Reference state for arbitrary $U$-consistent subspace

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Abstract

The reduced dynamics of the system $S$, interacting with the environment $E$, is not given by a linear map, in general. However, if it is given by a linear map, then this map is also Hermitian. In order that the reduced dynamics of the system is given by a linear Hermitian map, there must be some restrictions on the set of possible initial states of the system-environment or on the possible unitary evolutions of the whole $SE$. In this paper, adding an ancillary reference space $R$, we assign to each convex set of possible initial states of the system-environment $S$, for which the reduced dynamics is Hermitian, a tripartite state $\omega_{RSSE}$, which we call the reference state, such that the set $S$ is given as the steered states from the reference state $\omega_{RSSE}$. The set of possible initial states of the system is also given as the steered set from a bipartite reference state $\omega_{RS}$. The relation between these two reference states is as $\omega_{RSSE} = \text{id}_R \otimes \Lambda_S(\omega_{RS})$, where $\text{id}_R$ is the identity map on $R$ and $\Lambda_S$ is a Hermitian assignment map, from $S$ to $SE$. As an important consequence of introducing the reference state $\omega_{RSSE}$, we generalize the result of Buscemi (2014 Phys. Rev. Lett. 113 140502): we show that, for a $U$-consistent subspace, the reduced dynamics of the system is completely positive, for arbitrary unitary evolution of the whole system-environment $U$, if and only if the reference state $\omega_{RSSE}$ is a Markov state. In addition, we show that the evolution of the set of system-environment (system) states is determined by the evolution of the reference state $\omega_{RSSE} (\omega_{RS})$.

Keywords: open quantum systems, linear Hermitian maps, assignment map, steered states, completely positive maps

1. Introduction

Consider a closed finite dimensional quantum system which evolves as

$$\rho \rightarrow \rho' = \text{Ad}_U(\rho) \equiv U\rho U^\dagger,$$

(1)
where $\rho$ and $\rho'$ are the initial and final states (density operators) of the system, respectively, and $U$ is a unitary operator ($UU^\dagger = U^\dagger U = I$, where $I$ is the identity operator).

In general, the system is not closed and interacts with its environment. We can consider the whole system-environment as a closed quantum system which evolves as equation (1). So the reduced state of the system after the evolution is given by

$$\rho_S' = \text{Tr}_E \circ \text{Ad}_U(\rho_{SE}) = \text{Tr}_E \left(U\rho_{SE}U^\dagger\right),$$

(2)

where $\rho_{SE}$ is the initial state of the combined system-environment quantum system and $U$ acts on the whole Hilbert space of the system-environment.

In general, the relation between the initial state of the system $\rho_S = \text{Tr}_E(\rho_{SE})$ and its final state $\rho_S'$ is not given by a map [1, 2]. Even if it is given by a map, then, in general, this map is not a linear map [3, 4]. In order that equation (2) leads to a linear map from $\rho_S$ to $\rho_S'$, there must be some restrictions on the set of possible initial states of the system-environment $\{\rho_{SE}\}$ or on the possible unitary evolutions $U$ [2, 5].

However, if the reduced dynamics of the system from $\rho_S$ to $\rho_S'$ can be given by a linear map $\Psi$, then this $\Psi$ is also Hermitian, i.e. maps each Hermitian operator to a Hermitian operator. Now, an important result is that for each linear trace-preserving Hermitian map from $\rho_S$ to $\rho_S'$, there exists an operator sum representation in the following form:

$$\rho_S' = \sum_i e_i \tilde{E}_i \rho_S \tilde{E}_i^\dagger, \quad \sum_i e_i \tilde{E}_i^\dagger \tilde{E}_i = I_S,$$

(3)

where $\tilde{E}_i$ are linear operators and $e_i$ are real coefficients [2, 6, 7]. For the special case that all of the coefficients $e_i$ in equation (3) are positive, then we call the map completely positive (CP) [8] and rewrite equation (3) in the following form:

$$\rho_S' = \sum_i E_i \rho_S E_i^\dagger, \quad \sum_i E_i^\dagger E_i = I_S,$$

(4)

where $E_i \equiv \sqrt{e_i} \tilde{E}_i$.

A general framework for linear trace-preserving Hermitian maps, arisen from equation (2), when both the system and the environment are finite dimensional, has been developed in [2]. The starting point of this framework is to consider a convex set of initial states $\mathcal{S} = \{\rho_{SE}\}$, for the whole system-environment, i.e. if $\rho_{SE}^{(1)}, \rho_{SE}^{(2)} \in \mathcal{S}$, then $\rho_{SE} = p\rho_{SE}^{(1)} + (1 - p)\rho_{SE}^{(2)} \in \mathcal{S}$, for $0 \leq p \leq 1$.

As we will see in the next section, an straightforward way to construct a convex $\mathcal{S}$ is to consider the set of steered states from performing measurements on the part $R$ of a fixed tripartite state $\omega_{RSE}$, which is a state on the Hilbert space of the reference-system-environment $\mathcal{H}_R \otimes \mathcal{H}_S \otimes \mathcal{H}_E$.

We call $\omega_{RSE}$ the reference state and we will show that if it can be written as equation (7) below, then the reduced dynamics of the system is Hermitian. Interestingly, this result includes all the previously found sets $\mathcal{S}$, in [9–14], for which the reduced dynamics of the system is CP.

Then, we question whether it is possible to find such reference state $\omega_{RSE}$ for arbitrary convex set $\mathcal{S}$, for which the reduced dynamics is Hermitian. Fortunately, this is the case as we will show in section 3. The possibility of introducing the reference state $\omega_{RSE}$, for arbitrary $\mathcal{S}$, has an important consequence: in section 4, we generalize the result of [13], i.e. we show that, for arbitrary $\mathcal{S}$, when $\omega_{RSE}$ is not a so-called Markov state, then the the reduced dynamics of the system, for at least one $U$, is not CP.
Sections 3 and 4 are on the case that there is a one to one correspondence between the members of $\mathcal{S}$ and the members of $\text{Tr}_E\mathcal{S}$. The general case, where there is no such correspondence, is given in section 5.

In section 6, we consider the case studied in [4], as an example, to illustrate (a part of) our results, and finally, we will end this paper in section 7, with a summary of our results.

2. Reduced dynamics for a steered set

Assume that, for each $\rho_{SE} \in \mathcal{S}$, the reduced dynamics of the system is given by a map $\Psi$. So, for each $\rho_S \in \text{Tr}_E\mathcal{S}$, we have:

$$\rho_S' = \Psi(\rho_S) = \text{Tr}_E \circ \text{Ad}_U(\rho_{SE}).$$

(5)

The first obvious requirement that such a map $\Psi$ can be defined, is the $U$-consistency of the $\mathcal{S}$ [2], i.e. if for two states $\rho_{SE}^{(1)}, \rho_{SE}^{(2)} \in \mathcal{S}$, we have $\text{Tr}_E(\rho_{SE}^{(1)}) = \text{Tr}_E(\rho_{SE}^{(2)}) = \rho_S$, then we must have $\text{Tr}_E \circ \text{Ad}_U(\rho_{SE}^{(1)}) = \text{Tr}_E \circ \text{Ad}_U(\rho_{SE}^{(2)}) = \Psi(\rho_S)$.

