HOW GOOD IS YOUR LAPLACE APPROXIMATION OF THE BAYESIAN POSTERIOR? FINITE-SAMPLE COMPUTABLE ERROR BOUNDS FOR A VARIETY OF USEFUL DIVERGENCES

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The Laplace approximation is a popular method for providing posterior mean and variance estimates. But can we trust these estimates for practical use? One might consider using rate-of-convergence bounds for the Bayesian Central Limit Theorem (BCLT) to provide quality guarantees for the Laplace approximation. But the bounds in existing versions of the BCLT either: require knowing the true data-generating parameter, are asymptotic in the number of samples, do not control the Bayesian posterior mean, or apply only to narrow classes of models. Our work provides the first closed-form, finite-sample quality bounds for the Laplace approximation that simultaneously (1) do not require knowing the true parameter, (2) control posterior means and variances, and (3) apply generally to models that satisfy the conditions of the asymptotic BCLT. In fact, our bounds work even in the presence of misspecification. We compute exact constants in our bounds for a variety of standard models, including logistic regression, and numerically demonstrate their utility. We provide a framework for analysis of more complex models.

1. Introduction.

1.1. Motivation. Researchers and practitioners who use statistical inference for solving real-world problems need good point estimates and uncertainties. Bayesian inference provides a way of obtaining those through expectations calculated with respect to the posterior distribution – and in particular through the posterior mean and variance. Such expectations are, however, often intractable or costly to compute, which forces users to use approximations. An easy and fast approach is to approximate the posterior by a suitably chosen Gaussian distribution. This is known as the Laplace approximation in the approximate Bayesian inference literature and is grounded in the celebrated Bernstein–von Mises theorem. Laplace approximation is a commonly used tool in many communities [6, Section 4.4], [39, Section 4.6.8.2], [5]. Studies have shown its appealing empirical performance, for instance in the context of Bayesian neural networks [12]. Widely applicable, computable and rigorously justified theoretical guarantees on the quality of the Laplace approximation are, however, still not available in the literature. This leaves researchers and practitioners using it unable to tell with high confidence whether their inference is robust enough for their purposes. Moreover, convergence in the Bernstein–von Mises theorem is normally expressed in terms of the total variation distance, and the Laplace approximation is typically justified by the fact that the total variation distance between the rescaled posterior and the Gaussian vanishes in the limit. The total variation distance, however, does not control the difference of means or the difference of covariances in general. At the same time, exactly those quantities are most often

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reported by users of approximate Bayesian inference. Practitioners and researchers therefore need finite sample guarantees on the quality of Laplace approximation, expressed not only in terms of the total variation distance but also in terms of metrics that control the error in mean and covariance approximation.

1.2. **Our contribution.** Our work provides the first closed-form, fully computable, finite-sample bounds for the Laplace approximation that do not require the posterior to be log-concave. We quantify the control not only over the total variation distance but also

- the 1-Wasserstein distance, which controls the difference of means
- another integral probability metric that bounds the difference of covariances.

Our bounds are computable without access to the true parameter or integrals with respect to the posterior. They are expressed in terms of the data and work under any distribution of the data, also when the model is misspecified. In particular, our results are fully applicable to models involving generalized likelihoods and the resulting generalized posteriors (see [7, 37, 9]). Our assumptions on the generalized likelihood and the prior are standard and no stronger than the assumptions of the classical proofs of the Bernstein–von Mises Theorem (see e.g. [17, Section 1.4] or [37] for details). The sample-size and dimension dependence of our bounds on the total-variation and the 1-Wasserstein distance are tight and such that they cannot be improved in general, as we discuss in Section 4.2. As far as we are aware, our bounds’ dimension and sample-size dependence is also better than that appearing in all the previous works on the Laplace approximation. We compute our bounds explicitly for a variety of Bayesian models, including logistic regression with Student’s t prior.

Our contribution lies also in our proof techniques. In order to control the discrepancy inside a ball around the maximum likelihood estimator (MLE) or the maximum a posteriori (MAP), we use the log-Sobolev inequality or Stein’s method. In order to control the discrepancy over the rest of the parameter space, we carefully bound the tail growth using standard assumptions of the Bernstein–von Mises Theorem. We believe that this approach to proving computable non-asymptotic bounds could be extended so as to cover more general statistical models satisfying the conditions of the local asymptotic normality (LAN) theory [27, Chapters 1-3]. We consider this, however, a separate problem and leave it for future work.

1.3. **Related work.** A number of recent papers have studied the non-asymptotic properties of the Laplace approximation and the Bernstein–von Mises theorem for Bayesian posteriors. In this section we contrast the present work with these analyses in terms of the computability of the bounds, strength of the results, and restrictiveness of the assumptions. A brief summary of the differences can be found in Table 1.
The recent papers [26, Section 6.1], [13] and [46] have offered ways of obtaining guarantees on the quality of Laplace approximation under log-concavity of the posterior. In [26, Proposition 6.1] the authors assume that the posterior is strongly log-concave and obtain a computable bound on the 1- and 2-Wasserstein distances between the posterior and the approximating Gaussian. In [26, Proposition 6.2] the authors relax the assumption on the posterior to weak log-concavity but the bound they obtain is not computable in practice, for finite data. In [13], the author assumes weak, yet strict log-concavity of the posterior. As their main result [13, Theorem 5], they obtain a bound on the Kullback-Leibler (KL) divergence between the posterior and the Gaussian approximation. In their bound, only the leading terms are computable, while the higher order terms are presented using the big-O notation. In reference [46], the author assumes a Gaussian prior and a log-concave likelihood, which yields a strongly log-concave posterior. They relax this assumption only in a very specific case of a nonlinear inverse problem with a certain "warm start condition". Their bounds are on the total variation distance. They also provide a bound on the difference of means - yet this bound involves generic abstract constants and it is not clear to us whether it is computable in applications. Reference [44] has proved a qualitative, asymptotic result about convergence of the Laplace approximation for inverse problems. Motivated by this work, [23] has provided non-asymptotic bounds for the Laplace approximation on the total variation distance - in the context of Bayesian inverse problems and only under the assumption that the posterior distribution is sub-Gaussian. Moreover, the concurrent work [16], has provided bounds on the Gaussian approximation of the posterior, yet only for i.i.d. data coming from a regular $k$-parameter exponential family. The dimension dependence of the bounds in [23, 46, 13, 16] is worse than ours, as we describe in Section 4.2. Besides, the recent paper [22] proved interesting asymptotic convergence results for Bayesian posteriors in setups in which the maximum-a-posteriori (MAP) may not be not unique. We also mention two very recent pieces of work [28, 29], which appeared after the first version of the present article was put on ArXiv. The former studies the leading order contribution to the total variation distance between the posterior and the Laplace approximation and thus gives necessary conditions under which the Laplace approximation is valid. For the sufficient conditions, it refers the reader to the ArXiv version of the present article, in particular to our Theorems 3.1 and 3.2. Paper [29] proposes a skew adjustment to the Laplace approximation so as to improve its quality in high dimensions.

Non-asymptotic analyses of the Bernstein–von Mises (BvM) Theorem (discussed in Appendix B) have previously been performed in [41, 53]. The aim and focus of those analyses is however significantly different from that of the present paper, as we concentrate on the quality of the Laplace approximation (of the form of Equation (B.4), as discussed in Appendix B). In [41] the authors consider semiparametric inference and prove results whose purpose it is to provide insight into the critical dimension of the parameter for which the BvM Theorem holds. They prove their bounds only for the non-informative and the Gaussian prior. Computing their bounds requires knowledge of the true parameter, which is the object of inference. Additionally, the recent work [53] derives a Berry–Esseen-type bound (which depends on the true parameter value) on the total variation distance between the posterior and the approximating normal in the approximately linear regression model.

In contrast to the references mentioned above, our bounds hold and are computable, without access to the true parameter, for general posteriors satisfying assumptions analogous to the classical assumptions of the Bernstein–von Mises theorem (see e.g. [37, Section 4] for a recent reference or [17, Section 1.4] for a more classical one). In particular, we do not require (weak or strong) log-concavity or sub-Gaussianity of the posterior. Indeed, we compute our bounds explicitly for examples of commonly used non-log-concave and heavy-tailed posteriors in Section 7. The reason we can avoid imposing the assumption of log-concavity is that all
we need to control the tail behaviour of the posterior is the assumption of the strict optimality of the MLE or MAP (Assumption 9 or Assumption 6). This is much weaker than assuming log-concavity or strong unimodality of the posterior. The array of priors our bounds are available for is also much wider than just the Gaussian family. All we require is differentiability, boundedness and boundedness away from zero of the prior density in a small neighbourhood around the MAP or the MLE. We do not make any assumption about the true distribution of the data and cover generalized likelihoods not coming from any particular family. Moreover, we bound a variety of divergences including those that control means and variances.

Finally, it is worth noting that bounds similar to ours are not widely available for approximate Bayesian inference techniques in general. Indeed, they are not available for variational inference (VI) methods [8, 52], except for the recently studied case of Gaussian VI [30]. In the area of Markov Chain Monte Carlo methods, progress has recently been made on deriving convergence guarantees for the Unadjusted Langevin Algorithm under different sets of assumptions on the tail growth of the target distribution [10, 4, 14]. However, the popular Metropolis-adjusted Langevin Algorithm and Hamiltonian Monte Carlo are not equipped with such guarantees beyond the case of log-concave targets. Flexible and computable post-hoc checks measuring a discrepancy between the empirical distribution of a sample and the target distribution are given by graph and kernel Stein discrepancies. Graph Stein discrepancies [19, 18] metrize weak convergence and control the difference of means for distant dissipative targets. They are however not known to control the difference of variances. Graph diffusion Stein discrepancies [18] possess the same properties under a slightly weaker (yet technical) assumption of the underlying diffusion having a rapid Wasserstein decay rate. The fast and popular kernel Stein discrepancies [11, 35, 40, 20] metrize weak convergence for certain choices of kernels and for distant-dissipative targets. In comparison, we derive control over the rate of weak convergence and the differences of means and variances under interpretable assumptions which do not require any particular tail behavior of the target posterior.

1.4. Structure of the paper. In Section 2 we describe our setup, introduce the necessary notation and present our assumptions. We also give examples of popular models satisfying the assumptions. In Section 3 we present and discuss our main bounds. In Section 4 we provide a comprehensive discussion of the main results and their dependence on the sample-size, dimension and data, as well as computability. Section 5 discusses our proof techniques. In Section 6 we present results analogous to those of Section 3, yet focused on different types of approximations. In Section 7 we show how to compute our bounds for the (non-log-concave) posterior in the logistic regression model with Student’s t prior. We also present plots of our bounds computed numerically in this case. Moreover, we numerically compare our control over the difference of means and the difference of variances to the ground truth for certain conjugate prior models. In Section 8 we present conclusions of our work. The proofs of the results of Sections 3 and 6 are postponed to the appendices, included as supplementary material.

2. Setup, assumptions and notation.

2.1. Setup. Our setup and assumptions are similar to those found for instance in [37]. We fix $n \in \mathbb{N}$ and study probability measures on $\mathbb{R}^d$ having Lebesgue densities of the form:

\[
\Pi_n(\theta) = e^{L_n(\theta)} \pi(\theta)/z_n,
\]

where $\pi : \mathbb{R}^d \to \mathbb{R}$ is a Lebesgue probability density function, $L_n : \mathbb{R}^d \to \mathbb{R}$ and $z_n \in \mathbb{R}_+$ is the normalizing constant. Throughout the paper, we call $\pi$ the prior density (or simply the prior), $L_n$ the generalized log-likelihood and $\Pi_n$ the generalized posterior. By

\[
\overline{L}_n(\theta) := \log (\Pi_n(\theta))
\]
we denote the generalized log-posterior and let:
\[ \theta_n := \arg \max_{\theta \in \mathbb{R}^d} L_n(\theta), \quad \bar{\theta}_n = \arg \max_{\theta \in \mathbb{R}^d} \bar{L}_n(\theta), \]
whenever those quantities exist. If those quantities are unique, we call \( \hat{\theta}_n \) the maximum likelihood estimator (MLE) and \( \bar{\theta}_n \) the maximum a posteriori (MAP). Here and elsewhere, overbars will refer to quantities derived from the MAP \( (\bar{J}_n(\bar{\theta}_n), \bar{\delta}, \bar{M}_2, \text{etc.}) \), introduced below) and hats will refer to quantities derived from the MLE (for instance \( \hat{J}_n(\hat{\theta}_n) \), introduced below). For any twice-differentiable function \( f : \mathbb{R}^d \rightarrow \mathbb{R} \), we let \( f' \) stand for its gradient and \( f'' \) for its Hessian. We shall write
\[ \hat{J}_n(\theta) = -\frac{L_n''(\theta)}{n}, \quad \bar{J}_n(\theta) = -\frac{\bar{L}_n''(\theta)}{n}, \]
whenever those expressions make sense (i.e. when the Hessians exist). For any three times differentiable function \( f : \mathbb{R}^d \rightarrow \mathbb{R} \), we will also write \( f''' \) for its third (Frechét) derivative, defined as the following multilinear 3-form on \( \mathbb{R}^d \):
\[ f'''(\theta)[u, v, w] = \sum_{i,j,k=1}^d \frac{\partial^3 f}{\partial \theta_i \partial \theta_j \partial \theta_k}(\theta) u_i v_j w_k. \]
The norm \( \| \cdot \|_* \) of this third derivative will be defined in the following way:
\[ \| f'''(\theta) \|_* := \sup_{\| u \|\leq 1, \| v \|\leq 1, \| w \|\leq 1} \left| f'''(\theta)[u, v, w] \right|. \]
Throughout the paper, \( \| \cdot \| \) denotes the Euclidean norm. We will also let \( \lambda_{\min}(\theta_n) \) be the minimal eigenvalue of \( \hat{J}_n(\hat{\theta}_n) \) and \( \bar{\lambda}_{\min}(\bar{\theta}_n) \) be the minimal eigenvalue of \( \bar{J}_n(\bar{\theta}_n) \). Moreover, throughout the paper \( \| \cdot \|_{op} \) will denote the operator (i.e. spectral) norm, \( \langle \cdot, \cdot \rangle \) will be the Euclidean inner product and \( \hat{\theta}_n \) will always denote a random variable distributed according to the generalized posterior measure with density given by eq. (2.1). \( \mathcal{N}(\mu, \Sigma) \) will denote the normal distribution with mean \( \mu \) and covariance \( \Sigma \) and function \( \mathcal{L}(\cdot) \) will return the law of its argument. \( I_{d \times d} \) will always denote the \( d \)-dimensional identity matrix.

Our bounds will be derived for two types of approximations. The first type is what we call the MAP-centric approach. Within this approach, \( \mathcal{L}(\sqrt{n}(\theta_n - \bar{\theta}_n)) \) is approximated by \( \mathcal{N}(0, \hat{J}_n(\hat{\theta}_n))^{-1} \). On the other hand, what we call the MLE-centric approach is the approximation of \( \mathcal{L}(\sqrt{n}(\theta_n - \bar{\theta}_n)) \) by \( \mathcal{N}(0, \bar{\lambda}_{\min}(\bar{\theta}_n)^{-1}) \). The MLE-centric approach is similar to the classical Bernstein–von Mises Theorem (see Appendix B for details). The approximation it yields is with a Gaussian distribution whose parameters depend only on the likelihood while ignoring the prior completely. The MAP-centric approach is arguably more popular and standard among practitioners using the Laplace approximation. The approximating Gaussian’s parameters depend on the posterior, thus depending on the prior as well. Thus, this approximation may be expected to be more accurate for many commonly used models. We derive our results for both approaches in order to make them applicable in a variety of circumstances. Indeed, our bounds can be used by practitioners who are interested in assessing the applicability of one of those approximation approaches in real-life circumstances. At the same time, we believe they are also interesting for researchers who study theoretical aspects of Bernstein–von Mises - type results.

The bounds we obtain are on the following distances:

1. The Total Variation (TV) distance, which, for two probability measures \( \nu_1 \) and \( \nu_2 \) on a measurable space \( (\Omega, \mathcal{F}) \) is defined by
\[ TV(\nu_1, \nu_2) := \sup_{A \in \mathcal{F}} |\nu_1(A) - \nu_2(A)|. \]
2. The 1-Wasserstein distance, which, for probability measures \( \nu_1 \) and \( \nu_2 \) and the set \( \Gamma(\nu_1, \nu_2) \) of all couplings between them, is defined by

\[
W_1(\nu_1, \nu_2) := \inf_{\gamma \in \Gamma(\nu_1, \nu_2)} \int \|x - y\| d\gamma(x, y).
\]

Kantorovich duality (see, e.g. [51, Theorem 5.10]) provides an equivalent definition. Let \( \| \cdot \|_L \) return the Lipschitz constant of the input. Then

\[
W_1(\nu_1, \nu_2) = \sup_{f \text{ Lipschitz: } \|f\|_L = 1} |E_{\nu_1} f - E_{\nu_2} f|.
\]

3. The following integral probability metric, which, for \( Y_1 \sim \nu_1 \) and \( Y_2 \sim \nu_2 \) is defined by

\[
\sup_{v: \|v\| \leq 1} \left| E \langle v, Y_1 \rangle^2 - E \langle v, Y_2 \rangle^2 \right|
\]

For zero-mean \( \nu_1 \) and \( \nu_2 \), this integral probability metric controls the operator norm of the difference of the covariance matrices of \( \nu_1 \) and \( \nu_2 \). For more general \( \nu_1 \) and \( \nu_2 \), it may be combined with the 1-Wasserstein distance in order to provide control over the operator norm of the difference between the covariance matrices of \( \nu_1 \) and \( \nu_2 \). Our proof techniques allow us to upper-bound integral probability metrics. Obtaining a bound on the above integral probability metric lets us reach our goal of controlling the difference between the covariance of the posterior and that of the Laplace approximation.

2.2. Assumptions made throughout the paper. Now we list the assumptions that we will need to prove our finite-sample bounds and define constants used therein. We will first present those assumptions that will stand for both approaches described above and then others, which are divided between those relevant for the MAP-centric approach (Section 2.3) and the MLE-centric approach (Section 2.4). We reiterate that the conditions we require are similar to the classical assumptions of the Bernstein–von Mises theorem, as given in [17, Section 1.4] and [37, Theorem 5]. The first assumption that will be used throughout the paper is the following:

**Assumption 1.** There exists a unique MLE \( \hat{\theta}_n \). There also exists a real number \( \delta > 0 \) such that the generalized log-likelihood \( L_n \) is three times differentiable inside \( \{ \theta : \|\theta - \hat{\theta}_n\| \leq \delta \} \). For the same \( \delta > 0 \) there exists a real number \( M_2 > 0 \), such that:

\[
\sup_{\|\theta - \hat{\theta}_n\| \leq \delta} \|L'''_n(\theta)\|^* \leq M_2.
\]

**Remark 1.** Assumption 1 is needed to ensure that the posterior looks Gaussian inside the \( \delta \)-neighborhood around the MLE. Indeed, Assumption 1, combined with Taylor’s theorem, implies that \( L_n(n^{-1/2}\theta + \hat{\theta}_n) - L_n(\hat{\theta}_n) \) can be approximated by \( -\frac{1}{2} \theta^T J_n(\hat{\theta}_n) \theta \) for \( \theta \) in a neighborhood of \( \hat{\theta}_n \). Note that \( -\frac{1}{2} \theta^T J_n(\hat{\theta}_n) \theta \) is the logarithm of the \( \mathcal{N}(0, J_n(\hat{\theta}_n)) \) density (up to an additive constant), which makes this approximation particularly useful. When proving our bounds in the MLE-centric approach, we also use Assumption 1 to prove that the posterior satisfies the log-Sobolev inequality inside this neighborhood (see Appendix F for details).

Moreover, we make the following assumption on the prior:
**ASSUMPTION 2.** For the same \( \delta > 0 \) as in Assumption 1, there exists a real number \( \bar{M}_1 > 0 \), such that

\[
\sup_{\theta : \|\theta - \hat{\theta}_n\| \leq \delta} \left| \frac{1}{\pi(\theta)} \right| \leq \bar{M}_1.
\]

**REMARK 2.** Note that for Assumption 2 to be satisfied, it suffices to assume that \( \pi \) is continuous and positive in the \( \delta \)-ball around \( \hat{\theta}_n \). Assumption 2 essentially ensures that the prior puts a non-negligible amount of mass in the \( \delta \)-neighbourhood of the MLE.

2.3. *Additional assumptions in the MAP-centric approach.* In the MAP-centric approach we keep Assumption 1 and Assumption 2 and additionally assume the following:

**ASSUMPTION 3.** There exists a unique MAP \( \bar{\theta}_n \). There also exists a real number \( \bar{\delta} > 0 \), such that the log-prior, \( \log \pi \), is three times differentiable inside \( \{ \theta : \|\theta - \hat{\theta}_n\| \leq \bar{\delta} \} \). Moreover, for the same \( \delta \), there exists a real number \( \bar{M}_2 > 0 \), such that

\[
\sup_{\theta : \|\theta - \hat{\theta}_n\| \leq \delta} \frac{\|T_n''(\theta)^*\|}{n} \leq \bar{M}_2.
\]

**REMARK 3.** Assumption 3 is very similar to Assumption 1. The difference is that we now consider a ball around the MAP rather than the MLE and we require additional differentiability of the prior density inside this ball. Assumption 3 will be used in the MAP-centric approach (together with Assumption 5) to show that inside the \( \delta \)-ball around the MAP the posterior satisfies the log-Sobolev inequality.

In the MAP-centric both Assumptions 1 and 3 will be used. Assumption 1 will play an important role in the process of controlling the posterior in the region \( \{ \theta : \|\theta - \hat{\theta}_n\| > \bar{\delta} \} \). More specifically, we will use Assumption 1 to lower-bound the normalizing constant of the posterior after controlling it with the integral \( \int_{||I - \hat{\theta}_n|| \leq \bar{\delta}} \pi(t) e^{L_n(t) - L_n(\hat{\theta}_n)} dt \). See Appendix D.5 for more detail.

**ASSUMPTION 4.** For the same \( \delta > 0 \) as in Assumption 1 and for the same \( \bar{\delta} \) as in Assumption 3,

\[
\max \left\{ \|\hat{\theta}_n - \bar{\theta}_n\|, \frac{\text{Tr} \left[ J_n(\hat{\theta}_n)^{-1} \right]}{n} \right\} < \bar{\delta} \quad \text{and} \quad \sqrt{\text{Tr} \left[ \left( J_n(\hat{\theta}_n) + \frac{\delta M_2}{3} I_{d \times d} \right)^{-1} \right]} < \delta.
\]

**REMARK 4.** Assumption 4 is an assumption on the size of \( n \) and the choice of \( \bar{\delta} \) and \( \delta \). Indeed, as long as the MLE and MAP converge to the same limit (which is true in the majority of commonly used modelling setups), we expect \( \max \left\{ \|\hat{\theta}_n - \bar{\theta}_n\|, \sqrt{\frac{\text{Tr} \left[ J_n(\hat{\theta}_n)^{-1} \right]}{n}} \right\} \) to go to zero as \( n \to \infty \). We similarly expect \( \sqrt{\frac{\text{Tr} \left[ \left( J_n(\hat{\theta}_n) + \frac{\delta M_2}{3} I_{d \times d} \right)^{-1} \right]}{n}} \) to go to zero as \( n \to \infty \).

Moreover \( \sqrt{\frac{\text{Tr} \left[ \left( J_n(\hat{\theta}_n) + \frac{\delta M_2}{3} I_{d \times d} \right)^{-1} \right]}{n}} < \delta \) will be satisfied if \( \sqrt{\frac{\text{Tr} \left[ J_n(\hat{\theta}_n)^{-1} \right]}{n}} < \delta \), which might be an easier condition to check. Assumption 4 allows us to use appropriate Gaussian concentration inequalities in our proofs. Moreover, the assumption \( \|\hat{\theta}_n - \bar{\theta}_n\| < \bar{\delta} \) is necessary for Assumption 6 below to be satisfied.
**Assumption 5.** For the same $\delta > 0$ and $\overline{M}_2 > 0$ as in Assumption 3,
\begin{equation}
\overline{\lambda}_{\min}(\hat{\theta}_n) > \delta \overline{M}_2.
\end{equation}

**Remark 5.** If $J_n(\hat{\theta}_n)$ is positive definite and Assumption 1 is satisfied then one can adjust the choice of $\delta$ so that both eqs. (2.3) and (2.4) hold. Indeed, one can adjust the value of $\delta$ accordingly because decreasing the value of $\delta$ in Assumption 3 does not lead to an increase in the value of $\overline{M}_2$. At the same time, decreasing the value of $\delta$ in Assumption 5, while keeping $\overline{M}_2$ fixed, decreases the right-hand side of eq. (2.4).

Assumption 5, combined with Assumption 3 and with Taylor’s theorem, will allow us to prove that the posterior is strongly log-concave inside the $\delta$-neighborhood around the MAP. As a result, we will be able to show that the posterior satisfies the log-Sobolev inequality inside this neighborhood.

**Assumption 6.** For the same $\delta > 0$, as in Assumption 3, there exists $\kappa > 0$, such that
\begin{equation}
\sup_{\theta:||\theta - \hat{\theta}_n|| > \delta - ||\hat{\theta}_n - \bar{\theta}_n||} \frac{L_n(\theta) - L_n(\hat{\theta}_n)}{n} \leq -\kappa.
\end{equation}

**Remark 6.** Assumption 6 ensures that any local maxima of $L_n$ achieved outside of the $(\delta - ||\hat{\theta}_n - \bar{\theta}_n||)$-ball around the MLE do not get arbitrarily close to the global maximum achieved at the MLE. It also ensures that the posterior puts asymptotically negligible mass outside the $(\delta - ||\hat{\theta}_n - \bar{\theta}_n||)$-neighborhood around the MLE. As a result, only the locally Gaussian part around the MLE remains as the sample size $n$ grows. Note that for the vast majority of commonly used parametric models and data generating distributions, $||\hat{\theta}_n - \bar{\theta}_n||$, which appears in the expression for the radius of the ball, will tend to 0 as $n \to \infty$, a.s. Moreover, note that we do not strictly require that $\kappa$ not depend on the sample size $n$. Under certain conditions, our bounds will converge to zero as $n \to \infty$ even if $\kappa \xrightarrow{n \to \infty} 0$, as long as $\kappa$ vanishes strictly slower than $\frac{d \log n}{n}$ (see Sections 4.1 and 4.2 for more discussion).

This kind of assumption is standard in the discussion of the Bernstein–von Mises Theorem, both in classical references [17] and in more recent ones, [37]. Nevertheless, we note that there are references in which this assumption is replaced with certain weaker probabilistic separation conditions (such as uniformly consistent tests), see e.g. [48]. However, it is not clear how to adapt this type of conditions to our setup in which we seek to obtain computable non-asymptotic bounds. Similarly, the author of [37] remarked that it is already not clear how to do so when proving the asymptotic convergence of the posterior to Gaussianity when the studied convergence is almost sure, rather than in probability.

**Remark 7.** In Assumptions 2 and 4 – 6 we require the stated conditions to hold for the same $\delta$ as in Assumption 1 and for the same $\delta$ as in Assumption 3. In practice, we may, however, verify those assumptions separately and, for each of them, find the ranges for $\delta > 0$ and $\delta > 0$ for which it holds. We may then set the values of $\delta > 0$ and $\delta > 0$ equal to (one of) the values of $\delta > 0$ and $\delta > 0$ for which all the assumptions Assumptions 1 – 6 are satisfied.

**2.4. Additional assumptions in the MLE-centric approach.** Besides Assumption 1 and Assumption 2, in the MLE-centric approach, we have the following assumptions:

**Assumption 7.** For the same $\delta > 0$ as in Assumption 1,
\begin{equation}
\sqrt{\frac{\text{Tr} \left[ J_n(\hat{\theta}_n)^{-1} \right]}{n}} < \delta.
\end{equation}
Remark 8. Assumption 7 is an assumption on the size of $n$ and the choice of $\delta$. Indeed, for typical applications we expect $\sqrt{\frac{\text{Tr}[\hat{J}_n(\hat{\theta}_n)\hat{J}_n(\hat{\theta}_n)^{-1}]}{n}}$ to go to zero as $n \to \infty$. This is a technical assumption, necessary to ensure that the Gaussian concentration inequalities we use in our proofs are valid.

Assumption 8. For the same $\delta > 0$ and $M_2 > 0$ as in Assumption 1,

$$
\lambda_{\min}(\hat{\theta}_n) > \delta M_2.
$$

Remark 9. Assumption 8 is the analogue of Assumption 5 for the MLE-centric approach. Combined with Assumption 1 and with Taylor’s theorem, this assumption will allow us to prove that the likelihood is strongly log-concave inside the $\delta$-neighborhood around the MLE. As a result, we will be able to show that the posterior satisfies the log-Sobolev inequality inside this neighborhood.

Assumption 9. For the same $\delta > 0$, as in Assumption 1, there exists $\kappa > 0$, such that

$$
\sup_{\theta: \|\theta - \hat{\theta}_n\| > \delta} \frac{L_n(\theta) - L_n(\hat{\theta}_n)}{n} \leq -\kappa.
$$

Remark 10. Assumption 9 is similar to Assumption 6 and ensures that any local maxima of $L_n$ achieved outside of the $\delta$-ball around the MLE do not get arbitrarily close to the global maximum achieved at the MLE. In other words, this assumption ensures that the posterior puts asymptotically negligible mass outside the $\delta$-neighborhood around the MLE.

Assumption 10. For the same $\delta$ as in Assumption 1, there exist real numbers $M_1 > 0$ and $\tilde{M}_1 > 0$, such that

$$
\sup_{\theta: \|\theta - \hat{\theta}_n\| \leq \delta} \left| \frac{\pi'(\theta)}{\pi(\theta)} \right| \leq M_1 \quad \text{and} \quad \sup_{\theta: \|\theta - \hat{\theta}_n\| \leq \delta} |\pi(\theta)| \leq \tilde{M}_1.
$$

Remark 11. Note that for Assumption 10 to be satisfied, it suffices that $\pi$ is continuously differentiable and positive inside the $\delta$-ball around $\hat{\theta}_n$. Assumption 10 is a technical assumption that we use in the MLE-centric approach when showing that the posterior satisfies the log-Sobolev inequality inside the $\delta$-ball around the MLE.

Remark 12. As in Remark 7, we note that one may first verify Assumptions 1, 2 and 7 – 10 separately and then set the value of $\delta > 0$ equal to (one of) the values of $\delta > 0$ for which all of those assumptions are satisfied.

2.5. Additional notation. Certain quantities occur repeatedly in our bounds. By giving these quantities special symbols, we can express the bounds more compactly and readably.

