FEKETE-SZEGÖ PROBLEM FOR GENERALIZED BI-SUBORDINATE FUNCTIONS OF COMPLEX ORDER

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Abstract. In this paper, we obtain Fekete-Szegö inequality for the generalized bi-subordinate functions of complex order. The results, which are presented in this study, would generalize those in related works of several earlier authors.

1. INTRODUCTION

Let $A$ be the class of analytic functions in the open unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$ and let $S$ be the class of functions $f$ that are analytic and univalent in $D$ and are of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k.$$  

A function $f \in A$ is said to be subordinate to a function $g \in A$, denoted by $f \prec g$, if there exists a function $w \in B_0$ where

$$B_0 := \{w \in A : w(0) = 0, \ |w(z)| < 1, \ (z \in D)\},$$

such that

$$f(z) = g(w(z)), \quad (z \in D).$$

We let $S^*$ consist of starlike functions $f \in A$, that is $\Re\{zf'(z)/f(z)\} > 0$ in $D$ and $C$ consist of convex functions $f \in A$, that is $1 + \Re\{zf''(z)/f'(z)\} > 0$ in $D$. In terms of subordination, these conditions are, respectively, equivalent to

$$S^* \equiv \{f \in A : \frac{zf'(z)}{f(z)} \prec \frac{1+z}{1-z}\}$$

and

$$C \equiv \{f \in A : 1 + \frac{zf''(z)}{f'(z)} \prec \frac{1+z}{1-z}\}.$$ A generalization of the above two classes, according to Ma and Minda [23], are

$$S^*(\psi) \equiv \{f \in A : \frac{zf'(z)}{f(z)} \prec \psi(z)\}$$

and

$$C(\psi) \equiv \{f \in A : 1 + \frac{zf''(z)}{f'(z)} \prec \psi(z)\}$$

where $\psi$ is a positive real part function normalized by $\psi(0) = 1$, $\psi'(0) > 0$ and $\psi$ maps $D$ onto a region starlike with respect to 1 and symmetric with respect to the real axis. Obvious extensions of the above two classes (see [22]) are

$$S^*(\gamma;\psi) \equiv \{f \in A : 1 + \frac{zf'(z)}{f(z)} \prec \psi(z) ; \ \gamma \in \mathbb{C} \setminus \{0\}\}$$

and

$$C(\gamma;\psi) \equiv \{f \in A : 1 + \frac{zf''(z)}{f'(z)} \prec \psi(z) ; \ \gamma \in \mathbb{C} \setminus \{0\}\}.$$ In literature, the functions belonging to these classes are called Ma-Minda starlike and convex of complex order $\gamma$ ($\gamma \in \mathbb{C} \setminus \{0\}$), respectively.

Some of the special cases of the above two classes $S^*(\gamma;\psi)$ and $C(\gamma;\psi)$ are

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(1) $S^*\left((1, (1 + A)z)/(1 + Bz)\right) = S[A, B]$ and $C(1, (1 + A)z)/(1 + Bz)) = C[A, B], (-1 \leq B < A \leq 1)$ are classes of Janowski starlike and convex functions, respectively.

(2) $S^*\left((1 - \beta)e^{It} \cos \delta, (1 + z)/(1 - z)\right) = S^*[\beta, \delta]$ and $C((1 - \beta)e^{It} \cos \delta, (1 + z)/(1 - z)) = C[\beta, \delta]$,

\( (\beta) < \pi/2 \), $0 \leq \beta < 1$ are classes of $\delta$-spirallike and $\delta$-Robertson univalent functions of order $\beta$, respectively.

(3) $S^*\left((1, (1 + (1 - 2\beta)z)/z)/(1 - z)\right) = S^*[\beta]$ and $C((1, (1 + (1 - 2\beta)z)/z)/(1 - z)) = C[\beta]$ ($0 \leq \beta < 1$) are classes of starlike and convex functions of order $\beta$, respectively.

(4) $S^*\left(1, \left(\frac{1 + \beta}{1 - \beta}\right)\right) = S^*_\beta$ and $C\left(1, \left(\frac{1 + \beta}{1 - \beta}\right)\right) = C[\beta]$ are class of strongly starlike and convex functions of order $\beta$, respectively,

\begin{align*}
(5) \ S^*\left(1, \sqrt{1 + z}\right) = S^*_L = \left\{ f \in A : \left| \frac{zf'(z)}{f(z)} - 1 \right| < 1 \right\} \text{ is class of lemniscate starlike functions,} \\
(6) \ S^*\left(1, \gamma, (1 + z)/(1 - z)\right) = S^*[\gamma] \text{ and } C(1, \gamma, (1 + z)/(1 - z)) = C[\gamma] \quad (\gamma \in \mathbb{C}\setminus\{0\}) \text{ are classes of starlike and convex functions of complex order, respectively,} \\
(7) \ S^*\left(1, \gamma, (1 + z)/(1 - z)\right) = k - S^*_p = \left\{ f \in A : \Re\left(\frac{zf'(z)}{f(z)}\right) > k - 1 \right\} \text{ is class of } k-\text{parabolik starlike functions,} \\
(8) \ C(1, \gamma, (1 + z)/(1 - z)) = k - UCV = \left\{ f \in A : \Re\left(1 + \left|\frac{zf'(z)}{f(z)}\right|\right) > k - 1 \right\} \text{ is class of } k-\text{uniformly convex functions.}
\end{align*}

Here, for $0 \leq k < \infty$ the function $q_k : \mathbb{D} \to \{w = u + iv \in \mathbb{C} : \ u^2 > k^2 ((u - 1)^2 + v^2), \ u > 0\}$ has the form $q_k(z) = 1 + Q_1 z + Q_2 z^2 + \cdots$, $(z \in \mathbb{D})$ where

\begin{equation}
Q_1 = \begin{cases} 
\frac{2k^2}{1 - k}, & 0 \leq k < 1, \\
\frac{2k}{\sqrt{k^2 - 1}}, & k = 1, \\
\frac{2k}{\sqrt{(k^2 - 1)(1 + k^2 - 1)}} > k < 1, \\
\frac{2k}{\sqrt{(k^2 - 1)(1 + k^2)}} > k > 1,
\end{cases} \quad Q_2 = \begin{cases} 
\frac{(k^2 + 2)}{3}Q_1; & 0 \leq k < 1, \\
\frac{2k^2}{3}; & k = 1, \\
\frac{2k}{\sqrt{(k^2 - 1)(1 + k^2)}} > k > 1.
\end{cases}
\end{equation}

with $B = \frac{2}{\sqrt{\pi}} \arccos k$ and $K(t)$ is the complete elliptic integral of first kind (see [26]).

