Individual-centered partial information in social networks

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Summary.

Most existing statistical network analysis literature assumes a global view of the network, under which community detection, testing, and other statistical procedures are developed. Yet in the real world, people frequently make decisions based on their partial understanding of network information. As individuals barely know beyond friends’ friends, we assume that an individual of interest knows all paths of length up to $L = 2$ that originate from them. As a result, this individual’s perceived adjacency matrix $B$ differs significantly from the usual adjacency matrix $A$ based on the global information. The new individual-centered partial information framework sparks an array of fascinating endeavors from theory to practice. Key general properties on the eigenvalues and eigenvectors of $B_E$, a major term of $B$, are derived. These general results, coupled with the classic stochastic block model, lead to a new theory-backed spectral approach to detecting the community memberships based on an anchored individual’s partial information. Real data analysis delivers interesting insights that result from individuals’ heterogeneous knowledge, yet these insights cannot be obtained from global network analysis.

Keywords: Individual-centered, partial information, knowledge depth, social network, community detection, spectral approach

1. Introduction

Despite technological advances, most people do not have a clear understanding of the global pictures of the various networks in which they are embedded. More precisely, in contrast to the mainstream statistical network research that takes a bird-eye view of networks, individuals in society do not know much beyond a local neighborhood, e.g., friends’ friends. Yet we frequently make decisions based on such individual-centered partial network information. For instance, when we decide whether to pass certain sensitive or controversial messages to another person, our decision relies in part on our limited knowledge of the aggregated social network, which helps us make a judgment about that person’s loyalty to us.

In this work, we aim to understand the social networks from an individual’s perspective. To formalize individuals’ lack of knowledge about distant connections and to address people’s need to take action hinged on such partial knowledge, we introduce an individual-centered partial network analysis. 

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information framework to study social networks. Concretely, let $G = (V, E)$ denote the full (i.e., global) network of interest, where $V = \{1, \ldots, n\}$ is the index set of all individuals (a.k.a., vertices, or nodes) in the full network and $E$ is the set of unweighted and undirected edges between individuals. We characterize an individual’s partial knowledge (i.e., partial information) by their knowledge depth. We say an individual $i$ has knowledge depth $L$ if they know all paths of length (up to) $L$ that originate from them. Figure 1 illustrates the knowledge depth concept with a toy example. Suppose individual 1 is the individual of interest (i.e., centered-individual). The left panel is the full network. The left, middle and right panels show individual 1’s perceived networks based on knowledge depths $L = 3, 2, 1$, respectively. Note that in this example, individuals 5 and 6 are not within 1’s knowledge depth $L = 1$.

In the verbal discussion, we make a distinction between individual 1’s perceived network (adjacency matrix) based on knowledge depth $L$ and their perceived network within knowledge depth $L$. The former means collections of the individuals and the edges within the knowledge depth $L$, together with the rest of the individuals whose edges are not perceived by individual 1 within their knowledge depth $L$. In other words, the former includes all individuals in the full network and the edges within individual 1’s knowledge depth $L$. In contrast, the latter means the former with the perceived isolated individuals removed. For instance, in the toy example for $L = 1$, the former is the right panel of Figure 1 and the latter would have individuals 5 and 6 removed. As such, the former is associated with a $6 \times 6$ adjacency matrix and the latter a $4 \times 4$. Both the perceived network based on knowledge depth $L$ and that within knowledge depth $L$ have real-world relevance. For example, when the entire collection of individuals under investigation is available, such as Ph.D. students in the Statistics Department at a certain university or senators in congress, we can assume that a centered-individual knows the existence of everybody. In contrast, in a large network, such as the Facebook friendship network, it is not reasonable to assume that a centered-individual knows the existence of everyone on Facebook.

What do different knowledge depths $L$’s mean to individuals embedded in the network and to researchers? The $L = 1$ scenario is not that interesting, at least not from a statistical network analysis point of view. For example, except in trivial settings, there is no hope to perform community detection tasks with $L = 1$. Our everyday experience tells that $L > 3$ is beyond the scope of most people’s understanding of a network. Both $L = 2$ and $L = 3$ are interesting cases. Our current work addresses $L = 2$; we leave $L = 3$ for future research.

We start with a common problem in statistical network analysis: community detection. We

†We use “they”, “them” and “their” as gender neutral singular pronoun and its derivative forms.
begin by asking whether an individual can identify the community memberships of the individuals within their knowledge depth? This turns out to be a difficult mathematical question to answer because, under any generative model, the set of individuals within a given person’s knowledge depth is random, and we would have to deal with randomness in both the edge set and the node set. We bypass this mathematical challenge by switching to a conceptually more difficult but mathematically more convenient question: can the individual identify the community memberships of all the individuals in the full network? As we illustrate in simulations, when the connection probabilities between individuals are relatively large, there is little practical difference between the two questions, because almost all individuals are on a path of length 2 that originates from the individual of interest (i.e., within their knowledge depth $L = 2$). We adopt the spectral approach and develop our community detection algorithm from analysis on the perceived network based on knowledge depth $L = 2$ and its major term. In large sparse networks, as we demonstrate in the analysis of political blog data, our algorithm can be applied to just the perceived network within the knowledge depth, even though our algorithm is not developed from analyzing its associated matrix.

The individual-centered partial information framework brings new technical challenges. Let $A$ be the adjacency matrix of the full network $G$ and $B$ be the adjacency matrix of individual 1’s perceived network based on their knowledge depth $L = 2$. Both $A$ and $B$ are $n \times n$ matrices. In our setting, matrix $A$ is not available for constructing an algorithm. To adopt the spectral approach, we need to identify a major term of $B$. But unlike $EA$ is a major term for $A$, $EB$ is not a major term of $B$. We derived a major term $B_E$ of $B$, which will be introduced in the next section. $B_E$ has distinct characteristics including (1) $B_E$ is random; (2) if rank($EA$) = $K$, then rank($B_E$) = $2K$ with high probability. Through lengthy calculations, we first derived approximate expressions for $B_E$’s eigenvalues and eigenvectors. These approximate expressions suggest a nice decomposition of eigenvectors and motivated us to finally construct exact eigenvectors of $B_E$. The results for $B_E$ are generic as they do not depend on a specific network generation model.

To do community detection in the new individual-centered partial information framework, we apply the generic results for $B_E$ and adopt the simplest generative network model: the stochastic block model (SBM) [Holland et al., 1983] Wang and Wong, 1987 Abbe, 2017 for theory and algorithm development. Variants of the SBM, including the degree-corrected stochastic block model (DCSBM) [Karrer and Newman, 2011] and the mixed membership stochastic block model (MMSBM) [Airoldi et al., 2008], can be studied in future under the partial information structure. For a review of network models, the readers are referred to Newman and Peixoto (2015) and Newman (2018). The community detection problem is perhaps the most studied statistical network problem. There is ample literature on community detection which assumes the SBM [Bickel and Chen, 2009] Rohe et al., 2011; Lei et al., 2015; Wang et al., 2017 Abbe, 2017 and variants thereof [Zhao et al., 2012] Anandkumar et al., 2014; Jin, 2015 Jin et al., 2017]. Beyond community detection, much attention has been focused on the statistical inference of high-dimensional networks, including inference for the number of communities [Bickel and Sarkar...
In a good portion of these works, analyzing the spectral properties, such as the eigenvalues and the eigenvectors of the corresponding matrices (e.g., adjacency matrices, Laplacian matrices, and similarity matrices) is crucial. We adopt the spectral approach in this work. Common alternatives, such as the likelihood approach (Wang et al., 2017), could be explored in the future.

Next, we relate our framework with knowledge depth \( L = 2 \) to two existing “individual-centered” concepts in network science. First, the ego (a.k.a., ego-centered) network (Newman, 2018) is a subnetwork of a centered-individual’s perceived network within \( L = 2 \). Recall that an ego network consists of an “ego” (individual of interest), the “alters” (a.k.a., individuals adjacent to the ego), and the edges between the ego and the alters plus the edges, if any, among the alters. Second, an early stopping in snowball sampling gives rise to a centered-individual’s perceived network within \( L = 2 \). Recall that snowball sampling is an important network sampling procedure in practice to sample hidden populations such as drug users, which are difficult to access by usual sampling approaches. To implement snowball sampling, an experimenter chooses one individual as the “seed”. Then the seed picks up their neighbors into the sample. These neighbors can be regarded as the first “wave”. The individuals in the first “wave” pick up their neighbors into the sample to construct the second “wave”. The process is stopped until certain criteria are met. In this process, the seed’s perceived network within \( L = 2 \) is the network observed by this sampling procedure up to the second “wave”. Given the latent adjacency matrix, snowball sampling is a non-random sampling procedure; random variants, such as respondent-driven sampling (RDS), have been proposed (Salganik and Heckathorn, 2004; Volz and Heckathorn, 2008; Goel and Salganik, 2010; Lu et al., 2012; Rohe, 2019). One distinction between our framework and snowball sampling or RDS is that our framework primarily serves to discover the latent network structure while they are intended for good estimators of some population characteristics of the individuals.

In addition to being a new concept to describe (and a possible and feasible cause of) individuals’ heterogeneous knowledge of the networks, the new individual-centered partial information framework has been demonstrated, in real data analysis, to deliver insights not available previously from the analysis on the global network or the ego networks. For example, with the Zakary’s Karate club data, we show that social hubs, despite having the largest number of edges in their knowledge depth, do not understand the affiliation structure as well as some non-hubs do. Although we focus on knowledge depth \( L = 2 \) and on the community detection problem here, we believe that the new individual partial information framework can be explored in many interesting ways. Some candidates, such as fuzzy second-degree observations, will be discussed in the Discussion section.

The main contributions of this paper are summarized as follows:

- We propose a new individual-centered partial information framework to study social networks from an individual’s perspective. In this framework, knowledge depth \( L \) characterizes an individual’s understanding of a network.
We identify a major term of the associated perceived adjacency matrix based on \( L = 2 \) and derive its eigenvalue and eigenvector properties, which are instrumental for spectral approaches to studying an array of problems under the new framework.

With the SBM and \( L = 2 \), we derive the first theory-backed spectral algorithm for community detection based on individual-centered partial information and demonstrate from real data analysis that the new framework and algorithm lead to rich insights.

The rest of the paper is organized as follows. In Section 2, we introduce some notations and \( B_E \), a major term of the perceived adjacency matrix (based on knowledge depth \( L = 2 \)). In Section 3, we prepare the theory concerning \( B_E \) that motivates a new detection algorithm. Section 4 introduces our algorithm and its almost exact recovery property under certain conditions. Simulation and real data studies are conducted in Sections 5 and 6 respectively. Finally, we conclude with a discussion section. All proofs and additional simulation results are relegated to Supplementary Materials [Han and Tong 2021].

2. Notations and major term

We introduce some notations that will be used throughout the paper. For a matrix \( M = (m_{ij}) \), in which \( m_{ij} \) is the \((i, j)\)-th entry, denote the \( i \)th row of \( M \) by \( M(i) \). Let \( \|M\|_{\text{max}} = \max_{i,j} \{|m_{ij}|\} \) and \( \|M\| \) be the spectral norm of \( M \), which is the square root of the largest eigenvalue of \( MM^\top \). Moreover, we denote the Frobenious norm of \( M \) by \( \|M\|_F = \left(\text{tr}(MM^\top)\right)^{1/2} \). For any random matrix (or vector) \( M \), we use \( \mathbb{E}M \) to denote its expectation. We use \( \| \cdot \|_2 \) to denote the \( L_2 \) norm of a vector. If two positive sequences \( a_n \) and \( b_n \) satisfy \( \limsup_{n \to \infty} (a_n/b_n) < \infty \), we denote \( a_n \lesssim b_n \) or \( b_n \gtrsim a_n \); alternatively we write \( a_n = O(b_n) \). If \( a_n \lesssim b_n \) and \( b_n \gtrsim a_n \), we write \( a_n \sim b_n \). We write \( a_n \ll b_n \) or \( b_n \gg a_n \) if \( \lim_{n \to \infty} (a_n/b_n) = 0 \). For two symmetric matrices \( M_n \) and \( N_n \), if there exists a positive constant \( c \) such that \( N_n - cM_n \) is a semi-positive definite matrix, then we write \( M_n \lesssim N_n \); specifically, if \( c = 1 \), then we write \( M_n \preceq N_n \). If there exists a positive diverging sequence \( c_n \) (i.e., \( c_n \to \infty \)) such that \( N_n - c_nM_n \geq 0 \) for all \( n \) large, we write \( M_n \ll N_n \). We denote the \( i \)-th largest eigenvalue and singular value of \( M \) by \( \lambda_i(M) \) and \( \sigma_i(M) \), respectively. Denote \([n] = \{1, \ldots, n\} \), \([K] = \{1, \ldots, K\} \), \([-K] = \{-K, \ldots, -1\} \), and \([\pm K] = \{-K, \ldots, -1, 1, \ldots, K\} \). For any two non-negative real random sequences \( x_n \) and \( y_n \), if there exists a positive vanishing sequence \( \epsilon_n \) such that \( \mathbb{P}(x_n/y_n \leq \epsilon_n) \to 1 \), we denote \( x_n = a_p(y_n) \). We use \(| \cdot |\) to denote the cardinality of a set. Throughout the paper, \( c \) and \( C \) denote constants that may vary from line to line. When we say that an event occurs with high probability, it means the following:

**Definition 1.** We say an event \( \mathcal{A} \) holds with high probability if for any positive constant \( D \), there exists an \( n_0(D) \in \mathbb{N} \) such that for all \( n \geq n_0(D) \), \( \mathbb{P}(\mathcal{A}) \geq 1 - n^{-D} \).

Recall that \( A = (a_{ij}) \) is the \( n \times n \) adjacency matrix of \( G = (V, E) \), the full network of interest, in which \( a_{ij} = \begin{cases} 1, & (i, j) \in E, \\ 0, & (i, j) \notin E. \end{cases} \) Let \( K = \text{rank} (\mathbb{E}A) \) and assume that \( A = A^\top \) and \( \{a_{ij}\}_{1 \leq i \leq j \leq n} \)
are independent Bernoulli random variables. We assume that $K$ is a constant and denote the (reduced form) eigen decomposition of the real symmetric matrix $\mathbb{E}A$ by

$$\mathbb{E}A = V D V^\top,$$

where $D = \text{diag}(d_1, \ldots, d_K)$ with $d_i$ being the $i$-th largest eigenvalue (by magnitude) of $\mathbb{E}A$ and $V = (v_1, \ldots, v_K)$ is the corresponding eigenvector matrix.

In the methodology development, for consistency in language, we assume that the individual whose partial knowledge we are interested in is the first node in the graph $G$ (i.e., individual 1). Let $B = (b_{ij})$ be individual 1’s perceived adjacency matrix generated from $A$ based on their knowledge of depth $L = 2$. For example, $A$ and $B$ for the toy network example in the Introduction are respectively,

$$A = \begin{pmatrix}
0 & 1 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 0
\end{pmatrix},
B = \begin{pmatrix}
0 & 1 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0
\end{pmatrix}.$$

More generally, individual 1’s perceived adjacency matrix based on knowledge depth $L = 2$ has the $(i,j)$-th entry as

$$b_{ij} = a_{ij}(1 - \mathbb{I}(a_{1i} = 0)\mathbb{I}(a_{1j} = 0)), \ i, j \in [n],$$

where $\mathbb{I}(\cdot)$ is an indicator function. Then it follows that

$$B = -SAS + AS + SA,$$

where $S = \text{diag}(a_{11}, \ldots, a_{1n})$. (1)

For convenience, we define

$$B_E = -S(\mathbb{E}A)S + (\mathbb{E}A)S + S(\mathbb{E}A).$$

(2)

In Section 3.1, we will establish that the matrix $B_E$ is a major term of $B$. But before that, in Section 2.1, we will argue that some common suspects do not serve as a major term.