First, we define a set of matrices closely related to $\hat{J}_n(\hat{\theta}_n)$ and $\tilde{J}_n(\hat{\theta}_n)$.

$$
\hat{J}_n^p(\hat{\theta}_n, \delta) := \hat{J}_n(\hat{\theta}_n) + (\delta M_2/3) I_{d \times d} \quad \hat{J}_n^m(\hat{\theta}_n, \delta) := \hat{J}_n(\hat{\theta}_n) - (\delta M_2/3) I_{d \times d}
$$

$$
\tilde{J}_n^p(\tilde{\theta}_n, \delta) := \tilde{J}_n(\tilde{\theta}_n) + (\delta \tilde{M}_2/3) I_{d \times d} \quad \tilde{J}_n^m(\tilde{\theta}_n, \delta) := \tilde{J}_n(\tilde{\theta}_n) - (\delta \tilde{M}_2/3) I_{d \times d}.
$$

The superscript $p$ in those symbols refers to the plus sign appearing in the definition of the symbol and the superscript $m$ refers to the minus sign. Each of these matrices is positive definite by Assumptions 5 and 8.
We analogously define the minimum eigenvalues of each matrix in the preceding display:

$$
\hat{\lambda}_{\min}(\theta_n, \delta) := \left[ \| \hat{J}_n(\theta_n, \delta)^{-1} \|_{op} \right]^{-1} ; \quad \hat{\lambda}_{\min}^m(\theta_n, \delta) := \left[ \| \hat{J}_n^m(\theta_n, \delta)^{-1} \|_{op} \right]^{-1} ; \\
\bar{\lambda}_{\min}(\theta_n, \delta) := \left[ \| J_n(\theta_n, \delta)^{-1} \|_{op} \right]^{-1} ; \quad \bar{\lambda}_{\min}^m(\theta_n, \delta) := \left[ \| J_n^m(\theta_n, \delta)^{-1} \|_{op} \right]^{-1} .
$$

Finally, we define a set of quantities of the following form:

$$
\mathcal{D}(n, \delta) := \exp \left[ -\frac{1}{2} \left( \delta \sqrt{n} - \sqrt{\text{Tr} \left[ \hat{J}_n(\theta_n)^{-1} \right]} \right)^2 \bar{\lambda}_{\min}(\theta_n) \right] ; \\
\mathcal{D}^p(n, \delta) := \exp \left[ -\frac{1}{2} \left( \delta \sqrt{n} - \sqrt{\text{Tr} \left[ \hat{J}_n^p(\theta_n, \delta)^{-1} \right]} \right)^2 \bar{\lambda}_{\min}^p(\theta_n, \delta) \right] ; \\
\hat{\mathcal{D}}(n, \delta) := \exp \left[ -\frac{1}{2} \left( \delta \sqrt{n} - \sqrt{\text{Tr} \left[ \hat{J}_n(\theta_n)^{-1} \right]} \right)^2 \lambda_{\min}(\theta_n) \right] ; \\
\hat{\mathcal{D}}^p(n, \delta) := \exp \left[ -\frac{1}{2} \left( \delta \sqrt{n} - \sqrt{\text{Tr} \left[ \hat{J}_n^p(\theta_n, \delta)^{-1} \right]} \right)^2 \lambda_{\min}^p(\theta_n, \delta) \right] .
$$

The preceding terms are labeled with a script “D” for “decay,” because they decrease to zero exponentially in $n$ when all other quantities are held constant.

2.6. Examples of popular models satisfying the assumptions. Although our results are non-asymptotic, it is useful to identify the sources of $n$ dependence, both for intuition and to connect our results with asymptotic Bayesian CLTs. We now present results stating that, under standard regularity conditions, and for large enough $n$, exponential families and generalized linear models satisfy the assumptions listed above with constants that do not depend on $n$.

2.6.1. Exponential families. We have the following result, whose proof can be found in Appendix C.1.

**Proposition 2.1** (cf. [37, Theorem 12]). Consider an exponential family that is full, regular, nonempty, identifiable and in natural form. In particular, let this exponential family have density $q(y|\eta) = \exp \left( \eta^T s(y) - \alpha(\eta) \right)$ with respect to a sigma-finite Borel measure $\lambda$ on $\mathcal{Y} \subseteq \mathbb{R}^k$; where $s: \mathcal{Y} \rightarrow \mathbb{R}^d$, $\eta \in \mathbb{R}^d$ and $\alpha(\eta) = \log \int_{\mathcal{Y}} \exp(\eta^T s(y)) \lambda(dy)$. Let $Q_\eta(E) = \int_{E} q(y|\eta) \lambda(dy)$ and denote $E_\eta s(Y) = \int_{\mathcal{Y}} s(y) Q_\eta(dy)$. Let $\mathcal{E} := \{ \eta \in \mathbb{R}^d : |\alpha(\eta)| < \infty \}$. Assume that $\mathcal{E}$ is open and nonempty, let the parameter space be given by $\Theta := \mathcal{E}$ and assume that $\eta \rightarrow Q_\eta$ is one-to-one.

Suppose $Y_1, Y_2, \cdots \in \mathcal{Y}$ are i.i.d. random vectors, such that $E s(Y_1) = E_{\theta_0} s(Y)$ for some $\theta_0 \in \Theta := \mathcal{E}$. Let $L_n(\theta) = \sum_{i=1}^n \log q(Y_i|\theta)$. Then, almost surely, for large enough $n$, Assumptions 1, 7, 8 and 9 are satisfied, with constants $\delta, M_2, \kappa$ independent of $n$.

If, in addition, the prior density $p$ does not depend on $n$ and is continuous and positive in a neighborhood around $\theta_0$ then, almost surely, for large enough $n$, Assumption 2 is satisfied with constant $M_1$ independent of $n$.

If, in addition, the prior density $p$ is continuously differentiable in a neighborhood of $\theta_0$ then, almost surely, Assumption 10 is satisfied for large enough $n$, with $M_1, M_1$ independent of $n$. 


If, in addition, the prior density $\pi$ is thrice continuously differentiable on $\Theta$, then, almost surely, Assumptions 3 – 6 are satisfied for all large enough $n$, with constants $\delta, M_2, \kappa$ independent of $n$.

REMARK 13. Our Proposition 2.1 is very similar to [37, Theorem 12]. The assumptions of Proposition 2.1 are, firstly, that the exponential family is full, regular, nonempty, identifiable and in natural form. Such conditions are standard and hold for typical commonly used exponential families [38]. Secondly, we assume that $E s(Y_i) = E_{\theta_0} s(Y)$ for some $\theta_0$, which is a standard assumption and ensures that matching the expected sufficient statistics to the observed sufficient statistics is possible asymptotically. Note that we do not assume that the model is correctly specified.

2.6.2. Generalized Linear Models. We have the following result, whose proof can be found in Appendix C.2.

PROPOSITION 2.2 (cf. [37, Theorem 13]). Consider a regression model of the form $p(y_i|\theta, x_i) \propto q(y_i|\theta^T x_i)$ for covariates $x_i \in \mathcal{X} \subseteq \mathbb{R}^d$ and coefficients $\theta \in \Theta \subseteq \mathbb{R}^d$, where $q(y|\eta) = \exp(\eta s(y) - \alpha(\eta))$ is a one-parameter exponential family, with respect to a sigma-finite Borel measure $\lambda$ on $\mathcal{Y} \subseteq \mathbb{R}^d$. Note that proportionality $\propto q$ is with respect to $\lambda$, not $y_i$.

Let $s: \mathcal{Y} \to \mathbb{R}^d$, $\eta \in \mathbb{R}^d$ and $\alpha(\eta) = \log \int_{\mathcal{Y}} \exp(\eta^T s(y)) \lambda(dy)$. Moreover, let $Q_\eta(E) = \int_E q(y|\eta) \lambda(dy)$ and $\mathcal{E} := \{\eta \in \mathbb{R}^d : |\alpha(\eta)| < \infty\}$. Assume $\Theta$ is open, $\Theta$ is convex, and $\theta^T x \in \mathcal{E}$ for all $\theta \in \Theta$, $x \in \mathcal{X}$. Moreover, assume $\mathcal{E}$ is non-empty and open and $\eta \mapsto Q_\eta$ is one-to-one. Suppose $(X_1, Y_1), (X_2, Y_2), \cdots \in \mathcal{X} \times \mathcal{Y}$ are i.i.d. such that:

1. $f'(\theta_0) = 0$ for some $\theta_0 \in \Theta$, where $f(\theta) = E \log q(Y_i|\theta^T X_i)$
2. $E |X_i s(Y_i)| < \infty$ and $E |\alpha(\theta^T X_i)| < \infty$ for all $\theta \in \Theta$,
3. for all $a \in \mathbb{R}^d$, if $a^T X_i \overset{a.s.}{=} 0$ then $a = 0$
4. There is $\epsilon > 0$ such that for all $j, k, l \in \{1, \ldots, d\}$,

$$E \left[ \sup_{\theta:||\theta - \theta_0|| \leq \epsilon} |\alpha'''(\theta^T X_i) X_{ij} X_{ik} X_{il}| \right] < \infty.$$

Let $L_n(\theta) = \sum_{i=1}^n \log p(Y_i|\theta, X_i)$. Then, almost surely, for all large enough $n$, Assumptions 1, 7, 8 and 9 are satisfied with constants $\delta, M_2, \kappa$ independent of $n$.

If, in addition, the prior density $\pi$ does not depend on $n$ and is continuous and positive in a neighborhood around $\theta_0$ then, almost surely, Assumption 2 is satisfied for large enough $n$ and constant $M_1$ independent of $n$.

If, in addition, the prior density $\pi$ is continuously differentiable in a neighborhood of $\theta_0$ then, almost surely, Assumption 10 is satisfied for large enough $n$ and $M_1, M_1$ independent of $n$.

If, in addition, the prior density $\pi$ is thrice continuously differentiable on $\Theta$, then, almost surely, Assumptions 3 – 6 are satisfied for all large enough $n$ and for constants $\delta, M_2, \kappa$ independent of $n$.

REMARK 14. Our Proposition 2.2 is very similar to [37, Theorem 13]. Condition 1 in Proposition 2.2 says essentially that the MLE exists asymptotically. Conditions 2 and 4 are moment conditions. They will be satisfied in many situations - for instance if the covariates are bounded and $E s(Y_i)$ exists (since $\alpha \in C^\infty$). Condition 3 is necessary to ensure identifiability. When $E X_i X_i^T$ exists and is finite, condition 3 is equivalent to $E X_i X_i^T$ being non-singular, which is often assumed to ensure identifiability for GLMs [48, Example 16.8]. Note that Proposition 2.2 does not assume that the model is correctly specified.
3. Main results. In this section we present our bounds on the quality of Laplace approximation in the MAP-centric approach, as described in Section 2. This approach is arguably the most popular one among users of Laplace approximation (e.g. [6, Section 4.4], [39, Sections 4.6.8.2 and 10.5.1]). The proofs of the results from this section are presented in Appendices D and E. A discussion of our bounds can be found in Section 4 below. In particular, Section 4 discusses the dependence on the sample size $n$ and dimension $d$ of our bounds, their dependence on the data, their computability and the way in which they control credible sets, means and variances. Our bounds in the MLE-centric approach will be presented in Section 6.

3.1. Control over the total variation distance. We start with a bound over the total variation distance.

**Theorem 3.1.** Suppose that Assumptions 1 – 6 hold and retain the notation thereof. Let $TV$ denote the total variation distance. Then:

$$TV\left(\mathcal{L}\left(\sqrt{n}\left(\tilde{\theta}_n - \hat{\theta}_n\right)\right), \mathcal{N}(0, \bar{J}_n(\hat{\theta}_n)^{-1})\right) \leq A_1 n^{-1/2} + 2\bar{\varphi}(n, \bar{\delta}) + A_2 n^{d/2} e^{-n\bar{\delta}},$$

where

$$A_1 = \frac{\sqrt{3} \text{Tr} \left[ \bar{J}_n(\hat{\theta}_n)^{-1} \right]}{4\sqrt{\left(\lambda_{\min}(\hat{\theta}_n) - \delta M_2\right)(1 - \varphi(n, \delta))}}; \quad A_2 = \frac{2 \left| \det \left( \bar{J}_n^p(\hat{\theta}_n, \delta) \right) \right|^{1/2} \hat{M}_1}{(2\pi)^{d/2}\left(1 - \varphi^p(n, \delta)\right)}.$$

3.2. Control over the 1-Wasserstein distance. Now, we bound the 1-Wasserstein distance, which is known to control the difference of means.

**Theorem 3.2.** Suppose that Assumptions 1 – 6 hold and retain the notation thereof. Then:

$$W_1\left(\mathcal{L}\left(\sqrt{n}\left(\tilde{\theta}_n - \hat{\theta}_n\right)\right), \mathcal{N}(0, \bar{J}_n(\hat{\theta}_n)^{-1})\right) \leq B_1 n^{-1/2} + B_3 \left( B_2 + \sqrt{n} B_4 \right) n^{d/2} e^{-n\bar{\delta}} + \left( \bar{\delta} \sqrt{n} + \sqrt{\frac{2\pi}{\lambda_{\min}(\hat{\theta}_n)}} + B_2 \right) \bar{\varphi}(n, \bar{\delta}),$$

where

$$B_1 := \frac{\sqrt{3} \text{Tr} \left[ \bar{J}_n(\hat{\theta}_n)^{-1} \right]}{2(\lambda_{\min}(\hat{\theta}_n) - \delta M_2) \sqrt{1 - \varphi(n, \delta)}},$$

$$B_2 := \frac{\left| \det \left( \bar{J}_n^p(\hat{\theta}_n, \delta) \right) \right|^{1/2} \left| \det \left( \bar{J}_n^m(\hat{\theta}_n, \delta) \right) \right|^{-1/2} \text{Tr} \left[ \bar{J}_n^m(\hat{\theta}_n, \delta)^{-1} \right]}{1 - \varphi^p(n, \delta)};$$

$$B_3 := \frac{\left| \det \left( \bar{J}_n^p(\hat{\theta}_n, \delta) \right) \right|^{1/2} \hat{M}_1}{(2\pi)^{d/2}\left(1 - \varphi^p(n, \delta)\right)}; \quad B_4 := \int_{\|u - \tilde{\theta}_n\| > \delta - \|\hat{\theta}_n - \tilde{\theta}_n\|} \|u - \tilde{\theta}_n\| \pi(u) du.$$

**Remark 15.** Note that, in order for the bound in Theorem 3.2 to be finite, we need the following integral with respect to the prior to be finite: $\int_{\|u - \tilde{\theta}_n\| > \delta - \|\hat{\theta}_n - \tilde{\theta}_n\|} \|u - \tilde{\theta}_n\| \pi(u) du.$
3.3. Control over the difference of covariances. Finally, we upper bound an integral probability metric that lets us control the difference of covariances.

**Theorem 3.3.** Suppose that Assumptions 1–6 hold and retain the notation thereof. Let \( Z_n \sim \mathcal{N}(0, J_n(\theta_n)^{-1}) \) Then:

\[
\sup_{v: \|v\| \leq 1} \left| \mathbb{E} \left[ \langle v, \sqrt{n}(\hat{\theta}_n - \theta_n) \rangle^2 \right] - \mathbb{E} \left[ \langle v, Z_n \rangle^2 \right] \right| \\
\leq C_1 n^{-1/2} + \sqrt{\frac{3 \text{Tr} \left[ J_n(\theta_n)^{-1} \right]}{c \left( \lambda_{\min}(\theta_n) - \delta^2 M_2 \right)}} n^{-1} \left( C_1 + n C_4 \right) n^{d/2} e^{-n\kappa} + \left( \delta^2 n + \frac{\sqrt{2\pi}}{\lambda_{\min}(\theta_n)} + C_2 \right) \hat{\mathcal{D}}(n, \delta),
\]

for

\[
C_1 := \sqrt{3 \left( \text{Tr} \left[ J_n(\theta_n)^{-1} \right] \right)^{3/2} \frac{M_2}{c \left( \lambda_{\min}(\theta_n) - \delta^2 M_2 \right) \left( 1 - \hat{\mathcal{D}}(n, \delta) \right)}},
\]

\[
C_2 := \frac{\left| \det \left( J_n^p(\hat{\theta}_n, \delta) \right) \right|^{1/2}}{\left| \det \left( J_n^m(\hat{\theta}_n, \delta) \right) \right|^{1/2}} \frac{\text{Tr} \left[ J_n^m(\hat{\theta}_n, \delta)^{-1} \right]}{1 - \hat{\mathcal{D}}(n, \delta)},
\]

\[
C_3 := \frac{\left| \det \left( J_n^p(\hat{\theta}_n, \delta) \right) \right|^{1/2}}{\left( 2\pi \right)^{d/2} \left( 1 - \hat{\mathcal{D}}(n, \delta) \right)^{1/2}},
\]

\[
C_4 := \int_{\|u - \hat{\theta}_n\| > \delta - \|\hat{\theta}_n - \theta_n\|} \left\| u - \hat{\theta}_n \right\| 2\pi(u) \, du.
\]

**Remark 16.** Note that, in order for the bound in Theorem 3.3 to be finite, we need the following integral with respect to the prior to be finite: \( \int_{\|u - \hat{\theta}_n\| > \delta - \|\hat{\theta}_n - \theta_n\|} \left\| u - \hat{\theta}_n \right\| 2\pi(u) \, du \).

4. Discussion of the bounds. We make some remarks about the bounds presented in Section 3 and their applicability in approximate inference.

4.1. Dependence on the sample size \( n \). The quantities \( A_1, A_2, A_3, B_1, B_2, B_3, B_4, C_1, C_2, C_3, \) and \( C_4 \) appearing in the above bounds depend on \( n \) but, for models and data generating distributions that satisfy the assumptions of the Bernstein–von Mises theorem (as in [17, Theorem 1.4.2] or [37, Theorem 5]), they are bounded as \( n \) grows. In particular, they are bounded in \( n \), as long as the constants \( M_2, \bar{M}_2, \bar{M}_1 \) are bounded from above and \( \delta, \delta \) are bounded from below by a positive number. In this scenario, keeping the dimension \( d \) fixed, all our bounds will vanish as \( n \to \infty \) at the rate of \( \frac{1}{\sqrt{n}} \), as long as \( \kappa \gg \frac{d+3}{2} \frac{\log n}{n} \). We reiterate that our bounds are fully non-asymptotic and computable.

4.2. Dependence on the dimension \( d \). Several terms in our bounds depend on \( d \) and the exact dependence differs depending on the analysed model. This is not surprising—the fact that the dimension dependence of the Laplace approximation is model-dependent has already been observed by other authors, including [28]. Nevertheless, as observed in [28], which was written after the first version of the current paper appeared on ArXiv, the dimension dependence in our total-variation and 1-Wasserstein bounds is such that it cannot be improved.
in general. It is also better than the dimension dependence appearing in the previous works on this topic [23, 46, 13, 16]. Below, we elaborate more on our dimension dependence and on the comparison to the dependence other authors have achieved in the past.

Let us take a closer look at our bound on the total variation distance presented in Theorem 3.1. Our bound consists of three summands. The first summand is of the same order as 
\[
\frac{\text{Tr} \left( J_n(\theta_n)^{-1} \right)}{\sqrt{n} \sqrt{\lambda_{\text{min}}(\theta_n) - M_2 \delta}}
\]
as \(d\) and \(n\) grow. The second summand is of a lower order, provided, for instance, that \(\bar{\delta} \gg \frac{\sqrt{\log n}}{\sqrt{n \lambda_{\text{min}}(\theta_n)}}\). This condition is reasonable to expect as for models satisfying the assumptions of the Bernstein–von Mises theorem [17, Theorem 1.4.2], \(\bar{\delta}\) is required to be positive and constant in \(n\). The third summand is of the same order as \(M_1 \left| \det \left( \hat{J}_n^p(\delta) \right) \right|^{1/2} n^{d/2} e^{-n \delta} \). Therefore, if \(\bar{\kappa} \gg \frac{\log n}{n} \cdot \frac{d+1}{2}\) + \(\frac{1}{n} \log \left( \det \left( \hat{J}_n^p(\delta) \right) \right)^{1/2} \) and \(\bar{\delta} \gg \frac{\sqrt{\log n}}{\sqrt{n \lambda_{\text{min}}(\theta_n)}}\), then the first summand is of the leading order as \(n\) and \(d\) grow.

Let us then look at the dimension dependence of the first summand in Theorem 3.1 through the lens of the recent analysis presented in [28] and thus make a comparison to the previous work. Our first summand in Theorem 3.1 can be re-written as 
\[
C_d = \frac{\text{Tr} \left( J_n(\theta_n)^{-1} \right)}{d} \cdot \frac{M_2}{\sqrt{\lambda_{\text{min}}(\theta_n) - M_2 \delta}} \leq \frac{M_2}{\lambda_{\text{min}}(\theta_n) \sqrt{\lambda_{\text{min}}(\theta_n) - M_2 \delta}},
\]
where we recall from Assumption 3 that \(M_2\) controls the third derivative of the log posterior. As noted in [28], all the recent works [23, 46, 13] have obtained finite-sample bounds on the total variation distance of order \(c_d \sqrt{d^3 / n}\), where \(c_d\) is a ratio of the third derivative of the log posterior and the \((3/2)\)-power of the second derivative of the log-posterior, with the definition varying slightly from paper to paper\(^1\) (while the bounds in [16] are of a higher order). As described in Section 1.3, all the works [23, 46, 13, 16] obtain their bounds under assumptions significantly stronger than ours. Still, our bound offers a tighter dimension dependence as it is of order \(C_d \sqrt{d^2 / n}\), where \(C_d\) again represents the ratio of a bound on the third derivative of the log posterior and the \((3/2)\)-power of the second derivative of the log-posterior.

Moreover, we note that [28], which appeared after the first version of the current article was put on ArXiv, refers specifically to our paper and states that the order of our bound cannot be improved in general. Indeed, [28, Theorem V1] states that the condition \(\bar{C}_d d \ll \sqrt{n}\) is necessary for accurate Laplace approximation, where \(\bar{C}_d\) is a model-specific term involving ratios of derivatives of the log-posterior. We refer the interested reader to [28] for a more precise statement of this result and a more thorough discussion.

In Example 1, we discuss a popular model which satisfies the assumptions of [46] and for which our bound on the total variation distance is of smaller order than the one of [46].

**Example 1.** Let us consider the following example, inspired by [28, Section 3] and discussed in more detail in Appendix H.1. Suppose that \(X_i \overset{i.i.d.}{\sim} \mathcal{N}(0, I_{d \times d})\), \(Y_i | X_i \sim \mathcal{N}(d_{\text{eff}}^p(X_i), \bar{\delta})\), 
\(^1\)To be precise, we note that the bound appearing in [46] is actually expressed in terms of the effective dimension \(d_{\text{eff}}\), which may sometimes be smaller than the actual dimension \(d\) and depends on the strength of regularization by a Gaussian prior. The effective dimension \(d_{\text{eff}}\) however approaches the true dimension \(d\) as the sample size \(n\) goes to infinity, as long as the prior does not depend on \(n\). For such a prior, the asymptotic order of the total-variation-distance bound in [46] is still \(c_d \sqrt{d^3 / n}\).
Bernoulli \( s(\theta_0^T X_i) \) and

\[
Y_i = \begin{cases} 
1, & \text{if } \tilde{Y}_i = 1 \\
-1, & \text{if } \tilde{Y}_i = 0,
\end{cases}
\]

where \( s \) is the sigmoid \( s(t) = (1 + e^{-t})^{-1} \) and \( \theta_0 \) is the ground truth value of the parameter. For simplicity, take \( \theta_0 = (1, 0, \ldots, 0) \). Now, let \( \rho(t) = -\log s(t) \) and take \( L_n(\theta) = -\sum_{i=1}^n \rho \left( Y_i X_i^T \theta \right) \), which corresponds to logistic regression. Consider a standard Gaussian prior on \( \theta \) given by \( \pi(\theta) \propto \exp(-\|\theta\|^2/2) \) and assume \( 2d < n < e^{\sqrt{d}} \). This model satisfies the assumptions of Theorem 3.1 with high probability for sufficiently large \( n \) and the first summand in the bound in Theorem 3.1 is of the leading order, as long as \( d \frac{\log n}{n} \to 0 \). See Appendix H.1.3 for a discussion of those facts.

In [46], bounds are proved on the quality of the Laplace approximation for models in which the likelihood is log-concave and the prior is Gaussian. The model we consider here in this example (logistic regression with a Gaussian prior) satisfies these conditions as well as the additional local smoothness conditions imposed in [46]. The author of [46] notes in their paper that their bounds on the total-variation distance from the Laplace approximation converge to zero only if \( \sqrt{\frac{d_{eff}}{n}} \to 0 \). For logistic regression with a Gaussian prior, as we consider it here in this example, we have that:

\[
d_{eff} := \text{Tr} \left\{ \left( \hat{J}_n(\hat{\theta}_n) + \frac{(\log \pi)^{\nu}(\hat{\theta}_n)}{n} \right) \hat{J}_n(\hat{\theta}_n)^{-1} \right\} \geq d \left( 1 - \frac{1}{n \lambda_{\min}(\hat{\theta}_n)} \right),
\]

and \( \lambda_{\min}(\hat{\theta}_n) \) is lower-bounded by a positive constant not depending on \( n \) or \( d \) with high probability (see Appendix H.1.2 for more detail). Therefore \( d_{eff} \frac{n}{d} \to 1 \) in probability. As we show in Appendix H.1.1, our bound on the total variation distance is therefore of a smaller order than the one in [46] — our bound is of order \( \sqrt{\frac{d}{n}} \) with high probability. Moreover, we note that [28, Section 3] has shown numerically that this rate is optimal for logistic regression.

4.3. Dependence on the data. In the typical applications, the bounds in Theorems 3.1 – 3.3 depend on the data. For a practitioner, such dependence on data is very natural because the bound is computable using the data and therefore usable in practice. However, for a theoretical statistician, interested in Bernstein–von Mises phenomena, it might be interesting to assume the data come from a certain distribution. For instance, researchers investigating frequentist properties of Bayesian estimators may make such an assumption in order to compare the coverage of Bayesian credible sets with that of frequentist confidence sets. It might be interesting to make such an assumption for instance in order to investigate how fast the coverage of Bayesian credible sets approaches the coverage of frequentist confidence sets as \( n \to \infty \). Having made such an assumption, one can use our bounds in order to quantify the speed of almost sure convergence or convergence in probability of the distances between the prior and the Gaussian. To this end, one only needs to control the speed of the relevant mode of convergence of our bounds, which, in most cases, should be achievable using standard results, similar to those quantifying the rate of convergence in the law of large numbers.

4.4. Computability. Our bounds are computable from the constants and quantities introduced in Section 2, such as \( \hat{M}_1, \hat{M}_2, \overline{M}_2, \delta, \tilde{\kappa}, \hat{\theta}_n, \overline{\theta}_n, \hat{J}_n(\hat{\theta}_n), \overline{J}_n(\theta_n) \). Computing some of those constants and quantities, including the bounds on the third derivatives of the log-likelihood and log-posterior given by \( M_2 \) and \( \overline{M}_2 \), may require additional work. It is often possible to obtain such bounds analytically. Sharper bounds can typically be obtained numerically. A robust approach to doing so is to obtain an analytical bound on the fourth
derivative and then run a grid-search optimizer and apply the mean value theorem. Running a grid-search optimizer can, however, be very expensive computationally, especially in high dimensions. Another option is to run a simpler global optimizer. This is much faster but can occasionally return results that are inaccurate or incorrect. There is, therefore, a statistical–computational trade-off that needs to be taken into consideration when calculating our bounds in practice, on real data sets. Moreover, there is typically a range of values for $\delta$ and $\bar{\delta}$ for which the assumptions of Theorems 3.1 – 3.3 are satisfied. The user may find the optimal choice of $\delta$ and $\bar{\delta}$ within the appropriate ranges by running a numerical optimizer on the bounds. We present our bounds computed for several example Bayesian models in Section 7.

4.5. Our bounds control the quality of credible-set approximations. Our bound on the total variation distance provides quality guarantees on the approximate computation of posterior credible sets. Indeed, suppose, for instance, that one is interested in finding a value $b_\alpha$, such that $\mathbb{P}\left(\|\tilde{\theta}_n - \bar{\theta}_n\| \leq \frac{b_\alpha}{\sqrt{n}}\right) \geq 1 - \alpha$, for a fixed value $\alpha$, where the probability $\mathbb{P}$ is conditional on the observed data. Let $A(n)$ denote the value of our upper bound in Theorem 3.1. If $n$ is sufficiently large and $A(n)$ is smaller than $\alpha$, then one could choose $b_\alpha = \bar{b}_\alpha$, such that, for $Z_n \sim \mathcal{N}(0, \tilde{J}_n(\tilde{\theta}_n)^{-1})$, $\mathbb{P}\left(\|Z_n\| \leq \tilde{b}_\alpha\right) = 1 - \alpha + A(n)$. Our bound implies that:

$$\mathbb{P}\left(\|\tilde{\theta}_n - \bar{\theta}_n\| \leq \frac{\tilde{b}_\alpha}{\sqrt{n}}\right) - \mathbb{P}\left(\|Z_n\| \leq \tilde{b}_\alpha\right) \leq A(n)$$

and so $\mathbb{P}\left(\|\tilde{\theta}_n - \bar{\theta}_n\| \leq \frac{\tilde{b}_\alpha}{\sqrt{n}}\right) \geq 1 - \alpha$.

4.6. Our bounds control the difference of means. Our bound on the 1-Wasserstein distance controls the difference of means in the Laplace approximation, in the following way. The upper bound in Theorem 3.2 controls $\sqrt{n}\|\mathbb{E}[\tilde{\theta}_n] - \bar{\theta}_n\|$. In order to obtain an upper bound on $\|\mathbb{E}[\tilde{\theta}_n] - \bar{\theta}_n\|$, one needs to divide our bound from Theorem 3.2 by $\sqrt{n}$.

4.7. Our bounds control the difference of covariances. Theorem 3.3 together with Theorem 3.2 let us control the difference of covariances. Suppose, for instance, that we are interested in the operator norm of the difference of the posterior covariance matrix and the covariance matrix of the Laplace approximation. Let $B(n)$ denote the value of our bound from Theorem 3.2 and let $C(n)$ be the value of our bound from Theorem 3.3. Then, a straightforward calculation reveals that:

$$\left\|\text{Cov}(\tilde{\theta}_n) - \frac{\tilde{J}_n(\tilde{\theta}_n)^{-1}}{n}\right\|_{op} \leq \frac{1}{n} \left(B(n)^2 + C(n)\right).$$

5. Our proof techniques. Now, we briefly describe our proof strategy. In our proofs in the MAP-centric approach we compare the distribution of $\sqrt{n}\left(\tilde{\theta}_n - \bar{\theta}_n\right)$ to $\mathcal{N}(0, \tilde{J}_n(\tilde{\theta}_n)^{-1})$ by concentrating separately on the region $\{\theta : \|	heta\| \leq \bar{\delta}\sqrt{n}\}$ and the region $\{\theta : \|	heta\| > \bar{\delta}\sqrt{n}\}$. In order to compare the rescaled posterior with the Gaussian in the outer region $\{\theta : \|	heta\| > \sqrt{n}\bar{\delta}\}$, we simply upper bound tail integrals with respect to the posterior and with respect to the Gaussian. We use Assumption 6, together with Gaussian concentration inequalities, which let us upper-bound integrals with respect to the posterior with integrals with respect to the prior, multiplied by $n^{d/2}e^{-n\kappa}$ and suitable constants. Integrals with respect to the Gaussian are upper-bounded using Gaussian concentration inequalities.