A function $f \in A$ is said to be bi-univalent in $\mathbb{D}$ if both $f$ and its inverse map $f^{-1}$ are univalent in $\mathbb{D}$. Let $\sigma$ be the class functions $f \in S$ that are bi-univalent in $\mathbb{D}$. For a brief history and interesting examples of functions which are in (or are not in) the class $\sigma$, including various properties of such functions we refer the reader to the work of Srivastava et al. [3] and references therein. Bounds for the first few coefficients of various subclasses of bi-univalent functions were obtained by a variety of authors including [19, 13, 2], [1], [21] and references therein. Not much was known about the bounds of the general coefficients $a_n; n \geq 4$ of subclasses of $\sigma$ up until the publication of the article [11] by Jahangiri and Hamidi and followed by a number of related publications. Moreover, many author have considered the Fekete-Szegö problem for various subclasses of $A$, the upper bound for $|a_3 - \mu a_2^2|$ is investigated by many different authors (see [10, 15, 9, 8]). In this paper, we apply the Fekete-Szegö inequality to certain subclass of generalized bi-subordinate functions of complex order.

2. Coefficient Estimates

In the sequel, it is assumed that $\varphi$ is an analytic function with positive real part in the unit disk $\mathbb{D}$, satisfying $\varphi(0) = 1$, $\varphi'(0) > 0$, and $\varphi(\mathbb{D})$ is symmetric with respect to the real axis. Such a function is known to be typically real with the series expansion $\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$ where $B_1, B_2$ are real and $B_1 > 0$. Motivated by a class of functions defined by the first author [2], we define the following comprehensive class of analytic functions

$$S(\lambda, \gamma; \varphi) \equiv \left\{ f \in A : 1 + \frac{1}{\gamma} \left( \frac{zf'(z) + \lambda z^2 f''(z)}{(1 - \lambda)f(z) - \lambda z f'(z)} - 1 \right) < \varphi(z); \ 0 \leq \lambda \leq 1, \ \gamma \in \mathbb{C}\setminus\{0\} \right\}.$$ 

A function $f \in A$ is said to be bi-univalent in the domain $\mathbb{D}$ of complex order $\gamma$ and type $\lambda$ if both $f$ and its inverse map $g = f^{-1}$ are in $S(\lambda, \gamma; \varphi)$. As special cases of the class $S(\lambda, \gamma; \varphi)$ we have $S(0, \gamma; \varphi) \equiv S^* (\gamma; \varphi)$ and $S(1, \gamma; \varphi) \equiv C(\gamma; \varphi).

To prove our next theorems, we shall need the following well-known lemma (see [2]).

**Lemma 2.1.** (see [2]) Let the function $w \in B_0$ be given by

$$w(z) = c_1 z + c_2 z^2 + \cdots (z \in \mathbb{D}),$$
then for by every complex number \( s \),

\[
|c_2 - se^2| \leq 1 + (|s| - 1) |c_1|^2.
\]

In the following theorem, we consider functional \( |a_2 - \mu a_2^2| \) for \( \gamma \) nonzero complex number and \( \mu \in \mathbb{C} \).

**Theorem 2.1.** Let \( \gamma \) be a nonzero complex number, \( \mu \in \mathbb{C} \) and \( 0 \leq \lambda \leq 1 \). If both functions \( f \) of the form \((2.1)\) and its inverse maps \( g = f^{-1} \) are in \( S(\lambda, \gamma; \varphi) \), then we obtain,

\[
|a_2| \leq \frac{|\gamma| |B_1|}{(1 + \lambda)}
\]

(2.1)

\[
|a_3| \leq \frac{|\gamma| |B_1|}{4(1 + 2\lambda)} \max\{2, (|s| + |t|)\}
\]

(2.2)

and

\[
|a_3 - \mu a_2^2| \leq \left\{ \begin{array}{ll}
\frac{B_1|\gamma|}{2(1 + 2\lambda)} & \text{if } L \leq 2, \\
\frac{B_1|\gamma|}{4(1 + 2\lambda)} L & \text{if } L \geq 2,
\end{array} \right.
\]

(2.3)

where \( s = \frac{B_1}{B_4}, -\frac{4B_1(1 + 2\lambda)}{B_4(1 + \lambda)}, t = \frac{B_1}{B_3} \) and \( L = \frac{B_1}{B_2} + (1 - \mu) \frac{4B_1(1 + 2\lambda)}{B_4(1 + \lambda)} \).

**Proof.** Let \( f(z) \in S(\lambda, \gamma; \varphi) \) and \( g = f^{-1} \). Then there are two functions \( u(z) = c_1 z + c_2 z^2 + \cdots (z \in \mathbb{D}) \) and \( v(w) = d_1 w + d_2 w^2 + \cdots \), such that

\[
1 + \frac{1}{\gamma} \left( \frac{zf''(z) + \lambda z^2 f''(z)}{(1 - \lambda)f(z) - \lambda z f'(z)} - 1 \right) = \varphi(u(z))
\]

(2.4)

\[
= 1 + \frac{(1 + \lambda) a_2}{\gamma} z + \left[ \frac{2(1 + 2\lambda) a_3 - (1 - \lambda)^2 a_2^2}{\gamma} \right] z^2 + \cdots
\]

and

\[
1 + \frac{1}{\gamma} \left( \frac{ wg''(w) + \lambda w^2 g''(w)}{(1 - \lambda)g(w) - \lambda wg'(w)} - 1 \right) = \varphi(v(w))
\]

(2.5)

\[
= 1 - \frac{(1 + \lambda) a_2}{\gamma} w + \left[ -\frac{2(1 + 2\lambda) a_3 + (3 + 6\lambda - \lambda^2) a_2^2}{\gamma} \right] w^2 + \cdots
\]

Equating coefficients of both side of equations (2.4) and (2.5) yield

\[
\frac{(1 + \lambda) a_2}{\gamma} = B_1 c_1, \quad \frac{2(1 + 2\lambda) a_3 - (1 - \lambda)^2 a_2^2}{\gamma} = B_1 c_2 + B_2 c_1,
\]

(2.6)

\[
\frac{-(1 + \lambda) a_2}{\gamma} = B_1 d_1, \quad \frac{-2(1 + 2\lambda) a_3 + (3 + 6\lambda - \lambda^2) a_2^2}{\gamma} = B_1 d_2 + B_2 d_1,
\]

(2.7)

so that, on account of (2.6) and (2.7),

\[
c_1 = -d_1,
\]

(2.8)

\[
a_2 = \frac{\gamma B_1}{(1 + \lambda)} c_1
\]

