EXISTENCE AND CONTINUOUS-DISCRETE ASYMPTOTIC BEHAVIOUR FOR TIKHONOV-LIKE DYNAMICAL EQUILIBRIUM SYSTEMS

AICHA BALHAG, ZAKI CHIBANI AND HASSAN RIAHI

Laboratory LIBMA Mathematics, Faculty of Sciences Semlalia
Cadi Ayyad University
40000 Marrakech, Morocco

(Communicated by Hedy Attouch)

ABSTRACT. We consider the regularized Tikhonov-like dynamical equilibrium problem: find \( u : [0, +\infty[ \to H \) such that for a.e. \( t \geq 0 \) and every \( y \in K \),
\[
\langle \dot{u}(t), y - u(t) \rangle + F(u(t), y) + \varepsilon(t)(u(t), y - u(t)) \geq 0,
\]
where \( F : K \times K \to \mathbb{R} \) is a monotone bifunction, \( K \) is a closed convex set in Hilbert space \( H \) and the control function \( \varepsilon(t) \) is assumed to tend to 0 as \( t \to +\infty \). We first establish that the corresponding Cauchy problem admits a unique absolutely continuous solution. Under the hypothesis that
\[
\int_0^{+\infty} \varepsilon(t)dt < \infty,
\]
we obtain weak ergodic convergence of \( u(t) \) to \( x \in K \) solution of the following equilibrium problem
\[
F(x, y) \geq 0, \quad \forall y \in K.
\]
If in addition the bifunction is assumed demipositive, we show weak convergence of \( u(t) \) to the same solution. By using a slow control
\[
\int_0^{+\infty} \varepsilon(t)dt = \infty
\]
and assuming that the bifunction \( F \) is \( 3 \)-monotone, we show that the term \( \varepsilon(t)u(t) \) asymptotically acts as a Tikhonov regularization, which forces all the trajectories to converge strongly towards the element of minimal norm of the closed convex set of equilibrium points of \( F \). Also, in the case where \( \varepsilon \) has a slow control property and
\[
\int_0^{+\infty} |\dot{\varepsilon}(t)|dt < +\infty,
\]
we show that the strong convergence property of \( u(t) \) is satisfied. As applications, we propose a dynamical system to solve saddle-point problem and a neural dynamical model to handle a convex programming problem. In the last section, we propose two Tikhonov regularization methods for the proximal algorithm.

1. Introduction. Throughout the paper, \( H \) is a real Hilbert space which is endowed with the scalar product \( \langle \cdot, \cdot \rangle \), and the norm \( \|x\| = \sqrt{\langle x, x \rangle} \) for \( x \in H \). Let \( K \) be a nonempty closed convex subset of \( H \) and \( F : K \times K \to \mathbb{R} \) be a bifunction satisfying \( F(x, x) = 0 \), for every \( x \in K \). Recall that an equilibrium problem in the sense of Blum-Oettli [6] is the following Ky Fan minimax inequality [20]:
\[
\text{Find } \bar{x} \in K \text{ such that } F(\bar{x}, y) \geq 0, \quad \forall y \in K. \quad (EP)
\]

2000 Mathematics Subject Classification. Primary: 37N40, 46N10, 49J40, 90C33.

Key words and phrases. Monotone bifunction, Cauchy problem, demipositive bifunction, asymptotic behaviour, convex minimisation, saddle point problem.
Many authors had contributed to the study of existence for equilibrium problems when $K$ is assumed to be compact or some coerciveness assumptions are imposed on the bifunction $F$, we refer to [20, 6, 12, 10, 11, 1] and the bibliography therein. We denote by $S_F$ the solution set of $(EP)$, which is supposed nonempty and not necessarily reduced to a single point. To explain the interest in equilibrium problems, we can consider a number of particular cases, both classical and recently discovered, as optimization problems (minimization and saddle point), Nash equilibria, fixed points and variational inequalities (see [6, 12]). The interest of the equilibrium problem is that it unifies all above mentioned problems in a convenient way. Moreover, many methods devoted for solving one of these problems can be extended, with suitable modifications, to an equilibrium problem.

The Tikhonov regularization method [35] is a powerful tool in convex optimization to handle discrete or continuous ill-posed problems. It has been recently applied to equilibrium problems (see [27, 17, 29, 18, 23]). In this paper, we deal with Tikhonov regularization methods for dynamic monotone equilibrium problems. The basic idea of this method is to approach the bifunction of the following evolution differential equilibrium problem:

\[
\begin{cases}
\text{find } u \in C^0(0, +\infty; \mathcal{H}) \text{ such that for a.e. } t \geq 0, \\
\langle \dot{u}(t), y - u(t) \rangle + F(u(t), y) \geq 0, \quad \forall y \in K, \\
u(t) \in K \text{ for each } t \geq 0,
\end{cases}
\]

by a family of strongly monotone bifunctions $F_\varepsilon(u, v) := F(u, v) + \varepsilon(u, v - u)$ depending on a regularization real parameter $\varepsilon > 0$ to obtain a perturbed evolution equilibrium problem $(DEP_\varepsilon)$.

\[
\begin{cases}
\text{find } u \in C^0(0, +\infty; \mathcal{H}) \text{ such that for a.e. } t \geq 0, \\
\langle \dot{u}(t), y - u(t) \rangle + F(u(t), y) + \varepsilon \langle u(t), y - u(t) \rangle \geq 0, \quad \forall y \in K, \\
u(t) \in K \text{ for each } t \geq 0.
\end{cases}
\]

The resulting regularized evolution problem, see [15], has a unique solution $u \in C^0(0, +\infty; \mathcal{H})$. We show in Lemma 5.1 that this solution converges strongly, while $t$ goes to $+\infty$, to the unique solution $x_\varepsilon$ (that depends on the regularization parameter $\varepsilon$) of the following equilibrium problem:

\[
\text{find } x_\varepsilon \in K \text{ such that } F(x_\varepsilon, y) + \varepsilon \langle x_\varepsilon, y - x_\varepsilon \rangle \geq 0, \quad \forall y \in K.
\]

Next, passing to the limit as the parameter $\varepsilon$ goes to 0, we prove that the unique solution $x_\varepsilon$ of the regularized problem $(EP_\varepsilon)$ tends to the particular solution of minimum norm on the nonempty closed convex solution set of $(EP)$.

In this framework, for a measurable function $\varepsilon : \mathbb{R}_+ \to \mathbb{R}_+^*$, we consider the following Tikhonov regularized dynamical equilibrium problem: find $u \in C^0(0, +\infty; \mathcal{H})$ such that: for a.e. $t \geq 0$, $\forall y \in K$,

\[
\begin{cases}
\text{find } u \in C^0(0, +\infty; \mathcal{H}) \text{ such that for a.e. } t \geq 0, \\
\langle \dot{u}(t), y - u(t) \rangle + F(u(t), y) + \varepsilon(t) \langle u(t), y - u(t) \rangle \geq 0, \quad \forall y \in K, \\
u(t) \in K \text{ for each } t \geq 0,
\end{cases}
\]

where we denote the function $\varepsilon$ by $\varepsilon(\cdot)$ to avoid confusing both of these dynamic problems $(DEP_{\varepsilon(\cdot)})$ and $(DEP)$.

In the preliminaries section, we set up the abstract formulation of the resolvent and the associated Yosida approximate for the bifunction $F_\varepsilon$ and then summarize and prove some basic associate properties which are useful for further discussion.
Section 3 is devoted to existence and uniqueness of a strong solution for the evolution equilibrium problem \((\text{DEP}_{\varepsilon(t)})\) with an initial condition, i.e., an absolutely continuous function \(u : [0, +\infty] \to \mathcal{H}\), such that \((\text{DEP}_{\varepsilon(t)})\) holds for almost every \(t \geq 0\) and \(u(0) = u_0\).

In section 4, we discuss an introduced kind of rate of convergence for values of the dynamic equilibrium problems \((\text{DEP})\) and \((\text{DEP}_{\varepsilon(t)})\). More precisely, under suitable conditions in Theorems 4.2, 4.3, we obtain \(F(z, u(t)) = o\left(\frac{1}{t}\right)\) near the infinite value for \(t\), and moreover \(\lim_{t \to +\infty} t \varepsilon(t) \|u(t)\|^2 = 0\).

In the main section of this paper, we establish that the asymptotic convergence properties of trajectories for the dynamical equilibrium problem \((\text{DEP}_{\varepsilon(t)})\) depend on whether \(\varepsilon(\cdot)\) is in \(L^1([0, +\infty[)\) or not. For \(\varepsilon(\cdot) \in L^1([0, +\infty[)\), the authors in [9, 33] established that the following regularized dynamical monotone inclusion:

\[
\dot{u}(t) \in Au(t) + \varepsilon(t)u(t),
\]

reinforce the convergence properties of the initial problem \(\dot{u}(t) \in Au(t)\). This result remains true for our problem \((\text{DEP}_{\varepsilon(t)})\). In Theorem 5.4, we establish the weak ergodic convergence of \(u(t)\) to some \(x^\infty \in S\) for general maximal monotone bifunctions. In Theorem 5.7, we prove weak convergence of \(u(t)\) to some \(x^\infty \in S\), when more generally the monotone bifunction \(F\) is only assumed to be demipositive.

When \(\varepsilon(\cdot) \notin L^1([0, +\infty[)\), the strong convergence of the unique solution for the inclusion (1) to the least norm element of \(A^{-1}(0)\) were discussed in [8, 34] under the additional condition \(\varepsilon(\cdot)\) is nonincreasing to 0. We also refer to [16], where this strong convergence still holds without assuming \(\varepsilon(\cdot)\) to be nonincreasing. In Theorem 5.10 of this paper, we prove strong convergence of \(u : [0, +\infty] \to \mathcal{H}\), the unique solution of \((\text{DEP}_{\varepsilon(t)})\), to the least norm element of \(S_F\), as \(t \to +\infty\), when moreover \(F\) is 3-monotone. In Theorem 5.15, this strong convergence of \(u(t)\) remains true for general monotone bifunctions when moreover \(\int_0^{+\infty} |\varepsilon(t)|dt < +\infty\).

Using [3] we establish the link between solutions of \((\text{DEP}_{\varepsilon(t)})\) and those of the following evolution problem:

\[
\begin{align*}
\text{find } u &\in C^0([0, +\infty[; \mathcal{H}) \text{ such that for } a.e. \, t \geq 0, \\
\langle \dot{u}(t), y-u(t) \rangle + \beta(t) F(u(t), y) + \langle u(t), y-u(t) \rangle &\geq 0, \, \forall y \in K, \quad (\text{DEP}_{\beta(t)}) \\
u(t) &\in K, \, t \geq 0.
\end{align*}
\]

where \(\beta\) tends to \(+\infty\) as \(t\) goes to \(+\infty\). Then, according to Theorems 5.10 and 5.15 we prove, under corresponding assumptions on \(\beta\), i.e.,

\[
\lim_{t \to +\infty} \beta(t) = +\infty \quad \text{and} \quad \int_0^{+\infty} \frac{\dot{\beta}(t)}{\beta(t)^2}dt < +\infty,
\]

strong convergence of the solution of \((\text{DEP}_{\beta(t)})\) to the least norm element of \(S_F\) whenever \(t\) goes to \(+\infty\), see Theorem 6.2. The remaining part of this section deals with a comparison of Theorem 6.2 and existing results in [3, 4] and [13] where the Fitzpatrick transform, adapted for bifunctions in [2], is used and uniform control condition involving both \(F\) and \(S_F\) has been imposed, see (31) and (32).

Finally, in the last section, we provide the time discretization of this system \((\text{DEP}_{\varepsilon(t)})\). As we can see in [25], the authors proposed an approximation method which combines the Tikhonov method with the proximal point algorithm for a maximal monotone operator. This technique can be used in the case of a monotone bifunction to prove the strong convergence to least norm element of \(S_F\). We firstly
approximate the bifunction $F$ by the Tikhonov parametrized family of bifunctions

$$F_n(x, y) = F(x, y) + \varepsilon_n (x, y - x),$$

where $x, y \in K$ and $\{\varepsilon_n\}$ is a positive sequence. Observe that an implicit discretization of the dynamical system $(\text{DEP}_{\varepsilon(t)})$ gives, for $x_0 \in K$, $\{\lambda_n\}$ and $\{\varepsilon_n\}$ are two positive sequences:

$$\lambda_n F(x_{n+1}, y) + \lambda_n \varepsilon_n (x_{n+1}, y - x_{n+1}) + (x_{n+1} - x_n, y - x_n) \geq 0, \quad \forall y \in K,$$

which can be rewritten as the prox-penalization algorithm $x_{n+1} = J^F_{\lambda_n}(x_n)$, where $J^F_{\lambda_n}$ is the proximal point associated to the perturbed bifunction $F_n$.

For many applications, however, evaluating the resolvent $J^F_{\lambda_n}$ of the sum bifunction $F_n$ is much harder than evaluating the resolvent of $F$ and $G_n(x, y) := \varepsilon_n (x, y - x)$ individually. In fact, we propose the -proximal (forward-backward) algorithm:

$$\lambda_n F(x_{n+1}, y) + \lambda_n \varepsilon_n (x_{n+1}, y - x_{n+1}) + (x_{n+1} - x_n, y - x_n) \geq 0, \quad \forall y \in K,$$

which can be rewritten as the forward-backward algorithm $x_{n+1} = J^F_{\lambda_n}((1 - \lambda_n \varepsilon_n) x_n)$. When $\sum_{n=0}^{\infty} \lambda_n \varepsilon_n = \infty$, we give conditions (more general than those required in [25] for the operators), to ensure strong convergence of the generated sequence $\{x_n\}$ by algorithms (ProxPA) and (DProxA) to the least norm element of the solutions set $S_F$.

2. Resolvent and Yosida approximate of monotone bifunctions. We assume that $K$ is a non empty closed convex subset of $\mathcal{H}$, and $F : K \times K \to \mathbb{R}$ satisfies the following usual assumptions:

(H1) $F(x, x) = 0$ for each $x \in K$;

(H2) $F$ is a monotone bifunction, i.e.: $F(x, y) + F(y, x) \leq 0 \forall x, y \in K$;

(H3) For any $y \in K$, $x \to F(x, y)$ is upper hemicontinuous, i.e., upper semicontinuous on each line segment of $K$;

(H4) for each $x \in K$, $x \to F(x, y)$ is convex and lower semicontinuous.

Lemma 2.1. [14] Suppose that $F : K \times K \to \mathbb{R}$ satisfies $(H_1) - (H_4)$. Then, for each $x \in \mathcal{H}$ and $\lambda > 0$, there exists a unique $z_\lambda = J^F_\lambda(x) \in K$, called the resolvent of $F$ at $x$ such that

$$\lambda F(z_\lambda, y) + (y - z_\lambda, z_\lambda - x) \geq 0, \quad \forall y \in K. \tag{2}$$

Moreover, $x$ is an equilibrium point of $F \Leftrightarrow x = J^F_\lambda(x)$ for every $\lambda > 0 \Leftrightarrow x = J^F_\lambda(x)$ for some $\lambda > 0$.

Proof. The proof of the first step (see [12]) is based upon the generalized KKM-Fan’s lemma since $F$, satisfying $(H_1) - (H_4)$, is a maximal monotone bifunction. The second step relies on (2). □

Lemma 2.2. Suppose that $F$ satisfies $(H_1) - (H_4)$, and set, for $\varepsilon > 0$, $F_\varepsilon(x, y) = F(x, y) + \varepsilon (x, y - x)$. Then for all $\lambda > 0$ and for all $x, y \in \mathcal{H}$,

$$\|J^F_\lambda(x) - J^F_\lambda(y)\| \leq \frac{1}{1 + \lambda \varepsilon} \|x - y\|; \tag{3}$$

and if $x_\varepsilon$ is the unique equilibrium point of $F_\varepsilon$, then $x_\varepsilon = J^F_\lambda(x_\varepsilon) = J^F_{1/\varepsilon}(0)$. 

