Jordan derivations on $C^*$-ternary algebras for a Cauchy-Jensen functional equation

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Abstract

In this paper, we proved the generalized Hyers-Ulam stability of homomorphisms in $C^*$-ternary algebras and of derivations on $C^*$-ternary algebras for the following Cauchy- Jensen functional equation

$$3f\left(\frac{x+y+z}{3}\right) = 2f\left(\frac{x+y}{2}\right) + f(z).$$

These were applied to investigate isomorphisms between $C^*$-ternary algebras.
1 Introduction and preliminaries

Ternary structures and their generalization, the so-called \( n \)-ary structures, raise certain hopes in view of their applications in physics. Some significant physical applications are as follows (see [13, 14]):

(1) The algebra of ‘nonions’ generated by two matrices
\[
\begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{pmatrix}
\quad \&
\begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & \omega \\
\omega^2 & 0 & 0
\end{pmatrix}
\quad (\omega = e^{\frac{2\pi i}{3}})
\]

was introduced by Sylvester as a ternary analog of Hamilton’s quaternions (cf. [1]).

(2) The quark model inspired a particular brand of ternary algebraic systems. The so-called ‘Nambu mechanics’ is based on such structures (see [5]).

There are also some applications, although still hypothetical, in the fractional quantum Hall effect, the non-standard statistics, supersymmetric theory, and Yang–Baxter equation (cf. [1, 14, 46]).

A \( C^* \)-ternary algebra is a complex Banach space \( A \), equipped with a ternary product \( (x, y, z) \mapsto [x, y, z] \) of \( A^3 \) into \( A \), which is \( C \)-linear in the outer variables, conjugate \( C \)-linear in the middle variable, and associative in the sense that \([x, y, [z, w, v]] = [x, y, z], w, v\] satisfies \( \| [x, y, z] \| \leq \| x \| \cdot \| y \| \cdot \| z \| \) and \( \| [x, x, x] \| = \| x \|^3 \) (see [2, 47]). Every left Hilbert \( C^* \)-module is a \( C^* \)-ternary algebra via the ternary product \( [x, y, z] := \langle x, y \rangle z \).

If a \( C^* \)-ternary algebra \((A, [\cdot, \cdot, \cdot])\) has an identity, i.e., an element \( e \in A \) such that \( x = [x, e, e] = [e, e, x] \) for all \( x \in A \), then it is routine to verify that \( A \), endowed with \( x \circ y := [x, e, y] \) and \( x^* := [e, x, e] \), is a unital \( C^* \)-algebra. Conversely, if \((A, \circ)\) is a unital \( C^* \)-algebra, then \( [x, y, z] := x \circ y \circ z \) makes \( A \) into a \( C^* \)-ternary algebra.

A \( C \)-linear mapping \( H : A \to B \) is called a \( C^* \)-ternary algebra homomorphism if
\[
H([x, y, z]) = [H(x), H(y), H(z)]
\]
for all \( x, y, z \in A \). If, in addition, the mapping \( H \) is bijective, then the mapping \( H : A \to B \) is called a \( C^* \)-ternary algebra isomorphism. A \( C \)-linear mapping \( \delta : A \to A \) is called a \( C^* \)-ternary derivation if
\[
\delta([x, y, z]) = [\delta(x), y, z] + [x, \delta(y), z] + [x, y, \delta(z)]
\]
for all \( x, y, z \in A \) (see [2], [15]–[18]).

In 1940, S. M. Ulam [45] gave a talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of unsolved problems. Among these was the following question concerning the stability of homomorphisms.

We are given a group \( G \) and a metric group \( G' \) with metric \( \rho(\cdot, \cdot) \). Given \( \epsilon > 0 \), does there exist a \( \delta > 0 \) such that if \( f : G \to G' \) satisfies
\[
\rho(f(xy), f(x)f(y)) < \delta
\]
for all \( x, y \in G \), then a homomorphism \( h : G \to G' \) exists with
\[
\rho(f(x), h(x)) < \epsilon
\]
for all \( x \in G \)?

In 1941, D. H. Hyers [8] considered the case of approximately additive mappings \( f : E \to E' \), where \( E \) and \( E' \) are Banach spaces and \( f \) satisfies Hyers inequality
\[
\|f(x + y) - f(x) - f(y)\| \leq \epsilon
\]
for all \( x, y \in E \). It was shown that the limit
\[
L(x) = \lim_{n \to \infty} \frac{f(2^n x)}{2^n}
\]
exists for all \( x \in E \) and that \( L : E \to E' \) is the unique additive mapping satisfying
\[
\|f(x) - L(x)\| \leq \epsilon
\]
for all \( x \in E \).

In 1978, Th. M. Rassias [35] provided a generalization of the D. H. Hyers’ theorem which allows the Cauchy difference to be unbounded.

**Theorem 1.1.** (Th. M. Rassias) Let \( f : E \to E' \) be a mapping from a normed vector space \( E \) into a Banach space \( E' \) subject to the inequality
\[
\|f(x + y) - f(x) - f(y)\| \leq \epsilon(\|x\|^p + \|y\|^p)
\]
for all \( x, y \in E \), where \( \epsilon \) and \( p \) are constants with \( \epsilon > 0 \) and \( p < 1 \). Then the limit
\[
L(x) = \lim_{n \to \infty} \frac{f(2^n x)}{2^n}
\]
exists for all \( x \in E \) and \( L : E \to E' \) is the unique additive mapping which satisfies
\[
\|f(x) - L(x)\| \leq \frac{2\epsilon}{2 - 2^p} \|x\|^p
\]
for all \( x \in E \). If \( p < 0 \) then inequality (1) holds for \( x, y \neq 0 \) and (2) for \( x \neq 0 \).

