1. Introduction

Suppose that $R$ is a commutative ring with unit. Denote by $Sp_n(R)$ the group of automorphisms of $R^{2n}$ that preserve the unimodular alternating form given by the matrix

\[
\begin{pmatrix}
0 & I_n \\
-I_n & 0
\end{pmatrix}
\]

In this note we compute the rational cohomology ring of $Sp_3(\mathbb{Z})$, or equivalently, of $A_3$, the moduli space of principally polarized abelian 3-folds.

Denote the $\mathbb{Q}$-Hodge structure of dimension 1 and weight $-2n$ by $\mathbb{Q}(n)$. (For those not interested in Hodge theory, just interpret this as one copy of $\mathbb{Q}$.) Denote by $\lambda$ the first Chern class in $H^1(A_3; \mathbb{Q})$ of the Hodge bundle $\det \pi^* \Omega^1$ associated to the projection $\pi$ of the universal abelian 3-fold to $A_3$.

**Theorem 1.** The cohomology groups of $A_3$ are given by

\[
H^j(A_3; \mathbb{Q}) \cong H^j(Sp_3(\mathbb{Z}); \mathbb{Q}) \cong \begin{cases}
\mathbb{Q} & j = 0; \\
\mathbb{Q}(-1) & j = 2; \\
\mathbb{Q}(-2) & j = 4; \\
E & j = 6; \\
0 & \text{otherwise},
\end{cases}
\]

where $E$ is a two-dimensional mixed Hodge structure which is an extension

\[0 \to \mathbb{Q}(-3) \to E \to \mathbb{Q}(-6) \to 0.\]

The ring structure is determined by the condition that $\lambda^3 \neq 0$.

I do not know whether the mixed Hodge structure (MHS) $E$ on $H^6$ is split. Since $A_3$ is a smooth stack over $\text{Spec} \, \mathbb{Z}$, I expect it to be a multiple (possibly trivial) of the class

\[\zeta(3) \in \mathbb{C}/i\pi^3 \mathbb{Q} \cong \text{Ext}^1_{\mathcal{H}}(\mathbb{Q}, \mathbb{Q}(3))\]

given by the value of the Riemann zeta function at 3. Determining this class would be interesting.

As a corollary, we deduce the rational cohomology of $\overline{A_3}$, the Satake compactification of $A_3$.

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Theorem 2. The rational cohomology ring of \( \mathcal{A}_3 \) is given by
\[
H^j(\mathcal{A}_3; \mathbb{Q}) \cong \begin{cases} 
\mathbb{Q}(-n) & j = 2n, n \in \{0, 1, 2, 4, 5, 6\}; \\
B & j = 6; \\
0 & \text{otherwise,}
\end{cases}
\]
where \( B \) is a 3-dimensional mixed Hodge structure which is an extension
\[
0 \to \mathbb{Q}(0) \to B \to \mathbb{Q}(-3)^2 \to 0.
\]
The ring structure is determined by the condition that the cohomology ring contain the graded ring \( \mathbb{Q}[\lambda]/(\lambda^7) \), where \( \lambda \) has degree 2 and type \((1, 1)\).

Along the way, we compute the rational cohomology of \( \mathcal{A}_2 \), the Satake compactification of \( \mathcal{A}_2 \) as well.

Proposition 3. The rational cohomology ring of \( \mathcal{A}_2 \) is given by
\[
H^\ast(\mathcal{A}_2; \mathbb{Q}) \cong \mathbb{Q}[\lambda]/(\lambda^4).
\]
where \( \lambda \) is the first Chern class of the Hodge bundle.

Denote the moduli space of smooth projective curves over the complex numbers by \( \mathcal{M}_g \). Another consequence of the proof is the surjectivity of the homomorphism
\[
H^\ast(\mathcal{A}_3; \mathbb{Q}) \to H^\ast(\mathcal{M}_3; \mathbb{Q})
\]
induced by the period mapping \( \mathcal{M}_3 \to \mathcal{A}_3 \).

The computation of the rational cohomology of \( \mathcal{A}_1 \) is classical\(^1\) and follows from the fact that the quotient of the upper half plane by \( Sp_1(\mathbb{Z}) = SL_2(\mathbb{Z}) \) is a copy of the affine line. The computation of the rational cohomology of \( \mathcal{A}_2 \) is (essentially) due to Igusa\(^2\). Brownstein and Lee\(^3\) have computed the integral cohomology of \( Sp_2(\mathbb{Z}) \).

Suppose that \( g \geq 2 \). Recall that that the mapping class group \( \Gamma_g \) in genus \( g \) is the group of isotopy classes of orientation preserving diffeomorphisms of a closed, oriented surface \( S \) of genus \( g \). Its rational cohomology is isomorphic to that of \( \mathcal{M}_g \). The Torelli group \( T_g \) is defined to be the kernel of the natural homomorphism
\[
\Gamma_g \to Sp(H_1(S; \mathbb{Z}))
\]
where \( Sp \) denotes the symplectic group, and where \( H_1(S; \mathbb{Z}) \) is regarded as a symplectic module via its intersection form. Choosing a symplectic basis of \( H_1(S; \mathbb{Z}) \) gives an isomorphism \( Sp_g(\mathbb{Z}) \cong Sp(H_1(S; \mathbb{Z})) \). One then obtains the well known extension
\[
1 \to T_g \to \Gamma_g \to Sp_g(\mathbb{Z}) \to 1.
\]
The extended Torelli group \( \hat{T}_g \) is the preimage of the center \( \{\pm I\} \) of \( Sp(H_1(S; \mathbb{Z})) \) under \( (\text{I}) \). Equivalently, it is the group of isotopy classes of diffeomorphisms of \( S \) that act as \( \pm I \) on \( H_1(S) \). One has the extensions
\[
1 \to T_g \to \hat{T}_g \to \{\pm I\} \to 1
\]
and
\[
1 \to \hat{T}_g \to \Gamma_g \to PSp_g(\mathbb{Z}) \to 1
\]
\(^1\)It is trivial except in degree 0.
\(^2\)It is 1-dimensional in degrees 0 and 2, and trivial elsewhere.
where \( PSp_3(\mathbb{Z}) \) denotes the integral projective symplectic group \( Sp_3(\mathbb{Z})/\{\pm I\} \).

