RATIONAL POINTS ON THE FERMAT CUBIC SURFACE

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Abstract. We prove a lower bound that agrees with Manin’s prediction for the number of rational points of bounded height on the Fermat cubic surface. As an application we provide a simple counterexample to Manin’s conjecture over $\mathbb{Q}$.

1. Introduction

The subject of representing integers as a sum of two cubes has a rich history in number theory. Examples such as $91 = 6^3 - 5^3 - 4^3 + 3^3$ and $1729 = 1^3 + 12^3 = 9^3 + 10^3$ make one wonder how often integers with at least two essentially distinct representations occur and Euler had in fact showed that there are arbitrarily large such integers. Let $F = 0$ denote the Fermat cubic surface, where $F$ is given by

$$F := x_0^3 + x_1^3 + x_2^3 + x_3^3,$$

and notice that essentially non–distinct representations give rise to elements $x \in \mathbb{Z}_4^\text{prim}$ on the lines

$$x_0 + x_1 = x_2 + x_3 = 0, \quad x_0 + x_2 = x_1 + x_3 = 0, \quad x_0 + x_3 = x_1 + x_2 = 0$$

of the surface, where $\mathbb{Z}_4^\text{prim}$ denotes the set of integer vectors $x$ such that $\gcd(x_0, \ldots, x_3) = 1$. We are interested in estimating the growth of the counting function

$$N(B) = \# \{ x \in \mathbb{Z}_4^\text{prim}, F(x) = 0, x \text{ outside lines, } |x| \leq B \},$$

where $|\cdot|$ is the usual supremum norm.

Hooley [Hoo63], building upon work of Erdös [Erd39], showed that for almost all integers which are the sum of two cubes the representation is unique, by proving that $N(B) = o(B^2)$. In a subsequent revisit of the subject [Hoo80, Th.3 & Th.4] he used Deligne’s estimates along with sieve arguments to establish the stronger estimates

$$B \ll N(B) \ll B^{\frac{5}{4}+\varepsilon}$$

for all $\varepsilon > 0$ and $B \geq 6$. These estimates raised the question of evaluating the true asymptotic order of $N(B)$. A conjectural answer was provided by a special case of a very general conjecture due to Manin [FMT89].

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Whenever $f = 0$ is a general smooth cubic surface with a rational point, the conjecture states that

$$N_f(B) \sim c_f B (\log B)^{\rho_f - 1},$$

as $B \to \infty$, where $N_f(B)$ is defined similarly to $N(B)$ by excluding the 27 lines contained in $f = 0$, $\rho_f \in [1, 7]$ is the rank of the Picard group of the surface and $c_f$ is a positive constant. It is worth noticing that although the conjecture has been established for a large class of varieties, see for example [Bro07], it has never been established for a single smooth cubic surface.

In the case of the Fermat cubic surface the conjecture predicts the asymptotic behaviour

$$N(B) \sim c B (\log B)^3,$$

for some $c > 0$, since it is known that $\rho_F = 4$ (see §8.3.1 in [Bro09]). Wooley [Woo95] gave an elementary proof of the upper bound in (1.1) and Heath-Brown [HB97], building on this work, made the improvement

$$N(B) \leq B^{3+\varepsilon},$$

for any $\varepsilon > 0$, an estimate which actually applies to any smooth cubic surface with 3 rational coplanar lines. Regarding lower bounds, Slater and Swinnerton-Dyer [SSD98] used a secant and tangent process to obtain

$$N_f(B) \geq_f B (\log B)^{\rho_f - 1},$$

whenever the smooth cubic surface $f = 0$ contains two skew lines defined over the rationals. The result does not however cover the case of the Fermat cubic surface since its only skew lines are defined over $\mathbb{Q}(\sqrt{-3})$. Our main goal is to fill this gap and to improve optimally the lower bound in (1.1).

**Theorem 1.** We have the estimate

$$N(B) \geq B (\log B)^3,$$

for all $B \geq 6$.

Our key ingredient here is earlier work of the author [Sof13]. It allows us to cover the surface with a family of conics and count rational points on each conic individually. This approach leads to a sum of Hardy-Littlewood densities. We will show that this sum behaves as a divisor sum involving binary forms in §3. This approach can be used to prove (1.2) for other smooth cubic surfaces with a rational line, a topic that we intend to return to in the future. In fact, we note that our method, when fully extracted, is capable of proving

$$N_f(B) \geq \delta c_f B (\log B)^{\rho_f - 1},$$
for all large enough $B$, where $\delta > 0$ is a small absolute constant and $c_f$ is the Peyre constant $\text{Pey95}$.

We will use Theorem 1 in §6 in order to provide a simple counterexample over $\mathbb{Q}$ to the full version of Manin’s conjecture which is formulated for general Fano varieties. Let $k$ be a number field, $Z$ be a Fano variety over $k$ and $H$ be an anticanonical height function on $Z$. Suppose that $Z(k)$ is Zariski dense and define the counting function
\begin{equation}
N_k(U, B) := \# \{ x \in U(k) \mid H(x) \leq B \}
\end{equation}
for $B \geq 1$, where $U$ is a Zariski open subset of $Z$. A lack of the subscript $k$ will indicate that the counting is performed over the rationals.

Manin’s conjecture $\text{[FMT89]}$ describes the growth rate of $N_k(U, B)$ in terms of geometric invariants related to $Z$. It states that there exists a Zariski open $U \subset Z$ and a constant $c = c(U, k, H) > 0$ such that
\begin{equation}
N_k(U, B) \sim cB(\log B)^{\rho - 1},
\end{equation}
as $B \to \infty$, where $\rho$ is the rank of the Picard group of $Z$.