On the other hand, in 1982-1989, J. M. Rassias generalized the Hyers stability result by presenting a weaker condition controlled by a product of different powers of norms. The following is according to the J. M. Rassias’ theorem.

**Theorem 1.2.** (J. M. Rassias) If it is assumed that there exist constants \( \Theta \geq 0 \) and \( p_1, p_2 \in \mathbb{R} \) such that \( p = p_1 + p_2 \neq 1 \), and \( f : E \to E' \) is a mapping from a normed space \( E \) into a Banach space \( E' \) such that the inequality

\[
\|f(x + y) - f(x) - f(y)\| \leq \epsilon\|x\|^{p_1}\|y\|^{p_2}
\]

for all \( x, y \in E \), then there exists a unique additive mapping \( T : E \to E' \) such that

\[
\|f(x) - L(x)\| \leq \frac{\Theta}{2 - 2^p}\|x\|^p
\]

for all \( x \in E \).

In 1990, Th. M. Rassias [36] during the 27th International Symposium on Functional Equations asked the question whether such a theorem can also be proved for \( p \geq 1 \). In 1991, Z. Gajda [6] following the same approach as in Th. M. Rassias [35], gave an affirmative solution to this question for \( p > 1 \). It was shown by Z. Gajda [6], as well as by Th. M. Rassias and P. Šemrl [41] that one cannot prove a Th. M. Rassias’ type theorem when \( p = 1 \). The counterexamples of Z. Gajda [6], as well as of Th. M. Rassias and P. Šemrl [41] have stimulated several mathematicians to invent new definitions of approximately additive or approximately linear mappings, cf. P. Găvruţa [7], S.-M. Jung [12], who among others studied the Hyers-Ulam stability of functional equations. The inequality (1) that was introduced for the first time by Th. M. Rassias [35] provided a lot of influence in the development of a generalization of the Hyers-Ulam stability concept. This new concept is known as generalized Hyers-Ulam stability of functional equations (cf. the books of P. Czerwik [4], D. H. Hyers et al. [9]).

P. Găvruţa [7] provided a further generalization of Th. M. Rassias’ Theorem. In 1996, G. Isac and Th. M. Rassias [11] applied the generalized Hyers-Ulam stability theory to prove fixed point theorems and study some new applications in Nonlinear Analysis. In [10], D. H. Hyers et al. studied the asymptoticity aspect of Hyers-Ulam stability of mappings. Several papers have been published on various generalizations and applications of Hyers-Ulam stability and generalized Hyers-Ulam stability to a number of functional equations and mappings, for example: quadratic functional equation, invariant means, multiplicative mappings - superstability, bounded n-th
differences, convex functions, generalized orthogonality functional equation, Euler-Lagrange functional equation introduced by J. M. Rassias in 1992-1998, Navier-Stokes equations. Several mathematician have contributed works on these subjects (see [3], [19]–[44]).

In Section 2, we prove the generalized Hyers-Ulam stability of homomorphisms in $C^*$-ternary algebras for the Cauchy-Jensen additive mappings.

In Section 3, we investigate isomorphisms between unital $C^*$-ternary algebras associated with the Cauchy-Jensen additive mappings.

In Section 4, we prove the generalized Hyers-Ulam stability of derivations on $C^*$-ternary algebras for the Cauchy-Jensen additive mappings.

2 Stability of homomorphisms in $C^*$-ternary algebras

Throughout this section, assume that $A$ is a $C^*$-ternary algebra with norm $\| \cdot \|_A$ and that $B$ is a $C^*$-ternary algebra with norm $\| \cdot \|_B$.

For a given mapping $f : A \to B$, we define

$$D_\mu f(x, y, z) := 3f\left(\frac{\mu x + \mu y + \mu z}{3}\right) - 2\mu f\left(\frac{x + y}{2}\right) - \mu f(z)$$

for all $\mu \in T^1 := \{ \lambda \in \mathbb{C} \mid |\lambda| = 1\}$ and all $x, y, z \in A$.

Lemma 2.1. Let $f : A \to B$ be a mapping such that

$$D_\mu f(x, y, z) = 0$$

for all $\mu \in T^1$ and all $x, y, z \in A$. Then $f$ is $\mathbb{C}$-linear.

Proof. Letting $\mu = -1$ and $x = y = z = 0$ in (3), we gain $f(0) = 0$. Putting $\mu = 1$, $y = -x$ and $z = 2x$ in (3), we get $3f\left(\frac{x}{3}\right) = f(x)$ for all $x \in A$. So we have $3f(x) = f(3x)$ for all $x \in A$. Setting $\mu = 1$, $x = 0$ in (3), we gain

$$3f\left(\frac{y + z}{3}\right) = 2f\left(\frac{y}{2}\right) + f(z)$$

for all $y, z \in A$. So we get $f(y + z) = 2f\left(\frac{y}{2}\right) + f(z)$ for all $y, z \in A$. Taking $z = 0$ in the above equation, we have $f(y) = 2f\left(\frac{y}{2}\right)$ for all $y \in A$. Thus we obtain that $f(y + z) = f(y) + f(z)$ for all $y, z \in A$. Hence $f$ is additive.

