Non-linear equation: energy conservation and impact parameter dependence

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Abstract: In this paper we address two questions: how energy conservation affects the solution to the non-linear equation, and how impact parameter dependence influences the inclusive production. Answering the first question we solve the modified BK equation which takes into account energy conservation. In spite of the fact that we used the simplified kernel, we believe that the main result of the paper: the small (\( \leq 40\% \)) suppression of the inclusive production due to energy conservation, reflects a general feature. This result leads us to believe that the small value of the nuclear modification factor is of a non-perturbative nature. In the solution a new scale appears \( Q_{fr} = Q_s \exp(-1/(2\alpha_S)) \) and the production of dipoles with the size larger than \( 2/Q_{fr} \) is suppressed. Therefore, we can expect that the typical temperature for hadron production is about \( Q_{fr} (T \approx Q_{fr}) \). The simplified equation allows us to obtain a solution to Balitsky-Kovchegov equation taking into account the impact parameter dependence. We show that the impact parameter (b) dependence can be absorbed into the non-perturbative b dependence of the saturation scale. The solution of the BK equation, as well as of the modified BK equation without b dependence, is only accurate up to \( \pm 25\% \).

Keywords: BFKL Pomeron, non-linear equation, saturation, nuclear modification factor, impact parameter dependence.

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1. Introduction

High density QCD that has a long history [1–6], it is actually based on the idea that before summing the next-to-leading order correction to the BFKL Pomeron [7], we need to deal with different kinds of corrections. Indeed, the scattering amplitude in the leading order BFKL approach is proportional to

$$N_{\text{BFKL}} \propto \alpha_s^2 s^{\Delta_{\text{BFKL}}}$$

where $\Delta_{\text{BFKL}} \propto \alpha_s$. We can see two kind of corrections to the amplitude of Eq. (1.1): the first one is the next-to-leading order correction to the BFKL kernel that gives $\Delta_{\text{BFKL}} = \alpha_sC_1 + \alpha_s^2C_2$ where $C_1$ and $C_2$ are constants; and the second corrections due to the exchange of two BFKL Pomerons, namely,

$$N_{\text{2 BFKL Pomeron exchange}} \propto N_{\text{BFKL}}^2 \propto \alpha_s^4 s^{2\Delta_{\text{BFKL}}}$$ (1.2)

Comparing Eq. (1.1) and Eq. (1.2) one sees that for a wide range of energies

$$y = \ln s > \frac{2}{\alpha_s} \ln \left( \frac{1}{\alpha_s} \right)$$

(1.3)

One has also to include the exchange of two BFKL Pomerons that leads to non-linear corrections in the evolution equations.

For these energies the NLO corrections to the BFKL kernel are small. They becomes essential for higher energies for which $\alpha_s^2 y \approx 1$ or $y = \ln s > 1/\alpha_s$. Therefore, for energies in the kinematic region

$$\frac{1}{\alpha_s^2} \geq y \geq \frac{2}{\alpha_s} \ln \left( \frac{1}{\alpha_s} \right)$$

(1.4)

we have to include multi-Pomeron exchange and can neglect NLO corrections to the BFKL kernel.

Explicit calculations of the NLO corrections to the BFKL kernel [8], show that $\alpha_s^2$ corrections to $\Delta_{\text{BFKL}}$ are rather large, and have to be taken into account for any realistic and practical estimates. In terms of Eq. (1.4) the large NLO corrections mean that the kinematic range for the high density QCD is narrow. Fortunately, we know the NLO corrections to the linear BFKL equations [12] and to the nonlinear (Balitsky-Kovchegov) equation (see Ref. [13] for review and references) quite well. We also know that such corrections change the value and energy dependence of the saturation scale [14, 15]. Therefore, it is necessary to include the NLO correction to obtain reliable estimates of the non-linear effects.

It is not sufficient to solve the Balitsky-Kovchegov (BK) equation because together with the NLO corrections to the Balitsky-Kovchegov equation, we need to include in the calculating procedure the corrections referred to as Pomeron loops [10,16–22]. In spite of the fact, that we have learnt a lot about these corrections, no closed equation exists and the whole procedure turns out to be so complicated, that there is no hope of obtaining reliable estimates in the foreseeable future.

In such a situation the only thing that we can do, is to try to study the influence of different terms in NLO corrections to the Balitsky-Kovchegov equation, with the goal of finding the essential terms. After
that we can try to estimate the influence of the Pomeron loops and find out what is more important: loops or NLO corrections. We are presently at the first stage of such a study.

The clear example of such approach is the effect of running QCD coupling constant. The running QCD coupling constant is believed to be one of the most characteristic features of QCD, and it is important to find out how this phenomenon affects the value of the scattering amplitude. Fortunately, the form of the BK equation is known [23] and the numerical solution [24] shows that the solution with running QCD coupling constant is quite different from the one with fixed coupling.

The second example is the solution to more general JIMWLK equation which includes all $1/N_c$ correction to Balitsky-Kovchegov equation (see Ref. [25]). It turns out that the difference, between the solution of the JIMWLK equation and the BK one, is extremely small.

In this paper we wish to investigate the influence of the conservation of energy on the solution to the BK equation. As has been discussed the important observable for the interpretation of the nucleus-nucleus collision is the energy loss [28]. In LO the non linear equation is written neglecting the energy loss. Therefore, we need to generalize the BK equation in such way that it would respect energy conservation. Such a modified equation will allows us to discuss the value and energy and transverse momentum dependence of the nucleus modification factor (NMF). Since for single inclusive production the $k_t$ factorization has been proven (see Ref. [29]) the value of the NMF, that is originated from short distances, is determined by the BK equation. The generalized BK equation has been suggested in Ref. [30] and for the completeness of presentation we will discuss this equation in the next section. The main section of the paper is the third one in which we present the solution to this equation closely following the method proposed in Ref. [31]. We find no considerable effects due to energy conservation, and show that the dipole scattering amplitude in the region $r > 1/Q_s$ leads to the inclusive gluon production that is 20 - 40% smaller than that is predicted from the LO BK equation.

The second problem that we discuss in this paper is the impact parameter dependence of the solution both to Balitsky-Kovchegov equation and to the modified version of this equation. It is well known that $b$ dependence is one of the most difficult and challenging problems in QCD [32]. The massless gluon leads to power-like decrease of the scattering amplitude for large values of $b$ which results in the power-like increase of the interaction radius [32]. Such behavior of the interaction radius indicates that the main contribution to the solution of the non-linear equation stems from large values of $b$ making the entire approach, based on perturbative QCD, inconsistent. The attempts (only two [33] as far as we know) to solve BK equation numerically taking into account $b$ dependence, confirm these pessimistic expectations. The simplified version of BK equation allows one to obtain the analytical solution including $b$-dependence. It turns out that the non-perturbative $b$ dependence can be absorbed in the saturation scale. This observation gives us a guide for the solution of the impact parameter problem in high density QCD, but how to include this observation into general BK equation still remains unsolved. In section 4 we discuss the inclusive production for the solution with impact parameter dependence.
2. The modified non-linear equation

The non-linear (BK) equation has the following form for the dipole scattering amplitude \( N (r, Y; b) \)

\[
\frac{\partial N (r, Y; b)}{\partial Y} = \bar{\alpha}_S \, \pi \, \int d^2 r' \, K (r, r') \left( 2N \left(r', Y; \vec{b} - \frac{1}{2} (\vec{r} - \vec{r}') \right) - N \left(r, Y; \vec{b} \right) \right.
\]

\[
- N \left(r', Y; \vec{b} - \frac{1}{2} (\vec{r} - \vec{r}') \right) N \left(\vec{f} - \vec{r}', Y; \vec{b} - \frac{1}{2} \vec{r}' \right) \right)
\] (2.1)

where \( Y = \ln(1/x) \), \( \bar{\alpha}_S = \frac{C_F \alpha_S}{\pi} \), \( x \) denotes the Bjorken variable for the dipole-target scattering, and \( b \) is the impact parameter for the reaction. Kernel \( K (r, r') \) in the LO is equal to

\[
K (r, r') = \frac{r^2}{(\vec{r} - \vec{r}')^2 r'^2}
\] (2.2)

