PERIOD POLYNOMIALS AND IHARA BRACKETS

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Abstract. Schneps [J. Lie Theory 16 (2006), 19–37] has found surprising links
between Ihara brackets and even period polynomials. These results can be recovered
and generalized by considering some identities relating Ihara brackets and classical
Lie brackets. The period polynomials generated by this method are found to be
essentially the Kohnen-Zagier polynomials.

1. Introduction

The Ihara bracket is defined on a subspace of the free Lie algebra \( \mathbb{L} \) on two genera-
tors, whose elements are interpreted as special derivations of \( \mathbb{L} \). One can find relations
between the two brackets. This gives rise to interesting identities, which surprisingly,
turn out to be related to period polynomials [12].

The period polynomials \( r^\pm(f) \) of a modular form \( f \in M_k(SL_2(\mathbb{Z})) \) are defined by

\[
(1) \quad r(f)(t) = \int_0^{i\infty} f(z)(t - z)^w dz \quad (w = k - 2),
\]

and \( r = r^+ + ir^- \). These polynomials satisfy the period relations

\[
(2) \quad P(t) + t^w P \left( \frac{-1}{t} \right) = 0,
\]

\[
(3) \quad P(t) + t^w P \left( 1 - \frac{1}{t} \right) + (t - 1)^w P \left( \frac{1}{1 - t} \right) = 0.
\]

The solutions of this system are in one-to-one correspondence with cusp forms (except
for the \( t^w - 1 \) which are obtained from Eisenstein series). This is the Eichler-Shimura
correspondence (see, e.g., [9]).

Writing

\[
(4) \quad r(f)(t) = \sum_{n=0}^w i^{-n+1} \binom{w}{n} r_n(f),
\]

one defines linear forms \( r_n \) on \( M_k \). Since this is an inner product space for the
Petersson scalar product, there exist modular forms \( R_{n,k} \) such that for all \( f \in S_k \),

\[
(5) \quad r_n(f) = (f, R_{n,k}).
\]
The period polynomials of the $R_{n;k}$ have been explicitly computed by Kohnen and Zagier [9, Theorem 1, p. 208]. Their result can be stated as follows: let $B_n(t)$ be the Bernoulli polynomials and $b_n$ be the Bernoulli numbers defined by

\begin{equation}
ue^{tu}e^{-u} = \sum_{n \geq 0} B_n(t) \frac{u^n}{n!} \quad \text{and} \quad B_n(t) = \sum_{i=0}^{n} \binom{n}{i} b_i t^{n-i}.
\end{equation}

and let

\begin{equation}
P_{n;k}^\pm(t) := \pm \frac{1}{n+1} \left[ B_{n+1}(t) \mp t^w B_{n+1}(\frac{1}{t}) \right] + \frac{1}{p+1} \left[ B_{p+1}(t) \mp t^w B_{p+1}(\frac{1}{t}) \right],
\end{equation}

where $w = k - 2 = n + p$. Then [9],

\begin{equation}
P_{n;k}^\pm(t) = r^\pm(\lambda_{n;k} R_{n;k} + \mu_{n;k} G_k)
\end{equation}

where $G_k$ is the Eisenstein series and $\lambda_{n;k}^\pm, \mu_{n;k}^\pm$ are explicit constants that we shall not need.

Hence, any linear combination of the $P_{n;k}^\pm(t)$ for a given $k$ is a period polynomial. Such combinations arise naturally from the Ihara bracket. In [12], Schneps obtained a characterization of even period polynomials in terms of linear relations between certain brackets. In this note, we give a new proof of this result and obtain a similar characterization of odd period polynomials. We also give a simpler way to generate these polynomials.

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2. IHARA BRACKETS AND THE STABLE DERIVATION ALGEBRA

The stable derivation algebra $\mathfrak{F}$ appears in works by Ihara on Galois representations on $\pi_1(\mathbb{P}^1_Q \setminus \{0, 1, \infty\})$ [3, 4, 5, 6, 7, 8]. It plays also a crucial rôle in recent works on multiple zeta values [1, 2, 10].

