Combining Experts’ Causal Judgments

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Abstract

Consider a policymaker who wants to decide which intervention to perform in order to change a currently undesirable situation. The policymaker has at her disposal a team of experts, each with their own understanding of the causal dependencies between different factors contributing to the outcome. The policymaker has varying degrees of confidence in the experts’ opinions. She wants to combine their opinions in order to decide on the most effective intervention. We formally define the notion of an effective intervention, and then consider how experts’ causal judgments can be combined in order to determine the most effective intervention. We define a notion of two causal models being compatible, and show how compatible causal models can be merged. We then use it as the basis for combining experts’ causal judgments. We also provide definitions of partial compatibility and causal model decomposition to cater for cases when models are incompatible. We illustrate our approach on a number of real-life examples.

1. Introduction

Consider a policymaker who is trying to decide which intervention, that is, which actions, should be implemented in order to bring about a desired outcome, such as preventing violent behavior in prisons or reducing famine mortality in some country. The policymaker has access to various experts who can advise her on which interventions to consider. Some experts may be (in the policymaker’s view) more reliable than others; they may also have different areas of expertise; or may have perceived alternative factors in their analysis. The goal of the policymaker is to choose the best intervention, taking into account the experts’ advice.

There has been a great deal of work on combining experts’ probabilistic judgments. (Genest and Zidek \cite{8} provide a somewhat dated but still useful overview; Dawid \cite{5} and Fenton et al. \cite{6}, among others, give a Bayesian analysis.) We are interested in combining experts’ judgments in order to decide on the best intervention. Hence, we need more than probabilities. We need to have a causal understanding of the situation. Thus, we assume that the experts provide the policymaker with causal models. In general, these
models may involve different variables (since the experts may be focusing on different aspects of the problem). Even if two models both include variables $C_1$ and $C_2$, they may disagree on the relationships between them. For example, one expert may believe that $C_2$ is independent of $C_1$ while another may believe that $C_1$ causally depends on $C_2$. Yet somehow the policymaker wants to use the information in these causal models to reach her decision.

Despite the clear need for causal reasoning, and the examples in the literature and in practice where experts work with causal models (e.g., [2, 24]), there is surprisingly little work on combining causal judgments. Indeed, the only work that we are aware of that preceded our work is that of Bradley, Dietrich, and List [1] (BDL from now on), who prove an impossibility result. Specifically, they describe certain arguably reasonable desiderata, and show that there is no way of merging causal models so as to satisfy all their desiderata. They then discuss various weakenings of their assumptions to see the extent to which the impossibility can be avoided, none of which seem that satisfactory.

Following the conference version of our paper, Zennaro and Ivanovska 2018 examined the problem of merging causal models where the merged model must satisfy a fairness requirement (although the individual experts’ models may not be fair). They proposed a way of combining models based on ideas of BDL. Friedenberg and Halpern 2018 also considered the same problem of combining causal model of experts, but allowed for the possibility that experts disagree on the causal structure of variables due to having different focus areas.

There is also much work on the closely related problem of causal discovery: constructing a single causal model from a data set. A variety of techniques have been used to find the model that best describes how the data is generated (see, e.g., [3, 4, 16, 26, 27]; Triantafillou and Tsamardinos [27] provide a good overview of work in the area).

Of course, if we have the data that the experts used to generate their models, then we should apply the more refined techniques of the work on causal discovery. However, while the causals model constructed by experts are presumably based on data, the data itself is typically no longer available. Rather, the models represent the distillation of years of experience, obtained by querying the experts.

In this paper, we present an approach to combining experts’ causal models when sufficient data for discovering the overall causal model is not available. The key step in combining experts’ causal models lies in defining when two causal models are compatible. Causal models can be merged only if they are compatible. We start with a notion of strong compatibility, where the conditions say, among other things, that if both $M_1$ and $M_2$ involve variables $C_1$ and $C_2$, then they must agree on the causal relationship between $C_1$ and $C_2$. But that is not enough. Suppose that in both models $C_1$ depends on $C_2$, $C_3$, and $C_4$. Then in a precise sense, the two models must agree on how the dependence works, despite describing the world using possibly different sets of variables. Roughly speaking, this is the case when, for every variable $C$ that the two models have in common, we can designate one of the models as being “dominant” with respect to $C$, and use that model to determine the relationships for $C$. When $M_1$ and $M_2$ are compatible, we are able to construct a merged model $M_1 \oplus M_2$ that can be viewed as satisfying all but one of BDL’s desiderata (and we argue that the one it does not satisfy is unreasonable).

In a precise sense, all conclusions that hold in either of the models $M_1$ and $M_2$ also hold in the merged model (see Theorem 4.8(e) and Theorem 4.13(e)). In this way, the merged model takes advantage of the information supplied by all the experts (at least, to
the extent that the experts’ models are compatible), and can go beyond what we can do with either of the individual models (e.g., considering interventions that simultaneously act on variables that are in \(M_1\) but not in \(M_2\) and variables that are in \(M_2\) but not in \(M_1\)).

The set of constraints that need to be satisfied for models to be compatible is quite restrictive; as we show on real-life examples, models are often not compatible in this strong sense. We thus define two successively more general notions of compatibility, and demonstrate, by means of examples, how this allows for the merging a wider class of models, and reasoning about interventions under less strict conditions. But even with these more general notions, some experts’ models may still be incompatible due to disagreements about some parts of the model, even though possible interventions to be considered do not affect those parts of the model. We therefore introduce a notion of causal model decomposition to allow policymakers to “localize” the incompatibility between models, and merge the parts of the models that are compatible.

Having set out the formal foundation for merging causal models, we show how probabilities can be assigned to different reasonable ways of combining experts’ causal models based on the perceived reliability of the experts who proposed them, using relatively standard techniques. The policymaker will then have a probability on causal models that she can use to decide which interventions to implement. Specifically, we can use the probability on causal models to compute the probability that an intervention is efficacious. Combining that with the cost of implementing the intervention, the policymaker can compute the most effective intervention. As we shall see, although we work with the same causal structures used to define causality, interventions are different from (and actually simpler to analyze than) causes.

We draw on various examples from the literature (including real-world scenarios involving complex sociological phenomena) to illustrate our approach, including crime-prevention scenarios [24], radicalization in prisons [29], and child abuse [21].

These examples reinforce our belief that our approach provides a useful formal framework that can be applied to the determination of appropriate interventions for policymaking.

The rest of the paper is organized as follows. Section 2 provides some background material on causal models. We formally define our notion of intervention and compare it to causality in Section 3. We discuss our concept of compatibility and how causal models can be merged in Section 4. We discuss how the notions of interventions and of compatible models can be used by the policymakers to choose optimal interventions in Section 5. Finally, we summarize our results and outline future directions in Section 6.

2. Causal Models

In this section, we review the definition of causal models introduced by Halpern and Pearl [13]. The material in this section is largely taken from [11].

We assume that the world is described in terms of variables and their values. Some variables may have a causal influence on others. This influence is modeled by a set of structural equations. It is conceptually useful to split the variables into two sets: the exogenous variables, whose values are determined by factors outside the model, and the endogenous variables, whose values are ultimately determined by the exogenous variables.
For example, in a voting scenario, we could have endogenous variables that describe what the voters actually do (i.e., which candidate they vote for), exogenous variables that describe the factors that determine how the voters vote, and a variable describing the outcome (who wins). The structural equations describe how these values are determined (majority rules; a candidate wins if A and at least two of B, C, D, and E vote for him; etc.).

Formally, a causal model $M$ is a pair $(\mathcal{S}, \mathcal{F})$, where $\mathcal{S}$ is a signature, which explicitly lists the endogenous and exogenous variables and characterizes their possible values, and $\mathcal{F}$ defines a set of (modifiable) structural equations, relating the values of the variables. A signature $\mathcal{S}$ is a tuple $(\mathcal{U}, \mathcal{V}, \mathcal{R})$, where $\mathcal{U}$ is a set of exogenous variables, $\mathcal{V}$ is a set of endogenous variables, and $\mathcal{R}$ associates with every variable $Y \in \mathcal{U} \cup \mathcal{V}$ a nonempty set $\mathcal{R}(Y)$ of possible values for $Y$ (that is, the set of values over which $Y$ ranges). For simplicity, we assume here that $\mathcal{V}$ is finite, as is $\mathcal{R}(Y)$ for every endogenous variable $Y \in \mathcal{V}$. $\mathcal{F}$ associates with each endogenous variable $X \in \mathcal{V}$ a function denoted $F_X$ (i.e., $F_X = \mathcal{F}(X)$) such that $F_X : (\times_{U \in \mathcal{U}} \mathcal{R}(U)) \times (\times_{V \in \mathcal{V} \setminus \{X\}} \mathcal{R}(Y)) \rightarrow \mathcal{R}(X)$. This mathematical notation just makes precise the fact that $F_X$ determines the value of $X$, given the values of all the other variables in $\mathcal{U} \cup \mathcal{V}$. If there is one exogenous variable $U$ and three endogenous variables, $X$, $Y$, and $Z$, then $F_X$ defines the values of $X$ in terms of the values of $Y$, $Z$, and $U$. For example, we might have $F_X(u, y, z) = u + y$, which is usually written as $X = U + Y$. Thus, if $Y = 3$ and $U = 2$, then $X = 5$, regardless of how $Z$ is set.

The structural equations define what happens in the presence of external interventions. Setting the value of some variable $X$ to $x$ in a causal model $M = (\mathcal{S}, \mathcal{F})$ results in a new causal model, denoted $M_{X \leftarrow x}$, which is identical to $M$, except that the equation for $X$ in $\mathcal{F}$ is replaced by $X = x$.

The dependencies between variables in a causal model $M$ can be described using a causal network (or causal graph), whose nodes are labeled by the endogenous and exogenous variables in $M = ((\mathcal{U}, \mathcal{V}, \mathcal{R}), \mathcal{F})$, with one node for each variable in $\mathcal{U} \cup \mathcal{V}$. The roots of the graph are (labeled by) the exogenous variables. There is a directed edge from variable $X$ to $Y$ if $Y$ depends on $X$; this is the case if there is some setting of all the variables in $\mathcal{U} \cup \mathcal{V}$ other than $X$ and $Y$ such that varying the value of $X$ in that setting results in a variation in the value of $Y$; that is, there is a setting $\vec{z}$ of the variables other than $X$ and $Y$ and values $x$ and $x'$ of $X$ such that $F_Y(x, \vec{z}) \neq F_Y(x', \vec{z})$. A causal model $M$ is recursive (or acyclic) if its causal graph is acyclic. It should be clear that if $M$ is an acyclic causal model, then given a context, that is, a setting $\vec{u}$ for the exogenous variables in $\mathcal{U}$, the values of all the other variables are determined (i.e., there is a unique solution to all the equations). We can determine these values by starting at the top of the graph and working our way down. In this paper, following the literature, we restrict to acyclic models.

The following example, due to Lewis [19], describes a simple causal scenario.

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1The fact that $X$ is assigned $U + Y$ (i.e., the value of $X$ is the sum of the values of $U$ and $Y$) does not imply that $Y$ is assigned $X - U$; that is, $F_Y(U, X, Z) = X - U$ does not necessarily hold.

2Halpern [2016] calls this strongly recursive, and takes a recursive model to be one where, for each context $\vec{u}$, the dependency graph is acyclic (but it may be a different acyclic graph for context, so that in one context $A$ may be an ancestor of $B$, while in another, $B$ may be an ancestor of $A$).
Example 2.1. Suzy and Billy both pick up rocks and throw them at a bottle. Suzy’s rock gets there first, shattering the bottle. Since both throws are perfectly accurate, Billy’s would have shattered the bottle had it not been preempted by Suzy’s throw. Consider a model having an exogenous variable $U$ that encapsulates whatever background factors cause Suzy and Billy to decide to throw the rock (the details of $U$ do not matter, since we are interested only in the context where $U$’s value is such that both Suzy and Billy throw). Thus, $U$ has four possible values, depending on which of Suzy and Billy throw. We also have three binary variables: $ST$ for Suzy throws, $BT$ for Billy throws, and $BS$ for bottle shatters. $ST = 1$ means “Suzy throws”; $ST = 0$ means that she does not. We interpret $BT = 1$, $BT = 0$, $BS = 1$, and $BS = 0$ similarly. The values of $ST$ and $BT$ are determined by the context. The value of $BS$ is determined by the equation $F_{BS} (\vec{u}, ST, BT) = ST \lor BT$.

The causal graph corresponding to this model is depicted in Figure 1.

Figure 1: The causal graph for the rock-throwing example.

In many papers in the literature (e.g., [1, 24]) a causal model is defined simply by a causal graph indicating the dependencies, perhaps with an indication of whether a change has a positive or negative effect; that is, edges are annotated with + or −, so that an edge from $A$ to $B$ annotated with + means that an increase in $A$ results in an increase in $B$, while if it is annotated with a −, then an increase in $A$ results in a decrease in $B$ (where what constitutes an increase or decrease is determined by the model). Examples of these are shown in Section 4. Our models are more expressive, since the equations typically provide much more detailed information regarding the dependence between variables (as shown in Example 2.1); the causal graphs capture only part of this information. Of course, this extra information makes merging models more difficult (although, as the results of BDL show, the difficulties in merging models already arise with purely qualitative graphs).

To define interventions carefully, it is useful to have a language in which we can make statements about interventions. Given a signature $\mathcal{S} = (\mathcal{U}, \mathcal{V}, \mathcal{R})$, a primitive event is a formula of the form $X = x$, for $X \in \mathcal{V}$ and $x \in \mathcal{R}(X)$. A causal formula (over $\mathcal{S}$) is one of the form $[Y_1 \leftarrow y_1, \ldots, Y_k \leftarrow y_k] \varphi$, where $\varphi$ is a Boolean combination of primitive events, $Y_1, \ldots, Y_k$ are distinct variables in $\mathcal{U} \cup \mathcal{V}$, and $y_i \in \mathcal{R}(Y_i)$. Such a formula is

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3We could also model this problem using time-indexed variables that explicitly model the temporal order of events; see [11, 13]. The issues raised by this example could be made using either approach.

