Limitations of the classical phase-locked loop analysis

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Abstract—Nonlinear analysis of the classical phase-locked loop (PLL) is a challenging task. In classical engineering literature simplified mathematical models and simulation are widely used for its study. In this work the limitations of classical engineering phase-locked loop analysis are demonstrated, e.g., hidden oscillations, which cannot be found by simulation, are discussed. It is shown that the use of simplified dynamical models and the application of simulation may lead to wrong conclusions concerning the operability of PLL-based circuits.

I. INTRODUCTION

The Phase locked-loop (PLL) circuits were invented in the first half of the twentieth century and nowadays are widely used in modern telecommunications and computers. PLL is essentially a nonlinear control system and its real model is described by a nonlinear nonautonomous system of differential equations (mathematical model in the signal space). In practice, simulation and simplified mathematical models are widely used for the analysis of PLL-based circuits [1]–[3].

In the following it will be shown that 1) the use of simplified mathematical models and 2) the application of non-rigorous methods of analysis (e.g., simulation) may lead to wrong conclusions concerning the operability of real model of classical PLL.

II. SIMULATION OF THE CLASSICAL PHASE-LOCKED LOOP IN MatLab Simulink

Consider the classical PLL nonlinear models in the signal and signal’s phase spaces [4]–[9].

- Real model of the classical PLL in the signal space (Fig. 1) or its nonlinear mathematical model in the signal space (corresponds to the SPICE-level simulation):

\[ \dot{x} = Ax + b\phi(t), \quad \phi(t) = \sin(\theta_1(t))\cos(\theta_2(t)) \]
\[ \dot{\theta}_1 \equiv \omega_1, \quad \dot{\theta}_2 = \omega_2^\text{free} + L(c^*x) + Lh\phi(t). \]

- Model of the classical PLL in signal’s phase space (Fig. 2); system (1) with averaged \( \phi(t) \approx \phi(\theta_\Delta(t)) \) gives the nonlinear mathematical model in signal’s phase space:

\[ \dot{x} = Ax + b\phi(\theta_\Delta), \]
\[ \dot{\theta}_\Delta = \omega_\Delta - L(c^*x) - Lh\phi(\theta_\Delta), \]
\[ \theta_\Delta(t) = \theta_1(t) - \theta_2(t), \quad \omega_\Delta \equiv \omega_1 - \omega_2^\text{free}. \]

Fig. 1. Real model of the classical PLL in the signal space

Fig. 2. Simplified model of the classical PLL in signal’s phase space

Let us construct MatLab Simulink model, which corresponds to the model in the signal space (see Fig. 3).

Here all elements are standard blocks from Simulink Library except for the VCO. The VCO subsystem is shown in Fig. 4.

The VCO subsystem consists of one input, which is amplified by \( L \) (Gain block). The integration of the sum of amplified input signal and the VCO free-running frequency \( \omega_\text{free} \) forms the phase of the VCO output. The VCO output corresponds to \( \cos(\cdot) \).

Now consider MatLab Simulink model, which corresponds to the model in signal’s phase space (see Fig. 5).
The PD subsystem is shown in Fig. 6.

The VCO subsystem in signal’s phase space is shown in Fig. 7. The VCO output in signal’s phase space corresponds to $\theta_2(t)$.

**A. Simulation parameters and examples**

Consider a passive lead-lag loop filter with the transfer function $F(s) = \frac{1 + \tau_2 s}{\tau_1 + \tau_2 s}$, $\tau_1 = 0.0448$, $\tau_2 = 0.0185$ and the corresponding parameters $A = -\frac{1}{\tau_1 + \tau_2}$, $b = 1 - \frac{\tau_2}{\tau_1 + \tau_2}$, $c = \frac{\tau_0}{\tau_1 + \tau_2}$. The input signal frequency is $\omega = 100000$, initial phase is zero: $\theta_1(0) = 0$, and the VCO input gain $L = 250$.

**Example 1:** This example shows the importance of initial state of filter (see Fig. 8); while the real model (see Fig. 1) with nonzero initial state of loop filter $x_0 = 0.18$ does not acquire lock (black color), the same real model with zero initial state of loop filter $x(0) = 0$ acquires lock (red color). Here the VCO free-running frequency $\omega_2^\text{free} = 100000 - 95$. The input signal frequency is $\omega = 100000$, initial phase is zero: $\theta_1(0) = 0$, and the VCO input gain $L = 250$.

**Example 2:** This example shows the importance of initial phase difference $\theta_1(0) - \theta_2(0)$ between the VCO signal and input signal may affect stability of the classical PLL. In Fig. 9 the real model (see Fig. 1) with zero initial phase difference acquire lock (red color), the same real model with nonzero initial phase difference $\theta_1(0) = \pi$ is out of lock (black color). Here the VCO free-running frequency $\omega_2^\text{free} = 100000 - 95$ and the initial state of loop filter is $x_0 = 0.18$.

Experiments 1 and 2 shows that while the term “initial frequency” (without an explanation) is sometimes used instead of the term “free-running frequency” in engineering definitions of various stability ranges, it may lead to a misunderstanding (see corresponding discussion in [10], [11]).

