Super-allowed beta-decay rates in $1d_{5/2}$ shell in Coriolis coupling model

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The expression for super-allowed beta-decay transition rates have been derived within the context of Coriolis coupling model. The derived expressions, valid for the beta-decay between any two mirror nuclei, has been applied to calculate super-allowed beta-decay transition rates of $^{21}$Na, $^{21}$Mg, $^{21}$Al, and $^{21}$Si. The calculated rates agree well with the data and the calculations done using the shell model with configuration admixture.

PACS Nos: 21.60Ev, 23.40.-S

1 Introduction

Both the shell model with configuration admixture [1–3] and the Coriolis coupling model [4] reasonably account for the low-lying level spectra, and electric quadrupole and magnetic dipole moments of ground states in many odd $1d_{3/2}$-nuclei reasonably well. In addition the Landford and Wildenthal’s shell model calculations [5] with configuration mixing, using the Kuo-Brown [3] two-body matrix elements with $^{16}$O as a core, account for the super-allowed beta-decay transition rates between a pair of mirror nuclei in many odd nuclei in $1d$-$2s$ shells. However, there has been no beta-decay transition rate calculations within the context of Coriolis coupling model for $1d_{3/2}$ and other nuclei so far. In this paper, we first derive the expressions for super-allowed beta-decay transition rates in Coriolis coupling model and apply those to calculate super-allowed transition rates between the pairs $^{21}$Na→$^{21}$Ne, $^{21}$Mg→$^{21}$Na, $^{25}$Al→$^{25}$Mg, and $^{27}$Si→$^{27}$Al. Kim [6] has calculated the super-allowed transition rates of $^{25}$Al→$^{25}$Mg and $^{27}$Si→$^{27}$Al using the Nilsson’s [7] model. His results for log$ft$ are somewhat lower than the observed data. The level spectra of these nuclei are reasonably affected by the Coriolis coupling. It is, therefore,
interesting to investigate the extent to which the band mixing influences the results of superallowed beta-decay.

In the next section, we present a brief outline of the Coriolis coupling model of [4] and the theoretical derivation of super-allowed transition within the context of that model. This is followed by a section dealing with the application to these nuclei.

2 Theory

A. Outline of the model

The model Hamiltonian of a nucleon moving in an axial symmetric anharmonic potential of a rotating nucleus is given by [8–10]

\[ H = \frac{\hbar^2}{2I} \left[ I^2 + j^2 - 2(Ij) \right] + H_p + H_c \] (1)

where \( I \) and \( j \) are, respectively, the total spin of the body and angular momentum of a nucleon, usually the valence one, \( J \) is the moment of inertia of the body, \( H_p \) and \( H_c \) are, respectively, the intrinsic Hamiltonian governing the motion of the nucleon in consideration and the other nucleons in the core in body-fixed coordinate system denoted by primes. The explicit form of \( H_c \) is usually not considered but \( H_p \) is given by the following Nilsson Hamiltonian [7]

\[ H_p = -\frac{\hbar^2}{2\mu} \nabla^2 + \frac{\mu}{2} [\omega_2^2 (x'^2 + y'^2) + \omega_z^2 z'^2] + C \mathbf{l}s + D \mathbf{l} \cdot \mathbf{l} \] (2)

In (2) \( \mu, \omega_2, \) and \( \omega_z \) are, respectively, the reduced mass, oscillator frequencies in \((x' \text{ and } y')\) and \( z' \) directions respectively. \( C \) and \( D \) are strengths of the spin-orbit and \( \mathbf{l} \cdot \mathbf{l} \) interaction.

The properly anti-symmetrical and normalized wavefunction of the Hamiltonian (1) is given by

\[ |I, M> = \left[ \frac{2I + 1}{16\pi^2} \right]^{1/2} \sum_{K=\Omega,\nu} C_{K,\nu} \left\{ D_{MK}^I(\theta)\chi_{\Omega,\nu} + (-1)^{I-1/2} D_{M,-K}^I(\theta)\chi_{-\Omega,\nu} \right\} \varphi_c \] (3)

In (3), \( D_{MK}^I \) is the eigenfunction of \( I^2 \) with space and body fixed \( z \)-projections \( M \) and \( K \), respectively and \( \theta \) is the three Euler angles. \( \varphi_c \) is the normalized wavefunction of \( H_c \). \( \chi_{\Omega,\nu} \) is the eigenfunction of the Nilsson Hamiltonian (2) and is generated from the spherical shell model wavefunction \( |j \Omega> \), \( \Omega \) being the \( z \)-projection of \( j \) in the body-fixed system as follows:

\[ \chi_{\Omega,\nu} = \sum_j c_{j,\Omega,\nu} |j \Omega> \] (4)
The index \( \nu \) distinguishes among the Nilsson states of the same \( \Omega \) originating from different orbitals at zero deformation.

