HOPF ALGEBRA EXTENSIONS OF GROUP ALGEBRAS AND TAMBARA-YAMAGAMI CATEGORIES

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Abstract. We determine the structure of Hopf algebras that admit an extension of a group algebra by the cyclic group of order 2. We study the corepresentation theory of such Hopf algebras, which provide a generalization, at the Hopf algebra level, of the so-called Tambara-Yamagami fusion categories. As a byproduct, we show that every semisimple Hopf algebra of dimension < 36 is necessarily group-theoretical; thus 36 is the smallest possible dimension where a non group-theoretical example occurs.

1. Introduction

Along this paper we shall work over an algebraically closed base field $k$ of characteristic zero.

An important problem related to the classification of semisimple Hopf algebras over $k$ was the question raised in the paper [6], whether all semisimple Hopf algebras over $k$ are group-theoretical; see [6, Section 8] for a discussion of group-theoretical categories and quasi-Hopf algebras. A positive answer to this question was obtained for certain cases in the paper [4]; in particular, semisimple Hopf algebras of dimension $p^n$, where $p$ is a prime number, are always group-theoretical.

In the general case, the question was answered negatively by Nikshych [23]. In fact, a family of semisimple Hopf algebras which are not group-theoretical is constructed in [23]: a Hopf algebra $H$ in this family fits into an exact sequence

\[(1.1) \quad k \to k\mathbb{Z}_2 \to H \to (kG)^J \to k,\]

where $G$ is a certain finite group and $J \in kG \otimes kG$ is an invertible twist. It turns out that these examples are all semisolvable in the sense of [17].

In this paper we consider semisimple Hopf algebras which fit into an exact sequence

\[(1.2) \quad k \to A \to H \to k\mathbb{Z}_2 \to k,\]
where $A$ is a triangular semisimple Hopf algebra. In view of the classification of semisimple triangular Hopf algebras, $A \cong (kG)^J$ as Hopf algebras, for some finite group $G$ and invertible twist $J \in kG \otimes kG$ [5].

Contrary to the situation for the extensions (1.1), those in (1.2) are always group-theoretical, despite of the symmetry in the form of the extensions. Moreover, we show in Theorem 4.11 that a Hopf algebra $H$ as in (1.2) is twist equivalent to an abelian extension, and a fortiori group-theoretical by the main result of [19].

As mentioned before, a Hopf algebra $H$ as in (1.2) is twist equivalent to a semisimple Hopf algebra whose group of group-like elements has index 2. Therefore, up to twist deformations, we are lead to considering extensions of the form

$$k \to kG \to H \to k\mathbb{Z}_2 \to k,$$

(1.3)

where $G$ is a finite group. We prove that if such a nontrivial $H$ exists, then $H$ fits also into an abelian (cocentral) exact sequence; in particular, the group $G \cong G(H)$ is necessarily an extension of an abelian group of order $d^2$, $d > 1$; see Propositions 4.10, 5.4.

In the situation of (1.3), we show that there is an equivalence of fusion categories $\text{Rep} H \cong (\text{Rep} G)^{\mathbb{Z}_2}$, where $(\text{Rep} G)^{\mathbb{Z}_2}$ is an appropriate $\mathbb{Z}_2$-equivariantization of $\text{Rep} G$. More generally, we show that the representation category of every cocentral Hopf algebra extension is an equivariantization; see Proposition 3.5.

On the other hand, the corepresentation category of a Hopf algebra $H$ as in (1.3) turns out to be a generalization, at the Hopf algebra level, of the so called Tambara-Yamagami categories [29]: these are fusion categories with isomorphism classes of simple objects parameterized by the set $\Gamma \cup \{x\}$, where $\Gamma$ is a finite (necessarily abelian) group, $x \notin \Gamma$, obeying the fusion rules

$$s \otimes t = st, \quad s, t \in \Gamma, \quad x \otimes x = \bigoplus_{s \in \Gamma} s.$$

Distinct features of Tambara-Yamagami fusion rules and some of their generalizations have attracted the interest of several authors in the last years, c.f., [2, 6, 23, 24, 25].

In the situation of (1.3) there may be more than one non-invertible object, nevertheless, all non-invertible objects have the same dimension $d$ and the product of any two of them is a sum of invertible objects belonging to a fixed normal abelian subgroup of order $d^2$ of $G$.

We prove the following classification theorem.

**Theorem 1.1.** Semisimple Hopf algebras $H$ with $[H : kG(H)] = 2$ are determined by triples $(\Gamma, F, \xi)$, where

1. $\Gamma$ is a finite abelian group,
2. $F$ is a finite group acting on $\Gamma$ by group automorphisms,
3. $\xi$ is an element of the group $\text{Opext}(k\Gamma, kF)$,
satisfying

\[(1.4) \quad [F : F_0] = 2, \text{ where } F_0 \text{ is the subgroup } F_0 := \{x \in F : [\tau_x] = 1\}, \text{ and} \]

\[(1.5) \quad \tau_x \text{ is a non-degenerate 2-cocycle, for all } x \in F \setminus F_0, \]

if \((\sigma, \tau)\) is a pair of compatible cocycles representing \(\xi\), where \(\tau_x \in Z^2(\hat{\Gamma}, k^\times)\)

is defined by \(\tau_x(s, t) = \tau(s, t)(x), x \in F, s, t \in \hat{\Gamma}\).

If \(H\) corresponds to the triple \((\Gamma, F, \xi)\), then \(G(H)\) is isomorphic to the crossed product \(\Gamma \rtimes_\sigma F_0\).

Theorem 1.1 is proved in Section 5. It extends the classification result [27, Theorem 3.5] of Tambara; see Remark 5.7.

The smallest example of a non group-theoretical semisimple Hopf algebra from the construction in [23] has dimension 36 and it is a semisolvable Hopf algebra.

We also show in this paper that every semisimple Hopf algebra of dimension < 36 is group-theoretical. So that, in fact, 36 is the smallest possible dimension that a non group-theoretical semisimple Hopf algebra can have. See Theorem 6.3. It is known that in dimension < 36 every semisimple Hopf algebra is upper and lower semisolvable; see [20]. Coincidentally, the first non semisolvable example also appears in dimension 36 [10].

Moreover, in dimension < 36, except for dimension 24, every semisimple Hopf algebra is either nilpotent (dimensions \(p, p^2, p^3, p^4, p^5\), where \(p\) is a prime number) or an abelian extension (dimensions 30, \(pq\), and \(pq^2\), where \(p\) and \(q\) are distinct prime numbers). Then all these Hopf algebras are group-theoretical, in view of [4, 19]. Therefore we only need to consider the case of dimension 24. We prove that, in fact, up to a cocycle twist of the multiplication or the comultiplication, every semisimple Hopf algebra of dimension 24 fits into an abelian exact sequence; see Proposition 6.1.

The paper is organized as follows. In Section 2 we recall some preliminary notions on Hopf algebra extensions needed later on. We also discuss in Subsection 2.2 some properties of group-theoretical Hopf algebras and their relation with Hopf algebra extensions. In Section 3 we describe the representation theory of cocentral extensions of a Hopf algebra. In Sections 4 and 5 we prove our main results on Hopf algebras with group of group-likes of index 2. In Section 6 we apply the above to prove our statement on semisimple Hopf algebras of dimension < 36. At the end of the paper we include an appendix where we give a sufficient condition for normality of a group-like Hopf subalgebra in a semisimple Hopf algebra.

**Conventions and notation.** The notation for Hopf algebras is standard: \(\Delta, \epsilon, S\), denote the comultiplication, counit and antipode, respectively. We refer the reader to [6, 12] for the terminology and notation on tensor categories and tensor functors used throughout.

Let \(d\) be a positive integer. The notation \(M_d(k)\) will indicate the matrix coalgebra of dimension \(d^2\). For a group \(G\), the group algebra of \(G\) and its dual
will be denoted by $kG$ and $k^G$, respectively. The group of one-dimensional characters of $G$ will be indicated by $\hat{G}$.