Letting $y = z = 0$ in (3), we gain $3f\left(\frac{\mu x}{3}\right) = 2\mu f\left(\frac{x}{3}\right)$ for all $\mu \in T^1$ and all $x \in A$. Since $f$ is additive, $f(\mu x) = f\left(3\frac{\mu x}{3}\right) = 3f\left(\frac{\mu x}{3}\right) = 2\mu f\left(\frac{x}{3}\right)$ for all $\mu \in T^1$ and all $x \in A$. Now let $\lambda \in \mathbb{C}$ and $M$
an integer greater than $2|\lambda|$. Since $|\frac{\lambda}{M}| < \frac{1}{2}$, there is $t \in \left(\frac{\pi}{2}, \frac{\pi}{2}\right]$ such that $|\frac{\lambda}{M}| = \cos t = \frac{e^{it} + e^{-it}}{2}$. Now $\frac{\lambda}{M} = |\frac{\lambda}{M}| \mu$ for some $\mu \in \mathbb{T}^1$. Thus we have

$$f(\lambda x) = f \left( M \frac{\lambda}{M} x \right) = M f \left( \frac{\lambda}{M} x \right) = M f \left( \frac{\lambda}{M} \mu x \right)$$

$$= M f \left( \frac{e^{it} + e^{-it}}{2} \mu x \right) = \frac{1}{2} M f(e^{it} \mu x + e^{-it} \mu x)$$

$$= \frac{1}{2} M \left[ e^{it} f(x) + e^{-it} f(x) \right] = \lambda f(x)$$

for all $x \in A$. So the mapping $f : A \to B$ is $C$-linear. \[ \square \]

We prove the generalized Hyers-Ulam stability of homomorphisms in $C^*$-ternary algebras for the functional equation $D_\mu f(x, y, z) = 0$.

**Theorem 2.2.** Let $r > 3$ and $\theta$ be positive real numbers, and let $f : A \to B$ be a mapping satisfying $f(0) = 0$ such that

\begin{align}
(4) \quad & \|D_\mu f(x, y, z)\|_B \leq \theta(\|x\|_A^r + \|y\|_A^r + \|z\|_A^r), \\
(5) \quad & \|f([x, y, z]) - [f(x), f(y), f(z)]\|_B \leq \theta(\|x\|_A^r + \|y\|_A^r + \|z\|_A^r)
\end{align}

for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in A$. Then there exists a unique $C^*$-ternary algebra homomorphism $H : A \to B$ such that

\begin{align}
(6) \quad & \|f(x) - H(x)\|_B \leq \theta \frac{3^r + 2}{3^r - 3} \|x\|_A^r
\end{align}

for all $x \in A$.

**Proof.** Letting $\mu = 1$ and $y = -x$ and $z = 3x$ in (4), we obtain

\begin{align}
(7) \quad & \|3f(x) - f(3x)\|_B \leq \theta(2 + 3^r) \|x\|_A^r
\end{align}

for all $x \in A$. So we get

\begin{align}
\left\| f(x) - 3f \left( \frac{x}{3} \right) \right\|_B \leq \theta \left( \frac{2}{3^r} + 1 \right) \|x\|_A^r
\end{align}

for all $x \in A$. Thus we have

\begin{align}
\left\| 3^l f \left( \frac{x}{3^l} \right) - 3^m f \left( \frac{x}{3^m} \right) \right\|_B \leq \sum_{j=l}^{m-1} \left\| 3^j f \left( \frac{x}{3^j} \right) - 3^{j+1} f \left( \frac{x}{3^{j+1}} \right) \right\|_B \\
\leq \theta \left( \frac{2}{3^r} + 1 \right) \sum_{j=l}^{m-1} 3^{j(1-r)} \|x\|_A^r = \theta \frac{3^r + 2}{3^r - 3} [3^{l(1-r)} - 3^{m(1-r)}] \|x\|_A^r
\end{align}

for all nonnegative integers $m$ and $l$ with $m > l$ and all $x \in A$. It follows from (8) that the sequence $\{3^m f \left( \frac{x}{3^m} \right)\}$ is a Cauchy sequence for all $x \in A$. 

Next we consider the case where $r = 1$. Since $\theta(2 + 3^r) \|x\|_A^r = \theta(2 + 3) \|x\|_A^r = \theta(2 + 2) \|x\|_A^r$, we obtain

\begin{align}
\left\| f(x) - 3f(x) \right\|_B \leq \theta(2 + 2) \|x\|_A^r
\end{align}

and the same conclusion holds as in (7).
Since $B$ is complete, the sequence $\{3^n f(\frac{1}{3^n})\}$ converges for all $x \in A$. Hence one can define a mapping $H : A \to B$ by

$$H(x) := \lim_{n \to \infty} 3^n f\left(\frac{x}{3^n}\right)$$

for all $x \in A$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (8), we get (6).

It follows from (4) that

$$\left\| 3H\left(\frac{x + y + z}{3}\right) - 2H\left(\frac{x + y}{2}\right) - H(z) \right\|_B = \lim_{n \to \infty} 3^n \left\| 3f\left(\frac{x + y + z}{3^n+1}\right) - 2f\left(\frac{x + y}{2 \cdot 3^n}\right) - f\left(\frac{z}{3^n}\right) \right\|_B \leq \lim_{n \to \infty} 3^n(1-r)\theta(||x||_A + ||y||_A + ||z||_A) = 0$$

for all $x, y, z \in A$. So we get

$$3H\left(\frac{x + y + z}{3}\right) = 2H\left(\frac{x + y}{2}\right) + H(z)$$

for all $x, y, z \in A$. Since $f(0) = 0$, by the same methods as in proof of Lemma 2.1, the mapping $H : A \to B$ is additive.