(2.9)

and

\[
a_3 = a_2^2 + \frac{\gamma}{4(1 + 2\lambda)} \left[ B_1 c_2 + B_2 c_1^2 - B_1 d_2 - B_2 d_1^2 \right], \quad |c_1| \leq 1.
\]

(2.10)

Taking into account (2.8), (2.9), (2.10) and Lemma 2.1, we obtain

\[
|a_2| = \left| \frac{\gamma B_1}{(1 + \lambda)} c_1 \right| \leq \frac{|\gamma| |B_1|}{(1 + \lambda)}
\]

(2.11)
and

\[
|a_3| = \left| a_2^2 + \frac{\gamma}{4(1 + 2\lambda)} \left[ B_1 c_2 + B_2 c_1^2 - B_1 d_2 - B_2 d_1^2 \right] \right|
\]

\[
= \left| \frac{\gamma^2 B_1^2}{(1 + \lambda)^2} c_2^2 + \frac{\gamma}{4(1 + 2\lambda)} \left[ (B_1 c_2 - B_2 d_1^2) - (B_1 d_2 - B_2 c_1^2) \right] \right|
\]

\[
= \left| \frac{\gamma^2 B_1^2}{(1 + \lambda)^2} c_2^2 + \frac{\gamma}{4(1 + 2\lambda)} \left[ (B_1 c_2 - B_2 c_1^2) - (B_1 d_2 - B_2 d_1^2) \right] \right|
\]

\[
= \frac{\gamma B_1}{4(1 + 2\lambda)} \left\{ c_2 - \left( \frac{B_2}{B_1} - \frac{4\gamma B_1 (1 + 2\lambda)}{(1 + \lambda)^2} \right) \right\} c_1^2 - \left[ d_2 - \frac{B_2}{B_1} d_1^2 \right] \right\}
\]

\[
\leq \frac{|\gamma| B_1}{4(1 + 2\lambda)} \left\{ c_2 - \left( \frac{B_2}{B_1} - \frac{4\gamma B_1 (1 + 2\lambda)}{(1 + \lambda)^2} \right) \right\} c_1^2 + \left[ d_2 - \frac{B_2}{B_1} d_1^2 \right] \right\}
\]

\[
\leq \frac{|\gamma| B_1}{4(1 + 2\lambda)} \left\{ 1 + (|s| - 1) |c_2^2| + 1 + (|t| - 1) |c_1^2| \right\}
\]

\[
= \frac{|\gamma| B_1}{4(1 + 2\lambda)} \max \{2, (|s| + |t|)\}
\]

Thus, we have

\[
|a_3| \leq \frac{|\gamma| B_1}{4(1 + 2\lambda)} \max \{2, (|s| + |t|)\},
\]

where \( s = \frac{B_2}{B_1} - \frac{4B_1 \gamma (1 + 2\lambda)}{(1 + \lambda)^2} \) and \( t = \frac{B_2}{B_1} \).

Furthermore,

\[
|a_3 - \mu a_2^2| = \left| (1 - \mu) a_2^2 + \frac{\gamma}{4(1 + 2\lambda)} \left[ B_1 c_2 + B_2 c_1^2 - B_1 d_2 - B_2 d_1^2 \right] \right|
\]

\[
= \frac{\gamma B_1}{4(1 + 2\lambda)} \left\{ c_2 - \left( \frac{B_2}{B_1} - (1 - \mu) \frac{4\gamma B_1 (1 + 2\lambda)}{(1 + \lambda)^2} \right) \right\} c_1^2 - \left[ d_2 - \frac{B_2}{B_1} d_1^2 \right] \right\}
\]

(2.12)

\[
\leq \frac{|\gamma| B_1}{4(1 + 2\lambda)} \left\{ 2 + \left( \frac{B_2}{B_1} - (1 - \mu) \frac{4\gamma B_1 (1 + 2\lambda)}{(1 + \lambda)^2} \right) + \frac{B_2}{B_1} - 2 \right\} \right\}
\]

As a result of this, we obtain

\[
|a_3 - \mu a_2^2| \leq \left\{ \begin{array}{ll}
\frac{B_1 (1 - \mu)}{2(1 + 2\lambda) L} & \text{if } \mathcal{L} < 2, \\
\frac{B_1 (1 - \mu) \frac{4B_1 \gamma (1 + 2\lambda)}{(1 + \lambda)^2}}{2(1 + 2\lambda) L} & \text{if } \mathcal{L} \geq 2.
\end{array} \right.
\]

where \( \mathcal{L} = \frac{B_2}{B_1} + (1 - \mu) \frac{4B_1 \gamma (1 + 2\lambda)}{(1 + \lambda)^2} \).

Thus the proof is completed.

We next consider the case, when \( \gamma \) and \( \mu \) are real. Then we have:

**Theorem 2.2.** Let \( \gamma > 0 \) and if both functions \( f \) of the form (1.1) and its inverse maps \( g = f^{-1} \) are in \( S(\lambda, \gamma; \varphi) \), then for \( \mu \in \mathbb{R} \),

(1) If \( |B_2| \geq B_1 \), we have

\[
|a_3 - \mu a_2^2| \leq \left\{ \begin{array}{ll}
\frac{|\gamma| B_2}{2(1 + 2\lambda)} - (\mu - 1) \frac{\gamma^2 B_1^2}{(1 + \lambda)^2} & \text{if } \mu \leq 1, \\
\frac{\gamma^2 B_1^2}{(1 + \lambda)^2} & \text{if } \mu > 1.
\end{array} \right.
\]

(2) If \( |B_2| < B_1 \), we have

\[
|a_3 - \mu a_2^2| \leq \left\{ \begin{array}{ll}
\frac{|\gamma| B_2}{2(1 + 2\lambda)} - (\mu - 1) \frac{\gamma^2 B_1^2}{(1 + \lambda)^2} & \text{if } \mu \leq 1 - \mathcal{F}, \\
\frac{\gamma^2 B_1^2}{2(1 + 2\lambda)} + (\mu - 1) \frac{\gamma^2 B_1^2}{(1 + \lambda)^2} & \text{if } 1 - \mathcal{F} < \mu < 1 + \mathcal{F}, \\
\frac{\gamma^2 B_1^2}{2(1 + 2\lambda)} + (\mu - 1) \frac{\gamma^2 B_1^2}{(1 + \lambda)^2} & \text{if } \mu \geq 1 + \mathcal{F}.
\end{array} \right.
\]
where \( F = \frac{(1 + \lambda)^2}{\zeta_1(B_1 - B_2)} \).

For each \( \mu \) there is a function \( f \in S(\lambda, \gamma; \varphi) \) such that equality holds.

