Higher displays arising from filtered de Rham-Witt complexes

Oli Gregory∗ and Andreas Langer

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Abstract

For a smooth projective scheme $X$ over a ring $R$ on which $p$ is nilpotent that meets some general assumptions we prove that the crystalline cohomology is equipped with the structure of a higher display which is a relative version of Fontaine’s strongly divisible lattices. Frobenius-divisibility is induced by the Nygaard filtration on the relative de Rham-Witt complex. For a nilpotent PD-thickening $S/R$ we also consider the associated relative display and can describe it explicitly by a relative version of the Nygaard filtration on the de Rham-Witt complex associated to a lifting of $X$ over $S$. We prove that there is a crystal of relative displays if moreover the mod $p$ reduction of $X$ has a smooth and versal deformation space.

1 Introduction

For a ring $R$ in which $p$ is nilpotent, we constructed in [LZ07] an exact tensor category of displays which contains the displays associated to $p$-divisible groups [Zin02] as a full subcategory. If $R = k$ is a perfect field, a display is a finitely generated free $W(k)$-module $M$ endowed with an injective Frobenius-linear map $F : M \to M$. In general, displays can be regarded as a relative version of Fontaine’s strongly divisible lattices [Fon83]. For a smooth projective scheme $X$ over Spec $R$ we also gave examples in [LZ07] when the crystalline cohomology $H^i_{\text{cris}}(X/W(R))$ can be equipped with the structure of a display. The following assumption was essential in the construction of such “geometric” displays:

(A1) The cohomology groups $H^i(X_n, \Omega^j_{X_n/W_n(R)})$ are for each $n, i$ and $j$ locally free $W_n(R)$-modules of finite type.

(A2) For each $n$ the de Rham spectral sequence degenerates at $E_1$

$E_1^{i,j} = H^i(X_n, \Omega^j_{X_n/W_n(R)}) \Rightarrow H^{i+j}(X_n, \Omega^\bullet_{X_n/W_n(R)})$

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(compare (*) and (**) in ([LZ07], p.150) and assumptions 5.2, 5.3 in [LZ07]).

Let \( I_R := VW(R) \) and \( WΩ_{X/R}^\bullet \) be the relative de Rham-Witt complex as constructed in [LZ04]. For \( r \geq 0 \) define the complex \( N^rWΩ_{X/R}^\bullet \) as follows:

\[
(WΩ_{X/R}^0)[F] \xrightarrow{d} (WΩ_{X/R}^1)[F] \xrightarrow{d} \cdots \xrightarrow{d} (WΩ_{X/R}^{r-1})[F] \xrightarrow{d} WΩ_{X/R}^r \xrightarrow{d} \cdots
\]

This is a complex of \( W(R) \)-modules where \( (WΩ_{X/R}^i)[F] \), for \( i < r \), is considered as a \( W(R) \)-module via restriction of scalars along \( W(R) \xrightarrow{F} W(R) \).

It was conjectured in ([LZ07], Conj. 5.8) that the predisplay structure on \( X_n \) forms a display structure \( P_0 := H^0_{\text{cris}}(X/W(R)) \), defined by the data \( P_r := H^n(X,N^rWΩ_{X/R}^\bullet) \) and maps \( \tilde{α}_r : I_R \otimes P_r \to P_{r+1}, \tilde{ι}_r : P_{r+1} \to P_r \) and \( F_r : P_r \to P_0 \) induced by the corresponding maps of complexes \( α_r : I_R \otimes N^rWΩ_{X/R}^\bullet \to N^{r+1}WΩ_{X/R}^\bullet, ι_r : N^{r+1}WΩ_{X/R}^\bullet \to N^rWΩ_{X/R}^\bullet \) and \( F_r : N^rWΩ_{X/R}^\bullet \to WΩ_{X/R}^\bullet \) given in ([LZ07], (5)) define a display structure on \( P_0 \).

In this note we essentially prove this conjecture under the assumption \( r \leq n < p \), which is a standard hypothesis in integral \( p \)-adic Hodge theory, and prove the following:

**Theorem 1.1.**

(a) Let \( R \) be a local ring in which \( p \) is nilpotent, and let \( X \) be a smooth projective scheme over \( \text{Spec} \, R \). Assume that there exists a compatible system of liftings \( X_n/\text{Spec} \, W_n(R) \) satisfying (A1) and (A2). Then the data \( (P_r, ι_r, α_r, F_r) \) forms a display structure \( P_R \) on \( H^n_{\text{cris}}(X/W(R)) \) for \( n < p \).

(b) Assume that there exists in addition a frame \( A \to R \) and a smooth projective \( p \)-adic lifting \( Y/\text{Spf} \, A \) such that the \( Y_n := Y \times_{\text{Spf} \, A} \text{Spf} \, A/p^n \) satisfy the analogous assumptions (A1) and (A2). Then the display structure on \( H^n_{\text{cris}}(X/W(R)) \) obtained by base change from the window structure on \( H^n_{\text{cris}}(X/A) \) (compare [LZ07] Theorem 5.5 and Corollary 5.6) is isomorphic to \( P_R \).

**Remarks.**

- Our assumption on the existence of liftings \( X_n \) satisfying (A1) and (A2) is stronger than the assumption made in ([LZ07], Conj. 5.8), but they are satisfied in all examples.

- Theorem 1.1(a) was shown for reduced rings \( R \) in ([LZ07], Theorem 5.7).

