Thermodynamic Gross-Neveu model under constant electromagnetic field

Shinya Kanemura\textsuperscript{a)}\textsuperscript{1}, Haru-Tada Sato\textsuperscript{b)}\textsuperscript{2} and Hiroshi Tochimura\textsuperscript{c)}\textsuperscript{3}

\textsuperscript{a)} Theory Group, KEK, Tsukuba, Ibaraki 305, Japan,
\textsuperscript{b)} The Niels Bohr Institute, University of Copenhagen, Blegdamsvej 17, DK-2100 Copenhagen, Denmark
\textsuperscript{c)} Department of Physics, Hiroshima University, Kagamiyama 1-3-1, Higashi-Hiroshima 739, Japan

Abstract

Analyzed is the effective potential of $D$-dimensional thermodynamic Gross-Neveu model (defined by large-$N$ leading order, $2 \leq D < 4$) under constant electromagnetic field ($\vec{E} \cdot \vec{B} = 0$). The potential is derived from a thermal analogy of the worldline formalism. In the magnetic case, the potential is expressed in terms of a discretized momentum sum instead of a continuum momentum integral, and it reproduces known critical values of $\mu$ and $T$ in the continuum limit. In the electric case, chiral symmetry is restored at finite $T$ (with arbitrary $\mu$) against instability of fermion vacuum. An electromagnetic duality is pointed out as well. Phase diagrams are obtained in both cases.

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\textsuperscript{1}e-mail: kanemu@theory.kek.jp
\textsuperscript{2}e-mail: sato@nbi.dk
\textsuperscript{3}e-mail: tochi@theo.phys.sci.hiroshima-u.ac.jp
1 Introduction

The Gross-Neveu (GN) model is known as the simple model endowed with the similar properties to QCD: the dynamical breaking of (discrete) chiral symmetry, dimensional transmutation, asymptotic freedom and renormalizability ($D < 4$). Indeed, a number of GN models (as well as the NJL models) have often been studied as the prototypes of dynamical symmetry breaking in various circumstances; for example, not only in finite temperature [1, 17, 19] or in electromagnetic field [2] - [10] but also in finite curvature [14, 15]. The high temperature behaviors of these models [7, 10, 13, 16] may potentially be interesting for exploring a dynamical symmetry breaking in (quark-gluon) plasma and in cosmological situations like early universe or a neutron star.

In this paper, we examine, under a magnetic $B$ (or electric $E$) background field in finite temperature and density, the effective potential and chiral phase structure of the GN model which is defined by the leading order of $1/N$-expansion and is analytically continued to arbitrary dimensions ($2 \leq D < 4$) through the dimensional regularization procedure.

In the case of $D = 4$ [13], correctly speaking $D$ being slightly below 4, our unthermalized effective potential corresponds to the effective potential of the NJL model as a cut-off theory, if we put one of auxiliary fields into zero. It may be interesting to apply to quark-gluon plasma in a magnetic/electric field (at high temperature). In the case of $D = 3$ [10], the physics of our model concerns planar condensed matter systems such as BCS theory: a magnetic field works as a catalyst of generating an energy gap in fermion spectrum, enhancing the interactions of fermions at small momenta even if the weakest (attractive) interaction. Gusynin et. al. suggested [10, 11] that this effect is universal, i.e., model independent, in $D = 3$ and should be considered in the phenomena such as quantum Hall effects and high temperature superconductivity. Similarly in a supersymmetric model, constant magnetic field enhances fermion’s dynamical mass [12].

Hence our first concern is the question how the effect of magnetic catalyst universal is, in particular, in the setting of both finite temperature $T$ and chemical potential $\mu$; in addition, how the magnetic catalyst effect differs in other dimensions under finite $T$ and
\( \mu \). Secondly, related to these questions, we are interested in what happens if an electric field exists as well as a magnetic field. In the case of \( D = 2 \), magnetic field can not exist, however the GN model by itself corresponds to a model of polyacetylene. The first order phase transition of polyacetylene from solitonic (broken) phase to metallic (symmetric) phase is formulated \[18\] as a function of doping concentration (chemical potential) within the large-\( N \) approximation. The critical value of chemical potential coincides with an experimental value, and recently a \( 1/N \)-correction to this result was obtained in \[24\].

Our study might explain some thermodynamic behaviour of a polyacetylene model under constant electric field in the finite temperature and density system.

In order to handle with these models all together, it is, of course, useful to analyze a dimensionally regulated effective potential: such observations were made in the cases of finite temperature \[19\], curvature \[14\] and their mixture \[16\]. We shall derive, in \( D \) dimensions, the effective potential as a function of \( T, \mu \) and the external field parameter \( \xi (= e \sqrt{|\vec{B}|^2 - |\vec{E}|^2} \), where we have assumed \( \vec{B} \cdot \vec{E} = 0 \)\) within the leading order of the \( 1/N \)-expansion. Then we present various interesting properties which can be derived from a universal expression of the effective potential. Although we have not completed our analysis yet, we confirm that our present results will be useful in order to establish a foundation for further developments in various branches. For instance, this might be a step to find out the potential which furnishes all background contents of curvature, electromagnetic fields, and temperature as well as chemical potential.

In sect.2, we present a simple and convenient derivation of the effective potential. In our derivation, there is no manifestation of an exact fermion propagator of finite temperature field theory; namely, we do not utilize its Feynman rule to evaluate a fermion loop. Instead, following the worldline formalism which was recently developed by Strassler \[20\] and later by Schmidt-Schubert \[21\], we generalize the formalism into the finite temperature case. Then we list fundamental (integral) representations of critical equations to determine a critical surface of second order in \( \xi-T-\mu \) space and a critical line of third order defined by a boundary between first and second order phase transitions.

In sect.3, we discuss the case that magnetic field is dominant, i.e., \( \xi \) is a real number. All the integral representations can be integrated in arbitrary \( D \) \((2 < D < 4)\), and we
further evaluate, for finite $\mu$ and $\xi$, exact zero-temperature limits of the effective potential and 1’st order critical equation. We recognize that the potential is a discretized version of the one presented in [19] as a natural embodiment of magnetic discretization. Taking $\xi \to 0$ (continuum limit), we reproduce the known potentials ($\xi = 0$) and 1’st order critical equations [17, 18, 19]. We then find that a constant magnetic field becomes the catalyst in $3 \leq D < 4$ in the situations of finite $T$ and $\mu$ as well, and we verify that the magnitude of catalyst effect becomes weaker as $D$ or $\mu$ increasing. First order phase transitions are observed in large $\mu$ regions on $\xi$-\(\mu\) plane, where no 2’nd order phase transition occurs.

In sect.4, we discuss the electric dominant case, i.e., $\xi$ is a pure imaginary number. First, analyzing the imaginary part of effective potential, we show that chiral symmetry can be restored in high $T$ situation ($\mu \geq 0$) in $2 \leq D < 4$, with vanishing imaginary part. This is a contrast to the $T = 0$ situation of $D = 2$ [3]. Then analytic expressions of real parts of the effective potential and gap equation at $T = \mu = 0$ are shown to be related to those obtained in sect.3 with the rotation $\xi \to i\rho$. In contrast with sect.3, the critical surface turns out to be a closed shape in the $\rho$-$T$-$\mu$ space; namely, small parameter regions are the broken phase. Conclusions and discussions are in sect.5.

2 The effective potential in $\xi$-$T$-$\mu$ space

The model Lagrangian is

$$L = \bar{\psi}i\gamma^\mu D_\mu \psi - \frac{N}{2\lambda}\sigma^2 - \sigma \bar{\psi}\psi,$$

(2.1)

where $\sigma$ is the auxiliary field and $N$ the number of flavors. The fermions $\psi = \{\psi_i\}, (i = 1, \cdots, N)$, are coupled to the background gauge fields via the covariant derivative $D_\mu = \partial_\mu - ieA_\mu$. The effective potential is given by

$$V(\sigma, \xi) = V(\sigma) - V(0),$$

(2.2)

with

$$V(\sigma) = \frac{1}{2\lambda}\sigma^2 - \ln \text{Det}(i\gamma^\mu D_\mu - \sigma).$$

(2.3)

This model does not possess a continuum chiral symmetry but a discrete one. After the effective potential is obtained as a function of the auxiliary field, the potential, as
mentioned previously, becomes just the same form as that of the NJL model, which has a continuum chiral invariance except for \( D = 2 \).

Our interest is to investigate properties of thermodynamic extension of this model and to search chiral phase structures in finite temperature/density situations under external gauge field in arbitrary dimension \( 2 \leq D < 4 \). The two cases; magnetic dominant and electric dominant cases, correspond to whether \( \xi \) is real or pure imaginary. We shall show in this section that \( D \)-dimensional thermodynamic effective potential is given by

\[
V(\sigma; \xi, \beta, \mu) = \frac{1}{2\lambda} \sigma^2 + \frac{\text{tr}[1]}{2\beta} \int_0^\infty ds \, \frac{\Theta_2(s^{2\mu}, is^{4\pi})}{s^{(D-1)/2}} \, s \xi \coth(s\xi) e^{s\mu^2} (e^{-s\sigma^2} - 1),
\]

where \( \Theta_2 \) is the elliptic theta function of second kind and \( \text{tr}[1] \) means the trace of gamma matrix unit (we do not need its explicit value). The \( D = 3 \) (magnetic dominant) case was derived in [10]. Also, by letting \( \beta (= 1/T) \) to be infinity as well as \( \mu \) to be zero, this potential reproduces the potentials derived in the \( D = 3 \) magnetic dominant cases [5, 7] and the electric dominant cases in \( D = 2 \) [3, 22], \( D = 3 \) [6] and \( D = 4 \) [4].

### 2.1 Derivation of the effective potential

In this subsection, we present a somewhat different derivation of the potential from that done in [10], where they evaluated it in the Matsubara (imaginary time) method using a fermion propagator under a constant magnetic field. We shall derive the effective potential (2.4), modifying the worldline formalism [20, 21] into the thermodynamic case.

Let us start with reviewing a few notes on \( V(\sigma) \) for convenience of notation. In the Schwinger proper time method [23], a basic expression for the potential (2.3) already exists in [3]

\[
V(\sigma) = \frac{1}{2\lambda} \sigma^2 + \frac{1}{2} \text{tr} \int_0^\infty ds \, \frac{1}{s (4\pi s)^{D/2}} e^{-s\sigma^2} \exp \left[ -\frac{1}{2} e s F_{\mu\nu} \sigma^{\mu\nu} - \frac{1}{2} \text{Tr} \ln \left( \frac{\sin(esF)}{esF} \right) \right],
\]

where \( \text{tr} \) and \( \text{Tr} \) mean the traces in the Gamma and Lorentz matrix spaces respectively. The trace of \( F_{\mu\nu}^{2n} \) is given by the following in respective dimensions: \( 2E^{2n} \) in \( D = 2 \), \( 2(E^2 - B^2)^n \) in \( D = 3 \) and \( 2[(\rho_+^2)^n + (\rho_-^2)^n] \) in \( D = 4 \), where

\[
\rho_\pm^2 = \frac{1}{2} \left[ \vec{E}^2 - \vec{B}^2 \pm \sqrt{(\vec{B}^2 - \vec{E}^2)^2 + 4(\vec{E} \cdot \vec{B})^2} \right].
\]
If we assume \( \vec{E} \cdot \vec{B} = 0 \) in \( D = 4 \), and if the notation \( \rho = e \sqrt{\vec{E}^2 - \vec{B}^2} \) is properly understood in each case (for example, \( \vec{B} \) is set to be zero in \( D = 2 \)), we can write these trace values in a unified form

\[
\text{Tr}\left[ (eF)^{2n} \right]_{\mu\nu} = 2 \rho^{2n}. \tag{2.6}
\]

Similarly, the eigenvalue of \( \frac{1}{2} e \sigma \cdot F \) can be written in \( \pm i \rho \) accompanied by the degeneracy \( \frac{1}{2} \text{Tr}[1] \) in each \( D \), where we have again used \( \vec{E} \cdot \vec{B} = 0 \) in \( D = 4 \). In this way, we express

\[
\text{Tr} \left[ \ln \frac{\sin(esF)}{esF} \right]_{\mu\nu} = 2 \ln \frac{\sinh(s\xi)}{s\xi}, \tag{2.8}
\]

where \( \xi = i \rho = e \sqrt{\vec{B}^2 - \vec{E}^2} \). When \( \xi \) is real, the case is magnetic dominant, while \( \xi \) is pure imaginary (\( \rho \) is real), the case is electric dominant. Note that \( \xi \) can not be real in \( D = 2 \) because of \( \vec{B} = 0 \). Thereby the potential (2.5) becomes

\[
V(\sigma) = \frac{1}{2\lambda} \sigma^2 + \frac{1}{2} \text{Tr}[1] \int_0^\infty \frac{ds}{(4\pi s)^{D/2}} e^{-s\sigma^2} s\coth(s\xi). \tag{2.9}
\]

An important thing here is that the external field dependence \( s\xi \coth s\xi \), the loop momentum contribution \( (4\pi s)^{-D/2} \) and the mass term \( e^{-s\sigma^2} \) are decoupled in this expression.

