Distributed Reinforcement Learning for Decentralized Linear Quadratic Control: A Derivative-Free Policy Optimization Approach

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Abstract
This paper considers a distributed reinforcement learning problem for decentralized linear quadratic control with partial state observations and local costs. We propose the Zero-Order Distributed Policy Optimization algorithm (ZODPO) that learns linear local controllers in a distributed fashion, leveraging the ideas of policy gradient, zero-order optimization and consensus algorithms. In ZODPO, each agent estimates the global cost by consensus, and then conducts local policy gradient in parallel based on zero-order gradient estimation. ZODPO only requires limited communication and storage even in large-scale systems. Further, we investigate the nonasymptotic performance of ZODPO and show that the sample complexity to approach a stationary point is polynomial with the error tolerance’s inverse and the problem dimensions, demonstrating the scalability of ZODPO. We also show that the controllers generated by ZODPO are stabilizing with high probability. Lastly, we numerically test ZODPO on a multi-zone HVAC system.

Keywords: learning-based control, multi-agent reinforcement learning, zero-order optimization

1 Introduction

Reinforcement learning (RL) has emerged as a promising tool for controller design for dynamical systems, especially when the system model is unknown or complex, e.g., robotics (Riedmiller et al., 2009), games (Silver et al., 2017), healthcare (Esteva et al., 2019), smart manufacturing (Wang and Usher, 2005), autonomous driving (Shah et al., 2018), energy systems (O’Neill et al., 2010). However, theoretical performance guarantees of RL are still under-developed across a wide range of problems, limiting the application of RL to real-world systems. Recently, there have been exciting results on the theoretical performance guarantees for learning-based control for (centralized) linear quadratic (LQ) control problems (Dean et al., 2017; Fazel et al., 2018; Ouyang et al., 2017). LQ control is one of the most well-studied optimal control problems, which considers optimal state feedback control for a linear dynamical system such that a quadratic cost on the states and control inputs is minimized over a finite or infinite horizon (Lewis et al., 2012).

Encouraged by the recent success of learning-based centralized LQ control, this paper aims to extend the current results and develop scalable learning algorithms for decentralized LQ control. In

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decentralized control, the global system is controlled by a group of individual agents with limited communication, each of which observes only a partial state of the global system (Bakule, 2008). Decentralized LQ control has many applications, including transportation (Bazan, 2009), power grids (Pipattanasomporn et al., 2009), robotics (Cao et al., 1997), smart buildings (Moroşan et al., 2010), etc. It is worth mentioning that partial observations and limited communication place major challenges on finding optimal decentralized controllers, even when the global system model is known (Witsenhausen, 1968; Rotkowitz and Lall, 2005).

Specifically, we consider the setting where there is an underlying linear dynamical system with a global state $x(t) \in \mathbb{R}^n$ and a global control action $u(t)$ at each time $t$. The global control action is composed of local control inputs, i.e., $u(t) = [u_1(t)^\top, \ldots, u_N(t)^\top]^\top$, where $u_i(t)$ is the control input of agent $i$. At time $t$, each agent $i$ directly observes a partial state $x_{I_i}(t)$ and a quadratic local cost $c_i(t)$ that could depend on the global state and control action. The dynamical system model is assumed to be unknown. The goal is to design a cooperative distributed learning scheme to find local control policies for agents such that the globally averaged cost among all agents is minimized over an infinite horizon. The local control policies are limited to those that only use local observations.

Our contributions. Firstly, we propose the Zero-Order Distributed Policy Optimization algorithm (ZODPO). ZODPO learns the local controllers in a distributed fashion by leveraging consensus algorithms and zero-order policy optimization. In ZODPO, each agent only shares a small number of scalars with its neighbors for policy search coordination, which requires limited communication. In addition, each agent only needs to store and update its local policy. These features ensure that ZODPO is applicable for large-scale systems.

Secondly, we analyze the nonasymptotic performance of ZODPO. For theoretical purposes, we consider a simple linear static policy class $u_i(t) = K_i x_{I_i}(t)$ for some matrix $K_i$ for each agent $i$, though the algorithm can be extended to more general control classes. We show that, to approach some stationary point, the required number of samples has polynomial dependence on the inverse of the error tolerance, the number of policy parameters and the number of agents, demonstrating the scalability of ZODPO. Further, all policies generated and implemented by ZODPO are stabilizing controllers with high probability, guaranteeing the safety during the learning process.

Finally, we conduct numerical experiments on a multi-zone HVAC system to test ZODPO.

1.1 Related work

Learning-based LQ control. Controller design without (accurate) model information has been studied in the field of adaptive control for a long time (Åström and Wittenmark, 2008), but most papers focus on stability and asymptotic performance. Recently, much progress has been made on algorithm design and nonasymptotic analysis for learning-based centralized (single-agent) LQ control for both the full observability case, e.g., model-free schemes (Fazel et al., 2018; Malik et al., 2018; Yang et al., 2019), identification-based controller design (Dean et al., 2017; Mania et al., 2019), Thompson sampling (Ouyang et al., 2017), etc., and the partial observability case (Oymak and Ozay, 2019; Mania et al., 2019). As for learning-based decentralized (multi-agent) LQ control, most studies either adopt a centralized learning scheme (Bu et al., 2019) or still focus on asymptotic analysis (Abouheaf et al., 2014; Zhang et al., 2016a, 2019). Though a recent paper (Gagrani and Nayyar, 2018) proposes a distributed learning algorithm with a nonasymptotic guarantee, the algorithm requires each agent to store and update the model of the whole system, which may be prohibitive for large-scale systems.

Our algorithm design and analysis are related with the policy gradient approach for centralized
LQ control (Fazel et al., 2018; Bu et al., 2019). Though policy gradient can reach the global optimum in the centralized setting thanks to the gradient dominance property (Fazel et al., 2018), the objective function for decentralized LQ control may lack such nice properties (Feng and Lavaei, 2019), and most papers only aim for stationary points as in nonconvex optimization (Bu et al., 2019). More studies on the optimization landscape of decentralized LQ control are interesting future directions.

**Decentralized control.** Even with model information, decentralized control is very challenging. For example, the optimal controller for general decentralized LQ problems may be nonlinear (Witsenhausen, 1968), and the computation of such optimal controllers mostly remains unsolved. Moreover, even for the special cases with linear optimal controllers, e.g., the quadratic invariance cases, one usually needs to optimize over an infinite dimensional space (Rotkowitz and Lall, 2005). For tractability, many papers, including this one, consider finite dimensional linear policy spaces and study suboptimal controller design (Märtnsson and Rantzer, 2009; Al Alam et al., 2011).  

**Multi-agent reinforcement learning (MARL).** There are various settings for MARL, e.g., cooperative (Bono et al., 2019) v.s. noncooperative settings (Littman, 1994), full observability (Zhang et al., 2018) v.s. partial observability (Claus and Boutilier, 1998; Lowe et al., 2017), etc. Our problem is similar to the cooperative setting with partial observability, also known as Dec-POMDP (Bernstein et al., 2002). Many MARL algorithms have been developed for Dec-POMDP (Peshkin et al., 2000; Foerster et al., 2018; Omidshafiei et al., 2017), but nonasymptotic analysis is usually lacking.

**Policy gradient approaches.** Policy gradient and its variants are popular algorithms in both RL (Sutton et al., 2000; Sutton and Barto, 1998; Silver et al., 2014) and MARL (Peshkin et al., 2000; Lowe et al., 2017; Foerster et al., 2018; Omidshafiei et al., 2017). Various gradient estimation schemes have been proposed, e.g., REINFORCE (Williams, 1992), policy gradient theorem (Sutton et al., 2000), deterministic policy gradient theorem (Silver et al., 2014), zero-order gradient estimation (Fazel et al., 2018), etc. This paper adopts the zero-order gradient estimation, which has been employed for learning centralized LQ control (Fazel et al., 2018; Malik et al., 2018).

**Zero-order optimization.** This aims to solve optimization without gradients by, e.g., estimating gradients based on function values (Flaxman et al., 2005; Nesterov and Spokoiny, 2017; Duchi et al., 2015; Tang and Li, 2019). This paper adopts the gradient estimator in Flaxman et al. (2005).

**Notation** Let \( \| \cdot \| \) denote the \( \ell_2 \) norm for vectors and matrices. Let \( \| \cdot \|_F \) and vec(\( \cdot \)) denote the Frobenious norm and vectorization of a matrix. Let \( 1 \) denote the vector with all one entries. The unit sphere \( \{ x \in \mathbb{R}^p : \| x \| = 1 \} \) is denoted by \( S_p \), and \( \text{Uni}(S_p) \) denotes the uniform distribution on \( S_p \).

## 2 Problem Formulation

Suppose there are \( N \) agents jointly controlling a discrete-time linear system of the form

\[
x(t + 1) = Ax(t) + Bu(t) + w(t), \quad t = 0, 1, 2, \ldots,
\]  

(1)

where \( x(t) \in \mathbb{R}^n \) denotes the state vector, \( u(t) \in \mathbb{R}^m \) denotes the joint control input, and \( w(t) \in \mathbb{R}^n \) denotes the random disturbance at time \( t \). We assume \( w(0), w(1), w(2), \ldots \) are i.i.d. following the Gaussian distribution \( \mathcal{N}(0, \Sigma_w) \) for some positive definite matrix \( \Sigma_w \). Each agent \( i \) is associated with a local control input \( u_i(t) \in \mathbb{R}^{m_i} \), which constitutes the global control input of the system (1) by

\[
u(t) = [u_1(t) \top \cdots \cdots \ u_N(t) \top] \top.
\]
We consider the case where each agent $i$ only observes a partial state, denoted by $x_{I_i}(t) \in \mathbb{R}^{n_i}$ at each time $t$, where $I_i$ is a fixed subset of $\{1, \ldots, n\}$ and $x_{I_i}(t)$ denotes the subvector of $x(t)$ with indices in $I_i$. The admissible local control policies are limited to the ones that only use the historical local observations. As a starting point, this paper only considers static linear policies that use the current observation, i.e., $u_i(t) = K_i x_{I_i}(t)$. For notational simplicity, we denote
\begin{equation}
K := \left[ \text{vec}(K_1)^\top \cdots \text{vec}(K_N)^\top \right]^\top \in \mathbb{R}^{n_K}, \quad n_K := \sum_{i=1}^{N} n_i m_i.
\end{equation}

It is straightforward to see that the global control policy is also a static linear policy on the current state. We use $M(K)$ to denote the global control gain, i.e., $u(t) = M(K)x(t)$. Note that $M(K)$ is often sparse in network control applications.

At each time step $t$, agent $i$ receives a quadratic local stage cost $c_i(t)$ given by
\[ c_i(t) = x(t)^\top Q_i x(t) + u(t)^\top R_i u(t), \]
which is allowed to depend on the global state $x(t)$ and control $u(t)$. The goal is to find a control policy that minimizes the infinite-horizon average cost among all agents, that is,
\begin{equation}
\min_K \quad J(K) := \lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} \mathbb{E} \left[ \frac{1}{N} \sum_{i=1}^{N} c_i(t) \right]
\end{equation}
\[ \text{s.t.} \quad x(t + 1) = Ax(t) + Bu(t) + w(t), \quad u_i(t) = K_i x_{I_i}(t), \quad \forall i, \forall t. \]

When the model parameters are known, the problem (3) can be viewed as a decentralized LQ control problem, which is known to be a challenging problem in general. Various heuristic or approximate methods have been proposed (see Section 1.1), but most of them require accurate model information that may be hard to obtain in practice. Recently, much progress has been made on learning centralized LQ control policies without model information. This motivates our study on learning-based decentralized control for (3), where each agent $i$ learns the local controller $K_i$ by utilizing the partial states $x_{I_i}(t)$ and local costs $c_i(t)$ observed along the system’s trajectories.

In many real-world applications of decentralized control, limited communication among agents is available via a communication network (Cao et al., 1997). Here, we consider an undirected communication network $\mathcal{G} = (\{1, \ldots, N\}, \mathcal{E})$, where each node represents an agent and $\mathcal{E}$ denotes the set of edges. At each time $t$, agent $i$ and $j$ can directly communicate a small number of scalars to each other if and only if $(i, j) \in \mathcal{E}$. Further, we introduce a doubly-stochastic communication matrix $W = [W_{ij}] \in \mathbb{R}^{N \times N}$ associated with the communication network $\mathcal{G}$, with $W_{ij} = 0$ if $(i, j) \notin \mathcal{E}$ for $i \neq j$ and $W_{ii} > 0$ for all $i$. The construction of the matrix $W$ has been extensively discussed in existing literature [see, for example, Xiao and Boyd (2004)].

Finally, we introduce the technical assumptions that will be imposed throughout the paper.

**Assumption 1.** The dynamical system $(A, B)$ is controllable. The cost matrices $Q_i, R_i$ are positive semidefinite for each $i$, and the global cost matrices $\frac{1}{N} \sum_{i=1}^{N} Q_i$ and $\frac{1}{N} \sum_{i=1}^{N} R_i$ are positive definite.

**Assumption 2.** There exists a control policy $K \in \mathbb{R}^{n_K}$ such that the resulting global dynamics $x(t + 1) = (A + B M(K)) x(t)$ is asymptotically stable.

\footnote{\(I_i\) and \(I_j\) may overlap. The results in this paper can be extended to more general observations, e.g., \(y_i(t) = C_i x(t)\).}

\footnote{The framework and algorithm can be extended to more general policy classes, but analysis is left as future work.}
Both assumptions are common in LQ control literature. Without Assumption 2, the problem (3) does not admit a reasonable solution even if all system parameters are known, let alone learning-based control. For ease of exposition, we denote $\mathcal{K}_{\text{st}}$ as the set of stabilizing controller, i.e.,

$$
\mathcal{K}_{\text{st}} := \{ K \in \mathbb{R}^{n_K} : A + B M(K) \text{ is asymptotically stable} \}.
$$

3 Algorithm Design

Preliminaries: zero-order policy optimization for centralized LQ control. To find a policy $K$ that minimizes $J(K)$, one commonly used approach is the policy gradient method, that is,

$$
K(s + 1) = K(s) - \eta \hat{g}(s), \quad s = 1, 2, \ldots, \quad K(1) = K_0,
$$

where $\hat{g}(s)$ is an estimator of the gradient $\nabla J(K(s))$, $\eta > 0$ is a stepsize, and $K_0$ is some known stabilizing controller. In (Fazel et al., 2018; Malik et al., 2018), the authors have proposed to employ gradient estimators from zero-order optimization, one version of which is given by

$$
G_r(K, D) := \frac{n_K}{r} J(K + r D) D
$$

for $K \in \mathcal{K}_{\text{st}}$ and $r > 0$ such that $K + r S_{n_K} \subseteq \mathcal{K}_{\text{st}}$, where $D \in \mathbb{R}^{n_K}$ is randomly sampled from $\text{Uni}(S_{n_K})$. The parameter $r$ is sometimes called the smoothing radius, and it can be shown that the bias $\| E_D \left[ G_r(K, D) \right] - \nabla J(K) \|$ can be controlled by $r$ under certain smoothness conditions on $J(K)$ (Malik et al., 2018). The policy gradient based on the estimator (4) is given by

$$
K(s + 1) = K(s) - \eta G_r(K(s), D(s)) = K(s) - \eta \cdot \frac{n_K}{r} J(K(s) + r D(s)) D(s),
$$

where $D(s), s = 1, 2, \ldots$ is a sequence of i.i.d. random vectors following the distribution $\text{Uni}(S_{n_K})$.