In our second main result we derive a relative version of Theorem 1.1 on relative displays using a modified version of the complexes \( N^rWΩ_{X/R}^\bullet \). For this, let \( S \to R \) be a homomorphism of rings in which \( p \) is nilpotent and such that the kernel \( a \) is equipped with nilpotent divided powers. In [LZ15] and [Gre16] we considered the Witt frames \( Ψ_S, Ψ_{S/R}, Ψ_R \) and the canonical homomorphisms of frames \( Ψ_S \xrightarrow{d} Ψ_{S/R} \) and \( Ψ_{S/R} \to Ψ_R \). Let \( X/\text{Spec} \, R \) be as before and let \( X_S \) be a smooth projective lifting of \( X \) over \( \text{Spec} \, S \), admitting liftings \( (X_S)_n \) over \( \text{Spec} \, W_n(S) \) that satisfy the assumptions (A1) and (A2). Let \( a \) be the logarithmic Teichmüller ideal in \( W(S) \), as defined in ([Zin02], 1.4). Then \( J := a \oplus VW(S) \) is the kernel of the composite map \( W(S) \to S \to R \) and is again equipped with a PD-structure (see [Zin02], 2.3). Define the complexes
\(N'_{\text{rel}/R}W^{\bullet}_{X/S} \) as follows:
\[
W^{\bullet}_{X} \oplus \hat{a}^rW^{\bullet}_{X} \overset{d \oplus d}{\longrightarrow} W^{\bullet}_{X} \oplus \hat{a}^{-r-1}W^{\bullet}_{X} \overset{d \oplus d}{\longrightarrow} \ldots
\]
\[
\ldots \overset{d \oplus d}{\longrightarrow} W^{\bullet-1}_{X} \oplus \hat{a}W^{\bullet-1}_{X} \overset{d \oplus d}{\longrightarrow} W^{\bullet}_{X} \overset{d}{\longrightarrow} \ldots
\]
The maps \( \hat{\alpha}_r, \hat{\iota}_r \) on \( N^rW^{\bullet}_{X/S} \) that define the predisplay structure on \( H^0_{\text{cris}}(X/W(S)) = H^0_{\text{cris}}(X_S/W(S)) \) can easily be extended to maps on the complexes \( N^r_{\text{rel}/R}W^{\bullet}_{X/S} \), where multiplication by \( p \) on \( W^{\bullet}_{X/S} \) is replaced by the map
\[
\pi : W^{\bullet}_{X/S} \oplus \hat{a}^{-r}W^{\bullet}_{X/S} \rightarrow W^{\bullet}_{X/S} \oplus \hat{a}^{-r-1}W^{\bullet}_{X/S}
\]
which is multiplication by \( p \) on \( W^{\bullet}_{X/S} \) and the inclusion on the other summand. The divided Frobenius \( \hat{F}_r \) is defined on the subcomplex \( N^rW^{\bullet}_{X/S} \) as before and on \( \hat{a}^{-r}W^{\bullet}_{X/S} \) it is defined to be the zero map. In analogy to (LZ07, (5)) we get induced maps of complexes
\[
\hat{\alpha}_r : J \otimes N^r_{\text{rel}/R}W^{\bullet}_{X/S} \rightarrow N^{r+1}_{\text{rel}/R}W^{\bullet}_{X/S}
\]
\[
\hat{\iota}_r : N^{r+1}_{\text{rel}/R}W^{\bullet}_{X/S} \rightarrow N^r_{\text{rel}/R}W^{\bullet}_{X/S}
\]
\[
\hat{F}_r : N^r_{\text{rel}/R}W^{\bullet}_{X/S} \rightarrow W^{\bullet}_{X/S}
\]
It is then easy to see that the above data form a predisplay \( \mathcal{P}_{S/R} \) over the relative Witt frame \( \mathbb{W}_{S/R} \) (see (LZ15, Def. 2). Then one has:

**Theorem 1.2.**

(a) Let \( \mathcal{P}_S \) be the display over \( S \) constructed in Theorem [1.1(a)]. Then the associated display \( u_{\mathcal{P}_S} \) under the homomorphism of frames \( \mathbb{W}_{S} \rightarrow \mathbb{W}_{S/R} \) is isomorphic to \( \mathcal{P}_{S/R} \). In particular, the predisplay \( \mathcal{P}_{S/R} \) is a display over \( \mathbb{W}_{S/R} \) in the sense of (LZ15, Def. 3).

(b) Assume that \( R \) is artinian local with perfect residue field \( k \). Assume that \( X_0 := X \times_R k \) has a smooth versal formal deformation space and that the assumptions (32) and (33) of [LZ15] (analogous to (A1) and (A2)) are satisfied. Then the display \( \mathcal{P}_{S/R} \) only depends - up to isomorphism - on \( X \) and \( S \), not on the lifting \( X_S \). The collection \( (\mathcal{P}_{S/R})_{S \rightarrow R} \) where \( S \rightarrow R \) are PD-morphisms defines a crystal of relative displays.

**Remark.**

Relative displays were first considered by Zink [Zin02] and Lau [Lau12] in their classification of p-divisible groups. Zink associated to a formal p-divisible group \( \mathcal{G} \) and a nilpotent PD-thickening \( S \rightarrow R \) a relative display \( \mathcal{P}_{S/R}(\mathcal{G}) \) (which he calls a triple in [Zin02]) which is - up to isomorphism - the unique relative display lifting the display associated to \( \mathcal{G} \) over \( R \) to \( S \). Using Dieudonné displays, i.e. displays defined over the small Witt ring \( \mathbb{Z}_{\text{in}} \), Zink extended the classification to all p-divisible groups over an artinian local ring with perfect residue field, see also [Lau12] and [Mes07]. Over slightly more general rings, called admissible rings, Lau [Lau12] constructed a unique functor from p-divisible groups to crystals of relative displays. The relative display is obtained by base change from a window structure associated to the universal p-divisible group over the deformation ring. The proof of Theorem [1.2(b)] was inspired by this construction.
2 Proof of the theorems

Theorem 1.1(a) is a consequence of the following result which was conjectured in ([LZ07], Conj. 4.1) and was recently proved by the second named author in ([Lan16] Thm. 0.2) under the assumption \( r < p \):

**Theorem 2.1.** Let \( R \) be a ring on which \( p \) is nilpotent and let \( X/\text{Spec } R \) be smooth projective, and \( X_n/\text{Spec } W_n(R) \) a compatible system of smooth liftings. Let \( \mathcal{F}^0\Omega_{X_n/W_n(R)}^\bullet \) be the following complex:

\[
I_R \mathcal{O}_{X_n} \longrightarrow I_R \otimes \Omega_{X_n/W_n(R)}^1 \longrightarrow \cdots \longrightarrow I_R \otimes \Omega_{X_n/W_n(R)}^{r-1} \longrightarrow \Omega_{X_n/W_n(R)}^r \longrightarrow \cdots
\]

where \( I_R := VW_n^{-1}(R) \), and let \( \mathcal{N}^r W_n\Omega_{X/R}^\bullet \) be the Nygaard complex, given as

\[
(W_{n-1}\Omega_{X/R}^0[F]) \longrightarrow (W_{n-1}\Omega_{X/R}^1[F]) \longrightarrow \cdots \longrightarrow (W_{n-1}\Omega_{X/R}^{r-1}[F]) \longrightarrow W_n\Omega_{X/R}^r \longrightarrow \cdots
\]

Then \( \mathcal{F}^0\Omega_{X_n/W_n(R)}^\bullet \) and \( \mathcal{N}^r W_n\Omega_{X/R}^\bullet \) are isomorphic in the derived category of \( W_n(R) \)-modules for \( r < p \).