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2. Hopf algebra extensions

Let $H$ be a Hopf algebra over $k$. The left (respectively, right) adjoint action of $H$ on itself is defined by $\text{ad}_h(x) = h_1 x S(h_2)$ (respectively, $\text{ad}_h^r(x) = S(h_1)xh_2$), $x, h \in H$. A Hopf subalgebra $K \subseteq H$ is called normal if it is stable under both adjoint actions; $H$ is called simple if it contains no proper normal Hopf subalgebras.

Let $K \subseteq H$ be a normal Hopf subalgebra. Then $B = H/HK$ is a quotient Hopf algebra and the sequence of Hopf algebra maps $k \rightarrow K \rightarrow H \rightarrow B \rightarrow k$ is an exact sequence of Hopf algebras. In this case $H$ is called an extension of $K$ by $B$. If the extension is cleft, then $H$ is isomorphic to a bicrossed product $H \cong K \# \sigma B$ as a Hopf algebra. See [1, 14, 15] and references therein.

Suppose $H$ is finite dimensional. Then $K$ is normal in $H$ if and only if $K$ is stable under the left adjoint action. In this case, the extension $K \subseteq H$ is always cleft. Moreover, in the finite dimensional context, the notion of simplicity is self-dual; that is, $H$ is simple if and only if $H^*$ is.

By [20, Corollary 1.4.3], if $K \subseteq H$ is a normal Hopf subalgebra, such that $\dim K$ is the least prime number dividing $\dim H$, then $K$ is central in $H$.

2.1. Abelian extensions. We refer the reader to [14, 15] for the notion of abelian exact sequence and, in particular, for the study of the cohomology theory underlying such exact sequence.

Suppose that $L = FT$ is an exact factorization of the finite group $L$, where $\Gamma$ and $F$ are subgroups of $L$. Equivalently, $F$ and $\Gamma$ form a matched pair of finite groups with the actions $\triangleleft: \Gamma \times F \rightarrow \Gamma$, $\triangleright: \Gamma \times F \rightarrow F$, defined by $sx = (x \triangleleft s)(x \triangleright s), x \in F, s \in \Gamma$.

Let $\sigma : F \times F \rightarrow (k\Gamma)^\times$, $\sigma(x, y) = \sum_s \sigma_s(x, y)e_s$, and $\tau : \Gamma \times \Gamma \rightarrow (k\Gamma)^\times$, $\tau(s, t) = \sum_x \tau_x(s, t)e_x$, be normalized 2-cocycles with the respect to the actions afforded, respectively, by $\triangleleft$ and $\triangleright$, subject to appropriate compatibility conditions [14]. Here, $e_y \in kF$, $y \in F$, are the canonical idempotents defined by $e_y(x) = \delta_{x,y}$, and similarly for $e_s \in k\Gamma$.

Consider the bicrossed product $H = k\Gamma \# \sigma kF$ corresponding to this data. Then $H$ is a Hopf algebra, with multiplication and comultiplication determined by

\[
\begin{align*}
(e_s \# x)(e_t \# y) &= \delta_{s \triangleleft x, t} \sigma_s(x, y)e_s \# xy, \\
\Delta(e_s \# x) &= \sum_{gh = s} \tau_x(g, h)e_y \# (h \triangleright x) \otimes e_h \# x,}
\end{align*}
\]
\[ s, t \in \Gamma, \quad x, y \in F, \] and there is an abelian exact sequence \[ k \to k^\Gamma \to H \to kF \to k. \] Moreover, every Hopf algebra fitting into such exact sequence can be described in this way. This gives a bijective correspondence between the equivalence classes of these Hopf algebra extensions and a certain abelian group \( \text{Opext}(k^\Gamma, kF) \) associated to the matched pair \((F, \Gamma)\).

**Remark 2.1.** Suppose that the extension \( k \to k^\Gamma \to H \to kF \to k \) is cocentral or, equivalently, that the action \( \triangleright : \Gamma \times F \to F \) is trivial, so that \( L = F \Gamma \cong \Gamma \times F \) is a semidirect product. By Proposition 3.5 below, there is an equivalence of tensor categories

\[ \text{Rep } H \cong (\text{Vec}^\Gamma)^F, \]

where \( \text{Vec}^\Gamma \) denotes the tensor category of \( \Gamma \)-graded vector spaces and the \( F \)-action is given by \( T_x(V)_s = V_{s \triangleleft x}, \ s \in \Gamma, \ x \in F. \)

The following lemma describes the group of group-likes in an abelian extension. Let \( F^\Gamma \subseteq F \) denote the (subgroup) of elements in \( F \) which are invariant under the action \( \triangleright. \)

Note that if \( x \in F^\Gamma \), then \( \tau_x : \Gamma \times \Gamma \to k^\times \) is a normalized 2-cocycle. We shall denote by \([\tau_x]\) its cohomology class in \( H^2(\Gamma, k^\times)\).

**Lemma 2.2.** The exact sequence of Hopf algebras \( k \to k^\Gamma \to H \to kF \to k \) induces by restriction an exact sequence of groups

\[ 1 \to \hat{\Gamma} \to G(H) \to F_0 \to 1, \]

where \( F_0 = \{ x \in F^\Gamma : [\tau_x] = 1 \}. \)

Therefore \( G(H) \) is isomorphic to a crossed product \( G(H) \cong \hat{\Gamma} \rtimes_{\triangleleft, \sigma} F_0. \)

**Proof.** The proof is straightforward using the formula for the comultiplication \( (2.2) \). Indeed, each group-like element of \( H \) is of the form \( \gamma \# x \), where \( 0 \neq \gamma \in k^\Gamma \) and \( x \in F^\Gamma \), such that \( \gamma(1) = 1 \) and \( \tau_x(s, t) = \gamma(s)\gamma(t)\gamma(st)^{-1}, \forall s, t \in \Gamma. \)

\[ \text{Remark 2.3.} \] Formula \( (2.2) \) implies that for all \( x \in F \), the subspace \( I_x = k^\Gamma \# x \subseteq H \) is a left coideal, and we have a decomposition \( H = \bigoplus_{x \in F} I_x. \)

Moreover, by \( (2.1) \), \( I_x I_y \subseteq I_{xy}, \forall x, y \in F. \)

Suppose \( x \in F^\Gamma \). Then \( I_x \) is a subcoalgebra of \( H \), and \( I_x = (k^\Gamma)^\times \# x, \) as coalgebras. In particular, \( I_x \cong k^\Gamma \) if and only if \( [\tau_x] = 1 \) and it is simple if and only if \( \tau_x \) is non-degenerate.

If \( y \in F^\Gamma \) is such that \( \tau_y \) is non-degenerate, then the relation \( I_1 I_y \subseteq I_y \) implies that \( \hat{\Gamma} = G(k^\Gamma) \subseteq G(H) \) stabilizes the simple subcoalgebra \( I_y. \)

**2.2. Relations with group-theoretical Hopf algebras.** Let \( G \) be a finite group and let \( \omega \) be a 3-cocycle on \( G \). Let \( \mathcal{C} = \mathcal{C}(G, \omega) \) be the category of \( G \)-graded vector spaces with associativity isomorphism given by \( \omega \). Let also \( F \) be a subgroup of \( G \) and \( \alpha \) a 2-cochain on \( F \). Suppose that \( \omega|_F = d\alpha; \) so that the twisted group algebra \( k_{\alpha} F \) is an algebra in \( \mathcal{C} \). Then the category
$\mathcal{C}(G, \omega, F, \alpha)$ of $k_\alpha F$-bimodules in $\mathcal{C}$ is a fusion category, called a *group-theoretical* category. A (quasi-)Hopf algebra is called group-theoretical if its category of representations is group-theoretical [6, 8.8].

In particular, the class of group-theoretical (quasi-)Hopf algebras is closed under twisting. Furthermore, it is shown in [6] that the class of group-theoretical categories is closed under duals, Drinfeld centers, taking full fusion subcategories and components in a quotient category.