By the same reasoning as in the proof of Theorem 2.1 in [21], the mapping $H : A \to B$ is $C$-linear. It follows from (5) and (8) that

$$\left\| H([x, y, z]) - [H(x), H(y), H(z)] \right\|_B = \lim_{n \to \infty} \left\| 3^n f\left(\frac{1}{3^n}[x, y, z]\right) - 3^n f\left(\frac{x}{3^n}\right), 3^n f\left(\frac{y}{3^n}\right), 3^n f\left(\frac{z}{3^n}\right) \right\|_B$$

$$= \lim_{n \to \infty} \left[ \left\| 3^n f\left(\frac{1}{3^n}[x, y, z]\right) - 3^{2n} f\left(\frac{1}{3^{2n}}[x, y, z]\right) \right\|_B + \left\| 3^{2n} f\left(\frac{1}{3^{2n}}[x, y, z]\right) - 3^{3n} f\left(\frac{1}{3^{3n}}[x, y, z]\right) \right\|_B + \left\| 3^{3n} f\left(\frac{x}{3^n}, \frac{y}{3^n}, \frac{z}{3^n}\right) - 3^n f\left(\frac{x}{3^n}\right), 3^n f\left(\frac{y}{3^n}\right), 3^n f\left(\frac{z}{3^n}\right) \right\|_B \right]$$
\[
\leq \lim_{n \to \infty} \left[ \theta \left( \frac{2}{3^r} + 1 \right) \sum_{j=n}^{2n-1} 3^j(1-r) \|[x, y, z]\|_A^r \\
+ \theta \left( \frac{2}{3^r} + 1 \right) \sum_{j=2n}^{2n-1} 3^j(1-r) \|[x, y, z]\|_A^r \\
+ 3^n(3-r) \theta \left( \|x\|_A^r + \|y\|_A^r + \|z\|_A^r \right) \right] \\
= \theta \left( \frac{3^r + 2}{3^r - 3} \|[x, y, z]\|_A^r \right) \lim_{n \to \infty} \left[ 3^n(1-r) - 3^{3n(1-r)} \right] \\
+ \theta \left( \|x\|_A^r + \|y\|_A^r + \|z\|_A^r \right) \lim_{n \to \infty} 3^n(3-r) \\
= 0
\]
for all \(x, y, z \in A\). So
\[
H([x, y, z]) = [H(x), H(y), H(z)]
\]
for all \(x, y, z \in A\).

Now, let \(T : A \to B\) be another additive mapping satisfying (6). Then we have
\[
\|H(x) - T(x)\|_B = 3^n \left[ \left\|H \left( \frac{x}{3^n} \right) - T \left( \frac{x}{3^n} \right) \right\|_B \right. \\
\leq 3^n \left[ \left\|H \left( \frac{x}{3^n} \right) \right\|_B + \left\|T \left( \frac{x}{3^n} \right) \right\|_B \right] \\
\leq \frac{2\theta}{3^n} \frac{3^r + 2}{3^r - 3} \|[x]\|_A^r,
\]
which tends to zero as \(n \to \infty\) for all \(x \in A\). So we can conclude that \(H(x) = T(x)\) for all \(x \in A\). This proves the uniqueness of \(H\). Thus the mapping \(H : A \to B\) is a unique \(C^*\)-ternary algebra homomorphism satisfying (6). \(\square\)

J. M. Rassias presents the following counterexample modified by the well-known counterexample of Z. Gajda [6] for the following Cauchy-Jensen functional equation:
\[
3f \left( \frac{x + y + z}{3} \right) = 2f \left( \frac{x + y}{2} \right) + f(z).
\]
Fix \(\theta > 0\) and put \(\mu := \frac{\theta}{6}\). Define a function \(f : \mathbb{R} \to \mathbb{R}\) given by
\[
f(x) := \sum_{n=0}^{\infty} \frac{\phi(2^n x)}{2^n}
\]
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for all \( x \in \mathbb{R} \), where

\[
\phi(x) := \begin{cases} 
\mu & \text{if } x \geq 1 \\
\mu x & \text{if } -1 < x < 1 \\
-\mu & \text{if } x \leq -1 
\end{cases}
\]

for all \( x \in \mathbb{R} \). It was proven in [6] that

\[
|f(x + y) - f(x) - f(y)| \leq \theta(|x| + |y|)
\]

for all \( x, y \in \mathbb{R} \). From the above inequality, one can obtain that

\[
|f(x + y + z) - f(x) - f(y) - f(z)|
\]

\[
\leq \frac{1}{3} \left[ |f(x + y + z) - f(x + y) - f(z)| + |f(x + y + z) - f(x + z) - f(y)| + |f(x + y + z) - f(y + z) - f(x)| + |f(x + y) - f(x) - f(y)| + |f(x + z) - f(x) - f(z)| + |f(y + z) - f(y) - f(z)| \right] 
\]

\[
\leq \frac{5}{3} \theta(|x| + |y| + |z|)
\]

