Scattering in Anti-de Sitter Space and Operator Product Expansion

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Abstract

We develop a formalism to evaluate generic scalar exchange diagrams in AdS\(_{d+1}\) relevant for the calculation of four-point functions in AdS/CFT correspondence. The result may be written as an infinite power series of functions of cross-ratios. Logarithmic singularities appear in all orders whenever the dimensions of involved operators satisfy certain relations. We show that the AdS\(_{d+1}\) amplitude can be written in a form recognisable as the conformal partial wave expansion of a four-point function in CFT\(_d\) and identify the spectrum of intermediate operators. We find that, in addition to the contribution of the scalar operator associated with the exchanged field in the AdS\(_{d+1}\) diagram, there are also contributions of some other operators which may possibly be identified with two-particle bound states in AdS\(_{d+1}\). The CFT\(_d\) interpretation also provides a useful way to “regularize” the logarithms appearing in AdS\(_{d+1}\) amplitude.

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1. Introduction

There has been a recent revival of interest in the connection between large $N$ Yang-Mills theory [1] and string theory [2] following the conjecture [3] that there is an exact correspondence [4,5] between IIB superstring theory on $AdS_5 \times S_5$ and $\mathcal{N} = 4$ Super-Yang-Mills theory in four dimension (see also [6]).

Under this proposal, correlation functions of $\mathcal{N} = 4$ super-Yang-Mills (SYM) with gauge group $SU(N)$ in the large-$N$ and large 't Hooft coupling limit can be obtained by evaluating scattering amplitudes of type IIB supergravity on $AdS_5 \times S_5$. Some ‘model’ and ‘realistic’ two-point and three-point functions have been computed in [7,8,9,10,11,12,13,14,15]. Since the structures of two- and three-point functions are severely restricted by conformal symmetry, in many cases the computations amount to fixing the overall constants. Four-point functions can be arbitrary functions of cross-ratios and thus encode more dynamical information. Recently some efforts have been made in this direction [16,17,18,19,20,21,22] aiming to understand more about the non-perturbative dynamics of $\mathcal{N} = 4$ SYM.

Considering a CFT$_d$ [6], we shall assume that there exists a closed operator algebra, which is a strong version of the Wilson operator product expansion,

$$O_i(x)O_j(0) = \sum_k C_{ij}^k(x)O_k(0). \quad (1.1)$$

Here the summation is over all operators and their coordinate derivatives, and $C_{ij}^k$ are c-number functions. From (1.1), a four-point function may be expanded in terms of conformal partial waves, e.g., when $x_{12}, x_{34} \to 0$, as an $s$–channel exchange,

$$< O_{i_1}(x_1)O_{i_2}(x_2)O_{i_3}(x_3)O_{i_4}(x_4) > = \sum_j C_{i_1i_2}^j(x_{12})C_{i_3i_4}^j(x_{34}) < O_j(x_2)O_j(x_4) > \quad (1.2)$$

Alternatively, we can also write the four-point function in terms of $t$– or $u$–channel exchanges in the limit $x_{14}, x_{23} \to 0$ or $x_{13}, x_{24} \to 0$. If the algebra (1.1) is complete and associative, all channels of exchange are equivalent.

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1 Though our prime interest is $\mathcal{N} = 4$ SYM, most of our discussion will apply to any $d$–dimensional conformal field theory appearing in $AdS_{d+1}/$CFT$_d$ correspondence.
We would like to examine whether a four-point function calculated from the scattering amplitude in AdS\(_{d+1}\) can be written in the form of (1.2) as we take the corresponding limits in cross-ratios. A positive answer would be a confirmation of the assumption of a closed algebra (1.1), which hitherto has been only known to hold in two dimensions. And we could further extract important non-perturbative information about CFT\(_d\) by identifying the spectra of intermediate operators in each channel. In the case of \(\mathcal{N} = 4\) SYM in large \(N\) and large \(g^2 N\) limit, knowledge of four-point functions would help us answer questions such as [16,17,18]:

1. Does \(\mathcal{N} = 4\) SYM in large \(N\) and large \(g^2 N\) limit have a closed algebra (1.1)?

2. If yes, what is the spectrum of operators? In particular, do chiral operators, which are in one-to-one correspondence with IIB supergravity modes on AdS\(_5\) \(\times\) S\(_5\), form a complete set?

In this paper, we shall make some preliminary progress in answering these questions. In particular, we shall find indications that there are operators in the spectrum which correspond to two-particle bound states in AdS\(_{d+1}\).

One of the obstacles in the computation of realistic four-point functions in \(\mathcal{N} = 4\) SYM has been the difficulty in evaluating exchange diagrams [3] in AdS space (see figure 1a), which involve very complicated integrals. Here we present a formalism to address this problem (see also [19]), providing explicit formulas for AdS integrals involving scalar fields of arbitrary mass needed to evaluate generic four-point functions. The result is written as a single inverse Mellin integral so that the analytic properties of the amplitudes become transparent. In particular, for the exchange diagram of fig. 1a, in the limit \(x_{12}, x_{34} \to 0\)
the scattering amplitude can be written as a contour integral

\[ S_\lambda = \int_C ds \Gamma(\frac{\lambda_1 + \lambda_2}{2} - s) \Gamma(\frac{\lambda_3 + \lambda_4}{2} - s) \Gamma(\frac{\lambda}{2} - s) H(s, \eta, \xi) . \]  

(1.3)

where \( \xi \) and \( \eta \) are independent cross-ratios and \( H \) is a function of complex variable \( s \) and \( \xi, \eta \). In (1.3) we have only explicitly written down the Gamma functions which generate poles inside the contour. \( S_\lambda \) can then be evaluated by the calculus of residues and written as a sum of residues of the integrand at three infinite pole sequences (see figure 2). We find that logarithms of cross ratios, first found \[18\] in leading order expansion of some contact diagrams, arise generically whenever the poles in (1.3) merge into double poles or triple poles, i.e. when

\[ \frac{\lambda_1 + \lambda_2 - \lambda_3 - \lambda_4}{2} \text{ or } \frac{\lambda_1 + \lambda_2 - \lambda}{2} \text{ or } \frac{\lambda_3 + \lambda_4 - \lambda}{2} = \text{integer} . \]  

(1.4)

They appear in all orders of the series. In particular, the contribution from a triple pole will contain a part proportional to

\[ (\log |x_{12}x_{34}|^2) . \]

Similarly, it can be shown in contact diagrams (see figure 1b) logarithms occur \[18\] when

\[ \frac{\lambda_1 + \lambda_2 - \lambda_3 - \lambda_4}{2} = \text{integer} . \]  

(1.5)

**Figure 2:** Poles and the contour of (1.3). There are three sequences of poles: 1) \( s = \frac{\lambda_1 + \lambda_4}{2} + n \) (represented by solid circles); 2) \( s = \frac{\lambda_3 + \lambda_4}{2} + n \) (circles); 3) \( s = \frac{\lambda}{2} + n \) (crosses), where \( n = 0, 1, 2, \cdots \).
In $\mathcal{N} = 4$ SYM, the dimensions of the chiral fields are protected by supersymmetry and take integer values. Since in IIB supergravity on $AdS_5 \times S_5$, all vertices are $SU(4)$ singlets, the scattering diagrams associated with four-point functions will in general satisfy (1.4) or (1.5). This implies that logarithms are universally present.\(^2\)

We then proceed to investigate whether the amplitude we find can be written as the conformal partial wave expansion (CPWE) (1.2). The Mellin integral representation (1.3), in which our results are presented, turns out to be particularly convenient to identify them with $s$–channel OPE exchanges in CFT\(_d\). The contribution of each pole sequence in (1.3) can be identified with the CPWE (1.2) of a conformal operator: the value of a pole corresponds to the scale dimension of a spin-0 descendant\(^3\) (we shall call it a sub-primary), while the residue at the pole may be identified with the CPWE contribution of a subset of descendants associated with the sub-primary. The pattern may be presented diagrammatically as

\[ \begin{array}{c}
\lambda_1 \\
\lambda_2
\end{array} + \begin{array}{c}
\lambda_1 \\
\lambda_3
\end{array} + \begin{array}{c}
\lambda_1 \\
\lambda_4
\end{array} + \begin{array}{c}
\lambda_2 \\
\lambda_3
\end{array} + \begin{array}{c}
\lambda_3 \\
\lambda_4
\end{array} \]

\[ = \begin{array}{c}
\lambda_1 \\
\lambda_2
\end{array} + \begin{array}{c}
\lambda_1 \\
\lambda_3
\end{array} + \begin{array}{c}
\lambda_1 \\
\lambda_4
\end{array} + \begin{array}{c}
\lambda_2 \\
\lambda_3
\end{array} + \begin{array}{c}
\lambda_3 \\
\lambda_4
\end{array} \]

**Figure 3:** $s$–channel OPE interpretation of an exchange diagram.

The first diagram on the right hand side corresponds to the exchange of a scalar primary operator of dimension $\lambda$, which may be interpreted as the operator (we shall call it $O_\lambda$) related to the exchanged field in $AdS_{d+1}$ by AdS/CFT correspondence. This result was expected earlier in [17] on the basis of indirect considerations. Here we identify the contributions of all the descendents of $O_\lambda$ and show that their relative OPE couplings (1.1) are consistent with those required by conformal symmetry. The second and third diagram on the right hand side correspond to the exchanges of operators of dimensions

\(^2\) It might still happen that when we add up all the diagrams contributing to a four-point function, logarithms will cancel.

\(^3\) Here we mean an $SO(d,2)$ descendant. A spin-0 descendant takes the form $(\partial^2)^n O$, where $O$ is the primary and $\partial^2$ is the Laplacian.
\[ \lambda_1 + \lambda_2 \text{ and } \lambda_3 + \lambda_4 \text{ respectively (which we shall call } O_{12} \text{ and } O_{34}). \] However, in these cases, there are some mismatches in the identifications. In fig. 3 we have used dotted lines in intermediate states to distinguish them from the first diagram. Although we have found contributions from operators having the same quantum numbers as the complete set of descendants of a primary operator of dimension \( \lambda_1 + \lambda_2 \) (and \( \lambda_3 + \lambda_4 \)), the relative OPE couplings (1.1) between the primary and descendants seem to be inconsistent with those required by conformal symmetry. The OPE couplings of these descendant operators have a peculiar pattern suggesting the mismatch may be due to some mixing among different operators. But we have not been able to make it precise in this paper.

\[ Figure \ 4: \ s-\text{channel OPE interpretation of a contact diagram.} \]

Similarly the contact diagram fig. 1b may be represented in terms of \( s-\text{channel} \) exchanges as in figure 4 and the identifications are also not complete in the sense as described in exchange case.

The identification of AdS\(_{d+1}\) diagrams with CPWE also sheds light on the appearance of logarithms. The conditions (1.4) and (1.5) are satisfied precisely when the quantum numbers (spin, scale dimensions, etc.) of certain descendants of \( O_\lambda \) or \( O_{12} \) or \( O_{34} \) become identical to one another. The OPE couplings in fig. 3 and 4 determined from (1.3) fall into the following pattern: when the quantum numbers of descendants of different operators are degenerate, their contributions to the conformal partial wave expansion become identical and cancel one another. The results are given by their derivatives, which contain logarithms. Thus by moving infinitesimally away from the degeneracy points (1.4) and (1.3) in parameter space, we see that the relations in fig. 3 and 4 provide a physically meaningful way to “regularize” logarithms.

\[ ^4 \text{ In a CFT}_d, \text{ the OPE coupling } (1.1) \text{ of a descendant is uniquely determined by that of the primary.} \]
Although our present analysis based on generic diagrams would not give a conclusive answer to the questions listed earlier, it nevertheless provides a starting point for further study. The relation we find here between an arbitrary exchange diagram in AdS$_{d+1}$ and CPWE appears to be universal and should be helpful for understanding the general structure of AdS/CFT correspondence. Before having a complete calculation of realistic four-point functions in a specific theory, it is probably premature to speculate about the relevance of operators $O_{12}$ and $O_{34}$ and the mismatches in their CFT$_d$ identification. However, if their contributions are indeed present in a realistic amplitude, it should imply the existence of new operators in the spectrum not seen in the Lagrangian of supergravity. In $\mathcal{N} = 4$ SYM they may be written as double-trace operators, while in AdS$_5$ supergravity they may be identified with two-particle bound states.

The plan of the paper is as follows. In section two and three we discuss the evaluation of scattering diagrams in AdS$_{d+1}$. In section four we review, for the convenience of comparing with AdS$_{d+1}$ results, the conformal partial wave expansion in CFT$_d$. In section five we discuss the CFT$_d$ interpretation of AdS$_{d+1}$ amplitude. We have included a number of appendices. In Appendix A we describe briefly the subtleties in evaluation of integrals using Mellin transform and analytic continuation. Appendices B, C are devoted to detailed evaluation of some integrals in the main text.

2. Scalar exchange in anti-de Sitter space

We consider tree-level scattering of four scalar fields in AdS$_{d+1}$ with masses $m_i$, $i = 1, \cdots, 4$ by exchanging a scalar field of mass $m$. According to AdS/CFT correspondence, a scalar field $\phi_i$ of mass $m_i$ in AdS$_{d+1}$ corresponds to a scalar operator $\Phi_{\lambda_i}$ in CFT$_d$, with conformal dimension $\lambda_i = \frac{d}{2} + \sqrt{m_i^2 + \frac{d^2}{4}} = \frac{d}{2} + \nu_i$. The scattering amplitude describes in CFT$_d$ the contribution of $\Phi_{\lambda}$ to the four-point function of scalar operators $\Phi_{\lambda_i}, i = 1, \cdots, 4$.

