VIRTUAL CALCULUS — PART II

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Abstract

A simultaneous extension of real numbers set and the class of real functions is discussed.
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I. Introduction

In Ref. 1 we presented a process of extending which can be applied to any set, including several sets simultaneously. In the first part of this work, we applied this process to the ordered field of real numbers $\mathbb{R}$, thus obtaining the set $\overline{\mathbb{R}}$ of virtual numbers, which contains infinitesimal and infinite quantities. That extension was then used in a reorganization of Infinitesimal Calculus.

We will now apply that extending process simultaneously to $\mathbb{R}$ and to the set of real functions, thus defining virtual functions as classes of sequences of real functions. By means of some identifications, we can consider these mathematical objects as maps between virtual numbers, i.e., as functions from subsets of $\mathbb{R}$ into $\mathbb{R}$.

Having done this, we can extend the constructions and techniques of Infinitesimal Calculus to the virtual functions. For example, we can define its derivatives and so prove that they can be calculated according to the usual derivation rules. We can also define the integration of a virtual function between two virtual numbers, which provides another virtual number as a result, and then generalize the Fundamental Theorem of Calculus to those integrals.

This procedure ends up broadening the reach of Calculus techniques, systematizing many ideas which have been used for the last decades in specific contexts. We can mention, for example, Delta Calculus created by Dirac, which has been object of various attempts of mathematical rigorous formalization. This application of virtual functions will be presented in a subsequent work.

In Sec. II we discuss the simultaneous virtual extension of real numbers and functions. In Sec. III, we show that the virtual objects thus obtained can be manipulated practically as if they were real. In Secs. IV, V and VI we extend the basic constructions of Calculus to the virtual functions. Sec. VII is dedicated to the Fundamental Theorem of Calculus. In Sec. VIII we introduce the concept of virtual sequence, and point out other possible applications of the virtual extension process to Calculus.

II. Virtual Functions

For any set $A$, we will represent the class of all functions whose domain and image are subsets of $A$ by $\mathcal{F}(A)$. In other words, $\mathcal{F}(A)$ is the set of functions $f: D \rightarrow A$ such that $D \subset A$. We will consider the empty relation $\emptyset$ as a member of $\mathcal{F}(A)$, for each set $A$ (the domain and the image of this function are both empty).
This way, $\mathcal{F}(\mathbb{R})$ denotes the set of real functions, i.e., the set of all functions whose domain and image are subsets of the ordered field $\mathbb{R}$ of real numbers. The empty function $\emptyset \in \mathcal{F}(\mathbb{R})$ should not be mistaken by the null real function, which constantly equals zero and whose domain is the whole real line.

We will now consider the simultaneous virtual extension of $\mathbb{R}$ and $\mathcal{F}(\mathbb{R})$, i.e., we will apply the process of virtual extension to the disjoint union $U$ of these two sets. The members of the virtual extension $\overline{\mathbb{R}}$ of the subset $\mathbb{R} \subset U$ will be called virtual numbers, and the members of the virtual extension $\overline{\mathcal{F}(\mathbb{R})}$ of the subset $\mathcal{F}(\mathbb{R}) \subset U$ will be called virtual functions.

A generic element of the extension $\overline{U}$ is an equivalence class of sequences on $U$, i.e., of sequences formed either by real numbers or real functions. The members of $\overline{\mathbb{R}}$ are the classes of sequences which end taking only values in $\mathbb{R}$, whereas the virtual functions are the classes of sequences which end taking only values in $\overline{\mathcal{F}(\mathbb{R})}$.

We will consider $\mathbb{R} \subset \overline{\mathbb{R}}$ according to the identification $\mathbb{R} = K(\mathbb{R})$, and $\mathcal{F}(\mathbb{R}) \subset \overline{\mathcal{F}(\mathbb{R})}$ according to the identification $\mathcal{F}(\mathbb{R}) = K[\mathcal{F}(\mathbb{R})]$, i.e., the element $x \in \mathbb{R}$ will be identified with the class $\langle x, x, \ldots \rangle$ of the constant sequence at $x$, and the function $f \in \mathcal{F}(\mathbb{R})$ will be identified with the class $\langle f, f, \ldots \rangle$ of the constant sequence at $f$. Thus, we will omit both the “bar” which distinguishes $x \in \mathbb{R}$ from $\overline{x} \in K(\mathbb{R})$ and that which distinguishes $f \in \mathcal{F}(\mathbb{R})$ from $\overline{f} \in K[\mathcal{F}(\mathbb{R})]$. In this manner, any real number can be understood as a virtual number, and every real function can be understood as a virtual function. Nevertheless, it is clear that the reciprocal statements do not hold.

Throughout this whole work, real numbers and functions will be denoted by low-case Latin letters ($x, a, f, g, \ldots$), whereas virtual numbers and functions will be denoted by low-case Greek letters ($\xi, \alpha, \phi, \psi, \ldots$). The letter ‘$\pi$’ is an exception keeping its usual mathematical meaning: $\pi \in \mathbb{R}$ is the constant ratio between the circumference and the diameter of a circle.

A virtual function $\phi \in \overline{\mathcal{F}(\mathbb{R})}$, as defined above, is not a relation between elements of two sets. Therefore, it has neither a domain nor an image according to the usual meaning of these terms. In spite of that, we can consider the virtual extension of the relation “is defined at” between real functions and real numbers, which is a relation between virtual functions and virtual numbers. That extension allows us to decide when $\phi \in \overline{\mathcal{F}(\mathbb{R})}$ is defined at $\xi \in \overline{\mathbb{R}}$. So, we will say that the domain of a virtual function $\phi \in \overline{\mathcal{F}(\mathbb{R})}$ is the set of all $\xi \in \overline{\mathbb{R}}$ which satisfy that condition. This set will be denoted by $\text{dom}\phi \subset \overline{\mathbb{R}}$, in
analogy with the notation dom$f$ $\subseteq \mathbb{R}$ for the domain of a function $f \in F(\mathbb{R})$. Thus, if $\phi$ is the class of the sequence $(f_1, f_2, \ldots) \in \Sigma[F(\mathbb{R})]$ and $\xi$ the class of $(x_1, x_2, \ldots) \in \Sigma(\mathbb{R})$, then $\xi \in \text{dom } \phi$ if and only if there exists $n \in \mathbb{N}$ such that, for every $i > n$, the function $f_i$ is defined at $x_i$.