With the tail integrals controlled, we can focus on the two distributions truncated to the region $\{\theta : \|	heta\| \leq \sqrt{n}\bar{\delta}\}$. As we will describe in more detail shortly, we prove our main
results, Theorems 3.1 – 3.3 by (1) controlling the Fisher divergence using a Taylor series expansion, (2) controlling the Kullback-Leibler (KL) divergence using strong log-concavity in the inner region together with the log-Sobolev inequality, and (3) controlling the total variation and Wasserstein distances in terms of the KL divergence using Pinsker’s inequality and the transportation-entropy (Talagrand) inequality.

We use the following notation to represent the truncated versions of the two probability measures. First, for any measure \( \mu \) and measurable set \( A \), let \( [\mu]_A \) denote \( \mu \) truncated (restricted) to set \( A \). Now, let \( B_0(\delta \sqrt{n}) := \{ \theta : \| \theta \| \leq \delta \sqrt{n} \} \) denote the inner region, and define the truncated posterior and normal measures respectively as \( \mu_n^\theta := \left[ \mathcal{L} \left( \sqrt{n} \left( \bar{\theta}_n - \bar{\theta}_n \right) \right) \right]_{B_0(\delta \sqrt{n})} \) and \( \mu_n^N = \mathcal{N} \left( 0, J_n(\bar{\theta}_n)^{-1} \right) \right]_{B_0(\delta \sqrt{n})} \), with densities with respect to the Lebesgue measure given respectively by \( f_n^\theta \) and \( f_n^N \). By Taylor expanding \( (\log f_n^\theta)' \) around \( \bar{\theta}_n \) and applying Assumption 3 we can successfully control the following \textit{Fisher divergence} between the two truncated distributions:

\[
(5.1)
\end{equation}

\[
Fisher \left( \mu_n^N \| \mu_n^\theta \right) := \int \left| (\log f_n^\theta)'(t) - (\log f_n^N)'(t) \right|^2 \mu_n^N(dt) \leq \frac{3 \left( \text{Tr} \left[ J_n(\bar{\theta}_n)^{-1} \right] \mathcal{M}_2 \right)^2}{4n}.
\]

Now, by Assumptions 3 and 5, combined with Taylor’s theorem, we show that the density of \( \mu_n^\theta \) is \( (\bar{\lambda}_\text{min}(\bar{\theta}_n) - \delta \mathcal{M}_2) \)-strongly log-concave on \( B_0(\delta \sqrt{n}) \) (i.e. the logarithm of its density is a strongly concave function). The celebrated Bakry-Émery criterion [3] (see also [45, Theorem A1]) says that any strongly log-concave measure satisfies the \textit{log-Sobolev inequality}, which is an inequality between the KL divergence (or relative entropy) and the Fisher divergence (see [2, Chapter 5] for a comprehensive treatment of the topic or [49, Section 2] for a brief and intuitive explanation of the main ideas). Specifically, in our context, the log-Sobolev inequality for \( \mu_n^\theta \) on the set \( B_0(\delta \sqrt{n}) \) implies that:

\[
KL \left( \mu_n^N \| \mu_n^\theta \right) := \int \log \left( \frac{f_n^N(x)}{f_n^\theta(x)} \right) \mu_n^N(dx) \leq \frac{\text{Fisher} \left( \mu_n^N \| \mu_n^\theta \right)}{2 (\bar{\lambda}_\text{min}(\bar{\theta}_n) - \delta \mathcal{M}_2)}.
\]

Note that we could also use the log-Sobolev inequality for \( \mu_n^N \) and obtain a bound on \( KL(\mu_n^\theta \| \mu_n^N) \) expressed in terms of \( \text{Fisher}(\mu_n^\theta \| \mu_n^N) \). However, such a bound would not be as useful as it would involve integrating with respect to \( \mu_n^\theta \), which is often intractable. This is why we intentionally use the log-Sobolev inequality for \( \mu_n^\theta \), which yields the upper bound (5.2) expressed in terms of \( \text{Fisher}(\mu_n^N \| \mu_n^\theta) \), thus involving only Gaussian integration. Because of that, we derive a bound expressed in terms of the right-hand side of (5.1).

Finally, we can then control the total variation and Wasserstein distances in terms of the KL divergence. By Pinsker’s inequality [36, Theorem 2.16] we have that

\[
\text{TV} \left( \mu_n^\theta, \mu_n^N \right) \leq \sqrt{\frac{1}{2} KL \left( \mu_n^N \| \mu_n^\theta \right)},
\]

which gives us the desired control over the total variation distance. In order to control the 1-Wasserstein distance and the integral probability metric introduced in Theorem 3.3, we bound the 2-Wasserstein distance given by:

\[
W_2 \left( \mu_n^\theta, \mu_n^N \right) := \inf_{\Gamma} \sqrt{\mathbb{E}_{\Gamma} \left[ \| X - Y \|^2 \right]},
\]

where the infimum is taken over all distributions \( \Gamma \) of \( (X, Y) \) with the correct marginals \( X \sim \mu_n^\theta \) and \( Y \sim \mu_n^N \). Indeed, the 2-Wasserstein distance is known to upper bound the 1-Wasserstein distance and a transformation of it upper-bounds the integral probability metric appearing in Theorem 3.3. We upper-bound the 2-Wasserstein distance by the KL divergence,
which we have previously controlled in (5.1). We use the transportation-entropy (Talagrand) inequality, which is implied by the log-Sobolev inequality (see [21, Theorem 4.1] for the specific result we use or [49, Section 2.2.1] for an intuitive explanation). In our context, this inequality says that

\[ W_2(\mu_n^\pi, \mu_n^\pi) \leq \sqrt{\frac{2\text{KL}[\mu_n^N||\mu_n^\pi]}{\lambda_{\min}(\theta_n) - \delta M_2}}. \]

Combined with (5.2) it yields the desired control over the 1-Wasserstein distance and the metric considered in Theorem 3.3, inside the inner region \( B_0(\hat{\delta}\sqrt{n}) \).

Our strategy for proving Theorems 6.1 – 6.3, which look at the MLE-centric approach, is very similar. We compare the distribution of \( \sqrt{n}(\hat{\theta}_n - \hat{\theta}_n) \) to \( N(0, \hat{J}_n(\hat{\theta}_n)^{-1}) \) by concentrating separately on the region \( \{ \theta : ||\theta|| \leq \delta\sqrt{n} \} \) and the region \( \{ \theta : ||\theta|| > \delta\sqrt{n} \} \). In the outer region, we use Assumption 9 and Gaussian concentration inequalities. In the inner region, we use the log-Sobolev inequality.

The supplementary material contains another useful result, Theorem A.1. Its proof also divides the domain into an inner and outer region, but proceeds using Stein’s method instead of via the Fisher divergence, using known properties of solutions to the Stein equation in one dimension. As with our control over the Fisher divergence, the key to our proof is using Taylor’s expansions and expressing the bound as an integral over the known measure \( \mu_n^N \).

All the proofs of our results, Theorems 3.1 – 3.3, 6.1 – 6.3 and A.1, are provided in full detail in Appendices D – G.

6. Further results: MLE-centered approach. In this section we present some further results on the quality of Laplace approximation in the MLE-centric approach. The proofs of Theorems 6.1 – 6.3 presented below can be found in Appendices D and F.

6.1. Control over the TV distance in the MLE-centric approach. We start by controlling the total variation distance.

**Theorem 6.1.** Suppose that Assumptions 1, 2 and 7 – 10 hold. Let TV denote the total variation distance. We have the following upper bound:

\[ TV\left( L\left( \sqrt{n}(\hat{\theta}_n - \hat{\theta}_n) \right), N(0, \hat{J}_n(\hat{\theta}_n)^{-1}) \right) \leq D_1 n^{-1/2} + D_2 n^{d/2} e^{-\eta_0} + 2\hat{\varpi}(n, \delta), \]

where

\[ D_1 := \frac{\sqrt{M_1 M_1}}{2\sqrt{\lambda_{\min}(\theta_n) - \delta M_2}} \left( \frac{\sqrt{3} \text{Tr} \left[ \hat{J}_n(\hat{\theta}_n)^{-1} \right]}{2\sqrt{1 - \hat{\varpi}(n, \delta)}} M_2 + M_1 \right); \]

\[ D_2 := \frac{2M_1 \left| \det \left[ \hat{J}_n(\hat{\theta}_n, \delta) \right] \right|^{1/2}}{(2\pi)^{d/2} \left( 1 - \hat{\varpi}(n, \delta) \right)}. \]

6.2. Control over the 1-Wasserstein distance in the MLE-centric approach.

**Theorem 6.2.** Suppose that Assumptions 1, 2 and 7 – 10 hold. We have the following upper bound:
\[ W_1 \left( \mathcal{L} \left( \sqrt{n} \left( \hat{\theta}_n - \hat{\theta}_n \right) \right), \mathcal{N}(0, \hat{J}_n(\hat{\theta}_n)^{-1}) \right) \]

\[ \leq E_1 n^{-1/2} + E_3 \left( E_2 + \sqrt{n} E_4 \right) n^{d/2} e^{-n\kappa} + \left( \delta \sqrt{n} + \sqrt{\frac{2\pi}{\lambda_{\min}(\hat{\theta}_n)}} + E_2 \right) \hat{\mathcal{Q}}(n, \delta), \]

where

\[ E_1 := \frac{\widetilde{M}_1 \hat{M}_1}{\lambda_{\min}(\hat{\theta}_n) - \delta M_2} \left( \frac{\sqrt{3} \text{Tr} \left[ \hat{J}_n(\hat{\theta}_n)^{-1} \right]}{2\sqrt{1 - \hat{\mathcal{Q}}(n, \delta)}} \right) \]

\[ E_2 := \frac{\hat{M}_1 \hat{M}_1 \left| \det \left( \hat{J}_n(\hat{\theta}_n, \delta) \right) \right|^{1/2} \left| \det \left( \hat{J}_n(\hat{\theta}_n, \delta) \right) \right|^{-1/2} \sqrt{\text{Tr} \left[ \hat{J}_n(\hat{\theta}_n, \delta)^{-1} \right]} \left( \sqrt{1 - \hat{\mathcal{Q}}(n, \delta)} \right)}{1 - \hat{\mathcal{Q}}(n, \delta)} \]

\[ E_3 = \frac{\hat{M}_1 \left| \det \left( \hat{J}_n(\hat{\theta}_n, \delta) \right) \right|^{1/2}}{(2\pi)^{d/2} \left( 1 - \hat{\mathcal{Q}}(n, \delta) \right)}; \quad E_4 := \int_{\|u - \hat{\theta}_n\| > \delta} \|u - \hat{\theta}_n\| \pi(u) du. \]

**Remark 17.** Note that, in order for the bound in Theorem 6.2 to be finite, we need the following integral with respect to the prior: \( \int_{\|u - \hat{\theta}_n\| > \delta} \|u - \hat{\theta}_n\| \pi(u) du \)

6.3. Control over the difference of covariances in the MLE-centric approach.

**Theorem 6.3.** Suppose that Assumptions 1, 2 and 7–10 hold. Let \( Z_n \sim \mathcal{N}(0, \hat{J}_n(\hat{\theta}_n)^{-1}) \). We have that

\[ \sup_{\|v\| \leq 1} \left| \mathbb{E} \left[ \left( v, \sqrt{n} \left( \hat{\theta}_n - \bar{\theta}_n \right) \right)^2 \right] - \mathbb{E} \left[ (v, Z_n)^2 \right] \right| \]

\[ \leq (F_1)^2 n^{-1} + F_1 F_2 n^{-1/2} + \frac{F_3(F_3)^2 + F_2 \sqrt{n} \lambda_{\min}(\hat{\theta}_n)}{(2\pi)^{d/2}} n^{d/2} e^{-n\kappa} \]

\[ + \left( \delta^2 n + \sqrt{\frac{2\pi}{\lambda_{\min}(\hat{\theta}_n)}} + F_3 F_5 \right) \hat{\mathcal{Q}}(n, \delta) \]

where

\[ F_1 := \frac{\widetilde{M}_1 \hat{M}_1}{\lambda_{\min}(\hat{\theta}_n) - \delta M_2} \left( \frac{\sqrt{3} \text{Tr} \left[ \hat{J}_n(\hat{\theta}_n)^{-1} \right]}{\sqrt{1 - \hat{\mathcal{Q}}(n, \delta)}} \right); \quad F_2 := \frac{2\sqrt{\text{Tr} \left[ \hat{J}_n(\hat{\theta}_n)^{-1} \right]}}{\sqrt{1 - \hat{\mathcal{Q}}(n, \delta)}}; \]

\[ F_3 := \frac{\hat{M}_1 \left| \det \left( \hat{J}_n(\hat{\theta}_n, \delta) \right) \right|^{1/2}}{1 - \hat{\mathcal{Q}}(n, \delta)}; \quad F_4 := \int_{\|u\| > \delta} \|u\|^2 \pi(u + \hat{\theta}_n) du; \]

\[ F_5 := \hat{M}_1 \left| \det \left( \hat{J}_n(\hat{\theta}_n, \delta) \right) \right|^{-1/2} \text{Tr} \left[ \hat{J}_n(\hat{\theta}_n, \delta)^{-1} \right]. \]

**Remark 18.** Note that, in order for the bound in Theorem 6.3 to be finite, we need the following integral with respect to the prior to be finite: \( \int_{\|u\| > \delta} \|u\|^2 \pi(u + \hat{\theta}_n) du \).
7. Example applications. Now, we present examples of our bounds computed for different models. None of those examples correspond to strongly log-concave posteriors. Indeed, the first one yields a weakly log-concave posterior and the second and third one produce non-log-concave posteriors. They all show that our bounds are explicitly computable for a variety of commonly used models, including heavy-tailed ones. They also show that the computations may be executed in a practical amount of time, as we point out in Section 7.3.2 below. Our bounds also go well below the true values of the mean and the norm of the covariance, for reasonable sample sizes. This indicates that they are applicable for practitioners who wish to assess how confident they should be in their mean and variance estimates.

7.1. Our bounds work under misspecification: Poisson likelihood with gamma prior and exponential data. First, we look at a one-dimensional conjugate model, for which we can compare our bounds on the difference of means and the difference of variances to the ground truth. We consider a Poisson likelihood and a gamma prior with shape equal to 0.1 and rate equal to 3. Our data are generated from the exponential distribution with mean 10. Figures 1a and 1b show that our bounds (in the MAP-centric approach) on the difference of means and the difference of variances get close to the true difference of means and the true difference of variances for sample sizes above around 250. More detail on how one can compute the constants appearing in the bounds can be found in Appendix H.2.

7.2. Our bounds work for non log-concave posteriors: Weibull likelihood with inverse-gamma prior. Now, we consider another conjugate model and compare our bounds to the ground truth. In this case, the posterior is not log-concave. In our experiment, we set the shape of the Weibull to $\frac{1}{2}$ and we make inference about the scale. The prior is inverse-gamma with shape equal to 3 and scale equal to 10. The data are Weibull with shape 1/2 and scale 1. Figures 2a and 2b demonstrate that our bounds on the difference of means and the difference of variances (in the MAP centric approach) get close to the true difference of means and the true difference of variances for sample sizes in low thousands. More detail on how one can compute the constants appearing in the bounds can be found in Appendix H.3.

The code for our experiments is available at https://github.com/mikkasprzak/laplace_approximation.git
7.3. Our bounds work for multivariate heavy-tailed posteriors: logistic regression with Student’s $t$ prior.

7.3.1. Setup. Suppose $X_1, \ldots, X_n \in \mathbb{R}^d$ and $Y_1, \ldots, Y_n \in \{-1, 1\}$. We will study the following log-likelihood:

$$L_n(\theta) := L_n(\theta | (Y_i)_{i=1}^n, (X_i)_{i=1}^n) = -\sum_{i=1}^n \log \left(1 + e^{-X_i^T \theta Y_i}\right).$$  \hfill (7.1)

For a covariance matrix $\Sigma$, a vector $\mu \in \mathbb{R}^d$ and hyperparameter $\nu > 0$, we consider a $d$-dimensional Student’s $t$ prior on $\theta$, given by

$$\pi(\theta) = \frac{\Gamma((\nu + d)/2)}{\Gamma(\nu/2) \nu^{d/2} \pi^{d/2} |\Sigma|^{1/2}} \left[1 + \frac{1}{\nu}(\theta - \mu)^T \Sigma^{-1}(\theta - \mu)\right]^{-(\nu+d)/2}. \hfill (7.2)$$

Combining the log-likelihood given by eq. (7.1) with the prior given by eq. (7.2) yields a posterior that is known not to be log-concave. Moreover, for certain data sets the posterior is heavy tailed. An easy one-dimensional example of such a data-set is one for which $X_i Y_i > 0$ for all $i$. Appendix H.4 contains calculations of constants involved in the bounds of Section 3.

7.3.2. Experiments. Figures 3a and 3b present our bounds obtained using the quantities derived in Appendix H.4. We concentrate on the MAP-centric approach. We derived the values for MAP and MLE numerically. Moreover, in order to sharpen our bounds, we ran a built-in global optimizer scipy.optimize.shgo to derive the maximum of the third derivative inside a ball around the MLE and the MAP, i.e. to derive $M_2$ and $\overline{M}_2$. As there was a certain degree of choice for $\delta$ and $\overline{\delta}$, we also optimized the choice thereof numerically. We performed our experiments for the 5-dimensional logistic regression with Student’s $t$ prior with mean zero, identity covariance matrix and 4 degrees of freedom. The data $(Y_i)_{i=1}^n$ we used came from logistic regression with parameter $(1, 1, 1, 1, 1)$, where the $(X_i)_{i=1}^n$ were simulated i.i.d. from the 5-dimensional standard normal distribution. Figures 3a and 3b demonstrate that our bounds on the 1-Wasserstein distance and the 2-norm of the difference of variances go well below the approximate values of the mean and 2-norm of the covariance, respectively, for sample sizes of about 1300 and 3000, respectively. Therefore, they also go below the true values of the mean and the 2-norm of the covariance for such sample sizes. We have also

Fig 2: Weibull likelihood with inverse-gamma prior (MAP-centric approach)
calculated the minimum number of data points required for our bounds to be available, for dimensions $d$ between 1 and 11. This is illustrated in Figure 4.

Because of the numerical optimization applied in order to calculate $\mathcal{M}_2$, the calculation of our bounds gets slower as the dimension increases. This is mainly caused by the global optimizer scipy.optimize.shgo becoming slow in higher dimensions. Figure 5 illustrates our bounds on the quality of the Laplace approximation of the posterior arising from $d$-dimensional logistic regression with Student’s t prior, for $d$ between 1 and 9. As before, the prior has mean zero, identity covariance and 4 degrees of freedom and the data $(Y_i)_{i=1}^n$ come from logistic regression with parameter $(1, \ldots, 1)$, where $(X_i)_{i=1}^n$ were simulated i.i.d. from the $d$-dimensional standard normal. We computed our bounds on the 1-Wasserstein distance and the 2-norm of the difference of covariances simultaneously, on 50 points spread uniformly on the logarithmic scale between the minimum sample size given by Figure 4 and the sample size of $10^5$. Performing this experiment on a MacBook Pro 13 Retina Z0Y6 for $d = 3$ took us about 9 minutes, for $d = 5$ about 35 minutes, for $d = 7$ about 150 minutes and for $d = 9$ about 505 minutes.
7.3.3. Strategies for obtaining the bounds. We derive the values of $M_2, \bar{M}_2, \kappa, \bar{\kappa}$ in Appendix H.4. They all, however depend on the MAP $\hat{\theta}_n$ or the MLE $\hat{\theta}_n$. Therefore, we need to have access to a good global optimizer in order to obtain values of the MLE and MAP numerically. The choice for $\delta$ and $\bar{\delta}$, may also be optimized numerically, using the same packages. In order to improve on the bounds, one can also run a numerical optimizer in order to derive tighter values of $M_2$ and $\bar{M}_2$ than those we derive analytically in Appendix H.4. A robust (yet slow) approach to deriving tighter bounds is via grid search, combined with the mean value theorem and a bound on the fourth derivative of the log-likelihood (or log-posterior). Another option is to run a faster built-in global optimizer to derive the maximum of the third derivative inside a ball around the MLE (or MAP). What is gained in terms of speed is lost in terms of robustness. Indeed, a global numerical optimizer might output a value smaller than the maximum, rather than producing the desired upper bound. Also, still, this approach is fast only in relatively low dimensions. In higher dimensions, any global optimizer inevitably gets slow. Nevertheless, this might be a useful approach for users of our bounds.

8. Conclusions and future work. We provide bounds on the quality of the Laplace approximation that are computable and hold under the standard assumptions of the Bernstein–von Mises Theorem. We control the total-variation distance, the 1-Wasserstein distance and another useful integral probability metric. Our bounds on the total-variation and 1-Wasserstein distance are such that their sample-size and dimension dependence cannot be improved in general. An interesting question is whether, for multivariate posteriors, we could derive bounds on more general integral probability metrics, in a way similar to our univariate Theorem A.1. We proved Theorem A.1 using Stein’s method and we present it as part of the supplementary material. Whether or not this result could be extended to the multivariate context is to a certain extent a question about the applicability of techniques like Stein’s method in dimension greater than one for measures truncated to a bounded convex set. So far we have struggled to find enough theory that would let us compute useful bounds using this approach.

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SUPPLEMENTARY MATERIAL

The supplementary material contains the appendices. Appendix A contains an additional bound in the MLE-centric approach, on the distance between expectations of general test functions, in dimension one. Appendix B contains a discussion of the Bernstein–von Mises Theorem, the Laplace approximation and connections between them. Appendix C proves the propositions stated in Section 4, as part of the discussion of our assumptions. Appendix D contains introductory arguments, relevant for the proofs of all the results of this paper. Appendix E contains proofs of the results of Section 3. Appendix F proves the three theorems of Section 6 and Appendix G proves the theorem of Appendix A. Appendix H contains additional detail concerning Example 1 and the examples discussed in Section 7.

REFERENCES

[1] BACH, F. (2021). Approximating integrals with Laplace’s method. https://francisbach.com/laplace-method/. Accessed: 2022-08-29.
[2] BAKRY, D., GENTIL, I. and LEDOUX, M. (2016). Analysis and Geometry of Markov Diffusion Operators. Springer Cham.
[3] BAKRY, D. and ÉMERY, M. (1985). Diffusions hypercontractives. Séminaire de probabilités de Strasbourg 19 177–206.
[4] BALASUBRAMANIAN, K., CHEWI, S., ERDOGDU, M. A., SALIM, A. and ZHANG, S. (2022). Towards a Theory of Non-Log-Concave Sampling: First-Order Stationarity Guarantees for Langevin Monte Carlo. In Proceedings of Thirty Fifth Conference on Learning Theory (P.-L. LOH and M. RAGinsky, eds.). Proceedings of Machine Learning Research 178 2896–2923. PMLR.
[5] BARBER, R. F., DARTON, M. and TAN, K. M. (2016). Laplace Approximation in High-Dimensional Bayesian Regression. In Statistical Analysis for High-Dimensional Data (A. FRIGESSI, P. BüHLMANN, I. K. GLAD, M. LANGAAS, S. RICHARDSON and M. VANNUCI, eds.) 15–36. Springer International Publishing, Cham.
[6] BISHOP, C. M. (2006). Pattern Recognition and Machine Learning. Information Science and Statistics. Springer New York, NY.
[7] BISSIRI, P. G., HOLMES, C. C. and WALKER, S. G. (2016). A general framework for updating belief distributions. Journal of the Royal Statistical Society. Series B (Statistical Methodology) 78 1103–1130.
[8] BLEI, D. M., KUCUKELBIR, A. and MACAULIFFE, J. D. (2017). Variational Inference: A Review for Statisticians. Journal of the American Statistical Association 112 859–877.
[9] CHERNOZHUKOV, V. and HONG, H. (2003). An MCMC approach to classical estimation. Journal of Econometrics 115 293–346.
[10] CHEWI, S., ERDOGDU, M. A., LI, M., SHEN, R. and ZHANG, S. (2022). Analysis of Langevin Monte Carlo from Poincare to Log-Sobolev. In Proceedings of Thirty Fifth Conference on Learning Theory (P.-L. LOH and M. RAGinsky, eds.). Proceedings of Machine Learning Research 178 1–2. PMLR.
[11] CHWIALKOWSKI, K., STRATHMANN, H. and GRETTON, A. (2016). A Kernel Test of Goodness of Fit. In Proceedings of The 33rd International Conference on Machine Learning (M. F. BALCAN and K. Q. WEINBERGER, eds.). Proceedings of Machine Learning Research 48 2606–2615. PMLR, New York, New York, USA.
[12] DAXBERGER, E., KRISTIADI, A., IMMERS, A., ESCHENHAGEN, R., BAUER, M. and HENNIG, P. (2021). Laplace Redux - Effortless Bayesian Deep Learning. In Advances in Neural Information Processing Systems (M. RANZATO, A. BEYGELZIMER, Y. DAUPHIN, P. S. LIANG and J. W. VAUGHAN, eds.) 34 20089–20103. Curran Associates, Inc.
[13] DEHAENE, G. (2019). A deterministic and computable Bernstein-von Mises theorem. arXiv:1904.02505v2.
[14] ERDOGDU, M. A., HOSEINZADEH, R. and ZHANG, S. (2022). Convergence of Langevin Monte Carlo in Chi-Squared and Rényi Divergence. In Proceedings of The 25th International Conference on Artificial Intelligence and Statistics (G. CAMPS-VALLS, F. J. R. RUIZ and I. VALERA, eds.). Proceedings of Machine Learning Research 151 8151–8175. PMLR.
APPENDIX A: AN ADDITIONAL GENERAL UNIVARIATE BOUND

Now we present the following bound, which works for pretty arbitrary test functions, but only in dimension one. The proof of this bound is different from the proofs of the results presented in the main body of the paper as it relies on Stein’s method instead of the log-Sobolev inequality. Stein’s method allows us to achieve control over the difference of expectations with respect to the rescaled posterior and with respect to the approximating Gaussian of very general test functions. The particular properties of Stein’s method we use, however, only yield bounds in dimension one. We believe it would be interesting and useful to derive such general bounds in higher dimensions in the future. The proof of Theorem A.1 presented below can be found in Appendices D and G.

**Theorem A.1.** Assume that we study a univariate posterior, i.e. that \( d = 1 \). Let \( \sigma_n^2 := \hat{J}_n(\hat{\theta}_n)^{-1} \). Suppose that Assumptions 1, 2 and 8 – 10 hold. Let \( Z_n \sim \mathcal{N}(0, \sigma_n^2) \). Then, for any function \( g : \mathbb{R} \to \mathbb{R} \) which is integrable with respect to the posterior and with respect to \( \mathcal{N}(0, \sigma_n^2) \),

\[
\left| \mathbb{E} \left[ g \left( \sqrt{n}(\hat{\theta}_n - \tilde{\theta}_n) \right) \right] - \mathbb{E} \left[ g(Z_n) \right] \right| \leq (G_1 + G_2)n^{-1/2} + \int_{|u| > \delta \sqrt{n}} \frac{g(u)e^{-u^2/(2\sigma_n^2)}}{\sqrt{2\pi \sigma_n^2}} du + G_3 \int_{|u| > \delta} |\pi(u + \hat{\theta}_n)du + G_4| n^{1/2}e^{-\delta u} + G_5 e^{-\delta^2 n/(2\sigma_n^2)},
\]

for

\[
C_n^{(1)} := \frac{1}{2\sigma_n^2} - \frac{\delta M_2}{3} > 0; \quad C_n^{(2)} := \left( \frac{1}{2\sigma_n^2} - \frac{\delta M_2}{6} \right) > 0;
\]

\[
C_n^{(3)} := \left( \frac{1}{2\sigma_n^2} + \frac{\delta M_2}{3} \right) > 0; \quad C_n^{(4)} := \left( \frac{1}{2\sigma_n^2} + \frac{\delta M_2}{6} \right) > 0.
\]
and

\begin{align*}
G_1 &= \frac{2\hat{M}_1 \hat{M}_1}{\sqrt{2\pi \sigma_n^2}} \int_{-\delta \sqrt{n}}^{\delta \sqrt{n}} |u g(u)| \left[ \left( M_1 + \frac{3}{\delta} \right) e^{-C_n^{(1)} u^2} - \frac{3}{\delta} e^{-C_n^{(2)} u^2} \right] du; \\
G_2 &= \frac{2 \sqrt{C_n^{(4)}} (\hat{M}_1 \hat{M}_1)^2 (M_1 + \frac{3}{\delta}) \int_{-\delta \sqrt{n}}^{\delta \sqrt{n}} |g(u)| e^{-C_n^{(2)} u^2} du}{C_n^{(1)} \pi \sqrt{\sigma_n^2} \left( 1 - 2 e^{-\delta^2 n C_n^{(4)}} \right)} \left( M_1 + \frac{3}{\delta} - \frac{3}{C_n^{(1)} - \delta C_n^{(2)}} \right); \\
G_3 &= \frac{\hat{M}_1 \sqrt{C_n^{(4)}}}{\sqrt{\pi} \left\{ 1 - 2 \exp \left[ -C_n^{(4)} \delta^2 n \right] \right\}}; \\
G_4 &= \frac{\hat{M}_1^2 \hat{M}_1 C_n^{(4)} \int_{|t| \leq \delta \sqrt{n}} |g(t)| e^{-C_n^{(2)} t^2} dt}{\pi \left\{ 1 - 2 \exp \left[ -C_n^{(4)} \delta^2 n \right] \right\}^2}; \\
G_5 &= \frac{2 \hat{M}_1 \hat{M}_1 \sqrt{C_n^{(4)}} \int_{|t| \leq \delta \sqrt{n}} |g(t)| e^{-C_n^{(2)} t^2} dt}{\sqrt{\pi} \left\{ 1 - 2 \exp \left[ -C_n^{(4)} \delta^2 n \right] \right\}}.
\end{align*}

**Remark 19.** The bound in Theorem A.1 is for the MLE-centric approach. However, its proof can easily be modified in order to yield analogous bounds on the quality of approximation in the MAP-centric approach, as described in Section 2 and presented in Section 3.

Constants $G_1, G_2, G_4, G_5$ appearing above may be straightforwardly controlled by controlling the involved Gaussian integrals. Moreover, the fact that $C_n^{(1)}$, $C_n^{(2)}$, $C_n^{(3)}$, $C_n^{(4)}$ are positive follows from Assumption 8.