In this section, we shall take the interacting vertices to be of the form

$$\mathcal{L} = \phi_1 \phi_2 \phi + \phi_3 \phi_4 \phi .$$

Scattering amplitudes resulting from more complicated vertices involving derivatives and contact vertices will be discussed in next section.
As in [5] we use the Euclidean (half-space) metric,
\[ ds^2 = g_{\mu\nu} du^\mu du^\nu = \frac{1}{u_0^2} (du_0^2 + du_i^2), \quad i = 1, 2, \cdots, d. \]  
(2.1)

The AdS$_{d+1}$ bulk indices will be denoted by $\mu, \nu, \cdots$ and will take values $0, 1, \ldots, d$. The points in the bulk are labelled by $u, v, \cdots$, while those on the boundary by $x, y, \cdots$. We also use shorthand notations $u = (u_0, \vec{u}), x = (\vec{x})$ and $x_{ij}^2 = |\vec{x}_i - \vec{x}_j|^2, |u - x_i|^2 = u_0^2 + |\vec{u} - \vec{x}_i|^2$.

The scattering amplitude can then be written as
\[ S_\lambda(x_1, x_2, x_3, x_4) = \int \frac{du_0 d^d u_0 dv_0 d^d v}{u_0^{d+1} v_0^{d+1}} K_{\lambda_1}(u, x_1) K_{\lambda_2}(u, x_2) G(u, v) K_{\lambda_3}(v, x_3) K_{\lambda_4}(v, x_4), \]  
(2.2)

where $K_{\lambda_i}(u, x_i), i = 1, \cdots, 4$ is the bulk-to-boundary propagator [5] for field $\phi_i$,
\[ K_{\lambda_i}(u, x_i) = c_{\lambda_i} (\frac{u_0}{|u - x_i|^2})^{\lambda_i}, \quad c_{\lambda_i} = \frac{\Gamma(\lambda_i)}{\pi^{\frac{d}{2}} \Gamma(\nu_i)} \]  
(2.3)

and $G(u, v)$ is the AdS bulk scalar propagator [23],
\[ G(u, v) = rt^{-\lambda} F(\lambda, \nu + \frac{1}{2}; 2\nu + 1, t^{-1}). \]  
(2.4)

In (2.4) $F$ is a hypergeometric function and
\[ r = \frac{\Gamma(\lambda)}{2^{2\lambda} \pi^\frac{d}{2}} \frac{1}{\Gamma(\nu + 1)}, \quad t = \frac{(u_0 + v_0)^2 + (\vec{u} - \vec{v})^2}{4u_0v_0}. \]

To evaluate (2.2), first we would like to get rid of the cross term of $u_0$ and $v_0$ in $t$ in (2.4), which complicates the integrals. This can be achieved by a quadratic transformation of the hypergeometric function in (2.4), after which the bulk propagator becomes,
\[ G(u, v) = \frac{\Gamma(\lambda)}{2^{2\lambda} \pi^\frac{d}{2}} \frac{1}{\Gamma(\nu + 1)} q^{-\lambda} F(\frac{\lambda + 1}{2}, \frac{\lambda}{2}; \nu + 1; \frac{1}{q^2}), \]  
(2.5)

The one we use here is:
\[ F(a, b; 2b; z) = (1 - \frac{z}{2})^{-a} F(\frac{1}{2} a, \frac{1}{2} (a + 1); b + \frac{1}{2}; z^2 (2 - z)^{-2}) \]
where
\[ q = \frac{u_0^2 + v_0^2 + |\bar{u} - \bar{v}|^2}{2u_0v_0}. \]

Now we use the Mellin-Barnes representation of a hypergeometric function
\[ F(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} ds \frac{\Gamma(a+s)\Gamma(b+s)}{\Gamma(c+s)} \Gamma(-s)(-z)^s \]  
(2.6)
in (2.3) and plug it into (2.2). This gives us,
\[ S_\lambda = C_1 \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} ds \frac{\Gamma(\lambda+1/2 + s)\Gamma(\lambda/2 + s)}{\Gamma(\nu + 1 + s)} \Gamma(-s)(-1)^s J(s) \]  
(2.7)
with
\[ J(s) = \int\frac{dudvdv}{u_0d_1v_0d_2} \left( \frac{u_0}{|u - x_1|^2} \right)^{\lambda_1} \left( \frac{u_0}{|u - x_2|^2} \right)^{\lambda_2} \left( \frac{u_0^2 + v_0^2 + |\bar{u} - \bar{v}|^2}{\lambda_0^2 + v_0^2} \right)^{\lambda + 2s} \]
(2.8)
and \( C_1 = \frac{1}{4\pi i} \Pi_{i=1}^{4} c_i \).

\( J(s) \) still involves quite complicated integrals. We present its detailed evaluation in Appendix B. The result can be written in terms of the cross ratios of the boundary points as an inverse Mellin type integral (for notations see Appendix B),
\[ J(s) = C_2 \frac{1}{2\pi i} \int_{\mathcal{C}} ds_1 \xi^{-s_1 - \frac{\Delta_{34}}{2} - s_1} \Gamma\left( \frac{\lambda_{12}}{2} - s_1 \right) \Gamma\left( \frac{\lambda_{34}}{2} - s_1 \right) \frac{F\left( \frac{\Delta_{34}}{2} + s_1, \frac{\Delta_{12}}{2} + s_1; 2s_1; 1 - \frac{\eta}{\xi} \right)}{\Gamma\left( \frac{\Delta_{12} + \Delta_{34}}{2} + s - s_1 \right)} \]
(2.9)
where \( \eta, \xi \) are cross ratios defined by,
\[ \eta = \frac{|x_{13}|^2|x_{24}|^2}{|x_{12}|^2|x_{34}|^2}, \quad \xi = \frac{|x_{14}|^2|x_{23}|^2}{|x_{12}|^2|x_{34}|^2}, \]
(2.10)
and
\[ C_2 = \frac{\pi^d}{4} \frac{\Gamma\left( \frac{\lambda_{12} + \Delta_{34}}{2} - 4 \right) \Gamma\left( s + \frac{\Delta_{12}}{2} \right) \Gamma\left( s + \frac{\Delta_{34}}{2} \right)}{\Gamma\left( \lambda_{12} \right) \Gamma\left( \lambda_{23} \right) \Gamma\left( \lambda_{34} \right) \Gamma\left( \lambda + 2s \right)} \frac{2^{\lambda + 2s}}{|x_{12}|^{\lambda_{12} + \Delta_{34}}|x_{14}|^{\Delta_{12} - \Delta_{34}}|x_{24}|^{\Delta_{21} - \Delta_{34}}|x_{34}|^{2\lambda_3}}. \]

The path of integration \( \mathcal{C} \) in (2.9) (see the last paragraph of Appendix B for a more precise description) is taken to be parallel to the imaginary \( s_1 \)-axis and is deformed if
necessary to separate the poles of ascending sequences (e.g. those of $\Gamma(\frac{\Delta_{12}}{2} - s)$) from the poles of descending sequences (e.g. those of $\Gamma(\frac{\Delta_{12}}{2} + s)$) of the integrand.

Plugging the expression for $J$ into (2.7), using the duplication formula for Gamma functions,

$$\Gamma(\lambda + 2s) = \frac{1}{2\pi i} 2^{\lambda + 2s} \Gamma(\frac{\lambda + 1}{2} + s) \Gamma(\frac{\lambda}{2} + s)$$

and regrouping the terms in the integrand, we find,

$$S_\lambda = C_3 \frac{1}{2\pi i} \int_C ds_1 \xi^{-s_1} \Gamma(\frac{\lambda_{12}}{2} - s_1) \Gamma(\frac{\lambda_{34}}{2} - s_1) F(\frac{\Delta_{34}}{2} + s_1, \frac{\Delta_{12}}{2} + s_1; 2s_1; 1 - \frac{\eta}{\xi})$$

$$\times \frac{\Gamma(\frac{\Delta_{34}}{2} + s_1) \Gamma(\frac{\Delta_{12}}{2} + s_1) \Gamma(\frac{\Delta_{34}}{2} + s_1) \Gamma(\frac{\Delta_{34}}{2} + s_1)}{\Gamma(2s_1)} I_1$$

with

$$I_1 = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} ds \Gamma(-s)(-1)^s \frac{\Gamma(\frac{\xi_{12}}{2} + s) \Gamma(\frac{\xi_{34}}{2} + s) \Gamma(\frac{\lambda}{2} + s - s_1)}{\Gamma(s - s_1 + \frac{\lambda_{12} + \xi_{34}}{2}) \Gamma(\nu + 1 + s)}$$

(2.11)

$$I_1$$ is nothing but the Mellin-Barnes representation of the generalised hypergeometric function $3F_2$ [24], which leads to,

$$I_1 = \frac{\Gamma(\frac{\xi_{12}}{2}) \Gamma(\frac{\xi_{34}}{2})}{\Gamma(\nu + 1)} \frac{\Gamma(\frac{\lambda}{2} - s_1)}{\Gamma(-s_1 + \frac{\lambda_{12} + \xi_{34}}{2})} 3F_2(\frac{\xi_{12}}{2}, \frac{\xi_{34}}{2}, \frac{\lambda}{2} - s_1; \frac{\lambda_{12} + \xi_{34}}{2} - s_1, \nu + 1; 1)$$

(2.12)

When the parameters of a generalised hypergeometric function $3F_2(a, b, c; e, f; z)$ satisfy the relation $e + f = a + b + c + 1$, the series will be said to be Saalschutzian. It is easy to check that the hypergeometric series in (2.13) is Saalschutzian. Saalschutz’s theorem states that $3F_2(a, b, c; e, f; z)$ satisfies,

$$3F_2(a, b, c; e, f; 1) = \frac{\Gamma(e) \Gamma(1 + a - f) \Gamma(1 + b - f) \Gamma(1 + c - f)}{\Gamma(1 - f) \Gamma(e - a) \Gamma(e - b) \Gamma(e - c)}$$

(2.14)

provided $e + f = a + b + c + 1$ and $a, b$ or $c$ is a negative integer.

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6 If necessary, the integration path in (2.12) should be deformed to separate the poles $s = 0, 1, \cdots$ from those poles in descending series.

7 The hypergeometric series $3F_2(a, b, c; e, f; z)$ converges when $|z| < 1$, also when $z = 1$ provided that $Re(e + f - a - b - c) > 0$. Thus we see a Saalschutzian series is convergent at $z = 1$.  

9
From eqs (2.11) and (2.13), we reach the final expression for $S_\lambda$,

$$S_\lambda = C \frac{1}{2\pi i} \int_C ds_1 \xi^{-s_1} \Gamma\left(\frac{\lambda_{12}}{2} - s_1\right) \Gamma\left(\frac{\lambda_{34}}{2} - s_1\right) \Gamma\left(\frac{\lambda}{2} - s_1\right) F\left(\frac{\Delta_{34}}{2} + s_1, \frac{\Delta_{12}}{2} + s_1; 2s_1; 1 - \frac{\eta}{\xi}\right)$$

$$\times \frac{\Gamma\left(\frac{\Delta_{34}}{2} + s_1\right) \Gamma\left(\frac{\Delta_{12}}{2} + s_1\right) \Gamma\left(\frac{\Delta_{21}}{2} + s_1\right)}{\Gamma\left(\frac{\lambda_{12} + \xi_{34}}{2} - s_1\right) \Gamma\left(2s_1\right)} \ _3F_2\left(\frac{\xi_{12}}{2}, \frac{\xi_{34}}{2}, \frac{\lambda_{12} + \xi_{34}}{2}; -s_1, \nu+1; 1, \nu+1; 1\right)$$

(2.15)

with

$$C = \frac{1}{8\pi \frac{d}{2} \Gamma(\nu_1) \Gamma(\nu_2) \Gamma(\nu_3) \Gamma(\nu_4) \Gamma(\nu + 1)} \frac{1}{|x_{12}|^{\xi_{12}} |x_{14}|^{\xi_{34}} |x_{21}|^{\Delta_{21}} |x_{23}|^{\Delta_{23}} |x_{34}|^{\lambda_{34}}}$$

Thus we have been able to reduce (2.2) to a single inverse Mellin type of integral in (2.15), which may be evaluated by choosing the appropriate contour in the complex $s_1$ plane and the calculus of residues.

Let us consider the $s-$channel OPE limit where $x_{12}, x_{34}$ are much smaller than other distances, i.e., $\xi, \eta \gg 1$ and $1 - \frac{\eta}{\xi} \ll 1$. In this case, we can take the integration path $C$ over a contour enclosing the right half plane and the integral is given by the sum of the residues of the integrand at the poles of ascending sequences.