The \( F_{\mu\nu} \) parts are an ensemble of two-point functions of world-line fields living on a closed loop [20]. This decomposition can similarly be understood from the following identity in the Euclidean space

\[
\int_0^\infty \frac{ds}{(4\pi s)^{D/2}} \exp \left[ -s\sigma^2 - \frac{(x - y)^2}{4s} \right] = \int \frac{d^Dk}{(2\pi)^D} e^{ik(x-y)} \sigma^2 + k^2, \tag{2.10}
\]

(To make a loop, we must put \( x = y \) and change the integration measure from \( ds \) to \( ds/s \), which is known as a one-loop world-line measure).

This structure is more clearly revealed in the worldline formalism. Let us consider the one-loop effective action of a spinor loop (with a mass \( \sigma \)) coupled to a background gauge (constant field strength) \( A_\mu = \frac{1}{2} x^\nu F_{\nu\mu} \). In the worldline formalism, the effective action is organized in a form of closed path integral of one-dimensional Euclidean action [20, 21]

\[
\Gamma[A] = \ln \text{Det}(i\gamma^\mu D_\mu - \sigma) = \frac{1}{2} \int_0^\infty \frac{dT}{T} e^{-\sigma^2 T} \oint \mathcal{D}x\mathcal{D}\psi
\times \exp \left[ -\int_0^T d\tau \left( \frac{1}{4} \dot{x}^2 + \frac{1}{2} \psi\dot{\psi} + \frac{1}{2} iexF\dot{x} - i\psi F\dot{\psi} \right) \right], \tag{2.11}
\]
where \( x = x^\mu(\tau) \) and \( \psi = \psi^\mu(\tau) \) are the periodic and anti-periodic world-line (real) fields which are defined on a closed loop parametrized by \( \tau \) (0 \( \leq \tau \leq T \)), and \( \dot{x} \) means \( \partial_\tau x(\tau) \) etc. The closed path integral amounts to the super determinant

\[
\text{SDet}^{-1/2} \left( \begin{array}{cc} -\frac{1}{4} \partial_\tau^2 + \frac{1}{2} i e F \partial_\tau & 0 \\ 0 & \frac{1}{2} \partial_\tau - i e F \end{array} \right) = \mathcal{N} \left[ \frac{\text{Det}(1 - 2 i e F \partial_\tau^{-1})_P}{\text{Det}(1 - 2 i e F \partial_\tau^{-1})_A} \right]^{-1/2},
\]

where \( \mathcal{N} \) is defined by the free part integral \[20\],

\[
\mathcal{N} = \oint D x D \psi \exp \left[ -\int_0^T \left( \frac{1}{4} \dot{x}^2 + \frac{1}{2} \psi \dot{\psi} \right) d\tau \right],
\]

and its explicit value is given by the formulae

\[
\oint D x \exp \left[ -\int_0^T \frac{1}{4} \dot{x}^2 d\tau \right] = \int \frac{d^D k}{(2\pi)^D} e^{-T k^2} = (4\pi T)^{-D/2},
\]

\[
\oint D \psi \exp \left[ -\int_0^T \frac{1}{2} \psi \dot{\psi} d\tau \right] = -\text{tr}[1].
\]

Thus the effective action yields

\[
\Gamma[A] = -\frac{1}{2} \text{tr}[1] \int \frac{dT}{T} (4\pi T)^{-D/2} e^{-\sigma^2 T} \left[ \frac{\text{Det}(1 - 2 i e F \partial_\tau^{-1})_A}{\text{Det}(1 - 2 i e F \partial_\tau^{-1})_P} \right]^{1/2}.
\]

The determinants can be evaluated on the Fourier bases \( \{ \exp[2\pi i k \tau/T] | k \in \mathbb{Z}, k \neq 0 \} \) for the periodic and \( \{ \exp[2\pi i (k + \frac{1}{2}) \tau/T] | k \in \mathbb{Z} \} \) for the anti-periodic functions \[21\]. Note that the zero mode in the periodic basis is excluded. We have

\[
\text{Det} \left( 1 - 2 i e F \partial_\tau^{-1} \right)_P = \text{Det}' \prod_{k=-\infty, \neq 0}^\infty \left( 1 - 2 i e F \frac{T}{2\pi i k} \right) = \text{Det}' \frac{\sin(e F T)}{e F T},
\]

and similarly

\[
\text{Det} \left( 1 - 2 i e F \partial_\tau^{-1} \right)_A = \text{Det}' \prod_{k=-\infty}^\infty \left( 1 - 2 i e F \frac{T}{2\pi i (k + 1/2)} \right) = \text{Det}' \cos(e F T),
\]

where \( \text{Det}' \) concerns the determinant on a Lorentz matrix. We then arrive at the expression \[21\]

\[
\Gamma[A] = -\frac{1}{2} \text{tr}[1] \int \frac{dT}{T} (4\pi T)^{-D/2} e^{-\sigma^2 T} \text{Det}' \left[ e F T \cot(e F T) \right]^{1/2},
\]

and this coincides with the second term of \eqref{eq:2.9} with the identification of \eqref{eq:2.6}.

\[4\]For the moment, we denote the proper time \( s \) by \( T \) following to the convention of worldline formalism.
Now let us proceed to the thermodynamic case. We have to replace $k_0$-integral according to the Matsubara formalism

$$ k_0 \to \frac{\pi}{\beta} (2n + 1) - i\mu, \quad \int \frac{dk_0}{2\pi} \to \frac{1}{\beta} \sum_n. $$

(2.20)

Namely, we change the path-integral normalization for $x_0$ through discretizing $k_0$-integral in (2.14)

$$ \oint D x_0 \exp\left[-\int_0^T \frac{1}{4} \dot{x}_0^2\right] \to \frac{1}{\beta} \sum_{n=-\infty}^{\infty} \exp\left[-T\left\{\frac{\pi}{\beta} (2n + 1) - i\mu\right\}^2\right] $$

$$ = \frac{1}{\beta} e^{T\mu^2} \Theta_2(T \frac{2\mu}{\beta}, iT \frac{4\pi}{\beta^2}), $$

(2.21)

and (2.14) is then replaced by

$$ \oint_{\beta,\mu} D x \exp\left[-\int_0^T \frac{1}{4} \dot{x}^2\right] = (4\pi T)^{-\frac{D-1}{2}} \frac{1}{\beta} e^{T\mu^2} \Theta_2(T \frac{2\mu}{\beta}, iT \frac{4\pi}{\beta^2}). $$

(2.22)

Since the zero temperature (continuum) limit $\beta \to \infty$ of RHS of (2.21) is

$$ \lim_{\beta \to \infty} \frac{1}{\beta} e^{T\mu^2} \Theta_2(T \frac{2\mu}{\beta}, iT \frac{4\pi}{\beta^2}) = \frac{1}{\sqrt{4\pi T}}, $$

(2.23)

(2.14) is reproduced from (2.22). Equivalently, it can be verified by decoupling the zero temperature normalization part $(4\pi T)^{-D/2}$,

$$ \oint_{\beta,\mu} D x \exp\left[-\int_0^T \frac{1}{4} \dot{x}^2\right] = \oint D x \exp\left[-\int_0^T \frac{1}{4} \dot{x}^2\right] $$

$$ + 2(4\pi T)^{-\frac{D}{2}} \sum_{n=1}^{\infty} (-1)^n e^{-n^2\beta^2/4T} \cosh(n\beta \mu), $$

(2.24)

where we have applied the transformation

$$ e^{\frac{s\mu^2}{\beta}} \sqrt{\frac{4\pi s}{\beta}} \Theta_2\left(s \frac{2\mu}{\beta}, is \frac{4\pi}{\beta^2}\right) = \Theta_4\left(i \frac{\beta \mu}{2\pi}, i \frac{\beta^2}{4\pi s}\right) $$

$$ = 1 + 2 \sum_{n=1}^{\infty} (-1)^n e^{-n^2\beta^2/4s} \cosh(n\beta \mu). $$

(2.25)

Let us remember that the thermal parameter $\beta^{-1}$ has nothing to do with the loop period $T$ which is the Schwinger proper time of the spinor loop $[23]$. All integrations on internal momenta ($k_i, k_0$) are now done, and thus the operator determinants do not change any more, since $\partial_r$ acts on the proper time coordinate and $F$ is a constant matrix. We therefore conclude

$$ \Gamma[A] = -\frac{1}{2} \text{tr}[1] \int \frac{dT}{T} \Theta_2(T \frac{2\mu}{\beta}, iT \frac{4\pi}{\beta^2}) e^{T(\mu^2-\sigma^2)/T} \cosh(T\xi). $$

(2.26)
Replacing the logarithmic part of (2.3) by this quantity, and with changing $T$ into $s$, we get

$$V_{\beta,\mu}(\sigma; \xi) = \frac{1}{2\lambda} \sigma^2 + \frac{\text{tr}[1]}{2\beta} \int_0^\infty ds \frac{\Theta_2(s^{2\mu}/\beta, is^{4\pi}/\beta^2)}{(4\pi s)^{(D-1)/2}} s \xi \coth(s\xi) e^{-s(\sigma^2 - \mu^2)}. \tag{2.27}$$

When $D = 3$, this thermodynamic potential coincides with the equation (B6) of Ref. [10]. Also, when $\xi = 0$, $V_{\beta,\mu}(\sigma; \xi = 0)$ coincides with the $D$-dimensional thermodynamic potential discussed in [19]. When $T = \mu = 0$, $V_{\beta,\mu}(\sigma; \xi)$ is reduced to (2.3) because of (2.23). Finally, subtracting the zero point value of the potential, we obtain (2.4)

$$V(\sigma; \xi, \beta, \mu) = V_{\beta,\mu}(\sigma; \xi) - V_{\beta,\mu}(0; \xi). \tag{2.28}$$

### 2.2 The critical equations

In the following, we list the critical equations that are entire bases of our numerical analyses of chiral phase structures.

The effective potential contains the ultra-violet divergence in $D = 2$ and is finite in $3 \leq D < 4$. In addition, we should remember that the effective potential has an imaginary part when $\xi$ is pure imaginary. The imaginary part describes the probability of a fermion pair creation from fermion vacuum per unit time and volume [22, 23]. Hence imposing the renormalization condition on the real part of potential (under a situation of vanishing imaginary part)

$$\lim_{T, \mu \to 0} \left| \frac{\partial^2}{\partial \sigma^2} \text{Re} V(\sigma; \xi, T, \mu) \right|_{\sigma = 1} = \frac{1}{\lambda_R}, \tag{2.29}$$

we define the renormalized coupling constant $\lambda_R$ in the following form

$$\frac{1}{\lambda} - \frac{1}{\lambda_R} = \text{tr}[1] \text{Re} \int_0^\infty \frac{ds}{(4\pi s)^{D/2}} e^{-s} (1 - 2s) s \xi \coth(s\xi). \tag{2.30}$$

The real part of renormalized effective potential is then expressed in terms of $\lambda_R$ as

$$\text{Re} V_R(\sigma; \xi, T, \mu) = \frac{1}{2\lambda_R} \sigma^2 + \frac{1}{2} \text{tr}[1] \text{Re} \int_0^\infty \frac{ds}{(4\pi s)^{D/2}} s \xi \coth(s\xi) \tag{2.31}$$

$$\times \left[ \frac{1}{s} (e^{-s\sigma^2} - 1) \sqrt{\frac{4\pi s}{\beta}} e^{s\mu^2/\beta} \Theta_2 \left( \frac{2\mu}{\beta}, is^{4\pi}/\beta^2 \right) + \sigma^2 e^{-s}(1 - 2s) \right].$$
The gap equation to determine a dynamical mass, i.e., the lowest energy configuration of \( \sigma \), is then

\[
0 = \frac{1}{\lambda_R} + \text{tr}[1] \text{Re} \int_0^\infty \frac{ds}{(4\pi s)^{D/2}} s \xi \coth(s\xi) \times \left[ -e^{-s(\sigma^2 - \mu^2)} \sqrt{\frac{4\pi s}{\beta}} \Theta_2 \left( s \frac{2\mu}{\beta}, is \frac{4\pi}{\beta^2} \right) + e^{-s}(1 - 2s) \right], \quad (2.32)
\]

where we have eliminated the trivial solution \( \sigma = 0 \).

