Fuzzy-Stochastic Partial Differential Equations

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Abstract

We introduce and study a new class of partial differential equations (PDEs) with hybrid fuzzy-stochastic parameters, coined fuzzy-stochastic PDEs. Compared to purely stochastic PDEs or purely fuzzy PDEs, which may treat either only random or only non-random uncertainty in physical systems, fuzzy-stochastic PDEs offer powerful models for accurate description and propagation of the hybrid random and non-random uncertainties inevitable in many real applications. We will use the level-set representation of fuzzy functions and define the solution to fuzzy-stochastic PDE problems through a corresponding parametric problem, and further present theoretical results on the well-posedness and regularity of such problems. We also propose a numerical strategy for computing output fuzzy-stochastic quantities, such as fuzzy failure probabilities and fuzzy probability distributions. We present two numerical examples to compute various fuzzy-stochastic quantities and to demonstrate the applicability of fuzzy-stochastic PDEs to complex engineering problems.

1 Introduction

Most viable uncertainty quantification (UQ) methodologies are set in a probabilistic framework; see e.g. [50, 17, 24, 55], where the underlying mathematical models are often PDEs with stochastic parameters. In such a framework, the forward propagation of uncertainty is often performed by Monte Carlo sampling techniques [21, 15, 27, 26, 37] or spectral stochastic techniques [9, 56, 38], and the inverse propagation of uncertainty is done by Bayesian inference [29, 49, 23]. All these approaches assume that the uncertainty in the model parameters is precisely known and can be described by precise probability distribution functions (PDFs). They are therefore suitable for treating aleatoric (or random) uncertainty, which arises from inherent randomness or variability in a system. There is yet another type of uncertainty, known as epistemic (or non-random) uncertainty, that arises from limited and/or inaccurate information about a system, for instance from insufficient data or an inaccurate model. It does not have a random nature and may not be accurately described by precise PDFs [53, 14]. One approach to represent non-random epistemic uncertainty is through interval analysis [36]. In this approach uncertain parameters are represented by closed intervals describing the incomplete knowledge of parameters. Another approach that

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generalizes interval analysis lies within the framework of fuzzy set theory \[60\]. In a fuzzy set elements can partially be in the set. This notion can be exploited to represent an epistemically uncertain parameter by a set of nested intervals with different membership degrees. In a fuzzy framework the underlying mathematical models are often PDEs with fuzzy parameters; see e.g. \[12\], \[16\], \[10\]. Solving fuzzy PDEs leads to fuzzy computations that involve interval arithmetic \[28\], \[36\] and optimization \[31\], \[40\] at different membership levels.

Many real-world problems indeed exhibit a mixture of aleatoric and epistemic uncertainties. A typical example is the dynamic response of composite materials, such as carbon fiber polymers, where uncertainty in material properties and damage parameters has contributions from both types \[3\]. On the one hand, variations in material properties (such as the modulus of elasticity) and the spatial distribution of fiber constituents are of random nature. On the other hand, material constants may be either difficult or impossible to measure. Moreover, materials may come from different manufacturers, and there may be large variations in their quality leading to large variations in the experimental measurements. When experiments cannot be performed, material constants must be obtained from the literature, such as handbooks and standards. There may be large disagreement in the literature for the values of these quantities. For example, see \[8\] that studies large variations in the thermal conductivity of stainless steel AISI 304, based on the data given in various sources \[1\], \[51\]. Such experimental and literature-based variations, which may be larger than the intrinsic random experimental noise, will introduce epistemic uncertainty.

A major difficulty that arises in modeling uncertainty in real-world problems is that there is often no clear-cut distinction between aleatoric and epistemic uncertainty. There may be a random quantity whose parameters are partially known, or there may be an epistemically uncertain quantity for which some values are more likely to occur than others. Consequently, it may not be possible to simply model aleatoric uncertainty by probability distributions and epistemic one by intervals or fuzzy sets. In such situations a hybrid model obtained by the synthesis of the two models, rather than simply adding them, is needed to model hybrid uncertainties.

One approach to describe hybrid uncertainties is to synthetize interval analysis and probability theory and build up interval-valued probability distributions \[54\]. This approach constructs a probability-box (or a p-box) consisting of a family of cumulative distribution functions (CDFs). The left and right envelopes of the family will form a box and bound the “unknown” distribution of the uncertain parameter from above and below. We also refer to a recent approach, known as optimal UQ \[42\], where the optimal distribution among the family is targeted and obtained as the candidate for the “unknown” distribution. Other related approaches in the framework of imprecise probability include coherent lower and upper previsions \[52\], second-order hierarchical probabilities \[23\], and the Dempster-Shafer theory of belief functions \[47\]. Another approach that goes beyond the framework of probability is to synthesize probability theory and fuzzy set theory, thereby building fuzzy probability distributions \[62\], \[35\], \[1\], \[17\]. We refer to \[53\], \[19\] for a general discussion of the subject.

In the present work we consider the hybrid fuzzy-probability approach to UQ, and introduce and study PDEs with fuzzy-stochastic parameters. Such hybrid PDEs will serve as the underlying mathematical models for physical systems subject to hybrid random and non-random uncertainties. Compared to purely stochastic PDEs and purely fuzzy PDEs, which may treat either only random and only non-random uncertainty in physical systems, respectively, fuzzy-stochastic PDEs offer powerful tools and models for accurate description and propagation of hybrid uncertainties. We
use the level-set representation of fuzzy functions and define the solution to a fuzzy-stochastic PDE problem through a corresponding parametric problem, and further develop theoretical results on the existence, uniqueness, and regularity of the solution. We then present a numerical approach for computing fuzzy-stochastic quantities, such as fuzzy failure probabilities and fuzzy probability distributions. Considering the notion of full interaction between fuzzy variables and incorporating it in the proposed numerical approach, we avoid overestimating the lower and upper bounds of output intervals, a problem known as dependency phenomenon \cite{36} that may occur in interval arithmetic and fuzzy computations. We present two numerical examples. In the first example, we will compute and visualize various types of fuzzy-stochastic quantities of interest (QoIs) being functionals of the PDE solution. In the second example, we will demonstrate the importance and applicability of fuzzy-stochastic PDEs for an engineering problem in materials science: the response of fiber-reinforced polymers to external forces.

The main contributions of this paper include: 1) introducing fuzzy-stochastic PDEs and defining their solution; 2) presenting rigorous well-posedness and regularity analysis of such PDEs; and 3) developing a numerical algorithm for computing fuzzy-stochastic quantities, taking into account the full interaction between fuzzy variables. The development of more efficient numerical methods for solving fuzzy-stochastic PDEs and more sophisticated numerical experiments are the subjects of our current work and will be presented elsewhere.

The rest of the paper is organized as follows. Section 2 provides the mathematical and computational foundations of fuzzy and fuzzy-stochastic quantities necessary for and relevant to the focus of this work. In Section 3 we present fuzzy-stochastic PDEs and define their solution. The well-posedness and regularity of the fuzzy-stochastic problems are discussed in Section 4. We present two numerical examples in Section 5.

2 Fuzzy and Fuzzy-Stochastic Quantities

This section provides the mathematical and computational foundations of fuzzy and fuzzy-stochastic quantities. Only the concepts relevant to the focus of this work are discussed here. We refer to \cite{18, 30, 44, 35, 11, 17} for a more general description of fuzzy set theory and fuzzy randomness from an engineering point of view.

2.1 Fuzzy variables

Fuzzy sets \cite{60}, or fuzzy variables, generalize the notion of crisp sets. In a crisp set, the membership of an element is given by the characteristic function, taking values either 0 (not a member) or 1 (a member). In a fuzzy set, elements can partially be in the set. Each element is given a membership degree, ranging from 0 to 1; see Figure 1.

**Definition 1.** A fuzzy variable is defined by a set of pairs

\[
\tilde{z} = \{(z, \mu_{\tilde{z}}(z)), z \in Z \subset \mathbb{R}, \mu_{\tilde{z}} : Z \rightarrow [0, 1]\},
\]

where $Z$, referred to as the universe, is a non-empty subset of the real line $\mathbb{R}$, and $\mu_{\tilde{z}}$ is a membership function defined on $Z$ with range $[0, 1]$. The set of all fuzzy variables defined on $Z$ is denoted by $\mathcal{F}(Z)$. 

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Figure 1: The notion of membership in crisp sets and fuzzy sets.

It is to be noted that in general the range of the membership function may be the subset of nonnegative real numbers whose supremum is finite. However, it is always possible to normalize the range to \([0,1]\). Such fuzzy variables, considered here, are sometimes referred to as normalized fuzzy variables. Throughout the present paper, a (normalized) fuzzy variable is denoted by the superimposition of a tilde over a letter.

An important notion in fuzzy set theory is the notion of \(\alpha\)-cuts (see e.g. [18, 30]), which allows one to decompose fuzzy computations into several interval computations.

**Definition 2.** Any membership function \(\tilde{\mu}_z\) of a fuzzy variable \(\tilde{z} \in \mathcal{F}(Z)\) can be represented by a family \(\{S^\tilde{z}_\alpha \subset Z, \alpha \in [0,1]\}\) of its \(\alpha\)-level sets, known as \(\alpha\)-cuts:

\[
S^\tilde{z}_0 = \text{closure}\{z \in Z | \tilde{\mu}_z(z) > 0\}, \quad \text{and} \quad S^\tilde{z}_\alpha = \{z \in Z | \tilde{\mu}_z(z) \geq \alpha\}, \quad \forall \alpha \in (0,1].
\]

In the present work we will need the following assumptions:

(A1) The universe \(Z \subset \mathbb{R}\) is a bounded, convex set.

(A2) The membership function is upper semicontinuous, i.e.

\[
\limsup_{z \to z_0} \tilde{\mu}_z(z) \leq \tilde{\mu}_z(z_0), \quad \forall z_0 \in Z.
\]

(A3) The membership function is quasi-concave, i.e.

\[
\tilde{\mu}_z(\lambda z_1 + (1-\lambda) z_2) \geq \min(\tilde{\mu}_z(z_1), \tilde{\mu}_z(z_2)), \quad \forall z_1, z_2 \in Z, \quad \forall \lambda \in [0,1].
\]

The boundedness of \(Z\) in (A1) implies that the \(\alpha\)-cuts are bounded sets. This assumption is natural for the physical quantities to be represented by fuzzy variables, as such quantities are usually bounded. The convexity of \(Z\) in (A1) ensures that \(\forall z_1, z_2 \in Z\) and \(\forall \lambda \in [0,1]\), we have \(\lambda z_1 + (1-\lambda) z_2 \in Z\), which is needed for (A3). Assumptions (A2)-(A3) imply that the \(\alpha\)-cuts are closed and convex sets, respectively. Hence, for a fuzzy variable \(\tilde{z} \in \mathcal{F}(Z)\) satisfying assumptions (A1)-(A3), the \(\alpha\)-cuts \(S^\tilde{z}_\alpha\) will be bounded, closed intervals with the inclusion property \(S^\tilde{z}_{\alpha_2} \subset S^\tilde{z}_{\alpha_1}\) for \(0 \leq \alpha_1 \leq \alpha_2 \leq 1\); see Figure [2]

We also define the following relational operators for fuzzy variables (see e.g. [18]), needed for the boundedness and positivity assumption [10] needed in Section [3].

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Definition 3. A fuzzy variable $\tilde{z} \in \mathcal{F}(Z)$ is greater than or equal to a real number $a$ if $\mu_{\tilde{z}}(z) = 0$, $\forall z \in Z$ such that $z < a$. This is denoted by $\tilde{z} \geq a$. Similarly, a fuzzy variable $\tilde{z} \in \mathcal{F}(Z)$ is smaller than or equal to a real number $a$ if $\mu_{\tilde{z}}(z) = 0$, $\forall z \in Z$ such that $z > a$. This is denoted by $\tilde{z} \leq a$.

We note that the above relations can also be expressed in terms of the zero-cut. A fuzzy variable is greater (respectively smaller) than a real number if the real number is smaller (respectively greater) than all points on the zero-cut of the fuzzy variable.

Fuzzy vectors. A fuzzy vector can be considered as the $n$-dimensional generalization of a fuzzy variable, with $n \geq 2$.

