Probabilities in the topos approach to branching space-time

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Abstract. There was previously proposed a topos unified view-point both to relativistic and quantum physics [4-5] on the basis of Belnapian branching space-time [3]. In the context of this approach, to any event of the branching space-time there can be associated a local orthocomplemented lattice (logic) of ‘physically reasonable’ propositions made by a localized observer. Elements of these local orthologics are special collections of the so-called Belnapian worlds (maximal upward directed subsets of events) which the local event belongs to. In the ordinary quantum setting, the counterpart of any local orthologic is the lattice of closed subspaces of Hilbert space of the observer’s environment. The present work suggests a natural scheme of inserting truth probabilities of the ‘physically reasonable’ propositions into the topos framework. To this end, the concept of Chu spaces is used. The role of hypothesis for the state of environment is played by an element of local ‘retro-causal’ orthologic. Sufficient condition of functorial character of the law by which Chu space is related to any event is pointed and proved, i.e. a functor is constructed from the branching space-time to the category of all Chu spaces. There presented a theorem on category nature of Bayesian reassessment of hypothesis likelihood made by the local observer.

1. Introduction
Fundamental phenomena of micro- and macro-worlds are presently described by different but equally successful physical theories -- quantum mechanics (QM) and general relativity (GR). Creation of a unified theory will definitely demand new frame of notions and new branch of mathematics as a basis of the theory. The machinery of linear operators in Hilbert spaces, the mathematics of quantum physics, as well as geometry of smooth manifolds, the basis of GR, should find natural places within this new still unknown construction.

Category theory is the most universal branch of mathematics. We may suppose it to be the basis of the desired physical theory in a broad sense. The notion of sets and accompanying Boolean logic lie in the foundation of almost all special branches of mathematics and physics. There is a notion of topos in category theory as a deep generalization of category of sets Set [1]. Theory of topos is considered for a long time as a powerful tool for unification of mathematics. Recent works [2] have demonstrated the topos theory being a promising instrument in physics -- in construction of a so-called neo-realistic approach to QM with consistent quantum gravity as a final goal.

Generalizing the notion of set and map between sets, topos provide their own non-classical intuitionistic logic. This logic must evidently be inherited by physical theories base on topos mathematics. In some a sense, one may expect the formation of logical paradigm in physics in parallel with geometric paradigm which was formulated a century ago.

The conceptual frame of the expected topos theory must naturally retain the main features both of quantum world with its immanent randomness and macroscopic theory of space-time with its non-
trivial structure of causal relations between events. Not long ago in the framework of purely logical (non-topos) approach to unification of the aforementioned basic features of quantum and relativistic physics Nuel Belnap proposed the model of branching space-time [3]. Partly ordered (by means of causal relations) set $C$ of all possible events lies in its basis. Not a single outcome which has been actualized but all possible ones of any quantum measurement belong to $C$. Event is a primitive of the Belnap's model; the main derived notion is a 'history' -- maximal causally directed set of events. In our approach we call it 'Belnapian world'. Being translated into the topos context, the model of branching space-time reveals new interesting features [4-5]. All the play takes place in the topos $\text{Set}^C$. It is the category of presheaves -- contravariant functors from the poset category $C$ into the category of sets. The basic object of the topos is $\text{Loc}$. The presheaf $\text{Loc}$ associates to any event $e$ the set $\text{Loc}_e$ of all Belnapian worlds which $e$ belongs to, and to any causal arrow (relation) $e_1 \xleftarrow{i} e_2$ the inclusion map $\text{Loc}_{e_2} \to \text{Loc}_{e_1}$ is associated.

