CONFORMING FINITE ELEMENT DIVDIV COMPLEXES AND
THE APPLICATION FOR THE LINEARIZED EINSTEIN-BIANCHI
SYSTEM

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Abstract. This paper presents the first family of conforming finite element div div complexes
on tetrahedral grids in three dimensions. In these complexes, finite element spaces of
\(H^{p}_{\text{div div}}(\Omega; S)\) are from a current preprint [Chen and Huang, arXiv: 2007.12399, 2020]
while finite element spaces of both \(H^{p}_{\text{sym curl}}(\Omega; T)\) and \(H^{1}(\Omega; \mathbb{R}^{3})\) are newly constructed here. It is proved that these finite
element complexes are exact. As a result, they can be used to discretize the linearized
Einstein-Bianchi system within the dual formulation.

Key words. divdiv complex, \(H(\text{symcurl})\) conforming finite element, linearized Einstein-Bianchi
system

1. Introduction. The linearized Einstein-Bianchi system from [16] reads
\[
\begin{align*}
\dot{E} + \text{curl } B &= 0, \quad \text{div } E = 0, \\
\dot{B} - \text{curl } E &= 0, \quad \text{div } B = 0
\end{align*}
\]
with symmetric and traceless tensor fields \(E\) and \(B\), respectively. By introducing
a new variable \(\sigma(t) = \int_{0}^{t} \text{div } E \, ds\), the linearized Einstein-Bianchi system can be
realized as a Hodge wave equation
\[
\begin{align*}
\dot{\sigma} &= \text{div div } E, \\
\dot{E} &= -\nabla \nabla \sigma - \text{sym curl } B, \\
\dot{B} &= \text{curl } E.
\end{align*}
\]
Given initial conditions \(\sigma(0), E(0)\) and \(B(0)\), and with appropriate boundary con-
ditions, (1) is well posed (see [16]). The weak formulation of (1) in [16] introduces
some Lagrange multiplies to reduce the constraint of the symmetry of the electric
field \(E\) and high derivatives of \(\sigma\). In [11], the first family of conforming finite element
spaces of \(H(\text{curl}, \Omega; S)\) on a bounded polyhedral domain \(\Omega \subset \mathbb{R}^{3}\) is constructed as well
as those finite element spaces associated with the following Gradgrad complex from
[2, 15]
\[
\begin{align*}
P_{1}(\Omega) \hookrightarrow H^{2}(\Omega) \overset{\nabla \nabla}{\longrightarrow} H(\text{curl}, \Omega; S) \overset{\text{curl}}{\longrightarrow} H(\text{div}, \Omega; T) \overset{\text{div}}{\longrightarrow} L^{2}(\Omega; \mathbb{R}^{3}) \rightarrow 0,
\end{align*}
\]
where the space \(H(\text{curl}, \Omega; S)\) consists of square-integrable tensors with square-integrable
curl, taking values in the space of \(S\) of symmetric matrices, and the space \(H(\text{div}, \Omega; T)\)
consists of square-integrable tensors with square-integrable divergence, taking value
in the space $\mathbb{T}$ of traceless matrices. The finite element Gradgrad complexes are utilized in [11] to discretize (1) with a weak formulation such that $E$ is sought in $C^0([0, T], H(\text{curl}, \Omega; \mathbb{S}))$ with the strongly symmetric constraint, $\sigma$ is in $C^0([0, T], H^2(\Omega))$ and the magnetic tensor field $B$ is in $C^1([0, T], L^2(\Omega; \mathbb{T}))$. The polynomial degree of the lowest order case for the pair $(\sigma, E, B)$ is $P_0(K) - P_1(K; S) - P_0(K; \mathbb{T})$ on each tetrahedron $K$. In [4], conforming and nonconforming virtual element Grad grad complexes are constructed on tetrahedral grids.

This paper considers the discretization of (1) in the dual weak formulation of [11] (see (33) below) with $E \in C^0([0, T], H(\text{div div}, \Omega; \mathbb{S}))$, $\sigma \in C^1([0, T], L^2(\Omega))$ and $B \in C^0([0, T], H(\text{sym curl}; \mathbb{T}))$. The associated finite element spaces for the linearized Einstein-Bianchi system is then closely related to the div div complex (dual to (2)) from [2,15]

\[ RT \subseteq H^1(\Omega; \mathbb{R}^3) \xrightarrow{\text{dev \nabla}} H(\text{sym curl}; \Omega; \mathbb{T}) \xrightarrow{\text{sym curl}} H(\text{div div}, \Omega; \mathbb{S}) \xrightarrow{\text{div div}} L^2(\Omega) \rightarrow 0. \]

Such a complex is exact provided that the domain is contractible and Lipschitz. A family of conforming finite element spaces $\Sigma_{k,h} \subseteq H(\text{div div}, \mathbb{S})$ has been constructed for $k \geq 3$ in [6] on tetrahedral grids (see [5] in two dimensions). Another family of conforming finite element spaces $\Sigma_{k,h} \subseteq H(\text{div div}, \mathbb{S})$ has been designed in [12] on simplicial grids for both two and three dimensions by using the $H(\text{div div}, \mathbb{S})$ conforming finite element spaces from [10,13,14]. The discontinuous Petrov-Galerkin (DPG) discretization of $H(\text{div div}, \Omega; \mathbb{S})$ can be found in [8,9]. The main contribution of the paper is to construct a family of conforming finite element spaces $\Lambda_{k,h} \subseteq H(\text{sym curl}; \mathbb{T})$ of order $k + 1$ such that the following finite element div div complexes

\[ RT \subseteq V_{k+2,h} \xrightarrow{\text{dev \nabla}} \Lambda_{k+1,h} \xrightarrow{\text{sym curl}} \Sigma_{k,h} \xrightarrow{\text{div div}} P_{k-1}(\mathcal{T}) \rightarrow 0 \]

are exact together with a family of $H^1$ vectorial conforming finite element spaces $V_{k+2,h}$ of order $k + 2$ and discontinuous piecewise polynomial spaces $P_{k-1}(\mathcal{T})$ on tetrahedral grids $\mathcal{T}$. The construction is based on the finite element de Rham complexes and the new finite element strain complexes in two dimensions. The associated finite elements of the strain complex have extra continuity at vertices compared with those elements in [5]. The construction shows some geometry decomposition in two and three dimensions as in [7] for the de Rham complex, see more references therein. These elements are used to discretize the linearized Einstein-Bianchi system, and the error estimates are provided in the end of the paper. The polynomial degree of the lowest order case for the pair $(\sigma, E, B)$ in this paper is $P_1(K) - P_3(K; S) - P_4(K; \mathbb{T})$. It is more practical compared with the finite element methods in [11].

Throughout this paper, denote the space of all $3 \times 3$ matrices by $\mathbb{M}$, all symmetric $3 \times 3$ matrices by $\mathbb{S}$, and all trace-free $3 \times 3$ matrices by $\mathbb{T}$. Standard notation in Sobolev spaces will be used such as $L^2(\omega; X)$ and $H^m(\omega; X)$, taking values in the finite-dimensional space $X$. Let $P_k(\omega; X)$ denote the set of all polynomials over $\omega$ of total degree not greater than $k$. The range space $X$ will be either $\mathbb{R}, \mathbb{R}^3, \mathbb{M}, \mathbb{T}, \mathbb{S}$ in three dimensions. If $X = \mathbb{R}$, then $L^2(\omega)$ abbreviates $L^2(\omega; X)$, similarly for $H^m(\omega)$ and $P_k(\omega)$.

The organization of the paper is as follows. Section 2 introduces some operators for vectors and tensors. Section 3 introduces the finite elements with extra continuity at vertices corresponding to the de Rham complex and strain complex, respectively. Section 4 first constructs a family of $H(\text{sym curl}; \mathbb{T})$ conforming finite elements. Second, the finite element div div complexes (4) are established for a contractible domain.
Section 5 uses the newly proposed finite element spaces to discretize the linearized Einstein-Bianchi system within the dual formulation and shows the error estimates.

2. Preliminaries. This section prepares some operators for vectors and tensors. More related results can be found in [6]. Let $I$ denote the $3 \times 3$ identity matrix. Given a matrix $A \in \mathbb{R}^{3 \times 3}$, define

$$\text{sym } A = \frac{1}{2}(A + A^T), \text{ dev } A = A - \frac{1}{3}\text{tr}(A)I.$$ 

For a vector $v = (v_1, v_2, v_3)^T$, define a skew-symmetric matrix as follows

$$\text{mspn } v = \begin{pmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{pmatrix}.$$ 

For a matrix function $A$, the curl and div operators apply row-wise to produce a matrix function $\text{curl } A$ and a vector function $\text{div } A$, respectively. For a vector function $v = (v_1, v_2, v_3)^T$, the gradient $\nabla$ applies row-wise to produce a matrix function

$$\nabla v = \begin{pmatrix} \partial_x v_1 & \partial_y v_1 & \partial_z v_1 \\ \partial_x v_2 & \partial_y v_2 & \partial_z v_2 \\ \partial_x v_3 & \partial_y v_3 & \partial_z v_3 \end{pmatrix}.$$ 

Define the symmetric gradient by

$$\epsilon(v) = \text{sym}(\nabla v).$$

Given a plane $f$ with the unit normal vector $n$, for a vector $v \in \mathbb{R}^3$, the following orthogonal decomposition holds

$$v = \Pi_n v + \Pi_f v := (v \cdot n)n + (n \times v) \times n.$$ 

Define the tangential derivatives and the surface curl operator for a scalar function $q$ as

$$\nabla_f q = \Pi_f \nabla q = (n \times \nabla q) \times n, \quad \text{curl}_f q = n \times \nabla q.$$ 

Define the surface rot operator for a vector function $v$ as

$$\text{rot}_f v = (n \times \nabla ) \cdot v = n \cdot (\text{curl } v)$$

and the surface divergence $\text{div}_f$ as

$$\text{div}_f v = \text{div}_f(\Pi_f v) = (n \times \nabla) \cdot (n \times v) = \text{rot}_f(n \times v).$$

The surface gradient $\nabla_f v$ applies row-wise to produce a matrix function $\nabla_f v$. Define the surface symmetric gradient $\epsilon_f$ by

$$\epsilon_f(v) = \text{sym}(\nabla_f(\Pi_f v)).$$

In particular, for $n = (0, 0, 1)^T$, $f$ is the $x - y$ plane. Then, these operators $\nabla_f, \text{rot}_f, \text{curl}_f, \text{div}_f, \epsilon_f$ are standard differential operators in two dimensions.
Given a matrix function $A$, $n \times A$ acts row-wise while $A \times n$ acts column-wise. The operators $\Pi_f$, $\rot_f A$ and $\div_f A$ act row-wise. Define the symmetric projection

$$\Pi_{f, \text{sym}} A = \text{sym}(\Pi_f A).$$

Given a vector function $v$ and a matrix function $A$, the vector products commute with differentiation as follows

$$(\nabla v)^T n = \nabla(v \cdot n),$$

$$\nabla v \times n = \nabla(v \times n),$$

$$(\curl A)^T n = \curl(A^T n).$$

3. Finite elements in two dimensions. This section constructs new finite elements on triangular grids in two dimensions for the de Rham complex and strain complex below with extra continuity at vertices. Some elements with extra continuity at vertices and along edges have been introduced in [7] for de Rham complexes in two and three dimensions. The finite elements presented in this section have weaker continuity along edges compared with those elements from [7]. The finite elements for the strain complex have stronger continuity at vertices compared with those finite elements in [5].

