Phase transitions of cold and dense quark matter in an external magnetic field

D C Duarte\textsuperscript{1} and R L S Farias\textsuperscript{2}

\textsuperscript{1}Departamento de Física, Instituto tecnológico de Aeronáutica, 12228-900 São José dos Campos, SP, Brazil
\textsuperscript{2}Departamento de Física, Universidade Federal de Santa Maria, 97105-900 Santa Maria, RS, Brazil
E-mail: dyana@ita.br

Abstract. In this work we study the effects of an external magnetic field in the competition of chiral and diquark order parameters in cold and dense quark matter, using a SU(2) version of the NJL model. We verify the influence of the magnetic field in the phase diagram, and perform a comparison of the results obtained using a Fermi-Dirac form factor regularization with ones obtained by using a method that makes a full separation of the finite magnetic contributions and the divergencies, the Magnetic Field Independent Regularization.

1. Introduction

One of the most intriguing problems in physics currently is the description of the quantum chromodynamics (QCD) phase diagram in $T \times \mu_B$ plane. Mainly in the region of intermediate to high baryon density, many efforts have been devoted in an attempt to understand the transition between hadronic and deconfined phases, but this mechanism is still poorly understood. One of the reasons for this scenario is that lattice first principle calculation, using Monte Carlo method, have serious problems in to deal with the region of nonzero chemical potential, since the fermion determinant that becomes complex \cite{1}. Moreover, motivated by the fact that strong magnetic fields may be produced in noncentral heavy-ion collisions \cite{2}, might have played an important role in the physics of the early universe \cite{3} and also may be present in the surface and core of magnetars \cite{4}, investigations of the effects produced by a magnetic field in the phase diagram of strongly interacting matter became a subject of great interest in recent years. Results from lattice simulation of three color QCD at zero baryon density shows that the background magnetic field strengthens the chiral symmetry breaking region at zero temperature, phenomenon called magnetic catalysis (MC) \cite{5}, while an inverse magnetic catalysis (IMC) takes place at finite temperature \cite{6}.

The astrophysical scenario may occur in regions of low temperature and intermediate density, where color superconducting phases are expected to exist. Due to the already mentioned sign problem, the most part of the knowledge of QCD phase structure at finite baryon density comes from effective models that preserves some of its symmetries. One of the most popular effective model used to study the color superconducting phases is the Nambu–Jona-Lasinio (NJL) model \cite{7, 8}, due to its simplicity to implementate and possibility to include and analyze different color pairing configurations. Many works have been dedicated to study the effects...
of magnetic fields in color superconducting phases (see [9, 10, 11, 12] and references therein), through chiral effective models in different contexts and approximations. In this paper we argue that the use of a nonrenormalizable model must be performed carefully, and devoting special attention to the regularization of divergencies. In this sense, the negligence in to cut medium contributions might lead to spurious results and interpretations, as we will discuss in Sec. 3.

This work is organized as follows. In Sec. 2 we present the NJL model in the presence of a constant external magnetic field, focusing in the parametrization and discussing two different regularization schemes. Sec. 3 is dedicated to present the results obtained and compare these two schemes, and in Sec. 4 we show some final remarks.

2. NJL Model and Parameters

We consider a NJL model including scalar-pseudoscalar and color pairing interactions, whose standard Lagrangian density in the presence of an external magnetic field is given by

\[
\mathcal{L} = \bar{\psi}(i\gamma^\mu \gamma_5 - m_\psi)\psi + G_s \left[ (\bar{\psi}\gamma^\mu)\psi \right]^2 + G_d \left( i\bar{\psi}\gamma^\mu (\gamma_5 \bar{\psi}) \gamma^\mu (\gamma_5 \bar{\psi}) \right) + \mathcal{L}_{\text{magnetic}}.
\]

where \(G_s\) and \(G_d\) are the scalar and diquark coupling constants, \(m_\psi\) is the current quark mass (that we choose to take the exact isospin symmetry limit, i.e., \(m_u = m_d = m_\psi\)), \(\vec{\tau}\) are Pauli matrices in flavor space, \((\bar{\tau}_a)^{\mu\nu} = (\bar{\tau}_a)^{\nu\mu}\) and \((\bar{\tau}_i)_{ij}\) are the antisymmetric matrices in color and flavor spaces, and \(D_\mu = \partial_\mu - ie\vec{Q}\vec{A}_\mu\). The rotated charge matrix \(\tilde{Q}\) is given by

\[
\tilde{Q} = Q_f \otimes 1_c - 1_f \otimes \left( \frac{\lambda_8}{2\sqrt{3}} \right)
\]