2. Local orthologics
The set $\text{Loc}_e$ is a local site used by an observer in $e$ when constructing the picture of his (her) surrounding world. Certain subsets in $\text{Loc}_e$ were shown in [5] to naturally form orthocomplemented lattice $\text{OL}_e$ as a model of some orthologic. This is a non-classical logical scheme known also as minimal quantum logic. In quantum physics to the proposition of the type ‘The value of observable $O$ lies in Borel set $\Delta$’ there associated some subspace $H_{o \in \Delta}$ of a total Hilbert space $H$. This subspace is twofold closed. First, $H_{o \in \Delta}$ is orthoclosed, i.e. $H_{o \in \Delta}^+ = (H_{o \in \Delta})^\perp$. Secondly, it is closed in the norm topology of $H$. Elements of $\text{OL}_e$ were shown to be twofold closed as well. First, they were constructed in such a way that any $A \in \text{OL}_e$ is orthoclosed by definition: $A = (A^\perp)^+$, where $\perp$ is local orthocomplementation. Secondly, there is a natural topology in $\text{OL}_e$ and elements of $\text{OL}_e$ are closed with respect to this topology [5]. The conspicuous similarity between subspaces of $H$ and elements of $\text{OL}_e$ let one identify the last ones with physical propositions made by the localized observer. It is worth to stress that the lattice $\text{OL}_e$ with its internal connectives acts as a prototype of the lattice of closed subspaces of some local Hilbert space $H$. The vectors of $H$ are states of the external world seen by the observer from $e$.

3. Probabilities in local orthologics
Developing the aforementioned analogy, we face the problem of ascribing truth values to propositions from $\text{OL}_e$. In QM the proposition ‘$O \in \Delta$’ is true with probability

$$\text{Prob}(O \in \Delta) = Tr \tilde{P}_{o \in \Delta} \tilde{n}_o,$$  \hspace{1cm} (1)

where $\tilde{P}_{o \in \Delta}$ is the projector onto the subspace $H_{o \in \Delta}$ and $\tilde{n}_o$ is the statistical operator (state) of the considered system. It encapsulates the entire observer’s knowledge on the system. Generally, the statistical operator may be presented as

$$\tilde{n}_o = \sum_{i \in I} w_i \tilde{P}^{(i)}$$  \hspace{1cm} (2)

Here $I$ is some set of indices, $\tilde{P}^{(i)}$ are projectors onto subspaces of $H$ and $w_i$ are weights (probabilities) ascribed to various $\tilde{P}^{(i)}$. Naturally, one has $w_i > 0$ and $\sum_{i \in I} w_i = 1$. Hence, both multipliers in the rhs of (1) have the same nature -- they are constructed of projectors. Yet there is a significant
difference between the forms of their time behavior. In the simplest case a unitary operator $\hat{U}$ gives the evolution of the system from zero time up to $t$. Truth probability of the proposition `$O \in \Delta$' at the moment $t$

$$Prob_O(O \in \Delta) = Tr\hat{n}_0\hat{P}_{O\Delta},$$

where the transformed state is the result of evolution:

$$\hat{n}_0 \mapsto \hat{n} = \hat{U}\hat{n}_0\hat{U}^*.$$

Note that the rhs of (3) may also be presented as

$$Prob_O(O \in \Delta) = Tr\hat{n}_0(\hat{U}^\dagger\hat{P}_{O\Delta}\hat{U}).$$

If (4) is interpreted as forward $t$-transformation (Schrodinger arrow) than the transformation

$$\hat{P}_{O\Delta} \mapsto \hat{U}^\dagger\hat{P}_{O\Delta}\hat{U}$$

is in this sense backward $t$-transformation (Heisenberg arrow). In the general case the transformations (4) and (6) are replaced by some weighted mixtures

$$\hat{n} \mapsto \sum a q^{(a)}\hat{U}_a\hat{n}_0\hat{U}_a^*, \hat{P} \mapsto \sum a q^{(a)}\hat{U}_a^\dagger\hat{P}\hat{U}_a^*,$$

where $q^{(a)} > 0, \sum a q^{(a)} = 1$ and $\hat{U}_a$ are unitary operators.