4In earlier work [11, 13], each $Y_i$ was taken to be an endogenous variable. For technical reasons (explained in Section 4), we also allow $Y$ to be exogenous.
abbreviated as $[\vec{Y} \leftarrow \vec{y}]\varphi$. The special case where $k = 0$ is abbreviated as $\varphi$. Intuitively, $[Y_1 \leftarrow y_1, \ldots, Y_k \leftarrow y_k]\varphi$ says that $\varphi$ would hold if $Y_i$ were set to $y_i$ for $i = 1, \ldots, k$.

We call a pair $(M, \vec{u})$ consisting of a causal model $M$ and a context $\vec{u}$ a (causal) setting. A causal formula $\psi$ is true or false in a setting. We write $(M, \vec{u}) \models \psi$ if the causal formula $\psi$ is true in the setting $(M, \vec{u})$. The $\models$ relation is defined inductively.

$(M, \vec{u}) \models X = x$ if the variable $X$ has value $x$ in the unique (since we are dealing with acyclic models) solution to the equations in $M$ in context $\vec{u}$ (that is, the unique vector of values for the exogenous variables that simultaneously satisfies all equations in $M$ with the variables in $\mathcal{U}$ set to $\vec{u}$). If $k \geq 1$ and $Y_k$ is an endogenous variable, then

$$(M, \vec{u}) \models [Y_1 \leftarrow y_1, \ldots, Y_k \leftarrow y_k] \varphi \iff (M_{Y_k \leftarrow y_k}, \vec{u}) \models [Y_1 \leftarrow y_1, \ldots, Y_{k-1} \leftarrow y_{k-1}] \varphi.$$  

If $Y_k$ is an exogenous variable, then

$$(M, \vec{u}) \models [Y_1 \leftarrow y_1, \ldots, Y_k \leftarrow y_k] \varphi \iff (M, \vec{u}[Y_k/y_k]) \models [Y_1 \leftarrow y_1, \ldots, Y_{k-1} \leftarrow y_{k-1}] \varphi,$$

where $\vec{u}[Y_k/y_k]$ is the result of replacing the value of $Y_k$ in $\vec{u}$ by $y_k$.

### 3. Interventions

In this section, we define (causal) interventions, and compare the notion of intervention to that of cause.

**Definition 3.1.** [Intervention] $\vec{X} = \vec{x}$ is an intervention leading to $\neg \varphi$ in $(M, \vec{u})$ if the following three conditions hold:

I1. $(M, \vec{u}) \models \varphi$.

I2. $(M, \vec{u}) \models [\vec{X} \leftarrow \vec{x}] \neg \varphi$.

I3. $\vec{X}$ is minimal; there is no strict subset $\vec{X}'$ of $\vec{X}$ and values $\vec{x}'$ such that $\vec{X}' = \vec{x}'$ satisfies I2.

I1 says $\varphi$ must be true in the current setting $(M, \vec{u})$, while I2 says that performing the intervention results in $\varphi$ no longer being true. I3 is a minimality condition. From a policymaker’s perspective, I2 is the key condition. It says that by making the appropriate changes, we can bring about a change in $\varphi$.

Our definition of an intervention leading to $\neg \varphi$ slightly generalizes others in the literature. Pearl [23] assumes that the causal model is first analyzed, and then a new intervention variable $I_V$ is added for each variable $V$ on which we want to intervene. If $I_V = 1$, then the appropriate intervention on $V$ takes place, independent of the values of the other parents of $V$; if $I_V = 0$, then $I_V$ has no effect, and the behavior of $V$ is determined by its parents, just as it was in the original model. Lu and Druzdzel [20], Korb et al. [13], and Woodward [30] take similar approaches.

We do not require special intervention variables; we just allow interventions directly on the variables in the model. But we can certainly assume as a special case that for each variable $V$ in the model there is a special intervention variable $I_V$ that works just like...
Pearl’s intervention variables, and thus recover the other approaches considered in the literature. Our definition also focuses on the outcome of the intervention, not just the intervention itself. In any case, it should be clear that all these definitions are trying to capture exactly the same intuitions, and differ only in minor ways.

Although there seems to be relatively little disagreement about how to capture intervention, the same cannot be said for causality. Even among definitions that involve counterfactuals and structural equations [9, 10, 13, 14, 15, 30], there are a number of subtle variations. Fortunately for us, the definition of intervention does not depend on how causality is defined. While we do not get into the details of causality here, it is instructive to compare the definitions of causality and intervention.

For definiteness, we focus on the definition of causality given by Halpern [10]. It has conditions AC1–3 that are analogous of I1–3. Specifically, AC1 says $\vec{X} = \vec{x}$ is a cause of $\phi$ in $(M, \vec{u})$ if $(M, \vec{u}) \models (\vec{X} = \vec{x}) \land \phi$ and AC3 is a minimality condition. AC2 is a more complicated condition; it says that there exist values $\vec{x}'$ for the variables in $\vec{X}$, a (possibly empty) subset $\vec{W}$ of variables, and values $\vec{w}$ for the variables in $\vec{W}$ such that $(M, \vec{u}) \models \vec{W} = \vec{w}$ and $(M, \vec{u}) \models [\vec{X} \leftarrow \vec{x}, \vec{W} \leftarrow \vec{w}] \neg \phi$. We do not attempt to explain or motivate AC2 here, since our focus is not causality.

Consider Example 2.1 again. Changing the value of Suzy’s throw by itself is not an intervention leading to the bottle not being shattered. Even if we prevent Suzy from throwing, the bottle will still shatter because of Billy’s throw. That is, although $ST = 1$ is a cause of the bottle shattering, $ST = 0$ is not an intervention leading to the bottle not being shattered; intervening on $ST$ alone does not change the outcome. On the other hand, $ST = 0 \land BT = 0$ is an intervention leading to the bottle not being shattered, but $ST = 1 \land BT = 1$ is not a cause of the bottle shattering; it violates the minimality condition AC3.

It is almost immediate from the definitions that we have the following relationship between interventions and causes:

**Proposition 3.2.** If $\vec{X} = \vec{x}$ is an intervention leading to $\neg \phi$ in $(M, \vec{u})$, then there is some subset $\vec{X}'$ of $\vec{X}$ such that $\vec{X}' = \vec{x}'$ is a cause of $\phi$ in $(M, \vec{u})$, where $\vec{x}'$ is such that $(M, \vec{u}) \models \vec{X}' = \vec{x}'$. Conversely, if $\vec{X} = \vec{x}$ is a cause of $\phi$ in $(M, \vec{u})$ then there is a superset $\vec{X}'$ of $\vec{X}$ and values $\vec{x}'$ such that $\vec{X}' = \vec{x}'$ is an intervention leading to $\neg \phi$.

Halpern [10] proved that (for his latest definition) the complexity of determining whether $\vec{X} = \vec{x}$ is a cause of $\phi$ in $(M, \vec{u})$ is DP-complete, where DP consists of those languages $L$ for which there exist a language $L_1$ in NP and a language $L_2$ in co-NP such that $L = L_1 \cap L_2$ [22]. It is well known that DP is at least as hard as NP and co-NP (and most likely strictly harder). The following theorem shows that the problem of determining whether $\vec{X} = \vec{x}$ is an intervention is in a lower complexity class.

**Theorem 3.3.** Given a causal model $M$, a context $\vec{u}$, and a Boolean formula $\phi$, the problem of determining whether $\vec{X} = \vec{x}$ is an intervention leading to $\neg \phi$ in $(M, \vec{u})$ is co-NP-complete.

**Proof.** First, we prove membership in co-NP. It is easy to see that checking conditions I1 and I2 of Definition 3.1 can be done in polynomial time by simply evaluating $\phi$ first in $(M, \vec{u})$ and then in the modified context where the values of $\vec{X}$ are set to $\vec{x}$. Checking whether I3 holds is in co-NP, because the complementary condition is in NP; indeed, we
simply have to guess a subset $\vec{X}'$ of $\vec{X}$ and values $\vec{\varphi}$ and verify that $I1$ and $I2$ hold for $\vec{X}' = \vec{X}$ and $\varphi$, which, as we observed, can be done in polynomial time.

For co-NP-hardness, we provide a reduction from UNSAT, which is the language of all unsatisfiable Boolean formulas, to the intervention problem. Given a formula $\psi$ that mentions the set $\vec{V}$ of variables, we construct a causal model $M_\psi$, context $\vec{u}$, and formula $\varphi$ such that $\vec{V} = 1$ is an intervention $5joe42$ leading to $\neg\varphi$ in $(M, \vec{u})$ iff $\psi$ is unsatisfiable.

The set of endogenous variables in $M$ is $\vec{V} \cup \{V', Y\}$, where $V'$ and $Y$ are fresh variables not in $\vec{V}$. Let $\vec{W} = \vec{V} \cup \{V'\}$. There is a single exogenous variable $U$ that determines the value of the variables in $\vec{W}$; we have the equation $V = U$ for each variable $V \in \vec{W}$. The equation for $Y$ is $Y = \vee_{V \in \vec{W}} (V = 0)$ (so $Y = 1$ if at least one variable in $\vec{W}$ is 0). Let $\varphi$ be $\neg\psi \land (Y = 1)$. We claim that $\vec{W} = \vec{I}$ is an intervention leading to $\neg\varphi$ in $(M_\psi, 0)$ iff $\psi \in UNSAT$.

Suppose that $\psi \in UNSAT$. Then, it is easy to see that $(M, 0) \models \psi$ (since $\neg\psi$ is valid) and $(M, 0) \models [\vec{W} \leftarrow \vec{I}] \neg\varphi$ (since $(M, 0) \models [\vec{W} \leftarrow \vec{I}] (Y = 0)$). To see that $I3$ holds, suppose by way of contradiction that $\vec{W}' \leftarrow \vec{w}'$ satisfies $I1$ and $I2$ for some strict subset $\vec{W}'$ of $\vec{W}$. In particular, we must have $(M, 0) \models [\vec{W}' \leftarrow \vec{w}'] \neg\varphi$. We clearly have $(M, 0) \models [\vec{W}' \leftarrow \vec{w}] (Y = 1)$, so we must have $(M, 0) \models [\vec{W}' \leftarrow \vec{w}'] \neg\varphi$, contradicting the assumption that $\psi \in UNSAT$. Thus, $\vec{W} \leftarrow \vec{I}$ is an intervention leading to $\neg\varphi$, as desired.

For the converse, suppose that $\vec{W} \leftarrow \vec{I}$ is an intervention leading to $\neg\varphi$. Then we must have $(M, 0) \models [\vec{W}' \leftarrow \vec{w}] \neg\psi$ for all strict subsets $\vec{W}'$ of $\vec{W}$ and all settings $\vec{w}'$ of the variables in $\vec{W}'$. Since, in particular, this is true for all subsets $\vec{W}'$ of $\vec{W}$ that do not involve $V'$, it follows that $\neg\psi$ is true for all truth assignments, so $\psi \in UNSAT$.

In practice, however, we rarely expect to face the co-NP complexity. For reasons of cost or practicality, we would expect a policymaker to consider interventions on at most $k$ variables, for some small $k$. The straightforward algorithm that, for a given $k$, checks all sets of variables of the model $M$ of size at most $k$ runs in time $O(|M|^k)$.

4. Merging Compatible Causal Models

This section presents our definition for compatibility of expert opinions. We consider each expert’s opinion to be represented by a causal model and, for simplicity, that each expert expresses her opinion with certainty. (We can easily extend our approach to allow the experts to have some uncertainty about the correct model; see the end of Section 5.)

We start with a strong notion of compatibility, and then consider generalizations of this notion that are more widely applicable.

4.1. Full compatibility

To specify what it means for a set of models to be compatible, we first define what it means for the causal model $M_1$ to contain at least as much information about variable $C$ as the causal model $M_2$, denoted $M_1 \succeq_C M_2$. Intuitively, $M_1$ contains at least as much information about $C$ as $M_2$ if $M_1$ and $M_2$ say the same things about the causal structure of $C$ as far as the variables that $M_1$ and $M_2$ share, but $M_1$ contains (possibly) more information about $C$, because, for example, there are additional variables in $M_1$ that affect $C$. We capture this property formally below. We say that $B$ is an immediate $M_2$-ancestor of $Y$ in $M_1$ if $B \in U_2 \cup V_2$, $B$ is an ancestor of $Y$ in $M_1$, and there is a path
from $B$ to $Y$ in $M_1$ that has no nodes in $U_2 \cup V_2$ other than $B$ and $Y$ (if $Y \in U_2 \cup V_2$). That is, $Y$ is the first node in $M_2$ after $B$ on a path from $B$ to $Y$ in $M_1$.

**Definition 4.1.** [Strong domination] Let $M_1 = ((U_1, V_1, R_1), F_1)$ and $M_2 = ((U_2, V_2, R_2), F_2)$. Let $\text{Par}_M(C)$ denote the variables that are parents of $C$ in $M$. $M_1$ strongly dominates $M_2$ with respect to $C$, denoted $M_1 \succeq_C M_2$, if the following conditions hold:

$\text{MI1}_{M_1, M_2, C}$. The parents of $C$ in $M_2$, $\text{Par}_{M_2}(C)$, are the immediate $M_2$-ancestors of $C$ in $M_1$.

$\text{MI2}_{M_1, M_2, C}$. Every path from an exogenous variable to $C$ in $M_1$ goes through a variable in $\text{Par}_{M_2}(C)$.