Our approach to computing the cohomology of \( Sp_3(\mathbb{Z}) \) is to analyse the spectral sequence of the extension (3). This entails knowing the cohomology of \( \Gamma_3 \) (or equivalently, \( \mathcal{M}_3 \)) and of \( \tilde{T}_3 \). Looijenga [12] computed the cohomology of \( \mathcal{M}_3 \) using the theory of Del-Pezzo surfaces:

\[
H^j(\mathcal{M}_3; \mathbb{Q}) \cong \begin{cases} 
\mathbb{Q} & j = 0; \\
\mathbb{Q}(-1) & j = 1; \\
\mathbb{Q}(-6) & j = 6; \\
0 & \text{otherwise.}
\end{cases}
\]

The cohomology of \( \tilde{T}_3 \) is computed in Section 3 using the stratified Morse Theory of Goresky and MacPherson [7]. We use their theory of non-proper Morse functions (see their Part II, Chapter 10) applied to the square of the distance to a point function restricted to the jacobian locus. This is an elaboration of a trick of Geoff Mess [13] which he used to show that the Torelli group in genus 2 is free of countable rank, and which was also used by Johnson and Millson (cf. [13]) to show that Torelli space in genus 3 does not have the homotopy type of a finite complex. Our use of Morse theory is analogous to Goresky and MacPherson’s treatment [7, Part III] of complements of affine subspaces of euclidean spaces. [Look at Mess’s paper.]

It seems to be a curious fact that in low genus (\( g = 2,3 \) so far), it is easier to compute the rational cohomology groups of \( \mathcal{M}_g \) than of \( \mathcal{A}_g \). This is perhaps a reflection of the richness of curve theory — it is a more powerful tool for understanding the geometry of \( \mathcal{M}_g \) than the theory of abelian varieties is as a tool for understanding the geometry of \( \mathcal{A}_g \). It will be interesting to know if this trend persists when \( g \geq 4 \), when \( \mathcal{M}_g \) is no longer dense in \( \mathcal{A}_g \). The cohomology of \( \mathcal{M}_4 \) is not yet known, nor does it seem tractable to compute the cohomology of the extended Torelli group in genus 4.

2. Preliminaries

General references for this section are [8, Chapt. 3] and [9].

A polarized abelian variety is a compact complex torus \( A \) together with a cohomology class

\[
\theta \in H^{1,1}(A) \cap H^2(A; \mathbb{Z})
\]

whose translation invariant representative is positive. The corresponding complex line bundle is ample. The polarization \( \theta \) can be regarded as a skew symmetric bilinear form on \( H_1(A; \mathbb{Z}) \). The polarization is principal if this form is unimodular.

Jacobians of curves are polarized by the intersection pairing on \( H_1(C; \mathbb{Z}) \cong H_1(\text{Jac} C; \mathbb{Z}) \). This form is unimodular, and so jacobians are canonically principally polarized abelian varieties.

A framed principally polarized abelian variety is a principally polarized abelian variety \( A \) together with a symplectic basis of \( H_1(A; \mathbb{Z}) \) with respect to the polarization \( \theta \).

Suppose that \( g \geq 1 \). The maximal compact subgroup of \( Sp_g(\mathbb{R}) \) is \( U(g) \). The symmetric space \( Sp_g(\mathbb{R})/U(g) \) is isomorphic to the rank \( g \) Siegel upper half space

\[
\mathfrak{h}_g = \left\{ \text{symmetric } g \times g \text{ complex matrices} \right\} \\
\text{with positive definite imaginary part}
\]
It has dimension \(g(g + 1)/2\). Taking a framed principally polarized abelian variety \((A; a_1, \ldots, a_g, b_1, \ldots, b_g)\) to the corresponding period matrix gives a bijection

\[
\mathfrak{h}_g \cong \left\{ \text{isomorphism classes of framed principally polarized abelian varieties} \right\}
\]

We will regard \(\mathfrak{h}_g\) as the (fine) moduli space of framed principally polarized abelian varieties of dimension \(g\).

Changing symplectic bases gives a natural left action of \(Sp_g(\mathbb{Z})\) on the moduli space of framed principally polarized abelian varieties. There is clearly a natural left \(Sp_g(\mathbb{Z})\) action on \(\mathfrak{h}_g = Sp_g(\mathbb{Z})/U(g)\). The bijection (4) is equivariant with respect to these actions.

The moduli space \(A_g\) of principally polarized abelian varieties of dimension \(g\) is the quotient \(Sp_g(\mathbb{Z})/\mathfrak{h}_g\). Since \(\mathfrak{h}_g\) is contractible and since \(Sp_g(\mathbb{Z})\) acts discontinuously and virtually freely on \(\mathfrak{h}_g\), it follows that there is a natural isomorphism

\[
H^*(Sp_g(\mathbb{Z}); \mathbb{Q}) \cong H^*(A_g; \mathbb{Q}).
\]

Taking a curve \(C\) to its jacobian \(\text{Jac}\) defines a morphism \(M_g \to A_g\) which is called the period mapping.

Now suppose that \(g \geq 2\). Denote Teichmüller space in genus \(g\) by \(X_g\). The mapping class group \(\Gamma_g\) acts properly discontinuously and virtually freely on \(X_g\) with quotient \(M_g\). It follows that there is a natural isomorphism

\[
H^*(\Gamma_g; \mathbb{Q}) \cong H^*(M_g; \mathbb{Q}).
\]

We shall need several moduli spaces that sit between \(X_g\) and \(M_g\). Denote the quotient of \(X_g\) by \(T_g\) by \(\mathcal{T}_g\). This space is known as Torelli space. Since \(X_g\) is contractible and \(T_g\) is torsion free, \(T_g\) acts freely on \(X_g\) and Torelli space is an Eilenberg-MacLane space with fundamental group \(T_g\). Consequently,

\[
H^*(T_g; \mathbb{Z}) \cong H^*(\mathcal{T}_g; \mathbb{Z}).
\]

By a framed Riemann surface of genus \(g\) we shall mean a compact Riemann surface \(C\) together with a symplectic basis \(a_1, \ldots, a_g, b_1, \ldots, b_g\) of \(H_1(C; \mathbb{Z})\) with respect to the intersection form. Torelli space \(\mathcal{T}_g\) is the moduli space of framed Riemann surfaces of genus \(g\); its points correspond to isomorphism classes of framed, genus \(g\) Riemann surfaces. The symplectic group \(Sp_g(\mathbb{Z})\) acts on the framings in the natural way; the quotient \(Sp_g(\mathbb{Z})/\mathcal{T}_g\) is \(M_g\).