The coefficients \( c_{j,\Omega,\nu} \) is obtained by diagonalizing (2) and the band mixing coefficients \( C_{K,\nu} \) are obtained by diagonalizing (1) among all single particle and hole excited bands within a major shell. We have used three Cs for three distinctly different quantities: \( C_{K,\nu} \)s for band mixing coefficients, \( c_{j,\Omega,\nu} \) for coefficients of the spherical shell model wave functions to obtain Nilsson wave-function, and \( C \) for the strength of the spin-orbit coupling. The coefficients \( c_{j,\Omega,\nu} \) are related to Nilsson’s coefficients \( a_{l,\Lambda} \) as follows (after suppressing the index \( \nu \)):

\[
c_{j,\Omega} = \sum_{l,\Lambda} \frac{1}{2} l \Lambda \Sigma |j\Omega > a_{l,\Lambda}
\]

The oscillator frequency at zero deformation, \( \hbar \omega \), is set to be 41 MeV/\( A^{1/3} \), “\( A \)” being the mass number. The parameters of moment of inertia, \( \mathcal{A} = \hbar^2/2I \), the deformation, \( \beta \), and the strengths of the spin-orbit and orbit-orbit coupling along with the core-overlap or quenching factor \( Q \) of the Coriolis coupling term (\( I \cdot j \)) in (1) are adjusted to reproduce the low lying observed level spectra. The band-head energy, \( E \), in the model, is not a free parameter but obtained by summing appropriately over the Nilsson’s eigenvalues, \( E_{\Omega,\nu} \) and is given by

\[
E = \sum_{\nu} \frac{1}{2} (1 + \mu/2M) E_{\Omega,\nu} - (\mu/4M) < C1s + D1d >
\]

In (6), \( M \) is the mass of a nucleon.

**B. Super-allowed beta-decay rate**

Under the assumption that the transition matrix \( S \) in beta-decay is energy independent, the comparative half-life, \( ft \), is given by [11]

\[
ft = \frac{2\pi^3 \hbar^7 c \ln 2}{(mc)^5 g^2 |S|^2}
\]

In (7), \( m, c, \) and \( g \) are, respectively, the rest mass of electron, the velocity of light in vacuum, and the coupling constant. And \( f \) is given by the following expression:

\[
f = \int_{1}^{W_0} F(Z, W)(W^2 - 1)^{1/2} (W_0 - W)^2 W dW
\]

In (8), \( W \) and \( W_0 \) are, respectively, the total and maximum energies of beta particle in the units of \( mc^2 \). \( F(Z, W) \) is the energy distribution function of beta-particle. Denoting the parities of the parent and the daughter states by \( \Pi_P \) and \( \Pi_D \), respectively and the magnitude of the orbital angular momentum of beta-particle by \( l_\beta \), the parity conservation in the decay process is given by

\[
\Pi_P = (-1)^{l_\beta} \Pi_D
\]
The matrix-element $|S|^2$ is usually expanded in terms of the ascending values of $l_\beta$. Thus,

$$|S|^2 = |S(l_\beta = 0)|^2 + |S(l_\beta = 1)|^2 + |S(l_\beta = 2)|^2 + \cdots$$

The matrix elements associated with $l_\beta = 0, 1, 2 \ldots$ etc. are, respectively, termed as allowed, first forbidden, second forbidden, etc.

For allowed transition (7) can be rewritten as follows:

$$f_t = \frac{F}{<1>^2 + (g_A/g_V)^2 <\sigma>^2}$$

From the analysis of the decay of free neutron $(g_A/g_V)^2$, the ratio of the coupling strength of the matrix elements of $<1>$ and $<\sigma>$, is determined to be $(1.239 \pm 0.011)$, and $F = 6150$. This is the value used by Landford and Wildenthal [5].

