Abstract

Requiring the soft supersymmetry-breaking (SSB) parameters in finite gauge-Yukawa unified models to be finite up to and including two-loop order, we derive a two-loop sum rule for the soft scalar-masses. It is shown that this sum rule coincides with that of a certain class of string models in which the massive string states are organized into $N = 4$ supermultiplets. We investigate the SSB sector of two finite $SU(5)$ models. Using the sum rule which allows the non-universality of the SSB terms and requiring that the lightest superparticle particle is neutral, we constrain the parameter space of the SSB sector in each model.

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1 Introduction

The standard model (SM) has a large number of free parameters whose values are determined only experimentally. To reduce the number of these free parameters, and thus render it more predictive, one is usually led to enlarge the symmetry of the SM. For instance, unification of the SM forces based on the $SU(5)$ GUT \[1\] was predicting one of the gauge couplings \[1\] as well as the mass of the bottom quark \[2\]. Now it seems that LEP data is suggesting that the symmetry of the unified theory should be further enlarged and become $N = 1$ globally supersymmetric \[3\].

Relations among gauge and Yukawa couplings, which are missing in ordinary GUTs, could be a consequence of a further unification provided by a more fundamental theory at the Planck scale. Moreover, it might be possible that some of these relations are renormalization group invariant (RGI) below the Planck scale so that they are exactly preserved down to the GUT scale $M_{\text{GUT}}$. In fact, one of our motivation in this paper is to point out such indication in the soft supersymmetry-breaking (SSB) sector in supersymmetric unified theories.

In our recent studies \[4\]-\[6\], we have been searching for RGI relations among gauge and Yukawa couplings in various supersymmetric GUTs. Thus, the idea of gauge-Yukawa unification (GYU) \[4\]-\[6\] relies not only on a symmetry principle, but also on the principle of reduction of couplings \[7, 8\] (see also \[1\]). This principle is based on the existence of RGI relations among couplings, which do not necessarily result from a symmetry, but nevertheless preserve perturbative renormalizability or even finiteness. Here we would like to focus on finite unified theories \[10\]-\[21\], \[4\], \[6\].

Supersymmetry seems to be essential for a successful GYU, but, as it is for any realistic supersymmetric model, the breaking of supersymmetry has to be understood. We recall that the SSB parameters have dimensions greater than or equal to one and it is possible to treat dimensional couplings along the line of GYU \[22, 23\], which shows that the SSB sector of a GYU model is controlled by the unified gaugino mass $M$. As for one- and two-loop finite SSB terms, only the universal solution for the SSB terms \[14, 19\] is known
so far. So another motivation of this paper is to re-investigate the conditions for the
two-loop finite SSB terms and to express them in terms of simple sum rules for these
parameters. We will indeed find that the universal solution can be relaxed for the SSB
terms to be finite up to and including the two-loop corrections, and we will derive the
two-loop corrected sum rule for the soft scalar-masses. We will comment on the possibility
of all-order-finite SSB terms.

The authors of [25, 26, 23] have pointed out that the universal soft scalar masses also
appear for dilaton-dominated supersymmetry breaking in 4D superstring models [27]-
[29]. Ibáñez [25] (see also [26]) gives a possible superstring interpretation to it. We shall
examine whether or not the two-loop corrected sum rule can also be obtained in some
string model. We will indeed find that there is a class of 4D orbifold models in which
exactly the same sum rule is satisfied. It may be worth-mentioning that not only in finite
GYU models, but also in nonfinite GYU models the same soft scalar-mass sum rule is
satisfied at the one-loop level [30]. In ref. [30] a possible answer to why this happens is
speculated.

Motivated by the fact that the universal choice for the SSB terms can be relaxed, we
will investigate the SSB sector of two finite $SU(5)$ models. The SSB parameters of these
models are constrained by the sum rule and also by the requirement that the electroweak
gauge symmetry is radiatively broken [31]. We will find that there is a parameter range
for each model in which the lightest superparticle (LSP) is a neutralino, which will be
compared with the case of the universal SSB parameters. The lightest Higgs turns out to
be $\sim 120$ GeV.
2 Two-loop finiteness and Soft scalar-mass sum rule

2.1 Two-loop finite SSB terms

Various groups [24, 19] have independently computed the coefficients of the two-loop RG functions for SSB parameters \(1\). Here we would like to use them to re-investigate their two-loop finiteness and derive the two-loop soft scalar-mass sum rule.

The superpotential is

\[
W = \frac{1}{6} Y^{ijk} \Phi_i \Phi_j \Phi_k + \frac{1}{2} \mu^{ij} \Phi_i \Phi_j ,
\]

along with the Lagrangian for SSB terms,

\[
- \mathcal{L}_{SB} = \frac{1}{6} h^{ijk} \phi_i \phi_j \phi_k + \frac{1}{2} b^{ij} \phi_i \phi_j + \frac{1}{2} (m^2)^j_i \phi^* i \phi_j + \frac{1}{2} M \lambda \lambda + \text{H.c.}
\]

Since we would like to consider only finite theories here, we assume that the gauge group is a simple group and the one-loop \(\beta\) function of the gauge coupling \(g\) (A.1) vanishes, i.e.,

\[
b \equiv T(R) - 3C(G) = 0 .
\]

We also assume that the reduction equation

\[
\beta_Y^{ijk} = \beta_g dY^{ijk} / dg
\]

admits power series solutions of the form

\[
Y^{ijk} = g \sum_{n=0} \rho^{ijk}_{(n)} g^{2n} ,
\]

where \(\beta_g\) and \(\beta_Y^{ijk}\) are \(\beta\) functions of \(g\) and \(Y^{ijk}\), respectively. According to the finiteness theorem of ref. [17], the theory is then finite \(\dagger\) to all orders in perturbation theory, if the one-loop anomalous dimensions \(\gamma_i^{(1)j}\) given in (A.2) vanish, i.e., if

\[
\frac{1}{2} \sum_{p,q} \rho_{pq(0)} \rho^{pq}_{(0)} - 2 \delta_i^j C(i) = 0
\]

\(\dagger\)The RG functions [11, 12, 24, 23, 19] are given in Appendix for completeness.

\(\dagger\)Finiteness here means only for dimensionless couplings, i.e. \(g\) and \(Y^{ijk}\).
is satisfied, where we have inserted $Y^{ijk}$ in (2) into $\gamma^{(1)}_{i}$. We recall that if the conditions (3) and (6) are satisfied, the two-loop expansion coefficients in (5), $\rho^{(1)}_{ijk}$, vanish [19]. (From (A.6) ad (A.7) we see that the two-loop coefficients $\beta^{(2)}_{g}$ and $\gamma^{(2)}_{ij}$ vanish if $\beta^{(1)}_{g}$ and $\gamma^{(1)}_{ij}$ vanish.) Further, the one- and two-loop finiteness for $h^{ijk}$ can be achieved by \[ h^{ijk} = -M Y^{ijk} + \ldots = -M \rho^{(1)}_{ijk} g + O(g^{5}) , \tag{7} \]

which can be seen from (A.9) if one uses eq. (6). Note further that the $O(g^{3})$ term is absent in (3). As for $b^{ij}$ there is no constraint; $b^{ij}$ is finite if eqs. (3) and (7) are satisfied, which can be seen from the one- and two-loop coefficients of the $\beta$ function for $b^{ij}(A.5)$ and (A.10).

Now, to obtain the two-loop sum rule for soft scalar masses, we assume that the lowest order coefficients $\rho^{(0)}_{ijk}$ and also $(m^{2})^{j}$ satisfy the diagonality relations \[ \rho_{ipq}^{(0)} \rho_{jpq}^{(0)} \propto \delta^{j}_{i} \text{ for all } p \text{ and } q \text{ and } (m^{2})^{j} = m^{2}_{j} \delta^{j}_{j} , \tag{8} \]

respectively. Then one finds that \[ \left[ \beta^{(1)}_{m^{2}} \right]^{j}_{i} = \rho_{ipq}^{(0)} \rho_{jpq}^{(0)} ( m^{2}_{i}/2 + m^{2}_{j}/2 + m^{2}_{p} + m^{2}_{q} ) g^{2} + (\rho_{ipq}^{(0)} \rho_{jpq}^{(0)} - 8 \delta^{j}_{i} C(i)) M M^{\dagger} g^{2} + O(g^{6}) , \tag{9} \]

where we have used $\rho_{ipq}^{(1)} = 0$ (which implies that the $O(g^{4})$ term in (3) is absent). Using the condition (3), the diagonality relations (3) and also the soft scalar-mass sum rule (which we are going to prove) \[ ( m^{2}_{i} + m^{2}_{j} + m^{2}_{k} )/M M^{\dagger} = 1 + \frac{g^{2}}{16\pi^{2}} \Delta^{(1)} + O(g^{4}) \text{ for } i, j, k \text{ with } \rho^{(0)}_{ijk} \neq 0 , \tag{10} \]

we find that eq. (3) can be written as \[ \left[ \beta^{(1)}_{m^{2}} \right]^{j}_{i} = 4 \delta^{j}_{i} M M^{\dagger} C(i) \Delta^{(1)} \frac{g^{4}}{16\pi^{2}} + O(g^{6}) . \tag{11} \]