where \(Q_f = \text{diag}(g_u, q_d)\), \(\lambda_8\) is the color quark matrix \(\lambda_8 = \text{diag}(1, 1, -2)/\sqrt{3}\) and \(A_\mu = \delta_{\mu 2}\vec{x}_1B\). It results in the different rotated charges \(\tilde{q}\) for each quark colors, namely, \(u_\nu = 1/2\), \(u_\rho = 1/2\), \(u_\delta = 1\), \(d_\nu = 1/2\), \(d_\rho = 1/2\) and \(d_\delta = 0\). In the presence of an external magnetic field the corresponding mean field thermodynamical potential in \(T \to 0\) limit is given by [13]

\[
\Omega(eB, \mu, \Delta) = \frac{(M - m_\psi)^2}{4G} + \frac{\Delta^2}{4G_d} + \sum_{\tilde{q}=0,1} \Omega_{\tilde{q}}
\]

(2)

where we have defined

\[
\Omega_0 = 2 \int d^3k (2\pi)^3 \left[ E_{k,0} + (\mu - E_{k,0})\theta(\mu - E_{k,0}) \right]
\]

(3)

\[
\Omega_1 = \frac{eB}{2\pi} \sum_{l=0}^{\infty} \alpha_l \int_{-\infty}^{\infty} \frac{dk_3}{2\pi} \left[ E_{k,1} + (\mu - E_{k,1})\theta(\mu - E_{k,1}) \right]
\]

(4)

\[
\Omega_\pm = \frac{eB}{2\pi} \sum_{l=0}^{\infty} \alpha_l \int_{-\infty}^{\infty} \frac{dk_3}{2\pi} \left( E_{k,\pm}^\Delta + E_{k,\pm}^\Delta \right)
\]

(5)

In these expressions, \(\alpha_l\) takes into account the degeneracy of Landau levels, and dispersion relations are given by

\[
E_{k,0}^{\pm} = \sqrt{k_3^2 + M^2} \pm \mu
\]

\[
E_{k,1}^{\pm} = \sqrt{k_3^2 + M^2 + 2eB\mu} \\
E_{k,\pm}^\Delta = \sqrt{\left( E_{k,\pm}^\Delta \right)^2 + \Delta^2}
\]
Due to its nonrenormalizability, NJL needs a proper regularization scheme to avoid the ultraviolet divergences. As a consequence it is necessary to introduce a cutoff parameter $\Lambda$, that becomes a scale of the model. Together with the scalar and diquark coupling constants $G_s$ and $G_d$ and the current quark mass $m_c$, $\Lambda$ is a parameter that must be specified to obtain the numerical results. These parameters are usually fixed such as to reproduce the empirical values of the pion mass $m_\pi$, the pion decay constant $f_\pi$ and the quark condensate $\langle \bar{q}q \rangle_0$. Besides that, Fierz transformation gives the value of $G_d = 0.75G_s$, but to investigate the competition between scalar and diquark condensates we choose to treat $G_d$ as a free parameter.

2.1. Regularization

Most part of the studies of magnetic field effects in color superconducting phases are based on form factors $U_\Lambda$ [10, 14, 15], that are implemented through the prescription

$$\sum_{l=0}^{\infty} \int_{-\infty}^{\infty} \frac{dk_3}{2\pi} \rightarrow \sum_{l=0}^{\infty} \int_{-\infty}^{\infty} \frac{dk_3}{2\pi} U_\Lambda \left( \sqrt{k_3^2 + 2|q_f|B} \right)$$

in the momentum integrals. Different smooth form factors have been used in the literature, as discussed in [13] and references therein. One of the most frequently used is the Fermi-Dirac type, given by

$$U_\Lambda^\alpha(x) = \frac{1}{2} \left[ 1 - \text{tanh} \left( \frac{x}{\Lambda} - \frac{1}{\alpha} \right) \right]$$

where $\alpha$ is a smoothness parameter.

By the other hand, many works have been claiming that the separation of medium contributions from divergent integrals is crucial to obtain the correct description of phase structure and behavior of physical quantities [16, 17, 22, 18, 19]. In this context we use the Magnetic Field Independent Regularization (MFIR), based in the complete separation of magnetic field dependent contributions and divergent terms [20, 13, 21, 23, 24, 25, 26], whose implementation is based in to add and subtract the $eB = \mu = 0$ contribution in $\Omega_1$ and $\mu = 0$ contributions in and $\Omega_2$. After some algebraic manipulations we obtain

$$\Omega_0 = 2 \int \frac{d^3k}{(2\pi)^3} [2E_{k,0} + (\mu - E_{k,0})\theta(\mu - E_{k,0})]$$

$$\Omega_1 = \frac{(eB)^2}{2\pi^2} \left[ \zeta'(-1, \chi) - \frac{x^2 - x}{2 \ln(x)} + \frac{x^2}{4} \right]$$

