Nonlinear superconformal symmetry of a fermion in the field of a Dirac monopole

Carlos Leiva\textsuperscript{a}* and Mikhail S. Plyushchay\textsuperscript{a,b†}

\textsuperscript{a}Departamento de Física, Universidad de Santiago de Chile, Casilla 307, Santiago 2, Chile

\textsuperscript{b}Institute for High Energy Physics, Protvino, Russia

Abstract

We study a longstanding problem of identification of the fermion-monopole symmetries. We show that the integrals of motion of the system generate a nonlinear classical $\mathbb{Z}_2$-graded Poisson, or quantum super-algebra, which may be treated as a nonlinear generalization of the $osp(2|2) \oplus su(2)$. In the nonlinear superalgebra, the shifted square of the full angular momentum plays the role of the central charge. Its square root is the even $osp(2|2)$ spin generating the $u(1)$ rotations of the supercharges. Classically, the central charge’s square root has an odd counterpart whose quantum analog is, in fact, the same $osp(2|2)$ spin operator. As an odd integral, the $osp(2|2)$ spin generates a nonlinear supersymmetry of De Jonghe, Macfarlane, Peeters and van Holten, and may be identified as a grading operator of the nonlinear superconformal algebra.

\textsuperscript{*}E-mail: caleiva@lauca.usach.cl

\textsuperscript{†}E-mail: mplyushc@lauca.usach.cl
1 Introduction

Hidden, or dynamical symmetries are behind the special properties of some classical and quantum mechanical systems. The best known example of the hidden symmetry is provided, probably, by the Kepler problem. Being associated with the Laplace-Runge-Lenz vector integral, it is responsible for the closed character of the finite orbits, fixing the orbits’ orientation in the both cases of a finite and an infinite motion. Also, it explains a degeneracy of the quantum spectrum.

The problem of identification of symmetries of the fermion-monopole system is a long-standing puzzle. More than twenty years ago, Jackiw found that the system of a scalar charged particle in the field of a Dirac magnetic monopole possesses a hidden conformal $so(1,2)$ symmetry [1]. Four years later D’Hoker and Vinet showed that the fermion-monopole system can be characterized by the $osp(1,2)$ supersymmetry [2]. Then, De Jonghe, Macfarlane, Peeters and van Holten [3] identified a new supersymmetry of the system which squares to the (shifted) Casimir invariant of the full rotation group. Their analysis, based on a general method which uses Killing-Yano tensors to generate additional supersymmetries [4], revealed that the full supersymmetry algebra of the fermion-monopole is, in fact, nonlinear. Lately, Spector [5] found that such a non-standard nonlinear supersymmetry associated with the Casimir invariant of the full rotation group may exist even when an additional potential term breaking a usual supersymmetry is included in the Pauli Hamiltonian. In both papers [3, 5], the supersymmetries were treated outside the context of conformal symmetry. Following their line, in Ref. [6] it was argued that the commutator of the usual (“square root from the Hamiltonian”) and of the non-standard (“square root from the rotational Casimir”) supercharges should also be treated as a new supercharge, which squares for the product of the Hamiltonian and rotational Casimir invariant\(^1\). This resembles the nonlinear structure of the Kepler problem, where the commutator of the Laplace-Runge-Lenz vector components takes a form of the product of the Hamiltonian and angular momentum operator [7].

The $osp(1,2)$ symmetry of the fermion-monopole system [2] supplied with the additional nonlinear supersymmetry of Refs. [3, 6] is in the obvious but puzzling contrast with the $osp(2,2)$ supersymmetry of the Akulov-Pashnev-Fubini-Rabinovici superconformal model [8, 9, 10, 11]. The difference is rather surprising since the both systems are characterized by the same Lie algebraic dynamical structure $so(1,2)$, and their quantum spaces associated with the spin degrees of freedom are exactly the same. Recently, however, it was observed [12, 13] that for the special (discrete) values of the fermion-boson coupling parameter, in addition to the usual $osp(2,2)$ symmetry, the superconformal model [8, 9] can be characterized by a hidden nonlinear superconformal symmetry.

In the present paper, we investigate the problem of identification of (super)symmetries of the fermion-monopole system by comparing its structure with realization of superconformal symmetry in the model [8, 9]. As a result, it will be shown that the system possesses a nonlinear superconformal symmetry. A nonlocal transformation applied to the states with the angular momentum different from the lowest one, reduces the full symmetry to the usual $osp(2,2)$ supersymmetry (plus a ‘decoupled’ rotational symmetry), in which the square root from the shifted rotational Casimir invariant plays the role of the central charge similar to

\(^1\)See Eq. (3.28) and the comment to it.
the boson-fermion coupling constant of the superconformal model [8, 9].

