AUTO-CORRELATION FUNCTIONS FOR UNITARY GROUPS

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Abstract. We compute the auto-correlation functions of order \( m \geq 1 \) for the characteristic polynomials of random matrices from certain subgroups of the unitary groups \( U(2) \) and \( U(3) \) by applying branching rules. These subgroups can be understood as analogs of Sato–Tate groups of \( USp(4) \) in our previous paper. This computation yields symmetric polynomial identities with \( m \)-variables involving irreducible characters of \( U(m) \) for all \( m \geq 1 \) in an explicit, uniform way.

1. Introduction

1.1. Auto-correlation functions. The distribution of characteristic polynomials of random matrices has been of great interest for their applications in mathematical physics and number theory. Since Keating and Snaith [KS00b, KS00a] computed averages of characteristic polynomials of random matrices in 2002 motivated in part by connections to number theory and in part by the importance of these averages in quantum chaos [AS95], it has become clear that averages of characteristic polynomials are fundamental for random matrix models [BDS03, BS06, BH00, BH01, FS02a, FS02b, MN01].

On the way of these developments, the auto-correlation functions of the distributions of characteristic polynomials in the compact classical groups were computed by Conrey, Farmer, Keating, Rubinstein and Snaith [CFK+03, CFK+05] and by Conrey, Farmer and Zirnbauer [CFZ08, CFZ05]. Later, Bump and Gamburd [BG06] obtained different derivations of the formulas starting from (analogues of) the dual Cauchy identity and adopting a representation-theoretic method. Their results show that the auto-correlation functions are actually combinations of characters of classical groups.

For example, for the symplectic groups, an analogue of the dual Cauchy identity is due to Jimbo–Miwa [JM85] and Howe [How95]:

\[
(1.1) \quad \prod_{i=1}^{m} \prod_{j=1}^{g} (x_i + x_i^{-1} + t_j + t_j^{-1}) = \sum_{\lambda \leq (g^m)} \chi^{Sp(2m)}_{\lambda}(x_1^{\pm 1}, \ldots, x_m^{\pm 1}) \chi^{Sp(2g)}_{\lambda'}(t_1^{\pm 1}, \ldots, t_g^{\pm 1}),
\]

where \( \chi^{Sp(2m)}_{\lambda} \) is the irreducible character of \( Sp(2m, \mathbb{C}) \) associated with the partition \( \lambda \leq (g^m) \) and we set \( \lambda = (m - \lambda'_g, \ldots, m - \lambda'_1) \) with \( \lambda' = (\lambda'_1, \ldots, \lambda'_g) \) the transpose of \( \lambda \). This identity can be considered as a reflection of Howe duality. Using this identity, Bump and Gamburd computed the auto-correlation functions to obtain

\[
(1.2) \quad \int_{USp(2g)} \left( \prod_{j=1}^{m} \det(I + x_j \gamma) \right) d\gamma = (x_1 \ldots x_m)^g \chi^{Sp(2m)}_{(g^m)}(x_1^{\pm 1}, \ldots, x_m^{\pm 1}).
\]

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1.2. **Sato–Tate groups.** The celebrated Sato–Tate conjecture for elliptic curves (i.e. genus 1 curves) predicts that the distribution of Euler factors of an elliptic curve is the same as the distribution of characteristic polynomials of random matrices from SU(2), U(1) or N(U(1)), where N(U(1)) is the normalizer of U(1) in SU(2). The conjecture is proven (under some conditions) by the works of R. Taylor, jointly with L. Clozel, M. Harris, and N. Shepherd-Barron [CHT08, Tay08, HSBT10]. For curves of higher genera, J.-P. Serre, N. Katz and P. Sarnak [Ser94, KS99] proposed a generalized Sato–Tate conjecture. Pursuing this direction, K. S. Kedlaya and A. V. Sutherland [KS09] and later together with F. Fité and V. Rotger [FKRS12] made a list of 52 compact subgroups of USp(4) called Sato–Tate groups that would classify all the distributions of Euler factors for abelian surfaces. Recently, Fité, Kedlaya and Sutherland showed that there are 410 Sato–Tate groups for abelian threefolds [FKS21].

1.3. **Our previous work.** Inspired by the approach of Bump and Gamburd, in a previous paper [LO20], the authors computed the auto-correlation functions of characteristic polynomials for Sato–Tate groups $H \leq \text{USp}(4)$, which appear in the generalized Sato–Tate conjecture for genus 2 curves. The result of [LO20] can be described as follows. Let $H \leq \text{USp}(4)$ be a Sato–Tate group. Then, for arbitrary $m \in \mathbb{Z}_{\geq 1}$, we have

$$(1.3) \quad \int_H \prod_{j=1}^{m} \det(I + x_j \gamma) d\gamma = (x_1 \cdots x_m)^2 \sum_{b=0}^{m} \sum_{z=0}^{\lfloor m-b \rfloor} m_{(b+2z,b)} \chi_{\text{Sp}(2m)}^{\text{Sp}(4)}(2m-b-2z,12z),$$

where the coefficients $m_{(b+2z,b)}$ are the multiplicities of the trivial representation in the restrictions $\chi_{(b+2z,b)}^{\text{Sp}(4)}|_H$ and are explicitly given in the paper [LO20] for all the Sato–Tate groups of abelian surfaces. Exploiting the representation-theoretic meaning of $m_{(b+2z,b)}$, the authors obtained this result by establishing branching rules for $\chi_{(b+2z,b)}^{\text{Sp}(4)}|_H$.

Moreover, since most of the Sato–Tate groups are disconnected, we can decompose the integral in (1.3) according to coset decompositions, and find that the characteristic polynomials over some cosets are independent of the elements of the cosets. Combining this observation with the computations of branching rules, we obtain families of non-trivial identities of irreducible characters of $\text{Sp}(2m, \mathbb{C})$ for all $m \in \mathbb{Z}_{\geq 1}$. For example, we have, for any $m \in \mathbb{Z}_{\geq 1},$

$$(1) \quad \prod_{i=1}^{m} (x_i^2 + x_i^{-2}) = \sum_{b=0}^{m} \sum_{z=0}^{\lfloor m-b \rfloor} \psi_4(z,b) \chi_{\text{Sp}(2m)}^{\text{Sp}(4)}(2m-b-2z,12z),$$

where $\psi_4(z,b)$ is defined on the congruence classes of $z$ and $b$ modulo 4 by the table

| $z$ \( \mod 4 $ | 0 | 1 | 2 | 3 |
|---|---|---|---|
| 0 | 1 | -1 | 0 |
| 1 | 0 | 1 | -1 |
| 2 | 1 | 0 | 0 |
| 3 | 0 | -1 | 1 |

Notice that the irreducible characters $\chi_{\lambda}^{\text{Sp}(2m)}$ are symmetric functions with the number of terms growing very fast as $m$ increases, but that the coefficients $\psi_4(z,b)$ are independent of $m$. In order to produce the left-hand side of the identities, there must be systematic cancelations in the right-hand side.
1.4. Schur functions. The Schur functions $S_\lambda$ form the distinguished self-dual basis of the ring of symmetric functions. They appear naturally in representation theory, algebraic combinatorics, enumerative combinatorics, algebraic geometry and quantum physics. In particular, (i) every Schur function corresponds to an irreducible character of the unitary group, which implies the ring of symmetric functions form the Grothendieck ring for unitary groups, (ii) it has various combinatorial realizations in various aspects of algebraic combinatorics. Hence understanding the properties of Schur functions takes a center stage in these research areas. One of the key features of understanding Schur functions is how other symmetric functions can be expressed in the basis of Schur functions; that is, computing the coefficients of $S_\lambda$, called Schur coefficients, in the expansions. One of the well-known instances is the (inverse of) Kostka matrix, which can be understood as Schur coefficients for monomial (complete) symmetric functions in algebraic combinatorics, and as composition multiplicities of $V(\lambda)$ in the permutation representation $W(\lambda)$ in representation theory. Recall that the (inverse of) Kostka matrix is a uni-upper triangular matrix with integer coefficients and entries in the Kostka matrix has a description in terms of semistandard Young tableaux. However, the closed-form formulas for entries in the (inverse of) Kostka matrix are not available in general. Another well-known instance is Littlewood–Richardson rule, which can be understood as Schur coefficients for a product of two Schur functions in algebraic combinatorics, and as composition multiplicities of $V(\lambda)$’s in the tensor product $V(\mu) \otimes V(\eta)$ in representation theory.

1.5. Main Result. In what follows, we describe the main result of this paper and its application.

(M) We compute explicitly the auto-correlation functions for $H = U(1) \leq U(g)$ $(g = 2, 3)$ and for the subgroups $H \leq U(g)$ $(g = 2, 3)$ defined in (1.5) below. Namely, for any $m \in \mathbb{Z}_{\geq 1}$, we obtain

$$\int_{H \leq U(g)} \prod_{i=1}^{m} \det(I + x_i \gamma) \, d\gamma = \sum_{\lambda \leq (g^m)} m_\lambda(H) S_\lambda^{U(m)}(x),$$

where the coefficients $m_\lambda(H)$ are completely determined. Here $S_\lambda^{U(m)}(x)$ denotes the character of the irreducible representation $V(\lambda)$ of the unitary group $U(m)$, which are Schur functions.

(A) As an application of the main result, we give closed-form formulas of Schur coefficients $c^\lambda_{g_1, g_2}$ for special infinite families of symmetric functions $t_{g_1, g_2}^{(m)}(x)$, which also have simple expansions in terms of monomial symmetric functions $\{m_\lambda^{(m)}(x)\}$. That is, for any $m \in \mathbb{Z}_{\geq 1}$ and $1 \leq g_1 + g_2 \leq 3$, we obtain

$$t_{g_1, g_2}^{(m)}(x) := \prod_{i=1}^{m} (1 \pm x_i^{g_1})(1 \pm x_i^{g_2}) = \sum_{\lambda \leq (g_1 + g_2)^m} c_{g_1, g_2}^\lambda S_\lambda^{U(m)}(x),$$

where the Schur coefficients $c_{g_1, g_2}^\lambda$ are given in closed-form formulas. Thus $c_{g_1, g_2}^\lambda$ can be understood as a simple combination of entries in the inverse of Kostka matrix as $t_{g_1, g_2}^{(m)}(x)$ has an expansion in $m_\lambda^{(m)}(x)$ with coefficients from $\{1, 0, -1\}$. Let us explain the main result and its application in more detail. To adopt the same strategy as in USp$(m)$, we first introduce several subgroups $H_g$, $H'_{g,4}$ and $H_{g,4}$ of $U(g)$ $(g = 2, 3)$ as follows, which play the role of Sato–Tate groups in [LO20]:

$$H_2 := \langle U(1), J_2 \rangle, \quad H'_{2,4} := \langle U(1), \zeta_{2,4} \rangle \leq H_{2,4} := \langle U(1), J_2, \zeta_{2,4} \rangle \leq U(2),$$

$$H_3 := \langle U(1), J_3 \rangle, \quad H'_{3,4} := \langle U(1), \zeta_{3,4} \rangle \leq H_{3,4} := \langle U(1), J_3, \zeta_{3,4} \rangle \leq U(3),$$

\[ 1.5 \]
where
\[ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \zeta_{2,4} := \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & \sqrt{-1} \end{pmatrix}, \quad \zeta_3 := \begin{pmatrix} 0 & 1 \\ -1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \zeta_{3,4} := \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & \sqrt{-1} \\ 0 & 0 \end{pmatrix}. \]

Considering these subgroups in \( U(g) \ (g = 2, 3) \), we present the main result more precisely.

**Main Theorem.** Let \( H \leq U(g) \ (g = 2, 3) \) be \( U(1) \) or a group in (1.5). Then, for any \( m \in \mathbb{Z}_{\geq 1} \), we have
\[
\int_{H \leq U(g)} \prod_{i=1}^{m} \det(I + x_i \gamma) \, d\gamma = \sum_{\lambda \leq (g^m)} m_{\lambda^{'}}(H) S_{\lambda}^U(m)(x)
\]
where the coefficient \( m_{\lambda^{'}}(H) \) are the *composition multiplicities* of the trivial representation in the restriction \( \chi_{U(g)}^{U(1)} \mid_H \) and are explicitly given in Theorem 3.1, Theorem 3.4, Theorem 4.6 and Theorem 4.14.

This theorem can be interpreted as a result on branching rules from \( U(g) \) to \( H \) in representation theory. To prove the above theorem, we analyze the representation structure of \( V(\lambda^{'}) \) over \( U(g) \) with respect to \( H \) and determine a certain list of linearly independent subsets in \( V(\lambda^{'}) \).

