Relation between worldline Green functions for scalar two-loop diagrams

Haru-Tada Sato

The Niels Bohr Institute, University of Copenhagen
Blegdamsvej 17, DK-2100 Copenhagen, Denmark

Abstract

We discuss a relation between two-loop bosonic worldline Green functions which are obtained by Schmidt and Schubert in two different parametrizations of a two-loop worldline. These Green functions are transformed into each other by some transformation rules based on reparametrizations of the proper time and worldline modular parameters.

1 Fellow of the Danish Research Academy, sato@nbivax.nbi.dk
One of interesting and important theoretical aspects of quantum field theories, in particular of gauge theories, is the Bern-Kosower formalism [1] which provides a reformulation of one-loop Feynman amplitudes. Their idea is very natural that amplitudes in ordinary field theory may be reproduced by the infinite string tension limit of superstring amplitudes. It results in a new set of rules instead of Feynman rules and enables several calculations: five point gluon [2], four point graviton amplitudes [3] and so on [4]. The advantage of this formalism is to get rid of handling a vast number of Feynman diagrams and of taking care of gauge cancelation of diagrams in gauge invariant theories. It might clarify underlying structure, which we have not been recognized yet in gauge theories; for example, various useful informations have been developed in worldline approaches [5]-[7].

Extension of this formalism to multi-loop diagrams has also been investigated recently [11],[12],[13] generalizing Strassler’s approach [8] to the Bern-Kosower rules. Strassler derived the worldline Green functions [9] of spinor, scalar and gauge fields for one-loop diagrams rewriting one-loop effective actions as path integrals of (supersymmetric) worldline actions. In one-loop case, the modular parametrization of a loop is unique and we can define the worldline Green functions in a unique way. However, this situation changes in multi-loop cases because we have a variety of choices of the parametrization due to node points, i.e., internal vertices. Schmidt and Schubert have actually obtained two expressions for the two-loop worldline Green function in the $\phi^3$ theory [11]. They explicitly checked that amplitudes derived from these respective Green functions coincide with at least three or four point functions from the Feynman rule calculations.

It is apparent that this kind of equivalence check between worldline Green functions in different forms becomes difficult in more general situations such as higher loop diagrams and a large number of external legs. In addition, we can not recognize which parametrization is natural or convenient one in a complicated case, and hence it is useful to know how to convert the Green functions to a differently parametrized form. It is therefore important to find a clear connection between multi-loop Green functions defined on a differently parametrized worldline.

In this report, we discuss a relationship between two-loop bosonic worldline Green
functions proposed by Schmidt and Schubert in the scalar $\phi^3$ theory. First let us survey
the vacuum amplitude at one-loop level in two different ways which are instructive to get
an insight into two-loop case. One is given by the path integral for a worldline action
with cyclic boundary condition and the other may be given by sewing two propagators
of path integral expression [8],[9];

$$I_0^{(1)} = \int_0^{\infty} \frac{dT}{T} N(T) \int_{x(0)=x(T)} Dx \exp[- \int_0^T d\tau \frac{1}{4} \dot{x}^2(\tau)] = \int \frac{dT}{T} (4\pi T)^{-D/2} \quad (1)$$

and

$$J_0^{(1)} = \int d^D x_a d^D x_b \prod_{i=1}^2 \int_0^{\infty} d{T_i} \tilde{N}_i(T_i) \int_{x_i(0)=x_a \at{x_i(T_i)=x_b}} D x_i \exp[- \int_0^{T_i} d\tau_i \frac{1}{4} \dot{x}_i^2(\tau)] \quad (2)$$

$$= \int d^D x_b \int d{T_1} d{T_2} [4\pi(T_1 + T_2)]^{-D/2},$$

where the path integral normalizations $N$ and $\tilde{N}$ are determined as the second equality
in each case. We omit the mass term $e^{-m^2 T}$ in (1) for simplicity. Comparing these
equations, we think of the transformation from $J_0^{(1)}$ to $I_0^{(1)}$ in the following form

$$T_1 = T(1 - u), \quad T_2 = Tu \quad (3)$$

which gives

$$\int_0^{\infty} d{T_1} d{T_2} = \int_0^{\infty} dT T \int_0^1 du. \quad (4)$$

RHS of the latter expression (2) becomes

$$\int d^D x_b \int_0^{\infty} \frac{dT}{T} T^2 \int_0^1 du (4\pi T)^{-D/2} = (2\pi)^D \delta^D(0) \cdot \int_0^{\infty} \frac{dT}{T} \int_0^{T} d\tau_a \int_0^{T} d\tau_b (4\pi T)^{-D/2}, \quad (5)$$