The domain of the virtual empty function $\emptyset = \langle \emptyset, \emptyset, \emptyset, \ldots \rangle \in F(\mathbb{R})$ is the empty subset $\emptyset \subseteq \overline{\mathbb{R}}$. It is important to note that, besides this one, there are many other virtual functions with an empty domain. For instance, if $\text{id}_{\mathbb{R}} \in F(\mathbb{R})$ is the identity function on $\mathbb{R}$, then the sequence of functions:

$$(\text{id}_{\mathbb{R}}, \emptyset, \text{id}_{\mathbb{R}}, \emptyset, \ldots) \in \Sigma[F(\mathbb{R})]$$

represents a virtual function, different from $\emptyset \in F(\mathbb{R})$ whose domain is empty. It is easy to verify that the domain of $\phi = \langle f_1, f_2, \ldots \rangle \in F(\mathbb{R})$ is empty when, for every $n \in \mathbb{N}$, there exists $i > n$ such that $f_i = \emptyset$. Also, it is not difficult to see that if the domain of $\phi = \langle f_1, f_2, \ldots \rangle$ is equal to $\overline{\mathbb{R}}$, then there exists $n \in \mathbb{N}$ such that $\text{dom } f_i = \mathbb{R}$ for every $i > n$.

We can consider the evaluation of a real function $f$ at a real number $x$ in its domain $\text{dom } f \subseteq \mathbb{R}$ as a map which sends each pair $(f, x)$, with $f \in F(\mathbb{R})$ and $x \in \text{dom } f$, to a member $f(x) \in \mathbb{R}$. The virtual extension of this map defines the evaluation of a virtual function, which allows us to compute the image by $\phi$ of any element in its domain. Putting it another away, if $\phi = \langle f_1, f_2, \ldots \rangle$ is defined at $\xi = \langle x_1, x_2, \ldots \rangle$, then $\phi(\xi) \in \overline{\mathbb{R}}$ is the class of sequences which end taking the values $f_i(x_i)$.

Following the convention introduced in the beginning of this section, we will denote the set of all functions whose domain and image are subsets of $\overline{\mathbb{R}}$ by $\overline{F(\mathbb{R})}$, i.e., $\overline{F(\mathbb{R})}$ is the set of functions between members of the virtual extension of $\overline{\mathbb{R}}$. This set should not be confused with the virtual extension $\overline{F(\mathbb{R})}$ of the set of functions between members of $\mathbb{R}$.

According to that notation, the above defined “virtual evaluation” assigns a function $[\phi] \in \overline{F(\mathbb{R})}$ to each virtual function $\phi \in \overline{F(\mathbb{R})}$. The domain of $[\phi]$ is exactly the set $\text{dom } \phi \subseteq \overline{\mathbb{R}}$:

$$[\phi] : \text{dom } \phi \to \overline{\mathbb{R}}.$$ 

This assignment is such that:

If $\phi$ and $\psi$ are two distinct virtual functions, both having a non-empty domain, then the functions $[\phi]$ and $[\psi]$ are also distinct.
Proof: Let \((f_1, f_2, \ldots) \in \Sigma[\mathcal{F}(\mathbb{R})]\) be a representative sequence of \(\phi\), and \((g_1, g_2, \ldots) \in \Sigma[\mathcal{F}(\mathbb{R})]\) a representative sequence of \(\psi\). If \(\phi \neq \psi\) then, for every \(n \in \mathbb{N}\), there exists \(i > n\) such that \(f_i \neq g_i\). Thus, for every \(n \in \mathbb{N}\), there exist \(i > n\) and \(x_i \in \mathbb{R}\) such that:

\[
\begin{align*}
    x_i &\in \text{dom} f_i \text{ and } x_i \notin \text{dom} g_i; \quad \text{or} \\
    x_i &\notin \text{dom} f_i \text{ and } x_i \in \text{dom} g_i; \quad \text{or} \\
    x_i &\in \text{dom} f_i \text{ and } x_i \in \text{dom} g_i \text{ and } f_i(x_i) \neq g_i(x_i).
\end{align*}
\]

Therefore, at least one of the three conditions below is satisfied:

(a) For every \(n \in \mathbb{N}\), there exist \(i > n\) and \(x_i \in \mathbb{R}\) such that \(x_i \in \text{dom} f_i\) and \(x_i \notin \text{dom} g_i\).

(b) For every \(n \in \mathbb{N}\), there exist \(i > n\) and \(x_i \in \mathbb{R}\) such that \(x_i \notin \text{dom} f_i\) and \(x_i \in \text{dom} g_i\).

(c) For every \(n \in \mathbb{N}\), there exist \(i > n\) and \(x_i \in \mathbb{R}\) such that \(x_i \in \text{dom} f_i\), \(x_i \in \text{dom} g_i\) and \(f_i(x_i) \neq g_i(x_i)\).

If (a) holds then there exists \(\xi \in \text{dom} \phi \neq \emptyset\) which does not belong to \(\text{dom} \psi\). On the other hand, if (b) holds then there exists \(\xi \in \text{dom} \psi \neq \emptyset\) which does not belong to \(\text{dom} \phi\). Finally, condition (c) implies the existence of a virtual number \(\xi\) which belongs to the domain of both functions, and for which \([\phi](\xi) \neq [\psi](\xi)\).

