THE SMALL-MASS LIMIT AND WHITE-NOISE LIMIT OF AN INFINITE DIMENSIONAL GENERALIZED LANGEVIN EQUATION

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Abstract. We study asymptotic properties of the Generalized Langevin Equation (GLE) in the presence of a wide class of external potential wells with a power-law decay memory kernel. When the memory can be expressed as a sum of exponentials, a class of Markovian systems in infinite-dimensional spaces is used to represent the GLE. The solutions are shown to converge in probability in the small-mass limit and the white-noise limit to appropriate systems under minimal assumptions, of which no Lipschitz condition is required on the potentials. With further assumptions about space regularity and potentials, we obtain $L^1$ convergence in the white-noise limit.

Keywords: Markov processes, power-law decay, memory kernel

1. Introduction

The Generalized Langevin Equation is a Stochastic Integro-Differential Equation that is commonly used to model the velocity $\{v(t)\}_{t \geq 0}$ of a microparticle in a thermally fluctuating viscoelastic fluid [31, 25, 18]. It can be written in the following form

$$\dot{x}(t) = v(t),$$

$$m \dot{v}(t) = -\gamma v(t) - \Phi'(x(t)) - \int_{-\infty}^{t} K(t-s)v(s) \, ds + F(t) + \sqrt{2\gamma} \dot{W}(t),$$

where $m > 0$ is the particle mass, $\gamma > 0$ is the viscous drag coefficient, $\Phi$ is a potential well and $K(t)$ is a phenomenological memory kernel that summarizes the delayed drag effects by the fluid on the particle. The noise has two components: $F(t)$ is a mean-zero, stationary Gaussian process with autocovariance $\mathbb{E}[F(t)F(s)] = K(|t-s|)$, and $W(t)$ is a standard two-sided Brownian motion. The appearance of $K(t)$ in the autocovariance of $F(t)$ is a manifestation of the Fluctuation-Dissipation relationship, originally stated in [27], see also [36] for a more systematic review.

When there are no external forces, the GLE has the form

$$\dot{x}(t) = v(t),$$

$$m \dot{v}(t) = -\gamma v(t) - \int_{-\infty}^{t} K(t-s)v(s) \, ds + F(t) + \sqrt{2\gamma} \dot{W}(t),$$

it was shown in [32] that with extra assumptions, when $K$ is integrable, the Mean-Squared Displacement (MSD) $\mathbb{E}x(t)^2$ satisfies $\mathbb{E}x(t)^2 \sim t$ as $t \to \infty$; otherwise, if $K(t) \sim t^{-\alpha}$,
\( \alpha \in (0,1) \), then \( \mathbb{E}x(t)^2 \sim t^\alpha \) as \( t \to \infty \). The former asymptotic behavior of the MSD is called \textit{diffusive} whereas the latter is called \textit{subdiffusive}. Here the notation \( f(t) \sim g(t) \) as \( t \to \infty \) means
\[
\frac{f(t)}{g(t)} \to c \in (0, \infty), \quad \text{as} \quad t \to \infty.
\]

It has been observed that when \( K(t) \) is written as a sum of exponential functions, by adding auxiliary terms, the non-Markovian GLE (1.1) can be mapped onto a multi-dimensional Markovian system [33, 38, 28, 29]. If \( K(t) \) has the form of a finite sum of exponentials, the resulting finite-dimensional SDE was studied extensively in e.g. [34, 36]. One can show that these systems admit a unique invariant structure with geometric ergodicity. Moreover, the marginal density of the pair \((x,v)\) is independent of \( K(t) \). It is also worthwhile to note that, in the case of the linear GLE (1.2), these memory kernels \( K(t) \) produce \textit{diffusive} MSD since they are integrable [32]. In order to include memory kernels that have a power-law decay, one has to consider an infinite sum of exponentials resulting in a corresponding infinite-dimensional system.

From now on, we shall adopt the notations from [12]. Let \( \alpha, \beta > 0 \) be given, and define constants \( c_k, \lambda_k, k = 1, 2, \ldots \), by
\[
c_k = \frac{1}{k^{1+\alpha\beta}}, \quad \lambda_k = \frac{1}{k^\beta}.
\]
We introduce the memory kernel \( K(t) \) given by
\[
K(t) = \sum_{k \geq 1} c_k e^{-\lambda_k t}.
\]
It is shown that (see Example 3.3 of [1]) with this choice of constants \( c_k \) and \( \lambda_k \), \( K(t) \) obeys a power-law decay, namely
\[
K(t) \sim t^{-\alpha} \quad \text{as} \quad t \to \infty,
\]
where \( \alpha \) is as in (1.3). The constant \( \beta \) is an auxiliary parameter that is only assumed to be positive. When \( \alpha > 1 \), as mentioned above in the linear GLE (1.2), \( K(t) \) is in the \textit{diffusive} regime, whereas for \( \alpha \in (0,1) \), \( K \) belongs to the \textit{subdiffusive} regime. There is however no claim regarding to the case \( \alpha = 1 \). For such reason, it is called the \textit{critical regime}. With \( K(t) \) defined as in (1.4), the GLE (1.1) is expressed as the following infinite-dimensional system
\[
\begin{align*}
&dx(t) = v(t) \, dt, \\
&m \, dv(t) = \left( -\gamma v(t) - \Phi'(x(t)) - \sum_{k \geq 1} \sqrt{c_k} z_k(t) \right) dt + \sqrt{2\gamma} \, dW_0(t), \\
&dz_k(t) = \left( -\lambda_k z_k(t) + \sqrt{c_k} v(t) \right) dt + \sqrt{2\lambda_k} \, dW_k(t), \quad k \geq 1,
\end{align*}
\]
where \( W_k \) are independent, standard Brownian motions. In [12], the well-posedness and the existence of invariant structures of (1.6) were studied for all \( \alpha > 0 \). Employing a recent advance tool called \textit{asymptotic coupling} [16, 13], it can be shown that (1.6) admits a unique
invariant distribution in the diffusive regime, \((\alpha > 1)\). However, ergodicity when \(\alpha \in (0, 1]\) remains an open question.

The goal of this note is to give an analysis of the behavior of (1.6) in two different limits. First, we are interested in the asymptotic behavior of (1.6) concerning the small-mass limit, namely taking \(m\) to zero on the LHS of the second line in (1.6). Due to the random perturbations, the velocity \(v(t)\) is fluctuating fast whereas the displacement \(x(t)\) is still moving slow. We hence would like to find a process \(u(t)\) such that on any compact interval \([0, T]\),

\[
\lim_{m \downarrow 0} \sup_{0 \leq r \leq t} |x(r) - u(r)| = 0,
\]

where the limit holds in an appropriate sense. Such statement is called Smoluchowski-Kramer approximation [10]. There is a literature of analyzing asymptotic behaviors for fast-slow processes when taking zero-mass limit. Earliest results in this direction seem to be the works of [26, 37]. For more recent studies in finite dimensional systems, we refer to [10] in which the convergence in probability is established with constant drag and multiplicative noise. Under stronger assumptions and using appropriate time rescaling, weak convergence is proved in [35] where the friction is also state-dependent. When the potential is assumed to be Lipschitz, one can obtain better results in \(L^p\), following the works of [19, 30]. Without such assumption, convergence in probability is established in [17] provided appropriate Lyapunov controls. In addition, limiting systems are observed numerically in [20, 21]. Similar analysis in infinite dimensional settings for semi linear wave equations are studied in a series of paper [4, 5, 6, 7]. The systems therein are shown to converge to a heat equation under different assumptions about non linear drifts. Motivated by [17, 19], in this note, we establish the convergence in probability for (1.6), cf. Theorem 4. The technique that we employ is inspired by those in [17].

Then, we study the white-noise limit of (1.6), namely as the random force \(F(t)\) in (1.1) converges to a white noise process. Under different conditions on the potential and space regularity, we aim to find a pair of processes \((u(t), p(t))\) that can be approximated by the \((x(t), v(t))\)-component in (1.6). While there is a rich history on the small-mass limit, the white-noise limit seems to receive less attention. Nevertheless, there have been many works on the asymptotics of deterministic systems with memories. To name a few in this direction, we refer the reader to [8, 11, 15]. With regards to the white-noise limit of our system, we establish the convergence in different modes for a wide class of potentials, that are not necessarily Lipschitz or bounded. While the proof of Theorem 5 concerning probability convergence shares the same arguments with that of Theorem 4, the result in Theorem 8 concerning strong topology requires more work, where we have to estimate a universal bound on the solutions of (1.6) using appropriate Lyapunov structures, cf. Proposition 13. To the best of our knowledge, these results seem to be new in infinite-dimensional stochastic differential equations with memory. Particularly, they (cf. Theorem 5 and Theorem 8) generalize analogous results for finite-dimensional settings in [34], where \(K(t)\) has a form of finite sum of exponentials.
The rest of this paper is organized as follows. We introduce notations and summarize our main results in Section 2. The small-mass limit is addressed rigorously in Section 3. We obtain a formula for the limiting system as a form of a Smoluchowski-Kramers equation. Finally, Section 4 studies the white-noise limit.

2. Summary of Results

Throughout this work, we will assume that the potential $\Phi$ satisfies the following growth condition \[12\].

**Assumption 1.** $\Phi \in C^\infty(\mathbb{R})$ and there exists a constant $c > 0$ such that for all $x \in \mathbb{R}$

$$c(\Phi(x) + 1) \geq x^2.$$ 

By adding a positive constant if necessary, we also assume that $\Phi$ is non-negative.

A typical class of potentials $\Phi$ that satisfies Assumption 1 is the class of polynomials of even degree whose leading coefficient is positive. Functions that grow faster than polynomials are also included, e.g. $e^{x^2}$.