Proof. By characterization of the resolvent mapping $J^F_\lambda$, we have for each $x, y \in \mathcal{H}$
\[ \lambda F(J^F_\lambda(x), J^F_\lambda(y)) + \lambda \varepsilon(J^F_\lambda(x) - J^F_\lambda(y)) + \langle J^F_\lambda(x) - x, J^F_\lambda(y) - J^F_\lambda(x) \rangle \geq 0, \]
and
\[ \lambda F(J^F_\lambda(y), J^F_\lambda(x)) + \lambda \varepsilon(J^F_\lambda(y) - J^F_\lambda(x)) + \langle J^F_\lambda(y) - y, J^F_\lambda(x) - J^F_\lambda(y) \rangle \geq 0. \]
By summing these two inequalities and using monotonicity of $F$, we obtain
\[ (1 + \lambda \varepsilon)||J^F_\lambda(x) - J^F_\lambda(y)||^2 \leq \langle y - x, J^F_\lambda(x) - J^F_\lambda(y) \rangle. \]
which leads to (3).

If $x_\varepsilon$ is the equilibrium point of $F_\varepsilon$, then $x_\varepsilon = J^F_\lambda(x_\varepsilon)$, which means:
\[ \frac{1}{2} F(x_\varepsilon, y) + \langle x_\varepsilon - 0, y - x_\varepsilon \rangle \geq 0 \text{ for all } y \in K, \text{i.e., } x_\varepsilon = J^F_{1/\varepsilon}(0). \]

Lemma 2.3. For each $\lambda, \mu \geq 0$ and $x \in \mathcal{H}$, we have
\[ J^F_\lambda(x) = J^F_\mu \left( \frac{\mu}{\lambda} x + \left( 1 - \frac{\mu}{\lambda} \right) J^F_\lambda(x) \right). \]

Proof. Set $z = J^F_\mu \left( \frac{\mu}{\lambda} x + \left( 1 - \frac{\mu}{\lambda} \right) J^F_\lambda(x) \right)$, then
\[ \lambda \mu F(z, J^F_\lambda(x)) + \langle J^F_\lambda(x) - z, \lambda z - \mu x - (\lambda - \mu) J^F_\lambda(x) \rangle \geq 0. \]
Also (2) for $y = z$ can be expressed as
\[ \lambda \mu F(J^F_\lambda(x), z) + \mu(z - J^F_\lambda(x), J^F_\lambda(x) - x) \geq 0. \]
Then summing and using monotonicity of $F$ we get $-\lambda||J^F_\lambda(x) - z||^2 \geq 0$ and thus $J^F_\lambda(x) = z$. \hfill \Box

Definition 2.4. Let $F : K \times K \to \mathbb{R}$. For each $\lambda > 0$, the associated Yosida $\lambda$-approximate to $F$ over $K$ is defined, by $B^F_\lambda := \frac{1}{\lambda}(I - J^F_\lambda)$.

Remark 1. [12, 28, 15] For each $\lambda > 0$, we have the resolvent $J^F_\lambda$ is firmly nonexpansive, namely
\[ \langle J^F_\lambda x - J^F_\lambda y, x - y \rangle \geq ||J^F_\lambda x - J^F_\lambda y||^2 \quad \forall x, y \in K. \]  \hfill (4)
As consequence, the Yosida $\lambda$-approximate is $\frac{1}{\lambda}$-Lipschitz continuous, that is
\[ ||B^F_\lambda x - B^F_\lambda y|| \leq \frac{1}{\lambda}||x - y|| \quad \forall x, y \in K. \]

The following lemmas play an important role in the analysis of existence of solutions of (DEP$\varepsilon$).

Lemma 2.5. Suppose that $F$ satisfies $(H_1) - (H_4)$, and set for $\varepsilon \geq 0$ and $x, y \in K$, $F_\varepsilon(x, y) = F(x, y) + \varepsilon(x, y - x)$. Suppose for each $x \in K$ there exists $z_{x, \varepsilon} \in \mathcal{H}$, such that
\[ F_\varepsilon(x, y) + \langle z_{x, \varepsilon}, x - y \rangle \geq 0 \quad \forall y \in K. \]  \hfill (5)
Then for each $\lambda > 0$ and $x \in K$ we have $||B^F_\lambda x|| \leq ||z_{x, \varepsilon}||$.

Proof. Fix $\varepsilon \geq 0, x \in K$ and $\lambda > 0$, there exists $z_{x, \varepsilon} \in \mathcal{H}$ satisfying (5). Also, by Lemma 2.1, we have
\[ \lambda F_\varepsilon(J^F_\lambda x, y) + \langle y - J^F_\lambda x, J^F_\lambda x - x \rangle \geq 0 \quad \forall y \in K. \]  \hfill (6)
By setting $y = J^F_\lambda x$ in (5) (respectively $y = x$ in (6)), summing and using monotonicity of $F_\varepsilon$, we get
\[ \langle \lambda z_{x, \varepsilon}, x - J^F_\lambda x \rangle + \langle x - J^F_\lambda x, J^F_\lambda x - x \rangle \geq 0. \]
Using $B_F^x = \frac{1}{\lambda} \left( x - J_{\lambda}^F x \right)$, we obtain $\|B_F^x x\|^2 \leq \langle x, x, B_F^x x \rangle$, which proves the claim. \hfill \Box

**Remark 2.** Consider the bifunction $F_x : K \times H \rightarrow \mathbb{R}$ defined by $F_x(x, y) = F_x(x, y)$ if $(x, y) \in K \times K$ and $F_x(x, y) = +\infty$ if $(x, y) \in K \times (H \setminus K)$. Then condition (5) is equivalent to $\partial F(x, y) \neq \emptyset$.

**Lemma 2.6.** Suppose $F$ satisfies conditions $(H_1) - (H_4)$. Then, for every $x, y \in K, \lambda > 0$ and $\varepsilon, \varepsilon' \geq 0$, we have

(i) $\lambda \|B_F^x x - B_F^x y\|^2 \leq \langle B_F^x x - B_F^x y, y - x \rangle$;

(ii) $\|B_F^x x - B_F^x y\| \leq |\varepsilon - \varepsilon'| \left( \lambda \|B_F^x x\| + \|x\| \right)$;

(iii) $\langle B_F^x x - B_F^x y, J_{\lambda}^F x - J_{\lambda}^F y \rangle \geq 0$.

**Proof.** We refer to [15, Lemma 2.5] for the assertion (i). For (ii), we use the associated $\lambda$-Yosida approximate to $F_x$ to have, for each $\varepsilon, \varepsilon' \geq 0$ and each $x \in K$

$$F_x(J_{\lambda}^F x, J_{\lambda}^F y) - \langle B_F^x x, J_{\lambda}^F x - J_{\lambda}^F y \rangle \geq 0$$

and

$$F_x(J_{\lambda}^F x, J_{\lambda}^F y) - \langle B_F^x y, J_{\lambda}^F y - J_{\lambda}^F x \rangle \geq 0.$$ 

By adding the two last inequalities and using monotonicity of $F$, we obtain

$$\langle B_F^x x - B_F^x y, J_{\lambda}^F x - J_{\lambda}^F y \rangle \geq - \left[ \varepsilon \langle J_{\lambda}^F x, J_{\lambda}^F x - J_{\lambda}^F y \rangle + \varepsilon' \langle J_{\lambda}^F y, J_{\lambda}^F x - J_{\lambda}^F x \rangle \right].$$

Since $J_{\lambda}^F x - J_{\lambda}^F y = -\lambda(B_F^x x - B_F^x y)$, we get

$$\lambda \|B_F^x x - B_F^x y\|^2 \leq \varepsilon \langle J_{\lambda}^F x, J_{\lambda}^F x - J_{\lambda}^F y \rangle + \varepsilon' \langle J_{\lambda}^F y, J_{\lambda}^F x - J_{\lambda}^F y \rangle$$

$$= \langle \varepsilon J_{\lambda}^F x - \varepsilon' J_{\lambda}^F y, J_{\lambda}^F x - J_{\lambda}^F y \rangle$$

$$= -\varepsilon \|J_{\lambda}^F x - J_{\lambda}^F y\|^2 + (\varepsilon - \varepsilon') \langle J_{\lambda}^F x, J_{\lambda}^F y \rangle \leq |\varepsilon - \varepsilon'| \left( \lambda \|J_{\lambda}^F x\| \|J_{\lambda}^F y\| \right)$$

and then

$$\|B_F^x x - B_F^x y\| \leq |\varepsilon - \varepsilon'| \left( \lambda \|B_F^x x\| + \|x\| \right).$$

(iii) Also using the definition of $J_{\lambda}^F x$ and $J_{\lambda}^F y$, and summing the associated two inequalities, the desired relation is obtained. \hfill \Box

3. Existence and uniqueness result for Tikhonov regularized dynamical equilibrium problem. Our main theorem on the existence and uniqueness for the Tikhonov regularized dynamical equilibrium problem $(DEP_{\varepsilon}())$ will exploit the following results.

**Lemma 3.1.** Suppose that $F : K \times K \rightarrow \mathbb{R}$ satisfies $(H_1) - (H_4)$, then $u$ is a solution of $(DEP_{\varepsilon}())$ if, and only if, for a.e. $t \geq 0$,

$$\left\langle \frac{du}{dt}(t), y - u(t) \right\rangle + \varepsilon(t) \langle u(t), y - u(t) \rangle \geq F(y, u(t)), \quad \forall y \in K.$$ 

**Lemma 3.2.** Assume that $K$ is a nonempty, closed and convex subset of $H$ and $J : [0, +\infty] \times K \rightarrow K$ satisfies Lipschitz condition:

$$\|J(t)x - J(t)y\| \leq \|x - y\|, \quad \forall x, y \in K, \quad t \geq 0.$$
Then, for every \( \lambda > 0 \) and every \( u_0 \in K \), the following differential equation:

\[
\begin{aligned}
\frac{du}{dt}(t) + \frac{1}{\lambda}(u(t) - J(t)u(t)) &= 0, \\
u(0) &= u_0.
\end{aligned}
\] (7)

admits a unique solution \( u_\lambda : [0, +\infty[ \to \mathcal{H} \), which is an absolutely continuous function, i.e., for each \( t > 0 \), we have

\[
u_\lambda(t) = u_0 + \int_0^t \dot{u}_\lambda(s)ds.
\] (8)

**Proof.** This Lemma is a direct consequence of the Cauchy-Lipschitz-Picard Theorem (see [7, Theorem I.4]). \(\square\)

**Theorem 3.3.** Suppose conditions \((H_1) - (H_4)\) and (5) above are satisfied. If \( \dot{\varepsilon} \in L^\infty_{loc}[0, +\infty[, \) then, for each \( u_0 \in K \), \((DEP_{\varepsilon}(\cdot))\) admits a unique strong solution, i.e., an absolutely continuous function \( u : [0, +\infty[ \to \mathcal{H} \), such that \((DEP_{\varepsilon}(\cdot))\) holds for almost every \( t \geq 0 \).

**Proof.** Fix \( \lambda_0 > 0 \), \( u_0 \in K \) and set \( F_t := F_{\varepsilon(t)} \) for each \( t \geq 0 \). For \( 0 < \lambda \leq \lambda_0 \), by (4), we have \( F_{\lambda t} \) is Lipschitz continuous, and according to Lemma 3.2, the following differential equation

\[
\text{for a.e. } t > 0, \quad \frac{du}{dt}(t) + \frac{1}{\lambda}(u(t) - J_{\lambda t}u(t)) = 0 \quad \text{and} \quad u(0) = u_0 \in K,
\] (9)

admits a unique absolutely continuous solution, denoted by \( u_\lambda \), i.e., satisfying: for each \( t > 0 \),

\[
u_\lambda(t) = u_0 + \int_0^t \dot{u}_\lambda(s)ds.
\] (10)

**Step 1.** We first prove, for each \( T > 0 \), the net \( \{u_\lambda : 0 < \lambda \leq \lambda_0\} \) is Cauchy in \( C^0([0, T], H) \), as \( \lambda \to 0 \). For this, let \( t \in (0, T) \), then for \( h > 0 \) sufficiently small, and by setting \( t + h \) and \( t \) in (9), we have:

\[
\frac{d}{dt}\|u_\lambda(t + h) - u_\lambda(t)\|^2 = 2\langle \dot{u}_\lambda(t + h) - \dot{u}_\lambda(t), u_\lambda(t + h) - u_\lambda(t) \rangle
= -2B_{\lambda}^{F_{\lambda t}}u_\lambda(t + h) - B_{\lambda}^{F_{\lambda t}}u_\lambda(t), u_\lambda(t + h) - u_\lambda(t))
= -2(B_{\lambda}^{F_{\lambda t}}u_\lambda(t + h) - B_{\lambda}^{F_{\lambda t}}u_\lambda(t), u_\lambda(t + h) - u_\lambda(t))
+ 2B_{\lambda}^{F_{\lambda t}}u_\lambda(t) - B_{\lambda}^{F_{\lambda t}}u_\lambda(t), u_\lambda(t + h) - u_\lambda(t)).
\]

By using Lemma 2.6 (i), we get:

\[
\frac{d}{dt}\|u_\lambda(t + h) - u_\lambda(t)\|^2 \leq 2(B_{\lambda}^{F_{\lambda t}}u_\lambda(t) - B_{\lambda}^{F_{\lambda t}}u_\lambda(t), u_\lambda(t + h) - u_\lambda(t)).
\]

So by integrating on \([0, t]\) and using Lemma 2.6 (ii), we deduce

\[
\|u_\lambda(t + h) - u_\lambda(t)\|^2 \leq 2\int_0^t (B_{\lambda}^{F_{\lambda s}}u_\lambda(s) - B_{\lambda}^{F_{\lambda s}}u_\lambda(s), u_\lambda(s + h) - u_\lambda(s))ds
\leq 2\int_0^t \|B_{\lambda}^{F_{\lambda s}}u_\lambda(s) - B_{\lambda}^{F_{\lambda s}}u_\lambda(s)\||u_\lambda(s + h) - u_\lambda(s)||ds
\leq 2\int_0^t \|\varepsilon(s + h) - \varepsilon(s)\| \left( \lambda\|B_{\lambda}^{F_{\lambda s}}u_\lambda(s)\| + \|u_\lambda(s)\| \right) ||u_\lambda(s + h) - u_\lambda(s)||ds
\]
which implies, see [7, Lemma A.5],
\[ \|u_\lambda(t + h) - u_\lambda(t)\| \leq \|u_\lambda(h) - u_\lambda(0)\| + \int_0^t |\varepsilon(s + h) - \varepsilon(s)|(\lambda\|B^{F_\lambda^\varepsilon}u_\lambda(s)\| + \|u_\lambda(s)\|)ds. \]

Dividing by \( h \) and letting \( h \to 0 \), we get
\[ \|\dot{u}_\lambda(t)\| \leq \|\dot{u}_\lambda(0)\| + \int_0^t \|\dot{\varepsilon}(s)(\lambda\|B^{F_\lambda^\varepsilon}u_\lambda(s)\| + \|u_\lambda(s)\|)ds \quad \text{for a.e. } t \in [0, T]. \]

Setting \( M = \sup_{0 \leq t \leq T} |\dot{\varepsilon}(s)| \), which is finite since \( \dot{\varepsilon} \in L^\infty_{\text{loc}}[0, +\infty[ \), and using
\[ \|\dot{u}_\lambda(t)\| = \|B^{F_\lambda^\varepsilon}u_\lambda(t)\|, \]
we get
\[ \|\dot{u}_\lambda(t)\| \leq \|\dot{u}_\lambda(0)\| + M \int_0^t (\lambda\|\dot{u}_\lambda(t)\| + \|u_\lambda(s)\|)ds \quad \text{for a.e. } t \in [0, T]. \]

By integrating on \([0, t]\) the norm of relation (10), we obtain
\[ \int_0^t \|u_\lambda(s)\|ds \leq \|u_0\|T + \int_0^t s\|\dot{u}_\lambda(s)\|ds, \tag{11} \]
and then, we get
\[ \|\dot{u}_\lambda(t)\| \leq (\|\dot{u}_\lambda(0)\| + M\|u_0\|T) + M \int_0^t (\lambda + s)\|\dot{u}_\lambda(s)\|ds \quad \forall t \in [0, T], \]

Using Gronwall’s lemma, we deduce, for a.e. \( t \in [0, T] \),
\[ \|\dot{u}_\lambda(t)\| \leq (\|\dot{u}_\lambda(0)\| + MT\|u_0\|)e^{M(\lambda+T)t} \leq (\|\dot{u}_\lambda(0)\| + MT\|u_0\|)e^{M(\lambda_0+T)t}. \]

By assumption (5) and Lemma 2.5, there exists \( z_{u_0} \in H \), such that \( \|B^{F_\lambda^\varepsilon}u_0\| \leq \|z_{u_0}\| \), and so \( \|\dot{u}_\lambda(0)\| = \|u_0\| \) which leads to
\[ \|B^{F_\lambda^\varepsilon}u_\lambda(t)\| = \|\dot{u}_\lambda(t)\| \leq (\|\dot{u}_\lambda(0)\| + \|u_0\|T)e^{M(\lambda_0+T)t} \leq (\|z_{u_0}\| + \|u_0\|T)e^{M(\lambda_0+T)t}. \tag{12} \]

We get
\[ \|B^{F_\lambda^\varepsilon}u_\lambda(t)\| \leq C \quad \forall t \in [0, T], \]
where \( C = (\|z_{u_0}\| + \|u_0\|T)e^{M(\lambda_0+T)t} \).

