Exact Formulae for the Fractional Partition Functions

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Abstract

The partition function $p(n)$ has been a testing ground for applications of analytic number theory to combinatorics. In particular, Hardy and Ramanujan invented the “circle method” to estimate the size of $p(n)$, which was later perfected by Rademacher who obtained an exact formula. Recently, Chan and Wang considered the fractional partition functions, defined for $\alpha \in \mathbb{Q}$ by $\sum_{n=0}^{\infty} p_\alpha(n)x^n := \prod_{k=1}^{\infty} (1 - x^k)^{-\alpha}$. In this paper we use the Rademacher circle method to find an exact formula for $p_\alpha(n)$ and study its implications, including log-concavity and the higher-order generalizations (i.e., the Turán inequalities) that $p_\alpha(n)$ satisfies.

1 Introduction and Statement of Results

A partition of a nonnegative integer $n$ is a non-increasing sequence of positive integers with sum $n$. We use $p(n)$ to denote the number of partitions of $n$. One powerful tool for analyzing the partition function is Euler’s generating function:

$$P(x) := \sum_{n=0}^{\infty} p(n)x^n = \prod_{k=1}^{\infty} \frac{1}{1 - x^k}. \quad (1.1)$$

The study of the size of $p(n)$ spurred the development of the “circle method,” which has had many applications, including the proof of the weak Goldbach conjecture [11]. In 1918, G. H. Hardy and S. Ramanujan [10] invented this method to obtain an infinite but divergent series expansion for $p(n)$ and the asymptotic formula:

$$p(n) \sim \frac{e^{\pi \sqrt{2n/3}}}{4n\sqrt{3}}. \quad (1.2)$$

This method was perfected by H. Rademacher [16], who determined the convergent exact formula

$$p(n) = \frac{2\pi}{(24n - 1)^{1/4}} \sum_{k=1}^{\infty} \frac{A_k(n)}{k} \cdot I_{3/2} \left( \frac{\pi}{6k}\sqrt{24n - 1} \right), \quad (1.2)$$

where

$$I_{\nu}(z) := \left( \frac{z}{2} \right)^{\nu} \sum_{k=0}^{\infty} \frac{(z/2)^{2k}}{k!\Gamma(\nu + k + 1)}$$

is the modified Bessel function of the first kind,

$$A_k(n) := \sum_{0 \leq h < k \atop \gcd(h,k) = 1} e^{\pi i s(h,k) - 2\pi i nh/k}$$

is a Kloosterman sum, and

$$s(h,k) := \sum_{r=1}^{k-1} r \left( \frac{hr}{k} - \left[ \frac{hr}{k} \right] - \frac{1}{2} \right) \quad (1.3)$$
is the usual Dedekind sum.

The partition function also satisfies certain congruences, which exhibit a great degree of structure. Ramanujan was the first to study these congruences, and he discovered examples including \( p(5n + 4) \equiv 0 \pmod{5} \). In a recent paper, Chan and Wang [4] defined for \( \alpha \in \mathbb{Q} \) the fractional partition function \( p_\alpha(n) \) in terms of its generating function

\[
P(x)^\alpha = \prod_{k=1}^{\infty} \frac{1}{(1 - x^k)^\alpha} =: \sum_{n=0}^{\infty} p_\alpha(n) x^n,
\]

and studied its congruences, showing, for instance, that \( p_{1/2}(29n + 26) \equiv 0 \pmod{29} \). A general theory of such congruences has recently been developed by Bevilacqua, Chandran, and Choi [2]. The discussion of congruences for \( p_\alpha(n) \) is possible because \( p_\alpha(n) \) is rational whenever \( \alpha \) is rational.

When \( \alpha \in \mathbb{Z}^+ \), \( p_\alpha(n) \) counts the number of partitions of \( n \) in which each term is labeled with one of \( \alpha \) different colors, where the order of the colors does not matter [12]. Moreover, in such cases, the function

\[
\eta(\tau)^{-\alpha} = q^{-\frac{\alpha}{24}} P(q)^\alpha
\]

is a weakly holomorphic modular form of weight \( -\alpha / 2 \in (1/2)\mathbb{Z} \), where \( \tau \) is in the upper half-plane, \( \eta(\tau) := q^{1/24} \prod_{n \geq 1} (1 - q^n) \) is the Dedekind eta function, and \( q := e^{2\pi i \tau} \). This makes it possible to compute the values of \( p_\alpha(n) \) using Maass-Poincaré series, as described by Bringmann et al. [3, §6.3], which give a Rademacher-type infinite series expansion that reduces to (1.2) when \( \alpha = 1 \). To do this, one computes the principal part of \( \eta(\tau)^{-\alpha} \), which correspond to the values \( p_\alpha(n) \) for \( 0 \leq n \leq \lfloor \alpha/24 \rfloor \). Then, using the fact that a weakly holomorphic modular form is determined by its weight and principal part, one can write it as a finite sum of Maass-Poincaré series and apply a known formula for the coefficients of such series. While these observations shed light on the case where \( \alpha \) is a positive integer, there is currently no known combinatorial or modular-form interpretation of \( p_\alpha(n) \) for arbitrary rational \( \alpha \).

In this paper, we extend the definition of \( p_\alpha(n) \) to arbitrary real \( \alpha \) via (1.4) and give exact formulas for \( p_\alpha(n) \) in the spirit of Rademacher. For real \( \alpha > 0 \), \( n > \alpha/24 \), and \( m \leq \alpha/24 \), we define the functions

\[
\nu_\alpha(n) := \sqrt{n - \frac{\alpha}{24}}, \quad \mu_\alpha(m) := \sqrt{\frac{\alpha}{24} - m}
\]

and the \( \alpha \)-Kloosterman sum

\[
A_\alpha^{(\alpha)}(n, m) := \sum_{0 \leq h < k \atop (h, k) = 1} e^{\alpha \pi is(h, k) + \frac{2\pi i}{k}(mH - nh)},
\]

where \( H \) denotes an inverse of \( h \) modulo \( k \) and \( s(h, k) \) is the Dedekind sum defined in (1.3). Our exact formulas for \( p_\alpha(n) \) are the content of the following theorem.

**Theorem 1.1.** For all \( \alpha > 0 \) and \( n > \alpha/24 \), we have

\[
p_\alpha(n) = \nu_\alpha(n)^{-\frac{n}{\alpha}} \sum_{m=0}^{\lfloor \alpha/24 \rfloor} \mu_\alpha(m)^{\alpha + 1} p_\alpha(m) \sum_{k=1}^{\infty} \frac{2\pi}{k} A_\alpha^{(\alpha)}(n, m) \mathcal{I}_{\alpha+1} \left( \frac{4\pi}{k} \nu_\alpha(n) \mu_\alpha(m) \right),
\]

where \( q := \frac{\alpha}{24} \).

Theorem 1.1 also enables the calculation of explicit error bounds for approximations of \( p_\alpha(n) \) obtained by truncating (1.8). These have several implications, including a simple description of the asymptotic behavior of \( p_\alpha(n) \) for large \( n \), given in Corollary 1.2.
Corollary 1.2. For all $\alpha > 0$, as $n \to \infty$, we have
\[ p_\alpha(n) \sim 2\pi I_{\frac{n}{\alpha} + 1} \left( \frac{\pi n}{\alpha} \lambda_\alpha(n) \right) \sim \sqrt{\frac{12}{\alpha}} \cdot \frac{\Gamma \left( \frac{\pi n}{\alpha} \right)}{\lambda_\alpha(n)^{\frac{\pi n}{\alpha} - 1}}, \]
where $\lambda_\alpha(n) := \sqrt{24n/\alpha - 1}$.