2.1. A major term of $B$: not the common suspects

As we plan to use spectral methods to study partial information networks, it is important to derive a major term of $B$, for which we will derive relevant theoretical properties of eigenvalues and eigenvectors. Unlike the full information network, for which $\mathbb{E}A$ is a major term of $A$, the partial information network has a more subtle situation. Corollary [2] (to be introduced in the next section) shows that under certain conditions, $\|B - B_E\| \ll$ the smallest (in magnitude) non-zero eigenvalue of $B_E$ ($B_E$ was defined in [2]). Therefore, $B_E$ is a major term of $B$ from the spectral point of view. On the other hand, the obvious candidates $\mathbb{E}B$ and $-(\mathbb{E}S)(\mathbb{E}A)(\mathbb{E}S) + (\mathbb{E}A)(\mathbb{E}S) + (\mathbb{E}S)(\mathbb{E}A)$ do not fit a major term role, as suggested by Lemma [1].
Lemma 1. In the simplest scenario that $\mathbb{P}(a_{ij} = 1) = p_n = o(1)$, for $i, j \in [n]$, we have

$$\| - \mathbb{E} (\mathbb{E} A) \mathbb{E} S + (\mathbb{E} A) \mathbb{E} S + \mathbb{E} S (\mathbb{E} A) \| + \| \mathbb{E} B \| = o_p(\| \mathbb{B}_E \|).$$  \hspace{1cm} (3)

It is well-known that the leading eigenvectors are key in spectral clustering. In view of Lemma 1, the leading eigenvalue of $\mathbb{B}_E$ is much larger than those of $\mathbb{E} B$ and $- (\mathbb{E} S (\mathbb{E} A) \mathbb{E} S + (\mathbb{E} A) \mathbb{E} S + \mathbb{E} S (\mathbb{E} A))$ when $p_n = o(1)$, a typical condition for large networks. In other words, $\mathbb{E} B$ and $- (\mathbb{E} S (\mathbb{E} A) \mathbb{E} S + (\mathbb{E} A) \mathbb{E} S + \mathbb{E} S (\mathbb{E} A))$ do not contribute to the leading eigenvectors.

Together with the fact that $\mathbb{B}_E$ is a major term, Lemma 1 helps us conclude that neither $\mathbb{E} B$ nor $- (\mathbb{E} S (\mathbb{E} A) \mathbb{E} S + (\mathbb{E} A) \mathbb{E} S + \mathbb{E} S (\mathbb{E} A))$ is a major term.

3. Theory preparation

In this section, we first study a few general theoretical properties of $\mathbb{B}_E$ and establish that $\mathbb{B}_E$ is major term of $\mathbb{B}$ (Section 3.1). Then we apply these results to the stochastic block model (SBM) and derive results that are insightful in motivating our community detection algorithm and in proving its properties (Section 3.2).

To illustrate the idea and highlight the difference between the partial information network and its corresponding full information network, we will focus on the fundamental understanding of the core partial information structure. With that focus, we leave boundary conditions, model extensions, and other theoretical issues for future studies.

3.1. Theoretical properties of $\mathbb{B}_E$

Recall $\mathbb{S} = \text{diag}(a_{11}, \ldots, a_{1n})$ and $\mathbb{E} A = \mathbb{V} \mathbb{D} \mathbb{V}^\top$, in which $\mathbb{V}$ and $\mathbb{D}$ are of dimensions $n \times K$ and $K \times K$ respectively.

**Theorem 1.** Suppose that $\mathbb{V}^\top \mathbb{S} \mathbb{V}$ and $\mathbb{I} - \mathbb{V}^\top \mathbb{S} \mathbb{V}$ are invertible. Denote $\mathbb{H}(x) = \mathbb{I} - x \mathbb{D} \mathbb{V}^\top \mathbb{S} \mathbb{V} - x^2 \mathbb{D} (\mathbb{I} - \mathbb{V}^\top \mathbb{S} \mathbb{V}) \mathbb{D} \mathbb{V}^\top \mathbb{S} \mathbb{V}$. Then the determinant equation

$$\det (\mathbb{H}(x)) = 0$$

has $2K$ non-zero real solutions; we denote them by $x_{-K}, \ldots, x_{-1}$ and $x_{1}, \ldots, x_{K}$, in which $x_i \leq x_j$ for all $i < j$. Moreover, for $i \in [\pm K]$, let $\mathbb{q}_{1i}$ be an eigenvector of $\mathbb{H}(x_i)$ corresponding to the zero eigenvalue, and

$$\mathbb{q}_{2i} = x_i \mathbb{D} \mathbb{V}^\top \mathbb{S} \mathbb{V} \mathbb{q}_{1i}. \hspace{1cm} (5)$$

Then $\mathbb{q}_i = \mathbb{S} \mathbb{V} \mathbb{q}_{1i} + (\mathbb{I} - \mathbb{S}) \mathbb{V} \mathbb{q}_{2i}$ satisfies

$$\mathbb{B}_E \mathbb{q}_i = x_i^{-1} \mathbb{q}_i. \hspace{1cm} (6)$$

Furthermore, $\mathbb{q}_i \neq 0$; hence $\mathbb{q}_i$ is an eigenvector of $\mathbb{B}_E$ corresponding to the eigenvalue $x_i^{-1}$.

**Theorem 2.** Suppose that $\mathbb{V}^\top \mathbb{S} \mathbb{V}$ and $\mathbb{I} - \mathbb{V}^\top \mathbb{S} \mathbb{V}$ are invertible. Then

$$\text{rank}(\mathbb{B}_E) = 2K.$$
If $q_0 \neq 0$ is an eigenvector of $B_E$ corresponding to a non-zero eigenvalue $x_0^{-1}$, then

$$\det (H(x_0)) = 0.$$ \hspace{1cm} (7)

Moreover, $q_0$ can be decomposed as

$$q_0 = SVq_{10} + (I - S)Vq_{20},$$ \hspace{1cm} (8)

where $q_{10}$ is an eigenvector of $H(x_0)$ corresponding to zero eigenvalue and $q_{20} = x_0DV^T SVq_{10}$.

**Remark 1.** For the setting of our problem, the invertibility assumption on $V^T SV$ and $I - V^T SV$ is not a stringent condition. To communicate some intuition, we write the eigen decomposition of $V^T SV$ as

$$V^T SV = \sum_{i=1}^{n} a_{1i} V^T (i) V(i).$$

Since $n$ is large and $V^T SV$ is a matrix of small dimensionality $K \times K$, the invertibility of $V^T SV$ should be satisfied if the direct connections from individual 1 to the others are not too rare. Similarly, the invertibility of $I - V^T SV$ can be ensured if the direct connections from individual 1 do not reach most of the other individuals in the network.

In the sequel, we introduce a few conditions to gain further understanding into the individual-centered partial information networks. Denote $p_n = \max_{i,j} P(a_{ij} = 1)$.

**Condition 1.** $\min_{j \geq 2} P(a_{1j} = 1) \sim p_n$ and $1 - c > p_n \gg \log n/n$ for some constant $c > 0$.

**Condition 2.** $\|V\|_{\text{max}} \leq C/\sqrt{n}$ for some constant $C > 0$.

**Condition 3.** $|d_1| \sim |d_2| \sim \ldots \sim |d_K| \sim np_n$.

Conditions 1-2 are sufficient to ensure the invertibility of matrices $V^T SV$ and $I - V^T SV$ with high probability, as we will prove in Lemma 2. We should also point out that Conditions 1-2 can be relaxed but we adopt these conditions in the paper for convenience and simplicity. Condition 3 is a strong condition to assume that the magnitude of the smallest non-zero eigenvalue of $EA$ has the same order as $\|EA\|_F$, which is to ensure a big enough gap between $|d_K|$ and $\|A - EA\|$ for more straightforward analysis. This condition could also be relaxed but we leave that analysis for future studies. The Bernstein inequality (Lemma 9 in Supplementary Materials) helps establishing the following lemma.

**Lemma 2.** Under Conditions 1 and 2, there exists a positive constant $c$ such that with high probability, we have

$$cp_n \left(1 - \sqrt{\frac{\log n}{np_n}}\right) I \leq V^T SV \leq p_n \left(1 + \sqrt{\frac{\log n}{np_n}}\right) I < \left(1 - \frac{c}{2}\right) I.$$ \hspace{1cm} (9)

Therefore with high probability, $V^T SV$ and $I - V^T SV$ are invertible.

Theorems 1 and 2 and Lemma 2 imply the following corollary.
Corollary 1. Under Conditions 1 and 2, for suitably chosen $q_l$, $l \in [\pm K]$, $q_l$’s, as defined in Theorem 1, satisfy with high probability that

$$q_i^\top q_j = 0, \quad i \neq j, \quad i, j \in [\pm K], \quad (10)$$

and that $\dim(\text{span}\{q_l, l \in [\pm K]\}) = \text{rank}(B_E) = 2K$.

Corollary 1 will play a role when we develop our algorithm under the stochastic block model. Note that the words “suitably chosen” in Corollary 1 do not require us to choose in the construction of our algorithm. We just need to calculate the eigenvectors of $B$ in our (to be proposed) community detection algorithm.

Condition 4. $p_n \to 0$.

Condition 5. $\mu_i / \mu_{i+1} \geq 1 + c$ for some positive constant $c$, $1 \leq i \leq K - 1$, where $\mu_1 \geq \ldots \geq \mu_K$ are the eigenvalues of $V^\top (IE) V D (I - V^\top (IE) V) D$.

Condition 4 means that as the network size goes to infinity, the connection probability between any two individuals should go to zero. Condition 5 gives an eigenvalue separation condition for a key matrix in the proofs. Under Conditions 1 and 2, both $V^\top (IE) V$ and $I - V^\top (IE) V$ are invertible; therefore the $\mu_K$ in Condition 5 is positive, when Conditions 1 and 2 are also assumed.

Theorem 3. Under Conditions 1, 2 and 3, with high probability, we have $|x_i|^{-1} \sim np_n^{3/2}$ for $i \in [\pm K]$. Further assuming Conditions 4 and 5 it holds with high probability that,

$$x_i^{-1} = \left(\lambda_i (D (I - V^\top (IE) V)DV^\top (IE) V) \right)^{1/2} (1 + o(\sqrt{p_n})) \quad , \quad i \in [K],$$

and

$$x_i^{-1} = -\left(\lambda_{K+i+1} (D (I - V^\top (IE) V)DV^\top (IE) V) \right)^{1/2} (1 + o(\sqrt{p_n})) \quad , \quad i \in [-K].$$

Theorem 3 has two parts. The first part points out the order of non-zero eigenvalues of $B_E$. The second part, with more stringent conditions, finds the approximate expressions for these eigenvalues. We note in advance that our community detection results will not depend on Conditions 1 and 3. But the second part of Theorem 3 is of standalone theoretical interests and can be useful to study other problems under the partial information framework.

The matrix Bernstein inequality (Lemma 10 in Supplementary Materials) implies an upper bound for the spectral norm of $B - B_E$.

Lemma 3. It holds with high probability that

$$\|B - B_E\| \lesssim \sqrt{(\log n)np_n}.$$  

Lemma 3 and Theorem 3 (first part) imply the following corollary, whose proof we omit. This corollary establishes that $B_E$ is a major term of $B$.

Corollary 2. Under Conditions 1, 2 and 3 assuming $p_n \gg \sqrt{\log n/n}$, then we have with high probability,

$$\|B - B_E\| \ll \min_{i \in [2K]} \sigma_i(B_E).$$
### 3.2. Stochastic block model

We illustrate the use of the generic results concerning $B_E$ derived in Section 3.1 under the stochastic block model (SBM) \cite{Holland:1983}, a canonical model in statistical network analysis. In the SBM, individuals are assumed to belong to one (and only one) of $K$ different communities: $[K]$. The connection probability between two individuals depends on their community memberships. Concretely, in the SBM with $K$ communities, $E A$ can be expressed as

$$
E A = P \Pi \Pi^\top,
$$

where $P = (p_{kl})$ is a symmetric $K \times K$ matrix in which the $(k, l)$-th entry $p_{kl}$ is the connection probability between communities $k$ and $l$, $\Pi = (\pi_1, \ldots, \pi_n)^\top \in \mathbb{R}^{n \times K}$ is the matrix of community membership vectors of individuals $1, \ldots, n$, and individual $i$’s membership vector $\pi_i \in \{e_1, \ldots, e_K\}$, where $e_k \in \mathbb{R}^K$ is a standard basis vector whose $k$th component is equal to one and whose other components are equal to zero. In this model, when individual $i$ belongs to community $k$ and individual $j$ belongs to community $l$, we have $E a_{ij} = \pi_i^\top P \pi_j = p_{kl}$. Strictly speaking, this model allows self-loops. The networks we wish to analyze do not have them. However, we can still study this model theoretically, as explained in Supplementary Materials.

We derive a few technical results that motivate our partial-information-based community detection algorithm. Without loss of generality, in the sequel we assume that individual 1 belongs to the first community (i.e., community 1). Recall that $p_n$ was defined by $p_n = \max_{i,j} I P(a_{ij} = 1)$.

**Condition 6.** \( \min_{k \in [K]} p_{1k} \sim p_n \cdot \min_{k \in [K]} \sum_{j \in [n]} I(\pi_j = e_k) \geq c_0 n \) and \( \sigma_K(P) \geq c_1 p_n \) for some positive constants $c_0$ and $c_1$. Moreover, for some $c > 0$, $1 - c \geq p_n \gg (\log n/n)^{1/2}$.

**Lemma 4.** Under the stochastic block model defined in (11), Condition 2 implies $\text{rank}(E A) = K$ and Conditions 1, 2, and 3. Moreover, there exists a $K \times K$ matrix $D$ such that

$$
V = \Pi D \quad \text{and} \quad DD^\top \geq \frac{1}{n} I.
$$

Under Condition 6, which validates Conditions 1 and 2 in view of Lemma 4, Corollary 1 implies that when we let $Q = (q_1, \ldots, q_K, q_{-1}, \ldots, q_{-K})$ and assume WLOG $\|q_i\|_2 = 1$, $i \in [\pm K]$, then we have

$$
Q^\top Q = I.
$$

Let $Q_1 = (q_{11}, \ldots, q_{1K}, q_{-1}, \ldots, q_{-K})$ and $Q_2 = (q_{21}, \ldots, q_{2K}, q_{2-1}, \ldots, q_{2-K})$. By the definition of $q_i$ above (6), we have

$$
Q = S \Pi D Q_1 + (I - S) \Pi D Q_2,
$$

which can be equivalently expressed as

$$
Q = \left( \frac{S \Pi}{\sqrt{p_n}}, (I - S) \Pi \right) Q, \quad \text{where} \quad Q = \left( \frac{\sqrt{p_n} D Q_1}{D Q_2} \right).
$$
Lemma 5. Under Condition \( \mathcal{C} \), with high probability, there exists some positive constant \( c_2 \) such that
\[
QQ^\top \geq (c_2 n)^{-1} \mathbf{I}, \quad DQ_2Q_2^\top D^\top \geq (c_2 n)^{-1} \mathbf{I}, \quad \text{and} \quad p_n DQ_1Q_1^\top D^\top \geq (c_2 n)^{-1} \mathbf{I}.
\] (16)

Lemma 5 is a technical result. Together with Corollary 1, it implies the following lemma.