Now, let \( \lambda, \beta \in (0, \lambda_0) \), then we have, for a.e. \( t \in [0, T] \),
\[ \frac{d}{dt}\|u_\lambda(t) - u_\beta(t)\|^2 = -2\langle B^{F_\lambda^\varepsilon}u_\lambda(t) - B^{F_\beta^\varepsilon}u_\lambda(t), u_\lambda(t) - u_\beta(t) \rangle. \]

By writing \( u_\lambda = \lambda B^{F_\lambda^\varepsilon}u_\lambda + J^{F_\lambda^\varepsilon}u_\lambda \) and likewise for \( u_\beta \), we get, for a.e. \( t \in [0, T] \),
\[ \langle B^{F_\lambda^\varepsilon}u_\lambda - B^{F_\beta^\varepsilon}u_\beta, u_\lambda - u_\beta \rangle = \langle B^{F_\lambda^\varepsilon}u_\lambda - B^{F_\beta^\varepsilon}u_\beta, \lambda B^{F_\lambda^\varepsilon}u_\lambda - \beta B^{F_\beta^\varepsilon}u_\beta \rangle \]
\[ \quad + \langle B^{F_\beta^\varepsilon}u_\beta - B^{F_\lambda^\varepsilon}u_\lambda, J^{F_\beta^\varepsilon}u_\beta - J^{F_\lambda^\varepsilon}u_\lambda \rangle. \tag{13} \]

Using Lemma 2.6 (iii), the right hand side of (13) is bounded below by
\[ \langle B^{F_\lambda^\varepsilon}u_\lambda - B^{F_\beta^\varepsilon}u_\beta, \lambda B^{F_\lambda^\varepsilon}u_\lambda - \beta B^{F_\beta^\varepsilon}u_\beta \rangle = \lambda \left( \|B^{F_\lambda^\varepsilon}u_\lambda\|^2 - \langle B^{F_\lambda^\varepsilon}u_\lambda, B^{F_\beta^\varepsilon}u_\beta \rangle \right) \]
\[ \quad + \beta \left( \|B^{F_\beta^\varepsilon}u_\beta\|^2 - \langle B^{F_\beta^\varepsilon}u_\beta, B^{F_\lambda^\varepsilon}u_\lambda \rangle \right) \]
\[ = \lambda\|B^{F_\lambda^\varepsilon}u_\lambda\|^2 - \frac{1}{2}\|B^{F_\beta^\varepsilon}u_\beta\|^2 + \beta\|B^{F_\beta^\varepsilon}u_\beta - \frac{\lambda}{\beta}B^{F_\lambda^\varepsilon}u_\lambda\|^2 \]
\[ - \frac{1}{2}\|B^{F_\lambda^\varepsilon}u_\lambda\|^2 - \frac{\beta}{4}\|B^{F_\beta^\varepsilon}u_\beta\|^2 \]
\[ \geq - \frac{\lambda}{4}\|B^{F_\lambda^\varepsilon}u_\lambda\|^2 - \frac{\beta}{4}\|B^{F_\beta^\varepsilon}u_\beta\|^2 \]
\[ \geq - \frac{(\lambda + 2\beta)}{4}C^2. \]
and then, for a.e. \( t \in [0, T] \),
\[
\frac{d}{dt} \left( \|u_\lambda(t) - u_\beta(t)\|^2 \right) \leq \frac{(\lambda + \beta)}{2} C^2.
\]
Thus, for each \( \lambda, \beta > 0 \) and \( t \in [0, T] \)
\[
\|u_\lambda(t) - u_\beta(t)\| \leq C \sqrt{\frac{\lambda + \beta}{2}} T.
\]
We deduce that, for \( \lambda \to 0 \), the family \( \{u_\lambda\} \) is a Cauchy net in \( C^0([0, T], \mathcal{H}) \).

**Step 2.** Strong convergence of \( \{u_\lambda\} \) to a solution \( u \) of \((DEP_{\varepsilon(t)})\), as \( \lambda \to 0 \).

Since \( \{u_\lambda\} \) is Cauchy in the Banach space \( C^0([0, T], \mathcal{H}) \), then \( u_\lambda \) strongly converges to some \( u \in C^0([0, T], \mathcal{H}) \) with
\[
\|u_\lambda(t) - u(t)\| \leq \|z_{\lambda_0}\| \sqrt{\frac{\lambda}{2}}, \text{ for } t \in [0, T] \text{ and } \lambda > 0.
\]
From the estimate \( \|J_{\lambda_k}^F u_\lambda(t) - u_\lambda(t)\| = \|\lambda B_{\lambda_k}^F u_\lambda(t)\| \leq \lambda \|z_{\lambda_0}\| \), we deduce that the function \( v_\lambda(t) := J_{\lambda_k}^F u_\lambda(t) \) strongly converges in \( C^0([0, T], \mathcal{H}) \) to \( u \).

Since \( (\dot{u}_\lambda) \) is bounded in the Hilbert space \( L^2([0, T], \mathcal{H}) \), there is a sequence \( (\lambda_k) \) for which \( (\dot{u}_{\lambda_k}) \) weakly converges. Passing to limits in \( u_{\lambda_k}(t) = u_0 + \int_0^t \dot{u}_{\lambda_k}(s) \, ds \)
we find that the weak limit of \( (\dot{u}_{\lambda_k}) \) in \( L^2([0, T], \mathcal{H}) \) is \( \dot{u} \), and then for a.e. \( t \in [0, T] \)
\[
w - \lim_{k \to +\infty} \dot{u}_{\lambda_k}(t) = \dot{u}(t) \text{ in } \mathcal{H}.
\]
Return to equation (9), we have for a.e. \( t > 0 \) and a.e. \( t \in [0, T] \)
\[
\dot{u}_{\lambda_k}(t) = \frac{1}{\lambda_k} \left( J_{\lambda_k}^F u_{\lambda_k}(t) - u_{\lambda_k}(t) \right) = 0 \text{ with } u_{\lambda_k}(0) = u_0,
\]
and by using (2), we obtain for a.e. \( t > 0 \) and all \( y \in K \)
\[
F(J_{\lambda_k}^F u_{\lambda_k}(t), y) + \varepsilon(t) \langle u_{\lambda_k}(t), y - J_{\lambda_k}^F u_{\lambda_k}(t) \rangle + \langle \dot{u}_{\lambda_k}(t), y - J_{\lambda_k}^F u_{\lambda_k}(t) \rangle \geq 0. \tag{14}
\]
Since \( F \) is monotone and \( F(y, \cdot) \) is convex and lower semicontinuous for every \( y \in K \), passing to the limit in (14), we obtain
\[
F(y, u(t)) \leq \liminf_{k \to +\infty} F(y, J_{\lambda_k}^F u_{\lambda_k}(t))
\leq \liminf_{k \to +\infty} -F(J_{\lambda_k}^F u_{\lambda_k}(t), y)
\leq \lim_{k \to +\infty} \left( \langle \dot{u}_{\lambda_k}(t), y - J_{\lambda_k}^F u_{\lambda_k}(t) \rangle \right)
= \langle \dot{u}(t), y - u(t) \rangle.
\]
By Lemma 3.1, we finally get for a.e. \( t > 0 \)
\[
F(u(t), y) + \varepsilon(t) \langle u(t), y - u(t) \rangle + \langle \dot{u}(t), y - u(t) \rangle \geq 0, \quad \forall y \in K, \tag{15}
\]
and then \( u \) is a solution \((DEP_{\varepsilon(t)})\).

**Step 3.** Uniqueness of solution for \((DEP_{\varepsilon(t)})\). For uniqueness, suppose \( v \) is another solution of \((DEP_{\varepsilon(t)})\), then for a.e. \( t > 0 \)
\[
F(v(t), u(t)) + \varepsilon(t) \langle v(t), u(t) - v(t) \rangle + \langle \dot{v}(t), v(t) - u(t) \rangle \geq 0. \tag{16}
\]
Summing these two inequalities (15) and (16), and using monotonicity of \( F \), we obtain: for a.e. \( t > 0 \)
\[
\varepsilon(t)\|v(t) - u(t)\|^2 + \langle \dot{v}(t) - \dot{u}(t), v(t) - u(t) \rangle \leq 0
\]
By setting $\theta(t) = \|v(t) - u(t)\|^2$, we deduce $2\varepsilon(t)\theta(t) + \dot{\theta}(t) \leq 0$, and then by differential form of Gronwall’s Lemma, we have

$$
\theta(t) \leq \theta(0) \exp\left(-\int_0^t 2\varepsilon(s) ds\right),
$$

which means $\theta(t) = 0$ for every $t > 0$ since $\theta(0) = 0$. Thus $u = v$ on $\mathbb{R}_+$, and then the solution of $(DEP_{\epsilon(t)})$ is unique.

4. Rate of convergence of the optimal values of the dynamic equilibrium systems. In this section, we discuss the rate of convergence of values of the dynamic equilibrium problems $(DEP)$ and $(DEP_{\epsilon(t)})$. To prove our results, we need the following lemma.

**Lemma 4.1.** Suppose $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ nonincreasing and $\int_0^{+\infty} \varphi(s) ds < +\infty$, then

$$
\lim_{t \to +\infty} t\varphi(t) = 0.
$$

**Proof.** We have $\int_0^{+\infty} \varphi(s) ds < +\infty$, which implies that $\lim \inf_{t \to +\infty} \varphi(t) = 0$, and since $\varphi$ is nonincreasing, then $\lim_{t \to +\infty} \varphi(t) = 0$.

Also, for each $0 < s < t$

$$
t\varphi(t) = \varphi(t) \int_0^t d\tau = \varphi(t) \int_0^s d\tau + \varphi(t) \int_s^t d\tau \\
\leq \varphi(t)s + \int_s^t \varphi(\tau) d\tau.
$$

By taking respectively $\lim \sup$ as $t \to +\infty$ and $\lim \inf$ as $s \to +\infty$, we get

$$
\lim_{t \to +\infty} t\varphi(t) \leq 0;
$$

and since $\varphi(t) \geq 0$, we conclude $\lim_{t \to +\infty} t\varphi(t) = 0$.

**Theorem 4.2.** Suppose that $F : K \times K \to \mathbb{R}$ satisfies the following condition: there exists $\delta \geq 0$ and $\alpha > 1$ such that

$$
F(x,z) \geq F(x,y) + F(y,z) - \delta\|z - y\|^{\alpha}, \quad \forall x \in S_F, \forall y,z \in K. \tag{17}
$$

Then, for each solution $u$ of $(DEP)$ and each $x \in S_F$, we have

$$
\text{for a.e. } t \geq 0, \quad \frac{d}{dt} F(x,u(t)) = -\|\dot{u}(t)\|^2 \text{ and } F(x,u(t)) = o\left(\frac{1}{t}\right).
$$

**Proof.** Fix $x \in S_F$ and, for $t \geq 0$, set $\varphi(t) = F(x,u(t))$. By using assumption (17), and $u$ is solution to $(DEP)$, we have for each $h \geq 0$ and a.e. $t \geq 0$,

$$
\varphi(t+h) - \varphi(t) = F(x,u(t+h)) - F(x,u(t)) \\
\geq F(u(t),u(t+h)) - \delta\|u(t+h) - u(t)\|^\alpha \\
\geq -\langle \dot{u}(t), u(t+h) - u(t) \rangle - \delta\|u(t+h) - u(t)\|^\alpha.
$$

Dividing this inequality by $h$ (positive and negative) and letting $h \to 0$, we obtain

$$
\text{for a.e. } t \geq 0, \quad \frac{d}{dt} F(x,u(t)) = \frac{d\varphi}{dt}(t) = -\|\dot{u}(t)\|^2
$$

which means also that $\varphi$ is nonincreasing.

Return to $(DEP)$, we have

$$
\varphi(t) = F(x,u(t)) \leq -\frac{1}{2} \frac{d}{dt} \|u(t) - x\|^2;
$$

$$
\dot{\varphi}(t) = -\frac{1}{2} \frac{d}{dt} \|u(t) - x\|^2
$$

$$
\|u(t) - x\| \leq \frac{1}{\sqrt{\frac{d}{dt}}}.
$$
By assumption (17), since 

\[ \varphi(t) dt \leq - \int_0^{+\infty} \frac{1}{2} \frac{d}{dt} \| u(t) - x \|^2 \leq \frac{1}{2} \| u(0) - x \|^2 < +\infty. \]

Therefore, Lemma 4.1 ensures

\[ \lim_{t \to +\infty} tF(x, u(t)) = \lim_{t \to +\infty} t\varphi(t) = 0, \]

which means \( F(x, u(t)) = o(\frac{1}{t}) \).

**Theorem 4.3.** Suppose that \( F : K \times K \to \mathbb{R} \) satisfies condition (17). Let \( u \) be a solution of \( (DEP_{\epsilon(t)}) \), where \( \epsilon \) is nonincreasing and satisfying \( \int_0^{+\infty} \epsilon(t) dt < +\infty \). Then, for each \( x \in S_F \), we have \( F(x, u(t)) = o \left( \frac{1}{t} \right) \) and \( \lim_{t \to +\infty} t\epsilon(t) \| u(t) \|^2 = 0. \)

**Proof.** By assumption (17), since \( u \) is a solution of \( (DEP_{\epsilon(t)}) \), we have for a.e. \( t > 0 \) each \( h > 0 \) and each \( x \in S_F \)

\[ F(x, u(t + h)) - F(x, u(t)) \geq F(u(t), u(t + h)) - \delta\| u(t + h) - u(t) \|^2 \]

\[ \geq - \langle \dot{u}(t), u(t + h) - u(t) \rangle - \delta\| u(t + h) - u(t) \|^2 \]

By setting \( \varphi(t) = F(x, u(t)) \) and dividing this inequality by \( h \) and letting \( h \to 0 \), we get

\[ \frac{d\varphi(t)}{dt} = - \| \dot{u}(t) \|^2 - \epsilon(t)\langle u(t), \dot{u}(t) \rangle = - \| \dot{u}(t) \|^2 - \frac{\epsilon(t)}{2} \frac{d}{dt} \left( \| u(t) \|^2 \right). \]

Since \( \epsilon \) is nonincreasing, then

\[ \epsilon(t) \frac{d}{dt} \left( \| u(t) \|^2 \right) = \frac{d}{dt} \left( \epsilon(t) \| u(t) \|^2 \right) - \epsilon(t)\| u(t) \|^2 \geq \frac{d}{dt} \left( \epsilon(t) \| u(t) \|^2 \right); \]

and therefore

\[ \frac{d}{dt} \left( \varphi(t) + \frac{\epsilon(t)}{2} \| u(t) \|^2 \right) \leq - \| \dot{u}(t) \|^2 \leq 0. \]

We conclude that the real function \( \varphi(t) + \epsilon(t) \| u(t) \|^2 \) is nonincreasing.