and

\[
|2f\left(\frac{x + y}{2}\right) - f(x) - f(y)|
\]

\[
\leq 2 \left| f\left(\frac{x}{2} + \frac{y}{2}\right) - f\left(\frac{x}{2}\right) - f\left(\frac{y}{2}\right) \right| + \left| f\left(\frac{x}{2} + \frac{y}{2}\right) - f\left(\frac{x}{2}\right) - f\left(\frac{x}{2}\right) \right| + \left| f\left(\frac{y}{2} + \frac{y}{2}\right) - f\left(\frac{y}{2}\right) - f\left(\frac{y}{2}\right) \right| 
\]

\[
\leq 2\theta(|x| + |y|)
\]

for all \( x, y, z \in \mathbb{R} \). By the inequality (9), we see that

\[
|3f\left(\frac{x + y + z}{3}\right) - f(x) - f(y) - f(z)|
\]
\[ \leq 3 \left| f\left(\frac{x}{3} + \frac{y}{3} + \frac{z}{3}\right) - f\left(\frac{x}{3}\right) - f\left(\frac{y}{3}\right) - f\left(\frac{z}{3}\right) \right| \\
+ \left| - \left[ f\left(\frac{x}{3} + \frac{y}{3} + \frac{z}{3}\right) - f\left(\frac{x}{3}\right) - f\left(\frac{y}{3}\right) - f\left(\frac{z}{3}\right) \right] \right| \\
+ \left| - \left[ f\left(\frac{x}{3} + \frac{y}{3} + \frac{y}{3}\right) - f\left(\frac{x}{3}\right) - f\left(\frac{y}{3}\right) - f\left(\frac{y}{3}\right) \right] \right| \\
+ \left| - \left[ f\left(\frac{z}{3} + \frac{z}{3} + \frac{z}{3}\right) - f\left(\frac{z}{3}\right) - f\left(\frac{z}{3}\right) - f\left(\frac{z}{3}\right) \right] \right| \\
\leq \frac{10}{3} \theta (|x| + |y| + |z|) \\
\]

for all \( x, y, z \in \mathbb{R} \). From the inequalities (10) and (11), we obtain that

\[ \left| 3f\left(\frac{x+y+z}{3}\right) - 2f\left(\frac{x+y}{2}\right) - f(z) \right| \]

\[ \leq \left| 3f\left(\frac{x+y+z}{3}\right) - f(x) - f(y) - f(z) \right| \]

\[ + \left| - \left[ 2f\left(\frac{x+y}{2}\right) - f(x) - f(y) \right] \right| \]

\[ \leq \frac{2}{3} \theta (8|x| + 8|y| + 5|z|) \leq \frac{16}{3} \theta (|x| + |y| + |z|) \]

for all \( x, y, z \in \mathbb{R} \). But we observe from [6] that

\[ \frac{f(x)}{x} \to \infty \text{ as } x \to \infty \]

and so

\[ \frac{|f(x) - g(x)|}{|x|} (x \neq 0) \text{ is unbounded,} \]

where \( g : \mathbb{R} \to \mathbb{R} \) is the function given by

\[ g(x) := \lim_{n \to \infty} 3^n f\left(\frac{x}{3^n}\right) \]

for all \( x \in \mathbb{R} \). Thus the function \( f \) provides an example to the effect that Theorem 2.2 fails to hold for \( r = 1 \).

**Theorem 2.3.** Let \( r < 1 \) and \( \theta \) be positive real numbers, and let \( f : A \to B \) be a mapping satisfying (4), (5) and \( f(0) = 0 \). Then there exists a unique \( C^* \) -ternary algebra homomorphism \( H : A \to B \) such that

\[ \|f(x) - H(x)\|_B \leq \theta \frac{2 + 3^r}{3 - 3^r} \|x\|^r_A \]

for all \( x \in A \).

**Proof.** It follows from (7) that

\[ \left\| f(x) - \frac{1}{3} f(3x) \right\|_B \leq \theta \frac{2 + 3^r}{3} \|x\|^r_A \]
for all \( x \in A \). So

\[
\left\| \frac{1}{3^j} f(3^j x) - \frac{1}{3^m} f(3^m x) \right\|_B \leq \sum_{j=l}^{m-1} \left\| \frac{1}{3^j} f(3^j x) - \frac{1}{3^{j+1}} f(3^{j+1} x) \right\|_B
\]

(13)

\[
\leq \theta \frac{2 + 3^r}{3} \sum_{j=l}^{m-1} 3^{j(r-1)} \| x \|_A^r = \theta \frac{2 + 3^r}{3 - 3^r} \left[ 3^{(r-1)} - 3^{m(r-1)} \right] \| x \|_A^r
\]

for all nonnegative integers \( m \) and \( l \) with \( m > l \) and all \( x \in A \). It follows from (13) that the sequence \( \left\{ \frac{1}{3^j} f(3^j x) \right\} \) is a Cauchy sequence for all \( x \in A \). Since \( B \) is complete, the sequence \( \left\{ \frac{1}{3^j} f(3^j x) \right\} \) converges for all \( x \in A \). So one can define the mapping \( H : A \to B \) by

\[
H(x) := \lim_{n \to \infty} \frac{1}{3^n} f(3^n x)
\]

for all \( x \in A \). Moreover, letting \( l = 0 \) and passing the limit \( m \to \infty \) in (13), we get (12).

