An Alternating Sequence Iteration’s Method for Computing Largest Real Part Eigenvalue of Essentially Positive Matrices: Collatz and Perron-Frobenius’ Approach

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Abstract

This paper describes a new numerical method for the numerical solution of eigenvalues with the largest real part of essentially positive matrices. Finally, a numerical discussion is given to derive the required number of mathematical operations of the new method. Comparisons between the new method and several well known ones, such as Power and QR methods, were discussed. The process consists of computing lower and upper bounds which are monotonically approximating the eigenvalue.

Keywords: Collatz’s theorem; Perron-Frobenius’ theorem; Eigenvalue

AMS subject classifications: 15A48; 15A18; 15A99; 65F10 and 65F15

Introduction

A variety of numerical methods for finding eigenvalues of non-negative irreducible matrices have been reported over the last decades, and the mathematical and numerical aspects of most of these methods are well reported [1-24]. In recent article of Tedja [19], it was presented the mathematical aspects of Collatz’s eigenvalue inclusion theorem for non-negative irreducible matrices. It is the purpose of this manuscript to present the numerical implementation of [19]. Indeed, there is the hope that developing new numerical method could lead to discovering properties that might be responsible for better numerical method in finding and estimating eigenvalues of non-negative irreducible matrices. Birkhoff and Varga [2] observed that the results of the Perron-Frobenius theory and consequently Collatz’s theorem could be slightly generalized by allowing the matrices considered to have negative diagonal elements. They introduced the terms “essentially non-negative for matrices, the off-diagonal elements of which are non-negative, and “essentially positive matrices” for essentially non-negative, irreducible matrices. The only significant changes is that whenever Perron-Frobenius theory and Collatz’s theorem refer to the spectral radius of a non-negative matrix A, the corresponding quantity for an essentially non-negative matrix A is the (real) eigenvalue of the maximal real part in the spectrum of A, also to be denoted by Λ[A]. Of course Λ[A] need not be positive and it is not necessary dominant among the eigenvalues in the absolute value sense.

Definition

Incidentally, if A is what you call an essentially positive matrix (so it is a real matrix with positive off-diagonal entries), then A + αI has positive entries for sufficiently large positive α, so A + αI has a Perron eigenvalue, say, with corresponding eigenvector v, say, having positive entries. But p is the only eigenvalue of A + αI for which there is a corresponding eigenvector with positive entries. Thus p - α is the only eigenvalue of A with a corresponding eigenvector having all its entries nonnegative, but p - α is real and need not be positive (since a could be greater than p). In this manuscript the eigenvalue corresponding to a positive eigenvector is real. There probably is a term already in the literature for “essentially positive” For example: Z-matrix, tau-matrix, M-matrix, and Metzler matrix all refer to matrices with off-diagonal entries all of the same sign, but having extra conditions.

Background

Let A be an n x n essentially positive matrix. The new method can be used to numerically determine the eigenvalue λ, with the largest real part and the corresponding positive eigenvector x[A] for essentially positive matrices. This numerical method is based on previous manuscript [16]. A matrix A=(a) is an essentially positive matrix if a≥ 0 for all i ≠ j, 1 ≤ i, j ≤ n, and A is irreducible.

Let x>0 be any positive n components vector [19]. Let

\[ z_i(x) = \sum_{j≠i} a_{ij} x_j \] (1)

\[ f(x) = \frac{(Ax)_i}{x_i} = \sum_{j≠i} a_{ij} x_j \quad (i \in N); \] (2)

\[ m(x) = \min_{i≠j} f(x) \] [6, 8, 20, 23]*

\[ M(x) = \max_{i≠j} f(x) \] (3)

\[ \Delta(x) = M(x) - m(x); \] (4)

\[ \|x\| = \sum_{i=1}^{n} x_i \quad * \text{In Ostrowsky [25] } m(x) \text{ is defined as:} \]

\[ \min_{x≠0} \frac{\sum_{i=1}^{n} a_{ii} x_i}{x_i}; \] (5)

The following theorem is an application of corollary 2.3 from [16]

