Riesz Wavelets, Tiling and Spectral Sets in LCA Groups

Azita Mayeli

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Abstract

This paper is devoted to the study of geometry properties of wavelet and Riesz wavelet sets on locally compact abelian groups. The catalyst for our research is a result by Wang ([32], Theorem 1.1) in the Euclidean wavelet theory. Here, we extend the result to wavelet and Riesz wavelet collection of sets in infinite locally compact abelian groups.

Keywords: LCA groups, wavelet set, Riesz wavelet collection of sets, spectral set, tiling with multiplicity,
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1 Introduction

Let \( \varphi \in L^2(\mathbb{R}^n) \), \( n \geq 1 \). \( \varphi \) is called a wavelet for \( L^2(\mathbb{R}^n) \) if there exists countable sets \( \mathcal{L} \subset \mathbb{R}^n \) and \( \mathcal{D} \subset GL(n, \mathbb{R}) \) such that the family

\[
\{| \det D |^{1/2} \varphi(D x - t) : D \in \mathcal{D}, t \in \mathcal{L} \}
\]

is an orthonormal basis for \( L^2(\mathbb{R}^n) \). In this case, the sets \( \mathcal{D} \) and \( \mathcal{L} \) are called dilation and translation sets, respectively. Wavelets are important tools in approximation theory due to their time and frequency localization property.

One type of wavelets with simplest structures are those which are supported on a Lebesgue measurable set in \( \mathbb{R}^n \) with finite and positive measure. The most basic one is the Haar wavelet \( \varphi(x) = 1_{[0,1/2]} - 1_{[1/2,1]} \), constructed by Alfred Haar in 1910, is a one-dimensional example of this kind and it serves as a prototypical wavelet. The dilation and translation sets associated to the Haar wavelet are given by \( \mathcal{D} = \{2^m : m \in \mathbb{Z}\} \) (dyadic dilation) and \( \mathcal{L} = \mathbb{Z} \). Another type of simple wavelets are those whose Fourier transforms
are characteristic functions of a Lebesgue measurable domain in $\mathbb{R}^n$ with finite measure. A typical example of these kind is Littlewood-Paley or Shannon wavelet whose Fourier transform is $1_{[-1,-1/2) \cup (1/2,1]}$. Shannon wavelet is a significant tool in sampling and reconstruction of functions. Haar wavelet and Shannon wavelet are situated at the opposite side of each other and "somehow" related. However, in this paper we shall only focus on Shannon-type wavelets, i.e., those with Fourier transform supported in a domain with finite measure.

The Fourier support of the Shannon wavelet $I := [-1, -1/2) \cup (1/2, 1]$ has this nice geometry property that its dyadic dilations by $\{2^n : n \in \mathbb{Z}\}$ as well as its integer translations tile $\mathbb{R}$. The sets with the similar tiling properties have been considered in the paper by Dai and Larson [10] where the authors proved the following results: Given a measurable set $E \subset \mathbb{R}$ with finite and positive measure, the function $\varphi$ with $\hat{\varphi} = 1_E$ is a wavelet with dyadic dilation and integer translations if and only if $E$ tiles $\mathbb{R}$ by dyadic dilations as well as by integer translations. This result and characterization of wavelets in terms of tilings properties of their Fourier supports was somehow generalized by Wang [32] in $n$-dimensional Euclidean spaces, where the dilation set is any countable subset of invertible matrices and the translation set is any countable subset in $\mathbb{R}^n$. Before we recall his result, we need several definitions.

**Definition 1.1.** A measurable subset $\Omega \subset \mathbb{R}^n$ with finite and positive measure is called wavelet set if for the function $\varphi \in L^2(\mathbb{R}^n)$ with $\hat{\varphi} = 1_\Omega$ the family (1.1) is an orthonormal basis for $L^2(\mathbb{R}^n)$.

These wavelets are also known as MSF wavelets (minimally supported frequency wavelets) or in older literatures as s-wavelets. The wavelet sets and minimally supported frequency wavelets were introduced by Fang and Wang ([13]) and studied exclusively by Hernandez, Wang and Weiss ([17, 18]). For a series of paper on wavelet sets see e.g., [2, 4, 11, 12] and references contained therein.

**Additive and multiplicative tilings.** We say a set $\Omega$ tiles $\mathbb{R}^n$ additively with respect to a countable set $\mathcal{L} \subset \mathbb{R}^n$ if

$$\sum_{\ell \in \mathcal{L}} 1_\Omega(x - \ell) = 1 \quad a.e. \quad x \in \mathbb{R}^n.$$ 

Equivalently, $\mathbb{R}^n = \cup_{\ell \in \mathcal{L}} \Omega + \ell$, where for any distinct $\ell$ and $\ell'$, the sets $\Omega + \ell$ and $\Omega + \ell'$ are disjoint in measure, i.e., $|\Omega + \ell \cap \Omega + \ell'| = 0$. Here, $|\Omega|$ denotes the Lebesgue measure. We say $\Omega$ is a multiplicative tiling set for $\mathbb{R}^n$ with respect to a collection of $n$ by $n$ invertible matrices $A$ if $\{\alpha(\Omega) : \alpha \in A\}$ is a partition for $\mathbb{R}^n$, i.e.,

$$\sum_{\alpha \in A} 1_{\alpha(\Omega)}(x) = 1 \quad a.e. \quad x \in \mathbb{R}^n.$$ 

This is also equivalent to say that $\mathbb{R}^n = \cup_{\alpha \in A} \alpha(\Omega)$ where for any $\alpha \neq \alpha'$ the two sets $\alpha(\Omega)$ and $\alpha'(\Omega)$ are disjoint in measure.

The characterization of the wavelet sets in terms of translation and multiplicative tiling was obtained by Wang (Theorem 1.1, [32]).
Theorem A. Let $\Omega \subset \mathbb{R}^n$ with finite and positive Lebesgue measure, $A \subset GL(n, \mathbb{R})$ and $L$ be a countable subset of $\mathbb{R}^n$. For $\alpha \in A$, let $\alpha^\tau$ denote the transpose matrix of $\alpha$. If the set $\{\alpha^\tau(\Omega) : \alpha \in A\}$ tiles $\mathbb{R}^n$ and the set of exponentials $E_L := \{e^{2\pi i l \cdot x} : l \in L\}$ is an orthogonal basis for $L^2(\Omega)$, then $\Omega$ is a wavelet set with respect to the dilation set $A$ and translation set $L$. The converse also holds, provided that $0 \in L$.

Here, $l \cdot x$ denote the inner product in $\mathbb{R}^n$ and “$x$” is generic.

In his paper [32], Wang shows that the additional condition $0 \in L$ cannot be dropped. Indeed, $\Omega = [-1, -1/2] \cup [1/2, 1]$ is a wavelet set with respect to the dilations $M = \{2^n\}_{n \in \mathbb{Z}}$ and translations $T = 2\mathbb{Z} + 1/6$. However, $\Omega$ nor tiles $\mathbb{R}$ multiplicatively by $M$ neither the exponentials $E_T$ is dense in $L^2(\Omega)$. The latter is true since, for example, the non-zero function $g(x) = e^{-\pi ix/3}$ is orthogonal to all elements of $E_T$ with inner product in $L^2(\Omega)$.

