The Boussinesq equation and Miura type transformations

Maxim Pavlov

Abstract. Several Miura type transformations for the Boussinesq equation are found and the corresponding integrable systems constructed.

Contents

1. Introduction 1
2. The cubic Miura type transformations 3
3. The quadratic Miura type transformations 5
4. Concluding remarks 6
Acknowledgments 7
References 7

1. Introduction

In [10] integrable equations that admit a scalar spectral problem were considered and an algorithm for constructing the Miura type transformations for these equations was described. The KdV equation was considered as an example. Another example, the Kaup-Boussinesq equation, was studied in [9]. The latter equation is related with the NLS equation by a pair of invertible differential first order substitutions. In fact, there exist more powerful methods that allow to describe modified equations, that is, the equations related with the initial ones by non-invertible differential substitutions. These substitutions are called the Miura type transformations. The approach based on constructing the dressing sequences of discrete symmetries was elaborated in the paper [2]. The KdV equation and the Bonnet equation served as examples in that paper (the Bonnet equation is also known as the sine-Gordon equation in the mathematical physics literature). The Kaup-Boussinesq equation was considered in the paper [3]. This method allows to construct multi-parametric integrable systems and does not depend on the form of the spectral problem, which can be either scalar or matrix. Therefore this method differs from the one proposed in [10]. One may think that the first approach was proposed in order to replicate integrable equations. This is not so. Conversely, new integrable equations were obtained as an auxiliary result while decreasing the orders of the Hamiltonian operators. Indeed, the reduction of the Hamiltonian operators to the canonical form “d/dx” was the
aim of that approach, which can be treated as a generalization of the Darboux theorem on reduction of a Hamiltonian operator to the constant operator for the infinite-dimensional case. Also, we emphasize that the method described in [10] is extremely simple from the technical viewpoint. Still, this procedure was not applied for the Boussinesq equation which is at the next level of complexity in comparison with the KdV equation. The present paper is devoted to solution of this problem. Since the Boussinesq equation is a two-component system of equations, then new field variables, which are used in construction of the modified equations, are not uniquely defined. Moreover, the third order spectral problem admits two sets of non-invertible differential substitutions, unlike the case of the KdV or the Kaup-Boussinesq equations, which are related with the second order spectral problem. We also see that any non-invertible differential substitution decreases the order of the Hamiltonian operator such that the canonical form “d/dx” for the KdV and the Kaup-Boussinesq equation is achieved. Now consider the Boussinesq equation. Then only one of two sets of the Miura type transformations is related with the Hamiltonian structures. Namely, this is the set defined by transformations that are quadratic in the field variables; another set is cubic. This situation illustrates a general concept: any scalar spectral problem of order $N$ admits a unique sequence of differential substitutions related with the Hamiltonian operators. Indeed, the field variables, which appear in powers not greater than two in the corresponding set, are conservation law densities for the modified equations, simultaneously. By ([10], [9], [11]), if the number of the field variables used in the quadratic Miura type transformation equals $M + 1$, where $M$ is the number of equations in the corresponding integrable system, then there are $M$ variables that annihilate the Poisson bracket which is related with the zero curvature metric (in case the coordinates are flat) or the constant curvature metric. If the number of summands in such quadratic transformation exceeds $M + 1$, then the corresponding Hamiltonian structure is related with the nonlocal Ferapontov brackets (see [4], [6]). In this paper, we do not concentrate on the reduction of the Hamiltonian operators to the canonical form “d/dx”. Indeed, all our attention is drawn to the description of all possible differential substitutions (similar to ones obtained for the KdV equation) in view of the specific features discussed above (e.g., the presence of two components in the system at hand and the third order scalar spectral problem).