By similar arguments to the proof of Theorem 2.2, the mapping \( H : A \to B \) is \( \mathbb{C} \)-linear. It follows from (5) and (13) that

\[
\| H([x, y, z]) - [H(x), H(y), H(z)] \|_B \\
\leq \theta \frac{2 + 3^r}{3 - 3^r} \| [x, y, z] \|_A^r \lim_{n \to \infty} \left[ 3^{n(r-1)} - 3^{3n(r-1)} \right] \\
+ \theta (\| x \|_A^r + \| y \|_A^r + \| z \|_A^r) \lim_{n \to \infty} 3^{n(r-3)} \\
= 0
\]

for all \( x, y, z \in A \). So

\[
H([x, y, z]) = [H(x), H(y), H(z)]
\]

for all \( x, y, z \in A \). Now, let \( T : A \to B \) be another additive mapping satisfying (12). Then we have

\[
\| H(x) - T(x) \|_B \leq 2 \theta 3^{3n(r-1)} \frac{2 + 3^r}{3 - 3^r} \| x \|_A^r,
\]

which tends to zero as \( n \to \infty \) for all \( x \in A \). So we can conclude that \( H(x) = T(x) \) for all \( x \in A \). This proves the uniqueness of \( H \). Thus the mapping \( H : A \to B \) is a unique \( C^* \)-ternary algebra homomorphism satisfying (12). □

**Theorem 2.4.** Let \( r > \frac{1}{3} \) and \( \theta \) be positive real numbers, and let \( f : A \to B \) be a mapping satisfying \( f(0) = 0 \) such that

(14) \[
\| D_\mu f(x, y, z) \|_B \leq \theta \cdot \| x \|_A^r \cdot \| y \|_A^r \cdot \| z \|_A^r,
\]

(15) \[
\| f([x, y, z]) - [f(x), f(y), f(z)] \|_B \leq \theta \cdot \| x \|_A^r \cdot \| y \|_A^r \cdot \| z \|_A^r
\]
for all $\mu \in T^1$ and all $x, y, z \in A$. Then there exists a unique $C^*$-ternary algebra homomorphism $H : A \to B$ such that

\[(16) \quad \|f(x) - H(x)\|_B \leq \frac{3^r \theta}{27^r - 3} \|x\|_A^{3r}\]

for all $x \in A$.

**Proof.** Letting $\mu = 1$ and $y = -x$ and $z = 3x$ in (14), we get

\[(17) \quad \|f(3x) - 3f(x)\|_B \leq 3^r \theta \|x\|_A^{3r}\]

for all $x \in A$. So

\[
\left\| f(x) - 3f\left(\frac{x}{3}\right) \right\|_B \leq \frac{\theta}{9^r} \|x\|_A^{3r}
\]

for all $x \in A$. Hence

\[
\left\| 3^l f\left(\frac{x}{3^l}\right) - 3^m f\left(\frac{x}{3^m}\right) \right\|_B \leq \sum_{j=l}^{m-1} \left\| 3^j f\left(\frac{x}{3^j}\right) - 3^{j+1} f\left(\frac{x}{3^{j+1}}\right) \right\|_B
\]

\[
\leq \frac{\theta}{9^r} \sum_{j=l}^{m-1} 3^{j(1-3r)} \|x\|_A^{3r} = \frac{\theta}{9^r - 3^{1-3r}} \left[ 3^{l(1-3r)} - 3^{m(1-3r)} \right] \|x\|_A^{3r}
\]

for all nonnegative integers $m$ and $l$ with $m > l$ and all $x \in A$. It follows from (18) that the sequence $\{3^n f\left(\frac{x}{3^n}\right)\}$ is a Cauchy sequence for all $x \in A$. Since $B$ is complete, the sequence $\{3^n f\left(\frac{x}{3^n}\right)\}$ converges. So one can define the mapping $H : A \to B$ by

\[H(x) := \lim_{n \to \infty} 3^n f\left(\frac{x}{3^n}\right)\]

for all $x \in A$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (18), we get (16).

The rest of the proof is similar to the proof of Theorem 2.2. □

**Theorem 2.5.** Let $r < \frac{1}{3}$ and $\theta$ be positive real numbers, and let $f : A \to B$ be a mapping satisfying (14), (15) and $f(0) = 0$. Then there exists a unique $C^*$-ternary algebra homomorphism $H : A \to B$ such that

\[(19) \quad \|f(x) - H(x)\|_B \leq \frac{3^r \theta}{3 - 27^r} \|x\|_A^{3r}\]

for all $x \in A$.

**Proof.** It follows from (17) that

\[
\left\| f(x) - \frac{1}{3} f(3x) \right\|_B \leq 3^{r-1} \theta \|x\|_A^{3r}
\]
for all $x \in A$. So

\[
\left\| \frac{1}{3^l} f(3^l x) - \frac{1}{3^m} f(3^m x) \right\|_B \leq \sum_{j=l}^{m-1} \left\| \frac{1}{3^j} f(3^j x) - \frac{1}{3^{j+1}} f(3^{j+1} x) \right\|_B
\]

(20)

\[
\leq 3^{r-1} \frac{\theta}{3^{3r-1}} \sum_{j=l}^{m-1} 3^j \|x\|^{3r} = \frac{3^{r-1} \theta}{1 - 3^{3r-1}} [3^{l(3r-1)} - 3^{m(3r-1)}] \|x\|^{3r}
\]

for all nonnegative integers $m$ and $l$ with $m > l$ and all $x \in A$. It follows from (20) that the sequence $\left\{ \frac{1}{3^m} f(3^m x) \right\}$ is a Cauchy sequence for all $x \in A$. Since $B$ is complete, the sequence $\left\{ \frac{1}{3^m} f(3^m x) \right\}$ converges for all $x \in A$. So one can define the mapping $H : A \to B$ by

\[
H(x) := \lim_{n \to \infty} \frac{1}{3^n} f(3^n x)
\]

for all $x \in A$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (20), we get (19).

