda, m) \) but includes counterterms only up to \( n - 1 \) loop order, admits an expansion of the form

\[
\hat{\Delta}_{hl}^{(n)}(\lambda, m) = C_0 m^{\delta} + C_1 \lambda m^{\delta-1} + \ldots + C_{\delta+k} \frac{\lambda^{\delta+k}}{(2m)^k} + O\left(g^{h+l-2} a_s^n \lambda^\delta (\lambda/m)^{k+1} \ln(n/\lambda)\right),
\]

(23)

can be accomplished by introducing local counterterms of order \( a_s^n \) and up to order \( 1/m^k \) into the effective Lagrangian. Since \( \omega = \delta + k \) is the maximal UV degree of divergence of the effective theory diagrams, the coefficients \( C_0 \) to \( C_{\delta+k} \) which translate into coefficients of counterterms are in general UV divergent. To make (22) hold, we have to include counterterms exactly for those diagrams with \( \omega \geq 0 \), so the set of matching conditions (22) uniquely defines a renormalization scheme in the effective theory.

If (23) did not hold, we would encounter IR divergences in evaluating \( \hat{\Delta}_{hl}^{(n)} \) or its derivatives for vanishing external momenta. These could come from regions where one or more loop momenta become small. We consider a particular diagram of the full theory together with its counterparts in the effective theory up to order \( 1/m^k \), including all counterterm insertions up to \( n - 1 \) loop order. Let us investigate the behaviour of the integrand in the region where the momenta of some subset of \( s \) massless propagators become small of order \( \lambda \). Since the \( s \) light propagators are the same in both theories, we can factor them out in the difference. The remaining expression corresponds to a Greens function — not necessarily connected — in lower order of the loop expansion. It is represented by the same set of diagrams, where the \( s \) light lines have been cut, their momenta being regarded as external. If the remaining \( n' \) loop momenta are integrated over, according to the induction hypothesis this Greens function satisfies

\[
\Delta_{hl}^{(n')} (\lambda, m) = O\left(g^{h+l-2} a_s^{n'} \lambda^{\delta'} (\lambda/m)^{k+1} \ln(n'/\lambda)\right),
\]

(24)

where \( \delta' = \delta + 2s - 4(n-n') \). The total IR degree of divergence in this integration region is therefore

\[
\rho = -[(\delta' + k + 1) - 2s + 4(n-n')] = -\delta - k - 1,
\]

(25)

and thus no IR singularities show up in the first \( \delta + k \) derivatives of \( \hat{\Delta}_{hl}^{(n)} \).
Any diagram with $\delta + k < 0$ automatically satisfies (22) without additional counterterms, since by the same reasoning the matching conditions for its subdiagrams exclude unexpected positive powers of $m$ coming from UV divergent subintegrations. Simple power counting then directly gives (22) for these diagrams.

We did not consider additional IR divergences from integration regions where some momenta of heavy quark lines become small. In fact there are none. If we cut a heavy quark line, the remaining subdiagram has a larger number of external heavy quark lines and thus it is part of a different sector of the theory. Violations of the IR power counting could occur if this subdiagram is UV divergent, which can happen first at order $1/m^2$. We are tempted to renormalize it by imposing matching conditions also for Greens functions with more than $h$ external heavy lines, but as explained in Sec. 5, their counterterms can contain divergent phases which should not appear in the Lagrangian. Fortunately, these terms give vanishing contributions to the final result, because in evaluating them, we have to close the heavy quark line again and get a heavy quark loop which is zero by definition. We conclude that the multiparticle sectors are completely irrelevant for the renormalization and matching of the one–particle sector, and all possible IR singularities have been accounted for in the previous paragraphs.

In practice, the counterterms needed in the effective Lagrangian are given by the $1/m$ expansion up to a certain order $1/m^k$ of the 1PI diagrams in the full theory, with all corresponding diagrams of the effective theory up to the same order subtracted. Since the effective theory does not contain any mass scale, the subtraction terms are defined uniquely by the requirement that the first $\delta + k$ derivatives of the subtracted diagram are IR finite, where $\delta$ is the mass dimension of the diagram, so that the effective theory is not needed explicitly in the matching calculation.

In evaluating the counterterms introduced by a particular 1PI diagram into the effective theory, we first subtract all counterterm diagrams necessary to make this diagram UV finite in the full theory. These may be taken in the $\overline{MS}$ scheme as usual, with one exception: In order to have a renormalization group invariant mass as expansion parameter, we must also include a counterterm for the finite part of the quark mass renormalization. In this way the self–energy diagram contains no constant term, and the $1/m$ expansion is done in terms of the pole mass which has to be redefined in each order of the perturbation expansion. Furthermore, the counterterms in the effective Lagrangian all have nonpositive mass dimension.