**Proof.** Using (2.12) and Lemma 2.1, we obtain

\[
|a_3 - \mu a_2^2| = \left| \frac{\gamma B_1}{4(1 + 2\lambda)} \left( c_2 - \left( \frac{B_2}{B_1} - (1 - \mu) \frac{4\gamma B_1 (1 + 2\lambda)}{(1 + \lambda)^2} \right) c_1^2 \right) - \left( d_2 - \frac{B_2}{B_1} d_1^2 \right) \right|
\]

\[
\leq \left| \frac{\gamma |B_1|}{4(1 + 2\lambda)} \right| \left( 2 + \left( \frac{B_2}{B_1} - (1 - \mu) \frac{4\gamma B_1 (1 + 2\lambda)}{(1 + \lambda)^2} \right) + \frac{B_2}{B_1} - 2 \right) c_1^2
\]

(2.13)

\[
= \frac{\gamma B_1}{2(1 + 2\lambda)} + \left( \frac{\gamma |B_2| - \gamma B_1}{2(1 + 2\lambda)} + (\mu - 1) \frac{\gamma^2 B_1^2}{(1 + \lambda)^2} \right) |c_1^2|
\]

Now, the proof will be presented in two cases by considering \( B_1, B_2 \in \mathbb{R} \) and \( B_1 > 0 \).

Firstly, we want to consider the case with \( |B_2| \geq B_1 \). Several possible cases need to further analyze.

**Case 1:** If \( \mu \leq 1 \), using (2.13) and Lemma 2.1 we obtain

\[
|a_3 - \mu a_2^2| \leq \frac{\gamma B_1}{2(1 + 2\lambda)} + \left( \frac{\gamma |B_2| - \gamma B_1}{2(1 + 2\lambda)} + (1 - \mu) \frac{\gamma^2 B_1^2}{(1 + \lambda)^2} \right) |c_1^2|
\]

\[
\leq \frac{\gamma B_1}{2(1 + 2\lambda)} + \left( \frac{\gamma |B_2| - B_1}{2(1 + 2\lambda)} + (1 - \mu) \frac{\gamma^2 B_1^2}{(1 + \lambda)^2} \right)
\]

\[
= \frac{\gamma |B_2|}{2(1 + 2\lambda)} + \frac{\gamma B_1}{2(1 + 2\lambda)} - \frac{\gamma B_1}{2(1 + 2\lambda)} + \frac{\gamma B_1}{2(1 + 2\lambda)} - \frac{\gamma^2 B_1^2}{(1 + \lambda)^2} + \mu \frac{\gamma^2 B_1^2}{(1 + \lambda)^2}
\]

\[
= \frac{\gamma |B_2|}{2(1 + 2\lambda)} + (\mu - 1) \frac{\gamma^2 B_1^2}{(1 + \lambda)^2}
\]

**Case 2:** If \( \mu > 1 \), again using (2.13) and Lemma 2.1 we obtain

\[
|a_3 - \mu a_2^2| \leq \frac{\gamma B_1}{2(1 + 2\lambda)} + \left( \frac{\gamma |B_2| - B_1}{2(1 + 2\lambda)} + (\mu - 1) \frac{\gamma^2 B_1^2}{(1 + \lambda)^2} \right) |c_1^2|
\]

\[
\leq \frac{\gamma B_1}{2(1 + 2\lambda)} + \left( \frac{\gamma |B_2| - B_1}{2(1 + 2\lambda)} + (1 - \mu) \frac{\gamma^2 B_1^2}{(1 + \lambda)^2} \right)
\]

\[
= \frac{\gamma |B_2|}{2(1 + 2\lambda)} + \frac{\gamma B_1}{2(1 + 2\lambda)} - \frac{\gamma B_1}{2(1 + 2\lambda)} + \frac{\gamma B_1}{2(1 + 2\lambda)} - \frac{\gamma^2 B_1^2}{(1 + \lambda)^2} + \mu \frac{\gamma^2 B_1^2}{(1 + \lambda)^2}
\]

\[
= \frac{\gamma |B_2|}{2(1 + 2\lambda)} + (\mu - 1) \frac{\gamma^2 B_1^2}{(1 + \lambda)^2}
\]

Finally, we want to consider the case with \( |B_2| < B_1 \). By a similar way, several possible cases need to further analyze.

(i) Let \( \mu \leq 1 - F \), using (2.13) and Lemma 2.1 we have

\[
|a_3 - \mu a_2^2| \leq \frac{\gamma B_1}{2(1 + 2\lambda)} + \left( \frac{\gamma |B_2| - B_1}{2(1 + 2\lambda)} + (1 - \mu) \frac{\gamma^2 B_1^2}{(1 + \lambda)^2} \right) |c_1^2|
\]

\[
\leq \frac{\gamma B_1}{2(1 + 2\lambda)} + \left( \frac{\gamma |B_2| - B_1}{2(1 + 2\lambda)} + (1 - \mu) \frac{\gamma^2 B_1^2}{(1 + \lambda)^2} \right)
\]

\[
= \frac{\gamma |B_2|}{2(1 + 2\lambda)} - (\mu - 1) \frac{\gamma^2 B_1^2}{(1 + \lambda)^2}
\]

(ii) Let \( 1 - F < \mu \leq 1 \), using (2.13) and Lemma 2.1 we obtain

\[
|a_3 - \mu a_2^2| \leq \frac{\gamma B_1}{2(1 + 2\lambda)} + \left( \frac{\gamma |B_2| - B_1}{2(1 + 2\lambda)} + (1 - \mu) \frac{\gamma^2 B_1^2}{(1 + \lambda)^2} \right) |c_1^2|
\]

\[
\leq \frac{\gamma B_1}{2(1 + 2\lambda)}
\]
(iii) Let $1 < \mu < 1 + \mathcal{F}$, using (2.13) and Lemma 2.1, we obtain

$$
|a_3 - \mu a_2^2| \leq \frac{\gamma B_1}{2 (1 + 2 \lambda)} + \left\{ \frac{\gamma (|B_2| - B_1)}{2 (1 + 2 \lambda)} + \frac{\mu - 1}{1 + \frac{2 \lambda}{4}} \right\} |c_1^2| 
$$

Thus the proof is completed.

