LOCAL-GLOBAL CONVERGENCE,
AN ANALYTIC AND STRUCTURAL APPROACH

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In memory Bohuslav Balcar.

ABSTRACT. Based on methods of structural convergence we provide a unifying view of local-global convergence, fitting to model theory and analysis. The general approach outlined here provides a possibility to extend the theory of local-global convergence to graphs with unbounded degrees. As an application, we extend previous results on continuous clustering of local convergent sequences and prove the existence of modeling quasi-limits for local-global convergent sequences of nowhere dense graphs.

The true logic of the world is in the calculus of probabilities

James Clerk Maxwell

1. INTRODUCTION

The study of graph limits recently gained a strong interest, motivated both by the study of large networks and the emerging studies of real evolving networks. The different notions of graph limit, and the basic notions of graph similarity on which they are based, opens a vast panorama. Among them, two notions have a particular importance:

- the notion of left convergence of graphs introduced by Lovász et al. [7, 8, 9, 10, 18], for which analytic limit objects (called graphon) are fully characterized, and which throws a bridge between homomorphism densities based similarity and cut metrics via Szemerédi’s regularity lemma. This setting is the natural one for the study of dense graph property testing.

- the notion of local convergence of graphs of bounded degrees introduced by Benjamini and Schramm [5], for which an analytic representation of the limit (called graphing) is known, and which establishes a deep connection with group theory and ergodic theory. In particular, the inverse problem of characterizing which graphing (conjecturally all) are local limits of finite graphs appears to be equivalent to the problem of characterizing finitely generated sofic groups. This is also the natural setting for the study of sparse (bounded degree) graph property testing.

These frameworks have in common to be built on statistics of locally defined motives when the vertices of the graphs in the sequence are sampled uniformly and independently. A unified framework for the study of convergence of structures has been introduced by the authors in [19]. In this setting the notion of convergence is, in essence, model theoretic, and relies on the the following notions:

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For a $\sigma$-structure $A$ and a first-order formula $\phi$ (in the language of $\sigma$, with free variables $x_1, \ldots, x_p$), we denote by $\phi(A)$ the satisfying set of $\phi$ in $A$:
\[
\phi(A) = \{(v_1, \ldots, v_p) \in A^p : A \models \phi(v_1, \ldots, v_p)\},
\]
and we define the Stone pairing of $\phi$ and $A$ as the probability
\[
\langle \phi, A \rangle = \frac{|\phi(A)|}{|A|^p}
\]
that $A$ satisfies $\phi$ for a (uniform independent) random interpretation of the random variables.

A sequence $A = (A_n)_{n \in \mathbb{N}}$ of finite $\sigma$-structures is $\text{FO-convergent}$ if the sequence $\langle \phi, A \rangle = (\langle \phi, A_n \rangle)_{n \in \mathbb{N}}$ converges for every first-order formula $\phi$.

It is important that one can define a weakened notion of $X$-convergence (for a fragment $X$ of first-order logic) by restricting the range of the test formulas $\phi$ to $X$. In particular, for a sequence of graphs with growing orders, the $\text{QF-convergence}$ (that is of convergence driven by the fragment QF of quantifier-free formulas) is equivalent to left convergence, and the $\text{FO}_{\text{local}}$-convergence (that is of convergence driven by the fragment FO$_{\text{local}}$ of the so-called local formulas) is equivalent to local convergence when restricting to sequences of graphs with bounded degrees [19]. The study of $X$-convergence of structures and the related problems is called shortly “structural limits”. It extends all previously considered types of non-geometric convergence of combinatorial structures (two above and also, e.g. permutations) to general relational structures. A survey of structural limits can be found in [22, 21].

The framework of structural limits proved to be useful and led to a new analytic characterization of the nowhere dense vs somewhere dense dichotomy for classes of graphs [20], as well as a new technique of continuous clustering [24]. Both aspects are considered here in an attempt to provide a proper setting for local-global convergence, extending the framework introduced in [6, 12] for sequences of bounded degree graphs. The study of local-global convergence is motivated by an attempt to refine the notion of local convergence to capture further important characteristics of networks, such as chromatic number and expansion properties.

In this paper we introduce the notion of lift-Hausdorff convergence, which is defined generally for graphs and relational structures, and prove a basic representation theorem for corresponding limits. In light of this general setting, a natural extension of the notion local-global convergence is proposed, keeping the name of “local-global convergence”, even in this more general setting. the power of this approach is demonstrated by applications to continuous clustering refining results in [24]. This also leads to a rich theory combining analytic and model theoretical techniques. It also seems to provide a proper setting for several questions considered in [19] and [12] (see Section 3.4).

The motivation for local-global convergence is to increase the sensitivity to global topological properties of networks. While connectivity itself is out of reach, expansion properties can be handled efficiently. This is in accordance with the approach of the authors taken in [24], which was described as “asymptotic connectivity” or “continuous clustering”.

This paper is organized as follows:

1. Introduction
2. Preliminaries
   2.1. Relational Structures and First-Order Logic
   2.2. Functional Analysis
   2.3. Sequences
   2.4. Basics of Structural Convergence
   2.5. The Representation Theorem for Structural Limits
3. From interpretation to lift convergence
   3.1. A Categorical Approach to Interpretations
Let us end this introduction by few remarks. The lifts involved in our local-global structural convergence are all monadic (and can be seen as coloring of vertices). It follows that the expressive power of such lifts is restricted to embeddings and classes which are hereditary. If we would consider more general lifts, such as coloring of the edges then we could represent monomorphisms (not induced substructures) which in turn leads to monotone classes. Monotone classes of graphs which have modeling limits (of which graphing limits are a particular case) were characterized in [20] and coincides with nowhere dense classes (of graphs). This also coincides (in the case of monotone classes of graphs) with the notion of NIP and stable classes [1] (See also [23]). For hereditary classes the structure theory and the existence of modeling limits is more complicated (see [22]) and local-global convergence seems to provide a useful framework.

2. PRELIMINARIES

2.1. Relational Structures and First-Order Logic. A signature $\sigma$ is a set of relation symbols with associated arities. In this paper we will consider countable signatures. A $\sigma$-structure $A$ is defined by its domain $A$, which is a set, and by interpreting each relation symbol $R \in \sigma$ of arity $k$ as a subset of $A^k$. We denote by Rel($\sigma$) the set of all finite $\sigma$-structures and by $\mathcal{R}$el($\sigma$) the class of all $\sigma$-structures.

A first-order formula $\phi$ in the language of $\sigma$-structures is a formula constructed using disjunction, conjunction, negation and quantification over elements, using the relations in $\sigma$ and the equality symbol. A variable used in a formula is free if it is not bound by a quantifier. We always assume that free variables are named $x_1, \ldots, x_n, \ldots$ and we consider formulas obtained by renaming the free variables as distinct. For instance, $x_1 = x_2$ and $x_2 = x_3$ are distinct formulas. We also consider two constants, 0 and 1 to denote the false and true statements. We denote by FO($\sigma$) the (countable) set of all first-order formulas in the language of $\sigma$-structures. The conjunction and disjunction of formulas are denoted by $\land$ and $\lor$, and the negation of $\phi$ is denoted by $\neg \phi$. We say that two formulas $\phi$ and $\psi$ are logically equivalent, which we denote by $\phi \iff \psi$, if one can infer one from the other (i.e. $\phi \vdash \psi$ and $\psi \vdash \phi$). Note that in first-order logic the notions of syntactic and semantic equivalence coincides. In this context we denote by $[\phi]$ the equivalence class of $\phi$ with respect to logical equivalence. It is easily checked that $B_{\sigma} = \text{FO($\sigma$)/$\iff$}$ is a countable Boolean algebra with minimum 0 and maximum 1, which is called the Lindenbaum-Tarski algebra of FO($\sigma$).

In this paper we consider special fragments of first-order logic (see Table 1). The Lindenbaum-Tarski algebra of a fragment $X \subseteq \text{FO($\sigma$)}$ will be denoted by $B_{\sigma}^X$. For instance, $B_{\sigma}^{\text{QF}} = \text{QF($\sigma$)/$\iff$}$.

2.2. Functional Analysis. Basic facts from Functional Analysis, which will be used in this paper, are recalled now.

A standard Borel space is a Borel space associated to a Polish space, i.e. a measurable space $(X, \Sigma)$ such that there exists a metric on $X$ making it a separable complete metric space with $\Sigma$ as its Borel $\sigma$-algebra. Typical examples of standard Borel spaces are $\mathbb{R}$ and the Cantor space. Note that according to Maharam’s theorem, all uncountable standard
Borel spaces are (Borel) isomorphic. (The authors cannot resist to the temptation to mention in this context the award winning work of Balcar [41].)

In this paper, we shall mainly consider compact separable metric spaces. Note that if \((M, d)\) is a compact separable metric space, then linear functionals on the space of real continuous functions on \(M\) can be represented, thanks to Riesz-Markov-Kakutani representation theorem, by Borel measures on \(M\). We denote by \(P(M)\) the space of all probability measures on \(M\). A sequence of probability measures \((\mu_n)\) is weakly convergent if \(\int f \, d\mu_n\) converges for every (real-valued) continuous function \(f\). Weak convergence defines the weak topology of \(P(M)\), and (as we assumed that \((M, d)\) is a compact separable metric space) this space is compact, separable, and metrizable by the Lévy-Prokhorov metric (based on the metric \(d\)):

\[
d_{LP}(\mu_1, \mu_2) = \inf \left\{ \epsilon > 0 \mid \mu_1(A) \leq \mu_2(A^\epsilon) + \epsilon \text{ and } \mu_2(A) \leq \mu_1(A^\epsilon) + \epsilon \text{ for all Borel } A \right\},
\]

where \(A^\epsilon = \{ x \in M \mid (\exists y \in M) \, d(x, y) < \epsilon \}\).

The Hausdorff metric is defined on the space of nonempty closed bounded subsets of a metric space. Consider a compact metric space \((M, d)\), and let \(\mathcal{C}_M\) be the space of nonempty closed subsets of \(M\) endowed with the Hausdorff metric defined by

\[
d_H(X, Y) = \max_{x \in X} \min_{y \in Y} d(x, y), \min_{x \in X} \max_{y \in Y} d(x, y).
\]

One of the most important properties of Hausdorff metric is that the space of non-empty closed subsets of a compact set is also compact (see [13], and [26] for an independent proof). Hence the space \((\mathcal{C}_M, d_H)\) is compact.

We can use the inverse function of a surjective continuous function from a compact metric space \((M, d)\) to a (thus compact) Hausdorff space \(\mathcal{T}\) to isometrically embed the space \(\mathcal{C}_T\) (of non-empty closed subsets of \(T\)) into \((\mathcal{C}_M, d_H)\). Then, using the natural injection \(i : T \to \mathcal{C}_T\) (defined by \(i(x) = \{ x \}\)) we pull back the Hausdorff distance on \(\mathcal{C}_M\) into \(T\):

\[
d_T(x, y) = d_H(f^{-1}(x), f^{-1}(y)).
\]

The situation is summarized in the following diagram.

\[
\begin{array}{c}
\mathcal{C}_M, d_H \xrightarrow{j^{-1}} \mathcal{C}_T \\
\downarrow \quad \downarrow i \\
(M, d) \xrightarrow{f} T
\end{array}
\]

In this diagram \(f^{-1}\) denotes the mapping from \(T\) to \(\mathcal{C}_M\) and \(j^{-1}\) the corresponding mapping from \(\mathcal{C}_T\) to \(\mathcal{C}_M\) defined by \(j^{-1}(X) = \{ y \mid f(y) \in X \}\). Also remark that the metric \(d_T\) defined on \(T\) is usually not compatible with the original topology of \(T\).