In the classical context, given $\Omega$ and $\Lambda$, the $(\Omega, \Lambda)$ is called spectral pair if the exponentials $E_\Lambda := \{e^{2\pi i \lambda \cdot x} : \lambda \in \Lambda\}$ from an orthogonal basis for $L^2(\Omega)$. In this case, $\Omega$ is spectral and $\Lambda$ is spectra (or spectrum). It has been shown by Fuglede ([15]) that when $\Lambda$ is a lattice in $\mathbb{R}^n$, then the existence of exponential orthogonal basis $E_\Lambda$ for $L^2(\Omega)$ is equivalent to say that $\Omega$ tiles $\mathbb{R}^n$ additively by the dual lattice $\Lambda^\perp$. The relation between exponential basis for spectral sets and their tiling property in general has been conjectured by Fuglede in ’74. The Fuglede Conjecture asserts that given a bounded set $\Omega \subset \mathbb{R}^d$ of positive Lebesgue measure, $L^2(\Omega)$ has an orthogonal basis of exponentials $E_\Lambda = \{e^{2\pi i \lambda \cdot x}\}_{\lambda \in \Lambda}$ for some countable subset $\Lambda \subset \mathbb{R}^n$ if and only if $\Omega$ tiles $\mathbb{R}^d$ by additively by some translations $T$. The Fuglede Conjecture led to considerable activity in the past three decades. For example, in 2001, Iosevich, Katz, Tao ([20]) proved the conjecture for complex planar domain. In [26] and [27], Laba established the conjecture for the union of two intervals on the real line and provided a series of connections between the study of orthonormal bases and interesting problems in algebraic number theory. However, in 2004, Tao ([31]) disproved the Fuglede conjecture for dimension 5 and higher, followed by [24] double check the reference, where Kolountzakis and Matolcsi also disprove the Fuglede Conjecture for dimensions 3 and higher. Yet the conjecture may still be true in several important special cases. In spite of the disproof of its general validity, the conjecture has still generated many interesting results. Most recently, it was proved that the Fuglede Conjecture holds in some abelian finite groups ([22]). For the link between tiling property and existence of general bases in locally compact abelian groups see [3].

$k$-additive tiling. Given an integer number $k \geq 1$, we say $\Omega$ multi-tiles $\mathbb{R}^d$ additively by a multiset $L$ with multiplicity $k$ (or is a $k$-additive tiling) when

$$\sum_{\ell \in L} 1_{\Omega}(x - \ell) = k \quad a.e. \ x \in \mathbb{R}^n.$$ 

By the additive multi-tiling property, almost every point in $\mathbb{R}^n$ can be covered by exactly $k$-translations of $\Omega$. In dimension $d = 2$, it is known that every $k$-additive tiling of $\mathbb{R}^2$ by a convex polygon, which is not a parallelogram, is a $k$-additive tiling with a finite union.
of two-dimensional lattices ([23]). When $k = 1$, then we have the tiling situation. The multi-tiling property of a bounded measurable set is a sufficient condition for existence of exponentials set $\mathcal{E}_A$ which is a Riesz bases for $L^2(\Omega)$. This result was proved by Kolountzakis ([25], Theorem 1).

An extension of the sufficiency part of Theorem A has been recently proved by Führ and Maus [14] where the authors replace the additive tiling by additive multi-tiling property, thus the orthogonality by Riesz bases condition. Their result is the following.

**Theorem B.** ([14], Theorem 1.2) Let $\Omega \subset \mathbb{R}^n$ be a bounded and measurable set with positive measure. Assume that $\Omega$ tiles $\mathbb{R}^n$ multiplicatively by a countable subset of $GL(n, \mathbb{R})$ and multi-tiles $\mathbb{R}^n$ additively by a lattice with multiplicity $k$ ($k \in \mathbb{N}$). Then $\Omega$ is a Riesz wavelet set in $\mathbb{R}^n$, i.e, the system (1.1) for $\varphi$ with $\hat{\varphi} = 1_\Omega$ is a Riesz basis for $L^2(\mathbb{R}^n)$.

The focus of this paper is to extend the results of Theorems A and B for infinite locally compact topological abelian (LCA) groups $(G, +)$ with identity $e$ and equipped with a Haar measure. Note that some LCA groups, like $p$-adic additive groups $\mathbb{Q}_p$, the field of $p$-adic rational numbers, which do not possess any lattice. Therefore, the classical definition of wavelet sets does not apply to such groups. It is also known that the finite abelian group do not possess any wavelet set in the traditional sense ([21]). Therefore, we exclude such cases here and assume that $G$ is infinite and admits a lattice, i.e., a discrete subgroup which is co-compact. For the notion of wavelet sets in $\mathbb{Q}_p$ and in a non-commutative setting we refer to [5] and [9], respectively.

For a characterization of Riesz wavelets generated by MRA see e.g., [6] and [16].

The purpose of this paper is to establish the necessary formalism to state and prove Theorem A and Theorem B in the setting of locally compact abelian groups. We shall also prove the converse of Theorem B, not only in Euclidean space but also in the general context.

In the classical setting of $\mathbb{R}^n$, the definition of a wavelet is associated to two type of unitary operators: dilations and translations. The translation operator for the functions on an LCA group can be defined by the group action analogous with the Euclidean setting. The dilation of a function defined on $\mathbb{R}^n$ is given by the action of an invertible matrix $A$ and the associated normalization factor $c = |\det A|^{n/2}$. Therefore the immediate feeling is that the dilation operator acting on the functions on a LCA group must be defined by using group automorphisms. The only obstacle here would be the normalization factor. Obviously, for discrete groups, the normalization factor is $c = 1$ since the Haar measure is the counting measure. Therefore we shall assume that our setting is not discrete. In this case we overcome the problem by using modular function $\Delta$, which is a positive homomorphism defined on the class of automorphisms of $G$. We will define this function in Section 4.
2 Statement of main results

Borrowing the notion of wavelet collection of sets for the Euclidean setting in ([4]), we have the following definition.

**Definition 2.1.** Let $G = (G, +)$ be an LCA group with identity $e$ and the dual group $\hat{G}$. We equip $G$ by a Haar measure. A collection of measurable subsets $\Omega_i$, $1 \leq i \leq m$, of $\hat{G}$ is called a (Riesz) wavelet collection of sets for $G$ if there are countable subset $A$ of $\text{Aut}(G)$ and countable subsets $\Lambda_i \subset G$ ($1 \leq i \leq m$) such that

$$\bigcup_{i=1}^{m} \{ \psi_{i,\lambda,a} : 1 \leq i \leq m, \lambda \in \Lambda_i, a \in A \}$$

is (Riesz) orthogonal basis for $L^2(G)$, where $\psi_{i,\lambda,a}(x) = \Delta(a)^{1/2}\psi_i(a(x) - \lambda)$ and $\hat{\psi}_i = 1_{\Omega_i}$. When $m = 1$, we may say $\Omega = \Omega_1$ is a (Riesz) wavelet set.

The study of wavelet sets is an interplay between group theory, geometry, operator theory and analysis. The following result is an extension form of Theorem A and Theorem 2.2 in [4]. In the sequel, we assume that $\Omega_1, \ldots, \Omega_m$ are $m$ mutual disjoint and measurable subsets of $\hat{G}$ with finite and non-zero Haar measure, and $\Omega := \bigcap_{i=1}^{m} \Omega_i$.

**Theorem 2.2.** Let $\Omega_1, \ldots, \Omega_m$ be given. Let $A \subset \text{Aut}(\hat{G})$ and $\Lambda_i \subset G$, $1 \leq i \leq m$, be countable and non-empty subsets. Assume that for every $1 \leq i \leq m$, the pair $(\Omega_i, \Lambda_i)$ is spectral and the set $\Omega$ tiles $\hat{G}$ multiplicatively by the automorphisms $A$. Then $\{ \Omega_1, \ldots, \Omega_m \}$ is a wavelet collection of sets for $L^2(G)$ with respect to the dilations $A$ and translations $\Lambda_i$. The converse also holds, provided that $e \in \bigcap_{i=1}^{m} \Lambda_i$.