We may put the results of the preceding paragraphs together in a formal language, and give an explicit expression for the effective theory counterterm of a particular diagram $\Gamma$ with mass dimension $\delta(\Gamma)$. For a one–loop diagram, it is

\footnote{The possibility of expanding in terms of a different mass, which introduces a residual mass term into the effective Lagrangian, has been considered in \cite{11}.}
given by the integral of the expression

\[ C_\Gamma = \tilde{T}_{\delta+k} I_\Gamma = (1 - J_\rho)T_{\delta+k} I_\Gamma, \tag{26} \]

where \( I_\Gamma \) is the integrand of the full theory diagram. The operator \( T_{\delta+k} \) gives the Taylor expansion of the integrand in terms of external momenta up to the required order \( \delta + k \). The operator \( J_\rho \) gives the first \( \rho + 1 \) terms of a Laurent expansion in terms of internal momenta, where \( \rho \) is the IR degree of divergence of the term on which \( J_\rho \) acts. This takes into account the diagrams of the effective theory. All expansions are done about zero momentum \( k = 0 \), where the heavy quark momenta are parameterized by \( p = mv + k \).

Because of the additional subtractions, the subtracted integrand \( C_\Gamma \) is no longer UV finite, but has an UV degree of divergence \( \omega = \delta + k \). Since it is a polynomial in external momenta of order \( \delta + k \) which is designed to cancel the leading terms in the difference of the full and effective theory diagrams, it also exactly cancels the UV divergences in this difference.

Evaluating the counterterms for \( n \)-loop graphs, we have to take into account all lower order counterterms in the effective theory. We observe that a recursive procedure emerges closely analogous to the well-known renormalization of UV divergences. Thus we may give the effective theory counterterm of any \( n \)-loop 1PI diagram in closed form as an extension of Zimmermann’s forest formula

\[ C_\Gamma = \tilde{T}_{\delta(\Gamma)+k} \sum_{F: \Gamma \notin F} \prod_{\gamma \in F} (-\tilde{T}_{\delta(\gamma)+k}) I_\Gamma, \tag{27} \]

where the modified Taylor operator \( \tilde{T} \) is defined as in (26), and the sum runs over all forests \( F \) of subdiagrams \( \gamma \) not containing \( \Gamma \) itself. The “renormalized” integrand

\[ R_\Gamma = \sum_{\text{all } F} \prod_{\gamma \in F} (-\tilde{T}_{\delta(\gamma)+k}) I_\Gamma \tag{28} \]

is the full theory diagram with all effective theory diagrams subtracted, i.e., a term contributing to the difference \( \Delta^{(n)}_{hl} \) which obeys the matching condition (22). Of course, we can apply these formulae also to the \( h = 0 \) case, where the IR subtractions first appear in two–loop order.

This treatment of the matching conditions is not just formal. In fact, it is convenient because all integrations are manifestly IR finite, and the integrals can be evaluated for vanishing external momenta. In dimensional regularization, the \( 1/\epsilon^n \) poles are due only to UV singularities.

In the effective theory we have the freedom left to rescale the heavy quark field by a finite amount \( z \). The matching condition (22) is then modified to

\[ G^{(n)}_{hl}(\lambda, m) - z^{h/2} \tilde{G}^{(n)}_{hl}(\lambda, 1/m) = O \left( g^{h+l-2} \alpha_s^n \lambda^\delta (\lambda/m)^{k+1} \ln^n(\lambda/m) \right). \tag{29} \]

If the renormalization scheme of the full theory is fixed, choosing a value for \( z \) and requiring that (29) holds uniquely defines a renormalization scheme for
the effective theory. Using the \( \overline{\text{MS}} \) scheme in the lowest order effective theory amounts to specifying a wave function renormalization constant \( z \) that is given by the finite part of the difference of the heavy quark self energy diagrams in the full and effective theory. In higher order of the \( 1/m \) expansion we have to insert the counterterms as given by \( (27) \) explicitely, with the self–energy contribution of the external lines factored out.