We will find shortly that the two-loop correction term $\Delta^{(1)}$ is given by \[ \Delta^{(1)} = -2 \sum_{l} \left[ (m^{2}_{l}/M M^{\dagger}) - (1/3) \right] T(R_{l}) . \tag{12} \]

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Therefore, the $\Delta^{(1)}$ vanishes for the universal choice
\[ m_i^2 = \kappa_i MM^\dagger \quad \text{with} \quad \kappa_i = \frac{1}{3} \quad \text{for all} \quad i , \quad (13) \]
in accord with the previous findings of refs. [13]. The result agrees also with that of ref. [10] on $N = 4$ theory; $N = 4$ theory contains three $N = 1$ chiral superfields in the adjoint representation, which means $T(R_i) = C(G)$ $(i = 1, 2, 3)$. If $\kappa_1 + \kappa_2 + \kappa_3 = 1$ is satisfied, we obtain
\[ \Delta^{(1)}(N = 4) = -2 \sum_{l=1}^{3} \left[ \kappa_l - (1/3) \right] C(G) = 0 . \quad (14) \]

To see that $\Delta^{(1)}$ is really given by eq. (12) for two-loop finiteness of $m_i^2$, we recall that the two-loop $\beta$ function for $m_i^2$ (A.11) can be nicely organized as [23]
\[ \beta^{(2)}_{m_i^2} = \left[ A_{(\gamma)in} \gamma_p^{(1)n} + A_{(m^2)in} \left[ \beta^{(1)}_{m_i^2} + A_{(g)i} \beta^{(1)}_g \right] + A_{(h)in} \left[ h^{nqr}Y_{lrq} + 4M \delta^n_p g^2 C(n) \right] + 4g^4 C(i) S' M M^\dagger \delta_i^j \right] + \text{H.c.}, \quad (15) \]
where
\[ S' = \sum_i (m_i^2/MM^\dagger) T(R_i) - C(G) \]
\[ = \sum_i [(m_i^2/MM^\dagger) - (1/3)] T(R_i) \quad \text{for} \quad \sum_i T(R_i) = 3C(G) , \quad (16) \]
and the coefficients $A$’s are given in (A.11). Using the one-loop finiteness conditions (which are ensured by eqs. (8), (9), (7) and (11)), we finally obtain
\[ \beta^{(2)}_{m_i^2} = 8g^4 C(i) M M^\dagger S' \delta_i^j . \quad (17) \]
It is now easy to see that this term can be canceled by the $O(g^4)$ contribution to $\beta^{(1)}_{m_i^2}$ (which is given in (11)) if $\Delta^{(1)}$ is exactly given by eq. (12). Note that we have not shown that the sum rule (11) is the unique solution for $\beta^{(2)}_{m_i^2}$. That is, we have only shown that the sum rule (11) is a solution to
\[ \rho_{ipq(0)\rho_{ipq}} \left[ \frac{m_i^2 + m_p^2 + m_q^2}{|M|^2 - 1} \right] = -8S' \delta_i^j C(i) , \quad (18) \]
but not in the opposite way. The question of whether the sum rule is the unique solution to (18) depends on the concrete model of course. We will address the question when discussing concrete finite models and find that the sum rule (10) is the unique solution for these models.

Since $S'$ will be of $O(C(G))$, the two-loop correction term in the sum rule (10) may be estimated as

$$\frac{g^2}{16\pi^2} \Delta^{(1)} \sim \frac{\alpha_{\text{GUT}}}{\pi} C(G).$$

(19)

If, however, the soft scalar masses are close to the universal one (13), the correction is small. In the concrete example of the $SU(5)$ finite models which we will consider below, it will turn out that the soft scalar masses should differ from the universal one if we require that the LSP is a neutralino. But the two-loop correction term $\Delta^{(1)}$ happens to vanish exactly, no matter how large the deviation from the universal choice of the soft scalar masses is.

2.2 Coincidence

It has been known [23, 25, 26] that the universal soft scalar masses which preserve their two-loop finiteness also appear for dilaton-dominated supersymmetry breaking in 4D superstring models [27]-[29]. Ibáñez [23] (see also [26]) gives a possible superstring interpretation and argues that for dilaton dominance to work, the soft SSB terms have to be independent of the particular choice of compactification and consistent with any possible compactification, in particular with a toroidal compactification preserving $N = 4$ supersymmetry. Given that the universality of the soft scalar masses can be relaxed (as we have shown above), we would like to examine whether or not the two-loop corrected sum rule (10) can also be obtained in some string model. To this end, we consider a specific class of orbifold models with three untwisted moduli $T_1, T_2, T_3$ (which exist for instance in $(0, 2)$ symmetric abelian orbifold construction always). We then assume that some non-perturbative superpotential which breaks supersymmetry exists and that the dilation $S$ and the moduli $T_a$ play the dominant role for supersymmetry breaking. The
Kähler potential $K$ and the gauge kinetic function $f$ in this case assume the generic form

$$K = -\ln(S + S^*) - \sum_{a=1}^{3} \ln(T_a + T_a^*) + \sum_i \Pi_i^{a=1}(T_a + T_a^*)^{n_a^i}|\Phi_i|^2, \quad f = kS, \quad (20)$$

where $n_a^i$ stand for modular weights and are fractional numbers, and $k$ is the Kac-Moody level $[32]-[34]$. The SSB parameters in this class of models are given by $[29], [36]-[38]$.

$$M = \sqrt{3}m_{3/2}\sin\theta, \quad m_i^2 = m_{3/2}^2(1 + 3\cos^2\theta \sum_{a=1}^{3} n_a^a\Theta_a^2), \quad (21)$$

$$h^{ijk} = -\sqrt{3}Y^{ij}m_{3/2}[\sin\theta + \cos\theta \sum_{a=1}^{3} \Theta_a(u_a^n + n_a^n + n_a^n)], \quad (22)$$

where $\theta$ and $\Theta_a$ (which parametrize the unknown mechanism of supersymmetry breaking $[29]$) are defined as $F^S/Y = \sqrt{3}m_{3/2}\sin\theta$ and $F^{T_a}/(T_a + T_a^*) = \sqrt{3}m_{3/2}\cos\theta\Theta_a$ with $\sum_{a=1}^{3} \Theta_a^2 = 1$. In eq. $(22)$ we have assumed $Y^{ijk}$ is independent of $S$ and $T_a$. It is straightforward to see that the tree-level form of the sum rule $(10)$ $[29, 37, 25, 26, 39]$ is satisfied, if

$$n_i + n_j + n_k = -u \equiv -(1, 1, 1). \quad (23)$$

Note that the condition $(23)$ ensures that $K + \ln|W|^2$ is invariant under the duality transformation,

$$T_a \rightarrow a_a T_a - ib_a \quad \frac{i c_a T_a + d_a}{}, \quad (24)$$

where $a, b, c_a$ satisfying $d_a$ are integers and $a_ad_a - b_ac_a = 1$. The Kähler potential $K$ $(20)$ belongs to the general class of the Kähler potentials that lead to the tree-level sum rule $[30]$. When gauge symmetries break, we generally have $D$-term contributions to the soft scalar masses. Such $D$-term contributions, however, do not appear in the sum rule, because each $D$-term contribution is proportional to the charge of the matter field $\Phi_i$ $[40]$.

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3Since the SSB parameter $b^{ij}$ are not constrained by two-loop finiteness, we do not consider it here.
4We call the soft scalar-mass sum rule $(10)$ without the two-loop correction term the tree-level sum rule.
We then would like to extend our discussion so as to include the two-loop correction in the sum rule (10). In superstrings, the correction to the tree level relations among the SSB terms can be computed by using the fact that the target-space modular anomaly \[41, 42, 27\] are canceled by the Green-Schwarz mechanism \[43\] and the threshold correction coming from the massive states \[44, 45\]. The Green-Schwarz mechanism induces a nontrivial transformation of $S$ under the duality transformation, which implies that the Kähler potential for the dilaton $S$ has to be modified to the duality-invariant Kähler potential \[41, 27\],

$$- \ln Y, \ Y \equiv S + S^* - \sum_{a=1}^{3} \frac{\delta_{GS}^a}{8\pi^2} \ln(T_a + T_a^*) \ ,$$

(25)

where $\delta_{GS}^a$ is the Green-Schwarz coefficient \[41, 27\]. This correction alters the tree-level formulae for $h^{ijk}$ and $m_i^2$, while the threshold correction coming from the massive states modifies the tree-level gauge kinetic function $f = S$ and hence changes the tree-level formula for the gaugino mass $M$. The requirement of the vanishing cosmological constant leads to the redefinition of the Goldstino parameters \[36\]-\[38\] as

$$\frac{1}{Y}(F^S - \sum_a \frac{\delta_{GS}^a/8\pi^2}{T_a + T_a^*})F^{T_a} = \sqrt{3}m_3/2 \sin \theta \ ,$$