The method proposed in [10] is based on the following reasoning. Consider the integrable system defined by an $\hat{L}\hat{A}$ pair such that the scalar differential operator $\hat{L}$ of order $N$ is polynomial in the spectral parameter $\lambda$ (Note that only four examples of such operators are known for the simplest case $N = 3$. They are: systems that can be reduced to the Boussinesq equation by using invertible differential substitutions, to the long-short resonance system, to the two-component NLS, and, probably, to the 3-wave system, see [1]). Consider the equation $\hat{L}\psi = 0$ and make the substitution

$$\psi = \exp[\int r dx]. \quad (1)$$

Hence we obtain a nonlinear differential equation, which is a generalization of the Riccati equation for $N > 2$. Now, decompose $r$ to the Laurent series in $\lambda$ in vicinity of the infinity. We get the infinite set of differential polynomials in the field variable. Each polynomial is a conservation law density. Our main idea is that one must consider the decomposition
of \( r \) to the Taylor series in \( \lambda \) in vicinity of zero! The initial coefficients of the series

\[ r = a + \lambda b + \lambda^2 c + \ldots \]

are the new field variables. Then we represent the modified integrable systems by using these variables, while the relations between the old and new field variables are precisely the Miura type transformations.

2. The cubic Miura type transformations

Represent the Boussinesq equation

\[ u_{tt} = \partial_x^2 \left( -\frac{1}{3} u_{xx} + \frac{2}{3} u^2 \right), \]

as the system of two evolution equations

\[ \begin{align*}
    u_t &= \partial_x \eta, \\
    \eta_t &= \partial_x \left( -\frac{1}{3} u_{xx} + \frac{2}{3} u^2 \right). 
\end{align*} \tag{2} \]

This equation is the compatibility condition for two linear differential equations

\[ \psi_{xxx} = u \psi_x + (\lambda^3 + \frac{1}{2} \eta + \frac{1}{2} u_x) \psi, \quad \psi_t = \psi_{xx} - \frac{2}{3} u \psi. \tag{3} \]

By using substitution (1) we rewrite system (3) in the form

\[ \begin{align*}
    r_{xx} + 3rr_x + r^3 &= ru + \frac{1}{2} \eta + \frac{1}{2} u_x + \lambda^3, \\
    r_t &= \partial_x [r_x + r^2 - \frac{2}{3} u], 
\end{align*} \tag{4} \]

Here the first equation is the generating function of conservation law densities (as \( \lambda \to \infty \)) and the second equation is the generating function of conservation laws. Now substitute the Taylor series

\[ r = a + \lambda^3 b + \lambda^6 c + \ldots \] \tag{5}

in the first equation among these two. Hence we obtain the Miura type transformations

\[ \begin{align*}
    a_{xx} + 3aa_x + a^3 &= ua + \frac{1}{2} \eta + \frac{1}{2} u_x, \\
    b_{xx} + 3ab_x + 3ba_x + 3a^2 b &= ub + 1, \\
    c_{xx} + 3ac_x + 3bb_x + 3ca_x + 3a^2 c + 3ab^2 &= uc, \ldots 
\end{align*} \tag{6} \]

Now substitute the Taylor series given by (5), in the second equation. Then we obtain the corresponding pseudononlocal conservation laws, that is, the conservation laws such that the densities and the fluxes are not expressed in terms of the field variables \( u, \eta \) and finite number of their derivatives: we have

\[ \begin{align*}
    a_t &= \partial_x [a_x + a^2 - \frac{2}{3} u], \\
    b_t &= \partial_x [b_x + 2ab], \\
    c_t &= \partial_x [c_x + 2ac + b^2], \ldots 
\end{align*} \tag{7} \]

for the Boussinesq equation. Now express \( \eta \) from the first equation in (6). We get

\[ \eta = 2a_{xx} + 6aa_x + 2a^3 - 2au - u_x, \]
Hence we obtain the modified Boussinesq equation (MB)

\[ a_t = \partial_x[a_x + a^2 - \frac{2}{3}u], \]  
\[ u_t = \partial_x[2a_{xx} + 6aa_x + 2a^3 - 2au - u_x]. \]  

Next, express \( u \) from the second equation in (6). We get

\[ u = 3(a_x + a^2) + \frac{b_{xx} + 3ab_x - 1}{b}, \]  

Hence we obtain the twice modified Boussinesq equation (TwMB)

\[ b_t = \partial_x[b_x + 2ab], \]  
\[ a_t = \partial_x[-a_x - a^2 - \frac{2}{3b}(b_{xx} + 3ab_x - 1)]. \]  

Now, express \( a \) from the third equation in (6)

\[ a = \frac{c - (cb_x - bc_x - b^3)_x}{3(cb_x - bc_x - b^3)}. \]  

