The Complete Extensions do not form a Complete Semilattice

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Abstract

In his seminal paper that inaugurated abstract argumentation, Dung proved that the set of complete extensions forms a complete semilattice with respect to set inclusion. In this note we demonstrate that this proof is incorrect with counterexamples. We then trace the error in the proof and explain why it arose. We then examine the implications for the grounded extension.

1 Introduction

Argumentation has been intensively studied by AI researchers over the last few decades. Dung’s seminal paper on abstract argumentation [7] has inspired much work on logical and computational models of argumentation, with applications spanning non-monotonic reasoning, multi-agent systems, decision support, and machine learning [9] [11]. Some reasons for the success of abstract argumentation are that it is simple, general and intuitive. Arguments are represented as nodes and disagreements between arguments are represented as directed edges. The justified arguments are the sets of arguments that do not attack each other and attack all of its attackers, respectively capturing the ideas of mutual consistency and being able to respond adequately to all counterarguments.

Many different notions of justified arguments have been devised [1] [4] [7], and families of such sets of justified arguments can have lattice-theoretic properties under set inclusion [6]. In this paper, we show with two counterexamples that the complete extensions do not form a complete semilattice [7, Theorem 25(3)]. We then trace the original error in the proof and explain why the error occurred. This error necessitates a more careful definition of the grounded extension, specifically in relating the two definitions of the grounded extension as the least fixed point of the characteristic function where we clarify both the proof of [7, Theorem 25(2)] and [5, Appendix, Proposition 1].

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1Here, we will call the characteristic function [7, Definition 16] the defence function.
In Section 2, we review necessary concepts from graph theory, lattice theory and abstract argumentation. In Section 3, we recap the proof of [7, Theorem 25(3)] and show that it is false with two counterexamples. In Section 4, we clarify the implicit steps of the proof in [7] and trace the error. In Section 5, we discuss the consequences of this error for the grounded extension. We conclude in Section 6.

2 Background

Recall from graph theory that a directed graph, a.k.a. digraph, is a pair \( \langle A, R \rangle \) where \( A \) is the set of nodes and \( R \subseteq A^2 \) is the set of (directed) edges. We use \( R(a, b) \) to denote \( (a, b) \in R \), for all \( a, b \in A \). For \( S \subseteq A \) we define its forward set to be \( S^+ := \{ b \in A \mid (\exists a \in S) R(a, b) \} \) and its backward set to be \( S^- := \{ b \in A \mid (\exists a \in S) R(b, a) \} \). In both cases, if \( S = \{ a \} \) for some \( a \in A \), then we define \( a^\pm := \{ a \}^\pm \). Clearly, if \( S \subseteq T \subseteq A \), then \( S^\pm \subseteq T^\pm \), while the converse is not in general true.

Recall from lattice theory [6] that a poset \( \langle P, \leq \rangle \) is a lattice iff every pair of elements \( x, y \in P \) has a supremum (a.k.a. join) \( x \lor y \) and an infimum (a.k.a. meet) \( x \land y \) in \( P \). A lattice is chain complete iff for every totally ordered subset of \( P \), the supremum of this subset is also in \( P \). A lattice is meet-complete iff for every subset of \( P \), its infimum is in \( P \). Dually, a lattice is join-complete iff for every subset of \( P \), its supremum is in \( P \). A complete semilattice is a lattice that is chain complete and meet-complete over non-empty subsets of elements [7, Page 330, Footnote 5]. We say that a poset is directed iff every pair of elements has an upper bound in the poset. A lattice is directed complete iff every directed subset of the lattice has a supremum in the lattice. A complete lattice is a lattice where every subset has a meet and a join. For any set \( X \), \( \langle P(X), \subseteq \rangle \) is a complete lattice with meet \( \cap \) and join \( \cup \).

In abstract argumentation, arguments are represented as nodes and attacks between arguments are represented as directed edges. The resulting digraph \( \langle A, R \rangle \) is called an (abstract) argumentation framework (AF). For the rest of the document we assume an arbitrary underlying AF.

