Reidemeister-Schreier's Algorithm for 2-Coverings of Seifert Manifolds

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Abstract. It is classical that given any Seifert structure on $N$, Reidemeister-Schreier’s algorithm produces a presentation of all index 2 subgroups of $\pi_1(N)$, described as the fundamental group of some Seifert manifolds. The new result of this article is concise formulas that gather all possible cases.

1. Introduction

The index 2 subgroups of $\pi_1(N)$ are kernel of epimorphisms $\varphi : \pi_1(N) \to \mathbb{Z}_2$. When $N$ is a Seifert manifold (described by its Seifert invariants) and one needs a list of all its 2-coverings, it is necessary to explicit a combinatoric way to gather together all the data. Theorems 1 and 3 give Reidemeister-Schreier concise answers, [2], [6].

The notations are given in the section after this introduction. Section 3 studies the situation where the morphism $\varphi$ maps the generator corresponding to the regular fiber to 1. This is Theorem 1. Theorem 3 stating the result when $\varphi$ maps the generator corresponding to the regular fiber to 0 is expressed in the fourth section. The following subsections prove this theorem. In the first subsection, a crucial lemma (Lemma 10) proves that if two morphisms from $\pi_1(N)$ to $\mathbb{Z}_2$ map $m \geq 0$ exceptional fibres to 1 and all the other generators to 0 then their kernels are isomorphic. This gives importance to Theorem 11 which explicits the kernel with these hypothesis. The second subsection studies the situation where $\varphi$ maps some generators corresponding to the basis to 1 and all the other generators to 0. Theorem 3 is proved.

Each index 2 subgroup of $\pi_1(N)$ is the fundamental group of a Seifert manifold $M$ and an associated free involution $\tau$. The motivation of this study came to us through the study of Borsuk-Ulam type theorem for $(M, \tau)$ [3], [1].

2. Seifert Invariants for the Kernel

$N$ is any Seifert manifold (orientable or not), as introduced in [5].

Following the notations of [4], from now on, $N$ will be a Seifert manifold described by a list of Seifert invariants

$\{e; (\epsilon, g); (a_1, b_1), \ldots, (a_n, b_n)\}$.

We do not need them to be “normalized” (as defined in [5] or [4]): we only assume that $e$ is an integer, the type $\epsilon$ is detailed below, $g$ is the genus of the base surface, and for each $k$, the integers $a_k, b_k$ are coprime and $a_k \neq 0$.

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*Mathematics Subject Classification 2010 : Primary 57M10, Secondary 57M05, 57S25

Keywords and phrases : covering, Seifert manifolds*
Such invariants give the following presentation of the fundamental group of $N$:

$$\pi_1(N) = \left\langle s_1, \ldots, s_n \mid [s_k, h] \text{ and } s_k^{a_k} h^{b_k}, \quad 1 \leq k \leq n ; v_1, \ldots, v_{g'}, \quad h v_j h^{-1} h^{-e_j}, \quad 1 \leq j \leq g' ; h^{-e} s_1 \ldots s_n V \right\rangle.$$ 

- The type $\epsilon$ of $N$ equals:
  - $o_1$ if both the base surface and the total space are orientable (which forces all $\varepsilon_j$’s to equal 1);
  - $o_2$ if the base is orientable and the total space is non-orientable, hence $g \geq 1$ (all $\varepsilon_j$’s are assumed to equal $-1$);
  - $n_1$ if both the base and the total space are non-orientable (hence $g \geq 1$) and moreover, all $\varepsilon_j$’s equal 1;
  - $n_2$ if the base is non-orientable (hence $g \geq 1$) and the total space is orientable (which forces all $\varepsilon_j$’s to equal $-1$);
  - $n_3$ if both the base and the total space are non-orientable and moreover, all $\varepsilon_j$’s equal $-1$ except $\varepsilon_1 = 1$, and $g \geq 2$;
  - $n_4$ if both the base and the total space are non-orientable and moreover, all $\varepsilon_j$’s equal $-1$ except $\varepsilon_1 = \varepsilon_2 = 1$, and $g \geq 3$.

- The orientability of the base and its genus $g$ determine the number $g'$ of the generators $v_j$’s and the word $V$ in the last relator of $\pi_1(N)$:
  - when the base is orientable, $g' = 2g$ and $V = [v_1, v_2] \ldots [v_{2g-1}, v_{2g}]$;
  - when the base is non-orientable, $g' = g$ and $V = v_1^2 \ldots v_g^2$.

- The generator $h$ corresponds to the generic regular fibre.
- The generators $s_k$ for $1 \leq k \leq n$ correspond to (possibly) exceptional fibres.

The subgroups of index 2 of $\pi_1(N)$ are the kernel of epimorphism $\varphi : \pi_1(N) \to \mathbb{Z}_2$. The two next subsections describe $\text{Ker}(\varphi)$ as the fundamental group of some Seifert manifold given by a similar list of invariants, when $\varphi(h) = 1$ (Theorem 1) and when $\varphi(h) = 0$ (Theorem 4).

### 3. If $\varphi$ Maps $h$ to 1

**Theorem 1.** If $\varphi$ maps $h$ to 1 then its kernel is the fundamental group of the Seifert manifold given by the following invariants:

$$\left\{ \frac{e - m}{2} - m' ; (\epsilon, g), (a_1, b'_1), \ldots, (a_n, b'_n) \right\},$$

where $b'_k = \begin{cases} \frac{b_k}{2} & \text{if } b_k \text{ is even} \\ \frac{a_k + b_k}{2} & \text{if } b_k \text{ is odd} \end{cases}$.

