Macroscopic Degeneracy and Order in a 3D Plaquette Ising model

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Plan of talk

A 3D plaquette Ising model with a highly degenerate low-T phase

Consequences for FSS at first order transition

Consequences for order parameter
A 3D Plaquette Ising action

3D cubic lattice, spins on vertices

\[ H = - \sum_{ijkl} \sigma_i \sigma_j \sigma_k \sigma_l \]

Not edges

\[ H = - \sum_{ijkl} U_{ij} U_{jk} U_{kl} U_{li} \]
Plaquette Hamiltonian Groundstates: Single cube
Ground states, Low T: Lattice

Persists into low temperature phase (Wegner, Pietig)

Degeneracy $2^{3L}$
Consequences: FSS
First order FSS: Boundary Conditions

Pirogov-Sinai Theory (Borgs/Kotecký)

\[ Z(\beta) = \left[ e^{-\beta L^d f_d} + q e^{-\beta L^d f_o} \right] [1 + \ldots] \]

Fixed boundaries \((1/L \text{ leading term})\)

\[ Z(\beta) = \left[ e^{-\beta (L^d f_d + L^{d-1} f_o)} + q e^{-\beta (L^d f_o + L^{d-1} f_d)} \right] \]
First order FSS: Exponential Degeneracy

Exponential degeneracy

\[ Z(\beta) = \left[ e^{-\beta L^d f_d} + 2^3 L e^{-\beta L^d f_o} \right] \]

\[ \frac{1}{L^2} \text{ in } d = 3 \]

\[ Z(\beta) = \left[ e^{-\beta L^d f_d} + e^{(3 \ln 2)L} e^{-\beta L^d f_o} \right] \]
FSS: Specific Heat

Probability of being in any of the states

\[ p_0 \propto e^{-\beta L^d \hat{f}_0} \quad \text{and} \quad p_d \propto e^{-\beta L^d \hat{f}_d} \]

Time spent in the ordered states \( \propto qp_0 \)

\[ W_o/W_d \simeq q e^{-L^d \beta \hat{f}_o} / e^{-\beta L^d \hat{f}_d} \]

Expand around \( \beta^\infty \)

\[ 0 = \ln q + L^d \Delta \hat{e} (\beta - \beta^\infty) + \ldots \]

Solve for specific heat peak

\[ \beta_{CV}^{\text{max}} (L) = \beta^\infty - \frac{\ln q}{L^d \Delta \hat{e}} + \ldots \]
With \( q \propto 2^{3L} = e^{(3 \ln 2) L} \)

\[
\beta^{C_{\text{max}}}_{V}(L) = \beta^\infty - \frac{\ln q}{L^d \Delta \hat{e}} + \ldots
\]

become

\[
\beta^{C_{\text{max}}}_{V}(L) = \beta^\infty - \frac{3 \ln 2}{L^{d-1} \Delta \hat{e}} + \ldots
\]
Plaquette Hamiltonian fits

\[ \beta^{\text{Cmax}} = 0.551221(11) \]
\[ \beta^{\text{Bmin}} = 0.551347(7) \]
\[ \beta^{\text{eqw}} = 0.551221(11) \]
\[ \beta^{\text{eqh}} = 0.551288(14) \]

\[ L = 13 \]
Consequences: Order parameter
Magnetization won’t do
Not a gauge theory
Consider anisotropic variant
Suzuki 1973, Jonsson/Savvidy 2000, Castelnovo et.al. 2010

\[ H_{\text{aniso}} = -J_x \sum_{yz} \sigma_i \sigma_j \sigma_k \sigma_l - J_y \sum_{xz} \sigma_i \sigma_j \sigma_k \sigma_l \]

Set \( J_z = 0, J_x = J_y = 1 \). No horizontal plaquettes
Anisotropic Model

Define new spins $\tau$ from pairs of $\sigma$'s

$$\tau_{x,y,z} = \sigma_{x,y,z} \sigma_{x,y,z} + 1$$
Anisotropic Model = Stack of $2D$ Ising

\[ H_{\text{fuki-nuke}} = - \sum_{x=1}^{L} \sum_{y=1}^{L} \sum_{z=1}^{L} (\tau_{x,y,z} \tau_{x+1,y,z} + \tau_{x,y,z} \tau_{x,y+1,z}) , \]
Anisotropic Model Order Parameter

Single layer

\[ m_{2d,z} = \left\langle \frac{1}{L^2} \sum_{x=1}^{L} \sum_{y=1}^{L} \tau_{x,y,z} \right\rangle \]

In terms of original variables

\[ m_{2d,z} = \left\langle \frac{1}{L^2} \sum_{x=1}^{L} \sum_{y=1}^{L} \sigma_{x,y,z} \sigma_{x,y,z+1} \right\rangle \]

Add 'em up

\[ m_{\text{abs}} = \left\langle \frac{1}{L^3} \sum_{z=1}^{L} \left| \sum_{x=1}^{L} \sum_{y=1}^{L} \sigma_{x,y,z} \sigma_{x,y,z+1} \right| \right\rangle , \]
Isotropic case

Hypothesis: Same order parameter works

Hashizume and Suzuki (2011)

Take layers, add 'em up

\[
\begin{align*}
m_{\text{abs}} &= \left\langle \frac{1}{L^3} \sum_{z=1}^{L} \left| \sum_{x=1}^{L} \sum_{y=1}^{L} \sigma_{x,y,z} \sigma_{x,y,z+1} \right| \rightangle \\
\end{align*}
\]

\[
\begin{align*}
m_{\text{sq}} &= \left\langle \frac{1}{L^5} \sum_{z=1}^{L} \left( \sum_{x=1}^{L} \sum_{y=1}^{L} \sigma_{x,y,z} \sigma_{x,y,z+1} \right)^2 \rightangle
\end{align*}
\]
Isotropic case: Effect of flips

\[ m_{\text{abs}}^{z} = \left\langle \frac{1}{L^3} \sum_{z=1}^{L} \left| \sum_{x=1}^{L} \sum_{y=1}^{L} \sigma_{x,y,z} \sigma_{x,y,z+1} \right| \right\rangle \]
Isotropic case: numerical investigation

Strong first order PT

Multicanonical simulation

Correct order parameter: predicts PT point? isotropic?

FSS properties?
Numerical results: order parameters

Order parameter $m^x_{abs}$

Order parameter $m^x_{sq}$

$m^y$ and $m^z$ identical to $m^x$ - isotropy restored
Numerical results: scaled susceptibilities

\[ \chi(\beta) = \beta L^3 \left( \langle m^2 \rangle(\beta) - \langle m \rangle(\beta)^2 \right) \]

\[ \beta \chi_{m_{abs}^x}(L) = 0.551 \, 37(3) - 2.46(3)/L^2 + 2.4(3)/L^3 \]

Susceptibility for \( m_{abs}^x \)  

Susceptibility for \( m_{sq}^x \)
Exponential degeneracy gives $\frac{1}{L^2}$ corrections in 3D

Suzuki was right - isotropic model displays Fuki-Nuke order

OP Scaling agrees well with energetic observables
Cases where *ground state* is exponentially degenerate but low-T phase is not: AFM Ising on FCC

Scaling at first order Quantum PTs
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