Now let us derive the equations to determine phase diagrams in the \( \xi-T-\mu \) parameter space. The critical surface of first order is governed by the simultaneous conditions

\[
\text{Re} V_R(\sigma = m, \xi, \beta, \mu) = 0, \quad (2.33)
\]

\[
\frac{\partial}{\partial \sigma} \text{Re} V_R(\sigma, \xi, \beta, \mu) \bigg|_{\sigma = m} = 0, \quad (2.34)
\]

where \( m \) denotes a non-trivial solution of (2.32). Combining these equations, we obtain

\[
0 = \text{Re} \int_0^\infty \frac{ds}{(4\pi s)^{D/2}} s \xi \coth(s\xi) \times \left[ m^2 e^{-sm^2} + \frac{1}{s} (e^{-sm^2} - 1) \right]. \quad (2.35)
\]

One can decouple RHS of this equation into \( V(m; \xi) \) and a pure thermal part using (2.25), however we stop writing further details, which will be discussed in sect.3.1. The necessary condition to determine the critical surface of second order is generally expressed by

\[
\lim_{\sigma \to 0} \frac{\partial}{\partial \sigma^2} \text{Re} V_R(\sigma, \xi, \beta, \mu) = 0, \quad (2.36)
\]

and this reads

\[
0 = \text{Re} \int_0^\infty \frac{ds}{(4\pi s)^{D/2}} \left[ s \xi \coth(s\xi) \left\{ -e^{s\mu^2} \sqrt{\frac{4\pi s}{\beta}} \Theta_2 \left( s \frac{2\mu}{\beta}, is \frac{4\pi}{\beta^2} \right) \right. \\
+ e^{-s}(1 - 2s) \left\} + 2se^{-s} \right]. \quad (2.37)
\]

Here, we have adopted the following value of renormalized coupling constant

\[
\frac{1}{\lambda_R} = \text{tr}[1] \int_0^\infty \frac{ds}{(4\pi s)^{D/2}} 2se^{-s}, \quad (2.38)
\]

which makes the chiral symmetry broken at \( T = \mu = \xi = 0 \) with generating the dynamical mass \( \sigma = 1 \). Note that a solution of (2.37) does not always represent a true second order
transition point if a first order transition occurs. We have to cut an irrelevant piece away from the true critical surface solving the equation of third order critical line. The third order critical line, which yields the boundary between the first and second order critical surfaces, is determined by the simultaneous conditions (2.36) and
\[
\lim_{\sigma \to 0} \left( \frac{\partial}{\partial \sigma^2} \right)^2 \text{Re} V_R(\sigma, \zeta, \beta, \mu) = 0.
\] (2.39)

The latter condition leads to
\[
0 = \text{Re} \int_0^\infty \frac{ds}{(4\pi s)^{D/2}} s \zeta \coth(s\zeta) e^{s\mu^2 \sqrt{4\pi s} / \beta} \Theta_2 \left( \frac{2\mu}{\beta}, is \frac{4\pi}{\beta^2} \right).
\] (2.40)

One can perform any combination of limits \(\xi, T\) and/or \(\mu \to 0\) in the 2'nd and 3'rd order critical equations (2.37) and (2.40). For instance, taking the limit \(\xi \to 0\) in each, we obtain the following respectively \([16, 19]\) (see Appendix A)
\[
\beta^{D-2} \Gamma \left( 1 - \frac{D}{2} \right) = \frac{2}{\sqrt{\pi}} (2\pi)^{D-2} \Gamma \left( \frac{3-D}{2} \right) \text{Re} \zeta(3-D, \frac{1}{2} + i\beta \mu / 2\pi),
\] (2.41)
and
\[
\text{Re} \zeta(5-D, \frac{1}{2} + i\beta \mu / 2\pi) = 0,
\] (2.42)
where \(\zeta\) is the generalized zeta function. It is shown in \([19]\) that these equations reproduce the critical equations derived in \([17]\). The limit \(\beta \to \infty\) can also be taken by applying (2.23) to (2.37) and (2.40), and the limits are valid unless any 1'st order phase transition takes place at \(T = 0\). In contrast with these, the effective potential on the \(\xi-\mu\) plane (and hence the 1'st order critical equation there) can not be obtained by naively applying (2.23). In order to reproduce a precise limit, we have to collect all contributions from under the Fermi surface at \(T = 0\). This can be achieved by an insertion of the step function \(\theta(\mu - k)\) in a loop momentum integration \([13]\) at a very beginning of computation. This prescription is, however, unclear in our present formulation, since we integrated out the loop momenta. We observe a method how to deal with this problem in the next section.

### 3 The magnetic dominant case

In this section, we discuss chiral phase structures in the magnetic dominant case for finite \(T\) and \(\mu\). In this case (of course except for \(D = 2\)), \(\xi^{-1}\) is relevant to the magnetic
length $eB$, which represents a radius of cyclotron motion of an electron in classical mechanical picture. Since $\xi$ is real, the effective potential does not possess an imaginary part, i.e.,

$$\text{Re} V_R(\sigma; \xi, \beta, \mu) = V_R(\sigma; \xi, \beta, \mu), \quad (3.1)$$

and there is no more intricacy of critical equation analysis. In [10], the $D = 3$ case ($\mu = 0, T \neq 0$) with a constant magnetic field was analyzed, and the magnetic field turns out to play a role of strong catalyst of chiral symmetry breaking. We shall observe this feature in the $\xi$-$T$-$\mu$ space for various values of $D$. Each critical surface is determined by numerically analyzing the 1’st, 2’nd and 3’rd order critical equations which have already been established in the previous section. Here, we formally include $D = 2$ case, because it will be useful when $\xi$ is rotated to $i\rho$ and is convenient to see a $D$-dependence of magnetic catalyst as a function of $D$ in the region $2 < D < 4$.

However, before proceeding to the phase structures, we would like to clarify a discrepancy concerning magnetic lattice structure that can not be seen in the theta function representations of effective potential (2.31) etc. We stress here that we can perform the $s$-integrations, and after that, we can see a magnetic discretization in the effective potential for arbitrary $T$, $\mu$ and $D$. We prove this point in sect. 3.1. Also, we present a method how to take the exact $T \to 0$ limit with keeping $\mu$ finite in the effective potential as announced before. Later on, we show phase diagrams in sect. 3.2.

### 3.1 The effective potential on $\xi$-$\mu$ plane

As mentioned at the end of sect.2, the special care is necessary in order to analyze the effective potential (not gap equation) on $\xi$-$\mu$ plane. It is useful to see how to deduce, from (2.31), precise expressions of effective potential and 1’st order critical equation at $T = 0$ in preliminary to the $E$-dominant (imaginary $\xi$) case. Expected in [8], we shall show that first order phase transitions exist on the $\xi$-$\mu$ plane at large values of $\mu$. It is convenient to first observe a manifestation of the role of $\xi$, which is a discrete momentum unit as a reflection of magnetic translation invariance, through a comparison between continuum ($\xi = 0$) and discrete effective potentials. The correspondence can also be seen in the first order critical equation on $\xi$-$\mu$ plane.
The effective potential (2.28) with the renormalization (2.30) can be written as the sum of the zero-temperature potential (2.2) and pure thermodynamic remainder by means of the formula (2.25):

\[ V(\sigma; \xi, \beta, \mu) = V(\sigma; \xi) + \tilde{V}(\sigma; \xi, \beta, \mu), \quad (3.2) \]

or

\[ V_{\beta, \mu}(\sigma; \xi) = V(\sigma) + \tilde{V}_{\beta, \mu}(\sigma; \xi), \quad (3.3) \]

where

\[ V(\sigma) = \frac{1}{2\lambda_R} \sigma^2 + \frac{1}{2} \int_0^\infty \frac{ds}{(4\pi s)^{D/2}} s^\xi \coth(s^\xi) \left[ \frac{1}{s} e^{-\sigma^2} + \sigma^2 e^{-s(1 - 2s)} \right], \quad (3.4) \]

\[ \tilde{V}_{\beta, \mu}(\sigma; \xi) = \text{tr}[1] \int_0^\infty \frac{ds}{(4\pi s)^{D/2}} s^\xi \coth(s^\xi) \cdot e^{-\sigma^2} \sum_{n=1}^\infty (-1)^n e^{-n^2 \beta^2/4s} \cosh(n\beta \mu). \quad (3.5) \]

Using the following formulae

\[ \coth(s^\xi) = 1 + 2 \sum_{l=1}^\infty e^{-2ls^\xi}, \quad (3.6) \]

\[ \int_0^\infty ds s^{\nu - 1} \exp\left[ -\frac{\beta}{s} - \gamma s \right] = 2 \left( \frac{\beta}{\gamma} \right)^{\frac{\nu}{2}} K_\nu(2\sqrt{\gamma}), \quad (3.7) \]

we can perform the \( s \)-integrations in (3.4) and (3.3) just similarly to the \( D = 3 \) case [10];

\[ V(\sigma) = \frac{\sigma^2}{2\lambda_R} + \frac{\text{tr}[1]_\xi}{2(4\pi)^{D/2}} \left[ 2(2\xi)^{D-1} \zeta(1 - \frac{D}{2}, 1 + \frac{\sigma^2}{2\xi}) \Gamma(1 - \frac{D}{2}) + \sigma^{D-2} \Gamma(1 - \frac{D}{2}) \right. \]

\[ + 2\sigma^2(2\xi)^{D-2} \zeta(2 - \frac{D}{2}, 1 + \frac{1}{2\xi}) \Gamma(2 - \frac{D}{2}) + \sigma^2 \Gamma(2 - \frac{D}{2}) \]

\[ - 4\sigma^2(2\xi)^{-3} \zeta(3 - \frac{D}{2}, 1 + \frac{1}{2\xi}) \Gamma(3 - \frac{D}{2}) - 2\sigma^2 \Gamma(3 - \frac{D}{2}) \] \quad (3.8) \]

and

\[ \tilde{V}_{\beta, \mu}(\sigma; \xi) = \frac{\text{tr}[1]_\xi}{(4\pi)^{D/2}} \left[ \frac{1}{2} \mathcal{O}_{\beta}(\sigma) + \sum_{l=1}^\infty \mathcal{O}_{\beta}(\sqrt{\sigma^2 + 2l^2}) \right], \quad (3.9) \]

where we have introduced the compact notation

\[ \mathcal{O}_{\beta}(\sigma) = 2 \int_0^\infty ds s^{D/2} \sum_{n=1}^{\infty} (-1)^n \cosh(n\beta \mu) \exp\left[ -(s\sigma^2 + \frac{n^2 \beta^2}{4s}) \right] \]

\[ = 4 \sum_{n=1}^{\infty} (-1)^n \cosh(n\beta \mu) \left( \frac{\beta n}{2\sigma} \right)^{1-D/2} K_{D/2-1}(n\beta \sigma). \quad (3.10) \]
However in order to observe the low temperature limit of $\mathcal{O}_\beta(\sigma)$, it is rather convenient to consider an integral representation through the formula

$$K_\nu(z) = \frac{\sqrt{\pi} \left( \frac{2}{z} \right)^\nu}{\Gamma(\nu + \frac{1}{2})} \int_1^\infty e^{-zt} (t^2 - 1)^{\nu - \frac{1}{2}} dt, \quad \text{Re} \nu > -\frac{1}{2}, \text{ Re } z > 0,$$

and we rewrite

$$\mathcal{O}_\beta(\sigma) = \frac{4\sqrt{\pi}}{\Gamma(D/2)} \sigma^{D-2} \sum_{n=1}^{\infty} (-)^n \cosh(n\beta\mu) \int_1^\infty e^{-n\beta\sigma(t^2 - 1)} \frac{dt}{(t^2 - 1)^{D/2}}.$$  \hspace{1cm} (3.11)