Lemma 6. Under Condition \( \mathcal{C} \), for any \( 2K \times 2K \) orthogonal matrix \( O \), it holds with high probability that for all \( i, j \in [n] \),
\[
\pi_i \neq \pi_j \implies \|Q(i)O - Q(j)O\|_2 \geq \frac{2}{c_2 n}, 
\] (17)
\[
\pi_i = \pi_j, a_{1i} \neq a_{1j} \implies \|Q(i)O - Q(j)O\|_2 \geq \sqrt{\frac{2}{c_2 n}},
\] (18)
\[
\pi_i = \pi_j, a_{1i} = a_{1j} \implies \|Q(i)O - Q(j)O\|_2 = 0.
\] (19)

For networks generated from the SBM, when one has access to the full network, one can make the top \( K \) eigenvectors of \( A \) as an \( n \times K \) matrix and apply the \( k \)-means algorithm to separate the \( n \) rows of that matrix into \( K \) groups. The rationale is that the distinct rows of the eigenvectors correspond to different communities. Unfortunately, such a simple approach does not work if we were to just replace \( A \) with \( B \). Indeed, equation (18) in Lemma 6 reveals that even if \( \pi_i = \pi_j \), \( Q(i) \) and \( Q(j) \) can be very different. Therefore, \( Q(i) \neq Q(j) \) does not imply that individuals \( i \) and \( j \) belong to different communities. On the other hand, equation (19) suggests that we should treat individuals who are adjacent to individual \( 1 \) and those who are NOT separately. In view of Lemma 6 we propose a novel detection algorithm in Section 4.

4. Detection algorithm and recovery properties

Let \( W = (w_1, \ldots, w_K, w_1, \ldots, w_K) \) be the collection of \( 2K \) unit eigenvectors corresponding to the \( 2K \) largest eigenvalues (in magnitude) of \( B \). Loosely, our approach is to first apply the \( k \)-means algorithm (with \( k = K \)) separately to the non-zero rows of \( SW \) and those of \( (I - S)W \) (Section 4.1) and then merge these \( 2K \) clusters into \( K \) communities (Section 4.2).

4.1. Divide and conquer

The rationale of the separation in the first step is that \( SQ \) and \( (I - S)Q \) contain the community membership information (in view of Lemma 6) and that \( W \) is “close” to \( Q \). The closeness between \( W \) and \( Q \) is established through the following lemma.

Lemma 7. Under Condition \( \mathcal{C} \), with high probability we have
\[
\|W - QO\|_F = O \left( \frac{\sqrt{\log n}}{\sqrt{n}p_n} \right),
\] (20)
where \( O = U_1 U_2^\top \), in which \( U_1 \) and \( U_2 \) are from the singular value decomposition \( (Q)^\top W = U_1 \Sigma U_2^\top \) such that \( \Sigma \) is the diagonal matrix with singular values.
Following Rohe et al. (2011), we define $M$. Lemma 8 implies that with high probability, $r = 1$. Algorithm 1

As $W$ is the empirical counterpart of $Q$, we will apply the $k$-means algorithm (with $k = K$) to the non-zero rows of the matrices $SW$ and $(I - S)W$ (i.e., $\{W(i) : i \in [n], a_{1i} = 1\}$ and $\{W(i) : i \in [n], a_{1i} = 0\}$), respectively. Here $S$ and $I - S$ separate the individuals into two different groups; in each group we get $K$ different clusters. This divide-and-conquer strategy is summarized as Algorithm 1.

**Algorithm 1**

1. Take matrices $S$ and $W$ as defined respectively in equation (1) and above Lemma 7.
2. Apply the $k$-means algorithm to the rows of $\{W(i) : i \in [n], a_{1i} = 1\}$ and $\{W(i) : i \in [n], a_{1i} = 0\}$, respectively, to separate each group into $K$ clusters.
3. Return (i) the $2K$ clusters and (ii) the centroid matrix $C = (c_1^T, \ldots, c_n^T)^T$ in which the length $2K$ row vector $c_i$ denotes the centroid associated with individual $i$.

Now we study theoretical properties of Algorithm 1. In the theoretical analysis, we assume for both $r = 0$ and $r = 1$,

$$\{c_i : a_{1i} = r\} = \arg\min_{\{x_i : x_i \text{ is 2K-dimensional row vector with } a_{1i} = r \text{ and } \|x_i\|_\pi \leq K\}} \sum_{i \in [n], a_{1i} = r} \|W(i) - x_i\|_2^2.$$  \hspace{1cm} (21)

**Lemma 8.** Under Condition 4, for any $1 \times 2K$ row vector $c$ and $i \in [n]$, if $\|c - Q(i)O\|_2 < 1/\sqrt{2cn}$, where $c_2$ is the same as in Lemma 5 and $O$ is defined in Lemma 7, then with high probability, $\|c - Q(j)O\|_2 > 1/\sqrt{2cn}$ for all $j \in [n]$ such that $\pi_j \neq \pi_i$.

For simplicity, we assume $|\{c_i, l \in [n]\}| = 2K$. When $c_i = c_j$ and $a_{1i} = a_{1j}$, Algorithm 1 assigns individuals $i$ and $j$ to the same cluster. For any $i \in [n]$, if

$$\|c_i - Q(i)O\|_2 < \frac{1}{\sqrt{2cn}},$$ \hspace{1cm} (22)

Lemma 8 implies that with high probability, $\|c_i - Q(j)O\|_2 > \frac{1}{\sqrt{2cn}}$ for any $j$ such that $\pi_j \neq \pi_i$. Following Rohe et al. (2011), we define $M$, a set of individuals that do not satisfy (22):

$$M = \left\{i \in [n] : \|c_i - Q(i)O\|_2 \geq \frac{1}{\sqrt{2cn}}\right\}.$$ \hspace{1cm} (23)

In Theorem 4 we will show that $|M|/n$ can actually control the misclustering rate under our final algorithm (Algorithm 2). For now, we have a result regarding $|M|$.

**Theorem 4.** Under Condition 4, let $\{c_i, i \in [n]\}$ be the centroids returned by (21), then for $M$ defined in (23), it holds with high probability that $|M| = O(\log n/p_n^2)$.
4.2. Merging 2K clusters and the final algorithm

Algorithm returns 2K clusters. Without loss of generality, we denote these 2K clusters by \( \{c_1, \ldots, c_K\} \) and \( \{d_1, \ldots, d_K\} \), belonging to the two groups \( \{W(i) : i \in [n], a_{ii} = 1\} \) and \( \{W(i) : i \in [n], a_{ii} = 0\} \), respectively. Our next step is to merge them into K communities. Concretely, we first construct two estimators of the connection probability matrix \( P \) defined in (11). Then we devise an innovative strategy to compare these two estimates and merge the clusters. Motivated by the relation that for \( i, j > 1 \) and \( i \neq j \),

\[
\mathbb{P}(b_{ij} = 1|a_{ii} = a_{jj} = 1) = \mathbb{P}(b_{ij} = 1|a_{ii} = 1 - a_{jj} = 1) = \mathbb{P}(a_{ij} = 1),
\]

we propose two estimators of \( P \), denoted respectively by \( \hat{P}^{SS} \) and \( \hat{P}^{SI-S} \), whose entries are

\[
\hat{P}^{SS}_{kl} = \frac{1}{|S^{(1)}_{k,l}|} \sum_{(i,j) \in S^{(1)}_{k,l}} b_{ij} \quad \text{and} \quad \hat{P}^{SI-S}_{kl} = \frac{1}{|S^{(2)}_{k,l}|} \sum_{(i,j) \in S^{(2)}_{k,l}} b_{ij},
\]

where \( S^{(1)}_{k,l} = \{(i, j) : a_{ii} = a_{jj} = 1, i \text{ and } j \text{ belong to clusters } c_k \text{ and } c_l \text{ respectively}\} \) and \( S^{(2)}_{k,l} = \{(i, j) : a_{ii} = 1 - a_{jj} = 1, i \text{ and } j \text{ belong to clusters } c_k \text{ and } c_l \text{ respectively}\} \). Given the estimators \( \hat{P}^{SS} \) and \( \hat{P}^{SI-S} \), a natural idea is to find a permutation of the memberships \( \{c_1, \ldots, c_K\} \) such that \( ||\hat{P}^{SS} - \hat{P}^{SI-S}|| \approx 0 \). Then we merge the community memberships according to this permutation. The following theorem provides a foundation for this strategy.

**Theorem 5.** Let \( C = \{c_1, \ldots, c_K\} \) and \( D = \{d_1, \ldots, d_K\} \) be two ordered sets of size K and let \( f_0 : [K] \to [K] \) be the unique permutation such that \( c_{f_0(i)} = d_i \) for all \( i \in [K] \). Suppose that the K elements in these ordered sets have been endowed with pairwise connection probabilities. Let \( C = (c_{ij}) \) be a \( K \times K \) matrix such that the \( (i, j) \)-th entry \( c_{ij} \) is the connection probability between \( c_i \) and \( c_j \), and \( D = (d_{ij}) \) be a \( K \times K \) matrix such that the \( (i, j) \)-th entry \( d_{ij} \) is the connection probability between \( c_i \) and \( d_j \). If \( \text{rank}(C) = K \), then for a permutation function \( f : [K] \to [K] \),

\[
f = f_0 \iff C_{(f,f)} = D_{(f,s)},
\]

where the \( (i, j) \)-th entries of \( C_{(f,f)} \) and \( D_{(f,s)} \) are \( c_{f(i)f(j)} \) and \( d_{f(i)j} \), respectively.

Motivated by Theorem 5 correctly combining the 2K clusters into K communities is equivalent to obtaining a permutation function \( f \) such that

\[
\hat{P}^{SS}_{(f,f)} \approx \hat{P}^{SI-S}_{(f,s)},
\]

where the \( (i, j) \)-th entry of \( \hat{P}^{SS}_{(f,f)} \) and \( \hat{P}^{SI-S}_{(f,s)} \) are the \( (f(i), f(j)) \)-th entry of \( \hat{P}^{SS} \) and the \( (f(i), j) \)-entry of \( \hat{P}^{SI-S} \) respectively. The previous divide-and-conquer strategy and this merging strategy together make our final algorithm: Algorithm 2. We exemplify how to merge these clusters in Step 5. Let \( \hat{f}_0 \) be the “best” permutation returned by Step 4. If \( \hat{f}_0(1) = 2 \), then the second cluster belonging to \( SW \) and the first cluster belonging to \( (I - S)W \) are merged into one community in Step 5.

To establish the theoretical property of our Algorithm 2, we introduce the following assumption.
Algorithm 2
1: Take matrices $S$ and $W$ as defined respectively in equation (1) and above Lemma 7.
2: Apply the k-means algorithm to the non-zero rows of the two groups $SW$ and $(I - S)W$ respectively to separate each group into $K$ clusters.
3: Based on the estimated clusters, calculate the connection probability matrix estimates $\hat{P}_{S,S}$ and $\hat{P}_{S,1-S}$ as in equation (24).
4: Let $\hat{f}_0 = \arg \min_f \| \hat{P}_{S,S}(f,f) - \hat{P}_{S,1-S}(f,*) \|_F$.
5: Merge the $2K$ clusters into $K$ communities corresponding to the permutation $\hat{f}_0$.
6: Return the community memberships.

Condition 7. There exists a positive constant $c$ such that for any $i_1, j_1, i_2, j_2 \in [K]$, either $p_{i_1,j_1} = p_{i_2,j_2}$ or $|p_{i_1,j_1} - p_{i_2,j_2}| \geq cp_n$.

As an estimated connection probability matrix $\hat{P}$ contains noise, Condition 7 ensures that the different values of $P$ have enough gap such that the noise $\| \hat{P} - P \|$ is smaller than this gap. Armed with this condition, the following theorem is a key precursor to the final misclustering result of Algorithm 2.

Theorem 6. Under Conditions 6–7 and $p_n \gg (\log n/n)^{1/4}$, let $\{c_i, i \in [n]\}$ be the centroids returned by (21). Then it holds with high probability, uniformly for $i, j \in M^c$, that

$$\text{individuals } i \text{ and } j \text{ are assigned to the same community by Algorithm 2 } \iff \pi_i = \pi_j.$$  

By Algorithm 2, we essentially propose an estimated membership matrix $\hat{\Pi} = (\hat{\pi}_1^\top, \ldots, \hat{\pi}_n^\top)^\top$, then the misclustering rate is defined by

$$\text{Misclustering rate} = \frac{\min_{Z: Z \text{ is } K \times K \text{ permutation matrix}} \sum_{j=1}^n \mathbb{I} (\hat{\pi}_j^\top Z \neq \pi_j^\top)}{n}.$$  

Theorem 7. Under Conditions 6–7 and $p_n \gg (\log n/n)^{1/4}$, with high probability,

$$\text{Misclustering rate of Algorithm 2 } \leq \frac{|M|}{n} = o(1).$$

In other words, Algorithm 2 has the almost exact recovery property (c.f. Def 4 of Abbe (2017)).

5. Simulation studies

In this section, we consider two SBM (defined in equation (11)) settings:

- Model 1 ($K = 2$): $P = \begin{pmatrix} 3q & q \\ q & 3q \end{pmatrix}$ and each group is of size $n/2$.
- Model 2 ($K = 3$): $P = \begin{pmatrix} 3q & 1.5q & q \\ 1.5q & 3q & 1.5q \\ q & 1.5q & 3q \end{pmatrix}$ and each group is of size $n/3$. 
For each model, we vary the number of individuals \(n\in\{300, 600, 900, 1200, 1500, 1800, 2100\}\) and \(q\in\{.1, \sqrt{\log n/n}, (\log n/n)^{1/4}/2, 1/\sqrt{n}\}\). Under each model, for every combination of \(n\) and \(q\), we simulate 100 datasets; Algorithm 2 is applied to each dataset. In Section 8.21 of Supplementary Materials, we have another model \((K=3)\) in which the connection probabilities \(p_{13}=p_{23}\). As the results are similar to those for Model 2, we omit them in the main text.

5.1. Results for Model 1

In Table 1, we report the ratio of individual 1’s observed edges out of the total edges. This table shows that for most combinations, the fraction of the edges observed by individual 1 is smaller than 50%. Meanwhile, Table 2 indicates that the fraction of individuals within individual 1’s knowledge depth is either 100% or close to 100%. Tables 1 and 2 together illustrate that in our simulation settings, although individual 1 can observe at least one edge of (almost) every other individual, the number of missing edges in their perspective is significant. Hence, their information is indeed “partial” compared to the full network.

We calculate the mean misclustering rates for each combination and report the results in Figure 2. Except for \(q = 1/\sqrt{n}\), all other \(q\)’s resulted in close to perfect clustering performance. As a comparison, we apply spectral clustering to \(A\) (i.e., full information network) and plot the results in Figure 3. We see in Figure 3 that for all \((n,q)\) combinations, misclustering rates are 0. This is not surprising, because the network literature (Abbe, 2017) has suggested the connection probability \(q \sim \log n/n\) for exact recovery (for the usual 2-block SBM with full information); clearly \(\log n/n \ll \min\{\sqrt{\log n/n}, 1/\sqrt{n}, (\log n/n)^{1/4}/2\}\) and \(\log(2100)/2100 < .1\).