Denote the locus in \(\mathfrak{h}_g\) consisting of jacobians of smooth curves by \(\mathcal{J}_g\). (Note that this is not closed in \(A_g\).) The period mapping \(\mathcal{T}_g \to \mathcal{J}_g\) is surjective by definition. Since minus the identity is an automorphism of every polarized abelian variety \(A\),

\[
(A; a_1, \ldots, b_g) \cong (A; -a_1, \ldots, -b_g).
\]

But if \(C\) is a genus \(g\) curve, then

\[
(C; a_1, \ldots, b_g) \cong (C; -a_1, \ldots, -b_g)
\]

if and only if \(C\) is hyperelliptic. It follows that, when \(g \geq 3\), the period mapping \(\mathcal{T}_g \to \mathcal{J}_g\) that takes a framed curve to its jacobian with the same framing is surjective and 2:1 except along the hyperelliptic locus, where it is 1:1. It follows that, when \(g \geq 3\),

\[
\mathcal{J}_g = \mathcal{T}_g/\mathcal{X}_g \text{ and } M_g = PSp_g(\mathbb{Z})/\mathcal{J}_g.
\]
The following diagram shows the coverings and their Galois group when \( g \geq 3 \):

\[
\begin{align*}
\xymatrix{ & X_g \ar[rr]^{T_g} \ar[dr]^{\hat{T}_g} & & \tilde{T}_g \ar[rr]^{S_p(Z)} & & \mathcal{M}_g \ar[dl]_{PSp(Z)} \ar[rr]^{pSp(Z)} & & h_g \ar[r] & A_g } \\
T_g & & & \mathbb{Z}/2\mathbb{Z} & & \mathcal{T}_g & & \mathcal{J}_g & & \mathcal{P} \ar[r] & \mathcal{M}_g \ar[r] & A_g }
\end{align*}
\]

\[ \text{(5)} \]

**Lemma 4.** If \( 1 \to \mathbb{Z}/2\mathbb{Z} \to E \to G \to 1 \) is a group extension, then the projection \( E \to G \) induces an isomorphism on homology and cohomology with 2-divisible coefficients. In particular

\[
H^\bullet (PSp_g(Z); \mathbb{Z}[1/2]) \to H^\bullet (Sp_g(Z); \mathbb{Z}[1/2])
\]

is an isomorphism.

**Proof.** This follows from the fact that

\[
H^j(\mathbb{Z}/2; \mathbb{Z}[1/2]) = 0 \quad j > 0
\]

using the Hochschild-Serre spectral sequence of the group extension. \( \square \)

**Proposition 5.** For all \( g \geq 2 \), there is a natural isomorphism

\[
H_\bullet (\mathcal{J}_g; \mathbb{Z}[1/2]) \cong H_\bullet (\tilde{T}_g; \mathbb{Z}[1/2]) \cong H_\bullet (T_g; \mathbb{Z}[1/2])^{\mathbb{Z}/2\mathbb{Z}}.
\]

There are similar isomorphisms for cohomology.

**Proof.** Since \( T_g \) is torsion free, \( T_g \) acts fixed point freely on Teichmüller space, and \( T_g \) is a model of the classifying space of \( T_g \). Recall that if \( X \) is a simplicial complex on which \( \mathbb{Z}/2 \) acts simplicially (but not necessarily fixed point freely), then the map

\[
p_* : H_\bullet (X/(\mathbb{Z}/2); \mathbb{Z}[1/2]) \to H_\bullet (X; \mathbb{Z}[1/2])^{\mathbb{Z}/2}
\]

induced by the projection \( p \) is an isomorphism, whose inverse is half the pullback map \( p^* \). Applying this twice gives isomorphisms

\[
H^\bullet (\mathcal{J}_g; \mathbb{Z}[1/2]) \cong H^\bullet (\tilde{T}_g; \mathbb{Z}[1/2])^{\mathbb{Z}/2} \cong H^\bullet (T_g; \mathbb{Z}[1/2])^{\mathbb{Z}/2} \cong H^\bullet (\tilde{T}_g; \mathbb{Z}[1/2]).
\]

\( \square \)

A theta divisor of a principally polarized abelian variety \( A \) is a divisor \( \Theta \) whose Poincaré dual is the polarization and which satisfies \( i^* \Theta = \Theta \), where \( i : x \mapsto -x \). Any two such divisors differ by translation by a point of order 2, and can be given as the zero locus of a theta function associated to a period matrix of \( A \). A principally polarized abelian variety \( A \) is reducible if it is isomorphic (as a polarized variety) to the product of two proper abelian subvarieties. If \( A = A_1 \times A_2 \), then any theta divisor of \( A \) is reducible:

\[
\Theta_A = (\Theta_{A_1} \times A_2) \cup (A_1 \times \Theta_{A_2}).
\]
Denote the locus of reducible abelian varieties in \( \mathcal{A}_g \) by \( \mathcal{A}^\text{red}_g \) and in \( \mathfrak{h}_g \) by \( \mathfrak{h}^\text{red}_g \). Elements of \( \mathfrak{h}^\text{red}_g \) are precisely those period matrices \( \Omega \) that can be written as a direct sum of two smaller period matrices.

**Proposition 6.** Denote the closure of \( \mathcal{J}_g \) in \( \mathfrak{h}_g \) by \( \overline{\mathcal{J}}_g \). If \( g \geq 2 \), then \( \mathcal{J}_g = \overline{\mathcal{J}}_g - (\overline{\mathcal{J}}_g \cap \mathfrak{h}^\text{red}_g) \).

**Proof.** The period mapping \( \mathcal{M}_g \to \mathcal{A}_g \) extends to a morphism \( \overline{\mathcal{M}}_g \to \overline{\mathcal{A}}_g \) from the Deligne-Mumford compactification of \( \mathcal{M}_g \) to the Satake compactification of \( \mathcal{A}_g \).

The inverse image of the boundary \( \overline{\mathcal{A}}_g - \mathcal{A}_g \) of \( \overline{\mathcal{A}}_g \) is the boundary divisor \( \Delta_0 \) of \( \overline{\mathcal{M}}_g \), whose generic point is an irreducible stable curve of genus \( g \) with one node. Denote the moduli space of curves of compact type \( \overline{\mathcal{M}}_g - \Delta_0 \) by \( \tilde{\mathcal{M}}_g \). Since \( \tilde{\mathcal{M}}_g \) is complete, it follows that the period mapping \( \tilde{\mathcal{M}}_g \to \mathcal{A}_g \) is proper and therefore has closed image in \( \mathcal{A}_g \). Since \( \mathcal{M}_g \) is dense in \( \tilde{\mathcal{M}}_g \) and has image \( \text{Sp}_g(\mathbb{Z}) \setminus \overline{\mathcal{J}}_g \) under the period mapping, it follows that the image of \( \mathcal{M}_g \) in \( \mathcal{A}_g \) is \( \text{Sp}_g(\mathbb{Z}) \setminus \overline{\mathcal{J}}_g \).

