Ratner’s property for special flows over irrational rotations under functions of bounded variation

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Abstract. We consider special flows over the rotation by an irrational \( \alpha \) under the roof functions of bounded variation without continuous, singular part in the Lebesgue decomposition and sum of jumps not equal to zero. We show that all such flows are weakly mixing. Under the additional assumption that \( \alpha \) has bounded partial quotients, we study the weak Ratner property. We establish this property whenever an additional condition (stable under sufficiently small perturbations) on the set of jumps is satisfied. While it is a classical result that the flows under consideration are not mixing, one more condition on the set of jumps turns out to be sufficient to obtain the absence of partial rigidity, hence mild mixing of such flows.

1. Introduction

In this paper we deal with measurable, measure-preserving flows on standard probability spaces. Hence, we are given a standard probability space \((X, \mathcal{B}, \mu)\) and an \(\mathbb{R}\)-representation \(\mathbb{R} \ni t \rightarrow S_t \in \text{Aut}(X, \mathcal{B}, \mu)\), where \(\text{Aut}(X, \mathcal{B}, \mu)\) denotes the group of measure-preserving automorphisms of \((X, \mathcal{B}, \mu)\). Moreover, we assume that this representation is weakly continuous, meaning that for each \(f, g \in L^2(X, \mathcal{B}, \mu)\) the function \(\mathbb{R} \ni t \rightarrow \langle f \circ S_t, g \rangle \in \mathbb{C}\) is continuous.

Among different classes of flows studied in ergodic theory, one of the most important is the class of horocycle flows acting on the unit tangent spaces of compact surfaces of constant negative curvature. Ratner [19–22] studied this class during the 1980s and established several ergodic rigidity phenomena in it. In particular, in [21], she discovered a special divergence of orbits (called, in [21], the \(H_p\) property) for horocycle flows and derived from it important dynamical consequences. The property discovered by Ratner was later named the \(R_p\) property or Ratner’s property (see [24]). Ratner’s property was further extended by Witte [26] to the so-called compact Ratner property and used in the study of the conjugacy problem for unipotent flows.

A natural question arises of whether there are flows satisfying Ratner’s property beyond the class of flows of algebraic origin. This question has been answered positively: Frączek
and Lemańczyk [5] have proved that Ratner’s property holds for each special flow given by an irrational rotation by \( \alpha \), with \( \alpha \) having bounded partial quotients, and the roof function piecewise absolutely continuous†, satisfying von Neumann’s condition (from [18])

\[
\int_{\mathbb{T}} f' \, d\lambda \neq 0,
\]

where \( \lambda \) stands for Lebesgue measure on \( \mathbb{T} \). As a matter of fact, Ratner’s property (from [21]) was slightly modified in [5] to the so-called finite Ratner property, although the dynamical consequences (the rigidity property of joinings; see Theorem 5.1 below) of the extended property remained valid. We emphasize that when dealing with the aforementioned class of special flows over irrational rotations, in fact we deal with flows different from (not isomorphic to) horocycle flows. Indeed, horocycle flows are mixing, in fact they have Lebesgue spectrum of infinite multiplicity (e.g. [12]). It now follows by the classical result due to Kochergin [14] that if we consider special flows over an irrational rotation with the roof function of bounded variation (and such are roof functions from the aforementioned class), then the resulting flows are not mixing. (We note in passing that if we want to be sure that we do not consider, up to isomorphism, horocycle flows, we need to put some smoothness condition on the roof function; otherwise, if we fix an irrational rotation by \( \alpha \) and let the roof function be arbitrary continuous then, up to isomorphism, we will obtain every horocycle flow; indeed, every horocycle flow is loosely Bernoulli [20].) More than that, it follows from [4] that every special flow over an irrational rotation with the roof function of bounded variation has singular spectrum and is spectrally disjoint from all mixing flows. This shows that flows considered in [5] are, from the dynamical point of view, completely different from horocycle flows.

Note that if \( f : \mathbb{T} \to \mathbb{R} \) is a function of bounded variation, it has only countably many discontinuity points, say, \( 0 \leq y_1, y_2, \ldots, \) all of them discontinuities of first order; let \( s_i \) denote the jump at \( y_i \). In fact, every countable subset of \( \mathbb{T} \), in particular dense subsets, can be the set of discontinuities of some \( f \) as above. In order to study Ratner’s property, we make two introductory remarks. First of all, it seems that the finite Ratner property is too strong to hold in such a general class of flows. That is why we need to weaken this property to the so-called weak Ratner property. We borrow the latter notion from [6], where it was used to study special flows over two-dimensional rotations‡.

**Definition 1.1.** [6] Let \( (X, d) \) be a \( \sigma \)-compact metric space, \( \mathcal{B} \) the \( \sigma \)-algebra of Borel subsets of \( X \), \( \mu \) a Borel probability measure on \( (X, d) \) and let \( S = (S_t)_{t \in \mathbb{R}} \) be a flow on \( (X, \mathcal{B}, \mu) \). Let \( P \subset \mathbb{R} \setminus \{0\} \) be a compact subset and \( t_0 \in \mathbb{R} \setminus \{0\} \). The flow \( (S_t)_{t \in \mathbb{R}} \) is said to have the property \( R(t_0, P) \) if for every \( \epsilon > 0 \) and \( n \in \mathbb{N} \) there exist \( \kappa = \kappa(\epsilon), \delta = \delta(\epsilon, N) > 0 \) and a subset \( Z = Z(\epsilon, N) \in \mathcal{B} \) with \( \mu(Z) > 1 - \epsilon \) such that if \( x, x' \in Z \), \( x' \) is not in the orbit of \( x \), and \( d(x, x') < \delta \), then there are \( M = M(x, x') \geq N, L = L(x, x') \geq N \) such that \( L/M \geq \kappa \) and there exists \( \rho = \rho(x, x') \in P \) such that

\[
\frac{1}{L} \left| \left\{ n \in \mathbb{Z} : d(S_{nt_0}(x), S_{nt_0+\rho}(x')) < \epsilon \right\} \right| > 1 - \epsilon.
\]

† \( f : [0, 1) \to \mathbb{R} \) is piecewise absolutely continuous if \( \{0, 1\} = \{a_0, a_1\} \cup \{a_1, a_2\} \cup \cdots \cup \{a_{N-1}, a_N\}, a_0 = 0, a_N = 1 \) and \( f|_{[a_i, a_{i+1})} \) is absolutely continuous, \( i = 0, \ldots, N - 1 \).

‡ Dynamical consequences (rigidity of joinings) hold for flows with the weak Ratner property.
Moreover, we say that \((S_t)_{t \in \mathbb{R}}\) has the weak Ratner (WR) property if the set of \(s \in \mathbb{R}\) such that the flow \((S_t)_{t \in \mathbb{R}}\) has the \(R(s, P)\) property is uncountable.

Secondly, note that \(\int_T f' \, d\lambda\) for piecewise absolutely continuous functions is equal to \(S(f)\), which is the (finite) sum of jumps. The number \(S(f) := \sum_{i=1}^{+\infty} s_i\) is well defined for each \(f\) of bounded variation (the series is absolutely convergent because \(f\) is of bounded variation). Therefore, it might seem that \(S(f) \neq 0\) (corresponding to (1) for piecewise absolutely continuous functions) is a ‘working’ condition in our general set-up. However, it turns out that the presence of a singular component in the Lebesgue decomposition

\[
f = f_j + f_a + f_s + S[\cdot],
\]

where \(f_j\) is the jump function, \(f_a\) is absolutely continuous on \(\mathbb{T}\), \(f_s\) is singular and continuous on \([0, 1]\) (see §5.1 for details), is another obstacle when studying the ergodic properties of such flows. Indeed, it follows from a result from [11] that for each \(\alpha\) with unbounded partial quotients there exists \(f = f_s, S(f_s) \neq 0\), and the corresponding special flow is not weakly mixing. More than that, this was strengthened in [25] to give, for each irrational rotation, examples of \(f = f_s, S(f_s) \neq 0\), isomorphic to the suspension flow.

With all this in mind, we restrict ourselves to the study of special flows with roof functions for which \(f_s = 0\), that is, \(f = f_j + f_a + S[\cdot]\) (note that \(f_s = 0\) for \(f\) piecewise absolutely continuous). Such functions form a Banach space \(\mathcal{V}\) with norm given by \(\|f\| = \text{Var}(f) + \|f\|_{L^1}\). Now, \(S(f) = \int_T f' \, d\lambda\) is a linear, continuous functional on \(\mathcal{V}\) and the von Neumann condition (1) is given by \(S(f) = S \neq 0\), where \(S\) comes from the decomposition (2). It follows that the set \(\mathcal{U}\) of functions in \(\mathcal{V}\) satisfying (1) is open in \(\mathcal{V}\); it is the complement of a hyperplane.

In order to describe the main results of this paper, recall that von Neumann in [18] proved that, for an arbitrary irrational \(\alpha\) and a piecewise absolutely continuous \(f\), (1) implies weak mixing of the corresponding special flow. We will generalize this result by showing (Proposition 6.2) that the special flow over the rotation by an arbitrary irrational \(\alpha\) with roof function in \(\mathcal{U}\) is weakly mixing.

To investigate stronger properties than weak mixing, we will constantly assume that \(\alpha\) has bounded partial quotients. Moreover, there will be some assumptions concerning the rate of convergence of the series of jumps of \(f\). This may look quite strange because the set of discontinuity points can be any countable subset of \(\mathbb{T}\). It will be shown, however, that we can introduce some well-ordering on the set of discontinuity points inherited from a natural well-ordering of the set of absolute values of the jumps. This ordering will be optimal when considering the speed of convergence of the series (see Remark 7.2). One of the main results of the paper is the following theorem.

