Signature of odd-\(\omega\) equal-spin triplet pairing in the Josephson current on the surface of Weyl nodal loop semimetals

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We theoretically predict proximity-induced odd-frequency (odd-\(\omega\)) pairing on the surface of a Weyl nodal loop semimetal (WNLS), characterized by a nodal loop Fermi surface and drumhead-like surface states (DSSs), attached to conventional spin-singlet \(s\)-wave superconducting leads. Due to the complete spin-polarization of the DSS, odd-\(\omega\) equal-spin triplet pairing is present, and we show that it gives rise to a finite Josephson current. Placing an additional ferromagnet in the junction can also generate odd-\(\omega\) mixed-spin triplet pairing, but the pairing and current is not affected if the magnetization is orthogonal to the DSS spin-polarization, which further confirms the equal-spin structure of the pairing.

Weyl nodal loop semimetals (WNLSs) are recently discovered topological semimetals, where the valence and conduction bands cross each other along a closed one-dimensional (1D) loop carrying a \(\pi\) Berry flux [1–8]. The band topology results in drumhead-like surface states (DSSs) whose boundary is defined by the projection of the Fermi nodal loop onto the two-dimensional (2D) surface Brillouin zone (BZ) [9, 10]. There already exist several proposals for WNLSs, including SrIrO\(_3\) based on crystal symmetry analysis [11], PbTaSe\(_2\) based on topology analysis [12], TiTaSe\(_2\) and HgCr\(_2\)Se\(_4\) based on first principles [13, 14], and tight-binding calculations [9]. There are also reports of experimental characterization, for PbTaSe\(_2\) using angle-resolved photoemission spectroscopy [12] and Ca\(_3\)P\(_2\) using X-ray diffraction [15].

The topology of WNLS makes it a very interesting candidate for unconventional superconductivity. WNLSs have already been shown to allow for fully-gapped chiral 3D bulk superconductivity [10], while \(p + ip\) chiral surface superconductivity has been proposed to be present in the DSS if the bulk has \(p\)-wave pairing [4]. On the other hand, the DSS is fully spin-polarized for all WNLSs breaking time-reversal symmetry, making it completely immune to proximity-induced spin-singlet pairing from a conventional \(s\)-wave superconductor (SC). There however exists evidence in the literature showing that instead odd-frequency (odd-\(\omega\)) spin-triplet \(s\)-wave pairing can appear in simple ferromagnetic materials [16, 17]. This motivates us to investigate if the DSS also induces odd-\(\omega\) pairing on the surface of a WNLS in proximity to a conventional SC.

Odd-\(\omega\) superconductivity refers to when two electrons in a Cooper pair are odd in the relative time coordinate, or equivalently frequency [16, 18]. It was first predicted by Berezinskii [19] in the context of \(^3\)He and later introduced for superconductivity [20–22] and also the Kondo lattice [23]. Research on systems hosting this unusual pairing has flourished during the last few decades [18]. One major reason behind this is that the fermionic nature of the Cooper pairs allows for unusual pairing symmetries, such as odd-\(\omega\) spin-triplet \(s\)-wave or spin-singlet \(p\)-wave pairing.

Odd-\(\omega\) superconductivity has mainly been considered for hybrid structures [24, 25] like ferromagnet (FM)-SC [16, 17, 26–30], normal metal (NM)-SC [31–33], topological insulator-SC [34–37], but also in multi-band SCs with finite inter-band hybridization [38, 39], and conventional Josephson junctions, [40] as well as for SCs subjected to time-dependent drives [41, 42]. Odd-\(\omega\) behavior is also present for systems with Majorana fermions [43]. Experimentally, this exotic pairing state can be captured through various experimental phenomena like Meissner effect [44–47], Josephson current [48], Majorana STM [49], Kerr effect [50, 51], and thermopower measurement [52].

In this work we use a Josephson junction setup on the surface of a WNLS to explore proximity-induced superconductivity, especially odd-\(\omega\) pairing. Odd-\(\omega\) pairing has very recently been discussed in WNLS in Ref. 8, but then with leads attached across the bulk of the WNLS, thus measuring the bulk effects. We instead use a much simpler setup on a single surface and show explicitly how only odd-\(\omega\) equal-spin triplet pairing is present on the surface, due to the complete spin-polarization of the DSS. We further prove that the odd-\(\omega\) pair amplitude is directly measurable by generating a finite Josephson current. Moreover, the flat-band dispersion of the DSS is also detectable in the Josephson current, as it forces part of the current to flow into sub-surface layers and reduces the overall magnitude. By adding a FM region to the junction we find additional mixed-spin triplet pairing, but only when the FM magnetization opposes the spin-polarization of the DSS. The finite proximity effect and Josephson current on WNLS surfaces can therefore be entirely attributed to odd-\(\omega\) pairing, with its equal-spin structure verified by an anisotropic behavior with an external magnetic field.