Return to \( u \) solution of \( (DEP_{\epsilon(t)}) \), then

\[ \varphi(t) + \frac{\epsilon(t)}{2} \| u(t) \|^2 \leq \langle \dot{u}(t), x - u(t) \rangle + \epsilon(t)\langle u(t), x - u(t) \rangle + \frac{\epsilon(t)}{2} \| u(t) \|^2 \]

\[ = - \frac{1}{2} \frac{d}{dt} \left( \| u(t) - x \|^2 \right) - \frac{\epsilon(t)}{2} \| u(t) \|^2 + \epsilon(t)\langle u(t), x \rangle \]

\[ = - \frac{1}{2} \frac{d}{dt} \left( \| u(t) - x \|^2 \right) - \frac{\epsilon(t)}{2} \| u(t) - x \|^2 + \frac{\epsilon(t)}{2} \| x \|^2 \]

After integrating, since \( \int_0^{+\infty} \epsilon(t) dt < +\infty \), we get

\[ \int_0^{+\infty} \left( \varphi(t) + \frac{\epsilon(t)}{2} \| u(t) \|^2 \right) dt \leq \frac{1}{2} \| u_0 - x \|^2 + \| x \|^2 \int_0^{+\infty} \epsilon(t) dt < +\infty \]

Return to Lemma 4.1, we conclude

\[ \lim_{t \to +\infty} t \left( F(x, u(t)) + \frac{\epsilon(t)}{2} \| u(t) \|^2 \right) = 0; \]

which yields the results. \( \square \)
Remark 3. • In the first example of bifunctions satisfying condition (17), we consider the bifunction $F(x, y) := f(y) - f(x)$ for $(x, y) \in K \times K$ where $f : K \rightarrow \mathbb{R}$ is a real function. Indeed, for each $\delta \geq 0$, $\alpha > 1$ and $x, y \in K$, we have

$$F(x, z) - F(x, y) - F(y, z) = 0 \geq -\delta\|z - y\|^\alpha.$$  

• As a second example, from [30], we consider $F(x, y) := -f(Ty - Tx)$, where $T$ maps $K$ into $\mathcal{H}$ and $f : \mathcal{H} \rightarrow \mathbb{R}$ is a subadditive real function on $\mathcal{H}$.

We have for $x, y, z \in K$,

$$F(x, z) = -f(Tz - Tx) = -f((Tz - Ty) + (Ty - Tx)) \geq -f(Tz - Ty) - f(Ty - Tx) = F(y, z) + F(x, y)$$

and then condition (17) is satisfied for $\delta = 0$.

Remark 4. Suppose that $F : K \times K \rightarrow \mathbb{R}$ satisfies: there exists $\delta_1, \delta_2 \geq 0, \alpha > 1$ such that

$$F(x, y) \leq F(x, z) + F(z, y) + \delta_1\|z - y\|^\alpha, \quad \forall x \in S_F, \forall y, z \in K,$$  

and the weak monotonicity condition

$$F(z, y) + F(y, z) \leq \delta_2\|z - y\|^\alpha, \quad \forall y, z \in K.$$  

Then condition (17) is also satisfied. It suffices to sum inequalities (18) and (19), and to take $\delta = \delta_1 + \delta_2$.

5. Asymptotic behavior of solutions. We assume that the equilibrium points set $S_F = \{x \in K : F(x, y) \geq 0, \forall y \in K\}$ is nonempty and we denote by $x^*$ the least norm element of $S_F$, i.e. $x^* = P_{S_F}(0)$ is the orthogonal projection of the point 0 onto $S_F$.

In the case where $\varepsilon$ is a constant parameter, we have

Lemma 5.1. For any fixed $\varepsilon > 0$ the perturbed bifunction $F_\varepsilon(x, y) = F(x, y) + \varepsilon\langle x, y - x \rangle$ is strongly monotone and each solution $u_\varepsilon(\cdot)$ of the dynamical equilibrium problem $(DEP_{\varepsilon})$ converges strongly, as $t \rightarrow +\infty$, to the unique equilibrium point $x_\varepsilon$ of $(EP_{\varepsilon})$.

Proof. The first assertion is justified as in Lemma 2.1, so for any fixed $\varepsilon > 0$, the equilibrium problem $(EP_{\varepsilon})$ has a unique equilibrium point $x_\varepsilon$. If we suppose that $u_\varepsilon(t)$ is a solution of $(DEP_{\varepsilon})$ with $u_\varepsilon(0) = u_0$, we have

$$\langle \dot{u}_\varepsilon(t) - u_\varepsilon(t) - x_\varepsilon\rangle \leq F_\varepsilon(u_\varepsilon(t), x_\varepsilon) \leq -F_\varepsilon(x_\varepsilon, u_\varepsilon) - \varepsilon\|u_\varepsilon(t) - x_\varepsilon\|^2,$$

and then

$$\frac{d}{dt}\left(\frac{1}{2}\|u_\varepsilon(t) - x_\varepsilon\|^2\right) + \frac{\varepsilon}{2}\|u_\varepsilon(t) - x_\varepsilon\|^2 \leq 0.$$  

Integration of this differential inequality gives $\|u_\varepsilon(t) - x_\varepsilon\|^2 \leq \|u_0 - x_\varepsilon\|^2 e^{-\varepsilon t}$, and therefore $u_\varepsilon(t)$ converges strongly to $x_\varepsilon$, when $t \rightarrow +\infty$. \hfill $\square$

 Afterwards, see Lemma 5.8, when passing to the limit as the parameter $\varepsilon$ tends to zero, the unique equilibrium point $x_\varepsilon$ converges strongly to the element of minimum norm of the closed convex nonempty set $S_F$.

To move to the least norm solution $x^*$ of $S_F$ in a single step, we consider the dynamic perturbation parameter setting $(DEP_{\varepsilon(t)})$. We will see that the convergence depends whether $\varepsilon(t)$ is in $L^1([0, +\infty[)$ or not.

In this section, when $\varepsilon \in L^1(\mathbb{R}^*_+)$, we study the weak ergodic convergence of general maximal monotone bifunctions.
5.1. Weak ergodic convergence of general maximal monotone bifunctions.

At first, we give some lemmas which we need in the sequel.

**Lemma 5.2 (Jensen’s inequality).** Let $H$ be a Hilbert space and $K$ be a nonempty closed convex set. Let $u : [0, +\infty) \to K$ be an integrable function and let $f : K \to \mathbb{R}$ be convex and lower semi-continuous. If $f \circ u$ is integrable, then for any $t > 0$,

$$f \left( \frac{1}{t} \int_0^t u(s)ds \right) \leq \frac{1}{t} \int_0^t f(u(s))ds.$$ 

**Lemma 5.3 (Opial-Passty [32]).** Let $H$ be a Hilbert space, let $S$ be a nonempty subset of $H$ and let $u : [0, +\infty) \to H$ be a function. For any $t > 0$, set $U(t) = \frac{1}{t} \int_0^t u(s)ds$, if we suppose that

i) $\forall z \in S$, $\lim_{t \to +\infty} \left\| u(t) - z \right\|$ exists.

ii) If $t_n \to +\infty$ and $U(t_n) \to U$ weakly in $H$, then $U \in S$.

Then, the whole weak limit $w = \lim_{t \to +\infty} U(t)$ exists and belongs to $S$.

**Theorem 5.4.** Suppose $F$ satisfies $(H_1) - (H_4)$ and $\varepsilon(t) \to 0$ as $t \to +\infty$. If $\int_0^t \varepsilon(t)dt < \infty$, then the unique solution $u(t)$ of $(DEP_{\varepsilon(t)})$, with initial condition $u(0) = u_0$, converges weakly in average to some $x^\infty \in S_F$, i.e., $\frac{1}{t} \int_0^t u(s)ds \to x^\infty$.

*Proof.* Let us verify conditions (i) and (ii) of the Opial-Passty Lemma.

For Condition (i), fix $z \in S_F$ and set, for every $t \geq 0$, $\theta(t) = \frac{1}{2} \| u(t) - z \|^2$. By monotonicity of $F$ we have:

$$\theta(t) = -\langle \dot{u}(t), z - u(t) \rangle \leq F(u(t), z) + \varepsilon(t) \langle u(t), z - u(t) \rangle$$

$$\leq -F(z, u(t)) + \varepsilon(t) \langle u(t), z - u(t) \rangle$$

$$\leq \varepsilon(t) \langle u(t), z - u(t) \rangle$$

$$= \frac{\varepsilon(t)}{2} \left[ \| z \|^2 - \| u(t) \|^2 - \| z - u(t) \|^2 \right]$$

from which, it follows that

$$\frac{d}{dt} \left( \theta(t) - \frac{\| z \|^2}{2} \int_0^t \varepsilon(s)ds \right) = \dot{\theta}(t) - \frac{1}{2} \| z \|^2 \varepsilon(t) \leq 0.$$ 

Thus the real function $\theta(t) - \frac{\| z \|^2}{2} \int_0^t \varepsilon(s)ds$ is nonincreasing and hence converges to a limit as $t \to +\infty$. Since $\int_0^{+\infty} \varepsilon(t)dt < \infty$, we conclude that $\theta(t)$ has a limit as $t \to +\infty$, and then $\lim_{t \to +\infty} \| z - u(t) \|$ exists.

For Condition (ii), set $U(t_n) = \frac{1}{t_n} \int_0^{t_n} u(s)ds$ and suppose $U(t_n) \to U$, for a sequence $t_n \to +\infty$. Since $u(t)$ is solution of $(DEP_{\varepsilon(t)})$, for almost every $t > 0$ and $y \in K$ we have,

$$F(y, u(t)) \leq \langle \dot{u}(t), y - u(t) \rangle + \varepsilon(t) \langle u(t), y - u(t) \rangle,$$

which implies that

$$F(y, u(t)) \leq \frac{1}{2} \frac{d}{dt} \| u(t) - y \|^2 + \frac{\varepsilon(t)}{2} \| y \|^2.$$ 

After integrating on $[0, t_n]$ and dividing by $t_n$, we obtain

$$\frac{1}{t_n} \int_0^{t_n} F(y, u(s)ds \leq \frac{1}{2t_n} \| u_0 - y \|^2 + \frac{1}{2t_n} \| y \|^2 \int_0^{t_n} \varepsilon(s)ds.$$ 

(21)

Since $F(y, \cdot)$ is convex and lower semi-continuous, then by Lemma 5.2, we get

$$F(y, U(t_n)) \leq \frac{1}{t_n} \int_0^{t_n} F(y, u(s)ds \leq \frac{1}{2t_n} \| u_0 - y \|^2 + \frac{1}{2t_n} \| y \|^2 \int_0^{t_n} \varepsilon(s)ds.$$ 

(22)
Passing to the limit as $t_n \to +\infty$ and using $\int_0^{+\infty} \varepsilon(t)dt < \infty$, we can easily conclude that for every $y \in K$,
\[
F(y, U^\infty) \leq \lim_{n \to +\infty} F(y, U(t_n)) \leq 0,
\]
and so $U^\infty \in S_F$. Therefore, all conditions of Opial-Passty Lemma 5.3 hold, and then the unique solution $u(t)$ of $(DEP_{\varepsilon(\cdot)})$ converges weakly in average to some $x^\infty \in S_F$.

5.2. Case of demipositive bifunctions. In this subsection, we study the Tikhonov regularization for a special class of demipositive monotone bifunctions.

**Definition 5.5.** A monotone bifunction $F : K \times K \to \mathbb{R}$ is called demipositive if there exists $z \in S$ such that for every sequence $\{u_n\} \subset K$ converging weakly to $u$, we have: $F(u_n, z) \to 0$ implies that $u \in S$.

The following proposition gives an important example for demipositive bifunctions, which ensure the strong convergence.

**Proposition 1 ([15]).** If we suppose that the monotone bifunction $F$ verifies $(H_1)$, $(H_3)$, $(H_4)$ and is also 3-monotone, i.e. $F(x,y) + F(y,z) + F(z,x) \leq 0$, for each $x, y, z \in K$, then $F$ is demipositive.

**Proof.** For $S \neq \emptyset$, consider some $z \in S$ and a sequence $\{u_n\} \subset K$ such that $u_n \to u$ and $\lim_{n \to +\infty} F(u_n, z) = 0$. According to Definition 5.5, it suffices to show that $u \in S$. Fix $y \in K$, since $F$ is a 3-monotone bifunction then
\[
F(u_n, z) + F(z, y) + F(y, u_n) \leq 0;
\]
and as $z \in S$, we find $F(y, u_n) \leq -F(u_n, z)$.

By passing to the limit when $n \to +\infty$, and using lower semicontinuity of $F(y, \cdot)$ we get
\[
F(y, u) \leq \liminf_{n \to +\infty} F(y, u_n) \leq - \lim_{n \to +\infty} F(u_n, z) = 0,
\]
then by Minty’s Lemma, we get $u \in S$.

**Remark 5.** This class of demipositive monotone bifunctions (see [15]) contains as particular cases, more than 3-monotone ones, strongly monotone bifunctions, those such that $\text{int}(S_F \neq \emptyset)$ and $\mu$-cocoercive differentiable for the second variable bifunctions.

**Lemma 5.6 (Opial [31]).** Let $\mathcal{H}$ be a Hilbert space, let $S$ be a nonempty subset of $\mathcal{H}$ and let $u : [0, +\infty[ \to \mathcal{H}$ be a function. if we suppose that

- $i)$ $\forall z \in S$, $\lim_{t \to +\infty} \|u(t) - z\|$ exists.
- $ii)$ If $t_n \to +\infty$ and $u(t_n) \to u^\infty$ weakly in $\mathcal{H}$, then $u^\infty \in S$.

Then, the whole weak limit $w - \lim_{t \to +\infty} u(t)$ exists and belongs to $S$.

**Theorem 5.7.** Suppose the monotone bifunction $F$ is demipositive and $u : [0, +\infty[ \to \mathcal{H}$ is the unique solution of $(DEP_{\varepsilon(\cdot)})$ for $u(0) = u_0$, where $\varepsilon(t) \to 0$ as $t \to +\infty$. If $\int_0^{+\infty} \varepsilon(t)dt < \infty$, then $u(t) \to x^\infty$, where $x^\infty \in S_F$.