On right half plane we have three pole series which come from $\Gamma\left(\frac{\lambda_{12}}{2} - s_1\right)$, $\Gamma\left(\frac{\lambda_{34}}{2} - s_1\right)$ and $\Gamma\left(\frac{\lambda}{2} - s_1\right)$ respectively,

(1) $s_1 = \frac{\lambda}{2} + n, \ m = 0, 1, 2, \cdots;$$
(2) s_1 = \frac{\lambda_{12}}{2} + n, \ m = 0, 1, 2, \cdots;$$
(3) s_1 = \frac{\lambda_{34}}{2} + n, \ m = 0, 1, 2, \cdots.$

(2.16)

We first consider the case that no pole series in above coincide with one another, i.e., none of $\frac{\xi_{12}}{2}, \frac{\xi_{34}}{2}$ and $\frac{\lambda_{12} - \lambda_{34}}{2}$ is an integer. Then we can write $S_\lambda$ as,

$$S_\lambda = \sum_{n=0}^{\infty} S_n^{(\lambda)} + \sum_{n=0}^{\infty} S_n^{(\lambda_{12})} + \sum_{n=0}^{\infty} S_n^{(\lambda_{34})}$$

where $S_n^{(\lambda)}$, $S_n^{(\lambda_{12})}$ and $S_n^{(\lambda_{34})}$ are the contributions from the n-th pole in each series.
2.1. series 1

Let us first look at the pole series \( s_1 = \frac{1}{2} + n \). In this case the third parameter in \( 3F_2 \left( \frac{\lambda_{12}}{2}, \frac{\lambda_{34}}{2}, s + \frac{\lambda_{12}}{2}; -s + \frac{\lambda_{12}}{2}; -s + \frac{\lambda_{12} + \lambda_{34}}{2}, \nu + 1; 1 \right) \) becomes a negative integer and we can use the Saalschütz’s theorem (2.14) to get,

\[
3F_2 \left( \frac{\lambda_{12}}{2}, \frac{\lambda_{34}}{2}, -n; -s + \frac{\lambda_{12}}{2}; -s + \frac{\lambda_{12} + \lambda_{34}}{2}, \nu + 1; 1 \right)
= \frac{\Gamma(\lambda_{12} + \lambda_{34} - n)}{\Gamma(-n)} \frac{\Gamma\left(\frac{\lambda_{12}}{2} - n\right) \Gamma\left(\frac{\lambda_{34}}{2} - n\right)}{\Gamma\left(\frac{\lambda_{12} + \lambda_{34}}{2} - n\right) \Gamma\left(\frac{\lambda_{12}}{2} - n\right) \Gamma\left(\frac{\lambda_{34}}{2} - n\right) \Gamma(1 + n + \nu)} \tag{2.17}
\]

where in the second identity we have used the relation: \( \Gamma(x) \Gamma(1 - x) = \frac{\pi}{\sin \pi x} \).

Now plugging (2.17) into (2.15), we get

\[
S^{(\lambda)}_n = A^{(\lambda)} \frac{\xi^{-n} \Gamma\left(\frac{\lambda_{12}}{2} - n\right) \Gamma\left(\frac{\lambda_{34}}{2} - n\right) \Gamma\left(\lambda_{12} + \lambda_{34} - n\right)}{\Gamma\left(\lambda_{12} + 2n\right) \Gamma(\nu + n + 1)} \Gamma(\nu_1) \Gamma(\nu_2) \Gamma(\nu_3) \Gamma(\nu_4) \frac{1}{|x_1|^\lambda |x_2|^\nu |x_3|^\eta |x_4|^{1-\eta}} \tag{2.18}
\]

with

\[
A^{(\lambda)} = \frac{1}{8\pi^{\frac{1}{2}d}} \frac{\Gamma\left(\frac{\lambda_{12}}{2}\right) \Gamma\left(\frac{\lambda_{34}}{2}\right) \Gamma\left(\frac{\lambda_{12}}{2} + \frac{\lambda_{34}}{2}\right) \Gamma\left(\frac{\lambda_{12}}{2} + n\right)}{\Gamma\left(\lambda_{12} + 2n\right) \Gamma(\nu_1) \Gamma(\nu_2) \Gamma(\nu_3) \Gamma(\nu_4)} \tag{2.20}
\]

Note that \( \Gamma(-s) \) has residue \((-1)^{n-1}/n!\) at its pole \( s = n \).

2.2. series 2

In this case we have \( s_1 = \frac{\lambda_{12}}{2} + n \), and

\[
S^{(\lambda_{12})}_n = A^{(\lambda_{12})} \frac{(-1)^n \xi^{-n} G_n \Gamma\left(\frac{\lambda_{12} + \Delta_{34}}{2} + n, \lambda_1 + n; \lambda_{12} + 2n; 1 - \frac{\eta}{\xi}\right) \times}{\Gamma\left(\frac{\lambda_{34} - \lambda_{12}}{2} - n\right)} \frac{\Gamma\left(\frac{\lambda_{12} + \lambda_{34} - \Delta_{34}}{2} + n\right) \Gamma\left(\lambda_{12} + \Delta_{34} - n\right) \Gamma\left(\lambda_{12} + \Delta_{34} - n\right) \Gamma\left(\lambda_{12} + n\right)}{\Gamma\left(\lambda_{12} + 2n\right)} \tag{2.19}
\]

with

\[
A^{(\lambda_{12})} = \frac{1}{8\pi^{\frac{1}{2}d}} \frac{\Gamma\left(\frac{\lambda_{34} + \lambda_{12} - d}{2}\right) \Gamma\left(\frac{\lambda_{12} + \Delta_{34}}{2} + n\right) \Gamma\left(\lambda_{12} + \Delta_{34} - n\right) \Gamma\left(\lambda_{12} + n\right) \Gamma\left(\lambda_{12} + \Delta_{34} - n\right) \Gamma\left(\lambda_{12} + \Delta_{34} - n\right) \Gamma\left(\lambda_{12} + n\right)}{\Gamma\left(\lambda_{12} + 2n\right) \Gamma(\nu_1) \Gamma(\nu_2) \Gamma(\nu_3) \Gamma(\nu_4) |x_1|^{2\lambda_1} |x_2|^\Delta_{21-34} |x_3|^{\lambda_{12}+\Delta_{34}} |x_4|^{\lambda_{34}-\lambda_{12}}} \tag{2.20}
\]
and
\[ G_n = \frac{\Gamma(\tilde{\omega}_{12}/2)\Gamma(\tilde{\omega}_{34}/2)\Gamma(-\tilde{\omega}_{12}/2 - n)}{\Gamma(\tilde{\omega}_{34}/2 - n)\Gamma(n + 1)} 3F_2\left(\tilde{\omega}_{12}/2, \tilde{\omega}_{34}/2, -\tilde{\omega}_{12}/2 - n; n, \nu + 1; 1\right) \]

\[ = \sum_{m=0}^{\infty} \frac{1}{m!} \frac{\Gamma(\tilde{\omega}_{12}/2 + m)\Gamma(\tilde{\omega}_{34}/2 + m)\Gamma(m - \tilde{\omega}_{12}/2 - n)}{\Gamma(\tilde{\omega}_{34}/2 - n + m)\Gamma(n + 1 + m)} \] (2.21)

By a transformation of $3F_2$ [24], (2.21) can be written in a form symmetric under $\lambda \to \tilde{\lambda}$ and given by a terminating series,
\[ G_n = -\frac{1}{\tilde{\omega}_{12}/2} 3F_2\left(\frac{\lambda_{12} + \lambda_{34} - d}{2}, -n; 1 + \tilde{\omega}_{12}/2, 1 + \frac{\epsilon_{12}}{2}, 1\right) \]
\[ = -\frac{\Gamma(\tilde{\omega}_{12}/2)\Gamma(\tilde{\omega}_{34}/2)}{\Gamma(\lambda_{12} + \lambda_{34} - d/2)} \sum_{m=0}^{n} (-1)^m \frac{n!}{(n - m)!} \frac{\Gamma(\lambda_{12} + \lambda_{34} - d/2 + m)}{\Gamma(1 + \frac{\epsilon_{12}}{2} + m)\Gamma(1 + \frac{\epsilon_{12}}{2} + m)}. \] (2.22)

The contribution from poles in series (3) can be obtained from (2.19) - (2.21) by exchanging 1, 2 and 3, 4.

2.3. Coinciding poles

When $\frac{\epsilon_{12}}{2}$, $\frac{\epsilon_{34}}{2}$ or $\frac{\lambda_{12} - \lambda_{34}}{2}$ become integers, the poles from different series in (2.16) may merge into double or triple poles. For example, when $\frac{\lambda_{12} - \lambda_{34}}{2}$ is an integer, apart from a finite number of them, all poles in series two and three in (2.16) will merge into double poles, while the poles in the first series remain untouched. The contribution from a double pole is given by the derivative of the integrand of (2.15). The expressions are quite complicated and we do not explicitly write them down here. We simply note that there will be terms proportional to $\ln \xi$ as a result of $\partial \xi^{-s}/\partial s = -\log \xi \xi^{-s}$. If all three parameters are integers, then apart from a finite number of simple and double poles all poles may merge into triple poles and their contributions are given by the second derivative of the integrand of (2.15). In these cases, among other things, we will have terms proportional to $(\log \xi)^2$ from the second derivative of $\xi^{-s}$.

We caution that in certain range of parameters, $3F_2$ in (2.15) may develop zeros at the poles and the pole structure may be different from what we naively read from (2.17). This happens when $\frac{\epsilon_{12}}{2}$ or $\frac{\epsilon_{34}}{2}$ is a positive integer.\footnote{The following discussion is partly motivated by the results in [25], where simplifications in some expressions in this range of parameters have been observed. I would like to thank D. Freedman for correspondence regarding this issue.} As an example let us take $\frac{\epsilon_{12}}{2} = k + 1$
with \( k \geq 0 \) an integer. By a transformation \([24]\) of generalized hypergeometric functions, \( _3F_2 \) in eq (2.15) can be rewritten as,

\[
_3F_2\left(\frac{\tilde{\epsilon}_{12}}{2}, \frac{\tilde{\epsilon}_{34}}{2}, \frac{\lambda}{2} - s_1, \frac{\lambda_{12} + \tilde{\epsilon}_{34}}{2} - s_1, \nu + 1; 1\right)
= \frac{\Gamma\left(\frac{\lambda_{12} + \tilde{\epsilon}_{34}}{2} - s_1\right)}{\Gamma\left(\frac{\lambda_{12}}{2} - s_1\right)\Gamma\left(1 + \frac{\tilde{\epsilon}_{34}}{2}\right)} _3F_2\left(\frac{\tilde{\epsilon}_{34}}{2}, 1 + \frac{\lambda - d}{2} + s_1, 1 - \frac{\epsilon_{12}}{2}; 1 + \frac{\tilde{\epsilon}_{34}}{2}, \nu + 1; 1\right).
\]

(2.23)

Since \( 1 - \frac{\epsilon_{12}}{2} = -k \), \( _3F_2\left(\frac{\tilde{\epsilon}_{34}}{2}, 1 + \frac{\lambda - d}{2} + s_1, 1 - \frac{\epsilon_{12}}{2}; 1 + \frac{\tilde{\epsilon}_{34}}{2}, \nu + 1; 1\right) \) on the right hand side of (2.23) is given by a terminating series,

\[
_3F_2\left(\frac{\tilde{\epsilon}_{34}}{2}, 1 + \frac{\lambda - d}{2} + s_1, 1 - \frac{\epsilon_{12}}{2}; 1 + \frac{\tilde{\epsilon}_{34}}{2}, \nu + 1; 1\right) = \sum_{m=0}^{k} \frac{(-1)^m k!}{m!(k-m)!} \frac{\Gamma(\nu+1)}{\Gamma(1+\frac{\lambda-d}{2}+s_1+m)} \frac{\Gamma(1+\frac{\lambda-d}{2}+s_1+m)}{\Gamma(\frac{\lambda_{34}}{2}+s_1)} F\left(\frac{\Delta_{34}}{2}+s_1, \frac{\Delta_{12}}{2}+s_1; 2s_1; 1-\eta \xi\right).
\]

(2.24)

from which we can see that it is convergent and has no poles inside the contour in (2.15).

Plugging (2.23) into (2.15), we get,

\[
S_{\lambda} = \frac{C}{\Gamma(1+\frac{\tilde{\epsilon}_{34}}{2})} \frac{1}{2\pi i} \int_C ds_1 \xi^{-s_1} \Gamma\left(\frac{\lambda_{34}}{2} - s_1\right) \Gamma\left(\frac{\lambda}{2} - s_1\right) H(\lambda, s_1)
\times _3F_2\left(\frac{\tilde{\epsilon}_{34}}{2}, 1 + \frac{\lambda - d}{2} + s_1, 1 - \frac{\epsilon_{12}}{2}; 1 + \frac{\tilde{\epsilon}_{34}}{2}, \nu + 1; 1\right)
\]

(2.25)

where \( C \) is given below eq (2.15) and \( H(\lambda, s_1) \) is defined by,

\[
H(\lambda, s_1) = \frac{\Gamma\left(\frac{\Delta_{34}}{2}+s_1\right)\Gamma\left(\frac{\Delta_{12}}{2}+s_1\right)\Gamma\left(\frac{\Delta_{21}}{2}+s_1\right)}{\Gamma(2s_1)} F\left(\frac{\Delta_{34}}{2}+s_1, \frac{\Delta_{12}}{2}+s_1; 2s_1; 1-\eta \xi\right).
\]

Naively we may expect from (2.15) that there are double poles at \( s_1 = \frac{\Delta_{12}}{2} + n, n = 0, 1, \cdots \). However, eq (2.25) indicates that they are actually simple poles. This result may also be seen indirectly from eqs (2.18) and (2.19) - (2.22): there is no singularity developed in either (2.18) or (2.19) when \( \frac{\epsilon_{12}}{2} \) approaches a positive integer. In fact it can be checked that the residue of the integrand of (2.23) at a pole \( s_1 = \frac{\Delta_{12}}{2} + n \) is equal to the sum of (2.18) and (2.19) at the corresponding pole. If further \( \frac{\epsilon_{12}}{2} \) is an integer, then from (2.23) the pole at \( s_1 = \frac{\lambda}{2} + k + 1 + m = \frac{\lambda_{12}}{2} + m = \frac{\lambda_{34}}{2} + n \) (\( m \) and \( n \) non-negative integers).
is a double pole instead of a triple pole. In particular, there is no terms proportional to 

$$ (\log \xi)^2 $$

here [25]. Similar analysis can be applied when $c_{ij}^{\pm}$ is a positive integer.

The appearance of logarithm in coinciding pole cases can be summarized as follows:

(i) Only one of $c_{ij}^{\pm}$, $c_{ij}^{\pm}$ or $\frac{\lambda_{ij} - \lambda_{kl}}{2}$ is an integer:

(i.1) $\frac{\lambda_{ij} - \lambda_{kl}}{2}$ is an integer: $\log \xi$ associated with the double poles at $s_1 = \frac{\lambda_{ij}}{2} + n$.