2.1 Proof of Theorem 1.1(a)

Under the assumptions (A1) and (A2) it follows from ([LZ07] Prop. 3.2 and 3.3 and the projection formula Proposition 3.1 that one has a degenerating spectral sequence \( E_1^{i,j} \Rightarrow H^{i+j}(X_n, \mathcal{F}^0\Omega_{X_n/W_n(R)}^\bullet) \) where

\[
E_1^{i,j} = \begin{cases} 
I_R \otimes H^j(X_n, \Omega_{X_n/W_n(R)}^i) & \text{for } i < r \\
H^j(X_n, \Omega_{X_n/W_n(R)}^i) & \text{for } i \geq r
\end{cases}
\]

It follows from the proof of Theorem 2.1 ([Lan16], Thm. 0.2) that the isomorphisms \( \mathcal{F}^0\Omega_{X_n/W_n(R)}^\bullet \cong \mathcal{N}^r W_n\Omega_{X/R}^\bullet \) are compatible for varying \( n \) and yield an isomorphism \( \mathcal{F}^0\Omega_{X/W(R)}^\bullet \cong \mathcal{N}^r W^\bullet\Omega_{X/R}^\bullet \) of procomplexes in \( D_{\text{pro},\text{Zar}}(X) \). This induces an isomorphism resp. a decomposition

\[
P_r = H^n(X, \mathcal{F}^0\Omega_{X/W(R)}^\bullet) \cong I_R L_0 \oplus I_R L_1 \oplus \cdots \oplus I_R L_{r-1} \oplus L_r \oplus \cdots \oplus L_n
\]

where \( L_i := H^{n-i}(X, \Omega_{X/W(R)}^i) \). Since the divided Frobenius \( F_r \) is defined on \( H^n(X, \mathcal{F}^0\Omega_{X/W(R)}^\bullet) \) via Theorem 2.1 we can define \( \Phi_r : L_r \rightarrow P_0 \) by \( \Phi_r := F_r L_r \).

To show that \( (P_r, \hat{F}_r, \hat{i}_r, \hat{\alpha}_r) \) defines a display on \( H^n_{\text{cris}}(X/W(R)) \) is equivalent to the condition that

\[
\bigoplus_{i=0}^n \Phi_i : \bigoplus_{i=0}^n L_i \rightarrow \bigoplus_{i=0}^n L_i
\]

is a \( \sigma \)-linear isomorphism, or equivalently that \( \det(\bigoplus_{i=0}^n \Phi_i) \in W(R)^\times \). This is reduced by base change to the case that \( R = k \) is a perfect field in the same way as in the proof of ([LZ07] Thm. 5.5), and then follows from ([Fon83], p.91) and ([Kat87], Prop. 2.5).
2.2 Proof of Theorem 1.2

We are going to explicitly construct displays over the relative Witt frames. For this, let \( S \to R \) be a homomorphism of rings in which \( p \) is nilpotent and such that the kernel \( \mathfrak{a} \) is equipped with divided powers. Then the kernel of \( W(S) \to R \) is \( \mathfrak{a} \oplus I_S \) where \( I_S := VW(S) \) and \( \mathfrak{a} \) is the logarithmic Teichmüller ideal. Then we consider the relative Witt frame \( \mathcal{W}_{S/R} \) as defined in [15].

We first prove a relative version of Theorem 2.1

**Theorem 2.2.** Let \( X_S/\text{Spec } S \) be a smooth projective lifting of \( X/\text{Spec } R \) and assume that \( X_S \) admits a compatible system of liftings \( (X_S)_n \) over \( \text{Spec } W_n(S) \). Set \( I_S := VW_{n-1}(S) \). Let \( \text{Fil}^r_{rel/R} \Omega^\bullet_{(X_S)_n/W_n(S)} \) be the following complex:

\[
I_S \Omega(X_S)_n \oplus \tilde{a}^r \Omega(X_S)_n \xrightarrow{pd\oplus d} I_S \Omega^1(X_S)_n/W_n(S) \oplus \tilde{a}^r \Omega^1(X_S)_n/W_n(S) \xrightarrow{pd\oplus d} \cdots
\]

and let \( N^r_{rel/R} W_n \Omega^\bullet_{X_S/S} \) be the following complex:

\[
(W_{n-1} \Omega(X_S)_n)[F] \oplus \tilde{a}^r W_n \Omega(X_S)_n \xrightarrow{d\oplus d} (W_{n-1} \Omega^1(X_S)_n)[F] \oplus \tilde{a}^r W_n \Omega^1(X_S)_n \xrightarrow{d\oplus d} \cdots
\]

Then for \( r < p \) the complexes \( \text{Fil}^r_{rel/R} \Omega^\bullet_{(X_S)_n/W_n(S)} \) and \( N^r_{rel/R} W_n \Omega^\bullet_{X_S/S} \) are quasi-isomorphic.

**Proof.** First notice that we may write \( \text{Fil}^r_{rel/R} \Omega^\bullet_{(X_S)_n/W_n(S)} \) and \( N^r_{rel/R} W_n \Omega^\bullet_{X_S/S} \) as the mapping fibres of certain morphisms of complexes:

\[
\text{Fil}^r_{rel/R} \Omega^\bullet_{(X_S)_n/W_n(S)} = \text{MF}(\tilde{a}^r \Omega^\bullet_{(X_S)_n/W_n(S)} \xrightarrow{f} \Omega^\bullet_{(X_S)_n/W_n(S)})
\]

and

\[
N^r_{rel/R} W_n \Omega^\bullet_{X_S/S} = \text{MF}(\tilde{a}^r W_n \Omega^\bullet_{X_S/S} \xrightarrow{g} \Omega^\bullet_{X_S/S})
\]

where \( \tilde{a}^r \Omega^\bullet_{(X_S)_n/W_n(S)} \) and \( \tilde{a}^r W_n \Omega^\bullet_{X_S/S} \) are the truncated complexes

\[
0 \to \tilde{a}^r \Omega(X_S)_n \xrightarrow{-d} \tilde{a}^r \Omega^1(X_S)_n/W_n(S) \xrightarrow{-d} \cdots \xrightarrow{-d} \tilde{a}^r \Omega^r(X_S)_n/W_n(S) \to 0 \to \cdots
\]

and

\[
0 \to \tilde{a}^r W_n \Omega(X_S)_n \xrightarrow{-d} \tilde{a}^r \Omega^1(X_S)_n \xrightarrow{-d} \cdots \xrightarrow{-d} \tilde{a}^r W_n \Omega^1(X_S)_n \to 0 \to \cdots
\]

respectively. The morphisms \( f \) and \( g \) are respectively given by

\[
\begin{array}{cccccccc}
0 & \xrightarrow{-d} & \tilde{a}^r \Omega & \xrightarrow{-d} & \tilde{a}^r \Omega^2 & \xrightarrow{-d} & \tilde{a}^r \Omega^3 & \cdots \\
I_S \Omega & \xrightarrow{pd} & I_S \Omega^1 & \xrightarrow{pd} & I_S \Omega^2 & \xrightarrow{pd} & I_S \Omega^3 & \xrightarrow{pd} \\
0 & \xrightarrow{0} & 0 & \xrightarrow{a} & 0 & \xrightarrow{d} & 0 & \xrightarrow{d} \\
\end{array}
\]