It was shown in [19] that every abelian extension of Hopf algebras is group-theoretical. However, the group-theoretical Hopf algebras are not closed under taking extensions. In the paper [23], Nikshych constructs examples of semisimple Hopf algebras $H$ which are not group-theoretical, and nevertheless fit into an exact sequence $k \to k^2 \to H \to (kG)^J \to k$, where $G$ is a certain finite group and $J \in kG \otimes kG$ is an invertible twist.

3. Representation category of a cocentral extension

3.1. $G$-equivariantization. We start this subsection by recalling the definition of equivariantization of a fusion category. This notion has been considered by different authors, see [3, 7, 23, 28]. We first present the construction for group actions on a linear category $\mathcal{C}$, that is, dropping the tensor structure in $\mathcal{C}$.

Let $F$ be a finite group and let $\mathcal{C}$ be a $k$-linear category. The group $F$ will also be regarded as a monoidal category, denoted by $\mathcal{F}$, whose objects are the elements of $F$, arrows are identities and tensor product is the multiplication in $F$.

Let $\text{Aut} \mathcal{C}$ denote the monoidal category whose objects are autoequivalences of $\mathcal{C}$, morphisms are isomorphisms of functors and tensor product is the composition of functors. An action of $F$ on $\mathcal{C}$ is a monoidal functor

$$(T, f) : \mathcal{F} \to \text{Aut} \mathcal{C}.$$  

In particular, for all $g, h \in F$, there are natural equivalences

$$(3.2) f_{g,h} : T_g \circ T_h \to T_{gh},$$

giving the tensor structure to the functor $T$, where $T_g = T(g), g \in F$.

Suppose that $(3.1)$ is an action of $F$ on $\mathcal{C}$. An $F$-equivariant object in $\mathcal{C}$ is a pair $(V, (u^V_g)_{g \in F})$, where $V$ is an object of $\mathcal{C}$ and $u^V_g : T_g(V) \to V$, $g \in F$, are isomorphisms compatible with the tensor structure on $T$ in the sense that, for all $g, h \in F$,

$$(3.3) u^V_g T_g(u^V_h) = u^V_{gh} f_{g,h}.$$  

An $F$-equivariant morphism $\phi : (U, u^U_g) \to (V, u^V_g)$ between $F$-equivariant objects $(U, u^U_g)$ and $(V, u^V_g)$, is a morphism $\phi : U \to V$ in $\mathcal{C}$ such that $\phi u^U_g = u^V_g \phi$, for all $g \in F$. 
The $F$-equivariantization of $\mathcal{C}$, denoted $\mathcal{C}^F$, is defined to be the category of $F$-equivariant objects and $F$-equivariant morphisms.

If $\mathcal{C}$ is a tensor (respectively, fusion) category, and $F$ acts by tensor autoequivalences, that is, the action \[(3.1)\]
takes values in the subcategory $\text{Aut}_{\otimes}\mathcal{C}$ of tensor autoequivalences of $\mathcal{C}$ and isomorphisms of tensor functors, then the equivariantization $\mathcal{C}^F$ is a tensor (respectively, fusion) category with tensor product inherited from $\mathcal{C}$.

More precisely, it is required in this case that $(T_g, j_g)$ be a monoidal functor, with $j_{g|U,V} : T_g(U \otimes V) \xrightarrow{\cong} T_g(U) \otimes T_g(V)$, for all $g \in F$. Then, for equivariant objects $(U, u_g^U)$ and $(V, u_g^V)$, their tensor product $(U \otimes V, u_g^{U \otimes V})$ is an equivariant object, where $u_g^{U \otimes V} = (u_g^U \otimes u_g^V)j_g|_{U \otimes V}$. Moreover, $\mathcal{C}^F$ is dual to a crossed product fusion category $\mathcal{C} \rtimes F$ with respect to the indecomposable module category $\mathcal{C}$. See \cite{16}.

**Remark 3.1.** Suppose $\mathcal{C} = H^\mathcal{M}$ is the category of (finite dimensional) left $H$-comodules over the Hopf algebra $H$. If $L$ is another Hopf algebra, it is known that isomorphism classes of equivalences of tensor categories $H^\mathcal{M} \rightarrow L^\mathcal{M}$ are in bijective correspondence with classes of $(L, H)$-bigalois extensions [26].

Let $\text{Bigal}(H)$ denote the set of classes of $(H, H)$-bigalois extensions. Then $\text{Bigal}(H)$ is a group under cotensor product, and there is an inclusion of tensor categories $\text{Bigal}(H) \rightarrow \text{Aut}_{\otimes}\mathcal{C}$. Moreover, every object of $\text{Aut}_{\otimes}\mathcal{C}$ is isomorphic to exactly one object of $\text{Bigal}(H)$.

### 3.2. Representations of crossed products.

Let $A$ be a finite dimensional $k$-algebra. Suppose that the finite group $F$ measures $A$ via $\cdot : kF \otimes A \rightarrow A$, $g \otimes a \mapsto g.a$ and let $\sigma : kF \otimes kF \rightarrow A$ be an invertible compatible 2-cocycle; that is, $\sigma(g, h)$ is invertible in $A$, and we have

\[
\begin{align*}
(3.4) & \quad g.(ab) = (g.a)(g.b), \quad g.1 = 1, \\
(3.5) & \quad (g.\sigma(h, t))\sigma(g, ht) = \sigma(gh, t)\sigma(g, h), \quad \sigma(1, g) = \sigma(g, 1) = 1, \\
(3.6) & \quad g.(h,a) = \sigma(g, h)(gh,a)\sigma(g, h)^{-1},
\end{align*}
\]

for all $g, h, t \in F$, $a, b \in A$. See [16] Chapter 7. In this situation, we shall also say that $\cdot$ is a weak action. Let $A\#_{\sigma} kF$ be the corresponding crossed product.

The category $\text{Rep} A$ of finite dimensional representations of $A$ admits an action of the group $F$, $(T, f) : F \rightarrow \text{Aut}(\text{Rep} A)$, defined as follows: $T_g(V) = V$, with $a.gv = (g.a)v$, $V \in \text{Rep} A$, $v \in V$. We let $T_g = \text{id}$ on morphisms, and

\[
f_{g,h} : T_g(T_h(V)) \rightarrow T_{gh}(V), \quad v \mapsto \sigma(g, h)^{-1}v,
\]

for all $g, h \in F$. Note that $f_{g,h}$ is an $A$-linear isomorphism thanks to (3.5).

Suppose $V$ is a $F$-equivariant object with respect to this action. This means that there are $A$-linear isomorphisms $u_g = u_g^V : V \rightarrow V$ satisfying

\[
(3.7) \quad (g.a)|_V = u_g^{-1}a|_Vu_g,
\]
for all \( g \in F, \ a \in A \). By \((3.3)\), we have in addition
\[
(3.8) \quad \sigma(g, h)^{-1}|_{V} = f_{g, h}|_{V} = u_{g h}^{-1} u_{g} u_{h},
\]
for all \( g, h \in F \).

It is straightforward to check that conditions \((3.7)\) and \((3.8)\) imply that there is a well-defined action of the crossed product \( A\#_{\sigma} kF \) on \( V \) determined by
\[
(3.9) \quad (a \# g). v = a u_{g}^{-1}(v),
\]
for \( a \in A, \ g \in F, \ v \in V \). Moreover, morphisms of \( F \)-equivariant objects are exactly morphisms of \( A \)-modules commuting with the action of \( F \) afforded by the \( u_{g} \)’s, so they are \( A\#_{\sigma} kF \)-morphisms. We get in this way a functor
\[
(3.10) \quad \mathcal{F} : (\text{Rep} A)^{F} \to \text{Rep}(A\#_{\sigma} F).
\]

**Proposition 3.2.** The functor \( \mathcal{F} : (\text{Rep} A)^{F} \to \text{Rep} H \) defines an equivalence of categories.