Interestingly, if $\mathcal{S}$ is convex and $U$-consistent, then the reduced dynamics of the system is given by a (linear trace-preserving) Hermitian map [2].

An straightforward way to construct a convex $\mathcal{S}$ is to consider the set of steered states from performing measurements on the part $R$ of a reference state $\omega_{RSE}$ [13, 15]:

$$\mathcal{S} = \left\{ \frac{\text{Tr}_E[(P_R \otimes I_{SE})\omega_{RSE}]}{\text{Tr}[(P_R \otimes I_{SE})\omega_{RSE}]} : P_R > 0 \right\},$$

(6)

where $P_R$ is arbitrary positive operator on $\mathcal{H}_R$ such that $\text{Tr}[(P_R \otimes I_{SE})\omega_{RSE}] > 0$ and $I_{SE}$ is the identity operator on $\mathcal{H}_S \otimes \mathcal{H}_E$. Note that, up to a positive factor, $P_R$ can be considered as an element of a POVM.

It can be shown simply that the set of initial states of the system-environment $\mathcal{S}$, in equation (6), is convex. So, if, in addition, it be a $U$-consistent set, then the reduced dynamics of the system, for all $\rho_{SE} \in \mathcal{S}$ ($\rho_S \in \text{Tr}_E\mathcal{S}$), is given by a Hermitian map, as equation (3).

Note that if there is a one to one correspondence between the members of $\mathcal{S}$ and the members of $\text{Tr}_E\mathcal{S}$, then, trivially, the $U$-consistency condition is satisfied. In other words, if, for each $\rho_S \in \text{Tr}_E\mathcal{S}$, there is only one $\rho_{SE} \in \mathcal{S}$ such that $\rho_S = \text{Tr}_E(\rho_{SE})$, then the set $\mathcal{S}$ is $U$-consistent, for any arbitrary unitary evolution of the whole system-environment $U$.

Now, let us consider the case that the reference state $\omega_{RSE}$ can be written as

$$\omega_{RSE} = \text{id}_R \otimes \Lambda_S(\omega_{RS}),$$

(7)

where $\omega_{RS} = \text{Tr}_E(\omega_{RSE})$, $\text{id}_R$ is the identity map on $\mathcal{L}(\mathcal{H}_R)$ and $\Lambda_S : \mathcal{L}(\mathcal{H}_S) \to \mathcal{L}(\mathcal{H}_S \otimes \mathcal{H}_E)$ is a Hermitian map. ($\mathcal{L}(\mathcal{H})$ is the space of linear operators on $\mathcal{H}$.) So, each $\rho_{SE}^{(i)} \in \mathcal{S}$, in equation (6), can be written as

$$\rho_{SE}^{(i)} = (\text{Tr}_R \circ \mathcal{P}_R^{(i)}) \otimes \Lambda_S(\omega_{RS})$$

$$= (\text{Tr}_R \circ \mathcal{P}_R^{(i)}) \otimes \text{id}_{SE}(\omega_{RSE}),$$

(8)

where, the map $\mathcal{P}_R^{(i)}$ on $\mathcal{L}(\mathcal{H}_R)$ is defined as $\mathcal{P}_R^{(i)}(A_R) = P_R^{(i)}A_R$, for each $A_R \in \mathcal{L}(\mathcal{H}_R)$. In addition, without loss of generality, we have considered only those $P_R^{(i)}$, in equation (6), for which we have $\text{Tr}[(P_R^{(i)} \otimes I_{SE})\omega_{RSE}] = 1$. Therefore
\begin{align}
\rho_S^{(i)} &= \text{Tr}_E(\rho_{SE}^{(i)}) \\
&= (\text{Tr}_R \circ \mathcal{P}_R^{(i)}) \otimes (\text{Tr}_E \circ \Lambda_S)(\omega_{RS}) \\
&= [\text{Tr}_R \circ \mathcal{P}_R^{(i)}] \otimes |id_S \otimes (\text{Tr}_E \circ \Lambda_S)|(\omega_{RS}) \\
&= (\text{Tr}_R \circ \mathcal{P}_R^{(i)}) \otimes |id_S(\omega_{RS})|, \tag{9}
\end{align}

where, in the fourth line, we have used this fact that, according to equation (7), we have

\[ |id_R \otimes (\text{Tr}_E \circ \Lambda_S)|(\omega_{RS}) = \text{Tr}_E[id_R \otimes \Lambda_S(\omega_{RS})] = \text{Tr}_E(\omega_{RSE}) = \omega_{RS}. \]

Next, assume that \{S_j\} is an orthonormal basis (according to the Hilbert–Schmidt inner product [8]) for \(L(H_S)\). So, we can decompose \(\omega_{RS}\) as

\[ \omega_{RS} = \sum_j R_j \otimes S_j, \tag{10} \]

where \(R_j\) are linear operators in \(L(H_R)\). Therefore, from equation (9), we have

\[ \rho_S^{(i)} = \sum_j \text{Tr}(P_R^{(i)} R_j)S_j = \sum_j a_j S_j, \tag{11} \]

where \(a_j = \text{Tr}(P_R^{(i)} R_j)\). From equations (7) and (10), we have

\[ \omega_{RSE} = \sum_j R_j \otimes \Lambda_S(S_j). \tag{12} \]

So, from equations (8) and (11), we get

\[ \rho_{SE}^{(i)} = \sum_j a_j \Lambda_S(S_j) = \Lambda_S(\rho_S^{(i)}). \tag{13} \]

Now, if \(\rho_{SE}^{(i)} \neq \rho_{SE}^{(l)}\), then, at least for one \(j\), we have \(a_j \neq a_l\). So, from equation (11), we conclude that \(\rho_S^{(i)} \neq \rho_S^{(l)}\). Therefore, there is a one to one correspondence between the members of \(S\) and the members of \(\text{Tr}_E S\), and so, the \(U\)-consistency condition is satisfied for the set \(S\), steered from the \(\omega_{RSE}\) in equation (7).

In summary, we have proved the following proposition:

**Proposition 1.** If the set of possible initial states of the whole system-environment is given by the set of steered states from the tripartite reference state \(\omega_{RSE}\) in equation (7), then the reduced dynamics of the system, for arbitrary unitary evolution of the whole system-environment \(U\), is given by a (linear trace-preserving) Hermitian map.

For the special case that \(\Lambda_S\) in equation (7) is a CP map, \(\omega_{RSE}\) is called a Markov state [16], and the reduced dynamics of the system, for arbitrary \(U\), is, therefore, CP [13]. In fact, the reverse is also true. In summary, we have [13]:

**Theorem 1.** For a set of steered states, from a tripartite reference state \(\omega_{RSE}\), as equation (6), the reduced dynamics of the system, for arbitrary \(U\), is CP if and only if \(\omega_{RSE}\) is a Markov state.

**Remark 1.** During the proof of theorem 1 in [13], it has been assumed that, in general, the dimensions of \(H_S\) and \(H_E\) can vary during the evolution, while the dimension of \(H_S \otimes H_E\) remains unchanged.