Definition 4. An $n$-dimensional fuzzy vector is defined by a set of pairs

$$\tilde{z} = \{(z, \mu_{\tilde{z}}(z)), \ z \in Z \subset \mathbb{R}^n, \ \mu_{\tilde{z}} : Z \to [0,1]\},$$

where the universe $Z$ is a non-empty subset of $\mathbb{R}^n$, and $\mu_{\tilde{z}}$ is a joint membership function with range $[0,1]$. The set of all fuzzy vectors $\tilde{z}$ on $Z$ is denoted by $\mathcal{F}(Z)$.

An fuzzy vector, written as $\tilde{z} = (\tilde{z}_1, \ldots, \tilde{z}_n)$, is in fact a collection of $n$ fuzzy variables $\tilde{z}_i = \{(z_i, \mu_{\tilde{z}_i}(z_i)), \ z_i \in Z_i \subset \mathbb{R}, \ \mu_{\tilde{z}_i} : Z_i \to [0,1]\}$, with $i = 1, \ldots, n$. The universe on which the fuzzy vector is defined is a subset of the Cartesian product of the one-dimensional universes, i.e. $Z \subset Z_1 \times \cdots \times Z_n$.

Analogous to one-dimensional $\alpha$-cuts we can define $\alpha$-cuts for fuzzy vectors.

Definition 5. Any joint membership function $\mu_{\tilde{z}}$ of an $n$-dimensional fuzzy vector $\tilde{z} \in \mathcal{F}(Z)$ can be identified with the one-parametric family $\{S_{\alpha} \subset Z, \ \alpha \in [0,1]\}$ of $n$-dimensional joint $\alpha$-cuts:

$$S_{\alpha} = \text{closure}\{z \in Z | \mu_{\tilde{z}}(z) > 0\}, \text{ and } S_{\alpha}^\emptyset = \{z \in Z | \mu_{\tilde{z}}(z) \geq \alpha\}, \ \forall \alpha \in (0,1]. \quad (1)$$

Similar to the case of fuzzy variables, we will need the following assumptions:

(A4) The universe $Z \subset \mathbb{R}^n$ is a bounded, convex set.

(A5) The joint membership function is upper semicontinuous.

(A6) The joint membership function is quasi-concave.
It is to be noted that for a fuzzy vector \( \tilde{z} \in F(Z) \) satisfying assumptions (A4)-(A6), the joint \( \alpha \)-cuts \([1]\) will be compact, convex sets satisfying the inclusion property:

\[
S_{\alpha_2}^{\tilde{z}} \subset S_{\alpha_1}^{\tilde{z}}, \quad 0 \leq \alpha_1 \leq \alpha_2 \leq 1. \tag{2}
\]

**Interaction.** In fuzzy arithmetic it is important to consider the *interaction* between fuzzy variables (analogous to the correlation between random variables). In general, one can distinguish between three types of interaction and split fuzzy variables into three types: 1) *non-interactive* variables; 2) *fully interactive* variables; and 3) *partially interactive* variables. Interaction can be defined in terms of the notion of \( \alpha \)-cuts.

**Definition 6.** Consider a fuzzy vector \( \tilde{z} \in F(Z) \) consisting of \( n \geq 2 \) fuzzy variables \( \tilde{z}_i \in F(Z_i) \), with \( i = 1, \ldots, n \), satisfying (A4)-(A6). Let \( S_{\alpha}^{\tilde{z}_i} \) be the one-dimensional (or marginal) \( \alpha \)-cut interval corresponding to each fuzzy variable \( \tilde{z}_i \). The fuzzy variables \( \{\tilde{z}_i\}_{i=1}^n \) are said to be non-interactive if their joint \( \alpha \)-cut \( S_{\alpha}^{\tilde{z}} \) is the \( n \)-dimensional hyperrectangle given by the Cartesian product of \( n \) marginal \( \alpha \)-cuts:

\[
S_{\alpha}^{\tilde{z}} = S_{\alpha}^{\tilde{z}_1} \times \ldots \times S_{\alpha}^{\tilde{z}_n} =: \prod_{i=1}^n S_{\alpha}^{\tilde{z}_i}, \quad \forall \alpha \in [0, 1].
\]

**Definition 7.** The fuzzy variables \( \{\tilde{z}_i\}_{i=1}^n \) in Definition 6 are said to be fully interactive if their joint \( \alpha \)-cut \( S_{\alpha}^{\tilde{z}} \) is a (possibly non-linear) continuous curve in the hyperrectangle \( \prod_{i=1}^n S_{\alpha}^{\tilde{z}_i} \subset \mathbb{R}^n \), satisfying the inclusion property \([2]\).

It is to be noted that since the joint \( \alpha \)-cut \( S_{\alpha}^{\tilde{z}} \) of fully interactive fuzzy variables is a continuous curve in \( \mathbb{R}^n \), there is a bijective mapping between \( S_{\alpha}^{\tilde{z}} \) and a one-dimensional closed, bounded interval \( I_\alpha = [0, L_\alpha] \subset \mathbb{R} \), with \( L_\alpha \) being the Euclidean length of the curve \( S_{\alpha}^{\tilde{z}} \). By the arc length parameterization of the curve we can therefore obtain a (possibly non-linear) bijective map \( \varphi_\alpha : [0, L_\alpha] \rightarrow \mathbb{R}^n \) so that \( \varphi_\alpha(s) \in S_{\alpha}^{\tilde{z}} \) for each arc length parameter \( s \in I_\alpha = [0, L_\alpha] \). Such mapping facilitates practical fuzzy computations; see Section 5.

Clearly, unlike the case of non-interactive fuzzy variables for which the inclusion property \([2]\) is automatically satisfied, in the case of fully-interactive variables the inclusion property must be imposed, because not every continuous curve in the hyperrectangle satisfies this property. A particular type of full interaction that can be easily handled in practical computations may be considered by setting the joint \( \alpha \)-cut to be the polygonal (i.e. continuous and piecewise linear) curve, given recursively by

\[
S_{1}^{\tilde{z}} = \text{diag}(\prod_{i=1}^n S_{\alpha_i}^{\tilde{z}_i}),
\]

\[
S_{\alpha_j}^{\tilde{z}} = S_{\alpha_{j+1}}^{\tilde{z}} \bigcup \text{diag}(\prod_{i=1}^n [S_{\alpha_i}^{\tilde{z}_i} \setminus S_{\alpha_{j+1}}^{\tilde{z}_i}]) \bigcup \text{diag}(\prod_{i=1}^n [S_{\alpha_i}^{\tilde{z}_i} \setminus S_{\alpha_{j+1}}^{\tilde{z}_i}]^r), \quad 0 \leq \alpha_j < \alpha_{j+1} \leq 1.
\]

Here, \( \text{diag}(\prod_{i=1}^n S^i) \) with \( S^i = [S^i, S^i] \) denotes the main space diagonal of the hyperrectangle \( S = \prod_{i=1}^n S^i \), i.e. the line segment between the vertices \( (S^1, \ldots, S^n) \) and \( (S^1, \ldots, S^n) \), and the hyperrectangles \( [S_{\alpha_{j+1}}^{\tilde{z}_i} \setminus S_{\alpha_j}^{\tilde{z}_i}]_l \) and \( [S_{\alpha_{j+1}}^{\tilde{z}_i} \setminus S_{\alpha_j}^{\tilde{z}_i}]_r \) are the left and right portions of the set \( S_{\alpha_{j+1}}^{\tilde{z}_i} \setminus S_{\alpha_j}^{\tilde{z}_i} \), respectively; see Figure 3. We notice that this setting ensures that the inclusion property \([2]\) holds.
Definition 8. The fuzzy variables \( \{ \tilde{z}_i \}_{i=1}^n \) in Definition 6 are said to be partially interactive if they are neither non-interactive nor fully interactive.

Due to the assumptions (A4)-(A6), Definition 8 implies that the joint \( \alpha \)-cut \( S^\alpha_{\tilde{z}_i} \) of partially interactive fuzzy variables is a strict subset of the Cartesian product of the marginal \( \alpha \)-cuts, that is, \( S^\alpha_{\tilde{z}_i} \subset \prod_{i=1}^n S^\alpha_{z_i} \), \( \forall \alpha \in [0,1] \). Moreover, it cannot be mapped into a one-dimensional interval. In practice the joint \( \alpha \)-cuts of partially interactive variables are geometrically more complicated than those of non-interactive and fully interactive variables. Efficient fuzzy computation in the case of partial interaction is a challenging task due to the need for solving constrained optimization problems over complicated joint \( \alpha \)-cuts. We refer to [46] for the treatment of particular types of partial interaction using triangular norms.

Importantly, partial interaction does not often occur in hybrid fuzzy-stochastic modeling, and hence we may not need to treat this case in a hybrid fuzzy-stochastic framework. Full interaction may however occur in hybrid fuzzy-stochastic models. An example is when the uncertain parameter is characterized by a random variable with fuzzy statistical moments using a set of measurement data. Since the moments are directly related to each other, i.e. higher moments are obtained from lower moments, the fuzzy moments may be fully interactive; see Section 5.2. Another example, which may occur even in a pure fuzzy framework, is when we perform mathematical operations on two functions with the same fuzzy arguments. In this case the two fuzzy functions are fully interactive.

Although in a hybrid framework we often face non-interactive and/or fully interactive fuzzy variables, we note that the mathematical definitions and results in the present paper are independent of the type of interaction between fuzzy variables. In the rest of the paper whenever the type of interaction between fuzzy variables is not specified, the fuzzy variables are understood as general fuzzy variables in \( \mathcal{F}(\mathbb{Z}) \).

2.2 Fuzzy functions

A fuzzy function is a generalization of the concept of a classical function. A classical function is a mapping from its domain of definition into its range. There are various generalizations in the literature on fuzzy calculus; see e.g. [18 35] and the references therein. Here, we consider only two cases: 1) a crisp map with fuzzy arguments, and 2) a crisp map with both fuzzy and nonfuzzy

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Figure 3: The intervals used in the definition of the joint \( \alpha \)-cuts of fully interactive fuzzy variables by polygonal nested curves. We have \( S^\alpha_{\tilde{z}_i} = S^\alpha_{\tilde{z}_i} \cup [S^\alpha_{\tilde{z}_i} \setminus S^\alpha_{\tilde{z}_i+1}]_l \cup [S^\alpha_{\tilde{z}_i} \setminus S^\alpha_{\tilde{z}_i+1}]_r \) with \( \alpha_j < \alpha_{j+1} \).
arguments.

**Definition 9.** Consider a function \( u : \mathbf{Z} \to V \) mapping every element of its domain \( \mathbf{Z} \subset \mathbb{R}^n \) to an element of its range \( V = \{ v \in \mathbb{R} | v = u(z), z \in \mathbf{Z} \} \subset \mathbb{R} \). Let further \( \mathbf{z} \in \mathcal{F}(\mathbf{Z}) \) be a fuzzy vector with a joint membership function \( \mu_{\mathbf{z}} : \mathbf{Z} \to [0,1] \). A function \( u \) of \( \mathbf{z} \), referred to as a fuzzy function, is then a mapping
\[
u : \mathcal{F}(\mathbf{Z}) \to \mathcal{F}(V), \quad V = \{ v \in \mathbb{R} | v = u(z), z \in \mathbf{Z} \} \subset \mathbb{R},
\]
so that \( \tilde{u} := u(\mathbf{z}) = \{ (v, \mu_{\mathbf{z}}(v)), v \in V \} \in \mathcal{F}(V) \) is a fuzzy variable with the membership function \( \mu_{\tilde{u}} \) given by the generalized extension principle \([13, 22]\):
\[
\mu_{\tilde{u}}(v) = \begin{cases} \sup_{z=u^{-1}(v)} \mu_{\mathbf{z}}(z) & u^{-1}(v) \neq \emptyset, \\ 0 & u^{-1}(v) = \emptyset, \end{cases} \quad \forall v \in V. \tag{3}
\]
Here, \( u^{-1}(v) \) is the inverse image of \( v = u(z) \in V \) and \( \emptyset \) is the empty set.