A number of people, particularly Wilkinson [12], has pointed out that the ratio of the coupling constant and the value of $F$ in beta-decay from nuclei are likely to differ from the beta-decay rates of a nucleon in isolation. From the analysis of data, Wilkinson obtained and $(g_A/g_V)^2 = 1.27$ and $F = (6162 \pm 14)(1 - 3.67 \times 10^{-4}Z + 1.30 \times 10^{-5}Z^2)$, $Z$ being the atomic number of parent nucleus. In this paper, we shall present results using both sets of constants.

The allowed transition between two iso-baric multiplets i.e., states having the same iso-spin $T$ but different $z$-component, $T_3$, are termed super-allowed transition. The matrix element of $<1>$ between a parent state of total spin $I$, $z$-projection $M$, iso-spin $T$ and its $z$-component $T_3$, $|I(M)(T(T_3))>$ to a daughter state of spin $I'$, its $z$-projection $M'$, iso-spin $T$ and its $z$-component $T'_3$ is given by

$$<1> \equiv <I'(M')T'(T'_3)|I(M)T(T_3)>$$

$$= \delta_{II'} \delta_{MM'} \delta_{T_3T'_3} \sqrt{T(T + 1) - T_3T'_3}$$

Since the wavefunction (13) is in body-fixed coordinate system ($r'$), to determine the matrix-element of $<\sigma>$, $<\sigma>$ being an operator of rank one $O^1$, the latter is to be transformed from space-fixed ($r$) to body-fixed axis using the following transformation

$$O^1_M(r) = \sum_\mu D^1_M(\theta) O^1_\mu(r')$$

In (11), $M$ and $\mu$ are $z$-projections in space-fixed and body-fixed coordinates and ($\theta$) are three Euler angles.

Super-allowed transitions are characterized by the selection rule $\Delta I = 0$ and $\Delta T = 0$, and the spin and spatial part of the wavefunction of the parent and its daughter are the same. After the integration over the Euler angles, the matrix
elements of Gamow-Teller transition becomes

\[
<\sigma> = \frac{1}{\sqrt{I(I+1)}} \sum_{K,\nu} C_{K,\nu} \left[ K C_{K,\nu} <K,\nu|\sigma_0|K,\nu> + \frac{1}{2} \sqrt{(I-K+1)(I+K)} \right]
\]

\[\{C_{K-1,\nu} + (-1)^{I-1/2} C_{1-K,\nu}\} < (K-1),\nu|\sigma_-|K,\nu> \]

\[+ \frac{1}{2} \sqrt{(I+K+1)(I-K)} C_{K+1,\nu} < (K+1),\nu|\sigma_+|K,\nu> \]

The prime over the summation indicates that the summation runs over all positive values of \(K\) and \(\nu\) i.e. \(K = 1/2, 3/2, \ldots I\). The integration over the single particle matrix elements leads to the following expression for the Gamow-Teller moment for super-allowed transition:

\[
<\sigma> = \sum_{K,\nu} \left\{ \frac{K}{\sqrt{I(I+1)}} C_{K,\nu}^2 \sum_j \left( \frac{K}{j} (c_{j,K,\nu}^+)^2 + \sqrt{1 - \frac{K^2}{j^2}} c_{j-1,K,\nu}^+ c_{j,K,\nu}^+ \right) \right. \\
+ \frac{1}{2} \sqrt{(I-K+1)(I+K)} C_{K,\nu} \left[ \sum_j C_{K-1,\nu} \left( \frac{c_{j,K,\nu}^+}{j} \right) \right. \\
\left[ \sqrt{j(j+1) - K(K-1)} c_{j,K-1,\nu}^+ - \sqrt{(j+K)(j+K-1)} c_{j-1,K-1,\nu}^+ \right] \\
\left. - \frac{c_{j,K,\nu}}{j+1} \left( \sqrt{j(j+1) - K(K-1)} c_{j,K-1,\nu}^+ - \sqrt{(j+K+1)(j+K+2)} c_{j+1,K-1,\nu}^+ \right) \right]\]

\[+ (-1)^{I+j+1} C_{1-K,\nu} \left( \frac{c_{j,K,\nu}^+}{j} \left\{ \sqrt{j(j+1) - K(K+1)} c_{j,K,\nu}^+ \right. \\
+ \frac{1}{2} \sqrt{(I+K+1)(I-K)} C_{K+1,\nu} C_{K,\nu} \sum_j \left( \frac{c_{j,K,\nu}}{j} \right) \right. \\
\left[ \sqrt{j(j+1) - K(K-1)} c_{j,K+1,\nu}^+ + \sqrt{(j-K+1)(j-K+2)} c_{j-1,K+1,\nu}^+ \right] \]
\[
\frac{c_{j,K,\nu}^-}{j+1} \left[ \sqrt{(j+K+1)(j+K+2)}c_{j+1,K+1,\nu}^+ + \sqrt{j+1-K(K+1)}c_{j,K+1,\nu}^- \right] \right) \right]
\]

