Topology Change in ISO(2,1) Chern-Simons Gravity

KAORU AMANO AND SABURO HIGUCHI

Department of Physics
Tokyo Institute of Technology
Meguro, Tokyo, 152 Japan

ABSTRACT

In 2+1 dimensional gravity, a dreibein and the compatible spin connection can represent a space-time containing a closed spacelike surface $\Sigma$ only if the associated SO(2,1) bundle restricted to $\Sigma$ has the same non-triviality (Euler class) as that of the tangent bundle of $\Sigma$. We impose this bundle condition on each external state of Witten’s topology-changing amplitude. The amplitude is non-vanishing only if the combination of the space topologies satisfies a certain selection rule. We construct a family of transition paths which reproduce all the allowed combinations of genus $g \geq 2$ spaces.

* shiguchi@cc.titech.ac.jp
1. Introduction

2+1 dimensional pure gravity as reformulated by Witten[1] may provide us with a tractable model of Einstein gravity with topology change[2]. The theory (without a cosmological constant) can be identified with a Chern-Simons gauge theory (CSGT) of structure group ISO(2,1) (the Poincaré group in 2+1 dimensions) with an appropriate Killing form. The path integral in this ISO(2,1) CSGT is computable for an arbitrary space-time topology with or without boundary. Accordingly not only propagators but vacuum amplitudes, Hartle-Hawking wave functions, and other amplitudes involving sectional topology changes can be constructed. They all can be brought into the form of the sum (integral) over the moduli space of flat connections up to gauge transformations. Whether an amplitude with a particular mode of topology change vanishes or not can be judged by seeing whether there exists a flat connection that satisfies the corresponding boundary condition.

However, Witten’s topology-changing amplitude may not entirely correspond to the change of spatial topology. There is no guarantee that the asymptotic space-time admits a spacelike slice. This implies that the in- or out-states may not be completely free from the interacting region, and then it is quite awkward to regard that amplitude as referring to observable spatial topologies. We propose to restrict the theory so that asymptotic states are only those with a spacelike slice. But how can this be achieved?

In his first paper on Chern-Simons gravity[1], Witten selects a special sector of the ISO(2,1) CSGT for the description of gravity. Based on a relation between flat SO(2,1) structure and conformal structure on closed surfaces, he adopts, in effect, the following restriction. In canonical theory on a closed 2-space Σ of genus \( g \geq 2 \), of all the flat SO(2,1) connections on Σ only those defined in an SO(2,1) bundle with Euler class \( \pm(2g - 2) \) are relevant to gravity. This condition is in fact equivalent to demanding that Σ can be embedded in the space-time as a spacelike hypersurface[3].
What we do is as follows. We require that the path integral should be taken only over connections in an ISO(2,1) bundle whose portion over each component of space-time boundary has the right Euler class. Then we ask what kind of topology change is possible. We will see that the condition for the existence of a bundle satisfying the above condition leads to a selection rule on the combination of 2-space topologies involved in the process. Then we construct a family of transition paths that cover a substantial part of allowed topology combinations.

As we do not actually work out the amplitude or check the consistency of the whole model, our investigation remains at a preliminary stage. Our work is comparable to refs.[4][5][6][7]. In particular, our selection rule has a striking formal resemblance to that of Sorkin[4], which arises from Lorentzian cobordism. Also the pull-back construction of Horowitz[5] guarantees the existence of a certain type of topology-changing paths in our approach.

Besides the issue of spatial topology change, this paper contains a contribution towards a geometric understanding of Chern-Simons gravity. There are two apparently different ways to associate an ISO(2,1) flat connection with a space-time geometry. One is to identify the connection with a dreibein-spin connection pair, and the other is to identify the holonomy of the connection with that of a flat Lorentzian structure. The equivalence with gravity in terms of the action and equations of motion thereof is based on the first, while the second is invoked in the original derivation of the restriction on the SO(2,1) bundle[1]. Whether these two principles of identification are compatible is a non-trivial question[8]. In the following section, we demonstrate the compatibility by deriving the second from the first. This strengthens the interpretation of the bundle condition as a condition for a spacelike space, which we base on the first principle.

This paper is organized as follows. In sect.2 we discuss how an ISO(2,1) gauge field dictates space-time geometry, and then clarify the condition for space-time to contain a spacelike slice. In sect.3 we derive the selection rule and then produce the examples of topology-changing paths. In sect.4 we discuss our results.
Convention
Throughout this work, by SO(2,1) and ISO(2,1) we actually mean the maximal connected subgroups instead of the full groups. Thus SO(2,1) in our convention is isomorphic to PSL(2,\mathbb{R})=SL(2,\mathbb{R})/\{\pm 1\}.

2. ISO(2,1) connections, space-time, and spacelike surfaces

We shall discuss how ISO(2,1) flat connections relate to space-time geometry. We will see that the the holonomy of a flat Lorentz structure arising from a flat connection is identical with that of the connection[1][8]. Also the condition for space-time to admit a closed 2-space is introduced[3].

2.1 The Einstein-Hilbert-Chern-Simons action

Let \( A \) be a connection in an ISO(2,1) flat bundle \( \hat{P} \) over an orientable 3-manifold \( M \). Locally \( A \) is a Lie-algebra-valued 1-form

\[
A = e^a P_a + \omega^a J_a, \tag{2.1}
\]

where \( P_a, J_a (a = 0, 1, 2) \) are a basis for the ISO(2,1) algebra with the commutators

\[
[P_a, P_b] = 0, \quad [J_a, P_b] = \epsilon_{abc} P^c, \quad \text{and} \quad [J_a, J_b] = \epsilon_{abc} J^c.
\]

The indices are raised and lowered with the Lorentz metric \((\eta^{ab}) = (\eta_{ab}) = \text{diag}[-1, 1, 1]\). \( A \) is expressed as eq.(2.1) with respect to local sections of \( \hat{P} \) over coordinate patches \( U_\alpha \) of \( M \). The local sections are related to each other by transition functions \( \psi_{\alpha\beta} \) on overlaps \( U_\alpha \cap U_\beta \). Correspondingly the local expressions of \( A \) satisfy the relation

\[
A_\beta = \psi^{-1}_{\alpha\beta} A_\alpha \psi_{\alpha\beta} + \psi^{-1}_{\alpha\beta} d\psi_{\alpha\beta}. \tag{2.2}
\]