**Proof.** Suppose that \( W \in \text{Rep}(A\#_{\sigma} F) \). Then \( W \) is a representation of \( A \) by restriction. Moreover, \( (W, u_{W}^{W}) \) is a \( F \)-equivariant object in \( \text{Rep} A \), letting \( u_{W}^{W} : W \to W \), be defined by \( u_{W}^{W}(w) = g^{-1}w \), for every \( g \in F \). We have thus a functor \( \mathcal{G} : \text{Rep}(A\#_{\sigma} F) \to (\text{Rep} A)^{F} \). It is clear that \( \mathcal{F} \) and \( \mathcal{G} \) are inverse equivalences of categories. This proves the proposition. \( \square \)

We shall use later (c. f. Section 5) the following description of irreducible representations of crossed products given in \([17, \text{Theorem 1.3}]\). Consider an irreducible \( A \)-module \( V \), and let \( F^{V} \) its stabilizer in \( F \), and \( \alpha \) the 2-cocycle of \( F^{V} \) associated to \( V \) as in \([17, \text{pp. 318}]\). Then there is an equivalence between the category of \( k_{\alpha} F^{V} \)-modules and the category of those \( A\#_{\sigma} kF \)-modules whose restriction to \( A \) is isomorphic to a direct sum of copies of conjugates of \( V \), which maps the \( k_{\alpha} F^{V} \)-module \( W \) to the induced module \( \text{Ind}_{A\#_{\sigma} kF}^{A\#_{\sigma} kF} U \otimes W \).

**3.3. Cocentral Hopf algebra extensions.** Suppose that
\[
(3.11) \quad k \to A \to H \xrightarrow{\pi} B \to k
\]
is a cleft exact sequence of Hopf algebras; that is, the projection \( \pi : H \to B \) admits a convolution invertible \( B \)-colinear section \( j : B \to H \). The sequence \((3.11)\) will be called **cocentral** if \( \pi(h_{1}) \otimes h_{2} = \pi(h_{2}) \otimes h_{1} \), for all \( h \in H \).

If \( B \) is finite dimensional, this is equivalent to saying that the dual inclusion \( \pi^{*} : B^{*} \to H^{0} \) is central. In this case \( B^{*} \) must be commutative, whence \( B \cong kF \) for some finite group \( F \).

We shall next give a description of the representation category of a cocentral extension \( H \) as in \((3.11)\), such that \( B \) is finite dimensional. That is, the Hopf algebra \( H \) fits into a cocentral cleft exact sequence
\[
(3.12) \quad k \to A \to H \to kF \to k.
\]
By the cleftness assumption, $H$ has the structure of a bicrossed product $H \cong A\#_\sigma kF$, with respect to a certain compatible datum $(.,\rho,\sigma,\tau)$, where $\cdot : kF \otimes A \to A$ is a weak action, $\sigma : kF \otimes kF 
rightarrow A$ is an invertible cocycle, $\rho : kF \to kF \otimes A$ is a weak coaction and $\tau : kF \to A \otimes A$ is an invertible dual cocycle, subject to the compatibility conditions in $[1$, Theorem 2.20].

**Lemma 3.3.** The exact sequence (3.12) is cocentral if and only if the afforded weak coaction $\rho$ is trivial.

**Proof.** The proof is straightforward. \hfill $\square$

In particular, $H$ is a crossed product $H \cong A\#_\sigma kF$ as an algebra. As in Subsection 3.2, this gives rise to an action of $F$ on $\text{Rep}A$.

**Lemma 3.4.** Suppose the exact sequence (3.12) is cocentral. Then the afforded $F$-action on $\text{Rep}A$ is by tensor autoequivalences, and the functor $\mathcal{F}$ in Proposition 3.2 is a tensor functor.

**Proof.** By Lemma 3.3 the afforded weak coaction is trivial. On the other hand, by condition (2.21) of compatibility between the product and the coproduct in $[1$, we have

$$\Delta_A(g.a) = \tau(g)(g.a_1 \otimes g.a_2)\tau(g)^{-1},$$

for all $g \in F$, $a \in A$. This implies that the action of $\tau(g)^{-1}$ gives a well-defined $A$-linear isomorphism $\tau(g)^{-1} : T_g(V \otimes W) \to T_g(V) \otimes T_g(W)$, natural in $V,W$, for all $g \in F$. Moreover, the dual cocycle condition $[1$, (2.18)] with respect to the trivial coaction $\rho$ implies that $\tau(g)^{-1}$ actually gives a monoidal functor structure to the autoequivalence $T_g$.

Let $g,h \in F$. By condition (2.21) in $[1$ and the assumption on $\rho$, we also get

$$\Delta(\sigma(g,h))\tau(gh) = \tau(g)(g.\tau(h)) (\sigma(g,h) \otimes \sigma(g,h)).$$

Inverting this identity, we find that the map $f_{g,h} : T_g \circ T_h \to T_{gh}$ is indeed an equivalence of tensor functors.

Formula $[1$, (2.15)] for the comultiplication in $H$ reads in this case

$$\Delta(a\#_{g}) = \Delta(a)\tau(g)(g \otimes g),$$

for all $a \in A$, $g \in F$. This implies that $\mathcal{F}$ is a monoidal functor with isomorphisms $\mathcal{F}(U \otimes V) \to \mathcal{F}(U) \otimes \mathcal{F}(V)$ given by identities. This finishes the proof of the lemma. \hfill $\square$

In conclusion, we obtain the following description of the category $\text{Rep}H$. It generalizes the statement in $[23$, Example 2.7].

**Proposition 3.5.** Suppose that the cleft exact sequence (3.12) is cocentral. Then the functor $\mathcal{F} : (\text{Rep}A)^F \to \text{Rep}H$ defines an equivalence of tensor categories.

$\square$
Remark 3.6. Suppose that $H$ is semisimple (hence finite dimensional). Observe that the dual Hopf algebra $H^*$ fits into a central extension $k \to k^F \to H^* \to A^* \to k$. By [9, Proof of Theorem 3.8], this amounts exactly to the fact that the category $R = \text{Rep} \, H^*$ is an $F$-graded fusion category, $R = \oplus_{g \in F} R_g$, with trivial component $R_e = \text{Rep} \, A^*$.

Therefore, Proposition 3.5 establishes a duality between $F$-graded fusion categories with trivial component $\text{Rep} \, A^*$ and $F$-equivariantizations of $\text{Rep} \, A$, for a semisimple Hopf algebra $A$.

4. Extensions of group algebras by a group of order 2

Let $G$ be a finite group. In this section we shall assume that $H$ is a semisimple Hopf algebra for which $G \subseteq G(H)$ as a subgroup, and the group algebra $kG$ has index 2 in $H$. Therefore $kG$ is a normal Hopf subalgebra of $H$ and we have an extension

$$k \to kG \to H \to k\mathbb{Z}_2 \to k. \quad (4.1)$$

This extension is necessarily cocentral, by [20, Corollary 1.4.3]. By Proposition 3.5 there is an equivalence of tensor categories

$$\text{Rep} \, H \cong (\text{Rep} \, G)^{\mathbb{Z}_2},$$

with respect to an appropriate action by tensor autoequivalences $\mathbb{Z}_2 \to \text{Aut}_{\otimes}(\text{Rep} \, G)$.

Examples 4.1. (1) Nontrivial examples of Hopf algebras $H$ as in (4.1) are the duals of those whose representation category is a Tambara-Yamagami category [29], as observed in [19]. In this case, the group $G$ is abelian.

(2) Other examples of such $H$ appear in the classification of Kac algebras of dimension 24 by Izumi and Kosaki [11]. More precisely, Kac algebras of dimension 24 with group of group-likes of order 12 are classified in [11, Theorem XIV.40-I]: for exactly two of the possible isomorphism classes, the group of group-likes is nonabelian and isomorphic to the alternating group $A_4$.

Our first lemma describes the coalgebra structure of $H$.

Lemma 4.2. Suppose $H$ is not cocommutative. Then, as a coalgebra, $H \cong kG \oplus M_{d_1}(k)^{n_1} \oplus \cdots \oplus M_{d_r}(k)^{n_r}$, where $d > 1$ and $|G| = d^2 n$.