$m'$ is the number of odd $b_k$’s, and

$$\begin{cases} m' = 0 & \text{if } \epsilon = o_1, n_2 \\ m' \equiv \sum \varphi(v_j) \text{(mod 2)} & \text{if } \epsilon = o_2, n_1 \\ m' \equiv \varphi(v_1) \text{(mod 2)} & \text{if } \epsilon = n_3 \\ m' \equiv \varphi(v_1) + \varphi(v_2) \text{(mod 2)} & \text{if } \epsilon = n_4. \end{cases}$$

Note that in the non-orientable cases, $m'$ is only determined modulo 2, which is sufficient to determine the Seifert manifold.

**Proof.** Necessarily, all $a_k$’s are odd, $\varphi(s_k) = b_k \text{(mod 2)}$, and $e + m$ is even. Let us choose a presentation of $\pi_1(N)$ adapted to $\varphi$ by keeping $h$ untouched but taking...
new generators $s'_k$, $v'_j$ mapped to 0 by $\varphi$:
\[
s'_k = \begin{cases} 
  s_k & \text{if } b_k \text{ is even} \\
  h^{-1}s_k & \text{if } b_k \text{ is odd}
\end{cases}
\]
\[
v'_k = \begin{cases} 
  v_k & \text{if } \varphi(v_k) = 0 \\
  h^{-1}v_k & \text{if } \varphi(v_k) = 1.
\end{cases}
\]

The new presentation of $\pi_1(N)$ corresponds to the Seifert invariants
\[
\{e - m - 2m'; (e, g); (a_1, 2b'_1), \ldots, (a_n, 2b'_n)\},
\]
where the $b'_k$'s and $m$ are as stated, and
\[
m' = \begin{cases} 
  0 & \text{if } \epsilon = o_1, n_2 \\
  \#\{j \text{ odd } | \varphi(v_j) = 1\} - \#\{j \text{ even } | \varphi(v_j) = 1\} & \text{if } \epsilon = o_2 \\
  \#\{j \mid \varphi(v_j) = 1 \text{ and } \varepsilon_j = 1\} & \text{if } \epsilon = n_1, n_2, n_3, n_4
\end{cases}
\]
(hence $m'$ fulfills the condition of the statement).

Choosing $q = h$, Reidemeister-Schreier’s algorithm produces a presentation of $\text{Ker}(\varphi)$ with
- **generators:**
  - for $1 \leq k \leq n$, $(y_k, y'_k) = (s'_k,q s'_k q^{-1})$
  - for $1 \leq j \leq g'$, $(x_j, x'_j) = (v'_j, q v'_j q^{-1})$
  - $(z, z') = (h^{-1}q, qh)$
- **relations:**
  - $z = 1$
  - for $1 \leq k \leq n$, $y'_k = y_k$, $[y_k, z'] = 1$ and $y_k^n z b'_k = 1$
  - for $1 \leq j \leq g'$, $x_j z x_j^{-1} z'^{-1} = 1$ and $x'_j = \begin{cases} 
  x_j & \text{if } \varepsilon_j = 1 \\
  z'x_j & \text{if } \varepsilon_j = -1
\end{cases}$
  - $y_1 \ldots y_n W = z^{(\epsilon + m + 2m')/2}$, with
    \[
    W = \begin{cases} 
      [x_1, x_2] \ldots [x_{2g-1}, x_{2g}] & \text{if } \epsilon = o_1, o_2 \\
      x_1^2 \ldots x_g^2 & \text{if } \epsilon = n_1, n_2, n_3, n_4
    \end{cases}
    \]

Eliminating the redundant generators $z, y'_k, x'_k$ yields the result. \hfill \Box

4. If $\varphi$ maps $h$ to 0

4.0.1. Results of the two next subsections. Denote by $m$ the number of $s_k$’s mapped to 1 by $\varphi$ and assume (if $m > 0$) that these $m$ $s_k$’s are the first ones. (This reordering of the $s_k$’s may be achieved by an obvious change of presentation of $\pi_1(N)$, using repeatedly the equation $ss' = (ss's^{-1})s$.) The next theorem announces the conclusion of Theorems 11 and 15, which will be proved in the two next subsections. The following notations will be used to state the results.

**Notation 2.** The notations $F_{OC}$ and $F_m$ will respectively denote
\[
F_{OC} = (a_1, b_1), (a_1, -b_1), (a_2, b_2), (a_2, -b_2), \ldots, (a_n, b_n), (a_n, -b_n)
\]
\[
F_m = (a_1/2, b_1), (a_2/2, b_2), \ldots, (a_m/2, b_m),
\]
\[
(a_{m+1}, b_{m+1}), (a_{m+1}, b_{m+1}), (a_{m+2}, b_{m+2}), (a_{m+2}, b_{m+2}), \ldots, (a_n, b_n), (a_n, b_n)
\]

**Theorem 3.** If $\varphi(h) = 0$, denoting by $m$ the number of $s_k$’s mapped to 1 by $\varphi$ and assuming these are the first ones, $\text{Ker}(\varphi)$ is the fundamental group of the Seifert manifold given by the following invariants:
- **(Orientation covers)** If $m = 0$ and
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- if $\epsilon = o_2$ and $\varphi$ maps all $v_j$’s to 1: $\{0; (o_4, 2g-1); F_{OC}\}$
- if $\epsilon = n_1$ and $\varphi$ maps all $v_j$’s to 1: $\{0; (o_1, g-1); F_{OC}\}$
- if $\epsilon = n_3$ and $\varphi$ sends only $v_1$ to 1, or if $\epsilon = n_4$ and $\varphi$ sends only $v_1, v_2$ to 1: $\{0; (n_2, 2g-2); F_{OC}\}$

