Group analysis of an ideal plasticity model

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Abstract
A group analysis of a system describing an ideal plastic flow is made in order to obtain analytical solutions. The complete Lie algebra of point symmetries of this system are given. Two of the infinitesimal generators that span the Lie algebra are original to this paper. A classification into conjugacy classes of all one- and two-dimensional subalgebras is performed. Invariant and partially invariant solutions corresponding to certain conjugacy classes are obtained using the symmetry reduction method. Solutions of algebraic, trigonometric, inverse trigonometric and elliptic type are provided as illustrations and other solutions expressed in terms of one or two arbitrary functions have also been found. For some of these solutions, a physical interpretation allows one to determine the shape of feasible extrusion dies corresponding to these solutions. The corresponding tools could be used to curve rods or slabs, or to shape a ring in an ideal plastic material by an extrusion process.

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1. Introduction
This work is an investigation of a system modelling the plane flow of an incompressible ideal plastic material [2–4], defined by the following four equations:

\begin{align}
\sigma_x - 2k(\theta_x \cos 2\theta + \theta_y \sin 2\theta) &= 0, \\
\sigma_y - 2k(\theta_x \sin 2\theta - \theta_y \cos 2\theta) &= 0, \\
(u_y + v_x) \sin 2\theta + (u_x - v_y) \cos 2\theta &= 0, \\
u_x + v_y &= 0,
\end{align}

where \( \sigma_x = \partial \sigma / \partial x \), etc. These partial differential equations (PDEs) form a hyperbolic system involving \( q = 4 \) dependent variables \( \sigma, \theta, u, v \) and \( p = 2 \) independent variables \( x \) and \( y \). The equilibrium equations (1a) and (1b) correspond to the Cauchy differential equations of
motion in a continuous medium, if we assume that the dependent variables do not depend on \(z\), and involve the quantities \(\sigma\) and \(\theta\). The quantity \(\sigma\) is the mean pressure and \(\theta\) is the angle measured from the \(x\)-axis in the counterclockwise direction minus \(\pi/4\). The equation (1c) relates \(\theta\) to the velocity components \(u\) and \(v\) along the \(x\)-axis and \(y\)-axis, respectively. This equation corresponds to the Saint-Venant–Von Mises plasticity theory equations in the planar case. The positive-definite constant \(k\) is called the yield limit and it is associated with the plastic material. Without loss of generality we assume that \(k = 1/2\) (this is the same as re-scaling the pressure \(\sigma\)).

In a recent work [20], the concept of homotopy of two functions has been used to construct two families of exact solutions for the system formed by the two first equations in (1). In [5, 6], the Nadaï solution [7] for a circular cavity under normal stress and shear and the Prandtl solution [8] for a bloc compressed between two plates has been mapped by elements of a symmetry group of the system consisting of (1a) and (1b), in order to calculate new solutions. In addition, in [9], simple and double Riemann wave solutions for the system (1) were found using the method of characteristics. However, as is often the case with this method, those solutions rely on numerical integration for obtaining the velocities \(u\) and \(v\). Symmetries of the system (1) were found in [1]. However, the Lie algebra of symmetries was not complete because of the absence of the generators \(B_1\) and \(K\) (or equivalents) defined by (4) in section 2 of this work. Moreover, we found two infinite-dimensional subalgebras spanned by generators \(X_1\) and \(X_2\) provided below in equation (5). The generator \(X_1\) was known [10] as a symmetry of equations (1a) and (1b) and \(X_2\) was known [11] as a symmetry of equations (1c) and (1d), but these are shown in this paper to also be symmetries of the complete system (1). A classification of the one-dimensional subalgebras was performed in [1]. Some invariant and partially invariant solutions were obtained in, see e.g. [1, 12], but, to our knowledge, there has been no systematic Lie group analysis based on a complete subalgebra classification into conjugacy classes under the action of the symmetry group \(G\) of the system (1) that includes the new found generators.

The objective of this work is to obtain analytical solutions of system (1). This is achieved by a systematic analysis of the Lie group of point symmetries \(G\) in order to find invariant and partially invariant (of structure defect \(\delta = 1\) in the sense defined by Ovsiannikov [13]) by the use of the symmetry reduction method (SRM). To ensure that the obtained solutions are non-equivalent in the sense that they cannot be obtained from one another by a transformation of \(G\) (the solutions are not in the same group orbit), a classification of the subalgebras of the Lie algebra \(\mathcal{L}\) associated to \(G\) into conjugacy classes [14–16] has been performed. Two subalgebras \(\mathcal{L}_i \subset \mathcal{L}\) and \(\mathcal{L}_i' \subset \mathcal{L}\) are conjugate if \(G\mathcal{L}_iG = \mathcal{L}_i'\). This classification consists of a list of representative subalgebras, one for each class. Each representative subalgebra can be used to apply the SRM in order to find non-equivalent solutions as discussed above. Many examples of algebraic and implicit solutions are constructed in this paper. For some of these solutions, we deduced the shape of a corresponding extrusion die. This deduction is based on the fact that the curves defining the contours of a die must coincide with the flow lines of the velocity field \((u, v)\) for a corresponding solution. In this paper, we consider extrusion dies fed rectilinearly at constant speed since they are more convenient for applications. So, the curve defining the beginning of the plasticity region is defined by the ordinary differential equation (ODE)

\[
\frac{dy}{dx} = \frac{V_0 - v(x, y)}{U_0 - u(x, y)}
\]

where \(U_0, V_0\) are components of the feeding velocity along the \(x\)-axis and \(y\)-axis, respectively. Equation (2) is a consequence of mass conservation and of the incompressibility of the material and also defines the limit of the plasticity region at the end of the die if \(U_0, V_0\) are interpreted
as a constant velocity at which the material is expelled from the die. One should note that the conditions (2) are reduced to those required on the limits of the plasticity region in [9] when \( V_0 = 0 \) and that the curves defining the limits coincide with slip lines (characteristics), which obey the equation

\[
\frac{dy}{dx} = \tan \theta(x, y) \quad \text{or} \quad \frac{dy}{dx} = -\cot \theta(x, y).
\]

Hence, equation (2) can be viewed as a relaxation of the boundary conditions given in [9]. These relaxed conditions can be explained by the fact that we choose the contours of the die to coincide with some flow lines of the solution instead of requiring the flow lines of the solution to be parallel to the contours. With the relaxed conditions (2), we can choose (with certain restrictions) the feeding speed and direction for a tool and this determines the limits of the plasticity region.

This paper is organized as follows. Section 2 deals with the symmetries of the system (1). More precisely, the infinitesimal generators spanning the Lie algebra of point symmetries and discrete transformations that leave the system (1) invariant are presented. This is followed by a description of the main steps of the classification procedure. In section 3, the SRM is applied to the system (1) and invariant as well as partially invariant solutions are constructed. We conclude this paper with a discussion on the obtained results and some future outlook.

### 2. Symmetry algebra and classification of its subalgebras

In this section we study the symmetries of the system (1). Following the standard algorithm [17], the Lie symmetry algebra \( \mathfrak{g} \) of the system has been determined. The Lie algebra \( \mathfrak{g} \) is spanned by the seven infinitesimal generators (where we have used the notation \( \partial_x = \partial/\partial_x \), etc)

\[
P_0 = \partial_x, \quad D_1 = x\partial_x + y\partial_y + u\partial_u + v\partial_v, \quad D_2 = x\partial_x + y\partial_y - u\partial_u - v\partial_v,
\]

\[
L = -y\partial_x + x\partial_y - v\partial_u + u\partial_v, \quad B_1 = -v\partial_x + u\partial_y, \quad B_2 = y\partial_x - x\partial_y,
\]

\[
K = \left( -\frac{1}{2} \cos 2\theta \left( \sigma + \frac{1}{2} \sin 2\theta \right) \right) \partial_x + \left( (\sigma - \frac{1}{2} \sin 2\theta) x + \frac{1}{2} y \cos 2\theta \right) \partial_y
\]

\[
+ \left( \frac{1}{2} \cos 2\theta + v \left( \frac{1}{2} \sin 2\theta - \sigma \right) \right) \partial_u + \left( (\sigma + \frac{1}{2} \sin 2\theta) u - \frac{1}{2} v \cos 2\theta \right) \partial_v
\]

\[
+ \theta \partial_\sigma + \sigma \partial_\eta,
\]

where the coefficients \( \xi \) and \( \eta \) must satisfy the two quasilinear first order PDEs

\[
\xi_x = \cos 2\theta \xi_\phi + \sin 2\theta \eta_\phi, \quad \xi_\phi = \cos 2\theta \xi_\sigma + \sin 2\theta \eta_\sigma.
\]

while the coefficients \( \phi \) and \( \psi \) must satisfy the two PDEs, of the same type as the previous ones,

\[
\phi_x = -\left( \cos 2\theta \phi_\phi + \sin 2\theta \psi_\phi \right), \quad \phi_\phi = -\left( \cos 2\theta \phi_\sigma + \sin 2\theta \psi_\sigma \right).
\]