**Theorem 1.2.** Let \(T : \mathbb{T} \to \mathbb{T}\) be the rotation by an irrational \(\alpha\) with bounded partial quotients, that is, \(C := \sup_i \{a_i\} + 1 < +\infty\), where \([0; a_1, a_2, \ldots]\) stands for the continued fraction expansion of \(\alpha\). Let \(f : \mathbb{T} \to \mathbb{R}_+\) be a function in \(\mathcal{U}\) bounded away from zero with

† In order to compare the present results with previous ones, we recall that in [8] \(f_j\) has finitely many discontinuity points and \(S \neq 0\) while in [7], \(S = 0, f_j\) has finitely many discontinuity points (with some connection to \(\alpha\)). Here \([x]\) denotes the fractional part of \(x \in \mathbb{R}\).
the set of jumps \( \{d_i\}_{i=1}^{+\infty} \) satisfying, for some \( j \in \mathbb{N} \),
\[
\sum_{i > j} |d_i| \leq \frac{|S|}{(2 + \theta)(2C + 1)((2C + 1)^j + 1)},
\]
for some \( \theta > 0 \). Then the special flow \( T_f^t \) has the \( R(\gamma, P) \) property\(^\dagger\) for every \( \gamma > 0 \), where \( P := (\text{sgn } S)p - D \), \( D := \{(\sum_{i=1}^{+\infty} n_id_i | 0 \leq n_i < 2C + 1) \) and \( p \in \mathbb{R} \) is such that \( (p - \eta, p + \eta) \subset (0, |S|) \setminus (D \cup -D) \) for some \( \eta > 0 \).

The second main result of the paper is the following.

**Theorem 1.3.** Assume \( T : \mathbb{T} \to \mathbb{T} \) is an ergodic rotation by \( \alpha \) having bounded partial quotients. Suppose that \( f \in U \) is a positive function, bounded away from zero. Moreover, assume that there exists a function \( \eta : \mathbb{R}_+ \to \mathbb{R}_+ \) such that \( \lim \inf_{\epsilon \to 0} \eta(\epsilon)\epsilon = 0 \) and
\[
\sum_{i > \eta(\epsilon)} |d_i| < \frac{\epsilon}{C_2/C_1 + 1}.
\]
Here \( C_1, C_2 > 0 \) are constants determined by diophantine approximation of \( \alpha \); see Lemma 4.1. Then the special flow \( (T_f^t)_{t \in \mathbb{R}} \) is not partially rigid.

Theorem 1.2 gives a stability result of the WR property in the class of piecewise absolutely continuous functions satisfying (1) in the space \( \mathcal{V} \). Namely, it will follow from Theorem 1.2 that there is an open set \( \mathcal{G} \subset \mathcal{V} \) of functions satisfying the WR property such that \( \mathcal{G} + \mathbb{R} \) is dense in \( \mathcal{V} \). In particular, the set of functions satisfying the WR property has non-empty interior in \( \mathcal{V} \) and contains every piecewise absolutely continuous function satisfying (1); see Remark 7.4.

The mild mixing property of a (finite) measure-preserving transformation was introduced by Furstenberg and Weiss in [10]. By definition, a transformation is mildly mixing if its Cartesian product with an arbitrary ergodic (finite or infinite not of type I) transformation remains ergodic. As is shown in [10], an equivalent condition is absence of non-trivial rigid factors, that is, \( \lim \inf_{n \to +\infty} \mu(T^{-n}B \triangle B) > 0 \) for every \( B \in \mathcal{B} \) with \( \mu(B) \notin \{0, \mu(X)\} \). The notion of mild mixing has been extensively studied by many authors; see, for example, [1, 9, 16, 17, 23]. A similar definition can be introduced and similar results hold for flows. It is immediate from the definition that the strong mixing property of a flow implies its mild mixing which implies the weak mixing property. It follows from [5] that for flows having the WR property, to prove mild mixing, it suffices to show the absence of partial rigidity (see §§2 and 3 for the definitions needed). Contrary to [5], where absence of partial rigidity is proved for all special flows under piecewise absolutely continuous functions and arbitrary irrational \( \alpha \), Theorem 1.3 is shown to hold for \( \alpha \) having bounded partial quotients.

2. Basic notions

Let \( \mathcal{S} = (S_t)_{t \in \mathbb{R}} \) be an ergodic (measurable) flow on a standard probability space \((X, \mathcal{B}, \mu)\). We recall that \( \mathcal{S} \) is called mixing if
\[
\lim_{t \to +\infty} \mu(S_tA \cap B) = \mu(A)\mu(B) \quad \text{for all } A, B \in \mathcal{B}.
\]
\(^\dagger\) This property is defined in §5.
If
\[
\lim_{T \to +\infty} \frac{1}{T} \int_0^T |\mu(S_t A \cap B) - \mu(A)\mu(B)| \, dt = 0,
\]
for all \(A, B \in \mathcal{B}\), then \((S_t)_{t \in \mathbb{R}}\) is called weakly mixing. Recall also that the weak mixing of \(\mathcal{S}\) is equivalent to the ergodicity of \((S_t \times S'_t)_{t \in \mathbb{R}}\) acting on \((X \times X', \mathcal{B} \otimes \mathcal{B}', \mu \times \mu')\) for each ergodic \((S'_t)_{t \in \mathbb{R}}\) acting on a standard probability space \((X', \mathcal{B}', \mu')\). It easily follows that mixing implies mild mixing which in turn implies weak mixing.

Assume that \(\mathcal{A} \subset \mathcal{B}\) is a factor of \((S_t)_{t \in \mathbb{R}}\), that is, \(\mathcal{A}\) is an \((S_t)_{t \in \mathbb{R}}\)-invariant sub-\(\sigma\)-algebra. Let \((t_n)_{n \in \mathbb{N}}\) be a sequence of real numbers such that \(t_n \to +\infty\). We say that the factor \(\mathcal{A}\) is rigid along \((t_n)\) if
\[
\lim_{n \to +\infty} \mu(A \cap S_{-t_n} A) = \mu(A)
\]
for every \(A \in \mathcal{A}\). In particular, \((S_t)\) is rigid along \((t_n)\) if \(\mathcal{A} = \mathcal{B}\). A flow \((S_t)_{t \in \mathbb{R}}\) is called partially rigid along \((t_n)\) if there exists \(0 < u \leq 1\) such that
\[
\lim \inf_{n \to +\infty} \mu(B \cap S_{-t_n} B) \geq u \mu(B)
\]
for every \(B \in \mathcal{B}\).

Assume that \(T\) is an ergodic automorphism on \((X, \mathcal{B}, \mu)\) and \(f : X \to \mathbb{R}\) a measurable function. One can define a \(\mathbb{Z}\)-cocycle \(f^{(\cdot)} : \mathbb{Z} \times X \to \mathbb{R}\) given by
\[
f^{(m)}(x) = \begin{cases} f(x) + f(Tx) + \cdots + f(T^{m-1}x) & \text{if } m > 0, \\ 0 & \text{if } m = 0, \\ -(f(T^mx) + \cdots + f(T^{-1}x)) & \text{if } m < 0. \end{cases}
\]

Now assume additionally that \(f : X \to \mathbb{R}\) is a strictly positive \(L^1\) function, and denote by \(\lambda\) the Lebesgue measure on \(\mathbb{R}\). Then we define a special flow \(T^f = (T^f_t)_{t \in \mathbb{R}}\) acting on \((X^f, \mathcal{B}^f, \mu^f)\), where \(X^f := \{(x, s) \in X \times \mathbb{R} : 0 \leq s < f(x)\}\), and \(\mathcal{B}^f (\mu^f)\) is the restriction of \(\mathcal{B} \otimes \mathcal{B}(\mathbb{R}) (\mu \times \lambda)\) to \(X^f\), by setting
\[
T^f_t(x, s) = (T^nx, s + t - f^{(n)}(x)).
\]
Here \(n \in \mathbb{Z}\) is unique such that \(f^{(n)}(x) \leq s + t < f^{(n+1)}(x)\).

It is well known (due to [18]) that the special flow \((T^f_t)\) is weakly mixing if and only if for every \(s \in \mathbb{R} \setminus \{0\}\) the equation
\[
\frac{\psi(Tx)}{\psi(x)} = e^{2\pi isf(x)}
\]
has no measurable solution \(\psi : X \to S^1 := \{z \in \mathbb{C} : |z| = 1\}\).

3. Joinings
Let \(\mathcal{S} = (S_t)_{t \in \mathbb{R}}, T = (T_t)_{t \in \mathbb{R}}\) be two ergodic flows defined on \((X, \mathcal{B}, \mu)\) and \((Y, \mathcal{C}, \nu)\), respectively. By a joining between \(\mathcal{S}\) and \(T\) we mean any probability \((S_t \times T_t)_{t \in \mathbb{R}}\)-invariant measure on \((X \times Y, \mathcal{B} \otimes \mathcal{C})\) whose projections on \(X\) and \(Y\) are equal to \(\mu\) and \(\nu\), respectively. The set of joinings between \(\mathcal{S}\) and \(T\) is denoted by \(J(\mathcal{S}, T)\). The subset of ergodic joinings is denoted by \(J^e(\mathcal{S}, T)\).
Given \( S = (S_t)_{t \in \mathbb{R}} \) and \( A \) a factor of it, one can define a measure \( \mu \times _A \mu \in J(S, S) \) by

\[
(\mu \times _A \mu)(D) = \int _{X/A} (\mu_{\tilde{x}} \times \mu_{\tilde{x}})(D) \, d\tilde{\mu}(\tilde{x})
\]

for \( D \in \mathcal{B} \otimes \mathcal{B} \), where \( \{ \mu_{\tilde{x}} : \tilde{x} \in X/A \} \) stands for the disintegration of the measure \( \mu \) over the image \( \tilde{\mu} \) of \( \mu \) via \( \Pi : X \to X/A \). The measure \( \mu \times _A \mu \) is called the relatively independent joining.

For every \( t \in \mathbb{R} \), we denote by \( \mu_{S_t} \in J^\epsilon(S, S) \) the graph joining determined by \( (S_t)_{t \in \mathbb{R}} \), that is, \( \mu_{S_t}(A \times B) = \mu(A \cap S_{-t}B) \) for \( A, B \in \mathcal{B} \). Then \( \mu_{S_t} \) is concentrated on the graph of \( S_t \).

Recall that in general the notions of (absence of) partial rigidity and mild mixing are not related. We have, however, the following result.

**Lemma 3.1.** [5] Let \( S \) be an ergodic flow on \((X, \mathcal{B}, \mu)\) which is a finite extension of each of its non-trivial factors. If the flow \( S \) is not partially rigid then it is mildly mixing.