We remark that because $p_\alpha(n)$ is rational for any $\alpha \in \mathbb{Q}$, Theorem 1.1 implies that the series in (1.8) converges to a rational number when $n \in \mathbb{Z}, n > \alpha/24$. We make use of this fact later in the paper (Corollary 4.2) to provide a finite formula for $p_\alpha(n)$ in the case where $\alpha \in \mathbb{Q}$.

When considering a sequence of real numbers, one is often interested in more than just its asymptotic behavior. One property that is often studied is log-concavity. A sequence $\{a(n)\}$ is called log-concave if we have
\[ a(n + 1)^2 - a(n)a(n + 2) \geq 0 \]
for all $n$. Nicolas [14] and DeSalvo and Pak [7] independently proved that $p(n)$ is log-concave for $n \geq 25$. In fact, the condition of log-concavity is a special case of what are known as the higher Turán inequalities [5]. One can show that a sequence satisfies the higher Turán inequalities of degree $d$ if and only if the Jensen polynomials
\[ J_{d,n}^d(x) := \sum_{j=0}^{d} \left( \begin{array}{c} d \\ j \end{array} \right) a(n + j)x^j \]
(1.9)
have strictly real roots for all $n$—we say that such a polynomial is hyperbolic [6]. Chen, Jia, and Wang [5] conjectured that for any fixed degree $d$, $J_{d,n}^d(x)$ is eventually hyperbolic, and proved this for $d = 3$; Larson and Wagner [13] independently proved this conjecture for $d \in \{3, 4, 5\}$. Griffin, Ono, Rolen, and Zagier [9] established the conjecture of Chen et al. for all $d$ by showing that, after suitable renormalization, the Jensen polynomials of $p(n)$ converge to the Hermite polynomials $H_d(x)$ as $n \to \infty$. We apply their methods to prove the analogue of Chen et al.’s conjecture for $p_\alpha(n)$.

Theorem 1.3. For $\alpha > 0$ and $d \in \mathbb{N}$, there exists $N_d(\alpha)$ such that $J_{p_\alpha}^{d,n}(X)$ is hyperbolic for all $n > N_d(\alpha)$.

Our paper is divided into five main sections. In Section 2, we establish some preliminary results, including a modified version of the Dedekind functional equation for $\eta(\tau)$. In Section 3, we use the circle method along with this identity to prove Theorem 1.1. In Section 4, we use Theorem 1.1 to prove more results about $p_\alpha(n)$, including the estimate given in Corollary 1.2. We also analyze the hyperbolicity of the Jensen polynomials associated with $p_\alpha(n)$. Finally, in Section 5 we provide numerical illustrations of our main theorems.

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2 Proof of the Functional Equation for $P(x)^\alpha$

In order to apply the circle method to $p_\alpha(n)$, we first require a precise statement of Dedekind’s functional equation for the eta function. We derive this from Iseki’s formula [1, §3.5]. For convenience, when $\text{Re}(x) > 0$, we set

$$\lambda(x) := \sum_{m=1}^{\infty} \frac{e^{-2\pi mx}}{m} = -\log(1 - e^{-2\pi x}).$$

**Remark.** Throughout this section, we let $\log z$ denote the branch of the logarithm with a branch cut along the negative imaginary axis and $\log 1 = 0$, and we define $\text{arg} z := \text{Im}(\log z)$.

2.1 Derivation of the Logarithmic Functional Equation from Iseki’s Formula

In order to derive the required modification of the functional equation for $\eta(\tau)$, we first prove a lemma which follows from Iseki’s formula [1, §3.5].

**Theorem 2.1** (Iseki’s Formula). For $\text{Re} z > 0$, $0 < \alpha < 1$, and $0 \leq \beta \leq 1$, let

$$\Lambda(\alpha, \beta, z) := \sum_{r=0}^{\infty} \left[ \lambda((r + \alpha)z - i\beta) + \lambda((r + 1 - \alpha)z + i\beta) \right].$$

Then we have

$$\Lambda(\alpha, \beta, z) = \Lambda(1 - \beta, \alpha, z^{-1}) - \pi z \left( \alpha^2 - \alpha + \frac{1}{6} \right) + \frac{\pi}{z} \left( \beta^2 - \beta + \frac{1}{6} \right) + 2\pi i \left( \alpha - \frac{1}{2} \right) \left( \beta - \frac{1}{2} \right). \tag{2.1}$$

**Lemma 2.2.** For $\text{Re} z > 0$, we have

$$\sum_{r=1}^{\infty} \lambda(rz) = \sum_{r=1}^{\infty} \lambda \left( \frac{r}{z} \right) + \frac{1}{2} \log z - \left( \frac{\pi z}{12} - \frac{\pi}{12z} \right). \tag{2.2}$$

**Proof.** Letting $\beta = 0$ in Iseki’s formula, we obtain

$$\Lambda(\alpha, 0, z) = \Lambda(1 - \alpha, z^{-1}) - \pi z \left( \alpha^2 - \alpha + \frac{1}{6} \right) + \frac{\pi}{6z} - \pi i \left( \alpha - \frac{1}{2} \right). \tag{2.3}$$

From here, bringing $\Lambda(1, \alpha, z^{-1})$ to the left side, reordering the summations, and setting $a(\alpha) := \lambda(\alpha z) - \lambda(i\alpha)$ and

$$b_r(\alpha) := \lambda((r + \alpha)z - i\beta) + \lambda((r + 1 - \alpha)z + i\beta)$$

yields

$$a(\alpha) + \sum_{r=1}^{\infty} b_r(\alpha) = -\pi z \left( \alpha^2 - \alpha + \frac{1}{6} \right) + \frac{\pi}{6z} - \pi i \left( \alpha - \frac{1}{2} \right). \tag{2.4}$$

The reordering is valid because the sum over each of the four terms in $b_r(\alpha)$ converges absolutely, since $\lambda(\gamma z) \sim e^{-2\pi\gamma z}$ as $\gamma \to \infty$. We proceed by taking the limit as $\alpha \to 0^+$. We start by observing that

$$\lim_{\alpha \to 0^+} a(\alpha) = \lim_{\alpha \to 0^+} \left[ \lambda(\alpha z) - \lambda(i\alpha) \right] = \lim_{\alpha \to 0^+} \left[ \log(1 - e^{-2\pi i\alpha}) - \log(1 - e^{-2\alpha z}) \right] = \lim_{\alpha \to 0^+} \log \left( \frac{1 - e^{-2\pi i\alpha}}{1 - e^{-2\alpha z}} \right), \tag{2.5}$$

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where the last step is justified because \( \arg(1 - e^{-2\pi i\alpha}) - \arg(1 - e^{-2\pi i\beta}) \in (-\pi, \pi) \) for \( \alpha > 0 \). By L'Hôpital's rule,

\[
\lim_{\alpha \to 0^+} \frac{1 - e^{-2\pi i\alpha}}{1 - e^{-2\pi i\beta}} = \lim_{\alpha \to 0^+} \frac{2\pi i e^{-2\pi i\alpha}}{2\pi i e^{-2\pi i\beta}} = \frac{i}{\beta},
\]

and so

\[
\lim_{\alpha \to 0^+} a(\alpha) = \log \left( \frac{i}{\beta} \right) = \frac{\pi i}{2} - \log \beta,
\]

using the fact that \( \arg(i/\beta) \in (0, \pi) \) and \( \log(1/\beta) = -\log \beta \) for our definition of the logarithm. We now show that

\[
\lim_{\alpha \to 0^+} \sum_{r=1}^{\infty} b_r(\alpha) = \sum_{r=1}^{\infty} \lim_{\alpha \to 0^+} b_r(\alpha) = \sum_{r=1}^{\infty} (2\lambda(r\beta) - 2\lambda(r/\beta)).
\]