In any of these three cases we have \([\phi] \neq [\psi]\). \(\blacksquare\)

Having that result in mind, we will go on by identifying all virtual functions with an empty domain with the empty virtual function \(\emptyset \in \overline{\mathcal{F}(\mathbb{R})}\). So, we will be allowed to say:

*Two virtual functions \(\phi\) and \(\psi\) are equal if and only if they have the same domain \(D \subset \overline{\mathbb{R}}\) and \([\phi](\xi) = [\psi](\xi), \text{ for each } \xi \in D\).*

Therefore, we can see there is no significant distinction between virtual functions \(\phi \in \overline{\mathcal{F}(\mathbb{R})}\) and the corresponding \([\phi] \in \mathcal{F}(\overline{\mathbb{R}})\) anymore. So, they will be identified to each other, not to overload notation needlessly. In other words, we will consider:

\[
\overline{\mathcal{F}(\mathbb{R})} \subset \mathcal{F}(\overline{\mathbb{R}}).
\]

In this sense, we will simply write:

\[
\phi: \text{dom} \phi \to \overline{\mathbb{R}}
\]

\[
\xi \mapsto \phi(\xi),
\]

5
and treat virtual functions as relations between virtual numbers. For example, we define the *image* of a virtual function $\phi$ as the set of virtual numbers which are the image of some member of its domain. That image will be denoted by $\text{im}\phi \subset \mathbb{R}$, as an analogy to the notation $\text{im}f \subset \mathbb{R}$ for the image of a real function $f$.

It is important to note, however, that the inclusion $\mathcal{F}(\mathbb{R}) \subset \mathcal{F}(\overline{\mathbb{R}})$ is proper, i.e., there exist functional relations in $\mathcal{F}(\overline{\mathbb{R}})$ which do not correspond, through the above identification, to any virtual function of $\mathcal{F}(\overline{\mathbb{R}})$.

To prove that statement, let us consider the function $\Phi: \mathbb{R} \to \mathbb{R}$ given by:

$$
\Phi(\xi) = \begin{cases} 
1 & \text{if } \xi = 2 \text{ or } \xi = -2; \\
0 & \text{otherwise.}
\end{cases}
$$

If $\Phi$ were a virtual function, then any representative sequence $(f_1, f_2, \ldots) \in \Sigma[\mathcal{F}(\mathbb{R})]$ must end taking the value 1 at $x = 2$ and $x = -2$, i.e., there would exist $n \in \mathbb{N}$ such that, for every $i > n$, $f_i(2) = f_i(-2) = 1$. But we would then have $\Phi(\pm 2) = 1$, and not $\Phi(\pm 2) = 0$. (According to the notation established in Ref. 2, the virtual $\pm 2 \in \overline{\mathbb{R}}$ is the class of the sequence $(-2, +2, -2, +2, \ldots) \in \Sigma(\mathbb{R})$.)

It is also easy to verify that the identifications:

$$
\mathcal{F}(\mathbb{R}) = K[\mathcal{F}(\mathbb{R})] \subset \mathcal{F}(\overline{\mathbb{R}}) \subset \mathcal{F}(\mathbb{R})
$$

are compatible with the virtual extension of real functions considered as relations between members of $\mathbb{R} \subset \mathbb{U}$, not as elements of $\mathcal{F}(\mathbb{R}) \subset \mathbb{U}$.

### III. Construction of Virtual Functions

The aim of this section is to show that the identification $\mathcal{F}(\overline{\mathbb{R}}) \subset \mathcal{F}(\overline{\mathbb{R}})$ is also compatible with the usual “pointwise” manipulations of real functions: compositions and algebraic operations. This compatibility will allow us to construct and represent virtual functions “as if they were real”.

We can consider the composition of real functions as a map which assigns to each pair $f, g \in \mathcal{F}(\mathbb{R})$ a third function $(f \circ g) \in \mathcal{F}(\mathbb{R})$ given by $(g \circ f)(x) = g[f(x)]$. The domain of that composite function is the set (eventually empty) of all values of $x \in \text{dom}f$ for which $f(x) \in \text{dom}g$. Thus, the virtual extension of this map defines the *composition of virtual functions*: an operation that assigns to each pair of virtual functions $\phi, \psi \in \mathcal{F}(\overline{\mathbb{R}})$ a third virtual function $(\psi \circ \phi) \in \mathcal{F}(\overline{\mathbb{R}})$, which will be called *composite of $\phi$ and $\psi*.
Since the composition of real functions is done “pointwisely”, the above definition of 
composition of virtual functions is perfectly compatible with the identifications:

\[ \mathcal{F}(\mathbb{R}) = \mathbb{R}[\mathcal{F}(\mathbb{R})] \subset \mathcal{F}(\mathbb{R}) \subset \mathcal{F}(\mathbb{R}) , \]

and the domain of the composite \( (\psi \circ \phi) \) is the set (eventually empty) of all values of \( \xi \in \text{dom}\phi \) for which \( \phi(\xi) \in \text{dom}\psi \). For these values we have \( (\psi \circ \phi)(\xi) = \psi[\phi(\xi)] \).

This shows that the subset \( \mathcal{F}(\mathbb{R}) \subset \mathcal{F}(\mathbb{R}) \) of virtual functions is closed with respect to the composition operation. It is also not difficult to verify that if a virtual function is \( \text{inversible} \) then its \( \text{inverse} \) is also a virtual function.

Analogously, we can consider the addition of real functions as a map which assigns to each pair \( f, g \in \mathcal{F}(\mathbb{R}) \) a third function \( (f + g) \in \mathcal{F}(\mathbb{R}) \) given by \( (f + g)(x) = f(x) + g(x) \). The domain of this sum function is the intersection (eventually empty) of the domains of \( f \) and \( g \). Thus, the virtual extension of this map defines an \( \text{addition of virtual functions} \): a map which assigns to each pair of virtual functions \( \phi, \psi \in \mathcal{F}(\mathbb{R}) \) a third virtual function \( (\phi + \psi) \in \mathcal{F}(\mathbb{R}) \), which will be called \( \text{sum of} \ \phi \ \text{and} \ \psi \).