The conjecture does not hold in full generality, the first counterexample having been provided by Batyrev and Tschinkel $\text{[BT96]}$. They consider the biprojective hypersurface Fano cubic bundle $Y \subset \mathbb{P}^3_k \times \mathbb{P}^3_k$ given by
\begin{equation}
\sum_{i=0}^{3} x_i y_i^3 = 0.
\end{equation}
It is shown that if $k$ contains a cube root of unity and $U$ is any nonempty Zariski open subset of $Y$ we have
\begin{equation*}
N_k(U, B) \gg B (\log B)^3.
\end{equation*}
This estimate disproves (1.4) since the Picard group of $Y$ is isomorphic to $\mathbb{Z}^2$, as shown in $\text{[BT96], Prop.1.3]}$.

This result however leaves a counterexample over $\mathbb{Q}$ to be desired. This was achieved by Loughran $\text{[Lou12]}$, where Weil restriction was used to provide implicit counterexamples over any number field $k$ and of arbitrarily large dimension. Further counterexamples related to the Peyre constant and the power of the logarithm in (1.4) are provided by Browning and Loughran $\text{[BL13]}$ and Le Rudulier $\text{[LR13]}$ respectively.

Our aim is to extend the Batyrev–Tschinkel counterexample over $\mathbb{Q}$.

**Theorem 2.** For any nonempty Zariski open $U \subset Y$, where $Y$ is the biprojective hypersurface given by (1.5), we have
\begin{equation*}
N(U, B) \gg B (\log B)^3,
\end{equation*}
where the counting is performed over $\mathbb{Q}$.

This estimate contradicts Manin’s conjecture (1.4) due to the incompatibility of logarithmic exponents. Although we will not give details,
our method is capable of proving that Manin’s conjecture is not valid for \(1.5\) over any number field.

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**Notation:** For any functions \(f, g : [1, \infty) \to \mathbb{C}\), the equivalent notations \(f(x) = O_S(g(x))\), and \(f(x) \ll_S g(x)\), will be used to denote the existence of a positive constant \(\lambda\), which depends at most on the set of parameters \(S\) such that for any \(x \geq 1\) we have \(|f(x)| \leq \lambda|g(x)|\). Throughout sections 3–5 the absence of a subscript \(S\) will indicate that the implied constant is absolute. As usually, we denote the Euler, Möbius and the divisor function by \(\phi, \mu\) and \(\tau\) respectively, and we let \(\omega(p^n)\) be the number of distinct prime divisors of \(n\). We shall make frequent use of the familiar estimate

\[
\tau(n) \ll_\varepsilon n^\varepsilon,
\]
valid for any \(\varepsilon > 0\).

2. Covering by conics

The identity \(x^3 + y^3 = \frac{1}{4}(x + y)((x + y)^2 + 3(x - y)^2)\) reveals that the Fermat surface \(F = 0\) is equivalent over \(\mathbb{Q}\) to

\[X : x_0(x_0^2 + 3x_1^2) = x_2(x_2^2 + 3x_3^2),\]

and for the purpose of proving Theorem 1 it will suffice to prove that the counting function \(N_X(B)\) associated to this equation satisfies

\[(2.1) \quad N_X(B) \gg B(\log B)^3.\]

Our proof makes use of the conic bundle structure present in \(X\). More specifically, \(X\) is equipped with a dominant morphism \(\pi : X \to \mathbb{P}^1_{\mathbb{Q}}\) such that

\[
\pi(x) = \begin{cases} [x_0 : x_2] & \text{if } (x_0, x_2) \neq 0 \\ [x_2^2 + 3x_3^2 : x_0^2 + 3x_1^2] & \text{if } (x_0^2 + 3x_1^2, x_2^2 + 3x_3^2) \neq 0. \end{cases}
\]

It can easily be verified that the fibres \(\pi^{-1}([s : t])\) are the diagonal conics given by

\[Q_{s,t} : (t^3 - s^3)x^2 + (-3s)y^2 + (3t)z^2 = 0.\]

whose discriminant equals

\[\Delta_{s,t} = -9(t^3 - s^3)st.\]
Define for any \((s, t) \in \mathbb{Z}^2\) the following norm on \(\mathbb{R}^3\),
\[
\|(x, y, z)\|_{s, t} := \max \left\{|sx|, |xt|, |y|, |z|\right\}
\]
and the following counting function
\[
M_{s, t}(B) := \sum \mathbb{1}\{x \in \mathbb{Z}^3, Q_{s, t}(x) = 0, \|x\|_{s, t} \leq B\}.
\]
It is evident that for any \(B \geq 1\) we have
\[
N_X(B) = \sum_{|s|, |t| \leq B}^{*} M_{s, t}(B),
\]
where \(\sum_{s, t}^{*}\) denotes summation over \((s, t) \in \mathbb{Z}^2\). Indeed, for any \(x\) counted by \(N_X(B)\) we let \(x := \gcd(x_0, x_2)\), thus getting \((s, t) \in \mathbb{Z}^2\) with \(x_0 = sx\) and \(x_2 = tx\). This shows that \((x, y, z)\) is a primitive integer zero of the conic \(Q_{s, t}\) with \(\|(x, y, z)\|_{s, t} \leq B\). The fact that zeros \(x\) on the 3 rational lines
\[
x_0 = x_2 = 0, \quad x_0 - x_2 = x_1 - x_3 = 0, \quad x_0 - x_2 = x_1 + x_3 = 0.
\]
of \(X\) correspond to the zeros \((x, y, z)\) of singular conics \(Q_{s, t}\) follows upon noticing that \(x\) lies on any of these lines if and only if \(x_0x_2 (x_2 - x_0) = 0\), which in turn is equivalent to the vanishing of the discriminant \(\Delta_{s, t}\).