We can choose local sections so that all \( \psi_{\alpha\beta} \) take values in the SO(2,1) subgroup generated by \( J_a \). This reduces \( \hat{P} \) to an SO(2,1) bundle \( P \). \( P \) is also flat by the semi-direct product nature of the group ISO(2,1). The decomposition (2.1) of \( A \) into \( \omega \) and \( e \) parts is now global, with \( \omega \) an SO(2,1) connection in the reduced bundle \( P \), and \( e \) an associated 1-form dreibein, which may be degenerate.
To make contact with gravity, we treat $e$ and $\omega$ as variables for 2+1 dimensional Einstein theory in the first-order formalism. Then the Einstein-Hilbert action,

$$S = \int_M e^a (d\omega^a + \frac{1}{2} \epsilon_{abc} \omega^b \omega^c),$$

(2.3)
is an ISO(2,1) Chern-Simons action[1]. $S$ is well-defined in our system of local sections and is invariant under any gauge transformations that reduce to SO(2,1) transformations on the boundary of $M$. By a gauge transformation, we mean the one globally defined: a set of ISO(2,1) valued functions $h_\alpha$ with the consistency condition $h_\alpha \psi_{\alpha\beta} = \psi_{\alpha\beta} h_\beta$ acting on $A$ as

$$A_\alpha \rightarrow A'_\alpha = h^{-1}_\alpha A_\alpha h_\alpha + h^{-1}_\alpha dh_\alpha.$$  

(2.4)

Chern-Simons action (2.3) gives equations of motion which demand that $A$ be flat,

$$dA + AA = 0.$$  

(2.5)
The existence of solutions is guaranteed by the flatness of $\hat{P}$.

2.2 Development and holonomy

The geometrical consequence of eq.(2.5) is that space-time $M$ with $A$ is locally Minkowski. That is, it can be mapped into Minkowski space $X$ preserving its local metric content. We shall construct such a map.

To begin with, we take a frame $u = (q, f_0, f_1, f_2)$ in $X$, where the tangents $f_a$ to $X$ at $q \in X$ are orthonormal, $f_a \cdot f_b = \eta_{ab}$. We let the group ISO(2,1) act on $u$ on the right by the rule,

$$(q, f_a) \cdot TL = (q + T^a f_a, f_b L^b_a),$$

(2.6)

with $T = e^{T^a P_a}$, $L J_a L^{-1} = J_b L^b_a$. Now the following first-order differential equation makes sense:

$$du = u \cdot A.$$  

(2.7)
The flatness (2.5) says (2.7) is locally integrable. Given a frame $u_0$ in $X$ and a
local section over point $\ast \in M$, we get a frame field $u(p)$ on a neighbourhood $U$ of $\ast$ by integrating eq.(2.7) along paths in $U$ starting from $\ast$ with the initial value $u(\ast) = u_0$. In other words, by associating $p = \ast$ with $u = u_0$ and moving them together according to the transportation law (2.7), we can map $U \subset M$ onto a set of frames of $X$. Then the map of $U$ into $X : p \mapsto u(p) \mapsto q(p)$ preserves the metric. This is evident if we write down eq.(2.7) in terms of $e$ and $\omega$:

$$dq = e^a f_a, \quad df_a = -\epsilon_{abc} \omega^b f^c.$$ (2.8)

Thus we obtained a local version of the map we have been looking for.

To get a global version, we extend the above construction to general paths from $\ast$, not necessarily confined to its neighbourhood. In the integration of eq.(2.7), we switch local sections over the path if necessary by the rule consistent with (2.2): $u_\beta = u_\alpha \cdot \psi_{\alpha\beta}$ on $U_\alpha \cap U_\beta$. At the switching, the frame $u$ just pivots, with $q$ maintaining its position and course, as our $\psi_{\alpha\beta}$ are all $SO(2,1)$-valued. To disentangle path dependence we lift the path to a universal cover $\tilde{M}$ of $M$. The end positions of $q$ put together give the development map\textsuperscript{[9]} $\phi : \tilde{M} \rightarrow X$.

The holonomy of connection $A$ arises naturally in the integration of eq.(2.7) in the form, $u(\gamma \cdot \ast) = u_0 \cdot \rho(\gamma)$ for $\gamma \in \pi_1(M, \ast)$. Let us denote by the same $\rho(\gamma)$ also the Poincaré transformation on $X$ whose differential map sends $u_0$ to $u_0 \cdot \rho(\gamma)$. Then for any $\gamma \in \pi_1(M, \ast)$ and $\tilde{p} \in \tilde{M}$, we have,

$$\phi(\gamma \cdot \tilde{p}) = \rho(\gamma)\phi(\tilde{p}).$$ (2.9)

Thus the holonomy of connection $A$ is also that of space-time geometry. The gauge transformation $h = \{h_\alpha\}$ acts on $u$ as $u_\alpha \rightarrow u'_\alpha = u_\alpha \cdot h_\alpha$ for each local section. This is intuitively a nonuniform Poincaré transformation. It deforms the development map $\phi$ rather arbitrarily except that the holonomy homomorphism $\rho : \pi_1(M, \ast) \rightarrow ISO(2,1)$ changes only by an overall conjugation, $\rho \rightarrow \rho' = h_0 \rho h_0^{-1}$ with some $h_0 \in ISO(2,1)$. 

6
2.3 The space condition

Let $\Sigma$ be a closed orientable surface (of genus $g$) embedded in $M$. Suppose $\Sigma$ is spacelike in a space-time associated with $A$, a flat connection in $\hat{P}$. That is, there exists a gauge choice such that the dreibein $e$ induces a positive definite metric on $\Sigma$. Then as we shall see (sect.2.3.2), the Euler class (sect.2.3.1) of $P|_\Sigma$ the restriction to $\Sigma$ of the reduced SO(2,1) bundle $P$ (sect.2.1), is given by \[ eul(P|_\Sigma) = \pm \chi_\Sigma, \] (2.10)

where $\chi_\Sigma = 2 - 2g$ is the Euler characteristic of $\Sigma$. In this statement $P$ can be replaced with the original ISO(2,1) bundle $\hat{P}$ without changing the content.