Since the real functions addition is also done “pointwisely”, the above definition of 
virtual functions addition is also perfectly compatible with the identifications:

\[ \mathcal{F}(\mathbb{R}) = \mathbb{R}[\mathcal{F}(\mathbb{R})] \subset \mathcal{F}(\mathbb{R}) \subset \mathcal{F}(\mathbb{R}) , \]

and the domain of the sum \( (\phi + \psi) \) is the intersection (eventually empty) of the domains of \( f \) and \( g \). So we have \( (\phi + \psi)(\xi) = \phi(\xi) + \psi(x) \), for every \( \xi \in \text{dom}(\phi + \psi) \).

This shows that the subset \( \mathcal{F}(\mathbb{R}) \subset \mathcal{F}(\mathbb{R}) \) of virtual functions is closed with respect to the addition operation.

Proceeding this way, we can define the remainder algebraic operations (subtraction, 
multiplication, division and exponentiation) with virtual functions, verify its compatibility 
with the above identifications, and the closure of the subset \( \mathcal{F}(\mathbb{R}) \subset \mathcal{F}(\mathbb{R}) \) of virtual 
functions with respect to these operations.

When we use Infinitesimal Calculus, most of the time we deal with functions con-
structed by successive applications of compositions and algebraic operations on a set of 
functions named \( \text{elementary} \): constant functions, exponentials and logarithmics, direct and 
inverse trigonometric functions.

The same procedure can be used to handle virtual functions: we take a set of virtual 
functions considered “elementary” and, from them, we construct others applying compositions 
and algebraic operations. That set of “elementary virtual functions” might include
the exponentials, logarithmics, trigonometrics (direct and inverse), considered virtual functions according to the identification:

$$\mathcal{F}(\mathbb{R}) = K[\mathcal{F}(\mathbb{R})] \subset \overline{\mathcal{F}(\mathbb{R})}$$

as well as the constant virtual functions, i.e., those which satisfy the virtual extension of the attribute (of real functions) “to be constant”. Those constant virtual functions are the classes in $\overline{\mathcal{F}(\mathbb{R})}$ which can be represented by sequences $(f_1, f_2, \ldots) \in \Sigma[\mathcal{F}(\mathbb{R})]$ such that all $f_i$ are constant functions, but not necessarily equal among them. In other words, a virtual function $\phi: \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ is constant when there exists a virtual number $\alpha \in \overline{\mathbb{R}}$ such that $\phi(\xi) = \alpha$, for every $\xi \in \overline{\mathbb{R}}$. For instance, $\kappa(\xi) = \infty$ defines a constant virtual function.

Having included these constant virtual functions among the elementary functions, we can easily construct many non-real functions of $\mathcal{F}(\overline{\mathbb{R}})$ which will automatically belong to $\overline{\mathcal{F}(\mathbb{R})}$. As an illustration, the expression:

$$\phi(\xi) = \frac{e^{(\xi^2 - \infty^2)}}{\cos(\pi \partial \xi)}$$

specifies a virtual function $\phi \in \overline{\mathcal{F}(\mathbb{R})}$ which has been formally defined as the class of the sequence $(f_1, f_2, \ldots)$ given by:

$$f_n(x) = \frac{e^{(x^2 - n^2)}}{\cos \left( \frac{\pi x}{n} \right)}.$$

We can calculate $\phi(\xi)$, for specific values of $\xi$, by direct substitution:

$$\phi(0) = \frac{e^{(0^2 - \infty^2)}}{\cos(\pi \partial 0)} = \frac{e^{-\infty^2}}{\cos 0} = e^{-\infty^2},$$

or:

$$\phi(\infty) = \frac{e^{(\infty^2 - \infty^2)}}{\cos(\pi \partial \infty)} = \frac{e^{0}}{\cos \pi} = -1.$$

However, this function $\phi$ is not defined at some virtual numbers $\xi \in \overline{\mathbb{R}}$. For example, if $\xi = \infty / 2$ then $\cos(\pi \partial \xi) = \cos(\pi / 2) = 0$, so the denominator of the above fraction nullifies itself.

We will adopt, exactly as we do for real functions, the following convention:

*Every time we define a virtual function by an expression, without explicitly indicating its domain, one should understand that it is the set of all virtual numbers for which that expression has mathematical meaning.*
This way, we will not need to explicitly state domains when this is not relevant. For example, let us consider the virtual function given simply by:

\[ \psi(\xi) = \frac{\infty}{1 + \infty^2 \xi^2}. \]

Since this expression is well defined for any \( \xi \in \mathbb{R} \), it is understood that:

\[ \psi: \mathbb{R} \rightarrow \mathbb{R}. \]

Formally, \( \psi \in \mathcal{F}(\mathbb{R}) \) is the class represented by the sequence \((g_1, g_2, \ldots)\) of the real functions:

\[ g_n(x) = \frac{n}{1 + n^2 x^2}. \]

In our “set of elementary virtual functions” we can also include other members of \( \mathcal{F}(\mathbb{R}) \) defined by explicit presentation of a representative sequence \((f_1, f_2, \ldots)\) of real functions. For example, we can take:

\[ f_n(x) = \begin{cases} n/2, & \text{if } |x| < 1/n; \\ 0, & \text{if } |x| \geq 1/n, \end{cases} \]

and then make \( \chi = (f_1, f_2, \ldots) \in \mathcal{F}(\mathbb{R}) \). For this virtual function \( \chi: \mathbb{R} \rightarrow \mathbb{R} \) we have:

\[ \chi(\xi) = \begin{cases} \infty/2, & \text{if } |\xi| < \partial; \\ 0, & \text{if } |\xi| \geq \partial. \end{cases} \]

This pair of clauses, nevertheless, is not sufficient to define \( \chi(\xi) \) for every \( \xi \in \mathbb{R} \), since there are virtual numbers \( \xi \) which do not satisfy either condition above.

IV. Continuity

We will now extend the notion of continuity to virtual functions.

We can interpret continuity of a function \( f \in \mathcal{F}(\mathbb{R}) \) at a point \( x \in \mathbb{R} \) as a relation between \( f \) and \( x \). So, the virtual extension of this relation allows us to define continuity of virtual function \( \phi \in \mathcal{F}(\mathbb{R}) \) at a point \( \xi \in \text{dom}\phi \). If \( \phi \) is the class of the sequence \((f_1, f_2, \ldots) \in \Sigma[\mathcal{F}(\mathbb{R})] \), and \( \xi \) the class of \((x_1, x_2, \ldots) \in \Sigma(\mathbb{R}) \), then \( \phi \) is continuous at \( \xi \) when there exists \( n \in \mathbb{N} \) such that \( f_i \) is continuous at \( x_i \), for every \( i > n \).