We estimate \(M_{s, t}(B)\) via [Sof13, Th.1], for any \((s, t)\) in the range with \(|s|, |t| \leq B^{\delta}\), where \(\delta \in (0, \frac{1}{40})\). In the notation of the theorem, we have \(\langle Q_{s, t}\rangle \leq B^{\frac{19}{20}}\) and \(K_{s, t} = 2\), thereby yielding
\[
M_{s, t}(B) = \frac{1}{2} \sigma_x(s, t) \prod_p \sigma_p(s, t) B + O \left( B^{\frac{19}{20}} \right),
\]
where the implied constant is absolute and the Hardy–Littlewood densities \(\sigma_x(s, t), \sigma_p(s, t)\) are defined in (3.1) and (3.2) respectively. To apply this to (2.2), notice that since we are interested in obtaining a lower bound for \(N_X(B)\), any extra conditions can be freely imposed on the summation over \(s\) and \(t\). Letting
\[
\mathfrak{G}(x) := \frac{1}{2} \sum_{|s|, |t| \leq x}^{*} \mathfrak{G}_{s, t}(s, t) \prod_p \sigma_p(s, t)
\]
we obtain
\[
N_X(B) \geq B \cdot \mathfrak{G}(B^{\frac{1}{20}}) + O(B).
\]
Now (2.1) implies that Theorem I would follow from
\[
(2.3) \quad \mathfrak{G}(x) \geq (\log x)^2,
\]
for all \(x \geq 2\). Establishing this estimate is the goal of §3–§5.
3. Passing from cubic surfaces to divisor sums

Our aim in this section is to show that the quantity \( G_{p, x, q} \) is approximated by a divisor sum. We will do so by finding explicit lower bounds for the Hardy–Littlewood densities uniformly with respect to \( s \) and \( t \). The following lemma will be used to facilitate the estimation of the \( p \)-adic densities.

**Lemma 1.** Let \( q(x) := \sum_{i=1}^{3} a_i x_i^2 \) be an integral smooth quadratic form and let \( p \) be an odd prime dividing exactly one coefficient, say \( a_1 \). Then for any \( n \geq 1 \), the number of solutions of \( q(x) \equiv 0 \pmod{p^n} \) such that \( p \) and \( x_3 \) are coprime is at least

\[
\left( 1 + \left( \frac{-a_2 a_3}{p} \right) \right) \left( 1 - \frac{1}{p} \right) p^{2n}.
\]

**Proof.** It suffices to consider the case in which \( -a_2 a_3 \) is a quadratic residue modulo \( p \) since otherwise the statement is trivial. Suppose therefore that there exists \( t \in \mathbb{Z}/p\mathbb{Z} \) such that \( a_2 t^2 + a_3 \equiv 0 \pmod{p} \). The proof is then completed via induction on \( n \).

Letting \( x_3 \) run through the values \( 1, \ldots, p-1 \) and \( x_2 = t x_3 \) we deduce the validity of the statement for \( n = 1 \). Let us observe that each solution \( x \pmod{p^n} \) of \( q(x) \equiv 0 \pmod{p^n} \) with \( p \nmid x_3 \) satisfies the hypothesis of Hensel’s lemma,

\[
p \nmid \nabla (q(x)) = 2 (a_1 x_1, a_2 x_2, a_3 x_3),
\]

and hence can be lifted to \( p^2 \) different solutions \( y \pmod{p^{n+1}} \) of the equation \( q(x) \equiv 0 \pmod{p^{n+1}} \) that must necessarily satisfy \( p \nmid y_3 \). \( \square \)

Let us define the arithmetic functions

\[
\tilde{\tau}(n) := \frac{\phi(n)}{n} 2^{\omega(n)}
\]

and

\[
\tilde{\tau}(n) := \frac{\phi(n)}{n} \prod_{p|n} \left( 1 + \chi_3(p) \right),
\]

where \( \chi_3 \) denotes the non–trivial character modulo 3, given by

\[
\chi_3(n) = \begin{cases} 
0 & \text{if } n \equiv 0 \pmod{3} \\
1 & \text{if } n \equiv 1 \pmod{3} \\
-1 & \text{if } n \equiv 2 \pmod{3}.
\end{cases}
\]

For each \( (s, t) \in \mathbb{Z}_{\text{prim}}^2 \) and every prime \( p \) we let

\[
N^*_{s, t}(p^n) := \sum_{\{x \pmod{p^n} \mid Q_{s, t}(x) \equiv 0 \pmod{p^n}, \ p \nmid x\}}
\]

denote the number of primitive zeros \( \pmod{p^n} \) of the quadratic form \( Q_{s, t} \). It was shown in [Sof13, Th.1] that the following limit exists, and its value is referred to as the Hardy–Littlewood \( p \)-adic density,

\[
\sigma_p(s, t) := \lim_{n \to \infty} N^*_{s, t}(p^n) p^{-2n}.
\]
The next lemma reveals that the evaluation of these densities naturally gives birth to the function $\tilde{r}$ which plays the role of the detector for the conics $Q_{s,t}$ that are isotropic over $\mathbb{Q}$.