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Received November 10, 2016; Accepted December 23, 2016; Published December 27, 2016

Citation: Oepomo TS (2016) An Alternating Sequence Iteration’s Method for Computing Largest Real Part Eigenvalue of Essentially Positive Matrices: Collatz and Perron-Frobenius’ Approach. J Appl Comput Math 5: 334. doi: 10.4172/2168-9679.1000334

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to the design of numerical method using the Perron-Frobenius-Collatz mini-max principle for the calculation of $x[A] \text{[10].}$

Let $[x^r] (p=0, 1, 2, \ldots)$ be a sequence of positive vectors and $x^r = [x^r_1, x^r_2, \ldots, x^r_n]^T$.

**Theorem 1** If the sequence $[x^r]$ $(p=0, 1, 2, \ldots)$ of positive unit vectors is such that either $m(x^r) \rightarrow A[x] = \lambda_1$ or $M(x^r) \rightarrow A[x] = \lambda_\nu$ as $p \rightarrow \infty$ then $x^r \rightarrow x[A] \equiv \xi$. Moreover, the sequence $[m(x^r)]$, $[x^r]$ are equi-convergent in the sense that an index $v$ and a constant $K > 0$ exist such that $\left| x^r_i - \xi_i \right| < K \left| \lambda - m(x^r) \right|$ if $1 \geq v$. Similar statements can be expressed if $M(x^r) \rightarrow A[x] = \lambda_\nu$ is known. ([19], theorem 2.4)

**Numerical Implication of Theorem 1**

We will now define a group of sequences, the "decreasing-sequence."

**Decreasing-sequence**

Let $Y_r(x) (r=1, 2, 3, \ldots)$ be an n component vector valued function such that the following equation is valid:

$$Y_r(x) = x_i \text{ if } i \neq r \text{ and } \Omega_r(x) \text{ if } i=r.$$  \hspace{1cm} (6)

Here $\Omega_r(x) (r=1, 2, 3, \ldots, n)$ are scalar valued functions which are having properties as follows:

- $\Omega(x)$ is a continuous positive valued function which maps the set of positive vectors $V_p$ into a set of real numbers $R$.
- $\Omega(x) \leq x_i$.
- $f_r(\Omega_r(x)) \equiv M(x).$  \hspace{1cm} (7)
- Equality in c) may be applicable only if $Y(x^r)$ converges to $x$; this yields that $f_r(\lambda) = M(x).$
- If for some $x > 0 \Omega(x) = x_i$ for each $r \in \{1, 2, 3, \ldots\}$ then
  $$f_r(x) = f_r(x) = \ldots = f_r(x) = \ldots = f_r(x) = \lambda.$$  \hspace{1cm} (9)

This will imply that $f_r(\lambda) = \lambda$ according to Collatz, hence $x = x[A]$. In Faddeev [25],

$$\lambda = \frac{1}{\lambda} \text{Min}_{x, \omega_r} \text{Max}_{x, \omega_r} \text{Min}_{x, \omega_r} \sum_{j=1}^{n} a_{ij} x_j.$$  \hspace{1cm} (10)

Then n-component vector valued function $Y(x)$ defined in equation (6) will be referred to as the Decreasing-functions. A sequence $[x^r]$ $(p=0, 1, 2, \ldots)$ of positive n-vectors is constructed which satisfy the conditions of theorem 1. The terms of the sequence $[x^r]$ are generated by the following recursive formula:

$$x^{r+1} = Y^{r+1}.$$  \hspace{1cm} (11)

Where $Y^{r}(x) = Y^{r \times r} (x)$ $(r=1, 2, 3, \ldots)$. If $x^r$ is given the sequence $[x^r]$ is completely defined. $x^{r+1}$ and $x^r$ differ only in the $r$th component where

$$r \equiv p + 1 \text{ (mod n)}.$$  \hspace{1cm} (12)

Such a sequence will be called a decreasing maximum ratio sequence or briefly decreasing-sequence.