The paper is organized as follows. In section 2 we analyze the dynamical symmetry of the superconformal mechanics model. The results of the analysis are applied in section 3 to find all the set of the integrals generating the dynamical symmetry of the fermion-monopole system and to identify the corresponding superalgebra at the classical and quantum levels. In section 4 we summarize the obtained results and discuss some open problems to be interesting for further investigation.

2 Dynamical symmetry of the 1D superconformal mechanics

We start with a short discussion of the symmetries of the 1D superconformal model [8, 9, 10, 11], that will be helpful for subsequent identification of the fermion-monopole symmetries.

The classical model [8, 9] is given by the Hamiltonian

$$H = \frac{1}{2} \left( p^2 + \frac{\alpha(\alpha + 2i\psi_1\psi_2)}{x^2} \right),$$

and by the fundamental Poisson brackets \( \{x, p\} = 1, \{\psi_a, \psi_b\} = -i\delta_{ab}, \) \( a, b = 1, 2. \) In addition to \( H, \) the system possesses the even,

$$D = \frac{1}{2} xp - tH, \quad K = \frac{1}{2} x^2 - 2tD - t^2 H, \quad \Sigma = -i\psi_1\psi_2,$$

and odd,

$$Q_a = p\psi_a + \frac{\alpha}{x} \epsilon_{ab}\psi_b, \quad \tilde{Q}_a = x\psi_a - tQ_a,$$

integrals of motion obeying the equation of the form \( \frac{d}{dt} I = \frac{\partial}{\partial t} I + \{I, H\} = 0. \) The set of integrals (2.1), (2.2), (2.3) generates the \( osp(2, 2) \) superalgebra given by the nontrivial Poisson bracket relations

$$\{H, K\} = -2D, \quad \{D, H\} = H, \quad \{D, K\} = -K,$$
$$\{Q_a, Q_b\} = -2i\delta_{ab}H, \quad \{\tilde{Q}_a, \tilde{Q}_b\} = -2i\delta_{ab}K,$$
$$\{\tilde{Q}_a, Q_b\} = -2i\delta_{ab}D + i\epsilon_{ab}(\Sigma + \alpha),$$

(2.4)

For convenience of comparison with the fermion-monopole system, we do not include a constant \( \alpha, \) playing a role of the central element of the superalgebra, in the definition of the \( osp(2|2) \) spin integral \( \Sigma \) generating the \( so(2) \cong u(1) \) rotations of the supercharges. The quantity

$$C = 4(D^2 - KH) - 2\alpha\Sigma + \alpha^2$$

is the Casimir element of the superalgebra (2.4) taking here the value \( C = 0. \)
Since the system is described by the two independent odd (Grassmann) variables $\psi_a$, $a = 1, 2$, not all the odd integrals (2.3) are independent. Using the explicit form of the integrals (2.1), (2.2), (2.3), one finds that they satisfy the odd and even relations

$$\alpha Q_a + 2\epsilon_{ab}(Q_b D - \tilde{Q}_b H) = 0, \quad \alpha \tilde{Q}_a + 2\epsilon_{ab}(Q_b K - \tilde{Q}_b D) = 0,$$

(2.6)

$$Q_1 Q_2 - 2i\Sigma H = 0, \quad Q_1 \tilde{Q}_2 - 2i\Sigma K = 0, \quad Q_a \tilde{Q}_b + i\delta_{ab}\alpha \Sigma - 2i\epsilon_{ab}\Sigma D = 0.$$

(2.7)

The set of quadratic combinations defined by Eqs. (2.6), (2.7) is transformed linearly under the action of the $osp(2, 2)$ generators. If the relations (2.6) are treated as a homogeneous linear set of equations for $Q_a$ and $S_a$, then due to the equality $C = 0$ and nilpotent character of $\Sigma$, its determinant $\Delta = (C + 2\alpha \Sigma)^2$ takes a zero value. And vice versa, the condition $\Delta = 0$ for the determinant of the homogeneous system of equations for $Q_a$ and $\tilde{Q}_a$ (2.6) fixes the value of the classical Casimir element: $C = 0$.