3.1. Notation in two dimensions. Throughout this section $x = (x, y)^T$ and $x^\perp = (-y, x)^T$. Suppose that $f$ is on the $x - y$ plane. Given a scalar function $q$ and a vector function $v = (v_1, v_2)^T$, the surface differential operators read $\nabla_f q = (\partial_2 q, \partial_1 q)^T$, $\curl_f v = (\partial_y v_1, \partial_x v_2, \partial_x v_1 - \partial_y v_2)$, $\rot_f v = \partial_x v_1 - \partial_y v_2$ and $\epsilon_f(v) = \frac{1}{2}(\nabla_f v + (\nabla_f v)^T)$. To distinguish the space $S$ of symmetric matrices in three dimensions, denote the space of symmetric matrices in two dimensions by $S_2 := \text{symmetric } \mathbb{R}^{2 \times 2}$.

Let $RT_f$ denote the lowest order Raviart-Thomas space in two dimensions, which reads

$$RT_f := \{a + bx \mid a \in \mathbb{R}^2, b \in \mathbb{R}\}.$$

Given a contractible domain $\omega \subset \mathbb{R}^2$, the de Rham complex [1] in two dimensions reads

$$\mathbb{R} \xhookrightarrow{\nabla_f} H^1(\omega) \xrightarrow{\rot_f} H(\rot_f, \omega; \mathbb{R}^2) \xrightarrow{\div_f} L^2(\omega) \rightarrow 0,$$

and the strain complex [5] reads

$$RT_f \xhookrightarrow{\epsilon_f} H^1(\omega) \xrightarrow{\rot_f \rot_f} H(\rot_f \rot_f, \omega; S) \xrightarrow{\div_f \rot_f \rot_f} L^2(\omega) \rightarrow 0.$$

The corresponding polynomial complexes read [1, 5]

$$\mathbb{R} \xhookrightarrow{\nabla_f} P_{k+2}(\omega) \xrightarrow{\rot_f} P_{k+1}(\omega; \mathbb{R}^2) \xrightarrow{\div_f} P_k(\omega) \rightarrow 0,$$

$$RT_f \xhookrightarrow{\epsilon_f} P_{k+2}(\omega; \mathbb{R}^2) \xrightarrow{\rot_f \rot_f} P_{k+1}(\omega; S) \xrightarrow{\div_f \rot_f \rot_f} P_{k-1}(\omega) \rightarrow 0.$$

Given a triangle $f$, let $E(f)$ denote the set of all edges of $f$. Given $e \in E(f)$, let $n = (n_1, n_2)^T$ denote the unit normal vector of $e$ and $t = (-n_2, n_1)^T$ denote the unit tangential vector of $e$. Let $\lambda_1, \lambda_2, \lambda_3$ denote the barycentric coordinates of $f$. 

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3.2. Finite element de Rham complexes in two dimensions. This subsection constructs the finite elements with respect to the de Rham complex (5).

The shape function space of the $H^1$ conforming element is $P_{k+2}(f)$ with $k \geq 3$. Define the set of polynomials of degrees $\leq k - 1$ with vanishing values at vertices of $f$ by

$$P_{k-1,0}(f) := \{ p \in P_{k-1}(f) \mid p \text{ vanishes at all vertices of } f \}.$$  

Following the idea from [11], the degrees of freedom of the $H^1$ conforming element with extra continuity at vertices are defined as follows

1a) function value and first and second order derivatives at each vertex $x$:

$$p(x), \nabla f p(x), \nabla^2 f p(x),$$

1b) moments of order $\leq k - 4$ on each edge $e$:

$$\int_e pq, \quad q \in P_{k-4}(e),$$

1c) the interior degrees of freedom in $f$ defined by

$$\int_f pq, \quad q \in P_{k-1,0}(f).$$

**Lemma 1.** The degrees of freedom (1a)–(1c) are unisolvent for $P_{k+2}(f)$.

**Proof.** It is easy to check that the number of the degrees of freedom is equal to the dimension of $P_{k+2}(f)$. Given $p \in P_{k+2}(f)$, if (1a)–(1b) vanish for $p$, then $p = 0$ on $\partial f$. This shows that there exists some $r \in P_{k-1}(f)$ such that

$$p = \lambda_1 \lambda_2 \lambda_3 r.$$  

Since the second order derivatives of $p$ vanish at each vertex of $f$, $r$ vanishes at each vertex of $f$ as well. The degrees of freedom (1c) with $P_{k-1,0}(f)$ from (9) lead to $r = 0$. This concludes the proof. 

The shape function space of the $H(\text{rot}_f)$ conforming element is $P_{k+1}(f; \mathbb{R}^2)$ with $k \geq 3$. Before introducing its degrees of freedom, define the polynomial space

$$P_{k-1,1}(f) := \{ q \in P_{k-1}(f) \mid \text{there exists some } r \in P_{k-1,0}(f) \text{ such that } \text{div}_f(qx) = r \}.$$  

The bijection $\text{div}_f : P_{k-1}(f)x \rightarrow P_{k-1}(f)$ [1] guarantees $\dim P_{k-1,1}(f) = \dim P_{k-1,0}(f)$. The degrees of freedom of the $H(\text{rot}_f)$ conforming element with extra continuity at vertices then read

2a) function value and first order derivatives at each vertex $x$:

$$u(x), \nabla f u(x),$$

2b) moments of order $\leq k - 3$ of the tangential component on each edge $e$:

$$\int_e u \cdot tq, \quad q \in P_{k-3}(e),$$

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(2c) moments of order $\leq k - 2$ of $\text{rot}_f$ on each edge $e$:
\[
\int_e \text{rot}_f u q, \quad q \in P_{k-2}(e),
\]
(2d) the interior degrees of freedom in $f$ defined by
\[
\int_f u \cdot v, \quad v \in \text{curl}_f P_{k-3}(f) + P_{k-1,1}(f) \mathbf{x}.
\]

**Lemma 2.** The degrees of freedom (2a)–(2c) are unisolvent for $P_{k+1}(f; \mathbb{R}^2)$.

**Proof.** The number of the degrees of freedom is equal to the dimension of $P_{k+1}(f; \mathbb{R}^2)$, namely
\[
18 + 3(k - 2) + 3(k - 1) + k^2 - k - 3 = (k + 2)(k + 3).
\]
If (2a)–(2c) vanish for some $u \in P_{k+1}(f; \mathbb{R}^2)$, then $u \cdot t$ and $\text{rot}_f u$ vanish on $\partial f$. This shows
\[
\text{rot}_f u = \lambda_1 \lambda_2 \lambda_3 r
\]
for some $r \in P_{k-3}(f)$. For any $q \in P_{k-3}(f)$, an integration by parts leads to
\[
\int_f u \cdot \text{curl}_f q = -\int_f \lambda_1 \lambda_2 \lambda_3 r q.
\]
This leads to $r = 0$ provided that (2d) vanishes for $u$ and thus $\text{rot}_f u = 0$. This and the zero tangential boundary conditions on $\partial f$ of $u$ show that there exists some $p \in P_{k-1}(f)$ such that
\[
u = \nabla_f (\lambda_1 \lambda_2 \lambda_3 p).
\]
Since the first order derivatives of $u$ vanish at vertices, $p$ vanishes at the vertices and $p \in P_{k-1,0}(f)$. Given $v \in P_{k-1,1}(f) \mathbf{x}$,
\[
\int_f u \cdot v = -\int_f \lambda_1 \lambda_2 \lambda_3 p \text{div}_f v.
\]
The degrees of freedom (2d) plus (10) lead to $p = 0$. This concludes $u = 0$. 

The shape function space of the $L^2$ element is $P_k(f)$. The following degrees of freedom coincide with the Lagrange element of order $k$ as

(3a) function value at each vertex $\mathbf{x}$:
\[
p(\mathbf{x}),
\]
(3b) moments of order $\leq k - 2$ on each edge $e$:
\[
\int_e pq, \quad q \in P_{k-2}(e),
\]
(3c) moments of order $\leq k - 3$ in $f$:
\[
\int_f pq, \quad q \in P_{k-3}(f).
\]
Let $B_{k+2,\nabla}(f)$ denote the $H^1$ bubble function space of $P_{k+2}(f)$ with vanishing degrees of freedom $(1a)-(1b)$, let $B_{k+1,\text{rot}}(f)$ denote the $H(\text{rot})$ bubble function space of $P_{k+1}(f;\mathbb{R}^2)$ with vanishing degrees of freedom $(2a)-(2c)$, and let $B_{k,0}(f)$ denote the $L^2$ bubble function space of $P_k(f)$ with vanishing degrees of freedom $(3a)-(3b)$. These bubble functions form exact complexes as in the following lemma.

**Lemma 3.** For $k \geq 3$, it holds that

$$0 \hookrightarrow B_{k+2,\nabla}(f) \xrightarrow{\nabla_f} B_{k+1,\text{rot}}(f) \xrightarrow{\text{rot}_f} B_{k,0}(f)/P_0(f) \rightarrow 0.$$

**Proof.** The proof of Lemma 2 leads to $B_{k+1,\text{rot}}(f) \cap \ker(\text{rot}_f) = \nabla_f B_{k+2,\nabla}(f)$. This results in

$$\dim \text{rot}_f B_{k+1,\text{rot}}(f) = \dim B_{k+1,\text{rot}}(f) - \dim B_{k+2,\nabla}(f)$$

$$= \dim P_{k-3}(f) - 1.$$

Since $B_{k,0}(f) = \lambda_1 \lambda_2 \lambda_3 P_{k-3}(f)$ and the degrees of freedom $(2a)-(2c)$ imply $\text{rot}_f B_{k+1,\text{rot}}(f) \subseteq B_{k,0}(f)/P_0(f)$, the combination with the previous identity concludes the proof. \qed

Given a bounded Lipschitz contractible polygonal domain $\omega \subset \mathbb{R}^2$, the finite elements with respect to the de Rham complex $(5)$ can be constructed with the above shape function spaces and degrees of freedom. The proof of the finite element de Rham complexes follows similar arguments as in [7, Theorem 1] and is omitted for brevity.

### 3.3. Finite element strain complexes in two dimensions

This subsection constructs the finite elements with respect to the strain complex $(6)$.