Interestingly, all the previous results, in this context, are special cases of the above result: all the previously found sets of the system-environment initial states in [9–12], for which the
3. Reference state for a $U$-consistent subspace

Let us denote the convex set of possible initial states of the system-environment as $S'$, and so, the convex set of possible initial states of the system as $S'_S = \text{Tr}_E S'$. Since the Hilbert space of the system $H_S$ is finite dimensional, one can find a set $S''_S \subset S'_S$ including a finite number of $\rho^{(j)}_{SE} \in S'_S$ which are linearly independent and other states in $S'_S$ can be decomposed as linear combinations of them: $S''_S = \{\rho^{(1)}_S, \rho^{(2)}_S, \ldots, \rho^{(m)}_S\}$, where $m$ is an integer and $m \leq (d_S)^2$. $d_S$ is the dimension of $H_S$, so $(d_S)^2$ is the dimension of $\mathcal{L}(H_S)$, and, for each $\rho_S \in S''_S$, we have $\rho_S = \sum_{j=1}^m b_j \rho^{(j)}_S$ with real $b_j$.

Consider the set $S'' = \{\rho^{(1)}_S, \rho^{(2)}_S, \ldots, \rho^{(m)}_S\}$, where $\text{Tr}_E(\rho^{(j)}_{SE}) = \rho^{(j)}_S \in S''_S$. So, $\rho^{(j)}_{SE}$ are also linearly independent. Now, there is a one to one correspondence between the members of $S'$ and the members of $S''$ if and only if each $\rho_{SE} \in S'$ can be decomposed as a linear combination of $\rho^{(j)}_{SE} \in S''$: $\rho_{SE} = \sum_{j=1}^m b_j \rho^{(j)}_{SE}$. (Note that the coefficients $b_j$ in the decomposition of $\rho_{SE}$ are the same as $b_j$ in the decomposition of $\rho_S = \text{Tr}_E(\rho_{SE})$.)

So, if the set $S''$ constructs a basis for the convex set $S'$, then $S'$ is, in addition, $U$-consistent for arbitrary $U$, and, as we will see in the following, the reduced dynamics of the system is Hermitian.

Now, we can define the linear trace-preserving Hermitian map $\Lambda_S$ as $\Lambda_S(\rho^{(j)}_S) = \rho^{(j)}_{SE}$, where $\rho^{(j)}_S \in S''_S$ and $\rho^{(j)}_{SE} \in S''$. Therefore, for each $\rho_S \in S''_S$, we have

$$
\Lambda_S(\rho_S) = \sum_{j=1}^m b_j \Lambda_S(\rho^{(j)}_S) = \sum_{j=1}^m b_j \rho^{(j)}_{SE} = \rho_{SE},
$$

where $\rho_{SE} \in S'$ such that $\text{Tr}_E(\rho_{SE}) = \rho_S$. The Hermitian map $\Lambda_S$ is called the assignment map $[2, 17]$. So, from equation (14), for arbitrary unitary evolution $U$ for the whole system-environment, we have

$$
\rho'_S = \text{Tr}_E \circ \text{Ad}_U(\rho_{SE})
= \sum_{j=1}^m b_j [\text{Tr}_E \circ \text{Ad}_U \circ \Lambda_S](\rho^{(j)}_S) = \mathcal{E}_S(\rho_S),
$$

where $\mathcal{E}_S = \text{Tr}_E \circ \text{Ad}_U \circ \Lambda_S$ is a Hermitian map on $\mathcal{L}(H_S)$, since $\text{Tr}_E$ and $\text{Ad}_U$ are completely positive $[8]$ and $\Lambda_S$ is Hermitian.

Now, our question is as follows: can we assign to the above convex $U$-consistent $S'$ a tripartite reference state $\omega_{\text{RSE}}$, such that $S'$ is the set of steered states from this $\omega_{\text{RSE}}$?

Without loss of generality, as we will show in the following, we consider a restricted set $S$, instead of $S'$, such that each $\rho_{SE} \in S$ can be decomposed as $\rho_{SE} = \sum_{j=1}^m p_j \rho^{(j)}_{SE}$, with $\rho^{(j)}_{SE} \in S''$, where $\{p_j\}$ is a probability distribution ($p_j \geq 0$ and $\sum p_j = 1$). As $S'$, the set $S$ is convex (and $U$-consistent) and so the set $\mathcal{S} = \text{Tr}_E S$ is also convex.
First, we define the bipartite state
\[ \omega_{RS} = \sum_{l=1}^{m} \frac{1}{m} |l_R \rangle \langle l_R| \otimes \rho_S^{(l)}, \]  
where \( \rho_S^{(l)} \in S''_R \) and \( \{|l_R \rangle\} \) is an orthonormal basis for the reference Hilbert space \( H_R \). So, using the assignment map \( \Lambda_S \) in equation (14), we have
\[ \omega_{RSE} \equiv \text{id}_R \otimes \Lambda_S(\omega_{RS}) = \sum_{l=1}^{m} \frac{1}{m} |l_R \rangle \langle l_R| \otimes \rho_{SE}^{(l)} \]  
where \( \rho_{SE}^{(l)} \in S'' \) such that \( \text{Tr}_E(\rho_{SE}^{(l)}) = \rho_S^{(l)} \). Therefore, using equation (6), we can write the set \( S \) as the steered set from the tripartite reference state \( \omega_{RSE} \), given in equation (17). It can be done, e.g. by considering the positive operators \( P_R \) in equation (6) as \( P_R = \sum_{l=1}^{m} \rho_R^{(l)} |{l_R \rangle \langle l_R|} \). Note that \( \omega_{RSE} \), in equation (17), is in the form of equation (7), with the Hermitian assignment map \( \Lambda_S \).

In summary, we have proved the following theorem:

**Theorem 2.** Consider a set of linearly independent states \( S''_R = \{ \rho_S^{(1)} , \rho_S^{(2)} , \ldots , \rho_S^{(m)} \} \). So, the set \( S'' = \{ \rho_{SE}^{(1)} , \rho_{SE}^{(2)} , \ldots , \rho_{SE}^{(m)} \} \), such that \( \text{Tr}_E(\rho_{SE}^{(l)}) = \rho_S^{(l)} \) (for each \( 1 \leq l \leq m \)), is also linearly independent. The set \( S \) of the convex combinations of \( \rho_{SE}^{(l)} \in S'' \) is convex and \( U \)-consistent, for arbitrary unitary evolution of the whole system-environment. Therefore, if the set of possible initial states of the system-environment is given by \( S \), then the reduced dynamics of the system is given by a Hermitian map \( E_S \). In addition, \( S \) can be written as the steered set from a tripartite reference state \( \omega_{RSE} \), given in equation (17), which is in the form of equation (7), with the Hermitian assignment map \( \Lambda_S \).

Next, let us define \( V \subseteq L(H_S \otimes H_E) \) as the subspace spanned by the states \( \rho_{SE}^{(l)} \in S'' \); i.e. for each \( X \in V \), we have \( X = \sum_i c_i \rho_{SE}^{(i)} \) with unique complex coefficients \( c_i \). Obviously \( S \subseteq S' \subset V \).