Given any AF \( \langle A, R \rangle \), we define the neutrality function to be

\[
n : \mathcal{P}(A) \to \mathcal{P}(A)
S \mapsto n(S) := A - S^+,
\]

where \( n(S) \) is the set of arguments that \( S \) is neutral to; it contains all arguments that \( S \) does not attack. We say \( S \subseteq A \) is conflict free (cf) iff \( S \subseteq n(S) \). Let \( CF \subseteq \mathcal{P}(A) \) denote the set of all cf sets. It can be shown that \( CF \) is non-empty, meet-complete over non-empty subsets (where meet is \( \cap \)), and directed complete. As directed complete posets are automatically chain complete, i.e. all non-empty subsets of elements of \( P \) have an infimum in \( P \).
complete because all chains are directed sets, we conclude that \( \langle CF, \subseteq \rangle \) is indeed a complete semilattice.

Given any AF \( \langle A, R \rangle \), we also define the defence function \( d \) to be

\[
d : \mathcal{P}(A) \rightarrow \mathcal{P}(A)
S \mapsto d(S)
\]

where the set of arguments that \( S \) defends, \( d(S) \) is defined as: \( a \in d(S) \iff a^- \subseteq S^+ \). The intuition is that each attacker of \( a \) is in turn attacked by \( S \), therefore \( S \) defends \( a \). We say \( S \subseteq A \) is self-defending (sd) iff \( S \subseteq d(S) \). Let \( SD \subseteq \mathcal{P}(A) \) denote the set of all sd sets. It can be shown that \( SD \) is non-empty and join-complete, and the latter implies that \( SD \) is directed complete and hence chain complete. Further, for an ordinal number \( \alpha \), \( d^\alpha \) denotes the \( \alpha \)th iterate of \( d \) on \( \mathcal{P}(A) \).

We say that \( S \subseteq A \) is admissible iff \( S \in CF \cap SD \). Let \( \text{ADM} := CF \cap SD \) be the set of all admissible extensions of \( \langle A, R \rangle \). It can be shown that \( \text{ADM} \) is non-empty and directed complete \([7, \text{Theorem 11}]\), where the latter implies that max \( \subseteq \text{ADM} \neq \emptyset \). Admissible sets represent sets of arguments that are collectively consistent and defends itself against all counterarguments.

One can strengthen the notion of admissible sets. We say \( S \subseteq A \) is a complete extension iff \( S \in CF \) and \( d(S) = S \). Let \( \text{COMP} \subseteq \mathcal{P}(A) \) denote the set of all complete extensions of \( \langle A, R \rangle \). The intuition for complete extensions is that if one could defend an argument then one is rationally compelled to believe in it. Finally, we call the \( \subseteq \)-least fixed point of \( d \) the grounded extension \([7, \text{Definition 20}]\). The grounded extension exists and is unique; this is because \( \langle \mathcal{P}(A), \subseteq \rangle \) is a complete lattice and \( d \) is a \( \subseteq \)-monotonic function from \( \mathcal{P}(A) \) to itself \([7, \text{Lemma 19}]\), so by the Knaster-Tarski theorem \([6, \text{Section 2.35}]\), the set of fixed points of \( d \) under \( \subseteq \) is also a complete lattice and hence has a least element. We will denote the grounded extension by \( G \).

### 3 Two Counterexamples

In Section 2 we have mentioned that \( \langle CF, \subseteq \rangle \) is a complete semilattice and \( \langle \text{ADM}, \subseteq \rangle \) is directed complete. What is the lattice-theoretic structure of \( \langle \text{COMP}, \subseteq \rangle \)?

**Theorem 1.** \([7, \text{Theorem 25(3)}]\) The complete extensions (of an AF \( \langle A, R \rangle \)) form a complete semilattice w.r.t. \( \subseteq \).

**Proof.** (Proof from \([7]\), mildly paraphrased) Let \( \emptyset \neq SE \subseteq \text{COMP} \). Define

\[
LB := \{ E \in \text{ADM} \mid (\forall E' \in SE) E \subseteq E' \},
\]

which is the set of admissible extensions that are lower bounds of all sets in \( SE \). Clearly, the grounded extension \( G \in LB \) so \( LB \neq \emptyset \). Let \( S := \bigcup LB \).

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3In \([7]\), this is called the characteristic function of an AF.
Clearly, $S \in ADM$. Let $E := \sup \{d^\alpha(S)\}$ for all ordinal numbers $\alpha$. Clearly, $E \in COMP$ and $E \in LB$. Therefore, $E = S$ and $E$ is the infimum of $SE$.

But Theorem 1 is not true. We offer the following counterexample.

