Remarks on the classification of a pair of commuting semilinear operators

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Abstract

Gelfand and Ponomarev [Functional Anal. Appl. 3 (1969) 325–326] proved that the problem of classifying pairs of commuting linear operators contains the problem of classifying \(k\)-tuples of linear operators for any \(k\). We prove an analogous statement for semilinear operators.

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1. Introduction

Gelfand and Ponomarev \cite{9} proved that the problem of classifying pairs of commuting linear operators on a vector space contains the problem of classifying \(k\)-tuples of linear operators for any \(k\) (that is, the solution of the former problem would imply the solution of the latter problem).

We prove an analogous statement for semilinear operators. A mapping \(\mathcal{A} : U \to V\) between two complex vector spaces is called semilinear if

\[ \mathcal{A}(u + u') = \mathcal{A}u + \mathcal{A}u', \quad \mathcal{A}(\alpha u) = \bar{\alpha} \mathcal{A}u \]

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for all \( u, u' \in U \) and \( \alpha \in \mathbb{C} \). We write \( \mathcal{A} : U \rightarrow V \) if \( \mathcal{A} \) is semilinear. If \( U = V \) then \( \mathcal{A} \) is called a *semilinear operator*. In Section 2 we recall some basic facts about semilinear mappings and describe all semilinear operators that commute with a given nilpotent semilinear operator.

In Sections 3 and 4 we prove the following theorem, which extends the results of [9] to semilinear operators.

**Theorem 1.** (a) *The problem of classifying pairs of commuting semilinear operators contains the problem of classifying pairs of arbitrary semilinear operators.*

(b) *The problem of classifying pairs of semilinear operators contains the problem of classifying \((p + q)\)-tuples consisting of \(p\) linear operators and \(q\) semilinear operators, in which \(p\) and \(q\) are arbitrary nonnegative integers.*

A similar statement for operators on unitary spaces was proved in [10, Lemma 2]: the problem of classifying semilinear operators on a unitary space contains the problem of classifying tuples of linear and semilinear operators on a unitary space.

Any tuple in Theorem 1(b) consists of operators acting on the same vector space. In Section 5 we generalize Theorem 1(b) to collections of mappings that act on different spaces. We use the notion of biquiver representations introduced in [14, Section 5], which generalizes the notion of quiver representations introduced by Gabriel [7]. A *biquiver* is a directed graph with full and dashed arrows; for example,

\begin{align}
\begin{array}{c}
\begin{tikzpicture}
  \node (1) at (0,1) {1};
  \node (2) at (0,0) {2};
  \node (3) at (1,0) {3};
  \node (4) at (0.5,0.5) {\alpha};
  \node (5) at (0.5,-0.5) {\beta};
  \node (6) at (1.5,0) {\gamma};

  \draw [->, thick] (1) -- (2);
  \draw [->, dashed] (2) -- (3);

  \end{tikzpicture}
\end{array}
\end{align}

Its *representation* is given by assigning to each vertex a complex vector space, to each full arrow a linear mapping, and to each dashed arrow a semilinear mapping of the corresponding vector spaces. Thus, a representation

\begin{align}
\begin{array}{c}
\begin{tikzpicture}
  \node (U) at (0,1) {U};
  \node (V) at (0,0) {V};
  \node (W) at (1,0) {W};
  \node (F) at (1,0) {F};
  \node (A) at (0.5,1) {A};
  \node (B) at (1.5,1) {B};
  \node (D) at (1.5,0) {D};
  \node (E) at (1.5,0) {E};

  \draw [->, thick] (U) -- (A);
  \draw [->, thick] (V) -- (D);
  \draw [->, thick] (D) -- (W);
  \draw [->, thick] (A) -- (B);
  \draw [->, dashed] (B) -- (F);
  \draw [->, dashed] (F) -- (W);
  \draw [->, dashed] (E) -- (F);

  \end{tikzpicture}
\end{array}
\end{align}
of (1) is formed by complex spaces $U, V, W$, linear mappings
\[ B : W \to U, \quad D : V \to W, \quad F : W \to W, \]
and semilinear mappings
\[ A : V \to U, \quad C : V \to V, \quad E : V \to W. \]
A biquiver without dashed arrows is a quiver and its representations are the quiver representations.