We now define a phase space for the infinite-dimensional process

$$X(t) = (x(t), v(t), z_1(t), z_2(t), \ldots).$$

Following [12], let $\mathcal{H}_{-s}, s \in \mathbb{R}$ denote the Hilbert space given by

\begin{equation}
\mathcal{H}_{-s} = \{X = (x, v, Z) = (x, v, z_1, z_2, \ldots) : \|X\|_{-s}^2 = x^2 + v^2 + \sum_{k \geq 1} k^{-2s} z_k^2 < \infty\},
\end{equation}

endowed with the usual inner product $\langle \cdot, \cdot \rangle_{-s}$,

\begin{equation}
\langle X, \tilde{X} \rangle_{-s} = \bar{x}\tilde{x} + \bar{v}\tilde{v} + \sum_{k \geq 1} k^{-2s} z_k\tilde{z}_k.
\end{equation}

With regards to kernel parameters $\alpha, \beta$ cf. (1.3), (1.4) and the phase space regularity parameter $s$, we assume that they satisfy the following condition.

**Assumption 2.** Let $\alpha, \beta > 0$ be as in (1.3) and $s \in \mathbb{R}$ as in (2.1). We assume that they satisfy either the asymptotically diffusive condition

\begin{equation}
(D) \quad \alpha > 1, \beta > \frac{1}{\alpha - 1} \quad \text{and} \quad 1 < 2s < (\alpha - 1)\beta;
\end{equation}

or the asymptotically subdiffusive condition

\begin{equation}
(SD) \quad 0 < \alpha < 1, \beta > \frac{1}{\alpha} \quad \text{and} \quad 1 < 2s < \alpha\beta;
\end{equation}

or the critical regime condition

\begin{equation}
(C) \quad \alpha = 1, \beta > 1 \quad \text{and} \quad 1 < 2s < \beta;
\end{equation}

Under Assumption 1 and Assumption 2, the well-posedness of (1.6) was studied rigorously in [12].
In regards to the small mass limit \((m \to 0)\), following [17], to determine the limiting system, one may formally set \(m = 0\) on the RHS of the second equation in (1.6) and substitute \(v(t)\) by \(dx(t)\) from the first equation to obtain
\[
\gamma dx(t) = \left[ -\Phi'(x(t)) - \sum_{k \geq 1} \sqrt{c_k} z_k(t) \right] dt + \sqrt{2\gamma} dW_0(t).
\]
The equation on \(z_k(t)\) in (1.6) still depends on \(v(t)\), but this can be circumvented by using Duhamel’s formula,
\[
z_k(t) = e^{-\lambda_k t} z_k(0) + \sqrt{c_k} \int_0^t e^{-\lambda_k (t-r)} v(r) dr + \sqrt{2\lambda_k} \int_0^t e^{-\lambda_k (t-r)} dW_k(r).
\]
By an integration by parts, we can transform the integral term involving \(v(r)\) to
\[
\int_0^t e^{-\lambda_k (t-r)} v(r) dr = x(t) - e^{-\lambda_k t} x(0) - \lambda_k \int_0^t e^{-\lambda_k (t-r)} x(r) dr.
\]
Plugging back into the formula for \(z_k(t)\), we find
\[
z_k(t) = e^{-\lambda_k t} (z_k(0) - \sqrt{c_k} x(0)) + \sqrt{c_k} x(t) - \lambda_k \sqrt{c_k} \int_0^t e^{-\lambda_k (t-r)} x(r) dr
\]
\[
+ \sqrt{2\lambda_k} \int_0^t e^{-\lambda_k (t-r)} dW_k(r).
\]
We now set \(f_k(t) := z_k(t) - \sqrt{c_k} x(t)\) and \(u(t) := x(t)\) and arrive at the following system
\[
\gamma du(t) = \left( -\Phi'(u(t)) - \sum_{k \geq 1} c_k u(t) - \sum_{k \geq 1} \sqrt{c_k} f_k(t) \right) dt + \sqrt{2\gamma} dW_0(t),
\]
\[
df_k(t) = (-\lambda_k f_k(t) - \lambda_k \sqrt{c_k} u(t)) dt + 2\lambda_k dW_k(t), \quad k = 1, 2, \ldots
\]
with the new shifted initial condition:
\[
u(0) = x \quad \text{and} \quad f_k(0) = z_k - \sqrt{c_k} x,
\]
where \(x, v, z_1, z_2, \ldots \in \mathcal{H}_{-s}\) is the initial condition for (1.6). The new phase space for the solution \(U(t) = (u(t), f_1(t), f_2(t), \ldots)\) of (2.3) is denoted by \(\mathcal{H}_{-s}\), \(s \in \mathbb{R}\),
\[
\mathcal{H}_{-s} = \left\{ U = (u, f_1, f_2, \ldots) : \|U\|_{\mathcal{H}_{-s}}^2 = u^2 + \sum_{k \geq 1} k^{-2s} f_k^2 < \infty \right\}
\]
edowed with the usual inner product. We can regard \(\mathcal{H}_{-s}\) as a subspace of \(\mathcal{H}_{-s}\) whose \(v\)-component is equal to zero. Recalling \(c_k\) in (1.3), it is straightforward to see that if \((x, v, z_1, z_2, \ldots) \in \mathcal{H}_{-s}\) then \((x, z_1 - \sqrt{c_1} x, z_2 - \sqrt{c_2} x, \ldots) \in \mathcal{H}_{-s}\).
From now on, we shall fix a stochastic basis \(\mathcal{S} = (\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \geq 0}, W)\) satisfying the usual conditions [24]. Here \(W\) is the cylindrical Wiener process defined on an auxiliary Wiener space \(W\) with the usual decomposition
\[
W(t) = e_0^W W_0(t) + e_1^W W_1(t) + \ldots,
\]
where \( \{ e_0^W, e_1^W, \ldots \} \) is the canonical basis of \( W \), and \( \{ W_k(t) \}_{k \geq 0} \) are independent one-dimensional Brownian Motions \([9]\). The well-posedness of (2.3) is guaranteed by the following result.

**Proposition 3.** Suppose that \( \Phi \) satisfies Assumption 1 and the constants \( \alpha, \beta, s \) satisfy Assumption 2. Then for all initial conditions \( U_0 \in \mathcal{H}_s \), there exists a unique pathwise solution \( U(\cdot, U_0) : \Omega \times [0, \infty) \rightarrow \mathcal{H}_s \) of (2.3) in the following sense: \( U(\cdot, U_0) \) is \( \mathcal{F}_t \)-adapted, \( U(\cdot, U_0) \in C([0, \infty), \mathcal{H}_s) \) almost surely and that if \( \tilde{U}(\cdot, U_0) \) is another solution then for every \( T \geq 0 \),

\[
\mathbb{P} \left\{ \forall t \in [0, T], U(t, U_0) = \tilde{U}(t, U_0) \right\} = 1.
\]

Moreover, for every \( U_0 \in \mathcal{H}_s \) and \( T \geq 0 \), there exists a constant \( C(T, U_0) > 0 \) such that

\[
\mathbb{E} \sup_{0 \leq t \leq T} \| U(t) \|_{\mathcal{H}_s}^2 \leq C(T, U_0).
\]

The proof of Proposition 3 is quite standard, similar to that of Proposition 6 in \([12]\) and will be briefly explained in Section 3. We now state the main result concerning the zero-mass limit.

**Theorem 4.** Suppose that \( \Phi \) satisfies Assumption 1 and the constants \( \alpha, \beta, s \) satisfy Assumption 2. Let \( X_m(t) = (x_m(t), v_m(t), z_{1,m}(t), \ldots) \) solve (1.6) with initial condition \( (x, v, z_1, z_2, \ldots) \in \mathcal{H}_s \) and \( U(t) = (u(t), f_1(t), \ldots) \) solve (2.3) with initial condition \( (x, z_1 - \sqrt{c_1}x, z_2 - \sqrt{c_2}x, \ldots) \in \mathcal{H}_s \). Then, for every \( T, \xi > 0 \), it holds that

\[
\mathbb{P} \left\{ \sup_{0 \leq t \leq T} | x_m(t) - u(t) | > \xi \right\} \rightarrow 0, \quad m \to 0.
\]

Next, we present our results on the asymptotical behavior of the \((x(t), v(t))\)-component of (1.6) in the diffusive regime by an appropriate scaling on the memory kernel \( K(t) \) in (1.4). For \( \epsilon > 0 \), we introduce \( K_\epsilon(t) \) given by

\[
K_\epsilon(t) = \frac{1}{\epsilon} K \left( \frac{t}{\epsilon} \right) = \sum_{k \geq 1} \frac{C_k}{\epsilon} e^{-\frac{\lambda_k}{\epsilon} |t|}.
\]

With \( K_\epsilon \) defined above, the corresponding system (1.6) becomes

\[
\begin{align*}
&dx_\epsilon(t) = v_\epsilon(t) \, dt, \\
&m \, dv_\epsilon(t) = \left( -\gamma v_\epsilon(t) - \Phi'(x_\epsilon(t)) - \sum_{k \geq 1} \sqrt{\frac{c_k}{\epsilon}} z_{k,\epsilon}(t) \right) \, dt + \sqrt{2\gamma} \, dW_0(t), \\
&dz_{k,\epsilon}(t) = \left( -\frac{\lambda_k}{\epsilon} z_{k,\epsilon}(t) + \sqrt{\frac{c_k}{\epsilon}} v_\epsilon(t) \right) \, dt + \sqrt{2\lambda_k} \, dW_k(t), \quad k \geq 1.
\end{align*}
\]
Inspired by [34], we consider the following system

\begin{equation}
du(t) = p(t) \, dt,
\end{equation}

\begin{equation}
mdp(t) = \left( -\left( \gamma + \sum_{k \geq 1} \frac{c_k}{\lambda_k} \right) p(t) - \Phi'(u(t)) \right) \, dt
\end{equation}

\begin{equation}
- \sum_{k \geq 1} \sqrt{\frac{2c_k}{\lambda_k}} dW_k(t) + \sqrt{2\gamma} dW_0(t).
\end{equation}

The well-posedness of (2.8) will be addressed briefly in Section 4. We then assert that \((x_\epsilon(t), v_\epsilon(t))\) converges to the solution \((u(t), p(t))\) of (2.8) in the following sense.