• (Exotic cases) If $m = 0$ and $\epsilon = n_2, n_3, n_4$ and $\varphi$ maps all $v_j$’s to 1:
  - if $\epsilon = n_2$: $\{2e; (o_1, g-1); F_0\}$
  - if $\epsilon = n_3, n_4$: $\{0; (o_2, g-1); F_0\}$

• (Ordinary cases) In all other cases: $\{e'; (e', G); F_m\}$ with

$$e' = \begin{cases} 2e & \text{if } \epsilon = o_1, n_2 \\ 0 & \text{if } \epsilon = o_2, n_1, n_3, n_4 \end{cases} \quad e' = \begin{cases} \epsilon & \text{if } \epsilon = o_1, o_2, n_1, n_2, n_4 \\ n_4 & \text{if } \epsilon = n_3 \end{cases}$$

and $G = \begin{cases} \frac{m}{2} - 1 + 2g & \text{if } \epsilon = o_1, o_2 \\ m - 2 + 2g & \text{if } \epsilon = n_1, n_2, n_3, n_4. \end{cases}$

4.0.2. If $\varphi$ maps $h$ to 0 but maps some $s_k$’s to 1. Later on (Lemma 10), we shall reorder the $v_j$’s in the same spirit as we did for the $s_k$’s, and show that the isomorphism type of $\text{Ker}(\varphi)$ is in fact independent from the values of $\varphi$ on the $v_j$’s, which reduces the computation of $\text{Ker}(\varphi)$ to the particular case where $\varphi$ vanishes on all $v_j$’s. But before performing such a reduction, we need to show that in that particular case, $\text{Ker}(\varphi)$ is the fundamental group a non-orientable Seifert manifold whenever $N$ is non-orientable.

So, let us first compute $\text{Ker} \varphi$ in the particular case where $\varphi$ vanishes on all $v_j$’s. The following lemma is an intermediate step for this computation: it gives a presentation of $\text{Ker} \varphi$ where the exceptional fibers gently appear, but where the long relation $W$ and the ± signs may still be of a “hybrid” form.

**Lemma 4.** If $\varphi$ maps $s_1, \ldots, s_m$ to 1 and all other generators to 0 then a presentation of its kernel is:

$$\text{Ker}(\varphi) = \left\langle s'_1, \ldots, s'_{n'} \mid \begin{array}{ll} s'_{k}, z & 1 \leq k \leq n' \\ v'_1, \ldots, v'_{g''} & z^{-2} s'_1 \ldots s'_m W \end{array} \right\rangle,$$

where

- $n' = m + 2(n - m)$,
- $(a'_k, b'_k) = (a_k/2, b_k)$ for $k \leq m$,
- $(a'_k, b'_k) = (a'_{k+n-m}, b'_{k+n-m}) = (a_k, b_k)$ for $m < k \leq n$,
- $g'' = (m - 2) + 2g'$
- $\varepsilon'_j = 1$ for $j \leq m - 2$,
- $\varepsilon'_{m-2+j} = \varepsilon'_{m-2+g'+j} = \varepsilon_j$ for $1 \leq j \leq g'$

• $W = \begin{cases} [v'_1, v'_2, \ldots, v'_{g'-1}, v'_{g''}] & \text{if } \epsilon = o_1, o_2, \\ [v'_1, v'_2, \ldots, v'_{m-3}, v'_{m-2}, v'_{m-1}, v'_{m}, \ldots, v'_{2}^{g''}] & \text{if } \epsilon = n_1, n_2, n_3, n_4. \end{cases}$

**Proof.** Choosing $q = s_1$, Reidemeister-Schreier’s algorithm produces a presentation of $\text{Ker}(\varphi)$ with

- generators:
  - for $1 \leq k \leq m$, $(y_k, y'_k) = \begin{cases} (s_k q^{-1}, q s_k) & \text{if } k \leq m \\ (s_k q s_k q^{-1}) & \text{if } k > m \end{cases}$
  - for $1 \leq j \leq g'$, $(x_j, x'_j) = (v_j, q v_j q^{-1})$
Note that $X$ tors in this presentation by suppressing $s$ (in particular, we are left with the following new presentation of $\ker (\phi)$:

- $y_1 = 1$ and $z' = z$
- $\forall k = 1, \ldots , n$,
  - $[y_k, z] = [y'_k, z] = 1$
  - $(y_k y_k)^{a_k/2} b_k = 1$ if $k \leq m$
  - $y'^{a_k/2} b_k = y'^{a_k} z^b_k = 1$ if $k > m$
- $x_j^2 j^2 \leq z^{-\epsilon_j} = x'_j z_j^{-1} \leq z^{-\epsilon_j} = 1$ ($\forall j = 1, \ldots , g'$)

(I) $y'^2 BCX = z^e$, where $B = y_3 y'_4 y_5 y'_6 \ldots y_{m-1} y'_m$, $C = y_{m+1} \ldots y_n$ and

$$X = \begin{cases} [x_1, x_2] \ldots [x_{2g-1}, x_{2g}] & \text{if } \epsilon = o_1, o_2 \\ x_1^2 \ldots x_g^2 & \text{if } \epsilon = n_1, n_2, n_3, n_4 \end{cases}$$

- (II) $y'_1 y'_2 y'_3 y'_4 \ldots y'_{m-1} y_m C' X' = z^e$, where $C' = y'_{m+1} \ldots y'_n$ and

$$X' = \begin{cases} [x'_1, x'_2] \ldots [x'_{2g-1}, x'_{2g}] & \text{if } \epsilon = o_1, o_2 \\ x'_1^2 \ldots x'_g^2 & \text{if } \epsilon = n_1, n_2, n_3, n_4. \end{cases}$$