Model and Method. The Hamiltonian for a minimal model of a 3D WNLS breaking time-reversal symmetry is [6, 7, 9]

\[
\mathbf{H}_\omega(k) = \sigma_x(6 - \alpha_1 - 2 \cos k_x - 2 \cos k_y - 2 \cos k_z) + 2\alpha_2 \sigma_y \sin k_z - \mu_w
\]  

(1)
where \( \mathbf{k}_i \) is the wave vector along the \( i \)-th \((x, y \text{ or } z)\) axis and the Pauli matrices, \( \sigma_i \) act in spin space. The first two terms give rise to a Fermi nodal loop with its shape set by \( \alpha_1 \) and \( \alpha_2 \) and protected by \( T \bar{T} \) symmetry and mirror symmetry with respect to \( k_x \rightarrow -k_x \) [4].

For finite chemical potential \( \mu_w \), the Fermi surface is instead a small torus-like surface surrounding the nodal loop. We here use \( \alpha_1 = \alpha_2 = 1 \) and \( \mu_w = 1 \), but the appearance of odd-\( \omega \) pairing is not restricted to these values. For surfaces perpendicular to the \( z \)-axis, the Fermi nodal loop projects to form a large DSS. To capture the DSS we therefore consider a finite cubic lattice system along the \( x \) and \( z \)-axes, but keep the periodicity along the \( y \)-axis. This leads to the nearest-neighbor hopping integrals \(-\sigma_x \) and \(-\sigma_y \), along the \( x \) and \( z \)-axes, respectively. We use a slab \( L_z = 21 \) layers thick along the \( z \)-axis for each layer.

To form a Josephson junction on the surface we attach two conventional spin-singlet \( s \)-wave SCs to the top layer \((n_x = 1)\) of the WNLS, at \( n_x = 1 \) and \( n_x = L_x \) sites, see Fig. 1. Each lead is described by a finite number of lattice sites along \( x \)-axis keeping the periodicity along \( y \)-axis to mimic bulk SCs. The left and right leads are characterized by the order parameter \( \Delta_s \) along \( x \) and \( y \)-axes, but keep the periodicity along the \( z \)-axis. We therefore only show the results for the top layer of the WNLS, with a very sharp peak at \( E = 0 \), which is completely spin-down polarized. Away from \( E = 0 \), the SLDOS becomes almost linear and spin-polarizations comparable signalling bulk contributions. A similar surface state is also found at the bottom layer but with up spin-polarization, and we also find the DSS spin-polarization analytically in SM [56]. In Fig. 2(b), we show the constant energy cut for \( E = 0 \) in the \( k_x - k_y \) plane, which confirms the Fermi nodal loop. The DSS is the projection of this nodal loop on the surface BZ, thus forming a drum-head structure.

**Fig. 1.** Schematic figure of WNLS with two SC leads attached to the surface. The spin-polarization of the DSSs (circular regions) of the top and bottom layers are shown by arrows.

**Odd-\( \omega \) pairing.** Next we attach the SC leads and study the proximity-induced Cooper pairs on the surface of WNLS. To analyze the nature and amplitude of the induced pairings, we plot in Fig. 3 the real and imaginary parts of the pair amplitude \( F \) as a function of frequency \( \omega \) on the surface of the WNLS. All possible spin configurations of the pairing function, spin-singlet \((\uparrow \downarrow - \downarrow \uparrow)\), mixed-spin triplet \((\uparrow \downarrow + \downarrow \uparrow)\), and equal-spin triplet \((\uparrow \uparrow , \downarrow \downarrow)\), are shown. We here only consider onsite \( s \)-wave pairing as we find the \( p \)-wave amplitudes to be negligible on the WNLS surface. Moreover, \( s \)-wave symmetry is very generally considered to be the most stable pairing in the presence of any disorder [57]. Due to the symmetrical positions of the SC leads on the surface, \( F \) is symmetric with respect to the middle site. We therefore only show the results for the first \((n_x = 1)\) and the middle site \((n_x = 8)\). We observe that spin-singlet pairing is present where the spin-singlet SC leads are located. However, this amplitude decays extremely quickly into the junction. Instead, the only pair amplitude present well inside the junction is the \( \downarrow \downarrow \) spin-triplet pairing, all other amplitudes are negligible. This is also true for sub-surface layers.