Theorem 2.2 along with a result of Fuglede [15] yields the following corollary, where the spectral condition has been replaced by additive tiling property.

**Corollary 2.3.** The set $\{ \Omega_1, \ldots, \Omega_m \}$ is a wavelet collection of sets for $L^2(G)$ with respect to the translations by lattices $\Lambda_i$ ($1 \leq i \leq m$) if and only if $\Omega := \bigcup_{i=1}^{m} \Omega_i$ tiles $\hat{G}$ multiplicatively and each $\Omega_i$ tiles $\hat{G}$ additively with $\Lambda_i$.

The following result is an extension form of Theorem 2.2 to the Riesz bases.

**Theorem 2.4.** Let $\Omega_1, \ldots, \Omega_m$ be as above and countable sets $\Lambda_i \subset \hat{G}$ be given such that for any $1 \leq i \leq m$ the pairs $(\Omega_i, \Lambda_i)$ are Riesz spectral with Riesz constants $L_i$ and $U_i$. Moreover, assume that $\Omega$ is a multiplicative tiling for $\hat{G}$ with respect to the automorphisms $A \subset \text{Aut}(\hat{G})$. Then $\Omega_1, \ldots, \Omega_m$ is a Riesz wavelet collection of sets for $L^2(G)$, i.e., the family

$$\bigcup_{i=1}^{m} \{ \psi_{i,\lambda,a} : \lambda \in \Lambda_i, a \in A \}$$

(2.1)

is a Riesz basis for $L^2(G)$ where $\hat{\psi}_i := 1_{\Omega_i}$. The Riesz constants in this case are given by $L := \min\{L_i\}$ and $U := \max\{U_i\}$.

A combination of Theorem 2.4 and [1, Theorem 4.1] yields the following result as an extension version of Theorem B for $m \geq 1$. 


Corollary 2.5. Let $\Omega_1, \ldots, \Omega_m$ be as above. Assume that each $\Omega_i$ multi-tiles $\hat{G}$ additively by some lattice $\Gamma_i \leq \hat{G}$ with multiplicity $k_i \in \mathbb{N}$, $k_i > 1$. Furthermore, assume that $\Omega = \bigcup_{i=1}^{m} \Omega_i$ tiles $\hat{G}$ multiplicatively with respect to a subset $A$ of $\text{Aut}(\hat{G})$. Then $\{\Omega_1, \ldots, \Omega_m\}$ is a Riesz wavelet collection of sets for $G$.

An inverse theorem for the results in Theorem 2.4 follows.

Theorem 2.6. Assume that $\Omega$ is a Riesz wavelet set with respect to the translations by $\Lambda$ and dilations by $A$. Then $\Omega$ is a multiplicative tiling for $\hat{G}$ with respect to the automorphisms $A$ if and only if $(\hat{\alpha}(\Omega), \alpha^{-1}(-\Lambda))$ is a Riesz spectral pair for all $\alpha \in A$.

Outline of the paper. The paper is organized as follows. After Section 3 on some open problems, in Section 4 we recall necessary backgrounds on the Fourier transform on locally compact abelian group $G$ and wavelet and Riesz bases in $L^2(G)$. Theorem 2.2 asserts that a collection of measurable sets with finite and positive measure in the dual space of a LCA group $G$ generate an orthogonal wavelet bases for $L^2(G)$ if and only if the sets are spectral and admit multiplicative tiling property, provided that the intersection of all translation sets contains $e$. In section 5 we give the proof of Theorem 2.2. In Section 6 we prove Theorems 2.4 and 2.6. These theorems are as an extension version of both necessary and sufficient part of Theorem 2.2. We conclude the paper by presenting some examples of wavelet and Riesz wavelet sets in Section 7.

3 Open problems

In the example constructed by Wang [32], we observed that the wavelet set $[-1, -1/2] \cup [1/2, 1]$ does tile the space $\mathbb{R}$ multiplicatively on $\{\pm 2^n\}_{n \in \mathbb{Z}}$. However, the set $\Omega$ has a multiplicative tiling property with respect to $M$ with multiplicity 2. In this relation we make the following definition.

Definition 3.1. Let $\Omega \subset G$. We say $\Omega$ tiles $G$ multiplicatively with multiplicity $k$ if there is a countable subset $A$ of automorphisms of $G$ such that

$$\sum_{\alpha \in A} 1_{\alpha(\Omega)}(x) = k \quad \text{a.e. } x \in G.$$  

The case $k = 1$ is the traditional multiplicative tiling.

Question 3.2. Suppose that $\Omega$ is a wavelet set in a locally compact abelian group $G$ with respect to a dilation set $A \subset \text{Aut}(G)$ and translation set $\Lambda \subset G$, where $e \notin \Lambda$. Must $\Omega$ tile $\hat{G}$ multiplicatively by some $A' \subset \text{Aut}(G)$, $A' \neq A$?

Notice that, when $e \in \Lambda$, then according to Theorem 2.2 the wavelet set is a multiplicative tiling and $A' = A$. A version of Question 3.2 for a non-separable wavelet index is the following.

Question 3.3. Let $B := \{(\lambda, a) : \lambda \in G, \ a \in \text{Auto}(G)\}$ be countable. Assume that for $\psi$ with $\hat{\psi} = 1_\Omega$, the set $\{\psi_{\lambda, a} : (\lambda, a) \in B\}$ is an orthogonal basis for $L^2(G)$, i.e., $\Omega$ is
a wavelet set with respect to $B$. Assume that $(e,a) \in B$ for some $a \in G$. Must $\Omega$ tile additively and multiplicatively?

In relation to Corollary 2.5 we have the following question.

**Question 3.4.** Suppose that $\{\Omega_i\}_{i=1}^m$ is a Riesz wavelet collection of sets for $L^2(G)$, which is not orthogonal. Let $\Omega := \cup_i \Omega_i$. Must $\{\alpha(\Omega)\}_{a \in A}$ be a multiplicative tiling for $G$ with multiplicity $k \geq 1$?

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## 4 Notations and Preliminaries

Following [30], let $(G,+)$ denote a locally compact abelian group with the group identity $e$. We assume $G$ is second countable and we equip $G$ with a normalized Haar measure $m_G$ which is non-zero and translation invariant. Let $\hat{G}$ denote the dual group of $G$ and $m_{\hat{G}}$ the Haar measure on $\hat{G}$. For any element $\xi$ in the dual group $\hat{G}$, we associate the character $\chi_\xi : G \to \mathbb{C}$ of $G$ and denote the duality by $\chi_\xi(x) := \langle \xi, x \rangle$, $x \in G$. When $G$ in $n$-dimensional Euclidean space $\mathbb{R}^n$, then $\mathbb{R}^n = \mathbb{R}^n$ as well and $\langle \xi, x \rangle = e^{2\pi i \xi \cdot x}$, $\xi, x \in \mathbb{R}^n$, is the canonical mapping.