The underlying vector space of $\mathfrak{F}$ can be identified with a subalgebra of the free Lie algebra over two letters $\mathfrak{F}^1$. Let $L$ be the free $\mathbb{Q}$-Lie algebra over two letters $a, b$. By Lazard elimination [11], the Lie algebra decomposes as

\begin{equation}L = \mathbb{Q}a \oplus \mathfrak{F}^1L\end{equation}

where the Lie algebra $\mathfrak{F}^1L$ is free over the infinite sequence

\begin{equation}\phi_{n+1} = \frac{1}{n!} \text{ad}_a^n(b) \quad (n \geq 0),\end{equation}

where ad denotes the adjoint representation.
Let $V$ be any vector space and Der$(L(V))$ be the Lie algebra of derivations of the free Lie algebra $L(V)$. The commutator of two inner derivations satisfies
\begin{equation}
[\text{ad}_f, \text{ad}_g] = \text{ad}_{[f,g]}
\end{equation}
where $f, g \in L$ and $[f, g]$ is the bracket in $L$. More generally, if $\pi : V \to V$ is a projector, one can associate with any element $f \in L(V)$ a derivation $D_f$ defined by
\begin{equation}
D_f(v) = [\pi(v), f],
\end{equation}
which satisfies
\begin{equation}
[D_f, D_g] = D_{\{f, g\}}
\end{equation}
where $\{f, g\} := [f, g] + D_g(f) - D_f(g)$.

In the special case where $V = \mathbb{Q}a \oplus \mathbb{Q}b$ and $\pi(a) = a, \pi(b) = 0$, the operation $\{f, g\}$ is called the Ihara bracket. Let $G$ be the Lie subalgebra of Der$(L)$ generated by the special derivations $D_f, f \in L$. It can be identified with the Ihara subalgebra $F^1L$ of $L$ generated by the $\phi_n$, so that it is also a Lie subalgebra of $L$ for the ordinary bracket. The stable derivation algebra $\mathfrak{F}$ is a subalgebra of $G$, spanned by the $D_f$ such that $f$ satisfies certain identities (8), see also (1).

In [1, 6], Furusho and Ihara consider the filtration
\begin{equation}
L = F^0L \supset F^1L \supset \cdots \supset F^nL \supset \cdots
\end{equation}
where
\begin{equation}
F^0L = L \text{ and } F^nL = [F^1L, F^{n-1}L] \text{ for } n \geq 2.
\end{equation}
This induces a filtration on $\mathfrak{F}$
\begin{equation}
\mathfrak{F} = F^1\mathfrak{F} \supset F^2\mathfrak{F} \supset \cdots \supset F^n\mathfrak{F} \supset \cdots,
\end{equation}
and the compatibility of the Ihara bracket with this filtration implies that the Lie algebra $\mathfrak{F}$ is generated by polynomials which do not belong to $F^2\mathfrak{F}$. From (8), $\dim \mathbb{Q}F^1\mathfrak{F}/F^2\mathfrak{F} = 1$ if $n$ is odd and greater than 1 and 0 otherwise. Hence, the Lie algebra $\mathfrak{F}$ is generated by a set
\begin{equation}
\{f_{2n+1} \in \mathfrak{F}_{2n+1} \mid n \geq 1\},
\end{equation}
such that
\begin{equation}
f_{2n+1} \equiv (2n)! \phi_{2n+1} \mod F^2\mathfrak{F}
\end{equation}
Therefore,
\begin{equation}
\{f_{2n+1}, f_{2n'+1}\} \equiv (2n)!(2n')! \{\phi_{2n+1}, \phi_{2n'+1}\} \mod F^3\mathfrak{F}
\end{equation}
which implies the equivalence between equalities of the form
\begin{equation}
\sum_{i+j=n, i,j \leq 3 \text{ odd}} a_{ij} \{f_i, f_j\} \equiv 0 \mod F^3\mathfrak{F}
\end{equation}
and
\[(21) \quad \sum_{i+j=n \leq 3 \text{ odd}} a_{ij} \frac{(i-1)!(j-1)!}{(i-1)!(j-1)!(t_i-1 - t_j-1)} \{ \phi_i, \phi_j \} = 0. \]