where we have used the fact that the integrand is independent of any $\tau$ variables. If we
consider $N$-point functions as discussed later, we see that $(2\pi)^D \delta^D(0)$ corresponds to the
conservation law of external momenta, which is implicit in (3). The integrals with respect
to $\tau_a$ and $\tau_b$ correspond to node point integrals which appear in multi-loop integrals. Here
we inevitably encounter the node point integrals because we need an additional leg at
a glue point of two propagators regardless of external or internal leg. Strictly speaking,
eq.(3) is not so much a one-loop vacuum amplitude as a two-point function with external
zero momenta or a kind of two-loop vacuum amplitude of one internal line of infinite
length (this will become clear later). We hence pick up pure one-loop contribution from (3) through dropping node point integrals by hand. In this way, we understand that (3) relates (1) with (2) which is symmetrized in $T_1$ and $T_2$.

We further proceed to $N$-point functions which concern the worldline Green functions. Evaluating path integrals with insertion of $N$ scalar vertex operators in both (1) and (2), we have the following two expressions which are seemingly different from each other

\[ I_N^{(1)} = \int_0^\infty \frac{dT}{T} (4\pi T)^{-D/2} \cdot \prod_{n=1}^N \int_0^T d\tau_n \exp \left[ \frac{1}{2} \sum_{j,k}^N p_j p_k G_B(\tau_j, \tau_k) \right] \] (6)

and

\[ J_N^{(1)} = (2\pi)^D \delta^D \left( \sum_{n,i}^a p_n^{(i)} \right) \int_0^\infty dT_1 dT_2 [4\pi(T_1 + T_2)]^{-D/2} \cdot \prod_{n=1}^{N_1} \int_0^{T_1} d\tau_n^{(1)} \prod_{n=1}^{N_2} \int_0^{T_2} d\tau_n^{(2)} \]

\[ \times \exp \left[ \frac{1}{2} \sum_{a=1}^2 \sum_{j,k}^N p_j^{(a)} p_k^{(a)} G_{aa}(\tau_j^{(a)}, \tau_k^{(a)}) + \sum_{j}^N \sum_{k}^N p_j^{(1)} p_k^{(2)} G_{12}(\tau_j^{(1)}, \tau_k^{(2)}) \right] \] (7)

where $G_B$ is the one-loop bosonic Green function \[8\],\[9\]

\[ G_B(x, y) = |x - y| - \frac{(x - y)^2}{T} \] (8)

and $G_{ij}$ are

\[ G_{11}(x, y) = G_{22}(x, y) = |x - y| - \frac{(x - y)^2}{T_1 + T_2} \] (9)

\[ G_{12}(x, y) = x + y - \frac{(x + y)^2}{T_1 + T_2}. \] (10)

In (8) and (11), $G_{ij}$ are rearranged from original path integral results with the use of momentum conservation in accordance with two-loop results \[11\]. Note that $\tau^{(2)}$ is defined to run in the opposite direction to $\tau$ and $\tau^{(1)}$ because we have chosen the boundary condition of path integral (3) as $x_i(0) = x_a$ and $x_i(T_i) = x_b$ for $i = 1, 2$. If we choose $x_1(0) = x_2(T_2) = x_a$ and $x_1(T_1) = x_2(0) = x_b$, we will obtain an expression with reversed sign for $\tau^{(2)}$ variable. Since we have seen a correspondence between $T$-parameters’ integral measures in (3), we do not have to repeat that here. Changing the sign of $\tau^{(2)}$ and renaming the variables \{\(\tau_n^{(1)}, \tau_n^{(2)}\)\} and \{\(p_n^{(1)}, p_n^{(2)}\)\} as $\tau_n$ and $p_n$, we obtain the same equation as (8) after omitting the delta function and node point integrals as
a result of the transformation (3). Here we have assumed that the transformation of \(\tau\)-integration measures follow a naive replacement

\[
N \prod_{n=1}^{N} \int_{T_0}^{T} d\tau_n \rightarrow \prod_{i=1}^{N_i} \int_{T_i}^{T_i} d\tau_n^{(i)}.
\]  

(11)