For this purpose, start by noting that for \( \Re x > 0 \), we have \( |\lambda(x)| \leq \lambda(\Re x) \) by the series expansion for \( \lambda \), and that \( \lambda(\Re x) \) is monotonically decreasing. In particular, we have

\[
|\lambda((r \pm \alpha)\beta)| \leq \lambda \left( \left( r - \frac{1}{2} \right) \Re \beta \right) \leq \lambda \left( r \cdot \frac{\Re \beta}{2} \right),
\]

and \( |\lambda(r\beta \pm i\alpha)| \leq \lambda(r\Re \beta) \). For \( x > 0 \), we can verify that \( \sum_{r=1}^{\infty} \lambda(r\beta) \) converges by the asymptotic behavior of \( \lambda(rx) \) as \( r \to \infty \). Consequently, by the discrete version of the dominated convergence theorem, we may exchange the order of the limit and the summation over \( b_r \). Thus, in the limit, (2.4) becomes

\[
\frac{\pi i}{2} - \log \beta + 2 \sum_{r=1}^{\infty} \lambda(r\beta) - 2 \sum_{r=1}^{\infty} \lambda \left( r \cdot \frac{\beta}{2} \right) = -\frac{\pi \beta}{6} + \frac{\pi}{6\beta} + \frac{\pi i}{2}.
\]

(2.6)

This is equivalent to (2.2).

For the main theorem of this section, we begin by citing a fact proven in [1, §3.6].

**Proposition 2.3.** Let \( \Re z > 0 \), let \( h, k \in \mathbb{Z} \) be coprime with \( k > 0 \), and choose \( H \) such that \( hH \equiv -1 \pmod{k} \). Then we have that

\[
\sum_{n=1}^{\infty} \lambda \left( \frac{n}{k}(z - i\beta) \right) = \sum_{n=1}^{\infty} \lambda \left( \frac{n}{k}(z^{-1} - iH) \right) + \frac{\pi \beta}{12} - \frac{\pi}{12} \left( 1 - \frac{1}{k} \right) + \pi i s(h, k). \tag{2.7}
\]

With this fact and Lemma 2.2, we may finally provide the desired logarithmic version of the functional equation for \( \eta(\tau) \).

**Theorem 2.4.** For \( \Re z > 0 \) and \( h, k, H \in \mathbb{Z} \) with \( k > 0 \), \( \gcd(h, k) = 1 \), and \( hH \equiv -1 \pmod{k} \), we have

\[
\sum_{n=1}^{\infty} \lambda \left( \frac{n}{k}(z - i\beta) \right) = \sum_{n=1}^{\infty} \lambda \left( \frac{n}{k}(z^{-1} - iH) \right) + \frac{1}{k} \left( \frac{\pi z}{12} - \frac{\pi}{12} \right) + \frac{1}{2} \log z + \pi i s(h, k). \tag{2.8}
\]

**Proof.** Using the periodicity of \( \lambda \), we note that

\[
\sum_{r=1}^{\infty} \lambda(rz) = \sum_{n=0}^{\infty} \lambda \left( \frac{n}{k}(z - i\beta) \right) \quad \text{and} \quad \sum_{r=1}^{\infty} \lambda \left( \frac{r}{z} \right) = \sum_{n=0}^{\infty} \lambda \left( \frac{n}{k}(z^{-1} - iH) \right).
\]

Substituting this into (2.2) and adding equation (2.7) yields the desired result. \( \square \)
2.2 Application of the Logarithmic Functional Equation to \( P(x)^\alpha \)

We recall the generating function

\[
P(x) := \prod_{k=1}^{\infty} \frac{1}{1-x^k} = \sum_{n=0}^{\infty} p(n)x^n,
\]

which is holomorphic for \( x \) in the open unit disk. In deriving the Hardy-Ramanujan-Rademacher series formula for the partition function, we rely on the fact that the equation above holds analytically as well as formally. We extend this observation to \( p_\alpha(n) \) by showing that the generating function \( P(x)^\alpha \) is well-defined.

**Lemma 2.5.** For \( x \) in the open unit disk and \( \alpha > 0 \), we have

\[
\sum_{n=0}^{\infty} p_\alpha(n)x^n = \prod_{k=1}^{\infty} e^{-\alpha \log(1-x^k)} = P(x)^\alpha.
\]  

**Proof.** Start by observing that our branch of the logarithm ensures that \( \exp(-\alpha \log(1-x^k)) \) is formally equivalent to \((1-x^k)^{-\alpha}\). Thus, because the \( p_\alpha(n) \) are defined in terms of the formal equivalence in \((12)\), it suffices to show that \( P(x)^\alpha \) as defined above is holomorphic for \(|x| < 1\). For this purpose, let \( 0 < r < 1 \), and observe that \(-\sum_{k=1}^{\infty} \alpha \log(1-x^k)\) converges uniformly for \(|x| \leq r\) by the ratio test, as

\[
\lim_{k \to \infty} \left| \frac{\log(1-x^{k+1})}{\log(1-x^k)} \right| = \lim_{k \to \infty} \left| \frac{-\log(x) x^{k+1}/(1-x^{k+1})}{-\log(x) x^k/(1-x^k)} \right| = \lim_{k \to \infty} \left| x \cdot \frac{1-x^k}{1-x^{k+1}} \right| = |x| \leq r.
\]

Thus, \( \prod_{k=1}^{\infty} e^{-\alpha \log(1-x^k)} \) converges uniformly for \(|x| \leq r\), from which it follows that \( P(x)^\alpha \) is holomorphic in every closed disk \(|x| \leq r\) and hence in the open unit disk \(|x| < 1\) as desired.

We are finally ready for the main result of this section, which expresses the functional equation for \( \eta(\tau) \) in terms of \( P(x)^\alpha \).

**Theorem 2.6** (Modified Functional Equation). For \( \text{Re } z > 0 \), \( \alpha > 0 \), \( h, k, H \in \mathbb{Z} \) with \( k > 0 \), \( \gcd(h,k) = 1 \), and \( hH \equiv -1 \pmod{k} \), we have

\[
P(x)^\alpha = e^{\pi i \alpha s(h,k)} \left( \frac{z}{k} \right)^{\alpha/2} \exp \left( \frac{\alpha \pi}{12k} \left( \frac{k}{z} - \frac{z}{k} \right) \right) P(x')^\alpha,
\]

where

\[
x := \exp \left( \frac{2\pi}{k} \left( iH - \frac{k}{z} \right) \right), \quad x' := \exp \left( \frac{2\pi}{k} \left( iH - \frac{k}{z} \right) \right),
\]

and real powers are given for the precise branch of the logarithm described in Section 2.1.