4. **Continued fraction expansion of an irrational number \( \alpha \)**

Let \( \mathbb{T} \) be the circle group given as the quotient space \( \mathbb{R}/\mathbb{Z} \) identified with \([0, 1)\) with addition mod 1. For \( \alpha \in \mathbb{R}\setminus \mathbb{Q} \), let \((a_n)_{n=0}^{+\infty}\) stand for the sequence of partial quotients in the continued fraction expansion of \( \alpha \) [13]. Let \((p_n)_{n=0}^{+\infty}, (q_n)_{n=0}^{+\infty}\) denote the sequence of numerators and denominators, respectively. By definition, we have the properties

\[
\frac{1}{2q_nq_{n+1}} < |\alpha - \frac{p_n}{q_n}| < \frac{1}{q_nq_{n+1}},
\]

where \( q_0 = 1, q_1 = a_1, q_{n+1} = a_{n+1}q_n + q_{n-1}, p_0 = 0, p_1 = 1, p_{n+1} = a_{n+1}p_n + p_{n-1} \).

One says that \( \alpha \) has bounded partial quotients if the sequence \((a_n)_{n=1}^{+\infty}\) is bounded. Setting \( C := \sup\{a_n : n \in \mathbb{N}\} + 1 \), we get \( q_{n+1} \leq Cq_n \) and

\[
\frac{1}{2Cq_n} \leq \frac{1}{2q_{n+1}} < \|q_n\alpha\| < \frac{1}{q_{n+1}} < \frac{1}{q_n}
\]

for each \( n \in \mathbb{N} \), where \( \|t\| \) denotes the distance of \( t \) to integer numbers. The following lemma is well known.

**Lemma 4.1.** Let \( \alpha \in \mathbb{T} \) be irrational with bounded partial quotients. Then there exist positive constants \( C_1, C_2 \) such that for every \( k \in \mathbb{N} \) the lengths \( |J_j| \) of the intervals \( J_1, \ldots, J_k \) arising from the partition of \( \mathbb{T} \) by \( 0, -\alpha, \ldots, -(k-1)\alpha \) satisfy \( C_2/k \leq |J_j| < C_1/k \) for each \( j = 1, \ldots, k \).

5. **The WR property**

In this section we recall the WR property (see the Introduction) and we list the results from [6] that are needed in what follows. As in [6] we will constantly assume that \((S_t)_{t \in \mathbb{R}}\) satisfies the following ‘almost continuity’ condition: for every \( \epsilon > 0 \) there exists \( X(\epsilon) \in \mathcal{B} \) with \( \mu(X(\epsilon)) > 1 - \epsilon \) such that for every \( \epsilon' > 0 \) there exists \( \epsilon_1 > 0 \) such that \( d(S_t x, S_{t'} x) < \epsilon' \) for all \( x \in X(\epsilon) \) and \( t, t' \in [-\epsilon_1, \epsilon_1] \).
Note that if \((T^f_t)_{t \in \mathbb{R}}\) is a special flow acting on \((X^f, \mathcal{B}^f, \mu^f)\) equipped with the metric 
\[d_1((x, t), (y, s)) = d(x, y) + |t - s|,\]
then the above condition holds.

**Theorem 5.1.** [6] Let \((X, d)\) be a \(\sigma\)-compact metric space, \(\mathcal{B}\) the \(\sigma\)-algebra of Borel subsets of \(X\) and \(\mu\) a probability Borel measure on \((X, d)\). Let \((S_t)_{t \in \mathbb{R}}\) be a weakly mixing flow on the space \((X, \mathcal{B}, \mu)\) that satisfies the \(R(P)\) property, where \(P \subset \mathbb{R}\setminus\{0\}\) is a non-empty compact set. Assume that \((S_t)_{t \in \mathbb{R}}\) satisfies the ‘almost continuity’ condition. Let \((T_t)_{t \in \mathbb{R}}\) be an ergodic flow on \((Y, \mathcal{C}, v)\) and let \(\rho\) be an ergodic joining of \((S_t)_{t \in \mathbb{R}}\) and \((T_t)_{t \in \mathbb{R}}\). Then either \(\rho = \mu \times v\) or \(\rho\) is a finite extension of \(v\).

**Remark 5.2.** It follows by [5, Remark 2] that if \(\mathcal{S}\) is an ergodic flow on \((X, \mathcal{B}, \mu)\) and for each ergodic flow \(T\) acting on \((Y, \mathcal{C}, v)\) an arbitrary ergodic joining \(\rho\) is either the product measure or a finite extension of \(v\), then \(\mathcal{S}\) is a finite extension of each of its non-trivial factors.

**Proposition 5.3.** [6] Let \((X, d)\) be a \(\sigma\)-compact metric space, \(\mathcal{B}\) be the \(\sigma\)-algebra of Borel subsets of \(X\) and \(\mu\) a probability Borel measure on \((X, d)\). Assume that \(T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)\) is an ergodic isometry and \(f : X \to \mathbb{R}\) is a bounded positive measurable function which is bounded away from zero. Let \(P \subset \mathbb{R}\setminus\{0\}\) be a compact set. Assume that for every \(\epsilon > 0\) and \(N \in \mathbb{N}\) there exist \(\kappa = \kappa(\epsilon) > 0\), \(\delta = \delta(\epsilon, N) > 0\) and a subset \(Z = Z(\epsilon, N) \in \mathcal{B}\) with \(\mu(Z) > 1 - \epsilon\) such that if \(x, x' \in Z\), and \(d(x, x') < \delta\), then there are \(M = M(x, x') \geq N\), \(L = L(x, x') \geq N\) such that \(L/M \geq \kappa\) and there exists \(p = p(x, x') \in P\) such that
\[
\frac{1}{L} |\{n \in Z \cap [M, M + L] : |f^{(n)}(x) - f^{(n)}(x') - p| < \epsilon\}| > 1 - \epsilon.
\]

Suppose that \(\gamma \in \mathbb{R}\) is a positive number such that the \(\gamma\)-time automorphism \(T^f_\gamma : X^f \to X^f\) is ergodic. Then the special flow \(T^f_\gamma\) has the \(R(\gamma, P)\) property.

5.1. **Properties of bounded variation functions without singular continuous component in the Lebesgue decomposition.** Let \(T : \mathbb{T} \to \mathbb{T}\) be an irrational rotation by \(\alpha\) with the sequence of denominators \((q_n)_{n=1}^{+\infty}\) and \(f : \mathbb{T} \to \mathbb{R}\) a function with bounded variation \((f \in BV(\mathbb{T}))^\dagger\). It is well known that \(f \in L^1(\mathbb{T})\) and the following inequality holds.

**Proposition 5.4.** (Denjoy–Koksma inequality; see [3, 15]) If \(f : \mathbb{T} \to \mathbb{R}\) is a function of bounded variation then
\[
\left| \sum_{k=0}^{q_n-1} f(x + k\alpha) - q_n \int_T f \, d\lambda \right| \leq \text{Var} \, f,
\]
for every \(x \in \mathbb{T}\) and \(n \in \mathbb{N}\).

Let us identify the function \(f : \mathbb{T} \to \mathbb{R}\) with the function \(\tilde{f} : [0, 1] \to \mathbb{R}\) by \(\tilde{f}(x) = f(x)\) if \(x \in [0, 1]\), and \(\tilde{f}(1) = \lim_{x \to -1^+} f(x)\) which exists since \(f \in BV(\mathbb{T})\).

\(^\dagger\) We constantly assume that all functions of bounded variation considered are left-continuous.
Then $\tilde{f} \in BV([0, 1])$. (Note that if $f$ is continuous from the left, so is $\tilde{f}$. ) By the Lebesgue decomposition (see, for example, [8]), it follows that

$$f = f^a + f^j + f_s + S\{\cdot\},$$

where $f^j := \tilde{f}^j|_{(0, 1)}$ is the jump function. Note that $\tilde{f}_j$ has countably many discontinuity points, say $\beta_2, \beta_3, \ldots$, and the series of jumps is absolutely convergent (since $\tilde{f} \in BV([0, 1])$). Furthermore, $\tilde{f}_s$ is singular and continuous, and $\tilde{f}^a$ absolutely continuous on $[0, 1]$. This decomposition is unique up to constants. It follows that we can decompose $f$ as

$$f = f^a + f^j + f_s + S\{\cdot\},$$

where $f^a := \tilde{f}^a|_{(0, 1)}$ is the jump function on $\mathbb{T}$ with the set of jumps $\{\beta_i\}^\infty_{i=1}$ at $\{\beta_i\}^\infty_{i=1}$ respectively ($\beta_1 = 0$ and $d_1$ can be equal to zero), $f^j := \tilde{f}^j|_{(0, 1)}$ is singular, continuous on $\mathbb{T}\setminus[0]$ (the only discontinuity point can be 0), and $f_s := \tilde{f}^s|_{(0, 1)} - S\{\cdot\}$ is absolutely continuous on $\mathbb{T}$ ($S = \tilde{f}^a(1) - \tilde{f}^a(0)$). Recall that $BV(\mathbb{T})$ is a Banach space with the norm $\|f\| := \text{Var } f + \|f\|_L$. Let us set $BV_{a+j}(\mathbb{T}) := \{f \in BV(\mathbb{T}) : f^a = 0\} (=U)$, which is a Banach space (it is a closed subspace of $BV(\mathbb{T})$). For the functions in $BV_{a+j}(\mathbb{T})$ we have that $S := \sum_{i=1}^{\infty} d_i$ is just the sum of the jumps of the function $f$ and $\text{Var } f = \sum_{i=1}^{\infty} |d_i| + \|f^a\|_L + 2S$. Consider the set $U := \{f \in BV_{a+j}(\mathbb{T}) : S \neq 0\}$. Notice that all piecewise absolutely continuous functions satisfying (1), which from now on will be called von Neumann’s functions, belong to $U$. Von Neumann’s functions are precisely all functions in $BV_{a+j}(\mathbb{T})$ for which there are only finitely many discontinuity points. Moreover, $U$ is an open set in $BV_{a+j}(\mathbb{T})$. In fact, it is the complement of a hyperplane.

