Tidal instability in a rotating and differentially heated ellipsoidal shell

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SUMMARY

The stability of a rotating flow in a triaxial ellipsoidal shell with an imposed temperature difference between inner and outer boundaries is studied numerically. We demonstrate that (i) a stable temperature field encourages the tidal instability, (ii) the tidal instability can grow on a convective flow, which confirms its relevance to geo- and astrophysical contexts and (iii) its growth rate decreases when the intensity of convection increases. Simple scaling laws characterizing the evolution of the heat flux based on a competition between viscous and thermal boundary layers are derived analytically and verified numerically. Our results confirm that thermal and tidal effects have to be simultaneously taken into account when studying geophysical and astrophysical flows.

Key words: Tides and planetary waves; Heat flow; Core, outer core and inner core.

1 INTRODUCTION

The tidal or elliptical instability comes from a triadic parametric resonance between two inertial waves of a rotating fluid and an imposed elliptic deformation (see for instance the review by Kerswell 2002, and references therein). It is a generic phenomenon in turbulence (e.g. Widnall et al. 1974; Moore & Saffman 1975) and vortex dynamics (e.g. Leweke & Williamson 1998; Le Dizès & Laporte 2002; Meunier et al. 2002). There, the elliptical deformation comes from interactions between adjacent vortices. However this instability is also studied in geophysical and astrophysical tidally deformed bodies: for example, its presence has been suggested (i) in binary stars (Rieutord 2003; Le Bars et al. 2010) and accretion discs (Goodman 1993; Ryu & Goodman 1994), where they could participate in the energy and angular momentum exchanges between neighbouring systems (Lubow et al. 1993), and (ii) in planetary cores (e.g. Aldridge et al. 1997; Cébron et al. 2010), where they could play an important role in the induction of a magnetic field (Kerswell & Malkus 1998; Lacaze et al. 2006; Herreman et al. 2009) and even in planetary dynamos (Malkus 1993). Previous studies of tidal instability have been performed for an isothermal fluid, starting from a solid body rotation or from an homogeneous isotropic turbulent rotating flow (Fabijonas & Holm 2003). Nevertheless, in all natural systems, temperature differences are also present, which lead either to stratification or to convection. The previous studies need then to be reinvestigated to quantify the impact of a temperature field on the elliptical instability. Conversely, from a thermal point of view, many studies have been performed regarding the convective flow of an incompressible homogeneous fluid in a rotating spherical shell (see for instance Aurnou 2007, and references therein). However, since most astrophysical bodies are tidally deformed, these studies have to be reinvestigated to account for the potential presence of an elliptical instability. Note that a link between tidal resonance of gravito-inertial waves and thermal effects has been proposed by Kumazawa et al. (1994) in a geophysical context. However, this study considers the resonant forcing of a given wave by tides, which is different from the elliptical instability possibly induced by parametric resonance. Then, to the best of our knowledge, the only previous work which combines thermal and elliptical effects is the theoretical study of Le Bars & Le Dizès (2006), who considered the influence of an established diffusive temperature profile. They then interpreted the thermo-elliptical instability in terms of gravito-inertial waves resonance and determined its linear growth rate by a local approach. To complete this work, a guideline of our study is given by the following questions: (i) How is the growth rate of the tidal instability modified by an imposed temperature difference in the spheroidal geometry? (ii) Does the tidal instability grow over an established convective flow? (iii) Is the heat flux modified by the instability, and what are the scaling laws involved? In this paper, we tackle these questions using a systematic numerical study of the thermo-elliptical rotating flow in a fixed triaxial ellipsoidal shell. The numerical model is presented in Section 2. The variations of the heat flux and the growth rate are systematically quantified and interpreted in Section 3. Finally, orders of magnitude and conclusions are drawn for geo- and astrophysical flows in Section 4.