Our algorithm: Zero-Order Distributed Policy Optimization (ZODPO). Now, let us consider the decentralized LQ control formulated in Section 2. Notice that the iterations (5) can be equivalently written in an almost decoupled way for each agent $i$:

$$
K_i(s + 1) = K_i(s) - \eta \cdot \frac{n_K}{r} J(K_i(s) + r D(s)) D_i(s), \quad \forall i = 1, \ldots, N
$$

where $D(s) \sim \text{Uni}(S_{n_K})$, and $K_i(s), D_i(s)$ are real $n_i \times m_i$ matrices such that

$$
K(s) = \begin{bmatrix} \text{vec}(K_1(s))^\top & \cdots & \text{vec}(K_N(s))^\top \end{bmatrix}^\top, \quad D(s) = \begin{bmatrix} \text{vec}(D_1(s))^\top & \cdots & \text{vec}(D_N(s))^\top \end{bmatrix}^\top.
$$

The formulation (7) suggests that, if each agent $i$ can sample $D_i(s)$ properly and obtain the value of the global objective $J(K(s) + r D(s))$, then the policy gradient (5) can be implemented in a decentralized fashion by letting each agent $i$ store and update its own policy $K_i$ in parallel according to (7). This key observation leads us to the ZODPO algorithm summarized in Algorithm 1.

Roughly speaking, ZODPO conducts distributed policy gradient iterations with two main steps:
Algorithm 1: Zero-Order Distributed Policy Optimization (ZODPO)

**Input:** smoothing radius $r$, step size $\eta$, $J > 0$, termination steps $T_G$ and $T_J$, $K_0 \in \mathcal{K}_{st}$.

1. Initialize $K(1) = K_0$.
2. for $s = 1, 2, \ldots, T_G$ do
   1. Local estimation of the global objective
      1. Sample $D_i(s) \in \mathbb{R}^{n_i \times m_i}$, $i = 1, \ldots, N$ such that $D(s)$ defined in (8) follows $\text{Uni}(\mathbb{S}_{m_K})$.
      2. Each agent $i$ implements the controller $K_i(s) + rD_i(s)$ and resets $\mu_i(0) \leftarrow 0$.
      3. Restart the dynamical system from $x(0) \leftarrow 0$.
      4. for $t = 1, 2, \ldots, T_J$ do
         1. Each agent $i$ sends $\mu_i(t-1)$ to its neighbors, receives $c_i(t)$, and updates $\mu_i(t)$ by
            \[
            \mu_i(t) = \frac{t-1}{t} \sum_{j=1}^{N} W_{ij} \mu_j(t-1) + \frac{1}{t} c_i(t).
            \] (6)
      5. end
      6. Each agent $i$ sets $\mathcal{J}_i(s) = \min\{\mu_i(T_J), \mathcal{J}\}$.
      7. Each agent $i$ updates $K_i(s+1)$ by
         \[
         K_i(s+1) = K_i(s) - \eta \cdot \frac{n_K}{r} \mathcal{J}_i(s) D_i(s).
         \]
   8. end
3. for $t = 1, 2, \ldots, T_S$ do
   1. Agent $i$ sends $q_i(t-1)$ to its neighbors and updates $q_i(t) = \sum_{j=1}^{N} W_{ij} q_j(t-1)$.
4. end

**Output:** $(V_i/\sqrt{N} q_i(T_S))_{i=1}^{N}$

- In Step 1, each agent $i$ estimates the global objective $J(K(s) + rD(s))$ by implementing the local policy $K_i(s) + rD_i(s)$ simultaneously for $T_J$ time steps. The quantity $\mu_i(t)$ records agent $i$’s estimation of $J(K(s) + rD(s))$ at time step $t$, and is updated based on the neighbors’ estimations $\mu_j(t-1)$ and the local stage cost $c_i(t)$. The updating rule (6) can be viewed as a combination of a consensus procedure via the communication matrix $W$ and an online implementation of computing the average $\frac{1}{t} \sum_{\tau=1}^{t} c_i(\tau)$. Our theoretical analysis justifies that $\mu_i(T_J) \approx J(K(s) + rD(s))$ for sufficiently large $T_J$. Further, we introduce an additional truncation $\mathcal{J}_i(s) = \min\{\mu_i(T_J), \mathcal{J}\}$ for some sufficiently large $J$, which guarantees the boundedness of the gradient estimator in Step 2 to help ensure the stability of our iterating policy $K(s+1)$ and simplify the associated analysis.

- In Step 2, each agent $i$ updates its local policy $K_i$ by policy gradient (7), where the global objective $J(K(s) + rD(s))$ is approximated by the individual estimation $\mathcal{J}_i(s)$ obtained in Step 1.

Algorithm 2: Sampling Jointly from the Unit Sphere

1. Each agent $i$ samples $V_i \in \mathbb{R}^{n_i \times m_i}$ with i.i.d. entries from $\mathcal{N}(0, 1)$, and lets $q_i(0) = ||V_i||^2_{\mathcal{F}}$.
2. for $t = 1, 2, \ldots, T_S$ do
3. Agent $i$ sends $q_i(t-1)$ to its neighbors and updates $q_i(t) = \sum_{j=1}^{N} W_{ij} q_j(t-1)$.
4. end

Next, we provide a distributed algorithm (Algorithm 2) to sample $D_i(s)$ jointly from the distribution $\text{Uni}(\mathbb{S}_{m_K})$ as needed in Line 3 of Algorithm 1. In Algorithm 2, each agent $i$ samples a Gaussian random matrix $V_i$ independently, and then employs a simple consensus procedure to
compute the averaged squared norm $\frac{1}{n} \sum_i \| V_i \|^2$. The algorithm’s outputs approximately follow the desired distribution Uni($S_{nK}$), since $N q_i(t) \to \sum_i \| V_i \|^2$ exponentially (Xiao and Boyd, 2004) and $z/\|z\| \sim \text{Uni}(S_{nK})$ for $z \sim \mathcal{N}(0, I_{nK})$ by the isotropy of the standard Gaussian distribution. For simplicity, we neglect the sampling errors in Section 4, as they are expected to cause no substantial changes in our theoretical results. Incorporating sampling errors in the analysis is left as future work.

Finally, notice that each agent $i$ only communicates a scalar $\mu_i(t)$ for global cost estimation in Algorithm 1 and a scalar $q_i(t)$ for joint sampling in Algorithm 2 per communication round, which demonstrates the applicability of ZODPO in the situations with limited communication. Besides, each agent $i$ only stores and updates the local policy $K_i$, indicating that only limited storage will be used even in large-scale systems.

4 Theoretical Analysis

In this section, we first discuss some properties of $J(K)$, and then provide the main result of our paper, a nonasymptotic performance guarantee of ZODPO, followed by some discussions.

As indicated by Feng and Lavaei (2019); Bu et al. (2019), the objective function $J(K)$ of decentralized LQ control can be nonconvex. Nevertheless, $J(K)$ satisfies some smoothness properties.

**Lemma 1** (Properties of $J(K)$). The function $J(K)$ is continuously differentiable over $K \in \mathcal{K}_{st}$. Further, given a nonempty sublevel set $G_\alpha := \{K \in \mathcal{K}_{st} : J(K) \leq \alpha\}$ and an arbitrary $\alpha' > \alpha$, there exist constants $\xi > 0$ and $\phi > 0$ such that, for any $K \in G_\alpha$ and $K'$ with $\|K' - K\| \leq \xi$, we have $K' \in G_{\alpha'}$ and $\|\nabla J(K') - \nabla J(K)\| \leq \phi \|K' - K\|$.

This lemma is a direct consequence of (Malik et al., 2018; Feng and Lavaei, 2019). Without loss of generality, we pick $\alpha = 10 J(K_0)$ and $\alpha' = 20 J(K_0)$ and denote the associated constants in Lemma 1 as $\xi_0$ and $\phi_0$. The constants $\xi_0$ and $\phi_0$ depend on $A$, $B$, $\Sigma_w$, $J(K_0)$ and $Q_i, R_i$ for all $i$.

**Theorem 2** (Main result). Suppose $K_0 \in \mathcal{K}_{st}$. Let $0 < \epsilon \leq 625 \min \{\phi_0^2, \xi_0 J(K_0)\}$, and suppose the algorithmic parameters of Algorithm 1 satisfy

$$0 < r \leq \frac{\sqrt{\epsilon}}{25 \phi_0}, \quad 0 < \eta \leq \min \left\{ \frac{\xi_0 r}{\sqrt{K}}, \frac{3 \epsilon r^2}{250 \phi_0 (40 J(K_0))^2 \cdot n_K^2} \right\}, \quad \bar{J} \geq 50 J(K_0),$$

and

$$T_G = \left\lceil \frac{60 J(K_0)}{\eta \epsilon} \right\rceil, \quad T_J \geq 5 \times 10^3 J(K_0) \frac{n_K}{r \sqrt{\epsilon}} \max \left\{ \frac{n_K^2}{\beta_0}, \frac{N}{1 - \rho_W} \right\},$$

where $\beta_0$ is a constant determined by $A$, $B$, $\Sigma_w$, $K_0$ and $Q_i, R_i$ for all $i$, and $\rho_W := \|W - 11^T / N\|$ captures the convergence rate of the consensus via $W$ and is known to be within $(0, 1)$ (Qu and Li, 2017). Then, the policies $K(1), \ldots, K(T_G)$ generated by Algorithm 1 are all stabilizing controllers and enjoy the following bound with probability at least 0.7:

$$\frac{1}{T_G} \sum_{s=1}^{T_G} \|\nabla J(K(s))\|^2 \leq \epsilon. \quad (9)$$

**Probabilistic bound.** Theorem 2 establishes the stability and optimality of the controllers generated by ZODPO in a “with high probability” sense. The probability 0.7 is not restrictive and can be
improved by, e.g., increasing the numerical factors of $T_G$ and $T_J$, repeating the learning processes for several times as discussed in Ghadimi and Lan (2013); Malik et al. (2018), etc.

**Output controller.** Due to the nonconvexity of $J(K)$, we evaluate the algorithm performance by the averaged squared norm of the gradients as in (9), which is commonly used in nonconvex optimization (Ghadimi and Lan, 2013; Reddi et al., 2016). To obtain a good output controller, one common way in nonconvex optimization is to randomly select a controller from $\{K(s)\}_{s=1}^{T_G}$, which usually enjoys similar performance guarantees. Our numerical experiments show that $K(T_G)$ with large $T_G$ also yields desirable performance, though lacking theoretical guarantees.

**Sample complexity.** The number of samples to guarantee (9) with high probability is given by

$$T_G T_J = \Theta\left(\frac{n_K^2}{\epsilon^4} \max\left\{ n_K^2, \frac{N}{1 - \rho_W} \right\}\right),$$

when the conditions in Theorem 2 are applied with equalities. The sample complexity (10) has explicit polynomial dependence on the error tolerance’s inverse $\epsilon^{-1}$, the number of controller parameters $n_K$ and the number of agents $N$, demonstrating the scalability of ZODPO. In particular, the sample complexity is proportional to the maximum over two terms, where $N/(1 - \rho_W)$ stems from the consensus procedure among $N$ agents and decreases with the rate of consensus, while $n_K^2$ stems from approximating the infinite-horizon averaged cost, which exists even for a single agent.

**Optimization landscape.** Unlike centralized LQ control with full observations, the global optimum is hard to acquire for general decentralized LQ control with partial observations. In some cases, the stabilizing region $K_{st}$ may even contain multiple connected components (Feng and Lavaei, 2019), while our algorithm can only explore the component that contains $K_0$. Consequently, the choice of $K_0$ heavily affects the algorithm performance. The exploration of other components and the computation of $K_0$, potentially based on domain or prior knowledge, are left as future work.

## 5 Numerical Studies

In this section, we conduct numerical experiments on building energy regulation of Heating Ventilation and Air Conditioning (HVAC) systems for a multi-zone building to test our algorithm.

Specifically, we consider a multi-zone building with a linear thermal dynamics model as studied in Zhang et al. (2016b), where the temperature of each zone is affected by the supply air of the HVAC system as well as the temperatures of the nearby zones and the outdoor environment due to convection and conduction. Here, we consider $N = 4$ zones, where each zone is equipped with a sensor measuring local temperatures and an actuator adjusting supply air flow rates. The goal is to ensure user comfort by maintaining a desired room temperature while improving energy efficiency. To achieve this, we consider cost function $c_i(t) = \frac{1}{2}(x_i(t) - \theta_i^*)^2 + \alpha u_i(t)^2$, where $x_i(t)$ and $\theta_i^*$ denote the actual and desired temperatures of zone $i$ respectively, $u_i(t)$ is the control input, and $\alpha$ is a trade-off parameter. In our experiments, we consider two scenarios: 1) a constant outdoor temperature $30^\circ$C; 2) varying outdoor temperatures collected by Harvard HouseZero Program.\(^3\)

We apply our ZODPO to the two scenarios with $T_J = 300$. In Scenario 1, we consider affine control policies $u_i(t) = K_i x_i(t) + b_i$. In Scenario 2, we consider policies that adapt to the varying outdoor temperature $\theta^0(t)$, i.e., $u_i(t) = K_i x_i(t) + K_i^0 \theta^0(t) + b_i$. Besides, we consider a time discretization resolution of 1 minute and set the desired temperature as $\theta_i^* = 22^\circ$C for all zones.