Note that \( X_1 \) and \( X_2 \) span infinite-dimensional subalgebras. The generators \( D_1 \) and \( D_2 \) generate dilations in the space of the independent variables \( \{x, y\} \) and dependent variables \( \{u, v\} \). The generator \( L \) corresponds to a rotation, \( P_0 \) to a translation, and \( B_1 \) and \( B_2 \) to boost-like transformations. Of these seven generators five were already known [1, 10, 11], but two are original to this paper. The newly found generators are \( B_1 \) and \( K \). In addition, the system (1) admits two infinite-dimensional subalgebras. The one spanned by \( X_1 \) was known [1] as a symmetry of the two first equations of the system (1) and the one spanned by \( X_2 \) was known [11] as a symmetry of the last two equations of the system (1), but they are still symmetries of
the entire system (1). One should note that the system (1) is also invariant under the discrete transformations:

\[ R_1 : x \mapsto -x, \ y \mapsto -y, \ \sigma \mapsto \sigma, \ \theta \mapsto \theta, \ u \mapsto u, \ v \mapsto v; \]

\[ R_2 : x \mapsto x, \ y \mapsto y, \ \sigma \mapsto \sigma, \ \theta \mapsto \theta, \ u \mapsto -u, \ v \mapsto -v. \]  

These transformations \( R_1 \) and \( R_2 \) are rotations by an angle \( \pi \) in the plane of independent variables \( x, y \) and of dependent variables \( u, v \), respectively, that induce the following automorphisms of the Lie algebra \( \mathcal{L} \):

\[ \mathcal{R}_1 : D_1 \mapsto D_1, \ D_2 \mapsto D_2, \ B_1 \mapsto -B_i, \ i = 1, 4, \ B_1 \mapsto B_i, \ i = 5, 6, \]

\[ P_1 \mapsto -P_i, \ i = 1, 2, \ P_1 \mapsto P_i, \ i = 3, 4, 5, \ L \mapsto L, \ K \mapsto K; \]

\[ \mathcal{R}_2 : D_1 \mapsto D_1, \ D_2 \mapsto D_2, \ B_1 \mapsto -B_i, \ i = 1, 2, 5, 6, \ B_1 \mapsto B_i, \ i = 3, 4, \]

\[ P_1 \mapsto P_i, \ i = 1, 2, 5 \ P_1 \mapsto -P_i, \ i = 3, 4, \ L \mapsto L, \ K \mapsto K. \]  

The symmetry Lie algebra \( \mathcal{L} \) of system (1) has the following semi-direct sum decomposition

\[ \mathcal{L} = \mathcal{S} \triangleright \mathcal{X}, \]  

where \( \mathcal{S} = \{B_1, D_2, B_2, K, L, P_0, D_1\} \) and \( \mathcal{X} = \{X_1, X_2\} \) is an infinite-dimensional Abelian ideal generated by all generators of the form (5) whose coefficients satisfy equations (6) and (7). Given that subalgebra \( \mathcal{S} \) is a factor of a semi-direct sum, the list of conjugation classes of \( \mathcal{S} \) under the action of \( \exp(\mathcal{S}) \) is included in the list of conjugation classes of \( \mathcal{L} \) under the action of \( \exp(\mathcal{L}) \). Indeed, since every element of \( \mathcal{L} \) can be written as the sum \( s + x \) of an element of \( s \in \mathcal{S} \) and \( x \in \mathcal{X} \), it is possible to consider the action through conjugation of an element \( \exp(s + x) \in \exp(\mathcal{L}) \) on the basis elements \( \{s_j\} \) of a subalgebra \( \mathcal{S}_i \subset \mathcal{S} \). By using the Baker–Campbell–Hausdorff identity, it is easily seen that

\[ \exp(-s + x)s_j \exp(s + x) = \exp(-s)s_j \exp(s + \tilde{x}_i), \quad \tilde{x}_i \in \mathcal{X}. \]

Therefore, a subalgebra \( \mathcal{S}_i \) is conjugate to a subalgebra \( \mathcal{S}_k \) only if the \( \tilde{x}_i \) terms cancel out for all basis elements, but in this case the effect of the action through conjugation of \( \exp(s + x) \) is the same as that of \( \exp(s) \). Consequently, two subalgebras \( \mathcal{S}_i, \mathcal{S}_k \subset \mathcal{S} \) are conjugate under the action of \( \exp(\mathcal{L}) \) only if they are conjugate under the action of \( \exp(\mathcal{S}) \). Since the usual classification procedure into conjugation classes of the subalgebra of a semi-direct sum \( \mathcal{S} \triangleright \mathcal{X} \) requires us to first classify the subalgebras of \( \mathcal{S} \) into conjugation classes, it is justified to perform the classification of the subalgebra \( \mathcal{S} \) under the action of \( \exp(\mathcal{S}) \) and of the automorphisms (9). The commutation relations for \( \mathcal{S} \) are given in table 1.

Moreover, it should be noted that equations (6) and (7) admit constant solutions for the coefficient \( \xi, \eta, \phi, \psi \). The generators associated with these constant solutions are

\[ P_1 = \partial_x, \quad P_2 = \partial_y, \quad P_3 = \partial_u, \quad P_4 = \partial_v. \]  

| \( \mathcal{S} \) | \( B_1 \) | \( D_2 \) | \( B_2 \) | \( K \) | \( L \) | \( P_0 \) | \( D_1 \) |
|---|---|---|---|---|---|---|---|
| \( B_1 \) | 0 | 2B_1 | -D_2 | 0 | 0 | 0 | 0 |
| \( D_2 \) | -2B_1 | 0 | 2B_2 | 0 | 0 | 0 | 0 |
| \( B_2 \) | D_2 | -2B_2 | 0 | 0 | 0 | 0 | 0 |
| \( K \) | 0 | 0 | 0 | -P_0 | -L | 0 | 0 |
| \( L \) | 0 | 0 | 0 | P_0 | L | 0 | 0 |
| \( P_0 \) | 0 | 0 | 0 | L | 0 | 0 | 0 |
| \( D_1 \) | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

Table 1. Commutation relations for the algebra \( \mathcal{S} \).
which represent translations. Next, using the fact that $[K, X_1] \in \mathcal{L}$ and $[K, X_2] \in \mathcal{L}$, four new generators are obtained

\begin{align*}
B_3 &= (\sigma + 1/2 \sin 2\theta) \partial_x - 1/2 \cos 2\theta \partial_y, \quad B_4 = -1/2 \cos 2\theta \partial_x + (\sigma - 1/2 \sin 2\theta) \partial_y, \\
B_5 &= (\sigma - 1/2 \sin 2\theta) \partial_x + 1/2 \cos 2\theta \partial_y, \quad B_6 = 1/2 \cos 2\theta \partial_x + (\sigma + 1/2 \sin 2\theta) \partial_y,
\end{align*}

where, $B_i, i = 1, \ldots, 4$ are associated with a type of boost. This procedure can be applied iteratively to generate four additional generators, and so on. Since the degree of the coefficients of the obtained generators increases at each stage, as a polynomial in $\sigma$, it is clear that this procedure produces an infinite-dimensional subalgebra $\mathcal{Z} \subset \mathcal{X}$. As an example, if we begin with the trivial solution

$$\xi(\sigma, \theta) = 1, \quad \eta(\sigma, \theta) = 0$$

of the system (6), we obtain that the translation $P_1 = \partial_x$ is a symmetry of the system. Next, replacing solution (13) into the commutator

$$[K, X_1] = (\theta \xi_x + \sigma \xi_v + 1/2 \xi \cos(2\theta) + \sigma \eta + 1/2 \eta \sin(2\theta)) \partial_x + (\theta \eta_x + \sigma \eta_v - \sigma \xi + 1/2 \xi \sin(2\theta) - 1/2 \eta \cos(2\theta)) \partial_y,$$

we obtain that $1/2 \cos(2\theta) \partial_x + (1/2 \sin(2\theta) - \sigma) \partial_y$ is an additional symmetry generator associated with $P_1$. Therefore, $\xi = 1/2 \cos(2\theta), \eta = 1/2 \sin(2\theta) - \sigma$ is another solution of (6) that we can itself replace into the commutator (14) in order to obtain yet another generator. It follows that a subalgebra which includes $K$ and $\partial_x$ is infinite-dimensional. A similar procedure can be applied to generator $X_3$ and system (7).

Let us now consider the structure of the subalgebra

$$\mathcal{L} = S \triangleright \mathcal{Z},$$

where $S$ is the factor algebra in the semi-direct sum decomposition (10) of $\mathcal{L}$ and $\mathcal{Z} \subset \mathcal{X}$ is the Abelian ideal generated by the generators defined in the way mentioned above, using the constant solution. It should be noted that the subalgebra $\mathcal{L}$ is the smallest subalgebra including $S$ that contains the translations $P_1-P_3$ and that subalgebras with such components often lead to physically interesting solutions through symmetry reduction. This justifies an enhanced study of subalgebra $\mathcal{L} \subset \mathcal{L}$. We denote:

$$\mathcal{Z} = \{Z_i^n, Z_2^n, Z_3^n, Z_4^n\}, \quad n \in \mathbb{N}^*.$$

and we suppose that the elements of $\mathcal{Z}$ satisfy the commutation relations

$$Z_i^{n+1} = [K, Z_i^n], \quad i = 1, \ldots, 4, \quad n \geq 1,$$

where $K$ is defined in equations (4). Assuming the commutation relations (17) and using those given in table 1, we determine the form of commutation relations of elements of $S$ with those of $\mathcal{Z}$. The result is that the commutation relations are defined recursively by the equations

\begin{align*}
[L, Z_i^n] &= [P_0, Z_i^{n-1}] + [K, [L, Z_i^{n-1}]], \\
[P_0, Z_i^n] &= [L, Z_i^{n-1}] + [K, [P_0, Z_i^{n-1}]], \\
[B_j, Z_i^n] &= [K, [B_j, Z_i^{n-1}]], \quad j = 1, 2, \\
[D_j, Z_i^n] &= [K, [D_j, Z_i^{n-1}]], \quad j = 1, 2,
\end{align*}

where $n > 1$ and $i = 1, \ldots, 4$. The recurrence relations are solved in terms of the first-order commutators. As a result, we obtain the commutation relations

$$[L, Z_i^n] = \sum_{m=0}^{n-1} a(n, m) (m)[K, [\Omega(n + m), Z_i^m]], \quad n > 1,$$
\[ [P_0, Z^n] = \sum_{m=0}^{n-1} \alpha(n, m) (m) [K, [\Omega(n + m + 1), Z^n]], \quad n > 1, \]
\[ [B_j, Z^n] = (n-1) [K, [B_j, Z^n]], \quad n > 0, \]
\[ [D_j, Z^n] = (n-1) [K, [B_j, Z^n]], \quad n > 0, \] (19)

where we denote

\[ \Omega(m) = (1 - \bar{m})P_0 + \bar{m}L, \quad \bar{m} = m \mod 2, \] (20)

\[ (n+1)[K, Y] = [K, \alpha(n) [K, Y]], \quad (1)[K, Y] = [K, Y], \quad (0)[K, Y] = Y, \] (21)

and \( \alpha(n, m) \) is a binomial coefficient defined by:

\[ \alpha(n, m) = \binom{n-1}{m}. \] (22)

