On $H = 1/2$ Surfaces in $\widetilde{PSL}_2(\mathbb{R}, \tau)$

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Abstract. We study the behavior of $H = 1/2$ surfaces immersed in $\widetilde{PSL}_2(\mathbb{R}, \tau)$.

Keywords. Constant mean curvature surfaces. Horizontal graphs.

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1 Introduction

In this paper we study complete constant mean curvature $H = 1/2$ surfaces in $\widetilde{PSL}_2(\mathbb{R}, \tau)$. Recall that in [3] the authors generalized to $\mathbb{H}^2 \times \mathbb{R}$ the half-space theorem of Hoffman and Meeks which says that a properly immersed minimal surface in $\mathbb{R}^3$ that lies in a half-space must be a plane. The main theorem in [3] says that a properly embedded constant mean curvature $H = 1/2$ surface in $\mathbb{H}^2 \times \mathbb{R}$ which is asymptotic to a horocylinder $C$ and on one side of $C$; such that the mean curvature vector of the surface has the same direction as that of $C$ at points of the surface converging to $C$, then the surface is equal to $C$ (or a subset of $C$ if the surface has non-empty boundary).

We follow these ideas to show an analogous result in the space $\widetilde{PSL}_2(\mathbb{R}, \tau)$. More precisely, our main theorem says:

Theorem 1.1. Let $\Sigma$ be a properly embedded constant mean curvature $H = 1/2$ surface in $\widetilde{PSL}_2(\mathbb{R}, \tau)$. Suppose $\Sigma$ is asymptotic to a horocylinder $C$, and on one side of $C$. If the mean curvature vector of $\Sigma$ has the same direction as that of $C$ at points of $\Sigma$ converging to $C$, then $\Sigma$ is equal to $C$ (or a subset of $C$ if $\partial \Sigma \neq \emptyset$).

This theorem lets us obtain (in the same sense as in [3]) the following result:

Theorem 1.2. Let $\Sigma$ be a complete immersed surface in $\widetilde{PSL}_2(\mathbb{R}, \tau)$ of constant mean curvature $H = 1/2$. If $\Sigma$ is transverse to $E_3 = \partial_t$ then $\Sigma$ is an entire vertical graph over $\mathbb{H}^2$.

2 The Laplacian and the Mean Curvature Equation for $H$ Surfaces in $\widetilde{PSL}_2(\mathbb{R}, \tau)$

The space $\widetilde{PSL}_2(\mathbb{R}, \tau)$ is a complete simply connected homogeneous manifold. Such a manifold is a Riemannian fibration over the 2-dimensional Hyperbolic space $\mathbb{H}^2$. That is, there

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exist a Riemannian submersion $\pi : \tilde{PSL}_2(\mathbb{R}, \tau) \to \mathbb{H}^2$, which also is a Killing submersion (see Definition 4.1).

The space $\tilde{PSL}_2(\mathbb{R}, \tau)$ is topologically $\mathbb{H}^2 \times \mathbb{R}$ each fiber is diffeomorphic to $\mathbb{R}$ (the real line) and the bundle curvature of the submersion is $\tau$.

It is well known, that the space $\tilde{PSL}_2(\mathbb{R}, \tau)$ is given by (see [1])

$$\tilde{PSL}_2(\mathbb{R}, \tau) = \{(x, y, t) \in \mathbb{R}^3; y > 0\}$$

endowed with the metric

$$g = \lambda^2(dx^2 + dy^2) + (-2\tau \lambda dx + dt)^2, \quad \lambda = \frac{1}{y}.$$  

There is a natural orthonormal frame $\{E_1, E_2, E_3\}$ given by (in coordinates $\{\partial_x, \partial_y, \partial_t\}$),

$$E_1 = \frac{\partial_x}{\lambda} + 2\tau \partial_t, \quad E_2 = \frac{\partial_y}{\lambda}, \quad E_3 = \partial_t.$$

$E_3$ is the killing field tangent to the fibers. The metric $g$ induces a Riemannian connection $\nabla$ given by:

$$\nabla_{E_1}E_1 = -\frac{\lambda_y}{\lambda^2}E_2 \quad \nabla_{E_1}E_2 = \frac{\lambda_y}{\lambda^2}E_1 + \tau E_3 \quad \nabla_{E_1}E_3 = -\tau E_2$$

$$\nabla_{E_2}E_1 = \frac{\lambda_x}{\lambda^2}E_2 - \tau E_3 \quad \nabla_{E_2}E_2 = -\frac{\lambda_x}{\lambda^2}E_1 \quad \nabla_{E_2}E_3 = \tau E_1$$

$$\nabla_{E_3}E_1 = -\tau E_2 \quad \nabla_{E_3}E_2 = \tau E_1 \quad \nabla_{E_3}E_3 = 0$$

$$[E_1, E_2] = \frac{\lambda_y}{\lambda^2}E_1 - \frac{\lambda_x}{\lambda^2}E_2 + 2\tau E_3 \quad [E_1, E_3] = 0 \quad [E_2, E_3] = 0$$

Following ideas of [3], we will consider horizontal graphs $y = f(x, t)$. Denoting by $S = \text{graph} f$, we have the following lemma:

**Lemma 2.1.** Denoting by $H$ the length of the mean curvature vector of $S$. Then, the function $f$ satisfies the equation

$$2H\lambda^2W^3 = (f^2 + f_t^2)f_{xx} - 2(f_xf_t - 2\tau f)f_{xt} + ((1 + 4\tau^2) + f_x^2)f_u + f(1 + f_t^2) + 2\tau f_xf_t$$

where $W = \sqrt{f^2 + f_t^2 + f_x^2(2\tau f)^2}$.