It is useful to look at the symmetries of the one-dimensional model (2.1) from the viewpoint of the planar system of a free spin-1/2 particle given by the action

$$A = \int \left(\frac{1}{2} \dot{x}_a^2 - \frac{i}{2} \dot{\psi}_a \psi_a\right) \, dt$$

(2.8)

and reduced to the surface of the fixed full angular momentum \[13\]. The system (2.8) can be characterized by the set of quadratic integrals

$$H = \frac{1}{2} p_a^2, \quad K = \frac{1}{2} X_a^2, \quad D = \frac{1}{2} X_a p_a, \quad \Sigma = -\frac{i}{2} \epsilon_{ab} \psi_a \psi_b,$$

(2.9)

$$L = \epsilon_{ab} X_a p_b,$$

$$Q_1 = p_a \psi_a, \quad Q_2 = \epsilon_{ab} p_a \psi_b, \quad \tilde{Q}_1 = X_a \psi_a, \quad \tilde{Q}_2 = \epsilon_{ab} X_a \psi_b$$

(2.10)

constructed from the obvious set of linear integrals of motion $p_a$, $X_a = x_a - p_a t$ and $\psi_a$. The integrals (2.9), (2.10) form a superalgebra of the form (2.4), in which instead of the parameter $\alpha$ the full angular momentum $J = L + \Sigma$ plays the role of the central element. As a consequence, the system (2.11) may be obtained by reducing the system (2.8) to the surface of the constraint

$$J - \alpha = 0.$$

(2.11)

Due to zero Poisson brackets of the quadratic integrals (2.9), (2.10) with the constraint (2.11), they are observables, and after reduction their superalgebra takes exactly the form (2.4), while the integrals themselves take the form of the corresponding integrals (2.1) – (2.3) (for the details see ref. \[13\]).

One can introduce into the system one more odd degree of freedom described by the Grassmann variable $\psi_3$ if to add the kinetic term $-\frac{i}{2} \dot{\psi}_3 \psi_3$ into the Lagrangian of the system (2.8). With such an extension, classically the system will be described by one more odd integral of motion $\Gamma = \psi_3$ being also, due to the relation $\{\psi_3, J\} = 0$, the observable variable. This additional odd integral satisfies the relation $\{\Gamma, \Gamma\} = -i \cdot 1$, and has zero
Poisson brackets with all the even and odd integrals (2.9), (2.10). With respect to the superalgebra \( osp(2|2) \), the additional integral \( \Gamma \) may be interpreted as a classical analog of the quantum grading operator. Indeed, at the quantum level, the Grassmann variables satisfying the Poisson bracket relations \( \{ \psi_a, \psi_b \} = -i\delta_{ab} \) are transformed into the generators of the Clifford algebra \( Cl_2 \) (for the initial system with \( \psi_a, a = 1, 2 \)), or of the \( Cl_3 \) (for the extended system with the three odd variables \( \psi_a, a = 1, 2, 3 \)). Here, in correspondence with the relation \( \dim Cl_{2n+1} = \dim Cl_{2n} = 2^n \) for the dimensions of irreducible representations, in both cases the quantum analogs of the odd variables can be realized in terms of Pauli matrices, \( \hat{\psi}_a = \sqrt{\frac{\hbar}{2}}\sigma_a \), and \( \sigma_3 \propto \hat{\psi}_3 \) is identified as a grading operator of the quantum \( osp(2|2) \) superalgebra.

Up to a numerical factor the integral \( \hat{\Sigma} = -\frac{i}{2}[\hat{\psi}_1, \hat{\psi}_2] = \frac{\hbar}{2}\sigma_3 \) coincides with the integral \( \hat{\psi}_3 \) of the extended system. Hence, the integral \( \hat{\Sigma} = \frac{\hbar}{2}\sigma_3 \) of the 1D superconformal model (2.1) may be treated not only as an even generator of the \( osp(2|2) \) superalgebra, but simultaneously it may be considered as a grading operator of the superconformal algebra.

Note that at the quantum level, the condition of existence of a nontrivial solution to the system of equations (2.6) is reduced to the equation

\[
\left( \hat{C} - \frac{3}{4}\hbar^2 \right) \left( \hat{C} - \frac{3}{4}\hbar^2 + 8\hbar\alpha\sigma_3 \right) = 0,
\]

which fixes the value of the quantum analog of the Casimir element (2.5),

\[
\hat{C} \equiv 4\hat{D}^2 - 2(\hat{K}\hat{H} + \hat{H}\hat{K}) + \alpha(\alpha - \hbar\sigma_3) = \frac{3}{4}\hbar^2. \tag{2.12}
\]

### 3 Dynamical symmetry of the fermion-monopole

The 3D fermion-monopole system is described by the Hamiltonian

\[
H = \frac{1}{2}P_i^2 - eB_iS_i \tag{3.1}
\]

with \( P_i = p_i - eA_i, \ B_i = \epsilon_{ijk}\partial_jA_k = g x_j/|x|^3, \ |x| = \sqrt{x_i x_i}, \ S_j = -\frac{1}{2}\epsilon_{jkl}\psi_l\psi_k \), and by the fundamental Poisson brackets \( \{ x_i, p_j \} = \delta_{ij}, \ \{ \psi_j, \psi_k \} = -i\delta_{jk} \). The Hamiltonian (3.1) and the quantities

\[
D = \frac{1}{2}X_i P_i + etB_i S_i = \frac{1}{2}x_i P_i - tH, \tag{3.2}
\]