(iv) Let $\mu \geq 1 + \mathcal{F}$, using (2.13) and Lemma 2.1, we have

$$
|a_3 - \mu a_2^2| \leq \frac{\gamma B_1}{2 (1 + 2 \lambda)} + \left\{ \frac{\gamma (|B_2| - B_1)}{2 (1 + 2 \lambda)} + \frac{\mu - 1}{1 + \frac{2 \lambda}{4}} \right\} |c_1^2| 
$$

Finally, we consider the case, when $\gamma$ nonzero complex number and $\mu \in \mathbb{C}$. Then we get:

**Theorem 2.3.** Let $\gamma$ be a nonzero complex number and let both functions $f$ of the form (1.4) and its inverse maps $g = f^{-1}$ are in $\mathcal{S}(\lambda, \gamma; \varphi)$. Then for $\mu \in \mathbb{R}$ we have

1. If $\frac{(1 + |\sin \theta|)|B_2|}{2B_1} \geq 1$, we have

$$
|a_3 - \mu a_2^2| \leq \left\{ \begin{array}{ll}
\frac{|\gamma|^2 B_1^2}{(1 + \lambda)^2} (1 - \mu - \Re (k_1)) + \frac{|\gamma|^2 |B_2| (1 + |\sin \theta|)}{4(1 + 2 \lambda)} & \text{if } \mu \leq 1 - \Re (k_1), \\
\frac{|\gamma|^2 |B_2| (1 + |\sin \theta|)}{4(1 + 2 \lambda)} - \frac{|\gamma|^2 B_1^2}{(1 + \lambda)^2} (1 - \mu - \Re (k_1)) & \text{if } \mu > 1 - \Re (k_1). 
\end{array} \right.
$$

2. If $\frac{(1 + |\sin \theta|)|B_2|}{2B_1} < 1$, we obtain

$$
|a_3 - \mu a_2^2| \leq \left\{ \begin{array}{ll}
\frac{|\gamma|^2 B_1^2}{(1 + \lambda)^2} (1 - \mu - \Re (k_1)) + \frac{|\gamma|^2 |B_2| (1 + |\sin \theta|)}{4(1 + 2 \lambda)} & \text{if } \mu \leq 1 - \Re (k_1) + \mathcal{N}, \\
\frac{|\gamma|^2 |B_2| (1 + |\sin \theta|)}{4(1 + 2 \lambda)} - \frac{|\gamma|^2 B_1^2}{(1 + \lambda)^2} (1 - \mu - \Re (k_1)) & \text{if } 1 - \Re (k_1) + \mathcal{N} < \mu < 1 - \Re (k_1) - \mathcal{N}, \\
& \text{if } \mu \geq 1 - \Re (k_1) - \mathcal{N},
\end{array} \right.
$$

where $k_1 = \frac{B_2 (1 + \lambda)^2 e^{i \theta}}{4B_1^2 |\gamma| (1 + 2 \lambda)}$, $l_1 = \frac{|B_2| - 2B_1 (1 + \lambda)^2}{4B_1^2 |\gamma| (1 + 2 \lambda)}$, $|\gamma| = \gamma e^{i \theta}$ and $\mathcal{N} = \frac{(1 + |\sin \theta|)|B_2| (1 + |\sin \theta|)}{4B_1^2 |\gamma| (1 + 2 \lambda)}$

For each $\mu$ there is a function $\mathcal{S}(\lambda, \gamma; \varphi)$ such that the equality holds.

**Proof.** Suppose $f (z) = z + \sum_{k=2}^{\infty} a_k z^k \in \mathcal{S}(\lambda, \gamma; \varphi)$, using (2.12) and Lemma 2.1, then we obtain

$$
|a_3 - \mu a_2^2| \leq \frac{|\gamma| B_1}{4 (1 + 2 \lambda)} \left\{ \begin{array}{l}
2 + \left| \frac{B_2 - B_1 - (1 - \mu) \frac{4 \gamma B_1 (1 + 2 \lambda)}{(1 + \lambda)^2}}{B_1} \right| + \left| \frac{B_2}{B_1} - 2 \right| |c_1^2| \\
\frac{|\gamma| B_1}{2 (1 + 2 \lambda)} + \frac{|\gamma|^2 B_1^2}{(1 + \lambda)^2} \left[ (1 - \mu) - \frac{B_2 (1 + |\sin \theta|)}{1 + \lambda} \right] \left| \frac{B_2}{B_1} - 2 \right| |c_1^2| 
\end{array} \right. 
$$

(2.14)

Thus the proof is completed. □
Taking $|\gamma| = \gamma e^{i\theta}$, $k_1 = \frac{B_3(1+\lambda)^2 e^{i\theta}}{4B_1^2(1+2\lambda)^2}$ and $l_1 = \frac{(B_2 - 2B_1)(1+\lambda)^2}{4B_1^2(1+2\lambda)^2}$, a direct calculation with (2.14) shows that

\[ |a_3 - \mu a_2^2| \leq \frac{|\gamma| B_1}{2(1+2\lambda)} + \left| \frac{|\gamma|^2 B_2^2}{(1+\lambda)^2} (1 - \mu - k_1) + l_1 \right| |c_1^2| \]

\[ \leq \frac{|\gamma| B_1}{2(1+2\lambda)} + \left| \frac{|\gamma|^2 B_2^2}{(1+\lambda)^2} (1 - \mu - \Re(k_1) - i(\Im(k_1)) + l_1 \right| |c_1^2| \]

\[ \leq \frac{|\gamma| B_1}{2(1+2\lambda)} + \left| \frac{|\gamma|^2 B_1^2}{(1+\lambda)^2} (1 - \mu - \Re(k_1)) + \frac{|B_2|(1+\lambda)^2 |\sin \theta|}{4B_1^2 |\gamma|(1+2\lambda)} + l_1 \right| |c_1^2| \]

\[ = \frac{|\gamma| B_1}{2(1+2\lambda)} + \left| \frac{|\gamma|^2 B_2^2}{(1+\lambda)^2} (1 - \mu - \Re(k_1)) + \frac{|B_2||\sin \theta|}{|B_2| - 2B_1} \right| |c_1^2| \]

\[ = \frac{|\gamma| B_1}{2(1+2\lambda)} + \left| \frac{|\gamma|^2 B_1^2}{(1+\lambda)^2} (1 - \mu - \Re(k_1)) + \frac{|\gamma||B_2|(1+|\sin \theta|) - 2B_1}{4(1+2\lambda)} \right| |c_1^2|. \]

(2.15)

Now, we will make some discussions for several different cases by considering $B_1, B_2 \in \mathbb{R}$ and $B_1 > 0$.

Firstly, we want to consider the case with $\frac{2B_1}{|B_2|} - |\sin \theta| < 1$. Several possible cases need to further analyze.