Consequently, all the commutation relations of elements of the subalgebra \( L \subset \mathcal{L} \) are expressed in terms of the first-order \( (n = 1) \) commutation relations. If we consider

\[ Z^1_1 = P_1, \quad Z^1_2 = P_2, \quad Z^3_1 = P_3, \quad Z^4_1 = P_4, \] (23)

then, through the formula (17), we obtain

\[ Z^2_1 = -B_4, \quad Z^2_2 = B_3, \quad Z^2_3 = -B_6, \quad Z^2_4 = B_5, \] (24)

where the \( B_i \) are given in equations (12). Now, using the commutation relations of generators \( P_1\ldots P_4 \) with the elements of \( S \):

\[ [L, P_1] = -P_2, \quad [L, P_2] = P_1, \quad [L, P_3] = -P_4, \quad [L, P_4] = P_3, \]
\[ [B_1, P_1] = [B_1, P_2] = 0, \quad [B_1, P_3] = -P_2, \quad [B_1, P_4] = P_1, \]
\[ [B_2, P_1] = -P_3, \quad [B_2, P_2] = -P_3, \quad [B_2, P_3] = [B_2, P_4] = 0, \]
\[ [P_0, P_i] = 0, \quad i = 1, \ldots, 4, \quad [D_i, P_1] = -P_i, \quad i = 1, \ldots, 4 \]
\[ [D_2, P_i] = -P_i, \quad i = 1, 2, \quad [D_2, P_i] = P_i, \quad i = 3, 4, \] (25)

we obtain, through the formulæ (19), the commutation relations for the generators \( B_3\ldots B_6 \) with form

\[ [L, B_3] = -B_4, \quad [L, B_4] = B_3, \quad [P_0, B_3] = P_1 \quad [P_0, B_4] = P_2, \]
\[ [L, B_5] = -B_6, \quad [L, B_6] = B_5, \quad [P_0, B_5] = P_1 \quad [P_0, B_6] = P_4, \]
\[ [B_1, B_3] = [B_1, B_4] = 0 \quad [B_1, B_5] = -B_4 \quad [B_1, B_6] = B_3, \]
\[ [B_2, B_3] = B_6, \quad [B_2, B_4] = -B_5 \quad [B_2, B_5] = B_6, \quad [B_2, B_6] = 0, \]
\[ [D_1, B_i] = -B_i, \quad i = 3, \ldots, 6, \]
\[ [D_2, B_i] = -B_i, \quad i = 3, 4, \quad [D_2, B_i] = B_i, \quad i = 5, 6. \] (26)

It is straightforward to show that the second-order generators \( (n = 2) \) defined by (12) satisfy the commutation relations (26). In addition, relations (25) and (26) satisfy the recurrence relations (18) for \( n = 2 \). Thus, the structure of the subalgebra \( \mathcal{L} \) defined by (15), with \( \mathcal{Z} \) defined by (16), (17) and (23), is completely established by the commutation relations in table 1, to which we add relations (19) and \( [Z^n_i, Z^j_i] = 0, \quad i, j = 1, \ldots, 4, \quad n, m \geq 1. \) In the following subsections, we describe the classification of classes having a representative subalgebra inside the subspace \( \mathcal{L}_2 \subset \mathcal{L} \) which are not subalgebras as for \( \mathcal{L} \).

Since we look for solutions that are invariant and partially invariant of defect structure \( \delta = 1 \), we only have to classify the subalgebras of codimension 1 and 2. We consider separately the subalgebras of \( \mathcal{S} \) and those contained in some subspace of \( \mathcal{L} \).
For the sake of classification, we decompose the seven-dimensional subalgebra \( S \) into the direct sum \( \mathcal{G} \oplus \{D_1\} \), with \( \mathcal{G} = \{B_1, D_2, B_2, K, L, P_0\} \), which is further decomposed as follows:

\[
\mathcal{G} = A \oplus M,
\]

where \( A \) is a simple algebra already classified in [18] and \( M = \{K, L, P_0\} \) is the ideal. Applying the method [14–16] (summarized in [19]), we proceed to classify all subalgebras of \( S \) into conjugacy classes under the action of the automorphisms generated by \( G \) and the discrete transformations (8). In practice, we can classify the subalgebras under the automorphisms \( G \) and decrease the range of the parameters that appear in the representative subalgebra of a class using the Lie algebra automorphisms (9). The classification results for \( S \) are shown in table 2 for subalgebras of codimension 1 and in table 3 for subalgebras of codimension 2.

### 2.2. Classification with representative subalgebras contained in a subspace \( \mathcal{L}_1 \subset \mathcal{L} \)

Denote by \( \mathcal{L}_0 = \{P_1, P_2, P_3, P_4\} \subset \mathcal{L} \), the subspace generated by the generators (11). The space

\[
\mathcal{L}_0 = \mathcal{S} \times \mathcal{Z}_0
\]

generated by the combined generators of \( \mathcal{S} \) and of \( \mathcal{Z}_0 \) is not a Lie algebra. The same is true for all subspaces of form \( \mathcal{L}_n = \mathcal{S} \times \mathcal{Z}_0 \times \cdots \times \mathcal{Z}_n \), where the subspace \( \mathcal{Z}_n \) is generated by the generators obtained from the generators of \( \mathcal{Z}_{n-1} \) through the commutation relations \([K, X_1]\) and \([K, X_2]\), as discussed previously. For example, \( \mathcal{Z}_1 \) is the subspace generated by generators (12). Such subspaces \( \mathcal{L}_n \) are not Lie algebra since the commutator \([K, z], z \in \mathcal{Z}_n\), is an element of \( \mathcal{L}_{n+1} \) with a non-zero component in \( \mathcal{Z}_{n+1} \). This, however, does not prevent us from carrying out a classification of the subalgebras of \( \mathcal{L} \) under the action of the symmetry group \( G = \text{exp}(\mathcal{L}) \) and of the automorphism (9) with the property that it contains a representative in a given subspace \( \mathcal{L}_n \). Given that conjugation involves the complete symmetry group, the subalgebra conjugacy classes contained in the space \( \mathcal{L}_n \) also appear in the list of subalgebra classes contained in the space \( \mathcal{L}_m, n < m \). Therefore, at each step, the classification of the subalgebras of the space \( \mathcal{L}_n \) is based on the classification of the subalgebras of \( \mathcal{L}_{n-1} \). The procedure for the classification of the subalgebras of a subspace \( \mathcal{L} \) (which is not a Lie algebra) of a Lie algebra \( \mathcal{L} \) into conjugacy classes under the action of \( G \) therefore remains the same. Evidently, the action of \( G \) can make a subalgebra contained in space \( \mathcal{L} \) conjugate to a subalgebra with non-zero components outside of \( \mathcal{L} \). In this case, we simply choose the representative contained in \( \mathcal{L} \).