Performing the summation over $n$ and changing the integration variable $t = 1 + t'/\sigma$, we obtain

$$\mathcal{O}_\beta(\sigma) = \frac{-2\sqrt{\pi}}{\Gamma(D-1)} \left[ \int_0^\infty (t^2 + 2\sigma t) \frac{dt}{1 + e^{\beta(t+\sigma+\mu)}} + (\mu \to \mu) \right].$$  \hspace{1cm} (3.12)

$\mathcal{O}_\beta(0)$ is given by (a derivation is in Appendix A)

$$\mathcal{O}_\beta(0) = \frac{2}{\sqrt{\pi}} \left( \frac{2\pi}{\beta} \right)^{D-2} \Gamma \left( \frac{3 - D}{2} \right) \text{Re} \zeta \left( 3 - D, \frac{1}{2} + \frac{i\beta\mu}{2\pi} \right).$$  \hspace{1cm} (3.13)

Here, we stress that (3.9) is a discrete version of the $\xi = 0$ potential obtained in [19]

$$\tilde{V}(\sigma; 0, \beta, \mu) = -\frac{\text{tr}[1]2\sqrt{\pi}}{(4\pi)^{D/2} \Gamma(D/2)\beta} \int_0^\infty dk k^{D-2} \ln \left( \frac{1 + e^{-\beta(\mu+\sqrt{\sigma^2+k^2})}}{(1 + e^{-\beta(\mu+k)})(1 + e^{-\beta(k-\mu)})} \right).$$  \hspace{1cm} (3.14)

This can be easily understood in the case of $D = 3$, since $\mathcal{O}_\beta(\sigma)$ takes the following simple form

$$\mathcal{O}_\beta(\sigma) = -2\sqrt{\pi} \beta \ln \left( (1 + e^{-\beta(\sigma+\mu)})(1 + e^{-\beta(\sigma-\mu)}) \right),$$  \hspace{1cm} (3.15)

and thereby

$$\tilde{V}(\sigma; \xi, \beta, \mu) = \tilde{V}_{\beta,\mu}(\sigma; \xi) - \tilde{V}_{\beta,\mu}(0; \xi)$$

$$= -\frac{\text{tr}[1]2\sqrt{\pi}}{(4\pi)^{3/2} \beta} \ln \left( \frac{1 + e^{\beta(\mu-\sigma)}}{1 + e^{\beta(\mu-\sigma)}(1 + e^{-\beta(\mu-\sigma)})} \right)$$

$$+ \sum_{l=1}^{\infty} \ln \left( \frac{1 + e^{\beta(\mu-\sqrt{\sigma^2+2l^2}\xi)}}{(1 + e^{\beta(\mu-\sqrt{\sigma^2+2l^2}\xi)})(1 + e^{-\beta(\mu+\sqrt{\sigma^2+2l^2}\xi)})} \right).$$  \hspace{1cm} (3.16)

This is exactly a discrete summation form of (3.13), and the correspondence to (3.15) is understood by the following translation rule from discrete spectrum to continuum one

$$2l\xi \to k^2,$$

$$\xi \to kdk,$$

$$\xi \sum_{l=1}^{\infty} \to \int_0^\infty kdk.$$  \hspace{1cm} (3.17)

$$\xi \to kdk,$$
Remember, in the continuum expression, we have to (i) insert a step function \( \theta(\mu - k) \) into the above momentum integration and have to (ii) keep \( \mu < \sigma \) in order to observe a correct potential behavior in \( \beta \to \infty \) with finite \( \mu \); namely \[19\]

\[
\lim_{\beta \to \infty} \tilde{V}(\sigma; 0, \beta, \mu) = \frac{\text{tr}[1][2\sqrt{\pi}]}{(4\pi)^{D/2}\Gamma\left(\frac{D-1}{2}\right)} \int_0^\mu dk k^{D-2}(\mu - k). \tag{3.19}
\]

Due to the correspondence \((3.18)\), we may similarly assume an upper bound on the discrete momentum sum; \( l < L \equiv \lfloor \mu^2/2\xi \rfloor \), where \( \lfloor \cdot \rfloor \) is the Gauss symbol,

\[
\lim_{\beta \to \infty} \tilde{V}(\sigma; \xi, \beta, \mu) = \frac{\text{tr}[1][2\sqrt{\pi}]}{(4\pi)^{3/2}} \left[ \frac{1}{2}\mu + \sum_{l=1}^L (\mu - \sqrt{2l\xi}) \right]. \tag{3.20}
\]

This potential is still clearly a discrete form of \((3.19)\) under the identification of \((3.18)\).

Let us return to the generic \( D \) case. The above cutting manipulation is also valid for the generic \( D \) case, because we have

\[
\mathcal{O}_\infty(\sigma) \equiv \lim_{\beta \to \infty} \mathcal{O}_\beta(\sigma) = -\frac{2\sqrt{\pi}}{\Gamma\left(\frac{D-1}{2}\right)} \theta(\mu - \sigma) \int_0^{\mu-\sigma} (t^2 + 2\sigma t)^{D-3} dt
\]

\[= -\frac{2\sqrt{\pi}}{\Gamma\left(\frac{D+1}{2}\right)} (2\sigma)^{D-3} (\mu - \sigma)^{D-3} F\left(\frac{3-D}{2}, \frac{D-1}{2}; \frac{D+1}{2}; \frac{\sigma - \mu}{\sigma}\right) \theta(\mu - \sigma),
\]

where we have applied the integration formula \((B.3)\). Owing to the presence of the step function \( \theta(\mu - \sigma) \), the upper bound \( L \) is naturally introduced here, and we figure out

\[
\lim_{\beta \to \infty} \tilde{V}(\sigma; \xi, \beta, \mu) = -\frac{\text{tr}[1][2\sqrt{\pi}]}{(4\pi)^{D/2}} \left[ \frac{1}{2}\mu + \sum_{l=1}^L \mathcal{O}_\infty(\sqrt{2l\xi}) \right]. \tag{3.22}
\]

This is the discrete form corresponding to \((3.19)\). The explicit form of \( \mathcal{O}_\infty(\sigma) \) in each \( D \) is

\[
\mathcal{O}_\infty(\sigma) = \frac{-2\sqrt{\pi}}{\Gamma\left(\frac{D-1}{2}\right)} \theta(\mu - \sigma) \times \begin{cases} 
\arccosh (\mu/\sigma) & \text{for } D = 2 \\
\mu - \sigma & \text{for } D = 3 \\
\frac{1}{2}\sqrt{u^2 - \sigma^2} - \frac{1}{2}\sigma^2 \arccosh (\mu/\sigma) & \text{for } D = 4,
\end{cases}
\]

and

\[
\mathcal{O}_\infty(0) = -\frac{2\sqrt{\pi}}{\Gamma\left(\frac{D-1}{2}\right)} \frac{\mu^{D-2}}{D-2}. \tag{3.24}
\]

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The $O_{\infty}(0)$ is singular when $D = 2$, however it vanishes under the 'physical' limit $\xi \to 0$. In Appendix B, we show how the discrete form (3.22) reproduces the continuum one (3.19) in the limit $\xi \to 0$ via eq. (3.18).

Now let us compute the 1'st order critical equation defined by (2.33) and (2.34). Since the pure thermal part (3.22) does not depend on $\sigma$ any more, the gap equation is the same as the one for the zero temperature potential $V(\sigma; \xi)$,

\[
0 = \frac{1}{\lambda_R} + \text{tr}[1] \int_0^{\infty} \frac{ds}{(4\pi s)^{D/2}} s \xi \coth(s\xi) \left[ e^{-s(1-2s)} - e^{-sm^2} \right]. \tag{3.25}
\]

(Of course, this can be derived from (2.32) with the use of (2.23).) Eliminating the renormalized coupling from (3.4) with using (3.25), and applying the following formula\(^5\)

\[
\int_0^{\infty} ds s^{\mu-1} e^{-\beta s} \coth(s) = \Gamma(\mu) \left[ 2^{1-\mu} \zeta(\mu, 1 + \frac{\beta}{2}) + \beta^{-\mu} \right], \quad \text{Re} \mu > 1 \tag{3.26}
\]

$V(\sigma; \xi)|_{\sigma=m}$ turns out to be

\[
V(m; \xi) = \frac{\text{tr}[1] 2^{D/2} \xi^{D-1}}{2(4\pi)^{D/2}} \Gamma(1 - \frac{D}{2}) \left[ \frac{1}{2} m^2 (1 - \frac{D}{2}) \zeta \left( 2 - \frac{D}{2}, 1 + \frac{m^2}{2\xi} \right) + \xi \left( 1 - \frac{D}{2}, 1 + \frac{m^2}{2\xi} \right) - \xi \zeta(1 - \frac{D}{2}) + \frac{m^2}{4} (2 - \frac{D}{2}) (\frac{2\xi}{m^2})^{2-D/2} \right]. \tag{3.27}
\]

Combining this and (3.22), we obtain the 1'st order critical equation on the $\xi-\mu$ plane

\[
0 = \frac{-2}{\Gamma(1 - \frac{D}{2})} (2\xi)^{2-D} \left[ \frac{1}{2} O_{\infty}(0) + \sum_{l=1}^{L} g_{\infty}(\sqrt{2l\xi}) \right] + m^2 (2 - D) \zeta(2 - D, 1 + \frac{m^2}{2\xi}) + \frac{4 - D}{2} (2\xi)^{2-D/2} m^{D-2} + 4\xi \zeta(1 - \frac{D}{2}), \tag{3.28}
\]

where $m$ should be determined from (3.23); for example, we can take $m = 1$ as a solution if we choose, instead of (2.38),

\[
\frac{1}{\lambda_R} = \text{tr}[1] \int_0^{\infty} \frac{ds}{(4\pi s)^{D/2}} s \xi \coth(s\xi) \cdot 2se^{-s}. \tag{3.29}
\]

This coupling choice makes our critical equations much simpler, however it is not essential to the numerical analyses of phase structures discussed later. Under the choice (2.38), the

---

\(^5\) This formula is derived from the expansion (3.6) and the definition of zeta function.
mass \[ \tilde{m} \] is determined by

\[
0 = 2 - \xi(3 - D) - \xi m^{D-4} - (2\xi)^{\frac{D}{2} - 1}\zeta\left(2 - \frac{D}{2}, 1 + \frac{m^2}{2\xi}\right) \\
+ (2\xi)^{\frac{D}{2} - 1}\zeta\left(2 - \frac{D}{2}, 1 + \frac{1}{2\xi}\right) - (2\xi)^{\frac{D}{2} - 2}(4 - D)\zeta\left(3 - \frac{D}{2}, 1 + \frac{1}{2\xi}\right).
\]

(3.30)

The dynamical mass determined by (3.30) grows as external magnetic field becoming strong (see Fig. 1), whose feature coincides with the one argued in [8].