(26)

$$\frac{F^{T_a}}{T_a + T_a^*} = \sqrt{3}m_3/2 \cos \theta \tilde{\Theta}_a \ ,$$

(27)

where

$$\tilde{\Theta}_a = (1 - \frac{\delta_{GS}^a/8\pi^2}{Y})^{-1/2} \Theta_a \ ,$$

(28)

and $\Theta_a$ is defined in (22). Note that the quantum modification (27) does not change the tree-level relation for $h^{ijk}$ (22) at all, which coincides the two-loop result (6). This motivates us to assume that the relation for $M$ also remains unchanged, which is true only if the contribution to the gauge kinetic function $f$ coming from the massive states \[45\] are absent. It is known \[45\] (see also \[27\]) that such situation appears for the class of orbifold models in which the massive states are organized into $N = 4$ supermultiplets \[5\] and we obtain One can easily convince oneself that if the condition (23) is satisfied, the

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\[5\]The absence of the threshold effects coming from $N = 4$ massive supermultiplets has been first observed in an $N = 4$ Yang-Mills theory with spontaneously broken gauge symmetry \[10\].
tree-level sum rule for $m_i^2$ is modified to

$$\frac{m_i^2 + m_j^2 + m_k^2}{|M|^2} - 1 = \frac{\cos^2 \theta}{\sin^2 \theta} (1 - \sum_{a=1}^{3} \tilde{\Theta}_a^2).$$

(29)

In this case the duality anomaly should be canceled only by the Green-Schwarz mechanism, implying that \[41, 27\]

$$\delta_{GS}^a = -C(G) + \sum_l T(R_l)(1 + 2n_l^a).$$

(30)

After a straightforward calculation one then finds two identities:

$$(1 - \sum_{a=1}^{3} \tilde{\Theta}_a^2)[1 - \frac{g^2}{24\pi^2} \sum T(R_l)] = -\frac{g^2}{16\pi^2} \left[ 2 \sum_l T(R_l)(\frac{1}{3} + \sum_{a=1}^{3} \tilde{\Theta}_a^2 n_l^a) 
+ \frac{b}{3} \sum_{a=1}^{3} \tilde{\Theta}_a^2 \right],$$

$$\sum_l T(R_l)(\frac{m_l^2}{|M|^2} - \frac{1}{3}) = \frac{\cos^2 \theta}{\sin^2 \theta} \sum_l T(R_l)(\frac{1}{3} + \sum_{a=1}^{3} \tilde{\Theta}_a^2 n_l^a),$$

(31)

(32)

where we have used $Y = 2/g^2$. Using these identities, one can convince oneself that the two-loop corrected sum rule (10) coincides with the sum rule (29) of the orbifold model up to and including $O(g^2)$ terms. For finite theories ($b = 0$) it is possible to express the sum rule (29) in terms of field theory quantities only:

$$\frac{m_i^2 + m_j^2 + m_k^2}{|M|^2} - 1 = \sum_l T(R_l)(m_l^2/|M|^2 - 1/3) \frac{C(G) - 8\pi^2/g^2}{C(G) - 8\pi^2/g^2}. $$

(33)

It is remarkable that in this combination of the SSB terms the quantities such as the Goldstino angle parameterizing unknown supersymmetry breaking disappear. Since the sum rule (33) can be seen as an exact result, we conjecture that the sum rule (33) and the tree-level form of the relation $h_{ijk} = -MY^{ijk}(g)$ are also exact results in field theory that result from the finiteness of the SSB parameters.

2.3 Comment

We next would like to comment on the possibility of having all-order finite SSB terms. To begin with we recall that the RG functions are renormalization-scheme dependent
starting at two-loop order. This is true, even if we assume that a mass-independent renormalization-scheme is employed, except for the gauge coupling $\beta$ function. Therefore, it could be possible to find a renormalization-scheme in which all the higher order coefficients of the $\beta$ functions (except for the gauge coupling $\beta$ function) vanish. Since we know explicitly the two-loop RG functions, we would like to ask whether we can find a renormalization scheme in which all the RG functions beyond the two-loop vanish. To simplify the problem, we assume that all the supersymmetric, massive parameters are set equal to zero and that $Y^{ijk}$ and $h^{ijk}$ have been reduced in favor of $g$ and $M$. Suppose that we have found reparametrizations of $g$, $M$ and $m^2$ such that the $\beta$ functions, except for $\beta_g$ and $\beta_{m^2}$, beyond the two-loop order vanish. We then ask ourselves whether or not it is possible to find a reparametrization of $m_i^2$’s of the form

$$m_i^2 \to m_i^2 + \frac{g^4}{16\pi^2}K_i \text{ with } K_i = r_{ij}m_j^2 + p_i|M|^2,$$  \hspace{1cm} (34)

where $r_{ij}$ and $p_i$ are numbers, such that the three-loop $\beta$ functions for $m_i^2$’s vanish.

Inserting (34) into the one-loop $\beta$ function (A.4), we see that the three-loop terms in the $\beta$ function should be canceled by the term

$$\rho_{i[pq(0)\rho_{j[pq]}(K_i + K_j + K_k),}$$  \hspace{1cm} (35)

where we have used eq. (4). Recall that because of the diagonality condition (8) the terms given above are proportional to $\delta^j_i$ and so the total number of these terms, $N$, is exactly the number of the chiral superfields present in the theory. It is clear that if these $N$ terms are linearly independent, the three-loop contributions in the $\beta$ functions for $m_i^2$’s can be canceled by them.

This algebraic question is very much related to the question of whether or not the sum rule is the unique solution to the two-loop finiteness, because it depends on the explicit form of $\rho^{ijk}_{(0)}$. One can convince oneself that if the sum rule is the unique solution to the two-loop finiteness and the sum rule does not fix $m_i^2/|M|^2$ completely, the $N$ terms given in (35) are not linearly independent. In this case, it is not clear from the beginning that

\footnote{It is possible to find a reparametrization of $m_i^2$ and then to make $\beta_{m^2}^{(2)}$ zero.}
three-loop terms in the \( \beta \) function can be canceled by (35); one has to compute explicitly the three-loop contributions to see it. In the concrete models we will consider later, these \( N \) terms (35) are not linearly independent. The string inspired result (33) should have a nontrivial meaning in this case; it suggests that the three-loop contributions can be canceled by a reparametrization of \( m_i^2 \), because the reparametrization defined by

\[
m_i^2 \rightarrow m_i'' = m_i^2 - \frac{1}{3} \sum_l T(R_l)(m_l^2 - |M|^2/3) C(G) - 8\pi^2/g^2
\]

(36)
can bring the "exact" result (33) into the tree-level form. If, on the other hand, the sum rule is the unique solution to the two-loop finiteness and the sum rule fixes \( m_i^2/|M|^2 \) completely, the \( N \) terms (35) are linearly independent. We can then cancel all the three-loop contributions, which then can be continued to arbitrary order.

3 Finite theories based on \( SU(5) \)

3.1 General comments

From the classification of theories with vanishing one-loop gauge \( \beta \) function [13], one can easily see that there exist only two candidate possibilities to construct \( SU(5) \) GUTs with three generations. These possibilities require that the theory should contain as matter fields the chiral supermultiplets \( 5, \bar{5}, 10, \bar{5}, 24 \) with the multiplicities \((6, 9, 4, 1, 0)\) and \((4, 7, 3, 0, 1)\), respectively. Only the second one contains a \( 24 \)-plet which can be used to provide the spontaneous symmetry breaking (SB) of \( SU(5) \) down to \( SU(3) \times SU(2) \times U(1) \).

For the first model one has to incorporate another way, such as the Wilson flux breaking mechanism to achieve the desired SB of \( SU(5) \) [4]. Therefore, for a self-consistent field theory discussion we would like to concentrate only on the second possibility.

It is clear, at least for the dimensionless couplings, that the matter content of a theory is only a necessary condition for all-order finiteness. Therefore, there exist, in principle, various finite models for a given matter content. However, during the early studies [14, 13], the theorem [17] that guarantees all-order finiteness and requires the existence of power series solution to any finite order in perturbation theory was not known.
The theorem introduces new constraints, in particular requires that the solution to the one-loop finiteness conditions should be non-degenerate and isolated. In most studies the freedom resulted as a consequence of the degeneracy in the one- and two-loop solutions has been used to make specific ansätze that could lead to phenomenologically acceptable predictions. Note that the existence of such freedom is incompatible with the power series solutions [7, 17].

Taking into account the new constraints an all-order finite $SU(5)$ model has been constructed [4], which among others successfully predicted the bottom and the top quark masses [4, 6]. The later is due to the Gauge-and-Yukawa-of-the-third-generation Unification [4]-[6] which has been achieved. In general the predictive power of a finite $SU(5)$ model depends on the structure of the superpotential and on the way the four pairs of Higgs quintets and anti-quintets mix to produce the two Higgs doublets of the minimal supersymmetric standard model (MSSM). Given that the finiteness conditions do not restrict the mass terms, there is a lot of freedom offered by this sector of the theory in mixing the four pairs of Higgs fields. As a result it was possible in the early studies (a) to provide the adequate doublet-triplet splitting in the pair of $\mathbf{5}$ and $\mathbf{\bar{5}}$ which couple to ordinary fermions so as to suppress the proton decay induced by the coloured triplets and (b) to introduce angles in the gauge-Yukawa relations suppressing in this way the strength of the Yukawa couplings. Concerning the requirement (b) one has to recall that at that time it was very unpleasant to have a top mass prediction at $O(150 - 200) \text{ GeV}$; the popular top quark mass was at $O(40) \text{ GeV}$. The above was most clearly stated in ref. [15] and has been revived [21] taking into account the recent data. However, it is clear that using the large freedom offered by the Higgs mass parameter space in requiring the condition (b) one strongly diminishes the beauty of a finite theory. Consequently, this freedom was abandoned in the recent studies of the all-loop finite $SU(5)$ model [4] and only the condition (a) was kept as a necessary requirement.
3.2 Models

A predictive Gauge-Yukawa unified SU(5) model which is finite to all orders, in addition to the requirements mentioned already, should also have the following properties.