### 2.2.1. Subalgebras contained in \( \mathcal{L}_0 \)

Consider first conjugacy classes with representative in \( \mathcal{L}_0 \). It should be noted that the conjugacy classes of \( \mathcal{S} \) found previously have representatives

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### Table 2. List of one-dimensional representative subalgebras of \( \mathcal{S} \). The parameters are \( a, \lambda, \lambda_0 \in \mathbb{R}, \lambda > 0, \lambda_0 \geq 0 \) and \( \epsilon, \epsilon_1, \epsilon_2 = \pm 1 \).

| \( S_{1,1} \) | \( S_{1,2} \) | \( S_{1,3} \) | \( S_{1,4} \) | \( S_{1,5} \) | \( S_{1,6} \) | \( S_{1,7} \) | \( S_{1,8} \) |
|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| \{B_1\}        | \{B_1 + K + aD_1\} | \{B_1 + L + aD_1\} | \{B_1 + L + \epsilon P_0 + aD_1\} | \{B_1 + P_0 + aD_1\} | \{B_1 + D_1\} | \{D_2 + \lambda K + aD_1\} | \{D_2 + \epsilon_1 L + \epsilon_2 P_3 + aD_1\} |
| \{D_1\}        | \{D_2 + \lambda L + aD_1\} | \{D_2 + \lambda D_1\} | \{B_1 - B_2 + \lambda_0 K + aD_1\} | \{B_1 - B_2 + L + \epsilon P_0 + aD_1\} | \{B_1 - B_2 + \lambda L + aD_1\} | \{B_1 - B_2 + \lambda D_1\} | \{B_1 - B_2 + \lambda_0 K + aD_1\} |
| \{K + aD_1\}   | \{L + \epsilon P_0\} | \{L + \epsilon_1 P_0 + \epsilon_2 D_1\} | \{L + aD_1\} | \{P_0 + aD_1\} | \{D_1\} | \{D_1\} | \{D_1\} |
| \{K\}          | \{L\}           | \{L\}           | \{L\}           | \{L\}           | \{L\}           | \{L\}           | \{L\}           |
\[
\begin{array}{ll}
S_{2.1} &= \{b_1, D_2, a_1, K + a_2, D_1\} \\
S_{2.2} &= \{b_1, D_1, \epsilon_1 (L + \epsilon \cdot P_3 + aD_1)\} \\
S_{2.3} &= \{b_1, D_1, \delta (L + \delta_1 D_1)\} \\
S_{2.4} &= \{b_1, D_1, \delta_0 (P_3 + aD_1)\} \\
S_{2.5} &= \{b_1, K + aD_1\} \\
S_{2.6} &= \{b_1, L + \epsilon \cdot P_3 + aD_1\} \\
S_{2.7} &= \{b_1, L + aD_1\} \\
S_{2.8} &= \{b_1, P_3 + aD_1\} \\
S_{2.9} &= \{b_1, D_1\} \\
S_{2.10} &= \{b_1 + K + \delta_1 D_1, L + \epsilon \cdot P_3\} \\
S_{2.11} &= \{b_1 + K, D_1\} \\
S_{2.12} &= \{b_1 + L, P_3\} \\
S_{2.13} &= \{b_1 + L, L + \epsilon \cdot P_3\} \\
S_{2.14} &= \{b_1 + L, D_1\} \\
S_{2.15} &= \{b_1 + L + \epsilon \cdot P_3, D_2 + 2 \epsilon_1 K + aD_1\} \\
S_{2.16} &= \{b_1 + L + \epsilon \cdot P_3, D_1\} \\
S_{2.17} &= \{b_1 + L + \delta_1 D_1, L + \epsilon \cdot P_3 + \delta_1 D_1\} \\
S_{2.18} &= \{b_1 + L + \delta_1 D_1, P_3 + \delta_1 D_1\} \\
S_{2.19} &= \{b_1 + P_3, L\} \\
S_{2.20} &= \{b_1 + P_3, D_1\} \\
S_{2.21} &= \{b_1 + P_3, \delta_1 D_1, L + \delta_1 D_1\} \\
S_{2.22} &= \{b_1 + D_1, L + aD_1\} \\
S_{2.23} &= \{b_1 + D_1, L + \epsilon \cdot P_3 + aD_1\} \\
S_{2.24} &= \{b_1 + D_1, P_3 + aD_1\} \\
S_{2.25} &= \{b_1 + K + aD_1\} \\
S_{2.26} &= \{b_1 + L + \epsilon \cdot P_3\} \\
S_{2.27} &= \{b_1 + L, D_1\} \\
S_{2.28} &= \{b_1 + P_3 + aD_1\} \\
S_{2.29} &= \{b_1 + D_1\} \\
S_{2.30} &= \{b_1 + K, D_1\} \\
S_{2.31} &= \{b_1 + K, D_1\} \\
S_{2.32} &= \{b_1 + L, L + \epsilon \cdot P_3\} \\
S_{2.33} &= \{b_1 + L, P_3\} \\
S_{2.34} &= \{b_1 + L, D_1\} \\
S_{2.35} &= \{b_1 + L + \epsilon \cdot P_3, D_1\} \\
S_{2.36} &= \{b_1 + L + \delta_1 D_1, P_3 + aL + \delta_2 D_1\} \\
S_{2.37} &= \{b_1 + \lambda P_3, L\} \\
S_{2.38} &= \{b_1 + \lambda P_3, D_1\} \\
S_{2.39} &= \{b_1 + \lambda P_3 + \delta_1 D_1, L + \delta_1 D_1\} \\
S_{2.40} &= \{b_1 + \lambda D_1, K + aD_1\} \\
S_{2.41} &= \{b_1 + \lambda D_1, L + aD_1\} \\
S_{2.42} &= \{b_1 + \lambda D_2, L + \epsilon P_3\} \\
S_{2.43} &= \{b_1 + \lambda D_2, L + \epsilon P_3 + \epsilon_2 D_1\} \\
S_{2.44} &= \{b_1 + \lambda D_2, P_3 + aD_1\} \\
S_{2.45} &= \{b_1 - B_2, K + aD_1\} \\
S_{2.46} &= \{b_1 - B_2, L + aD_1\} \\
S_{2.47} &= \{b_1 - B_3, P_3 + aD_1\} \\
S_{2.48} &= \{b_1 - B_3, D_1\} \\
S_{2.49} &= \{b_1 - B_2 + \lambda K, D_1\} \\
S_{2.50} &= \{b_1 - B_2 - \lambda D_2, L + \epsilon P_3\} \\
S_{2.51} &= \{b_1 - \delta_1 D_1, L + \epsilon \cdot P_3 + \delta_1 D_1\} \\
S_{2.52} &= \{b_1 - B_2 + \lambda D_1, P_3\} \\
S_{2.53} &= \{b_1 - B_2 - \lambda L, D_1\} \\
S_{2.54} &= \{b_1 - B_2 + L + \epsilon P_3, D_1\} \\
S_{2.55} &= \{b_1 - B_2 - \lambda L + \delta_1 D_1, L + \epsilon P_3 + \delta_2 D_1\} \\
S_{2.56} &= \{b_1 - B_2 + \lambda L + \delta_1 D_1, P_3 + \delta_2 D_1\} \\
S_{2.57} &= \{b_1 - B_2 + \lambda P_3, L\} \\
S_{2.58} &= \{b_1 - B_2 + \lambda P_3, D_1\} \\
S_{2.59} &= \{b_1 - B_2 + \lambda P_3 + \delta_1 D_1, L + \delta_2 D_1\} \\
S_{2.60} &= \{b_1 - \lambda D_2, K + aD_1\} \\
S_{2.61} &= \{b_1 - B_2 + \lambda D_1, L + aD_1\} \\
S_{2.62} &= \{b_1 - B_2 + \lambda D_1, L + \epsilon P_3\} \\
S_{2.63} &= \{b_1 - B_2 + \lambda D_1, P_3 + aD_1\} \\
S_{2.64} &= \{b_1 - B_2 + \lambda D_1, P_3 + aD_1\} \\
S_{2.65} &= \{b_1 - B_2 + \lambda D_1, P_3 + aD_1\} \\
S_{2.66} &= \{b_1 - B_2 + \lambda D_1, L + \epsilon P_3\} \\
S_{2.67} &= \{b_1 - B_2 + \lambda D_1, L + aD_1\} \\
S_{2.68} &= \{b_1 - B_2 + \lambda D_1, L + aD_1\} \\
S_{2.69} &= \{b_1 - B_2 + \lambda D_1, L + \epsilon P_3\} \\
S_{2.70} &= \{b_1 - B_2 + \lambda D_1, L + aD_1\} \\
\end{array}
\]

Table 3. List of two-dimensional representative subalgebras of $S$. The parameters are $\epsilon, \epsilon_1, \epsilon_2 = \pm 1$ and $a, \lambda, \delta, \delta_1, \delta_2 \in \mathbb{R}, \delta, \delta_1^2 + \delta_2^2 \neq 0, \lambda > 0$.

in $\mathcal{L}_0$. It therefore remains to find the subalgebra classes having non-zero component in $Z_0$ which are not conjugate to a subalgebra $S_i \subset S$ by the action of the symmetry group $G$. In order to do this, we start with the classification of $S$. Following the usual method, we begin by forming splitting subalgebras of the form $\mathcal{L}_0, \mathcal{J} = S \triangleright Z_0, \mathcal{J}$ from the subalgebras $S_i \subset S$ and $Z_0, \mathcal{J} \subset Z$ (since $Z$ is Abelian, all subspaces are Lie subalgebras). Next, we simplify the parameters as much as possible by acting through conjugation by $G$ and using the transformation (9). It will remain for us to find the nonsplitting subalgebras by the usual procedure from the established list of splitting subalgebras. This procedure result in the addition (to the subalgebras of tables 2 and 3) of the classes given in tables 4 and 5.