2.1 The linear equation in NLO

The linear part of Eq. (2.1) can be easily written in the double Mellin transform

\[
N (\xi = \ln(r^2/R^2), Y = \ln(1/x); b) = \int \frac{d\omega}{2\pi i} \int \frac{d\gamma}{2\pi i} \, N(\omega, \gamma; b) \, e^{\omega Y + \gamma \xi}
\] (2.3)

It has the form

\[
\omega N(\omega, \gamma; b) = (\bar{\alpha}_S \chi_{LO}(\gamma) + \alpha_S^2 \chi_{NLO}(\gamma)) \, N(\omega, \gamma; b) \quad \text{or} \quad \omega = \omega(\gamma) = \bar{\alpha}_S \chi_{LO}(\gamma) + \alpha_S^2 \chi_{NLO}(\gamma). \]

(2.4)

For \( \chi_{LO}(\gamma) \) we have the well known expression [7]:

\[
\chi_{LO}(\gamma) = 2 \psi(1) - \psi(\gamma) - \psi(1-\gamma) = \frac{1}{\gamma} + \frac{1}{1-\gamma} + \chi_{HT}^{LO}(\gamma),
\]

(2.5)

with \( \psi = d \ln \Gamma(\gamma)/d\gamma \) and \( \Gamma \) is the Euler Gamma function. We will discuss the form of \( \chi_{NLO}(\gamma) \) later. \( \chi_{HT}^{LO}(\gamma) = 2 \psi(1) - \psi(1+\gamma) - \psi(2-\gamma) \) denotes the contribution of the higher twist which we will discuss below.

The NLO BFKL kernel is known [12, 26], following Ref. [30] we use the simplified form of the NLO kernel:

- In Ref. [15] the following form of the NLO BFKL kernel is proposed

\[
\bar{\alpha}_S \chi_{NLO}(\gamma) = \left( \frac{1 + \omega A_1(\omega)}{\gamma} - \frac{1}{\gamma} + \frac{1 + \omega A_1(\omega)}{1-\gamma + \omega} - \frac{1}{1-\gamma} \right) - \omega \chi_{HT}^{LO}(\gamma),
\]

(2.6)

where,

\[
A(\omega) = -11/12 + O(\omega) + n_F \left( \frac{\bar{\alpha}_S}{4 N_c^2 \gamma} P_{Gq}(\omega) P_{qG}(\omega) - 1/3 \right),
\]

(2.7)

with \( P(\omega) \) being the DGLAP kernel.
The singularities in Eq. (2.6) describe the different branches of evolution corresponding to the sizes of the interacting dipole (or the transverse momenta of partons). The pole at $\gamma = 0$ corresponds to the normal twist-2 DGLAP contribution, with the ordering in the transverse parton momenta $Q > \ldots k_{t,i} > k_{t,i+1} > \ldots > Q_0$, where $Q_0$ is the typical virtuality of the target $Q_0 \approx 1/R$. The pole at $\gamma = 1$ in Eq. (2.6) corresponds to inverse $k_t$ ordering ($k_{t,i} < k_{t,i+1} < \ldots Q_0$). The other poles, at $\gamma = -1, -2, \ldots (\gamma = 2, 3, \ldots)$, are the higher twists contributions due to the gluon reggeization.

As can be seen in Eq. (2.6), the main changes in the NLO kernel to the ordinary DGLAP evolution are introduced, so as to account for the DGLAP anomalous dimension, and to the part of the kernel that describes the inverse evolution. This branch of evolution is moderated by the non-linear effect, rather than by the NLO corrections. Indeed, it was shown in Ref. [31] that the term $1/(1 - \gamma)$ leads to exponentially small corrections in the saturation region. Consequently we can neglect changes in the inverse evolution, and keep the BFKL kernel without a shift from $\gamma$ to $\gamma - \omega$.

We use the observation of Ref. [34], according to which Eq. (2.7) for $A_1(\omega)$ can be approximated to an accuracy ($> 95\%$), by $A_1(\omega) = 1$.

Hence, we can rewrite the NLO BFKL kernel in a very simple form:

$$\chi_{NLO}(\gamma) = -\omega \chi_{LO}(\gamma). \quad (2.8)$$

Using Eq. (2.8) we obtain the full kernel for the linear equation in the form:

$$\omega(\gamma) = \bar{\alpha}_S \chi_{LO}(\gamma) (1 - \omega) \quad (2.9)$$

This kernel imposes energy conservation (see Ref. [34]), and describes the NLO BFKL kernel. It does not include the contribution coming from inverse ordering, which should be suppressed in the solution to the non-linear equation.

Note that substituting $\chi_{LO} = \bar{\alpha}_S / \gamma$ yields

$$\gamma(\omega) = \bar{\alpha}_S \left( \frac{1}{\omega} - 1 \right), \quad (2.10)$$

which approximates the DGLAP anomalous dimension in leading order so well, that the difference between this anomalous dimension and Eq. (2.10) turns out to be less than $5\%$ [34].

Eq. (2.10) corresponds to Eq. (2.6) where $A_1(\omega) = 1$, and the high twist terms are neglected.

### 2.2 The non linear equation in NLO

The main idea of Ref. [30] is to use Eq. (2.9) as the kernel for the non-linear equation.

First one can see that Eq. (2.9) can be rewritten in the coordinate space as
\[
\frac{\partial N (r, Y; b)}{\partial Y} = \frac{C_F \alpha_S}{\pi^2} \int d^2 r' K_{LO} (r, r') \left\{ 1 - \frac{\partial}{\partial Y} \right\} \\
\left( 2N \left( r', Y; \vec{b} - \frac{1}{2} (\vec{r} - \vec{r}') \right) - N \left( r, Y; \vec{b} \right) \right) .
\]

As is shown in Refs. [3, 21, 27] the non-linear evolution equation has a very simple probabilistic interpretation and can be written as the equation for the Markov chain or, in other words, can be viewed as the process of the birth (non-linear term) and death (linear term) of the colorless dipoles. Assuming that this interpretation is correct in NLO order, we obtain the following equation [30]:

\[
\frac{\partial N (r, Y; b)}{\partial Y} = \frac{C_F \alpha_S}{\pi^2} \int d^2 r' K_{LO} (r, r') \left\{ 1 - \frac{\partial}{\partial Y} \right\} \\
\left( 2N \left( r', Y; \vec{b} - \frac{1}{2} (\vec{r} - \vec{r}') \right) - N \left( r, Y; \vec{b} \right) - N \left( r', Y; \vec{b} - \frac{1}{2} (\vec{r} - \vec{r}') \right) N \left( \vec{r} - \vec{r}', \vec{b} - \frac{1}{2} \vec{r}' \right) \right) .
\]

(2.11)

(2.12)

The advantage of this equation is that energy is conserved by the non-linear term, as well as by the linear one. The linear term has been discussed above. We need only to rewrite the non-linear term in Mellin transform (see Eq. (2.3)), namely,

\[
N \left( r', Y; \vec{b} - \frac{1}{2} (\vec{r} - \vec{r}') \right) N \left( \vec{r} - \vec{r}', Y; \vec{b} - \frac{1}{2} \vec{r}' \right) =
\]

\[
\int \frac{d\omega \, d\omega'}{(2\pi i)^2} e^{(\omega - \omega') Y + \omega' Y} N \left( r', \omega - \omega'; \vec{b} - \frac{1}{2} (\vec{r} - \vec{r}') \right) N \left( \vec{r} - \vec{r}', \omega'; \vec{b} - \frac{1}{2} \vec{r}' \right)
\]

(2.13)

to understand why this term conserves energy, as the derivative with respect to \( Y \) leads to extra factor \( \omega \) in the non-linear term.