\[
K = \frac{1}{2}X_i^2 - et^2B_i S_i = \frac{1}{2}x_i^2 - 2tD - t^2 H \tag{3.3}
\]

together with the full angular momentum \( J_i \), given by the relations

\[
J_i = L_i - \nu n_i + S_i, \quad L_i = \epsilon_{ijk}x_j P_k, \quad n_i = \frac{x_i}{|x|}, \quad \nu = eg, \tag{3.4}
\]

constitute the set of integrals of motion generating the \( so(1,2) \oplus so(3) \) symmetry [1, 14]. Here, the \( so(1,2) \) Casimir element \( D^2 - KH \) and the \( so(3) \) rotational invariant are related by the equation

\[
C = 4(D^2 - KH) + J - 2\mathcal{Q}_L = 0, \tag{3.5}
\]
\[ J = J_i^2 - \nu^2, \quad (3.6) \]

\[ Q_L = L_i S_i, \quad (3.7) \]

The \( Q_L \) is the integral of motion commuting with all the generators of the algebra \( so(1, 2) \oplus so(3) \), and extending it to the \( so(1, 2) \oplus so(3) \oplus u(1) \).

The odd, \( \psi_i \), and the even (the fermion’s spin), \( S_i \), nilpotent vectors satisfy the relations

\[ \{ S_i, \psi_j \} = \epsilon_{ijk} \psi_k, \quad \{ S_i, S_j \} = \epsilon_{ijk} S_k, \quad \psi_i S_j = \frac{1}{3} \delta_{ij} (S_k \psi_k), \quad (3.8) \]

and their evolution is described by the same precession motion,

\[ \frac{d}{dt} \psi_i = \epsilon \epsilon_{ijk} \psi_j B_k, \quad \frac{d}{dt} S_i = \epsilon \epsilon_{ijk} S_j B_k. \quad (3.9) \]

Classically, they may be treated as “parallel” vectors, \( \epsilon \epsilon_{ijk} \psi_j S_k = 0 \), and from eq. (3.9) one finds that their scalar product

\[ Q_S = S_i \psi_i \quad (3.10) \]

is the odd integral of motion. Analogously to the 1D superconformal model, in irreducible representation the quantum operators \( \hat{\psi}_i \) and \( \hat{S}_i \) can be realized in terms of the Pauli matrices,

\[ \hat{\psi}_i = \sqrt{\frac{\hbar}{2}} \sigma_i, \quad \hat{S}_i = \frac{\hbar}{2} \sigma_i, \]

i.e., up to a numerical factor they are represented by the same operators \( \sigma_i \). Hence, the quantum analog of the odd integral of motion (3.10) is reduced to a pure quantum constant: \( \hat{Q}_S = 3(\hbar/2)^{5/2} \).

Defining the vector \( X_i = x_i - t P_i \), which in the free case \( eg = 0 \) is reduced to the integral generating the Galilean boosts, one can construct the pair of odd quantities

\[ Q_P = P_i \psi_i, \quad Q_X = X_i \psi_i \quad (3.11) \]

satisfying the Poisson bracket relations

\[ \{ Q_P, Q_P \} = -2i H, \quad \{ Q_X, Q_X \} = -2i K, \quad \{ Q_X, Q_P \} = -2i D. \quad (3.12) \]

For the components of the vector \( P_i \) the classical relation

\[ \epsilon_{ijk} \{ P_i, \{ P_j, P_k \} \} \propto \delta^{(3)}(x) \quad (3.13) \]

takes place (see also refs. [15, 16, 17]), and the quantities (3.11) are the odd integrals of motion if the point \( x_i = 0 \) is excluded from the configuration space. In a physical context, this can be achieved by some sort of ‘regularization’, e.g., by adding into the Hamiltonian a spherical reflecting barrier potential \( V(|x|) = +\infty \) for \( |x| \leq a \), and \( V(|x|) = 0 \) for \( |x| \geq a > 0 \), and by taking subsequently a limit \( a \to 0 \); for alternative regularization see refs. [16, 18, 19]. Here we are not interested in regularization and related problem of bounded states [16, 18, 19, 20, 21], and in what follows will assume simply that the point \( x_i = 0 \) is excluded.

