Deterministic Fault-Tolerant Connectivity Labeling Scheme

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Abstract

The \emph{$f$-fault-tolerant connectivity labeling (f-FTC labeling)} is a scheme of assigning each vertex and edge with a small-size label such that one can determine the connectivity of two vertices $s$ and $t$ under the presence of at most $f$ faulty edges only from the labels of $s$, $t$, and the faulty edges. This paper presents a new deterministic $f$-FTC labeling scheme attaining $O(f^2 \text{polylog}(n))$-bit label size and a polynomial construction time, which settles the open problem left by Dory and Parter [DP21]. The key ingredient of our construction is to develop a deterministic counterpart of the graph sketch technique by Ahn, Guha, and McGregor [AGM12b], via some natural connection with the theory of error-correcting codes. This technique removes one major obstacle in de-randomizing the Dory-Parter scheme. The whole scheme is obtained by combining this technique with a new deterministic graph sparsification algorithm derived from the seminal $\epsilon$-net theory, which is also of independent interest. As byproducts, our result deduces the first deterministic fault-tolerant approximate distance labeling scheme with a non-trivial performance guarantee and an improved deterministic fault-tolerant compact routing. The authors believe that our new technique is potentially useful in the future exploration of more efficient FTC labeling schemes and other related applications based on graph sketches.

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1 Introduction

1.1 Motivation and Background

Most message-passing distributed systems are modeled by graphs. By the nature of distributed computing, nodes in the network must cooperatively solve a given task without rich access to the whole topological information. In addition, the network is typically prone to faults, i.e., some of the vertices and/or links can be down by faults. Hence the distributed and compact representation of some property of the network (e.g., connectivity) adapting to topology modification is potentially useful for applications in distributed environments. The \( f \)-fault-tolerant connectivity labeling (\( f \)-FTC labeling) is a scheme of assigning each vertex and edge with a small-size label. For any two vertices \( s \) and \( t \), and an edge set \( F \) of \( |F| \leq f \), it determines the connectivity of two vertices \( s \) and \( t \) under the deletion of edges \( F \) only from the labels of \( s \), \( t \), and the edges in \( F \). The concept of \( f \)-FTC labeling schemes (precisely, more general fault-tolerant distance labeling schemes returning the \( s-t \) distance rather than the \( s-t \) connectivity) has been initiated explicitly by Courcelle and Twigg [CT07], following an earlier work by Feigenbaum, Karger, Mirrokni, and Sami [FKMS07]. The feature of FTC labeling schemes as a distributed data structure yields efficient structural algorithms for the forbidden set routing which routes packets avoiding a given set of faulty edges, and for more general fault-tolerant compact routing [CT07, Che11, CLPR12, DP21, Raj12] where the faulty edge set is initially unknown.

1.2 Our Result

While all of early results [ACG12, ACGP16, BCG+21, CT07, CT10] mainly focus on the construction of small-sized labels for restricted graph classes, \( f \)-FTC labeling schemes for general graphs were recently proposed by Dory and Parter [DP21]. They propose two randomized \( f \)-FTC labeling schemes of \( O(f + \log n) \)-bit and \( O(\log^3 n) \)-bit label sizes respectively, which guarantee the weaker form of the correctness that the response to each single query is correct with high probability. In other words, they guarantee the correctness only for \( 1 - 1/\tilde{O}(\text{poly}(n)) \) fraction of all possible queries. We refer to this type of correctness criteria as “whp query support”, in contrast with the standard criteria of “full query support” ensuring correct answers for all possible queries with high probability. The authors of [DP21] also mention how the presented two schemes are converted to the ones with full query support, allowing the blow-up of their label sizes into \( O(f \log n) \) bits and \( O(f \log^3 n) \) bits respectively (see the footnote 4 of [DP21]). In total, the paper [DP21] presents the four randomized schemes, two of which attain full query support and the other two attain only whp query support. They leave as an open problem the polynomial-time deterministic construction of compact FTC labeling schemes for general graphs. The main contribution of this paper is to settle this open problem:

\textbf{Theorem 1} There exist two deterministic \( f \)-FTC labeling schemes for any graph \( G \) of \( n \) vertices, \( m \) edges, and diameter \( D \), which respectively attain the following bounds:

- The label size is \( O(\log n) \) bits per vertex, and \( O(f^2 (\log^2 n) \log \log n) \) bits per edge. The query processing time is \( \tilde{O}(|F|^4) \), where \( F \) is the set of queried edges satisfying \( |F| \leq f \). The construction time is polynomial of \( m \).

- The label size is \( O(\log n) \) bits per vertex, and \( O(f^2 \log^3 n) \) bits per edge. The query processing time is \( \tilde{O}(|F|^4) \). The construction time is near linear, i.e., \( \tilde{O}(mf^2) \). In addition, there exists

\footnote{The \( \tilde{O}(\cdot) \) notation hides polylog(n) factors.}
a deterministic CONGEST distributed algorithm of constructing the labels in $\tilde{O}(\sqrt{mD} + f^2)$ rounds.

Note that every deterministic scheme inherently achieves full query support. We emphasize that de-randomizing any of two original schemes of Dory and Parter is a highly non-trivial challenge. Those schemes are based on the other labeling schemes representing a sort of cut structures, whose construction heavily depends on randomness. Our deterministic construction is based on the second scheme of Dory and Parter [DP21] utilizing the graph sketch technique by Ahn, Guha, and McGregor [AGM12b] as the key structure. Informally, the graph sketch is a labeling scheme to edges, admitting the detection of an outgoing edge for a given vertex set $S \subseteq V_G$ from the bitwise XOR of all labels of the edges incident to vertices in $S$. The technical highlight of our result is to develop a deterministic counterpart of the graph sketch technique via some natural connection with the theory of error-correcting codes. This technique is very simple, and completely removes one of two major obstacles in de-randomizing the outgoing edge detection by graph sketches. Yet another obstacle is the sparsification of the input graph. The sketch-based outgoing edge detection, including ours, works only when the input vertex set $S$ has a small number of outgoing edges. To handle the case with many outgoing edges, the original approach prepares a collection of spanning subgraphs, where for each possible input $S$ with a non-empty outgoing edge set, there exists at least one subgraph in the collection such that $S$ has exactly one outgoing edge. The construction of such a collection follows random sampling of edges. Our second contribution is a novel de-randomization technique for this graph sparsification process based on the seminal $\epsilon$-net theory in computational geometry [HW87]. On this part, we present two different algorithms respectively corresponding to the schemes presented in Theorem 1.

In addition to the key technical ideas above, the construction in Theorem 1 are developed on the top of a few more notable features: First, our result is presented as a general framework with good modularity, and thus one can easily transform our deterministic scheme into an efficient randomized FTC labeling scheme with full query support, just by replacing the graph sparsification part with the conventional random edge sampling. The construction time and label size of this randomized scheme are competitive with the original sketch-based scheme in [DP21]. Second, we propose a new query optimization strategy. A drawback of our deterministic outgoing edge detection technique requires the decoding time roughly quadratic of the label size. Since the sketch-based $f$-FTC labeling scheme requires $|F|$ iterations of the outgoing edge detection for processing a single query, the straightforward implementations result in the $\tilde{O}(f^4|F|)$ time for the deterministic case, and $\tilde{O}(f^2|F|)$ time for the randomized case. Our query processing algorithm shaves off this additional $|F|$ factor, as well as getting rid of the dependency on $f$ in the outgoing edge detection. Consequently, we obtain a slightly improved randomized $f$-FTC labeling scheme of $\tilde{O}(|F|^2)$ decoding time. While the improvement of replacing $f$ by $|F|$ is very straightforward and easily applicable to any scheme not limited to ours, it is practically an intriguing feature because in typical scenarios the actual number of faults $|F|$ is substantially smaller than the upper bound $f$.

The detailed comparison between the schemes in [DP21] and our schemes are summarized in Table 1.

### 1.3 Applications

Our deterministic replacement of graph sketches provides a clearer insight to known outgoing edge detection techniques. It is simple, versatile, and potentially useful in the future exploration of other applications not limited to FTC-labeling schemes (e.g., [AGM12a, GK18, GP16, GKKT15, ...].
Table 1: Comparison between the schemes in [DP21] and our results. The dagger mark † implies that the complexity is not explicitly stated in the original paper, and thus based on our analyses. For any scheme, the dependency on \( f \) in query processing time is easily replaced by \(|F|\) utilizing the technique proposed in this paper.

| label size | query time | Det./Rand. | correctness | construction |
|------------|------------|------------|-------------|--------------|
| 1st (whp) [DP21] | \( O(f + \log n) \) | \( O(f^3) \) | Rand. | whp | \( \tilde{O}(fm) \) |
| 2nd (whp) [DP21] | \( O(\log^3 n) \) | \( \tilde{O}(|F|) \) | Rand. | whp | \( \tilde{O}(fm) \) |
| 1st (full)[DP21] | \( O(f \log n) \) | \( \tilde{O}(f^3) \dagger \) | Rand. | full | \( \tilde{O}(fm) \) |
| 2nd (full)[DP21] | \( O(f \log^3 n) \) | \( \tilde{O}(f|F|) \dagger \) | Rand. | full | \( \tilde{O}(fm) \) |
| This paper | \( O(f^2 \log^3 n) \) | \( O(|F|^4) \) | Det. | full | \( \tilde{O}(fm) \) |
| This paper | \( O(f^2 \log^2 n \log \log n) \) | \( O(|F|^4) \) | Det. | full | \( \tilde{O}(fm) \) |
| This paper | \( O(f \log^3 n) \) | \( O(|F|^2) \) | Rand. | full | \( \tilde{O}(fm) \) |

Additional content:

GP18, HPP+15, JN18, KLM+14, KW14, KKM13, KKT15, MK21]. Actually, we obtain several non-trivial de-randomization results in related topics. It has been shown in [DP21] that one can deduce the approximate distance labeling scheme, which provides an approximate \( s-t \) distance in \( G - F \) given the labels of \( s, t \) and the edges in \( F \), utilizing any \( f \)-FTC labeling scheme in the black-box manner. In addition, such an approximate distance labeling scheme further deduces an efficient fault-tolerant compact routing scheme. Since the deduction parts are deterministically implemented, our deterministic \( f \)-FTC labeling schemes also de-randomize the construction of the schemes above. More precisely, we obtain the following applications as corollaries of Theorem 1.

**Corollary 1** Assume that the input graph is any weighted undirected graph with polynomially bounded edge weights. For any positive integers \( k > 0 \) and \( f > 0 \), there exists a \( f \)-fault tolerant \( O(|F|^k) \)-approximate distance labeling scheme which achieves \( \tilde{O}(f^2 n^{1/k}) \)-bit label size and \( \tilde{O}(|F|^4) \) query time.

**Corollary 2** For any positive integer \( k > 0 \) and \( f > 0 \), there exist two deterministic fault-tolerant compact routing schemes which achieve the stretch factor of \( O(|F|^2) \) and one of the following table-size bounds:

- \( \tilde{O}(f^2 n^{1+1/k}) \)-bit total table size and \( \tilde{O}(f^2 n^{1+1/k}) \)-bit maximum local table size.
- \( \tilde{O}(f^5 n^{1/k}) \)-bit maximum local table size.

The result of Corollary 1 is the first deterministic scheme for general graphs achieving a non-trivial performance guarantee. On Corollary 2, the prior work by Chechik [Che11], which attains \( O(|F|^2(|F|+\log^2 n) k) \) stretch factor with a smaller table size, is also implemented deterministically, and thus our result is not the first deterministic solution. However, our scheme takes an advantage with respect to stretch factors. Since the proofs completely follow the reduction techniques proposed in [DP21], this paper does not present the precise formalism on these corollaries. See [DP21] for details.

### 1.4 Related Work

As mentioned in Section 1.1, FTC-labeling schemes, and more general fault-tolerant (approximate) distance labeling schemes are introduced in the literature of *forbidden set routing*, which is the
routing scheme avoiding non-adaptive faulty edges/vertices (i.e., the set of faults is not specified at
the construction of routing tables, but given at the beginning of packet routing). The first result
by Courcelle and Twigg [CT07] presents a FTC-labeling scheme of \( O(k^2 \log n) \)-bit labels for graphs
of treewidth \( k \), as well as its application to forbidden-set routing. A few results following this line
exist [ACG12, ACGP16, BCG+21, CT07, CT10], but all of them are interested in the construction of
compact labels for specific graph classes. The result for general graphs is not much addressed
until the result by Dory and Parter [DP21]. In the context of deterministic construction, many of
the results for restricted graphs stated above are deterministic, but the deterministic construction
for general graphs is not known so far. In the paper of Dory and Parter, two randomized FTC
labeling schemes relying on different techniques are presented. While the second scheme is based
on graph sketches as we mentioned, the first one relies on the cut-verification labeling by Pritchard
and Thurimella [PT11].

There are many works in the literature of the centralized version of connectivity oracles and (ap-
proximate) distance oracles supporting edge/vertex deletion [BGLP16, BCG+18, BK09, CLPR12,
CCFK17, DT02, DP09, DP10, DP20, GR21, GW19]. One of the major setting on this line is the
case of \( f = 1 \), which is known as the replacement path problem [GW19, GR21, BK09], or distance
sensitivity oracle [DT02, BK09, BGLP16, BCG+18]. Roughly, the replacement path problem com-
putes all pair (approximate) shortest path distances for every possible single edge/vertex fault. The
sensitivity oracle is further required to store the information of replacement paths into a compact
data structure. Their single-source variants are also investigated [BGLP16, BK13]. The sensitivity
oracles for multiple faults are considered mainly in the context of fault-tolerant compact routing,
which is a generalization of forbidden-set routing addressing adaptive faults (i.e., the set of faults
is not explicitly given at the beginning of packet routing) [Che11, CLPR12, DP21, Raj12]. There are also a few results considering the oracles specific to connectivity [PT07, DP09, DP10, DP20],
which are seen as centralized counterparts of FTC labeling schemes. Obviously, any \( f \)-FTC la-
beling scheme is also usable as a centralized oracle with the space complexity of \( m \times \text{label size} \). More general dynamic connectivity of undirected graphs aims to develop the data struc-
ture of supporting the operations of inserting/deleting edges as well as the connectivity query.
By definition, such a data structure can be used as a fault-tolerant connectivity oracle with the
query processing time of \( O(|F| \cdot \text{operation cost}) \). While there is a long history of this prob-
lem [Tho00, HdLT01, HK99, CGL+20, WN, PD06, KKM13, GKT15], all the known results with
polylog\((n)\)-time operation cost rely on amortized analyses or the correctness criteria of whp guaran-
tee. Focusing on deterministic construction, the best known results are the algorithm by Chuzhoy,
Gao, Li, Nanongkai, Peng and Saranurak for dynamic connectivity achieving \( n^{o(1)} \) operation cost
per one edge deletion [CGL+20], and the connectivity oracle by Pătraşcu and Throup [PT07]
achieving \( \tilde{O}(|F|) \) query processing time. The deterministic algorithm for dynamic connectivity
with worst-case polylog\((n)\)-time operation is a major open problem in this research field.