1. One-loop anomalous dimensions are diagonal, i.e., $\gamma_{ij}^{(1)} \propto \delta^i_j$, according to the assumption (8).

2. Three fermion generations, $\bar{\psi}_i$ ($i = 1, 2, 3$), obviously should not couple to $24$. This can be achieved for instance by imposing $B - L$ conservation.

3. The two Higgs doublets of the MSSM should mostly be made out of a pair of Higgs quintet and anti-quintet, which couple to the third generation.

In the following we discuss two versions of the all-order finite model.

A: The model of ref. [4].

B: A slight variation of the model A, which can also be obtained from the class of the models suggested by Kazakov et al. [20] with a modification to suppress non-diagonal anomalous dimensions.

The quark mixing can be accommodated in these models, but for simplicity we neglect the intergenerational mixing and postpone the interesting problem of predicting the mixings to a future publication.

The superpotential which describe the two models takes the form [4, 20]

$$W = \frac{1}{2} g_i^u \, 10_i 10_i H_1 + g_i^d \, 10_i \bar{\psi}_i \, \overline{H}_i + g_{23}^u \, 10_2 10_3 H_4$$

$$+ g_{23}^d \, 10_2 \bar{\psi}_3 \overline{H}_4 + g_{32}^d \, 10_3 \bar{\psi}_2 \overline{H}_4 + \sum_{a=1}^{4} g_a^l \, H_a \, 24 \overline{H}_a + \frac{g^\lambda}{3} (24)^3,$$

(37)

where $H_a$ and $\overline{H}_a$ ($a = 1, \ldots, 4$) stand for the Higgs quintets and anti-quintets. Given the superpotential $W$, we can compute now the $\gamma$ functions of the model, from which we then compute the $\beta$ functions. We find:

$$\gamma_{10}^{(1)} = \frac{1}{16\pi^2} \left[ \frac{36}{5} g^2 + 3 (g_1^u)^2 + 2 (g_1^d)^2 \right].$$
\[\gamma_{10_2}^{(1)} = \frac{1}{16\pi^2} \left[ -\frac{36}{5} g^2 + 3 (g_2^u)^2 + 2 (g_2^d)^2 + 3 (g_2^{d_3})^2 + 2 (g_{23}^d)^2 \right],\]
\[\gamma_{10_3}^{(1)} = \frac{1}{16\pi^2} \left[ -\frac{36}{5} g^2 + 3 (g_3^u)^2 + 2 (g_3^d)^2 + 3 (g_{23}^d)^2 + 2 (g_{32}^d)^2 \right],\]
\[\gamma_{3_1}^{(1)} = \frac{1}{16\pi^2} \left[ -\frac{24}{5} g^2 + 4 (g_1^d)^2 \right],\]
\[\gamma_{3_2}^{(1)} = \frac{1}{16\pi^2} \left[ -\frac{24}{5} g^2 + 4 (g_2^d)^2 + 4 (g_{32}^d)^2 \right],\]
\[\gamma_{3_3}^{(1)} = \frac{1}{16\pi^2} \left[ -\frac{24}{5} g^2 + 4 (g_3^d)^2 + 4 (g_{32}^d)^2 \right],\]
\[\gamma_{H_i}^{(1)} = \frac{1}{16\pi^2} \left[ -\frac{24}{5} g^2 + 3 (g_i^u)^2 + \frac{24}{5} (g_i^f)^2 \right] \quad i = 1, 2, 3, \]
\[\gamma_{\Pi_i}^{(1)} = \frac{1}{16\pi^2} \left[ -\frac{24}{5} g^2 + 3 (g_i^u)^2 + \frac{24}{5} (g_i^f)^2 \right] \quad i = 1, 2, 3, \]
\[\gamma_{H_4}^{(1)} = \frac{1}{16\pi^2} \left[ -\frac{24}{5} g^2 + 6 (g_2^{d_3})^2 + \frac{24}{5} (g_i^f)^2 \right],\]
\[\gamma_{\Pi_4}^{(1)} = \frac{1}{16\pi^2} \left[ -\frac{24}{5} g^2 + 4 (g_{23}^d)^2 + 4 (g_{32}^d)^2 + \frac{24}{5} (g_i^f)^2 \right],\]
\[\gamma_{24}^{(1)} = \frac{1}{16\pi^2} \left[ -10 g^2 + \sum_{a=1}^{4} (g_a^u)^2 + \frac{21}{5} (g_1^u)^2 \right].\]

The non-degenerate and isolated solutions to \(\gamma_i^{(1)} = 0\) for the models \{A, B\} are:

\[(g_1^u)^2 = \left\{ \frac{8}{5}, \frac{8}{5} \right\} g^2, \quad (g_1^d)^2 = \left\{ \frac{6}{5}, \frac{6}{5} \right\} g^2, \quad (g_2^u)^2 = (g_3^u)^2 = \left\{ \frac{8}{5}, \frac{4}{5} \right\} g^2,\]
\[(g_2^d)^2 = \left\{ \frac{6}{5}, \frac{3}{5} \right\} g^2, \quad (g_2^{d_3})^2 = \left\{ 0, \frac{4}{5} \right\} g^2, \quad (g_{23}^d)^2 = (g_{32}^d)^2 = \left\{ 0, \frac{3}{5} \right\} g^2,\]
\[(g_i^u)^2 = \left\{ \frac{15}{7} g^2, \quad (g_i^f)^2 = (g_i^{d_3})^2 = \left\{ 0, \frac{1}{2} \right\} g^2, \quad (g_i^f)^2 = 0, \quad (g_4^f)^2 = \left\{ 1, 0 \right\} g^2.\]

We have explicitly checked that these solutions (39) are also the solutions of the reduction equation (4) and that they can be uniquely extended to the corresponding power series solutions (4) \(^7\). Consequently, these models are finite to all orders.

After the reduction of couplings (33) the symmetry of \(W\) (37) is enhanced: For the model A one finds that the superpotential has the \(Z_7 \times Z_3 \times Z_2\) discrete symmetry

\[
\begin{align*}
\mathbf{5}_1 & : \ (4, 0, 1) \ , \ \mathbf{5}_2 : \ (1, 0, 1) \ , \ \mathbf{5}_3 : \ (2, 0, 1) \ , \\
\mathbf{10}_1 & : \ (1, 1, 1) \ , \ \mathbf{10}_2 : \ (2, 2, 1) \ , \ \mathbf{10}_3 : \ (4, 0, 1) \ , \\
H_1 & : \ (5, 1, 0) \ , \ H_2 : \ (3, 2, 0) \ , \ H_3 : \ (6, 0, 0) \ .
\end{align*}
\]

\(^7\)The coefficients in (39) are slightly different from those in models considered in refs. \[20\].
$$\mathcal{H}_1 : (-5, -1, 0), \mathcal{H}_2 : (-3, -2, 0), \mathcal{H}_3 : (-6, 0, 0),$$

$$H_4 : (0, 0, 0), \mathcal{H}_4 : (0, 0, 0), 24 : (0, 0, 0),$$

while for the model $B$ one finds $Z_4 \times Z_4 \times Z_4$ defined as

$$\mathcal{5}_1 : (1, 0, 0), \mathcal{5}_2 : (0, 1, 0), \mathcal{5}_3 : (0, 0, 1),$$

$$10_1 : (1, 0, 0), 10_2 : (0, 1, 0), 10_3 : (0, 0, 1),$$

$$H_1 : (2, 0, 0), H_2 : (0, 2, 0), H_3 : (0, 0, 2),$$

$$\mathcal{H}_1 : (-2, 0, 0), \mathcal{H}_2 : (0, -2, 0), \mathcal{H}_3 : (0, 0, -2),$$

$$H_4 : (0, 3, 3), \mathcal{H}_4 : (0, -3, -3), 24 : (0, 0, 0),$$

where the numbers in the parenthesis stand for the charges under the discrete symmetries.

The main difference of the models $A$ and $B$ is that three pairs of Higgs quintets and anti-quintets couple to the $24$ for $B$ so that it is not necessary to mix them with $H_4$ and $\mathcal{H}_4$ in order to achieve the triplet-doublet splitting after SB of $SU(5)$. This enhances the predictivity, because then the mixing of the three pairs of Higgses are strongly constrained to fit the phenomenology of the first two generations.