The odd integrals \( Q_P \) and \( Q_X \), being analogs of the integrals \( Q_1 \) and \( \tilde{Q}_1 \) of the superconformal model (cf. the structure of the integrals (3.11) with that of the integrals \( Q_1 \) and
Given by eq. (2.10), generate together with the $H, K, D$ and $J_i$ the $osp(1,2) \oplus so(3)$ superalgebra with the Casimir element \[ (3.5) \], which was discussed by D’Hoker and Vinet [2]. In the fermion-monopole system, however, there is also the nontrivial classical odd integral \[ (3.10) \]. Its bracket with the odd integrals \[ (3.11) \] generates new even integrals

$$ Q_P = P_i S_i, \quad Q_X = X_i S_i. \quad (3.14) $$

The operators corresponding to the classical quantities \[ (3.11) \] coincide up to a numerical factor with the quantum analogs of \[ (3.14) \]. But classically the sets \[ (3.11) \] and \[ (3.14) \] are different, and their mutual nontrivial Poisson brackets produce via the relation

$$ \{Q_X, Q_P\} = \{Q_X, Q_P\} = Q_L + Q_S $$

a new integral of motion

$$ Q_L = L_i \psi_i \quad (3.15) $$

being the odd counterpart of the even integral \[ (3.7) \]. Again, these two, $Q_L$ and $Q_L$, are related by the Poisson bracket with the integral \[ (3.10) \], \[ \{Q_S, Q_L\} = -3i Q_L \], and their quantum analogs are different only in a numerical factor proportional to $\hbar^{1/2}$.

Let us change the integral $Q_L$ for the linear combination of $Q_L$ and $Q_S$,

$$ Q_Y = Q_L + \frac{2}{3} Q_S. \quad (3.16) $$

It is this odd integral, satisfying the relation

$$ \{Q_Y, Q_Y\} = -iJ, $$

that was treated in Ref. [3] as a generator of a new supersymmetry. It, unlike the $Q_L$, has zero Poisson bracket relations (anticommutes at the quantum level) with the odd integrals \[ (3.11) \]. On the other hand, the Poisson brackets of the odd integrals \[ (3.11) \] with the even integral $Q_L$, which due to the relation \[ \{Q_S, Q_S\} = 0 \] may be treated as an even counterpart of \[ (3.10) \] as well, generate the two new odd integrals of motion,

$$ \{Q_L, Q_P\} = Q_P, \quad \{Q_L, Q_X\} = Q_X, $$

where

$$ Q_P = P_i \psi_i, \quad Q_X = X_i \psi_i, \quad (3.17) $$

$$ P_i = \epsilon_{ijk} L_j P_k + \frac{2}{3} \nu |x|^{-1} S_i, \quad X_i = \epsilon_{ijk} L_j X_k - \frac{2}{3} t \nu |x|^{-1} S_i. \quad (3.18) $$

The brackets of the integrals \[ (3.17) \] with \[ (3.10) \] produce their even counterparts

$$ Q_P = P_i S_i, \quad Q_X = X_i S_i. $$

We list all the brackets of the nilpotent integrals, except the trivial brackets

$$ \{Q_S, Q_S\} = \{Q_S, Q_L\} = 0, $$
The quantities $Q_S$, $Q_Y$ and $Q_L$ have zero Poisson brackets with the $so(1,2)$ generators (3.1), (3.2), (3.3), while the nontrivial brackets of the odd integrals (3.11) with the $so(1,2)$ generators are

$$
\{D, Q_P\} = \frac{1}{2} Q_P, \quad \{D, Q_X\} = -\frac{1}{2} Q_X, \quad \{K, Q_P\} = Q_X, \quad \{H, Q_X\} = -Q_P. \quad (3.19)
$$

The pairs $(Q_P, Q_X)$, $(Q_P, Q_X)$, and $(Q_P, Q_X)$, like the pair $(Q_P, Q_X)$, form the spin-1/2 representations of the $so(1,2)$, i.e. their corresponding brackets with the $H$, $K$ and $D$ have a form similar to (3.19).

The odd integrals $Q_P$, $Q_X$, $Q_P$ and $Q_X$ satisfy the relations to be analogous to the relations (2.6), (2.7):

$$
Q_P J + 2(Q_P D - Q_X H) = 0, \quad Q_X J - 2(Q_X D - Q_P K) = 0,
Q_P - 2(Q_P D - Q_X H) = 0, \quad Q_X + 2(Q_X D - Q_P K) = 0, \quad (3.20)
$$

$$
Q_P Q_P - 2i H Q_L = 0, \quad Q_X Q_X - 2i K Q_L = 0, \quad Q_P Q_X + 2i Q_L = 0,
Q_P Q_X - i Q_L, J = 0, \quad Q_P Q_X + 2i Q_L D = 0, \quad Q_P Q_X - 2i Q_L D = 0. \quad (3.21)
$$

Having in mind this similarity, and comparing the structure of the Poisson brackets of the integrals $Q_P$, $Q_X$, $Q_P$, $Q_X$, $H$, $K$, $D$ and $Q_L$ of the fermion-monopole system with that for the integrals $Q_1$, $Q_2$, $\tilde{Q}_2$, $H$, $K$, $D$ and $\Sigma$ of the superconformal model, we find that the former set of the integrals forms a nonlinear $\mathbb{Z}_2$-graded Poisson algebra which can be considered as a nonlinear generalization of the superconformal algebra $osp(2,2)$.