2.2.2. Subalgebras contained in $\mathcal{L}_1$. The classification of the conjugation classes can be expanded by considering the subalgebras contained in $\mathcal{L}_1 = S \times Z_0 \times Z_1$. The procedure is the same as for $\mathcal{L}_0$ with the exception that we can now ignore subalgebras having a zero component in $Z_1$, since these are already in the classification of $\mathcal{L}_0$. The additional classes are given in tables 6 and 7.
3. Invariant and partially invariant solutions

Since equations (1a) and (1b) do not involve the velocity components $u$ and $v$, they can first be solved for $\theta$ and $\sigma$. Next, the result is introduced into the system formed by equations (1c) and (1d) and we look for the solution of this system for the velocity components $u$ and $v$. This system always admits the particular solution

$$u = b_1 y + b_2, \quad v = -b_1 x + b_3, \quad b_1, b_2, b_3 \in \mathbb{R},$$

(27)

obtained by requiring that the coefficients of the trigonometric functions in (1c) vanish. The velocity field defined by (27) forms concentric circles. Therefore, this solution does not
Table 7. List of two-dimensional representative subalgebras contained in \( \mathcal{L}_1 \). The parameters are \( a, a_i, \lambda, \delta \in \mathbb{R}, i = 1, \ldots, 9 \) and \( \lambda > 0, \delta \neq 0 \).

\[
\begin{array}{ll}
L_{2.30} &= \{ B_i, D_i - D_j + B_k + aP_i \} \\
L_{2.31} &= \{ B_i, K + aD_i + P_j + aP_k \} \\
L_{2.32} &= \{ B_i, K + aD_i + P_j \} \\
L_{2.33} &= \{ B_i + P_j + B_k + aB_i, a_k, B_l - P_j + a_2P_i \} \\
L_{2.34} &= \{ B_i + P_j + B_k + aB_i, P_j \} \\
L_{2.35} &= \{ B_i + P_j + B_k, B_l - B_j + a_3P_i \} \\
L_{2.36} &= \{ B_i + P_j + B_k, P_j \} \\
L_{2.37} &= \{ B_i + B_4 + aB_i + a_2P_i + a_3P_i, B_j + a_4P_i \} \\
L_{2.38} &= \{ B_i + B_4 + aB_i + a_3P_i, P_i \} \\
L_{2.39} &= \{ B_i + B_4 + aP_i, D_j + 3D_i \} \\
L_{2.40} &= \{ B_i + B_4 + aP_i + a_3P_i, B_j + a_3P_j \} \\
L_{2.41} &= \{ B_i + B_4 + aP_i, P_j \} \\
L_{2.42} &= \{ D_i - D_j + B_k + aB_i, aP_i + a_2P_i, B_3 + a_3P_i \} \\
L_{2.43} &= \{ D_i - D_j + B_k + aB_i, a_2P_i, P_i \} \\
L_{2.44} &= \{ D_i - D_j + B_k, P_i \} \\
L_{2.45} &= \{ D_i - D_j + B_k + aP_i + a_2P_i, B_j + a_3P_i \} \\
L_{2.46} &= \{ D_i + D_j + B_k + aB_i, aP_i + a_3P_i, B_j + a_3P_j \} \\
L_{2.47} &= \{ D_i + D_j + B_k + aB_i, a_3P_i, P_i \} \\
L_{2.48} &= \{ D_i + D_j + B_k + aP_i, P_i \} \\
L_{2.49} &= \{ D_i + D_j + B_k + aP_i, P_j \} \\
L_{2.50} &= \{ B_i - B_j + L + \lambda B_i, aB_i, B_j + a(P_j - P_k) \} \\
L_{2.51} &= \{ B_i - B_j + L + \lambda B_i, B_k - B_j + a(P_i - P_k) \} \\
L_{2.52} &= \{ B_i + B_j + aP_i + a_3P_i, B_k + aB_3 + a_4B_i + a_5P_i + a_6P_3 + a_7P_3 + a_8P_4 \} \\
L_{2.53} &= \{ B_i + aP_i + a_3P_i, B_k + B_1 + a_2B_5 + a_3B_6 + a_4P_i + a_5P_3 + a_6P_5 + a_7P_6 + a_8P_4 \} \\
L_{2.54} &= \{ B_i + aP_i + a_3P_i, B_k + B_1 + a_2B_5 + a_4P_i + a_5P_3 + a_6P_5 + a_7P_6 + a_8P_4 \} \\
L_{2.55} &= \{ B_i + aP_i + a_3P_i, B_k + a_2B_5 + a_4P_i + a_5P_3 + a_6P_5 + a_7P_6 + a_8P_4 \} \\
L_{2.56} &= \{ B_i + aP_i + a_3P_i, B_k + a_2B_5 + a_4P_i + a_5P_3 + a_6P_5 + a_7P_6 + a_8P_4 \} \\
L_{2.57} &= \{ B_i + aP_i + a_3P_i, B_k + a_2B_5 + a_4P_i + a_5P_3 + a_7P_6 + a_8P_4 \} \\
L_{2.58} &= \{ B_i + aP_i + a_3P_i, B_k + a_2B_5 + a_4P_i + a_5P_3 + a_6P_5 \} \\
L_{2.59} &= \{ B_i + aP_i + a_3P_i, B_k + a_2B_5 + a_4P_i + a_5P_3 + a_6P_5 + a_7P_6 \} \\
L_{2.60} &= \{ B_i + aP_i + a_3P_i, B_k + a_2B_5 + a_4P_i + a_5P_3 + a_6P_5 \} \\
L_{2.61} &= \{ B_i + aP_i + a_3P_i, B_k + a_2B_5 + a_4P_i + a_5P_3 \} \\
L_{2.62} &= \{ B_i + aP_i, B_4 + a_2P_3 + a_3P_j \} \\
L_{2.63} &= \{ B_i + aP_i, B_4 + a_2P_3 + a_3P_j \} \\
L_{2.64} &= \{ B_i + aP_i + P_2 + a_3P_i + a_4P_3 + a_5P_5 \} \\
L_{2.65} &= \{ B_i + aP_i + P_2 + a_3P_j + a_4P_3 + a_5P_5 \} \\
L_{2.66} &= \{ B_i + aP_i + P_2 + a_3P_j + a_4P_3 + a_5P_5 \} \\
L_{2.67} &= \{ B_i + aP_i + P_2 + B_4 + a_3P_6 + a_4P_5 \} \\
L_{2.68} &= \{ B_i + aP_i + P_2 + B_4 + a_3P_6 + a_4P_5 \} \\
L_{2.69} &= \{ B_i + aP_i + P_2 + B_4 + a_3P_6 + a_4P_5 \} \\
L_{2.70} &= \{ B_i + aP_i + P_2 + B_4 + a_3P_6 + a_4P_5 \} \\
L_{2.71} &= \{ B_i + aP_i + P_2 + B_4 + a_3P_6 + a_4P_5 \} \\
L_{2.72} &= \{ B_i + aP_i + P_2 + B_4 + a_3P_6 + a_4P_5 \} \\
\end{array}
\]
establish any relation between the flow velocity and the strain involved in the plastic material. Therefore, it is not an interesting result by itself from the physical point of view. Nevertheless, since the PDEs (1c) and (1d) are linear (assuming that θ is known), they admit a linear superposition principle. Then, we can add the solution (27) to any solution, for the velocity components, of the system (1c), (1d), corresponding to given solutions θ and σ of the system (1a), (1b). Consequently, we can satisfy a much broader family of boundary conditions.

3.1. Symmetry reduction for the representative subalgebra $B_1$

Consider for illustration the one-dimensional representative subalgebra generated by the infinitesimal generator

$$B_1 = -v \partial_v + u \partial_u.$$  (28)

Since no derivative with respect to variables $u, v, \theta, \sigma$ appears in $B_1$, it follows that these variables are all invariants of the subalgebra generated by $B_1$. In order to obtain a complete set of functionally independent invariants, one can also include the symmetry variable

$$\xi = u x - v y.$$  (29)

We look for a solution of the form

$$u = F(\xi), \quad v = G(\xi), \quad \theta = T(\xi), \quad \sigma = S(\xi),$$  (30)

where $\xi$ is defined by (29). Replacing (30) into the original system (28) and assuming that $1 - x F'(\xi) - y G'(\xi) \neq 0$, where $F'(\xi) = dF(\xi)/d\xi$, etc, so that we can use the implicit function theorem, we obtain a system of equations with a reduced number of independent variables, where the functions $F, G, T, S$ are to be determined. This system takes the form of four coupled ODEs:

$$F(\xi)S'(\xi) - (\cos(2T(\xi))F(\xi) + \sin(2T(\xi))G(\xi)) T'(\xi) = 0,$$

$$G(\xi)S'(\xi) - (\sin(2T(\xi))F(\xi) - \cos(2T(\xi))G(\xi)) T'(\xi) = 0,$$

$$(G(\xi)F'(\xi) + F(\xi)G'(\xi)) \sin(2T(\xi)) + (F(\xi)F'(\xi) - G(\xi)G'(\xi)) \cos(2T(\xi)) = 0,$$

$$(F(\xi)G'(\xi) + G(\xi)G'(\xi)) T'(\xi) = 0.$$  (31)

The solution of this system is

$$F(\xi) = c_1 \cos(T(\xi)), \quad G(\xi) = c_1 \sin(T(\xi)), \quad S(\xi) = T(\xi) + c_2,$$  (32)

where $T(\xi)$ is an arbitrary function of a single variable and $c_1, c_2$ are constants of integration. The solution is obtained by replacing expressions (32) for $F, G, T, S$ into equation (30). The solution of the original system is given implicitly by relations

$$u = c_1 \cos(T(ux + vy)), \quad v = c_1 \sin(T(ux + vy)),$$  (33)

while the angle $\theta$ and the pressure $\sigma$ are defined by the choice of the arbitrary function $T$ as follows:

$$\theta = T(ux + vy), \quad \sigma = T(ux + vy) + c_2.$$  (34)

Since by defining $\theta$ through a certain choice of $T$, we also determine $\sigma$, it follows that equation (34) is a relation defining the pressure $\sigma$ in terms of the angle $\theta$ or vice versa. Moreover, we can see from equation (33) that the sum of squares of the velocity components $u$ and $v$, is constant. Therefore, since the material is incompressible, the solution preserves the kinetic energy of the plastic material, i.e.