(2.3)
and

\[
\begin{array}{ccccccccc}
0 & \to & \tilde{a}^rW_n\mathcal{O} & \overset{d}{\to} & \cdots & \overset{d}{\to} & \tilde{a}^2W_n\Omega^{r-2} & \overset{d}{\to} & \tilde{a}W_n\Omega^{r-1} & \to & 0 & \cdots \\
(\mathcal{O}[F])_n & \overset{d}{\to} & (W_n\mathcal{O})[F]_n & \overset{d}{\to} & \cdots & \overset{d}{\to} & (W_n\mathcal{O})^{r-1}[F]_n & \overset{dV}{\to} & W_n\Omega^r & \overset{d}{\to} & W_n\Omega^{r+1} & \cdots \\
\end{array}
\]

(we briefly omitted the subscripts for typographical reasons). We therefore have a morphism of distinguished triangles in the derived category

\[
\begin{array}{ccccccccc}
\tilde{a}^{(r)}\Omega^\bullet_{(X_n)_n/W_n(S)} & \overset{f}{\to} & \mathcal{F}^\bullet\Omega^\bullet_{(X_n)_n/W_n(S)} & \overset{\mathcal{F}\Omega^\bullet_{rel/R}}{\to} & \mathcal{O}^\bullet_{rel/R\Omega^\bullet_{(X_n)_n/W_n(S)}} \\
\tilde{a}^{(r)}W_n\Omega^\bullet_{X/S} & \overset{g}{\to} & \mathcal{N}^rW_n\Omega^\bullet_{X/S} & \overset{\mathcal{N}^r\Omega^\bullet_{rel/R\Omega^\bullet_{X/S}}}{\to} & \mathcal{O}^\bullet_{rel/R\Omega^\bullet_{X/S}} \\
\end{array}
\]

(2.4)

where the middle vertical arrow is the quasi-isomorphism of Theorem 2.1 and the left vertical map is defined as follows:

Assume first that there exists a closed embedding \((X_n)_n \hookrightarrow Z_n\) into a projective smooth \(W_n(S)\)-scheme which is a Witt lift of \(Z_n\times\text{Spec} \ W_n(S)\text{Spec} S = Z\) in the sense of [LZ04] Definition 3.3. Such a Witt lift always exists locally and induces maps \(\mathcal{O}_{Z_n} \to W_\mathcal{O} \to W_\mathcal{O}(X_n)\). Let \(\mathcal{O}_{D_n}\) be the PD-envelope of \(i\) and let \(\mathcal{J}^{[r]}\) be the divided power ideal with \(J = \ker(\mathcal{O}_{Z_n} \to \mathcal{O}_{(X_n)_n})\). The comparison with crystalline cohomology yields a chain of quasi-isomorphisms

\[
\Omega^\bullet_{(X_n)_n/W_n(S)} \cong \Omega^\bullet_{D_n/W_n(S)} \overset{\sim}{\to} W_n\Omega^\bullet_{X/S} \\
\]

(2.5)

We construct a complex \(\tilde{a}^{(r)}\Omega^\bullet_{D_n/W_n(S)}\) together with a diagram of maps

\[
\tilde{a}^{(r)}\Omega^\bullet_{D_n/W_n(S)} \to \tilde{a}^{(r)}\Omega^\bullet_{D_n/W_n(S)} \to \tilde{a}^{(r)}W_n\Omega^\bullet_{X/S} \\
\]

(2.7)

The argument is very similar to the proof of [Lan16] Theorem 0.2. Consider the following diagram for \(r < p\)

\[
\begin{array}{ccccccccc}
\tilde{a}^r\mathcal{O}_{D_n} & \overset{d}{\to} & \tilde{a}^r\Omega^\bullet_{D_n} & \overset{d}{\to} & \cdots & \overset{d}{\to} & \tilde{a}^r\Omega^{r-1} & \overset{d}{\to} & \tilde{a}^r\Omega^r & \overset{d}{\to} & 0 \\
\tilde{a}^r\mathcal{J}^{[r-1]} & \overset{d}{\to} & \tilde{a}^r\mathcal{J}^{[r-2]} & \overset{d}{\to} & \cdots & \overset{d}{\to} & \tilde{a}^r\mathcal{J}^{[r-3]} & \overset{d}{\to} & \cdots & \overset{d}{\to} & 0 \\
\tilde{a}^r\mathcal{J}^{[r-2]} & \overset{d}{\to} & \tilde{a}^r\mathcal{J}^{[r-3]} & \overset{d}{\to} & \cdots & \overset{d}{\to} & \tilde{a}^r\mathcal{J}^{[r-4]} & \overset{d}{\to} & \cdots & \overset{d}{\to} & 0 \\
\tilde{a}^r\mathcal{J}^{[r-3]} & \overset{d}{\to} & \tilde{a}^r\mathcal{J}^{[r-4]} & \overset{d}{\to} & \cdots & \overset{d}{\to} & \tilde{a}^r\mathcal{J}^{[r-5]} & \overset{d}{\to} & \cdots & \overset{d}{\to} & 0 \\
\end{array}
\]

(2.8)

The complex \(\mathcal{J}^{[r]} \to \mathcal{J}^{[r-1]} \to \cdots \to \mathcal{J}^{[0]} \to \mathcal{J}^{[-1]}\) is exact - up to \(\Omega^\bullet_{D_n}\) - and quasi-isomorphic to \(\Omega^\bullet_{(X_n)_n/W_n(S)[−s]}\) by [BO78] Theorem 7.2. Since all \(\mathcal{J}^{[r]}\mathcal{O}_{D_n}\) are
- locally - free $\mathcal{O}_S$-modules by [BO78] Prop. 3.32, the complexes $\mathbb{J}^{[s-\epsilon]}_n \Omega^n_{X/S}$ remain exact after $\otimes W_n(S) \mathbb{R}$ and $\otimes W_n(S) R$; since $\mathbb{a}$ is a direct summand of $\ker(W_n(S) \to R)$ the lower horizontal sequence in $(2.8)$ is exact - up to the diagonal term - and quasi-isomorphic to $\mathbb{a} \Omega^{n-1}_{(X/S),n}/W_n(S) \to \mathbb{a} \Omega^{n-1}_{(X/S),n}/W_n(S)[-(r-1)]$. By an easy induction argument (replace $S$ by $S/\mathbb{a}$) one sees that the other horizontal sequences in $(2.8)$ are - up to the diagonal term - exact as well. The degree wise sum of the two lower sequences is quasi-isomorphic to

\[
\left( \mathbb{a}^2 \Omega^{n-2}_{(X/S),n}/W_n(S) \xrightarrow{\partial} \mathbb{a} \Omega^{n-1}_{(X/S),n}/W_n(S) \right)[-(r-2)]
\]

Finally, the degree wise sum of all horizontal sequences yields a complex denoted by $\mathbb{a}^{(r)} \Omega^\bullet_{(X/S),n}/W_n(S)$ which is quasi-isomorphic to $\mathbb{a}^{(r)} \Omega^\bullet_{(X/S),n}/W_n(S)$.