2.3.1 Euler class

Let $G$ be one of ISO(2,1), SO(2,1), GL$^+(2,\mathbb{R})$, SO(2), or any other groups which have the same homotopy type as the circle $S^1$, and let $E$ be a $G$ bundle over $\Sigma$. The Euler class $\text{eul}(E) \in \mathbb{Z} \approx \mathbb{H}^2(\Sigma, \mathbb{Z})$ classifies $E$ completely (up to isomorphism). It gives a one-to-one correspondence between integers and the bundles with fixed $G$ and $\Sigma$. Another qualification is that $\text{eul}(E)$ measures the obstruction to taking a global section of $E$. To give a more definite idea, we remove a disc from $\Sigma$ and cover the hole by a larger open disc $D$, so that $\Sigma = C \cup D$ where $C$ is the compliment of the smaller disc. We take sections over the patches $C$ and $D$, obtaining transition functions $\psi_{CD}$ and $\psi_{DC} = \psi_{CD}^{-1}$ on $C \cap D$. The integer $\text{eul}(E)$ is given by counting how many times $\psi_{CD}$ winds in $G \simeq S^1$ as we go round the annulus $C \cap D$.

A vanishing $\text{eul}(E)$ will mean that the $E$ admits a global section, i.e., it is trivial. In other cases the sign of $\text{eul}$ depends on the orientation of $\Sigma$. Also the following should be obvious. If $G$ bundle $E$ reduces to an SO(2) bundle $E_0$, i.e., $E$ and $E_0$ share a common system of transition functions, then $\text{eul}(E) = \text{eul}(E_0)$. Structure group $G$ is reducible to its SO(2) subgroup. So there is an $E_0$ for every $E$. 

7
This allows us to rephrase the above procedure into a Gauss-Bonnet type formula:

Picking an arbitrary connection \( \nu \) in \( E_0 \), we have,

\[
eul(E) = \frac{1}{2\pi i} \int_{\Sigma} d\nu,
\]

(2.11)

where we used the \( U(1) \) notation, \( \nu = i\nu_{12} \).

We will need the following fact shortly: \( \eul(\theta_{\Sigma}) = \chi_{\Sigma}[10] \). This is the special case of eq.(2.11) when \( E = \theta_{\Sigma} \) is the \( GL^+(2,\mathbb{R}) \) bundle defined by the transition functions \( \psi_{\alpha\beta} = (\partial\phi^i_{\alpha}/\partial\phi^j_{\beta}) \), the Jacobian matrices for a consistently oriented system of coordinates \( \{\phi_{\alpha}\} \) of \( \Sigma \). \( \theta_{\Sigma} \) is associated with the tangent bundle \( T\Sigma \), and is itself called the tangent bundle of \( \Sigma \).

2.3.2 2.3.2. Proof of eq.(2.10)

The assertion is that \( P|_{\Sigma} \) and the tangent bundle \( \theta_{\Sigma} \) have the same Euler class up to a sign.

We may think of \( \tilde{\Sigma} \) as embedded in Minkowski space \( X \), spacelike and with \( SO(2,1) \) local frame fields \( f_a \) on it, in accordance with the construction of sect.2.2. The local sections of \( P|_{\Sigma} \) can be adjusted so that \( f_0 \) is normal to \( \tilde{\Sigma} \) everywhere. Then \( f_0 \) is common to all the sections over overlapping patches, and \( f_1 \) and \( f_2 \) constitute an \( SO(2) \) frame tangent to \( \tilde{\Sigma} \). The transition functions for \( P|_{\Sigma} \) are now also those for the tangent bundle \( \theta_{\Sigma} \) of \( \Sigma \) or of the same surface with a reversed orientation. Thus these two bundles reduce to a common \( SO(2) \) bundle, and therefore have a common Euler class. So \( \eul(P|_{\Sigma}) = \pm\eul(\theta_{\Sigma}) = \pm\chi_{\Sigma} \). We are done.
2.3.3 Comments

In the above we used the notation from the construction of a development map, but it is not essential. With a little modification the proof applies to the bundle of SO(2,1) frames tangent to a not necessarily flat Lorentzian 3-manifold with a spacelike hypersurface $\Sigma$. Thus eq.(2.10) holds in a more general situation than suggested before.

Sticking with our original motivation, we require $P$ to be flat. $P|_{\Sigma}$ is also flat then. For $g \geq 2$, flat bundles $E$ over $\Sigma$ with structure group SO(2,1) are those with $\chi_{\Sigma} \leq \text{eul}(E) \leq -\chi_{\Sigma}$ \cite{11}\cite{3}. $P|_{\Sigma}$ must belong to this range and the space condition (2.10) picks out the ones with the maximal absolute value. For $g = 0, 1$, flat $E$ is necessarily trivial, eul($E$) = 0. So for $g = 0$, no $P$ is allowed. This reflects the obvious fact that sphere $\tilde{\Sigma} = \Sigma$ cannot be embedded in $X$ spacelike.

The condition (2.10) is necessary for $\Sigma$ to become spacelike. A natural question is then, is it sufficient? To give a partial answer, we restrict ourselves to the topology $M = \Sigma \times \mathbb{R}$. This is a suitable space-time topology for canonical formulation, taking $\Sigma$ at some time as an initial surface. $P$ is determined by its restriction to the initial surface. The condition (2.10) specifies $P$. The holonomy homomorphism $\rho : \pi_1(M) = \pi_1(\Sigma) \to \text{ISO}(2,1)$ should contain all gauge invariant information on the flat connection. (This view entails the assumption that gauge transformation at infinity $t \to \pm\infty$ is subject to no restrictions.) Then we may say the condition (2.10) is sufficient if for any flat connection in the selected $P$ there is an appropriate flat Lorentzian manifold that admits spacelike $\Sigma$ and has the holonomy corresponding to the connection. For it implies that in some gauges the connection reproduces that Lorentzian manifold with $\Sigma$ embedded in the desired way. Mess\cite{3} classifies flat Lorentzian manifolds containing a spacelike hypersurface by their holonomy. He establishes that for $g \geq 2$, the moduli space of flat connections in the eul = $\pm\chi_{\Sigma}$ bundles parametrizes a certain family of reasonable space-times with a spacelike slice. Thus the space condition (2.10) is sufficient in the above sense for $g \geq 2$. For $g = 1$, the only allowed $P$ is a trivial bundle. In
this bundle some flat connections correspond to space-times with spacelike $\Sigma$ but others do not. For the necessary restriction on the space of flat connections, see refs.[3][12].