2 NUMERICAL MODEL AND ITS VALIDATION

As sketched in Fig. 1, we consider the rotating flow inside a fixed triaxial ellipsoid of axes \((a, b, c)\), with a constant tangential velocity \(U \sqrt{1 - (z/c)^2}\) imposed along the outer boundary in each plane of coordinate \(z\) perpendicular to the rotation axis \((Oz)\), where \(U\) is the imposed boundary velocity at the equator. An homothetic ellipsoidal core, in a ratio \(\eta = 0.3\), is placed at the centre, with also
an imposed constant tangential velocity \( \eta \) along its boundary. In the following, we use the mean equatorial radius \( R_{eq} = (a + b)/2 \) as a lengthscale and we introduce the timescale \( \Omega^{-1} \) by writing the tangential velocity along the deformed outer boundary at the equator \( U = \Omega R_{eq} \). Then, the problem is fully described by the adimensionalized Navier–Stokes equations with four dimensionless numbers: the eccentricity \( \epsilon = (b^2 - a^2)/a(b^2) \), the inner to outer core ratio \( \eta \), the aspect ratio \( c/b \) and the Reynolds number \( Re = \Omega R_{eq}^2/\nu \), or its inverse the Ekman number \( E_k \), where \( \nu \) is the kinematic viscosity of the fluid. In all the computations presented here, we choose \( c \) equal to the mean equatorial radius \( R_{eq} = (a + b)/2 \), as in classical experimental setups (see e.g. Lacaze et al. 2005). This allows us to focus on the well-known stationary spin-over mode of the tidal instability. Note that for lots of tidally deformed planetary and stellar bodies, the c-axis is the shortest axis because of the rotational flattening, whereas the longest axis is the one pointing towards the tide-raising body. In this configuration, the first selected mode of the tidal instability close to threshold may give rise to oscillatory motions. Nevertheless, we expect the main physical processes described here to remain valid for these other modes.

In addition to these purely hydrodynamic considerations, we also solve the temperature equation with constant and uniform temperatures \( T_i \) and \( T_e \) applied, respectively, at the inner and outer boundaries. To model the buoyancy, the density is assumed to vary linearly with the temperature and the Boussinesq approximation is used. Then, the gravity field shown here in the equatorial plane is calculated explicitly, assuming that the outer boundary is an isopotential.

The numerical procedure is the same for all the computations presented in this paper: starting from a fluid at rest in a spherical shell of radius \( R_{eq} \) with a diffusive temperature profile, the shell is set in rotation at a constant angular velocity \( \Omega \) and a first statistically stationary state is obtained by solving the Navier–Stokes and temperature equations; then, the spherical shell is instantaneously deformed to the chosen triaxial ellipsoidal shape and we follow the dynamics. Note that this instantaneous deformation allows to study the growth rate of the elliptical instability on a well-known conductive or convective base flow. However we have also performed computations with a gradual increase of the deformation, as well as computations starting from a deformed shell with a fluid at rest before imposing the rotation at the boundaries: the classical isothermal tidal instability then grows, which is only slightly modified by then imposing a temperature difference. In all cases, the final steady-state is the same.

Two quantities are systematically studied: (i) the growth rate \( \sigma \) of the tidal instability, which quantifies the feedback of the temperature field on the elliptical instability and (ii) the Nusselt number \( Nu \), which quantifies the feedback of the elliptical instability on the thermal properties of the system. Since the elliptical

\[ \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \frac{1}{Re} \Delta \mathbf{u} - \frac{Ra}{Pr Re^2(1 - \eta)} T g, \]

\[ \nabla \cdot \mathbf{u} = 0, \]

\[ \frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T = \frac{1}{Pr Re} \Delta T, \]

\[ \Delta \phi = 3, \]

\[ g = -\nabla \phi, \]

where \( T \) is the dimensionless temperature anomaly \( T = (T_e - T_i)/(T_e - T_i) \) and all boundary conditions have been described earlier.

This numerical setup is a simple model for a tidally deformed liquid planetary core with a solid inner core and an imposed temperature contrast. Note that in a geophysical context, because of the compressibility, the dimensionless temperature difference \( \Delta T \) is linear for the pressure field but quadratic for the velocity field. The temporal solver is IDA (Hindmarsh et al. 2005), based on backward differencing formulae. At each time step the system is solved with the sparse direct linear solver PARDISO.