Note that under the canonical map $\mathcal{O}_{D_0} \to W_n(\mathcal{O}_X)$ the image of $J$ is contained in $VW_{n-1}(\mathcal{O}_X)$ and hence the image of $a : J$ is zero in $W_n(\mathcal{O}_X)$. Hence the map $\mathcal{O}_{D_0} \to W_n(\mathcal{O}_X)$, compatible with Frobenius, induces a well-defined map

$$
\mathbb{a}^{(r)} \Omega^\bullet_{D_0}/W_n(S) \to \mathbb{a}^{(r)} W_n \Omega^\bullet_{X/S}/S
$$

If one has two embeddings $(X/S)_n \to Z_n$, $(X/S)_n' \to Z_n'$ into Witt lifts then by considering the product embedding $(X/S)_n \to Z_n \times Z_n'$ we get a well-defined map in the derived category

$$
\mathbb{a}^{(r)} \Omega^\bullet_{(X/S),n}/W_n(S) \to \mathbb{a}^{(r)} W_n \Omega^\bullet_{X/S}/S
$$

To show that it is a quasi-isomorphism is a Zariski-local question on $X_S$, so it suffices to check the case where $X_S = \text{Spec} B$ is affine and $B$ is étale over a polynomial algebra $A := S[T_1, \ldots, T_d]$. Set $A_n := W_n(S)[T_1, \ldots, T_d]$ and let $\phi_n : A_n \to A_{n-1}$ be the map extending $F : W_n(S) \to W_{n-1}(S)$ given by setting $\phi_n(T_i) = T_i^n$, and let $\delta_n : A_n \to W_n(A)$ be the unique $W_n(S)$-algebra homomorphism which sends each $T_i$ to its Teichmüller representative. Then the data $(A_n, \phi_n, \delta_n)$ is a Frobenius lift of $A$ to $W(S)$ (see §3 of [LZ04]). Since $A \to B$ is étale, there exists a unique set of liftings $B_n$ of $B$ which are each étale over $A_n$, and homomorphisms $\psi_n : B_n \to B_{n-1}$, $\epsilon_n : B_n \to W_n(B)$ which are compatible with $\phi_n, \delta_n$. The morphism $\mathbb{a}^{(r)} \Omega^\bullet_{B_n}/W_n(S) \to \mathbb{a}^{(r)} W_n \Omega^\bullet_{B/S}$ is the one induced by the $\epsilon_n$.

First let us treat the special case that $B = A = S[T_1, \ldots, T_d]$; the quasi-isomorphism is easy to establish because in this case the de Rham-Witt complex has a rather explicit description (originally due to Illusie in the case of a perfect field, and by §2 of [LZ04] in our generality). Indeed, the de Rham-Witt complex decomposes into a direct sum of an integral part and an acyclic fractional part ([LZ04], (3.9)) and the fractional part is contained in the image of $V$ resp. $dV$. We must check that the fractional part is still acyclic after multiplying by the logarithmic Teichmüller ideal $\mathbb{a}$. But multiplying anything in the image of $V$ by an element $a$ means applying Frobenius to $a$, and Frobenius kills $\mathbb{a}$, so we conclude that the fractional part of the de Rham-Witt complex is annihilated by $\mathbb{a}$.

Now we return to the general case where $X_S = \text{Spec} B$ for $B$ étale over $A = S[T_1, \ldots, T_d]$. Choose an integer $m$ such that $p^m W_n(S) = 0$ and set $\phi^m := \phi_{m+n} \circ \cdots \circ \phi_{n+1} : A_{m+n} \to A_n$. Then for each $l$ we have an isomorphism

$${\Omega}^l_{B_n}/W_n(S) \cong B_n \otimes_{A_n} {\Omega}^l_{A_n}/W_n(S) \cong B_{m+n} \otimes_{A_{m+n}} \phi^m {\Omega}^l_{A_n}/W_n(S)$$
and, likewise, for each \( j, l \) an isomorphism
\[
\tilde{a}^j \Omega^l_{A_n/W_n(S)} \cong B_n \otimes A_n \tilde{a}^j \Omega^l_{A_n/W_n(S)} \cong B_{m+n} \otimes A_{m+n} \phi^m \tilde{a}^j \Omega^l_{A_n/W_n(S)}
\]
and this gives an isomorphism of complexes
\[
\tilde{a}^{(r)} \Omega^l_{A_n/W_n(S)} \cong B_{m+n} \otimes A_{m+n} \phi^m \tilde{a}^{(r)} \Omega^l_{A_n/W_n(S)}
\]
if we give the right-hand side the differential \( 1 \otimes -d \).

Let
\[
W_n \Omega^i_{A/S} = W_n \Omega^i_{A/S} \oplus W_n \Omega^{\text{frac} \cdot i}_{A/S}
\]
be the decomposition into integral and fractional parts, as mentioned above. Note that this is a direct sum decomposition of complexes of \( A_{m+n} \)-modules via restriction of scalars \( A_{m+n} \xrightarrow{\phi^m} A_n \). Then
\[
W_n \Omega^i_{B/S} = (B_{m+n} \otimes A_{m+n} W_n \Omega^{\text{int} \cdot i}_{A/S}) \oplus (B_{m+n} \otimes A_{m+n} W_n \Omega^{\text{frac} \cdot i}_{A/S})
\]
Since \( W_n \Omega^{\text{frac} \cdot i}_{A/S} \) is acyclic and \( B_{m+n} \) is a flat \( A_{m+n} \)-module, \( W_n \Omega^{\text{frac} \cdot i}_{B/S} \) is acyclic too.