For an application of **Main Theorem**, we observe (i) \( H^{'s} \) are decomposed into disconnected cosets
\[
\begin{align*}
H_2 &= U(1) \sqcup J_2 U(1), \\
H_3 &= U(1) \sqcup J_3 U(1), \\
H_{2,4} &= H_{2,4}^{' \sqcup J_2 U(1) \sqcup \zeta_{2,4} J U(1)}, \\
H_{3,4} &= H_{3,4}^{' \sqcup J_3 U(1) \sqcup \zeta_{3,4} J U(1)},
\end{align*}
\]
and (ii) the characteristic polynomials in the underlined cosets of \( H \) are independent of elements of the coset. These observations enable us to obtain closed-form identities involving Schur functions.

**Application.** For arbitrary \( m \in \mathbb{Z}_{\geq 1} \), we have the following identities:

(A) \[
\prod_{i=1}^{m} (1 + x_i^2) = \sum_{b=0}^{m} \sum_{j=0}^{\lfloor \frac{m-b}{2} \rfloor} (-1)^j S_{(2^b,1^j)}^{U(m)}(x) \quad \text{(derived from } H_2, \text{ Theorem 3.1)},
\]

(B) \[
\frac{1}{2} \left( \prod_{i=1}^{m} (1 + x_i^2) + \prod_{i=1}^{m} (1 - x_i^2) \right) = \sum_{(b+2j,b) \leq (m^2)} (-1)^j S_{(2^b,1^j)}^{U(m)}(x) \quad \text{(derived from } H_{2,4} \text{ and } H_{2,4}^{'}, \text{ Theorem 3.4)},
\]

(C) \[
\prod_{i=1}^{m} (1 + x_i)(1 + x_i^2) = \sum_{\lambda \leq (3^m) \leq (3^k,2^h,1^s)} \tau(z,b) S_{\lambda}^{U(m)}(x) \quad \text{(derived from } H_3, \text{ Theorem 4.6)},
\]

(D) \[
\frac{1}{2} \left( \prod_{i=1}^{m} (1 + x_i)(1 + x_i^2) + \prod_{i=1}^{m} (1 + x_i)(1 - x_i^2) \right) = \sum_{\lambda \leq (3^m)} \omega(z,b') S_{\lambda}^{U(m)}(x), \quad \text{(derived from } H_{3,4} \text{ and } H_{3,4}^{'}, \text{ Theorem 4.14)},
\]

where

(i) \( \tau(z,b) \in \{0,1\} \) is defined on the congruence of \( z \) and \( b \) modulo 4 as in the following table
\[
\begin{array}{cccc}
1 & 0 & 1 & 2 & 3 \\
0 & 1 & 1 & 0 & 0 \\
1 & 1 & 0 & -1 & 0 \\
2 & 0 & -1 & -1 & 0 \\
3 & 0 & 0 & 0 & 0
\end{array}
\]

\[\tau(z, b) = \begin{bmatrix}
1 + \frac{1}{2} & 0 & 1 & 2 & 3 \\
0 & 1 & 1 & 0 & 0 \\
1 & 1 & 0 & -1 & 0 \\
2 & 0 & -1 & -1 & 0 \\
3 & 0 & 0 & 0 & 0
\end{bmatrix}\]

(ii) \(\omega_i(z, b')\) depends on the congruence of \(z\) and \(b\) modulo 2, 4 and given in Corollary 4.13.

Combining the identities in Application and replacing \(x_i\) with \(-x_i\), we obtain all the other identities in (1.4) (see (3.6), Corollary 4.17, Corollary 4.18 and Remark 4.19).

Note that the identities obtained involves negative Schur coefficients and the number of terms in Schur functions grows enormously as \(m\) increases. However, our result implies that such combinations of Schur functions have miraculous cancellations and yield symmetric functions with positive coefficients. Furthermore, the identities state that the Schur coefficients do not depend on \(m\) (see Example 3.3, Example 4.8 and Example 4.16). These identities seem intriguing from the viewpoint of representation theory and algebraic combinatorics. It might have been difficult for us to expect that such identities exist, without regard to the auto-correlation functions and branching rules of the newly introduced groups in (1.5).

1.6. **Organization of the paper.** In Section 2, we review the necessary backgrounds for auto-correlation functions and the dual Cauchy identity. In Section 3, we compute the auto-correlation functions of \(H\)'s of \(U(2)\) and establish the corresponding identities involving Schur functions. In Section 4, we present the auto-correlation functions of \(H\)'s of \(U(3)\) and consider the corresponding identities by analyzing the representation structure of \(V(\lambda)\) with respect to \(H\)'s. But we postpone a part of the proof to Section 5, which is devoted to determine the composition multiplicities of trivial representations in \(\chi^{U(g)}_{\lambda} |_{\mu}\)'s. This amounts to the proof for \(U(3)\). We convert this problem into counting the pairs of integers encoding certain information from representation theory. By expressing the cardinalities as closed-form formulas, we complete the proof.

**Convention 1.1.** Throughout this paper, we keep the following conventions.

(i) For a statement \(P\), the notation \(\delta(P)\) is equal to 1 or 0 according to whether \(P\) is true or not.

(ii) For \(m, m' \in \mathbb{Z}\) and \(k \in \mathbb{Z}_{>0}\), we write \(m \equiv_k m'\) if \(k\) divides \(m - m'\), and \(m \not\equiv_k m'\) otherwise.

2. **Dual Cauchy identity and auto-correlation functions**

In this section, we fix notations and review the dual Cauchy identity and establish a general formula for the auto-correlation functions of characteristic polynomials.

A partition \(\lambda = (\lambda_1 \geq \cdots \geq \lambda_k)\) is a non-increasing sequence of non-negative integers \(\lambda_i\). Define \(|\lambda| = \sum_{i=1}^{k} \lambda_i\) and \(\ell(\lambda) = k\). We write \(\lambda = (m^k)\) when \(m = m_1 = \cdots = m_k\). More generally, a partition is written as \((m_1^{k_1}, m_2^{k_2}, \ldots, m_s^{k_s})\) for \(m_1 > m_2 > \cdots > m_s\) and \(k_i \geq 1\). For two partitions \(\lambda = (\lambda_1 \geq \cdots \geq \lambda_k)\) and \(\mu = (\mu_1 \geq \cdots \geq \mu_l)\), we define a partial order \(\lambda \preceq \mu\) if \(k \leq l\) and \(\lambda_i \leq \mu_i\) for all \(i = 1, 2, \ldots, k\). A partition \(\lambda\) corresponds to a Young diagram, and the transpose \(\lambda'\) is defined to be the partition corresponding to the transpose of the Young diagram of \(\lambda\).

Let \(U(g)\) be the unitary group for \(g \geq 1\). For a partition \(\lambda\) with at most \(g\) parts, let \(S^{U(g)}_\lambda\) be the Schur function associated with \(\lambda\). It is well-known that \(S^{U(g)}_\lambda\) is the irreducible character of \(U(g)\)
with highest weight \( \lambda \). Denote by \( V_g(\lambda) \) the representation space of \( S_\lambda^{U(g)} \). When \( g \) is clear from the context, we will simply write \( V(\lambda) \).

**Definition 2.1.** Let \( H \) be a closed subgroup of \( U(g) \). Define \( m_\lambda(H) \) to be the multiplicity of the trivial representation \( 1_H \) in the restriction of \( V(\lambda) \) to \( H \).

### 2.1. Dual Cauchy identity

Let us recall the dual Cauchy identity (see, e.g., [BG06, (9)]) :

**Lemma 2.2.** For any \( m \geq 1 \) and \( g \geq 1 \), we have

\[
\prod_{i=1}^{m} \prod_{j=1}^{g} (1 + x_i t_j) = \sum_{\lambda \leq (g^m)} S_\lambda^{U(m)}(x) S_\lambda^{U(g)}(t),
\]

where \( x = (x_1, \ldots, x_m) \) and \( t = (t_1, \ldots, t_g) \). We also have

\[
\prod_{i=1}^{m} \prod_{j=1}^{g} (1 - x_i t_j) = \sum_{\lambda \leq (g^m)} (-1)^{|\lambda|} S_\lambda^{U(m)}(x) S_\lambda^{U(g)}(t),
\]

by replacing \( x_i \) to \( -x_i \).

Following Proposition 3.3 in [LO20], we obtain a formula for the auto-correlation functions.

**Proposition 2.3.** Let \( H \) be a subgroup of \( U(g) \) and \( d\gamma \) be the probability Haar measure on \( H \). Then, for each \( m \geq 1 \), the auto-correlation function for the distribution of characteristic polynomials of \( H \) is given by

\[
(2.1) \quad \int_H \prod_{i=1}^{m} \det(I + x_i \gamma) d\gamma = \sum_{\lambda \leq (g^m)} m_\lambda(H) S_\lambda^{U(m)}(x),
\]

\[
(2.2) \quad \int_H \prod_{i=1}^{m} \det(I - x_i \gamma) d\gamma = \sum_{\lambda \leq (g^m)} (-1)^{|\lambda|} m_\lambda(H) S_\lambda^{U(m)}(x).
\]

**Proof.** Let \( t_1, \ldots, t_g \) be the eigenvalues of \( \gamma \in H \). Since we have

\[
\det(I + x_i \gamma) = \prod_{j=1}^{g} (1 + x_i t_j),
\]

it follows from Lemma 2.2 that

\[
\int_H \prod_{j=1}^{m} \det(I + x_j \gamma) d\gamma = \int_H \prod_{i=1}^{g} \prod_{j=1}^{g} (1 + x_i t_j) d\gamma = \int_H \sum_{\lambda \leq (g^m)} S_\lambda^{U(m)}(x) S_\lambda^{U(g)}(t) d\gamma = \sum_{\lambda \leq (g^m)} S_\lambda^{U(m)}(x) \int_H S_\lambda^{U(g)}(t) d\gamma.
\]

From Schur orthogonality (for example, [Bum13]), the integral \( \int_H S_\lambda^{U(g)}(t) d\gamma \) is equal to the multiplicity of the trivial representation \( 1_H \) of \( H \) in the restriction of \( S_\lambda^{U(g)} \) to \( H \), which is equal to \( m_\lambda(H) \) by Definition 2.1. This establishes the first identity. The second identity follows from replacing \( x_i \) with \(-x_i \) in the first identity. \( \square \)

**Corollary 2.4.** Suppose that \(-I \in H \). Then \( m_\lambda(H) = 0 \) whenever \( |\lambda'| \) is odd.
Proof. If \(-I \in H\), we have
\[
\int_H \prod_{j=1}^m \det(I + x_j \gamma) d\gamma = \int_H \prod_{j=1}^m \det(I - x_j \gamma) d\gamma.
\]
Our assertion follows from (2.1) and (2.2) by comparing the right-hand sides, since the Schur functions are linearly independent. 

We recall the classical branching rule from \(U(g)\) to \(U(g - 1)\) for \(g \geq 2\).

**Proposition 2.5.** Let \(V_g(\lambda)\) be the irreducible representation of \(U(g)\) with highest weight \(\lambda\). Then we have
\[
[V_g(\lambda) : V_{g-1}(\mu)] \leq 1
\]
for any partition \(\mu\) with at most \(g - 1\) parts. Furthermore, \([V_g(\lambda) : V_{g-1}(\mu)] = 1\) precisely when
\[
\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \mu_{g-1} \geq \lambda_g.
\]

3. Identities for \(g = 2\)

In this section, we consider some disconnected subgroups \(H\) of \(U(2)\) and compute \(m_{\lambda^\prime}(H)\) for \(\lambda \leq (2^m)\) in (2.1). This computation produces identities involving Schur functions \(S_{\lambda}^{U(m)}(x)\) for all \(m \in \mathbb{Z}_{\geq 1}\).

We identify \(U(1)\) with the subgroup \(\left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} : t \in \mathbb{C}, |t| = 1 \right\} \leq U(2)\).