3. Solution to the modified B-K equation

3.1 Simplified kernel

3.1.1 General approach

As was suggested in Ref. [31] we replace the full BFKL kernel in LO by the following simplified kernel

\[
\omega_{LO} (\gamma) = \bar{\alpha}_S \begin{cases} 
\frac{1}{\gamma} & \text{for } z = \ln (r^2 Q_s^2) \leq 0 ; \\
\frac{1}{1 - \gamma} & \text{for } z = \ln (r^2 Q_s^2) > 0 ;
\end{cases}
\]

(3.1)

This kernel leads to the leading twist contribution and gives the natural generalization of the DGLAP equation that includes two different kind of logs: for \( z < 0 \) it sums \((\bar{\alpha}_S \ln (r^2 Q_{CD}))^n\) while for \( z > 0 \) the new type of appears, namely, \((\bar{\alpha}_S \ln (r^2 Q_s^2))^n\).
For the NLO kernel we assume that for $z = \ln \left( r^2 Q_s^2 \right) \leq 0$ we have Eq. (2.10) which can be rewritten as

$$\omega_{NLO}(\gamma) = \bar{\alpha}_S \frac{1}{\gamma + \bar{\alpha}_S}$$  \hspace{1cm} (3.2)

This form of the kernel shows that in the NLO, we sum, in addition to $\left( \bar{\alpha}_S \ln \left( 1 / (r^2 \Lambda_{QCD}) \right) \right)^n$ terms, also non-logarithmic corrections. Recall that log contribution stems from the singularity $(1/\gamma)$ in the kernel.

For $z = \ln \left( r^2 Q_s^2 \right) \geq 0$ we parameterize the NLO kernel as the solution to the following equation:

$$\omega_{NLO}(\gamma) = \bar{\alpha}_S (1 - \omega_{NLO}(\gamma)) \left\{ \frac{1}{1 - \gamma} + \alpha_S \kappa \right\}$$  \hspace{1cm} (3.3)

where we determine the constant $\kappa$ from the condition

$$\omega_{NLO} \left( \gamma \xrightarrow{\gamma \leq \gamma_{cr}} \gamma_{cr} \right) = \omega_{NLO} \left( \gamma \xrightarrow{\gamma \geq \gamma_{cr}} \gamma_{cr} \right)$$  \hspace{1cm} (3.4)

Therefore, to find $\kappa$ it is necessary to discuss the critical value of the anomalous dimension.

### 3.1.2 Critical anomalous dimension and saturation momentum

Using Eq. (3.1) one can calculate the critical anomalous dimension. It is well known that for this calculation one only needs to know the kernel for $z < 0$ (see Refs. [1,14,35,36]). The equation for the critical anomalous dimension ($\gamma_{cr}$) has the following form

$$- \frac{\partial \omega(\gamma_{cr})}{\partial \gamma_{cr}} = \frac{\omega(\gamma_{cr})}{1 - \gamma_{cr}}$$  \hspace{1cm} (3.5)

Inserting Eq. (3.1) in Eq. (3.5) one obtains

$$\gamma_{cr} = \frac{1}{2} (1 - \bar{\alpha}_S)$$  \hspace{1cm} (3.6)

he equation for the saturation momentum has the form [1,14,35,36]:

$$\ln \left( Q_s^2 / Q_0^2 \right) = \frac{4\bar{\alpha}_S}{(1 + \bar{\alpha}_S)^2} Y$$  \hspace{1cm} (3.7)

where $Y = \ln(1/x)$, and $Q_0$ is the scale associated with the initial condition at low energies.

In the vicinity of the saturation scale $z \to 1$ the behavior of the dipole amplitude has the form [14,37]

$$N(Y; r) \sim (r^2 Q_s^2)^{1 - \gamma_{cr}} = e^{(1 - \gamma_{cr}) z}$$  \hspace{1cm} (3.8)
3.1.3 The NLO kernel for $\gamma \geq \gamma_{cr}$

Using Eq. (3.6) we can calculate

$$\omega_{NLO}(\gamma \rightarrow \gamma_{cr}) = \frac{2\bar{\alpha}_S}{1 + \alpha_S}$$

and Eq. (3.4) leads to the following value of $\kappa$:

$$\kappa = \frac{4\bar{\alpha}_S}{1 - \alpha_S}$$

Finally, the NLO kernel has the form:

$$\omega_{NLO}(\gamma) = \bar{\alpha}_S \left\{ \begin{array}{ll}
\frac{1}{\gamma + \alpha_S} & \text{for } z = \ln \left( r^2 Q_z^2 \right) \leq 0 ; \\
\frac{1 - \gamma + \bar{\alpha}_S \kappa}{(1 - \gamma)(1 + \alpha_S \kappa) + \alpha_S} & \text{for } z = \ln \left( r^2 Q_z^2 \right) > 0 ;
\end{array} \right. \quad (3.11)$$

It is worthwhile mentioning that the kernel of Eq. (3.11) describes the LO anomalous dimension of the DGLAP equations for $z \geq 0$, and it takes into account energy conservation. Therefore, we consider it as a model that can lead to a reliable predictions.

3.2 The simplified equation in the coordinate space

3.2.1 $z < 0$

In this kinematic region we can simplify $K_{LO}(r,r')$ in Eq. (2.12) in the following way [31], since $r' \gg r$ and $|\vec{r} - \vec{r}'| > r$

$$\int d^2r' K_{LO}(r,r') \rightarrow \pi r^2 \int_{r^2}^{\Lambda_{QCD}^2} \frac{dr'^2}{r'^4} \quad (3.12)$$

Introducing $n(r,Y;b) = N(r,Y;b)/r^2$ we obtain

$$\frac{\partial^2 n(r,Y;b)}{\partial Y \partial \ln \left( 1/(r^2 \Lambda_{QCD}^2) \right)} = \bar{\alpha}_S \left\{ 1 - \frac{\partial}{\partial Y} \right\} \left( 2n(r,Y;b) - r^2 \Lambda_{QCD}^2 n^2(r,Y;b) \right) \quad (3.13)$$

3.2.2 $z > 0$

The main contribution in this kinematic region originates from the decay of the large size dipole into one small size dipole and one large size dipole. However, the size of the small dipole is still larger than $1/Q_s$

This observation can be translated in the following form of the kernel

$$\int d^2r' K_{LO}(r,r') \rightarrow \pi \int_{1/Q_s^2(Y,b)}^{r^2} \frac{dr'^2}{r'^4} + \pi \int_{1/Q_s^2(Y,b)}^{r^2} \frac{d|\vec{r} - \vec{r}'|^2}{|\vec{r} - \vec{r}'|^2} \quad (3.14)$$
One can see that this kernel leads to the $\ln (r^2 Q_s^2)$-contribution. Introducing a new function $\tilde{N}(r, Y; b) = \int r^2 \, dr^2 \, N(r, Y; b) / r^2$ one obtain the following equation

$$\frac{\partial^2 \tilde{N}(r, Y; b)}{\partial Y \, \partial \ln r^2} = 2\tilde{\alpha}_S \left\{ 1 - \frac{\partial}{\partial Y} \right\} \left( \left( 1 + \tilde{\alpha}_S \kappa \frac{\partial}{\partial \ln r^2} - \frac{\partial \tilde{N}(r, Y; b)}{\partial \ln r^2} \right) \tilde{N}(r, Y; b) \right)$$

(3.15)

This equation does not depend on the value of the saturation momentum. Because of this we can assume that the solution of this equation depends on one variable $z = \ln (r^2 Q_s^2)$. The result that the solution in the saturation region depends on only one variable, has been proven (see Ref. [35]). The value of the saturation momentum depends on two variables: on rapidity $Y$ and impact parameter $b$. Since the equation does not depend explicitly on $Q_s$ and $b$, the only information that we have on these variables stems from the matching with the equation at $z < 0$ on the critical line. Transforming from variable $Y$ and $r$ in Eq. (3.15) to variable $z = \ln (r^2 Q_s^2)$ we have

$$\frac{d^2 \tilde{N}(z)}{dz^2} = \frac{(1 + \tilde{\alpha}_S)^2}{4} \left( 1 + \frac{4\tilde{\alpha}_S}{1 - \tilde{\alpha}_S^2} \frac{d}{dz} - \frac{d\tilde{N}(z)}{dz} \right) \tilde{N}(z) - \tilde{\alpha}_S \frac{d}{dz} \left\{ \left( 1 - \frac{d\tilde{N}(z)}{dz} \right) \tilde{N}(z) \right\}$$

(3.16)

It worthwhile mentioning that the BK equation in this kinematic region has the form

$$\frac{d^2 \tilde{N}(z)}{dz^2} = \frac{1}{4} \left( 1 - \frac{d\tilde{N}(z)}{dz} \right) \tilde{N}(z)$$

(3.17)