Let us promote the above one-loop idea to the two-loop case. Two-loop \(N\)-point functions with different parametrizations from each other are obtained through insertion of one propagator parametrized by length \(T_3\)

\[
\int_{0}^{\infty} dT_3 N_3(T_3) \int_{x_3(0)=x_a}^{x_3(T_3)=x_b} D x_3 \exp[-\int_{T_0}^{T_3} d\tau \frac{1}{4} \dot{\tau}_3^2(\tau)].
\]  

(12)

The insertion of this propagator and scalar vertex operators into (11) and (2) leads to

\[
I^{(2)}_N = \int_{0}^{\infty} \frac{dT}{T} \int_{0}^{\infty} dT_3 (4\pi)^{-D} \int_{T_0}^{T} d\tau_a \int_{T_0}^{T} d\tau_b \left[ TT_3 + TG_B(\tau_a, \tau_b) \right]^{-D/2} \prod_{n=1}^{N_1} \int_{T_0}^{T} d\tau_n
\times \exp\left[ \frac{1}{2} \sum_{j,k} p_j p_k G_B^{(1)}(\tau_j, \tau_k) \right]
\]  

(13)

and

\[
J^{(2)}_N = 3 \prod_{a=1}^{N_a} \int_{0}^{\infty} dT_a (4\pi)^{-D} (T_1 T_2 + T_2 T_3 + T_3 T_1)^{-D/2} \prod_{n=1}^{N_1} \int_{T_0}^{T} d\tau_n^{(1)} \prod_{n=1}^{N_2} \int_{T_0}^{T} d\tau_n^{(2)}
\times \exp\left[ \frac{1}{2} \sum_{a=1}^{N_a} \sum_{j,k} p_j^{(a)} p_k^{(a)} G_{aa}^{sym}(\tau_j^{(a)}, \tau_k^{(a)}) + \sum_{j}^{N_1} \sum_{k}^{N_2} p_j^{(1)} p_k^{(2)} G_{12}^{sym}(\tau_j^{(1)}, \tau_k^{(2)}) \right],
\]  

(14)

where our \(G_B^{(1)}\) and \(G_{ij}^{sym}\) are same ones in (11) denoted by \(\tilde{G}_B^{(1)}\) and \(\tilde{G}_{ij}^{sym}\); namely,

\[
G_B^{(1)}(x, y) = G_B(x, y) - \frac{1}{4} \frac{(G_B(x, \tau_a) - G_B(x, \tau_b) - G_B(y, \tau_a) + G_B(y, \tau_b))^2}{T_3 + G_B(\tau_a, \tau_b)}
\]  

(15)

and

\[
G_{aa}^{sym}(x, y) = |x - y| - \frac{T_{a+1} + T_{a+2}}{T_1 T_2 + T_2 T_3 + T_3 T_1} (x - y)^2
\]  

(16)

\[
G_{aa+1}^{sym}(x, y) = x + y - \frac{x^2 T_{a+1} + y^2 T_{a+2} + (x + y)^2 T_{a+2}}{T_1 T_2 + T_2 T_3 + T_3 T_1}.
\]  

(17)

The subscripts on \(G\) mean the superscripts of first and second arguments as easily understood in (14) and \(T_4 \equiv T_1\). It is worth noticing that the integrand of (14) coincides
with that of (7) in the limit $T_3 \to \infty$. This implies the previous note that $J_{0}^{(1)}$ may be regarded as a two-loop contribution from an infinite internal line.

Similarly to the one-loop case, we begin with the vacuum ($N = 0$) case. According to the change of variables (3) and (4), eq.(14) becomes

$$J_{0}^{(2)} = \int_{0}^{\infty} dT_3 dT \int_{0}^{1} du (4\pi)^{-D} [T T_3 + T^2 u (1 - u)]^{-D/2}. \quad (18)$$

We then transform the variable $u$ in order to reproduce the Green function $G_B(\tau_a, \tau_b)$ which can be seen in (13)

$$u = \frac{\tau_a - \tau_b}{T}. \quad (19)$$

Taking account of this transformation, eq.(18) coincides with the vacuum amplitude $I_{0}^{(2)}$

$$J_{0}^{(2)} = I_{0}^{(2)} = \int_{0}^{\infty} dT_3 \frac{dT}{T} \int_{0}^{T} dT_{\tau_a} \int_{0}^{T} dT_{\tau_b} (4\pi)^{-D} [T T_3 + T G_B(\tau_a, \tau_b)]^{-D/2}. \quad (20)$$

We here observe that (19) is needed in addition to (3) differently from one-loop case (4) where $u$ disappears.