This is precisely the kind of linear relations giving rise to period polynomials. Those found by Schneps [12] are obtained by substituting (22) \(\{ \phi_i, \phi_j \} \rightarrow \frac{1}{(i-1)!(j-1)!}(t^{i-1} - t^{j-1})\) in the left-hand side of relations like (21). Let us note that if we set \(\varphi_i(t) = \frac{1}{(i-1)!}t^{i-1}\) and \(\langle f(t), g(t) \rangle = f(t)g(1) - f(1)g(t)\) (this is a Lie bracket), then (22) amounts to replace \(\{ \phi_i, \phi_j \}\) by \(\langle \varphi_i(t), \varphi_j(t) \rangle\).

All the calculations presented in this paper aim at finding explicit linear relations between the \([\varphi_m, \varphi_n], \{ \varphi_m, \varphi_n \}\) and \(D_{\varphi_m}(\varphi_n)\). As a general rule, the coefficients are expressed in terms of Bernoulli numbers and can be related to the Kohnen-Zagier polynomials.

3. Computation of Ihara brackets

The aim of this section is to express the Ihara bracket in terms of classical brackets. Our first step is to compute the action of the special derivations on the generators \(\phi_n\). Let us start with the simple equality
\[(23) \quad D_{\phi_n} a = -n\phi_{n+1}. \]

Since \(D_{\phi_n}\) is a derivation, it follows from (23) that
\[(24) \quad D_{\phi_n}(\phi_p) = -\frac{n}{(p-1)!} \sum_{i=1}^{p-1} (p-i-1)! \text{ad}_{\phi_n}^{-1} [\varphi_{n+1}, \varphi_{p-i}]. \]

The Leibniz formula gives
\[(25) \quad \text{ad}_{\phi_n}^{-1} [\varphi_{n+1}, \varphi_{p-i}] = \sum_{j=0}^{i-1} \binom{i-1}{j} \frac{(n+j)!(p-j-2)!}{n!(p-i-1)!} [\varphi_{n+j+1}, \varphi_{p-j-1}]. \]

Substituting in (24) and rearranging the sums, one gets, for all \(n, p \geq 1\)
\[(26) \quad D_{\phi_n}(\phi_p) = \sum_{j=1}^{p-1} \binom{n+j-1}{j} [\varphi_{p-j}, \varphi_{n+j}] \]
\[= \sum_{k=1}^{p-1} \binom{n+p-1-k}{p-k} [\varphi_k, \varphi_{p+n-k}]. \]
Hence, from the definition of Ihara bracket, one has, for all \( n, p \geq 1 \)

\[
\{ \phi_n, \phi_p \} = [\phi_n, \phi_p]
+ \sum_{k=1}^{n-1} \binom{n+p-1-k}{n-k}[\phi_k, \phi_{n+p-k}]
- \sum_{k=1}^{p-1} \binom{n+p-1-k}{p-k}[\phi_k, \phi_{n+p-k}].
\]

(27)

This equation can be rewritten as

\[
\{ \phi_n, \phi_p \} = \max(n-1, p-1) \sum_{k=1}^{\max(n-1, p-1)} \left( \binom{n+p-1-k}{n-k} - \binom{n+p-1-k}{p-k} \right)[\phi_k, \phi_{n+p-k}],
\]

(28)

for all \( n, p \geq 1 \) as one can check on all cases \( n < p \), \( n = p \), and \( n > p \).

In terms of generating series, both Equations (26) and (28) have simple expressions.