In particular the horocylinders $f(x, t) = \text{cte}$ have constant mean curvature $H = 1/2$.

**Proof.** The surface $S = \text{grah} f$ is parameterized by $\varphi(x, t) = (x, f(x, t), t)$, so the adapted frame to $S$ is given by:

$$\varphi_x = \lambda(E_1 + f_xE_2 - 2\tau E_3)$$
$$\varphi_t = \lambda f_tE_2 + E_3$$
$$N = \frac{-(f_x + 2\tau f_x f_t)E_1 + E_2 - \lambda f_tE_3}{\sqrt{1 + (f_x + 2\tau f_x f_t)^2 + \lambda^2 f_t^2}}$$
\( N \) is the unit normal to \( S \). Denoting by \( g_{ij} \) and \( b_{ij} \) the coefficients of the first and second fundamental form respectively we have that the function \( H \) satisfies the equation:

\[
2H = \frac{b_{11}g_{22} + b_{22}g_{11} - 2b_{12}g_{12}}{g_{11}g_{22} - g_{12}^2}
\]

Since,

\[
\nabla_{\varphi} \varphi_x = -\lambda^2 f_x(2 + 4\tau^2)E_1 + [\lambda f_{xx} + \lambda^2((1 + 4\tau^2) - f_x)]E_2 + 2\tau \lambda^2 f_xE_3
\]

\[
\nabla_{\varphi} \varphi_x = [\tau \lambda f_x - \lambda^2 f_x(1 + 2\tau^2)]E_1 + [\lambda f_{xt} - \lambda^2 f_x f_t - \lambda \tau]E_2 + \lambda^2 \tau f_tE_3
\]

\[
\nabla_{\varphi} \varphi_t = 2\tau \lambda f_t E_1 + (\lambda f_{tt} - \lambda^2 f_t^2)E_2
\]

and,

\[
\begin{align*}
    b_{11} &= \lambda f_{xx} + \lambda^2(1 + 4\tau^2)f_x^2 + 2\tau \lambda^2(1 + 4\tau^2)f_x f_t + \lambda^2(1 + 4\tau^2) \\
    b_{12} &= \lambda f_{xt} - \tau \lambda f_x^2 + 2\tau \lambda^3 \left( \frac{1}{2} + 2\tau^2 \right) f_t^2 - \tau \lambda \\
    b_{22} &= \lambda f_{tt} - 2\tau \lambda f_x f_t - \lambda^2 f_t^2(1 + 4\tau^2)
\end{align*}
\]

and,

\[
\begin{align*}
    g_{11} &= \lambda^2[(1 + 4\tau^2) + f_x^2] \\
    g_{12} &= \lambda^2 f_x f_t - 2\tau \lambda \\
    g_{22} &= 1 + \lambda^2 f_t^2
\end{align*}
\]

a straightforward computation gives the result. \( \square \)

An interesting formula for the Laplacian is given in the next Lemma.

**Lemma 2.2.** Considering \( H = 1/2 \) then:

The mean curvature equation is given by

\[
1 = \frac{f^2}{W^3}[(f^2 + f_t^2)f_{xx} - 2(f_x f_t - 2\tau f) f_{xt} + ((1 + 4\tau^2) + f_x^2)f_{tt} + f(1 + f_x^2) + 2\tau f_x f_t]
\]

And

\[
\begin{align*}
    \Delta_S f &= \frac{f^2}{W} \left( 1 - \frac{f}{W} + \frac{f f_t^2 + 2\tau f_x f_t}{W} \right) \\
    \Delta_S \left( \frac{1}{f} \right) &= \frac{W - f}{f W} + \frac{f_t^2 + 2\tau f_x f_t + 2\tau f_t^2}{W}
\end{align*}
\]

**Proof.** The proof follows from a hard computation by considering

\[
\Delta_S = \frac{1}{\sqrt{g}} \sum_{ij} \partial_{x_i}(\sqrt{g} g^{ij} \partial_{x_j})
\]

where \( g \) is the determinant of the first fundamental form and \( (g^{ij}) = (g_{ij})^{-1} \).

Observe that:

\[
\Delta_S f = \frac{1}{\sqrt{g} W^{3/2}} \left[ f^2[(f^2 + f_t^2)f_{xx} + 2(2\tau f - f_x f_t)f_{xt} + (f_x^2 + (1 + 4\tau^2)) f_{tt} + (\lambda^2 + f_x^2)f_x + (a f_x - (1 + 4\tau^2) f f_t)f_t] \right]
\]

where \( a = f f_x + 2\tau f_t \) and \( W^2 = f^2 + f_t^2 + (f f_x + 2\tau f_t)^2 \) \( \square \)
In particular when $H = 1/2$, we see from lemma 2.2 that $f$ satisfies:

$$(f^2 + f_t^2)_{xx} - 2(f_x f_t - 2\tau f) f_{xt} + (f_x^2 + (1 + 4\tau^2)) f_{tt} = -f(1 + f_x^2) - 2\tau f_x f_t + \frac{W^3}{f^2}$$

(1)

Observe that we can rewriting equation (1) in the form:

$$(f^2 + f_t^2)_{xx} - 2(f_x f_t - 2\tau f) f_{xt} + (f_x^2 + (1 + 4\tau^2)) f_{tt} + 2\tau f_t f_x^2 f_t + \frac{W^3}{f^2} = 0$$

Setting $a(x, t) = f + 2\tau f_x f_t - \frac{W^3}{f^2}$. The equation (2) hold, by observing that:

$$a(x, t) = -\left[\frac{W^3 - f^3 - 2\tau f^2 f_x f_t}{f^2}\right]$$

$$= -\left[\frac{W^3 - f^2 W^2 + f W^2 - f^3 - 2\tau f^2 f_x f_t}{f^2}\right]$$

$$= -\left[\frac{W^2(W - f)}{f^2} + \frac{W^2 - f^2 - 2\tau f f_x f_t}{f}\right]$$

$$= -\left[\frac{W^2}{f^2(W + f)} + \frac{1}{f}\right] (W^2 - f^2 + 2\tau f f_x f_t)$$

$$= \left[2\tau f_t - \left(\frac{W^2}{f^2(W + f)}\right) f_x - \left[(1 + 4\tau^2) f_t + 4\tau f f_x f_t\right] \left(\frac{W^2}{f^2(W + f)} + \frac{1}{f}\right) f_t$$

3 The First Theorem

Before proving the first theorem, we prove the theorem 3.1.