(Noe that $X$ and $X'$ always commute with $z$.) Let us make a change of generators in this presentation by suppressing $y'_1, y'_2, y'_3, y'_4, \ldots , y'_m, y_m$ and introducing instead new generators $s'_1, s'_2, \ldots , s'_m$ defined by:

$$s'_k = \begin{cases} y_2^{-1} y_3 y_4^{-1} y_5 y_6 \ldots y_{k-1} (y_k y_k) y_{k-1} \ldots y_5 y'_4 y'_3 y'_2 & \text{if } k \text{ is odd} \\ y_2^{-1} y_3 y_4 y_5 y_6 \ldots y_{k-1} (y_k y_k) y_{k-1} \ldots y_5 y'_4 y'_3 y'_2 & \text{if } k \text{ is even} \end{cases}$$

(in particular, $s'_1 = y'_1$ and $s'_2 = y'_2$). These new generators still commute with $z$, the relations $(y_k y_k)^{a_k/2} z^b = 1$ become $s'_k^{a_k/2} z^b = 1$, and the relation (II) becomes

(III): $y_2^{-1} B' C' X' = z^e$, where $A = s'_1 \ldots s'_m$ and $B' = y_3 y'_4 y_5 y'_6 \ldots y_{m-1} y'_m$.

Using (I) to eliminate the generator $y'_2$ from (III), which will become (IV) below, we are left with the following new presentation of $\ker (\phi)$:

- generators:
  - $s'_k$ for $1 \leq k \leq m$
  - $y_3, y'_4, y_5, y'_6, \ldots , y_{m-1}, y'_m$
  - $y_k, y'_k$ for $m < k \leq n$
  - $x_j, x'_j$ for $1 \leq j \leq g'$
  - $z$

- relations:
  - $[s'_k, z] = 1$ and $s_k^{a_k/2} z^b_k = 1$ for $1 \leq k \leq m$
  - $[y_k, z] = [y'_k, z] = 1$ and $y'^{a_k} z^b_k = y'^{a_k} z^b_k$ for $m < k \leq n$
  - $[y_3, z] = \ldots = [y_m, z] = 1$
  - $x_j^2 j^2 \leq z^{-\epsilon_j} = x'_j z_j^{-1} \leq z^{-\epsilon_j} = 1$ for $1 \leq j \leq g'$
  - (IV) $ABCX B'C' X' = z^{2e}$.

This last relation (IV) may be reordered by a new change of generators (replacing some of the generators by conjugates thereof) so as to become $ACC' BX B'X' = z^{2e}$, which we rewrite $ACC'(BX'B'X')XX' = z^{2e}$. The parenthesis $BX'B'X'^{-1}$ is the product of $m - 1$ elements, followed by the product (in the same order) of their inverses. It can be transformed into a product of $\frac{m}{2} - 1$ commutators, by another change of generators given by Lemma 5 below.
Provided that next (classical) lemma, this concludes the proof of Lemma 4 up to a renaming of the generators.

**Lemma 5.** Let $F_{2k+1}$ be the free group over $g_0, \ldots , g_{2k}$. There exist elements $h_0, \ldots , h_{2k-1} \in F_{2k+1}$ such that:

$$g_0 g_1 \cdots g_{2k} g_0^{-1} g_1^{-1} \cdots g_{2k}^{-1} = [h_0, h_1][h_2, h_3] \cdots [h_{2k-2}, h_{2k-1}]$$

and $F_{2k+1}$ is the free group over $h_0, \ldots , h_{2k-1}, g_{2k}$.

**Proof.** Let $U_i = g_{2i} \cdots g_{2k}$ and $V_i = g_{2i}^{-1} \cdots g_{2k}^{-1}$. Then,

$$U_i V_k = 1 \quad \text{and} \quad U_i V_i = [g_{2i} g_{2i+1}, U_{i+1} g_{2i+1}^{-1}] U_{i+1} V_{i+1}.$$ 

Lemma 5 produces a standard presentation of $\text{Ker} (\varphi)$ only when $\epsilon = o_1$, or when $m = 2$ and $\epsilon \neq n_4$. In other cases, the following Lemmas 6, 7 and 8 tell how to normalize both $W$ (which must not be a mixture of commutators and squares) and the list of the $\epsilon_i$'s (for which the numbers of 1's and −1's are partially prescribed).

**Lemma 6.** Let $F_3$ be the free group over $x, y, z$ and $\epsilon : F_3 \to \{1, -1\}$ the morphism defined by

$$\epsilon(x) = \epsilon(y) = 1 \quad \text{and} \quad \epsilon(z) = -1.$$ 

There exist elements $u, v, w$ such that

$$\epsilon(u) = \epsilon(v) = \epsilon(w) = -1 \quad \text{and} \quad [x, y] z^2 = u^2 v^2 w^2$$

and $F_3$ is the free group over $u, v, w$.

**Proof.** Take for instance $u = xz$, $v = (zxz)^{-1}yz$ and $w = (yz)^{-1}z^2$.

**Lemma 7.** Let $F_4$ be the free group over $x, y, z, t$ and $\epsilon : F_4 \to \{1, -1\}$ the morphism defined by

$$\epsilon(x) = \epsilon(y) = 1 \quad \text{and} \quad \epsilon(z) = \epsilon(t) = -1.$$ 

There exist elements $x', y', z', t' \in F_4$ such that

$$\epsilon(x') = \epsilon(y') = \epsilon(z') = \epsilon(t') = -1 \quad \text{and} \quad [x, y][z, t] = [x', y'][z', t']$$

and $F_4$ is the free group over $t', u', v', w'$.