**Theorem 5.** Suppose that \(\Phi\) satisfies Assumption 1 and the constants \(\alpha, \beta, s\) satisfy Condition (D) of Assumption 2. Let \(X_\epsilon(t) = (x_\epsilon(t), v_\epsilon(t), z_1, z_2, \ldots)\) be the solution of (2.7) with initial condition \((x, v, z_1, z_2, \ldots) \in \mathcal{H}_{-s}\) and \((u(t), p(t))\) be the solution of (2.8) with initial condition \((x, v)\). Then, for every \(T, \xi > 0\),

\[ P\left\{ \sup_{0 \leq t \leq T} \left| x_\epsilon(t) - u(t) \right| + \left| v_\epsilon(t) - p(t) \right| > \xi \right\} \to 0, \quad \epsilon \to 0. \]

The reader may wonder why the convergence result of Theorem 5 is restricted to the diffusive regime, namely \(\alpha > 1\) according to Condition (D) of Assumption 2. Heuristically, since the memory kernel \(K(t)\) decays like \(t^{-\alpha}\) as \(t \to \infty\), we see that

\[ K_\epsilon(t) = \frac{1}{\epsilon} K\left( \frac{t}{\epsilon} \right) \sim \epsilon^{\alpha-1} t^{-\alpha}, \quad t \to \infty. \]

By shrinking \(\epsilon\) further to zero, if \(\alpha > 1\), \(K_\epsilon(t)\) does not behave “badly” at infinity. In fact, it is not difficult to show that as \(\epsilon \downarrow 0\), \(K_\epsilon\) converges to the Dirac function \(\delta_0\) centered at the origin, in the sense of tempered distribution, namely, for every \(\varphi \in \mathcal{S}\), the Schwartz space on \(\mathbb{R}\), it holds that

\[ \int_{\mathbb{R}} K_\epsilon(t) \varphi(t) dt \to [K]_{L^1(\mathbb{R})} \varphi(0). \]

which implies that the random force \(F(t)\) in (1.1) converges to a white noise process in the sense of random distribution, cf. [22], hence the so called “white-noise limit”.

Finally, if \(\Phi\) and parameters \(\alpha, \beta\) satisfy stronger assumptions, then we are able to obtain better convergence than the result in Theorem 5. To be precise, we assume the following condition on \(\Phi\).

**Assumption 6.** There exist constants \(n, c > 0\) such that for every \(x, y \in \mathbb{R}\),

\[ \Phi'(x)y \leq c(\Phi(x) + |y|^n + 1). \]

The assumption above is again a requirement about the growth of \(\Phi'\) that guarantees a universal bound independent of \(\epsilon\) on the solution \((x_\epsilon(t), v_\epsilon(t))\) of (2.7), cf. Proposition 13. It is worthwhile to note that the class of polynomials of even degree satisfies Assumption 6. However, functions growing exponentially fast, e.g. \(e^{x^2}\), do not. Also, we assume the following condition about parameters \(\alpha, \beta\).
Assumption 7. Let $\alpha, \beta > 0$ be as in (1.3). We assume that they satisfy
\[
\alpha > 2, \quad \text{and} \quad (\alpha - 2)\beta > 1.
\]
We then have the following important result.

Theorem 8. Suppose that $\Phi$ satisfies Assumption 1 and Assumption 6 and that the constants $\alpha, \beta, s$ satisfy Condition (D) of Assumption 2 and Assumption 7. Let $X_\epsilon(t) = (x_\epsilon(t), v_\epsilon(t), z_{1,\epsilon}(t), \ldots)$ be the solution of (2.7) with initial condition $(x, v, z_1, z_2, \ldots) \in H_{-s}$ and $(u(t), p(t))$ be the solution of (2.8) with initial condition $(x, v)$. Then, for every $T > 0$, $1 \leq q < 2$, it holds that
\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} |x_\epsilon(t) - u(t)|^q + |v_\epsilon(t) - p(t)|^q \right] \to 0, \quad \epsilon \to 0.
\]

Theorem 8 strengthens a previous result from [34], where the potential $\Phi'$ is assumed to be bounded. The proofs of Theorem 5 and Theorem 8 will be carried out in Section 4.

3. Zero-mass limit
Throughout the rest of the paper, $C, c$ denote generic positive constants. The important parameters that they depend on will be indicated in parenthesis, e.g. $c(T, q)$ depends on parameters $T$ and $q$.

In this section, for notation simplicity, we shall omit the subscript $m$ in $X_m(t) = (x_m(t), v_m(t), z_{1,m}(t), \ldots)$. We begin by addressing the well-posedness of (2.3) whose proof follows a standard Lyapunov-type argument that was also used to establish the well-posedness of (1.6) in [12]. The technique is classical and has been employed previously in literature [2, 14, 23]. We shall omit specific details and briefly summarize the main steps.

Sketch of the proof of Proposition 3. For $R > 0$, let $\theta^R \in C^\infty(\mathbb{R}, [0, 1])$ satisfy
\[
\theta^R(x) = \begin{cases} 
1 & \text{if } |x| \leq R, \\
0 & \text{if } |x| \geq R + 1.
\end{cases}
\]
We consider the “cutoff” equation corresponding to (2.3)
\[
\gamma du(t) = \left( -\Phi'(u(t))\theta^R(u(t)) - \sum_{k \geq 1} c_k u(t) - \frac{1}{\sqrt{c_k}} f_k(t) \right) dt + \sqrt{2\gamma} dW_0(t),
\]
\[
df_k(t) = \left( -\lambda_k f_k(t) - \lambda_k \sqrt{c_k} u(t) \right) dt + \sqrt{2\lambda_k} dW_k(t), \quad k = 1, 2, \ldots
\]
We observe that in (3.2), the drift term is globally Lipschitz and the noise is additive. Thus, by using a standard Banach fixed point argument, the corresponding global (in time) solution $U^R$ exists and is unique. Next, define the stopping time
\[
\tau_R = \inf \left\{ t > 0 : \|U(t)\|_{H_{-s}} > R \right\}.
\]
Note that, for all times $t < \tau_R$, $U^R$ solves (2.3). Consequently, the solution (2.3) exists and is unique up until the time of explosion $\tau_\infty = \lim_{R \to \infty} \tau_R$, which is possibly finite on a set of positive probability. We finally introduce the Lyapunov function

$$\Psi(U) := \frac{1}{\gamma} \left( \Phi(u) + \left( \sum_{k \geq 1} c_k \frac{u^2}{2} \right) + \frac{1}{2} \sum_{k \geq 1} k^{-2s} f_k^2 \right).$$

It is clear that $\Psi(U)$ dominates $\|U\|_{H^{-s}}^2$. Applying Ito’s formula to $\Psi(U)$, one can derive a global bound on the solutions $U^R(t)$ that is independent of $R$, namely, there exists a constant $C(U_0, T)$ such that

$$\mathbb{E} \left[ \sup_{0 \leq t \leq T} \|U(t) \cap \tau_R\|_{H^{-s}}^2 \right] \leq C(T, U_0).$$

Sending $R$ to infinity, it follows from Fatou’s Lemma that

$$\mathbb{E} \left[ \sup_{0 \leq t \leq T} \|U(t) \cap \tau_\infty\|_{H^{-s}}^2 \right] \leq C(U_0, T),$$

implying $\mathbb{P} \{ T < \tau_\infty \} = 1$ for any $T > 0$. Taking $T$ to infinity, we see that $\mathbb{P} \{ \tau_\infty = \infty \} = 1$, thereby obtaining the global solution of (2.3).

Although the construction of the global solution $U(t)$ of (2.3) via the local solutions $U^R(t)$ of (3.2) is quite standard, the proof of Theorem 4 will make use of a non trivial observation on these local solutions. The arguments that we are going to employ are inspired from the work of [17]. Before diving into detail, we briefly explain the main idea, which is a two-fold: first, we show that the result holds for $\Phi'$ being Lipschitz. In particular, we obtain the convergence in sup norm for the local solutions, namely, for all $R, T > 0$, we have

$$\mathbb{E} \left[ \sup_{0 \leq t \leq T} |x^R(t) - u^R(t)| \right] \to 0, \quad m \to 0,$$

where $x^R(t)$ is in the following cut-off system for (1.6)

$$dx(t) = v(t) \, dt,$$

$$m \, dv(t) = \left( -\gamma v(t) - \Phi'(x(t)) \delta^R(x(t)) - \sum_{k \geq 1} \sqrt{c_k} z_k(t) \right) \, dt + \sqrt{2\gamma} \, dW_0(t),$$

$$dz_k(t) = \left( -\lambda_k z_k(t) + \sqrt{c_k} v(t) \right) \, dt + \sqrt{2\lambda_k} \, dW_k(t), \quad k \geq 1.$$

Then, by taking $R$ necessarily large, we obtain the desired result.

We now proceed by showing that the result holds true in a simpler setting where $\Phi'$ is globally Lipschitz. The proof is adapted from that of Theorem 1 of [19].

**Proposition 9.** Suppose that $\Phi'$ is globally Lipschitz and that the constants $\alpha, \beta, s$ satisfy Assumption 2. Let $X(t) = (x(t), v(t), z_1(t), \ldots)$ solve (1.6) with initial condition
\[(x, v, z_1, z_2, \ldots) \in \mathcal{H}_{-s} \text{ and } U(t) = (u(t), f_1(t), \ldots) \text{ solve } (2.3) \text{ with initial condition } (x, z_1 - \sqrt{c_1} x, z_2 - \sqrt{c_2} x, \ldots) \in \mathcal{H}_{-s}. \text{ Then, for every } T, q > 0, \text{ it holds that}
\[\mathbb{E} \sup_{0 \leq t \leq T} |x(t) - u(t)|^q \to 0, \quad m \to 0.\]