**Theorem 3.1.** Let $U$ be the annulus $U = B_{R_2} \setminus B_{R_1}$ with $R_2 \geq 2R_1$. Then for $\epsilon > 0$ sufficiently small (depending only on $R_1$), there exist constant mean curvature $H = 1/2$ horizontal graphs $f^+$ and $f^-$, satisfying equation (1) in $U$ with Dirichlet boundary data $f^\pm = 1 \pm \epsilon$ on $\partial B_{R_1}$, $f^\pm = 1$ on $B_{R_2}$. Moreover $f^\pm$ tends to $1 \pm \epsilon$ uniformly on compact subsets as $R_2$ tends to $\infty$.

**Proof.** Let $U = B_{R_2} \setminus B_{R_1}$ be an annulus with $R_2 \geq 4R_1$ and fix $h = 1 \pm \frac{\epsilon}{\log(R_2/R_1)} \log\left(\frac{R_2}{R_1}\right)$, where $r^2 = x^2 + t^2$.

We expect the solution $f$ to be close to $h$, so we define the weighted $C^{2,\alpha}$ norm:

$$|v|_{2,\alpha;U}^2 = \sup_{X} \{|v(X)| + r(X)|Dv(X)| + r^2(X)|D^2v(X)| + r_{X}^{2+\alpha}|D^2v|_{\alpha;X}\}$$

where $X = (x, t)$ and $|D^2v|_{\alpha;X}$ is the Hölder coefficient of $D^2v$ at $X$. 

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Definition 3.1. We say \( f \) is an admissible solution of \( (1) \) if \( f \in \mathcal{A}_\varepsilon \), where:

\[
\mathcal{A}_\varepsilon = \{ f \in C^{2,\alpha}(U), f = h \text{ on } U : |f - h|_{2,\alpha,U} \leq \sqrt{\varepsilon} \}
\]

We note that \( \mathcal{A}_\varepsilon \) is convex and compact subset of the Banach space \( \mathfrak{B} = C^{2,\beta}(U) \), \( \beta < \alpha \). We will reformulate our existence problem as a fixed point of a continuous operator \( T : \mathcal{A}_\varepsilon \rightarrow \mathcal{A}_\varepsilon \) by rewriting the equation \((2)\) in the form:

\[
(f^2 + f_x^2)f_{xx} - 2(f_xf_t - 2\tau f)f_{xt} + (f_x^2 + (1 + 4\tau^2))f_{tt} + 2\tau f_t - f^2 f_x \left( \frac{W^2}{f^2(W + f)} + \frac{1}{f} \right)
\]

Remark 3.1. Note that this least equation implies that any solution \( f^\pm \) solving the Dirichlet problem of Theorem 3.1 satisfies \( 1 - \varepsilon \leq f^- \leq 1 \) and \( 1 \leq f^+ \leq 1 + \varepsilon \) on \( U \).

We now define the operator \( w = Tf \) as the solution of the linear Dirichlet problem

\[
\begin{align*}
L_f w := & aw_{xx} + bw_{xt} + cw_{tt} + dw_x + ew_t = 0, & \text{in } U; \\
w = & h, & \text{on } \partial U.
\end{align*}
\]

where:

\[
\begin{align*}
a &= f^2 + f_x^2 \\
b &= 2\tau f - f_x f_t \\
c &= f_x^2 + (1 + 4\tau^2) \\
d &= ff_x + 2\tau f_t - \left[ \frac{W^2}{f^2(W + f)} + \frac{1}{f} \right] f_x^2 f_x \\
e &= -[4\tau ff_x + (1 + 4\tau^2)f_t] \left[ \frac{W^2}{f^2(W + f)} + \frac{1}{f} \right]
\end{align*}
\]

Definition 3.2. \( D \subset \overline{U} \) is of scale \( R \), if \( X \in D \), then \( c_1 R \leq |X| \leq c_2 R \) for uniform constant \( c_1, c_2 \).

Note that for \( f \in \mathcal{A}_\varepsilon \), if \( L_f u = F \) in \( D \subset \overline{U} \) where \( D \) is of scale \( R \), then \( \tilde{u} = u(RX) \) satisfies:

\[
\tilde{L}\tilde{u} = \tilde{a} u_{xx} + 2\tilde{b} u_{xt} + \tilde{c} u_{tt} + R\tilde{d} u_x + R\tilde{e} u_t = R^2 \tilde{F} \quad \text{in } \tilde{D}
\]

where \( \tilde{D} \) is of scale 1 and \( \tilde{a}(X) = a(RX), \tilde{b}(X) = b(RX) \), etc. Observe that, in this case \( \tilde{u} \) is defined on \( \tilde{D} = \{ RX; c_1 \leq |X| \leq c_2 \} \).

Hence for \( \varepsilon \) sufficiently small, \( \tilde{L} \) is uniform close to \( \Delta \) with Hölder continuous coefficients.