### 3.3 Solution to BK equation with the simplified kernel

For the sake of completeness we start by recalling the solution to BK equation in the form of Eq. (3.17), obtained in Ref. [31]. Introducing

$$\tilde{N}(z) = \int^z dz' \left( 1 - e^{-\phi(z')} \right)$$

(3.18)

Eq. (3.17) can be rewritten in the form

$$\frac{d\phi(z)}{dz} = \frac{1}{4} \int^z dz' \left( 1 - e^{-\phi(z')} \right)$$

(3.19)

Considering $d\phi(z') / dz = D(\phi)$ we can rewrite Eq. (3.19) as

$$D(\phi) = \frac{1}{4} \int_0^\phi \frac{d\phi'}{D(\phi')} \left( 1 - e^{-\phi'} \right)$$

(3.20)

Differentiating Eq. (3.20) with respect to $\phi$ we have

$$\frac{dD(\phi)}{d\phi} = \frac{1}{4} \frac{1}{D(\phi)} \left( 1 - e^{-\phi} \right)$$

(3.21)
which leads to

\[ D^2(\phi) = \frac{1}{2} \left( \phi + e^{-\phi} - 1 \right) \]  

(3.22)

and finally,

\[ z = \sqrt{2} \int_{\phi_0}^{\phi} \frac{d\phi'}{\sqrt{\phi + e^{-\phi} - 1}} \]  

(3.23)

The boundary conditions at \( z = 0 \) have the form \([1, 31, 37]\)

\[ \phi(z) \xrightarrow{z \to 0^-} \phi_0 \quad \text{and} \quad \phi(z) \xrightarrow{z \to 0^+} \phi(z) ; \quad \frac{d\ln(\phi(z))}{dz} \xrightarrow{z \to 0^-} \frac{1}{2} \xrightarrow{z \to 0^+} \frac{d\ln(\phi(z))}{dz} ; \]  

(3.24)

(3.25)

Eq. (3.25) follows from Eq. (3.8) for the kernel given by Eq. (3.1)

3.4 Solution to the modified BK equation

Using Eq. (3.18) we can reduce Eq. (3.16) to the form

\[ \phi'(z) e^{-\phi(z)} = \frac{(1 + \bar{\alpha}S)^2}{4} e^{-\phi(z)} \tilde{N}(z) + \bar{\alpha}S \frac{(1 + \bar{\alpha}S)}{1 - \bar{\alpha}S} \tilde{N}_z - \bar{\alpha}S \left( e^{-\phi(z)} \tilde{N}(z) \right)' \]  

(3.26)

or

\[ \phi'_z \left( 1 - \bar{\alpha}S \tilde{N} \right) = \frac{(1 + \bar{\alpha}S)^2}{4} \tilde{N}(z) + \bar{\alpha}S \left( 1 - e^{-\phi(z)} \right)^2 \]  

(3.27)

Considering \( \phi \) being function on \( \tilde{N} \) and neglecting the terms that are proportional to \( \bar{\alpha}_S^2 \), we can rewrite Eq. (3.28) as

\[ \phi'_\tilde{N} = \frac{1}{1 - \bar{\alpha}S \tilde{N}} \left( \frac{(1 + \bar{\alpha}S)^2}{4} \frac{\tilde{N}}{1 - \exp\left\{ -\phi\left( \tilde{N} \right) \right\}} \right) + \bar{\alpha}S \left( 1 - \exp\left\{ -\phi\left( \tilde{N} \right) \right\} \right) \]  

(3.28)

The BK equation for \( \phi(\tilde{N}) \) has the form

\[ \phi'_\tilde{N} = \frac{1}{4} \frac{\tilde{N}}{1 - \exp\left\{ -\phi\left( \tilde{N} \right) \right\}} \]  

(3.29)

In comparison with the BK equation Eq. (3.28) (see Eq. (3.29)) this contains a new feature: \( \phi'_\tilde{N} \to \infty \) at \( \tilde{N} \to 1/\bar{\alpha}S \). In the vicinity of \( \tilde{N} = 1/\bar{\alpha}S \) we can simplify Eq. (3.28), namely,

\[ \phi'_\tilde{N} = \frac{1}{1 - \bar{\alpha}S \tilde{N}} \left( \frac{(1 + \bar{\alpha}S)^2}{4} \frac{\tilde{N}}{1 - \exp\left\{ -\phi\left( \tilde{N} \right) \right\}} \right) \]  

(3.30)

since the first term in the bracket in Eq. (3.28) is of the order of \( \geq 1/\bar{\alpha}S \).
The solution to Eq. (3.30) has the form
\[ \phi + e^{-\phi} - 1 = \frac{(1 + \bar{\alpha}s)^2}{4} \frac{1}{\bar{\alpha}s} \left( \ln \left( 1 - \bar{\alpha}s\tilde{N} \right) + \bar{\alpha}s \right) \] (3.31)

From Eq. (3.31) we can see that \( \phi \) is large in the region where \( \bar{\alpha}s\tilde{N} \ll 1 \). Having this in mind we can replace Eq. (3.28) by the following equation
\[ \phi'_{\tilde{N}} = \frac{1}{1 - \bar{\alpha}s\tilde{N}} \left( \frac{(1 + \bar{\alpha}s)^2}{4} \tilde{N} + \bar{\alpha}s \right) \] (3.32)
which gives
\[ \phi \left( \tilde{N} \right) = \frac{1}{\bar{\alpha}s^2} \left( 1 - \frac{(1 + \bar{\alpha}s)^2}{4} \tilde{N} - \frac{(1 + \bar{\alpha}s)^2}{4} \right) \ln \left( 1 - \bar{\alpha}s\tilde{N} \right) \] (3.33)

In Fig. 1 we illustrate the behavior of \( \phi \left( \tilde{N} \right) \) for the solution to Eq. (3.28) and Eq. (3.29). The difference between modified equation and BK one is clearly seen, and it is well reproduced by the simplified solutions of Eq. (3.31) and Eq. (3.33). However, for small values of \( \tilde{N} \) we have to solve equation numerically to determine the accuracy of the estimates. Having \( \phi \left( \tilde{N} \right) \) we can find \( \tilde{N} \) as function of \( z \). Indeed,
\[ \frac{d\tilde{N}(z)}{dz} = 1 - \exp \left( -\phi \left( \tilde{N} \right) \right) \] (3.34)
and the following equation gives \( \tilde{N} \) as function of \( z \)
\[ \int_{\tilde{N}_0}^{\tilde{N}} \frac{d\tilde{N}'}{1 - \exp \left( -\phi \left( \tilde{N}' \right) \right)} = z \] (3.35)

The value of \( \tilde{N} \left( z = 0 \right) = \tilde{N}_0 \) can estimated from Eq. (3.38). Indeed, in Ref. [31,37] was shown that for \( z < 0 \) but in the vicinity of \( z = 0 \) \( \tilde{N} = \int_{-\infty}^{z} dz' \left( 1 - \exp \left( -\phi_0 \exp \left( (1 - \gamma_{cr})z \right) \right) \right) \).

Taking this integral we obtain the value for \( \tilde{N}_0 \).

In Fig. 2 the solutions for modified equation and for the BK equation are plotted. The comparison between them is illustrated in Fig. 3 in which the ratio \( N^{(1)}/N^{(0)} \) is shown, where \( N^{(1)} \) and \( N^{(0)} \) are solutions to Eq. (3.28) and Eq. (3.29), respectively. One can see that the effect could be as large as 20% (see Fig. 3).
Figure 2: Dependence of function $\phi(z)$ (Fig. 2-a) and the scattering amplitude $N(z)$ (Fig. 2-b) versus $z$ for solution to Eq. (3.28) ($\tilde{N}^{(1)}$) and to Eq. (3.29) ($\tilde{N}^{(0)}$). $\bar{\alpha}_S = 0.2, \phi_0 = 0.3, \tilde{N}_0 = 0.6$.