**Proof.** Applying Theorem 2.4 with \( z/k \) in place of \( z \) and multiplying by \( \alpha \), we obtain

\[
\alpha \sum_{n=1}^{\infty} \lambda \left( \frac{n}{k} \left( \frac{z}{k} - iH \right) \right) = \alpha \frac{\pi}{12k} \left( \frac{k}{z} - \frac{z}{k} \right) + \alpha \frac{\pi}{2} \log \left( \frac{z}{k} \right) + \pi i \alpha s(h,k) + \alpha \sum_{n=1}^{\infty} \lambda \left( \frac{n}{k} \left( \frac{k}{z} - iH \right) \right).
\]

Exponentiating both sides yields

\[
\prod_{n=1}^{\infty} \exp(-\alpha \log(x)) = \exp \left( \frac{\alpha \pi}{12k} \left( \frac{k}{z} - \frac{z}{k} \right) \right) \exp \left( \frac{\alpha \pi}{2} \log \left( \frac{z}{k} \right) \right) e^{\pi i \alpha s(h,k)} \prod_{n=1}^{\infty} \exp(-\alpha \log(x'))
\]

for \( x \) and \( x' \) defined above, which is equivalent to \((2.10)\).
3 Proof of the Series Formula for $p_\alpha(n)$

In this section, we use Radamacher’s circle method to prove the series formula for $p_\alpha(n)$. We closely follow Apostol’s proof of the $\alpha = 1$ case \cite[§5.7]{1}.

Proof of Theorem 1.1. Using Cauchy’s residue theorem and Lemma 2.5, we can write

$$p_\alpha(n) = \frac{1}{2\pi i} \int_C \frac{P(x)^\alpha}{x^{n+1}} \, dx,$$

where $C$ is any simple closed contour in the unit disk which encloses the origin. To evaluate this, we consider the change of variables $x = e^{2\pi i \tau}$, under which the closed unit disk $|x| \leq 1$ is the image of the infinite vertical strip $\{ \tau : 0 \leq \text{Re} \tau \leq 1, \ 0 \leq \text{Im} \tau \}$. We start by recalling the Farey sequences $F_N$, defined by enumerating the rational numbers in $[0,1]$ with reduced denominators at most $N$. In addition, for $\gcd(h,k) = 1$, we let $C(h,k)$ denote the Ford circle associated with $h/k$, which has center $h/k + i/(2k^2)$ and radius $1/(2k^2)$ (details are given in \cite[§5.6]{1}). As in Rademacher’s original work, we integrate along the Rademacher paths $R(N)$ in the $\tau$-plane, consisting of the upper arcs of the Ford circles associated with $F_N$, with the intent to later take the limit as $N \to \infty$ (depicted in Figure 2). For $N \geq 1$, we write

$$p_\alpha(n) = \hat{R}(N) P(x)^\alpha e^{-2\pi i n \tau} d\tau.$$ (3.2)

Decomposing $R(N)$ into its component arcs, we may write the above integral as

$$\int_{\gamma(h,k)} \hat{z}_{2(h,k)} - \hat{z}_{1(h,k)} e^{2\pi i n \tau} P(x)^\alpha dz,$$ (3.3)

where we define the right side as a shorthand for the double sum over $h$ and $k$, and $\gamma(h,k)$ is the upper arc of the Ford circle $C(h,k)$ of radius $1/(2k^2)$ tangent to the real axis at $h/k$.

We now introduce a second change of variables given by

$$z = -ik^2 \left( \tau - \frac{h}{k} \right),$$ (3.4)

which maps the circle $C(h,k)$ onto the circle $K$ of radius $1/2$ centered at $1/2$. Let $z_1(h,k)$ and $z_2(h,k)$ be the respective endpoints of the image of $\gamma(h,k)$, and let $x$ and $x'$ be defined as in Theorem 2.6. Then

$$p_\alpha(n) = \sum_{h,k} ik^{-2} e^{-2\pi i nh} \int_{z_1(h,k)}^{z_2(h,k)} e^{2\pi iz} P(x)^\alpha dz,$$

from which the modified functional equation from Theorem 2.6 yields

$$p_\alpha(n) = \sum_{h,k} \frac{\alpha}{2} - 2 e^{-2\pi i nh} \omega^{(\alpha)}(h,k) \int_{z_1(h,k)}^{z_2(h,k)} e^{2\pi iz} \Psi_k^{(\alpha)}(z) P(x')^\alpha dz,$$

where

$$\omega^{(\alpha)}(h,k) := e^{\alpha \pi i s(h,k)}, \quad \Psi_k^{(\alpha)}(z) := z^{\frac{\alpha}{2}} \exp\left(\frac{\alpha \pi z}{12z} - \alpha \pi z \frac{1}{12k^2}\right).$$

Let $q = \lfloor \alpha/24 \rfloor$, and define

$$Q^{(\alpha)}(x) := \sum_{m=0}^{q} p_\alpha(m)x^m.$$
We proceed by separating out a part of the integral that corresponds to \( Q^{(\alpha)}(x) \) and showing that the remaining part goes to zero as \( N \to \infty \). In particular, we write

\[
I_1(h, k) = \int_{z_1(h, k)}^{z_2(h, k)} \Psi_k^{(\alpha)}(z) e^{\frac{2\pi i n}{k} Q^{(\alpha)}(x')} dz
\]

and

\[
I_2(h, k) = \int_{z_1(h, k)}^{z_2(h, k)} \Psi_k^{(\alpha)}(z) e^{\frac{2\pi i n}{k} (P^{(\alpha)}x' - Q^{(\alpha)}(x'))} dz
\]

to obtain

\[
p_{\alpha}(n) = \sum_{h, k} ik^{-2 - \frac{\alpha}{2}} e^{-\frac{2\pi i m h}{k} \omega^{(\alpha)}(h, k)} \cdot (I_1(h, k) + I_2(h, k)). \tag{3.5}
\]

We now show that \( I_2(h, k) \) is “small” for large \( N \) by considering the integral along the chord in the \( z \)-plane joining \( z_1(h, k) \) and \( z_2(h, k) \). Because \( 0 < \text{Re } z \leq 1 \) and \( \text{Re}(z^{-1}) \geq 1 \) for \( z \) on the path of integration, we can write

\[
\left| \Psi_k^{(\alpha)}(z) \cdot e^{\frac{2\pi i n}{k} \cdot \{ P^{(\alpha)}x' - Q^{(\alpha)}(x') \}} \right| \tag{3.6}
\]

\[
= |z|^\frac{\alpha}{2} \exp \left( \frac{\alpha \pi}{12} \text{Re}(z^{-1}) - \frac{\alpha \pi}{12 k^2} \text{Re } z + \frac{2n\pi i}{k^2} \text{Re } z \right) \cdot \left| \sum_{m=q+1}^{\infty} p_{\alpha}(m) \exp \left( \frac{2\pi i H m}{k} - \frac{2\pi m}{z} \right) \right| \leq |z|^\frac{\alpha}{2} \exp \left( \frac{\alpha \pi}{12} \text{Re}(z^{-1}) + \frac{2n\pi i}{k^2} \right) \sum_{m=q+1}^{\infty} p_{\alpha}(m) e^{-2\pi m \text{Re}(z^{-1})} \tag{3.7}
\]

