Darboux Coordinates on Coadjoint Orbits of Lie Algebras †

M.R. Adams1, J. Harnad2, J. Hurtubise3

1 Department of Mathematics
University of Georgia
Athens, GA 30602 USA
e-mail: adams@alpha.math.uga.edu

2 Department of Mathematics and Statistics
Concordia University
7141 Sherbrooke W., Montréal, Canada H4B 1R6,
and
Centre de recherches mathématiques
Université de Montréal
C. P. 6128-A, Montréal, Canada H3C 3J7
e-mail: harnad@alcor.concordia.ca or harnad@crm.umontreal.ca

3 Department of Mathematics
McGill University
Montréal, Canada H3A 2K6
e-mail: hurtubis@gauss.math.mcgill.ca

Abstract. The method of constructing spectral Darboux coordinates on finite dimensional coadjoint orbits in duals of loop algebras is applied to the one pole case, where the orbit is identified with a coadjoint orbit in the dual of a finite dimensional Lie algebra. The constructions are carried out explicitly when the Lie algebra is \( \mathfrak{sl}(2, \mathbb{R}) \), \( \mathfrak{sl}(3, \mathbb{R}) \), and \( \mathfrak{so}(3, \mathbb{R}) \), and for rank two orbits in \( \mathfrak{so}(n, \mathbb{R}) \). A new feature that appears is the possibility of identifying spectral Darboux coordinates associated to “dynamical” choices of sections of the associated eigenvector line bundles; i.e. sections that depend on the point within the given orbit.

Keywords. Integrable systems, loop algebras, isospectral flow, spectral Darboux coordinates.

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Many finite dimensional integrable Hamiltonian systems have been realized through symplectic reduction [AM] or collectivization [GS] from integrable Hamiltonian systems on coadjoint orbits in the dual of Lie algebras. The classic example of this is the motion of a free rigid body, whose natural phase space $T^*SO(3)$ reduces to $\mathfrak{so}(3)^*$ after factoring out the symmetries given by the left $SO(3)$ action. The reduced dynamics are given by the Euler equations on $\mathfrak{so}(3)^*$ [Ar]. Other well-known examples include the symmetric tops (i.e. rigid bodies in external force fields), constrained oscillators on quadrics, and Toda lattices. In view of such examples it is interesting to make a general study of integrable Hamiltonian systems on coadjoint orbits. (This is also interesting from the geometric quantization point of view since a completely integrable system yields a possibly singular real polarization of the coadjoint orbit.)

There are two widely used methods for producing completely integrable systems on coadjoint orbits: the Mischenko, Fomenko technique [MF], and the Adler, Kostant, Symes (AKS) theorem [Ad], [K], [S]. In fact, the former can be understood as a special case of the latter. It relies on the observation that if two functions $F_1, F_2$ Poisson commute on $\mathfrak{g}^*$, and $\mu_0 \in \mathfrak{g}^*$ then the family of shifted functions $F^\lambda_i$ defined by $F^\lambda_i(\nu) = F_i(\lambda \mu_0 + \nu)$ Poisson commute for all scalars $\lambda$. The AKS theorem gives Poisson commuting functions on the dual of a Lie algebra $\mathfrak{k}^*$ when $\mathfrak{g}$ splits into a vector space direct sum $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{l}$ with both $\mathfrak{k}$ and $\mathfrak{l}$ Lie subalgebras. It states that the $Ad^*$-invariant functions on $\mathfrak{g}^*$ then restrict to Poisson commuting functions on $\mathfrak{k}^*$. More generally, if the $Ad^*$-invariant functions are shifted by an infinitesimal character of $\mathfrak{l}$ (in $\mathfrak{l}^*$), one still obtains a ring of Poisson commuting functions on $\mathfrak{k}^*$. To treat periodic Toda lattices and constrained oscillators in such a framework requires use of coadjoint orbits in loop algebras. This places such systems in the same context as the invariant “finite dimensional” solutions to integrable systems of PDE’s such as the KdV equation.

Most of the above mentioned families of integrable systems can be studied as shifted AKS type flows on finite dimensional orbits of loop algebras (see e.g. [AvM], [R], [AHP], [AHH3]). In particular if $\mathfrak{g}$ denotes a Lie algebra of matrices we let $\tilde{\mathfrak{g}}$ denote the formal loop algebra

$$\tilde{\mathfrak{g}} = \left\{ \sum_{i=-\infty}^{n} X_i \lambda^i \mid X_i \in \mathfrak{g} \right\}$$