3.5 Inclusive production

An interesting case is to compare the inclusive production for both equations. The inclusive production can be calculated using $k_t$ factorization, this was proven for the case of hadron-nucleus interaction in Ref. [29].

\begin{equation}
\frac{d\sigma}{dy d^2p_T} = \frac{2C_F}{\alpha_s(2\pi)^4} \frac{1}{p_T^2} \int d^2\vec{b} d^2\vec{B} d^2\vec{r}_T e^{j\vec{r}_T \cdot \vec{r}_T} \nabla_T^2 N_G \left(y_1 = \ln(1/x_1); r_T; \vec{b}\right) \nabla_T^2 N_G \left(y_2 = \ln(1/x_2); r_T; \vec{b} - \vec{B}\right). \tag{3.36}
\end{equation}

In the saturation region $\nabla^2 N_G = \nabla^2 (2N - N^2) \propto \exp(-2\phi(z))$ and, therefore,

\begin{equation}
\frac{d\sigma}{dy d^2p_T} \propto e^{-4\phi(z)} \tag{3.37}
\end{equation}

Using Fig. 2-a we can see that the effect is large, and practically for $z > 3$ the ratio of $\exp(-4\phi)$ is negligibly small. In other words, the production of the gluons with transverse momenta smaller than $Q_s \exp(-1.5)$ is small.

To estimate the inclusive production we need to know $\nabla^2 N_G$. It is easy to see that $\nabla^2 N_G = (1/r^2)d^2N_G/dz^2$. Eq. (3.27), Eq. (3.27), Eq. (3.34) and Eq. (3.35) allow us to calculate $\phi'_z$ and $\phi''_zz$. 

\[\phi''_zz\]
Indeed,

\[ \frac{d\phi(0)}{dz} = \frac{1}{\sqrt{2}} \sqrt{\phi(0)(z) + e^{-\phi(0)(z)}} - 1; \]  \hspace{1cm} (3.38) 

\[ \frac{d^2\phi(0)}{dz^2} = \frac{1}{4} \left( 1 - e^{-\phi(0)(z)} \right); \]  \hspace{1cm} (3.39) 

\[ \frac{d\phi(1)}{dz} = \frac{(1+\bar{\alpha}_S)^2 N(z) - \bar{\alpha}_S \left( 1 - e^{-\phi(1)(z)} \right)^2}{1 - \bar{\alpha}_S N(z)}; \]  \hspace{1cm} (3.40) 

\[ \frac{d^2\phi(1)}{dz^2} = \bar{\alpha}_S \left( 1 - e^{-\phi(1)(z)} \right) \left\{ \frac{(1+\bar{\alpha}_S)^2 N(z) - \bar{\alpha}_S \left( 1 - e^{-\phi(1)(z)} \right)^2}{1 - \bar{\alpha}_S N(z)} \right\} 
+ \frac{(1+\bar{\alpha}_S)^2}{4} \left( 1 - e^{-\phi(1)(z)} \right) - 2 \bar{\alpha}_S \frac{d\phi(1)}{dz} e^{-\phi(1)(z)} \left( 1 - e^{-\phi(1)(z)} \right) \right) 
\frac{1}{1 - \bar{\alpha}_S N(z)}; \]  \hspace{1cm} (3.41) 

![Figure 3: Dependence of the ratio $N^{(1)}/N^{(0)}$ versus $z$ ($\bar{\alpha}_S = 0.2, \phi_0 = 0.15, \tilde{N}_0 = 0.6$)](image)

In Fig. 3, the solutions for modified equation and for the BK one are plotted. The comparison between them is more illustrative.
For $y_1 = y_2$ Eq. (3.36) can be re-written in the form
\[ \frac{d\sigma}{dy d^2p_T} = \frac{2C_F}{\alpha_s(2\pi)^3} \frac{1}{x^2} \int d^2b \, d^2B \int_{-\infty}^{+\infty} dz \, e^{-z} \, J_0 \left( e^{\frac{1}{2}z} x_{\perp} \right) \frac{d^2N_G(z;b)}{dz^2} \frac{d^2N_G(z;\bar{b} - \bar{B})}{dz^2}. \] (3.42)

The integral over $z$ in Eq. (3.42) includes the region for $z < 0$, where the function $\phi(z)$ behaves as is shown in Eq. (3.8), however the kinematic region where we can trust Eq. (3.8) is rather narrow [37]. Using Eq. (3.1) and Eq. (3.11) we can find the full solution for $\phi(z)$ at $z < 0$:
\[ \phi^{(0)}(z; z < 0) = \phi_0 \exp \left( \sqrt{L(L+z)} - L - z \right); \] (3.43)
\[ \phi^{(1)}(z; z < 0) = \phi_0 \exp \left( (1 + \bar{\alpha}_S) \sqrt{L(L+z)} - (1 + \bar{\alpha}_S)(L+z) \right); \] (3.44)

where $x_{\perp} \equiv p_T/Q_s(Y)$ and $L = 4\bar{\alpha}_S y$ for BK equation and $L = 4\bar{\alpha}_S/(1 + \bar{\alpha}_S)^2 y$ for the modified BK equation.

Using these expressions we can calculate $d^2N/dz^2$ for both cases at $z \to 0$ for $z < 0$:

\[ \frac{d^2N^{(0)}}{dz^2} \xrightarrow{z \to 0^-} \phi_0 \frac{1}{4} \left( 1 + \frac{1}{L} \right) \neq \phi_0 \frac{1}{4} \xrightarrow{z \to 0^+} \frac{d^2N^{(0)}}{dz^2} \] (3.45)
\[ \frac{d^2N^{(1)}}{dz^2} \xrightarrow{z \to 0^-} \phi_0 \frac{1 + \bar{\alpha}_S}{4} \left( 1 + \frac{1}{L} \right) \neq \phi_0 \frac{1 + \bar{\alpha}_S}{4} \xrightarrow{z \to 0^+} \frac{d^2N^{(1)}}{dz^2} \] (3.46)

From Eq. (3.45) and Eq. (3.46) we have that the double derivatives at $z = 0$ only match at large values of $L$. The calculations for the ratio
\[ R = \frac{\frac{d\sigma^{(1)}}{dy d^2p_T}}{\frac{d\sigma^{(0)}}{dy d^2p_T}} \] (3.47)
are shown in Fig. 3. One can see that the energy conservation could lead to the suppression of the inclusive production by a factor of two.

### 3.6 New scale for freeze-out

The most striking result of the modified BK equation is the appearance of a new scale. Indeed, Eq. (3.32) and Eq. (3.33) show that for $z$ when $\bar{\alpha}_S \tilde{N} \to 1$, the value of $\phi$ increases. Formally speaking for $\bar{\alpha}_S \tilde{N} > 1$ $\phi$ becomes an oscillating function which gives negligible contribution to the inclusive cross section (see Fig. 3). Since in vicinity $\bar{\alpha}_S \tilde{N} \approx 1$ the value of $\phi \propto 1/\bar{\alpha}_S^2 \geq 1$, we can conclude from Eq. (3.18), that $\tilde{N} = z$. Therefore, the equation $\bar{\alpha}_S \tilde{N} = 1$ leads to $z = 1/\bar{\alpha}_S$, which gives a new scale
\[ Q_{fr} = Q_s \exp \left( -\frac{1}{2\bar{\alpha}_S} \right) \] (3.48)

The physical meaning is clear: the dipoles with the size larger than $2/Q_{fr}$ will not produced. In other words, in the collisions we produce dipoles with the size $2/Q_{fr}$ with the typical temperature $T = Q_{fr}$. 

\[ -14- \]
Figure 4: Dependence of $d^2N/dz^2$ versus $z$ for the solution to BK equation (dashed line) and to modified BK equation at different values of $\alpha_s$.

It should be stressed that we reconstruct the form of the non-linear term in the modified BK equation assuming the probability interpretation of the parton cascade, which was proven for the LO BK, however we believe on physical grounds that it will also be correct in the NLO. However, the NLO corrections also lead to the transition one dipole decays into three dipoles and induced by this transition term in the linear part of the equation. We neglected these terms since they induce the $N^3$ term in the equation. This term is small in the vicinity of the critical line ($N \propto \bar{\alpha}_S$ on it [31]) and only this kinematic region contributes to inclusive production.

The failure to obtain large corrections from energy conservation forces us to believe that non-perturbative contribution will be the reason for the small value of the NMF (nuclear modification factor). It should be stressed that the dipole rescatterings has been taken into account in Eq. (3.36). A separate question is, could we trust the $k_t$ factorization formula of Eq. (3.36) in the case of nucleus-nucleus collisions? The rigorous answer is no since there is no proof of the $k_t$ factorization for this case. However, it is difficult to believe that the violation of the $k_t$ factorization will be responsible for the value of NMF. Indeed, in Eq. (1.36) and Eq. (3.42) the main contribution stems from the vicinity of the saturation scale (the largest one in nucleus-nucleus scattering) and in the case of the two scale problem: high $p_T$ and the saturation scale, $k_t$ factorization has been proven in Ref. [29].