Proof. As a coalgebra, $H \cong kG \oplus M_{d_1}(k)^{n_1} \oplus \cdots \oplus M_{d_r}(k)^{n_r}$, where $d_i > 1$ and $n_i \geq 1$, $i = 1, \ldots, r$. Each subcoalgebra of the form $M_{d_i}(k)^{n_1}$ is stable under left multiplication by $kG$ and is thus a left $(kG, H)$-Hopf module. Therefore, by [22], $|G|$ divides $n_i d_i^2$, for all $i = 1, \ldots, r$.

Then $\dim H = |G| + \sum_{i=1}^r n_i d_i^2 \geq (r + 1)|G|$. Since, on the other hand, $\dim H = 2|G|$, we get that $r = 1$, and $|G| = d^2 n$, $d = d_1$, $n = n_1$, as claimed.

Corollary 4.3. Suppose $|G|$ is square-free. Then $H$ is cocommutative.
The corollary implies that there is no semisimple Hopf algebra $H$ with $[H : kG(H)] = 2$ and such that $|G(H)|$ is square-free.

Let $X = \{\chi_1, \ldots, \chi_n\}$ denote the set of irreducible characters of degree $d$.

Let $\chi, \psi$ be characters corresponding to the $H$-comodules $V$ and $W$, respectively. Then the product $\chi \psi$ is the character of the tensor product $V \otimes W$. We may write $\chi \psi = \sum \mu \langle \mu, \chi \psi \rangle \mu$, where $\mu$ runs over the set of irreducible characters and $\langle \mu, \chi \psi \rangle$ denotes the multiplicity of $\mu$ in the product $\chi \psi$.

Consider the action $G \times X \to X$ given by left multiplication. Let $\chi \in X$ be an irreducible character, and let $G[\chi] \subseteq G$ be the stabilizer of $\chi$, that is, $G[\chi]$ is the subgroup consisting of all $g \in G$ such that $g\chi = \chi$. In view of [21, Theorem 10], $g \in G[\chi]$ if and only if $\langle g, \chi \rangle > 0$, if and only if $m(g, \chi^g) = 1$. Therefore

$$\chi \chi^g = \sum_{g \in G[\chi]} g + \sum_{\mu \in \chi \chi^g} m(\mu, \chi^g) \mu.$$  \hfill (4.2)

Note that $G[\chi] = \{g \in G : \chi g = \chi\}$. On the other hand, $G[\chi] = G[\chi]$, $G[\chi] = gG[\chi]g^{-1}$, for all $g \in G$.

**Lemma 4.4.** The group $G$ acts transitively on $X$ by left multiplication.

**Proof.** Let $\chi \in X$ be the irreducible character corresponding to the simple subcoalgebra $C$. We have $|G| = |G[\chi]|G[\chi]|$, where $G[\chi] \subseteq X$ denotes the orbit of $\chi$. By [22], $|G[\chi]|$ divides $d^2 = \dim C$, hence $|G| = d^2 n$ divides $|G[\chi]|d^2$. This implies that $n$ divides $|G[\chi]|$ and thus $|X| = n = |G[\chi]|$ as claimed. \hfill $\square$

**Corollary 4.5.** Let $\chi \in X$. Then $|G[\chi]| = d^2$. In particular, $\chi \chi^g = \sum_{g \in G[\chi]} g$, for all $\chi \in X$.

**Proof.** By Lemma 4.4, $|G[\chi]| = d^2$ for all $\chi \in X$. Combined with formula (4.2) we get the claimed expression for the product $\chi \chi^g$. \hfill $\square$

**Remark 4.6.** Note that all arguments used so far can also be applied to the right action of $G$ on $X$ given by right multiplication. In particular, the right action is also transitive.

As a consequence of this, we note the following. Let $\chi \in X$. Then we may write $\chi^g = \chi g$, and therefore $G[\chi^g] = G[\chi]$. This implies that $\chi \chi^g = \sum_{g \in G[\chi]} g = \chi^g \chi$.

**Corollary 4.7.** We have $G[\chi] = G[\psi]$, for all $\chi, \psi \in X$.

The common stabilizer $G[\chi]$, $\chi \in X$, will be denoted by $\Gamma$.

**Proof.** This follows from the fact that, by Remark 4.6, the right action of $G$ is also transitive. Thus $\psi = \chi g$ and $G[\psi] = G[\chi]$. \hfill $\square$

**Proposition 4.8.** The stabilizer $\Gamma$ is a normal abelian subgroup of $G$ which admits a non-degenerate 2-cocycle.
Recall that a 2-cocycle $\alpha$ on $\Gamma$ is called non-degenerate if the twisted group algebra $k_\alpha \Gamma$ is simple.

**Proof.** Proposition 3.4.4 of [20] implies that $\Gamma$ is an abelian subgroup and admits a non-degenerate 2-cocycle.

Let $g \in G$ and fix $\chi \in X$. By Corollary 4.7, $G[g\chi] = G[\chi] = \Gamma$. On the other hand, $G[g\chi] = gG[\chi]g^{-1} = g\Gamma g^{-1}$. Then $\Gamma$ is normal in $G$ as claimed.

**Lemma 4.9.** Let $\chi, \psi \in X$. Then $\psi \chi = \sum_{g \in \Gamma} a_g b_g$, where $a, b \in G$ are such that $\psi = a \chi$ and $\chi = \chi^* b$. In particular, $\psi \chi \in kG$, for all $\psi, \chi \in X$.

**Proof.** It is clear.

**Proposition 4.10.** The group algebra $k\Gamma$ is a normal Hopf subalgebra of $H$. There is an (abelian) exact sequence of Hopf algebras

$$k \to k\hat{\Gamma} \to H \to kF \to k,$$

where $F$ is a group of order $2n$.

**Proof.** Consider the group algebra $A = k\Gamma \simeq k\hat{\Gamma}$. It satisfies $AC = C = CA$, for all simple subcoalgebras $C \subseteq H$, and also $\dim A = \dim C$. By [20, Proposition 3.2.6] (see also Lemma 7.3 in the Appendix), $A = k\Gamma$ is a normal Hopf subalgebra of $H$.

By [20, Corollary 3.3.2], the quotient Hopf algebra $H/H(k\Gamma)^+$ is co-commutative, hence $H/H(k\Gamma)^+ \simeq kF$ for some finite group $F$ of order $[H : k\Gamma] = 2n$.

**Theorem 4.11.** Let $A$ be a semisimple triangular Hopf algebra. Let also $k \to A \to K \to k\mathbb{Z}_2 \to k$ be an exact sequence of Hopf algebras. Then $K$ is group-theoretical.

**Proof.** By the classification results of Etingof and Gelaki, $A \simeq (kG)^J$ as a Hopf algebra, for some finite group $G$ and $J \in kG \otimes kG$, a twist. Let $H = K^{J^{-1}}$, so that $H$ fits into an exact sequence $k \to kG \to H \to k\mathbb{Z}_2 \to k$.

By Proposition 4.10, $H$ fits into an abelian extension as well. Hence $H$, and also $A$, must be group-theoretical, in view of [19].

5. **Classification**

Let $H$ be a nontrivial Hopf algebra that fits into an exact sequence $\{4.11\}$. The aim of the present section is to determine the structure of these Hopf algebras. We keep the notation from previous sections.

According to what we have proven in Section 4, the isomorphism classes of simple objects in the fusion category $\text{Rep } H^*$ are parameterized by the set $G \cup \{x_1, \ldots, x_n\}$, where $n \geq 1$, $G$ is a group of order $d^2 n$, and $x_i$ are non-invertible objects of dimension $d > 1$, satisfying

$$g \otimes h = gh, \quad x_i \otimes x_i^* = \oplus_{s \in \Gamma}s, \quad x_i \otimes x_j \in kG,$$

where $x_i^* = x_i^*$, for all $g, h \in G$, $1 \leq i, j \leq n$. 
Lemma 5.1. Suppose, conversely, that \( H \) is a noncocommutative semisimple Hopf algebra such that \( \text{Rep} H^\ast \) has fusion rules as described in (5.1). Then \( G(H) \simeq G \) has index 2 in \( H \) and \( H \) fits into an exact sequence (4.1).

