On stopping Fock-space processes

Alexander C. R. Belton
Department of Mathematics and Statistics
Lancaster University, United Kingdom
a.belton@lancaster.ac.uk

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Abstract

We consider the theory of stopping times within the framework of Hudson–Parthasarathy quantum stochastic calculus. Coquio’s method of stopping (J. Funct. Anal. 238:149–180, 2006) is modified for the vacuum-adapted setting, where certain results, including the proof of the optional-sampling theorem, take a more natural form.

Key words: quantum stopping time; quantum stochastic calculus.

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1 Introduction

The extension of the notion of stopping time from classical to non-commutative probability is straightforward, with the earliest definition in the literature due to Hudson [16]. The idea was developed in the setting of the Clifford probability gauge space by Barnett and Lyons [7], and for abstract filtered von Neumann algebras by Barnett and Thakrar [8, 9], and Barnett and Wilde [10, 11]; see also [25], where Sauvageot initiated a programme to solve a C∗-algebraic version of the Dirichlet problem, and recent work by Luczak [22].

In the Fock-space context, a quantum stopping time S is a projection-valued measure on the extended half line [0, ∞], such that S([0, ∞]) = I, the identity operator on F, and t → S([0, t]) is an identity-adapted process, i.e., S([0, t]) ∈ B(ℓ2(R+; k)) for all t ≥ 0. (Here the Boson Fock space F is identified with F(ℓ2(R+; k)) has the factorisation F(S) ⊗ F(S); where the spaces F(S) and F(S) are pre-S and post-S spaces; this provides a form of strong Markov property that generalises Hudson’s result [16]. Further contributions in this setting have been made by Meyer [23], Applebaum [2], Accardi and Sinha [1], Attal and Sinha [6], Sinha [26], Attal and Coquio [4], Coquio [15] and Hudson [17, 18]; quantum stopping times are applied to the CCR flow in [14].

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The primary object associated with a quantum stopping time S is its time projection E_S, and this is introduced in Section 3 if S is deterministic, with S(\{t\}) = I, then E_S is the orthogonal
projection $E_t = I_t \otimes P_0^\mathcal{D}$ onto $\mathcal{F}_t \otimes \mathcal{E}(0)$, where here $\mathcal{E}(0)$ is the vacuum vector in $\mathcal{F}_0$. The time projection $E_S$ can be represented as a quantum stochastic integral (Theorem 3.6):

$$E_S = I + \int_0^\infty I_t \otimes S((s, \infty]) E_s \, d\Lambda_s.$$  \hfill (1.1)

The integrand is a vacuum-adapted process, which shows that this form of adaptedness appears naturally when considering quantum stopping times.

If $S$ is a quantum stopping time and $X$ is a process (i.e., a family of operators on $\mathcal{F}$ satisfying suitable measurability and adaptedness conditions) then there are three natural approaches to stopping $X$ at $S$:

\[ S \cdot X := S(\{0\}) X_0 + \lim_{\pi \to \infty} \sum_{j=0}^\infty S((\pi_j - 1, \pi_j]) X_{\pi_j} \quad \text{(left)}, \]

\[ X \cdot S := X_0 S(\{0\}) + \lim_{\pi \to \infty} \sum_{j=0}^\infty X_{\pi_j} S((\pi_j - 1, \pi_j]) \quad \text{(right)} \]

and

\[ S \cdot X \cdot S := S(\{0\}) X_0 S(\{0\}) + \lim_{\pi \to \infty} \sum_{j=0}^\infty S((\pi_j - 1, \pi_j]) X_{\pi_j} S((\pi_j - 1, \pi_j]) \quad \text{(double)}, \]

assuming these limits, taken over partitions of $\mathbb{R}_+$, exist in some sense. (To establish convergence is often difficult, or even impossible in general.) The time projection $E_S$ is the result of stopping the vacuum-adapted process $(E_t)_{t \in \mathbb{R}_+}$ in any of these three senses; each of the sums yields the same orthogonal projection and convergence holds in the strong operator topology, since these projections form a decreasing family as the partition is refined: see Theorem 3.3 below.

In the vacuum-adapted setting, the value at time $t$ of a martingale $M$ closed by the operator $M_\infty$ is simply $E_t M_\infty E_t$; see Section 5 for the definitions of vacuum-adapted and identity-adapted martingales. Thus it is natural to define $M_S$, the value of the martingale $M$ stopped in a vacuum-adapted manner at $S$, to be $E_S M_\infty E_S$; it is easy to see that this equals $S \cdot M \cdot S$, the result of double stopping $M$ at $S$. The issue of convergence becomes that of the existence of $E_S$, which is long established, and this definition has various good properties: see Sections 4 and 7 below. In particular, the optional-sampling theorem, Theorem 7.4, holds, and is an immediate consequence of the identity $E_S \wedge E_T = E_{S \wedge T}$, which is true for any two quantum stopping times $S$ and $T$ (Theorem 3.11).

In [15], Coquio has proposed a method of stopping for identity-adapted processes which is not obviously one of the forms given above. She begins by working with discrete stopping times, i.e., those with finite support: if $T$ is a quantum stopping time with support $\{t_1 < \cdots < t_n\} \subseteq \mathbb{R}_+$ and $X$ is a process then

\[ X_{\widehat{T}} := \sum_{i,j=1}^n \widehat{\pi}(E_{t_i \vee t_j}) T(\{t_i\}) \widehat{\pi}(X_{t_i \vee t_j})_{t_i \vee t_j} T(\{t_j\}) \widehat{\pi}(E_{t_j})_{t_i \vee t_j} \quad \text{(1.2)} \]
is the result of applying identity-adapted stopping to \( X \) at \( T \), where \( \hat{\pi}(Es)_t = I_s \otimes P_{s,t}^\Omega \otimes I_t \) maps \( F_{[s,t]} \) onto the vacuum subspace and acts as the identity on \( F_s \) and \( F_t \), with \( F \) identified with \( F_s \otimes F_{[s,t]} \otimes F_t \). (This notation is explained further in Section 5). In Section 6 the vacuum-adapted version of this definition is introduced and various consequences are derived; in particular, Lemma 6.9 shows that, for a closed martingale, it agrees with the natural definition of \( \hat{M}_P \) described above.

From the definition (1.2), Coquio derives an integral formula for \( \hat{M}_P \) (Lemma 7.6) when \( M \) is a closed martingale, and uses this to extend the definition of \( \hat{M}_P \) to arbitrary stopping times. The key step in this result, Theorem 7.8 is to show that, given any quantum stopping time \( S \) and any operator \( Z \in B(F) \), the sum
\[
E_S Z E_S + \int_0^\infty I_k \otimes S([0,s]) \hat{\pi}(E_S Z E_S) S([0,s]) \, d\Lambda_s
\]
extends to a bounded operator \( Z_S \); we provide a somewhat shorter version of Coquio’s proof in Section 7. It follows that the difference between stopping a closed martingale \( M \) in the identity-adapted and vacuum-adapted senses at an arbitrary quantum stopping time \( S \) is given by a gauge integral:
\[
M_S - M_S = \int_0^\infty I_k \otimes S([0,s]) \hat{\pi}(M_S)_s S([0,s]) \, d\Lambda_s;
\]
in particular, the integral in Coquio’s definition of \( M_S \) can be seen as an artifact produced by working with identity adaptedness.

In Section 8, vacuum-adapted stopping is extended from discrete to arbitrary times for FV processes and for semimartingales; the former are processes \( Y \) of the form \( Y_t = \int_0^t H_s \, ds \), and the latter are sums of martingales and FV processes. Again we follow Coquio [15, Proposition 3.15].

Given sufficient regularity, a semimartingale may be written as the sum of four quantum stochastic integrals. Such a process is called a regular semimartingale (if identity adapted) or a regular \( \Omega \)-semimartingale (if vacuum adapted). The integral formula (1.1) for \( E_S \) is used in Section 9 to show that the class of regular \( \Omega \)-semimartingales is closed under vacuum-adapted stopping (Theorem 9.1); Coquio has obtained the analogous result for regular semimartingales [15, Proposition 3.16].

An appendix, Section \( \text{A} \), is included to gather the necessary results on quantum stochastic integration.

### 1.1 Notation and conventions

The term “increasing” applies in the weak sense. All Hilbert spaces have complex scalar fields; inner products are linear in the second argument. The indicator function of the set \( A \) is denoted by \( 1_A \). The complement of an orthogonal projection \( P \) is denoted by \( P^\perp \). The set of natural numbers \( \mathbb{N} = \{1, 2, 3, \ldots\} \), the set of non-negative integers \( \mathbb{Z}_+ = \mathbb{N} \cup \{0\} \) and the set of non-negative real numbers \( \mathbb{R}_+ = [0, \infty) \). The von Neumann algebra of bounded operators on a Hilbert space \( H \) is denoted by \( B(H) \).
2 Quantum stopping times

Notation 2.1. Let \( \mathcal{F}_A = \mathcal{F}_+(L^2(A; k)) \) denote the Boson Fock space over \( L^2(A; k) \), where \( A \) is a subinterval of \( \mathbb{R}_+ \) and \( k \) is a complex Hilbert space. For brevity, let \( \mathcal{F} := \mathcal{F}_{\mathbb{R}_+} \) and \( \mathcal{F}_t := \mathcal{F}_{[0, t]} \) for all \( t \in (0, \infty) \) and \( \mathcal{F}_t := \mathcal{F}_{[t, \infty)} \) for all \( t \in \mathbb{R}_+ \), with similar abbreviations \( I_t \) and \( I_{[0, t]} \) for the identity operators on these spaces.

Recall that the set of exponential vectors \( \{ e(f) : f \in L^2(\mathbb{R}_+; k) \} \) is linearly independent and total in \( \mathcal{F} \), with \( \langle e(f), e(g) \rangle = \exp(f, g) \) for all \( f, g \in L^2(\mathbb{R}_+; k) \); let \( E \) denote the linear span of this set.

Definition 2.2. A spectral measure on \( \mathcal{B}[0, \infty] \), the Borel \( \sigma \)-algebra on the extended half-line \([0, \infty] := \mathbb{R}_+ \cup \{ \infty \} \), is a map

\[
S : \mathcal{B}[0, \infty] \to B(\mathcal{H}),
\]

where \( \mathcal{H} \) is a complex Hilbert space, such that

(i) the operator \( S(A) \) is an orthogonal projection for all \( A \in \mathcal{B}[0, \infty] \);

(ii) the map \( \mathcal{B}[0, \infty] \to \mathbb{C} ; A \mapsto \langle x, S(A)y \rangle \) is a complex measure for all \( x, y \in \mathcal{F} \);

(iii) the total measure \( S([0, \infty]) = I \).

The spectral measure \( S \) is a quantum stopping time if \( \mathcal{H} = \mathcal{F} \) and \( S \) is identity adapted in the following sense:

(iv) the operator \( S(\{ \{0\} \}) \in \mathbb{C}I \) and \( S([0, t]) \in B(\mathcal{F}_t) \otimes I_{[0, t]} \) for all \( t \in (0, \infty) \).

Remark 2.3. A spectral measure on \( \mathcal{B}[0, \infty] \) may also be defined to be an increasing family of orthogonal projections \( (S_t)_{t \in [0, \infty]} \) in \( B(\mathcal{H}) \) such that \( S_\infty = I \). The equivalence of these definitions is ensured by the spectral theorem for unbounded self-adjoint operators.

Proposition 2.4. Let \( S \) be a spectral measure on \( \mathcal{B}[0, \infty] \) and let \( A, B \in \mathcal{B}[0, \infty] \).

(i) The operator \( S(\emptyset) = 0 \).

(ii) If \( A \subseteq B \) then \( S(A)S(B) = S(A) = S(B)S(A) \).

(iii) If \( A \) and \( B \) are disjoint then \( S(A \cup B) = S(A) + S(B) \) and \( S(A)S(B) = 0 \).

(iv) In general, \( S(A)S(B) = S(A \cap B) = S(B)S(A) \).

Proof. The first statement is an immediate consequence of Definition 2.2 as is the first part of the third. The second statement holds because \( S(B) = S(A) + S(B \setminus A) \geq S(A) \). To see the second part of the third, note that if \( A \) and \( B \) are disjoint then

\[
S(A) = S(A)S(A \cup B) = S(A)^2 + S(A)S(B).
\]

The fourth statement now follows by writing \( S(B) \) as \( S(A \cap B) + S(B \setminus A) \). □
Lemma 2.5. If $S : B[0, \infty] \to B(H)$ is a spectral measure then the map $t \mapsto S([0, t])x$ is strongly measurable for all $x \in H$.

Proof. Let $x \in H$. If $t > s \geq 0$ then

$$\|S([0, t])x\|^2 - \|S([0, s])x\|^2 = \|S([0, t])x - S([0, s])x\|^2 = \|S((s, t])x\|^2 \geq 0,$$

so $t \mapsto \|S([0, t])x\|$ is increasing and therefore has a countable set of discontinuities; the second equality above now shows that $t \mapsto S([0, t])x$ also has a countable set of discontinuities, so is strongly measurable.

The following result will be used without comment. Its proof may be shortened by assuming that $k$ is separable, but this hypothesis is unnecessary.

Corollary 2.6. If $S$ is a spectral measure as in Lemma 2.5 and $F : [0, \infty) \to B(H)$ is continuous in the strong operator topology then the maps $t \mapsto S([0, t])F(t)x$ and $t \mapsto S([0, t])F(t)S([0, t])x$ are strongly measurable for all $x \in H$.

Proof. Note first that $t \mapsto F_1x$ is continuous, so $\{F_1x : t \in \mathbb{R}_+\}$ is separable: let $\{y_n : n \in \mathbb{N}\}$ be dense in this set. By Pettis’s theorem, for all $n \in \mathbb{N}$ there exists a null set $N_n \subseteq \mathbb{R}_+$ such that $\{S([0, t])y_n : t \in \mathbb{R}_+ \setminus N_n\}$ is separable, therefore the closed linear span of

$$\bigcup_{n \in \mathbb{N}} \{S([0, t])y_n : t \in \mathbb{R}_+ \setminus N_n\}$$

is also separable and $t \mapsto S([0, t])F_1x$ is separably valued almost everywhere. Approximating the map $t \mapsto S([0, t])y$ by step functions shows that $t \mapsto \langle S([0, t])y, F_1x \rangle$ is measurable for all $y \in H$, so another application of Pettis’s theorem gives that $t \mapsto S([0, t])F_1x$ is strongly measurable.