Recall that the theta divisor \( \Theta_C \subset \text{Jac} C \) of a smooth genus \( g \) curve \( C \) is (up to a translate by a point of order 2) the image of mapping

\[
C^{g-1} \to \text{Pic}^{g-1} C \to \text{Jac} C
\]

that takes \((x_1, \ldots, x_{g-1})\) to \( x_1 + \cdots + x_{g-1} - \alpha \), where \( \alpha \) is a square root of the canonical bundle of \( C \). (See, for example, [8, p. 338].) It follows that \( \Theta_C \) is irreducible. On the other hand, if \( C \) is a reducible, stable, curve of compact type, its jacobian is the product of the components of its irreducible components, and is therefore reducible. The result follows. \( \Box \)

**Corollary 7.** We have \( \mathcal{J}_3 = \mathfrak{h}_3 - \mathfrak{h}^\text{red}_3 \).

**Proof.** Both \( \mathcal{M}_3 \) and \( \mathcal{A}_3 \) have dimension 6. Since the period mapping \( \mathcal{M}_3 \to \mathcal{A}_3 \) is generically of maximal rank, \( \mathcal{M}_3 \to \mathcal{A}_3 \) is surjective. This implies that \( \overline{\mathcal{J}}_3 = \mathfrak{h}_3 \), from which the result follows. \( \Box \)

### 3. The Homology of \( \mathcal{J}_3 \) and \( \tilde{\mathcal{J}}_3 \)

Denote the singular locus of an analytic variety \( Z \) by \( Z^{\text{sing}} \). We shall compute the homology of \( \mathcal{J}_3 \) by applying stratified Morse theory to the stratification

\[
\mathfrak{h}_3 \supseteq \mathfrak{h}^\text{red}_3 \supseteq \mathfrak{h}^\text{red,sing}_3
\]

of \( \mathfrak{h}_3 \). The top stratum is, by Corollary 7, \( \overline{\mathcal{J}}_3 \). Note that there are natural inclusions

\[
\mathfrak{h}_1 \times \mathfrak{h}_2 \hookrightarrow \mathfrak{h}^\text{red}_3 \quad \text{and} \quad \mathfrak{h}_1 \times \mathfrak{h}_1 \times \mathfrak{h}_1 \hookrightarrow \mathfrak{h}^\text{red,sing}_3
\]

defined by

\[
(\tau, \Omega) \mapsto \begin{pmatrix} \tau & 0 \\ 0 & \Omega \end{pmatrix} \quad \text{and} \quad (\tau_1, \tau_2, \tau_3) \mapsto \begin{pmatrix} \tau_1 & 0 & 0 \\ 0 & \tau_2 & 0 \\ 0 & 0 & \tau_3 \end{pmatrix}
\]

respectively. Note that the image of \( \mathfrak{h}_1 \times \mathfrak{h}_2 \) is stabilized in \( \text{Sp}_3(\mathbb{R}) \times \text{SL}_2(\mathbb{R}) \), and the image of \( (\mathfrak{h}_1)^3 \) by \( \Sigma_3 \ltimes \text{SL}_2(\mathbb{R})^3 \), where \( \Sigma_3 \) is identified with the subgroup

\[
\{ a_j \mapsto a_{\sigma(j)} \text{ and } b_j \mapsto b_{\sigma(j)} : \sigma \text{ is a permutation of } \{1, 2, 3\} \}
\]

of \( \text{Sp}_3(\mathbb{R}) \). Here \( a_1, \ldots, b_6 \) is the distinguished framing of the first homology of the corresponding abelian variety.
Proposition 8. The stratification (4) satisfies Whitney’s conditions (A) and (B) (cf. [7, p. 37]). Moreover
\[ h_3^{\text{red}} = \bigcup_{g \in Sp_3(Z)} g(h_1 \times h_2) = \bigcup_{g \in Sp_3(Z)/(SL_2(Z) \times Sp_2(Z))} g(h_1 \times h_2) \]
and
\[ h_3^{\text{red,sing}} = \bigcup_{g \in Sp_3(Z)} g(h_1 \times h_1 \times h_1) \]
In particular, $h_3^{\text{red}}$ is a locally finite union of totally geodesic complex submanifolds of $h_3$ of complex codimension 2 and $h_3^{\text{sing,red}}$ is a countable disjoint union of totally geodesic complex submanifolds of $h_3$ of complex codimension 3.

Proof. Since every reducible abelian variety is the product (as polarized varieties) of an elliptic curve and an abelian surface, $A_1 \times A_2 \rightarrow A_3^{\text{red}}$ is surjective. Lifting to $h_3$, this implies that the $Sp_3(Z)$ acts transitively on the components of $h_3^{\text{red}}$, and that $h_3^{\text{red}}$ is the $Sp_3(Z)$-orbit of $h_1 \times h_2$ in $h_3$. Since $SL_2(\mathbb{R}) \times Sp_2(\mathbb{R})$ acts transitively on $h_1 \times h_2$, the stabilizer in $Sp_3(Z)$ of $h_1 \times h_2$ is $SL_2(Z) \times Sp_2(Z)$. Assertion (8) follows.

The components of $h_3^{\text{red}}$ are smooth, so the $h_3^{\text{red,sing}}$ is the locus where two or more components of $h_3^{\text{red}}$ intersect. This is precisely the preimage of the locus in $A_3$ of products of 3 elliptic curves. Since this locus is irreducible (it is the image of $(A_1)^3 \rightarrow A_3$), $h_3^{\text{red,sing}}$ is the $Sp_3(Z)$-orbit of $(h_1)^3$. By the semi-simplicity of polarized abelian varieties, there is a unique way to decompose an element of $h_3^{\text{red,sing}}$ as a product of three elliptic curves. This, and the fact that $SL_2(\mathbb{R})^3$ acts transitively on $(h_1)^3$, imply the stabilizer in $Sp_3(Z)$ of $(h_1)^3$ is $\Sigma_3 \times SL_2(Z)^3$ and that each component of $h_3^{\text{red,sing}}$ is smooth. This proves (9).

Whitney’s condition (A) is automatic as each component of $h_3^{\text{red,sing}}$ is a homogeneous submanifold of the closure of each component of $h_3^{\text{red}}$. Condition (B) is well known to be a consequence of condition (A). \[ \square \]

Integrating the Riemannian metric along geodesics gives an $Sp_p(\mathbb{R})$-invariant distance function $d$ on $X$. For a point $p \in X$, let $D_p : X \rightarrow \mathbb{R}$ be the square of the distance to $p$:
\[ D_p(x) = d(x, p)^2. \]

The following result can be proved, either by appealing to [7, I.2.2.3] or by an elementary and direct argument.