We may collect all integrands \( C_\Gamma (27) \) that contribute to the counterterm of some 1PI Greens function with \( h \) heavy–quark and \( l \) gluon legs. After integrating over internal momenta, this becomes a function \( C_{hl} \) which is a polynomial of order \( \delta + k \) in external momenta. Fourier transforming back into coordinate space and inserting gluon fields (we do not write out colour indices), we define the quantity

\[
C_h = \sum_l \frac{1}{l!} C_{hl}^{\mu_1 \cdots \mu_l} (i\partial) A_{\mu_1} \cdots A_{\mu_l}
\]  

which looks like a generating functional of 1PI vertices, but does not contain the heavy quark fields yet. In contrast to the effective action of the full theory, it can be expanded as a series of powers of \( \partial/m \). By definition, we also include the tree–level contribution into \( C_2 \) which is just the ordinary vertex \( gA \).

In the following paragraphs, we consider only the case \( h \leq 2 \), the one–particle case. With the caveat discussed at the end of Sec. 5, the generalization to multiparticle Greens functions is straightforward.

The effective vertices which are summarized in \( C_2 \) can be linked together by propagator projections in the same way as the vertices \( gA \) at tree–level. Repeating the reasoning of Sec. 2, we find that we can incorporate all loop effects in the effective Lagrangian by defining a generalized covariant derivative

\[
i\Phi = i\phi + C_2 = i\phi + O(\alpha_s)
\]

with projections

\[
iD^+ = P_v i\Phi P^+_v = (iv \cdot D) P^+_v + O(\alpha_s),
\]
\[
iD^\perp = P_v i\Phi P^\perp_v + P^- v i\Phi P^+_v = iD^\perp_v + O(\alpha_s),
\]
\[
iD^- = P^- v i\Phi P^-_v = -(iv \cdot D) P^-_v + O(\alpha_s).
\]

The effective Lagrangian which generalizes \( (9) \) then reads

\[
\mathcal{L}_v = \bar{h}_v iD^+_v h_v + (\bar{h}_v iD^\perp_v + R_v) \frac{1}{2m - iD^-_v} (iD^\perp_v h_v + R_v) + C_0
\]
\[
+ \bar{\rho}_v h_v + \bar{h}_v \rho_v,
\]

where \( C_0 \) summarizes the counterterms without heavy quarks. This effective Lagrangian is valid to arbitrary order in the \( 1/m \) and loop expansions.
It is straightforward to apply this procedure to operator insertions. For instance, a heavy–light current $K = ar{q} \Gamma \psi$, where $q$ is some light quark field, is matched onto

$$K_v = \bar{q} \mathcal{K} \left[ \frac{1}{2m - iD} (iD_v h_v + R_v) + h_v \right],$$

(34)

where $\mathcal{K}$ stands for the sum of all 1PI diagrams of the full theory that involve one insertion of $K$, with an arbitrary number of gluons, where the IR subtractions as in (27) have been carried out.

The form of $\mathcal{L}_v$ and $K_v$ is restricted if BRS invariance holds in the matching. One would expect the generalized covariant derivative $\mathcal{D}$ to depend on the gluon field only through the ordinary covariant derivative $i\mathcal{D} = i\partial + gA$, at least in the background field gauge [12]. This has been shown for the lowest order ($1/m^0$) theory in [16].

Furthermore, there are restrictions following from reparameterization invariance [13]. They emerge from the fact that the Greens functions of the full theory do not depend on the heavy–quark velocity $v$ and the residual momentum $k$ separately, but only in the combination $p = mv + k$. In the effective theory, the Lagrangian therefore depends on the velocity and the covariant derivative only in the combination $V = v + iD/m$. This introduces relations among operator coefficients in different orders of the $1/m$ expansion.

4 Coefficients to order $1/m$ and $\alpha_s$

As an example, in this section we shall discuss the one–loop matching up to order $1/m$. By combining denominators and algebraically reducing vector and tensor integrals, all quantities that occur in the one–loop matching calculations to arbitrary order in the $1/m$ expansion can be expressed in terms of IR–finite integrals in the form

$$\Xi_{abc}(m) = \int \frac{dk}{(k^2)^a(v \cdot k)^b(v \cdot k + k^2/2m)^c},$$

(35)

of which only a few have to be evaluated because of the relations

$$\frac{d}{dm} \Xi_{abc}(m) = \frac{c}{2m^2} \Xi_{a-1,b,c+1}(m)$$

(36)

and

$$\Xi_{abc}(m) = 2m (\Xi_{a+1,b,c-1}(m) - \Xi_{a+1,b-1,c}(m)).$$

(37)

To order $1/m$ there are three independent operators involving two heavy quarks possible which are usually written in the form

$$\frac{1}{2m} \bar{h}_v (iD)^2 h_v, \quad \frac{1}{2m} \bar{h}_v (iv \cdot D)^2 h_v, \quad \frac{g}{4m} \bar{h}_v \sigma^{\mu\nu} G_{\mu\nu} h_v.$$
The second operator vanishes when sandwiched between physical states due to the heavy–quark equations of motion.