**Proof.** Take for instance

$$x' = xyz, \quad y' = z^{-1} x^{-1}, \quad z' = (y^{-1} z)^{-1} z(y^{-1} z) \quad \text{and} \quad t' = t z^{-1}(y^{-1} z).$$

**Lemma 8.** Let $F_4$ be the free group over $t, u, v, w$ and $\epsilon : F_4 \to \{1, -1\}$ the morphism defined by

$$\epsilon(t) = \epsilon(u) = \epsilon(v) = 1 \quad \text{and} \quad \epsilon(w) = -1.$$ 

There exist elements $t', u', v', w' \in F_4$ such that

$$\epsilon(t') = 1, \quad \epsilon(u') = \epsilon(v') = \epsilon(w') = -1 \quad \text{and} \quad t^2 u^2 v^2 w^2 = t^2 u'^2 v'^2 w'^2$$

and $F_4$ is the free group over $x', y', z', t'$.

**Proof.** Take for instance $t', u', w', v'$ successively defined by:

$$t' = tu^2v^{-1}, \quad t'u' = u^2 v w, \quad u'w' = uvw^2 \quad \text{and} \quad u'v'w' = w.$$
Proposition 9. If \( \varphi \) maps \( s_1, \ldots, s_m \) to 1 and all other generators to 0 then its kernel is the fundamental group of the Seifert manifold given by

\[
\{ e'; (e', G); F_m \}
\]

where

\[
e' = \begin{cases} 
2e & \text{if } \epsilon = o_1, n_2 \\
0 & \text{if } \epsilon = o_2, n_1, n_3, n_4 
\end{cases}
\]

\[
e' = \begin{cases} 
\epsilon & \text{if } \epsilon = o_1, o_2, n_1, n_2, n_4 \\
n_4 & \text{if } \epsilon = n_3 
\end{cases}
\]

\[
G = \begin{cases} 
m - 1 + 2g & \text{if } \epsilon = o_1 \\
m - 2 + 2g & \text{if } \epsilon = o_2 \\
m - 2 + 2g & \text{if } \epsilon = n_1, n_2, n_3, n_4 
\end{cases}
\]

and \( F_m \) as defined in Notation 2.

Proof. When \( \epsilon = o_1 \), Lemma 4 directly gives the result and \( e' = o_1 \), with \( 2G = g'' = m - 2 + 2g' = m - 2 + 4g \). When \( \epsilon = o_2 \), the number of \( \epsilon' \)'s equal to \(-1\) is \( g' = 2g > 0 \) hence by Lemma 4 all \( \epsilon' \)'s can be replaced by \(-1\), so \( e' = o_2 \) and \( 2G = m - 2 + 4g \) as in the previous case. When \( \epsilon = n_1 \), Lemma 5 may be applied if necessary (i.e. if \( m > 2 \)) to replace each commutator by a product of two squares (only a weak form of the Lemma is used here, forgetting about the morphism \( \varepsilon \) of its statement). Thus we get \( e' = n_1 \) and \( G = g'' = m - 2 + 2g' = m - 2 + 2g \).

When \( \epsilon = n_2 \), applying again Lemma 5 if necessary (in its strong form) gives the result and \( e' = n_2 \) with \( G = m - 2 + 2g \) as in the previous case. When \( \epsilon = n_3 \), \( W = [v_1', v_2'] \ldots [v_{m-3}', v_{m-2}']v_{m-1}'v_{m-1}' \ldots v_{m-2}'v_{m-2}'v_{m-1}'v_{m-1}' \) and among \( \varepsilon'_{m-1}', \ldots, \varepsilon'_{m-2}2-g \), there are two 1's, but the number of \(-1\)'s is \( 2g - 2 > 0 \) and we may reorder these \( 2g \) \( \varepsilon' \)'s to put the two 1's first (by a repeated change of variable, using that \( u^2v^2 = (u^2v^2-2u^2) \)). Lemma 5 again gives the conclusion. When \( \epsilon = n_4 \) the same method allows to transform \( W \) into a product of squares but we are left with four 1's instead of two. Reordering again to put these four 1's at the beginning, Lemma 5 allows to reduce this number of 1's to two, hence (again) \( e' = n_4 \) and \( G = m - 2 + 2g \). \( \square \)

The next lemma will be used in Theorem 11 to extend the result of the previous proposition to the general case, where \( \varphi \) does not necessarily vanish on all \( v_j \)'s.

Lemma 10. If two morphisms from \( \pi_1(N) \) to \( \mathbb{Z}_2 \) map \( s_1, \ldots, s_m \) \( (m > 0) \) to 1 and \( h, s_{m+1}, \ldots, s_n \) to 0 then their kernels are isomorphic.

Proof.

When \( \epsilon = n_1, n_2, n_3, n_4 \), the idea is to change the presentation of \( \pi_1(N) \), using \( \varphi(s_1) = 1 \) to “kill” all the \( \varphi(v_j) \)'s, one after another. The formulas are simpler if we prepare each of these “murders” by temporarily permuting the \( \varphi(v_j) \)'s, to put at the end the one whose value we want to switch from 1 to 0.

The values \( \varphi(v_j) \) and \( \varphi(v_{j+1}) \) can be exchanged by the following change of generators (leaving the other generators untouched):

\[ v'_j = v_j^2v_{j+1}v_j^{-2}, \quad v'_{j+1} = v_j. \]

Combining such transpositions, we can temporarily reorder the \( v_j \)'s (it may affect the conventional ordering of the \( v_j \)'s when \( \epsilon = n_3, n_4 \), but this is temporary hence harmless).