The second claim follows similarly. Let $N_0 \subseteq \mathbb{R}_+$ be a null set such that $\{S([0, t])x : t \in \mathbb{R}_+ \setminus N_0\}$ is separable, and let $\{z_n : n \in \mathbb{N}\}$ be dense in this set. Then $\{F(t)z_n : n \in \mathbb{N}, t \in \mathbb{R}_+\}$ is separable; let $\{w_n : n \in \mathbb{N}\}$ be dense in this set and, for all $n \in \mathbb{N}$, let $N_n \subseteq \mathbb{R}_+$ be a null set such that $\{S([0, t])w_n : t \in \mathbb{R}_+ \setminus N_n\}$ is separable. If $t \not\in \bigcup_{n \geq 0} N_n$ then $S([0, t])F_1S([0, t])x$ is in the closed linear span of $\{S([0, t])w_n : n \in \mathbb{N}, t \in \mathbb{R}_+ \setminus N_n\}$, which is separable, so the map $t \mapsto S([0, t])F_1S([0, t])x$ is separably valued almost everywhere. Weak measurability of this function holds by a similar argument to the previous one, noting that

$$\|F_1S([0, t])x - F_1S([0, s])x\| \leq \|(F_1 - F_s)S([0, t])x\| + \|F_s\| \|S([0, t])x - S([0, s])x\| \to 0$$

as $t \to s$ for almost every $s \in \mathbb{R}_+$.

Example 2.7. For all $t \geq 0$, setting

$$t : B[0, \infty] \to B(F); \quad A \mapsto \begin{cases} 0 & \text{if } t \not\in A, \\ I & \text{if } t \in A \end{cases}$$

defines a quantum stopping time which corresponds to the non-random time $t \in [0, \infty]$. 5
**Example 2.8.** Let $B = (B_t)_{t \in \mathbb{R}_+}$ be a standard Brownian motion and, taking $k = \mathbb{C}$, use the Wiener–Itô–Segal transform to identify the Fock space with Wiener space $L^2(\Omega, \mathcal{A}, \mathbb{P})$. If $\tau$ is a classical stopping time for $B$ then

$$S : \mathcal{B}[0, \infty] \to \mathcal{B}(\mathcal{F}); \ A \mapsto 1_{\{\tau \in A\}}$$

is a quantum stopping time, where $1_{\{\tau \in A\}}$ acts by multiplication on $L^2(\Omega, \mathcal{A}, \mathbb{P})$. The same applies with $B$ replaced by any classical process with the chaotic representation property, e.g., the classical or monotone Poisson processes, or Azéma’s martingale [3, Section II.1].

**Example 2.9.** [4, pp.508–510] Let $\nu(t)_{t \in \mathbb{R}_+}$ be a standard Poisson process with intensity 1 and unit jumps on the probability space $(\Omega, \mathcal{A}, \mathbb{P})$, where $\mathcal{A}$ is complete and generated by $\nu$; for all $n \in \mathbb{Z}_+$, let

$$\tau_n := \inf\{t \in \mathbb{R}_+: \nu_t = n\}$$

be the $n$th jump time. The fact that

$$\int_0^t \phi_s(\omega) \, d\nu_s(\omega) = \sum_{k=1}^{\nu_t(\omega)} \phi_{\tau_k}(\omega) \quad \text{for all } \omega \in \Omega,$$

where $\phi$ is any process, together with the identity $\{\nu_t \geq n\} = \{\tau_n \leq t\}$, implies that

$$\int_0^t (1_{\{\tau_{n-1} < s\}}(\omega) - 1_{\{\tau_n < s\}}(\omega)) \, d\nu_s(\omega) = \int_0^t \int_0^s 1_{\{\nu_t(\omega) \geq n\}}(s) \, d\nu_s(\omega) = 1_{\{\tau_n \leq t\}}(\omega)$$

for all $\omega \in \Omega$, $n \in \mathbb{N}$ and $t \geq 0$. Since $(\nu_t - t)_{t \in \mathbb{R}_+}$ is a normal martingale and $\tau_n$ has the gamma distribution with mean and variance $m$, it holds that

$$\mathbb{E}\left[\left(\int_0^t 1_{\{\tau_n = s\}} \, d(\nu_s - s)\right)^2\right] = \mathbb{E}\left[\int_0^t 1_{\{\tau_n = s\}} \, ds\right] = \int_0^t \mathbb{P}(\tau_n = s) \, ds = 0$$

and therefore, as elements of $L^2(\Omega, \mathcal{A}, \mathbb{P})$,

$$1_{\{\tau_n \leq t\}} = \int_0^t (1_{\{\tau_{n-1} < s\}} - 1_{\{\tau_n < s\}}) \, d\nu_s \quad \text{for all } t \geq 0. \quad (2.1)$$

With $k = \mathbb{C}$, let $N_t = A_t + A_t^j + tI$ for all $t \in \mathbb{R}_+$, so that $N$ is the usual quantum stochastic representation of the Poisson process $\nu$: as is well known [19, Theorems 6.1–2], there exists an isometric isomorphism $U_P : L^2(\Omega, \mathcal{A}, \mathbb{P}) \to \mathcal{F}$ such that $U_P^* N_t U_P$ is an essentially self-adjoint operator, the closure of which corresponds to multiplication by $\nu_t$, for all $t \in \mathbb{R}_+$, and $U_P 1 = \varepsilon(0)$. It follows from Lemma [A.10] that

$$\int_0^t T_n([0, s]) \, dN_s = U_P \int_0^t 1_{\{\tau_n \leq s\}} \, d\nu_s U_P^* \quad \text{on } \mathcal{E} \quad (2.2)$$

for all $t \in \mathbb{R}_+$, where the quantum stopping time $T_n$ is defined by setting

$$T_n(A) := U_P 1_{\{\tau_n \in A\}} U_P^* \quad \text{for all } n \in \mathbb{Z}_+ \text{ and } A \in \mathcal{B}[0, \infty),$$
as in Example 2.8 and the integral on the right-hand side of (2.2) acts by multiplication on $L^2(\Omega, \mathcal{A}, \mathbb{P})$. In particular, $T_n$ satisfies the quantum stochastic differential equation

$$T_n([0, t]) = \int_0^t \left( T_{n-1}([0, s]) - T_n([0, s]) \right) dN_s \quad \text{for all } t \in \mathbb{R}_+ \text{ and } n \in \mathbb{N}.$$ 

**Definition 2.10.** A partial order is defined on quantum stopping times in the following manner: if $S$ and $T$ are quantum stopping times then $S \leq T$ if and only if $S([0, t]) \geq T([0, t])$ for all $t \geq 0$. The definition agrees with the classical ordering which applies to Examples 2.8 and 2.9 in the latter, $T_m \leq T_n$ for all $m, n \in \mathbb{Z}_+$ such that $m \leq n$.

**Theorem 2.11.** If $S$ is a quantum stopping time and $s \geq 0$ then $S \wedge s$ is a quantum stopping time, where

$$(S \wedge s)([0, t]) = \begin{cases} S([0, t]) & \text{if } t < s, \\ I & \text{if } t \geq s \end{cases}$$

for all $t \geq 0$. Furthermore, if $t \geq s \geq 0$ then

$$S \wedge s \leq S \wedge t \leq S, \quad S \wedge s \leq s \quad \text{and} \quad (S \wedge t) \wedge s = S \wedge s.$$ 

**Proof.** This is a straightforward exercise. \hfill \qed

**Remark 2.12.** More generally, quantum stopping times $S \wedge T$ and $S \vee T$ are defined for any pair of quantum stopping times $S$ and $T$ by setting

$$(S \wedge T)([0, t]) = S([0, t]) \vee T([0, t])$$

and

$$(S \vee T)([0, t]) = S([0, t]) \wedge T([0, t]) \quad \text{for all } t \geq 0.$$ 

By definition, $S \wedge T \leq S \leq S \vee T$ and $S \wedge T \leq T \leq S \vee T$.

If $S([0, t])$ commutes with $T([0, t])$ for all $t \in \mathbb{R}_+$ then

$$(S \vee T)([0, t]) = S([0, t]) T([0, t])$$

and

$$(S \wedge T)([0, t]) = S([0, t]) + T([0, t]) - S([0, t]) T([0, t]) \quad \text{for all } t \geq 0.$$ 

### 3 Time projections

**Definition 3.1.** Let $E_0 \in B(\mathcal{F})$ be the orthogonal projection onto $\mathbb{C} \varepsilon(0)$, let $E_t \in B(\mathcal{F})$ be the orthogonal projection onto $\mathcal{F}_t \otimes \varepsilon(0|_{t, \infty})$, considered as a subspace of $\mathcal{F}$, for all $t \in (0, \infty)$ and let $E_\infty := I$. Then

$$E_t \varepsilon(f) = \varepsilon(1_{[0, t]} f) \quad \text{for all } t \geq 0 \text{ and } f \in L^2(\mathbb{R}_+; k).$$
Given a quantum stopping time $S$, the time projection

$$E_S = \int_{[0, \infty]} S(ds)E_{s+} = \int_{[0, \infty]} E_{s+}S(ds),$$

where these integrals are strongly convergent limits of Riemann sums: see Theorem 3.3. Note that left, right and double stopping $E = (E_t)_{t \in [0, \infty]}$ at $S$ produce the same result, as observed in the Introduction.

**Definition 3.2.** A partition of $\mathbb{R}_+$ is a strictly increasing sequence $\pi = (\pi_j)_{j \in \mathbb{Z}_+}$ with $\pi_0 = 0$ and $\lim_{j \to \infty} \pi_j = \infty$. A partition $\pi'$ is a refinement of $\pi$ if $(\pi_j)_{j \in \mathbb{Z}_+}$ is a subsequence of $(\pi'_j)_{j \in \mathbb{Z}_+}$.

The following theorem dates back at least as far as [9, Theorem 2.3].

**Theorem 3.3.** Let $S$ be a quantum stopping time and let $\pi$ be a partition of $\mathbb{R}_+$. The series

$$E^\pi_S := S(\{0\})E_0 + \sum_{j=1}^{\infty} S((\pi_{j-1}, \pi_j])E_{\pi_j} + S(\{\infty\})$$

(3.1)

converges in the strong operator topology to an orthogonal projection. If $\pi'$ is a refinement of $\pi$ then $E^\pi_S \leq E^{\pi'}_S$, so $E_S := \operatorname{st.lim}_\pi E^\pi_S$ exists and is an orthogonal projection such that $E_S \leq E^\pi_S$ for all $\pi$.

**Proof.** Note first that if $s, t \in \mathbb{R}_+$ are such that $s < t$ then

$$S((s, t]) = S([0, t]) - S([0, s]) \in B(\mathcal{F}_I) \otimes I_t \quad \text{and} \quad E_t \in I_0 \otimes B(\mathcal{F}_I).$$

Hence if $m, n \in \mathbb{Z}_+$ are such that $m \leq n$ then

$$\left\| \sum_{j=m+1}^{n} S((\pi_{j-1}, \pi_j])E_{\pi_j}x \right\|^2 = \sum_{j=m+1}^{n} \|S((\pi_{j-1}, \pi_j])E_{\pi_j}x\|^2 \leq \sum_{j=m+1}^{n} \|S((\pi_{j-1}, \pi_j])x\|^2 \leq \|S((\pi_m, \infty])x\|^2$$

for all $x \in \mathcal{F}$, which gives the first claim; that $E^\pi_S$ is an orthogonal projection is immediately verified. For the second, note first that if $s, t \in \mathbb{R}_+$ are such that $0 < s \leq t$ then

$$(E_t - E_s)^2 = E_t^2 + E_s^2 - 2E_t - E_s \in I_s \otimes B(\mathcal{F}_s).$$

For all $j \in \mathbb{Z}_+$, let $k_j \geq j$ be such that $\pi^j_{k_j} = \pi_j$ and let $l_j \geq 1$ be such that $\pi^j_{k_j+l_j} = \pi_{j+1}$. Then

$$\langle x, (E^\pi_S - E^{\pi'}_S)x \rangle = \sum_{j=0}^{\infty} \sum_{l=1}^{l_j} \langle x, S((\pi^j_{k_j+l-1}, \pi^j_{k_j+l}])(E_{\pi^j_{k_j+l_j}} - E_{\pi^j_{k_j+l+1}})x \rangle \geq 0$$

for all $x \in \mathcal{F}$, as required. \qed
Remark 3.4. If the quantum stopping time $S$ corresponds to the non-random time $t \in [0, \infty]$, so that $S\{t\} = I$, then $E_S = E_t$.

Example 3.5. Let $M$ be a normal martingale with the chaotic-representation property, such as standard Brownian motion, and identity the Fock space $\mathcal{F}$ with $L^2(\Omega, \mathcal{A}, \mathbb{P})$, as in Examples 2.8 and 2.9. If $\tau$ is a classical stopping time for $M$ then

$$E_T X = \mathbb{E}[X|A_\tau] \quad \text{for all } X \in L^2(\Omega, \mathcal{A}, \mathbb{P}),$$

where $T$ is the quantum stopping time corresponding to $\tau$, as in Example 2.8, and $A_\tau$ is the $\sigma$-algebra at the stopping time $\tau$. To see this, note first that, in this interpretation of Fock space, the exponential vector $\varepsilon(f)$ is a stochastic exponential and satisfies the stochastic differential equation

$$\varepsilon(f) = 1 + \int_0^\infty f(t) \varepsilon([0,t]) dM_t \quad \text{for all } f \in L^2(\mathbb{R}_+; k),$$

therefore

$$\mathbb{E}[\varepsilon(f)|A_t] = 1 + \int_0^t f(s) \varepsilon([0,s]) dM_s = \varepsilon([0,t]) = E_t \varepsilon(f) \quad \text{for all } t \in \mathbb{R}_+.$$

Hence if $\tau$ takes values in the set $\{t_1 < \cdots < t_n\}$ then

$$\mathbb{E}[X|A_\tau] = \sum_{i=1}^n 1_{\{\tau = t_i\}} \mathbb{E}[X|A_{t_i}] = \sum_{i=1}^n 1_{\{\tau = t_i\}} E_{t_i} X = E_T X \quad \text{for all } X \in L^2(\Omega, \mathcal{A}, \mathbb{P}),$$

and the general case follows by approximation: given $X \in L^2(\Omega, \mathcal{A}, \mathbb{P})$, let $X_t := \mathbb{E}[X|A_t]$ for all $t \in \mathbb{R}_+$ and note that, given a sequence of classical stopping times $(\tau_n)_{n \in \mathbb{N}}$ such that $\tau_n \to \tau$ almost surely as $n \to \infty$, then, by optional sampling,

$$\mathbb{E}[X|A_\tau] = X_{\tau_n} \to X_\tau = \mathbb{E}[X|A_\tau] \quad \text{as } n \to \infty,$$

almost surely and in $L^2(\Omega, \mathcal{A}, \mathbb{P})$.

The following theorem has its origins in work of Meyer [23, equation (12) on p.74]: see also [6, Proposition 6], [4, Theorem 6.2] and [15, Theorem 2.5]. This integral representation shows that $(E_{S(\cdot)})_{S \in \mathcal{I}}$ is a regular $\Omega$-martingale closed by $E_S$; this observation is very useful when combined with the quantum Itô product formula. For the necessary details about quantum stochastic integration, see Section A.