The results in Fig. 3 has to be understood in the context of the complete spin-down polarization of the DSS. This spin-polarization strongly opposes spin-singlet pairing, although exactly at the SC leads spin-singlet pairing still survives to some extent, as also observed in SC-FM junctions [58]. Instead, the DSS only supports \( \downarrow \downarrow \) pairing as tunnelling of down spin is heavily favored compared to up spin from the perspective of energy cost. Follo-
FIG. 3. Real (upper) and imaginary parts (lower) of $F$ as a function of frequency $\omega$ for two sites in the top layer ($n_z=1$) using $t_{w,sc}=0.5$.

In using Fermi-Dirac statistics, any spin-triplet $s$-wave state necessarily has to have an odd-$\omega$ dependence. This is confirmed in Fig. 3, where all spin-triplet pair amplitudes, both real and imaginary parts, are odd functions of frequency.

Odd-$\omega$ spin-triplet pairing is well established in SC-FM heterostructures [16, 17, 27], including SC-half metal structures [58, 59]. As such, the appearance of odd-$\omega$ pairing on the WNLS surface is not surprising. However, for the long-range equal-spin triplet pairing to appear in any of these previously studied structures, two magnetization directions have to be present: one magnetization direction rotates the spin-singlet state to a mixed-spin triplet state, while the second direction, not parallel to the first, generates equal-spin pairing. Alternatively, spin-orbit coupling can substitute for one of the magnetic fields, where for example an FM-SC interface with spin-orbit coupling can be shown to be sufficient to generate equal-spin triplet pairing [58, 60, 61]. In contrast, for a Josephson junction on the surface of a WNLS, the DSS is only polarized in a single direction, quite along the $z$-direction, but still equal-spin triplet pairing completely dominates. We can attribute this to the existence of spin-orbit coupling present in the bulk of the WNLS, which is in proximity to the DSS. The WNLS surface Josephson junction therefore provide a unique system for generating long-range odd-$\omega$ pairing, as it does not require engineering a spin-active interface, nor any application of external magnetic fields, and still the pairing is exclusively of odd-$\omega$ equal-spin nature. This is one of the main results of this work. We also note that this result is distinctive from the recently found odd-$\omega$ pairing in the bulk of WNLS, as spin-orbit coupling is there automatically present inside the Josephson junction [8]. As $H_w(k)$ is symmetric with respect to the $x$ and $y$-axes, we conclude that odd-$\omega$ pairing is dominating irrespective of the orientation of the Josephson junction.

**Ferromagnetic junction.** Having established the existence of only odd-$\omega$ equal-spin pairing in the DSS, we next add a FM island on the top layer of WNLS to further study the pairing structure. In Fig. 4, we show the variation of the absolute values of $F$ as a function of the position in the junction $n_z$ in both the absence and presence of a FM. With no FM (i.e. $m_n=0$), the odd-$\omega$ spin-triplet pairing completely dominates over all other pairings by an order of magnitude or more. As we move away from the SC leads towards the middle of the junction, the pair amplitude first increases due to the large spin-polarization of the DSS, but as the distance from the SC lead increases the pair amplitude eventually decays. For a FM with $P$ vector along the $x$-axis this behavior is unaffected. We here show the result for $m_x=0.5$, but this is true for all values of $m_x$ and also $m_y$. However, the pairing changes if we set $P$ in the $+z$-direction, i.e. opposite to the DSS spin-polarization. The amplitude of the $\downarrow\downarrow$ pairing is then reduced. Instead, spin-singlet and also $\uparrow\uparrow$ spin-triplet pairing grows close to the SC leads. But, most importantly, the amplitude of mixed-spin triplet pairing strongly increases and eventually becomes comparable to that of the down equal-spin pairing throughout the junction when we increase $m_z$.

We can understand the behavior of the WNLS with an added FM from the traditional FM Josephson junction, where the spin-singlet pairing of the SC is transformed to mixed-spin triplet pairing in the FM region [16, 17, 27]. But in the WNLS we see this effect only for a FM with $P$ along $+z$, because then the FM counteracts the spin-polarization of the DSS such that the down equal-spin pairing is reduced, leaving space for other pairing symmetries. The reduced overall spin-polarization also causes the spin-singlet pairing to increase close to
the SCs. On the other hand, for $P$ along the $x$ or $y$-directions, the FM acts on both spins equally, and thus the DSS can unperturbed continue to generate the down equal-spin pairing. As a consequence, the transformation from even-frequency (even-ω) spin-singlet to odd-ω spin-triplet pairings depends on the value of $m_n$ as well as the direction of $P$. Most notably, the mixed-spin triplet state only appears if a magnetization is present to counteract the intrinsic DSS spin-polarization, otherwise only equal-spin pairing is present, which is a very different behavior from traditional FM Josephson junctions. For the results in Fig. 4 we choose a particular $\omega$ where the amplitude of odd-ω is reasonably high, but the results do not change for other $\omega$, see SM [56].