3. Quantization, transition paths, and a selection rule

We regard gravity as dynamics of space geometry, and restrict the species of spaces to be closed orientable surfaces. The space-time metric must be such that the spaces are spacelike by the metric induced on them. In the present theory the classical space-time to accommodate space dynamics seems to invariably have a constant spatial topology. The space-time is flat Lorentzian, and time-orientable because our structure group is a connected subgroup of the Poincaré group. Mess[3] proves that a compact, flat, time-orientable Lorentzian manifold with spacelike boundary necessarily has a topology $M = \Sigma \times [0, 1]$, with $\Sigma$ spacelike at each time $t \in [0, 1]$. We interpret this as demonstrating that topology change is not allowed in the ISO(2,1) gravity at classical level. To put it the other way round, our classical topology change is what is ruled out by Mess’ theorem: either the space-time has a topology different from $\Sigma \times [0, 1]$ or defies a spacelike time-slicing. Note that we are assuming the non-degeneracy of space-time metric. Without this assumption, there do exist solutions in the ISO(2,1) CSGT that can almost be called space-times with spatial topology change[5]. Nonetheless those solutions are presumably more suitable as intermediate paths than as full-fledged classical space-times. We prefer to consider such solutions in the context of quantum theory.

3.1 Quantization[1][2]

The ISO(2,1) CSGT owes much of its simplicity to the fact that the structure group $\hat{G} = \text{ISO}(2,1)$ is equal to the total space of the tangent bundle to $G = \text{SO}(2,1)$ as a Lie group: $\hat{G} = TG$. This leads to a relation of the form $\hat{A} = T\mathcal{A}$, where $\hat{A}$ is any of the spaces of connections, flat connections in the ISO(2,1) bundle $\hat{P}$, or the corresponding moduli spaces, and $\mathcal{A}$ the counterpart for the reduced
SO(2,1) bundle $P$. (Of course we cannot expect the relations to be precise when $\mathcal{A}$ is not a manifold.)

In canonical formulation on the surface $\Sigma$, the relation $\hat{\mathcal{M}} = T\mathcal{M}$ is particularly important, where $\hat{\mathcal{M}}$ is the moduli space of flat connections in $\hat{P}$ and $\mathcal{M}$ counterpart for $P$. The $\hat{P}$ and $P$ here can be thought of as either bundles over the space-time manifold $M = \Sigma \times \mathbb{R}$, or those over the 2-space $\Sigma$. $\hat{\mathcal{M}}$, the moduli space of classical solutions, is the physical phase space. The symplectic structure from the action (2.3) says $e$ and $\omega$ are canonically conjugate. The relation $\hat{\mathcal{M}} = T\mathcal{M}$ suggests we should take a gauge equivalence class $\mathrm{cls} \, \omega \in \mathcal{M}$ as ‘coordinates’ and tangents to $\mathcal{M}$ as ‘momenta.’ Correspondingly a physical state in canonical quantization is represented by a function on $\hat{\mathcal{M}}$, or equivalently by a gauge-invariant function of connection $\omega$ in $P$. For $g = 1$, some modification is necessary since the space of flat connections must be restricted$[12]$. For $g \geq 2$, with $P$ satisfying the space condition (2.10), $\mathcal{M}$ can be identified with Teichmüller space of $\Sigma$. This arises from the fact that the holonomy homomorphism of a flat connection in $P$ gives a discrete embedding of $\pi_1(\Sigma)$ into $\text{SO}(2,1) \approx \text{PSL}(2,\mathbb{R})$, and vice versa$[13]$. In particular, to provide $\Sigma$ with the structure of a Riemann surface is effectively to give a flat connection in $P$ up to gauge transformation. We will need these facts later.

To be precise the above procedure does not complete canonical quantization. For we would have to discuss observables and the measure on $\mathcal{M}$, which includes taking care of the mapping class group on $\Sigma[12][8][14]$. We will not go into these issues.

Now we proceed to the path integral approach. We take space-time 3-manifold $\hat{M}$ to be a compact orientable manifold with boundary consisting of connected components $\Sigma_1, \ldots, \Sigma_N$, each of which is closed and orientable. With some SO(2,1) flat connections $\omega_j$ given on $\Sigma_j$, we consider the path integral,

$$I_M(\omega_1, \ldots, \omega_N) = \int_{\omega = \omega_j \text{ on } \Sigma_j} D\omega \int De \exp iS.$$  \hspace{1cm} (3.1)
The first integral is over all SO(2,1) connections $\omega$ on $M$ that gives $\omega_j$ when restricted to $\Sigma_j$, and the second over all dreibeins $e$ associated with the same bundle as $\omega$ belong to. The integral region has a tangent bundle structure with its base restricted by the boundary conditions. The integral on $e$ over fibres can easily be done to give,

$$I_M(\omega_1, \ldots, \omega_N) = \int_{\omega=\omega_j \text{ on } \Sigma_j} D\omega \delta[d\omega + \omega\omega],$$

(3.2)

where the delta functional has its support on flat $\omega$. Let $P_j$ denote the SO(2,1) bundle over $\Sigma_j$ to which $\omega_j$ belongs. The integral (3.2) vanishes unless $\omega_j$ have some flat $\omega$ as a common extension. Let us assume the existence of such an extension. Let $P$ be the flat SO(2,1) bundle over $M$ for the extension, and $I_P$ the restriction of the integrand (3.2) to connections in $P$. If there is more than one $P$ then $I_M$ is the sum of $I_P$. $I_P$ has an unphysical divergence arising from gauge invariance, which is removed by an appropriate gauge fixing. Another possible source of divergence is variations of flat $\omega$ that keep $\omega$ flat but are transverse to gauge orbits. In other words, $I_P$ diverges if the moduli space of flat connections in $P$ has at least one dimension. Geometrically this will occur if the interior of $M$ has a homotopically non-trivial loop ‘of its own’ in the sense that the holonomy around it cannot be controlled by the holonomy on the boundary. Physically it is an infrared divergence due to a portion of space-time escaping from the Planckian into the classical dimensions[2].

We would like $I_M$ to represent the amplitude of the process involving 2-spaces $\Sigma_j$ mediated by the 3-manifold $M$. For this we would like the state attached to each boundary component $\Sigma_j$ to admit an interpretation as a space-time with a spacelike slice. We therefore require $P_j$ to have Euler class $\pm \chi_j$ where $\chi_j = \chi_{\Sigma_j}$. This effectively demands the space condition (2.10) for the $P$ over $M$ that accompanies the extension of $\omega_j$. This does not mean however, that we seek to interpret the whole $M$ as a Lorentzian manifold with spacelike boundary. This is usually impossible as noted earlier. Our basic view is that the space-time geometry of asymptotic states is observable but not is that of intermediate ones.
For convenience we will say a state on a surface $\Sigma$ is in the spatial sector, or just spatial, if the corresponding $P|_{\Sigma}$ satisfies the requirement $\text{eul}(P|_{\Sigma}) = \pm \chi_{\Sigma}$. Similarly an amplitude is said to be in the spatial sector when all the external states are spatial.