We think that the most interesting outcome of this solution is the appearance of the new scale
$Q_{fr} = Q_s \exp(-1/(2\bar{\alpha}_s))$ which characterize the temperature of the produced hadron ($T \approx Q_{fr}$). The production of dipoles with the size larger than $2/Q_{fr}$ are suppressed. Much more work is needed to answer the question whether this scale is an artifact of our simplification or a general feature of the energy conservation.

4. Impact parameter dependence

In this section we address the problem of the impact parameter ($b$) dependence for both BK equation and modified BK equation. It is well known that $b$ dependence is one of the most challenging questions in perturbative QCD [32]. Indeed, in QCD we have the massless gluon, and because of this at large values of $b$ we are doomed to have power-like decrease of the scattering amplitude in both linear and non-linear equation. Such a decrease results in power-like increase of the radius of interaction $R \propto s^\lambda$, which indicates that the large values of $b$ are dominant in the equation, which has been derived in perturbative QCD [32]. Therefore, at first sight the approach based on non-linear equation appears to be inconsistent. Unfortunately, as far as we know there exist only two papers (see Ref. [33]) where the numerical solutions were found to BK equation including $b$ dependence. These solutions confirm the result that large values of $b$ become essential and some modifications of the approach based on the BK equation, are required.

In such a situation the analytical solution that includes $b$ dependence even for the BK equation with simplified kernel, can be instructive.

4.1 Solution with the impact parameter dependence

For simplified kernels of Eq. (3.1) and Eq. (3.11) the $b$-dependence can be easily found following Ref. [31]. Indeed, in both equations the kernel for $r^2 Q_s^2 < 1$ coincides with the DGLAP kernel. In DGLAP evolution we sum the log of transverse momenta of the partons. In this integral the momentum transfer $q$ enters only as the low limit of integration over transverse momentum (say $k_\perp$). The integral has the following form

$$\ln \left( \frac{Q^2}{q^2} \right) = \int_q^Q \frac{d^2 k_\perp}{k_\perp^2}$$

(4.1)

However, the low limit in Eq. (4.1) is $q$ only in the region where $q > Q_0$ where $Q_0$ is the scale from with which we can apply perturbative QCD. Therefore, in the kinematic region where $q < Q_0$ no dependence of $q$ appears in the QCD evolution. However, the non-perturbative corrections generate the non-perturbative form factor, and the general form of the solution for say, gluon density, has the following form

$$N (z < 0; r, b; Y) = S(b) N(z < 0; r; Y) \xrightarrow{z \to 0} S(b) e^{(1-\gamma_{cr})z}$$

(4.2)

In the vicinity of the saturation scale Eq. (4.2) together with Eq. (3.8) leads to

$$N \left( z \xrightarrow{z < 0} 0; r, b; Y \right) = \left( r^2 Q_s^2(Y) \right)^{1-\gamma_{cr}} S(b) = \left( r^2 Q_s^2(Y; b) \right)^{1-\gamma_{cr}}$$

(4.3)
Figure 5: Dependence of the ratio \( R = \frac{d\sigma^{(i)}}{dy d^2p_T} / \frac{d\sigma^{(0)}}{dy d^2p_T} \) for \( y_1 = y_2 \) versus \( x_\perp = p_T/Q_s \) and different values of \( L \) where \( L = 4 \bar{\alpha}_S Y \) for the solution of BK equation and \( L = 4 \bar{\alpha}_S / (1 + \bar{\alpha}_S)^2 \) \( Y \) for the solution of the modified BK equation. In the picture the value of \( L \) for the BK equation is shown. \((\bar{\alpha}_S = 0.2, \phi_0 = 0.15, \tilde{N}_0 = 0.6)\)

where

\[
Q_s^2(Y; b) = Q_s^2(Y) S^{\frac{1}{\gamma_{cr}}} (b) \tag{4.4}
\]

The kernel of the BK equation as well as the modified BK equation does not depend on \( b \). Therefore, the only information about \( b \)-dependence stems from the \( b \)-dependence on the critical line which can be absorbed in the dependence of the saturation scale on \( b \) as is given by Eq. (4.4). Finally, the solution of the equation will be the same as without \( b \)-dependence but with new variable

\[
\tilde{z}(z; b) = \ln \left( r^2 Q_s^2(Y; b) \right) = z + \frac{1}{1 - \gamma_{cr}} \ln (S(b)) \tag{4.5}
\]

This new variable alters the estimates for the new scale for freeze-out as one can see from Fig. 5 that shows \( \tilde{N} \) as function of \( z \) at different values of \( b \). One can see that the value of \( z \) at which \( \tilde{N} \rightarrow 1/\bar{\alpha}_S \) is different for different \( b \). In terms of ion-ion collisions this dependence can be translated to dependence on the centrality cuts leading to temperature dependence for different centrality.

### 4.2 Inclusive production

Eq. (4.4) can be re-written in the form (for \( y_1 = y_2 \))

\[
\frac{d\sigma}{dy d^2p_T} = \frac{2C_F}{\alpha S(2\pi)^3} \frac{1}{x_\perp^2} \int_{-\infty}^{+\infty} dz e^{-z} J_0 \left( e^{\frac{1}{2}z} x_\perp \right) \left\{ \int d^2b \frac{d^2N_G(z; b)}{dz^2} \right\}^2 \tag{4.6}
\]
Figure 6: $\tilde{N}$ versus $z$ and for solution to modified BK equation at different values of $b$ (see Eq. (4.7)). The dotted line shows $\tilde{N} = 1/\tilde{\alpha}_S$ for $\tilde{\alpha}_S = 0.2$.

where $x_\perp$ is defined in Eq. (4.42).

As we have mentioned in section 3 we calculated the inclusive cross section assuming that $N \propto \Theta (R - b)$ Here, we choose the exponential dependence for $S (b)$

$$S (b) = \exp \left( -\frac{b^2}{R^2} \right) \quad (4.7)$$

We take $R^2 = 11 GeV^{-2}$ for the nucleon and for nuclei we choose $R^2 = (2/5) R_{WS}^2$ where $R_{WS}$ is the radius of the nucleus in Wood-Saxon parameterizations ($R_{WS} = 6 A^{1/3} GeV^{-1}$).

Changing variable $b \to \rho = b/R$ one obtains

$$\frac{1}{(\pi R^2)^2} \frac{d\sigma}{dy d^2p_T} = \frac{2C_F}{\alpha_S^2 (2\pi)^3} \frac{1}{x_\perp} \int_{-\infty}^{+\infty} dz \ e^{-z} \ J_0 \left( e^{\frac{1}{2}z} x_\perp \right) \left\{ \int d\rho^2 \frac{d^2N_G \left( \tilde{z} (z; \rho) \right)}{dz^2} \right\}^2 = T (x_\perp) \quad (4.8)$$

Function $T (x_\perp)$ does not depend on the target. Therefore, Eq. (4.8) shows that for the scattering of two identical nuclei at $y_1 = y_2$ we expect the scaling behavior: the inclusive production depends only on variable $x_\perp$, instead of dependence on the number of nucleons ($A$), rapidity $y = y_1 = y_2$ and $p_T$.

Fig. 7 shows the influence of $b$ dependence of the inclusive production.

In Fig. 7 the ratio

$$R (b) = T \left( \text{with } b \text{ dependence} \right) / T \left( \text{without } b \text{ dependence} \right) \quad (4.9)$$

is plotted.
Figure 7: The ratio $R(b)$ (see Eq. 4.9) versus $x_\perp$ for the solution of the modified BK equation (curve 1) and for the BK equation (curve 2). Curve 3 gives the ratio of $T(x_\perp)$ for BK equation to the modified BK equation without $b$-dependence while curve 4 gives the same ratio taking into account $b$-dependence.