Case 1: Let $\mu \leq 1 - \Re(k_1)$. Then from (2.15) and Lemma 2.1 we obtain

\[ |a_3 - \mu a_2^2| \leq \frac{|\gamma| B_1}{2(1+2\lambda)} + \left| \frac{|\gamma|^2 B_2^2}{(1+\lambda)^2} (1 - \mu - \Re(k_1)) + \frac{|\gamma||B_2|(1+|\sin \theta|) - 2B_1}{4(1+2\lambda)} \right| |c_1^2| \]

\[ \leq \frac{|\gamma| B_1}{2(1+2\lambda)} + \left| \frac{|\gamma|^2 B_2^2}{(1+\lambda)^2} (1 - \mu - \Re(k_1)) + \frac{|\gamma||B_2|(1+|\sin \theta|) - 2B_1}{4(1+2\lambda)} \right| |c_1^2| \]

\[ = \frac{|\gamma| B_1}{2(1+2\lambda)} + \left| \frac{|\gamma|^2 B_2^2}{(1+\lambda)^2} (1 - \mu - \Re(k_1)) + \frac{|\gamma||B_2|(1+|\sin \theta|) - 2B_1}{4(1+2\lambda)} \right| \frac{2(1+2\lambda)}{2(1+2\lambda)} \]

Case 2: Let $\mu > 1 - \Re(k_1)$, then from (2.15) and Lemma 2.1 we yield

\[ |a_3 - \mu a_2^2| \leq \frac{|\gamma| B_1}{2(1+2\lambda)} + \left| \frac{|\gamma|^2 B_2^2}{(1+\lambda)^2} (1 - \mu - \Re(k_1)) + \frac{|\gamma||B_2|(1+|\sin \theta|) - 2B_1}{4(1+2\lambda)} \right| |c_1^2| \]

\[ \leq \frac{|\gamma| B_1}{2(1+2\lambda)} + \left| \frac{|\gamma|^2 B_2^2}{(1+\lambda)^2} (\mu + \Re(k_1) - 1) + \frac{|\gamma||B_2|(1+|\sin \theta|) - 2B_1}{4(1+2\lambda)} \right| \frac{2(1+2\lambda)}{2(1+2\lambda)} \]

\[ = \frac{|\gamma||B_2|(1+|\sin \theta|) - 2B_1}{4(1+2\lambda)} \left| \frac{|\gamma|^2 B_2^2}{(1+\lambda)^2} (1 - \mu - \Re(k_1)) \right| \frac{2(1+2\lambda)}{2(1+2\lambda)} \]

Finally, we want to consider the case with $\frac{2B_1}{|B_2|} - |\sin \theta| > 1$. By a similar approximation, several possible cases need to further analyze.
(i) Let $\mu \leq 1 - \Re(k_1) + \mathcal{N}$, using (2.15) and Lemma 2.1 we have
\[
|a_3 - \mu a_2| \leq \frac{|\gamma|}{2(1 + 2\lambda)} \left[ |\gamma|^2 B_2^2 \frac{(1 + |\sin \theta|)}{(1 + \lambda)^2} \right] |c_1| + \frac{|\gamma|}{4(1 + 2\lambda)} \left[ |\gamma| ||B_2| (1 + |\sin \theta|) - 2B_1| \right] |c_1|
\]
\[
\leq \frac{|\gamma|}{2(1 + 2\lambda)} \left[ \frac{|\gamma|^2 B_2^2}{(1 + \lambda)^2} \right] \cdot |1 - \mu - \Re(k_1)| + \frac{|\gamma|}{4(1 + 2\lambda)} \left[ |\gamma| ||B_2| (1 + |\sin \theta|) - 2B_1| \right] |c_1|
\]
\[
= \frac{|\gamma|^2 B_2^2}{(1 + \lambda)^2} \cdot |1 - \mu - \Re(k_1)| + \frac{|\gamma|}{4(1 + 2\lambda)} \left[ |\gamma| ||B_2| (1 + |\sin \theta|) - 2B_1| \right] |c_1|
\]
\[
= \frac{|\gamma|^2 B_2^2}{(1 + \lambda)^2} \cdot (1 - \mu - \Re(k_1)) + \frac{|\gamma|}{4(1 + 2\lambda)} \left[ |\gamma| ||B_2| (1 + |\sin \theta|) - 2B_1| \right] |c_1|.
\]

(ii) Let $1 - \Re(k_1) + \mathcal{N} < \mu \leq 1 - \Re(k_1)$, using (2.15) and Lemma 2.1 we obtain
\[
|a_3 - \mu a_2| \leq \frac{|\gamma|}{2(1 + 2\lambda)} \left[ |\gamma|^2 B_2^2 \frac{(1 + |\sin \theta|)}{(1 + \lambda)^2} \right] |c_1| + \frac{|\gamma|}{4(1 + 2\lambda)} \left[ |\gamma| ||B_2| (1 + |\sin \theta|) - 2B_1| \right] |c_1|
\]
\[
\leq \frac{|\gamma|}{2(1 + 2\lambda)} \left[ \frac{|\gamma|^2 B_2^2}{(1 + \lambda)^2} \right] \cdot |1 - \mu - \Re(k_1)| + \frac{|\gamma|}{4(1 + 2\lambda)} \left[ |\gamma| ||B_2| (1 + |\sin \theta|) - 2B_1| \right] |c_1|
\]
\[
= \frac{|\gamma|^2 B_2^2}{(1 + \lambda)^2} \cdot |1 - \mu - \Re(k_1)| + \frac{|\gamma|}{4(1 + 2\lambda)} \left[ |\gamma| ||B_2| (1 + |\sin \theta|) - 2B_1| \right] |c_1|
\]
\[
= \frac{|\gamma|^2 B_2^2}{(1 + \lambda)^2} \cdot (1 - \mu - \Re(k_1)) + \frac{|\gamma|}{4(1 + 2\lambda)} \left[ |\gamma| ||B_2| (1 + |\sin \theta|) - 2B_1| \right] |c_1|.
\]

(iii) Let $1 - \Re(k_1) < \mu < 1 - \Re(k_1) - \mathcal{N}$, using (2.15) and Lemma 2.1 we obtain
\[
|a_3 - \mu a_2| \leq \frac{|\gamma|}{2(1 + 2\lambda)} \left[ |\gamma|^2 B_2^2 \frac{(1 + |\sin \theta|)}{(1 + \lambda)^2} \right] |c_1| + \frac{|\gamma|}{4(1 + 2\lambda)} \left[ |\gamma| ||B_2| (1 + |\sin \theta|) - 2B_1| \right] |c_1|
\]
\[
\leq \frac{|\gamma|}{2(1 + 2\lambda)} \left[ \frac{|\gamma|^2 B_2^2}{(1 + \lambda)^2} \right] \cdot |1 - \mu - \Re(k_1)| + \frac{|\gamma|}{4(1 + 2\lambda)} \left[ |\gamma| ||B_2| (1 + |\sin \theta|) - 2B_1| \right] |c_1|
\]
\[
= \frac{|\gamma|^2 B_2^2}{(1 + \lambda)^2} \cdot |1 - \mu - \Re(k_1)| + \frac{|\gamma|}{4(1 + 2\lambda)} \left[ |\gamma| ||B_2| (1 + |\sin \theta|) - 2B_1| \right] |c_1|
\]
\[
= \frac{|\gamma|^2 B_2^2}{(1 + \lambda)^2} \cdot (1 - \mu - \Re(k_1)) + \frac{|\gamma|}{4(1 + 2\lambda)} \left[ |\gamma| ||B_2| (1 + |\sin \theta|) - 2B_1| \right] |c_1|.
\]