In Section 5 we prove the following generalization of Theorem 1(b).

**Theorem 2.** The problem of classifying pairs of semilinear operators contains the problem of classifying representations of any biquiver.

The results in [9] ensure that the problem of classifying pairs of linear operators over any field $F$ contains the problem of classifying $k$-tuples of linear operators. This implies that it contains the problem of classifying representations of an arbitrary $k$-dimensional algebra $\Lambda$ over $F$ by operators of a vector space $V$. Thus, the problem of classifying pairs of linear operators contains the problem of classifying representations of any quiver. A direct proof of this inclusion is given in [13, Sect. 3.1] and [3]. The problem of classifying pairs of linear operators also contains the problem of classifying any system of linear mappings and bilinear or sesquilinear forms because the latter problem can be reduced to the problem of classifying quiver representations (see [12, 14, 15]).

For this reason, the problem of classifying pairs of linear operators is used in representation theory as a measure of complexity: all classification problems split into two types: tame (or classifiable) and wild (containing the problem of classifying pairs of linear operators); wild problems are considered as hopeless. These terms were introduced by Donovan and Freislich [4] in analogy with the partition of animals into tame and wild ones. It follows from Theorem 2 that the problem of classifying pairs of semilinear operators plays the same role in the theory of systems of linear and semilinear mappings.

---

1. The exists an isomorphism from $\Lambda$ to a factor algebra $F(x_1, \ldots, x_t)/J$ of the free algebra of noncommutative polynomials in $x_1, \ldots, x_t$. Let $g_1, \ldots, g_r$ be generators of $J$, then each representation of $\Lambda$ is a $k$-tuple of linear operators $(A_1, \ldots, A_k)$ satisfying $g_i(A_1, \ldots, A_k) = 0$ for all $i = 1, \ldots, r$.

2. The representations of a quiver can be identified with the representations of its path algebra.
2. Semilinear operators commuting with a nilpotent semilinear operator

In this section, we describe all semilinear operators that commute with a given nilpotent semilinear operator, but first we recall basic facts about semilinear mappings. All vector spaces and matrices that we consider are over the field of complex numbers.

We denote by $\bar{a}$ the complex conjugate of $a \in \mathbb{C}$, by $[v]_e$ the coordinate vector of $v$ in a basis $e_1, \ldots, e_n$, and by $S_{e \to e'}$ the transition matrix from a basis $e_1, \ldots, e_n$ to a basis $e'_1, \ldots, e'_n$. If $A = [a_{ij}]$ then $\bar{A} := [\bar{a}_{ij}]$.

Let $A : U \to V$ be a semilinear mapping. We say that an $m \times n$ matrix $A_{ee}$ is the matrix of $A$ in bases $e_1, \ldots, e_n$ of $U$ and $f_1, \ldots, f_m$ of $V$ if

$$[Au]_f = \bar{A}_{fe}[u]_e$$

for all $u \in U$. (3)

Therefore, the columns of $A_{ee}$ are $[Ae_1]_f, \ldots, [Ae_n]_f$. We write $A_e$ instead of $A_{ee}$ if $U = V$.

If $e'_1, \ldots, e'_n$ and $f'_1, \ldots, f'_m$ are other bases of $U$ and $V$, then

$$A_{f'e'} = \bar{S}_{f' \to f}^1 A_{fe} S_{e \to e'}$$

since the right hand matrix satisfies (3) with $e', f'$ instead of $e, f$:

$$\bar{S}_{f' \to f}^1 A_{fe} S_{e \to e'}[v]_{e'} = S_{f' \to f}^1 A_{fe}[v]_e = S_{f' \to f}^{-1}[Av]_f = [Av]_{f'}$$