**APPENDIX B: INTRODUCTION TO THE LAPLACE APPROXIMATION AND THE BERNSTEIN–VON MISES THEOREM**

The foundations of Laplace approximation date back to the work on Laplace [33] (see [34] for an English translation and [1] for an intuitive discussion). It was originally introduced as a method of approximating integrals of the form

$$\text{Int}(n) := \int_{\mathbb{R}^d} e^{-nf(x)} dx, \quad n \in \mathbb{N},$$

where $K$ is a subset of $\mathbb{R}^d$ and $f$ is a real-valued function on $\mathbb{R}^d$. Suppose that $x^* \in K$ is a strict global maximizer of $f$ on $K$. Heuristically, under appropriate smoothness assumptions on $f$, we can use Taylor’s expansion to obtain:

$$f(x) \approx f(x^*) + f'(x^*)(x - x^*) + \frac{1}{2} (x - x^*)^T f''(x^*)(x - x^*)$$

$$= f(x^*) + \frac{1}{2} (x - x^*)^T f''(x^*)(x - x^*).$$

We therefore have:

$$\text{Int}(n) \approx \int_{K} \exp \left[ -nf(x^*) - \frac{n}{2} (x - x^*)^T f''(x^*)(x - x^*) \right] dx$$

As a result, heuristically, up to a constant not depending on $K$, $\text{Int}(n)$ can be approximated by the integral of the density of the Gaussian measure with mean $x^*$ and covariance matrix.
given by $\frac{1}{n} f''(x^*)^{-1}$. Now, suppose that $e^{-n f(x)}$ is an unnormalized posterior density, i.e. that $\Pi_n(\cdot) \propto e^{-n f(\cdot)}$ for some posterior $\Pi_n$. Writing $\Pi_n$ for the posterior probability measure and $\hat{\theta}_n$ for the posterior mode (i.e. the maximum a posteriori or MAP), the above statement suggests that

(B.1) \[ \Pi_n \approx N(\bar{\theta}_n, (\log(\Pi_n)^\prime(\bar{\theta}_n))^{-1}), \]

where $N(\mu, \Sigma)$ denotes the normal law with mean $\mu$ and covariance $\Sigma$. The computation of the mean and covariance of the above Gaussian is in the majority of cases easy numerically. It can be achieved using standard optimization schemes and does not require access to integrals with respect to the posterior, the normalizing constant of the posterior density or the true parameter. This is why Laplace approximation is a popular tool in approximate Bayesian inference.

While the above heuristic considerations may be turned into rigorous statements under certain conditions, a proper probabilistic grounding for the Laplace approximation is provided by the Bernstein–von Mises (BvM) theorem. As described in numerous classical references, including [17, Section 1.4] or [48, Section 10.2], the BvM theorem says that under mild assumptions on the likelihood and the prior, the posterior distribution converges to a Gaussian law in the following sense. Suppose that $\theta_n$ is distributed according to the posterior, obtained after observing $n$ data points. Let $\theta_0$ be the true parameter, $\hat{\theta}_n$ be the MLE and $I(\cdot)$ be the Fisher information matrix. Let $TV$ denote the total variation distance and let the function $L(\cdot)$ return the law of its argument. Then, if the model is well-specified and certain regularity conditions are satisfied,

(B.2) \[ TV( L(\sqrt{n}(\hat{\theta}_n - \hat{\theta}_n)), N(0, I(\theta_0)^{-1})) \xrightarrow{P} 0, \text{ as } n \to \infty, \]

where $n$ is the number of data and the convergence occurs in probability with respect to the law of the data.

While the model being well-specified is a crucial assumption in the above statement, the authors of [31] proved its modified version, under model misspecification. The main difference is in the limiting covariance matrix. Specifically, in this context, the authors assume the model is of the form $\theta \to p_0$ and the observations are sampled from a density $p_0$ that is not necessarily of the form $p_0 \theta_0$ for some $\theta_0$. They show that under certain regularity conditions,

(B.3) \[ TV( L(\sqrt{n}(\hat{\theta}_n - \hat{\theta}_n)), N(0, V(\theta^*)^{-1})) \xrightarrow{P} 0, \text{ as } n \to \infty, \]

where $\theta^*$ minimizes the Kullback-Leibler divergence $\theta \to \int \log(p_0(x)/p_0(x))p_0(x)dx$ and $V(\theta^*)$ is minus the second derivative of this map, evaluated at $\theta^*$.

Let us now denote by $L_n$ the generalized log-likelihood. A closer look at the classical proofs, including Le Cam’s one (see e.g. [17, Section 1.4]), or more recent ones, including that of [37, Appendix B], reveals that, under standard regularity conditions:

(B.4) \[ TV( L(\sqrt{n}(\hat{\theta}_n - \hat{\theta}_n)), N(0, \left[ -\frac{L_n''(\hat{\theta}_n)}{n} \right]^{-1}) \xrightarrow{P} 0, \text{ as } n \to \infty, \]

no matter if the model is well-specified or not. It is known that under mild assumptions the MLE $\hat{\theta}_n$ and the maximum a posteriori (MAP) $\tilde{\theta}_n$ get arbitrarily close to each other as the number of data $n$ goes to infinity. Similarly, denoting by $T_n$ the logarithm of the posterior density, $\tilde{L}_n$ and $L_n$ get arbitrarily close as $n$ goes to infinity. It can be shown, in a similar fashion to eq. (B.4), that under standard regularity assumptions,

(B.5) \[ TV( L(\sqrt{n}(\tilde{\theta}_n - \hat{\theta}_n)), N(0, \left[ -\frac{T_n''(\tilde{\theta}_n)}{n} \right]^{-1}) \xrightarrow{P} 0, \text{ as } n \to \infty. \]
Equation (B.5) gives a rigorous justification and meaning to the Laplace approximation given by eq. (B.1) and eq. (B.4) provides its alternative version. The approximating covariance in both eqs. (B.4) and (B.5) is computable without access to the posterior normalizing constant or the true parameter.

The recent paper [37] proves almost sure versions of the statements given in eqs. (B.2) and (B.3) for a large collection of commonly used models. Naturally, similar almost sure convergence statements can be obtained for the approximations appearing in eqs. (B.4) and (B.5).

Appendix C: Proofs of Propositions 2.1 and 2.2

C.1. Proof of Proposition 2.1. The proof is inspired by the proof of [37, Theorem 12]. Note that $L_n(\theta) = -n\alpha(\theta) + \theta^T S_n$, where $S_n = \sum_{i=1}^n s(Y_i)$. By standard exponential family theory [38, Proposition 19], $\alpha$ is $C^\infty$, strictly convex on $\Theta$, $\alpha'(\theta) = \mathbb{E}_{\theta}s(Y)$ and $\alpha''(\theta)$ is symmetric positive definite for all $\theta \in \Theta$. Let $s_0 := \mathbb{E}_{\theta_0}s(Y)$. By the strong law of large numbers, for all $\theta \in \Theta$,

$$
\frac{L_n(\theta)}{n} \xrightarrow{n \to \infty} -\alpha(\theta) + \theta^T s_0 =: f(\theta), \quad \text{almost surely}.
$$

Note that due to the almost sure convergence of the sufficient statistics, we actually have a stronger statement. Indeed, it holds that, almost surely, for all $\theta \in \Theta$, $\frac{L_n(\theta)}{n} \xrightarrow{n \to \infty} f(\theta)$.

Now, note that $f'(\theta_0) = 0$. Note also that the MLE $\hat{\theta}_n$ satisfying $L_n'(\hat{\theta}_n) = 0$ is unique (if it exists) because $L_n$ is strictly concave. By [37, Theorems 12 and 5], we have that the MLE $\hat{\theta}_n$ almost surely exists and

$$
\hat{\theta}_n \to \theta_0 \quad \text{almost surely} \quad \text{and} \quad \frac{L_n(\hat{\theta}_n)}{n} \xrightarrow{n \to \infty} f(\theta_0) \quad \text{almost surely.}
$$

Let $\delta > 0$ and $\bar{E} := \{\theta : \|\theta - \theta_0\| \leq 2\delta\}$ be such that $\bar{E} \subseteq \Theta$. Then $\alpha'''$ is bounded on $\bar{E}$, since $\alpha'''$ is continuous and $\bar{E}$ is compact. Hence, $L_n'''(\theta) \leq \frac{L_n'''(\theta_0)}{n}$ uniformly bounded on $\bar{E}$ because $L_n'''(\theta) = -n\alpha'''(\theta)$. Therefore, almost surely, Assumption 1 is satisfied for large enough $n$ and small enough $\delta$ with $M_2$ not depending on $n$. This is because $\hat{\theta}_n \to \theta_0$ almost surely and so, almost surely, for large enough $n$, $\{\theta : \|\theta - \hat{\theta}_n\| \leq \delta\} \subseteq \bar{E}$. Now, let $\xi_{\theta}$ be the point on the line connecting $\theta$ and $\theta_0$ that lies on $\{t : \|t - \theta_0\| \leq \delta/2\}$. The strict concavity of $L_n$ implies that, almost surely,

$$
\limsup_{n \to \infty} \sup_{\theta, \|\theta - \theta_0\| > \delta} \frac{L_n(\theta) - L_n(\hat{\theta}_n)}{n} \leq \limsup_{n \to \infty} \sup_{\theta, \|\theta - \theta_0\| > \delta/2} \frac{L_n(\theta) - L_n(\theta_0)}{n} \leq \limsup_{n \to \infty} \sup_{\theta, \|\theta - \theta_0\| > \delta/2} \frac{\|\theta_0 - \theta\| \cdot L_n(\xi_{\theta}) - L_n(\theta_0)}{\delta^2/2} \leq \limsup_{n \to \infty} \sup_{t, \|t - \theta_0\| = \delta/2} \frac{L_n(t) - L_n(\theta_0)}{n} = \limsup_{n \to \infty} \sup_{t, \|t - \theta_0\| = \delta/2} \left( -\alpha(t) + \frac{t^T S_n}{n} \right) - f(\theta_0) \xrightarrow{(*)} \sup_{t, \|t - \theta_0\| = \delta/2} f(t) - f(\theta_0) \quad (**)
$$

where

$$
(\ast) \quad f(t) - f(\theta_0) \leq 0.
$$
Equality (*) follows from the fact that the sequence of functions \( (t \mapsto t^n S_n) \) almost surely converges to \( (t \mapsto t^n s_0) \) uniformly on the set \( \{ t : \|t - \theta_0\| = \delta / 2 \} \). Now, we note that the function \( t \mapsto f(t) - f(\theta_0) \) is continuous and negative on the set \( \{ t : \|t - \theta_0\| = \delta / 2 \} \), by [37, Lemma 27 (1)]. Therefore, its supremum on the compact set \( \{ t : \|t - \theta_0\| = \delta / 2 \} \) is negative, which justifies the last inequality (**). Therefore almost surely Assumption 9 is satisfied for large enough \( n \), for a \( \kappa \) not depending on \( n \).

Note that the above considerations are valid for any fixed \( \delta > 0 \), such that \( \bar{E} \subseteq \Theta \). Since \( \alpha'' \) is continuous on \( \Theta \) and \( \alpha''(\theta) \) is symmetric, positive definite for all \( \theta \) and \( \hat{\theta}_n \to \theta_0 \), almost surely, Assumption 8 is also satisfied for \( n \) large enough and for \( \delta > 0 \) small enough, almost surely. Moreover, note that \( \text{Tr} \left( \hat{J}_n(\theta_n)^{-1} \right) = \text{Tr} \left( \alpha''(\hat{\theta}_n)^{-1} \right) \leq \frac{d}{\lambda_{\min}(\theta_n)} \). Since \( \alpha'' \) is continuous, \( \alpha''(\theta) \) is positive definite for all \( \theta \in \Theta \) and \( \hat{\theta}_n \to \theta_0 \) almost surely, we have that almost surely for large \( n \), \( \frac{d}{\lambda_{\min}(\theta_n)} \) is uniformly bounded from above. Therefore, Assumption 7 is satisfied for large enough \( n \), almost surely.

Furthermore, Assumption 2 is satisfied immediately for large \( n \) (with \( M_1 \) independent of \( n \)) if the prior density \( \pi \) is continuous and positive in a neighborhood around \( \theta_0 \) and if \( \hat{\theta}_n \to \theta_0 \), which holds almost surely (eq. (C.1)). Similarly Assumption 10 is immediately satisfied for large \( n \) (with \( M_1 \) and \( M_1 \) independent of \( n \)) if, additionally, \( \pi \) is continuously differentiable in a neighborhood of \( \theta_0 \).

Now, assume that \( \pi \) is thrice continuously differentiable on \( \Theta \). Then \( T_n \) is upper-bounded and has at least one global maximum \( \theta_n^* \in \Theta \). Using eq. (C.2), we note that \( \|\theta_n^* - \hat{\theta}_n\| \xrightarrow{n \to \infty} 0 \) almost surely (as \( \pi \) is uniformly upper bounded on \( \Theta \)). We also note that, almost surely, for any fixed neighborhood of \( \theta_0 \) and for sufficiently large \( n \), \( \frac{\alpha''(\theta)}{n} \) is strictly positive definite for \( \theta \) in that neighborhood. This means that, almost surely, for sufficiently large \( n \), the MAP \( \hat{\theta}_n \) is unique. Moreover, \( \|\hat{\theta}_n - \hat{\theta}_n\| \xrightarrow{n \to \infty} 0 \) almost surely, which implies that \( \hat{\theta}_n \to \theta_0 \) almost surely, by eq. (C.1). Moreover, note that

\[
\frac{T_n''(\theta)}{n} = -\alpha''(\theta) + \frac{(\log \pi)''(\theta)}{n}.
\]

Note that \( (\log \pi)'' \) is continuous in a neighbourhood of \( \theta_0 \). Then, almost surely, for large enough \( n \) and small enough \( \delta \), \( T_n''(\theta) \) is uniformly bounded inside \( \{ \theta : \|\theta - \theta_n^*\| \leq \delta \} \) (which follows from the fact that Assumption 1 is satisfied). Thus, Assumption 3 is satisfied for large enough \( n \) and small enough \( \delta \), almost surely. Using the same assumption that \( (\log \pi)'' \) is continuous in a neighbourhood of \( \theta_0 \), we have that \( \alpha'' - \frac{(\log \pi)''}{n} \) is continuous in a neighbourhood of \( \theta_0 \) and for large enough \( n \), \( \alpha''(\theta) - \frac{(\log \pi)''(\theta)}{n} \) is symmetric, positive definite for all \( \theta \) in a (potentially smaller) neighborhood of \( \theta_0 \). Moreover, as we showed above, \( \hat{\theta}_n \to \theta_0 \), almost surely. Therefore, Assumption 5 is satisfied for \( n \) large enough and for \( \delta > 0 \) small enough, almost surely.

Now, keeping the same assumptions, note that \( \max \left\{ \|\hat{\theta}_n - \hat{\theta}_n\|, \frac{\text{Tr}(J_n(\theta_n)^{-1})}{n} \right\} < \delta \) for large enough \( n \) and small enough \( \delta \), almost surely. This is because if \( \lambda_{\min}(\hat{\theta}_n) > 0 \) then \( \text{Tr} (J_n(\theta_n)^{-1}) \leq \frac{d}{\lambda_{\min}(\theta_n)} \), which is uniformly bounded from above for large enough \( n \). Note also that \( \frac{\text{Tr} (J_n(\theta_n) + \frac{2n^2}{n} I_{d \times d})^{-1}}{n} < \delta \) follows from Assumption 7, which holds for large \( n \) and small \( \delta > 0 \), almost surely, as we showed above. Therefore, Assumption 4 is satisfied for large enough \( n \) and small enough \( \delta \) and \( \delta \), almost surely. Moreover, as \( \hat{\theta}_n \to \theta_0 \) almost surely, Assumption 6 is satisfied for large enough \( n \) and small enough \( \delta \), almost surely, by an argument analogous to eq. (C.2).
C.2. Proof of Proposition 2.2. The proof is inspired by the proof of [37, Theorem 13]. Note that, for all \( \theta \in \Theta \), \( L_n(\theta) = -\sum_{i=1}^{n} \alpha(\theta^T X_i) + \theta^T S_n \), where \( S_n = \sum_{i=1}^{n} X_i s(Y_i) \). Thus, \( L_n \) is \( C^\infty \) on \( \Theta \) by the chain rule, since \( \alpha \) is \( C^\infty \) on \( E \) by [38, Proposition 19]. Also, \( L_n \) is strictly concave since \( \alpha \) is strictly convex ([38, Proposition 19]). Now, let \( n \rightarrow \infty \) for all \( \theta \in \Theta \), \( \frac{L_n(\theta)}{n} \rightarrow f(\theta) \) almost surely (by our assumed condition 2). This implies that, almost surely, \( \frac{L_n(\theta)}{n} \rightarrow f(\theta) \), for all \( \theta \in \Theta \). In order to show this implication, let us fix a countable dense subset \( C \) of \( \Theta \). Then, almost surely, \( \frac{L_n(\theta)}{n} \rightarrow f(\theta) \) for all \( \theta \in C \). Since \( \frac{L_n}{n} \) is concave, it follows from [42, Theorem 10.8] that, almost surely, the limit \( \tilde{f}(\theta) := \lim_{n \rightarrow \infty} \frac{L_n(\theta)}{n} \) exists and is finite for all \( \theta \in \Theta \) and \( \tilde{f} \) is concave. As \( f \) is also concave, then \( \tilde{f} \) and \( f \) are continuous functions [42, Theorem 10.1] that agree on a dense subset of \( \Theta \) so they are equal on \( \Theta \).

Now, note that, by [37, Theorem 5 and Theorem 13], almost surely, there exists MLE \( \hat{\theta}_n \), such that \( \hat{\theta}_n \rightarrow \theta_0 \) almost surely.

(C.3) \[ \hat{\theta}_n \rightarrow \theta_0, \quad \text{almost surely}. \]

As \( L_n \) is strictly concave, this MLE is almost surely unique.

We will show that, with probability 1, \( \frac{L_n'}{n} \) is uniformly bounded on \( \tilde{E} = \{ \theta : \|\theta - \theta_0\| \leq \epsilon \} \), where \( \epsilon > 0 \) satisfies condition 4 of Proposition 2.2. Fix \( j, k, l \in \{1, \ldots, d\} \) and define \( T(\theta, x) = \alpha''(\theta^T x) x_j x_k x_l \) for \( \theta \in \Theta \), \( x \in \mathcal{X} \). For all \( x \in \mathcal{X} \), \( \theta \rightarrow T(\theta, x) \) is continuous and measurable. Since \( L_n''(\theta) j, k, l = -\sum_{i=1}^{n} T(\theta, X_i) \), condition 4 above implies that with probability 1, \( \frac{L_n''(\theta) j, k, l}{n} \) is uniformly bounded on \( \tilde{E} \), by the uniform law of large numbers [17, Theorem 1.3.3]. Letting \( C_{j, k, l}(X_1, X_2, \ldots) \) be such a uniform bound for each \( j, k, l \), we have that with probability 1, for all \( n \in \mathbb{N} \), \( \theta \in \tilde{E} \), \( \left\| \frac{L_n''(\theta)}{n} \right\|^2 = \sum_{j, k, l} \frac{L_n''(\theta) j, k, l}{n^2} \leq \sum_{j, k, l} C_{j, k, l} \left\| X_1, X_2, \ldots \right\|^2 < \infty \). Hence \( \frac{L_n''}{n} \) is uniformly bounded on \( \tilde{E} \). Therefore, almost surely, for large enough \( n \) and small enough \( \delta \), Assumption 1 is satisfied with a constant \( M_2 \) not depending on \( n \). This is because \( \hat{\theta}_n \rightarrow \theta_0 \) almost surely and so, almost surely, for large enough \( n \) and \( \delta = \epsilon/2 \), \( \{ \theta : \|\theta - \theta_0\| \leq \delta \} \subseteq \tilde{E} \).

Now, using [37, Theorem 7],

(C.4) \[ f''(\theta_0) \stackrel{a.s.}{=} \lim_{n \rightarrow \infty} \frac{L_n''(\theta_0)}{n} = \lim_{n \rightarrow \infty} \left( -\frac{1}{n} \sum_{i=1}^{n} \alpha''(\theta_0^T X_i) X_i X_i^T \right). \]

This limit exists and is finite almost surely. Therefore, by the strong law of large numbers,

\[ f''(\theta_0) \stackrel{a.s.}{=} -\mathbb{E} \left( \alpha''(\theta_0^T X_i) X_i X_i^T \right). \]

Therefore, \( -f''(\theta_0) \) is positive definite almost surely because \( \alpha''(\eta) \) is positive definite for any \( \eta \) and so for any nonzero \( a \in \mathbb{R}^d \), we have

\[ \mathbb{E} \left[ \alpha''(\theta_0^T X_i) a^T X_i X_i^T a \right] > 0 \]

by our assumption that \( a^T X_i \) is distinct from zero with a positive probability. Now, by [37, Theorem 7], \( \frac{L_n''}{n} \) is \( L \)-equi-Lipschitz for some \( L < \infty \). Therefore, we have that:

(C.5) \[ \left\| \frac{L_n''(\hat{\theta}_n)}{n} - f''(\theta_0) \right\| \leq L \left\| \hat{\theta}_n - \theta_0 \right\| + \left\| \frac{L_n''(\theta_0)}{n} - f''(\theta_0) \right\|. \]

It follows that \( \frac{L_n''(\hat{\theta}_n)}{n} \rightarrow f''(\theta_0) \) almost surely by eqs. (C.3) and (C.4). This means that, almost surely, for large enough \( n \), the minimum eigenvalue of \( -\frac{L_n''(\hat{\theta}_n)}{n} \) stays positive and
lower bounded by a positive number, not depending on \( n \), as \( f''(\theta_0) \) is positive definite. Therefore, almost surely, Assumption 8 is satisfied for large enough \( n \) and small enough \( \delta \). Moreover, for the same reason, Assumption 7 is satisfied, for large enough \( n \) and small enough \( \delta > 0 \), almost surely. Indeed, note that \( \text{Tr}\left(\hat{J}_n(\hat{\theta}_n)^{-1}\right) = \text{Tr}(\alpha''(\hat{\theta}_n)^{-1}) \leq \frac{d}{\lambda_{\min}(\hat{\theta}_n)} \). Since \( \frac{L_n''(\hat{\theta}_n)}{n} \to f''(\theta_0) \) almost surely, we have that, almost surely, for large \( n \), \( \frac{d}{\lambda_{\min}(\hat{\theta}_n)} \) is uniformly bounded from above.

Now, let \( \xi_\theta \) be the point on the line connecting \( \theta \) and \( \theta_0 \) that lies on \( \{ t : \| t - \theta_0 \| = \delta/2 \} \). The strict concavity of \( L_n \) implies that, almost surely,

\[
\limsup_{n \to \infty} \sup_{\| \theta - \theta_0 \| > \delta} \frac{L_n(\theta) - L_n(\hat{\theta}_n)}{n} \\
\leq \limsup_{n \to \infty} \sup_{\| \theta - \theta_0 \| > \delta/2} \frac{L_n(\theta) - L_n(\theta_0)}{n} \\
\leq \limsup_{n \to \infty} \sup_{\| \theta - \theta_0 \| > \delta/2} \frac{\| \theta_0 - \theta \|. L_n(\xi_\theta) - L_n(\theta_0)}{\delta/2} \\
\leq \limsup_{n \to \infty} \sup_{t : \| t - \theta_0 \| = \delta/2} \frac{L_n(t) - L_n(\theta_0)}{n} \\
= \limsup_{n \to \infty} \sup_{t : \| t - \theta_0 \| = \delta/2} \frac{L_n(t)}{n} - f(\theta_0) \\
= \sup_{t : \| t - \theta_0 \| = \delta/2} f(t) - f(\theta_0) < 0.
\]

Equality (\(*)(n)\) follows from the fact that, by [37, Theorem 7], the sequence of functions \( \left( \frac{L_n}{n} \right) \) almost surely converges to \( f \) uniformly on the set \( \{ t : \| t - \theta_0 \| = \delta/2 \} \). Now, we note that the function \( t \mapsto f(t) - f(\theta_0) \) is continuous and negative on the set \( \{ t : \| t - \theta_0 \| = \delta/2 \} \), by [37, Lemma 27 (1)]. Therefore, its supremum on the compact set \( \{ t : \| t - \theta_0 \| = \delta/2 \} \) is negative, which justifies the last inequality. Therefore, almost surely, Assumption 9 is satisfied, for large enough \( n \), small enough \( \delta \) and for some \( \kappa > 0 \) not depending on \( n \).

Furthermore, Assumption 2 is satisfied immediately, almost surely for large \( n \) and small enough \( \delta \), if the prior density \( \pi \) is continuous and positive in a neighborhood around \( \theta_0 \) since \( \hat{\theta}_n \to \theta_0 \) almost surely (eq. (C.3)). Similarly Assumption 10 is immediately satisfied almost surely if, additionally, \( \pi \) is continuously differentiable in a neighborhood of \( \theta_0 \).

Now, assume that \( \pi \) is thrice continuously differentiable on \( \Theta \). Then \( L_{\theta_0} \) is upper-bounded and has at least one global maximum \( \theta_n^* \in \Theta \). Using eq. (C.6), we note that \( \| \theta_n^* - \hat{\theta}_n \| \to 0 \) almost surely (as \( \pi \) is uniformly upper bounded on \( \Theta \)). We also note that, almost surely, for any fixed neighborhood of \( \theta_0 \) and for sufficiently large \( n \), \( \frac{L_n''(\theta)}{n} \) is strictly positive definite for \( \theta \) in that neighborhood. This means that, almost surely, for sufficiently large \( n \), the MAP \( \hat{\theta}_n \) is unique. Moreover, \( \| \hat{\theta}_n - \theta_0 \| \to 0 \) almost surely, which implies that \( \hat{\theta}_n \to \theta_0 \) almost surely. Moreover, note that

\[
\frac{L_n''(\theta)}{n} = \frac{L_n'''(\theta)}{n} + \frac{(\log \pi)''''(\theta)}{n}.
\]

Hence, if \( (\log \pi)'''' \) is continuous in a neighbourhood of \( \theta_0 \) then, almost surely, for large enough \( n \) and small enough \( \delta \), \( \frac{L_n''(\theta)}{n} \) is uniformly bounded inside \( \{ \theta : \| \theta - \theta_n \| \leq \delta \} \) (which follows from the fact that Assumption 1 holds almost surely and that \( \| \hat{\theta}_n - \theta_0 \| \to 0 \) almost


with $\delta$ and large enough $L$ and so $\bar{n}$ surely). Thus, almost surely, Assumption 3 is satisfied for large enough $n$ and small enough $\delta$ with $M_2$ independent of $n$.

Now, as in eq. (C.5), we have that
\[
\left\| \frac{L''(\hat{\theta}_n)}{n} - f''(\theta_0) \right\| 
\leq L \left\| \hat{\theta}_n - \theta_0 \right\| + \left\| \frac{L''(\theta_0)}{n} - f''(\theta_0) \right\|
\]
and so $\frac{L''(\hat{\theta}_n)}{n} \xrightarrow{n \to \infty} f''(\theta_0)$ almost surely. As a result, almost surely, for large enough $n$, the minimum eigenvalue of $-\frac{L''(\hat{\theta}_n)}{n}$ stays lower bounded by a positive number not depending on $n$. It follows that, if $(\log \pi)^{'''}(\theta_0)$ is continuous in a neighbourhood of $\theta_0$, then, almost surely, for large $n$, the minimum eigenvalue of $-\frac{L''(\hat{\theta}_n)}{n} - (\log \pi)^{'''}(\hat{\theta}_n)$ stays lower bounded by a positive number not depending on $n$. Therefore, almost surely, Assumption 5 is also satisfied, for $n$ large enough and for $\delta > 0$ small enough.

Still assuming that $(\log \pi)^{'''}(\theta_0)$ is continuous in a neighborhood of $\theta_0$, note that, almost surely, $\max \left\{ \|\hat{\theta}_n - \hat{n}t\|, \sqrt{\frac{\text{Tr}(\hat{J}_n(\hat{n}t_{\min})^{-1})}{n}} \right\} < \bar{\delta}$ for large enough $n$ and small enough $\bar{\delta}$. This is because $\text{Tr}(\hat{J}_n(\hat{n}t_{\min})^{-1}) \leq \frac{d}{\lambda_{\min}(\hat{\theta}_n)}$ and $\|\hat{\theta}_n - \hat{n}t\| \to 0$ almost surely. Moreover, note that $\sqrt{\frac{\text{Tr}\left((\hat{J}_n(\hat{n}t_{\min}) + \frac{d\lambda_2}{t} I_{d \times d})^{-1}\right)}{n}} < \delta$ if Assumption 7 is satisfied. Therefore, almost surely, Assumption 4 is satisfied for large enough $n$ and small enough $\delta$ and $\bar{\delta}$. Furthermore, as $\hat{\theta}_n \to \theta_0$ almost surely then also almost surely Assumption 6 is satisfied for large enough $n$ and small enough $\bar{\delta}$, by an argument analogous to eq. (C.6).

**APPENDIX D: INTRODUCTORY ARGUMENTS FOR THE PROOFS OF THEOREMS 3.1 – 3.3, 6.1 – 6.3 AND A.1**

**D.1. Introduction.** In order to prove the main results of this paper, we let $g : \mathbb{R}^d \to \mathbb{R}$ and consider:

(D.1) $D_g^{MLE} := \mathbb{E} \left[ g\left(\sqrt{n}(\hat{\theta}_n - \hat{n}t)\right) - \mathbb{E}_{Z_n \sim N(0, \hat{J}_n(\hat{n}t_{\min})^{-1})} \left[ g(Z_n) \right] \right]$;

(D.2) $D_g^{MAP} := \mathbb{E} \left[ g\left(\sqrt{n}(\hat{\theta}_n - \hat{n}t)\right) - \mathbb{E}_{Z_n \sim N(0, \hat{J}_n(\hat{n}t_{\min})^{-1})} \left[ g(Z_n) \right] \right]$.

The following lemma will be useful in the sequel:

**LEMMA D.1.** Let $Z \sim N(0, \Sigma)$. Then, for any $t > 0$,

(D.3) $\mathbb{P} \left[ \|Z\| - \sqrt{\text{Tr}(\Sigma)} \geq \sqrt{2\|\Sigma\|_{op} t} \right] \leq e^{-t}$;

(D.4) $\mathbb{P} \left[ \|Z\|^2 - \text{Tr}(\Sigma) \geq 2\sqrt{\text{Tr}(\Sigma^2) t + 2\|\Sigma\|_{op} t} \right] \leq e^{-t}$.

**PROOF.** The proof follows an argument similar to the one of [50, page 135]. Specifically, eq. (D.4) comes from [25, Proposition 1]. In order to prove eq. (D.3), we note that

$\mathbb{P} \left[ \|Z\| - \sqrt{\text{Tr}(\Sigma)} \geq \sqrt{2\|\Sigma\|_{op} t} \right] = \mathbb{P} \left[ \|Z\|^2 - \text{Tr}(\Sigma) \geq 2\|\Sigma\|_{op} t + 2\sqrt{2\text{Tr}(\Sigma) \|\Sigma\|_{op} t} \right] \leq \mathbb{P} \left[ \|Z\|^2 - \text{Tr}(\Sigma) \geq 2\sqrt{\text{Tr}(\Sigma^2) t + 2\|\Sigma\|_{op} t} \right] \leq e^{-t}$. 