Set

\[
\Phi(x) := \sum_{n \geq 1} \phi_n x^{n-1}.
\]

(29)

Then,

\[
D_{\Phi(x)}\Phi(y) := \sum_{n,p \geq 1} D_{\phi_n}(\phi_p) x^{n-1} y^{p-1}
= \sum_{n,p \geq 1} \sum_{k=1}^{p-1} \binom{n+p-1-k}{p-k}[\phi_k y^{k-1}, \phi_{n+p-k} x^{n-1} y^{p-k}]
= \sum_{k \geq 1} \phi_k y^{k-1} \sum_{s \geq 1} \phi_s \sum_{n \geq 1} \binom{s-1}{n-1} x^{n-1} y^{s-n}
= [\Phi(y), \sum_{s \geq 1} \phi_s ((x+y)^{s-1} - x^{s-1})]
= [\Phi(y), \Phi(x+y) - \Phi(x)]
= [\Phi(x), \Phi(y)] + [\Phi(y), \Phi(x+y)].
\]

(30)

Hence,

\[
D_{\Phi(x)}\Phi(y) = [\Phi(x), \Phi(y)] + [\Phi(y), \Phi(x+y)]
\]

(31)

and

\[
\{ \Phi(x), \Phi(y) \} = [\Phi(y), \Phi(x)] + [\Phi(x) - \Phi(y), \Phi(x+y)].
\]

(32)
4. Inversion of Equation (31) and period polynomials

4.1. Generic inversion of (31). Let \((F_{i,j})_{i,j \geq 1}\) and \((G_{i,j})_{i,j \geq 1}\) be two bi-indexed sequences of elements of some vector space whose generating series

\[
F(x, y) = \sum_{i,j \geq 1} F_{i,j} x^{i-1} y^{j-1}, \quad G(x, y) = \sum_{i,j \geq 1} G_{i,j} x^{i-1} y^{j-1},
\]

satisfy

\[
F(x, y) = G(x, y) - G(x + y, y) = (1 - e^{y\partial_x}) G(x, y).
\]

Thanks to Equation (31), this is the case of

\[
F(x, y) = D_{\Phi(x)} \Phi(y) \quad \text{and} \quad G(x, y) = [\Phi(x), \Phi(y)].
\]

If \(F(x, y)\) is given, this formula determines \(G(x, y)\) up to a function of \(y\):

\[
y \partial_x G(x, y) = \frac{y \partial_x}{1 - e^{y\partial_x}} F(x, y)
\]

\[
= - \sum_{k \geq 0} \frac{b_k}{k!} y^k \partial_x \sum_{i,j \geq 1} F_{i,j} x^{i-1} y^{j-1}
\]

\[
= - \sum_{k \geq 0} \frac{b_k}{k!} \sum_{i \geq k+1, j \geq 1} (i-1) \cdots (i-k) F_{i,j} x^{i-1-k} y^{j-1+k}
\]

\[
= - \sum_{k \geq 0} \sum_{i \geq k+1, j \geq 1} \binom{i-1}{k} b_k F_{i,j} x^{i-1-k} y^{j-1+k}
\]

\[
= - \sum_{i \geq 1} \sum_{k=0}^{i-1} \binom{i-1}{k} b_k x^{i-1-k} y^k \sum_{j \geq 1} F_{i,j} y^{j-1}
\]

\[
= - \sum_{i \geq 0} \sum_{k=0}^{i} \binom{i}{k} b_k \left( \frac{x}{y} \right)^{i-k} y^i \sum_{j \geq 1} F_{i+1,j} y^{j-1}
\]

so that, finally

\[
y \partial_x G(x, y) = - \sum_{i \geq 0} B_i \left( \frac{x}{y} \right) y^i \sum_{j \geq 1} F_{i+1,j} y^{j-1}.
\]
Then, comparing the coefficients of \(x^n y^p\) (with \(n \geq 0\) and \(p \geq 1\)) on both sides of Equation (37), one obtains

\[
(n + 1)G_{n+2,p} = -\sum_{i=n}^{n+p} \binom{i}{i-n} b_{i-n} F_{i+1,n+p+1-i}
\]

\[
= -\sum_{i=0}^{p} \binom{n+i}{i} b_{i} F_{n+1+i,p+1-i},
\]

so that

**Proposition 4.1.** For all \(n \geq 2\) and \(p \geq 1\),

\[
G_{n,p} = -\frac{1}{n-1} \sum_{i=0}^{p} \binom{n-2+i}{n-2} b_{i} F_{n-1+i,p+1-i}.
\]