In particular, if $U = V$, then

$$A_{e'} = \bar{S}_{e' \to e}^{-1} A_e S_{e \to e'}$$

and so $A_{e'}$ and $A_e$ are consimilar: recall that two matrices $A$ and $B$ are consimilar if there exists a nonsingular matrix $S$ such that $\bar{S}^{-1}AS = B$ (see [6, Section 4.6]). Two pairs $(A_1, A_2)$ and $(B_1, B_2)$ of $n \times n$ matrices are called consimilar if there exists a nonsingular matrix $S$ such that

$$\bar{S}^{-1}(A_1, A_2)S := (\bar{S}^{-1}A_1S, \bar{S}^{-1}A_2S) = (B_1, B_2).$$

Thus, the problem of classifying pairs of semilinear operators reduces to the problem of classifying matrix pairs up to consimilarity.

**Lemma 3.** The composition of two semilinear operators $A : U \to U$ and $B : U \to U$ is a linear operator and its matrix in a basis $e_1, \ldots, e_n$ of $U$ is

$$(AB)_e = \bar{A}_e B_e$$

(4)
Proof. The identity (6) follows from observing that $A\bar{B}$ is a linear operator and
\[
\bar{A}_eB_e[u]_e = \bar{A}_e[Bu]_e = [A(Bu)]_e = [(A\bar{B})u]_e \quad \text{for each } u \in U.
\]

A canonical form of a matrix under consimilarity is given in [5, Theorem 3.1]. In particular, each nilpotent matrix is consimilar to a nilpotent Jordan matrix that is determined uniquely up to permutation of Jordan blocks. Each nilpotent Jordan matrix is *permutationally similar* (i.e., is reduced by simultaneous permutations of rows and columns) to the form
\[
J := J_{p_1}(0_{q_1}) \oplus \cdots \oplus J_{p_t}(0_{q_t}), \quad p_i \neq p_j \text{ if } i \neq j,
\]
in which
\[
J_{p_i}(0_{q_i}) := \begin{bmatrix} 0_{q_i} & I_{q_i} & 0 \\ 0_{q_i} & \ddots & \vdots \\ \vdots & \ddots & I_{q_i} \\ 0 & \cdots & 0_{q_i} \end{bmatrix} \quad (p_i \times p_i \text{ subblocks of size } q_i \times q_i).
\]

We consider $J$ as a block matrix $[J_{ij}]_{i,j=1}^t$; each block $J_{ij}$ is $p_i q_i \times p_j q_j$ and is partitioned into $p_i \times p_j$ subblocks of size $q_i \times q_j$.

All matrices that commute with a given square matrix are described in [8, Sect. VIII, §2]. In the following lemma, we give an analogous description of all matrices $S$ satisfying $\bar{S}J = JS$.

**Lemma 4.** (a) For each nilpotent semilinear operator $\bar{J} : U \longrightarrow U$ there exists a basis in which its matrix has the form (5). If $S : U \longrightarrow U$ is another semilinear operator and $S$ is its matrix in the same basis, then $\bar{S}J = JS$ if and only if $\bar{S}J = JS$.

(b) Let $J$ be the matrix (5), let $S$ be a matrix of the same size, and let $S$ be partitioned into blocks and subblocks conformally to the partition of $J$. Then $\bar{S}J = JS$ if and only if $S = [S_{ij}]_{i,j=1}^t$, in which every $S_{ij}$ is a $p_i q_i \times p_j q_j$
block of the form

$$S_{ij} = \begin{cases} 
    
    \begin{bmatrix}
    C_{ij} & C'_{ij} & C''_{ij} & \cdots & C^{(p_j-1)}_{ij} \\
    \bar{C}_{ij} & \bar{C}'_{ij} & \bar{C}''_{ij} & \cdots & \bar{C}^{(p_j-2)}_{ij} \\
    C_{ij} & C'_{ij} & C''_{ij} & \cdots & C^{(p_j-3)}_{ij} \\
    \bar{C}_{ij} & \bar{C}'_{ij} & \bar{C}''_{ij} & \cdots & \bar{C}^{(p_j-4)}_{ij} \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    \hat{C}_{ij} & \hat{C}'_{ij} & \hat{C}''_{ij} & \cdots & \hat{C}^{(p_j-2)}_{ij} \\
\end{bmatrix} & \text{if } p_i \leq p_j, \\
0 & \text{if } p_i \geq p_j 
\end{cases}$$

and

$$\hat{C}_{ij} = \begin{cases} 
    C_{ij} & \text{if } \min(p_i, p_j) \text{ is odd}, \\
    \bar{C}_{ij} & \text{if } \min(p_i, p_j) \text{ is even.}
\end{cases}$$