Because of BRS invariance one only has to calculate the two one–loop vertex diagrams of the full theory, with the appropriate IR subtractions, to obtain the matching coefficients up to order $\alpha_s$ and $1/m$. Applying the formulae (26,30,33), and expanding the result up to order $1/m$, we obtain the one–loop effective Lagrangian at the matching scale $\mu = m$ in the background field gauge with $\xi = 1$ (Feynman gauge)

$$\mathcal{L}^{(1)}_{v} = Z_{\psi}^{-1} Z_h \bar{h}_v \left[ i v \cdot D + Z_1 \frac{(iD)^2}{2m} - Z_2 \frac{(i v \cdot D)^2}{2m} + Z_3 \frac{g}{4m} \sigma_{\mu\nu} G^{\mu\nu} \right] h_v,$$

where the matching renormalization constants are given by $[1/\hat{\epsilon} = 1/\epsilon + \ln(4\pi)/2 - \gamma_E/2$, where the space–time dimension is $4 - \epsilon]$

$$Z_h = 1 + \frac{\alpha_s}{\pi} \left[ \frac{1}{\hat{\epsilon}} + 1 \right] C_F,$$

$$Z_1 = 1,$$

$$Z_2 = 1 + \frac{\alpha_s}{\pi} \left[ \frac{3}{\hat{\epsilon}} + \frac{1}{2} \right] C_F,$$

$$Z_3 = 1 + \frac{\alpha_s}{\pi} \left[ \frac{1}{2} C_F + \left( \frac{1}{2\hat{\epsilon}} + \frac{1}{2} \right) C_A \right],$$

and the wave–function renormalization of the full theory is in the $\overline{\text{MS}}$ scheme

$$Z_\psi = 1 + \frac{\alpha_s}{\pi} \left( \frac{1}{2\hat{\epsilon}} \right) C_F.$$  

The values of $Z_h$, $Z_1$, and $Z_2$ are also obtained by computing the one–loop self–energy to the required order. In addition, this gives the mass renormalization

$$\delta m = \frac{\alpha_s}{\pi} C_F \left( -\frac{3}{2\hat{\epsilon}} - 1 \right) m_0$$

so that the $1/m$–expansion is done in terms of the pole mass

$$m = m_0 - \delta m.$$

The fact that $Z_1 = 1$ is a consequence of reparameterization invariance [13]. BRS invariance ensures that no gauge–variant operators appear, and that only the unphysical constants $Z_\psi$, $Z_h$, and $Z_2$ depend on the gauge parameter. Our results for the one–loop matching renormalization constants agree with those given in [14]. The calculational method of [14], where dimensional regularization has been used for both UV and IR singularities, is thus justified by employing a scheme that is manifestly free of IR divergences.

In order to obtain the effective Lagrangian valid at scales $\mu < m$, one may sum up the leading mass logarithms by evaluating the UV–divergent parts of the loop
diagrams in the effective theory. The leading log calculation for the Lagrangian \((39)\) has been performed in \([15]\). In fact, the one–loop anomalous dimensions are determined by the \(1/\epsilon\)–poles of the renormalization constants \([10]\). The finite terms in \((10)\) become relevant when subleading logarithms are summed.

5 Hadronic States

In a typical application of HqEFT one considers the matrix element of some operator \(O\), describing e.g. the weak decay of a heavy quark, between hadronic states \(|in\rangle\) and \(|out\rangle\) involving heavy quarks. As in ordinary perturbation theory, the full QCD eigenstates are evolved from the corresponding lowest order HqEFT eigenstates \(|in(out)H\rangle\) by adiabatically switching on the \(1/m\) terms in the effective Lagrangian:

\[
\langle out | O | in \rangle = \langle out H | T \left[ \exp(-iS'_v) \tilde{O} \right] | in H \rangle, \quad (44)
\]

where \(S'_v = \int dx L'_v(x)\) is given by the effective Lagrangian \((33)\) excluding terms of order \(1/m^0\), and \(\tilde{O}\) is the result of a matching calculation such as \((34)\).