In Figure 4, we provide visualization support of Algorithm 2 with a scatter plot in which the axes are the two eigenvectors corresponding to the positive eigenvalues of \(B\). Recall that...
we have shown in previous sections that $B$ for $K = 2$ has two positive eigenvalues and two negative ones. The outlier point (a blue point) close to the vertical axis between .01 and .02 represents individual 1. The blue points are the individuals not adjacent to individual 1, and the red points represent those who are adjacent to 1. Using the same dataset, we color the points by their true community memberships in Figure 5. Comparing Figure 4 with Figure 5, one can see that it makes sense to develop a strategy to first apply $k$-means respectively to the two groups of individuals separated by whether they are adjacent to individual 1 and then merge the corresponding clusters across groups.
Fig. 4. Scatter plot of the two eigenvectors corresponding to the positive eigenvalues of B for one dataset from Model 1 when $q = (\log n/n)^{1/4}/2$ and $n = 2100$. The blue points are not adjacent to individual 1, while the red points are adjacent to individual 1.

5.2. Results for Model 2

Similar to their counterparts for Model 1, Tables 3 and 4 show that in Model 2 ($K = 3$), although individual 1 can observe at least one edge of (almost) every other individual in the network, the proportion of total missing edges in their perspective is large. The visualization of Model 2 data in 3-D, similar to that of Model 1 in 2-D, to support Algorithm 2 is attached in Section 5.20 of Supplementary Materials. We report the misclustering results in Figure 6. This figure shows that $K = 3$ is a more challenging situation compared to $K = 2$. Algorithm 2 for $q = 1/\sqrt{n}$ with Model 2 works worse than with Model 1. Also note that $q = \sqrt{\log n/n}$ in Model 1 delivers almost perfect clustering results, but the trend in Figure 6 suggests that even as $n$ goes to infinity, the misclustering rate does not go down to zero. The rate $p_n \sim q = \sqrt{\log n/n}$ is smaller than the rate in the theoretic Condition 7 but at this rate, Algorithm 2 works well for Model 1 while its performance is acceptable for Model 2. As a comparison, we report in Figure 7 the simulation results based on the adjacency matrix $A$ with the usual spectral clustering algorithm. For larger $n$ in each combination, the misclustering rate is very close to 0, which is theoretically guaranteed by a few published studies (c.f. Abbe (2017)). Although we did not work on the boundary condition, we conjecture that the boundary condition under the new partial information framework for almost exact recovery is at least of order $\sqrt{\log n/n}$. 
Fig. 5. Scatter plot of the first two eigenvectors corresponding to positive eigenvalues of $B$ for one dataset from Model 1 when $q = (\log n/n)^{1/4}/2$ and $n = 2100$. The blue points belong to community 1 and the red points belong to community 2.

Table 3. The ratio of the edges observed by individual 1 out of the full networks for Model 2, averaged over 100 datasets for each $(n, q)$ combination.

| $q \setminus n$ | 300 | 600 | 900 | 1200 | 1500 | 1800 | 2100 |
|-----------------|-----|-----|-----|------|------|------|------|
| $1$             | .3298 | .3302 | .3309 | .3314 | .3286 | .3308 | .3294 |
| $\sqrt{\log n/n}$ | .4387 | .3407 | .2932 | .2619 | .2361 | .2221 | .2069 |
| $(\log n/n)^{1/4}/2$ | .5531 | .4941 | .4623 | .4413 | .4199 | .4064 | .3948 |
| $1/\sqrt{n}$    | .2018 | .1441 | .1204 | .1047 | .0930 | .0849 | .0797 |

Table 4. The fraction of the individuals within individual 1’s knowledge depth for Model 2, averaged over 100 datasets for each $(n, q)$ combination.

| $q \setminus n$ | 300 | 600 | 900 | 1200 | 1500 | 1800 | 2100 |
|-----------------|-----|-----|-----|------|------|------|------|
| $1$             | .9999 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\sqrt{\log n/n}$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $(\log n/n)^{1/4}/2$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $1/\sqrt{n}$    | .9653 | .9632 | .9619 | .9642 | .9632 | .9656 | .9648 |
6. Real data analysis

In this section, we investigate the performance of our algorithm on two well-known data sets: Zachary’s karate club data and political blog data.

6.1. Zachary’s karate club data

Zachary’s karate club data, originally introduced in [Zachary (1977)], has become a canonical dataset in statistical network analysis. This dataset is about a university karate club which consists of 34 members. The club friendship network has 78 edges in total; an edge links
two individuals if they spent significant time together outside the club. The topology of the network is plotted in Figure 8. It was reported that following conflicts between the instructor “Mr. Hi” (i.e., “H”) and the administrator “John A.” (i.e., “A”), the 34 members split into two communities, colored yellow and blue respectively, in Figure 8. One can identify “H” and “A” as clear hubs of the network and “centers” of the communities. The existence of two hubs suggests that this network does not fit the stochastic block model (SBM) well. Nonetheless, our SBM-derived Algorithm 2 delivers surprisingly interesting results on this dataset.

As an illustration, we pick six individuals, including the two hubs. One by one, we treat each of them as the individual of interest (i.e., “individual 1” in our methodology development and simulation studies) to obtain their perceived network based on knowledge depth $L = 2$. Then we apply Algorithm 2 to calculate the number of the wrongly estimated memberships. Table 5 indicates that different people have vastly different capacities to correctly identify the memberships. Individual 3, with about 30% edges missing from view, does an OK job. Individual 2, with about 49% edges missing, performs very poorly. The hubs “H” and “A” perform poorly even though both of them have high degrees. On the other hand, individuals 20 and 32, who do not have a particular advantage in the number of observed edges, do particularly well. We conjecture that this is because they each are adjacent to both hubs.

In general, we hypothesize that the individuals whose partial information networks are most
Table 6. Detection results for chosen individuals after deleting the edge between "A" and 20.

| individual of interest | H | 2 | 3 | A | 20 | 32 |
|------------------------|---|---|---|---|----|----|
| # of the wrongly estimated memberships | 11 | 14 | 5 | 8 | 10 | 0 |

effective in detecting community memberships are those adjacent to the “center” members of different communities. To partially evaluate this hypothesis, we delete individual 20’s connection to “A” and investigate again 20’s perceived network based on $L = 2$. Table 6 reports the number of wrongly estimated memberships after this edge deletion. As expected, after losing the connection to “A”, individual 20’s partial information is less powerful. However, individual 20 only misidentifies ten (out of thirty-four) people, while having only eighteen people within their knowledge depth after deletion. They misidentify fewer people than the number of their perceived isolated individuals (c.f. $10 < 16 = 34 - 18$). This interesting phenomenon is worth further investigation. Other than individual 20, this change in the network structure does not change the detection capacity of individuals “H”, 2, 3, and 32, while “A” appears to perform slightly better. Admittedly, this kind of deletion exercise does not provide a comprehensive picture of which network position is equipped with the “best” partial information structure. Identifying the individuals whose partial information should be explored is an interesting endeavor for future research.

6.2. Political blog data

A political blog network (Adamic and Glance, 2005) records hyperlinks between 1222 web blogs in the run-up to the 2004 U.S. presidential election. The entire network has 1494 nodes, but we only evaluated the largest (weakly) connected component, as did most previous network studies. As hyperlinks have directions, the blog network here is properly represented as a directed graph. However, as in most previous community detection studies, we treat the hyperlinks as undirected edges. Adamic and Glance (2005) studied the political outlooks of these blogs and assigned each blog a party preference: Democratic or Republican. In a similar approach to that we used for Zachary’s club data, we chose six ‘individuals/web blogs of interest’. Because this is a large network with relatively sparse edges, most people’s $L = 2$ knowledge depth does not reach all blogs; therefore we apply Algorithm 2 to perceived networks within one’s knowledge depth (i.e., the individuals with no edges observed are removed). Table 7 summarizes the performance of the algorithm. The misclustering rate is calculated as the ratio of wrongly estimated memberships out of the individuals within knowledge depth. The ratio of the edges observed is calculated from the number of edges of the subnetwork of the full network consisting of the individuals within the knowledge depth (not the total number of edges in the full network). From Table 7 we can see that individuals 1073 and 1074 perform poorly while the other individuals have acceptable performance, with individual 1077 has the lowest rate of error (less than 7%). These observations again demonstrate the importance of deciding whose partial information to look for. Moreover, it shows that even though partial information networks may lose much information
on the full network, restricting oneself to individuals within knowledge depth may achieve good local results. Another interesting observation is that although the degree corrected stochastic block model is widely believed to be more suitable for this data than the simple SBM (c.f. Karrer and Newman (2011) and Zhao et al. (2012)), our algorithm, based upon the SBM, works well for some individuals’ partial information networks. We speculate that the reason might be that partial information networks tend to decrease the degrees, thereby decreasing degree heterogeneity. For example, Figure 9 illustrates that the degree distribution of 1077’s perceived network within $L = 2$ is more concentrated than that of the full network.

7. Discussion

For theory and methodology development, we analyze a centered-individual’s perceived network based on their knowledge depth $L = 2$. In practice, when the network is sparse, we can apply our algorithm to the perceived network within the knowledge depth, as we did for the political blog data analysis. An interesting question is whether we can analytically show that our algorithm (or some variant thereof) has good recovery results for individuals within the knowledge depth when connection probabilities are lower than the current setting?

In both simulation and real data studies, we report misclustering rates. But a good misclustering rate does not answer an important question: did the individuals of interest correctly identify their own memberships? The answers are fixed. Specifically, in simulation, our algorithm almost 100% identifies the correct community membership of individual 1; however in practice the results were more mixed. For the Zachary’s karate club data, we can correctly identify the memberships of four (out of six) individuals of interest: “H”, “A”, 20, and 32. While for political blog data, only individuals 1075 and 1076 are correctly identified. From a practical
point of view, this is not necessarily a problem, because after the individuals of interest have assigned everybody else into $K$ communities, they usually have some side information to decide (correct) their own affiliations. Alternatively, we could analyze the asymptotic properties of the eigenvectors of $B$ and design a correction step accordingly. This is a point that we would like to pursue in future studies.

The new individual-centered partial information network framework provides one possible cause for individuals’ heterogeneous information and ample opportunities to investigate its consequences. We end our article with five questions to stimulate thought: (1) how can one choose the number of communities $K$ adaptively from data? One possible solution is to estimate the number of communities by studying the differences between spiked eigenvalues and non-spiked eigenvalues (e.g., Fan et al. (2019); Cai et al. (2020)). But are there better alternatives? (2) suppose the individual of interest knows their edges accurately but does not know the immediate neighbors’ edges accurately (e.g., can only identify those edges with a high probability). Can we design new algorithms that achieve almost exact recovery, or is only partial recovery possible? (3) what new insight can we get by studying the more expansive $L = 3$ partial information networks? (4) in addition to community detection, how can we study other problems, such as link prediction, membership profile inference, importance ranking of individuals, and construction of centrality measures, under the individual-centered partial network information structure? (5) how can we combine multiple individuals’ partial information and collectively gain a better understanding of the social networks?
8. Supplement to “Individual-centered partial information in social networks”

The supplementary materials consist of proofs, additional simulation results, as well as some established results we cite in the proof for the readers’ convenience.

8.1. Proof of Lemma

To show [3], it suffices to prove

$$\| - \mathbf{E} \mathbf{S} (\mathbf{E} \mathbf{A}) \mathbf{E} \mathbf{S} + (\mathbf{E} \mathbf{A}) \mathbf{E} \mathbf{S} + \mathbf{E} \mathbf{S} (\mathbf{E} \mathbf{A}) \| = \alpha_p(\| \mathbf{B}_E \|), \quad \| \mathbf{E} \mathbf{B} \| = \alpha_p(\| \mathbf{B}_E \|). $$

By the definition of $\mathbf{S}$, $a_{ij}$ is independent of $\mathbf{S}$ for $i \geq 2$ and $j \geq 2$. By [2],

$$\mathbf{E} b_{ij} = \mathbf{E} a_{ij} (1 - (1 - \mathbf{E} a_{1i})(1 - \mathbf{E} a_{1j})), \quad 2 \leq i \neq j \leq n.$$  

Therefore $\mathbf{E} \mathbf{B}$ is equal to $- (\mathbf{E} \mathbf{S}) (\mathbf{E} \mathbf{A}) (\mathbf{E} \mathbf{S}) + (\mathbf{E} \mathbf{A}) (\mathbf{E} \mathbf{S}) + (\mathbf{E} \mathbf{S}) (\mathbf{E} \mathbf{A})$ except for the diagonal entries, the first row and first column. Then we have

$$\| \mathbf{E} \mathbf{B} - (\mathbf{E} \mathbf{S}) (\mathbf{E} \mathbf{A}) (\mathbf{E} \mathbf{S}) + (\mathbf{E} \mathbf{A}) (\mathbf{E} \mathbf{S}) + (\mathbf{E} \mathbf{S}) (\mathbf{E} \mathbf{A}) \| \lesssim \left( \sum_{j=1}^n (\mathbf{E} a_{1j})^2 \right)^{1/2} + p_n \leq (\sqrt{n} + 1)p_n.$$  

Therefore, it suffices to prove the first inequality of (25).