Note that in the first equation, the summation can be taken up to \(p - 1\) since \(D_{\phi_i}(\phi_1) = D_{\phi_i}(b) = 0\) for all \(i\). Now, since the Lie bracket is antisymmetric, one gets the following relations between the \(D_{\phi_i}(\phi_j)\):

**Corollary 4.3.**

(i) For any \(n > 1\),

\[
\sum_{i=0}^{n} \binom{n-2+i}{n-2} b_{i} D_{\phi_{n-1+i}}(\phi_{n+1-i}) = 0,
\]

(ii) Setting \(b_i = 0\) if \(i < 0\), one has, for all \(n, p \geq 2\)

\[
\sum_{i=1}^{n+p-1} \left( \binom{i-1}{i-p+1} \frac{b_{i-p+1}}{p-1} + \binom{i-1}{i-n+1} \frac{b_{i-n+1}}{n-1} \right) D_{\phi_i}(\phi_{n+p-i}) = 0.
\]
4.2. Period polynomials. Let us now consider the specialization

\[ F_{i,j} = \frac{1}{(i-1)!} \frac{1}{(j-1)!} (t^{i-1} - \epsilon t^{j-1}). \]

Then, Equation (39) gives

\[ G_{n,p} = -\frac{1}{n-1} \sum_{i=0}^{p} \binom{n-2+i}{n-2} b_i \frac{t^{n-2+i} - \epsilon t^{p-i}}{(n-2-i)!((p-i)!)} \]

\[ = -\frac{1}{(n-1)!p!} \sum_{i=0}^{p} \binom{p}{i} b_i (t^{n-2+i} - \epsilon t^{p-i}) \]

\[ = -\frac{1}{(n-1)!p!} \left( t^{n+p-2}B_p \left( \frac{1}{t} \right) - \epsilon B_p(t) \right) \]

\[ = -\frac{1}{(n-1)!p!} \left( \frac{1}{p} \left( \epsilon B_p(t) - \epsilon^{n+p-2}B_p \left( \frac{1}{t} \right) \right) \right) \].

Note that in terms of generating series, the expressions of \( F \) and \( G \) are simple:

\[ F(x, y) = e^{tx+y} - \epsilon e^{x+ty} \quad \text{and} \quad G(x, y) = f(y, t) + \frac{e^{tx+y} - \epsilon e^{x+ty}}{1-\epsilon y}. \]

One recognizes in the coefficients \( G_{n,p} \) the building blocks of the period polynomials introduced by Kohnen and Zagier (9, Theorem 1). It follows from their results that, with \( \epsilon = 1 \),

\[ n! p! \left( G_{n+1,p+1} + G_{p+1,n+1} \right) = P_{n,n+p+2}^+(t) \]

is an even period polynomial for \( n, p \) even, and with \( \epsilon = -1 \),

\[ n! p! \left( G_{n+1,p+1} - G_{p+1,n+1} \right) = P_{n,n+p+2}^-(t) \]

is an odd period polynomial for \( n, p \) odd.

Note that in the case of even period polynomials, this amounts to substituting

\[ D_{\phi_i}(\phi_j) \mapsto \frac{1}{(i-1)!}(t^{i-1} - t^{j-1}) \]

in the left-hand sides of the linear relations (41) and (42), which is analogous to the result of Schneps [12].