The rest of the proof is similar to the proof of Theorem 2.2. □

3 Isomorphisms between $C^*$-ternary algebras

Throughout this section, assume that $A$ is a unital $C^*$-ternary algebra with norm $\| \cdot \|_A$ and unit $e$, and that $B$ is a unital $C^*$-ternary algebra with norm $\| \cdot \|_B$ and unit $e'$.

We investigate isomorphisms between $C^*$-ternary algebras associated with the functional equation $D_\mu f(x, y, z) = 0$.

**Theorem 3.1.** Let $r > 1$ and $\theta$ be positive real numbers, and let $f : A \to B$ be a bijective mapping satisfying (4) and $f(0) = 0$ such that

(21)

\[
f([x, y, z]) = [f(x), f(y), f(z)]
\]

for all $x, y, z \in A$. If $\lim_{n \to \infty} 3^n f(x) = e'$, then the mapping $f : A \to B$ is a $C^*$-ternary algebra isomorphism.

**Proof.** By the same argument as in the proof of Theorem 2.2, one can obtain a $\mathbb{C}$-linear mapping $H : A \to B$ satisfying (6). The mapping $H$ is given by

\[
H(x) := \lim_{n \to \infty} 3^n f \left( \frac{x}{3^n} \right)
\]

for all $x \in A$. 

Since \( f([x, y, z]) = [f(x), f(y), f(z)] \) for all \( x, y, z \in A \),

\[
H([x, y, z]) = \lim_{n \to \infty} 3^n f \left( \frac{1}{3^n} [x, y, z] \right) = \lim_{n \to \infty} 3^n f \left( \left[ \frac{x}{3^n}, \frac{y}{3^n}, \frac{z}{3^n} \right] \right)
\]

\[
= \lim_{n \to \infty} \left[ 3^n f \left( \frac{x}{3^n} \right), 3^n f \left( \frac{y}{3^n} \right), 3^n f \left( \frac{z}{3^n} \right) \right]
\]  

\[= [H(x), H(y), H(z)]\]

for all \( x, y, z \in A \). So the mapping \( H : A \to B \) is a \( C^* \)-ternary algebra homomorphism.

It follows from (21) that

\[
H(x) = H([e, e, x]) = \lim_{n \to \infty} 3^n f \left( \frac{e}{3^n}, e, x \right) = \lim_{n \to \infty} 3^n f \left( \left[ \frac{e}{3^n}, \frac{e}{3^n}, x \right] \right)
\]

\[
= \lim_{n \to \infty} \left[ 3^n f \left( \frac{e}{3^n} \right), 3^n f \left( \frac{e}{3^n} \right), f(x) \right] = [e', e', f(x)] = f(x)
\]

for all \( x \in A \). Hence the bijective mapping \( f : A \to B \) is a \( C^* \)-ternary algebra isomorphism. \( \square \)

**Theorem 3.2.** Let \( r < 1 \) and \( \theta \) be positive real numbers, and let \( f : A \to B \) be a bijective mapping satisfying (4), (21) and \( f(0) = 0 \). If

\[
\lim_{n \to \infty} 3^n f \left( \frac{1}{3^n} e \right) = e',
\]

then the mapping \( f : A \to B \) is a \( C^* \)-ternary algebra isomorphism.

**Proof.** By the same argument as in the proof of Theorem 2.3, one can obtain a \( \mathbb{C} \)-linear mapping \( H : A \to B \) satisfying (12).

The rest of the proof is similar to the proof of Theorem 3.1. \( \square \)

**Theorem 3.3.** Let \( r > \frac{1}{3} \) and \( \theta \) be positive real numbers, and let \( f : A \to B \) be a bijective mapping satisfying (14), (21) and \( f(0) = 0 \). If

\[
\lim_{n \to \infty} 3^n f \left( \frac{e}{3^n} \right) = e',
\]

then the mapping \( f : A \to B \) is a \( C^* \)-ternary algebra isomorphism.

**Proof.** By the same argument as in the proof of Theorem 2.4, one can obtain a \( \mathbb{C} \)-linear mapping \( H : A \to B \) satisfying (16).

The rest of the proof is similar to the proof of Theorem 3.1. \( \square \)

**Theorem 3.4.** Let \( r < \frac{1}{3} \) and \( \theta \) be positive real numbers, and let \( f : A \to B \) be a bijective mapping satisfying (14), (21) and \( f(0) = 0 \). If

\[
\lim_{n \to \infty} 3^n f \left( \frac{1}{3^n} e \right) = e',
\]

then the mapping \( f : A \to B \) is a \( C^* \)-ternary algebra isomorphism.

**Proof.** By the same argument as in the proof of Theorem 2.5, one can obtain a \( \mathbb{C} \)-linear mapping \( H : A \to B \) satisfying (19).
The rest of the proof is similar to the proof of Theorem 3.1.