Proposition 9. There is an open dense subset $U$ of $X$ such that for all $p \in U$, $D_p : X \rightarrow \mathbb{R}$ is a Morse function in the sense of Goresky and MacPherson [7, p. 52], all of whose critical points are “nondepraved.” \[ \square \]

Since each stratum is a union of totally geodesic subspaces, and since the symmetric space metric is complete with non-positive curvature it follows that there is a unique critical point on each component of each stratum. Since the Morse data for each critical point is a product of the normal and tangential Morse data [7, p. 61], we only need compute the normal Morse data at each critical point. There are two types of these: those that lie on a translate of $h_1 \times h_2$ and those that lie on a translate of $(h_1)^3$. The Morse data at each depends only on its type.
Proposition 10. If \( x \in \mathfrak{h}_3^{\text{red}} - \mathfrak{h}_3^{\text{red, sing}} \) is a critical point of \( D_p \), then the normal Morse data at \( x \) is homotopy equivalent to \((S^3, *)\). In particular, \( D_p \) is perfect at such critical points.

Proof. Since the normal slice at a smooth point of \( \mathfrak{h}_3^{\text{red}} \) is a complex 2-ball, the normal Morse data at \( x \) is \((S^3, *)\). Since \( H_\ast(S^3) \to H_\ast(S^3, *) \) is surjective, \( D_p \) is perfect at \( x \).

Lemma 11. At each point of \( \mathfrak{h}_3^{\text{red, sing}} \), there is a normal slice with coordinates \((z_1, z_2, z_3)\) such that \( \mathfrak{h}_3^{\text{red}} \) has three components with equations

\[
z_2 = z_3 = 0, \quad z_1 = z_3 = 0, \quad z_1 = z_2 = 0.
\]

Proof. There are three obvious ways to deform the product \( A = E \times E' \times E'' \) of three elliptic curves, preserving the polarization, into \( \mathfrak{h}_3^{\text{red}} \). Namely, one can deform one of the elliptic curves in \( \mathfrak{h}_1 \), and deform the product of the other two into \( \mathfrak{h}_2 - \mathfrak{h}_2^{\text{red}} \). The semi-simplicity of abelian varieties implies that each component of \( \mathfrak{h}_3^{\text{red, sing}} \) is smooth and there are no other ways to deform \( A \) into \( \mathfrak{h}_3^{\text{red}} \). It follows that 3 components of \( \mathfrak{h}_3^{\text{red}} \) intersect at each point of \( \mathfrak{h}_3^{\text{red, sing}} \).

Since each component of \( \mathfrak{h}_3^{\text{red, sing}} \) is locally homogeneous, and since all are conjugate under the action of \( Sp_3(Z) \), to determine the local structure of \( \mathfrak{h}_3^{\text{red}} \) near \( \mathfrak{h}_3^{\text{red, sing}} \), it suffices to write down local equations along the component \((\mathfrak{h}_1)^3\) of \( \mathfrak{h}_3^{\text{red}} \). Here we can use the coordinates

\[
\begin{pmatrix}
\tau_1 & z_3 & z_2 \\
z_3 & \tau_2 & z_1 \\
z_2 & z_1 & \tau_3
\end{pmatrix}
\]
where each $\tau_j \in h_1$, and $(z_1, z_2, z_3)$ lies in a neighbourhood of the origin in $\mathbb{C}^3$ small enough to guarantee that this matrix has positive definite imaginary part. In these coordinates, the three components of $h_3^{\text{red}}$ have equations:

$$z_2 = z_3 = 0, \quad z_1 = z_3 = 0, \quad z_1 = z_2 = 0.$$ 

View $S^5$ as the unit sphere in $\mathbb{C}^3$. The intersection of each coordinate axis with $S^5$ is a linearly imbedded $S^1$. Together these give an imbedding

$$k : S^1 \amalg S^1 \amalg S^1 \hookrightarrow S^5$$

where $\amalg$ denotes disjoint union. The boundary of a small tubular neighbourhood of each $S^1$ is a trivial $S^3$ bundle over $S^1$. Taking one fiber of each and connecting them to an arbitrary point of $S^5 - (S^1 \cup S^1 \cup S^1)$ gives an imbedding

$$i : S^3 \vee S^3 \vee S^3 \hookrightarrow S^5.$$ 

**Proposition 12.** If $x \in h_3^{\text{red,sing}}$ is a critical point of $D_p$, then the normal Morse data at $x$ is homotopy equivalent to

$$\left( S^5 - k(S^1 \amalg S^1 \amalg S^1), i(S^3 \vee S^3 \vee S^3) \right).$$

**Proof.** By the previous Lemma, the normal Morse data is the same as that for the distance squared to a point function for

$$\mathbb{C}^3 - \text{the union of the 3 coordinate axes}.$$ 

This is computed using [7, I.3.11.2] as explained in [7, III.3.3].

To compute the homology of $J_3$, we need to compute the homology of each kind of Morse data. Denote the quotient of $Z_3$ by the diagonal subgroup by $V$. It is isomorphic to $\mathbb{Z}^2$.

**Proposition 13.** The homology of the Morse data at a critical point of $h_3^{\text{red,sing}}$ is

$$H_j(S^5 - k(S^1 \amalg S^1 \amalg S^1), i(S^3 \vee S^3 \vee S^3); \mathbb{Z}) = \begin{cases} V & j = 4; \\ 0 & \text{otherwise.} \end{cases}$$

The homology is generated by the boundaries of tubular neighbourhoods of the three imbedded $S^1$'s, which are subject to the relation that their sum is zero. In particular, $D_p$ is perfect at such critical points.

**Proof.** The computation is elementary. The Morse function is perfect at such critical points because the relative homology is generated by absolute cycles.

The natural action of the symmetric group $\Sigma_3$ on $\mathbb{Z}^3$ (by permuting the coordinates) preserves the diagonal subgroup and therefore descends to an action on $V$. We view $V$ as a $\Sigma_3 \ltimes SL_2(\mathbb{Z})^3$-module via the projection $\Sigma_3 \ltimes SL_2(\mathbb{Z})^3 \rightarrow \Sigma_3$.

Recall that if $R$ is a commutative ring, $K$ a subgroup of $G$, and $M$ a $RK$ module, then the $G$-module induced from $M$ is defined by

$$\text{Ind}_K^G M := RG \otimes_{RK} M.$$
Theorem 14. We have,

$$H_j(\mathcal{F}_3; \mathbb{Z}) \cong \begin{cases} 
\mathbb{Z} & j = 0; \\
\text{Ind}_{\Sigma^3 \ltimes SL_2(\mathbb{Z})^3}^{Sp_3(\mathbb{Z})} \mathbb{Z} & j = 3; \\
\text{Ind}_{\Sigma_3 \ltimes SL_2(\mathbb{Z})^3}^{Sp_3(\mathbb{Z})} V & j = 4; \\
0 & \text{otherwise.}
\end{cases}$$

Proof. Each critical point has trivial tangential Morse data, as the critical points are minima on each stratum. It follows from [1, 3.7] that the Morse data at each critical point is homotopy equivalent to its normal Morse data. Since the Morse function is perfect, the homology of \( \mathcal{F}_3 \) is the sum of the relative homologies of the normal Morse data at each critical point. The action of \( Sp_3(\mathbb{Z}) \) follows from the description of the \( Sp_3(\mathbb{Z}) \) action on the strata in Proposition 8.