The shape function space of the $H^1$ vectorial conforming element is $P_{k+2}(f;\mathbb{R}^2)$ with $k \geq 3$. Recall the space $P_{k-1,0}(f)$ of polynomials degree $\leq k-1$ and vanishing at vertices of $f$ from $(9)$. Define $P_{k-1,0}(f;X) := \{v \in P_{k-1}(f;X) \mid \text{each component of } v \text{ is in } P_{k-1,0}(f)\}$ for $X = \mathbb{R}^2$ or $\mathbb{R}^3$. The degrees of freedom read

1. function value and first order derivatives at each vertex $x$:
   $$u(x), \nabla_f u(x), \nabla^2_f u(x),$$
2. moments of order $k-4$ of each component on each edge $e$:
   $$\int_e u \cdot v, \quad v \in P_{k-4}(e;\mathbb{R}^2),$$
3. the interior degrees of freedom in $f$ defined by
   $$\int_f u \cdot v, \quad v \in P_{k-1,0}(f;\mathbb{R}^2).$$

**Lemma 4.** The degrees of freedom $(4a)-(4c)$ are unisolvent for $P_{k+2}(f;\mathbb{R}^2)$.

**Proof.** The proof follows the same arguments as in Lemma 1. \qed

The shape function space of the $H(\text{rot}_f) \otimes (S_2)$ element is $P_{k+1}(f;S_2)$ with $k \geq 3$. Define

$$P_{k-1,2}(f;\mathbb{R}^2) := \{v \in P_{k-1}(f;\mathbb{R}^2) \mid \text{there exists some } w \in P_{k-1,0}(f;\mathbb{R}^2) \\ \text{such that } \text{div}_f(\text{sym}(v)) = w\}.$$

The bijection $\text{div}_f : \text{sym}(P_{k-1}(f;\mathbb{R}^2)x^T) \rightarrow P_{k-1}(f;\mathbb{R}^2)$ guarantees $\dim P_{k-1,2}(f;\mathbb{R}^2) = \dim P_{k-1,0}(f;\mathbb{R}^2)$ (see [5, Lemma 3.6] up to a rotation). The degrees of freedom are given by

$$\dim \text{div}_f B_{k+1,\text{rot}}(f) = \dim B_{k+1,\text{rot}}(f) - \dim B_{k+2,\nabla}(f)$$

$$= \dim P_{k-3}(f) - 1.$$
(5a) function value and first order derivatives at each vertex $x$:

$$\tau(x), \nabla_f \tau(x),$$

(5b) moments of order $\leq k - 3$ of the tangential tangential component on each edge $e$:

$$\int_e t^T \tau q, \quad q \in P_{k-3}(e),$$

(5c) moments of order $\leq k - 2$ of the following derivative on each edge $e$:

$$\int_e (-\epsilon_i (n^T \tau t) + t^T \text{rot}_f \tau)q, \quad q \in P_{k-2}(e),$$

(5d) the interior degrees of freedom in $f$ defined by

$$\int_f \tau : \xi, \quad \xi \in \text{curl}_f \text{curl}_f P_{k-1}(f) + \text{sym}(P_{k-1,2}(f; \mathbb{R}^2)x^T).$$

**Remark 5.** The above finite element space is a modified finite element space of $H(\text{rot})$ as in [5] with additional continuity at vertices.

**Lemma 6.** The degrees of freedom (5a)–(5d) are unisolvent for $P_{k+1}(f; \mathbb{S}_2)$.

**Proof.** A direction computation shows that the number of the degrees of freedom is equal to the dimension $P_{k+1}(f; \mathbb{S}_2)$ with

$$27 + 3(k - 2) + 3(k - 1) + \frac{3k(k + 1)}{2} - 9 = \frac{3(k + 2)(k + 3)}{2}.$$ 

Suppose that (5a)–(5d) vanish for some $\tau \in P_{k+1}(f; \mathbb{S}_2)$. Similar arguments as in [5, Lemma 3.8] show

$$\tau = \epsilon_f (\lambda_1 \lambda_2 \lambda_3 u)$$

for some $u \in P_{k-1}(f; \mathbb{R}^2)$. Since the first order derivatives of $\tau$ vanish at each vertex, this shows $u \in P_{k-1,0}(f; \mathbb{R}^2)$. For any $\xi \in \text{sym}(P_{k-1,2}(f; \mathbb{R}^2)x^T)$, an integration by parts leads to

$$\int_f \tau : \xi = -\int_f \lambda_1 \lambda_2 \lambda_3 u \cdot \text{div}_f \xi.$$ 

The combination with vanishing (5d) and (11) leads to $u = 0$. This concludes the proof. \hfill \qed

Let $B_{k+2,\epsilon_f}(f)$ denote the $H^1$ vectorial bubble function space of $P_{k+2}(f; \mathbb{R}^2)$ with vanishing degrees of freedom (4a)–(4c). Let $B_{k+1,\text{rot}_f \text{rot}_f}(f)$ denote the $H(\text{rot}_f \text{rot}_f, \mathbb{S}_2)$ bubble function space of $P_{k+1}(f; \mathbb{S}_2)$ with vanishing degrees of freedom (5a)–(5c). The bubble function spaces form exact complexes as in the following lemma.

**Lemma 7.** Suppose $k \geq 3$. The complexes

$$0 \xhookrightarrow{} B_{k+2,\epsilon_f}(f) \xhookrightarrow{} B_{k+1,\text{rot}_f \text{rot}_f}(f) \xrightarrow{\text{rot}_f \text{rot}_f} P_{k-1}(f)/P_1(f) \rightarrow 0$$

are exact.
Proof. The proof follows similar arguments as in [5, Lemma 3.9]. Lemma 6 shows $B_{k+1,\text{rot}_f}(f) \cap \ker(\text{rot}_f \text{rot}_f) = \epsilon_f(B_{k+2,\epsilon_f}(f))$. This also means
\[
\dim(\text{rot}_f \text{rot}_f B_{k+1,\text{rot}_f}(f)) = \dim B_{k+1,\text{rot}_f}(f) - \dim B_{k+2,\epsilon_f}(f)
= \frac{1}{2}k(k + 1) - 3 = \dim P_{k-1}(f)/P_1(f).
\]

Given a bounded Lipschitz contractible polygonal domain $\omega \subset \mathbb{R}^2$, the finite elements with respect to the strain complex (6) can be constructed with the above shape function spaces and degrees of freedom. The proof of the finite element strain complexes follows similar arguments as in [5, Lemma 3.10] and is omitted here. A rotation will lead to the finite element $\text{div div}$ complexes in two dimensions which are modified ones from [5].

4. Finite element spaces in three dimensions. This section constructs a family of $H(\text{sym curl}, \mathbb{T})$ tensor conforming finite elements on tetrahedral grids. The definitions of the degrees of freedom on faces will employ the polynomial spaces defined in Section 3. Although those spaces are defined when $f$ is on the $x-y$ plane, they can be extended to a three dimensional face $f$ with $\mathbf{x}$ replaced by $\Pi_f \mathbf{x}$ and $\mathbf{x}^\perp$ replaced by $\mathbf{n} \times \mathbf{x}$. It is proved that the $H(\text{sym curl}, \mathbb{T})$ tensor conforming finite elements form finite element $\text{div div}$ complexes with the $H(\text{div div}, \mathbb{S})$ tensor conforming elements in [6] and the newly constructed $H^1$ vectorial conforming elements in this paper.

4.1. Further notation for three dimensions. The space $RT$ in three dimensions reads as
\[
RT := \{a + b\mathbf{x} \mid a \in \mathbb{R}^3, b \in \mathbb{R}\}.
\]
Suppose that $\Omega$ is a bounded, strong Lipschitz and contractible domain. Define
\[
\begin{align*}
H(\text{sym curl}, \Omega; \mathbb{T}) &:= \{\tau \in L^2(\Omega; \mathbb{T}) \mid \text{sym curl} \tau \in L^2(\Omega; \mathbb{S})\}, \\
H(\text{div div}, \Omega; \mathbb{S}) &:= \{\tau \in L^2(\Omega; \mathbb{S}) \mid \text{div} \tau \in L^2(\Omega)\}.
\end{align*}
\]
Recall the $\text{div div}$ complex in three dimensions [2, 15] from (3)
\[
(12) \quad RT \xrightarrow{\text{dev} \nabla} H^1(\Omega; \mathbb{R}^3) \xrightarrow{\text{sym curl}} H(\text{sym curl}, \Omega; \mathbb{T}) \xrightarrow{\text{sym curl}} H(\text{div div}, \Omega; \mathbb{S}) \xrightarrow{\text{div} \text{div}} L^2(\Omega) \to 0,
\]
and the polynomial complex [6]
\[
(13) \quad RT \xleftarrow{\text{dev} \nabla} P_{k+2}(\Omega; \mathbb{R}^3) \xrightarrow{\text{sym curl}} P_{k+1}(\Omega; \mathbb{T}) \xrightarrow{\text{sym curl}} P_k(\Omega; \mathbb{S}) \xrightarrow{\text{div} \text{div}} P_{k-2}(\Omega) \to 0.
\]
Let $\mathcal{T}$ be a shape regular triangulation of $\Omega \subset \mathbb{R}^3$ into tetrahedra. Let $\mathcal{V}$ denote the set of all vertices, $\mathcal{E}$ the set of all edges and $\mathcal{F}$ the set of all faces. Given $e \in \mathcal{E}$, let $\mathbf{t}$ denote the unit tangential vector along $e$, and let $\mathbf{n}_1$ and $\mathbf{n}_2$ denote two independent unit normal vectors such that $\mathbf{n}_1 \times \mathbf{n}_2 = \mathbf{t}$. Given $f \in \mathcal{F}$, let $\mathbf{n}$ denote the unit normal vector of $f$, and let $\mathbf{n}_{\partial f}$ denote the outward normal vector of $\partial f$ on $f$ and $\mathbf{t}_{\partial f}$ denote the unit tangential vector of $\partial f$ such that $\mathbf{n}_{\partial f} \times \mathbf{t}_{\partial f} = \mathbf{n}$. Given $K \in \mathcal{T}$, let $\lambda_j$ with $1 \leq j \leq 4$ denote the barycentric coordinates of $K$.  

4.2. $H(\text{sym curl}, \mathbb{T})$ conforming finite elements. Recall the space $\mathbb{T} = \{\tau \in \mathbb{R}^{3 \times 3} \mid \text{tr}(\tau) = 0\}$ of trace free matrices. The shape function space of the $H(\text{sym curl}, \mathbb{T})$ element is $P_{k+1}(K; \mathbb{T})$ with $k \geq 3$. Recall the operators $\Pi_f$ and $\Pi_f, \text{sym}$ from Section 2
for face \( f \in \mathcal{F} \) with the unit normal vector \( n \). The degrees of freedom on the faces require the polynomial spaces \( P_{k-1,0}(f) \) from (9), \( P_{k-1,1}(f) \) from (10) and \( P_{k-1,2}(f; \mathbb{R}^2) \) from (11), which are extended to three dimensional faces as follows

\[
P_{k-1,1}(f) := \{ q \in P_{k-1}(f) \mid \text{there exists some } r \in P_{k-1,0}(f) \text{ such that } \text{div}_f(q\Pi_f x) = r \},
\]

and

\[
P_{k-1,2}(f; \mathbb{R}^2) := \{ v \in \Pi_f P_{k-1}(f; \mathbb{R}^3) \mid \text{there exists some } w \in \Pi_f P_{k-1,0}(f; \mathbb{R}^3) \text{ such that } \text{div}_f(\text{sym}(v(\Pi_f x)^T)) = w \}.
\]

The degrees of freedom of the \( H(\text{sym curl}, \mathbb{T}) \) element are defined as follows:

- (7a) function value and first order derivatives of each component at each vertex \( x \in \mathcal{V} \):
  \[
  \tau(x), \nabla \tau(x),
  \]

- (7b) moments of order \( \leq k - 3 \) of the following components on each edge \( e \in \mathcal{E} \):
  \[
  \int_e n_i^T \tau q, \quad q \in P_{k-3}(e), i = 1, 2,
  \]