For example, if $J = J_4(0_q) \oplus J_2(0_{q'})$ and $\bar{S}J = JS$, then

$$(7)$$

(unspecified blocks are zero).
Proof of Lemma 4. (a) This statement follows from Lemma 3 and the canonical form of a matrix under consimilarity [5].

(b) We have \( \tilde{S}J = JS \) if and only if
\[
\tilde{S}_{ij}J_{p_j}(0_{q_j}) = J_{p_i}(0_{q_i})S_{ij} \quad \text{for all } i, j = 1, \ldots, t. \tag{8}
\]

Assuming (8), we verify that each \( S_{ij} \) has the form (6) as follows: divide \( S_{ij} \) into \( p_i \times p_j \) subblocks of size \( q_i \times q_j \) and compare subblocks in the identity \( \tilde{S}_{ij}J_{p_j}(0_{q_j}) = J_{p_i}(0_{q_i})S_{ij} \) starting in subblock \( (p_i, 1) \), moving along vertical strips from bottom to up, and finishing in subblock \( (1, p_j) \).

Conversely, if all \( S_{ij} \) have the form (6), then (8) holds. \( \square \)

Let \( M \) be an arbitrary block matrix partitioned into strips and substrips such that all diagonal blocks and subblocks are square. We index the \( \alpha \)th substrip of \( i \)th strip by the pair \( \alpha, i \) (as in (7)). Denote by \( M^\# \) the block matrix obtained from \( M \) by permuting its substrips so that their index pairs form a lexicographically ordered sequence. For example, if \( J \) and \( S \) are the block matrices (7), then
\[
J^\# = \begin{array}{cccccc}
0_q & I_q & 0_{q'} & I_{q'} & 1.1 & 1.2 \\
0_{q'} & I_q & 0_{q'} & 2.1 & 2.2 \\
0_q & I_q & 3.1 & 4.1 & 3.1 \\
0_{q'} & I_{q'} & 4.1 & 3.1 \\
\end{array}
\quad S^\# = \begin{array}{cccccc}
C & D & C_1 & D_1 & C_2 & C_3 & 1.1 & 1.2 \\
F & F_1 & E & E_1 & 2.1 & 2.2 \\
\bar{C} & \bar{D} & \bar{C}_1 & \bar{D}_1 & 2.1 & 2.2 \\
\bar{F} & \bar{E} & \bar{E}_1 & \bar{E}_1 & 3.1 & 4.1 \\
\end{array}
\tag{9}
\]

The block matrix \( M^\# \) can be obtained from \( M \) as follows: we gather at the top the first substrips of all horizontal strips, we dispose all second substrips under them, and so on. Finally, we make the same permutation of vertical substrips.

Suppose that the direct summands in (5) are numbered so that
\[
p_1 > p_2 > \cdots > p_t. \tag{10}
\]

Then the block matrix \( J^\# \) (which is permutationally similar to a nilpotent Jordan matrix) is a nilpotent Weyr matrix; see [11] or [13]. The second matrix in (9) is block triangular; in the following lemma we prove that \( S^\# \)
is block triangular for all nilpotent Weyr matrices. This property is a minor modification (in the nilpotent case) of the most important property of Weyr matrices, which was discovered by Belitskii [1] (see also [2, 13]): all matrices commuting with a Weyr matrix are block triangular.

Lemma 5. (a) Let $J^\#$ be a nilpotent Weyr matrix. Then a matrix $X$ satisfies $\bar{X}J^\# = J^\#X$ if and only if $X = S^\#$ for some block matrix $S$ of the form described in Lemma 4. The matrix $S^\#$ is upper block triangular with respect to the partition obtained from the partition of $S$ by the above-described permutation of substrips.