**Example 2.** (Counterexample to [7, Theorem 25(3)]) Consider floating reinstatement [11, Figure 2], which has the following argument framework:

$$A = \{a, b, c, d\} \text{ and } R = \{(a, b), (b, a), (a, c), (b, c), (c, d)\}.$$ This argument framework is depicted in Figure 1.

![Figure 1: The argument framework for floating reinstatement](image)

It can be shown that:

- $CF = \{\emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{a, d\}, \{b, d\}\}$.
- $SD = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, d\}, \{b, d\} \}$.
- $ADM = \{\emptyset, \{a\}, \{b\}, \{a, d\}, \{b, d\}\}$.

Further, it can be shown that [10, Example 4]:

$$COMP = \{\emptyset, \{a, d\}, \{b, d\}\}.$$ (7)

Clearly, $(COMP, \subseteq)$ is not a complete semilattice because it is not closed under $\cap$. This is because $\{a, d\} \cap \{b, d\} = \{d\} \notin COMP$.

One can implement the steps in the proof of Theorem 1 to generate further counterexamples based on finite AFs. The idea is to repeatedly generate Erdős-Rényi digraphs up to a given number of nodes and a given edge probability. For each such graph, seen as an AF, one would calculate $COMP$. Further, for all given $\emptyset \neq SE \subseteq COMP$, one would calculate $LB$, $S$ and $E$ as specified in the above proof, checking for each graph whether $S = E$ and whether $E$ is indeed

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4In practice one takes the supremum of the sequence $\{d^\alpha(S)\}_{\alpha < \beta}$ over some sufficiently large ordinal $\beta$ depending on $|A|$, e.g. $\beta = \omega$ for $|A|$ finite, and $\beta$ being the first uncountable ordinal for $|A| = \aleph_0$, assuming the truth of the continuum hypothesis [8]. As $d$ is $\subseteq$-monotonic, this sequence will stabilise at a fixed point of $d$ after transfinite iterations.

5The proof that $(COMP, \subseteq)$ is chain complete and therefore a complete semilattice has been omitted in [7].

6In this example we have relabeled the arguments from [10, Figure 2].

7Here, $\cap$ is the meet associated with the partial order $\subseteq$. 

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Whenever $E \neq \cap SE$, we save the corresponding AF. This procedure is capable of providing more counterexamples, such as:

**Example 3.** Consider the following AF: $A = \{a, b, c, d, e\}$ and

$$R = \{(a, d), (b, a), (c, d), (e, c), (d, a), (d, e), (e, b), (e, c)\}.$$ 

This argument framework is depicted in Figure 2.

![Figure 2: The argument framework for the second counter example.](image)

It can be shown that:

$$COMP = \{\emptyset, \{b, c\}, \{b, d\}, \{a, e\}\}.$$

Clearly, $\langle COMP, \subseteq \rangle$ is not a complete semilattice because it is not closed under $\cap$. This is because $\{b, c\} \cap \{b, d\} = \{b\} \notin COMP$. Indeed, $COMP$ in this case is not a lattice at all, because $\{\{b, c\}, \{b, d\}\}$ has no infimum or supremum. However, it is known that the union of cf sets may not be cf, and hence it is unsurprising to expect that unions of complete extensions are not complete.

### 4 The Error in the Proof

If Theorem 1 is not true, then where is the error in the proof? Let us repeat the proof of Theorem 1 and write out the implicit steps.

**Proof.** We elaborate the proof of Theorem 1 (i.e. [7, Theorem 25(3)]) from [7].

1. Given an arbitrary AF $(A, R)$, let $\emptyset \neq SE \subseteq COMP$ and define $LB := \{E \in ADM \mid (\forall E' \in SE) E \subseteq E'\}$ as before.

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8One cannot check failure of chain completeness in the definition of complete semilattices because chain completeness is automatically satisfied by finite AFs.
2. We prove that \( LB \neq \emptyset \). It is easy to see that \( \emptyset \in LB \) iff \((\forall E' \in SE) \emptyset \subseteq E'\), which is true, and that \( \emptyset \in ADM \). Therefore, \( \emptyset \in LB \) and hence \( LB \neq \emptyset \).

3. We prove that \( d \) is closed on \( LB \). Let \( U \in LB \) be arbitrary, which is equivalent to \((\forall E' \in SE) U \subseteq E'\). However, for \( E' \in SE \) arbitrary, \( d(U) \subseteq d(E') = E' \) because \( E' \in SE \subseteq COMP \) and \( d \) is \( \subseteq \)-monotonic. Therefore, \( d(U) \in LB \) as well. As \( U \) is arbitrary, this shows that \( d : LB \rightarrow LB \).

