A REAL OPEN BOOK NOT FILLABLE BY A REAL LEFSCHETZ FIBRATION

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Abstract. A real 3- or 4-manifold has by definition an orientation preserving smooth involution acting on it. We consider Lefschetz fibrations of 4-dimensional manifolds-with-boundary and open book decompositions on their boundary in the existence of a real structure. We prove that there is a real open book which cannot be filled by a real Lefschetz fibration, although it is filled by non-real Lefschetz fibrations.

1. Introduction and basic definitions

There is a very close relation between open book decompositions and Lefschetz fibrations over $D^2$, with fibers surfaces with boundary. It is known that any open book decomposition is filled by a Lefschetz fibration if its monodromy can be factorized as a product of positive Dehn twists (work of Eliashberg, Giroux, Gompf, Loi-Piergallini, Akbulut-Ozbagci etc. See e.g. [4]). Moreover, two fibrations filling the same open book are isomorphic if and only if the two such factorizations are Hurwitz equivalent [3].

In the present work we consider Lefschetz fibrations and open book decompositions on manifolds with a real structure. A real structure on an oriented 4-manifold (respectively 3-manifold) with boundary (possibly empty) is defined to be an involution which is orientation preserving and the fixed point set of which is of dimension 2 (respectively 1), if it is not empty. Hence, if $M$ is the oriented boundary of an oriented 4-manifold $X$, a real structure on $X$ restricts to a real structure on $M$. We call a manifold $X$ together with a real structure a real manifold and denote by $c_X$ the real structure on it.

Let $(L,\pi)$ be an open book decomposition of a real 3-manifold $(M,c_M)$. Here $L \subset M$ is the binding and $\pi : M-L \to S^1$ is a fibration with fibers the page surfaces. We say that $(L,\pi)$ is a real open book decomposition, if $\rho \circ \pi = \pi \circ c_M|_{M-L}$ where $\rho : S^1 \to S^1$ is a reflection [6].

Let $p : E \to B$ be a genus-$g$ Lefschetz fibration of a 4-manifold $E$. A real Lefschetz fibration is a Lefschetz fibration together with a pair of real structures $c_E : E \to E$ and $c_B : B \to B$ commuting with the fiber structure [8].

In this article we answer a natural question that relates real open books and real Lefschetz fibrations. We show that there is a Lefschetz fibration that does not admit a real structure, while the canonical open book on its boundary is real. We also show that this real open book cannot be filled by any real Lefschetz fibration with the same fiber topology and with arbitrary number of singular fibers (Theorem [9]).

Acknowledgements. The second author would like to thank A. Degtyarev for helpful discussions. The work in this article is supported by the Scientific and Technological Research Council of Turkey [TUBITAK-ARDEB-109T671]. The second
2. Construction

Let $E$ be an oriented smooth 4-manifold and $B$ an oriented smooth surface. A genus-$g$ Lefschetz fibration of $E$ is a proper smooth projection $p : E \to B$ such that $p$ has only finitely many critical points in $\text{int}(E)$ with pairwise distinct images around which one can choose complex charts such that the projection takes the form $(z_1, z_2) \to z_1^2 + z_2^2$. Moreover, the inverse image of a regular value is a closed oriented smooth surface of genus $g$. It follows from the definition that if $\partial B \neq \emptyset$, then $\partial E = p^{-1}(\partial B)$ is a fiber bundle over $\partial B$. The notion of Lefschetz fibration can be slightly generalized to cover the case of fibers with boundary. Then $E$ turns into a manifold with corners and its boundary, $\partial E$, becomes naturally divided into two parts: $p^{-1}(\partial B)$ and $\partial B \times B$. In this case, if, in particular, $B = D^2$, then $\partial E$ admits a canonical open book decomposition with binding $p^{-1}(0)$ and projection $p|_{\partial E}$. A real Lefschetz fibration is a fibration together with a pair of real structures $c_E : E \to E$ and $c_B : B \to B$ commuting with the fiber structure.

It is known that around a singular fiber, Lefschetz fibrations are determined by the monodromy that is a single positive Dehn twist around a simple closed curve, the vanishing cycle. Algebraically, Lefschetz fibrations over $B = D^2$ can be encoded by the factorization $(t_{a_1}, \ldots, t_{a_n})$ of the monodromy $f = t_{a_n} \circ \ldots \circ t_{a_1}$ (because of the composition notation the order is reversed) along $\partial D^2$ into a product of positive Dehn twists considered up to Hurwitz equivalence. Namely, two such factorizations are called Hurwitz equivalent if one can get from one factorization to the other by a finite sequence of Hurwitz moves:

\[
\begin{align*}
&(\ldots, t_{a_{i}}, t_{a_{i+1}}, \ldots) \to (\ldots, t_{a_{i}}^{-1} \circ t_{a_{i+1}} \circ t_{a_{i}}, t_{a_{i}}, \ldots), \\
&(\ldots, t_{a_{i}}, t_{a_{i+1}}, \ldots) \to (\ldots, t_{a_{i+1}}^{-1} \circ t_{a_{i}} \circ t_{a_{i+1}}, t_{a_{i}}, \ldots);
\end{align*}
\]

and possibly a global conjugation.