One can see that the solution of the BK equation with $b$ dependence, differs from the solution of the same equation without taking into account the $b$-dependence, by $+25\%$ at small values of $x_\perp$ and by $-25\%$ at large values of $x_\perp$. It is surprising that the solution of the modified BK equation leads to the same inclusive production at large $x_\perp$ both for the cases with $b$ and without $b$ dependence. However, at small $x_\perp$ the solution with $b$ dependence exceed the one without $b$-dependence by $35\%$. The differences between solutions to modified BK and BK equations are shown in curves 3 and 4 in Fig. 7. The solutions with $b$-dependence could differ by a factor of three at large values of $x_\perp$.

$T(x_\perp)$ as a function of $x_\perp$ is plotted in Fig. 8. Note that the nuclear modification factor (NMF) can be easily re-written through $T(x_\perp)$, namely,

$$\text{NMF} \equiv \frac{1}{N_{\text{coll}}} \frac{d^2N_{AA}}{dxd^2p_\perp} = \frac{1}{A^{2/3}} \frac{T(x_\perp)}{T\left(x_\perp \frac{Q_{s,A}}{Q_{s,N}}\right)}$$

(4.10)

where $N_{\text{coll}}$ is the number of collisions. Using the calculated $T(x_\perp)$ (see Fig. 8) and the fact that $Q_{s,A}/Q_{s,N} = A^{1/6}$ we see that for the gold NMF = $(0.2 \div 0.3) \times 1.7 = 0.3 \div 0.5$ for $x_\perp = 0.1 \div 10$ ( $p_\perp = 0.14 \div 14 \text{GeV}$ ) which is in about 2 times larger than the experimental NMF (see RHIC data [38–41]). However, the ratio of the radii ($R_A^4/R_N^4 = 1.7$) as well as Gaussian parametrization cannot be considered as
reliable. Therefore, we can conclude that both BK equation and the modified BK equation lead to rather small NMF but perhaps, this suppression is not sufficient to describe the experimental data.

![Graph showing T(x⊥) versus x⊥ = p⊥/Qs for the solution of BK equation that takes into account b-dependence.]

**Figure 8:** $T(x⊥)$ (see Eq. (4.8) versus $x⊥ = p⊥/Qs$ for the solution of BK equation that takes into account $b$-dependence.

5. Conclusions

In this paper we study two questions: the influence of energy conservation on the solution to the non-linear equation; and the $b$-dependence of the solution. The influence of energy conservation on the solution of the BK equation was investigated taking the impact parameter dependence in the form of $\Theta (b - R)$. The answer to this question is that the energy conservation reduces the value of the inclusive cross section by 20-40% at reasonable values of the QCD coupling ($\bar{\alpha}_S \approx 0.2$). These estimates are based on the modified BK equation (see Eq. (2.11)) and on the form of the kernel given in Eq. (3.11), which were derived from the form of the NLO corrections to the BFKL kernel, given in Eq. (2.8). The derivation of Eq. (2.8) stems from two observations: (i) the fact that Eq. (2.10) describes the exact value of the anomalous dimension of the DGLAP evolution with a very good accuracy; and (ii) the suppression of anti-DGLAP evolution of the transverse momenta of partons in the non-linear equation (see section 2.1 of this paper).

It should be stressed that we reconstruct the form of the non-linear term in the modified BK equation assuming the probability interpretation of the parton cascade which was proven for the LO BK but has so
clear physical meaning that we believe that it will be correct in the NLO. However, the NLO corrections also lead to the transition of one dipole decaying into three dipoles, and induced by this transition term in the linear part of the equation. We neglected these terms since they induce the $N^3$ term in the equation. This term is small in the vicinity of the critical line ($N \propto \bar{\alpha}_S$ on it [31]), and only this kinematic region contribute to inclusive production.

The failure to obtain the large corrections from the energy conservation suggests that the non-perturbative contribution is the reason for the small value of the NMF (nuclear modification factor). We wish to stress that the dipole rescatterings has been taken into account in Eq. (3.36). The separate question is can we trust the $k_t$ factorization formula of the case of nucleus-nucleus collisions? The rigorous answer is no, since there is no proof of the $k_t$ factorization for this case. However, it is difficult to believe that the violation of the $k_t$ factorization will be responsible for the value of NMF. Indeed, in Eq. (3.36) and Eq. (3.42) the main contribution stems from the vicinity of the saturation scale (the largest one in nucleus-nucleus scattering) and in the case of two scale problems: high $p_T$ and the saturation scale the $k_t$ factorization has been proven in Ref. [29].

In our opinion the most interesting result of this solution is the appearance of the new scale $Q_{fr} = Q_s \exp(-1/(2\bar{\alpha}_S))$, which characterize the temperature of the produced hadron ($T \approx Q_{fr}$). The production of dipoles with the sizes larger than $2/Q_{fr}$ are suppressed. Much more work is needed to answer the question whether this scale is the artifact of our simplification or a general feature of energy conservation.

As far as $b$ dependence is concerned, we found that the BK equation without $b$-dependence cannot guarantee an accuracy of calculation better than ±25%. We consider an important observation that in our simplified equation the entire $b$-dependence can be absorbed in the non-perturbative behavior of the saturation scale. We hope that this fact can lead to a general scheme for taking into account the impact parameter dependence in the framework of high density QCD, however to rewrite the non-linear equation that absorbs the new non-perturbative scale, still needs more work.

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References

[1] L. V. Gribov, E. M. Levin and M. G. Ryskin, Phys. Rep. 100 (1983) 1.
[2] A. H. Mueller and J. Qiu, Nucl. Phys. B268 (1986) 427.
[3] A. H. Mueller, Nucl. Phys. B 415, 373 (1994); Nucl. Phys. B 437 (1995) 107 [arXiv:hep-ph/9408245].
[4] L. McLerran and R. Venugopalan, Phys. Rev. D 49 (1994) 2233, 3352; D 50 (1994) 2225, D 53 (1996) 458, D 59 (1999) 09400.
[5] Ia. Balitsky, Nucl.Phys. B463 (1996) 99; Yu. Kovchegov, Phys. Rev. D60 (2000) 034008.
[6] J. Jalilian-Marian, A. Kovner, A. Leonidov and H. Weigert, Phys. Rev. D59, 014014 (1999), [arXiv:hep-ph/9706377]; Nucl. Phys. B504, 415 (1997), [arXiv:hep-ph/9701284]; J. Jalilian-Marian, A. Kovner and H. Weigert, Phys. Rev. D59, 014015 (1999), [arXiv:hep-ph/9709432]; A. Kovner, J. G. Milhano and H. Weigert, Phys. Rev. D62, 114005 (2000), [arXiv:hep-ph/0004014]; E. Iancu, A. Leonidov and L. D. McLerran, Phys. Lett. B510, 133 (2001); [arXiv:hep-ph/0102009]; Nucl. Phys. A692, 583 (2001), [arXiv:hep-ph/0011241]; E. Ferreiro, E. Iancu, A. Leonidov and L. McLerran, Nucl. Phys. A703, 489 (2002), [arXiv:hep-ph/0109115]; H. Weigert, Nucl. Phys. A703, 823 (2002), [arXiv:hep-ph/0004044].

[7] E. A. Kuraev, L. N. Lipatov, and F. S. Fadin, Sov. Phys. JETP 45 (1977) 199 ; Ya. Ya. Balitsky and L. N. Lipatov, Sov. J. Nucl. Phys. 28 (1978) 22 .

[8] V.S. Fadin and L.N. Lipatov, Phys. Lett. B429 (1998) 127
G. Camici and M. Ciafaloni, Phys. Lett. B430 (1998) 349.

[9] E. Levin and M. Lublinsky, Nucl. Phys. A730 (2004) 191 [arXiv:hep-ph/0308279].

[10] A. H. Mueller and A. I. Shoshi, Nucl. Phys. B692 (2004) 175 [arXiv:hep-ph/0402193].

[11] V. N. Gribov and L. N. Lipatov, Sov. J. Nucl. Phys 15 (1972) 438;
G. Altarelli and G. Parisi, Nucl. Phys. B 126 (1977) 298;
Yu. I. Dokshitzer, Sov. Phys. JETP 46 (1977) 641.