and split $\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}^+ \oplus \tilde{\mathfrak{g}}_-$ where $\tilde{\mathfrak{g}}^+$ consists of those loops that are polynomial in $\lambda$ (i.e. matricial polynomials) and $\tilde{\mathfrak{g}}_-$ consists of loops with only strictly negative
powers of \( \lambda \). If \( g \) has a nondegenerate bilinear form \( <, > \) we may use it to give an identification
\[
g \sim g^*. \quad (1.2)
\]
We can extend \( <, > \) to \( \tilde{g} \) by computing \( <X(\lambda), Y(\lambda)> \) as a formal power series and then taking the coefficient of \( \lambda^{-1} \). This gives an identification
\[
(\tilde{g}^+)^* \sim \tilde{g}_. \quad (1.3)
\]
If \( <, > \) is \( Ad \) invariant one can compute the \( ad^* \) action of \( \tilde{g}^+ \) on \( \tilde{g}_- \) and thus determine the symplectic leaves in \( \tilde{g}_- \) with respect to the Poisson structure of \( (\tilde{g}^+)^* \). In particular, there is a Poisson embedding of the finite dimensional space, \( g^* \) into \( (\tilde{g}^+)^* \) which under the identifications (1.2) and (1.3) is simply
\[
X \rightarrow \frac{1}{\lambda}X \quad (1.4)
\]
for \( X \in g \). In the next section we explain how Mischenko - Fomenko type Hamiltonians on \( g^* \) can be realized as shifted AKS Hamiltonians on \( (\tilde{g}^+)^* \) by means of the map (1.4). Generalizing to twisted loop algebras, AKS type Hamiltonians on \( g^* \) can also be derived in this manner.

In [AHH2] the shifted AKS flows were studied on finite dimensional orbits in \( (\tilde{g}^+)^* \) through points of the form
\[
N_0(\lambda) = \sum_{i=1}^{n} \frac{N_i}{\lambda - \alpha_i}, \quad (1.5)
\]
where \( N_i \in g \) and the \( \alpha_i \)'s are scalar. It was shown how the spectral data on these orbits led directly to algebraic Darboux coordinates that are particularly suited to integrating the flows of shifted AKS type Hamiltonians. Since these coordinates are obtained from a divisor of a section of the dual eigenvector line bundle over the invariant spectral curve defined by the characteristic equation of \( N_0(\lambda) \), we call them spectral Darboux coordinates. In section 3 of this letter we show how to obtain spectral Darboux coordinates on coadjoint orbits of finite dimensional Lie algebras and we illustrate the construction explicitly for \( g = sl(2), sl(3) \) and \( so(3) \). Since the Euler equations for rigid body motion are Mischenko - Fomenko type flows on \( so(3)^* \), this case produces Darboux coordinates particularly suited to integrating rigid body flow. More generally, these techniques may be used to study the \( so(n) \) generalization of rigid body motion. Unfortunately, as the size of the matrix increases so do the the degrees of the polynomials whose roots describe the Darboux coordinates, so these coordinates are no longer computable for large \( n \). On the other hand, it was pointed out by
Moser [Mo] that certain rigid body flows in $\mathfrak{so}(n)^*$ (i.e. flows on very special coadjoint orbits) can be dealt with explicitly in terms of rank two perturbations of an $n \times n$ matrix. In [AHH3] it was shown that this is an example of a general phenomenon, referred to as duality, relating spectral curves of two matricial polynomials of different size. In section 4 of this letter we describe the Darboux coordinates on this special orbit for the $\mathfrak{so}(n)$ rigid body and compare them with those obtained through duality.

2 Mischenko Fomenko Flows as Shifted AKS Flows

For convenience we take $\mathfrak{g} = \mathfrak{sl}(r)$ and $G = SL(r)$. The $Ad^*$ - invariant inner product can then be taken to be

$$<X, Y> = tr(XY^T)$$

and the $Ad^*$ - invariant functions are generated by

$$\sigma_k(X) := \frac{1}{k} tr(X^k).$$

Fixing $Y \in \mathfrak{g}$ and $\mu \in \mathbb{R}$, Hamilton’s equation for the Mischenko - Fomenko Hamiltonian $\sigma_k(X + \mu Y)$ is

$$\frac{d}{dt} X = [(X + \mu Y)^{k-1}, X].$$

On the other hand, the $Ad^*$ - invariant functions on $\tilde{\mathfrak{g}}^*$ (identified with $\tilde{\mathfrak{g}}$ by (1.3)) are generated by

$$S_{k,j}(X(\lambda)) := \frac{1}{k} tr(X(\lambda)^k))_j$$

where the subscript $j$ denotes taking the coefficient of $\lambda^j$. Shifting $S_{k,-j}$ by $Y \in \mathfrak{g}$ and restricting to the union of orbits $\{X(\lambda) = \frac{1}{X}X \} \subset \tilde{\mathfrak{g}}_-$ by equation (1.4), Hamilton’s equations are of the form

$$\frac{d}{dt} X = [((\lambda Y + X)^{k-1}\lambda^{-j})_+, (\lambda Y + X)]$$

where the subscript $+$ denotes taking the polynomial part of the Laurent polynomial $(\lambda Y + X)^{k-1}\lambda^{-j}$. To realize the equation (2.3) as one of AKS - type we must find a Hamiltonian in the AKS ring of spectral invariants which gives the equation (2.3). Let the $\lambda^i$ coefficient of $(\lambda Y + X)^{k-1}$ be denoted by $A_i$. Let

