The limit theorem for maximum of partial sums of exchangeable random variables

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Abstract

We obtain the analogue of the classical result by Erdős and Kac on the limiting distribution of the maximum of partial sums for exchangeable random variables with zero mean and variance one. We show that, if the conditions of the central limit theorem of Blum et al. hold, the limit coincides with the classical one. Under more general assumptions, the probability of the random variables having conditional negative drift appears in the limit.

Keywords: exchangeable random variables, limit theorem, maximum, de Finetti’s theorem

MSC: 60G09, 60F05, 60G70

1. Introduction

Erdős and Kac established in [14] some fundamental results on the distribution of the maximum of partial sums $S_k := \sum_{i=1}^{k} X_i$, where $\{X_n\}_{n \in \mathbb{N}}$ is a sequence of independent, identically distributed (i.i.d.) centered random variables with variance one. In particular, they proved that the limiting distribution of $n^{-\frac{1}{2}} \max_{1 \leq k \leq n} S_k$ is given by $(2\Phi(x) - 1)1_{[0,\infty)}(x)$, where $\Phi(\cdot)$ denotes the probability distribution function (p.d.f.) of the standard normal distribution.

Our interest in studying the (rescaled) maximum of partial sums is motivated by its manifold applications. On the one hand, it is directly related to first passage times of random walks and renewal theory [17, 23]. On the other hand, in the classical i.i.d. setting, this statistic has since long been employed in numerous research areas such as hydrology [6], reservoir storage [18] and change-point analysis [19]. Moreover, as a matter of study in extreme value theory, this type of limit theorems are of especial relevance, for instance in finance (see [21] and references therein).

The purpose of this paper is to generalize the original result of Erdős and Kac to exchangeable sequences of random variables and thereby extend the mentioned statistic to further stochastic models. Exchangeable random variables, introduced by de Finetti in [12], are random variables with the property of being conditionally independent. Equivalently, one can think of them as mixtures of i.i.d. random variables directed by a random measure. The study of classical results of probability theory in the exchangeable setting started with the Central Limit Theorem (CLT)
by Blum, Chernoff, Rosenblatt and Teicher in [6] and it led to a series of works [13, 14, 24] that continues expanding (see e.g. [3, 9, 15, 21]). Exchangeable random variables are of great interest due to their versatility as stochastic models [3, 22, 2] and their wide applicability in genetics [20], Bayesian analysis [10] and many other branches of statistical analysis [16, 26, 8].

The limit theorem considered in this paper contributes to extend results of extreme value theory to the exchangeable context. In this direction, Berman obtained in [4] the limiting distribution of the maximum of an exchangeable sequence of random variables.

Our results show that, if the classical conditions of the CLT of Blum et al. hold, one obtains the original statement of Erd"os and Kac in the i.i.d. setting, c.f. Proposition 1. Dropping off the assumption on the variance of the directing random measure gives rise in Theorem 2 to a limiting distribution that, in the non-degenerate case, resembles the previous result and involves the distribution function of a mixture of Gaussians. Consequently, we discover in Corollary 1 that, when no assumptions are imposed to the directing random measure, the limit of the distribution of $n^{-1/2} \max_{1 \leq k \leq n} S_k$ depends on the conditional drift and the conditional variance of the random variables. In particular, we see that the probability of the random variables having negative drift makes a substantial contribution to this limit.

The paper is organized as follows: In Section 2 we fix notation and briefly review basic results of the theory of exchangeable random variables. Section 3 is devoted to presenting and proving the different generalizations of the limit distribution of Erd"os and Kac. Finally, these results are furnished with examples in Section 4.

2. Definitions and auxiliary results

Let $\Pi(n)$ denote the set of permutations of $\{1, \ldots, n\}$. A sequence of random variables $\{X_n\}_{n \in \mathbb{N}}$ is said to be exchangeable if for any $n \in \mathbb{N}$, $X_1, \ldots, X_n$ are exchangeable, i.e., for any permutation $\pi \in \Pi(n)$,

$$\text{Law}(X_1, \ldots, X_n) = \text{Law}(X_{\pi(1)}, \ldots, X_{\pi(n)}).$$

Alternatively we write $(X_1, \ldots, X_n) \overset{d}{=} (X_{\pi(1)}, \ldots, X_{\pi(n)})$. The concept of exchangeability was introduced by de Finetti in [12], who in particular proved that such a sequence is conditionally i.i.d. given the $\sigma$-field of permutable events.