5. Ordinary brackets in terms of Ihara brackets

We shall now give a partial inversion of Equation (28), i.e., express the Lie brackets \([\phi_{2n}, \phi_k]\) as a linear combination of Ihara brackets. Here are some examples:

\[ [\phi_2, \phi_k] = \frac{1}{n-1} \{\phi_3, \phi_{k-1}\} - \frac{1}{2} \{\phi_2, \phi_k\}. \]

\[ [\phi_4, \phi_k] = \frac{1}{17} \{\phi_3, \phi_{k+1}\} - \frac{1}{2} \{\phi_4, \phi_k\} + \frac{1}{17} \{\phi_5, \phi_{k-1}\}. \]
Rearranging the sum, one obtains

\[
[φ_6, φ_k] = - \frac{(k+2)(k+1)}{20} \{φ_3, φ_{k+3}\} + \frac{i}{k} \{φ_5, φ_{k+1}\} - \frac{1}{2} \{φ_6, φ_k\}.
\]

(51)

\[
[φ_8, φ_k] = \frac{(k+4)(k+3)(k+2)(k+1)}{30240} \{φ_3, φ_{k+5}\} - \frac{(k+2)(k+1)}{120} \{φ_5, φ_{k+3}\}
\]

\[
+ \frac{i}{k} \{φ_7, φ_{k+1}\} - \frac{1}{2} \{φ_8, φ_k\} + \frac{1}{60} \{φ_9, φ_{k-1}\}.
\]

(52)

The general formula is as follows.

**Proposition 5.1.** For each \(k \geq 2\) and \(n \geq 1\), one has

\[
[φ_{2n}, φ_k] = \sum_{i=0}^{2n} \binom{k-1+i}{k-1} \frac{b_i}{i!} \{φ_{2n-i+1}, φ_{k+i-1}\}.
\]

(53)

**Proof –** The generating series of the right-hand side of (53) is

\[
S(x, y) = \sum_{n \geq 1} \sum_{k \geq 2} x^{2n-1} y^{k-1} \sum_{i=0}^{2n-1} \frac{(k+i-2)! b_i}{(k-1)!} \{φ_{2n-i+1}, φ_{k+i-1}\}.
\]

(54)

Rearranging the sum, one obtains

\[
x ∂_y S(x, y) = \sum_{n \geq 1} \sum_{k \geq 2} x^{2n} (k-1)y^{k-2} \sum_{i=0}^{2n} \frac{(k+i-2)! b_i}{(k-1)!} \{φ_{2n-i+1}, φ_{k+i-1}\}
\]

\[
= \sum_{n \geq 1} \sum_{k \geq 2} \sum_{i=0}^{2n} \frac{(k+i-2)! b_i}{(k-2)! i!} \{φ_{2n-i+1} x^{2n-i}, φ_{k+i-1} y^{k-2}\}
\]

(55)

\[
= \sum_{i \geq 0} \frac{x b_i}{i!} \left\{ \sum_{n \geq [i/2]} φ_{2n-i+1} x^{2n-i} , \sum_{k \geq 2} \frac{(k+i-2)!}{(k-2)!} y^{k-2} φ_{k+i-1} \right\}
\]

\[
= \sum_{i \geq 0} \frac{x b_i}{i!} \left\{ \frac{1}{2} (φ(x) + (-1)^i φ(-x)), \sum_{k \geq 2} \partial_y^i y^{k+i-2} φ_{k+i-1} \right\}
\]

\[
= \frac{1}{2} \sum_{i \geq 0} (x ∂_y)^i \frac{b_i}{i!} \{φ(x) + (-1)^i φ(-x), φ(y)\},
\]

so that

\[
2x ∂_y S(x, y) = \frac{x ∂_y}{e^{x ∂_y} - 1} \{φ(x), φ(y)\} + \frac{-x ∂_y}{e^{-x ∂_y} - 1} \{φ(-x), φ(y)\}.
\]

(56)

Equation (52) gives

\[
\{φ(x), φ(y)\} = (e^{x ∂_y} + e^{y ∂_x} - 1) [φ(x), φ(y)].
\]