$$q_j(\lambda) = ((\lambda Y + X)^{k-1}\lambda^{-j})_+$$
and notice that 
\[ q_j(0) = A_{k-1-j} \]  
so in particular 
\[ \sum_{j=0}^{k-1} q_{k-j-1}(0) \mu^j = (\mu Y + X)^{k-1}. \]  
Thus if we take 
\[ H(\lambda, \mu, k) = \sum_{j=0}^{k-1} S_{k,k-j-1} \mu^j, \]  
we get a Lax pair which when evaluated at \( \lambda = 0 \) gives the Mischenko-Fomenko equation (2.3).

The proof for a more general Lie algebra \( \mathfrak{g} \) goes through *mutatis mutandis* with \( \sigma_k \) replaced by a general \( \text{Ad}^* \)-invariant function \( \Phi \) and \( S_{k,j} \) replaced by \( \Phi(X(\lambda))_j \).

### 3 Spectral Darboux Coordinates

We begin by recalling some of the results of [AHH2]. For the present, we continue with \( \mathfrak{g} = \mathfrak{sl}(r), \ G = SL(r), \) and the pairing given by (2.1). Given \( \mathcal{N}_0(\lambda) \) of the form (1.5) the coadjoint orbit through \( \mathcal{N}_0(\lambda) \) is given by 
\[ \mathcal{O}_{\mathcal{N}_0} = \left\{ \sum_{i=1}^{n} \frac{g_i N_i g_i^{-1}}{\lambda_\alpha_i} \middle| g_i \in G \right\} \]  
which may be identified with a product of \( n \) coadjoint orbits in \( \mathfrak{sl}(r)^* \). For fixed \( Y \in \mathfrak{g} \) we consider shifted AKS Hamiltonians of the type 
\[ \phi(\mu) = \Phi(Y + \mu), \]  
where \( \mu \in \mathcal{O}_{\mathcal{N}_0} \) and \( \Phi \) is an element of the ring \( I(\tilde{\mathfrak{g}}^*) \) of \( \text{Ad}^* \) invariant functions on \( \tilde{\mathfrak{g}}^* \). Hamilton’s equations for such functions are given by 
\[ \frac{d\mathcal{N}(\lambda)}{dt} = [d\Phi(\mathcal{N}(\lambda))_+, \mathcal{N}(\lambda)] \]  
where \( \mathcal{N}(\lambda) \) has the form \( Y + \mu \) and the + subscript denotes projection to \( \tilde{\mathfrak{g}}^+ \).

Setting 
\[ \mathcal{L}(\lambda) := a(\lambda)\mathcal{N}(\lambda) \]  
with 
\[ a(\lambda) = \prod_{i=1}^{n} (\lambda_\alpha_i), \]
it is evident that the curve

\[ S_0 = \{ (\lambda, z) \mid \det(\mathcal{L}(\lambda) - zI) = 0 \} \]  

(3.6)
is invariant under the flows. By the method of Krichever (see e.g. [KN], [Du], [AHH1]) the eigenspaces of \( \mathcal{L}^T(\lambda) \) define a line bundle \( E^* \) over a suitable compactification \( S \) of \( S_0 \) and this bundle moves linearly in a component of \( Pic(S) \) as \( \mathcal{L}(\lambda) \) evolves by (3.3).

This geometric description of the linearization suggests the following prescription for computing Darboux coordinates on \( \mathcal{O}_{N_0} \). First, let \( \mathcal{K}(\lambda, z) \) be the classical adjoint matrix of \( \mathcal{L}(\lambda) - zI \); i.e. the matrix of cofactors transposed. The rows of \( \mathcal{K}(\lambda, z) \) give sections of the eigenvector line bundle \( E^* \) (over \( S_0 \)). Taking the section of the dual bundle \( E \) obtained by restriction of a fixed section \( (\lambda, z) \mapsto ((\lambda, z), V_0) \) of the trivial bundle \( S_0 \times \mathbb{C}^r \), the zeros \( \{ (\lambda_\mu, z_\mu) : \mu = 1, \ldots, d \} \) of \( \mathcal{K}(\lambda, z)V_0 \) give the associated (spectral) divisor which, generically is of degree \( g+r-1 \). If \( V_0 \) is not orthogonal to any of the eigenvectors of \( \mathcal{L}(\lambda)^T \) over \( \lambda = \infty \), this divisor is given by the \( d = g + r - 1 \) finite solutions of the polynomial equations

\[ \mathcal{K}(\lambda, z)V_0 = 0. \]  

(3.7)

However, one must be careful if \( V_0 \) is an eigenvector of the shift matrix \( Y \). It then follows that \( r - 1 \) of the points in the divisor lie over \( \lambda = \infty \). Now let

\[ \zeta_\mu := \frac{z_\mu}{a(\lambda_\mu)} \]  

(3.8)

and consider

\[ \{ (\lambda_\mu, \zeta_\mu) : \mu = 1, \ldots, d \} \]  

(3.9)
to be functions on the orbit \( \mathcal{O}_{N_0} \). A straightforward computation [AHH2] shows that these functions satisfy canonical relations, i.e.

\[ \{ \lambda_\mu, \lambda_\nu \} = 0 \]  

(3.10a)

\[ \{ \zeta_\mu, \zeta_\nu \} = 0 \]  

(3.10b)

\[ \{ \lambda_\mu, \zeta_\nu \} = \delta_{\mu\nu}. \]  

(3.10c)

This fact, together with some dimension counts yield the following theorem (see[AHH2]).