Proposition 3.1. Let \( w = Tf \) for \( f \in \mathcal{A}_\varepsilon \). Then for \( \varepsilon \) sufficiently small, \( w \in \mathcal{A}_\varepsilon \).

Proof. Set \( u = \epsilon - h \), then:

\[
L_f u = [(1 - f^2 - f_x^2)h_{xx} + 2f_xf_th_{xt} - f_x^2h_{tt} - dh_x - eh_t] := F.
\]

By the maximum principle [1, Theorem 3.1(pag. 32)], \( 1 \leq w \leq 1 + \varepsilon \) (or \( 1 - \varepsilon \leq w \leq 1 \)) so \( |u| \leq \sqrt{\varepsilon} \).

To see this, observe that for example \( h^+ = 1 + \frac{\epsilon}{\log(R_2/R_1)} \log\left(\frac{R_2}{R_1}\right) \), so:

\[
1 \leq w \leq 1 + \varepsilon \Leftrightarrow 1 - h^+ \leq u = w - h^+ \leq 1 + \varepsilon + h^+
\]
which is equivalent to:

\[-\epsilon \leq u \leq \epsilon\]

if and only if, \(0 \leq \frac{\log(R_2/r)}{\log(R_2/R_1)} \leq 1\), which is true when \(R_1 \leq r \leq R_2\).

Since \(R_2 \geq 4R_1\), there is a positive number \(m_0\), such that, \(R_2 = m_0R_1\). Without loss of generality, we can suppose that \(m_0\) is not a rational number. We now write \(U = U_1 \cup U_2 \cup U_3\) where

\[
U_1 = \{X; R_1 \leq |X| \leq \frac{m_0 + 2}{3}R_1\}
\]

\[
U_2 = \{X; \frac{m_0 + 2}{3}R_1 \leq |X| \leq \frac{m_0 + 1}{3}R_1\}
\]

\[
U_3 = \{X; \frac{m_0 + 1}{3}R_1 \leq |X| \leq R_2\}
\]

Thus, each domain is of scale \(R = R_1\) and we can apply Schauder interior or boundary estimates to \(\tilde{L}\tilde{u} = R^2\tilde{F}\) in \(\tilde{D}\) to obtain: (see [4, Theorem 6.6(page 98)])

\[
\|\tilde{u}\|_{2,\alpha;\tilde{D}} \leq C(\|\tilde{u}\|_{0;\tilde{D}} + \|R^2\tilde{F}\|_{0,\alpha;\tilde{D}}).
\]

Observe that \(|\tilde{u}| \leq \epsilon\) implies \(\|\tilde{u}\|_{0;\tilde{D}} \leq \epsilon\). From equation (4) follows \(\|\tilde{F}\|_{0,\alpha;\tilde{D}} \leq C\epsilon^{3/2}\). This implies:

\[
\|\tilde{u}\|_{2,\alpha;\tilde{D}} \leq C(\|\tilde{u}\|_{0;\tilde{D}} + \|R^2\tilde{F}\|_{0,\alpha;\tilde{D}}) \leq C\epsilon. \tag{5}
\]

This implies, (see [4, Equation 4.17(page 61)])

\[
\|\tilde{u}\|_{2,\alpha;\tilde{D}} \leq C\epsilon.
\]

Undoing the scaling gives

\[
\|u\|_{2,\alpha;D}^* \leq C\epsilon.
\]

Since \(u = w - h\), it follows that for \(\epsilon\) small enough, \(w \in A_\epsilon\) and the proposition is proved.

We are now in a position to apply the Schauder fixed point theorem to our operator \(w = Tf\) to find a solution \(f^\pm \in A_\epsilon\) to (2) which is equivalent to our original equation (1).

Now, we show that \(f^\pm\) tends to \(1 \pm \epsilon\) uniformly on compact subsets as \(R_2\) tends to \(\infty\). To see this, observe that:

\[
h^\pm(r) = 1 \pm \frac{\epsilon}{\log(R_2/R_1)} \log(R_2/r)
\]

tends to \(1 \pm \epsilon\) uniformly on compact subsets as \(R_2\) goes to \(\infty\). In fact, without loss of generality, we can suppose that \(R_2 = nR_1\), where \(n\) is a positive integer (sufficiently large). Thus, we consider,

\[
h_n(r) = \frac{\epsilon}{\log(R_2/R_1)} \log(R_2/r) = \frac{\epsilon}{\log(n)} \log(nR_1/r) = 1 + \frac{\log(R_1/r)}{\log(n)}
\]

where \(R_1 \leq r\). Consider a compact domain \(D\), then there exist constants \(M_1, M_2 > 0\) such that,

\[-M_1 \leq \log(R_1/r) \leq -M_2\]
for all \( r \in [R_1, R_2] \) (closed interval). This implies,

\[
\frac{-M_1}{\log(n)} \leq \frac{\log(R_1/r)}{\log(n)} \leq \frac{-M_2}{\log(n)}
\]

Since, the function \( \frac{1}{\log(n)} \) tends to 0 uniformly on compact subsets, we conclude that, \( h^\pm \) tends to \( 1 \pm \epsilon \) uniformly on compact sets as \( R_2 \) tends to \( \infty \). As \( f^\pm \) is \( C^{2,\alpha} \) close to \( h^\pm \), we conclude the second affirmation of the theorem.

\[ \square \]

**Theorem 3.2.** Let \( \Sigma \) be a properly embedded constant mean curvature \( H = 1/2 \) surface in \( \text{PSL}_2(\mathbb{R}, \tau) \). Suppose \( \Sigma \) is asymptotic to a horocylinder \( C \), and one side of \( C \). If the mean curvature vector of \( \Sigma \) has the same direction as that of \( C \) at points of \( \Sigma \) converging to \( C \), then \( \Sigma \) is equal to \( C \) (or a subset of \( C \) if \( \partial \Sigma \neq \phi \)).