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1 We do not have to assume that \(f\) has compact support as we assumed that \((M, d)\) is compact.
For the topology defined by the metric $d_T$, one can define the compactification of $T$, which may be identified with the closure of the image of $T$ (by $f^{-1}(o)$) in $C_M$.

We shall make use of the following folklore result, which we prove here for completeness.

**Lemma 2.2.1.** Let $X, Y$ be compact standard Borel spaces and let $f : X \to Y$ be continuous. Let $P(X)$ and $P(Y)$ denote the metric space of probability measures on $X$ and $Y$ (with Lévy-Prokhorov metric).

Then the pushforward by $f$, that is the mapping $f_* : P(X) \to P(Y)$ defined by $f_*(\mu) = \mu \circ f^{-1}$, is continuous.

**Proof.** Assume $\mu_n \Rightarrow \mu$ is a weakly convergent sequence of measures in $P(X)$. Then for every continuous function $g : Y \to \mathbb{R}$ it holds

$$\int_Y g(y) \, df_*(\mu)(y) = \int_X g \circ f(x) \, d\mu(x)$$

$$= \lim_{n \to \infty} \int_X g \circ f(x) \, d\mu_n(x)$$

$$= \lim_{n \to \infty} \int_Y g(y) \, df_*(\mu_n)(y)$$

Hence $f_*(\mu_n) \Rightarrow f_*(\mu)$. \hfill \Box

### 2.3. Sequences.

In this paper we denoted sequences by sans serif letters. In particular, we denote by $A$ a sequence of structures $A = (A_n)_{n \in \mathbb{N}}$, and by $X = (X_n)_{n \in \mathbb{N}}$ a sequence of sets $X_n$, where $X_n$ is a subset of the domain $A_n$ of $A_n$.

Subsequences will be denoted by $A_f$ and $X_f$, where $f$ is meant to be a strictly increasing function $f : \mathbb{N} \to \mathbb{N}$, and represent the sequences $(A_{f(n)})_{n \in \mathbb{N}}$ and $(X_{f(n)})_{n \in \mathbb{N}}$. Note that $(A_f)_n = A_{g \circ f}$.

In order to simplify the notations, we extend binary relations and standard constructions to sequences by applying them component-wise. For instance $X \subseteq Y$ means $(\forall n \in \mathbb{N}) X_n \subseteq Y_n$, $X \cap Y$ represents the sequence $(X_n \cap Y_n)_{n \in \mathbb{N}}$, and if $f : \text{Rel}(\sigma) \to \mathbb{R}$ is a mapping then $f(A)$ represents the sequence $(f(A_n))_{n \in \mathbb{N}}$.

We find these notations extremely helpful for our purposes.

### 2.4. Basics of Structural Convergence.

Let $\sigma$ be a countable signature, let $X$ be a fragment of FO($\sigma$). For $\phi \in X$ with free variables within $x_1, \ldots, x_p$ and $A$, we denote by $\langle \phi, A \rangle$ the probability that $\phi$ is satisfied in $A$ for a random assignment of elements of $A$ to the free variables of $\phi$ (for an independent and uniform random choice of the assigned elements), that is:

$$\langle \phi, A \rangle = \frac{\left| \{ \overline{v} \in A^p \mid A \vDash \phi(\overline{v}) \} \right|}{|A|^p}.$$ 

(Note that the presence of unused free variables does not change the value in the next equation.) In the special case where $\phi$ is a sentence, we get

$$\langle \phi, A \rangle = \begin{cases} 1 & \text{if } A \vDash \phi, \\ 0 & \text{otherwise} \end{cases}$$

Two $\sigma$-structures $A$ and $B$ are $X$-equivalent, what we denote by $A \equiv_X B$, if we have $\langle \phi, A \rangle = \langle \phi, B \rangle$ for every $\phi \in X$.

**Example 2.4.1.** Let QF$^-$ be the fragment of quantifier-free formulas that do not use equality.

If a graph $G$ is obtained from a graph $H$ by blowing each vertex into $k$ vertices (i.e. if $G$ is the lexicographic product of $H$ by an edgeless graph of order $k$) then $G \equiv_{\text{QF}^-} H$.

If $X$ is a fragment including QF or FO$_0$, then $\equiv_X$ is trivial on finite relational structures.
The case of $\text{FO}_{\text{local}}$-equivalence is settled by the next proposition.

**Proposition 2.4.2.** For any two finite $\sigma$-structures $A$ and $B$ we have that $A \equiv_{\text{FO}_{\text{local}}} B$ if and only if there exists a finite $\sigma$-structure $C$ and two positive integers $a$ and $b$ such that $A$ is isomorphic to a copies of $C$ and $B$ is isomorphic to $b$ copies of $C$.

**Proof.** Let $F_1, \ldots, F_n, \ldots$ be an enumeration of the finite $\sigma$-structures (up to isomorphism), and let $\varphi_i(x)$ be a local formula expressing that the connected component of $x$ is isomorphic to $F_i$ (i.e. that the ball of radius $|F_i| + 1$ around $x$ is isomorphic to $F_i$). Then $\langle \varphi_i, A \rangle$ is equal to the product of $|F_i|/|A|$ by the number of connected components of $A$ isomorphic to $F_i$. Thus there exists a positive integer $q$ and non-negative integers $p_1, \ldots, p_n, \ldots$ such that $\langle \varphi_i, A \rangle = p_i/q$ and the set of all positive $p_i$ values is setwise coprime. Then if $C$ consists in this union (over $i$) of $p_i$ copies of $F_i$, it is immediate that $A$ and $B$ consists in a positive number of copies of $C$. □

A sequence $A$ of $\sigma$-structures is $X$-convergent if $\langle \langle \varphi_i, A \rangle \rangle_{n \in \mathbb{N}}$ converges for each $\varphi \in X$. This provides a unifying to left and local convergence, as mentioned in the introduction: left convergence coincides with QF$^\omega$-convergence and local convergence with FO$^\text{local}$-convergence (when restricted to graphs with bounded degrees). The term of structural convergence covers the general notions of X-convergence.

The basic result of [19], which is going to provide us a guideline for a proper generalization of local-global convergence is the representation theorem for structural limits in terms of probability measures. We adopt [19] to the setting of this paper.

2.5. **The Representation Theorem for Structural Limits.** For a countable signature $\sigma$ and a fragment $X$ of $\text{FO}(\sigma)$ we denote by $S^\sigma_X$ the Stone dual of the Lindenbaum-Tarski $B^\sigma_X$ of $X$, which is a compact Polish space. Recall that the points of $B^\sigma_X$ are the maximal consistent subsets of $X$ (or equivalently the ultrafilters on $B^\sigma_X$). The topology of $S^\sigma_X$ is generated by the base of the clopen subsets of $S^\sigma_X$, which are in bijection with the formulas in $X$ by

$$\varphi \mapsto K(\varphi) = \{ t \in S^\sigma_X \mid \varphi \in t \}.$$ 

In the setting of this paper we work with metric (and, notably, pseudo-metric) spaces. First note that the topology of $S^\sigma_X$ is metrizable by the several metrics, including the metrics we introduce now.

A *chain covering* of $X$ is an increasing sequence $\mathcal{X} = (X_1, X_2, \ldots)$ of finite sets (i.e. $X_1 \subset X_2 \subset \ldots \subset X_n \subset \ldots$) such that every formula in $X$ is logically equivalent to a formula in $\bigcup_{i \geq 1} X_i$. The metric $\delta_{\mathcal{X}}$ induced by $\mathcal{X}$ on $S^\sigma_X$ is defined by

$$\delta_{\mathcal{X}}(t_1, t_2) = \inf \{ 1/n \mid (t_1 \triangle t_2) \cap X_n = \emptyset \},$$

where $t_1 \triangle t_2$ stands for the symmetric difference of the sets $t_1$ and $t_2$.

First-order limits (shortly FO-limits) and, more generally, $X$-limits can be uniquely represented by a probability measure $\mu$ on the Stone space $\mathcal{S}$ dual to the Lindenbaum-Tarski algebra of the formulas. This can be formulated as follows.

**Theorem 2.5.1 ([19]).** Let $\sigma$ be a countable signature, let $X$ be a fragment of $\text{FO}(\sigma)$ closed under disjunction, conjunction and negation, let $B^\sigma_X$ be the Lindenbaum-Tarski algebra of $X$, and let $S^\sigma_X$ be the Stone dual of $B^\sigma_X$.

Then there is a map $T^\sigma_X$ from the space Rel($\sigma$) of finite $\sigma$-structures to the space of $P(S^\sigma_X)$ of probability measures on the Stone space $S^\sigma_X$, such that for every $A \in \text{Rel}(\sigma)$ and
every \( \phi \in X \) we have

\[
\langle \phi, A \rangle = \int_{S^X_\sigma} 1_{\phi}(t) \, d\mu_A(t),
\]

where \( \mu_A = T^X_\sigma(A) \) and \( 1_{\phi} \) is the indicator function of the clopen subset \( K(\phi) \) of \( S^X_\sigma \) dual to the formula \( \phi \in X \) in Stone duality, i.e.

\[
1_{\phi}(t) = \begin{cases} 
1 & \text{if } \phi \in t \\
0 & \text{otherwise}
\end{cases}.
\]

Additionally\(^2\) if the fragment \( X \) includes \( \mathrm{FO}_0 \) or \( \mathrm{QF} \) then the mapping \( T^X_\sigma \) is one-to-one.

In this setting, a sequence \( A \) of finite \( \sigma \)-structures is \( X \)-convergent if and only if the measures \( T^X_\sigma(A_n) \) converge weakly to some measure \( \mu \). Then for every first-order formula \( \phi \in X \) we have

\[
\int_{S^X_\sigma} 1_{\phi}(t) \, d(t) = \lim_{n \to \infty} \int_{S^X_\sigma} 1_{\phi}(t) \, d\mu_A(n) = \lim_{n \to \infty} \langle \phi, A_n \rangle,
\]

where \( \mu_{A_n} = T^X_\sigma(A_n) \).

Assume that a subgroup \( \Gamma \) of the group \( \mathbb{S}_{\infty} \) of permutations of \( \mathbb{N} \) acts on the first-order formulas in \( X \) by permuting the free variables. Then this action induces an action on \( S^X_\sigma \), and the probability measure \( T^X_\sigma(A) \) associated with a finite structure \( A \) is obviously \( \Gamma \)-invariant, thus so is the weak limit \( \mu \) of a sequence \( (T^X_\sigma(A_n))_{n \in \mathbb{N}} \) of probability measures associated with the finite structures of an \( X \)-convergent sequence. It follows that the measure \( \mu \) appearing in (3) has the property to be \( \Gamma \)-invariant.

This theorem generalizes the representation of the limit of a left-convergent sequence of graphs by an infinite exchangeable random graph \([2, 16]\) and the representation of the limit of a local-convergent sequence of bounded degree graphs by a unimodular distribution \([5]\).

Figure 1 schematically depicts some of the notions related to the representation theorem.

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\(^2\)Note that in \([19]\) the condition on \( X \) was erroneously omitted.
This metric in turn uniquely defines a pseudometric $d_{\mathcal{X}}$ on $\text{Rel}(\sigma)$ such that the mapping $T^X_\sigma$ induces an isometric embedding of $\text{Rel}(\sigma)/\equiv_X$ into $P(S^X_\sigma)$:

$$
\text{dist}_\mathcal{X}(A, B) = d_{\mathcal{X}}(T^X_\sigma(A), T^X_\sigma(B)).
$$

Note that we have the following expression for $\text{dist}_\mathcal{X}$:

$$
\text{dist}_\mathcal{X}(A, B) = \inf_{n \in \mathbb{N}} \left\{ \max \left( \frac{1}{n}, \max_{\phi \in \mathcal{X}_n} \left| \int_{S^X_\sigma} 1_\phi(t) d(\mu_1 - \mu_2)(t) \right| \right) \right\}.
$$

It is easily checked that, as expected, a sequence $A$ is $X$-convergent if and only if it is Cauchy for $\text{dist}_\mathcal{X}$.