While this paper focuses only on edge faults, it is also an interesting research direction to con-
sider vertex faults. Despite its similarity, vertex fault-tolerance often exhibits a technical difficulty
quite different from edge fault-tolerance. A trivial approach is to reduce the failure of a vertex \( v \)
into the failure of all the edges incident to \( v \), which results in a \( f \)-vertex fault tolerant connectivity
labeling scheme of \( \tilde{O}(\Delta f) \)-bit label size (where \( \Delta \) is the maximum degree of the input graph).
Unfortunately, this approach does not provide a good worst-case bound because \( \Delta \) could become
\( \Omega(n) \). Recently, vertex fault tolerant labeling schemes for small \( f \) are proposed by Parter and
Petruschka [PP22b]. They provide two schemes respectively attaining a polylogarithmic label size
for \( f = 2 \) and a sublinear size for \( f = o(\log \log n) \).

The graph sketch technique is first presented by Ahn, Guha, and McGregor [AGM12b], aiming
to develop space-efficient algorithms for graph stream [AGM12a, KW14, KLM+14]. There are a
variety of applications not limited to graph stream, such as distributed computation [GK18, GP16, GP18, HPP+15, JN18, KKT15, MK21] and dynamic algorithms [GKKT15, KKM13].

1.5 Roadmap
Following the introduction of necessary notations and definitions, we first explain the high-level idea of our framework in Section 3, including the explanation of the sub-components constituting our scheme. Section 4 explains the key technical ideas of our de-randomization technique. Following the summary of whole structure in Section 5, we present a further query optimization technique in Section 6. Section 7 explains the details of our construction. We consider the distributed construction of our labeling schemes in Section 8. Finally, we conclude this paper in Section 9, as well as a few promising future research directions.

2 Notations and Terminologies
We denote the vertex set and edge set of a graph \( G \) respectively by \( V_G \) and \( E_G \). We use the notation \( H \subseteq G \) to represent that \( H \) is a subgraph of \( G \). For any edge subset \( E' \subseteq E_G \), we define \( G - E' \) as the graph obtained from \( G \) by removing all the edges in \( E' \). Given a vertex subset \( S \subseteq V_G \), a \textit{outgoing edge} of \( S \) is the edge having exactly one endpoint in \( S \). We define \( \partial_G(S) \) as the set of all outgoing edges of \( S \) in \( G \). For any \( E' \subseteq E_G \), we also define \( \partial_{E'}(S) = \partial_G(S) \cap E' \).

Given a rooted tree \( T \) and vertex \( v \in V_T \), we denote by \( T(v) \) the subtree of \( T \) rooted by \( v \). Given an edge \( e \in E_T \), we also denote by \( T(e) \) the subtree of \( T \) rooted by the lower vertex (i.e., the endpoint farther from the root than the other) of \( e \).

An \( f \)-\textit{FTC labeling scheme} for a given input graph \( G \) consists of a labeling function \( L^\text{con}_{G,f} \) and a universal decoding function \( D^\text{con}_f \). The labeling function assigns each of vertices and edges \( x \in V_G \cup E_G \) with a label \( L^\text{con}_{G,f}(x) \) (i.e., a binary string). Let \( s, t \in V_G \) be any two vertices and \( F \subseteq E \) be any edge subset of size at most \( f \). The decoder function \( D^\text{con}_f \) correctly answers the connectivity between \( s \) and \( t \) in \( G - F \) only with the information of \( L^\text{con}_{G,f}(s) \), \( L^\text{con}_{G,f}(t) \), and \( \{L^\text{con}_{G,f}(e) \mid e \in F\} \). Note that the decoder function \( D^\text{con}_f \) is universal for all \( G \), and cannot have any direct access to the information of \( G \). The detailed formalism of \( f \)-\textit{FTC labeling scheme} is given in Section 7.1.

3 Construction Framework
We first introduce a general framework of constructing the \( f \)-\textit{FTC labeling scheme}. The technical core of this framework relies on the scheme by Dory and Parter [DP21], but some additional techniques and abstractions are newly introduced. Let \( G \) be the undirected input graph of \( n \) vertices and \( m \) edges. Throughout this paper, we fix an arbitrary rooted spanning tree \( T \) of \( G \). The framework consists of two technical components. The first one is a weaker variant of \( f \)-\textit{FTC labeling schemes} which supports only the query \((s, t, F)\) satisfying \( F \subseteq E_T \) (i.e., only the edges in \( T \) can be faulty). We refer to this scheme as the \textit{tree edge} \( f \)-\textit{FTC labeling scheme}. Our tree edge \( f \)-\textit{FTC labeling scheme} is implemented with two other labeling schemes, respectively referred to as the \textit{ancestry labeling scheme} and the \textit{\( S \)-outdetect labeling scheme} (explained in the next section). The second component is the very simple transformer which deduces an \( f \)-\textit{FTC labeling scheme} from any tree edge \( f \)-\textit{FTC labeling scheme} with no blow up of label size. In the following sections we explain the outline of each component.
3.1 Tree Edge $f$-FTC Labeling Scheme

First, we state the informal specifications of the two sub-schemes. The formal definitions of these sub-schemes are also presented in Section 7.1.

- **Ancestry Labeling Scheme:** Let $T$ be any tree. This scheme assigns each vertex $v \in V_T$ with a label $L^\text{anc}_T(v)$. Given two labels $L^\text{anc}_T(u)$ and $L^\text{anc}_T(v)$ of distinct vertices $u, v \in V_T$, one can determine if $u$ is the ancestor of $v$, the descendant of $v$, or otherwise. There exists a linear-time deterministic algorithm which provides the ancestry labeling of $O(\log n)$ bits [KNR92].

- **$S$-Outdetect Labeling Scheme:** We assume that each edge $e \in E_G$ is assigned with a unique ID from some domain $\mathcal{E}$. Let $S \subseteq 2^V$ be a collection of vertex subsets. An $S$-outdetect labeling scheme assigns each vertex $v \in V_G$ with a label $L^\text{out}_G(v)$. For any vertex subset $S \in S$ such that $\partial_G(S)$ is nonempty, one can compute an outgoing edge $e \in \partial_G(S)$ only from the bitwise XOR sum $\bigoplus_{v \in S} L^\text{out}_G(v)$ of all the labels assigned to vertices in $S$. If $\partial_G(S)$ is empty, the scheme also detects it. The graph sketch technique [ACG12] provides a randomized $S$-outdetect labeling scheme with $O((\log |S| \cdot \text{polylog}(n)))$-bit label size.

We define $S_{f,t}$ as the collection of all vertex subsets $S$ satisfying $\partial_T(S) \leq f$. Note that $S$ is not required to induce a connected subtree of $T$. Roughly, our tree edge $f$-FTC labeling scheme is the combination of the ancestry labeling scheme of $T$ and the $S_{f,t}$-outdetect labeling scheme of $G - E_T$ for an appropriate edge ID domain $\mathcal{E}$ (explained later). Each vertex $u$ is assigned with the ancestry label of $u$, and each tree edge $e = (u, v) \in E_T$ is assigned with the concatenation of the ancestry labels of $u$ and $v$, and the XOR sum of the $S_{f,t}$-outdetect labels over all the descendant vertices of $e$. We do not have to assign any label to non-tree edges because we focus on the construction of the tree edge $f$-FTC labeling scheme.

Given a query $(s, t, F)$ satisfying $F \subseteq E_T$ and $|F| \leq f$, the spanning tree $T$ is split into $|F| + 1$ subtrees by removing all the edges in $F$. We refer to the vertex set of each split subtree as a fragment. Let $S$ be the fragment of containing $s$. The query processing algorithm iteratively grows $S$ by detecting an outgoing edge $e \in \partial_{G - E_T}(S)$. If such an edge is found, the fragment that $e$ reaches from $S$ is merged into $S$. This process terminates until no outgoing edge is found or the fragment with $t$ is merged. If no outgoing edge is found, one can conclude that $s$ and $t$ are not connected in $G - F$, or connected otherwise. Our framework detects an outgoing edge of $S$ in $G - E_T$ by the $S_{f,t}$-outdetect labeling scheme (recall $S \in S_{f,t}$ by definition). With the support of the ancestry labeling scheme, one can detect the ancestor-descendant relationship between any entities in $s$, $t$, and $F$, which provides the information of the edge set $\partial_T(S')$ for all fragments $S'$. To compute $\bigoplus_{v \in S'} L^\text{out}_{G - E_T}(v)$ for each fragment $S'$, it suffices to compute the XOR sum of the $S_{f,t}$-outdetect labels assigned to the edges in $\partial_T(S')$. Since the outdetect label of each edge in $T$ is the XOR sum of all descendants’ outdetect labels, it appropriately cancels out the labels assigned with the vertices not in $S'$. The fragment merging is simple but has a point to be careful. Let $e = (u, v)$ be an outgoing edge of $S$ (assuming $u \in S$), and $S'$ be the fragment containing $v$. Then the XOR sum over all the labels of $S \cup S'$ is easily calculated by the XOR sum of the two computed sums for $S$ and $S'$. However, how can we identify the fragment $S'$ containing $v$? This problem is resolved by embedding the ancestry labels of $u$ and $v$ into the edge ID of $e$. That is, as a preprocessing step, we assign each edge $(u, v)$ with the pair of $(L^\text{anc}_T(u), L^\text{anc}_T(v))$ as the edge ID, and the outdetect labeling is constructed for the edge domain by this assignment. Then the decoding of an edge ID immediately yields the information of the fragments containing its endpoints. We formalize our framework explained above into the following lemma.
**Lemma 1** Assume any deterministic $S_{f,T}$-outdetect labeling scheme $(L_{\text{out}}^G, D_{\text{out}})$ of label size $\alpha$ and decoding time $\beta$. Then there exists a deterministic tree edge $f$-FTC labeling scheme of $(\alpha + O(\log n))$-bit label size and $O(|F|/ \beta + \log |F|)$ decoding time, where $F$ is a set of faulty edges given by a query.

The proof details are given in Section 7.2.

**3.2 Transformation to General Scheme**

The second component reduces the construction of general $f$-FTC labeling schemes into that of tree edge $f$-FTC labeling schemes. For the input graph $G$ and its rooted spanning tree $T$, our transformation constructs an auxiliary graph $G'$ by subdividing all non-tree edges $e \in E_G \setminus E_T$ into two edges, respectively referred to $e$ and $e'$ (see Figure 1). The spanning tree $T'$ of $G'$ is also obtained by adding $e$ to the tree $T$. This input transformation naturally defines an injective (but not onto) mapping $\sigma : E_G \rightarrow E_{T'}$, where every edge $e \in E_G$ is mapped to the corresponding edge in $T'$ with the same name. A query $(s, t, F)$ for $G$ is also naturally interpreted to the query $(s, t, \{\sigma(e) | e \in F\})$ for $G'$. It is easy to see that $s$ and $t$ are connected in $G - F$ if and only if they are connected in $G' - \{\sigma(e) | e \in F\}$. Hence the following proposition obviously holds:

**Proposition 1** Let $G$ and $T$ be the input graph and its rooted spanning tree, and $G'$, $T'$, and $\sigma$ be the graphs and the mapping as defined above. Assume any tree edge $f$-FTC labeling scheme $(L_{\text{tree}}^{G',f}, D_{\text{tree}}^f)$ for $G'$ and $T'$. Then we define the labeling function $L_{G,f}^{\text{con}}$ for $G$ as follows:

$$L_{G,f}^{\text{con}}(x) = \begin{cases} L_{G',f}^{\text{tree}}(x) & \text{if } x \in V_G \cup E_T \\ L_{G',f}^{\text{tree}}(\sigma(x)) & \text{otherwise.} \end{cases}$$

Then $(L_{G,f}^{\text{con}}, D_{\text{tree}}^f)$ is a $f$-FTC labeling scheme for $G$.