3.1. **Subgroup** \(\langle U(1), J \rangle\). Let us consider the subgroup \(H_2\) of \(U(2)\) generated by \(U(1)\) and \(J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\), i.e.
\[
H_2 = \langle U(1), J \rangle \leq U(2).
\]

Then we have
\[
H_2 = U(1) \sqcup J U(1).
\]

**Theorem 3.1.** For any partition \((a,b) \leq (m^2)\), we have
\[
m_{(a,b)}(U(1)) = \delta(a \equiv_2 b) \quad \text{and} \quad m_{(a,b)}(H_2) = \delta(a \equiv_4 b).
\]

Furthermore, for any \(m \in \mathbb{Z}_{\geq 1}\), we have
\[
\prod_{i=1}^m (1 + x_i^2) = \sum_{\lambda \leq (2^m) \atop \lambda_N = (b+2j,b)} (-1)^j S_{\lambda}^{U(m)}(x) = \sum_{b=0}^m \sum_{j=0}^{\lfloor \frac{m-b}{2} \rfloor} (-1)^j S_{\lambda}^{U(m)}(2^b,1^2j)(x),
\]
where we set \(j := (a - b)/2\).
Proof. For any $\gamma = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \in U(1)$, we have $\det(I + xJ\gamma) = 1 + x^2$. Let $du = du(\gamma)$ be the probability Haar measure on $U(1) \leq U(2)$. By Proposition 2.3 and (3.1), we have

$$
\sum_{\lambda \leq (2^m)} m_\lambda(H_2) S_\lambda^{U(m)}(x) = \int_{H_2} \prod_{i=1}^m \det(I + x_i \gamma) d\gamma
$$

$$
= \frac{1}{2} \int_{U(1)} \prod_{i=1}^m \det(I + x_i \gamma) du + \frac{1}{2} \int_{U(1)} \prod_{i=1}^m \det(I + x_i J \gamma) du
$$

$$
= \frac{1}{2} \sum_{\lambda \leq (2^m)} m_\lambda(U(1)) S_\lambda^{U(m)}(x) + \frac{1}{2} \int_{U(1)} \prod_{i=1}^m (1 + x_i^2) du
$$

(3.3)

Let $v_1 = (1,0)$ and $v_2 = (0,1)$ be the standard unit vectors of $V := \mathbb{C}^2$, and consider the standard representation of $U(2)$ on $V$, and let $\det$ be the one-dimensional representation of $U(2)$ defined by the determinant. For $\lambda' = (a,b) \leq (m^2)$, we have $V(\lambda') \cong \det^b \otimes \text{Sym}^{a-b}(V)$. Thus the trivial $U(1)$-module is generated by $v_1^j v_2^j$ only when $a - b$ is even, where we set $j := (a-b)/2$. In other word, we have

$$
m_{(a,b)}(U(1)) = \delta(a \equiv_2 b).
$$

Furthermore, since $J$ sends $v_1 \mapsto -v_2$ and $v_2 \mapsto v_1$, we see that $v_1^j v_2^j$ is fixed by $J$ when $j$ is even. Therefore,

$$
m_{(a,b)}(H_2) = \delta(j \equiv_2 0) = \delta(a \equiv_4 b).
$$

Note that, when $\lambda' = (b + 2j, b)$, we have $\lambda = (2^b, 1^{2j})$. Now it follows from (3.3) that

$$
\prod_{i=1}^m (1 + x_i^2) = \sum_{b=0}^m \sum_{j=0}^{\lfloor m-b \rfloor} (2\delta(j \equiv_2 0) - 1) S_{(2^b, 1^{2j})}^{U(m)}(x) = \sum_{b=0}^m \sum_{j=0}^{\lfloor m-b \rfloor} (-1)^j S_{(2^b, 1^{2j})}^{U(m)}(x).
$$

\[\square\]

Remark 3.2. The left hand side of (3.2) can be written as a simple combination of the monomial symmetric functions. Namely, we have

$$
\prod_{i=1}^m (1 + x_i^2) = \sum_{k=0}^m m_{(2^k)}^{(m)}(x),
$$

where $m_\lambda^{(m)}$ denote the monomial symmetric functions in $m$-variables associated with partitions $\lambda$ with $\ell(\lambda) \leq m$.

Example 3.3. Let us see an example for the case $m = 10$ in Theorem 3.1. We have

$$
\prod_{i=1}^{10} (1 + x_i^2) = \sum_{b=0}^{10} \sum_{j=0}^{\lfloor 10-b \rfloor} (-1)^j S_{(2^b, 1^{2j})}^{U(10)}(x).
$$

(3.4)

Note that $S_{(2^1, 1^2)}^{U(10)}(x)$ appears in the right hand side of (3.4) with the coefficient $-1$ since $j = 1$. As a polynomial itself, $S_{(2^1, 1^2)}^{U(10)}(x)$ contains 8701 monomial terms and $S_{(2^1, 1^2)}^{U(10)}(1) = 29700$, where
1 = (1, 1, . . . , 1). Actually, there are 15 Schur functions with negative coefficient −1 in the right hand side of (3.4) including $S_{(2^4, 1^2)}^{U(10)}(x)$. After amazing cancellations among Schur functions, we obtain a symmetric function in the left hand side of (3.4), which contains only 1024 monomial terms with coefficients all 1 in its expansion.

3.2. Subgroup $\langle U(1), J, \zeta_4 \rangle$. Set

$$\zeta_4 := \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & \sqrt{-1} \end{pmatrix} \in U(2),$$

and denote by $H_{2,4}$ the subgroup of $U(2)$ generated by $U(1)$, $J$ and $\zeta_4$. That is, we define $H_{2,4} := \langle U(1), J, \zeta_4 \rangle \leq U(2)$.

Then we have

$$H_{2,4} = U(1) \sqcup J U(1) \sqcup \zeta_4 U(1) \sqcup \zeta_4 J \sqcup U(1).$$

Let $H'_{2,4}$ be the subgroup of $H_{2,4}$ generated by $U(1)$ and $\zeta_4$.

Theorem 3.4. For any partition $(a, b) \leq (m^2)$, we have

$$m_{(a,b)}(H'_{2,4}) = \delta(a + b \equiv 4 \ 0) \quad \text{and} \quad m_{(a,b)}(H_{2,4}) = \delta(a + b \equiv 4 \ 0)\delta(a - b \equiv 4 \ 0).$$

Moreover, for any $m \in \mathbb{Z}_{\geq 1}$, we have

$$\frac{1}{2} \left( \prod_{i=1}^{m} (1 + x_i^2) + \prod_{i=1}^{m} (1 - x_i^2) \right) = \sum_{(a,b) \leq (m^2) \atop a+b \equiv 0} (-1)^{\delta(a \neq 4 b)} S_{(2^b,1^{a-b})}^{U(m)}(x)$$

$$= \sum_{(b+2j, b) \leq (m^2) \atop b+j \equiv 0} (-1)^{j} S_{(2^b, 1^{2j})}^{U(m)}(x),$$

where we set $j := (a - b)/2$ as before.

Proof. Let $\lambda' = (a,b) \leq (m^2)$. We keep the notations in the proof of Theorem 3.1 for $V(\lambda') \cong \det^b \otimes \text{Sym}^{a-b}(V)$. A vector in $V(\lambda')$ is fixed by $U(1)$ if it is of the form $v_1^j v_2^j$ up to scalar multiplication. Since $\det(\zeta_4) = -1$, we get

$$\zeta_4 v_1^j v_2^j = (-1)^{b+j} v_1^j v_2^j.$$

We see that $b + j \equiv 2 \ 0 \iff a + b \equiv 4 \ 0$, and if $a + b \equiv 4 \ 0$ then $a - b \equiv 2 \ 0$. Thus we obtain

$$m_{(a,b)}(H'_{2,4}) = \delta(a + b \equiv 4 \ 0).$$

As observed in the proof of Theorem 3.1, the vector $v_1^j v_2^j$ is fixed by $J$ if and only if $a - b \equiv 4 \ 0$. Therefore we have

$$m_{(a,b)}(H_{2,4}) = \delta(a + b \equiv 4 \ 0)\delta(a - b \equiv 4 \ 0).$$

Let $d\gamma' = d\gamma'/(\gamma)$ be the probability Haar measure on $H'_{2,4}$. Since

$$\det(I + xJ\gamma) = 1 + x^2 \quad \text{and} \quad \det(I + x\zeta_4 J\gamma) = 1 - x^2$$
for all $\gamma \in U(1)$, we have
\[
\int_{H_{2,4}} \Delta(\gamma) d\gamma = \frac{1}{2} \int_{H_{2,4}'} \Delta(\gamma) d\gamma' + \frac{1}{4} \int_{U(1)} \Delta(\gamma) d\gamma + \frac{1}{4} \int_{\xi_4 U(1)} \Delta(\gamma) d\gamma
\]
\[
= \frac{1}{2} \int_{H_{2,4}'} \Delta(\gamma) d\gamma' + \frac{1}{4} \prod_{i=1}^{m} (1 + x_i^2) + \frac{1}{4} \prod_{i=1}^{m} (1 - x_i^2),
\]
where we write $\Delta(\gamma) = \prod_{i=1}^{m} \det(I + x_i \gamma)$ for convenience. Applying Proposition 2.3 to the integrals, we obtain
\[
\frac{1}{2} \left( \prod_{i=1}^{m} (1 + x_i^2) + \prod_{i=1}^{m} (1 - x_i^2) \right) = \sum_{(a,b) \leq (m^2)} (2m_{(a,b)}(H_{2,4}) - m_{(a,b)}(H_{2,4}')) S_{(2^b,1^a)}^U(x)
\]
\[
= \sum_{(a,b) \leq (m^2)} (-1)^{(a\neq b)} S_{(2^b,1^a)}^U(x) = \sum_{(b+j) \equiv 0} (-1)^j S_{(2^b,1^j)}^U(x). \tag{3.5} \square
\]

**Remark 3.5.** (1) The identity (3.5) can be derived from (3.2). We will consider the alternate proof in Section 3.3.

(2) As in Remark (3.2), we observe that the left hand side of (3.5) is a simple combination of the monomial symmetric functions in $m$-variables:
\[
\frac{1}{2} \left( \prod_{i=1}^{m} (1 + x_i^2) + \prod_{i=1}^{m} (1 - x_i^2) \right) = \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} m_{(2k)}^m(x).
\]

3.3. **Pieri’s rule.** One can see that
\[
\prod_{i=1}^{m} (1 - x_i^2) = \sum_{\lambda \leq (2^m)} \sum_{\lambda' = (b+2j,b)} (-1)^b S_{(2^b,1^2j)}^U(x) = \sum_{k=0}^{m} (-1)^k m_{(2k)}^m(x). \tag{3.6}
\]

Indeed, by replacing $x_i$ with $\sqrt{-1} x_i$ in (3.2), we obtain
\[
\prod_{i=1}^{m} (1 - x_i^2) = \sum_{\lambda \leq (2^m)} \sum_{\lambda' = (b+2j,b)} (-1)^j (-1)^{j+b} S_{(2^b,1^2j)}^U(x) = \sum_{\lambda \leq (2^m)} \sum_{\lambda' = (b+2j,b)} (-1)^b S_{(2^b,1^2j)}^U(x),
\]

and the second equality in (3.6) follows from the definition of the monomial symmetric function $m_{(2^k)}^m(x)$. Combining (3.2) with (3.6) yields the identity (3.5). If we combine (3.2) and (3.6) in a different way, we obtain
\[
\frac{1}{2} \left( \prod_{i=1}^{m} (1 + x_i^2) - \prod_{i=1}^{m} (1 - x_i^2) \right) = \sum_{(a,b) \leq (m^2)} (-1)^{(a\neq b)} S_{(2^b,1^a)}^U(x) = \sum_{k=0}^{\lfloor \frac{m-1}{2} \rfloor} m_{(2k+1)}^{(m)}(x).
\]

Moreover, (3.6) can also be proven using Pieri’s rule. Since the idea can be used in other cases, let us see the proof. We first consider the trivial case $g = 1$ to use the results for the case $g = 2$. Let $m_{\lambda}^m$ be the monomial symmetric function of $m$-variables associated to a partition $\lambda$ with $\ell(\lambda) \leq m$, as before. Then, in particular, we have
\[
m_{(1^k)}^m = S_{(1^k)}^U \quad \text{for } k \leq m.
and obtain

\[(3.7)\]
\[
\prod_{i=1}^{m}(1 + x_i) = \sum_{k=0}^{m} m^{(m)}_{(1^k)}(x) = \sum_{\lambda \leq (1^m)} S^{U(m)}_{\lambda}(x).
\]

By replacing \(x_i\) with \(-x_i\), we have

\[(3.8)\]
\[
\prod_{i=1}^{m}(1 - x_i) = \sum_{k=0}^{m} (-1)^k m^{(m)}_{(1^k)}(x) = \sum_{\lambda \leq (1^m)} (-1)^{\lambda} S^{U(m)}_{\lambda}(x).
\]

Recall Pieri’s rule from, e.g., Macdonald’s book [Mac98, (5.17)]:

\[(3.9)\]
\[
S^{U(m)}_{\lambda}(x) \times \prod_{i=1}^{m}(1 + x_i) = S^{U(m)}_{\lambda}(x) \times \left(\sum_{i=0}^{m} e_i(x)\right) = \sum_{\lambda \leq \mu \leq \lambda + (1^m)} S^{U(m)}_{\mu}(x),
\]

where \(e_i(x)\) denotes the elementary symmetric function of partition \((l)\) of length 1 and \(\lambda + (1^m) = (\lambda_1 + 1, \lambda_2 + 2, \ldots, \lambda_m + 1)\) for \(\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)\).