The oppositely directed transformations will serve us as a hint when introducing probabilities in the topos approach where physical propositions made in $e$ are elements of $\text{OL}_\epsilon$. To any causal arrow $e' \hat{1} e$ a natural map $\text{OL}_\epsilon \rightarrow \text{OL}_{\epsilon'}$ can be associated. We clarify its structure by transforming some $A \in \text{OL}_\epsilon$. By virtue of the presheaf $\text{Loc}$ we have $A \subset \text{Loc}_{\epsilon'}$. $A$ might be not an element of $\text{OL}_{\epsilon'}$ but its local algebraic orthoclosure

$$A \mapsto \text{CL}_{\epsilon'}(A) =_{\#}(A^{1_{\epsilon'}})^{1_{\epsilon'}}$$

pertains to $\text{OL}_{\epsilon'}$. It is this operation which will be used as a map associated to $e' \hat{1} e$:

$$\text{OL}_{\epsilon'} : \text{OL}_\epsilon \rightarrow \text{OL}_{\epsilon'},$$

where $\text{OL}_\epsilon^{\epsilon'} =_{\#} \text{CL}_{\epsilon'}$.

The behavior of local observer in the branching space-time is conditioned by his (or her) conjectures about the external world. These conjectures along with weights-probabilities which the observer ascribes to them make up a prototype of quantum state. So, looking at the resemblance between propositions and states in QM, it is reasonable to expect some similarity of their nature with that of propositions from $\text{OL}_\epsilon$ (the local character first) but with the mentioned peculiarity in the directions of $t$-transforms. That means that for any causal arrow $e' \hat{1} e$ we should expect to have a map for the sets of local hypothesis (conjectures) in reverse direction with respect to (8). Keeping this fact in the mind, it is natural to construct the system of local hypothesis by means of opposed category $\text{C}^\text{op}$ with reversed directions of all causal arrows. The main topos is $\text{Set}^{\text{C}^\text{op}} = \text{Set}^\text{C}$ -- the category of all covariant functors from $\text{C} \rightarrow \text{Set}$. Objects of $\text{Set}^\text{C}$ and derived structures will get the same notations as their counterparts in $\text{Set}^{\text{C}^\text{op}}$ but with over-lines. For instance, $\text{Loc}$ is the presheaf of Belnapianretroworlds -- maximal directed subsets of $\text{C}^\text{op}$. The sets of local conjectures are $\overline{\text{OL}}_{\epsilon'}$, i.e. orthocomplemented lattices which are akin to $\text{OL}_\epsilon$ in their construction. In $\text{P}(\overline{\text{Loc}}_{\epsilon'})$ the closure $\overline{\text{CL}}_\epsilon$ operates. It gives the map
\[ \overline{\text{OL}_{e'}} : \text{OL}_{e'} \rightarrow \text{OL}_{e}, \] (9)

which resembles (8) in all respects but the direction (along the arrow \( e' \overset{1}{\rightarrow} e \)).

Assuming a hypothesis \( H_e \in \overline{\text{OL}_{e'}} \), the observer in \( e \) gets the conditional probability \( p_e(A_e \mid H_e) \) of a chosen physical proposition \( A_e \) being true. This probability is every bit as objective as the probability (1) is. Taking into account the mutually opposite directions of the maps (8) and (9), we subordinate the conditional probability to a restriction (just in parallel with (3) and (5)). Namely, for any \( e' \overset{1}{\rightarrow} e \), \( H_e \in \overline{\text{OL}_{e'}} \) and \( A_e \in \text{OL}_{e} \) we assume

\[ p_e(A_e \mid \overline{\text{OL}_{e'}}(H_e)) = p_e(\text{OL}_{e'}(A_e) \mid H_e). \] (10)