Combining Lemma 1 and Proposition 1, we obtain the following corollary:

**Corollary 3** Assume any deterministic $S_{f,T}$-outdetect labeling scheme $(L_{G}^{\text{out}}, D_{\text{out}})$ of label size $\alpha$ whose decoding time is $\beta$. Then there exists a deterministic $f$-FTC labeling scheme of $(\alpha + O(\log n))$-bit label size and $O(|F|/ \beta + \log |F|)$ decoding time, where $F$ is a set of faulty edges given by a query.
4 Technical Outline of Our Approach

4.1 Obstacles in De-Randomization

Corollary 3 implies that the difficulty of de-randomization lies only at the implementation of the deterministic $S_f,T$-outdetect labeling scheme. The known $S$-outdetect labeling scheme based on the graph sketch includes two major points relying on random bits, which are summarized as follows:

- The first is at the computation of vertex labels. Let $I_G(v)$ be the set of incident edges of $v$ in $G$. The graph sketch first prepares some function $g : E \rightarrow \{0,1\}^k$, where $E$ is the edge ID domain and $k$ is the label length, and define the label $L_{G}^{\text{out}}(v)$ of vertex $v$ as the bitwise XOR sum of $g(e)$ for all the edges $e \in I_G(v)$. When computing $\bigoplus_{e \in S} L_{G}^{\text{out}}(v)$ for a given subset $S \subseteq V_G$, the value $g(e)$ for any $e$ lying at the inside of $S$ are canceled out because the term $g(e)$ appears exactly twice in the sum $\bigoplus_{e \in S} L_{G}^{\text{out}}(v) = \bigoplus_{e \in S}(\bigoplus_{e \in I_G(v)} g(e))$. That is, $\bigoplus_{e \in S} L_{G}^{\text{out}}(v) = \bigoplus_{e \in \partial_G(S)} g(e)$ holds. For clarifying the essence of the first point, we consider the simple case such that $|\partial_G(S)| = 1$ holds (the general case is addressed in the second point). In this case, $\bigoplus_{e \in S} L_{G}^{\text{out}}(v) = g(e)$ obviously holds for the unique outgoing edge $e$ of $S$. Hence one can extract the outgoing edge ID from $\bigoplus_{e \in S} L_{G}^{\text{out}}(v)$, provided that there exists a way of computing the inverse $g^{-1}$. However, if $g$ is not well-designed, some subset $S' \in S_f,T$ which does not have $e$ as an outgoing edge might accidentally satisfy $\bigoplus_{e \in S'} L_{G}^{\text{out}}(v) = g(e)$. Then, $e$ is wrongly detected as an outgoing edge of $S'$. To avoid it, the graph sketch needs to guarantee that $\bigoplus_{e \in S'} L_{G}^{\text{out}}(v)$ becomes different from the value $g(e)$ of any edge $e \in E_G$ if $|\partial_G(S')| \neq 1$. The first point of utilizing random bits is to attain this condition by taking a random hash function $g$.

- As explained above, the graph sketch provides the $S$-outdetect labeling scheme working only for $S \in S$ satisfying $|\partial_G(S)| = 1$. To cover the case of $\partial_G(S) > 1$, the original scheme prepares the collection $G$ of spanning subgraphs of $G$ such that for any $S \in S$ there exists a corresponding $H \in G$ which satisfies $\partial_H(S) = 1$. The label to vertex $v$ is then obtained by the concatenation of $L_{H}^{\text{out}}(v)$ for all $H \in G$. Roughly, each spanning subgraph in $G$ must be (almost everywhere) sparser than the original input $G$. The construction of each graph in that collection follows a stochastic sampling of edges, which is the second point relying on randomization.

The technical highlight of our deterministic scheme is twofold, which respectively resolve the two issues above. We explain the outline of each technique in the remainder of this section. The formal argument is provided in Section 7.3.

4.2 First Technique: Deterministic $k$-Threshold Outdetect Labeling Scheme

The first technique is a deterministic function $g$ replacing the random function of the graph sketch, based on the theory of error-correcting codes. To explain it, we first present a very concise review of coding theory: A linear code $W$ is a $y$-dimensional linear subspace of $\text{GF}(2)^x$, where $\text{GF}(2)$ is the finite field of two elements (i.e., the element set $\{0,1\}$ and every calculation is done in modulo 2), $y$ is the length of source data, and $x$ is the length of codewords. We abuse $W$ as the set of all codewords. The minimum distance of a linear code is the minimum Hamming distance over all pairs of the codewords in $W$. In principle, any linear code with minimum distance $k$ can correct any error of less than $k/2$ symbols (but it does not necessarily imply that there exists an efficient algorithm of correcting errors). One of the standard approaches of correcting errors is the syndrome decoding based on parity check matrices. The parity check matrix $C$ of $W$ is the full-rank $x \times (x-y)$
matrix satisfying \( w \cdot C = 0 \) for any codeword \( w \in W \). Since the parity check matrix of \( W \) is uniquely determined from \( W \), linear codes are often defined by the corresponding parity check matrices. A key property of the parity check matrix is that given a codeword with noise \( w + \delta \), where \( w \in W(A) \) and \( \delta \) is a noise vector, \( (w + \delta) \cdot C = \delta \cdot C \) holds. The syndrome of a received (noisy) codeword \( w + \delta \) is the vector \( (w + \delta) \cdot C \), and the syndrome decoding is the process of recovering \( \delta \) from the syndrome \( (w + \delta) \cdot C = \delta \cdot C \). If \( \delta \) is correctly recovered, the noiseless codeword \( w \) is also recovered by adding \( \delta \) to the received codeword \( w + \delta \).

The background idea of our first technique is as follows: We treat \( g : \mathcal{E} \to \{0,1\}^\ell \) as the mapping from \( \mathcal{E} \) to \( \ell \)-dimensional row vectors over \( \text{GF}(2) \) (where \( \ell \) is the label size), and define the \( |\mathcal{E}| \times \ell \) matrix \( C = (c_{e,i})_{e \in \mathcal{E}, i \in [0,\ell-1]} \), where \( c_{e,i} \) is the \( i \)-th bit of \( g(e) \). Let \( w(X) = (w_e)_{e \in \mathcal{E}} \) be the characteristic row vector for \( X \subseteq \mathcal{E} \), i.e., \( w_e = 1 \) if \( e \in X \), or zero otherwise. Then the following equality holds for any \( S \subseteq V \):

\[
w(\partial_G(S)) \cdot C = \sum_{e \in \partial_G(S)} g(e) = \sum_{v \in S} L^\text{out}_G(v).
\]

Note that the summation \( \sum \) is the sum over \( \text{GF}(2) \), and thus that operation is equivalent to the bitwise XOR. What we need is the recovery of one non-zero entry in \( w(\partial_G(S)) \) from the right-hand side sum. This task can be interpreted into the following scenario: Consider the linear code whose parity check matrix is \( C \). Then recover the noise vector \( w(\partial_G(S)) \) from the syndrome \( w(\partial_G(S))C = \sum_{v \in S} L^\text{out}_G(v) \). If the linear code defined by \( C \) has a minimum distance \( k > 0 \), one can obtain the complete recovery of \( w(\partial_G(S)) \) for any \( S \) satisfying \( |\partial_G(S)| < k/2 \). Since fixing \( C \) implies fixing \( g \) (and thus the labeling function \( L^\text{out}_G \)), one can obtain the deterministic function \( g \) from the parity check matrix of any linear code. We show that Reed-Solomon code nicely fits our objective, \(^2\) which provides the outdetect labeling of \( O(k \log n) \) bits supporting the detection of all outgoing edges of a given subset \( S \in 2^V \) in \( O(k^2) \) time if \( |\partial_G(S)| \leq k \) holds. We refer to such a scheme as the \( k \)-threshold outdetect labeling scheme hereafter. Let \( (L^\text{RS}(k), D^\text{RS}(k)) \) be the \( k \)-threshold outdetect labeling scheme for \( H \subseteq G \) defined by the \( |\mathcal{E}| \times 2k \) parity check matrix of Reed-Solomon code. It satisfies the following properties:

**Proposition 2** The \( k \)-threshold outdetect labeling scheme \( (L^\text{RS}(k), D^\text{RS}(k)) \) satisfies the following conditions:

- The label size is \( O(k \log n) \) bits.
- The time taken to assign the labels \( L^\text{RS}(k)_H(v) \) to all vertices \( v \in V_H \) is \( O(mk) \). The time of computing \( D^\text{RS}(k)_H(L^\text{RS}(k)_H(S)) \) for given \( L^\text{RS}(k)_H(S) \) is always bounded by \( O(k^2) \).
- Given \( L^\text{RS}(k)_H(S) \), the output of the decoding function is the IDs of all edges in \( \partial_H(S) \) if \( |\partial_H(S)| \leq k \) holds. If \( |\partial_H(S)| > k \), the returned value is unspecified. That is, an arbitrary value can be returned.

In contrast with the single edge detection capability of the original graph sketch, it is a great advantage that our technique admits the detection of at most \( k \) outgoing edges, which made the construction of the collection \( \mathcal{G} \) much easier. It suffices to construct the sparsification hierarchy \( E_G \supseteq E_T = E_0 \supseteq E_1 \supseteq E_2 \supseteq \ldots \supseteq E_h = \emptyset \) for \( h = O(\log n) \) such that every \( S \in S_{LT} \) satisfying \( \partial_G(S) \neq \emptyset \) admits a graph \( G_i = (V, E_i) \) satisfying \( 0 < |\partial_{G_i}(S)| \leq k \). We define such a hierarchy as a \((S,k)\)-good hierarchy:
4.3 Second Technique: Deterministic Construction of $(S_{f,T}, k)$-good Hierarchy

A crucial requirement of the logarithmic-depth hierarchy $E_0 \supseteq E_1 \supseteq E_2 \supseteq \ldots \supseteq E_h$ is to avoid the set $S \in S_{f,T}$ such that $|\partial E_{i+1}(S)| = 0$ holds despite $|\partial E_i(S)| > k$ and to make $|E_{i+1}|$ substantially smaller than $|E_i|$. It implies that $E_{i+1} \subseteq E_i$ must be a hitting set of the family of edge sets $Z_{i,f,k} = \{\partial E_i(S) \mid S \in S_{f,T}, |\partial E_i(S)| > k\}$ of a constant fraction size. Allowing randomization, it suffices to construct $E_{i+1}$ by the independent edge sub-sampling from $E_i$ with probability $1/2$. Such a construction satisfies the desired property for $k = O(f \log n)$ with high probability (see the appendix A for details). While the standard greedy algorithm can deterministically construct the hitting set with the same guarantee, such an approach is not tractable because the size of $Z_{i,f,k}$ could be super-polynomial, i.e., $|Z_{i,f,k}| = \Theta(|S_{f,T}|) = \Theta(n^f)$ can hold. The second key ingredient of our construction is to provide a polynomial-time deterministic algorithm of constructing the hitting set for $k = \tilde{O}(f^2)$ through the geometric representation based on the Euler-tour structure by Duan and Pettie [DP10]. In this structure, each undirected edge $e$ in $T$ is replaced by two directed edges with opposite orientations. We refer to the tree $T$ after the replacement as $\bar{T}$, and extend the definition of $\partial_T(S)$ into the directed case $\partial_T(S)$, which consists of all the directed edges obtained by the replacement of an edge in $\partial_T(S)$. All the edges in $\bar{T}$ are ordered by any Euler tour $ET$ of $\bar{T}$ starting from the root $r$, and each vertex in the tree is assigned with the smallest order of the incident in-edge (i.e. the edge coming from its parent) as its one-dimensional coordinate in the range $[1, 2n - 2]$. We denote the one-dimensional coordinate of a vertex $v \in V_T$ by $c(v)$. Then, one can map each non-tree edge $e = (u, v)$ into the 2D-point $(c(u), c(v))$ in the range $[1, 2n - 2] \times [1, 2n - 2]$ (assuming the $x$-coordinate is always smaller than the $y$-coordinate to make the mapping well-defined). An example of the geometric representation for the instance of Figure 1 is presented in Figure 2. For any $a \in [1, 2n - 2]$ and $z \in \{x, y\}$, let $hs(z, a)$ be the axis-aligned halfspace defined by $z \geq a$. Then it is observed that the point set $\partial E_i(S)$ for any $S \in S_{f,T}$ lies in the symmetric difference of at most $4f$ axis-aligned halfspaces. More precisely, the following lemma holds:

\[\text{Definition 1} \text{ Let } S \subseteq 2^V_G \text{ and a } k \text{ be a positive integer. A } (S, k)\text{-good hierarchy of } E_G - E_T \text{ is the hierarchical edge set } E_G - E_T = E_0 \supseteq E_1 \supseteq E_2 \supseteq \ldots \supseteq E_h = \emptyset \text{ satisfying the following conditions}
\]

- The subset $E_{i+1} \subseteq E_i$ is a constant fraction size\(^3\) for any $i \in [0, h - 1]$. Note that this condition inherently deduces the property of $h = O(\log n)$.
- For any $S \in S$ such that $|\partial E_i(S)| > k$, $|\partial E_{i+1}(S)| > 0$ holds.

The collection of $k$-threshold outdetect labeling schemes for all $G_i = (V_G, E_i)$ forms a $S_{f,T}$-outdetect labeling scheme for $G - E_T$ with $O(k \log^2 n)$-bit label size and $O(k^2 \log n)$ decoding time. The decoding process tries to obtain the edge(s) in $\partial G_i(S)$ in the decreasing order of $i$. For the largest $i$ such that $\partial G_i(S)$ is non-empty, the corresponding $k$-threshold outdetect labeling returns a subset of $\partial G(S)$ correctly. Formally, the following lemma holds:

\[\text{Lemma 2} \text{ Assume that any } k\text{-threshold outdetect labeling scheme } (\hat{L}^\text{out}, \hat{\rho}^\text{out}) \text{ of label size } \alpha \text{ and query processing time } \beta \text{ is available. If there exists an algorithm of constructing a } (S, k)\text{-good hierarchy for } S \subseteq 2^V_G \text{ and } E_G - E_T, \text{ there exists an } S\text{-outdetect labeling scheme } (L^\text{out}_{G - E_T}, D^\text{out}) \text{ for } G - E_T \text{ whose label size is } O(\alpha \log n) \text{ bits and query processing time is } O(\beta \log n)\]
Lemma 3 For any vertex subset $S \subseteq V_G$ and edge subset $E' \subseteq E_G^*$, the following equality holds.

$$\partial_{E'}(S) = E' \cap \left( \Delta_{e \in \partial_T(S), z \in \{x,y\}} \; hs(z, e) \right).$$

where $\Delta$ represents the symmetric difference of sets.

This lemma implies that every $\partial_{E_i}(S)$ for $S \in S_{f,T}$ is associated with a “checkered shape” in the plane with at most $2f$ vertical (or horizontal) alternations. In Figure 2, the region colored by white corresponds to the outgoing edges of $S$ such that $\partial_T(S)$ consists of two directed edges with numbers 3 and 18. Then the problem of constructing the hitting set is seen as the construction of $\epsilon$-nets [HW87].

Definition 2 ($\epsilon$-nets) Let $Z$ be a class of geometric shapes (e.g., rectangles or disks) in some space, and $X$ be a set of points in the space. An $\epsilon$-net for $(X, Z)$ is a subset $X' \subseteq X$ such that for any $Z \in Z$, $Z \cap X' \neq \emptyset$ holds if $|Z \cap X| \geq \epsilon |X|$.