Before we go to present our analysis on low-energy predictions of the models, we would like to discuss the structure of the sum rule for the soft scalar masses for each case. According to (8), we recall that they are supposed to be diagonal. From the one-loop finiteness for the soft scalar masses, we obtain (there are $\{10, 13\}$ equations for 15 unknown $\kappa^{(0)}$s):

$$\kappa^{(0)}_{H_1} = 1 - 2\kappa^{(0)}_{10_1}, \kappa^{(0)}_{H_2} = 1 - \kappa^{(0)}_{10_2} - \kappa^{(0)}_{\mathcal{5}_3} (i = 1, 2, 3),$$

$$\kappa^{(0)}_{H_4} = \frac{2}{3} - \kappa^{(0)}_{\mathcal{H}_4}, \kappa^{(0)}_{24} = \frac{1}{3} \text{ for } A,$$

and

$$\kappa^{(0)}_{H_1} = 1 - 2\kappa^{(0)}_{10_1}, \kappa^{(0)}_{H_2} = \kappa^{(0)}_{H_3} = \kappa^{(0)}_{H_4} = 1 - 2\kappa^{(0)}_{10_3},$$

$$\kappa^{(0)}_{\mathcal{H}_1} = 1 - \kappa^{(0)}_{10_1} - \kappa^{(0)}_{\mathcal{5}_3}, \kappa^{(0)}_{\mathcal{H}_2} = \kappa^{(0)}_{\mathcal{H}_3} = \kappa^{(0)}_{\mathcal{H}_4} = -\frac{1}{3} + 2\kappa^{(0)}_{10_1},$$

$$\kappa^{(0)}_{\mathcal{5}_2} = \kappa^{(0)}_{\mathcal{5}_3} = \frac{4}{3} - 3\kappa^{(0)}_{10_3}, \kappa^{(0)}_{10_2} = \kappa^{(0)}_{10_3}, \kappa^{(0)}_{24} = \frac{1}{3} \text{ for } B,$$
where we have defined
\[ \frac{m_i^2}{|M|^2} = \kappa_i^{(0)} + \frac{g^2}{16\pi^2} \kappa_i^{(1)} + \cdots, \ i = 10_1, 10_2, \ldots, 24. \] (44)
We then use the solution (39) to calculate the actual value for \( S' \) by using eq. (16), which express the two-loop correction to the sum rule. Surprisingly, it turns out for both models that
\[ S' = 0. \] (45)
That is, the one-loop sum rule in the present models is not corrected in two-loop order.

Next we would like to address the question of whether the sum rule (10) is the unique solution to the two-loop finiteness. To this end, we recall that the two-loop finiteness for the soft scalar masses follows if eq. (18), i.e.
\[ \rho_{ijpq}(\rho_{ijpq}^{(0)} (\kappa_i^{(1)} + \kappa_p^{(1)} + \kappa_q^{(1)})) = -8C(i) \sum_l [\kappa_p^{(0)} - (1/3)] T(R_l) = -8C(i) S', \] is satisfied. There are 15 equations for 15 unknown \( \kappa^{(1)} \)'s. We find that the solution is not unique; it can be parameterized by \{7, 4\} parameters for a given \( S' \) which is zero for the present models. For instance,
\[ \kappa_{H_1}^{(1)} = -2S' - 2\kappa_{10_1}^{(1)}, \kappa_{H_2}^{(1)} = -2S' - \kappa_{10_1}^{(1)}, \kappa_{H_3}^{(1)} = -2S' - \kappa_{10_2}^{(1)}, \kappa_{H_4}^{(1)} = -2S' - \kappa_{10_3}^{(1)}, \] (47)\[ \kappa_{10_1}^{(1)} = -\frac{4S'}{3} - \kappa_{10_2}^{(1)}, \kappa_{10_3}^{(1)} = -\frac{2S'}{3} \] for \( A \),
and
\[ \kappa_{H_1}^{(1)} = -2S' - 2\kappa_{10_1}^{(1)}, \kappa_{H_2}^{(1)} = -2S' - 2\kappa_{10_1}^{(1)}, \kappa_{H_3}^{(1)} = -2S' - \kappa_{10_2}^{(1)}, \kappa_{H_4}^{(1)} = -2S' - \kappa_{10_3}^{(1)} , \] (48)\[ \kappa_{10_1}^{(1)} = -\frac{4S'}{3} - \kappa_{10_2}^{(1)}, \kappa_{10_3}^{(1)} = \frac{2S'}{3} \] for \( B \).

As one can easily see that \( \kappa^{(1)} \)'s satisfy
\[ \kappa_i^{(1)} + \kappa_j^{(1)} + \kappa_k^{(1)} = -2S' = 0, \] (49)
which shows that the sum rule (10) in the present models is the unique solution to two-loop finiteness.
4 Predictions of Low Energy Parameters

Since the gauge symmetry is spontaneously broken below $M_{\text{GUT}}$, the finiteness conditions do not restrict the renormalization property at low energies, and all it remains are boundary conditions on the gauge and Yukawa couplings (39) and the $h = -MY$ relation (47) and the soft scalar-mass sum rule (10) at $M_{\text{GUT}}$. So we examine the evolution of these parameters according to their renormalization group equations at two-loop for dimensionless parameters and at one-loop for dimensional ones with these boundary conditions. Below $M_{\text{GUT}}$ their evolution is assumed to be governed by the MSSM. We further assume a unique supersymmetry breaking scale $M_s$ so that below $M_s$ the SM is the correct effective theory.

We recall that $\tan \beta$ is usually determined in the Higgs sector. However, it has turned out that in the case of GYU models it is convenient to define $\tan \beta$ by using the matching condition at $M_s$ [47],

$$
\alpha_i^{\text{SM}} = \alpha_i \sin^2 \beta, \quad \alpha_b^{\text{SM}} = \alpha_b \cos^2 \beta, \quad \alpha_r^{\text{SM}} = \alpha_r \cos^2 \beta, \\
\alpha_\lambda = \frac{1}{4} \left( \frac{3}{5} \alpha_1 + \alpha_2 \right) \cos^2 2\beta,
$$

(50)

where $\alpha_i^{\text{SM}} (i = t, b, \tau)$ are the SM Yukawa couplings and $\alpha_\lambda$ is the Higgs coupling ($\alpha_I = g_I^2/4\pi^2$). With a given set of the input parameters [48],

$$
M_\tau = 1.777 \text{ GeV}, \quad M_Z = 91.188 \text{ GeV},
$$

(51)

with [49]

$$
\alpha_{\text{EM}}^{-1}(M_Z) = 127.9 + \frac{8}{9\pi} \log \frac{M_t}{M_Z}, \\
\sin^2 \theta_W(M_Z) = 0.2319 - 3.03 \times 10^{-5}T - 8.4 \times 10^{-8}T^2,
$$

(52)

$$
T = M_t/[\text{GeV}] - 165,
$$

the matching condition (50) and the GYU boundary condition at $M_{\text{GUT}}$ can be satisfied only for a specific value of $\tan \beta$. Here $M_\tau, M_t, M_Z$ are pole masses, and the couplings above are defined in the $\overline{\text{MS}}$ scheme with six flavors. Under the assumptions specified
above, it is possible without knowing the details of the scalar sector of the MSSM to predict various parameters such as the top quark mass \([4]-[6]\). We present them for the model A in table 1 and for the model B in table 2.

Comparing, for instance, the \(M_t\) predictions above with the most recent experimental value \([50]\),

\[
M_t = (175.6 \pm 5.5) \text{ GeV} ,
\]

and recalling that the theoretical values for \(M_t\) given in the tables may suffer from a correction of less than \(\sim 4\%\) \([3]\), we see that they are consistent with the experimental data. (For more details, see ref. \([3]\), where various corrections on the predictions of GYU models such as the MSSM threshold corrections are estimated \([4]\)).

Now we come to the SSB sector. As mentioned, we impose at \(M_{\text{GUT}}\) the \(h = -MY\) relation \([4]\) and the soft scalar-mass sum rule \([10]\), i.e. \([12]\) and \([17]\) for the models A and \([43]\) and \([48]\) for the model B, and calculate their low-energy values. To make

\[\text{Table 1: Table 1: The predictions for different } M_s \text{ for } A\]

| \(M_s [\text{GeV}]\) | \(\alpha_{3(5f)}(M_Z)\) | \(\tan \beta\) | \(M_{\text{GUT}} [\text{GeV}]\) | \(M_b [\text{GeV}]\) | \(M_t [\text{GeV}]\) |
|---------------------|------------------|----------|----------------|-------------|----------|
| 300                 | 0.123            | 54.1     | \(2.2 \times 10^{16}\) | 5.3         | 183      |
| 500                 | 0.122            | 54.2     | \(1.9 \times 10^{16}\) | 5.3         | 183      |
| \(10^3\)           | 0.120            | 54.3     | \(1.5 \times 10^{16}\) | 5.2         | 184      |

\[\text{Table 2: Table 2: The predictions for different } M_s \text{ for } B\]

| \(M_s [\text{GeV}]\) | \(\alpha_{3(5f)}(M_Z)\) | \(\tan \beta\) | \(M_{\text{GUT}} [\text{GeV}]\) | \(M_b [\text{GeV}]\) | \(M_t [\text{GeV}]\) |
|---------------------|------------------|----------|----------------|-------------|----------|
| 800                 | 0.120            | 48.2     | \(1.5 \times 10^{16}\) | 5.4         | 174      |
| \(10^3\)           | 0.119            | 48.2     | \(1.4 \times 10^{16}\) | 5.4         | 174      |
| \(1.2 \times 10^3\) | 0.118            | 48.2     | \(1.3 \times 10^{16}\) | 5.4         | 174      |
our unification idea and its consequence transparent, we shall make an oversimplifying assumption that the unique supersymmetry breaking scale $M_s$ can be set equal to the unified gaugino mass $M$ at $M_{GUT}$. That is, we calculate the SSB parameters at $M_s = M$ from which we then compute the spectrum of the superpartners by using the tree-level formulae. Since $\tan \beta$ by the dimension-zero sector because of GYU, one should examine each time whether GYU and the sum rule are consistent with the radiative breaking of the electroweak symmetry [31]. This concitency can be achieved, though not always, by using the freedom to fix the $b$ term and the supersymmetric mass term $\mu$ which remain unconstrained by finiteness.