**Theorem 3.1.**

a) If \( V_0 \) is not orthogonal to any eigenvector of \( Y^T \), \( d = g + r - 1 \) and the functions \( \{ (\lambda_\mu, \zeta_\mu) \} \) give Darboux coordinates on a dense open set of \( \mathcal{O}_{N_0}(\lambda) \).
b) If $V_0 = (1, 0, 0, ..., 0)^T$ and is an eigenvector of $Y$, then the functions $\{\lambda_\mu, \zeta_\mu\}$ together with

$$\zeta_i = (L_0)_{ii}, \quad \lambda_i = \ln(L_1)_{i1}$$

(3.11)

$i = 1, ...n - 1$, give Darboux coordinates. Here $L_0$ and $L_1$ are determined by

$$L(\lambda) = a(\lambda)Y + L_0\lambda^{n-1} + L_1\lambda^{n-2} + ... .$$

(3.12)

We remark that part b) of this theorem may be stated for more general $V_0$ but a change of basis may always be used to transform it to this form.

To illustrate how this result applies to semisimple Lie algebras, we consider the case when $\mathcal{N}_0(\lambda)$ has the form (1.4) with $X$ in $\mathfrak{sl}(2), \mathfrak{sl}(3)$ or $\mathfrak{so}(3)$. Here $a(\lambda) = \lambda$ so

$$\zeta_i = z_i/\lambda_i.$$  

(3.13)

Note that although the theorem is proved for $\mathfrak{sl}(r, \mathbb{C})$ or $\mathfrak{sl}(r, \mathbb{R})$, it may be applied to subalgebras defined as fixed points of a finite group of automorphisms, provided the resulting coordinates are invariant under this group.

(a.1) Take $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})$. The dimension of a generic orbit is $\dim \mathcal{O}_{\mathcal{N}_0} = 2$. We parametrize $\mathcal{N}_0(\lambda)$ as follows:

$$\mathcal{N}_0(\lambda) = N_1/\lambda := \frac{1}{\lambda} \begin{pmatrix} -a & r \\ u & a \end{pmatrix},$$

(3.14)

and choose

$$Y := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad V_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$  

(3.15)

The characteristic equation is then

$$\det (L(\lambda) - z\mathbb{I}_r) = z^2 - \lambda^2 + 2\lambda a - a^2 - ur = 0.$$  

(3.16)

In this case, $V_0$ is an eigenvector of $Y$ and the genus of the spectral curve is $g = 0$, so there are no $\{\lambda_\mu, \zeta_\mu\}$’s. The single pair of spectral Darboux coordinates is thus

$$q_2 = \ln u, \quad P_2 = a.$$  

(3.17)

It is easily verified that, relative to the Lie Poisson structure, they satisfy

$$\{q_2, P_2\} = 1.$$  

(3.18)

(a.2) Consider the same orbit as in (a.1), but choose

$$Y := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$  

(3.19)
In this case, \( V_0 \) is not an eigenvector of \( Y \). The genus is still 0 but the line bundle does have a finite divisor point, giving the Darboux coordinate pair

\[
\lambda_1 = -u, \quad \zeta_1 = -\frac{a}{u}.
\]  

(3.20)

These are verified to also satisfy

\[
\{\lambda_1, \zeta_1\} = 1.
\]  

(3.21)

(b.1) Take \( g = \mathfrak{sl}(3, \mathbb{R}) \). The dimension of a generic orbit is \( \dim \mathcal{O}_{\mathcal{N}_0} = 6 \). We parametrize \( \mathcal{N}_0(\lambda) \) as:

\[
\mathcal{N}_0(\lambda) := \frac{1}{\lambda} \begin{pmatrix} -a - b & r & s \\ u & a & e \\ v & f & b \end{pmatrix},
\]  

(3.22)

and choose

\[
Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad V_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.
\]  

(3.23)

