Bounding ramification with covers and curves

Hélène Esnault, joint with Vasudevan Srinivas

Upstate NY NT Seminar, September 28, 2020
Lefschetz theorem: topology

$X$ sm q-proj var over $\mathbb{C}$, $\pi_1^{\text{top}}(X, x)$ top fund gr based at $x \in X(\mathbb{C})$.

**Theorem (Lefschetz)**

$\exists$ sm curve $C \to X$, $C \ni x$, st $\pi_1^{\text{top}}(C, x) \to \pi_1^{\text{top}}(X, x)$.

given $X \hookrightarrow \bar{X}$ a good compactification, any ci of sm ample divisors in good position wrt $\bar{X} \setminus X$ does it.
X sm q-proj var over $\mathbb{C}$, $\pi_1^{\text{top}}(X, x)$ top fund gr based at $x \in X(\mathbb{C})$.

**Theorem (Lefschetz)**

$\exists$ sm curve $C \to X$, $C \ni x$, st $\pi_1^{\text{top}}(C, x) \twoheadrightarrow \pi_1^{\text{top}}(X, x)$.

given $X \hookrightarrow \bar{X}$ a good compactification, any ci of sm ample divisors in good position wrt $\bar{X} \setminus X$ does it.

$\leadsto$ same thm on $\pi_1^{\text{alg}}(X, x)$ pro-alg. completion: $\forall \mathcal{V}$ cplx loc syst, the restriction $\mathcal{V}|_C$ keeps the same monodromy.
Lefschetz theorem: topology

$X$ sm q-proj var over $\mathbb{C}$, $\pi_1^{\text{top}}(X, x)$ top fund gr based at $x \in X(\mathbb{C})$.

**Theorem (Lefschetz)**

$\exists$ sm curve $C \to X$, $C \ni x$, st $\pi_1^{\text{top}}(C, x) \twoheadrightarrow \pi_1^{\text{top}}(X, x)$.

given $X \hookrightarrow \bar{X}$ a good compactification, any ci of sm ample divisors in good position wrt $\bar{X} \setminus X$ does it.

$\leadsto$ same thm on $\pi_1^{\text{alg}}(X, x)$ pro-alg. completion: $\forall \mathcal{V}$ cplx loc syst, the restriction $\mathcal{V}|_C$ keeps the same monodromy.

$\mathbb{C} \leadsto k$ alg. cl. of char. 0, $\pi_1^{\text{top}}(X, x) \leadsto \pi_1(X, x)$ Grothendieck’s étale fundamental gr $\leadsto$ same thm (and tiny rmk).
No Lefschetz thm: eg $X = \mathbb{A}^2$, Artin-Schreier cover $t^p - t = f$, $f \in \mathcal{O}(\mathbb{A}^2)$ splits on curve $C : f = 0$. So $\nexists C$ with $\pi_1(C, x) \to \pi_1(X, x)$. 

No Lefschetz /$k$ of char. $p > 0$
Tameness: Kerz-Schmidt’s definition

Recall: \( R \) complete (or henselian) dvr, finite Galois ext \( R \subset S \) of such, perfect res fields, Galois gr \( G \), then \( G = \exists G_0 \supset G_1 \supset \ldots G_{\exists N \geq 1} = 0 \) with \( G_0/G_1 \subset \text{Frac}(S)^\times \) cyclic of order prime to \( p \), \( G_i/G_{i+1} = \text{fin pr} \) of cyclic gr of order \( p \).

Definition

1) \( \text{Sw} \ (S/R) \leq n \) iff \( N \leq n + 1 \); \( \text{Sw} \ (S/R) = 0 \) iff \( S/R \) tame.
2) [Kerz-Schmidt] \( X/k \) sm, \( k \) perfect, \( Y \rightarrow X \) fin étale is tame if \( \forall \) sm curve \( C \rightarrow X \), \( Y \times_X C \rightarrow C \) is tame.
3) \( \leadsto \pi_1(X, x) \rightarrow \pi_1^t(X, x) \) tame quotient.

- tame allows non-perfect res fields: res field ext should be sep and ram index prime to \( p \)
- if has good comp \( X \leftrightarrow \bar{X} \), defn agrees with Grothendieck’s defn: tame at the codim 1 points in \( \bar{X} \setminus X \)
Theorem (Drinfeld)

$X/k$ sm quasi-proj, $\exists C \rightarrow X$, $x \in C$ sm curve st $\pi^t_1(C, x) \rightarrow \pi^t_1(X, x)$.

if $X \hookrightarrow \bar{X}$ good compactification, then any ci of sm ample divisors in good position wrt $\bar{X} \setminus X$ does it (E-Kindler)
Ramification in geometry: definition

Definition

Given $X \hookrightarrow \bar{X}$ normal comp $/k$ perfect, $D$ eff div supp in $\bar{X} \setminus X$, then

1) $Y \to X$ finite étale has \textit{ramification bounded by} $D$ if $\forall C \to X$ sm curve, $\text{Sw} \left( Y \times_X C/C \right) \leq D \times \bar{X}$.