*Proof.* After an isometry, we can assume that, there is a sequence of points \( p_i = (x_i, y_i, t_i) \in \Sigma \) with \( y_i \to 1 \) and \( \Sigma \) is asymptotic to \( C(1) \) when \( t \) goes to \( +\infty \) (here \( C(\eta) \) denote the 1/2 horocylinder \( y = \eta \)). In this case \( \langle H, \partial_y \rangle > 0 \). Thus, either \( \Sigma \) is contained in the set \( \{ y < 1 \} \) or \( \Sigma \) is contained in the set \( \{ y > 1 \} \). First, we suppose that \( \Sigma \) is contained in the set \( \{ y < 1 \} \).

For \( \epsilon > 0 \) we consider the slab \( S^- \) bounded by \( C(1-\epsilon) \) and \( C(1) \), then by the maximum principle \( \Sigma^- = \Sigma \cap S^- \) has a non compact component with boundary \( \partial \Sigma \subset C(1-\epsilon) \).

Let \( D(\eta, R) \) denote the disk in \( C(\eta) \) defined by \( D(\eta, R) = \{(x, \eta, t); x^2 + t^2 \leq R^2 \} \). By considering vertical translation, we can find a disk \( D(1,3R_1) \times [1 - \epsilon, 1] \cap \Sigma^- = \phi \).

By Theorem 3.1 for each \( R \geq 2R_1 \), there exist a horizontal graph \( f^\prime_R \) defined on the annulus \( U = B_{R_1} \setminus B_{R_2} \), this horizontal graph converge to \( C(1-\epsilon) \), when \( R \) goes to \( +\infty \).

Now, consider \( R \) large, such that the graph of \( f^\prime_R \) (which we denote by \( \Gamma^- \)), satisfies \( \Sigma^- \cap \Gamma^- \neq \phi \). By considering translations along the geodesic \( \{ x = 0, t = 0 \} \) together with vertical translation, the translated surface of \( \Gamma^- \) does not touch \( \Sigma^- \), that is, there is a translated surface of \( \Gamma^- \) (which we denote by \( \Gamma^-_1 \)) such that \( \Gamma^-_1 \) and \( \Sigma^- \) has an interior contact point. Since the mean curvature vectors are pointing up, his violates the maximum principle and \( \Sigma^- \) cannot exist.

In the second case, we redo exactly the same argument exchanging the roles of \( C(1+\epsilon) \) and \( C(1-\epsilon) \).

\[ \square \]

### 4 H Sections in \( \text{PSL}_2(\mathbb{R}, \tau) \)

We give the definition of Killing Submersion which is due to H. Rosenberg, R. Souam, and E. Toubiana see [2].

Consider a Riemannian 3-manifolds \( (M^3, g) \) which fibers over a Riemannian surface \( (M^2, h) \), where \( g \) and \( h \) denote the Riemannian metrics respectively

**Definition 4.1.** A Riemannian submersion \( \pi : (M^3, g) \to (M^2, h) \) such that:

1. each fiber is a complete geodesic,
2. the fibers of the fibration are the integral curves of a unit Killing vector field $\xi$ on $M^3$.

will be called a Killing submersion.

**Definition 4.2.** Let $\pi : (M^3, g) \rightarrow (M^2, h)$ be a Killing submersion.

1. Let $\Omega \subset M^2$ be a domain. An $H$-section over $\Omega$ is an $H$-surface which is the image of a section.

2. Let $\gamma \subset M^2$ be a smooth curve with geodesic curvature $2H$. Observe that the surface $\pi^{-1}(\gamma) \subset M^3$ has mean curvature $H$. We call such a surface a vertical $H$—cylinder.

Observe that, $\pi : \widetilde{PSL}_2(\mathbb{R}, \tau) \rightarrow \mathbb{H}^2$ given by $\pi(x, y, t) = (x, y)$ is a Killing submersion. Since, $\pi : \widetilde{PSL}_2(\mathbb{R}, \tau) \rightarrow \mathbb{H}^2$ is a Killing submersion, we can consider graphs in $\widetilde{PSL}_2(\mathbb{R}, \tau)$.

**Definition 4.3.** A graph in $\widetilde{PSL}_2(\mathbb{R}, \tau)$ over a domain $\Omega$ of $\mathbb{H}^2$ is the image of a section $s_0 : \Omega \subset \mathbb{H}^2 \rightarrow \widetilde{PSL}_2(\mathbb{R}, \tau)$.

Given a domain $\Omega \subset \mathbb{H}^2$ we also denote by $\Omega$ its lift to $\mathbb{H}^2 \times 0$, with this identification we have that the graph (vertical graph) $\Sigma(u)$ of $u \in (C^0(\partial \Omega) \cap C^\infty(\Omega))$ is given by:

$$\Sigma(u) = \{(x, y, u(x, y)) \in \widetilde{PSL}_2(\mathbb{R}, \tau); (x, y) \in \Omega\}$$

5 The Second Theorem

In this section our second main result concerns complete $H = 1/2$ surfaces in $\widetilde{PSL}_2(\mathbb{R}, \tau)$ transverse to the vertical Killing field $E_3 = \partial_t$. We prove such surfaces are entire graphs.

**Theorem 5.1.** Let $\Sigma$ be a complete immersed surface in $\widetilde{PSL}_2(\mathbb{R}, \tau)$ of constant mean curvature $H = 1/2$. If $\Sigma$ is transverse to $E_3$ then $\Sigma$ is an entire vertical graph over $\mathbb{H}^2$.