(13)

In the above, \(c^+\) is the coefficient where \(j = l+1\) and \(c^-\) when \(j = l-1\).

The expression for beta decay in Nilsson model of the above Gamow-Teller moment can be obtained by setting \(K = 1\), and \(C_{K,\nu} = 1\) and, omitting the summation over \(K\):

\[
<\sigma> = \sqrt{\frac{I}{I+1}} \left[ \sum_{j} \left\{ \frac{I^2}{j^2} c_{j,I}^+ (c_{j,I}^+)^2 + \frac{I^2}{j+1} (c_{j,I}^-)^2 \right\} + \frac{(j+1)}{j+1} \right] \left[ \sqrt{j(j+1) - I(I+1)}c_{j,I}^+ + \sqrt{(j+I)(j+I-1)}c_{j-1,I}^- \right] \right] \right) + \delta_{I,1/2} (-1)^{j-1/2} \left\{ \frac{c_{j,I}^+}{j} \left[ \sqrt{j(j+1) - I(I+1)}c_{j,I}^+ + \sqrt{(j+I)(j+I-1)}c_{j-1,I}^- \right] \right) \right] \right) - \frac{c_{j,I}^-}{j+1} \left[ \sqrt{j(j+1) - I(I-1)}c_{j,I}^- + \sqrt{(j-I+1)(j-I+2)}c_{j+1,I}^- \right] \right) \right] \right) \right] \right) \right]
\]

(14)

The above expression can be rewritten in terms of Nilsson’s [7] coefficient \(a_{l,\Lambda}\) using (5). One, then, obtains the following expression for transitions between two Nilsson level

\[
<\sigma> = \sqrt{\frac{I}{I+1}} \sum_{l} (a_{l,\Lambda}^2 - a_{l,\Lambda+1/2}^2) \delta_{I,1/2} \frac{2}{\sqrt{3}} \Pi_\chi \sum_{l} a_{l,0}^2
\]

(15)

The above is identical to Kim’s [6] expression for allowed transition. \(\Pi_\chi\) in the above equation is the parity of the wavefunction.

3 Application and Discussion

We present here calculations for the super allowed transition rates between the mirror pairs, \(^{21}\text{Na}, \ ^{19}\text{Ne}\), \(^{23}\text{Mg}, \ ^{21}\text{Na}\), \(^{25}\text{Al}, \ ^{23}\text{Mg}\), and \(^{27}\text{Si}, \ ^{25}\text{Al}\) and compare our results with the calculations of the Nilsson model i.e., without band mixing and the shell model.

The parameters of the model, \(A = (\hbar/2\chi)\), \(C\), \(D\), \(Q\), and \(\beta\), have been determined from the fits to the observed low lying level spectra of these nuclei except for \(^{21}\text{Na}\) and \(^{21}\text{Ne}\) in [4] and those for \(^{21}\text{Na}\) and \(^{21}\text{Ne}\) in this paper. Calculated
as well as observed level spectra up to approximately 3 MeV excitation energy for these nuclei are shown in Figs. 1 to 4 and the agreements are reasonable. As a further test to the model, the ground state magnetic and quadrupole moments of these nuclei have been calculated using the appropriate expressions for these quantities presented in [9]. These calculations along with the observed moments are presented in Table 1 and the agreement is reasonable.

In Table 2 we present calculated super-allowed ground state to ground state transition rates using both Wilkinson’s and Landford and Wildenthal’s [5] parameters for $ft$ expression along with the measured one for these four pairs of mirror nuclei. The theoretical results are in good agreement with the observed one, particularly using Wilkinson’s value for $ft$. The Fermi moment, $<1>$, in these calculations has been set to 1.