(b) A matrix $S$ of the form described in Lemma 4 is nonsingular if and only if all diagonal subblocks $C_{ii}$ on its main diagonal

\[(C_{11}, \bar{C}_{11}, \ldots | C_{22}, \bar{C}_{22}, \ldots | \ldots | C_{tt}, \bar{C}_{tt}, \ldots)\]

are nonsingular.

Proof. (a) Let $\bar{X}J^\# = J^\#X$. Since $J^\#$ is permutationally similar to $J$, there is a permutation matrix $P$ such that $J^\# = P^{-1}JP$. Since

\[P\bar{X}P^{-1}PJ^\#P^{-1} = PJ^\#P^{-1}PX^{-1},\]

we have $\bar{S}J = JS$, in which $S = PXP^{-1}$. Then $X = P^{-1}SP = S^\#$ and $S$ has the form described in Lemma 4(b). Only subblocks $C_{ij}^{(k)} (k = 0, 1, \ldots)$ of $S$ can be nonzero. Each subblock $C_{ij}^{(k)}$ is at the intersection of horizontal and vertical substrips indexed by pairs $\alpha, i$ and $\beta, j$ in which $\beta = \alpha + k$, hence $\alpha \leq \beta$. If $\alpha = \beta$ then $i \leq j$ by (10), which proves that $S^\#$ is upper block triangular.

(b) Each matrix $S$ of the form described in Lemma 4 is nonsingular if and only if its diagonal subblocks $C_{ii}$ and $\bar{C}_{ii}$ are nonsingular. □

3. Proof of Theorem 1(a)

The matrices

\[
J := \begin{bmatrix}
0 & I & 0 & 0 & 0 \\
0 & 0 & I & 0 & 0 \\
0 & 0 & 0 & I & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}, \quad M := \begin{bmatrix}
0 & 0 & X & 0 & Y \\
0 & 0 & 0 & \bar{X} & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & I
\end{bmatrix}, \quad (11)
\]
in which all blocks are $n$-by-$n$ and the blocks $X$ and $Y$ are arbitrary, satisfy $MJ = JM$. They define commuting semilinear operators by Lemma 4(a).

Write

$$M' := \begin{bmatrix}
0 & 0 & X' & 0 & Y' \\
0 & 0 & 0 & \bar{X}' & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & I & 0
\end{bmatrix}. \tag{12}$$

The following lemma completes the proof of Theorem 1(a).

**Lemma 6.** The pairs $(J, M)$ and $(J, M')$ defined in (11) and (12) are consimilar if and only if $(X, Y)$ and $(X', Y')$ are consimilar.

**Proof.** Suppose that there is a nonsingular $S$ such that $\bar{S}^{-1}(J, M)S = (J, M')$. Then $JS = \bar{S}J$, and by Lemma 4(b)

$$S = \begin{bmatrix}
C & C_1 & C_2 & C_3 & D \\
0 & \bar{C} & C_1 & \bar{C}_2 & \bar{C}_3 & 0 \\
0 & 0 & C & C_1 & 0 \\
0 & 0 & 0 & \bar{C} & 0 \\
0 & 0 & 0 & E & F
\end{bmatrix}.$$ 

Since $MS = \bar{S}M'$, we have

$$\begin{bmatrix}
0 & 0 & XC & XC_1+YE & YF \\
0 & 0 & 0 & \bar{XC} & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \bar{C} & 0 \\
0 & 0 & 0 & \bar{C} & 0
\end{bmatrix} = \begin{bmatrix}
0 & 0 & \bar{CX}' & \bar{C}X_1+\bar{D} & \bar{CY}' \\
0 & 0 & 0 & \bar{CX}' & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & F & 0 \\
0 & 0 & 0 & \bar{C} & 0
\end{bmatrix},$$

which implies $XC = \bar{CX}'$, $YF = \bar{CY}'$, and $\bar{C} = F$. Hence, $(X, Y)C = \bar{C}(X', Y')$.