- (7c) moments of order \( \leq k - 2 \) of the following derivatives on each edge \( e \in \mathcal{E} \):
  \[
  \int_e n_i^T \text{sym curl } \tau n_j q, \quad q \in P_{k-2}(e), i, j = 1, 2,
  \]
  \[
  \int_e (n_i^T \text{curl } \tau n_2 - \partial_i(t^T \tau)) q, \quad q \in P_{k-2}(e),
  \]

- (7d) degrees of freedom on each face \( f \in \mathcal{F} \) defined by
  \[
  \int_f \Pi_f(\tau^T n) \cdot v, \quad v \in \text{curl}_f P_{k-3}(f) + P_{k-1,1}(f)\Pi_f x,
  \]

- (7e) degrees of freedom on each face \( f \in \mathcal{F} \) defined by
  \[
  \int_f \Pi_{f, \text{sym}}(\tau \times n) : \xi, \quad \xi \in \text{curl}_f \text{curl}_f P_{k-1}(f) + \text{sym}(P_{k-1,2}(f; \mathbb{R}^2)(\Pi_f x)^T),
  \]

- (7f) interior degrees of freedom in each element \( K \in \mathcal{T} \) defined by
  \[
  \int_K \text{sym curl } \tau : \xi, \quad \xi \in \text{sym}(x \times P_{k-2}(K; \mathbb{T})),
  \]
  \[
  \int_{f_1} \text{sym curl } \tau n \cdot v, \quad v \in P_{k-2}(f_1)(n \times x) \text{ for an arbitrarily but fixed face } f_1,
  \]
  \[
  \int_K \tau : \xi, \quad \xi \in \text{dev}(P_{k-2}(K; \mathbb{R}^3)x^T).
  \]

**Remark 8.** The first degrees of freedom in (7c) plus (7a) imply the continuity of \( n_i^T \text{sym curl } \tau n_j^T \). This, the second degrees of freedom in (7c) plus (7a) imply the continuity of \( n_i^T \text{curl } \tau n_2 - \partial_i(t^T \tau) \) and \( n_i^T \text{curl } \tau n_1 + \partial_i(t^T \tau) \). Besides, they are independent of the choices of the normal vectors. Suppose that there are another two
unit normal vectors \( n_1' = c_1 n_1 + c_2 n_2 \) and \( n_2' = -c_2 n_1 + c_1 n_2 \) such that \( n_1' \times n_2' = t \). An elementary computation leads to the continuity of \((n_1')^T \text{curl} \tau n_2'\). The continuity of \((n_1')^T \text{curl} \tau n_2' - \partial_t (t^T \tau t)\) follows from

\[
(n_1')^T \text{curl} \tau n_2' - \partial_t (t^T \tau t) = -c_1 c_2 n_1^T \text{curl} \tau n_1 + c_1 c_2 n_2^T \text{curl} \tau n_2 + c_1^2 n_1^T \text{curl} \tau n_2 - c_2^2 n_2^T \text{curl} \tau n_1 - (c_1^2 + c_2^2) \partial_t (t^T \tau t).
\]

The first two degrees of freedom in (7f) are motivated from those of the \(H(\text{div} \text{div}, S)\) conforming finite elements in \( [6] \).

The proof of unisolvence of the degrees of freedom (7a)-(7f) requires the following three lemmas. The identities in the first two lemmas present the restrictions of functions and operators on faces and show some connections with the finite elements in two dimensions from Section 3. The third lemma from [4] will be used to deal with interior degrees of freedom.

**Lemma 9.** Given \( f \) with two unit tangential vectors \( t_1 \) and \( t_2 \) such that \( t_1 \times t_2 = n \), it holds that

\[
\begin{align*}
(14) & \quad \Pi_f (\tau^T n) \cdot t_2 = n^T \tau t_2, \\
(15) & \quad \text{rot}_f \Pi_f (\tau^T n) = n^T \text{curl} \tau n, \\
(16) & \quad t_2^T \Pi_{f, \text{sym}} (\tau \times n) t_2 = -t_1^T \tau t_2, \\
(17) & \quad -\partial_{t_2} (t_1^T \Pi_{f, \text{sym}} (\tau \times n) t_2) + t_2^T \text{rot}_f \Pi_{f, \text{sym}} (\tau \times n) = -t_1^T \text{curl} \tau n - \partial_{t_2} (t_2^T \tau t_2).
\end{align*}
\]

**Proof.** Let \( w = \Pi_f (\tau^T n) \). A direct computation shows

\[
\begin{align*}
w \cdot t_2 & = \tau^T n \cdot t_2 = n^T \tau t_2, \\
\text{rot}_f w & = n \cdot \text{curl}(\tau^T n) = n^T \text{curl} \tau n.
\end{align*}
\]

This proves (14)–(15). Let \( \zeta = \Pi_{f, \text{sym}} (\tau \times n) \). The cross product rule plus \( t_1 \times t_2 = n \) show

\[
t_2^T \zeta t_2 = t_2^T (\tau \times n) t_2 = -t_1^T \tau t_2.
\]

This proves (16). Some elementary computations lead to

\[
\begin{align*}
t_1^T \zeta t_2 & = \frac{1}{2} (t_1^T \tau t_2 - t_1^T \tau t_1), \\
t_2^T \text{rot}_f (\Pi_f (\tau \times n)) & = \text{rot}_f ((\Pi_f (\tau \times n))^T t_2) \\
& = -\text{rot}_f (\tau^T t_1) = -t_1^T \text{curl} \tau n, \\
t_2^T \text{rot}_f ((\Pi_f (\tau \times n))^T) & = \text{rot}_f ((\Pi_f (\tau \times n)) t_2) = \text{rot}_f (\tau t_2 \times n) \\
& = -\text{div}_f (\tau t_2) = -\partial_{t_2} (t_2^T \tau t_2) - \partial_{t_2} (t_1^T \tau t_2).
\end{align*}
\]

The previous three identities plus \( \text{rot}_f (\tau^T t_1) = -\partial_{t_2} (t_1^T \tau t_1) + \partial_{t_1} (t_1^T \tau t_2) \) lead to

\[
-\partial_{t_2} (t_1^T \zeta t_2) + t_2^T \text{rot}_f \zeta = -t_1^T \text{curl} \tau n - \partial_{t_2} (t_2^T \tau t_2).
\]

This proves (17).

**Lemma 10.** Let \( f \in \mathcal{F} \) with the unit normal vector \( n \). (a) Suppose \( \xi = \text{sym} \text{curl} \tau \).

Then,

\[
\begin{align*}
(18) & \quad n^T \xi n = \text{rot}_f (\Pi_f (\tau^T n)), \\
(19) & \quad 2 \text{div}_f (\xi n) + \partial_n (n^T \xi n) = -\text{rot}_f \text{rot}_f \Pi_{f, \text{sym}} (\tau \times n).
\end{align*}
\]
(b) Suppose $\boldsymbol{\tau} = \text{dev} \nabla \mathbf{v}$. Then,

$$
\text{(20)} \quad \Pi_f(\tau^T \mathbf{n}) = \nabla_f(\mathbf{v} \cdot \mathbf{n}),
$$

$$
\text{(21)} \quad \Pi_{f,\text{sym}}(\boldsymbol{\tau} \times \mathbf{n}) = \epsilon_f(\mathbf{v} \times \mathbf{n}).
$$

**Proof.** Proof of (a). The first identity (18) follows from the definition of rot$\frown$ with

$$\mathbf{n}^T \text{curl} \mathbf{\tau n} = \text{curl}(\tau^T \mathbf{n}) \cdot \mathbf{n} = \text{rot}_f(\tau^T \mathbf{n}) = \text{rot}_f(\Pi_f(\tau^T \mathbf{n})).$$

The second term on the left-hand side of (19) satisfies

$$\partial_n(\mathbf{n}^T \text{curl} \mathbf{\tau n}) = \partial_n(\text{rot}_f(\tau^T \mathbf{n})) = \text{rot}_f(\partial_n(\tau^T \mathbf{n})).$$

For curl $\mathbf{\tau n}$,

$$\text{div}_f(\text{curl} \mathbf{\tau n}) = \text{rot}_f(\mathbf{n} \times \text{rot}_f \tau) = -\text{rot}_f \text{rot}_f(\tau \times \mathbf{n}).$$

As for $(\text{curl} \mathbf{\tau})^T \mathbf{n} = \text{curl}(\tau^T \mathbf{n})$, the cross product rule leads to

$$\text{div}_f((\text{curl} \mathbf{\tau})^T \mathbf{n}) = \text{rot}_f(\mathbf{n} \times \text{curl}(\tau^T \mathbf{n}))$$

$$= \text{rot}_f(\nabla(\mathbf{n}^T \mathbf{\tau n}) - \partial_n(\tau^T \mathbf{n})) = -\text{rot}_f(\partial_n(\tau^T \mathbf{n})).$$

The previous arguments lead to

$$2 \text{div}_f(\mathbf{\xi n}) + \partial_n(\mathbf{n}^T \mathbf{\xi n}) = -\text{rot}_f \text{rot}_f(\mathbf{\tau} \times \mathbf{n}).$$

Since

$$\text{rot}_f \text{rot}_f(\mathbf{\tau} \times \mathbf{n}) = \text{rot}_f \text{rot}_f(\Pi_f(\mathbf{\tau} \times \mathbf{n})) = \text{rot}_f \text{rot}_f((\Pi_f(\mathbf{\tau} \times \mathbf{n}))^T),$$

the combination with the previous identity concludes (19).

**Proof of (b).** Since $\Pi_f \mathbf{n} = 0$, it holds that

$$\Pi_f(\tau^T \mathbf{n}) = \Pi_f(\nabla(\mathbf{\tau}^T \mathbf{n} = \Pi_f(\nabla(\mathbf{\tau} \cdot \mathbf{n}) = \nabla_f(\mathbf{\tau} \cdot \mathbf{n}).$$

This proves (20). Since $I \times \mathbf{n} = \mathbf{n} \times I = -(\mathbf{n} \times I)^T$, this leads to

$$\Pi_{f,\text{sym}}(\mathbf{\tau} \times \mathbf{n}) = \Pi_{f,\text{sym}}(\nabla(\mathbf{\tau} \times \mathbf{n}) = \Pi_{f,\text{sym}}(\nabla(\mathbf{\tau} \times \mathbf{n})) = \epsilon_f(\mathbf{\tau} \times \mathbf{n}).$$

This proves (21).

**Lemma 11 ([14]).** It holds that

$$\text{div} : \text{dev}(P_k(\Omega; \mathbb{R}^3)) \xrightarrow{x^T} \Omega(\Omega; \mathbb{R}^3).$$

is a bijection.