**Lemma 2.** Suppose that the integers $s$ and $t$ satisfy $0 < s < t$, are coprime, and both equivalent to $1$ modulo $8$. Then we have

$$\prod_p \sigma_p(s, t) \geq \tilde{r}(t^3 - s^3) \tilde{r}(st),$$

where the implied constant is absolute.

**Proof.** We begin by calculating the $p$-adic densities for every prime $p \neq 2, 3$ dividing the discriminant of the conic $Q_{s,t}$. The coprimality of $s$ and $t$ ensures that no two coefficients of the conic are divisible by $p$ and hence we are allowed to use Lemma 1.

In the case that $p$ divides $s^3 - t^3$, we compute that

$$\left(\frac{-3 (t - 3) s}{p}\right) = \left(\frac{st}{p}\right) = \left(\frac{s^3 t}{p}\right) = \left(\frac{t^4}{p}\right) = 1,$$

and hence we are provided with the estimate $\sigma_p(s, t) \geq \tilde{r}(p)$.

The cases where $p|s$ or $p|t$ are symmetric and we therefore focus on the latter. A similar computation yields

$$\left(\frac{-3 (t^3 - s^3) (-3s)}{p}\right) = \left(\frac{-3s^4}{p}\right) = \left(\frac{-3}{p}\right).$$

Alluding to quadratic reciprocity reveals that the value of the Legendre symbol $\left(\frac{-3}{p}\right)$ equals $\chi_3(p)$, thereby yielding the estimate $\sigma_p(s, t) \geq \tilde{r}(p)$.

We recall at this point that if the conic $Q_{s,t}$ has zeros over $\mathbb{Q}_p$, then the $p$-adic density $\sigma_p(s, t)$ does not vanish, in which case the estimate

$$\sigma_p(s, t) \geq 1 - \frac{1}{p^2}$$

follows from [Sof13, Eq.(5.2)] or via Hensel lifting. Therefore the validity of the lemma would follow upon showing that the conic $Q_{s,t}$ is isotropic over $\mathbb{Q}_p$, for all primes $p$ not considered so far.

If the conic is nonsingular over $\mathbb{Q}_p$, i.e. $p \nmid \Delta_{s,t}$, then it is a well-known fact that it is isotropic over $\mathbb{Q}_p$. We are thus left with considering the cases $p = 2$ and $3$. Hilbert’s product formula, implies that it suffices to consider solubility merely over $\mathbb{Q}_2$. Recalling that $1 + 8\mathbb{Z}_2 \subseteq \mathbb{Q}_2^\times$ shows that $9st = u^2$ for some $u \in \mathbb{Q}_2^\times$. We observe that $(0, u, 3s)$ is a zero of $Q_{s,t}$ over $\mathbb{Q}_2$, which concludes the proof of the lemma.  

We next turn our attention to the evaluation of the archimedean Hardy–Littlewood density. It is defined as

$$(3.2) \quad \sigma_\infty(s, t) := \lim_{\varepsilon \to 0^+} \frac{1}{2\varepsilon} \int_{\|x\| \leq \varepsilon} \int_{Q_{s,t}(x)} 1 \ dx.$$ 

Let $B := \left(\frac{1}{4}, \frac{1}{2}\right) \times \left(\frac{1}{2}, 1\right)$ be an interval of $\mathbb{R}^2$. 
Lemma 3. Suppose that \((s, t) \in \mathbb{Z}_{\text{prim}}^2\) belongs to \(B2^n\) for some natural number \(n \geq 2\). We have the estimate
\[ \sigma_{\infty}(s, t) \geq 4^{-n}, \]
with an absolute implied constant.

Proof. Assume that \(\varepsilon \in (0, \frac{1}{2})\) throughout the proof and notice that the condition \((s, t) \in B2^n\) implies that \(0 < \frac{1}{4} < s < \frac{1}{2}\). It is clear that whenever \(y \in [0, \frac{1}{4}]\) and \((x, z)\) lies in the interior of the region \(R(y, \varepsilon)\) defined by
\[ (3s)y^2 - \varepsilon \leq (t^3 - s^3)x^2 + (3t)z^2 \leq (3s)y^2 + \varepsilon \]
then \(tx\) and \(z\) are both bounded in modulus by 1, thus leading to
\[ \int_{\{Q_{s,t}(x)\mid \|x\| \leq \varepsilon \}} 1 dx \geq \int_{0}^{\frac{1}{4}} \text{vol}(R(y, \varepsilon)) \, dy. \]
Noting that for \(a, b > 0\) the area of the ellipse \(ax^2 + by^2 = 1\) is equal to \(\pi(ab)^{-\frac{1}{2}}\), we find that the volume of \(R(y, \varepsilon)\) is at least \(\varepsilon t^{-2}\). We therefore deduce that
\[ \frac{1}{2\varepsilon} \int_{\{Q_{s,t}(x)\mid \|x\| \leq \varepsilon \}} 1 dx \geq \frac{1}{8t^2} \]
which leads to the desired result. \(\square\)

Recall that \(\sum_{s,t}^*\) denotes summation over coprime integers \(s\) and \(t\) and define the sum
\[ D(x) := \sum_{(s,t) \in Bx \atop s,t \equiv 1 \pmod{8}} \tilde{\gamma}(t^3 - s^3) \tilde{\gamma}(st) \]
for any \(x \geq 2\). The following result is obvious.