Conversely, if $(X, Y)C = \bar{C}(X', Y')$ for some nonsingular $S$, then $(J, M)S = \bar{S}(J, M')$ for $S := \text{diag}(C, \bar{C}, C, \bar{C}, C)$. \qed
4. Proof of Theorem II(b)

Let \( p \) and \( q \) be nonnegative integers and let \( X_1, \ldots, X_p, Y_1, \ldots, Y_q \) be \( n \times n \) matrices. Define the block matrix

\[
M_{X,Y} := \\
\begin{bmatrix}
0 & X_1 & 0 \\
0 & \ddots & \ddots \\
0 & \ddots & X_p \\
0 & 0 & 0
\end{bmatrix} \oplus \begin{cases} 
Y_1 \oplus Y_2 \oplus \cdots \oplus Y_q & \text{if } p \text{ is odd}, \\
0 \oplus Y_1 \oplus Y_2 \oplus \cdots \oplus Y_q & \text{if } p \text{ is even},
\end{cases}
\]

in which all blocks are \( n \times n \). Define the block matrix

\[
J := \\
\begin{bmatrix}
0 & I_n & 0 \\
0 & \ddots & \ddots \\
0 & \ddots & I_n \\
0 & 0 & 0
\end{bmatrix}
\]

of the same size. Denote by \( M_{X',Y'} \) the matrix obtained from \( M_{X,Y} \) by replacing all \( X_i \) and \( Y_j \) with \( X'_i \) and \( Y'_j \).

Theorem II(b) is a consequence of the following lemma.

**Lemma 7.** The matrix pairs \((J, M_{X,Y})\) and \((J, M_{X',Y'})\) are consimilar if and only if there exists a nonsingular \( C \) such that

(i) all \( X_{2i} \) are similar to \( X'_{2i} \) via \( C \),

(ii) all \( X_{2i+1} \) are similar to \( X'_{2i+1} \) via \( \bar{C} \),

(iii) all \( Y_{2i+1} \) are consimilar to \( Y'_{2i+1} \) via \( C \), and

(iv) all \( Y_{2i} \) are consimilar to \( Y'_{2i} \) via \( \bar{C} \).

**Proof.** \( \implies \). Suppose that there is an \( S \) such that \( \bar{S}^{-1}(J, M_{X,Y})S = (J, M_{X',Y'}) \). By Lemma II(a), all matrices \( S \) satisfying \( \bar{S}J = JS \) have the form

\[
S = \\
\begin{bmatrix}
C & C_1 & C_2 & C_3 & \cdots \\
\bar{C} & \bar{C}_1 & \bar{C}_2 & \cdots \\
C & \bar{C}_1 & \cdots \\
0 & \bar{C} & \cdots \\
\end{bmatrix}
\]
and so $S^{-1}M_{X,Y}S = M'_{X',Y'}$ implies

\[
\tilde{C}^{-1}X_1\tilde{C} = X'_1, \quad C^{-1}X_2C = X'_2, \quad \tilde{C}^{-1}X_3\tilde{C} = X'_3, \ldots \\
\tilde{C}^{-1}Y_1\tilde{C} = Y'_1, \quad C^{-1}Y_2C = Y'_2, \quad \tilde{C}^{-1}Y_3\tilde{C} = Y'_3, \ldots
\]

which ensures the validity (i)–(iv).

$\iff$. Let (i)–(iv) hold for some matrix $C$. Then $(J, M_{X,Y})$ and $(J, M'_{X',Y'})$ are consimilar via $S := C \oplus \tilde{C} \oplus C \oplus \tilde{C} \oplus \cdots$.

\[\square\]

5. Proof of Theorem 2

In this section, we prove that for each biquiver $Q$, the problem of classifying pairs of semilinear operators contains the problem of classifying representations of $Q$. (13)

To make the proof clear, we first establish that (13) holds for all representations of the biquiver (1). Its arbitrary representation $R$ has the form (2); let the mappings $A, B, \ldots, G$ be given by matrices $A, B, \ldots, G$ in some bases of the spaces $U, V, W$. Changing the bases, we can reduce these matrices by transformations

\[\text{(14)}\]

in which $S_1, S_2, S_3$ are the change of basis matrices.