**Proof.** The proof is the same as this one to $\mathbb{H}^2 \times \mathbb{R}$ (see [3]). By completeness we give it. From now on we identify $\mathbb{H}^2$ with its lift $\mathbb{H}^2 \times \{0\}$. The mean curvature vector of $\Sigma$ never vanish so $\Sigma$ is orientable. Let $\nu$ be a unit vector field along $\Sigma$ in $\widetilde{PSL}_2(\mathbb{R}, \tau)$. The function $u = \langle \nu, E_3 \rangle$ is a non-zero Jacobi function on $\Sigma$, so $\Sigma$ is strongly stable and thus has bounded curvature. We can assume $u > 0$ and $\langle \nu, \widetilde{H} \rangle > 0$.

Here there is $\delta > 0$ such that for each $p \in \Sigma$, $\Sigma$ is a graph (in exponential coordinates) over the disk $D_\delta \subset T_p \Sigma$ of radius $\delta$, centered at the origin of $T_p \Sigma$. This graph, denoted by $G(p)$, has bounded geometry. The $\delta$ is independent of $p$ and the bound on the geometry of $G(p)$ is uniform as well (see [2]).

We denote by $F(p)$ the surface $G(p)$ translated to the origin $i \in \mathbb{H}^2 \approx \mathbb{H}^2 \times \{0\}$. (The translation that takes $p$ to $i$).

For $q \in \mathbb{H}^2$, we denote by $\Gamma_\delta(q)$ a horizontal horocycle arc of length $2\delta$, centered at $q$.

**Claim1:** Let $p_n \in \Sigma$, satisfy $u(p_n) \rightarrow 0$ as $n \rightarrow \infty$ ($T_{p_n}(\Sigma)$ are becoming vertical). There is a subsequence of $p_n$ (which we also denote by $\{p_n\}$) such that $F(p_n)$ converges to $\Gamma_\delta(i) \times [-\delta, \delta]$, for some horocycle $\Gamma_\delta(i)$. The convergence is in the $C^2$-topology.
Proof of Claim 1. Choose a subsequence \( p_n \) so that the oriented tangent planes \( T_i(F(p_n)) \) converge to a vertical plane \( P \). Let \( \Gamma_\delta(i) \) be the horocycle arc through \( i \in \mathbb{H}^2 \approx \mathbb{H}^2 \times 0 \) whose curvature vector has the same direction as the curvature vector of the (limit) curvature vectors of \( F(p_n) \).

Since the \( F(p_n) \) have bounded geometry and they are graphs over \( D_\delta(p_n) \subset T_{p_n}(F(p_n)) \), the surfaces \( F(p_n) \) are bounded horizontal graphs over \( \Gamma_\delta(i) \times [-\delta, \delta] \) for \( n \) large. Thus a subsequence of these graphs converges to an \( H = 1/2 \) surface \( F \); \( F \) is tangent to \( \Gamma_\delta(i) \times [-\delta, \delta] \) at \( i \) and a horizontal graphs over this. It suffices to show \( F = \Gamma_\delta(i) \times [-\delta, \delta] \).

Were this not the case, then the intersection near \( i \), of \( F \) and \( \Gamma_\delta(i) \times [-\delta, \delta] \) would consist of \( m \) smooth curves passing through \( i \), \( m \geq 2 \), meeting transversally at \( i \). In a neighborhood of \( i \), theses curves separate \( F \) into \( 2m \) components. Adjacents components lie on opposite sides of \( \Gamma_\delta(i) \times [-\delta, \delta] \).

Hence in a neighborhood of \( i \in F \), the mean curvature vector of \( F \) alternates from pointing up in \( \mathbb{H}^2 \) to pointing down (or vice-versa), as one goes from one component to the other. But \( F(p_n) \) converges to \( F \) in the \( C^2 \)-topology, so \( F(p_n) \) converges to a vertical plane \( P \) as \( n \) large, would also have points where the mean curvature vector point up and down in \( \mathbb{H}^2 \). This contradicts that \( F(p_n) \) is transverse to \( E_3 \), and claim 1 is proved. Notice that we have proved that whenever \( F(p_n) \) converges to a local surface \( F \), \( F \) is necessarily some \( \Gamma_\delta(i) \times [-\delta, \delta] \). This prove Claim 1.

Now let \( p \in \Sigma \) and assume \( \Sigma \) in a neighborhood of \( p \) is an \( H \) section, that is, in a neighborhood of \( p \), \( \Sigma \) is a vertical graph of a function \( f \) defined on \( B_R \), \( B_R \) the open ball of radius \( R \) of \( \mathbb{H}^2 \), centered at \( i \in \mathbb{H}^2 \). Denote by \( S(R) \) the graph of \( f \) over \( B_R \). If \( \Sigma \) is not an entire graph then we let \( R \) be the largest such \( R \) so that \( f \) exist. Since \( \Sigma \) has constant mean curvature, \( f \) has bounded gradient on relatively compact subsets of \( B_R \), see [2, Theorem 3.5].

Let \( q \in \partial B_R \) be such that \( f \) does not extend to any neighborhood of \( q \) (to an \( H = 1/2 \) graph).

Claim 2: For any sequence \( q_n \in B_R \) converging to \( q \), the tangent planes \( T_{p_n}(S(R)) \) converge to a vertical plane \( P \). \( P \) is tangent to \( \partial B_R \) at \( q \) (after vertical translation to height zero in \( \mathbb{H}^2 \)).

Proof of Claim 2. Let \( F(n) \) denote the image of \( G(p_n) \) under the vertical translation taking \( p_n \) to \( q_n \). Observe first, that \( T_{q_n}(F(n)) \) converges to the vertical , for any subsequence of the \( q_n \). Otherwise the graph of bounded geometry \( G(p_n) \), would extend to a vertical graph beyond \( q \), for \( q_n \) close enough to \( q \), hence \( f \) would extend; a contradiction.