Crucially, the generalized extension principle \([3]\) uses the general form of the input joint membership function \( \mu_{\mathbf{z}} \) and hence is valid for both non-interactive and interactive input fuzzy variables.

**Remark 1.** This notion of fuzzy functions was originally introduced by Zadeh \([60, 61]\) for non-interactive input fuzzy variables, where \( \mu_{\mathbf{z}} \) is given by the minimum of the marginal memberships \( \mu_{\mathbf{z}}(z) = \min(\mu_{z_1}(z_1), \ldots, \mu_{z_n}(z_n)) \), \( \forall z = (z_1, \ldots, z_n) \in \mathbf{Z} \), and the generalized extension principle \([3]\) reduces to Zadeh’s sup-min extension principle. In \([18]\) this type of mapping is called “fuzzy extension of a nonfuzzy function”.

In order to define fuzzy and fuzzy-stochastic fields that appear in the study of fuzzy-stochastic PDEs, we further need to consider crisp maps with both fuzzy and nonfuzzy arguments. We closely follow the extension of classical functions to Sobolev space-valued functions that arise in the study of time-dependent PDEs; see e.g. \([20]\), and extend the notion of fuzzy functions to fuzzy Sobolev space-valued functions. In the study of time-dependent PDEs, a function \( u \) of space \( x \in D \subset \mathbb{R}^d \) and time \( t \in I \subset \mathbb{R} \) may be viewed as a function of \( t \) taking values in a function space \( H(D) \). A mapping \( u : I \to H(D) \) can then be defined by \( [u(t)](x) := u(x,t) \), \( \forall t \in I \), \( \forall x \in D \). When \( H(D) \) is a Sobolev space of functions defined on \( D \), the function \( u \) is referred to as a “Sobolev space-valued function”. This representation is not limited to functions with spatial and temporal arguments and can be generalized to include both fuzzy arguments and nonfuzzy arguments, such as spatial, temporal, and random variables.

**Definition 10.** Consider a real-valued function \( u : \mathbf{X} \times \mathbf{Z} \to V \) mapping every element of its domain \( \mathbf{X} \times \mathbf{Z} \), with \( \mathbf{X} \subset \mathbb{R}^p \) and \( \mathbf{Z} \subset \mathbb{R}^n \), to an element of its range \( V = \{ v \in \mathbb{R} | v = u(\mathbf{k}, z), \mathbf{k} \in \mathbf{X}, z \in \mathbf{Z} \} \subset \mathbb{R} \). Let \( \mathbf{z} \in \mathcal{F}(\mathbf{Z}) \) be a fuzzy vector with a joint membership function \( \mu_{\mathbf{z}} : \mathbf{Z} \to [0,1] \). A function \( u \) of \( \mathbf{k} \in \mathbf{X} \) and \( \mathbf{z} \in \mathcal{F}(\mathbf{Z}) \), written as
\[
[u(\mathbf{z})](\mathbf{k}) := u(\mathbf{k}, \mathbf{z}), \quad \forall \mathbf{k} \in \mathbf{X}, \quad \mathbf{z} \in \mathcal{F}(\mathbf{Z}), \tag{4}
\]
is defined by an infinite set of fuzzy variables \( \{ \tilde{u}(\mathbf{k}), \mathbf{k} \in \mathbf{X} \} \). Each element of this set is a fuzzy variable \( \tilde{u}(\mathbf{k}) := [u(\mathbf{z})](\mathbf{k}) \) corresponding to a fixed \( \mathbf{k} \in \mathbf{X} \), given by
\[
\tilde{u}(\mathbf{k}) = \{ (v, \mu_{\tilde{u}(\mathbf{k})}(v)), v \in V(\mathbf{k}) \}, \quad V(\mathbf{k}) = \{ v = u(\mathbf{k}, z), z \in \mathbf{Z} \} \subset \mathbb{R}, \quad \forall \mathbf{k} \in \mathbf{X},
\]

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with the membership function \( \mu_{\tilde{u}(\kappa)} \) given by the generalized extension principle \( \text{[3]} \). The restriction of \( [4] \) to \( Z \) is a function of \( z \in Z \) taking values in a function space \( H(X) \), i.e. \( u : Z \rightarrow H(X) \). If \( H(X) \) is a Sobolev space, the function \( [4] \) may then be viewed as a fuzzy function taking values in a Sobolev space. We call such a function a fuzzy Sobolev space-valued function, denoted by the mapping \( u : \mathcal{F}(Z) \rightarrow \mathcal{F}(H(X)) \).

It is to be noted that the function space \( H \) may in general be of any type. The main reason that we consider Sobolev spaces here is that in the present work such fuzzy functions appear as the coefficients, data, and solutions to PDE problems.

**Remark 2.** A fuzzy Sobolev space-valued function is closely related to a fuzzy map with nonfuzzy arguments discussed in \( \text{[32]} \). A fuzzy mapping on the nonfuzzy variables \( \kappa \), denoted by \( \tilde{u}(\kappa) \), may be formulated as a crisp mapping \( u(\kappa, \tilde{z}) \) on the nonfuzzy variables \( \kappa \) and a fuzzy vector \( \tilde{z} \) referred to as “fuzzy bunch parameters”, i.e. \( \tilde{u}(\kappa) = u(\kappa, \tilde{z}) \). In \( \text{[32]} \) this is called the bunch parameter representation of fuzzy functions. Fuzzy Sobolev space-valued functions are also related to “fuzzifying functions” discussed in \( \text{[13]} \).

**Computation of fuzzy functions.** The computation of a fuzzy function \( \tilde{u} \) amounts to the computation of its output membership function \( \mu_{\tilde{u}} \). Computing \( \mu_{\tilde{u}}(v) \) for all \( v = u(z) \in V \) by a direct application of the generalized extension principle \( \text{[3]} \) can be quite complicated and numerically cumbersome, as there is no efficient method to evaluate the supremum of \( \mu_{\tilde{u}}(z) \) over all \( z \) for which \( u(z) = v \). The computations can substantially be simplified using the \( \alpha \)-cut representation of \( \mu_{\tilde{u}} \), thanks to the following important result, referred to as the function-set identity \( \text{[13, 22]} \), extending the earlier work of Nguyen \( \text{[41]} \).

**Theorem 1.** (Function-set identity \( \text{[13, 22]} \)) Let \( \tilde{z} \in \mathcal{F}(Z) \) be a fuzzy vector with a joint membership function \( \mu_{\tilde{z}} : Z \rightarrow [0, 1] \) and corresponding joint \( \alpha \)-cuts \( S^\alpha_{\tilde{z}} \), satisfying the assumptions \( \text{(A4)-(A6)} \). Let further \( u : Z \rightarrow V \) be a continuous map, where \( V = \{ v \in \mathbb{R} | v = u(z), z \in Z \} \subset \mathbb{R} \). Then the \( \alpha \)-cuts \( S^\alpha_{\tilde{u}} \) corresponding to the output membership function \( \mu_{\tilde{u}} \) of the fuzzy function \( \tilde{u} = u(\tilde{z}) \in \mathcal{F}(V) \) is given by:

\[
S^\alpha_{\tilde{u}} = u(S^\alpha_{\tilde{z}}) = [\min_{z \in S^\alpha_{\tilde{z}}} u(z), \max_{z \in S^\alpha_{\tilde{z}}} u(z)], \quad \forall \alpha \in [0, 1].
\]  

(5)

It is to be noted that two conditions must be satisfied for \( [5] \) to hold: 1) the map \( u : Z \rightarrow V \) is continuous, and 2) the fuzzy input vector \( \tilde{z} \in \mathcal{F}(Z) \) satisfies the assumptions \( \text{(A4)-(A6)} \). Under these two conditions, the \( \alpha \)-cuts \( S^\alpha_{\tilde{u}} \) will be compact intervals given by \( [5] \). The continuity assumption holds when, for instance, the function \( \tilde{u} \) is the solution to a differential equation under appropriate assumptions on the data. This important observation will be later utilized for the analysis and computation of fuzzy-stochastic PDEs in this paper.

Crucially, Theorem \( [\text{1}] \) allows us to decompose fuzzy computations into several interval computations. Motivated by this, we present a numerical approach, outlined in Algorithm \( [\text{1}] \) for computing fuzzy functions.

The optimization problems in step 2 can be numerically solved for instance by iterative methods; see e.g. \( \text{[31, 40, 45, 46]} \). The choice of the method would depend on the dimension and the complexity of \( S^\alpha_{\tilde{u}} \) and the regularity of \( u \) with respect to \( z \).

Similarly, the output membership function of a fuzzy Sobolve space-valued function (see Definition \( [10] \)) may be computed by Algorithm \( [\text{1}] \) pointwise in \( \kappa \in X \).
Algorithm 1 Computation of fuzzy functions

0. Given a fuzzy vector \( \tilde{z} \in \mathcal{F}(\mathbb{Z}) \) satisfying (A4)-(A6) and a continuous map \( u : \mathbb{Z} \to V \), where \( V = \{v \in \mathbb{R} | v = u(z), z \in \mathbb{Z} \} \subset \mathbb{R} \), the output membership function of the fuzzy function \( \tilde{u} = u(\tilde{z}) \in \mathcal{F}(V) \) is computed as follows.

1. Interaction: Find the input joint \( \alpha \)-cut \( S^Z_\alpha \) for a fixed \( \alpha \in [0, 1] \) based on the interaction between the input fuzzy variables.

2. Optimization: Obtain the output \( \alpha \)-cut \( S^Z_\alpha = [u, \tilde{u}] \) by computing two global optimization problems: \( y := \min_{z \in S^Z_\alpha} u(z) \) and \( \tilde{u} := \max_{z \in S^Z_\alpha} u(z) \).

3. Repeat steps 1-2 for various \( \alpha \) and form the output membership function \( \mu_{\tilde{u}} \).

2.3 Fuzzy fields

A scalar fuzzy field is a particular type of a fuzzy Sobolev space-valued function. It is a crisp map with spatial variables and a fuzzy vector as arguments generating an infinite set of fuzzy variables.

**Definition 11.** Let \( D \subset \mathbb{R}^d \) be a compact spatial domain, with \( d = 1, 2, 3 \), and consider a vector of spatial variables \( x \in D \). Let further \( \tilde{z} \in \mathcal{F}(\mathbb{Z}) \) be a fuzzy vector on \( \mathbb{Z} = \mathbb{R}^n \). A scalar fuzzy field, written as \( \tilde{u}(x) = u(x, \tilde{z}), \forall x \in D \), is a fuzzy Sobolev space-valued function \( u : \mathcal{F}(\mathbb{Z}) \to \mathcal{F}(H(D)) \), where \( H(D) \) is a Sobolev space on \( D \).

A typical example of \( H(D) \) in the context of PDEs is the Hilbert space of functions whose weak derivatives up to order \( s \geq 0 \) are square integrable, denoted by \( H^s(D) \).