Naturally, we get the thrice modified Boussinesq equation (ThrMB)

\[ c_t = \partial_x[c_x + b^2 + \frac{c + 3b^2b_x + bc_{xx} - cb_{xx}}{3(cb_x - bc_x - b^3)}], \]  
\[ b_t = \partial_x[b_x + 2b\frac{c + 3b^2b_x + bc_{xx} - cb_{xx}}{3(cb_x - bc_x - b^3)}]. \]  

From general theory of linear ordinary differential equations it is well known that the Wronskian of three linearly independent solutions \((\psi, \psi^-, \psi^+)\) to the first equation in (3) is equal to a constant:

\[(s_{xx} - us)s_x - s_x\psi' + s\psi'' = \varepsilon.\]  

Here \( s = \psi^-\psi^+ - \psi^+\psi^- \) is a solution of a conjugate equation (see first equation in (3))

\[ s_{xxx} = us_x - (\lambda^3 + \frac{1}{2}\eta - \frac{1}{2}u_x)s. \]

**Theorem 1:** The function \( \varphi = s\psi \) is a solution of the ordinary differential equation

\[ \varphi_{xx} - 3r\varphi_x + (3r^2 - u)\varphi = \varepsilon \]  

Also, this function is the generating function of conservation law densities for the Boussinesq equation

\[ \varphi_t = \partial_x[2r\varphi - \varphi_x]. \]  

The proof follows from (3) by a straightforward calculation.

**Remark 1:** The function \( \varphi \) is not a new generating function for conservation law densities. Indeed,

\[ \frac{\delta R}{\delta \eta} = \frac{1}{2\varepsilon}\varphi, \]

where \( R = \int rdx \). The latter equality means that the Euler derivative \( \delta/\delta \eta \) acts as the shift operator of the space of conservation law densities

\[ \frac{\delta H_{k+1}}{\delta \eta} = h_k. \]
where \( H_k = \int h_k dx \). Therefore, the coefficients of the Laurent series in \( \lambda \) around infinity for the function \( \varphi \) (see (12) and (13)) differ from the coefficients of the Laurent series for the function \( r \) by some constant factors only (see also (4)).

Recall that for the KdV equation it was sufficient to express the old field variable by the new one. Next, consider the Kaup-Boussinesq equation, which is a two-component system. Then this procedure is already not sufficient for the modified system to be uniquely defined \([2]\). Indeed, there are two possible variants in this case (see (12)). The first case is
\[
 u = 3r^2 + \frac{\varphi_{xx}}{\varphi} - 3r \frac{\varphi_x}{\varphi} - \varepsilon,
\]
The first modified Boussinesq equation (MB\(_1\))
\[
 r_t = \partial_x[r_x - r^2 + 2r \frac{\varphi_x}{\varphi} - \frac{2\varphi_{xx}}{3\varphi} + \frac{2\varepsilon}{3\varphi}], \quad \varphi_t = \partial_x[2r\varphi - \varphi_x] \quad (14)
\]
is of orders 3 and 2 with respect to the derivatives. Still, the second modified Boussinesq equation (MB\(_2\))
\[
 u_t = \partial_x[2r_{xx} + 6rr_x + 2r^3 - 2ru - u_x], \quad \varphi_t = \partial_x[2r\varphi - \varphi_x],
\]
where
\[
 r = \frac{\varphi_x}{2\varphi} \pm \sqrt{-\frac{\varphi_{xx}}{3\varphi} + \frac{\varphi_x^2}{4\varphi^2} + \frac{\varepsilon}{3\varphi} + \frac{1}{3}u},
\]
is of orders 5 and 3, respectively. The leading order with respect to derivatives is preserved by the Miura type transformations for the scalar KdV type equations. We see that this is not true even in the two-component case. The explicit formulas are huge and therefore omitted.