Since $J$ has zero Poisson brackets with all the integrals, it plays the role of the central charge of the nonlinear superconformal symmetry to be analogous to that for the $\alpha^2$ in the superconformal model. From the comparison it follows also that the odd integral $Q_Y$ is

\[
\begin{array}{|c|c|c|c|c|c|c|c|c|}
\hline
  & Q_X & Q_P & Q_X & Q_P & Q_Y & Q_X & Q_P & Q_X & Q_P \\
\hline
Q_X & -2iK & -2iD & 0 & i\Lambda & 0 & 0 & Q_+ & 2K Q_Y & 2D Q_Y \\
Q_P & -2iD & -2iH & -i\Lambda & 0 & 0 & -Q_+ & 0 & 2D Q_Y & 2H Q_Y \\
Q_X & 0 & -i\Lambda & -2i J K & -2i J D & 0 & -2K Q_Y & -2D Q_Y & 0 & J Q_+ \\
Q_P & i\Lambda & 0 & -2i J D & -2i J H & 0 & -2D Q_Y & -2H Q_Y & -J Q_+ & 0 \\
Q_0 & 0 & 0 & 0 & 0 & -i J & -Q_X & 0 & -J Q_X & -J Q_+ \\
Q_L & Q_X & Q_P & -J Q_X & -J Q_P & 0 & Q_X & Q_P & -J Q_X & -J Q_+ \\
Q_S & -3i Q_X & -3i Q_P & -3i Q_X & -3i Q_P & -3i Q_L & 0 & 0 & 0 & 0 \\
Q_X & 0 & Q_+ & 2K Q_Y & 2D Q_Y & -Q_X & 0 & Q_L & 2K Q_L & 2D Q_L \\
Q_P & -Q_+ & 0 & 2D Q_Y & 2H Q_Y & -Q_P & -Q_L & 0 & 2D Q_L & 2H Q_L \\
Q_X & -2K Q_Y & -2D Q_Y & 0 & -J Q_+ & J Q_X & -2K Q_L & -2D Q_L & 0 & J Q_L \\
Q_P & -2D Q_Y & -2H Q_Y & J Q_+ & 0 & J Q_P & -2D Q_L & -2H Q_L & -J Q_L & 0 \\
\hline
\end{array}
\]
similar to the odd integral $\Gamma$ playing the role of the classical analog of the grading operator for the superconformal model. For $L_i^2 \neq 0$, the rescaling

$$Q_p \rightarrow Q_p J^{-1/2}, \quad Q_X \rightarrow Q_X J^{-1/2}, \quad Q_L \rightarrow Q_L J^{-1/2} \quad (3.22)$$

transforms the nonlinear superconformal algebra of the fermion-monopole system into the $osp(2,2)$ Lie superalgebra \[24\], in which the role of the parameter $\alpha$ is played by the $J^{1/2}$.

For $L_i^2 \neq 0$, one can define the vector

$$N_i = \left(1 - \frac{2\nu}{L_i^2} S_k n_k\right) Y_i, \quad Y_i = L_i + \frac{2}{3} S_i.$$

Then, using Eq. \[3.31\], the equality $S_i S_j = 0$ and the last relation from Eq. \[3.8\], one can represent the integrals $Q_p, Q_X, Q_Y$ and $Q_L$ in the form

$$Q_p = i_{ijk} N_i P_j \psi_k, \quad Q_X = i_{ijk} N_i X_j \psi_k, \quad Q_L = -\frac{i}{2} i_{ijk} N_i \psi_j \psi_k, \quad Q_Y = N_i \psi_i. \quad (3.23)$$

The 'extended' superconformal model may also be represented in a 3D form by introducing the notations $x_i = (x_1, x_2, 0)$, $p_i = (p_1, p_2, 0)$, $\psi_i = (\psi_1, \psi_2, \psi_3)$ and $N_i = (0, 0, 1)$. Then the integrals \[2.9\], \[2.10\] can be rewritten in the form of 3D scalar products, similar to the integrals of motion of the fermion-monopole system. In particular, the classical analog of the grading operator, $\Gamma = \psi_3$, takes a form similar to $Q_Y$ from \[3.23\], $\Gamma = N_i \psi_i$.

Identifying the $\mathbb{Z}_2$-graded Poisson algebra of the integrals of motion as a nonlinear superconformal algebra (plus the 'decoupled' $so(3) \cong su(2)$), we have omitted from consideration the even nilpotent integrals $Q_X, Q_P, Q_X$ and $Q_P$. This was done having in mind the quantum case, where these integrals are different from the quantum analogs of the corresponding odd integrals only in a simple numerical factor $\sqrt{\hbar/2}$.