Then, the value of \( \varphi(v_g) \) can be switched from 1 to 0 by:

\[ v'_g = v_g s_1, \quad s'_1 = v'_g^{-1}s_1v'_g. \]
When \( v_g \) anticommutes with \( h \), this is in fact (like the previous one) an automorphism of \( \pi_1(N) \), but when \( v_g \) commutes with \( h \), it is only a change of presentation, since \( b_1 \) is changed into its opposite. Thus, after “killing” all \( \varphi(v_j)'s \) and then applying Proposition 9, we get the same invariants for \( \text{Ker} (\varphi) \) as if all \( \varphi(v_j)'s \) were already 0, up to a possible change of sign of \( b_1 \) in the type \((a_1/2, b_1)\) of the first exceptional fiber. However, this may only happen when some \( \varepsilon_j \)'s are equal to 1, i.e. when \( \varepsilon = n_1, n_3, n_4 \). But we see from Proposition 9 that \( \text{Ker} (\varphi) \) inherits this non-orientability, which allows to replace in the final result such an \((a_1/2, -b_1)\) (if it occurs) by \((a_1/2, b_1)\).

When \( \varepsilon = o_1, o_2 \), the idea is the same:

the value of \( (\varphi(v_{2i-1}), \varphi(v_{2i})) \) can be exchanged with that of \( (\varphi(v_{2i+1}), \varphi(v_{2i+2})) \) by:

\[
v'_{2i-1} = [v_{2i-1}, v_{2i}]v_{2i+1}[v_{2i-1}, v_{2i}]^{-1}, \quad v'_{2i} = [v_{2i-1}, v_{2i}]v_{2i+2}[v_{2i+2}, v_{2i}]^{-1},
\]

and the values of \( \varphi(v_j) \) and \( \varphi(v_{j+1}) \) when \( j \) is odd can be exchanged by:

\[v'_j = v_jv_{j+1}v_j^{-1}, \quad v'_{j+1} = v_{j+1}^{-1}.
\]

The value of \( \varphi(v_{2g}) \) can be switched from 1 to 0 by:

\[s'_1 = v_{2g-1}s_1v_{2g}^{-1}, \quad v'_{2g-1} = s'_1v_{2g-1}s_1^{-1}, \quad v'_{2g} = s'_1s_1^{-1}v_{2g}^{-1}.
\]

When \( \varepsilon = o_2 \), this again changes \( b_1 \) to \(-b_1\) but it can be cured if necessary in the final result, by the same argument.

From Proposition 9 and Lemma 10 we immediately deduce:

**Theorem 11.** If \( \varphi \) map \( s_1, \ldots, s_m \) \((m > 0)\) to 1 and \( h, s_{m+1}, \ldots, s_n \) to 0 then its kernel is the fundamental group of the Seifert manifold given by

\[\{e'; (e', G); F_m\}\]

where

\[e' = \begin{cases} 2e & \text{if } \varepsilon = o_1, n_2 \\ 0 & \text{if } \varepsilon = o_2, n_1, n_3, n_4 \end{cases}\]

\[e' = \begin{cases} \varepsilon & \text{if } \varepsilon = o_1, o_2, n_1, n_2, n_4 \\ n_4 & \text{if } \varepsilon = n_3 \end{cases}\]

\[G = \begin{cases} m - 1 + 2g & \text{if } \varepsilon = o_1, o_2 \\ m - 2 + 2g & \text{if } \varepsilon = n_1, n_2, n_3, n_4 \end{cases}\]

and \( F_m \) as defined in Notation 3.

4.0.3. If \( \varphi \) maps some \( v_j \)'s to 1 and all other generators 0. In this subsection, \( \varphi(h) = \varphi(s_1) = \ldots = \varphi(s_n) = 0 \). Apart from orientation covers and some “exotic” cases, \( \text{Ker} (\varphi) \) will have the same description as in Theorem 11 with \( m \) replaced by 0. The two cases \( \varepsilon = o_1, o_2 \) and \( \varepsilon = n_1, n_2, n_3, n_4 \) will be treated separately (Propositions 12 and 13) and the global result will be rephrased in Theorem 15.

**Proposition 12.** When \( \varphi \) maps \( r > 0 \) generators \( v_j \)'s to 1 and all other generators to 0 and \( \varepsilon = o_1 \) or \( o_2 \), \( \text{Ker} (\varphi) \) is the fundamental group of the Seifert manifold given by the following invariants, with \( F_{OC} \) and \( F_0 \) as defined in Notation 2:

- if \( \varepsilon = o_1 \): \( \{2e; (o_1, 2g - 1); F_0\} \)
- if \( \varepsilon = o_2 \) and \( r = g \) (orientation cover): \( \{0; (o_1, 2g - 1); F_{OC}\} \)
Proof. Let us reorder \( r \) the number of \( v_j \)'s mapped to 1 in such a way that \( \varphi(v_j) = 1 \Rightarrow j \leq r \) (by the same method as in Lemma 10). Let \( \varepsilon \) be the common value of the \( \varepsilon_j \)'s, i.e. \( \varepsilon = 1 \) if \( \varepsilon = o_1 \) and to \( \varepsilon = -1 \) if \( \varepsilon = o_2 \).

Choosing \( q = v_r \), Reidemeister-Schreier’s algorithm produces a presentation of \( \text{Ker}(\varphi) \) with

- generators:
  - for \( 1 \leq k \leq n \), \( (y_k, y'_k) = (s_k, qskq^{-1}) \)
  - for \( 1 \leq j \leq 2g \),
    \[
    (x_j, x'_j) = \begin{cases}
    (v_jg^{-1}, qv_j) & \text{if } j \leq r \\
    (v_j, qv_jg^{-1}) & \text{if } j > r
    \end{cases}
    \]
    \( - (z, z') = (h, qhq^{-1}) \)