For a subgroup \( G \) of \( Sp_3(\mathbb{Z}) \) that contains \(-I\), define \( PG \) to be the subgroup \( G/(\pm I) \) of \( PSp_3(\mathbb{Z}) \).

Corollary 15. For all \( j \geq 0 \),

$$H_j(\hat{T}_3; \mathbb{Z}[1/2]) \cong H_k(T_3; \mathbb{Z}[1/2])^{\mathbb{Z}/2\mathbb{Z}} \cong \begin{cases} 
\mathbb{Z}[1/2] & j = 0; \\
\text{Ind}_{\Sigma^3 \ltimes SL_2(\mathbb{Z})^3}^{Sp_3(\mathbb{Z})} \mathbb{Z}[1/2] & j = 3; \\
\text{Ind}_{\Sigma_3 \ltimes SL_2(\mathbb{Z})^3}^{Sp_3(\mathbb{Z})} V \otimes \mathbb{Z}[1/2] & j = 4; \\
0 & \text{otherwise.}
\end{cases}$$

4. THE SPECTRAL SEQUENCE

We shall compute the homology spectral sequence

$$H_i(PSp_3(\mathbb{Z}); H_j(\hat{T}_3; \mathbb{Q})) \Rightarrow H_{i+j}(\Gamma_3; \mathbb{Q}).$$

Thanks to Shapiro’s Lemma (see, for example, [1]) the \( E^2 \) term of the spectral sequence can be computed.

Lemma 16. We have

$$H_i(PSp_3(\mathbb{Z}); H_j(\hat{T}_3; \mathbb{Q})) \cong \begin{cases} 
\mathbb{Q} & i = 0, 2 \text{ and } j = 3; \\
0 & j > 0 \text{ and } j \neq 3.
\end{cases}$$

Proof. Applying Shapiro’s Lemma, Lemma 8, the Kunneth Theorem, and the fact that the rational homology of \( SL_2(\mathbb{Z}) \) is that of a point, we have:

$$H_i(PSp_3(\mathbb{Z}); \text{Ind}_{P(SL_2(\mathbb{Z}) \times Sp_2(\mathbb{Z}))}^{PSp_3(\mathbb{Z})} \mathbb{Q}) \cong H_i(P(SL_2(\mathbb{Z}) \times Sp_2(\mathbb{Z})); \mathbb{Q})
\cong H_i(SL_2(\mathbb{Z}) \times Sp_2(\mathbb{Z}); \mathbb{Q})
\cong H_i(Sp_2(\mathbb{Z}); \mathbb{Q}).$$

This is \( \mathbb{Q} \) in when \( i = 0, 2 \) and 0 otherwise by Igusa’s computation.
Let $V_Q = V \otimes \mathbb{Q}$. This is the unique 2-dimensional irreducible representation of $\Sigma_3$. Since $V_Q$ is divisible and has no coinvariants, $H_\bullet(\Sigma_3; V_Q)$ vanishes in all degrees. Arguing as above, we have

$$H_4(PSp_3(\mathbb{Z}) \backslash \text{Ind}_{P(\Sigma_3 \ltimes SL_2(\mathbb{Z}))}^{PSp_3(\mathbb{Z})} V_Q) \cong H_4(P(\Sigma_3 \ltimes SL_2(\mathbb{Z})) \backslash V_Q) \cong H_4(\Sigma_3 \ltimes SL_2(\mathbb{Z})^3; V_Q) \cong H_4(\Sigma_3; V_Q) \cong 0.$$

\[\square\]

**Proposition 17.** For $2 \leq r \leq 4$, the $E^r$-term of the spectral sequence \([3]\) is

| deg | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|-----|---|---|---|---|---|---|---|---|
| 4   | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 3   | $\mathbb{Q}$ | 0 | $\mathbb{Q}$ | 0 | 0 | 0 | 0 | 0 |
| 2   | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1   | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0   | $\mathbb{Q}$ | 0 | $\mathbb{Q}$ | 0 | $\mathbb{Q}$ | 0 | $\mathbb{Q}^2$ | 0 |

(All terms not shown are zero.) In addition, the differentials

$$d^4 : E^4_{1,0} \to E^4_{0,3} \text{ and } d^4 : E^4_{6,0} \to E^4_{2,3}$$

are both surjective, and $E^5 = E^\infty$.

**Proof.** The computation of $E^2_{s,t}$ for $t > 0$ follows from the previous lemma. It implies that $d^r = 0$ when $2 \leq r < 4$ and $E^5 = E^\infty$. By Looijenga’s computation of the rational homology of $\Gamma_3$ we know that $H_3(\Gamma_3; \mathbb{Q})$ and $H_5(\Gamma_3; \mathbb{Q})$ both vanish. This implies that the differentials

$$d^4 : E^4_{1,0} \to E^4_{0,3} \text{ and } d^4 : E^4_{6,0} \to E^4_{2,3}$$

must be surjective. Since $H_4(\Gamma_3; \mathbb{Q}) = 0$, and since $H_6(\Gamma_3; \mathbb{Q})$ is one dimensional, the first of these is an isomorphism and the second has one dimensional kernel. The result follows. \[\square\]

**Proof of Theorem 1.** The computation of the rational homology (and therefore the rational cohomology) of $A_3$ follows from Proposition 17.

Denote the first Chern class in $H^2(A_3; \mathbb{Q})$ of the Hodge bundle by $\lambda$. It is the class of an ample line bundle. A standard argument, that uses the fact that the Satake compactification of $A_3$ has boundary of codimension 3, shows that there is a complete surface in $A_3$. This implies that $\lambda^3 \neq 0$ in $H^4(A_3; \mathbb{Q})$.

The proof that $\lambda^3$ does not vanish in $H^6(A_3; \mathbb{Q})$ is more subtle, and is due to van der Geer \([3]\). We will give a topological proof of this fact in the next section. The key point in van der Geer’s argument is that there is a complete subvariety of the characteristic p version $A_{3, p}$ of $A_3$. This implies that $\lambda^3$ is not zero in $H^6_{\text{et}}(A_{3, p}; \mathbb{Q}_p)$, where $p$ is a prime where $A_3$ has good reduction and $\ell$ is a prime distinct from $p$. Standard comparison theorems imply that this last group is isomorphic to $H^6(A_3; \mathbb{Q}_p)$, which gives the desired non-vanishing of $\lambda^3$.