$\square$
D.2. Initial decomposition of the distances $D_{g}^{MLE}$ and $D_{g}^{MAP}$. Now, let

$$h_{g}^{MLE}(u) := g(u) - \frac{n^{-d/2}}{C_{n}^{MLE}} \int_{|t| \leq \delta \sqrt{n}} g(t) \Pi_{n}(n^{-1/2} t + \bar{\theta}_{n}) dt,$$

$$h_{g}^{MAP}(u) := g(u) - \frac{n^{-d/2}}{C_{n}^{MAP}} \int_{|t| \leq \delta \sqrt{n}} g(t) \Pi_{n}(n^{-1/2} t + \bar{\theta}_{n}) dt,$$

for

$$C_{n}^{MLE} := n^{-d/2} \int_{|u| \leq \delta \sqrt{n}} \Pi_{n}(n^{-1/2} u + \bar{\theta}_{n}) du$$

$$C_{n}^{MAP} := n^{-d/2} \int_{|u| \leq \delta \sqrt{n}} \Pi_{n}(n^{-1/2} u + \bar{\theta}_{n}) du.$$

Note that, for

$$F_{n}^{MLE} := \int_{|u| \leq \delta \sqrt{n}} \sqrt{\det \left( \hat{J}_{h}(\bar{\theta}_{n}) \right)} \left| e^{-u^{T} \hat{J}_{n}(\bar{\theta}_{n}) u/2} \right| \left( \frac{1}{2\pi} \right)^{d/2} du,$$

we have

$$D_{g}^{MLE}$$

$$\leq \frac{1}{F_{n}^{MLE}} \int_{|u| \leq \delta \sqrt{n}} h_{g}^{MLE}(u) \left| \sqrt{\det \left( \hat{J}_{h}(\bar{\theta}_{n}) \right)} \left| e^{-u^{T} \hat{J}_{n}(\bar{\theta}_{n}) u/2} \right| \left( \frac{1}{2\pi} \right)^{d/2} \right| du$$

$$+ \int_{|u| > \delta \sqrt{n}} h_{g}^{MLE}(u) \left[ \sqrt{\det \left( \hat{J}_{h}(\bar{\theta}_{n}) \right)} \left| e^{-u^{T} \hat{J}_{n}(\bar{\theta}_{n}) u/2} \right| \left( \frac{1}{2\pi} \right)^{d/2} - n^{-d/2} \Pi_{n}(n^{-1/2} u + \bar{\theta}_{n}) \right] du$$

$$= \int_{|u| \leq \delta \sqrt{n}} g(u) \left| \sqrt{\det \left( \hat{J}_{h}(\bar{\theta}_{n}) \right)} \left| e^{-u^{T} \hat{J}_{n}(\bar{\theta}_{n}) u/2} \right| \left( \frac{1}{2\pi} \right)^{d/2} \right| du$$

$$+ \frac{n^{-d/2}}{C_{n}^{MLE}} \int_{|u| \leq \delta \sqrt{n}} g(u) \Pi_{n}(n^{-1/2} u + \bar{\theta}_{n}) du$$

$$+ \int_{|u| > \delta \sqrt{n}} h_{g}^{MLE}(u) \left[ \sqrt{\det \left( \hat{J}_{h}(\bar{\theta}_{n}) \right)} \left| e^{-u^{T} \hat{J}_{n}(\bar{\theta}_{n}) u/2} \right| \left( \frac{1}{2\pi} \right)^{d/2} - n^{-d/2} \Pi_{n}(n^{-1/2} u + \bar{\theta}_{n}) \right] du.$$
\( I_{\text{MLE}} + I_{\text{MAP}} \).

In a similar manner, for
\[
F_n^{\text{MAP}} := \int_{\|u\| \leq \delta \sqrt{n}} \frac{\sqrt{\det (\bar{J}_n(\bar{\theta}_n))} e^{-u^T \bar{J}_n(\bar{\theta}_n) u/2}}{(2\pi)^{d/2}} du,
\]
we have
\[
D_{\text{MAP}}^g \leq \int_{\|u\| \leq \delta \sqrt{n}} g(u) \frac{\sqrt{\det (\bar{J}_n(\bar{\theta}_n))} e^{-u^T \bar{J}_n(\bar{\theta}_n) u/2}}{F_n^{\text{MAP}}(2\pi)^{d/2}} du
\]
\[
- \frac{n^{-d/2}}{C_{\text{MAP}}^n} \int_{\|u\| \leq \delta \sqrt{n}} g(u) \Pi_n(n^{-1/2}u + \bar{\theta}_n) du
\]
\[
+ \int_{\|u\| > \delta \sqrt{n}} h_g^{\text{MAP}}(u) \left[ \frac{\sqrt{\det (\bar{J}_n(\bar{\theta}_n))} e^{-u^T \bar{J}_n(\bar{\theta}_n) u/2}}{(2\pi)^{d/2}} - n^{-d/2} \Pi_n(n^{-1/2}u + \bar{\theta}_n) \right] du
\]
\( =: I_{\text{MAP}} + I_{\text{MAP}}^2 \).

**D.3. Strategies for controlling terms** \( I_{\text{MLE}}^1 \) and \( I_{\text{MAP}}^1 \). In order to control \( I_{\text{MAP}}^1 \) in Theorems 3.1 – 3.3 and \( I_{\text{MLE}}^1 \) in Theorems 6.1 – 6.3, we apply the log-Sobolev inequality and the associated transportation inequalities. Before we introduce those concepts, we define the following Kullback-Leibler divergence (or relative entropy, see e.g. [6, Section 1.6.1]) between two measures \( \nu \) and \( \mu \) such that \( \nu \ll \mu \):
\[
\text{KL}(\nu\|\mu) = \int \log \left( \frac{d\nu}{d\mu} \right) (x) \nu(dx).
\]
We also define the following Fisher divergence between two measures \( \nu \) and \( \mu \) such that \( \nu \ll \mu \):
\[
\text{Fisher}(\nu\|\mu) = \int \left\| \left( \log \left( \frac{d\nu}{d\mu} \right) \right)' (x) \right\|^2 \nu(dx).
\]
The following definition will play a central role in our proofs:

**Definition D.2** (Log-Sobolev inequality (LSI), [2, Chapter 5], [49, Section 2.2]). Let \( \mu \) be a probability measure on \( \Omega \subset \mathbb{R}^d \) and let \( \alpha > 0 \). We say that \( \mu \) satisfies the log-Sobolev inequality with constant \( \alpha \), or LSI(\( \alpha \)) for short, if
\[
\int f^2 \log f^2 d\mu - \int f^2 d\mu \log \left( \int f^2 d\mu \right) \leq \frac{2}{\alpha} \int \|f'\|^2 d\mu, \quad \text{for all } f : \Omega \to \mathbb{R}^+.
\]
Equivalently, we say that \( \mu \) satisfies LSI(\( \alpha \)) if for any probability measure \( \nu \) on \( \Omega \), such that \( \mu \ll \nu \) and \( \nu \ll \mu \),
\[
\text{KL}(\nu\|\mu) \leq \frac{1}{2\alpha} \text{Fisher}(\nu\|\mu).
\]
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Remark 20. The equivalence between the two definitions of the log-Sobolev inequality LSI(\(\alpha\)), given by eqs. (D.12) and (D.13), follows by the following argument. If \(\nu\) is a probability measure on \(\Omega\) such that \(\mu\) and \(\nu\) are equivalent then we obtain eq. (D.13) by setting \(f^2 = \frac{d\nu}{d\mu}\) in eq. (D.12). On the other hand, for a function \(f : \Omega \to \mathbb{R}^+\), let \(\nu\) be a probability measure on \(\Omega\) with density \(\nu(dx) \propto f^2(x)\mu(dx)\). Plugging this \(\nu\) into eq. (D.13) yields eq. (D.12).

One of the fundamental results we will use is the Bakry-Émery criterion. Its first general version was proved in [3]. Below we state its version for measures on \(\mathbb{R}^d\) truncated to convex sets:

**Proposition D.1** (Bakry-Émery criterion, [32, Theorem 2.1], [45, Theorem A1]). Let \(\Omega \subset \mathbb{R}^d\) be convex, let \(H : \Omega \to \mathbb{R}\) and consider a probability measure \(\mu(dx) \propto e^{-H(x)}\mathbb{1}_\Omega(x)dx\). Assume that \(H''(x) \geq \alpha I_{d \times d}\), for some \(\alpha > 0\). Then \(\mu\) satisfies LSI(\(\alpha\)) (see Definition D.2).

The following Holley-Stroock perturbation principle will be crucial in our proofs concerning the MLE-centric approach:

**Proposition D.2** (Holley-Stroock perturbation principle, [24, page 1184], [45, Theorem A2]). Let \(\Omega \subset \mathbb{R}^d\) and \(H : \Omega \to \mathbb{R}\). Let \(\psi : \Omega \to \mathbb{R}^d\) be a bounded function. Let \(\mu\) and \(\tilde{\mu}\) be probability measures with densities of the form \(\mu(dx) \propto e^{-H(x)}\mathbb{1}_\Omega(x)dx\) and \(\tilde{\mu}(dx) \propto e^{-H(x)-\psi(x)}\mathbb{1}_\Omega(x)dx\). Suppose that \(\mu\) satisfies the LSI(\(\alpha\)) (see Definition D.2). Then \(\tilde{\mu}\) satisfies LSI\(\left(e^{-\text{osc}\psi} \alpha\right)\), where \(\text{osc}\psi := \sup_{\Omega} \psi - \inf_{\Omega} \psi\).

In order to control \(I_{\text{MAP}}\) and \(I_{\text{MLE}}\) in our proofs, we control the Fisher divergence between the rescaled posterior and the Gaussian inside the ball around the MAP or the MLE, establish the log-Sobolev inequality inside that ball for the rescaled posterior and thus control the KL divergence. In order to obtain our bounds on the total variation distance in Theorems 3.1 and 6.1, we will use the following result:

**Proposition D.3** (Pinsker’s inequality, e.g. [36, Theorem 2.16]). For any two probability measures \(\mu\) and \(\nu\) such that \(\nu \ll \mu\), we have that

\[
TV(\mu, \nu) \leq \sqrt{\frac{1}{2} KL(\nu\|\mu)}.
\]

In order to control the 1-Wasserstein distance in Theorems 3.2 and 6.2 and the integral probability metric appearing in Theorems 3.3 and 6.3, we first control \(I_{\text{MAP}}^1\) or \(I_{\text{MLE}}^1\) by controlling the 2-Wasserstein distance inside the appropriate neighborhood of the MAP or the MLE. The 2-Wasserstein distance for two measures \(\mu\) and \(\nu\) on the same measurable space is defined in the following way:

\[
W_2(\mu, \nu) = \inf_{\Gamma} \sqrt{\mathbb{E}_{\Gamma} \left[ \|X - Y\|^2 \right]}
\]

(D.14)

where the infimum is over all distributions \(\Gamma\) of \((X, Y)\) with the correct marginals \(X \sim \nu\), \(Y \sim \mu\). It is an easy consequence of Jensen’s inequality that for any probability measures \(\mu\), \(\nu\),

\[
W_1(\mu, \nu) \leq W_2(\mu, \nu).
\]

(D.15)

We also have the following Talagrand inequality:
Proposition D.4. Let \( \mu \) be a probability measure on a closed ball \( \{ u : \| u \| \leq \eta \} \) for some \( \eta > 0 \) such that \( \mu \) is absolutely continuous with respect to the Lebesgue measure. Suppose that \( \mu \) satisfies LSI(\( \alpha \)) on \( \{ u : \| u \| \leq \eta \} \). Then, for any probability measure \( \nu \ll \mu \) on \( \{ u : \| u \| \leq \eta \} \), we have that

\[
W_2(\mu, \nu)^2 \leq \frac{2}{\alpha} KL(\nu\| \mu).
\]

Proof. This result follows from [21, Theorem 4.1] by an argument analogous to the one that let the authors prove [21, Corollary 4.2]. Indeed, for a natural number \( n \geq 1 \) and \( x = (x_1, \ldots, x_n) \in (\mathbb{R}^d)^n \), consider \( L_n^x = n^{-1} \sum_{i=1}^n \delta_{x_i} \), where \( \delta_{x_i} \)'s are Dirac deltas. In the proof of [21, Corollary 4.2] it is shown that the function \( x \mapsto W_2(L_n^x, \mu) \) is \( \frac{1}{\sqrt{n}} \)-Lipschitz for the Euclidean distance. When restricted to the set \( \{ u : \| u \| \leq \eta \} \subset (\mathbb{R}^d)^n \), this function is still \( \frac{1}{\sqrt{n}} \)-Lipschitz. As in the proof of [21, Corollary 4.2], we note that, by Rademacher’s theorem the function \( F_n(x) = W_2(L_n^x, \mu) \) is differentiable Lebesgue-almost everywhere on \( (\mathbb{R}^d)^n \). When restricted to \( \{ u : \| u \| \leq \eta \} \subset (\mathbb{R}^d)^n \) this function stays differentiable Lebesgue-almost everywhere. Therefore, the condition

\[
\sum_{i=1}^n \| \nabla_i F_n \|^2 (x) \leq \frac{1}{n} \quad \text{for } \mu^n\text{-almost every } x \in \{ u : \| u \| \leq \eta \} \subset (\mathbb{R}^d)^n
\]

is fulfilled, as required by [21, Theorem 4.1]. \( \square \)

Proposition D.4 together with eq. (D.15) and the log-Sobolev inequality will let us control \( I_1^{\text{MAP}} \) and \( I_1^{\text{MLE}} \) in the proofs of Theorems 3.2, 3.3, 6.2 and 6.3.

D.4. Controlling term \( I_2^{\text{MLE}} \). Note that

\[
I_2^{\text{MLE}} \leq \left| \int_{\| u \| > \delta \sqrt{n}} g(u) \left( n^{-d/2} \Pi_n(n^{-1/2} u + \hat{\theta}_n) - \frac{\sqrt{\det(J_n(\hat{\theta}_n))} e^{-u^T J_n(\hat{\theta}_n) u/2}}{(2\pi)^{d/2}} \right) du \right|
\]

\[
+ \frac{n^{-d/2}}{C_{\text{MLE}}^n} \int_{\| t \| \leq \delta \sqrt{n}} |g(t)| \Pi_n(n^{-1/2} t + \hat{\theta}_n) dt
\]

\[
= I_{2,1}^{\text{MLE}} + I_{2,2}^{\text{MLE}}.
\]

Now, note that \( L_n'(\hat{\theta}_n) = 0 \). Therefore, for \( \| t \| \leq \delta \sqrt{n} \), Assumption 1 implies that

\[
|L_n(n^{-1/2} t + \hat{\theta}_n) - L_n(\hat{\theta}_n) + \frac{1}{2} t^T \hat{J}_n(\hat{\theta}_n) | t | \leq \frac{1}{6} n^{-1/2} M_2 \| t \|^3 \leq \frac{\delta M_2}{6} \| t \|^2.
\]

Therefore, under Assumptions 1, 2 and 9, and using the notation of Section 2.5,

\[
\int_{\| u \| > \delta \sqrt{n}} |g(u)| n^{-d/2} \Pi_n(n^{-1/2} u + \hat{\theta}_n) du
\]
where the last inequality follows from Lemma D.1. A bound on $I_{2,2}$ can be obtained by combining eq. (D.19) with eq. (D.18) applied to $g = 1$. Indeed, we thus obtain:

\[ I_{2,2}^{MLE} \leq \frac{M_1 M_1}{M_1 M_1} \left| \det \left( \hat{J} \hat{P} \left( \hat{\theta}, \hat{\delta} \right) \right) \right| \left| \int_{\|u\| > \delta \sqrt{n}} g(u) \left| \frac{1}{2} e^{-\frac{1}{2} t^T \left( \hat{J} \hat{P} \left( \hat{\theta}, \hat{\delta} \right) \right) t} dt \right| \right| \left( \frac{2\pi}{d/2} \left( 1 - \hat{\theta} \hat{P} \left( \theta, \delta \right) \right) \right) \]
where we used Lemma D.1. A bound on $I_{2,1}^{MLE}$ is obtained by adding together the bounds on $I_{2,1}^{MLE}$ (eq. (D.18)) and $I_{2,2}^{MLE}$ (eq. (D.20)).

Remark 21. Note that, for $g$, such that $|g| \leq U$, for some $U > 0$, we have that

$$\int_{|t| \leq \delta \sqrt{n}} \frac{n^{-d/2}}{C_n^{MLE}} |g(t)| \Pi_n(n^{-1/2} t + \tilde{\theta}_n) dt \leq U.$$ 

Therefore, for $|g| \leq U$, the same argument as above yields a simpler bound:

$$I_{2,2}^{MLE} \leq U \left\{ \tilde{\mathcal{G}}(n, \delta) + \frac{n^{-d/2} e^{-\alpha_k \hat{M}_1}}{(2\pi)^{d/2}} \left| \det \left( \hat{J}_n^p(\hat{\theta}_n, \delta) \right) \right|^{1/2} \right\}.$$ 

D.5. Controlling term $I_2^{MAP}$. Note that

$$I_2^{MAP} \leq \left| \int_{|u| > \delta \sqrt{n}} g(u) \left( n^{-d/2} \Pi_n(n^{-1/2} u + \bar{\theta}_n) - \frac{\sqrt{\left| \det \hat{J}_n(\hat{\theta}_n) \right| e^{-u^T \hat{J}_n(\hat{\theta}_n) u/2} \left| \det \hat{J}_n(\hat{\theta}_n) \right| e^{-u^T \hat{J}_n(\hat{\theta}_n) u/2} \right)}{(2\pi)^{d/2}} \right| du \right| + \frac{n^{-d/2}}{C_n^{MAP}} I_{2,2}^{MLE}$$

where

$$\int_{|u| > \delta \sqrt{n}} \frac{n^{-d/2}}{C_n^{MAP}} \left| \det \hat{J}_n(\hat{\theta}_n) \right|^{1/2} \Pi_n(n^{-1/2} u + \bar{\theta}_n) du$$

Note that, using Assumptions 1, 2 and 6, a calculation similar to eq. (D.17), yields

$$\int_{|u| > \delta \sqrt{n}} |g(u)| n^{-d/2} \Pi_n(n^{-1/2} u + \bar{\theta}_n) du$$
if \( n > \frac{\text{Tr}[J_n^T(\delta_n, \delta)^{-1}]}{\delta^2} \), where the last inequality follows from Lemma D.1. Therefore, under Assumptions 1, 2 and 6 and if \( n > \frac{\text{Tr}[J_n^T(\delta_n, \delta)^{-1}]}{\delta^2} \),

\[
I_{2,1}^{MAP} \leq \left| \int_{\|u\| > \delta \sqrt{n}} g(u) \frac{\sqrt{\det J_n(\bar{\theta}_n)}}{e^{-\frac{1}{2}u^T J_n(\bar{\theta}_n)u/2}} \frac{1}{(2\pi)^{d/2}} du \right|
\]

(D.21)

\[
n^{d/2}e^{-n\kappa} \tilde{M}_1 \left| \frac{1}{\sqrt{2\pi}} \int_{\|v - \bar{\theta}_n\| > \delta - \|\bar{\theta}_n - \theta_n\|} \sqrt{n} |g(\sqrt{n}(v - \bar{\theta}_n))| \pi(v) du \right|
\]

\[
+ \frac{n^{-d/2}}{C_n^{MAP}} \int_{\|t\| \leq \delta \sqrt{n}} |g(t)| \Pi_n(n^{-1/2}t + \bar{\theta}_n) dt
\]

\[
\leq \frac{1}{\int_{\|u\| \leq \delta \sqrt{n}} e^{\bar{J}_n(\bar{\theta}_n) - \frac{2\kappa}{d} \bar{J}_n(\bar{\theta}_n) du}}
\]

\[
\leq \frac{1}{\int_{\|u\| \leq \delta \sqrt{n}} e^{\bar{J}_n(\bar{\theta}_n) - \frac{2\kappa}{d} \bar{J}_n(\bar{\theta}_n) du}}
\]

\[
\leq \left| \det \left( \bar{J}_n^p(\bar{\theta}_n, \delta) \right) \right| \frac{1}{\sqrt{2\pi}} \int_{\|u\| \leq \delta \sqrt{n}} |g(u)| e^{-\frac{1}{2}u^T J_n^p(\bar{\theta}_n, \delta)u} du
\]

\[
\leq \frac{1}{\sqrt{2\pi}} \int_{\|u\| \leq \delta \sqrt{n}} |g(u)| e^{-\frac{1}{2}u^T J_n(\bar{\theta}_n)u} du
\]

\[
\leq \frac{1}{\sqrt{2\pi}} \int_{\|u\| \leq \delta \sqrt{n}} |g(u)| e^{-\frac{1}{2}u^T J_n(\bar{\theta}_n)u} du
\]

(D.22)

where the last inequality follows from Lemma D.1. A bound on \( I_{2,2}^{MAP} \) can be obtained by combining eq. (D.22) with eq. (D.21) applied to \( g = 1 \). Indeed, we obtain:

\[
I_{2,2}^{MAP} \leq \left\{ \varphi(n, \delta) + \frac{\tilde{M}_1}{\sqrt{2\pi}} \frac{1}{\left| \det \left( \bar{J}_n^p(\bar{\theta}_n, \delta) \right) \right|^{1/2}} \right\}
\]

(D.23)

where we applied Lemma D.1.

**Remark 22.** As in Remark 21, our bound gets simpler if \( |g| \leq U \), for some \( U > 0 \). In that case, instead of eq. (D.23), we can write:

\[
I_{2,2}^{MAP} \leq U \left\{ \varphi(n, \delta) + \frac{\tilde{M}_1}{\sqrt{2\pi}} \frac{1}{\left| \det \left( \bar{J}_n^p(\bar{\theta}_n, \delta) \right) \right|^{1/2}} \right\}
\]

**Appendix E: Proofs of Theorems 3.1 – 3.3**

Throughout this section we adopt the notation of Appendix D. Additionally, for any probability measure \( \mu \), we let \( [\mu]_{B_n(\delta \sqrt{n})} \) denote its restriction (truncation) to the ball of radius \( \delta \sqrt{n} \) around 0.
In all the proofs below, we wish to control the quantity \( D_{g,\text{MAP}} \) of eq. (D.2) for all functions \( g \) that satisfy certain prescribed criteria. In the proof of Theorem 3.1, we look at functions \( g \) that are indicators of measurable sets, in the proof of Theorem 3.2 we look at 1-Lipschitz functions \( g \) and in the proof of Theorem 3.3 at those that are of the form \( g(x) = \langle v, x \rangle^2 \) for some \( v \in \mathbb{R}^d \) with \( \|v\| = 1 \). In order to prove Theorems 3.1 – 3.3, we will bound terms \( I_{2,\text{MAP}} \) and \( I_{1,\text{MAP}} \) of eq. (D.9) separately.

### E.1. Proof of Theorem 3.1.

#### E.1.1. Controlling term \( I_{2,\text{MAP}} \).

We wish to obtain a uniform bound on \( I_{2,\text{MAP}} \) for all functions \( g \) that are indicators of measurable sets. Every indicator function is upper-bounded by one, so we can use Remark 22 to obtain:

\[
I_{2,2,\text{MAP}} \leq \tilde{\mathcal{G}}(n, \delta) + \frac{n^{d/2} e^{-n \kappa} \hat{M}_1}{(2\pi)^{d/2} (1 - \hat{\mathcal{G}}_p(n, \delta))} \det \left( \hat{J}_n^p(\bar{\theta}_n, \delta) \right)^{1/2}.
\]

Similarly, since \( |g| \leq 1 \), we can use eq. (D.21) and Lemma D.1 to obtain:

\[
I_{2,1,\text{MAP}} \leq \tilde{\mathcal{G}}(n, \delta) + \frac{n^{d/2} e^{-n \kappa} \hat{M}_1}{(2\pi)^{d/2} (1 - \hat{\mathcal{G}}_p(n, \delta))} \det \left( \hat{J}_n^p(\bar{\theta}_n, \delta) \right)^{1/2}.
\]

From eqs. (E.1) and (E.2), it follows that

\[
I_{2,\text{MAP}} \leq 2 \tilde{\mathcal{G}}(n, \delta) + \frac{2n^{d/2} e^{-n \kappa} \hat{M}_1}{(2\pi)^{d/2} (1 - \hat{\mathcal{G}}_p(n, \delta))} \det \left( \hat{J}_n^p(\bar{\theta}_n, \delta) \right)^{1/2}.
\]

#### E.1.2. Controlling term \( I_{1,\text{MAP}} \) using the log-Sobolev inequality.

Let \( \text{KL}(\cdot \| \cdot) \) denote the Kullback-Leibler divergence (eq. (D.10)) and Fisher (eq. (E.2))

\[
\lambda \text{Sobolev inequality} \quad \text{LSI} \left( \hat{\lambda}_{\text{min}}(\bar{\theta}_n) - \delta \hat{M}_2 \right) \left( \sqrt{n} \right)
\]

This means that, inside the convex set \( \{ t \in \mathbb{R}^d : \|t\| < \sqrt{n} \delta \} \), the density of \( \sqrt{n}(\tilde{\theta}_n - \bar{\theta}_n) \) is \( \left( \hat{\lambda}_{\text{min}}(\bar{\theta}_n) - \delta \hat{M}_2 \right) \)-strongly log-concave (see e.g. [43]). Using Proposition D.1 (the Bakry-Émery criterion), we have that \( \left[ \mathcal{L} \left( \sqrt{n} \left( \tilde{\theta}_n - \bar{\theta}_n \right) \right) \right]_{B_n(\delta \sqrt{n})} \) satisfies the log-Sobolev inequality \( \text{LSI} \left( \hat{\lambda}_{\text{min}}(\bar{\theta}_n) - \delta \hat{M}_2 \right) \left( \sqrt{n} \right) \) (see Definition D.2). By combining the log-Sobolev inequality with Pinsker’s inequality (Proposition D.3) we obtain that, for all functions \( g \), which are indicators of measurable sets,

\[
I_{1,\text{MAP}} \leq \text{TV} \left( \left[ \mathcal{L} \left( \sqrt{n} \left( \tilde{\theta}_n - \bar{\theta}_n \right) \right) \right]_{B_n(\delta \sqrt{n})}, \left[ \mathcal{N}(0, \hat{J}_n(\bar{\theta}_n)^{-1}) \right]_{B_n(\delta \sqrt{n})} \right)
\]

\[
\text{Pinsker's inequality} \quad \frac{1}{2} \text{KL} \left( \left[ \mathcal{N}(0, \hat{J}_n(\bar{\theta}_n)^{-1}) \right]_{B_n(\delta \sqrt{n})} \right) \left( \left[ \mathcal{L} \left( \sqrt{n} \left( \tilde{\theta}_n - \bar{\theta}_n \right) \right) \right]_{B_n(\delta \sqrt{n})} \right)
\]

\[
\text{log-Sobolev inequality} \quad \frac{1}{2} \text{Fisher} \left( \left[ \mathcal{N}(0, \hat{J}_n(\bar{\theta}_n)^{-1}) \right]_{B_n(\delta \sqrt{n})} \right) \left( \left[ \mathcal{L} \left( \sqrt{n} \left( \tilde{\theta}_n - \bar{\theta}_n \right) \right) \right]_{B_n(\delta \sqrt{n})} \right)
\]

\[
2 \sqrt{\hat{\lambda}_{\text{min}}(\bar{\theta}_n) - \delta \hat{M}_2}
\]
\[
\frac{1}{2\sqrt{\lambda_{\text{min}}(\bar{\theta}_n)} - \delta M_2} \cdot \int_{\|u\| \leq \delta \sqrt{n}} \sqrt{\det(\bar{J}_n(\bar{\theta}_n))} e^{-u^T J_n(\bar{\theta}_n)u/2} F_n^{\text{MAP}}(2\pi)^{d/2} \left\| \bar{J}_n(\bar{\theta}_n)u + \frac{\mathbf{T}_n(n^{-1/2}u + \bar{\theta}_n)}{\sqrt{n}} \right\|^2 du
\]

Taylor’s theorem \[\leq \frac{\sqrt{3} \text{Tr} \left[ J_n(\bar{\theta}_n)^{-1} \right] M_2}{4n \left( \lambda_{\text{min}}(\bar{\theta}_n) - \delta M_2 \right) (1 - \mathcal{O}(n, \delta))}, \]

as long as \[n > \frac{\text{Tr} \left[ J_n(\bar{\theta}_n)^{-1} \right]}{\delta^2}, \]

where the last inequality follows from Lemma D.1 and the fact that, for \[U \sim \mathcal{N}(0, \bar{J}_n(\bar{\theta}_n)^{-1}), \] \[\mathbb{E}\|U\|^4 \leq 3 \text{Tr} \left[ \bar{J}_n(\bar{\theta}_n)^{-1} \right]^2. \] To see this last fact, use the spectral theorem and write \[\bar{J}_n(\bar{\theta}_n)^{-1} = P^T \Lambda P, \]

for an orthogonal matrix \(P\) and diagonal matrix \(\Lambda\). Then, it follows that, for \[N \sim \mathcal{N}(0, I_{d \times d}), \]

\[U^T U = N^T P^T \Lambda P N = (PN)^T \Lambda (PN). \]

Since \[(PN) \sim \mathcal{N}(0, I_{d \times d}), \]

note that \[\mathbb{E}(U^T U)^2 = \mathbb{E} \left[ \left( \sum_{i=1}^d \lambda_i^2 \right)^2 \right] \leq 3 \left( \sum_{i=1}^d \lambda_i \right)^2, \]

where \(\lambda_i\)’s are the diagonal elements of \(\Lambda\) and \(\mathcal{N}_\lambda\)’s are independent such that \[N_\lambda \sim \mathcal{N}(0, \lambda_\lambda)\]

for \(\lambda = 1, \ldots, d\).

E.1.3. Conclusion. The result now follows from adding together bounds in eqs. (E.3) and (E.4).

E.2. Proof of Theorem 3.2.