Kim’s calculations using the Nilsson’s model, i.e., without taking into consideration the band mixing caused by the Coriolis coupling, yields an $ft$ value of 3.49 for $^{25}\text{Al} \rightarrow ^{25}\text{Mg}$ and $^{27}\text{Si} \rightarrow ^{27}\text{Al}$ transitions, which deviates significantly from the observed value of 3.6 in both cases. Thus, the band mixing affects significantly the calculations of super-allowed transition.

As is well known, the elementary single particle shell model can not predict
Table 1: Calculated, noted as Cal, and experimental, noted as Exp, magnetic dipole, \( \mu \), and electric quadrupole, \( Q \), moments. The data are from National Nuclear Data Center, Database version of January 22, 2009.

| NUCLEUS | \( \mu \) in nm | \( Q \) in eb |
|---------|-----------------|--------------|
|         | Cal | Exp | Cal | Exp |
| \( ^{21} \text{Na} \) | 2.15 | 2.386 | 0.075 | 0.05 |
| \( ^{21} \text{Ne} \) | -0.564 | -0.662 | 0.072 | 0.103 |
| \( ^{23} \text{Mg} \) | -0.582 | 2.218 | 0.100 | - |
| \( ^{23} \text{Na} \) | 2.172 | - | 0.084 | 0.101 |
| \( ^{25} \text{Al} \) | 3.399 | 3.646 | 0.101 | - |
| \( ^{25} \text{Mg} \) | -0.779 | -0.855 | 0.112 | 0.20 |
| \( ^{27} \text{Si} \) | -0.778 | - | 0.122 | - |
| \( ^{27} \text{Al} \) | 3.399 | 3.642 | 0.092 | 0.140 |
Table 2: Calculated and observed super-allowed transitions between ground states of mirror nuclear pairs noted in the first column. Second to sixth columns represent the spin of initial and final states, calculated value of Gamow-Teller moment, log ft calculated with Wilkinson parameter, log ft calculated with Landford-Wildenthal’s parameter and the observed log ft respectively. The data are from National Nuclear Data Center, Database version of January 22, 2009.

| TRANSITION       | $I_i \rightarrow I_f$ | $<\sigma>$ | log ft          |
|------------------|------------------------|------------|-----------------|
|                  |                        |            | Cal(Wi) | Cal(LW) | Exp |
| $^{21}_{11}$Na $\rightarrow$ $^{21}_{10}$Ne | $3/2^+ \rightarrow 3/2^+$ | 0.553 | 3.65 | 3.63 | 3.61 |
| $^{23}_{12}$Mg $\rightarrow$ $^{23}_{11}$Na | $3/2^+ \rightarrow 3/2^+$ | 0.443 | 3.65 | 3.63 | 3.67 |
| $^{25}_{13}$Al $\rightarrow$ $^{25}_{12}$Mg | $5/2^+ \rightarrow 5/2^+$ | 0.731 | 3.56 | 3.54 | 3.57 |
| $^{27}_{14}$Si $\rightarrow$ $^{27}_{13}$Al | $5/2^+ \rightarrow 5/2^+$ | 0.692 | 3.58 | 3.51 | 3.62 |

the correct ground state spins of $^{21}$Na, $^{21}$Ne, $^{23}$Mg, and $^{23}$Na and hence, can not account for the observed log ft values. The consideration of residual interaction in the shell model calculations rectifies the situation. Landford and Wildenthal have calculated the level spectra of many of these nuclei using $^{16}$O as a core and configuration mixture among nucleons outside the core. Their calculated log ft values for super allowed transitions $^{21}_{11}$Na$\rightarrow^{21}_{10}$Ne, $^{23}_{12}$Mg$\rightarrow^{23}_{11}$Na, and $^{27}_{14}$Si$\rightarrow^{27}_{13}$Al are, respectively, 3.56, 3.68, and 3.51 (they did not report $^{25}_{13}$Al$\rightarrow^{25}_{12}$Mg rate), which are close to our and observed values.

4 Conclusion

We have derived here the expression for super-allowed transition for the Coriolis coupling model of Malik and Scholz, which are valid for transition between any pair of mirror nuclei. The application to four pairs of $1d_{1/2}$-nuclei indicate that the band mixing affects the rate. Whereas calculations based on the Nilsson model are at variance with the data, calculations presented here and based on the shell model with configuration mixing presented in [5] can account for the observed rates.

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