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instability induces a strong 3-D destabilization of the initial flow, the growth rate \( \sigma \) is determined as the time constant of the exponential growth of the mean value of the vertical velocity \( W = \frac{1}{L} \int_J \vert u \vert \mathrm{d}r \), where \( w \) stands for the dimensionless vertical velocity and \( V \) for the volume of the ellipsoidal shell. The Nusselt number is calculated by dividing the measured mean heat flux at the outer boundary (once a statistically stationary state is reached) by the theoretical flux and the numerically calculated flux for a purely conductive ellipsoid is below 2 per cent for our all simulations. It has systematically been checked that the difference between this analytically known flux and the numerically calculated flux for a purely conductive ellipsoid is below 2 per cent for our all simulations.

The purely hydrodynamic part of the numerical simulation has already been validated in Cébron et al. (2010). A complementary validation of the resolution of thermal effects has been done by computing the heat flux in a rotating spherical shell: in Fig. 2(a), our results at \( \text{Re} = 344 \) are compared with the compilation of results given in Aurnou (2007) for the modified Nusselt number defined by \( Nu^* = \frac{Nu}{E} \), where \( E = \frac{\nu}{\kappa \alpha} \) is the Ekman number based on the gap width, that is, \( E = \frac{1}{(1 - \eta)^2 \text{Re}} \). Note that our numerical results fill an area with few data and agree with the general trend. Besides, our results are compared in Fig. 2(b) with the scaling law proposed by Christensen (2002) obtained from numerical simulations

\[
Nu^* = 0.077 (Ra_0^{8/9})^{1/2}, \tag{6}
\]

where \( Ra_0 = RaNuE^3/Pr^2 \) is the Rayleigh number based on the heat flux instead of the temperature contrast and which does not take into account any viscous or thermal diffusive effect. An excellent quantitative agreement is found.

3 INTERACTION BETWEEN THE ELLIPTICAL INSTABILITY AND THE THERMAL FIELD

In their analytical WKB (Wentzel–Kramers–Brillouin) [Correction added after online publication 2010 July 30: this expansion of ‘WKB’ has been corrected] analysis of the linear stability of elliptical streamlines in the presence of a diffusive temperature field, Le Bars & Le Dizès (2006) found an analytical expression of the inviscid growth rate of the elliptical instability in the limit of small deformations as

\[
\sigma = \frac{9 - 3Ra}{16 - 4Ra} \epsilon, \tag{7}
\]

where \( Ra = \frac{\nu - 1}{\nu + 1} \frac{\kappa}{\alpha} \epsilon^2 \) is a modified Rayleigh number. The expression 7 reflects the modification by thermal effects of the elliptical instability, which in this context corresponds to the parametric resonance of two gravito-inertial waves with the tidal deformation. Le Bars & Le Dizès (2006) also showed that the elliptical instability cannot grow for \( Ra > 3 \) because no resonance between inertial modes is then possible. As shown in Fig. 3(a), the same decreasing trend with \( Ra \) is recovered in the spherical shell, even if the results differ quantitatively from this theoretical local estimate. The first noticeable result from our complete calculations is that the growth rate of the elliptical instability is significantly enhanced by the thermal stratification (i.e. \( Ra < 0 \)), as first suggested by Le Bars & Le Dizès (2006). It may seem at first surprising that an \( a \ priorei \) stabilizing effect leads to an encouragement of flow destabilization, but such a behaviour has already been shown for instance in a Taylor–Couette flow, where a stable configuration can be destabilized by the presence of an axial stratification, leading to the so-called strato-rotational instability (see for instance the recent experiments by Le Bars & Le Gall 2007). The second result of planetary relevance is that the tidal instability can still grow on an established convective flow. As shown in Fig. 4, the organization of the flow is then completely changed. Indeed, in the absence of tidal instability, the convective flow organizes in a classical columnar structure shown in Fig. 4(a) (the so-called Busse columns), where motions are mainly bidimensional and vertical velocities remain small, coming only from the Ekman pumping inside the columns. On the contrary in Fig. 4(b), Busse columns are completely destroyed by the tidal instability, whose velocity field is fully 3-D. The growth rate \( \sigma \) of the elliptical instability progressively decreases when the Rayleigh number increases as shown in Fig. 3(a). Indeed, the thermal convective motions can be seen as a perturbation superimposed on the classical isothermal base flow of the elliptical instability, corresponding to a rotation along elliptical streamlines: when the Rayleigh number increases, the typical convective velocity increases and the ellipticity of the base flow streamlines is less and less felt by the fluid particles. The dimensionless version of this model can be quantified in defining the typical Rossby number of the flow, that is, the ratio between the rms convective velocity and the typical rotating velocity \( \Omega Re_0^* \), where \( \Omega \) is the dimensionless velocity in the rotating frame. One would expect to find a critical Rossby number of order one for the
disappearance of the tidal instability. As shown in Fig. 3(b), the critical value $Ro \approx 0.25$, where the growth rate is zero, is indeed found for $Pr = 1$ and $Re = 344$, which is near the threshold of the tidal instability. The critical Rossby number then seems to increase with the Reynolds number (see Fig. 3a for $Re = 688$), if it exists at all. Indeed, when the Rayleigh number increases, the typical convective velocity also increases whereas the typical lengthscale of convection $L_{con}$ simultaneously decreases. In the limit of very large Rayleigh numbers, $L_{con}$ is very small and turbulent motions become fully tridimensional. We then expect thermal convective motions to be seen as an effective turbulent viscosity, where the results of Fabijonas & Holm (2003) predict an encouragement of the tidal destabilization. The extensive study of this regime is very interesting, but currently out of reach of our numerical resources. Nevertheless, our current results allow to use the critical Rossby number $Ro_c = 0.25$ determined previously for $Re = 344$ as a lower bound for flow at larger Reynolds numbers. This result is sufficient for the planetary applications that we are interested in, as will be shown in the next section.