(i.2) $c_{ij}^{\pm}$ or $c_{ij}^{\pm}$ is a positive integer: all poles are simple poles, no logarithm.

(i.3) $c_{ij}^{\pm}$ or $c_{ij}^{\pm}$ is zero or a negative integer: $\log \xi$ associated with the double poles at $s_1 = \frac{\lambda_{ij}}{2} + n$.

(ii) $c_{ij}^{\pm}$, $c_{ij}^{\pm}$ and $\frac{\lambda_{ij} - \lambda_{kl}}{2}$ are all integers:

(ii.1) at least one of $c_{ij}^{\pm}$ and $c_{ij}^{\pm}$ is positive: except for a finite number of simple poles, all poles are double poles with $\log \xi$.

(ii.2) $c_{ij}^{\pm}$ and $c_{ij}^{\pm}$ are zero or negative: except for a finite number of them, all poles are triple poles with $(\log \xi)^2$.

3. Scattering amplitudes from generic vertices and contact terms

Scattering amplitudes from more complicated interaction vertices such as $\phi \partial_{\mu} \phi_1 \partial^\mu \phi_2$ and $\phi D_{\mu} \partial_{\mu} \phi_1 D_{\mu} \partial^\mu \phi_2$ can be reduced to (2.2) and contact-type interactions by integration by part [17] or field redefinitions [14]. For example, the amplitude resulting from vertices $\phi \partial \phi_1 \partial \phi_2$ and $\phi \phi_3 \phi_4$ can be written as ($d\alpha$ and $d\beta$ denote the integration measures as in (2.2)),

$$ \int d\alpha \, d\beta \, \partial K_1 \partial K_2 \, G(x, y) \, K_3 K_4 $$

$$ = \frac{1}{2} \int d\alpha \, d\beta \, \left[ \partial^2 (K_1 K_2) - \partial^2 K_1 K_2 - \partial^2 K_2 K_1 \right] G(x, y) \, K_3 K_4 \quad (3.1) $$

$$ = -\frac{1}{2} \int d\alpha \, d\beta \, K_1 K_2 K_3 K_4 + \frac{1}{2} (m^2 - m_1^2 - m_2^2) \int d\alpha \, d\beta \, K_1 K_2 G(x, y) \, K_3 K_4 $$

Note the coefficient of the second term $\frac{1}{2} (m^2 - m_1^2 - m_2^2)$ is precisely the ratio between coefficients of $< \Phi_{\lambda} \Phi_{\lambda'} \Phi_{\lambda''} >$ calculated from two types of interactions $\phi \partial \phi_1 \partial \phi_2$ and $\phi \phi_1 \phi_2$ [10].

In general we can consider the following Lagrangian of scalar fields,

$$ \mathcal{L} = \frac{1}{2} (\partial \phi_i)^2 + \frac{1}{2} m_i^2 \phi_i^2 + A_{ijk}^{(0)} \phi_i \phi_j \phi_k + A_{ijk}^{(1)} \phi_i \phi_j D_{\mu} \phi_k + \cdots + A_{ijk}^{(n)} \phi_i D^{(n)} \phi_j D^{(n)} \phi_k \quad (3.2) $$
where $D^{(m)} \phi_i$ is defined by
\[ D^{(m)} \phi_i = D_{\mu_1} D_{\mu_2} \cdots D_{\mu_m} \phi_i. \] (3.3)
\[
\{ \} \text{ in (3.3) denotes that the indices are symmetrized and traces are removed.}
\]
For the purpose of tree-level four-particle scattering we can eliminate those vertices with derivatives by a field redefinition,
\[ \phi_i = \phi_i' + B_{ijk}^{(0)} \phi_j' \phi_k' + \cdots + B_{ijk}^{(n-1)} D^{(n-1)} \phi_j' D^{(n-1)} \phi_k'. \] (3.4)

$B$’s in (3.4) can be found by plugging (3.4) into (3.2) and setting to zero the coefficients of the cubic derivative vertices. The resulting Lagrangian can be written as,
\[ \mathcal{L} = \frac{1}{2} (\partial \phi_i')^2 + \frac{1}{2} m_i^2 (\phi_i')^2 + \lambda_{ijk} \phi_i' \phi_j' \phi_k' + \text{contact vertices of quartic or higher order} \] (3.5)

For example for $n = 2$ in AdS_{d+1}, $B$’s can be found to be [14],
\[ B_{ijk}^{(1)} = \frac{1}{2} A_{ijk}^{(2)}; \quad B_{ijk}^{(0)} = \frac{1}{2} A_{ijk}^{(1)} + \frac{1}{4} A_{ijk}^{(2)} (m_i^2 - m_j^2 - m_k^2 + 2d) \]

and
\[ \lambda_{ijk} = A_{ijk}^{(0)} + B_{ijk}^{(0)} (m_i^2 - m_j^2 - m_k^2) - \frac{2}{d+1} m_j^2 m_k^2 B_{ijk}^{(1)}. \] (3.6)

Thus for generic interactions, the scattering amplitude can be written as,
\[ A_\lambda = S_\lambda + S^{(4)} \] (3.7)

where $S_\lambda$ is given by (2.15) with normalised vertices (3.6) and $S^{(4)}$ is given by quartic vertices in (3.3).

Let us now look at the contribution from contact terms. We observe that by repeatedly using the identity $(J_{\mu\nu}(x) = \delta_{\mu\nu} - 2x^\mu x^\nu/|x|^2)$,
\[ D^\mu K_{\lambda_i}(u, x_i) D_\mu K_{\lambda_j}(u, x_j) = c_{\lambda_i} c_{\lambda_j} u_0^2 \partial_\mu \left( \frac{u_0}{|u - x_i|^2} \right) \lambda_i \partial_\mu \left( \frac{u_0}{|u - x_j|^2} \right) \lambda_j \]
\[ = \lambda_i \lambda_j \ K_{\lambda_i}(u, x_i) K_{\lambda_j}(u, x_j) J_{\mu 0}(u - x_i) J_{\mu 0}(u - x_j) \]
\[ = \lambda_i \lambda_j \ K_{\lambda_i}(u, x_i) K_{\lambda_j}(u, x_j) - 2 \nu_i \nu_j x_{ij}^2 K_{\lambda_i+1}(u, x_i) K_{\lambda_j+1}(u, x_j) \] (3.8)

\[ \text{We can use } D^2 \phi = m^2 \phi + \cdots \text{ to reduce terms containing traces of indices to lower order terms. Similarly the commutators of derivatives } [D_\mu, D_\nu] \propto R \text{ also reduce to lower order terms, where } R \text{ is the constant curvature.} \]
and $D^2 \mathcal{K}_i = m_i^2 \mathcal{K}_i$, we can put a generic quartic contribution into a sum of terms without derivatives,

\[
S_c = \int \frac{dudv}{u_0^{d+1}} \mathcal{K}_{\lambda_1}(u, x_1)\mathcal{K}_{\lambda_2}(u, x_2)\mathcal{K}_{\lambda_3}(u, x_3)\mathcal{K}_{\lambda_4}(u, x_4)
= \Pi_i \mathcal{K}_i \int \frac{dudv}{u_0^{d+1}} \left( \frac{u_0}{|u-x_1|^2} \right)^{\lambda_1} \left( \frac{u_0}{|u-x_2|^2} \right)^{\lambda_2} \left( \frac{u_0}{|u-x_3|^2} \right)^{\lambda_3} \left( \frac{u_0}{|u-x_4|^2} \right)^{\lambda_4}.
\]

(3.9)

Thus it is enough to look at (3.9).

Contact contribution (3.9) in AdS$_{d+1}$ have been discussed before in [8] and [18] (see also [20]). In particular in [18] it was pointed out that when $\lambda_{12} = \lambda_{34}$ the leading term in short distance limit $x_{12}, x_{34} \to 0$ is given by a logarithmic contribution. Here we give a more thorough analysis of the analytic properties of (3.9), presenting the result in a way suitable for our later discussion of its CFT$_d$ interpretation.

In Appendix C, we show that similarly to the exchange amplitude, the contact contribution (3.9) can also be written as an inverse Mellin integral,

\[
S_c = C_c \frac{1}{2\pi i} \int \frac{ds}{s} \xi^{-s} \Gamma(\frac{\lambda_{12}}{2} - s) \Gamma(\frac{\lambda_{34}}{2} - s) F(\frac{\Delta_{34}}{2} + s, \frac{\Delta_{12}}{2} + s; 2s; 1 - \eta \xi) \times \frac{\Gamma(\frac{\Delta_{34}}{2} + s)\Gamma(\frac{\Delta_{12}}{2} + s)\Gamma(\frac{\Delta_{21}}{2} + s)}{\Gamma(2s)}\]

(3.10)

with

\[
C_c = \frac{1}{2\pi^{3d/2}} \frac{\Gamma(\frac{\lambda_{12} + \lambda_{34} - d}{2})}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\nu_3)\Gamma(\nu_4) |x_{12}|^{\lambda_{12}} |x_{14}|^{\lambda_{12}} |x_{24}|^{\lambda_{12}} |x_{23}|^{\lambda_{34}} |x_{34}|^{\lambda_{34}}}
\]

where the integration path $C$ should be understood in the same sense as that in (2.9) (see the remark below (2.9)). Thus in the $s$-channel limit $\eta, \xi \gg 1$, (3.10) can be written as,

\[
S_c = \sum_{n=0}^{\infty} S^c_{\lambda_{12}} + \sum_{n=0}^{\infty} S^c_{\lambda_{34}}
\]

where,

\[
S^c_{\lambda_{12}} = 4 A^{\lambda_{12}} (-1)^n n! \xi^{-n} \frac{\Gamma(\frac{\lambda_{12} + \Delta_{34}}{2} + n, \lambda_1 + n; \lambda_{12} + 2n; 1 - \eta \xi)}{\Gamma(\lambda_{12} + 2n)}
\]

\[
\Gamma(\frac{\lambda_{34} - \lambda_{12}}{2} - n) \frac{\Gamma(\frac{\lambda_{12} + \Delta_{34}}{2} + n)\Gamma(\frac{\lambda_{12} - \Delta_{34}}{2} + n)\Gamma(\lambda_1 + n)\Gamma(\lambda_2 + n)}{\Gamma(\lambda_{12} + 2n)}
\]

(3.11)
and $S_{cn}^{(\lambda_{34})}$ can be obtained from (3.11) by taking $1, 2 \rightarrow 3, 4$. Note $A^{(\lambda_{12})}$ in above is given by eq (2.20) and except for the extra $G_n$ in (2.19), (3.11) is almost identical to (2.19).

When $\lambda_{12} = \lambda_{34}$ is an integer, except for a finite number of poles the two ascending simple-pole sequences of the integrand in (3.10) will merge into a double-pole sequence. Again as in the case of exchange amplitude, the double-pole contribution will contain $\log \xi$. In particular, when $\lambda_{12} = \lambda_{34}$ all ascending poles become double poles and the leading contribution contains a $\log \xi$.

Note that since (3.9) is symmetric under exchanges of its four boundary propagators, its expansion in $u$–channel limit $x_{13}, x_{24} \rightarrow 0$ can be obtained by exchanging 2 and 3 in (3.10) and (3.11) and $\xi \rightarrow \frac{\xi}{\eta}$ and $\eta \rightarrow \frac{1}{\eta}$.

4. Four-point functions and conformal partial wave expansion in CFT

To seek a CFT$_d$ interpretation of the AdS$_{d+1}$ amplitudes discussed in last two sections, in this section we review the conformal partial wave expansion (CPWE) approach to the calculation of four-point functions in CFT [26,27] (for a review see [28,29], see also [30] for some recent discussions).

In CFT$_d$, the states generated by acting by a product of the conformal operators on the vacuum can be decomposed into a direct sum of irreducible representations of the conformal group,

$$\Phi_1(x_1)\Phi_2(x_2)|0> = \sum_k \int \! d^d x \, Q_{12k}(x|x_1, x_2)|k, x > ,$$  \hspace{1cm} (4.1)

where $k$ sums over all the irreducible representations in the Hilbert space and states $|k, x > = \Phi_k(x)|0> \Phi_k(x)$ span the space of an irreducible representation of the conformal group. (4.1) can be further lifted into an operator equation,

$$\Phi_1(x_1)\Phi_2(x_2) = \sum_k \int \! d^d x \, Q_{12k}(x|x_1, x_2)\Phi_k(x) ,$$  \hspace{1cm} (4.2)

understood as a relation between correlation functions. The summation in (4.2) is over primary fields (non-derivatives) only and the integration over all space effectively incorporated

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10 I would like to thank A. Petkou for bringing these references to my attention.
the contribution of their $SO(d,2)$ descendants (fields with derivatives). The short-distance OPE can be obtained from (4.2) in small $|x_{12}|$ limit by expanding the integrand in terms of $x_1 - x_2$. When $\Phi$’s are orthogonal to each other, it can be seen from (4.1) that $Q$’s are given by the amputated three-point functions.

Applying (4.1) to a four-point function we find

$$W_{1234}(x_1, x_2, x_3, x_4) = \langle 0 | \Phi_1(x_1)\Phi_2(x_2)\Phi_3(x_3)\Phi_4(x_4) | 0 \rangle > = \sum_k \int d^d x d^d y \; Q_{12k}(x_1, x_2|x) \; W_k(x - y) \; Q_{k34}(y|x_3, x_4). \tag{4.3}$$

where $W_k(x - y) = \langle 0 | \Phi_k(x)\Phi_k(y) | 0 \rangle >$.