First of all, we look at the matrix

$$(\mathbf{E} \mathbf{S}) (\mathbf{E} \mathbf{A}) (\mathbf{E} \mathbf{S}).$$

Since $(\mathbf{E} \mathbf{S}) (\mathbf{E} \mathbf{A}) (\mathbf{E} \mathbf{S}) = (\mathbf{E} \mathbf{S}) \mathbf{V} \mathbf{D} \mathbf{V}^\top (\mathbf{E} \mathbf{S}), (\mathbf{E} \mathbf{S}) (\mathbf{E} \mathbf{A}) (\mathbf{E} \mathbf{S})$ has the same non-zero eigenvalues as $\mathbf{V}^\top (\mathbf{E} \mathbf{S})^2 \mathbf{V}$. The counterpart of $(\mathbf{E} \mathbf{S}) (\mathbf{E} \mathbf{A}) (\mathbf{E} \mathbf{S})$ in $\mathbf{B}_E$ is $\mathbf{S} (\mathbf{E} \mathbf{A}) \mathbf{S}$, whose non-zero eigenvalues are the same as $\mathbf{V}^\top \mathbf{S}^2 \mathbf{V}$. In this case $\mathbf{V}^\top (\mathbf{E} \mathbf{S})^2 \mathbf{V} = p_n^2 \mathbf{I}$ and $\mathbf{E} (\mathbf{V}^\top \mathbf{S}^2 \mathbf{V}) = p_n \mathbf{I}$, which means that we cannot replace $\mathbf{S}$ by $\mathbf{E} \mathbf{S}$ for $\mathbf{V}^\top \mathbf{S}^2 \mathbf{V}$. Similarly, we can show that $\mathbf{E} \| (\mathbf{E} \mathbf{A}) \mathbf{S} \|^2_F = \text{tr}(\mathbf{E} \mathbf{A} \mathbf{E} (\mathbf{S}^2) \mathbf{E} \mathbf{A}) = n^2 p_n^3 \gtrsim \| (\mathbf{E} \mathbf{A}) \mathbf{E} \mathbf{S} \|^2_F = n^2 p_n^4$. These insights combined with Condition $p_n = o(1)$ imply that

$$\| - \mathbf{E} \mathbf{S} (\mathbf{E} \mathbf{A}) \mathbf{E} \mathbf{S} + (\mathbf{E} \mathbf{A}) \mathbf{E} \mathbf{S} + \mathbf{E} \mathbf{S} (\mathbf{E} \mathbf{A}) \| = \alpha_p(\| \mathbf{B}_E \|).$$

8.2. Proof of Theorem

Note that $\det(y^2 \mathbf{I} - y \mathbf{D} \mathbf{V}^\top \mathbf{S} \mathbf{V} - \mathbf{D} (\mathbf{I} - \mathbf{V}^\top \mathbf{S} \mathbf{V}) \mathbf{D} \mathbf{V}^\top \mathbf{S} \mathbf{V})$ is a polynomial of $y$ with degree $2K$. Hence the equation $\det(y^2 \mathbf{I} - y \mathbf{D} \mathbf{V}^\top \mathbf{S} \mathbf{V} - \mathbf{D} (\mathbf{I} - \mathbf{V}^\top \mathbf{S} \mathbf{V}) \mathbf{D} \mathbf{V}^\top \mathbf{S} \mathbf{V}) = 0$ has $2K$ solutions in $y \in \mathbb{C}$. Moreover, as $\mathbf{V}^\top \mathbf{S} \mathbf{V}$ and $\mathbf{I} - \mathbf{V}^\top \mathbf{S} \mathbf{V}$ are invertible, $\det(\mathbf{D} (\mathbf{I} - \mathbf{V}^\top \mathbf{S} \mathbf{V}) \mathbf{D} \mathbf{V}^\top \mathbf{S} \mathbf{V}) \neq 0$; hence $y = 0$ is NOT a solution. Let $x = y^{-1}$, then $x^{-2K} \times \det (\mathbf{H}(x)) = \det (y^2 \mathbf{I} - y \mathbf{D} \mathbf{V}^\top \mathbf{S} \mathbf{V} - \mathbf{D} (\mathbf{I} - \mathbf{V}^\top \mathbf{S} \mathbf{V}) \mathbf{D} \mathbf{V}^\top \mathbf{S} \mathbf{V})$. Hence there are $2K$ non-zero solutions to $\det (\mathbf{H}(x)) = 0$ (i.e., [4]). Denote these solutions by $x_{-K}, \ldots, x_{-1}, x_1, \ldots, x_K$.

Then it remains to prove that for each $i \in [\pm K]$, $q_i$ satisfies [4], $q_i \neq 0$, and $x_i \in \mathbb{R}$. By the definitions of $q_{1i}$ and $q_{2i}$, we have

$$\left( - S(\mathbf{E} \mathbf{A}) \mathbf{S} + (\mathbf{E} \mathbf{A}) \mathbf{S} + S(\mathbf{E} \mathbf{A}) \right) (\mathbf{S} \mathbf{V} q_{1i} + (I - S) \mathbf{V} q_{2i})$$

$$= \mathbf{V} \mathbf{D} \mathbf{V}^\top \mathbf{S} \mathbf{V} q_{1i} + x_i \mathbf{S} \mathbf{V} (\mathbf{I} - \mathbf{V}^\top \mathbf{S} \mathbf{V}) \mathbf{D} \mathbf{V}^\top \mathbf{S} \mathbf{V} q_{1i}$$

$$= x_i^{-1} (\mathbf{V} q_{2i} + \mathbf{S} (\mathbf{q}_{1i} - \mathbf{q}_{2i})) = x_i^{-1} (\mathbf{S} \mathbf{V} q_{1i} + (I - S) \mathbf{V} q_{2i}).$$

(26)
where the second equation follows from
\[ q_{1i} - x_i D V^\top S V q_{1i} - x_i^2 D (I - V^\top S V) D V^\top S V q_{1i} = 0. \]  
(27)

Therefore \( q_i = SVq_{1i} + (I - S)Vq_{2i} \) is the eigenvector of \( -S(EA)S + (EA)S + S(EA) \) corresponding to the eigenvalue \( x_i^{-1} \) if \( q_i \neq 0 \). We prove \( q_i \neq 0 \) by contradiction. Actually, if \( q_i = 0 \), by the definition of \( q_i \), we have
\[
0 = \|q_i\|_2^2 \geq q_{1i}^\top V^\top S V q_{1i} = 0.
\]
Then \( (V^\top S V)^{1/2} q_{1i} = 0 \) and by (27), we have
\[
q_{1i} = x_i D V^\top S V q_{1i} + x_i^2 D (I - V^\top S V) D V^\top S V q_{1i} = 0,
\]
which contradicts with \( \|q_{1i}\|_2 = 1! \) Finally, since \( -S(EA)S + (EA)S + S(EA) \) is a real symmetric matrix, its eigenvalues \( x_{-K}, \ldots, x_{-1}, x_1, \ldots, x_K \) are real numbers. Without loss of generality, we can take \( x_i \leq x_j \) for all \( i < j \).

8.3. Proof of Theorem 2

Recalling that \( E A = VD V^\top \) we have
\[
E A - (I - S)E A (I - S) = (V, (I - S)V) \text{diag}(D, -D) (V, (I - S)V)^\top.
\]
(28)

By simple algebra, the non-zero eigenvalues of \( (V, (I - S)V) \text{diag}(D, -D) (V, (I - S)V)^\top \) are equal to the non-zero eigenvalues of
\[
\text{diag}(D, -D) (V, (I - S)V)^\top (V, (I - S)V) = \begin{pmatrix} D & 0 \\ 0 & -D \end{pmatrix} \begin{pmatrix} I & V^\top (I - S)V \\ V^\top (I - S)V & V^\top (I - S)V \end{pmatrix}.
\]
Since \( V^\top S V \) and \( V^\top (I - S)V \) are invertible, we have
\[
\det \left( \begin{pmatrix} I & V^\top (I - S)V \\ V^\top (I - S)V & V^\top (I - S)V \end{pmatrix} \right) = \det(V^\top (I - S)V - (V^\top (I - S)V)^2) = \det(V^\top S V V^\top (I - S)V) \neq 0.
\]

Combining this with (28), we have
\[
\text{rank} \left( -S(EA)S + (EA)S + S(EA) \right) = 2K.
\]

We will show that there exists \( q_{10} \) and \( q_{20} \) such that
\[
q_0 = S V q_{10} + (I - S)V q_{20}.
\]
(29)

Indeed, we have
\[
x_0^{-1} q_0 = B_E q_0 = \left( -S V D^\top S + V D^\top S + S V D^\top \right) q_0.
\]
(30)

Multiplying both sides of (30) by \( S \), we have
\[
x_0^{-1} S q_0 = S V (D^\top q_0).
\]
(31)
Similarly, multiplying both sides of (30) by \((I - S)\), we have
\[
x_{0}^{-1}(I - S)q_{0} = (I - S)V(DV^\top S)q_{0}.
\] (32)
Notice that \(q_{0} = Sq_{0} + (I - S)q_{0}\), by (31) and (32), (29) holds by defining \(q_{10} = x_{0}DV^\top q_{0}\) and \(q_{20} = x_{0}DV^\top S q_{0}\). Now it is ready for us to show that (7) and the statement below (8) hold. Substituting \(q_{0} = SVq_{10} + (I - S) V q_{20}\) and \(EA = VDV^\top\) into the eigenvalue definition
\[
B_{E}q_{0} = x_{0}^{-1}q_{0},
\]
we have the following equality
\[
VDV^\top SVq_{10} + SVD(I - V^\top SV)q_{20} = x_{0}^{-1}[SVq_{10} + (I - S) V q_{20}].
\] (33)
Multiplying \(V^\top (I - S)\) to both sides of (33), we have
\[
V^\top (I - S) VDV^\top SVq_{10} = x_{0}^{-1}V^\top (I - S) V q_{20}.
\]
This means that \(q_{20} = x_{0}DV^\top SV q_{10}\) if \(V^\top (I - S) V = I - V^\top SV\) is invertible. Substituting \(q_{20} = x_{0}DV^\top SV q_{10}\) into (33) and multiply both sides of (33) by \(V^\top S\), we see that \(q_{10}\) is should be an eigenvector of \(H(x_{0})\) corresponding to the zero eigenvalue if \(V^\top SV\) is invertible, therefore \(\det(H(x_{0})) = 0.\) Summarizing the arguments above, we finish our proof.

### 8.4. Proof of Lemma 2

By Condition 1 for sufficiently large \(n\), there exists a positive constant \(c\) such that
\[
V^\top (ES)V \leq p_{n}I \quad \text{and} \quad 2cp_{n}I \leq V^\top (ES + (p_{n} - EE_{11})e_{1}e_{1}^\top)V,
\]
in which \(e_{1} = (1, 0, \ldots, 0)^{\top} \in \mathbb{R}^{n}\). Condition 2 implies that
\[
\|V^\top(p_{n} - EE_{11})e_{1}e_{1}^\top V\|_{F} \leq \frac{p_{n}KC^{2}}{n}.
\]
Combining the three above inequalities together, we have
\[
2cp_{n}I \leq V^\top (ES)V \leq p_{n}I.
\] (34)
Then we study the relation between \(V^\top (ES)V\) and \(V^\top SV\). By Lemma 9 we have for any \(t > 0\),
\[
\mathbb{P}\left(\left|V_{i}^\top (S - ES)v_{j}\right| \geq t\right) \leq \exp\left(-\frac{t^{2}/2}{\sum_{l=1}^{n}v_{il}^{2}v_{jl}^{2}\text{var}(S_{il})} + \frac{t^{2}}{\bar{t}}\right),
\] (35)
in which \(L = \frac{C^{2}}{n}\) by Condition 2. It follows from Conditions 1 and 2 that,
\[
\sum_{l=1}^{n}v_{il}^{2}v_{jl}^{2}\text{var}(S_{il}) \leq p_{n} \sum_{l=1}^{n}v_{il}^{2}v_{jl}^{2} \leq \frac{C^{2}p_{n}}{n}.
\]
We choose \(t = 2c_{1}\sqrt{\log n} \cdot \sqrt{p_{n}\frac{C^{2}}{n}}\) for some constant \(c_{1} > 1\). With this choice of \(t\), under Condition 1 it holds for sufficiently large \(n\) that
\[
\frac{t^{2}/2}{\sum_{l=1}^{n}v_{il}^{2}v_{jl}^{2}\text{var}(S_{il})} + \frac{t^{2}}{\bar{t}} \geq c_{1}\log n.
\]
It follows from Condition 1 that \( p_n \sqrt{\frac{\log n}{np_n}} \gg t \). Moreover, since \( c_1 \) can any positive constant, by (35), with high probability we have

\[
-c_n \left( \sqrt{\frac{\log n}{np_n}} \right) I \leq V^T (S - E) V \leq p_n \left( \sqrt{\frac{\log n}{np_n}} \right) I. \tag{36}
\]

This combined with (34) implies that with high probability,

\[
c_n \left( 1 - \sqrt{\frac{\log n}{np_n}} \right) I \leq V^T S V \leq p_n \left( 1 + \sqrt{\frac{\log n}{np_n}} \right) I < \left( 1 - \frac{c}{2} \right) I.
\]

8.5. Proof of Corollary 1

By Lemma 2 with high probability, both \( V^T S V \) and \( I - V^T S V \) are invertible and the inequalities (9) hold. In the rest, we restrict ourselves to this high probability event \( A_1 \).

We will show that for suitably chosen \( q_i \), \( \dim(\text{span}\{q_i, l \in [\pm K]\}) = 2K \). Concretely, for any given pair \( i_0 \neq j_0 \), we consider two scenarios (I) and (II).

(I) \( x_{i_0} \neq x_{j_0} \). Note that \( -S(EEA)S + (EEA)S + SEAE \) is a real symmetric matrix and \( q_{i_0} \) and \( q_{j_0} \) are eigenvectors corresponding to distinct eigenvalues, then \( q_{i_0}^T q_{j_0} = 0 \).

(II) \( x_{i_0} = x_{j_0} \). In this scenario, the argument above does not directly apply. However, we can perturb the entries of \( A \) and \( S \) and replicate the argument, and then make the perturbation vanish in the limit. Concretely, we replace \( a_{ij} \) by \( \hat{a}_{ij} = a_{ij} + e^{-m} g_{ij} \), where \( m \geq n \), \( g_{ij} = g_{ji} \) and \( g_{ij} \) follows i.i.d. standard gaussian distribution for \( i \leq j \). Then the entries of \( \hat{A} = (\hat{a}_{ij}) \) are absolute continuous random variables. Then the entries of the matrix \( -\hat{S}(EEA)\hat{S} + (EEA)\hat{S} \) are invertible and the inequalities (9) are not equal almost surely (c.f. Knowles and Yin 2013). Similar to \( EAA = EEA = VDV^T \). By the tail probability of standard gaussian distribution, \( \max_j |\hat{a}_{1j} - a_{1j}| \leq \frac{p_n}{\log n} \) with high probability. This combined with Lemma 2 implies that \( V^T \hat{S} V \) and \( I - V^T \hat{S} V \) are invertible with high probability. Denote this high probability event by \( A_2 \). In the following, we restrict ourselves to \( A_1 \cap A_2 \). Note that with \( a_{ij} \) replaced by \( (\hat{a}_{ij}) \), counterparts of Theorems 1 and 2 hold by following exact the same proof, and we use notations \( \hat{q}_{i_1}, \hat{q}_{i_2}, \) and \( \hat{q}_i \) accordingly. By Theorem 2, \( \|\hat{q}_{i_1}\|_2 \neq 0, i \in [\pm K] \). Without loss of generality, we assume \( \|\hat{q}_{i_1}\|_2 = 1, i \in [\pm K] \). We denote the non-zero eigenvalues of \( -\hat{S}(EEA)\hat{S} + (EEA)\hat{S} + SEAE \) by \( \hat{x}_{-j}, j \in \{1, \ldots, \pm K\} \) and \( \hat{x}_j^{-1} \leq \hat{x}_j^{-1} \) for \( i < j \). Moreover, as almost surely, the non-zero real eigenvalues of \( -\hat{S}(EEA)\hat{S} + (EEA)\hat{S} + SEAE \) are not equal, we can just ignore the measure zero set and take \( \hat{x}_i^{-1} \leq \hat{x}_j^{-1} \) for \( i < j \). For the particular indexes \( i_0 \) and \( j_0 \), we have \( \hat{q}_{i_0}^T \hat{q}_{j_0} = 0 \).