So, the subspace \( V_S = \text{Tr}_E(V) \subseteq L(H_S) \) is spanned by the states \( \rho_S^{(l)} \in S''_R \); for each \( X \in V \), we have \( x = \text{Tr}_E(X) = \sum_i c_i \rho_S^{(i)} \) with the same coefficients \( c_i \) as in the decomposition of \( X \).

Note that, since there is a one to one correspondence between the \( x \in V_S \) and the \( X \in V \), the whole subspace \( V \) is \( U \)-consistent, for arbitrary \( U \).

In addition, we can write the subspace \( V \) as
\[ V = \{ \text{Tr}_R[(A_R \otimes \mathbb{1}_E)\omega_{RSE}] | A_R \in L(H_R) \} \],
where \( A_R \) is arbitrary linear operator in \( L(H_R) \) and \( \omega_{RSE} \) is the reference state, given in equation (17). We will call the above set the generalized steered set from the reference state \( \omega_{RSE} \).

Using this fact that if the subspace \( V \) is \( U \)-consistent, for arbitrary \( U \), then there is a one to one correspondence between the \( x \in V_S \) and the \( X \in V \) [2], we can write the above result in the following form:

**Corollary 1.** Consider the subspace \( V \subseteq L(H_S \otimes H_E) \), which is spanned by states. If \( V \) is \( U \)-consistent, for arbitrary \( U \), then it can be written as the generalized steered set from the reference state \( \omega_{RSE} \), as equation (18).

Note that, since \( S' \subset V \), \( S' \) can be written as (a subset of) equation (18). In our discussion, leading to the reference state \( \omega_{RSE} \) in equation (17), we have restricted ourselves to the set \( S' \).
instead of $S'$. Now, as stated before, we see that this restriction does not lose the generality of our discussion.

The next observation is that the evolution of the system subspace $V_S$ and the whole system-environment subspace $V$ can be given from the evolution of $\omega_{RS}$ in equation (16), and $\omega_{RSE}$ in equation (17), respectively. So, we can call $\omega_{RS}$ as the reference state of the system and $\omega_{RSE}$ as the reference state of the whole system-environment. Note that these two reference states are related to each other as equation (7).

Assume that the unitary time evolution of the whole system-environment, from the initial instant to the time $t$, is given by $U(t)$. So, $\omega_{RSE}$ evolves as

$$\omega_{RSE}(t) = \text{id}_R \otimes \text{Ad}_{U(t)}(\omega_{RSE}(0)),$$

where $\omega_{RSE}(0)$ is given in equation (17). As stated before, each $X \in V = V(0)$ can be written as $X = X(0) = \sum_c c_i \rho_{SE}^{(i)} = \text{Tr}_E[(\Lambda_R \otimes I_S)\omega_{RSE}(0)]$. So, $X(t) = \sum_c c_i \text{Ad}_{U(t)}(\rho_{SE}^{(i)})$ and therefore

$$V(t) = \{X(t)\} = \{\text{Tr}[\Lambda_R \otimes I_S(\omega_{RSE}(t))]\},$$

where $\Lambda_R$ is an arbitrary linear operator in $\mathcal{L}(H_R)$ and $\omega_{RSE}(t)$ is the reference state of the system-environment, given in equation (19).

Similarly, $\omega_{RS}$ evolves as

$$\omega_{RS}(t) = \text{id}_R \otimes \mathcal{E}(t)(\omega_{RS}(0)),$$

where $\omega_{RS}(0)$ is given in equation (16) and $\mathcal{E}(t) = \text{Tr}_E \circ \text{Ad}_{U(t)} \circ \Lambda_S$ is a Hermitian map on $\mathcal{L}(H_S)$. Each $x = x(0) = \text{Tr}_E(X(0))$ can be decomposed as $x(0) = \sum_c c_i \rho_S^{(i)}$. So, $x(t) = \sum_c c_i \mathcal{E}(t)(\rho_S^{(i)}) = \mathcal{E}(t)(x(0))$ and therefore

$$V_S(t) = \{x(t)\} = \{\text{Tr}[(\Lambda_R \otimes I_S)\omega_{RS}(t)]\},$$

where $\Lambda_R$ is an arbitrary linear operator in $\mathcal{L}(H_R)$ and $\omega_{RS}(t)$ is the reference state of the system, given in equation (21).

In summary,

**Corollary 2.** Consider the subspace $V(0) \subseteq \mathcal{L}(H_S \otimes H_E)$, which is spanned by states. If $V(0)$ is $U$-consistent, for arbitrary $U$, then $V(t)$ and $V_S(t)$ can be written as the generalized steered sets, from the reference states $\omega_{RSE}(t)$, in equation (19), and $\omega_{RS}(t)$, in equation (21), respectively.

In general, there are more than one possible assignment maps $\Lambda_S$. So, it may be possible, by choosing an appropriate $\Lambda_S$, to write the reduced dynamics of the system $S$ as a CP map. Note that, from equations (16) and (21), we have

$$\omega_{RS}(t) = \sum_{i=1}^{m} \frac{1}{m} |l_i\rangle \langle l_i | \otimes \rho_{S}^{(i)}(t).$$

(23)

Now, if the time evolution of the system can be written as a CP map, then, since $\rho_{S}^{(i)}(t) = \mathcal{E}_{S}^{(CP)}(t)(\rho_{S}^{(i)}(0))$, where $\mathcal{E}_{S}^{(CP)}(t)$ is a CP map on $\mathcal{L}(H_S)$, we have

$$\omega_{RS}(t) = \text{id}_R \otimes \mathcal{E}_{S}^{(CP)}(t)(\omega_{RS}(0)).$$

(24)

i.e. even if we have used a $\Lambda_E$ which leads to equation (21), with a non-CP map $\mathcal{E}_{S}(t)$, $\omega_{RS}(t)$ can be written as equation (24) too.
Reversely, if $\omega_{RSE}(t)$, in equation (23), can be written as equation (24), then, using this fact that $m(|k\rangle\omega_{RSE}(t)|k\rangle) = \rho^{(i)}_S(t)$, we, simply, conclude that $\rho^{(i)}_S(t) = E^{(CP)}_S(t)(\rho^{(i)}_S(t))$, and so, the reduced dynamics of the system is given by the CP map $E^{(CP)}_S(t)$. In summary,

**Theorem 3.** The reduced dynamics of the system can be written as a CP map if and only if the reference state $\omega_{RSE}(t)$, in equation (23), evolves as equation (24), with a CP map $E^{(CP)}_S(t)$.

### 4. Markovianity of the reference state and the complete positivity of the reduced dynamics

Theorem 1 states the relation between the Markovianity of the reference state $\omega_{RSE}$ and the CP-ness of the reduced dynamics, for a steered set as equation (6). In the previous section, we have seen that, for an arbitrary $U$-consistent subspace $\mathcal{V}$, we can also introduce a reference state as equation (17), such that $\mathcal{V}$ can be written as the generalized steered set from it. Therefore, we conjecture that theorem 1 can be generalized to arbitrary $U$-consistent subspace $\mathcal{V}$. Fortunately, this is the case, as we will show in this section.