Theorem 3.6. Let $S$ be a quantum stopping time. Then

$$E_S = I - \int_0^\infty I_k \otimes S([0,s]) E_s d\Lambda_s = E_0 + \int_0^\infty I_k \otimes S((s,\infty]) E_s d\Lambda_s. \quad (3.2)$$
Proof. Note first that

\[ \langle \varepsilon(f), \int_t^\infty I_k \otimes E_s \, d\Lambda_s \varepsilon(g) \rangle = \int_t^\infty \langle f(s), g(s) \rangle \langle \varepsilon(f), E_s \varepsilon(g) \rangle \, ds = \int_t^\infty \frac{d}{dt} \langle \varepsilon(f), E_s \varepsilon(g) \rangle \, ds = \langle \varepsilon(f), (I - E_t) \varepsilon(g) \rangle, \]

so

\[ E_t = I - \int_t^\infty I_k \otimes E_s \, d\Lambda_s = E_0 + \int_t^t I_k \otimes E_s \, d\Lambda_s \quad \text{for all } t \geq 0, \tag{3.3} \]

where the second identity follows from the first because

\[ \int_0^t I_k \otimes E_s \, d\Lambda_s = \int_0^\infty I_k \otimes E_s \, d\Lambda_s - \int_0^\infty I_k \otimes E_s \, d\Lambda_s = I - E_0 + E_t - I = E_t - E_0. \]

If \( S(\{0\}) = I \) then \( E_S = E_0 \) and the identities hold as claimed. Now suppose \( S(\{0\}) = 0 \) and let \( \pi = (\pi_j)_{j=0}^{\infty} \) be a partition of \( \mathbb{R}_+ \). If \( f, g \in L^2(\mathbb{R}_+; k) \) and \( g \) has support in \( [0, \pi_n] \) then

\[ \langle \varepsilon(f), (I - E_{S,\pi}) \varepsilon(g) \rangle = \sum_{j=1}^{\infty} \langle \varepsilon(f), S((\pi_{j-1}, \pi_j))(I - E_{\pi_j}) \varepsilon(g) \rangle \]

\[ = \sum_{j=1}^{n-1} \sum_{k=j}^{n-1} \langle \varepsilon(f), S((\pi_{j-1}, \pi_j))(E_{\pi_{k+1}} - E_{\pi_k}) \varepsilon(g) \rangle \]

\[ = \sum_{k=1}^{n-1} \langle \varepsilon(f), S([0, \pi_k]) \int_{\pi_k}^{\pi_{k+1}} I_k \otimes E_s \, d\Lambda_s \varepsilon(f) \rangle \]

\[ = \langle \varepsilon(f), \left( \int_0^\infty I_k \otimes S([0, s]) E_s \, d\Lambda_s - R_\pi \right) \varepsilon(g) \rangle, \]

by Lemma A.6, where

\[ R_\pi := \int_0^\infty \sum_{k=0}^{\infty} 1_{(\pi_k, \pi_{k+1})}(s) I_k \otimes S((\pi_k, s)) E_s \, d\Lambda_s. \]

Finally, if \( h \in L^2(\mathbb{R}_+; k) \) then

\[ \| R_\pi \varepsilon(h) \|^2 \leq \int_0^\infty \| \sum_{k=0}^{\infty} 1_{(\pi_k, \pi_{k+1})}(s) S((\pi_k, s)) E_s \varepsilon(h) \|^2 \| h(s) \|^2 \, ds \]

\[ = \int_0^\infty \sum_{k=0}^{\infty} 1_{(\pi_k, \pi_{k+1})}(s) \| S((\pi_k, s)) E_s \varepsilon(h) \|^2 \| h(s) \|^2 \, ds \to 0 \]
as $\pi$ is refined, by the dominated-convergence theorem. To see this, note that if $s \in (\pi_k, \pi_{k+1}]$ then
\[
\|S((\pi_k, s]) E_s \varepsilon(h)\|^2 = \|S((\pi_k, s]) \varepsilon(h)\|^2 \exp(-\|s, \infty\| h)^2 \|
\leq \|S((\pi_k, s]) \varepsilon(h)\|^2
= \|S([0, s]) \varepsilon(h)\|^2 - \|S([0, \pi_k]) \varepsilon(h)\|^2 \to 0
\]
as $s \to \pi_k$, as long as $\pi_k$ is not a point of discontinuity of $s \mapsto S([0, s]) \varepsilon(h)$; furthermore,
\[
\sum_{k=0}^{\infty} 1_{(\pi_k, \pi_{k+1}]}(s) \|S((\pi_k, s]) E_s \varepsilon(h)\|^2 \|h(s)\|^2 \leq \|E_s \varepsilon(h)\|^2 \|h(s)\|^2 \quad \text{for all } s \in \mathbb{R}_+.
\]
The first identity is now established, and the second may be obtained by writing
\[
I - E_0 = \int_0^\infty I_k \otimes E_s dA_s = \int_0^\infty I_k \otimes S([0, s]) E_s dA_s + \int_0^\infty I_k \otimes S((s, \infty]) E_s dA_s.
\]

**Remark 3.7.** Let $S$ be a quantum stopping time. It follows from [5, Section 2.2] that
\[
\|E_S x\|^2 = \|E_0 x\|^2 + \int_0^\infty \|I_k \otimes S((s, \infty]) D_s x\|^2 ds \quad \text{for all } x \in \mathcal{F},
\]
where
\[
D : \mathcal{F} \to L^2(\mathbb{R}_+; k \otimes \mathcal{F}); \quad (D \varepsilon(f))(t) = D_t \varepsilon(f) := f(t)\varepsilon[1_{[0,t]} f]
\]
is the adapted gradient [5, Proposition 2.3]. When $S$ is non-random, this identity [5, Proposition 2.3] is a key tool for establishin\text{g the existence of quantum stochastic integrals, particularly in the vacuum-adapted setting [12].}

**Remark 3.8.** If $S$ and $T$ are quantum stopping times then (3.2) implies that
\[
\|(E_S - E_T) x\|^2 = \int_0^\infty \|I_k \otimes (S([0, s]) - T([0, s])) D_s x\|^2 ds \quad \text{for all } x \in \mathcal{F}.
\]
It follows that the map $S \mapsto E_S$ is continuous when the set of time projections is equipped with the strong operator topology and a net of quantum stopping times $(S_\lambda)$ is defined to converge to a spectral measure $S$ (which must then be a quantum stopping time) if and only if $S_\lambda([0, t]) \to S([0, t])$ in the strong operator topology for almost every $t \in [0, \infty]$; this situation will be denoted by “$S_\lambda \Rightarrow S$”.

Any quantum stopping time $S$ is the limit, in this sense, of a decreasing sequence of discrete quantum stopping times $(S_n)_{n \in \mathbb{N}}$ [24 Proposition 3.3], [10 Proposition 2.3]; a quantum stopping time $S$ is discrete if there exists a finite set $A \subseteq [0, \infty]$, the support of $S$, such that $S(A) = I$ and $S(B) \neq I$ if $B$ is any proper subset of $A$. Note that $S \wedge s \to S$ as $s \to \infty$, so $E_{S \wedge s} \to E_S$ in the strong operator topology.

(Parthasarathy and Sinha [21] use a weaker notion of discreteness (allowing the support of $S$ to be countably infinite) and a stronger notion of convergence (requiring that $S_\lambda([0, t]) \to S([0, t])$ in the strong operator topology for all $t \in [0, \infty]$ such that $S([t]) = 0$.)
Example 3.9. [4, pp.507–508], [24, pp.323–324] Recall that Boson Fock space $\mathcal{F}_+(H)$ has a chaos decomposition, so that

$$\mathcal{F}_+(H) = \bigoplus_{n=0}^{\infty} H^\otimes n$$

and

$$\varepsilon(f) = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} f^\otimes n$$

for all $f \in H$.

where $H$ is any complex Hilbert space and $\otimes_s$ denotes the symmetric tensor product.

Fix $n \in \mathbb{Z}_+$ and, for all $t \in (0, \infty)$, let $P_{n,t} \in B(\mathcal{F})$ be the orthogonal projection onto

$$\bigoplus_{j=0}^{n} L^2([0,t); \mathbb{C}^j) \otimes \mathcal{F}_{[t]} \otimes \mathcal{F}_{[t]} = \mathcal{F};$$

let $P_{n,0} = I$, let $P_{n,\infty} = 0$ and let $P_n$ be the orthogonal projection onto $\bigoplus_{j=0}^{n} L^2(\mathbb{R}_+; \mathbb{C}^j) \otimes \mathcal{F}_{[t]} \otimes \mathcal{F}_{[t]} = \mathcal{F}$.

Note that $P_{n,s} \geq P_{n,t}$ for all $s, t \geq 0$ such that $s \leq t$, so setting

$$S_n([0,t]) := P_{n,t}^\perp = I - P_{n,t} \quad \text{for all } t \geq 0$$

defines a stopping time. Furthermore,

$$S_n((t, \infty]) E_t \varepsilon(f) = P_{n,t} \varepsilon(1_{[0,t]}f) = P_n E_t \varepsilon(f) \quad \text{for all } t \in \mathbb{R}_+ \text{ and } f \in L^2(\mathbb{R}_+; \mathbb{C}),$$

so

$$\langle \varepsilon(f), S_n \varepsilon(g) \rangle = 1 + \int_{0}^{t} \langle \varepsilon(f), P_n E_s \varepsilon(g) \rangle \langle f(s), g(s) \rangle \, ds$$

$$= 1 + \sum_{k=0}^{n} \frac{1}{k!} \int_{0}^{t} \left( \int_{0}^{s} \langle f(r), g(r) \rangle \, dr \right)^k \langle f(s), g(s) \rangle \, ds$$

$$= 1 + \sum_{k=0}^{n} \frac{1}{(k+1)!} \left( \int_{0}^{t} \langle f(s), g(s) \rangle \, ds \right)^{k+1}$$

$$= \langle \varepsilon(f), P_{n+1} \varepsilon(g) \rangle$$

for all $f, g \in L^2(\mathbb{R}_+; \mathbb{C})$ and therefore $E_{S_n} = P_{n+1}$.

Finally, note that $P_{n,t} \leq P_{n+1,t}$ for all $t \geq 0$, so $S_n \leq S_{n+1}$ for all $n \in \mathbb{Z}_+$.

Definition 3.10. Given a quantum stopping time $S$, let the pre-$S$ space $\mathcal{F}_S := E_S(\mathcal{F})$.

Theorem 3.11. Let $S$ and $T$ be quantum stopping times.

(i) If $S \leq T$ then $E_S \leq E_T$ and $\mathcal{F}_S \subseteq \mathcal{F}_T$.

(ii) The time projections $E_{S \wedge T} = E_S \wedge E_T$ and $E_{S \vee T} = E_S \vee E_T$.

(iii) If $S([0, t])$ and $T([0, t])$ commute for all $t \in \mathbb{R}_+$ then so do $E_S$ and $E_T$.

(iv) If $s \in \mathbb{R}_+$ then $E_SE_s = E_{S\wedge s} = E_s E_S$. 

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Proof. Some of these may be obtained by working from the definitions, but Theorem 3.5 and the quantum Itô product formula, Theorem A.2, provide a slicker means of establishing them.

(i) As $S([0, s]) T([0, s]) = T([0, s])$ for all $s \geq 0$, it follows that

$$E_S E_T = \left( I - \int_0^\infty I_k \otimes S([0, s]) E_s \, d\Lambda_s \right) \left( I - \int_0^\infty I_k \otimes T([0, s]) E_s \, d\Lambda_s \right)$$

$$= I - \int_0^\infty I_k \otimes (S([0, s]) E_s + T([0, s]) E_s - S([0, s]) T([0, s]) E_s) \, d\Lambda_s$$

$$= I - \int_0^\infty I_k \otimes S([0, s]) E_s \, d\Lambda_s$$

$$= E_S.$$

(ii) (Cf. [10, Theorem 3.5].) By the quantum Itô product formula and von Neumann’s method of alternating projections,

$$\left( E_S^\frac{1}{n} E_T^\frac{1}{n} \right)^n = \int_0^\infty I_k \otimes \left( (S([0, s]) T([0, s]) \right)^n E_s \, d\Lambda_s \to \int_0^\infty I_k \otimes (S \wedge T)([0, s]) E_s \, d\Lambda_s = E_{S \vee T}^\frac{1}{n}$$

as $n \to \infty$, so $E_{S \vee T} = (E_S^\frac{1}{n} E_T^\frac{1}{n})^n = E_S \vee E_T$.

For the second identity, note first that $0 \leq S$ and $0 \leq T$, by (i), so $E_S - E_0$ and $E_T - E_0$ are orthogonal projections and, as $n \to \infty$,

$$\left((E_S - E_0)(E_T - E_0)\right)^n = \int_0^\infty I_k \otimes \left( S([s, \infty]) T([s, \infty]) \right)^n E_s \, d\Lambda_s$$

$$\to \int_0^\infty I_k \otimes (S \wedge T)([s, \infty]) E_s \, d\Lambda_s = E_{S \wedge T} - E_0.$$

However, as $E_S E_0 = E_0 E_S = E_0 = E_0 E_T = E_T E_0$, we also have that

$$\left((E_S - E_0)(E_T - E_0)\right)^n = (E_S E_T - E_0)^n = (E_S E_T)^n - E_0 \to E_S \wedge E_T - E_0,$$

which gives the result.

(iii) As $S([0, s]) T([0, s]) = T([0, s]) S([0, s])$ for all $s \in \mathbb{R}_+$, it follows that

$$E_S E_T = I - \int_0^\infty I_k \otimes \left( S([0, s]) + T([0, s]) - S([0, s]) T([0, s]) \right) E_s \, d\Lambda_s = E_T E_S.$$

(iv) By (ii) and (iii), as $S$ and $s$ commute in the necessary sense, so $E_S \wedge E_s = E_S E_s$. □

Proposition 3.12. For all $s \geq 0$, it holds that

$$E_{S \wedge s} = S([0, s]) E_S + S([s, \infty]) E_s.$$
Proof. Without loss of generality, let \( s \in \mathbb{R}_+ \) and let \( \pi \) be a partition of \( \mathbb{R}_+ \) with \( s = \pi_n \) for some \( n \in \mathbb{Z}_+ \). Then

\[
E_S E_s = S(\{0\}) E_0 + \sum_{j=1}^{n} S((\pi_{j-1}, \pi_j]) E_{\pi_j} + \sum_{j=n+1}^{\infty} S((\pi_{j-1}, \pi_j]) E_s + S([\infty]) E_s
\]

\[
= S([0, s]) E_S + S((s, \infty]) E_s.
\]

The claim now follows by refining \( \pi \), since \( E_S E_s = E_{S \land s} \) by Theorem \ref{thm:preparation}(iii).

4 The stopping algebras

Definition 4.1. Given \( Z \in \mathcal{B}(\mathcal{F}) \) and a quantum stopping time \( S \), let

\[
Z_S := E_S Z E_S \in \mathcal{B}(\mathcal{F}).
\]

Note that \( E_S Z_S E_S = Z_S E_S \), so \( Z_S \) maps \( \mathcal{F}_S \) to itself. Remark \ref{rem:preparation} implies that \( (Z, S) \mapsto Z_S \) is jointly continuous on the product of any bounded subset of \( \mathcal{B}(\mathcal{F}) \) with the collection of all quantum stopping times, when \( \mathcal{B}(\mathcal{F}) \) is equipped with the strong operator topology and a net \( (S_\lambda) \) of quantum stopping times converges to the quantum stopping time \( S \) if and only if \( S_\lambda \Rightarrow S \).