\(^3\)https://snohetta.com/projects/413-harvard-housezero
Figure 1: The solid lines represent the temperature dynamics of the 4 zones given controllers generated by ZODPO after $T_G$ policy iterations for the two scenarios. The blue dashed lines mark the desired temperature, and the black dashed line in (d) shows the varying outdoor temperature.

Figures 1(a)–(c) plot the temperature dynamics of the four zones in Scenario 1 by implementing the controllers generated by ZODPO at policy gradient iterations $T_G = 50$, 150 and 250 respectively. It can be observed that with more iterations, the controllers generated by ZODPO stabilize the system faster and steer the room temperature closer to the desired temperature. Figure 1(d) plots the temperature dynamics in Scenario 2 by implementing the controller generated by ZODPO at policy iteration $T_G = 250$. The figure shows that even with varying outdoor temperatures, ZODPO is still able to find a controller that roughly maintains the room temperature at the desired level.

6 Conclusions and Future Work

This paper proposed a distributed learning method ZODPO for decentralized linear quadratic control and analyzed its sample complexity. This work may serve as a preliminary step for distributed learning for decentralized control and has many future directions, e.g., i) dealing with imperfect sampling from $\text{Uni}(S_{nK})$, ii) variance reduction for gradient estimation, iii) improving the sample complexity and studying the fundamental limit, iv) considering more general policy classes, e.g., linear dynamic controllers, v) off-policy and actor-critic algorithm design, vi) exploring optimization landscapes, vii) escaping the saddle points and establishing global convergence, etc.

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Overview of the Appendix

The appendix consists of the following parts:
1. Further discussions on the design of the proposed algorithm and the theoretical guarantee (Appendix A).
2. Theoretical analysis of Algorithm 1 (Appendices B to E).
3. Detailed settings of the HVAC test case (Appendix F).

The discussions and theoretical analysis provided in the appendix consider a more general version of Algorithm 1 that incorporates the mini-batch technique for variance reduction:

**Algorithm 1**: Zero-Order Distributed Policy Optimization (ZODPO)

**Input**: smoothing radius \( r \), step size \( \eta \), uniform bound \( J > 0 \), termination steps \( T_G, T_J, T_B \)

1. \( K(1) \leftarrow K_0 \) (known stabilizing controller)
2. for \( s = 1, 2, \ldots, T_G \) do
3.   for \( b = 1, 2, \ldots, T_B \) do
4.     Sample \( D_i(s, b) \in \mathbb{R}^{n_i \times m_i}, i = 1, \ldots, N \) such that \( \text{vec}(D_i(s, b)) \) satisfy the joint distribution \( \text{Uni}(S_{\sigma_K}) \).
5.     \( (\tilde{J}_i(s, b))_{i=1}^{N} \leftarrow \text{GlobalCostConsensus}((K_i(s) + rD_i(s, b))_{i=1}^{N}, T_J) \).
6.   end
7.   foreach \( i \in \{1, \ldots, N\} \) do
8.     Agent \( i \) estimates the gradient by
9.     \[ \hat{g}_i(s) = \frac{1}{T_B} \sum_{b=1}^{T_B} \frac{n_K}{r} \tilde{J}_i(s, b)D_i(s, b), \text{ where } \tilde{J}_i(s, b) = \min\{\tilde{J}_i(s, b), \bar{J}\}. \] (11)
10. Agent \( i \) updates \( K_i(s + 1) = K_i(s) - \eta \hat{g}_i(s) \).
11. end
12. end

Here we extract part of the algorithm as the subroutine **GlobalCostConsensus**:

**Function** \( \text{GlobalCostConsensus}((K_i)_{i=1}^{N}, T_J) \):

Reset the state of the linear dynamical system to \( x(0) = 0 \).
Each agent \( i \) implements \( K_i \), and set \( \mu_i(0) \leftarrow 0 \).
for \( t = 1, 2, \ldots, T_J \) do
  foreach \( i \in \{1, \ldots, N\} \) do
    Agent \( i \) observes the local cost \( c_i(t) \) and updates
    \[ \mu_i(t) = \frac{t - 1}{t} \sum_{j=1}^{N} W_{ij} \mu_j(t - 1) + \frac{1}{t} c_i(t). \] (12)
    Agent \( i \) sends \( \mu_i(t) \) to its neighbors.
  end
end
return \( (\mu_1(T_J), \ldots, \mu_N(T_J)) \)

Note that the only difference from Algorithm 1 is that Algorithm 1’ now employs \( T_B \) independent estimates of the gradient and takes their average to construct \( \hat{g}_i(s) \), where \( T_B \) is the batch
size. The theoretical performance guarantee of Algorithm 1′, which will be proved later, is given as follows.

**Theorem 3.** Let $0 < \epsilon \leq 625 \min \{\phi_0^2 \xi_0^2, \phi_0 J(K_0)\}$. Suppose $K_0 \in K_{st}$ and the algorithmic parameters of Algorithm 1′ satisfy

\[0 < r \leq \frac{\sqrt{\epsilon}}{25\phi_0}, \quad 0 < \eta \leq \min \left\{ \frac{\xi_0}{JnK}, \frac{1}{6\phi_0} \right\}, \quad \bar{J} \geq 50J(K_0),\]

and

\[T_G = \left\lceil \frac{60J(K_0)}{\eta \epsilon} \right\rceil, \quad T_J \geq 5 \times 10^3 J(K_0) \frac{nK}{r \sqrt{\epsilon}} \max \left\{ \frac{n\beta_0^2}{3}, \frac{N}{1 - \rho_W} \right\},\]

\[T_B \geq \frac{250\phi_0 (40J(K_0))^2}{3} \frac{\eta n_K^2}{\epsilon r^2},\]

where $\beta_0$ is a constant determined by $A, B, Q, R, \Sigma_w$ and $K_0$, and we denote $\rho_W := \|W - N^{-1}11^T\|$. Then the controllers generated by Algorithm 1′, $K(1), \ldots, K(T_G)$, are all stabilizing controllers and enjoy the following bound

\[\frac{1}{T_G} \sum_{s=1}^{T_G} \|\nabla J(K(s))\|^2 \leq \epsilon\]

(13)

with probability at least 0.7.

It can be seen that Theorem 2 is a direct consequence of Theorem 3 when we set $T_B = 1$ and

\[\frac{250\phi_0 (40J(K_0))^2}{3} \frac{\eta n_K^2}{\epsilon r^2} \leq 1.\]

Thus, we only need to prove Theorem 3.

**A Additional Remarks and Discussions**

**Additional remarks on the proposed algorithms**

1. Every time we begin the subroutine GlobalCostConsensus, the linear system is reset and restarts evolving from the origin. This eliminates the correlations between $\hat{J}_i(s, b)$ for different $s$ and $b$, which simplifies theoretical analysis. We suspect that removing this reset step will not change the theoretical guarantees of the algorithm much, but detailed analysis is left for future work.

2. The truncation step $\hat{J}_i(s, b) = \min \left\{ \hat{J}_i(s, b), \bar{J} \right\}$ ensures that the gradient estimation $\hat{g}_i(s)$ can be uniformly bounded by some constant, which simplifies associated analysis regarding the stability of $K(s + 1)$. In simulation, ZODPO works well without this truncation. The theoretical explanation for this is left as future work.

3. In Algorithm 1′, we assume that the underlying discrete-time dynamical system evolves one time step per communication round. A more general setting is that we allow the underlying dynamical system to evolve multiple steps before a new round of communication is carried out, whose theoretical performance can be analyzed similarly but is omitted here for simplicity.
4. We point out that the consensus procedure in Algorithm 2 can be carried out simultaneously with (12) in Algorithm 1′ if we set \( T_S = T_J \). We present them separately for clarity of exposition.

5. Algorithm 1′ only utilizes the cost information but does not explicitly use the state information in finding an optimal controller. In this sense, Algorithm 1 shares some similarities with extremum-seeking control (Ariyur and Krstic, 2003).

6. While the main focus of this paper is on the analysis of Algorithm 1′ for distributed learning of decentralized LQ control, the proposed algorithms can be adapted to more general situations where historical observations can be utilized or where the underlying system has nonlinear dynamics. The performance analysis for more general settings, however, might be more complicated, which is left as future work.

**Further discussions on Theorem 2 / Theorem 3**

1. Theorems 2 and 3 neglect the inaccuracy incurred by the sampling procedure that draws \( D(s) \) from \( \text{Uni}(\mathbb{S}_n) \) in a distributed fashion. One major reason for imposing this simplification is that in Algorithm 2, \( q_i(t) \) converges exponentially fast, while our analysis indicates that \( \mu_i(t) \) converges at a sublinear rate. Therefore, if we run the consensus procedure in Algorithm 2 simultaneously with (12) and let \( T_S = T_J \), it is reasonable to believe that the inaccuracy incurred by Algorithm 2 is almost negligible compared to the estimation error of the global objective function \( J(K(s) + rD(s)) \), and that the sampling complexity may not have substantial changes even if we take this inaccuracy into account. Detailed analysis will be left for future work.

2. It is interesting to mention that Theorem 3 provides an equality condition for \( T_G \), suggesting that \( T_G \) should not be too large. The intuition behind it is that as \( T_G \) increases, the probability that Algorithm 1 produces a controller \( K(s) \) that escapes \( K_{\text{st}} \) can also increase due to the nonzero bias and variance of gradient estimation. Nevertheless, the condition on \( T_G \) is very conservative and it is usually okay to have a relatively large \( T_G \) in practice.

3. Some further discussions on the sample complexity are as follows:

   (a) The sample complexity (10) is proportional to \( n_K^3 \). Detailed analysis reveals that the variance of the single-point gradient estimation contributes a dependence of \( n_K^2 \), which also accords with the theoretical lower bound for zero-order optimization in Shamir (2013). The additional \( n_K \) comes from the non-zero bias of the global cost estimation.

   (b) The effect of the network structure is characterized through the term \( N/(1 - \rho_W) \) in (10). We see that the sample complexity is non-decreasing in \( \rho_W \), as a larger \( \rho_W \) indicates a slower rate of achieving consensus. The factor \( N \) might be a proof artifact but its improvement is left for future work.

   (c) While there is an explicit linear asymptotic dependence on the state vector dimension \( n \) in (10), we point out that the quantities \( \beta_0, J(K_0), \phi_0, \xi_0 \) are also implicitly affected by \( n \) as they are determined by \( A, B, Q, R, \Sigma_w \) and \( K_0 \). Therefore, the actual dependence of the sample complexity on \( n \) is complicated and not straightforward to summarize.

**B Proof of Lemma 1, Additional Notations and Auxiliary Lemmas**

In this section, we introduce some notations and auxiliary lemmas to prepare for the proof of Theorem 3.
For any matrix $M \in \mathbb{R}^{p \times q}$, its spectral norm will be denoted by $\|M\|$, its Frobenius norm will be denoted by $\|M\|_F$, and its spectral radius will be denoted by $\rho(M)$. The trace of $M$ will be denoted by $\text{tr}(M)$. We point out that the norm $\|\cdot\|_F$ can be induced from the inner product $\langle M_1, M_2 \rangle = \text{tr}(M_1^\top M_2)$ on the linear space $\mathbb{R}^{p \times q}$. The $p \times p$ identity matrix will be denoted by $I_p$.

For simplicity of notation, we denote $Q := \frac{1}{N} \sum_{i=1}^{N} Q_i$, $R := \frac{1}{N} \sum_{i=1}^{N} R_i$, and $A_K := A + BM(K)$, $Q_{i,K} := Q_i + M(K)^\top R_i M(K)$, $Q_K := \frac{1}{N} \sum_{i=1}^{N} Q_{i,K}$.

We point out that $M$ is an injective linear map from $\mathbb{R}^{n_K}$ to $\mathbb{R}^{n \times m}$ satisfying $\|M(K)\|_F = \|K\|$ for any $K \in \mathbb{R}^{n_K}$.

It is known from the theory of linear dynamical systems that the objective function $J(K)$ can be represented as

$$J(K) = \text{tr}(Q_K \Sigma_{K,\infty}),$$

where

$$\Sigma_{K,\infty} := \sum_{t=0}^{\infty} A_K^t \Sigma_w (A_K^\top)^t$$

is the covariance matrix of the state variable under the stationary distribution of the linear dynamical system.

In addition, we introduce an auxiliary function $\mathcal{J}(K)$ defined by

$$\mathcal{J}(K) := \text{tr}\left[ (Q + K^\top R K) \sum_{t=0}^{\infty} (A + BK)^t \Sigma_w ((A + BK)^\top)^t \right]$$

for any $K \in \mathbb{R}^{n \times m}$ with $\rho(A + BK) < 1$. The function $\mathcal{J}(K)$ is in fact the standard objective function of LQR with the (global) feedback gain $K$ being the variable. We can readily recognize that $J(K) = \mathcal{J}(M(K))$.

Firstly, the following lemma shows the compactness of the sublevel sets of $J(K)$.

**Lemma 4.** Suppose $R$ and $\Sigma_w$ are positive definite. Then the sublevel set

$$G_\alpha := \{K \in \mathcal{K}_{\text{nst}} : J(K) \leq \alpha\}$$

is compact for any $\alpha > 0$ as long as $G_\alpha \neq \emptyset$.

**Proof.** It has been shown in Fazel et al. (2018) that $\mathcal{J}(K)$ is a continuously differentiable function over its domain. Then, since $J(K) = \mathcal{J}(M(K))$, we have

$$G_\alpha = J^{-1}(-\infty, \alpha] = \mathcal{M}^{-1}(\mathcal{J}^{-1}(-\infty, \alpha]),$$

from which we see that $G_\alpha$ is a closed subset of $\mathbb{R}^{n_K}$. Therefore, we only need to prove that $G_\alpha$ is bounded.
Now, notice that

\[ J(K) = \text{tr}\left( (Q + \mathcal{M}(K)^\top R \mathcal{M}(K)) \Sigma_{K,\infty} \right) \]
\[ \geq \lambda_{\min}(\Sigma_{K,\infty}) \text{tr}\left( Q + \mathcal{M}(K)^\top R \mathcal{M}(K) \right) \]
\[ \geq \lambda_{\min}(\Sigma_w) \left( \text{tr}(Q) + \lambda_{\min}(R)\|K\|^2 \right) \]

where we used the fact that tr\((M_1 M_2) \geq \lambda_{\min}(M_1) \lambda_{\min}(M_2)\) for any positive semidefinite matrices \(M_1\) and \(M_2\), that \(\lambda_{\min}(\Sigma_w) \leq \lambda_{\min}(\Sigma_{K,\infty})\), and that \(\text{tr}(\mathcal{M}(K) \mathcal{M}(K)^\top) = \|\mathcal{M}(K)\|^2_F = \|K\|^2\). As a result, for any \(K \in G_\alpha\),

\[ \|K\|^2 \leq \frac{1}{\lambda_{\min}(R)} \left( \frac{J(K)}{\lambda_{\min}(\Sigma_w)} - \text{tr}(Q) \right) \leq \frac{1}{\lambda_{\min}(R)} \left( \frac{\alpha}{\lambda_{\min}(\Sigma_w)} - \text{tr}(Q) \right), \quad (14) \]

which implies that \(G_\alpha\) is bounded.