When thermal and tidal instabilities are simultaneously present, heat transfers are determined by the competition between the natural thermal convection and the forced convection due to the elliptical instability. In the latter case, the flow driven by the instability can be schematically described by three distinct regions: a fully mixed and
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Figure 3. (a) Influence of the Rayleigh number on the growth rate of the tidal instability for $\epsilon = 0.317$, $c = 4.45$, $Pr = 1$ and $Re = 344$ or $Re = 688$. The growth rate given by eq. (7) is also shown, taking into account a classical damping by a surface viscous term $-K/\sqrt{E_k}$, where $K$ is a constant of order 1 (Lacaze et al. 2005). The values of $K$ have been determined by matching the theoretical value with the numerical one at $Ra = 0$, giving $K = 3.07$ for $Re = 344$ and $K = 2.91$ for $Re = 688$. (b) Evolution of the growth rate of the tidal instability with the Rossby number, calculated following the scaling law given by Christensen & Aubert (2006) $Ro = 0.85(Ra^*Q)^{0.41}$. For $Re = 344$, the instability disappears for $Ro \approx 0.25$.

isothermal bulk, and viscous boundary layers at the outer and inner boundaries, where the temperature is purely diffusive. The advective heat flux driven by the tidal instability is then determined by the size of the viscous boundary layers $\delta_v$, following $Nu - 1 \sim d/\delta_v$. With the classical scaling law of viscous boundary layers, we obtain $Nu - 1 = \alpha_1/\sqrt{E^*}$, where $\alpha_1$ is a constant, $E^* = u_\epsilon/\nu$ the Ekman number associated with the flow driven by the instability and $u_\epsilon$ the fluid velocity due to the elliptical instability. As shown in Cébron et al. (2010), this velocity $u_\epsilon$ may be computed numerically in the absence of temperature gradient by subtracting to the computed flow the theoretical base flow corresponding to an imposed rotation along elliptical streamlines. $\frac{Nu}{\sqrt{Re_\epsilon}}$ generally scales as $\alpha_2(Re/Re_\epsilon - 1)^{1/4}/\sqrt{Re}$. [Correction added after online publication 2010 July 30: this term has been corrected] near the threshold of instability, where $\alpha_2 \approx 0.036$ has been determined numerically and where $Re_\epsilon$ is the critical Reynolds number for the onset of tidal instability equal to $(5.24/\epsilon)^2$.

Our numerical results shown in Fig. 5(a) validate this scaling with $\alpha_1 \approx 0.01$. Far from threshold, $E^* = E$ and the scaling law becomes $Nu = \alpha_1/\sqrt{E}$.