An essential tool in our proofs is de Finetti’s theorem. Let $\mathcal{F}$ denote the collection of all p.d.f.s on $\mathbb{R}$ with the topology of weak convergence of distribution functions. De Finetti’s theorem states that for an infinite sequence of exchangeable random variables $\{X_n\}_{n \in \mathbb{N}}$, there exists a unique probability measure $\mu$ on the Borel $\sigma$-field $\mathcal{A}$ of subsets of $\mathcal{F}$ such that for any $n \geq 1$,

$$P(g(X_1, \ldots, X_n) \in B) = \int_{\mathcal{F}} P_F(g(X_1, \ldots, X_n) \in B) \mu(dF)$$

(1)
holds for any Borel set $B \in \mathcal{B}(\mathbb{R})$ and any Borel function $g : \mathbb{R}^n \to \mathbb{R}$. Here, $P_F(g(X_1, \ldots, X_n) \in B)$ is the probability of the event under the assumption that the random variables $X_1, \ldots, X_n$ are independent with common p.d.f. $F$. The mean $E_F g(X_1, \ldots, X_n)$ is obtained by integrating $g$ with respect to the probability measure $F$. Let us now denote by $F : \Omega \to \mathcal{F}$ a random variable whose probability distribution is given by the measure $\mu$ from de Finetti’s theorem. The conditional mean $E_F g(X_1, \ldots, X_n)$ is defined analogously to $E_F g(X_1, \ldots, X_n)$ and is itself a random variable because the p.d.f. $F$ is random. It should be noted that de Finetti’s theorem fails for finite collections of exchangeable random variables. We refer to [1] for further details on this subject.

The law of large numbers (LLN) for exchangeable sequences was established by Hu and Taylor in [24]. They showed that for an exchangeable sequence $\{X_n\}_{n \in \mathbb{N}}$ such that $E|X_1| < \infty$, a.s.,

$$\frac{1}{n}S_n \xrightarrow{a.s.} E_F X_1 \quad \text{if and only if} \quad EX_1X_2 = 0. \quad (2)$$

It is not difficult to see that $EX_1X_2 = 0$ is equivalent to $E_F X_1 = 0$ a.s. As already mentioned in the introduction, Blum, Chernoff, Rosenblatt and Teicher proved in [6] that for an exchangeable sequence with zero mean and variance one the CLT holds if and only if

$$EX_iX_j = 0 \quad \text{and} \quad EX_i^2X_j^2 = 1 \quad \forall \ i \neq j. \quad (3)$$

In general, it is possible to obtain limit theorems for sums of exchangeable sequences under weaker assumptions. De Finetti’s theorem can be rephrased (see [1, Theorem 3.1]) by saying that the infinite exchangeable sequence $\{X_n\}_{n \in \mathbb{N}}$ is a mixture of i.i.d. random variables directed by the random measure $F$, whose probability distribution $\mu$ is given in (1). With this notation,

$$P_F((X_1, \ldots, X_n) \in A) = \prod_{i=1}^{n} F(A_i), \quad A = A_1 \times \cdots \times A_n \in \mathcal{B}(\mathbb{R}^n),$$

and $E_F X_1 = \int_{\mathbb{R}} x F(dx)$. Moreover, if $E|X_1| < \infty$, then the LLN

$$\frac{1}{n}S_n \xrightarrow{a.s.} E_F X_1 \quad \text{as} \quad n \to \infty \quad (4)$$

holds (see e.g. [1, p.17]). This directly implies the necessity of (2) since $EX_1X_2 = 0$ means that the directing random measure $F$ has zero mean, i.e. $E_F X_1 = 0$ a.s. Furthermore, we have that if $0 < EX_1^2 < \infty$, then the CLT

$$\frac{S_n - nE_F X_1}{\sqrt{n\sigma_F^2}} \xrightarrow{d} N(0,1) \quad (5)$$

holds, where $\sigma_F^2 := E_F(X_1 - E_F X_1)^2$. This formulation generalizes the necessity of (3) because again $F$ has zero mean and $EX_1^2X_2^2 = 1$ is equivalent to the fact that $F$ has variance one, i.e. $\sigma_F^2 = 1$ a.s. These limit theorems can also be obtained in terms of conditional characteristic functions, see [28].
3. Limit theorem for maximum of sums of exchangeable random variables