By (3.9) and (3.8), we have

\[(3.10)\]
\[
\prod_{i=1}^{m}(1 - x_i^2) = \prod_{i=1}^{m}(1 + x_i) \times \sum_{(1^k) \leq (1^m)} (-1)^k S^{U(m)}_{(1^k)}(x) = \sum_{k=0}^{m} \sum_{(1^k) \leq \mu \leq (2^k,1^{m-k})} (-1)^k S^{U(m)}_{\mu}(x).
\]

Thus for any partition \((2^k,1^{a-b}) \leq (2^m)\), the coefficient of \(S^{U(m-1)}_{(2^k,1^{a-b})}(x)\) in (3.10) is given by

\[
\sum_{s=b}^{a} (-1)^s = \begin{cases} 
0 & \text{if } a - b \equiv 2 \text{ 1}, \\
1 & \text{if } a - b \equiv 2 \text{ 0 and } b \equiv 2 \text{ 0}, \\
-1 & \text{if } a - b \equiv 2 \text{ 0 and } b \equiv 2 \text{ 1}.
\end{cases}
\]

When \(a - b \equiv 2 \text{ 0}\), write \(a - b = 2j\). Then we have

\[(3.11)\]
\[
\prod_{i=1}^{m}(1 - x_i^2) = \sum_{\lambda \leq (2^{m})} (-1)^{\lambda} S^{U(m)}_{(2^k,1^{2j})}(x)
\]
as desired.

Using a similar argument, we obtain two more identities.

**Proposition 3.6.** For any \(m \in \mathbb{Z}_{\geq 1}\), we have

\[(3.11)\]
\[
\prod_{i=1}^{m}(1 + 2x_i + x_i^2) = \sum_{(a,b) \leq (m^2)} (a - b + 1) S^{U(m)}_{(2^k,1^{a-b})}(x),
\]

\[(3.12)\]
\[
\prod_{i=1}^{m}(1 - 2x_i + x_i^2) = \sum_{(a,b) \leq (m^2)} (-1)^{a-b}(a - b + 1) S^{U(m)}_{(2^k,1^{a-b})}(x).
\]
Proof. By (3.9) and (3.7), we have

\[
\prod_{i=1}^{m}(1 + x_i)^2 = \prod_{i=1}^{m}(1 + x_i) \times \sum_{(\lambda) \preceq (1^m)} S_{U(m)}^{U(m)}(x)
\]

(3.13)

\[
= \sum_{k=0}^{m} \sum_{(\mu) \preceq (2^k, 1^{m-k})} S_{\mu}^{U(m)}(x)
\]

Thus for any partition \((2^b, 1^{a-b}) \preceq (2^m)\), the coefficient of \(S_{U(m)}^{U(m)}(x)\) in (3.13) is equal to

\[
\sum_{s=b}^{a} \mu_s = a - b + 1,
\]

and the identity (3.11) follows. By replacing \(x_i\) with \(-x_i\) in (3.11), we obtain (3.12). \(\square\)

Remark 3.7. The above use of Pieri’s rule may not be applicable, in general, if one can try to obtain closed-form formulas for \(g \geq 3\). For instance, based on Theorem 3.1 about \(g = 2\), one can check the formula in Theorem 4.6 about \(g = 3\) below using Pieri’s rule, for first several small values of \(m\). But, when \(g \geq 3\), obtaining closed-form formula for the coefficient of \(S_{U(m)}^{U(m)}(x)\), \(m \geq 1\), seems not easy in this approach.

4. Identities for \(g = 3\)

In this section, we consider some disconnected subgroups \(H\) of U(3) and compute \(m_\lambda(H)\) for \(\lambda \preceq (3^m)\) in (2.1). As with the case \(g = 2\), our computation yields identities involving Schur functions \(S_{\lambda}^{U(m)}\) for all \(m \in \mathbb{Z}_{\geq 1}\).

To begin with, we embed U(1) into U(3) via

\[
U(1) \simeq \left\{ \begin{pmatrix} t & 0 & 0 \\ 0 & t^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} : t \in \mathbb{C}, |t| = 1 \right\},
\]

and U(2) into U(3) via \(A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}\) for \(A \in U(2)\).

4.1. Subgroup \((U(1), J)\). Define

\[
J := \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in U(3).
\]

Note that

\[
J^2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in U(1) \quad \text{and} \quad J^4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I.
\]

Consider the subgroup \(H_3\) of U(3) generated by \(J\) and U(1), i.e.

\[
H_3 := \langle U(1), J \rangle \leq U(3).
\]
Then one can easily check that \( J \) normalizes \( U(1) \), and \( H_3 = U(1) \sqcup J U(1) \). Note that

\[
J \begin{pmatrix} t & 0 & 0 \\ 0 & t^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & t^{-1} & 0 \\ -t & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\] and \( \det(I + xJ\gamma) = (1 + x)(1 + x^2) \)

for all \( \gamma \in U(1) \).

We prove a useful lemma.

**Lemma 4.1.** For any \( k \in \mathbb{Z}_{\geq 0} \), we have

\[
m_{(a+k,b+k,k)}(U(1)) = m_{(a,b,0)}(U(1)) \quad \text{and} \quad m_{(a+k,b+k,k)}(H_3) = m_{(a,b,0)}(H_3).
\]

**Proof.** Let \( \det \) be the one-dimensional representation of \( U(3) \) defined by the determinant. Then we have

\[
V(a+k, b+k, k) = \det^k \otimes V(a, b, 0).
\]

Since \( \det(A) = 1 \) for any \( A \in U(1) \) and \( \det(J) = 1 \), the assertion follows. \( \square \)

Thanks to Lemma 4.1, we need to consider the irreducible representations \( V(\lambda') \) of \( U(3) \) only for \( \lambda' = (a, b, 0) \). In what follows, we assume \( \lambda' = (a, b, 0) \) and freely write \( V(a, b) \) as the irreducible representation of \( \mathfrak{sl}(3, \mathbb{C}) \) with highest weight \( (a, b) \). More precisely, define \( h_1, h_2 \in \mathfrak{sl}(3, \mathbb{C}) \) by

\[
h_1 = \text{diag}(1, -1, 0) \quad \text{and} \quad h_2 = \text{diag}(0, 1, -1),
\]

and denote by \( \mathfrak{h} \) the subspace of \( \mathfrak{sl}(3, \mathbb{C}) \) spanned by \( h_1 \) and \( h_2 \). We regard any partition \( \mu = (\mu_1, \mu_2) \) as an element of \( \mathfrak{h}^* \) by setting

\[
\mu(h_1) = \mu_1 - \mu_2 \quad \text{and} \quad \mu(h_2) = \mu_2,
\]

and \( \mu \) is a weight of \( \mathfrak{sl}(3, \mathbb{C}) \).

Let \( V \) and \( W = \wedge^2 V \) be the fundamental representations of \( U(3) \). Take a basis \( \{v_1, v_2, v_3\} \) of \( V \) such that

\[
Jv_1 = -v_2, \quad Jv_2 = v_1 \quad \text{and} \quad Jv_3 = v_3.
\]

Write

\[
w_{12} = v_1 \wedge v_2, \quad w_{13} = v_1 \wedge v_3, \quad w_{23} = v_2 \wedge v_3.
\]

Then \( \{w_{12}, w_{13}, w_{23}\} \) is a basis for \( W \), and we have

\[
Jw_{12} = w_{12}, \quad Jw_{13} = -w_{23} \quad \text{and} \quad Jw_{23} = w_{13}.
\]

As \( \mathfrak{sl}(3, \mathbb{C}) \)-representations, \( V \) and \( W \) are equivalent and can be described as follows:

\[
V : \begin{array}{c} 1 \\ 2 \end{array} \rightarrow \begin{array}{c} 1 \\ 3 \end{array} \quad \text{and} \quad W : \begin{array}{c} 2 \\ 1 \end{array} \rightarrow \begin{array}{c} 1 \\ 3 \end{array}.
\]

The diagrams mean

\[
f_1 v_1 = v_2, \quad f_2 v_2 = v_3, \quad f_2 w_{12} = w_{13}, \quad f_1 w_{13} = w_{23}
\]
and \( f_i v_j = 0 \) and \( f_i w_{jk} = 0 \) for other choices of \( i, j, k \), where we set \( f_1 := \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \), \( f_2 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \) in \( \mathfrak{sl}(3, \mathbb{C}) \). Equivalently, if we set \( e_1 := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \), \( e_2 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \) in \( \mathfrak{sl}(3, \mathbb{C}) \), we have
\[
e_2 v_3 = v_2, \quad e_1 v_2 = v_1, \quad e_1 w_{23} = w_{13}, \quad e_2 w_{13} = w_{12}
\]
and \( e_i v_j = 0 \) and \( e_i w_{jk} = 0 \) for other choices of \( i, j, k \). The embedding \( U(2) \hookrightarrow U(3) \) corresponds to \( \langle e_1, h_1, f_1 \rangle \cong \mathfrak{sl}(2, \mathbb{C}) \hookrightarrow \mathfrak{sl}(3, \mathbb{C}) \).

We realize the representation \( V(a, b) \) for a partition \((a, b)\) as the irreducible component of \( \text{Sym}^{a-b} V \otimes \text{Sym}^b W \) generated by the highest weight vector \( v_1^{a-b} \otimes w_1^b \). In particular, \( V = V(1, 0) \) and \( W = V(1, 1) \). We identify \( V(a + 1, b + 1) \) with the image of the embedding
\[(4.4) \quad \iota_{a+1,b+1} : V(a + 1, b + 1) \hookrightarrow V(a, b) \otimes V(1, 1)
\]
given by
\[
v_1^{a-b} \otimes w_1^b \hookrightarrow (v_1^{a-b} \otimes w_1^b) \otimes w_{12}.
\]

For a partition \((a, b)\), set \( z := a - b \) and define a set of partitions which interlace with \((a, b)\):
\[
\Phi(a, b) := \{ (p, q) \mid a \geq p \geq b \geq q \geq 0 \}.
\]
Clearly, we have
\[
|\Phi(a, b)| = (a - b + 1)(b + 1) = (z + 1)(b + 1).
\]
It follows from Proposition 2.5 that
\[(4.5) \quad \text{the set } \Phi(a, b) \text{ is exactly the set of } U(2)\text{-highest weights in the restriction } V(a, b)|_{U(2)}.
\]

Example 4.2.

(a) \( \Phi(3, 2) = \{(3, 0), (2, 0), (3, 1), (2, 1), (3, 2), (2, 2)\} \).
(b) \( \Phi(4, 3) = \{(4, 0), (3, 0), (4, 1), (3, 1), (4, 2), (3, 2), (4, 3), (3, 3)\} \).

For our purpose, we need to precisely describe \( U(2)\)-highest weight vectors in the restriction \( V(a, b)|_{U(2)} \). In what follows, we specify such vectors. We freely use the \( \mathfrak{sl}(3, \mathbb{C})\)-representation structure on \( V(a, b) \) and apply actions of \( e_i, f_i \) \((i = 1, 2)\) on vectors of \( V(a, b) \).

For \( V(a, 0) \), define
\[
v_{(k,0,a,0)} := v_1^k v_3^{a-k} \quad \text{for } (k, 0) \in \Phi(a, 0) \quad \text{(or equivalently, for } 0 \leq k \leq a).\]

By considering \( \mathfrak{sl}(2, \mathbb{C}) \) action from (4.3), one can see that \( v_{(k,0,a,0)} \) are \( U(2)\)-highest weight vectors with highest weights \((k,0)\).

Next let us consider \( U(2)\)-highest weight vectors of \( V(a + 1, 1) \) via the embedding \( \iota_{a+1,1} \) in (4.4). Obviously, the vectors
\[
v_{(k+1,1:a+1,1)} := v_{(k,0:a,0)} \otimes w_{12}
\]
are contained in \( V(a + 1, 1) \) from the construction of \( V(a + 1, 1) \), and each of them is a \( U(2)\)-highest weight vector of \( V(a + 1, 1) \), which generates a \( (k+1)\)-dimensional \( U(2)\)-module. Thus they correspond to the partitions \((k + 1, 1)\) in \( \Phi(a + 1, 1) \). Similarly, the vectors
\[
v_{(k+1,0:a+1,1)} := f_2 \left( v_{(k,0:a,0)} \otimes w_{12} \right) = v_{(k,0:a,0)} \otimes w_{13}
\]
are contained in \( V(a+1,1) \) and are \( U(2) \)-highest weight vectors of \( V(a+1,1) \). Each of them generates a \((k+2)\)-dimensional \( U(2) \)-module. Hence they correspond to the partitions \((k+1,0)\) in \( \Phi(a+1,1) \).