Next, we provide a proof of Lemma 1 based on the properties of \(\mathcal{J}(K)\) established in Fazel et al. (2018) and Malik et al. (2018).

**Proof of Lemma 1.** Note that \(J(K) = \mathcal{J}(\mathcal{M}(K))\) and \(\|\mathcal{M}(K)\|_F = \|K\|\). Also, it’s not hard to see that

\[ \mathcal{M}(\nabla J(K)) = \mathcal{P}_\mathcal{M}[\nabla \mathcal{J}(\mathcal{M}(K))], \]

where \(\mathcal{P}_\mathcal{M}\) denotes orthogonal projection onto the range space of \(\mathcal{M}\); see also Bu et al. (2019, Lemma 7.3). This implies that

\[ \|\nabla J(K') - \nabla J(K)\| = \|\mathcal{M}(\nabla J(K') - \nabla J(K))\|_F \]
\[ = \|\mathcal{P}_\mathcal{M} [\nabla \mathcal{J}(\mathcal{M}(K')) - \nabla \mathcal{J}(\mathcal{M}(K))]\|_F \]
\[ \leq \|\nabla \mathcal{J}(\mathcal{M}(K')) - \nabla \mathcal{J}(\mathcal{M}(K))\|_F. \]

Then we can use the results in the proof of Malik et al. (2018, Lemma 1) to show that

\[ |J(K') - J(K)| \leq h_1(K)\|K' - K\|, \]
\[ \|\nabla J(K') - \nabla J(K)\| \leq h_2(K)\|K' - K\| \]

for any \(K'\) with \(\|K' - K\| \leq h_0(K)\), where \(h_0(K)\), \(h_1(K)\) and \(h_2(K)\) are some continuous positive functions over \(K \in \mathcal{K}_{\text{st}}\) (we refer to Malik et al. (2018) for explicit expressions of these functions). Then, by letting

\[ \phi = \sup_{K \in G_\alpha} h_2(K), \quad \xi = \min\left\{ \frac{\alpha' - \alpha}{\sup_{K \in G_\alpha} h_1(K)}, \frac{\sup_{K \in G_\alpha} h_0(K)}{\sup_{K \in G_\alpha} h_1(K)} \right\}, \]

it can be verified that the desired properties of Lemma 1 will be satisfied. Notice that \(\phi\) and \(\xi\) have well-defined finite values as \(h_0(K)\), \(h_1(K)\) and \(h_2(K)\) are continuous and \(G_\alpha\) is compact.

The following lemma bounds the spectral radius of \(A_K\) by the associated cost \(J(K)\).
Lemma 5. For any $K \in \mathcal{K}_{st}$, we have

$$\rho(A_K) \leq \sqrt{\frac{\lambda_{\min} \left(\frac{1}{2}Q\Sigma_{kw}^{-1}\right)}{J(K)}}.$$

Proof. We have

$$J(K) = \text{tr}_K \left[ Q_K \left( \sum_{t=0}^{\infty} A_K^{\top} (A_K^{\top})^t \right) \right]$$

$$= \text{tr} \left[ \Sigma_{kw}^{1/2} Q_K \Sigma_{kw}^{1/2} \sum_{t=0}^{\infty} \left( \Sigma_{kw}^{-1/2} A_K \Sigma_{kw}^{1/2} \right)^t \left( \Sigma_{kw}^{1/2} A_K^{\top} \Sigma_{kw}^{-1/2} \right)^t \right]$$

$$\geq \lambda_{\min} \left( \Sigma_{kw}^{1/2} Q_K \Sigma_{kw}^{1/2} \right) \left\| \sum_{t=0}^{\infty} \left( \Sigma_{kw}^{-1/2} A_K \Sigma_{kw}^{1/2} \right)^t \left( \Sigma_{kw}^{1/2} A_K^{\top} \Sigma_{kw}^{-1/2} \right)^t \right\|.$$

By Gahinet et al. (1990, Theorem 5.4),\(^4\) we have

$$\left\| \sum_{t=0}^{\infty} \left( \Sigma_{kw}^{-1/2} A_K \Sigma_{kw}^{1/2} \right)^t \left( \Sigma_{kw}^{1/2} A_K^{\top} \Sigma_{kw}^{-1/2} \right)^t \right\| \geq \frac{1}{1 - \rho \left( \Sigma_{kw}^{-1/2} A_K \Sigma_{kw}^{1/2} \right)^2} = \frac{1}{1 - \rho(A_K)^2},$$

which then leads to the desired bound.

The following lemma shows that the norm of $\left( \Sigma_{K,\infty}^{-1/2} A_K \Sigma_{K,\infty}^{1/2} \right)^t$ decays exponentially as $t$ increases.

Lemma 6. There exists a continuous function $\varphi : \mathcal{K}_{st} \to [1, +\infty)$ such that

$$\left\| \left( \Sigma_{K,\infty}^{-1/2} A_K \Sigma_{K,\infty}^{1/2} \right)^t \right\| \leq \varphi(K) \left( \frac{1 + \rho(A_K)}{2} \right)^t$$

for any $t \in \mathbb{N}$ and any $K \in \mathcal{K}_{st}$.

Proof. Denote $\tilde{\rho}(A_K) = (1 + \rho(A_K))/2$. Since $\rho \left( \Sigma_{K,\infty}^{-1/2} A_K \Sigma_{K,\infty}^{1/2} \right) = \rho(A_K) < \tilde{\rho}(A_K)$, it can be seen that the matrix

$$\tilde{P}_K = \sum_{t=0}^{\infty} \tilde{\rho}(A_K)^{-2t} \left( \Sigma_{K,\infty}^{-1/2} A_K \Sigma_{K,\infty}^{1/2} \right)^t \left( \Sigma_{K,\infty}^{1/2} A_K^{\top} \Sigma_{K,\infty}^{-1/2} \right)^t$$

converges, and satisfies the Lyapunov equation

$$\tilde{\rho}(A_K)^{-2} \left( \Sigma_{K,\infty}^{-1/2} A_K \Sigma_{K,\infty}^{1/2} \right) \tilde{P}_K \left( \Sigma_{K,\infty}^{1/2} A_K^{\top} \Sigma_{K,\infty}^{-1/2} \right) + I = \tilde{P}_K,$$

\(^4\) The inequality we employ here holds even when $A_K$ is not diagonalizable. See the remark after Theorem 5.4 of Gahinet et al. (1990).
which further implies
\[
\left( \tilde{P}_K^{-1/2} \Sigma_{K,\infty}^{-1/2} A_K \Sigma_{K,\infty}^{1/2} \tilde{P}_K^{1/2} \right) \left( \tilde{P}_K^{-1/2} \Sigma_{K,\infty}^{-1/2} A_K \Sigma_{K,\infty}^{1/2} \tilde{P}_K^{1/2} \right)^\top \preceq \tilde{\rho}(A_K)^2 I.
\]

Denoting
\[
\tilde{A}_K = \tilde{P}_K^{-1/2} \Sigma_{K,\infty}^{-1/2} A_K \Sigma_{K,\infty}^{1/2} \tilde{P}_K^{1/2},
\]
we see that \(\|A_K\| \leq \tilde{\rho}(A_K)\), and
\[
\left\| \left( \Sigma_{K,\infty}^{-1/2} A_K \right) \right\| = \left\| \left( \tilde{P}_K^{1/2} A \tilde{P}_K^{1/2} \right) \right\| \leq \left\| \tilde{P}_K^{1/2} \right\| \left\| \tilde{A}_K \right\| \leq \tilde{P}_K^{1/2} \left\| \tilde{P}_K^{1/2} \right\| \tilde{\rho}(A_K)^t =: \varphi(K) \tilde{\rho}(A_K)^t,
\]
where we denote
\[
\varphi(K) := \left\| \tilde{P}_K^{1/2} \right\| \left\| \tilde{P}_K^{1/2} \right\|.
\]

It’s easy to see that \(\varphi(K) \geq \left\| \tilde{P}_K^{1/2} \tilde{P}_K^{1/2} \right\| = 1\), and by the results of perturbation analysis of Lyapunov equations (Gahinet et al., 1990), we can see that \(\varphi(K)\) is a continuous function over \(K \in K_{st}\).

C Analysis of the Subroutine GlobalCostConsensus

Let us suppose that the input of the subroutine GlobalCostConsensus is \((K_i)_{i=1}^N\) and \(T_J\), and let \(K \in \mathbb{R}^{nK}\) denote the vector that concatenates \(\text{vec}(K_1), \ldots, \text{vec}(K_N)\) as usual. We assume that \(\rho(A_K) < 1\).

We denote \(\Sigma_{K,t}\) as the covariance matrix of the state vector at time \(t\). We have
\[
\Sigma_{K,t} = \sum_{\tau=0}^{t-1} A_K^\top \Sigma_{w}(A_K^\top)^\tau.
\]

Obviously \(\Sigma_{K,t} \preceq \Sigma_{K,\infty}\), and since we restart from \(x(0) = 0\), we also have
\[
\frac{1}{N} \sum_{i=1}^N \mathbb{E}[c_i(t)] = \mathbb{E} \left[ x(t)^\top Q_K x(t) \right] = \text{tr}(Q_K \Sigma_{K,t}).
\]

To analyze the statistics of \(\mu_i(T_J)\), we represent \(\mu_i(T_J)\) as a quadratic form of some standard Gaussian random vector as follows. First, since we restart from \(x(0) = 0\), it can be checked by induction that
\[
x(t) = \sum_{\tau=1}^t A_K^{t-\tau} w(\tau - 1).
\]
We introduce the following auxiliary quantities

\[
\varpi(T_J) := \begin{bmatrix}
\Sigma_w^{-\frac{1}{2}} w(0) \\
\Sigma_w^{-\frac{1}{2}} w(1) \\
\vdots \\
\Sigma_w^{-\frac{1}{2}} w(T_J-1)
\end{bmatrix}, \quad \Psi(T_J) := \begin{bmatrix}
I_n \\
\Sigma_{K,\infty}^{-\frac{1}{2}} A_{K} \Sigma_{w}^{-\frac{1}{2}} \\
\vdots \\
\Sigma_{K,\infty}^{-\frac{1}{2}} A_{K}^{T_J-1} \Sigma_{w}^{-\frac{1}{2}} \\
\Sigma_{K,\infty}^{-\frac{1}{2}} A_{K}^{T_J-2} \Sigma_{w}^{-\frac{1}{2}} \\
\vdots \\
I_n
\end{bmatrix},
\]

and

\[
\Phi^{(T_J)}_{\gamma} := \Psi(T_J)^\top \cdot \text{blkdiag} \left[ \left( \gamma T_J^{-l} \Sigma_{K,\infty}^{\frac{1}{2}} Q_{K} \Sigma_{K,\infty}^{\frac{1}{2}} \right)_{l=1}^{T_J} \right] \cdot \Psi(T_J),
\]

\[
\Phi^{(T_J)}_{W,i} := \Psi(T_J)^\top \cdot \text{blkdiag} \left[ \left( \Sigma_{w}^{\frac{1}{2}} \left( \sum_{j=1}^{N} \tilde{W}_{ij}^{(T_J^{-l})} Q_{j,K} \right) \Sigma_{K,\infty}^{\frac{1}{2}} \right)_{l=1}^{T_J} \right] \cdot \Psi(T_J),
\]

where \( \gamma \in [0, 1] \) and

\[
\tilde{W}_{ij}^{(t)} := \left[ \left( W - N^{-1} 1 1^\top \right)^t \right]_{ij},
\]

and the notation \( \text{blkdiag}[(M_{l})_{l=1}^{p}] \) denotes the block-diagonal matrix

\[
\text{blkdiag}[(M_{l})_{l=1}^{p}] = \begin{bmatrix} M_1 \\
\vdots \\
M_p \end{bmatrix}.
\]

We also denote \( \Phi^{(T_J)} := \Phi^{(T_J)}_{1} \). We can then see that

\[
T_J \mu_i(T_J) = \sum_{t=1}^{T_J} \sum_{j=1}^{N} \left[ W^{T_J-t} \right]_{ij} x(t)^\top Q_{j,K} x(t)
\]

\[
= \sum_{t=1}^{T_J} \sum_{\tau=1}^{t} \sum_{\tau'=1}^{t} w(\tau-1)^\top (A_{K})^{t-\tau} \left( \sum_{j=1}^{N} \left[ W^{T_J-t} \right]_{ij} Q_{j,K} \right) A_{K}^{t-\tau'} w(\tau'-1)
\]

\[
= \sum_{t=1}^{T_J} \sum_{\tau=1}^{t} \sum_{\tau'=1}^{t} w(\tau-1)^\top (A_{K})^{t-\tau} \left( \sum_{j=1}^{N} \tilde{W}_{ij}^{(T_J-t)} Q_{j,K} + Q_{K} \right) A_{K}^{t-\tau'} w(\tau'-1)
\]

\[
= \varpi(T_J)^\top \left( \Phi_{W,i}^{(T_J)} + \Phi^{(T_J)} \right) \varpi(T_J), \tag{15}
\]

and similarly

\[
\sum_{t=1}^{T_J} x(t)^\top Q_{K} x(t) = \varpi(T_J)^\top \Phi^{(T_J)} \varpi(T_J). \tag{16}
\]

**Lemma 7.** Let \( z \) be a \( p \)-dimensional Gaussian random vector with distribution \( N(0, I_p) \), let \( M \in \mathbb{R}^{p \times p} \) be any symmetric matrix.
1. (Seber and Lee, 2003, Theorems 1.5 & 1.6) We have $\mathbb{E}[z^T M z] = \text{tr}(M)$ and $\text{Var}(z^T M z) = 2\|M\|^2_F$.