Lemma 4. We have the estimate
\[ G(x) \gg \sum_{n \in \mathbb{N}} 4^{-n} D(2^n), \]
for all \(x \geq 4\) with an absolute implied constant.

In light of (2.3), we are led to the conclusion that Theorem 1 would follow upon proving the estimate
\[ D(x) \gg x^2 (\log x)^3, \]
for all \(x \geq 4\). Notice that this is a lower bound of the correct order of magnitude.
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4. Estimation of divisor sums

We gather level of distribution results regarding functions related to \( r(n) \) before evaluating \( D(x) \) in \( \S 5 \). We denote the summation over residue classes \( a \equiv 0 \pmod{q} \) which are coprime to \( q \) by \( \sum_{a \equiv 0 \pmod{q}}^{*} \) and the inverse of \( a \pmod{q} \) by \( \bar{a} \). The Kloosterman sums \( S(a, b; c) \) are defined by

\[
S(a, b; c) = \sum_{x \equiv 0 \pmod{c}}^{*} e\left(\frac{ax + bx}{c}\right),
\]

for any integers \( a, b, c \), where we use the notation \( e(z) := e^{2\pi iz} \). The Weil bound [IK04, Eq.(1.60)] states that

\[
|S(a, b; c)| \leq \tau(c) \gcd(a, b, c)^{1/2} |c|^{1/2}.
\]

Lemma 5. For any integers \( a, q \) with \( q \) coprime to \( 3a \) and any \( X \geq 1 \), we have

\[
\sum_{\substack{n \leq X \\mod{q} \equiv a}} (1*\chi_3)(n) = \frac{\pi}{3^{2}} \tilde{c}(q) \frac{X}{q} + O\left(X^{\frac{1}{4}} q^{\frac{1}{2}} \tau(q)\right),
\]

where \( * \) denotes the Dirichlet convolution, the implied constant is absolute and

\[
\tilde{c}(q) := \sum_{d|q} \chi_3(d) \frac{\mu(d)}{d}.
\]

Proof. We use the function \( g \) defined in the proof of [IK04, Cor.4.9]. Introducing additive characters to detect the congruence \( n \equiv a \pmod{q} \), we arrive at the following upper bound for the sum in the statement of the lemma,

\[
\sum_{\substack{n \leq X \\mod{q} \equiv a}} (1*\chi_3)(n) g(n) = \frac{1}{q} \sum_{m \equiv 0 \pmod{q}} e\left(-\frac{ma}{q}\right) \sum_{n=1}^{\infty} (1*\chi_3)(n) e\left(\frac{mn}{q}\right) g(n).
\]

We partition the summation over \( m \) according to the value of the \( \gcd(m, q) = \frac{q}{d} \) which leads to

\[
\frac{1}{q} \sum_{d|q} \sum_{k \equiv 0 \pmod{d}}^{*} e\left(-\frac{ka}{d}\right) \sum_{n=1}^{\infty} (1*\chi_3)(n) e\left(\frac{kn}{d}\right) g(n).
\]

We will use [IK04, Eq.(4.70)] to estimate the inner sum. Notice that although the function \( g(x) \) is not smooth, the statement is still valid.
Recall that the values of the \( L \)-function and the Gauss sum corresponding to the character \( \chi_3 \) are \( \frac{\pi}{3^2} \) and \( i3^2 \) respectively. This yields
\[
\sum_{n=1}^{\infty} (1*\chi_3)(n)e\left(\frac{kn}{d}\right) g(n) = \frac{\pi}{3^2} \chi_3(d) \int_0^{\infty} g(x) \, dx \\
- \frac{2\pi i}{3^2} \chi_3(d) \sum_{n=1}^{\infty} (1*\chi_3)(n) e\left(\frac{3kn}{d}\right) h\left(\frac{4n}{3d^2}\right),
\]
where \( h \) is defined in [IK04, Eq.(4.36)] and denotes the Hankel type transform of \( g \). Interchanging the order of summation gives rise to Kloosterman sums,
\[
\sum_{k \pmod{d}} e\left(-\frac{ka}{d}\right) \sum_{n=1}^{\infty} (1*\chi_3)(n) h\left(\frac{4n}{3d^2}\right) e\left(\frac{(3k)n}{d}\right) \\
= \sum_{n=1}^{\infty} (1*\chi_3)(n) h\left(\frac{4n}{3d^2}\right) S(-a, 3n; d).
\]
The bound [IK04, Eq.(4.44)] when combined with (4.1) proves that the last expression is \( \ll \tau(d) d^2(X/Y)^{\frac{1}{2}} \). Putting everything together yields
\[
\sum_{\substack{n \leq X \atop n \equiv a \pmod{q}}} (1*\chi_3)(n) g(n) = \frac{\pi}{3^2} \frac{1}{q} \left(\int_0^{\infty} g(x) \, dx\right) \sum_{d|q} \chi_3(d) \sum_{k \pmod{d}} e\left(-\frac{ka}{d}\right) \\
+ O\left((X/Y)^{\frac{1}{2}} q^{\frac{1}{2}} \tau^2(q)\right).
\]
This with \( \sum_{k \pmod{d}} e\left(-\frac{ka}{d}\right) = \mu(d) \) and \( \int_0^{\infty} g(x) \, dx = X + \frac{Y+1}{2} \) proves the one-side estimate required for this lemma, on taking \( Y = X^\frac{1}{2} \). The desired lower bound is obtained in an identical way by using the weight function \( g(x) = \max\{x, 1, (X-x) Y^{-1}\} \) instead. \( \square \)

We will later need to estimate averages over arithmetic progressions of functions belonging to a rather large class of multiplicative functions. Our results are best formulated in terms of the group of functions
\[
G := \left\{ f : \mathbb{N} \to \mathbb{R}_{\geq 0}, f \text{ multiplicative, } f(p) = 1 + O_f\left(\frac{1}{p}\right) \text{ for all primes } p \right\}.
\]
Note that \( G \) is an abelian group with respect to pointwise multiplication and contains elements such as \( \phi(q)/q \) and the function \( \tilde{c}(q) \) defined in Lemma 5. The bound (1.16) reveals that each element \( f \in G \) satisfies \( \mu^2(n) f(n) \ll_{\varepsilon, f} n^\varepsilon \) for any \( \varepsilon > 0 \).