Let us define the class $\mathcal{H}_q$ which consists of all the shapes formed by the symmetric difference of at most $q$ horizontal halfspaces $hs(y, a_0), hs(y, a_1), \ldots, hs(y, a_{q-1})$ ($q' \leq q$, $a_i \in [1, 2n - 2]$) and the corresponding vertical halfspaces $hs(x, a_0), hs(x, a_1), \ldots, hs(x, a_{q'-1})$. Recalling that the construction of $E_{i+1}$ is equivalent to the computation of a hitting set of a constant fraction size for the family $Z_{i,f,k} = \{\partial_{E_i}(S) \mid S \in S_{f,T}, |\partial_{E_i}(S)| > k\}$, our goal is to construct the hitting set of a constant fraction size for $\{Z \cap E_i \mid Z \in \mathcal{H}_{2f}, |Z \cap E_i| \geq k\}$, i.e., to construct an $\epsilon$-net of a constant fraction size for $(E_i, \mathcal{H}_{2f})$ and $\epsilon = k/|E_i|$.

While there are a few deterministic polynomial-time algorithms of constructing nearly-optimal $\epsilon$-nets for a given class of shapes [Mat96, CM96], their running times exponentially depend on the VC dimension of the given class. The VC dimension of $\mathcal{H}_{2f}$ is $\Omega(f)$, and thus those algorithms cannot be applied. To circumvent this issue, we regard any shape in $\mathcal{H}_{2f}$ as the union of $(2f+1)^2/2$ disjoint axis-aligned rectangles. For any $H \subseteq \mathcal{H}_{2f}$ containing at least $\gamma(2f+1)^2/2$ points ($\gamma \geq 1$), there exists at least one axis-aligned rectangle as a subset of $H$ which contains at least $\gamma$ points.

\[\text{More precisely, they take } \Omega(1/\epsilon)^d \text{ time for the class with VC dimension } d. \text{ This would efficiently work if } 1/\epsilon \text{ is small, but in our use } 1/\epsilon \text{ is roughly close to } m/f, \text{ and thus they are not tractable for } f = \omega(1).\]

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Hence the construction of an $O(\gamma/|E_i|)$-net of a constant fraction size for all axis-aligned rectangles deduces an $O(\gamma f^2/|E_i|)$-net of a constant fraction size for $H_{2f}$. In other words, we can obtain a deterministic polynomial-time algorithm of constructing $(S_{f,T}, O(\gamma f^2))$-good hierarchy from any deterministic polynomial-time algorithm of constructing a $(\gamma/N)$-net of a constant fraction size for $N$ points and all axis-aligned rectangles.

Since the VC-dimension of axis-aligned rectangles is a constant, it is possible to use the general de-randomization technique as stated above. The optimal $\epsilon$-net for $N$ points and axis-aligned rectangles is of size $O((\log \log N)/\epsilon)$ (i.e., $O((\log \log N)/N)$-net of a constant fraction size), which is known to be deterministically constructed [MDG18]. However, the construction time takes a high-exponent polynomial. Hence we also present a simpler alternative construction which provides an $O((\log \log N)/N)$-net of a constant fraction size for axis-aligned rectangles in a near linear time. In summary, we obtain the following lemma:

**Lemma 4** (Partly By Moustafa, Dutta, and Ghosh [MDG18]) There exist two deterministic algorithms of constructing an $O(\gamma/N)$-net of a constant fraction size for any $N$ points and all axis-aligned rectangles, each of which attains the following performance guarantee.

- $\gamma = \log \log N$ and the construction time is $\text{poly}(N)$.
- $\gamma = \log N$ and the construction time is $\tilde{O}(N)$.

By the argument above, we obtain the following lemma.

**Lemma 5** There exist two deterministic algorithms respectively constructing a $(S_{f,T}, k)$-good hierarchy with the following performance guarantees:

- $k = O(f^2 \log n)$ and the construction time is $\tilde{O}(m)$.
- $k = O(f^2 \log \log n)$ and the construction time is $\text{poly}(m)$.

## 5 Wrap-Up

We summarize how all the components are combined into the $f$-FTC labeling scheme. We present below the case of the $f$-FTC labeling scheme of label size $O(f^2 \log^3 n)$. Yet another scheme of label size $O(f^2 (\log^2) \log \log n)$ is constructed in the same way. Consider any input graph $G$ of $n$ vertices and $m$ edges. The whole construction algorithm works as the following steps:

1. Construct any spanning tree $T$ of $G$, and transform $G$ and $T$ into the auxiliary graph $G'$ and its spanning tree $T'$ explained in Section 3.2. Note that the graph $G'$ satisfies $|V_{G'}| = O(m)$ and $|E_{G'}| = O(m)$.

2. Utilizing the algorithm of Lemma 5, construct a $(S_{f,T}, cf^2 \log n)$-good hierarchy $E_0 \supseteq E_1 \supseteq E_2 \supseteq \ldots \supseteq E_h$ for $E_{G'} - E_{T'}$, where $c$ is a hidden constant. The construction time is $\tilde{O}(|E_{G'}|) = O(m)$.

3. Let $G_i = (V_{G'}, E_i)$ ($0 \leq i \leq h$). Construct $L_{G_i}^{\text{RS}(cf^2 \log n)}$ for all $i$. By the construction of step 2 and Lemma 2, we obtain the $S_{f,T}$-outdetect labeling scheme of $O(f^2 \log^3 n)$-bit label size. The construction time is in $\tilde{O}(mkh) = \tilde{O}(mf^2)$, and the decoding time is $\tilde{O}(f^4)$ due to Proposition 2 and Lemma 2.

4. Construct the ancestry labeling scheme for $T'$ by the algorithm of [KNR92], which takes $O(m)$ time.
5. By Lemma 1, the labels constructed in the steps 3 and 4 form a tree edge \( f \)-FTC labeling scheme for \( G' \) and \( T' \). Then we also obtain the \( f \)-FTC labeling scheme for \( G \) by Corollary 3.

Finally we have the following theorem.

**Theorem 2** There exist two deterministic \( f \)-FTC labeling schemes for any graph \( G \) of \( n \) vertices and \( m \) edges which respectively attain the following bounds:

- The label size is \( O(\log n) \) bits per vertex, and \( O(f^2(\log^2 n) \log \log n) \) bits per edge. The query processing time is \( \tilde{O}(|F|^4) \), where \( F \) is the set of queried edges satisfying \( |F| \leq f \). The construction time is polynomial of \( m \).

- The label size is \( O(\log n) \) bits per vertex, and \( O(f^2 \log^3 n) \) bits per edge. The query processing time is \( \tilde{O}(|F|^4) \). The construction time is near linear, i.e., \( \tilde{O}(mf^5) \).

This theorem is a weaker form of Theorem 1. In the next section we present how one can improve this to Theorem 1 claiming faster query processing time.

### 6 Improving Query Processing Time

The algorithmic idea of this improvement is twofold: The first idea attains the adaptiveness of \( S_{f,T} \)-outdetect labeling scheme, i.e., to get rid of the dependency on \( f \) in the decoding time. There is a simple technique of transforming any \( S_{f,T} \)-outdetect labeling scheme with \( \tilde{O}(f^c) \) decoding time into the one with \( \tilde{O}(|\partial_T(S)|^c) \) decoding time for any given query \( S \in S_{f,T} \): Instead of single labeling, we prepare the multiple instances of the (non-adaptive) \( S_{f',T} \)-outdetect labeling scheme for \( f' = 2, 4, \ldots, f \). If the original labeling scheme has a \( \Omega(f) \)-bit label size, this transformation does not cause any asymptotic blow-up of label size. Assume that a query \( S \subseteq S_{f,T} \) is given. Since \( S \in S_{|\partial_T(S)|,T} \) necessarily holds, the adaptive scheme can find the outgoing edge of \( S \) by utilizing the \( S_{f',T} \)-outdetect labeling scheme for \( f' \) such that \( f'/2 < |\partial_T(S)| \leq f' \) holds, which runs in \( \tilde{O}(|\partial_T(S)|^c) \) time \(^5\).

The second idea is to utilize the adaptive scheme for further acceleration of the decoding time. In processing the query of \((s, t, F)\), every query \( S \subseteq V_G \) issued to the \( S_{f,T} \)-outdetect labeling scheme necessarily belongs to \( S_{|F|,T} \). Hence The adaptive decoding of the \( S_{f,T} \)-outdetect labeling scheme always runs in \( \tilde{O}(|F|^4) \) time for the deterministic cases, and in \( \tilde{O}(|F|^2) \) time for the randomized case. Since the decoding time of the tree edge \( f \)-FTC labeling scheme is dominated by \( |F| \) queries to the \( S_{f,T} \)-outdetect labeling scheme, the straightforward implementation respectively results in the decoding time of \( \tilde{O}(|F|^5) \) and \( \tilde{O}(|F|^3) \). We shave off this extra \( |F| \) factor by a simple refinement of the decoding process of the tree edge \( f \)-FTC labeling scheme: In the refined process, the outgoing edge detection for merging fragments is always applied to the fragment \( S \) such that \( |\partial_T(S)| \) is the smallest of all the fragments currently managed, while the original process always applies it to the fragment with \( s \). By a careful analysis, we obtain the following lemma:

**Lemma 6** Assume that there exists a \( S_{f,T} \)-outdetect labeling scheme of label size \( \alpha = \tilde{O}(f^b) \) and decoding time \( \beta = \tilde{O}(f^c) \). Then there exists a \( f \)-FTC labeling scheme of \( O(\alpha + \log n) \)-bit label size and \( \tilde{O}(|F|^{b+1} + |F|^c) \) decoding time. The resultant \( f \)-FTC labeling scheme is deterministic if the corresponding \( S_{f,T} \)-outdetect labeling scheme is deterministic.

\(^5\)In reality, this transformation is not necessary if we utilize our deterministic \( S_{f,T} \)-outdetect labeling scheme based on the Reed-Solomon code. More precisely, it inherently admits the adaptive decoding without any modification of the label construction. See the appendix B for details.
The three $\mathcal{S}_{f,T}$-outdetect labeling schemes we presented in this paper (including the randomized case) attain $(\alpha, \beta) = (O(f^2 (\log^3 n) \log \log n), \tilde{O}(f^4)), (O(f^2 \log^3 n), \tilde{O}(f^4))$, and $O(f \log^3 n), \tilde{O}(f^2))$. By this lemma, they respectively deduce the schemes as claimed in Table 1.

7 Technical Details

7.1 Formal Specification of Labeling Schemes

Fault-Tolerant Connectivity Labeling A $f$-fault-tolerant connectivity labeling scheme ($f$-FTC labeling scheme) for a given input graph $G$ consists of a labeling function $L_{G,f}^\text{con} : V_G \cup E_G \to \{0,1\}^*$ and a universal decoding function $D_{f}^\text{con} : \{0,1\}^* \times \{0,1\}^* \times \{0,1\}^* \to \{0,1\}$. For an edge subset $F = \{e_1,e_2,\ldots,e_{|F|}\} \subseteq E_G$, let $L_{G,f}^\text{con}(F)$ be the concatenation of the labels $L_{G,f}^\text{con}(e_1) \circ L_{G,f}^\text{con}(e_2) \circ \cdots \circ L_{G,f}^\text{con}(e_{|F|})$ in an arbitrary order. For a query $(s,t,F)$ of $s,t \in V_G$ and an edge subset $F \subseteq E_G$ of cardinality at most $f$, the decoder function returns the $s$-$t$ connectivity in $G - F$ by giving the labels of $s$, $t$ and all the edges in $F$, i.e., $D^\text{con}$ satisfies that $D^\text{con}(L_{G,s}^\text{con}(s), L_{G,t}^\text{con}(t), L_{G}^\text{con}(F)) = 1$ if and only if $s$ and $t$ are connected in $G - F$. The label size of the scheme is defined as the maximum length of the labels assigned to vertices and edges.

Ancestry labeling Given any rooted tree $T$, this labeling scheme assigns all vertices with the labels such that the ancestor-descendant relationship between any two vertices is determined only from their labels. More precisely, the ancestry labeling scheme for $T$ consists of a labeling function $L_T^\text{anc} : V_T \to \{0,1\}^*$ and a universal decoding function $D^\text{anc} : \{0,1\}^* \times \{0,1\}^* \to \{-1,0,1\}$ (not dependent on $T$). It determines if two given vertices $x, y \in V_T$ have the ancestor-descendant relationship or not from their labels, i.e., $D^\text{anc}(L_T^\text{anc}(x), L_T^\text{anc}(y)) = 1$ if $x$ is an ancestor of $y$, $-1$ if $y$ is an ancestor of $x$, or $0$ otherwise (including the case of $x = y$). The following lemma is well-known:

Lemma 7 (Kannan, Naor, and Rudich [KNR92]) Let $T$ be a rooted tree of $n$ vertices. There exists a deterministic ancestry labeling scheme with label size of $O(\log n)$ bits. Computing $L_{T}^\text{anc}(x)$ for all $x \in V_T$ takes $O(n)$ time, and $D^\text{anc}(L_{T}^\text{anc}(x), L_{T}^\text{anc}(y))$ for each $x, y \in V_T$ takes $O(1)$ time. The labeling function $L_T^\text{anc}$ is injective for any $T$, i.e., a unique label assignment.

$\mathcal{S}$-Outdetect Labeling Let $\mathbb{F}$ be any finite field of characteristic two whose addition and multiplication operators are respectively denoted by “+” and “$\cdot$”$^6$ and $\mathcal{E}$ be the domain of unique edge IDs not depending on $G$$^7$. An $\mathcal{S}$-outdetect labeling scheme consists of a vertex labeling function $L_G^\text{out} : V_G \to \mathbb{F}^\ell$ and a universal decoding function $D^\text{out} : \mathbb{F}^\ell \to \mathcal{E}$, where $\ell$ is a positive integer representing the size of labels. It must satisfy the following two conditions:

- For any $S \subseteq \mathcal{S}$, $D^\text{out}(\sum_{v \in S} L_G^\text{out}(v))$ returns the ID of an outgoing edge of $S$ in $G$ if $\partial_G(S)$ is nonempty.