In accordance with the definition of Chu spaces [6], the triples \( (p_e, \text{OL}_{e'}, \overline{\text{OL}_{e'}}) \) and \( (p_e, \text{OL}_{e}, \overline{\text{OL}}_{e}) \), the ordered pair of maps \( (\overline{\text{OL}}_{e'}, \text{OL}_{e'}) \) provide the example of Chu space and a morphism between them:

\[ (\overline{\text{OL}}_{e'}, \text{OL}_{e'}) : (p_e, \text{OL}_{e'}, \overline{\text{OL}}_{e'}) \rightarrow (p_e, \text{OL}_{e}, \overline{\text{OL}}_{e}). \] (11)

The maps (8) and (9) are associated to the arrow \( e' \overset{1}{\rightarrow} e \). It is worth to find conditions which endow this association with functorial property, i.e. which make the composition of causal arrows to be transformed into the composition of respective associated maps. Hence, for the chain of three events \( e_1 \overset{1}{\rightarrow} e_2 \overset{1}{\rightarrow} e_3 \) one has to get

\[ \text{OL}^{e_3}_{n_3} = \text{OL}^{e_2}_{n_2} \text{OL}^{e_1}_{n_1}, \overline{\text{OL}}^{e_3}_{n_3} = \overline{\text{OL}}^{e_2}_{n_2} \overline{\text{OL}}^{e_1}_{n_1}. \] (12)

We shall consider the first relation. It is equivalent to

\[ \text{Cl}_{e_1} \left[ \text{Cl}_{e_2}(A) \right] = \text{Cl}_{e_3}(A) \] (13)

for any \( A \in \text{OL}_{e_1} \). The inclusion \( \text{Cl}_{e_2}(A) \subseteq \text{Cl}_{e_1} \left[ \text{Cl}_{e_2}(A) \right] \) is trivially fulfilled, since \( A \subseteq \text{Cl}_{e_1}(A) \). We shall explore the conditions of the inverse inclusion

\[ \text{Cl}_{e_1} \left[ \text{Cl}_{e_2}(A) \right] \subseteq \text{Cl}_{e_3}(A). \] (14)

To this end, we are to take into account the structure of closure operation (see [5]). There is used a special presheaf \( \text{Acc} \) from \( \text{Set}^{\text{op}} \). For any event \( e \), \( \text{Acc}_e \subseteq \text{Loc} \times \text{Loc} \) is the so-called accessibility relation for Belnapian worlds: for \( w_i \) and \( w_j \) from \( \text{Loc}_e \) one has \( \langle w_i, w_j \rangle \in \text{Acc} \) if \( e \) is not the last common event in these worlds. The following expression for \( \text{Cl}_e \) was evaluated in [5]:

\[ \text{Cl}_e(X) = \bigcup \{ Y \subseteq \text{Loc}_e : Y \cdot \text{Acc}_e \subseteq X \cdot \text{Acc}_e \}, \] (15)

where \( \cdot \) stands for relational product:

\[ X' \cdot \text{Acc}_e = \{ w \in \text{Loc}_e : \exists w' (w' \in X', \langle w, w' \rangle \in \text{Acc}_e) \}. \] (16)

The inclusion (14) takes place if \( w \in \text{Cl}_{e_1} \left[ \text{Cl}_{e_2}(A) \right] \) (i.e. \( \{ w \} \cdot \text{Acc}_{e_1} \subseteq \text{Cl}_{e_2}(A) \cdot \text{Acc}_{e_2} \)) implies \( w \in \text{Cl}_{e_3}(A) \) (i.e. \( \{ w \} \cdot \text{Acc}_{e_3} \subseteq A \cdot \text{Acc}_{e_3} \)).