**Proposition 5.5.** The set of von Neumann’s functions is dense in $U$.

**Proof.** Let $f \in U$ be a function with the set of discontinuities $\{\beta_i\}^\infty_{i=1}$ and the set of corresponding jumps $\{d_i\}^\infty_{i=1}$. We will construct a sequence $(f_n)^\infty_{n=1} \subset U$ of von Neumann’s functions which tends to $f$ as $n \to +\infty$. The series $\sum_{i=1}^{\infty} |d_i|$ is convergent, so there exists $j_n \in \mathbb{N}$ such that $\sum_{i=j_n}^{\infty} |d_i| < 1/3n$. Let us consider the discontinuity points $\beta_i$, $i = 1, \ldots, j_n$. By permuting them, write $0 \leq \beta_{k_1} < \beta_{k_2} < \cdots < \beta_{k_{j_n}}$, where $\{1, \ldots, f_n\} = \{k_1, \ldots, k_{j_n}\}$. Let $g_{j_n}(x) = \sum_{i=k}^{j_n} d_i$ if $\beta_{k_i} \leq x < \beta_{k_{i+1}}$, $r = 1, \ldots, j_n$, $\beta_{j_n+1} = \beta_{k_1}$ and set $S_n := \sum_{i=1}^{j_n} d_i$. Then let

$$f_n(x) := f^a(x) + g_{j_n}(x) + S_n\{x\} \in BV_{a+j}(\mathbb{T}).$$

It follows that the only points of discontinuity of $f_n$ are $\beta_{k_1}, \ldots, \beta_{k_{j_n}}$ (perhaps $f_n$ is continuous at $\beta_{k_i}$). Moreover, $|S - S_n| < 1/3n$. Since $S \neq 0$, there exists $n_0 \in \mathbb{N}$ such that $S_n \neq 0$ if $n \geq n_0$. Thus, $f_n$ is a von Neumann’s function. Now consider $(f - f_n)(x) = (f^j - g_{j_n})(x) + (S - S_n)(x)$. We have

$$\text{Var}(f - f_n) \leq \text{Var}((S - S_n)\{\cdot\}) + \text{Var}((f^j - g_{j_n})) = 2|S - S_n| + \sum_{i>j_n} |d_i| \leq \frac{1}{n},$$

so $f_n \to f$ in $BV_{a+j}(\mathbb{T})$ (obviously $f_n \to f$ in $L^1$). $\square$

$f(y^\pm) := \lim_{z \to y^\pm} f(z)$.
As an arbitrary \( f \in \mathcal{U} \) is left-continuous, \n\[ f := f_{ac} + f_{pl}, \]\nwhere \( f_{ac} : \mathbb{T} \rightarrow \mathbb{R} \) is absolutely continuous with zero mean and \( f_{pl} : \mathbb{T} \rightarrow \mathbb{R} \) belongs to \( \mathcal{U} \) and \( f'_{pl}(x) = S \) for all \( x \in \mathbb{T} \setminus \{ \beta_i \}_{i=1}^{+\infty} \). Indeed, we set \n\[ f_{pl}(x) := \sum_{i=1}^{+\infty} d_i [x - \beta_i] + c, \]
for some constant \( c \in \mathbb{R} \) (\( c := \int_T f_a(x) \, d\lambda \), cf. (2)). Note that the points of discontinuity and the size of jumps of \( f \) and \( f_{pl} \) are the same. Indeed, this is an easy consequence of the decomposition (2) \( f_{pl}(x) = f_j(x) + S(x) + c \).

We consider \( f \in BV_{a+j}(\mathbb{T}) \). Then \( \int_T f' \, d\lambda = S \). Moreover, for each interval \([a, b] \subset [0, 1] \) we have \n\[
\int_a^b f'(x) \, d\lambda = f(b^+) - f(a^+) - \sum_{\{i: \beta_i \in [a, b]\}} d_i
= f(b^+) - f(a^+) - \sum_{x \in [a, b]} f_j(x^+) - f_j(x). \tag{6}
\]

Indeed, \n\[
\int_a^b f' \, d\lambda = \int_a^b f'_a \, d\lambda + \int_a^b S \, d\lambda = f_a(b^-) - f_a(a^+) + S(b-a)
= f_a(b^-) - f_a(a^+) + S(b-a) + f_j(b^-) - f_j(a^+) - (f_j(b^-) - f_j(a^+))
= f_a(b^-) - f_a(a^+) - \sum_{\{i: \beta_i \in [a, b]\}} d_i.
\]

5.2. Von Neumann’s functions considered up to cohomology do not exhaust \( \mathcal{U} \). We say that \( f : \mathbb{T} \rightarrow \mathbb{R} \) is a coboundary if there exists a measurable function \( j : \mathbb{T} \rightarrow \mathbb{R} \) such that \( f(x) = j(x) - j(Tx) \) for almost every \( x \in \mathbb{T} \). One says that \( f \) and \( g \) are cohomologous if their difference is a coboundary. In the previous section, we have shown that von Neumann’s functions are dense in \( \mathcal{U} \), whence a natural question arises of whether there are some functions \( f \in \mathcal{U} \) which are not cohomologous to any von Neumann’s function.

To answer this question positively, let us consider a function \( f \in \mathcal{U} \) with an infinite set of discontinuities \{\( \beta_i \)\}_{i=1}^{+\infty} \subset \mathbb{Q} \) such that \( \beta_i - \beta_j \notin \mathbb{Z} + \mathbb{Z} \alpha \) whenever \( i \neq j \). Let the jump \( d_i \) at \( \beta_i \) be positive for each \( i \in \mathbb{N} \). Moreover, we assume that for every \( \epsilon > 0 \) there exists \( N_\epsilon \) such that \n\[
\sum_{i > N_\epsilon} d_i \leq \epsilon \min\{d_1, \ldots, d_{N_\epsilon}\}. \tag{7}
\]

Take a von Neumann’s function \( g \) with the set of discontinuities \{\( \gamma_j \)\}_{j=1}^{k} \) and the corresponding set of jumps \{\( m_j \)\}_{j=1}^{k} \), \( S(g) = \int_T g' \, d\lambda = \sum_{j=1}^{k} m_j \neq 0 \), and suppose that \( (f - g)(x) = j(x) - j(Tx) \) for some measurable function \( j : \mathbb{T} \rightarrow \mathbb{R} \). Let \( (q_n) \) be the sequence of denominators of \( \alpha \). Then \( (f - g)^{(q_n)} \rightarrow 0 \) in measure as \( n \rightarrow +\infty \). We must have that \( S(f) = S(g) \), otherwise \( f - g \in \mathcal{U} \) and by Theorem 6.2 below we get
a contradiction. By (2), $f - g = (f_j - g_j) + (f_a - g_a) = (f_j - g_j + c) + (f_a - g_a - c)$\footnote{$c := \int f_a - g_a \, d\lambda$.}. Using the Denjoy–Koksma inequality (Proposition 5.4), $(f_a - g_a - c)_{(q_n)} \to 0$ uniformly as $n \to +\infty$. It follows that $(f_j - g_j + c)_{(q_n)} \to 0$ in measure as $n \to +\infty$, and we will show that this is impossible under our additional assumption (7).

Remark 5.6. (e.g. [2]) Without loss of generality we can assume that the difference between any two discontinuity points of $f_j - g_j$ is not a multiple of $\alpha$. Indeed, if $\delta - \delta' \in \mathbb{Z} + \mathbb{Z}\alpha$, there are still infinitely many points of discontinuity of $q_n$ be the intervals with the right $+\beta \geq \rightarrow -\infty$ arising from $s - f I$ in $I \beta$.

Since the number of discontinuities of $f$ is infinite and $\beta_i - \beta_j \notin \mathbb{Z} + \mathbb{Z}\alpha$, after applying Remark 5.6, there are still infinitely many points of discontinuity of $f_j - g_j$ left and the difference between any two of them is not a multiple of $\alpha$. Without loss of generality, we can assume that the set of points of discontinuity is $\{\beta_i\}_{i=1}^{+\infty} \cup \{y_j\}_{j=1}^{k}$.

Lemma 5.7. [7, Lemma 2.3] Let $\alpha$ be irrational with bounded partial quotients and let $\beta \in (\mathbb{Q} + \mathbb{Q}\alpha) \setminus (\mathbb{Z} + \mathbb{Z}\alpha)$. Then there exists $c > 0$ such that for each $m \in \mathbb{N}$ the length of each interval in the partition of $\mathbb{T}$ arising from $0, -\alpha, \ldots, -(m - 1)\alpha, \beta, \beta - \alpha, \ldots, \beta - (m - 1)\alpha$ is at least $c/m$.

Let us fix $\epsilon_0 > 0$ such that $N_{\epsilon_0} \geq 4k$ and, for $m \in \mathbb{N}$, let $I_m$ be the partition of the circle given by the points $\beta_i - j\alpha$, $i = 1, \ldots, N_{\epsilon_0}$, $j = 0, \ldots, m - 1$.

Remark 5.8. By the above lemma, there exists a constant $c' := c'(\beta_1, \ldots, \beta_{N_{\epsilon_0}}) > 0$ such that the length of each interval in partition $I_m$ is at least $c'/m$. It also follows by Lemma 4.1 that the length of each interval in $I_m$ is at most $C_1/m^{\frac{\epsilon}{2}}$.

Let us consider the points $y_i - s\alpha$, $i = 1, \ldots, k$, $s = 0, \ldots, q_n - 1$; the number of such points is $kq_n$. The number of intervals in $I_{q_n}$ is at least $4kq_n$ (because $N_{\epsilon_0} \geq 4k$). Fix $1 \leq i \leq N_{\epsilon_0}$. For $s = 0, \ldots, q_n - 1$ let $I_{i,s}, I'_{i,s} \subseteq I_{q_n}$ be the intervals with the right endpoint $\beta_i - s\alpha$ and the left endpoint $\beta_i - s\alpha$, respectively. Then there exists $1 \leq i' \leq N_{\epsilon_0}$ such that $|A_{q_n,i'}| > q_n/2$, where

$$A_{q_n,i'} := \left\{ 0 \leq s \leq q_n - 1 : \left( \bigcup_{j=1}^{k} \{y_j - r\alpha\}_{r=0}^{q_n-1} \right) \cap (I'_{i,s} \cup I'_{i,s}) = \emptyset \right\}.$$  

Set $H_{q_n}(i') := \bigcup_{s \in A_{q_n,i'}} I'_{i,s}$ and $H'_{q_n}(i') := \bigcup_{s \in A_{q_n,i'}} I'_{i,s}$. By definition,

$$\left( \bigcup_{s=1}^{q_n-1} \{y_i - s\alpha\}_{i=1}^{k} \right) \cap H_{q_n}(i') = \emptyset, \quad \left( \bigcup_{s=1}^{q_n-1} \{y_i - s\alpha\}_{i=1}^{k} \right) \cap H'_{q_n}(i') = \emptyset. \quad (8)$$