Note that we cannot take the limit \( m \to 0 \) without circumventing divergence for \( D < 4 \) in this equation. This means there is no 2’nd order phase transition on the \( \xi-\mu \) plane. However, 1’st order transitions exist (see Fig. 2) as determined from (3.28) and (3.30). The equation (3.28) can be shown to coincide with the following equation [19] in the limit \( \xi \to 0 \) (The proof is in Appendix C),

\[
\left(\frac{\mu}{m}\right)^D = \frac{D-1}{2}(1 - \frac{D}{2})B\left(1 - \frac{D}{2}, \frac{D-1}{2}\right),
\]

(3.31)

which has the solutions \( \mu = m/\sqrt{2} \) for \( D = 2 \) and \( \mu = m \) for \( D = 3 \) [17, 18].

\[\begin{align*}
\text{Fig. 1:} & \quad \text{The dynamical mass } m \text{ for magnetic dominant cases on } T = 0 \text{ plane.} \\
\text{Fig. 2:} & \quad \text{The values of critical chemical potential } \mu_c \text{ on the } T = 0 \text{ plane as a function of the external magnetic field } \xi.
\end{align*}\]

\[6\text{For example, a solution is } m = 1.02107 \text{ for } \xi = 0.3 \text{ in } D = 3.\]
3.2 The phase structures

In Fig.3, we illustrate several critical curves which represent the sections of second order critical surface in $\xi$-$T$-$\mu$ space for $D = 3$. In this case, the point $(\xi, T, \mu) = (0, 0, 1)$ is the unique tricritical point, since (2.42) has a solution $(\beta \mu)^{-1} = 0$, which means $T = 0$ (for $\mu \neq 0$). Fig.4 is the case of $D = 4 - \epsilon$, where we take $\epsilon = 0.5$. No tricritical point appears in this case. We could not approach $\epsilon$ to be smaller due to a singular situation of numerical computations, however we can easily read off signs of magnetic catalyst and of disappearance of a tricritical point, if we inspect behaviors of lower $D$-values as well.

First, we observe the $D = 2.5$ diagrams in Fig.5, where a tricritical point starts appearing around $0.75 < \mu < 0.76$. The tricritical points are determined from intersections between two curves solved by (2.37) and (2.40) (for example, see Fig.6). Next, let us watch the $D = 2$ diagrams (Fig.7). The tricritical line (points) and first order transitions are observed for $0.61 < \mu$ similarly to Fig.5. It is interesting that the critical surface does not intersect with the $\xi$ axis. It means that a remnant of magnetic catalyst still exists even in these unphysical dimensions $2 \leq D < 3$. Now we gather the tricritical points of each dimension in Fig.8. It clearly shows that the tricritical line moves down to a lower temperature region as $D$ increasing. Extrapolating these data, we thereby deduce that there is no tricritical point in $D = 4 - \epsilon$ for small $\epsilon$ with observing the magnetic catalyst in finite $T$ and $\mu$.

Comparing phase diagrams, for example within the diagrams of $D = 3$ or of $D = 3.5$, we can read that the critical curve on $\xi$-$\mu$ plane for each $T$ has a common structure except for two things seen in the $T = 0$ diagram: One thing is that the curve is composed of 1st order phase transitions; The other is that $\xi = 0$ is not a continuous limit but an asymptotic limit, in other words, singular and discontinuous. The common structure is that a large $\mu$ tends to revive chiral symmetry, and the critical $\mu$ grows as $\xi$ increasing.
Fig. 3. The critical lines for magnetic dominant case in $D = 3$. The point $(\xi, T, \mu) = (0, 0, 1)$ is the unique tricritical point.

Fig. 4: The critical lines for magnetic dominant case in $D = 4 - \epsilon$, where $\epsilon$ is chosen to be 0.5.

Fig. 5: The critical lines for magnetic dominant case in $D = 2.5$. The solid lines are the second order critical lines and the dashed lines are the first order critical lines.

Fig. 6: How to determine the tricritical point $A$ for magnetic dominant case (e.g. $\mu = 0.7, D = 2$). The $C2$ and $C3$ are the solutions of eqs. (2.37) and (2.40). The $C1$ represents the solution of (2.35).
Fig. 7: The critical lines in $D = 2$ for magnetic-like external field. The solid (dashed) lines are the second (first) order critical lines. The $T = 0$ plane is singular.

Fig. 8: The tricritical points for magnetic dominant case projected on the $T - \xi$ plane.

4 The electric dominant case

4.1 The imaginary part of effective potential

In contrast with the $B$-dominant case, the effective potential (2.4) gives rise to an imaginary part due to the instability of fermion vacuum in the $E$-dominant case [3, 22, 23]. In fact, the function $s\xi \coth(s\xi)$ in the integrand of (2.4) is now $s\rho \cot(s\rho)$ with changing $\xi = i\rho$, and the integrand becomes to contain an infinite number of the singular points $s = n\pi/\rho, (n = 1, 2, \cdots)$ accordingly. Then we must displace the contour of integration ($-\infty < s < \infty$) by $i\epsilon$ along the imaginary axis of the complex $s$-plane. The contribution of each singular point is taken into account by making use of the $i\epsilon$ rule (for example, [22])

$$\frac{1}{s + i\epsilon - \frac{n\pi}{\rho}} = \frac{P}{s - \frac{n\pi}{\rho}} - i\pi \delta\left(s - \frac{n\pi}{\rho}\right), \quad (4.1)$$

where $P$ stands for the principal-value symbol. By virtue of this rule, the imaginary part
of the effective potential consists of

$$\text{Im} V_{\beta,\mu}(\sigma; \rho) = -\text{tr}[1] \frac{\rho^{(D-1)/2}}{2^{D-2} \pi^{D-2} \beta} \sum_{n=1}^{\infty} n^D e^{-n\pi(\sigma^2-\mu^2)/\rho} \Theta_2 \left( \frac{2n\pi \mu}{\rho \beta}, i \frac{4n\pi^2}{\rho \beta^2} \right)$$

$$= -\text{tr}[1] \frac{\rho^{(D-1)/2}}{2^{D-1} \pi^{D-2} \beta} \text{Re} \sum_{m=0}^{\infty} \exp \left[ -\frac{\pi}{\rho} \left( \sigma^2 + \left( \frac{2m+1}{\beta} + i\mu \right)^2 \right) \right]$$

$$\times \Phi \left( \frac{D-1}{2}, \frac{i}{2\rho} \left( \sigma^2 + \left( \frac{2m+1}{\beta} + i\mu \right)^2 \right), 1 \right), \quad \text{(4.2)}$$

where $\Phi(z; s, a)$ is the Lerch transcendent function defined by (D.7). This can be regarded as a probability, per unit space-time volume, for a fermion pair-production from the vacuum by an external $E$ field. Taking the $T = \mu = 0$ limit

$$\lim_{T, \mu \to 0} \text{Im} V_{\beta,\mu}(\sigma; \rho) = -\text{tr}[1] \frac{\rho^{D/2}}{2^{D+1} \pi^D} e^{-\pi\sigma^2/\rho} \Phi \left( \frac{D}{2}, i \pi \frac{\sigma^2}{2\rho}, 1 \right), \quad \text{(4.3)}$$

we reproduce the known results [3, 7]

$$\text{Im} V(\sigma) = \begin{cases} \text{tr}[1] \frac{\rho}{(8\pi)} \cdot \ln(1 - e^{-\sigma^2/\rho}) & \text{for } D = 2 \\ -\text{tr}[1] \frac{\rho^{3/2}}{4(\pi)^2} e^{-\pi\sigma^2/\rho} \Phi \left( \frac{3}{2}, i \pi \frac{\sigma^2}{2\rho}, 1 \right) & \text{for } D = 3. \end{cases} \quad \text{(4.4)}$$

One can rewrite (4.3) in the following form using the Lerch transformation (D.3),

$$\text{Im} V(\sigma) = -\text{tr}[1] \frac{\rho^{D/2}}{4(\pi)^{D/2}} \Gamma \left( 1 - \frac{D}{2} \right) \left[ \exp \left[ \frac{\pi}{2} \left( (1 - \frac{D}{2}) \right) \right] \zeta \left( 1 - \frac{D}{2}, i \pi \frac{\sigma^2}{2\rho} \right) \right]$$

$$+ \exp \left[ -\frac{\pi}{2} \left( (1 - \frac{D}{2}) \right) \right] \zeta \left( 1 - \frac{D}{2}, 1 - i \pi \frac{\sigma^2}{2\rho} \right) \right), \quad \text{(4.5)}$$

We shall search for the lowest energy configuration of $\sigma$ solving the critical surface equations derived in sect.2. This ground state solution $\sigma(E)$ minimizes the real part of the effective potential, and the chiral symmetry is broken accordingly. On the contrary, we expect that a strong external field might induce a restoration of chiral symmetry. However, we can not conclude immediately whether or not the chiral symmetry is restored even if the fermion mass vanishes, because the effective potential possesses the imaginary part that represents a decay of "symmetry vacuum" to a fermion pair. Indeed, in the case of $T = \mu = 0$, the chiral mass term $\sigma(E) \langle \bar{\psi} \psi \rangle$ does not vanish even when $\sigma(E) = 0$ in $D = 2$ [3].

Hence we adopt the similar criterion of chiral symmetry restoration as in [3, 4]; i.e.,

$$\lim_{\rho \to \rho_c} \sigma(\rho) \text{Im} V'(\sigma(\rho); \rho, \beta, \mu) = 0, \quad \text{(4.6)}$$
where we have applied the fact that $\langle \bar{\psi} \psi \rangle$ is related to the $\text{Im} V'$ by

\[
\langle \bar{\psi} \psi \rangle_{\sigma=\sigma(\rho)} = iN \text{Im} V'(\sigma(\rho); \rho, \beta, \mu) - \frac{N}{\lambda} \sigma(\rho),
\]

(4.7)

and $\sigma(\rho)$ is a solution of gap equation, i.e., $\text{Re} V' = 0$. To check this criterion, it is sufficient to see whether or not the $\text{Im} V'(\sigma; \rho, \beta, \mu)$ is finite as $\sigma \to 0$ (for example, one can see that $\text{Im} V'(\sigma)$ diverges for $T = \mu = 0$ when $D = 2$). From (4.2), we have

\[
\text{Im} V'(\sigma; \rho, \beta, \mu) = \sigma \text{tr}[1] \rho (D - 3) / 2 \rho \pi \sum_{n=1}^{\infty} \frac{n^{3-D}}{\pi^{D-3}} e^{-\frac{\pi}{\rho}(\sigma^2 - \mu^2)} \Theta_2 \left( \frac{2n\pi\mu}{\rho\beta}, i \frac{4n\pi^2}{\rho\beta^2} \right).
\]

(4.8)

The upper bound of the summation in (4.8) can be estimated in the following way

\[
\sum_{n=1}^{\infty} \frac{n^{3-D}}{\pi^{D-3}} e^{-\frac{\pi}{\rho}(\sigma^2 - \mu^2)} \Theta_2 \left( \frac{2n\pi\mu}{\rho\beta}, i \frac{4n\pi^2}{\rho\beta^2} \right) \\
= \sum_{n=1}^{\infty} \frac{n^{3-D}}{\pi^{D-3}} \sum_{m=-\infty}^{\infty} \exp \left[ -\frac{\pi}{\rho} \left\{ \frac{\sigma^2}{\pi} + \left( \frac{2m+1}{\beta} - i\mu \right)^2 \right\} \right]
\leq \int_{0}^{\infty} d\xi x^{3-D} \sum_{m=-\infty}^{\infty} \exp \left[ -\frac{\pi}{\rho} \left\{ \frac{\sigma^2}{\pi} + \left( \frac{2m+1}{\beta} - i\mu \right)^2 \right\} \right].
\]

(4.9)

Performing the $x$-integration

\[
\text{RHS (4.9)} = \Gamma \left( \frac{5-D}{2} \right) \left( \frac{\rho}{\pi} \right)^{\frac{5-D}{2}} \sum_{m=-\infty}^{\infty} \left[ \sigma^2 + \left( \frac{2m+1}{\beta} - i\mu \right)^2 \right]^{\frac{D-5}{2}},
\]

(4.10)

we find it finite as $\sigma \to 0$

\[
\lim_{\sigma \to 0} \text{RHS (4.9)} = 2\Gamma \left( \frac{5-D}{2} \right) \left( \frac{2\pi^2}{\rho\beta} \right)^{\frac{D-4}{2}} \text{Re} \zeta \left( \frac{5-D}{2}, \frac{1}{2} + i\frac{\beta\mu}{2\pi} \right). \]

(4.11)

The criterion is therefore satisfied

\[
\lim_{\sigma \to 0} \sigma \text{Im} V'(\sigma; \rho, \beta, \mu) = 0,
\]

(4.12)

and we conclude that the chiral symmetry is restored for finite $T$ and $\mu$ in $2 \leq D < 4$.