**Proof.** Using Duhamel’s formula, \(z_k\) from (1.6) can be solved explicitly as
\[z_k(t) = e^{-\lambda_k t} z_k + \sqrt{c_k} \int_0^t e^{-\lambda_k (t-r)} v(r) dr + \sqrt{2\lambda_k} \int_0^t e^{-\lambda_k (t-r)} dW_k(r),\]
which is equivalent to
\[z_k(t) = e^{-\lambda_k t} (z_k - \sqrt{c_k} x) + \sqrt{c_k} x(t) - \sqrt{c_k \lambda_k} \int_0^t e^{-\lambda_k t} x(r) dr + \sqrt{2\lambda_k} \int_0^t e^{-\lambda_k (t-r)} dW_k(r),\]
where we have used an integration by parts on the term \(\int_0^t e^{-\lambda_k (t-r)} v(r) dr\) in the first equality. Substituting into the second equation of (1.6), we arrive at
\[m dv(t) + \gamma dx(t) = \left( -\Phi'(x(t)) - \sum_{k \geq 1} \sqrt{c_k} e^{-\lambda_k t} (z_k - \sqrt{c_k} x) - \left( \sum_{k \geq 1} c_k \right) x(t) \right. \]
\[+ \sum_{k \geq 1} c_k \lambda_k \int_0^t e^{-\lambda_k (t-r)} x(r) dr - \sum_{k \geq 1} \sqrt{2c_k \lambda_k} \int_0^t e^{-\lambda_k (t-r)} dW_k(r) \bigg) dt\]
\[+ \sqrt{2\gamma} dW_0(t).\]
Likewise, we obtain the following equation from (2.3)
\[\gamma du(t) = \left( -\Phi'(u(t)) - \left( \sum_{k \geq 1} c_k \right) u(t) - \sum_{k \geq 1} \sqrt{c_k} e^{-\lambda_k t} (z_k - \sqrt{c_k} x) \right. \]
\[+ \sum_{k \geq 1} c_k \lambda_k \int_0^t e^{-\lambda_k (t-r)} u(r) dr - \sum_{k \geq 1} \sqrt{2c_k \lambda_k} \int_0^t e^{-\lambda_k (t-r)} dW_k(r) \bigg) dt\]
\[+ \sqrt{2\gamma} dW_0(t).\]
Subtracting (3.7) from (3.6) and setting \(\varpi(t) = x(t) - u(t)\), we find that
\[m dv(t) + \gamma d\varpi(t) = \left( - \left[ \Phi'(x(t)) - \Phi'(u(t)) \right] - \left( \sum_{k \geq 1} c_k \right) \varpi(t) + \sum_{k \geq 1} c_k \lambda_k \int_0^t e^{-\lambda_k (t-r)} \varpi(r) dr \right) dt\]
\[\leq c \left( 1 + \sum_{k \geq 1} c_k \right) \sup_{0 \leq r \leq t} |\varpi(r)| dt,\]
where \( c > 0 \) is a Lipschitz constant for \( \Phi' \). Recalling \( c_k \) from (1.3), we apply Gronwall’s inequality to estimate for all \( T, q > 0 \)
\[
\mathbb{E} \sup_{0 \leq t \leq T} |x(t)|^q \leq m^q \mathbb{E} \sup_{0 \leq t \leq T} |v(t) - v|^q e^{c(T)}.
\]
The result now follows immediately from Proposition 10 below. \( \square \)

**Proposition 10.** Under the same Hypothesis of Theorem 4, suppose further that \( \Phi'(x) \) is globally Lipschitz. Let \( X(t) = (x(t), v(t), z_1(t), \ldots) \) solve (1.6) with initial condition \((x, v, z_1, z_2, \ldots) \in \mathcal{H}_{-s}\). Then, for every \( T > 0, q > 1 \), it holds that
\[
m^q \mathbb{E} \sup_{0 \leq t \leq T} |v(t)|^q \to 0, \quad m \to 0.
\]

In order to prove Proposition 10, we need the following important lemma whose proof is based on Lemma 3.19, [3] and Lemma 2, [19]. It will be also useful later in Section 4.

**Lemma 11.** Given \( \kappa, \eta > 0 \), let \( f(t) = \sqrt{2\kappa} \int_0^t e^{-\eta(t-r)} dW(r) \) where \( W(t) \) is a standard Brownian Motion. Then, for all \( T > 0, q > 1 \), there exists a constant \( C(T, q) > 0 \) such that
\[
\mathbb{E} \sup_{0 \leq t \leq T} f(t)^2 q \leq \frac{\kappa^q}{\eta^{q-1}} C(T, q).
\]

**Remark 12.** The estimate in (3.8) is sharper than the usual exponential martingale estimate. In finite-dimensional settings, it is sufficient to bound the LHS of (3.8) by \( C(T, q, \eta, \kappa) \), cf. [3, 19, 30]. In our setting, we have to keep track explicitly in term of \( \eta \) and \( \kappa \), hence the RHS of (3.8).

The proof of Lemma 11 is similar to that of Lemma 2, [19]. We include it here for the sake of completeness.

**Proof of Lemma 11.** In view of Lemma 3.19, [3], we have the following estimate for \( A > 0 \),
\[
\mathbb{P}\left\{ \sup_{0 \leq t \leq T} f(t)^2 \geq A \right\} \leq \frac{\eta T}{\int_0^{\sqrt{\eta A/\kappa}} e^{r^2/2} \int_0^r e^{-r^2/2} dr dr}.
\]

We proceed to find a lower bound for the above denominator. To this end, we first claim that for \( r \geq 0 \),
\[
\int_0^r e^{-r^2/2} dr \geq \frac{r e^{-r^2/4}}{2}.
\]

Indeed, on one hand, if \( r \geq 1 \), then
\[
e^{r^2/4} \int_0^r e^{-r^2/2} dr \geq e^{r^2/4} \int_1^r e^{-r^2/2} dr \geq e^{r^2/4} \int_0^1 e^{-1/2} dr \geq \frac{e^{r^2/4}}{2} \geq \frac{r}{2}.
\]

On the other hand, if \( 0 \leq r \leq 1 \), then
\[
e^{r^2/4} \int_0^r e^{-r^2/2} dr \geq \int_0^r 1 - \frac{r^2}{2} dr = r (1 - \frac{r^2}{6}) \geq \frac{r}{2}.
\]
With this observation, we find
\[
\int_0^{\eta A/\kappa} e^{x^2/2} \int_0^r e^{-\ell^2/2} d\ell \, dr \geq \int_0^{\eta A/\kappa} e^{x^2/2} \frac{e^{-\ell^2/2}}{2} \, dr = e^{\eta A/4\kappa} - 1 \geq \frac{\eta A}{4\kappa} e^{\eta A/8\kappa},
\]
where in the last implication, we have used the following inequality for every \( r \geq 0, \)
\[
e^r - 1 \geq re^{r/2}.
\]
Putting everything together, we obtain
\[
P\left\{ \sup_{0 \leq t \leq T} f(t)^2 \geq A \right\} \leq \frac{\eta T}{4\kappa e^{\eta A/8\kappa}} = \frac{4\kappa T}{A} e^{-\eta A/8\kappa}.
\]
It follows that for \( q > 2, \)
\[
E \sup_{0 \leq t \leq T} f(t)^{2q} = \int_0^\infty q A^{q-1} \eta q^{\frac{\gamma q}{2}} \left\{ \sup_{0 \leq t \leq T} f(t)^2 \geq A \right\} dA \\
\leq \int_0^\infty q A^{q-1} 4\kappa T e^{-\eta A/8\kappa} dA \\
= C(T, q) \int_0^\infty A^{q-2} e^{-\eta A/8\kappa} dA \\
= C(T, q) \frac{\kappa^q}{\eta^q-1},
\]
which completes the proof. \( \square \)

With Lemma 11 in hand, we are ready to give the proof of Proposition 10.

**Proof of Proposition 10.** We only have to prove the result for \( q > 0 \) sufficiently large. Note that \( v(t) \) from (1.6) is written as
\[
m v(t) = me^{-\frac{\gamma}{m}t}v(0) - \int_0^t e^{-\frac{2}{m}(t-\ell)} \Phi'(x(\ell)) d\ell - \sum_{k \geq 1} \sqrt{c_k} \int_0^t e^{-\frac{2}{m}(t-\ell)} z_k(\ell) d\ell \\
+ \sqrt{2\gamma} \int_0^t e^{-\frac{2}{m}(t-\ell)} dW_0(\ell).
\]
Substituting $z_k(t)$ from (3.5), we have

$$m v(t) = me^{-\frac{\nu}{m} t} v(0) - \int_0^t e^{-\frac{\nu}{m} (t-r)} \Phi'(x(r)) dr - \sum_{k \geq 1} c_k \sqrt{\frac{c_k}{m}} \int_0^t e^{-\frac{\nu}{m} (t-r)} e^{-\lambda_k r} z_k(0) dr$$

$$- \sum_{k \geq 1} c_k \int_0^t e^{-\frac{\nu}{m} (t-r)} \int_0^r e^{-\lambda_k (r-\ell)} v(\ell) d\ell \; dr$$

$$- \sum_{k \geq 1} \sqrt{2c_k \lambda_k} \int_0^t e^{-\frac{\nu}{m} (t-r)} \int_0^r e^{-\lambda_k (r-\ell)} W_k(\ell) \; d\ell \; dr$$

$$+ \sqrt{2\gamma} \int_0^t e^{-\frac{\nu}{m} (t-r)} W_0(r).$$

For every $q$ sufficiently large, we invoke the assumption that $\Phi'$ is Lipschitz to estimate

$$m^{2q} \mathbb{E} \sup_{0 \leq t \leq T} v(t)^{2q}$$

$$\leq c(q, \nu) \left[ \frac{1}{q} + \mathbb{E} \sup_{0 \leq t \leq T} x(t)^{2q} + \left( \sum_{k \geq 1} c_k^2 \right)^{2q} \right]$$

$$- \sum_{k \geq 1} c_k \int_0^t \mathbb{E} \sup_{0 \leq r \leq t} v(r)^{2q} dt$$

$$+ \left( \sum_{k \geq 1} c_k \right)^{(1/2)q} \left( \mathbb{E} \sup_{0 \leq t \leq T} \left( \int_0^t e^{-\lambda_k (t-r)} W_k(r) \right)^{2q} \right)$$

$$+ \mathbb{E} \sup_{0 \leq t \leq T} \left( \int_0^t e^{-\frac{\nu}{m} (t-r)} W_0(r) \right)^{2q},$$

where in the third line, we have used Holder’s inequality with $\frac{1}{2} + \frac{1}{2q} = 1$. Also, note that from the first equation of (1.6), it holds that

$$\mathbb{E} \sup_{0 \leq t \leq T} x(t)^{2q} \leq c(q) \left( \frac{1}{q} + \int_0^T \mathbb{E} \sup_{0 \leq r \leq t} v(r)^{2q} dt \right),$$