\[
\leq |z|^\frac{\alpha}{2} \sum_{m=q+1}^{\infty} p_{\alpha}(m) e^{-2\pi (m - \frac{\alpha}{2}) \text{Re}(z^{-1})} \leq |z|^\frac{\alpha}{2} \sum_{m=q+1}^{\infty} p_{\alpha}(m) e^{-2\pi (m - \frac{\alpha}{2})} = |z|^\frac{\alpha}{2} \phi \left( P(e^{-2\pi})^{\alpha} - Q^{(\alpha)}(e^{-2\pi}) \right). \tag{3.8}
\]
Since $|z| < \sqrt{2k/N}$ for $z$ on the chord from $z_1(h, k)$ to $z_2(h, k)$, the integrand is less than $C(k/N)^{\alpha/2}$ for some constant $C$ not depending on $N$. Thus, because the length of the chord is at most $2\sqrt{2k/N}$, we have

$$|I_2(h, k)| < \frac{Ck^{\alpha+1}}{N^{\alpha+1}}. \quad (3.9)$$

Substituting this bound into the sum of the $I_2$ terms in (3.5) yields

$$\left| \sum_{h,k} ik^{-\frac{\alpha}{2}} - 2e^{-\frac{2\pi i n h}{k}} \omega^{(\alpha)}(h, k) I_2(h, k) \right| < \sum_{k=1}^{N} \sum_{0 \leq h < k \atop (h, k) = 1} Ck^{-1}N^{-\frac{\alpha}{2}} \leq C N^{-\frac{\alpha}{2}} - 1 \sum_{k=1}^{N} 1 = C N^{-\frac{\alpha}{2}}.$$

Thus, we have

$$p_\alpha(n) = \left( \sum_{k=1}^{N} \sum_{0 \leq h < k \atop (h, k) = 1} ik^{-\frac{\alpha}{2}} - 2e^{-\frac{2\pi i n h}{k}} \omega^{(\alpha)}(h, k) I_1(h, k) \right) + O(N^{-\frac{\alpha}{2}}). \quad (3.10)$$

Next we consider $I_1(h, k)$. We can write

$$I_1(h, k) = \int_{-K}^{z_1(h, k)} - \int_{0}^{z_2(h, k)} = : \int_{-K}^{z_1(h, k)} - J_1 - J_2, \quad (3.11)$$

where we omit the integrands for brevity, and where $-K$ indicates that we integrate in the negative direction along $K$. Because $|z| \leq \sqrt{2k/N}$ on the paths of integration, we can bound the integrands of $J_1$ and $J_2$ by

$$\left| \psi^{(\alpha)}_k(z) e^{\frac{2\pi i n h}{k^2}} Q^{(\alpha)}(x') \right| \leq \left| z^{\frac{\alpha}{2}} \exp \left( \frac{\alpha \pi}{12} \right) - \frac{2n \pi}{k^2} \right| \left| \sum_{m=0}^{q} p_\alpha(m) e^{\frac{2\pi i H m}{k}} \frac{2m}{z} \right| \left| \sum_{m=0}^{q} p_\alpha(m) e^{-2\pi m} \right| \leq \left| z^{\frac{\alpha}{2}} \exp \left( \frac{2\pi i H m}{k} \right) \right| \left| \sum_{m=0}^{q} p_\alpha(m) e^{-2\pi m} \right| \quad (3.12)$$

$$\leq \frac{e^{2\pi i H m/k^2} \sum_{m=0}^{q} p_\alpha(m) e^{-2\pi m}}{N^{\frac{\alpha}{2}}} \quad (3.13)$$

The lengths of the arcs from 0 to $z_1(h, k)$ and $z_2(h, k)$ are less than $\pi |z_1(h, k)|$ and $\pi |z_2(h, k)|$, respectively, and both of these are bounded by $\pi \sqrt{2k/N}$, so we get that $|J_1|, |J_2| < C_1 k^{\frac{\alpha}{2} + 1} N^{-\frac{\alpha}{2}}$ for some constant $C_1$.

Combining (3.10), (3.11), and the bounds for $J_1$ and $J_2$ above, we find that

$$p_\alpha(n) = \sum_{k=1}^{N} \sum_{0 \leq h < k \atop (h, k) = 1} ik^{-\frac{\alpha}{2}} - 2e^{-\frac{2\pi i n h}{k}} \omega^{(\alpha)}(h, k) \int_{-K}^{z_1(h, k)} \psi^{(\alpha)}_k(z) e^{\frac{2\pi i n h}{k^2}} Q^{(\alpha)}(x') \, dz + O(N^{-\frac{\alpha}{2}}), \quad (3.14)$$
which in the limit as $N$ goes to infinity becomes

$$p_\alpha(n) = \sum_{m=0}^{q} p_\alpha(m) \sum_{k=1}^{\infty} \sum_{0 \leq h < k}^{(h,k)=1} ik^{-\frac{\alpha}{2}} - 2 \frac{\sin h}{k} \omega^{(\alpha)}(h,k)$$

\[ \cdot \int_{-K}^{K} z^\frac{\alpha}{2} \exp \left( \frac{2 \pi n z}{k^2} + \frac{\alpha \pi}{12z} - \frac{\alpha \pi z}{12k^2} + \frac{2 \pi i m H}{k} - \frac{2 \pi m z}{z} \right) \, dz \]

$$= \sum_{m=0}^{q} p_\alpha(m) \sum_{k=1}^{\infty} \sum_{0 \leq h < k}^{(h,k)=1} ik^{-\frac{\alpha}{2}} - 2 \frac{\sin h}{k} (mH - n\alpha) \omega^{(\alpha)}(h,k)$$

\[ \cdot \int_{-K}^{K} z^\frac{\alpha}{2} \exp \left( \frac{2 \pi z}{k^2} \nu_\alpha(n)^2 + \frac{2 \pi z}{z} \mu_\alpha(m)^2 \right) \, dz \]

$$= \sum_{m=0}^{q} p_\alpha(m) \sum_{k=1}^{\infty} \sum_{0 \leq h < k}^{(h,k)=1} i \frac{A_k^{(\alpha)}(n,m)}{k^{\frac{\alpha}{2} + 2}} \int_{-K}^{K} z^\frac{\alpha}{2} \exp \left( \frac{2 \pi z}{k^2} \nu_\alpha(n)^2 + \frac{2 \pi z}{z} \mu_\alpha(m)^2 \right) \, dz. \]

To evaluate the integral on the right, we make the change of variables $t = 2\pi(\alpha/24 - m)/z$ to obtain

$$p_\alpha(n) = 2\pi \sum_{m=0}^{q} p_\alpha(m) \sum_{k=1}^{\infty} \frac{A_k^{(\alpha)}(n,m)}{k^{\frac{\alpha}{2} + 2}} \left[ 2\pi \mu_\alpha(n)^2 \right]^{\frac{\alpha}{2} + 1}$$

\[ \cdot \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} t^{\frac{\alpha}{2} - 2} \exp \left( t + \frac{2 \pi \nu_\alpha(n) \mu_\alpha(m)}{k} \right) \frac{1}{t} \, dt, \]

where $c = \alpha/12$. Now recall that the modified Bessel function of the first kind satisfies

$$I_\beta(z) = \frac{(z/2)^\beta}{2\pi i} \int_{c-i\infty}^{c+i\infty} t^{-\beta - 1} e^{t + \frac{2 \pi i}{t}} \, dt \quad (3.15)$$

for $c > 0, \text{Re}(\nu) > 0$ [17, p. 181]. Consequently, for $n \geq \alpha/24$, we find that

$$p_\alpha(n) = 2\pi \sum_{m=0}^{q} p_\alpha(m) \sum_{k=1}^{\infty} \frac{A_k^{(\alpha)}(n,m)}{k^{\frac{\alpha}{2} + 2}} \left( 2\pi \mu_\alpha(m)^2 \right)^{\frac{\alpha}{2} + 1} \left( \frac{2 \pi \nu_\alpha(n) \mu_\alpha(m)}{k} \right)^{-\frac{\alpha}{2} - 1} I_{\frac{\alpha}{2} + 1} \left( \frac{4 \pi}{k} \nu_\alpha(n) \mu_\alpha(m) \right)\quad (3.16)$$

$$= \nu_\alpha(n)^{-\frac{\alpha}{2} - 1} \sum_{m=0}^{q} \mu_\alpha(m)^{\frac{\alpha}{2} + 1} p_\alpha(m) \sum_{k=1}^{\infty} \frac{2 \pi}{k} A_k^{(\alpha)}(n,m) I_{\frac{\alpha}{2} + 1} \left( \frac{4 \pi}{k} \nu_\alpha(n) \mu_\alpha(m) \right). \quad (3.17)$$

\[ \square \]