**Josephson current.** To be able to measure the effect of odd-ω pairing, we calculate the maximum Josephson current $J$, which appears for phase difference $\Delta \phi = \pi / 2$ between the two leads, see SM [56]. We express $J$ in units of $t_{w} \hbar / e a$ and plot it as a function of $t_{w}$ in Fig. 5. With increasing $t_{w}$, $J$ grows due to the enhancement of the induced SC pairing, but ultimately saturates.

In the absence of a FM, the current can only be carried by the odd-ω down spin pairing, as that is the only pairing present well inside the junction. From the previous studies of both conventional FM Josephson junctions (with two magnetization directions) [27] and bulk WNLS junctions [8], we know that odd-ω equal-spin pairing can carry a significant current. Based on the large DOS of the DSS we could also naively expect a very large current for a WNLS surface Josephson junction. To investigate this, we compare with a normal metal (NM) junction using the same SC leads, see SM for further details [56]. From Fig. 5 we see that the current is actually suppressed in the WNLS junction compared to the NM junction, despite the much larger normal-state DOS for the WNLS. We attribute this effect to the (almost) flat-band nature of the DSS. With the carrier velocity being proportional to the band dispersion, $\partial E / \partial k_x$, the flat-band nature of the DSS limits the Josephson current, despite the large number of available carriers.

To further illustrate how the flat dispersion of the DSS suppresses the current, we divide in the inset of Fig. 5 the total current into contributions from each layer of the WNLS slab along the $z$-direction. Normally we expect the current to be strongly concentrated to the first layers closest to the surface, as proximity-induced pairing quickly decays when moving away from the SC leads. However, in the middle of the junction ($n_z=8$) the current stays approximately constant for many sub-surface layers, even slightly increasing. This demonstrates that the DSS does not conduct the majority of the current, but the current instead flows through sub-surface layers, where a proximity effect is still present and the carrier velocity is larger. For the site just below the SC ($n_z=1$), the presence of the SC mitigates the effects of the DSS and we see a more conventional behavior. This constitutes the other important main result in the work: measuring a finite Josephson current proves the presence of odd-ω pairing, as no current would be present if only even-ω pairing was allowed. In addition, a majority of current in sub-surface layers, with an overall reduced magnitude, verify the existence of extremely low-velocity carriers in the DSS.

![Fig. 5. Maximum Josephson current Log$_{10}J$ as a function of $t_{w}$ for a WNLS Josephson junction with and without a FM island, and for a NM junction.](image)

**Summary.** In this work we show that the superconducting proximity-effect on the surface of WNLS is significant and consists entirely of odd-ω equal-spin triplet pairing, due to the complete spin-polarization of the DSS. We further show that this odd-ω pairing is directly measured by the presence of a finite Josephson current. The flat-band dispersion of the DSS is also clearly visible by forcing some current into sub-surface layers and causing a reduction of the total Josephson current. Placing a FM island in the Josephson junction additionally generates mixed-spin triplet pairing when the FM magnetization is opposing that of the DSS polarization. However, for FM magnetization perpendicular to the DSS spin-
polarization, the pairing and Josephson current remain unaffected, further corroborating the equal-spin triplet nature of the pairing in WNLS Josephson junctions.

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SUPPLEMENTARY MATERIAL

In this supplementary material we provide additional information supporting the results in the main text. In order, we provide a more in-depth account on the model and method, analytical calculation of the DSS spin-order, we provide a more in-depth account on the model and method used in this work in order to make a self-sufficient report, beyond referring to earlier references. In this section we provide a detailed account of the model and method, analytical calculation of the DSS spin-order, and data on the anisotropic nature of the current with the rotation of the polarization vector.

Theoretical background

In this section we provide a detailed account of the model and method used in this work in order to make for a self-sufficient report, beyond referring to earlier references.