Then we define \( \tilde{a}^{(r)} W_n \Omega^i_{B/S}, \tilde{a}^{(r)} W_n \Omega^{\text{int} \cdot i}_{B/S} \) and \( \tilde{a}^{(r)} W_n \Omega^{\text{frac} \cdot i}_{B/S} \) as before. Evidently we get a direct sum decomposition
\[
\tilde{a}^{(r)} W_n \Omega^i_{B/S} = \tilde{a}^{(r)} W_n \Omega^{\text{int} \cdot i}_{B/S} \oplus \tilde{a}^{(r)} W_n \Omega^{\text{frac} \cdot i}_{B/S}
\]
Since \( W_n \Omega^{\text{int} \cdot i}_{A/S} \cong \Omega^{\text{int} \cdot i}_{A_n/W_n(S)} \), we get an isomorphism
\[
\tilde{a}^{(r)} W_n \Omega^{\text{int} \cdot i}_{B/S} \cong B_{m+n} \otimes A_{m+n} \phi^m \tilde{a}^{(r)} W_n \Omega^{\text{int} \cdot i}_{A/S}
\cong \tilde{a}^{(r)} \Omega^i_{B_n/W_n(S)}
\]
Since \( \tilde{a} \) annihilates the fractional part \( W_n \Omega^{\text{frac} \cdot i}_{A/S} \) as observed above, we get that \( \tilde{a}^{(r)} W_n \Omega^{\text{frac} \cdot i}_{B/S} \) vanishes. Hence we obtain an isomorphism
\[
\tilde{a}^{(r)} W_n \Omega^i_{B/S} \cong \tilde{a}^{(r)} \Omega^i_{B_n/W_n(S)}
\]
as desired. In the absence of a global embedding into a Witt lift one proceeds by simplicial methods as in the proof of [Lan16] Theorem 0.2, [LZ04] §3.2 and [Il79] II.1. to obtain Theorem 2.2.

We now prove Theorem 2.2(a). As in Theorem 2.1 the isomorphisms between the complexes in Theorem 2.2 are compatible for varying \( n \). One first assumes the existence of a compatible system of embeddings into Witt lifts; in the general case one uses again simplicial methods as outlined in [Lan16] to obtain an isomorphism of procomplexes \( \mathcal{F}il^r_{rel/R} \Omega^\bullet_{(X_S)_r/diag(S)} \cong \mathcal{N}^r_{rel/R} W_n \Omega^\bullet_{X_S/S} \).

Let
\[
(P_{S/R})_r = \mathcal{H}^n(S, \mathcal{N}^r_{rel/R} W_n \Omega^\bullet_{X_S/S}) \cong \mathcal{H}^n(S, \mathcal{F}il^r_{rel/R} \Omega^\bullet_{(X_S)_r/diag(S)})
\]
Using the same argument as in the proof of Theorem 1.1(a), we see that the $E_1$-spectral sequence associated to the complex $\mathcal{F}il^r_{rel/R} \Omega^{\ast}(X_S)_{n/W_n(S)}$ degenerates. This implies a decomposition

$$(\mathcal{P}_{S/R})_r = \mathcal{J}_r L_0 \oplus \mathcal{J}_{r-1} L_1 \oplus \cdots \oplus \mathcal{J}_{L_r-1} L_r \oplus \cdots \oplus L_n$$

where $L_i = H^{n-i}((X_S)_{\bullet}, \Omega^{\ast}(X_S)_{n/W_n(S)})$ and $\mathcal{J}_i = \tilde{\mathfrak{a}}^i \oplus I_S$ (compare the construction of standard displays over the relative Witt frame $\mathcal{W}_{S/R}$ in [LZ15]).

The maps $\tilde{F}_r : (\mathcal{P}_{S/R})_r \to (\mathcal{P}_{S/R})_0 = H^{n}_{\text{cris}}(X/W(S))$ induce maps $\Phi_r : L_r \to (\mathcal{P}_S)_0$ by $\Phi_r = \tilde{F}_r | L_r$. To show that $((\mathcal{P}_{S/R})_r, \tilde{F}_r, i_r, \alpha_r)$ defines a relative display on $H^{n}_{\text{cris}}(X/W(S))$ is equivalent to the condition that

$$\bigoplus_{i=0}^{n} \Phi_i : L_0 \oplus \cdots \oplus L_n \to (\mathcal{P}_S)_0$$

is a $\sigma$-linear isomorphism. The argument is the same as in the proof of Theorem 1.1(a). On the level of standard displays (i.e. displays given by standard data), the relative display associated to the display $\mathcal{P}_S$ is given by the inclusions $I_S L_0 \oplus \cdots \oplus I_S L_{r-1} \oplus L_r \oplus \cdots \oplus L_n \hookrightarrow \mathcal{J}_r L_0 \oplus \cdots \oplus \mathcal{J}_{L_r-1} L_r \oplus \cdots \oplus L_n$

Since the chain of quasi-isomorphisms between $\mathcal{F}il^r_{rel/R} \Omega^{\ast}(X_S)_{n/W_n(S)}$ and between $\mathcal{F}il^r_{rel/R} \Omega^{\ast}(X_S)_{n/W_n(S)}$ and $\mathcal{F}il^r_{rel/R} W_n \Omega^{\ast}(X_S)_{n/S}$ are compatible under the inclusion maps

$$\mathcal{F}il^r_{rel/R} \Omega^{\ast}(X_S)_{n/W_n(S)} \hookrightarrow \mathcal{F}il^r_{rel/R} \Omega^{\ast}(X_S)_{n/W_n(S)}$$

and

$$\mathcal{F}il^r_{rel/R} W_n \Omega^{\ast}(X_S)_{n/S} \hookrightarrow \mathcal{F}il^r_{rel/R} W_n \Omega^{\ast}(X_S)_{n/S}$$

we conclude that $u_* \mathcal{P}_S = \mathcal{P}_{S/R}$. This proves Theorem 1.2.

2.3 Proof of Theorem 1.1(b)

Let $A \to R$ be a frame for $R$ such that the kernel is equipped with divided powers, by definition $A$ is equipped with a lifting $\sigma : A \to A$ of the Frobenius $A/pA \to A/pA$. We consider the Lazard map $A \to W(A)$ into the Witt ring. Then $A \to R$ factors through $A \to W(A) \to W(R) \to R$

The kernel $\mathcal{J}$ of $W(A) \to R$ is $\tilde{\mathfrak{a}} \oplus VW(A) = \tilde{\mathfrak{a}} \oplus I_A$, where $\tilde{\mathfrak{a}}$ is the logarithmic Teichmüller ideal, equipped again with divided powers. We then get a second frame $(W(A), \mathcal{J}, \sigma, \hat{\sigma})$ for $R$, where $\sigma$ is the Frobenius on $W(A)$ and $\hat{\sigma} : \mathcal{J} \to W(A)$, $a + V\xi \mapsto \xi$. This is the definition of the relative Witt frame $\mathcal{W}_{A/R}$.