In this section, we investigate the limiting distribution of the largest partial sum of an exchangeable sequence of random variables. In a first step, this limit is obtained under the assumption that the directing random measure $F$ has zero mean and variance one. Secondly, the variance-one assumption is removed and the corresponding limiting theorem is derived. Finally, the latter result is applied to analyze the limit of the probability of the maximum of partial sums for a sequence with a general directing random measure.

Let us start by recalling the original result of Erdős and Kac.

**Theorem 1.** \([14]\) Let \(\{X_n\}_{n\in\mathbb{N}}\) be a sequence of i.i.d. random variables with zero mean and variance one, and let \(S_k := \sum_{i=1}^{k} X_i\). Then,

\[
\lim_{n \to \infty} P(\max(S_1, \ldots, S_n) < x\sqrt{n}) = G(x),
\]

where \(G: \mathbb{R} \to \mathbb{R}\) is given by

\[
G(x) := (2\Phi(x) - 1)1_{[0,\infty)}(x)
\]

and \(\Phi\) denotes the p.d.f. of the standard normal distribution.

A direct extension of this theorem in the exchangeable setting is obtained when we assume that the conditions for the classical CLT given in (3) are satisfied.

**Proposition 1.** Let \(\{X_n\}_{n\in\mathbb{N}}\) be an exchangeable sequence of random variables with zero mean and variance one satisfying (3). Then,

\[
\lim_{n \to \infty} P(\max(S_1, \ldots, S_n) < x\sqrt{n}) = G(x),
\]

where \(G: \mathbb{R} \to \mathbb{R}\) is given by (6).

**Proof.** Since \(P_F(\max(S_1, \ldots, S_n) < x\sqrt{n})\) is uniformly bounded by one and \(\mu\) is a probability measure, applying de Finetti’s theorem, Lebesgue dominated convergence theorem and Theorem 1 to the conditional probability yields

\[
\lim_{n \to \infty} P(\max(S_1, \ldots, S_n) < x\sqrt{n}) = \int F \lim_{n \to \infty} P_F(\max(S_1, \ldots, S_n) < x\sqrt{n}) \mu(dF) = G(x).
\]

\(\square\)

**Remark 1.** Notice that in fact, every limiting result originally proved by Erdős and Kac in [14] can be obtained in the same fashion.
The next natural step to generalize Proposition 1 consists in considering an exchangeable sequence \( \{X_n\}_{n \in \mathbb{N}} \) whose directing random measure \( F \) only satisfies the zero-mean condition. From (5) we know that
\[
\frac{1}{\sqrt{n}} S_n \xrightarrow{d} Z \cdot \sigma_F,
\]
where \( Z \sim \mathcal{N}(0, 1) \) is independent of \( \sigma_F \) and \( F \) has distribution measure \( \mu \) from de Finetti’s theorem. Let us now define \( G \mu : \mathbb{R} \to \mathbb{R} \) as
\[
G \mu(x) := \int_{\mathbb{R}} 1_{(0, \infty)}(\sigma_F^2) G(x/\sigma_F) \mu(dF),
\]
where \( G \) was given (6). Then we have the following result.

**Theorem 2.** Let \( \{X_n\}_{n \in \mathbb{N}} \) be an exchangeable sequence of random variables with zero mean and variance \( 0 < \text{EX}_1^2 < \infty \) such that \( \text{EX}_1X_2 = 0 \). Then,
\[
\lim_{n \to \infty} \mathbb{P}(\max(S_1, \ldots, S_n) < x\sqrt{n}) = \mathbb{P}(\sigma_F^2 = 0)1_{[0, \infty)}(x) + G \mu(x),
\]
where \( G \mu : \mathbb{R} \to \mathbb{R} \) is given in (8) and \( \mu \) is the distribution of the directing random measure of the sequence \( \{X_n\}_{n \in \mathbb{N}} \).