Again, \( V_0 \) is an eigenvector of \( Y \), but the spectral curve has genus \( g = 1 \), and is realized as a 3–fold branched cover of \( \mathbb{P}^1 \). We therefore find one Darboux coordinate pair \((\lambda_1, \zeta_1)\), corresponding to a finite zero of the eigenvector components, plus two further pairs, \((q_2, P_2, q_3, P_3)\), corresponding to zeros over \( \lambda = \infty \):

\[
\lambda_1 = \frac{1}{2} \left( -a - b - \frac{uv}{u} + \frac{uf}{v} \right), \quad \zeta_1 = \frac{wva + uvb - ev^2 - fu^2}{-wva + uvb - ev^2 + fu^2}
\]

\[
q_2 = \ln u, \quad q_3 = \ln v, \quad P_2 = a, \quad P_3 = b.
\]  

(3.24)

Again, it is easily verified directly that these form a Darboux system, with nonvanishing Lie Poisson brackets

\[
\{\lambda_1, \zeta_1\} = 1, \quad \{q_2, P_2\} = 1, \quad \{q_3, P_3\} = 1.
\]  

(3.25)

(b.2) Consider the same orbit as in (b.1), but take

\[
Y = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad V_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.
\]  

(3.26)

The genus of the curve is still 1, but the number of finite divisor points is now 3. The divisor coordinates may be obtained by solving the pair of equations

\[
z^2 + z(v - a - b) + \lambda(u - e) + ab - ef + fu - av = 0
\]  

(3.27a)
\[\lambda z + uz + (v - b)\lambda + ev - bu = 0,\]  
\((3.27b)\)

which reduces to a cubic equation for \(z\), with generically distinct roots \((z_1, z_2, z_3)\). Setting

\[\lambda_i = z_i^2 + (v - a - b)z_i + ab - av + fu - fe \over u - e, \quad \zeta_i = z_i\over \lambda_i, \quad i = 1, 2, 3 \]  
\((3.28)\)

gives Darboux coordinates.

(c) Take \(g = \mathfrak{so}(3, \mathbb{R})\). (The proof of Theorem 3.1 was given only in the case \(g = \mathfrak{sl}(r)\), but we include this example to illustrate that it can be extended to a more general setting. The explicit results can be readily checked by direct computation.) The dimension of a generic orbit is \(\dim \mathcal{O}_{N_0} = 2\). We parametrize \(N_0(\lambda)\) as:

\[N_0(\lambda) := \frac{1}{\lambda} \begin{pmatrix} 0 & r & s \\ -r & 0 & e \\ -s & -e & 0 \end{pmatrix}.\]
\((3.29)\)

Using \(Y\) and \(V_0\) as in \((3.23)\) we get a spectral curve of genus 1 and one pair of divisor coordinates given by making the appropriate restrictions of \((\lambda_1, \zeta_1)\) in \((3.24)\). Namely,

\[\lambda_1 = -\frac{1}{2} \frac{e(s^2 + r^2)}{sr}, \quad \zeta_1 = \frac{s^2 - r^2}{r^2 + s^2}.\]
\((3.30)\)

A direct computation shows that for the Lie Poisson structure on \(\mathfrak{so}(3, \mathbb{R})\) one has

\[\{\lambda_1, \zeta_1\} = 2.\]
\((3.31)\)

Thus, one adjusts \(\zeta_1\) by a factor of \(1/2\) to get a canonical pair. (This is a remnant of the fact that we used the pairing \((1.2)\) to identify \(g\) with \(g^*\), while here the appropriate pairing is one half of the pairing \((1.2)\).)

More generally, it is interesting to compute the Darboux coordinates one gets on these orbits using

\[Y = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix},\]
\((3.32)\)

with distinct \(\alpha, \beta, \gamma\). Rigid body motion in 3 dimensions can be written in the Lax pair form

\[\frac{d\mathcal{L}}{dt} = [\mathcal{A}, \mathcal{L}]\]
\((3.33)\)

where

\[\mathcal{L} = \lambda Y + \Lambda\]
\((3.34)\)
with the diagonal matrix
\[ Y = \text{diag}(\alpha, \beta, \gamma) \] (3.35)
given by the inverses of the principal moments of inertia
\[ \alpha = \frac{1}{I_1}, \quad \beta = \frac{1}{I_2}, \quad \gamma = \frac{1}{I_3} \] (3.36)
and \( \Lambda \in \mathfrak{o}(3) \) the components of the angular momentum vector relative to the principal axes of inertia:
\[ \Lambda = \begin{pmatrix} 0 & L_3 & -L_2 \\ -L_3 & 0 & L_1 \\ L_2 & -L_1 & 0 \end{pmatrix}, \] (3.37)
while
\[ \mathcal{A} = \frac{1}{\lambda} \Lambda^2. \] (3.38)
Thus, identifying
\[ e = L_1, \quad s = -L_2, \quad r = L_3, \] (3.39)
we have
\[ \mathcal{L} = \lambda(Y + N_0(\lambda)). \] (3.40)
The spectral curve for this case is given by
\[
z^3 - z^2 \lambda(\alpha + \beta + \gamma) + z\lambda^2(\alpha\beta + \alpha\gamma + \beta\gamma) - \lambda^3(\alpha^2 + \beta^2 + \gamma^2) - \lambda^2(\alpha e^2 + \gamma r^2 + \beta s^2) + z(r^2 + s^2 + e^2) = 0.
\] (3.41)
The coefficient of the linear terms in \( z \) and \( \lambda \) are just the square of the angular momentum vector
\[ |\mathbf{L}|^2 = e^2 + s^2 + r^2 = L_1^2 + L_2^2 + L_3^2 \] (3.42)
and the Hamiltonian for the Euler top
\[ H = \frac{1}{2}(\alpha e^2 + \beta s^2 + \gamma r^2) = \frac{1}{2} \left[ \frac{L_1^2}{I_1} + \frac{L_2^2}{I_2} + \frac{L_3^2}{I_3} \right]. \] (3.43)
Choosing the vector
\[ V_0 = (1, 0, 0)^T \] (3.44)
and noting that the pairing needed to identify \( \mathfrak{o}(3) \) with its dual is
\[ <X, Y> = -\frac{1}{2} tr(XY), \] (3.45)
the Darboux coordinates provided by theorem (3.1) are:

$$\frac{1}{2} \lambda_1 = \frac{e(r^2 + s^2)}{2rs(\beta - \gamma)}, \quad \zeta_1 = \frac{\beta s^2 + \gamma r^2}{(r^2 + s^2)}.$$  

(Note that the factor $\frac{1}{2}$ in $\lambda$ is again required because we are not dealing with the full $\mathfrak{sl}(3)$ algebra, but the subalgebra consisting of fixed points under the involution $X(\lambda) \to -X(-\lambda)^T$. Note also that, although $V_0$ is an eigenvector of $Y$, in this case the orbits are 2–dimensional, and part (b) of theorem 3.1 provides no additional coordinates.)

On the level sets given by fixing the values of the invariants $H$ and $|L|^2$ as

$$H = E, \quad |L|^2 = \ell^2, \quad (3.47)$$

the coordinate function $\zeta_1$ is just

$$\zeta_1 = \frac{2E - \alpha \ell_1^2}{\ell^2 - L_1^2}. \quad (3.48)$$

For purposes of integration, it is convenient to parametrize the spectral curve in a more standard form that makes it evident it is elliptic. In terms of $(\lambda, \zeta = \frac{\zeta}{\lambda})$ we may express eq. (3.41) as:

$$\lambda^2 a(\zeta) + \zeta\ell^2 - 2E = 0, \quad (3.49)$$

where

$$a(\zeta) := (\zeta - \alpha)(\zeta - \beta)(\zeta - \gamma). \quad (3.50)$$

Then, defining the meromorphic function

$$y := \frac{2E - \ell^2 \zeta}{\lambda}. \quad (3.51)$$

the expression (3.47) is equivalent to

$$y^2 = (2E - \ell^2 \zeta)a(\zeta). \quad (3.52)$$

We may therefore interpret the spectral curve $S_0$ as the Riemann surface of the function $y$.

The flow may then be computed through the Liouville method using the Darboux coordinates $(\lambda_1/2, \zeta_1)$. Restricting the 1–form

$$\theta = -\frac{\lambda_1}{2} d\zeta_1 \quad (3.53)$$
to the Lagrangian leaf defined by eq. (3.47) and integrating, the Liouville generating function is

\[ S(\zeta_1, E, \ell) = -\frac{1}{2} \int_{(\zeta_0, \lambda_0)}^{(\zeta_1, \lambda_1)} \lambda d\zeta = -\frac{1}{2} \int_{(\zeta_0, \lambda_0)}^{(\zeta_1, \lambda_1)} \frac{(2E - \ell^2 \zeta)}{y} d\zeta \]  

(3.54)

(where we have used the fact that the point \((\lambda_1, z_1 = \zeta_1 \lambda_1)\) lies on the spectral curve). Differentiating with respect to \(E\) then gives the conjugate coordinate to \(H\), which evolves linearly in time

\[ Q := \frac{\partial S}{\partial E} = -\frac{1}{2} \int_{(\zeta_0, \lambda_0)}^{(\zeta_1, \lambda_1)} \frac{d\zeta}{\sqrt{(2E - \ell^2 \zeta) (a(\zeta))}} = t - t_0, \]  

(3.55)

where a base point \((\zeta_0, \lambda_0) = (\zeta_1(t_0), \lambda_1(t_0))\) has been chosen. As usual, the elliptic integral appearing in eq. (3.55) may be inverted to determine \(\zeta_1(t)\), and hence \((L_1(t), L_2(t), L_3(t))\) in terms of Jacobi elliptic functions [W].

4 Duality

We now extend this method to the case of spectral Darboux coordinates on rank two orbits in \(\mathfrak{so}(n)^*\). First we describe these using \(n \times n\) matrices and then, using duality and a judicious choice of \(V_0\), we show that the method can be used to deduce the hyperelliptic coordinates used to integrate Euler flow on these orbits as in [Mo].