Since
\[
\Phi(a+1,1) = \{(k+1,1) \mid 0 \leq k \leq a\} \cup \{(k+1,0) \mid 0 \leq k \leq a\},
\]
we have obtained all the \( U(2) \)-highest weight vectors of the restriction \( V(a+1,1)|_{U(2)} \).

Generally, for \((p,q) \in \Phi(a,b)\), define
\[
\nu((p,q,a,b)) := \nu_2^{p-b} \nu_3^{a-p} \otimes \nu_{12}^q \nu_{13}^{b-q}, \quad b \leq p \leq a.
\]

**Lemma 4.3.** The vectors \( \nu((p,q,a,b)) \) are \( U(2) \)-highest weight vectors of \( V(a,b) \) for \((p,q) \in \Phi(a,b)\).

**Proof.** It is straightforward to check that \( \nu((p,q,a,b)) \) are \( U(2) \)-highest weight vectors. For induction, assume that \( \nu((p,q,a-1,b-1)) \) are contained in \( V(a-1,b-1) \). Then, by the construction of \( V(a,b) \), we have
\[
\nu((p,q,a-1,b-1)) \otimes \nu_{12} = \left( \nu_1^k \nu_3^{a-b-k} \otimes \nu_{12}^{b-1} \right) \otimes \nu_{12} = \nu_1^k \nu_3^{a-b-k} \otimes \nu_{12} = \nu((p,q,a,b)) \in V(a,b)
\]
for \( 0 \leq k \leq a - b \). Since
\[
f_2 \nu((p,q,a,b)) = \frac{b!}{(b-l)!} \nu_1^k \nu_3^{a-b-k} \otimes \nu_{12}^{b-l} \nu_{13}^{l} = \frac{b!}{(b-l)!} \nu((p,q,a,b)) \quad \text{for } 0 \leq l \leq b,
\]
we have \( \nu((p,q,a,b)) \in V(a,b) \) for any \((p,q) \in \Phi(a,b)\). \( \square \)

The vectors in the above lemma are distinct, linearly independent and exhaust all the \( U(2) \)-highest weight vectors in \( V(a,b) \). Note that

\[J(v((p,q,a,b))) = J(v_2^{p-b} v_3^{a-p} \otimes \nu_{12}^q \nu_{13}^{b-q}) = (-1)^p v_2^{p-b} v_3^{a-p} \otimes \nu_{12}^q \nu_{13}^{b-q},\]

which by (4.6) is a \( U(2) \)-lowest vector in the \( U(2) \)-representation generated by \( v((p,q,a,b)) \).

**Proposition 4.4.** For a partition \( \lambda = (a,b,0) \), the multiplicity \( m_{\lambda'}(U(1)) \) of the trivial representation in \( V(\lambda)|_{U(1)} \) is equal to the cardinality of the set
\[
\Phi^{(2)}(a,b) := \{(p,q) \in \Phi(a,b) \mid p \equiv q \bmod 2\},
\]
and the multiplicity \( m_{\lambda'}(H_3) \) of the trivial representation in \( V(\lambda)|_{H_3} \) is equal to the cardinality of the set
\[
\Phi^{(4)}(a,b) := \{(p,q) \in \Phi(a,b) \mid p \equiv q \bmod 4\}.
\]

**Proof.** From the embeddings of \( U(1) \) into \( U(3) \), we see that the multiplicity \( m_{\lambda'}(U(1)) \) is equal to the number of linearly independent vectors in \( V(\lambda) \) with weight \( \mu \) such that \( \mu(h_1) = 0 \).

Similarly, the multiplicity \( m_{\lambda'}(H_3) \) is equal to the number of linearly independent vectors \( v \) in \( V(\lambda) \) with weight \( \mu \) such that
\[
\mu(h_1) = 0 \quad \text{and} \quad Jv = v.
\]

If we consider the restriction \( V(\lambda)|_{U(2)} \), then the condition \( \mu(h_1) = 0 \) means weight 0. A weight 0 vector occurs in \( V_2(p,q) \) precisely when \( p - q \equiv 0 \) with multiplicity 1, where \( V_2(p,q) \) is the irreducible representation of \( U(2) \) with highest weight \( (p,q) \). Thus the first assertion follows from (4.5).
Write $p - q = 2k$. Then $v := f^k_1 v_{(p,q,a,b)}$ is a weight 0 vector by Lemma 4.3. Using (J1) and (J2) in (4.6), we obtain

$$J(v) = J f^k_1 (v_1^{p-b} v_3^{a-p} \otimes w^{q-b}_{12} w^{b-q}_{13}) = (-1)^k e^k_1 J (v_1^{p-b} v_3^{a-p} \otimes w^{q-b}_{12} w^{b-q}_{13})$$

$$= (-1)^k e^k_1 (v_2^{p-b} v_3^{a-p} \otimes w^{q-b}_{12} w^{b-q}_{23}) = (-1)^k f^k (v_1^{p-b} v_3^{a-p} \otimes w^{q-b}_{12} w^{b-q}_{13}) = (-1)^k v.$$ 

Thus $v$ is fixed only when $k$ is even. Thus the second assertion follows. \qed

The cardinalities of the sets $\Phi^{(2)}$ and $\Phi^{(4)}$ are computed in the following proposition.

**Proposition 4.5.** For a partition $(a, b)$, write $z := a - b$. Then we have

$$|\Phi^{(2)}(a, b)| = \lceil (z + 1)(b + 1)/2 \rceil,$$

and

$$|\Phi^{(4)}(a, b)| = \frac{1}{2} \left( \lceil (z + 1)(b + 1)/2 \rceil + \tau(z, b) \right)$$

where $\tau(z, b) \in \{0, \pm 1\}$ is defined on the congruence classes of $z$ and $b$ modulo 4 as follows:

$$
\begin{array}{cccc}
 z \mod 4 & 0 & 1 & 2 & 3 \\
 \tau(z, b) & 0 & 1 & -1 & 0 \\
 1 & 0 & 0 & 0 & 0 \\
 2 & -1 & -1 & 0 & 0 \\
 3 & 0 & 0 & 0 & 0 \\
\end{array}
$$

**Proof.** The elements $(p, q)$ in $\Phi(a, b)$ and the corresponding dimensions $p - q + 1$ can be each arranged into an array of size $(z + 1) \times (b + 1)$ as follows, where we put $(p, q)$ in the left and its corresponding dimensions in the right:

$$(a, 0) \quad (a, 1) \quad \cdots \quad (a, b)$$

$$\begin{array}{cccc}
 (a-1, 0) & (a-1, 1) & \cdots & (a-1, b) \\
 \vdots & \vdots & \cdots & \vdots \\
 (b, 0) & (b, 1) & \cdots & (b, b) \\
\end{array}
$$

By counting the number of odd integers in the right array, we obtain

$$|\Phi^{(2)}(a, b)| = \lceil (z + 1)(b + 1)/2 \rceil,$$

and by counting the number of integers congruent to 1 modulo 4 in the right array, we get

$$|\Phi^{(4)}(a, b)| = \frac{1}{2} \left( \lceil (z + 1)(b + 1)/2 \rceil + \tau(z, b) \right). \quad \Box$$

Now we state and prove the main theorem of this subsection.

**Theorem 4.6.** For a partition $\lambda' = (a + k, b + k, k)$, $k \in \mathbb{Z}_{\geq 0}$, we have

$$m_{\lambda'}(U(1)) = \lceil (z + 1)(b + 1)/2 \rceil \quad \text{and} \quad m_{\lambda'}(H_3) = \frac{1}{2} \left( \lceil (z + 1)(b + 1)/2 \rceil + \tau(z, b) \right).$$
where we set $z := a - b$. Furthermore, for any $m \in \mathbb{Z}_{\geq 1}$, we have

\begin{equation}
\prod_{i=1}^{m} (1 + x_i)(1 + x_i^2) = \sum_{\lambda \leq (3^m)} \tau(z, b) S^{U(m)}_{\lambda}(x), 
\end{equation}

\begin{equation}
\prod_{i=1}^{m} (1 - x_i)(1 + x_i^2) = \sum_{\lambda \leq (3^m)} (-1)^{|\lambda|} \tau(z, b) S^{U(m)}_{\lambda}(x).
\end{equation}

**Proof.** From Proposition 4.4, we obtain

$$m_{\lambda}(U(1)) = |\Phi^{(2)}(a, b)| \quad \text{and} \quad m_{\lambda}(H_3) = |\Phi^{(4)}(a, b)|,$$

and the formulas for $m_{\lambda}(U(1))$ and $m_{\lambda}(H_3)$ are from Proposition 4.5.

Let $d_\gamma, d_{\gamma_1}$ be the probability Haar measures on $H_3, U(1) \leq U(3)$, respectively. Since $H_3 = U(1) \sqcup JU(1)$, we use (4.1) and Proposition 2.3 to obtain

$$\sum_{\lambda \leq (3^m)} m_{\lambda}(H_3) S^{U(m)}_{\lambda}(x) = \int_{H_3} \prod_{i=1}^{m} \det(I + x_i \gamma) d_\gamma$$

$$= \frac{1}{2} \int_{U(1)} \prod_{i=1}^{m} \det(1 + x_i \gamma) d_\gamma + \frac{1}{2} \int_{JU(1)} \prod_{i=1}^{m} (1 + x_i)(1 + x_i^2) d_\gamma_1$$

$$= \frac{1}{2} \sum_{\lambda \leq (3^m)} m_{\lambda}(U(1)) S^{U(m)}_{\lambda}(x) + \frac{1}{2} \prod_{i=1}^{m} (1 + x_i)(1 + x_i^2).$$

Hence, using Lemma 4.1,

$$\prod_{i=1}^{m} (1 + x_i)(1 + x_i^2) = \sum_{\lambda \leq (3^m)} (2m_{\lambda}(H_3) - m_{\lambda}(U(1))) S^{U(m)}_{\lambda}(x) = \sum_{\lambda \leq (3^m)} \tau(z, b) S^{U(m)}_{\lambda}(x).$$

The identity (4.9) follows from (4.8) by replacing $x_i$ with $-x_i$. \qed

**Remark 4.7.** The left hand side of (4.8) is a simple combination of the monomial symmetric functions $m_{\lambda}^{(m)}$ associated with partitions $\lambda$ in $m$-variables:

$$\prod_{i=1}^{m} (1 + x_i)(1 + x_i^2) = \sum_{\lambda \leq (3^m)} m_{\lambda}^{(m)}(x).$$

**Example 4.8.** (1) Let us see an example for the case $m = 7$ in Theorem 4.6. We have

\begin{equation}
\prod_{i=1}^{7} (1 + x_i)(1 + x_i^2) = \sum_{\lambda \leq (3^7)} \tau(z, b) S^{U(7)}_{\lambda}(x)
\end{equation}

One can see that $S^{U(7)}_{(3^5, 2^2, 1^1)}(x)$ appears in the right hand side of (4.10) with the coefficient $-1$ as $\tau(1, 2) = -1$. As a polynomial, $S^{U(7)}_{(3^5, 2^2, 1^1)}(x)$ contains 1778 monomial terms and $S^{U(7)}_{(3^5, 2^2, 1^1)}(1) = 7560$, where $1 = (1, 1, \ldots, 1)$. There are 18 Schur functions with coefficient $-1$ in the right hand side of (4.10) including $S^{U(7)}_{(3^5, 2^2, 1^1)}(x)$. On the other hand, we obtain a polynomial in the left hand side of (4.10),
which contains 16384 monomial terms with coefficient all 1. One can check this, for example, using SageMath.

(2) Let us consider the case \( m = 20 \) in Theorem 4.6. We have

\[
\prod_{i=1}^{20} (1 + x_i)(1 + x_i^2) = \sum_{\lambda \subseteq (3^{20})} \tau(z, b) S_{\lambda}^{U(20)}(x)
\]

Here \( S_{\lambda}^{U(20)}(x) \) appears in the right hand side of (4.11) with the coefficient \(-1\) as \( \tau(5, 6) = \tau(1, 2) = -1 \). Using WeylCharacterRing in SageMath, one can check that \( S_{(2^6, 1^5)}^{U(20)}(1) = 4557090720 \).