(iv) Let $\mu \geq 1 - \Re(k_1) - \mathcal{N}$, using (2.15) and Lemma 2.1 we have
\[
|a_3 - \mu a_2| \leq \frac{|\gamma|}{2(1 + 2\lambda)} \left[ |\gamma|^2 B_2^2 \frac{(1 + |\sin \theta|)}{(1 + \lambda)^2} \right] |c_1| + \frac{|\gamma|}{4(1 + 2\lambda)} \left[ |\gamma| ||B_2| (1 + |\sin \theta|) - 2B_1| \right] |c_1|
\]
\[
\leq \frac{|\gamma|}{2(1 + 2\lambda)} \left[ \frac{|\gamma|^2 B_2^2}{(1 + \lambda)^2} \right] \cdot |1 - \mu - \Re(k_1)| + \frac{|\gamma|}{4(1 + 2\lambda)} \left[ |\gamma| ||B_2| (1 + |\sin \theta|) - 2B_1| \right] |c_1|
\]
\[
= \frac{|\gamma|^2 B_2^2}{(1 + \lambda)^2} \cdot |1 - \mu - \Re(k_1)| + \frac{|\gamma|}{4(1 + 2\lambda)} \left[ |\gamma| ||B_2| (1 + |\sin \theta|) - 2B_1| \right] |c_1|
\]
\[
= \frac{|\gamma|^2 B_2^2}{(1 + \lambda)^2} \cdot (1 - \mu - \Re(k_1)) + \frac{|\gamma|}{4(1 + 2\lambda)} \left[ |\gamma| ||B_2| (1 + |\sin \theta|) - 2B_1| \right] |c_1|.
\]

Thus the proof is completed. \(\square\)

**Corollary 2.4.** Let $\gamma = 1$ and $\lambda = 0$. If both functions $f$ of the form (1.1) and its inverse maps $g = f^{-1}$ are in $\mathcal{S}[A,B]$, then using Theorem (2.7), (2.23) and (2.24), we obtain

(1) For $\gamma \in \mathbb{C}\setminus\{0\}$ and $\mu \in \mathbb{C}$,
\[
|a_3 - \mu a_2| \leq \left\{ \begin{array}{ll}
\frac{A-B}{2} & |B| + |4(1 - \mu)(A-B) - B| < 2,
\frac{|B(A-B)|}{2} & |B| + |4(1 - \mu)(A-B) - B| \geq 2.
\end{array} \right.
\]

(2) For $\gamma > 0$ and $\mu \in \mathbb{R}$,
\[
|a_3 - \mu a_2| \leq \left\{ \begin{array}{ll}
\frac{|B(A-B)|}{2} & (\mu - 1) (A-B)^2 \quad \text{if } |B| + |4(1 - \mu)(A-B) - B| < 2,
\frac{A-B}{2} & (\mu - 1) (A-B)^2 \quad \text{if } |B| + |4(1 - \mu)(A-B) - B| \geq 2.
\end{array} \right.
\]

(3) For $\gamma \in \mathbb{C}\setminus\{0\}$ and $\mu \in \mathbb{R}$,
\[
|a_3 - \mu a_2| \leq \left\{ \begin{array}{ll}
\frac{(A-B)^2}{2} (1 - \mu) + \frac{|B(A-B)(1 + |\sin \theta| - |\cos \theta|)}{4} & \text{if } |B| + |4(1 - \mu)(A-B) - B| < 2,
\frac{A-B}{2} & \text{if } |B| + |4(1 - \mu)(A-B) - B| \geq 2.
\end{array} \right.
\]

(4) if $|B| + |4(1 - \mu)(A-B) - B| < 2$,
\[
|a_3 - \mu a_2| \leq \frac{A-B}{2} - (\mu - 1) (A-B)^2 \quad \text{if } |B| + |4(1 - \mu)(A-B) - B| \geq 2.
\]

(5) if $\mu \leq 1 + \psi_1(A,B,\theta)$,
\[
|a_3 - \mu a_2| \leq \frac{|B(A-B)|}{2} + (\mu - 1) (A-B)^2 \quad \text{if } |B| + |4(1 - \mu)(A-B) - B| < 2,
\]

(6) if $\mu \geq 1 - \psi_2(A,B,\theta)$,
\[
|a_3 - \mu a_2| \leq \frac{A-B}{2} - (\mu - 1) (A-B)^2 \quad \text{if } |B| + |4(1 - \mu)(A-B) - B| \geq 2.
\]
where \( B_1 = A - B, B_2 = -B (A - B), -1 \leq B < A \leq 1, \psi_1 (A, B, \theta) = \frac{[(1 + |\sin \theta| - \cos \theta)]}{2(|A - B|)} \) and 
\[ \psi_2 (A, B, \theta) = \frac{[(1 + |\sin \theta| + \cos \theta)]}{2(|A - B|)}. \]

**Corollary 2.5.** Let \( \gamma = 1 \) and \( \lambda = 1 \). If both functions \( f \) of the form (1.1) and its inverse maps \( g = f^{-1} \) are in \( \mathcal{C}[A, B] \), then using Theorem (2.7), (2.8), and (2.9), we have

(i) For \( \gamma \in \mathbb{C} \setminus \{0\} \) and \( \mu, \nu \in \mathbb{C}, \)
\[ |a_3 - \mu a_2^2| \leq \left\{ \begin{array}{ll}
\frac{|B|}{\sqrt{2}} |1 + (1 - \mu) (A - B)| & \text{if } |B| + |3 (1 - \mu) (A - B)| < 2,
\frac{|B|}{\sqrt{2}} |1 + (1 - \mu) (A - B)| & \text{if } |B| + |3 (1 - \mu) (A - B)| \geq 2.
\end{array} \right. \]