3.2 A selection rule

If there is no restriction on the external states, then for any $M$, $I_M$ is not zero for some states. This is because there exists at least one $\text{SO}(2,1)$ flat bundle over $M$, the trivial bundle for instance, hence a flat connection $\omega$. The amplitude does not vanish for $\omega_j = \omega|_{\Sigma_j}$. Thus topology change is arbitrary. When restricted to the spatial sector however, the situation is quite different.

Consider $M$ with a topology obtained by removing from a $g = 2$ handlebody (solid double-torus) two $g = 1$ handlebodies (solid tori) so that the two handles are hollow (fig.1). The boundary of $M$ consists of $g = 1$ surfaces (tori) $\Sigma_1, \Sigma_2$, and a $g = 2$ surface (double-torus) $\Sigma_3$. $I_M$ does not have infrared divergences since the holonomy of flat $\omega$ on $\Sigma_3$ completely determines that on $M$. We now claim that on the support of $I_M$ the state on $\Sigma_3$ cannot be in the spatial sector. Look at path $\gamma$ in $\Sigma_3$ depicted in fig.1. This closed path is homotopically non-trivial in $\Sigma_3$ (with the basepoint $*$) but is trivial in $M$ as it can be shrunk to $*$. If an $\omega_3$ on $\Sigma_3$ extends to a flat $\omega$ on $M$, then its holonomy round $\gamma$ is necessarily trivial: with its holonomy homomorphism $\rho_3$, $\rho_3(\gamma) = 1 \in \text{SO}(2,1)$. This is not compatible with the space condition. For this condition requires $\rho_3$ to be a discrete embedding of $\pi_1(\Sigma)$ into $\text{SO}(2,1) \approx \text{PSL}(2,\mathbb{R})[13]$. In particular, $\rho_3(\gamma) \neq 1$ for $\gamma \neq 1 \in \pi_1(\Sigma,*)$. Thus we find that the $M$ is irrelevant to the processes for the spatial sector. This generalizes to any $M$ that has a boundary component of $g \geq 2$ with a loop contractible in $M$ but non-contractible in the boundary.

In the above we saw one way to identify processes suppressed by the space condition, working on particular $M$. We now ask instead what combinations of spatial topologies can give a non-vanishing amplitude, without specifying the interpolating manifold $M$. We answer this by the following.
**Selection rule.** Suppose the disjoint union \( \Sigma_1 \sqcup \Sigma_2 \sqcup \ldots \sqcup \Sigma_N \) of genus \( g_j \geq 1 \) surfaces \( \Sigma_j \) bounds some space-time manifold \( M \) such that the amplitude \( I_M \) is not identically zero in the spatial sector. Then the Euler characteristics \( \chi_j = 2 - 2g_j \) of the closed surfaces \( \Sigma_j \) satisfy the relation,

\[
\sum_{j=1}^{N} \epsilon_j \chi_j = 0,
\]

(3.3)

with some sign assignment \( \epsilon_j = \pm 1 \).

This follows from the observation that \( I_M \) can be non-zero only if the bundle \( \sqcup_{j=1}^{N} P_j \) over the boundary \( \sqcup_{j=1}^{N} \Sigma_j \) extends to an \( \text{SO}(2,1) \) bundle \( P \) over \( M \). In fact, take such a \( P \) and reduce it to an \( \text{SO}(2) \approx \text{U}(1) \) bundle \( Q \). We pick an arbitrary \( \text{U}(1) \) connection \( \nu \) in \( Q \) and apply the formula (2.11) over the boundary components \( \Sigma_j \). If we use the orientation of \( \Sigma_j \) induced from \( M \), the sum of euler classes of \( P_j \) becomes as follows:

\[
\sum_{j=1}^{N} \text{eul}(P_j) = \frac{1}{2\pi i} \int_{\partial M} d\nu = 0.
\]

(3.4)

The integral vanishes by Stokes’ theorem since the curvature \( d\nu \) is a closed 2-form over \( M \). Eq.(3.3) follows from eq.(3.4) with the space condition \( \text{eul}(P_j) = \pm \chi_j \).

Note that the flatness of \( P \) was not used in the proof. Actually the validity of the constraint (2.10) on spatial boundaries extends to three-dimensional Lorentzian gravity in general. See sect.4 for a comparison with Lorentzian cobordism.

The topology combination corresponding to the example of fig.1 gives \( \chi_1 = \chi_2 = 0, \chi_3 = -2 \). This is forbidden no matter what the \( M \) is. A generalization of this situation is given by \( \Sigma_j \) of \( g_j \geq 1 \), with \( g_N = g_1 + g_2 + \cdots + g_{N-1}, \ N \geq 3 \). Then \( |\chi_N| > \sum_{j=1}^{N-1} |\chi_j| \), so eq.(3.3) cannot be satisfied. There are also countless examples that satisfy eq.(3.3). In the case of three spaces, for example, eq.(3.3) states that the largest genus is one less than the sum of the other two, \( g_3 = g_1 + g_2 - 1 \). We can get examples by taking arbitrary positive integers for \( g_1 \) and \( g_2 \).
The existence of solutions to eq.(3.3) does not readily mean that the corresponding amplitudes are non-zero. However, we will see in the next subsection that eq.(3.3) does not leave much room for improvement, as long as we ask only the spatial topologies but no further details of the boundary states.

3.3 Transition paths

We shall present examples of transition paths for topology changes. Here to give such a path means to give the pair \((M, \omega)\) of a space-time manifold \(M\) and a flat connection \(\omega\) in an \(SO(2,1)\) bundle \(P\) over \(M\) satisfying the space condition (2.10) for every boundary component \(\Sigma_j\) of \(M\). The amplitude \(I_M\) is non-vanishing for \(\omega_j = \omega|_{\Sigma_j}\), and is within the spatial sector.