2. (Hsu et al., 2012) If $M$ is positive semidefinite, then for any $\delta \geq 0$, the following inequality holds:

$$
\mathbb{P}(z^T M z > \text{tr} M + 2\|M\|_F \sqrt{\delta} + 2\|\delta\|) \leq e^{-\delta}.
$$

(17)

**Lemma 8.** We have

$$
\|\Phi(T_J)\| \leq J(K) \left( \frac{2\varphi(K)}{1 - \rho(A_K)} \right)^2, \quad \|\Phi(T_J)\|_F^2 \leq J(K)^2 \cdot n T_J \left( \frac{2\varphi(K)}{1 - \rho(A_K)} \right)^4.
$$

and

$$
\text{tr} \left( \Phi(T_J)^T \right) \leq \frac{J(K)}{1 - \rho_W}.
$$

**Proof.** For $\Phi(T_J)$, notice that

$$
\|\Phi(T_J)\|^2 \leq \left\| \Sigma_{K,\infty}^{1/2} Q_K \Sigma_{K,\infty}^{1/2} \right\| \left\| \Psi(T_J) \right\|^2 \leq J(K) \left\| \Psi(T_J) \right\|^2.
$$

Now for any $z = (z^T_0, z^T_1, \ldots, z^T_{T_J-1})^T \in \mathbb{R}^{n T_J}$, we have

$$
\|\Psi(T_J) z\|^2 \leq \sum_{t=1}^T \sum_{\tau=1}^T \sum_{\tau' = 1}^T \left\| z_{\tau-1} \Sigma_{\tau,\infty}^{1/2} \Sigma_{\tau',\infty}^{1/2} \right\| \left( \Sigma_{K,\infty}^{1/2} A_K^T \Sigma_{K,\infty}^{1/2} \right)^{t-\tau} \left( \Sigma_{K,\infty}^{1/2} A_K \Sigma_{K,\infty}^{1/2} \right)^{t-\tau'} \left\| z_{\tau'-1} \right\|,
$$

where we denote $\tilde{\rho}(A_K) = (1 + \rho(A_K))/2$, and used $\left\| \Sigma_{K,\infty}^{1/2} \Sigma_{w}^{1/2} \right\| \leq 1$ since $\Sigma_{K,\infty} \succeq \Sigma_w$. We further notice that

$$
\sum_{t=1}^T \sum_{\tau=1}^T \sum_{\tau' = 1}^T \left\| z_{\tau-1} \right\| \cdot \tilde{\rho}(A_K)^{t-\tau} \cdot \tilde{\rho}(A_K)^{t-\tau'} \cdot \left\| z_{\tau'-1} \right\|
$$

$$
\leq \left\| \frac{1}{1 - \tilde{\rho}(A_K)e^{-j\omega}} \right\|_{\mathcal{H}_\infty}^2 \|z\|,
$$

where

$$
\left\| \frac{1}{1 - \tilde{\rho}(A_K)e^{-j\omega}} \right\|_{\mathcal{H}_\infty}
$$

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denotes the $\mathcal{H}_\infty$ norm of the filter whose transfer function is $1/(1 - \tilde{\rho}(A_K)e^{-j\omega})$, and can be shown to be equal to $1/(1 - \tilde{\rho}(A_K))$. Therefore

$$
\| \Phi(T_j) \| \leq \frac{\varphi(K)}{1 - \tilde{\rho}(A_K)} = \frac{2\varphi(K)}{1 - \rho(A_K)},
$$

(18)

and we get

$$
\| \Phi(T_j) \| \leq \left( \frac{2\varphi(K)}{1 - \rho(A_K)} \right)^2 J(K).
$$

(19)

By using $\| \Phi(T_j) \|_F^2 \leq nT_j \| \Phi(T_j) \|_F^2$, we get the desired bound on $\| \Phi(T_j) \|_F^2$.

Now for $\Phi_{\rho_W}^{(T_j)}$, we have

$$
\text{tr} \left( \Phi_{\rho_W}^{(T_j)} \right) = \mathbb{E} \left[ \sum_{t=1}^{T_j} \rho_W^{T_j-t} x(t)^\top Q_K x(t) \right] = \sum_{t=1}^{T_j} \rho_W^{T_j-t} \text{tr} \left( Q_K \Sigma_{K,t} \right)
$$

$$
\leq J(K) \sum_{t=1}^{T_j} \rho_W^{T_j-t} \leq \frac{J(K)}{1 - \rho_W}.
$$

Lemma 9. For any $i \in \{1, \ldots, N\}$, we have

$$
|\mathbb{E}[\mu_i(T_j)] - J(K)| \leq \frac{J(K)}{T_j} \left[ \frac{N}{1 - \rho_W} + \left( \frac{2\varphi(K)}{1 - \rho(A_K)} \right)^2 \right],
$$

(20)

and

$$
\mathbb{E} \left[ (\mu_i(T_j) - J(K))^2 \right] \leq \frac{6nJ(K)^2}{T_j} \left( \frac{2\varphi(K)}{1 - \rho(A_K)} \right)^4 + \frac{8J(K)^2}{T_j^2} \left( \frac{N}{1 - \rho_W} \right)^2.
$$

(21)

Proof. First we notice that for any $v_1, \ldots, v_{T_j} \in \mathbb{R}^p$,

$$
\begin{align*}
\sum_{l=1}^{T_j} v_l^\top \left( \sum_{j=1}^{N} \tilde{W}_{ij}^{T_j-l} Q_{j,K} \right) v_l & \leq \sum_{l=1}^{T_j} \sum_{j=1}^{N} \tilde{W}_{ij}^{T_j-l} v_l^\top Q_{j,K} v_l \\
& \leq \sum_{l=1}^{T_j} \rho_W^{T_j-l} \sum_{j=1}^{N} (v_l^\top Q_{j,K} v_l)^2 \\
& \leq \sum_{l=1}^{T_j} \rho_W^{T_j-l} \sum_{j=1}^{N} v_l^\top Q_{j,K} v_l \\
& = N \sum_{l=1}^{T_j} \rho_W^{T_j-l} v_l^\top Q_K v_l,
\end{align*}
$$

where in the second and third inequalities we used $\|(W - N^{-1}11^\top)v\|_\infty \leq \|(W - N^{-1}11^\top)v\| \leq \rho_W \|v\|$ and $\|v\| \leq 1^\top v$ for any vector $v \in \mathbb{R}^N$ with nonnegative entries. This implies that

$$
\left| \mathcal{W}(T_j)^\top \Phi_{\rho_W}^{(T_j)} \mathcal{W}(T_j) \right| \leq N \mathcal{W}(T_j)^\top \Phi_{\rho_W}^{(T_j)} \mathcal{W}(T_j).
$$

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Analysis of the bias. For the bias, we have

\[ |\mathbb{E}[\mu_i(T_j)] - J(K)| = \left| \frac{1}{T_j} \sum_{t=1}^{T_j} \mathbb{E} \left[ \varpi(T_j) \right] \right| \]

\[ \leq \frac{1}{T_j} \sum_{t=1}^{T_j} \mathbb{E} \left[ \varpi(T_j) \Phi^{(T_j)}_W (\phi^{(T_j)}_W) \right] - J(K) \]

\[ \leq \frac{1}{T_j} \sum_{t=1}^{T_j} \mathbb{E} \left[ \varpi(T_j) \Phi^{(T_j)}_W (\phi^{(T_j)}_W) \right] - J(K) \]

\[ = \frac{1}{T_j} \sum_{t=1}^{T_j} \mathbb{E} \left[ \Phi^{(T_j)}_W \right] + \frac{1}{T_j} \sum_{t=1}^{T_j} \mathbb{E} \left[ \varpi(T_j) \Phi^{(T_j)}_W (\phi^{(T_j)}_W) \right] - J(K) \]

It can be seen from Lemma 8 that the first term can be upper bounded by \( N \mathbb{J}(K)/(T_j(1 - \rho_W)) \).

Now for the second term, we notice that

\[ J(K) - \frac{1}{T_j} \sum_{t=1}^{T_j} \mathbb{E} \left[ x(t)^\top Q_K x(t) \right] = \frac{1}{T_j} \sum_{t=1}^{T_j} \text{tr} (Q_K (\Sigma_{K,\infty} - \Sigma_{K,t})) \]

which is always nonnegative as \( \Sigma_{K,\infty} \succeq \Sigma_{K,t} \). Thus

\[ J(K) - \frac{1}{T_j} \sum_{t=1}^{T_j} \mathbb{E} \left[ x(t)^\top Q_K x(t) \right] = \frac{1}{T_j} \sum_{t=1}^{T_j} \text{tr} \left( Q_K (\Sigma_{K,\infty} - \Sigma_{K,t}) \right) \]

Then by Lemma 6 and \( \Sigma_{K,\infty} \succeq \Sigma_w \), we have

\[ \left\| I - \Sigma_{K,\infty}^{-1/2} \Sigma_{K,t} \Sigma_{K,\infty}^{-1/2} \right\| = \left\| \Sigma_{K,\infty}^{-1/2} \left( \sum_{t=1}^{\infty} A_t^\top \Sigma_w (A_t^\top) \right) \Sigma_{K,\infty}^{-1/2} \right\| \]

\[ \leq \left\| \sum_{t=1}^{\infty} \left( \Sigma_{K,\infty}^{-1/2} A_t^\top \Sigma_{K,\infty}^{-1/2} \right) \right\| \left\| \Sigma_{K,\infty}^{-1/2} \Sigma_w \Sigma_{K,\infty}^{-1/2} \right\| \]

\[ \leq \sum_{t=1}^{\infty} \varphi(K) \left( \frac{1 + \rho(A_K)}{2} \right)^{2t} 2 \varphi(K)^2 \left( \frac{1 + \rho(A_K)}{2} \right)^{2t} \]

and therefore

\[ J(K) - \frac{1}{T_j} \sum_{t=1}^{T_j} \mathbb{E} \left[ x(t)^\top Q_K x(t) \right] \leq \frac{\text{tr}(Q_K \Sigma_{K,\infty})}{T_j} \cdot \frac{2 \varphi(K)^2}{1 - \rho(A_K)} \sum_{t=1}^{T_j} \left( \frac{1 + \rho(A_K)}{2} \right)^{2t} \]

\[ \leq \frac{J(K)}{T_j} \left( \varphi(K) \frac{1 + \rho(A_K)}{1 - \rho(A_K)} \right)^2 \leq \frac{J(K)}{T_j} \left( \frac{2 \varphi(K)}{1 - \rho(A_K)} \right)^2. \]

We now get the inequality (20).
Analysis of \( \mathbb{E} \left[ (\mu_i(T_J) - J(K))^2 \right] \). We first bound the variance of \( \mu_i(T_J) \). Notice that

\[
\left\| \Phi_{W,i}^{(T_J)} \right\|_F^2 = \frac{1}{2} \text{Var} \left( (\varpi(T_J)^\top \Phi_{W,i}^{(T_J)} \varpi(T_J)) \right) \\
\leq \frac{1}{2} \mathbb{E} \left[ (\varpi(T_J)^\top \Phi_{W,i}^{(T_J)} \varpi(T_J))^2 \right] \leq \frac{1}{2} N^2 \mathbb{E} \left[ (\varpi(T_J)^\top \Phi_{\rho_W}^{(T_J)} \varpi(T_J))^2 \right] \\
= \frac{1}{2} N^2 \left( 2 \left\| \Phi_{\rho_W}^{(T_J)} \right\|_F^2 + \text{tr} (\Phi_{\rho_W}^{(T_J)})^2 \right) \\
\leq \frac{3}{2} N^2 \text{tr} (\Phi_{\rho_W}^{(T_J)})^2 \leq \frac{3}{2} \left( \frac{N J(K)}{1 - \rho_W} \right)^2
\]

by Lemma 7, Lemma 8 and the fact that \( \|M\|_F^2 \leq \text{tr}(M)^2 \) for any positive semidefinite matrix \( M \).

We then get

\[
\text{Var}(\mu_i(T_J)) = \frac{2}{T_J^2} \left\| \Phi_{W,i}^{(T_J)} + \Phi^{(T_J)} \right\|_F^2 \leq \frac{4}{T_J^2} \left( \left\| \Phi_{W,i}^{(T_J)} \right\|_F^2 + \left\| \Phi^{(T_J)} \right\|_F^2 \right) \\
\leq 6J(K)^2 \left( \frac{N}{1 - \rho_W} \right)^2 + \frac{4 n J(K)^2}{T_J} \left( \frac{2 \varphi(K)}{1 - \rho(A_K)} \right)^4.
\]

Now, the bound on \( \mathbb{E} \left[ (\mu_i(T_J) - J(K))^2 \right] \) follows from

\[
\mathbb{E} \left[ (\mu_i(T_J) - J(K))^2 \right] = (\mathbb{E}[\mu_i(T_J)] - J(K))^2 + \text{Var}(\mu_i(T_J)) \\
\leq \frac{J(K)^2}{T_J^2} \left[ \frac{N}{1 - \rho_W} + \left( \frac{2 \varphi(K)}{1 - \rho(A_K)} \right)^2 \right] + \text{Var}(\mu_i(T_J)) \\
\leq \frac{J(K)^2}{T_J^2} \left[ 2 \left( \frac{N}{1 - \rho_W} \right)^2 + 2 \left( \frac{2 \varphi(K)}{1 - \rho(A_K)} \right)^4 \right] + \text{Var}(\mu_i(T_J)).
\]

\[\square\]

**Lemma 10.** Let \( \bar{J} \) be any given constant such that

\[
\frac{\bar{J}}{J(K)} \geq \max \left\{ \frac{5}{2}, \frac{5N}{T_J(1 - \rho_W)} \right\}.
\]

Then

\[
0 \leq \mathbb{E}[\mu_i(T_J) - \min\{\mu_i(T_J), \bar{J}\}] \leq \frac{90J(K)}{T_J^2} \left[ n^2 \left( \frac{2 \varphi(K)}{1 - \rho(A_K)} \right)^8 + \frac{N^2}{(1 - \rho_W)^2} \right].
\]

**Proof.** We have shown in the proof of Lemma 9 that

\[
\left| \varpi(T_J)^\top \Phi_{W,i}^{(T_J)} \varpi(T_J) \right| \leq N \varpi(T_J)^\top \Phi_{\rho_W}^{(T_J)} \varpi(T_J).
\]
Therefore for any $\varepsilon_1 \geq 1$ and $\varepsilon_2 \geq 0$, we have

\[
\mathbb{P} \left( \mu_i(T_J) > (\varepsilon_1 + \varepsilon_2)J(K) \right) \\
\leq \mathbb{P} \left( \frac{1}{T_J} \varpi(T_J)^\top \left( \Phi(T_J) + N\Phi_{\rho_w}^{(T_J)} \right) \varpi(T_J) > (\varepsilon_1 + \varepsilon_2)J(K) \right) \\
\leq \mathbb{P} \left( \frac{1}{T_J} \varpi(T_J)^\top \Phi(T_J) \varpi(T_J) > \varepsilon_1 J(K) + \frac{1}{T_J} \text{tr} \left( \Phi(T_J) \right) - J(K) \right) \\
+ \mathbb{P} \left( \frac{N}{T_J} \varpi(T_J)^\top \Phi_{\rho_w}^{(T_J)} \varpi(T_J) > \varepsilon_2 J(K) + J(K) - \frac{1}{T_J} \text{tr} \left( \Phi(T_J) \right) \right) \\
\leq \mathbb{P} \left( \frac{1}{T_J} \varpi(T_J)^\top \Phi(T_J) \varpi(T_J) > \varepsilon_1 J(K) + \frac{1}{T_J} \text{tr} \left( \Phi(T_J) \right) - J(K) \right) \\
+ \mathbb{P} \left( \frac{N}{T_J} \varpi(T_J)^\top \Phi_{\rho_w}^{(T_J)} \varpi(T_J) > \varepsilon_2 J(K) \right),
\]

where we used $J(K) \geq T_J^{-1} \text{tr} \left( \Phi(T_J) \right)$ by (16), (22), Lemma 7 and the fact that $\Sigma_{K,\infty} \succeq \Sigma_{K,t}$.