A tripartite state $\rho_{RSE}$ is called a Markov state if it can be written as $\rho_{RSE} = \text{id}_R \otimes \Lambda_S(\rho_{RS})$, where $\rho_{RS} = \text{Tr}_E(\rho_{RSE})$, and $\Lambda_S : \mathcal{L}(\mathcal{H}_S) \rightarrow \mathcal{L}(\mathcal{H}_S \otimes \mathcal{H}_E)$ is a CP assignment map [16]. Now, it has been shown in [16] that if $\rho_{RSE}$ is a Markov state, then there exists a decomposition of the Hilbert space of the system $S$ as $\mathcal{H}_S = \bigoplus \mathcal{H}_{a_k} = \bigoplus \mathcal{H}_{\lambda_k} \otimes \mathcal{H}_{E'}$ such that

$$\rho_{RSE} = \bigoplus_k \lambda_k \rho_{R\lambda_k} \otimes \rho_{E'E},$$

(25)

where $\{\lambda_k\}$ is a probability distribution, $\rho_{R\lambda_k}$ is a state on $\mathcal{H}_R \otimes \mathcal{H}_{\lambda_k}$, and $\rho_{E'E}$ is a state on $\mathcal{H}_{E'} \otimes \mathcal{H}_{E'}$.

Consider a set $\mathcal{S}''$ which spans the subspace $\mathcal{V}$. In general, we can consider different assignment maps $\Lambda_S$ such that, for all of them, $\Lambda_S(\rho^{(i)}_S) = \rho^{(i)}_{SE}$, where $\rho^{(i)}_S \in \mathcal{S}_S''$ and $\rho^{(i)}_{SE} \in \mathcal{S}''$, and so, we can write equation (14), for all of them. Different assignment maps $\Lambda_S$ can lead to different reduced dynamics $E_S$, in equation (15).

Therefore, it is possible that we choose a Hermitian (non-CP) assignment map $\Lambda_S$ to construct the reference state $\omega_{RSE}$ in equation (17), while there is another CP assignment map which could be used instead. So, the reduced dynamics could be written as a CP map, while we write it as a non-CP map. How can we avoid such inappropriate choosing?

Note that if there is a CP assignment map $\Lambda_S$, such that, for all $\rho^{(i)}_S \in \mathcal{S}_S''$, we have $\Lambda_S(\rho^{(i)}_S) = \rho^{(i)}_{SE}$, then $\omega_{RSE}$ in equation (17) is a Markov state, even if we have used a non-CP assignment map $\Lambda_S$ to construct it. So, we can check whether $\omega_{RSE}$ can be written as equation (25), or not. If it can be written so, then the reference state $\omega_{RSE}$ is a Markov state, and the reduced dynamics is CP, for arbitrary $U$.

But if $\omega_{RSE}$ cannot be written as equation (25), then we conclude that there is no CP assignment map which can map all $\rho^{(i)}_S \in \mathcal{S}_S''$ to $\rho^{(i)}_{SE} \in \mathcal{S}''$. In other words, though there may be more than one possible assignment maps $\Lambda_S$, but none of them is CP.

Also note that, for the subspace $\mathcal{V}$, we can construct different reference states $\omega_{RSE}$ as equation (17), in general: by choosing a different set $\mathcal{S}''$, which also spans $\mathcal{V}$, we can construct a different $\omega_{RSE}$. Interestingly, if the previously constructed reference state is non-Markovian, this new reference state is not a Markov state, too; otherwise, there is a CP assignment map which maps all $\rho_S \in \mathcal{V}_S$ to $\rho_{RSE} \in \mathcal{V}$.
In summary, we have proved the following theorem:

**Theorem 4.** Consider the subspace \( V \subseteq \mathcal{L}(\mathcal{H}_S \otimes \mathcal{H}_E) \), which is spanned by states and is \( U \)-consistent, for arbitrary \( U \). One can find, at least, one CP assignment map \( \Lambda_S : \mathcal{L}(\mathcal{H}_S) \rightarrow \mathcal{L}(\mathcal{H}_S \otimes \mathcal{H}_E) \), which maps \( \mathcal{V}_S = \text{Tr}_E V \) to \( V \), if and only if any reference state \( \omega_{RSE} \), which is constructed as equation (17), is a Markov state, as equation (25).

Next, consider the case that the reference state \( \omega_{RSE} \), in equation (17), is not a Markov state. Construct the set of steered states from \( \omega_{RSE} \), i.e. the set \( S \) in theorem 2. Using theorem 1, we conclude that the reduced dynamics of the system is non-CP, for at least one \( U \). Since \( S \subseteq V \), the non-CP-ness of the reduced dynamics for \( S \) results in the non-CP-ness of the reduced dynamics for \( V \). In other words,

**Theorem 5.** Consider the subspace \( V \subseteq \mathcal{L}(\mathcal{H}_S \otimes \mathcal{H}_E) \), which is \( U \)-consistent, for arbitrary \( U \), and can be written as the generalized steered set, from the reference state \( \omega_{RSE} \), in equation (17). When \( \omega_{RSE} \) is not a Markov state as equation (25), then the reduced dynamics of the system, for at least one \( U \), is non-CP.

The above theorem, states that, not only for a steered set of initial states of the system-environment as equation (6), but also, for any arbitrary subspace \( V \), which is \( U \)-consistent for all \( U \), the reduced dynamics of the system, for arbitrary \( U \), is CP if and only if the reference state \( \omega_{RSE} \) is a Markov state. This is the generalization of theorem 1, to arbitrary \( U \)-consistent subspace \( V \).

The following point is also worth noting:

**Corollary 3.** Theorems 4 and 5 state that the impossibility of a CP assignment map is equivalent to non-CP-ness of the reduced dynamics, for at least one \( U \).

5. **Generalization to arbitrary \( G \)-consistent subspace**

Until now, our discussion was restricted to the case that there is a one to one correspondence between the members of \( V \) and \( \mathcal{V}_S \). We can generalize our discussion to include the general case (with no such correspondence), straightforwardly.

Consider the subspace \( V \subseteq \mathcal{L}(\mathcal{H}_S \otimes \mathcal{H}_E) \), which is spanned by states. If there is not a one to one correspondence between the members of \( V \) and the members of \( \mathcal{V}_S = \text{Tr}_E V \), then \( V \) is \( U \)-consistent only for a restricted set \( G \subseteq \mathcal{U}(\mathcal{H}_S \otimes \mathcal{H}_E)(\mathcal{U}(\mathcal{H}_S \otimes \mathcal{H}_E)) \), is the set of all unitary \( U \in \mathcal{L}(\mathcal{H}_S \otimes \mathcal{H}_E) \) [2]. In such case, the subspace \( V \) is called a \( G \)-consistent subspace.