**Proof.** For every $z \in S_F$, set $\theta(t) = \frac{1}{2}\|u(t) - z\|^2$ for $t > 0$.

Following the proof of Theorem 5.4, we get the same inequality (20), which means, since $\int_0^{+\infty} \varepsilon(t)dt < \infty$, that the limit $\lim_{t \to +\infty} \|u(t) - z\|$ exists. Invoking Opial’s Lemma 5.6, one concludes if we show that every weak accumulation point of $u(t)$ belongs to $S_F$. 
Consider \( t_n \to +\infty \) for which \( u(t_n) \to y \). The fact that \( u \) is absolutely continuous implies that \( \theta(t) \) is also absolutely continuous, and then
\[
\int_0^\infty |\dot{\theta}(t)|dt \leq \frac{\|z\|^2}{2} \int_0^\infty \varepsilon(t)dt < \infty.
\]
Hence \( \dot{\theta}(t) \in L^1[0, +\infty[ \) and thus there is a subsequence \( (t_{n_k}) \) of \( (t_n) \) such that \( \dot{\theta}(t_{n_k}) \to 0 \) as \( k \to \infty \).

Since \( u \) is the solution of \((\text{DEP}_\varepsilon)\) and \( z \in S_F \), we have:
\[
(u(t_{n_k}), z - u(t_{n_k})) + F(u(t_{n_k}), z) + \varepsilon(t_{n_k})(u(t_{n_k}), z - u(t_{n_k})) \geq 0,
\]
which means
\[
-\dot{\theta}(t_{n_k}) + \varepsilon(t_{n_k})(u(t_{n_k}), z - u(t_{n_k})) \geq -F(u(t_{n_k}), z) \geq 0.
\]
Passing to the limit when \( k \to \infty \), we obtain \( \lim_{n \to \infty} F(u(t_{n_k}), z) = 0 \). Since \( u(t_{n_k}) \to y \) and \( F \) is demipositive, we conclude that \( y \in S_F \), and thus every weak accumulation point of \( u(t) \) belongs to \( S_F \).

To prove our result of strong asymptotic convergence when \( \varepsilon \not\in L^1(\mathbb{R}^+) \), we need the following two lemmas.

**Lemma 5.8.** If the solution set \( S_F \) of the problem \((\text{EP})\) is nonempty, then the net \( \{x_\varepsilon\} \) of unique solutions of the regularized problems \((\text{EP}_\varepsilon)\) converges strongly, as \( \varepsilon \to 0 \), to the element \( x^* \) of minimum norm of the closed convex \( S_F \).

**Proof.** Since \( x^* \in S_F \) and \( x_\varepsilon \) resolves \((\text{EP}_\varepsilon)\), monotonicity of \( F \) gives
\[
\varepsilon(||x_\varepsilon||^2 - \|x^*\| \cdot ||x_\varepsilon||) \leq \varepsilon(||x_\varepsilon||^2 -\langle x_\varepsilon, x^* \rangle) \leq -\varepsilon\langle x_\varepsilon, x^* - x_\varepsilon \rangle - F(x_\varepsilon, x^*) \leq 0,
\]
and then \( \langle x_\varepsilon \rangle \) remains bounded: \( ||x_\varepsilon|| \leq ||x^*|| \) for every \( \varepsilon > 0 \).

If we take \( x_\varepsilon \) a weak limit point of \( x_\varepsilon \), then there exists a sequence \( \varepsilon_k \to 0 \) such that \( x_{\varepsilon_k} \to x_\infty \). Since \( x_{\varepsilon_k} \) is an equilibrium point of \( F_\varepsilon \), then for any \( y \in K \) we have
\[
F(x_{\varepsilon_k}, y) + \varepsilon\langle x_{\varepsilon_k}, y - x_{\varepsilon_k} \rangle \geq 0,
\]
which implies by monotonicity of \( F \), \( F(y, x_{\varepsilon_k}) \leq \varepsilon\langle x_{\varepsilon_k}, y - x_{\varepsilon_k} \rangle \). Going to the limit when \( \varepsilon_k \to 0 \), we obtain
\[
F(y, x_\infty) \leq \liminf_{\varepsilon_k \to 0} F(y, x_{\varepsilon_k}) \leq \lim_{\varepsilon_k \to 0} \varepsilon\langle x_{\varepsilon_k}, y - x_{\varepsilon_k} \rangle = 0.
\]
By Minty’s Lemma, we conclude that \( x_\infty \) belong to \( S_F \), and using the weak lower semicontinuity of the norm, the inequality \( ||x_{\varepsilon_k}|| \leq ||x^*|| \) for every \( k \) gives \( ||x_\infty|| \leq ||x^*|| \), and then \( x_\infty = x^* \). We conclude \( x^* \) is the unique weak cluster point of the bounded family \( \{x_\varepsilon\} \), and then \( x_\varepsilon \to x^* \).

Since the norm is weakly lower semicontinuous, we also have
\[
||x^*|| \leq \liminf_{\varepsilon \to 0} ||x_\varepsilon|| \leq \limsup_{\varepsilon \to 0} ||x_\varepsilon|| \leq ||x^*||.
\]
which implies that \( ||x_\varepsilon|| \to ||x^*|| \), and then the \( (x_\varepsilon) \) converges strongly to \( x^* \). \( \square \)

**Lemma 5.9** ([16]). Let \( h : [0, +\infty[ \to \mathbb{R} \) be a bounded function and \( \varepsilon : [0, +\infty[ \to \mathbb{R}^+ \) with \( \varepsilon \in L^1_{\text{loc}}(\mathbb{R}^+) \). Suppose \( \theta : [0, +\infty[ \to \mathbb{R} \) is an absolutely continuous function with \( \theta + \varepsilon(t)\theta(t) \leq \varepsilon(t)\theta(t) \) for almost all \( t \geq 0 \), then \( \theta(t) \) is bounded.

If moreover \( \lim_{t \to +\infty} \varepsilon(t)dt = \infty \), then \( \limsup_{t \to +\infty} \theta(t) \leq \limsup_{t \to +\infty} h(t) \).
Theorem 5.10. Suppose $F$ verifies $(H_1)-(H_4)$ and is $3$-monotone, and let $u : [0, +\infty) \to H$ be the unique solution of $(DEP_{\varepsilon(t)})$ with $\varepsilon(t) \to 0$ as $t \to +\infty$. If $\int_0^{+\infty} \varepsilon(t) dt = \infty$, then $(u(t))$ strongly converges to $x^*$ as $t \to +\infty$.

Proof. Set again $\theta(t) = \frac{1}{2} \|u(t) - x^*\|^2$ for $t > 0$, then

$$\dot{\theta}(t) = \langle \dot{u}(t), u(t) - x^* \rangle \leq F(u(t), x^*) + \varepsilon(t) \langle u(t), x^* - u(t) \rangle. \quad (23)$$

Since $F$ is $3$-monotone, then for each $\varepsilon > 0$, $F(u(t), x^*) + F(x^*, x^*) + F(x^*, u(t)) \leq 0$, where $x^*$ is the unique solutions of the regularized problems $(EP_{\varepsilon(t)})$. Taking $\varepsilon = \varepsilon(t)$, then

$$F(u(t), x^*) + \varepsilon(t) \langle u(t), x^* - u(t) \rangle \leq -F(x^*, x(t)) + F(x(t), u(t)) + \varepsilon(t) \langle u(t), x^* - u(t) \rangle. \quad (24)$$

As $F(x^*, x(t)) \geq 0$ and $-F(x(t), u(t)) \leq \varepsilon(t) \langle x(t), u(t) - x(t) \rangle$, we conclude via (23) and (24) that

$$\dot{\theta}(t) \leq -F(x^*, x(t)) - F(x(t), u(t)) + \varepsilon(t) \langle u(t), x^* - u(t) \rangle$$

$$= \frac{\varepsilon(t)}{2} \left( \|u(t)\|^2 - \|x(t)\|^2 - \|u(t) - x(t)\|^2 \right) + \frac{\varepsilon(t)}{2} \left( \|x^*\|^2 - \|u(t)\|^2 - \|u(t) - x^*\|^2 \right)$$

$$\leq -\varepsilon(t) \theta(t) + \frac{\varepsilon(t)}{2} \left( \|x^*\|^2 - \|x(t)\|^2 \right),$$

and thus

$$\dot{\theta}(t) + \varepsilon(t) \theta(t) \leq \frac{\varepsilon(t)}{2} \left( \|x^*\|^2 - \|x(t)\|^2 \right).$$

Now, invoking Lemma 5.8, and since $\varepsilon(t) \to 0$ as $t \to +\infty$, we deduce that $(x(t))$ converges strongly to $x^*$. According to Lemma 5.9 with $h(t) = \frac{1}{2} \left( \|x^*\|^2 - \|x(t)\|^2 \right)$, we conclude that $\limsup_{t \to +\infty} \theta(t) \leq 0$, and thus $u(t) \to x^*$, which is the claim of the theorem. \hfill \Box

Example 5.11 (Tikhonov dynamic system for convex minimization).

Let $f : K \to \mathbb{R}$ be a convex and lower semicontinuous function, and consider the minimization problem

$$\min_{x \in K} f(x), \quad (MP)$$

where the optimal solution set $S = \{x \in K : f(x) \leq f(y), \forall y \in K\}$ is assumed nonempty. The Tikhonov regularization concerning this problem has been studied in paper [16], by formulating the problem $(MP)$ as an equilibrium problem $(EP)$ by setting $F(x, y) = f(y) - f(x)$. It is clear that $F$ satisfies all conditions $(H_1)-(H_4)$ and is $3$-monotone; also the problem $(DEP_{\varepsilon(t)})$ becomes

$$\langle u(t) + \varepsilon(t) u(t), y - u(t) \rangle + f(y) - f(u(t)) \leq 0, \forall y \in K. \quad (MP_{\varepsilon(t)})$$

Using Theorems 4.3, 5.7 and 5.10, we get:

Corollary 1. Suppose $f : K \to \mathbb{R}$ is convex lsc, the solution set of $(MP)$ is nonempty, and consider $u(t)$ the unique solution of $(MP_{\varepsilon(t)})$.

- If $\varepsilon$ is nonincreasing and $\int_0^{+\infty} \varepsilon(t) dt < \infty$, then $f(u(t)) - \inf_K f = o\left(\frac{1}{t}\right)$ and $\lim_{t \to +\infty} t \varepsilon(t) \|u(t)\|^2 = 0$. 

• If \( \int_0^{+\infty} \varepsilon(t) dt = \infty \), then \( u(t) \) converges strongly to the least-norm element of \( S_F \).

• If \( \int_0^{+\infty} \varepsilon(t) dt < \infty \), then \( u(t) \) converges weakly towards a point in \( S \).

**Remark 6.** Notice that in this framework, we generalize the result obtained by [16] about the strong convergence when \( \varepsilon \notin L^1(\mathbb{R}_+) \). We get this convergence not only for the Fenchel-Moreau subdifferential of convex functions but also for all 3-monotone bifunctions; this is one benefit of using bifunction’s method.

In Example 5.13, we treat a 3-monotone bifunction which satisfies conditions \((H_1)-(H_4)\) and is not cyclically monotone.

**Definition 5.12.** For \( n = 2, 3, \ldots \), we say that a bifunction \( F : K \times K \to \mathbb{R} \) is \( n \)-cyclically monotone if for every \( x_1, x_2, \ldots, x_n \in K \) and for \( x_{n+1} = x_1 \), one has \( \sum_{k=1}^{n} F(x_k, x_{k+1}) \leq 0 \).

\( F \) is said to be cyclically monotone if it is \( n \)-cyclically monotone for each integer \( n \geq 2 \).

Remark that a bifunction \( F \) is monotone if it is 2-cyclically monotone, and \( F \) is 3-monotone if it is 3-cyclically monotone.

**Remark 7.** Suppose \( F \) is \( n \)-cyclically monotone, then according to Theorem 5.7 (for \( n = 2 \)) and Theorem (5.10) (for \( n = 3 \)), we obtain respectively the weak convergence and the strong convergence to a solution of \((MP)\) of the Tikhonov evolution problem \((MP_{\varepsilon(t)})\).

**Example 5.13** (Tikhonov dynamic system for 3-monotone bifunction which is not cyclically monotone).

According to Example 4.6 in [5], let \( K = \mathbb{R}^2 \) and for \( n \in \{2, 3, \ldots\} \), denote by \( R_n \) the matrix corresponding to counter-clockwise \( \frac{\pi}{n} \)-rotation, i.e.,

\[
R_n = \begin{pmatrix}
\cos(\pi/n) & -\sin(\pi/n) \\
\sin(\pi/n) & \cos(\pi/n)
\end{pmatrix}.
\]

Then, as an operator from \( K \) to \( K \), \( R_n \) is maximal monotone and \( n \)-cyclically monotone, but it is not \((n+1)\)-cyclically monotone. This implies that the bifunction \( G_{R_n} \) defined on \( \mathbb{R}^2 \times \mathbb{R}^2 \) by \( G_{R_n}(X, Y) = \langle R_n X, Y - X \rangle_{\mathbb{R}^2} \) satisfies conditions \((H_1)-(H_4)\) and is \( n \)-cyclically monotone, nevertheless it is not \((n+1)\)-cyclically monotone.

For \( n = 3 \), \( x = (x_1, x_2) \in \mathbb{R}^2 \) and \( y = (y_1, y_2) \in \mathbb{R}^2 \), we have

\[
G_{R_3}(x, y) = \frac{1}{2} \left[(x_1 - \sqrt{3}x_2)(y_1 - x_1) + (\sqrt{3}x_1 + x_2)(y_2 - x_2)\right].
\]

In particular, \( G_{R_3} \) is 3-monotone but is not cyclically monotone. According to Theorem 5.10 (respectively Theorem 5.7), the associate Tikhonov regularization:

\[
\langle \dot{u}(t), y - u(t) \rangle + G_{R_3}(u(t), y) + \varepsilon(t) \langle u(t), y - u(t) \rangle \geq 0, \forall y \in K = \mathbb{R}^2
\]

has a unique solution \( u(t) \) which converges strongly (resp. weakly) as \( \varepsilon(t) \to 0 \) to the least-norm element of \( S_F \), the solution set of \( G_{R_3} \), if \( \int_0^{+\infty} \varepsilon(t) dt = +\infty \) (resp. \( \int_0^{+\infty} \varepsilon(t) dt < +\infty \)).
5.3. Case where $\varepsilon(\cdot)$ has bounded variations. As already treated in [15], when $\int_0^{+\infty} \varepsilon(t)dt < +\infty$ the trajectories of $(\text{DEP}_{\varepsilon(t)})$ converge (weakly or strongly) to a point in $S_E$ if, and only if, the corresponding property holds for unperturbed evolution problem $(\text{DEP})$. Suppose $\int_0^{+\infty} \varepsilon(t)dt = \infty$. Can this condition ensure the strong convergence of the trajectories of $(\text{DEP}_{\varepsilon(t)})$ to $x^*$?

In this subsection, similarly as in [16] for maximal monotone operators Tikhonov’s dynamic systems, we study the strong convergence of the solution of $(D_{\varepsilon(t)})$ for any monotone bifunction in the case where $\varepsilon \notin L^1([0,+\infty[)$, but has bounded variations, i.e., $\int_0^{+\infty} |\varepsilon(t)|dt < +\infty$.

The following lemma characterizes the strong convergence of the solutions of $(\text{DEP}_{\varepsilon(t)})$.

**Lemma 5.14.** Under conditions above, the unique solution $u(t)$ of $(\text{DEP}_{\varepsilon(t)})$ is bounded. If, moreover $\int_0^{+\infty} \varepsilon(t)dt = \infty$, then for $t \to +\infty$ the following properties are equivalent:

(a) every weak cluster point of $u(t)$ belongs to $S_E$;
(b) $\liminf_{t \to +\infty} \|u(t)\| \geq \|x^*\|$;
(c) $u(t)$ converges strongly to $x^*$.