$\text{MI3}_{M_1, M_2, C}$. Let $X = ((U_1 \cup V_1) \cap (U_2 \cup V_2)) - \{C\}$. Then for all settings $\bar{x}$ of the variables in $\bar{X}$, all values $c$ of $C$, all contexts $\bar{u}_1$ for $M_1$, and all contexts $\bar{u}_2$ for $M_2$, we have

$$(M_1, \bar{u}_1) \models [\bar{X} \leftarrow \bar{x}](C = c) \text{ iff } (M_2, \bar{u}_2) \models [\bar{X} \leftarrow \bar{x}](C = c).$$

If $\text{MI1}_{M_1, M_2, C}$ holds and, for example, $B$ is a parent of $C$ in $M_2$, then there may be a variable $B'$ on the path from $B$ to $C$ in $M_1$. Thus, $M_1$ has in a sense more detailed information than $M_2$ about the causal paths leading to $C$. $\text{MI1}_{M_1, M_2, C}$ is not by itself enough to say that $M_1$ and $M_2$ agree on the causal relations for $C$. This is guaranteed by $\text{MI2}_{M_1, M_2, C}$ and $\text{MI3}_{M_1, M_2, C}$. $\text{MI2}_{M_1, M_2, C}$ says that the variables in $\text{Par}_{M_2}(C)$ screen off $C$ from the exogenous variables in $M_1$. (Clearly the variables in $\text{Par}_{M_2}(C)$ also screen off $C$ from the exogenous variables in $M_2$.) It follows that if $(M_1, \bar{u}_1) \models [\text{Par}_{M_2}(C) \leftarrow \bar{x}](C = c)$ for some context $\bar{u}_1$, then $(M_1, \bar{u}_1) \models [\text{Par}_{M_2}(C) \leftarrow \bar{x}](C = c)$ for all contexts $\bar{u}$ in $M_1$, and similarly for $M_2$. In light of this observation, it follows that $\text{MI3}_{M_1, M_2, C}$ assures us that $C$ satisfies the same causal relations in both models. We write $M_1 \not\preceq_C M_2$ if any of the conditions above does not hold.

Two technical comments regarding Definition 4.1. First, note that in $\text{MI3}$ we used the fact that we allow the $\bar{X}$ in formulas of the form $[\bar{X} \leftarrow \bar{x}](C = c)$ to include exogenous variables, since some of the parents of $C$ may be exogenous. Second, despite the suggestive notation, $\succeq_C$ is not a partial order. In particular, it is not hard to construct examples showing that it is not transitive. However, $\succeq_C$ is a partial order on compatible models (see the proof of Proposition 4.8), which is the only context in which we are interested in transitivity, so the abuse of notation is somewhat justified.

Note that we have a relation $\succeq_C$ for each variable $C$ that appears in both $M_1$ and $M_2$. Model $M_1$ may be more informative than $M_2$ with respect to $C$ whereas $M_2$ may be more informative than $M_1$ with respect to another variable $C'$. Roughly speaking, $M_1$ and $M_2$ are fully compatible if for each variable $C \in V_1 \cap V_2$, either $M_1 \succeq_C M_2$ or $M_2 \succeq_C M_1$. We then merge $M_1$ and $M_2$ by taking the equations for $C$ to be determined by the model that has more information about $C$. Consider another example demonstrating the notion of strong dominance, taken from [1].

**Example 4.2.** [1] An aid agency consults two experts about causes of famine in a region. Both experts agree that the amount of rainfall ($R$) affects crop yield ($Y$). Specifically, a shortage of rainfall leads to poor crop yield. Expert 2 says that political conflict ($P$) can also directly affect famine. Expert 1, on the other hand, says that $P$ affects $F$ only via
Y. The experts’ causal graphs are depicted in Figure 2 where the graph on the left, $M_1$, describes expert 1’s model, while the graph on the right, $M_2$, describes expert 2’s model. These graphs already appear in the work of BDL. In these graphs (as in many other causal graphs in the literature), the exogenous variables are omitted; all the variables are taken to be endogenous. Neither $MI_{M_1,M_2,F}$ nor $MI_{M_2,M_1,F}$ holds, since $P$ is not an $M_2$-immediate ancestor of $F$ in $M_1$. Similarly, neither $MI_{M_1,M_2,Y}$ nor $MI_{M_2,M_1,Y}$ holds, since $P$ is not an $M_1$-immediate ancestor of $Y$ in $M_2$ (indeed, it is not an ancestor at all). $MI_{M_1,M_2,F}$ holds since every path in $M_1$ from an exogenous variable to $F$ goes through a variable that is a parent of $F$ in $M_2$ (namely, $Y$); $MI_{M_2,M_1,F}$ does not hold (there is a path in $M_2$ to $F$ via $P$ that does not go through a parent of $F$ in $M_1$). Although we are not given the equations, we also know that $MI_{M_1,M_2,F}$ does not hold. Since $P$ is a parent of $F$ in $M_2$ according to expert 2, there must be a setting $y$ of $Y$ such that the value of $F$ changes depending on the value of $P$ if $Y = y$. This cannot be the case in $M_1$, since $Y$ screens off $P$ from $F$. It easily follows that taking $X = (P,Y)$ we get a counterexample to $MI_{M_1,M_2,F}$. Therefore, we have neither $M_1 \succeq_F M_2$ nor $M_2 \succeq_F M_1$.

While the definition of dominance given above is useful, it does not cover all cases where a policymaker may want to merge models. Consider the following example, taken from the work of Sampson et al. [24].

**Example 4.3.** Two experts have provided causal models regarding the causes of domestic violence. According to the first expert, an appropriate arrest policy (AP) may affect both an offender’s belief that his partner would report any abuse to police (PLS) and the amount of domestic violence (DV). The amount of domestic violence also affects the likelihood of a victim calling to report abuse (C), which in turn affects the likelihood of there being a random arrest (A). (Decisions on whether to arrest the offender in cases of domestic violence were randomized.)

According to the second expert, DV affects A directly, while A affects the amount of repeated violence (RV) through both formal sanction (FS) and informal sanction on socially embedded individuals (IS). Sampson et al. [24] use the causal graphs shown in Figure 3, which are annotated with the direction of the influence (the only information provided by the experts) to describe the expert’s opinions.

For the two common variables $DV$ and $A$, $MI_{M_1,M_2,DV}$ and $MI_{M_1,M_2,A}$ both hold. If the only variables that have exogenous parents are $AP$ in $M_1$ and $DV$ in $M_2$, and the set
of parents of $AP$ in $M_1$ is a subset of the set of parents of $DV$ in $M_2$, then $MI_{2M_1,M_2,DV}$ holds. Sampson et al. seem to be implicitly assuming this, and that $MI_{3}$ holds, so they merge $M_1$ and $M_2$ to get the causal graph shown in Figure 4.

Sampson et al. do not provide structural equations. Moreover, for edges that do not have a $+$ or $-$ annotation, such as the edge from $DV$ to $A$ in Figure 3, we do not even know qualitatively what the impact of interventions is. Presumably, the lack of annotation represents the expert’s uncertainty. We can capture this uncertainty by viewing the expert as having a probability on two models that disagree on the direction of $DV$’s influence on $A$ (and thus are incompatible because they disagree on the equations). We discuss in Section 5 how such uncertainty can be handled.

Suppose that some parent of $AP$ (or $AP$ itself) in $M_1$ is not a parent of $DV$ in $M_2$. Then, in $M_1$, it may be possible to change the value of $DV$ by intervening on $AP$, while keeping the values of all the exogenous variables that are parents of $DV$ in $M_2$ fixed. This will seem like an inexplicable change in the value of $DV$ from the perspective of the
second expert. If the second expert had been aware of such possible changes, she surely 
would have added additional variables to $M_2$ to capture this situation. One explanation 
of the fact that no changes were observed is that the second expert was working in a 
setting where the values of all variables that she cannot affect by an intervention are 
determined by some default setting of exogenous variables of which she is not aware (or 
not modeling).

Given models $M_1$ and $M_2$, we now want to define a notion of weak domination relative 
to a default setting $\vec{v}^*$ of the exogenous variables in $U_1 \cup U_2$. We say that contexts $\vec{u}_1$ for 
$M_1$ and $\vec{u}_2$ for $M_2$ are compatible with $\vec{v}^*$ if $\vec{u}_1$ and $\vec{u}_2$ agree on the variables in $U_1 \cap U_2$, 
$\vec{u}_1$ agrees with $\vec{v}^*$ on the variables in $U_1 - U_2$, and $\vec{u}_2$ agrees with $\vec{v}^*$ on the variables in 
$U_2 - U_1$.

**Definition 4.4.** [Domination relative to a default setting] Let $\vec{v}^*$ be a default setting for 
the variables in $M_1$ and $M_2$. $M_1$ dominates $M_2$ with respect to $C$, denoted $M_1 \succeq_C M_2$, if $\text{MI}_{M_1,M_2,C}$ holds, and, in addition, the following condition (which can 
be viewed as a replacement for $\text{MI}_{2M_1,M_2,C}$ and $\text{MI}_{2M_1,M_2,C}$) holds:

\[
\text{MI}_{M_1,M_2,C,\vec{v}^*} \quad \text{Let } \vec{X} = (U_1 \cup V_1) \cap (U_2 \cup V_2) - \{C\}. \text{ Then for all settings } \vec{x} \text{ of the } 
\text{variables in } \vec{X}, \text{ all values } c \text{ of } C, \text{ and all contexts } \vec{u}_1 \text{ for } M_1 \text{ and } \vec{u}_2 \text{ for } M_2 \text{ that } 
\text{are compatible with } \vec{v}^*, \text{ we have } 
\]

\[
(M_1, \vec{u}_1) \models [\vec{X} \leftarrow \vec{x}](C = c) \text{ iff } (M_2, \vec{u}_2) \models [\vec{X} \leftarrow \vec{x}](C = c). 
\]

It is easy to see that $\succeq_C$ is a special case of $\succeq_{\vec{v}^*}$:

**Lemma 4.5.** If $M_1 \succeq_C M_2$, then, for all default settings $\vec{v}^*$ of the variables in $M_1$ and 
$M_2$, we have $M_1 \succeq_{\vec{v}^*} M_2$.

**Proof.** Suppose that $M_1 \succeq_C M_2$. Fix default values $\vec{v}^*$. Clearly $\text{MI}_{M_1,M_2,C,\vec{v}^*}$ is a 
special case of $\text{MI}_{2M_1,M_2,C}$. Thus, $M_1 \succeq_{\vec{v}^*} M_2$. [1]

In light of Lemma 4.5, we give all the definitions in the remainder of the paper using 
$\succeq_{\vec{v}^*}$. All the technical results hold if we replace $\succeq_{\vec{v}^*}$ by $\succeq_C$ throughout.

**Definition 4.6.** [Full compatibility of causal models] If $M_1 = ((U_1, V_1, R_1), F_1)$ and 
$M_2 = ((U_2, V_2, R_2), F_2)$, then $M_1$ and $M_2$ are fully compatible with respect to default 
setting $\vec{v}^*$ if (1) for all variables $C \in (U_1 \cup V_1) \cap (U_2 \cup V_2)$, we have $R_1(C) = R_2(C)$ 
and (2) for all variables $C \in (V_1 \cap V_2) \cup (V_1 \cap U_2) \cup (V_2 \cap U_1)$, either $M_1 \succeq_{\vec{v}^*} M_2$ or 
$M_2 \succeq_{\vec{v}^*} M_1$. We say that $M_1$ and $M_2$ are fully compatible if $U_1 = U_2$ (so we can ignore 
the default setting).

**Definition 4.7.** [Merging fully compatible models] If $M_1 = ((U_1, V_1, R_1), F_1)$ and $M_2 = 
((U_2, V_2, R_2), F_2)$ are fully compatible with respect to $\vec{v}^*$, then the merged model $M_1 \oplus_{\vec{v}^*} M_2$ 
is the causal model $((U, V, R), F)$, where

- $U = U_1 \cup U_2 - (V_1 \cup V_2)$;
- $V = V_1 \cup V_2$;
- if $C \in U_1 \cup V_1$, then $R(C) = R_1(C)$, and if $C \in U_2 \cup V_2$, then $R(C) = R_2(C)$;
• if \( C \in V_1 - V_2 \) or if both \( C \in V_1 \cap V_2 \) and \( M_1 \succeq^C M_2 \), then \( F(C) = F_1(C) \); if \( C \in V_2 - V_1 \) or if both \( C \in V_1 \cap V_2 \) and \( M_2 \succeq^C M_1 \), then \( F(C) = F_2(C) \).

We write \( M_1 \oplus M_2 \) if \( U_1 = U_2 \).

Note that we assume that when experts use the same variable, they are referring to the same phenomenon within the same domain. Our approach does not deal with the possibility of two experts using the same variable name to refer to different phenomenon.

Returning to Example 4.3 assume that either \( \text{MI2}_{M_1,M_2,DV}, \text{MI2}_{M_1,M_2,A}, \text{MI3}_{M_1,M_2,DV} \), and \( \text{MI3}_{M_1,M_2,A} \) all hold, or there is a default setting \( \vec{v}^* \) such that Then \( M_1 \oplus M_2 \) has the causal graph described in Figure 1 that is, even though Sampson et al. [24] do not have a formal theory for merging models, they actually merge models in just the way that we are suggesting.

Let \( M_1 \sim^C_1 M_2 \) be an abbreviation for \( M_1 \succeq^C_1 M_2 \) and \( M_2 \succeq^C_1 M_1 \). We also write \( M_1 \succeq^C_2 M_2 \) if \( M_1 \succeq^C_1 M_2 \) and \( M_2 \succeq^C_1 M_1 \).

The next theorem provides evidence that Definition 4.6 is reasonable and captures our intuitions. Part (b) says that it is well defined, so that in the clauses in the definition where there might be potential conflict, such as in the definition of \( F(C) \) when \( C \in V_1 \cap V_2 \) and \( M_1 \sim^C_1 M_2 \), there is in fact no conflict; part (a) is a technical result needed to prove part (b). Part (c) says that the merged model is guaranteed to be acyclic. Part (d) says that causal paths in \( M_1 \) are preserved in \( M_1 \oplus^\vec{v}^* M_2 \), while part (e) says that at least as far as formulas involving the variables in \( M_1 \) go, \( M_1 \oplus^\vec{v}^* M_2 \) and \( M_1 \) agree, provided that the default values are used for the exogenous variables not in \( U_1 \cap U_2 \). Parts (d) and (e) can be viewed as saying that the essential causal structure of \( M_1 \) and \( M_2 \) is preserved in \( M_1 \oplus^\vec{v}^* M_2 \). All conclusions that can be drawn in \( M_1 \) and \( M_2 \) individually can be drawn in \( M_1 \oplus^\vec{v}^* M_2 \). (In the language of Halpern [12], part (e) says that \( M_1 \oplus^\vec{v}^* M_2 \) is essentially a conservative extension of \( M_1 \).) But it is important to note that \( M_1 \oplus^\vec{v}^* M_2 \) lets us go beyond \( M_1 \) and \( M_2 \), since we can, for example, consider interventions that simultaneously affect variables in \( M_1 \) that are not in \( M_2 \) and variables in \( M_2 \) that are not in \( M_1 \). Finally, parts (f) and (g) say that \( \oplus^\vec{v}^* \) is commutative and associative over its domain.