Including \( S_{(2^6, 1^5)}^{U(20)} \), there are 315 Schur functions with coefficient \(-1\) in the right hand side of (4.11) and the specialization of the left hand side of (4.11) at 1 is equal to \( 420 = 1099511627776 \). However, checking whether the right hand side of (4.11) coincides with the left hand side of (4.11) may well go beyond the capacity of a regular personal computer.

### 4.2. Subgroup \( \langle U(1), J, \zeta_4 \rangle \)

Let us set

\[
\zeta_4 := \begin{pmatrix}
\sqrt{-1} & 0 & 0 \\
0 & \sqrt{-1} & 0 \\
0 & 0 & 1
\end{pmatrix} \in U(3) \setminus U(1).
\]

Then we have

\[
det(I + x\zeta_4 J\gamma) = (1 + x)(1 - x^2) \quad \text{and} \quad det(I + xJ\gamma) = (1 + x)(1 + x^2).
\]

Define \( H_{3,4} \) to be the subgroup generated by \( U(1), J, \zeta_4 \), i.e.

\[
H_{3,4} := \langle U(1), J, \zeta_4 \rangle \leq U(3).
\]

Then we have the coset decomposition

\[
H_{3,4} = U(1) \sqcup JU(1) \sqcup \zeta_4 U(1) \sqcup \zeta_4 JU(1).
\]

Let \( H'_{3,4} \) be the subgroup of \( H_{3,4} \) generated by \( U(1), \zeta_4 \). Note that

\[
H_{3,4} = H'_{3,4} \sqcup JH'_{3,4}.
\]

**Lemma 4.9.** For any \( c \in \mathbb{Z}_{\geq 2} \), we have

\[
m_{(a, b, c)}(H_{3,4}) = m_{(a-2b-2c-2)}(H_{3,4}) \quad \text{and} \quad m_{(a, b, c)}(H'_{3,4}) = m_{(a-2b-2c-2)}(H'_{3,4}).
\]

**Proof.** As before, let \( \text{det} \) be the one-dimensional representation of \( U(3) \) defined by the determinant. Then we have

\[
V(a, b, c) = \text{det}^2 \otimes V(a - 2, b - 2, c - 2).
\]

Since \( \text{det}(A) = 1 \) for any \( A \in U(1) \), \( \text{det}(J) = 1 \) and \( \text{det}(\zeta_4) = -1 \), the assertion follows. \( \square \)

By Lemma 4.9, it suffices to consider partitions of the form

\[(a, b, \epsilon), \quad \epsilon \in \{0, 1\}.
\]

For a partition \((a, b, \epsilon) \in \{0, 1\}\), define

\[
\Phi(a, b, \epsilon) := \{(p, q) \in \mathbb{Z}^2 \mid a \geq p \geq b \geq q \geq \epsilon\}.
\]
For example, $\Phi(3, 2, 1) = \{(3, 2), (2, 2), (3, 1), (2, 1)\}$. From Proposition 2.5, we see that

(4.15) the set $\Phi(a, b, \epsilon)$ is exactly the set of $U(2)$-highest weights in the restriction $V(a, b, \epsilon)|_{U(2)}$.

**Proposition 4.10.** For a partition $\lambda' = (a, b, \epsilon)$ ($\epsilon \in \{0, 1\}$), the multiplicity $m_{\lambda'}(H_{3, 4}')$ is the same as the cardinality of the set

$$\Phi^{(2, 4)}(\lambda') = \{(p, q) \in \Phi(\lambda') \mid p + q \equiv 0 \},$$

and the multiplicity $m_{\lambda'}(H_{3, 4}')$ is the same as the cardinality of the set

$$\Phi^{(4, 4)}(\lambda') = \{(p, q) \in \Phi(\lambda') \mid p - q \equiv 0, p + q \equiv 0 \}.$$

**Proof.** Let $v$ be a weight vector fixed by $H_{3, 4}'$. Then, in particular, $v$ is fixed by $U(1)$, and we may write $v = f^k_{1}v_{(p, q, a, b)}$ for $p - q = 2k$ as in the proof of Proposition 4.4. Since

$$v = f^k_{1}(v_{1}^{p-b+k}a-p \otimes w_{12}^{q}w_{13}^{b-q}) = C v_{1}^{p-b-k}v_{2}^{k}e_{3}^{a-p} \otimes w_{12}^{q}w_{13}^{b-q} + \text{(other terms with the same weight)},$$

for some constant $C$, the number of $v_{1}$ factors and the number of $v_{2}$ factors are the same, which is equal to $k + q = (p + q)/2$, and we have

$$\zeta_{4}v = (\sqrt{-1})^{p+q}v = v.$$

Thus we have $p + q \equiv 0$, which implies $p - q \equiv 0$.

If $v$ is also fixed by $J$, the we obtain an additional condition $p - q \equiv 0$ as in (the proof of) Proposition 4.4.

The cardinalities of the sets $\Phi^{(2, 4)}(\lambda')$ and $\Phi^{(4, 4)}(\lambda')$ will be computed in the next section, and we present the resulting formulas in the following two propositions.

**Proposition 4.11.** For a partition $\lambda' = (a, b, \epsilon)$ ($\epsilon \in \{0, 1\}$), write $z := a - b$ and $b' := b - \epsilon$. Then we have

(4.16) $|\Phi^{(2, 4)}(\lambda')| = \frac{(z + 1)(b' + 1) + \kappa(z, b')}{4},$

where $\kappa(z, b')$ are defined on the congruence classes of $z$ and $b'$ modulo 4 by

$$\kappa_{0}(z, b) = \begin{array}{cccc} z \backslash b & 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 0 \\ 2 & 1 & 0 & 1 \\ 3 & 0 & 1 & 2 \end{array} \quad \text{and} \quad \kappa_{1}(z, b) = \begin{array}{cccc} z \backslash b & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 \end{array}.$$

**Proposition 4.12.** For a partition $\lambda' = (a, b, \epsilon)$ ($\epsilon \in \{0, 1\}$), write $z := a - b$ and $b' := b - \epsilon$. Then we have

(4.17) $|\Phi^{(4, 4)}(\lambda')| = \frac{b'z + \eta(z, b') \cdot (b', z) + \xi(z, b')}{8},$

where $(x, y) \cdot (b', z) = xb' + yz$, and $\eta(z, b')$ and $\xi(z, b')$ are defined on the congruence classes of $z$ and $b'$ by

$$\eta_{0}(z, b) = \begin{array}{cccc} z \backslash b & 0 & 1 & 2 \\ 0 & (2, 2) & (0, 1) & (1, 2) \\ 1 & (1, 2) & (1, 1) & (0, 1) \end{array}$$

$\quad \text{and} \quad \eta_{1}(z, b') = \begin{array}{cccc} z \backslash b' & 0 & 1 & 2 \\ 0 & (0, 0) & (2, 1) & (0, 1) \\ 1 & (1, 0) & (1, 1) & (0, 1) \end{array}.$
\[ \begin{array}{c|cccc} z \backslash b & 0 & 1 & 2 & 3 \\ \\
0 & 8 & 0 & 4 & 0 \\ \\
1 & 6 & -3 & 2 & 1 \\ \\
2 & 4 & -4 & 4 & 0 \\ \\
3 & 2 & 1 & 2 & 1 \\ \\
\end{array} \quad \text{and} \quad \begin{array}{c|cccc} z \backslash b' & 0 & 1 & 2 & 3 \\ \\
0 & 0 & 0 & 0 & 2 \\ \\
1 & 0 & 3 & -4 & 1 \\ \\
2 & 0 & 2 & -4 & 2 \\ \\
3 & 0 & 1 & 0 & 1 \\ \\
\end{array} \]

Corollary 4.13. For each partition \( \lambda' = (a, b, \epsilon) \ (\epsilon \in \{0, 1\}) \), write \( z := a - b \) and \( b' := b - \epsilon \). Then we have

\[ \omega_\epsilon(z, b') := 2|\Phi^{(4,4)}(\lambda')| - |\Phi^{(2,4)}(\lambda')| = \frac{\alpha_\epsilon(z, b') \cdot (b', z) + \beta_\epsilon(z, b')}{4}, \]

where \( (x, y) \cdot (b', z) = xb' + yz \), and \( \alpha_\epsilon(z, b') \) and \( \beta_\epsilon(z, b') \) are defined on the congruence classes of \( z \) and \( b' \) by

\[ \begin{array}{c|cccc} z \backslash b & 0 & 1 & 2 & 3 \\ \\
0 & (1, 1) & (-1, 0) & \cdot \ & \cdot \ \\ \\
1 & (0, 1) & (0, 0) & \cdot \ & \cdot \ \\ \\
\end{array} \quad \text{and} \quad \begin{array}{c|cccc} z \backslash b' & 0 & 1 & 2 & 3 \\ \\
0 & (-1, -1) & (1, 0) & \cdot \ & \cdot \ \\ \\
1 & (0, -1) & (0, 0) & \cdot \ & \cdot \ \\ \\
\end{array} \]

Now we state and prove the main result of this subsection.

Theorem 4.14. For a partition \( \lambda' = (a + 2k, b + 2k, \epsilon + 2k) \ (k \in \mathbb{Z}_{\geq 0}, \epsilon \in \{0, 1\}) \), write \( z := a - b \) and \( b' := b - \epsilon \). Then we have

\[ m_{\lambda'}(H_{3,4}^k) = \frac{(z + 1)(b' + 1) + \kappa_\epsilon(z, b')}{4} \quad \text{and} \quad m_{\lambda'}(H_{3,4}) = \frac{b'z + \tau_\epsilon(z, b') \cdot (b', z) + \xi_\epsilon(z, b')}{8}. \]

Furthermore, for any \( m \in \mathbb{Z}_{\geq 1} \), we have

\[ \frac{1}{2} \left( \prod_{i=1}^{m} (1 + x_i)(1 + x_i^2) + \prod_{i=1}^{m} (1 + x_i)(1 - x_i^2) \right) = \sum_{\lambda \subseteq (3^m)} \omega_\epsilon(z, b') S^U_{\lambda}(x), \]

where \( z, b', \epsilon \) are determined by the transpose \( \lambda' \) of \( \lambda \subseteq (3^m) \).

Proof. The first assertion follows from Propositions 4.10, 4.11, 4.12 and Lemma 4.9. Let \( d_\gamma, d_{\gamma_1}, d_{\gamma_2} \) be the probability Haar measures on \( H_{3,4}, H_{3,4}', U(1) \leq U(3) \), respectively. We use the coset decompositions (4.13) and (4.14) and the computation (4.12) and Proposition 2.3 to obtain

\[ \sum_{\lambda \subseteq (3^m)} m_{\lambda'}(H_{3,4}) S^U_{\lambda}(x) = \int_{H_{3,4}} \prod_{i=1}^{m} \det(I + x_i \gamma) d_\gamma \]

\[ = \frac{1}{2} \int_{H_{3,4}'} \prod_{i=1}^{m} \det(I + x_i \gamma) d_{\gamma_1} + \frac{1}{4} \int_{U(1)} \prod_{i=1}^{m} \det(I + x_i \gamma) d_{\gamma_2} + \frac{1}{4} \int_{\xi_{\lambda} U(1)} \prod_{i=1}^{m} \det(I + x_i \gamma) d_{\gamma_2} \]

\[ = \frac{1}{2} \sum_{\lambda \subseteq (3^m)} m_{\lambda'}(H_{3,4}^k) S^U_{\lambda}(x) + \frac{1}{4} \prod_{i=1}^{m} (1 + x_i)(1 + x_i^2) + \frac{1}{4} \prod_{i=1}^{m} (1 + x_i)(1 - x_i^2). \]
Hence, using Corollary 4.13,
\[
\frac{1}{2} \left( \prod_{i=1}^{m} (1 + x_i)(1 + x_i^2) + \prod_{i=1}^{m} (1 + x_i)(1 - x_i^2) \right) = \sum_{\lambda \leq (3^m)} (2m_{\lambda'}(H_{3,4}) - m_{\lambda'}(H'_{3,4})) S^U_{\lambda}(x) \\
= \sum_{\lambda \leq (3^m)} \omega_\lambda(z, b') S^U_{\lambda}(x). \quad \square
\]

**Remark 4.15.** (1) The left hand side of (4.18) can be written as a simple combination of the monomial symmetric functions in \(m\)-variables:
\[
\frac{1}{2} \left( \prod_{i=1}^{m} (1 + x_i)(1 + x_i^2) + \prod_{i=1}^{m} (1 + x_i)(1 - x_i^2) \right) = \sum_{\lambda' \leq (m^3)} m_{\lambda}(x).
\]
where \(\lambda' = (\lambda_1', \lambda_2', \lambda_3').\)

(2) By replacing \(x_i\) by \(-x_i\) in (4.18), we obtain
\[
\frac{1}{2} \left( \prod_{i=1}^{m} (1 - x_i)(1 + x_i^2) + \prod_{i=1}^{m} (1 - x_i)(1 - x_i^2) \right) = \sum_{\lambda \leq (3^m)} (-1)^{\lambda_1} \omega_\lambda(z, b') S^U_{\lambda}(x).
\]

**Example 4.16.** (1) Let us see the case \(m = 7\) in Theorem 4.14. Then we have
\[
(4.19) \quad \frac{1}{2} \left( \prod_{i=1}^{7} (1 + x_i)(1 + x_i^2) + \prod_{i=1}^{7} (1 + x_i)(1 - x_i^2) \right) = \sum_{\lambda \leq (3^7)} \omega_\lambda(z, b') S^U_{\lambda}(x)
\]

The function \(S^U_{(3,2^2,1)}(x)\) appears in the right hand side of (4.19) with the coefficient \(-2\) as \(\lambda' = (4, 3, 1)\) and hence \(z = 1\) and \(b' = 2\). As a polynomial itself, \(S^U_{(3,2^2,1)}(x)\) contains 1239 monomial terms and \(S^U_{(3,2^2,1)}(1) = 3402\). There are 36 Schur functions with negative coefficients in the right hand side of (4.19) including \(S^U_{(3,2^2,1)}(x)\). However, we obtain a polynomial of 8192 monomial terms with all coefficient 1 in the left hand side of (4.19).