We denote by $\mathcal{M}^X_\sigma \subseteq P(S^X_\sigma)$ the space of probability measures on $S^X_\sigma$ associated to finite $\sigma$-structures:

$$
\mathcal{M}^X_\sigma = \{ T^X_\sigma(A) \mid A \in \text{Rel}(\sigma) \}.
$$

We denote by $\overline{\mathcal{M}^X_\sigma}$ the weak closure of $\mathcal{M}^X_\sigma$ in $P(S^X_\sigma)$ and by $(\text{Rel}(\sigma)/\equiv_X, \text{dist}_\mathcal{X})$ the completion of the pseudometric space $(\text{Rel}(\sigma), \text{dist}_\mathcal{X})$. Note that $(\text{Rel}(\sigma)/\equiv_X, \text{dist}_\mathcal{X})$ has a dense subspace naturally identified with $(\text{Rel}(\sigma)/\equiv_X, \text{dist}_\mathcal{X})$, and $T^X_\sigma$ induces an isometric isomorphism of $(\text{Rel}(\sigma)/\equiv_X, \text{dist}_\mathcal{X})$ and $(\overline{\mathcal{M}^X_\sigma}, d_{\mathcal{X}})$. Consequently both spaces are separable compact metric spaces.

3. FROM INTERPRETATION TO LIFT CONVERGENCE

Our basic approach to local-global convergence is by means of lifts of structures, which demands a change of signature. By doing so we still have to preserve some functorial properties and this is done by means of interpretations.

3.1. A Categorical Approach to Interpretations. Interpretations of classes of relational structures in other classes of relational structures are a useful and powerful technique to transfer properties from one class of structures to another (with possibly a different signature).

First, we define interpretations syntactically (in the spirit of [17]), which allows us to organize them as a category. This functorial view will be particularly useful in our setting.

Let $\tau, \sigma$ be countable relational signatures. An interpretation $I$ of $\sigma$-structures in $\tau$-structures is a triple $(\nu, \eta, (\rho_R)_{R \in \sigma})$, where:

- $\nu(\overline{x}) \in \text{FO}(\tau)$ is a formula defined on $p$ tuples of variables $\overline{x}$;
- $\eta(\overline{x}, \overline{y}) \in \text{FO}(\tau)$ is a formula defining an equivalence relation on $p$-tuples (satisfying $\nu$);
- for each relation $R \in \sigma$ of arity $k$, the formula $\rho_R(\overline{x}_1, \ldots, \overline{x}_k) \in \text{FO}(\tau)$ (with $|\overline{x}_1| = \cdots = |\overline{x}_k| = p$) is compatible with $\eta$, meaning
  $$
  \bigwedge_{i=1}^k \eta(\overline{x}_i, \overline{y}_i) \vdash \rho_R(\overline{x}_1, \ldots, \overline{x}_k) \leftrightarrow \rho_R(\overline{y}_1, \ldots, \overline{y}_k).
  $$

- the formula $\eta(\overline{x}, \overline{y})$ is a formula defining an equivalence relation on $p$-tuples (satisfying $\nu$);
- for each relation $R \in \sigma$ of arity $k$, the formula $\rho_R(\overline{x}_1, \ldots, \overline{x}_k) \in \text{FO}(\tau)$ (with $|\overline{x}_1| = \cdots = |\overline{x}_k| = p$) is compatible with $\eta$, meaning
  $$
  \bigwedge_{i=1}^k \eta(\overline{x}_i, \overline{y}_i) \vdash \rho_R(\overline{x}_1, \ldots, \overline{x}_k) \leftrightarrow \rho_R(\overline{y}_1, \ldots, \overline{y}_k).
  $$

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By replacing equality by $\eta$, relation $R$ by $\rho_R$ and by conditioning quantifications using $\nu$ one easily checks that the interpretation $I$ allows to associate to each formula $\phi(x_1, \ldots, x_k) \in \text{FO}(\sigma)$ a formula $\hat{\phi}(\overline{x}_1, \ldots, \overline{x}_k) \in \text{FO}(\tau)$. We define

$$L(I) : \text{FO}(\sigma) \rightarrow \text{FO}(\tau)$$

as the mapping $\phi \mapsto \hat{\phi}$.

Note that we have $L(I)(1) = \nu$, $L(I)(x_1 = x_2) = \eta$, and $L(I)(R) = \rho_R$ for every $R \in \sigma$. Hence $L(I)$ fully determines $I$.

This definition allows us to consider interpretations $I : \tau \rightarrow \sigma$ as morphisms in a category of interpretations. The objects of this category are all countable relational signatures (here denoted by $\sigma, \tau, \ldots$) and morphisms $I : \tau \rightarrow \sigma$ are triples $(\nu, \eta, (\rho_R)_{R \in \sigma})$ forming an interpretation as above. Morphisms compose as if $I : \tau \rightarrow \sigma$ and $J : \sigma \rightarrow \kappa$ are interpretations then we can define

$$J \circ I = (L(J \circ I)(x_1 = x_1), L(J \circ I)(x_1 = x_2), (L(J \circ I)(S))_{S \in \kappa}).$$

The identity (for $\sigma$) is provided by the morphism $(x_1 = x_1, x_1 = x_2, (R)_{R \in \sigma})$. Thus we indeed have a category of interpretations.

A basic interpretation $[\underline{22}]$ is an interpretation $(\nu, \eta, (\rho_R)_{R \in \sigma})$ such that $\nu(\overline{x}) := (\overline{x} = \overline{x})$ and $\eta(\overline{x}_1, \overline{x}_2) := (\overline{x}_1 = \overline{x}_2)$. (For instance the identity interpretation defined above is a basic interpretation.)

Note that every basic interpretation $I : \tau \rightarrow \sigma$ induces a homomorphism

$$H(I) : B_\tau \rightarrow B_\sigma \quad \text{defined by} \quad H(I)([\phi]) = [L(I)(\phi)],$$

where $[\phi]$ denotes the class of $\phi$ for logical equivalence. The mapping $H$ is actually a contravariant functor from the category of interpretations to the category of Boolean algebras.

By Stone duality theorem, the interpretation $I$ also defines a continuous function

$$F(I) : S_\tau \rightarrow S_\sigma \quad \text{defined by} \quad F(I)(\iota) = \{ \phi \mid L(I)(\phi) \in \iota \}.$$ 

Note that $F$ is a covariant functor from the category of interpretations to the category of Stone spaces.

Finally, the interpretation $I$ also defines a mapping

$$P(I) : \text{Rel}(\tau) \rightarrow \text{Rel}(\sigma)$$

as follows:

- The domain of $P(I)(A)$ is $\nu(A)/\eta$, that is all the $\eta$-equivalence classes of $p$-tuples in $\nu(A)$.
- For every relational symbol $R \in \sigma$ with arity $k$ (and associated formula $\rho_R$) we have

$$P(I)(A) \vDash R([\overline{v}_1], \ldots, [\overline{v}_k]) \iff A \vDash \rho_R(\overline{v}_1, \ldots, \overline{v}_k).$$

(Note that this does not depend on the choice of the representatives $\overline{v}_1, \ldots, \overline{v}_k$ of the $\eta$-equivalence classes $[\overline{v}_1], \ldots, [\overline{v}_k]$.)

This mapping $P(I) : \text{Rel}(\tau) \rightarrow \text{Rel}(\sigma)$ is what is usually meant by an interpretation (of $\sigma$-structures in $\tau$-structures, see [14]). It is easily checked that the mapping $pSI$ has the property that for every formula $\phi \in \text{FO}(\sigma)$ with $k$ free variables and every $[\overline{v}_1], \ldots, [\overline{v}_k]$ in $P(I)(A)$ we have

$$P(I)(A) \vDash \phi([\overline{v}_1], \ldots, [\overline{v}_k]) \iff A \vDash L(I)(\phi)(\overline{v}_1, \ldots, \overline{v}_k).$$

The interpretations, which we shall the most frequently consider, belong to the following types of basic interpretations (which are easily checked to be basic interpretations):

- forgetful interpretations that simply forget some of the relations,
- renaming interpretations that bijectively map a signature to another, mapping a relational symbol to a relational symbol with same arity,
- projecting interpretations that forget some symbols and rename others.
Our categorical approach allows us to obtain a more functorial point of view:

\[
\begin{array}{c|c|c|c|c|c}
\tau & \text{FO}(\tau) & B_\tau & \text{Stone duality} & S_\tau & \text{Rel}(\tau) \\
\hline
\tau \rightarrow & \text{L}(\tau) & H(\tau) & \text{Stone duality} & F(\tau) & P(\tau)
\end{array}
\]

In this diagram the mapping \( F(\tau) \) is the pushforward defined by \( F(\tau) \) (see Lemma 3.2.1 below).

One can also consider the case where we do not consider all first-order formulas. Let \( X \) be a fragment of \( \text{FO}(\tau) \) and let \( Y = L(\tau)^{-1}(X) \). (Note that if \( X \) is closed by disjunction, conjunction and negation, so is \( L(\tau)^{-1}(X) \).) The basic interpretation \( \text{I} \) then defines a homomorphism

\[ H(\text{I}) : B_\sigma^Y \rightarrow B_\tau^X, \]

which is the restriction of \( H(\text{I}) \) to \( B_\sigma^Y \). By duality, this homomorphism defines a continuous mapping

\[ \hat{F}(\text{I}) : S_\sigma^X \rightarrow S_\tau^Y. \]

In particular, if \( K_\phi \) is the clopen subset of \( S_\sigma^Y \) defined by \( \phi \in Y \) then \( \hat{F}(\text{I})^{-1}(K_\phi) \) is the clopen subset \( K_{L(I)(\phi)} \) of \( S_\tau^X \). Note that we have \( F(\text{I}) = \hat{F}(\text{I}) \circ \Pi_\tau^X \), where \( \Pi_\tau^X \) is the natural projection from \( S_\tau \) to \( S_\tau^X \).

3.2. Metric properties of interpretations. We have seen in the previous section that interpretations define continuous functions between Stone spaces. This property can be used to transfer convergence from one signature to another. This is done in a very general setting we introduce now.

Let \( I : \tau \rightarrow \sigma \) be a basic interpretation, let \( X \) be a fragment of \( \text{FO}(\tau) \), let \( Y = L(\tau)^{-1}(X) \), let \( \mathcal{X} \) be a chain coverings of \( X \), and let \( \mathcal{Y} \) be a chain covering of \( Y \) such that every formula in \( L(\tau)(Y_n) \) is logically equivalent to a formula in \( X_n \).

Let us explain this choice of \( Y \).

In \( \text{Im} \, P(\text{I}) \) we should not distinguish two finite structures \( P(\text{I})(\text{A}) \) and \( P(\text{I})(\text{B}) \) if there exist a chain \( C_1, \ldots, C_{2n+1} \) of finite structures such that \( C_1 = \text{A}, C_{2n+1} = \text{B}, C_{2i-1} \equiv_X X_{2i} \), and \( P(\text{I})(C_{2i}) = P(\text{I})(C_{2i+1}) \) for \( i = 1, \ldots, n \). But \( P(\text{I})(\text{A}') = P(\text{I})(\text{B}') \) holds if and only if \( \langle \phi, \text{A}' \rangle = \langle \phi, \text{B}' \rangle \) for every \( \phi \in L(\tau)(\text{FO}(\sigma)) \). Hence the conditions can be rewritten as

\[
\forall \phi \in X \quad \langle \phi, C_{2i-1} \rangle = \langle \phi, C_{2i} \rangle \\
\forall \phi \in L(\tau)(\text{FO}(\sigma)) \quad \langle \phi, C_{2i} \rangle = \langle \phi, C_{2i+1} \rangle
\]

A necessary (but maybe not sufficient) condition is obviously that

\[
\forall \phi \in X \cap L(\tau)(\text{FO}(\sigma)) \quad \langle \phi, \text{A} \rangle = \langle \phi, \text{B} \rangle,
\]

that is:

\[
\forall \phi \in L(\tau)^{-1}(X) \quad \langle \phi, P(\text{I})(\text{A}) \rangle = \langle \phi, P(\text{I})(\text{B}) \rangle,
\]

which we can rewrite as \( P(\text{I})(\text{A}) \equiv_X P(\text{I})(\text{B}) \). This shows that the fragment \( Y \) is sufficiently small to ensure the continuity of \( P(\text{I}) \). By our choice of the chain covering \( \mathcal{Y} \) we further get that \( P(\text{I}) \) induces a short map (that is a 1-Lipschitz function). We summarize this in the following lemma.