(ii) For \( \gamma > 0 \) and \( \mu \in \mathbb{R}, \)
\[ |a_3 - \mu a_2^2| \leq \frac{|B|}{\sqrt{2}} |1 + (1 - \mu) (A - B)| \quad \text{if } |B| + |3 (1 - \mu) (A - B)| < 2, \]
\[ \frac{|B|}{\sqrt{2}} |1 + (1 - \mu) (A - B)| \quad \text{if } |B| + |3 (1 - \mu) (A - B)| \geq 2. \]

(iii) For \( \gamma \in \mathbb{C} \setminus \{0\} \) and \( \mu, \nu \in \mathbb{C}, \)
\[ |a_3 - \mu a_2^2| \leq \left\{ \begin{array}{ll}
\frac{|B(1 + |\sin \theta| - \cos \theta)|}{2} |1 + (1 - \mu) (A - B)| & \text{if } |B(1 + |\sin \theta| - \cos \theta)| + |3 (1 - \mu) (A - B)| < 2,
\frac{|B(1 + |\sin \theta| + \cos \theta)|}{2} |1 + (1 - \mu) (A - B)| & \text{if } |B(1 + |\sin \theta| + \cos \theta)| + |3 (1 - \mu) (A - B)| \geq 2.
\end{array} \right. \]

**Corollary 2.6.** Let \( \gamma \in \mathbb{C} \setminus \{0\} \) and \( \lambda = 0 \). If both functions \( f \) of the form (1.1) and its inverse maps \( g = f^{-1} \) are in \( S^\ast[\gamma] \), then similarly, using Theorem (2.7), (2.8), and (2.9), we obtain

(i) For \( \gamma \in \mathbb{C} \setminus \{0\} \) and \( \mu, \nu \in \mathbb{C}, \)
\[ |a_3 - \mu a_2^2| \leq \left\{ \begin{array}{ll}
\frac{|\gamma|}{2} |1 + (1 - \mu) 8 \gamma| + 1 & \text{if } |1 + (1 - \mu) 8 \gamma| < 1,
\frac{|\gamma|}{2} |1 + (1 - \mu) 8 \gamma| + 1 & \text{if } |1 + (1 - \mu) 8 \gamma| \geq 1.
\end{array} \right. \]

(ii) For \( \gamma > 0 \) and \( \mu \in \mathbb{R}, \)
\[ |a_3 - \mu a_2^2| \leq \left\{ \begin{array}{ll}
\frac{|\gamma|}{2} |1 + (1 - \mu) 8 \gamma| + 1 & \text{if } |1 + (1 - \mu) 8 \gamma| < 1,
\frac{|\gamma|}{2} |1 + (1 - \mu) 8 \gamma| + 1 & \text{if } |1 + (1 - \mu) 8 \gamma| \geq 1.
\end{array} \right. \]

(iii) For \( \gamma \in \mathbb{C} \setminus \{0\} \) and \( \mu, \nu \in \mathbb{C}, \)
\[ |a_3 - \mu a_2^2| \leq \left\{ \begin{array}{ll}
\frac{4 |\gamma|^2 (1 - \mu) + |\gamma|(1 + |\sin \theta| - \cos \theta)|}{2} & \text{if } \mu \leq 1 + \psi_1 (\gamma, \theta),
\frac{4 |\gamma|^2 (1 - \mu) + |\gamma|(1 + |\sin \theta| - \cos \theta)|}{2} & \text{if } \mu \geq 1 + \psi_1 (\gamma, \theta),
\end{array} \right. \]

where \( B_1 = A - B, B_2 = -B (A - B), -1 \leq B < A \leq 1, \phi_1 (A, B, \theta) = \frac{[(1 + |\sin \theta| - \cos \theta)]}{2(|A - B|)}, \) and 
\[ \phi_2 (A, B, \theta) = \frac{[(1 + |\sin \theta| + \cos \theta)]}{2(|A - B|)}. \]

**Corollary 2.7.** Let \( \gamma \in \mathbb{C} \setminus \{0\} \) and \( \lambda = 1 \). Let both functions \( f \) of the form (1.1) and its inverse maps \( g = f^{-1} \) are in \( \mathcal{C}[\gamma] \). Then similarly, using Theorem (2.7), (2.8), and (2.9), we have

(i) For \( \gamma \in \mathbb{C} \setminus \{0\} \) and \( \mu, \nu \in \mathbb{C}, \)
\[ |a_3 - \mu a_2^2| \leq \left\{ \begin{array}{ll}
\frac{|\gamma|}{2} |1 + (1 - \mu) 8 \gamma| + 1 & \text{if } |1 + (1 - \mu) 6 \gamma| < 1,
\frac{|\gamma|}{2} |1 + (1 - \mu) 8 \gamma| + 1 & \text{if } |1 + (1 - \mu) 6 \gamma| \geq 1.
\end{array} \right. \]

(ii) For \( \gamma > 0 \) and \( \mu \in \mathbb{R}, \)
\[ |a_3 - \mu a_2^2| \leq \left\{ \begin{array}{ll}
\frac{|\gamma|}{2} |1 + (1 - \mu) 8 \gamma| + 1 & \text{if } |1 + (1 - \mu) 6 \gamma| < 1,
\frac{|\gamma|}{2} |1 + (1 - \mu) 8 \gamma| + 1 & \text{if } |1 + (1 - \mu) 6 \gamma| \geq 1.
\end{array} \right. \]

(iii) For \( \gamma \in \mathbb{C} \setminus \{0\} \) and \( \mu, \nu \in \mathbb{C}, \)
\[ |a_3 - \mu a_2^2| \leq \left\{ \begin{array}{ll}
\frac{|\gamma|^2 (1 - \mu) + |\gamma|(1 + |\sin \theta| - \cos \theta)|}{6} & \text{if } \mu \leq 1 + \psi_1 (\gamma, \theta),
\frac{|\gamma|^2 (1 - \mu) + |\gamma|(1 + |\sin \theta| - \cos \theta)|}{6} & \text{if } \mu \geq 1 + \psi_1 (\gamma, \theta),
\end{array} \right. \]

where \( B_1 = 2, B_2 = 2, \psi_1 (\gamma, \theta) = \frac{[(1 + |\sin \theta| - \cos \theta)]}{8|\gamma|}, \) and 
\[ \psi_2 (\gamma, \theta) = \frac{[(1 + |\sin \theta| - \cos \theta)]}{|\gamma|}. \]
where $B_1 = 2$, $B_2 = 2$, $\phi_1(\gamma, \theta) = \frac{(|\sin \theta | - \cos \theta - 1)}{6|\gamma|}$ and $\phi_2(\gamma, \theta) = \frac{|\sin \theta + \cos \theta - 1)}{6|\gamma|}$.

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