By the properties of characters, we have
\[
\chi_\xi(-x) = \chi_{-\xi}(x) = \overline{\chi_\xi(x)}. \tag{4.1}
\]

For any $p > 0$, analogously to the Euclidean case, we define the space $L^p(G)$ with respect to the measure $m_G$ by
\[
L^p(G) = \{ f : G \to \mathbb{C}, \text{ f $m_G$-measurable and } \int_G |f(x)|^p dm_G(x) < \infty \}. \tag{4.2}
\]

For any $f \in L^1(G) \cap L^2(G)$, let $\hat{f}$ denote the Fourier transform of $f$ given by
\[
\hat{f}(\xi) = \int_G f(x)\chi_\xi(x) dm_G(x), \quad \xi \in \hat{G} \tag{4.3}
\]

By the Plancherel theorem the definition of the Fourier transform can be extend uniquely to the functions in $L^2(G)$ (See, for example, 1.6.1 in [30]), so that for any $f,g \in L^2(G)$ the Parseval identity holds:
\[
\int_G f(x)\overline{g(x)} dm_G(x) = \int_{\hat{G}} \hat{f}(\xi)\overline{\hat{g}(\xi)} dm_{\hat{G}}(\xi). \tag{4.4}
\]
We denote topological automorphisms of $G$ onto itself by $\text{Aut}(G)$. For $\alpha \in \text{Aut}(G)$, the adjoint homomorphism $\hat{\alpha} : \hat{G} \to \hat{G}$ is given as following: For any character $\chi_\xi$, $\hat{\alpha}(\chi_\xi)$ is a character and its duality is given by

$$\hat{\alpha}(\chi_\xi)(x) := \chi_\xi(\alpha(x)) = \langle \xi, \alpha(x) \rangle \quad x \in G.$$ 

It is easy to show that $\hat{\alpha}$ is an automorphism (one-to-one homomorphism) of $\hat{G}$ onto $\hat{G}$ and its inverse $\hat{\alpha}^{-1}$ acts on $\hat{G}$ by

$$\hat{\alpha}^{-1}(\chi_\xi)(x) := \chi_\xi(\alpha^{-1}(x)) = \langle \xi, \alpha^{-1}(x) \rangle.$$ 

This implies that $\hat{\alpha}^{-1} = \hat{\alpha}^{-1}$. By the definition of the adjoint homomorphism, the following result immediately follows.

**Lemma 4.1.** Given any automorphism $\alpha : G \to G$, $\xi \in \hat{G}$ and $x \in G$

$$\hat{\alpha}(\chi_\xi)(x) = \chi_{\hat{\alpha}(\xi)}(x) \quad \text{and} \quad \hat{\alpha}^{-1}(\chi_\xi)(x) = \chi_{\hat{\alpha}^{-1}(\xi)}(x).$$

A subgroup $\Lambda$ of $G$ is called a lattice if it is discrete and co-compact, i.e., the quotient group $G/\Lambda$ is compact. $G$ is second countable, therefore any lattice in $G$ is countable ([29, Section 12, Example 17]). The annihilator or dual lattice of a lattice $\Lambda$ is given by

$$\Lambda^\perp = \{ \xi \in \hat{G} : \chi_\xi(\lambda) = 1 \quad \forall \lambda \in \Lambda \}.$$ 

Then $\Lambda^\perp$ is also a lattice in $\hat{G}$. By the duality theorem [30, Lemma 2.1.3], $\Lambda^\perp$ is topologically isomorphic to the dual of $G/\Lambda$, that is, $\Lambda^\perp \cong (G/\Lambda)$.

Let $\alpha : G \to G$ be an automorphism. Then

$$\hat{\alpha}(\Lambda^\perp) = (\alpha^{-1}(\Lambda))^\perp.$$ 

Given a countable set $\Lambda$ in $G$ and a non-zero measurable set $\Omega \subset G$, we say $\Omega$ tiles $G$ additively by $\Lambda$ with multiplicity $k \in \mathbb{N}$ if

$$\sum_{\lambda \in \Lambda} 1_\Omega(x - \lambda) = k \quad a.e. \ x \in G. \quad (4.5)$$

If $k = 1$, then we simply say $\Omega$ tiles $G$ additively by $\Lambda$.

Given $\Omega \subset \hat{G}$ with non-zero measure and a countable set $T \subset G$, we say $(\Omega, T)$ is a spectral pair if the countable family $\mathcal{E}_T := \{ \chi_t : t \in T \}$ is an orthogonal basis for $L^2(\Omega)$. The pair is called Riesz spectral pair if $\mathcal{E}_T$ constitute a Riesz basis for $L^2(\Omega)$. The set $\Omega \subset G$ is multiplicative tiling if there is a set of automorphisms $\mathcal{A}$ of $G$ such that

$$\sum_{\alpha \in \mathcal{A}} 1_{\alpha(\Omega)}(x) = 1 \quad a.e. \ x \in \mathbb{G}.$$
For general definition of multiplicative tiling sets in a measure space \((X, \mu)\) see [12].

**Modular function.** Associated to a locally compact abelian group \(G\) with the Haar measure \(m_G\), the modular function \(\Delta : \text{Aut}(G) \to (0, \infty)\) is a continuous homomorphism such that for any measurable set \(\Omega \subset G\) there holds

\[
m_G(\alpha(\Omega)) = \Delta(\alpha)m_G(\Omega).
\]

(4.6)

It is well-known that the modular function \(\Delta\) exists. For this, consult [19, 26.21]. (The existence of such map for a larger class of continuous group homomorphisms of \(G\) onto \(G\) has also been proved in [7, Theorem 6.2].) The relation (4.6) implies that for any integrable function \(f\) on \(G\) with respect to the Haar measure \(m_G\) the equation

\[
\int_G f(\alpha(x))m_G(x) = \Delta(\alpha)^{-1}\int_G f(x)m_G(x)
\]

(4.7)

holds for all automorphism \(\alpha\) of \(G\). For our purpose, we shall assume that \(\Delta\) is not identical to one.

**Dilations and translations.** Let \(G\) be a LCA group. There are two unitary operators associated to the definition of wavelets on the group: translation and dilation. For \(x \in G\), the translation operator \(\tau_x\) is defined on function \(f : G \to \mathbb{C}\) and given by \(\tau_x f(y) = f(y - x)\). By the translation invariant property of Haar measure, \(\tau_x\) is unitary restricted to the functions in \(L^2(G)\). For a fixed automorphism \(\alpha \in \text{Aut}(G)\), we define \(\delta_\alpha\) on \(f\) by

\[
\delta_\alpha f(x) = \Delta(\alpha)^{1/2}f(\alpha(x)), \ \forall \ x \in G.
\]

When \(\Delta(\alpha) \neq 1\), we call \(\delta_\alpha\) dilation operator. In the sequel we shall assume that the operator \(\delta_\alpha\) is a dilation. It is easy to see that by (4.7) the operator \(\delta_\alpha\) is an unitary operator. For a mapping \(f : G \to \mathbb{C}\), \(\lambda \in \Lambda\) and \(\alpha \in \text{Aut}(G)\) we denote by \(f_{\alpha,\lambda}\) the “dilation and translation copy” of \(f\) with respect to \(\alpha\) and \(\lambda\), respectively, and define it by

\[
f_{\alpha,\lambda}(x) := \delta_\alpha \tau_\lambda f(x) = \Delta(\alpha)^{1/2}f(\alpha(x) - \lambda).
\]

For \(x \in G\) and \(g : \hat{G} \to \mathbb{C}\), the modulation operator \(M_x\) on \(g\) is given by

\[
M_x g(\xi) = \langle \xi, x \rangle g(\xi) \quad \xi \in \hat{G}.
\]

The following result is immediate by the group Fourier transform.