Usually, \((44)\) cannot be evaluated completely in terms of Feynman diagrams because perturbation theory breaks down at low scales. One rather uses perturbation theory to sum up the leading (and subleading) logarithms \(\log(m/\mu)\), where \(\mu\) is some low hadronic scale, and expresses the result in terms of some operator \(O(\mu)\) which has the same matrix element between scaled–down states \(|in(out)H,\mu\rangle\). These are defined in such a way that the matrix element is \(\mu\) independent:

\[
\langle out H | T \left[ \exp(-iS'_v) \tilde{O} \right] | in H \rangle = \langle out H, \mu | O(\mu) | in H, \mu \rangle. \quad (45)
\]

Obviously, \(O(\mu)\) contains nonlocal operators (time–ordered products) in its \(1/m\) expansion. In the solution of the renormalization group equations they mix with local operators.

If we want to include into this treatment also states involving more than one heavy quark at a time, we have to extend the heavy–quark Lagrangian accordingly. We can apply HqEFT to asymptotic states where the heavy quarks end up in different hadrons, so each one can be assigned a separate velocity. The Lagrangian then contains a sum over all heavy quark and antiquark flavours and velocities that occur in the problem

\[
\mathcal{L} = \sum_{f,v} \left\{ \bar{h}_v^f iD_v^+ h_v^f + (\bar{h}_v^f iD_v^+ + \bar{R}_v^f) \frac{1}{2m - iD_v^-(iD_v^+ h_v^f + R_v^f)} \right. \\
+ \left. \tilde{\rho}_v^f h_v^f + \tilde{\rho}_v^f \rho_v^f \right\} + \sum_f C_0^f + \text{(counterterms with } h > 2) , \quad (46)
\]

where the projections of the generalized covariant derivative \(\mathcal{D}_v\) \((31)\) have been defined in \((32)\). The counterterms without heavy quarks \(C_0\) are inserted only
once for each quark flavour. The bilinear matching corrections that are incorporated in $\mathcal{D}$ are needed once for each flavor and velocity. In addition, one expects counterterms involving more heavy quarks at higher order in the $1/m$ expansion if there is more than one heavy quark present in the initial or final states. These counterterms can link different velocity and flavour sectors. They are needed only if they involve no more heavy quarks than the process under consideration: Any diagram which contains an operator with more heavy fields than there are available as external lines necessarily contains a heavy quark loop and thus vanishes.

If more than one heavy quark is present at the same time, some Feynman amplitudes develop UV divergent phases analogous to the well-known Coulomb phases of QED \cite{16}. They can be absorbed into the definition of the multiparticle states itself \cite{7}. In particular, the phase of the two-particle final state is related to the static interquark potential \cite{7}. If two velocities of different heavy quarks become equal, these phases become infinite, so that the whole expression is ill-defined. However, one could go back into coordinate space and consider the case of two separated heavy quarks which act as static colour sources. This notion was taken in the first perturbative evaluation of the interquark potential \cite{17}.

6 Conclusions

In the present paper we have extended the construction of HqEFT to arbitrary order in both expansion parameters $1/m$ and $\alpha_s$. Although our arguments as given in Sec. 2 are merely heuristic, we believe that they suffice to show that the effective theory reproduces the behaviour of full QCD near mass shell to any required accuracy. The formulae (27) and (33) provide a method to obtain the terms in the effective Lagrangian directly from the full theory. The fact that no IR regulators need to be introduced simplifies practical calculations. As an example, we have rederived the coefficients of the terms in the effective Lagrangian up to order $1/m$ and $\alpha_s$.

For the construction of the complete effective Lagrangian (33) one needs in principle the knowledge of all diagrams of the full theory. One might argue that for this reason the effective theory is without physical content. However, its main power lies in the possibility of summing up large logarithms of the heavy quark mass, which are difficult to extract from the full theory expressions, and thus reorganizing the perturbation series. The method for calculating anomalous dimensions and solving the renormalization group equations within the effective theory is well known. The Lagrangian (33) provides in closed form the initial conditions valid at the matching scale $\mu = m$. It may also serve as a starting point for theoretical considerations, e.g., the proof of BRS invariance to all orders.

In the last section we have considered the extension of HqEFT to sectors with more than one heavy quark. It should have become clear that these sectors do not
introduce difficulties if the heavy quarks remain well separated in the asymptotic states, because the UV–divergent phases that appear in these sectors can then be absorbed in the definition of the multiparticle states.

All arguments in the present paper have been in the context of perturbation theory. We did not address the question whether a nonperturbative approach to the $1/m$ expansion such as lattice gauge theory introduces any new problems. However, the perturbative matching of the continuum theory to the lattice regulated version should proceed mainly along the same lines as we have discussed.

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