**Lemma 3.2.1.** In the above setting and notation we have:

\[
\text{dist}_Y(P(\text{I})(\text{A}), P(\text{I})(\text{B})) \leq \text{dist}_X(\text{A}, \text{B}).
\]
This can be restated as follows: Let \((\mathsf{Rel}(\tau)/\equiv_X, \mathsf{dist}_X)\) and \((\mathsf{Rel}(\sigma)/\equiv_Y, \mathsf{dist}_Y)\) be the quotient metric spaces induced by the pseudometric spaces \((\mathsf{Rel}(\tau), \mathsf{dist}_X)\) and \((\mathsf{Rel}(\sigma), \mathsf{dist}_Y)\). Then the unique continuous function \(\hat{P}(\mathfrak{T}) : (\mathsf{Rel}(\tau)/\equiv_X, \mathsf{dist}_X) \rightarrow (\mathsf{Rel}(\sigma)/\equiv_Y, \mathsf{dist}_Y)\) such that \(P(I)([A]_X) = [P(I)(A)]_Y\) is a short map.

**Proof.** For every pair \(A, B\) au \(\tau\)-structures we have

\[
\mathsf{dist}_Y(P(I)(A), P(I)(B)) = \inf_{n \in \mathbb{N}} \left\{ \max \left\{ \frac{1}{n}, \max_{\phi \in \mathcal{Y}_n} \| (\phi, P(I)(A)) - (\phi, P(I)(B)) \| \right\} \right\}
\]

\[
= \inf_{n \in \mathbb{N}} \left\{ \max \left\{ \frac{1}{n}, \max_{\psi \in \mathcal{Y} \cap \mathcal{I}^+_{\mathcal{M}}} \| (\phi, A) - (\phi, B) \| \right\} \right\}
\]

\[
\leq \inf_{n \in \mathbb{N}} \left\{ \max \left\{ \frac{1}{n}, \max_{\psi \in \mathcal{X}_n} \| (\phi, A) - (\phi, B) \| \right\} \right\}
\]

\[
= \mathsf{dist}_X(A, B).
\]

(In particular \(A \equiv_X B\) imply that \(\text{dist}_Y(P(I)(A), P(I)(B)) = 0\) thus \(P(I)(A) \equiv_Y P(I)(B)\) hence \(P(I)\) descends to the quotient and there exists a unique map

\[
\hat{P}(\mathfrak{T}) : (\mathsf{Rel}(\tau)/\equiv_X, \mathsf{dist}_X) \rightarrow (\mathsf{Rel}(\sigma)/\equiv_Y, \mathsf{dist}_Y)
\]

such that \(\hat{P}(I)([A]_X) = [P(I)(A)]_Y\).}

Let \(\mathcal{M}_{\mathcal{X}, I} = F(I_n(\mathcal{M}_X))\) and let \(\overline{\mathcal{M}_{\mathcal{X}, I}} = F(I_n(\overline{\mathcal{M}_X}))\) be the closure of \(\mathcal{M}_{\mathcal{X}, I}\) in \((P(S_n^X), \mathsf{d}_n^P)\). We tried to summarize in Fig. 2 the relations between the different (pseudo)metric spaces defined from signatures, fragments, and interpretations.

**Figure 2.** The considered (pseudo)metric spaces and their relations. Unlabeled arrows correspond to inclusions (\(\rightarrow\)) or isometric embeddings (\(\rightarrow\)). In this diagram the space \((\mathsf{Im} P(I), \mathsf{dist}_Y)\) is the completion of the pseudometric space \((\mathsf{Im} P(I), \mathsf{dist}_Y)\).
3.3. **Lift-Hausdorff convergence.** We now show how all the above constructions nicely fit in the definition of the lift-Hausdorff convergence. We first show how the definition derives from the preceding notions dealing with general basic interpretations.

Let \( I : \tau \rightarrow \sigma \) be a fixed interpretation, let \( X \) be a fixed fragment of \( \text{FO}(\tau) \), let \( \mathfrak{X} \) be a fixed cover chain of \( X \), and let \( \mathcal{Y} \) be a chain covering of \( Y \) such that every formula in \( L(I)(Y) \) is logically equivalent to a formula in \( X \).

According to Lemma 2.2.1 the pushforward mapping \( F(I) : P(S_\tau^X) \rightarrow P(S_\sigma^Y) \). Then the Lévy-Prokhorov distance \( d^P_\tau \) on \( \mathcal{M}_\tau^X \) defines a Hausdorff distance \( d^H_\tau \) on the space \( \mathcal{M}_\tau^X \) of non-empty closed subsets of \( \mathcal{M}_\tau^X \):

\[
(7) \quad d^H_\tau(M_1, M_2) = \max \left( \sup_{\mu_1 \in M_1} \inf_{\mu_2 \in M_2} d^P_\tau(\mu_1, \mu_2), \sup_{\mu_2 \in M_2} \inf_{\mu_1 \in M_1} d^P_\tau(\mu_1, \mu_2) \right).
\]

Also the pseudometric \( \text{dist}_X \) on \( \text{Rel}(\tau) \) defines a Hausdorff pseudometric \( d^H_\tau \) on the space of non-empty closed subsets of \( \text{Rel}(\tau) \) (for the topology induced by the pseudometric \( \text{dist}_X \)):

\[
(8) \quad \text{dist}_X^H(F_1, F_2) = \max \left( \sup_{A_1 \in F_1} \inf_{A_2 \in F_2} \text{dist}_X(A_1, A_2), \sup_{A_2 \in F_2} \inf_{A_1 \in F_1} \text{dist}_X(A_1, A_2) \right).
\]

These (pseudo)metrics are related by the following equation (where \( F_1 \) and \( F_2 \) denote non-empty closed subsets of \( \text{Rel}(\tau) \):

\[
\text{dist}_X^H(F_1, F_2) = d^H_\tau(T_\tau^X(F_1), T_\tau^X(F_2))
\]

Using the injective mapping \( F(I)^{-1} \) we can transfer to \( \mathcal{M}_{X,1} \) the Hausdorff distance \( d^H_\tau \) defined on \( \mathcal{M}_\tau^X \), thus defining a distance \( d^H_\tau \) on \( \mathcal{M}_{X,1} \):

\[
d^H_\tau(\mu_1, \mu_2) = d^H_\tau(F(I)^{-1}(\mu_1), F(I)^{-1}(\mu_2)).
\]

(Note that this metric usually does not define the same topology as \( d^P_\tau \).)

Using the mapping \( T_\sigma^Y \) we can transfer to \( \text{Im} P(I) \) the metric \( d^H_{X,1} \) just defined on \( \mathcal{M}_{X,1} \). As \( T_\sigma^Y \) is not injective in general we get this way a pseudometric \( d^H_{X,1} \) on \( \text{Im} P(I) \):

\[
\text{dist}_{X,1}(A, B) = \max\left( d^H_{X,1}(F(I)^{-1}(T_\sigma^Y(A)), F(I)^{-1}(T_\sigma^Y(B))), d^H_{X,1}(P(I)^{-1}(A), P(I)^{-1}(B)) \right).
\]

Hence we have
\begin{align}
\text{dist}^H_{X,I}(A, B) &= \text{dist}^H_I(P(I)^{-1}(A), P(I)^{-1}(B)).
\end{align}

The situation is summarized in the following diagram:

\begin{center}
\begin{tikzcd}
(\mathcal{X} \rightarrow \mathcal{X}, d^H_X) \arrow{r}{\tau^X} \arrow{d}{F(I)^{-1}} & (\mathcal{X} \rightarrow \mathcal{X}, \text{dist}^H_X) \arrow{d}{P(I)^{-1}} \\
(\mathcal{M}_{X,I}, \mathcal{H}^H_{X,I}) \arrow{r}{\tau^Y} & (\text{Im } P(I), \text{dist}^H_{X,I})
\end{tikzcd}
\end{center}

It follows from Lemma [3.2.1] that for every \(A, B \in \text{Im } P(I)\) we have

\begin{align}
\text{dist}^H_{X,I}(A, B) &\geq \text{dist}_\varphi(A, B).
\end{align}

(In particular the topology defined by the pseudometric \(\text{dist}_{X,I}\) is finer that the topology defined by the pseudometric \(\text{dist}_\varphi\).)

Our basic notion of convergence with respect to an interpretation is the following.

**Definition 3.3.1** (Lift-Hausdorff convergence). Let \(I : \tau \rightarrow \sigma\) be a basic interpretation and let \(X\) be a fragment of \(\text{FO}(\tau)\). A sequence \(A\) of finite \(\sigma\)-structures in \(\text{Im } P(I)\) is \(I_\tau(X)\)-convergent if it is Cauchy for \(\text{dist}^H_{X,I}\).

We refer to the general notion of \(I_\tau(X)\)-convergence as lift-Hausdorff convergence (or simply lift convergence).

By construction, the \(I_\tau(X)\)-limit of a sequence of \(\sigma\)-structures can be uniquely represented by means of a non-empty compact subset of \(\mathcal{M}^X\).

This convergence notion may seem complicated at first glance, but \(I_\tau(X)\)-convergence admits a concise and practical equivalent definition:

**Theorem 3.3.2** (Subsequence completion). A sequence \(A\) of \(\sigma\)-structures in \(\text{Im } P(I)\) is \(I_\tau(X)\)-convergent if and only if for every \(X\)-convergent subsequence \(B_f\) (i.e. for \((B_{f(n)}))_{n \in \mathbb{N}}\) with \(f\) monotone increasing) of \(\tau\)-structures such that \(P(I)(B_{f}) = A_f\) (i.e. \(P(I)(B_{f(i)}) = A_{f(i)}\) for every \(i \in \mathbb{N}\)) there exists an \(X\)-convergent sequence \(C\) of \(\tau\)-structures, such that \(C_f = B_f\) and \(P(I)(C) = A\).

**Proof.** We consider the two implications.

First assume that the sequence \(A\) of \(\sigma\)-structures in \(\text{Im } P(I)\) is \(I_\tau(X)\)-convergent and let \(B_f\) be an \(X\)-convergent subsequence such that \(P(I)(B_f) = A_f\). For every positive integer \(m\), let \(N(m)\) be minimum integer such that \(f(N(m)) \geq m\). Let \(C_m\) be a \(\tau\)-structure in \(P(I)^{-1}(A_{N(m)})\) such that \(\text{dist}_X(C_m, B_{f(N(m))})\) is minimum. Note that the minimum is attained as \(P(I)^{-1}(A_{N(m)})\) is compact. By definition we have

\[
\text{dist}_X(C_m, B_{f(N(m))}) \leq \text{dist}_{X,I}(A_m, A_{f(N(m))}).
\]

As \(A\) is Cauchy for \(\text{dist}_{X,I}\) and \(B_f\) is Cauchy for \(\text{dist}_X\) it directly follows that \(C\) is Cauchy for \(\text{dist}_{X,I}\), i.e. that \(C\) is \(X\)-convergent.