- If $\partial_G(S) = \emptyset$, $D^\text{out}(\sum_{v \in S} L_G^\text{out}(v))$ returns formal zero, which is a special value in $\mathcal{E}$ never assigned to actual edges.

$^6$A field $\mathbb{F}$ has characteristic two if and only if any element $x \in \mathbb{F}$ satisfies $x + x = 0$, or equivalently $x = x^{-1}$ (where $0$ is the unit element and $x^{-1}$ is the inverse element of $x$).

$^7$Since the size of this domain inherently depends on $n$, it should be defined precisely as $\mathcal{E}_n$, which is the universal domain valid for all the graphs with at most $n$ vertices. But we intentionally omit such a dependency for avoiding non-essential complication.
For short, we use the notation \( L^\text{out}_G(S) \) to represent \( \sum_{v \in S} L^\text{out}_G(v) \) in the following argument. By definition, \( D^\text{out}(L^\text{out}_G(V_G)) = 0 \) must hold. The *k*-threshold outdetect labeling scheme for \( G \) is a restricted variant of \( 2^{V_G} \)-outdetect labeling scheme, which guarantees that \( D^\text{out}(L^\text{out}_G(S)) \) returns an outgoing edge of \( S \) only if \( 0 < |\partial_G(S)| \leq k \) holds, or returns zero if \( |\partial_G(S)| = 0 \). In the case of \( |\partial_G(S)| > k \), the returned value is undefined, i.e., an arbitrary value is returned.

7.2 Proof of Lemma 1

Let \( G^* = G - E_T \) for short. The tree edge \( f \)-FTC labeling scheme is implemented by any \( S_{f,T} \)-outdetect labeling scheme for \( G^* \) and any ancestry labeling scheme for \( T \). Let \((L^\text{anc}_T, D^\text{anc})\) be the ancestry labeling scheme. Without loss of generality, we assume that \( L^\text{anc}_T(v) \) for all \( v \) has a fixed bit length \( p = O(\log n) \). We assign the edge ID \( L^\text{anc}_T(u) \circ L^\text{anc}_T(v) \in \{0, 1\}^{2p} \) to each edge \((u, v) \in E_{G^*}\). The uniqueness of the edge ID follows the uniqueness of the ancestry labeling guaranteed in Lemma 7. For the edge ID domain \( E = \{0, 1\}^{2p} \), we construct the \( S_{f,T} \)-outdetect labeling scheme \((L^\text{out}_G, D^\text{out})\). The labeling function \( L^\text{out}_G(v) \) of our tree edge \( f \)-FTC labeling scheme is defined as follows:

- \( L^\text{out}_G(v) = L^\text{anc}_T(v) \).
- For any \( e = (u, v) \in E_T \), \( L^\text{out}_G(e) = L^\text{anc}_T(u) \circ L^\text{anc}_T(v) \circ L^\text{out}_G(V_{T(e)}) \).

We see how to implement the decoding function. Assume that a query \((s, t, F)\) of \(|F| \leq f\) is given. Let \( \mathcal{C}(F) = \{C_0, C_1, \ldots, C_{|F|}\} \) be the collection of the fragments (i.e., the vertex subset inducing a connected component of \( T - F \)). Let \( V(F) \) be the set of the endpoints of the edges in \( F \). By introducing any total order over \( \{0, 1\}^p \), we define the ID of \( C_i \in \mathcal{C}(F) \) as the maximum ancestry label in \( V_{C_i} \cap V(F) \), and abuse \( C_i \) itself as the ID of \( C_i \). The component graph \( H(\mathcal{C}(F)) \) of a spanning subgraph \( H \subseteq G \) is the multigraph obtained from \( H \) by contracting each vertex set \( C_i \) \((0 \leq i \leq |F|)\) into a single vertex and then removing self-loops. The following proposition is known:

**Proposition 3** (Claim 3.14 in [DP21]) Let \((s, t, F)\) be any given query, and \(|F| \leq f\). Then \( T/\mathcal{C}(F) \) is computed deterministically in \( O(|F| \log |F|) \) time. In addition, there exists a deterministic algorithm which is given \( L^\text{anc}_T(v) \) and returns in \( O(\log |F|) \) time the ID of \( C_i \in \mathcal{C}(F) \) containing \( v \).

The proposition above admits the detection of the two connected components in \( T - F \) respectively containing \( s \) and \( t \). We assume \( s \in C_0 \) and \( t \in C_1 \) without loss of generality. Starting from \( C_0 \), the decoding procedure grows the component containing \( s \) iteratively by finding its outgoing edge: Initially, let \( S = C_0 \). The procedure detects an outgoing edge \((u, v)\) of \( S \) in \( G^* \), where \( u \) is the vertex in \( S \) and \( v \in C_j \) for some \( j \in [0, f'] \), until no outgoing edge is found. When the edge \((u, v)\) is found, \( S \) is updated with \( S \cup \{C_j\} \). Obviously, \( s \) and \( t \) are connected in \( G - F \) if and only if this process merges \( C_1 \) with \( S \). Throughout this process, \( \partial_T(S) \subseteq F \) obviously holds and thus \( S \in S_{f,T} \) always holds. Hence one can use the \( S_{f,T} \)-outdetect labeling scheme to find an outgoing edge \((u, v)\). The primary matter is how to manage \( L^\text{out}_G(S) \). We resolve it with a technique similar to [DP21] (Claim 3.15). The following proposition holds.

**Proposition 4** For any \( X \subseteq V_T \), \( L^\text{out}_G(X) = \sum_{e \in \partial_T(X)} L^\text{out}_G(V_{T(e)}) \) holds.

**Proof.** Let \( k = |\partial_T(X)| \). The proof is based on the induction on \( k \). (Basis) \( k = 0 \): Then \( X = V_G \) holds. Since we have \( L^\text{out}_G(V_T) = 0 \) by the definition of the \( S_{f,T} \)-outdetect labeling scheme, the proposition obviously holds. (Inductive step) Suppose that the proposition holds for any \( X' \) such
that $|\partial T(X')| = k$, and consider $X$ such that $|\partial T(X)| = k + 1$ holds. Let $\partial T(X) = \{e_1, e_2, \ldots, e_{k+1}\}$. Without loss of generality, we assume that there is no descendant of $e_{k+1}$ in $\partial T(X)$, i.e., $T(e_{k+1})$ does not contain any edge in $\partial T(X)$. Let $e_{k+1} = (u, v)$ where $v$ is the lower vertex of $e_{k+1}$. We consider the case of $v \in X$. Let $W = X \setminus V_{T(e_{k+1})}$. Since we have $\partial T(W) = \partial T(X) \setminus \{e_{k+1}\}$, $|\partial T(W)| = k$ holds. Thus by the induction hypothesis we obtain $L_{G^*}^{\text{out}}(W) = \sum_{e \in \partial T(W)} L_{G^*}^{\text{out}}(V_{T(e)})$. Since $W$ and $V_{T(e)}$ are disjoint by definition, we have

$$L_{G^*}^{\text{out}}(X) = L_{G^*}^{\text{out}}(W) + L_{G^*}^{\text{out}}(V_{T(e_{k+1})})$$

$$= \sum_{e \in \partial T(W)} L_{G^*}^{\text{out}}(V_{T(e)}) + L_{G^*}^{\text{out}}(V_{T(e_{k+1})})$$

$$= \sum_{e \in \partial T(X)} L_{G^*}^{\text{out}}(V_{T(e)}).$$

The case of $v \not\in X$ is proved similarly.\[\]

We prove Lemma 1 by the proposition above.

**Lemma 1** Assume any deterministic $S_{f,T}$-outdetect labeling scheme $(L_{G}^{\text{out}}, D^{\text{out}})$ of label size $\alpha$ and decoding time $\beta$. Then there exists a deterministic tree edge $f$-FTC labeling scheme of $(\alpha + O(\log n))$-bit label size and $O(|F|(\beta + \log |F|))$ decoding time, where $F$ is a set of faulty edges given by a query.

**Proof.** By Proposition 4, one can compute the $L_{G}^{\text{out}}(C_i)$ for all $i \in [0, |F|]$ from the labels of the edges in $F$. Assume an outgoing edge $e = (u, v)$ of $S$ ($v \not\in S$) is detected by the $S_{f,T}$-outdetect labeling. Since we can obtain the ID $L_{T}^{\text{anc}}(u) \circ L_{T}^{\text{anc}}(v)$ of $e$, the ancestry label $L_{T}^{\text{anc}}(v)$ is available. By Proposition 3, it also gives the ID of the component in $C(F)$ which contains $v$. Let us assume $v \in C_j$. When merging $C_j$ into $S$, we have known both $L_{G}^{\text{out}}(S)$ and $L_{G^*}^{\text{out}}(C_j)$. Thus the label $L_{G}^{\text{out}}(S)$ is updated by adding $L_{G^*}^{\text{out}}(C_j)$. The query processing time is spent for $|F|$ times of querying the $S_{f,T}$-outdetect labeling scheme, which takes $O(|F|(\beta + \log |F|))$ time in total. The initial set-up takes $O(|F|\log |F| + |F|\beta)$ time, where the term $|F|\log |F|$ is for computing the component graph, and $|F|\beta$ is for computing the $S_{f,T}$-outdetect labels of all fragments. The label size is obviously $O(\alpha + \log n)$.\[\]

### 7.3 Construction of Deterministic $S_{f,T}$-Outdetect Labeling Scheme

**Lemma 2** Assume that any $k$-threshold outdetect labeling scheme $(\hat{L}_H^{\text{out}}, \hat{D}^{\text{out}})$ of label size $\alpha$ and query processing time $\beta$ is available. If there exists an algorithm of constructing a $(\mathcal{S}, k)$-good hierarchy for $S \subseteq 2^{V_G}$ and $E_G - E_T$, there exists an $\mathcal{S}$-outdetect labeling scheme $(L_{G^*}^{\text{out}}, D^{\text{out}})$ for $G - E_T$ whose label size is $O(\alpha \log n)$ bits and query processing time is $O(\beta \log n)$.

**Proof.** Assume that a $(\mathcal{S}, k)$-good hierarchy $E_0 \supseteq E_1 \supseteq E_2 \supseteq \ldots \supseteq E_h$ is obtained. Let $G_i = (V_G, E_i)$. The labeling function $L_{G^*}^{\text{out}}$ is defined as the concatenation of the labels by $\hat{L}_G^{\text{out}}$ for all $i \in [0, h]$, i.e., $L_{G^* - E_T}^{\text{out}}(v) = \hat{L}_{G_0}^{\text{out}}(v) \circ \hat{L}_{G_1}^{\text{out}}(v) \circ \ldots \circ \hat{L}_{G_h}^{\text{out}}(v)$ (where $\circ$ is the binary operator of concatenating two strings). To detect an outgoing edge of $S \in \mathcal{S}$, it suffices to compute the value of $\hat{D}^{\text{out}}(\bigoplus_{v \in S} \hat{L}_{G_j}^{\text{out}}(v))$ such that it returns a non-zero value and $\hat{D}^{\text{out}}(\bigoplus_{v \in S} L_{G_j}^{\text{out}}(v))$ for any $j > i$ returns zero. The condition of the hierarchy implies $0 < |E_i \cap \partial_G(S)| \leq k$, and thus $\hat{D}^{\text{out}}(\bigoplus_{v \in S} \hat{L}_{G_j}^{\text{out}}(v))$ correctly returns all the outgoing edges of $S$ in $E_i \subseteq E_G$. The bounds for the label size and the query processing time are obvious.\[\]
7.4 \( k \)-threshold outdetect labeling: Choice of Codes

Following the construction presented in Section 4.2, any linear code of the minimum distance \( 2k \) naturally induces a \( k \)-threshold outdetect labeling scheme for any graph \( G \). The label size is determined by the number of columns of the parity check matrix \( C \) of the code. Obtaining both a smaller number of rows and a larger minimum distance is roughly equivalent to achieving a good code rate. Hence, as a general principle, any high-rate code would provide a good scheme. In addition, we need to care other additional criteria for efficient implementation of the outdetect labeling scheme, which are stated as follows:

- We use a parity check matrix which has \(|E|\) rows, i.e., the codeword length is \(|E|\). Thus it is not appropriate to use the error-correcting codes whose decoding time depends on the codeword length. Ideally, the decoding time should depend only on the length of the syndrome (i.e., the length of labels).

- The construction of the label for an edge \( e \) corresponds to the computation of the row vector of \( C \) corresponding to \( e \). Since the number of rows \(|E|\) could be much larger than the actual number of edges \(|E_G|\), computing the whole matrix \( C \) can result in slower computation of edge labels to all \( e \in E_G \). To complete the label assignment in time dependent only on the actual number \(|E_G|\) of edges but not on \(|E|\), \( C \) must admit efficient “local” computation of a specified row.

One of the error-correcting codes addressing the issues above is Reed-Solomon code. Reed-Solomon code is a non-binary code whose alphabet is a finite field \( F \). Let \( C \) be chosen arbitrarily. Since \(|E|\) is a polynomial of \( n \) in our application, each code symbol is encoded with \( O(\log n) \) bits, and the addition and multiplication over \( F \) takes \( O(1) \) time in the standard word-RAM model. Let \( C_{2k} \) be the \(|E| \times 2k \) parity check matrix of Reed-Solomon code. We have the following nice features:

- The minimum distance of the code defined by \( C_{2k} \) is equal to \( 2k \). That is, a \( k \)-threshold outdetect labeling scheme is deduced from \( C_{2k} \). Since each column vector of \( C_{2k} \) is encoded by \( O(k \log |E|) \) bits, the label size is \( O(k \log n) \) bits.

- Let \( w \) be any \(|E|\)-dimensional vector over \( F \) which contains at most \( k \) nonzero elements. There exists a deterministic algorithm of computing all non-zero elements in \( w \) (in the form of the pairs of value and position) from \( w \cdot C_{2k} \), which runs in \( O(k^2) \) time in the standard word-RAM model [DORS08]. The recovery of \( w \) necessarily succeeds if \( w \) contains at most \( k \) non-zero elements, but the result can become arbitrary if \( w \) contains more than \( k \) non-zero elements.