Proposition 4.2. The map \( Z \mapsto Z_S \) is a conditional expectation from \( \mathcal{B}(\mathcal{F}) \) onto the norm-closed \( * \)-subalgebra \( \mathcal{B}_{\hat{S}} := \{ E_S Z E_S : Z \in \mathcal{B}(\mathcal{F}) \} \) which preserves the vacuum state

\[
\mathbb{E}_{\Omega} : \mathcal{B}(\mathcal{F}) \to \mathbb{C}; \ Z \mapsto \langle \varepsilon(0), Z \varepsilon(0) \rangle.
\]

Proof. This is a straightforward exercise.

Remark 4.3. The collection of stopped algebras \( \{ B_{\hat{S}} : S \text{ is a quantum stopping time} \} \) has the following properties.

(i) If \( S \) is a quantum stopping time and \( Z \in B_{\hat{S}} \) then \( E_S Z E_S = Z E_S \), so \( Z \) preserves the pre-\( S \) space \( \mathcal{F}_S \).

(ii) If the quantum stopping times \( S \) and \( T \) are such that \( S \leq T \) then \( B_{\hat{S}} \subseteq B_{\hat{T}} \), by Theorem \ref{thm:preparation}(i).

(iii) For deterministic stopping times,

\[
B_0 = \text{im} \ P_0^\Omega, \quad B_t = B(\mathcal{F}_t) \otimes P_0^\Omega \quad \text{and} \quad B_\infty = B(\mathcal{F})
\]

for all \( t \in (0, \infty) \), where \( P_0^\Omega \in B(\mathcal{F}_0) \) is the orthogonal projection onto the vacuum subspace \( \mathbb{C} \varepsilon(0|_{s, \infty}) \) for all \( s \in \mathbb{R}_+ \).

As noted by Coquio, this is impossible if we work instead in the identity-adapted setting with the natural analogue of (iii) \cite[Proposition 2.2]{Coquio}.
5 Processes and martingales

Definition 5.1. A process is a family \( X = (X_t)_{t \in \mathbb{R}_+} \subseteq B(\mathcal{F}) \). (We have no need to impose any measurability conditions at this point.) Two processes \( X \) and \( Y \) are equal if and only if \( X_t = Y_t \) for all \( t \in \mathbb{R}_+ \). The set of processes is an algebra, where addition and multiplication are defined pointwise.

Given \( Z \in B(\mathcal{F}) \), let \( \tilde{\pi}(Z) \) and \( \hat{\pi}(Z) \) be the processes with initial values
\[
\tilde{\pi}(Z)_0 = \mathbb{E}_\Omega[Z] E_0 \quad \text{and} \quad \hat{\pi}(Z)_0 = \mathbb{E}_\Omega[Z] I,
\]
and such that
\[
\tilde{\pi}(Z)_t = E_t Z E_t \quad \text{and} \quad \hat{\pi}(Z)_t = E_t Z |_{\mathcal{F}_t} \otimes I_t \quad \text{for all } t \in (0, \infty).
\]
Note that \( t \mapsto \tilde{\pi}(Z)_t \) and \( t \mapsto \hat{\pi}(Z)_t \) are continuous on \([0, \infty]\) in the strong operator topology, where \( \tilde{\pi}(Z)_\infty = \tilde{\pi}(Z)_\infty := Z \), for any \( Z \in B(\mathcal{F}) \). Note also that \( Z \mapsto \tilde{\pi}(Z) \) and \( Z \mapsto \hat{\pi}(Z) \) are algebra homomorphisms. Extend these definitions from operators to processes by setting
\[
\tilde{\pi}(X)_t := \tilde{\pi}(X_t)_t \quad \text{and} \quad \hat{\pi}(X)_t := \hat{\pi}(X_t)_t \quad \text{for all } t \geq 0,
\]
where \( X \) is an arbitrary process.

The process \( X \) is adapted if \( X_t E_t = E_t X_t \) for all \( t \in \mathbb{R}_+ \).

The process \( X \) is vacuum adapted if \( X = \tilde{\pi}(X) \) or, equivalently, if \( X_t E_t = X_t = E_t X_t \) for all \( t \in \mathbb{R}_+ \). Note that \( \tilde{\pi}(Z) \) is a vacuum-adapted process for any \( Z \in B(\mathcal{F}) \).

The process \( X \) is identity adapted if \( X = \hat{\pi}(X) \). Note that \( \hat{\pi}(Z) \) is an identity-adapted process for any \( Z \in B(\mathcal{F}) \).

Note that vacuum-adapted and identity-adapted processes are adapted, and the sets of adapted processes, vacuum-adapted processes and identity-adapted processes are subalgebras of the algebra of processes.

The process \( M \) is a martingale if it is adapted and \( E_s M_t E_s = M_s E_s \) for all \( s, t \in \mathbb{R}_+ \) with \( s \leq t \).

The martingale \( M \) is closed if there exists \( M_\infty \in B(\mathcal{F}) \) such that \( \text{st.} \lim_{t \to \infty} M_t = M_\infty \), where “st.\lim” denotes the limit in the strong operator topology.

The sets of martingales and closed martingales are subspaces of the algebra of adapted processes.

Proposition 5.2. The process \( X \) is a vacuum-adapted martingale closed by \( X_\infty \) if and only if \( X = \tilde{\pi}(X_\infty) \).

Proof. Suppose \( X = \tilde{\pi}(X_\infty) \). Then \( X \) is vacuum adapted, and if \( s, t \in \mathbb{R}_+ \) are such that \( s \leq t \) then
\[
E_s X_t E_s = E_s E_t X_s E_t E_s = E_s X_\infty E_s = X_s = X_s E_s,
\]
so \( X \) is a martingale. Furthermore, \( X_t = \tilde{\pi}(X_\infty)_t \to \tilde{\pi}(X_\infty)_\infty = X_\infty \) in the strong operator topology as \( t \to \infty \), so \( X \) is closed by \( X_\infty \).
Conversely, if $X$ is a vacuum-adapted martingale such that $X_\infty = \lim_{t \to \infty} X_t$, then, for all $s \in \mathbb{R}_+$,

$$
\hat{\pi}(X_\infty)_s = E_s X_\infty E_s = \lim_{t \to \infty} E_s X_t E_s = \lim_{t \to \infty} X_s E_s = X_s.
$$

\[\square\]

**Proposition 5.3.** The process $X$ is an identity-adapted martingale closed by $X_\infty$ if and only if $X = \hat{\pi}(X_\infty)$.

**Proof.** Suppose $X = \hat{\pi}(X_\infty)$. Then $X$ is identity adapted, and if $s, t \in \mathbb{R}_+$ are such that $s \leq t$ then

$$
E_s X_t E_s = E_s \hat{\pi}(X_\infty)_t E_s = E_s X_\infty E_s = \hat{\pi}(X_\infty)_s E_s,
$$

so $X$ is a martingale. Furthermore, $X_t = \hat{\pi}(X_\infty)_t \rightarrow \hat{\pi}(X_\infty)_\infty = X_\infty$ in the strong operator topology as $t \to \infty$, so $X$ is closed by $X_\infty$.

Conversely, if $X$ is an identity-adapted martingale such that $X_\infty = \lim_{t \to \infty} X_t$ then, for all $s \in \mathbb{R}_+$,

$$
\hat{\pi}(X_\infty)_s E_s = E_s X_\infty E_s = \lim_{t \to \infty} E_s X_t E_s = X_s E_s
$$

and

$$
\hat{\pi}(X_\infty)_s = \hat{\pi}(\hat{\pi}(X_\infty)_s E_s)_s = \hat{\pi}(X_s E_s)_s = \hat{\pi}(X_s)_s = X_s.
$$

\[\square\]

**Proposition 5.4.** If $Z \in B(F)$ and $S$ is a quantum stopping time then $Z^{\hat{S}} := (Z_{S,t})_{t \in \mathbb{R}_+}$ is a vacuum-adapted martingale which is closed by $Z^{\hat{S}}$.

**Proof.** If $t \in \mathbb{R}_+$ then Theorem 3.11(ii) implies that

$$
E_t Z_{S} E_t = E_t E_S Z E_t E_t = E_{S \land t} Z E_{S \land t} = Z_t^{\hat{S}},
$$

so the result follows from Proposition 5.2. \[\square\]

**Remark 5.5.** The process $Z^{\hat{S}}$ remains a vacuum-adapted martingale if $(E_t)_{t \in \mathbb{R}_+}$ is replaced by $(E_{S \land t})_{t \in \mathbb{R}_+}$ in Definition 5.1 since the analogue of Proposition 5.2 holds.

6 Stopping processes at discrete times

**Definition 6.1.** A quantum stopping time $T$ is said to be discrete if there exists a finite set $\{t_1 < \ldots < t_n\} \subseteq [0, \infty]$, called the support of $T$, such that $T(\{t_i\}) \neq 0$ for $i = 1, \ldots, n$ and $T(\{t_1, \ldots, t_n\}) = I$. Note that

$$
T([0, t]) = \begin{cases} 
0 & \text{if } t < t_1, \\
T(\{t_1, \ldots, t_m\}) & \text{if } t \in [t_m, t_{m+1}), \\
I & \text{if } t \geq t_n
\end{cases} \quad (m = 1, \ldots, n - 1),
$$

for all $t \geq 0$. In particular, $t_1 \leq T \leq t_n$. 

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**Definition 6.2.** If $T$ is a discrete quantum stopping time with support $\{t_1, \ldots, t_n\}$ and $X$ is a process then Coquio stops $X$ at $T$ by setting [15, Definition 3.2(2)]

$$X_T := \sum_{i,j=1}^{n} \pi(E_{t_i})_j T_i \pi(X_{t_i \lor t_j})_j T_j \hat{\pi}(E_{t_j})_j T_j,$$

where $T_i := T(\{t_i\})$ for $i = 1, \ldots, n$. For definiteness, if $t_n = \infty$ then $X_\infty := 0$ if it is not otherwise defined; in practice, $X$ will either be closed or constant, so that $X_\infty$ exists.

Note that if $t \geq 0$ then $X_T = \hat{\pi}(X)_t$. Furthermore, considering $I$ as a constant process, $I_T = I$.

**Remark 6.3.** Coquio only considers identity-adapted processes, for which her definition agrees with Definition 6.2 above. She prefers the notation $M_T(Z)$ when stopping an operator; as we do not need processes to be adapted, so may identify operators with constant processes, our choice of notation seems more appropriate.

**Definition 6.4.** Imitating Coquio, if $X$ is a process and $T$ is a discrete quantum stopping time with support $\{t_1 < \ldots < t_n\}$ then the result of applying vacuum-adapted stopping to the process $X$ at $T$ is defined to be

$$X_T := \sum_{i,j=1}^{n} E_{t_i} T_i X_{t_i \lor t_j} T_j E_{t_j} = \sum_{i,j=1}^{n} T_i E_{t_i} X_{t_i \lor t_j} E_{t_j} T_j,$$

(6.1)

where $T_i := T(\{t_i\})$ for $i = 1, \ldots, n$; again, let $X_\infty := 0$ if necessary.

Note that if $t \geq 0$ then $X_T = \hat{\pi}(X)_t$. Furthermore, if the process $X$ has the constant value $Z$ then $X_T = E_T Z E_T = Z_T$ in the sense of Definition 6.4 in particular, $I_T = E_T$.

**Proposition 6.5.** If $X$ is a process and $T$ is a discrete quantum stopping time then

$$(X_T)_T = X_T, \quad (X_T)_T = X_T, \quad (X_T)_T = X_T \quad \text{and} \quad (X_T)_T = X_T,$$

where $X_T$ and $X_T$ are stopped by regarding them as constant processes.

**Proof.** This is immediate from the definitions and the fact that $E_r \hat{\pi}(E_r)_s = E_r = \hat{\pi}(E_r)_s E_r$ for all $r \leq s$ and $E_r \hat{\pi}(X_{r \lor s})_r E_s = E_r X_{r \lor s} E_s$ for all $r, s > 0$. $\square$

**Proposition 6.6.** If $T$ is a discrete quantum stopping time and $X$ is a process then

$$T([0, t]) X_T T([0, t]) = T([0, t]) X_{T T} T([0, t]),$$

whereas

$$X_T E_T = E_T X_T \quad \text{and} \quad T([0, t]) X_T T([0, t]) = T([0, t]) X_{T_T} T([0, t]),$$

for all $t \geq 0$.
Proof. Suppose without loss of generality that \( t \in [t_k, t_{k+1}) \), where \( t_0 := 0 \) and \( t_{n+1} := \infty \). Since \( T \land t \) has support with maximum element \( t \) and \( T([0, t]) (T \land t)(\{t\}) = T(\{t\}) \), it follows that
\[
T([0, t]) X_T T([0, t]) = \sum_{i,j=1}^k E_t T_i X_{t_i \land t_j} T_j E_{t_j} = T([0, t]) X_{T \land t} T([0, t]).
\]
The other identities are contained in [15 Properties 3.3(1–2)].

Proposition 6.7. If \( T \) is a discrete quantum stopping time and \( X \) is a process then
\[
E_t X_{T \land t} E_t = X_{T \land t} \quad \text{and} \quad \hat{\pi}(X_{T \land t})_t = X_{T \land t} \quad \text{for all} \ t \geq 0,
\]
so the processes \( X^{\hat{T}} := (X_{T \land t})_{t \in \mathbb{R}_+} \) and \( X^{\hat{T}} := (X_{T \land t})_{t \in \mathbb{R}_+} \) are vacuum adapted and identity adapted, respectively.

Proof. Since \( T \land t \) is discrete and \( T \land t \leq t \), by Theorem 2.11 it follows from the first part of Proposition 6.6 and Theorem 3.11(i) that
\[
E_t X_{T \land t} E_t = E_t E_{T \land t} X_{T \land t} E_{T \land t} E_t = E_{T \land t} X_{T \land t} E_{T \land t} = X_{T \land t}.
\]
The second claim is contained in [15 Properties 3.3(2)].

Remark 6.8. The definitions of \( X^{\hat{T}} \) in Proposition 6.7 and that of \( Z^{\hat{S}} \) in Proposition 5.4 are consistent: they agree when when \( S = T \) is discrete and \( Z \) is regarded as a constant process \( X \), by the final remark in Definition 6.3.

Lemma 6.9. If \( T \) is a discrete quantum stopping time with support \( \{t_1 < \cdots < t_n\} \) and \( M \) is a martingale then \( M^{\hat{T}} = E_T M_t E_T \) for all \( t \in [t_n, \infty) \).

Proof. With the notation of Definition 6.4 if \( t \in [t_n, \infty) \) then
\[
M^{\hat{T}} = \sum_{i,j=1}^n T_i E_t E_{t_i \land t_j} M_t E_{t_i \land t_j} E_{t_j} T_j = \sum_{i,j=1}^n T_i E_t M_t E_{t_i \land t_j} E_{t_j} T_j = E_T M_t E_T.
\]

Theorem 6.10. If \( T \) is a discrete quantum stopping time and \( M \) is a martingale then
\[
E_t M^{\hat{T}} E_t = M^{\hat{T}} \quad \text{and} \quad \hat{\pi}(M^{\hat{T}})_t = M^{\hat{T}} \quad \text{for all} \ t \geq 0;
\]
in particular, the processes \( M^{\hat{T}} \) and \( M^{\hat{T}} \) are martingales closed by \( M^{\hat{T}} \).