It follows that $|H_{q_n}(i')|, |H'_{q_n}(i')| \geq q_n/2$ (each interval in the above families has length at least $c'/q_n$). Let us fix $I_{i',s} \subseteq H_{q_n}(i')$ and $I'_{i',s} \subseteq H'_{q_n}(i')$. Take any $x \in I_{i',s}$, $y \in I'_{i',s}$. By Remark 5.8, the length of the interval $[x, y]$ is at most $2/q_n$. Moreover, the only points of discontinuity of $(f - g)_{(q_n)}$ in $[x, y]$ are $\beta_{i'} - s\alpha$ and some of the
form $\beta_i - r\alpha$, $i > N_\epsilon$, $r = 0, \ldots, q_n - 1$. Let us fix $i_1 > N_\epsilon$ and consider the points $\beta_{i_1} - t\alpha, \beta_{i_1} - r\alpha$ for some $t \neq r$. Then

$$\|(\beta_{i_1} - t\alpha) - (\beta_{i_1} - r\alpha)\| \geq \|q_{n-1}\alpha\| \geq 1/(2q_n).$$

It follows that the number of discontinuities of $(f - g)(q_n)$ in $[x, y)$ of the form $\beta_{i_1} - r\alpha$ for some $r = 0, \ldots, q_n - 1$ is at most 5. Thus

$$(f - g)(q_n)(y) - (f - g)(q_n)(x) = \sum_{i=1}^{+\infty} \left( \sum_{r=0}^{q_n-1} \chi_{[x, y)}(\{\beta_i - r\alpha\}) \right) d_i$$

$$+ \sum_{j=1}^{k} \left( \sum_{r=0}^{q_n-1} \chi_{[x, y)}(\{\gamma_j - r\alpha\}) \right) m_j$$

$$= d_i + \sum_{i > N_\epsilon} n_i d_i,$$

for some integers $|n_i| \leq 5$. We get that $(f - g)(q_n)(y) - (f - g)(q_n)(x) \geq (1 - 5\epsilon_0)d_{i'}$, which is a contradiction with $((f - g)(q_n))_n \lambda \to \delta_0$ as $n \to +\infty$.

6. **Weak mixing of special flows when the roof function is in $U$**

Let $T$ be an ergodic automorphism acting on $(X, \mathcal{B}, \mu)$ and $f : X \to \mathbb{R}_+$ in $L^1(X, \mathcal{B}, \mu)$. Assume that $T$ is rigid and let $(q_n)_{n=1}^{+\infty}$ be a rigid sequence. We will state a criterion for weak mixing of special flows over rigid systems.

**Lemma 6.1.** (Cf. [6, Proposition 2.1]) Under the above assumptions, assume additionally that there exists $0 < c < 1$ such that

$$\limsup_{n \to +\infty} \left| \int_X e^{2\pi i r f(q_n)(x)} \, d\mu(x) \right| < c,$$

for all $|r| \in \mathbb{R}$ large enough. Then the special flow $(T_t^f)_{t \in \mathbb{R}}$ is weakly mixing.

**Proof.** Suppose to the contrary that for some $s \neq 0$ and a measurable $\psi : X \to S^1$,

$$\frac{\psi(T x)}{\psi(x)} = e^{2\pi is f(x)}.$$

Then, for all $k \in \mathbb{Z} \setminus \{0\}$, 

$$\left| \int_X (\psi(T^{q_n x}) \overline{\psi(x)})^k \, d\mu(x) \right| = \left| \int_X e^{2\pi i k s f(q_n)(x)} \, d\mu(x) \right| < c < 1,$$

for $n, k$ large enough. Since clearly $(\psi \circ T^{q_n})^k \psi^k \to 1$ in measure as $n \to +\infty$, we obtain a contradiction.

We will now prove that all special flows under the roof functions from $U$ are weakly mixing. The proof of this fact is based on the proof of Theorem 3 from [17] concerning the ergodicity of real cocycles over irrational rotations.

**Proposition 6.2.** Let $T : \mathbb{T} \to \mathbb{T}$ be an arbitrary irrational rotation by $\alpha$. Let $f : \mathbb{T} \to \mathbb{R}_+$, $f \in U$. Then the special flow $T^f$ is weakly mixing.
Proof. By Proposition 5.5, there exists a sequence \((f_n)_{n=1}^{\infty}\) of von Neumann’s functions which tends to \(f\) in \(BV_{a+j}(\mathbb{T})\). It follows that there exists \(n_0\) such that for \(n \geq n_0\), \(|S(f_n)| \neq 0\) and \(\text{Var}(f - f_n) < |S(f_n)|\) (this holds because \(\text{Var}(f - f_n) \to 0\) and \(S(f_n) \to S(f) \neq 0\)). Let \(g := f_{n_0+1}\) and \(S := |S(g)|\). Then \(g\) is a von Neumann’s function; let \(K\) denote its number of discontinuities. We will prove that if \(c\) is any constant satisfying \(\text{Var}(f - g)/S < c < 1\), then

\[
\limsup_{n \to +\infty} \left| \int_{\mathbb{T}} e^{2\pi irf^{(q_n)}(x)} d\lambda(x) \right| < c
\]

for every \(|r|\) large enough. This will show weak mixing in view of Lemma 6.1.

Recall (see (5)) that \(f(x) = f_{pl}(x) + f_{ac}(x)\), where \(f_{ac} : \mathbb{T} \to \mathbb{R}\) is absolutely continuous with zero mean and \(f_{pl}(x) = f_j(x) + S(f)\{x\} + c' (c' = \int_{\mathbb{T}} f_{ac}(x) d\lambda(x))\). It follows from the Denjoy–Koksma inequality that \(f_{ac}^{(q_n)} \to 0\) uniformly as \(n \to +\infty\), so to prove (9), without loss of generality, assume that \(f = f_{pl}\). By the proof of Proposition 5.5, \(g\) is piecewise linear \((g(x) = g_j(x) + S\{x\} + c')\). Let \(h := f - g\); then, by definition, \(\text{Var}(h) < S\). Let

\[
I_{r,q} = \int_{\mathbb{T}} e^{2\pi ir(g^{(q)}(x) + h^{(q)}(x))} d\lambda(x)
\]

for \(r \neq 0\) and \(q \geq 1\).

Since \(g\) is piecewise linear and \(g' = S\), we get \((1/q)|g^{(q)}| = |S|\) so, for all \(q\),

\[
|g^{(q)}(x)| = |S||q|
\]

for almost every \(x \in \mathbb{T}\).

Denote by \(x_1 \leq \cdots \leq x_qK\) the points of discontinuity of \(g^{(q)}\). They divide the interval \([0, 1]\) into subintervals \([x_j, x_{j+1})\), \(j = 1, \ldots, qK\). It may happen that some of these intervals are empty. However, if \([x_j, x_{j+1})\) is not degenerate then \(g^{(q)}\)'s \((x_j, x_{j+1})\) is absolutely continuous and \((g^{(q)}')'(x) = qS\) for all \(x \in (x_j, x_{j+1})\). So

\[
\int_{x_j}^{x_{j+1}} e^{2\pi ir(g^{(q)}(x) + h^{(q)}(x))} d\lambda(x) = \int_{x_j}^{x_{j+1}} \frac{e^{2\pi irh^{(q)}}}{2\pi i g^{(q)}} d(e^{2\pi i rh^{(q)}}).
\]

Integrating by parts on each interval \((x_j, x_{j+1})\), we obtain

\[
I_{r,q} = \frac{1}{2\pi ir} \sum_{j=1}^{Kq} \left( \frac{e^{2\pi irg^{(q)}(x_{j+1}) + h^{(q)}(x_{j+1})}}{g^{(q)}_-(x_{j+1})} - \frac{e^{2\pi irg^{(q)}(x_j) + h^{(q)}(x_j)}}{g^{(q)}_+(x_j)} \right)
\]

\[\vphantom{\sum_{j=1}^{Kq}} - \frac{1}{2\pi ir} \int_0^1 e^{2\pi irg^{(q)}} d(e^{2\pi irh^{(q)}}/g^{(q)})\]

It follows that (cf. the proof of Theorem 3 from [17])

\[
|I_{r,q}| \leq \frac{1}{2\pi |r|} \left( \frac{2Kq}{Sq} + \text{Var}(e^{2\pi irh^{(q)}}/g^{(q)}) \right)
\]
and

\[ \text{Var}(e^{2\pi irh(q)}/g'(q)) \leq \sup \left( \frac{1}{|g'(q)|} \right) \text{Var}(e^{2\pi irh(q)}) + \text{Var}(1/g'(q)) \]

\[ \leq \frac{1}{S^2} 2\pi |r| \text{Var}(h(q)) + \frac{1}{S^2 q^2} \text{Var}(g'(q)) \]

\[ \leq \frac{2\pi |r|}{S} \text{Var}(h) + \frac{1}{S^2 q} \text{Var}(g'). \]

Finally,

\[ |I_{r,q}| \leq \frac{K}{\pi |r|S} + \frac{\text{Var}(h)}{S} + \frac{\text{Var}(g')}{2\pi |r|S^2 q} = \frac{\text{Var}(h)}{S} + O(1/r) \]

uniformly in \( q \). Since \( \text{Var}(h)/S < c \), we get \( \limsup_{q \to +\infty} |I_{r,q}| \leq c < 1 \) for \( r \) large enough. \( \square \)

7. The WR property for roof functions from a subfamily of \( \mathcal{U} \)
We will prove the WR property of the special flow \((T_i^f)_{i \in \mathbb{R}} \) with \( f \) belonging to some subfamily of \( \mathcal{U} \) (§5.1) which will be specified below. From now on, we assume that \( \alpha \) has bounded partial quotients and let \( C := \sup \{a_n\}_{n=1}^{+\infty} + 1 \). First, let us recall the following result.