By Weyl’s inequality, \( \lim_{m \to \infty} \hat{x}_{i_0} = x_{i_0} \) and \( \lim_{m \to \infty} \hat{x}_{j_0} = x_{j_0} \). Without loss of generality, assume that the limits \( \lim_{m \to \infty} \hat{q}_{i_0} \) and \( \lim_{m \to \infty} \hat{q}_{j_0} \) exist. Otherwise, because \( \|\hat{q}_{i_1}\|_2 = \|\hat{q}_{j_1}\|_2 = 1 < \infty \), we can always find a subsequence of \( \{m, m + 1, \ldots\} \) and take the limits on this subsequence. Denote by \( q_{i_0} = \lim_{m \to \infty} \hat{q}_{i_0} \) and \( q_{j_0} = \lim_{m \to \infty} \hat{q}_{j_0} \). It can be shown easily that \( q_{i_0} \) and \( q_{j_0} \) are unit eigenvectors of \( H(x_{i_0}) \) and \( H(x_{j_0}) \), respectively. Let \( q_{i_0} = S V q_{i_0} + (I - S) V q_{2i_0} \) and \( q_{j_0} = S V q_{j_0} + (I - S) V q_{2j_0} \). By the definition of \( \hat{q}_{i_0} \), we have

\[
[-\hat{S}(EEA)\hat{S} + (EEA)\hat{S} + SEAE] \hat{q}_{i_0} = \hat{x}_i^{-1} \hat{q}_{i_0}.
\]
Then

\[-S(EA)S + (EA)S + SEAS_q = \lim_{m \to \infty} \left[-\hat{S}(EA)S + (EA)S + SE\hat{A}\right]q_i = \lim_{m \to \infty} \left[\hat{x}_m^{-1} \hat{q}_i\right] = x_i^{-1} q_i.\]

Combining this with Lemma \[2\] \(q_i\) (\(q_j\)) is not equal to \(0\) and it is the eigenvector of

\[-S(EA)S + (EA)S + SE(A)\]

corresponding to \(x_i^{-1} (x_j^{-1})\). Moreover,

\[q_i^\top q_j = (\lim_{m \to \infty} \hat{q}_i) \hat{q}_j = \lim_{m \to \infty} \hat{q}_i \hat{q}_j = 0, \text{ a.s.}\]

In the above, one should note that the limit is taken on \(m\) while \(n\) is fixed.

Therefore we can finish our proof of the first statement \[\[10\].\] The second statement follows from \[\[10\]\] directly.

### 8.6. Proof of Theorem 3

Because \(\det(I - AB) = \det(I - BA)\), \(\det(H(x)) = 0\) (i.e., \[3\]) is equivalent to

\[\det(I - x(V^\top SV)^{1/2}D(V^\top SV)^{1/2} - x^2 (V^\top SV)^{1/2}D(I - V^\top SV)D(V^\top SV)^{1/2}) = 0.\]  

(37)

Let \(A_1 = (V^\top SV)^{1/2}D(V^\top SV)^{1/2}\), \(A_2 = (V^\top SV)^{1/2}D(I - V^\top SV)D(V^\top SV)^{1/2}\) and \(y = 1/x\). Then \[37\] becomes

\[\det(y^2 I - yA_1 - A_2) = 0.\]  

(38)

By Lemma \[2\] and Condition \[3\] with high probability we have

\[\|A_1\| \leq |d_1| : \|V^\top SV\| \leq |d_1| \cdot p_n \left(1 + \sqrt{\frac{\log n}{n p_n}}\right) \lesssim n p_n^2.\]  

(39)

Also by Lemma \[2\] and Condition \[3\] it holds with high probability that

\[n^2 p_n^3 I \lesssim \lambda_K(V^\top SV) \lambda_K(D^2) I \lesssim A_2 \leq \lambda_1(V^\top SV) \lambda_1(D^2) I \lesssim n^2 p_n^3 I.\]  

(40)

Combining \[39\] and \[40\], we have with high probability,

\[A_1 \lesssim n p_n^2 I \lesssim \frac{1}{n p_n} A_2.\]  

(41)

Similarly, we also have

\[A_1 \gtrsim n p_n^2 I.\]  

(42)

For \(y\) to be a solution for \[38\], it must hold that

\[\lambda_K(yA_1 + A_2) I \leq y^2 I \leq \lambda_1(yA_1 + A_2) I.\]

Combining this with \[40\] and \[41\], we have

\[|y|n p_n^2 I + n^2 p_n^3 I \lesssim y^2 I \lesssim |y|n p_n^2 I + n^2 p_n^3 I.\]
Therefore we have

\[-|y_i|np_n^2 + n^2p_n^3 \lesssim y_i^2 \lesssim |y_i|np_n^2 + n^2p_n^3, \quad i = \pm 1, \ldots, \pm K.\]  

(43)

For the right inequality that \( y_i^2 \lesssim |y_i|np_n^2 + n^2p_n^3 \), by the solutions to quadratic equation in one unknown, we imply that

\[|y_i| \lesssim np_n^2 + \sqrt{n^2p_n^4 + 4n^2p_n^3} \lesssim n^{3/2}.\]

Similarly considering the left inequality of (43), we have

\[|y_i| \gtrsim -np_n^2 + \sqrt{n^2p_n^4 + 4n^2p_n^3} \gtrsim n^{3/2}.\]

It follows from the two above inequalities that \(|x_i|^{-1} = |y_i| \sim n^{3/2}\). This conclude the proof of the first claim.

When \( p_n \to 0 \), by (41), we have \( |y_i|A_1 \lesssim p_n^{1/2}A_2 \ll A_2 \). Then to approximately solve for \( y \) in (38), we solve \( z \) in the following determinant equation

\[\det(z^2I - A_2) = 0.\]

(44)

The left hand side of (44) is a \( 2K \) polynomial of \( z \); therefore (44) has \( 2K \) solutions in \( z \). A straightforward calculation shows that (44) has \( 2K \) solutions \( \pm \sqrt{\lambda_i(A_2)} \), \( i \in [K] \). By Lemma 2 and an intermediate step (36) in its proof, it holds with high probability that

\[|\lambda_i(V^\top(ES)V)D(I - V^\top(ES)V)D) - \lambda_i(A_2)|

= |\lambda_i(V^\top(ES)V)D(I - V^\top(ES)V)D) - \lambda_i(V^\top SVD(I - V^\top SVD))| \lesssim n^2p_n^3 \left(\sqrt{\frac{\log n}{np_n}}\right).

Combining this with Condition 5, we have

\[1 + c_1 \leq \frac{\lambda_{i-1}(A_2)}{\lambda_i(A_2)}, \quad i = 2, \ldots, K,\]  

(45)

for some positive constant \( c_1 \). For \( i = 1, \ldots, K \), let \( y_{1i} = \sqrt{\lambda_1(A_2) + p_n^{5/6}n^{1/2}}/\|A_1\| \) and \( y_{2i} = \sqrt{\lambda_i(A_2) - p_n^{5/6}n^{1/2}}/\|A_1\| \), by (40), (41), (42), (45) and Weyl’s inequality, it can be shown that with high probability

\[\lambda_{i-1}(y_{1i}A_1 + A_2) \geq \lambda_{i-1}(A_2) - |y_{1i}||A_1| \geq (1 + c_1)\lambda_i(A_2) - n^2p_n^{7/2-1/100} > y_{1i}^2 \sim \lambda_i(A_2) - (-1)^l n^2p_n^{10/3},\]  

(46)

\[\lambda_{i+1}(y_{1i}A_1 + A_2) \leq \lambda_{i+1}(A_2) + |y_{1i}||A_1| \leq (1 + c_1)^{-1}\lambda_i(A_2) + n^2p_n^{7/2-1/100} < y_{1i}^2 \sim \lambda_i(A_2) - (-1)^l n^2p_n^{10/3}, l = 1, 2.\]  

(47)

By Weyl’s inequality and (46), (47), we have

\[\lambda_i(y_{1i}A_1 + A_2) \leq \lambda_i(A_2) + |y_{1i}||A_1| < y_{1i}^2, \quad \lambda_i(y_{2i}A_1 + A_2) > \lambda_i(A_2) - |y_{2i}||A_1| > y_{2i}^2.\]  

(48)
It follows from \cite{16}-\cite{18} that
\[
\det(y_i^2I - y_{1i}A_1 - A_2) \cdot \det(y_i^2I - y_{2i}A_1 - A_2)
= \prod_{j=1}^{K} \left( y_{1i}^2 - \lambda_j(y_{1i}A_1 + A_2) \right) \prod_{j=1}^{K} \left( y_{2i}^2 - \lambda_j(y_{2i}A_1 + A_2) \right) < 0.
\]
(49)

Since \(\det(y_i^2I - yA_1 - A_2)\) is a continuous function of \(y_i\), there exists one \(y_i \in [y_{2i}, y_{1i}]\) satisfying \(37\), or equivalently \(38\). Moreover, by \(40\), \(41\), \(42\) and \(45\), the intervals \([y_{2i}, y_{1i}]\) are non-overlapping for different \(i\). Hence the second claim of Theorem \(3\) holds for \(i \in [K]\). Similarly, the second claim of Theorem \(3\) holds for \(\lambda < 0\) by almost the same proof if we define \(y_{1i} = -\sqrt{\lambda K+1}(A_2) + p_i^{5/6} n^{1/2} \sqrt{\|A_1\|}\) and \(y_{2i} = -\sqrt{\lambda K+1}(A_2) - p_i^{5/6} n^{1/2} \sqrt{\|A_1\|}\).

8.7. Proof of Lemma \(3\)

Notice that the adjacency matrix with global information can be decomposed by
\[
A = \sum_{1 \leq i \leq j \leq n} \frac{1 + \mathbb{1}(i \neq j)}{2} a_{ij} \left( e_i^{(n)}(e_j^{(n)})^\top + e_j^{(n)}(e_i^{(n)})^\top \right),
\]
where \(e_i^{(n)}\) the \(i\)th element of the standard basis of \(\mathbb{R}^n\), and that \(\{a_{ij}\}_{1 \leq i \leq j \leq n}\) are independent. Let
\[
X_{ij} = \frac{1 + \mathbb{1}(i \neq j)}{2} (a_{ij} - E a_{ij}) \left( e_i^{(n)}(e_j^{(n)})^\top + e_j^{(n)}(e_i^{(n)})^\top \right),
\]
and \(A_{ij}^2 = \frac{1 + \mathbb{1}(i \neq j)}{2} \text{var}(a_{ij}) \left( e_i^{(n)}(e_i^{(n)})^\top + e_j^{(n)}(e_j^{(n)})^\top \right), \ i \leq j. \) Then \(A - E A = \sum_{i \leq j} X_{ij}\). Notice that \(\|X_{ij}\| \leq 1\), \(E \left( X_{ij}^p \right) \leq \frac{p^2}{2} \cdot \sigma^2 = \frac{1}{2} \cdot \sigma^2\) for \(1 \leq i \leq j \leq n\), \(R = 1\), \(p \geq 2\) and \(\|\sum_{i \leq j} A_{ij}^2\| = \sigma^2 = \max_{1 \leq i \leq n} \sum_{j=1}^{n} \text{var}(a_{ij})\), hence Lemma \(10\) holds. With high probability we have
\[
\|A - E A\| \lesssim \sqrt{(\log n)n p_n}.
\]
(50)

Combining this with the fact that \(\|S\| \leq 1\), it holds with high probability that
\[
\|B - B_E\| \lesssim \sqrt{(\log n)n p_n}.
\]

8.8. The matrix \(A\) and no-self loop

In network analysis, if there is no self-loop, then the diagonal entries of the adjacency matrix are 0’s. In this case we should analyze \(\tilde{A} = A - \text{diag}(a_{11}, \ldots, a_{nn})\) instead of \(A\) and therefore individual 1’s perceived adjacency matrix is \(\tilde{B} = -\tilde{S}A + \tilde{A}S + \tilde{A} \tilde{S}\), where \(\tilde{S} = S - \text{diag}(a_{11},0, \ldots, 0)\). In this case, \(E \tilde{A}\) may not be a low-rank matrix, while \(E A\) is. For instance, we look at a simple SBM with \(n = 4\), where the corresponding \(4 \times 4\) expected adjacency matrix with self-loop can be expressed as
\[
E A = \begin{pmatrix}
   p_{11} & p_{11} & p_{12} & p_{12} \\
p_{11} & p_{11} & p_{12} & p_{12} \\
p_{21} & p_{21} & p_{22} & p_{22} \\
p_{21} & p_{21} & p_{22} & p_{22}
\end{pmatrix}
\]
in which \( p_{12} = p_{21} < p_{11} = p_{22} \), and \( p_{ij} \in (0, 1) \). Clearly \( \text{rank}(\mathbf{EA}) = 2 \). In contrast, the expectation of the matrix \( \tilde{\mathbf{A}} \) is

\[
\mathbf{E}\tilde{\mathbf{A}} = \begin{pmatrix}
0 & p_{11} & p_{12} & p_{12} \\
p_{11} & 0 & p_{12} & p_{12} \\
p_{21} & p_{21} & 0 & p_{22} \\
p_{21} & p_{21} & p_{22} & 0
\end{pmatrix}.
\]

It is easy to show that \( \text{rank}(\mathbf{E}\tilde{\mathbf{A}}) = 4 \), so \( \mathbf{E}\tilde{\mathbf{A}} \) has the full rank.

There is a rich line of network literature that assumes a low-rank structure of \( \mathbf{A} \), including Zhao et al. (2012), Abbe (2017), Abbe et al. (2017), and Zhang et al. (2020). Moreover, in this paper we consider the case that the largest degree of the nodes are bounded. By the equation \( \tilde{\mathbf{A}} = \mathbf{EA} + (\tilde{\mathbf{A}} - \mathbf{EA}) \) and \( \|\tilde{\mathbf{A}} - \mathbf{EA}\| \sim \|\tilde{\mathbf{A}} - \mathbf{EA}\| \to \infty \), \( \tilde{\mathbf{A}} - \mathbf{EA} \) can be regarded as the “noise matrix” of the model. Hence the noise level (measured by spectral norm) of \( \tilde{\mathbf{A}} \) is not changed compared to \( \mathbf{A} \).

In other words, the signal matrix of \( \tilde{\mathbf{A}} \) is essentially \( \mathbf{EA} \). Therefore a major term of \( \tilde{\mathbf{B}} \) is also a major term of \( \mathbf{B} \), and Theorems\[1\]\[8\] can be applied too. On the other hand, by the definition of Bernoulli random variables, \( \|\text{diag}(a_{11}, a_{22}, \ldots, a_{nn})\| \leq 1 \). By checking the proofs carefully, the community detection results from Theorem\[4\] to Theorem\[7\] mainly rely on the order of the gap between \( \|\mathbf{B} - \mathbf{B}_E\| \) and the smallest non-zero eigenvalue (in magnitude) of \( \mathbf{B}_E \). It is essentially the same as the gap between \( \|\mathbf{B} - \mathbf{B}_E\| \) and the smallest non-zero eigenvalue (in magnitude) of \( \mathbf{B}_E \), where the difference between \( \mathbf{S} \) and \( \hat{\mathbf{S}} \) can be shown to have a negligible effect on this gap. Therefore Theorems\[1\]\[7\] hold for the stochastic block model without self-loop.

Given the above arguments, throughout this paper, we only consider \( \mathbf{A} \) (instead of \( \tilde{\mathbf{A}} \)) for convenience.

### 8.9. Proof of Lemma\[4\]

As we have assumed that individual 1 belongs to community 1, \( \mathbb{P}(a_{1l} = 1) = \pi_i^\top \mathbf{P} \pi_l = \mathbf{e}_l^\top \mathbf{P} \pi_l \), for \( l \in [n] \). Then in view of the definition of \( p_n \) and \( \min_{k \in [K]} p_{lk} \sim p_n \), it follows that \( \min_{l \geq 2} \mathbb{P}(a_{1l} = 1) \sim p_n \).

Moreover, \( 1 - c \geq p_n \gg (\log n/n)^{1/2} \gg \log n/n \). Therefore Condition\[1\]\[4\] is validated.