2.4 Fuzzy-stochastic variables

A fuzzy-stochastic variable, introduced in \([32, 33]\), is a generalization of a random variable; see also \([62, 35, 11, 17]\). Let \((\Omega, \Sigma, P)\) be a probability space, where \( \Omega \) is a sample space, \( \Sigma \) is a non-empty sigma-field on \( \Omega \), and \( P \) is a probability measure assigned to each measurable subset of \( \Omega \) and satisfying Kolmogorov’s axioms \([25]\). A random variable \( y : \Omega \to \mathbb{R} \) is a real-valued measurable function defined on \((\Omega, \Sigma, P)\). Every realization of a random variable \( y(\omega) \), for some \( \omega \in \Omega \), is a real number. If the probability measure \( P \) is absolutely continuous \([25]\), it can be described by a single CDF denoted by \( F \), or a single PDF denoted by \( \pi \),

\[
F(y_0) = P(y \leq y_0) = \int_{-\infty}^{y_0} \pi(\tau) \, d\tau, \quad y_0 \in \mathbb{R}.
\]

The CDF and PDF are usually presented as functions of \( y_0 \in \mathbb{R} \) and a set of \( n \) crisp parameters collected in a parameter vector \( \theta \in \mathbb{R}^n \):

\[
F(y_0; \theta) = \int_{-\infty}^{y_0} \pi(\tau; \theta) \, d\tau, \quad y_0 \in \mathbb{R}, \quad \theta \in \mathbb{R}^n.
\]  

(6)

For instance, for a normal random variable \( y \sim \mathcal{N}(\theta_1, \theta_2^2) \) with two parameters \( (\theta_1, \theta_2) \) being the mean and standard deviation, a parameterized CDF is specified as

\[
F(y_0; \theta) = \frac{1}{\sqrt{2\pi} \theta_2} \int_{-\infty}^{y_0} e^{-(\tau-\theta_1)^2 / 2\theta_2^2} \, d\tau, \quad y_0 \in \mathbb{R}, \quad \theta = (\theta_1, \theta_2) \in \mathbb{R}^2.
\]

This concept can be generalized to define fuzzy-stochastic variables as follows.
Definition 12. A fuzzy-stochastic variable \( \tilde{y} : \Omega \to \mathcal{F}(V) \), with \( V \subset \mathbb{R} \), is a fuzzy-valued measurable function on a sample space \( \Omega \). Every realization of a fuzzy-stochastic variable \( \tilde{y}(\omega) \), for some \( \omega \in \Omega \), is a fuzzy variable, rather than a real number, given by a set of pairs

\[
\tilde{y}(\omega) = \{(y(\omega), \mu_{\tilde{y}}(y(\omega))) : y(\omega) \in V \subset \mathbb{R}, \mu_{\tilde{y}} : V \to [0, 1] \}, \quad \forall \omega \in \Omega.
\]

The fuzzy-valued probability measure \( \tilde{P} \) corresponding to \( \tilde{y} \) is described by a fuzzy CDF, denoted by \( \tilde{F} \), and defined in Definition 13 and Definition 15.

We consider and define fuzzy CDFs for two types of fuzzy-stochastic variables:

Type I: random variables with fuzzy parameters;

Type II: outputs of crisp functions with input random and fuzzy variables.

We note that the first type is a special case of the second type. We will first define the notion of fuzzy CDFs for type-I fuzzy-stochastic variables. The definition of fuzzy CDFs for type-II fuzzy-stochastic variables will be presented in Section 2.5.

Definition 13. (Type-I fuzzy CDF) Consider a type-I fuzzy-stochastic variable \( \tilde{y} \), consisting of a random variable with \( n \) fuzzy parameters \( \tilde{\theta} \in \mathcal{F}(Z) \), where \( Z \subset \mathbb{R}^n \). Let \( S_\alpha^{\tilde{\theta}} \subset \mathbb{R}^n \) be the joint \( \alpha \)-cut of \( \tilde{\theta} \). For every fixed \( \theta \in S_\alpha^{\tilde{\theta}} \), let the parameterized CDF of the corresponding random variable be given by \( (6) \). Consider the family of all parameterized CDFs \( (6) \) over \( S_\alpha^{\tilde{\theta}} \) for a fixed \( y_0 \in \mathbb{R} \):

\[
\{ F(y_0; \theta) : \theta \in S_\alpha^{\tilde{\theta}} \}, \quad y_0 \in \mathbb{R}.
\]

At any fixed \( \alpha \)-level, let \( F^L_\alpha(y_0) \) and \( F^R_\alpha(y_0) \) be the extrema of the family of parameterized CDFs over the joint \( \alpha \)-cut \( S_\alpha^{\tilde{\theta}} \), referred to as the left (upper) and right (lower) bounds:

\[
F^L_\alpha(y_0) = \max_{\theta \in S_\alpha^{\tilde{\theta}}} F(y_0; \theta), \quad F^R_\alpha(y_0) = \min_{\theta \in S_\alpha^{\tilde{\theta}}} F(y_0; \theta), \quad \forall \alpha \in [0, 1].
\]

The fuzzy CDF of \( \tilde{y} \), evaluated at \( y_0 \in \mathbb{R} \) and denoted by \( \tilde{F}(y_0) = F(y_0; \tilde{\theta}) \), is then defined by a nested set of left and right bounds at different \( \alpha \)-levels:

\[
\tilde{F}(y_0) = F(y_0; \tilde{\theta}) = \{ (F^L_\alpha(y_0), F^R_\alpha(y_0)) : \alpha \in [0, 1] \}.
\]

Interpretation of fuzzy CDFs. For any fixed \( \alpha \)-level, the set of all left and right bounds \( (7) \) corresponding to all points \( y_0 \) will constitute two left and right envelopes forming a p-box. It is important to note that the left and right envelopes are not necessarily two single CDFs. In fact, for different values of \( y_0 \in \mathbb{R} \), there may exist different maximizers and/or minimizers. Hence, different distributions on different regions may constitute the two envelopes. Fuzzy CDFs provide a far more comprehensive representation of uncertainty, compared to a class of imprecise probabilistic models such as p-boxes \( [54] \), coherent lower and upper previsions \( [52, 53] \), and optimal UQ \( [42] \) which provide only crisp lower and upper bounds from a set of admissible distributions. A fuzzy CDF can indeed be thought of as a nested set of p-boxes at different levels of possibility (corresponding to different \( \alpha \)-levels); see the numerical examples in Section 5.
2.5 Fuzzy-stochastic functions

A fuzzy-stochastic function is a particular type of a fuzzy Sobolev space-valued function. It is a crisp map with a random vector and a fuzzy vector as arguments generating an output fuzzy-stochastic variable.

**Definition 14.** Let \( y \in \Gamma \subset \mathbb{R}^m \) be a random vector and \( \tilde{z} \in \mathcal{F}(Z) \) be a fuzzy vector on \( Z \subset \mathbb{R}^n \). A fuzzy-stochastic function, written as \( \tilde{u}(y) = u(y, \tilde{z}) \), \( \forall y \in \Gamma \), is a fuzzy Sobolev space-valued function \( u: \mathcal{F}(Z) \rightarrow \mathcal{F}(H(\Gamma)) \), with \( H(\Gamma) \) being a Sobolev space of random functions. The fuzzy-valued probability measure \( \hat{P} \) corresponding to \( \tilde{u} \) is described by a type-II fuzzy CDF \( \tilde{F} \) defined in Definition 15.

A typical example of \( H(\Gamma) \) is the space of random functions with bounded second moments, denoted by \( L^2(\Gamma) \).

**Definition 15.** (Type-II fuzzy CDF) Consider a type-II fuzzy-stochastic variable \( \tilde{u} \), being the output of a fuzzy-stochastic function defined in Definition 14. For every fixed \( z \in S^\alpha \), the parameterized CDF of the corresponding random variable, evaluated at any point \( u_0 \in \mathbb{R} \), will be determined by the PDF of the input random vector \( \pi = \pi(y) \) as

\[
F(u_0; z) = \int_{\{\tau : u(\tau, z) \leq u_0\}} \pi(\tau) \, d\tau.
\]

At any fixed \( \alpha \)-level, let \( F^L_\alpha \) and \( F^R_\alpha \) be the extrema of the family of parameterized CDFs over the joint \( \alpha \)-cut \( S^\alpha \):

\[
F^L_\alpha(u_0) = \max_{z \in S^\alpha_i} F(y_0; z), \quad F^R_\alpha(u_0) = \min_{z \in S^\alpha_i} F(y_0; z), \quad \forall \alpha \in [0, 1].
\]

The fuzzy CDF of \( \tilde{u} \), evaluated at \( u_0 \in \mathbb{R} \) and denoted by \( \tilde{F}(u_0) = F(u_0; \tilde{z}) \), is then defined by a nested set of left and right bounds at different \( \alpha \)-levels:

\[
\tilde{F}(u_0) = F(u_0; \tilde{z}) = \left\{ (F^L_\alpha(u_0), F^R_\alpha(u_0)) \right\}, \quad \forall \alpha \in [0, 1].
\]

**Computation of fuzzy-stochastic functions.** Let \( \tilde{u}(y) = u(y, \tilde{z}) \) be a fuzzy-stochastic function. Assume that we want to compute a fuzzy QoI, denoted by \( \hat{Q} \), given in terms of \( \tilde{u}(y) \). Three important examples of \( \hat{Q} \) include:

- \( \hat{Q} = \mathbb{E}[u^r(y, \tilde{z})] \): the \( r \)-th fuzzy moment of \( \tilde{u}(y) \), where \( r \) is a positive integer.
- \( \hat{Q} = \mathbb{E} [\mathbb{I}_{u(y, \tilde{z}) \leq u_0}] \): the fuzzy CDF of \( \tilde{u}(y) \) evaluated at a fixed point \( u_0 \in \mathbb{R} \), where \( \mathbb{I}_{[\cdot]} \) is the indicator function taking the value 1 or 0 if the event [\( \cdot \)] is “true” or “false”, respectively.
- \( \hat{Q} = \mathbb{E} [\mathbb{I}_{g(\tilde{u}(y)) \leq 0}] \): the fuzzy failure probability assuming that failure occurs when \( g(\tilde{u}(y)) \leq 0 \), where \( g \) is a differential and/or integral operator on \( \tilde{u}(y) \).

Each of the above fuzzy QoIs is the expectation of a fuzzy-stochastic function, say \( \hat{Q} = \mathbb{E}[q(y, \tilde{z})] \), where \( q = u^r(y, \tilde{z}) \) in the first example, \( q = \mathbb{I}_{u(y, \tilde{z}) \leq u_0} \) in the second example, and \( q = \mathbb{I}_{g(\tilde{u}(y)) \leq 0} \) in the third example above. Algorithm 2 outlines a numerical approach for computing \( \hat{Q} \).
Algorithm 2 Computation of fuzzy-stochastic functions

0. Given a random vector \( y \in \Gamma \), a fuzzy vector \( \tilde{z} \in \mathcal{F}(\mathcal{Z}) \) satisfying (A4)-(A6), and a fuzzy-stochastic function \( q : \mathcal{F}(\mathcal{Z}) \to \mathcal{F}(H(\Gamma)) \), we compute the fuzzy QoI \( \tilde{Q} = Q(\tilde{z}) = \mathbb{E}[q(y, \tilde{z})] \) as follows.

1. **Interaction**: For a fixed \( \alpha \in [0, 1] \), find the input joint \( \alpha \)-cut \( S_{\alpha}^{\tilde{z}} \) based on the interaction between the input fuzzy variables.

2. **Optimization**: Obtain the lower and upper bounds of the output \( \alpha \)-cut \( S_{\alpha}^{Q} \) by computing two global optimization problems:

\[
Q := \min_{z \in S_{\alpha}^{\tilde{z}}} Q(z), \quad \bar{Q} := \max_{z \in S_{\alpha}^{\tilde{z}}} Q(z), \quad Q(z) = \mathbb{E}[q(y, z)].
\]

An iterative optimization algorithm requires \( M_f \) function evaluations \( Q(z^{(k)}) \) at \( M_f \) fixed points \( \{z^{(k)}\}_{k=1}^{M_f} \in S_{\alpha}^{\tilde{z}} \). Each function evaluation amounts to computing the expectation \( \mathbb{E}[q(y, z^{(k)})] \) which may be done by a Monte Carlo sampling strategy or a spectral stochastic method, depending on the regularity of \( q \) with respect to \( y \).

3. Repeat steps 1-2 for various levels of \( \alpha \in [0, 1] \).

2.6 Fuzzy-stochastic fields

Since the solution of fuzzy-stochastic PDEs are functions of space/time, in addition to being functions of stochastic and fuzzy vectors, the notion of fuzzy-stochastic functions needs to be extended to include the dependency on space/time. A scalar fuzzy-stochastic field is indeed a particular type of a fuzzy Sobolev space-valued function with a vector of spatial variables, a random vector, and a fuzzy vector as arguments. Similarly, one can include a temporal variable as argument and define fuzzy-stochastic processes.

**Definition 16.** Let \( D \subset \mathbb{R}^d \) be a compact spatial domain, with \( d = 1, 2, 3 \), and consider a vector of spatial variables \( x \in D \). Let further \( y \in \Gamma \) and \( \tilde{z} \in \mathcal{F}(\mathcal{Z}) \) be a random vector and a fuzzy vector, respectively. A scalar fuzzy-stochastic field, written as \( \tilde{u}(x, y) = u(x, y, \tilde{z}) \), \( \forall x \in D, \forall y \in \Gamma \), is a fuzzy Sobolev space-valued function \( u : \mathcal{F}(\mathcal{Z}) \to \mathcal{F}(H(D \times \Gamma)) \), where \( H(D \times \Gamma) \) is a Sobolev space of functions on \( D \times \Gamma \).