Let us note here that the even nilpotent integrals for the model of superconformal mechanics extended by the odd integral $\psi_3$ might also be constructed in an obvious way. Having in mind the correspondence $\alpha^2 \sim J$, $\Sigma \sim \frac{1}{3} Q_S$, $\alpha \Gamma \sim Q_Y$, $\alpha \Sigma \sim Q_L$ and relations \[3.22\], one finds that the even nilpotent quantities $i\Gamma Q_2$, $-i\alpha \Gamma Q_3$, $i\Gamma \hat{Q}_2$ and $-i\alpha \Gamma \hat{Q}_3$ would be the analogs of the integrals $Q_p, Q_P, Q_X$ and $Q_X$, respectively, which, with taking into account Eq. \[2.7\], would generate the nonlinear Poisson bracket relations of the form presented in the Table. One could treat the complete set of these classical 'commutation' relations of the all even and odd integrals as some nonlinear Poisson superalgebra $\mathcal{G} \[7\]. However, its nature is essentially different from that of the nonlinear generalization of the $osp(2|2) \oplus su(2)$ generated by the integrals $H, K, D, Q_L, Q_P, Q_X, Q_P, Q_X$ and $J_i$ (plus the Yano supercharge $Q_Y \[3\]$ being the classical analog of the grading operator of the quantum version of this superalgebra, see below). The difference is that the nonlinearity of the generalized $osp(2|2) \oplus su(2)$ is encoded in the central charge $J$ appearing additively and multiplicatively in the Poisson bracket relations, while the nontrivial $so(1,2)$ generators $H, D$ and $K$ appear as multiplicative factors in the Poisson superalgebra $\mathcal{G}$. One of the consequences of this is that there exists no analog of the rescaling procedure \[3.22\] which would transform $\mathcal{G}$ into some Lie superalgebra.

The role of the quantum $osp(2|2)$ spin operator is played by

$$\hat{Q}_L = \frac{\hbar}{2} \left(\hat{L}_i \sigma_i + \hbar\right) \quad (3.24)$$
In correspondence with the classical Poisson bracket relations, it rotates the supercharges (cf. with the two last relations from (2.4)),
\[ [\hat{Q}_L, \hat{Q}_X] = i\hbar \hat{Q}_X, \quad [\hat{Q}_L, \hat{Q}_P] = i\hbar \hat{Q}_P, \]
\[ [\hat{Q}_L, \hat{Q}_X] = -i\hbar \hat{J} \hat{Q}_X, \quad [\hat{Q}_L, \hat{Q}_P] = -i\hbar \hat{J} \hat{Q}_P. \] (3.25)

Here
\[ \hat{J} = \hat{J}_i^2 - \nu^2 + \frac{1}{4}\hbar^2 \]
is the quantum analog of (3.6) with the quantized \( \nu \), \( |\nu| = \hbar n/2 \), \( n \in \mathbb{N} \), and the quantum analogs of \( Q_X \) and \( Q_P \) are obtained from (3.17), (3.18) by a simple antisymmetrization of the noncommuting factors \( \hat{L}_j \) and \( \hat{P}_k \), or \( \hat{L}_j \) and \( \hat{X}_k \).

The only difference of the nonlinear superalgebra formed by the fermion-monopole integrals of motion in comparison with the quantum version of the \( \text{osp}(2|2) \) superalgebra (2.4) consists in the presence of the additional factor \( \hat{J} \) in commutators (3.25) and in anticommutators of \( \hat{Q}_P \) and \( \hat{Q}_X \) (see the table). In accordance with the classical Poisson bracket relations we have also
\[ [\hat{Q}_P, \hat{Q}_X] = [\hat{Q}_P, \hat{Q}_X] = \hbar (\hat{Q}_L + \hat{J}). \]

The integral \( \hat{Q}_L \) commutes with all the even generators of the nonlinear generalization of the \( \text{osp}(2|2) \oplus su(2) \), and in addition, due to the relation
\[ \hat{Q}_L = \sqrt{\frac{\hbar}{2}} \hat{Q}_Y \] (3.26)
anticommutates with the odd supercharges. Satisfying the relation
\[ \hat{Q}_L^2 = \frac{\hbar^2}{4} \hat{J}, \] (3.27)
it is the square root from the central element \( \hat{J} \) of the nonlinear superalgebra, and may be treated simultaneously as the grading operator of the nonlinear version of the \( \text{osp}(2|2) \oplus su(2) \).