E.2.1. Controlling term \(I_2^{MAP}\). Now we wish to control \(I_2^{\text{MAP}}\) uniformly over all functions \(g\) which are \(1\)-Lipschitz. Let us fix a function \(g\) that is \(1\)-Lipschitz and WLOG set \(g(0) = 0\). In that case \[|g(u)| \leq \|u\| \text{ and, using the notation of Appendix D and eq. (D.21),}\]

\[
I_2^{\text{MAP}} \leq \int_{\|u\| > \delta \sqrt{n}} \|u\| \frac{\sqrt{\det(\bar{J}_n(\bar{\theta}_n))} e^{-u^T J_n(\bar{\theta}_n)u/2}}{(2\pi)^{d/2}} du
\]

\[
+ \frac{n^{d/2+1/2} e^{-n\bar{\kappa} M_1} \left| \det(\bar{J}_n(\bar{\theta}_n, \delta)) \right|^{1/2} \int_{\|u\| > \delta - \|\bar{\theta}_n - \bar{\theta}_n\| \|v - \bar{\theta}_n\| \pi(v)}{(2\pi)^{d/2} (1 - \mathcal{O}(n, \delta))} \]

Now, for \[U \sim \mathcal{N}(0, \bar{J}_n(\bar{\theta}_n)^{-1}), \]

and assuming that \[n > \frac{\text{Tr}[\bar{J}_n(\bar{\theta}_n)^{-1}]}{\delta^2}, \]

\[
\int_{\|u\| > \delta \sqrt{n}} \|u\| \frac{\sqrt{\det(\bar{J}_n(\bar{\theta}_n))} e^{-u^T J_n(\bar{\theta}_n)u/2}}{(2\pi)^{d/2}} du
\]

\[
= \int_0^\infty \mathbb{P} \left[ \|U\|_1 > t \right] dt
\]
\[
\int_0^\infty P \left[ \| U \| > \max(t, \bar{\delta}\sqrt{n}) \right] dt
\]

**Lemma D.1**

\[
\int_0^\infty \exp \left[ -\frac{1}{2} \left( t - \sqrt{\text{Tr} \left( \mathcal{J}_n(\bar{\theta}_n) \right)^{-1} \bar{\lambda}_{\min}(\bar{\theta}_n) \right)^2 \right] \] 
\[
+ \bar{\delta}\sqrt{n} \exp \left[ -\frac{1}{2} \left( \bar{\delta}\sqrt{n} - \sqrt{\text{Tr} \left( \mathcal{J}_n(\bar{\theta}_n) \right)^{-1} \bar{\lambda}_{\min}(\bar{\theta}_n) \right)^2 \right] 
\]
\[
\leq \left( \bar{\delta}\sqrt{n} + \frac{2\pi}{\bar{\lambda}_{\min}(\bar{\theta}_n)} \right) \exp \left[ -\frac{1}{2} \left( \bar{\delta}\sqrt{n} - \sqrt{\text{Tr} \left( \mathcal{J}_n(\bar{\theta}_n) \right)^{-1} \bar{\lambda}_{\min}(\bar{\theta}_n) \right)^2 \right] .
\]

This means that

\[
I_{2,1}^{MAP} \leq \left( \bar{\delta}\sqrt{n} + \frac{2\pi}{\bar{\lambda}_{\min}(\bar{\theta}_n)} \right) \bar{G}(n, \bar{\delta})
\]

\[
+ \frac{n^{d/2 + 1/2} e^{-n\bar{\kappa}} \bar{M}_1 \det \left( \mathcal{J}_{n}^p(\hat{\bar{\theta}}_n, \bar{\delta}) \right)^{1/2} \int_{\| v - \hat{\bar{\theta}}_n \| > \bar{\delta} - \| \bar{\theta}_n \|} \| v - \bar{\theta}_n \| \pi(v) du}{(2\pi)^{d/2} \left( 1 - \bar{G}(n, \bar{\delta}) \right)}.
\]

Now, using eq. (D.23), we have

\[
I_{2,2}^{MAP} \leq \frac{\det \left( \mathcal{J}_{n}^p(\hat{\bar{\theta}}_n, \bar{\delta}) \right)^{1/2} \det \left( \mathcal{J}_{n}^m(\hat{\bar{\theta}}_n, \bar{\delta}) \right)^{-1/2} \sqrt{\text{Tr} \left[ \mathcal{J}_{n}^m(\hat{\bar{\theta}}_n, \bar{\delta})^{-1} \right]}}{1 - \bar{G}(n, \bar{\delta})}
\]

\[
\cdot \left\{ \bar{G}(n, \bar{\delta}) + \frac{n^{d/2} e^{-n\bar{\kappa}} \bar{M}_1 \det \left( \mathcal{J}_{n}^p(\hat{\bar{\theta}}_n, \bar{\delta}) \right)^{1/2}}{(2\pi)^{d/2} \left( 1 - \bar{G}(n, \bar{\delta}) \right)} \right\}.
\]

Adding together bounds in Equations (E.6) and (E.7) now yields a bound on \( I_2^{MAP} \).

**E.2.2. Controlling term \( I_1^{MAP} \) using the log-Sobolev inequality and the transportation-entropy inequality.** As in Appendix E.1.2, we shall use the log-Sobolev inequality \( \text{LSI}(\bar{\lambda}_{\min}(\bar{\theta}_n) - \bar{\delta}\sqrt{\bar{M}_2}) \) for the measure \( \mathcal{L} \left( \sqrt{n} \left( \bar{\theta}_n - \hat{\bar{\theta}}_n \right) \right) \), implied by Proposition D.1. A consequence of the log-Sobolev inequality is that we can apply the transportation-entropy inequality for \( \mathcal{L} \left( \sqrt{n} \left( \bar{\theta}_n - \hat{\bar{\theta}}_n \right) \right) \), given by Proposition D.4. It lets us upper bound the 1- and 2-Wasserstein distances (see eq. (D.14)) by a constant times the KL divergence. The KL divergence is in turn bounded by a constant times the Fisher divergence by the log-Sobolev inequality. Let \( W_2(\cdot, \cdot) \) denote the 2-Wasserstein distance and \( W_1(\cdot, \cdot) \) denote the 1-Wasserstein distance. We have that for all 1-Lipschitz functions \( g \),

\[
I_1^{MAP} \leq W_1 \left( \mathcal{L} \left( \sqrt{n} \left( \bar{\theta}_n - \hat{\bar{\theta}}_n \right) \right) \right)_{B_0(\bar{\delta}\sqrt{n})}, \left[ N(0, \mathcal{J}_n(\hat{\bar{\theta}}_n)^{-1}) \right]_{B_0(\bar{\delta}\sqrt{n})}
\]

\[
eq \text{eq. (D.15)} \leq W_2 \left( \mathcal{L} \left( \sqrt{n} \left( \bar{\theta}_n - \hat{\bar{\theta}}_n \right) \right) \right)_{B_0(\bar{\delta}\sqrt{n})}, \left[ N(0, \mathcal{J}_n(\hat{\bar{\theta}}_n)^{-1}) \right]_{B_0(\bar{\delta}\sqrt{n})}
\]
transportation-entropy inequality
\[\sqrt{2\text{KL} \left( N(0, \bar{J}_n(\bar{\theta}_n)^{-1}) \right)}_{B_o(\delta \sqrt{n})} \left\| \mathcal{L} \left( \sqrt{n} \left( \bar{\theta}_n - \hat{\theta}_n \right) \right) \right\|_{B_o(\delta \sqrt{n})} \]

log-Sobolev inequality
\[\sqrt{\text{Fisher} \left( N(0, \bar{J}_n(\bar{\theta}_n)^{-1}) \right)}_{B_o(\delta \sqrt{n})} \left\| \mathcal{L} \left( \sqrt{n} \left( \bar{\theta}_n - \hat{\theta}_n \right) \right) \right\|_{B_o(\delta \sqrt{n})} \]

\[\leq \frac{1}{\bar{\lambda}_{\text{min}}(\theta_n) - \delta M_2} \int_{\|u\| \leq \delta \sqrt{n}} \sqrt{\text{det} \left( \bar{J}_n(\theta_n) \right)} e^{-u^T \bar{J}_n(\bar{\theta}_n) u/2} \left\| \bar{J}_n(\theta_n) u + \frac{T_n(n^{-1/2} u + \bar{\theta}_n)}{\sqrt{n}} \right\|^2 du \]

Taylor’s theorem
\[\leq \frac{\sqrt{3} \text{Tr} \left[ \bar{J}_n(\theta_n)^{-1} \right] M_2}{2 (\bar{\lambda}_{\text{min}}(\theta_n) - \delta M_2) \sqrt{n (1 - \mathcal{D}(n, \delta))}} \]

as long as \(n > \frac{\text{Tr} \left[ \bar{J}_n(\theta_n)^{-1} \right]}{\delta^2} \), where the last inequality follows from Lemma D.1.

E.2.3. Conclusion. The result now follows from adding together bounds in Equations (E.6) – (E.8).

E.3. Proof of Theorem 3.3.

E.3.1. Controlling term \(I_{2,1}^{MAP}\). Now we wish to control \(I_{2,1}^{MAP}\) uniformly over all functions \(g\) which are of the form \(g(u) = \langle v, u \rangle^2\) for some \(v \in \mathbb{R}^d\) with \(\|v\| = 1\). Such functions satisfy the following property: \(|g(u)| \leq \|u\|^2\). Using the notation of Appendix D and eq. (D.21), we therefore have that:

\[I_{2,1}^{MAP} \leq \int_{\|u\| > \delta \sqrt{n}} \|u\|^2 \sqrt{\text{det} \left( \bar{J}_n(\theta_n) \right)} e^{-u^T \bar{J}_n(\bar{\theta}_n) u/2} \left\| \bar{J}_n(\theta_n) u + \frac{T_n(n^{-1/2} u + \bar{\theta}_n)}{\sqrt{n}} \right\|^2 du \]

\[+ \frac{n^{d/2+1} e^{-n \bar{M}_1} \text{det} \left( \bar{J}_n(\theta_n) \right)}{(2\pi)^{d/2}} \int_{\|v - \bar{\theta}_n\| > \delta - \|\hat{\theta}_n - \bar{\theta}_n\|} \|v - \bar{\theta}_n\|^2 \pi(v) du \]

Now, for \(U \sim N(0, \bar{J}_n(\bar{\theta}_n)^{-1})\), and assuming that \(n > \frac{\text{Tr} \left[ \bar{J}_n(\theta_n)^{-1} \right]}{\delta^2}\),

\[\int_{\|u\| > \delta \sqrt{n}} \|u\|^2 \sqrt{\text{det} \left( \bar{J}_n(\theta_n) \right)} e^{-u^T \bar{J}_n(\bar{\theta}_n) u/2} \left\| \bar{J}_n(\theta_n) u + \frac{T_n(n^{-1/2} u + \bar{\theta}_n)}{\sqrt{n}} \right\|^2 du \]
of all couplings between $(E.9)$

where we have used Lemma D.1. This means that $(E.12)$

Now, let us fix two random vectors: $X$ and $Y$. Let $(\bar{X}, \bar{Y}) \in \gamma$ be such that

\[
\inf_{(Z_1, Z_2) \in \gamma} \mathbb{E} \left[ \| Z_1 - Z_2 \|^2 \right] = \mathbb{E} \left[ \| \bar{X} - \bar{Y} \|^2 \right].
\]
It follows that:
\[
\mathbb{E} \left[ \langle v, X \rangle^2 \right] - \mathbb{E} \left[ \langle v, Y \rangle^2 \right] = \mathbb{E} \left[ \langle v, \bar{X} - \bar{Y} \rangle \langle v, \bar{X} + \bar{Y} \rangle \right] \\
= \mathbb{E} \left[ \langle v, \bar{X} - \bar{Y} \rangle^2 \right] + 2\mathbb{E} \left[ \langle v, \bar{X} - \bar{Y} \rangle \langle v, \bar{Y} \rangle \right] \\
\leq \mathbb{E} \left[ \| \bar{X} - \bar{Y} \|^2 \right] + 2\sqrt{\mathbb{E} \left[ \| \bar{X} - \bar{Y} \|^2 \right] \sqrt{\mathbb{E} \left[ \| Y \|^2 \right]}}.
\]

Therefore, for all functions \( g \) which are of the form \( g(u) = \langle v, u \rangle^2 \) for some \( v \in \mathbb{R}^d \) with \( \|v\| = 1 \), we have that
\[
I_1^{MAP} \leq W_2 \left( \left[ \mathcal{L} \left( \sqrt{n} \left( \hat{\theta}_n - \hat{\theta}_n \right) \right) \right]_{B_0(\delta\sqrt{n})}, \left[ \mathcal{N}(0, J_n(\hat{\theta}_n)^{-1}) \right]_{B_0(\delta\sqrt{n})} \right)^2 \\
+ 2W_2 \left( \left[ \mathcal{L} \left( \sqrt{n} \left( \hat{\theta}_n - \hat{\theta}_n \right) \right) \right]_{B_0(\delta\sqrt{n})}, \left[ \mathcal{N}(0, J_n(\hat{\theta}_n)^{-1}) \right]_{B_0(\delta\sqrt{n})} \right) \sqrt{\text{Tr} \left[ J_n(\hat{\theta}_n)^{-1} \right]} \sqrt{1 - \mathcal{H}(n, \delta)},
\]
if \( n > \frac{\text{Tr}[J_n(\hat{\theta}_n)^{-1}]}{\delta^2} \). The final bound on \( I_1^{MAP} \) may now be obtained by using eq. (E.13).

E.3.3. Conclusion. The result now follows by combining eq. (E.14) with eq. (E.13) and then summing together with eqs. (E.11) and (E.12).

APPENDIX F: PROOFS OF THEOREMS 6.1 – 6.3

Throughout this section we adopt the notation of Appendix D. For any probability measure \( \mu \), we let \( [\mu]_{B_0(\delta\sqrt{n})} \) denote its restriction (truncation) to the ball of radius \( \delta\sqrt{n} \) around 0. In all the proofs below, we wish to control the quantity \( D_2^{M LE} \) of eq. (D.1) for all functions \( g \) that satisfy certain prescribed criteria. In the proof of Theorem 3.1, we look at functions \( g \) that are indicators of measurable sets, in the proof of Theorem 3.2 we look at 1-Lipschitz functions \( g \) and in the proof of Theorem 3.3 at those which are of the form \( g(x) = \langle v, x \rangle^2 \) for some \( v \in \mathbb{R}^d \) with \( \|v\| = 1 \). In order to prove Theorems 6.1 – 6.3, we will bound terms \( I_2^{MLE} \) and \( I_1^{MAP} \) of eq. (D.7) separately.

F.1. Proof of Theorem 6.1.

F.1.1. Controlling term \( I_2^{MLE} \). We wish to obtain a uniform bound on \( I_2^{MLE} \) for all functions \( g \) which are indicators of measurable sets. Every indicator function is upper-bounded by one, so we can use Remark 21 to obtain:
\[
I_2^{MLE} \leq \tilde{\mathcal{G}}(n, \delta) + \frac{n^{d/2}e^{-n\kappa}M_1}{(2\pi)^{d/2}} \left[ \det \left( \hat{J}_n^p(\hat{\theta}_n, \delta) \right) \right]^{1/2}.
\]

Similarly, since \( |g| \leq 1 \), we can use eq. (D.18) and Lemma D.1 to obtain
\[
I_2^{MLE} \leq \tilde{\mathcal{G}}(n, \delta) + \frac{n^{d/2}e^{-n\kappa}M_1}{(2\pi)^{d/2}} \left[ \det \left( \hat{J}_n^p(\hat{\theta}_n, \delta) \right) \right]^{1/2},
\]

which implies that

\[
I_2^{MLE} \leq 2 \sqrt{\tilde{\Theta}(n, \delta)} + \frac{2n^{d/2}e^{-n\kappa} \tilde{M}_1}{(2\pi)^{d/2}(1 - \tilde{\Theta}(n, \delta))^{1/2}} \left| \det \left( J_n^p(\hat{\theta}_n, \delta) \right) \right|^{1/2}.
\]

F.1.2. **Controlling term \( I_1^{MLE} \) using the log-Sobolev inequality.** We shall proceed as we did in Appendix E.1.2. Let \( \text{KL}(\cdot; \cdot) \) denote the Kullback-Leibler divergence (eq. (D.10)) and Fisher \((\cdot; \cdot)\) denote the Fisher divergence (eq. (D.11)). Let \( F_n^{MLE} \) be given by eq. (D.6).

Note that, by Assumption 1, for \( t \) such that \( \|t\| < \sqrt{n} \delta \), we have

\[-n^{-1} L_n''(\hat{\theta}_n + n^{-1/2} t) \geq J_n(\hat{\theta}_n) - \delta M_2 I_{d\times d} \geq (\lambda_{\min}(\hat{\theta}_n) - \delta M_2) I_{d\times d}.
\]

This means that, inside the convex set \( \{ t \in \mathbb{R}^d : \|t\| < \sqrt{n} \delta \} \), the measure whose density, up to a normalizing constant, is given by \( t \mapsto e^{L_n(\hat{\theta}_n + n^{-1/2} t) - \frac{1}{2} \|t\|^2} \chi_{\|t\| < \sqrt{n} \delta}(t) \), is \((\lambda_{\min}(\hat{\theta}_n) - \delta M_2)\)-strongly log-concave (see e.g. [43]). Also, note that

\[
\sup_{\|t\| \leq \delta \sqrt{n}} \pi(n^{-1/2} t + \hat{\theta}_n) \leq \tilde{M}_1 \tilde{M}_1.
\]

Using the **Bakry-Émery criterion**, given by Proposition D.1, and the **Holley-Stroock perturbation principle**, given by Proposition D.2, we therefore have that \( \left[ \sqrt{n} (\hat{\theta}_n - \hat{\theta}_n) \right]_{B_0(\delta \sqrt{n})} \) satisfies the log-Sobolev inequality \( \text{LSI}_{(\lambda_{\min}(\hat{\theta}_n) - \delta M_2)} \) (see Definition D.2). By combining the log-Sobolev inequality with ** Pinsker’s inequality** (Proposition D.3) we obtain that, for all functions \( g \), which are indicators of measurable sets,

\[
I_1^{MLE} \leq \text{TV} \left( \left[ L \left( \sqrt{n} (\hat{\theta}_n - \hat{\theta}_n) \right) \right]_{B_0(\delta \sqrt{n})}, \left[ \mathcal{N}(0, J_n(\hat{\theta}_n)^{-1}) \right]_{B_0(\delta \sqrt{n})} \right) \leq \sqrt{\frac{\tilde{M}_1 \tilde{M}_1}{2 \lambda_{\min}(\hat{\theta}_n) - \delta M_2}} \sqrt{\text{Fisher} \left( \left[ \mathcal{N}(0, J_n(\hat{\theta}_n)^{-1}) \right]_{B_0(\delta \sqrt{n})}, \left[ L \left( \sqrt{n} (\hat{\theta}_n - \hat{\theta}_n) \right) \right]_{B_0(\delta \sqrt{n})} \right)}
\]

\[
\leq \sqrt{\frac{\tilde{M}_1 \tilde{M}_1}{2 \lambda_{\min}(\hat{\theta}_n) - \delta M_2}} \sqrt{\int_{\|u\| \leq \delta \sqrt{n}} \frac{\sqrt{\left| \det \left( J_n(\hat{\theta}_n) \right) \right| e^{-\|J_n(\hat{\theta}_n) u\|^2 / 2} F_n^{MLE}(2\pi)^{d/2}}}{\left| J_n(\hat{\theta}_n) u + L_n(n^{-1/2} u + \hat{\theta}_n) / \sqrt{n} \right|^2} du}
\]

\[
+ \sqrt{\frac{\tilde{M}_1 \tilde{M}_1}{2n (\lambda_{\min}(\hat{\theta}_n) - \delta M_2)}} \]

\[
\text{TV} \left( \left[ L \left( \sqrt{n} (\hat{\theta}_n - \hat{\theta}_n) \right) \right]_{B_0(\delta \sqrt{n})}, \left[ \mathcal{N}(0, J_n(\hat{\theta}_n)^{-1}) \right]_{B_0(\delta \sqrt{n})} \right) \leq \sqrt{\frac{\tilde{M}_1 \tilde{M}_1}{2 \lambda_{\min}(\hat{\theta}_n) - \delta M_2}} \sqrt{\text{Fisher} \left( \left[ \mathcal{N}(0, J_n(\hat{\theta}_n)^{-1}) \right]_{B_0(\delta \sqrt{n})}, \left[ L \left( \sqrt{n} (\hat{\theta}_n - \hat{\theta}_n) \right) \right]_{B_0(\delta \sqrt{n})} \right)}
\]

\[
\leq \sqrt{\frac{\tilde{M}_1 \tilde{M}_1}{2 \lambda_{\min}(\hat{\theta}_n) - \delta M_2}} \sqrt{\int_{\|u\| \leq \delta \sqrt{n}} \frac{\sqrt{\left| \det \left( J_n(\hat{\theta}_n) \right) \right| e^{-\|J_n(\hat{\theta}_n) u\|^2 / 2} F_n^{MLE}(2\pi)^{d/2}}}{\left| J_n(\hat{\theta}_n) u + L_n(n^{-1/2} u + \hat{\theta}_n) / \sqrt{n} \right|^2} du}
\]

\[
+ \sqrt{\frac{\tilde{M}_1 \tilde{M}_1}{2n (\lambda_{\min}(\hat{\theta}_n) - \delta M_2)}} \]
and so

\[ (F.2) \]

\[ \frac{1}{\sqrt{\lambda_{\min}(\hat{\theta}) - \delta M_2}} \sqrt{\frac{\text{Tr} \left[ \tilde{J}_n(\hat{\theta})^{-1} \right]}{M_1^2 M_2^2}} \sqrt{\frac{\sqrt{\det (\hat{J}_n(\hat{\theta}))}}{F_n^{MLE}(2\pi)^{d/2}}} \mathbb{E}\left[ e^{-u^T \tilde{J}_n(\hat{\theta}) u/2} \right] \left\| u \right\|^4 \, du \]

Taylor’s theorem,

\[ F.1.3. \] \textbf{Conclusion.} The result now follows by adding together the bounds in eqs. (F.1) and (F.2).

\[ F.2. \] \textbf{Proof of Theorem 6.2.}

\[ F.2.1. \] \textbf{Controlling term } I_{2,1}^{MLE}. Now we wish to control \( I_{2,1}^{MLE} \) uniformly over all functions \( g \) which are 1-Lipschitz and WLOG set \( g(0) = 0 \). It follows that \( |g(u)| \leq \|u\| \) and, using the notation of Appendix D and eq. (D.18),

\[ I_{2,1}^{MLE} \leq \int_{\|u\| > \delta \sqrt{n}} \left\| \mathbf{M}_1^2 \right\|^{1/2} \left\| \mathbf{M}_1 \right\|^{1/2} \left\| \mathbf{M}_1 \right\|^{1/2} \frac{\sqrt{\det (\hat{J}_n(\hat{\theta}))}}{F_n^{MLE}(2\pi)^{d/2}} \mathbb{E}\left[ e^{-u^T \tilde{J}_n(\hat{\theta}) u/2} \right] \left\| u \right\|^4 \, du.

A calculation similar to eq. (E.5) reveals that

\[ \int_{\|u\| > \delta \sqrt{n}} \left\| u \right\| \frac{\sqrt{\det (\hat{J}_n(\hat{\theta}))}}{F_n^{MLE}(2\pi)^{d/2}} \mathbb{E}\left[ e^{-u^T \tilde{J}_n(\hat{\theta}) u/2} \right] \left\| u \right\| \, du \leq \left( \delta \sqrt{n} + \sqrt{\frac{2\pi}{\lambda_{\min}(\hat{\theta})}} \right) \mathbb{E}\left[ e^{-u^T \tilde{J}_n(\hat{\theta}) u/2} \right] \left\| u \right\|^4 \, du.

and so

\[ I_{2,1}^{MLE} \leq \left( \delta \sqrt{n} + \sqrt{\frac{2\pi}{\lambda_{\min}(\hat{\theta})}} \right) \mathbb{E}\left[ e^{-u^T \tilde{J}_n(\hat{\theta}) u/2} \right] \left\| u \right\|^4 \, du + \frac{n^{d/2+1/2} e^{-\|u\|^2}}{F_n^{MLE}(2\pi)^{d/2}} \mathbb{E}\left[ e^{-u^T \tilde{J}_n(\hat{\theta}) u/2} \right] \left\| u \right\|^4 \, du.

(F.3)
Now, using eq. (D.20), we obtain
\[
I_{2,2}^{MLE} \leq \frac{\tilde{M}_1 \tilde{M}_1 \det \left( \tilde{J}_n^p(\hat{\theta}_n, \delta) \right)^{1/2} \det \left( \tilde{J}_n^m(\hat{\theta}_n, \delta) \right)^{1/2} \sqrt{\text{Tr} \left[ \tilde{J}_n^m(\hat{\theta}_n, \delta)^{-1} \right]}}{1 - \mathscr{D}p(n, \delta)}
\]
(F.4)
\[
\left\{ \hat{h}(n, \delta) + \frac{n^{d/2} e^{-n\kappa} \tilde{M}_1 \det \left( \tilde{J}_n^p(\hat{\theta}_n, \delta) \right)^{1/2}}{(2\pi)^{d/2} (1 - \mathscr{D}p(n, \delta))} \right\}.
\]

Adding together bounds from eqs. (F.3) and (F.4) yields a bound on \(I_2^{MLE}\).

F.2.2. Controlling term \(I_1^{MLE}\) using the log-Sobolev inequality and the transportation-entropy inequality. As in Appendix F.1.2, we shall use the log-Sobolev inequality for the measure \(\left[ \mathcal{L} \left( \sqrt{n} \left( \hat{\theta}_n - \bar{\theta}_n \right) \right) \right]_{B_0(\delta\sqrt{n})}\). A consequence of the log-Sobolev inequality is that we can apply the transportation-entropy inequality for \(\left[ \mathcal{L} \left( \sqrt{n} \left( \hat{\theta}_n - \bar{\theta}_n \right) \right) \right]_{B_0(\delta\sqrt{n})}\) (Proposition D.4), which lets us upper bound the 1- and 2-Wasserstein distances by a constant times the KL divergence. The log-Sobolev inequality then upper-bounds the KL divergence in terms of the Fisher divergence. Let \(W_2(\cdot, \cdot)\) denote the 2-Wasserstein distance and \(W_1(\cdot, \cdot)\) denote the 1-Wasserstein distance. We have that, for all 1-Lipschitz test functions \(g\),

\[
I_1^{MLE} \leq W_1 \left( \left[ \sqrt{n} \left( \hat{\theta}_n - \bar{\theta}_n \right) \right]_{B_0(\delta\sqrt{n})}, \left[ \mathcal{N}(0, \tilde{J}_n(\hat{\theta}_n)^{-1}) \right]_{B_0(\delta\sqrt{n})} \right)
\]

\[
eq W_2 \left( \left[ \sqrt{n} \left( \hat{\theta}_n - \bar{\theta}_n \right) \right]_{B_0(\delta\sqrt{n})}, \left[ \mathcal{N}(0, \tilde{J}_n(\hat{\theta}_n)^{-1}) \right]_{B_0(\delta\sqrt{n})} \right)
\]

\[
\leq \sqrt{2 \tilde{M}_1 \tilde{M}_1 \text{KL} \left( \left[ \mathcal{N}(0, \tilde{J}_n(\hat{\theta}_n)^{-1}) \right]_{B_0(\delta\sqrt{n})}, \left[ \mathcal{L} \left( \sqrt{n} \left( \bar{\theta}_n - \hat{\theta}_n \right) \right) \right]_{B_0(\delta\sqrt{n})} \right)}
\]

\[
\leq \sqrt{\tilde{M}_1 \tilde{M}_1 \text{Fisher} \left( \left[ \mathcal{N}(0, \tilde{J}_n(\hat{\theta}_n)^{-1}) \right]_{B_0(\delta\sqrt{n})}, \left[ \mathcal{L} \left( \sqrt{n} \left( \bar{\theta}_n - \hat{\theta}_n \right) \right) \right]_{B_0(\delta\sqrt{n})} \right)}
\]

\[
\leq \frac{\tilde{M}_1 \tilde{M}_1}{\lambda_{\min}(\hat{\theta}_n) - \delta M_2}
\]

\[
\cdot \int_{\|u\| \leq \delta\sqrt{n}} \sqrt{\det \left( \tilde{J}_n(\hat{\theta}_n)^{-1} \right)} e^{-u^\top \tilde{J}_n(\hat{\theta}_n)u/2} \left\| \tilde{J}_n(\hat{\theta}_n)^{\top} u + \frac{L_n'(n^{-1/2}u + \hat{\theta}_n)}{\sqrt{n}} \right\|^2 du
\]

\[
+ \frac{\tilde{M}_1 \tilde{M}_1}{\sqrt{n}(\lambda_{\min}(\hat{\theta}_n) - \delta M_2)}
\]

\[
\cdot \int_{\|u\| \leq \delta\sqrt{n}} \sqrt{\det \left( \tilde{J}_n(\hat{\theta}_n)^{-1} \right)} e^{-u^\top \tilde{J}_n(\hat{\theta}_n)u/2} \left\| \frac{\pi'(n^{-1/2}u + \hat{\theta}_n)}{\pi(n^{-1/2}u + \hat{\theta}_n)} \right\|^2 du
\]
\[
\begin{align*}
\text{Taylor's theorem} & \leq 
\frac{\hat{M}_1 \hat{M}_1 M_2}{2\sqrt{n} \left( \lambda_{\min}(\hat{\theta}_n) - \delta M_2 \right)} + 
\frac{M_1 \hat{M}_1 M_1}{\sqrt{n} \left( \lambda_{\min}(\hat{\theta}_n) - \delta M_2 \right)} \\
& \leq \sqrt{3} \text{ Tr} \left[ \hat{J}_n(\hat{\theta}_n)^{-1} \right] \frac{\hat{M}_1 \hat{M}_1 M_2}{2 \left( \lambda_{\min}(\hat{\theta}_n) - \delta M_2 \right)} + \frac{M_1 \hat{M}_1 M_1}{\sqrt{n} \left( \lambda_{\min}(\hat{\theta}_n) - \delta M_2 \right)}.
\end{align*}
\]

(F.5)

F.2.3. Conclusion. The result now follows from adding together the bounds in eqs. (F.3) – (F.5).

F.3. Proof of Theorem 6.3.

F.3.1. Controlling term \( I_2^{MLE} \). Now we want to control \( I_2^{MLE} \) uniformly over all functions \( g \) which are of the form \( g(u) = (v, u) \), for some \( v \in \mathbb{R}^d \) with \( \|v\| = 1 \). For such functions we have that \( |g(u)| \leq \|u\|^2 \). Using the notation of Appendix D and eq. (D.18), we have that

\[
I_2^{MLE} \leq \int_{\|u\|>\sqrt{n}} \|u\|^2 \left( \sqrt{\det \hat{J}_n(\hat{\theta}_n)} e^{-u^T \hat{J}_n(\hat{\theta}_n) u/2} \right) \frac{\left( 2\pi \right)^{d/2}}{\left( 2\pi \right)^{d/2}(1 - \hat{\gamma}(n, \delta))} du
\]

\[
+ \frac{n^{d/2+1} e^{-n\kappa} \hat{M}_1}{\left( 2\pi \right)^{d/2}(1 - \hat{\gamma}(n, \delta))} \left( \int_{\|u\|>\delta \sqrt{n}} \|u\|^2 \pi(u + \hat{\theta}_n) du \right)
\]

A calculation similar to eq. (E.10) reveals that

\[
\int_{\|u\|>\sqrt{n}} \|u\|^2 \left( \sqrt{\det \hat{J}_n(\hat{\theta}_n)} e^{-u^T \hat{J}_n(\hat{\theta}_n) u/2} \right) \frac{\left( 2\pi \right)^{d/2}}{\left( 2\pi \right)^{d/2}(1 - \hat{\gamma}(n, \delta))} du \leq \left( \delta^2 n + \frac{2\pi}{\lambda_{\min}(\hat{\theta}_n)} \right) \hat{\gamma}(n, \delta)
\]

and so

\[
I_2^{MLE} \leq \left( \delta^2 n + \sqrt{\frac{2\pi}{\lambda_{\min}(\hat{\theta}_n)}} \right) \hat{\gamma}(n, \delta)
\]

(F.6)

Now, using eq. (D.20), we obtain

\[
\begin{align*}
I_2^{MLE} & \leq \frac{\hat{M}_1 \hat{M}_1 \left( \det \hat{J}_n(\hat{\theta}_n, \delta) \right)^{1/2} \det \left( \hat{J}_n^m(\hat{\theta}_n, \delta) \right)^{-1/2} \text{ Tr} \left[ \hat{J}_n^m(\hat{\theta}_n, \delta)^{-1} \right]}{1 - \hat{\gamma}(n, \delta)}
\end{align*}
\]
Adding together bounds from eqs. (F.6) and (F.7) yields a bound on $I_2^{MLE}$.