In what follows, we consider a genus-1 Lefschetz fibration $p : X \to D^2$ with exactly two singular fibers. Choose a base point $d \in \partial D^2$ and consider a basis $(\gamma_1, \gamma_2)$ of $\pi_1(D^2 \setminus \{\text{critical values}\}, d)$ obtained by connecting the base point $d$ to the positively oriented simple loops, each surrounding the corresponding critical value once. Fix an identification of $F_d = p^{-1}(d)$ with $T^2 = \mathbb{R}^2/\mathbb{Z}^2$. Denote by $a$, the class on $T^2$ of $(1, 0) \in \mathbb{R}^2$ and by $b$, the class of $(0, 1)$ and consider the curves $u, v$ represented respectively by $3a + 5b$ and $a$ on $T^2$. We assume that the monodromy along $\gamma_1$ and $\gamma_2$ are given respectively by the positive Dehn twists $t_u$ and $t_v$. In other words, the corresponding singular fibers are obtained from $T^2$ by pinching the curves $u$ and $v$. The total monodromy of the fibration, thus, becomes the composition $f = t_v \circ t_u$.

Recall that $f \mapsto f_*$ defines an isomorphism from the mapping class group $\text{Map}(T^2)$ (the identity component of the space of diffeomorphisms) of the torus to the group of automorphisms of $H_1(T^2, \mathbb{Z}) \cong \mathbb{Z}a \oplus \mathbb{Z}b$. The latter is isomorphic to

\[
\text{SL}(2, \mathbb{Z}) = \left\{ [t_a] = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, [t_b] = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} : [t_a][t_b][t_a] = [t_b][t_a][t_b] \right\}
\]

The author is supported by the European Community’s Seventh Framework Programme ([FP7/2007-2013] [FP7/2007-2011]) under grant agreement no [258204].
where \([t_u]\) refers to the matrix representation of the automorphism \(t_u\). With respect to the above presentation, \([f] = [t_v][t_u]\) is given by the matrix \[
\begin{pmatrix}
-39 & 25 \\
-25 & 16
\end{pmatrix}
\].

**Proposition 1.** \(p : X \to D^2\) does not admit a real structure.

**Proof:** Suppose that \(p : X \to D^2\) admits a real structure. That is to say, there exist real structures \(c_X : X \to X\) and \(c_{D^2} : D^2 \to D^2\) such that \(p \circ c_X = c_{D^2} \circ p\). Therefore, the critical points as well as their images are invariant under the action of the real structures \(c_X\) and \(c_{D^2}\), respectively, so either they are both real or they are interchanged by the real structures. By definition of real Lefschetz fibrations, the decomposition \((t_v, t_u)\) associated to \((\gamma_1, \gamma_2)\) is Hurwitz equivalent to a decomposition \((t_{c_X(v)}, t_{c_X(u)})\) associated to \((c_{D^2}(\gamma_2), c_{D^2}(\gamma_1))\). Although, the positions of \((\gamma_1, \gamma_2)\) can be arbitrary with respect to the real structure \(c_{D^2}\), by conjugation by a power of \(f\) (such a conjugation preserves Hurwitz classes) and by moving, if necessary, the base point on the boundary, we can assume that we have only the two positions shown in Figure 1. In the former case (the case shown on the left in Figure 1), the fiber \(F_d\) can be endowed with a real structure \(c\) by pulling a real structure on a real fiber between the two real singular fibers. Up to isotopy, we can assume that vanishing cycles \(u, v\) are invariant under the action of the real structure \(c\). Lemma 2 concerns the intersection number of invariant curves and prohibits this case, (since the intersection number of \(u\) and \(v\) is equal to 5).