2) $\bar{Q}_\ell$- loc sys $\mathcal{V}_\rho$ defined by $\rho : \pi_1(X, x) \to GL_r(\bar{\mathbb{Z}}_\ell) \subset GL_r(\bar{Q}_\ell)$ has ramification bounded by $D$ iff Galois cover $\pi : X_{\bar{\rho}} \to X$ defined by $\bar{\rho} : \pi_1(X, x) \to GL_r(\bar{\mathbb{F}}_\ell)$ has ramification bounded by $D$ (depends only on $(\bar{\rho})^{ss}$).

3) $\pi^* \mathcal{V}_\rho$ tame: say $\pi$ \textit{tamifies} $\rho$.

4) A sm curve $C \to X$ is a \textit{Lefschetz curve for} a family $S = \{ \mathcal{V} \}$ if $\mathcal{V}|_C$ keeps the same monodromy $\forall \mathcal{V} \in S$. 

Hélène Esnault, joint with Vasudevan Srinivas

Bounding ramification

Upstate NY NT Seminar, September 28, 2020
Theorem (L. Lafforgue dim 1, Deligne in higher dim, cor Langlands corr)

$X$ sm q-proj/$k$, then $\exists$ only finitely many $\tilde{\mathbb{Q}}_\ell$-simple loc sys $\mathcal{V}$ with $(r, D)$ bounded, up to twist by a char. of $k$.

Analog of the Hermite-Minkowski thm: $\#$ field $K$, $\exists$ only fin many ext $L/K$ of bounded deg and disc
$k = \mathbb{F}_q$ finite field

**Theorem (L. Lafforgue dim 1, Deligne in higher dim, cor Langlands corr)**

If $X$ is a smooth $q$-projective over $k$, then there exist only finitely many $\mathbb{Q}_\ell$-simple local systems $V$ with $(r, D)$ bounded, up to twist by a character of $k$.

Analog of the Hermite-Minkowski thm: If $K$ is a field, then there exist only finitely many extensions $L/K$ of bounded degree and discriminant.

**Corollary**

1) If $(r, D)$ is bounded, then there exists a finite étale cover $\pi : Y \to X$ which tamifies all $V$ with $(r, D)$ bounded ('covers' from title).

2) Given $(r, D)$, there exists a Lefschetz curve for all $V$ with bounded $(r, D)$ ('curves' from title).
on Proof of Corollary

1) take cover $\pi: X_{\oplus_{\text{fin}}} \rightarrow X$
on Proof of Corollary

1) take cover $\pi : X_{\oplus_{\text{fin}}} \rho \to X$

2) a) top gr th: $\pi_1(C, x) \to \pi_1(X, x) \to I \subset GL_r(O_E) \text{ surj } (E/\mathbb{Q}_\ell \text{ finite})$ iff $\pi_1(C, x) \to \pi_1(X, x) \to \tilde{I} \subset GL_r(O_E/m_E^2) \text{ surj } (O_E/m_4 \text{ for } \ell = 2)$;
1) take cover $\pi : X_{\oplus_{\text{fin}}} \rightarrow X$

2a) top gr th: $\pi_1(C, x) \rightarrow \pi_1(X, x) \rightarrow I \subset GL_r(O_E)$ surj $(E/\mathbb{Q}_\ell$ finite) iff $\pi_1(C, x) \rightarrow \pi_1(X, x) \rightarrow \overline{I} \subset GL_r(O_E/m^2_E)$ surj $(O_E/m^4$ for $\ell = 2)$;

2b) Hilbert irreducibility (or Bertini if we allow ext $\mathbb{F}_q^m \supset \mathbb{F}_q$) $\Rightarrow \exists C$. 
How to bound the ramification if $k = \bar{k}$?

The notion of ramification bounded by $D$ is purely geometric, i.e. depends only on cover $(Y \rightarrow X)_{\bar{k}}$ or $V|_{\pi_1(x_{\bar{k}}, x)}$. 
How to bound the ramification if $k = \bar{k}$?

The notion of ramification bounded by $D$ is purely geometric, i.e. depends only on cover $(Y \to X)_{\bar{k}}$ or $\mathcal{V}|_{\pi_1(x_{\bar{k}}, x)}$.

To 1) ‘covers’: $/k = \bar{k}$, $(r, D)$ bounded, then $\exists \pi : Y \to X$ finite étale which tamifies all simple $\mathcal{V}$ with $(r, D)$ bounded: given $Sw$, Witt-artin-Schreier covers with Galois gr $\mathbb{Z}/p^n \ \forall n \geq 1$ with this $Sw$ exist (Brylinski-Kato).
How to bound the ramification if $k = \overline{k}$?

The notion of ramification bounded by $D$ is purely geometric, i.e. depends only on cover $(Y \to X)_{\overline{k}}$ or $\mathcal{V}|_{\pi_1(X_{\overline{k}}, x)}$.