The statement about weights follows as $\lambda$ is of type $(1, 1)$. Since $\lambda^3 \neq 0$, this implies that $H^4(A_3; \mathbb{Q})$ is generated by $\lambda^2$ and has type $(2, 2)$, and that $H^6(A_3; \mathbb{Q})$
contains a copy of $\mathbb{Q}(-3)$ spanned by $\lambda^3$. On the other hand, the spectral sequence in Proposition 17 implies that the restriction mapping

$$H^6(A_3; \mathbb{Q}) \to H^6(M_3; \mathbb{Q}) \cong \mathbb{Q}(-6)$$

is surjective, which completes the proof. □

5. Cycles

In this section, we give a topological proof that $\lambda^3$ is non-zero in $H^6(A_3; \mathbb{Q})$. The approach is to construct a topological 6-cycle in $A_3$ and then show that the value of $\lambda^3$ on it is non-zero.

Fix an imbedding of the Satake compactification $\overline{A}_3$ of $A_3$ in some projective space. Take a generic codimension 3 linear section of $\overline{A}_3$ that avoids the boundary $\overline{A}_3 - A_3$, is transverse to $A_3$, and intersects $A_3$ transversally. This section is a complete curve $X$ in $A_3$, smooth (in the orbifold sense) away from its intersection with $A_3$. At each point $x$ where it intersects $A_3$, it has three branches. Set $X' = X - (X \cap \overline{A}_3)$.

![Figure 2](image_url)

The next step is to construct a 5-cycle in $M_3$ which is an $S^3$ bundle over $X$ away from the triple points, and where the three branches of this bundle at each triple point $x$ are plumbed together using the normal Morse data at $x$.

Denote the moduli space of principally polarized abelian 3-folds with a level $\ell$ structure by $A_3[\ell]$. This is the quotient of $h_3$ by the level $\ell$ subgroup $Sp_3(\mathbb{Z})[\ell]$ of $Sp_3(\mathbb{Z})$, and is smooth when $\ell \geq 3$. Fix an $\ell \geq 3$. The symmetric space metric on $h_3$ descends to $A_3[\ell]$. With the help of the metric, the normal bundle of any stratum can be viewed as a subbundle of the tangent bundle of $A_3[\ell]$. Using the exponential mapping, we can identify a neighbourhood of the zero section of the normal bundle as being imbedded in $A_3[\ell]$. Let $Y$ and $Y'$ be the inverse images of $X$ and $X'$ in $A_3[\ell]$.

Choose a positive real number $\epsilon$ such that the exponential mapping is an imbedding on the $\epsilon/4$-ball $B$ of the normal bundle of $A_3[\ell]$ restricted to $Y'$, and also on the $\epsilon$-ball $B'$ of the normal bundle of $A_3[\ell]$ at each point of $Y - Y'$. Set $\tilde{D} = B \cup B'$ and $\tilde{W} = \partial \tilde{D}$. Denote their pushforwards to $A_3$ by $D$ and $W$. Then $W$ is a 5-cycle in $M_3$ which is generically an $S^3$ bundle over $X'$. Note that $D$ is a 6-chain in $A_3$ with $\partial D = W$.

On the other hand, Looijenga’s computation of the rational homology of $M_3$, implies that $W$ bounds a rational 6-chain $E$ in $M_3$. Set $Z = D - E$. This is a rational 6-cycle in $A_3$. By construction, we have:

...
**Proposition 18.** The cycle $Z$ intersects $\mathcal{A}_3^{\text{red,sing}}$ transversally (in the orbifold sense), and the intersection number of $Z$ with $\mathcal{A}_3^{\text{red,sing}}$ is non-zero.

**Corollary 19.** The class of $Z$ is non-trivial in $H_6(\mathcal{A}_3; \mathbb{Q})$ and the class of $\mathcal{A}_3^{\text{red,sing}}$ is non-trivial in $H^4(\mathcal{A}_3; \mathbb{Q})$.

The proof of the non-triviality of $\lambda^3$ is completed by the following result.

**Proposition 20.** The class of $\mathcal{A}_3^{\text{red,sing}}$ in $H^6(\mathcal{A}_3; \mathbb{Q})$ is a non-zero multiple of $\lambda^3$.

**Proof.** Denote the closure in $\overline{\mathcal{A}}_3$ of a subvariety $X$ of $\mathcal{A}_3$ by $\overline{X}$. Set $\partial X = \overline{X} - X$. Since $\mathcal{A}_3$ is a rational homology manifold, the sequence

$$H_8(\overline{\mathcal{A}}_3^\text{red}, \partial \overline{\mathcal{A}}_3^\text{red}; \mathbb{Q}) \to H^4(\mathcal{A}_3; \mathbb{Q}) \to H^4(\mathcal{M}_3; \mathbb{Q})$$

is exact. Since $\mathcal{A}_3^{\text{red}}$ is irreducible of dimension 4, the left hand group is one-dimensional and spanned by the fundamental class of $\overline{\mathcal{A}}_3^\text{red}$. Since the middle group is one-dimensional and the right hand group trivial, we see that the class of $\mathcal{A}_3^{\text{red}}$ spans $H^4(\mathcal{A}_3; \mathbb{Q})$. On the other hand, since $\lambda$ is ample, and $\mathcal{A}_3$ contains a complete surface, $\lambda^2$ also spans $H^4(\mathcal{A}_3; \mathbb{Q})$. It follows that there is a non-zero rational number $c$ such that $\lambda^2 = c[\mathcal{A}_3^{\text{red}}]$ in $H^4(\mathcal{A}_3; \mathbb{Q})$.

Denote the determinant of the Hodge bundle by $L$. The class $\lambda^3$ is represented by the divisor of a section of the restriction of $L$ to $\mathcal{A}_3^{\text{red}}$. This can be computed by pulling back along the mapping $A_1 \times A_2 \to \mathcal{A}_3^{\text{red}}$. Since the Picard group of $A_1$ is torsion, we see that the pullback of $L$ is represented mod-torsion by $A_1 \times D$, where $D$ is a cycle representing $\lambda$ on $A_2$. But the cusp form $\chi_{10}$ of weight 10 on $A_2$ is a section of $L^{10}$, and has divisor supported on $A_2^{\text{red}}$, we see that $\lambda$ is represented by a non-zero rational multiple of the cycle $A_1 \times A_2^{\text{red}}$ on $A_1 \times A_2$, and by a non-zero multiple of $\mathcal{A}_3^{\text{red,sing}}$ in $H^4(\mathcal{A}_3^{\text{red}}; \mathbb{Q})$. This implies that $\lambda^3$ is represented by a non-zero rational multiple of $\mathcal{A}_3^{\text{red,sing}}$ in $H^4(\mathcal{A}_3; \mathbb{Q})$.