In the following, we shall look at the contribution of an intermediate scalar operator $\Phi_\lambda$ (with dimension $\lambda$) to the four-point function of four scalar operators $\Phi_{\lambda_i}, i = 1, \cdots, 4$ (with dimensions $\lambda_i$ respectively),

$$S_\lambda = \int d^d x d^d y \; Q_{\lambda_1, \lambda_2}(x_1, x_2|x) \; W_\lambda(x - y) \; Q_{\lambda_3, \lambda_4}(y|x_3, x_4). \tag{4.4}$$

In Euclidean signature, the two- and three-point functions are given by,

$$G_\lambda(x - y) = \langle \Phi_\lambda(x)\Phi_\lambda(y) \rangle > = \frac{c}{|x - y|^{2\lambda}},$$

$$G_{\lambda_1, \lambda_2}(x, x_1, x_2) = \langle \Phi_\lambda(x)\Phi_{\lambda_1}(x_1)\Phi_{\lambda_2}(x_2) \rangle > = f_{\lambda_1, \lambda_2} A_{\lambda_1, \lambda_2}(x, x_1, x_2),$$

$$G_{\lambda_3, \lambda_4}(x, x_3, x_4) = \langle \Phi_\lambda(x)\Phi_{\lambda_3}(x_3)\Phi_{\lambda_4}(x_4) \rangle > = f_{\lambda_3, \lambda_4} A_{\lambda_3, \lambda_4}(x, x_3, x_4)$$

where function $A_{abc}(x, y, z)$ is defined by,

$$A_{abc}(x, y, z) = \frac{1}{|x - y|^{a+b-c}|z - y|^{c+b-a}|x - z|^{a+c-b}}.$$

The normalisation constants $c$ and $f$’s will be taken to be those given by AdS calculations (4.5), i.e.

$$c = \frac{\Gamma(\lambda)}{\pi^{d/2}\Gamma(\nu)}(2\lambda - d), \quad f_{\lambda_1, \lambda_2} = \frac{\Gamma(\frac{4\nu_2}{2})\Gamma(\frac{\nu_1}{2})\Gamma(\frac{4\nu_1}{2})\Gamma(\frac{\nu_2}{2})}{2\pi^d \Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\nu)} \tag{4.5}$$

Here the normalisation for three-point functions is given by interaction vertex $\phi_1\phi_2\phi_3$ in $AdS_{d+1}$. When considering more complicated vertices, an additional normalisation factor (3.6) should be taken into account.
and a similar expression for $f_{\lambda\lambda_3\lambda_4}$ obtained from $f_{\lambda\lambda_1\lambda_2}$ by taking $1, 2 \to 3, 4$.

In Minkowski signature, due to the spectrality condition, it is more convenient to work in momentum space, where the two- and three-point functions are given by

$$W(p) = -i\text{Disc } G(p)|_{p^i = -ip^0} = \frac{2\pi}{\Gamma(1 + \nu)\Gamma(-\nu)} \theta(p^0) \theta(-p_{Min}^2) G(p)|_{p^i = -ip^0}$$

(4.6)

$$W(p|x_1, x_2) = -i\text{Disc } (p|x_1, x_2)|_{p^i = -ip^0}$$

(4.7)

where $G(p)$ and $G(p|x_1, x_2)$ are Euclidean two- and three-point functions in momentum space,

$$G(p) = c \int d^d y e^{-ip\cdot y} \frac{1}{|y|^{2\lambda}} = c \frac{\pi^{d/2} \Gamma(-\nu)}{2^{2\nu} \Gamma(\lambda)} p^{2\nu}$$

$$G_{\lambda\lambda_1\lambda_2}(p|x_1, x_2) = \int d^d y e^{-ip\cdot y} G_{\lambda\lambda_1\lambda_2}(y, x_1, x_2) = f_{\lambda\lambda_1\lambda_2} \frac{2\pi^{d/2}}{\Gamma(\frac{d}{2})\Gamma(\frac{d}{2} + \nu)} \frac{1}{x_{12}^{1/2}(4x_{12})^{d/2}} \times$$

$$\int_0^1 du \Delta_{12}^{\frac{d}{2} + \frac{d}{2} - 1} (1 - u) \Delta_{21}^{\frac{d}{2} + \frac{d}{2} - 1} e^{-ip[ux_1 + (1 - u)x_2]} K_\nu(\sqrt{u(1 - u)p^2x_{12}^2})$$

(4.8)

The amputated three-point function $Q$ can then be found to be,

$$Q_{\lambda\lambda_1\lambda_2}(p|x_1, x_2) = W_\lambda^{-1}(p) W_{\lambda\lambda_1\lambda_2}(p|x_1, x_2) = c^{-1} f_{\lambda\lambda_1\lambda_2} \frac{2^\nu \Gamma(\lambda)\Gamma(1 + \nu)}{\Gamma(\frac{d}{2})\Gamma(\frac{d}{2} + \nu)} \frac{p^{\nu - \nu}}{x_{12}^{1/2}} \times$$

$$\int_0^1 du \Delta_{12}^{\frac{d}{2} + \frac{d}{2} - 1} (1 - u) \Delta_{21}^{\frac{d}{2} + \frac{d}{2} - 1} e^{-ip[ux_1 + (1 - u)x_2]} I_\nu(\sqrt{u(1 - u)p^2x_{12}^2})$$

(4.9)

In (4.8) and (4.9) $K_\nu$ and $I_\nu$ are modified Bessel functions.

Plugging eqs (4.6) and (4.9) into (4.4), we have, in momentum space with Minkowskian signature,

$$W_\lambda = \frac{1}{2\pi^d} \int d^d p Q_{\lambda\lambda_1\lambda_2}(p|x_1, x_2) W_\lambda(p) Q_{\lambda\lambda_3\lambda_4}(p|x_3, x_4)$$

(4.10)

The integrals in (4.10) were explicitly computed in [26] and the result can written as an inverse Mellin integral,

$$W_\lambda = c^{-1} f_{\lambda\lambda_1\lambda_2} f_{\lambda\lambda_3\lambda_4} \frac{\Gamma(\lambda)\Gamma(1 + \nu)}{\Gamma(\frac{d}{2})\Gamma(\frac{d}{2} + s)\Gamma(\frac{d}{2} + s)} \frac{1}{x_{12}^{1/2} x_{14}^{1/2} x_{24}^{1/2} x_{23}^{1/2} x_{34}^{1/2}} \times$$

$$\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} ds (-\xi)^{-s} \Gamma(-s) \frac{\Gamma(\frac{d}{2} + s)\Gamma(\frac{d}{2} + s)\Gamma(\frac{d}{2} + s)}{\Gamma(\lambda + 2s)\Gamma(\nu + s + 1)} F\left(\frac{d}{2} + s, \frac{d}{2} + s; \lambda + 2s; 1 - \frac{\eta}{\xi}\right)$$

(4.11)
where $\xi$ and $\eta$ are cross ratios defined in (2.10) and the Mellin integral should be understood in the same sense as the ones in previous sections. Again when $\eta, \xi > 1$, (4.11) can be written as an expansion,

$$W_\lambda = \frac{1}{8\pi \frac{d}{2} d_{\nu 1} d_{\nu 2} d_{\nu 3} d_{\nu 4}} \times \
\sum_{n=0}^{\infty} \frac{1}{n!} \sum_{n=0}^{\infty} \frac{1}{\xi^n} \sum_{n=0}^{\infty} \frac{1}{\eta^n} \frac{\Gamma(\frac{\delta_1}{2} + n) \Gamma(\frac{\delta_2}{2} + n) \Gamma(\frac{\delta_3}{2} + n) \Gamma(\frac{\delta_4}{2} + n)}{\Gamma(\lambda + 2n) \Gamma(\nu + n + 1)} F\left(\frac{\delta_3}{2} + n, \frac{\delta_1}{2} + n; \lambda + 2n; 1 - \frac{\eta}{\xi}\right) \quad (4.12)$$

We notice that (4.12) agree precisely with (2.18) including the numerical coefficient.

5. CFT$_d$ interpretation of AdS$_{d+1}$ amplitudes

In previous sections, we have managed to express all our results as inverse Mellin integrals and when $\xi, \eta > 1$ write them in terms of inverse power series of $\xi, \eta$ as a sum of residues of the integrand. When the pole sequences in (2.16) and (3.10) do not coincide with one another, in all cases (see eqs (2.18),(2.19),(3.11)) the contribution from a pole sequence can be written in a similar pattern as the CPWE expression (4.12),

$$\sum_{n=0}^{\infty} a_n H(\Lambda + 2n) \quad (5.1)$$

where each term in the summation is given by the residue at the pole $\frac{\Lambda}{2} + n$. In (5.1), $a_n$ are numerical coefficients and $H$ is a function defined by,

$$H(\alpha) = \frac{1}{|x_{12}|^{\lambda_{12} - \alpha} |x_{14}|^{\alpha + \Delta_{12}} |x_{24}|^{\Delta_{21} - \Delta_{34}} |x_{23}|^{\alpha + \Delta_{34}} |x_{34}|^{\lambda_{34} - \alpha}} \times \frac{\Gamma(\frac{\Delta_{12}}{2} + \frac{\alpha}{2}) \Gamma(\frac{\Delta_{21}}{2} + \frac{\alpha}{2}) \Gamma(\frac{\Delta_{34}}{2} + \frac{\alpha}{2}) \Gamma(\frac{\Delta_{43}}{2} + \frac{\alpha}{2})}{\Gamma(\alpha)} F\left(\frac{\Delta_{34}}{2} + \frac{\alpha}{2}, \frac{\Delta_{12}}{2} + \frac{\alpha}{2}; \alpha; 1 - \frac{\eta}{\xi}\right) \quad (5.2)$$

In (4.12),

$$a_n = \frac{1}{8\pi \frac{d}{2} d_{\nu 1} d_{\nu 2} d_{\nu 3} d_{\nu 4}} \frac{\Gamma(\frac{\Delta_{12} - \lambda}{2}) \Gamma(\frac{\Delta_{21} - \lambda}{2}) \Gamma(\frac{\Delta_{34} + \Delta - d}{2}) \Gamma(\frac{\Delta_{43} + \Delta - d}{2})}{\Gamma(n + 1) \Gamma(\Delta - \frac{d}{2} + n + 1)} \quad (5.3)$$

The contact amplitude (3.9), may be written as,

$$S_c = S_c^{(\lambda_{12})} + S_c^{(\lambda_{34})} \quad (5.4)$$
where $S_c^{(\lambda_{12})}$, $S_c^{(\lambda_{34})}$ are of the form of (5.1) with $\Lambda = \lambda_{12}, \lambda_{34}$. For $S_c^{(\lambda_{12})}$,

$$a_n = \frac{1}{2\pi^{3d/2}} \frac{\Gamma(\lambda_{12} + \lambda_{34} - d)}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\nu_3)\Gamma(\nu_4)\Gamma(n + 1)} \frac{(-1)^n}{\Gamma(\lambda_{34} - \lambda_{12} + d - 2n)}$$

(5.5)

with those of $S_c^{(\lambda_{34})}$ given by (5.3) with $\lambda_{12} \leftrightarrow \lambda_{34}$. Similarly, the exchange amplitude (2.2) can be written in terms of (5.1) with $\Lambda = \lambda, \lambda_{12}, \lambda_{34}$,

$$S_{ex} = S^{(\lambda)} + S^{(\lambda_{12})} + S^{(\lambda_{34})}$$

(5.6)

where $a_n^{(\lambda)}$ are the same as those of CPWE (5.3) and

$$a_n^{(\lambda_{12})} = \frac{1}{2\pi^{3d/2}} \frac{\Gamma(\lambda_{12} + \lambda_{34} - d)}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\nu_3)\Gamma(\nu_4)\Gamma(n + 1)} \frac{(-1)^n}{\Gamma(\lambda_{34} - \lambda_{12} + d - 2n)} G_n$$

(5.7)

with $G_n$ given by (2.21) or (2.22). $a_n^{(\lambda_{34})}$ may be obtained from (5.7) with 1, 2 and 3, 4 exchanged.

To have a more precise picture of what we have found so far, we would like to understand the physical meaning of the poles and the residues of each pole. For this purpose, let us go back to the contribution of a primary operator $\Phi_\Lambda$ to the operator algebra (4.2),

$$\Phi_{\lambda_1}(x_1)\Phi_{\lambda_2}(x_2) = \int d^d x \; Q_{\lambda_1,\lambda_2}(x|\lambda_1, \lambda_2)\Phi_\Lambda(x),$$

(5.8)

where

$$Q_{\lambda_1,\lambda_2}(x|x_1, x_2) = \int d^d p \; e^{ip \cdot x} Q_{\lambda_1,\lambda_2}(p|x_1, x_2)$$

(5.9)

and $Q_{\lambda_1,\lambda_2}(p|x_1, x_2)$ is given by (4.9). For simplicity we will look at the analytic continuation of (4.9) to Euclidean space. Plugging (4.9) and (5.3) into (5.8), we find that

$$\Phi_{\lambda_1}(x_1)\Phi_{\lambda_2}(x_2) = \sum_{n=0}^{\infty} b_n \frac{1}{x_{12}^{\lambda_{12} - \lambda - 2n - d}} \int_0^1 du \; u^{\frac{d}{2} + n - 1} (1 - u)^{\frac{d}{2} + n - 1} e^{u x_{12} \cdot \partial} \Phi_\Lambda^{(n)}(x_2)$$

$$= \sum_{n=0}^{\infty} b_n F_n$$

(5.10)

\[12\] This is not the completely right thing to do, as in Minkowski signature integration over momenta involves the spectrality conditions $p_0 > 0, p^2 < 0$, in which case the expressions are rather complicated. For illustrative purpose, we shall use an Euclidean expression.
where \(b_n\) are numerical factors and we have defined operators,

\[
\Phi^{(n)}_\Lambda(x_2) = (\partial^2)^n \Phi(x_2), \quad n = 0, 1, 2, \ldots
\]  

(5.11)

which we will call sub-primary operators (\(n=0\) is primary). To reach (5.10), we have used the series expansion for Bessel function. \(F_n\) in (5.10) denotes the contribution to OPE from a sub-primary operator \(\Phi^{(n)}_\Lambda\) and its diagonal descendants. By diagonal descendants we mean the states in weight diagram diagonally generated from a sub-primary (see figure 5).