**Lemma 4.2.** Let \(\alpha\) be an automorphism of \(G\) and \(\lambda \in G\). Then for any \(f \in L^1(G) \cap L^2(G)\)

\[
\hat{f}_{\alpha,\lambda}(\xi) = \delta_\alpha \tau_\lambda \hat{f}(\xi) = \Delta(\alpha)^{-1/2}(M_{-\lambda} \hat{f})(\hat{\alpha}^{-1}(\xi)), \quad \xi \in \hat{G}.
\]

**Proof.** The result obtains from the application of the Fourier transform on the dilation
and translation operators, as follows. Let \( \xi \in \hat{G} \). Then by (4.3), the following holds.

\[
\hat{f}_{\alpha,\lambda}(\xi) = \int_G f_{\alpha,\lambda}(x) \chi_\xi(x) dm_G(x)
\]

(4.8)

\[
= \Delta(\alpha)^{1/2} \int_G f(\alpha(x) - \lambda) \chi_\xi(x) dm_G(x)
\]

\[
= \Delta(\alpha)^{-1/2} \int_G f(x) \chi_\xi(\alpha^{-1}(x + \lambda)) dm_G(x)
\]

\[
= \Delta(\alpha)^{-1/2} \chi_\xi(\alpha^{-1}(\lambda)) \int_G f(x) \chi_\xi(\alpha^{-1}x) dm_G(x)
\]

\[
= \Delta(\alpha)^{-1/2} \chi_\xi(\alpha^{-1}(\lambda)) \hat{f}(\hat{\alpha}^{-1}(\xi))
\]

\[
= \Delta(\alpha)^{-1/2} (M_{-\lambda}\hat{f})(\hat{\alpha}^{-1}(\xi)).
\]

This completes the proof. Note that within the above lines we used the property \( \chi_\xi(\alpha^{-1}(x)) = \chi_{\hat{\alpha}^{-1}(\xi)}(x) \) proved in Lemma 4.1 and the equation \( \chi_\xi(x) = \chi_\xi(-x) \) by (4.1).

Assume that \( f \in L^2(G) \) with \( \hat{f} = 1_\Omega \). As a result of the previous lemma one has

\[
\hat{f}_{\alpha,\lambda}(\xi) = \Delta(\alpha)^{-1/2} \chi_{\hat{\alpha}^{-1}(\xi)}(-\lambda) 1_\hat{\alpha}(\xi), \quad \xi \in \hat{G}.
\]

(4.9)

**Convention.** In the sequel, we shall use the notation “exponential” \( e_x(\xi) := \chi_\xi(x) \) for \( x \in G \) and \( \xi \in \hat{G} \).

**Riesz bases.** A countable sequence \( \{x_n\}_{n \in I} \) in a Hilbert space \( \mathcal{H} \) is a Riesz basis for \( \mathcal{H} \) if the sequence is of the form \( \{U e_n\}_{n \in I} \) for some orthonormal basis \( \{e_n\}_{n \in I} \) for \( \mathcal{H} \) and an invertible and bounded linear operator \( U : \mathcal{H} \to \mathcal{H} \). This definition is equivalent to say that the sequence \( \{x_n\}_{n \in I} \) is dense in \( \mathcal{H} \) and there is finite and positive constants \( A, B \) such that for any finite sequence \( \{c_n\}_{n \in I} \) we have

\[
A \sum_{n \in I} |c_n|^2 \leq \left\| \sum_{n \in I} c_n x_n \right\|^2_{\mathcal{H}} \leq B \sum_{n \in I} |c_n|^2.
\]

(4.10)

It is obvious that if the upper and lower estimations in (4.10) hold for any finite sequence \( \{c_n\}_{n \in I} \), then they also hold for any sequence \( \{c_n\}_{n \in I} \in \ell^2(I) \). For more about Riesz bases for any Hilbert space we refer to [8].

The following result is well-known, but we give a proof below to keep the presentation as self-contained as possible.

**Lemma 4.3.** Let \( \{x_n\}_{n \in I} \) be a Riesz basis for a Hilbert space \( \mathcal{H} \). Assume that for any finite sequence \( \{c_n\} \) the identity holds:

\[
\left\| \sum_n c_n x_n \right\| = \left( \sum_n |c_n|^2 \right)^{1/2}
\]

(4.11)

Then \( \{x_n\} \) is an orthonormal basis.
Proof. We only need to show that for any \( m, n \in I \), \( \langle x_n, x_m \rangle = \delta_{m,n} \). The identity (4.11) implies that \( \|x_M\| = 1 \). For the orthogonality of the vectors we proceed as follows: By (4.11), it is immediate that \( \|x_n + x_m\|^2 = 2 \) for any \( n \neq m \). From the other hand \( \|x_n + x_m\|^2 = 2 + 2\text{Re}\langle x_n, x_m \rangle \). This implies that we must have \( \text{Re}\langle x_n, x_m \rangle = 0 \). By replacing \( x_n \) by \( ix_n \), we also obtain \( \text{Im}\langle x_n, x_m \rangle = 0 \). These imply that \( x_n \) and \( x_m \) are orthogonal, and we are done. \( \square \)

Definition 4.4. Let \( H \) be an infinite-dimensional Hilbert space. An infinite collection \( \{x_n\} \) of vectors in \( H \) is \( w \)-linearly independent if for any sequence \( \{c_n\} \) such that \( \sum_n c_n x_n \) converges to zero in the norm of \( H \) is identically zero.

It is well-known that any \( w \)-linearly independent sequence is linearly independent. However, the converse does not always hold.

Lemma 4.5. A sequence \( \{x_n\} \) in a Hilbert space is a Riesz basis for \( H \) if and only if \( \{x_n\} \) is \( w \)-linearly independent and there are constants \( 0 < A \leq B < \infty \) such that for any vector \( x \in H \),

\[
A\|x\|^2 \leq \sum_n |\langle x, x_n \rangle|^2 \leq B\|x\|^2.
\]

The inequality in the preceding lemma is called the frame inequality. The \( w \)-linearly independency of Riesz basis follows from the fact that any Riesz basis has a biorthogonal sequence. For the proof we refer to [8].

5 Proof of Theorem 2.2

For the rest of this paper we fix the following notations unless it is stated otherwise: We let \( A \) denote a countable subset of \( \text{Aut}(G) \), the group of automorphisms of \( G \) onto \( G \) with \( \Delta(\alpha) \neq 1 \), \( \alpha \in \text{Aut}(G) \), \( \Lambda \) be a countable subset in \( G \), and \( \Omega \) be a subset of \( \hat{G} \) with finite and non-zero Haar measure.

The first result of this section shows that the spectral property of a set is preserved under the action of automorphisms.

Lemma 5.1. Assume that \( (\Omega, \Lambda) \) is a spectral pair and \( \alpha \in A \). Then the pair \( (\hat{\alpha}(\Omega), \alpha^{-1}(\Lambda^{-1})) \) is spectral, where \( \Lambda^{-1} = \{-\lambda : \lambda \in \Lambda\} \).