Assume that the set of linearly independent states \( S'' = \{ \rho_S^{(1)}, \rho_S^{(2)}, \ldots, \rho_S^{(M)} \} \), where \( M \) is an integer such that \( M \leq (d_Sd_E)^2 \) (\( d_S \) and \( d_E \) are the dimensions of \( \mathcal{H}_S \) and \( \mathcal{H}_E \), respectively), spans the subspace \( V \). Without loss of generality, we can assume that only \( \rho_S^{(l)} = \text{Tr}_E (\rho_{SE}^{(l)}) \), for \( 1 \leq l \leq M \) (where the integer \( M \leq (d_S)^2 \) is, in addition, less than \( M \), are linearly independent. So, the subspace \( \mathcal{V}_S \) is spanned by the set of states \( S'' = \{ \rho_S^{(1)}, \rho_S^{(2)}, \ldots, \rho_S^{(M)} \} \).

As before, we can define the (linear trace-preserving) Hermitian assignment map \( \Lambda_S \) as \( \Lambda_S(\rho_S^{(j)}) = \rho_{SE}^{(j)} \), where \( \rho_S^{(j)} \in S_S', \rho_{SE}^{(j)} \in S'' \) and \( 1 \leq j \leq M \), and so, we can write a similar relation as equation (14), for each \( x \in \mathcal{V}_S \). Therefore, the assignment map \( \Lambda_S \) maps \( \mathcal{V}_S \) to a subspace \( \mathcal{V}' \subset V \), which is spanned by \( \{ \rho_{SE}^{(1)}, \rho_{SE}^{(2)}, \ldots, \rho_{SE}^{(m)} \} \).

Note that

\[ V = V' \oplus V_0, \]  

(26)
where, for each $Y \in \mathcal{V}_0$, we have $\text{Tr}_E(Y) = 0$. So, the most general possible assignment map is as

$$\tilde{\Lambda}_S = \Lambda_S + \mathcal{V}_0,$$

where $\mathcal{V}_0$ denotes arbitrary $Y \in \mathcal{V}_0$.

Each $U \in \mathcal{G}$ maps $\mathcal{V}_0$ to $\ker \text{Tr}_E$, the set of all $Z \in \mathcal{L}(\mathcal{H}_S \otimes \mathcal{H}_E)$ for which we have $\text{Tr}_E(Z) = 0$, and so, $\mathcal{V}$ is $U$-consistent under all $U \in \mathcal{G}$ [2]. Therefore, for a unitary time evolution $U(t) \in \mathcal{G}$, from equation (26), we have

$$\mathcal{V}(t) = \mathcal{V}'(t) \oplus \mathcal{V}_0(t),$$

where $\mathcal{V}_0(t) \subseteq \ker \text{Tr}_E$ and $\mathcal{V}'(t)$ is given as equation (20), i.e. as the generalized steered set from the reference state $\omega_{\text{RSE}}(t)$ in equation (19).

In addition, for each $x = x(0) \in \mathcal{V}_3 = \mathcal{V}_3(0)$ and each $U(t) \in \mathcal{G}$, we have

$$x(t) = [\text{Tr}_E \circ \text{Ad}_{U(t)} \circ \tilde{\Lambda}_S](x) = [\text{Tr}_E \circ \text{Ad}_{U(t)} \circ \Lambda_S](x) = \mathcal{E}_S(t)(x).$$

Therefore, as before, $\mathcal{V}_3(t)$ can be written as equation (22), i.e. as the generalized steered set from the reference state $\omega_{\text{RSE}}(t)$ in equation (21).

So, we have proved the following proposition:

**Proposition 2.** Consider the $\mathcal{G}$-consistent subspace $\mathcal{V}(0) \subseteq \mathcal{L}(\mathcal{H}_S \otimes \mathcal{H}_E)$, which is spanned by states. For each $U(t) \in \mathcal{G}$, $\mathcal{V}_3(t)$, even simply, can be written as the generalized steered set from the reference state $\omega_{\text{RSE}}(t)$ in equation (19).

In addition, for each $x = x(0) \in \mathcal{V}_3 = \mathcal{V}_3(0)$ and each $U(t) \in \mathcal{G}$, we have $x(t) = [\text{Tr}_E \circ \text{Ad}_{U(t)} \circ \tilde{\Lambda}_S](x) = [\text{Tr}_E \circ \text{Ad}_{U(t)} \circ \Lambda_S](x) = \mathcal{E}_S(t)(x).$ Therefore, as before, $\mathcal{V}_3(t)$ can be written as equation (22), i.e. as the generalized steered set from the reference state $\omega_{\text{RSE}}(t)$ in equation (21).

Note that theorem 3 is valid for a $\mathcal{G}$-consistent subspace, too, since, even simply, the reduced dynamics of the system is determined by the evolution of the reference state $\omega_{\text{RSE}}(t)$.

In the following, we discuss about the generalization of the results given in the previous section, to a $\mathcal{G}$-consistent subspace $\mathcal{V}$. First, theorem 4 is changed as below:

**Theorem 4’.** Consider a $\mathcal{G}$-consistent subspace $\mathcal{V}$, which is spanned by states. There exists, at least, one CP assignment map $\tilde{\Lambda}_S$ if and only if, at least, one reference state $\omega_{\text{RSE}}$, as equation (17), is a Markov state, as equation (25).

Note that when there exists a CP assignment map $\tilde{\Lambda}_S$, then using this $\tilde{\Lambda}_S$ in equation (17), we can construct a Markov reference state $\omega_{\text{RSE}}$. But, from the CP-ness of $\tilde{\Lambda}_S$, we cannot, in general, conclude that $\tilde{\Lambda}_S = \Lambda_S + \mathcal{V}_0$ is also CP. So, in general, one can construct other reference states which are not Markov states. However, if, for our $\mathcal{G}$-consistent subspace $\mathcal{V}$, we can find a reference state $\omega_{\text{RSE}}$, as equation (17), which is a Markov state, as equation (25), then the reduced dynamics of the system is CP, for any arbitrary $U \in \mathcal{G}$.

Unfortunately, theorem 5 cannot be generalized to a $\mathcal{G}$-consistent subspace $\mathcal{V}$, in general. Assume that the reduced dynamics of the system $\mathcal{E}_S$ is CP, for any arbitrary $U \in \mathcal{G}$. The CP-ness of the reduced dynamics, for any $\rho_{\text{RSE}} \in \mathcal{V}$, results in the CP-ness of the reduced dynamics, for any convex set of initial states $\mathcal{S} = \{\rho_{\text{RSE}}\} \subseteq \mathcal{V}$. Therefore, for the steered set $\mathcal{S}$, from any reference state $\omega_{\text{RSE}}$, constructed as equation (17), the reduced dynamics is CP, for any arbitrary $U \in \mathcal{G}$. But, from this result, we cannot (in general) conclude that $\omega_{\text{RSE}}$ is a Markov state, unless $\mathcal{G} = \mathcal{U}(\mathcal{H}_S \otimes \mathcal{H}_E)$, which is the case considered in the previous section. In fact, as we will see in the next section, the reduced dynamics can be CP, for some (but not all) $U$, even though $\omega_{\text{RSE}}$ is not a Markov state.
6. Example

In [4], a two-qubit case, one as the system $S$ and the other as the environment $E$, has been considered. First, note that an arbitrary state of the system can be written as

$$\rho_S = \frac{1}{2}(I_S + \vec{\alpha} \cdot \vec{\sigma}_S),$$

where $\vec{\sigma} = (\sigma_S^{(1)}, \sigma_S^{(2)}, \sigma_S^{(3)})$, $\sigma_S^{(i)}$ are the Pauli operators, and the Bloch vector $\vec{\alpha} = (\alpha^{(1)}, \alpha^{(2)}, \alpha^{(3)})$ is a real 3D vector such that $|\vec{\alpha}| \leq 1$ [8].