The Euler flow on \(\mathfrak{so}(n)^*\) is generated by a quadratic spectral invariant of the matrix

\[ M(z) = A + \frac{1}{z} L, \quad L \in \mathfrak{so}(n), \]  

(4.1)

where \(A\) is a diagonal \(n \times n\) matrix. If we take \(L\) to have rank two, we may write it in the form

\[ L = X \wedge Y = XY^T - YX^T \]  

(4.2)

for some \(X, Y \in \mathbb{R}^n\). This provides a map from \(\mathbb{R}^{2n}\) to the space of rank two \(n \times n\) skew matrices. It is easily checked that this is a Poisson map into the one parameter family of rank two coadjoint orbits in \(\mathfrak{so}(n)\). The only nonvanishing coadjoint invariant is

\[ tr(L^2) = 2(|X \cdot Y|^2 - |X|^2 |Y|^2) \]  

(4.3)

and the fiber of this map is given by the orbits of the \(\mathfrak{sl}(2)\) action on \(\mathbb{R}^{2n}\) given by

\[ g \cdot (X, Y) = (aX + bY, cX + dY) \]  

(4.4)
where
\[ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \] (4.5)

Thus the coadjoint orbit is \(2n - 4\) dimensional. In the following we will describe the divisor coordinates for \(M(z)\) as functions in the variables \((X, Y)\) that project under the \(\mathfrak{sl}(2)\) quotient to the orbits of the form (4.2).

If we let \(\widetilde{M(z)} - \lambda I\) denote the transposed matrix of cofactors of \(M(z) - \lambda I\) and choose \(V_0 \in \mathbb{R}^n\) then, as above, the divisor Darboux coordinates are given by the zeros of \((\widetilde{M(z)} - \lambda I)V_0\). Note that
\[ \widetilde{M(z)} - \lambda I = (\widetilde{A} - \lambda I)(I + \frac{1}{z}\widetilde{L(A - \lambda I)^{-1}}) \] (4.6)

so
\[ (\widetilde{M(z)} - \lambda I)V_0 = 0 \] (4.7)

if and only if
\[ (I + \frac{1}{z}\widetilde{L(A - \lambda I)^{-1}})V_0 = 0. \] (4.8)

On these orbits, we can compute \(I + \frac{1}{z}\widetilde{L(A - \lambda I)^{-1}}\) explicitly. First, by direct computation, one sees that
\[ (\widetilde{L(A - \lambda I)^{-1}})^3 = \gamma L(A - \lambda I)^{-1} \] (4.9)

where
\[ \gamma = Q(X, Y)^2 - Q(X, X)Q(Y, Y) \] (4.10)

with
\[ Q(U, V) = U^T (A - \lambda I)^{-1} V. \] (4.11)

Thus, the minimal polynomial for \(I + \frac{1}{z}\widetilde{L(A - \lambda I)^{-1}}\) is given by
\[ u^3 - 3u^2 + (3 - \frac{\gamma}{z^2})u = (1 - \frac{\gamma}{z^2}). \] (4.12)

From (4.12), it follows that
\[ (I + \frac{1}{z}\widetilde{L(A - \lambda I)^{-1}}) = \frac{1}{z^2}(L(A - \lambda I)^{-1})^2 - \frac{1}{z}L(A - \lambda I)^{-1} + (1 - \frac{\gamma}{z^2})I. \] (4.13)

Applying this matrix to \(V_0\), we obtain
\[ (I + \frac{1}{z}\widetilde{L(A - \lambda I)^{-1}})V_0 = \]
\[ (1 - \frac{\gamma}{z^2})V_0 + \left[ \frac{1}{z^2}(Q(X, Y)Q(Y, V_0) - Q(Y, Y)Q(X, V_0)) - \frac{1}{z}Q(Y, V_0) \right]X \]
\[ + \left[ \frac{1}{z^2}(Q(X, Y)Q(X, V_0) - Q(X, X)Q(Y, V_0)) + \frac{1}{z}Q(X, V_0) \right]Y. \] (4.14)
For fixed $V_0$ and generic values of $X$ and $Y$, these three terms are linearly independent, so the above expression vanishes if

$$\gamma = z^2,$$

$$Q(X,Y)Q(Y,V_0) - Q(Y,Y)Q(X,V_0) - zQ(Y,V_0) = 0, \tag{4.15a}$$

$$Q(X,Y)Q(X,V_0) - Q(X,X)Q(Y,V_0) + zQ(X,V_0) = 0. \tag{4.15c}$$

The first of these follows from the last two, and these two are equivalent to

$$Q(X,X)Q(Y,V_0)^2 - 2Q(X,Y)Q(X,V_0)Q(Y,V_0) + Q(Y,Y)Q(X,V_0)^2 = 0 \tag{4.16a}$$

$$z = Q(X,Y) - Q(Y,Y) \frac{Q(X,V_0)}{Q(Y,V_0)}. \tag{4.16b}$$

Equation (4.16a) determines the $\lambda_{\mu}$'s as roots of a polynomial, and substituting these into (4.16b) gives the $z_{\mu}$'s. For example, if we take $V_0 = (1,0,0,\ldots,0)^T$, then (4.16a) is just