**Proof.** In view of (7), de Finetti’s theorem, Lebesgue dominated convergence theorem and Theorem 1 lead to
\[
\lim_{n \to \infty} \mathbb{P}(\max(S_1, \ldots, S_n) < x\sqrt{n})
= \mathbb{P}(\sigma_F^2 = 0)1_{[0, \infty)}(x) + \int_{\mathbb{R}} \lim_{n \to \infty} 1_{(0, \infty)}(\sigma_F^2) \mathbb{P}_F(\max(S_1, \ldots, S_n) < x\sqrt{n}) \mu(dF)
= \mathbb{P}(\sigma_F^2 = 0)1_{[0, \infty)}(x) + \int_{\mathbb{R}} 1_{(0, \infty)}(\sigma_F^2) G(x/\sigma_F) \mu(dF).
\]
\( \square \)

**Remark 2.** Notice that in the exchangeable setting one may encounter sequences of non constant random variables with \( \mathbb{P}(\sigma_F^2 = 0) > 0 \) (see Example 2). In particular, Theorem 2 shows that if the sequence is non-degenerated in the sense that the conditional variance is almost surely positive, i.e. \( \mathbb{P}(\sigma_F^2 > 0) = 1 \), then the limiting distribution in (9) becomes the mixture
\[
\int_{\mathbb{R}} G(x/\sigma_F) \mu(dF).
\]

**Remark 3.** The exchangeable counterparts of the limiting distributions in [14] under the assumptions of Theorem 2 can be derived in the same fashion.
By de Finetti’s theorem,

**Corollary 1.** Let \( \{X_n\}_{n \in \mathbb{N}} \) be an exchangeable sequence of random variables with zero mean and variance \( 0 < \mathbb{E}X_1^2 < \infty \). Then,

\[
\lim_{n \to \infty} \mathbb{P}(\max(S_1, \ldots, S_n) < x\sqrt{n}) = \mathbb{P}(\mathbb{E}_F X_1 < 0) + \mathbb{P}(\mathbb{E}_F X_1 = 0, \sigma_F^2 = 0)1_{(0, \infty)}(x) + G'(x),
\]

where \( G' : \mathbb{R} \to \mathbb{R} \) is given by

\[
G'(x) = \int_{\mathbb{F}} 1_{\{0\}}(\mathbb{E}_F X_1)1_{(0, \infty)}(\sigma_F^2) G(x/\sigma_F) \mu(dF)
\]

and \( G \) is defined in (6).

**Proof.** By de Finetti’s theorem,

\[
\mathbb{P}(\max(S_1, \ldots, S_n) < x\sqrt{n}) = \int_{\mathbb{F}^*} \mathbb{P}_F(\max_{1 \leq k \leq n} S_k < x\sqrt{n}) \mu(dF)
\]

\[
+ \int_{\mathbb{F}_0} \mathbb{P}_F(\max_{1 \leq k \leq n} S_k < x\sqrt{n}) \mu(dF) + \int_{\mathbb{F}_-} \mathbb{P}_F(\max_{1 \leq k \leq n} S_k < x\sqrt{n}) \mu(dF)
\]

\[
=: I_{n,+}(x) + I_{n,0}(x) + I_{n,-}(x),
\]

where \( \mathbb{F}_+ := \{F \in \mathbb{F} | \mathbb{E}_F X_1 > 0\}, \mathbb{F}_0 := \{F \in \mathbb{F} | \mathbb{E}_F X_1 = 0\}, \) and \( \mathbb{F}_- := \{F \in \mathbb{F} | \mathbb{E}_F X_1 < 0\}. \)

On the one hand, for any \( F \in \mathbb{F}_+ \),

\[
\mathbb{P}_F(\max_{1 \leq k \leq n} S_k < x\sqrt{n}) \leq \mathbb{P}_F(S_n - n\mathbb{E}_F X_1 < x\sqrt{n}) \leq \frac{\sigma_F^2}{nx^2}
\]

which tends to zero as \( n \to \infty \). Lebesgue dominated convergence theorem thus yields \( \lim_{n \to \infty} I_{n,+}(x) = 0 \). On the other hand, following the proof of Theorem 2 we have that

\[
\lim_{n \to \infty} I_{n,0}(x) = \int_{\mathbb{F}_0} 1_{(0, \infty)}(\sigma_F)1_{[0, \infty)}(x) \mu(dF) + \int_{\mathbb{F}_0} 1_{(0, \infty)}(\sigma_F) G(x/\sigma_F) \mu(dF)
\]

\[
= \mathbb{P}(\mathbb{E}_F X_1 = 0, \sigma_F^2 = 0)1_{(0, \infty)}(x) + G'(x).
\]