- relations:
  - \( x_r = 1, z' = z^\varepsilon \)
  - \( z \) commutes with all \( y_k \)'s and \( y'_k \)'s, and with \( x_j \) and \( x'_j \) for \( j \leq r \)
  - \( x_jx^{-1}z^{-\varepsilon} = x'_jx^{-1}z^{-\varepsilon} = 1 \) for \( j > r \)
  - \( y_ka_k^b = y_ka_k^b = 1 \) \( (\forall k = 1, \ldots, n) \)
  - (I) \( YB = z^\varepsilon \), (II) \( Y'B' = z^\varepsilon \varepsilon' \), where \( Y = y_1 \ldots y_n \), \( Y' = y'_1 \ldots y'_n \), and \( B, B' \) are described as follows:
    - if \( r \) is odd, \( B = \prod x'_{r+1}x^{-1}_{r+1}x_r \) and \( B' = \prod x'_{r+1}x^{-1}_{r+1}x_r \),
    - if \( r \) is even, \( B = \prod x'_{r+1}x^{-1}_{r+1}x_r \) and \( B' = \prod x'_{r+1}x^{-1}_{r+1}x_r \),

Assume first that \( r \) is odd. Using (II) to eliminate \( x'_{r+1} \) in (I), and replacing some of the generators by conjugates thereof (without altering the previous relations) to reorder the subexpressions of the resulting relation, (I) becomes:

\[
YXW = z^{(1+\varepsilon)e}, \quad \text{with} \quad W = \prod z'^{x'_{r+1}x^{-1}_{r+1}x}X'.
\]

Transforming \( ZZ' \) by lemma 13 below (which is stated informally but whose proof gives explicit formulas), \( W \) becomes a product of \( (r-1)+1+2(2g-r-1) \) commutators of new generators, whose associated \( \varepsilon' \)'s are equal to 1 for the first \( 2(r-1) + 1 \) of them and to \( \varepsilon \) for the \( 1+2(2g-r-1) \) last ones. We thus get the following Seifert invariants for \( \text{Ker}(\varphi) \):

- if \( \varepsilon = o_1 \): \( \{0; (o_1, 2g - 1); F_0\} \)
- if \( \varepsilon = o_2 \) (noting that \( 1+2(2g-r-1) > 0 \)): \( \{0; (o_2, 2g - 1); F_{OC}\} \).
In the $o_2$ (non-orientable) case, all $-b_k$’s can be replaced by their opposites, i.e. $F_{OC}$ replaced by $F_0$, which concludes the odd case.

Assume now that $r$ is even. Using (I) to eliminate $x'_{r-1}$ and performing similar transformations, (II) becomes:

$$ W = Z'[x_{r-1}, x']XX'. $$

Using lemma 13 again, $W$ becomes a product $(r - 2) + 1 + (2g - r)$ commutators and the first $2(r - 2) + 2 \varepsilon_j$’s are equal to 1, the $2(2g - r)$ last ones being equal to $\varepsilon$. Hence the conclusion is the same as in the odd case, except when $\varepsilon = -1$ and $2(2g - r) = 0$, which corresponds to the orientation cover case of the statement. This concludes the proof of Proposition 12 provided the next Lemma.

Lemma 13. In any group, an expression of the form

$$(a_1b_1c_1d_1)\ldots(a_kb_kc_kd_k)(c_1^{-1}d_1^{-1}a_1^{-1}b_1^{-1})\ldots(c_k^{-1}d_k^{-1}a_k^{-1}b_k^{-1})$$

is the product of $2k$ commutators.

Proof. Let $U_k = (a_1b_1c_1d_1)\ldots(a_kb_kc_kd_k)$ and $V_k = (c_1^{-1}d_1^{-1}a_1^{-1}b_1^{-1})\ldots(c_k^{-1}d_k^{-1}a_k^{-1}b_k^{-1})$. Then $U_0V_0 = 1$, and $U_{k+1}V_{k+1}$ is the product of $U_kV_k$ (which by induction hypothesis is a product of $2k$ commutators) by

$$ [V_k^{-1}a_{k+1}b_{k+1}c_{k+1}d_{k+1}V_k^{-1}d_{k+1}^{-1}a_{k+1}^{-1}b_{k+1}^{-1}] = [V_k^{-1}a_{k+1}b_{k+1}V_k](b_{k+1}^{-1}a_{k+1})[V_k^{-1}c_{k+1}d_{k+1}V_k](b_{k+1}^{-1}a_{k+1})^{-1}. $$

Proposition 12 dealt with the case $\varepsilon = o_1, o_2$. The next proposition deals with the other case, $\varepsilon = n_1, n_2, n_3, n_4$.

Proposition 14. When $\varphi$ maps some $v_j$’s to 1 and all other generators to 0 and $\varepsilon = n_1, n_2, n_3$ or $n_4$, $\text{Ker}(\varphi)$ is the fundamental group of the Seifert manifold given by the following invariants, whith $F_{OC}$ and $F_0$ as defined in Notation 2.

- (Ordinary cases) in all other cases: $\{\varepsilon'; (e', 2g - 2); F_0\}$ with

$$ e' = \begin{cases} 
\varepsilon & \text{if } \varepsilon = n_1, n_2, n_4 \\
\varepsilon & \text{if } \varepsilon = n_3, n_4 \\
\varepsilon' & \text{if } \varepsilon = n_1, n_3, n_4.
\end{cases} $$

Proof. Assume, like in the proof of Proposition 12 that $\varphi(v_j) = 1 \iff j \leq r$. We shall have to be cautious about a new phenomenon: this reordering of the $\varepsilon_j$’s may affect the ordering of the $\varepsilon_j$’s, i.e. we shall try to maintain the convention that the $\varepsilon_j$’s equal to 1 are always $\varepsilon_1$ when $\varepsilon = n_3$ and $\varepsilon_1, \varepsilon_2$ when $\varepsilon = n_4$, but we shall locally drop this convention inside the present proof whenever it is not compatible with our reordering of the $v_j$’s.