**Example 3:** This example shows that the PLL model in signal’s phase space may not be equivalent to the PLL real model in the signal space. In Fig. 10 the real model (see Fig. 1) does not acquire lock (red color), the equivalent signal’s phase space model acquires lock (black color). Here the VCO free-running frequency $\omega_2^\text{free} = 100000 - 95$, the initial state of loop filter is $x_0 = 0.017$, and the initial phase difference $\theta_1(0) = 2.276$.

**Example 4:** These examples shows the importance of analytic methods for investigation of PLL stability. More precisely, it is shown that the simulation may lead to wrong results. In Fig. 11 the PLL model in signal’s phase space simulated with relative tolerance “1e-3” does not acquire lock (black color), but the PLL model in signal’s phase space simulated with standard parameters (relative tolerance set to “auto”) acquires lock (red color). Here the input signal frequency is $10000$, the VCO free-running frequency $\omega_2^\text{free} = 100000 - 178.9$, the VCO input gain is $L = 500$, the initial state of loop filter is $x_0 = 0.1318$, and the initial phase difference is $\theta_1(0) = 0$. Consider now a phase portrait (the loop filter state $x$ versus the phase difference $\theta_A$)

1. See, e.g., the corresponding internal time step parameter in PSpice [http://www.stuffle.net/references/PSpice](http://www.stuffle.net/references/PSpice). In [12] the SIMetrix SPICE model of the two-phase PLL with lead-lag filter gives two essentially different results with default sampling step and minimum sampling step set to $1m$. 

Fig. 5. Simulink realization of the model in signal’s phase space

Fig. 6. Simulink realization of the PD in signal’s phase space

Fig. 7. Simulink realization of the VCO in signal’s phase space

Fig. 8. Loop filter output $g(t)$ for real model with nonzero initial state of loop filter (red), real model with zero initial state of loop filter (black).

Fig. 9. Loop filter output $g(t)$ for real model with nonzero initial phase difference (black), real model with zero initial phase difference (red).
corresponding to signal’s phase model (see Fig. 12). The solid blue line in Fig. 12 corresponds to the trajectory with the loop filter initial state $x(0) = 0.2206$ and the VCO phase shift $-6.808$ rad. This line tends to the periodic trajectory, therefore it will not acquire lock.

The solid red line corresponds to the trajectory with the loop filter initial state $x(0) = 0.187386983333130$ and the VCO initial phase $12.938118990628919$. This trajectory lies just under the unstable periodic trajectory and tends to a stable equilibrium. In this case PLL acquires lock.

All trajectories between stable and unstable periodic trajectories tend to the stable one (see, e.g., a solid green line). Therefore, if the gap between stable and unstable periodic trajectories is smaller than the discretization step, the numerical procedure may slip through the stable trajectory. In other words, the simulation will show that the PLL acquires lock, but in reality it is not the case. The considered case corresponds to the coexisting attractors (one of which is so-called hidden oscillation) and the bifurcation of birth of semistable trajectory [13], [14].

An oscillation in a dynamical system can be easily localized numerically if the initial conditions from its open neighborhood lead to long-time behavior that approaches the oscillation. Thus, from a computational point of view, it is natural to suggest the following classification of attractors, based on the simplicity of finding the basin of attraction in the phase space [13], [15]–[17]:

An attractor is called a hidden attractor if its basin of attraction does not intersect with small neighborhoods of equilibria, otherwise it is called a self-excited attractor.

For a self-excited attractor its basin of attraction is connected with an unstable equilibrium and, therefore, self-excited attractors can be localized numerically by the standard computational procedure, in which after a transient process a trajectory, started from a point of an unstable manifold in a neighborhood of an unstable equilibrium, is attracted to the state of oscillation and traces it. Thus self-excited attractors can be easily visualized.

In contrast, for a hidden attractor its basin of attraction is not connected with unstable equilibria. For example, hidden attractors are attractors in the systems with no equilibria or with only one stable equilibrium (a special case of multistable systems and coexistence of attractors).

### III. Conclusion

The derivation of mathematical model in signal’s phase space and the use of the results of its analysis to draw conclusions about the behavior of real model in the signal space have need for a rigorous foundation. But the attempts to justify analytically the reliability of conclusions, based on such engineering approaches, and to study the nonlinear models of PLL-based circuits are quite rare in the modern engendering literature [18]. One of the reasons is that “nonlinear analysis techniques are well beyond the scope of most undergraduate courses in communication theory” [3].

The examples considered in the paper are the motivation to use rigorous analytical methods for the analysis of nonlinear PLL models [1], [2]. Some analytical tools can be found in [8], [19]–[24].

Note once more that various simplifications and the analysis of linearized models of control systems may result in incorrect conclusions (see, e.g., the counterexamples to the filter hypothesis, Aizerman’s and Kalman’s conjectures on
the absolute stability of nonlinear control systems [13], [25],
and the Perron effects of the largest Lyapunov exponent sign
reversals [26], etc.).
In the work it is shown that 1) the consideration
of simplified models, constructed intuitively by engineers
and 2) the application of non-rigorous methods of analysis
(e.g., simulation and linearization) can lead to wrong
conclusions concerning the operability of the classical
phase-locked loop. Similar examples for nonlinear Costas
loop models can be found in [27]–[31].

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