In the case of $N$-point functions, we need a further consideration. We use the fact that to deal only with $\tau_a \leq \tau_b$ is guaranteed by the symmetry of (13) under $\tau_a \leftrightarrow \tau_b$.

The reversal of $\tau^{(2)}$ considered in the one-loop case should be modified taking account of dependence on the proper time position of a node point;

$$\tau_{n}^{(1)} = x_n - \tau_b, \quad \tau_{n}^{(2)} = \tau_b - y_n. \quad (21)$$

This is the $\tau^{(2)}$ reversal transformation with respect to $\tau_b$ which is regarded as the origin of $\tau^{(1)}$ coordinate shifted by $\tau_b$ accordingly. In one-loop case, we can put the origin $\tau_b = 0$ in safety (because of no $u$ dependence). Notice also that to adopt $\tau_a$ as the origin of the reversal transformation instead of $\tau_b$ corresponds to consider $\tau_b \leq \tau_a$ exchanging $\tau_a \leftrightarrow \tau_b$.

From (3) and $0 \leq \tau^{(i)} \leq T_i$, we see also

$$\tau_a \leq y_n \leq \tau_b \leq x_n. \quad (22)$$

It is convenient to write down $G_B^{(1)}$ explicitly according to this sequence (22),

$$G_B^{(1)}(x_n, x_m) = G_B(x_n, x_m) - \frac{|\tau_a - \tau_b|^2}{T_3 + G_B(\tau_a, \tau_b)} \frac{(x_n - x_m)^2}{T^2}$$
\[ G_B^{(1)}(y_n, y_m) = G_B(y_n, y_m) - \frac{(T^2 - |\tau_a - \tau_b|^2)(y_n - y_m)^2}{T_3 + G_B(\tau_a, \tau_b)} \]  
\[ G_B^{(1)}(x, y) = G_B(x, y) - \frac{1}{T_3 + G_B(\tau_a, \tau_b)}(\tau_b - y - |\tau_a - \tau_b|\frac{x - y}{T})^2. \]

With substitution of (3), (19), (21) in (16), we verify that \( G_{ij} \) for \( i, j = 1, 2 \) are transformed into \( G_B^{(1)} \) as follows:

\[ G_{11}^{\text{sym}}(\tau_n^{(1)}, \tau_m^{(1)}) = G_B^{(1)}(x_n, x_m), \]
\[ G_{22}^{\text{sym}}(\tau_n^{(2)}, \tau_m^{(2)}) = G_B^{(1)}(y_n, y_m), \]
\[ G_{12}^{\text{sym}}(\tau_n^{(1)}, \tau_m^{(2)}) = G_B^{(1)}(x_n, y_m). \]  

Finally, we apply our transformation to another set of worldline Green functions which can be obtained from inserting vertex operators also into the \( T_3 \) propagator part (12). The exponential parts including Green functions of (13) and (14) are generalized to the following respectively

\[ \exp\left[ \frac{1}{2} \sum_{jk}^N p_j p_k G_{11}^{(1)}(\tau_j, \tau_k) + \frac{1}{2} \sum_{jk}^N p_j^{(3)} p_k^{(3)} G_{33}^{(1)}(\tau_j^{(3)}, \tau_k^{(3)}) + \sum_{j}^N \sum_{k}^N p_j p_k^{(3)} G_{13}^{(1)}(\tau_j, \tau_k^{(3)}) \right] \]  

and

\[ \exp\left[ \frac{1}{2} \sum_{a=1}^3 \sum_{j,k}^N p_j^{(a)} p_k^{(a)} G_{aa}^{\text{sym}}(\tau_j^{(a)}, \tau_k^{(a)}) + \sum_{a=1}^3 \sum_{j}^N \sum_{k}^{N_a+1} p_j^{(a)} p_k^{(a+1)} G_{aa+1}^{\text{sym}}(\tau_j^{(a)}, \tau_k^{(a+1)}) \right], \]