**Theorem 4.8.** Suppose that \( M_1, M_2, \) and \( M_3 \) are pairwise fully compatible with respect to default setting \( \vec{v}^* \). Then the following conditions hold.

(a) If \( M_1 \sim^C_1 M_2 \) then (i) \( \text{Par}_{M_1}(C) = \text{Par}_{M_2}(C) \) and (ii) \( F_1(C) = F_2(C) \);

(b) \( M_1 \oplus^\vec{v}^* M_2 \) is well defined.

(c) \( M_1 \oplus^\vec{v}^* M_2 \) is acyclic.

(d) If \( A \) and \( B \) are variables in \( M_1 \), then \( A \) is an ancestor of \( B \) in \( M_1 \) iff \( A \) is an ancestor of \( B \) in \( M_1 \oplus^\vec{v}^* M_2 \).

(e) If \( \varphi \) is a formula that mentions only the endogenous variables in \( M_1 \), \( \bar{u} \) is a context for \( M_1 \oplus^\vec{v}^* M_2 \), \( \bar{u}_1 \) is a context for \( M_1 \), and \( \bar{u} \) and \( \bar{u}_1 \) are compatible with \( \vec{v}^* \), then \( (M_1,\bar{u}_1) \models \varphi \) iff \( (M_1 \oplus^\vec{v}^* M_2, \bar{u}) \models \varphi \).

---

\(^5\)We are abusing notation here and viewing \( F_i(C) \) as a function from the values of the parents of \( C \) in \( M_i \) to the value of \( C \), as opposed to a function from all the values of all variables other than \( C \) to the value of \( C \).
(f) \( M_1 \odot v^* M_2 = M_2 \odot v^* M_1 \).

(g) If \( M_3 \) is fully compatible with \( M_1 \odot v^* M_2 \) with respect to \( v^* \) and \( M_1 \) is fully compatible with \( M_2 \odot v^* M_3 \) with respect to \( v^* \), then \( M_1 \odot v^* (M_2 \odot v^* M_3) = (M_1 \odot v^* M_2) \odot v^* M_3 \).

The proof of Theorem 4.8 is rather involved; the details can be found in Appendix A.

We define what it means for a collection \( \mathcal{M} = \{M_1, \ldots, M_n\} \) of causal models to be mutually compatible with respect to default setting \( v^* \) (for all the exogenous variables that are not common to \( M_1, \ldots, M_n \)) by induction on the cardinality of \( \mathcal{M} \). If \( |\mathcal{M}| = 1 \), then mutual compatibility trivially holds. If \( |\mathcal{M}| = 2 \), then the models in \( \mathcal{M} \) are mutually compatible with respect to \( v^* \) if they are fully compatible with respect to \( v^* \) according to Definition 4.6. If \( |\mathcal{M}| = n \), then the models in \( \mathcal{M} \) are mutually compatible with respect to \( v^* \) if the models in every subset of \( \mathcal{M} \) of cardinality \( n - 1 \) are mutually compatible with respect to \( v^* \), and for each model \( M \in \mathcal{M} \), \( M \) is fully compatible with \( \odot_{M \neq M'} v^* M' \) with respect to \( v^* \). By Theorem 4.8, if \( M_1, \ldots, M_n \) are mutually compatible with respect to \( v^* \), then the causal model \( M_1 \odot v^* \cdots \odot v^* M_n \) is well defined; we do not have to worry about parenthesization, nor the order in which the settings are combined. Thus, the model \( \odot_{M \neq M'} v^* M' \) considered in the definition is also well defined. Theorem 4.8(e) also tells us that \( M_1 \odot v^* \cdots \odot v^* M_n \) contains, in a precise sense, at least as much information as each model \( M_i \) individually. Thus, by merging mutually compatible models, we are maximizing our use of information.

This approach to merging models is our main contribution. Using it, we show in Section 5 how experts’ models can be combined to reason about interventions.

We now discuss the extent to which our approach to merging models \( M_1 \) and \( M_2 \) satisfies BDL’s desiderata. Recall that BDL considered only causal networks, not causal models in our sense; they also assume that all models mention the same set of variables. They consider four desiderata. We briefly describe them and their status in our setting. Since BDL do not consider contexts explicitly, we assume for simplicity in this discussion that the context is the same for all models, and talk only about \( \oplus \) rather than \( \odot v^* \).

- **Universal Domain:** the rule for combining models accepts all possible inputs and can output any logically possible model. This clearly holds for us.

- **Acyclicity:** the result of merging \( M_1 \) and \( M_2 \) is acyclic. This follows from Theorem 4.8(c), provided that \( M_1 \oplus M_2 \) is defined.

- **Unbiasedness:** if \( M_1 \oplus M_2 \) is defined, and \( M_1 \) and \( M_2 \) mention the same variables, then whether \( B \) is a parent of \( C \) in \( M_1 \oplus M_2 \) depends only on whether \( B \) is a parent of \( C \) in \( M_1 \) and in \( M_2 \). This property holds trivially for us, since if \( B \) and \( C \) are in both \( M_1 \) and \( M_2 \) and \( M_1 \oplus M_2 \) is defined, then \( B \) is a parent of \( C \) in \( M_1 \oplus M_2 \) iff \( B \) is a parent of \( C \) in both \( M_1 \) and \( M_2 \). (The version of this requirement given by BDL does not say “if \( M_1 \oplus M_2 \) is defined”, since they assume that arbitrary models can be merged.) BDL also have a neutrality requirement as part of unbiasedness. Unfortunately, an aggregation rule that says that \( B \) is a parent of \( C \) in \( M_1 \oplus M_2 \) iff \( B \) is a parent of \( C \) in both \( M_1 \) and \( M_2 \) (which seems quite reasonable to us) is not neutral in their sense, so we do not satisfy neutrality (nor, in light of the observation above, do we consider it a reasonable requirement to satisfy).
• Non-dictatorship: no single expert determines the aggregation. This clearly holds for us.

We conclude this section with a characterization of the complexity of determining whether two causal models are fully compatible. Determining whether $M_1$ and $M_2$ are fully compatible requires checking whether the conditions of Definition 4.6 hold. This amounts to checking the conditions $MI_{M_1,M_2,C}$ and $MI_{M_1,M_2,C,v^*}$ for all variables $C \in (U_1 \cup V_1) \cap (U_2 \cup V_2)$.

How hard this is to do depends in part on how the models are presented. If the models are presented explicitly, which means that, for each variable $C$, the equation for $C$ is described as a (huge) table, giving the value of $C$ for each possible setting of all the other variables, the problem is polynomial in the sizes of the input models. However, the size of the model will be exponential in the number of variables.

In this case, checking whether $MI_{M_1,M_2,C}$ holds for all $C$ amounts to checking whether the parents of $C$ in $M_2$ are the immediate $M_2$-ancestors of $C$ in $M_1$. To solve this, we need to determine, for each pair of endogenous variables $X$ and $Y$ in $M_i$ for $i = 1, 2$, whether $X$ depends on $Y$. With this information, we can construct the causal graphs for $M_1$ and $M_2$, and then quickly determine whether $MI_{M_1,M_2,C}$ holds.

If the model is given explicitly, then determining whether $X$ depends on $Y$ amounts to finding two rows in the table of values of $F_X$ that differ only in the value of $Y$ and in the outcome. As the number of pairs of rows is quadratic in the size of the table, this is polynomial in the size of the input. Thus, we can determine if $MI_{M_1,M_2,C}$ holds in polynomial time.

Checking if $MI_{M_1,M_2,C,v^*}$ holds amounts to checking whether $(M_1, \vec{u}_1) \models [\vec{X} \leftarrow \vec{x}](C = c)$, iff $(M_2, \vec{u}_2) \models [\vec{X} \leftarrow \vec{x}](C = c)$.

For a specific setting $\vec{u}$ and choice of $\vec{X}$ and $\vec{x}$, we can easily compute the value of $C$ in a context $\vec{u}$ if $\vec{X}$ is set to $\vec{x}$ (even if the model is not given explicitly). Since the number of possible settings is smaller than the size of an explicitly presented model, we can also determine whether $MI_{M_1,M_2,C,v^*}$ holds in polynomial time if the model is presented explicitly.

On the other hand, if the models are presented in a more compact way, with the structural equations, and hence the (descriptions of the) models, being of size polynomial in the number of variables in the model, checking full compatibility is in a higher complexity class, as we now show.

**Proposition 4.9.** Given two causal models $M_1$ and $M_2$ of size polynomial in the number of variables, determining whether they are fully compatible with respect to a default setting $\vec{v}^*$ is in $P_{||}$ and is co-NP-hard in the sizes of $M_1$ and $M_2$.

**Proof.** We prove a slightly stronger claim: that checking $MI_{M_1,M_2,C,v^*}$ is co-NP-complete in the sizes of $M_1$ and $M_2$, and that checking $MI_{M_1,M_2,C}$ is in $P_{||}$. The complexity class $P_{||}$ consists of all decision problems that can be solved in polynomial time with parallel (i.e., non-adaptive) queries to an NP oracle (see [? ? ? ?]).

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We start by showing that checking that \( \text{MI}_4_{M_1, M_2, C, \vec{v}} \) holds is in co-NP by showing that the complementary problem, namely demonstrating that \( \text{MI}_4_{M_1, M_2, C, \vec{v}} \) does not hold, is in NP. To do this, we guess a witness: a setting \( \vec{x} \) for the common variables \( \vec{X} \) of \( M_1 \) and \( M_2 \) other than \( C \), a value \( c \) of \( C \), and contexts \( \vec{u}_1 \) and \( \vec{u}_2 \) for \( M_1 \) and \( M_2 \), respectively, such that \( \vec{u}_1 \) and \( \vec{u}_2 \) are compatible with \( \vec{v}^* \):

\[
(M_1, \vec{u}_1) \models [\vec{X} \leftarrow \vec{x}](C = c),
\]

but

\[
(M_2, \vec{u}_2) \not\models [\vec{X} \leftarrow \vec{x}](C \neq c)
\]

(or vice versa). A witness can be verified in polynomial time in the size of the model, as it amounts to assigning values to all variables in the models and checking the value of \( C \).

The proof that the problem is co-NP hard is by reduction from the known co-NP-complete problem \textit{Tautology}: determining whether a Boolean formula \( \varphi \) is a tautology. Let \( \varphi \) be a Boolean formula over the variables \( \{Y_1, \ldots, Y_n\} \). We construct a causal model \( M_1 \) as follows:

1. \( \mathcal{U}_1 = \{U_1, \ldots, U_n\} \);
2. \( \mathcal{V}_1 = \{Y_1, \ldots, Y_n, C\} \);
3. \( \mathcal{R}_1(X) = \{0, 1\} \) for all \( X \in \mathcal{V}_1 \);
4. the equations are \( Y_i = U_i \) for \( i = 1, \ldots, n \) and \( C = \varphi \).

We note that since the set of exogenous variables is the same in \( M_1 \) and in \( M_2 \), we don’t need to define the default setting \( \vec{v}^* \). In other words, the variables \( \{Y_1, \ldots, Y_n\} \) are binary variables in \( M_1 \), and the value of \( C \) is determined by \( \varphi \).

The second causal model \( M_2 \) is constructed as follows:

1. \( \mathcal{U}_2 = \{U_1, \ldots, U_n\} \);
2. \( \mathcal{V}_2 = \{Y_1, \ldots, Y_n, C\} \);
3. \( \mathcal{R}_2(X) = \{0, 1\} \) for all \( X \in \mathcal{V}_2 \);
4. the equations are \( Y_i = U_i \) for \( i = 1, \ldots, n \), and \( C = 1 \).

\( \text{MI}_4_{M_1, M_2, C, \vec{v}} \) holds iff \( \varphi \) is a tautology. Indeed, if \( \varphi \) is a tautology, then \( \text{MI}_4_{M_1, M_2, C, \vec{v}} \) holds trivially. On the other hand, if \( \varphi \) is not a tautology, then it is easy to see that \( \text{MI}_4_{M_1, M_2, C, \vec{v}} \) does not hold, since there is some setting of the variables \( Y_1, \ldots, Y_n \) that makes \( C = 0 \).

To prove membership of \( \text{MI}_{M_1, M_2, C} \) in \( \mathbb{P}^{NP} \), we describe a polynomial-time algorithm for deciding \( \text{MI}_{M_1, M_2, C} \) that makes parallel queries to an NP oracle. We define an oracle \( O^{Dep}(M, X, Y) \) as follows: for a causal model \( M \) and two variables \( X \) and \( Y \) of \( M \), it answers “yes” if \( F_X \) depends on the variable \( Y \) in \( M \) and “no” otherwise. It is easy to see that \( O^{Dep}(M, X, Y) \) is in NP, since a witness for the positive answer is a pair of assignments to the variables of \( F_X \) that differ only in the value of \( Y \) and in the result. A witness is clearly verifiable in polynomial time; we simply instantiate \( F_X \) on these two assignments and verify that the results are different. (We have implicitly assumed here that the equation \( F_X \) can be computed in polynomial time, as it is a part of \( M \).) By querying the oracle \( O^{Dep}(M_i, X, Y) \) for all endogenous variables \( X \) and \( Y \) in \( M_i \), for \( i = 1, 2 \), we can determine the causal graphs of \( M_1 \) and \( M_2 \), and thus whether \( \text{MI}_{M_1, M_2, C} \) holds. The number of queries is at most quadratic in the sizes of \( M_1 \) and \( M_2 \), hence the algorithm terminates in polynomial time.
4.2. Partial compatibility

While the notion of dominance used in Definition 4.6 is useful, it still does not cover many cases of interest. We briefly describe an example here on causal models for the emergence of radicalization in US prisons. The material is taken from Useem and Clayton [28]. Although Useem and Clayton do not provide causal models, we construct these based on the description provided. We do not provide a detailed explanation of all the variables and their dependencies here (details are provided in the full paper); for our purposes, it suffices to focus on the structure of these models.