Define the matrices

\[J := J_2(0_{q_1}) \oplus J_7(0_{q_2}) \oplus J_4(0_{q_3}),\]
and denote by $M'$ the matrix obtained from $M$ by replacing $A, B, C, D, E, F$ with $A', B', C', D', E', F'$.

The statement (13) is valid for representations of the biquiver (11) due to the following lemma.

**Lemma 8.** Let $J$, $M$, and $M'$ be the matrices defined above. The following statements are equivalent:

(i) The matrix pairs $(J, M)$ and $(J, M')$ are consimilar.

(ii) There exist nonsingular matrices $S_1, S_2, S_3$ such that

\[
AS_2 = \bar{S}_1 A', \quad BS_3 = S_1 B', \quad CS_2 = \bar{S}_2 C', \\
DS_2 = S_3 D', \quad ES_2 = \bar{S}_3 E', \quad FS_3 = S_3 F'.
\]

(iii) The matrix tuples $(A, B, C, D, E, F)$ and $(A', B', C', D', E', F')$ give the same representation (11) of the biquiver (11) in different bases; see (14).

**Proof.** (i) $\implies$ (ii). Let $(J, M)$ and $(J, M')$ be consimilar; that is, there exists a nonsingular matrix $S$ such that

\[
JS = \bar{S} J, \quad MS = \bar{S} M'.
\]

Applying Lemma 4(b) to the first equality in (16), we partition $S$ into blocks and subblocks conformally to the partition of $J$ and find that the diagonal subblocks of $S$ form a sequence of the form

$$(S_1, \bar{S}_1 | S_2, \bar{S}_2, S_2, \bar{S}_2, S_2, \bar{S}_2, S_2 | S_3, \bar{S}_3, S_3, \bar{S}_3).$$
Lemma 5(b) ensures that the subblocks $S_1, S_2, S_3$ are nonsingular. Each of the horizontal and vertical substrips of $M$ and $M'$ has at most one nonzero subblock; we obtain the equalities (15) from the second equality in (16) by equating the corresponding subblocks on the positions of subblocks $A, B, C, D, E, F$.

(i) $\iff$ (ii). Suppose that there are nonsingular matrices $S_1, S_2, S_3$ that satisfy the equations (15). Then the equations (16) are satisfied if we choose

$$
S := (S_1 \oplus \bar{S}_1) \oplus (S_2 \oplus \bar{S}_2 \oplus S_2 \oplus \bar{S}_2) \oplus (S_3 \oplus \bar{S}_3 \oplus S_3 \oplus \bar{S}_3).
$$

It follows that $(J, M)$ and $(J, M')$ are consimilar.

(ii) $\iff$ (iii). This equivalence follows from (14).

Proof of Theorem 2. Let us prove (13) for an arbitrary biquiver $Q$ with vertices $1, \ldots, t$. Let $R$ be a representation of $Q$. Denote by $R_i$ the vector space that is assigned to a vertex $i$ and by $R_\alpha$ the linear or semilinear mapping that is assigned to an arrow $\alpha$. Choose bases in the spaces $R_1, \ldots, R_t$ and denote by $R_\alpha$ the matrix of $R_\alpha$ in these bases. Changing the bases, we can reduce all $R_\alpha$ by transformations

$$
R_\alpha \mapsto \begin{cases} 
S_j^{-1}R_\alpha S_i & \text{if } \alpha : i \to j, \\
\bar{S}_j^{-1}R_\alpha S_i & \text{if } \alpha : i \dashrightarrow j,
\end{cases}
$$

in which $S_1, \ldots, S_t$ are the change of basis matrices.