Now we can prove \( T_{q_n}(F(n)) \) converges to the vertical plane \( P \) passing through \( q \) and tangent to \( \partial B_R \) at \( q \). Suppose some subsequence \( q_n \) satisfies \( T_{q_n}(F_n) \) converges to a vertical plane \( Q \), \( Q \neq P \), \( q \in Q \). By Claim 1, the \( F_n \) converge in the \( C^2 \)-topology, to \( \Gamma_\delta \times [-\delta, \delta] \), where \( \Gamma_\delta(q) \) is a horocycle arc centered at \( q \). Since \( Q \neq P \), and \( \Gamma_\delta \) is tangent to \( Q \) at \( q \), there are points of \( \Gamma_\delta \) in \( B_R \). Such a point is the limit of points on \( F_n \). Then the gradient of \( f \) at these points of \( F_n \) diverges, which contradicts interior gradient estimates of \( f \). This proves Claim 2.

Now applying Claim 1 and Claim 2, we know that for any sequence \( q_n \in B_R \) converging to \( q \), the \( F(q_n) \) converge to \( \Gamma_\delta(q) \times [-\delta, \delta] \).

Claim 3: For any \( q_n \to q \), \( q_n \in B_R \), we have \( f(q_n) \to +\infty \) or \( f(q_n) \to -\infty \).
Proof of Claim 3. Let $\gamma$ be a compact horizontal geodesic of length $\epsilon$ starting at $q$, entering $B_R$ at $q$, and orthogonal to $\partial B_R$ at $q$. Let $C$ be the graph of $f$ over $\gamma$. Notice that $C$ has no horizontal tangents at points near $q$ since the tangent planes of $S(R)$ are converging to $P$. So assume $f$ is increasing along $\gamma$ as one converges to $q$. If $f$ were bounded above, then $C$ would have a finite limit point $(q,c)$; a contradiction. This prove Claim 3.

Now choose $q_n \in \gamma$, $q_n \to q$, and $F(q_n)$ converges to $\Gamma_\delta(q) \times [-\delta, \delta]$. Let $\Gamma$ be the horocycle containing $\Gamma_\delta(q)$, and parameterize $\Gamma$ by arc length; denote $q(s) \in \Gamma$ the point at distance $s$ on $\Gamma$ from $q = q(0)$, $-\infty < s < +\infty$. Denote by $\gamma(s)$ a horizontal geodesic arc orthogonal to $\Gamma$ at $q(s)$, $q(s)$ the mid-point of $\gamma(s)$. Assume the length of each $\gamma(s)$ is $2\epsilon$ and $\bigcup_{s \in R} \gamma(s) = N_\epsilon(\Gamma)$ is the $\epsilon$-tubular neighborhood of $\Gamma$.

Let $\gamma^+(s)$ be the part of $\gamma(s)$ on the mean convex side of $\Gamma$; so $\gamma = \gamma^+(0)$. More precisely, the mean curvature vector of $\Sigma$ points up in $\overline{PSL_2(\mathbb{R}, \tau)}$, and $f \to +\infty$ as one approaches $q$ along $\gamma$, so $\Gamma$ is convex towards $B_R$.

Claim 4: For $n$ large, each $F(q_n)$ is disjoint from $\Gamma \times \mathbb{R}$. Also, for $|s| \leq \delta$, $F(q_n) \cap (\gamma^+(s) \times \mathbb{R})$ is a vertical graph over an interval of $\gamma^+(s)$.

Proof of Claim 4. Choose $n_0$ such that for $n \geq n_0$, $C_n(s) = F(q_n) \cap (\gamma(s) \times \mathbb{R})$ is one connected curve of transverse intersection, for each $s \in [-\delta, \delta]$. Since the $F(q_n)$ are $C^2$-close to $\Gamma_\delta(q) \times [-\delta, \delta]$, $C_n(s)$ has no horizontal or vertical tangents and is a graph over an interval in $\gamma(s)$.

We now show this interval is in $\gamma^+(s) - q(s)$. Suppose not, so $C_n(s)$ goes beyond $\Gamma \times \mathbb{R}$ on the concave side. Recall that $C = \gamma \cap P^\perp$ is the graph of $f$ and $f \to +\infty$ as one goes up on $C$. We have $p_n = (q_n, f(q_n))$. Fix $n \geq n_0$ and choose new points $q_k$, $k \geq n$, so that $f(q_{k+1}) - f(q_k) = \delta$; clearly $q_k \to q$ as $k \to +\infty$. Lift each $C_k(s)$ to $G(p_k)$ by the vertical translation of $F(q_n)$ by $f(q_n)$. By construction, $C_{k+1}(s)$ is the analytic continuation of $C_k(s)$ in $\Sigma \cap (\gamma(s) \times \mathbb{R})$, for each $s \in [-\delta, \delta]$ and for all $k \geq n + 1$. The curve $C(s) = \bigcup_{k \geq n} C_k(s)$ is a vertical graph over an interval in $\gamma(s)$. It has points on the concave side of $\Gamma \times \mathbb{R}$ for some $s_0 \in [-\delta, \delta]$. For $s = 0$, $C(0) = C$ stays on the convex side of $\Gamma \times \mathbb{R}$. So for some $s_1$, $0 < s_1 \leq s_0$, $C(s_1)$ has a point on $\Gamma \times \mathbb{R}$ and also inside the concave side of $\Gamma \times \mathbb{R}$.