An example of \( H(D \times \Gamma) \) in the context of fuzzy-stochastic PDEs is the Sobolev space of functions formed by the tensor product of two Sobolev spaces \( H^s(D) \otimes L^2_{\mu}(\Gamma) \).

3 Fuzzy-Stochastic PDEs

In general, we refer to PDEs with fuzzy-stochastic parameters, including coefficients, force terms, and boundary/initial data, as **fuzzy-stochastic PDEs**. Without loss of generality, in the present work, we consider only the case where the PDE coefficient is a fuzzy-stochastic field and assume that the forcing and data functions are deterministic.
3.1 A fuzzy-stochastic elliptic model problem

Let $D \subset \mathbb{R}^d$ be a bounded, convex, Lipschitz spatial domain, with $d = 1, 2, 3$. Consider the following fuzzy-stochastic elliptic boundary value problem:

$$
- \nabla_x \cdot (a(x, y, \tilde{z}) \nabla_x u(x, y, \tilde{z})) = f(x), \quad (x, y) \in D \times \Gamma, \quad \tilde{z} \in \mathcal{F}(Z),
$$

$$
u(x, y, \tilde{z}) = 0, \quad (x, y) \in \partial D \times \Gamma, \quad \tilde{z} \in \mathcal{F}(Z),$$

(8)

where $x = (x_1, \ldots, x_d) \in D$ is the vector of spatial variables, $y = (y_1, \ldots, y_m) \in \Gamma \subset \mathbb{R}^m$ is a random vector with a bounded joint PDF $\pi = \pi(y) : \Gamma \to \mathbb{R}^+$, and $\tilde{z} = (\tilde{z}_1, \ldots, \tilde{z}_n) \in \mathcal{F}(Z)$ is a fuzzy vector on $Z \subset \mathbb{R}^n$ satisfying assumptions (A4)-(A6) and with a family of joint $\alpha$-cuts $S_\alpha^z \subset Z$ with $\alpha \in [0, 1]$. The only source of uncertainty is assumed to be the parameter $a$ characterized by a fuzzy-stochastic field $\tilde{a}(x, y) = a(x, y, \tilde{z})$. This implies that the PDE solution $\tilde{u}(x, y) = u(x, y, \tilde{z})$ is a fuzzy-stochastic field; see Section 3.2 for the definition of the PDE solution.

We assume that $m$ and $n$ are finite numbers. We further assume

$$f \in L^2(D),$$

(9)

$$0 < a_{\min} \leq \tilde{a}(x, y) \leq a_{\max} < \infty, \quad \forall x \in D, \; \forall y \in \Gamma.$$  

(10)

Assumption (9) states that the forcing function $f$ is square integrable, and assumption (10) states that the PDE coefficient at every fixed $(x, y) \in D \times \Gamma$ is a uniformly positive and bounded fuzzy variable in the sense of Definition 3.

3.2 Solution of the fuzzy-stochastic problem

Following Definition 10 and Definition 16, we interpret the solution $\tilde{u}(x, y) = u(x, y, \tilde{z})$ to (8), under assumptions (9) - (10), as a fuzzy Sobolev space-valued function:

$$u : \mathcal{F}(Z) \to \mathcal{F}(H(D \times \Gamma)), \quad \text{where} \quad H(D \times \Gamma) = H^1_0(D) \otimes L^2_\pi(\Gamma).$$

(11)

Here, the function space $H(D \times \Gamma)$ is formed by the tensor product of two Sobolev spaces: $H^1_0(D)$ is the closure of the space of smooth functions with compact support in the Sobolev space of functions whose first weak derivatives are square integrable; and $L^2_\pi(\Gamma)$ is the Sobolev space of random functions with bounded second moments.

For the convenience of both analysis and computation, and thanks to the function-set identity [3], which will be shown to hold (see Theorem 2) due to the continuity of the mapping in (11), we will define the solution to (8) through the corresponding parametric problem:

$$- \nabla_x \cdot (a(x, y, z) \nabla_x u(x, y, z)) = f(x), \quad \text{in} \quad D \times \Gamma \times S_\alpha^z,$$

$$u(x, y, z) = 0, \quad \text{on} \quad \partial D \times \Gamma \times S_\alpha^z,$$

(12)

where, following the assumption (10),

$$0 < a_{\min} \leq a(x, y, z) \leq a_{\max} < \infty, \quad \forall x \in D, \; \forall y \in \Gamma, \; \forall z \in S_\alpha^z.$$  

(13)

Corresponding to the interpretation (11), we interpret the solution $u(x, y, z)$ to the parametric problem (12) as a Sobolev space-valued function on $S_\alpha^z$:

$$u : S_\alpha^z \to H(D \times \Gamma), \quad \text{where} \quad H(D \times \Gamma) = H^1_0(D) \otimes L^2_\pi(\Gamma).$$

(14)
As we will show in Section 4, the mapping in (14) is continuous and \( u \) is uniformly bounded on \( S_\alpha \), i.e. \( u \in L^\infty(S_\alpha; H^1_0(D) \otimes L^2_\pi(\Gamma)) \). This suggests that we may obtain the \( \alpha \)-cuts of the solution (11) to the fuzzy-stochastic problem (8) from the extrema of the solution (14) to the parametric problem (12) on the input joint \( \alpha \)-cuts \( S_\alpha \):

\[
S_\alpha(x, y) = \left[ \min_{x \in S_\alpha} u(x, y, z), \max_{x \in S_\alpha} u(x, y, z) \right] =: [u_\alpha(x, y), \bar{u}_\alpha(x, y)], \quad \alpha \in [0, 1].
\]

Note that the lower and upper limits of the \( \alpha \)-cuts in (15) are stochastic fields. We also notice that the solution \( u \) to (12) is \( \alpha \)-dependent. However, for ease of notation, we omit the explicit dependence on \( \alpha \) when no ambiguity arises.

The interpretation of the solution to fuzzy-stochastic PDE problems through Sobolev space-valued functions, i.e. the mappings (11) and (14), simplifies the analysis of such problems. Indeed, it transforms the original problem into a parametric one, as done in the case of pure stochastic and pure fuzzy PDEs; see e.g. \([5, 38, 16]\). We can therefore extend the proofs for well-posedness and regularity of deterministic (see e.g. \([20]\)) and stochastic (see e.g. \([5, 38, 39]\)) problems to fuzzy-stochastic problems.

4 Well-posedness and Regularity Analysis

In this section we will address the well-posedness and regularity of the fuzzy-stochastic problem (8) with a forcing function satisfying (9) and a PDE coefficient satisfying (10).

4.1 Well-posedness

We base the well-posedness analysis on the parametric representation (12) of problem (8) and consider the following weak formulation of problem (12) pointwise in \( z \in S_\alpha \).

Weak formulation I. Find \( u : S_\alpha \rightarrow H^1_0(D) \otimes L^2_\pi(\Gamma) \) such that \( \forall z \in S_\alpha \) and for all test functions \( v \in H^1_0(D) \otimes L^2_\pi(\Gamma) \) the following holds:

\[
\int_{D \times \Gamma} a(x, y, z) \nabla_x u(x, y, z) \cdot \nabla_x v(x, y) \pi(y) dy dx = \int_{D \times \Gamma} f(x) v(x, y) \pi(y) dy dx.
\]

Such a solution, provided it exists, is referred to as a weak solution to (12).

Theorem 2. Under the assumptions (9) and (13), there exists a unique weak solution \( u \in C^0(S_\alpha; H^1_0(D) \otimes L^2_\pi(\Gamma)) \) to the parametric problem (12). Moreover, the solution depends continuously on the data.

Proof. Thanks to the uniform positivity assumption (13), we have \( \forall z \in S_\alpha \):

\[
a_{\min} \int_{D \times \Gamma} |\nabla_x u(x, y, z)|^2 \pi(y) dy dx \leq \int_{D \times \Gamma} a(x, y, z) |\nabla_x u(x, y, z)|^2 \pi(y) dy dx,
\]

and hence, using the notation \( \| \cdot \|_H := \| \cdot \|_{H^1_0(D) \otimes L^2_\pi(\Gamma)} \) for the norm in \( H^1_0(D) \otimes L^2_\pi(\Gamma) \),

\[
\|u(z)\|_H^2 \leq \frac{1}{a_{\min}} \int_{D \times \Gamma} a(x, y, z) |\nabla_x u(x, y, z)|^2 \pi(y) dy dx, \quad \forall z \in S_\alpha.
\]
Moreover, due to the uniform boundedness assumption \( (13) \) and by Hölder inequality,
\[
\left| \int_{D \times \Gamma} a(x, y, z) \nabla_x u(x, y, z) \cdot \nabla_x v(x, y) \pi(y) \, dy \, dx \right| \leq a_{\text{max}} \| u(z) \|_H \| v \|_H.
\]
Hence, by the Lax-Milgram theorem \([20]\), there is a unique solution \( u(z) \in H^1_0(D) \otimes L^2_\pi(\Gamma) \) that satisfies \( (16) \). By setting \( v = u(z) \) in \( (16) \) and using Hölder and Poincaré inequalities on the right hand side, we obtain \( \forall z \in S^\alpha \),
\[
a_{\text{min}} \| u(z) \|_H^2 \leq \int_{D \times \Gamma} a(x, y, z) |\nabla_x u(x, y, z)|^2 \pi(y) \, dy \, dx
\]
\[
= \int_{D \times \Gamma} f(x) u(x, y, z) \pi(y) \, dy \, dx
\]
\[
\leq \| f \|_{L^2(D)} \| u(z) \|_{H^1_0(D)} \| L^2(\Gamma) \|_{L^2(\Gamma)} \leq C \| f \|_{L^2(D)} \| u(z) \|_H,
\]
where \( C \) is the Poincaré constant:
\[
\| u(x, y, z) \|_{L^2(D)} \leq C \| u(x, y, z) \|_{H^1_0(D)}, \quad \forall y \in \Gamma, \quad \forall z \in S^\alpha.
\]
This gives the energy estimate
\[
\| u(z) \|_H \leq \frac{C}{a_{\text{min}}} \| f \|_{L^2(D)}, \quad \forall z \in S^\alpha.
\]
Hence, thanks to assumption \( (13) \), the mapping \( u : S^\alpha \to H^1_0(D) \otimes L^2_\pi(\Gamma) \) is continuous and uniformly bounded \( u \in L^\infty(S^\alpha, H^0_0(D) \otimes L^2_\pi(\Gamma)) \). This completes the proof. \( \square \)

By Theorem 1 and the continuity of the mapping \( u : S^\alpha \to H^1_0(D) \otimes L^2_\pi(\Gamma) \) by Theorem 2, we have the function-set identity
\[
S^\alpha(x, y) = u(x, y, S^\alpha), \quad \forall x \in D, \quad \forall y \in \Gamma.
\]
The lower and upper limits of the \( \alpha \)-cuts of solution \( (11) \) to the fuzzy-stochastic problem \( (8) \) may then be obtained from the extrema of the solution \( (14) \) to the parametric problem \( (12) \). In particular, provided the solution \( u = u(x, y, z) \) is a continuous function for every fixed point \( (x, y) \in D \times \Gamma \), its \( \alpha \)-cuts \( S^\alpha(x, y) \) will be compact, nested intervals given by \( (15) \) and satisfying \( S^\alpha_{\alpha_2} \subset S^\alpha_{\alpha_1} \) with \( 0 \leq \alpha_1 \leq \alpha_2 \leq 1 \). We notice that in the absence of continuity, an \( \alpha \)-cut may be the union of disjoint intervals, and \( (15) \) may not hold.

As a corollary of Theorem 2 we have the following result.

**Corollary 1.** Consider the fuzzy-stochastic PDE problem \( (8) \) under the assumptions \( (9) \) - \( (10) \). There exists a unique solution \( \tilde{u} \in F(H^1_0(D) \otimes L^2_\pi(\Gamma)) \) that depends continuously on the data.