Proof. Let \( T \) have support \( \{t_1 < \cdots < t_n\} \); Lemma 6.9 and Theorem 3.11(ii) imply that
\[
E_t M^{\hat{T}} E_t = E_t E_T M_{t_n} E_T E_t = E_{T \land t} M_{t_n} E_{T \land t} = M^{\hat{T}}_T,
\]
since \( T \land t \) has support \( \{t_1 \land t, \ldots, t_n \land t\} \). The second claim is [15 Properties 3.3(3)], and the final remark follows from Propositions 5.2 and 5.3.
Theorem 6.11. An adapted process $X$ is a martingale if and only if $\mathbb{E}^\Omega[X_T] = \mathbb{E}^\Omega[X_0]$ for every discrete quantum stopping time $T$, where $\mathbb{E}^\Omega$ is the vacuum state \[\text{[15].}\]

**Proof.** We follow the proof of \[\text{[15, Proposition 3.10].}\] If $X$ is a martingale and the discrete quantum stopping time $T$ has support $\{t_1, \ldots, t_n\}$ then Lemma \[\text{[6.9].}\] implies that
\[
\mathbb{E}^\Omega[X_T] = \langle \varepsilon(0), E_0 E_T X_{t_n} E_T E_0 \varepsilon(0) \rangle = \langle \varepsilon(0), E_0 X_{t_n} E_0 \varepsilon(0) \rangle = \mathbb{E}^\Omega[X_0];
\]

note that $T \wedge 0 = 0$ and $E_0 E_T = E_0 = E_T E_0$, by Theorem \[\text{3.11(ii).}\]

Conversely, let $T$ have support $\{s < t\}$, where $s, t \in \mathbb{R}_+$, let $T(\{s\}) = P$ and note that
\[
\mathbb{E}^\Omega[X_P] = \mathbb{E}^\Omega[PX_s P] + \mathbb{E}^\Omega[PX_t P] + \mathbb{E}^\Omega[P_\perp X_s P] + \mathbb{E}^\Omega[P_\perp X_t P] = \mathbb{E}^\Omega[P(X_s - X_t)P + X_t],
\]
so if $\mathbb{E}^\Omega[X_P] = \mathbb{E}^\Omega[X_0] = \mathbb{E}^\Omega[X_t] = \mathbb{E}^\Omega[X_t]$ then
\[
\langle P\varepsilon(0), (X_t - X_s)P\varepsilon(0) \rangle = 0
\]
for any orthogonal projection $P \in B(\mathcal{F}_s) \otimes I_{[s]}$, so for any operator $P \in B(\mathcal{F}_s) \otimes I_{[s]}$. Hence $E_s(X_t - X_s)E_s = 0$ and the result follows. \hfill \Box

**Remark 6.12.** Since $\mathbb{E}^\Omega[X_T] = \mathbb{E}^\Omega[X_T]$ for any process $X$ and any discrete quantum stopping time $T$, the identity-adapted version of Theorem \[\text{6.11 holds: Coquio’s work contains a similar result \[\text{[15, Proposition 3.10].}\]}\]

### 7 Stopping closed martingales

**Definition 7.1.** If $M = (M_t)_{t \in \mathbb{R}_+}$ is a martingale closed by $M_\infty$ and $T$ is a discrete quantum stopping time with support $\{t_1, \ldots, t_n\}$ then Lemma \[\text{[6.9].}\] implies that
\[
M_T = E_T M_{t_n} E_T = E_T M_{t_n} E_{t_n} E_T = E_T E_{t_n} M_\infty E_{t_n} E_T = E_T M_\infty E_T.
\]

Hence the result of applying vacuum-adapted stopping to a martingale $M$ closed by $M_\infty$ at a quantum stopping time $S$ is defined to be
\[
M_S := E_S M_\infty E_S.
\]

Note that $(E_t)_{t \in \mathbb{R}_+}$ is a vacuum-adapted martingale closed by $E_\infty = I$ and $I_S = E_S$ for any quantum stopping time $S$. Note also that $S \mapsto M_S$ is continuous when the collection of quantum stopping times has the topology described in Remark \[\text{3.8}\] and $B(\mathcal{F})$ is equipped with the strong operator topology.

**Theorem 7.2.** Let $M$ be a closed martingale and let $S$ be a quantum stopping time.

(i) $M_S = E_S M_S E_S$.

(ii) $E_t M_{S \wedge t} E_t = M_{S \wedge t}$ for all $t \geq 0$.  

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(iii) The process $M^\tilde{S} := (M^\tilde{S}_t)_{t \in \mathbb{R}_+}$ is a vacuum-adapted martingale closed by $M^\tilde{S}$ and such that $M^\tilde{S}_t = E_{S\lambda t} M^\tilde{S}_{S\lambda t}$ for all $t \in \mathbb{R}_+$.

(iv) $S([0, t]) M^\tilde{S} S([0, t]) = S([0, t]) M^\tilde{S}_{\lambda t} S([0, t])$ for all $t \in \mathbb{R}_+$.

Proof. The first claim is immediate, since $E_S$ is idempotent. Furthermore, if $t \in \mathbb{R}_+$ then

$$E_t M^\tilde{S}_{\lambda t} E_t = E_t E_{S\lambda t} M^\infty_{S\lambda t} E_{S\lambda t} E_t = E_{S\lambda t} M^\infty_{S\lambda t} E_{S\lambda t} = M^\tilde{S}_{\lambda t},$$

by Theorems 3.11(ii) and 2.11. Theorem 3.11(ii) also implies that

$$E_t M^\tilde{S} E_t = E_t E_S M^\infty E_S E_t = E_{S\lambda t} M^\infty_{S\lambda t} E_{S\lambda t} = M^\tilde{S}_{\lambda t},$$

so $M^\tilde{S}$ is a vacuum-adapted martingale closed by $M^\tilde{S}$, by Proposition 5.2. If $t \in \mathbb{R}_+$ then

$$E_{S\lambda t} M^\tilde{S} E_{S\lambda t} = E_{S\lambda t} E_S M^\infty E_S E_{S\lambda t} = E_{S\lambda t} M^\infty_{S\lambda t} E_{S\lambda t} = M^\tilde{S}_{\lambda t},$$

by Theorems 2.11 and 3.11(i), so (iii) holds. Finally, Proposition 3.12 implies that

$$S([0, t]) E_{S\lambda t} M^\infty E_S S([0, t]) = S([0, t]) E_{S\lambda t} M^\infty_{S\lambda t} E_S S([0, t]).$$

Proposition 7.3. If $M$ is a martingale closed by $M^\infty$ then $E_\Omega [M_T] = E_\Omega [M_0]$ for any quantum stopping time $T$, where $E_\Omega$ is the vacuum state (4.1).

Proof. Note that

$$E_\Omega [M_T] = \langle \varepsilon(0), E_T M^\infty E_T \varepsilon(0) \rangle = \langle \varepsilon(0), E_0 E_T M^\infty E_T E_0 \varepsilon(0) \rangle = \langle \varepsilon(0), M_0 \varepsilon(0) \rangle = E_\Omega [M_0].$$

Theorem 7.4 (Optional Sampling). Let $M$ be a closed martingale and let $S$ and $T$ be quantum stopping times with $S \leq T$. Then

$$(M^\tilde{S})_T = M^\tilde{S} = (M^\tilde{S})_T.$$ 

Proof. Suppose $M$ is closed by $M^\infty$. Theorem 3.11(i) implies that

$$(M^\tilde{S})_T = E_S E_T M^\infty E_T E_S = E_S M^\infty E_S = M^\tilde{S}$$

and

$$(M^\tilde{S})_T = E_T E_S M^\infty E_S E_T = E_S M^\infty E_S = M^\tilde{S}.$$ 

The following result [15, Lemma 3.6] expresses the relationship between the vacuum-adapted and identity-adapted martingales closed by the same operator.

Proposition 7.5. If $Z \in B(F)$ and $t \geq 0$ then

$$\tilde{\pi}(Z)_t = \pi(Z)_t + \int_t^\infty I_k \otimes \tilde{\pi}(\tilde{\pi}(Z)_t) d\Lambda_s = \pi(Z)_t + \int_t^\infty I_k \otimes \tilde{\pi}(\tilde{\pi}(Z)_t) d\Lambda_s.$$
Proof. Applying Theorem [A.8] to the first identity in (3.3), it follows that
\[ I - E_t = \hat{\pi}(I - E_t)_{\infty} = \int_t^{\infty} I_k \otimes (I - \hat{\pi}(I - E_t)_s) \, d\Lambda_s = \int_t^{\infty} I_k \otimes \hat{\pi}(E_t)_s \, d\Lambda_s. \]
Hence, by Lemma [A.6]
\[ \hat{\pi}(Z)_t = \hat{\pi}(Z)_t \left( E_t + \int_t^{\infty} I_k \otimes \hat{\pi}(E_t)_s \, d\Lambda_s \right) = \hat{\pi}(Z)_t + \int_t^{\infty} I_k \otimes \hat{\pi}(\hat{\pi}(Z)_t)_s \, d\Lambda_s. \]
Similar working, but using the first identity in (3.3) directly, gives the second claim. \( \square \)

The following lemma extends Proposition 7.5 from non-random times to discrete quantum stopping times. The identity-adapted version was obtained by Coquio [15, Theorem 3.5].

Lemma 7.6. If \( T \) is a discrete quantum stopping time and \( X \) is a closed martingale or constant process then \( \langle T(0, s) \rangle \hat{\pi}(X^\pi)_{\pi} \langle T(0, s) \rangle_{\pi} \in \mathbb{C} \) is an identity-adapted \( \mathbb{C} - \mathbb{C} \) process with uniformly bounded norm and

\[ X_{\hat{\pi}} = X_{\hat{\pi}} + \int_0^{\infty} I_k \otimes T([0, s]) \hat{\pi}(X^\pi)_{\pi} T([0, s]) \, d\Lambda_s \quad (7.1) \]
\[ = X_{\hat{\pi}} + \int_0^{\infty} I_k \otimes T([0, s]) \hat{\pi}(X^\pi)_{\pi} T([0, s]) \, d\Lambda_s. \quad (7.2) \]

Proof. Suppose \( T \) has support \( \{t_1 < \cdots < t_n\} \), let \( T_{i} = T(\{t_i\}) \) for \( i = 1, \ldots, n \) and either let \( X_{\infty} \) close \( X \) (if \( X \) is a martingale) or let \( X_{\infty} \) equal \( X \) (if \( X \) is a constant process). Then, by Proposition 7.5 and Lemma [A.6]

\[ X_{\hat{\pi}} = \sum_{i,j=1}^{n} T_i \hat{\pi}(E_{t_i})_{t_i \lor t_j} \hat{\pi}(X_{\infty})_{t_i \lor t_j} \hat{\pi}(E_{t_j})_{t_i \lor t_j} \]
\[ = \sum_{i,j=1}^{n} \left( T_i \hat{\pi}(E_{t_i})_{t_i \lor t_j} E_{t_i \lor t_j} X_{\infty} E_{t_i \lor t_j} \hat{\pi}(E_{t_j})_{t_i \lor t_j} \right) \]
\[ + \int_0^{\infty} I_k \otimes T_{i} \hat{\pi}(E_{t_i})_{t_i \lor t_j} E_{t_i \lor t_j} X_{\infty} E_{t_i \lor t_j} \hat{\pi}(E_{t_j})_{t_i \lor t_j} \, d\Lambda_s \]
\[ = \sum_{i,j=1}^{n} \left( T_i E_{t_i} X_{\infty} E_{t_j} T_j + \int_0^{\infty} I_k \otimes 1_{(t_i, \infty)}(s) 1_{(t_j, \infty)}(s) T_i \hat{\pi}(E_{t_i})_{s} \hat{\pi}(X_{\infty})_{s} \hat{\pi}(E_{t_j})_{s} T_j \, d\Lambda_s \right) \]
\[ = X_{\hat{\pi}} + \int_0^{\infty} I_k \otimes T([0, s]) \hat{\pi}(X^\pi)_{\pi} T([0, s]) \, d\Lambda_s; \]
the penultimate equality holds because \( \hat{\pi}(E_{r})_{s} \hat{\pi}(E_{s})_{t} = \hat{\pi}(E_{r})_{t} \) whenever \( r \leq s \leq t \), and the final equality holds because

\[ \sum_{i=1}^{n} 1_{(t_i, \infty)}(s) T_i \hat{\pi}(E_{t_i})_{s} = \sum_{i,k=1}^{n} 1_{(t_i, \infty)}(s) T_k \hat{\pi}(T_k E_{t_k})_{s} = T([0, s]) \hat{\pi}(E_{T})_{s} \quad \text{for all } s \in \mathbb{R}_+. \]
Similarly,

\[
X_{\tilde{T}} - X_T = \sum_{i,j=1}^{n} T_i \tilde{\pi}(E_{t_i} E_{t_j}) \int_{t_i \vee t_j}^{\infty} I_k \otimes \tilde{\pi}(\tilde{\pi}(X_{\infty})_{t_i \vee t_j}) s \, d\Lambda_s \tilde{\pi}(E_{t_j})_{t_i \vee t_j} T_j
\]

\[
= \sum_{i,j=1}^{n} \int_{0}^{\infty} I_k \otimes 1_{[t_i, \infty)}(s) 1_{[t_j, \infty)}(s) T_i \tilde{\pi}(\tilde{\pi}(E_{t_i} X_{\infty} E_{t_j})_{t_i \vee t_j}) s \, d\Lambda_s \tilde{T}_s
\]

\[
= \int_{0}^{\infty} I_k \otimes T([0, s]) \tilde{\pi}(X_{\tilde{T}} s) \, d\Lambda_s.
\]

**Remark 7.7.** The integrals on the right-hand sides of (7.1) and (7.2) appear to be artifacts produced by working with identity-adapted processes rather than vacuum-adapted ones.

Lemma 7.6 motivates the definition in the following theorem, which was established by Coquio [15, Theorem 3.5]; the proof given here is a shorter version of hers.

**Theorem 7.8.** Let \( Z \in B(\mathcal{F}) \) and let \( S \) be a quantum stopping time. Then the family of operators \( (S([0, s]) \tilde{\pi}(Z_{\tilde{S}}) s S([0, s]))_{s \in \mathbb{R}_+} \) is an identity-adapted \( \mathbb{C} - \mathbb{C} \) process with uniformly bounded norm and

\[
Z_{\tilde{S}} := Z_{\tilde{S}} + \int_{0}^{\infty} I_k \otimes S([0, s]) \tilde{\pi}(Z_{\tilde{S}}) s S([0, s]) \, d\Lambda_s
\]

(7.3)

extends to an element of \( B(\mathcal{F}) \), denoted in the same manner, such that \( \| Z_{\tilde{S}} \| \leq \| Z \| \).