**Lemma 6.** Let \( f \) be a positive function such that either \( f \in G \) or
\[
f(n) = g(n) \prod_{p|n} (1 + \chi_3(n))
\]
for some $g \in G$. Then there exists a function $\hat{f} \in G$ and a positive constant $c \leq f^1$, both of which depend on $f$, such that for any integers $q,a,k$ with $q$ coprime to $3ak$ and $x \geq 1, \varepsilon > 0$ we have

$$\sum_{n \leq x} \frac{f(n)}{\phi(k)} \frac{x}{p} + O_{\varepsilon,f}(\frac{1}{p^2}).$$

Proof. The fact that $g \in G$ implies that in the second case we have $f(p) = 1 + \chi_3(p) + O(\frac{1}{p})$

for all primes $p$. Hence there exists $\delta \in \{0, 1\}$ such that the quantity

$$M := 1 + \sup_p \left| f(p) - 1 - \delta \chi_3(p) \right|$$

is well-defined. Define the function $\theta$ by letting

$$\theta(n) := \begin{cases} \frac{M}{p} \quad &\text{if } k = 1 \text{ and } p \nmid a \\ 3 + 2M \quad &\text{if } k = 2, 3 \text{ and } p \nmid a \\ 2 \quad &\text{if } k = 1, 2 \text{ and } p | a \\ 0 \quad &\text{if } k = 3 \text{ and } p | a \\ 0 \quad &\text{if } k \geq 4. \end{cases}$$

Hence for each $\varepsilon > 0$ we have the estimate

$$\sum_{n \leq x} \frac{\theta(n)}{n^{1/2 + \varepsilon}} \ll \prod_{p \leq x} \left( 1 + \frac{3 + 3M}{p^{3/2 + \varepsilon}} + \frac{3 + 2M}{p^{1+2\varepsilon}} \right) \prod_{p | a} \left( 1 + \frac{4}{p^{2+\varepsilon}} \right) \ll_{\varepsilon,M} \tau(a).$$

Therefore the case $\delta = 0$ of the lemma follows by [BBB97, Lem.2] with $k = \frac{1}{2} + \varepsilon$. Hence let $\delta = 1$. Since $\chi_3$ is the inverse of $\chi_3$, we deduce that $\theta = \theta' \ast \chi_3$, which leads to

$$\sum_{n \leq x} \theta(n) = \sum_{n \leq x} \theta'(n) \sum_{n \leq x} \sum_{m \leq x/n} (1 \ast \chi_3)(m).$$
An application of Lemma 5 yields
\[ \frac{\pi}{3^2} \frac{x}{q} \tilde{\varepsilon}(q) \left( \sum_{\substack{n \leq x \\gcd(n,q)=1}} \frac{\theta'(n)}{n} \right) + O_{\varepsilon,M} \left( x^{\frac{1}{2}+\varepsilon} q^{\frac{1}{2}+\varepsilon} (aq) \right) \]
valid for any \( \varepsilon > 0 \). The sum over \( n \) is absolutely convergent and extending the summation to infinity introduces a negligible error term. Factoring the series into Euler products reveals that the statement of the lemma holds with the constant
\[ c := \frac{\pi}{3^2} \prod_{p} \left( 1 + \frac{f(p)}{p} \right) \left( 1 - \frac{1}{p} \right) \left( 1 - \frac{\chi_3(p)}{p} \right) \]
and the function
\[ \hat{f}(n) := \tilde{\varepsilon}(n) \prod_{p|n} \left( 1 + \frac{f(p)}{p} \right)^{-1} \left( 1 - \frac{1}{p} \right)^{-1} \left( 1 - \frac{\chi_3(p)}{p} \right)^{-1} \].

\[ \square \]

We will later consider divisor sums involving binary forms for which a two-dimensional version of the previous lemma shall be required.

**Lemma 7.** Let \( f_1 \) and \( f_2 \) be functions satisfying the assumption of Lemma 5. Then there exists a function \( \hat{f} \in G \) and a positive constant \( c \), both of which depend on \( f_1 \) and \( f_2 \), such that for any \( x, y \geq 1, \varepsilon > 0 \) and integers \( q, a, k, \sigma, \tau \) with \( q \) coprime to \( 3ak\sigma\tau \) we have
\[ \sum_{s \leq y, t \leq x} f_1(s) f_2(t) = c \hat{f}(kq) \left( \frac{\phi(k)}{k} \right)^2 \frac{xy}{q^2} + O_{\varepsilon,f_1,f_2} \left( \frac{xy}{\min(x,y)^{2-\varepsilon} q^3 k^\varepsilon} \right). \]