and that by Lemma 11, we have

$$\sum_{k \geq 1} \mathbb{E} \sup_{0 \leq t \leq T} \left( \int_0^t e^{-\lambda_k (t-r)} W_k(r) \right)^{2q} \leq c(T, q) \sum_{k \geq 1} c_k \lambda_k,$$

and

$$\mathbb{E} \sup_{0 \leq t \leq T} \left( \int_0^t e^{-\frac{\nu}{m} (t-r)} W_0(r) \right)^{2q} \leq c(T, q) \gamma^q m^{q-1}.$$
Furthermore, recalling $c_k$ from (1.3), we see that for $q > 0$ sufficiently large
\[
\sum_{k \geq 1} c_k^{(1/2-1/2q)q*} = \sum_{k \geq 1} \frac{1}{k(1+\alpha\beta)(1/2-1/2q)q*} < \infty,
\]
thanks to the fact that $q* > 1$ and $\alpha\beta > 1$, where the latter follows from the conditions about $\alpha$, $\beta$ in Assumption 2. Also, recalling $\lambda_k$ from (1.3) and the norm $\| \cdot \|_{-s}$ from (2.1), it is straightforward to verify that the sums $\sum_{k \geq 1} \sqrt{c_k}\mathbb{E}k$, $\sum_{k \geq 1} c_k$, and $\sum_{k \geq 1} c_k\lambda_k$ are absolutely convergent. Putting everything together, we find
\[
m^{2q}\mathbb{E} \sup_{0 \leq t \leq T} v(t)^{2q} \leq c(T, q, X_0)\left[m^{2q} + m^q - 1 + m^{2q} \int_0^T \mathbb{E} \sup_{0 \leq r \leq t} v(r)^{2q} \, dt\right],
\]
where $c(T, q, X_0) > 0$ is independent with $m$. Gronwall’s inequality now implies
\[
m^{2q}\mathbb{E} \sup_{0 \leq t \leq T} v(t)^{2q} \leq c(T, q, X_0)(m^{2q} + m^q - 1) \to 0, \quad m \to 0.
\]
The proof is thus complete. \(\square\)

We now turn our attention to Theorem 4. The proof is a slightly modification from that of Theorem 2.4 of [17]. The key observation is that instead of controlling the exiting time of the process $x(t)$ as $m \to 0$, we are able to control $u(t)$ since $u(t)$ is independent of $m$.

**Proof of Theorem 4.** For $R, m > 0$, define the following stopping times
\[
(3.9) \quad \sigma^R = \inf_{t \geq 0}\{|u(t)| \geq R\}, \quad \text{and} \quad \sigma^R_m = \inf_{t \geq 0}\{|x(t)| \geq R\},
\]
and recall
\[
\tau^R = \inf_{t \geq 0}\{|U(t)|_{\mathcal{H}_{-s}} \geq R\}, \quad \text{and} \quad \tau^R_m = \inf_{t \geq 0}\{|X(t)|_{\mathcal{H}_{-s}} \geq R\}.
\]
By the definitions of the norms in $\mathcal{H}_{-s}$, cf. (2.4), we see that $\tau^R \leq \sigma^R$ a.s. From the proof of Proposition 3, it is straightforward to verify that for all $T > 0$,
\[
\mathbb{P}\{\sigma^R < T\} \leq \mathbb{P}\{\tau^R < T\} \to 0, \quad R \to \infty.
\]
For $R, T, m, \xi > 0$, we have
\[
\mathbb{P}\left\{\sup_{0 \leq t \leq T}|x(t) - u(t)| > \xi\right\} \leq \mathbb{P}\left\{\sup_{0 \leq t \leq T}|x(t) - u(t)| > \xi, \sigma^R \land \sigma^R_m \geq T\right\} + \mathbb{P}\{\sigma^R \land \sigma^R_m < T\}.
\]
To control the first term on the above RHS, we note that for $0 \leq t \leq \sigma^R \land \sigma^R_m$, $u(t) = u^R(t)$ and $x(t) = x^R(t)$ a.s. We thus obtain the bound
\[
\mathbb{P}\left\{\sup_{0 \leq t \leq T}|x(t) - u(t)| > \xi, \sigma^R \land \sigma^R_m \geq T\right\} \leq \mathbb{P}\left\{\sup_{0 \leq t \leq T}|x^R(t) - u^R(t)| > \xi\right\} \to 0, \quad m \to 0,
\]
where the last convergence in probability follows immediately from Proposition 9. We are left to estimate \( P \{ \sigma^R \land \sigma^R_m < T \} \). To this end, we have that
\[
P \{ \sigma^R \land \sigma^R_m < T \} \leq P \left\{ \sup_{0 \leq t \leq T} |x^R(t) - u^R(t)| \leq \xi, \sigma^R \land \sigma^R_m < T \right\}
+ P \left\{ \sup_{0 \leq t \leq T} |x^R(t) - u^R(t)| > \xi \right\}
\leq P \left\{ \sup_{0 \leq t \leq T} |x^R(t) - u^R(t)| \leq \xi, \sigma^R < T \leq \sigma^R \right\}
+ P \left\{ \sup_{0 \leq t \leq T} |x^R(t) - u^R(t)| > \xi \right\}.
\]

Note that for \( R > 1 \) and \( \xi \in (0, 1) \), a chain of event implications is derived as follows.
\[
\left\{ \sup_{0 \leq t \leq T} |x^R(t) - u^R(t)| \leq \xi, \sigma^R_m < T \leq \sigma^R \right\}
= \left\{ \sup_{0 \leq t \leq T} |x^R(t) - u(t)| \leq \xi, \sup_{0 \leq t \leq T} |x^R(t)| \geq R, \sigma^R_m < T \leq \sigma^R \right\}
\subseteq \left\{ \sup_{0 \leq t \leq T} |u(t)| > R - 1, \sigma^R_m < T \leq \sigma^R \right\}
\subseteq \{ \sigma^{R-1} < T \},
\]
which implies that
\[
P \left\{ \sup_{0 \leq t \leq T} |x^R(t) - u^R(t)| < \xi, \sigma^R_m < T \leq \sigma^R \right\} \leq P \{ \sigma^{R-1} < T \}.
\]

Finally, putting everything together, for \( R > 1 > \xi > 0, T, m > 0 \), we obtain the estimate
\[
P \left\{ \sup_{0 \leq t \leq T} |x(t) - u(t)| > \xi \right\}
\leq 2P \left\{ \sup_{0 \leq t \leq T} |x^R(t) - u^R(t)| > \xi \right\} + P \{ \sigma^{R-1} < T \} + P \{ \sigma^R < T \}
\leq 2P \left\{ \sup_{0 \leq t \leq T} |x^R(t) - u^R(t)| > \xi \right\} + P \{ \sigma^{R-1} < T \} + P \{ \sigma^R < T \}.
\]

By taking \( R \) sufficiently large and then shrinking \( m \) further to zero, we obtain the result, thus completing the proof.

\[\square\]

4. White-noise limit

For notation simplicity, in this section, we shall omit the subscript \( \epsilon \) in
\[
X_{\epsilon}(t) = (x_{\epsilon}(t), v_{\epsilon}(t), z_{1,\epsilon}(t), \ldots).
\]
With regards to the well-posedness of (2.8), recalling \( c_k, \lambda_k \) from (1.3), we see that the noise term is well-defined thanks to Condition (D) of Assumption 2, namely

\[
\mathbb{E} \left( \int_0^T \sum_{k \geq 1} \sqrt{2c_k/\lambda_k} \, dW_k(t) \right)^2 = 2T \sum_{k \geq 1} \frac{c_k}{\lambda_k} = 2T \sum_{k \geq 1} \frac{1}{k^{1+(\alpha-1)\beta}} < \infty.
\]

The solution \((u(t), p(t))\) of (2.8) then is constructed using similar arguments as in the proof of Proposition 3 in Section 3 via stopping times \( \tau^R, R > 0 \), given by

\[
\tau^R = \inf_{t \geq 0} \{ u(t)^2 + p(t)^2 \geq R^2 \},
\]

and the local solutions

\[
du^R(t) = p^R(t) \, dt,
\]

\[
mp^R(t) = \left( -\left( \gamma + \sum_{k \geq 1} \frac{c_k}{\lambda_k} \right) p^R(t) - \Phi'(u^R(t)) \theta_R(u^R(t)) \right) dt
\]

\[
\quad + \sum_{k \geq 1} \sqrt{\frac{2c_k}{\lambda_k}} \, dW_k(t) + \sqrt{2\gamma} \, dW_0(t),
\]

where \( \theta^R \) is the cut-off function defined in (3.1). Furthermore, we have the following bound: for every \( T > 0 \) and \((u_0, p_0) \in \mathbb{R}^2\), it holds that

\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} u(t)^2 + p(t)^2 \right] \leq C(T, u_0, p_0).
\]

This estimate will be useful later in the proof of Theorem 8. The solution \( X_\epsilon(t) \) is constructed using the stopping time \( \tau^R_\epsilon \) given by

\[
\tau^R_\epsilon = \inf_{t \geq 0} \{ \|X(t)\|_{\mathcal{H}_{-\epsilon}} \geq R \},
\]

and the local solutions of the cut-off system obtained from (2.7)

\[
dx^R(t) = u^R(t) \, dt,
\]

\[
mdv^R(t) = \left( -\gamma v^R(t) - \Phi'(x^R(t)) \theta^R(x^R(t)) - \sum_{k \geq 1} \sqrt{\frac{c_k}{\epsilon^k}} z^R_k(t) \right) dt
\]

\[
\quad + \sqrt{2\gamma} \, dW_0(t),
\]

\[
dz^R_k(t) = \left( -\frac{\lambda_k}{\epsilon^k} z^R_k(t) + \sqrt{\frac{c_k}{\epsilon^k}} v^R(t) \right) dt + \sqrt{\frac{2\lambda_k}{\epsilon}} \, dW_k(t), \quad k \geq 1.
\]

We now turn to the proof of Theorem 5. Similar to the proof of Theorem 4, it will make use of the local solutions \((u^R(t), p^R(t))\) from (4.3) and \((x^R(t), v^R(t))\) from (4.6). As mentioned previously in Section 3, the idea essentially consists of two major steps: first, fixing \( R > 0 \), we show that the corresponding local solution \((x^R(t), v^R(t))\) in (4.6) converges to \((u^R(t), p^R(t))\) in (4.3). Then, taking \( R \) sufficiently large, we obtain the convergence in probability of the original solutions by using appropriate bounds on stopping times when
(u(t), p(t)) exits the ball of radius R centered at origin. We begin by the following important result giving a uniform bound on the pair (x(t), v(t)).