**Lemma 7.1.** [5, Lemma 6.1] Let \( T : \mathbb{T} \to \mathbb{T} \) be the rotation by \( \alpha \) with bounded partial quotients and let \( f : \mathbb{T} \to \mathbb{R} \) be absolutely continuous with zero mean. Then

\[ \sup_{0 \leq n < q_{i+1}} \sup_{\|y-x\| < 1/q_i} |f^{(n)}(y) - f^{(n)}(x)| \to 0 \quad \text{as} \quad s \to +\infty. \]

Let \( f : \mathbb{T} \to \mathbb{R} \) be a positive function in \( \mathcal{U} \) with the set of discontinuities \( \{\beta_i\}_{i=1}^{+\infty} \) and the corresponding set of jumps \( \{d_i\}_{i=1}^{+\infty} \). We make an additional assumption (cf. Theorem 1.2) on the jumps of \( f \). Namely, we assume that there exist \( j \in \mathbb{N} \) and \( \theta > 0 \) such that

\[ \sum_{i > j} |d_i| \leq \frac{|S|}{(2 + \theta)(2C + 1)((2C + 1)^j + 1)}. \]

**Remark 7.2.** There is some natural well-ordering on the set of points \( \{\beta_i\}_{i=1}^{+\infty} \) optimal from the point of view of (10). We may assume that \( |d_i| \geq |d_{i+1}| \) for \( i = 1, 2, \ldots \). Indeed, let \( \sigma : \mathbb{N} \to \mathbb{N} \) be a permutation such that \( |d_{\sigma(i)}| \geq |d_{\sigma(i+1)}| \) for all \( i \in \mathbb{N} \). Let \( S' = \sum_{i=1}^{+\infty} |d_i| \). Then

\[ \sum_{i > j} |d_{\sigma(i)}| = S' - \sum_{i=1}^{j} |d_{\sigma(i)}| \leq S' - \sum_{i=1}^{j} |d_i| = \sum_{i > j} |d_i| < \frac{|S|}{(2 + \theta)(2C + 1)^j + 1}. \]

It follows that if (10) is satisfied for some permutation of jumps, it is also satisfied for \( d_i \geq d_{i+1}, \) where \( i \in \mathbb{N} \).

Let us now define

\[ D := \left\{ \sum_{i=1}^{+\infty} m_i d_i : 0 \leq m_i < 2C + 1 \right\}. \]
LEMMA 7.3. Under the above assumptions there exist \( p, \eta > 0 \) such that \( (p - \eta, p + \eta) \subset (0, |S|) \setminus (D \cup -D) \), and consequently the set \( \{ \text{sgn} \, S \} p - D \) is bounded away from 0.

Proof. Let \( j, \theta \) satisfy (10). We consider the set of sums \( A := \{ \sum_{i=1}^{j} m_i d_i, \, 0 \leq m_i < 2C + 1, \, i = 1, \ldots, j \} \) and \( B := A \cup -A \) \( (|B| \leq 2(2C + 1))^j \) and let \( \xi := |S|/((2 + \theta)(2C + 1))^j + 1) \). Then there exist \( p \in (0, |S|) \) and \( \eta > 0 \) such that \( (p - \eta, p + \eta) \cap (B + (-\xi, \xi)) = \emptyset \). Indeed, the number of points from \( B \) in the interval \( (0, |S|) \) is at most \( (2C + 1)^j \) (because \( B \) is symmetric), so there exists an interval \( I = [a, b] \subset (0, |S|) \) of length at least \( |S|/((2 + \theta)(2C + 1))^j + 1 \) and \( B \cap I = \emptyset \). Hence, if we take \( p \) to be midpoint of \( I \), and any \( 0 < \eta < |S|/((2 + \theta)(2C + 1))^j + 1 \) then \( p - \eta > a + \xi, \, p + \eta < b - \xi \), so \( (p - \eta, p + \eta) \subset (a + \xi, b - \xi) \). But from the definition of \( I \) we get \( (a + \xi, b - \xi) \cap (B + (-\xi, \xi)) = \emptyset \), and from (10), \( D \cup -D \subset B + (-\xi, \xi) \). Finally, \( (p - \eta, p + \eta) \subset (0, |S|) \setminus (D \cup -D) \). □

We now recall (see Theorem 1.2) that \( P := \text{cl} \{ \text{sgn} \, (S)p - D \} \). Then \( 0 \notin P \) and \( P \) is compact.

Proof of Theorem 1.2. (The proof contains some ideas from [5].) By (5),
\[
    f(x) = f_{pl}(x) + f_{ac}(x) = \sum_{i=1}^{+\infty} d_i \{ x - \beta_i \} + c + f_{ac}(x),
\]
for some constant \( c \in \mathbb{R} \).

Let us fix \( \varepsilon > 0 \) and \( N \in \mathbb{N} \). Because the series \( \sum_{i=1}^{+\infty} |d_i| \) converges, there exists a number \( m(\varepsilon) \in \mathbb{N} \) such that
\[
    \sum_{i>m(\varepsilon)} |d_i| < \frac{\varepsilon}{4(2C + 1)}.
\]
Let \( \kappa(\varepsilon) := 1/(m(\varepsilon)(2C + 1)) \min[\varepsilon/2pC, 1/C^2] \).

From Lemma 7.1 it follows that there exists \( s_0 \) such that, for all \( s \geq s_0 \),
\[
    \sup_{0 \leq n < q_{s+1}} \sup_{\|y-x\| \leq 1/q_s} |f_{ac}^{(n)}(y) - f_{ac}^{(n)}(x)| < \varepsilon/4, \quad (11)
\]
and \( \min[\kappa(\varepsilon), 1]q_{s_0} > N \). Let \( \delta := \varepsilon(N, q_{s_0}) = p/|S|q_{s_0+1} \) and take \( x, y \in \mathbb{T} \) with \( 0 < \|x - y\| < \delta \). Let \( s \) denote the unique natural number such that
\[
    \frac{p}{|S|q_{s+1}} < \|x - y\| \leq \frac{p}{|S|q_{s}}; \quad (12)
\]
then \( s \geq s_0 \). Without loss of generality assume that \( x < y \) and \( S > 0 \); the proof for \( S < 0 \) is analogous. We will consider the sequence \( (f_{pl}^{(n)}(y) - f_{pl}^{(n)}(x))_{n \in \mathbb{N}} \). We have
\[
    f_{pl}^{(n+1)}(y) - f_{pl}^{(n+1)}(x) = f_{pl}^{(n)}(y) - f_{pl}^{(n)}(x) + \sum_{i=0}^{+\infty} d_i(\{ y + n\alpha - \beta_i \} - \{ x + n\alpha - \beta_i \})
\]
\[
    = f_{pl}^{(n)}(y) - f_{pl}^{(n)}(x) + \sum_{i=0}^{+\infty} d_i(y - x - \chi_{(x,y)}(\{\beta_i - n\alpha\})). \quad (13)
\]
So setting \( m_i = m_i(n) = \sum_{j=0}^{n-1} \chi_{(x,y)}(\{\beta_i - j\alpha\}) \), we get
\[
    f_{pl}^{(n)}(y) - f_{pl}^{(n)}(x) = nS(y - x) - \bar{d}_n, \quad (14)
\]
where

$$\overline{d}_n := d_n(x, y) = \sum_{i=1}^{\infty} m_i d_i.$$  

Let us now fix $i \in \mathbb{N}$ and assume that $\{\beta_i - \ell \alpha\}, \{\beta_i - r \alpha\} \in (x, y)$, with $0 \leq \ell, r < q_{s+1}$, $\ell \neq r$. Then

$$\|\{\beta_i - \ell \alpha\} - \{\beta_i - r \alpha\}\| > \frac{1}{2q_{s+1}} \geq \frac{1}{2Cq_s}.$$  

So the number $m_i = m_i(q_{s+1})$ of discontinuities of the function $f_{pl}^{(q_{s+1})}$ in the interval $(x, y)$ which are of the form $\{\beta_i - j \alpha\}$, for some $0 \leq j < q_{s+1}$, is smaller than

$$2Cq_s |y - x| + 1 \leq 2C \frac{p}{|S|} + 1 \leq 2C + 1.$$  

Hence $(\overline{d}_n) \in D, n = 1, \ldots, q_{s+1}$. Because of (14) and (12), we have

$$f_{pl}^{(q_s)}(y) - f_{pl}^{(q_s)}(x) + \overline{d}_s = q_s S(y - x) \leq p,$$

$$f_{pl}^{(q_{s+1})}(y) - f_{pl}^{(q_{s+1})}(x) + \overline{d}_{s+1} = q_{s+1} S(y - x) > p.$$  

Moreover, for every natural $n$,

$$0 < f_{pl}^{(n+1)}(y) - f_{pl}^{(n+1)}(x) + \overline{d}_{n+1} - (f_{pl}^{(n)}(y) - f_{pl}^{(n)}(x) + \overline{d}_n) = S(y - x) \leq \frac{p}{q_s}.$$  

Hence, there exists an integer interval $I \subset [q_s, q_{s+1}]$ such that

$$|f_{pl}^{(n)}(y) - f_{pl}^{(n)}(x) + \overline{d}_n - p| < \frac{\epsilon}{2} \quad \text{for } n \in I,$$

and

$$|I| \geq \min \left( \frac{\epsilon}{2p}, q_s, q_{s+1} - q_s \right) \geq \min \left( \frac{\epsilon}{2pC}, \frac{1}{C^2} \right) q_{s+1}.$$  

Since $s \geq s_0$, by (11) we have

$$|f^{(n)}(y) - f^{(n)}(x) + \overline{d}_n - p| < \frac{3\epsilon}{4} \quad \text{for } n \in I.$$  

Let us consider all $i, 1 \leq i \leq m(\epsilon)$. Then the number of discontinuities of $f^{(q_{s+1})}$ coming from $\beta_i - j \alpha$, $0 \leq j < q_{s+1}$, in the interval $(x, y)$ is at most $(2C + 1)m(\epsilon)$. It follows that we can split the interval $I$ into at most $m(\epsilon)(2C + 1)$ integer intervals on which the discontinuities come only from $\beta_i - j \alpha$, with $i > m(\epsilon)$ and $j = 0, \ldots, q_{s+1}$. It follows that if $a, b$ are in any such integer interval then $m_i(a) = m_i(b), i = 1, \ldots, m(\epsilon)$. Thus we can choose an integer subinterval $J \subset I$ such that if $\min J := M$ and if $M, M + r \in J$ then by (10),

$$\overline{d}_{M+r} - \overline{d}_M = \sum_{i=1}^{+\infty} (m_i(M + r) - m_i(M))d_i \leq \sum_{i>m(\epsilon)} (2C + 1)|d_i| \leq \frac{\epsilon}{4}$$  

and, moreover,

$$|J| > \frac{1}{m(\epsilon)(2C + 1)} \min \left( \frac{\epsilon}{2pC}, \frac{1}{C^2} \right) q_{s+1} = \kappa(\epsilon)q_{s+1}. $$
Set \( d = d_M \). Then, in view of (15),
\[
|f^{(m)}(y) - f^{(m)}(x)| - p - d| \leq |f^{(m)}(y) - f^{(m)}(x)| - p - d_m| + |d_m - d| < \epsilon,
\]
\( m \in J \). Now let \( J = [M, M + L] \cap \mathbb{Z} \); then
\[
\frac{L}{M} \geq \frac{|J|}{q_{s+1}} \geq \kappa(\epsilon), \quad M \geq q_s \geq q_{s_0} > N, \quad L \geq |J| \geq \kappa(\epsilon)q_{s+1} > \kappa(\epsilon)q_{s_0} > N.
\]

Because the special flow \( T^\gamma \) is weakly mixing by Proposition 6.2, the automorphism \( T^\gamma \)
is ergodic for all \( \gamma \neq 0 \), and an application of Proposition 5.3 completes the proof.