Consider the following (linear trace-preserving) Hermitian assignment map $\Lambda_S$:

$$\Lambda_S(\sigma_S^{(i)}) = \frac{1}{2} \sigma_S^{(i)} \otimes I_E \equiv X^{(i)} \quad (i = 1, 2, 3),$$

$$\Lambda_S(I_S) = \frac{1}{2} \left( I_{SE} + a \sum_{i=1}^{3} \sigma_S^{(i)} \otimes \sigma_E^{(i)} \right) \equiv X^{(4)},$$

where $a$ is a fixed real constant. For the special case that $a = 0$, we have $\Lambda_S(x) = x \otimes (I/2)$, for each $x \in \mathcal{L}(H_S)$, i.e. $\Lambda_S$ is a CP map, in the form first introduced by Pechukas [6, 17]. We denote this special case of $\Lambda_S$ as $\Lambda_S^{(CP)}$. But, for $a \neq 0$, $\Lambda_S$ is not CP. We have

$$\tau_{SE} \equiv \Lambda_S(\rho_S)$$

$$= \frac{1}{4} \left( I_{SE} + \sum_{i=1}^{3} \alpha^{(i)} \sigma_S^{(i)} \otimes I_E + a \sum_{i=1}^{3} \sigma_S^{(i)} \otimes \sigma_E^{(i)} \right).$$

When $a \geq 0$, $\tau_{SE}$ is positive for $|\vec{\alpha}| \leq \sqrt{(1 + a)(1 - 3a)}$, and when $a \leq 0$, $\tau_{SE}$ is positive for $|\vec{\alpha}| \leq (1 + a)$ [2, 4]. Therefore, for $a \neq 0$, $\Lambda_S$ is not even a positive map and, consequently, it is not a CP map.

Within the positivity domain of $\tau_{SE}$, i.e. $-1 < a < \frac{1}{3}$, we can apply the framework of [2]. We can construct $\mathcal{V}$ as [2]

$$\mathcal{V} = \text{Span}_c\{X^{(i)}\},$$

i.e. each $X \in \mathcal{V}$ can be decomposed as $X = \sum_{i=1}^{3} c_i X^{(i)}$, with complex coefficients $c_i$. (Out of the positivity domain, $\tau_{SE}$, in equation (31), is not a state. In other words, $\mathcal{V}$ does not contain any state, and so, is not spanned by states.) Therefore,

$$\mathcal{V}_S = \text{Span}_c\{\sigma_S^{(1)}, \sigma_S^{(2)}, \sigma_S^{(3)}, I_S\} = \mathcal{L}(H_S).$$

From equations (32) and (33), we see that there is a one to one correspondence between the members of $\mathcal{V}$ and the members of $\mathcal{V}_S$. Therefore, $\mathcal{V}$ is a $U$-consistent subspace, for arbitrary unitary evolution of the whole system-environment $U$, and so, the reduced dynamics of the system, from equation (2) (when $\tau_{SE}$, in equation (31), is positive), is given by

$$\rho'_{SE} = \text{Tr}_E \circ \text{Ad}_U(\tau_{SE}) = \text{Tr}_E \circ \text{Ad}_U \circ \Lambda_S(\rho_S) = \mathcal{E}_S(\rho_S).$$

Since $\Lambda_S$ is Hermitian (and not CP), we expect that the reduced dynamics $\mathcal{E}_S$ be so, in general. But, interestingly, when $U$ commutes with $\sum \sigma_S^{(i)} \otimes \sigma_E^{(i)}$, $\mathcal{E}_S$ is CP [4]. For such $U$, we have

$$\mathcal{E}_S = \text{Tr}_E \circ \text{Ad}_U \circ \Lambda_S = \text{Tr}_E \circ \text{Ad}_U \circ \Lambda_S^{(CP)},$$
which is a CP map. An interesting question is whether this result can be generalized to other $U$ or whether we can find at least one $U$, for which the reduced dynamics $\mathcal{E}_S$ is not CP.

This question can be answered simply, using theorem 5. For an $a$ within the positivity domain $-1 < a < \frac{1}{9}$, we, first, choose four states $\rho_S^{(l)}$, which can span $\mathcal{V}_S$:

$$\rho_S^{(l)} = \frac{1}{2}(I_S + \alpha^{(l)} \sigma_S^{(l)}) \quad (l = 1, 2, 3),$$

$$\rho_S^{(4)} = \frac{1}{2}I_S,$$



where $\alpha^{(l)}$ is an arbitrary real constant such that, for $a \geq 0$, $0 < |\alpha^{(l)}| \leq \sqrt{(1 + a)(1 - 3a)}$, and for $a \leq 0$, $0 < |\alpha^{(l)}| \leq (1 + a)$. Therefore, from equation (16), we can construct the reference state $\omega_{RS}$ as

$$\omega_{RS} = \sum_{l=1}^{4} \frac{1}{4} |l_k\rangle \langle l_k| \otimes \rho_S^{(l)}$$

$$= \sum_{l=1}^{3} \frac{1}{8} |l_k\rangle \langle l_k| \otimes (I_S + \alpha^{(l)} \sigma_S^{(l)}) + \frac{1}{8} |4_k\rangle \langle 4_k| \otimes I_S. \quad (37)$$

Next, using equations (31) and (36), we can construct four states $\rho_{SE}^{(l)} = \Lambda_S(\rho_S^{(l)})$, which span $\mathcal{V}$:

$$\rho_{SE}^{(l)} = \frac{1}{4}(I_{SE} + \alpha^{(l)} \sigma_S^{(l)} \otimes I_E$$

$$+ a \sum_{i=1}^{3} \sigma_S^{(i)} \otimes \sigma_E^{(i)}) \quad (l = 1, 2, 3),$$

$$\rho_{SE}^{(4)} = \frac{1}{4}(I_{SE} + a \sum_{i=1}^{3} \sigma_S^{(i)} \otimes \sigma_E^{(i)}). \quad (38)$$

So, from equation (17), the reference state $\omega_{RSE} = I_{SE} \otimes \Lambda_S(\omega_{RS})$ is

$$\omega_{RSE} = \sum_{i=1}^{3} \frac{1}{16} |l_k\rangle \langle l_k|$$

$$\otimes \left( I_{SE} + \alpha^{(l)} \sigma_S^{(l)} \otimes I_E + a \sum_{i=1}^{3} \sigma_S^{(i)} \otimes \sigma_E^{(i)} \right)$$

$$+ \frac{1}{16} |4_k\rangle \langle 4_k| \otimes (I_{SE} + a \sum_{i=1}^{3} \sigma_S^{(i)} \otimes \sigma_E^{(i)}). \quad (39)$$