Proof. Define \( T_\alpha : L^2(\Omega) \to L^2(\hat{\alpha}(\Omega)) \) by \( T_\alpha(g)(\xi) = \Delta(\hat{\alpha})^{-1/2}g(\hat{\alpha}^{-1}(\xi)), \xi \in \hat{\alpha}(\Omega) \). \( T \) is linear and

\[
T_\alpha(e_\lambda) = \Delta(\hat{\alpha})^{-1/2}e_{\alpha^{-1}(-\lambda)}, \quad \lambda \in \Lambda
\]

\( T \) is a unitary isomorphism and \( (\Omega, \Lambda) \) is a spectral pair, then \( \{\Delta(\hat{\alpha})^{-1/2}e_{\alpha^{-1}(-\lambda)}\}_{\lambda \in \Lambda} \) is an orthogonal basis for \( L^2(\hat{\alpha}(\Omega)) \). This completes the proof of the lemma. \( \square \)
Proof of Theorem 2.2. We shall “somehow” adjust the proof of Theorem A for our situation. Fix \( A \subset \mathcal{Aut}(G) \) and \( \Lambda_i \subset G \) and let \( \{\Omega_i\}_{i=1}^m \) be a collection of disjoint subsets in \( \hat{G} \) such that \((\Omega_i, \Lambda_i)\) are spectral pairs.

Let \( \varphi_i \in L^2(G) \) be the inverse Fourier transform of \( 1_{\Omega_i} \), i.e., \( \hat{\varphi}_i = 1_{\Omega_i} \). Define

\[
\mathcal{W} := \bigcup_{i=1}^m \{ \varphi_{i,\alpha,\lambda} : \alpha \in A, \lambda \in \Lambda_i \}. \tag{5.1}
\]

Then by an application of the Fourier transform and the equation (4.9), \( \mathcal{W} \) is an orthogonal basis for \( L^2(G) \) if and only if the system \( \hat{\mathcal{W}} \) is an orthogonal basis for \( L^2(\hat{G}) \):

\[
\hat{\mathcal{W}} = \bigcup_{i=1}^m \{ \hat{\varphi}_{i,\alpha,\lambda} : \alpha \in A, \lambda \in \Lambda_i \} = \bigcup_{i=1}^m \{ \Delta(\alpha)^{-1/2}e_{\alpha^{-1}(-\lambda)}1_{\hat{\alpha}(\Omega_i)} : \alpha \in A, \lambda \in \Lambda_i \}. \tag{5.2}
\]

By the assumption, each pair \((\Omega_i, \Lambda_i)\) is a spectral. Therefore, for any \( \alpha \in A \), by Lemma 5.1 the collection \( \{e_{\alpha^{-1}(-\lambda)}1_{\hat{\alpha}(\Omega_i)} : \lambda \in \Lambda_i\} \) is an orthogonal basis for \( L^2(\hat{\alpha}(\Omega_i)) \). From the other hand, by multiplicative tiling property of \( \{\Omega_i\}_{i=1}^m \), for any \( \alpha, \beta \in A \) and \( 1 \leq i, j \leq m \) we have

\[
m_G(\hat{\alpha}(\Omega_i) \cap \hat{\beta}(\Omega_j)) = m_G(\hat{\alpha}(\Omega_i))\delta_{\alpha,\beta}\delta_{i,j}. \tag{5.3}
\]

Thus the elements of \( \hat{\mathcal{W}} \) are mutual orthogonal. The completeness of the system \( \hat{\mathcal{W}} \) in \( L^2(\hat{G}) \) holds by the \( A \)-multiplicative tiling of \( \hat{G} \) and the decomposition

\[
L^2(\hat{G}) = \bigoplus_{\alpha \in A} \bigoplus_{1 \leq i \leq m} L^2(\hat{\alpha}(\Omega_i)). \tag{5.4}
\]

Conversely, let \( \{\Omega_i\}_{i=1}^m \) be a wavelet collection of sets with respect to the automorphisms \( A \) and translations \( \Lambda_i, 1 \leq i \leq m \). Assume that \( e \in \Lambda_i \) for all \( i \). Then the elements of the collection \( \{\varphi_{i,\alpha} := \varphi_{i,\alpha,0} : \alpha \in A, 1 \leq i \leq m\} \) are mutual orthogonal and for any \( (i, \alpha_1) \neq (j, \alpha_2) \) we have

\[
0 = \int_G \varphi_{i,\alpha_1}(x)\overline{\varphi_{j,\alpha_2}(x)}dm_G(x) \tag{5.5}
\]

\[
= \int_G \hat{\varphi}_{i,\alpha_1}(\xi)\overline{\hat{\varphi}_{j,\alpha_2}(\xi)}dm_G(\xi)
\]

\[
= \int \hat{1}_{\hat{\alpha_1}(\Omega_i)}(\xi)\overline{\hat{1}_{\hat{\alpha_2}(\Omega_j)}(\xi)}dm_G(\xi)
\]

\[
= m_G(\hat{\alpha_1}(\Omega_i) \cap \hat{\alpha_2}(\Omega_j)).
\]

This implies that for \( \Omega = \bigcup_{i=1}^m \Omega_i \), the subsets \( \hat{\alpha}(\Omega), \alpha \in A \) are mutual disjoint and form a covering for \( \hat{G} \). Indeed, assume that there is a subset \( W \subset \hat{G} \) of positive and finite Haar measure such that \( |W \cap \hat{\alpha}(\Omega_i)| = 0 \) for all \( \alpha \in A \). Then \( |W \cap \hat{\alpha}(\Omega_i)| = 0 \) for all \( 1 \leq i \leq m \). Then for the function \( f : G \to \mathbb{C} \) with \( f = 1_W \) is \( \langle f, \varphi_{i,\alpha,\lambda} \rangle = \langle 1_W, \hat{\varphi}_{i,\alpha,\lambda} \rangle = 0 \).
for all \( i, \alpha \) and \( \lambda \in \Lambda_i \). From the other hand, \( \{ \Omega_i \}_{i=1}^m \) is a wavelet collection of sets. Then \( f \) must be identical to zero. This contradicts the assumption that \( |W| > 0 \), hence \( \Omega \) tiles \( \hat{G} \) multiplicatively by \( A \).

For the rest, we prove that the pairs \((\Omega_i, \Lambda_i)\) are spectral. To this end, according to Lemma 5.1, it is sufficient to show that for any \( \alpha \in A \) each pair \((\hat{\alpha}(\Omega_i), \alpha^{-1}(-\Lambda_i))\) is a spectral. Or, equivalently, the system

\[
\{ \Delta(\alpha)^{-1/2}e_{\alpha^{-1}(-\lambda)}1_{\hat{\alpha}(\Omega_i)} : \lambda \in \Lambda_i \}
\]

is orthogonal and complete in \( L^2(\hat{\alpha}(\Omega_i)) \). The orthogonality holds from the assumption that \( \{ \Omega_i \}_{i=1}^m \) is a wavelet collection of sets. For the completeness, assume that \( g \in L^2(\hat{\alpha}(\Omega_i)) \) such that \( \langle g, e_{\alpha^{-1}(-\lambda)} \rangle_{L^2(\hat{\alpha}(\Omega_i))} = 0 \) for all \( \lambda \in \Lambda_i \). This, along the mutual disjointness of \( \{ \hat{\alpha}(\Omega_j) : \alpha \in A, j \} \), yields

\[
\|g\|^2 = \sum_{\beta \in A} \sum_{j=1}^m \sum_{\lambda \in \Lambda_j} |\langle g1_{\hat{\alpha}(\Omega_i)} , \Delta(\beta)^{-1/2}1_{\hat{\beta}(\Omega_j)}e_{\beta^{-1}(-\lambda)} \rangle|^2 = 0,
\]

which implies that \( g \) must be zero. Thus \((\hat{\alpha}(\Omega_i), \alpha^{-1}(\Lambda^{-1}))\) is a spectral pair and we are done. \( \square \)

**Remark.** Note that the multiplicative tiling property of the set \( \Omega \) for \( \hat{G} \) is required for the completeness of the wavelet system in \( L^2(G) \). However, the results hold for a subspace of \( L^2(G) \) if we only assume that \( \Omega \) is a multiplicative tiling for a subset of \( \hat{G} \).