For the first term, by using (17) and the bound $\|\Phi(T_J)\|_F \leq \sqrt{nT_J} \|\Phi(T_J)\|$, we get

\[
\mathbb{P} \left( \varpi(T_J)^\top \Phi(T_J) \varpi(T_J) > \text{tr} \left( \Phi(T_J) \right) + 2\|\Phi(T_J)\| \sqrt{nT_J} \varepsilon + 2\|\Phi(T_J)\| \varepsilon \right) \leq e^{-\varepsilon},
\]

and by letting $\varepsilon$ satisfy

\[
2\|\Phi(T_J)\| \sqrt{nT_J} \varepsilon + 2\|\Phi(T_J)\| \varepsilon = (\varepsilon_1 - 1)T_J J(K)
\]

for $\varepsilon_1 \geq 1$, we can get

\[
\mathbb{P} \left( \frac{1}{T_J} \varpi(T_J)^\top \Phi(T_J) \varpi(T_J) > \varepsilon_1 J(K) + \frac{1}{T_J} \text{tr} \left( \Phi(T_J) \right) - J(K) \right) \\
\leq \exp \left[ -\frac{1}{4} \left( \sqrt{\frac{\varepsilon_1 - 1}{\|\Phi(T_J)\|}} T_J J(K) + nT_J - \sqrt{nT_J} \right)^2 \right] \\
\leq \exp \left[ -\frac{1}{4} \min \left\{ \frac{\varepsilon_1 - 1}{\|\Phi(T_J)\|}, \frac{\varepsilon_1^2 - 2\varepsilon_1 T_J J(K)^2}{4n\|\Phi(T_J)\|^2} \right\} \right] \\
\leq \exp \left( -\frac{\varepsilon_1 - 1}{4\|\Phi(T_J)\|} T_J J(K) \right) + \exp \left( -\frac{(\varepsilon_1 - 1)^2 T_J J(K)^2}{16n\|\Phi(T_J)\|^2} \right),
\]

where we used

\[
\left( \sqrt{2\delta + nT_J} - \sqrt{nT_J} \right)^2 \geq \min \left\{ \delta, \frac{\delta^2}{4nT_J} \right\}, \quad \forall \delta \geq 0
\]

in the second inequality. For the second term, by using (17) and the bound $\|\Phi_{\rho_w}^{(T_J)}\| \leq \|\Phi_{\rho_w}^{(T_J)}\|_F \leq \text{tr} \left( \Phi_{\rho_w}^{(T_J)} \right)$, we get

\[
\mathbb{P} \left( \varpi(T_J)^\top \Phi_{\rho_w}^{(T_J)} \varpi(T_J) > \text{tr} \left( \Phi_{\rho_w}^{(T_J)} \right) \left( 1 + 2\sqrt{\varepsilon} + 2\varepsilon \right) \right) \leq e^{-\varepsilon},
\]

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and by letting
\[ \varepsilon = \frac{1}{4} \left( \sqrt{\frac{2T_J \varepsilon_2 J(K)}{N \text{tr} \left( \Phi_{\rho W}^{(T_J)} \right)} - 1} \right)^2 \]

for \( \varepsilon_2 \geq N \text{tr} \left( \Phi_{\rho W}^{(T_J)} \right) / (T_J J(K)) \), we get

\[ \mathbb{P} \left( \frac{N}{T_J} \omega (T_J) \Phi_{\rho W}^{(T_J)} \omega (T_J) > \varepsilon J(K) \right) \leq \exp \left[ - \frac{1}{4} \left( \sqrt{\frac{2 \varepsilon_2 T_J J(K)}{N \text{tr} \left( \Phi_{\rho W}^{(T_J)} \right)} - 1} \right)^2 \right] \leq \exp \left[ - \frac{1}{3} \left( \frac{\varepsilon_2 T_J J(K)}{N \text{tr} \left( \Phi_{\rho W}^{(T_J)} \right)} - 2 \right) \right] , \]

where we used
\[ \left( \sqrt{2\delta - 1} - 1 \right)^2 \geq \frac{4}{3} (\delta - 2), \quad \forall \delta > 1 \]
in the last inequality.

Therefore by letting \( \varepsilon_1 = 4 \varepsilon / 5 \) and \( \varepsilon_2 = \varepsilon / 5 \), we get

\[ \mathbb{P} (\mu_i(T_J) > \varepsilon J(K)) \leq \exp \left( - \frac{(4 \varepsilon / 5 - 1) T_J J(K)}{4 \| \Phi^{(T_J)} \|} \right) + \exp \left( - \frac{(4 \varepsilon / 5 - 1)^2 T_J J(K)^2}{16 n \| \Phi^{(T_J)} \|^2} \right) \]

\[ + \exp \left[ - \frac{1}{3} \left( \frac{\varepsilon J(K)}{5 N \text{tr} \left( \Phi_{\rho W}^{(T_J)} \right)} - 2 \right) \right] \]

for \( \varepsilon \geq 5 N \text{tr} \left( \Phi_{\rho W}^{(T_J)} \right) / (T_J J(K)) \). Now we have

\[ \mathbb{E} \left[ \mu_i(T_J) - \min \{ \mu_i(T_J), \bar{J} \} \right] \]

\[ = \int_0^{+\infty} \mathbb{P} (\mu_i(T_J) - \min \{ \mu_i(T_J), \bar{J} \} \geq x) \, dx \]

\[ = \int_0^{+\infty} \mathbb{P} (\mu_i(T_J) \geq \bar{J} + x) \, dx = J(K) \int_{J/\bar{J}}^{+\infty} \mathbb{P} (\mu_i(T_J) \geq \varepsilon J(K)) \, d\varepsilon. \]

By using the inequalities
\[ e^{-x} < \frac{1}{2x} \quad \text{and} \quad \int_x^{+\infty} e^{-u^2} \, du < \frac{e^{-x^2}}{2x} \quad \forall x > 0, \]

we can see that

\[ \int_{J/\bar{J}}^{+\infty} \exp \left( - \frac{(4 \varepsilon / 5 - 1) T_J J(K)}{4 \| \Phi^{(T_J)} \|} \right) \, d\varepsilon = \frac{5 \| \Phi^{(T_J)} \|}{T_J J(K)} \exp \left[ - \frac{T_J J(K)}{4 \| \Phi^{(T_J)} \|} \left( \frac{4 \bar{J}}{5J(K)} - 1 \right) \right] \]

\[ < \frac{10 \| \Phi^{(T_J)} \|^2}{T_J J(K)^2} \cdot \frac{1}{4J} \frac{1}{5J(K) - 1}, \]
\[
\int_{J/J(K)}^{+\infty} \exp \left( -\frac{(4\varepsilon/5 - 1)^2 T_J J(K)^2}{16n\|\Phi(T_J)\|^2} \right) \, d\varepsilon < \frac{10n\|\Phi(T_J)\|^2}{T_J J(K)^2} \exp \left[ -\frac{T_J J(K)^2}{16n\|\Phi(T_J)\|^2} \left( \frac{4J}{5J(K)} - 1 \right)^2 \right] \\
< \frac{80n^2\|\Phi(T_J)\|^4}{T_J^2 J(K)^4} \left( \frac{4J}{5J(K) - 1} \right)^3,
\]

and
\[
\int_{J/J(K)}^{+\infty} \exp \left( -\frac{1}{3} \left( \varepsilon T_J J(K) \left( \Phi(T_J) \right) - 2 \right) \right) \, d\varepsilon = \frac{15n^{2\rho_w} \text{tr} \left( \Phi(T_J) \right)}{T_J J(K)} \exp \left( -\frac{\bar{J} T_J}{15N \text{tr} \left( \Phi(T_J) \right)} \right) \\
< \frac{225N^2 \text{tr} \left( \Phi(T_J) \right)^2}{T_J^2 J(K)^2} \cdot \frac{J(K)}{J}.
\]

Finally, by Lemma 8 and the condition on \( \bar{J} \), we see that
\[
\mathbb{E} \left[ \mu_i(T_J) - \min \{ \mu_i(T_J), \bar{J} \} \right] \\
\leq \frac{J(K)}{T_J^2} \left[ 10 \left( \frac{2\varphi(K)}{1 - \rho(A_K)} \right)^4 + 80n^2 \left( \frac{2\varphi(K)}{1 - \rho(A_K)} \right)^8 + \frac{90N^2}{(1 - \rho_w)^2} \right] \\
\leq \frac{90J(K)}{T_J^2} \left[ n^2 \left( \frac{2\varphi(K)}{1 - \rho(A_K)} \right)^8 + \frac{N^2}{(1 - \rho_w)^2} \right].
\]

The inequality \( \mathbb{E} \left[ \mu_i(T_J) - \min \{ \mu_i(T_J), \bar{J} \} \right] \geq 0 \) is obvious.

\[\square\]

### D Preliminary Results on the Zero-Order Gradient Estimator

For simplicity of notation, we will denote the sublevel sets in Lemma 1 with \( \alpha = 10J(K) \) and \( \alpha' = 20J(K_0) \) as \( G^0 \) and \( G^1 \), i.e.,
\[
G^0 := \{ K \in \mathcal{K}_\text{st} : J(K) \leq 10J(K_0) \}, \quad G^1 := \{ K \in \mathcal{K}_\text{st} : J(K) \leq 20J(K_0) \}.
\]

We then define the constant \( \beta_0 \) that will appear in subsequent derivations by
\[
\beta_0 := \sup_{K \in G^1} \left( \frac{2\varphi(K)}{1 - \rho(A_K)} \right)^2.
\]

Lemmas 4, 5 and 6 ensure that \( \beta_0 \) is finite. We mention that \( \beta_0 \) will depend on the system parameters \( A, B, \Sigma_w, Q, R \) as well as the initial cost \( J(K_0) \).

We introduce the smoothed version of \( J(K) \) defined by
\[
J'(K) := \mathbb{E}_U[J(K + rU)], \quad U \sim \text{Uni}(\mathbb{B}_{n_K})
\]
for any $K \in \mathcal{K}_{\text{st}}$ such that $K + rB_n \subseteq \mathcal{K}_{\text{st}}$, where $B_n$ denotes the closed unit ball $\{ x \in \mathbb{R}^n : \|x\| \leq 1 \}$. The quantity $r$ will be referred to as the smoothing radius. By Flaxman et al. (2005, Lemma 1), we have

$$\nabla J^r(K) = \mathbb{E}_D \left[ \frac{nK}{r} J(K + rD)D \right], \quad D \sim \text{Uni}(S_n).$$

We’ll denote

$$\frac{\partial J^r}{\partial K_i}(K) := \mathbb{E}_D \left[ \frac{nK}{r} J(K + rD)D_i \right] \in \mathbb{R}^{n_i \times m_i}$$

where each $D_i$ denotes an $n_i \times m_i$ matrix such that

$$D = \begin{bmatrix} \text{vec}(D_1) \\ \vdots \\ \text{vec}(D_N) \end{bmatrix}.$$ 

This notation involving $D$ and $D_i$ will be used throughout the paper. 

Now, for any $K \in \mathcal{K}_{\text{st}}$, we define

$$(\tilde{J}_1(K), \ldots, \tilde{J}_N(K)) := \text{GlobalCostConsensus}((K_i)_{i=1}^N, T_J),
\tilde{J}_i(K) := \min \{ \tilde{J}_i(K), \tilde{J} \}$$

We also define

$$G^r_i(K, D) := \frac{nK}{r} \tilde{J}_i(K + rD)D_i, \quad G^r(K, D) := \begin{bmatrix} \text{vec}(G_1(K, D)) \\ \vdots \\ \text{vec}(G_N(K, D)) \end{bmatrix}$$

for any $K \in \mathcal{K}_{\text{st}}$ and $D \in \mathbb{R}^{nK}$. We can see that

$$\hat{g}_i(s) = \frac{1}{T_B} \sum_{b=1}^{T_B} G^r_i(K(s), D(s, b))D_i(s, b), \quad (23)$$

where $\hat{g}_i(s), K(s)$ and

$$D(s, b) := \begin{bmatrix} \text{vec}(D_1(s, b)) \\ \vdots \\ \text{vec}(D_N(s, b)) \end{bmatrix}$$

are the corresponding quantities generated by Algorithm $1'$.

We now continue the theoretical analysis, and first focus on the bias of the estimator $G^r(K, D)$ with respect to $\nabla J(K)$.

**Lemma 11.** Let $K \in G^0$ be arbitrary. We have

$$\| \nabla J^r(K) - \nabla J(K) \| \leq \phi_0 r$$

for any $r \leq \xi_0$. 