(2) Let us see the case \(m = 20\) in Theorem 4.14:
\[
(4.20) \quad \frac{1}{2} \left( \prod_{i=1}^{20} (1 + x_i)(1 + x_i^2) + \prod_{i=1}^{20} (1 + x_i)(1 - x_i^2) \right) = \sum_{\lambda \leq (3^m)} \omega_\lambda(z, b') S^U_{\lambda}(x)
\]

Here \(S^U_{(2^9,1^2)}(x)\) appears in the right hand side of (4.20) with the coefficient \(-3\) as \(\lambda' = (11, 9)\) and hence \(z = 2\) and \(b' \equiv 1\). One can check that \(S^U_{(2^9,1^2)}(1) = 12342120700\). There are 590 Schur functions with negative coefficients in the right hand side of (4.20), and the specialization of the right hand side of (4.20) at 1 is equal to \(2^{39}\). This shows some systematic, even miraculous, cancellations of monomial terms in Schur functions involved in this example.

Combining Theorems 4.6 and 4.14, we obtain the following identity.
Corollary 4.17. For each partition \( \lambda' = (a + 2k, b + 2k, \epsilon + 2k) \) \((k \in \mathbb{Z}_{\geq 0}, \epsilon \in \{0, 1\}\)), write \( z := a - b \) and \( b' := b - \epsilon \). Then we have
\[
\prod_{i=1}^{m}(1 + x_i)(1 - x_i^2) = \sum_{(\lambda_1, \lambda_2, \lambda_3) \leq (m^3)} (-1)^{\delta(\lambda_2 \equiv 21)} m_{\lambda}^{(m)}(x) = \sum_{\lambda \leq (3m)} \tilde{\omega}_\epsilon(z, b') S_{\lambda}^{U(m)}(x),
\]
where
\[
\tilde{\omega}_\epsilon(z, b') := \frac{\alpha_\epsilon(z, b') \cdot (b', z) + \beta_\epsilon(z, b')}{2}.
\]
Here \( \beta_\epsilon(z, b') \) are defined on the congruence classes of \( z \) and \( b' \) by
\[
\tilde{\beta}_0(z, b) = \begin{vmatrix} \bar{z} \mid \bar{b} & 0 & 1 & 2 & 3 \\ 0 & 2 & -1 & 1 & 0 \\ 1 & 1 & 0 & -3 & 0 \\ 2 & -2 & -1 & 4 & 1 \\ 3 & -1 & 0 & -1 & 0 \end{vmatrix}
\text{ and } \tilde{\beta}_1(z, b') = \begin{vmatrix} \bar{z} \mid \bar{b}' & 0 & 1 & 2 & 3 \\ 0 & 4 & 1 & 2 & -1 \\ 1 & 3 & 0 & -1 & 0 \\ 2 & 2 & -3 & 0 & -1 \\ 3 & 1 & 0 & 1 & 0 \end{vmatrix}
\]

Similarly, combining Theorem 4.6 and Corollary 4.17, we obtain another identity below.

Corollary 4.18. For each partition \( \lambda' = (a + 2k, b + 2k, \epsilon + 2k) \) \((k \in \mathbb{Z}_{\geq 0}, \epsilon \in \{0, 1\}\)), write \( z := a - b \) and \( b' := b - \epsilon \). Then we have
\[
\frac{1}{2} \left( \prod_{i=1}^{m}(1 + x_i)(1 + x_i^2) - \prod_{i=1}^{m}(1 + x_i)(1 - x_i^2) \right) = \sum_{\lambda \leq (3m)} m_{\lambda}^{(m)}(x) = \sum_{\lambda \leq (3m)} \tilde{\omega}_\epsilon(z, b') S_{\lambda}^{U(m)}(x),
\]
where
\[
\tilde{\omega}_\epsilon(z, b') := \frac{-\alpha_\epsilon(z, b') \cdot (b', z) + \beta_\epsilon(z, b')}{4}.
\]
Here \( \beta_\epsilon(z, b') \) are defined on the congruence classes of \( z \) and \( b' \) by
\[
\tilde{\beta}_0(z, b) = \begin{vmatrix} \bar{z} \mid \bar{b} & 0 & 1 & 2 & 3 \\ 0 & 0 & 3 & -2 & 1 \\ 1 & 1 & 0 & -3 & 0 \\ 2 & -1 & -1 & -4 & 1 \\ 3 & 1 & -1 & -1 & 0 \end{vmatrix}
\text{ and } \tilde{\beta}_1(z, b') = \begin{vmatrix} \bar{z} \mid \bar{b}' & 0 & 1 & 2 & 3 \\ 0 & 4 & 1 & 2 & -1 \\ 1 & 3 & 0 & -1 & 0 \\ 2 & 2 & -3 & 0 & -1 \\ 3 & 1 & 0 & 1 & 0 \end{vmatrix}
\]

Remark 4.19. Replacing \( x_i \) with \(-x_i\), we obtain identities for
\[
\prod_{i=1}^{m}(1 - x_i)(1 + x_i^2) \quad \text{ and } \quad \prod_{i=1}^{m}(1 - x_i)(1 - x_i^2)
\]
and hence identities for
\[
\frac{1}{2} \left( \prod_{i=1}^{m}(1 - x_i)(1 + x_i^2) \pm \prod_{i=1}^{m}(1 - x_i)(1 - x_i^2) \right).
\]

5. PROOFS FOR THE CARDINALITIES OF \( \Phi^{(2, 4)} \) AND \( \Phi^{(4, 4)} \)

In this section, we will prove the explicit closed-form formulas of \( \Phi^{(2, 4)} \) and \( \Phi^{(4, 4)} \), which are presented in Propositions 4.10 and 4.11, respectively. Throughout this section, we will explain the reason why the functions \( \kappa_\epsilon(z, b), \eta_\epsilon(z, b) \) and \( \xi_\epsilon(z, b) \) are defined with respect to congruence classes modulo 2 or 4. We start this section with an example.

Let us consider the partition \( \nu' = (a, b, 0) = (a, b) = (7, 2) \) of length 2. Note that
\[
\bullet \quad z = a - b = 5 \quad \text{and} \quad z + 1 = 6, \ b + 1 = 3,
\]
among the partitions \((a, b)\) with
\[ a - b = 5 \quad \text{and} \quad b \equiv 4 \pmod{2}, \]
the partition \(7, 2\) is the smallest one.

Thus the set \(\Phi(7, 2)\) can be displayed by the following \((6 \times 3)\)-array as in (4.7):

\[
\begin{array}{ccc}
(7, 2) & (7, 1) & (7, 0) \\
(6, 2) & (6, 1) & (6, 0) \\
(5, 2) & (5, 1) & (5, 0) \\
(4, 2) & (4, 1) & (4, 0) \\
(3, 2) & (3, 1) & (3, 0) \\
(2, 2) & (2, 1) & (2, 0) \\
\end{array}
\begin{array}{cccc}
(z + b, b) & (z + b, b - 1) & (z + b, 0) \\
(z + b - 1, b) & (z + b - 1, b - 1) & (z + b - 1, 0) \\
\cdots & \cdots & \cdots \\
(b + 1, b) & (b + 1, 2) & (b + 1, 0) \\
(b, b) & (b + 1, 1) & (b, 0) \\
\end{array}
\]

Motivated by (5.2), for \((k, l) \in \mathbb{Z}^2\) and \(z, b \in \mathbb{Z}_{\geq 0}\), we define the set \(\{(k, l)\}^z_b\) of pairs of integers:
\[
\{(k, l)\}^z_b := \{(p, q) \in \mathbb{Z}^2 \mid k + b \geq p \geq k, \quad z + l \geq q \geq l\},
\]
whose cardinality is \((z + 1) \times (b + 1)\).

In (5.2), one can check that, for \(\nu' = (7, 2)\), we have
- \(\Phi^{(2, 4)}(\nu') = \{(4, 0), (3, 1), (7, 1), (6, 2), (2, 2)\}\),
- \(\Phi^{(4, 4)}(\nu') = \{(4, 0), (6, 2), (2, 2)\}\),

and hence \(\Phi^{(2, 4)}(\nu') \setminus \Phi^{(4, 4)}(\nu') = \{(3, 1), (7, 1)\}\). Here one can notice that \((p, q) \in \Phi^{(2, 4)}(\nu') \setminus \Phi^{(4, 4)}(\nu')\) are located in the second column of (5.2), since
\[
q \equiv 4 \pmod{4} \quad \text{and hence} \quad p \equiv 4 \pmod{4}, \quad \text{yielding} \quad p - q \not\equiv 4 \pmod{4}.
\]

Let us consider a partition \(\mu' = (11, 6)\) which is the second smallest case in the sense of (5.1). Then its corresponding \(\Phi(11, 6)\) can be represented by the following \((6 \times (3 + 4))\)-array as follows:

\[
\begin{array}{cccc}
(11, 6) & (11, 5) & (11, 4) & (11, 3) \\
(10, 6) & (10, 5) & (10, 4) & (10, 3) \\
(9, 6) & (9, 5) & (9, 4) & (9, 3) \\
(8, 6) & (8, 5) & (8, 4) & (8, 3) \\
(7, 6) & (7, 5) & (7, 4) & (7, 3) \\
(6, 6) & (6, 5) & (6, 4) & (6, 3) \\
\end{array}
\begin{array}{cccc}
\cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
\end{array}
\]

Here the left (resp. right) part of (5.5) coincides with \(\left\lfloor b + 4, 4 \right\rfloor \begin{array}{cccc} z \end{array}\) (resp. \(\left\lceil b + 4, 0 \right\rceil \begin{array}{cccc} z \end{array}\)).

Then one can see that (i) the left part of (5.5) can be obtained from (5.2) by adding \((4, 4)\) for each entry and (ii) the right part of (5.5) consists of four columns. Since the number of \((p, q)\)'s satisfying \(p + q \equiv 4 \pmod{4}\) in \(\left\lfloor b + 4, 4 \right\rfloor \begin{array}{cccc} z \end{array}\) is the same as \(\Phi^{(2, 4)}(\nu')\) by (i), and each row in \(\left\lceil b + 4, 0 \right\rceil \begin{array}{cccc} z \end{array}\) contains \((p, q)\) with \(p + q \equiv 4 \pmod{4}\) exactly once by (ii), we have
\[
|\Phi^{(2, 4)}(\mu')| = |\Phi^{(2, 4)}(\nu')| + (z + 1) = 5 + 6.
\]

For the same reason as in (5.4), the \((p, q)\)'s with \(p + q \equiv 4 \pmod{4}\) cannot satisfy the condition \(p - q \equiv 4 \pmod{4}\). Thus we obtain
\[
|\Phi^{(4, 4)}(\mu')| = |\Phi^{(4, 4)}(\nu')| + \frac{(z + 1)}{2} = 3 + 3.
\]
Since
\[ \Phi^{(4,4)}(c + 4, d + 4) = \left( \Phi^{(4,4)}(c, d) + (4, 4) \right) \sqcup (d + 4, 0) \]
and the number of \((p, q)\)'s satisfying \(p + q \equiv 4 \mod 0\) (resp. \(p + q \equiv 1 \mod 0\) and \(p - q \equiv 4 \mod 0\)) in \((d + 4, 0)\) does not change for any \((c, d)\) with \(c - d = 5\) and \(d \equiv 4 \mod 2\), the closed-form formulas are written in (4.16) and (4.17), which are determined by \(|\Phi^{(k,4)}(\nu')|\), \(|\Phi^{(k,4)}(\mu')|\) \(-|\Phi^{(k,4)}(\nu')|\) \((k = 2, 4)\) depending on \(z\) and on the congruent classes of \(z\) and \(b\) modulo 4.