(57)
Substituting this expression in (56), one gets

\[ 2x \partial_y S(x, y) = x \partial_y \left( 1 + \frac{1}{e^{x \partial_y} - 1} e^{y \partial_y} \right) [\Phi(x), \Phi(y)] \]

\[ - x \partial_y \left( 1 + \frac{1}{e^{-x \partial_y} - 1} e^{-y \partial_y} \right) [\Phi(-x), \Phi(y)]. \]

\[ = x \partial_y \left( [\Phi(x) - \Phi(-x), \Phi(y)] + \frac{1}{e^{x \partial_y} - 1} [\Phi(x + y), \Phi(y)] \right) \]

\[ - \frac{1}{e^{-x \partial_y} - 1} [\Phi(-x + y), \Phi(y)] \]

\[ = x \partial_y \left( [\Phi(x) - \Phi(-x), \Phi(y)] + \frac{1}{e^{x \partial_y} - 1} ([\Phi(x + y), \Phi(y)] \right) \]

\[ + e^{x \partial_y} [\Phi(-x + y), \Phi(y)] \]

\[ = x \partial_y \left( [\Phi(x) - \Phi(-x), \Phi(y)] \right). \]

Hence,

\[ x \partial_y S(x, y) = \frac{1}{2} x \partial_y [\Phi(x) - \Phi(-x), \Phi(y)] \]

\[ = \sum_{n \geq 1} \sum_{k \geq 2} (k - 1) [\phi_{2n}, \phi_k] x^{2n} y^{k-2}. \]

Comparing the coefficients of \( x^{2n-1} y^{k-1} \) in (54) and of \( x^{2n} y^{k-2} \) in (59) for \( n \geq 1 \) and \( k \geq 2 \), one obtains (53). \( \blacksquare \)

Note that Formula (53) is very similar to Formula (40) when substituting \( n = k \) and \( p = 2n \):

\[ [\phi_{2n}, \phi_k] = \sum_{i=0}^{2n} \binom{k - 1 + i}{k - 1} \frac{b_i}{k - 1 + i} \{ \phi_{2n - i + 1}, \phi_{k + i - 1} \}. \]

\[ [\phi_{2n}, \phi_k] = \sum_{i=0}^{2n} \binom{k - 1 + i}{k - 1} \frac{b_i}{k - 1 + i} D_{\phi_{k-1+i}}(\phi_{2n+1-i}). \]

Thus, we obtain without further calculations the following analogs of (41) and (42):

**Corollary 5.2.** For all \( n \geq 1 \),

\[ \sum_{i=0}^{2n} \binom{2n - 2 + i}{2n - 2} b_i \{ \phi_{2n - i + 1}, \phi_{2n + i - 1} \} = 0. \]
For all \( n \geq 1 \) and \( p \geq 1 \),
\[
\sum_{i=1}^{2n+2p-1} \left( \binom{i-1}{i-2p+1} \frac{b_{i-2p+1}}{2p-1} + \binom{i-1}{i-2n+1} \frac{b_{i-2n+1}}{2n-1} \right) \{\phi_{2n+2p-i}, \phi_i\} = 0.
\]

Note that the left-hand sides of Equations (62) and (63) are only composed of brackets of odd \( \phi \). For example, the first equation gives
\[
0 = 9 \{\phi_5, \phi_7\} - 14 \{\phi_3, \phi_9\},
0 = 11 \{\phi_7, \phi_9\} - 21 \{\phi_5, \phi_{11}\} + 66 \{\phi_3, \phi_{13}\},
\]
whereas the second one gives
\[
0 = 195 \{\phi_{11}, \phi_7\} - 825 \{\phi_{13}, \phi_5\} - 4004 \{\phi_{15}, \phi_3\},
0 = 85 \{\phi_{13}, \phi_9\} - 442 \{\phi_{15}, \phi_7\} + 2730 \{\phi_{17}, \phi_5\} - 21216 \{\phi_{19}, \phi_3\},
\]