The quantum analog of the classical relation (3.5) is
\[ \hat{C} = 4\hat{D}^2 - 2(\hat{K} \hat{H} + \hat{H} \hat{K}) + \hat{J} - 2\hat{Q}_L = \frac{3}{4}\hbar^2. \]

This quantum relation fixing the value of \( \hat{C} \) (cf. with (2.12)) appears as the condition of existence of nontrivial solutions to the system of homogeneous equations being the quantum analog of (3.21).

In representation where the squared full angular momentum operator is diagonal, \( \hat{J}_i^2 = j(j + 1)\hbar^2, \ j + \frac{1}{2} = \hbar^{-1}|\nu| + m, \ m = 0, 1, 2, ..., \) we have \( \hat{J} = (|\nu| + m)^2 - \nu^2 \) \[2, 14\]. Then, in accordance with relation (3.27), for the states with \( m > 0 \) the appropriately normalized operators \( \hat{Q}_L, \hat{Q}_X \) and \( \hat{Q}_P \) (see eq. (3.22)) together with the rest of integrals of motion generate the Lie superalgebra \( \text{osp}(2|2) \oplus su(2) \). However, as in the case of the Kepler problem, such a normalization procedure has a hidden nonlocal nature. Following ref. \[2\], one can show that in the sector \( m = 0 \), corresponding classically to the phase space surface given.
by the equations \( L_i = 0, S_j n_j = 0 \), the symmetry of the system is reduced to the conformal symmetry \( so(1,2) \).

Note that the analogy with the Kepler problem, mentioned in the Introduction, is given by the quantum relations (3.27),

\[
[\hat{Q}_P, \hat{Q}_P]_+ = 2\hbar\hat{H}, \quad [\hat{Q}_L, \hat{Q}_P] = i\hbar\hat{Q}_P, \quad [\hat{Q}_P, \hat{Q}_P]_+ = 2\hbar\hat{J}\hat{H},
\]

(3.28)

see Eq. (3.26) and the Table. Here, however, it is necessary to stress that the relation (3.27) is satisfied by the \( \hat{Q}_L \) not as by the even generator of the nonlinear superconformal algebra, but as its grading operator which coincides (up to the quantum factor, see Eq. (3.26)) with the Yano supercharge \( \hat{Q}_Y \).

4 Discussion and outlook

To conclude, let us summarize the obtained results and discuss shortly some problems that deserve a further attention.

Comparing the system of the charged fermion in the field of the Dirac magnetic monopole with the model of superconformal mechanics, we have showed that its integrals of motion generate a nonlinear \( Z_2 \)-graded Poisson algebra, or quantum superalgebra, which may be treated as a nonlinear generalization of the \( osp(2|2) \oplus su(2) \). In this nonlinear superalgebra, containing the \( osp(1|2) \) as a Lie sub-superalgebra, the shifted square of the full angular momentum of the system plays the role of the central charge appearing additively and multiplicatively in the quantum (anti)commutation and classical Poisson bracket relations. The square root from the central charge is the \( osp(2|2) \) spin generating the \( u(1) \) rotations of the odd supercharges. Classically, it has an odd counterpart, whose quantum analog is, up to a numerical factor \( \sqrt{\hbar/2} \), the same \( osp(2|2) \) spin operator. As an odd integral, it generates a nonlinear supersymmetry discussed earlier in [3, 5, 6]. Since it anticommutes with all other odd supercharges, and commutes with all the even integrals of the nonlinearly generalized \( osp(2|2) \oplus su(2) \) (including itself), it may be identified as a grading operator of the superalgebra. Note also that the form of the nonlinear superalgebra can be simplified a little bit by shifting the \( osp(2|2) \) spin for the central charge, \( \hat{Q}_L \rightarrow \hat{Q}_L + \hat{J} \equiv \Lambda \) (see the table).

The natural question is what are the possible physical consequences of the described nonlinear superconformal symmetry of the charge-monopole system? Having in mind the mentioned analogy with the hidden symmetry of the Kepler problem, it would be interesting to look at the charge-monopole scattering problem [16] from the perspective of the revealed symmetry.

In refs. [12, 13] it was observed that the change of a boson-fermion coupling constant \( \alpha \rightarrow n\alpha \), \( n \in \mathbb{N} \), in the superconformal mechanics model corresponds to the change of the particle’s spin \( \hbar/2 \) for \( n\hbar/2 \), and that the modified superconformal model is characterized by the nonlinear superconformal symmetry \( osp(2|2)_n \), in which the set of \( 2(n + 1) \) odd integrals constitute the spin-\( n/2 \) representation of the \( so(1,2) \). Proceeding from the close similarity between the fermion-monopole system and superconformal mechanics model, one could expect the appearance of some generalization of the nonlinear superconformal
symmetry $osp(2|2)_n$ of refs. \[12\] \[13\] as a symmetry for a higher spin charged particle in the field of the Dirac monopole.

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