Now we generalize the argument above. We begin with a definition.

**Definition 5.1.** Fix \(\epsilon \in \{0, 1\}\), \(b' \in \{0, 1, 2, 3\}\) and \(z \in \mathbb{Z}_{\geq 0}\). We call the partition
\[ \nu'(z, b', \epsilon) := (z + b' + \epsilon, b' + \epsilon, \epsilon) \]
the **base** partition for the triple \((z, b', \epsilon)\).

For any finite set \(X\) of pairs of integers, we denote by \(\phi^{(2,4)}(X)\) (resp. \(\phi^{(4,4)}(X)\)) the number of \((p, q)\)'s in \(X\) satisfying \(p + q \equiv 4 \mod 0\) (resp. \(p + q \equiv 1 \mod 0\) and \(p - q \equiv 4 \mod 0\)). In particular, we write
\[ \phi^{(u,4)}(\nu') := \phi^{(u,4)}(\Phi(\nu')) = |\Phi^{(u,4)}(\nu')| \quad (u = 2, 4) \]
for a partition \(\nu'\) of the form \((z + b' + \epsilon, b' + \epsilon, \epsilon)\) \((z, b' \in \mathbb{Z}_{\geq 0}, \epsilon \in \{0, 1\})\). Then the following lemma is obvious:

**Lemma 5.2.** For any \(\left[ (k, l) \right]_b^z\) and \((m, n) \in \mathbb{Z}^2\), we have
\[ \phi^{(u,4)}(\left[ (k, l) \right]_b^z) = \phi^{(u,4)}(\left[ (k, l) \right]_b^{z + (4m, 4n)}) \quad (u = 2, 4). \]

**Lemma 5.3.** For any partition \((a, b, \epsilon)\) with \(a - b = z \in \mathbb{Z}_{\geq 0}\) and \(\epsilon \in \{0, 1\}\), we have
\[ \phi^{(2,4)}(a + 4, b + 4, \epsilon) - \phi^{(2,4)}(a, b, \epsilon) = z + 1. \]

**Proof.** Let \(z = a - b\) and \(b' = b - \epsilon\). Then we have
\[ \Phi(a + 4, b + 4, \epsilon) = \left( b + 4, \epsilon \right)_{b'+4}, \quad \Phi(a, b, \epsilon) = \left( b, \epsilon \right)_y' \]
and hence
\[ \Phi(a + 4, b + 4, \epsilon) = \Phi(a, b, \epsilon) + (4, 4) \sqcup \left( b, \epsilon \right)_3^z. \]

By Lemma 5.2, \(\phi^{(u,4)}(a, b, \epsilon) = \phi^{(u,4)}(\left( b, \epsilon \right)_3 + (4, 4)) \quad (u = 2, 4)\) and hence
\[ \phi^{(u,4)}(a + 4, b + 4, \epsilon) - \phi^{(2,4)}(a, b, \epsilon) = \phi^{(u,4)}(\left( b, \epsilon \right)_3^z) \quad (u = 2, 4). \]

Recall that \(\left( b, \epsilon \right)_3^z\) consists of four columns with height \(z + 1\). Thus we have
\[ \phi^{(2,4)}(\left( b, \epsilon \right)_3^z) = z + 1, \]
because each row in \(\left( b, \epsilon \right)_3^z\) contains \((p, q)\) with \(p + q \equiv 4 \mod 0\) exactly once. \(\Box\)

**Lemma 5.4.** For any base partition \(\nu' = \nu'(z, b', \epsilon)\), we have
\[ \phi^{(2,4)}(\nu') = \frac{(b' + 1)(z + 1) + \kappa(z, b')}{4}. \]
Proof. (1) Note that, when \( b' = 3 \) (resp. \( z \equiv_3 4 \)), the integer \( \kappa_\epsilon(z,b') = 0 \). The assertion for this case comes from the fact that

\[
\Phi(\nu') = \frac{(b' + \epsilon, \epsilon)}{b'}
\]

consists of 4-columns with height \( z + 1 \) (resp. \( (b' + 1) \)-columns with height \( z + 1 = 4k \) for some \( k \in \mathbb{Z}_{\geq 1} \)), implying that each row in the set contains \((p,q)\) with \( p + q \equiv_4 0 \) exactly once (resp. each column in the set contains \((p,q)\)'s with \( p + q \equiv_4 0 \) exactly \( k \)-times).

(2) Now let us consider the case when \( b' = 2, z \equiv_4 1 \) and \( \epsilon = 1 \). Write \( z = 4k + 1 \) for some \( k \in \mathbb{Z}_{\geq 0} \).

Then \( \left[ (b' + \epsilon, \epsilon) \right]_2 \) can be described as follows:

\[
\begin{array}{cccc}
(b' + z + 1, 3) & (b' + z + 1, 2) & (b' + z + 1, 1) \\
(b' + z, 3) & (b' + z, 2) & (b' + z, 1) \\
(b' + z - 1, 3) & (b' + z - 1, 2) & (b' + z - 1, 1) \\
\vdots & \vdots & \vdots \\
(b' + 3, 3) & (b' + 3, 2) & (b' + 3, 1) \\
(b' + 2, 3) & (b' + 2, 2) & (b' + 2, 1) \\
(b' + 1, 3) & (b' + 1, 2) & (b' + 1, 1) \\
\end{array}
\]

(5.7)

Here the underlined pairs \((p,q)\) of integers satisfy the condition that \( p + q \equiv_4 0 \). One sees that the number \( t_i \) of underlined \((p,q)\)'s in the column with \( q = i \) (\( i = 0, 1, 2 \)) is given by

\[
t_0 = k + 1, \quad t_1 = k \quad \text{and} \quad t_2 = k.
\]

Thus we have

\[
\phi^{(2,4)}(\nu') = 3k + 1 = \frac{3 \times (4k + 2) - 2}{4}
\]

which implies \( \kappa_1(1,2) = -2 \).

(3) For the remaining cases, one can prove the formula by a similar argument. \(\square\)

Proof of Proposition 4.11. By Lemma 5.3 and Lemma 5.4, for a partition \((a,b,\epsilon)\) of the form

\[
(z + b' + 4k + \epsilon, b' + 4k + \epsilon, \epsilon) \quad (z = a - b, k \in \mathbb{Z}_{\geq 0} \text{ and } b' \in \{0, 1, 2, 3\}),
\]

we have

\[
\phi^{(2,4)}(a,b,\epsilon) = \frac{(b' + 1)(z + 1) + \kappa_\epsilon(z,b')}{4} + k \times (z + 1)
\]

\[
= \frac{(b' + 1)(z + 1) + \kappa_\epsilon(z,b')}{4} \quad (\because b' = b - \epsilon),
\]

as we desired. \(\square\)

Lemma 5.5. For any partition \((a,b,\epsilon)\) with \( a - b = z \in \mathbb{Z}_{\geq 0} \), we have

\[
\phi^{(4,4)}(a+4,b+4,\epsilon) - \phi^{(4,4)}(a,b,\epsilon) = \frac{z + \eta_\epsilon(z,b)_1}{2},
\]

where \( \eta_\epsilon(z,b)_1 \) denotes the first component of \( \eta_\epsilon(z,b) \).

Proof. By (5.6), it suffices to show that

\[
\phi^{(4,4)} \left( \left[ (b, \epsilon) \right]_3 \right) = \frac{z + \eta_\epsilon(z,b)_1}{2},
\]
where \( \phi^{(2,4)} \left( \begin{pmatrix} b & \epsilon \end{pmatrix}_{3} \right) \) \( \equiv z + 1 \) by Lemma 5.3 and the set \( \begin{pmatrix} b & \epsilon \end{pmatrix}_{3} \) can be described as follows:

\[
\begin{pmatrix}
(b + z, \epsilon + 3) & (b + z, \epsilon + 2) & (b + z, \epsilon) \\
(b + z - 1, \epsilon + 3) & (b + z - 1, \epsilon + 2) & (b + z - 1, \epsilon) \\
(b + z - 2, \epsilon + 3) & (b + z - 2, \epsilon + 2) & (b + z - 2, \epsilon) \\
\vdots & \vdots & \vdots \\
(b + 2, \epsilon + 3) & (b + 2, \epsilon + 2) & (b + 2, \epsilon) \\
(b + 1, \epsilon + 3) & (b + 1, \epsilon + 2) & (b + 1, \epsilon) \\
(b, \epsilon + 3) & (b, \epsilon + 2) & (b, \epsilon)
\end{pmatrix}
\]

Then, for each adjacent two rows, there are exactly two pairs of integers \((p_i, q_i) \ (i = 1, 2)\) satisfying \(p_i + q_i \equiv 0\), where exactly one of them does not satisfy the condition \(p - q \equiv 4 \) and the other satisfies the condition, as in (5.4).

1. For \( z \equiv 1 \), we have always \( \eta_e(z, b) = 1 \) and hence \( \frac{z + \eta_e(z, b)}{2} = z + 1 \). Thus the assertion for this case follows from the fact that \( z + 1 \equiv 2 \).

2. Let us consider the case when \( z \equiv 0, b \equiv 1 \) and \( \epsilon = 0 \). Then it suffices to consider the bottom row in (5.9):

\[
\begin{pmatrix}
(b, 3) & (b, 2) & (b, 1) & (b, 0)
\end{pmatrix}
\]

Then exactly one of \((b, 1)\) and \((b, 3)\) satisfies the condition that \(p + q \equiv 4 \) but it does not satisfy the condition that \(p - q \equiv 4 \), since \( q \equiv 1, 3 \). Hence

\[
\phi^{(4,4)} \left( \begin{pmatrix} b & \epsilon \end{pmatrix}_{3} \right) = \frac{z}{2}.
\]

3. The remaining 3-cases can be proved by a similar argument. □

**Lemma 5.6.** For any base partition \( \nu' = \nu'(z, b', \epsilon) \), we have

\[
\phi^{(4,4)}(\nu') = \frac{b'z + \eta_e(z, b') \cdot (b', z) + \xi_e(z, b')}{8}.
\]

**Proof.** (1) As in Lemma 5.4, let us consider the case when \( b' = 2, z \equiv 1 \) and \( \epsilon = 1 \). Write \( z = 4k + 1 \) for some \( k \in \mathbb{Z}_{\geq 0} \). Then the underlined pairs \((p, q)\) of integers in columns with \( q = 1 \) or \( 3 \) in (5.7) cannot satisfy the condition \( p - q \equiv 4 \), while the ones in the column with \( q = 2 \) in (5.7) satisfy the condition as \( p \equiv 4 \). Thus \( \phi^{(2,4)}(\nu') = k \) by (5.8). Since

\[
k = \frac{2(4k + 1) \cdot (1, 0) \cdot (2, 4k + 1) - 4}{8},
\]

our assertion holds in this case.

(2) For the remaining cases, one can prove the formula by a similar argument. □

**Proof of Proposition 4.12.** By Lemma 5.5 and Lemma 5.6, for a partition \((a, b, \epsilon)\) of the form

\[
(z + b' + 4k + \epsilon, b' + 4k + \epsilon, \epsilon) \quad (z = a - b, k \in \mathbb{Z}_{\geq 0} \text{ and } b' \in \{0, 1, 2, 3\}),
\]

we have

\[
\phi^{(2,4)}(a, b, \epsilon) = \frac{b'z + \eta_e(z, b') \cdot (b', z) + \xi_e(z, b')}{8} + k \times \frac{z + \eta_e(z, b)}{2} + \frac{b'z + \eta_e(z, b') \cdot (b', z) + \xi_e(z, b')}{8} \quad (\because b' = b - \epsilon),
\]

as we desired. □
Remark 5.7. After the first version of this paper was posted on the arXiv, Ronald C. King informed us that identities (A)-(C) in Application can be extracted from the work of Yang–Wybourne, Lascoux–Pragacz and King–Wybourne–Yang [YW, LP, KWY] that appeared in the mathematical physics literature in the 1980’s. Their methods are purely combinatorial and do not have direct connections to auto-correlation functions.

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