**F.3.2. Controlling term $I_1^{MLE}$ using the log-Sobolev inequality and the transportation-entropy inequality.** Note that the calculation in eq. (F.5) yields that

$$W_2 \left( \sqrt{n} \left( \begin{array}{c} 0 \\ \hat{\phi}(\hat{\theta}_n - \hat{\theta}_n) \end{array} \right) \right) \leq \frac{\sqrt{3} \text{Tr} \left[ \hat{J}_n(\hat{\theta}_n)^{-1} \right]}{2 \left( \lambda_{\min}(\hat{\theta}_n) - \delta M_2 \right) \sqrt{n \left( 1 - \hat{\phi}(n, \delta) \right)}} + \frac{1}{\sqrt{n} \left( \lambda_{\min}(\hat{\theta}_n) - \delta M_2 \right)}$$

(F.8)

Therefore, for all functions $g$ which are of the form $g(u) = \langle v, u \rangle^2$ for some $v \in \mathbb{R}^d$ with $\|v\| = 1$, an argument similar to the one that led to eq. (E.14) yields:

$$I_1^{MLE} \leq W_2 \left( \left[ L \left( \sqrt{n} \left( \begin{array}{c} 0 \\ \hat{\phi}(\hat{\theta}_n - \hat{\theta}_n) \end{array} \right) \right) \right]_{B_0(\delta \sqrt{n})}, \left[ N(0, \hat{J}_n(\hat{\theta}_n)^{-1}) \right]_{B_0(\delta \sqrt{n})} \right)^2$$

(F.9)

and the final bound on $I_1^{MLE}$ follows from eq. (F.8).

**F.3.3. Conclusion.** The result now follows from combining eqs. (F.8) and (F.9) and adding together with eqs. (F.6) and (F.7).

**APPENDIX G: PROOF OF THEOREM A.1**

In this section, we concentrate on the univariate context (i.e., on $d = 1$). We shall apply Stein’s method, in the framework described in [15, Section 2.1]. Before we do that, however, let us recall that we want to upper-bound the quantity $D_g^{MLE}$ given by eq. (D.1) for all functions $g$ for which the two expectations in eq. (D.1) exist. Recall the definition of $C_n^{MLE}$ from eq. (D.5) and let:

$$h(t) = h_g^{MLE}(t) = g(t) - \frac{n^{-1/2}}{C_n} \int_{-\delta \sqrt{n}}^{\delta \sqrt{n}} g(u) \Pi_n(n^{-1/2} u + \hat{\theta}_n) du.$$

We can repeat the calculation leading to eq. (D.7), without dividing the first term after the first inequality by $I_n^{MLE}$. We then obtain:

$$D_g^{MLE} \leq \left| \int_{-\delta \sqrt{n}}^{\delta \sqrt{n}} h(u) \frac{e^{-t^2/(2\sigma_n^2)}}{\sqrt{2\pi \sigma_n^2}} du \right| + \left| \int_{-\delta \sqrt{n}}^{\delta \sqrt{n}} h(u) \left[ \frac{e^{-u^2/(2\sigma_n^2)}}{\sqrt{2\pi \sigma_n^2}} - n^{-1/2} \Pi_n(n^{-1/2} u + \hat{\theta}_n) \right] du \right|$$

$$=: \hat{I}_1 + \hat{I}_2.$$

We will bound $\hat{I}_1$ and $\hat{I}_2$ separately.
G.1. Controlling term \( \tilde{I}_2 \). Note that \( \tilde{I}_2 \) is the same as \( I_2^{MLE} \) defined by eq. (D.7), for \( d = 1 \). We will use the calculations leading to eqs. (D.18) and (D.19). Instead of using Lemma D.1, we will, however, apply the standard one-dimensional Gaussian concentration inequality, which says that, for \( Z_n \sim \mathcal{N}(0, \sigma_n^2) \),

\[
\mathbb{P} \left[ |Z_n| > \delta \sqrt{n} \right] \leq 2e^{-\delta^2 n/(2\sigma_n^2)}.
\]

We obtain

\[
\tilde{I}_2 \leq \left| \int_{|u| > \delta \sqrt{n}} g(u) e^{-u^2/(2\sigma_n^2)} \sqrt{2\pi \sigma_n^2} \, du \right|
+ \frac{n^{1/2} e^{-nk} \hat{M}_1 \left( \frac{1}{\sigma_n^2} + \frac{5M \delta}{3} \right)^{1/2} \int_{|u| > \delta} |g(u \sqrt{n})| \pi(u + \hat{\theta}_n) \, du}{\sqrt{2\pi} \left\{ 1 - 2 \exp \left[ -\frac{1}{2} \left( \frac{1}{\sigma_n^2} + \frac{M \delta}{3} \right) \delta^2 n \right] \right\}}
+ \frac{\hat{M}_1 \hat{M}_1 \left( \frac{1}{\sigma_n^2} + \frac{\delta M \delta}{3} \right)^{1/2} \int_{|t| \leq \delta \sqrt{n}} |g(t)| e^{-\frac{1}{2} \left( \frac{1}{\sigma_n^2} + \frac{M \delta}{3} \right) t^2} \, dt}{\sqrt{2\pi} \left\{ 1 - 2 \exp \left[ -\frac{1}{2} \left( \frac{1}{\sigma_n^2} + \frac{M \delta}{3} \right) \delta^2 n \right] \right\}}
\]

(G.2)

\[
\left\{ 2e^{-\delta^2 n/(2\sigma_n^2)} + \frac{n^{1/2} e^{-nk} \hat{M}_1 \left( \frac{1}{\sigma_n^2} + \frac{\delta M \delta}{3} \right)^{1/2}}{\sqrt{2\pi} \left\{ 1 - 2 \exp \left[ -\frac{1}{2} \left( \frac{1}{\sigma_n^2} + \frac{M \delta}{3} \right) \delta^2 n \right] \right\}} \right\}.
\]

G.2. Controlling term \( \tilde{I}_1 \) using Stein’s method. In this section we will use Stein’s method in the framework of [15, Section 2.1]. Note that, by integration by parts, for all continuous functions \( f : [-\delta \sqrt{n}, \delta \sqrt{n}] \to \mathbb{R} \) which are differentiable on \( (-\delta \sqrt{n}, \delta \sqrt{n}) \) and satisfy \( f(-\delta \sqrt{n}) = f(\delta \sqrt{n}) \), we have

\[
\int_{-\delta \sqrt{n}}^{\delta \sqrt{n}} f'(t) \frac{e^{-t^2/(2\sigma_n^2)}}{\sqrt{2\pi \sigma_n^2}} \, dt = \int_{-\delta \sqrt{n}}^{\delta \sqrt{n}} t f(t) \frac{e^{-t^2/(2\sigma_n^2)}}{\sigma_n^2 \sqrt{2\pi \sigma_n^2}} \, dt.
\]

Now, for our function \( h \), given by eq. (G.1), let

\[
f(t) := \begin{cases} 
\frac{1}{\Pi_n(n^{-1/2}t + \hat{\theta}_n)} t^{\frac{1}{2}} - \delta \sqrt{n} h(u) \Pi_n(n^{-1/2}u + \hat{\theta}_n) \, du, & \text{if } t \in (-\delta \sqrt{n}, \delta \sqrt{n}) \\
0, & \text{otherwise}.
\end{cases}
\]

Now, by Taylor’s theorem, we obtain that, for some \( c \in (0, 1) \),

\[
\tilde{I}_1 \leq \int_{-\delta \sqrt{n}}^{\delta \sqrt{n}} \left[ f'(t) + f(t) \left( \frac{d}{dt} L_n(n^{-1/2}t + \hat{\theta}_n) + \frac{d}{dt} \log \Pi_n(n^{-1/2}t + \hat{\theta}_n) \right) \right] \frac{e^{-t^2/(2\sigma_n^2)}}{\sqrt{2\pi \sigma_n^2}} \, dt
\]

\[
\leq \int_{-\delta \sqrt{n}}^{\delta \sqrt{n}} \left[ f'(t) + f(t) \left( \frac{L_n'(\hat{\theta}_n)}{\sqrt{n}} + \frac{tL_n''(\hat{\theta}_n)}{n} + \frac{t^2}{2n^{3/2}} L_n'''(\hat{\theta}_n + cn^{-1/2}t) \right) \right] \frac{e^{-t^2/(2\sigma_n^2)}}{\sqrt{2\pi \sigma_n^2}} \, dt
\]
\begin{align*}
&\quad + \frac{1}{\sqrt{n}} \left| \int_{-\delta \sqrt{n}}^{\delta \sqrt{n}} f(t) \frac{\pi'(n^{-1/2} t + \hat{\theta}_n)}{\pi(n^{-1/2} t + \hat{\theta}_n)} e^{-t^2/(2\sigma_n^2)} dt \right| \\
&\quad = \left| \int_{-\delta \sqrt{n}}^{\delta \sqrt{n}} \left[ f'(t) + f(t) \left( -\frac{t}{\sigma_n^2} + \frac{t^2}{2\sigma_n^2} \right) \right] e^{-t^2/(2\sigma_n^2)} dt \right| \\
&\quad \quad + \frac{1}{\sqrt{n}} \left| \int_{-\delta \sqrt{n}}^{\delta \sqrt{n}} f(t) \frac{\pi'(n^{-1/2} t + \hat{\theta}_n)}{\pi(n^{-1/2} t + \hat{\theta}_n)} e^{-t^2/(2\sigma_n^2)} dt \right| \\
&\quad \leq \left| \int_{-\delta \sqrt{n}}^{\delta \sqrt{n}} \left[ f'(t) - \frac{tf''(n^{-1/2} t + \hat{\theta}_n)}{\sigma_n^2} \right] e^{-t^2/(2\sigma_n^2)} dt \right| \\
&\quad \quad + M_2 \left| \int_{-\delta \sqrt{n}}^{\delta \sqrt{n}} |t|^2 f(t) e^{-t^2/(2\sigma_n^2)} dt \right| \\
&\quad \quad + \frac{M_1}{\sqrt{n}} \left| \int_{-\delta \sqrt{n}}^{\delta \sqrt{n}} f(t) e^{-t^2/(2\sigma_n^2)} dt \right| \\
&\quad =: I_{1.1} + I_{1.2} + I_{1.3} + I_{1.4}.
\end{align*}

Now, note that, for some \( c_1, c_2 \in (0, 1), \)
\begin{align*}
I_{1.1} &\leq \frac{M_2}{2\sqrt{2\pi\sigma_n^2 n}} \int_{-\delta \sqrt{n}}^{0} \left| t^2 e^{-t^2/(2\sigma_n^2)} \left( \frac{\pi'(n^{-1/2} t + \hat{\theta}_n)}{\pi(n^{-1/2} t + \hat{\theta}_n)} \right) \int_{-\delta \sqrt{n}}^{t} h(u) \Pi_n(n^{-1/2} u + \hat{\theta}_n) du \right| dt \\
&\quad + \frac{M_1}{2\sqrt{2\pi\sigma_n^2 n}} \int_{-\delta \sqrt{n}}^{0} \left| h(u) \int_{-\delta \sqrt{n}}^{t} \left( \frac{\pi'(n^{-1/2} t + \hat{\theta}_n)}{\pi(n^{-1/2} t + \hat{\theta}_n)} \right) \right| dt \\
&\quad \leq \frac{M_1 M_2}{2\sqrt{2\pi\sigma_n^2 n}} \int_{-\delta \sqrt{n}}^{0} \left| h(u) \int_{-\delta \sqrt{n}}^{t} t^2 e^{-t^2/(2\sigma_n^2)} \exp \left[ L_n(\hat{\theta}_n) + \frac{u}{\sqrt{n}} L_n'(\hat{\theta}_n) + \frac{u^2}{2n} L_n''(\hat{\theta}_n) + \frac{u^3}{6n^{3/2}} L_n'''(\hat{\theta}_n + c_1 n^{-1/2} u) \right] dt \\
&\quad \quad + \frac{u^2}{2n} L_n''(\hat{\theta}_n) \right| du.
\end{align*}
\[ \exp \left[ -L_n(\hat{\theta}_n) - \frac{t}{\sqrt{n}} L'_n(\hat{\theta}_n) - \frac{t^2}{2n} L''_n(\hat{\theta}_n) - \frac{t^3}{6n^{3/2}} L'''_n(\hat{\theta}_n + c_2n^{-1/2}t) \right] dt \, du \\
\leq \frac{\tilde{M}_1 \tilde{M}_1 M_2}{2\sqrt{2\pi \sigma_n^2}} \int_{-\sqrt{n}}^{\sqrt{n}} |h(u)|e^{-u^2/(2\sigma_n^2)}\, e^{\delta M_2 u^2/6} \int_u^0 t^2 e^{\delta M_2 t^2/6} \, dt \, du \\
\leq \frac{3\tilde{M}_1 \tilde{M}_1}{\delta \sqrt{2\pi \sigma_n^2}} \int_{-\sqrt{n}}^{\sqrt{n}} |uh(u)| \left( e^{-\left(\frac{1}{2\pi} - \frac{\delta M_2}{3}\right)u^2} - e^{-\left(\frac{1}{2\pi} - \frac{\delta M_2}{6}\right)u^2} \right) du. \]

By a similar argument,
\[ \tilde{I}_{1,2} \leq \frac{3\tilde{M}_1 \tilde{M}_1}{\delta \sqrt{2\pi \sigma_n^2}} \int_{-\sqrt{n}}^{\sqrt{n}} |uh(u)| \left( e^{-\left(\frac{1}{2\pi} - \frac{\delta M_2}{3}\right)u^2} - e^{-\left(\frac{1}{2\pi} - \frac{\delta M_2}{6}\right)u^2} \right) du. \]
\[ \tilde{I}_{1,3} \leq \frac{\tilde{M}_1 \tilde{M}_1 M_1}{\sqrt{2\pi \sigma_n^2}} \int_{-\sqrt{n}}^{\sqrt{n}} |uh(u)| e^{-\left(\frac{1}{2\pi} - \frac{\delta M_2}{3}\right)u^2} du. \]
\[ \tilde{I}_{1,4} \leq \frac{\tilde{M}_1 \tilde{M}_1 M_1}{\sqrt{2\pi \sigma_n^2}} \int_{-\sqrt{n}}^{\sqrt{n}} |uh(u)| e^{-\left(\frac{1}{2\pi} - \frac{\delta M_2}{6}\right)u^2} du. \]

Therefore,
\[ \tilde{I}_1 \leq \frac{2\tilde{M}_1 \tilde{M}_1 (M_1 + \frac{3}{\delta})}{\sqrt{2\pi \sigma_n^2}} \int_{-\sqrt{n}}^{\sqrt{n}} |uh(u)| e^{-\left(\frac{1}{2\pi} - \frac{\delta M_2}{3}\right)u^2} du - \frac{6\tilde{M}_1 \tilde{M}_1}{\delta \sqrt{2\pi \sigma_n^2}} \int_{-\sqrt{n}}^{\sqrt{n}} |uh(u)| e^{-\left(\frac{1}{2\pi} - \frac{\delta M_2}{6}\right)u^2} du. \] (G.6)

Now, by Taylor’s expansion:
\[ \frac{n^{-1/2}}{C_{MLE}^n} \left| \int_{-\sqrt{n}}^{\sqrt{n}} g(t) \Pi_n(n^{-1/2}t + \hat{\theta}_n) dt \right| \leq \frac{\int_{-\sqrt{n}}^{\sqrt{n}} |g(t)| \Pi_n(n^{-1/2}t + \hat{\theta}_n) dt}{\int_{-\sqrt{n}}^{\sqrt{n}} \Pi_n(n^{-1/2}u + \hat{\theta}_n) du} \leq M_1 \tilde{M}_1 \int_{-\sqrt{n}}^{\sqrt{n}} |g(t)| e^{-(1/(2\sigma_n^2) - \delta M_2/6)u^2} dt \frac{\int_{-\sqrt{n}}^{\sqrt{n}} e^{-(1/(2\sigma_n^2) + \delta M_2/6)u^2} du}{\sqrt{2\pi} \left( 1 - 2e^{-\delta^2(n^{-1/(2\sigma_n^2)}) + \delta M_2/6} \right)} \] (G.7)

Equations (G.6) and (G.7), together with a standard expression for the normal first absolute moment now yield that
\[ \tilde{I}_1 \leq \frac{2\tilde{M}_1 \tilde{M}_1}{\sqrt{2\pi \sigma_n^2}} \int_{-\sqrt{n}}^{\sqrt{n}} |ug(u)| \left[ \left( M_1 + \frac{3}{\delta} \right) e^{-\left(\frac{1}{2\pi} - \frac{\delta M_2}{3}\right)u^2} - \frac{3}{\delta} e^{-\left(\frac{1}{2\pi} - \frac{\delta M_2}{6}\right)u^2} \right] du + 2\frac{\left( \frac{1}{2\sigma_n^2} + \frac{\delta M_2}{6} \right) M_1 \tilde{M}_1}{\pi \sqrt{\sigma_n^2}} \int_{-\sqrt{n}}^{\sqrt{n}} |g(u)| e^{-(1/(2\sigma_n^2) - \delta M_2/6)u^2} du \left( \frac{1}{2\sigma_n^2} + \frac{\delta M_2}{6} \right) \pi \sqrt{\sigma_n^2} \left( 1 - 2e^{-\delta^2(n^{-1/(2\sigma_n^2)}) + \delta M_2/6} \right) \sqrt{n} \] (G.8)
G.3. Conclusion. The final bound now follows from eqs. (G.2) and (G.8).

APPENDIX H: MORE DETAIL ON THE EXAMPLES

H.1. More detail on Example 1. Note that the function $L_n$ is concave, so its global maximum (i.e. the MLE) $\hat{\theta}_n$ exists as long as the data are not linearly separable. The log-posterior is given by

$$L_n(\theta) = -\sum_{i=1}^{n} \rho(Y_i X_i^T \theta) - \frac{\|\theta\|^2}{2} - \frac{d}{2} \log(2\pi).$$

Note that $L_n$ is also concave so its global maximum (i.e. the MAP) $\bar{\theta}_n$ exists as long as the data are not linearly separable. Note also that $L_n''(\theta) = -\sum_{i=1}^{n} \rho''(Y_i X_i^T \theta) X_i X_i^T$ and $(\log \pi)''(\theta) = -I_{d \times d}$.

H.1.1. Deriving the order of our total variation distance bound. Consider the following two results:

**Lemma H.1** (cf. [47, Lemma 4]). Suppose that $d/n < 1$. Let $H(\epsilon) = -\epsilon \log \epsilon - (1 - \epsilon) \log(1 - \epsilon)$. Then there exists a constant $\epsilon_0$ such that for all $0 \leq \epsilon \leq \epsilon_0$, with probability at least $1 - 2 \exp(-nH(\epsilon)) - 2 \exp(-n/2)$, the following matrix inequality

$$-L_n''(\theta) \succeq \frac{\exp(3\|\theta\|/\sqrt{\epsilon})}{(1 + \exp(3\|\theta\|/\sqrt{\epsilon}))^2} \left( \sqrt{1 - \frac{d}{n}} - 2 \frac{H(\epsilon)}{1 - \epsilon} \right) I_{d \times d}$$

holds simultaneously for all $\theta \in \mathbb{R}^d$.

**Proof.** This follows directly from [47, Lemma 4], using the fact that

$$\inf_{z:|z| \leq \frac{3\|\theta\|}{\sqrt{\epsilon}}} \rho''(z) = \frac{\exp(3\|\theta\|/\sqrt{\epsilon})}{(1 + \exp(3\|\theta\|/\sqrt{\epsilon}))^2}.$$

**Lemma H.2** ([47, Theorem 4b]). Fix any small constant $\epsilon > 0$. Then, there exist universal constants $c_1, c_2, C_2 > 0$, such that if $\frac{d}{n} < \frac{1}{2} - c_1 \epsilon^{3/4}$ then the MLE $\hat{\theta}_n$ associated to $L_n$ satisfies

$$\|\hat{\theta}_n\| < \frac{4 \log 2}{\epsilon^2}$$

with probability at least $1 - C_2 \exp(-c_2 \epsilon^2 n)$.

Note that functions $L_n$ and $\frac{G_n}{n}$ are continuously differentiable and strictly concave and converge almost surely to the same (concave) limit as $n \to \infty$. Therefore, if $\|\bar{\theta}_n\|$ is uniformly bounded, then so is $\|\hat{\theta}_n\|$. Therefore, by Lemma H.2, if $\frac{d}{n} < \frac{1}{2}$ then $\|\hat{\theta}_n\|$ is upper-bounded, uniformly in $n$ and $d$, with high probability. As $\frac{G_n}{n}(\theta) = L_n''(\theta) - I_{d \times d}$, we can now apply Lemma H.1 to conclude that, when $\frac{d}{n} < \frac{1}{2}$, then $\lambda_{\min}(\bar{\theta}_n)$ is lower bounded uniformly in $d$ and $n$, with high probability.

Now, consider the following result, which is a direct consequence of [28, Lemma 3.2]:
Lemma H.3 (cf. [28, Lemma 3.2]). Suppose that \( d \leq n \leq e^{\sqrt{d}} \). Then, there exist absolute constants \( B_1, B_2, C > 0 \), such that

\[
\sup_{\theta \in \mathbb{R}^d} \left\| \frac{L_n''(\theta)}{n} \right\| \leq C \left( 1 + \frac{d^{3/2}}{n} \right);
\]

\[
\sup_{\theta \in \mathbb{R}^d} \left\| \frac{L_n''(\theta)}{n} \right\| \leq C \left( 1 + \frac{d^{3/2}}{n} \right)
\]

with probability at least \( 1 - B_1 \exp\left(-B_2 \sqrt{nd}/\log(2n/d)\right) \).

As discussed in Section 4.2 and in Appendix H.1.3 below, as long as \( \frac{d \log n}{n} \to \infty \), the leading term in our bound on the total variation distance from Theorem 3.1 is of order \( C_d \sqrt{d^2/n} \), where \( C_d \leq \frac{M_2}{\lambda_{\min}(\theta_0) \sqrt{\lambda_{\min}(\theta_0) - \delta M_2}} \). As described above, we have established via Lemmas H.1 and H.2 that \( \lambda_{\min}(\theta_0) \) is lower-bounded with high probability. We have also established via Lemma H.3 that \( M_2 \) is upper-bounded with high probability. Moreover, we can choose \( \delta > 0 \) to be independent of \( n \) and \( d \) and arbitrarily small. Therefore \( C_d \) is with high probability upper-bounded by a finite constant not depending on \( d \) or \( n \). It follows that the leading order term in our bound on the total variation distance from Theorem 3.1 is with high probability upper bounded by a universal constant multiplied by \( \sqrt{d^2/n} \).

H.1.2. Lower bound on the effective dimension. Now, we shall show that, in the setup of Example 1,

\[
d_{\text{eff}} := \text{Tr}\left\{ \left( J_n(\hat{\theta}_n) + \frac{(\log \pi)'(\hat{\theta}_n)}{n} \right) J_n(\hat{\theta}_n)^{-1} \right\} \geq d \left( 1 - \frac{1}{n \lambda_{\min}(\hat{\theta}_n)} \right),
\]

where \( d_{\text{eff}} \) denotes the effective dimension introduced in [46]. Note that

\[
d_{\text{eff}} = \text{Tr}\left\{ \left( J_n(\hat{\theta}_n) - \frac{1}{n} I_{d \times d} \right) J_n(\hat{\theta}_n)^{-1} \right\}
\]

\[
= \text{Tr}\left\{ I_{d \times d} - \frac{1}{n} J_n(\hat{\theta}_n)^{-1} \right\}
\]

\[
= d - \frac{1}{n} \text{Tr}\left\{ J_n(\hat{\theta}_n)^{-1} \right\}
\]

\[
\geq d \left( 1 - \frac{1}{n \lambda_{\min}(\hat{\theta}_n)} \right).
\]

The fact that \( \lambda_{\min}(\hat{\theta}_n) \) is lower-bounded by a positive number not depending on \( n \) of \( d \) with high probability follows from Lemmas H.1 and H.2 presented above and is discussed directly below Lemma H.2 in Appendix H.1.1 above.

H.1.3. Checking that the model satisfies the assumptions of Theorem 3.1. We retain the assumption that \( 2d < n < e^{\sqrt{d}} \) and assume that \( \frac{d \log n}{n} \to \infty \). As discussed above, the MLE \( \hat{\theta}_n \) and MAP \( \hat{\theta}_n \) exist and are bounded with high probability by Lemma H.2. Assumption 1 is satisfied with high probability, (for instance with \( \delta = 1 \)), with \( M_2 \) not depending on \( n \) or \( d \), by Lemma H.3. The Gaussian prior trivially satisfies Assumption 2 for any \( \delta \leq 1 \). Moreover, \( M_1 \) is with high probability bounded by a constant multiplied by \( (2\pi)^{d/2} \), as \( ||\hat{\theta}_n|| \) is upper bounded by a constant independent of \( n \) and \( d \) with high probability. By Lemma H.3, with high probability Assumption 3 is also satisfied for any fixed \( \delta \) not depending on \( n \) or \( d \) and with \( M_2 \) not depending on \( n \) or \( d \).
As discussed above, $\frac{\lambda_n}{n}$ and $\frac{L_n}{n}$ are continuously differentiable and concave and converge almost surely to the same (concave) limit as $n \to \infty$. By Lemma H.2, $\hat{\theta}_n$ and $\bar{\theta}_n$ are bounded with high probability. Therefore, with high probability, $\|\theta_n - \bar{\theta}_n\| < \delta$, for any fixed $\delta > 0$ and for large enough $n$. Moreover, with high probability, $\bar{\lambda}_{\min}(\bar{\theta}_n)$ and $\lambda_{\min}(\bar{\theta}_n)$ are lower bounded by a positive number not depending on $n$ or $d$, by Lemmas H.1 and H.2. Moreover $\frac{\text{Tr}[J_n(\theta_n)^{-1}]}{n} \leq \frac{d}{n\lambda_{\min}(\theta_n)}$ and $\frac{\text{Tr}[J_n(\bar{\theta}_n)^{-1}]}{n} \leq \frac{d}{n\lambda_{\min}(\theta_n)}$. Therefore, with high probability, Assumption 4 is satisfied, for large enough $n$ and for any fixed $\delta$ and $\bar{\delta}$, not depending on $n$ or $d$.

Recall that $\bar{\lambda}_{\min}(\theta_n)$ is lower bounded by a positive number not depending on $n$ or $d$ with high probability, if $d < \frac{n}{2}$. Therefore, Assumption 5 is satisfied for small enough $\bar{\delta}$. Assume $n$ is large enough so that $\bar{\delta} - \|\hat{\theta}_n - \bar{\theta}_n\| > 0$. Assumption 6 is satisfied with high probability by the following reasoning, which uses the strict concavity of $L_n$:

$$
\sup_{\theta: \|\theta - \hat{\theta}_n\| > \delta - \|\hat{\theta}_n - \bar{\theta}_n\|} \frac{L_n(\theta) - L_n(\hat{\theta}_n)}{n} \leq \sup_{\theta: \|\theta - \hat{\theta}_n\| = \delta - \|\hat{\theta}_n - \bar{\theta}_n\|} \frac{L_n(\theta) - L_n(\hat{\theta}_n)}{n} \\
\leq \sup_{\theta: \|\theta - \hat{\theta}_n\| = \delta - \|\hat{\theta}_n - \bar{\theta}_n\|} \left\{ -\frac{1}{2} (\theta - \hat{\theta}_n)^T J_n(\hat{\theta}_n) (\theta - \hat{\theta}_n) \right\} + \frac{M_2 (\bar{\delta} - \|\hat{\theta}_n - \bar{\theta}_n\|)^3}{6} \\
\leq -\frac{1}{2} \bar{\lambda}_{\min}(\theta_n) (\bar{\delta} - \|\hat{\theta}_n - \bar{\theta}_n\|)^2 + \frac{M_2 (\bar{\delta} - \|\hat{\theta}_n - \bar{\theta}_n\|)^3}{6} =: -\bar{\kappa}.
$$

With high probability, $\bar{\kappa}$ is lower bounded by a positive number not depending on $n$ or $d$, for any fixed $\bar{\delta}$ independent of $n$ and $d$, as long as $n$ is large enough.

Now, we have shown that we can make choices of $\bar{\delta}$ and $\bar{\kappa}$ independent of $n$ and $d$, such that with high probability the assumptions of Theorem 3.1 hold for large enough $n$. For such a choice of $\bar{\delta}$ we have that $\bar{\delta} \gg \frac{\sqrt{\log n}}{\sqrt{n\lambda_{\min}(\theta_n)}}$. By our assumption $\frac{d \log n}{n} \overset{n \to \infty}{\to} 0$, we also have that $\bar{\kappa} \gg \frac{\log n}{n} \cdot \frac{d+1}{2}$. Moreover, we have established that $\tilde{M}_1$ is with high probability bounded by a universal constant multiplied by $(2\pi)^{d/2}$. Also, note that

$$
\left| \det \left( J_n^p(\hat{\theta}_n, \delta) \right) \right| = \left| \det \left( \sum_{i=1}^{n} \frac{\rho''(Y_i X_i^T \hat{\theta}_n) X_i X_i^T}{n} + (\delta M_2/3) I_{d \times d} \right) \right|
$$

Note that

$$
\left\| \sum_{i=1}^{n} \frac{\rho''(Y_i X_i^T \hat{\theta}_n) X_i X_i^T}{n} \right\|_{op} \leq \frac{1}{4n} \sum_{i=1}^{n} \|X_i X_i^T\|_{op} \leq \frac{1}{4n} \sum_{i=1}^{n} \|X_i\|^2.
$$

Therefore,

$$
\log \left[ \left| \det \left( J_n^p(\hat{\theta}_n, \delta) \right) \right|^{1/2} \right] \leq \frac{d}{2} \log \left[ \frac{1}{4n} \sum_{i=1}^{n} \|X_i\|^2 + \delta M_2/3 \right].
$$

It follows that $\bar{\kappa} \gg \frac{1}{n} \log \left( \tilde{M}_1 \left| \det \left( J_n^p(\hat{\theta}_n, \delta) \right) \right|^{1/2} \right)$ with high probability. Hence, by the discussion of Section 4.2, with high probability, the bound in Theorem 3.1 is of the same order as that of the first summand $A_1 n^{-1/2}$.