6. THE RATIONAL COHOMOLOGY OF $\overline{\mathcal{A}}_2$ AND $\overline{\mathcal{A}}_3$.

Note that $\overline{\mathcal{A}}_0$ is just a point. Suppose now that $g > 0$. Then

$$\overline{\mathcal{A}}_g = A_g \amalg \overline{\mathcal{A}}_{g-1}.$$
Choose a triangulation of $\mathcal{A}_g$ such that $\mathcal{A}_{g-1}$ is a subcomplex. Let $U_g$ be a regular PL neighbourhood of $\mathcal{A}_{g-1}$ in $\mathcal{A}_g$. Set $U^*_g = U_g - \mathcal{A}_{g-1}$. This has the homotopy type of $\partial U_g$, which is a rational homology manifold (as $\mathcal{A}_g[3]$ is smooth) of dimension $2d_g - 1$, where $d_g = g(g + 1)/2$ is the dimension of $\mathcal{A}_g$. The cohomology of $U^*_g$ has a mixed Hodge structure and the cup product

$$H^{k-1}(U^*_g; \mathbb{Q}) \otimes H^{2d_g-k}(U^*_g; \mathbb{Q}) \to H^{2d_g-1}(U^*_g; \mathbb{Q}) \cong \mathbb{Q}(-d_g)$$

is a perfect pairing of mixed Hodge structures (MHSs) (see [3], for example).

Since $\mathcal{A}_g$ is a rational homology manifold, Lefschetz duality gives an isomorphism

$$H^k_c(\mathcal{A}_g, \mathbb{Q}) \cong \text{Hom}(H^{2d_g-k}(\mathcal{A}_g), \mathbb{Q}(-d_g)).$$

The standard long exact sequence

$$\cdots \to H^{k-1}(U^*_g) \to H^k_c(\mathcal{A}_g; \mathbb{Q}) \to H^k(A_g; \mathbb{Q}) \to H^k(U^*_g; \mathbb{Q}) \to \cdots$$

is exact in the category of MHSs. These facts will allow us to compute the cohomology of $U^*_g$ and $U^*_3$.

The second step in the computation will be to use the Mayer-Vietoris sequence associated to the covering $\mathcal{A}_g = \mathcal{A}_g \cup U_g$

$$\cdots \to H^k(\mathcal{A}_g) \to H^k(\mathcal{A}_g) \oplus H^k(\mathcal{A}_g) \to H^k(U^*_g) \to H^{k+1}(\mathcal{A}_g) \to \cdots$$

associated to the covering $\mathcal{A}_g = \mathcal{A}_g \cup U_g$, which is exact in the category of MHS, to compute the cohomology of $\mathcal{A}_g$ and $\mathcal{A}_3$.

For determining the ring structure and also for seeing that some maps in these long exact sequences are non-trivial, it is useful to note that since $\lambda$ is the class of an ample line bundle on $\mathcal{A}_g$, the rational cohomology ring of $\mathcal{A}_g$ contains the ring $\mathbb{Q}[\lambda]/(\lambda^{d+1})$.

6.1. **Proof of Proposition 3.** Since $H^*(\mathcal{A}_2; \mathbb{Q})$ is $\mathbb{Q}$ in degree 0, $\mathbb{Q}(-1)$ in degree 2, and 0 otherwise, $H^*(\mathcal{A}_2; \mathbb{Q})$ is $\mathbb{Q}(-2)$ in degree 4, $\mathbb{Q}(-3)$ in degree 6, and 0 otherwise. Using the sequence (10) and the fact that the cohomology of $U^*_2$ satisfies Poincaré duality, we have

$$H^j(U^*_2; \mathbb{Q}) \cong \begin{cases} \mathbb{Q} & j = 0; \\
\mathbb{Q}(-1) & j = 2; \\
\mathbb{Q}(-2) & j = 3; \\
\mathbb{Q}(-3) & j = 5. \end{cases}$$

Putting this into the Mayer-Vietoris sequence (11), and using the fact that $\mathcal{A}_1$ is $\mathbb{P}^1$, we obtain the result.

6.2. **Proof of Proposition 4.** By duality, $H^*(\mathcal{A}_3; \mathbb{Q})$ is an extension of $\mathbb{Q}(-3)$ by $\mathbb{Q}(0)$ in degree 6, $\mathbb{Q}(-4)$ in degree 8, $\mathbb{Q}(-5)$ in degree 10, and $\mathbb{Q}(-6)$ in degree 12. Using the sequence (10) and the facts that $H^j(\mathcal{A}_3; \mathbb{Q})$ vanishes when $j \geq 7$ and $H^j_c(\mathcal{A}_3; \mathbb{Q})$ vanishes when $j \leq 5$, we have

$$H^j(U^*_3; \mathbb{Q}) \cong \begin{cases} H^j(\mathcal{A}_3; \mathbb{Q}) & j < 5; \\
H^{j+1}(\mathcal{A}_3; \mathbb{Q}) & j \geq 7. \end{cases}$$

In degrees 5 and 6 we have the exact sequence

$$0 \to H^5(U^*_3; \mathbb{Q}) \to H^6_c(\mathcal{A}_3; \mathbb{Q}) \to H^6(\mathcal{A}_3; \mathbb{Q}) \to H^6(U^*_3; \mathbb{Q}) \to 0.$$
It follows from [4, 8.2.2] that the image of $\alpha$ is all of $W_6 H^6(A_3; \mathbb{Q})$, so that $\alpha$ is non-zero. Since the sequence is exact in the category of MHSs, it follows that $H^5(U_3^*; \mathbb{Q})$ is $\mathbb{Q}(0)$ and $H^6(U_3^*; \mathbb{Q})$ is $\mathbb{Q}(-6)$.

Since $\mathcal{A}_3$ is projective and $\lambda$ is the class of a projective imbedding, the restriction of $\lambda_j$ to $U_3$ is non-zero when $j = 1, 2, 3$. It follows from this and the computations above that $\lambda$ and $\lambda^2$ restrict to non-trivial classes in the rational cohomology of $U_3^*$.

Putting all of this into the Mayer-Vietoris sequence (11), easily gives the computation of $H^j(A_3; \mathbb{Q})$ when $j \neq 6$. The computation of $H^6(A_3; \mathbb{Q})$ follows as the sequence

$$0 \rightarrow H^5(U_3^*; \mathbb{Q}) \rightarrow H^6(\mathcal{A}_3; \mathbb{Q}) \rightarrow H^6(A_3; \mathbb{Q}) \oplus H^6(\mathcal{A}_2; \mathbb{Q}) \rightarrow H^6(U_3^*; \mathbb{Q}) \rightarrow 0$$

is exact in the category of MHSs.

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