But the $F(q_k)$ converge uniformly to $\Gamma_\delta(q) \times [-\delta, \delta]$ as $k \to \infty$, so the curve $C(s_1)$ converges to $q(s_1) \times \mathbb{R}$ as the height goes to $\infty$. This obliges $C(s_1)$ to have a vertical tangent on the concave side of $\Gamma \times \mathbb{R}$, a contradiction. This prove Claim 4.

Now we choose $\epsilon_1 < \epsilon$ (which we call $\epsilon$ as well) so that $\bigcup_{s \in [-\delta, \delta]} C(s)$ is a vertical graph of a function $g$ on $\bigcup_{s \in [-\delta, \delta]} (\gamma^+(s) - q(s))$, (the $\gamma^+(s)$ now have length $\epsilon_1$); $g$ is an extension of $f$.

The graph of $g$ on each $\gamma^+ \times \mathbb{R}$ is the curve $C(s)$, and the graph of $g$ converges to $\Gamma_\delta(q) \times \mathbb{R}$ as the height goes to infinity.

Now we begin this process again replacing $C$ by the curves $C(\delta)$ and then $C(-\delta)$. Analytic continuation yields an extension $h$ of $g$ to a domain $\Omega$ contained in the open $\epsilon$-tubular neighborhood of $\Gamma \times \mathbb{R}$, on the convex side of $\Gamma$. $\Omega$ is an open neighborhood of $\Gamma$ in this mean convex side. The graph $h \to \infty$ as one approaches $\Gamma$ in $\Omega$; it converges to $\Gamma \times \mathbb{R}$ as the height goes to infinity.

Claim 5: There is an $\epsilon > 0$, such that $\Omega$ contains the $\epsilon$ tubular neighborhood of $\Gamma$ on the
convex side.

Proof of Claim 5. We know there is a domain $\Omega$ on the convex side of $\Gamma$; $\Omega$ is a neighborhood of $\Gamma$ on the convex side. Also the surface $\Sigma$ contains a graph over $\Omega$, composed of curves $C(q)$, $q \in \Gamma$, where each curve $C(q)$ is a graph over an interval $\gamma^+(q)$, $\gamma^+(q)$ orthogonal to $\gamma$ at $q$. Also $C(q)$ is a strictly monotone increasing graph with no horizontal tangents and $C(q)$ converges to $\{q\} \times \mathbb{R}^+$, as one goes up to $+\infty$.

The graph over $\Omega$ is converging uniformly to $\Gamma \times \mathbb{R}^+$ as one goes up.

Now suppose that for some $q \in \Gamma$, $\gamma^+(q)$ is of length less than $\epsilon$. Then $C(q)$ diverges to $-\infty$ as one approaches the end-point $\bar{q}$ of $\gamma^+(q)$, $\bar{q} \neq q$.

The previous discussion where we showed the graph over $\Omega$ exists and converges to $\Gamma \times \mathbb{R}^+$, now applies to show that there is a horocycle $\tilde{\Gamma}$ passing through $\bar{q}$, $C(q)$ converges to $\{\bar{q}\} \times \mathbb{R}^-$ as one tends to $\bar{q}$ on $\gamma^+(q)$. Also a $\delta$-neighborhood of $C(q)$ in $\Sigma$, converges uniformly to $\tilde{\gamma}_\delta(\bar{q}) \times \mathbb{R}^-$, as one goes down to $-\infty$. We know this $\delta$-neighborhood of $C(q)$ in $\Sigma$, converges uniformly to $\Gamma_\delta(q) \times \mathbb{R}^+$, as one goes up to $+\infty$.

For each $q(s) \in \Gamma$, a distance $s$ from $q$ on $\Gamma$, $|s| \leq \delta$, the curve $C(q(s))$ converges uniformly to some $\{\bar{q}(s) \times \mathbb{R}^-\}$, as one goes down to $-\infty$. By analytic continuation of the $\delta$-neighborhoods, one continues this process along $\gamma$.

If $\Gamma \cup \tilde{\Gamma} = \phi$, then the process continues along all $\Gamma$ and $\Omega$ is the region bounded by $\Gamma \cup \tilde{\Gamma}$. This suffices to prove Claim 5 since each $\gamma^+(q)$, $q \in \Gamma$, has the same length.

So we can assume $\Gamma \cup \tilde{\Gamma} = \{p\}$. Consider the curves $C(q(s))$, as $q(s)$ goes from $q$ to $p$ along $\Gamma$. They are graphs that become vertical both at $+\infty$ and $-\infty$. Hence the graphs $C(q(s))$ become vertical at every point as $q(s) \to p$.

Consider the point of $C(q(s))$ at height 0 in $\widetilde{PSL_2}(\mathbb{R}, \tau)$. As $q(s) \to p$, these converge to a point of $\Sigma$ and the tangent plane of $\Sigma$ is vertical at this point; a contradiction.

We remark that in this case $f(q_n) \to -\infty$ (see Claim 3), one works on the concave side of the horocycle $\Gamma(q)$ and Claims 4 and 5 show there is an $\epsilon > 0$ and a graph $G \subset \Sigma$ over the domain $\Omega(\epsilon)$ between $\Gamma(\epsilon)$ (the equidistant horocycle to $\Gamma$ on the concave side of $\Gamma$) and $\Gamma$. The graph $G$ converges uniformly to $\Gamma \times \mathbb{R}$ as one approaches $\Gamma$ in $\Omega(\epsilon)$.

To complete the proof of the theorem we apply the half-space theorem to show no such $H = 1/2$ graph exist. Strictly speaking we can not apply the half-space theorem directly since the $G$ does not have a compact boundary. But this graph is proper in the tubular neighborhood of the horocylinder, so the proof shows the graph can not exist.

\[ \Box \]

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