Assuming the existence of liftings $\mathcal{Y}/\text{Spf}X$ of $A$ that satisfy (A1) and (A2), we get by base change liftings $\mathcal{Y}/\text{Spf}W(A)$ that also satisfy (A1) and (A2). It is therefore enough to show Theorem 1.1(b) by working with the relative Witt frame $\mathcal{W}_{A/R}$ and the lifting $\mathcal{Y}$. Then a window over $W(A)$ ([LZ07], Def. 5.1) is the same as a display $\mathcal{P}_{A/R}$ over the relative Witt frame $\mathcal{W}_{A/R}$. We denote now...
by $\mathcal{P}_{A/R}$ the display associated to the lifting $\tilde{Y}$ that exists by [LZ07, Thm. 5.5]. Let $Y_{m,*} := \tilde{Y} \times_{W(A)} W_s(A/p^m)$. Then

\[
\langle \mathcal{P}_{A/R} \rangle_r = \lim_{\overset{s}{\longrightarrow}} H^{n}_{\text{cris}}(X, \mathcal{J}^{[r]}_X/W_s(A/p^m))
\]

is equipped with a divided Frobenius $F_r = \frac{p^r}{p}$ where $F$ is the Frobenius on crystalline cohomology. Assume $A \to R$ factors through $A/p^m \to R$. By [BO78, Thm. 7.2] the groups

\[
H^{n}_{\text{cris}}(X, \mathcal{J}^{[r]}_X/W(A/p^m)) = \lim_{\overset{s}{\longrightarrow}} H^{n}_{\text{cris}}(X, \mathcal{J}^{[r]}_X/W_s(A/p^m))
\]

are the hypercohomology groups of the procomplexes $F\tilde{\mathcal{H}}^{[r]}_Y \Omega_{Y,m/W_s(A/p^m)}^\bullet$ defined as follows:

\[
\begin{align*}
(\tilde{a}^r_m \oplus p^{r-1}I_A/p^m)\Omega^0_{Y,m/W_s(A/p^m)} & \xrightarrow{d} (\tilde{a}^{r-1}_m \oplus p^{r-2}I_A/p^m)\Omega^0_{Y,m/W_s(A/p^m)} \xrightarrow{d} \cdots \\
& \xrightarrow{d} \cdots \xrightarrow{d} (\tilde{a}_m \oplus I_A/p^m)\Omega^{r-1}_{Y,m/W_s(A/p^m)} \xrightarrow{d} \Omega^r_{Y,m/W_s(A/p^m)} \xrightarrow{d} \cdots \\
& \xrightarrow{d} \cdots
\end{align*}
\]

where $\tilde{a}_m$ is the logarithmic Teichmüller ideal associated to $a_m := \ker(A/p^m \to R)$ and $I_{A/p^m} := VW(A/p^m)$, and we have used that $\tilde{a}_m : I_{A/p^m} = 0$.

As $A$ and $W(A)$ are $p$-torsion free, a modified version of ([LZ07, Prop. 4.4]) yields that the procomplexes $F\tilde{\mathcal{H}}^{[r]}_Y \Omega_{Y,m/W_s(A/p^m)}^\bullet$ and $F\tilde{\mathcal{H}}^{[r]}_Y \Omega_{Y,m/W_s(A/p^m)}^\bullet$ defined as

\[
\begin{align*}
(\tilde{a}^r_m \oplus I_A/p^m)\Omega^0_{Y,m/W_s(A/p^m)} & \xrightarrow{d+pd} (\tilde{a}^{r-1}_m \oplus I_A/p^m)\Omega^0_{Y,m/W_s(A/p^m)} \xrightarrow{d+pd} \cdots \\
& \xrightarrow{d+pd} \cdots \xrightarrow{d+pd} (\tilde{a}_m \oplus I_A/p^m)\Omega^{r-1}_{Y,m/W_s(A/p^m)} \xrightarrow{d+pd} \Omega^r_{Y,m/W_s(A/p^m)} \xrightarrow{d+pd} \cdots \\
& \xrightarrow{d+pd} \cdots
\end{align*}
\]

are quasi-isomorphic. By Theorem [22] the procomplexes $F\tilde{\mathcal{H}}^{[r]}_Y \Omega_{Y,S/W_s(A/p^m)}^\bullet$ and $\mathcal{N}_{rel/R} W_{\Omega_{Y,S/W_s(A/p^m)}}^\bullet$, defined as

\[
\begin{align*}
\tilde{a}^r_m W_s \mathcal{O}_Y \oplus W_s \mathcal{O}_Y & \xrightarrow{d} \tilde{a}^{r-1}_m W_s \mathcal{O}_Y \oplus W_s \mathcal{O}_Y \xrightarrow{d} \cdots \\
& \xrightarrow{d} \cdots \xrightarrow{d} \tilde{a}_m W_s \mathcal{O}_Y^{r-1} \oplus W_s \mathcal{O}_Y^{r-1} \xrightarrow{d+dv} W_s \mathcal{O}_Y^r \xrightarrow{d} \cdots \\
& \xrightarrow{d} \cdots
\end{align*}
\]

are quasi-isomorphic. This implies that

\[
\langle \mathcal{P}_{A/R} \rangle_r = H^p(Y_s, \mathcal{N}_{rel/R} W_{\Omega_{Y,S/W_s(A/p^m)}}^\bullet)
\]

and the divided Frobenius on $\langle \mathcal{P}_{A/R} \rangle_r$ is induced by the divided Frobenius on $\mathcal{N}_{rel/R} W_{\Omega_{Y,S/W_s(A/p^m)}}^\bullet$. It is unique because $A$ and $W(A)$ are $p$-torsion free.

The morphism of frames $\mathcal{W}_{A/R} \xrightarrow{\epsilon} \mathcal{W}_R$ induces a base change $\epsilon_s$ on displays. Let $\tilde{X} := \tilde{Y} \times_{\text{Spf } W(A)} \text{Spec } W(R)$ be the induced lifting of $X$ over $W(R)$. The standard display defined on $\tilde{L}_0 \oplus \cdots \oplus \tilde{L}_n$ with

\[
\tilde{L}_i = H^i(\tilde{Y}, \Omega^{n-i}_{\tilde{Y}/\text{Spf } W(A)})
\]

is transformed into the standard display on $L_0 \oplus \cdots \oplus L_n$ with

\[
L_i = \tilde{L}_i \oplus_{W(A)} W(R) = H^i(\tilde{X}, \Omega^{n-i}_{\tilde{X}/W(R)})
\]
Note that under the composite map $\kappa : A \to W(A) \to W(R)$, $\kappa(a) \in I_R$ for $a \in a$, hence the image of $\hat{a} \oplus I_A$ in $W(R)$ is $I_R$.