Discretization of WNLS Hamiltonian

After taking inverse Fourier transformation in the x- and z-directions of WNLS Hamiltonian as written in Eq. (1) in the main text, we arrive at

\[ H_w(k_y) = \sum_r \left[ c_{r,k_y}^\dagger (\sigma_x (6 - \alpha_1 - 2 \cos k_y) - \mu_w \sigma_0) c_{r,k_y} + t_{r,k_y} \sigma_x c_{r+\delta z,k_y} - t_{r,k_y} \sigma_x c_{r+\delta x,k_y} + H.c. \right], \]

where \( r+\delta x \) (\( \delta z \)) represents the position of the nearest neighbor site along \( x \) (\( z \))-direction. \( c_{r,k_y}^\dagger \) (\( c_{r,k_y} \)) is the creation (annihilation) operator for the electrons on site \( r \) on a square lattice (in the \( x-z \) plane) with momentum \( k_y \). Note that we use \( \mu_w=0 \) throughout the manuscript, but qualitatively, our main results remain valid also for finite \( \mu_w \) values.

Modelling of superconductor leads and coupling

We model each SC lead by a 2D Hamiltonian

\[ H_{sc}(k) = -\mu_{sc} + 2t_{sc}(2 - \cos k_x - \cos k_y) \]

where \( \mu_{sc} \) and \( t_{sc} \) are the chemical potential and hopping amplitude. Then, we take inverse Fourier transformation along the \( x \)-direction and arrive at

\[ H_{sc}(k_y) = \sum_{r_x} \left[ (-\mu_{sc} + 2t_{sc}(2 - \cos k_y)) b_{r_x,k_y}^\dagger b_{r_x,k_y} + t_{sc} b_{r_x,k_y}^\dagger b_{r_x+\delta x,k_y} + H.c. \right], \]

where \( r_x \) denotes a lattice site index along \( x \)-direction within SC lead and \( b_{r,k_y}^\dagger \) (\( b_{r,k_y} \)) is the creation (annihilation) operator for the electrons in SC. In order to get a higher current using a smaller system size for computational reasons, we keep \( \Delta_s \) a little bit higher compared to the value in a realistic material. To use the realistic \( \Delta_s \), we have to increase the system size more to get a significant Josephson current. This assumption does not affect the qualitative behavior of our results. We show all the results for 20 number of lattice sites of SC, which is large enough to model bulk SC leads. Throughout the work, we fix \( t_{sc}=1 \) and \( \mu_{sc}=2 \), again with no qualitative effects on the results.

The coupling between each SC lead and WNLS is given by the hopping Hamiltonian

\[ H_{w-sc}(k_y) = t_{w-sc} \left( c_{1,k_y}^\dagger b_{L1,k_y} + c_{L1,k_y}^\dagger b_{R1,k_y} + H.c. \right), \]

where \( t_{w-sc} \) is the coupling strength between WNLS and each SC lead. In our model, we couple the first site (1) of the left lead to the first (\( n=1 \)) site of the top layer of WNLS and similarly, the first site (1) of the right lead to \( L_x \)-th site of the top layer of WNLS.

Normal metal Hamiltonian

To compare the Josephson current on the surface of the WNLS with that of a normal metal Josephson junction, we model a normal metal as,

\[ H_{NM}(k_y) = \sum_r \left[ (6 - t_1 - 2 \cos k_y - \mu_{NM}) c_{r,k_y} + t_2 (c_{r,k_y}^\dagger c_{r+\delta z,k_y} + c_{r,k_y}^\dagger c_{r+\delta x,k_y} + H.c.) \right], \]

where the hopping integral both along \( x \) and \( z \)-directions are \( t_2 \). We set \( t_1=t_2=1 \) and \( \mu_{NM}=0 \) to keep the symmetry with the WNLS Hamiltonian. This is the simplest possible normal metal state, which is also directly comparable with the WNLS Hamiltonian.
Calculation of SLDOS

We use the retarded Green’s function of the bare WNLS to define the SLDOS as

$$\rho_{r,\sigma}(E) = -\frac{1}{\pi} \sum_{k_y} \text{Im}[G^R_{r\sigma\sigma}(E, k_y)]$$

(7)

where $G^R_{r\sigma\sigma}(E, k_y)$ is the element of the whole retarded Green’s function given by

$$G^R(E, k_y) = [(E + i\delta)I - H_w(k_y)]^{-1}$$

(8)

and $r (= \{x, z\})$ is the lattice site index, $E$ is the energy, and $H_w(k_y)$ is given in Eq. (2). We use Eq. (7) to calculate SLDOS of the upper surface layer in Fig. 2 of the main text as well as the lower surface layer in the following section.