As we can see from (42) and (43), the structure of the sum rules for the two models is different. Recall that the MSSM Higgs doublets, $H_u$ and $H_d$, mostly stem from the third Higgsess $H_3$ and $\overline{H}_3$. Therefore, the scalar masses $m_i^2$ with $i = H_1, H_2, H_1, H_2$ do not enter into the low-energy sector, implying that $m_{10}^{2}, m_{5_1}^{2}, m_{10}^{2}$ and $m_{5_2}^{2}$ for the model A, and $m_{10}^{2}$ and $m_{5_1}^{2}$ for the B, respectively, are free parameters. So in following discussions we would like to focus on the third-generation scalar-masses. The relevant sum rules at the GUT scale are thus given by

\begin{align}
m_{H_u}^2 + 2 m_{10}^2 &= m_{H_d}^2 + m_{5}^2 + m_{10}^2 = M^2 \quad \text{for A}, \quad (54) \\
m_{H_u}^2 + 2 m_{10}^2 &= M^2, \quad m_{H_d}^2 - 2 m_{10}^2 = -\frac{M^2}{3}, \quad m_{5}^2 + 3 m_{10}^2 = \frac{4 M^2}{3} \quad \text{for B}, \quad (55)
\end{align}

where we use as free parameters $m_{5} \equiv m_{5_3}$ and $m_{10} \equiv m_{10_3}$ for the model A, and $m_{10}$ for B, in addition to $M$.

First we present the result for the model A. We look for the parameter space in which the lighter s-tau mass squared $m_{\tilde{\tau}}^2$ is larger than the lightest neutralino mass squared $m_{\chi}^2$ (which is the LSP). In fig. 1, 2 and 3 we show this region in the $m_{5} - m_{10}$ plane for $M = M_s = 0.3, 0.5$ and 1 TeV, respectively. The region with open squares does not lead to a successful radiative electroweak symmetry breaking, and the region with dots and

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9 For the lightest Higgs mass we include radiative corrections.

10 For the model A, this is an assumption as we have discussed, while for B this is a consequence of the unitarity of the mixing matrix of the three Higgsess [20].
crosses defines the region with $m_\tau^2 < 0$ and $m_\tau^2 < m_\chi^2$, respectively.

Fig. 1: The region without squares, dots and crosses yields a neutralino as the LSP for the model A with $M = 0.3$ TeV.

Fig. 2: The same as fig. 1 with $M = 0.5$ TeV.
Fig. 3: The same as fig. 1 with $M = 1$ TeV.

Fig. 4: $m_\tilde{\tau}^2$ and $m_\tilde{\chi}^2$ for the universal choice of the soft scalar masses.

In fig. 4 we show $m_\tilde{\tau}^2$ and $m_\tilde{\chi}^2$ for the universal choice $m_\tilde{\tau}^2 = m_\tilde{\chi}^2 = M^2/3$ at $M_{\text{GUT}}$. We find that there is no region of $M_s = M$ below $O(\text{few})$ TeV in which $m_\tilde{\tau}^2 > m_\tilde{\chi}^2$ is satisfied. In table 3 we present the s-spectrum and the lightest Higgs mass $m_h$ of the model A with $M = 0.5$ TeV, $m_\tilde{\tau} = 0.3$ TeV and $m_{10} = 0.5$ TeV. (Radiative corrections are included in $m_h$.)
Table 3: A representative example of the predictions for the s-spectrum for the model A.

| $m_\chi = m_{\chi_1}$ (TeV) | $m_{\tilde{\chi}_2}$ (TeV) |
|----------------------------|---------------------------|
| 0.22                      | 1.06                      |
| $m_{\chi_2}$ (TeV)        | $m_{\tilde{\tau}} = m_{\tilde{\tau}_1}$ (TeV) | 0.33     |
| 0.41                      | 0.54                      |
| $m_{\chi_3}$ (TeV)        | $m_{\tilde{\tau}_2}$ (TeV) | 0.54     |
| 0.93                      | 0.94                      |
| $m_{\chi_4}$ (TeV)        | $m_{\tilde{\nu}_1}$ (TeV) | 0.41     |
| 0.94                      | 0.41                      |
| $m_{\chi_\pm}$ (TeV)      | $m_A$ (TeV)               | 0.44     |
| 0.41                      | 0.44                      |
| $m_{\chi_\pm}$ (TeV)      | $m_{H^\pm}$ (TeV)         | 0.45     |
| 0.94                      | 0.45                      |
| $m_{\tilde{\tau}_1}$ (TeV)| $m_H$ (TeV)               | 0.44     |
| 0.94                      | 0.44                      |
| $m_{\tilde{\tau}_2}$ (TeV)| $m_h$ (TeV)               | 0.12     |
| 1.09                      | 0.12                      |
| $m_{\tilde{b}_1}$ (TeV)   |                           | 0.86     |

The model B has only two free SSB parameters $m_{10}$ and $M = (M_s)$. For a fixed $M$, the neutralino masses are independent of $m_{10}$, while $m_{\tilde{\tau}}$ depends on it. Shown are $m_{\tilde{\tau}}^2$ and $m_\chi^2$ as function of $m_{10}$ in fig. 5, 6 and 7 for $M = 0.5, 0.8$ and 1 TeV.

![Graph](image.png)

Fig. 5: $m_{\tilde{\tau}}^2$ and $m_\chi^2$ against $m_{10}^2$ for $M = 0.5$ TeV.

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Fig. 6: The same as fig. 4 for $M = 0.8$ TeV.

Fig. 7: The same as fig. 4 for $M = 1$ TeV.

In fig. 8 we plot the maximal value of $m_\tau^2$, denoted by $\text{Max}(m_\tau^2)$, and $m_\chi^2$ for different values of $M$, which should be compared with fig. 9 in which we plot the case of the universal choice of the scalar masses.
As fig. 8 shows, $M$ has to be relatively large to satisfy the constraint $m_{\tilde{\tau}}^2 < m_{\chi}^2$ for the model B. We find, for this model too, that there is no region of $M$ below $O$(few) TeV for the universal choice in which $m_{\tilde{\tau}}^2 < m_{\chi}^2$ is satisfied. In Table 4 we give a representative prediction for the s-spectrum for the model B, where we have used: $M = 1$ TeV and $m_{10} = 0.65$ TeV.
Table 4: A representative example of the predictions of the s-spectrum for the model B.

| $m_{\chi_1}$ (TeV) | $m_{\tilde{b}_2}$ (TeV) | 1.79 |
|---------------------|-------------------------|------|
| $m_{\chi_2}$ (TeV)  | $m_{\tilde{t}_1}$ (TeV) | 0.47 |
| $m_{\chi_3}$ (TeV)  | $m_{\tilde{\tau}_2}$ (TeV) | 0.69 |
| $m_{\chi_4}$ (TeV)  | $m_{\tilde{\tau}_1}$ (TeV) | 0.62 |
| $m_{\chi_1^\pm}$ (TeV) | $m_A$ (TeV) | 0.74 |
| $m_{\chi_2^\pm}$ (TeV) | $m_{H^\pm}$ (TeV) | 0.75 |
| $m_{\tilde{t}_1}$ (TeV) | $m_{H}$ (TeV) | 0.74 |
| $m_{\tilde{b}_1}$ (TeV) | $m_h$ (TeV) | 0.12 |
| $m_{\tilde{b}_1}$ (TeV) | 1.56 |

5 Conclusion

In this paper we have re-investigated the two-loop finiteness conditions for the SSB parameters in softly broken $N = 1$ supersymmetric Yang-Mills theories with a simple gauge group and found that the previously known result [11, 19] on the \( h = -MY \) relation (7) is necessary while the universal solution for the soft scalar masses can be continuously deformed to the sum rule (10).

Since it has been known [25, 26, 23] that the universal soft scalar masses appear for dilaton-dominated supersymmetry breaking in 4D superstring models, we have examined whether or not the two-loop corrected soft scalar-mass sum rule can also be obtained in some string model. We have indeed found that the same sum rule is satisfied in a certain class of string models in which the massive string states are organized into $N = 4$ supermultiplets so that they do not contribute to the quantum modification of the gauge kinetic function. Since not only in finite GYU models, but also in nonfinite GYU models the same soft scalar-mass sum rule is satisfied at least at the one-loop level [30], we believe there exists something non-trivial behind these coincidences.