4 Applications of the Series Formula for $p_\alpha(n)$

4.1 Estimates of $p_\alpha(n)$

In this section, we consider the error of the approximation

$$p_\alpha(n; \delta) := \nu_\alpha(n)^{-\frac{\alpha}{2} - 1} \sum_{m=0}^{q} \mu_\alpha(m)^{\frac{\alpha}{2} + 1} p_\alpha(m) \sum_{1 \leq h < \frac{\alpha}{2} \mu_\alpha(m)}^{\text{for} \ 1 \leq k < \frac{\alpha}{2} \mu_\alpha(m)} \frac{2 \pi}{k} A_k^{(\alpha)}(n,m) I_{\frac{\alpha}{2} + 1} \left( \frac{4 \pi}{k} \nu_\alpha(n) \mu_\alpha(m) \right).$$

(4.1)

for $p_\alpha(n)$. Note in particular that in the limit as $\delta \to 0^+$, we have $p_\alpha(n; \delta) \to p_\alpha(n)$. 

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\textbf{Theorem 4.1.} For all \(\alpha > 0\), \(0 < \delta < 2\pi \mu_\alpha(0)\), and \(n > \alpha/24\), we have

\[|p_\alpha(n) - p_\alpha(n; \delta)| < C \frac{I_{\frac{\alpha}{2} + 1}(2\delta \nu_\alpha(n))}{\nu_\alpha(n)\frac{\alpha}{2} + 1} < C\delta^2 \frac{I_{\frac{\alpha}{2} + 1}(4\pi \mu_\alpha(0)\nu_\alpha(n))}{(2\pi \mu_\alpha(0)\nu_\alpha(n))\frac{\alpha}{2} + 1},\]  

(4.2)

where

\[C := 4\pi^2 \left(1 + \frac{2}{\alpha}\right) \mu_\alpha(0) \sum_{m=0}^{q} \mu_\alpha(m)\frac{\alpha}{2} + 1 p_\alpha(m).\]

\textbf{Proof.} Start by noting that

\[\left|A_k^{(\alpha)}(n, m)\right| \leq \sum_{0 \leq h < k, (h, k) = 1} |e^{\alpha \pi i (h, k)} + 2\pi m (mH - nh)| = \sum_{0 \leq h < k, (h, k) = 1} 1 \leq k. \]  

(4.3)

Moreover, using the fact from [15] that for \(0 < x < y\) and \(\nu > 1\), the modified Bessel function of the first kind satisfies

\[\frac{I_\nu(x)}{I_\nu(y)} < \left(\frac{x}{y}\right)\nu,\]  

(4.4)

we have that

\[\sum_{k \geq \frac{2\pi}{\alpha} \mu_\alpha(m)} \frac{I_{\frac{\alpha}{2} + 1}(\frac{2\pi}{\alpha} \mu_\alpha(n)\nu_\alpha(m))}{I_{\frac{\alpha}{2} + 1}(2\delta \nu_\alpha(n))} < \sum_{k \geq \frac{2\pi}{\alpha} \mu_\alpha(m)} \left(\frac{2\pi}{k\delta} \mu_\alpha(m)\right)^{\frac{\alpha}{2} + 1} \]

\[< 1 + \int_{\frac{2\pi}{\alpha} \mu_\alpha(m)}^{\infty} \left(\frac{2\pi}{t\delta} \mu_\alpha(m)\right)^{\frac{\alpha}{2} + 1} dt \]

\[= 1 + \frac{4\pi}{\alpha\delta} \mu_\alpha(m)\]

for \(0 \leq m \leq q\). Thus, we find that

\[\nu_\alpha(n)^{\frac{\alpha}{2} + 1}\left|p_\alpha(n) - p_\alpha(n; \delta)\right| \leq 2\pi \sum_{m=0}^{q} \mu_\alpha(m)^{\frac{\alpha}{2} + 1} p_\alpha(m) \sum_{k \geq \frac{2\pi}{\alpha} \mu_\alpha(m)} \frac{I_{\frac{\alpha}{2} + 1}(4\pi \mu_\alpha(0)\nu_\alpha(m))}{(2\pi \mu_\alpha(0)\nu_\alpha(n))^{\frac{\alpha}{2} + 1}} \]

\[< 2\pi I_{\frac{\alpha}{2} + 1}(2\delta \nu_\alpha(n)) \sum_{m=0}^{q} \mu_\alpha(m)^{\frac{\alpha}{2} + 1} p_\alpha(m) \left[1 + \frac{4\pi}{\alpha\delta} \mu_\alpha(m)\right].\]

Since \(1 < \frac{2\pi}{\alpha} \mu_\alpha(0)\) and \(\mu_\alpha(m) \leq \mu_\alpha(0)\), it follows that

\[|p_\alpha(n) - p_\alpha(n; \delta)| < \frac{4\pi^2}{\delta} \frac{I_{\frac{\alpha}{2} + 1}(2\delta \nu_\alpha(n))}{\nu_\alpha(n)^{\frac{\alpha}{2} + 1}} \left(1 + \frac{2}{\alpha}\right) \mu_\alpha(0) \sum_{m=0}^{q} \mu_\alpha(m)^{\frac{\alpha}{2} + 1} p_\alpha(m),\]

or applying the Paris inequality a second time using \(2\delta \nu_\alpha(n) < 4\pi \mu_\alpha(0)\nu_\alpha(n)\),

\[|p_\alpha(n) - p_\alpha(n; \delta)| < \frac{4\pi^2}{\delta} \nu_\alpha(n)^{\frac{\alpha}{2} + 1} \left(1 + \frac{2}{\alpha}\right) \mu_\alpha(0) \sum_{m=0}^{q} \mu_\alpha(m)^{\frac{\alpha}{2} + 1} p_\alpha(m).\]

\[\Box\]

We are now in a position to prove the simple asymptotic formula for \(p_\alpha(n)\) stated in the introduction.
Proof of Corollary 1.2. Observe that since $\frac{4\pi}{\delta} \mu_\alpha(m)$ is strictly increasing in $m$, there exists a $0 < \delta < 2\pi \mu_\alpha(0)$ such that $\frac{4\pi}{\delta} \mu_\alpha(m) \leq 2$ for $0 < m \leq q$ and so

$$p_\alpha(n; \delta) = 2\pi \left( \frac{\mu_\alpha(0)}{\nu_\alpha(n)} \right)^{\frac{2}{\alpha} + 1} I_{\frac{2}{\alpha} + 1} \left( 4\pi \nu_\alpha(n) \mu_\alpha(0) \right) = \frac{2\pi}{\lambda_\alpha(n)} \frac{I_{\frac{2}{\alpha} + 1} \left( \frac{\pi\alpha}{6} \lambda_\alpha(n) \right)}{\lambda_\alpha(n)^{\frac{2}{\alpha} + 1}}.$$ 