4. We prove that \( LB \) is closed under arbitrary unions of non-empty families of subsets. Let \( \{U_i\}_{i \in I} \subseteq \mathcal{P}(LB) \) be an arbitrary set of subsets of \( LB \), where \( I \neq \emptyset \) is an arbitrary index set. We have that

\[
(\forall i \in I) U_i \in LB \Leftrightarrow (\forall i \in I) (\forall E' \in SE) U_i \subseteq E'
\]

\[
\Leftrightarrow (\forall E' \in SE) (\forall i \in I) U_i \subseteq E'
\]

\[
\Rightarrow (\forall E' \in SE) \bigcup_{i \in I} U_i \subseteq E'
\]

\[
\Leftrightarrow \bigcup_{i \in I} U_i \in LB.
\]

5. Let \( S := \bigcup LB \in LB \). Clearly \( S \neq \emptyset \) because \( LB \neq \emptyset \). To show that \( S \in ADM \), we need to show that \( S \in CF \) and \( S \in SD \).

(a) To show that \( S \in CF \), assume for contradiction that \( S \notin CF \). Then there are \( a, b \in S \) such that \( R(a, b) \). It follows that there are \( U, V \in LB \) such that \( a \in U \) and \( b \in V \), respectively. However, \( a, b \in U \cup V \in LB \subseteq ADM \) because \( LB \) is closed under unions, this implies that \( U \cup V \in ADM \) is not \( \subseteq \)-contradiction. Therefore, \( S \in CF \).

(b) To show that \( S \in SD \), let \( a \in S \) and let \( b \in a^- \). As \( a \in S \), there exists some \( U \in LB \) such that \( U \in SD \) and hence \( b \in U^+ \). But clearly, \( U \subseteq S \) and hence \( U^+ \subseteq S^+ \), so \( b \in S^+ \). As \( b \) is arbitrary, we conclude that \( S \subseteq d(S) \) and hence \( S \in SD \).

We conclude that \( S \in ADM \).

6. Furthermore, as \( S \in SD \), we have that \( S \subseteq d(S) \). However, we have just shown that \( d : LB \rightarrow LB \) and hence \( d(S) \in LB \). As \( S \) is the \( \subseteq \)-greatest element of \( LB \), we must have \( d(S) \subseteq S \). Therefore, \( d(S) = S \) and hence \( S \in COMP \).

7. Now let \( E := \sup \{ d^n(S) \} \) for all ordinal numbers \( \alpha \). But as \( S \) is a fixed point of \( d \), it must be the case that \( S = E \in COMP \). As \( S \in LB \), we also have \( E \in LB \).

Then the very last statement of the proof, that \( E = \bigcap SE \), is not correct, as Examples \ref{ex:2} and \ref{ex:3} have shown. \[\square\]

\textsuperscript{9}If \( a^- = \emptyset \) then it can be shown that \( a \in d(S) \), for any \( S \subseteq A \).
An alternative proof can highlight the error, again using Example 2. Given any AF and $\langle \text{COMP}, \subseteq \rangle$, let $\varnothing \neq \text{SE} \subseteq \text{COMP}$. Is it the case that $\bigcap \text{SE} \in \text{COMP}$? To establish this, we need to show that $d(\bigcap \text{SE}) = \bigcap \text{SE}$ and $\bigcap \text{SE} \in \text{CF}$. The latter is true because $\text{CF}$ is closed under arbitrary intersections of non-empty families of sets. Now, for simplicity, let $\text{SE} = \{C_i\}_{i \in I}$ for some arbitrary index set $I \neq \varnothing$. Therefore, $\bigcap \text{SE} = \bigcap_{i \in I} C_i$. We have that:

$$d(\bigcap \text{SE}) = d\left(\bigcap_{i \in I} C_i\right) \subseteq \bigcap_{i \in I} d(C_i) = \bigcap_{i \in I} C_i = \bigcap \text{SE}. \quad (9)$$

This establishes that $d(\bigcap \text{SE}) \subseteq \bigcap \text{SE}$. However, equality is not true in general. In Example 2, we have for $\text{SE} = \{\{a, d\}, \{b, d\}\} \neq \varnothing$, $\bigcap \text{SE} = \{d\}$. However, $d(\bigcap \text{SE}) = \varnothing \subseteq \{d\}$, but $\{d\} \not\subseteq \varnothing$. This means in this case $\bigcap \text{SE}$ is not a fixed point of $d$ and hence cannot be a complete extension. Therefore, $\text{COMP}$ is not in general closed under intersection, and hence cannot be a complete semilattice.