Motivated by these facts, we have investigated the SSB sector of two finite $SU(5)$ models A and B. We have found out that the two-loop corrections to the sum rule is
absent in these models. Since we do not know why this happens, it is an accident to us. Finally we have investigated the low-energy sector of these models. Using the sum rule and requiring that the LSP is neutral, we have constrained the parameter space of the low-energy SSB sector in each model and calculated the spectrum of the superparticles. We have found that the model \textbf{A} allows relatively light superparticles while in the model \textbf{B} they are heavier than $\sim 0.5$ TeV. The mass of the lightest Higgs is $\sim 120$ GeV.

Taking into account all these results, we would like to conclude that the finite models we have considered are not only academically attractive, but also making interesting predictions which are consistent with the present experimental knowledge.

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Appendix

The RG functions which we have used in the text are defined as:

\[
\frac{d}{dt} g = \beta_g = \sum_{n=1} \frac{1}{(16\pi^2)^n} \beta_g^{(n)} , \quad \frac{d}{dt} M = \beta_M = \sum_{n=1} \frac{1}{(16\pi^2)^n} \beta_M^{(n)} ,
\]

\[
\frac{d}{dt} Y_{ij}^k = \beta_{Y_{ij}^k} = Y_{ij}^p \sum_{n=1} \frac{1}{(16\pi^2)^n} \gamma_p^{(n)k} + (k \leftrightarrow i) + (k \leftrightarrow j) ,
\]

\[
\frac{d}{dt} h_{ij}^k = \beta_{h_{ij}^k} = \sum_{n=1} \frac{1}{(16\pi^2)^n} [\beta_h^{(n)}]_{ij}^k ,
\]

\[
\frac{d}{dt} (m^2)_i^j = [\beta_{m^2}]_i^j = \sum_{n=1} \frac{1}{(16\pi^2)^n} [\beta_{m^2}^{(n)}]_i^j ,
\]

\[
\frac{d}{dt} b_{ij} = \beta_{b_{ij}} = \sum_{n=1} \frac{1}{(16\pi^2)^n} \beta_{b^{(n)ij}} ,
\]

where we assume that the gauge group is a simple group. The coefficients of the one- and two-loop RG functions \[11, 12, 24, 23, 19\] are:

\[
\beta_g^{(1)} = g^3 [T(R) - 3C(G)] , \quad \beta_M^{(1)} = 2M \beta_g^{(1)}/g , \quad (A.1)
\]

\[
\gamma_i^{(1)j} = (1/2)Y_{ipq}Y_{jpq} - 2\delta_i^j g^2 C(i) , \quad X_j^{(1)i} = h_{mnn}Y_{jmn} + 4Mg^2C(i)\delta_j^i , \quad (A.2)
\]

\[
[\beta_h^{(1)}]_{ij}^k = (1/2)h_{ij}Y_{lmn}Y_{mnk} + Y_{ijk}Y_{lmn}h_{mnk} - 2(h_{ij} - 2MY_{ij}) g^2 C(k)
\]

\[+ (k \leftrightarrow i) + (k \leftrightarrow j) , \quad (A.3)\]

\[
[\beta_{m^2}]_i^j = (1/2)Y_{ipq}Y_{pqn}(m^2)_i^n + (1/2)Y_{jpq}Y_{pnq}(m^2)_i^n + 2Y_{ipq}Y_{pqr}(m^2)_i^r
\]

\[+ h_{ipq}h_{jpr} - 8\delta_i^j MM^1 g^2 C(i) , \quad (A.4)\]

\[
\beta_{b_{ij}}^{(1)} = b_i^t \gamma_i^{(1)j} + \mu_i^t \chi_i^{(1)j} + (i \leftrightarrow j) , \quad (A.5)
\]

\[
\beta_g^{(2)} = 2g^2C(G)\beta_g^{(1)} - 2g^3d^{-1}(G) \sum_i C(i)\gamma_i^{(1)i} , \quad (A.6)
\]

\[
\gamma_j^{(2)i} = 2gC(i)\delta_j^i \beta_g^{(1)} - [ Y_{jmn}Y_{ mpi} + 2g^2C(j)\delta_j^p \delta_n^i ] \gamma_p^{(1)n} , \quad (A.7)
\]

\[
\beta_M^{(2)} = 8gC(G)\beta_g^{(1)}M + g^2d^{-1}(G) \sum_i C(i) [ - 4\gamma_i^{(1)i} M + 2\chi_i^{(1)i} ] , \quad (A.8)
\]
\begin{align}
\beta_{h}^{(2)ijk} &= - \left[ h_{ijl}Y_{lmn}Y_{mpk} + 2Y_{ijl}Y_{lmn}h_{mpk} - 4g^2MY_{ijp}C(n)\delta_{n}^{(1)n} \right] \gamma_{p}^{(1)n} \\
&\quad - 2g^2 \left[ h_{ijl}\gamma_{l}^{(1)k} + Y_{ijl}\chi_{l}^{(1)k} \right] C(k) + g(2h_{ijk} - 8MY_{ijk})C(k)\beta_{g}^{(1)} \\
&\quad - Y_{ijl}Y_{lmn}Y_{mpk}\gamma_{p}^{(1)n} + (k \leftrightarrow i) + (k \leftrightarrow j) , \quad (A.9) \\
\beta_{b}^{(2)ij} &= \left[ - b_{ilj}Y_{lmn}Y_{mpj} - 2\mu_{ilj}Y_{lmn}h_{mpj} - Y_{ijl}Y_{lmn}h_{mpj} + 4g^2MC(i)\mu_{ipj}^{(1)}n \right] \gamma_{p}^{(1)n} \\
&\quad - \left[ \mu_{ilj}Y_{lmn}Y_{mpj} + \frac{1}{2}Y_{ijl}Y_{lmn}\mu_{mpj}^{(1)} \right] \chi_{p}^{(1)n} - 2g^2C(i) \left[ b_{ij}^{(1)}i + \mu_{ij}^{(1)}j \right] \\
&\quad + 2gC(i)\beta_{g}^{(1)} \left[ b_{ij}^{(1)}i - 2\mu_{ij}^{(1)}j \right] + (i \leftrightarrow j) . \quad (A.10) \\
\left[ \beta_{m}^{(2)i} \right]_{i} &= \left[ \left( m_{i}^2 \right)_{i}Y_{lmn}Y_{mpj} + \frac{1}{2}Y_{ilm}Y_{jmp} \left( m_{i}^2 \right)_{i} + \frac{1}{2}Y_{ilm}Y_{jln} \left( m_{i}^2 \right)_{i} + Y_{ilm}Y_{jln} \left( m_{i}^2 \right)_{i} \right] \\
&\quad + h_{ilm}h_{jlp} + 4g^2|M|^2C(j)\delta_{n}^{i}\delta_{p}^{j} + 2g^2\sum_{A} \left( R_{A} \right)_{i}^{j} \left( R_{A} \right)_{n}^{p} \gamma_{p}^{(1)n} \\
&\quad + \left[ 2g^2M^4C(i)\delta_{n}^{j}\delta_{p}^{i} - h_{ilm}Y_{jlp} \right] \chi_{p}^{(1)n} - \frac{1}{2}Y_{ilm}Y_{jlp} + 2g^2C(i)\delta_{n}^{j}\delta_{p}^{i} \left[ \beta_{m}^{(1)n} \right]_{p} \\
&\quad + 4|M|^2C(i)\delta_{n}^{i} \left[ 3g\beta_{g}^{(1)} + g^4S' \right] + H.c. , \quad (A.11)
\end{align}

where $S'$ is defined in eq. (16). Further references may be found for instance in ref. [47].
References

[1] H. Georgi and S.L. Glashow, Phys. Rev. Lett. 32 (1974) 438; H. Georgi, H. Quinn, S. Weinberg, Phys. Rev. Lett. 33 (1974) 451.

[2] A. Buras, J. Ellis, M.K. Gaillard and D. Nanopoulos, Nucl. Phys. B135 (1978) 66.

[3] J. Ellis, S. Kelly and D.V. Nanopoulos, Phys. Lett. B260 (1991) 131; U. Amaldi, W. de Boer and H. Fürstenau, Phys. Lett. B260 (1991) 447.

[4] D. Kapetanakis, M. Mondragón and G. Zoupanos, Zeit. f. Phys. C60 (1993) 181; M. Mondragón and G. Zoupanos, Nucl. Phys. B (Proc. Suppl) 37C (1995) 98.

[5] J. Kubo, M. Mondragón and G. Zoupanos, Nucl. Phys. B424 (1994) 291; J. Kubo, M. Mondragón, N.D. Tracas and G. Zoupanos, Phys. Lett. B342 (1995) 155; J. Kubo, M. Mondragón, S. Shoda and G. Zoupanos, Nucl. Phys. B469 (1996) 3.

[6] J. Kubo, M. Mondragón, M. Olechowski and G. Zoupanos, Nucl. Phys. B479 (1996) 25.

[7] W. Zimmermann, Com. Math. Phys. 97 (1985) 211; R. Oehme and W. Zimmermann Com. Math. Phys. 97 (1985) 569; R. Oehme, K. Sibold and W. Zimmermann, Phys. Lett. B147 (1984) 117; B153 (1985) 142.