Moreover, by Theorem 4.1, we have

$$|p_\alpha(n) - p_\alpha(n; \delta)| \leq C \frac{I_{\frac{2}{\alpha} + 1} \left( 2\delta \nu_\alpha(n) \right)}{\nu_\alpha(n)^{\frac{2}{\alpha} + 1}},$$

for some constant $C$. Using the fact that $I_\nu(z) \sim e^z / \sqrt{2\pi z}$ from [8, 10.30.4], we easily verify that

$$C_\nu \alpha(n) - e^{2\pi \nu_\alpha(n)} \sim \frac{e^{\frac{\pi\alpha}{6} \lambda_\alpha(n)}}{\lambda_\alpha(n)^{\frac{2}{\alpha} + 1}}.$$ 

Theorem 4.1 also allows us to derive a finite exact formula for $p_\alpha(n)$ when $\alpha$ is rational. This is made possible by a formula for the denominator of $p_\alpha(n)$ from [4], which states that if $\alpha = a/b$ for coprime $a, b \in \mathbb{Z}$ with $b > 0$, then

$$\text{denom}(p_\alpha(n)) := b^n \prod_{p|b} p^{\text{ord}_p(n)},$$

where $\text{ord}_p(n)$ denotes the multiplicity of a prime $p$ as a factor of $n$.

Corollary 4.2. Let $\alpha, \varepsilon > 0$ and $n > \alpha/24$ with $\alpha$ rational. Then

$$p_\alpha(n) = \frac{\lfloor Dp_\alpha(n; \delta) \rfloor}{D},$$

where $D = \text{denom}(p_\alpha(n))$ and

$$\delta := \left( \frac{(2\pi \mu_\alpha(0) \nu_\alpha(n))^{\frac{2}{\alpha} + 1}}{2DCI_{\frac{2}{\alpha} + 1} \left( 4\pi \mu_\alpha(0) \nu_\alpha(n) \right)} \right)^{\frac{2}{\alpha}},$$

with $C$ defined as in Theorem 4.1.

Proof. Observe that by Theorem 4.1, we have

$$|p_\alpha(n) - p_\alpha(n; \delta)| < C_\delta \frac{I_{\frac{2}{\alpha} + 1} \left( 4\pi \mu_\alpha(0) \nu_\alpha(n) \right)}{(2\pi \mu_\alpha(0) \nu_\alpha(n))^{\frac{2}{\alpha} + 1}} = \frac{1}{2D}.$$ 

Thus, $D|p_\alpha(n) - p_\alpha(n; \delta)| < 1/2$, implying that $Dp_\alpha(n)$ is the nearest integer to $Dp_\alpha(n; \delta)$. □

4.2 Hyperbolicity of the Jensen Polynomials of $p_\alpha(n)$

In this section, we demonstrate how the asymptotics of $p_\alpha(n)$ in this paper can be used to generalize a recent hyperbolicity result for the usual partition function.
Proof of Theorem 1.3. Set
\[ m = \frac{\alpha}{24} \] and \[ c_0 = \log \left( \sqrt{\frac{12}{\alpha}} \cdot \left( \frac{\alpha}{24} \right)^{\frac{\alpha}{4}} \right). \]

Then by Corollary 1.2,
\[ p_\alpha(n) \sim e^{c_0 + 4\pi \sqrt{mn} - \frac{\alpha}{4}}. \]

Thus, as in [9, §3], we have
\[ \log \left( \frac{p_\alpha(n + j)}{p_\alpha(n)} \right) \sim 4\pi \sqrt{m} \sum_{i=1}^{\infty} \frac{1/2}{i} \cdot (-1)^{i-1} j^i. \]

from which it is clear that \( p_\alpha(n) \) satisfies the conditions of Theorem 3 from [9] with \( A(n) = 2\pi \sqrt{m/n + O(1/n)} \) and \( \delta(n) = (\pi/2)^{1/2} m^{1/4} n^{-3/4} + O(n^{-5/4}). \) It follows immediately that for all \( d \) the Jensen polynomials associated with \( p_\alpha(n) \) are hyperbolic for sufficiently large \( n. \]

Remark. The proof of Theorem 1.3 follows [9, §3]. In particular, we consider the renormalization of the Jensen polynomials given by
\[ \hat{J}_{d,n}^{p_\alpha}(X) = \frac{\delta(n)-d}{p_\alpha(n)} \cdot J_{d,n}^{p_\alpha}(\delta(n)X - \frac{1}{\exp(A(n))}). \]

Theorem 1.3 follows from the fact that for fixed \( d \),
\[ \lim_{n \to \infty} \hat{J}_{d,n}^{p_\alpha}(X) = H_d(x), \]
where \( H_d(x) \) is the degree \( d \) renormalized Hermite polynomial in [9].

5 Numerical Data

In this section, we illustrate the theorems of the previous sections using numerical examples. For simplicity, we limit our examples to cases where \( 0 < \alpha < 24 \). For such \( \alpha \), it will be convenient to define
\[ r_\alpha(n; m) = \frac{\text{Re} \left( p_\alpha \left( n; \frac{2\pi \mu_0(0)}{m+1} \right) \right)}{p_\alpha(n)}, \]
the ratio between the real part of the \( m \)-term approximation to \( p_\alpha(n) \) and the actual value. Note that a value of \( r_\alpha(n; m) \) closer to 1 indicates that the \( m \)-term approximation to \( r_\alpha(n) \) is more accurate.

By Corollary 1.2, we know that \( p_\alpha(n) \) is asymptotically equivalent to the first term in the series expansion in Theorem 1.1 as \( n \) goes to infinity. Table I displays the accuracy of the first-term expansion for \( \alpha = e \) and \( n \) varying from 1 to 10. Table II shows the ratio of both the first-term and the five-term approximation to \( p_\alpha(n) \) where \( \alpha = 1/\pi \) and \( \alpha = 5 \). Note that the sign of the error term \( |p_\alpha(n) - p_\alpha(n; m)| \) is usually periodic with period \( m + 1 \). This is a consequence of the periodicity of the Kloosterman sums.

Table III displays how \( p_\alpha(n, m) \) converges to \( p_\alpha(n) \) for \( \alpha = 1/e, n = 50, \) and \( 1 \leq m \leq 10 \). Table IV displays the ratio of the \( m \)-term approximation of \( p_\alpha(n) \) to the actual value for \( n = 100 \) and various
Table 1: Accuracy of first-term approximations to $p_e(n)$

| $n$ | $p_e(n)$ | Re($p_e(n; 1)$) | $r_e(n; 1)$ |
|-----|----------|-----------------|-------------|
| 1   | 2.71     | 2.83            | 1.04253     |
| 2   | 7.77     | 7.65            | 0.98444     |
| 3   | 18.05    | 18.23           | 1.01014     |
| 4   | 40.26    | 39.96           | 0.99263     |
| 5   | 81.84    | 82.28           | 1.00543     |
| 6   | 161.99   | 161.41          | 0.9964      |
| 7   | 303.75   | 304.41          | 1.00217     |
| 8   | 556.32   | 555.61          | 0.99873     |
| 9   | 985.41   | 986.27          | 1.00086     |
| 10  | 1710.31  | 1709.07         | 0.99927     |