Why would the conclusion be drawn that $E = \bigcap \text{SE}$ in the final step of the proof of Theorem 1? The set $\text{LB}$ contains the admissible sets that are lower bounds of all complete extensions in $\text{SE}$. Further, $S$ is the $\subseteq$-greatest of the admissible extensions in $\text{SE}$, and is also complete. So it seems reasonable to conclude that $S$ (which is equal to $E$) is the greatest lower bound of $\text{SE}$, especially that $S \in \text{COMP}$. This is not correct because this assumes that the underlying poset is complete. The underlying poset in the proof is $\langle \text{ADM}, \subseteq \rangle$, which is a directed-complete, but certainly not complete.

5 Consequences for the Grounded Extension

If $\text{COMP}$ may not be a complete semilattice or a lattice in general, then what are the consequences? Recall from [7, Definition 20] (and also [3, Definition 2.4.2]) that the grounded extension, $G \subseteq A$, is initially defined to be the $\subseteq$-least fixed point of $d$, which exists and is unique by the Knaster-Tarski theorem. However, [7, Theorem 25(2)] claims that the grounded extension is also the $\subseteq$-least complete extension, and the proof is “obvious”. Some other papers (e.g. [5, Definition 3]) define $G := \bigcap \text{COMP}$. However, as $\text{COMP}$ is no longer closed under intersections in general, we cannot infer from the latter definition of $G$ that it is complete.

We now present a proof that fills in the gap of the omitted proof of [7, Theorem 25(2)] and clarifies the proof of [3, Appendix, Proposition 1], from which it follows that $\bigcap \text{COMP} \in \text{COMP}$ is true for all AFs and hence the grounded extension is complete.

**Theorem 4.** Let $\langle A, R \rangle$ be an arbitrary AF with defence function $d : \mathcal{P}(A) \to \mathcal{P}(A)$. Let $F_d := \{S \subseteq A | d(S) = S\}$ be the set of fixed points of $d$. Clearly $\langle F_d, \subseteq \rangle$ is a complete lattice. The following two statements are equivalent.

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10 It can be shown that, for any AF and for any set of subsets of arguments, $\{S_i\}_{i \in I} \subseteq \mathcal{P}(A)$, $d(\bigcap_{i \in I} S_i) \subseteq \bigcap_{i \in I} d(S_i)$, and the converse is not in general true.

11 $\langle F_d, \subseteq \rangle$ is a well-defined complete lattice by the Knaster-Tarski theorem.
1. \( S \) is the \( \subseteq \)-least complete extension.

2. \( S \) is the \( \subseteq \)-least element of \( F_d \).

Proof. (1 \( \Rightarrow \) 2, contrapositive) Assume that \( S \) is not the \( \subseteq \)-least element of \( F_d \). Either \( S \notin F_d \) or \( S \in F_d \) and \( S \) is not the \( \subseteq \)-least element of \( F_d \).

1. \( S \notin F_d \), then \( d(S) \neq S \) and hence \( S \notin \text{COMP} \). Therefore, \( S \) cannot be the \( \subseteq \)-least complete extension.

2. If \( S \in F_d \) and \( S \) is not the \( \subseteq \)-least element of \( F_d \), then \( \exists T \subset S \) \( T \in F_d \). Either \( T \in \text{CF} \) or \( T \notin \text{CF} \).

   (a) If \( T \in \text{CF} \), then \( T \in \text{COMP} \). As \( T \subset S \), \( S \) cannot be the \( \subseteq \)-least complete extension.

   (b) If \( T \notin \text{CF} \), then as \( T \subset S \), \( S \notin \text{CF} \) either so \( S \notin \text{COMP} \). Therefore, \( S \) cannot be the \( \subseteq \)-least complete extension.

In all cases, \( S \) cannot be the \( \subseteq \)-least complete extension.

(2 \( \Rightarrow \) 1, contrapositive) Assume \( S \) is not the \( \subseteq \)-least complete extension, then either \( S \notin \text{COMP} \), or \( S \in \text{COMP} \) and \( S \) is not \( \subseteq \)-least.