We now consider the other direction. Assume that for every \(X\)-convergent subsequence \(B_f\) such that \(P(I)(B_f) = A_f\) there exists a sequence \(C\) such that \(C_f = B_f\) and \(P(I)(C) = A\), and assume for contradiction that the sequence \(A\) is not \(I_\tau(X)\)-convergent. Then there
exists \( \alpha > 0 \), such that for every integer \( N \) there exist integers \( n, m > N \) and \( B_m \in P(I)^{-1}(A_m) \) such that for every \( C_n \in P(I)^{-1}(A_n) \) we have \( \text{dist}_X(B_m, C_n) > \alpha \). This allows to construct subsequence \( B_f \) and \( C_f \) (where \( (f(i), g(i)) \) correspond to a pair of admissible values of \( m \) and \( n \) with \( \min(m,n) > \max(f(i)-1, g(i)-1) \)). Moreover, we can assume that \( B_f \) is \( X \)-convergent. By assumption the subsequence \( B_f \) can be extended into a full \( X \)-convergent sequence, which we (still) denote by \( B \) such that \( P(I)(B) = A \). In particular, there exist some \( N \) such that for every \( n, m > N \) we have \( \text{dist}_X(B_n, B_m) < \alpha \). In particular, \( \text{dist}_X(B_{f(n)}, B_{g(n)}) < \alpha \), what contradicts the minimality hypothesis on \( \text{dist}_X(B_{f(n)}, C_{g(n)}) \).

This lemma gives an easy proof of the following result.

**Proposition 3.3.3.** Let \( C \) be a class of \( \sigma \)-structures, let \( I : \tau \to \sigma \) be an interpretation, and let \( X, Y \) be fragments of \( \text{FO}(\tau) \).

If \( X \)-convergence implies \( Y \)-convergence in the class \( D = \{ B \mid P(I)(B) \in C \} \) of \( \tau \)-structures then \( I_*(X) \)-convergence implies \( I_*(Y) \)-convergence in the class \( C \).

**Proof.** Let \( A \) be an \( I_*(X) \)-convergent sequence of \( \tau \)-structures in \( C \) and let \( B_f \) be a \( Y \)-converging subsequence of \( \tau \)-structures (in \( D \)) such that \( P(I)(B_{f(n)}) = A_{f(n)} \). Let \( B_{g,f} \) be an \( X \)-converging subsequence of \( B_f \). Then there exists, according to Theorem 3.3.2 an \( X \)-convergent sequence \( C \) such that \( C_{g,f} = B_{g,f} \) and \( P(I)(C) = A \) (hence \( C \) is in \( D \)). As \( X \)-convergence implies \( Y \)-convergence on \( D \) the sequence \( D \) is \( Y \)-convergent, and has the same \( Y \)-limit as the \( Y \)-convergent sequence \( B_f \) as they share infinitely many elements. It follows that the sequence \( D \) defined by

\[
D_n = \begin{cases} B_n & \text{if } (\exists i) \ n = f(i) \\ C_n & \text{otherwise} \end{cases}
\]

as the property that \( D_f = B_f \) and \( P(I)(D) = A \). By Theorem 3.3.2 we deduce that \( A \) is \( I_*(Y) \)-convergent.

Here are some more remarks indicating convenient properties of \( I_*(X) \)-convergence.

First note that if \( I : \sigma \to \sigma \) is the identity interpretation, then \( \text{dist}_X I = \text{dist}_X I_*(X) \)-convergence is the same as \( X \)-convergence. Also, we have that every sequence \( A \) in \( \text{Im} P(I) \) has an \( I_*(X) \)-convergent subsequence. Finally, let us remark that for every \( I : \tau \to \sigma \), \( I_*(\text{FO}) \)-convergence implies \( \text{FO} \)-convergence.

Let \( \tilde{\tau} \) be the signature obtained from \( \tau \) by duplicating each relation symbol countably many times, which we denote by \( \tilde{\tau} = \kappa \tilde{\tau} \). To each symbol \( R \in \tau \) correspond the symbols \( R' \) in \( \tilde{\tau} \) (for \( i \in \mathbb{N} \)). We define the interpretation \( I_\tilde{\tau} \) obtained from \( I \) by replacing relations \( R \) by \( R' \) (\( I_\tilde{\tau} \) is a clone of \( I \) based on the relations \( R' \)).

**Proposition 3.3.4** (Almost \( I_*(X) \)-limit probability measure). Let \( A \) be an \( I_*(X) \)-convergent sequence of finite \( \sigma \)-structures.

There exists a probability measure \( \tilde{\mu} \in \bar{\mathscr{M}}^X_{\tilde{\tau}} \) such that for every \( \epsilon > 0 \) and for every \( C \) such that \( P(I)(C) = A \) there exists \( i \in \mathbb{N} \) such that

\[
d_{I_\tilde{\tau}}^X(F(I)_\epsilon(\tilde{\mu}), \lim_{n \to \infty} T_{\tau}^X(C_n)) < \epsilon,
\]

where \( \lim_{n \to \infty} \) stands for the weak limit of probability measures.

**Proof.** For \( i \in \mathbb{N} \) we choose \( B_{n,i} \) such that \( P(I)(B_{n,i}) = A_n \). We construct the \( \tilde{\tau} \)-structure \( \tilde{B}_n \) by amalgamating all the relations of all the \( B_{n,i} \). We denote by \( \tilde{S}_i \) the interpreting projection \( \tilde{B}_n \to B_{n,i} \). Note that \( I_\tilde{\tau} = I \circ S_i \). Then we have
Then we consider an \(\hat{X}\)-convergent subsequence \(\hat{\mathbf{B}}\) of \(\hat{\mathbf{B}}\), the limit of which is represented by the probability measure \(\hat{\mu} \in \mathcal{M}_{\hat{X}}\). The measure \(\hat{\mu}\) has obviously the claimed property. \(\square\)

3.4. Local Global Convergence. In this section we show how the abstract framework of Section 3.3 provides a proper setting for local-global convergence.

The notion of local-global convergence of graphs with bounded degrees has been introduced by Bollobás and Riordan [6] based on a colored neighborhood metric. In [12], Hatami, Lovász, and Szegedy gave the following equivalent definition:

**Definition 3.4.1** ([12]). A graph sequence \(\mathbf{G}\) of graphs with maximum degree \(D\) is local-global convergent if for every \(r, k \in \mathbb{N}\) and \(\epsilon > 0\) there is an index \(l\) such that if \(n, m > l\), then for every coloring of the vertices of \(G_n\) with \(k\) colors, there is a coloring of the vertices of \(G_m\) with \(k\) colors such that the total variation distance between the distributions of colored neighborhoods of radius \(r\) in \(G_n\) and \(G_m\) is at most \(\epsilon > 0\).

The following is the principal result which relates local-global convergence to a lift-Hausdorff convergence.

Let us consider a fixed countable signature \(\sigma\) and the signature obtained from \(\sigma\) by adding countably many unary symbols. Thus \(\sigma \subseteq \tau\). Let

\[
\text{Sh} : \sigma \rightarrow \tau
\]

be the forgetful interpretation (Sh for “Shadow”). This means \(\text{Sh} = (\nu, \eta, (\rho_R)_{R \in \sigma})\), where

\[
\nu(x_1) := (x_1 = x_1), \quad \eta(x_1, x_2) := (x_1 = x_2), \quad \text{and} \quad \rho_R(x_1, \ldots, x_p) := R(x_1, \ldots, x_p)
\]

for \(R \in \sigma\) with arity \(p\). Then, for instance:

- for a \(\tau\)-structure \(\mathbf{A}\), the \(\sigma\)-structure \(P(\text{Sh})(\mathbf{A})\) is obtained from \(\mathbf{A}\) by forgetting all unary relations in \(\tau \setminus \sigma\);
- for a formula \(\phi \in \text{FO}(\sigma)\), we have \(L(\text{Sh})(\phi) = \phi\);
- for \(t \in S_2\) we have we have \(F(\text{Sh})(t) = t \cap \text{FO}(\sigma)\).

By [19] we know that \(\text{FO}^\text{local}_{1}\)-convergence coincides with \(\text{FO}^\text{local}\)-convergence for graphs with bounded degree. By Proposition 3.3.3 the notions of \(\text{Sh}_\epsilon(\text{FO}^\text{local})\)-convergence and \(\text{Sh}_\epsilon(\text{FO}^\text{local}_{1})\)-convergence will also coincide for graphs with bounded degrees. These notions actually coincide with the notion of local-global convergence of graphs with bounded degrees:

**Proposition 3.4.2.** Let \(\mathbf{G}\) be a sequence of graphs with maximum degree \(D\). Then the following are equivalent:

1. \(\mathbf{G}\) is local-global convergent,
2. \(\mathbf{G}\) is \(\text{Sh}_\epsilon(\text{FO}^\text{local})\)-convergent,
3. \(\mathbf{G}\) is \(\text{Sh}_\epsilon(\text{FO}^\text{local}_{1})\)-convergent.

**Proof.** For classes of colored graphs with degree at most \(D\), \(\text{FO}^\text{local}_{1}\)-convergence is equivalent to \(\text{FO}^\text{local}\)-convergence (see [19]). It follows from Proposition 3.3.3 that for these
graphs $\text{Sh}_n(\text{FO}^\text{local})$-convergence is equivalent to $\text{Sh}_n(\text{FO}^\text{local})$-convergence. Thus we only have to prove the equivalence of local-global convergence and $\text{Sh}_n(\text{FO}^\text{local})$-convergence.

We consider the fragment $X \subset \text{FO}^\text{local}_1$ of formulas consistent with the property of having maximum degree $D$. Consider a cover chain $X = (X_r)_{r \in \mathbb{N}}$ of $X$ where $X_r$ contains (one representative of the equivalence class of) each formula in $X$ that is $r$-local and use only the $r$ first unary predicates. (Note that $|X_r|$ is finite.)

It is easily checked that every $r$-local formula $\phi \in X_r$ is equivalent (on graphs with maximum degree $D$) to a formula $\psi$ of the form $\sqrt{f_{B, B'}}(\zeta_{B, B'}(x))$ where $\zeta_{B, B'}(x)$ expresses that the ball of radius $r$ rooted at $x$ is isomorphic to the rooted graph $B$, and $f_{B, B'}$ is a finite set of rooted graphs of radius at most $r$. It easily follows that the maximum of $|\{\phi, G_1\} - \{\phi, G_2\}|$ over $\phi \in X_r$ equals the total variation distance of the distributions of $r$-balls in $G_1$ and $G_2$ where we consider only the $r$ first colors, which we denote by $d_{TV}(r)(G_1, G_2)$. Then we have

\begin{equation}
\text{dist}_X(G_1, G_2) = \inf_{r \in \mathbb{N}} \left\{ \max \left\{ \frac{1}{r}, d_{TV}(r)(G_1, G_2) \right\} \right\}
\end{equation}

As one easily checks that $d_{TV}(r')(G_1, G_2) \geq d_{TV}(r)(G_1, G_2)$ if $r' \geq r$ we have that for every fixed integer $r$ we have

\begin{equation}
\min \left\{ \frac{1}{r}, d_{TV}(r)(G_1, G_2) \right\} \leq \text{dist}_X(G_1, G_2) \leq \max \left\{ \frac{1}{r}, d_{TV}(r)(G_1, G_2) \right\}.
\end{equation}

Now assume $G$ is $\text{Sh}_n(\text{FO}^\text{local})$-convergent. Let $k, r$ be a fixed integer. Then $\text{Sh}_n(\text{FO}^\text{local})$-convergence of $G$ easily implies the convergence of the lifts of $G_n$ by $k$ colors, which means that for every $\epsilon > 0$ there is an index $l$ such that if $n, m > l$, then for every coloring $G_n^+ + G_m^+$ of $G_n$ with $k$ colors, there is a coloring $G_n^+$ of $G_m^+$ with $k$ colors such that $\text{dist}_X(G_n^+, G_m^+) < \epsilon$ hence by \[\text{(12)}\] the total variation distance between the distributions of colored neighborhoods of radius $r$ in $G_n$ and $G_m$ is at most $\epsilon > 0$, provided that $\epsilon < 1/r$. Hence $G$ is local-global convergent.