$$u^2 + v^2 = c_1^2.$$
For the purpose of illustration, consider the function:

$$T(\xi) = \frac{1}{2} \arcsin(\xi).$$

This particular choice of $T$ allows us to solve relations (33) in order to find the velocities $u$ and $v$ explicitly in terms of $x$ and $y$. The obtained formulae can be expressed in terms of radicals and are very involved. Therefore, they are omitted here. Nevertheless, these formulae can be used to trace the vector fields corresponding to solution (33), where $T$ is defined by (35). An example of such a tool is presented in figure 1 for a flow velocity $c_1 = 5$. The feeding velocity used is $(U_0, V_0) = (4.30, 2.55)$ and the extraction speed is $(U_1, V_1) = (-4.30, 2.55)$. The boundaries of the extrusion die are chosen in such a way that they coincide with the flow lines of the velocity field. Therefore, they are solutions of the equation

$$\frac{dy}{dx} = v(x, y)/u(x, y).$$

For figure 1, the inner boundary corresponds to the initial value $(x_0, y_0) = (-0.5, -0.35)$ and the outer boundary to the initial value $(x_0, y_0) = (-0.43, -0.46)$. The curves $C_1$ and $C_2$ are the limits of the plasticity region with respect to the entrance and exit of the extrusion die. They are solutions of equation (2), where $(U_0, V_0)$ is replaced by $(U_1, V_1)$ for $C_2$. In order to define the limit of the plasticity region at the ends of the boundary of the tool, the initial data used to trace the curve $C_1$ are $(x_0, y_0) = (-0.5, -0.35)$ while for $C_2$ they are $(x_0, y_0) = (-0.5, 0.35)$. Numerical integration has been used to identify the boundary of the tool and the limits of the region of plasticity. This type of extrusion die can be used to bend a rectangular rod or a slab of an ideal plastic material. The average pressure and the angle $\theta$, which define the strain tensor inside the tool, are evaluated by formulae (34), where $T$ is defined by (35).
3.2. Symmetry reduction for the representative subalgebra $K$

As an example, we find a partially invariant solution corresponding to the subalgebra generated by generator $K$ which admits the following invariants:

$$\xi = xy \cos(2\theta) - (1/2)(x^2 - y^2) \sin(2\theta), \quad F = xu + yv.$$  \hspace{1cm} (36)

and

$$S = \theta^2 - \sigma^2, \quad G = uv \cos(2\theta) - (1/2)(u^2 - v^2) \sin(2\theta).$$

In order to obtain a PIS, we use only the two invariants given by (36). We begin by inverting the first relation in (36) in order to find $\theta$ as a function of $\xi$. Next, we introduce the result in the first two equations of system (1). Then, comparing the values of the mixed derivatives of $\sigma(x,y)$ with respect to $x$ and $y$, we obtain the following PDE for the quantity $\xi$

$$(\xi_x - \xi_y)((x^2 + y^2)^2 - 4\xi^2)(\xi(x^2 - y^2) - xy\sqrt{(x^2 + y^2)^2} - 4\xi^2)$$

$$-4xy\xi + (x^2 - y^2)\sqrt{(x^2 + y^2)^2} - 4\xi^2\xi_{xy} + (x^2 + y^2)^2<(x+y)\xi_x - (x-y)\xi_0>$$

$$\times(x-y)\xi_1 + (x+y)\xi_0 - 4(x^2 + y^2)^2\xi(x\xi_x + y\xi_y - \xi) = 0.$$  \hspace{1cm} (37)

There are two particular solutions for $\xi$ to equation (37) defined by

$$\xi = \frac{1}{2}(x^2 + ey^2), \quad e = \pm 1.$$  \hspace{1cm} (38)

Let us now consider the case $e = 1$ and introduce this solution into the first relation (36). Solving for $\theta$, we obtain:

$$\theta = -\frac{1}{2} \arctan \left( \frac{x^2 - y^2}{xy} \right).$$  \hspace{1cm} (39)

The mean pressure $\sigma$ is found by quadrature from the first two equations (1) in which we have introduced the solution (39) for $\theta$. The result for $\sigma$ is:

$$\sigma = -(1/2) \ln (x^2 + y^2) + c_1,$$  \hspace{1cm} (40)

where $c_1$ is a real integration constant.

Using the form of the second invariant in (36), we look for a solution for the components $u$ and $v$ of the velocity, with the form

$$u = (y/x)v - F(\xi),$$  \hspace{1cm} (41)

where the symmetry variable $\xi$ is given by (38). By replacing $\theta$ given by (39) and $u$ by (41) into the system composed of the last two equations in (1), then using the compatibility condition of the mixed derivatives of $v$ with respect to $x$ and $y$, we obtain the condition that $F = c_2$, where $c_2$ is a real constant. The solution for $u$ and $v$ is then:

$$u = \frac{c_2x}{x^2 + y^2} + c_3y + c_4, \quad v = \frac{c_2y}{x^2 + y^2} - c_3x + c_5,$$  \hspace{1cm} (42)

where $c_3, \ldots, c_5$ are real constants of integration. Note that in the case when $c_4 = c_5 = 0$ and $c_2 \neq 0 \neq c_3$, the flow lines form logarithmic spirals centred at the origin.

An example of velocity fields is given in figure 2 for parameters $c_2 = -1$, $c_3 = -2$, $c_4 = 4$ and $c_5 = 1$ for solution (42). The chosen region, $[-1, 1] \times [-1, 1]$, includes the singularity at the origin. Corresponding to this solution for the same parameters, an extrusion tool is given in figure 3 for the feeding and extraction velocities $(U_0, V_0) = (5.5, 0)$ and $(U_1, V_1) = (3, 3)$ respectively. The curve $C_1$ is the limit of the plasticity region at the entrance of the extrusion die and $C_2$ has the same significance at the exit of the extrusion die. The upper contour of the extrusion die is a solution of $dy/dx = v/u$, with $u$ and $v$ defined by (42), for an initial value $y(-0.5) = -0.8$ while, for the lower contour, we have used the initial value $y(-0.7) = -0.95$.  

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3.3. Similarity solution for the angle $\theta$ and corresponding pressure $\sigma$

Let us now study partially invariant solutions. The approach is based on the observation that

$$\xi = \frac{y}{x}$$  \hspace{1cm} (43)

and $\theta$ are invariants of the subalgebras $\mathcal{L}_{1,21}$–$\mathcal{L}_{1,24}$. So, we are looking for a solution of the system (1) such that $\theta$ is in the form

$$\theta(x, y) = J(\xi),$$  \hspace{1cm} (44)
where \( J \) is a function to be defined. Introducing (44) in (1a) and (1b), we are led to the system
\[
\begin{align*}
\sigma_x & = (\xi_x \cos(2J(\xi))) + \xi_y \sin(2J(\xi)) J'(\xi), \\
\sigma_y & = (\xi_x \sin(2J(\xi)) - \xi_y \cos(2J(\xi))) J'(\xi).
\end{align*}
\] (45)

Thereafter, the requirement that the compatibility condition on mixed partial derivatives of \( \sigma \) in respect to \( x \) and \( y \) must be satisfied implies that the function \( J \) obeys a nonlinear second-order ODE. This ODE is omitted here since, in what follows, we will only work with its first integral given by
\[
((\xi^2 - 1) \sin(2J(\xi)) + 2\xi \cos(2J(\xi))) J'(\xi) = c_1, \quad c_1 \in \mathbb{R}.
\] (46)

Assuming that the function \( J \) satisfies the equation (46), the system (45) is solved by quadrature for the quantity \( \sigma \). We find the pressure
\[
\sigma = -\int \xi \sin(2J(\xi)) J'(\xi) \, d\xi - \frac{1}{2} \sin(2J(\xi)) - c_1 \ln(x) + c_0, \quad c_0 \in \mathbb{R}.
\] (47)

The solution \( J \) to the PDE (46) depends on whether or not \( c_1 \) vanishes.

(i) If \( c_1 \neq 0 \), the solution for \( J \) is defined implicitly by the equation
\[
\frac{(\tan(J(\xi)) - \xi) \sqrt{c_1^2 - 1}}{(\tan(J(\bar{\xi})) J(\bar{\xi}) + 1) c_1 - \xi + \tan(J(\bar{\xi}))} - \tan\left(\frac{\sqrt{c_1^2 - 1}(c_2 - J(\xi))}{c_1}\right) = 0,
\] (48)

where \( c_2 \) is an integration constant. Hence, \( \sigma \) given by (47) together, with \( \theta \) given by (44), form a solution of the system (1a), (1b) if the function \( J \) satisfies the equation (48).