We may now interpret that each pole in (4.11) represents the dimension of a sub-primary and the residue at the pole corresponds to the contribution of the sub-primary and its diagonal descendants.

\[0, 1, 2, 3, 4, 5\]
\[\lambda_1^+, \lambda_2^+, \lambda_3^+, \lambda_4^+, \lambda_5^+\]

\[l\]

\[\lambda, \lambda+1, \lambda+2, \lambda+3, \lambda+4, \lambda+5\]

\[l=0, 1, 2, 3, 4, 5\]

**Figure 5:** Weight diagram for \(SO(d,2)\) representation \(D(\lambda,0)\). The horizontal axis \(l\) represents the set of indices for \(SO(d)\) subgroup, and the perpendicular axis represents the scale dimension. A sub-primary is a state lying on the line with \(l = 0\). The states connected to a sub-primary by dotted lines are descendants diagonally generated from the sub-primary.

By comparing (5.6) and (5.4) with CPWE (5.1) and (5.3), we see that for the case of non-coinciding poles:

i) \(S^{(\lambda)}\) in exchange amplitude (5.6), which arises from the first pole sequence \(\frac{\lambda}{2} + n\) in (2.16) agrees precisely with the result from conformal partial wave expansion (4.12), including the overall numerical coefficient. This indicates that \(S^{(\lambda)}\) has a CFT\(_d\) interpretation in terms of exchange of a primary operator of dimension \(\lambda\).

ii) In both exchange and contact amplitudes, the contributions from a pole at \(\frac{\lambda_{12}}{2} + n\) (or \(\frac{\lambda_{34}}{2} + n\)) agree exactly with that of a sub-primary operator with dimension \(\lambda_{12} + 2n\)
(λ_{34} + 2n) and its diagonal descendants. But the numerical coefficients (5.7) and (5.3) at each pole are different from that predicted by CPWE (5.3), in particular, in exchange amplitude the coefficient involves a somewhat complicated factor G_n (2.21).

The above identification of (5.4) and (5.6) in terms of exchanges of sub-primary operators also cast light on the conditions (1.4) and (1.5) for the occurrence of logarithms, which are satisfied precisely when sub-primaries of different operators become degenerate. For example, consider a contact diagram with \( \lambda_{12} - \lambda_{34} = 2k \) and \( k \) a positive integer. The dimension \( \lambda_{12} + 2n \) of a sub-primary operator \( O_{12}^{(n)} \) of \( O_{12} \) will then be the same as that of the sub-primary \( O_{34}^{(k+n)} \). In addition, the quantum numbers of all the diagonal descendants generated from \( O_{12}^{(n)} \) and \( O_{34}^{(k+n)} \) will be identical (see figure 5). To see this more explicitly, let us move slightly off the degeneracy point, i.e. consider, \( \lambda_{12} = \lambda_{34} + 2k + 2\epsilon \) where \( 0 < \epsilon < 1 \). Then (5.3) may be written as (below we will omit the overall constant),

\[
a_n^{12} = (-1)^k \frac{\pi}{\sin \epsilon \pi} \frac{1}{\Gamma(n+1)\Gamma(1+n+k+\epsilon)}
\]

where we have used \( \Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x} \). And (5.4) may be written as,

\[
S_c = \sum_{n=0}^{k-1} \frac{(-1)^n}{n!} \Gamma(k-n+\epsilon) H(\lambda_{34}+2n)
\]

\[
+ (-1)^k \frac{\pi}{\sin \epsilon \pi} \sum_{m=0}^{\infty} \left[ \frac{H(\lambda_{34}+2m+2k+2\epsilon)}{\Gamma(m+1)\Gamma(1+m+k+\epsilon)} - \frac{H(\lambda_{34}+2m+2k)}{\Gamma(m+1-\epsilon)\Gamma(1+m+k)} \right]
\]

As we take \( \epsilon \) to zero, the conflicting contributions from \( O_{12}^{(m)} \) and \( O_{34}^{(m+k)} \) in the square bracket become degenerate and the result is given by their derivatives over \( \epsilon \), which contain logarithms. The above discussion suggests that by turning on a very small \( \epsilon \) at degeneracy points, (5.12) provides a useful way to “regularize” logarithms.

Since a contact diagram is symmetric under exchanging its external legs, it’s \( t \)- or \( u \)-channel expansion can be simply obtained by taking \( 2 \leftrightarrow 4 \) or \( 2 \leftrightarrow 3 \) in (5.4) and (5.3). For example, it can be represented as \( t \)-channel exchange in CFT\(_d\) as in figure 6.

**Discussions**

We note that conformal symmetry imposes strong restrictions on the coefficients \( C_{ij}^k \) in (1.1); that of primary determines those of their descendants. The structure of (5.10) and
\[\lambda_1 \lambda_4 = \lambda_1 \lambda_3 + \lambda_2 \lambda_3\]

**Figure 6:** $t$–channel OPE interpretation of a contact diagram.

(4.12), including the numerical coefficients $a_n$ in (5.8) and $b_n$ in (5.10) in the summation, are uniquely fixed up to an overall constant by the fact that $\Phi_\Lambda$ and its derivatives fill an irreducible representation of the conformal group. Although we have found in (5.4) and (5.6) the contributions from a complete set of sub-primary operators of dimensions $\lambda_{12}$ and $\lambda_{34}$, their relative OPE coefficients (1.1) are not consistent with those required by conformal symmetry, in other words, these sub-primaries do not seem to fill the same irreducible multiplets.

It is probably not surprising that we do not find a complete CFT$_d$ identification in (5.4) and (5.6). After all, we are only looking at a generic diagram in AdS$_{d+1}$, which hardly makes too much sense before we specify a particular theory and add up all the diagrams contributing to a realistic amplitude. The encouraging message seems to be that we are indeed able to find a relation between an arbitrary scattering diagram and OPE, which indicates some kind of universality between a theory in AdS$_{d+1}$ and CFT$_d$.

It is not clear at the present time how much we see here will survive in the final expression of a realistic amplitude, in particular, whether operators $O_{12}$ and $O_{34}$ will have a consistent CFT$_d$ interpretation when we add up all the diagrams. Let us now consider what could be these operators if their contribution do survive in the final expression. A clue comes from the consideration of free theory, where the operator product expansion takes the form (see e.g. [32]),

\[A(z)B(w) = \underbrace{A(z)B(w)}_{\text{contraction}} + :A(z)B(w): \quad (5.13)\]

where the first term on the right hand side denotes contraction and $:A(z)B(w):$ stands for normal-ordered operator whose explicit form can be obtained from a Taylor expansion,

\[:A(z)B(w): = \sum_{k=0}^{\infty} \frac{(z-w)^k}{k!} (\partial^k AB)(w)\]
In free theory the conformal dimension for : $A(z)B(w)$ : is just $\lambda_A + \lambda_B$. Thus naturally we may expect that $O_{12}$ and $O_{34}$ should be the counter-parts of : $O_{\lambda_1}O_{\lambda_2} :$ and : $O_{\lambda_3}O_{\lambda_4} :$ in interacting theory. In $\mathcal{N} = 4$ Super-Yang-Mills with gauge group $SU(N)$, $O_{12}$ and $O_{34}$ may be interpreted as double-trace operators, i.e. operators of type $TrF^2TrF^2(x)$. Since we do not see a continuous spectrum of dimensions in (5.6) and (5.4), we would expect $O_{12}$ and $O_{34}$ to correspond in AdS$_{d+1}$ to two-particle bound states of supergravity/string theory. This is consistent with the expectation [33] that, to lowest order in $1/N$, there cannot be any two-particle cut in Yang-Mills four-point functions.

Finally, we note that since (5.5) and (5.7) involve dimensions of other operators (e.g. $\lambda$ and $\lambda_{34}$), it suggests a possibility that the mismatch in our OPE identification may be due to certain underlying mixing and interaction between different operators at sub-primary level. Moreover, when $\epsilon \to 0$, the pattern indicated in (5.12) for the degeneracy of sub-primaries strongly reminds us of the behaviour of a two-level system. Similarly, by examining the exchange amplitude (5.6) and (5.5), we also find that the pattern near degeneracy points is rather like a three-state system.

Appendix A. Mellin transformation and analytic continuation

Here we give an brief introduction to the Mellin transformation and how to use it to evaluate integrals.\footnote{I would like to thank T.W.B. Kibble, F.G. Leppington, and A.B. Zamolodchikov for discussions related to the content of this appendix.}

The Mellin transform of a function $g(x)$ is

$$h(s) = \int_0^\infty dx \, g(x) \, x^{s-1} \quad (A.1)$$

If the set of convergence of the integral (A.1) has a nonempty interior $\alpha < Re(s) < \beta$ and $h(s)$ is analytic in this strip, we can have the inverse Mellin transformation,

$$g(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} h(s) x^{-s} ds \quad (A.2)$$

for all $c$ such that $\alpha < c < \beta$. 
Some well-known integral representations of higher transcendental functions can be interpreted as (inverse) Mellin transforms, for example,

\[ \Gamma(s) = \int_0^\infty dx \, x^{s-1} e^{-x}, \quad \Gamma(s)\zeta(s) = \int_0^\infty dx \, x^{s-1} (e^x - 1)^{-1}. \]

and the Mellin-Barnes representation of a hypergeometric function,

\[ F(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} ds \, \frac{\Gamma(a+s)\Gamma(b+s)}{\Gamma(c+s)} \Gamma(-s)(-z)^s. \quad (A.3) \]

Taking \( b = c \) in \((A.3)\) we get,

\[ F(a, b; b; z) = (1 - z)^{-a} = \frac{1}{\Gamma(a)} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} ds \, \Gamma(a+s)\Gamma(-s)(-z)^s. \quad (A.4) \]

Generally, to evaluate an integral:

\[ I = \int dx \, g(x) f(x) \quad (A.5) \]

we can first plug into \((A.5)\) the inverse Mellin transform \((A.2)\) of \( g(x) \), then do the \( x \)-integral and finally inverse-Mellin transform back,

\[ I = \int_{c-i\infty}^{c+i\infty} ds \, h(s) J(s), \quad J(s) = \int dx \, x^s f(x). \quad (A.6) \]

Normally Mellin transform \((A.1)\), \((A.2)\) is not as convenient as Fourier or Laplace transform as it requires the functions to be transformed have reasonable “good behaviours” both at zero and infinity to ensure the existence of the strip where the transform can be defined. But for those functions \( g \) in \((A.1)\) which have a convenient power series expansion (such as hypergeometric functions) Mellin representation is more powerful since the indices and the coefficients of the expansion are represented by the poles and the corresponding residues of \( h(s) \) in the complex \( s \)-plane.

Let us look at a simple example,

\[ I = \int_0^\infty dx \, x^{\nu-1} (1 + x)^{-\mu} (x + t)^{-\rho} \quad (A.7) \]

The integral is defined when \( \text{Re}(\nu) > 0, \text{Re}(\mu + \rho - \nu) > 0 \), and \( \arg(t) < \pi \) which we assume is the case. For convenience we will also take \( |t| < 1 \). The result of \((A.7)\) is well known, given by a hypergeometric function,

\[ I = t^{\nu-\rho} B(\nu, \mu - \nu + \rho) F(\mu, \nu; \mu + \rho; 1 - t) \quad (A.8) \]
where $B(\nu, \mu - \nu + \rho)$ is a Beta function.

Here we would like to reproduce the result by using the Mellin transformation technique of (A.6). For the moment we first assume $\text{Re}(\rho) > 0$. Note that from (A.4),

$$(x + t)^{-\rho} = \frac{1}{\Gamma(\rho)} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} ds \Gamma(\rho + s)\Gamma(-s) t^s x^{-\rho-s}$$  \hspace{1cm} (A.9)

where $-\text{Re}(\rho) < c < 0$. Plugging the above expression into (A.7), we get,

$$I = \frac{1}{\Gamma(\rho)} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} ds \Gamma(\rho + s)\Gamma(-s) t^s \int_0^\infty dx \ x^{\nu-\rho-s-1}(1 + x)^{-\mu}$$

$$\hspace{1cm} = \frac{1}{\Gamma(\rho)\Gamma(\mu)} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} ds t^s \Gamma(\rho + s)\Gamma(-s)\Gamma(\nu - \rho - s)\Gamma(\mu + \rho - \nu + s)$$  \hspace{1cm} (A.10)

and the convergence of $x$-integral requires:

$$-\text{Re}(\mu + \rho - \nu) < \text{Re}(s) < \text{Re}(\nu - \rho)$$  \hspace{1cm} (A.11)

Since $x$-integral in (A.10) has generated new pole sequences in the complex $s$-plane, we have to check that they won’t cause any ambiguity in carrying the inverse-Mellin integral in the second line of (A.10).

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{flag.png}
\caption{Pole structures and contours. a: $\text{Re}(\mu) > \text{Re}(\nu) > \text{Re}(\rho) > 0$; b: $\text{Re}(\rho) > \text{Re}(\nu) > \text{Re}(\mu) > 0$.}
\end{figure}

A.1. $\text{Re}(\mu) > \text{Re}(\nu) > \text{Re}(\rho) > 0$

In this case, it is easy to check that there is no overlap between descending and ascending pole sequences and the new pole sequences sit outside the strip $-\text{Re}(\rho) < c < 0$
Figure 8: Pole structure and analytic continuation. a: $Re(\rho) > Re(\nu) > 0,$ $Re(\mu) < 0$; b: general values of parameters.