Assume $G$ is local-global convergent. Then for every $\epsilon > 0$, letting $r = \lceil 1/\epsilon \rceil$, there exists an index $l$ such that if $n, m > l$, then for every coloring $G_n^+$ of $G_m^+$ with $r$ colors, there is a coloring $G_n^+$ of $G_m^+$ with $r$ colors such that the total variation distance between the distributions of colored neighborhoods of radius $r$ in $G_n$ and $G_m$ is at most $\epsilon$. Hence by \[\text{(12)}\] we have $\text{dist}_X(G_n^+, G_m^+) < \max(\epsilon, d_{TV}(r)(G_n, G_m)) \leq \epsilon$. (Note that we do not need to use any of the colors with index greater than $r$.) It follows that $G$ is $\text{Sh}_n(\text{FO}^\text{local})$-convergent. \hfill \Box

Motivated by this theorem we can extend the definition of local-global convergence to general graphs and relational structures:

**Definition 3.4.3** (Local-global convergence). A sequence $\mathbf{A}$ is local-global convergent if it is $\text{Sh}_n(\text{FO}^\text{local})$-convergent.

The stronger notion of $\text{Sh}_n(\text{FO})$-convergence also implies convergence of some graph invariants in an interesting way. This is, for instance, the case of the relative size of the stability number.

**Proposition 3.4.4.** Let $G$ be an $\text{Sh}_n(\text{FO})$-convergent sequence of graphs. The the independence ratio $a(G_n)/|G_n|$ converges.

**Proof.** Let $a = \liminf a(G_n)/|G_n|$. Let $G_n^+$ be obtained by marking (by $M$) a maximum independent set in $G_n$. (Thus $G_n = P(\text{Sh}(G_n^+))$. We extract a subsequence of $G^+$ with limit measure of $M(G_n^+)$ equal to $a$, then an FO-convergent subsequence. According to the lifting property, this subsequence can be extended into a full sequence $G^*$. The sentences $\forall x, y \ (M(x) \land M(y)) \rightarrow \neg \text{Adj}(x, y)$ and $\forall x \ (\neg M(x) \rightarrow (\exists y \text{Adj}(x, y) \land M(y)))$ are true from some index thus, by modifying first lifts if necessary, we can assume that $M$ always marks a maximal independent set in $G_n^+$ and $\lim_{n} \langle M, G_n^+ \rangle = \limsup a(G_n)/|G_n|$ as $\langle M, G_n^+ \rangle = a(G_n)/|G_n|$ infinitely often. As $\langle M, G_n^+ \rangle \leq a(G_n)/|G_n|$ (as $M$ marks an...
independent set) we get \( \lim (M, G_n^*) \leq \liminf \frac{a(G_n)}{|G_n|} \). Thus \( \frac{a(G_n)}{|G_n|} \) converges. □

In such a context it is not possible to distinguish (at the limit) a maximal independent set from a near maximal independent set. The only test one can do is that \( M \) marks a maximal independent set of a union of connected components with global measure 1. Of course this does not change the property that \( \frac{a(G_n)}{|G_n|} \) converges nor the measure of the (near) maximal independent set found in the limit.

Although \( \text{Sh}(\text{FO}) \)-convergence is quite strong, as witnessed by the above proposition, it is remarquable that \( \text{lift-Hausdorff} \)-convergence \( \text{Sh}(\text{QF}^-) \)-convergence is not stronger that \( \text{QF}^- \)-convergence (i.e. left convergence), and that both convergence modes admits the same limit objects (graphons, hypergraphons, …).

We end this section by giving an example showing that not every graphing is a local-global limit of a sequence of finite graphs (thus solving a problem posed in [12]).

**Example 3.4.5.** Consider the graphing \( G \) with domain \([0, 1)\) (with standard Borel measure) and edge set \( \{ (x, y) \mid x = y \pm \alpha \text{ mod } 1 \} \) for some fixed irrational number \( \alpha \). It is easily checked that \( G \) is a local limit for paths (for instance). Let \( G = (G_n)_{n \in \mathbb{N}} \) be a sequence of graphs having \( G \) as a local limit. Then there exists a negligible sequence \( X = (X_n)_{n \in \mathbb{N}} \) of subsets of vertices \( (X_n \subseteq G_n) \) such that \( \Delta(G_n \setminus X_n) = 2 \) and such that the minimum order of connected component of \( G_n \setminus X_n \) tends to infinity with \( n \). (By negligible, we mean that for every integer \( d \) we have \( \lim_{n \to \infty} N_d(X_n)/|G_n| = 0 \), see Section [1]).

However, \( G \) is not a local-global limit.

Indeed: each connected component of \( G_n \setminus X_n \) can be colored by colors 1 and 2 in such a way that each monochromatic induced subgraph is a path (order half of the order of the connected component) and at most 2 vertices of color 1 are adjacent to a vertex colored 2. The vertices in \( X_n \) are colored arbitrarily. Then for every integer \( r \) the distribution of the colored \( r \)-balls converge to the distribution with a rooted line colored 1 (with probability 1/2) and a rooted line colored 2 (with probability 1/2). Assume we can find a corresponding measurable coloring of \( G \). Note that the limit distribution is concentrated on monochromatic graphs. At the limit, let \( X \) be the union of all monochromatic connected components colored 1. Then \( X \) is obtained by removing from the Borel set of measure 1/2 of all vertices colored 1 the union over \( n \) of all the vertices at distance at most \( n \) to a vertex colored 2 (which is a Borel set of measure 0). Hence \( X \) is a Borel subset of measure 1/2 that is invariant by the mapping \( x \mapsto x + 1 \text{ mod } \alpha \), what contradicts the ergodicity of the dynamical system defined by this measure preserving transformation. Remark that this construction extends to any arbitrary degree \( d > 2 \) by considering a 2-coloring of the graphing with domain \([0, 1) \times \{1, \ldots, d-1\} \) where \((x, i)\) and \((y, j)\) are adjacent whenever \((x = y \text{ and } i \neq j)\) or \((y = x + \alpha \text{ mod } 1 \text{ and } i = j)\), see Fig [3].
4. Applications

4.1. Clustering. Monadic lifts (i.e. lifts by unary relations) were considered in [24] in the context of continuous clustering of the structures in an $\text{FO}^{\text{local}}$-convergent sequence. One of the main results (see Theorem 4.1.1 bellow) expresses that every $\text{FO}^{\text{local}}$-convergent sequence has monadic lift tracing components while preserving $\text{FO}^{\text{local}}$-convergence. This will be refined in this section under the stronger assumption of $\text{Sh}_e(\text{FO}^{\text{local}})$-convergence (see Theorem 4.1.6).

The analysis in [24] leads to interesting notions: globular cluster (corresponding to a limit non-zero measure connected component), residual cluster (corresponding to all the zero-measure connected components taken as a whole), and negligible cluster (corresponding to the stretched part connecting the other clusters, which eventually disappears at the limit).

Negligible sets intuitively correspond to parts of the graph one can remove, without a great modification of the statistics of the graph: A sequence $X \subseteq A$ is negligible in a local-convergent sequence $A$ if

$$\forall d \in \mathbb{N} : \limsup_{n \to \infty} \nu_{A_n}(N^d_{A_n}(X_n)) = 0.$$  

This we simply formulate as $\forall d \in \mathbb{N} : \limsup_{n \to \infty} \nu_{A}(N^d_{A}(X)) = 0$.

Two sequences $X$ and $Y$ of subsets are equivalent in $A$ if the sequence $X \Delta Y = (X_n \Delta Y_n)_{n \in \mathbb{N}}$ is negligible in $A$. This will be denoted by $X \approx Y$. We denote by $0$ the sequence of empty subsets. Hence $X \approx 0$ is equivalent to the property that $X$ is negligible. We further define a partial order on sequences of subsets by $X \preceq Y$ if the sequence $X \setminus Y = (X_n \setminus Y_n)_{n \in \mathbb{N}}$ is negligible in $A$. Hence $\preceq$ has $0$ for its minimum and $X \approx Y$ if $X \preceq Y$ and $Y \preceq X$.

The notion of cluster of a local-convergent sequence is a weak analog of the notion of union of connected components, or more precisely of the topological notion of “clopen subset”. A sequence $X$ of subsets of a local-convergent sequence $A$ is a cluster of $A$ if the following conditions hold:

1. the lifted sequence $L_X(A)$ obtained by marking set $X_n$ in $A_n$ by a new unary relation $M_X$ is local-convergent;
are residual clusters: A cluster $X$ open clusters converse does not hold as witnessed, for instance, by sequence of expanders. The strongly asymptotically no difference.) Every globular cluster is clearly strongly atomic, but the

A cluster $X$ is atomic if, for every cluster $Y$ of $A$ such that $Y \subseteq X$ either $Y \approx \emptyset$ or $Y \approx X$; the cluster $X$ is strongly atomic if $X_f$ is an atomic cluster of $A_f$ for every increasing function $f : \mathbb{N} \to \mathbb{N}$. To the opposite, the cluster $X$ is a nebula if, for every increasing function $f : \mathbb{N} \to \mathbb{N}$, every atomic cluster $Y_f$ of $A_f$ with $Y_f \subseteq X_f$ is trivial (i.e. $Y_f \approx \emptyset$). Finally, a cluster $X$ is universal for $A$ if $X$ is a cluster of every conservative lift of $A$.

Two clusters $X$ and $Y$ of a local-convergent sequence $A$ are interweaving, and we note $X \nleq Y$ if every sequence $Z$ with $Z_n \in \{X_n, Y_n\}$ is a cluster of $A$.

We say that two clusters $C_1$ and $C_2$ are:

- weakly disjoint if $C_1 \cap C_2 \approx \emptyset$;
- disjoint if $C_1 \cap C_2 = \emptyset$;
- strongly disjoint if $(N_{A}(C_1) \cap C_2) \cup (C_1 \cap N_{A}(C_2)) = \emptyset$.

A cluster $C$ of a local-convergent sequence $A$ is globular if, for every $\epsilon > 0$ there exists $d \in \mathbb{N}$ such that

$$\lim \inf_{n \to \infty} \sup_{v_n \in C_n} v_{A_n} (N_{A_n}^d (v_n)) > (1 - \epsilon) \lim_{n \to \infty} v_{A_n} (C_n).$$

In other words, a cluster $C$ is globular if, for every $\epsilon > 0$ and sufficiently large $n$, $\epsilon$-almost all elements of $C_n$ are included in some ball of radius at most $d$ in $C_n$, for some fixed $d$. (Note that for a cluster $C$ and $v_n \in C_n$, considering $v_{A_n} (N_{A_n}^d (v_n))$ or $v_{A_n} (N_{A_n \setminus C_n}^d (v_n))$ makes asymptotically no difference.) Every globular cluster is clearly strongly atomic, but the converse does not hold as witnessed, for instance, by sequence of expanders. The strongly atomic clusters that are not globular are called open clusters. Opposite to globular clusters are residual clusters: A cluster $X$ of $A$ is residual if for every $d \in \mathbb{N}$ it holds

$$\lim \sup_{n \to \infty} \sup_{v_n \in A_n} v_{A_n} (N_{A_n}^d (v_n)) = 0.$$

**Theorem 4.1.1 (\cite{12}).** Let $A$ be a local convergent sequence of $\sigma$-structures. Then there exists a signature $\sigma^+$ obtained from $\sigma$ by the addition of countably many unary symbols $M_{R}$ and $M_{i,j}$ ($i \in \mathbb{N}$, $1 \leq j \leq N_i$) and a clustering $A^+$ of $A$ with the following properties:

- For every $i \in \mathbb{N}$, $\bigcup_{j=1}^{N_i} M_{i,j} (A)$ is a universal cluster;
- For every $i \in \mathbb{N}$ and every $1 \leq j \leq N_i$, $M_{i,j} (A)$ is a globular cluster;
- Two clusters $M_{i,j} (A)$ and $M_{i',j'} (A)$ are interweaving if and only if $i = i'$;
- $M_R (A)$ is a residual cluster.