- Given any \( e \in E \), the row vector of \( C_{2k} \) corresponding to \( e \) is deterministically computed in \( O(k) \) time in the standard word-RAM model.

Let \( (L^{RS(k)}_H, D^{RS(k)}) \) be the \( k \)-threshold outdetect labeling scheme for \( H \subseteq G \) defined by the \(|E| \times 2k \) parity check matrix of Reed-Solomon code. The features above obviously deduces Proposition 2.

7.5 Deterministic Construction of Good Hierarchy

We first focus on the proof of Lemma 3. Since \( \partial_{G'}(S) = \partial_{G'}(V_G \setminus S) \) holds for any \( S \subseteq V_G \) and any spanning subgraph \( G' \) of \( G \), we assume \( S \) always contains the root \( r \) wlog. We use notations \( T, ET, c(v), hs(z, a), \) and \( H_{2f} \) as defined in Section 4.3. In addition, we introduce a few additional notations. Let \( ET = e_1, e_2, \ldots, e_i, \ldots, e_{2n-2} \). The prefix \( e_1, e_2, \ldots, e_i \) of \( ET \) up to the \( i \)-th element is denoted by \( ET(i) \). For the proof, we introduce a few auxiliary lemmas.
Lemma 8 (Claim 3.3 in [DP21]) Let $F$ be any induced cutset of $G$, and let $n_v(F)$ be the number of the edges in $F$ lying on the path from the root to $v$ in $T$. Let $V_0 = \{v \in V_G \mid n_v(F) \text{ is even}\}$ and $V_1 = \{v \in V_G \mid n_v(F) \text{ is odd}\}$. Then $(V_0, V_1)$ is the cut of $G$ induced by $F$.

Lemma 9 Assume that $S \subseteq V_T$ contains the root $r$ of $T$. For any vertex $v \in S$, $|ET(c(v)) \cap \partial_T(S)|$ is even if $v \in S$, or odd otherwise.

Proof. We first show that any vertex $v$ satisfies that $n_v(\partial_T(S))$ is even if and only if $v \in S$. Let $F' \subseteq E_G$ be the cutset associated with the cut $(S, V_G \setminus S)$. By Lemma 8, for any vertex $v \in S$, $n_v(F')$ has the same parity. Since $n_r(F)$ is obviously even, we can conclude that $n_v(F')$ is even for any $v \in S$. It also implies that $n_v(F')$ has the odd parity for any $v \in V_G \setminus S$. By definition, $n_v(F') = n_v(\partial_T(S))$ obviously holds.

Next, we prove the statement of the lemma. For any undirected edge $e \in ET$, let $e^+$ and $e^-$ be the downward and upward directed edges in $E_T$ corresponding to $e$ respectively. Since $e^+$ always precedes $e^-$ in $ET$, only the following three cases can occur: (1) both $e^+$ and $e^-$ precedes $e_{c(v)}$ in $ET$, (2) both $e^+$ or $e^-$ follows $e_{c(v)}$ in $ET$, or (3) $e^+$ precedes or is equal to $e_{c(v)}$ in $ET$ and $e^-$ follows. Since the edge $e \in \partial_T(S)$ to which the case (1) or (2) applies does not affect the parity of $|ET(c(v)) \cap \partial_T(S)|$, the parity of $|ET(c(v)) \cap \partial_T(S)|$ is equal to the number of the edges $e \in \partial_T(S)$ to which the third case applies. Since the case (3) applies to an edge $e$ if and only if $e$ is on the path from $r$ to $v$, we obtain $|ET(c(v)) \cap \partial_T(S)| = n_v(\partial_T(S))$. That is, it has the even parity. The lemma is proved. \qed

Now we are ready to prove Lemma 3.

Lemma 3 For any vertex subset $S \subseteq V_G$ and edge subset $E' \subseteq E_{G^*}$, the following equality holds.

\[ \partial_{E'}(S) = E' \cap \left( \Delta_{e \in \partial_T(S), z \in \{x, y\}} hs(z, c(e)) \right). \]

Proof. Let $Q_x = \{hs(x, c(e)) \mid e \in \partial_T(S)\}$, $Q_y = \{hs(y, c(e)) \mid e \in \partial_T(S)\}$, and $Q = \Delta Q \in Q_x \cup Q_y Q'$. For any $(u, v) \in \partial_{E'}(S)$, exactly one of $u$ and $v$ belongs to $S$ and the other one belongs to $V_G \setminus S$. By symmetry, we assume $u \in S$ and $c(u) < c(v)$ wlog. Lemma 9 implies $|ET(c(u)) \cap \partial_T(S)|$ is even, and $|ET(c(v)) \cap \partial_T(S)|$ is odd. It implies that $(u, v)$ lies in the two regions respectively defined as the intersection of an even number of halfspaces in $Q_y$ and as the intersection of an odd number of halfspaces in $Q_x$. Since $Q_x$ and $Q_y$ are disjoint, $(u, v)$ lies in the region defined as the intersection of an odd number of halfspaces in $Q_x \cup Q_y$, i.e., it is contained in $Q$. Similarly, if $(u, v)$ is not an edge in $\partial_{E'}(S)$, the parities of $|ET(c(u)) \cap \partial_T(S)|$ and $|ET(c(v)) \cap \partial_T(S)|$ becomes the same, and thus $e$ lies in the region defined as the intersection of an even number of halfspaces in $Q_x \cup Q_y$, and thus not contained in $Q$. The lemma is proved. \qed

Next, we focus on the deterministic $\epsilon$-net construction. We first quote a known deterministic construction for axis-aligned rectangles.

Lemma 10 (Mustafa, Dutta, and Ghosh [MDG18]) Let $\epsilon > 0$. There exists a deterministic polynomial-time algorithm of constructing an $\epsilon$-net of size $O(\log \log N/\epsilon)$ for any $N$ point set and all axis-aligned rectangles.

As we mentioned in Section 4.3, the polynomial of the construction time has a high exponent, and thus we consider a slightly weaker but much faster solution. Let $P$ be any set of points in a 2D-range $R = [a_1, a_2] \times [b_1, b_2]$. For simplicity, we assume that $|P|$ is a power of two in the lemma below, but it is not essential.
Lemma 11 Let $P$ be any point set in $R = [a_1, a_2] \times [b_1, b_2]$, $0 < \epsilon < 1$, and $M \in [a_1, a_2]$. There exists a $O(|P| \log |P|)$-time algorithm of computing a subset $P^* \subseteq P$ such that any axis-aligned rectangle $X$ crossing the vertical line $x = M$ and satisfying $|X \cap P| \geq 6/\epsilon$ necessarily contains at least one point in $P^*$.

Proof. The construction follows the technique by Kulkarni and Govindarajan [KG10]. For simplicity, we assume that any two points in $P$ have different $y$-coordinates, $2/\epsilon$ is an integer, and $|P|$ is divisible by $2/\epsilon$. Let $Y_P$ be the sequence of points in $P$ sorted by their $y$-coordinates. We split $Y_P$ into $\epsilon|P|/2$ subsequences $Y_P^1, Y_P^2, \ldots$ of the length $2/\epsilon$. For each $Y_P^i$, we define $p_i^- \in Y_P^i$ as the point with the maximum $x$-coordinate not exceeding $M$, and $p_i^+ \in Y_P^i$ as the one with the minimum $x$-coordinate not lower than or equal to $M$. We construct $P^*$ as the union of $\{y_i^+, y_i^-\}$ for all $i$.

It is easy to check that the constructed point set $P^*$ satisfies the condition of the lemma: The size of $P^*$ is obviously bounded by $\epsilon|P|$. We define the $y$-range $[b_1, b_2]$ of $Y_P$ as the minimal interval containing the $y$-coordinates of all the points in $Y_P^i$. Consider any rectangle $X = [a_1, a_2] \times [b_1, b_2]$ such that $|X \cap P| \geq 6/\epsilon$. Then there exists at least one subsequence $Y_P^i$ such that at least one point in $Y_P^i$ is contained in $X \cap P$ and the $y$-range of $Y_P^i$ is covered by $[b_1, b_2]$. For such an $i$, either $p_i^-$ or $p_i^+$ must be contained in $X$. ◻

We obtain the deterministic algorithm of constructing a $O(\log |P|/|P|)$-net of a constant fraction size for any point set $P$ and axis-aligned rectangles in near linear time.

Lemma 12 Let $P$ be any set of points in $[a_1, a_2] \times [b_1, b_2]$, and $N$ be any upper bound of $|P|$. There exists a deterministic algorithm $\text{NetFind}(N, P)$ which constructs a $(12 \log N/|P|)$-net of size at most $|P| \log |P|/(2 \log N)$ for $P$ and all axis-aligned rectangles in $O(|P| \log |P| \log N)$ time.

Proof. We first present the algorithm. It is based on the divide and conquer approach as follows:

1. If $|P| \geq 12 \log N$: find the vertical line $x = M$ bisecting $P$ into two equal-size subsets $P_0$ and $P_1$ (with an arbitrary tie-breaking rule for the points on $x = M$). Let $R_0 = [a_1, M] \times [b_1, b_2]$ and $R_1 = [M, a_2] \times [b_1, b_2]$. Then $\text{NetFind}(N, P)$ outputs the union of the following four subsets of $P$:
   - The outputs of $\text{NetFind}(N, P_0)$ and $\text{NetFind}(N, P_1)$.
   - The point set $P^*$ obtained from $P$ by Lemma 11 for $\epsilon = 1/(2 \log N)$ and $x = M$.

2. If $|P| < 12 \log N$: output the empty set.

Let $P'$ be the output in the run of $\text{NetFind}(N, P)$. The proof is based on the induction on the size of $P$.

(Basis) $|P| < 12 \log N$ holds: Then the case 2 of the algorithm applies. The output size is trivially bounded by $|P| \log |P|/(2 \log N) \geq 0$. Since $|P| < 12 \log N$ holds, there is no axis-aligned rectangle containing more than or equal to $12 \log N$ points. Hence the constructed output (i.e., the empty set) is a $(12 \log N/|P|)$-net.

(Inductive Step): Let $P$ be the set of points such that $|P| \geq 12 \log N$ holds. Consider any axis-aligned rectangle $X$ such that $|X \cap P| \geq 12 \log N$ holds. We show that $X$ necessarily contains a point in $P'$. By the induction hypothesis, both $\text{NetFind}(N, P_0)$ and $\text{NetFind}(N, P_1)$ correctly computes a $(12 \log N/|P_0|)$-net and a $(12 \log N/|P_1|)$-net. Hence if $X$ is contained either $R_0$ or $R_1$, $X$ necessarily contains a point in $P'$. If $X$ intersects $x = M$, it also intersects $P'$ by Lemma 11. Hence the constructed $P'$ is a $(12 \log N/|P|)$-net. We bound the output size $|P'|$. By the induction
of the output sizes of NetFind\((N, P_0)\) and NetFind\((N, P_1)\) are respectively bounded by \(|P| \log(|P|/2)/(4 \log N)\). The size of \(P^*\) is bounded by \(|P|/(2 \log N)\). Summing up them, we obtain

\[
|P'| \leq 2 \cdot \left(\frac{|P| \log |P|}{4 \log N} + \frac{|P|}{2 \log N}\right) \\
\leq \frac{|P| (\log |P| - 1)}{2 \log N} + \frac{|P|}{2 \log N} \\
\leq \frac{|P| \log |P|}{2 \log N}.
\]

It is easy to bound the running time because the total running time of all recursive calls at the same depth is bounded by \(O(|P| \log N)\). The lemma is proved.

\[\square\]

Lemma 4 is obviously obtained from Lemma 10 and Lemma 12. We finally show the main lemma of this section.

**Lemma 5** There exists two deterministic algorithms respectively constructing a \((S_{f,T}, k)\)-good hierarchy with the following performance guarantees:

- \(k = O(f^2 \log n)\) and the construction time is \(\tilde{O}(m)\).
- \(k = O(f^2 \log \log n)\) and the construction time is \(\text{poly}(m)\).

**Proof.** We only show that first construction, but the second construction is proved in the same way. Consider the construction of \(E_{i+1}\) from \(E_i\). The algorithm first maps all the edges in \(E_i\) into the space \([1, 2n - 2]^2\). Applying the algorithm of Lemma 12 to \(E_i\) with \(P = E_i\) and \(N = |E_i|\), we obtain an \((6 \log |E_i|/|E_i|)\)-net \(E_{i+1}\) of a constant fraction size for \(E_i\) and all axis-aligned rectangles. As mentioned in Section 4.3, \(E_{i+1}\) works as a \((6(2f + 1)^2 \log n)\)-net for \(\mathcal{H}_{2f}\) and thus it satisfies the condition of \((S_{f,T}, 6(2f + 1)^2 \log n)\)-good edge hierarchy. Since \(E_{i+1}\) is a subset of \(E_i\) with a constant fraction size, the depth \(h\) of hierarchy is bounded by \(O(\log n)\). The construction time of \(E_{i+1}\) from \(E_i\) follows the running time of the algorithm of Lemma 12, i.e., it takes \(\tilde{O}(m)\) time. Hence the total running time is \(\tilde{O}(m)\). \[\square\]

### 7.6 Fast Query Processing

We construct a refined query processing algorithm for our tree edge \(f\)-FTC labeling scheme. Let \(G^* = G - E_T\) for short. Throughout this section, we assume that the tree edge \(f\)-FTC labeling scheme is implemented with any adaptive \(S_{f,T}\)-outdetect labeling scheme \((L^\text{out}_{G^*}, D^\text{out})\) of \(\tilde{O}(f^b)\)-bit label size which admits \(\tilde{O}(|\partial_T(S)|^c)\) decoding time for a given query \(S \subseteq V_{G^*}\), as explained in Section 6. Similarly as the original one, the refined algorithm also iteratively merges the vertices of the component graph \(T/C(F)\). It manages a collection \(\mathcal{X}\) of disjoint subsets of \(C(F)\) throughout the procedure. A subset \(S \subseteq C(F)\) in \(\mathcal{X}\) is called a **component fragment** (note that \(S\) is not a subset of \(V_G\)). For any component fragment \(S \subseteq C(F)\), we define \(V(S) \subseteq V_G\) as \(V(S) = \bigcup_{C \in S} C\). As a loop invariant, the algorithm guarantees that \(V(S)\) for any component fragment \(S \in \mathcal{X}\) induces a connected subgraph of \(G - F\). Each component fragment \(S \subseteq C(F)\) is stored as the triple \((S, \partial_T(V(S)), L^\text{out}_{G^*(V(S))})\) in \(\mathcal{X}\). We denote this triple associated with \(S \subseteq C(F)\) by \(\tau(S)\). The whole structure of the algorithm is stated below:

1. Initially, we set \(\mathcal{X} = \{\tau(\{C\}) \mid C \in C(F)\}\).

2. In each iteration, we pick up \(\tau(S)\) such that \(|\partial_T(V(S))|\) is the smallest, and find an outgoing edge of \(S\) by decoding \(L^\text{out}_{G^*}(V(S))\), which is obtained from \(L^\text{out}_{G^*,K}(V(S))\) stored in \(\tau(S)\).
3. If no outgoing edge of \( S \) is found, we remove \( \tau(S) \) from \( \mathcal{X} \) and go to the next iteration. Otherwise, let \( S' \) be the component fragment the outgoing edge from \( S \) reaches.