**Proposition 13.** Suppose that α, β, s satisfy Condition (D) of Assumption 2. We assume further that either

(a) \( \Phi' \) is globally Lipschitz,

or

(b) \( \Phi \) satisfies Assumptions 1 and Assumptions 6, and α, β satisfy Assumption 7.

Let \( X(t) \) solve (2.7) with initial condition \( X_0 = (x, v, z_1, z_2, \ldots) \in H_{-s} \). Then, for every \( T > 0 \), there exists a finite constant \( C(T, X_0) \) such that

\[
\sup_\epsilon \mathbb{E} \left[ \sup_{0 \leq t \leq T} x(t)^2 + v(t)^2 \right] \leq C(T, X_0).
\]

**Proof.** We begin by applying Duhamel’s formula on \( z_k(t) \) from (2.7) to see that

\[
z_k(t) = e^{-\frac{\lambda_k}{\epsilon} t} z_k(0) + \sqrt{\frac{c_k}{\epsilon}} \int_0^t e^{-\frac{\lambda_k}{\epsilon} (t-r)} v(r) dr + \sqrt{\frac{2c_k}{\epsilon}} \int_0^t e^{-\frac{\lambda_k}{\epsilon} (t-r)} dW_k(r).
\]

Substituting \( z_k \) by the formula above in the second equation from (2.7) in integral form, we obtain

\[
mv(t) = mv(0) + \int_0^t -\gamma v(r) - \Phi'(x(r)) dr + \sqrt{2\gamma} \int_0^t dW_0(r)
\]

\[
- \sum_{k \geq 1} \sqrt{\frac{c_k}{\epsilon}} \int_0^t e^{-\frac{\lambda_k}{\epsilon} r} z_k(0) dr - \sum_{k \geq 1} \frac{c_k}{\epsilon} \int_0^t \int_0^r e^{-\frac{\lambda_k}{\epsilon} (r-\ell)} v(\ell) d\ell dr
\]

\[
- \sum_{k \geq 1} \frac{2c_k \lambda_k}{\epsilon} \int_0^t \int_0^r e^{-\frac{\lambda_k}{\epsilon} (r-\ell)} dW_k(\ell) dr.
\]

It is important to note that using integration by parts, the last noise term above can be written as

\[
- \frac{2c_k \lambda_k}{\epsilon} \int_0^t \int_0^r e^{-\frac{\lambda_k}{\epsilon} (r-\ell)} dW_k(\ell) dr = \sqrt{\frac{2c_k}{\lambda_k}} \int_0^t e^{-\frac{\lambda_k}{\epsilon} (t-r)} dW_k(r) - \sqrt{\frac{2c_k}{\lambda_k}} \int_0^t dW_k(r).
\]
Suppose that Condition (a) holds, i.e., $\Phi'$ is Lipschitz. In view of (4.8) and (4.9), we have the following estimate for every $q > 1$ and $0 \leq t \leq T$,

$$v(t)^{2q} \leq c(q) \left[ |v(0)|^{2q} + \int_0^T \sup_{0 \leq r \leq t} v(r)^{2q} + x(r)^{2q} dr + \left( \sum_{k \geq 1} \sqrt{c_k} \left( 1 - e^{-\frac{\lambda_k}{q}} \right) |z_k(0)| \right)^{2q} \right]$$

$$+ c(T) \left( \sum_{k \geq 1} \frac{c_k}{\lambda_k} \right)^{2q} \int_0^T \sup_{0 \leq r \leq t} v(r)^{2q} dr$$

$$+ \sum_{0 \leq t \leq T} \left( \sqrt{2\gamma} \int_0^t dW_0(r) - \sum_{k \geq 1} \sqrt{\frac{2c_k}{\lambda_k}} \int_0^t dW_k(r) \right)^{2q}$$

$$+ \left( \sum_{k \geq 1} k^{-q*} \right)^{2q/q*} \sum_{0 \leq t \leq T} \left| \frac{2c_k k^{2s}}{\lambda_k} \int_0^t e^{-\frac{\lambda_k}{c} (t-r)} dW_k(r) \right|^2,$$

where in the last line, we have used Holder’s inequality with $\frac{1}{2q} + \frac{1}{q*} = 1$. Note that for every $x \geq 0$, we have $1 - e^{-x} \geq \sqrt{x}$. Using this inequality, we estimate

$$\sum_{k \geq 1} \sqrt{c_k} \left( 1 - e^{-\frac{\lambda_k}{q}} \right) |z_k(0)| \leq \sum_{k \geq 1} \sqrt{\frac{c_k}{\lambda_k}} |z_k(0)|$$

(4.10)

$$\leq \left( \sum_{k \geq 1} \frac{c_k k^{2s}}{\lambda_k} \sum_{k \geq 1} k^{-2s} z_k(0)^2 \right)^{1/2}.$$

Recalling $c_k$, $\lambda_k$ from (1.3) and the norm $\| \cdot \|_{\mathcal{H}_{-s}}$ from (2.1), thanks to Condition (D) of Assumption 2, we see that the above RHS is finite and so is the sum $\sum_{k \geq 1} c_k/\lambda_k$. In addition, using Burkholder-Davis-Gundy’s inequality, we have

$$\mathbb{E} \sup_{0 \leq t \leq T} \left( \sqrt{2\gamma} \int_0^t dW_0(r) - \sum_{k \geq 1} \sqrt{\frac{2c_k}{\lambda_k}} \int_0^t dW_k(r) \right)^{2q}$$

$$\leq c(q) \mathbb{E} \left( 2\gamma \int_0^T dr + \sum_{k \geq 1} \frac{2c_k}{\lambda_k} \int_0^T dr \right)^q = c(T, q) < \infty.$$

Finally, we invoke Lemma 11 again to find

$$\mathbb{E} \left( \frac{2c_k k^{2s}}{\lambda_k} \int_0^t e^{-\frac{\lambda_k}{c} (t-r)} dW_k(r) \right)^{2q}$$

(4.11)

$$\leq c(T, q) \sum_{k \geq 1} \frac{c_k^{q-1} k^{2sq}}{\lambda_k^{2q-1}}$$

$$= c(T, q) e^{q-1} \sum_{k \geq 1} \frac{1}{k^{q+(q_0-2q+1)\beta-2sq}},$$
Note that for \( \alpha > 1, s > 1/2 \) and \( \beta > 0 \) satisfying Condition (D) of Assumption 2, there exist constants \( q > 1 \) and \( 0 < q^* < 2 \) such that

\[
q + (q\alpha - 2q + 1)\beta - 2sq > 1,
\]

\( sq^* > 1, \) and \( \frac{1}{2q} + \frac{1}{q^*} = 1. \)

Consequently, the sums \( \sum_{k \geq 1} k^{-[q+(q\alpha-2q+1)\beta-2sq]} \) and \( \sum_{k \geq 1} k^{-sq^*} \) are both finite. Combining everything together, we infer

\[
\mathbb{E}\left(\sup_{0 \leq t \leq T} x(t)^{2q} + v(t)^{2q}\right) \leq c(T, q, X_0)\left[1 + \int_0^T \mathbb{E}\left(\sup_{0 \leq r \leq t} x(r)^{2q} + v(r)^{2q}\right) dt\right].
\]

Choosing such \( q \), we finally obtain the following estimate using Gronwall’s inequality

(4.12) \[
\mathbb{E}\left(\sup_{0 \leq t \leq T} x(t)^{2q} + v(t)^{2q}\right) \leq c(T, q, X_0),
\]

which proves the result for Condition (a) since \( q > 1. \)

Now suppose that Condition (b) holds. To simplify notation, we set

\[
g_k(t) := \int_0^t e^{-\frac{\lambda_k(t-r)}{m}} v(r) dr, \quad \text{and} \quad w_k(t) := \sqrt{2} \int_0^t e^{-\frac{k^2r}{m}} dW_k(r).
\]

Following (4.8) and (4.9), the equation on \( v(t) \) is written as

\[
d\left(m v(t) - \sum_{k \geq 1} \sqrt{\frac{c_k}{\lambda_k}} w_k(t)\right) = \left(-\gamma v(t) - \Phi'(x(t)) - \sum_{k \geq 1} \sqrt{\frac{c_k}{m}} e^{-\frac{\lambda_k}{m^2} t} z_k(0) - \sum_{k \geq 1} \frac{c_k}{m} g_k(t)\right) dt
\]

\[
+ \sqrt{2}\gamma dW_0(t) - \sum_{k \geq 1} \sqrt{\frac{2c_k}{\lambda_k}} dW_k(t).
\]