By analogy with the proof of (13) for the biquiver (1), we construct a matrix pair $(J, M)$ as follows:

- The matrix $J$ is any matrix of the form

$$
J = J_{p_1}(0_{q_1}) \oplus \cdots \oplus J_{p_t}(0_{q_t}), \quad p_i \neq p_j \text{ if } i \neq j, \quad q_i := \dim R_i,
$$

in which all $p_i$ are large enough (it suffices to take $p_i \geq 2n(i)$ in which $n(i)$ is the number of arrows leaving or entering the vertex $i$ with loops being counted twice). The matrix $J$ is divided into $t$ horizontal and $t$ vertical strips of sizes $p_1q_1, \ldots, p tq_t$; the $i$th strip is divided into $p_i$ substrips of size $q_i$.

- The matrix $M$ is any matrix that satisfies the following conditions:
- $M$ and $J$ have the same size and the same partition into horizontal and vertical strips and substrips,
- every substrip of $M$ has at most one nonzero subblock,
- the nonzero subblocks of $M$ are all the nonzero matrices $R_\alpha$,
- if $\alpha$ is an arrow from a vertex $i$ to a vertex $j$ and $R_\alpha$ is at the intersection of substrip $k$ of horizontal strip $i$ with substrip $l$ of vertical strip $j$, then $k$ is even if $\alpha : i \rightarrow j$ and odd if $\alpha : i \rightarrow j$; $l$ is odd.

Reasoning as in the case of the biquiver (1), one can prove that if $(J, M)$ is reduced by consimilarity transformations that preserve $J$:

$$(J, M) \mapsto S^{-1}(J, M)S, \quad S^{-1}JS = J, \quad S \text{ is nonsingular},$$

then the blocks $R_\alpha$ of $M$ are transformed as in (17). \[\square\]

References

[1] G.R. Belitskiï, Normal forms in a space of matrices, in: V.A. Marchenko (Ed.), Analysis in Infinite-Dimensional Spaces and Operator Theory, Naukova Dumka, Kiev, 1983, pp. 3–15 (in Russian).

[2] G. Belitskii, Normal forms in matrix spaces, Integral Equations and Operator Theory, 38 (2000), no. 3, 251-283.

[3] G.R. Belitskii, V.V. Sergeichuk, Complexity of matrix problems, Linear Algebra Appl. 361 (2003) 203–222.

[4] P. Donovan, M.R. Freislich, Some evidence for an extension of the Brauer–Thrall conjecture, Sonderforschungsbereich Theor. Math. 40 (1972) 24–26.

[5] Y.P. Hong, R.A. Horn, A canonical form for matrices under consimilarity, Linear Algebra Appl. 102 (1988) 143–168.

[6] R.A. Horn, C.R. Johnson, Matrix Analysis, Cambridge U. P., Cambridge, 1985.

[7] P. Gabriel, Unzerlegbare Darstellungen I, Manuscripta Math. 6 (1972) 71–103.
[8] F.R. Gantmacher, The Theory of Matrices, Vol. 1, AMS Chelsea Publishing, 2000.

[9] I.M. Gelfand, V.A. Ponomarev, Remarks on the classification of a pair of commuting linear transformations in a finite dimensional vector space, Funct. Anal. Appl. 3 (1969) 325–326.

[10] T.G. Gerasimova, R.A. Horn, V.V. Sergeichuk, Simultaneous unitary equivalences, to appear in Linear Algebra Appl.

[11] K.C. O’Meara, J. Clark, C.I. Vinsonhaler, Advanced Topics in Linear Algebra: Weaving Matrix Problems Through the Weyr Form, Oxford University Press, New York, 2011.

[12] V.V. Sergeichuk, Classification problems for systems of forms and linear mappings, Math. USSR-Izv. 31 (no. 3) (1988) 481–501.

[13] V.V. Sergeichuk, Canonical matrices for linear matrix problems, Linear Algebra Appl. 317 (2000) 53–102.

[14] V.V. Sergeichuk, Linearization method in classification problems of linear algebra. São Paulo J. Math. Sci. 1 (no. 2) (2007) 219–240.

[15] V.V. Sergeichuk, Canonical matrices of isometric operators on indefinite inner product spaces, Linear Algebra Appl. 428 (2008) 154–192.