Moreover, we will say simply that a virtual function is continuous when it is continuous at each point of its domain (as we do for real functions). It is easy to see that this attribute
of virtual functions is equivalent to the virtual extension of the attribute “is continuous” for real functions.

We know from Calculus that the composite of two continuous real functions is also a continuous (real) function. Hence, we have from the Virtual Extension Theorem (VET, Ref. 1) that the composite of two continuous virtual functions is also a continuous virtual function.

The VET also shows that a virtual function obtained from two other continuous virtual functions by an algebraic operation is necessarily continuous.

The attribute of virtual functions “to be constant”, defined in the previous section, is the virtual extension of the attribute “to be constant” applied to real functions. Since any constant real function is continuous, the VET guarantees that every constant virtual function is continuous, even when the constant is a virtual number.

Still from the VET, we know that the exponential, logarithmic, and trigonometric functions (direct and inverse), considered as virtual functions by the identification $\mathcal{F}(\mathbb{R}) = K[\mathcal{F}(\mathbb{R})] \subset \overline{\mathcal{F}(\mathbb{R})}$, are all continuous.

Therefore, all virtual functions constructed from these “elementary” ones, through successive applications of compositions and algebraic operations, are continuous.

For instance, the virtual functions $\phi$ and $\psi$ viewed in the previous section are continuous:

$$\phi(\xi) = \frac{e^{(\xi^2 - \infty^2)}}{\cos(\pi \partial \xi)} \quad \text{and} \quad \psi(\xi) = \frac{\infty}{1 + \infty^2 \xi^2}.$$

This latter function $\psi : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ is continuous at each point of its domain, including at real values of its argument, for which:

$$\psi(x) \approx \begin{cases} \infty, & \text{if } x = 0; \\ 0, & \text{if } x \neq 0. \end{cases}$$

However, the virtual function $\chi : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ (also seen in the previous section) is not continuous at $\xi = \partial$, since the corresponding functions $f_n$ are not continuous at $x = 1/n$.

The continuity of a real function $f$ at a point $x$ of its domain was defined in the first part of this work by the condition:

$$\alpha \approx x \Rightarrow f(\alpha) \approx f(x).$$

It is important to note that continuity of virtual functions, as defined in this section, does not require that this condition hold. For example, evaluating the function $\psi$ at the
infinitesimal $\alpha = \sqrt{\partial}$ we get:

$$\psi(\sqrt{\partial}) = \frac{\infty}{1 + \infty^2 \partial} = \frac{\infty}{1 + \infty} \approx 1,$$

i.e., $\sqrt{\partial} \approx 0$ but $\psi(\sqrt{\partial}) \neq \psi(0)$.

Analogously, defining uniform continuity of a virtual function $\phi$ by the extension of the attribute “to be uniformly continuous”, we will not have necessarily that:

$$\alpha \approx \beta \Rightarrow \phi(\alpha) \approx \phi(\beta).$$

V. Derivation

Our goal in this section is to define the derivative of virtual functions, and show that the usual derivation algorithm of Calculus can be used “as if virtual functions were real”.

We can interpret the derivation process of functions in $F(\mathbb{R})$ as a map which assigns a function $f' \in F(\mathbb{R})$ to each $f \in F(\mathbb{R})$, the domain of $f'$ being the set (eventually empty) of points at which $f$ is derivable. Thus, the virtual extension of this map allows us to define the derivative $\phi' \in \overline{F(\mathbb{R})}$ of any virtual function $\phi \in \overline{F(\mathbb{R})}$. If $\phi$ is the class of the sequence $(f_1, f_2, \ldots) \in \Sigma[F(\mathbb{R})]$ then its derivative $\phi' \in \overline{\mathbb{R}}$ is the class of the sequence $(f'_1, f'_2, \ldots) \in \Sigma[F(\mathbb{R})]$.

We will say that $\phi \in \overline{F(\mathbb{R})}$ is derivable at $\xi \in \overline{\mathbb{R}}$ when $\xi \in \text{dom}\phi'$. We can alternatively interpret the derivability of a function $f \in F(\mathbb{R})$ at a point $x \in \mathbb{R}$ as a relation between $f$ and $x$, and then define the derivability of a virtual function $\phi \in \overline{F(\mathbb{R})}$ at a point $\xi \in \text{dom}\phi$ through the virtual extension of this relation. These two definitions are equivalent: if $\phi$ is the class of the sequence $(f_1, f_2, \ldots) \in \Sigma[F(\mathbb{R})]$, and $\xi$ is the class of $(x_1, x_2, \ldots) \in \Sigma(\mathbb{R})$, then $\phi$ is derivable at $\xi$ when there exists $n \in \mathbb{N}$ such that $f_i$ is derivable at $x_i$ for every $i > n$. In this case, $\phi'(\xi) \in \overline{\mathbb{R}}$ is exactly the class of sequences which end taking the values $f'_i(x_i)$.

Furthermore, we will say that a virtual function is derivable when it is derivable at each point in its domain (as we do for real functions). Clearly, this definition is also equivalent to the direct virtual extension of the attribute “to be derivable”.}

In the first part of this work we saw a stronger condition of differentiability for real functions than simple derivability, and we proved that this condition is equivalent to the demand for continuity of the derivative at the considered point. We define the attribute “to be differentiable” for virtual functions through the extension of the corresponding attribute for real functions. So, the VET shows that a virtual function is differentiable at a point
ξ in its domain if and only if it is derivable at ξ, and its derivative is continuous at this point.