5.1. Period polynomials. It follows from the discussion of Section 4.2 that if one writes (62) and (63) as
\[
\sum_{i,j \text{ odd}} a_{i,j} \{\phi_i, \phi_j\} = 0,
\]
then
\[
\sum_{i,j \text{ odd}} a_{i,j} \frac{1}{(i-1)!(j-1)!} (t^{i-1} - t^{j-1})
\]
is a period polynomial as first shown by Schneps [12]. Actually, Schneps has shown that a relation of the type (66) holds iff (67) is a period polynomial. Let us recall the explanation: Ihara and Takao have proved that the space of linear relations of the form (67) has the same dimension as the space of cusp forms \( S_n(SL_2(\mathbb{Z})) \). We have seen that all the even period polynomials \( P_{2n,2m}^+(t) \) can be obtained in this way. Hence, all linear relations between the \( \{\phi_i, \phi_j\} \) (with \( i, j \) odd) are consequences of (66), so that we recover the result of Schneps.

References

[1] Furusho, H. *The multiple Zeta value algebra and the stable derivation algebra*, Publ. Res. Inst. Math. Sci. 39 n.4 (2003) 695–720.
[2] Goncharov, A. *The dihedral Lie algebras and Galois symmetries of \( \pi_1^{(t)}(\mathbb{P}^1 - \{0, \infty, \mu_n\}) \)*, Duke Math. J. 110 n.3 (2001) 397–487.
[3] Ihara, Y. *The Galois representation arising from $\mathbb{P}^1 - \{0,1,\infty\}$ and Tate twists of even degree*, Galois groups over $\mathbb{Q}$ (Berkeley, CA, 1987), 299–313 Math. Sci. Res. Inst. Publ. 16 (1989) Springer New-York-Berlin.

[4] Ihara, Y. *Braids, Galois groups and some arithmetical functions*, Proc. Int. Cong. Math., Vol I, II (Kyoto 1990), 99–120.

[5] Ihara, Y. *Automorphisms of pure sphere braid groups and Galois representations*, The Grothendieck Festschrift, Vol. II, 353–373, Prog. Math., 87, Birkhäuser Boston, Bosto, MA, 1990.

[6] Ihara, Y. *On the stable derivation algebra associated with some braid groups*, Israel J. Math. 80 (1992), no 1-2, 135–153.

[7] Ihara, Y. *On the embedding of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ into $\hat{GT}$*. London Math. Soc., Lecture Note Ser., 200, The Grothendieck theory of dessins d’enfants (Luminy 1993), 289-321, Cambridge Univ. Press, Cambridge 1994.

[8] Ihara, Y. *Some arithmetic aspects of Galois actions on the pro-p fundamental group of $\mathbb{P} - \{0,1,\infty\}$*, Proc. Symp. Pure Math. 70, 247-273, AMS Publ., 2002.

[9] Kohnen, W.; Zagier, D. *Modular Forms with Rational Periods*, Ch. 9, Modular Forms, ed. R. Rankin, Ellis Horwood, 1984.

[10] Racinet, G. *Séries génératrices non commutatives de polyzétas et associateurs de Drinfeld*, PHD Thesis, Amiens, 2000.

[11] Reutenauer, C. *Free Lie algebras*, Oxford University Press, 1993.

[12] Schneps, L. *On the Poisson bracket on the free Lie algebra in two generators*, Journal of Lie Theory 16 (2006), 019–037.

[13] Viennot, G. *Algèbres de Lie libres et Monoïdes Libres*, Lecture Notes in Mathematics, 691, 1978.

[14] Zagier, D. *Quelques conséquences surprenantes de la cohomologie de $SL_2(\mathbb{Z})$*, Leçons de mathématiques d’aujourd’hui, E. Charpentier and N. Nikolski, eds., Cassini, Paris, 2000, 99–123.

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