**Proof.** Consider $\theta(t) = \frac{1}{2} \|u(t) - x^*\|^2$ and using the monotonicity of $F$ we obtain

$$
\dot{\theta}(t) = - \langle \dot{u}(t), x^* - u(t) \rangle \leq F(u(t), x^*) + \varepsilon(t) \langle u(t), x^* - u(t) \rangle \\
\leq -F(x^*, u(t)) + \varepsilon(t) \langle u(t), x^* - u(t) \rangle \\
\leq \varepsilon(t) \langle u(t), x^* - u(t) \rangle \\
= \frac{\varepsilon(t)}{2} \left[ \|x^*\|^2 - \|u(t)\|^2 \right].
$$

By the same argument as in [16, Proposition 6], we get equivalence between the properties (a), (b) and (c).

By exploiting Lemma 5.14 we can prove the following strong convergence result.

**Theorem 5.15.** Suppose $F$ satisfies $(H_1)$ -- $(H_4)$, $\lim_{t \to +\infty} \varepsilon(t) = 0$, $\int_0^{+\infty} \varepsilon(t)dt = \infty$ and $\int_0^{+\infty} |\dot{\varepsilon}(t)|dt < +\infty$. Then the unique solution $u : [0, +\infty[ \to H$ of $(\text{DEP}_{\varepsilon(t)})$ converges strongly, as $\varepsilon \to 0$, to the element of minimum norm $x^*$ of the solution set $S_E$ of the problem $(EP)$.

**Proof.** We adopt the scheme used by [16, Theorem 9]

**Step 1.** There is a zero Lebesgue measure set outside of which $\dot{u}(t) \to 0$ as $t \to +\infty$.

Set, for $h > 0$, $\theta_h(t) = \frac{1}{2} \|u(t+h) - u(t)\|^2$ with $t > 0$. By monotonicity of $F$ we have

$$
\dot{\theta}_h(t) = \langle \dot{u}(t+h) - \dot{u}(t), u(t+h) - u(t) \rangle \\
= -\langle \dot{u}(t+h), u(t) - u(t+h) \rangle - \langle \dot{u}(t), u(t+h) - u(t) \rangle \\
\leq F(u(t+h), u(t)) + \varepsilon(t+h) \langle u(t+h), u(t) - u(t+h) \rangle + F(u(t), u(t+h)) \\
= \varepsilon(t+h) \langle u(t+h), u(t) - u(t+h) \rangle + \varepsilon(t) \langle u(t), u(t+h) - u(t) \rangle \\
= -\varepsilon(t) \langle u(t+h), u(t) - u(t+h) \rangle \cdot \left( \|u(t+h)\|^2 - \|u(t)\|^2 \right).
$$
Multiplying this inequality by $e^{\alpha_h(t)}$, where $\alpha_h(t) := \int_0^t (\varepsilon(s + h) + \varepsilon(s)) \, ds$, and integrating on $[t', t]$, we obtain

$$\frac{2\theta_h(t) e^{\alpha_h(t)}}{h^2} \leq \frac{2\theta_h(t') e^{\alpha_h(t')}}{h^2} + \frac{1}{h^2} \int_{t'}^t e^{\alpha_h(s)} (\varepsilon(s) - \varepsilon(s + h)) \left(\|u(s + h)\|^2 - \|u(s)\|^2\right) \, ds.$$  

(25)

Since the absolutely continuous function $u$ is a solution of $\text{(DEP}_\varepsilon\text{)}$, there exists a set $N \subset [0, +\infty]$ of zero Lebesgue measure such that $u$ is differentiable on the dense set $D = ]0, +\infty[ \setminus N$, and then for $M = \sup_{s \geq 0} \|u(s)\|^2$, passing to the limit in (25) if $h \to 0$, we obtain for each $t, t' \in D$ with $t' > t$

$$e^{\alpha_0(t)} \|\hat{u}(t)\|^2 \leq e^{\alpha_0(t')} \|\hat{u}(t')\|^2 - 2 \int_{t'}^t e^{\alpha_0(s)} \hat{\varepsilon}(s) \langle \hat{u}(s), u(s) \rangle \, ds$$

$$\leq e^{\alpha_0(t')} \|\hat{u}(t')\|^2 + \int_{t'}^t e^{\alpha_0(s)} \|\hat{\varepsilon}(s)\| \left(\|\hat{u}(s)\|^2 + \|u(s)\|^2\right) \, ds$$

$$\leq e^{\alpha_0(t')} \|\hat{u}(t')\|^2 + \int_{t'}^t \|\hat{\varepsilon}(s)\| e^{\alpha_0(s)} \|\hat{u}(s)\|^2 \, ds + M \int_{t'}^t e^{\alpha_0(s)} \|\hat{\varepsilon}(s)\| \, ds.$$  

By Gronwall’s Lemma, we get

$$e^{\alpha_0(t)} \|\hat{u}(t)\|^2 \leq \left(e^{\alpha_0(t')} \|\hat{u}(t')\|^2 + M \int_{t'}^t e^{\alpha_0(s)} \|\hat{\varepsilon}(s)\| \, ds\right) \exp\left(\int_{t'}^t \|\hat{\varepsilon}(s)\| \, ds\right).$$

Dividing this inequality by $e^{\alpha_0(t)}$ and letting $t \to +\infty$ with $t \in D$, we obtain for every $t' \in D$

$$\limsup_{t \to +\infty} \|\hat{u}(t)\|^2 \leq M \int_{t'}^t \|\hat{\varepsilon}(s)\| \, ds \exp\left(\int_{t'}^t \|\hat{\varepsilon}(s)\| \, ds\right).$$

With condition $\int_0^{+\infty} \|\hat{\varepsilon}(t)\| \, dt < +\infty$, the last inequality states that

$$\limsup_{t \to +\infty} \|\hat{u}(t)\|^2 \leq \lim_{t' \to +\infty} \limsup_{t \to +\infty} \int_{t'}^t \|\hat{\varepsilon}(s)\| \, ds \exp\left(\int_{t'}^{+\infty} \|\hat{\varepsilon}(s)\| \, ds\right) = 0;$$

and thus $\hat{u}(t)$ strongly converges to 0 as $t \to +\infty$ in $D$.

**Step 2.** Every weak cluster point of $u(t)$ lies in $S_F$.

Let $\bar{x}$ be a weak cluster point of $u(t)$, and choose $t_k \to +\infty$ such that $u(t_k) \to \bar{x}$. Since $u(.)$ is continuous and $D$ is dense in $[0, +\infty[$, one may find $\bar{t}_k \in D$ close enough to $t_k$ so that $\|u(\bar{t}_k) - u(t_k)\| \leq \frac{1}{k}$, and therefore the sequence $(u(\bar{t}_k))$ also weakly converges to $\bar{x}$.

Using the first step above, we affirm that $\hat{u}(\bar{t}_k) \to 0$, and since $\varepsilon(t) \to 0$ and $u(t)$ is bounded, it follows that $\hat{u}(\bar{t}_k) + \varepsilon(\bar{t}_k) u(\bar{t}_k)$ converges strongly to 0. Now, using conditions $(H_2)$ and $(H_4)$ on $F$, we obtain: for each $y \in K$

$$F(y, \bar{x}) \leq \liminf_{k \to +\infty} F(y, u(\bar{t}_k))$$

$$\leq \liminf_{k \to +\infty} -F(u(\bar{t}_k), y)$$

$$\leq \lim_{k \to +\infty} \langle \hat{u}(\bar{t}_k) + \varepsilon(\bar{t}_k) u(\bar{t}_k), y - u(\bar{t}_k) \rangle = 0.$$
By using the Minty’s Lemma, we conclude that \( \bar{x} \in S \).

**Step 3.** Combining Step 2 with \((a) \Rightarrow (c)\) in Lemma 5.14, we deduce the strong convergence of the solution \( u(t) \) of \((DEP_{\epsilon(t)})\) to \( x^* \).

**Remark 8.** The demipositivity is an important condition to get the weak convergence of the solution \( u(t) \) of \((DEP)\), and the solution \( u(t) \) of \((DEP_{\epsilon(t)})\). Theorem 5.15 offers a way for dealing with a non necessarily demipositive monotone bifunction to get strong convergence of \( u(t) \) to the least norm element of \( S_F \), whenever only \( \epsilon \) satisfies the conditions:

\[
\int_0^{+\infty} \epsilon(t)dt = +\infty \quad \text{and} \quad \int_0^{+\infty} |\dot{\epsilon}(t)|dt < +\infty
\]

5.3.1. Application to saddle points. Consider \( U \) (respectively \( V \)) a nonempty closed convex subset of the Hilbert space \( H_1 \) (respectively \( H_2 \)), and \( L : U \times V \to \mathbb{R} \) a closed convex-concave real function, i.e., \( L(u, v) \) is convex lower semicontinuous in \( u \) and concave upper semicontinuous in \( v \). We consider the saddle-point problem:

Find \((\bar{u}, \bar{v}) \in U \times V\), such that

\[
L(\bar{u}, v) \leq L(\bar{u}, \bar{v}) \leq L(u, \bar{v}) \quad \text{for each } (u, v) \in U \times V \qquad (SP)
\]

By setting the bifunction

\[ F_L((u, v), (u', v')) := L(u', v) - L(u, v') \quad \text{for each } (u, v), (u', v') \in K = U \times V, \]

we can see [12] that the problems \((SP)\) and \((SP)\) are equivalent, i.e. \((\bar{u}, \bar{v})\) is a solution of \((SP)\) iff it is a solution of \((EP)\). Then the regularization problem \((DEP_{\epsilon(t)})\) of \((SP)\) becomes:

\[
L(x, v(t)) - L(u(t), y) + \langle \dot{u}(t), x - u(t) \rangle + \langle \dot{v}(t), y - v(t) \rangle + \epsilon(t)\langle v(t), y - v(t) \rangle \geq 0.
\]

\((DSP_{\epsilon(t)})\)

As in [15], when we take in \((DSP_{\epsilon(t)})\) respectively \( y = v \) and \( x = u \), we deduce the nonlinear dynamical system

\[
\begin{cases}
-\dot{u} \in \partial(L(., v) + \delta_U)(u) + \epsilon u \\
-\dot{v} \in \partial(-L(u, .) + \delta_V)(v) + \epsilon v
\end{cases}
\]

\((DS)\)

Conversely, one can easily go back from \((DS)\) to \((DSP_{\epsilon(t)})\), and then these two problems are equivalent. Thus, according to Theorem 5.15, we conclude

**Corollary 2.** Under conditions above, if \( \int_0^{+\infty} \epsilon(s)ds = +\infty \) and \( \int_0^{+\infty} |\dot{\epsilon}(s)|ds < +\infty \), then the unique solution of \((DS)\) strongly converges to the least norm element of the set of all saddle points of \( L \).

**Remark 9.** In the following example, we justify that the strong convergence of the trajectory \( x(t) \) in Corollary 2 may be obtained without imposing the condition \( \int_0^{+\infty} |\dot{\epsilon}(t)|dt < +\infty \).

**Example 5.16.**

Consider \( L(u, v) = uv \) on \( \mathbb{R}^2 \), then the unique saddle point of \( L \) is \( (\bar{u}, \bar{v}) = (0, 0) \).

The associated bifunction \( F_L \) defined on \( \mathbb{R}^2 \times \mathbb{R}^2 \) by

\[ F_L((u, v), (u', v')) = u'v - uv' \]

satisfies all conditions \((H_1)-(H_4)\) but is not demipositive (see [15]).
The dynamical system \((DS)\) becomes
\[
\dot{u} = -v + \varepsilon u \quad \text{and} \quad \dot{v} = u + \varepsilon v.
\] (26)
Then, for the initial condition \((u(0), v(0)) = (u_0, v_0)\), the solution of (26) is
\[
x(t) := (u(t), v(t)) = e^{-\int_0^t \varepsilon(s) ds} (u_0 \cos(t) - v_0 \sin(t)), u_0 \sin(t) + v_0 \cos(t));
\]
which means on \([0, +\infty[\]
\[
\|x(t)\|_2 = e^{-\int_0^t \varepsilon(s) ds} \|(u_0, v_0)\|_2.
\]
When, we only suppose \(\int_0^{+\infty} \varepsilon(s) ds = +\infty\), the trajectory \(x(t)\) converges to the unique saddle point \((0, 0)\).
If \(\int_0^{+\infty} \varepsilon(s) ds < +\infty\) then the trajectory \(x(t) = (u(t), v(t))\) attempts to reach from outside the circle where the center is the origin and the radius is \(e^{-\int_0^{+\infty} \varepsilon(s) ds}\) and can never converge.

5.3.2. Neural model for convex programming. The linear Programming Problems has received considerable research attention from the neural networks community. The first solution of the linear programming problem was proposed by Tank and Hopfield wherein they used the continuous-time Hopfield network \([22]\), afterwards, many researchers were inspired by their work, see \([36, 26, 19]\) for a historical and bibliographical study of these problems. We consider here the convex programming problem:
\[
\min f(x) \text{ subject to } x \geq 0 \text{ and } g(x) \leq 0, \quad (CP)
\]
where \(f\) and \(g_i\), for \(i = 1 \cdots, m\), are convex lower semicontinuous real-valued functions on the non-negative orthant \(\mathbb{R}_+^n\).
In \([15]\) Chbani-Riahi proposed a neural dynamical model for solving \((CP)\) and study the asymptotic behavior of the solution of an associated dynamical equilibrium problem generated by the associate bifunction \(F_L\) where \(L\) is the Lagrangian saddle-function defined by:
\[
L(x, \lambda) = f(x) + \sum_{i=1}^m \ln(1 + \lambda_i)g_i(x), \text{ for } x \in \mathbb{R}_+^n \text{ and } \lambda \in \mathbb{R}_+^m. \quad (27)
\]
Their proof is based on the fact that \((\bar{x}, \bar{\lambda})\) is a saddle point of \(L\), i.e. an equilibrium point of \(F_L\), implies \(\bar{x}\) is an optimum vector for \((CP)\). By using a Tikhonov regularization we consider the following neural dynamical model for solving \((CP)\):
\[
\begin{cases}
\dot{x}(t) + \nabla f(x(t)) + \sum_{i=1}^m \ln(1 + \lambda_i(t)) \nabla g_i(x(t)) + \varepsilon(t) x(t), z - x(t) \geq 0, \\
\sum_{i=1}^m \left( \lambda_i(t) - \frac{1}{1 + \lambda_i(t)} g_i(x(t)) + \varepsilon(t) \lambda_i(t) \right) (\beta_i - \lambda_i(t)) \geq 0, \\
\forall (z, \beta) \in \mathbb{R}_+^n \times \mathbb{R}_+^m \text{ and } (x(0), \lambda(0)) = (x_0, \lambda_0).
\end{cases}
\] (28)
which equivalent to
\[
\begin{aligned}
\dot{x}(t) + \nabla f(x(t)) + \sum_{i=1}^{m} \ln(1 + \lambda_i(t)) \nabla g_i(x(t)) + \varepsilon(t)x(t) &\in -N_{\mathbb{R}_+^n}(x(t)) \\
\dot{\lambda}(t) - \left[ \frac{1}{1 + \lambda_i(t)} g_i(x(t)) \right]_i + \varepsilon(t)\lambda(t) &\in -N_{\mathbb{R}_+^m}(\lambda(t)) \\
(x(0), \lambda(0)) &=(x_0, \lambda_0).
\end{aligned}
\]

(MN_{\varepsilon(t)})

Since $L$ is a closed convex-concave saddle function on $\mathbb{R}_+^n \times \mathbb{R}_+^m$, Corollary 2 ensures

**Corollary 3.** Suppose $f, g_i$, for $i = 1 \cdots m$, are convex lower semicontinuous real-valued functions on $\mathbb{R}_+^n$, $\int_0^{+\infty} \varepsilon(s)ds = +\infty$ and conditions $\int_0^{+\infty} |\dot{\varepsilon}(s)|ds < +\infty$. The the unique solution of the nonlinear dynamical system $(MN_{\varepsilon(t)})$ converges to the least norm element of the set of the solutions of $(CP)$.