We first define 3-manifold \(M_k\), with \(k\) an integer \(\geq 2\). The aim is to give paths connecting \(k + 1\) spaces \(\Sigma_1, \ldots, \Sigma_{k+1}\) consisting of \(g = 2\) surfaces \(\Sigma_1, \ldots, \Sigma_k\) and a \(g = k + 1\) surface \(\Sigma_{k+1}\). This topology combination satisfies the selection rule (3.3): \(\chi_{k+1} = -2k = \sum_{j=1}^{k} \chi_j\). We represent \(M_k\) as \(M_k = \Lambda_k \cup V^k\). Fig.2 depicts \(\Lambda_k\) and \(V^k\) separately. \(\Lambda_k\) is obtained from a \(g = 2k\) handlebody (bounded by \(\Gamma_k\)) by removing \(g = 2\) handlebodies (bounded by \(\Sigma_1, \ldots, \Sigma_k\)), while \(V^k\) from a reflected copy of the \(g = 2k\) handlebody (bounded by \(L_k\)) by removing a \(g = k + 1\) handlebody (bounded by \(\Sigma_{k+1}\)). In this, the two-handled surfaces \(\Sigma_j\) (1 \(\leq j \leq k\)) pair the holes of \(\Gamma_k\) (numbered from 1 to 2\(k\) in fig.2) each handle linking one hole like \((1|2)(3|4) \cdots (2k-1|2k)\), while the relation of the handles of \(\Sigma_{k+1}\) to the holes of \(L_k\) is like \((1|2, 3|4, 5) \cdots |2k - 2, 2k - 1|2k)\). The outer surfaces \(\Gamma_k\) and \(L_k\), one being a reflected copy of the other, are identified so that they represent a single surface in the interior of \(M\).

So we have manifold \(M_k\). Before giving an example of flat connection \(\omega\), we show that \(I_M\) is not divergent in the spatial sector. For simplicity we will work on the \(k = 2\) case. See fig.3. The fundamental group of \(M_2\) can be described by the generators \(\alpha_1, \ldots, \alpha_8 \in \pi_1(M_2, \ast)\), represented by the paths numbered in fig.3,
and the relations,

\[ [\alpha_1, \alpha_2][\alpha_3, \alpha_4] = [\alpha_5, \alpha_6][\alpha_7, \alpha_8] = 1, \quad (35a) \]

\[ \alpha_5 = \alpha_4 \alpha_3 \alpha_4^{-1}, \quad (35b) \]

where the commutator \([,]\) is defined by \([\alpha, \beta] = \alpha \beta \alpha^{-1} \beta^{-1}\). Let \(\rho : \pi_1(M, \ast) \to \SO(2,1)\) be a holonomy homomorphism of a flat connection \(\omega\) on \(M\), and let \(\rho_j\) denote the image of \(\alpha_j\) under \(\rho\): \(\rho_j = \rho(\alpha_j) \in \SO(2,1)\). The gauge equivalence class of the ordered set \((\rho_1, \rho_2, \rho_3, \rho_4)\) is determined by \(\omega_1\) the restriction of \(\omega\) to \(\Sigma_1\). Similarly, \(\text{cls}(\rho_5, \rho_6, \rho_7, \rho_8)\) is determined by \(\omega_2\), and \(\text{cls}(\rho_1, \rho_2, \rho_3, \rho_6 \rho_4, \rho_7, \rho_8)\) by \(\omega_3\). If \(\Sigma_1\) with \(\omega_1\) corresponds to a state in the spatial sector, then its holonomy homomorphism gives a discrete embedding of \(\pi_1(\Sigma)\) into \(\PSL(2,\mathbb{R}) \cong \SO(2,1)\). From this it can be shown that if \(\gamma \in \PSL(2,\mathbb{R})\) satisfies \(\gamma \rho \gamma^{-1} = \rho_1\) and \(\gamma \rho_2 \gamma^{-1} = \rho_2\), then \(\gamma = 1\). Hence with the help of \((35b)\), \(\omega_1\) and \(\omega_3\) together determine \(\text{cls}(\rho_j|1 \leq j \leq 8)\) and therefore \(\text{cls}\omega\), completely. In particular, for given \(\omega_1, \omega_2, \omega_3\) in the spatial sector, \(\text{cls}\omega\) is unique if it exists. Hence \(I_M\) is divergence-free in the spatial sector.

We now give an example of \(\omega\), again concentrating on \(k = 2\). We leave \(\omega_1\), or \(\text{cls}(\rho_1, \ldots, \rho_4)\), arbitrary except that it belongs to the spatial sector. With a representative \((\rho_1, \ldots, \rho_4)\), we set

\[ \rho_5 = \rho_4 \rho_3 \rho_4^{-1}, \quad \rho_6 = \rho_4, \]

\[ \rho_7 = \rho_4 \rho_1 \rho_4^{-1}, \quad \rho_8 = \rho_4 \rho_2 \rho_4^{-1}. \quad (3.6) \]

Then \(\rho_j\), \(1 \leq j \leq 8\), satisfy all the relations \((35)\). We have to check that \(\omega_2\) and \(\omega_3\) are in the spatial sector. It is easily done for \(\omega_2\), since \(\text{cls}(\rho_1, \ldots, \rho_4) = \text{cls}(\rho_5, \ldots, \rho_8)\) by \((3.6)\), which means that \(\Sigma_1\) and \(\Sigma_2\) are ‘equivalent’ under \(\omega\). (The definition of the equivalence is as follows: \(\omega_1\) is gauge equivalent to the pull-back of \(\omega_2\) by some diffeomorphism of \(\Sigma_1\) onto \(\Sigma_2\) that preserves the orientation relative to that of \(\partial M\).) To check on \(\Sigma_3\), the easiest way would be to infer \(\text{eul}(P_j)\) for \(j = 3\) from the knowledge of the other two via \(\text{eq.}(3.4)\). Here we instead resort to the
relation of $\text{PSL}(2,\mathbb{R})$ flat connections on $\Sigma$ of $g \geq 2$ to the Teichmüller space of $\Sigma$. A $\text{PSL}(2,\mathbb{R})$ connection on $\Sigma$ belongs to an $\text{eul} = \pm \chi_\Sigma$ bundle if and only if its holonomy group $\subset \text{PSL}(2,\mathbb{R})$ is the cover group of the Riemann surface $\Sigma$ with some complex structure. By eqs.(35) we have

$$\text{cls}(\rho_1, \rho_2, \rho_3, \rho_6\rho_4, \rho_7, \rho_8) = \text{cls}(\rho_1, \rho_2, \rho_3, \rho_4^2, \rho_4\rho_1\rho_4^{-1}, \rho_4\rho_2\rho_4^{-1}).$$  \hspace{1cm} (3.7)$$

The left-hand side corresponds to the generators of the holonomy group of $\omega_3$. Since $\Sigma_1$ with $\omega_1$ is spatial, $\rho_1, \ldots, \rho_4$ are the generators of the cover group of $\Sigma_1$ as a Riemann surface. Eq.(3.7) implies that the holonomy group of $\omega_3$ is the cover group of a $g = 3$ Riemann surface, namely a double cover of $\Sigma_1$ (the one associated with a cut along a path in the path class $\alpha_3$ if the basepoint $*$ is chosen on $\Sigma_1$). Hence $\Sigma_3$ is also in the spatial sector. We also see that $\Sigma_3$ is ‘equivalent’ to a $(-2)$-fold cover of $\Sigma_1$ under $\omega$, by which we mean $\omega_3$ is gauge equivalent to the pull-back of $\omega_2$ by a double-cover map of $\Sigma_3$ onto $\Sigma_1$ that reverses the orientation relative to that of $\partial M$.