The compactness and the inclusion property of the \( \alpha \)-cuts \( (15) \) is crucial for efficient computations in fuzzy space for two reasons. First, it will allow us to restrict fuzzy computations to only a few \( \alpha \in [0, 1] \) levels, for example \( \alpha = 0, 0.25, 0.5, 0.75, 1 \). After computing \( S^\alpha \) for these \( \alpha \) values, the output membership function can be constructed by interpolation. Secondly, since the zero-cut \( S^\emptyset \) contains all other \( \alpha \)-cuts, i.e., \( S^\emptyset \subset S^\alpha, \forall \alpha \in (0, 1) \), we will need to construct the response surface of the solution \( u(x, y) \) only over the zero-cut. Hence we solve the parametric problem \( (12) \) over the zero-cut. The response surface of the solution over any desired \( \alpha \)-cut may then be obtained by restricting the zero-cut response surface to the desired \( \alpha \)-cut.
4.2 Parametric regularity

The convergence rate of spectral methods, such as sparse collocation, depends on the regularity of the solution to the parametric problem \((12)\) with respect to parameters both in stochastic space and in fuzzy space. We will therefore combine the stochastic and fuzzy spaces and let \(\xi = (y, z)\) be the parameter vector in the combined stochastic-fuzzy space

\[ \Xi := \Gamma \times S^\mathbb{R}_0 \subset \mathbb{R}^N, \quad N = m + n, \]

and study the \(\xi\)-regularity of the solution \(u(x, \xi)\) to \((12)\). We note that since the solution over any desired \(\alpha\)-cut \(S^\mathbb{R}_0\), with \(\alpha \in [0, 1]\), can be obtained by restricting the zero-cut solution to the desired \(\alpha\)-cut, we need to consider only the zero-cut \(S^\mathbb{R}_0\) in \(\Xi\). Indeed, the regularity of the solution over the zero-cut will determine the regularity of the solution over all other \(\alpha\)-cuts.

We view the solution to the parametric problem \((12)\) as a function of \(\xi \in \Xi\) taking values in a Sobolev space \(H^1_0(D)\) and study the regularity of the mapping \(u : \Xi \to H^1_0(D)\). In the light of this interpretation we consider the following weak formulation of problem \((12)\) pointwise in \(\xi \in \Xi\).

**Weak formulation II.** Find \(u : \Xi \to H^1_0(D)\) such that \(\forall \xi \in \Xi\) and for all test functions \(v \in H^1_0(D)\) the following holds:

\[ B[u, v] = f(v), \]

\[ B[u, v] := \int_D a(x, \xi) \nabla_x u(x, \xi) \cdot \nabla_x v(x) \, dx, \quad f(v) = \int_D f(x) \, v(x) \, dx. \]

By assumption \((13)\), the bilinear from \(B\) in \((18)\) is uniformly coercive and bounded, that is, \(\forall \xi \in \Xi\),

\[ |B[u(\xi), u(\xi)]| \geq a_{\min} \|u(\xi)\|_{H^1_0(D)}, \quad |B[u(\xi), v]| \leq a_{\max} \|u(\xi)\|_{H^1_0(D)} \|v\|_{H^1_0(D)}. \]

Moreover, by assumption \((9)\) and employing Hölder and Poincaré inequalities, the linear functional \(f(v)\) in \((18)\) is bounded in \(H^1_0(D)\),

\[ \|f\|_{H^{-1}(D)} = \sup_{v \in H^1_0(D)} \frac{|f(v)|}{\|v\|_{H^1_0(D)}} \leq \frac{\|f\|_{L^2(D)} \|v\|_{L^2(D)}}{\|v\|_{H^1_0(D)}} \leq C \|f\|_{L^2(D)} \leq \infty. \]

Hence, by the Lax-Milgram theorem, there exist a unique solution \(u \in L^\infty(\Xi; H^1_0(D))\).

For regularity analysis we will also need some regularity assumptions on the \(\xi\)-regularity of the PDE coefficient. Let \(k = (k_1, \ldots, k_N) \in \mathbb{N}^N\) be a multi-index with \(|k| = k_1 + \ldots + k_N\) and \(\mathbb{N}\) denoting the set of all non-negative integers including zero. We make the following regularity assumption on the PDE coefficient,

\[ \partial^{|k|}_\xi a(\cdot, \xi) := \frac{\partial^{k_1}_{\xi_1} \ldots \partial^{k_N}_{\xi_N} a(\cdot, \xi)}{\partial \xi_1 \ldots \partial \xi_N} \in L^\infty(D), \quad 0 \leq |k| \leq s, \quad \forall \xi \in \Xi, \]

where \(s \in \mathbb{N}\). The assumption \((21)\) states that \(a\) has \(s\) bounded mixed \(\xi\)-derivatives.

We now present the main regularity results. First, we state a component-wise result on the \(\xi_i\)-regularity of the solution for every component of \(\xi = (\xi_1, \ldots, \xi_N)\).

**Theorem 3.** For the solution of \((12)\) with the forcing term satisfying \((9)\) and the coefficient satisfying \((13)\) and \((21)\), we have for \(i = 1, \ldots, N\),

\[ \partial^k_{\xi_i} u \in L^\infty(\Xi; H^1_0(D)), \quad k \in \mathbb{N}, \quad 0 \leq k \leq s. \]
Proof. The case $k = 0$ follows directly from the weak formulation (17)-(18) and the Lax-Milgram theorem, thanks to (19) and (20). We now let $1 \leq k \leq s$ and $k$-times differentiate the weak formulation (17) with respect to the parameter $\xi_i$ and obtain

$$B[\partial_{\xi_i}^k u, v] = F_k(v),$$

where $B$ is the bilinear from given in (18) with $u$ replaced by $\partial_{\xi_i}^k u$, and $F_k(v)$ is a linear functional of $v$ which reads

$$F_k(v) := -\sum_{\ell=1}^{k} \binom{k}{\ell} \int_D \partial_{\xi_i}^\ell a(x, \xi) \cdot \nabla_x \partial_{\xi_i}^{k-\ell} u(x, \xi) \cdot \nabla_x v(x) \, dx. \tag{23}$$

The weak formulation (22)-(23) has a similar form to (17)-(18) with a slightly different right hand side. Hence, the existence of the weak solution $\partial_{\xi_i}^k u$, which determines the $\xi_i$-regularity of $u$ follows from the boundedness of the functional (23) in $\tilde{H}_0^1(D)$, which can easily be shown by induction on $k$.

We now state the following result on the mixed $\xi$-derivative of the solution.

**Theorem 4.** For the solution of the parametric problem (12) with the forcing term satisfying (9) and the coefficient satisfying (13) and (21), we have,

$$\frac{\partial^{\kappa^1} u}{\partial^{\kappa^1}_{\xi_1} \ldots \partial^{\kappa^N}_{\xi_N}} \in L^\infty(\Xi; H_0^1(D)), \quad \kappa \in \mathbb{N}^N, \quad 0 \leq |\kappa| \leq s.$$

**Proof.** The proof is an easy generalization of the previous theorem, following the arguments in the proof of Theorem 6 in [38].

### 5 Numerical Experiments

In this section we present two numerical examples. In both examples, we consider the following fuzzy-stochastic elliptic problem:

$$\frac{d}{dx} \left( a(x, y, \bar{z}) \frac{du}{dx}(x, y, \bar{z}) \right) = 0, \quad x \in [0, L], \quad y \in \Gamma, \quad \bar{z} \in \mathcal{F}(Z), \tag{24a}$$

$$u(0, y, \bar{z}) = 0, \quad a(L, y, \bar{z}) \frac{du}{dx}(L, y, \bar{z}) = 1. \tag{24b}$$

Here, the source of uncertainty is the parameter $a$ characterized by a fuzzy-stochastic field; see the two examples below. The solution to (24) is analytically given by

$$u(x, y, \bar{z}) = \int_0^x a^{-1}(\xi, y, \bar{z}) \, d\xi. \tag{25}$$

We note that in more complex problems in higher dimensions, the PDE problem needs to be discretized on the spatial domain by a numerical method. Here, we consider the one-dimensional problem (24) and focus on computations in fuzzy-stochastic spaces. More sophisticated numerical examples will be presented in forthcoming papers.
5.1 Numerical example 1

As an illustrative example, we let \( L = 2 \) and consider (24) with the fuzzy-stochastic parameter

\[
a(x, y, \tilde{z}) = a_1(x) a_2(y, \tilde{z}) = (2 + \sin(2\pi x/L)) e^{\tilde{z}_1 + y \tilde{z}_2}, \quad y \sim \mathcal{N}(0, 1), \quad \tilde{z} = (\tilde{z}_1, \tilde{z}_2).
\]

Here, \( a \) is a fuzzy-stochastic field, given by the product of a deterministic function \( a_1(x) \) and a fuzzy-stochastic function \( a_2(y, \tilde{z}) \), being a lognormal random variable with a fuzzy mean \( \tilde{z}_1 \) and a fuzzy standard deviation \( \tilde{z}_2 \), i.e. \( a_2(y, \tilde{z}) \sim \ln \mathcal{N}[\tilde{z}_1, \tilde{z}_2^2] \). We assume that \( \tilde{z}_1 \) and \( \tilde{z}_2 \) are triangular numbers, that is, they have triangular-shaped membership functions, uniquely described by triples \( (z_1^l, z_1^m, z_1^r) \), where \( z_1^l < z_1^m < z_1^r \) and such that \( \mu_{\tilde{z}_i}(z_1^l) = \mu_{\tilde{z}_i}(z_1^r) = 0 \) and \( \mu_{\tilde{z}_i}(z_1^m) = 1 \), with \( i = 1, 2 \).

The marginal \( \alpha \)-cuts of the two fuzzy variables are then given by

\[
S_{\alpha}^{\tilde{z}_1} = [z_1^l + \alpha (z_1^m - z_1^l), z_1^r - \alpha (z_1^r - z_1^m)] =: [a_\alpha, b_\alpha],
\]

\[
S_{\alpha}^{\tilde{z}_2} = [z_2^l + \alpha (z_2^m - z_2^l), z_2^r - \alpha (z_2^r - z_2^m)] =: [c_\alpha, d_\alpha].
\]

We consider two cases of non-interactive and fully interactive fuzzy variables. If \( \tilde{z}_1 \) and \( \tilde{z}_2 \) are non-interactive, following Definition 6, their joint \( \alpha \)-cut is

\[
S_{\alpha} = [a_\alpha, b_\alpha] \times [c_\alpha, d_\alpha].
\]

If \( \tilde{z}_1 \) and \( \tilde{z}_2 \) are fully interactive, following Definition 7, their joint \( \alpha \)-cut may be considered to be a piecewise linear curve in \( \mathbb{R}^2 \) with the Euclidean length

\[
L_\alpha = \sqrt{(a_\alpha - z_1^m)^2 + (c_\alpha - z_2^m)^2} + \sqrt{(b_\alpha - z_1^m)^2 + (d_\alpha - z_2^m)^2} =: L_{1,\alpha} + L_{2,\alpha},
\]

given by the collection of points

\[
S_{\alpha} = \{(z_1, z_2) = \varphi(s), s \in [0, L_\alpha]\},
\]

with the piecewise linear map

\[
\varphi(s) = \begin{cases} 
(a_\alpha + s|z_1^m - a_\alpha|/L_{1,\alpha}, c_\alpha + s|z_2^m - c_\alpha|/L_{1,\alpha}) & s \in [0, L_{1,\alpha}] \\
(z_1^m + s|b_\alpha - z_1^m|/L_{2,\alpha}, z_2^m + s|d_\alpha - z_2^m|/L_{2,\alpha}) & s \in [L_{1,\alpha}, L_\alpha]
\end{cases}.
\]

We consider the following QoIs

\[
\hat{Q}_1 = Q_1(\tilde{z}) = \mathbb{E}[u(L, y, \tilde{z})],
\]

\[
\hat{Q}_2(x) = Q_2(x, \tilde{z}) = \mathbb{E}[u(x, y, \tilde{z})],
\]

\[
\hat{Q}_3(y) = Q_3(y, \tilde{z}) = u(L, y, \tilde{z}).
\]

These QoIs cover a wide range of fuzzy quantities: \( \hat{Q}_1 \) is a fuzzy function; \( \hat{Q}_2 \) is a fuzzy field; and \( \hat{Q}_3 \) is a fuzzy-stochastic function. We now discuss the computation and visualization of each quantity in turn, based on Algorithm 1 and Algorithm 2.