Example 4.10. Consider the two causal models in Figure 5. $M_1$ represents expert 1's opinion about emerging radicalization ($R$) in the State Correctional Institution Camp Hill in Pennsylvania. $M_2$ represents expert 2’s opinion about the causes of emergence in the Texas Department of Corrections and Rehabilitation. Both experts agree on the structural equations for $R$. However, they differ on the structural equations for $PD$, $CB$ and $AM$. The authors point to three main factors upon which the emergence of radicalization settings in both prison settings is dependent: “order in prisons” ($PD$), “a boundary between the prison and potentially radicalizing communities” ($CB$), and “having missionary leadership within the prison organizations” ($AM$). The also both share the same outcome—emerging radicalization ($R$). As can be observed from the descriptions provided, some variables and their dependency relations are specific to a prison. In $M_1$, $PD$ is attributed to corruption ($CG$) and lax management ($LM$) in the prison’s staff together with prisons being allowed to roam freely ($FM$). $CB$ is viewed as a result of religious leaders within the facilities being permitted to provide religious services freely ($IL$) and by prisoners showing membership within a prison community ($CM$) which in turn is signalled by prisoners being allowed to wear distinguished street clothing ($SC$). Prison authorities’ exercising of internal punishments, such as administrative segregation ($AS$), away from external oversight, and $IL$ are considered to directly contribute to $AM$. $M_2$ instead considers $PD$ to be linked to the rapid growth in inmate numbers ($RG$), inmates being allowed to assist authorities in maintaining order ($AA$), and inmates feeling significantly deprived ($D$) within the prisons—the latter as a result of being forced to engage in unpaid work ($W$) and having limited contact with visitors ($C$). For the common variables $CB$ and $AM$, assuming default values for $CG$, $LM$ and $FM$ in $SCICH$ and for $AA$, $D$ and $RG$ in $M_2$, we can show that $M_1 \succeq_{CB} M_2$ and $M_1 \succeq_{AM} M_2$. However, neither model dominates the other with respect to $PD$; neither $MI_1,M_2,PD$ nor $MI_1,M_2,PD$ holds. Therefore the models are not fully compatible according to Definition 4.6.

Although the models are not fully compatible according to our definition, the incompatibility is “localized” to the variable $PD$, a point we make precise in Section 4.3. Moreover, it is not even clear that there is disagreement with regard to $PD$: the experts could just be focusing on different variables. (The issue of what the experts are focusing on and how this might affect the issue of combining experts’ models is discussed by Friedenberg and Halpern [7].) In a richer model, $PD$ might have six parents. The trouble is, knowing the two models does not give us any insight into what the equations for $PD$ would be in the richer model.

Definition 4.11 (Weak domination). Let $\vec{v}^*$ be a default setting for the variables in $M_1$.
and $M_2$. We say that $M_1$ weakly dominates $M_2$ with respect to a node $C$ relative to $\vec{v}$, written $M_1 \geq^w_\vec{v} M_2$, if $M_1 \succeq^w_\vec{v} M_2$ and the following weakening of $\text{MI1}_{M_1,M_2,C}$ hold:

$\text{MI1}'_{M_1,M_2,C}$: If $A$ is a node in both $M_1$ and $M_2$ then $A$ is an immediate $M_1$-ancestor of $C$ in $M_2$ iff $A$ is a parent of $C$ in $M_1$.

Note that in Example 4.2, neither $M_1$ nor $M_2$ weakly dominates the other with respect to $F$: $P$ is a parent of $F$ in $M_2$ and is not an immediate $M_1$ ancestor of $F$ in $M_1$, so $M_1$ does not weakly dominate $M_2$ with respect to $F$, while $P$ is an immediate $M_1$ ancestor of $F$ in $M_2$ and is not a parent of $F$ in $M_1$, so $M_2$ does not weakly dominate $M_1$ either. Also note that $\text{MI1}$ implies $\text{MI1}'$; $\text{MI1}'$ is a strictly weaker condition than $\text{MI1}$, since it allows $M_1$ to weakly dominate $M_2$ with respect to $C$ if $C$ has parents in $M_1$ that are not in $M_2$ at all.

Definition 4.12. [Partial compatibility] If $M_1 = ((U_1, V_1, R_1), F_1)$ and $M_2 = ((U_2, V_2, R_2), F_2)$, then $M_1$ and $M_2$ are partially compatible with respect to default setting $\vec{v}$ iff (1) for all variables $C \in \{U_1 \cup V_1\} \cap \{U_2 \cup V_2\}$, we have $R_1(C) = R_2(C)$ and (2) for all variables $C \in (V_1 \cap V_2) \cup (V_1 \cap U_2) \cup (V_2 \cap U_1)$, either $M_1 \succeq^w_\vec{v} M_2$ or $M_2 \succeq^w_\vec{v} M_1$. If $M_1$ and $M_2$ are partially compatible with respect to $\vec{v}$, then $M_1 \oplus^\vec{v} M_2 = ((U, V, R), F)$ $M_1 \oplus M_2 = ((U, V, R), F)$ is defined as follows:

- $U$, $V$, $R$ are defined just as in Definition 4.6.

- For $F$, if $C \in V_1 - V_2$ then $F(C) = F_1(C)$, and if $C \in V_2 - V_1$ then $F(C) = F_2(C)$. If $C \in V_1 \cap V_2$ and $M_1 \succeq^w_\vec{v} M_2$, then let $\vec{p}_1$ consist of the parents of $C$ in $M_1$ and let $\vec{p}_2$ consist of the parents of $C$ in $M_2$ that are not in $M_1$. Then the parents of $C$ in $M_1 \oplus M_2$ are the nodes $\vec{p}_1 \cup \vec{p}_2$. Let $\vec{v}_2$ consist be the values of the variables in $\vec{p}_2$ when the exogenous variables in $M_2$ have their default value in $\vec{v}$. Given
an arbitrary setting \( \bar{x} \) of the variables in \( \bar{P}_1 \), we define \( F(C)(\bar{x}, \bar{v}_2) = F_2(C)(\bar{x}) \). Symmetrically, if \( C \in V_2 - V_1 \) or both \( C \in V_1 \cap V_2 \) and \( M_1 \not\succeq_{w,C}^* M_2 \), then let \( \bar{P}_1 \) consist of the parents of \( C \) in \( M_1 \) and let \( \bar{P}_2 \) consist of the parents of \( C \) in \( M_2 \) that are not in \( M_1 \). Then the parents of \( C \) in \( M_1 \oplus M_2 \) are the nodes \( \bar{P}_1 \cup \bar{P}_2 \). Let \( \bar{v}_2 \) consist be the values of the variables in \( \bar{P}_2 \) when the exogenous variables in \( M_2 \) have their default value in \( \bar{v}^* \). Given an arbitrary setting \( \bar{x} \) of the variables in \( \bar{P}_1 \), we define \( F(C)(\bar{x}, \bar{v}_2) = F_2(C)(\bar{x}) \). If \( C \in V_1 \cap V_2 \) and \( M_1 \not\succeq_{w,C}^* M_2 \), then let \( \bar{P}_1 \) consist of the parents of \( C \) in \( M_1 \) and let \( \bar{P}_2 \) consist of the parents of \( C \) in \( M_2 \) that are not in \( M_1 \). Then the parents of \( C \) in \( M_1 \oplus M_2 \) are the nodes \( \bar{P}_1 \cup \bar{P}_2 \). Let \( \bar{v}_2 \) consist be the values of the variables in \( \bar{P}_2 \) when the exogenous variables in \( M_2 \) have their default value in \( \bar{v}^* \). Given an arbitrary setting \( \bar{x} \) of the variables in \( \bar{P}_1 \), we define \( F(C)(\bar{x}, \bar{v}_2) = F_2(C)(\bar{x}) \). We have a symmetric definition of \( F(C) \) if \( C \in V_1 \cap V_2 \) and \( M_2 \not\succeq_{w,C}^* M_1 \). Again, we write \( M_1 \oplus M_2 \) if \( U_1 = U_2 \).

This definition does not define \( F(C) \) for all possible values of the parents of \( C \). For example, if \( C \in V_1 \cap V_2 \) and \( M_1 \not\succeq_{w}^* M_2 \), we have not defined \( F(C)(\bar{x}, \bar{y}) \) if \( \bar{y} \) is a setting of the variables in \( \bar{P}_2 \) other than \( \bar{v}_2 \). Intuitively, this is because the experts have not given us the information to determine \( F(C) \) in these cases. We can think if \( M_1 \oplus M_2 \) as a partial causal model. Intuitively, we cannot define \( \models \) in \( M_1 \oplus M_2 \) since we will not be able to define value of \( (M_1 \oplus M_2, \bar{u}) \models C = c \) for some setting \( \bar{u} \). Say that causal model \( M^* = (\{U^*, V^*, R^*\}, F^*) \) extends \( M_1 \oplus M_2 \) if \( (U^*, V^*, R^*) = (U, V, R) \) and \( F^*(C) = F(C) \) whenever \( F(C) \) is defined. We now define a 3-valued version of \( \models \) in \( M_1 \oplus M_2 \) by taking \( (M_1 \oplus M_2, \bar{u}) \models \varphi \) iff \( (M^*, \bar{u}) \models \varphi \) for all (complete) causal models \( M^* \) extending \( M_1 \oplus M_2 \) and taking \( (M_1 \oplus M_2, \bar{u}) \) to be undefined if neither \( (M_1 \oplus M_2, \bar{u}) \models \varphi \) nor \( (M_1 \oplus M_2, \bar{u}) \models \neg \varphi \) holds.

Note that, according to the definition above, the two models in Figure 5 are partially compatible. We can now prove a generalization of Theorem 4.8.

**Theorem 4.13.** Suppose that \( M_1, M_2, \) and \( M_3 \) are pairwise partially compatible with respect to \( \bar{v}^* \). Then the following conditions hold.

(a) If \( M_1 \not\succeq_{w,C}^* M_2 \) then (i) \( \text{Par}_{M_1}(C) = \text{Par}_{M_2}(C) \) and (ii) \( F_1(C) = F_2(C) \).

(b) \( M_1 \oplus \bar{v}^* M_2 \) is well defined.

(c) \( M_1 \oplus \bar{v}^* M_2 \) is acyclic.

(d) If \( A \) and \( B \) are variables in \( M_1 \), then \( A \) is an ancestor of \( B \) in \( M_1 \) iff \( A \) is an ancestor of \( B \) in \( M_1 \oplus \bar{v}^* M_2 \).

(e) If \( \varphi \) is a formula that mentions only the endogenous variables in \( M_1 \), \( \bar{u} \) is a context for \( M_1 \oplus \bar{v}^* M_2 \), \( \bar{u}_1 \) is a context for \( M_1 \), and \( \bar{u} \) and \( \bar{u}_1 \) are compatible with \( \bar{v}^* \), then \( (M_1, \bar{u}_1) \models \varphi \) iff \( (M_1 \oplus \bar{v}^* M_2, \bar{u}) \models \varphi \).

(f) \( M_1 \oplus \bar{v}^* M_2 = M_2 \oplus \bar{v}^* M_1 \).

(g) If \( M_3 \) is partially compatible with \( M_1 \oplus \bar{v}^* M_2 \) and \( M_1 \) is partially compatible with \( M_2 \oplus \bar{v}^* M_3 \), then \( M_1 \oplus \bar{v}^*(M_2 \oplus \bar{v}^* M_3) = (M_1 \oplus \bar{v}^* M_2) \oplus \bar{v}^* M_3 \).
The proof is almost identical to that of Theorem 4.8, so we omit the details here.

It is easy to see that the problem of determining whether \( M_1 \) and \( M_2 \) are partially compatible is of the same complexity as the problem of determining full compatibility by the following argument. Clearly, if \( M_1 \) and \( M_2 \) are fully compatible, they are also partially compatible, so the problem of determining partial compatibility is at least as hard as the problem of determining full compatibility. On the other hand, by Definition 4.12, the scope of compatibility is defined by the set of the common variables, hence given two models \( M_1 \) and \( M_2 \), it is easy to determine the subset of variables with respect to which we need to check compatibility. Therefore, checking partial compatibility is of the same complexity as checking full compatibility.

4.3. Decomposition of causal models

Even with the generalised notions of compatibility introduced above, it may still be the case that two expert’s models are not compatible. But we would expect that these models have submodels that are compatible. Finding such submodels has several advantages. First, consider the situation where the policymaker is given several different causal models that are not fully compatible according to Definition 4.6. If we could decompose the models, we might be able to “localize” the incompatibility, and merge the parts of the models that are fully compatible. Doing so may suggest effective interventions. Another advantage of decomposing a model is that it allows the policymaker to reason about each submodel in isolation. Since the problem of computing causes is DP-complete and the problem of computing interventions is co-NP-complete, having a smaller model to reason about could have a significant impact on the complexity of the problem.

In order to define the notion of decomposition, we need some preliminary definitions.

Definition 4.14. [Order-preserving partition] A sequence \( \langle V_1, \ldots, V_k \rangle \) of subsets of variables in \( V \) variables in a causal model \( M \) is an order-preserving partition if \( V_i \cap V_j = \emptyset \) for \( i \neq j \), \( \bigcup_{i=1}^{k} V_i = V \) (so \( \{V_1, \ldots, V_k\} \) is a partition of \( V \)), and for all \( i, j \) with \( i < j \), no variable in \( V_j \) is an ancestor of a variable in \( V_i \).