1. If \( S \in \text{COMP} \) and \( S \) is not \( \subseteq \)-least, then \( \exists T \subset S \) \( T \in \text{COMP} \), but as \( S, T \in \text{COMP} \), we have \( S, T \in F_d \) and hence \( S \) is not the \( \subseteq \)-least element of \( F_d \).

2. If \( S \notin \text{COMP} \), then either \( S \notin \text{CF} \) or \( S \notin F_d \).

   (a) If \( S \notin F_d \), then \( S \) cannot be the \( \subseteq \)-least element of \( F_d \).

   (b) If \( S \notin \text{CF} \), then assume for contradiction that \( S \) is the \( \subseteq \)-least fixed point of \( d \). Therefore, \( S \in F_d \) and \( \forall T \in F_d \) \( S \subseteq T \). It follows that \( \forall T \in F_d \) \( T \notin \text{CF} \), because any superset of a non-cf set cannot be cf. It follows that \( F_d \cap \text{CF} = \emptyset \), which means \( \text{COMP} = \emptyset \). However, we have assumed that the underlying AF is arbitrary. It cannot be true that \( \text{COMP} = \emptyset \) for arbitrary AFs. For example, in Example 2 we have an AF where \( \text{COMP} \neq \emptyset \). Therefore, \( S \) is not the \( \subseteq \)-least fixed point of \( d \).

In all cases, \( S \) is not the \( \subseteq \)-least fixed point of \( d \).

The result follows.

One criticism of Theorem 4 is that it is folklore and hence not new. Further, such a result is obvious because it is easy to show that the defence function \( d \) maps \( \text{CF} \) to itself. Further, one can show (trivially) that \( d \) also maps the sets \( \text{SD}, \text{ADM} \) and \( \text{COMP} \) to themselves.
well-defined \( CF \)-sequence by transfinite induction, and its supremum is also in \( CF \) and is the least fixed point of \( d \), i.e. the grounded extension is complete. We concede that the claim and proof of Theorem \( 4 \) is not new, because it is a clarification of \( 5 \) Appendix Proposition \( 1 \), and likely folklore as many mentions of this result in the literature have their proofs omitted. However, our purpose in providing this version of the proof was to re-examine the result in the light of the observation that the complete extensions do not have to form a complete semilattice. Furthermore, it is not \textit{a priori} guaranteed that iterating \( d \) from \( \emptyset \) will converge to the grounded extension unless the underlying AF is finitary \( 7 \) Definition \( 27 \), which implies that \( d \) is \( \omega \)-continuous \( 7 \) Lemma \( 28 \) and hence by Kleene's fixed point theorem \( 6 \) Section \( 8.15 \) the \( CF \)-sequence \( \{d^\alpha(\emptyset)\}_{\alpha<\beta} \) does converge to the least fixed point of \( d \), which is the grounded extension. Theorem \( 4 \) does not assume that the underlying AF is finite, finitary or infinite. This clarification puts both the definitions of the grounded extension on a more symmetric footing, and that the grounded extension is still complete even when the complete extensions do not form a complete semilattice. This will ensure that claims such as the grounded extension as the \( \subseteq \)-least complete extension exists and is unique for infinite AFs (e.g. \( 2 \) Section \( 4.2 \)) would still be true.

6 Conclusions

We have shown via two counterexamples that the complete extensions of abstract argument frameworks do not have to form a complete semilattice; this shows that \( 7 \) Theorem \( 25(3) \) is incorrect. We have elaborated the original proof in \( 7 \) and the reason for the error was that the supremum of the lower bounds does not equal the infimum of a given non-empty set of complete extensions, because \( (ADM, \subseteq) \) is not a complete lattice. One consequence is that given any AF, \( COMP \) does not have to be closed under intersection, therefore \( \bigcap COMP \), i.e. the grounded extension, may not be complete. We have carefully related both the definitions of the grounded extension given in \( 7 \) and clarified the relevant proofs \( (7 \) Theorem \( 25(2) \) and \( 5 \) Appendix, Proposition \( 1) \), such that even though \( COMP \) is no longer closed under intersection in general, \( \bigcap COMP \in COMP \) still and hence the grounded extension is still complete regardless of whether the underlying AF is finite or infinite.

In future work, we seek to classify the AF structures that guarantee that \( COMP \) is a complete semilattice (e.g. \( 3 \) Example \( 2.2.2) \), or where \( COMP \) is certainly not a complete semilattice. There may also be further consequences for abstract argumentation yet to be considered.

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