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Proof. We have
\[
\| \nabla J^r(K) - \nabla J(K) \| = \left\| \nabla \mathbb{E}_{U \sim \text{Uni}(\mathbb{B}_{nK})}[J(K + rU) - J(K)] \right\|
\]
\[
= \left\| \mathbb{E}_{U \sim \text{Uni}(\mathbb{B}_{nK})}[\nabla J(K + rU) - \nabla J(K)] \right\|
\]
\[
\leq \mathbb{E}_{U \sim \text{Uni}(\mathbb{B}_{nK})}[\| \nabla J(K + rU) - \nabla J(K) \|] \leq \phi_0 r.
\]
In the second equality we interchange the derivative with the expectation, which follows from the dominated convergence theorem as \( \nabla J \) is continuous and \( \mathbb{B}_{nK} \) is compact. \( \square \)

Lemma 12 (Bias of the gradient estimator). Suppose \( r \leq \xi_0, \tilde{J} \geq 50J(K_0) \) and
\[
T_J \geq \frac{10}{\delta} \max \left\{ n\beta_0^2, \frac{N}{1 - \rho_W} \right\},
\]
where \( \delta \in (0, 1) \) is arbitrary. Then for any \( K \in G^0 \), we have
\[
\| \mathbb{E}_D[\nabla^r(K, D)] - \nabla J(K) \|^2 \leq 5 \left[ \phi_0^2 r^2 + \left( \frac{20J(K_0) nK}{r^2} \right)^2 \right]
\]
where \( \mathbb{E}_D \) denotes expectation with respect to \( D \sim \text{Uni}(\mathcal{S}_{nK}) \).

Proof. Notice that
\[
\| \mathbb{E}_D[\nabla^r(K, D)] - \nabla J(K) \|^2 \leq \frac{5}{4} \| \mathbb{E}_D[\nabla^r(K, D)] - \nabla J(K) \|^2 + 5 \| \nabla^r(K) - \nabla J(K) \|^2
\]
\[
\leq \frac{5}{4} \sum_{i=1}^N \left\| \mathbb{E}_D[\nabla^r_i(K, D)] - \frac{\partial J^r}{\partial K_i}(K) \right\|^2_F + 5\phi_0^2 r^2,
\]
where the last inequality is by Lemma 11.

For the first term, we have
\[
\sum_{i=1}^N \left\| \mathbb{E}_D[\nabla^r_i(K, D)] - \frac{\partial J^r}{\partial K_i}(K) \right\|^2_F
\]
\[
= \sum_{i=1}^N \left\| \mathbb{E}_D \left[ \frac{nK}{r} \cdot \mathbb{E}_w \left[ \hat{J}_i(K + rD) - J(K + rD) \mid D \right] \cdot D_i \right] \right\|^2_F
\]
\[
\leq \frac{n^2 K}{r^2} \sum_{i=1}^N \sup \left\{ \left\| \mathbb{E}_w \left[ \hat{J}_i(K') \right] - J(K') \right\|^2 : K' \in K + r\mathcal{S}_{nK} \right\} \cdot \mathbb{E}_D[\|D_i\|^2_F]
\]
\[
\leq \frac{n^2 K}{r^2} \max_{1 \leq i \leq N} \left\{ \left\| \mathbb{E}_w \left[ \hat{J}_i(K') \right] - J(K') \right\|^2 : K' \in K + r\mathcal{S}_{nK} \right\},
\]
where \( \mathbb{E}_w \) denotes expectation with respect to the noise process of the dynamical system in the subroutine \texttt{GlobalCostConsensus}. 30
We then proceed to provide a uniform upper bound on \(|E_w[\tilde{J}_i(K')] - J(K')|^2\) for \(K' \in K + rS_n\) for \(1 \leq i \leq N\). Since \(r \leq \xi_0\) and \(K \in G^0\), we have \(K' \in K + rS_n \subseteq G^1 \subseteq K_{st}\). By applying Lemma 9, for any \(i\) and \(K' \in G^1\), we have

\[
\left| E_w[\tilde{J}_i(K')] - J(K') \right| 
\leq \left| E_w[\tilde{J}_i(K') - E\tilde{J}_i(K')] \right| + \left| E_w[\tilde{J}_i(K')] - J(K') \right|
\leq \frac{90J(K')}{T_J^2} \left[ n^2 \left( \frac{2\varphi(K')}{1 - \rho(A_{K'})} \right)^8 + \frac{N^2}{(1 - \rho W)^2} \right] + \frac{J(K')}{T_J} \left[ \left( \frac{2\varphi(K')}{1 - \rho(A_{K'})} \right)^2 + \frac{N}{1 - \rho W} \right]
\leq \frac{90J(K')}{T_J^2} \left( n^2 \beta_0^4 + \frac{N^2}{(1 - \rho W)^2} \right) + \frac{J(K')}{T_J} \left( \beta_0 + \frac{N}{1 - \rho W} \right)
\]

where we used Lemma 10 and the bound (20) in Lemma 9 in the second inequality, and the definition of \(\beta_0\) in the last inequality. Finally, by the condition (24) on \(T_J\) and \(J(K') \leq 20J(K_0)\) for \(K' \in K + rS_n\), we get

\[
\left| E_w[\tilde{J}_i(K')] - J(K') \right| 
\leq \frac{90J(K')}{T_J} \left( n\beta_0^2 + \frac{N}{1 - \rho W} \right) + \frac{J(K')}{T_J} \left( \beta_0 + \frac{N}{1 - \rho W} \right)
\leq \frac{20J(K_0)}{T_J} \left( (9n + 1)\beta_0 + (9\delta + 1)\frac{N}{1 - \rho W} \right)
\leq \frac{200J(K_0)}{T_J} \left( n\beta_0^2 + \frac{N}{1 - \rho W} \right) \leq 40J(K_0)\delta.
\]

By combining it with previous results, we obtain the desired bound. \(\square\)

**Lemma 13** (Second moment of the gradient estimator). Suppose \(r \leq \xi_0\), \(\bar{J} \geq 50J(K_0)\) and

\[
T_J \geq \frac{10}{\delta} \max \left\{ n\beta_0^2, \frac{N}{1 - \rho W} \right\},\]

where \(\delta \in (0, 1)\) is arbitrary. Then for any \(K \in G^0\), we have

\[
E_D \left[ \|G'(K, D)\|^2 \right] \leq \left( 40J(K_0)\frac{nK}{r} \right)^2,
\]

where \(E_D\) denotes expectation with respect to \(D \sim \text{Uni}(S_n)\).

**Proof.** It can be seen that

\[
E_D \left[ \|G'(K, D)\|^2 \right] = \sum_{i=1}^{N} E_D \left[ \|G'_i(K, D)\|^2 \right] = \sum_{i=1}^{N} E_D \left[ \frac{nK}{r^2} \|D_i\|^2_F E_w[\tilde{J}_i(K+rD)^2 \mid D] \right]
\leq \frac{nK}{r^2} \max_{1 \leq i \leq N} \left\{ E_w[\tilde{J}_i(K')^2] : K' \in K + rS_n \right\} \cdot E_D \left[ \sum_{i=1}^{N} \|D_i\|^2_F \right]
\leq \frac{nK}{r^2} \max_{1 \leq i \leq N} \left\{ E_w[\tilde{J}_i(K')^2] : K' \in K + rS_n \right\}.
\]
where \( \mathbb{E}_w \) denotes expectation with respect to the noise process of the dynamical system in the subroutine \( \text{GlobalCostConsensus} \), and the last inequality holds since \( 0 \leq \hat{J}_i(K') \leq J_i(K) \). Then, since for any \( K' \in K + rS_{nK} \) and any \( K \in \mathcal{G}^o \), we have

\[
\mathbb{E}_w[\hat{J}_i(K')^2] \leq 2 \mathbb{E}_w[(\hat{J}_i(K') - J(K'))^2] + 2J(K')^2
\]

\[
\leq 2 \left[ \frac{6nJ(K')^2}{T_j} \left( \frac{2\varphi(K')}{1 - \rho(A_{K'})} \right)^4 + \frac{8J(K')^2}{T_j^2} \left( \frac{N}{1 - \rho_W} \right)^2 \right] + 2J(K')^2
\]

\[
\leq 2 \left[ \frac{6n \cdot (20J(K_0))^2}{T_j} \beta_0^2 + \frac{8 \cdot (20J(K_0))^2}{T_j^2} \left( \frac{N}{1 - \rho_W} \right)^2 + (20J(K_0))^2 \right]
\]

\[
\leq 2 \cdot (20J(K_0))^2 \left[ \frac{6\delta}{10} + \frac{8\delta^2}{100} + 1 \right] < (40J(K_0))^2,
\]

where we used (21) in Lemma 9 in the second inequality, and used the definition of \( \beta_0 \) and the fact that \( J(K') \leq 20J(K_0) \) for any \( K' \in K + rS_{nK} \) in the third inequality, and the last two inequalities follows from the conditions of the lemma. By combining this bound with previous results, we get the bound on the second moment of \( G'(K, D) \).

Recalling that \( g_i(s) \) can be represented as in (23), we immediately have the following corollary regarding the conditional second moment of \( g_i(s) \):

**Corollary 14.** Let \( \mathcal{F}_s \) denote the filtration generated by \( (K_i(s') : s' \leq s) \). Suppose \( r \leq \xi_0, J \geq 50J(K_0) \) and

\[
T_j \geq \frac{10}{\delta} \max \left\{ n\beta_0^2, \frac{N}{1 - \rho_W} \right\},
\]

where \( \delta \in (0, 1) \) is arbitrary. Then

\[
\mathbb{E}[\|g_i(s)\|^2 | \mathcal{F}_s] \leq \frac{1}{T_B} \left( 40J(K_0) \frac{nK}{r} \right)^2 + 10 \left[ \beta_0^2r^2 + \left( \frac{20J(K_0)nK}{r} \delta \right)^2 \right] + 2\|\nabla J(K)\|^2
\]

**Proof.** Notice that

\[
\|\mathbb{E}_{D}[G'(K, D)]\|^2 \leq 2 \|\mathbb{E}_{D}[G'(K, D)] - \nabla J(K)\|^2 + 2\|\nabla J(K)\|^2.
\]

The conclusion then follows from Lemmas 12, 13 and the fact that for any i.i.d. random vectors \( X_1, \ldots, X_b \) with finite second moments,

\[
\mathbb{E} \left( \frac{1}{b} \sum_{i=1}^b X_i \right)^2 = \frac{1}{b} \mathbb{E} [\|X_1\|^2] + \frac{b - 1}{b} \|\mathbb{E}[X_1]\|^2 \leq \frac{1}{b} \mathbb{E} [\|X_1\|^2] + \|\mathbb{E}[X_1]\|^2.
\]

\[\square\]

## E Analysis of the Policy Gradient Procedure

Firstly, we introduce the following lemma which characterizes the improvement of one iteration of zeroth order policy gradient update. This lemma is essential to our proof.
**Lemma 15 (Key Lemma).** Let $\mathcal{F}_s$ denote the filtration generated by $(K_i(s') : s' \leq s)$, and suppose

$$0 < r \leq \xi_0, \quad 0 < \eta \leq \min \left\{ \frac{\xi_0 r}{n_K J}, \frac{1}{6\phi_0} \right\}, \quad \bar{J} \geq 50J(K_0)$$

and

$$T_J \geq 10 \frac{1}{\delta} \max \left\{ \frac{n\beta^2_0}{N}, \frac{N}{1 - p_W} \right\},$$

where $\delta \in (0, 1]$ is arbitrary. Then, as long as $K(s) \in G^0$, we will have $K(s + 1) \in G^1$ and

$$\mathbb{E}[J(K(s + 1)) | \mathcal{F}_s] \leq J(K(s)) - \frac{\eta}{3} \|\nabla J(K(s))\|_F^2 + \frac{\eta}{3} Z_\delta$$

where

$$Z_\delta := 10 \left[ \phi_0^2 \sqrt{2} + \left( \frac{20J(K_0)n_K \delta}{r} \right)^2 \right] + \frac{3\phi_0 \eta}{2T_B} \left( 40J(K_0) \frac{n_K}{r} \right)^2. \quad (25)$$

**Proof.** Firstly, since

$$\|K(s + 1) - K(s)\|^2 = \eta^2 \|\hat{g}(s)\|^2 = \eta^2 \sum_{i=1}^{N} \|\hat{g}_i(s)\|_F^2$$

$$\leq \eta^2 \sum_{i=1}^{N} \frac{1}{T_B} \sum_{b=1}^{n_K J} \frac{n_K J}{r^2} \|D_i(s, b)\|_F^2$$

$$= \eta^2 \left( \frac{n_K J}{r} \right)^2 \leq \xi_0^2.$$ 

we can see that $K(s + 1) \in G^1$ as long as $K(s) \in G^0$.

Secondly, since $K(s) \in G^0$, by locally smoothness in Lemma 1, we have

$$J(K(s + 1)) \leq J(K(s)) - \eta \langle \nabla J(K(s)), \hat{g}(s) \rangle + \frac{\phi_0}{2} \eta^2 \|\hat{g}(s)\|^2.$$

Taking expectation conditioning on the filtration $\mathcal{F}_s$ yields

$$\mathbb{E}[J(K(s + 1)) | \mathcal{F}_s] \leq J(K(s)) - \eta \langle \nabla J(K(s)), \mathbb{E}[\hat{g}(s) | \mathcal{F}_s] \rangle + \frac{\phi_0}{2} \eta^2 \mathbb{E} \left[ \|\hat{g}(s)\|^2 | \mathcal{F}_s \right]$$

$$= J(K(s)) - \eta \|\nabla J(K(s))\|^2 + \frac{\phi_0}{2} \eta^2 \mathbb{E} \left[ \|\hat{g}(s)\|^2 | \mathcal{F}_s \right]$$

$$+ \eta \langle \nabla J(K(s)), \nabla J(K(s)) - \mathbb{E}[\hat{g}(K(s)) | \mathcal{F}_s] \rangle$$

$$\leq J(K(s)) - \frac{\eta}{2} \|\nabla J(K(s))\|^2 + \frac{\phi_0}{2} \eta^2 \mathbb{E} \left[ \|\hat{g}(K(s))\|^2 | \mathcal{F}_s \right]$$

$$+ \frac{\eta}{2} \|\nabla J(K(s)) - \mathbb{E}[\hat{g}(s) | \mathcal{F}_s]\|^2.$$ 

By plugging in the bounds of Lemma 12 and Corollary 14, we get

$$\mathbb{E}[J(K(s + 1)) | \mathcal{F}_s]$$

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\[ \leq J(K(s)) - \frac{\eta}{2} \| \nabla J(K(s)) \|^2 + \frac{5\eta}{2} \left[ \phi_0^2 r^2 + \left( \frac{20J(K_0)n_K \delta}{r} \right)^2 \right] \\
+ \frac{\phi_0}{2} \eta \left( \frac{1}{T_B} \left( 40J(K_0) \frac{n_K}{r} \right)^2 + 10 \left[ \phi_0^2 r^2 + \left( \frac{20J(K_0)n_K \delta}{r} \right)^2 \right] + 2\| \nabla J(K) \|^2 \right) \\
\leq J(K(s)) - \frac{\eta}{3} \| \nabla J(K(s)) \|^2 + \frac{10\eta}{3} \left[ \phi_0^2 r^2 + \left( \frac{20J(K_0)n_K \delta}{r} \right)^2 \right] + \frac{\phi_0 \eta^2}{2T_B} \left( 40J(K_0) \frac{n_K}{r} \right)^2, \]

where we used the condition that \( \eta \phi_0 \leq 1/6 \).