H.2. Calculations for Section 7.1.
H.2.1. The MLE-centric approach. Let \( X_1, \ldots, X_n \geq 0 \) be our data and assume that their sum is positive. We have

\[
L_n(\theta) = -n \theta + (\log \theta) \left( \sum_{i=1}^{n} X_i \right) - \sum_{i=1}^{n} \log (X_i!)
\]

The MLE is given by \( \hat{\theta}_n = \bar{X}_n := \frac{1}{n} \sum_{i=1}^{n} X_i \). Also,

\[
L'_n(\theta) = -n + \frac{n \bar{X}_n}{\theta}, \quad L''_n(\theta) = -\frac{n \bar{X}_n}{\theta^2}, \quad L'''_n(\theta) = \frac{2n \bar{X}_n}{\theta^3}.
\]

We have \( \sigma_n^2 := \hat{J}_n(\hat{\theta}_n)^{-1} = \frac{\hat{\theta}_n^2}{\bar{X}_n} = |\bar{X}_n| \).

Now, for \( c \in (0, 1) \) and \( \delta = \epsilon \hat{\theta}_n \), we have that for \( \theta \in (\hat{\theta}_n - \delta, \hat{\theta}_n + \delta) = (\bar{X}_n - c\bar{X}_n, \bar{X}_n + c\bar{X}_n) \),

\[
\frac{|L''_n(\theta)|}{n} = \frac{2\bar{X}_n}{|\theta|^3} \leq \frac{2}{(1-c)^3 \bar{X}_n^2} =: M_2
\]

Moreover, for \( \theta \), such that \( \left| \theta - \hat{\theta}_n \right| > \delta = \epsilon \bar{X}_n \), i.e. for \( \theta > \bar{X}_n + c\bar{X}_n \) or \( \theta < \bar{X}_n - c\bar{X}_n \),

\[
\frac{L_n(\theta) - L_n(\hat{\theta}_n)}{n} \leq \max \left\{ \frac{L_n((1+c)\bar{X}_n) - L_n(\bar{X}_n)}{n}, \frac{L_n((1-c)\bar{X}_n) - L_n(\bar{X}_n)}{n} \right\}
\]

\[
\leq \bar{X}_n \cdot \max \{ \log(1+c) - c, \log(1-c) + c \} = \left[ \log(1+c) - c \right] \bar{X}_n =: -\kappa.
\]

We also need to make sure that \( \hat{J}_n(\theta_n) > \delta M_2 \), i.e. that \( \frac{1}{\bar{X}_n} > \frac{2c}{\sqrt{n} \bar{X}_n^2} \), which is true for all \( 0 < c \leq 0.229 \).

Now, the gamma prior with shape \( \alpha \) and rate \( \beta \) satisfies

\[
\pi'(\theta) = \frac{\beta^\alpha}{\Gamma(\alpha)} \left[ (\alpha - 1)\theta^{\alpha - 2} e^{-\beta \theta} - \beta \theta^{\alpha - 1} e^{-\beta \theta} \right].
\]

Suppose that \( \alpha < 1 \), then

\[
\sup_{\theta \in ((1-c)\bar{X}_n, (1+c)\bar{X}_n)} |\pi'(\theta)| \leq \frac{\beta^\alpha}{\Gamma(\alpha)} e^{-\beta(1-c)\bar{X}_n} \bar{X}_n^{\alpha - 2} (1-c)^{\alpha - 2} \left[ (1-\alpha) + \beta(1-c) \bar{X}_n \right]
\]

\[
\sup_{\theta \in ((1-c)\bar{X}_n, (1+c)\bar{X}_n)} |\pi(\theta)| \leq \frac{\beta^\alpha}{\Gamma(\alpha)} (1-c)^{\alpha - 1} \bar{X}_n^{\alpha - 1} e^{-\beta(1-c)\bar{X}_n}
\]

\[
\sup_{\theta \in ((1-c)\bar{X}_n, (1+c)\bar{X}_n)} \frac{1}{|\pi(\theta)|} \leq \frac{\Gamma(\alpha)}{\beta^\alpha} (1+c)^{1-\alpha} \bar{X}_n^{1-\alpha} e^{\beta(1+c)\bar{X}_n}.
\]

Finally, the bounds in the MLE-centric approach are computed under the assumption \( \sqrt{n} \hat{J}_n(\hat{\theta}_n)^{-1} < \delta \). In our case, it says \( \sqrt{n} \bar{X}_n < c \bar{X}_n \), i.e. that \( c > \frac{1}{\sqrt{n} \bar{X}_n} \). Therefore, assuming \( \frac{1}{\sqrt{n} \bar{X}_n} < 0.299 \), letting \( c \in \left( \frac{1}{\sqrt{n} \bar{X}_n}, 0.229 \right) \), and assuming the shape \( \alpha \) of the gamma prior is smaller than one, we can set

a) \( \delta = c \bar{X}_n \)

b) \( M_1 = (1-c)^{\alpha - 2} (1+c)^{1-\alpha} \bar{X}_n^{-1} e^{2\beta c \bar{X}_n} \left[ (1-\alpha) + \beta(1-c) \bar{X}_n \right] \)

c) \( \tilde{M}_1 = \frac{\beta^\alpha}{\Gamma(\alpha)} (1-c)^{\alpha - 1} \bar{X}_n^{\alpha - 1} e^{-\beta(1-c)\bar{X}_n} \)

d) \( \tilde{M}_1 = \frac{\Gamma(\alpha)}{\beta^\alpha} (1+c)^{1-\alpha} \bar{X}_n^{1-\alpha} e^{\beta(1+c)\bar{X}_n} \)
e) \( M_2 = \frac{2}{(1-c)^3(X_n)^2} \)

f) \( J_n(\theta_n) = X_n^{-1} \)

g) \( \kappa = [c - \log(1 + c)]X_n. \)

The concrete choice of \( c \) may be optimized numerically.

**H.2.2. The MAP-centric approach.** Let us still assume \( \alpha < 1 \). We note that

\[
L_n(\theta) = -n\theta + n(\log \theta)X_n - \sum_{i=1}^{n} \log(X_i!) + \alpha \log(\beta) - \log(\Gamma(\alpha)) + (\alpha - 1) \log \theta - \beta \theta;
\]

\[
L_n'(\theta) = -n + \frac{nX_n}{\theta} + \frac{(\alpha - 1)}{\theta} - \beta = 0 \quad \text{iff} \quad \theta = \bar{\theta} := \frac{nX_n + (\alpha - 1)}{(n + \beta)};
\]

\[
L_n''(\theta) = -\frac{nX_n + \alpha - 1}{\theta^2}
\]

and so for \( \bar{\theta} \in (0, 1) \), \( \hat{\theta} = \bar{\theta} \theta_n \) and \( \theta \in (\bar{\theta} - \delta, \bar{\theta} + \delta) = ((1 - \bar{\theta})\theta_n, (1 + \bar{\theta})\theta_n) \), we have that

\[
\frac{1}{n} |L_n'''(\theta)| \leq \frac{2(n + \beta)^3}{n(nX_n + \alpha - 1)(1 - \bar{\theta})^3} =: \bar{M}_2.
\]

Now, we require that \( \bar{J}_n(\bar{\theta}_n) > \delta \bar{M}_2 \), i.e. that

\[
\frac{(n + \beta)^2}{n(nX_n + \alpha - 1)} > \frac{2\bar{\theta}X_n + \alpha - 1(1 - \bar{\theta})^3}{n(nX_n + \alpha - 1)(1 - \bar{\theta})^3}, \quad \text{which holds for } \bar{\theta} \in (0, 0.229), \text{ if } nX_n > 1 - \alpha.
\]

We will also want to make sure that \( \delta = \bar{\theta}X_n + (\alpha - 1) + (1 + \bar{\theta})X_n \geq \| \bar{\theta}_n - \hat{\theta}_n \| = \frac{\bar{\theta}X_n + 1 - \alpha}{n + \beta} \). Assuming that \( nX_n > 1 - \alpha \), this translates to \( \bar{\theta} \geq \frac{1}{n} \cdot \frac{\bar{\theta}X_n + 1 - \alpha}{X_n + (\alpha - 1)/n} \). Moreover, we require that \( \delta = \bar{\theta}X_n + (\alpha - 1) > \sqrt{\frac{1}{nJ_n(\bar{\theta}_n)}} = \sqrt{\frac{nX_n + \alpha - 1}{(nX_n + \alpha - 1)^2}}, \) which is equivalent to saying that \( \bar{\theta} > \sqrt{\frac{1}{nX_n + \alpha - 1}} \).

Finally, the value of \( \kappa \) may be obtained in the following way:

\[
\bar{\kappa} := - \max \left\{ - \bar{\delta} + \| \bar{\theta}_n - \hat{\theta}_n \| + \hat{\theta}_n \log \left( \frac{\bar{\theta}_n + \bar{\delta} - \| \bar{\theta}_n - \hat{\theta}_n \|}{\hat{\theta}_n} \right), \right.
\]

\[
\left. \bar{\delta} - \| \bar{\theta}_n - \hat{\theta}_n \| + \hat{\theta}_n \log \left( \frac{\bar{\theta}_n - \bar{\delta} + \| \bar{\theta}_n - \hat{\theta}_n \|}{\hat{\theta}_n} \right) \right\}
\]

\[
= -X_n \left\{ \frac{\beta - \bar{\theta}n + (1 - \alpha)(1 + \bar{\theta})X_n}{n + \beta} + \log \left( 1 + \frac{\beta - \bar{\theta}n + (1 - \alpha)(1 + \bar{\theta})X_n}{n + \beta} \right) \right\}.
\]

Therefore, in addition to the values we listed at the end of Appendix H.2.2, we have the following. We assume that \( \alpha < 1 \), \( nX_n > 1 - \alpha \) and \( \max \left\{ \sqrt{\frac{1}{nX_n + \alpha - 1}}, \frac{1}{n} \cdot \frac{\bar{\theta}X_n + 1 - \alpha}{X_n + (\alpha - 1)/n} \right\} < 0.229 \). We let \( \bar{c} \in \left( \frac{1}{n} \cdot \frac{\bar{\theta}X_n + 1 - \alpha}{X_n + (\alpha - 1)/n}, 0.229 \right) \cap \left( \sqrt{\frac{1}{nX_n + \alpha - 1}}, 0.229 \right) \). Then

i) \( \delta = \bar{c}nX_n + (\alpha - 1) \)

ii) \( \bar{J}_n(\bar{\theta}_n) = \frac{(n + \beta)^2}{n(nX_n + \alpha - 1)} \)
iii) $\bar{M}_2 = \frac{2(n+\beta)^3}{(1-\epsilon)^3 n(n+\alpha-1)}$

iv) $\bar{\kappa} = -\bar{X}_n \left\{ \frac{\theta^\alpha (1-\alpha)}{n+\beta} + \log \left( 1 + \frac{\epsilon n-\beta-(1-\alpha)(1+\epsilon)}{n+\beta} \right) \right\}$.

**H.3. Calculations for Section 7.2: the MAP-centric approach.** Let $k$ be the shape of the Weibull and let $X_1, \ldots, X_n \geq 0$ be our data. Our log-likelihood is given by:

$$L_n(\theta) = n \left[ \log(k) - \log(\theta) \right] + (k-1) \sum_{i=1}^{n} \log(X_i) - \frac{\sum_{i=1}^{n} X_i^k}{\theta}$$

**H.3.1. Calculating $\hat{\theta}_n$ and $\hat{J}_n(\hat{\theta}_n)$.** Now

$$L'_n(\theta) = -\frac{n}{\theta} + \frac{\sum_{i=1}^{n} X_i^k}{\theta^2}, \quad L''_n(\theta) = \frac{n}{\theta^2} - \frac{2 \sum_{i=1}^{n} X_i^k}{\theta^3}$$

The MLE is $\hat{\theta}_n = \frac{\sum_{i=1}^{n} X_i^k}{n} =: \bar{X}(n)$. We have that

$$\hat{J}_n(\hat{\theta}_n) = -\frac{1}{\theta_n^2} + \frac{2}{\theta_n^3} = \frac{1}{\theta_n^2}.$$

**H.3.2. Calculating $M_2$.** Now, note that

$$L'''_n(\theta) = \frac{6 \sum_{i=1}^{n} X_i^k - 2n\theta}{\theta^4}, \quad L^{(4)}_n(\theta) = \frac{6n\theta - 24 \sum_{i=1}^{n} X_i^k}{\theta^5}, \quad L^{(5)}_n(\theta) = \frac{120 \sum_{i=1}^{n} X_i^k - 24n\theta}{\theta^6}.$$ 

Therefore, for $0 < \theta < 2\hat{\theta}_n$, $L'''_n$ is decreasing and positive. This means that, if we let $0 < \delta < \hat{\theta}_n$, then

$$\sup_{\theta \in (\hat{\theta}_n-\delta, \hat{\theta}_n+\delta)} \left| \frac{L'''_n(\theta) - \delta}{n} \right| \leq M_2.$$ 

**H.3.3. Calculating $M_1$.** Now, for a given shape $\alpha > 0$ and scale $\beta > 0$,

$$\pi(\theta) = \frac{\beta^\alpha}{\Gamma(\alpha)} \left( \frac{1}{\theta} \right)^{\alpha+1} \exp(-\beta/\theta), \quad \pi'(\theta) = \frac{\beta^\alpha \exp(-\beta/\theta)}{\Gamma(\alpha) \theta^{\alpha+2}} \left( -\alpha - 1 + \frac{\beta}{\theta} \right)$$

and it follows that, for $\theta > \frac{\beta}{\alpha+1}$, $\frac{1}{\pi(\theta)}$ is increasing and it is decreasing otherwise. Therefore:

$$\sup_{\theta \in (\hat{\theta}_n-\delta, \hat{\theta}_n+\delta)} \left| \frac{1}{\pi(\theta)} \right| \leq \max \left\{ \frac{1}{\pi(\theta_n-\delta)}, \frac{1}{\pi(\theta_n+\delta)} \right\} =: M_1.$$ 

**H.3.4. Calculating $\bar{\theta}_n$ and $\bar{J}_n(\bar{\theta}_n)$.** Now

$$\bar{L}_n(\theta) = n \left[ \log(k) - \log(\theta) \right] + (k-1) \sum_{i=1}^{n} \log(X_i) - \frac{\sum_{i=1}^{n} X_i^k}{\theta} - (\alpha + 1) \log(\theta) - \frac{\beta}{\theta}.$$ 

Therefore,

$$\bar{L}_n(\theta)' = -\frac{n}{\theta} + \frac{\sum_{i=1}^{n} X_i^k}{\theta^2} - \frac{\alpha + 1}{\theta} + \frac{\beta}{\theta^2}$$

$$\bar{L}_n(\theta)'' = \frac{n}{\theta^2} - \frac{2 \sum_{i=1}^{n} X_i^k}{\theta^3} + \frac{\alpha + 1}{\theta^2} - \frac{2\beta}{\theta^3}.$$
and the MAP $\tilde{\theta}_n$ is given by

$$(n + \alpha + 1)\tilde{\theta}_n = \beta + \sum_{i=1}^{n} X_i^k \quad \iff \quad \tilde{\theta}_n = \frac{\beta + \sum_{i=1}^{n} X_i^k}{n + \alpha + 1}.$$ 

Moreover,

$$\tilde{J}_n(\tilde{\theta}_n) = -\frac{1}{\tilde{\theta}_n^2} + \frac{2\sum_{i=1}^{n} X_i^k}{n\tilde{\theta}_n^3} - \frac{\alpha + 1}{n\tilde{\theta}_n^2} + \frac{2\beta}{n\tilde{\theta}_n^2}.$$ 

H.3.5. Calculating $M_2$. Now, note that

$$\mathcal{T}_{n}''(\theta) = -\frac{2n}{\theta^4} + \frac{6\sum_{i=1}^{n} X_i^k}{\theta^3} - \frac{2(\alpha + 1)}{\theta^2} + \frac{6\beta}{\theta}, \quad \mathcal{T}_{n}^{(4)}(\theta) = \frac{6n}{\theta^4} - \frac{24\sum_{i=1}^{n} X_i^k}{\theta^5} + \frac{6(\alpha + 1)}{\theta^4} - \frac{24\beta}{\theta^5}.$$ 

Therefore $\mathcal{T}_{n}'''$ is increasing if and only if

$$6(n + \alpha + 1)\theta > 24 \left( \beta + \sum_{i=1}^{n} X_i^k \right) \quad \iff \quad \theta > 4\tilde{\theta}_n.$$ 

This means that, for $\delta \in (\tilde{\theta}_n, 0)$ and $\theta \in (\tilde{\theta}_n - \delta, \tilde{\theta}_n + \delta)$, $\mathcal{T}_{n}'''$ is decreasing and

$$\sup_{|\theta - \tilde{\theta}_n| < \delta} \left\{ \frac{\mathcal{T}_{n}'''(\theta)}{n} \right\} \leq \frac{\mathcal{T}_{n}'''(\tilde{\theta}_n - \delta)}{n} =: M_2.$$ 

H.3.6. Calculating $\bar{\kappa}$. Now, note that, for $\theta < \tilde{\theta}_n$, $L_n$ is increasing and otherwise it’s decreasing. This means that,

$$\sup_{|\theta - \tilde{\theta}_n| > \|\theta - \tilde{\theta}_n\|} \left\{ \frac{L_n(\theta) - L_n(\tilde{\theta}_n)}{n} \right\} \leq \max \left\{ \frac{L_n(\theta - \hat{\delta} - \tilde{\theta}_n) - L_n(\tilde{\theta}_n)}{n}, \frac{L_n(\tilde{\theta}_n + \hat{\delta} - \tilde{\theta}_n) - L_n(\tilde{\theta}_n)}{n} \right\} =: -\bar{\kappa}.$$ 

H.3.7. Constraints on $\hat{\delta}$ and $\delta$. Finally, we need the derive the constraints on $\hat{\delta}$. We first assume that $0 < \hat{\delta} < \tilde{\theta}_n$. We also suppose that $\hat{\delta} > \max \left( \|\theta - \tilde{\theta}_n\|, \frac{1}{\sqrt{nL_n(\tilde{\theta}_n)}} \right)$ and conditions on how large $n$ needs to be for this to hold are easy to obtain numerically. We also require $\hat{\delta} < \frac{\lambda(\theta_1)}{M_2}$. Looking closer at this last condition, we require that:

$$\hat{\delta} < \lambda_{\min}(\theta) (\tilde{\theta}_n - \hat{\delta})^4 \left[ -2 - \frac{2\alpha + 2}{n} \right] (\tilde{\theta}_n - \hat{\delta}) + 6 \left( \frac{\beta}{n} + \tilde{\theta}_n \right)^{-1}$$

Letting $0 < \delta := \tilde{\theta}_n - \hat{\delta} < \tilde{\theta}_n$, we therefore require

$$\hat{\delta} + \lambda_{\min}(\theta)\delta^4 \left[ -2 - \frac{2\alpha + 2}{n} \right] \delta + 6 \left( \frac{\beta}{n} + \tilde{\theta}_n \right)^{-1} > \tilde{\theta}_n$$

which is equivalent to:

$$\hat{\delta} + \delta^4 \lambda_{\min}(\theta) \left( 2 + \frac{2\alpha + 2}{n} \right)^{-1} \left[ 3\tilde{\theta}_n - \delta \right]^{-1} > \tilde{\theta}_n.$$
This condition will be satisfied if
\[ \delta > \sqrt{n J_0(\bar{\theta}_n)} \left( 2 + \frac{2 \alpha + 2}{n} \right)^{-1} \left[ 3 \bar{\theta}_n \right]^{-1} > \bar{\theta}_n. \] (H.1)

The left-hand side of eq. (H.1) is increasing in \( \delta \) and is clearly strictly greater than \( \bar{\theta}_n \) for \( \delta = \bar{\theta}_n \). This means that there exists a choice of \( \delta \) that yields \( \delta < \frac{\sum_{n} \min(\theta, \lambda)}{M_1} \) and the set of such choices can be obtained by solving eq. (H.1) numerically. Finally, in order to make sure that the condition on \( \delta \) is satisfied, we just need to check numerically how large \( n \) needs to be so that \( \delta > \sqrt{n J_0(\bar{\theta}_n)} \).

**H.4. Calculations for Section 7.3.**

**H.4.1. Calculating \( \hat{J}_n(\bar{\theta}_n) \) and \( \bar{J}_n(\bar{\theta}_n) \).** Note that:

\[ \hat{J}_n(\bar{\theta}_n) = \frac{1}{n} \sum_{k=1}^{n} \left( \frac{e^{-X_k^t \bar{\theta}_n Y_k}}{1 + e^{-X_k^t \bar{\theta}_n Y_k}} \right)^2 X_k (X_k)^T; \]

\[ \bar{J}_n(\bar{\theta}_n) = \frac{1}{n} \sum_{k=1}^{n} \left( \frac{e^{-X_k^t \bar{\theta}_n Y_k}}{1 + e^{-X_k^t \bar{\theta}_n Y_k}} \right)^2 X_k (X_k)^T \]
\[ + \frac{\nu + d}{2 \mu n} \left[ 1 + \frac{1}{\nu} (\bar{\theta}_n - \mu) \Sigma^{-1} (\bar{\theta}_n - \mu) \right]^{-1} (\Sigma^{-1} + \text{diag} (\Sigma^{-1})) \]
\[ - \frac{\nu + d}{2 \nu^2 n} \left[ 1 + \frac{1}{\nu} (\bar{\theta}_n - \mu) \Sigma^{-1} (\bar{\theta}_n - \mu) \right]^{-2} \cdot \left[ (\Sigma^{-1} + \text{diag} (\Sigma^{-1})) (\bar{\theta}_n - \mu) \right] \left[ (\Sigma^{-1} + \text{diag} (\Sigma^{-1})) (\bar{\theta}_n - \mu) \right]^T. \]

**H.4.2. Calculating \( M_1, \bar{M}_1 \) and \( \bar{M}_1 \).** Note that

\[ \pi' (\theta) = \frac{\Gamma((\nu + d)/2)}{\Gamma(\nu/2) \nu^{d/2} \pi^{d/2} |\Sigma|^{1/2}} \left( -\frac{\nu + d}{2} \right) \left[ 1 + \frac{1}{\nu} (\theta - \mu)^T \Sigma^{-1} (\theta - \mu) \right]^{-(\nu + d)/2 - 1} \]
\[ \cdot \left[ \frac{1}{\nu} (\Sigma^{-1} + \text{diag} (\Sigma^{-1})) \right] (\theta - \mu). \] (H.2)

It follows from the expressions in eqs. (7.2) and (H.2) that

\[ \sup_{\theta: ||\theta - \bar{\theta}|| < \delta} \left| \frac{\pi(\theta)}{\pi(\bar{\theta})} \right| \leq \frac{\Gamma((\nu + d)/2)}{\Gamma(\nu/2) \nu^{d/2} \pi^{d/2} |\Sigma|^{1/2}} \left[ 1 + \frac{1}{\nu \lambda_{\min}(\Sigma)} ||\theta - \mu||^2 \right]^{(\nu + d)/2} \]
\[ \leq \frac{\Gamma(\nu/2) \nu^{d/2} \pi^{d/2} |\Sigma|^{1/2}}{\Gamma((\nu + d)/2)} \left[ 1 + 2 \delta^2 + 2 ||\hat{\theta} - \mu||^2 \right]^{(\nu + d)/2} =: \bar{M}_1; \]

\[ \sup_{\theta: ||\theta - \bar{\theta}|| < \delta} \left| \frac{\pi'(\theta)}{\pi(\theta)} \right| = \sup_{\theta: ||\theta - \bar{\theta}|| < \delta} \left( \frac{\nu + d}{2} \right) \left[ \frac{1}{\nu} (\Sigma^{-1} + \text{diag} (\Sigma^{-1})) (\theta - \mu) \right] \left[ 1 + \frac{1}{\nu} (\theta - \mu)^T \Sigma^{-1} (\theta - \mu) \right]^{(\nu + d)/2} \]
\[
\leq \left( \frac{\nu + d}{2} \right) \left( \delta + \| \bar{\theta} - \mu \| \right) \left\| \Sigma^{-1} + \text{diag} \left( \Sigma_{1,1}^{-1}, \ldots, \Sigma_{d,d}^{-1} \right) \right\| \nu
\]

\[
\leq \frac{(\nu + d) \left( \delta + \| \bar{\theta} - \mu \| \right)}{\nu \lambda_{\min}(\Sigma)} =: M_1.
\]

H.4.3. Calculating \( M_2 \) and \( \bar{M}_2 \). Note that, for all \( \theta \in \mathbb{R}^d \),

\[
L''_n(\theta) [u_1, u_2, u_3] = n \sum_{i=1}^n Y_i^3 \sum_{j,k,l=1}^d e^{X_i^T \theta Y_i} \frac{e^{X_i^T \theta Y_i} - 1}{(1 + e^{X_i^T \theta Y_i})^3} X_i^{(j)} X_i^{(k)} u_1^{(j)} u_1^{(k)} u_1^{(l)}
\]

and therefore, for all \( \theta \in \mathbb{R}^d \),

\[
(\text{H.3}) \quad \frac{1}{n} \| L''_n(\theta) \| \leq \frac{1}{n} \sum_{k=1}^n \left\| X_k \right\|^3 \frac{e^{X_i^T \theta Y_i} e^{X_i^T \theta Y_k} - 1}{(1 + e^{X_i^T \theta Y_k})^3} \leq \frac{1}{6 \sqrt{3} n} \sum_{k=1}^n \left\| X_k \right\|^3 =: M_2.
\]

Now, a straightforward calculation reveals that, for \(\| u \| \leq 1, \| v \| \leq 1, \| w \| \leq 1\) and \(\delta \leq 1\),

\[
\sup_{\| \theta - \bar{\theta}_n \| \leq \delta} \left\| \sum_{i,j,k=1}^d \left( \frac{\partial^3}{\partial \theta_j \partial \theta_k \partial \theta_i} \log \pi(\theta) \right) u_i v_j w_k \right\|
\]

\[
\leq \sup_{\| \theta - \bar{\theta}_n \| \leq \delta} \left\{ \frac{3(\nu + d)}{[\nu + (\theta - \mu)^T \Sigma^{-1}(\theta - \mu)]^2} \left\| \Sigma^{-1} + \text{diag}(\Sigma^{-1}) \right\|_{op}^2 \| \theta - \mu \|
\right.
\]

\[
\left. + \frac{2(\nu + d)}{[\nu + (\theta - \mu)^T \Sigma^{-1}(\theta - \mu)]^3} \left\| \Sigma^{-1} + \text{diag}(\Sigma^{-1}) \right\|_{op}^3 \| \theta - \mu \|^3 \right\}
\]

\[
\leq \frac{3(\nu + d)}{\nu^2} \left\| \Sigma^{-1} + \text{diag}(\Sigma^{-1}) \right\|_{op}^2 (1 + \| \bar{\theta}_n - \mu \|)
\]

\[
+ \frac{2(\nu + d)}{\nu^2} \left\| \Sigma^{-1} + \text{diag}(\Sigma^{-1}) \right\|_{op}^3 (1 + \| \bar{\theta}_n - \mu \|)^3.
\]

Combining this with eq. (H.3), we obtain

\[
\frac{1}{n} \left\| T''_n(\bar{\theta}_n) \right\| \leq \frac{1}{6 \sqrt{3} n} \sum_{k=1}^n \left\| X_k \right\|^3 + \frac{3(\nu + d)}{\nu^2 n} \left\| \Sigma^{-1} + \text{diag}(\Sigma^{-1}) \right\|_{op}^2 (1 + \| \bar{\theta}_n - \mu \|)
\]

\[
+ \frac{2(\nu + d)}{\nu^2} \left\| \Sigma^{-1} + \text{diag}(\Sigma^{-1}) \right\|_{op}^3 \| \bar{\theta}_n - \mu \|^3 =: \bar{M}_2.
\]

H.4.4. Calculating \( \kappa \) and \( \bar{\kappa} \). Note that \( L_n \) is strictly concave. Therefore,

\[
\sup_{\theta: \| \theta - \bar{\theta}_n \| \geq \delta} \frac{L_n(\theta) - L_n(\bar{\theta})}{n} \leq \sup_{\theta: \| \theta - \bar{\theta}_n \| = \delta} \frac{L_n(\theta) - L_n(\bar{\theta})}{n}
\]

\[
\leq \sup_{\theta: \| \theta - \bar{\theta}_n \| = \delta} \left\{ \left( \theta - \bar{\theta}_n \right)^T \tilde{J}_n(\bar{\theta}_n) \left( \theta - \bar{\theta}_n \right) \right\} + \frac{M_2 \delta^3}{2}
\]

\[
\leq -\frac{1}{2} \lambda_{\min}(\bar{\theta}_n) \delta^2 + \frac{M_2 \delta^3}{2} =: -\kappa.
\]
Since $M_2$ in eq. (H.3) provides a uniform bound on $\frac{1}{n} \|L''(\theta)\|$ over $\theta \in \mathbb{R}^d$, a similar calculation shows:

$$
\sup_{\theta : \|\theta - \hat{\theta}_n\| > \delta - \|\hat{\theta}_n - \bar{\theta}_n\|} \frac{L_n(\theta) - L_n(\hat{\theta}_n)}{n} \\
\leq -\frac{1}{2} \lambda_{\min}(\hat{\theta}_n) \left( \delta - \|\hat{\theta}_n - \bar{\theta}_n\| \right)^2 + \frac{M_2 \left( \delta - \|\hat{\theta}_n - \bar{\theta}_n\| \right)^3}{2} =: -\bar{\kappa}.
$$

**H.4.5. Finding appropriate values of $\delta$ and $\bar{\delta}$.** In order to apply our results in the MLE-centric approach, we need

$$
\sqrt{\frac{\text{Tr} \left[ J_n(\hat{\theta}_n)^{-1} \right]}{n}} < \delta < \frac{\lambda_{\min}(\hat{\theta}_n)}{M_2}.
$$

This assumption also ensures that $\kappa$ from Appendix H.4.4 is positive. In the MAP-centric approach we also require

$$
\max \left\{ \|\hat{\theta}_n - \bar{\theta}_n\|, \sqrt{\frac{\text{Tr} \left( J_n(\hat{\theta}_n)^{-1} \right)}{n}} \right\} < \delta < \frac{\lambda_{\min}(\hat{\theta}_n)}{M_2},
$$

which also ensures that $\bar{\kappa}$ from Appendix H.4.4 is positive. Such choices of $\bar{\delta}$ and $\delta$ will be available for sufficiently large $n$. In order to choose the appropriate concrete values of $\bar{\delta}$ and $\delta$ one can run a numerical optimization scheme.