The computation of \( \hat{Q}_1 \) amounts to computing its \( \alpha \)-cuts \( S_{\alpha}^{\hat{Q}_1} \) at various levels \( \alpha \in [0, 1] \). It requires evaluating the PDE solution (25) at a fixed point \( x = L \) and for \( M_s \) realizations \( \{y^{(i)}\}_{i=1}^{M_s} \) of \( y \sim \mathcal{N}(0, 1) \). The solution (25) at \( x = L \) and a fixed realization \( y^{(i)} \) is the integral of a fuzzy
field $a^{-1}(\xi, y(i), \tilde{z})$, with $\xi \in [0, L]$, over a crisp interval $[0, L]$. We approximate the crisp integral by a quadrature, such as the midpoint rule, and write

$$u(L, y(i), \tilde{z}) = \int_0^L a^{-1}(\xi, y(i), \tilde{z}) \, d\xi \approx h \sum_{j=1}^{N_h} a^{-1}(x_j, y(i), \tilde{z}), \quad x_j = (j - \frac{1}{2})h, \quad h = \frac{L}{N_h}.$$ 

Following Algorithm 2, we employ the standard Monte Carlo sampling and write

$$Q_1(\tilde{z}) = \mathbb{E}[u(L, y(i), \tilde{z})] \approx \frac{1}{M_s} \sum_{i=1}^{M_s} u(L, y(i), \tilde{z}) \approx \frac{h}{M_s} \sum_{i=1}^{M_s} \sum_{j=1}^{N_h} a^{-1}(x_j, y(i), \tilde{z}).$$

Note that we can alternatively employ other Monte Carlo sampling strategies [15, 27, 26, 37] or spectral stochastic techniques [57, 9, 56, 38] to approximate the expectation. Finally, following Algorithm 1, we perform the addition of $M_sN_h$ fuzzy functions $\{a^{-1}(x_j, y(i), \tilde{z})\}$, with $j = 1, \ldots, N_h$ and $i = 1, \ldots, M_s$, to get:

$$S^Q_{\alpha} = \left[ \min_{z \in S^h_\alpha} \left( \frac{h}{M_s} \sum_{i=1}^{M_s} \sum_{j=1}^{N_h} a^{-1}(x_j, y(i), z) \right), \max_{z \in S^h_\alpha} \left( \frac{h}{M_s} \sum_{i=1}^{M_s} \sum_{j=1}^{N_h} a^{-1}(x_j, y(i), z) \right) \right].$$

(26)

We notice that since all $M_sN_h$ fuzzy functions $\{a^{-1}(x_j, y(i), \tilde{z})\}$ are fully interactive, i.e. they are all functions of the same fuzzy vector $\tilde{z}$, the $\alpha$-cuts are obtained by the extrema of the sum of the terms, rather than the sums of the extrema. The latter would give conservative intervals overestimating the true $\alpha$-cuts. After computing various $\alpha$-cuts (26) at different $\alpha$ levels we can construct the membership function of $\tilde{Q}_1$ by interpolation. Figure 4 shows the membership functions of $\tilde{Q}_1$ for two cases of interaction, and with

$$\tilde{z}_1 = (z^l_1, z^m_1, z^r_1) = (1.00, 1.06, 1.20), \quad \tilde{z}_2 = (z^l_2, z^m_2, z^r_2) = (0.10, 0.13, 0.20).$$

(27)

We observe that $\mu_{\tilde{Q}_1}$, corresponding to non-interactive input fuzzy variables contains $\mu_{\tilde{Q}_1}$ corresponding to fully interactive fuzzy variables, as expected.

Figure 4: Membership functions of $\tilde{Q}_1$ when the input fuzzy variables are non-interactive (blue thick curves) and fully interactive (black think curves). The former contains the latter, as expected.

The computation of the fuzzy field $\tilde{Q}_2$ is similar to that of $\tilde{Q}_1$ for different $x$ values. Figure 5 shows the fuzzy field $\tilde{Q}_2$ versus $x \in [1.8, 2]$ with gray-scale colors representing the membership degrees, ranging from 0 (white color) to 1 (black color), in both non-interactive (left) and fully interactive (right) cases. We use the same values of parameters as those in (27). While both cases result in similar fuzzy fields, the field obtained by non-interactive fuzzy variables does contain the field obtained by fully interactive fuzzy variables, as expected.
The computation of the fuzzy-stochastic function $\tilde{Q}_3(y)$ amounts to computing its fuzzy CDF. We follow the approach outlined in Algorithm 2. For each fixed $Q_{3,0} \in [0.2, 0.6]$, we compute the $\alpha$-cuts $S^F_\alpha$ of $\tilde{F}(Q_{3,0}) = \mathbb{E}[\mathbb{1}_{[0.2, 0.6]}]$ and then construct the fuzzy CDF of $\tilde{Q}_3$. This corresponds to $q(y, \tilde{z}) = \mathbb{1}_{\tilde{Q}_3(y, \tilde{z}) \leq Q_{3,0}}$ in Algorithm 2. We use the parameter values in (27). Figure 6 shows the fuzzy CDF of $\tilde{Q}_3$ with gray-scale colors representing the membership degrees, ranging from 0 (white color) to 1 (black color), in both non-interactive (left) and fully interactive (right) cases. To each membership degree (or $\alpha$-level), there corresponds one lower and one upper envelope, forming a p-box. We observe a nested set of p-boxes at different $\alpha$-levels: p-boxes at upper levels of plausibility/possibility (higher $\alpha$-levels) are contained inside p-boxes at lower levels of plausibility/possibility (lower $\alpha$-levels). Again, we observe that $\tilde{F}(Q_3)$ corresponding to non-interactive input fuzzy variables contains $\tilde{F}(Q_3)$ corresponding to fully interactive fuzzy variables.

5.2 Numerical example 2

We next consider an engineering problem in materials science: the response of fiber-reinforced polymers to external forces. This example demonstrates the applicability of fuzzy-stochastic PDEs.
to real-world problems. We will in particular show how fuzzy-stochastic PDE parameters can be constructed based on real measurement data. The construction will be justified and validated by showing that the PDE-generated outputs accurately capture variations in the true quantities obtained by real data. To this end, we consider a small piece of HTA/6376 fiber composite [2, 4, 5], consisting of four plies containing 13688 carbon fibers with a volume fraction of 63% in epoxy matrix. Figure 7(top) shows a map of fibers in an orthogonal cross section of the composite obtained by an optical microscope. The modulus of elasticity of the composite constituents, given by the manufacturer, are $a_{\text{fiber}} = 24$ [GPa] and $a_{\text{matrix}} = 3.6$ [GPa]. We process this binary map and convert it into a form suitable for statistical analysis, based on which the modulus of elasticity of the composite $a$, which appears in [24], will be characterized. We follow [3] and discretize the rectangular cross section of the composite into a uniform mesh of square pixels of size $1 \times 1 \mu m^2$. We then construct a binary data structure for the composite’s modulus of elasticity, where we mark the presence or absence of fiber at every pixel by 1 ($a = a_{\text{fiber}}$) or 0 ($a = a_{\text{matrix}}$), respectively, assuming that fibers are perfectly circular. We next divide the rectangular domain into 50 thin horizontal strips (or bars) of width $10 \mu m$. This gives us 50 thin bars of length 1700 $\mu m$, labeled $i = 1, \ldots, 50$. Each bar is divided into 170 square elements of size $10 \times 10 \mu m^2$, labeled $j = 1, \ldots, 170$. On each element $j$, we take the harmonic average over its $10 \times 10$ pixels and compute a value $a_i(x_j)$ for modulus of elasticity. We repeat the process for all 50 bars and all 170 elements of each bar and obtain 50 one-dimensional discrete samples $\{a_i(x_j)\}_{i=1}^{50}$ of the uncertain parameter $a$ at the discrete points $\{x_j\}_{j=1}^{170}$; see Figure 7(bottom). The above process is accurate within 1% in predicting the overall volume fraction obtained by an analytic approach. We refer to [3] for details.

Figure 7: Top: a binary optical image of a small piece of a fiber composite. Bottom: modulus of elasticity of composite over a regular mesh of $170 \times 50$ square elements of size $10 \times 10 \mu m^2$.

Motivated by the form of the exact solution [25], we perform statistical analysis directly on the compliance $b = a^{-1}$. We approximate the first four moments of $b$ by sample averaging using the samples $\{b_i(x_j)\}_{i=1}^{50} = \{a_i^{-1}(x_j)\}_{i=1}^{50}$. At each discrete point $x_1, \ldots, x_{170}$, we use these 50 samples and compute the sample mean $z_1(x_j)$, sample standard deviation $z_2(x_j)$, sample skewness $z_3(x_j)$, and sample excess kurtosis $z_4(x_j)$. Figure 8 shows the sample moments of $b$ versus $x$ and their histograms.

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A common engineering practice is to model material parameters, such as the compliance $b$, by stationary Gaussian random fields; see e.g. [58, 59, 34, 43]. However, as Figure 8 shows, the moments are not constant and vary in $x$, and hence the parameter $b$ cannot be accurately represented by stationary random fields. Moreover, the field is not Gaussian, since its skewness is not zero. One option within the framework of precise probability is to construct a non-stationary non-Gaussian random field. This option would heavily rely on the availability of abundant high-quality data to correctly capture the highly oscillatory moments. In reality such data are not available, for example when characterizing permeability of porous rock layers or compliance of composites containing millions of fibers. Even if—in the non-realistic absence of epistemic uncertainty—we do represent $b(x)$ by a non-stationary random field, we would face a stochastic multiscale problem that may not be tractable. This is due to the well-known fact that stochastic homogenization, necessary to treat random multiscale parameters, is applicable only to stationary fields [48]. A second option is to construct an imprecise probabilistic model by considering a family of distributions, as done for example in interval probability [54] and optimal UQ [42]. Although these approaches can handle epistemic uncertainty, they may suffer from the loss of information and the non-propagation of uncertainty across multiple scales. Intuitively, this is because such models may not capture all input information that is available to us and hence cannot propagate the whole information. From the sample moments in Figure 8 it is obvious that one would lose information if the moments are modeled by intervals. For instance if one models the first moment with the interval $[0.123, 0.156]$, then the information that the value 0.135 is more possible/plausible would be lost. See also [42] for
an illustrative example of the non-propagation of uncertainty across multiple scales. Among the imprecise probabilistic models, second-order hierarchical models \cite{23} may be capable of treating this multiscale problem. In this case one may be able to model \( b \) by a random field with random moments. Here, we propose another alternative beyond the framework of probability. In order to accurately model and propagate uncertainty and afford multiscale strategies, we propose to model the parameter \( b \) by a fuzzy-stationary random field as follows.

We first fuzzify the moments of \( b \): we use the histograms of the sample moments to construct membership functions \( \mu_{\tilde{z}_1}, \mu_{\tilde{z}_2}, \mu_{\tilde{z}_3}, \mu_{\tilde{z}_4} \). This can be done, for instance, by the method of least squares and with piecewise-linear regression functions (the thick blue lines in Figure 8). We then normalize the regression functions so that the maximum membership function value is one. It is to be noted that this procedure generates an initial draft for membership functions. We may need to conduct a subsequent modification and make additional corrections, for instance if the initial draft is not quasi-concave. Here, we use five \( \alpha \)-levels \( (0, 0.25, 0.5, 0.75, 1) \) for the construction and obtain four decagonal fuzzy variables, described by their ten vertices

\[
\begin{align*}
\tilde{z}_1 &= (0.1222, 0.1249, 0.1277, 0.1304, 0.1330, 0.1360, 0.1388, 0.1445, 0.1502, 0.1559), \\
\tilde{z}_2 &= (0.0200, 0.0217, 0.0236, 0.0236, 0.0285, 0.0345, 0.0360, 0.0360, 0.0408, 0.0430), \\
\tilde{z}_3 &= (0, 0.25, 0.50, 0.75, 1.00, 1.20, 1.25, 1.50, 1.75, 2.00), \\
\tilde{z}_4 &= (-1.00, -0.55, -0.20, 0, 0.50, 1.00, 1.50, 2.00, 3.30, 4.50).
\end{align*}
\]

We note that the four fuzzy variables are fully interactive, because the four moments are obtained from the same set of data \( \{b_i(x_j)\}_{i=1}^{50} \) and hence are directly related to each other, that is, higher moments are obtained from lower moments. This will result in a reduction in fuzzy space dimension. While we have a vector of four fuzzy variables \( \tilde{z} = (\tilde{z}_1, \tilde{z}_2, \tilde{z}_3, \tilde{z}_4) \), their joint \( \alpha \)-cut \( S_{\alpha}^2 \) is a piecewise linear one-dimensional curve embedded in \( \mathbb{R}^4 \). Similar to the numerical example 1 and using the arc length parameterization of the curve, we can represent \( S_{\alpha}^2 \) by a piecewise linear map.