Finally, if \( F \in \mathbb{F}_- \), the Háyek-Rényi inequality leads to

\[
\mathbb{P}_F(\max_{1 \leq k \leq n} S_k < x\sqrt{n}) \geq 1 - \sum_{k=1}^{n} \frac{\sigma_F^2}{(x\sqrt{n} - k\mathbb{E}_F X_1)^2} \geq 1 - \frac{\sigma_F^2}{\mathbb{E}_F X_1 x\sqrt{n}},
\]

which tends to one as \( n \to \infty \). By Lebesgue dominated convergence theorem, \( \lim_{n \to \infty} I_{n,-}(x) = \mathbb{P}(\mathbb{E}_F X_1 < 0) \) and the result follows. \( \square \)
Remark 4. Notice that the limit appearing in Corollary 4 is not a distribution function unless the assumptions reduce to those of Theorem 2. Thus, convergence in distribution holds in Proposition 1 and Theorem 2, but not in the general situation of Corollary 1.

4. Examples

We finish our discussion with some examples that furnish the results presented in the previous section.

Example 1. Let \( \{Y_n\}_{n \in \mathbb{N}} \) be an exchangeable sequence of random variables with \( \text{E}Y_1Y_2 = 0 \) and \( \text{E}Y_1^2Y_2^2 = 1 \) that take values in \( \{-1, 1\} \), and let \( \{Z_n\}_{n \in \mathbb{N}} \) be a sequence of i.i.d. standard normal distributed random variables independent of \( \{Y_n\}_{n \in \mathbb{N}} \). The sequence \( \{X_n\}_{n \in \mathbb{N}} := \{Y_n + Z_n\}_{n \in \mathbb{N}} \) is exchangeable and the process \( S_n := \sum_{k=1}^n X_k \) may be called exchangeable random walk plus noise. This model appears for instance in Bayesian dynamic modeling [11, Chapter 8]. For this sequence, \( \text{E}X_n = \text{E}Y_n + \text{E}Z_n = 0 \) and

\[
\text{E}X_1X_2 = \text{E}(Y_1 + Z_1)(Y_2 + Z_2) = \text{E}Y_1Y_2 + \text{E}Y_1^2\text{E}Z_2^2 + \text{E}Z_1\text{E}Y_2^2 + \text{E}Z_1\text{E}Z_2^2 = 0.
\]

Moreover, it is non-degenerate in the sense that

\[
P(\sigma^2 F > 0) = P(\text{E}_FX_1^2 > 0) \geq P(\text{E}_F Y_1^2 > 0) = 1,
\]

and therefore \( P(\sigma^2 F = 0) = 0 \). Thus, we can investigate the asymptotic distribution of the stopping times

\[
T_n(x) := \inf\{k \geq 1 : S_k > x\sqrt{n}\}
\]

by studying the asymptotic distribution of the maximum of partial sums \( S_k \). Applying Theorem 2 we get

\[
\lim_{n \to \infty} P(T_n(x) \leq n) = \lim_{n \to \infty} P(\max(S_1, \ldots, S_n) > x\sqrt{n}) = 1 - \lim_{n \to \infty} P(\max(S_1, \ldots, S_n) \leq x\sqrt{n}) = 1 - G_\mu(x),
\]

where \( G_\mu \) is given by (10).