We have canonical reduction maps

$$F^\bullet \Omega^{\bullet}_{X/W(R)} \to F^\bullet \Omega^{\bullet}_{X/W(R)}$$

and

$$N^r_{\text{rel}/R} W^\bullet \Omega_{X/R}^{\bullet} \to N^r W^\bullet \Omega_{X/R}^{\bullet}$$

Under the base change of displays $\epsilon_* \mathcal{P}_{A/R}$ is a display over $R$ with $(\epsilon_* \mathcal{P}_{A/R})$, given by the hypercohomology of $F^\bullet \Omega^{\bullet}_{X/W(R)}$. Since the quasi-isomorphisms between $F^\bullet \Omega^{\bullet}_{X/W(R)}$ and $N^r W^\bullet \Omega_{X/R}^{\bullet}$ are compatible under the canonical reduction maps, we see that the divided Frobenius $F_\tau$ on $(\epsilon_* \mathcal{P}_{A/R})$, obtained by base change coincides with the divided Frobenius on the Nygaard complexes $N^r W^\bullet \Omega_{X/R}^{\bullet}$. This finishes the proof of Theorem 1.2(b).

2.4 Proof of Theorem 1.2(b)

As in the theorem, we assume that $R$ is an artinian local $W(k)$-algebras with residue field $k$, and that the special fibre $X_0$ is a smooth projective variety with smooth versal deformation space $\mathcal{S}$. Write $X/\mathcal{S}$ for the versal family. Then $\mathcal{S} \cong \text{Spf } A$, where $A = W(k)[t_1, \ldots, t_h]$ is a formal power series algebra over $W(k)$.

Consider the Frobenius $\sigma : A \to A$ given by $\sigma = F$ on $W(k)$ and by $T_i \mapsto T_i^p$. Set $C_m := A/(T_1^{m}, \cdots, T_h^{m})$ and $R_m := C_m/p^m C_m$, and write $\sigma : C_m \to C_m$ for the endomorphism induced by $\sigma$. Then $C_m \to R_m$ is a frame for $R_m$. By Theorem 1.1 we have a display structure on $H^{n}_{\text{cris}}(X \times_R R_m/W(R_m))$ for $n < p$. For $m$ large enough there is a homomorphism $R_m \to R$, and hence a frame homomorphism $W_{R_m} \to W_R$. The display $\mathcal{P}_{R}$ on $H^{n}_{\text{cris}}(X/W(R))$ is the base change of the display on $H^{n}_{\text{cris}}(X \times_R R_m/W(R_m))$ along this frame homomorphism.

Suppose now that $X_S$ is a deformation of $X/R$ over a PD-thickening $S \to R$ and let us write $\mathcal{P}_{S}(X_S)$ for the $W_S$-display structure on $H^{n}_{\text{cris}}(X/W(S))$. Write $\nu : W_S \to W_{S/R}$ for the frame homomorphism. Then we must prove that the relative display $\mathcal{P}_{S/R} = \nu_* \mathcal{P}_{S}$ does not depend on the lifting $X_S$. That is to say, given another deformation $X'_S$ of $X$ over $S$ with associated display $\mathcal{P}_{S/R}(X'_S)$, the relative displays $\mathcal{P}_{S/R}(X_S) := \nu_* \mathcal{P}_{S}(X_S)$ and $\mathcal{P}_{S/R}(X'_S) := \nu_* \mathcal{P}_{S}(X'_S)$ coincide.

By the versality of $\mathcal{S}$, the deformations $X_S$ and $X'_S$ are induced by two $W(k)$-algebra homomorphisms $A \to S$. Let $A_{\text{triv}} = (A, 0, A, \sigma/p)$ be the trivial frame for $A$ and write $\mathcal{P}_{A}^{\text{triv}}$ for the $A_{\text{triv}}$-display structure on the versal family. Then $\mathcal{P}_{S/R}(X_S)$ and $\mathcal{P}_{S/R}(X'_S)$ are the base change of $\mathcal{P}_{A}^{\text{triv}}$ along the two induced frame homomorphisms

$$A_{\text{triv}} \xrightarrow{\nu} W_{S/R}$$

That is $\mathcal{P}_{S/R}(X_S) = x_\nu \mathcal{P}_{A}^{\text{triv}}$ and $\mathcal{P}_{S/R}(X'_S) = y_\nu \mathcal{P}_{A}^{\text{triv}}$.

Now consider the following diagram
Write $D_B(J)$ for the PD-envelope of $(B, J)$. Similarly, set $A_0 := W[T_1, \ldots, T_h]$, $B_0 := A_0 \otimes_W A_0$, $J_0 := \ker(B_0 \to A_0)$ and write $D_{B_0}(J_0)$ for the PD-envelope of $(B_0, J_0)$. Then $D_{B_0}(J_0)$ is the PD-polynomial algebra over $B_0$ in $h$ variables. Since $A = W(k)[[t_1, \ldots, t_h]]$ is flat over $A_0 = W(k)[t_1, \ldots, t_h]$, [BO78] Prop. 3.21 gives that $D_B(J)$ is the PD-polynomial algebra over $B$ in $h$ variables. In particular, $D_B(J)$ is a flat $D_{B_0}(J_0)$-module, so is certainly $p$-torsion free. We get a diagram

$$
\begin{array}{ccc}
0 & \to & J & \to & B := A \otimes_W A \to A & \to 0 \\
& & \downarrow & & \downarrow & \\
& & S & \to & R & 
\end{array}
$$

Let $\hat{D_B(J)} := \varprojlim D_B(J)/p^n$ denote the $p$-adic completion of $D_B(J)$. Then $A = \{\hat{D_B(J)} \to A\}$ is a frame for $A$. By the universal property of PD-envelopes, the two homomorphisms $A \to S$ factor as

$$A \to D_B(J) \to S$$

and we get induced factorisations of the frame morphisms

$$A_{\text{triv}} \xrightarrow{x} A \xrightarrow{z} W_{S/R} \quad \text{and} \quad A_{\text{triv}} \xrightarrow{y} A \xrightarrow{z} W_{S/R}$$

We may now conclude since the $A$-display $\mathcal{P}_A$ is the base change of $\mathcal{P}_A^{\text{triv}}$ with respect to both $x$ and $y$, and hence $x_{\ast} \mathcal{P}_A^{\text{triv}} = y_{\ast} \mathcal{P}_A^{\text{triv}}$.

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Oli Gregory
Technische Universität München
Zentrum Mathematik - M11
Boltzmannstraße 3
85748 Garching bei München, Germany
deadline: oli.gregory@tum.de

Andreas Langer
University of Exeter
Mathematics
Exeter EX4 4QF
Devon, UK
deadline: a.langer@exeter.ac.uk