The VET also guarantees that the “virtual derivation” process applied to real functions provides the same result of “real derivation”. Besides, every constant real function is derivable and its derivative is the null function. So every constant virtual function is derivable and its derivative is the null function, even when the constant is a virtual number. For example, if κ(ξ) = ∞ for every ξ ∈ \(\mathbb{R}\), then κ′(ξ) = 0 for every ξ ∈ \(\mathbb{R}\).

The derivation rules are assertions about the derivation process considered as a map from \(\mathcal{F}(\mathbb{R})\) into \(\mathcal{F}(\mathbb{R})\). According to the preceding definitions, the VET guarantees that all of them also hold for virtual functions. For instance:

\[
(\phi + \psi)' = \phi' + \psi'
\]

\[
(\phi \psi)' = \phi' \psi + \phi \psi'
\]

\[
(\phi \circ \psi)' = (\phi' \circ \psi)\psi'.
\]

Therefore, the usual derivation algorithm of Calculus can be used for virtual functions exactly as if they were real. Doing so, we should treat virtual constants (present in an expression which defines a virtual function) exactly as we do real constants (present in expressions which define real function). Thus, if

\[
\psi(\xi) = \partial \xi^∞ + \infty^2
\]

then

\[
\psi'(\xi) = \partial \infty \xi^{(\infty - 1)} + 0 = \xi^{(\infty - 1)}.
\]

As an illustration involving many derivation rules, the chain rule inclusive, we have:

\[
\phi(\xi) = \frac{e^{(\xi^2 - \infty^2)}}{\cos(\pi \partial \xi)}
\]

implies:

\[
\phi'(\xi) = \frac{e^{(\xi^2 - \infty^2)}(2\xi)\cos(\pi \partial \xi) + e^{(\xi^2 - \infty^2)} \sin(\pi \partial \xi)\pi \partial}{\cos^2(\pi \partial \xi)}
\]

\[
= \frac{e^{(\xi^2 - \infty^2)}}{\cos^2(\pi \partial \xi)} [2\xi \cos(\pi \partial \xi) + \pi \partial \sin(\pi \partial \xi)].
\]

In Ref. 2 we observed that the Leibnizian notation for derivatives of real functions can be interpreted in \(\mathbb{R}\) as a quotient between infinitesimals. This notation can also be
generalized to represent the virtual derivation process earlier defined: if a virtual function
is specified by an expression which provides the values of its dependent virtual variable \( v \)
from the values of its independent virtual variable \( \xi \), then the symbol:

\[
\frac{dv}{d\xi}
\]

will be used to denote the dependent virtual variable of its derivative. For instance, if

\[ v = \frac{\infty}{1 + \infty^2 \xi^2} \]

then:

\[
\frac{dv}{d\xi} = \frac{-\infty}{(1 + \infty^2 \xi^2)^2} \left(2\infty^2 \xi \right) = \frac{-2\infty^3 \xi}{(1 + \infty^2 \xi^2)^2}.
\]

However, it is important to notice that the symbols \( dv \) and \( d\xi \) do not denote “in-
finitesimal variations” of \( v \) or \( \xi \), since these variables are virtual numbers themselves.
So, if we are using the Leibnizian notation for the derivative of a virtual function \( v = \phi(\xi) \)
then we should interpret the “quotient” \( (dv/d\xi) \) in a purely symbolic way, and not as
a quotient between infinitesimals (contrary to the \( dx \) e \( dy \) present in the Leibnizian
notation for the derivatives of real functions\(^2\)). Therefore, we write:

\[
\frac{dv}{d\xi} = \phi'(\xi) \quad \text{and not} \quad \frac{dv}{d\xi} \approx \phi'(\xi),
\]

whereas for the derivative of a generic real function \( y = f(x) \) we write:

\[
\frac{dy}{dx} \approx f'(x) \quad \text{and not} \quad \frac{dy}{dx} = f'(x).
\]

The derivative of a real function \( f \) at a point \( x \) in its domain was defined in the first
part of this work as being the real number \( f'(x) \) which satisfies the condition:

\[
\alpha \sim x \Rightarrow f'(x) \approx \frac{f(\alpha) - f(x)}{\alpha - x}.
\]

Nevertheless, the derivative of a virtual function at a point in its domain, as defined in
this very section, does not necessarily satisfy this condition. For example, calculating the
derivative of the above function \( v = \phi(\xi) \) at \( \xi = 0 \) we get:

\[
\phi'(0) = \left. \frac{dv}{d\xi} \right|_{\xi=0} = \frac{-2\infty^3 0}{(1 + \infty^2 0^2)^2} = 0,
\]

13
(which was expected, since $\phi$ is an even function) whereas, for $x = 0$ and $\alpha = \partial$, we have:

$$\frac{\phi(\alpha) - \phi(x)}{\alpha - x} = \frac{\frac{\infty}{1 + \infty^2 \partial^2} - \frac{\infty}{1 + \infty^2 0^2}}{\partial - 0}$$

$$= \frac{\infty}{1 + 1} - \frac{\infty}{1 + 0}$$

$$= \infty \left( \frac{\infty}{2} - \infty \right)$$

$$= -\frac{\infty^2}{2}.$$

So $\partial \sim 0$ but

$$\phi'(0) \neq \frac{\phi(\partial) - \phi(0)}{\partial - 0}.$$

VI. Integration

We will now define the integral of virtual functions. The integration theory on the real line due to Riemann is sufficient for our purposes. So, from here on, integrability and integrals of real functions between two real numbers should be understood in this sense.

For any real number $a$, we will consider that a real function is Riemann-integrable "between $a$ and $a$" when $f$ is defined at $a$, and in this case we shall have:

$$\int_{a}^{a} f(x) \, dx = 0.$$ 

Besides, we will not require that the lower integration limit be less than the upper one. We will always suppose that:

$$\int_{a}^{b} f(x) \, dx = -\int_{b}^{a} f(x) \, dx.$$ 

We can consider the integrability of a real function $f \in \mathcal{F}(\mathbb{R})$ between two points $a, b \in \mathbb{R}$ as a relation between $f$ and those two real numbers, and so define the integrability of a virtual function $\phi \in \overline{\mathcal{F}}(\mathbb{R})$ between two virtual points $\alpha, \beta \in \overline{\mathbb{R}}$ through the virtual extension of this relation. In other words, if $\phi$ is the class of the sequence $(f_1, f_2, \ldots) \in \Sigma[\mathcal{F}(\mathbb{R})]$, $\alpha$ is the class of $(a_1, a_2, \ldots) \in \Sigma(\mathbb{R})$, and $\beta$ is the class of $(b_1, b_2, \ldots) \in \Sigma(\mathbb{R})$, then $\phi$ is integrable between $\alpha$ and $\beta$ when there exists $n \in \mathbb{N}$ such that, for every $i > n$, the function $f_i$ is integrable between $a_i$ and $b_i$.