Proof. The relation for the product \( x_i \otimes x_i^\ast \) implies that all \( x_i^\ast \)'s have the same dimension \( d \). Hence, \( \dim H = |G| + d^2 n \). Moreover, since \( H \) is not cocommutative, then \( d > 1 \) and \( G(H) \simeq G \).

Condition \( x_i \otimes x_j \in kG \), for all \( 1 \leq i, j \leq n \), implies that the natural action of \( G \) on the set \( \{x_1, \ldots, x_n\} \) is transitive; see [21]. Moreover, since \( x_i \otimes x_i^\ast = \oplus_{s \in \Gamma} s, \forall i \), then the stabilizer of \( x_i \) with respect to this action is the subgroup \( \Gamma \) and it has order \( d^2 \).

Therefore \( |G| = d^2 n \), implying that \( G(H) \simeq G \) has index 2 in \( H \) and thus \( H \) fits into an exact sequence (4.1). \( \square \)

Equations (5.1) generalize the fusion rules considered by Tambara and Yamagami in [29], that correspond in our setting to the case \( n = 1 \).

For the case \( n = 1 \) the classification appears in the paper [28], where, moreover, a necessary and sufficient condition is given in order that a fusion category with the fusion rules in [29] admit a fiber functor, and hence be equivalent to the representation category of some semisimple Hopf algebra. The fact that, in this case, \( H \) is an abelian extension as in Proposition 4.10 was observed in [19].

By Proposition 4.10, \( H \) fits into an abelian exact sequence

\[
(5.2) \quad k \to k\hat{\Gamma} \to H \to kF \to k,
\]

where \( \hat{\Gamma} \) is a normal abelian subgroup of \( G \) of order \( d^2 \) possessing a non-degenerate 2-cocycle, and \( F \) is a group of order \( 2n \).

In particular, as a Hopf algebra, \( H \) is isomorphic to a bicrossed product \( H \simeq k\hat{\Gamma}^\ast \#_\sigma kF \), corresponding to fixed actions \( \triangleright : \hat{\Gamma} \times F \to F \), \( \triangleleft : F \times \hat{\Gamma} \to \hat{\Gamma} \), and compatible cocycles \( \sigma : F \times F \to (k\hat{\Gamma})^\times \), \( \tau : \hat{\Gamma} \times \hat{\Gamma} \to (kF)^\times \).

We shall now compare the description of the corepresentation theory of \( H \) given by Lemma 4.2 with the corepresentation theory of crossed products given by Clifford theory. See [13, Section 3], [17].

Let \( x \in F \) and let \( \hat{\Gamma}^x \subseteq \hat{\Gamma} \) denote the isotropy subgroup of \( x \) with respect to the action \( \triangleright \).

For all \( s, t \in \hat{\Gamma} \), write \( \tau(s,t) = \sum_{y \in F} \tau_y(s,t)e_y \), where \( e_y, y \in F \), are the canonical idempotents in \( kF \), and \( \tau_y(s,t) \in k^\times \). The restriction of \( \tau_x \) defines a normalized 2-cocycle \( \tau_x : \hat{\Gamma}^x \times \hat{\Gamma}^x \to k^\times \). Let \( k_{\hat{\Gamma}^x} \) denote the corresponding twisted group algebra.

Since \( H^\ast \simeq k\hat{\Gamma}^\ast \#_\tau k\hat{\Gamma} \) is a crossed product as an algebra, the isomorphism classes of irreducible \( H^\ast \)-modules (= \( H \)-comodules) are parameterized by the modules

\[
(5.3) \quad V_{x,W} = \text{Ind}_{k\hat{\Gamma}^\ast \#_\tau k\hat{\Gamma}^x} x \otimes W = H^\ast \otimes_{k\hat{\Gamma}^\ast \#_\tau k\hat{\Gamma}^x} (x \otimes W),
\]
where $x$ runs over a set of representatives of the orbits of $\hat{\Gamma}$ on $F$, and $W$ runs over a system of representatives of isomorphism classes of irreducible left $k_{\tau_x}\hat{\Gamma}^x$-modules. We have $\dim V_{x,W} = [\hat{\Gamma}:\hat{\Gamma}^x] \dim W$.

**Proposition 5.2.** The exact sequence (5.2) is cocentral. In other words, the action $:\hat{\Gamma} \times F \to F$ is trivial.

**Proof.** Let $\mathcal{C}$ denote the fusion category of representations of the dual Hopf algebra $H^*$, and view $\text{Rep} k\hat{\Gamma}$ as a fusion subcategory of $\mathcal{C}$. For any simple object $x \in \mathcal{C}$ we have $xx^* = \sum_{s \in \Gamma} s$. Hence $\mathcal{C}_{\text{ad}} = \text{Rep} k\hat{\Gamma}$, where $\mathcal{C}_{\text{ad}}$ is the adjoint subcategory as defined in [8, Section 8.5].

Let $U(\mathcal{C})$ be the universal grading group of $\mathcal{C}$, in the terminology of [8, Section 3.2]. Then the category $\mathcal{C}$ is $U(\mathcal{C})$-graded with trivial component equal to $\mathcal{C}_{\text{ad}}$.

By [8, Theorem 3.8], $\mathcal{C}_{\text{ad}} = \text{Rep}(H^*/H^*K^+)$, where $K = kU(\mathcal{C})$ is the maximal Hopf subalgebra of $H^*$ which is contained in its center. Exactness of the sequence (5.2) now implies that $U(\mathcal{C})$ is isomorphic to $F$ and, moreover, the inclusion $kF \subseteq H^*$ is central, as claimed. \qed

**Remark 5.3.** Propositions 5.2 and 5.4 imply that the category $\text{Rep} H$ is tensor equivalent to an $F$-equivariantization $(\text{Vec}^\Gamma)^F$. See Remark 2.1.

Consider the subgroup $F_0 \subseteq F$ defined by $F_0 = \{ x \in F : |\tau_x| = 1 \}$.

**Proposition 5.4.** There is an exact sequence of groups $1 \to \Gamma \to G \to F_0 \to 1$. In particular, the subgroup $F_0 \subseteq F$ is of order $n$.

**Proof.** It follows from Lemma 2.2 since $F\hat{\Gamma} = F$. By exactness of the sequence $1 \to \Gamma \to G \to F_0 \to 1$, $|F_0| = n$ is a necessary and sufficient condition in order that $[H : kG] = 2$. \qed

**Remark 5.5.** Proposition 5.4 can be proved alternatively as follows. Consider the simple $H^*$-modules (5.3). Note that $\dim V_{x,W} = [\hat{\Gamma} : \hat{\Gamma}^x] \dim W = 1$, if and only if $x \in F$ is a fixed point under the action of $\hat{\Gamma}$ and $W$ is an irreducible representation of $k_{\tau_x}\hat{\Gamma} = k_{\tau_x}\hat{\Gamma}^x$ of dimension 1. If such $W$ exists, then the class of $\tau_x$ is trivial (the twisted group algebra $k_{\tau_x}\hat{\Gamma}$ being augmented), and therefore $k_{\tau_x}\hat{\Gamma} \cong k\hat{\Gamma}$.

Thus, since $\Gamma$ is abelian, every element $x \in F$ such that $|\tau_x| = 1$ gives exactly $|\hat{\Gamma}| = d^2$ one-dimensional $H$-comodules $V_{x,W}$’s. Hence $|G(H)| = |F_0|d^2$. On the other hand, by Lemma 4.2, $H$ has exactly $|G| = d^2n$ irreducible co-modules of dimension 1. Therefore $|F_0| = n$, as claimed.

**Proposition 5.6.** For all $x \in F \setminus F_0$, the 2-cocycle $\tau_x \in H^2(\hat{\Gamma}, k^x)$ is non-degenerate.

**Proof.** Let $x \in F \setminus F_0$ and let $V_{x,W}$ be the simple $H$-comodule described in (5.3), where $W$ is a simple $k_{\tau_x}\hat{\Gamma}^x = k_{\tau_x}\hat{\Gamma}$-module. In view of Lemma 4.2 and Proposition 5.3 we have $\dim V_{x,W} = \dim W = d$. This implies that $\tau_x$ is nondegenerate, as claimed. \qed
We next prove our main classification result.