By \( \min_{k \in [K]} \sum_{j=1}^n 1(\pi_j = \mathbf{e}_k) \geq c_0 n, \sigma_K(\mathbf{P}) \geq c_1 p_n \), we have

\[
\text{rank}(\mathbf{EA}) = \text{rank}(\mathbf{P}\Pi\Pi^\top) = K.
\]

Recall the eigen decomposition \( \mathbf{EA} = \mathbf{VDV}^\top \), in which \( \mathbf{V} = (\mathbf{v}_1, \ldots, \mathbf{v}_K) \). By the structure of the stochastic block model and \( \text{rank}(\mathbf{EA}) = K \), there are \( K \) different rows in \( \mathbf{V} \) corresponding to the communities and hence there are \( K^2 \) different values in \( \mathbf{V} \) at most. Indeed, let \( \mathbf{v}_k(l) \) be the \( l \)-th entry of \( \mathbf{v}_k \), by the definition of eigenvector, we have

\[
(\mathbf{EA})\mathbf{v}_k = d_k \mathbf{v}_k, \quad \text{for } k \in [K],
\]

and therefore

\[
\sum_{j=1}^n (\pi_l^\top \mathbf{P} \pi_j \mathbf{v}_k(j)) = d_k \mathbf{v}_k(l), \quad \text{for } k \in [K] \text{ and } l \in [n].
\]
For any \( l_1 \neq l_2 \) with \( \pi_{l_1} = \pi_{l_2} \), we have \( \pi_{l_1}^T P \pi_j = \pi_{l_2}^T P \pi_j \), \( j = 1, \ldots, n \) and therefore
\[
\sum_{j=1}^{n} \left( \pi_{l_1}^T P \pi_j v_k(j) \right) = \sum_{j=1}^{n} \left( \pi_{l_2}^T P \pi_j v_k(j) \right) = d_k v_k(l_1) = d_k v_k(l_2).
\]

Notice that \( \pi_i \in \{e_1, \ldots, e_K\}, \ i \in [n] \). Then we conclude that \( V \) has at most \( K \) different rows and \( \{v_k(l), k \in [K], l \in [n]\} \) only has at most \( K^2 \) distinct values. Moreover, as \( \pi_{l_1} \neq \pi_{l_2} \) means that \( l_1 \) and \( l_2 \) belong to different communities, the rows of \( V \) with distinct values are corresponding to different communities. Since \( \text{rank}(E A) = \text{rank}(V D V^T) = K \), \( \text{rank}(V) = K \) and therefore \( V \) contains exactly \( K \) different rows.

Without loss of generality, assume that the first \( K \) rows of \( V \) are different and we denote this \( K \times K \) matrix by \( V^{(K)} \). Since distinct row values are corresponding to different communities, the first \( K \) rows of \( \Pi \) are different. Noticing that \( \pi_i \in \{e_1, \ldots, e_K\}, \ i \in [n] \): without loss of generality, we assume the first \( K \) rows of \( \Pi \) equal to \( I \). Let \( D = V^{(K)} \). Then it follows that
\[
V = \Pi D. \tag{51}
\]

Because \( D^T \Pi^T \Pi D = V^T V = I \), we have \( DD^T \geq \frac{1}{P} I \), where \( P = \max_{k \in [K]} \sum_{j=1}^{n} 1(\pi_j = e_k) \). Therefore, (12) is proved. Moreover, by the condition that \( \min_{k \in [K]} \sum_{j=1}^{n} 1(\pi_j = e_k) \geq c_0 n \), we have
\[
DD^T \leq \frac{1}{c_0 n} I.
\]
Therefore we have
\[
\|D\|_{\text{max}} \leq \frac{1}{\sqrt{c_0 n}}.
\]
Combining this with (51), Condition 2 holds. Now we prove Condition 3. Notice that \( \Pi^T \Pi = \pi_1 \pi_1^T + \ldots + \pi_n \pi_n^T \) is a \( K \times K \) diagonal matrix whose diagonal elements are \( \sum_{j=1}^{n} 1(\pi_j = e_k) \), \( k \in [K] \). By Condition 6 we have
\[
d_k^2 = \sigma_k^2(\Pi D) = \sigma_k^2(\Pi P \Pi^T) = \lambda K(\Pi P \Pi^T)^2 \\
\geq \lambda K(\Pi^T \Pi) \lambda K(P^2) = \sigma_k^2(\Pi^T \Pi) \sigma_k^2(P) \gtrsim n^2 w_n^2.
\]
This, combined with
\[
\|E A\|_F = \left[ \sum_{j=1}^{n} \sum_{i=1}^{n} (P(a_{ij} = 1))^2 \right]^{1/2} \leq np_n,
\]
implies that \( |d_1| \sim \ldots \sim |d_K| \sim np_n \), which is Condition 3.

8.10. Proof of Lemma 5
Substituting (15) into (13) we have,
\[
I = Q^T Q = Q^T \text{diag} \left( \Pi^T S P_n \Pi, \Pi^T (I - S) \Pi \right) Q. \tag{52}
\]
By Lemma 2 and 12 in Lemma 3 there exists a positive constant $c_2$ such that with high probability, 

$$\text{diag}\left(\Pi^T \frac{S}{p_n} \Pi, \Pi^T (I - S) \Pi\right) = \text{diag}(D^T, D)^{-1}\text{diag}\left(V^T \frac{S}{p_n} V, V^T (I - S) V\right) \text{diag}(D, D)^{-1} \leq c_2nI.$$  

By (52) and (53), with high probability we have

$$QQ^\top \geq (c_2n)^{-1}I.$$

Then by the definition of $Q$ in (15), we have

$$DQ_2Q_2^\top D^\top \geq (c_2n)^{-1}I \quad \text{and} \quad p_nDQ_1Q_1^\top D^\top \geq (c_2n)^{-1}I.$$

8.11. Proof of Lemma 6

By 14 and Corollary 1, $S\Pi I$ and $(I - S) \Pi$ contain $K$ different non-zero rows in each matrix; without loss of generality, these rows can be rearranged as $2K \times 2K$ identity matrix $I_{2K} = \text{diag}(I_K, I_K)$, where the two $I_K$’s correspond to the different non-zero rows of $S\Pi I$ and $(I - S) \Pi$ respectively. For (17), without loss of generality, assume that $\pi_i = e_1$ and $\pi_j = e_2$, $a_{ii} = a_{jj} = 1$. The other cases $a_{ii} \neq a_{jj}$ and $a_{ii} = a_{jj} = 0$ can be proved similarly.

By (16) in Lemma 5, with high probability, uniformly for $j$ we have

$$\|Q(i)O - Q(j)O\|_2 \geq \|e^{(n)}_i - e^{(n)}_j\|^2 = \frac{1}{c_2n}.$$ 

For (18), without loss of generality, assume that $\pi_i = \pi_j = e_1$, $a_{ii} = 1 - a_{jj} = 1$. Similar to the inequality above, we have

$$\|Q(i)O - Q(j)O\|_2 \geq \|e^{(2K)}_1 - e^{(2K)}_2\|^2 = \frac{1}{c_2n}.$$ 

The implication (19) follows from the expression of $Q$ in (14) and the fact that $\|Q(i)O - Q(j)O\|_2 = \|Q(i) - Q(j)\|_2$ for an orthogonal matrix $O$.

8.12. Proof of Lemma 7

By Lemma 3 with high probability we have

$$\|B - B_E\| \leq \sqrt{(\log n)np_n}.$$ 

Note that $Q$ consists of unit eigenvectors of $B_E$ and Lemma 4 validates the first statement in Theorem 3. Then by Davis-Kahan theorem in Davis and Kahan (1970) (c.f. Theorem 10 in Cai et al. (2013)), with high probability we have
\[ \|WW^T - QQ^T\|_F = O \left( \frac{\sqrt{\log n} np_n}{np_n^{3/2}} \right) = O \left( \frac{\sqrt{\log n}}{np_n} \right). \]

Moreover, it follows from the definition of \( O \) that
\[
\|W - QO\|_F^2 = \text{tr} \left[ (W - QU_1U_2^T)(W - QU_1U_2^T) \right]
= \text{tr} \left( W^T W + U_2U_1^T QU_1U_2^T - 2U_2U_1^T QW \right)
= 4K - 2\text{tr}(U_2U_1^T Q^T W) \leq 4K - 2\text{tr}(Q^T WW^T Q)
= \|WW^T - QQ^T\|_F^2,
\]
where the inequality follows from \( \|\Sigma\| = \|Q^T W\| \leq 1 \) and \( \text{tr}(U_2U_1^T Q^T W) = \text{tr}(\Sigma) \geq \text{tr}(\Sigma^2) = \text{tr}(Q^T WW^T Q) \). Therefore (20) is proved.

8.13. Proof of Lemma 6
In view of Lemma 6, we have
\[
\|c - Q(j)O\|_2 \geq \|Q(i)O - Q(j)O\|_2 - \|c - Q(i)O\|_2 > \sqrt{2c^2n} - \frac{1}{\sqrt{2c^2n}} = \frac{1}{\sqrt{2c^2n}}.
\]

8.14. Proof of Theorem 4
First, we show that \( S\Pi \) has \( K \) different nonzero rows with high probability. Concretely, we will show that
\[
\mathbb{P}(\exists k \in [K]: a_{1i} = 0 \text{ for all } i \in [n] \text{ such that } \pi_i = e_k) = O(n^{-D}),
\]
for sufficiently large \( n \), where \( D \) is any positive constant. In fact, by Condition 6 for sufficiently large \( n \) depending on \( D \), (55) follows from the inequality that
\[
\text{L.H.S of (55)} \leq \sum_{k \in [K]} \mathbb{P}(a_{1i} = 0 \text{ for all } i \in [n] \text{ such that } \pi_i = e_k)
\leq \sum_{k \in [K]} \left[ 1 - \min_i \mathbb{P}(a_{1i} = 1) \right] \sum_{i=1}^{n} e^{-\frac{\sum_{i=1}^{n} \|\pi_i = e_k\|}{\sqrt{n}}}
= O \left( e^{-\frac{\sum_{i=1}^{n} \|\pi_i = e_k\|}{\sqrt{n}}} \right) = O(n^{-D}).
\]
By the decomposition equation \([14]\) of \( Q \), we have \( SQ = S\Pi DQ_1 \). In view of Lemma 5 \( DQ_1(DQ_1)^T \) is a \( K \times K \) positive definite matrix, and so \( SQ \) has \( K \) different non-zero rows with high probability. Then, as \( O \) is an orthogonal matrix, \( SQO \) has the same property. Similarly, we can show that \( (I - S)QO \) has \( K \) different non-zero rows with high probability.

Then, as \( \{c_i, i \in [n]\} \) are the centroids returned by \([21]\), we have with high probability,
\[
\|SW - SC\|_F \leq \|SW - SQO\|_F,
\]
and
\[
\|(I - S)W - (I - S)C\|_F \leq \|(I - S)W - (I - S)QO\|_F.
\]
Then it follows
\[
\|SC - SQO\|_F \leq \|SW - SC\|_F + \|SW - SQO\|_F \leq 2\|SW - SQO\|_F,
\]
and
\[
\|(I - S)C - (I - S)QO\|_F \leq \|(I - S)W - (I - S)C\|_F + \|(I - S)W - (I - S)QO\|_F \\
\leq 2\|(I - S)W - (I - S)QO\|_F.
\]
Combining this with Lemma 7, we conclude that with high probability
\[
|\mathcal{M}| = \sum_{i \in \mathcal{M}} 1 \leq 2c_0n \sum_{i \in \mathcal{M}} \|c_i - Q(i)O\|_F^2 \leq 2c_0n \sum_{i \in [n]} \|c_i - Q(i)O\|_F^2 \\
= 2c_0n \left( \|SC - SQO\|_F^2 + \|(I - S)C - (I - S)QO\|_F^2 \right) \\
\leq 8c_0n \left( \|SW - SQO\|_F^2 + \|(I - S)W - (I - S)QO\|_F^2 \right) \\
= O \left( n\|SW - SQO\|_F^2 + \|(I - S)W - (I - S)QO\|_F^2 \right) \\
= O \left( \log n/p_n^2 \right). \tag{56}
\]

8.15. Proof of Theorem 5

As \(c_{f_0(i)} = d_i\) for \(i \in [K]\), the matrix \(D\) is formed by a column permutation of \(C\). Therefore, we have
\[
\text{rank}(D) = \text{rank}(C) = K,
\]
which implies that columns of \(D\) are distinct. By the definition of \(f_0\), clearly we have
\[
f = f_0 \implies C(f_0, f_0) = D(f_0, *).
\]
On the other hand, if \(f \neq f_0\), there exists a \(j_0 \in [K]\) such that
\[
f(j_0) \neq f_0(j_0).
\]
This, combined with the fact the columns of \(D\) are all distinct, implies
\[
(c_{f(1)}f(j_0), \ldots, c_{f(K)}f(j_0))^T \neq (d_{f(1)}j_0, \ldots, d_{f(K)}j_0)^T.
\]
Hence there exists \(i_0 \in [K]\) such that
\[
c_{f(i_0)}f(j_0) \neq d_{f(i_0)}j_0.
\]
In other words, \(C(f, f) \neq D(f, *)\).

8.16. Proof of Theorem 6

Since \(p_n \gg (\log n/n)^{1/4}\), we have \(np_n \gg \log n/p_n^2\). This, together with Lemma 9, Theorem 4 and \(\min_{k \in [K]} \sum_{j \in [n]} \|\pi_j = e_k \geq c_0n\) (in Condition 6), implies that for each \(k \in [K]\), any
positive constant $D$ and sufficiently large $n$, we have

$$\mathbb{P}(\mathcal{M}^c \cap \{ i \in [n] : a_{1i} = 1, \pi_i = e_k \} = \emptyset) \leq \mathbb{P}(\{ i \in [n] : a_{1i} = 1, \pi_i = e_k \} \leq |\mathcal{M}|)$$

$$= \mathbb{P} \left( \sum_{i \in [n]: \pi_i = e_k} a_{1i} \leq |\mathcal{M}| \right) \leq \mathbb{P} \left( \sum_{i \in [n]: \pi_i = e_k} a_{1i} \leq C \cdot \frac{\log n}{p_n} \right) \leq n^{-D}. \quad (57)$$

Similarly, we have

$$\mathbb{P}(\mathcal{M}^c \cap \{ i \in [n] : a_{1i} = 0, \pi_i = e_k \} = \emptyset) \leq n^{-D}. \quad (58)$$

Therefore, there exist $j_1 \in \mathcal{M}^c \cap \{ i \in [n] : a_{1i} = 1 \}$ and $j_2 \in \mathcal{M}^c \cap \{ i \in [n] : a_{1i} = 0 \}$ such that $\pi_{j_1} = \pi_{j_2} = e_k$ with high probability. In other words, for both $r = 0$ and $r = 1$, the set $\{ \pi_i : i \in \mathcal{M}^c \text{ and } a_{1i} = r \}$ contains $K$ different vectors with high probability. Therefore, by Lemma 8, $\{ c_i : i \in \mathcal{M}^c \}$ contains $2K$ different vectors. Moreover for $i, j \in \mathcal{M}^c$,

$$c_i \neq c_j, a_{1i} = a_{1j} \iff \pi_i \neq \pi_j, a_{1i} = a_{1j}, \quad (59)$$

which means that in Step 2, for $i, j \in \mathcal{M}^c$, with high probability $i$ and $j$ are assigned to the same cluster if and only if $a_{1i} = a_{1j}$ and $\pi_i = \pi_j$. By [4], $Q$ has $2K$ different rows

$$R = \{ e_i^\top DQ_1, e_i^\top DQ_2, \ldots, e_K^\top DQ_1, e_K^\top DQ_2 \}. \quad \text{(59)}$$

Notice that, for $l = 1, 2$, respectively, $e_k$ reflects the membership of individual $i$ if $\pi_i = e_k$; hence $\{ e_i^\top DQ_l, \ldots, e_K^\top DQ_l \} \subset R$ can be regarded as the “membership” vectors for the individuals $\{ i \in [n] : a_{1i} = 2 - l \}$. By [50], with high probability, in step 2 of Algorithm 2, $i \in \mathcal{M}^c \cap \{ i \in n : a_{1i} = 2 - l \}$ is assigned to the cluster associated with $e_i^\top DQ_l$ if $\pi_i = e_k$.