4. If \( S \) and \( S' \) contains \( s \) and \( t \) respectively, the procedure terminates with returning true. Otherwise, the algorithm deletes \( \tau(S) \) and \( \tau(S') \) from \( \mathcal{X} \), and newly insert the entry of \( \tau(S'') \) for \( S'' = S \cup S' \). The entry \( \tau(S'') \) is computed as \( \tau(S'') = (S'', (\partial_T(V(S)) \cup \partial_T(V(S'))) \setminus (\partial_T(V(S)) \cap \partial_T(V(S'))) \cup \partial_T(V(S')) \cup \partial_T(V(S')).) \). After the insertion, the algorithm proceeds to the next iteration unless \( |\mathcal{X}| = 1 \) holds. If \( |\mathcal{X}| = 1 \) holds, the algorithm terminates with returning false (this case occurs only when the component fragment containing \( s \) or \( t \) is discarded).

To implement the algorithm above efficiently, we manage \( \mathcal{X} \) by the heap which supports \( O(\log |\mathcal{X}|) \)-time insert, delete, and search of the element having the minimum cutset. Each cutset associated with an element in \( \mathcal{X} \) is stored as the bit vector of length \( |F| \) and the additional integer value representing the size of the stored cutset. This data structure obviously supports union and intersection in \( O(|F|) \) time, as well as getting the cutset size in \( O(1) \) time. Each fragment \( S \) associated with a triple in \( \mathcal{X} \) is managed by any disjoint-set data structure (e.g., union-find) over \( C(F) \). Combining this structure with Proposition 3, one can determine the fragment \( S \in \mathcal{X} \) containing a given vertex \( u \in V_G \) from \( L_T^{\text{inc}}(u) \) in \( O(\log |F|) \) time (i.e., identify \( C \in C(F) \) containing \( u \) first, and then identify \( S \in \mathcal{X} \) containing \( C \)). With support of all the data structures above, we can implement one iteration of the refined procedure in \( \tilde{O}(|F|^b + |\partial_T(V(S))|^c) \) time. The initialization of \( \mathcal{X} \) is implemented in \( \tilde{O}(|F|^{b+1}) \) time. We show that the refined algorithm runs in \( \tilde{O}(|F|^c) \) time in total.

**Lemma 6** Assume that there exists a \( S_f,T \)-outdetect labeling scheme of label size \( \alpha = \tilde{O}(f^b) \) and decoding time \( \beta = \tilde{O}(f^c) \). Then there exists a \( f \)-FTC labeling scheme of \( O(\alpha \log n) \)-bit label size and \( \tilde{O}(|F|^b + |F|^c) \) decoding time. The resultant \( f \)-FTC labeling scheme is deterministic if the corresponding \( S_f,T \)-outdetect labeling scheme is deterministic.

**Proof.** Consider the refined query processing algorithm above. We denote by \( \mathcal{X}_i \) the set \( \mathcal{X} \) at the beginning of the \( i \)-th iteration, and let \( \mathcal{Y}_i = \{ S \mid (S, \ldots) \in \mathcal{X}_i \} \). Since \( \mathcal{Y}_i \) is a disjoint collection of component fragments and each \( S \in \mathcal{Y} \) satisfies \( \partial_T(V(S)) \subseteq F \), we have \( \sum_{S \in \mathcal{Y}_i} |\partial_T(S)| \leq 2|F| \). Let \( S_1, S_2, \ldots, S_x \) be the component fragments chosen in each iteration \( (x \leq |F|) \). Since the algorithm chooses \( S_i \) minimizing \( |\partial_T(V(S_i))| \), we have \( |\partial_T(V(S_i))| \leq 2|F|/|\mathcal{Y}_i| \). As discussed in this section, the detection of an outgoing edge of \( S_i \) takes \( \tilde{O}(|\partial_T(V(S_i))|^c) \) time. Since at least one component in \( \mathcal{Y}_i \) is merged or discarded in the \( i \)-th iteration, we have \( |\mathcal{Y}_{i+1}| \leq |\mathcal{Y}_i| - 1 \). The computation time excluding that for the outgoing edge detection is \( O(|F|^b \log n) \) per one iteration. The total running time is bounded as follows:

\[
\sum_{1 \leq i \leq |F|} \tilde{O}\left(\left(\frac{|F|^c}{i}\right) + |F|^b\right) \\
\leq \tilde{O}(|F|^c) \cdot \sum_{1 \leq i \leq |F|} \frac{1}{i^c} + \tilde{O}(|F|^{b+1}) \\
\leq \tilde{O}(|F|^c + |F|^{b+1}).
\]

\( \square \)

8 Distributed Construction

In this section, we explain how our deterministic \( f \)-FCT labeling scheme is constructed in the standard CONGEST model, i.e., the round-based synchronous system with the \( O(\log n) \)-bit message
size bound. For the input graph \( G \), we fix \( T \) as its BFS tree for an arbitrary chosen root. Since the corresponding auxiliary graph \( G' \) is easily simulated on the top of the original graph, the behavior of the proposed algorithm is described as the message passing on \( G' \). The spanning tree of \( G' \) transformed from \( T \) is denoted by \( T' \).

**Construction of Ancestry Labels** The construction by Kannan, Naor, and Rudich is to assign each node and edge with the pair of its pre-order and post-order in the Euler-tour traversal of \( T' \) starting from the root. For any two labels \((a, b)\) and \((c, d)\) assigned to \( u \) and \( v \), \( u \) is an ancestor of \( v \) if and only if the interval \([a, b]\) contains \([c, d]\). In the following argument, we focus on the computation of the pre-orders and post-orders of edges in \( T' \). The computation of vertex orders is processed similarly. For every edge \( e \), the algorithm computes the number of edges in the subtree \( T'(e) \), which is implemented by the subtree-sum aggregation over \( T' \), taking \( O(D) \) rounds. The twice of the computed value, denoted by \( \text{gap}(e) \), is equal to the gap between the pre-order and the post-order of \( e \). Then the algorithm determines the pre-orders and post-orders of all tree edges from the root side. Assume that we have fixed the orders \((a, b)\) of an edge \( e \), and let \( e_1, e_2, \ldots, e_j \) be the set of children edges of \( e \). Then the pre-order of \( e_1 \) is obviously \( a + 1 \), and its post-order is \( a + 2 + \text{gap}(e_1) \). The orders of \( e_2, e_3, e_4, \ldots, e_j \) are decided similarly.

**Construction of Outdetect Labels** Given a \((S_{f,T'}, O(f^2 \log n))\)-good hierarchy, it is trivial to compute \( k \)-threshold outdetect labels. The label \( g(e) \) assigned to each non-tree edge is computed locally, and the label to each vertex is computed by the subtree-sum aggregation of message size \( \tilde{O}(f^2) \). Using the standard pipeline technique, it is processed in \( \tilde{O}(D+f^2) \) rounds.

**Distributed Construction of a \((S_{f,T'}, O(f^2 \log n))\)-good hierarchy** To construct such a good hierarchy, it suffices to implement NetFind in the CONGEST model. As a preprocessing, the algorithm first computes the \( x-y \) coordinates of all non-tree edges, which is realized in the same way as the construction of the ancestry labels. After this preprocessing, one can assume that both of the endpoints of any non-tree edge \( e \) knows the coordinate of \( e \). For simplicity of the argument, we assume that \( m' = |E_{G'}| - |E_{T'}| \) is a power of 2. Our implementation processes the calls of NetFind at the same recursion level in parallel. It is easy to check that the following two conditions are satisfied:

- For any call of NetFind\((N, P)\) at the recursion level \( j > (\log m')/2 \), we have \( |P| \leq \sqrt{m'} \). In addition, the information of the points (edges) in \( P \) is stored at nodes on a consecutive subsequence (denoted by \( \text{seq}(P) \)) of the Euler-tour traversal of \( T' \).
- For any two calls of NetFind\((N, P_1)\) and NetFind\((N, P_2)\) at the same recursion level, \( \text{seq}(P_1) \) and \( \text{seq}(P_2) \) are edge disjoint under the treatment of two edges \( (u,v) \) and \( (v,u) \) as distinct ones.

Since \( \text{seq}(P) \) induces a connected subtree of \( T' \), for the invocation of NetFind\((N, P)\), every node in \( \text{seq}(P) \) can aggregates whole information of \( P \) in \( O(\sqrt{m'}+D) \) rounds. Hence each node can execute the centralized version of NetFind locally. Due to the second condition above, the aggregation tasks for all \( P \) at the same recursion level is efficiently processed in parallel: each edge \( e \in E_{T'} \) is contained at most two induced subtrees, and thus the total running time is still bounded by \( O(\sqrt{m'+D}) \). Consequently, all the invocations of NetFind\((N, P)\) at the same recursion level \( j > (\log m')/2 \) are processed in \( O(\sqrt{m'}+D) \) rounds. For the recursion level \( j < (\log m')/2 \), the total number of invocations is \( O(\sqrt{m'}) \), and thus the algorithm sequentially processes each invocation. The main body of NetFind\((N, P)\) is the construction of the point set \( P' \) shown in Lemma 11. To identify the
set \(Y_1, Y_2, \ldots, Y_{p}\), it suffices to compute the order of each non-tree edge in the sequence \(Y\), which is computed in \(O(D)\) rounds similarly with the construction of ancestry labels. Next, the algorithm must decide \(p^i_\epsilon\) and \(p^+\) for each \(Y_p\), which is implemented by the information exchange within the subgraph induced by \(\text{seq}(Y_p)\). This part also needs only \(O(D + \epsilon^{-1})\) rounds. We use this construction with \(\epsilon = O(1/\log m')\), and thus the running time of NetFind\((N, P)\) except for the recursive calls is \(\tilde{O}(D)\). In total, the running time of NetFind\((N, P)\) for \(P = E_G \searrow E_T\) is \(\tilde{O}(\sqrt{mD})\).

To construct whole \((S_f, T)\)-good hierarchy, \(O(\log n)\) repetition of invoking NetFind suffices. Consequently, we obtain the following lemma

**Lemma 13** Let \(T'\) be any BFS tree of the auxiliary graph \(G'\) of the input graph \(G\). There exists a deterministic CONGEST algorithm of constructing a \((S_f, T), O(f^2 \log n)\)-good hierarchy in \(\tilde{O}(\sqrt{mD})\) rounds.

This lemma obviously deduces the theorem below:

**Theorem 3** There exists a deterministic CONGEST algorithm of constructing \(f\)-FTC labels for all vertices and edges in \(\tilde{O}(\sqrt{mD} + f^2)\) rounds.

### 9 Concluding Remarks

This paper presented a new deterministic \(f\)-FTC labeling scheme which attains \(O(f^2 \text{polylog}(n))\)-bit label size, polynomial-time construction, and \(O(\text{poly}(|F|))\)-time query processing time for a given faulty edge set \(F\). This is the first deterministic and polynomial-time \(f\)-FTC labeling scheme with a non-trivial label size. The scheme is developed on the top of a general framework, and only by the modification of graph sparsification, we can also obtain a randomized \(f\)-FTC labeling scheme which is competitive to the original Dory-Parter scheme and attains an adaptive query processing time. The key technical ingredient is a new deterministic \(S\)-outdetect labeling scheme based on error-correcting codes. From the authors’ perspective, our results pose a few promising future research directions. We conclude this paper with summarizing them.

- Is it possible to develop a deterministic algorithm yielding better edge hierarchies, i.e., the hierarchy such that for any \(S\) there exists \(i\) satisfying \(0 < |\partial_E(S)| = o(f^2 \log n)\)? Our framework automatically deduces a deterministic \(f\)-FTC labeling scheme with an improved label size if such an algorithm is found.

- With respect to the construction time in the CONGEST model, our deterministic scheme still has a large gap with the known randomized construction, only taking \(\tilde{O}(f + D)\) rounds. Is it possible to obtain the deterministic \(f\)-FTC labeling scheme of \(O(\text{poly}(f, \log n))\)-bit label size which is implemented in the CONGEST model with \(\tilde{O}(\text{poly}(f) + D)\) or \(\tilde{O}(\text{poly}(f) \cdot D)\) rounds?

- Can we obtain any non-trivial lower bound for the label size of \(f\)-FTC labeling schemes with full query support? It seems plausible that the \(\Omega(f)\)-bit lower bound holds, but no promising way of proving this is found so far.

- Can we use our technique to obtain a (deterministic) fault-tolerant connectivity labeling scheme for vertex faults? As pointed out in [PP22b], there exists a large technical gap between edge fault tolerance and vertex fault tolerance. It is still open to obtain a scheme with a label size sublinear of \(n\), even for moderately small \(f\) (e.g., \(f = O(\log n)\)).
Can our technique be exported to other applications of $S$-outdetect labeling schemes, such as centralized fault-tolerant connectivity oracles [DP20], distributed computation of sparse spanning subgraphs [KKT15, GP18, GK18, MK21] or small cut detection [PT11, PP22a], and dynamic algorithms [KKM13, GKT15], for obtaining any improved result?