**Remark 7.4.** The above theorem yields a stability result for the roof functions belonging to the set \( \mathcal{U} \) and satisfying (10). Namely, consider a function \( f \in \mathcal{U} \) with the set of discontinuity points \( \{\beta_j\}_{j=1}^{+\infty} \) and the corresponding set of jumps \( \{d_j\}_{j=1}^{+\infty} \) satisfying (10) with \( \theta_f, j_f \) and a function \( g \in BV_{a+j}(\mathbb{T}) \) such that
\[
\text{Var} \ g \leq \min \left\{ \frac{|S(f)|}{(2 + \eta_g)(2C + 1)((2C + 1)j_f + 1)}, \ d_{j_f} \right\} \quad (15)
\]
for some \( \eta_g > (\theta_f + 7)/\theta_f \). Then \( f + g \in \mathcal{U} \) and \( f + g \) satisfies (10) for some \( \theta_{f+g}, j_{f+g} \). Indeed, notice that \( |S(f + g)| = |S(f) + S(g)| \geq |S(f)| - |S(g)| \geq |S(f)| - \text{Var} \ g > 0 \), so \( f + g \in \mathcal{U} \). Moreover, let \( \{r_i\}_{i=1}^{+\infty} \) denote the set of discontinuities of \( f + g \) and \( \{m_i\}_{i=1}^{+\infty} \) the set of jumps of the function \( g \) and we reorder the set of discontinuities of \( f + g \) to get a decreasing set (see Remark 7.2). By setting \( j_{f+g} = j_f \), taking any \( 0 < \theta_{f+g} < 1/(4 + \theta_f + \eta_g) \) and using consecutively (15), (10) and (15), the definition of \( \theta_{f+g} \), and again (15), we obtain
\[
\sum_{i > j_{f+g}} |r_i| \leq \sum_{i > j_f} |d_i| + \sum_{i=1}^{+\infty} |m_i| \\
\leq \frac{|S(f)|}{(2 + \theta_f)(2C + 1)((2C + 1)j_f + 1)} + \frac{|S(f)|}{(2 + \eta_g)(2C + 1)((2C + 1)j_f + 1)} \\
\leq \frac{|S(f)|}{(2 + \theta_{f+g})(2C + 1)((2C + 1)j_{f+g} + 1)} \\
\leq \frac{|S(g)|}{(2 + \theta_{f+g})(2C + 1)((2C + 1)j_{f+g} + 1)} \\
\leq \frac{|S(f + g)|}{(2 + \theta_{f+g})(2C + 1)((2C + 1)j_{f+g} + 1)}.
\]

8. **Absence of partial rigidity for the roof functions from a subfamily of \( \mathcal{U} \)**

In this section, we will show the absence of partial rigidity of special flows over irrational rotation by \( \alpha \) having bounded partial quotients \( (C := \sup \{a_n\} + 1) \) and under roof functions \( f : \mathbb{T} \rightarrow \mathbb{R}_+, \ f \in \mathcal{U} \). We will use the following general lemma.
Lemma 8.1. [5, Lemma 7.1] Let \( S: (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu) \) be an ergodic automorphism and \( g \in L^1(X, \mu) \) such that \( g \geq c > 0 \). Suppose that the special flow \((S^t g)_t \in \mathbb{R}\) is partially rigid along a sequence \((t_n)_n, t_n \to +\infty\). Then there exists \( 0 < u \leq 1 \) such that, for every \( 0 < \epsilon < c \),
\[
\liminf_{n \to +\infty} \mu(\{x \in X : \exists j \in \mathbb{N} \mid |g^{(j)}(x) - t_n| < \epsilon\}) \geq u.
\]

Recall that \( \{\beta_i\}_{i=1}^{+\infty} \) denotes the set of discontinuity points and the corresponding set of jumps is \( \{d_i\}_{i=1}^{+\infty} \). Let \( C_1, C_2 \) be as in Lemma 4.1. By assumption (3) (see Theorem 1.3) there exists a function \( \eta: \mathbb{R}_+ \to \mathbb{R}_+ \) such that \( \liminf_{\epsilon \to 0} \eta(\epsilon) \epsilon = 0 \) and
\[
\sum_{i > \eta(\epsilon)} |d_i| < \frac{\epsilon}{C_2/C_1 + 1}.
\]

We note that if such an \( \eta \) exists it also exists for a monotonic permutation of the set of discontinuities of \( f \) (cf. Remark 7.2). Under the above assumption we prove the following result.

Proof of Theorem 1.3. Let \( m, M \) be positive numbers such that \( 0 < m \leq f(x) \leq M \) for every \( x \in \mathbb{T} \). We will proceed by contradiction assuming that \((t_n)_n, t_n \to +\infty\) is a partial rigidity time for \((T_f^t)_t \in \mathbb{R}\). By Lemma 8.1, there exists \( 0 < u \leq 1 \) such that, for every \( 0 < \epsilon < m \),
\[
\liminf_{n \to +\infty} \lambda(\{x \in \mathbb{T} : \exists j \in \mathbb{N} \mid |f^{(j)}(x) - t_n| < \epsilon\}) \geq u.
\]

Let us fix \( \epsilon > 0 \) such that \( \epsilon < m/10 \) and
\[
0 < \eta(\epsilon) \epsilon < \frac{|S|m^2}{48M(m + \text{Var } f)} + |S|m^2u.
\]

Since \( f' \in L^1(\mathbb{T}, \lambda) \), there exists \( 0 < \delta < \epsilon \) such that
\[
\lambda(A) < \delta \quad \text{implies} \quad \int_A |f'| \, d\lambda < \epsilon.
\]

Moreover, by the Egorov theorem and the ergodicity of \( T \) (and recalling that \( S = \int_{\mathbb{T}} f' \, d\lambda \)) it follows that there exist \( A_\epsilon \subset \mathbb{T} \) with \( \lambda(A_\epsilon) > 1 - \delta \) and \( m_0 \in \mathbb{N} \) such that
\[
\frac{S}{2} \leq \frac{1}{k} f^{(k)}(x) \quad \text{if } S > 0,
\]
\[
\frac{S}{2} \geq \frac{1}{k} f^{(k)}(x) \quad \text{if } S < 0,
\]
for all \( k \geq m_0 \) and \( x \in A_\epsilon \).

Take any \( n \in \mathbb{N} \) such that \( t_n/2M \geq m_0, t_n > 2\epsilon \). Let us consider the set \( J_{n,\epsilon} \) of all \( j \in \mathbb{N} \) such that \( |f^{(j)}(x) - t_n| < \epsilon \) for some \( x \in \mathbb{T} \). For such \( j \) and \( x \),
\[
t_n + \epsilon \geq f^{(j)}(x) \geq mj \quad \text{and} \quad t_n - \epsilon < f^{(j)}(x) \leq Mj,
\]
whence
\[
\frac{t_n}{2M} \leq \frac{t_n - \epsilon}{M} < j < \frac{t_n + \epsilon}{m} \leq \frac{2t_n}{m}
\]
for any \( j \in J_{n,\epsilon} \); in particular, \( j \in J_{n,\epsilon} \) implies \( j \geq m_0 \).
Now let \( j_n := \max J_{n,e} \). Let us consider the points of discontinuity of \( f^{(j_n)} \), that is, \{\beta_i - j\alpha\}, \( i = 1, \ldots, +\infty, 0 \leq j < j_n \). Consider first the points for which \( i \leq \eta(\epsilon) \). They divide \( \mathbb{T} \) into subintervals \( I^n_1, \ldots, I^n_{\eta(\epsilon)j_n} \). Some of these intervals may be empty. Note that the only discontinuities of \( f^{(j_n)} \) that are contained in the interiors of \( I^n_1, \ldots, I^n_{\eta(\epsilon)j_n} \) come from the set \{\beta_t\}_{t > \eta(\epsilon)} \). For every fixed \( k \in \mathbb{N} \), the number of points of the form \{\beta_k - r\alpha\}, \( 0 \leq r < j_n \), in the interval \( I^n_j \) is not bigger than \( C_2/C_1 + 1 \). Indeed, if \{\beta_k - t\alpha\}, \{\beta_k - r\alpha\} \( \in I^n_j, t \neq r \), then (see Lemma 4.1) 

\[
\|\{\beta_k - t\alpha\} - \{\beta_k - r\alpha\}\| \geq \frac{C_1}{j_n}.
\]

so the number of such points is not bigger than \((j_n/C_1)|I^n_j| + 1 \leq C_2/C_1 + 1 \). Let us now fix \( 1 \leq i \leq \eta(\epsilon) j_n \). For every \( j \in J_{n,e} \), let \( I^n_{i,j} \) stand for the minimal closed interval of \( \overline{I^n_i} \) which includes the set \( \{x \in I^n_i : |f^{(j)}(x) - t_n| < \epsilon\} \). Of course, \( I^n_{i,j} \) may be empty.

Moreover, for every \( k \geq 1 \), let \( m_k = m_k(n, w, i, j) = \sum_{s=0}^{w-1} \chi_{I^n_{i,s}}((\beta_k - s\alpha)) \). It follows that \( m_k(n, j_n, i, j) \leq C_2/C_1 + 1 \).

If \( I^n_{i,j} = [z_1, z_2] \) is not empty then, using (6) and (3),

\[
\frac{1}{j} \int_{I^n_{i,j}} f^{(j)} d\lambda \leq \frac{|f^{(j)}(z_1^+) - f^{(j)}(z_2^-)|}{j} + \frac{\sum_{k > \eta(\epsilon)} m_k(n, j, i, j) d_k}{j} \\
\leq \frac{|f^{(j)}(z_1^+) - t_n| + |t_n - f^{(j)}(z_2^-)|}{j} + \frac{\sum_{k > \eta(\epsilon)} m_k(n, j, i, j) d_k}{j} \\
\leq \frac{2\epsilon}{j} + \frac{(C_2/C_1 + 1)\epsilon}{C_2/C_1 + 1} \leq \frac{3\epsilon}{j} \leq \frac{6\epsilon}{t_n} \tag{21}
\]

because the only discontinuity points of \( f^{(j)} \) in \( I^n_{i,j} \) come from \( \beta_k - s\alpha, k > \eta(\epsilon) \), \( 0 \leq s < j \).