Table 2: Accuracy of $m$-term approximations to $p_{1/e}(50; m) = 357.1225$

| $m$ | Re($p_{1/e}(50; m)$) | $r_{1/e}(50; m)$ |
|-----|----------------------|------------------|
| 1   | 356.2898             | 0.997668         |
| 2   | 357.2586             | 1.000381         |
| 3   | 357.1278             | 1.000014         |
| 4   | 357.053              | 0.999805         |
| 5   | 357.1236             | 1.000003         |
| 6   | 357.1195             | 0.999991         |
| 7   | 357.1169             | 0.999994         |
| 8   | 357.1208             | 0.999995         |
| 9   | 357.1210             | 0.999993         |
| 10  | 357.1296             | 1.000002         |

Table 3: Accuracy of approximation to $p_\alpha(n)$ as $n$ increases

| $n$ | $r_{1/e}(\pi; 1)$ | $r_{1/e}(\pi; 5)$ | $r_5(n; 1)$ | $r_5(n; 5)$ |
|-----|-------------------|-------------------|-------------|-------------|
| 1   | 1.294180591       | 0.953980957       | 1.015286846 | 1.000097277 |
| 2   | 0.970982400       | 0.982523054       | 0.994583967 | 1.000042848 |
| 3   | 1.083673986       | 1.018088216       | 1.002732222 | 1.00007177 |
| 4   | 0.923295102       | 1.02170408        | 0.999524823 | 1.00000000 |
| 5   | 1.124698668       | 1.016001474       | 1.000871244 | 0.99999382 |
| 6   | 0.897139773       | 1.004350338       | 1.00093623  | 0.99999982 |
| 7   | 1.108496000       | 0.978153497       | 1.002556555 | 1.00000217 |
| 8   | 0.943494666       | 1.002688299       | 0.999854031 | 1.00000092 |
| 9   | 1.034408356       | 1.003218418       | 1.000093623 | 0.99999982 |
| 10  | 0.961090657       | 1.005487344       | 0.999935881 | 0.99999997 |
| 11  | 1.076769973       | 0.993996646       | 1.00043109  | 0.99999968 |
| 12  | 0.923558631       | 1.005369386       | 0.999972215 | 1.00000007 |
| 13  | 1.058750442       | 0.996292489       | 1.00017874  | 1.00000008 |
| 14  | 0.980265489       | 0.993723758       | 0.999987986 | 1.00000000 |

Table 4: Accuracy of approximation to $p_\alpha(n)$ as number of terms in series increases

| $m$ | $r_{0.01}(100; m)$ | $r_{0.1}(100; m)$ | $r_1(100; m)$ | $r_{10}(100; m)$ |
|-----|-------------------|-------------------|---------------|-----------------|
| 1   | 0.846079580       | 0.988058877       | 0.999998178   | 1.0000000000    |
| 2   | 0.969774117       | 0.999386989       | 1.000000009   | 1.0000000000    |
| 3   | 0.920711483       | 0.997246602       | 0.999999995   | 1.0000000000    |
| 4   | 0.973881495       | 0.999016179       | 0.999999999   | 1.0000000000    |
| 5   | 1.040636574       | 1.000923931       | 1.000000000   | 1.0000000000    |
| 6   | 1.028999226       | 1.000579623       | 1.000000000   | 1.0000000000    |
| 7   | 1.020829553       | 1.000421863       | 1.000000000   | 1.0000000000    |
| 8   | 0.995326778       | 0.999817677       | 1.000000000   | 1.0000000000    |
| 9   | 0.995461037       | 0.999846688       | 1.000000000   | 1.0000000000    |
| 10  | 1.011689149       | 1.000211135       | 1.000000000   | 1.0000000000    |
values of $\alpha$ and $m$. As we increase $\alpha$, we see that the relative error of the approximation for $p_\alpha(n)$ decreases.

Table 5 depicts the convergence of $\widetilde{J}_{2,\alpha}^n(X)$ to the Hermite polynomial $H_2(x) = x^2 - 2$, and the convergence of $\widetilde{J}_{3,\alpha}^n(X)$ to the Hermite polynomial $H_3(x) = x^3 - 6x$. Here,

$$A(n) = 2\pi \sqrt{\frac{\alpha}{24n - \alpha}} - \frac{24}{24n - \alpha}, \quad \text{and} \quad \delta(n) = \sqrt{\frac{12\pi\alpha^2}{(24n - \alpha)^2} - \frac{288\alpha}{(24n - \alpha)^2}}$$

as in Theorem 1.3 for $\sqrt{3}$. To compute $p_\alpha(n)$ for large $n$, we used the 100-term approximation of our series formula; this is valid for our purposes because by Theorem 4.1, the relative error $|r_{\sqrt{3}}(n, 100) - 1|$ is bounded by $10^{-75}$ for the values of $n$ we consider.

| $n$ | $\widetilde{J}_{2,\alpha}^n(x)$ | $\widetilde{J}_{3,\alpha}^n(x)$ |
|-----|-----------------|-----------------|
| 10000 | $0.999598x^2 + 0.120905x - 2.03828$ | $0.999942x^3 + 0.0939817x^2 - 6.03526x - 0.648632$ |
| 20000 | $0.999804x^2 + 0.0966267x - 2.02711$ | $0.999971x^3 + 0.0767061x^2 - 6.02522x - 0.543473$ |
| 30000 | $0.999871x^2 + 0.0852795x - 2.02216$ | $0.999981x^3 + 0.0683801x^2 - 6.0207x - 0.495049$ |
| 40000 | $0.999904x^2 + 0.0782302x - 2.0192$ | $0.999986x^3 + 0.0631174x^2 - 6.01799x - 0.435239$ |
| 50000 | $0.999923x^2 + 0.0732538x - 2.01719$ | $1.00252x^3 + 0.0595086x^2 - 6.03131x - 0.429626$ |
| : | $x^2 - 2$ | : |
| $\infty$ | $x^3 - 6x$ | : |

Table 5: Convergence to the Hermite polynomial of degree 2, $x^2 - 2$, and of degree 3, $x^3 - 6x$

In Table 6, we provide the actual value of $p_{51/7}(n)$ alongside the minimum number $M_{51/7}(n)$ for which Corollary 4.2 guarantees that $p_{51/7}(n)$ is given by a suitable rounding of $p_\alpha \left( n; \frac{2\pi p_\alpha(0)}{M_{51/7}(n)+1} \right)$, which has $M_{51/7}(n)$ terms. We also provide $M_{51/7}^*(n)$, the minimum number of terms such that this is numerically true.

| $n$ | $p_{51/7}(n)$ | $M_{51/7}(n; D)$ | $M_{51/7}^*(n; D)$ |
|-----|-----------------|-----------------|-----------------|
| 1   | 51/7            | 2               | 1               |
| 2   | 1836/49         | 3               | 2               |
| 3   | 52751/343       | 5               | 3               |
| 4   | 1322226/2401    | 8               | 4               |
| 5   | 29852442/16807  | 14              | 7               |
| 6   | 623075585/117649| 23              | 10              |
| 7   | 85346705106/5764801 | 67             | 26              |
| 8   | 1583888229297/40353607 | 114            | 43              |
| 9   | 28093059550223/282475249 | 194            | 63              |
| 10  | 479246612549889/1977326743 | 330            | 109             |

Table 6: Number of terms for exact solution for $p_{51/7}(n)$

\(^1\) All computations in this section were done with Wolfram Mathematica.
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