**Theorem 12.** The degrees of freedom (7a)-(7f) are unisolvent for $P_{k+1} (K; T)$.

**Proof.** Note that $\dim(\text{sym}(x \times P_{k-2}(K; T))) = \frac{k(k-1)(5k+14)}{12}$ from [6, Lemma 4.6].

A direct computation shows that the number of the degrees of freedom is equal to the dimension of $P_{k+1} (K; T)$, namely

$$128 + 12(k - 2) + 24(k - 1) + 4(k^2 - k - 3) + 4 \left( \frac{3k(k + 1)}{2} - 9 \right)$$

$$+ \frac{k(k - 1)(5k + 14)}{6} + \frac{k(k - 1)}{2} + \frac{(k - 1)k(k + 1)}{2} = 4(k + 2)(k + 3)(k + 4).$$

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It suffices to prove if (7a)–(7e) vanish for \( \tau \in P_{k+1}(K; T) \) then \( \tau = 0 \). The degrees of freedom (7a)–(7b) show, on each edge \( e \in \mathcal{E}(K) \),

\[
(22) \quad n_i^T \tau t = 0, \quad i = 1, 2.
\]

Given \( f \in \mathcal{F}(K) \), let \( w = \Pi_f(\tau^T n) \) and \( \zeta = \Pi_f,\text{sym}(\tau \times n) \). Lemma 9 with \( t_1 = n_{\partial f} \) and \( t_2 = t_{\partial f} \) shows

\[
w \cdot t_{\partial f} = n_i^T \tau t_{\partial f} \quad \text{and} \quad \text{rot}_f w = n_i^T \text{curl } \tau n = n_i^T \text{sym curl } \tau n.
\]

The combination with (22), (7a), (7b) and the first degrees of freedom in (7c) leads to

\[
(23) \quad w \cdot t_{\partial f} = 0 \quad \text{and} \quad \text{rot}_f w = 0 \text{ on } \partial f.
\]

The degrees of freedom (7d) combined with similar arguments as in Lemma 2 for \( f \) on the \( x-y \) plane lead to \( w = 0 \) on \( f \). On the other hand, (16)–(17) show

\[
t_{\partial f}^T \zeta t_{\partial f} = -n_{\partial f}^T \tau t_{\partial f},
\]

\[
-\partial_{t_{\partial f}}(n_{\partial f}^T \zeta t_{\partial f}) + t_{\partial f}^T \text{rot}_f \zeta = -n_{\partial f}^T \text{curl } \tau n - \partial_{t_{\partial f}}(t_{\partial f}^T \tau t_{\partial f}).
\]

Hence the former identity plus (7a) and (7b) show \( t_{\partial f}^T \zeta t_{\partial f} = 0 \), and the latter identity plus (7a) and (7c) show \( -\partial_{t_{\partial f}}(n_{\partial f}^T \zeta t_{\partial f}) + t_{\partial f}^T \text{rot}_f \zeta = 0 \) on \( \partial f \). The degrees of freedom (7e) combined with similar arguments as in Lemma 6 lead to \( \zeta = 0 \). The cross product rules show

\[
(n \times \tau + (n \times \tau)^T) n = n \times (\tau^T n),
\]

\[
\text{and} \quad n \times (n \times \tau + (n \times \tau)^T) = n = -\Pi_f(\tau \times n) - (\Pi_f(\tau \times n))^T.
\]

The combination with \( w = 0 \) and \( \zeta = 0 \) implies

\[
(24) \quad n \times \tau + (n \times \tau)^T = 0 \text{ on } f.
\]

Let \( \xi = \text{sym curl } \tau \). For any \( f \in \mathcal{F}(K) \) with the unit normal vector \( n \), it follows from Lemma 10 that

\[
n_i^T \xi n = \text{rot}_f(\Pi_f(\tau^T n)) = 0,
\]

\[
2 \text{div}_f(\xi n) + \partial_n(n_i^T \xi n) = -\text{rot}_f \text{rot}_f \Pi_f,\text{sym}(\tau \times n) = 0.
\]

For any \( e \in \mathcal{E}(K) \),

\[
n_i^T \xi n_j = n_i^T \text{sym curl } \tau n_j = 0, \quad i, j = 1, 2.
\]

Hence \( \xi \) is a \( H(\text{div div}) \) bubble function and \( \text{div div } \xi = 0 \). The first two degrees of freedom in (7f) and similar arguments as in [6, Lemma 4.6] lead to \( \xi = 0 \). Then the polynomial complex (13) shows that there exists \( u \in P_{k+2}(K; \mathbb{R}^3) \) such that

\[
(25) \quad \tau = \text{dev } \nabla u \quad \text{and} \quad Q_0^I(u \cdot n) = 0 \text{ for all } f \in \mathcal{F}(K)
\]

with the \( L^2 \) projection \( Q_0^I \) onto \( P_0(f) \).

Given \( f \in \mathcal{F}(K) \) with the unit normal vector \( n \), the boundary conditions \( \Pi_f(\tau^T n) = 0 \) and \( \Pi_{f,\text{sym}}(\tau \times n) = 0 \) for \( \tau \), Lemma 10 and (25) lead to \( u \cdot n = 0 \), and \( \epsilon_f(u \times n) = 0 \)
on $f$. This shows $u = 0$ at each vertex of $K$ and $u \times n$ is a linear function on $f$. Furthermore, $u \times n = 0$ on $f$ and thus $u = 0$ on $\partial K$. This leads to the existence of some $v \in P_{k-2}(K; \mathbb{R}^3)$ with

$$u = \lambda_1 \lambda_2 \lambda_3 \lambda_4 v.$$  

For any $\chi \in \text{dev}(P_{k-2}(K; \mathbb{R}^3)x^T)$, this and an integration by parts lead to

$$\int_K \tau : \chi = - \int_K \lambda_1 \lambda_2 \lambda_3 \lambda_4 v \cdot \text{div} \chi.$$  

The combination with the bijection in Lemma 11 and the third vanishing degrees of freedom in (7f) implies $v = 0$ and hence $\tau = 0$. This concludes the proof. \hfill \Box

**Remark 13.** Let $t_1$ and $t_2$ denote two independent unit tangential vectors of $f$. Define the space of traceless matrices related to the face $f$ as follows

$$T_f := \text{span}\{t_1 n^T, t_2 n^T, nn^T - \frac{1}{3} I\}.$$  

Then the $H(\text{sym curl}, \mathbb{T})$ bubble function space with respect to the degrees of freedom (7a)–(7e) on $K$ reads

$$B_{k+1, \text{sym curl}}(K) := \lambda_1 \lambda_2 \lambda_3 \lambda_4 P_{k-3}(K; \mathbb{T}) + \sum_{f \in \mathcal{F}(K)} \lambda_{f,1} \lambda_{f,2} \lambda_{f,3} P_{k-2}(f) T_f$$  

with the barycentric coordinates $\lambda_{f,1}, \lambda_{f,1}, \lambda_{f,3}$ with respect to $f$. In fact, it follows from (24) that $\tau = 0$ on all edges $e \in \mathcal{E}(K)$. Then

$$\tau \in \sum_{1 \leq i < j < k \leq 4} \lambda_i \lambda_j \lambda_k P_{k-2}(K; \mathbb{T}).$$

If $\tau \neq 0$ on some $f$, then $\tau|_f \in \lambda_{f,1} \lambda_{f,2} \lambda_{f,3} P_{k-2}(f; \mathbb{T})$. Since $n \times \tau + (n \times \tau)^T|_f = 0$, this leads to $\tau|_f \in \lambda_{f,1} \lambda_{f,2} \lambda_{f,3} P_{k-3}(f)$ for all $f \in \mathcal{F}(K)$, then $\tau \in \lambda_1 \lambda_2 \lambda_3 \lambda_4 P_{k-4}(K; \mathbb{T})$.

The proof of (24) in Theorem 12 implies $(n \times \tau) + (n \times \tau)^T$ is continuous across faces if the degrees of freedom (7a)–(7f) are single-valued. This allows the definition of the following $H(\text{sym curl}, \mathbb{T})$ conforming finite element space $\Lambda_{k+1,b}$ with $k \geq 3$ by

$$\Lambda_{k+1,b} := \{\tau_h \in H(\text{sym curl}, \Omega; \mathbb{T}) \mid \tau_h|_K \in P_{k+1}(K; \mathbb{T}) \text{ for all } K \in \mathcal{T}, \text{ all the degrees of freedom (7a)–(7f) are single-valued}\}.$$  

**4.3. Finite element div div complexes.** Recall the $H(\text{div div})$ finite element spaces from [6]. The shape function space is $P_k(K; \mathcal{S})$ with $k \geq 3$ and the degrees of freedom are defined by

- (8a) function value at each vertex $x \in \mathcal{V}$:

  $$\tau(x),$$

- (8b) moments of order $\leq k - 2$ of the following components on each edge $e \in \mathcal{E}$:

  $$\int_e n_i^T \tau n_j g, \quad q \in P_{k-2}(e), i, j = 1, 2,$$
(8c) moments of order $\leq k - 3$ of the normal normal component on each face $f \in \mathcal{F}$:

$$\int_f n^T \tau q, \quad q \in P_{k-3}(f),$$

(8d) moments of order $\leq k - 1$ of the following derivative on each face $f \in \mathcal{F}$:

$$\int_f (2 \text{div}_f(\tau n) + \partial_n(n^T \tau n))q, \quad q \in P_{k-1}(f),$$

(8e) interior degrees of freedom in each element $K \in \mathcal{T}$ defined by

$$\int_K \tau : \xi, \quad \xi \in \nabla^2 P_{k-2}(K) + \text{sym}(x \times P_{k-2}(K; \mathbb{T})), $$

(8f) interior degrees of freedom in each element $K \in \mathcal{T}$ defined by

$$\int_{f_1} \tau n \cdot v, \quad v \in P_{k-2}(f_1)(n \times x) \text{ for an arbitrarily but fixed face } f_1.$$
It suffices to prove if (9a)–(9d) vanish for \( u \in P_{m+1}(K; \mathbb{R}^3) \) then \( u = 0 \). For each \( f \in \mathcal{F}(K) \), due to \( u \in P_{k+2}(f; \mathbb{R}^3) \), similar arguments as in Lemma 1 plus (9a)–(9c) lead to \( u = 0 \) on \( f \). The zero boundary condition \( u = 0 \) on \( \partial K \) shows that there exists some \( w \in P_{k-2}(K; \mathbb{R}^3) \) such that
\[
 u = \lambda_1 \lambda_2 \lambda_3 \lambda_4 w.
\]
This and the degrees of freedom in (9d) lead to \( u \equiv 0 \) in \( K \). This concludes the proof.

The conforming finite element space \( V_{k+2,h} \subset H^1(\Omega; \mathbb{R}^3) \) is defined as
\[
 V_{k+2,h} := \{ v_h \in H^1(\Omega; \mathbb{R}^3) \mid v_h|_K \in P_{k+2}(K; \mathbb{R}^3) \text{ for all } K \in \mathcal{T}, \text{ all the degrees of freedom (9a)–(9d) are single-valued} \}.
\]

Before establishing the conforming finite element \( \text{div div} \) complexes with respect to (4), the following theorem proves that the bubble function spaces on each element \( K \) form exact complexes. Recall the bubble function space \( B_{k+1, \text{sym curl}}(K) \) of \( H(\text{sym curl}, \mathcal{T}) \) from (26). Let \( B_{k+2, \text{dev}} \mathcal{V}(K) \) denote the bubble function space of the vectorial \( H^1 \) space with vanishing degrees of freedom (9a)–(9c). It is easy to check that
\[
 B_{k+2, \text{dev}} \mathcal{V}(K) := \lambda_1 \lambda_2 \lambda_3 \lambda_4 P_{k-2}(K; \mathbb{R}^3).
\]
Let \( B_{k, \text{div div}}(K) \) denote the bubble function space of \( H(\text{div div}, \mathcal{S}) \) with vanishing degrees of freedom (8a)–(8c). The following lemma plus (b) of Lemma 10 imply the inclusion \( \text{div} \mathcal{V} \subset B_{k+1, \text{sym curl}}(K) \).