Note that RHS of (4.11) vanishes right at a tricritical point, whose necessary condition is (2.42). This is not the case with 2’nd order transitions and suggests that the multi-criticality can be mightier than the vacuum instability. As a by-product, we obtain a necessary condition for the tri-criticality on the imaginary part

\[
\lim_{\sigma \to 0} \frac{1}{\sigma} \text{Im} V'(\sigma; \rho, \beta, \mu) = 0,
\]

(4.13)
or it leads to
\[
\frac{1}{\beta} \sum_{n=1}^{\infty} n^{\frac{d-D}{2}} e^{\frac{n\pi^2}{\rho^2}} \Theta_{2}\left(\frac{2n\pi\mu}{\rho\beta}, i\frac{4n\pi^2}{\rho^2}\right) = 0.
\] (4.14)

It is interesting to note that the powers of \(\sigma\) in front of \(\text{Im} V'\) in (4.12) and (4.13) differ by two from each other. Hence (4.12) and (4.13) may be called a kind of discrete versions of (2.36) and (2.39).

For small \(T\), in particular \(\beta \to \infty\) (with \(\mu = 0\)), using
\[
\lim_{\beta \to \infty} \frac{1}{\beta} \sum_{m=-\infty}^{\infty} \exp\left[-\frac{\pi}{\rho} \left(\sigma^2 + \left(\frac{2m+1}{\beta}\right)^2\right)\right] = e^{-\frac{\pi^2}{4\pi x}},
\] we obtain
\[
\lim_{\beta \to \infty} \frac{1}{\beta} \text{RHS (4.9)} = \frac{1}{2} \frac{\pi^{d-3} \rho^{\frac{d-3}{2}} \Gamma(2 - \frac{D}{2})}{\sigma^{D-4}},
\] (4.16)
and its power dependence on \(\sigma\) exactly coincides with known results [3, 6]
\[
\lim_{T, \mu \to 0} \text{Im} V'(\sigma; \rho, \beta, \mu) \sim \begin{cases} \sigma^{-1} & \text{for } D = 2, \\ \text{const.} & \text{for } D = 3, \\ \sigma^{1-\epsilon} & \text{for } D = 4. \end{cases}
\] (4.17)
Namely, chiral symmetry is restored in \(D = 3\) [6] and \(4 - \epsilon\), and broken in \(D = 2\) [3].

4.2 The duality and phase structures

In this subsection, we proceed in the evaluation of real parts of the critical equations (2.37) and (2.40), and we point out a dual relation between magnetic dominant effective potential and electric one under the rotation \(\xi \to i\rho\). Later on, we present phase diagrams. In order to evaluate the real parts (principal values), we follow the same method employed in [3, 6].

Let us begin with analyzing the real part of effective potential (2.31), i.e.,
\[
\text{Re} V_R(\sigma; \rho, \beta, \mu) = \frac{1}{2\lambda_R} \sigma^2 + \frac{\text{tr}[1]}{2(4\pi)^{D/2}} \left(I_1(\sigma) - I_1(0)\right),
\] (4.18)
where
\[
I_1(\sigma) = P \int_0^\infty \frac{ds}{s^{D/2}} s \rho \cot s \rho \cdot f(s, \beta, \mu),
\] (4.19)
and
\[
f(s, \beta, \mu) = \frac{1}{s} e^{-s^{\sigma^2}} e^{\mu^2} \frac{\sqrt{4\pi s}}{\beta} \Theta_2\left(\frac{s^{2\mu}}{\beta^2}, i\frac{4\pi s}{\beta^2}\right) + \sigma^2 e^{-s}(1 - 2s).
\] (4.20)
We further split $I_1(\sigma) = A(\sigma) + B(\sigma)$ into the following two pieces for convenience:

$$A(\sigma) = \rho^{D/2-1} \int_0^{\pi/2} \frac{d\tau}{(4\pi)^{D/2}} \tau^{1-D/2} \cot \tau \cdot f(\frac{\tau}{\rho}, \beta, \mu)$$  (4.21)

$$B(\sigma) = \rho^{D/2-1} \sum_{n=1}^{\infty} P \int_{n\pi-\pi/2}^{n\pi+\pi/2} \frac{d\tau}{(4\pi)^{D/2}} \tau^{1-D/2} \cot \tau \cdot f(\frac{\tau+n\pi}{\rho}, \beta, \mu),$$

$$= \rho^{D/2-1} \sum_{n=1}^{\infty} \int_0^{\pi/2} \frac{d\tau}{(4\pi)^{D/2}} \cot \tau \cdot \left[ f(\frac{\tau+n\pi}{\rho}, \beta, \mu) - (\tau \to -\tau) \right].$$  (4.22)

The integration in (4.21) can be performed

$$A(\sigma) = -2 \sum_{k=0}^{\infty} \zeta(2k) \left( \frac{2\sqrt{4\pi}}{\beta} \right)^{2k} \rho^{2} \left[ \frac{\sigma^2 + (\pi \frac{2m+1}{\beta} + i\mu)^2}{2} \right]^{2k+\frac{D-1}{2}}$$

$$\times \gamma \left( 1 - \frac{D}{2} + 2k, \frac{\pi}{2\rho} \left( \sigma^2 + \frac{2m+1}{\beta} + i\mu^2 \right) \right) + \sigma^2 \gamma \left( 1 - \frac{D}{2}, \frac{\pi}{2\rho} - 2\sigma^2 \gamma \left( 2 - \frac{D}{2} + 2k, \frac{\pi}{2\rho} \right) \right).$$  (4.23)

Here we have applied the formula

$$\int_0^{\pi/2} dx x^z e^{-ax} \cot x = -2 \sum_{k=0}^{\infty} \zeta(2k) \frac{\pi}{2k} \alpha^{-2k-z} \gamma \left( z + 2k, \frac{\pi}{2} \alpha \right),$$  (4.24)

where $\gamma(z, p)$ is the incomplete gamma function defined by

$$\gamma(z, p) = \int_0^p e^{-t} t^{z-1} dt \quad \text{Re} \ z > 0.$$  (4.25)

The summation in (4.22) can be performed

$$B(\sigma) = \int_0^{\pi/2} d\tau \cot \tau \cdot \left[ \frac{\sqrt{4\pi}}{\beta} 2\Re \sum_{m=0}^{\infty} \exp \left[ -\frac{\tau + \pi}{\rho} \left( \sigma^2 + \frac{2m+1}{\beta} + i\mu^2 \right) \right] \right]$$

$$\times \left( \frac{\rho}{\pi} \right)^{\frac{D-1}{2}} \Phi \left( \frac{D - 1}{2}; \frac{i}{2\rho} \left( \sigma^2 + \frac{2m+1}{\beta} + i\mu^2 \right), 1 + \frac{\tau}{\pi} \right)$$

$$+ \sigma^2 \left( \frac{\rho}{\pi} \right)^{\frac{D-1}{2}} e^{-\left(\tau+\pi\right)/\rho} \Phi \left( \frac{D}{2}; \frac{i}{2\rho}, 1 + \frac{\tau}{\pi} \right)$$

$$- 2\sigma^2 \left( \frac{\rho}{\pi} \right)^{\frac{D-2}{2}} e^{-\left(\tau+\pi\right)/\rho} \Phi \left( \frac{D}{2}; 2; \frac{i}{2\rho}, 1 + \frac{\tau}{\pi} - (\tau \to -\tau) \right).$$  (4.26)

Similarly, we rewrite the second and third order critical equations (2.37) and (2.40) in a straightforward way,

$$I_2(\sigma) + 2\Gamma(2 - \frac{D}{2}) = 0.$$  (4.27)
\[ I_3(\sigma) = 0, \quad (4.28) \]

where the explicit forms of \( I_2 \) and \( I_3 \) are listed in Appendix D.

Now, we rather concentrate on the \( T = \mu = 0 \) case for the purpose to observe a dual relation under \( \xi \rightarrow i\rho \). In this case, (4.20) simplifies because of (2.23) (If one wants to consider the \( \mu \neq 0 \) situation, it is necessary to follow the similar procedure analyzed in sect.3). Let us write down \( I_1 \) and \( I_2 \):

\[
I_1(\sigma) = \rho^{D/2-1} \int_0^{\pi/2} d\tau \cot \tau \cdot \left[ \rho \tau^{-D/2} e^{-\tau\sigma^2/\rho} + \sigma^2 \tau^{1-D/2} e^{-\tau/\rho}(1 - \frac{2\tau}{\rho}) \right]
+ \pi^{-D/2} \left\{ \rho e^{-(\tau+\pi)\sigma^2/\rho} \Phi\left(\frac{D}{2}; \frac{\sigma^2}{2\rho}, 1 + \frac{\tau}{\pi}\right) \right. \\
+ \sigma^2 \pi e^{-(\tau+\pi)/\rho} \Phi\left(\frac{D}{2} - 1; \frac{i}{2\rho}, 1 + \frac{\tau}{\pi}\right) \\
- \left. \sigma^2 \frac{2\pi}{\rho} e^{-(\tau+\pi)/\rho} \Phi\left(\frac{D}{2} - 2; \frac{i}{2\rho}, 1 + \frac{\tau}{\pi}\right) \right\}, \\
\]

\[ I_1(0) = \rho^{D/2-1} \int_0^{\pi/2} d\tau \cot \tau \cdot \left[ \tau^{-D/2} + \pi^{-D/2} \left\{ \zeta\left(\frac{D}{2}, 1 + \frac{\tau}{\pi}\right) - \zeta\left(\frac{D}{2} - 1 - \frac{\tau}{\pi}\right) \right\} \right], \quad (4.30) \]

and differentiating \( I_1 \),

\[
I_2(\sigma) = \frac{\partial}{\partial \sigma^2} I_1(\sigma) \bigg|_{\sigma=0} \\
= \rho^{D/2-1} \int_0^{\pi/2} d\tau \cot \tau \cdot \left[ \tau^{-D/2} \left\{ -1 + e^{-\tau/\rho}(1 - \frac{2\tau}{\rho}) \right\} \right] \\
+ \pi^{1-D/2} \left\{ -\zeta\left(\frac{D}{2} - 1, 1 + \frac{\tau}{\pi}\right) + e^{-(\tau+\pi)/\rho} \Phi\left(\frac{D}{2} - 1; \frac{i}{2\rho}, 1 + \frac{\tau}{\pi}\right) \\
- \frac{2\pi}{\rho} e^{-(\tau+\pi)/\rho} \Phi\left(\frac{D}{2} - 2; \frac{i}{2\rho}, 1 + \frac{\tau}{\pi}\right) \right\}. \quad (4.31) \]

We obtain the following values as solutions of (4.27) with (4.31)

\[ \rho_c = \begin{cases} 
0.583 & \text{for} \quad D = 2 \\
1.238 & \text{for} \quad D = 3 \\
3.394 & \text{for} \quad D = 3.5 
\end{cases} \quad (4.32) \]

and we verified that the real part of potential, given by (4.18) with (4.29) and (4.30), behaves consistently. Here we attach a few remarks on these results. From \( D = 2 \) to \( D = 3.3, \) whose \( \rho_c = 1.965 \), all \( \rho_c \)'s excellently coincide with solutions of the following
analytic equation, which still exhibits a good agreement \[ \text{at } D = 3.5 \ (\rho_c = 3.395); \]

\[
0 = 2 - (2\rho)^{\frac{D}{2}-1}\zeta(2 - \frac{D}{2})\cos\left[\frac{\pi}{4}(D - 2)\right] 
+ \text{Re}\left[\left((2i\rho)^{\frac{D}{2}-1}\zeta(2 - \frac{D}{2}, 1 - i\frac{\sigma}{2\rho}) + (2i\rho)^{\frac{D}{2}-2}(D - 4)\zeta(3 - \frac{D}{2}, 1 - \frac{i}{2\rho})\right)\right].
\]

We have found this equation from (3.30) with rotating \( \xi \rightarrow i\rho \), taking real part and the limit \( m \rightarrow 0 \). The divergence stemming from \( m \rightarrow 0 \) now drops off since it thereby becomes a pure imaginary number. Similarly, we verify that those critical values are consistent with the potential described by the same rotation procedure to (3.8); i.e.,

\[
V(\sigma; \rho) = \frac{\sigma^2}{2\lambda_R} + \text{Re}\left[\frac{\text{tr}[1]}{2(4\pi)^{D/2}}\left[2\sigma^2(2i\rho)^{\frac{D}{2}-2}\zeta(2 - \frac{D}{2}, 1 - i\frac{\sigma}{2\rho})\Gamma(2 - \frac{D}{2})
- 4\sigma^2(2i\rho)^{\frac{D}{2}-3}\zeta(3 - \frac{D}{2}, 1 - i\frac{\sigma}{2\rho})\Gamma(3 - \frac{D}{2})
+ 2(2i\rho)^{\frac{D}{2}-1}\Gamma(1 - \frac{D}{2})\left\{\zeta(1 - \frac{D}{2}, 1 - \frac{i\sigma^2}{2\rho}) - \zeta(1 - \frac{D}{2})\right\}\right]\right].
\]

This rotation procedure is not always allowed in general cases (for example in the thermodynamic part \( \tilde{V} \)), because we have used the expansion (3.6) (or the formula (3.26) derived from (3.6)), which is not valid for a complex analytic function. In this sense, the above coincidences are non-trivial results (although they are rather trivial before the expansion is applied).