References

[ACG12] Ittai Abraham, Shiri Chechik, and Cyril Gavoille. Fully dynamic approximate distance oracles for planar graphs via forbidden-set distance labels. In Proc. of the 44th Annual ACM Symposium on Theory of Computing (STOC), page 1199–1218, 2012.

[ACGP16] Ittai Abraham, Shiri Chechik, Cyril Gavoille, and David Peleg. Forbidden-set distance labels for graphs of bounded doubling dimension. ACM Transactions on Algorithms, 12(2), 2016.

[AGM12a] Kook Jin Ahn, Sudipto Guha, and Andrew McGregor. Analyzing graph structure via linear measurements. In Proc. of the Twenty-Third Annual ACM-SIAM Symposium on Discrete Algorithms (SODA), pages 459–467. SIAM, 2012.

[AGM12b] Kook Jin Ahn, Sudipto Guha, and Andrew McGregor. Graph sketches: Sparsification, spanners, and subgraphs. In Proc. of the 31st ACM SIGMOD-SIGACT-SIGAI Symposium on Principles of Database Systems (PODS), page 5–14, 2012.

[BCG+18] Davide Bilò, Keerti Choudhary, Luciano Gualà, Stefano Leucci, Merav Parter, and Guido Proietti. Efficient Oracles and Routing Schemes for Replacement Paths. In Proc. of 35th Symposium on Theoretical Aspects of Computer Science (STACS), volume 96 of Leibniz International Proceedings in Informatics (LIPIcs), pages 13:1–13:15, 2018.

[BCG+21] Aviv Bar-Natan, Panagiotis Charalampopoulos, Pawel Gawrychowski, Shay Mozes, and Oren Weimann. Fault-tolerant distance labeling for planar graphs. In In Proc. of International Colloquium on Structural Information and Communication Complexity (SIROCCO), volume 12810 of Lecture Notes in Computer Science, pages 315–333, 2021.

[BGLP16] Davide Bilò, Luciano Gualà, Stefano Leucci, and Guido Proietti. Compact and fast sensitivity oracles for single-source distances. In Piotr Sankowski and Christos D. Zaroliagis, editors, Proc. of 24th Annual European Symposium on Algorithms (ESA), volume 57 of Leibniz International Proceedings in Informatics (LIPIcs), pages 13:1–13:14, 2016.

[BK09] Aaron Bernstein and David Karger. A nearly optimal oracle for avoiding failed vertices and edges. In Proc. of the 41st Annual ACM Symposium on Theory of Computing (STOC), page 101–110, 2009.

[BK13] Surender Baswana and Neelesh Khanna. Approximate shortest paths avoiding a failed vertex: Near optimal data structures for undirected unweighted graphs. Algorithmica, 66(1):18–50, 2013.

[CCFK17] Shiri Chechik, Sarel Cohen, Amos Fiat, and Haim Kaplan. $(1 + \epsilon)$-approximate $f$-sensitive distance oracles. In Proc. of the 2017 Annual ACM-SIAM Symposium on Discrete Algorithms (SODA), pages 1479–1496, 2017.
[CGL+20] Julia Chuzhoy, Yu Gao, Jason Li, Danupon Nanongkai, Richard Peng, and Thatchaphol Saranurak. A deterministic algorithm for balanced cut with applications to dynamic connectivity, flows, and beyond. In 61st IEEE Annual Symposium on Foundations of Computer Science (FOCS), pages 1158–1167. IEEE, 2020.

[Che11] Shiri Chechik. Fault-tolerant compact routing schemes for general graphs. In 38th International Colloquium on Automata, Languages, and Programming, (ICALP), pages 101–112, 2011.

[CLPR12] Shiri Chechik, Michael Langberg, David Peleg, and Liam Roditty. $f$-sensitivity distance oracles and routing schemes. Algorithmica, 63(4):861–882, 2012.

[CM96] Bernard Chazelle and Jiří Matoušek. On linear-time deterministic algorithms for optimization problems in fixed dimension. Journal of Algorithms, 21(3):579–597, 1996.

[CT07] Bruno Courcelle and Andrew Twigg. Compact forbidden-set routing. In Proceedings of the 24th Annual Conference on Theoretical Aspects of Computer Science, STACS’07, page 37–48, 2007.

[CT10] Bruno Courcelle and Andrew Twigg. Constrained-path labellings on graphs of bounded clique-width. Theory of Computing Systems, 47(2), 2010.

[DORS08] Yevgeniy Dodis, Rafail Ostrovsky, Leonid Reyzin, and Adam Smith. Fuzzy extractors: How to generate strong keys from biometrics and other noisy data. SIAM Journal on Computing, 38(1):97–139, 2008.

[DP09] Ran Duan and Seth Pettie. Dual-failure distance and connectivity oracles. In Proc. of the 20th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA), page 506–515, 2009.

[DP10] Ran Duan and Seth Pettie. Connectivity oracles for failure prone graphs. In Proc. of the 42nd ACM Symposium on Theory of Computing (STOC), page 465–474, 2010.

[DP20] Ran Duan and Seth Pettie. Connectivity oracles for graphs subject to vertex failures. SIAM Journal on Computing, 49(6):1363–1396, 2020.

[DP21] Michal Dory and Merav Parter. Fault-tolerant labeling and compact routing schemes. In Proc. of the 2021 ACM Symposium on Principles of Distributed Computing (PODC), page 445–455, 2021.

[DT02] Camil Demetrescu and Mikkel Thorup. Oracles for distances avoiding a link-failure. In Proc. of the 13th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA), page 838–843, 2002.

[FKMS07] Joan Feigenbaum, David R. Karger, Vahab S. Mirrokni, and Rahul Sami. Subjective-cost policy routing. Theoretical Computer Science, 378(2):175–189, 2007.

[GK18] Mohsen Ghaffari and Fabian Kuhn. Distributed MST and broadcast with fewer messages, and faster gossiping. In Proc. of 32nd International Symposium on Distributed Computing (DISC), volume 121 of LIPIcs, pages 30:1–30:12, 2018.

[GKKT15] David Gibb, Bruce M. Kapron, Valerie King, and Nolan Thorn. Dynamic graph connectivity with improved worst case update time and sublinear space. CoRR, abs/1509.06464, 2015.
[GP16] Mohsen Ghaffari and Merav Parter. Mst in log-star rounds of congested clique. In Proc. of the 2016 ACM Symposium on Principles of Distributed Computing (PODC), page 19–28, 2016.

[GP18] Robert Gmyr and Gopal Pandurangan. Time-message trade-offs in distributed algorithms. In Proc. of 32nd International Symposium on Distributed Computing (DISC), volume 121 of LIPIcs, pages 32:1–32:18, 2018.

[GR21] Yong Gu and Hanlin Ren. Constructing a distance sensitivity oracle in $o(n^{2.5794}m)$ time. In Nikhil Bansal, Emanuela Merelli, and James Worrell, editors, 48th International Colloquium on Automata, Languages, and Programming, (ICALP), volume 198 of LIPIcs, pages 76:1–76:20, 2021.

[GW19] Fabrizio Grandoni and Virginia Vassilevska Williams. Faster replacement paths and distance sensitivity oracles. ACM Transactions on Algorithms, 16(1), 2019.

[HdlLT01] Jacob Holm, Kristian de Lichtenberg, and Mikkel Thorup. Poly-logarithmic deterministic fully-dynamic algorithms for connectivity, minimum spanning tree, 2-edge, and biconnectivity. Journal of the ACM, 48(4):723–760, 2001.

[HK99] Monika R. Henzinger and Valerie King. Randomized fully dynamic graph algorithms with polylogarithmic time per operation. Journal of the ACM, 46(4):502–516, 1999.

[HPP+15] James W. Hegeman, Gopal Pandurangan, Sriram V. Pemmaraju, Vivek B. Sardeshmukh, and Michele Scquizzato. Toward optimal bounds in the congested clique: Graph connectivity and mst. In Proc. of the 2015 ACM Symposium on Principles of Distributed Computing (PODC), page 91–100, 2015.

[HW87] David Haussler and Emo Welzl. epsilon-nets and simplex range queries. Discrete Computational Geometry, 2:127–151, 1987.

[JN18] Tomasz Jurdzinski and Krzysztof Nowicki. MST in $O(1)$ rounds of the congested clique. In Proc. of the 29th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA), pages 2620–2632, 2018.

[KG10] Janardhan Kulkarni and Sathish Govindarajan. New epsilon-net constructions. In 22nd Annual Canadian Conference on Computational Geometry (CCCG), pages 159–162, 2010.

[KKM13] Bruce M. Kapron, Valerie King, and Ben Mountjoy. Dynamic graph connectivity in polylogarithmic worst case time. In Proc. of the 24th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA), pages 1131–1142, 2013.

[KKT15] Valerie King, Shay Kutten, and Mikkel Thorup. Construction and impromptu repair of an mst in a distributed network with $o(m)$ communication. In Proc. of the ACM Symposium on Principles of Distributed Computing (PODC), page 71–80, 2015.

[KLM+14] Michael Kapralov, Yin Tat Lee, Cameron Musco, Christopher Musco, and Aaron Sidford. Single pass spectral sparsification in dynamic streams. In Proc. of 55th IEEE Annual Symposium on Foundations of Computer Science (FOCS), pages 561–570, 2014.

[KNR92] Sampath Kannan, Moni Naor, and Steven Rudich. Implicit representation of graphs. SIAM Journal on Discrete Mathematics, 5(4):596–603, 1992.
A Randomized Construction of Edge Set Hierarchy

Comparing the quality of labeling schemes with the label size, decoding time, and construction time, our deterministic construction is competitive but certainly worse than the known randomized scheme. However, most of high costs incurred by our construction is derived from the construction of edge set hierarchies. As mentioned in Section 4.3, a simple edge sub-sampling strategy suffices to construct good hierarchies.
Proposition 5 Let \( E^* = E_0 \supseteq E_1 \supseteq E_2 \supseteq \ldots \supseteq E_h = \emptyset \) be the edge set hierarchy such that \( E_{i+1} \) is obtained by sampling each edge in \( E_i \) independently with probability 1/2. Then with probability \( 1 - 1/n^{O(1)} \), this hierarchy is \((S_{f,T}, 5f \log n)\)-good.

**Proof.** Consider the construction of \( E_{i+1} \) from \( E_i \). If \(|E_i| \leq 5f \log n\), we decide \( E_{i+1} = \emptyset \). Otherwise, sample each edge in \( E_i \) with probability 1/2. We show that with high probability, \(|\partial_{E_{i+1}}(S)| > 0\) holds for any \( S \in S_{f,T} \) satisfying \(|\partial_{E_i}(S)| > 5f \log n\). The probability that no edge in \( \partial_{E_i}(S) \) is added to \( E_{i+1} \) is at most \((1/2)^{5f \log n} = 1/n^{5f}\). Since the cardinality of \( S_{f,T} \) is bounded by \( |S_{f,T}| = O(n^f) \), by the union bound argument, we conclude that \(|\partial_{E_{i+1}}(S)| > 0\) holds for any \( S \in S_{f,T} \) satisfying \(|\partial_{E_i}(S)| > 5f \log n\) with probability \( 1 - O(1/n^5) \). This statement holds for all possible \( c \) with probability at least \( 1 - O(1/n^5) \). That is, the probability that \( E_{i+1} \) does not satisfy the second condition of Definition 1 is \( O(1/n^4) \). The first condition is satisfied with high probability because one can show \(|E_{i+1}| \leq 3|E_i|/4\) with probability \( 1 - O(1/n^2) \) by the straightforward application of Chernoff bound. Applying the union bound again on the failing events of the first and second conditions for all \( i \), we obtain the proposition. \( \square \)

B Adaptive Decoding of Deterministic \( S_{f,T} \)-outdetect labeling scheme based on the Reed-Solomon Code

In this appendix, we show that our \( S_{f,T} \)-outdetect labeling scheme attains the adaptiveness without any modification or transformation. The key idea is a nice property of Reed-Solomon Code: We define \( L^{RS(k)}_{H,k'} \) for \( k' \leq k \) as the labeling function which assign \( v \in V_H \) with the prefix of \( L^{RS(k)}_{H}(v) \) up to the \( k' \)-th element. Then the following proposition holds:

**Proposition 6** For any \( k' \leq k \), \( L^{RS(k)}_{H,k'} = L^{RS(k')}_{H} \) holds.

**Proof.** The proof trivially follows the definition of the parity check matrix \( C_{2k} \). Let \( c_{ij} \) be the \((i,j)\)-element of \( C_{2k} \) \((0 \leq i \leq |E|, 0 \leq j \leq 2k - 1) \) and \( \omega \) be a primitive element of \( \mathbb{F} \). Then the each element of \( C_{2k} \) is defined as \( c_{ij} = \omega^j \). That is, the submatrix formed by the first \( 2k' \) columns of \( C_{2k} \) is is equal to \( C_{2k'} \). \( \square \)

Intuitively, the proposition above implies that the \( O(k' \log n) \)-bit prefixes of the labels assigned by our \( k \)-threshold outdetect labeling scheme also work as the labels of the \( k' \)-threshold outdetect labeling scheme. It is also possible to make the sparsification hierarchy adaptive because our construction does not use the upper bound \( f \) at all, i.e., the construction is universal for every \( f \): We explained in the previous section that \( E_{i+1} \) is constructed in the way that it becomes the hitting set of \( \mathcal{Z}_{i,f,c,f'^2 \log n} \) (where \( c \) is a hidden constant). In reality, it also becomes the hitting set of \( \mathcal{Z}_{i,f',c,f'^2} \) for any \( f' > 0 \). Hence it is guaranteed that for any \( S \in S_{|\partial_T(S)|,T} \) there exists an index \( i \) such that \( 0 < |\partial_{G_i}(S)| \leq c|\partial_T(S)|^2 \log n \) holds (in the deterministic case).