Next, we introduce a stopping time \( \tau \) defined as the first time step when \( K(s) \) escapes \( G^0 \),

\[
\tau := \min \{ s \in \{1, \ldots, T_G + 1 \} : J(K(s)) > 10J(K_0) \}. \tag{26}
\]

We also introduce the notation of the indicator function \( 1_S \) of an event \( S \) such that \( 1_S = 1 \) when the event \( S \) occurs and \( 1_S = 0 \) when \( S \) does not occur.

Now we are ready for the following lemma. Roughly speaking, (27) of the lemma implies that either the averaged squared norm of the gradient in expectation is small, or the probability that \( K(s) \in G^0 \) for all \( s \) is small. The inequality (28) that the probability that \( K(s) \in G^0 \) for all \( s \) is large. We shall see later that combining the two inequalities yields a proof of Theorem 3.

**Lemma 16.** Suppose

\[ 0 < r \leq \frac{\sqrt{\epsilon}}{25\phi_0}, \quad 0 < \eta \leq \min \left\{ \frac{\xi_0 r}{n_K J}, \frac{1}{6\phi_0} \right\}, \quad \bar{J} \geq 50J(K_0), \]

and

\[ T_G = \left\lfloor \frac{60J(K_0)}{\eta \epsilon} \right\rfloor \]
\[ T_J \geq 5 \times 10^3 J(K_0) n_K \max \left\{ \frac{1}{n_K \beta_0^2}, \frac{N}{1 - \rho W} \right\}, \]
\[ T_B \geq \frac{250\eta \phi_0 n_K^2}{3\epsilon r^2} \left( 40J(K_0) \right)^2, \]

where \( \epsilon \) is an arbitrary positive number satisfying

\[ \epsilon \leq 625\phi_0 \cdot \min \left\{ \phi_0 \xi_0^2, 20J(K_0)n_K \right\} . \]

Then we have

\[
\mathbb{E} \left[ \left( \frac{1}{T_G} \sum_{s=1}^{T_G} \| \nabla J(K(s)) \|^2 \right) 1_{\{ \tau > T_G \}} \right] \leq \frac{\epsilon}{10}, \tag{27}
\]

and

\[
\mathbb{P}(\tau \leq T_G) \leq \frac{1}{5}. \tag{28}
\]
Proof. Denote $J^* := \inf_{K \in K^*} J(K)$ and
\[
\Delta(s) := J(K(s)) - J^*
\]
for $1 \leq s \leq T_G + 1$.

To prove (27), we establish the following inequality for the optimality gap:
\[
E[\Delta(s + 1)1_{\{\tau > s+1\}} \mid \mathcal{F}_s] \leq \Delta(s)1_{\{\tau > s\}} - \frac{\eta}{3} \|\nabla J(K(s))\|^2 1_{\{\tau > s\}} + Z_\delta,
\]
where we set
\[
\delta = \frac{r\sqrt{\epsilon}}{500J(K_0)n_K}.
\]

We consider two scenarios:

**Scenario 1:** $\tau > s$. By the definition of $\tau$, we have $K(s) \in G^0$, and it can also be checked that the other conditions of Lemma 15 hold. Therefore $1_{\{\tau > s\}} = 1$, which leads to
\[
E[\Delta(s + 1)1_{\{\tau > s+1\}} \mid \mathcal{F}_s] \leq E[\Delta(s)1_{\{\tau > s\}} - \frac{\eta}{3} \|\nabla J(K(s))\|^2 1_{\{\tau > s\}} + \frac{\eta}{3}Z_\delta,
\]

**Scenario 2:** $\tau \leq s$. In this case $1_{\{\tau > s\}} = 1_{\{\tau > s+1\}} = 0$. Since $Z > 0$, we trivially have
\[
E[\Delta(s + 1)1_{\{\tau > s+1\}} \mid \mathcal{F}_s] = 0 = \Delta(s)1_{\{\tau > s\}} - \frac{\eta}{3} \|\nabla J(K(s))\|^2 1_{\{\tau > s\}} + \frac{\eta}{3}Z_\delta.
\]

Summarizing the two scenarios, we have proved (29).

By taking the total expectation of (29), we have
\[
E[\Delta(s + 1)1_{\{\tau > s+1\}}] \leq E[\Delta(s)1_{\{\tau > s\}} - \frac{\eta}{3} \|\nabla J(K(s))\|^2 1_{\{\tau > s\}}] + \frac{\eta}{3}Z_\delta.
\]

By reorganizing terms, we get
\[
E[\|\nabla J(K(s))\|^2 1_{\{\tau > s\}}] \leq \frac{3}{\eta} E[\Delta(s)1_{\{\tau > s\}} - \Delta(s + 1)1_{\{\tau > s+1\}}] + Z_\delta,
\]
and by taking the telescoping sum, we obtain
\[
\frac{1}{T_G} \sum_{s=1}^{T_G} E[\|\nabla J(K(s))\|^2 1_{\{\tau > T_G\}}] \leq \frac{3}{\eta} \frac{1}{T_G} \sum_{s=1}^{T_G} E[\Delta(s)1_{\{\tau > s\}} - \Delta(s + 1)1_{\{\tau > s+1\}}] + Z_\delta
\]
\[
\frac{1}{T_G} \sum_{s=1}^{T_G} E[\|\nabla J(K(s))\|^2 1_{\{\tau > s\}}] \leq \frac{3}{\eta T_G} \Delta(1) + Z_\delta
\]
It’s not hard to see that
\[
\frac{3}{\eta T_G} \Delta(1) \leq \frac{3}{\eta} \cdot \frac{\eta e}{60J(K_0)} \Delta(1) \leq \frac{e}{20}.
\]
and we also have
\[
Z_\delta \leq 10 \left[ \phi_0^2 \eta^2 + \left( \frac{20J(K_0)n_K}{r} \frac{r \sqrt{\epsilon}}{500J(K_0)n_K} \right)^2 \right] \\
+ \frac{3 \phi_0 \eta}{2} \left( \frac{40J(K_0)n_K}{r} \right)^2 \cdot \frac{3 \epsilon r^2}{250 \eta \phi_0 n_K^2 (40J(K_0))^2} \\
\leq 10 \left( \frac{\epsilon}{625} + \frac{\epsilon}{625} \right) + \frac{9 \epsilon}{500} = \frac{\epsilon}{20}
\]
Consequently, we have established (27).

To prove (28), we define a nonnegative supermartingale \( Y(s) \) as follows. For \( 1 \leq s \leq T_G \), let
\[
Y(s) := J(K(\min\{s, \tau\})) + (T_G - s) \cdot \frac{\eta}{3} Z_\delta.
\]
It is straightforward to verify that \( Y(s) \in [0, +\infty) \) for \( 1 \leq s \leq T_G \). To verify that it is a supermartingale, we notice that when \( \tau > s \),
\[
\mathbb{E}[Y(s + 1) \mid \mathcal{F}_s] = \mathbb{E}[J(K(s + 1)) \mid \mathcal{F}_s] + (T_G - s - 1) \cdot \frac{\eta}{3} Z_\delta \\
\leq J(K(s)) - \frac{\eta}{3} \|\nabla J(K(s))\|^2 + \frac{\eta}{3} Z_\delta + (T_G - 1 - s) \cdot \frac{\eta}{3} Z_\delta \\
\leq Y(s)
\]
and when \( \tau \leq s \),
\[
\mathbb{E}[Y(s + 1) \mid \mathcal{F}_s] = \mathbb{E}[J(K(\tau)) \mid \mathcal{F}_s] + (T_G - s) \cdot \frac{\eta}{3} Z_\delta \\
\leq \mathbb{E}[J(K(\tau)) \mid \mathcal{F}_s] + (T_G - s) \cdot \frac{\eta}{3} Z_\delta = Y(s).
\]
Now, by the monotonicity and Doob’s maximal inequality for supermartingales, we have
\[
P(\tau \leq T_G) \leq P \left( \max_{s=1,\ldots,T_G} Y(s) > 10J(K_0) \right) \leq \frac{\mathbb{E}[Y(1)]}{10J(K_0)} \\
= \frac{J(K_0) + (T_G - 1) \epsilon Z_\delta / 3}{10J(K_0)} \leq \frac{1}{5},
\]
where the last inequality follows from
\[
(T_G - 1) \frac{\eta}{3} Z_\delta = \left( \frac{60}{\eta \epsilon} J(K_0) \right) - 1 \right) \cdot \frac{\eta}{3} Z_\delta \leq \frac{60}{\eta \epsilon} J(K_0) \cdot \frac{\eta \epsilon}{60} = J(K_0).
\]

Given Lemma 16, the proof of Theorem 3 is straightforward.

**Proof of Theorem 3.** We have
\[
P \left( \frac{1}{T_G} \sum_{s=1}^{T_G} \|\nabla J(K(s))\|^2 \geq \epsilon \right)
\]
\[ \begin{align*}
&= \mathbb{P} \left( \frac{1}{T_G} \sum_{s=1}^{T_G} \| \nabla J(K(s)) \|^2 \geq \epsilon, \tau > T_G \right) + \mathbb{P} \left( \frac{1}{T_G} \sum_{s=1}^{T_G} \| \nabla J(K(s)) \|^2 \geq \epsilon, \tau \leq T_G \right) \\
&\leq \mathbb{P} \left( \frac{1}{T_G} \sum_{s=1}^{T_G} \| \nabla J(K(s)) \|^2 1_{\{\tau > T_G\}} \geq \epsilon \right) + \mathbb{P} (\tau \leq T_G) \\
&\leq \frac{1}{\epsilon} \mathbb{E} \left[ \frac{1}{T_G} \sum_{s=1}^{T_G} \| \nabla J(K(s)) \|^2 1_{\{\tau > T_G\}} \right] + \frac{1}{5} \leq \frac{3}{10},
\end{align*} \]

where we used Markov’s inequality in the second inequality. \( \square \)

\section{F Details of the Numerical Experiments}

In this section we provide detailed settings of the numerical experiments carried out in Section 5.

Consider a multi-zone building with HVAC systems that can supply cooling air to each zone with adjustable air flow rates. Each zone is equipped with a sensor that can measure the local temperatures, and can adjust the supply air flow rate of its associated HVAC system. The dynamics of the room’s temperatures we employ are a discrete-time version of the linear model in Zhang et al. (2016b) given by

\[ v_i(x_i(t+1) - x_i(t)) = \frac{\theta^o(t) - x_i(t)}{\zeta_i} + \sum_{j=1}^{N} \frac{x_j(t) - x_i(t)}{\zeta_{ij}} + u_i(t) + \pi_i \Delta + w_i(t) \cdot \sqrt{\Delta}. \]  

(30)

Here \( x_i(t) \) denotes the temperature of zone \( i \) at time \( t \), \( u_i(t) \) controls the air flow rate of the HVAC system in zone \( i \), \( \theta^o(t) \) denotes the outdoor temperature, \( \pi_i \) represents a constant heat from external sources for zone \( i \), and \( w_i(t) \) represents random fluctuations. The quantities \( v_i, \zeta_i \) are constants derived from laws of physics, and \( \Delta > 0 \) represents the time discretization resolution. The local costs are given by

\[ c_i(t) = \frac{1}{2} (x_i(t) - \theta^*_i)^2 + \alpha u_i(t)^2, \quad i = 1, \ldots, N, \]  

(31)

where \( \theta^*_i \) denotes the desired temperature set by users, and \( \alpha > 0 \) is a trade-off parameter.

The numerical experiment is carried out on an \( N = 4 \) test case with \( \theta^*_i = 22 \, ^\circ\text{C} \) for all 4 rooms, and we set \( \alpha = 0.01 \, \text{C}^2/(kJ^2 \cdot s^2) \). The values of the model parameters are as follows:

\[ v_i = 200 \, \text{kJ}/\text{C}, \quad \frac{1}{\zeta_i} = 1 \, \text{kJ}/(\text{C} \cdot \text{s}), \quad \begin{bmatrix} 1 \\ \zeta_{ij} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \, \text{kJ}/(\text{C} \cdot \text{s}), \]

and we let \([\pi_1, \ldots, \pi_4] = [1, 2, 3, 4] \, \text{kJ/s} \) and \( \Delta = 60s \). Besides, we let \( w_i(t) \) follows the Gaussian distribution with zero mean and variance \( 2.5 \, \text{kJ}^2/\text{s} \). The communication matrix \( W \) employed in ZODPO is

\[ W = \begin{bmatrix} 1/2 & 1/4 & 1/4 & 0 \\ 1/4 & 1/2 & 0 & 1/4 \\ 1/4 & 0 & 1/2 & 1/4 \\ 0 & 1/4 & 1/4 & 1/2 \end{bmatrix}. \]
Scenario 1: Constant outdoor temperature. In this scenario, we set the outdoor temperature to be $\theta_o(t) = 30^\circ\text{C}$ for all $t$. The decentralized controller of each agent is an affine controller of the form $u_i(t) = K_i x_i(t) + b_i$, where $K_i$ and $b_i$ are the controller parameters to be determined. We incorporate a constant term $b_i$ in the controller to deal with the constant term in the thermal dynamics (30) and the temperature setpoint in the cost function (31). It is straightforward to adapt ZODPO to affine controllers. We set $T_J = 300$, $T_B = 1$. The step size in ZODPO is set to be $\eta = 2.5 \times 10^{-5}$, and we omit the truncation step in (11).

Scenario 2: Varying outdoor temperature. In this scenario, we consider time-varying outdoor temperatures. We consider local controllers that adapt to the time-varying temperatures, i.e. $u_i(t) = K_i x_i(t) + K_i^o \theta_o(t) + b_i$, where $\theta_o(t)$ represents the outdoor temperature. When learning the controller parameters $K_i$, $K_i^o$ and $b_i$, we consider different outdoor temperatures in different policy gradient iterations, but keep the outdoor temperature fixed within one iteration for better training. When plotting the figure, we implement the output controller on real outdoor temperature data collected by Harvard HouseZero Program, which may change every minute. We set $T_J = 300$, $T_B = 1$, $\eta = 5 \times 10^{-7}$ in this scenario, and omit the truncation step in (11).