4 Stability of $C^*$-ternary derivations on $C^*$-ternary algebras

Throughout this section, assume that $A$ is a $C^*$-ternary algebra with norm $\| \cdot \|_A$.

We prove the generalized Hyers-Ulam stability of $C^*$-ternary derivations on $C^*$-ternary algebras for the functional equation $D_\mu f(x, y, z) = 0$.

**Theorem 4.1.** Let $r > 3$ and $\theta$ be positive real numbers, and let $f : A \to A$ be a mapping satisfying $f(0) = 0$ such that

\[
\| D_\mu f(x, y, z) \|_A \leq \theta(\| x \|_A^r + \| y \|_A^r + \| z \|_A^r),
\]

\[
\| f(x, y, z) - [f(x), y, z] - [x, f(y), z] - [x, y, f(z)] \|_A \leq \theta(\| x \|_A^r + \| y \|_A^r + \| z \|_A^r)
\]

for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in A$. Then there exists a unique $C^*$-ternary derivation $\delta : A \to A$ such that

\[
\| f(x) - \delta(x) \|_A \leq \theta \frac{3^r + 2}{3^r - 3} \| x \|_A^r
\]

for all $x \in A$.

**Proof.** By the same argument as in the proof of Theorem 2.2, one can obtain a $\mathbb{C}$-linear mapping $\delta : A \to B$ satisfying (24). The mapping $\delta$ is given by

\[
\delta(x) := \lim_{n \to \infty} 3^n f\left(\frac{x}{3^n}\right)
\]

for all $x \in A$.

By the same reasoning as in the proof of Theorem 2.1 of [21], the mapping $\delta : A \to A$ is $\mathbb{C}$-linear.

It follows from (23) that

\[
\| \delta([x, y, z]) - [\delta(x), y, z] - [x, \delta(y), z] - [x, y, \delta(z)] \|_A
= \lim_{n \to \infty} 3^n f\left(\frac{x, y, z}{3^n}\right) - 3^n f\left(\frac{x}{3^n}, \frac{y}{3^n}, \frac{z}{3^n}\right)
\]

\[
-3^n f\left(\frac{x}{3^n}, \frac{y}{3^n}, \frac{z}{3^n}\right) - 3^n f\left(\frac{x}{3^n}, \frac{y}{3^n}, \frac{z}{3^n}\right)
\]

\[
\leq \lim_{n \to \infty} 3^{n(3-r)} \theta(\| x \|_A^r + \| y \|_A^r + \| z \|_A^r) = 0
\]

for all $x, y, z \in A$. So

\[
\delta([x, y, z]) = [\delta(x), y, z] + [x, \delta(y), z] + [x, y, \delta(z)]
\]
for all $x, y, z \in A$.

By the same argument as in the proof of Theorem 2.2, the uniqueness of $\delta$ is proved. Thus the mapping $\delta$ is a unique $C^*$-ternary derivation satisfying (24).

**Theorem 4.2.** Let $r < 1$ and $\theta$ be positive real numbers, and let $f : A \to A$ be a mapping satisfying (22), (23) and $f(0) = 0$. Then there exists a unique $C^*$-ternary derivation $\delta : A \to A$ such that

$$(25) \quad \|f(x) - \delta(x)\|_A \leq \frac{\theta \cdot 2 + 3^r}{3 - 3^r} \|x\|^r_A$$

for all $x \in A$.

**Proof.** By the same argument as in the proof of Theorem 2.3, one can obtain a $C^*$-linear mapping $\delta : A \to B$ satisfying (25).

The rest of the proof is similar to the proof of Theorem 4.1. \qed

**Theorem 4.3.** Let $r > \frac{1}{3}$ and $\theta$ be positive real numbers, and let $f : A \to A$ be a mapping satisfying $f(0) = 0$ such that

$$(26) \quad \|D_\mu f(x, y, z)\|_A \leq \theta \cdot \|x\|^r_A \cdot \|y\|^r_A \cdot \|z\|^r_A,$$

$$\|f([x, y, z]) - [f(x), y, z] - [x, f(y), z] - [x, y, f(z)]\|_A \leq \theta \cdot \|x\|^r_A \cdot \|y\|^r_A \cdot \|z\|^r_A$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in A$. Then there exists a unique $C^*$-ternary derivation $\delta : A \to A$ such that

$$(27) \quad \|f(x) - \delta(x)\|_B \leq \frac{3^r \theta}{3 - 27^r} \|x\|^3$$

for all $x \in A$.

**Proof.** By the same argument as in the proof of Theorem 2.4, one can obtain a $C^*$-linear mapping $\delta : A \to B$ satisfying (27).

The rest of the proof is similar to the proof of Theorem 4.1. \qed

**Theorem 4.4.** Let $r < \frac{1}{3}$ and $\theta$ be positive real numbers, and let $f : A \to A$ be a mapping satisfying (26), (27) and $f(0) = 0$. Then there exists a unique $C^*$-ternary derivation $\delta : A \to A$ such that

$$(28) \quad \|f(x) - \delta(x)\|_B \leq \frac{3^r \theta}{3 - 27^r} \|x\|^3$$

for all $x \in A$. 


Proof. By the same argument as in the proof of Theorem 2.5, one can obtain a \( \mathbb{C} \)-linear mapping \( \delta : A \to B \) satisfying (29).

The rest of the proof is similar to the proofs of Theorems 4.1. \( \square \)

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