**Lemma 2.** The number of intersection points of two invariant curves on a real torus can only be 0, 1 or 2.

**Proof:** The real structure restricted to an invariant curve defines an action on the curve. This action can be the identity, a reflection or an antipodal involution; let us call those curves a real curve, a reflection curve and an antipodal curve.
respectively. Recall that up to equivariant diffeomorphisms, the real structure on a torus are distinguished by the number of real components that can be 0, 1 or 2. For each real structure we have

- if $c$ has no real components, then there exist two $c$-equivariant isotopy classes of antipodal curves and no classes of other types;
- if $c$ has one real component, then $T^2$ contains a unique $c$-equivariant isotopy class of non-contractible reflection curves, a unique class of antipodal curves, and a unique real curve;
- if $c$ has two real components, then $T^2$ contains no $c$-equivariant isotopy class of antipodal curves, a unique class of reflection curves, and two classes of real curves.

The result follows from the following observations: a reflection curve has two real points, so the intersection with a real curve must be at 0, 1 or 2 points. Moreover, a reflection curve must intersect an antipodal curve at 0 or 2 points, and real curves and antipodal curves are disjoint. □

**Proposition 3.** Let $M = \partial X$; then $M$ as a surface bundle is real.

**Proof:** By [7, Proposition 4], it is enough to check whether its monodromy is real, i.e. it admits a decomposition into a product of two real structures. Real structures on $T^2$ correspond to involutive elements of determinant $-1$ in $\text{GL}(2, \mathbb{Z})$. Here we have $[f] = \begin{bmatrix} -39 & 25 \\ -25 & 16 \end{bmatrix} = \begin{bmatrix} 5 & -3 \\ 8 & -5 \end{bmatrix} \begin{bmatrix} -120 & 77 \\ -187 & 120 \end{bmatrix}$, a product of two involutive elements of determinant $-1$. □

**Remark 1.** In [7], an explicit classification of real elements of $\text{SL}(2, \mathbb{Z})$ as well as a particular decomposition of a product of two real structures for each conjugacy class is given. According to [7, Proposition 4], a hyperbolic element, i.e. an element with absolute value of trace greater than 2, of $\text{SL}(2, \mathbb{Z})$ is real if and only if its cutting period cycle $[a_1, \ldots, a_{2k}]$ is odd-bipalindromic. For hyperbolic matrices, the cutting period cycle together with the sign of the trace is a complete invariant of conjugacy classes and it can be recovered from the trace and the periodic tail of the continued fraction expansion of the slope of the eigenvectors. In our specific example the cutting period cycle of $f$ is found to be $[1313]$ which is odd-bipalindromic, i.e. up to cyclic ordering, it has two palindromic pieces 1, 313 of odd length.

So far we have constructed a $T^2$-fibered Lefschetz fibration cannot be real while the $T^2$ bundle at its boundary is real. Now we show that a similar construction can be cooked up to obtain a real open book at the boundary.

Denote by $\tilde{p} : \tilde{X} \to D^2$ the Lefschetz fibration with boundary obtained from $p : X \to D^2$ above by taking out a neighborhood of a section. Note that such a section always exists for genus 1 Lefschetz fibrations [5]. The regular fibers of $\tilde{p}$ are tori with one boundary component. The fibration $\tilde{p}$ has no real structure, since otherwise $p$ would have one. However, we have:

**Proposition 4.** The canonical open book of $\tilde{M} = \partial \tilde{X}$ admits a real structure.

**Proof:** As a consequence of Lemma 1 in [3] it is enough to show that the monodromy of $\tilde{p}|_{\tilde{M}}$ is real. Note that the monodromy of $\tilde{p}|_{\tilde{M}}$ is an element of $\text{Map}(T^2 \setminus \text{int}D^2, \partial)$,
the group of relative isotopy classes of orientation preserving diffeomorphisms which are identity on the boundary.

The crucial observation is that being real in \( \text{Map}(T^2) \) is equivalent to being real in \( \text{Map}(T^2 \setminus \text{int}D^2, \partial) \). Namely, there is a short exact sequence

\[
0 \to \langle t_\partial \rangle \to \text{Map}(T^2 \setminus \text{int}D^2, \partial) \to \text{Map}(T^2) \to 0
\]
given by the central extension of \( \text{Map}(T^2) \), where \( t_\partial \) is the Dehn twist along the boundary component. Obviously, images of real elements of \( \text{Map}(T^2 \setminus \text{int}D^2, \partial) \) are real. For the converse, suppose \( \text{im}[f] = [c][c'] \) for some \( [f] \in \text{Map}(T^2 \setminus \text{int}D^2, \partial) \) and real structures \( [c], [c'] \in \text{GL}(2, \mathbb{Z}) \). If necessary, by replacing \( c \) by \( c \circ g \) and \( c' \) by \( g^{-1} \circ c' \) for some diffeomorphism \( g \), we can assume that \( c \) and hence \( c' \) leaves \( \partial T^2 \) invariant. We have \( t_\partial = c' t_\partial^{-1} c' \). Thus \( t_\partial^{-1} c \) is a real structure which we denote by \( c'' \).