**Lemma 15.** Suppose \( \tau = \text{dev} \nabla v \). Then, on edge \( e \) with the unit tangential vector \( t = n_1 \times n_2 \),
\[
 (30) \quad n_i^T \text{sym curl} \tau n_j = 0, \quad i, j = 1, 2,
\]
\[
 (31) \quad n_i^T \text{curl} \tau n_2 - \partial_i (t^T \tau t) = -\partial_i^2 (v \cdot t).
\]

**Proof.** Given \( e \), \( \text{sym curl} \tau = 0 \) results in (30). Note that
\[
 n_i^T \text{curl} \tau n_2 - \partial_i (t^T \tau t) = \frac{1}{3} n_i^T \text{curl}(\text{div} \mathcal{V}) n_2 - \partial_i (t^T \nabla v t - \frac{1}{3} \text{div} v).
\]
The cross product rule plus \( t = n_1 \times n_2 \) lead to
\[
 n_i^T \text{curl}(\text{div} \mathcal{V}) n_2 = \text{curl} ((\text{div} v) n_1) \cdot n_2 = \text{div} ((\text{div} v) n_2 \times n_1)
\]
\[
 = -\text{div}((\text{div} v) t) = -\partial_i (\text{div} v).
\]
The previous two identities conclude (31).

**Theorem 16.** For \( k \geq 3 \), it holds that
\[
 0 \subset B_{k+2, \text{dev}} \mathcal{V}(K) \xrightarrow{\text{div} \nabla} B_{k+1, \text{sym curl}}(K) \xrightarrow{\text{sym curl}} B_{k, \text{div div}}(K) \xrightarrow{\text{div div}} P_{k-1}(K)/P_1(K) \rightarrow 0.
\]

**Proof.** Lemma 15 plus (b) of Lemma 10 show the inclusion \( \text{div} \mathcal{V} \subset B_{k+1, \text{sym curl}}(K) \) and the proof of Theorem 12 shows the inclusion \( \text{sym curl} B_{k+1, \text{sym curl}}(K) \subset B_{k, \text{div div}}(K) \). Suppose \( \tau \in \Lambda_{k+1,0}(K) \) and \( \text{sym curl} \tau = 0 \). The proof of Theorem 12 shows that there exists some \( u \in B_{k+2, \text{dev}} \mathcal{V}(K) \) such that \( \tau = \text{dev} \nabla u \).
The previous arguments show \( B_{k+1,\text{sym curl}}(K) \cap \ker(\text{sym curl}) = \text{dev} \nabla B_{k+2,\text{dev}} \nabla(K) \). This also means
\[
\dim \text{sym curl} B_{k+1,\text{sym curl}}(K) = \dim B_{k+1,\text{sym curl}}(K) - \dim B_{k+2,\text{dev}} \nabla(K) = \frac{k(k-1)(5k+14)}{6} + \frac{k(k-1)}{2}.
\]
It has been proved in [6] that \( \text{div div} B_{k,\text{div div}}(K) = P_{k-2}(K)/P_1(K) \). This implies
\[
\dim(B_{k,\text{div div}}(K) \cap \ker(\text{div div})) = \dim B_{k,\text{div div}}(K) - \dim P_{k-2}(K)/P_1(K) = \frac{k(k-1)(5k+14)}{6} + \frac{k(k-1)}{2}.
\]
This concludes that the complexes are exact.

**Theorem 17.** For \( k \geq 3 \), it holds that
\[
RT \subseteq V_{k+2,h} \xrightarrow{\text{dev} \nabla} \Lambda_{k+1,h} \xrightarrow{\text{sym curl}} \Sigma_{k,h} \xrightarrow{\text{div div}} P_{k-2}(T) \to 0.
\]

**Proof.** Similar arguments as in the proof of Theorem 16 show the inclusions \( \nabla V_{k+2,h} \subset \Lambda_{k+1,h} \) and \( \text{sym curl} \Lambda_{k+1,h} \subset \Sigma_{k,h} \). Suppose \( \tau \in \Lambda_{k+1,h} \) and \( \text{sym curl} \tau = 0 \). The continuous complex (12) shows that there exists \( u \in H^1(\Omega; \mathbb{R}^3)/RT \) such that
\[
\tau = \text{dev} \nabla u.
\]
As shown in the proof of [6, Lemma 3.2], \( u \in P_{k+2}(K; \mathbb{R}^3) \) for any \( K \in T \). Since \( u \) is a discrete function, \( u \) is continuous at vertices, and on edges and faces. It only remains to prove the extra continuity of the first and second order derivatives of \( u \) at vertices. It is shown in [6, Lemma 3.2]
\[
\text{mspn} (\nabla \text{div} u) = 3 \text{curl} \tau.
\]
This leads to the continuity of \( \nabla(\text{div} u) \) at vertices and hence leads to the continuity of \( \nabla^2 u \) at vertices. Given \( f \in \mathcal{F} \), the continuity of \( \nabla f(u \cdot n) \) and \( \epsilon_f(u \times n) \) imply the continuity of \( \nabla u \) at vertices. The previous arguments lead to \( u \in V_{k+2,h} \) and consequently \( \Lambda_{k+1,h} \cap \ker(\text{sym curl}) = \text{dev} \nabla V_{k+2,h} \). This also means
\[
\dim \text{sym curl} \Lambda_{k+1,h} = \dim \Lambda_{k+1,h} - \dim V_{k+2,h}/RT = 2\#\mathcal{V} + (3k+1)\#\mathcal{E} + (k^2-k-3)\#\mathcal{F} + \frac{5k^3 + 12k^2 - 17k}{6} \#\mathcal{T} + 4.
\]
It has been proved in [6] that \( \text{div div} \Sigma_{k,h} = P_{k-2}(T) \). This shows
\[
\dim(\Sigma_{k,h} \cap \ker(\text{div div})) = \dim \Sigma_{k,h} - \dim Q_{k-2,h} = 6\#\mathcal{V} + (3k-3)\#\mathcal{E} + (k^2 - k + 1)\#\mathcal{F} + \frac{5k^3 + 12k^2 - 17k - 24}{6} \#\mathcal{T}.
\]
The Euler’s formula is \( \#V - \#E + \#F - \#K - 1 = 0 \) for simple domains. Since \( \text{sym curl} \Lambda_{k+1,h} \subseteq \Sigma_{k,h} \cap \ker(\text{div div}) \), this concludes that the complexes are exact.

5. The Dual formulation of the linearized Einstein-Bianchi system.
This section considers the discretization of (1) in a weak formulation, which is dual to the formulation in [11]. The error analysis is provided by following similar arguments as in [11].

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5.1. Weak formulation. The dual formulation of the linearized Einstein-Bianchi system introduced in [11] reads: Find
\[
\begin{align*}
\sigma & \in C^1([0, T], L^2(\Omega)), \\
E & \in C^0([0, T], H(\text{div div}, \Omega; \mathbb{S})) \cap C^1([0, T], L^2(\Omega; \mathbb{S})), \\
B & \in C^0([0, T], H(\text{sym curl}, T)) \cap C^1([0, T], L^2(\Omega; T)),
\end{align*}
\]
such that
\[
\begin{align*}
\left\langle (\dot{\sigma}, q) \right\rangle &= (\text{div div} E, q), \quad \text{for any } q \in L^2(\Omega), \\
\left\langle (\dot{E}, \xi) \right\rangle &= -\left\langle (\sigma, \text{div div} \xi) - (\text{sym curl} B, \xi) \right\rangle, \quad \text{for any } \xi \in H(\text{div div}, \Omega; \mathbb{S}), \\
\left\langle (B, \zeta) \right\rangle &= (E, \text{sym curl} \zeta), \quad \text{for any } \zeta \in H(\text{sym curl}, \Omega; T).
\end{align*}
\]

For \( k \geq 3 \), the semidiscretization of (33) finds
\[
\sigma_h \in C^1([0, T], P_{k-2}(T)), \quad E_h \in C^0([0, T], c_{k,h}) \text{ and } B_h \in C^0([0, T], c_{k+1,h})
\]
such that
\[
\begin{align*}
\left\langle (\dot{\sigma}_h, q) \right\rangle &= (\text{div} E_h, q), \quad \text{for any } q \in P_{k-2}(T), \\
\left\langle (\dot{E}_h, \xi) \right\rangle &= -\left\langle (\sigma_h, \text{div div} \xi) - (\text{sym curl} B_h, \xi) \right\rangle, \quad \text{for any } \xi \in c_{k,h}, \\
\left\langle (B_h, \zeta) \right\rangle &= (E_h, \text{sym curl} \zeta), \quad \text{for any } \zeta \in c_{k+1,h},
\end{align*}
\]
for all \( t \in (0, T] \), with given initial data.

**Theorem 18.** There exists a unique solution to (34).

Proof. The proof follows the same argument as in [11, Theorem 6.1]. 

Below investigates the convergence of the discrete solutions. Let \( V := L^2(\Omega) \times H(\text{div div}, \Omega; \mathbb{S}) \times H(\text{sym curl}, \Omega; T) \) with the norm
\[
\| (q, \xi, \zeta) \|_V := \| q \|_{L^2(\Omega)} + \| \xi \|_{H(\text{div div}, \Omega)} + \| \zeta \|_{H(\text{sym curl}, \Omega)}
\]
for \( (q, \xi, \zeta) \in V \). Let \( V_h := P_{k-2}(T) \times c_{k,h} \times c_{k+1,h} \). Define the bilinear form
\[
A(\sigma, E, B; q, \xi, \zeta) = \left\langle (\sigma, q) + (E, \xi) + (B, \zeta) - (\text{div div} E, q) + (\sigma, \text{div div} \xi) + (\text{sym curl} B, \xi) - (E, \text{sym curl} \zeta) \right\rangle,
\]
The following theorem shows the inf-sup condition of \( A \) in \( V_h \times V_h \).

**Theorem 19.** The bilinear form \( A \) satisfies the inf-sup condition
\[
\inf_{0 \neq (\sigma, E, B) \in V_h} \sup_{0 \neq (q, \xi, \zeta) \in V_h} \frac{A(\sigma, E, B; q, \xi, \zeta)}{\| (\sigma, E, B) \|_V \| (q, \xi, \zeta) \|_V} = \beta > 0
\]
with constant \( \beta \) independent of \( h \).

Proof. For any \( (\sigma, E, B) \in V_h \), the finite element complexes in Theorem 17 show
\[
(\sigma, E, B) = (\sigma - \text{div div} E, E + \text{sym curl} B, B) \in V_h.
\]
Thus, there exists some positive constant \( C \) such that
\[
A(\sigma, E, B; q, \xi, \zeta) = (\sigma, \sigma) + (E, E) + (B, B) - (\sigma, \text{div div} E) + (E, \text{sym curl} B)
+ (\text{div div} E, \text{div div} E) + (\text{sym curl} B, \text{sym curl} B)
\geq \frac{1}{2} \| \sigma \|^2_{L^2(\Omega)} + \| E \|^2_{L^2(\Omega)} + \| B \|^2_{L^2(\Omega)} + \| \text{div div} E \|^2_{L^2(\Omega)} + \| \text{sym curl} B \|^2_{L^2(\Omega)}
\geq C \| (\sigma, E, B) \|^2_V.
\]
Since $\| (q, \xi, \zeta) \|_V \leq C \| (\sigma, E, B) \|_V$, this concludes the inf-sup condition with some constant $\beta > 0$ independent of $h$.