This structural theorem is assuming the local convergence of the sequence. If we assume local-global convergence we get stronger results (Theorem 4.1.6 bellow) involving expanding properties which we will define now. This is pleasing as the decomposition into expanders was one of the motivating examples \cite{6} and \cite{12}.

The following is a sequential version of expansion property: A structure $A$ is $(d, \epsilon, \delta)$-expanding if, for every $X \subset A$ it holds

$$\epsilon < v_{A}(X) < 1 - \epsilon \implies v_{A}(N_{A}^d (X)) > (1 + \delta) v_{A}(X).$$

This condition may be reformulated as:

$$\inf \left\{ \frac{v_{A} (N_{A}^d (X) \setminus X)}{v_{A}(X)} : \epsilon < v_{A}(X) < 1 - \epsilon \right\} > \delta.$$
Note that the left hand size of the above inequality is similar to the magnification introduced in [3], which is the isoperimetric constant \( h_{\text{out}} \) defined by
\[
h_{\text{out}} = \inf \left\{ \frac{|N_\delta(X) \setminus X|}{|X|} : 0 < \frac{|X|}{|A|} < 1/2 \right\}.
\]

A local-convergent sequence \( A \) is expanding if, for every \( \epsilon > 0 \) there exist \( d, t \in \mathbb{N} \) and \( \delta > 0 \) such that every \( A_n \) with \( n \geq t \) is \((d, \epsilon, \delta)\)-expanding. A non-trivial cluster \( X \) of \( A \) is expanding of \( A \) if \( A[X] \) is expanding. We have the following equivalent formulations of this concept:

**Lemma 4.1.2** ([24]). Let \( X \not\approx 0 \) be a cluster of a local convergent sequence \( A \). The following conditions are equivalent:

1. \( X \) is an expanding cluster of \( A \);
2. for every \( \epsilon > 0 \) there exists \( d, t \in \mathbb{N} \) such that for every \( Z \subseteq X \) with \( \nu_A(Z) > \epsilon \nu_A(X) \) it holds
   \[
   \nu_A(N^d_\delta(Z)) > (1 - \epsilon) \nu_A(X);
   \]
3. the sequence \( X \) is a strongly atomic cluster of \( A \);
4. for every \( \epsilon > 0 \) there exists no \( Y \subseteq X \) such that \( \partial_A Y \approx 0 \) and
   \[
   \epsilon < \liminf \nu_A(Y) < \lim \nu_A(X) - \epsilon.
   \]

Note that for local-global convergent sequences, the notions of atomic, strongly atomic, and expanding clusters are equivalent.

The case of bounded degree graphs is particularly interesting and our definitions capture this as well. Recall that a sequence \( G \) of graphs is a vertex expander if there exists \( a > 0 \) such that \( \liminf h_{\text{out}}(G_n) \geq a \). (For more information on expanders we refer the reader to [15].)

**Lemma 4.1.3** ([24]). Let \( G \) be a sequence of graphs with maximum degree at most \( \Delta \) and let \( C \not\approx 0 \) be a cluster of \( G \). The following are equivalent:

- \( C \) is an expanding cluster;
- for every \( \epsilon > 0 \) there exists \( X \subseteq C \) such that for every \( n \in \mathbb{N} \) it holds \( |X_n| < \epsilon |C_n| \) and \( G[C \setminus X] \) is a vertex expander.

We consider a fixed enumeration \( \phi_1, \phi_2, \ldots \) of \( \text{FO}^{\text{local}} \). The profile \( \text{Prof}(C) \) of a cluster \( C \) is the sequence formed by \( \lim \nu_A(C) \) followed by the values \( \lim(\phi_i, A|C|) \) for \( i \in \mathbb{N} \). The lexicographic order on the profiles is denoted by \( \leq \).

In [24] it was proved that two expanding clusters are either weakly disjoint or interweaving. We now prove a lemma with similar flavor.

**Lemma 4.1.4.** Let \( C_1 \) be an expanding cluster of a local-convergent sequence \( A \) and let \( C_2 \) be a cluster of \( A \).

Then the limit set of \( \nu_A(C_1 \cap C_2) \) is included in \( \{0, \lim \nu_A(C_1)\} \).

**Proof.** Let \( X = C_1 \cap C_2 \). Assume for contradiction that there exists \( 0 < \alpha < \lim \nu_A(C_1) \) and a subsequence \( A_f \) such that \( L_{\infty}^{\alpha}(A_f) \) is local convergent and \( \lim \nu_{A_f}(X_f) = \alpha \). As \( \delta_X \subseteq \delta_A C_1 \cup \delta_A C_2 \) we deduce that \( X_f \) is a cluster of \( A_f \). But \( X_f \subseteq (C_1)_f, X_f \not\approx 0 \), and \( X_f \not\approx C_1 \) (as \( \lim \nu_{A_f}(X_f) \not\in \{0, \lim \nu_A(C_1)\} \)), which contradicts the hypothesis that \( C_1 \) is expanding hence strongly atomic (see Lemma 4.1.2). \( \square \)

The following lemma is a restated version of a Lemma proved in [24].

**Lemma 4.1.5.** Two non-negligible clusters \( C_1 \) and \( C_2 \) are interweaving if and only if \( \text{Profile}(C_1) = \text{Profile}(C_2) \). \( \square \)

Our main result in this section reveals the expanding structure of local-global convergent sequences.
Theorem 4.1.6. Let $\sigma$ be a countable relational signature, let $\sigma^+$ be the extension of $\sigma$ by countably many unary symbols $\mathbb{N}$ and $U_i$ ($i \in \mathbb{N}$), and let $\sigma^*$ be the extension of $\sigma$ by countably many unary symbols $\mathbb{N}$ and $M_{ij}$. Let $\Gamma : \sigma^* \rightarrow \sigma^+$ be the basic interpretation defined by $\Gamma_U(x) := \sqrt{M_{ij}(x)}$ (and all other relations unchanged), and let $\text{Sh}^* : \sigma^* \rightarrow \sigma$ and $\text{Sh}^+ : \sigma^* \rightarrow \sigma$ be the natural forgetful interpretations.

Then for every local-global convergent sequence $A$ of $\sigma$-structure there exists a local-convergent sequence $A^+$ such that

- $P(\text{Sh}^+(A^*)) = A$,
- for every $i, j \in \mathbb{N}$, $M_{ij}(A^*)$ is either null or an atomic cluster of $A^*$, which is interweaving with $M_{ij}^*(A^*)$ if and only if $i = i'$,
- $N(A^*)$ is a nebula cluster of $A^*$,

and such that $A^* = P(I)(A^*)$ has the following properties:

- $A^+$ is a local-global convergent sequence such that $P(\text{Sh}^+)(A^+) = A$,
- for every $i \in \mathbb{N}$, $U_i(A^*)$ is either null or a cluster of $A^+$, which can be covered by (finitely many) interweaving atomic clusters,
- $N(A^*)$ is a nebula cluster of $A^+$.

Note that this result is in agreement with the intuition: The $\sigma^*$-lift is “finer” than the $\sigma^+$-lift and thus less likely to be local-global convergent. For instance, we can refer to the subsequence extension theorem (Theorem 3.3.2) or directly to the definition of lift-Hausdorff convergence.

Proof. Let $A$ be a local-global convergent sequence. We select inductively clusters expanding clusters $C^{ij}$ of $A$ as follows: We start with $Z = 0$, $i = 1$, $j = 1$ and let $P$ be the maximum profile of an expanding cluster of $A$. Then we repeat the following procedure as long as there exists an expanding cluster of $A$ that is weakly disjoint from $Z$.

- If there exists an expanding cluster in $A$ with profile $P$ that is weakly disjoint from $Z$ we select one as $C^{ij}$, we let $Z = Z \cup C^{ij}$, and we increase $j$ by 1.
- Otherwise, we select one with maximum profile as $C^{i+1,1}$, $e$ let $Z = Z \cup C^{i+1,1}$, we let $P$ be the profile of $C^{i+1,1}$, we increase $i$ by 1, and let $j = 1$.

It is easily checked that by modifying marginally the clusters $C^{ij}$ we can make them disjoint and such that $\mathbb{N} = A \cup \bigcup_{ij} C^{ij}$ is a nebula cluster. Then by Corollary 5 lifting $A$ by marking $M_{ij}$ the cluster $C^{ij}$ and $N$ the cluster $\mathbb{N}$ we get a local-convergent sequence $A^*$, which obviously satisfies the conditions stated in the Theorem.

Let $A^+ = P(I)(A^*)$. The only property we still have to prove is that $A^+$ is local-global convergent. According to Theorem 3.3.2 this boils down to proving that every local-convergent subsequence $B^+_j$ of lifts of $A^+$ can be extended into a full local-convergent sequence of lifts of $A^+$. We can transfer the relations $M_{ij}$ from $A^*$ to $B^+_j$. This way we obtain a subsequence $B^+_j$ of lifts of $A^*$ (which does not need to be local convergent), such that $P(I)(B^+_j) = B^+_j$. Let $B^*_h = P(I)(B^*_j)$ and $A$ is local-global convergent there exists a local-convergent sequence $D^*$ of lifts of $A$ extending $B^*_h$, that is: $P(\text{Sh}^*)(D^*) = A$ and $D^*_h = B^*_h$. Let $\hat{C}^{ij} = M_{ij}(D^*)$.

As $(\hat{C}^{ij})_h = (C^{ij})_h$ we get that $\hat{C}^{ij}$ is a cluster of $A$ with same profile as $C^{ij}$. According to Lemma 4.4.1 the limit set of $\nu_A(C^{ij} \cap C^{ij'})$ is included in $(0, m)$, where $m = \lim \nu_A(C^{ij}) = \lim \nu_A(C^{ij'})$. It follows that either $\nu(C^{ij}) \leq \nu(C^{ij'})$ or there exists a subsequence $A_h$ of $A$ such that $(\hat{C}^{ij})_h$ is weakly disjoint from the cluster $(\bigcup_j C^{ij'})_h$. Marking all the clusters $C^{ij}$ and $\hat{C}^{ij}$ in $A_h$ we get a local-convergent subsequence of lifts, which can be extended into full local-convergent sequence of lifts of $A$. In this sequence, the marks corresponding to the extension of $(\hat{C}^{ij})_h$ will correspond to a cluster of $A$ disjoint from all the clusters $C^{ij}$ but with the same profile, which contradicts the construction procedure of
the clusters $C^{i,j}$. Thus $\hat{C}^{i,j} \subseteq \bigcup_{j'} C^{i,j'}$, and $\bigcup_j \hat{C}^{i,j} \subseteq \bigcup_j C^{i,j'}$. As these two clusters have same limit measure we have $\bigcup_j \hat{C}^{i,j} \approx \bigcup_j C^{i,j'}$. This means that $P(I)(D^*)$ and $P(I)(B^*)$ are sufficiently close, so that if we consider the lifts of $A^+$ defined by $B^+$ for indices of the form $f(n)$ for some $n \in \mathbb{N}$ and by $P(I)(D^*)$ for the other indices, we get a local-convergent sequence of lifts of $A^+$ which extends $B^*_f$. It follows that $A^+$ is local-global convergent, what concludes our proof. □

4.2. Local Global Quasi-Limits. Let us finish this paper in an ambitious way. In [25] we defined the notion of modeling as a limit object from structural convergence.