(which means the convergence condition (A.11) is trivially satisfied), where the Inverse Mellin transform (A.9) is defined. Thus there is no ambiguity in defining the integral in (A.10) and we can take the integral around a contour $C$, which consists of the path in (A.10) and encloses the right half plane. See figure 7a for the pole structure and the contour. Since $|t| < 1$ the contribution from part of the contour other than (A.10) vanishes and (A.10) can be written by the calculus of residues as the sum of the residues of the integrand at the poles $s = 0, 1, \cdots$ and $s = \nu - \rho + n, n = 0, 1, \cdots$. It is easy to see the sum gives us (A.8).

A.2. $Re(\mu), Re(\nu), Re(\rho) > 0$

In this case, there is still no overlap between descending and ascending pole sequences, but there are poles sitting inside the strip $-Re(\rho) < c < 0$ if $\mu < \nu$ or $\nu < \rho$, which seems to cause ambiguity in the choice of $c$ in (A.9) as they may enclose different poles inside the strip. However, the convergent condition (A.11) requires that we squeeze the integration path in (A.10) into a smaller strip $\text{Max}[-Re(\mu + \rho - \nu), -Re(\rho)] < c < \text{Min}[0, Re(\nu - \rho)]$. It is clear that in this refined strip there is indeed no ambiguity to define the integral and again we get the desired result. See figure 7b for the pole structure and the contour for the case: $Re(\rho) > Re(\nu) > Re(\mu) > 0$

A.3. $Re(\mu) < 0, Re(\nu), Re(\rho) > 0$ and others

Now the convergent condition (A.11) can no longer be satisfied. There is an overlap
between the ascending poles from $\Gamma(\nu - \rho - s)$ and descending poles from $\Gamma(\mu + \rho - \nu + s)$ and there does not exist a uniform strip that the inverse Mellin integral (A.10) is well defined. In this case we can define the integral by analytic continuation from the convergent region of $\mu$. It is clear that as we vary $\mu$ continuously from $\mu > 0$ to $\mu < 0$, the only way to avoid the sudden jump of the value of integral by crossing the poles from $\Gamma(\mu + \rho - \nu + s)$ is to deform the integration path so that it still separates the descending and ascending pole sequences (see figure 8a).

It is obvious that by repeating the above procedure of deforming the path of (A.10) (see figure 8b) we can analytically continue the integral (A.7) to arbitrary complex values of $\mu, \nu, \rho$ except for some discrete surfaces in the space of $\mu, \nu, \rho$ where one or more of $\rho, \mu$ and $\nu, \mu + \rho - \nu$ become non-positive integers. In these cases there are coincidences between the ascending poles and descending poles and it is no longer possible to separate them. The analytic continuation breaks down at these surfaces. The pathology at $\mu, \rho = -k, k = 0, 1, 2, \cdots$ may be attributed to the method we are using (see eq (A.9) \(^{14}\)), while at $\nu, \mu + \rho - \nu = -k, k = 0, 1, 2, \cdots$ the analytic continuation truly breaks down (similar to the poles in Gamma functions).

Appendix B. Detailed evaluation of $J$

Here we present the details of the calculation leading from (2.8) to (2.9). To avoid making formulas too long, we will suppress the prefactors (numerical constants and powers of $x_{ij}$) of the integrals and give their final expression only at the end. We use the following definitions,

\[
\tilde{\lambda} = d - \lambda, \quad \nu = \lambda - \frac{d}{2}, \quad \nu_i = \lambda_i - \frac{d}{2}, \quad i = 1, \cdots, 4,
\]

\[
\lambda_{ij} = \lambda_i + \lambda_j, \quad \Delta_{ij} = \lambda_i - \lambda_j, \quad \epsilon_{ij} = \lambda_{ij} - \lambda, \quad \tilde{\epsilon}_{ij} = \lambda_{ij} - \tilde{\lambda},
\]

\[
\delta_1 = \lambda + \Delta_{12}, \quad \delta_2 = \lambda + \Delta_{21}, \quad \delta_3 = \lambda + \Delta_{34}, \quad \delta_4 = \lambda + \Delta_{43},
\]

\[
\tilde{\delta}_1 = \tilde{\lambda} + \Delta_{12}, \quad \tilde{\delta}_2 = \tilde{\lambda} + \Delta_{21}, \quad \tilde{\delta}_3 = \tilde{\lambda} + \Delta_{34}, \quad \tilde{\delta}_4 = \tilde{\lambda} + \Delta_{43}.
\]

\(^{14}\) When $\mu, \rho = -k, k = 0, 1, 2, \cdots$, $(1 + x)^{-\mu}$ or $(x + t)^{-\rho}$ becomes finite series and can be expanded directly to evaluate the integral. The result can be expressed in terms of the terminating series of hypergeometric functions.
\( J(s) \) in eq (2.8) can be further simplified by applying the inversion trick \(^{10}\): set \( x_4 = 0 \), then use a simultaneous inversion of the external coordinates and the integration variables, \( u_\mu \to u_\mu/|u|^2, \ v_\mu \to v_\mu/|v|^2, \ \bar{x}_i = \bar{x}'_i/|\bar{x}'_i|^2, \ i = 1, 2, 3, \) after which \( J \) becomes,

\[
J(s) = \frac{2^\lambda+2s}{|x_1|^{2\lambda_1}|x_2|^{2\lambda_2}|x_3|^{2\lambda_3}} \int \frac{du_0dv_0d^dv}{u_0^{d+1}v_0^{d+1}} \left( \frac{u_0}{|u-x'_1|^2} \right)^{\lambda_1} \left( \frac{v_0}{|v-x'_2|^2} \right)^{\lambda_2} \times \left( \frac{u_0v_0}{u_0^2+v_0^2+|\bar{u}-\bar{v}|^2} \right)^{\lambda+2s} \left( \frac{v_0}{|v-x'_3|^2} \right)^{\lambda_3} v^{\lambda_4}. \tag{B.2}
\]

Using

\[
\frac{\Gamma(\lambda)}{z^\lambda} = \int_0^\infty dp \rho^\lambda e^{-\rho z} \tag{B.3}
\]

we can rewrite equation (B.2) as,

\[
J(s) = \int_0^\infty d\rho_1d\rho_2d\rho_3dp \rho_1^{\lambda_1-1}\rho_2^{\lambda_2-1}\rho_3^{\lambda_3-1}\rho^{\lambda+2s-1} \int du_0dv_0d\bar{u} \bar{u}_0^{\xi_{12}+2s-1}\bar{v}_0^{\xi_{34}+2s-1} \times e^{-(\rho_1+\rho_2+\rho)}u_0^2 e^{-(\rho+\rho_3)v_0^2} \exp\left\{ -\rho_1|\bar{u}-\bar{x}'_1|^2 - \rho_2|\bar{u}-\bar{x}'_2|^2 - \rho|\bar{u}-\bar{v}|^2 - \rho_3|\bar{v}-\bar{x}'_3|^2 \right\}
\]

Integrating over \( u_0, v_0 \) we get \(^{13}\),

\[
J(s) = \int_0^\infty d\rho_1d\rho_2d\rho_3dp \rho_1^{\lambda_1-1}\rho_2^{\lambda_2-1}\rho_3^{\lambda_3-1}\rho^{\lambda+2s-1} \left( \frac{1}{\rho_1+\rho_2+\rho} \right)^{\xi_{12}+s} \left( \frac{1}{\rho_3+\rho} \right)^{\xi_{34}+s} \times \int d\bar{u}d\bar{v} \exp\left\{ -\rho_1|\bar{u}-\bar{x}'_1|^2 - \rho_2|\bar{u}-\bar{x}'_2|^2 - \rho|\bar{u}-\bar{v}|^2 - \rho_3|\bar{v}-\bar{x}'_3|^2 \right\}
\]

Now we use the following expression

\[
\int d^d\bar{u} \exp\left\{ -\sum_i \rho_i|\bar{u}-\bar{x}'_i|^2 \right\} = \left( \frac{\pi}{\sum_i \rho_i} \right)^{d/2} \exp\left\{ -\sum_{i<j} \rho_i\rho_j x'_{ij}^2 \right\} \tag{B.4}
\]

to integrate over \( \bar{u}, \bar{v}, \) which leads to,

\[
J(s) = \int_0^\infty d\rho_1d\rho_2d\rho_3dp \rho_1^{\lambda_1-1}\rho_2^{\lambda_2-1}\rho_3^{\lambda_3-1}\rho^{\lambda+2s-1} \left( \frac{1}{\rho_1+\rho_2+\rho} \right)^{\xi_{12}+s} \left( \frac{1}{\rho_3+\rho} \right)^{\xi_{34}+s} \times \left( \frac{1}{\rho_1+\rho_2+\rho} + \rho(\rho_1+\rho_2) \right)^{d/2} \exp\left\{ -\frac{\rho_2\rho_3|\bar{x}'_{23}|^2 + \rho\rho_1\rho_3|\bar{x}'_{13}|^2 + \rho_1\rho_2(\rho_3+\rho)|\bar{x}'_{12}|^2}{\rho_3(\rho_1+\rho_2+\rho) + \rho(\rho_1+\rho_2)} \right\}
\]

where \( |x'_{ij}| = |\bar{x}'_i - \bar{x}'_j| \).

\(^{15}\) The convergence of \( u_0, v_0 \) integrals require \( Re(\xi_{12} + 2s) > 0 \) and \( Re(\xi_{34} + 2s) > 0 \). Since \( Re(s) \sim 0 \), the convergence conditions are indeed satisfied with \( \lambda_i, \lambda > d/2 \).
Let \( \rho_i \to \rho \rho_i, \ i = 1, 2, 3 \) and integrate over \( \rho \),

\[
J(s) = \int_0^\infty d\rho_1 d\rho_2 d\rho_3 \rho_1^{\lambda_1-1} \rho_2^{\lambda_2-1} \rho_3^{\lambda_3-1} \left( \frac{1}{1 + \rho_1 + \rho_2} \right)^{\frac{\lambda_2 + \Delta_{34}}{2}} \times \\
\left[ (\rho_1 + \rho_2)(1 + \rho_3) + \rho_3 \right] \frac{1}{\rho_2 \rho_3 |x'_{23}|^2 + \rho_1 \rho_3 |x'_{13}|^2 + \rho_1 \rho_2 (\rho_3 + 1) |x'_{12}|^2}^{\frac{\lambda_1 + \Delta_{34}}{2}}
\]

Note that the convergence of \( \rho \)-integral requires \( \lambda_{12} + \Delta_{34} > 0 \).

Now define new variables, \( \rho_1 = \sigma u, \rho_2 = \sigma (1 - u), \rho_3 = \rho \), and

\[
\eta = \frac{|x'_{13}|^2}{|x'_{12}|^2}, \quad \xi = \frac{|x'_{23}|^2}{|x'_{12}|^2}, \quad z = \frac{\eta}{1 - u} + \frac{\xi}{u}, \quad (B.5)
\]

after which the integrals become,

\[
J(s) = \int_0^1 du \frac{\Delta_{12} - \Delta_{34}}{2} - 1 (1 - u) \frac{\Delta_{21} - \Delta_{34}}{2} - 1 \int d\sigma d\rho \frac{\lambda_{12} - \Delta_{34}}{2} - 1 (1 + \sigma) - \frac{\lambda_2}{2} - s
\times \rho^{\lambda_3 - 1} (1 + \rho)^{\frac{\Delta_{23} + \Delta_{34}}{2}} (\sigma + \rho + \sigma \rho)^{\frac{\lambda_{12} + \Delta_{34} - d}{2}} (\sigma (1 + \rho) + \rho z)^{\frac{\lambda_{12} + \Delta_{34}}{2}}
\]

Further define \( t = \frac{\rho}{1 + \rho} \), so that

\[
J = \int_0^1 du \frac{\Delta_{12} - \Delta_{34}}{2} - 1 (1 - u) \frac{\Delta_{21} - \Delta_{34}}{2} - 1 \int_0^\infty d\sigma \int_0^1 dt \sigma^{\frac{\lambda_{12} - \Delta_{34}}{2} - 1} (1 + \sigma)^{\frac{\lambda_2}{2} - s} \times t^{\lambda_3 - 1} (1 - t)^{\frac{\Delta_{23} + \Delta_{34}}{2} - 1} (\sigma + t)^{\frac{\lambda_{12} + \Delta_{34} - d}{2}} (\sigma + tz)^{\frac{\lambda_{12} + \Delta_{34}}{2}} \quad (B.6)
\]

Our next step is to use the inverse Mellin transformation eq (A.4),

\[
\left( \frac{1}{1 + x} \right)^\alpha = \frac{1}{\Gamma(\alpha)} \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} ds \Gamma(-s) \Gamma(\alpha + s) x^s 
\]

in \( (\sigma + tz)^{-\frac{\lambda_{12} + \Delta_{34}}{2}} \) in eq (B.6). Then the \( \sigma-t \) part of the integrals in eq (B.6) becomes

\[
\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} ds_1 \Gamma(-s_1) \Gamma(\frac{\lambda_{12} + \Delta_{34}}{2} + s_1) z^{-s_1 - \frac{\lambda_{12} + \Delta_{34}}{2}} J_1 
\]

with

\[
J_1 = \int_0^\infty d\sigma \int_0^1 dt t^{\frac{\Delta_{34} - \lambda_{12}}{2} - s_1} - 1 (1 - t)^{\frac{\lambda_{23} - \lambda_{12}}{2} + s_1} (1 + \sigma)^{-\frac{\lambda_2}{2} - s} (\sigma + t)^{\frac{\lambda_{12} + \Delta_{34} - d}{2}} 
\]

After \( \sigma \)-integration we get,

\[
J_1 = B(s_1 + \frac{\lambda_{12} - \Delta_{34}}{2}, s - s_1 - \frac{\lambda_{12}}{2}) \times 
\]

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\[
\int_0^1 dt \ t^{\frac{\lambda_4 + \lambda_2 - d}{2} - 1} F\left(\frac{\xi_{12}}{2} + s, s_1 + \frac{\lambda_{12} - \Delta_{34}}{2}; s + \frac{\delta_4}{2}, t\right).
\]