Calculation of anomalous Green’s function

In Nambu basis, each part of the SC-WNLS-SC junction can be described by Bogoliubov-deGennes (BdG) equation as

$$H_{\text{BdG}}(k_y) \Psi(r, k_y) = E \Psi(r, k_y)$$

(9)

where

$$H_{\text{BdG}}(k_y) = \left( \begin{array}{cc} H_y(k_y) & \Delta e^{i\phi} \sigma_y \\ \Delta e^{-i\phi} \sigma_y & -H_y^*(k_y) \end{array} \right)$$

(10)

$H_y(k_y)$ may be the WNLS ($H_w(k_y)$) or either of the two SC leads, $H_{sc}(k_y)$, as well as the coupling in-between. Here $\Psi(r, k_y)$ is the four component eigenstates given in the basis $|\psi^L_y(k_y), \psi^R_y(k_y), \psi^T_x(k_y), \psi^T_x(k_y)\rangle^T$. $\Delta$ is the SC gap parameter (zero for WNLS, $\Delta_s$ for the SC leads) and $\phi$ is the SC phase. We use $\phi_L$ and $\phi_R$ to denote the phase factors of the left and right SC lead, respectively. We use equal phases $\Delta \phi$ ($\phi_L = \phi_R = 0$) for all the plots of the anomalous Green’s function.

We define the retarded Green’s function in Nambu basis as

$$G^R(\omega, k_y) = [(\omega + i\delta)I - H_{\text{BdG}}(k_y)]^{-1}.$$ 

(11)

Following Eq. (10), $G^R$ can be expressed in a 2x2 block-matrix form like

$$G^R(\omega, k_y) = \begin{pmatrix} G^R_{yy} & G^R_{y\sigma} \\ G^R_{\sigma y} & G^R_{\sigma\sigma} \end{pmatrix}.$$

(12)

Each component of $G^R$ is a 2x2 matrices ($N = L_xL_z$) dimensional matrix. The off-diagonal component, $G^R_{y\sigma}$, is used to calculate the amplitude of the induced pairing on the surface of WNLS [45]. For each site $r$, we can express the anomalous Green’s function of the SC-WNLS-SC system as

$$F(\omega) = \sum_{k_y} G^R_{ch}(\omega, k_y) = \begin{pmatrix} G^R_{ch} & G^R_{ch}^T \\ G^R_{ch}^T & -G^R_{ch} \end{pmatrix}$$

(13)

where the diagonal components correspond to the equal-spin triplet pairing and the off-diagonals provide the information for mixed-spin-triplet ($[G^R_{ch}]_{\uparrow\downarrow} + [G^R_{ch}]_{\downarrow\uparrow}$) and spin-singlet pairing ($[G^R_{ch}]_{\uparrow\downarrow} - [G^R_{ch}]_{\downarrow\uparrow}$). Note that due to the periodicity along y-direction, we here sum over all $k_y$ within the first BZ.

Calculation of Josephson current

We define a local number density operator for each site $r$ as $n_r(k_y) = \psi^\dagger_r(k_y)\psi_r(k_y)$ which provides the information regarding the number of particles at site $r$. It necessarily has to obey the continuity equation given by [53–55],

$$\nabla \cdot J + e \frac{\partial n_r(k_y)}{\partial t} = 0$$

(14)

where $J$ is the current density vector and $e$ is the electronic charge. The time rate of change of the number density operator can be found by using the Heisenberg equation

$$\frac{\partial n_r(k_y)}{\partial t} = i \hbar \left[ H(k_y), n_r(k_y) \right],$$

(15)

where $H(k_y)$ ($H_w(k_y)$+$H_{Lsc}(k_y)$+$H_{Rsc}(k_y)$+$H_{w-sc}(k_y)$) is the total Hamiltonian for the whole SC-WNLS-SC system. Here, $H_{Lsc}(k_y)$ and $+H_{Rsc}(k_y)$ refer to the left and right SC lead as given in Eq. 4. To calculate the expectation value of the time evolution of the number density operator we consider all the occupied levels of WNLS. The expectation value of the time evolution of the number density operator gives us two different terms proportional to the incoming ($\psi^\dagger_r(k_y)\psi_{r-x}(k_y)$ terms) and outgoing ($\psi^\dagger_y(k_y)\psi_{r+x}(k_y)$ terms) current through the nearest neighbor bonds (since all terms in the full Hamiltonian are either on-site or nearest neighbor couplings) allowing us to write the expression $\nabla \cdot J$ as $(J_{\text{out}} - J_{\text{in}})/a$ where $a$ is the lattice constant of the unit cell.

Note here that the sum of the currents flowing between any two neighboring site $n_x$ to $n_x+1$ of a particular layer $n_z$, denoted by $J_{n_x,n_x}(k_y)$

$$J_{n_x} = \sum_{n_z=1}^{L_z} \sum_{k_y} J_{n_x,n_z}(k_y).$$

(16)

is constant throughout the WNLS since the current has no sinks or sources in the WNLS (the current is driven by an imposed phase difference in the SC leads only).
Since $J_{n_z}$ is same for each $n_z$ after summing over the layers, we can set the total current $J=J_{n_z}$. For the results of the current in each WNLS layer, we do not perform the summation in Eq. (16) to arrive at $J_{n_z}$. We take a summation over all $k_y$ within the first Brillouin zone for both of them.