For any $(\sigma, E, B) \in V$, define the projection $\Pi_h(\sigma, E, B) \in V_h$ such that

$$A(\Pi_h \sigma, \Pi_h E, \Pi_h B; q, \xi, \zeta) = A(\sigma, E, B; q, \xi, \zeta)$$

for any $(q, \xi, \zeta) \in V_h$.

with the full equivalent formulation

$$
\begin{cases}
(\Pi_h \sigma, q) - (\text{div} \text{ div} \Pi_h E, q) = (\sigma, q) - (\text{div} E, q), & \text{for any } q \in P_{k-2}(T), \\
(\Pi_h E, \xi) + (\Pi_h \sigma, \text{div div} \xi) + (\text{sym curl} \Pi_h B, \xi) = (E, \xi) + (\sigma, \text{div div} \xi) + (\text{sym curl} B, \xi), & \text{for any } \xi \in \Sigma_{k,h}, \\
(\Pi_h B, \zeta) - (\text{sym curl} \zeta, \zeta) = (B, \zeta) - (E, \text{sym curl} \zeta), & \text{for any } \zeta \in \Lambda_{k+1,h}.
\end{cases}
$$

The following error estimate holds from the Babuška theory \cite{Bab}

$$
\inf \| (\sigma, E, B) - \Pi_h(\sigma, E, B) \|_V \lesssim \inf_{(q, \xi, \zeta) \in V_h} \| (\sigma, E, B) - (q, \xi, \zeta) \|_V
$$

Let $P_V : L^2(\Omega) \to P_{k-2}(T)$ denote the $L^2$ projection onto $P_{k-2}(T)$. For any $0 \leq s \leq k - 1$, the estimate holds

$$
\| q - P_V q \|_{L^2(\Omega)} \lesssim h^s \| q \|_{H^s(\Omega)} \text{ for any } q \in H^s(\Omega).
$$

Let $P_\Sigma : H(\text{div div}, \Omega; \mathbb{S}) \cap H^2(\Omega; \mathbb{S}) \to \Sigma_{k,h}$ with $k \geq 3$ denote the interpolation indicated by the degrees of freedom (8a)–(8f) (the values at the vertices are obtained by averaging). An alternative interpolation can be found in \cite{Dzi} with the commuting property. The following error estimate holds, for any $0 \leq s \leq k - 1$, that

$$
\| \xi - P_\Sigma \xi \|_{H(\text{sym curl}, \Omega; \mathbb{T})} \lesssim h^s \| \xi \|_{H^{s+2}(\Omega)} \text{ for any } \xi \in H^{s+2}(\Omega).
$$

Let $P_\Lambda : H(\text{sym curl}, \Omega; \mathbb{T}) \cap H^3(\Omega; \mathbb{T}) \to \Lambda_{k+1,h}$ with $k \geq 3$ denote the interpolation by the degrees of freedom (7a)–(7f) (the values at the vertices are obtained by averaging). The following error estimate holds, for any $2 \leq s \leq k + 1$, that

$$
\| \zeta - P_\Lambda \zeta \|_{H(\text{sym curl}, \Omega; \mathbb{T})} \lesssim h^s \| \zeta \|_{H^{s+1}(\Omega)} \text{ for any } \zeta \in H^{s+1}(\Omega; \mathbb{T}).
$$

Suppose $(\sigma, E, B) \in H^{k-1}(\Omega) \times H^{k+1}(\Omega; \mathbb{S}) \times H^{k}(\Omega)$ for $k \geq 3$. The combination of (38)–(40) with (37) leads to

$$
\| (\sigma, E, B) - \Pi_h(\sigma, E, B) \|_V \lesssim h^{k-1} \| \sigma \|_{H^{k-1}(\Omega)} + \| E \|_{H^{k+1}(\Omega)} + \| B \|_{H^k(\Omega)}.
$$

### 5.2. The solution of the fully discrete system and error estimates

This subsection uses some notation in \cite{Chi}. Suppose that $T = N\Delta t$ with a positive integer $N$. Let $p^j$ denote the function $p(t_j)$ with $t_j = j\Delta t$ for $j = 0, 1, \cdots, N$. Define

$$
\hat{\partial}_t p^{j+\frac{1}{2}} = \frac{p^{j+1} - p^j}{\Delta t}, \quad \hat{p}^{j+\frac{1}{2}} = \frac{p^{j+1} + p^j}{2}.
$$

The time variable will be discretized by the Crank-Nicolson scheme. Denote $(\sigma_h^0, E_h^0, B_h^0) \in V_h$ the approximation of solution $(\sigma, E, B)$ of (33) at $t_j$. Given the initial data $(\sigma_h^0, E_h^0, B_h^0) \in V_h$, for $0 \leq j \leq N - 1$, the approximation $(\sigma_h^{j+1}, E_h^{j+1}, B_h^{j+1})$
The second terms in the above have been estimated in (42).

\[
\begin{align*}
(\sigma^j_{h+1}, q) &= \text{(div div } \tilde{E}^j_{h+1/2}, q), \quad \text{for any } q \in P_{k-2}(T), \\
(\tilde{E}^j_{h+1/2}, \xi) &= -\left(\tilde{\sigma}^j_{h+1/2}, \text{div div } \tilde{\zeta}^j_{h+1/2}, \xi\right), \quad \text{for any } \xi \in \Sigma_{k,h}, \\
(\tilde{\zeta}^j_{h+1/2}, \xi) &= \left(\tilde{E}^j_{h+1/2}, \text{sym curl } \xi\right), \quad \text{for any } \xi \in \Lambda_{k+1,h}.
\end{align*}
\]

This can be written as

\[
\begin{align*}
(\sigma^j_{h+1}, q) &= \text{(div div } \tilde{E}^j_{h+1}, q), \quad \text{for any } q \in P_{k-2}(T), \\
(\tilde{E}^j_{h+1}, \xi) &= -\left(\tilde{\sigma}^j_{h+1}, \text{div div } \tilde{\zeta}^j_{h+1}, \xi\right), \quad \text{for any } \xi \in \Sigma_{k,h}, \\
(\tilde{\zeta}^j_{h+1}, \xi) &= \left(\tilde{E}^j_{h+1}, \text{sym curl } \xi\right), \quad \text{for any } \xi \in \Lambda_{k+1,h}.
\end{align*}
\]

The system is nonsingular as in [11]. The error estimates mimic the proof of Theorem 6.3 of [11] and are stated in the following theorem.

**Theorem 20.** Suppose \(k \geq 3\). Let \((\sigma, \tilde{E}, \tilde{B})\) solve (33) and let \((\sigma^j_{h}, \tilde{E}^j_{h}, \tilde{B}^j_{h})\) solve (42), let the initial data \((\sigma_h, \tilde{E}_h^0, \tilde{B}_h^0) = \Pi_h(\sigma(0), \tilde{E}(0), \tilde{B}(0))\). Assume

\[
\begin{align*}
\sigma &\in W^{1,1}([0,T], H^{k-1}(\Omega)) \cap W^{3,1}([0,T], L^2(\Omega)) \cap L^\infty([0,T], H^{k-1}(\Omega)), \\
\tilde{E} &\in W^{1,1}([0,T], H^{k+1}(\Omega)) \cap W^{3,1}([0,T], L^2(\Omega)) \cap L^\infty([0,T], H^{k+1}(\Omega)), \\
\tilde{B} &\in W^{1,1}([0,T], H^{k}(\Omega)) \cap W^{3,1}([0,T], L^2(\Omega)) \cap L^\infty([0,T], H^{k}(\Omega)).
\end{align*}
\]

It holds that, for \(1 \leq j \leq N\)

\[
\|\sigma^j - \sigma_h^j\|_{L^2(\Omega)} + \|\tilde{E}^j - \tilde{E}_h^j\|_{L^2(\Omega)} + \|\tilde{B}^j - \tilde{B}_h^j\|_{L^2(\Omega)} \\
\leq (k^{-1} + \Delta_t^2)(\|\sigma\|_{W^{1,1}(H^{k-1})} + \|\tilde{E}\|_{W^{1,1}(H^{k+1})} + \|\tilde{B}\|_{W^{1,1}(H^{k})})
\]

**Proof.** There exists the following decomposition of the errors

\[
\begin{align*}
\delta^j_{\sigma} := \sigma^j - \sigma^j_h = (\sigma^j - \Pi_h \sigma^j) + (\Pi_h \sigma^j - \sigma^j) := \theta^j_{\sigma} + p^j_{\sigma}, \\
\delta^j_{\tilde{E}} := \tilde{E}^j_h - \tilde{E}^j = (\tilde{E}^j - \Pi_h \tilde{E}^j) + (\Pi_h \tilde{E}^j - \tilde{E}^j) := \theta^j_{\tilde{E}} + p^j_{\tilde{E}}, \\
\delta^j_{\tilde{B}} := \tilde{B}^j_h - \tilde{B}^j = (\tilde{B}^j - \Pi_h \tilde{B}^j) + (\Pi_h \tilde{B}^j - \tilde{B}^j) := \theta^j_{\tilde{B}} + p^j_{\tilde{B}}.
\end{align*}
\]

The second terms in the above have been estimated in (41). It remains to analyze the errors \((\theta^j_{\sigma}, \theta^j_{\tilde{E}}, \theta^j_{\tilde{B}})\).

The choices of \(t = t_j\) and \(t = t_{j+1}\) in (33) lead to

\[
\begin{align*}
(\tilde{\sigma}^{j+1/2}_h, q) &= (\text{div div } \tilde{E}^{j+1/2}_h, q), \quad \text{for any } \tau \in P_{k-2}(T), \\
(\tilde{E}^{j+1/2}_h, \xi) &= -\left(\tilde{\sigma}^{j+1/2}_h, \text{div div } \tilde{\zeta}^{j+1/2}_h, \xi\right), \quad \text{for any } \xi \in \Sigma_{k,h}, \\
(\tilde{\zeta}^{j+1/2}_h, \xi) &= \left(\tilde{E}^{j+1/2}_h, \text{sym curl } \xi\right), \quad \text{for any } \xi \in \Lambda_{k+1,h}.
\end{align*}
\]