Notice that the power in \(t\) coincides with the third parameter of the hypergeometric function inside the integral. In this case the integration over \(t\) can be done very easily and we get

\[
J_1 = \frac{\Gamma(\frac{\lambda_4 + \lambda_2 - d}{2})}{\Gamma(\lambda_4)} \frac{\Gamma(s_1 + \frac{\lambda_{12} - \Delta_{34}}{2}) \Gamma(s - s_1 + \frac{\xi_{12}}{2}) \Gamma(\lambda_{34} - \lambda_{12}) - s_1)}{\Gamma(s - s_1 + \frac{\xi_{12}}{2})} (B.9)
\]

Plugging (B.9) back into (B.8), (B.6) and rearranging the integrals

\[
J(s) = C_2 \frac{1}{2 \pi i} \int_{-i \infty}^{i \infty} ds_1 \Gamma(-s_1) \frac{\Gamma(\frac{\lambda_{12} + \Delta_{34}}{2} - s_1) \Gamma(\lambda_{34} - \lambda_{12} - s_1)}{\Gamma(s - s_1 + \frac{\xi_{12}}{2})} \int_0^1 duu \frac{\Delta_{12}^{\frac{\lambda_{12} - \Delta_{34}}{2} - 1}(1 - u) \Delta_{21}^{\Delta_{21} - \Delta_{34} - 1} - \lambda_{12} + \Delta_{34}}{\lambda_{12} + \Delta_{34} - s_1} (B.10)
\]

The integral in the second line gives us a hypergeometric function and the final expression for \(J\) is (we shifted \(s_1\) by \(s_1 + \lambda_{12} \rightarrow s_1\)),

\[
J(s) = C_2 \frac{1}{2 \pi i} \int_{-i \infty}^{i \infty} ds_1 \xi^{-s_1} \frac{\Gamma(\frac{\lambda_{12} + \Delta_{34}}{2} - s_1) \Gamma(\lambda_{34} - \lambda_{12} - s_1) F\left(\frac{\Delta_{34}}{2} + s_1, \frac{\Delta_{12}}{2} + s_1; 2s_1, 1 - \frac{\eta}{\xi}\right) \times}{\Gamma(\lambda_{12} + \frac{\lambda_{34} - \Delta_{34}}{2}) \Gamma(2s_1)} \frac{\Gamma(\Delta_{12} + s_1) \Gamma(\Delta_{21} + s_1)}{\Gamma(\Delta_{12} + \frac{\lambda_{34} + \Delta_{34} - s}{2})} (B.11)
\]

Now restore \(x_4\) by taking \(x_i \rightarrow x_i - x_4, i = 1, 2, 3\). \(\eta\) and \(\xi\) defined in (B.3) become the cross ratios,

\[
\eta = \frac{|x''_{14} - x''_{34}|^2}{|x''_{14} - x''_{24}|^2} = \frac{|x_{13}|^2 |x_{24}|^2}{|x_{12}|^2 |x_{34}|^2}, \quad \xi = \frac{|x''_{24} - x''_{34}|^2}{|x''_{14} - x''_{24}|^2} = \frac{|x_{14}|^2 |x_{23}|^2}{|x_{12}|^2 |x_{34}|^2} (B.12)
\]

Finally, the prefactor \(C_2\) is given by,

\[
C_2 = \frac{\pi^d}{4} \frac{\Gamma(\lambda_{12} + \lambda_{34} - d) \Gamma(s + \xi_{12}) \Gamma(s + \xi_{34})}{\Gamma(\lambda_1) \Gamma(\lambda_2) \Gamma(\lambda_3) \Gamma(\lambda_4) \Gamma(\lambda + 2s)} \frac{2^{\lambda + 2s}}{|x_{12}|^{\lambda_{12} + \Delta_{34}} |x_{14}|^{\Delta_{12} - \Delta_{34}} |x_{24}|^{\Delta_{21} - \Delta_{34}} |x_{34}|^{2\lambda_3}}
\]

We note that it can be checked that the integrals in intermediate steps from (B.2) to (B.11) are convergent only when the parameters satisfy the following conditions (\(\text{Re}(s) \sim 0\)),

\[
\frac{\delta_1}{2}, \frac{\delta_2}{2}, \frac{\delta_3}{2}, \frac{\delta_4}{2} > 0, \quad \frac{\lambda_{12} \pm \Delta_{34}}{2} > 0, \quad \frac{\lambda_{34} \pm \Delta_{12}}{2} > 0 (B.13)
\]
For the above range of parameters there is no overlap between ascending and descending poles in (B.11) and the $s_1$-integral can be unambiguously defined by squeezing (also determined by the convergence of the intermediate integrals) the integration path $C$ to lie inside the strip:

$$\text{Max}\left[\frac{\Delta_{12}}{2}, \frac{\Delta_{21}}{2}, \frac{\Delta_{34}}{2}, \frac{\Delta_{34}}{2}\right] < \text{Re}(s_1) < \text{Min}\left[\frac{\lambda_{12}}{2}, \frac{\lambda_{12}}{2}, \frac{\lambda_{34}}{2}\right]$$

When parameters are outside the range of (B.13), some integrals in intermediate steps may not be convergent and can only be defined by analytic continuation. Further there are overlaps between the ascending and descending pole sequences in (B.11) and there does not exist a uniform strip in which the Mellin integral can be well defined. In this case we can define the integral in (B.11) by analytic continuation by deforming the integration path $C$ so that it separates the ascending and descending poles. In Appendix A we have given a detailed discussion of this procedure in a simpler example. The whole discussion can be applied to this more complicated case without change. We note that when some poles from ascending and descending series coincide with one another, e.g. if one or more of $\frac{\lambda_{12} + \Delta_{12}}{2}, \frac{\lambda_{34} + \Delta_{34}}{2}, \frac{\Delta_{12}}{2}, i = 1, \ldots, 4$ are non-positive integers, there is no way to separate the ascending and descending poles. The analytic continuation breaks down at these points.

**Appendix C. Evaluation of contact contribution**

Here we would like evaluate eq (3.9). Again we will suppress the prefactor of the integrals most of the time and follow the notations defined in (B.1).

As in Appendix B, using eq (B.3) and integrating over $u_0$, we get,

$$S_c = \int \Pi_{i=1}^4 d\rho_i \rho_i^{\lambda_i-1} \frac{1}{(\rho_1 + \rho_2 + \rho_3 + \rho_4)^{\frac{\Delta_{12} + \Delta_{34} - d}{2}}} \int d\bar{u} \exp \left[-\sum_i \rho_i |\bar{u} - \bar{x}_i|^2\right]$$

$$= \int \Pi_{i=1}^4 d\rho_i \rho_i^{\lambda_i-1} \frac{1}{(\rho_1 + \rho_2 + \rho_3 + \rho_4)^{\frac{\Delta_{12} + \Delta_{34}}{2}}} \exp \left\{ -\sum_{i<j}^{4} \rho_i \rho_j |x_{ij}|^2 \right\}$$

(C.1)

where in the second line we have used eq (B.4). Now let $\rho_i' = \rho_i(\rho_1 + \rho_2 + \rho_3 + \rho_4)^{-\frac{d}{2}}$ and note that $det(\frac{\partial \rho_i}{\partial \rho_i'}) = 2(\sum_{i=1}^{4} \rho_i')^4$,

$$S_c = \frac{1}{\pi^{3d/2}} \frac{\Gamma(\frac{\Delta_{12} + \Delta_{34} - d}{2})}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\nu_3)\Gamma(\nu_4)} K(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$$

(C.2)
where we have included the numerical prefactor and \(K\) is defined by,

\[
K(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = \int_0^\infty dp_1 dp_2 dp_3 dp_4 \Pi_i \rho_i^{\lambda_i-1} \exp \left[ - \sum_{i<j=1}^4 \rho_i \rho_j |x_{ij}|^2 \right].
\]  

(C.3)

Since \(K\) has translational invariance, we can take \(x_4 = 0\). Defining \(\rho'_i = \rho_i |x_i|^2\), \(i = 1, 2, 3\), we find that,

\[
\sum_{i<j=1}^4 \rho_i \rho_j |x_{ij}|^2 = \rho_4 (\rho'_1 + \rho'_2 + \rho'_3) + \sum_{i<j=1}^4 \rho'_i \rho'_j |x'_{ij}|^2
\]

where \(x'_{ij} = x'_i - x'_j\) and \(x'_i = \frac{x_i}{|x_i|^2}\). Then in terms of \(\rho'_i\) (we omit primes on \(\rho_i\) below) and \(x'_i\), \(J\) becomes,

\[
K = \frac{1}{|x_1|^2 \lambda_1 |x_2|^2 \lambda_2 |x_3|^2 \lambda_3} \int_0^\infty \Pi_i \rho_i^{\lambda_i-1} \exp \left[ - \rho_4 (\rho'_1 + \rho'_2 + \rho'_3) - \sum_{i<j=1}^3 \rho_i \rho_j |x'_{ij}|^2 \right]
\]

\[
= \frac{\Gamma(\lambda_4)}{|x_1|^2 \lambda_1 |x_2|^2 \lambda_2 |x_3|^2 \lambda_3} \int_0^\infty \Pi_i \rho_i^{\lambda_i-1} \frac{1}{(\rho'_1 + \rho'_2 + \rho'_3 \lambda_4)} \exp \left[ - \sum_{i<j=1}^3 \rho_i \rho_j |x'_{ij}|^2 \right]
\]

(C.4)

Now let \(\rho_3 = \alpha\), \(\rho_1 = \alpha \beta\), \(\rho_2 = \alpha \gamma\),

\[
K = \int d\alpha d\beta d\gamma \frac{\beta^{\lambda_1-1} \gamma^{\lambda_2-1} \alpha^{\lambda_1+\Delta_{34}-1}}{(1 + \beta + \gamma)^{\lambda_4}} \exp \left[ - \alpha^2 (\beta |x'_{13}|^2 + \gamma |x'_{23}|^2 + \beta \gamma |x'_{12}|^2) \right]
\]

(C.5)

Further let \(\beta = \sigma u\), \(\gamma = \sigma (1 - u)\) and define,

\[
\eta = \frac{|x'_{13}|^2}{|x'_{12}|^2}, \quad \xi = \frac{|x'_{23}|^2}{|x'_{12}|^2}, \quad z = \frac{\eta}{1-u} + \frac{\xi}{u},
\]

after which \(K\) becomes,

\[
K = \int_0^1 du u^{\Delta_{12} - \Delta_{34} - 1} (1 - u)^{\Delta_{21} - \Delta_{34} - 1} \int_0^\infty d\sigma \sigma^{\frac{\lambda_2 - \Delta_{34} - 1}{2}} (1 + \sigma)^{-\lambda_4} (\sigma + z)^{-\Delta_{12} - \Delta_{34}}
\]

(C.6)

\(\sigma\)-integral above is nothing but the familiar integral representation of a hypergeometric function.
We now restore $x_4$, after which $\xi$ and $\eta$ become the cross ratios defined in (B.12).

Including the prefactor of the integral, we get the final expression for $K$,

$$K = \frac{1}{2} \frac{\Gamma(\lambda_3)\Gamma(\lambda_4)\Gamma(\frac{\lambda_{12} - \Delta_{34}}{2})\Gamma(\frac{\lambda_{12} + \Delta_{34}}{2})}{\Gamma(\frac{\lambda_{12}}{2} + \frac{\lambda_{34}}{2})} \frac{1}{|x_{12}|^{\lambda_{12}}|x_{14}|^{\lambda_{12} - \Delta_{34}}|x_{24}|^{\Delta_{21} - \Delta_{34}}|x_{34}|^{\lambda_{34}}^2} \times$$

$$\int_0^1 du u \frac{\Delta_{12} - \Delta_{34} - 1}{(1-u)^{\Delta_{21} - \Delta_{34} - 1}} z^{-\Delta_{34}} F\left(\frac{\lambda_{12} + \Delta_{34}}{2}, \frac{\lambda_{12} - \Delta_{34}}{2}; \frac{\lambda_{12} + \lambda_{34}}{2}; 1 - \frac{1}{z}\right).$$

(C.7)

Alternatively we may use the Mellin-Barnes representation for a hypergeometric function,

$$F(a, b; c; 1-z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c-a)\Gamma(c-b)} \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} ds z^s \Gamma(-s)\Gamma(c-a-b-s)\Gamma(a+s)\Gamma(b+s)$$

in eq (C.7), and then integrate over $u$. In this form $S_c$ can be written as,

$$S_c = C_c \frac{1}{2\pi i} \int_{\lambda_{12} - i\infty}^{\lambda_{12} + i\infty} ds \xi^{-s} \Gamma\left(\frac{\lambda_{12}}{2} - s\right)\Gamma\left(\frac{\lambda_{34}}{2} - s\right) F\left(\frac{\Delta_{34}}{2} + s, \frac{\Delta_{12}}{2} + s; 2s; 1 - \frac{\eta}{\xi}\right) \times$$

$$\frac{\Gamma(\frac{\Delta_{12}}{2} + s)\Gamma(\frac{\Delta_{34}}{2} + s)\Gamma(\lambda_{12})\Gamma(\lambda_{34})}{\Gamma(2s)}$$

with

$$C_c = \frac{1}{2\pi^{3d/2}} \frac{\Gamma(\frac{\lambda_{12} + \lambda_{34} - d}{2})}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\nu_3)\Gamma(\nu_4)} \frac{1}{|x_{12}|^{\lambda_{12}}|x_{14}|^{\lambda_{12}}|x_{24}|^{\Delta_{21} - \Delta_{34}}|x_{34}|^{\Delta_{34}}|x_{34}|^{\lambda_{34}}}$$

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