**Remark 2**: Invertible differential substitutions \( a = p - b_x/(2b) \) for TwMB (10) and \( q = r - \varphi_x/(2\varphi) \) for the MB\(_1\) (14) yield the same modified system
\[
 b_t = \partial_x(2bp), \quad \varphi_t = \partial_x(2\varphi q),
\]
\[
 p_t = \partial_x[-p^2 + \frac{2}{3b} + \frac{b^2}{4b^3} - \frac{b_{xx}}{6b}], \quad q_t = \partial_x[-q^2 + \frac{2\varepsilon}{3\varphi} + \frac{\varphi_x^2}{4\varphi^2} - \frac{\varphi_{xx}}{6\varphi}].
\]

3. The **quadratic** Miura type transformations

In the Introduction we noted that only quadratic Miura type transformations are related with the Hamiltonian structures. Now, consider the factorization of the first equation in scalar spectral problem (3). We have
\[
 (\partial_x + a + \bar{a})(\partial_x - \bar{a})(\partial_x - a)\psi = \lambda^3\psi
\]
This factorization supplies the well-known Miura type transformation (see [5])
\[
 u = 2a_x + \bar{a}_x + a^2 + a\bar{a} + \bar{a}^2. \quad (15)
\]
Therefore, the third modified Boussinesq equation (MB\(_3\))
\[
 a_t = \frac{1}{3}\partial_x[a^2 - 2a\bar{a} - 2\bar{a}^2 - a_x - 2\bar{a}_x], \quad (16)
\]
\[
 \bar{a}_t = \frac{1}{3}\partial_x[-2a^2 - 2a\bar{a} + \bar{a}^2 + 2a_x + \bar{a}_x]
We conclude that MB
where

Consider the variables (see for instance, [8]): we have

\[ u_t = \partial_x \frac{\delta H_4}{\delta \eta}, \quad \eta_t = \partial_x \frac{\delta H_4}{\delta u}, \] (17)

such that the Hamiltonian is

\[ H_4 = \int \left[ \frac{1}{2} \eta^2 + \frac{1}{6} u_x^2 + \frac{2}{5} u^3 \right] dx, \]

and two annihilators (Casimirs) are

\[ H_2 = \int \eta dx, \quad H_1 = \int u dx. \]

The other Hamiltonian structure is

\[
\begin{align*}
 u_t &= \partial_x \left[ \frac{3}{2} \frac{\delta H}{\delta \eta} + (-\partial_x^2 + u) \frac{\delta H}{\delta u} \right] - \frac{1}{2} \left( \frac{\delta H}{\delta \eta} \frac{\delta H}{\delta u} + \frac{\delta H}{\delta u} \frac{\delta H}{\delta u} \right), \\
 \eta_t &= \partial_x \left( \frac{1}{3} \left[ \partial_x^4 - 5u \partial_x^2 - \frac{5}{2} u_x \partial_x + 2(-u_{xx} + 2u^2) \right] \frac{\delta H}{\delta \eta} + \frac{3}{2} \frac{\delta H}{\delta u} \frac{\delta H}{\delta u} \right) - \left[ \frac{\delta H}{\delta u} \frac{\delta H_4}{\delta \eta} + \frac{\delta H}{\delta \eta} \frac{\delta H_4}{\delta u} \right].
\end{align*}
\]

Consider the variables \((a, \bar{a})\). Then the second structure acquires the canonical form

\[ a_t = \frac{1}{3} \partial_x \left[ -2 \frac{\delta H_2}{\delta a} + \frac{\delta H_2}{\delta \bar{a}} \right], \quad \bar{a}_t = \frac{1}{3} \partial_x \left[ \frac{\delta H_2}{\delta a} - 2 \frac{\delta H_2}{\delta \bar{a}} \right], \] (18)

such that the Hamiltonian is

\[ H_2 = -\frac{1}{2} \int \eta dx = \int a(\bar{a}(a + \bar{a}) + \bar{a}_x) dx, \]

the momentum is

\[ H_1 = -\int u dx = -\int [a^2 + \bar{a}a + \bar{a}^2] dx, \]

and two annihilators are

\[ \bar{H}_1 = \int \bar{a} dx, \quad \bar{H}_1 = \int \bar{a} dx. \]

We conclude that MB\(_3\) admits local Hamiltonian structure (18).

Finally, we describe another modified Boussinesq equation ThrMB\(_2\)

\[ b_t = \partial_x [bx + 2ab], \quad s_t = \partial_x [s(s + 2a) + (s + 2a)_x], \]

where \(s = \bar{a} - a\). The Miura type transformation

\[ a = \frac{-b_{xx} + (s^2 + s_x)b + 1}{b_x - sb} \]

is obtained by excluding \(u\) in (9) and (15). Therefore, ThrMB\(_2\) is of orders 3 and 4 with respect to the higher variables in its first and second components, respectively.