**Corollary 1**: Any decreasing-sequence converges to $x_r$.  

Proof: From equation (6) and $0 \leq \Omega(x) \leq x$, yield to imply $0 \leq Y(x) \leq x \forall x \in V_p$. Finally, it follows from inductive construction in equation (8) that $x^0 \geq x^1 \geq x^2 \geq \ldots \geq 0$. Therefore convergence of $x^p$ as $p \rightarrow \infty$ is always guaranteed. The condition that $0 \leq x^p < x^{p+1} < \ldots x^0$ necessarily implies that $\inf ||x^p|| < \infty$ is satisfied.

Note: In any case, if we have an increasing bounded sequence, the limit always exists and is finite (the sup is an upper bound! the sequence is bounded from above by a supremum). It does converge to the supremum. This criterion is also true for a decreasing bounded sequence and is bounded below by an infimum. It will converge to the infimum. But we need to be very careful to define the term bounded, it means bounded from above and bounded from below. These statements hold for sets of real numbers. We will now define a second group of sequences, the "Increasing-sequence".

**Increasing-sequence**

Let $y(x) (r=1, 2, 3, \ldots, n)$ be an $n$ component vector valued function such that the following equation is valid:

$$y_r(x) = x_i \text{ if } i = r \text{ and } \omega_r(x) \text{ if } i=r.$$  \hspace{1cm} (13)

Here $\omega_r(x) (r=1, 2, 3, \ldots, n)$ are scalar valued functions which are having properties as follows:

- $\omega(x)$ is a continuous positive valued function and bounded in $\omega(x) = \lambda$, $0 \leq \omega(x) \geq x_0$.
- $0 \leq \omega(x) \leq m(x)$.

Equality in c) may be applicable only if $y(x^r)$ converges to $x$; this yields that $fr(\omega(x)) = m(x)$.  \hspace{1cm} (14)

If for some $x > 0 \omega(x) = x_i$ for each $r \in \{1, 2, 3, \ldots\}$ then

$$f_r(x) = f_r(x) = \ldots = f_r(x) = \omega = \lambda.$$  \hspace{1cm} (15)

This will imply that $f_r(\omega) = \lambda$ according to Collatz, hence $x = x[A]$. In Faddeev [25],

$$\lambda = \frac{1}{\lambda} \text{Min}_{x, \omega_r} \text{Max}_{x, \omega_r} \text{Min}_{x, \omega_r} \sum_{j=1}^{n} a_{ij} x_j.$$  \hspace{1cm} (16)

Then n-component vector valued function $Y(x)$ defined in equation (9) will be referred to as the Decreasing-functions. A sequence $[x^r]$ $(p=0, 1, 2, \ldots)$ of positive n-vectors is constructed which satisfy the conditions of theorem 1. The terms of the sequence $[x^r]$ are generated by the following recursive formula:

$$x^{r+1} = Y^{r+1}.$$  \hspace{1cm} (17)

Where $Y^{r}(x) = Y^{r \times r} (x)$ $(r=1, 2, 3, \ldots)$. If $x^r$ is given the sequence $[x^r]$ is completely defined. Such a sequence will be called an increasing minimum ratio sequence or briefly increasing-sequence.

**Corollary 2**: Any increasing-sequence converges to $x_r$.  

Proof: Since $\omega_r(x) \geq x_i$, so that $y_r(x) \geq x_i$. This yields to imply that, starting with $x^0 \geq x$, we have $x_{r+1} = y^{r+1}(x) \geq x_r \geq \ldots x_0 \geq 0$, so that $x^p$ automatically converges as long a sup $||x^p|| < \infty$.

Numerical tests indicate that an alternation of the application of the decreasing and increasing sequences will converge faster than either the decreasing or increasing sequence separately. Therefore, we will define a sequence of vectors $[x^r]$ which are constructed by alternating methods of the decreasing or increasing type functions.