**Proof.** Suppose first that \( S \) is discrete and let \( Y \) denote the integral on the right-hand side of (7.3). From Proposition 6.6 and the remark at the end of Definition 6.4, it follows that

\[
Z_{\tilde{S}} = E_{\tilde{S}} Z_{\tilde{S}} = E_{\til{S}Z_{\til{S}}} + E_{\til{S}Y},
\]

so \( E_{\til{S}Y} = 0 \) and

\[
\til{\pi}(E_{\til{S}}) \til{\pi}(Y)_t = \til{\pi}(E_{\til{S}})_t \int_{0}^{t} I_k \otimes S([0, s]) \til{\pi}(Z_{\til{S}}) s S([0, s]) \, d\Lambda_s = 0 \quad \text{for all } t \geq 0.
\]

Thus if \( \theta \in \mathcal{E} \) then, as \( S([0, s]) \) commutes with \( \til{\pi}(E_{\til{S}}) s \) for all \( s \geq 0 \), the weak form of the quantum Itô product formula (Theorem A.4) for gauge integrals implies that

\[
\| Z_{\til{S}} \theta \|^2 - \| Z_{\til{S}} \theta \|^2 = \| (Z_{\til{S}} - Z_{\til{S}}) \theta \|^2 = \int_{0}^{\infty} \| (I_k \otimes S([0, s]) \til{\pi}(Z_{\til{S}}) s S([0, s])) \nabla_s \theta \|^2 \, ds
\]

\[
\leq \| Z \|^2 \int_{0}^{\infty} \| (I_k \otimes S([0, s]) \til{\pi}(E_{\til{S}}) s S([0, s])) \nabla_s \theta \|^2 \, ds
\]

\[
= \| Z \|^2 (\| \theta \|^2 - \| E_{\til{S}} \theta \|^2),
\]

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where \( \nabla : \mathcal{E} \rightarrow L^2(\mathbb{R}_+; k \otimes \mathcal{F}); \ (\nabla \varepsilon(f))(t) = \nabla_t \varepsilon(f) := f(t) \varepsilon(f) \) is the gradient operator and the final identity follows from the first line by taking \( Z = I \). Hence

\[
\| Z_S^\theta \|^2 \leq \| Z \|^2 (\| E_S^\theta \|^2 + \| \theta \|^2 - \| E_S^\theta \|^2) = \| Z \|^2 \| \theta \|^2,
\]

and \( Z_S \) extends as claimed.

For a general quantum stopping time \( S \), let \( S_n \) be a sequence of discrete quantum stopping times such that \( S_n \Rightarrow S \). Then \( Z_{S_n} \rightarrow Z_S \) in the strong operator topology

\[
\int_0^\infty I_k \otimes S_n([0, s]) \hat{\pi}(Z_{S_n})_s S_n([0, s]) \, d\Lambda_s \rightarrow \int_0^\infty I_k \otimes S([0, s]) \hat{\pi}(Z_S)_s S([0, s]) \, d\Lambda_s
\]

in the strong operator topology on \( \mathcal{E} \), by Lemma \( \text{A.7} \). Hence \( \| Z_S^\theta \| \leq \| Z \| \| \theta \| \) for all \( \theta \in \mathcal{E} \) and the result follows.

**Remark 7.9.** It is readily verified that the map \( (Z, S) \mapsto Z_S \) is jointly continuous on the product of any bounded subset of \( B(\mathcal{F}) \) with the collection of all quantum stopping times, when \( B(\mathcal{F}) \) is equipped with the strong operator topology and the collection of all quantum stopping times is given the topology of Remark \( \text{3.8} \).

Consequently,

\[
Z_S = Z_S + \int_0^\infty I_k \otimes S([0, s]) \hat{\pi}(E_S) S([0, s]) \, d\Lambda_s
\]

for any \( Z \in B(\mathcal{F}) \) and any quantum stopping time \( S \), since this identity holds when \( S \) is discrete and extends to the general case by approximation.

Furthermore, as \( I_{\hat{T}} = I \) for any discrete quantum stopping time \( T \), so

\[
I = I_\hat{S} = E_S + \int_0^\infty I_k \otimes S([0, s]) \hat{\pi}(E_S) S([0, s]) \, d\Lambda_s
\]

\[
\iff E_S = I - \int_0^\infty I_k \otimes S([0, s]) \hat{\pi}(E_S) S([0, s]) \, d\Lambda_s \quad (7.4)
\]

for any quantum stopping time \( S \). This identity, which is believed to be novel, expresses the time projection \( E_S \) in terms of an identity-adapted gauge integral; it should be compared with the first identity in \( (3.2) \).

The following result \( \text{[13] Proposition 3.11} \) is the identity-adapted counterpart of Theorem \( \text{3.6} \).

**Proposition 7.10.** Let \( S \) be a quantum stopping time. Then

\[
\hat{\pi}(E_S)_t = I - \int_0^t I_k \otimes S([0, s]) \hat{\pi}(E_S)_s d\Lambda_s = E_S + \int_t^\infty I_k \otimes S([0, s]) \hat{\pi}(E_S)_s d\Lambda_s \quad \text{for all } t \geq 0.
\]
Proof. The first identity follows from (7.4) and the fact that the gauge integral of an identity-adapted process is an identity-adapted martingale. Alternatively, applying Theorem A.8 to (3.2) gives that

\[ \hat{\pi}(E_S)_t - I = \hat{\pi}(E_S - I)_t = -\int_0^t I_k \otimes (S([0, s]) + \hat{\pi}(E_S - I)_s) \, d\Lambda_s, \]

so

\[ \hat{\pi}(E_S)_t = I - \int_0^t I_k \otimes (S([0, s])\hat{\pi}(E_S)_s + S((s, \infty])(I - \hat{\pi}(E_S)_s)) \, d\Lambda_s. \]

This gives the first claim, since Theorem 3.11(iv) and Proposition 3.12 imply that

\[ S((s, \infty))\hat{\pi}(E_S)_s = S((s, \infty])(I - \hat{\pi}(E_S)_s) = S((s, \infty)]. \tag{7.5} \]

For the second, note that the first claim with \( t = \infty \) yields the identity

\[ E_S + \int_t^\infty I_k \otimes S([0, s])\hat{\pi}(E_S)_s \, d\Lambda_s = I - \int_0^t I_k \otimes S([0, s])\hat{\pi}(E_S)_s \, d\Lambda_s \quad \text{for all } t \geq 0. \]

**Proposition 7.11.** If \( Z \in B(F) \) and \( S \) is a quantum stopping time then

\[ (Z_{\hat{S}})_{\hat{S}} = Z_{\hat{S}}, \quad (Z_{\hat{S}})_{\hat{S}} = Z_{\hat{S}}, \quad (Z_{\hat{S}})_{\hat{S}} = Z_{\hat{S}} \quad \text{and} \quad (Z_{\hat{S}})_{\hat{S}} = Z_{\hat{S}}. \]

Furthermore, \( E_S Z_{\hat{S}} = Z_{\hat{S}} \).

**Proof.** The first identity is trivial, and the third follows immediately from it. The fifth is true if \( S \) is discrete, as noted at the beginning of the proof of Theorem 7.8, so holds in general by approximation. The second now follows, since

\[ E_S \int_0^\infty I_k \otimes S([0, s])\hat{\pi}(Z_{\hat{S}})_s S([0, s]) \, d\Lambda_s = E_S(Z_{\hat{S}} - Z_{\hat{S}}) = 0, \]

and the fourth identity follows from the second. \( \square \)

**Remark 7.12.** If \( Z, W \in B(F) \) and \( S \) is a quantum stopping time then working as in the proof of Theorem 7.8 gives that

\[ Z_{\hat{S}}W_{\hat{S}} - (Z_{\hat{S}}W)_{\hat{S}} = -\int_0^\infty I_k \otimes S([0, s])\hat{\pi}(Z_{\hat{S}})_s S((s, \infty))\hat{\pi}(W_{\hat{S}})_s S([0, s]) \, d\Lambda_s. \]

(The formula given by Coquio [15, Remark 3.13] seems not to be quite correct.) Thus, as noted by Coquio, the map \( Z \mapsto Z_{\hat{S}} \) is not, in general, a conditional expectation on \( B(F) \), in contrast to vacuum-adapted stopping: see Proposition 4.2.

**Definition 7.13.** The result of identity-adapted stopping a martingale \( M \) closed by \( M_\infty \) at a quantum stopping time \( S \) is

\[ M_{\hat{S}} := (M_\infty)_{\hat{S}} = M_{\hat{S}} + \int_0^\infty I_k \otimes S([0, s])\hat{\pi}(M_{\hat{S}})_s S([0, s]) \, d\Lambda_s. \]
The following result is due to Coquio [15, Proposition 3.9].

**Theorem 7.14** (Optional Sampling). Let $M$ be a closed martingale and let $S$ and $T$ be quantum stopping times with $S \leq T$. Then

$$(M_T)_S = M_{\hat{S}}.$$ 

**Proof.** Note first that, since $E_S E_T = E_S$, so

$$(M_T)_S = E_S M_T E_S = E_S E_T M_T E_S = E_S M_{\hat{T}} E_S = M_{\hat{S}},$$

where the penultimate equality follows from Proposition 7.11. Hence

$$(M_T)_S = (M_T)_S + \int_0^\infty I_k \otimes S([0, s]) \hat{\pi}((M_T)_S) S([0, s]) d\Lambda_s$$

$$= M_{\hat{S}} + \int_0^\infty I_k \otimes S([0, s]) \hat{\pi}(M_{\hat{S}}) S([0, s]) d\Lambda_s$$

$$= M_{\hat{S}}. \quad \square$$

**Question 7.15.** Does $(Z_{\hat{S}})_T = Z_{\hat{S}}$ if $Z \in B(F)$ and the quantum stopping times $S$ and $T$ are such that $S \leq T$? If this is not true in general, is it possible to characterise the pairs of quantum stopping times $S$ and $T$ for which this identity holds?

### 8 Stopping closed FV processes

**Definition 8.1.** The process $Y = (Y_t)_{t \in \mathbb{R}_+}$ is an **FV process** if there exists an **integrand process** $H$ such that $s \mapsto H_s x$ is strongly measurable for all $x \in F$ and

$$Y_t = \int_0^t H_s ds \quad \text{for all } t \in \mathbb{R}_+,$$

where the integral exists pointwise in the strong (i.e., Bochner) sense; we write $Y = \int H dt$ to denote this. If $Y_\infty := \int_0^\infty H_t dt$ exists then the FV process $Y$ is **closed** by $Y_\infty$.

The sets of FV processes and closed FV processes are subalgebras of the algebra of processes. If the integrand process $H$ is identity adapted or vacuum adapted then $Y$ has the same property.

**Definition 8.2.** If $Y = \int H dt$ is an FV process which is closed by $Y_\infty$ and $T$ is a discrete quantum stopping time with support $\{t_1 < \cdots < t_n\}$ then, letting $T_i := T(\{t_i\})$,

$$Y_{\hat{T}} = \sum_{i,j=1}^n T_i E_{t_i} Y_{t_i \lor t_j} E_{t_j} T_j = E_T Y_\infty E_T - \sum_{i,j=1}^n \int_{t_i \lor t_j}^\infty T_i E_{t_i} H_s E_{t_j} T_j ds$$

$$= E_T Y_\infty E_T - \int_0^\infty \sum_{i,j=1}^n 1_{[t_i, \infty)}(s) T_i E_{t_i} H_s 1_{[t_j, \infty)}(s) E_{t_j} T_j ds$$

$$= E_T Y_\infty E_T - E_T \int_0^\infty T([0, s]) E_s H_s E_s T([0, s]) ds E_T.$$
Hence, for any quantum stopping time $S$ we define
\[
Y_{\tilde{S}} := E_S \left( Y_\infty - \int_0^\infty S([0, s]) E_s H_s E_s S([0, s]) \, ds \right) E_S
\]
\[
= \int_0^\infty E_S \left( H_s - S([0, s]) E_s H_s E_s S([0, s]) \right) E_S \, ds.
\]
Note that $E_S Y_{\tilde{S}} E_S = Y_{\tilde{S}}$. Furthermore, if $S_\lambda \Rightarrow S$ in the sense of Remark 3.8 then $Y_{\tilde{S}_{\lambda}} \to Y_{\tilde{S}}$ in the strong operator topology.

**Remark 8.3.** If $S$ corresponds to a classical stopping time $\tau$ and $Y = \int H \, dt$ is a classical FV process then, formally,
\[
\int_0^\infty \left( H_s - S([0, s]) E_s H_s E_s S([0, s]) \right) \omega \, ds = \int_0^\infty \left( S((s, \infty)) E_s H_s \right) \omega \, ds = \int_0^{\tau(\omega)} H_s(\omega) \, ds = (Y_\tau)(\omega).
\]

**Proposition 8.4.** If $S$ is a quantum stopping time, $Y$ is an FV process which is closed by $Y_\infty$ and $t \in \mathbb{R}_+$ then
\[
S([0, t]) Y_{\tilde{S}} S([0, t]) = S([0, t]) Y_{\tilde{S}_{\lambda}} S([0, t]).
\]

**Proof.** Note that, by the definition of $Y_{\tilde{S}}$ and Proposition 3.12,
\[
S([0, t]) Y_{\tilde{S}} S([0, t]) = S([0, t]) E_{S_{\lambda}} Y_\infty E_{S_{\lambda}} S([0, t])
- E_{S_{\lambda}} \int_0^\infty S([0, s \wedge t]) E_s H_s E_s S([0, s \wedge t]) \, ds E_{S_{\lambda}}
= S([0, t]) Y_{\tilde{S}_{\lambda}} S([0, t]),
\]
since $S([0, s \wedge t]) = S([0, t]) (S \wedge t)([0, s])$ for all $s \in \mathbb{R}_+$. \qed

**Definition 8.5.** A *semimartingale* is a process of the form $X = M + Y$, where $M$ is a martingale closed by $M_\infty$, the process $Y = \int H \, dt$ is an FV process closed by $Y_\infty$ and the integrand process $H$ is identity adapted or vacuum adapted. (Strictly speaking, this is a *closed* semimartingale, but no other sort of semimartingale will be considered.)

For any quantum stopping time $S$, the stopped semimartingale
\[
X_{\tilde{S}} := M_{\tilde{S}} + Y_{\tilde{S}}
= E_S (M_\infty + Y_\infty) E_S + \int_0^\infty E_S S([0, s]) E_s H_s E_s S([0, s]) E_S \, ds.
\]
The following lemma shows that the decomposition $X = M + Y$ is unique; it follows that this is a good definition.

**Lemma 8.6.** Let $M$ be a martingale, let $Y = \int H \, dt$ be an FV process and suppose the integrand process $H$ is identity adapted or vacuum adapted. If $M + Y = 0$ then $M = 0$ and $Y = 0$. 

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Proposition 3.12, it follows that almost everywhere and the claim follows.

Since $E_s M_t E_s = M_s E_s = E_s M_s$ for all $s, t \in \mathbb{R}_+$ such that $s \leq t$.