We apply Ito’s formula to \( \left(v(t) - \frac{1}{m} \sum_{k \geq 1} \sqrt{\frac{c_k}{\lambda_k}} w_k(t)\right)^2 \) to \( \Phi(x(t))/m \) to see that

\[
d\left[\left(\frac{1}{m} \sum_{k \geq 1} \sqrt{\frac{c_k}{\lambda_k}} w_k(t)\right)^2 \right] \frac{2}{m} + \Phi(x(t))/m
\]

\[
= \left(v(t) - \frac{1}{m} \sum_{k \geq 1} \sqrt{\frac{c_k}{\lambda_k}} w_k(t)\right) \left(-\frac{\gamma}{m} v(t) - \sum_{k \geq 1} \sqrt{\frac{c_k}{m^2}} e^{-\frac{\lambda_k}{m} t} z_k(0) - \sum_{k \geq 1} \frac{c_k}{m} g_k(t)\right) dt
\]

\[
+ \left(v(t) - \frac{1}{m} \sum_{k \geq 1} \sqrt{\frac{c_k}{\lambda_k}} w_k(t)\right) \left(\sqrt{2}\gamma m dW_0(t) - \sum_{k \geq 1} \sqrt{\frac{2c_k}{\lambda_k}} dW_k(t)\right)
\]

\[
+ \left(\frac{\Phi'(x(t))}{m} \sum_{k \geq 1} \sqrt{\frac{c_k}{\lambda_k}} w_k(t) + \frac{\gamma}{m^2} + \sum_{k \geq 1} \frac{c_k}{\lambda_k}\right) dt.
\]
We proceed to estimate the above RHS. Firstly, we invoke estimate \((4.10)\) to find
\[
\int_0^t \left( v(r) - \frac{1}{m} \sum_{k \geq 1} \sqrt{\frac{c_k}{\lambda_k}} w_k(r) \right) \left( - \sum_{k \geq 1} \sqrt{\frac{c_k}{m^2}} e^{-\lambda k r} \right) dr 
\]
\[
\leq \sup_{0 \leq r \leq t} \left| v(r) - \frac{1}{m} \sum_{k \geq 1} \sqrt{\frac{c_k}{\lambda_k}} w_k(r) \right| \left( \sum_{k \geq 1} \sqrt{\frac{c_k}{\lambda_k}} (1 - e^{-\lambda k t}) |z_k| \right) 
\]
\[
\leq \sup_{0 \leq r \leq t} \left| v(r) - \frac{1}{m} \sum_{k \geq 1} \sqrt{\frac{c_k}{\lambda_k}} w_k(r) \right| \left( \sum_{k \geq 1} \frac{c_k k^{2s}}{\lambda_k} \sum_{k \geq 1} k^{-2s} z_k^2 \right)^{1/2} 
\]
\[
\leq \frac{1}{2} \sup_{0 \leq r \leq t} \left| v(r) - \frac{1}{m} \sum_{k \geq 1} \sqrt{\frac{c_k}{\lambda_k}} w_k(r) \right|^2 + 2 \sum_{k \geq 1} \frac{c_k k^{2s}}{\lambda_k} \sum_{k \geq 1} k^{-2s} z_k^2. 
\]

Similarly, we have
\[
g_k(r) = \int_0^r e^{-\frac{\lambda k}{2} (r - \ell)} v(\ell) d\ell \leq \sup_{0 \leq \ell \leq r} |v(\ell)| \frac{e}{\lambda_k},
\]
which implies that
\[
\int_0^t \left( v(r) - \frac{1}{m} \sum_{k \geq 1} \sqrt{\frac{c_k}{\lambda_k}} w_k(r) \right) \left( - \sum_{k \geq 1} \frac{c_k}{m e} g_k(r) \right) dr 
\]
\[
\leq c \left( \sum_{k \geq 1} \frac{c_k}{\lambda_k} \right) \int_0^t \sup_{0 \leq \ell \leq r} v(\ell)^2 + \sup_{0 \leq \ell \leq r} \left( \sum_{k \geq 1} \sqrt{\frac{c_k}{\lambda_k}} w_k(\ell) \right)^2 dr.
\]

With regard to the martingale term, we invoke Burkholder-Davis-Gundy’s inequality to estimate
\[
\mathbb{E} \sup_{0 \leq r \leq t} \left| \int_0^r \left( v(\ell) - \frac{1}{m} \sum_{k \geq 1} \sqrt{\frac{c_k}{\lambda_k}} w_k(\ell) \right) \left( \frac{2 \gamma}{m} dW_0(\ell) - \sum_{k \geq 1} \sqrt{\frac{2 c_k}{m^2 \lambda_k}} dW_k(\ell) \right) \right| 
\]
\[
\leq c \left[ \left( \frac{2 \gamma}{m^2} + \sum_{k \geq 1} \frac{2 c_k}{m^2 \lambda_k} \right) \int_0^t \mathbb{E} \left( v(r) - \frac{1}{m} \sum_{k \geq 1} \sqrt{\frac{c_k}{\lambda_k}} w_k(r) \right)^2 dr + 1 \right].
\]

Lastly, we employ Assumption 6 to infer
\[
\int_0^t \Phi'(x(r)) \sum_{k \geq 1} \sqrt{\frac{c_k}{\lambda_k}} w_k(r) dr \leq c \int_0^t \Phi(x(r)) + \left( \sum_{k \geq 1} \sqrt{\frac{c_k}{\lambda_k}} w_k(r) \right)^n + 1 dr.
\]

Putting everything together, we arrive at the following inequality
\[
\mathbb{E} \sup_{0 \leq t \leq T} v(t)^2 + \Phi(x(t)) \leq c(T) \left[ 1 + \int_0^T \mathbb{E} \sup_{0 \leq r \leq t} v(r)^2 + \Phi(x(r)) dt 
\right.
\]
\[
+ \mathbb{E} \sup_{0 \leq t \leq T} \left( \sum_{k \geq 1} \sqrt{\frac{c_k}{\lambda_k}} w_k(r) \right)^2 + \mathbb{E} \sup_{0 \leq t \leq T} \left( \sum_{k \geq 1} \sqrt{\frac{c_k}{\lambda_k}} w_k(r) \right)^n \right].
\]
The result now follows immediately from Gronwall’s inequality if we can show that the last two terms on the above RHS is finite and independent of $\epsilon$. To this end, we claim that for every $T > 0$ and $q > 2$, there exists a finite constant $C(T,q) > 0$ such that

\[
\mathbb{E} \sup_{0 \leq t \leq T} \left( \sum_{k \geq 1} \sqrt{\frac{c_k}{\lambda_k}} w_k(r) \right)^{2q} \leq c(T,q).
\]

Recalling $w_k(t) := \sqrt{2} \int_0^t e^{-\frac{2k}{\lambda_k}(t-r)} dW_k(r)$, similar to (4.11), we employ Holder’s inequality and Lemma 11 to see that

\[
\mathbb{E} \sup_{0 \leq t \leq T} \left( \sum_{k \geq 1} \sqrt{\frac{c_k}{\lambda_k}} w_k(r) \right)^{2q} \leq \left( \sum_{k \geq 1} k^{-q_1 q^*} \right)^{2q/q^*} \sum_{k \geq 1} \mathbb{E} \sup_{0 \leq t \leq T} \left| \sqrt{\frac{c_k k^{2q_1}}{\lambda_k}} w_k(t) \right|^{2q} \leq c(T,q) \left( \sum_{k \geq 1} k^{-q_1 q^*} \right)^{2q/q^*} \left( \sum_{k \geq 1} k^{-1} \right)^{2q/q^*} \epsilon q_1 q > 1.
\]

Solving the above inequalities for $q_1$, we find

\[
\frac{1 + (\alpha - 2)\beta}{2q} + \frac{\beta}{2q} - \frac{1}{2q} > q_1 > 1 - \frac{1}{2q},
\]

which is always possible thanks to the second part of Condition (b), namely, $(\alpha - 2)\beta > 1$. The proof is thus complete. \qed

Remark 14. The trick of using integration by part in (4.9) was previously employed in [19, 34].

Proposition 15. Under the same Hypothesis of Theorem 5, assume further that $\Phi'(x)$ is globally Lipschitz. Let $X(t) = (x(t), v(t), z_1(t), \ldots)$ be the solution of (2.7) with initial condition $(x, v, z_1, z_2, \ldots) \in \mathcal{H}_{-s}$ and $(u(t), p(t))$ be the solution of (2.8) with initial condition $(u(0), p(0)) = (x, v)$. Then, for every $T > 0$,

\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} |x(t) - u(t)|^2 + \sup_{0 \leq t \leq T} |v(t) - p(t)|^2 \right] \rightarrow 0, \quad \epsilon \rightarrow 0.
\]

The proof of Proposition 15 is based on that of Theorem 2.6 in [34], adapted to our infinite-dimensional setting.
Proof. Setting \( \overline{x}(t) := x(t) - u(t) \), \( \overline{v}(t) := v(t) - p(t) \), we see that from (2.7), (2.8), \( (\overline{x}(t), \overline{v}(t)) \) satisfies the following system

\[
\overline{x}(t) = \int_0^t \overline{v}(r) \, dr,
\]

\[
m \overline{v}(t) = \int_0^t \left( -\gamma \overline{v}(r) + \left( \sum_{k \geq 1} \frac{c_k}{\lambda_k} \right) p(r) - \left[ \Phi'(x(r)) - \Phi'(u(r)) \right] - \sum_{k \geq 1} \sqrt{\frac{c_k}{\epsilon}} z_k(r) \right) \, dr
\]

\[
+ \int_0^t \sum_{k \geq 1} \sqrt{\frac{2c_k}{\lambda_k}} dW_k(r)
\]

with the initial condition \( (\overline{x}(0), \overline{v}(0)) = (0, 0) \). Regardig \( z_k(t) \) terms, we integrate with respect to time the third equation in (2.7) to find that

\[
\sqrt{\frac{\epsilon c_k}{\lambda_k}} (z_k(t) - z_k(0)) - \frac{c_k}{\lambda_k} \int_0^t v(r) \, dr - \sqrt{\frac{c_k}{\lambda_k}} \int_0^t dW_k(r) = -\sqrt{\frac{c_k}{\epsilon}} \int_0^t z_k(r).
\]

With these observations, the system of integral equations on \( (\overline{x}(t), \overline{v}(t)) \) becomes

\[
\overline{x}(t) = \int_0^t \overline{v}(r) \, dr,
\]

\[
m \overline{v}(t) = \int_0^t \left( -\gamma + \sum_{k \geq 1} \frac{c_k}{\lambda_k} \right) \overline{v}(r) - \left[ \Phi'(x(r)) - \Phi'(u(r)) \right] \, dr
\]

\[
+ \sqrt{\epsilon} \sum_{k \geq 1} \frac{\sqrt{c_k}}{\lambda_k} (z_k(t) - z_k(0)).
\]