Note that according to this model, the Nusselt number is a measure of the amplitude of the instability. Then, the agreement between our results and the scaling law confirms that the amplitude of the instability is not inhibited by a strong stratification, provided that the Reynolds number is far enough from the threshold.

The systematic evolution of the Nusselt number with the Rayleigh number is shown in Fig. 5(b). In the absence of tidal and convective instabilities, $Nu$ remains equal to 1. In the presence of tidal instability, the heat flux remains constant at a value larger than 1 up to a

Figure 4. Comparison between the convective flows in the absence of tidal instability (a) $\epsilon = 0$ and in the presence of tidal instability (b) $\epsilon = 0.317$ for the same values of the Reynolds number $Re = 344$, the Rayleigh number $Ra = 10762$ and the Prandtl number $Pr = 1$. The vertical velocity is shown in three different planes. Also shown in (b) is an isosurface of the norm of the velocity $||u|| = 0.19$, which clearly exhibits the ‘S’ shape of the spin-over mode.

in the isothermal case with a stationary bulge. $\frac{Nu}{\sqrt{Re_\epsilon}}$ then saturates towards 1 far from the threshold. Keeping the same scaling in the presence of thermal effects, we expect the Nusselt number to vary as $Nu - 1 = \alpha_1(1 - \eta)\left(\frac{Re}{Re_\epsilon} - 1\right)^{1/4}/\sqrt{Re}$. (8)
transition Rayleigh number which depends on $Re$, where the natural convection becomes more efficient than the forced one. Indeed, (8) and (9) describe the heat flux by the forced convection only, and are valid in particular for stably stratified regimes and before the onset of convection $Ra < Ra$. However following King et al. (2009), once $Ra > Ra$, and convective and tidal instabilities are competing, the heat transfer is determined by the most efficient mechanism. Following scaling laws (6) and (9), the transition between the forced convection and the natural convection appears for the transition Nusselt number $Nu^* = \frac{NuE}{Pr} = 0.077 (Ra)^{5/9} = \alpha_1 \frac{E}{Pr}$, which corresponds to the transition Rayleigh number

$$Ra_t \approx 2.5 E^{-8/5} Pr^{1/5}.$$  \hspace{1cm} (10)

For $Ra > Ra_t$, heat transfers are controlled by the natural convective motions, whereas for $Ra < Ra_t$, heat transfers are controlled by the tidal instability. For planetary applications, it is more convenient to write this result in term of flux Rayleigh number defined by $Ra_f = Ra Nu$, for which the transition appears at

$$Ra_f \approx 0.025 E^{-21/10} Pr^{1/5}. \hspace{1cm} (11)$$

### 4 PLANETARY AND STELLAR APPLICATIONS

As explained in detail by Sumita & Yoshida (2002), a stable density stratification may have existed in the whole early Earth outer core and could even have been sufficiently large to prevent...

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**Figure 5.** (a) Influence of the Reynolds number on the Nusselt number for a convective flow with $\varepsilon = 0.317$, $c = \frac{a+b}{2}$, $Pr = 1$ and $Ra = 18762$. (b) Evolution of the Nusselt number with the Rayleigh number in spherical and ellipsoidal ($\varepsilon = 0.317$, $c = \frac{a+b}{2}$) shells for $Pr = 1$ and $Re = 344$ or $Re = 688$. Note that the classical dependance of the critical Rayleigh for the convection $Ra_c \propto Re^{4/3}$ [see Tilgner & Busse (1997) for instance] can be observed: the decrease of the Reynolds number modifies the threshold and then, increases the convective motions.
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5 CONCLUSION

This paper presents the first numerical study of the interaction between the tidal instability and a thermal field. Even if the range of parameters (e.g. the Reynolds number) accessible to our numerical tool is relatively limited, we have validated general physical processes of direct relevance for planetary dynamics. We have shown that a stable temperature field encourages the tidal instability. We have demonstrated also that the tidal instability can grow on a convective flow, and may disrupt the famous Busse columns in planetary cores. Finally, we have demonstrated that the heat flux at planetary scales may be controlled by the forced convection due to this tidal instability, which in any case plays a fundamental role in the organization of fluid motions.

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