Definition 4.15. [Decomposable causal models] \( M = (\langle U, V, R \rangle, F) \) is decomposable if there exist \( k \geq 1 \) fully compatible causal models \( \{M_i = (\langle U_i, V_i, R_i \rangle, F_i) : 1 \leq i \leq k \} \), such that \( \langle V_1, \ldots, V_k \rangle \) is an order-preserving partition of \( V \), \( F_i \subseteq F \) is the set of structural equations that assign the values to the variables in \( V_i \), and for each model \( M_i \), the set \( U_i \) consists of all the endogenous and exogenous variables in \( M \) not in \( V_i \) that participate in the structural equations for the variables of \( V_i \). \( M_1, \ldots, M_k \) is called a decomposition of \( M \).

Lemma 4.16. If \( M_1, \ldots, M_k \) is a decomposition of \( M \), then \( M_1 \oplus \cdots \oplus M_k = M \).

Proof. The proof is immediate given the observation that we do not change any of the structural equations of \( M \) when decomposing it into submodels.

It is easy to see that, for a given model, there can be many ways to decompose it into a set of submodels according to Definition 4.15. Moreover, all models are decomposable by Definition 4.15. Indeed, any model \( M \) can be trivially decomposed to \( |V| \) submodels, each of which consists of exactly one endogenous variable. Of course, such a decomposition is useless for practical purposes; the decompositions we consider are those that help in either
analysing the model or reducing the complexity of computing causes. In Example 4.17 below, we demonstrate a nontrivial decomposition.

**Example 4.17.** Consider the causal models in Figure 5 from the prison example (Example 4.10). We can decompose $M_1$ according to Definition 4.15 into $M_{11}$, $M_{12}$, $M_{13}$, and $M_{14}$, where $M_{12} = (\{u_{ij}, v_{ij}, r_{ij}\}, f_{ij})$, $V_{11} = \{FM, CG, LM, PD\}$, $U_{11}$ consists of all the exogenous variables in $M_1$ (which are not explicitly given in Figure 5) that are ancestors of the variables in $V_{11}$, $V_{12} = \{SC, CM, CB, IL\}$, $U_{12}$ consists of all the exogenous variables in $M_1$ that are ancestors of the variables in $V_{12}$ together $FM$, $V_{13} = \{AS, AM\}$, $U_{13}$ consists of all the exogenous variables in $M_1$ that are ancestors of the variables in $V_{13}$ together $IL$, $V_{14} = \{R\}$, and $U_{14} = \{PD, CB, AM\}$. Similarly we can decompose $M_2$ into four submodels $M_{21}$, $M_{22}$, $M_{23}$, where $M_{24}$, where $M_{ij} = (\{u_{ij}, v_{ij}, r_{ij}\}, f_{ij})$, $V_{21} = \{C, W, AA, D, RG, PD\}$, $U_{21}$ consists of all the exogenous variables in $M_2$ that are ancestors of the variables in $V_{21}$, $V_{22} = \{CM, CB\}$, $U_{22}$ consists of all the exogenous variables in $M_2$ that are ancestors of the variables in $V_{22}$, $V_{23} = \{AS, AM\}$, $U_{23}$ consists of all the exogenous variables in $M_2$ that are ancestors of the variables in $V_{23}$, $V_{24} = \{R\}$, and $U_{24} = \{PD, CB, AM\}$. Figures 6 and Figures 7 show the four submodels resulting from these decompositions. There is some flexibility in how we do the decomposition. For example, we could move $IL$ from $V_{12}$ and $U_{13}$ to $V_{13}$ and $U_{12}$. However, we cannot, for example, move $CB$ into $V_{11}$, for then $\langle V_{11}, V_{12}, V_{13}, V_{14} \rangle$ would not be an order-preserving partition (since $FM$ is an ancestor of $CM$, which is an ancestor of $CB$).

![Figure 6: Decomposition of the model $M_1$ from Example 4.11](image)

We observe that decomposing incompatible models into smaller submodels can in some cases help determine common interventions over shared outcomes in the original models in spite of their incompatibility. Consider, for example, the two incompatible models $M_1$ and $M_2$ in the prison example. Although the two are incompatible (as observed in Example 4.10), the submodels $M_{12}$ and $M_{22}$ in Figures 6 and 7 respectively, obtained from their decomposition, are fully compatible according to Definition 4.16. We have $M_{12} \succeq_{CB} M_{22}$ and $M_{12} \succeq_{CM} M_{22}$. The composition of the two submodels yields a merged model similar to $M_{12}$. Given this, it may be concluded that interventions over $SC$ or $IL$ make it possible to change the value of $CB$ in the two models $M_1$ and $M_2$
and ultimately $R$ (assuming the structural equation for $R$ in both models $M_1$ and $M_2$ to be the conjunction of $PD$, $CB$ and $AM$)—a similar conclusion reached by considering partial compatibility between the models $M_1$ and $M_2$ in Section 4.2. We conjecture that reasoning using model decomposition and full compatibility of submodels is an alternative to reasoning using partial compatibility.

Another advantage of decomposing a causal model $M$ into a set of smaller submodels is that we can reason about each submodel separately. In particular, we can compute the set of causes and possible interventions for a given outcome. However, in order to use these results to reason about the whole model, we need to perform additional calculations. Informally, when decomposing $M$ into a set of smaller submodels, we can view each submodel as a black box, with inputs and outputs being the exogenous variables of the submodel and the leaves in the causal graph of the submodel, respectively. We can then put connect these variables into an abstract causal graph for the original model, essentially ignoring the internal variables. If the submodels are fairly large, the graph of submodels will be significantly smaller than the causal network of $M$. We can then apply causal reasoning to the abstract graph, which will result in a set of submodels being causes for the outcome. For these submodels, we can calculate the causes of their outcomes for each submodel separately. As causality is $DP$-complete, and computing interventions is co-$NP$-complete, solving a set of smaller problems instead of a large problem is cheaper.

We note that, in fact, interesting decompositions (that is, decompositions of a large model into a set of submodels of a reasonable size with relatively few interconnections between them, which means that we can analyse causality both within a submodel and between submodels relatively easily) are possible only in models that are somewhat loosely connected. Such a decomposition can often be done for real-life cases; see Example 4.18. We believe that, in practice, analyzing the effect of interventions in a model will be difficult precisely when a model is highly connected, so that there are many causal paths. We expect the causal models that arise in practice to be much more loosely connected, and thus amenable to useful decompositions. Hence, the computation of causes and interventions in practice should not be as bad as what is suggested by our worst-case

Figure 7: Decomposition of the model $M_2$ from Example 4.10.
analysis.

Below, we briefly discuss the relevant aspects of two cases of child abuse that resulted in the death of a child: the “Baby P” case and the Victoria Climbiè case. In these cases, experts’ opinions were in fact only partially compatible, and there were natural ways to decompose the causal model.

**Example 4.18** (The cases of Baby P and Victoria Climbiè). Baby P (Peter Connelly) died in 2007 after suffering physical abuse over an extended period of time [21]. The court ultimately found the three adults living in a home with baby Peter guilty of “causing or allowing [Peter’s] death” [25]. After baby Peter’s death, there was an extensive inquiry into practices, training, and governance in each of the involved professionals and organizations separately.

As shown by Chockler et al. [2], the complete causal model for the Baby P case is complex, involving many variables and interactions between them. There were several authorities involved in the legal proceedings, specifically social services, the police, the medical system, and the court. In addition, the legal proceedings considered the family situation of Baby P. Roughly speaking, the causal model can be viewed as having the schematic breakdown presented in Figure 8.

![Figure 8: Schematic representation of causal submodels in the Baby P case.](image)

Each of the experts involved in the legal inquest and enquiry had expertise that corresponded to one of the boxes in Figure 8 (i.e., there were no experts with expertise that covered more than one box). The figure suggests that we might divide the causal model into submodels corresponding to each box. The schematic representation in Figure 8 does not take into account the interaction between submodels. In reality, there were numerous interactions between, for example, the social services and the court submodels, leading to court hearings, which in turn determined the course of action taken by the social services and the police after the court decision. Once we model these interactions more carefully, we need a somewhat more refined decomposition.

We give a decomposition in Figure 9 that takes into account the interactions for part of the case, namely, the part that concerns the social services, the court, the police, and family life. To make the decomposition consistent with Definition 4.15, we break up social services into two submodels, for reasons explained below.

The variables in the figure are: $FV$ for whether there was a family visit from the social services; $PR$ for whether there was a police report; $CH$ for whether there was a court...
hearing; RFH for whether the child was removed from home; CP for whether the child was put on the Child Protection Register; SR for whether there was a social services report; CS for whether the child was declared safe in the family home; MA, PA, and OA for whether the child was abused by his mother, the mother’s partner, or another adult in the house, respectively; CA for whether the child was abused; and, finally, D for whether the final outcome was death (of Baby P) due to abuse. Note that, as usual, we have omitted exogenous variables of the full model in the figure; it shows only the endogenous variables. Thus, we do not have the exogenous variables that determine FV, PR, MA, PA or OA. The dotted rectangles in Figure 9 determine a decomposition. Each rectangle consists of the endogenous variables of one submodel. The exogenous variables of the SocialServices#2 and Outcome submodels are those parent variables appearing in the other submodels. Thus, for example, in the Outcome submodel, the exogenous variables are CS, RFH, and CA. The submodels are described in Figure 10. The dotted rectangles in Figure 9 can be viewed as compact representations of the submodels in Figure 10.

Of course, there is more than one way to decompose the model of Figure 9. For example, the submodel currently standing for the court and the police can be decomposed into two smaller submodels, one for the court and one for the police. However, it is critical that we have decomposed social services into two submodels. The variable CH depends on FV, and the variable CP in turn depends on CH, hence FV and CP cannot be in the same submodel (or else we would violate the requirement of Definition 4.15 that the sets of endogenous variables of each submodel form an order-preserving partition of the endogenous variables of the original model).

We consider another case of child abuse that resulted in child’s death: Victoria Climbié [21]. Victoria died in her house from hypothermia, 18 months after arriving
in the UK from the Ivory Coast to live with her great-aunt. Her great-aunt and the great-aunt’s boyfriend were found guilty of Victoria’s murder (in contrast with the Baby P case, where the adults in the house were found guilty of causing or allowing his death).

The inquiry into the circumstances of Victoria’s death placed the blame on social workers, who failed to notice Victoria’s injuries, paediatricians, who accepted the explanation of Victoria’s great-aunt that Victoria’s injuries were self-inflicted, and the metropolitan police. In addition, the inquiry noted that the pastors in the church to which Victoria’s great-aunt belonged, had concerns about the child’s well-being but failed to contact any child protection services. The inquiry suggested several interventions on the procedures of social workers and paediatricians. These interventions turned out to be inadequate, as the death of Baby P occurred under somewhat similar circumstances and his abuse also went unnoticed until his death.

Although there were some similarities between the Baby P case and the Victoria Climbie’e cases, there were also some differences. For example, while Victioria Climbie died at home, Baby P died in the hospital. Thus, the causal models for these two cases differ somewhat. However, the causal model for the Victoria Climbie case is also decomposable into fully compatible submodels in the sense of Section 4.3. Moreover, some of the submodels in the decomposition are identical to those in the causal model for Baby P. Specifically, there are submodels for the police, the medical system, the family system, and the courts, just as in the case of Baby P, as well as a submodel for the church. The schematic breakdown is presented in Figure 11. Although we do not provide the causal model in detail here, this discussion already illustrates a major advantage of decomposition: it allows us to reuse causal models that were developed in one case and apply them to another, thus saving a lot of effort. Moreover, if the same submodel appears in several different cases, such as the social services submodel in these examples, this suggests that
the policymaker should prefer interventions that address the problems demonstrated by this submodel, as they are likely to affect several cases. In fact, the cases of child abuse that remains undetected due to problems in the social services sadly continue to occur (see, for example, the recently published cases discussed in [17]). Even though the causal models for different cases will undoubtedly be different, we can still take advantage of the common submodels. We expect that this will be the case in many other situations as well.

From a practical perspective, Example 4.18 demonstrates one benefit of decomposition: the decomposition allows us to capture different aspects of the case, each requiring different expertise. This facilitates different experts working on each of the submodels independently. The process also works in the other direction: a policymaker often has a crude idea of the general structure of the causal model, and what components are involved in the decision-making process. She can then decompose her initial causal model into submodels and, guided by these submodels, decide which areas of expertise are critical.

A further benefit of decomposition illustrated by these examples is that, although different, the causal models had some common submodels. Thus, decomposition supports a form of modularity in the analysis, and enables results of earlier analyses to be reused.

5. Combining Experts’ Opinions

In this section, we show how we can combine experts’ causal options. Suppose that we have a collection of pairs \((M_1, p_1), \ldots, (M_n, p_n)\), with \(p_i \in (0, 1]\); we can think of \(M_i\) as the model that expert \(i\) thinks is the right one and \(p_i\) as the policymaker’s prior degree of confidence that expert \(i\) is correct. (The reason we say “prior” here will be clear shortly.) Our goal is to combine the expert’s models. We present one way of doing so, that uses relatively standard techniques. The idea is to treat the probabilities \(p_1, \ldots, p_n\) as mutually independent. Thus, if \(I\) is a subset of \(\{1, \ldots, n\}\), the prior probability that exactly the experts in \(I\) are right, which we denote \(p_I\), is \(p_I = \prod_{i \in I} p_i * \prod_{j \not\in I} (1 - p_j)\). Now we have some information regarding whether all the experts in \(I\) are right. Intuitively, we want to condition on this information. We proceed as follows.

Fix a default setting \(\vec{v}^*\) of the exogeneous variables that are not common to \(M_1, \ldots, M_n\). Let \(\text{Compat}^{\vec{v}^*} = \{ I \subseteq \{1, \ldots, n\} : \text{the models in } \{M_i : i \in I\} \text{ are mutually compatible} \)
with respect to \( \vec{v}^* \). For \( I \in \text{Compat}^{\vec{v}^*} \), define \( M_I = \bigoplus_{i \in I} M_i \). By Proposition 4.8, \( M_I \) is well defined. The models in \( \mathcal{M}_{\text{Compat}^{\vec{v}^*}} = \{ M_I : I \in \text{Compat}^{\vec{v}^*} \} \) are the candidate merged models that the policymaker should consider. \( M_I \) is the “right” model provided that exactly the experts in \( I \) are right. But even if \( M_I \in \mathcal{M}_{\text{Compat}^{\vec{v}^*}} \), it may not be the “right” model, since it may be the case that not all the expert in \( I \) are right. The probability that the policymaker should give \( M_I \) is \( p_I/N \), where \( N = \sum_{I \in \text{Compat}^{\vec{v}^*}} p_I \) is a normalization factor.