where \( N' = N_1 + N_2, N_4 = N_1 \) etc., and \( G_{ij}^{(1)} \) are

\[ G_{11}^{(1)}(x, y) = G_B^{(1)}(x, y), \]
\[ G_{33}^{(1)}(z_1, z_2) = |z_1 - z_2| - \frac{(z_1 - z_2)^2}{T_3 + G_B(\tau_a, \tau_b)}, \]
\[ G_{13}^{(1)}(x, z) = G_B^{(1)}(x, \tau_a) + \frac{1}{T_3 + G_B(\tau_a, \tau_b)} \left( T_3 z - z^2 + z[G_B(x, \tau_b) - G_B(x, \tau_a)] \right) \]
\[ \text{or} \]
\[ G_{13}^{(1)}(x, z) = G_B^{(1)}(x, \tau_b) + \frac{1}{T_3 + G_B(\tau_a, \tau_b)} \left( T_3 z - z^2 + z[G_B(x, \tau_a) - G_B(x, \tau_b)] \right). \]
Eqs. (29) and (30) are different from each other, however they are essentially same under the exchange of integral variables $\tau_a \leftrightarrow \tau_b$. If we follow the rule of transformations (3), (21) and (22) for $\tau_a \leq \tau_b$, we have to choose (30) which can be derived from boundary conditions $x(0) = x(\tau_b)$ and $x(T_3) = x(\tau_a)$ in the path integral (12), where $\tau_b$ is regarded as the origin of $\tau$ parameter. If we want to discuss (29), we have to use the transformations (obtained from $\tau_a \leftrightarrow \tau_b$) for the reverse case $\tau_b \leq \tau_a$ as mentioned above. This situation is completely same as the third equation in (23). As for (27) and (28), we have no attention to such choice as whether (29) or (30) because they are already symmetric form in $\tau_a$ and $\tau_b$.

Eq. (27) has been checked in (24) to be transformed into $G_{11}^{\text{sym}}, G_{22}^{\text{sym}}$ and $G_{12}^{\text{sym}}$. Eq. (28) also coincides with $G_{33}^{\text{sym}}$ under our transformation rules discussed above

\[
G_{33}^{(1)}(\tau_n^{(3)}, \tau_m^{(3)}) = G_{33}^{\text{sym}}(\tau_n^{(3)}, \tau_m^{(3)}).
\]  

(31)

Now, it is enough to show the correspondence of (30) to the rest $G_{23}^{\text{sym}}$ and $G_{31}^{\text{sym}}$. First, writing down $G_{13}^{(1)}(x_n, z)$ and $G_{13}^{(1)}(y_n, z)$ according to (22), and then using (3) and (19), we can rearrange them into similar forms as $G_{31}^{\text{sym}}$ or $G_{23}^{\text{sym}}$; for example,

\[
G_{13}^{(1)}(y_n, z) = b - y_n + z - \frac{1}{T_3 + G_B(\tau_a, \tau_b)} \left[ T_3(b - y_n)^2 + Tuz^2 + T(1 - u)(b - y_n + z)^2 \right].
\]  

(32)

With the identification (21) and $z = \tau^{(3)}$, we get the following relations

\[
G_{13}^{(1)}(x, z) = G_{31}^{\text{sym}}(\tau^{(3)}, \tau^{(1)}),
\]  

(33)

\[
G_{13}^{(1)}(y, z) = G_{23}^{\text{sym}}(\tau^{(2)}, \tau^{(3)}).
\]  

(34)

In this paper, we have discussed the relation between different worldline Green functions $G_{ij}^{(1)}$ and $G_{ij}^{\text{sym}}$. It is shown that both sets of these Green functions are transformed into each other under the relations (3), (13) and (21). Also the integral measures on the modular parameters $T_1$, $T_2$ and those on node points $\tau_a$, $\tau_b$ and $T$ are in correspondence to each other. From (3), $\tau$ is related to $\tau^{(i)}$ as

\[
\tau = (Tu - \tau^{(2)})\theta(Tu - \tau) + (\tau^{(1)} + Tu)\theta(\tau - Tu)
\]  

(35)
and this yields
\[ \int_0^T d\tau = \int_0^{T_u} d\tau^{(2)} + \int_0^{T(1-u)} d\tau^{(1)}, \] (36)
which produces combinatorial factors in (34), however such factors should be removed taking account of a constraint on proper times similarly to the analysis in [10].

Our transformation rules are found through generalization of simple one-loop case. Our discussion on two-loop Green functions might be useful for other theories; for example, scalar and spinor QED diagrams to which super Green functions are applied [12] (see also [13] where the same Green function as (15) and multi-loop diagrams are discussed in scalar QED). We finally anticipate that our analysis may be a help to clarification of multi-loop constructions in the worldline path integral approach.

Acknowledgments

The author would like to thank J.W. van Holten, L. Magnea, K. Roland, M.G. Schmidt for useful suggestions and T. Onogi for discussions in the early stage of this work.
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