Moreover, we will merely say that a virtual function is integrable when it is integrable between any pair of virtual numbers in its domain.
If $\phi \in \mathcal{F}(\mathbb{R})$ is integrable between $\alpha$ and $\beta$, then we define the virtual integral

$$\int_{\alpha}^{\beta} \phi(\xi) \, d\xi$$

through the virtual extension of the map which assigns the number

$$\int_{a}^{b} f(x) \, dx$$

to each triple $(f, a, b)$ with $f$ integrable between $a$ and $b$. The use of a Greek letter as an integration variable indicates the integral is virtual.

It is easy to see that if $\phi = \langle f_1, f_2, \ldots \rangle$ is integrable between $\alpha = \langle a_1, a_2, \ldots \rangle$ and $\beta = \langle b_1, b_2, \ldots \rangle$ then the integral of $\phi$ between $\alpha$ and $\beta$ is the class of sequences which end taking the values:

$$\int_{a_i}^{b_i} f_i(x) \, dx.$$  

Thus, it should be clear that if $f$ is a real function integrable between $a$ and $b$ then:

$$\int_{a}^{b} f(\xi) \, d\xi = \int_{a}^{b} f(x) \, dx,$$

i.e., it does not matter whether we use a real (Latin letter) or a virtual (Greek letter) integration variable when the integrand and the integration limits are real.

We will say that a virtual function is defined between two virtual numbers when those three objects satisfy the virtual extension of the corresponding relation between a real function and two real numbers. Analogously, we will say that a virtual function is continuous between two virtual numbers when those three objects satisfy the virtual extension of the corresponding relation between a real function and two real numbers.

We know that if a real function is defined and continuous between two real numbers then it is integrable between those two numbers. So, the VET guarantees that every virtual function defined and continuous between two virtual numbers is integrable between those two numbers.

Therefore, the symbols:

$$\int_{0}^{\theta} \frac{\infty}{1 + \infty^2 \xi^2} \, d\xi$$

and

$$\int_{-\infty}^{\infty} e^{\theta \tau} \, d\tau$$
are perfectly defined (they are just two particular examples). The last one should be understood as a virtual variable which depends on $\xi$ through a virtual function implicitly indicated. That function is the class of sequences which end equals to $(f_1, f_2, \ldots) \in \Sigma[\mathcal{F}(\mathbb{R})]$ given by:

$$f_n(x) = \int_{-n^2}^{n^2} e^{nt} dt.$$  

It is also known that the sum and the product of two integrable real functions are also integrable. So, the VET guarantees that the sum and the product of two integrable virtual functions are also integrable.

In the same way, we can use the VET to show that virtual integrals satisfy many of the properties expected from an integral, as, for example, the additivity with respect to the integration interval:

$$\int_{\alpha}^{\gamma} \phi(\xi) \, d\xi = \int_{\alpha}^{\beta} \phi(\xi) \, d\xi + \int_{\beta}^{\gamma} \phi(\xi) \, d\xi.$$  

We will not list all those properties here, since it is more convenient to use the VET (which is always at hand) whenever these properties are required. However, the basic fact from Differential and Integral Calculus which relates derivatives and integrals deserves a more detailed discussion.

VII. The Fundamental Theorem of Calculus

We will say that $\psi \in \mathcal{F}(\mathbb{R})$ is a primitive of $\phi \in \mathcal{F}(\mathbb{R})$ when $\psi' = \phi$. The set of all primitives of a virtual function will be called its indefinite integral (as we do for real functions).

The VET guarantees that two primitives of a continuous virtual function $\phi: \mathbb{R} \to \mathbb{R}$ differ by a (virtual) additive constant. Thus, we will represent the indefinite integral of that function by:

$$\int \phi(\xi) \, d\xi = \psi(\xi) + \kappa,$$

where $\psi: \mathbb{R} \to \mathbb{R}$ is a particular primitive of $\phi$ and $\kappa \in \mathbb{R}$ a generic constant, being understood that we are referring to the set of virtual functions which can be written in that manner.

According to those definitions and conventions, the VET shows that the integration techniques of traditional Calculus extend to virtual functions, as well as the traditional notations associated to them, for which it is enough to change from Latin to Greek letters.
For instance, to calculate
\[ \int \frac{\infty}{1 + \infty^2 \xi^2} \, d\xi \]
we make the substitution of variables \( \mu = \infty \xi \), so \( d\mu = \infty d\xi \) and:
\[ \int \frac{\infty}{1 + \infty^2 \xi^2} \, d\xi = \int \frac{d\mu}{1 + \mu^2} = \arctan \mu + \kappa = \arctan(\infty \xi) + \kappa. \]

Applying the VET to the first form of the Fundamental Theorem of Calculus we get:
*If \( \phi \in \mathcal{F}(\mathbb{R}) \) is defined and continuous between \( \alpha \) and \( \beta \), then, for every \( \xi \) between \( \alpha \) and \( \beta \):
\[
\frac{d}{d\xi} \int_{\alpha}^{\xi} \phi(\tau) \, d\tau = \phi(\xi).
\]
*In other words, the virtual function defined by:
\[
\psi(\xi) = \int_{\alpha}^{\xi} f(\tau) \, d\tau
\]
is a primitive of \( \phi \). In particular, every continuous virtual function \( \phi: \mathbb{R} \to \mathbb{R} \) admits a primitive.