This approach gives the policymaker a distribution over causal models. This can be used to compute, for each context, which interventions affect the outcome \( \varphi \) of interest, and then compute the probability that a particular intervention is effective (which can be done summing the probability of the models \( M_I \) in \( \mathcal{M}_{\text{Compat}^{\vec{v}^*}} \) where it is effective, which in turn can be computed as described in Section 3). Note that our calculation implicitly conditions on the fact that at least one expert is right, but allows for the possibility that only some subset of the experts in \( I \) is right even if \( I \in \text{Compat}^{\vec{v}^*} \); we place positive probability on \( M_I \) even if \( I' \) is a strict subset of some \( I \in \text{Compat}^{\vec{v}^*} \). This method of combining experts’ judgments is similar in spirit to the method proposed by Dawid [5] and Fenton et al. [6].

To get a sense of how this works, consider a variant of Example 4.2 in which a third expert provides her view on causes of famine and thinks that government corruption is an indirect cause via its effect on political conflict (see Figure 12); call this model \( M_3 \). For simplicity, we assume that all models have the same set of exogenous variables. According to the compatibility definition in Section 4, the models \( M_2 \) and \( M_3 \) are fully compatible (assuming that MB3 holds), but \( M_1 \) and \( M_3 \) are not. We have \( \mathcal{M}_{\text{Compat}} = \{ \{M_1\}, \{M_2\}, \{M_3\}, \{M_{2,3}\} \} \) with \( M_{2,3} = M_2 \oplus M_3 = M_3 \). Suppose that experts are assigned the confidence values as follows: \( (M_1, 0.4), (M_2, 0.6) \) and \( (M_3, 0.5) \) respectively. Then the probability on \( M_{2,3} \) is the probability of \( M_2 \) and \( M_3 \) being right (i.e., \( 0.6 \cdot 0.5 \)) and \( M_1 \) being wrong (i.e., \( 1 - 0.4 = 0.6 \)). So we have

\[
\begin{align*}
p_1 &= 0.4 \cdot 0.4 \cdot 0.5/0.56 = 0.14 \\
p_2 &= 0.6 \cdot 0.6 \cdot 0.5/0.56 = 0.32 \\
p_3 &= 0.6 \cdot 0.4 \cdot 0.5/0.56 = 0.21 \\
p_{2,3} &= 0.6 \cdot 0.5 \cdot 0.6/0.56 = 0.32
\end{align*}
\]

where \( 0.08 + 0.18 + 0.12 + 0.18 = 0.56 \) is the normalization factor \( N \).

Let us consider the Sampson’s domestic violence models as another point of illustration. The model shown in Figure 11 is the result of merging the two fully compatible models given in Figure 8. We thus have \( \mathcal{M}_{\text{Compat}} = \{ \{M_1\}, \{M_2\}, \{M_{1,2}\} \} \) with \( M_{1,2} = M_1 \oplus M_2 \)
as given in Figure 4. Assuming that expert 1 is assigned a confidence value 0.6 and expert 2 is assigned 0.7, then we have

\[
\begin{align*}
    p_1 &= 0.6 \times 0.3 \times 0.58/0.44 = 0.23 \\
    p_2 &= 0.4 \times 0.7 \times 0.58/0.44 = 0.36 \\
    p_{1,2} &= 0.6 \times 0.7 \times 0.42/0.44 = 0.41
\end{align*}
\]

Note that the number of models in \( M_{Compat} \) may be exponential in the number of experts. For example, if the experts’ models are mutually compatible, then \( Compat \) consists of all subsets of \( \{1, \ldots, n\} \). The straightforward computation of interventions per model is exponential in the number of variables in the model. Since the number of variables in a merged model is at most the sum of the variables in each one, the problem is exponential in the number of experts and the total number of variables in the experts’ models. In practice, however, we do not expect this to pose a problem. For the problems we are interested in, there are typically few experts involved; moreover, as we argued in Section 3, policymakers, in practice, restrict their attention to interventions on a small set of variables. Thus, we expect that the computation involved to be manageable.

Up to now, we have assumed that each expert proposes only one deterministic causal model. An expert uncertain about the model can propose several (typically incompatible) models, with a probability distribution on them. We can easily extend our framework to handle this.

Suppose that expert \( i \), with probability \( p_i \) of being correct, proposes \( m \) models \( M_{i1}, \ldots, M_{im} \), where model \( M_{ij} \) has probability \( q_j \) of being the right one, according to \( i \). To handle this, we simply replace expert \( i \) by \( m \) experts, \( i_1, \ldots, i_m \), where expert \( i_j \) proposes model \( M_{ij} \) with probability \( p_i q_j \) of being correct. As long as each of a few experts has a probability on only a few models, this will continue to be tractable.

6. Conclusions

We have provided a method for combining causal models whenever possible, and used that as a basis for combining experts’ causal judgments in a way that gets around the impossibility result of Bradley et al. [1]. We provided a gradual weakening of our definition of full compatibility, allowing us to merge models that only agree on some of their parts. Our approach can be viewed as a formalization of what was done informally in earlier work [2, 24]. While our requirements for compatibility are certainly nontrivial, the examples that we have considered do suggest that our approach is quite applicable. That said, it would be interesting to consider alternative approaches to combining experts’ models. The approach considered by Friedenberg and Halpern [7] is one such approach; there may well be others.

In any case, we believe that using causal models as a way of formalizing experts’ judgments, and then providing a technique for combining these judgments, will prove to be a powerful tool with which to approach the problem of finding the best intervention(s) that can be performed to ameliorate a situation.

Appendix A. Proof of Proposition 4.8

Proof. For part (a), suppose that \( M_1 \sim_{C}^{\mu} M_2 \) and, by way of contradiction, that \( \text{Par}_{M_1}(C) \neq \text{Par}_{M_2}(C) \). We can assume without loss of generality that there is some
variable \( Y \in \text{Par}_{M_1}(C) - \text{Par}_{M_2}(C) \). Let \( \bar{Z} = \text{Par}_{M_1}(C) - \{ Y \} \). Since \( Y \) is a parent of \( C \) in \( M_1 \), there must be some setting \( \bar{z} \) of the variables in \( \bar{Z} \) and values \( y \) and \( y' \) for \( Y \) such that \( F_{\bar{z}}(y, \bar{z}) \neq F_{\bar{z}}(y', \bar{z}) \) in \( M_1 \), where \( F_{\bar{z}} = F_1(C) \). Suppose that \( F_{\bar{z}}(y, \bar{z}) = c \) and \( F_{\bar{z}}(y', \bar{z}) = c' \). Let \( \bar{X} = ((U_1 \cup V_1) \cap (U_2 \cup V_2)) \). By MI1, \( \text{Par}_{M_1}(C) \cup \text{Par}_{M_2}(C) \subseteq \bar{X} \).

Let \( \bar{x} \) be a setting of the variables in \( \bar{X} - \{ C \} \) such that \( \bar{x} \) agrees with \( \bar{z} \) for the variables in \( \bar{Z} \) and \( \bar{x} \) assigns \( y \) to \( Y \). Let \( \bar{x}' \) be identical to \( \bar{x} \) except that it assigns \( y' \) to \( Y \). Since the values of the variables in \( \text{Par}_{M_1}(C) \) determine the value of \( C \) in \( M_1 \), for all contexts \( \bar{u}_1 \) for \( M_1 \), we have \( (M_1, \bar{u}_1) \models \bar{X} \leftarrow \bar{x}(C = c) \) and \( (M_1, \bar{u}_1) \models \bar{X} \leftarrow \bar{x}'(C = c') \).

Since \( \bar{x} \) and \( \bar{x}' \) assign the same values to all the variables in \( \text{Par}_{M_2}(C) \), we must have \( (M_2, \bar{u}_2) \models \bar{X} \leftarrow \bar{x}(C = c) \) iff \( (M_2, \bar{u}_2) \models \bar{X} \leftarrow \bar{x}'(C = c) \) for all contexts \( \bar{u}_2 \) for \( M_2 \).

Thus, we get a contradiction to \( \text{MI}_{M_1,M_2,C,\bar{x}} \). It follows that \( \text{Par}_{M_1}(C) = \text{Par}_{M_2}(C) \).

The fact that \( F_1(C) = F_2(C) \) also follows from \( \text{MI}_{M_1,M_2,C,\bar{x}} \). For suppose that \( \bar{z} \) is a setting of the variables in \( \text{Par}_{1}(C) = \text{Par}_{2}(C) \) and \( \bar{x} \) is a setting of the variables in \( \bar{X} = X - \{ C \} \) that agrees with \( \bar{z} \) on the variables in \( \text{Par}_{1}(C) \). Then, for all contexts \( \bar{u}_1 \) for \( M_1 \) and \( \bar{u}_2 \) for \( M_2 \) such that \( \bar{u}_1 \) and \( \bar{u}_2 \) agree on the variables in \( U_1 \cap U_2 \), we have \( F_{\bar{z}}(\bar{z}) = c \) iff \( (M_1, \bar{u}_1) \models \bar{X} \leftarrow \bar{x}(C = c) \) iff \( (M_2, \bar{u}_2) \models \bar{X} \leftarrow \bar{x}(C = c) \) (by \( \text{MI}_{M_1,M_2,C,\bar{x}} \)) iff \( F_{\bar{z}}(\bar{z}) = c \). Thus, \( F_1(C) = F_2(C) \).

For part (b), note that \( M_1 \otimes^\alpha M_2 \) is well defined unless (i) \( R_1(C) \neq R_2(C) \) for some \( C \in (V_1 \cup V_2) \) or (ii) \( M_1 \sim^\alpha M_2 \) but \( F_1(C) \neq F_2(C) \) for some \( C \in V_1 \cap V_2 \).

Since \( M_1 \) and \( M_2 \) are fully compatible with respect to \( \bar{x} \), (i) cannot happen; by part (a), (ii) cannot happen.

For part (c), we first show part (d): if \( A \) and \( B \) are both nodes in \( M_1 \) (i.e., \( A \) and \( B \) are in \( U_1 \cup V_1 \)), then (the node labeled) \( A \) is an ancestor of (the node labeled) \( B \) in (the causal graph corresponding to) \( M_1 \) iff \( A \) is an ancestor of \( B \) in \( M_1 \otimes^\alpha M_2 \), and similarly for \( M_2 \).

Suppose that \( A \) is an ancestor of \( B \) in \( M_1 \). Then there is a finite path \( A_0, \ldots, A_n \) in the causal graph for \( M_1 \), where \( A_0 = A \) and \( A_n = B \). We first show that if \( A_0, \ldots, A_n \) is an arbitrary sequence of nodes in \( M_1 \) such that none of the intermediate nodes (i.e., \( A_1, \ldots, A_{n-1} \)) are in \( M_2 \), and either \( A_0 = A_n \) or at most one of \( A_0 \) and \( A_n \) is in \( M_2 \), then \( A_0, \ldots, A_n \) is a path in \( M_1 \) iff \( A_0, \ldots, A_n \) is a path in \( M_1 \otimes^\alpha M_2 \). We proceed by induction on \( n \), the length of the path. Since all the nodes in \( M_1 \) are nodes in \( M_1 \otimes^\alpha M_2 \), the result clearly holds if \( n = 0 \). Suppose that \( n > 0 \) and the result holds for \( n - 1 \); we prove it for \( n \). We cannot have \( A_n \in U_1 - V_2 \), since then \( A_n \) has no parents in \( M_1 \) or \( M_1 \otimes^\alpha M_2 \). If \( A_n \in V_1 - V_2 \) or \( A_n \in V_1 \cap V_2 \) and \( M_1 \otimes^\alpha M_2 \), then \( F_{1,2}(A_n) = F_1(A_n) \), so the parents of \( A_n \) in \( M_1 \) are also the parents of \( A_n \) in \( M_1 \otimes^\alpha M_2 \). In particular, \( A_{n-1} \) is a parent of \( A_n \) in \( M_1 \otimes^\alpha M_2 \) iff \( A_{n-1} \) is a parent of \( A_n \) in \( M_1 \otimes^\alpha M_2 \), and the result follows from the induction hypothesis. Finally, if \( A_n \in (U_1 \cup V_1) \cap V_2 \) and \( M_1 \otimes^\alpha M_1 \), then \( F_{1,2}(A_n) = F_2(A_n) \), so \( A_{n-1} \) must be in \( M_2 \). But this contradicts our assumption, that no intermediate nodes are in \( M_2 \) and at most one of \( A_0 \) and \( A_n \) is in \( M_2 \). This completes the argument. Note that the same argument applies if we reverse the roles of \( M_1 \) and \( M_2 \).