Modeling limits generalize graphing limits and thus it follows from [19] that FO-convergent sequences of graphs with bounded degrees have modeling limits. In [25] we constructed modeling limits for FO-convergent sequences of graphs with bounded tree-depth, and extended the construction to FO-convergent sequences of trees in [22]. Then existence of modelings for FO-convergent sequences has been proved for graphs with bounded path-width [11] and eventually for sequences of graphs in an arbitrary nowhere dense class [20], which is best possible when considering monotone classes of graphs [22]. In fact this provides us with a high level analytic characterization of nowhere dense classes.

For local-global convergence, it was proved in [12] that graphings still suffice as limit objects. We don’t know, however, if every local-global convergent sequence of graphs in a nowhere dense class has a modeling local-global limit. We close this paper by proving that this is almost the case, in the sense that every local-global convergent sequence of graphs in a nowhere dense class has a modeling local-global quasi-limit.

We consider a fixed countable signature $\sigma$ and the signature $\tau$ obtained by adding countably many unary symbols $M_1, M_2, \ldots, M_n, \ldots$ to $\sigma$, and the forgetful interpretation $Sh : \tau \to \sigma$. As before we understand local-global convergence as $Sh_*(\mathrm{FO}_{\text{local}})$-convergence.

We fix a chain covering $X$ of $\mathrm{FO}_{\text{local}}(\tau)$ (see Section 2.5), from which we derive metrics and pseudo-metrics as in Sections 3.2 and 3.3. We also fix a bijection $\beta : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ (for Hilbert hotel argument) and let $Z_\beta$ be the renaming interpretation which renames $M_{\beta(c,i)}$ as $M_i$ and forget all the marks not being renamed.

**Lemma 4.2.1.** There exists a function $h : (0, 1) \to \mathbb{N}$ with the following property:

For every local-global convergent sequence $A$ of $\sigma$-structures there exists a local convergent sequence $B$ of $\tau$-structures with $P(Sh)(B) = A$, such that for every $\epsilon > 0$ there exists some integer $n_0$ such that

for every $n \geq n_0$ and every $C \in P(Sh)^{-1}(A_n)$ there exists $1 \leq c \leq h(\epsilon)$ with $\operatorname{dist}_X(C, P(Z_\beta)(B_n)) < \epsilon$.

**Proof.** As the space $\operatorname{Rel}(\tau)$ is totally bounded there exists a mapping $g : (0, 1) \to \mathbb{N}$ such that for each $\epsilon > 0$ and each $\sigma$-structure $A$ there is a subset $B_{A,\epsilon}$ of $P(Sh)^{-1}(A)$ of cardinality at most $g(\epsilon)$ with the property that every $C \in P(Sh)^{-1}(A)$ is at distance at most $\epsilon$ from a $\tau$-structure in $B_{A,\epsilon}$. (Such a set may be called an $\epsilon$-covering.) We construct an infinite sequence $(A^{(i)})_{i \in \mathbb{N}}$ of $\tau$-structures by listing all the structures in $B_{A,1/4}$ then all the structures in $B_{A,1/8}$, etc.

We now construct a $\tau$-structure $A^+ \in P(Sh)^{-1}(A)$ by letting $M_{\beta(c,i)}(A^+) = M_i(A^{(i)})$. Hence $A^{(i)} = P(Z_\beta)(A^+)$. We say that $A^+$ is a universal lift of $A$.

Define the function $h : (0, 1) \to \mathbb{N}$ by

$$h(x) = \sum_{i=1}^{\lceil -\log x \rceil + 1} g(2^{-i}).$$

Then for every $\epsilon > 0$ and every $B \in P(Sh)^{-1}(A)$ there is an index $c \leq h(\epsilon)$ such that $\operatorname{dist}_X(C, A^{(c)}) < \epsilon/2$, that is such that $\operatorname{dist}_X(C, P(Z_\beta)(A^+)) < \epsilon/2$. 

Now consider the local-global convergent sequence $A$ and a sequence $A^+$ where $A^+_n$ is a universal lift of $A_n$. This last sequence has a local convergent subsequence $A^+_f$, which we extend into a sequence $B$ lifting $A$.

Let $\epsilon > 0$. According to local-global convergence of $A$ and local convergence of $B$ there exists $n_0$ such that for every $n, m \geq n_0$ we have $\lim_{i \to \infty}\|A_n - \limsup_{i \to \infty} A_m\| < \epsilon/4$ and $\lim_{i \to \infty}\|B_n - \limsup_{i \to \infty} B_m\| < \alpha$, where $\alpha$ is such that for every $i \leq h(\epsilon/2)$ we have

$$\lim_{i \to \infty} \|A_i - \limsup_{i \to \infty} A_m\| < \alpha \Rightarrow \lim_{i \to \infty} \|B_i - \limsup_{i \to \infty} B_m\| < \epsilon/4.$$ 

Let $n \geq n_0$ (hence $f(n) \geq n_0$). Let $C \in P(Sh)^{-1}(A_n)$. Then there exists $C' \in P(Sh)^{-1}(A_{f(n)})$ such that $\lim_{i \to \infty} \|C_i - \limsup_{i \to \infty} C_m\| < \epsilon/4$. As $B_{f(n)} = A^+_f$ is a universal lift of $A_{f(n)}$ there exists $\epsilon \leq h(\epsilon)$ such that $\lim_{i \to \infty} \|P(Z_i)(B_i) - P(Z_i)(C_i)\| < \epsilon/2$. As $\lim_{i \to \infty} \|B_i - \limsup_{i \to \infty} B_m\| < \alpha$ we have $\lim_{i \to \infty} \|P(Z_i)(B_i) - P(Z_i)(C_i)\| < \epsilon/4$. Altogether, we get $\lim_{i \to \infty} \|P(Z_i)(B_i) - P(Z_i)(C_i)\| < \epsilon$ as wanted. \hfill \Box

**Definition 4.2.2.** A $\sigma$-modeling is a quasi-limits of a local global convergent sequence $A$ of $\sigma$-structures if, for every local convergent sequence $A^+$ of $\tau$-structures with $P(Sh)(A^+) = A$ and for every $\epsilon > 0$ there exists a $\tau$-modeling $L^+$ with $P(Sh)(L^+) = L$ such that $\lim \sup_{i \to \infty} \|L_i - L^+\| < \epsilon$.

In other words, for every local-global convergent sequence $A$ there is a modeling $L$ such that any local convergent sequence lifting $A$ has a limit which is $\epsilon$-close to an admissible lifting of $L$. (By admissible, we mean that the lift of $L$ is itself a modeling.)

**Theorem 4.2.3.** Every local-global convergent sequence of graphs in a nowhere dense class has a modeling quasi-limit.

**Proof.** Let $A$ be a local-global convergent of graphs in a nowhere dense class. According to Lemma 4.2.1 there exists a local convergent sequence $B$ of marked graphs with $P(Sh)(B) = A$, such that for every $\epsilon > 0$ there exists some integer $n_0$ such that for every $n \geq n_0$ and every $C \in P(Sh)^{-1}(A_n)$ there exists $1 \leq c \leq h(\epsilon)$ with

$$\lim_{i \to \infty} \|P(Z_i)(B_i)\| < \epsilon.$$ 

According to [20] the sequence $B$ has a modeling limit $L^+$. Then $L^+ = P(Sh)(L^+)$ is a modeling quasi-limit of $A$. \hfill \Box

**Conjecture 4.2.4.** Every local-global convergent sequence of graphs in a nowhere dense class has a modeling limit.

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**REFERENCES**

[1] H. Adler and I. Adler, Interpreting nowhere dense graph classes as a classical notion of model theory, European Journal of Combinatorics 36 (2014), 322–330.

[2] D. Aldous, Representations for partially exchangeable arrays of random variables, Journal of Multivariate Analysis 11 (1981), 581–598.

[3] N. Alon, Eigenvalues and expanders, Combinatorica 6 (1986), no. 2, 83–96.

[4] B. Balcar, T. Jech, and P. Pazák, Complete CCC Boolean algebras, the order sequential topology, and a problem of Von Neumann, Bulletin of the London Mathematical Society 37 (2005), no. 6, 885–898.

[5] I. Benjamini and O. Schramm, Recurrence of distributional limits of finite planar graphs, Electronic Journal of Probability 6 (2001), no. 23, 13pp.

[6] B. Bollobás and O. Riordan, Sparse graphs: metrics and random models, Random Structures & Algorithms 39 (2011), no. 1, 1–38.

[7] C. Borgs, J. T. Chayes, and L. Lovász, Moments of two-variable functions and the uniqueness of graph limits, Geometric And Functional Analysis 19 (2012), no. 6, 1597–1619.

[8] C. Borgs, J. T. Chayes, L. Lovász, V. T. Sós, B. Szegedy, and K. Vesztergombi, Graph limits and parameter testing, STOC’06: Proceedings of the 38th Annual ACM Symposium on Theory of Computing, 2006, pp. 261–270.

[9] C. Borgs, J. T. Chayes, L. Lovász, V. T. Sós, and K. Vesztergombi, Convergent sequences of dense graphs I: Subgraph frequencies, metric properties and testing, Advances in Mathematics 219 (2008), no. 6, 1801–1851.
[10] Convergent sequences of dense graphs II: Multiway cuts and statistical physics, Annals of Mathematics 176 (2012), 151–219.

[11] J. Gajarský, P. Hliněný, T. Kaiser, D. Kráľ, M. Kupec, J Obradžálek, S. Ordyniak, and V. Tůma, First order limits of sparse graphs: plane trees and path-width, Random Structures and Algorithms (2016).

[12] H. Hatami, L. Lovász, and B. Szegedy, Limits of locally–globally convergent graph sequences, Geometric and Functional Analysis 24 (2014), no. 1, 269–296.

[13] F. Hausdorff, Set theory, vol. 119, American Mathematical Soc., 1962.

[14] W. Hodges, A shorter model theory, Cambridge University Press, 1997.

[15] S. Hoory, N. Linial, and A. Wigderson, Expander graphs and their applications, Bulletin of the American Mathematical Society 43 (2006), no. 4, 439–561.

[16] D. Hoover, Relations on probability spaces and arrays of random variables, Tech. report, Institute for Advanced Study, Princeton, NJ, 1979.

[17] D. Lascar, La théorie des modèles en peu de maux, Cassini, 2009.

[18] L. Lovász and B. Szegedy, Limits of dense graph sequences, Journal of Combinatorial Theory, Series B 96 (2006), 933–957.

[19] J. Nešetřil and P. Ossona de Mendez, A model theory approach to structural limits, Commentationes Mathematicae Universitatis Carolinae 53 (2012), no. 4, 581–603.

[20] Existence of modeling limits for sequences of sparse structures, [arXiv:1608.00146 [math.CO]], 2016.

[21] First-order limits, an analytical perspective, European Journal of Combinatorics 52 Part B (2016), 368–388.

[22] Modeling limits in hereditary classes: Reduction and application to trees, Electronic Journal of Combinatorics 23 (2016), no. 2, #P2.52.

[23] Structural sparsity, Uspekhi Matematicheskikh Nauk 71 (2016), no. 1, 85–116, (Russian Math. Surveys 71:1 79-107).

[24] Cluster analysis of local convergent sequences of structures, Random Structures & Algorithms 51 (2017), no. 4, 674–728.

[25] A unified approach to structural limits (with application to the study of limits of graphs with bounded tree-depth), Memoirs of the American Mathematical Society (2017), 117 pages; accepted.

[26] P. Urysohn, Works on topology and other areas of mathematics 1, 2, State Publ. of Technical and Theoretical Literature, Moscow (1951).