As pointed out in Remark 2, it is possible to have an exchangeable sequence of non constant random variables with \( P(\sigma^2 F = 0) > 0 \). The following example illustrates how this situation may arise in applications.
Example 2. In financial modelling, the risk of a financial asset can be represented by a sequence \( \{\xi_n\}_{n \in \mathbb{N}} \) of i.i.d. real-valued random variables, for instance with zero mean and positive variance \( \sigma^2 > 0 \). The quantity \( \max_{1 \leq k \leq n} \sum_{i=1}^{k} \xi_i \) thus expresses the maximal loss over a certain period. Exchangeable models appear in this setting for instance by introducing an independent default indicator \( Y \) such that \( P(Y = 1) = 1 - P(Y = 0) = p, \ p \in (0, 1) \). In this case, the sequence 

\[
\{X_n\}_{n \in \mathbb{N}} := \{Y \xi_n\}_{n \in \mathbb{N}}
\]

is exchangeable with \( \mathbb{E}X_1 = p \mathbb{E}\xi_1 = 0 \) and \( \mathbb{E}X_2 = p \mathbb{E}\xi_2 = p \sigma^2 > 0 \). This example is especially illustrative because it is possible to verify Theorem 2 by calculating the limiting probability of the maximal loss directly. Since \( \xi_n \) are i.i.d. we have that 

\[
\mathbb{E}X_1X_2 = \mathbb{E}Y^2\xi_1\xi_2 = p \mathbb{E}\xi_1\xi_2 = p(\mathbb{E}\xi_1)^2 = 0 \text{ and } \mathbb{E}X_1^2X_2^2 = \mathbb{E}Y^4\xi_1^2\xi_2^2 = p \mathbb{E}\xi_1^2\xi_2^2 > 0,
\]

which might not necessarily be one.

For the partial sums \( S_n := \sum_{i=1}^{n} X_i \), the limiting probability 

\[
\lim_{n \to \infty} P(\max(S_1, \ldots, S_n) \leq x \sqrt{n})
\]

can be obtained as follows: Let \( \tilde{S}_n := \sum_{k=1}^{n} \xi_k \). By conditioning on \( Y \) and applying the classical result of Erdős and Kac to \( \tilde{S}_n/\sigma \) we have that 

\[
\lim_{n \to \infty} P(\max_{1 \leq k \leq n} S_k \leq x \sqrt{n}) = p G(x/\sigma) + (1 - p) 1_{[0, \infty)}(x).
\]

Let us now see that this coincides with the limiting expression in our Theorem. In this case, the directing measure \( \mathbb{F} \) is discrete and takes the values \( \mathbb{F}_1 \) and \( \mathbb{F}_2 \) with probability \( p \) and \( 1 - p \) respectively, and 

\[
X_n = \begin{cases} 
\xi_n & \text{under } \mathbb{F}_1, \\
0 & \text{under } \mathbb{F}_2,
\end{cases}
\]

for each \( n \in \mathbb{N} \). Thus, \( E_{\mathbb{F}_1} X_1 = E\xi_1 = 0 = E_{\mathbb{F}_2} X_2 \) and \( \text{var}_{\mathbb{F}_1} X_1 = \sigma^2, \ \text{var}_{\mathbb{F}_2} X_1 = 0 \).

In particular \( X_1 \equiv 0 \) under \( \mathbb{F}_2 \), hence 

\[
P(E_{\mathbb{F}_1} X_1 = 0, \sigma^2_{\mathbb{F}} = 0) = P(\mathbb{F} = \mathbb{F}_2) = 1 - p
\]

and 

\[
G_{\mu}(x) = \int_{\delta} 1_{[0, \infty)}(E_{\mathbb{F}_1} X_1) 1_{[0, \infty)}(\sigma^2_{\mathbb{F}}) G(x/\sigma_{\mathbb{F}}) \mu(d\mathbb{F}) = p G(x/\sigma)
\]

as desired.

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References

[1] D. J. Aldous, *Exchangeability and related topics*, École d’été de probabilités de Saint-Flour, XIII—1983, Lecture Notes in Math., vol. 1117, Springer, Berlin, 1985, pp. 1–198.

[2] ______, *Exchangeability and continuum limits of discrete random structures*, Proceedings of the International Congress of Mathematicians. Volume I, Hindustan Book Agency, New Delhi, 2010, pp. 141–153.

[3] T. Austin, *On exchangeable random variables and the statistics of large graphs and hypergraphs*, Probab. Surv. 5 (2008), 80–145.

[4] S. M. Berman, *Limiting distribution of the maximum term in sequences of dependent random variables*, Ann. Math. Statist. 33 (1962), 894–908.