**Spin-polarization of drumhead-like surface states**

In this section we present an additional result as well as an analytical calculation for the spin-polarization of the surface states of WNLS to further support the discussion of the spin density of states in the main text.

First, we show the behavior of SLDOS with the variation of energy for the middle site ($L_z/2=8$) of the very bottom layer ($n_z=21$) of the WNLS in Fig. 6. Here the SLDOS for the up spin is very high compared to that of the down spin close to the Fermi energy. This is in complete contrast to that of the top layer. Comparing Fig. 2 and 6, we conclude that the surface states of the top layer are completely down spin-polarized whereas those in the bottom layer are completely up spin-polarized. We also check the same for other sites of the bottom layer. The spin-polarization is much higher if we move towards the middle site from the outer boundary.

Next we find analytical solutions for the DSS for the Hamiltonian of Eq.(1). Keeping both two directions $x$ and $y$ periodic, we can represent $H_w(k||,z)$ as an effective 1D Hamiltonian as [9],

$$H_w(k||, z) = -\sigma_x(\partial_z^2 + \alpha'_1) - 2i\alpha_2\sigma_y\partial_z$$  

(17)

where $k|| = (k_x, k_y)$, being good quantum numbers, are absorbed in $\alpha'_1$. Further, we approximate cosine and sine functions of Eq.(1) to leading order terms and replace $k_z$ by $(-i\partial_z)$.

We set two open boundary at $z=0$ and $z=L_z$ in order to calculate the zero-energy end state solution. The boundary conditions are expressed as

$$\psi_{k||}(z)|_{z=0} = 0 \quad \text{and} \quad \psi_{k||}(z)|_{z=L_z} = 0.$$  

(18)

In the end we arrive at an equation for any zero energy states

$$[\sigma_x(\partial_z^2 + \alpha'_1) + 2i\alpha_2\sigma_y\partial_z] \psi_{k||}(z) = 0.$$  

(19)

We operate $\sigma_y$ from the left side and obtain

$$\partial_z\psi_{k||}(z) - \frac{\sigma_y}{2\alpha_2}(\partial_z^2 + \alpha'_1)\psi_{k||}(z) = 0.$$  

(20)

Now, we look for the eigenstate of $\sigma_z$ operator and separate the spatial and spin parts of the wavefunction as $\psi_{k||}(z)=\phi_{k||}(z)\chi_{\nu}$ where the spin part satisfies the equation

$$\sigma_y\chi_{\nu} = \nu\chi_{\nu},$$  

with $\nu=\pm1$. Considering the ansatz $\phi_{k||}(z)e^{-\eta z}$, we find the secular equation as

$$\eta^2\phi_{k||}(z) + 2\nu\alpha'_2\phi_{k||}(z) - 2\nu\alpha_1\alpha'_2\phi_{k||}(z) = 0.$$  

(21)

We have two boundaries one at $z=0$ and other one at $L_z$ as mentioned in Eq. (18). To find the solutions for the two surface states, we imagine two different situations. In one case, we consider it as semi-infinite having a cut-off at the top layer at $z=0$. For $L_z$ infinitely large we can write $\phi_{k||}|_{L_z\rightarrow\infty} = 0$. To satisfy this condition, the product of the two solutions $\eta_1$ and $\eta_2$ of the secular equation must be positive but we have $\eta_1\eta_2 = -2\nu\alpha_1\alpha'_2$. For positive $\alpha_1$ and $\alpha'_2$, we must have $\nu=-1$. This is true for all the states encircled by the projection of the bulk nodal loop on the $k||$ surface. Therefore, DSS of the top layer of WNLS is polarized along $-z$-direction or down spin-polarized.

Similarly, for the bottom layer, we imagine the cut-off at $z=0$ and semi-infinite along opposite direction i.e., $\phi_{k||}|_{L_z\rightarrow\infty} = 0$. Following the similar argument, the spin-polarization will be along $+z$-direction.

**Frequency dependence of $F$ in presence of FM**

In this section we provide additional data for the behavior of the superconding pair amplitudes in the presence of a FM island in the WNLS Josephson junction. In particular, the complete frequency behavior. In Fig. 4 in the main text we show results for different $P$, but choose the particular value $\omega=0.025$, as the $\downarrow\downarrow$ odd-$\omega$ pair amplitude is reasonably high at this value. We confirm the full frequency dependence we plot the real and imaginary part of $F$ at the middle site of the top layer as a function of $\omega$ for two different values of $m_z$ in Fig. 7. From both the real and imaginary parts of $F$, it is clear that all the spin-triplet components, both equal and mixed spin, are odd in $\omega$.