4. Concluding remarks

In this paper, we have found all “obvious” Miura type transformations that preserve the conservative form of the modified equations. Suppose the latter restriction is omitted. Then the Miura type transformations are still few, although their number increases (e.g., the KdV equation admits two transformations that preserve the conservative form and one that spoils it). Moreover, suppose that transformations of the independent variables \((x, t)\) are also allowed. Then we can continue the sequence of the modified equations, see
Such procedure will be described elsewhere. Also full set of multi-parametric modified Boussinesq equations by approach given in [2] should be found.

In [7] classification of integrable systems
\[ u_t = u_{xx} + F(u, w, u_x, w_x), \quad -w_t = w_{xx} + G(u, w, u_x, w_x) \]
was presented. The Boussinesq equation belongs to this class (by virtue of invertible differential substitution \( \eta = \rho \pm i u_x/\sqrt{3} \)). It will be nice to prove that all above modified Boussinesq equations up to module of invertible differential substitutions also belong to this class.

Acknowledgments

I would like to thank Dr. Arthemy Kiselev who suggested me to write this paper (it was a talk at International Conference on ”Integrable Systems: Solutions and Transformations” Guardamar (Alicante, Spain) at 15-19 June 1998).

I would like to thank the Loughborough University, UK for their financial support and hospitality when this work was made.

References

[1] M. Antonowicz, A.P. Fordy, Q.P. Liu, Energy-dependent third-order Lax operators, Nonlinearity 4 (1991) 669-684.
[2] A.B. Borisov, S.A. Zykov, The dressing chain of discrete symmetries and the proliferation of nonlinear equations. Theor. Math. Phys. 115, No. 2 (1998) 530–541.
[3] A.B. Borisov, M.V. Pavlov, S.A. Zykov, Proliferation scheme for the Kaup-Boussinesq system, Physica D 152/153 (2001) 104-109.
[4] E.V. Ferapontov, Differential geometry of nonlocal Hamiltonian operators of hydrodynamic type. Funct. Anal. Appl. 25, No. 3 (1991) 195–204.
[5] A.P. Fordy, J. Gibbons, Comm. Math. Phys. 77, No. 1 (1981) pp. 21-30. V.V. Sokolov, A.B. Shabat, LA-pairs and Riccati type substitution. (Russian) Func. Anal. Appl. 14, No. 2 (1980) 79–80.
[6] A. Ya. Maltsev, S.P. Novikov, On the local systems Hamiltonian in the weakly non-local Poisson brackets, Physica D 156 (2001) 53-80. A. Ya. Maltsev, On the compatible weakly non-local Poisson brackets of the hydrodynamic type, Intern. J. of Math. and Math. Sci. 32, No. 10 (2002) 587-614.
[7] A.V. Mikhailov, A.B. Shabat, R.I. Yamilov, Extension of the module of invertible transformations. Classification of integrable systems. Comm. Math. Phys. 115, No. 1 (1988) 1–19.
[8] P.J. Olver, Applications of Lie groups to differential equations. Second edition. Graduate Texts in Mathematics, 107 Springer-Verlag, New York (1993) 513 pp.
[9] M.V. Pavlov, Integrable systems and metrics of constant curvature, Journal of Nonlinear Mathematical Physics. No. 9, Supplement 1 (2002) 173-191.
[10] M.V. Pavlov, Relationships between differential substitutions and Hamiltonian structures of the Korteweg-de Vries equation. Phys. Lett. A 243, No. 5-6 (1998) 295-300.
[11] M.V. Pavlov, S.P. Tsarev, Tri-Hamiltonian structures for the Egorov systems of the hydrodynamical type. Func. Anal. Appl. 37, No. 1 (2003) 32-45.
[12] S.I. Svinolupov, V.V. Sokolov, R.I. Yamilov, Bäcklund transformations for integrable evolution equations. (Russian) Dokl. Akad. Nauk SSSR 271, No. 4 (1983) 802–805.

E-mail address: m.v.pavlov@lboro.ac.uk

Institute of Mathematics, Academia Sinica, Taipei 11529, Taiwan