Now suppose that there are \( m > 0 \) nodes in \( M_2 \) on the path from \( A \) to \( B \) in \( M_1 \), say \( C_1, \ldots, C_m \), in that order. We show that (i) \( C_m \) is an ancestor of \( B \) in \( M_1 \otimes^\alpha M_2 \), (ii) \( A \) is an ancestor of \( C_1 \) in \( M_1 \otimes^\alpha M_2 \), and (iii) \( C_1 \) is an ancestor of \( C_m \) in \( M_1 \otimes^\alpha M_2 \). Parts (i) and (ii) follow from the earlier argument, since there are no intermediate nodes in \( M_2 \) on the path from \( C_m \) to \( B \) or on the path from \( A \) to \( C_1 \). So it remains to prove
part (iii). We proceed by induction on $m$. If $m = 1$, the result is trivially true, since $C_1$ is a node in $M_1 \oplus^{\psi} M_2$. So suppose that $m > 1$. Since $M_1$ and $M_2$ are fully compatible with respect to $\vec{v}^* \oplus \vec{v}^*$ and $C_2$ is a node in both $M_1$ and $M_2$, for $j > 1$, we must have either $M_1 \geq^{\psi} C_2$ or $M_2 \geq^{\psi} C_2$. If $M_1 \geq^{\psi} C_2$ then the parents of $C_2$ in $M_1$ are the parents of $C_2$ in $M_1 \oplus^{\psi} M_2$. In particular, if $D$ is the parent of $C_2$ on the path from $C_1$ to $C_2$ in $M_1$, then $D$ is a parent of $C_2$ in $M_1 \oplus^{\psi} M_2$. Since none of the intermediate nodes on the path from $C_1$ to $D$ in $M_1$ are in $M_2$ except for $C_1$, it follows by our earlier argument than the path from $C_1$ to $D$ in $M_1$ is also a path from $C_1$ to $D$ in $M_1 \oplus^{\psi} M_2$. Thus, $C_1$ is an ancestor of $C_2$ in $M_1 \oplus^{\psi} M_2$. If $M_2 \geq^{\psi} C_2$, then the parents of $C_2$ in $M_1$ must also be in $M_2$ (in fact, they must be $M_1$-immediate ancestors of $C_2$ in $M_2$). Since none of the intermediate nodes on the path from $C_1$ to $C_2$ is in $M_2$, it must be the case that the path from $C_1$ to $C_2$ has length 1, and $C_1$ is a parent of $C_2$ in $M_1$. By $M_{12}$, there is a path from $C_1$ to $C_2$ in $M_2$ none of whose intermediate nodes is in $M_1$. Then the same argument given for the case that $M_1 \geq^{\psi} C_2$ shows that this path in $M_2$ also exists in $M_1 \oplus^{\psi} M_2$. Thus, $C_1$ is an ancestor of $C_2$ in $M_1 \oplus^{\psi} M_2$ in this case as well. The fact that $C_2$ is an ancestor of $C_m$ in $M_1 \oplus^{\psi} M_2$ follows from the induction hypothesis. Thus, $C_1$ is an ancestor of $C_m$ in $M_1 \oplus^{\psi} M_2$.

For the converse, suppose that $A$ and $B$ are nodes in $M_1$ and $A$ is an ancestor of $B$ in $M_1 \oplus^{\psi} M_2$. We want to show that $A$ is an ancestor of $B$ in $M_1$. The argument is similar to that above, but slightly simpler. Again, there is a finite path $A_0, \ldots, A_n$ in the causal graph for $M_1 \oplus^{\psi} M_2$, where $A_0 = A$ and $A_n = B$. If none of the intermediate nodes on the path are in $M_2$ and at most one of $A_0$ and $A_n$ is in $M_2$, then our initial argument shows that this path also exists in $M_1$.

Now suppose that there are $m > 0$ nodes in $M_2$ on the path from $A$ to $B$ in $M_1 \oplus^{\psi} M_2$, say $C_1, \ldots, C_m$, in that order. Much like before, we show that (i) $C_m$ is an ancestor of $B$ in $M_1$, (ii) $A$ is an ancestor of $C_1$ in $M_1$, and (iii) $C_1$ is an ancestor of $C_m$ in $M_1$. And again, parts (i) and (ii) follow from the earlier argument, since there are no intermediate nodes in $M_2$ on the path from $C_m$ to $B$ or the path from $A$ to $C_1$. For part (iii), we again proceed by induction on $m$. If $m = 1$, the result is trivially true. So suppose that $m > 1$. Since $M_1$ and $M_2$ are fully compatible with respect to $\vec{v}^* \oplus \vec{v}^*$ and $C_2$ is a node in both $M_1$ and $M_2$ for $j > 1$, we must have either $M_1 \geq^{\psi} C_2$ or $M_2 \geq^{\psi} C_2$. If $M_1 \geq^{\psi} C_2$, then the parents of $C_2$ in $M_1$ are the parents of $C_2$ in $M_1 \oplus^{\psi} M_2$, so if $D$ is the parent of $C_2$ on the path from $C_1$ to $C_2$ in $M_1 \oplus^{\psi} M_2$, $D$ is a parent of $C_2$ in $M_1$. Since the path from $C_1$ to $D$ in $M_1 \oplus^{\psi} M_2$ has no intermediate nodes in $M_2$, we can apply earlier argument to show that there is a path from $C_1$ to $D$ in $M_1$ and complete the proof as before. If $M_2 \geq^{\psi} C_2$, then all the parents of $C_2$ in $M_1 \oplus^{\psi} M_2$ must be in $M_2$, so the path has length 1 and $C_1$ is a parent of $C_2$ in $M_1 \oplus^{\psi} M_2$ and in $M_2$. Thus, $C_1$ is an immediate $M_1$-ancestor of $C_2$ in $M_2$. $M_{1M_2}M_{12}$ implies that $C_1$ must be a parent of $C_2$ in $M_1$. Again, we can complete the proof as before.

The acyclicity of $M_1 \oplus^{\psi} M_2$ is now almost immediate. For suppose that there is a cycle $A_0, \ldots, A_n$ in the causal graph for $M_1 \oplus^{\psi} M_2$, where $A_0 = A_n$ and $n > 0$. Either $A_0$ and $A_{n-1}$ are both in $M_1$ (if $F_{1,2}(A_0) = F_{1}(A_n)$) or they are both in $M_2$ (if $F_{1,2}(A_n) = F_{2}(A_n)$). Suppose that they are both in $M_1$. Then, since $A_{n-1}$ is an ancestor of $A_n$ in $M_1 \oplus^{\psi} M_2$ and $A_n$ is an ancestor of $A_{n-1}$ in $M_1 \oplus^{\psi} M_2$, by the preceding argument, $A_{n-1}$ is an ancestor of $A_n$ in $M_1$ and $A_n$ is an ancestor of $A_{n-1}$ in $M_1$, contradicting the acyclicity of $M_1$. A similar argument applies if both $A_{n-1}$ and $A_n$
are in $M_2$.

For part (e), suppose that $\vec{u}_1$ and $\vec{u}_2$ are compatible with $\vec{v}^*$. It clearly suffices to show that $(M_1, \vec{u}_1) \models \varphi$ iff $(M_1 \oplus \vec{v}^* M_2, \vec{u}) \models \varphi$ if $\varphi$ has the form $[\vec{X} \leftarrow \vec{z}](Y = y)$, where $(\vec{X} \cup \{Y\}) \subseteq V_1$. To show this, it suffices to show that $((M_1)_{\vec{X}=\vec{w}}, \vec{u}_1) \models (Y = y)$ iff $((M_1 \oplus \vec{v}^* M_2)_{\vec{X}=\vec{w}}, \vec{u}) \models (Y = y)$. Define the depth of a variable $Y$ in a causal graph to be the length of the longest path from an exogenous variable to $Y$ in the graph. We prove, by induction on the depth of the variable $Y$ in the causal graph of $M_1 \oplus \vec{v}^* M_2$, that for all contexts $\vec{u}_1$ in $M_1$, $\vec{u}_2$ in $M_2$, and $\vec{u}$ in $M_1 \oplus \vec{v}^* M_2$, (i) if $Y \in \mathcal{U}_1 \cup \mathcal{V}_1$, $\vec{X} \subseteq \mathcal{V}_1$, and $\vec{u}$ and $\vec{u}_1$ are compatible with $\vec{v}^*$, then $((M_1)_{\vec{X}=\vec{w}}, \vec{u}_1) \models (Y = y)$ iff $((M_1 \oplus \vec{v}^* M_2)_{\vec{X}=\vec{w}}, \vec{u}) \models (Y = y)$, and (ii) if $Y \in \mathcal{U}_2 \cup \mathcal{V}_2$, $\vec{X} \subseteq \mathcal{V}_2$, $\vec{u}$ and $\vec{u}_2$ are compatible with $\vec{v}^*$, then $((M_2)_{\vec{X}=\vec{w}}, \vec{u}_2) \models (Y = y)$ iff $((M_1 \oplus \vec{v}^* M_2)_{\vec{X}=\vec{w}}, \vec{u}) \models (Y = y)$. (Note that if $Y \in (\mathcal{U}_1 \cup \mathcal{V}_1) \cap (\mathcal{U}_2 \cup \mathcal{V}_2)$, then it must satisfy both (i) and (ii).)

If $Y$ has depth 0, then $Y$ is an exogenous variable, and the result is immediate. Suppose that $Y$ has depth $d > 0$. If $Y \in \mathcal{V}_1 - (\mathcal{U}_2 \cup \mathcal{V}_2)$, then the parents of $Y$ in $M_1 \oplus \vec{v}^* M_2$ are the same as the parents of $Y$ in $M_1$; (i) is then immediate from the induction hypothesis and (ii) is vacuously true. Similarly, if $Y \in \mathcal{V}_2 - (\mathcal{U}_1 \cup \mathcal{V}_1)$, then (ii) is immediate from the induction hypothesis and (i) is vacuously true. If $Y \in (\mathcal{U}_1 \cup \mathcal{V}_1) \cap (\mathcal{U}_2 \cup \mathcal{V}_2)$ and $M_1 \supseteq M_2$, then again, the parents of $Y$ in $M_1 \oplus \vec{v}^* M_2$ are the same as the parents of $Y$ in $M_1$, so (i) is immediate from the induction hypothesis. To show that (ii) holds, fix appropriate contexts $\vec{u}_2$ and $\vec{u}$. Now the parents of $Y$ in $M_2$ are the immediate $M_2$-ancestors of $Y$ in $M_1$. Let $\vec{Z} = \text{Par}_{M_2}(Y)$. It follows from the arguments for part (c) that for all $Z \in \text{Par}_{M_2}(Y)$, all the paths from $Z$ to $Y$ in $M_1$ also exist in $M_1 \oplus \vec{v}^* M_2$ and the parents of $Y$ in $M_2$ are exactly the immediate $M_2$-ancestors of $Y$ in $M_1 \oplus \vec{v}^* M_2$. That is, $\vec{Z}$ screens $Y$ from all other variables in $M_2$ not only in $M_2$, but also in $M_1$ and $M_1 \oplus \vec{v}^* M_2$. Suppose that $((M_2)_{\vec{X}=\vec{w}}, \vec{u}_2) \models \vec{Z} = \vec{z}$. It follows from the induction hypothesis that $((M_1 \oplus \vec{v}^* M_2)_{\vec{X}=\vec{w}}, \vec{u}) \models \vec{Z} = \vec{z}$. Let $\vec{W} = ((\mathcal{U}_1 \cup \mathcal{V}_1) \cap (\mathcal{U}_2 \cup \mathcal{V}_2)) - \{Y\}$. Let $\vec{w}$ be a setting for $\vec{W}$ that agrees with $\vec{z}$ on the variables in $\vec{Z}$. Then we have the following chain of equivalences:

\[
\begin{align*}
((M_2)_{\vec{X}=\vec{w}}, \vec{u}_2) &\models \vec{Z} = \vec{z} & \text{if} & ((M_1)_{\vec{X}=\vec{w}}, \vec{u}_1) \models \vec{Z} = \vec{z} \\
((M_2)_{\vec{X}=\vec{w}}, \vec{u}_2) &\models [\vec{W} \leftarrow \vec{w}](Y = y) & \text{if} & ((M_2, \vec{u}_2) \models [\vec{W} \leftarrow \vec{w}](Y = y)\text{ [by MI4M_1,M_2,Y,\vec{v}^*]} \\
((M_2)_{\vec{X}=\vec{w}}, \vec{u}_2) &\models [\vec{W} \leftarrow \vec{w}](Y = y) & \text{if} & ((M_1, \vec{u}_1) \models [\vec{Z} \leftarrow \vec{z}](Y = y) \\
((M_1)_{\vec{X}=\vec{w}}, \vec{u}_1) &\models \vec{Z} = \vec{z} & \text{if} & ((M_1)_{\vec{X}=\vec{w}}, \vec{u}_1) \models \vec{Z} = \vec{z} \text{ [already shown]}
\end{align*}
\]

This completes the proof of (d).

Part (f) is immediate from the definitions.

For part (g), suppose that $M_1 = (\mathcal{U}_1, \mathcal{V}_1, \mathcal{R}_1, \mathcal{F}_1)$, $M_2 = (\mathcal{U}_2, \mathcal{V}_2, \mathcal{R}_2, \mathcal{F}_2)$, $M_3 = (\mathcal{U}_3, \mathcal{V}_3, \mathcal{R}_3, \mathcal{F}_3)$, $M_4 = (\mathcal{U}_4, \mathcal{V}_4, \mathcal{R}_4, \mathcal{F}_4)$, $M_5 = (\mathcal{U}_5, \mathcal{V}_5, \mathcal{R}_5, \mathcal{F}_5)$, $M_6 = (\mathcal{U}_6, \mathcal{V}_6, \mathcal{R}_6, \mathcal{F}_6)$, $M_7 = (\mathcal{U}_7, \mathcal{V}_7, \mathcal{R}_7, \mathcal{F}_7)$, $M_8 = (\mathcal{U}_8, \mathcal{V}_8, \mathcal{R}_8, \mathcal{F}_8)$, $M_9 = (\mathcal{U}_9, \mathcal{V}_9, \mathcal{R}_9, \mathcal{F}_9)$, $M_{10} = (\mathcal{U}_{10}, \mathcal{V}_{10}, \mathcal{R}_{10}, \mathcal{F}_{10})$, $M_{11} = (\mathcal{U}_{11}, \mathcal{V}_{11}, \mathcal{R}_{11}, \mathcal{F}_{11})$, $M_{12} = (\mathcal{U}_{12}, \mathcal{V}_{12}, \mathcal{R}_{12}, \mathcal{F}_{12})$, $M_{13} = (\mathcal{U}_{13}, \mathcal{V}_{13}, \mathcal{R}_{13}, \mathcal{F}_{13})$, $M_{14} = (\mathcal{U}_{14}, \mathcal{V}_{14}, \mathcal{R}_{14}, \mathcal{F}_{14})$, $M_{15} = (\mathcal{U}_{15}, \mathcal{V}_{15}, \mathcal{R}_{15}, \mathcal{F}_{15})$, $M_{16} = (\mathcal{U}_{16}, \mathcal{V}_{16}, \mathcal{R}_{16}, \mathcal{F}_{16})$.
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