Therefore, with high probability we have

$$|i \in [n] : \{ \text{In step 2, } i \text{ is not assigned to the cluster associated with } e_i^\top DQ_1 \text{ or } e_i^\top DQ_2 \}| \leq |\mathcal{M}|. \quad \text{(60)}$$

By Theorem 4, under Condition 6, we have

$$|\mathcal{M}| = O \left( \frac{\log n}{p_n^2} \right). \quad (60)$$

We say the 1st group is $\{ i \in [n] : a_{1i} = 1 \}$ and the 2nd group is $\{ i \in [n] : a_{1i} = 0 \}$. Let $S_l(k)$ be the collection of individuals belonging to $k$-th community in the $l$-th group, and $\hat{S}_l(k)$ be the $k$-th cluster of the $l$-th group returned by step 2. Let $\hat{p}_{k_1,k_2}^{(1)}$ and $\hat{p}_{k_1,k_2}^{(2)}$ be the $(k_1,k_2)$-th entry of $\hat{P}_{S,S}$ and $\hat{P}_{S,S}$ (defined by equation (24)) respectively. Let $P_{S,S} = (p_{k_1,k_2}^{(1)})$ and $P_{S,S-1-S} = (p_{k_1,k_2}^{(2)})$ be the corresponding population versions. Note that there exists a unique permutation function $f_0$ such that

$$P_{f_0,f_0}^{S,S} = P_{f_0,*}^{S,S-1-S}. \quad (61)$$

Without loss of generality, we assume $P_{S,S-1-S} = P$. Then

$$\hat{p}_{k_1,k_2}^{(l)} - p_{k_1,k_2}^{(l)} = \frac{1}{|S_1(k_1)||S_1(k_2)|} \left[ \sum_{i \in S_1(k_1) \cap \hat{S}_1(k_1) \cap S_1(k_2) \cap \hat{S}_1(k_2)} (b_{ij} - \hat{p}_{k_1,k_2}^{(l)}) + \sum_{i \in \hat{S}_1(k_1) \setminus S_1(k_1) \cap S_1(k_2) \cap \hat{S}_1(k_2)} (b_{ij} - \hat{p}_{k_1,k_2}^{(l)}) + \sum_{i \in \hat{S}_1(k_1) \setminus S_1(k_1) \cap \hat{S}_1(k_2)} (b_{ij} - \hat{p}_{k_1,k_2}^{(l)}) \right], \quad (62)$$

where
Moreover, by (60), with high probability we have

\[ \frac{1}{|S_1(k_1)||S_1(k_2)|} \sum_{i \in S_1(k_1), j \in S_1(k_2)} (b_{ij} - p_{k_1,k_2}^{(l)}) = \frac{1}{|S_1(k_1)||S_1(k_2)|} \left[ \sum_{i \in S_1(k_1) \cap S_1(k_1), j \in S_1(k_2) \cap S_1(k_2)} (b_{ij} - p_{k_1,k_2}^{(l)}) + \sum_{i \in S_1(k_1) \setminus S_1(k_1), j \in S_1(k_2) \cap S_1(k_2)} (b_{ij} - p_{k_1,k_2}^{(l)}) + \sum_{i \in S_1(k_1) \cap S_1(k_1), j \in S_1(k_2) \setminus S_1(k_2)} (b_{ij} - p_{k_1,k_2}^{(l)}) \right]. \] (63)

By Lemma 9 with high probability, we have

\[ |S_1(k)| = \sum_{i \in [n], \pi_i = e_k} a_{1i} \sim \sum_{i \in [n], \pi_i = e_k} \mathbb{E} a_{1i} \sim np_n, \quad k \in [K], \] (64)

and

\[ |S_2(k)| = \sum_{i \in [n], \pi_i = e_k} (1 - a_{1i}) \sim \sum_{i \in [n], \pi_i = e_k} \mathbb{E} (1 - a_{1i}) \sim n, \quad k \in [K]. \] (65)

Moreover, by (60), with high probability we have

\[ \left| \bar{S}_l(k) - |S_l(k)| \right| \leq |S_l(k) \setminus \bar{S}_l(k)| + |S_l(k) \setminus \bar{S}_l(k)| \leq |M| = O\left( \frac{\log n}{p_n^2} \right), \quad l = 1, 2, \quad k \in [K]. \] (66)

It follows from (64)–(66) with high probability, the first term in (62) is bounded from above by

\[ \frac{1}{|S_1(k_1)||S_1(k_2)|} = \frac{1}{|S_1(k_1)||S_1(k_2)|} + O\left( \frac{\log n}{n^t p_n^2} \right). \] (67)

By Lemma 9, \( p_n \gg (\frac{\log n}{n})^{1/4} \) and (67) with high probability, the first term in (62) is bounded from above by

\[ \frac{1}{|S_1(k_1)||S_1(k_2)|} \sum_{i \in S_1(k_1), j \in S_1(k_2)} (b_{ij} - p_{k_1,k_2}^{(l)}) = O\left( \frac{\sqrt{\log n}}{np_n} \right) \ll p_n. \] (68)

where we let \( t = C n \sqrt{p_n \log n} \) with sufficiently large positive constant \( C \) for \( \sum_{i \in S_1(k_1), j \in S_1(k_2)} (b_{ij} - p_{k_1,k_2}^{(l)}) \) in Lemma 9.

By (60), the condition that \( p_n \gg (\frac{\log n}{n})^{1/4} \) and \( |b_{ij} - p_{k_1,k_2}^{(l)}| \leq 2 \), we have

\[ \frac{1}{|S_1(k_1)||S_1(k_2)|} \sum_{i \in S_1(k_1) \setminus S_1(k_1), j \in S_1(k_2)} |b_{ij} - p_{k_1,k_2}^{(l)}| \leq 2 \frac{|\bar{S}_1(k_1) \setminus S_1(k_1)||S_1(k_2)|}{|S_1(k_1)||S_1(k_2)|} = O\left( \frac{\log n}{np_n^3} \right) \ll p_n. \] (69)

Similar to (69), with high probability we have

\[ \frac{1}{|S_1(k_1)||S_1(k_2)|} \sum_{i \in S_1(k_1), j \in S_1(k_2) \setminus S_1(k_2)} |b_{ij} - p_{k_1,k_2}| = O\left( \frac{\log n}{np_n^3} \right) \ll p_n, \] (70)
\[
\left| \frac{1}{|S_1(k_1)||S_1(k_2)|} \sum_{i \in S_1(k_1) \setminus S_1(k_2), \ j \in S_1(k_2) \setminus S_1(k_2)} |b_{ij} - p_{k_1,k_2}| = O\left( \frac{\log n}{np_n^3} \right) \ll p_n. \quad (71)
\]

and
\[
\begin{align*}
\left| \frac{1}{|S_1(k_1)||S_1(k_2)|} \sum_{i \in S_1(k_1) \setminus S_1(k_2), \ j \in S_1(k_2) \cap S_1(k_1)} |b_{ij} - p_{k_1,k_2}^{(l)}| + \sum_{i \in S_1(k_1) \cap S_1(k_1), \ j \in S_1(k_2) \setminus S_1(k_2)} |b_{ij} - p_{k_1,k_2}^{(l)}| \\
+ \sum_{i \in S_1(k_1) \setminus S_1(k_1), \ j \in S_1(k_2) \setminus S_1(k_2)} |b_{ij} - p_{k_1,k_2}^{(l)}| \right| = O\left( \frac{\log n}{np_n^3} \right) \ll p_n. \quad (72)
\end{align*}
\]

By (63) and (72), we have
\[
\left| \frac{1}{|S_1(k_1)||S_1(k_2)|} \sum_{i \in S_1(k_1) \cap S_1(k_1), \ j \in S_1(k_2) \cap S_1(k_1)} (b_{ij} - p_{k_1,k_2}^{(l)}) \right| \ll p_n. \quad (73)
\]

Therefore, by (62), (68)–(71) and (73), we have \( \max_{i,j \in [K]} |p_{k_1,k_2}^{(l)} - p_{k_1,k_2}^{(l)}| \ll p_n. \) In view of this result, if \( \hat{f}_0 = f_0, \) it follows from (61) that
\[
\| \hat{\text{P}}^{S,S}_{(f_0,f_o)} - \hat{\text{P}}^{S,I-S}_{(f_0,*)} \|_F \ll p_n.
\]

Otherwise if \( \hat{f}_0 \neq f_0, \) by Condition 7 and Theorem 5, we have
\[
\| \hat{\text{P}}^{S,S}_{(f_0,f_o)} - \hat{\text{P}}^{S,I-S}_{(f_0,*)} \|_F \sim p_n.
\]

Recall that \( \hat{f}_0 \) is defined by \( \hat{f}_0 = \arg\min_f \| \hat{\text{P}}^{S,S}_{(f,f)} - \hat{\text{P}}^{S,I-S}_{(f,*)} \|_F. \) Hence, with high probability, we have \( \hat{f}_0 = f_0. \) Therefore, by Algorithm 2 with high probability, the set \( R \) can be merged into
\[
R = \left\{ \{e_1^T D Q_1, e_1^T D Q_2\}, \ldots, \{e_K^T D Q_1, e_K^T D Q_2\} \right\}.
\]

Notice that the \( 2K \) clusters are merged into \( K \) communities According to \( \tilde{R}. \) Then it follows from (59) that with high probability, for \( i, j \in M^c, \)
\[
\text{Individuals } i \text{ and } j \text{ are assigned to the same community } \iff \pi_i = \pi_j.
\]

**8.17. Proof of Theorem 7**

It holds with high probability that
\[
1 - \text{Misclustering rate} \geq \left| \frac{1}{n} \sum_{i \in M^c : i \text{ is assigned to group } k \text{ such that } \pi_i = e_k} \right| = \left| M^c \right| n \quad (\text{in which the first equality follows from Theorem 6}).
\]

By Theorem 4, we have with high probability \( \left| M \right| = O(\log n/p_n^3). \) Then \( p_n \gg (\log n/n)^{1/2} \) implies that \( \left| M \right| / n = o(1). \)
8.18. Bernstein inequality

Lemma 9 (Bernstein inequality). Suppose that \{y_i\}_{i=1}^n are independent Bernoulli random variables, then for any non-random series \{a_i\}_{i=1}^n such that |a_i| \leq L for some positive constant L, we have

\[ \text{IP} \left( \left| \sum_{i=1}^n a_i(y_i - \mathbb{E}y_i) \right| \geq t \right) \leq \exp \left( -\frac{t^2/2}{\sum_{i=1}^n (a_i^2 \mathbb{E}(y_i - \mathbb{E}y_i)^2) + \frac{Lt^3}{3}} \right), \quad t > 0. \quad (74) \]

8.19. Matrix Bernstein inequality

Lemma 10. [Theorem 6.2 of Tropp (2012)] Consider a finite sequence \{X_k\} of independent, random, self-adjoint \(d \times d\) matrices. Assume that \(\mathbb{E}X_k = 0\) and \(\mathbb{E}(X_k^p) \leq \frac{p!}{2} \cdot R^{p-2} A_k^2\) for \(p = 2, 3, 4, \ldots\)

Compute the variance parameter

\[ \sigma^2 := \left\| \sum_k A_k^2 \right\|. \]

Then the following chain of inequalities holds for all \(t \geq 0\)

\[ \text{IP} \left\{ \lambda_{\text{max}} \left( \sum_k X_k \right) \geq t \right\} \leq d \cdot \exp \left( \frac{-t^2/2}{\sigma^2 + Rt} \right) \leq \begin{cases} d \cdot \exp \left( -\frac{t^2}{4\sigma^2} \right) & \text{for } t \leq \sigma^2/R, \\ d \cdot \exp(-t/4R) & \text{for } t \geq \sigma^2/R. \end{cases} \]

8.20. Additional plot for Model 2

The 3-D scatter plots 10, 11 and 12 provide visual support to Algorithm 2 in Model 2.
Fig. 10. Scatter plot of the three eigenvectors corresponding to positive eigenvalues of $B$ for Model 2 when $q = (\log n/n)^{1/4}/2$ and $n = 2100$. The blue points are individuals are not adjacent to individual 1, while the red points adjacent to individual 1.

Fig. 11. Scatter plot of the three eigenvectors corresponding to positive eigenvalues of $B$ for Model 2 when $q = (\log n/n)^{1/4}/2$ and $n = 2100$. The blue points are individuals belonging to community 1, the red points belong to community 2, and the light blue points belong to community 3.
Fig. 12. Scatter plot of the three eigenvectors corresponding to positive eigenvalues of $\mathbf{B}$ for Model 2 when $q = (\log n/n)^{1/4}/2$ and $n = 2100$. The blue points represent the estimated community 1, the red points the estimated community 2 and the light blue points the estimated community 3.

Fig. 13. Misclustering rate for Model 3 ($L = 2$), averaged over 100 datasets for each combination.

8.21. Model 3

Model 3 ($K = 3$): $\mathbf{P} = \begin{pmatrix} 3q & q & q \\ q & 3q & q \\ q & q & 3q \end{pmatrix}$ and each group has size $n/3$.

Table 8 summarizes the ratio of observed edges in each combination. Figure 13 illustrates
Table 8. The ratio of the edges observed by individual 1 out of the full networks for Model 3, averaged over 100 datasets for each combination.

| $q \backslash n$ | 300 | 600 | 900 | 1200 | 1500 | 1800 | 2100 |
|-----------------|-----|-----|-----|------|------|------|------|
| .1              | .3063 | .3047 | .3050 | .3014 | .3056 | .3025 | .3031 |
| $\sqrt{\log \frac{n}{n}}$ | .4076 | .3094 | .2660 | .2397 | .2167 | .2048 | .1914 |
| $(\log \frac{n}{n})^{1/4}/2$ | .5124 | .4526 | .4255 | .4039 | .3868 | .3742 | .3620 |
| $1/\sqrt{n}$   | .1833 | .1328 | .1097 | .0959 | .0855 | .0780 | .0722 |

the community detection performance. Figures 14-16 provide visual support to Algorithm 2 in Model 3.

Fig. 14. Scatter plot of the three eigenvectors corresponding to positive eigenvalues of $B$ for Model 3 when $q = (\log n/n)^{1/4}/2$ and $n = 2100$. The blue points are individuals are not adjacent to individual 1, while the red points adjacent to individual 1.
Fig. 15. Scatter plot of the three eigenvectors corresponding to positive eigenvalues of B for Model 3 when $q = (\log n/n)^{1/4}/2$ and $n = 2100$. The blue points are individuals belonging to community 1, the red points belong to community 2, and the light blue points belong to community 3.
Fig. 16. Scatter plot of the three eigenvectors corresponding to positive eigenvalues of $B$ for Model 3 when $q = (\log n/n)^{1/4}/2$ and $n = 2100$. The blue points represent the estimated community 1, the red points the estimated community 2 and the light blue points the estimated community 3.

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