**Proof of Theorem 1.1.** We need to show that a semisimple Hopf algebra $H$ with $[H : kG(H)] = 2$ is determined by a triple $(\Gamma, F, \xi)$, where $F$ is a finite group acting on the abelian group $\Gamma$ by automorphisms, and $\xi = [(\sigma, \tau)] \in \text{Opext}(k\hat{F}, kF)$, such that the subgroup $F_0 := \{x \in F : [\tau_x] = 1\}$ has index 2 in $F$, and $\tau_x$ is non-degenerate for all $x \in F \setminus F_0$.

Suppose first given a semisimple Hopf algebra $H$ with $[H : kG(H)] = 2$. Then $H$ fits into an exact sequence (4.1), with $G = G(H)$. By the results in Section 4, $H$ is an abelian extension $k \to k\hat{F} \to H \to kF \to k$ associated to a matched pair $(\hat{F}, F)$ where the action $\hat{F} \times F \to F$ is trivial. Thus the action $\hat{F} \times F \to \hat{G}$ is by group automorphisms, giving by transposition an action by group automorphisms $F \to \text{Aut}(\hat{F})$.

Let $\sigma : F \times F \to (k\hat{F})^\times$, $\tau : \hat{F} \times \hat{F} \to (k\hat{F})^\times$ be the associated 2-cocycles. By Proposition 5.1 $\sigma \times \tau : \hat{F} \times \hat{F} \to (k\hat{F})^\times$ be the associated 2-cocycles. By Proposition 5.6 $\tau_x$ is a non-degenerate 2-cocycle, for all $x \in F \setminus F_0$. In this way, we obtain a triple $(\Gamma, F, \xi)$ satisfying the requirements. Furthermore, we have in this case $G(H) \simeq \Gamma \rtimes \sigma F_0$.

Conversely, suppose given such a triple $(\Gamma, F, \xi)$. Then we have a matched pair $(\hat{F}, F)$ with respect to the transpose action $\hat{F} \times F \to \hat{F}$. These data give rise to a Hopf algebra $H = k\hat{F} \#_\sigma kF$ as in (4.1) By Lemma 2.2 and the assumption $[F : F_0] = 2$, we find that $[H : kG(H)] = 2$.

Finally, suppose that the triples $(\Gamma, F, \xi)$, $(\Gamma', F', \xi')$ give rise to isomorphic Hopf algebras $H$ and $H'$, respectively.

Let $(\sigma, \tau)$ be a pair of compatible cocycles representing $\xi$. Since $[H : kG(H)] = 2$, then the coalgebra structure of $H$ is as described in Lemma 4.2. From Remark 2.3, in view of the assumption that $\tau_x$ is non-degenerate for all $x \notin F_0$, we get that $\Gamma \subseteq \Gamma_0$, where $\Gamma_0$ is the (common) stabilizer in $G(H)$ of simple subcoalgebras of dimension $> 1$.

By construction, there is a cocentral exact sequence $k \to k\hat{F} \to H \to kF \to k$. This implies that there is a faithful $F$-grading on the category $C$ of finite dimensional $H$-comodules with trivial component $C_e = k\Gamma$ - comod. Then necessarily $C_{ad} \subseteq k\Gamma$ - comod. See [9].

We have seen in the proof of Proposition 5.2 that $C_{ad} = k\Gamma_0$ - comod. Hence $\Gamma_0 \subseteq \Gamma$, and thus $\Gamma = \Gamma_0$. Similarly, $\Gamma'$ coincides with the stabilizer $\Gamma'_0$ of simple subcoalgebras of dimension $> 1$ in $G(H')$.

Therefore, a Hopf algebra isomorphism $H \to H'$ must send $\Gamma$ to $\Gamma'$. Then it induces an isomorphism of the corresponding exact sequences. The rest of the theorem follows from the fact that such exact sequences are classified by the group $\text{Opext}(k\hat{F}, kF)$.

**Remark 5.7.** Suppose that $F_0 = 1$. This corresponds to the case $n = 1$ in Lemma 4.2, so that $\text{Rep} H^*$ is a Tambara-Yamagami category with $\Gamma$ as the group of invertible objects.
Consider the invariant $(\Gamma, F, \xi)$ of $H$ given by Theorem 1.1, and identify $\Gamma \simeq \hat{\Gamma}$. The group $F$ is cyclic of order 2, hence the action of $F$ on $\Gamma$ reduces to an automorphism $T \in \text{Aut}(\Gamma)$ of order 2.

Let $(\sigma, \tau)$ be a pair of cocycles representing $\xi$. Since $F = \mathbb{Z}_2$, by [18, Proposition 1.2.6], we may assume that $\sigma = 1$. Thus $\xi$ reduces to a 2-cocycle $\xi = \tau_x : \Gamma \times \Gamma \to k^\times$, where $1 \neq x \in \mathbb{Z}_2$, such that $T^*(\xi) = \xi^{-1}$. Moreover, $\xi$ is non-degenerate.

The non-degenerate symmetric bilinear form $\chi : \Gamma \times \Gamma \to k^\times$ in [29] is given in this case by $\chi(a, b) = \alpha(a, T(b))$, for all $a, b \in \Gamma$, where $\alpha : \Gamma \times \Gamma \to k^\times$ is the non-degenerate alternating form corresponding to $\xi$ under the isomorphism $H^2(\Gamma, k^\times) \simeq \text{Hom}(\Lambda^2 \Gamma, k^\times)$.

Since the form defined by $\xi(T^*\xi_{21})^{-1}$ on $\Gamma$ is symmetric, then there exists $\nu : \Gamma \to k^\times$ such that $d\nu = \xi$.

We thus recover the invariants given by Tambara in [27, Proposition 3.2 and Theorem 3.5].

6. Semisimple Hopf algebras of low dimension

In [23] a family of examples of non group-theoretical semisimple Hopf algebras has been presented, answering a question raised in [6]. The smallest such example has dimension 36 and it is a semisolvable Hopf algebra.

We shall show in this section that every semisimple Hopf algebra of dimension $< 36$ is group-theoretical, so that in fact 36 is the smallest possible dimension for that a non group-theoretical semisimple Hopf algebra can have.

As mentioned in the introduction, except for dimension 24, every semisimple Hopf algebra of dimension $< 36$ is either nilpotent (dimension $p$, $p^2$, $p^3$, $p^4$, $p^5$, where $p$ is a prime number) or an abelian extension (dimensions 30 and $pq^2$, where $p$ and $q$ are prime numbers). Therefore all these Hopf algebras are group-theoretical, in view of [4, 19]. We may then restrict our analysis to the case of dimension 24.

In what follows we suppose that $H$ is a semisimple Hopf algebra of dimension 24 over $k$.

**Proposition 6.1.** Up to a cocycle twist of the multiplication or the comultiplication, $H$ fits into an abelian extension $k \to k^\Gamma \to H \to kF \to k$, where $|\Gamma||F| = 24$.

In particular, $H$ is group-theoretical, by [19].

**Proof.** We keep the notation and conventions in [20, Lemma 6.2.1]. By the results in *loc. cit.*, $H$ fits into an extension

\[
(6.1) \quad k \to A \to H \to \overline{H} \to k,
\]

with $\dim A, \dim \overline{H} > 1$. We shall prove that, unless $H$ satisfies the claim, $\overline{H}$ is necessarily cocommutative. Dualizing this (observe that the statement in Proposition 6.1 is of self-dual nature), we also get that $A$ is necessarily
commutative, whence the exact sequence (6.1) can be supposed to be abelian itself, and the claim is established.

Suppose $\overline{\mathcal{P}}$ is not cocommutative. Then $\dim \overline{\mathcal{P}} = 6, 8$ or 12.

First of all, if $|G(H)| = 12$, that is, $|H : kG(H)| = 2$, then $H$ fits into an abelian extension, by Proposition 4.10. Hence we may assume that neither $H$ nor $H^*$ have group of group-likes of index 2.

Also, if $H$ contains a commutative normal Hopf subalgebra of dimension 12, then $H$ is an abelian extension, since the quotient has dimension 2.