(ii) If \( c_1 = 0 \) and \( J'(\xi) \neq 0 \), the solution to (46) is
\[
J(\xi) = -\frac{1}{2} \arctan\left(\frac{2\xi}{\xi^2 - 1}\right).
\] (49)

In this case, it results from the introduction of the solution \( J \) into (47) and (44) that the solution for \( \sigma \) and \( \theta \) of the system (1a), (1b), is defined by
\[
\theta(x, y) = \frac{1}{2} \arctan\left(\frac{2xy}{x^2 - y^2}\right), \quad \sigma(x, y) = -\arctan\left(\frac{y}{x}\right) + c_0.
\] (50)

### 3.3.1. Additive separation for the velocities.
Having found the solution for \( \sigma, \theta \), we proceed to determine the velocity components \( u, v \) assuming that they are separable in the additive form
\[
\begin{align*}
u(x, y) & = f(x, y) + F(\xi), \\
v(x, y) & = g(x, y) + G(\xi),
\end{align*}
\] (51)

with \( \xi \) given by (43). Introducing (51) into the system (1c), (1d), we obtain
\[
\begin{align*}
&\text{with } \xi \text{ given by (43). Introducing (51) into the system (1c), (1d), we obtain}
\end{align*}
\] (52)

Next, we must find the conditions which ensure that the hypotheses (51) reduce the PDEs system (1c), (1d), to a system of ODEs for \( F \) and \( G \) in term of \( \xi \). Consider the annihilator of \( \xi = y/x \) given by
\[
A_\xi = \frac{1}{2} (D_1 + D_2).
\] (54)
in terms of the infinitesimal generators $D_1$ and $D_2$ given in (4). We apply $A_x$ to the equations (52) and (53), to eliminate the presence of the functions $F$ and $G$ in order to obtain conditions on the functions $f$ and $g$. This results in some preliminary conditions which reduce to,

\begin{align}
  f_x &= -g_y + \xi_1(\xi)x^{-1}, \\
  f_y &= -g_x + g_{x2}(\xi) + \xi_3(\xi)x^{-1}, \\
  (g_y + 2A_x(g_x)) (\xi_2(\xi) \sin(2J(\xi)) - 2 \cos(2J(\xi))) &= 0, \\
\end{align}

if we assume that $f_x + g_y \neq 0$ (otherwise we obtain the trivial solution for $u$, $v$). The functions of one variable $\xi_i$, $i = 1, 2, 3$ are arbitrary. Since equation (57) is factored, we must consider following two possibilities.

(a) We first suppose that

$$g_y + 2A_x(g_x) = 0.$$  

(58)

After a convenient choice of the functions $\xi_2(\xi)$, $\xi_3(\xi)$, such that the compatibility condition on mixed derivatives of $f$ relative to $x$ and $y$ is satisfied, we find that the functions $f$ and $g$ take the form

\begin{align}
  f(x, y) &= -\int_{\xi(x,y)}^{\xi(x,y)} \frac{\xi_1(\xi) - \xi'_1(\xi)}{\xi} d\xi + \omega_4 \ln(y) - \omega_1 y + \omega_5, \\
  g(x, y) &= \xi_4(\xi) + \omega_2 \ln(x) + \omega_1 x + \omega_3, \\
\end{align}

where the functions $\xi_1(\xi)$, $\xi_4(\xi)$ are arbitrary.

(b) Suppose now that the condition (57) is satisfied by requiring

$$\xi_2(\xi) = \cot(2J(\xi)).$$  

(60)

Then, applying the compatibility condition on mixed derivatives of $f$ relative to $x$ and $y$ to the equations (55), (56), and considering $\xi_2$ given by (60), we conclude that the function $g$ must solve the equation

\begin{align}
  g_{xx}(x, y) + 2 \cot(2J(\xi))g_{xy}(x, y) &= g_{xy}(x, y) - 4\xi(x \sin^2(2J(\xi)))^{-1}J'(\xi) g_y(x, y) \\
  &+ x^{-2}(\xi_1(\xi)x + \xi_3(\xi)) = 0. \\
\end{align}

(61)

This is a hyperbolic equation everywhere in the domain where $J$ is defined. So, we introduce the change of variable

\begin{align}
  \phi(x, y) &= x \exp \left( \int_{\xi(x,y)}^{\xi(x,y)} \frac{\sin(2J(\xi))}{1 + \cos(2J(\xi)) + \xi \sin(2J(\xi))} d\xi \right), \\
  \psi(x, y) &= x \exp \left( \int_{\xi(x,y)}^{\xi(x,y)} \frac{-\sin(2J(\xi))}{-1 + \cos(2J(\xi)) + \xi \sin(2J(\xi))} d\xi \right), \\
\end{align}

which brings the equation (61) to the simplified form

\begin{align}
  g_{\phi \psi} + \frac{c_1}{2} \left( \frac{\sin(2J(\phi, \psi))g_y}{\psi(2J(\phi, \psi))} + 1 - \frac{\sin(2J(\phi, \psi))g_y}{\phi(2J(\phi, \psi)) - 1} \right) \\
  - \frac{1}{4} \sin(2J(\phi, \psi)) \xi^2 &+ \sin(2J(\phi, \psi)) + 2 \cos(2J(\phi, \psi)) \left( \xi_3(\xi) + \xi'_1(\xi) + \xi_3(\xi) \right) = 0, \\
\end{align}

where $J(\phi, \psi)$ is defined by

$$J = (c_1/4)(\psi - \phi).$$  

(64)

To solve the equation (63) more easily, we define the function $\xi_3$ by

$$\xi_3(\xi) = \xi^{-1} (-\xi_1(\xi) + J(\xi) + \omega_1).$$  

(65)
We substitute (70) into equations (1) and the solution of the system (71) for either (48) or (49). The solution of the system (71) for (48) is defined by (48). In this case \( F(J(\dot{1})) \cos(2J(y/x)) - (1/4)c_1^{-1}\sin(2J(y/x)) + \omega_2 \) can neglect the terms corresponding to the particular solution (27) and the functions of a redefinition of \( f \). Now, we search for a solution for the velocity components into the form

\[
g(x, y) = -(1/2) \left( \omega_1 - (1/2)c_1^{-1}J(\dot{1}) \right) \cos(2J(y/x)) - (1/4)c_1^{-1}\sin(2J(y/x)) + \omega_2.
\]

We substitute (70) into equations (1c), (1d) in order to obtain the following system of ODEs for \( F \) and \( G \):

\[
(\xi^2 - 1) \sin(2J(\dot{1})) + 2\xi \cos(2J(\dot{1}))F'(\dot{1}) - \xi^{-1}(a_1 + a_2\xi) \sin(2J(\dot{1})) = 0,
\]

\[
G(\dot{1}) = \xi F'(\dot{1}).
\]

The solution of the system (71) for \( F \) and \( G \) clearly depends on the choice of the solution \( J(\dot{1}) \), either (48) or (49).

(i) Suppose that \( J(\dot{1}) \) is defined by (48). In this case \( c_1 \neq 0 \) in the first integral (46) and the coefficient in front of \( F'(\dot{1}) \), in the first equation of (71), does not vanish. It follows that the solution for \( F \) and \( G \) is

\[
F(\dot{1}) = -\frac{1}{2}c_1^{-1}a_2 \cos(2J(\dot{1})) + c_1^{-1}a_1 \int \xi^{-1}\sin(2J(\dot{1})))J'(\dot{1}) \, d\xi + c_3,
\]

\[
G(\dot{1}) = -\frac{1}{2}c_1^{-1}a_2 \cos(2J(\dot{1})) + c_1^{-1}a_2 \int \xi \sin(2J(\dot{1})))J'(\dot{1}) \, d\xi + c_4.
\]

where \( c_3, c_4 \) are integration constants. Thus, the introduction of (72) into (70) gives

\[
u(x, y) = a_2 \ln(x) - \frac{1}{2}c_1^{-1}a_2 \cos(2J(\dot{1})) + c_1^{-1}a_2 \int \xi^{-1}\sin(2J(\dot{1})))J'(\dot{1}) \, d\xi + c_3,
\]

\[
u(x, y) = a_2 \ln(x) - \frac{1}{2}c_1^{-1}a_1 \cos(2J(\dot{1})) + c_1^{-1}a_2 \int \xi \sin(2J(\dot{1})))J'(\dot{1}) \, d\xi + c_4,
\]

which, together with \( \theta \) and \( \sigma \) given respectively by (44), (48) and (47), form a solution of the system (1).
(ii) Consider now the case where \( J(\xi) \) is defined by (49). In this case, the vanishing of the coefficient in front of \( F'('\xi) \) in the first equation of (71) implies that \( a_1 = a_2 = 0 \). So, the assumptions (70) become equivalent to requiring that the velocity components be in the form \( u(x, y) = F(\xi), v(x, y) = G(\xi) \). The corresponding solutions for \( u \) and \( v \) are

\[
\begin{align*}
  u(x, y) &= H'(\xi), \\
  v(x, y) &= -H(\xi) + \xi H'(\xi),
\end{align*}
\]

(74)

where \( H \) is an arbitrary function of one variable and \( \xi \) is given by (43). The velocities (74) together with \( \sigma \) and \( \theta \) defined by (50) form a solution of the system (1). For example, if we choose the arbitrary function to be an elliptic function, that is

\[
H(\xi) = \text{cn}((1 + \cosh(\arctan(b_2 \xi))))^{-1}, \varrho, \quad 0 < \varrho^2 < 1,
\]

and we set the parameters as \( b_1 = 4\pi, c_2 = 0, \rho = 1/2 \), then we can trace (see figure 4) an extrusion die for a feeding speed of \((U_0, V_0) = (0, -0.94)\) and an extraction speed \((U_1, V_1) = (0, -0.94)\). The curve \( C_1 \) on the figure 4 constitutes a boundary of the plasticity region at the mouth of the tool, while \( C_2 \) does the same for the output of the tool. This type of tool could be used to undulate a plate. We can shape the tool by varying the parameters. For example, we can spread the bump by decreasing the parameter \( b_1 \). Moreover, one should note that if the modulus \( \varrho \) of the elliptic function is such that \( 0 \leq \varrho^2 \leq 1 \), then the solution has one purely real and one purely imaginary period. For a real argument \( \chi \), we have the relations

\[
-1 \leq \text{cn}(\chi, \varrho) \leq 1.
\]