To 1) ‘covers’ /$k = \overline{k}$, $(r, D)$ bounded, then $\not\exists \pi : Y \to X$ finite étale which tamifies all simple $\mathcal{V}$ with $(r, D)$ bounded: given $S_w$, Witt-artin-Schreier covers with Galois $gr \mathbb{Z}/p^n \forall n \geq 1$ with this $S_w$ exist (Brylinski-Kato).

To 2) ‘curves’ (Deligne): /$k = \overline{k}$, $X, (r, D)$, $\exists$ Lefschetz curve for all $\mathcal{V}$ with bounded $(r, D)$?
To 'covers': Tamifying up to codim 2

**Definition (E-S)**

\[ \pi : Y \to X \] finite connected tamifies \( V \) outside of codim 2 if there is a normal compactification \( Y \to \bar{Y} \) st \( \pi^* V \) is tame at codim 1 points of \( \bar{Y} \).
To 'covers': Tamifying up to codim 2

Definition (E-S)
\( \pi : Y \to X \) finite connected \textit{tamifies} \( V \) \textit{outside of codim} 2 if there is a normal compactification \( Y \hookrightarrow \bar{Y} \) st \( \pi^*V \) is tame at codim 1 points of \( \bar{Y} \).

Theorem (E-S)
\( X \) sm q-proj \( /k = \bar{k} \), given \( (r, D) \), \( \exists n \in \mathbb{N} \), \( \forall V \) with rank \( \leq r \) and ramification bounded by \( D \), \( \exists \pi_V : Y_V \to X \) of deg \( \leq n \) which tamifies \( V \) up to codim 2.

For \( X \leadsto R \), \( R \) complete dvr with res field \( k \), (E-Kindler-S)
On Proof of Thm

1) reduce to $X$ affine; via $X \to \mathbb{A}^d$ finite étale, reduce to

$$X = \mathbb{A}^d \hookrightarrow \bar{X} = \mathbb{P}^d;$$

2) prove local thm on $k(Z)[[t]]$ using (E-K-S), $Z = \mathbb{P}^d \setminus \mathbb{A}^d$ to produce a finite étale extension of $k(Z)((t))$ tamifying $\mathcal{V}|_{k(Z)((t))}$;

3) use Harbater-Katz-Gabber to extend to a finite étale cover of $\mathbb{G}_m/k(Z)$;

4) close it up to get the normal finite cover of $\mathbb{P}^d$, then of $X$. 
To ‘curves’: rank 1 case

Theorem (Kerz-S.Saito if $X \hookrightarrow \bar{X}$ good compactification, E-S in general) \[ /k = \bar{k}, X \hookrightarrow \bar{X} \text{ normal compactification, } D, \exists \text{ Lefschetz curve for } (1, D). \]
To ‘curves’: rank 1 case

Theorem (Kerz-S. Saito if $X \hookrightarrow \tilde{X}$ good compactification, E-S in general)

$/k = \overline{k}, X \hookrightarrow \tilde{X}$ normal compactification, $D$, $\exists$ Lefschetz curve for $(1, D)$.

On Proof.

1) reduce to Artin-Schreier;
2) $\{\mathcal{V}\}$ with $(r, D) \subset \{\mathcal{W}\}$ with $(r, D \cap X^{\text{reg}})$ (less curves to test).
3) use coh description (Kerz-S. Saito) on $X^{\text{reg}}$ and finiteness of Frobenius invariant $\mathcal{O}$-modules of local coh gr along $\tilde{X} \setminus X^{\text{reg}}$ to prove: $\exists N \geq 1$ so $\{\mathcal{W}\}$ with $(r, D \cap X^{\text{reg}}) \subset \{\mathcal{V}\}$ with $(r, ND)$.  \qed
Application of classical Bertini theorem

Theorem (E-S)

$\exists K/k$ alg. cl. of purely tr. fin. gen. field $/k$, $C_K \to X$ curve $/K$ st

$\pi_1(C_K, x) \to \pi_1(X, x)$. 

It is an illustration of the fact that if $C$ is not proper,

1) $\pi_1(C_k, x)$ does not satisfy base change;

2) there is no specialization map $\pi_1(C, x) \to \pi_1(C_k, x)$ for a specialization $K \to k$. 

Hélène Esnault, joint with Vasudevan Srinivas

Bounding ramification

Upstate NY NT Seminar, September 28, 2020
Application of classical Bertini theorem

Theorem (E-S)

\[ \exists K/k \text{ alg. cl. of purely tr. fin. gen. field } /k, C_K \rightarrow X \text{ curve } /K \text{ st } \]

\[ \pi_1(C_K, x) \rightarrow \pi_1(X, x). \]

It is an illustration of the fact that if \( C \) is not proper,

1) \( \pi_1(C, x) \) does not satisfy base change;
2) there is no specialization map \( \pi_1(C, x) \rightarrow \pi_1(C_k, x_k) \) for a specialization \( K \rightsquigarrow k \).