$$+ \sum_{l=0}^{l_{\text{max}}} \frac{eB}{4\pi^2} \left[ \mu \sqrt{\mu^2 - s^2} - s^2 \ln \left( \frac{\mu + \sqrt{\mu^2 - s^2}}{s} \right) \right]$$

and also

$$\Omega_2 = 4 \sum_{j=\pm 1} \int \frac{d^3k}{(2\pi)^3} \sqrt{(E_k + j\mu)^2 + \Delta^2}$$

$$- \frac{(eB)^2}{2\pi^2} \left[ \zeta'(-1, x) + \frac{1}{2} \left( x^2 - x \right) \ln(x) + \frac{x^2}{4} \right]$$

$$+ \frac{eB}{\pi^2} \int_0^\infty dk_3 \frac{F(k_3^2)}{2} + \frac{eB}{\pi^2} \int_0^\infty dk_3 \left\{ \sum_{l=1}^{\infty} F(k_3^2 + eBl) + \int_0^\infty dy \frac{F}{y} F(k_3^2 + eBy) \right\}$$
with $\chi = M^2/2eB$, $s = \sqrt{M^2 + 2eB l_{\max}}$, $l_{\max} = (\mu^2 - M^2)/2eB$, $x = (M^2 + \Delta^2)/eB$ and

$$F(z^2) = \sum_{j=\pm 1} \left[ \sqrt{\left( \sqrt{z^2 + m^2 + j\mu} \right)^2 + \Delta^2 - \sqrt{z^2 + m^2 + \Delta^2}} \right]$$

Note that from $\Omega_1$ and $\Omega_2$ we extract purely magnetic contribution, ensuring that no $eB$ contribution will be regularized with the divergences of the theory.

3. Numerical results and discussion

In this work we have obtained $G_s = 4.75$ GeV$^{-2}$, $m_c = 4.99$ MeV and $\Lambda = 660$ MeV, that reproduces $m_\pi = 135$ MeV, $f_\pi = 92.3$ MeV and $\langle \bar{q}q \rangle^{1/3} = -250$ MeV. For both regularization schemes the vacuum quark mass $M_0 \sim 302$ MeV, and we use the smoothness parameter $\alpha = 0.01$ for FD. Numerical results are obtained by minimizing the thermodynamical potential (2) in each regularization scheme with respect to $\Delta$ and $M$.

It is well known that at zero chemical potential (and consequently $\Delta = 0$) the chiral condensate and constituent quark mass increases with the magnetic field presents a magnetic catalysis. From Fig. 1 one may see that for both schemes the magnetic catalysis is observed, but when using FD there are strong non-physical oscillations that becomes more pronounced with the increase of the magnetic field, which does not happen in MFIR case. These oscillations are usually confused with the well-known Van Alphen-de Haas oscillations, that are related with discontinuities in the densities. Clearly this is not the case here, since $\mu = 0$. By examining the thermodynamic potential, the only contribution that can generate these oscillations is $\Omega_1$, that contains the theta function whose argument depends on $eB$, stablishing a upper limit to the Landau levels sum. While $q = 0$ contribution does not depends on the magnetic field, in $q = 1/2$ the coupling of quark species to $\Delta$ eliminates the theta function from the sum, i.e., the density is nonzero for all Landau levels while $\Delta \neq 0$.

In Figs. (2) and (3) we compare the order parameters obtained with FD and MFIR as functions of the magnetic field for $\mu = 0.4$ GeV. One may see now the vAdH oscillations also in MFIR, but note that the behavior of the curves is quite different for both schemes. In MFIR both $\Delta$ and $M$ are decreasing functions of the magnetic field (IMC) and present smooth oscillations, while using FD both quantities seems to increase with $eB$, besides the strong non-physical oscillations. Figs. (4) and (5) shows the MFIR results for $\Delta$ and $M$ as functions of $eB$, using $\mu = 0.4$ GeV and for different values of the diquark coupling constant. It is possible to see from both panels that the vAdH oscillations becomes smaller and curves becomes smooth with the increase of $G_d$. This is related to the fact that the value of diquark condensate increase while chiral condensate decreases with $G_d$, causing a supression of the oscillations even with the increase of the magnetic field.

Figure 1. Constituent quark mass $M$ as a function of $eB$ for $G_d = \mu = 0$, comparing MFIR and FD.