Now suppose that \( x \) is the endpoint of \( I^n_{i,j} \) and \( y \) is the endpoint of \( I^n_{i,j} \) with \( j \neq j' \).

Then, by (6) and (3), it follows that

\[
\int_x^y |f'|^{(j_n)} d\lambda \geq \int_x^y f^{(j)} d\lambda \geq |f^{(j)}(y^-) - f^{(j)}(x^+)| - \sum_{k > \eta(\epsilon)} m_k(n, j_n, i, j) d_k | \\
\geq |f^{(j)}(y^-) - f^{(j')}(y^-)| - |f^{(j')}(y^-) - t_n| - |f^{(j)}(x^+) - t_n| - \epsilon \\
\geq m - |f^{(j')}(y^-) - t_n| - |f^{(j)}(x^-) - t_n| - |f^{(j)}(x^+) - f^{(j)}(x)| - \epsilon \\
\geq m - 4\epsilon \geq \frac{m}{2} \tag{22}
\]

Let \( K_i = \{j \in J_{n,e} : I^n_{i,j} \neq \emptyset\} \) and denote \( |K_i| = r \geq 1 \). Then there exist \( r - 1 \) pairwise disjoint subintervals \( H_i \subset I^n_i, i = 1, \ldots, r - 1 \), which are disjoint from \( I^n_{i,j}, j \in K_i \), and fill up the space between those intervals. In view of (22),

\[
\int_{H_i} |f'|^{(j_n)} d\lambda \geq \frac{m}{2} \tag{23}
\]
for $t = 1, \ldots, r - 1$. Therefore, in view of (21) and (23),
\[
\left| \sum_{j \in K_t} \int_{I_{i,j}^n} \frac{f^{(j)}}{j} \, d\lambda \right| \leq 6M \epsilon \left( \frac{6M \epsilon}{t_n} + \frac{12M \epsilon}{m} \frac{r - 1}{2} \right) \\
\leq 6M \epsilon \left( \frac{6M \epsilon}{t_n} + \frac{12M \epsilon}{m} \sum_{t=1}^{r-1} \int_{H_i} |f'(j_n)| \, d\lambda \right) \\
\leq 6M \epsilon \left( \frac{6M \epsilon}{t_n} + \frac{12M \epsilon}{m} \int_{I_{i}^n} |f'(j_n)| \, d\lambda \right). \quad (24)
\]

Since $\lambda(A_{\epsilon}^c) < \delta$, and $1/j < 2M/t_n$,
\[
\sum_{i=1}^{\eta(\epsilon)j_n} \sum_{j \in K_t} \int_{I_{i,j}^n \cap A_{\epsilon}} \frac{f^{(j)}}{j} \, d\lambda \leq \frac{2M \eta(\epsilon)j_n}{t_n} \sum_{i=1}^{\eta(\epsilon)j_n} \sum_{j \in K_t} \int_{I_{i,j}^n \cap A_{\epsilon}} |f'(j_n)| \, d\lambda \\
\leq \frac{2M}{t_n} \int_{A_{\epsilon}^c} |f'(j_n)| \, d\lambda \\
\leq \frac{2M}{t_n} j_n \epsilon \leq \frac{4M}{m} \epsilon. \quad (25)
\]

Note that (see (17))
\[
B_n := \{ x \in \mathbb{T} : \exists j \in \mathbb{N} | f^{(j)}(x) - t_n | < \epsilon \} \subset \bigcup_{i=1}^{\eta(\epsilon)j_n} \bigcup_{j \in K_t} I_{i,j}^n.
\]

Now we can conclude as in [5] namely, using (20), (24), (25) that
\[
\frac{|S|}{2} \lambda(B_n \cap A_{\epsilon}) \leq \sum_{i=1}^{\eta(\epsilon)j_n} \sum_{j \in K_t} \int_{I_{i,j}^n \cap A_{\epsilon}} \frac{|f^{(j)}|}{j} \, d\lambda \leq \left| \sum_{i=1}^{\eta(\epsilon)j_n} \sum_{j \in K_t} \int_{I_{i,j}^n} \frac{f^{(j)}}{j} \, d\lambda \right| \\
+ \sum_{i=1}^{\eta(\epsilon)j_n} \sum_{j \in K_t} \int_{I_{i,j}^n \cap A_{\epsilon}} \frac{|f'(j_n)|}{j} \, d\lambda \\
\leq \eta(\epsilon) \frac{6M \epsilon}{t_n} \frac{12M \epsilon}{m} \int_{\mathbb{T}} |f'(j_n)| \, d\lambda + \frac{4M \epsilon}{m} \\
\leq \frac{12\eta(\epsilon)M \epsilon}{m} + \frac{4M \epsilon}{m^2} + \frac{24M \epsilon}{m} \| f' \|_{L^1} \leq \frac{48\eta(\epsilon)M}{m^2} (m + \text{Var } f) \epsilon + \epsilon < u.
\]

Finally, using (18), we obtain
\[
\lambda(B_n) \leq \lambda(B_n \cap A_{\epsilon}) + \lambda(A_{\epsilon}^c) < \frac{48\eta(\epsilon)M}{|S|m^2} (m + \text{Var } f) \epsilon + \epsilon < u.
\]

The yields a contradiction with (17) which completes the proof.

From Theorems 1.2, 1.3 and Lemma 3.1 we get the following result.

**Corollary 8.2.** Suppose that $T : \mathbb{T} \to \mathbb{T}$ is the rotation by an irrational number $\alpha$ with bounded partial quotients and $f : \mathbb{T} \to \mathbb{R}$ is a positive function from the set $\mathcal{U}$, bounded away from zero, satisfying conditions (10) and (3). Then the special flow $(T_t^f)_{t \in \mathbb{R}}$ is mildly mixing.
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References

[1] J. Aaronson, M. Lin and B. Weiss. Mixing properties of Markov operators and ergodic transformations, and ergodicity of Cartesian products. Israel J. Math. 33 (1979), 198–224.
[2] J. P. Conze and A. Piękniewska. On multiple ergodicity of affine cocycles over irrational rotations. Preprint, 2012, arXiv:1209.3798.
[3] I. P. Cornfield, S. V. Fomin and Ya. G. Sinai. Ergodic Theory. Springer, New York, 1982.
[4] K. Frączek and M. Lemańczyk. A class of special flows over irrational rotations which is disjoint from mixing flows. Ergod. Th. & Dynam. Sys. 24 (2004), 1083–1095.
[5] K. Frączek and M. Lemańczyk. On mild mixing of special flows over irrational rotations under piecewise smooth functions. Ergod. Th. & Dynam. Sys. 26 (2006), 1–21.
[6] K. Frączek and M. Lemańczyk. Ratner’s property and mild mixing for special flows over two-dimensional rotations. J. Mod. Dyn. 4 (2010), 609–635.
[7] K. Frączek, M. Lemańczyk and E. Lesigne. Mild mixing property for special flows under piecewise constant functions. Discrete Contin. Dyn. Syst. 19 (2007), 691–710.
[8] D. H. Fremlin. Measure Theory. Vol. 2. Cambridge University Press, London, 1974.
[9] H. Furstenberg. IP-systems in ergodic theory. Conference in Modern Analysis and Probability (New Heaven, CT, 1982) (Contemporary Mathematics, 26). American Mathematical Society, Providence, RI, 1984, pp. 131–148.
[10] H. Furstenberg and B. Weiss. The finite multipliers of infinite ergodic transformations. The Structure of Attractors in Dynamical Systems (Proc. Conf., North Dakota State University, Fargo, ND, 1977) (Lecture Notes in Mathematics, 668). Springer, Berlin, 1978, pp. 127–132.
[11] A. Iwanik, M. Lemańczyk and D. Rudolph. Absolutely continuous over irrational rotations. Israel J. Math. 83 (1993), 73–95.
[12] A. Katok and J.-P. Thouvenot. Spectral properties and combinatorial constructions in ergodic theory. Handbook of Dynamical Systems. Vol. 1B. Elsevier, Amsterdam, 2006, pp. 649–743.
[13] Y. Khintchin. Continued Fractions. Chicago University Press, Chicago, 1960.
[14] A. V. Kochergin. On the absence of mixing in special flows over the rotation of a circle and in flows on a two-dimensional torus. Dokl. Akad. Nauk SSSR 205 (1972), 949–952.
[15] L. Kuipers and H. Niederreiter. Uniform Distribution of Sequences. Wiley, London, 1975.
[16] M. Lemańczyk and E. Lesigne. Ergodicity of Rokhlin cocycles. J. Anal. Math. 85 (2001), 43–86.
[17] A. Iwanik, M. Lemańczyk and C. Mauduit. Piecewise absolutely continuous cocycles over irrational rotations. J. Lond. Math. Soc. (2) 59 (1999), 171–187.
[18] J. von Neumann. Zur Operatorenmethode in der klassischen Mechanik. Ann. of Math. (2) 33 (1932), 587–642.
[19] M. Ratner. Factors of horocycle flows. Ergod. Th. & Dynam. Sys. 2 (1982), 465–489.
[20] M. Ratner. Horocycle flows are loosely Bernoulli. Israel J. Math. 31 (1978), 122–132.
[21] M. Ratner. Horocycle flows, joinings and rigidity of products. Ann. of Math. (2) 118 (1983), 277–313.
[22] M. Ratner. Rigidity of horocycle flows. Ann. of Math. (2) 115 (1982), 597–614.
[23] K. Schmidt and P. Walters. Mildly mixing actions of locally compact groups. Proc. Lond. Math. Soc. (3) 45 (1982), 506–518.
[24] J.-P. Thouvenot. Some properties and applications of joinings in ergodic theory. Ergodic Theory and its Connections with Harmonic Analysis (Alexandria, 1993) (London Mathematical Society Lecture Note Series, 205). Cambridge University Press, Cambridge, 1995, pp. 207–235.
[25] D. Volny. BV coboundaries over irrational rotations. Studia Math. 126 (1997), 253–271.
[26] D. Witte. Rigidity of some translations on homogeneous spaces. Invent. Math. 81 (1985), 1–27.