**Example 5.2.** We shall use the multiplicative tiling property of the Shannon set to construct a simple example of one dimensional orthogonal wavelet basis for a subspace of \( L^2(\mathbb{R}) \). Let \( \Omega_1 := [-1, -1/2 - \epsilon_1) \) and \( \Omega_2 := (1/2 + \epsilon_2, 1] \) where \( 0 \leq \epsilon_i < 1/2 \). Then \((\Omega_i, \alpha_i \mathbb{Z})\) is a spectral pair with \( \alpha_i = 2(1 - 2\epsilon_i)^{-1} \). The set \( \Omega = \Omega_1 \cup \Omega_2 \) is a multiplicative tiling for some subset \( U \subset \mathbb{R} \) with respect to the dyadic dilations \( \{ 2^n : n \in \mathbb{Z} \} \). This along with the result of Theorem 2.2 implies that the collection \( \hat{W}(5.2) \) is an orthogonal basis for \( L^2(U) \).

# 6 Proof of Theorems 2.4 and 2.6

To prove Theorem 2.4, we need to prove an analogous result to Lemma 5.1 for Riesz spectral pairs, as follows.

**Lemma 6.1.** Let \((\Omega, \Lambda)\) be a Riesz spectra with Riesz constants \( A \) and \( B \). Then for any \( \alpha \in A \), the pair \((\hat{\alpha}(\Omega), \alpha^{-1}(\Lambda^{-1}))\) is a Riesz spectral with unified Riesz constants.

**Proof.** The proof is straightforward and uses an application of the operator \( T_\alpha \) in Lemma 5.1, and the fact the image of a Riesz basis under any unitary map is Riesz basis with the unified constants. \( \square \)
Proof of Theorem 2.4. By unitary Fourier transform it is sufficient to show that
\[
\hat{\mathcal{W}} := \bigcup_{i=1}^{m} \{ \hat{\psi}_{i\alpha,\lambda_i} \} = \bigcup_{i=1}^{m} \{ \Delta(\alpha)^{-1/2}c_{\alpha^{-1}(-\lambda_i)}(\xi)\hat{G}(\Omega_i) : \alpha \in A, \lambda_i \in \Lambda_i \}
\]
is a Riesz basis for \( L^2(\hat{G}) \). By the assumptions, for any \( i \), the pair \((\Omega_i, \Lambda_i)\) is Riesz spectral. Therefore by Lemma 6.1 for any \( \alpha \in A \) the pair \((\hat{\alpha}(\Omega_i), \alpha^{-1}(-\Lambda_i))\) is Riesz spectral with the unified constants. Then the completeness of the system \( \mathcal{W}_{A,T} \) in \( L^2(\hat{G}) \) holds by the \( A \) multiplicative tiling assumption of \( \Omega = \cup_i \Omega_i \) for \( \hat{G} \) and the decomposition \( L^2(\hat{G}) = \bigoplus_{\alpha \in A} L^2(\hat{\alpha}(\Omega)) = \bigoplus_{\alpha \in A, 1 \leq i \leq m} L^2(\hat{\alpha}(\Omega_i)) \). To prove the Riesz sequence approximation inequalities, we continue as follows: Let \( \{ c_{\alpha,\lambda_i} : \alpha \in A, \lambda_i \in \Lambda_i, i = 1, \ldots, m \} \) be a finite collection of numbers. Then, due the disjointness of \( \alpha(\Omega_i), \lambda_i \) and that the pairs \((\alpha(\Omega_i), \Lambda_i)\) are Riesz spectral with upper bound \( U_i \), we can write the following:
\[
\| \sum_{\alpha, i, \lambda_i} c_{\alpha, \lambda_i} \hat{\psi}_{i\alpha, \lambda_i} \|^2_{L^2(\hat{G})} = \sum_{\alpha} \sum_{i=1}^{m} \| \sum_{\lambda_i} c_{\alpha, \lambda_i} \hat{\psi}_{i\alpha, \lambda_i} \|^2_{L^2(\hat{\alpha}(\Omega_i))} \leq \sum_{i=1}^{m} U_i \sum_{\alpha} \sum_{\lambda_i} |c_{\alpha, \lambda_i}|^2 = U \sum_{\alpha, i, \lambda_i} |c_{\alpha, \lambda_i}|^2 ,
\]
with \( U = \max \{ U_i : 1 \leq i \leq m \} \). The lower estimate for Riesz sequence can be obtained in the same fashion. This completes the proof of the theorem.

Note that the completeness of the Riesz system in the previous theorem can also be obtained by a different approach: Let \( f \in L^2(G) \) such that \( \langle f, \hat{\psi}_{i\alpha, \lambda} \rangle = 0 \) for all \( \lambda, \alpha \) and \( 1 \leq i \leq m \). Fix \( \alpha \) and \( i \). Then \( \langle f1_{\alpha(\Omega_i)}, \hat{\psi}_{i\alpha, \lambda} \rangle = 0 \) for all \( \lambda \in \Lambda_i \). Since \((\Omega_i, \Lambda_i)\) is a Riesz spectral, and \( \{ \hat{\psi}_{i\alpha, \lambda} \} \) is complete in \( L^2(\hat{\alpha}(\Omega_i)) \), then we must have \( f1_{\alpha(\Omega_i)} = 0 \). But \( f = \sum_{\alpha} 1_{\alpha(\Omega_i)}f1_{\alpha(\Omega_i)} \), which implies that \( f \) must be zero.

Proof of Corollary 2.5. Assume that \( \Omega_i \) tiles \( \hat{G} \) additively by some lattice \( \Gamma_i \) with multiplicity \( k_i \). Then by [1, Theorem 4.1], for each \( i \), there is a countable set \( \Lambda_i \subset G \) such that \((\Omega_i, \Lambda_i)\) is a Riesz spectral pair in \( \hat{G} \). The rest of the proof is immediate from Theorem 2.4.

Example 6.2. Let \( \Lambda \) be a lattice such that, \( \Omega_1, \Omega_2, \cdots, \Omega_k \), be \( k \) mutual disjoint fundamental domains for \( \Lambda \). Then \( \Omega = \bigcup_{i=1}^{k} \Omega_i \) is an additive tiling set for \( \hat{G} \) with multiplicity \( k \) with respect to translations \( \Lambda \). Moreover, assume that \( \Omega \) is a multiplicative tiling with respect to a subset of automorphisms. Then \( \Omega \) is a Riesz wavelet set. In this connection, see Example 5.2 in Section 5.

Remark. Let \((\Omega_i, \Lambda)\), \( 1 \leq i \leq m \), be spectral pair, i.e., the exponentials \( \mathcal{E}_\Lambda \) form an orthogonal basis for \( L^2(\Omega_i) \). If \( \{\Omega_i\} \) are mutual disjoint, then \( \Omega = \bigcup_{i=1}^{m} \Omega_i \) is a Riesz
spectral set with spectra $\Lambda$. To see this, note that for any finite $\{c_\lambda\}_{\lambda \in \Lambda}$ we have
\[
\| \sum_\lambda c_\lambda \epsilon_\lambda \|_{L^2(\Omega)}^2 = \sum_{1 \leq i \leq m} \| \sum_\lambda c_\lambda \epsilon_\lambda \|_{L^2(\Omega_i)}^2 = \sum_{1 \leq i \leq m} \sum_\lambda |c_\lambda|^2.
\] (6.4)

The completeness of $\mathcal{E}_\Lambda$ in $L^2(\Omega_i)$ for disjoint sets $\Omega_i$, $1 \leq i \leq m$, implies that $\mathcal{E}_\Lambda$ is complete in $L^2(\Omega)$, too. This along the equation (6.4) and Lemma 4.11 completes the proof.