**Remark 10.** In contrary to [15, Theorem 6.3.], we need no strict convexity condition on the functions $f$ and $g_i$ to reach a solution of $(CP)$ via the asymptotic behavior of a solution of the dynamical system $(MN_{\varepsilon(t)})$.

6. Interchange of the penalty setting (Multiscale aspects). The purpose of this section is to establish the link between solutions of the following two Tikhonov’s regularization evolution problems:

\[
\langle \dot{u}(t), y - u(t) \rangle + F(u(t), y) + \varepsilon(t)\langle u(t), y - u(t) \rangle \geq 0 \quad \forall y \in K, \quad (DEP_{\varepsilon(t)})
\]

and

\[
\langle \dot{v}(t), y - v(t) \rangle + \beta(t)F(v(t), y) + \langle v(t), y - v(t) \rangle \geq 0 \quad \forall y \in K, \quad (DEP_{\beta(t)})
\]

where the positive function $\beta(t)$ (respectively $\varepsilon(t)$) converges to $+\infty$ (respectively to zero) as $t \to +\infty$.

Let us first mention that any solution of one of the two following problems

\[
F(u, y) + \varepsilon(t)(u, y - u) \geq 0 \quad \forall y \in K \quad \text{and} \quad \beta(t)F(v, y) + \langle v, y - v \rangle \geq 0 \quad \forall y \in K,
\]

is also a solution of the other; simply use $\varepsilon(t)\beta(t) = 1$.

For the link between solutions of the problems $(DEP_{\varepsilon(t)})$ and $(DEP_{\beta(t)})$, consider two $C^1$ functions $\beta : [0, T_\beta] \to \mathbb{R}_+$ and $\varepsilon : [0, T_\varepsilon] \to \mathbb{R}_+$, where $T_\beta, T_\varepsilon \in \mathbb{R}_+ \cup \{+\infty\}$.

Implicitly define the real functions $t_\beta : [0, T_\beta] \to [0, T_\beta]$ and $t_\varepsilon : [0, T_\varepsilon] \to [0, T_\varepsilon]$ by

\[
\int_0^{t_\beta(t)} \beta(s)ds = t \quad \text{and} \quad \int_0^{t_\varepsilon(t)} \varepsilon(s)ds = t.
\]

**Lemma 6.1.** [3, Lemma 4.1] Suppose $\beta(t)\varepsilon(t_\varepsilon(t)) = 1$ for every $t \geq 0$, then

(i) if $t_\varepsilon \circ t_\beta = id_{[0, T_\varepsilon]}$, $t_\beta \circ t_\varepsilon = id_{[0, T_\beta]}$, $t_\varepsilon = \int_0^{T_\beta} \beta(t)dt$ and $T_\beta = \int_0^{T_\varepsilon} \varepsilon(t)dt$;

(ii) the evolution problems $(DEP_{\varepsilon(t)})$ and $(DEP_{\beta(t)})$ are equivalent, i.e., if $u$ is a solution of $(DEP_{\varepsilon(t)})$ then $v = u \circ t_\varepsilon$ is a solution of $(DEP_{\beta(t)})$, and conversely, if $v$ is a solution of $(DEP_{\beta(t)})$, then $u = v \circ t_\beta$ is a solution of $(DEP_{\varepsilon(t)})$.

Accordingly, all our results may be established for the problem $(DEP_{\beta(t)})$. Before doing so, let us treat the corresponding assumption on $\beta$ for strong convergence to the least norm element of $S_F$. 

Remark 11. Setting \( \zeta(t) = \int_0^t \varepsilon(s)ds \), then from the definition of \( t_\varepsilon \) we have
\[
\zeta(t_\varepsilon(t)) = \int_0^{t_\varepsilon(t)} \varepsilon(s)ds = t.
\]
Since \( \int_0^{+\infty} \varepsilon(s)ds = +\infty \) and \( \zeta \) is a strictly increasing function from \( [0, +\infty[ \) onto \( [0, +\infty[ \), we have \( \lim_{t \to +\infty} t_\varepsilon(t) = \lim_{t \to +\infty} \zeta^{-1}(t) = +\infty \), and then for \( u \) a solution of \( (DEP_{\varepsilon(t)}) \) and \( v \) a solution of \( (DEP_{\beta(t)}) \), we deduce
\[
\lim_{t \to +\infty} u(t) \text{ exists and equal to } \ell \iff \lim_{s \to +\infty} v(s) \text{ exists and equal to } \ell.
\]

Remark 12. Take \( \beta \) and \( \varepsilon \) as in Lemma 6.1 with \( T_\beta = T_\varepsilon = +\infty \), then
\[
\int_0^{+\infty} |\dot{\xi}(t)|dt = \int_0^{+\infty} i_\varepsilon(s)|\dot{\xi}(t_\varepsilon(s))|ds \quad \text{by setting } t = t_\varepsilon(s)
\]
\[
= \int_0^{+\infty} \left| \frac{d}{ds} (\varepsilon \circ t_\varepsilon)(s) \right| ds \quad \text{by using } \dot{t}_\varepsilon(t_\varepsilon(t)) = 1
\]
\[
= \int_0^{+\infty} \frac{d}{ds} \left( \frac{1}{\beta(s)} \right) ds
\]
\[
= \int_0^{+\infty} \frac{\beta(s)}{\beta(s)^2} ds.
\]

Theorem 6.2. Consider \( v(t) \) a solution of \( (DEP_{\beta(t)}) \). Suppose \( \lim_{t \to +\infty} \beta(t) = +\infty \)

and assume moreover that either \( F \) is 3-monotone or \( \int_0^{+\infty} \frac{|\dot{\beta}(t)|}{\beta(t)^2} dt < +\infty \). Then \( (v(t)) \) strongly converges to \( x^* \) the least norm element of \( S_F \).

Proof. This is a consequence of Theorems 5.10 and 5.15, Lemma 6.1 and Remarks 11 and 12.  \( \square \)

Example 6.3. For \( \beta(t) = (t + 1)^\alpha \), where \( \alpha > 0 \), we have
\[
\int_0^{+\infty} \frac{|\dot{\beta}(t)|}{\beta(t)^2} dt = \int_0^{+\infty} \frac{\alpha}{(t + 1)^{\alpha+1}} dt < +\infty;
\]

and then conditions of Theorem 6.2 are satisfied.

Let us consider the differential inclusion:
\[
\dot{v}(t) + \beta(t)\partial \psi(v(t)) + v(t) \ni 0,
\]
where \( \psi : H \to \mathbb{R} \cup \{+\infty\} \) is a convex lsc function with domain \( K \) a closed convex set in \( H \). This problem can be seen as \( (DEP_{\beta(t)}) \) with \( F(x, y) = \psi(y) - \psi(x) \). Then according to Theorem 6.2, the solution of equation (29) converges strongly to the least norm element of argmin \( \psi \), the minimum set of \( \psi \).

We note that (29) is a particular case of the following asymptotic monotone inclusion
\[
\dot{v}(t) + \beta(t)\partial \psi(v(t)) + Av(t) \ni 0
\]
where \( A \) is a strongly monotone operator.

In order to ensure the strong convergence of the solution of (30) to the unique solution of the hierarchical problem \( 0 \in (A + N_C)(x^*) \), the authors in [3] use the following condition: for every \( p \) belonging to the range of \( N_C \), where \( C = \text{argmin } \psi \),
\[
\int_0^{+\infty} \beta(t) \left[ \psi^*(\frac{p}{\beta(t)}) - \sigma_C \left( \frac{p}{\beta(t)} \right) \right] dt < +\infty.
\]
For bifunctions, using results of [2] and [4], the authors in [13] used the Fitzpatrick transform of parametrized family of bifunctions to study the strong convergence of trajectories of a more general dynamical equilibrium system than \((DEP_{\beta(t)})\), where the quadratic form \((x, y) \rightarrow \langle x, y - x \rangle\), is replaced by a more general strongly monotone bifunction. More precisely, Theorem 4.4 in [13] use the following condition

\[\forall u \in S_F, \forall p \in N_{S_F}(u), \quad \int_0^{+\infty} \beta(t) \left[ F^*_u(p, \frac{1}{\beta(t)}) - \sigma_{S_F} \left( \frac{p}{\beta(t)} \right) \right] dt < +\infty. \tag{32} \]

In our setting, without condition (32), we present the same result in Theorem 6.2 by only assuming \(\lim_{t \to +\infty} \beta(t) = +\infty\) and either \(F\) is 3-monotone or \(\int_0^{+\infty} \frac{|\beta(t)|}{\beta(t)} dt < +\infty\).

**Example 6.4.** In the case when \(F(x, y) = \varphi(y) - \varphi(x)\), for \(x, y \in K\) and \(\varphi\) is a closed convex function on \(K\), \(F\) is a 3-monotone bifunction, and according to Theorem 6.2, without assumption (32), the solution of \((29)\) strongly converges to the element of minimal norm of \(S_F = \text{argmin}_{K} \varphi\).

**Example 6.5.** When \(F\) is not 3-monotone bifunction, take the next example for comparing between conditions (32) and \(\int_0^{+\infty} \frac{|\beta(t)|}{\beta(t)^2} dt < +\infty\).

For \(K = [0, 1] \times [0, 1]\), consider

\(F((x_1, x_2), (y_1, y_2)) = e^{x_2^2}(y_2^2 - x_2^2)\).

We have \(F\) is a maximal monotone bifunction and the solution set of equilibrium points is \(S_F = [0, 1] \times \{0\}\). To check Condition (32), we need to calculate for each \((u, v) \in S_F, N_{S_F}(u, v), F^*_u(p, q, t)\) and \(\sigma_{S_F} \left( \frac{1}{\beta(t)} (p, q) \right)\).

For \(p \in \mathbb{R} \times \mathbb{R}\) and \(u \in [0, 1]\) we have

\((p, q) \in N_{[0,1]\times\{0\}}(u, 0) \iff (p, q)(s - u, t) \leq 0, \forall (s, t) \in [0, 1] \times \{0\} \iff p(s - u) \leq 0, \forall s \in [0, 1];\)

so that

\[N_{S_F}(u, v) = \begin{cases} \mathbb{R} \times \mathbb{R} & \text{if } (u, v) = (0, 0), \\ \mathbb{R} \times \mathbb{R} & \text{if } (u, v) = (0, 0), \\ \{0\} \times \mathbb{R} & \text{if } u \in [0, 1], v = 0. \end{cases} \]

Also, we have

\[\sigma_{S_F} \left( \frac{p}{\beta(t)} : \frac{q}{\beta(t)} \right) = \sup_{u \in [0, 1]} \left\{ u \cdot \frac{p}{\beta(t)} \right\}, \]

and

\[F^*_u(p, q, t) = \sup_{(s, v) \in [0, 1] \times [0, 1]} \left\{ \left( \frac{p}{\beta(t)} : \frac{q}{\beta(t)} \right) (s, v) - F((u, 0), (s, v)) \right\} = \sup_{(s, v) \in [0, 1] \times [0, 1]} \left\{ s \cdot \frac{p}{\beta(t)} + v \cdot \frac{q}{\beta(t)} - v^2 \right\} = \sup_{s \in [0, 1]} \left\{ s \cdot \frac{p}{\beta(t)} \right\} + \sup_{v \in [0, 1]} \left\{ v \cdot \frac{q}{\beta(t)} - v^2 \right\} = \sigma_{S_F} \left( \frac{p}{\beta(t)} : \frac{q}{\beta(t)} \right) + \frac{q^2}{\beta^2(t)}\]

Therefore

\[\int_0^{+\infty} \beta(t) \left[ F^*_u(p, q, t) - \sigma_{S_F} \left( \frac{p}{\beta(t)} : \frac{q}{\beta(t)} \right) \right] dt = q^2 \int_0^{+\infty} \frac{dt}{\beta(t)}.\]
We conclude the condition (32) is verified iff \( \int_0^{+\infty} \frac{dt}{\beta(t)} < +\infty \).

Consider the function \( \beta(t) = (1 + t)^{\alpha} \), then our condition \( \int_0^{+\infty} \frac{\beta(t)}{\beta(t)^2} dt < +\infty \) is verified for each \( \alpha \geq 0 \), while the assumption (32) is satisfied iff \( \alpha > 1 \), which means that conditions in our Theorem 6.2 are wider than those used in [13, Theorem 4.4].

7. Strong convergence of the Prox-Tikhonov and forward-backward algorithms. A time discretization of dynamical systems is used to link between algorithms and continuous dynamical systems, and their asymptotic analysis. We firstly use an implicit discretization (the prox-penalization algorithm) (ProxPA) of the dynamical system (DEP_{\alpha(t)}) by starting from point \( x_0 \in K \) and iterating to go from \( x_n \in K \) to \( x_{n+1} = J_{\lambda_n}^F(x_n) \), where \( \{\varepsilon_n\} \) are proximal parameters, and \( \{\lambda_n\} \) is a positive sequence of proximal parameters. Afterwards, we propose the descent-proximal (forward-backward) algorithm (DProxA): \( x_{n+1} = J_{\lambda_n}^F((1 - \lambda_n \varepsilon_n)x_n) \).

We consider the sequence \( \{x_{\varepsilon_n}\}_{n \in \mathbb{N}} \) where \( x_{\varepsilon_n} \) is the unique equilibrium point of \( F_n \), which existence and uniqueness is ensured by conditions (H1) – (H4) and strong monotonicity of \( F_n \). Then from Lemma 5.8, we have \( \lim_{n \to +\infty} ||x_{\varepsilon_n} - x^*|| = 0 \), where \( x^* \) the least norm element of the set of equilibrium points \( S_F \).

To study the strong convergence of the algorithms (ProxPA) and (DProxA), we need the following classical convergence lemma for real sequences.

**Lemma 7.1.** Suppose \( \{a_n\} \subset \mathbb{R}_+ \) satisfies for each \( n \in \mathbb{N}, \ a_{n+1} \leq (1 - \alpha_n) a_n + \beta_n \), where \( \alpha_n \in (0, 1], \ \beta_n \geq 0 \) and \( \lim_{n \to +\infty} \frac{\beta_n}{a_n} = 0 \). Then \( \lim_{n \to +\infty} a_n \) exists. If moreover \( \sum_{k=1}^{+\infty} \alpha_k = +\infty \), then \( \lim_{n \to +\infty} a_n = 0 \).

**Theorem 7.2.** Suppose that \( F \) verifies (H1) – (H4) and is 3-monotone.

If \( \sum_{n=0}^{+\infty} \lambda_n \varepsilon_n = +\infty \), then the sequence \( \{x_{\varepsilon_n}\}, \) generated by (ProxPA), strongly converges to \( x^* \) the orthogonal projection of \( 0 \) onto \( S_F \).