Choosing $q = v_1$, Reidemeister-Schreier’s algorithm produces a presentation of $\text{Ker}(\varphi)$ with
RS’s algorithm for 2-coverings of Seifert manifolds

• generators:
  
  for 1 ≤ k ≤ n, \((y_k; y_k^{-1}) = (s_k, q s_k q^{-1})\)
  
  for 1 ≤ j ≤ g,

  \[
  (x_j, x_j') = \begin{cases} (v_j q^{-1}, q v_j) & \text{if } j \leq r \\ (v_j, q v_j q^{-1}) & \text{if } j > r \end{cases}
  \]

  \((z, z') = (h, q h q^{-1})\)

• relations:
  
  \(x_1 = 1, z' = z^{-1}\)
  
  \(z\) commutes with all \(y_k\)’s and \(y'_k\)’s
  
  \(x_j z x_j^{-1} = x'_j z x'_j^{-1} = 1\) with \(z' = \begin{cases} \varepsilon_j \varepsilon_1 & \text{if } j \leq r \\ \varepsilon_j & \text{if } j > r \end{cases}\)
  
  \(- y_k^{a_k} z b_k = y'_k^{a_k} z b_k = 1 \quad (\forall k = 1, \ldots, n)\)
  
  (I) \(Y' X' Z X = z^{\varepsilon_1},\) where \(Y = y_1 \ldots y_n, X = x_{r+1} \ldots x_n^2\) and \(Z = x_{2} x_{r} \ldots x_r\)

  (II) \(Y' X' Z' X' = z^{\varepsilon_1},\) where \(Y' = y'_1 \ldots y'_n, X' = x'_{r+1} \ldots x'_n\) and \(Z' = x'_{2} x'_{r} \ldots x'_{r} x_r\).

Eliminating \(x'_1,\) (I) and (II) join to become (III): \(Y Y^{r-1} X' Z^{r-1} Z X = z^{(1-\varepsilon_1)}\) and \(Z^{r-1} Z\) is a product of \(r-1\) commutators (of conjugates of inverses of \(x_2, x'_2, \ldots, x_r, x'_r,\) having the same \(\varepsilon'_j\)’s).

When \(r = g,\) \(X\) and \(X'\) are empty products hence \(\varepsilon' = \varepsilon_1 \) or \(\varepsilon_2.\) Moreover when \(\varepsilon = n_3, n_4,\) since we find \(\varepsilon' = \varepsilon_2,\) we can replace the \(-b_k\)’s which occur by their opposites. More precisely Seifert invariants for \(\text{Ker}(\varphi)\) when \(r = g\) are:

• if \(\varepsilon = n_1: \{0; (o_1, g-1); F_{OC}\}\)
• if \(\varepsilon = n_2: \{2; (o_1, g-1); F_{0}\}\)
• if \(\varepsilon = n_3, n_4: \{0; (o_2, g-1); F_{0}\}\)

When \(r < g,\) (III) contains a product of \(2(g-r)\) squares and \(r-1\) commutators, which can be converted to a product of \(2g-2\) squares (using Lemma 6 and taking care of the \(\varepsilon'_j\)’s to determine \(\varepsilon'\)). Moreover, when the \(\varepsilon'\) we find corresponds to a non-orientable manifold, all \(b'_k = -b_k\)’s (if any) can be replaced by \(b'_k = b_k.\) Hence Seifert invariants for \(\text{Ker}(\varphi)\) when \(r < g\) are:

• if \(\varepsilon = n_2: \{2; (n_2, 2g-2); F_{0}\}\)
• if \(\varepsilon = n_3\) and \(\varphi\) sends only \(v_1\) to 1, or if \(\varepsilon = n_4\) and \(\varphi\) sends only \(v_1, v_2\) to 1: \(\{0; (n_2, 2g-2); F_{OC}\}\)
• in all other cases: \(\{0; \varepsilon', 2g-2; F_{0}\}\) with \(\varepsilon' = n_1\) if \(\varepsilon = n_1,\) and \(\varepsilon' = n_4\) if \(\varepsilon = n_3, n_4.\)

The following theorem is a synthesis of Propositions 12 and 14

**Theorem 15.** When \(\varphi\) maps some \(v_j\)’s to 1 and all other generators to 0, \(\text{Ker}(\varphi)\) is the fundamental group of the Seifert manifold given by the following invariants, with \(F_{OC}\) and and \(F_0\) as defined in Notation 3

• (Orientation covers)
  
  if \(\varepsilon = o_2\) and \(\varphi\) maps all \(v_j\)’s to 1: \(\{0; (o_1, 2g-1); F_{OC}\}\)
  
  if \(\varepsilon = n_1\) and \(\varphi\) maps all \(v_j\)’s to 1: \(\{0; (o_1, g-1); F_{OC}\}\)
- if $\epsilon = n_3$ and $\varphi$ sends only $v_1$ to 1, or if $\epsilon = n_4$ and $\varphi$ sends only $v_1, v_2$ to 1: $\{0; (n_2, 2g - 2); F_{OC}\}$
- (Exotic cases) if $\epsilon = n_2, n_3, n_4$ and $\varphi$ maps all $v_i$’s to 1:
  - if $\epsilon = n_2$: $\{2e; (o_1, g - 1); F_0\}$
  - if $\epsilon = n_3, n_4$: $\{0; (o_2, g - 1); F_0\}$
- (Ordinary cases) in all other cases: $\{e'; (e', G); F_0\}$ with $e' = \begin{cases} 2e & \text{if } \epsilon = o_1, n_2 \\ 0 & \text{if } \epsilon = o_2, n_1, n_3, n_4 \end{cases}$
and $G = \begin{cases} 2g - 1 & \text{if } \epsilon = o_1, o_2 \\ 2g - 2 & \text{if } \epsilon = n_1, n_2, n_3, n_4 \end{cases}$.

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