We then construct a fuzzy-stochastic translation field to model the compliance:

\[
b(x, y, \tilde{z}) = \Psi^{-1}(\tilde{z}) \circ \Phi(G(x, y)). \tag{28}
\]

Here, \( \Psi(\tilde{z}) \) is the CDF of a four-parameter beta distribution determined by the four fuzzy moments, \( \Phi \) is the standard normal CDF, and \( G(x, y) \) is a standard Gaussian field, approximated by the truncated KL expansion: \( G(x, y) \approx \sum_{j=1}^{m} \sqrt{\lambda_j} \phi_j(x) y_j \), with \( y_j \sim \mathcal{N}(0,1) \) and the eigenpairs \( \{(\lambda_j, \phi_j(x))\}_{j=1}^{m} \) of the deterministic covariance

\[
C(x_1, x_2) = \exp \left( -\frac{|x_1 - x_2|^p}{2\ell^2} \right), \quad p = 2, \quad \ell = 20 \mu m. \tag{29}
\]

We note that the selection of the covariance function and its parameters, such as the exponent \( p \) and correlation length \( \ell \), must be based on a systematic calibration-validation strategy; see \cite{6}. As we will see in Figure 9, the choice (29) here delivers output quantities which fit the true quantities. Here, we choose \( m = 27 \) KL terms to preserve 90\% of the unit variance of the Gaussian field \( G \).

The construction (28) has several advantages. First, it benefits from the simplicity of working with a stationary Gaussian field \( G(x, y) \). Moreover, by applying the inverse of \( \Psi \) on \( \Phi(G) \in \Xi \).
For each group we then compute precise probability. For instance for \( N = 15 \) groups of samples, we obtain a field that achieves the target marginal fuzzy CDF \( \Psi(\tilde{z}) \). Finally, since the fuzzy moments are \( x \)-independent, the field (28) may be thought of as a fuzzy-stationary random field. One can hence employ global-local homogenization methods [3] and perform multiscale computations if needed.

We now let \( L = 1.7 \times 10^{-3} \text{m} \) and consider the problem (24) with the fuzzy-stochastic parameter \( a = b^{-1} \) given by (28). Hence, the analytical solution (25) reads

\[
\tilde{Q}_j(x, \tilde{y}, \tilde{z}) = \int_0^1 b(\xi, y, \tilde{z}) d\xi.
\]

We consider the following QoIs

\[
\tilde{Q}_4(x) = Q_4(x, \tilde{z}) = \mathbb{E}[u(x, y, \tilde{z})],
\]

\[
\tilde{Q}_5(y) = Q_5(y, \tilde{z}) = u(L/4, y, \tilde{z}),
\]

\[
\tilde{Q}_6 = Q_6(\tilde{z}) = P(u(L/4, y, \tilde{z}) \geq u_{cr}).
\]

Here, \( \tilde{Q}_4 \) is a fuzzy field, \( \tilde{Q}_5 \) is a fuzzy-stochastic function, and \( \tilde{Q}_6 \) is a fuzzy failure probability. We now discuss the computation of the above three quantities.

The computation of \( \tilde{Q}_4 \) and \( \tilde{Q}_5 \) is similar to that of \( \tilde{Q}_2 \) and \( \tilde{Q}_3 \) in Section 5.1. Figure 9 shows the fuzzy field \( \tilde{Q}_4(x) \) versus \( x \in [0,1000] \mu \text{m} \) (left) and the fuzzy CDF of \( \tilde{Q}_5 \) for three membership degrees \( \alpha = 0,0.5,1 \). We also compute and plot the “true” quantities directly obtained by the real data, i.e. the 50 discrete samples, as follows. First, we choose \( N_b = 20 \) groups of samples, where each group consists of \( M_b = 15 \) different, randomly selected samples out of 50 discrete samples. For each group we then compute \( M_b \) samples of the true quantity and then obtain their expected value (to compare with \( \tilde{Q}_4 \)) and their CDF (to compare with \( \tilde{Q}_5 \)). This gives us a set of \( N_b \) benchmark solutions, referred to as the “truth”. It is to be noted that the variations in true quantities reflect the presence of non-random uncertainty and justify the need for models beyond precise probability. For instance for \( Q_5 \) we obtain a range of distributions, hence forming a nested set of p-boxes, instead of one single distribution that one may obtain in the absence of non-random uncertainty. Figure 9 shows how accurately the computed quantities obtained by the proposed fuzzy-stochastic PDE model capture the variations in the true quantities.

![Figure 9](image-url)

Figure 9: The fuzzy field \( \tilde{Q}_4(x) \) versus \( x \) (left) and the fuzzy CDF of \( \tilde{Q}_5 \) (right). For comparison, the true quantities (thin turquoise curves) are included.

The quantity \( \tilde{Q}_6 \) is the fuzzy probability of failure that would occur when the displacement \( u \) at \( x = L/4 \) reaches a critical value \( u_{cr} \). At every fixed \( \alpha \)-level, we first uniformly discretize the one-dimensional joint \( \alpha \)-cut \( S^2_\alpha \) into \( M_f \) discrete points \( \{ z^{(k)} \}_{k=1}^{M_f} \in S^2_\alpha \). We next set \( g(y, \tilde{z}) := u_{cr} - u(L/4, y, \tilde{z}) \) and follow Algorithm [2] with \( q(y, \tilde{z}) = \mathbb{I}_{g(y, \tilde{z}) \leq 0} \). We use Monte Carlo sampling with \( M_s \) realizations \( \{ y^{(i)} \}_{i=1}^{M_s} \) to approximate...
\[ Q_6(z^{(k)}) = \mathbb{E}[\mathbb{I}_{[q(y,z^{(k)}) \leq 0]}] \approx \frac{1}{M_s} \sum_{i=1}^{M_s} \mathbb{I}_{[q(y^{(i)},z^{(k)}) \leq 0]}, \quad k = 1, \ldots, M_f. \]  

The output \( \alpha \)-cut for \( \tilde{Q}_6 \) is then obtained by

\[ S_{\alpha}^{\tilde{Q}_6} = \left[ \min_k Q_6(z^{(k)}), \max_k Q_6(z^{(k)}) \right]. \]  

Figure 10 shows the membership function of \( \tilde{Q}_6 \) obtained from the \( \alpha \)-cuts given in (30)-(31) computed for five \( \alpha \) levels \( \alpha = 0, 0.25, 0.75, 1 \), with \( u_{ct} = 6.9 \times 10^{-5} \mu \text{m}, M_f = 181, \) and \( M_s = 10^4 \).

Figure 10: The membership function of the fuzzy failure probability \( \tilde{Q}_6 \) and its five \( \alpha \)-cuts.

Note that the additional nuanced information given through nested intervals at different levels of possibility of \( \tilde{Q}_6 \) is a direct result of the propagation of additional nuanced information available in the statistical moments in Figure 8. Such additional information may not be accounted for and hence would not be propagated though other imprecise probabilistic models, such as interval probabilities and optimal UQ. This is particularly important in “certification problems”, where we need to certify or decertify a system of interest. To illustrate this, let \( \varepsilon_{\text{TOL}} = 0.1 \) be the greatest acceptable failure probability \( \tilde{Q}_6 \), that is, the system is safe if \( \tilde{Q}_6 \leq \varepsilon_{\text{TOL}} \) and unsafe if \( \tilde{Q}_6 > \varepsilon_{\text{TOL}} \). Suppose that the lower and upper bounds of the zero-cut of \( \tilde{Q}_6 \), i.e. \( 0 \) and \( 0.2284 \), represents the crisp lower and upper bounds obtained by an imprecise probabilistic approach. In this case since \( 0 < \varepsilon_{\text{TOL}} = 0.1 < 0.224 \), then we cannot decide on the safety of the system, unless additional information will be provided. However, the additional nuanced information provided by the lower and upper bounds at different levels of possibility (i.e. different \( \alpha \)-levels) may help decision-makers. In fact the highest level of possibility (the most possible scenario) corresponding to the 1-cut in Figure 10 suggests that the system may be safe.

5.3 Computational cost

Consider a fuzzy-stochastic function \( q(y, \tilde{z}) \), for example obtained by applying a combination of algebraic, integral, and differential operators on the solution \( u(x, y, \tilde{z}) \) to a fuzzy-stochastic PDE problem. Assume that we are interested in computing a fuzzy quantity \( \tilde{Q} = \mathbb{E}[q(y, \tilde{z})] \). The computation of one \( \alpha \)-cut \( S_{\alpha}^{\tilde{Q}} \) requires \( M_f \) function evaluations \( Q(z^{(k)}) = \mathbb{E}[q(y, z^{(k)})] \) at \( M_f \) discrete points \( \{z^{(k)}\}_{k=1}^{M_f} \subseteq S_{\alpha}^{\tilde{Q}} \). At each discrete point the expectation of \( q \) needs to be approximated by a sampling technique using \( M_s \) samples. In total we need to solve \( M = M_f M_s \) deterministic PDE problems. The size of \( M \) depends mainly on the number of random variables \( m \), the number of
fuzzy variables \( n \), and the regularity of \( q \) with respect to \( y \) and \( z \). When \( M \) is very large, e.g. in the absence of high regularity or when \( m \) and \( n \) are large, the computations may be prohibitively expensive. There are however practical situations where fuzzy-stochastic computations are feasible:

1. In many applications we have a low-dimensional fuzzy space, i.e. \( n \ll m \). A typical example is when we model an uncertain parameter, such as the compliance, by a hybrid fuzzy-stochastic field. In this case, \( n \) is usually 1 (if the moments are fully interactive) or 2-4 (if we use 2-4 non-interactive moments), while \( m \) may be rather large depending on the correlation length of the field. As a result, fuzzy-stochastic computations are usually not much more expensive compared to solving purely stochastic problems.

2. When \( q \) is highly regular with respect to \((y, z)\) we can employ spectral methods on sparse grids instead of Monte Carlo sampling strategies in order to speed up the computations. The stochastic and fuzzy spaces are different considering we extract statistical information in the stochastic space and perform optimization in the fuzzy space. Therefore, thanks to high regularity, we may build a surrogate model of \( q(y, z) \) on the tensor product of two separate sparse grids (one on each space) using sparse interpolating polynomials; see e.g. \cite{38, 39}. This type of separation affords efficient extraction of statistical information and sparse optimization \cite{28, 31}.

3. Current probabilistic models are usually not applicable to multiscale problems with highly oscillatory uncertain parameters. It is well known that stochastic homogenization, necessary to treat multiscale stochastic parameters, is applicable only to stationary random fields \cite{48}, while in many multiscale problems the moments are not constant. Another related problem with imprecise probabilistic models is the non-propagation of uncertainty across multiple scales \cite{12}. In such cases, a hybrid fuzzy-stochastic model may be considered as a feasible and accurate approach to treat the problem; see \cite{3} where a fuzzy-stochastic multiscale approach is presented for fiber composite polymers.

4. Due to the non-intrusiveness of the numerical methods, the \( M \) deterministic problems can be distributed and solved independently on parallel computers.

We are currently working on the development of efficient numerical methods for solving fuzzy-stochastic PDE problems. We are also exploring more applications involving time-dependent PDEs to be presented elsewhere.

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