Third, we will show that the $\omega_{RSE}$, in the above equation, is not a Markov state, as equation (25). For our case, where $S$ is a qubit, there are only three possibilities for decomposing $\mathcal{H}_S$. $\mathcal{H}_S = \mathcal{H}_f$. $\mathcal{H}_S = \mathcal{H}_\alpha$, and $\mathcal{H}_S = \mathcal{H}_\beta \oplus \mathcal{H}_\gamma$, where $\mathcal{H}_\alpha$ and $\mathcal{H}_\beta$ are 1D. Therefore, a tripartite state $\rho_{RSE}$ is a Markov state if it can be written as $\rho_{RS} \otimes \rho_E$, where $\rho_{RS} = \Tr_E(\rho_{RSE})$ and $\rho_E = \Tr_S(\rho_{RSE})$, or as $\rho_R \otimes \rho_{SE}$, where $\rho_R = \Tr_{SE}(\rho_{RSE})$ and $\rho_{SE} = \Tr_E(\rho_{RSE})$, or as

$$\rho_{RSE} = \lambda_1 \rho_R^{(1)} \otimes |1_S\rangle \langle 1_S| \otimes \rho_E^{(1)}$$

$$+ \lambda_2 \rho_R^{(2)} \otimes |2_S\rangle \langle 2_S| \otimes \rho_E^{(2)}, \quad (40)$$
where \( \{ \lambda_1, \lambda_2 \} \) is a probability distribution, \( \rho_R^{(i)} \) are states on \( \mathcal{H}_R \), \( \rho_E^{(i)} \) are states on \( \mathcal{H}_E \), and \( \{|1\rangle_S, |2\rangle_S\} \) is an orthonormal basis for \( \mathcal{H}_S \).

Now, from equation (39), we can verify simply that, for \( a \neq 0 \), \( \omega_{RSE} \) can not be written as \( \omega_E \otimes \omega_E \) or \( \omega_R \otimes \omega_{SE} \). (For \( a = 0 \), from equations (37) and (39), we see that \( \omega_{RSE} = \omega_S \otimes \frac{1}{2} \mathbb{I}_E = \omega_{RS} \otimes \omega_E \), i.e. \( \omega_{RSE} \) is a Markov state.)

In addition, we cannot write \( \omega_{RSE} \) as equation (40). For a \( \rho_{RSE} \), which can be written as equation (40), we have

\[
\rho_{RS} = \lambda_1 \rho_R^{(1)} \otimes |1\rangle_S \langle 1| + \lambda_2 \rho_R^{(2)} \otimes |2\rangle_S \langle 2|.
\]

From equation (37), we see that \( \langle l_R | \omega_{RS} | l_R \rangle = \frac{1}{2} \rho_S^{(i)} \). On the other hand, if \( \omega_{RS} \) can be written as equation (41), we have

\[
\langle l_R | \omega_{RS} | l_R \rangle = q_1 |1\rangle_S \langle 1| + q_2 |2\rangle_S \langle 2|,
\]

where \( q_i = \lambda_i \langle l_R | \rho_R^{(i)} | l_R \rangle \). So, all \( \rho_S^{(i)} \) must commute with each other. But, from equation (36), we see that this is not the case. Therefore, \( \omega_{RSE} \) cannot be written as equation (40). Finally, we conclude that, for \( a \neq 0 \), the reference state \( \omega_{RSE} \), in equation (39), is not a Markov state, as equation (25).

Theorem 5 states that the non-Markovianity of \( \omega_{RSE} \) leads to the non-CP-ness of the reduced dynamics, for, at least, one \( U \). This is in agreement with the result of [2, 4]. In [4], a class of unitary operators as

\[
U = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \cos \theta & \sin \theta & 0 \\
0 & -\sin \theta & \cos \theta & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

has been introduced, where, for some values of \( \theta \), the reduced dynamics of the system is non-CP [2, 4]. Note that, even if one shows that the non-CP-ness of the reduced dynamics, for the above \( U \), is due to inappropriate choosing the assignment map \( \Lambda_S \) as equation (30), theorem 5 assures that there exists, at least, one other \( U \), for which the reduced dynamics is non-CP, with any possible assignment map \( \Lambda_S \) (with any possible reference state \( \omega_{RSE} \)).

It is also worth noting that the above example shows that, even when the reference state \( \omega_{RSE} \) is not a Markov state, the reduced dynamics can be CP for some (but not all) \( U \), in our case, at least, all \( U \) which commute with \( \sum \sigma_S^{(i)} \otimes \sigma_E^{(i)} \).

7. Summary

An straightforward way to construct a convex set of initial states of the system-environment \( \mathcal{S} = \{ \rho_{SE} \} \) is to consider the set of steered states, from a reference state \( \omega_{RSE} \). In section 2, we have shown that if \( \omega_{RSE} \) can be written as equation (7), then the reduced dynamics of the system is Hermitian. For the special case that the assignment map \( \Lambda_S \), in equation (7), is CP, the reduced dynamics is so CP. Interestingly, this includes all the previous results in this context, in [9–14].

The convex set of initial states \( \mathcal{S} = \{ \rho_{SE} \} \) is the starting point of the framework introduced in [2]. From this \( \mathcal{S} \), we can construct the subspace \( \mathcal{V} \subseteq \mathcal{L}(\mathcal{H}_S \otimes \mathcal{H}_E) \). Now, in section 3 (section 5), we have shown that \( \mathcal{V} \) (\( \mathcal{V}^\prime \)) and \( \mathcal{V}_S = \text{Tr}_E \mathcal{V} \) can be written as the generalized steered sets, from the reference states \( \omega_{RSE} \) and \( \omega_{RS} \), in equations (16) and (17), respectively. The relation between \( \omega_{RSE} \) and \( \omega_{RS} \) is as equation (7). Therefore, the steered set, from a
reference state as equation (7), gives us the most general set (within the framework of [2]) for which the reduced dynamics is Hermitian.

In addition, the evolution of the system-environment (system) states is given by the evolution of the reference state, in equation (19) (equation (21)). Interestingly, for a unitary evolution of the system-environment $U$, the reduced dynamics of the system is CP if and only if $\omega_{RS}(t)$ can be written as equation (24), with a CP map $E^{(CP)}_S(t)$.

This fact that we can construct reference state $\omega_{RSE}$, for arbitrary $U$-consistent subspace, leads us to an important result, i.e. the generalization of the result of [13], to arbitrary $U(\mathcal{H}_S \otimes \mathcal{H}_E)$-consistent $V$: the reduced dynamics of the system, for arbitrary system-environment unitary evolution $U$, is CP if and only if the reference state $\omega_{RSE}$, in equation (17), is a Markov state, as equation (25).

Finally, in section 6, we have considered the case studied in [4]. This example illustrates this result that when the reference state $\omega_{RSE}$ is not a Markov state, then the reduced dynamics is non-CP, for at least one $U$. In addition, this example shows that, even when $\omega_{RSE}$ is not a Markov state, the CP-ness of the reduced dynamics, for some (but not all) $U$, is possible.

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