In Fig.9, we depict phase diagrams for several \( \mu \) values in the case of \( D = 2 \). When \( T = \mu = 0 \), it is shown in [3] that the vacuum state is not invariant under a chiral transformation due to the instability of vacuum. In this sense, the critical value \( \rho_c \) divides a phase between massive states (broken phase) and quasi-stationary states (quasi-symmetric phase), and the total probability of decay per volume is given by [3]

\[
\Gamma = - \lim_{\rho \to \rho_c} 2\sigma \text{Im} V'(\sigma; \rho, \infty, 0) = \frac{\rho_c}{\pi}.
\]

The similar situation is expected for finite \( \mu \) (with \( T = 0 \)) as well, and the \( \Gamma \) changes into \( \Gamma + \Delta\Gamma \), where \( \Delta\Gamma \) is formally given by

\[
\Delta\Gamma = - \lim_{\rho \to \rho_c} 2\sigma \text{Im} \tilde{V}'(\sigma; \rho, \infty, \mu),
\]

\footnote{The deviation of \( \rho_c \) grows as \( D \) increasing in the region \( D > 3.5 \). We believe this is related to a singular problem in numerical computation as encountered in sect. 3.2.}
although we have not estimated its explicit value yet. On the other hand, as discussed before, the chiral symmetry is restored (i.e., $\Gamma = 0$) when both of $T$ and $\mu$ are finite.

In Figs. 10 and 11, the cases of $D = 3$ and $4 - \epsilon$ ($\epsilon = 0.5$) are illustrated. As shown in (1.17), these cases have a stable (chiral symmetric) vacuum when $T = \mu = 0$. Also, in the situation of finite $T$ and $\mu$, the chiral symmetry is restored similarly to the $D = 2$ case. In each dimension, the critical surface goes down to a low temperature region as $\mu$ increasing. We could not find any tricritical point within a region of significant numerical data. In particular the computation did not work well below around $T \cong 0.1$, and we hence extrapolate the critical lines in Figs.9-11.

**Fig. 9:** The critical lines for electric dominant case in $D = 2$.

**Fig. 10:** The critical lines for electric dominant case in $D = 3$. 


Fig. 11: The critical lines for electric dominant case in $D = 4 - \epsilon$, where $\epsilon$ is chosen to be 0.5.

5 Conclusion and discussion

In this paper, we examined various interesting properties of the thermodynamic effective potential of large-$N$ Gross-Neveu model coupled to constant electromagnetic fields which are perpendicular to each other: worldline approach, magnetic discretization, electromagnetic duality, critical equations and chiral phase structures.

First, we presented in sect.2 a new derivation of the effective potential from a point of the worldline approach to effective action developed in recent years [20, 21]. Although the similar formulation had been applied to a constant curvature cases in [16], where we could not clarify whether or not a loop momentum contribution is included, to some extent, in the external field function such as $\xi \coth \xi$. We emphasize that it is interesting that our method proposed in sect.2 leads to the exact result [10]. Another spectacular of our potential can be seen in sect.3. Our potential reveals the role of $\xi$ as a discrete momentum unit even in the situations under finite temperature with keeping $\mu$ finite, and in particular, the discrete potential of $D = 3$ much resembles to the continuum potential [19]. In other words, our $D$-dimensional thermodynamic effective potential is composed of the sum over discrete energy spectra $\epsilon_n = \pm \sqrt{m^2 + 2n\xi}$, $n \in \mathbb{Z}$. We also argued how
to take the $\beta \to \infty$ limit of potential with $\xi$ and $\mu$ kept finite, and we described a precise continuum limit $\xi \to 0$ giving rise to known results of $\mu_c$. It means $\mu$ is properly taken into account in our derivation.

Secondly, we found in sect.4 the tri-criticality condition for the imaginary part of effective potential, and we also found the analytic expressions which are dual to the magnetic dominant effective potential and its gap equation. When the effective potential possesses the imaginary part, the vanishing conditions on the first and second derivatives of $\text{Re}V$ are not sufficient for defining second order critical and tricritical points. It is necessary for second order critical points to satisfy the condition (4.12), which is a vanishing condition on the chiral mass term, i.e., $\sigma \text{Im}V''$. Similarly, (4.13), $\sigma^{-1}\text{Im}V'' = 0$, should be satisfied at a tricritical point as well. These vanishing conditions on $\text{Im}V''$ may be regarded as a discrete version of the real part conditions (2.36) and (2.39).

Thirdly, we elucidated the phase structures of the effective potentials in various cases. In sect.3, we showed the phase diagrams of magnetic dominant cases, and observed a strong catalyst effect of magnetic field in each case. When $\beta = \infty$, there is no chiral symmetry restoration of second order even if $\mu \neq 0$. However (for example in $D = 3$ and 3.5), 1st order transitions certainly exist at large $\mu$ values. Similarly, For finite $\beta$ (with $\mu \neq 0$), the chiral symmetry becomes broken under a strong magnetic field. These features do not change if we formally extend equations, as functions of $D$, into $2 < D < 3$ region, from which we actually got some insights toward a region $D$ being near to 4. As pointed out in [23], the effective potential of $D = 2$ by itself resembles to an electrostatic potential problem of the charge distribution $s \text{coth} s$ along a one-dimensional line. It means that the electric field of such distribution rather acts as a "magnetic" catalyst, if we could realize the distribution in one dimensional laboratory.

In sect.4, we analyzed the electric dominant cases with taking care of imaginary part of effective potential. When $T = \mu = 0$ in $2 < D < 4$, we showed that the chiral symmetry is restored under the criterion whether the chiral mass term $\sigma \langle \bar{\psi}\psi \rangle$ vanishes or not, i.e., (4.0). Equivalently, it means that the imaginary part of effective potential vanishes as $\sigma$ goes to zero. This is obvious by integrating (4.17) w.r.t. $\sigma$. Namely we see that the power behavior $\text{Im}V(\sigma) \sim \sigma^{D-2}$ goes to zero as a solution of gap equation (defined by
real part of effective potential) approaching zero. While $D = 2$, the situation is different as discussed in [3]. The $\text{Im} V'$ diverges as $\sigma \to 0$, and the imaginary part $\text{Im} V$, which never vanishes, corresponds to the decay of fermion vacuum to a fermion pair creation.

However, in finite $\beta$ and $\mu$, we showed that the chiral symmetry is restored in $2 \leq D < 4$ based on the same criterion. This suggests that a symmetrization effect by high temperature can be stronger than the vacuum instability of pair creation by an external field. As to $\rho$-$\mu$ plane, we have not yet succeeded in formulating $T = 0$ limit with $\mu$ being kept finite (In this sense, the achievement of the magnetic case was remarkable). When the fermion density is large, a density effect might enhance the chiral symmetry, and consequently, the instability would be suppressed similarly to the magnetic case. In order to address this question, it is necessary to study the 1st order critical equation as done in (3.28). An interesting problem is whether or not a dual relation like the one between (3.30) and (4.33) holds in this case, because it includes and depends on a non-trivial part which consists of the thermodynamic quantity $\tilde{V}$.

A A proof of (3.14)

We present a proof of (3.14), after viewing how $(2.37)$ and $(2.40)$ reproduce $(2.41)$ and $(2.42)$.

First, in the limit $\xi \to 0$, $(2.37)$ becomes $(2.41)$ in the following way

\[
0 = \int_0^\infty ds \left[ e^{-s} - e^{s \mu^2 \frac{\sqrt{4 \pi} s}{\beta}} \Theta_2 \left( \frac{2 \mu}{\beta}, \frac{i s}{\beta^2} \right) \right] 
\]

\[
= \Gamma(1 - D/2) - 2 \frac{\sqrt{4 \pi}}{\beta} \int_0^\infty ds s^{1-D/2} \text{Re} \sum_{n=1}^\infty \exp \left[ -s \frac{4 \pi^2}{\beta^2} (n - \frac{1}{2} + i \frac{\beta \mu}{2 \pi})^2 \right] 
\]

\[
= \Gamma(1 - D/2) - 2 \frac{\sqrt{4 \pi}}{\beta} \left( \frac{2 \pi}{\beta} \right)^{D-3} \Gamma \left( \frac{3-D}{2} \right) \text{Re} \sum_{n=0}^\infty (n + \frac{1}{2} + i \frac{\beta \mu}{2 \pi})^{D-3} 
\]

\[
= \Gamma(1 - D/2) - 2 \frac{\sqrt{4 \pi}}{\beta} \left( \frac{2 \pi}{\beta} \right)^{D-2} \Gamma \left( \frac{3-D}{2} \right) \text{Re} \zeta \left( 3 - D, \frac{1}{2} + i \frac{\beta \mu}{2 \pi} \right), 
\]

and similarly $(2.40)$ reads

\[
0 = \int_0^\infty ds \left[ e^{-s} - e^{s \mu^2 \frac{\sqrt{4 \pi} s}{\beta}} \Theta_2 \left( \frac{2 \mu}{\beta}, \frac{i s}{\beta^2} \right) \right] 
\]
\[
= \frac{2}{\sqrt{\pi}} \left( \frac{2\pi}{\beta} \right)^{D-4} \Gamma \left( \frac{5-D}{2} \right) \Re \zeta \left( 5-D, \frac{1}{2} + i\frac{\beta \mu}{2\pi} \right).
\]

Let us turn to the proof of (3.14) making use of the above calculation. Recalling the definition of \( \mathcal{O}_\beta(\sigma) \) by (3.10), and using (2.25), we have
\[
\mathcal{O}_\beta(\sigma) = \int_0^\infty ds s^{-D/2} e^{-s^2} \left[ \Theta_4 \left( i\frac{\beta \mu}{2\pi}, i\frac{\beta^2}{4\pi s} \right) - 1 \right].
\]

The top line of (A.1) can also be written in the form
\[
0 = \int_0^\infty ds s^{-D/2} \left[ e^{-s} - \Theta_4 \left( i\frac{\beta \mu}{2\pi}, i\frac{\beta^2}{4\pi s} \right) \right].
\]

If \( 2 < D < 4 \), the following integral is convergent
\[
\int_0^\infty ds s^{-D/2} (e^{-s} - 1) = \frac{1}{1 - \frac{D}{2}} \int_0^\infty ds s^{1-D/2} e^{-s} = \Gamma \left( 1 - \frac{D}{2} \right),
\]
and therefore (A.3) amounts to
\[
0 = \Gamma \left( 1 - \frac{D}{2} \right) - \mathcal{O}_\beta(0).
\]

Comparing the RHS of this equation with the bottom line of (A.1), we have proven (3.14).