[5] P. Berti and P. Rigo, *A Glivenko-Cantelli theorem for exchangeable random variables*, Statist. Probab. Lett. 32 (1997), no. 4, 385–391.

[6] J. R. Blum, H. Chernoff, M. Rosenblatt, and H. Teicher, *Central limit theorems for interchangeable processes*, Canad. J. Math. 10 (1958), 222–229.

[7] T. A. Buishand, *Some methods for testing the homogeneity of rainfall records*, Journal of Hydrology 58 (1982), no. 1, 11 – 27.

[8] A. Bulinski and A. Rakitko, *Simulation and analytical approach to the identification of significant factors*, Communications in Statistics Part B: Simulation and Computation 44 (2015), 1–23.

[9] S. Chatterjee, *A generalization of the Lindeberg principle*, Ann. Probab. 34 (2006), no. 6, 2061–2076.

[10] A. Coen and R. H Mena, *Ruin probabilities for Bayesian exchangeable claims processes*, Journal of Statistical Planning and Inference (2015), in press.

[11] P. Damien, P. Dellaportas, N. G. Polson, and D. A. Stephens (eds.), *Bayesian theory and applications*, Oxford University Press, Oxford, 2013.

[12] B. De Finetti, *Funzione caratteristica di un fenomeno aleatorio*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Nat. 4 (1930), 86–133.

[13] G. K. Eagleson, *Weak limit theorems for exchangeable random variables*, Exchangeability in probability and statistics (Rome, 1981), North-Holland, Amsterdam-New York, 1982, pp. 251–268.

[14] P. Erdős and M. Kac, *On certain limit theorems of the theory of probability*, Bull. Amer. Math. Soc. 52 (1946), 292–302.
[15] S. Fortini, L. Ladelli, and E. Regazzini, *Central limit theorem with exchangeable summands and mixtures of stable laws as limits*, Boll. Unione Mat. Ital. (9) 5 (2012), no. 3, 515–542.

[16] A. Gerardi, F. Spizzichino, and B. Torti, *Exchangeable mixture models for lifetimes: the role of “occupation numbers”*, Statist. Probab. Lett. 49 (2000), no. 4, 365–375.

[17] C. C. Heyde, *Asymptotic renewal results for a natural generalization of classical renewal theory*, J. Roy. Statist. Soc. Ser. B 29 (1967), 141–150.

[18] H. E. Hurst, *Long-term storage capacity of reservoirs*, Trans. Am. Soc. Eng. 116 (1951), no. 1, 770–799.

[19] B. James, K. L. James, and D. Siegmund, *Tests for a change-point*, Biometrika 74 (1987), no. 1, 71–83.

[20] J. F. C. Kingman, *Mathematics of genetic diversity*, CBMS-NSF Regional Conference Series in Applied Mathematics, vol. 34, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, Pa., 1980.

[21] S. Y. Novak, *Extreme value methods with applications to finance*, Monographs on Statistics and Applied Probability, vol. 122, CRC Press, Boca Raton, FL, 2012.

[22] B. L. S. Prakasa Rao, *Conditional independence, conditional mixing and conditional association*, Ann. Inst. Statist. Math. 61 (2009), no. 2, 441–460.

[23] D. Siegmund, *On the asymptotic normality of one-sided stopping rules*, Ann. Math. Statist 39 (1968), 1493–1497.

[24] R. L. Taylor and T. C. Hu, *On laws of large numbers for exchangeable random variables*, Stochastic Anal. Appl. 5 (1987), no. 3, 323–334.

[25] N. C. Weber, *A martingale approach to central limit theorems for exchangeable random variables*, J. Appl. Probab. 17 (1980), no. 3, 662–673.

[26] C. Yu and D. Zelterman, *Sums of exchangeable Bernoulli random variables for family and litter frequency data*, Computational Statistics and Data Analysis 52 (2008), no. 3, 1636 – 1649.

[27] D. Yuan and S. Li, *Extensions of several classical results for independent and identically distributed random variables to conditional cases*, J. Korean Math. Soc. 52 (2015), no. 2, 431–445.

[28] D. Yuan, L. Wei, and L. Lei, *Conditional central limit theorems for a sequence of conditional independent random variables*, J. Korean Math. Soc. 51 (2014), no. 1, 1–15.