As \( [f] = [c][c'][t_\partial]^k \), then for some integer \( k \), \( [f] = [c][c'][c'' \circ c''']^k = [c][c'' \circ (c'' \circ c''')^k] \). Being conjugate to a real structure, \( c'' \circ (c'' \circ c''')^k \) is a real structure. Thus, \( [f] \) is real.

With similar hands-on approach as the one above, we prove the following further result.

**Proposition 5.** Up to isomorphism preserving the identification \( T^2 \to F_d \), there are exactly two Lefschetz fibrations with exactly two singular fibers filling the canonical open book on \( M \); moreover, neither of the fibrations admit a real structure.

**Proof:** Let \( u', v' \) be two simple curves on \( T^2 \) such that \( t_{u'} \circ t_{v'} = f \). We have \( [f] = [t_{u'}][t_{v'}] = \begin{pmatrix} -39 & 25 \\ -25 & 16 \end{pmatrix} \). Suppose \( v' \) is the curve defined by \( aa + \beta b \), then a simple calculation gives \( [t_{u'}] = \begin{pmatrix} 1 - \alpha \beta & \alpha^2 \\ -\beta^2 & 1 + \alpha \beta \end{pmatrix} \). All Dehn twists are represented by matrices whose traces have absolute value equal to 2. Therefore, from \( [t_{u'}] = [t_{u'}^{-1} f] \), we get the identity \( 25\alpha^2 + 25\beta^2 - 55\alpha\beta = 25 \). This quadratic Diophantine equation has solutions \((\pm 1, 0), (0, \pm 1)\) and (the transpose of) these vectors multiplied from left by the powers of the matrix \( S = \begin{pmatrix} -3 & 5 \\ -5 & 8 \end{pmatrix} \) (see e.g the step-by-step computation of the online application [2]). Note that \( u \) and \( v \) are in the solution set. Note also that \( u', v' \), the curves with class \( b \) and \( 5a + 8b \) respectively, are in the solution set too. The pairs \( (t_{u'}, t_{v'}) \) and \( (t_{u'}, t_{v'}) \) are not Hurwitz equivalent. Indeed in that case there would be an invertible matrix \( K \) with determinant 1, such that either

\[
K^{-1}[t_{u'}^{-1}][t_{u'}][t_{u'}] = [t_{u'}] \quad \text{and} \quad K^{-1}[t_{u'}][K = [t_{u'}]
\]
or

\[
K^{-1}[t_{v'}^{-1}][t_{u'}][K = [t_{u'}] \quad \text{and} \quad K^{-1}[t_{u'}][K = [t_{u'}].
\]
However, by straightforward calculation one can show that there exists no such \( K \).

Note also that the matrices \( S \) and \( [f] \) commute; hence for any pair \((x, y)\) such that \( f = t_y \circ t_x \), the factorization \((S^k x, S^k y)\) is Hurwitz equivalent (with a fixed identification) to the factorization \((x, y)\) for every \( k \in \mathbb{Z} \).

As a consequence, we have two Hurwitz equivalence classes given by the pairs \((t_{u'}, t_{v'})\) and \((t_{u'}, t_{v'})\). To finish, note that the number of intersection points of \( u' \) and \( v' \) is 5, so the proof of Proposition 1 applies to \((u', v')\) to show that the fibrations
Theorem 6. The canonical open book on $\tilde{M}$ cannot be filled by any real Lefschetz fibration with the same fiber topology and with arbitrary number of singular fibers.

Proof: We will show that $\tilde{M}$ cannot be filled by a (real or non-real) Lefschetz fibration with the number of singular fibers different from 2. Recall that
\[ \text{Map}(T^2 \setminus \text{int}D^2, \partial) = \{ [t_a], [t_b] : [t_a][t_b][t_a] = [t_b][t_a][t_b] \} . \]
Therefore, the homomorphism
\[ \text{deg} : \text{Map}(T^2 \setminus \text{int}D^2, \partial) \to \mathbb{Z} \]
\[ t_a, t_b \mapsto 1 \]
is well-defined; hence the number of Dehn twists that may constitute a given element $h$ equals $\text{deg}(h)$. In our case, $\text{deg}(f) = 2$, so it cannot admit a factorization as a product of an arbitrary number of positive Dehn twists other than 2. Moreover, all possible factorizations into a product of two Dehn twists are shown above to be non-real. □

Remark 2. Elements admitting a factorization into a product of two Dehn twists will be studied in [1]. There the classification of such factorizations up to Hurwitz equivalence are presented and their relation to real structures are also elaborated.

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