**Proof.** As in the proof of the previous lemma, there exist \( \delta_1, \delta_2 \in \{0,1\} \) such that the following quantity is well-defined
\[ M := 1 + \max_{i=1,2} \sup_p \left| p \left( f_i(p) - 1 - \delta_i \chi_3(p) \right) \right|. \]
We may assume \( y \leq x \) without loss of generality. Using Möbius inversion to deal with the coprimality of \( s \) and \( t \) implies that the sum appearing in the lemma equals
\[ \sum_{m \leq y/2} \mu(m) f_1(m) f_2(m) \sum_{s \leq y/m} f_1(s) \sum_{t \leq x/m} f_2(t). \]
Let us observe that the previous lemma supplies the estimates
\[ \mu^2(n) f_i(n), \mu^2(n) \hat{f}_i(n) \ll_{\varepsilon,M} n^{\varepsilon}, |c_i| \ll_M 1 \text{ for } i = 1, 2, \]
valid for any $\varepsilon > 0$, due to the definition of $G$ and \((5.1)\). We are therefore provided with an error term bounded by $\ll_{\varepsilon,M} x y^{\frac{3}{2} + \varepsilon} q^{2 + \varepsilon} k^\varepsilon$ and with the main term

$$c_1 c_2 \hat{f}_1(kq) \hat{f}_2(kq) \left( \frac{\phi(k)}{k} \right)^2 \frac{xy}{q^2} \times \sum_{\substack{m \leq y/2 \\ m = \text{squarefree}}} \frac{\mu(m)}{m^2} f_1(m) f_2(m) \hat{f}_1(m) \hat{f}_2(m) \left( \frac{\phi(m)}{m} \right)^2.$$

The bounds \((4.2)\) enable us to extend the summation over $m$ to infinity, introducing a negligible error. Comparing the ensuing Euler factors concludes the proof of the lemma. \qed

5. Proof of Theorem \ref{thm:main}

We dedicate this section to the proof of \((3.3)\). Denote the sum over residue classes $\sigma, \tau \in [0,d)$ for which $\gcd(\sigma, \tau, d) = 1$ by $\sum_{(\sigma,\tau) \equiv (0,d)}^*$: Defining the function $\hat{\tau}(n) := \mu^2(n) \prod_{p|n} \left( 1 - \frac{2}{p} \right)$ allows us to write $\hat{\tau}(n) = \sum_{\substack{d|n}} \hat{\tau}(d)$. We insert this into $D(x)$ and invert the order of summation. The non-negativity of the values assumed by $\hat{\tau}$ enables us to restrict the summation, arriving at

\begin{equation}
\label{eq:5.1}
D(x) \geq \sum_{\substack{d_1, d_2 \leq x^\varepsilon \\ \gcd(d_1, d_2, 6) = 1}} \hat{\tau}(d_1 d_2) \sum_{(\sigma,\tau) \equiv (0,d)}^* \sum_{\substack{s, t \text{ squarefree} \\ (s, t) \equiv (\sigma,\tau) \pmod{8d_1 d_2}}^* \hat{\tau}(st),
\end{equation}

valid for any $0 < \varepsilon < 1$. We use Lemma \ref{lem:main} with $f_1 = f_2 = \hat{\tau}$ to estimate the sum over $s$ and $t$. We are allowed to do so since $\phi(n)/n$ is an element of the group $G$ and therefore $\hat{\tau}$ satisfies the hypotheses of Lemma \ref{lem:main}. We are thus led to

$$\sum_{\substack{(s, t) \equiv (\sigma,\tau) \pmod{8d_1 d_2} \\ \text{s.t squarefree}}} \hat{\tau}(st) = c_0 f_0(d_1 d_2) \frac{x^2}{d_1 d_2} + O_\varepsilon \left( x^{\frac{7}{4} + \varepsilon} d_1^3 d_2^3 \right)$$

for any $\varepsilon > 0$, where $f_0 \in G$ and $c_0$ is an absolute positive constant. Taking into account that the number of solutions of $x^2 + x + 1 \equiv 0 \pmod{p}$ equals $1 + \left( \frac{-3}{p} \right)$ for any prime $p$ and inserting the previous estimate into \((5.1)\) leads to

\begin{equation}
\label{eq:5.2}
D(x) \geq c_1 x^2 \sum_{\substack{d_1, d_2 \leq x^\varepsilon \\ \gcd(d_1, d_2, 6) = 1}} \frac{\hat{\tau}(d_1 d_2)}{d_1 d_2} f_0(d_1 d_2) \prod_{p|d_1} (1 + \chi_3(p)) + O_\varepsilon \left( x^{\frac{7}{4} + 10\varepsilon} \right),
\end{equation}
where \( c_1 \) is an absolute positive constant. We observe that the functions \( \tilde{\Gamma} \) and \( f_0 \) are elements of the group \( G \) and so does their product. Hence the functions

\[
f_1(n) = \tilde{\Gamma}(n) f_0(n) \prod_{p|n} (1 + \chi_3(n)), \quad f_2(n) = \tilde{\Gamma}(n) f_0(n),
\]

fulfill the hypotheses of Lemma 7. Therefore for any \( \varepsilon > 0 \) we obtain

\[
\sum_{y/2 < d_1 \leq y \atop x/2 < d_2 \leq x \atop \gcd(d_1 d_2, 6) = 1} \frac{\tilde{\Gamma}(d_1 d_2)}{d_1 d_2} f_0(d_1 d_2) \prod_{p|d_1} (1 + \chi_3(p)) = c_2 xy + O_\varepsilon \left( \frac{xy}{\min^{4-\varepsilon}(x, y)} \right)
\]

with an absolute positive constant \( c_2 \). Partitioning in dyadic intervals shows that the sum appearing in (5.2) is larger than

\[
\sum_{1 \leq i < j \leq \log_2 x} 2^{-i-j} \sum_{2^{-i} < d_1 \leq 2^j \atop 2^{-i} < d_2 \leq 2^j \atop \gcd(d_1 d_2, 6) = 1} \tilde{\Gamma}(d_1 d_2) \frac{\tilde{\Gamma}(d_1 d_2)}{d_1 d_2} f_0(d_1 d_2) \prod_{p|d_1} (1 + \chi_3(p)).
\]

Using the previous estimate for each inner sum proves (3.3) from which Theorem 1 follows.