**B The \( \xi \to 0 \) limit of (3.22)**

We prove that (3.22) is reduced to (3.19) in the limit \( \xi \to 0 \). Obviously, the 1'\textquoteleft t term \( \mathcal{O}_\infty(0) \) in (3.22) drops in \( \xi \to 0 \), since \( \mathcal{O}_\infty(0) \) possesses no dependence on \( \xi \). Hence, we have only to prove the following equality
\[
\lim_{\xi \to 0} \frac{1}{i} \sum_{i=1}^L \xi \int_0^{\mu - \sqrt{2i\xi}} (t^2 + 2t \sqrt{2i\xi})^{\frac{D-3}{2}} dt = \frac{\mu^D}{D(D-1)}.
\]

Under the correspondence (3.18), LHS of the above equation can be transformed into
\[
\text{LHS (B.1)} = \int_0^\mu k dk \int_0^{\mu-k} (t^2 + 2kt)^{\frac{D-3}{2}} dt = \int_0^\mu dt \int_0^{\mu-t} k(t^2 + 2kt)^{\frac{D-3}{2}} dk = \int_0^\mu dt t^{D-1} \int_0^{(\mu-t)/t} k(1 + 2k)^{\frac{D-3}{2}} dk,
\]
and applying the formula 3.194.1 [20]
\[
\int_0^u x^{\mu-1}(1 + \beta x)^{-\nu} dx = \frac{1}{\mu} u^\mu F(\nu, \mu; 1 + \mu; -\beta u), \quad |\arg(1 + \beta u)| < \pi, \quad \Re \mu > 0,
\]
(B.3)
we get
\[ \text{LHS (B.1)} = \frac{1}{2} \int_0^\mu t^{D-3}(\mu - t)^2 F\left(\frac{3-D}{2}, 2; 3; -2\frac{\mu - t}{t}\right). \] (B.4)

Now changing the integration variable, \( t = 2\mu/(w + 2) \),
\[ \text{LHS (B.1)} = \mu^D 2^{D-3} \int_0^\infty dw w^2(w + 2)^{-1-D} F\left(\frac{3-D}{2}, 2; 3; -w\right) \] (B.5)
and applying the formula 7.512.10 \[26\]
\[ \int_0^\infty x^{\gamma-1}(x + z)^{-\sigma} F(\alpha, \beta; \gamma; -x) dx = \frac{\Gamma(\gamma)\Gamma(\alpha - \gamma + \sigma)\Gamma(\beta - \gamma + \sigma)}{\Gamma(\sigma)\Gamma(\alpha + \beta - \gamma + \sigma)}, \] (B.6)
(where \( \text{Re} \gamma > 0, \text{Re} \alpha - \gamma + \sigma > 0, \text{Re} \beta - \gamma + \sigma > 0, |\text{arg} z| < \pi \)), we find
\[ \text{LHS (B.1)} = \mu^D 2^{D-3} \mu(3)\Gamma\left(\frac{D-1}{2}\right)\Gamma\left(\frac{D}{2} + 1\right) F\left(\frac{D-1}{2}, D; \frac{D + 3}{2}, -1\right). \] (B.7)

Using a Kummer transformation to make the argument of hypergeometric function from \(-1\) to \(1/2\) and the following formula
\[ F(\alpha, 1-\alpha; \gamma; \frac{1}{2}) = \frac{2^{1-\gamma}\Gamma(\gamma)\sqrt{\pi}}{\Gamma\left(\gamma + \frac{1}{2}\right)\Gamma\left(1-\gamma + \frac{1}{2}\right)}, \] (B.8)
we have
\[ F\left(\frac{D-1}{2}, D; \frac{3 + D}{2}; -1\right) = \frac{2^{1-D}\Gamma\left(\frac{3+D}{2}\right)}{\Gamma\left(\frac{D+1}{2}\right)}. \] (B.9)

Finally substituting this into (B.7), the proof ends
\[ \text{LHS (B.1)} = \mu^D \frac{1}{4} \frac{\Gamma(3)\Gamma(D)\Gamma\left(\frac{D-1}{2}\right)}{\Gamma(D + 1)\Gamma\left(\frac{D+1}{2}\right)} = \frac{\mu^D}{D(D - 1)}. \] (B.10)

C  The \( \xi \to 0 \) limit of (3.28)

We prove (3.28) goes to (3.31) in the limit \( \xi \to 0 \). Eq.(3.28) is nothing but
\[ 0 = V(m; \xi) + \tilde{V}(m; \xi, \infty, \mu), \] (C.1)
where \( V(m, \xi) \) is given by (3.27), and \( \tilde{V}(m; \xi, \infty, \mu) \) is given by RHS of (3.22), because it does not depend on \( \sigma \) value. According to the result of Appendix B, we hence have,
\[ \lim_{\xi \to 0} \tilde{V}(m; \xi, \infty, \mu) = \frac{\text{tr}|1|2\sqrt{\pi}}{(4\pi)^{D/2}\Gamma\left(\frac{D-1}{2}\right)} \frac{\mu^D}{D(D - 1)}. \] (C.2)
Let us consider the limit of \( V(m; \xi) \). In the situation of \( \xi \to 0 \), three terms of zeta function on RHS of (3.27) only survive

\[
V(m; \xi) \approx \operatorname{tr}[1] \xi \left[ 2(2\xi)^{D/2} - m^2 \Gamma(2 - \frac{D}{2}) \sum_{n=0}^{\infty} \frac{1}{(n + 1 + m^2/2\xi)^{-D/2}} \right]
+ 2(2\xi)^{D/2 - 1} \Gamma(1 - \frac{D}{2}) \sum_{n=0}^{\infty} \left\{ \frac{1}{(n + 1 + m^2/2\xi)^{-1 + D/2}} - \frac{1}{(n + 1)^{-1 - D/2}} \right\},
\]

and we adopt the following continuum limit as \( \xi \to 0 \),

\[
(n + 1)2\xi \to x, \quad 2\xi \sum_{n=0}^{\infty} \to \int_{0}^{\infty} dx.
\]

The limit of (C.3) is therefore led to the following

\[
\lim_{\xi \to 0} V(m; \xi) = \frac{\operatorname{tr}[1]}{2(4\pi)^{D/2}} \int_{0}^{\infty} dx \left[ -m^2 \Gamma(2 - \frac{D}{2}) (x + m^2)^{D/2} - \Gamma(1 - \frac{D}{2}) (x + m^2)^{D/2} - \frac{2}{D} x^{D/2} \right]_0
\]

\[
= \frac{\operatorname{tr}[1]}{2(4\pi)^{D/2}} \Gamma(1 - \frac{D}{2}) (1 - \frac{2}{D}) m^D.
\]

Combining (C.2) and (C.5), the limit of (C.1) turns out to be

\[
\frac{1}{2} \Gamma(1 - \frac{D}{2}) (1 - \frac{2}{D}) m^D + \frac{2\sqrt{\pi}}{\Gamma(\frac{D}{2})} \mu^D (D - 1) = 0.
\]

This is equivalent to (3.31), and the proof ends.

\section*{D \ The explicit forms of \( I_2(\sigma) \) and \( I_3(\sigma) \)}

We write down explicit expressions for \( I_2(\sigma) \) and \( I_3(\sigma) \), which are defined by

\[
I_2(\sigma) = \left. \frac{\partial}{\partial \sigma^2} A(\sigma) \right|_{\sigma = 0},
I_3(\sigma) = \left. \frac{\partial^2}{\partial (\sigma^2)^2} I_1(\sigma) \right|_{\sigma = 0}.
\]

\( I_2(\sigma) \) is composed of a sum of the following two terms:

\[
\left. \frac{\partial}{\partial \sigma^2} A(\sigma) \right|_{\sigma = 0} = \sum_{k=0}^{\infty} \frac{\zeta(2k)}{2^{2k} \pi} \left[ 2 \frac{\sqrt{4\pi}}{\beta} \operatorname{Re} \sum_{m=0}^{\infty} \left( \frac{2m + 1}{\beta} + i\mu \right)^{D - 2 - 4k} \right.
\]

\[
\times \frac{\gamma(3 - \frac{D}{2} + 2k, \frac{i}{2\rho}) - \gamma(1 - \frac{D}{2} + 2k, \frac{i}{2\rho})}{2\rho \left( \frac{2m + 1}{\beta} + i\mu \right)^2}.
\]

This is equivalent to (3.31), and the proof ends.
\[ \frac{\partial}{\partial \sigma^2} B(\sigma) \bigg|_{\sigma=0} = \int_0^{\pi/2} d\tau \cot \tau \left[ -2\sqrt{\frac{4\pi}{\beta}} \left( \frac{\rho}{\pi} \right)^{\frac{D-5}{2}} \Re \sum_{m=0}^\infty \exp\left[ -\frac{\tau + \pi}{\rho} \left( \frac{2m+1}{\beta} + i\mu \right) \right] \right] \times \Phi \left( \frac{D-3}{2}; \frac{i}{2\rho} \left( \frac{2m+1}{\beta} + i\mu \right), 1 + \frac{\tau}{\pi} \right) \\
+ \left( \frac{\rho}{\pi} \right)^{\frac{D-1}{2}} e^{-\left( \tau+\pi \right)/\rho} \Phi \left( \frac{D}{2} - 1; \frac{i}{2\rho}, 1 + \frac{\tau}{\pi} \right) \\
- 2 \left( \frac{\rho}{\pi} \right)^{\frac{D-2}{2}} e^{-\left( \tau+\pi \right)/\rho} \Phi \left( \frac{D}{2} - 2; \frac{i}{2\rho}, 1 + \frac{\tau}{\pi} \right) - (\tau \to -\tau) \right]. \tag{D.4} \]

And similarly, \( I_3(\sigma) \) is composed of:

\[ \frac{\partial^2}{\partial (\sigma^2)^2} A(\sigma) \bigg|_{\sigma=0} = -4\sqrt{\frac{4\pi}{\beta}} \sum_{k=0}^\infty \zeta(2k) \left( \frac{\rho}{\pi} \right)^{2k} \Re \sum_{m=0}^\infty \exp\left[ -\frac{\tau + \pi}{\rho} \left( \frac{2m+1}{\beta} + i\mu \right) \right] \]
\times \gamma \left( \frac{5-D}{2} + 2k; \frac{\pi}{2\rho} \left( \frac{2m+1}{\beta} + i\mu \right)^2 \right), \tag{D.5} \]

and

\[ \frac{\partial^2}{\partial (\sigma^2)^2} B(\sigma) \bigg|_{\sigma=0} = \frac{2\sqrt{4\pi}}{\beta} \left( \frac{\rho}{\pi} \right)^{\frac{D-5}{2}} \int_0^{\pi/2} d\tau \cot \tau \cdot \Re \sum_{m=0}^\infty \exp\left[ -\frac{\tau + \pi}{\rho} \left( \frac{2m+1}{\beta} + i\mu \right) \right] \]
\times \Phi \left( \frac{D-5}{2}; \frac{i}{2\rho} \left( \frac{2m+1}{\beta} + i\mu \right), 1 + \frac{\tau}{\pi} \right) - (\tau \to -\tau) \right]. \tag{D.6} \]

Our notation for the Lerch transcendental \( \Phi(z; s, a) \) is

\[ \Phi(z; s, a) = \sum_{n=0}^\infty \frac{e^{2\pi i ns}}{(n+a)^z}, \quad \Re z > 1, \tag{D.7} \]

and a Lerch transformation is then expressed by

\[ (2\pi)^z \Phi(1-z; s, a) = \Gamma(z) \left[ \exp \left[ \pi i \left( \frac{z}{2} - 2as \right) \right] \Phi(z; -a, s) \\
+ \exp \left[ \pi i \left( -\frac{z}{2} + 2a(1-s) \right) \right] \Phi(z; a, 1-s) \right]. \tag{D.8} \]

Taking \( a = 1 \), we have

\[ e^{2\pi is} \Phi(1-z; s, 1) = (2\pi)^{-z} \Gamma(z) \left[ e^{\frac{\pi i z}{4}} \zeta(z, s) + e^{-\frac{\pi i z}{4}} \zeta(z, 1-s) \right]. \tag{D.9} \]
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