In the above system, we have implicitly re-arranged infinitely many terms, resulting in the cancellation of noise terms. Recalling \( c_k, \lambda_k \) from (1.3) and the norm \( \| \cdot \|_{\mathcal{H}_\omega} \) from (2.1), this re-arrangement is possible following from (4.1) and the estimate

\[
\sum_{k \geq 1} \frac{\sqrt{c_k}}{\lambda_k} (z_k(t) - z_k(0)) \leq \sum_{k \geq 1} \frac{c_k}{\lambda_k} \int_0^t |v(r)| \, dr + \sum_{k \geq 1} \frac{1}{\sqrt{\epsilon}} \int_0^t \sum_{k \geq 1} \sqrt{c_k} |z_k(r)| \, dr
\]

\[
+ \sum_{k \geq 1} \sqrt{\frac{c_k}{\lambda_k}} \int_0^t |dW_k(r)|
\]

\[
< \infty, \text{ a.s.}
\]

thanks to condition (D) of Assumption 2. We invoke the assumption that \( \Phi' \) is globally Lipschitz and Gronwall’s inequality to deduce from (4.14)

\[
\mathbb{E} \sup_{0 \leq t \leq T} \overline{x}(t) + |\overline{v}(t)| \leq C(T) \sqrt{\epsilon} \mathbb{E} \sup_{0 \leq t \leq T} \left[ \sum_{k \geq 1} \frac{\sqrt{c_k}}{\lambda_k} (z_k(t) - z_k(0)) \right].
\]

The result now follows immediately from Proposition 16 below. \( \square \)
Proposition 16. Under the same Hypothesis of Proposition 15, suppose that \( X(t) = (x(t), v(t), z_1(t), \ldots) \) solves (2.7) with initial condition \((x, v, z_1, z_2, \ldots) \in H_{-s}\). Then,

\[
\sqrt{\epsilon} \mathbb{E} \sup_{0 \leq t \leq T} \left| \sum_{k \geq 1} \frac{\sqrt{\epsilon c_k}}{\lambda_k} (z_k(t) - z_k) \right| \to 0, \quad \epsilon \to 0.
\]

Proof. From (4.7), we see that

\[
\sum_{k \geq 1} \frac{\sqrt{\epsilon c_k}}{\lambda_k} |z_k(t) - z_k| \leq \sum_{k \geq 1} \frac{\sqrt{\epsilon c_k}}{\lambda_k} (e^{-\frac{\lambda_k}{\epsilon} t} - 1) |z_k|
\]

\[
+ \sum_{k \geq 1} \frac{c_k}{\lambda_k} \int_0^t e^{-\frac{\lambda_k}{\epsilon} (t-r)} |v(r)| dr
\]

\[
+ \sum_{k \geq 1} \frac{2c_k}{\lambda_k} \left| \int_0^t e^{-\frac{\lambda_k}{\epsilon} (t-r)} dW_k(r) \right|.
\]

We aim to show that each series on the above RHS converges to zero in expectation as \( \epsilon \downarrow 0 \). We note that the convergence to zero of the last series follows immediately from (4.11). For the other two terms, we shall make use of the following inequality: for \( q > 0 \), there exists \( c(q) > 0 \) such that for every \( x \geq 0 \), it holds that

\[
1 - e^{-x} \leq c(q)x^q.
\]

For a positive \( q_1 < \frac{1}{2} \) (to be chosen later), we estimate the first sum on the RHS of (4.15) as follows.

\[
\sum_{k \geq 1} \sup_{0 \leq t \leq T} \frac{\sqrt{\epsilon c_k}}{\lambda_k} (e^{-\frac{\lambda_k}{\epsilon} t} - 1) z_k \leq c(T, q_1) \epsilon^{1/2 - q_1} \frac{c_k^{1/2}}{\lambda_k^{1/2 - q_1}} z_k
\]

\[
\leq c(T, q_1) \epsilon^{1/2 - q_1} \left( \sum_{k \geq 1} \frac{c_k k^{2s}}{\lambda_k^{2 - 2q_1}} \right)^{1/2} \left( \sum_{k \geq 1} k^{-2s} z_k^2 \right)^{1/2},
\]

where we have used (4.16) on the first line and Holder’s inequality on the second line, respectively. Recalling (1.3), we have

\[
\sum_{k \geq 1} \frac{c_k k^{2s}}{\lambda_k^{2 - 2q_1}} = \sum_{k \geq 1} \frac{1}{k^{1 + (\alpha + 2q_1 - 2)\beta - 2s}}.
\]

In view of Condition (D) of Assumption 2, there always exists a constant \( q_1 \in (0, 1/2) \) such that \((2q_1 - 1)\beta + (\alpha - 1)\beta - 2s > 0\), which implies that the above RHS is finite. Similarly,
we have
\[
\sum_{k \geq 1} \sup_{0 \leq t \leq T} e^{-\frac{\lambda k}{\lambda} (t-r)} |v(r)| dr \leq \sum_{k \geq 1} \sup_{0 \leq t \leq T} \frac{c_k}{\lambda_k} \left( 1 - e^{-\frac{\lambda k}{\lambda} t} \right) \sup_{0 \leq t \leq T} |v(t)|
\]
\[
\leq c(T, q_2) \epsilon^{1-q_2} \sum_{k \geq 1} \frac{c_k}{\lambda^2-k^2} \sup_{0 \leq t \leq T} |v(t)|
\]
\[
= c(T, q_2) \epsilon^{1-q_2} \sum_{k \geq 1} \frac{1}{k^{1+(\alpha-2+q_2)/\beta}} \sup_{0 \leq t \leq T} |v(t)|.
\]

We invoke Condition (D) from Assumption 2 again to see that there exists a positive $q_2 \in (0, 1)$ such that $\alpha - 2 + q_2 > 0$. Choosing such $q_2$ implies that the series on the above RHS is convergent. We thus obtain the estimate
\[
\mathbb{E} \sum_{k \geq 1} \sup_{0 \leq t \leq T} \frac{c_k}{\lambda_k} \int_0^t e^{-\frac{\lambda k}{\lambda_0} (t-r)} |v(r)| dr \leq c(T, q_2) \epsilon^{1-q_2} \mathbb{E} \sup_{0 \leq t \leq T} |v(t)| \leq c(T, q_2) \epsilon^{1-q_2},
\]
where the last implication follows from Proposition 13. Putting everything together, we obtain the result. $\square$

Since we will make use of exiting times, with a slightly abuse of notation, it is convenient to recall from (3.9) for $R > 0$
\[
\sigma^R = \inf_{t \geq 0} \{|u(t)| \geq R\}, \quad \text{and} \quad \sigma^R_\epsilon = \inf_{t \geq 0} \{|x(t)| \geq R\}.
\]
With Proposition 15 in hand, we give the proof of Theorem 5.

Proof of Theorem 5. The arguments are almost the same as those in the proof of Theorem 4 and hence omitted. The only difference here is the appearance of the term $|v(t) - p(t)|$. Nevertheless, we note that for $0 \leq t \leq \sigma^R \wedge \sigma^R_\epsilon$,
\[
(u(t), p(t)) = (u^R(t), p^R(t)) \quad \text{and} \quad (x(t), v(t)) = (x^R(t), v^R(t)),
\]
and thus the proof of Theorem 4 is applicable. $\square$

We finally turn our attention to Theorem 8. The proof is relatively short and will make use of Condition (b) in Proposition 13.

Proof of Theorem 8. For given $R > 0$, let $\sigma^R, \sigma^R_\epsilon$ be defined as in (3.9). As mentioned above, for $0 \leq t \leq \sigma^R \wedge \sigma^R_\epsilon$,
\[
(u(t), p(t)) = (u^R(t), p^R(t)) \quad \text{and} \quad (x(t), v(t)) = (x^R(t), v^R(t)).
\]
We then have a chain of implications
\[
\mathbb{E}\left[ \sup_{0 \leq t \leq T} |x(t) - u(t)|^q + |v(t) - p(t)|^q \right] \\
= \mathbb{E}\left[ \left( \sup_{0 \leq t \leq T} |x(t) - u(t)|^q + |v(t) - p(t)|^q \right) 1_{\{\sigma^R \wedge \sigma^R < T\}} \right] \\
\quad + \mathbb{E}\left[ \left( \sup_{0 \leq t \leq T} |x(t) - u(t)|^q + |v(t) - p(t)|^q \right) 1_{\{\sigma^R \wedge \sigma^R > T\}} \right] \\
\leq \mathbb{E}\left[ \left( \sup_{0 \leq t \leq T} |x(t) - u(t)|^q + |v(t) - p(t)|^q \right) 1_{\{\sigma^R \wedge \sigma^R < T\}} \right] \\
\quad + \mathbb{E}\left[ \sup_{0 \leq t \leq T} |x^R(t) - u^R(t)|^q + |v^R(t) - p^R(t)|^q \right].
\]

On one hand, in view of Proposition 15, since \(1 \leq q < 2\), it holds that
\[
\mathbb{E}\left[ \sup_{0 \leq t \leq T} |x^R(t) - u^R(t)| + |v^R(t) - p^R(t)| \right] \to 0, \quad \epsilon \to 0.
\]

On the other hand, we invoke Holder’s inequality with \(\frac{q}{2} + \frac{1}{q^*} = 1\) to estimate
\[
\mathbb{E}\left[ \left( \sup_{0 \leq t \leq T} |x(t) - u(t)|^q + |v(t) - p(t)|^q \right) 1_{\{\sigma^R \wedge \sigma^R < T\}} \right] \\
\leq c \left( \mathbb{E}\left[ \sup_{0 \leq t \leq T} x(t)^2 + v(t)^2 \right] + \mathbb{E}\left[ \sup_{0 \leq t \leq T} u(t)^2 + p(t)^2 \right] \right)^{q/2} \left( \mathbb{P}\left\{ \sigma^R \wedge \sigma^R < T \right\} \right)^{1/q^*}.
\]

Notice that by Markov’s inequality, we have
\[
\mathbb{P}\left\{ \sigma^R \wedge \sigma^R < T \right\} \leq \mathbb{P}\left\{ \sigma^R < T \right\} + \mathbb{P}\left\{ \sigma^R < T \right\} \\
\leq \mathbb{P}\left\{ \sup_{0 \leq t \leq T} |u(t)| \geq R \right\} + \mathbb{P}\left\{ \sup_{0 \leq t \leq T} |x(t)| \geq R \right\} \\
\leq \mathbb{E}\left[ \sup_{0 \leq t \leq T} |u(t)|^2 \right] + \mathbb{E}\left[ \sup_{0 \leq t \leq T} x(t)^2 \right] \frac{1}{R^2}.
\]

The result now follows immediately from (4.4) and Proposition 13 by first taking \(R\) sufficiently large and then shrinking \(\epsilon\) further to zero. The proof is thus complete. \(\square\)

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