In the same way, we can define examples of flat connections on $M_k$ for general $k \geq 2$ that satisfy the spatial condition for the boundary. In these examples, $\Sigma_1$ can be set in an arbitrary spatial state by selecting $\omega$, but under $\omega$ the $g = 2$ surface $\Sigma_j$, $1 \leq j \leq k$, are all equivalent and the $g = k + 1$ surface $\Sigma_{k+1}$ is equivalent to a $(-k)$-fold cover of $\Sigma_1$.

We step further and put together the above transition paths to seek more variety in the combinations of spatial topologies. Take an arbitrary set of spaces $\Sigma^{(\ell)}$ (1 $\leq \ell \leq N$) of genus $g_\ell \geq 2$ that satisfy the selection rule (3.3). We construct a transition path for these spaces. We define $K(\ell) = g_\ell - 1$, so that eq.(2.10) reads

$$\sum_{\ell=1}^{N} \epsilon_\ell K(\ell) = 0. \hspace{1cm} (3.8)$$

For each $\ell$ with $K(\ell) \geq 2$, we take a copy of $M_{K(\ell)}$, denote it by $M^{(\ell)}$, and identify $\Sigma^{(\ell)}$ with the $g = K(\ell) + 1$ boundary component of $M^{(\ell)}$: $\Sigma^{(\ell)} = \Sigma_{K(\ell)+1}^{(\ell)}$. A
set of copies of the previous transition paths \((M^{(\ell)}, \omega^{(\ell)}), K(\ell) \geq 2\), can be chosen so that they have diffeomorphisms \(\varphi_j^{(\ell)}\) from a standard \(g = 2\) surface \(\Sigma_\infty\) onto \(\Sigma_j^{(\ell)}, 1 \leq j \leq K(\ell)\), which have the following properties.

(i) With respect to a fiducial orientation of \(\Sigma_\infty\), the orientation induced from that of \(\partial M^{(\ell)}\) via \(\varphi_j^{(\ell)} : \Sigma_\infty \rightarrow \Sigma_j^{(\ell)}\) is positive or negative according to whether \(\epsilon_{\ell} = +1\) or \(-1\).

(ii) The flat connections \(\varphi_j^{(\ell)}_* \omega^{(\ell)}\) on \(\Sigma_\infty\) are mutually gauge equivalent for all possible \(j\) and \(\ell\) \((1 \leq j \leq K(\ell), K(\ell) \geq 2)\).

Then for any \(\ell, m\) with \(\epsilon_{\ell} \neq \epsilon_m\), we can join \((M^{(\ell)}, \omega^{(\ell)})\) and \((M^{(m)}, \omega^{(m)})\) by pasting together a pair of \(g = 2\) boundary components \(\Sigma_i^{(\ell)}\) and \(\Sigma_j^{(m)}\) say, by the identification map \(\varphi_j^{(m)} \circ \varphi_i^{(\ell)}\) and identifying \(\omega^{(\ell)}\) and \(\omega^{(m)}\) across the junction by appropriate transition functions. Eq.(3.8) ensures that we can construct a transition path \((M, \omega)\) for the spaces \(\Sigma^{(1)}, \ldots, \Sigma^{(N)}\) by first joining \((M^{(\ell)}, \omega^{(\ell)})\) at some pairs of the \(g = 2\) boundary components with opposite \(\epsilon_{\ell}\) as above, and then identifying \(\Sigma^{(\ell)}\) of \(K(\ell) = 1\) with the remaining \(g = 2\) components.

Thus we came by a fairly systematic way to construct a family of transition paths which cover all the combinations of spatial topologies that satisfy eq.(3.3) and \(\chi_j \leq -2\) \((g_j \geq 2)\). In particular we proved that for \(g_j \geq 2\) surfaces the selection rule (3.3) is sufficient for \(g_j \geq 2\) surfaces to admit an amplitude \(\mathcal{I}_M\) not identically zero in the spatial sector. We do not claim that the \(\mathcal{I}_M\) is finite. However, in the \(\mathcal{I}_P\) with \(P\) arising in our examples, infrared divergences can occur only through intermediate states in the spatial sector. This is no more than a fancy way of saying that \(P\) breaks down into \(P^{(\ell)}\) over \(M^{(\ell)}\) \((1 \leq \ell \leq N)\) with which each \(\mathcal{I}_{P^{(\ell)}}\) is free of infrared divergences. Finally we note that the existence of transition paths can alternatively be proved by Horowitz’ pull-back construction[5]. The topology of the spaces and flat connections on them expected from this construction contain those we obtained above. This is remarkable, but we do not know the precise relation of our examples to the theoretical construction of Horowitz’.
4. Discussions

We start with comparing the selection rule (3.3) with that of Sorkin’s[4]. The latter states that in a Lorentzian cobordism the Euler characteristics of the initial and final spaces are the same:

\[ \chi_{\text{in}} = \chi_{\text{out}}. \]  \hspace{1cm} (4.1)

A Lorentzian cobordism for spaces \( S_{\text{in}}, S_{\text{out}} \) in Sorkin’s definition consists of an interpolating compact 3-manifold \( M \) and a (non-degenerate) time-oriented Lorentzian metric with respect to which \( S_{\text{in}} \) and \( S_{\text{out}} \) are respectively initial and final spatial boundaries. The initial and final surfaces may have more than one connected component, but each component is required to be closed. The metric is not required to solve the Einstein equation. Lorentzian cobordisms may be regarded as the paths to be summed over in quantum gravity in the metric formulation. Eq.(4.1) is the condition for the transition amplitude with \( S_{\text{in}}, S_{\text{out}} \) to be non-zero in this formulation. To compare it with our result, we restrict ourselves to the case in which \( M \) is orientable. Obviously our rule (3.3) is observed under (4.1). We can understand this in the following way. With the metric in a Lorentzian cobordism, we can construct a bundle \( P \) of SO(2,1) frames of tangents to \( M \). Then the proof for (3.3) applies to \( P \). The assumption that \( P \) satisfies the space condition for the boundary is justified by a slightly modified version of the argument in sect.2.3.2. Furthermore a little inspection of the same argument shows that whether a component belongs to the initial side or to the final one is decided by the sign of \( \text{eul}(P_j) \) that appears in eq.(3.4). Thus (3.3) is valid here and Sorkin’s result can be reproduced when the boundary components are \( g \geq 2 \) surfaces.