The VET also provides the “virtual second form” of the Fundamental Theorem of Calculus:
*If \( \phi \in \mathcal{F}(\mathbb{R}) \) is defined and continuous between \( \alpha \) and \( \beta \), and \( \psi \) is a primitive of \( \phi \), then:
\[
\int_{\alpha}^{\beta} \phi(\xi) \, d\xi = \psi(\beta) - \psi(\alpha).
\]

Some examples:
\[
\frac{d}{d\xi} \int_{-\infty}^{\xi} e^{\partial \tau} \, d\tau = e^{\partial \xi} \frac{d}{d\xi} (\infty) = e^{\infty - 1} \infty \ln \infty.
\]
\[
\int_{0}^{\beta} \frac{\infty}{1 + \infty^2 \xi^2} \, d\xi = [\arctan(\infty \xi)]_{0}^{\beta} = \arctan(\infty \beta) - \arctan(\infty 0) = \arctan 1 - \arctan 0 = \frac{\pi}{4}.
\]

In Sec. VI we saw that it does not matter whether we use a real (Latin letter) or a virtual (Greek letter) integration variable when the integrand and the integration limits are real. When this is not the case, the symbol
\[
\int_{\alpha}^{\beta} \phi(x) \, dx
\]
has not yet been defined.

We will say that a virtual integral

\[ \int_{\alpha}^{\beta} \phi(\xi) \, d\xi \]

is reducible when it is near some real number. In this case, we will use the symbol

\[ \int_{\alpha}^{\beta} \phi(x) \, dx \]

to denote this real number, which will be called reduced integral of function \( \phi \) between \( \alpha \) and \( \beta \). This symbol will not be used if the virtual integral is not reducible, so an integral with a real (Latin letter) integration variable is always a real number, if it exists.

For any reducible integral, we have:

\[ \int_{\alpha}^{\beta} \phi(x) \, dx \approx \int_{a}^{b} \phi(\xi) \, d\xi, \]

but, in general, it is not true that:

\[ \int_{\alpha}^{\beta} \phi(x) \, dx = \int_{a}^{b} \phi(\xi) \, d\xi. \]

For instance:

\[ \int_{1}^{\infty} \frac{d\xi}{\xi^2} = \left[ -\frac{1}{\xi} \right]_{1}^{\infty} = -\partial + 1, \]

so

\[ \int_{1}^{\infty} \frac{dx}{x^2} = 1, \]

and

\[ \int_{1}^{\infty} \frac{dx}{x^2} \neq \int_{1}^{\infty} \frac{d\xi}{\xi^2}. \]

Analogously:

\[ \int_{\partial}^{1} \frac{d\xi}{\sqrt{\xi}} = \left[ 2\sqrt{\xi} \right]_{\partial}^{1} = 2 - 2\sqrt{\partial}, \]

so

\[ \int_{\partial}^{1} \frac{dx}{\sqrt{x}} = 2, \]

hence:

\[ \int_{\partial}^{1} \frac{dx}{\sqrt{x}} \neq \int_{\partial}^{1} \frac{d\xi}{\sqrt{\xi}}. \]
VIII. Virtual Sequences

The virtual functions whose domain is the set $\overline{\mathbb{N}} \subset \overline{\mathbb{R}}$ will be called virtual sequences, and they will be generically denoted by $(\alpha_\nu)$, where the virtual index $\nu$ ranges over the set $\overline{\mathbb{N}}$ of all virtual natural numbers. Since we can calculate

$$\sum_{n=1}^{k} a_n$$

for every $k \in \mathbb{N}$, the sum

$$\sum_{\nu=1}^{\kappa} \alpha_\nu$$

is always a well defined virtual number, for any virtual sequence $(\alpha_\nu)$ and any $\kappa \in \overline{\mathbb{N}}$. Thus, we can add infinitely many virtual numbers:

$$\sum_{\nu=1}^{\infty} \alpha_\nu$$

without worrying about “technical” convergence questions.

As an example of application, we can use Riemann sums of infinitely many infinitesimal terms to integrate a real function $f$ defined between two real numbers $a$ and $b$. To do so, it is enough to consider an infinitely fine partition of that interval as a virtual sequence $(\alpha_\nu)$ such that both $\alpha_1 = a$ and $\alpha_\infty = b$, as well as:

$$\alpha_{\nu+1} \approx \alpha_\nu, \quad \text{for every } \nu \in \overline{\mathbb{N}}.$$

As an illustration, the virtual sequence:

$$\alpha_\nu = \frac{\nu}{\infty}$$

is an infinitely fine partition of the real interval $[0, 1]$ according to that definition.

At this point, the general mechanism which allows us to transport the constructions and techniques of Calculus to virtual functions and sequences should be clear. After having understood this mechanism, it is easy to see that we could have defined virtual subsets of the real line, as well as the notion of virtual relation, from the very beginning. Through obvious identifications, these objects could have been considered, respectively, as subsets of $\overline{\mathbb{R}}$ and relations between virtual numbers.
We would then have, for instance, that the set of all virtuals between \( \alpha \) and \( \beta \) would be a virtual subset of \( \mathbb{R} \), for any pair \( \alpha, \beta \in \mathbb{R} \). More generally, the domain of any virtual function would always be a virtual set.

That general mechanism also allows us to construct *virtual differential equations*, which can have virtual functions as coefficients or initial conditions. The *unknown* of such an equation is a virtual function, and we can talk about general or particular *virtual solutions*.

It is not difficult to see that we can also reformulate the Multivariable Calculus as we did in the first part of this work for just one variable, and so “virtualize” many other notions, like differential partial equations and their respective solutions, for example.

In the context of Functional Analysis, or of Linear Algebra in general, the idea of virtual sequence of vectors can be used to define *infinite linear combinations*, which leads to the notion of *virtual base* of a vector space. Perhaps this concept reduces the “technic-alities” involved in the representation of functions by functional series, like Fourier’s, for instance.

We do not intend to explore all those possibilities here, but our hope is to have clearly indicated a way which might enlarge Infinitesimal Calculus usefulness and reach.

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