Therefore we may assume that $H$ is not of type $(1, 12; 2, 3)$ as a coalgebra.

If $\dim \overline{\mathcal{P}} = 6$ and $\overline{\mathcal{P}}$ is not cocommutative, then $\overline{\mathcal{P}} = kS_3$, and 6 divides $|G(H^*)|$. Then we may assume that $H^*$ is of type $(1, 6; 3, 2)$ as coalgebra, by [20, Lemma 6.1.1]. In this case, $H^*$ has a unique Hopf subalgebra $B$ of dimension 3, which coincides with the group algebra of the stabilizer of a simple subcoalgebra of dimension 9.

By [20, Lemma 6.1.1 and Remark 6.1.2(ii)], $H$ has a Hopf subalgebra $K$ of dimension 8 (since $\dim A = 4$ must divide $|G(H^*)|$). Then necessarily $(H^*)^{coK^*} = B$ and $B$ is normal in $H^*$. The coalgebra type of $H^*$ implies that $H^*/H^*B^+$ is cocommutative [20, Remark 3.2.7 and Corollary 3.3.2]. Therefore, the extension $k \to B \to H^* \to K^* \to k$ is an abelian extension.

If $\dim \overline{\mathcal{P}} = 8$, then $\dim A = 3$ and by [20, Lemma 6.1.1] $H$ is of type $(1, 6; 3, 2)$ as a coalgebra, since this is the only remaining possibility with group-like elements of order 3.

In this case $G(H)$ has a unique (normal) subgroup $G$ of order 3, which must coincide with the stabilizer of all simple subcoalgebras of dimension 9, such that $A = kG$. As before, the quotient $\overline{\mathcal{P}} = H/H(kG)^+$ is cocommutative.

Suppose that $\dim \overline{\mathcal{P}} = 12$ and $\overline{\mathcal{P}}$ is not cocommutative. If $\overline{\mathcal{P}}$ is commutative, then 12 divides $|G(H^*)|$ and we are done.

We may therefore assume that $\overline{\mathcal{P}}$ is not cocommutative and not commutative. By the classification in dimension 12 [8, $\overline{\mathcal{P}}$ is isomorphic to one of the self-dual Hopf algebras $A_0$ or $A_1$, in the notation of [20, 5.2].

If $\overline{\mathcal{P}} \simeq A_0$, then $\overline{\mathcal{P}}$ is a twisting of a group algebra; see [20, Proposition 5.2.1]. Then $H^*$ is twist equivalent to a Hopf algebra with group of group-likes of order 12, which must be an abelian extension.

Suppose that $\overline{\mathcal{P}} \simeq A_1$. Then $\overline{\mathcal{P}}$ is of type $(1, 4; 2, 2)$ as a coalgebra and $G(\overline{\mathcal{P}}) \simeq \mathbb{Z}_4$ is cyclic of order 4. Also, $\overline{\mathcal{P}} \simeq \overline{\mathcal{P}}$ is a Hopf subalgebra of $H^*$, and thus 4 divides $|G(H^*)|$.

Combining these with [20, Lemmas 6.1.1 and 6.1.7], since we may assume that $|G(H^*)| \neq 12$, we get that the possible coalgebra types for $H^*$ are $(1, 4; 2, 5)$ and $(1, 8; 2, 4)$. 
In the first case, $G(H^*) = G(H^*)$ is cyclic. By [20, Proposition 2.1.3], $H^*$ contains a Hopf subalgebra $K$ of dimension 8, which is not cocommutative and such that $G(H^*) \subseteq G(K)$. But, since $K$ is not cocommutative, we must have $G(K) \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$, which is a contradiction, because $G(H^*) \simeq \mathbb{Z}_4$. This discards this possibility.

Finally, consider the case where $H^*$ is of type $(1,8;2,4)$ as a coalgebra. Again by dimension, we have $H^* = k[H^*, G(H^*)]$. Let $F \subseteq G(H^*)$ be the unique subgroup of order 2. We have $F \subseteq Z(H^*)$. Also, $G(H^*)$ is normal in $G(H^*)$. Therefore $F$ is stable under the adjoint action of $G(H^*)$ and thus central in $G(H^*)$. Let $p : H^* \to B$ be the projection to the quotient Hopf algebra $B = H^*/H^*(kF)^+$. As before, $p(H^*)$ and $p(kG(H^*))$ are cocommutative and they generate $B$ as an algebra. Then $B$ is cocommutative and $H$ is an abelian extension. This finishes the proof of the proposition.

**Remark 6.2.** Suppose $k = \mathbb{C}$ is the field of complex numbers. It follows from Proposition 6.1 that, after twisting the multiplication or the comultiplication, $H$ turns into a Kac algebra, therefore belonging to the list of Izumi and Kosaki [11, Chapter XIV].

We have thus proved the following:

**Theorem 6.3.** Let $H$ be a semisimple Hopf algebra of dimension < 36. Then $H$ is group-theoretical.

In particular, 36 is the smallest possible dimension that a non group-theoretical semisimple Hopf algebra can have.

### 7. Appendix: Characters and normality

Let $H$ be a semisimple Hopf algebra and let $A \subseteq H$ be a Hopf subalgebra. In this Appendix we discuss some sufficient conditions, in terms of the character multiplication of $H$, in order that $A$ be normal in $H$.

**Lemma 7.1.** Suppose $\chi A \chi^* \subseteq A$, for all irreducible characters $\chi \in H$. Then $A$ is normal in $H$.

**Proof.** Let $C \subseteq H$ be the simple subcoalgebra with character $\chi$; so that $S(C)$ is the simple subcoalgebra with character $\chi^*$.

Write $A = \bigoplus_{\lambda \in \Lambda} C_\lambda$, where $\Lambda$ is the set of irreducible characters of $A$, and $C_\lambda \subseteq A$ is a simple subcoalgebra with character $\lambda \in \Lambda$. The character of $A$, as a left $H$-comodule, is $\chi_A = \sum_{\lambda \in \Lambda} \epsilon(\lambda)\lambda$.

By assumption, $\chi \lambda \chi^* \in A$ is a cocommutative element, and therefore it can be written as a sum $\chi \lambda \chi^* = \sum_{\lambda \in \Lambda} n_{\lambda, \lambda} \lambda$, for some $n_{\lambda, \lambda} \geq 0$, $\lambda \in \Lambda$. This implies that, for every $\lambda \in \Lambda$, $CC \lambda S(C) \subseteq A$, because the multiplication map $H \otimes H \to H$ is a left $H$-comodule map.

Therefore, $C \lambda S(C) \subseteq A$, implying that for all $c \in C$, $c_1 A S(c_2) \subseteq A$. Since the simple subcoalgebras of $H$ span $H$, this proves that $A$ is normal. 


Remark 7.2. Suppose that $C_1, \ldots, C_r$ is a set of simple subcoalgebras of $H$ that generate $H$ as an algebra. Then the assumption $\chi A \chi^* \subseteq A$, for all irreducible character $\chi \in H$, in Lemma [1] may be replaced by $\chi_i A \chi_i^* \subseteq A$, for all $i = 1, \ldots, r$, where $\chi_i$ is the character of $C_i$.

Lemma 7.3. Let $\chi \in H$ be an irreducible character corresponding to the simple subcoalgebra $C \subseteq H$.

Suppose $G[\chi] = G[\chi^*]$ and $|G[\chi]| = \chi(1)^2$. Then $kG[\chi]$ is normal in $k[C]$.

Here, $k[C] \subseteq H$ denotes the subalgebra generated by $C$, which is a Hopf subalgebra of $H$.

Proof. The assumption $|G[\chi]| = \chi(1)^2$ implies that $\chi \chi^* = \sum_{g \in G[\chi]} g$. On the other hand, since $G[\chi] = G[\chi^*]$, we also have $\chi g = \chi$, for all $g \in G[\chi]$. Then, for all $g \in G[\chi]$, $\chi g \chi^* = \chi \chi^* \in kG[\chi]$. By Lemma [1] and Remark [2] $kG[\chi]$ is normal in $k[C]$. □

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