$$0 = Q(X,X)y_1^2 - 2Q(X,Y)x_1y_1 + Q(Y,Y)x_1^2 \tag{4.17}$$

which, when multiplied by $\prod_{j=2}^{n}(\lambda - \alpha_j)$, gives a polynomial equation of degree $n - 2$ in $\lambda$. Notice that (4.16a,b) are invariant under the $\mathfrak{sl}(2)$ action on $\mathbb{R}^{2n}$, so the $n - 2$ pairs $(\lambda_{\mu}, z_{\mu})$ reduce to the quotient, which is identified with the coadjoint orbit through $L$.

We now turn to the dual description [AHH3] of divisor coordinates for these rank two coadjoint orbits in $\mathfrak{so}(n)^*$. Instead of the map $(X,Y) \rightarrow M(z)$ considered above we take the map $\mathbb{R}^{2n} \rightarrow \tilde{\mathfrak{sl}}(2,\mathbb{R})^*$ given by

$$(X,Y) \rightarrow N(\lambda) = \sum_{i=1}^{n} \frac{1}{\lambda - \alpha_i} \begin{pmatrix} -x_i y_i & y_i^2 \\ -y_i^2 & x_i y_i \end{pmatrix}. \tag{4.18}$$

This is a Poisson map whose generic fibers are the $2^n$ points in an orbit of the group that sends $(x_i, y_i)$ to $(\pm x_i, \pm y_i)$. We can identify the coadjoint orbit through $M(z) \in \mathfrak{so}(n)^*_+$ with the symplectic quotient by the $\mathfrak{sl}(2)$ action on the coadjoint orbit through $N(\lambda) \in \mathfrak{sl}(2)^*_+$. Indeed, the $\mathfrak{sl}(2)$ moment map is given by

$$N_0 = \sum_{i=1}^{n} \begin{pmatrix} -x_i y_i & y_i^2 \\ -y_i^2 & x_i y_i \end{pmatrix}. \tag{4.19}$$
so, for instance, the orbit in $\mathfrak{so}(n)$ with $\text{tr}(L^2) = -2$ can be realized as the quotient of the set

$$\{(X, Y) \in \mathbb{R}^{2n} | |X| = |Y| = 1, X \cdot Y = 0\} \quad (4.20)$$

by the action of $\mathfrak{so}(2) \subset \mathfrak{sl}(2)$.

To get divisor coordinates on the orbit through $N_0(\lambda)$ it is standard to choose $V_0 = (1, 0)$, giving the equations

$$Q(X, X) = 0, \quad z = Q(X, Y). \quad (4.21)$$

The first of these equations gives hyperellipsoidal coordinates on the sphere. However, these coordinates are not suited for studying the rigid body equations since they are not invariant under the $\mathfrak{so}(2)$ action and hence do not reduce to the $\mathfrak{so}(n)$ orbit. To circumvent this problem we must choose $V_0$ so that the resulting divisor coordinates are $\mathfrak{so}(2)$ invariant. One such choice is

$$V_0 = (y_1, x_1), \quad (4.22)$$

which leads to the equation (4.17). Since equation (3.7) for this choice of $V_0$ is invariant under the $\mathfrak{so}(2)$ action

$$(X, Y) \mapsto (\cos \theta X + \sin \theta Y, -\sin \theta X + \cos \theta Y), \quad (4.23)$$

these spectral divisor coordinates project to the reduced space. This is a non-standard choice of $V_0$ since it is dependent on the point in the coadjoint orbit. However it still defines a section of the dual eigenvector line bundle over the spectral curve, and its zeros still give Darboux coordinates on the coadjoint orbit.

Another way to assure invariance under the $\mathfrak{so}(2)$ action is to use the fixed complex vector

$$V_0 = (1, i) \quad (4.24)$$

which leads to the equations

$$Q(X + iY, X + iY) = 0, \quad z = Q(X + iY, Y). \quad (4.25)$$

Since the action of an element of $\mathfrak{so}(2)$ on $V_0$ simply amounts to multiplication of $V_0$ by a phase $\exp(i\theta)$, the resulting divisor coordinates are again $\mathfrak{so}(2)$ invariant, and thus reduce to the $\mathfrak{so}(n)$ orbit. Finally, we remark that equations (4.25) can be found in the $n \times n$ setting by choosing $V_0 = X + iY$ in equations (4.16a,b). This is again a “dynamical” choice of section of $E$ (i.e., it depends on the point in the orbit determined by the pair $(X, Y)$), just as the choice $V_0 = (y_1, x_1)$ was in the $2 \times 2$ setting. But, by equivalence of these dual formulations, it again provides a valid spectral Darboux system on the orbit.
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