Then, for such $s$ and $t$,

$$0 = E_s (M_t + Y_t - M_s - Y_s) E_s = E_s (Y_t - Y_s) E_s = \int_s^t E_s H_r E_s \, dr;$$

given $x \in F$ it follows that $\tilde{\pi}(H)_s x = E_s H_s E_s x = 0$ for almost all $s \in \mathbb{R}_+$, so $s \mapsto H_s x = 0$ almost everywhere and the claim follows.

The next result has been observed by Coquio [15, Proof of Proposition 3.15].

**Lemma 8.7.** Let the FV process $Y = \int H \, dt$ be closed by $Y_\infty$. Then

$$\hat{\pi}(Y_t)_t = E_t Y_\infty E_t + \int_t^\infty I_k \otimes \hat{\pi}(E_t Y_s E_t)_s \, d\Lambda_s - \int_t^\infty \hat{\pi}(E_t H_s E_t)_s \, ds \quad \text{for all } t \geq 0.$$

**Proof.** If $f, g \in L^2(\mathbb{R}_+; k)$ then

$$\langle \varepsilon(f), \int_t^\infty I_k \otimes \hat{\pi}(E_t Y_s E_t)_s \, d\Lambda_s \varepsilon(g) \rangle$$

$$= \int_t^\infty \langle f(s), g(s) \rangle \langle \varepsilon(1_{[0,t]}) f, Y_s \varepsilon(1_{[0,t]} g) \rangle \langle \varepsilon(1_{[s,\infty]} f), \varepsilon(1_{[s,\infty]} g) \rangle \, ds$$

$$= - \left[ \langle \varepsilon(1_{[0,t]} f), Y_s \varepsilon(1_{[0,t]} g) \rangle \langle \varepsilon(1_{[s,\infty]} f), \varepsilon(1_{[s,\infty]} g) \rangle \right]_t$$

$$+ \int_t^\infty \langle \varepsilon(1_{[0,t]} f), H_s \varepsilon(1_{[0,t]} g) \rangle \langle \varepsilon(1_{[s,\infty]} f), \varepsilon(1_{[s,\infty]} g) \rangle \, ds$$

$$= \langle \varepsilon(f), \hat{\pi}(Y_t)_t - E_t Y_\infty E_t \varepsilon(g) \rangle + \langle \varepsilon(f), \int_t^\infty \hat{\pi}(E_t H_s E_t)_s \, ds \varepsilon(g) \rangle.$$

**Remark 8.8.** It follows from Definition 6.2 and Lemma 8.7 that if $T$ is a discrete quantum stopping time and the FV process $Y = \int H \, dt$ is closed by $Y_\infty$ then

$$Y_T = E_T Y_\infty E_T + \int_0^\infty I_k \otimes T([0,s]) \hat{\pi}(E_T Y_s E_T)_s T([0,s]) \, d\Lambda_s$$

$$- \int_0^\infty T([0,s]) \hat{\pi}(E_T H_s E_T)_s T([0,s]) \, ds. \quad (8.1)$$

The identity (8.1) is used by Coquio to perform identity-adapted stopping of semimartingales [15, Proposition 3.15] at an arbitrary quantum stopping time $T$.

Since $E_T Y_T = Y_T$, by Proposition 6.6 and $E_T T([0,s]) = T([0,s]) E_T E_s$ for all $s \geq 0$, by Proposition 3.12, it follows that

$$E_T \int_0^\infty T([0,s]) \hat{\pi}(E_T H_s E_T)_s T([0,s]) \, ds = \int_0^\infty T([0,s]) E_s E_T H_s E_T E_s E_T T([0,s]) \, ds,$$

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so
\[ E_T \int_0^\infty I_k \otimes T([0, s]) \tilde{\pi}(E_T) s \tilde{\pi}(E_T) s T([0, s]) \, d\Lambda_s = 0 \]  
(8.2)
and the gauge integral can again be seen as an artifact produced by identity adaptedness. The identity (8.2) holds for an arbitrary quantum stopping time \( T \), by approximation.

9 Stopping regular \( \Omega \)-semimartingales

For the terminology in this section, see the appendix, Section A.

The class of regular \( \Omega \)-semimartingales is closed under vacuum-adapted stopping; Coquio proved the analogous result for regular quantum semimartingales [15, Proposition 3.16].

**Theorem 9.1.** Suppose \( X \) is a regular \( \Omega \)-semimartingale, with \( M = \int N \, d\Lambda + P \, dA + Q \, dA^\dagger \) its martingale part and \( Y = \int R \, dt \) its FV part. If \( S \) is a quantum stopping time then
\[
X_{\tilde{S}} = \int_0^\infty \tilde{S} N_s \, d\Lambda_s + \int_0^\infty \tilde{S} P_s \, dA_s + \int_0^\infty \tilde{S} Q_s \, dA^\dagger_s + \int_0^\infty \tilde{S} R_s \, ds,
\]
where, for all \( s \in \mathbb{R}_+ \),
\[
\tilde{S} N_s := (I_k \otimes S((s, \infty])) N_s (I_k \otimes S((s, \infty))) , \qquad \tilde{S} P_s := E_S P_s (I_k \otimes S((s, \infty)))
\]
\[
\tilde{S} Q_s := (I_k \otimes S((s, \infty))) Q_s E_S \quad \text{and} \quad \tilde{S} R_s := E_S R_s E_S - S([0, s]) E_S R_s E_S S([0, s]).
\]

**Proof.** If \( X = \int N \, d\Lambda + P \, dA + Q \, dA^\dagger + R \, dt \) and \( X' = \int N' \, d\Lambda \) are regular \( \Omega \)-semimartingales then Theorem A.2 implies that
\[
XX' = \int NN' \, d\Lambda + PN' \, dA + QX' \, dA^\dagger + RX' \, dt
\]
\[
X'X = \int N'N \, d\Lambda + X'P \, dA + N'Q \, dA^\dagger + X'R \, dt
\]
and
\[
X'XX' = \int N'NN' \, d\Lambda + X'PN' \, dA + N'QX' \, dA^\dagger + X'RX' \, dt,
\]
so
\[
X'XX' - X'X - XX' + X
\]
\[
= \int (N'NN' - N'N - NN' + N) \, d\Lambda + (X'PN' - X'P - PN' + P) \, dA
+ (N'QX' - N'Q - QX' + Q) \, dA^\dagger + (X'RX' - X'R - RX' + R) \, dt
\]
\[
= \int (I_k \otimes F - N')N(I_k \otimes F - N') \, d\Lambda + (I - X')P(I_k \otimes F - N') \, dA
+ (I_k \otimes F - N')Q(I - X') \, dA^\dagger + (I - X')R(I - X') \, dt.
\]
Taking
\[ X'_t = I - E_{S\Lambda t} = \int_0^t I_k \otimes S([0, s]) E_s \, d\Lambda_s \quad \text{for all } t \geq 0 \]
now gives the result, since
\[
E_S X_\infty E_S = (X'XX' - X'X - XX' + X)_{\infty}
\]
\[ = \int_0^\infty (I_k \otimes S([s, \infty])) N_s (I_k \otimes S([s, \infty])) \, d\Lambda_s + E_S P_s (I_k \otimes S([s, \infty])) \, dA_s \\
+ (I_k \otimes S([s, \infty])) Q_s E_S \, dA_s^\dag + E_S R_s E_S \, ds. \]

**Remark 9.2.** If \( N, P \) and \( Q \) are suitably bounded vacuum-adapted processes then
\[
E_S \int_0^\infty (I_k \otimes S([0, s])) N_s \, d\Lambda_s = \left( E_0 + \int_0^\infty I_k \otimes E_s S([s, \infty]) \, dA_s \right) \int_0^\infty (I_k \otimes S([0, s])) N_s \, d\Lambda_s \\
= 0 + \int_0^\infty (I_k \otimes (E_s S([s, \infty]) S([0, s]))) N_s \, d\Lambda_s \\
= 0,
\]
\[
\int_0^\infty P_s (I_k \otimes S([0, s])) \, dA_s E_S = \int_0^\infty P_s (I_k \otimes S([0, s])) \, dA_s \left( E_0 + \int_0^\infty I_k \otimes S([s, \infty]) E_s \, d\Lambda_s \right) \\
= 0 + \int_0^\infty P_s (I_k \otimes S([0, s])) S((s, \infty]) E_s \, dA \\
= 0
\]
and
\[
E_S \int_0^\infty (I_k \otimes S([0, s])) Q_s \, dA_s^\dag = \left( E_0 + \int_0^\infty I_k \otimes E_s S((s, \infty]) \, d\Lambda_s \right) \int_0^\infty (I_k \otimes S([0, s])) Q_s \, dA_s^\dag \\
= 0 + \int_0^\infty (I_k \otimes (E_s S((s, \infty]) S([0, s]))) Q_s \, dA_s^\dag \\
= 0.
\]

These identities give another proof of Theorem 9.1.

**Remark 9.3.** Let
\[
\hat{\pi}(X) = \int \hat{\pi}(N) \, d\Lambda + \hat{\pi}(P) \, dA + \hat{\pi}(Q) \, dA^\dag + \hat{\pi}(R) \, dt
\]
be a regular quantum semimartingale, with \( X, N, P, Q \) and \( R \) vacuum-adapted processes. Theorem A.8 implies that
\[
X = \int (N + I_k \otimes X) \, d\Lambda + P \, dA + Q \, dA^\dag + R \, dt,
\]

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and therefore, by Theorem 9.11

\[ X_{\tilde{g}} = \int \tilde{S}(N + I_k \otimes X) \, d\Lambda + \tilde{S}P \, dA + \tilde{S}Q \, dA^t + \tilde{S}R \, dt. \]

Applying Theorem A.8 again gives that \( \hat{\pi}(X_{\tilde{g}}) \) is a regular quantum semimartingale, with

\[ \hat{\pi}(X_{\tilde{g}}) = \int \hat{\pi}(\tilde{S}(N + I_k \otimes X) - I_k \otimes X_{\tilde{g}}) \, d\Lambda + \hat{\pi}(\tilde{S}P) \, dA + \hat{\pi}(\tilde{S}Q) \, dA^t + \hat{\pi}(\tilde{S}R) \, dt, \]

where

\[ \hat{\pi}(\tilde{S}(N + I_k \otimes X) - I_k \otimes X_{\tilde{g}})_t = (I_k \otimes S((t, \infty])) \hat{\pi}(N + I_k \otimes X)_t(I_k \otimes S((t, \infty])) - I_k \otimes \hat{\pi}(X_{\tilde{g}})_t \]

\[ = (I_k \otimes S((t, \infty])) \hat{\pi}(N)_t(I_k \otimes S((t, \infty])) + I_k \otimes \hat{\pi}(S((t, \infty))) XS((t, \infty)) - X_{\tilde{g}})_t \]

\[ = (I_k \otimes S((t, \infty])) \hat{\pi}(N)_t(I_k \otimes S((t, \infty))) - I_k \otimes \hat{\pi}(S((t, \infty))) XS((t, \infty)) - X_{\tilde{g}})_t \]

\[ = (I_k \otimes S((t, \infty])) \hat{\pi}(N)_t(I_k \otimes S((t, \infty))) - I_k \otimes \hat{\pi}(S((t, \infty))) XS((t, \infty)) - X_{\tilde{g}})_t \]

\[ \hat{\pi}(\tilde{S}P)_t = \hat{\pi}(E_S)_t \hat{\pi}(P)_t(I_k \otimes S((t, \infty))), \]

\[ \hat{\pi}(\tilde{S}Q)_t = (I_k \otimes S((t, \infty])) \hat{\pi}(Q)_t \hat{\pi}(E_S)_t \]

and \( \hat{\pi}(\tilde{S}R)_t = \hat{\pi}(E_S R E_S)_t - S([0, t]) \hat{\pi}(E_S R E_S)_t S([0, t]) \)

\[ = \hat{\pi}(E_S)_t \hat{\pi}(R)_t S((t, \infty)) + S((t, \infty]) \hat{\pi}(R)_t \hat{\pi}(E_S)_t \]

\[ - S((t, \infty]) \hat{\pi}(R)_t S((t, \infty]) \]

for all \( t \in \mathbb{R}_+ \); the final equalities for the gauge and time integrands hold by (7.5). Comparing these expressions for the integrands with those in [15 Proposition 3.16], it follows that

\[ \hat{\pi}(\hat{\pi}(X_{\tilde{g}}))_t = \hat{\pi}(X_{\tilde{g}})_t + \int_0^t I_k \otimes S([0, s]) \hat{\pi}(X_{\tilde{g}})_s S([0, s]) \, d\Lambda_s \quad \text{for all } t \geq 0. \]

In particular, taking \( t = \infty \) recovers the identity (7.1).

A Quantum stochastic calculus

**Definition A.1.** Let \( k_1 \) and \( k_2 \) be non-zero subspaces of \( k \). A \( k_1 - k_2 \) process \( X \) is a family of bounded operators \( (X_t)_{t \in \mathbb{R}_+} \subseteq B(k_1 \otimes \mathcal{F}; k_2 \otimes \mathcal{F}) \) such that \( t \mapsto X_t \in \mathcal{F} \) is strongly measurable for all \( f \in L^2(\mathbb{R}_+; k) \).
A \( k_1 - k_2 \) process \( X \) is vacuum adapted if \((I_{k_2} \otimes E_t)X_t(I_{k_1} \otimes E_t) = X_t \) for all \( t \in \mathbb{R}_+ \).

Let \( N, P, Q \) and \( R \) be vacuum adapted \( k - k, k - C, C - k \) and \( C - C \) processes, respectively, such that

\[
\|N\|_\infty := \text{ess sup}\{\|N_t\| : t \in \mathbb{R}_+\} < \infty \quad \text{and} \quad \int_0^\infty (\|P_s\|^2 + \|Q_s\|^2 + \|R_s\|) \, ds < \infty.
\]

The gauge integral \( \int N \, d\Lambda = (\int_0^t N_s \, d\Lambda_s)_{t \in \mathbb{R}_+} \), annihilation integral \( \int P \, dA = (\int_0^t P_s \, dA_s)_{t \in \mathbb{R}_+} \), creation integral \( \int Q \, dA^\dagger = (\int_0^t Q_s \, dA^\dagger_s)_{t \in \mathbb{R}_+} \) and time integral \( \int R \, dt = (\int_0^t R_s \, ds)_{t \in \mathbb{R}_+} \) are the unique vacuum-adapted \( C - C \) processes such that

\[
\langle \varepsilon(f), \int_0^t N_s \, d\Lambda_s \varepsilon(g) \rangle = \int_0^t \langle f(s) \otimes \varepsilon(f), N_s(g(s) \otimes \varepsilon(g)) \rangle \, ds \quad (A.1)
\]
\[
\langle \varepsilon(f), \int_0^t P_s \, dA_s \varepsilon(g) \rangle = \int_0^t \langle \varepsilon(f), P_s(g(s) \otimes \varepsilon(g)) \rangle \, ds \quad (A.2)
\]
\[
\langle \varepsilon(f), \int_0^t Q_s \, dA^\dagger_s \varepsilon(g) \rangle = \int_0^t \langle f(s) \otimes \varepsilon(f), Q_s \varepsilon(g) \rangle \, ds \quad (A.3)
\]

and

\[
\langle \varepsilon(f), \int_0^t R_s \, ds \varepsilon(g) \rangle = \int_0^t \langle \varepsilon(f), R_s \varepsilon(g) \rangle \, ds, \quad (A.4)
\]

respectively, for all \( f, g \in L^2(\mathbb{R}_+; k) \) and \( t \geq 0 \).