[12] M. Ciafaloni, Nucl. Phys. Proc. Suppl. 146 (2005) 129; In the Proceedings of QCD @ Work 2003: 2nd International Workshop on Quantum Chromodynamics: Theory and Experiment, Conversano, Italy, 14-18 Jun 2003, pp 013, [arXiv:hep-ph/0310110]; M. Ciafaloni, D. Colferai, G.P. Salam and A.M. Stasto, Phys. Rev. D68 (2003) 114003; M. Ciafaloni, D. Colferai, G. P. Salam and A. M. Stasto, Phys. Lett. B541, 314 (2002) [arXiv:hep-ph/0204287]; Phys. Rev. D66, 054014 (2002) [arXiv:hep-ph/0204282]; J. R. Forshaw, D. A. Ross and A. Sabio-Vera, Phys. Lett. B498, 149 (2001) [arXiv:hep-ph/0011047]; M. Ciafaloni, M. Taiuti and A. H. Mueller, Nucl. Phys. B616, 349 (2001) [arXiv:hep-ph/0107009]; S.J. Brodsky, V.S. Fadin, V.T. Kim, L.N. Lipatov and G.B. Pivovarov, JETP Lett. 70 (1999) 155; D. A. Ross, Phys. Lett. 431, 161 (1998) [arXiv:hep-ph/9804332]; E. Levin, Nucl. Phys. B545 (1999) 481. [arXiv:hep-ph/9806228]; N. Armesto, J. Bartels and M. A. Braun, Phys. Lett. 442, 459 (1998) [arXiv:hep-ph/9808340]; Y. V. Kovchegov and A. H. Mueller, Phys. Lett. B439 (1998) 428 [arXiv:hep-ph/9805208].

[13] I. Balitsky, arXiv:1004.0057 [hep-ph].

[14] A. H. Mueller and D. N. Triantafyllopoulos, Nucl. Phys. B640 (2002) 331 [arXiv:hep-ph/0205167]; D. N. Triantafyllopoulos, Nucl. Phys. B648 (2003) 293 [arXiv:hep-ph/0209121].

[15] V. A. Khoze, A. D. Martin, M. G. Ryskin and W. J. Stirling, Phys. Rev. D 70 (2004) 074013 [arXiv:hep-ph/0406135].

[16] A. H. Mueller, A. I. Shoshi and S. M. H. Wong, Nucl. Phys. B715, 440 (2005) [arXiv:hep-ph/0501088].

[17] E. Levin and M. Lublinsky, Nucl. Phys. A 763, 172 (2005) [arXiv:hep-ph/0501173].

[18] A. Kovner and M. Lublinsky, Phys. Rev. D 71, 085004 (2005) [arXiv:hep-ph/0501198].

[19] Y. Hatta, E. Iancu, L. McLerran, A. Stasto and D. N. Triantafyllopoulos, Nucl. Phys. A 764, 423 (2006) [arXiv:hep-ph/0504182].

[20] T. Altinoluk, A. Kovner, M. Lublinsky and J. Peressutti, JHEP 0903 (2009) 109 [arXiv:0901.2559 [hep-ph]].
[21] A. H. Mueller and B. Patel, *Nucl. Phys.* B425, 471 (1994); A. H. Mueller and G. P. Salam, *Nucl. Phys.* B475, 293 (1996), [arXiv:hep-ph/9605302]; G. P. Salam, *Nucl. Phys.* B461, 512 (1996); E. Iancu and A. H. Mueller, *Nucl. Phys.* A730 (2004) 460, 494, [arXiv:hep-ph/0308315],[arXiv:hep-ph/0309276].

[22] E. Levin, J. Miller and A. Prygarin, Nucl. Phys. A 806 (2008) 245 [arXiv:0706.2944 [hep-ph]].

[23] E. Levin, Nucl. Phys. B453 (1995) 303; [arXiv:hep-ph/9412345]; M. A. Braun, Phys. Lett. B 348 (1995) 190 [arXiv:hep-ph/9408261]; Y. V. Kovchegov and H. Weigert, Nucl. Phys. A 784 (2007) 188 [arXiv:hep-ph/0609090]; I. Balitsky, Phys. Rev. D 75 (2007) 014001 [arXiv:hep-ph/0609105].

[24] J. L. Albacete and Y. V. Kovchegov, Phys. Rev. D 75 (2007) 125021 [arXiv:0704.0612 [hep-ph]].

[25] Y. V. Kovchegov, J. Kuokkanen, K. Rummukainen and H. Weigert, Nucl. Phys. A 823 (2009) 47 [arXiv:0812.3238 [hep-ph]].

[26] G. P. Salam, *JHEP* 9807 (1998) 019; Acta Phys. Pol. B30 (1999) 3679; M. Ciafaloni, D. Colferai and G. P. Salam, *Phys. Lett.* B452 (1999) 372; *Phys. Rev.* D60 (1999) 114036.

[27] E. Levin and M. Lublinsky, Phys. Lett. B 607 (2005) 131 [arXiv:hep-ph/0411242].

[28] Y. L. Dokshitzer and D. E. Kharzeev, Phys. Lett. B 519 (2001) 199 [arXiv:hep-ph/0102202]; R. Baier, Y. L. Dokshitzer, A. H. Mueller and D. Schiff, Nucl. Phys. B 531 (1998) 403 [arXiv:hep-ph/9804212]; B. G. Zakharov, JETP Lett. 63 (1996) 952 [arXiv:hep-ph/9607440]; JETP Lett. 65 (1997) 615 [arXiv:hep-ph/9704255]; M. Gyulassy, P. Levai and I. Vitev, Phys. Rev. Lett. 85 (2000) 5535 [arXiv:nucl-th/0005032]; Nucl. Phys. B 594 (2001) 371 [arXiv:nucl-th/0006010]; C. A. Salgado and U. A. Wiedemann, Phys. Rev. D 68 (2003) 014008 [arXiv:hep-ph/0302184].

[29] Y. V. Kovchegov and K. Tuchin, Phys. Rev. D 65 (2002) 074026 [arXiv:hep-ph/0111362].

[30] E. Gotsman, E. Levin, U. Maor and E. Naftali, Nucl. Phys. A 750 (2005) 391 [arXiv:hep-ph/0411242].

[31] E. Levin and K. Tuchin, *Nucl. Phys.* A693 (2001) 787, [arXiv:hep-ph/0101275]; A691 (2001) 779,[arXiv:hep-ph/0012167]; B573 (2000) 833, [arXiv:hep-ph/9908317].

[32] A. Kovner and U. A. Wiedemann, Phys. Lett. B 551 (2003) 311 [arXiv:hep-ph/0207335]; Phys. Rev. D 66 (2002) 034031 [arXiv:hep-ph/0204277]; Phys. Rev. D 66 (2002) 051502 [arXiv:hep-ph/0112140].

[33] E. Gotsman, M. Kozlov, E. Levin, U. Maor and E. Naftali, Nucl. Phys. A 742 (2004) 55 [arXiv:hep-ph/0401021]; K. J. Golec-Biernat and A. M. Stasto, Nucl. Phys. B 668 (2003) 345 [arXiv:hep-ph/0306279].

[34] R. K. Ellis, Z. Kunszt and E. M. Levin, *Nucl. Phys.* B420 (1994) 517 [Erratum-ibid. B433 (1995) 498].

[35] J. Bartels and E. Levin, Nucl. Phys. B 387 (1992) 617.

[36] S. Munier and R. B. Peschanski, Phys. Rev. D 70 (2004) 077503 [arXiv:hep-ph/0401215]; Phys. Rev. D 69 (2004) 034008 [arXiv:hep-ph/0310357]; Phys. Rev. Lett. 91 (2003) 232001 [arXiv:hep-ph/0309177].

[37] E. Iancu, K. Itakura and L. McLerran, Nucl. Phys. A 708 (2002) 327 [arXiv:hep-ph/0203137].

[38] I. Arsene et al. [BRAHMS Collaboration], Phys. Rev. Lett. 91 (2003) 07305.

[39] I. Arsene et al. [BRAHMS Collaboration], Phys. Rev. Lett. 93 (2004) 242303 [arXiv:nucl-ex/0403005].

[40] I. Arsene et al. [BRAHMS Collaboration], Phys. Lett. B 650 (2007) 219 [arXiv:nucl-ex/0610021].

[41] S. S. Adler et al. [PHENIX Collaboration], Phys. Rev. Lett. 96 (2006) 032301 [arXiv:nucl-ex/05100