Subtracting (42) from the above equations shows

\[
\begin{align*}
(\tilde{\sigma}^{j+1/2}_h, q) + (\tilde{\sigma}_{\sigma}^{j+1/2}_h - \tilde{\sigma}^{j+1/2}_h, q) &= (\text{div div } \tilde{\sigma}^{j+1/2}_h, q), \quad \text{for any } q \in P_{k-2}(T), \\
(\tilde{E}^{j+1/2}_h, \xi) + (\tilde{\sigma}_{\tilde{E}}^{j+1/2}_h - \tilde{E}^{j+1/2}_h, \xi) &= -\left(\tilde{\sigma}^{j+1/2}_h, \text{div div } \xi\right), \quad \text{for any } \xi \in \Sigma_{k,h}, \\
(\tilde{\zeta}^{j+1/2}_h, \xi) + (\tilde{\sigma}_{\tilde{B}}^{j+1/2}_h - \tilde{\zeta}^{j+1/2}_h, \xi) &= \left(\tilde{E}^{j+1/2}_h, \text{sym curl } \xi\right), \quad \text{for any } \xi \in \Lambda_{k+1,h}.
\end{align*}
\]
It follows from (36) that

\[
\begin{align*}
(\tilde{p}^{\sigma + \frac{1}{2}}, q) &= (\text{div div } \tilde{p}^{\sigma + \frac{1}{2}}, q), \\
(\tilde{p}^{\sigma + \frac{1}{2}}, \xi) &= -(\tilde{p}^{\sigma + \frac{1}{2}}, \text{div div } \xi) - (\text{sym curl } \tilde{p}^{\sigma + \frac{1}{2}}, \xi), \\
(\hat{p}^{\sigma + \frac{1}{2}}, \zeta) &= (\hat{p}^{\sigma + \frac{1}{2}}, \text{sym curl } \zeta), \\
(\hat{p}^{\sigma + \frac{1}{2}}, q) &= (\hat{p}^{\sigma + \frac{1}{2}}, q),
\end{align*}
\]

for any \(q \in P_{k-2}(T),\)
for any \(\xi \in \Sigma_{k,h},\)
for any \(\zeta \in \Lambda_{k+1,h}.
\]

The choices of \(q = \tilde{\theta}^{\sigma + \frac{1}{2}}, \xi = \tilde{\theta}^{\sigma + \frac{1}{2}}, \zeta = \hat{\theta}^{\sigma + \frac{1}{2}}\) in (44) and the above equations plus an elementary computation lead to

\[
\begin{align*}
&\left(\|\theta^{\sigma + 1}\|_{L^2(\Omega)}^2 + \|\theta^{\sigma + 1}_E\|_{L^2(\Omega)}^2 + \|\theta^{\sigma + 1}_B\|_{L^2(\Omega)}^2\right) - \left(\|\theta^{\sigma}_E\|_{L^2(\Omega)}^2 + \|\theta^{\sigma}_B\|_{L^2(\Omega)}^2\right) \\
&= 2\Delta t\left( -\tilde{c}_t p^{\sigma + \frac{1}{2}} - (\tilde{c}_t \tilde{\sigma}^{\sigma + \frac{1}{2}} - \tilde{\sigma}^{\sigma + \frac{1}{2}}) + \tilde{p}^{\sigma + \frac{1}{2}} + \tilde{\theta}^{\sigma + \frac{1}{2}} \right) \\
&+ 2\Delta t\left( -\tilde{c}_t p^{\sigma + \frac{1}{2}} - (\tilde{c}_t E^{\sigma + \frac{1}{2}} - \tilde{E}^{\sigma + \frac{1}{2}}) + \tilde{p}^{\sigma + \frac{1}{2}}, \tilde{\theta}^{\sigma + \frac{1}{2}} \right) \\
&+ 2\Delta t\left( -\tilde{c}_t p^{\sigma + \frac{1}{2}} - (\tilde{c}_t B^{\sigma + \frac{1}{2}} - \tilde{B}^{\sigma + \frac{1}{2}}) + \tilde{p}^{\sigma + \frac{1}{2}}, \tilde{\theta}^{\sigma + \frac{1}{2}} \right).
\end{align*}
\]

An application of the Cauchy-Schwarz inequality proves

\[
\begin{align*}
&\left(\|\theta^{\sigma + 1}\|_{L^2(\Omega)}^2 + \|\theta^{\sigma + 1}_E\|_{L^2(\Omega)}^2 + \|\theta^{\sigma + 1}_B\|_{L^2(\Omega)}^2\right) - \left(\|\theta^{\sigma}_E\|_{L^2(\Omega)}^2 + \|\theta^{\sigma}_B\|_{L^2(\Omega)}^2\right) \\
&\leq \Delta t\left( \|\tilde{c}_t p^{\sigma + \frac{1}{2}}\|_{L^2(\Omega)} + \|\tilde{c}_t \tilde{\sigma}^{\sigma + \frac{1}{2}} - \tilde{\sigma}^{\sigma + \frac{1}{2}}\|_{L^2(\Omega)} + \|\tilde{p}^{\sigma + \frac{1}{2}}\|_{L^2(\Omega)} \\
&+ \|\tilde{c}_t E^{\sigma + \frac{1}{2}} - \tilde{E}^{\sigma + \frac{1}{2}}\|_{L^2(\Omega)} + \|\tilde{p}^{\sigma + \frac{1}{2}}\|_{L^2(\Omega)} \\
&+ \|\tilde{c}_t B^{\sigma + \frac{1}{2}} - \tilde{B}^{\sigma + \frac{1}{2}}\|_{L^2(\Omega)} + \|\tilde{p}^{\sigma + \frac{1}{2}}\|_{L^2(\Omega)} \right).
\end{align*}
\]

Given \(g \in C^3[0, T],\) the Taylor expansion of \(g\) reads

\[
\Delta t \|\tilde{c}_t g^{\sigma + \frac{1}{2}}\|_{L^2(\Omega)} = \left\| \int_{t_j}^{t_{j+1}} \tilde{g} dt \right\|_{L^2(\Omega)} \leq \int_{t_j}^{t_{j+1}} \|\tilde{g}\|_{L^2(\Omega)} dt,
\]

\[
\Delta t \|\tilde{c}_t g^{\sigma + \frac{1}{2}} - \tilde{g}^{\sigma + \frac{1}{2}}\|_{L^2(\Omega)} = \frac{1}{2} \|2\tilde{g}^{\sigma + 1} - 2\tilde{g} - \Delta t \tilde{g}^{\sigma + 1} - \Delta t \tilde{g}^{\sigma + \frac{1}{2}}\|_{L^2(\Omega)} \\
\leq \Delta t^2 \int_{t_j}^{t_{j+1}} \|\tilde{g}\|_{L^2(\Omega)} dt.
\]

A summation of (45) over all the time intervals plus the above estimates lead to

\[
\begin{align*}
&\left(\|\theta^{\sigma + 1}\|_{L^2(\Omega)}^2 + \|\theta^{\sigma + 1}_E\|_{L^2(\Omega)}^2 + \|\theta^{\sigma + 1}_B\|_{L^2(\Omega)}^2\right) - \left(\|\theta^{\sigma}_E\|_{L^2(\Omega)}^2 + \|\theta^{\sigma}_B\|_{L^2(\Omega)}^2\right) \\
&\leq \int_0^{t_{j+1}} \left( \|\tilde{c}_t p\|_{L^2(\Omega)} + \|\tilde{c}_t E\|_{L^2(\Omega)} + \|\tilde{c}_t B\|_{L^2(\Omega)} \right) dt \\
&+ \Delta t^2 \int_0^{t_{j+1}} \left( \|\tilde{g}\|_{L^2(\Omega)} + \|\tilde{E}\|_{L^2(\Omega)} + \|\tilde{B}\|_{L^2(\Omega)} \right) dt \\
&+ \Delta t \sum_{m=0}^{\frac{j-1}{m+1}} \left( \|\tilde{p}^{\sigma m}_E\|_{L^2(\Omega)} + \|\tilde{p}^{\sigma m}_B\|_{L^2(\Omega)} + \|\tilde{p}^{\sigma m}_B\|_{L^2(\Omega)} \right).
\end{align*}
\]
Since the initial data \((\sigma_0^0, \mathbf{E}_0^0, \mathbf{B}_0^0) = \Pi_0(\sigma(0), \mathbf{E}(0), \mathbf{B}(0))\), it implies that \((\theta_0^0, \theta_E^0, \theta_B^0)\) vanishes. By the estimates of the projection errors in (41), this shows that

\[
\begin{align*}
&\left(\|\theta^{j+1}_E\|_{L^2(\Omega)}^2 + \|\theta^{j+1}_{B}\|_{L^2(\Omega)}^2 + \|\theta^{j+1}_E\|_{L^2(\Omega)}^2\right)^{\frac{1}{2}} \\
\leq & h^{k-1} \int_0^{j+1} \left(\|\bar{\sigma}\|_{H^{k-1}(\Omega)} + \|\bar{\mathbf{E}}\|_{H^k(\Omega)} + \|\bar{\mathbf{B}}\|_{H^k(\Omega)}\right) dt \\
& + \Delta t^2 \int_0^{j+1} \left(\|\bar{\sigma}\|_{L^2(\Omega)} + \|\bar{\mathbf{E}}\|_{L^2(\Omega)} + \|\bar{\mathbf{B}}\|_{L^2(\Omega)}\right) dt \\
& + j\Delta t h^{k-1} \left(\|\sigma\|_{L^2(\Omega)} + \|\mathbf{E}\|_{L^2(\Omega)} + \|\mathbf{B}\|_{L^2(\Omega)}\right).
\end{align*}
\]

A combination of this and the estimates of the projection errors in (41) completes the proof. 

REFERENCES

[1] D. N. Arnold, R. S. Falk, and R. Winther, Finite element exterior calculus, homological techniques, and applications, Acta Numer., 15 (2006), pp. 1–155, doi:10.1017/S0962492906210018.
[2] D. N. Arnold and K. Hu, Complexes from complexes, 2021, arXiv:2005.12437v2.
[3] D. Boffi, F. Brezzi, and M. Fortin, Mixed finite element methods and applications, Springer, Heidelberg, 2013.
[4] L. Chen and X. Huang, Discrete Hessian complexes in three dimensions, 2020, arXiv:2012.10914.
[5] L. Chen and X. Huang, Finite elements for divdiv-conforming symmetric tensors, (2020), arXiv:2005.01271v1.
[6] L. Chen and X. Huang, Finite elements for divdiv-conforming symmetric tensors in three dimensions, (2020), arXiv:2007.12309.
[7] S. H. Christiansen, J. Hu, and K. Hu, Nodal finite element de rham complexes, Numer. Math., 139 (2018), pp. 411–446.
[8] T. Führer and N. Heuer, Fully discrete DPG methods for the Kirchhoff-Love plate bending model, Comput. Methods in Appl. Mech. Eng., 343 (2019), pp. 550–571, doi:10.1016/j.cma.2018.08.041.
[9] T. Führer, N. Heuer, and A. H. Niemi, An ultraweak formulation of the Kirchhoff-Love plate bending model and DPG approximation, Math. Comp., 88 (2019), pp. 1587–1619.
[10] J. Hu, Finite element approximations of symmetric tensors on simplicial grids in \(\mathbb{R}^n\): The higher order case, J. Comput. Math., 33 (2015), pp. 283–296.
[11] J. Hu and Y. Liang, Conforming discrete gradgrad-complexes in three dimensions, 2020, arXiv:2008.00497.
[12] J. Hu, R. Ma, and M. Zhang, A family of mixed finite elements for the biharmonic equations on triangular and tetrahedral grids, (2020), arXiv:2010.02638.
[13] J. Hu and S. Zhang, A family of conforming mixed finite elements for linear elasticity on triangular grids, arXiv, 1406.7457 (2014).
[14] J. Hu and S. Zhang, A family of symmetric mixed finite elements for linear elasticity on tetrahedral grids, Sci. China Math., 58 (2015), pp. 297–307.
[15] D. Pauly and W. Zulehner, The divdiv-complex and applications to biharmonic equations, Applicable Analysis, 99 (2020), pp. 1579–1630, doi:10.1080/00036811.2018.1542685.
[16] V. Quenneville-Bélair, A new approach to finite element simulation of general relativity, PhD thesis, University of Minnesota, Minneapolis, 2015.