A vacuum-adapted \( C - C \) process \( X = (X_t)_{t \in \mathbb{R}_+} \) is a regular \( \Omega \)-semimartingale if there exist vacuum-adapted processes \( N, P, Q, R \) as above and such that

\[
X_t = \int_0^t N_s \, d\Lambda_s + \int_0^t P_s \, dA_s + \int_0^t Q_s \, dA^\dagger_s + \int_0^t R_s \, ds \quad \text{for all } t \in \mathbb{R}_+;
\]

we write \( X = \int N \, d\Lambda + P \, dA + Q \, dA^\dagger + R \, dt \) to denote this.

The martingale part of \( X \) is the vacuum-adapted martingale \( M = \int N \, d\Lambda + P \, dA + Q \, dA^\dagger \) which is closed by \( M_\infty \), where

\[
M_t = \int_0^t N_s \, d\Lambda_s + \int_0^t P_s \, dA_s + \int_0^t Q_s \, dA^\dagger_s \quad \text{for all } t \geq 0,
\]

and the FV part of \( X \) is the vacuum-adapted FV process \( Y = \int R \, dt \) which is closed by

\[
Y_\infty = \int_0^\infty R_s \, ds.
\]

**Theorem A.2.** Let \( X = \int N \, d\Lambda + P \, dA + Q \, dA^\dagger + R \, dt \) and \( X' = \int N' \, d\Lambda + P' \, dA + Q' \, dA^\dagger + R' \, dt \) be regular \( \Omega \)-semimartingales. The process \( XX' = (X_t X'_t)_{t \in \mathbb{R}_+} \) is a regular \( \Omega \)-semimartingale, with

\[
XX' = \int NN' \, d\Lambda + (XP' + PN') \, dA + (QX' + NQ') \, dA^\dagger + (XR' + RX' + PQ') \, dt.
\]
**Definition A.3.** A $k_1 - k_2$ process $X$ is identity adapted if and only if

$$\langle \varepsilon(f), X_t \varepsilon(g) \rangle = \langle \varepsilon(1_{[0,t]} f), X_t \varepsilon(1_{[0,t]} g) \rangle \langle \varepsilon(1_{[t,\infty)} f), \varepsilon(1_{[t,\infty)} g) \rangle$$

for all $f, g \in L^2(\mathbb{R}^+; k)$ and $t \in \mathbb{R}^+$; equivalently, $X_t = X_t \otimes I_{[t]}$, where $X_t \in B(\mathcal{F}_t)$, for all $t \in \mathbb{R}^+$.

Given a quadruple of processes $N, P, Q$ and $R$ as in Definition A.1 but identity adapted rather than vacuum adapted, there exist identity-adapted gauge, annihilation, creation and time integrals which satisfy the same inner-product identities. However, integration may not preserve boundedness in this case, other than for the time integral: the integrals will exist as time integrals which satisfy the same inner-product identities. However, integration may not preserve boundedness in this case, other than for the time integral: the integrals will exist as linear operators with domains containing $\mathcal{E}$ such that the identities \(A.1 - A.4\) hold. If

$$M_t = \int_0^t N_s \, d\Lambda_s + \int_0^t P_s \, dA_s + \int_0^t Q_s \, dA^+_s$$

is a bounded operator for all $t \geq 0$ then the identity-adapted $\mathbb{C} - \mathbb{C}$ process $X = M + Y$, where $Y = \int H \, dt$, is a *regular quantum semimartingale*.

A weak form of Itô product formula holds for quantum semimartingales which are not necessarily regular. We only need the version for gauge integrals, which is as follows.

**Theorem A.4.** Let $N$ and $N'$ be identity-adapted $k - k$ processes such that $\|N\|_{\infty} < \infty$ and $\|N'\|_{\infty} < \infty$. If $X_t = \int_0^t N_s \, d\Lambda_s$ and $X'_t = \int_0^t N'_s \, d\Lambda_s$ then

$$\langle X_t x, X'_t x' \rangle = \int_0^t \left( \langle (I_k \otimes X_s) \nabla_s x, N'_s \nabla_s x' \rangle + \langle N_s \nabla_s x, (I_k \otimes X'_s) \nabla_s x' \rangle + \langle N_s \nabla_s x, N'_s \nabla_s x' \rangle \right) \, ds$$

for all $t \geq 0$ and $x, x' \in \mathcal{F}$, where

$$\nabla : \mathcal{E} \to L^2(\mathbb{R}^+; k \otimes \mathcal{F}); \quad (\nabla \varepsilon(f))(t) = \nabla \varepsilon(f) := f(t) \varepsilon(f)$$

is the linear gradient operator.

**Proof.** See [21, Theorem 3.15].

**Remark A.5.** The decomposition $X = M + Y$ is unique for both regular $\Omega$-semimartingales and regular quantum semimartingales, by Lemma A.6. In fact, more may be shown: the quantum stochastic integrators are independent, in the sense that if

$$\int_0^\infty N_s \, d\Lambda_s + \int_0^\infty P_s \, dA_s + \int_0^\infty Q_s \, dA^+_s + \int_0^\infty R_s \, ds = 0$$

then $N_s = P_s = Q_s = R_s = 0$ for almost all $s \in \mathbb{R}^+$ [20].

**Lemma A.6.** Let $t \in \mathbb{R}^+$ and $Z, W \in B(\mathcal{F}_t) \otimes I_{[t]}$. If $(N_s)_{s \in \mathbb{R}^+}$ is a vacuum-adapted or identity-adapted $k - k$ process such that $\|N\|_{\infty} < \infty$ then so is $\left(1_{[0,\infty)}(s)(I_k \otimes Z) N_s (I_k \otimes W) \right)_{s \in \mathbb{R}^+}$, with the same type of adaptedness, and

$$Z \int_t^\infty N_s \, d\Lambda_s W = \int_t^\infty (I_k \otimes Z) N_s (I_k \otimes W) \, d\Lambda_s.$$
Proof. The first claim is immediate. For the second, note that if \( Z = W(f) \), the Weyl operator corresponding to \( f \), so that
\[
W(f)\varepsilon(g) = \exp\left(-\frac{1}{2}\|f\|^2 - \langle f, g \rangle\right)\varepsilon(g + f)
\]
for all \( g \in L^2(\mathbb{R}_+; k) \), then \( Z^* = W(-f) \) and, if \( f \in L^2(\mathbb{R}_+; k) \) has support in \([0, t]\),
\[
\langle \varepsilon(g), Z \int_t^\infty N_s d\Lambda_s \varepsilon(h) \rangle
= \exp\left(-\frac{1}{2}\|f\|^2 + \langle f, g \rangle\right) \int_t^\infty N_s d\Lambda_s \varepsilon(h)) \]
\[
= \exp\left(-\frac{1}{2}\|f\|^2 + \langle f, g \rangle\right) \int_t^\infty \left((g - f)(s) \otimes \varepsilon(g - f), N_s(h(s) \otimes \varepsilon(h))\right) ds
\]
\[
= \int_t^\infty \langle g(s) \otimes Z^* \varepsilon(g), N_s(h(s) \otimes \varepsilon(h)) \rangle ds
\]
\[
= \langle \varepsilon(g), \int_t^\infty (I_k \otimes Z)N_s d\Lambda_s \varepsilon(h) \rangle
\]
for all \( g, h \in L^2(\mathbb{R}_+; k) \). Since such Weyl operators are dense in \( B(\mathcal{F}_t) \otimes I_k \) for the strong operator topology, the result holds for general \( Z \) and \( W = I \); the full version may be obtained by taking adjoints.

Lemma A.7. If \((N_s)_{s \in \mathbb{R}_+}\) is a vacuum-adapted or identity-adapted \( k - k \) process such that \( \|N\|_\infty < \infty \) then
\[
\|\int_0^\infty N_s d\Lambda_s \varepsilon(f)\|^2 \leq C_f \int_0^\infty \|N_s(f(s) \otimes \varepsilon(f))\|^2 ds
\]
for all \( f \in L^2(\mathbb{R}_+; k) \), where \( C_f : = 1 \) if \( N \) is vacuum adapted and \( C_f : = \|f\| + \sqrt{1 + \|f\|^2} \) if \( N \) is identity adapted.

Proof. See [13, Proof of Theorem 18] and [21, Theorem 3.13], respectively.

Theorem A.8. If \( X = \int N d\Lambda + P dA + Q dA^\dagger + R dt \) is a regular \( \Omega \)-semimartingale then \( \hat{\pi}(X) \) is a regular quantum semimartingale such that
\[
\hat{\pi}(X)_t = \int_0^t \hat{\pi}(N - I_k \otimes X)_s d\Lambda_s + \int_0^t \hat{\pi}(P)_s dA_s + \int_0^t \hat{\pi}(Q)_s dA^\dagger_s + \int_0^t \hat{\pi}(R)_s ds
\]
for all \( t \in \mathbb{R}_+ \).

Proof. See [13, Corollaries 31 and 40]; the extension to a non-separable multiplicity space \( k \) is straightforward.
Notation A.9. Let $k = \mathbb{C}$, let $\nu = (\nu_t)_{t \in \mathbb{R}_+}$ be a standard Poisson process on the probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and let $U_P : L^2(\Omega, \mathcal{A}, \mathbb{P}) \to \mathcal{F}$ be the isometric isomorphism such that $U_P 1 = \varepsilon(0)$ and the closure of $N_t := \Lambda_t + A_t + A^\dagger_{\tau(t)} + tI$ equals $U_P \nu_t U^*_P$ for all $t \geq 0$ [19, Theorems 6.1–2]. Recall that
\[
\zeta(f) := U_P \varepsilon(f) = 1 + \int_0^\infty f(t) \zeta(1_{[0, t]} f) \, d\chi_t \quad \text{for all } f \in L^2(\mathbb{R}_+),
\]
where $\chi$ is the normal martingale such that $\chi_t = \nu_t - t$ for all $t \in \mathbb{R}_+$ [3, Section II.1].

Lemma A.10. With the conventions of Notation A.9, let $\phi$ be a bounded process on $(\Omega, \mathcal{A}, \mathbb{P})$ adapted to the Poisson filtration and let $F_t = U_P \phi_t U^*_P \in \mathcal{B}(\mathcal{F})$ for all $t \in \mathbb{R}_+$, where $\phi_t$ acts as multiplication by $\phi_t$. Then
\[
U_P \int_0^\infty \phi_t \, d\chi_t U^*_P = \int_0^\infty F_t \, d(N_t - t) \quad \text{on } \mathcal{E}.
\]
(A.5)

Proof. For all $f \in L^2(\mathbb{R})$ and $t \in \mathbb{R}_+$, let $E_t \zeta(f) := \zeta(1_{[0, t]} f)$ and $D_t \zeta(f) := f(t) E_t \zeta(f)$. Since the quadratic variation $[\chi]_t = \chi_t + t$ for all $t \in \mathbb{R}_+$, if $f, g \in L^2(\mathbb{R}_+)$ then
\[
\mathbb{E}[\zeta(f) \int_0^\infty \phi_t \, d\chi_t \zeta(g)]
\]
\[
= \mathbb{E}
\left[
(1 + \int_0^\infty D_t \zeta(f) \, d\chi_t) \int_0^\infty \phi_t \, d\chi_t \zeta(g)
\right]
\]
\[
= \mathbb{E}
\left[
(\int_0^\infty \phi_t \, d\chi_t + \int_0^\infty D_t \zeta(f) \, d\chi_t + \int_0^t \phi_s \, d\chi_s \, d\chi_t
\right.
\]
\[
+ \int_0^\infty \int_0^t D_s \zeta(f) \, d\chi_s \phi_t \, d\chi_t + \int_0^\infty D_t \zeta(f) \, d\chi_t (1 + \int_0^\infty D_t \zeta(g) \, d\chi_t)
\]
\[
= \int_0^\infty \mathbb{E}
\left[
D_t \zeta(f) \phi_t + \phi_t D_t \zeta(g) + D_t \zeta(f) \int_0^t \phi_s \, d\chi_s \, D_t \zeta(g) + \int_0^t D_s \zeta(f) \, d\chi_s \phi_t D_t \zeta(g)
\right.
\]
\[
+ \int_0^\infty \int_0^t D_s \zeta(f) \, d\chi_s \phi_t \, d\chi_t + \int_0^\infty D_t \zeta(f) \phi_t \int_0^t D_s \zeta(g) \, d\chi_s \, ds
\]
\[
= \int_0^\infty \mathbb{E}
\left[
D_t \zeta(f) \phi_t E_s \zeta(g) + E_t \zeta(f) \phi_t D_t \zeta(g) + D_t \zeta(f) \phi_t \int_0^t \phi_s \, d\chi_s \, D_t \zeta(g)
\right.
\]
\[
+ \int_0^\infty \mathbb{E}
\left[
E_t \zeta(f) \phi_t + \phi_t g(t) + f(t) \left[ \phi_t \int_0^t \phi_s \, d\chi_s \right] g(t) \right] E_t \zeta(g)
\]
\[
\right)
\]
\[
\int_0^\infty \mathbb{E}[E_t \zeta(f) \phi_t + \phi_t g(t) + f(t) \left[ \phi_t \int_0^t \phi_s \, d\chi_s \right] g(t)] \, dt.
\]
Replacing $f$, $g$ and $\phi$ by $1_{[0, t]} f$, $1_{[0, t]} g$ and $1_{[0, t]} \phi$, respectively, it follows that
\[
\alpha_t := \mathbb{E}
\left[
\zeta(1_{[0, t]} f)
\right.
\]
\[
= \int_0^t \left( f(s) + g(s) + f(s) g(s) \right) \mathbb{E}[E_s \zeta(f) \phi_s \zeta(g)] + \int_0^t f(s) g(s) \alpha_s \, ds,
\]

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so
\[
\frac{d}{dt}(\alpha_t \exp(\int^{\infty}_{0} f(s)g(s) \, ds)) = (\overline{f(t)} + g(t) + \overline{f(s)}g(s)) \mathbb{E}[\overline{\zeta(f)} \phi_t \zeta(g)]
\]
and
\[
\mathbb{E}[\overline{\zeta(f)} \int^{\infty}_{0} \phi_t \, d\chi_t \zeta(g)] = \int^{\infty}_{0} (\overline{f(t)} + g(t) + \overline{f(t)}g(t)) \mathbb{E}[\overline{\zeta(f)} \phi_t \zeta(g)] \, dt
\]
\[
= \langle \varepsilon(f), \int^{\infty}_{0} F_t \, d(N_t - t) \varepsilon(g) \rangle.
\]

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