1 For more details relative to MFIR implementation see [13, 12]
4. Final Remarks

We study the effects of the inclusion of an external magnetic field in the phase structure and in the behavior of order parameters in a SU(2) version of the NJL model with diquark interaction, giving special attention to the regularization scheme. From the results obtained one may see that FD produces strong non physical oscillation in $\Delta$ and $M$, that are not related to the well known Van Alphen-de Haas, but to the incorrect regularization of integrals that depends on the magnetic field. When these oscillations are present they have a physical meaning, and are related with discontinuities in the density, represented by the theta functions. Using MFIR we were able to observe the real vAdH oscillations and obtain the correct behavior of the aforementioned quantities, and also notice that the oscillations becomes less pronounced with the increasing of the diquark coupling constant $G_d$. This is related to the increase of the value of $\Delta$ and decreasing of $M$ with $G_d$, that supress the oscillations when the magnetic field increases. Ref. [27] has shown that increase of $G_d$ causes the first-order transition of the chiral and diquark gaps becomes crossover through a second-order phase transition at $eB = 0$, while Ref. [15] shows, using FD, that the presence of the external magnetic field also provokes the change from a smooth crossover to a first order phase transition, and also changes the critical baryon chemical potential. It would be useful to perform a complete study of the phase diagram in the presence of an external magnetic field using MFIR, to verify and compare these results.
Acknowledgments
This work was partially supported by Fundação de Amparo à Pesquisa do Estado de São Paulo (FAPESP) under grant 2017/26111-4 (DCD) and by Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq) under grant Nos. 304758/2017-5 (RLSF).

References
[1] Karsch F 2002 Lattice QCD at High Temperature and Density Lectures on Quark Matter (Lecture Notes in Physics vol 583) ed Plessas W and Mathelitsch L (Springer Berlin Heidelberg) pp 209–249 ISBN 978-3-540-43234-0
[2] Fukushima K, Kharzeev D E and Warringa H J 2008 Phys. Rev. D 78(7) 074033
[3] Vachaspati T 1991 Physics Letters B 265 258–261 ISSN 0370-2693
[4] C K, S D, T S and J v P e a 1998 Nature 393 235–237 ISSN 0028-0836 10.1038/30410
[5] Shovkovy I 2013 Magnetic Catalysis: A Review Strongly Interacting Matter in Magnetic Fields (Lecture Notes in Physics vol 871) ed Kharzeev D, Landsteiner K, Schmitt A and Yee H U (Springer Berlin Heidelberg) pp 13–49 ISBN 978-3-642-37304-6
[6] Bali G S, Bruckmann F, Endrödi G, Fodor Z, Katz S D and Schäfer A 2012 Phys. Rev. D 86(7) 071502
[7] Klevansky S P 1992 Rev. Mod. Phys. 64(3) 649–708
[8] Buballa M 2005 Physics Reports 407 205–376 ISSN 0370-1573
[9] Noronha J L and Shovkovy I A 2007 Phys. Rev. D 76(10) 105030
[10] Fukushima K and Warringa H J 2008 Phys. Rev. Lett. 100(3) 032007
[11] Ferrer E J and de la Incera V 2013 Magnetism in Dense Quark Matter (Berlin, Heidelberg: Springer Berlin Heidelberg) pp 399–432 ISBN 978-3-642-37305-3
[12] Duarte D C, Allen P G, Farias R L S, Manso P H A, Ramos R O and Scoccola N N 2016 Phys. Rev. D 93(2) 025017
[13] Allen P, Grunfeld A G and Scoccola N N 2015 Phys. Rev. D 92(7) 074041
[14] Fayazbakhsh S and Sadooghi N 2010 Phys. Rev. D 82(4) 045010
[15] Mandal T and Jaikumar P 2017 Adv. High Energy Phys. 2017 6472909
[16] Farias R L S, Dallabona G, Krein G and Battistel O A 2006 Phys. Rev. C 73(1) 018201
[17] Farias R L S, Dallabona G, Krein G and Battistel O A 2008 Phys. Rev. C 77 065201
[18] Farias R L S, Duarte D C, Krein G and Ramos R O 2016 Phys. Rev. D 94(7) 074011
[19] Coppola M, Allen P, Grunfeld A G and Scoccola N N 2017 Phys. Rev. D 96 056013
[20] Ebert D and Klimenko K 2003 Nuclear Physics A 728 203–225 ISSN 0375-9474
[21] Menezes D P, Pinto M B, Avancini S S, Martínez A P and Providência C 2009 Phys. Rev. C 79(3) 035807
[22] Farias R L S, Gomes K P, Krein G I and Pinto M B 2014 Phys. Rev. C 90 025203
[23] Farias R L S, Timoteo V S, Avancini S S, Pinto M B and Krein G 2017 Eur. Phys. J. A 53 101
[24] Avancini S S, Farias R L S, Benghi Pinto M, Tavares W R and Timóteo V S 2017 Phys. Lett. B 767 247–252
[25] Avancini S S, Dexeheimer V, Farias R L S and Timóteo V S 2018 Phys. Rev. C 97 035207
[26] Avancini S S, Farias R L S and Tavares W R 2018 (Preprint 1812.00945)
[27] Huang M Z P C W 2002 Physical Review D 65(7)