What we have found in sect.3 is that the selection rule from Lorentzian cobordism still holds in our formulation save for the distinction between in-coming and out-going states. Actually we did not make an issue of the way of dividing the external states into in- and out-states. Since our quantum space-time does not admit a Lorentzian metric, we cannot implement an in-out distinction the way
Sorkin does. Of course we can define the distinction by the sign of \( \text{eul}(P_j) \) with the orientation induced from \( \partial M \). The justification is not easy to find, however. There is no guarantee that the orientation of a frame associated with \( P \) relates to the orientation of the base manifold \( M \), in contrast to the case of Lorentzian cobordism where frames can be regarded as tangent to \( M \) everywhere. A plausible alternative is to use the asymmetry of the asymptotic space-time in the temporal direction. A space-time with the topology \( \Sigma \times \mathbb{R} \) either has an initial singularity and is future complete or has the same features reversed in time\(^3\). The asymptotic space-times complete in the direction away from the interacting region may be called out-going, and the others in-coming. This way of grouping will surely make sense in the classical limit. We do not know if it is well-defined in quantum theory. Even if it is, the direction of time in this sense may not be diagonal in our representation. In fact, the pull-back construction of Horowitz\(^5\) can be used to show that \( \text{SO}(2,1) \) flat connections on the boundary do not fix the direction of time for the asymptotic space-times.

Restrictions on the way of dividing the external states into in- and out-groups, if they are justified, may mean a great deal to topology change. Just for the sake of illustration, let us assume a grouping that would lead to eq.(4.1). Then a creation of \( g \geq 2 \) surfaces from nothing or a combination of \( g = 1 \) surfaces is forbidden. A more subtle example is provided by the paths with \( M_k \) in sect.3.3. Take \( M = M_2 \), for simplicity. In terms of the spatial topology, only two processes are allowed under (4.1): \( \Sigma_1 \sqcup \Sigma_2 \to \Sigma_3 \), and its reverse \( \Sigma_3 \to \Sigma_1 \sqcup \Sigma_2 \). The amplitude for the first process almost always vanishes. This is seen by noting that the relation (3.5b) requires \( \rho_3 \) and \( \rho_5 \) to be mutually conjugate. The states on \( \Sigma_1 \) and \( \Sigma_2 \) are related on the support of \( \mathcal{I}_M \), and the subspace of such states has a non-zero codimension in the space of all the possible initial states. Thus the initial state must be prepared with an infinite degree of precision for the transition to occur. To put it the other way, the space \( \Sigma_1 \sqcup \Sigma_2 \) is practically stable as far as the above process is concerned. The same observation applies to the reversed process as well. On the other hand, if we remove the restriction in the grouping of the
external states, the transition path \((M_2, \omega)\) in sect.3.3 implies that a \(g = 2\) surface has a decay mode \(\Sigma_1 \rightarrow \Sigma_2 \sqcup \Sigma_3 \) (or \(\Sigma_2 \rightarrow \Sigma_1 \sqcup \Sigma_3 \)) with a finite probability for any state on the initial surface. Transition paths with \(M_k\) are the paths for topology change no matter what way we divide the external states. The point is that whether they represent topology changing paths starting from reasonably generic states depends on which external states are on the initial side.

The Gibbons-Hartle approach[6] applied on 2+1 dimensions as done by Fujiwara, Higuchi, Hosoya, Mishima, and Siino[7], also gives some results on topology change. In a critical path for a tunneling, the junction of the Riemannian and Lorentzian regions can be made only across totally geodesic surfaces. Without a cosmological constant, this means that \(g \geq 2\) surfaces do not emerge from the interacting region. This may have something to do with the restriction we found on such spacelike surfaces. The agreement is not so sharp since we did find non-vanishing amplitudes, but we have to allow for the fact that the tunneling approach is only an approximation. In the case of a negative cosmological constant, critical paths are possible only with \(g \geq 2\) surfaces but a large number of examples have been found[7]. Some of the examples violate the selection rule (3.3). This may be reflecting the fact that the approach has its foundation in Euclidean theory rather than Lorentzian. The discrepancy may not be so great however, since the critical paths are possible only for a countable number of points in the moduli space for hyperbolic surfaces.

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FIGURE CAPTIONS

**Figure 1** A 3-manifold $M$ for which $I_M$ is zero in the spatial sector. It is represented as a handlebody of genus $g = 2$ with two $g = 1$ handlebodies removed. Double-torus $\Sigma_3$ is on the outer boundary of $M$, while the tori $\Sigma_1$ and $\Sigma_2$ inside $\Sigma_3$ constitute the inner boundary. The amplitude vanishes essentially because in $M$ the loop $\gamma$ on $\Sigma_3$ shrinks to a point.

**Figure 2** The interpolating manifold $M_k$ with a non-zero transition amplitude within the spatial sector. $M_k$ is obtained by gluing $V^k$ and $\Lambda_k$ together around boundary components $L^k$ and $\Gamma_k$.

**Figure 3** The manifold $M_2$ drawn in the same way as in fig. 2. The closed paths which represent the generators of $\pi_1(M_2, \ast)$ are also shown.

**Figure 4** A picture of an interpolating manifold for spaces $\Sigma(\ell)$, $1 \leq \ell \leq 5$, with $K(1) = K(3) = 4, K(2) = K(4) = 1, K(5) = 2$. We take a copy of $M_2$ and two copies of $M_4$ and then sewing together the manifolds and the connections on them appropriately across $g = 2$ surfaces. The sign of $\epsilon_\ell$ for $\ell = 1, 3, 5$ is indicated in the centre of corresponding $M_\ell$. 