6. PROOF OF THEOREM 2

Recall the definition of \( Y \) in (1.5). The anticanonical divisor on \( Y \) is \( -K_Y = \mathcal{O}(3, 1) \) which provides the height \( H \) defined as follows. For a point \( (x, y) \in \mathbb{P}^3_\mathbb{Q} \times \mathbb{P}^3_\mathbb{Q} \), we choose \( x, y \in \mathbb{Z}^4_{\text{prim}} \), unique up to sign, so that \( (x, y) = ([x], [y]) \) and we let

\[
H(x, y) := |x|^3|y|,
\]

where \( |\cdot| \) denotes the usual sup norm in \( \mathbb{R}^4 \). The counting function, defined in (1.3), takes the following shape. For any Zariski open subset \( U \) of \( Y \) we set

\[
(6.1) \quad N(U, B) = \frac{1}{4} \left\{ (x, y) \in \mathbb{Z}^4_{\text{prim}} : ([x], [y]) \in U, |x|^3|y| \leq B \right\}.
\]

Define the map \( \tilde{\pi} : Y \to \mathbb{P}^3_\mathbb{Q} \) by \( \tilde{\pi}(x, y) = x \). The image of \( U \) under \( \tilde{\pi} \) forms a Zariski open set and it therefore intersects the Zariski dense subset of \( \mathbb{P}^3_\mathbb{Q} \) given by

\[
\left\{ [t_0^3 : \ldots : t_3^3] : t_0, \ldots, t_3 \in \mathbb{Q}^* \right\}.
\]

Letting \( Y_t \) stand for the corresponding cubic surface

\[
Y_t := \sum_{i=0}^3 t_i^3 y_i^3 = 0,
\]

we are provided with some \( t = (t_0, \ldots, t_3) \in \mathbb{Z}^4_{\text{prim}} \), with \( \prod_{i=0}^3 t_i \neq 0 \), such that \( U_t := U \cap Y_t \) is non–empty. We fix the choice of \( t \) for the rest of this section. The surface \( Y_t \) is irreducible since \( \prod_{i=0}^3 t_i \neq 0 \),
which implies that the closed subvariety $Y_t \setminus U_t$ is a finite union $\bigcup_{i=1}^r C_i$ of curves or points in $\mathbb{P}^3$. Letting
\[ N(U_t, B) = \sharp \{ y \in U_t(\mathbb{Q}), H(y) \leq B \} \]
we deduce that
\[ N(U, B) \gg N\left( U_t, \frac{B}{|t|^9} \right) \]
for all $B \geq |t|^9$. It therefore suffices for the purpose of establishing Theorem 2 to prove the lower bound
\[ N(U_t, B) \gg B (\log B)^3 \]
for all $B \geq 3$. We can therefore assume without loss of generality that $U_t$ contains none of the lines of $Y_t$. Letting $Y'_t$ denote the Zariski open subset of $Y_t$ where the lines have been excluded, we deduce that
\[ N(U_t, B) \geq N(Y'_t, B) - \sum_{1 \leq i \leq r \atop C_i \neq \text{line}} N(C_i, B) . \]
Hence if $C_i$ is a curve, its degree will be at least 2 in which case Theorem 1.1 in [Val13] shows that $N(C_i, B) \ll_{C_i} B$. Noting that this estimate trivially holds if $C_i$ is a point, we have shown that
\[ N(U_t, B) \geq N(Y'_t, B) + O(B) , \]
where the implied constant depends at most on $r, t$ and $C_i$.

**Lemma 8.** We have
\[ N(Y'_t, B) \gg_t B (\log B)^3 \]
for all $B \geq 3$.

**Proof.** Let $T := |t_0 t_1 t_2 t_3| \neq 0$ and recall that $N(B)$ denotes the counting function associated to the Fermat cubic surface. Alluding to Theorem 1 implies that it suffices to prove that
\[ N(Y'_t, B) \geq N\left( \frac{B}{T} \right) \]
for all $B \geq T$. For any $z \in \mathbb{Z}_{prim}^4$ counted by $N(B)$ define $[y] \in \mathbb{P}^3_{\mathbb{Q}}$ via $y_i := \frac{t_i}{t}$ and notice that $y$ lies on $Y'_t$. We have
\[ [y] = [z_0 t_1 t_2 t_3 : \ldots : z_3 t_0 t_1 t_2] \]
and we observe that for $d := \gcd(z_0 t_1 t_2 t_3, \ldots, z_3 t_0 t_1 t_2)$, we get
\[ H(y) = \frac{\max\{|z_0 t_1 t_2 t_3|, \ldots, |z_3 t_0 t_1 t_2|\}}{d} \leq BT , \]
which proves (6.4). \qed

Combining the estimate (6.3) and Lemma 8 proves (6.2) from which the validity of Theorem 2 is inferred.
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