When $P$ is set along $z$-axis, the FM polarization is partly counteracting the DSS spin-polarization. This allows for both spin-singlet as well as mixed-spin triplet
pairing to increase whereas, the amplitude of the \(\downarrow\) spin-triplet pairing is also finite due to the still strong spin-polarization of DSS. There is a competition between them and the dominating pairing is determined by the value of \(m_z\). When we increase \(m_z\), the amplitude of the \(\downarrow\) spin-triplet pairing decreases, while the amplitude of mixed-spin triplet pairing start increasing such that they eventually become comparable to each other. The behaviors of real and imaginary parts of \(F\) are oscillatory with the variation of \(\omega\) for all the spin-triplet pairings, but clearly the result in Fig. 4 in the main text is always qualitatively valid.

**Current-phase relation**

In this section we show that the maximum Josephson current is achieved at a phase difference \(\Delta\phi=\phi_L-\phi_R=\pi/2\) as used in the main text. In Fig. 8, we display the full \(J-\Delta\phi\) relationship. We notice that the current is maximum when the phase difference is \(\Delta\phi=\pi/2\), following closely the behavior of conventional Josephson junctions. We thus use this phase difference for all the plots of the maximum current.

**Anisotropy of \(J\) in presence of FM**

In this section we report on the anisotropy of the Josephson current as function of the magnetization of the FM island inside the WNLS Josephson junction. As stated in the main text, there is no change in the current for polarization vectors of FM \(P\) fully in the \(xy\)-plane. Only for \(P\) vectors with a component along the \(z\)-direction do we find a change of the current. In the main text we give the current for \(m_z=0.2, 0.8\) as a function of the coupling between the SC and WNLS, which shows that there are some minor changes of the current. In Fig. 9 we show how the pair amplitudes for these two FM islands behave in the weak and strong coupling limits, respectively. As seen, there is no qualitative difference in the nature of the pairing amplitudes between these limits: for both weak and strong coupling the mixed-spin triplet pairing increases with \(m_z\). In the strong coupling limit we see overall larger pair amplitudes compared to the no FM case, which explains the slightly larger currents in this regime. In conclusion, we see that the small changes in the current are not to be attributed to any fundamental change of the pairing structure.

**Fig. 7.** Real (upper panel) and imaginary (lower panel) parts of \(F\) as a function of frequency \(\omega\) for the middle (8th) site of the top layer \((n_z=1)\) for two values of the magnetization of the FM along the \(z\)-axis. Other parameters parameters are the same as in Figs. 3 and 4 of the main text.

**Fig. 8.** Josephson current \(J\) as a function of superconducting phase difference \(\Delta\phi\) for \(t_{w-sc}=0.5\) and in the absence of a FM island \((m_n=0)\).

**Fig. 9.** Magnitude of pair amplitude \(|F|\) as a function of frequency \(\omega\) at the middle site \((n_z=8)\) of the top layer \((n_z=1)\) of the WNLS in the absence \((m_n=0)\) and in the presence of FM set along \(z\)-axis. The left and right columns correspond to the weak and strong coupling regime, \(t_{w-sc}=0.2\) and \(t_{w-sc}=0.8\), respectively. Other parameter values are the same as in Fig. 3 of the main text.
Finally, we report how the current changes as the magnetization is tuned from the \( z \)-direction to the \( x \)-direction, given by varying \( \theta_F \) from 0 to \( \pi/2 \). In Fig. 10, we plot \( J \) as a function of \( \theta_F \) for \( m_n=0.5 \) and \( t_{\text{q-sc}}=0.5 \). We notice that with the increase of \( \theta_F \), \( J \) initially increases, but for \( \mathbf{P} \) getting closer to the \( xy \)-plane the current again decreases and settles to the value found in the absence of a FM (as shown in Fig. 5 of the main text). This non-monotonic behavior cannot simply be explained from the qualitative behavior of the pair amplitudes alone, but is the result of a combination of currents carried by the equal-spin and mixed-spin pairing. When \( \theta_F \) increases, so does the \( \downarrow\downarrow \) pair amplitude rather dramatically, while at the same time the mixed-spin amplitude is slowly decaying. Finally at \( \theta_F=\pi/2 \) there are only \( \downarrow\downarrow \) pairs, and the current is thus slightly reduced.