Another interesting situation occurs if we suppose that \( \zeta_2(\xi) \) is defined by (60) and

\[
\zeta_3(\xi) = \frac{\zeta_1(\xi)}{\xi}, 
\]

(75)

In this case, the condition (57) is identically satisfied and, from the mixed derivatives compatibility condition of \( f \) applied to the equations (55) and (56), we obtain the following ODE for the function \( g \):

\[
ge_{xx} - \frac{2 \cot(2J(\xi))g_{yy}}{x \sin^2(2J(\xi))} = \frac{4\xi J'(\xi)g_y}{x \sin^2(2J(\xi))} = 0. 
\]

(76)

We simplify the last equation by the introduction of new variables

\[
\xi = y/x, \quad \eta = x^2 + y^2.
\]

(77)

This reduces the PDE (76) to the simpler PDE in term of \( \xi \) and \( \eta \),

\[
g_{\xi\eta} + \frac{\xi g_\eta}{\xi^2 + 1} = 0,
\]

(78)
Figure 5. Extrusion die corresponding to the solution (50), (82).

which has the solution
\[ g(\xi, \eta) = \zeta_4(\xi) + \frac{\zeta_5(\eta)}{\sqrt{\xi^2 + 1}} \] (79)

where \( \zeta_4 \) and \( \zeta_5 \) are arbitrary functions of one variable. Then, we find the solution for \( f \) by integration of the PDE (55), (56), with \( \zeta_3 \) given by (75),
\[ f(x, y) = -\int \frac{\zeta_1(\xi) - \zeta_4'(\xi)}{\xi} \, d\xi - \frac{\eta\zeta_3(\eta)}{\sqrt{\eta}} + c_2. \] (80)

By the substitution of (79) and (80) into the equations (52) and (53), we find that \( F \) is an arbitrary function of one variable and \( G \) is defined by
\[ G(\xi(x, y)) = \int (-\zeta_1(\xi) + F'(\xi)\xi) \, d\xi + c_3. \] (81)

So, we introduce (79)–(81) into (51) and after an appropriate redefinition of \( \zeta_1, \zeta_4 \) and \( \zeta_5 \), the solution of (1c), (1d) is provided by
\[ u(x, y) = K'(y/x) - yH(x^2 + y^2) + c_2, \quad v(x, y) = -K'(y/x) + \xi K(\xi) + xH(x^2 + y^2) + c_1, \] (82)

where \( H, K \) are arbitrary functions of one variable. The velocities (82), together with the angle and pressure defined by (50), solve the system (1). For example, a tool corresponding to the solution (82) with \( H(\eta) = 2 \exp(-0.1\eta), K(\xi) = \xi \) and for feeding and extraction speed given respectively by \( (U_0, V_0) = (1.05, 0) \) and \( (U_1, V_1) = (1.05, 0) \). The extrusion die is shown in figure 5. The plasticity region limits correspond to the curves \( C_1 \) and \( C_2 \). This tool is symmetric under the reflection \( x \mapsto -x \). Moreover, the top contour of the tool almost makes a complete loop, and this lets one suppose that we could make a ring in a material by extrusion.

4. Final remarks

In this paper, we have obtained the infinitesimal generators which generate the Lie algebra of symmetries for the system (1) describing a planar flow of an ideal plastic material. The existence of generators \( P_0 \) to \( P_3 \), \( D_1, D_2, L, B_2 \), given in equation (4), together with generator \( X_1 \) of an infinite subalgebra, defined by (5), were already known in the literature [1]. However,
we have shown that the symmetry group is completed by the addition of generators $B_1$ and $K$, defined by (4) and by $X_1$ and $X_2$ given by equation (5), which generate an infinite-dimensional subalgebra. We have seen that it is possible to include the generator $K$ in the basis of a finite-dimensional Lie subalgebra only if no generator of $\chi$ are included. We have performed a classification of the subalgebras into conjugation classes under the action of the symmetry group using the methods described in [14, 15] (see section 2). This classification is an important tool in the analysis of invariant and partially invariant solutions. A classification of the symmetry subalgebras of (1) has been performed in the past [1] for one-dimensional subalgebras. However, the classification presented here is more complete in the sense that it includes new infinitesimal generators and a classification of two-dimensional subalgebras, which can be used to obtain partially invariant solutions. In section 3, we have performed (as an example) symmetry reductions corresponding to one-dimensional subalgebras represented by newly found generators. For the first reduction, we have used the generator $B_1$ and for the second reduction the generator $K$, both defined in the list of generators (4). The symmetry reduction, using the invariants of $B_1$, leads to a new solution (see (33) and (34)) defined in terms of an arbitrary function of $\xi = xu + yv$ and where the velocity fields are implicitly defined. For a particular choice of the arbitrary function, we have traced (in figure 1) the shape of an extrusion die corresponding to this solution. The obtained solution has the particularity that the kinetic energy is constant along the flow. A similar analysis has been performed for the generator $K$ in order to obtain a partially invariant solution. For this solution, the invariants given in equation (36) were used to add constraints which allow us to obtain a solution more easily. These considerations were illustrated by finding a particular solution of system (1) defined by equations (39), (40) and (42). An example of a velocity vector field and an example of an extrusion die have been traced respectively in figures 2 and 3. It should be noted that, to the vector field in figure 2 (i.e. for the corresponding parameter values), we can associate a large family of extrusion dies, of which the one in figure 3 is a particular choice. The contours have to be chosen in such a way that they correspond to flow lines to the extent that it is possible to trace two curves linking them which satisfy equation (2). These curves are the limit of the plasticity region.

An interesting observation concerning the generator $K$ is that if we take the commutator of $K$ with the generators $\{P_1, P_2, P_3, P_4\}$, we obtain the generators $\{B_3, B_4, B_5, B_6\}$. Repeating the procedure with generators $\{B_3, B_4, B_5, B_6\}$, we generate four new generators, and so on. An interesting fact is that, at each step, the new obtained generators can complete the base of the subalgebra $L_2$ generated at the previous stage in order to form a new higher-dimensional (but still finite-dimensional) subalgebra (ensured by excluding $K$ from the base). Consequently, it is always possible to enlarge a finite-dimensional subalgebra $L_2$ that excludes $K$ by increasing its base with the result of commutators $[K, Z]$ with $Z \in L_i$. This gives us a chain of finite-subalgebras of the form $L_0 \subset L_1 \subset L_2 \subset \ldots \subset L_i \subset \ldots$. Some preliminary results have been obtained in this work, but this subject will be addressed more thoroughly in the future.

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References

[1] Annin B, Bytev V O and Senashov S I 1985 Group Properties of Equations of Elasticity and Plasticity (Novosibirsk: Nauka) pp 143 (in Russian)
[2] Katchanov L 1975 Éléments de la théorie de la plasticité (Moscov: Éditions Mir)
[3] Hill R 1998 The Mathematical Theory of Plasticity (Oxford: Oxford University Press)
[4] Chakrabarty J 2006 Theory of Plasticity (Oxford: Elsevier)
[5] Senashov S I and Yakho A 2007 Reproduction of solutions of bidimensional ideal plasticity Int. J. Non-Linear Mech. 42 500–3
[6] Senashov S I, Yakho A and Yakho L 2009 Deformation of characteristic curves of the plane ideal plasticity equations by point symmetries Nonlinear Anal. 71 e1274–84
[7] Nada Ī A 1924 Über die gleit-und verweigungsflächen einiger gleichgewichtszustände bildsamer massen und die nachspannungen bleibend verzenter körper Z. Phys. 30 106–38
[8] Prandtl L 1923 Anwendungsbeispeide zu einem henckychen satz über das plastische gleichwich ZAMM 3 401–6
[9] Czyz J 1974 Construction of a flow of an ideal plastic material in a die, on the basis of the method of Riemann invariants Arch. Mech. 26 589–616
[10] Senashov S I and Vinogradov A M 1988 Symmetries and conservation laws of 2-dimensional ideal plasticity Proc. Edinb. Math. Soc. 31 415–39
[11] Gomonova O V and Senashov S I 2008 Symmetries of equations modelling the velocity flow of 2-dimensional platicity deformation. Contemporary mathematical problems and educational mathematics Herzen Lectures Conf. Proc. on Material Sciences pp 20–25 (in Russian)
[12] Lamothe V 2012 Symmetry group analysis of an ideal plastic flow J. Math. Phys. 53 3 (accepted for publication)
[13] Ovsiannikov L V 1982 Group Analysis of Differential Equations (New York: Academic)
[14] Winternitz P, Patera J and Zassenhaus H 1975 Continuous subgroups of the fundamental groups of physics: I. General method and the Poincaré group J. Math. Phys. 16 1597–1615
[15] Sharp R T, Winternitz P, Patera J and Zassenhaus H 1977 Continuous subgroup of the fundamental groups of physics: III. The de Sitter groups J. Math. Phys. 18 2259
[16] Winternitz P 1990 Partially Integrable Evolution Equations in Physics ed R Conte (Dordrecht: Kluwer) pp 515–67
[17] Ollier P J 1986 Applications of Lie Groups to Differential Equations (New York: Springer)
[18] Patera J and Winternitz P 1977 Subalgebras of real three- and four-dimensional Lie algebras J. Math. Phys. 18 1449
[19] Winternitz P 1993 Lie groups and solutions of nonlinear partial differential equations Number CRM—1841 Centre de Recharches Mathématiques, Université de Montréal
[20] Yakho A and Yakho L 2010 Homotopy of Prandtl and Nada solutions Int. J. Non-Linear Mech. 45 793–9