[8] J. Kubo, K. Sibold and W. Zimmermann, Nucl. Phys. B259 (1985) 331; Phys. Lett. B200 (1989) 185.

[9] N.-P. Chang, Phys. Rev. D10 (1974) 2706; N.-P. Chang, A. Das and J. Perez-Mercader, Phys. Rev. D22 (1980) 1829; E.S. Fradkin and O.K. Kalashnikov, J. Phys. A8 (1975) 1814; Phys. Lett. 59B (1975) 159; ibid 64B (1976) 177; E. Ma, Phys. Rev. D11 (1975) 322; ibid D17 (1978) 623; ibid D31 (1985) 1143; Prog. Theor. Phys. 54 (1975) 1828; Phys. Lett. 62B (1976) 347; Nucl. Phys. B116 (1976) 195; D.I. Kazakov and D.V. Shirkov, Singular Solutions of Renormalization Group Equations and Symmetry of the Lagrangian, in Proc. of the 1995 Smolence Conference on High Energy Particle Interactions, eds. D. Krupa and J. Pisut (VEDA, Publishing House of the Slovak Academy of Sciences, Bratislava 1976); G. Grunberg, Phys. Rev. Lett. 58 (1987) 1180; K.S. Babu and S. Nandi, Prediction of the top and fourth generation fermion masses, Oklahoma State University Preprint, OSU-RN-202 (1988); W.J. Marciano, Phys. Rev. Lett. 62 (1989) 2793; M. Bastero-Gil and J. Perez-Mercader, Phys. Lett. B247 (1990) 346.
[10] A. Parkes and P. West, Nucl. Phys. B222 (1983) 269.

[11] D.R.T. Jones, L. Mezincescu and Y.-P. Yao, Phys. Lett. B148 (1984) 317.

[12] A.J. Parkes and P.C. West, Phys. Lett. B138 (1984) 99; Nucl. Phys. B256 (1985) 340; P. West, Phys. Lett. B137 (1984) 371; D.R.T. Jones and A.J. Parkes, Phys. Lett. B160 (1985) 267; D.R.T. Jones and L. Mezincescu, Phys. Lett. B136 (1984) 242; B138 (1984) 293; A.J. Parkes, Phys. Lett. B156 (1985) 73.

[13] S. Hamidi, J. Patera and J.H. Schwarz, Phys. Lett. B141 (1984) 349; X.D. Jiang and X.J. Zhou, Phys. Lett. B197 (1987) 156; B216 (1985) 160.

[14] S. Hamidi and J.H. Schwarz, Phys. Lett. B147 (1984) 301; D.R.T. Jones and S. Raby, Phys. Lett. B143 (1984) 137; J.E. Bjorkman, D.R.T. Jones and S. Raby, Nucl. Phys. B259 (1985) 503.

[15] J. León et al., Phys. Lett. B156 (1985) 66.

[16] A.V. Ermushev, D.I. Kazakov and O.V. Tarasov, Nucl. Phys. B281 (1987) 72; D.I. Kazakov, Mod. Phys. Lett. A2 (1987) 663; Phys. Lett. B179 (1986) 352; D.I. Kazakov and I.N. Kondrashuk, Int. J. Mod. Phys. A7 (1992) 3869.

[17] C. Lucchesi, O. Piguet and K. Sibold, Helv. Phys. Acta 61 (1988) 321.

[18] O. Piguet and K. Sibold, Int. J. Mod. Phys. A1 (1986) 913; Phys. Lett. B177 (1986) 373.

[19] I. Jack and D.R.T. Jones, Phys.Lett. B333 (1994) 372.

[20] D.I. Kazakov, M.Yu. Kalmykov, I.N. Kondrashuk and A.V. Gladyshev, Nucl. Phys. B471 (1996) 387.

[21] K. Yoshioka, Finite SUSY GUT Revisted, hep-ph/9705449.

[22] O. Piguet and K. Sibold, Phys. Lett. B229 (1989) 83; D. Maison, unpublished; W. Zimmermann, unpublished; J. Kubo, M. Mondragón and G. Zoupanos, Phys. Lett. B389 (1996) 523.

[23] I. Jack and D.R.T. Jones, Phys. Lett. B349 (1995) 294; I. Jack, D.R.T. Jones and K.L. Roberts, Nucl. Phys. B455 (1995) 83.
[24] S.P. Martin and M.T. Vaughn, Phys. Lett. **B318** (1993) 331; Phys. Rev. **D50** (1994) 2282; Y. Yamada, Phys. Lett. **B316** (1993) 109; Phys. Rev. Lett. **72** (1994) 25; Phys. Rev. **D50** (1994) 3537; I. Jack, D.R.T. Jones, S.P. Martin, M.T. Vaughn and Y. Yamada, Phys. Rev. **D50** (1994) R5481.

[25] L.E. Ibáñez, *Strings, unification and dilaton/moduli-induced SUSY breaking*, [hep-th/9505098](http://arxiv.org/abs/hep-th/9505098).

[26] C. Muñoz, *Soft Terms from Strings*, [hep-ph/9601325](http://arxiv.org/abs/hep-ph/9601325).

[27] L.E. Ibáñez and D. Lüst, Nucl. Phys. **B382** (1992) 305.

[28] V.S. Kaplunovsky and J. Louis, Phys. Lett. **B306** (1993) 269.

[29] A. Brignole, L.E. Ibáñez and C. Muñoz, Nucl. Phys. **B422** (1994) 125; [Erratum: **B436** (1995) 747].

[30] Y. Kawamura, T. Kobayashi and J. Kubo, *Soft Scalar-Mass Sum Rule in Gauge-Yukawa Unified Models and Its Superstring Interpretation*, [hep-ph/9703320](http://arxiv.org/abs/hep-ph/9703320), to be published in Phys. Lett. **B**.

[31] K. Inoue, A. Kakuto, H. Komatsu and S. Takeshita, Prog. Theor. Phys. **68** (1982) 927; L.E. Ibáñez and G.G. Ross, Phys. Lett. **B110** (1982) 215.

[32] E. Witten, Phys. Lett. **B155** (1985) 151.

[33] S. Ferrara, C. Kounnas and M. Porrati, Phys. Lett. **B181** (1986) 263; M. Cvetic, J. Louis and B. Ovrut, Phys. Lett. **B206** (1988) 227; M. Cvetic, J. Molera and B. Ovrut, Phys. Rev. **D40** (1989) 1140.

[34] S. Ferrara, D. Lüst, A. Shapere and S. Theisen, Phys. Lett. **B225** (1989) 363; S. Ferrara, D. Lüst and S. Theisen, Phys. Lett. **B233** (1989) 147.

[35] L.J. Dixon, V.S. Kaplunovsky and J. Louis, Nucl. Phys. **B329** (1990) 27.

[36] T. Kobayashi, D. Suematsu, K. Yamada and Y. Yamagishi, Phys. Lett. **B348** (1995) 402.
[37] A. Brignole, L.E. Ibáñez, C. Muñoz and C. Scheich, Z. f. Phys. C74 (1997) 157; A. Brignole, L.E. Ibáñez and C. Muñoz, Soft supersymmetry breaking terms from supergravity and superstring models, hep-ph/9707209.

[38] D. Suematsu, Phys. Rev. D54 (1996) 5715.

[39] H.B. Kim and C. Muñoz, Z. f. Phys. C75 (1997) 367; Mod. Phys. Lett. A12 (1997) 315.

[40] Y. Kawamura, Phys. Rev. D53 (1996) 3779; Y. Kawamura and T. Kobayashi, Phys. Lett. B375 (1996) 141; Generic formulae of soft scalar masses in string models hep-ph/9608233, to be published in Phys. Rev. D; Y. Kawamura, T. Kobayashi and T. Komatsu, Phys. Lett. B400 (1997) 284.

[41] J.-P. Derendinger, S. Ferrara, C. Kounnas and F. Zwirner, Nucl. Phys. B372 (1992) 145.

[42] G. Lopes Cardoso and B. Ovrut, Nucl. Phys. B369 (1992) 351.

[43] M.B. Green and J.H. Schwarz, Phys. Lett. B149 (1984) 117.

[44] V.S. Kaplunovsky, Nucl. Phys. B307 (1988) 145 [Erratum: Nucl. Phys. B382 (1992) 436].

[45] L.J. Dixon, V.S. Kaplunovsky and J. Louis, Nucl. Phys. B355 (1991) 649; I. Antoniadis, K.S. Narain and T.R. Taylor Phys. Lett. B267 (1991) 37.

[46] J. Kubo and B. Milewski, Nucl. Phys. B354 (1985) 367.

[47] V. Barger, M.S. Berger and P. Ohmann, Phys. Rev. D47 (1993) 1093.

[48] Particle Data Group, L. Montanet et al., Phys. Rev. D50 (1994) 1173.

[49] P.H. Chankowski, Z. Pluciennik and S. Pokorski, Nucl. Phys. B439 (1995) 23.

[50] D0 and CDF Collaborations, Top Quark Mass Measurements from the Tevatron, hep-ex/9706011.