**Proof.** Set \( y = x^* \) in (ProxPA), then

\[
\lambda_n F(x_{n+1}, x^*) \geq \lambda_n \varepsilon_n (x_{n+1}, x_{n+1} - x^*) + \langle x_{n+1} - x_n, x_{n+1} - x^* \rangle. \tag{33}
\]

Furthermore, \( F \) is 3-monotone then \( F(x_{n+1}, x^*) + F(x^*, x_{n+1}) + F(x_{\varepsilon_n}, x_{n+1}) \leq 0 \). Combining this with \( F(x^*, x_{\varepsilon_n}) \geq 0 \) and \( F_n(x_{\varepsilon_n}, x_{n+1}) \geq 0 \), we obtain

\[
\lambda_n F(x_{n+1}, x^*) \leq \lambda_n \varepsilon_n \langle x_{\varepsilon_n}, x_{n+1} - x_{\varepsilon_n} \rangle. \tag{34}
\]

Summing (33) and (34), we conclude

\[
\lambda_n \varepsilon_n \langle x_{n+1}, x_{n+1} - x^* \rangle + \langle x_{n+1} - x_n, x_{n+1} - x^* \rangle \leq \lambda_n \varepsilon_n \langle x_{n+1}, x_{n+1} - x_{\varepsilon_n} \rangle,
\]

and then

\[
\|x_{n+1} - x_n\|^2 + \|x_{n+1} - x^*\|^2 - \|x_n - x^*\|^2 = 2\langle x_{n+1} - x_n, x_{n+1} - x^* \rangle \\
\leq 2\lambda_n \varepsilon_n \langle x_{n+1}, x_{n+1} - x_{\varepsilon_n} \rangle + \langle x_{n+1}, x^* - x_{n+1} \rangle \\
\leq \lambda_n \varepsilon_n (\|x_{n+1}\|^2 - \|x_{\varepsilon_n}\|^2 - \|x_{n+1} - x_{n+1}\|^2) \\
+ \lambda_n \varepsilon_n (\|x^*\|^2 - \|x_{n+1}\|^2 - \|x_{n+1} - x^*\|^2) \\
\leq \lambda_n \varepsilon_n (\|x^*\|^2 - \|x_{\varepsilon_n}\|^2 - \|x_{n+1} - x^*\|^2).
\]
We first observe that

$$\text{Proof.}$$

Then the sequence $$\{a_n\}$$ is bounded above, then the condition

$$\sum_{n=0}^{\infty} \lambda_n \varepsilon_n = +\infty$$

ensures that $$\sum_{n=0}^{\infty} \lambda_n \varepsilon_n = +\infty$$; otherwise, there exists a subsequence

$$\{\lambda_{k_n} \varepsilon_{k_n}\}$$

converging to $$+\infty$$, and then

$$\sum_{n=0}^{\infty} \lambda_n \varepsilon_n \geq \sum_{n=0}^{\infty} \alpha_{k_n} = \sum_{n=0}^{\infty} \frac{1}{1 + \lambda_{k_n} \varepsilon_{k_n}} = +\infty.$$ 

Finally, we have $$\sum_{n=0}^{\infty} \lambda_n \varepsilon_n = +\infty$$, and then by Lemma 7.1, we get strong convergence of the sequence $$\{x_n\}$$ to $$x^*$$. 

In the next theorems, we avoid 3-monotonicity of $$F$$ employed in Theorem 7.2 and reinforce the hypotheses on the parameters $$\lambda_n$$ and $$\varepsilon_n$$, in order to ensure the strong convergence of $$x_n$$ to $$x^*$$. 

**Theorem 7.3.** Consider $$F$$ satisfying conditions (H1) – (H4). Suppose that the following conditions are satisfied for the parameters $$\lambda_n$$ and $$\varepsilon_n$$:

(i) $$\lim_{n \to +\infty} \varepsilon_n = 0$$, $$\sum_{n=1}^{+\infty} \lambda_n \varepsilon_n = +\infty$$ and

(ii) $$\lim_{n \to +\infty} \left( \frac{1}{\varepsilon_{n+1}} - \frac{1}{\varepsilon_n} \right) \left( \frac{1}{\lambda_n} + \varepsilon_n \right) = 0.$$

Then the sequence $$\{x_n\}$$ generated by (ProxPA) strongly converges to the least norm element $$x^*$$ of $$S_F$$.

**Proof.** We first observe that

$$\|x_n - x^*\| \leq \|x_n - x_{\varepsilon_n}\| + \|x_{\varepsilon_n} - x^*\|$$

and $$\lim_{n \to +\infty} \|x_{\varepsilon_n} - x^*\| = 0.$$

To conclude $$x_n \to x^*$$, it remains to prove that $$\lim_{n \to +\infty} \|x_n - x_{\varepsilon_n}\| = 0$$. 

Set $$a_n = \|x_n - x_{\varepsilon_n}\|$$, then

$$a_{n+1} = \|x_{n+1} - x_{\varepsilon_n+1}\| \leq \|x_{n+1} - x_{\varepsilon_n}\| + \|x_{\varepsilon_n} - x_{\varepsilon_n+1}\|.$$ (36)

Next, we derive respectively from Lemmas 2.2 and 2.3, that

$$\|x_{n+1} - x_{\varepsilon_n}\| = \|J_{\lambda_n}^F(x_n) - J_{\lambda_n}^F(x_{\varepsilon_n})\| \leq (1 + \lambda_n \varepsilon_n)^{-1} \|x_n - x_{\varepsilon_n}\|,$$ (37)

$$x_{\varepsilon_n} = J_{\frac{x_n}{\varepsilon_n}}^F(0) = J_{\frac{1}{\varepsilon_{n+1}}}^F \left( \frac{1 - \varepsilon_n}{\varepsilon_{n+1}} x_{\varepsilon_n} \right).$$
and then
\[
\|x_{\epsilon_n} - x_{\epsilon_{n+1}}\| = \left\| J_{\frac{\epsilon_n}{\epsilon_n+1}}^F \left( 1 - \frac{\epsilon_n}{\epsilon_n+1} \right) x_{\epsilon_n} - J_{\frac{\epsilon_n}{\epsilon_n+1}}^F (0) \right\|
\]
\[
\leq \left| 1 - \frac{\epsilon_n}{\epsilon_n+1} \right| \|x_{\epsilon_n}\| \leq \left| \frac{\epsilon_{n+1} - \epsilon_n}{\epsilon_{n+1}} \right| \|x^*\|. \tag{38}
\]
Combining (36), (37) and (38), for \( \alpha_n = \frac{\lambda_n \epsilon_n}{1 + \lambda_n \epsilon_n} \) and \( \beta_k = \frac{\epsilon_{k+1} - \epsilon_k}{\epsilon_{k+1}} \|x^*\| \), we conclude
\[
a_{n+1} \leq (1 - \alpha_n) a_n + \beta_n. \tag{39}
\]
The rest of the proof of this result proceeds as in the proof of Theorem 7.2 by using
\[
\sum_{n=0}^{\infty} \lambda_n \epsilon_n = +\infty \text{ implies } \sum_{n=0}^{\infty} \alpha_n = +\infty. \]

**Theorem 7.4.** Under assumptions of Theorem 7.3, suppose instead of (ii) the following condition
\[
(iii) \quad \lim_{n \to +\infty} \left( \frac{1}{\epsilon_{n+1}} - \frac{1}{\epsilon_n} \right) \frac{1}{\lambda_n} = 0 \quad \text{and} \quad \lambda_n \epsilon_n \leq 1 \quad \text{for each } n.
\]
Then the sequence \( \{x_n\} \) generated by (DPProxA) strongly converges to the least norm element \( x^* \) of \( S_F \).

**Proof.** Setting \( a_n = \|x_n - x_{\epsilon_n}\| \) and using Lemmas 2.2 and 2.3, we have
\[
a_{n+1} = \|x_{n+1} - x_{\epsilon_{n+1}}\| \leq \|x_{n+1} - x_{\epsilon_n}\| + \|x_{\epsilon_n} - x_{\epsilon_{n+1}}\|
\]
\[
= \left\| J_{\frac{\epsilon_n}{\epsilon_n+1}}^F ((1 - \lambda_n \epsilon_n) x_{\epsilon_n}) - J_{\frac{\epsilon_n}{\epsilon_n+1}}^F ((1 - \lambda_n \epsilon_n) x_{\epsilon_n}) \right\| + \|x_{\epsilon_n} - x_{\epsilon_{n+1}}\|
\]
\[
\leq \left| 1 - \lambda_n \epsilon_n \right| \|x_n - x_{\epsilon_n}\| + \left| \frac{\epsilon_{n+1} - \epsilon_n}{\epsilon_{n+1}} \right| \|x^*\| \quad \text{since } \lambda_n \epsilon_n \leq 1. \tag{40}
\]
Set \( \alpha_n = \lambda_n \epsilon_n \) and \( \beta_k = \left| \frac{\epsilon_{k+1} - \epsilon_k}{\epsilon_{k+1}} \right| \|x^*\| \), conditions (i), (iii) and Lemma 7.1 ensure \( \{x_n\} \) strongly converges to \( x^* \) .

**Remark 13.** Take for example \( \epsilon_n = \frac{1}{n^\alpha} \) and \( \lambda_n = n^\alpha \), where \( \alpha > 0 \); then conditions (i) and (ii) of Theorems 7.3 and 7.4 above are satisfied if, and only if, \( \alpha > 0 \). Indeed, if \( \alpha \leq 0 \), then the common condition (i) is not met; and conversely, if \( \alpha > 0 \), then \( \lambda_n \epsilon_n = 1 \),
\[
\left( \frac{1}{\epsilon_{n+1}} - \frac{1}{\epsilon_n} \right) \left( \frac{1}{\lambda_n} + \epsilon_n \right) \approx \frac{2\alpha}{n} \quad \text{and} \quad \left( \frac{1}{\epsilon_{n+1}} - \frac{1}{\epsilon_n} \right) \frac{1}{\lambda_n} \approx \frac{\alpha}{n}.
\]
However, only condition \( \lambda_n \epsilon_n \leq 1 \) in (ii) of 7.4 does not hold when \( \lambda_n = cn^\alpha \) for each \( c > 1 \).

**REFERENCES**

[1] M. Ait Mansour, Z. Chbani and H. Riahi, Recession bifuntion and solvability of noncoercive equilibrium problems, Comm. Appl. Anal., 7 (2003), 369–377.

[2] M. H. Alizadeh, Monotone and Generalized Monotone Bifunctions and their Application to Operator Theory, Ph.D. Thesis, University of The Aegean, 2012.

[3] H. Attouch and M. O. Czarnecki, Asymptotic behavior of coupled dynamical systems with multiscale aspects, J. Differential Equations, 248 (2010), 1315–1344.
400 AICHA BALHAG, ZAKI CHBANI AND HASSAN RIAHI

[4] H. Attouch, A. Cabot and M. O. Czarnecki, Asymptotic behavior of nonautonomous monotone and subgradient evolution equations, Trans. Amer. Math. Soc., 370 (2018), 755–790.

[5] S. Bartz, H. H. Bauschke, J. Borwein, S. Reich and X. Wang, Fitzpatrick functions, cyclic monotonicity and Rockafellar’s antiderivative, Nonlinear Anal., 66 (2007), 1198–1223.

[6] E. Blum and W. Oettli, From optimization and variational inequalities to equilibrium problems, Math. Student, 63 (1994), 123–145.

[7] H. Brézis, Opérateurs maximaux monotones dans les espaces de Hilbert et équations d’évolution, Lecture Notes, vol. 5, North-Holland, 1972.

[8] F. E. Browder, Nonlinear operators and nonlinear equations of evolution in Banach spaces, Proc. Sympos. Appl. Math., vol. 18 (part 2), Amer. Math. Soc., Providence, RI, 1976, 1–308.

[9] R. E. Bruck, Asymptotic convergence of nonlinear contraction semigroups in Hilbert space, J. Funct. Anal., 18 (1975), 15–26.

[10] O. Chadli, Z. Chbani and H. Riahi, Recession methods for equilibrium problems and applications to variational and hemivariational inequalities, Discrete Contin. Dyn. Syst., 5 (1999), 185–196.

[11] O. Chadli, Z. Chbani and H. Riahi, Equilibrium problems and noncoercive variational inequalities, Optimization, 50 (2001), 17–27.

[12] O. Chadli, Z. Chbani and H. Riahi, Equilibrium problems with generalized monotone bifunctions and Applications to Variational inequalities, J. Optim. Theory Appl., 105 (2000), 299–323.

[13] Z. Chbani, Z. Mazgouri and H. Riahi, From convergence of dynamical equilibrium systems to bilevel hierarchical Ky Fan minimax inequalities and applications, Accepted in Minimax Theory Appl. 04 (2019), No. 2.

[14] Z. Chbani and H. Riahi, Variational principle for monotone and maximal bifunctions, Serdica Math. J., 29 (2003), 159–166.

[15] Z. Chbani and H. Riahi, Existence and asymptotic behaviour for solutions of dynamical equilibrium systems, Evol. Equ. Control Theory, 3 (2014), 1–14.

[16] R. Cominetti, J. Peypouquet and S. Sorin, Strong asymptotic convergence of evolution equations governed by maximal monotone operators with Tikhonov regularization, J. Differential Equations, 245 (2008), 3753–3763.

[17] X. P. Ding, Auxiliary principle and algorithm for mixed equilibrium problems and bilevel mixed equilibrium problems in Banach spaces, J. Optim. Theory Appl., 146 (2010), 347–357.

[18] B. V. Dinh and L. D. Muu, On penalty and gap function methods for bilevel equilibrium problems, J. Appl. Math., (2011), Art. ID 646452, 14 pp.

[19] S. Efati and M. Baymani, A new nonlinear neural network for solving convex nonlinear programming problems, Appl. Math. Comput., 168 (2005), 1370–1379.

[20] K. Fan, A minimax inequality and application, in Inequalities, III (Proc. Third Sympos., UCLA, 1969. Dedicated to the Memory of T. S. Motzkin; O. Shisha, Ed.), Academic Press, New York, (1972), 103–113.

[21] N. Hadjisavvas and H. Khatibzadeh, Maximal monotonicity of bifunctions, Optimization, 59 (2010), 147–160.

[22] J. J. Hopfield and D. W. Tank, Neural computation of decisions in optimization problems, Biol. Cybernet., 52 (1985), 141–152.

[23] P. G. Hung and L. D. Muu, The Tikhonov regularization extended to equilibrium problems involving pseudomonotone bifunctions, Nonlinear Anal., 74 (2011), 6121–6129.

[24] H. Khatibzadeh and S. Ranjbar, On the strong convergence of halpern type proximal point algorithm, J. Optim. Theory Appl., 158 (2013), 385–396.

[25] N. Lehdimi and A. Moudafi, Combining the proximal algorithm and Tikhonov regularization, Optimization, 37 (1996), 239–252.

[26] F. Li, Delayed Lagrangian neural networks for solving convex programming problems, Neural Comput., 73 (2010), 2266–2273.

[27] G. Mastroeni, Gap functions for equilibrium problems, J. Global Optim., 27 (2003), 411–426.

[28] A. Moudafi, A recession notion for a class of monotone bivariate functions, Serdica Math. J., 26 (2000), 207–220.

[29] A. Moudafi, Proximal methods for a class of bilevel monotone equilibrium problems, J. Global Optim., 47 (2010), 287–292.

[30] W. Oettli and M. Théra, Equivalents of Ekeland’s principle, Bull. Austral. Math. Soc., 48 (1993), 385–392.
[31] Z. Opial, Weak convergence of the sequence of successive approximations for nonexpansive mappings, *Bull. Amer. Math. Soc.*, **73** (1967), 591–597.

[32] G. B. Passty, Ergodic convergence to a zero of the sum of monotone operators in Hilbert spaces, *J. Math. Anal. Appl.*, **72** (1979), 383–390.

[33] J. Peypouquet, Analyse asymptotique de systèmes d’évolution et applications en optimisation, Ph.D. Thesis, UPMC Paris 6 and U. de Chile, 2007.

[34] S. Reich, Nonlinear evolution equations and nonlinear ergodic theorems, *Nonlinear Anal.*, **1** (1976), 319–330.

[35] A. N. Tikhonov and V. Y. Arsenin, *Solutions of Ill-Posed Problems*, Winston, New York, 1977.

[36] Y. Xia and J. Wang, A recurrent neural network for solving nonlinear convex programs subject to linear constraints, *IEEE Trans. Circuits Syst. I. Regul. Pap.*, **51** (2004), 1385–1394.

Received February 2017; revised March 2018.

E-mail address: aichabalhag@gmail.com
E-mail address: chbaniz@uca.ac.ma
E-mail address: h-riahi@uca.ac.ma