Let us consider the conditions which guarantee
\[ Cl_{e_i} (A) \cdot Acc_{e_i} \subseteq A \cdot Acc_{e_i}. \] (17)

By virtue of (15) we have the equivalence
\[ (w' \in Cl_{e_i}) \Leftrightarrow (\{w'\} \cdot Acc_{e_i} \subseteq A \cdot Acc_{e_i}). \] (18)

Hence, the inclusion (17) may be given the form of implication
\[ (\{w'\} \cdot Acc_{e_i} \subseteq A \cdot Acc_{e_i}) \Rightarrow (\{w'\} \cdot Acc_{e_i} \subseteq A \cdot Acc_{e_i}). \] (19)

Now let us assume Acc to be the presheaf of equivalences. Then the lhs of (19) asserts that the equivalence class of the world \( w' \) belongs to a set of classes. Since Acc\( e_i \subseteq Acc_{e_i} \), substitution of Acc\( e_i \) by Acc\( e_i \) expands the basis set (\( Loc_{e_i} \subseteq Loc_{e_i} \)) and generally expands any equivalence class. The affiliation of some class to a set of classes stays invariant. This is asserted by the rhs of (19). So, if Acc is the presheaf of equivalences, the law of association \( e \rightarrow OL \) (forgetting the orthocomplemented lattice structure of OL\( e_i \)) can be written as a contravariant functor
\[ OL : C^{op} \rightarrow Set, \] (20)

i.e. OL is an object of the main topos Set\( ^{op} \). In a similar manner, if \( \overline{Acc} \) is the presheaf of equivalences of retroworlds, we get the covariant functor
\[ \overline{OL} : C \rightarrow Set, \] (21)

i.e. \( \overline{OL} \) is an object of Set\( ^{C} \). Since both functor appear in the Chu spaces \( (p, OL, \overline{OL}) \), we have proven the following

3.1. Proposition
If Acc and \( \overline{Acc} \) are presheaves of equivalences and if (10) is fulfilled for local Chu spaces \( (p, OL, \overline{OL}) \), we have contravariant functor from C into the category of all Chu spaces.

4. Bayesianism in the branching space-time
Now it is worth to discuss the model of local observer in the branching space-time. The picture of external world is formulated in the observer’s mind as a set of hypothesis -- conjectures to which the observer ascribes weights -- probabilities. These subjective probabilities are taken into account by the observer when he faces the problem of making choice between various activities. The result of the observer’s acts provides him with a new information and make him re-evaluate the subjective probabilities. So, the proposed model of the observer in a great extent follows the well-known Bayesian concept [7]. Local nature of propositions and conjectures is the main peculiarity caused by the branching. This peculiarity will be proven to get adequate and natural representation in the topos approach. Yet the new generalized form of Bayesian rule is of independent interest beyond the the context of the topos approach. Hence, it is reasonable to give the most general form to this modified Bayesian rule. To this end, we consider two Chu spaces and some morphism between them:
\[ (f^*, f^*_1) : (p_1, A_1, H_1) \rightarrow (p_2, A_2, H_2), \] (22)

where \( f_1 : H_1 \rightarrow H_2 \), \( f^*_1 : A_2 \rightarrow A_1 \) and
\[ p_2(A_2 \mid f_1(H_1)) = p_1(f^*_1(A_2) \mid H_1) \] (23)

for all \( H_1 \in H_1 \) and \( A_2 \in A_2 \).
Probability distributions over the collection of all conjectures \( H \) form a set \( D(H) \), where \( D \) is the monad of distributions (special endofunctor \( D : Set \to Set \) in the category of sets) [8]. The original Bayesian rule for re-evaluation of subjective probabilities upon a proposition \( A \in A \) truth revealing will be denoted by the same symbol:

\[
A : D(H) \to D(H).
\]  

(24)

Under this map, any element \( q \in D(H) \) is transformed into \( q' \in D(H) \) by the Bayesian rule

\[
q'(H) = p(A \mid H)q(H)\left[\sum_{H' \in H} p(A \mid H')q(H')\right]^{-1}.
\]

(25)

The map \( f_* : H_1 \to H_2 \) is transformed by \( D \) into \( Df_* : D(H_1) \to D(H_2) \) where

\[
[Df_*(q)](H_2) = \sum_{f_*(H_1) = H_2} q_1(H_1)
\]

(26)

for any distribution \( q_1 \) over \( H_1 \).