**Proof of Theorem 2.6.** To prove the “if” part, let $\alpha \neq \beta$ and let $M$ be any subset of $\hat{\alpha}(\Omega) \cap \hat{\beta}(\Omega)$. Put $f := 1_M$. We show that $M$ must have zero measure. Since $f \in L^2(\hat{G})$ and also belongs to the spaces $L^2(\hat{\alpha}(\Omega))$ and $L^2(\hat{\beta}(\Omega))$, then by the theorem’s assumptions, there are $l^2$ sequences $\{c_{\theta,\lambda} : \theta \in A, \lambda \in \Lambda\}$, $\{d_{\lambda}^\alpha : \lambda \in \Lambda\}$ and $\{d_{\lambda}^\beta : \lambda \in \Lambda\}$ for which we can write
\[
1_M = \sum_{\theta,\lambda} c_{\theta,\lambda} \hat{\alpha}_\theta \hat{\tau}_\lambda \psi = \sum_\lambda d_{\lambda}^\alpha \hat{\alpha}_\lambda \hat{\tau}_\lambda \psi = \sum_\lambda d_{\lambda}^\beta \hat{\beta}_\lambda \hat{\tau}_\lambda \psi.
\]

Since the Riesz bases are $w$-linearly independent (Lemma 4.5), the preceding first and second equalities imply that we must have $c_{\alpha,\lambda} = d_{\lambda}^\alpha$ and $c_{\beta,\lambda} = 0$ for $\theta \neq \alpha$. With a similar argument, by the first and third equalities we obtain $c_{\beta,\lambda} = d_{\lambda}^\beta$ and $c_{\alpha,\lambda} = 0$ for $\theta \neq \beta$. These conclude that $d_{\lambda}^\alpha = d_{\lambda}^\beta = 0$ for all $\lambda$, hence $1_M$ is the zero function and $|M| = 0$. This implies the measure disjointness of $\alpha(\Omega)$. To prove that $\{\hat{\alpha}(\Omega) : \alpha \in A\}$ is a cover for $\hat{G}$, we shall use a contradiction approach and apply the frame inequality (4.12) in Lemma 4.5 for the wavelet Riesz basis $\mathcal{W} = \{\delta_{\alpha,\tau,\psi} : \alpha \in A, \lambda \in \Lambda\}$, respectively.

To prove the “only if” part, notice that the disjointness of sets $\hat{\alpha}(\Omega)$ with the frame inequality (4.12) imply that for any given $\alpha \in A$, the exponentials $\mathcal{E}_\Lambda$ are complete in $L^2(\hat{\alpha}(\Omega))$. The Riesz inequality for $\mathcal{E}_\Lambda$ is a direct implication of the Riesz inequality for the wavelet system $\mathcal{W}$. \hfill \Box

**Remark:** Note that in Theorem 2.6, only the $w$-linearly independency of the system $\{\delta_{\alpha,\tau,\psi}\}$ yields the disjointness of sets $\hat{\alpha}(\Omega)$.

### 7 Examples

**Example 7.1.** Let $\Omega = [-1, 1]^d \setminus [-1/2, 1/2]^d$. Fix a lattice $\Lambda$ and write $\Omega$ as a finite union of $\Omega_i$ such that each $\Omega_i$ is a tiling with respect the lattice $\Lambda$. For example, when $d = 2$, one can take $\Lambda = \{(m/2, n/2) : m, n \in \mathbb{Z}\}$. It is easy to see that $\{\Omega_i\}$ is a wavelet collection of sets. Indeed, each pair $(\Omega_i, \Lambda^\perp)$ is a spectral pair and $\Omega$ is a multiplicative tiling set for $\mathbb{R}^d$ by the automorphisms $A = \{2^n I : n \in \mathbb{Z}\}$, where $I$ is the $d \times d$ identity matrix. The result now follows by Theorem 2.2. (For more examples of wavelet sets in $\mathbb{R}^d$ see e.g. [4].)
Example 7.2. Let $\Omega_i$ are given as in Example 7.1. Assume that $\Omega := \bigcup_i \Omega_i$ is a multiplicative tiling with respect to the dyadic dilations $\{2^n I : n \in \mathbb{Z}\}$. Take $\Lambda := 4^{-1} \mathbb{Z}^d$. Then each pair $(\Omega_i, \Lambda)$ tiles $\mathbb{R}^d$ additively with multiplicity $4$, i.e., almost every point in $\mathbb{R}^d$ is covered four times by $\Lambda$ translations of each $\Omega_i$. Therefore $\Omega_i$ is a Riesz spectral set with spectra $\Lambda$. Theorem 2.4 implies that $\{\Omega_i\}$ is a Riesz wavelet collection of sets in $\mathbb{R}^d$.

Example 7.3. Let $\Omega_1$ be a symmetric polygon in $\mathbb{R}^2$ centered at origin. Take $\Omega_2 = 2^{-1} \Omega_1$, the dilation of $\Omega_1$ by scale $2^{-1}$. Define $\Omega := \Omega_1 \setminus \Omega_2$. Then it is easy to check that $\Omega$ tiles the plane multiplicatively with respect to the dyadic dilations $\{2^n I : n \in \mathbb{Z}\}$. Assume that $\Omega_1, \cdots, \Omega_m$ be $m$ convex tiles of $\Omega$ which are pairwise disjoint. Furthermore, we assume that each $\Omega_i$ is a symmetric polygon with respect to a basis for $\mathbb{R}^2$. Since every convex and symmetric polygon in $\mathbb{R}^2$ has a Riesz spectra ([28]), then each $\Omega_i$ is a Riesz spectral set with spectra $\Lambda_i$. Theorem 2.4 implies that $\{\Omega_i\}_1^m$ is a Riesz wavelet collection of sets for $L^2(\mathbb{R}^2)$.

The following example illustrates an approach to the construction of an orthogonal wavelet basis on $G$ induced by a wavelet set in a subgroup $K$.

Example 7.4. Let $G$ be a locally compact abelian group which is topologically isomorphic to $\mathbb{R}^n \times D \times K$, where $D$ discrete abelian group and $K$ is compact. (Indeed, by Theorem 24.30 of [19], every LCA group has this form.) Assume that $n$ is a non-negative integer and $D$ is a finite direct sum of finite abelian cyclic groups of prime power order, i.e., $p^r$, $p$ prime. Let $T := \hat{K}$ denote the Pontryagin dual group of $K$. $T$ is discrete and a spectra for $K$. Therefore, for given any spectral set $S$ in $\mathbb{R}^d$ with spectra $\Lambda$, $\Omega := S \times D \times K$ is spectral for $G$ with spectra $\Lambda \times D \times T$.

Furthermore, assume that $S$ tiles $\mathbb{R}^d$ multiplicatively with respect to a subset of automorphisms $A \subset GL(\mathbb{R}, d)$. For any $\alpha \in A$ define $t_\alpha : G \rightarrow G$ with $t_\alpha(x, d, k) := (\alpha(x), k)$. Then $t_\alpha$, $\alpha \in A$, is an automorphisms of $G$ and $\{ t_\alpha(S \times D \times K) : \alpha \in A \}$ is a mutual disjoint tiling (a partition) for $G$. This implies that $\Omega$ is a wavelet set for $L^2(G)$.

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