The introduced notations are used in the following

4.1. Theorem

For any pair of Chu spaces (22) and any morphism between them the following diagram is commutative for any proposition \( A_2 \in A_2 \):

\[
\begin{array}{ccc}
D(H_1) & \xrightarrow{Df_*} & D(H_2) \\
\downarrow{f_*(A_2)} & & \downarrow{A_2} \\
D(H_1) & \xrightarrow{Df_*} & D(H_2)
\end{array}
\]

Proof is straightforward. Let \( q_2 = Df_*(q_1) \) and \( q_2' = (A_2 - Df_*)\) \( (q_1) \). By virtue of (25), we have

\[
q_2'(H_2) = p_2(A_2 \mid H_2)q_2(H_2)\left[\sum_{H_1 = H_2} p_2(A_2 \mid H_1' q_1(H_1')\right]^{-1}.
\]

(27)

From (26) we get

\[
q_2'(H_2) = \sum_{f_*(H_1) = H_2} p_2(A_2 \mid H_2)q_1(H_1)\left[\sum_{H_1' = H_2} \sum_{f_*(H_1) = H_1'} p_2(A_2 \mid H_1' q_1(H_1')\right]^{-1} =
\]

\[
= \sum_{f_*(H_1) = H_2} p_2(A_2 \mid f_*(H_1))q_1(H_1)\left[\sum_{H_1' = H_2} \sum_{f_*(H_1) = H_1'} p_2(A_2 \mid f_*(H_1'))q_1(H_1')\right]^{-1}.
\]

(28)

The relation (23) let us transform the last equation into

\[
q_2'(H_2) = \sum_{f_*(H_1) = H_2} p_1(f^*(A_2) \mid H_1)q_1(H_1)\left[\sum_{H_1' = H_2} p_1(f^*(A_2) \mid H_1')q_1(H_1')\right]^{-1} =
\]

\[
= \left[(Df_* \cdot f^*(A_2))(q_1)\right](H_2).
\]

(29)

So, we get

\[
A_2Df_* = Df_* f^*(A_2).
\]

(30)

In the context of the elaborated topos approach this theorem states the commutativity of the diagram
for the causal arrow $e' \uparrow e$ and any physical proposition $A$ from $\text{OL}_e$. The commutativity may be interpreted as unambiguously defined generalized Bayesian rule (the main diagonal of the diagram):

$$
D(\text{OL}_{e'}) \xrightarrow{D(\text{OL}_{e'})} D(\text{OL}_e)
$$

for $e_\text{OL}$ and any physical proposition $A$ from $\text{OL}_e$. The commutativity may be interpreted as unambiguously defined generalized Bayesian rule (the main diagonal of the diagram):

$$
D(\text{OL}_{e'}) \xrightarrow{D(\text{OL}_{e'})} D(\text{OL}_e)
$$

for re-evaluation of subjectively estimated likelihood of the local observer’s conjectures about the external world.

5. Conclusion

There are some reasons to consider the notion of probability as a part of logic. Along with logic itself, this notion emerged in evolution. This assertion may in full measure be referred to the Bayesian rule. In the present work a natural way is suggested of introducing the notion of probability into the topos approach to the Belnar’s model of branching space-time. Heuristic considerations were delivered by quantum theory, since local orthologic $\text{OL}_e$ may be treated as prototype of lattices of subspaces of quantum states. The notion of Chu spaces appeared to be very helpful as well as constructions based on `retroworlds' in space-time with reversed causal relations.

The first important result is the sufficient condition of the functorial character of the local orthologics $\text{OL}$ and $\overline{\text{OL}}$. This provides us with a functor from the branching space-time to the category of Chu spaces, which encompasses the entire probability structure of our topos approach. The second important result deals with generalized Bayesian rule of re-evaluation of subjective probabilities ascribed by local observer to his (her) conjectures about the external world.

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