We will describe a sequence of $n$ steps which generate $x^{r+1} \ldots x^{r+n}$ $(p=0, n, 2n, \ldots)$ in an iteration. If the decreasing functions $\{y^r(x), r=1, 2, \ldots, n\}$ are used to generate the $n$ terms of the sequence $[x^r]$ during
iteration as defined in equation (8) then we will say that the iteration is in decreasing mode. Similarly, the iteration is in the increasing mode if increasing functions are used as defined in equation (12). Successive terms of the sequence \( x^k \) can be defined recursively in the following respects:

\[
x^{k+1} = \frac{1}{\lambda} \left( x_k \right) \quad \text{for } k = 0, 1, 2, \ldots \quad \text{or} \quad x^{k+1} = x^{k+1} \left( x^k \right) \quad \text{for } k = 0, 1, 2, \ldots
\]  

(13)

Where \( k = 0 \) corresponds to the input vector. The first iteration could be either in the increasing or decreasing mode. We also define the sequence of real number \( \{t\} \) and \( \{T\} \) as follows: \( t_0 = m(x^0) \) and \( T_0 = M(x^0) \). At the end of each iteration we consider the following inequalities:

\[
m(x^m) \geq t_{k-1} \quad \text{and} \quad M(x^m) \leq T_{k-1}
\]  

(14)

Where \( (k=1, 2, 3, \ldots) \) are indexes of the iteration. If inequalities (14-1) and (14-2) are met, we may set

\[
t_k = m(x^m) \quad \text{and} \quad T_k = M(x^m)
\]  

(15)

And the mode or sequence of the next \( (k+1) \) iteration will be different from the \( k \) iteration, i.e. the sequence of the \( k \) iteration is different from the \( (k+1) \) iteration. If either inequality (14-1) or (14-2) is not satisfied then the mode or sequence of the \( (k+1) \) iteration is the same as that of the \( k \) iteration unchanged and we set: \( t_k = t_{k-1} \) and \( T_k = T_{k-1} \).

A sequence having the above mentioned properties will be called the alternating sequence iteration.

**Corollary 3:** Any alternating sequence iteration converges to \( x_* \).

Proof: If inequalities (14-1) and (14-2) are satisfied, it means that the estimated result is located between the upper bound, \( M(x^{m+1}) \), and lower bound, \( m(x^{m+1}) \), so no change in the iteration sequence is required. Otherwise, the iteration sequence is required to be switched. Since if only one of those inequalities is met then it means that the result is not between \( M(x^{m+1}) \) and \( m(x^{m+1}) \). The condition stated in inequalities (14-1) and (14-2) will ensure that the iteration results are located between the upper and lower bound. As the iterations are continuing, eventually, \( M(x^{m+1}) \) and \( m(x^{m+1}) \) are getting closer and closer. This condition will accelerate the rate of convergence. At the end, equation (4) would be almost zero.

Corollary 1, 2, and 3 described above lay the foundation of the procedure of an iterative method for the determination of the positive eigenvector of essentially positive matrices. The choices of the functions \( \Omega(x) \), \( w(x) \) are open, but are subject to the restrictions specified in connection with the decreasing and increasing sequences. In theorem 2 and theorem 3 which follow, a possible choice for \( \Omega(x) \) and \( w(x) \) is given.

**Theorem 2**

Let \( H(x) \) \( (r=1, 2, 3, \ldots, n) \) denote continuous, positive valued functions which map the set of positive vectors \( V \), into a set of real number \( R \) such that \( m(x) \leq H(x) \leq M(x) \) (16) and equality may hold on either side of equation (16) only if \( m(x)=M(x)=H(x) \).

For \( r \epsilon N \) \( (N=1, 2, 3, \ldots, n) \), let \( \Omega(x) = x \) if \( f_r > H_r \) or \( \frac{z_r}{H_r - a} \) if \( f_r < H_r \),

(17)

Where \( H_r \equiv H_r \left( x, f_r \right) \), \( f_r \equiv f_r \left( x \right) \), and all notations are defined in equations (1, 2, 3, 4, and 5). Then the functions \( \Omega(x) \) (together with a starting vector \( x^0 \)) define a decreasing sequence.

**Proof:** We will first show that if \( f_r < H_r \), the term \( H_r - a_r \) in equation (17) is always positive. By definition from equations (1) and (2)

\[
f_r = \frac{z_r}{x_r} + a_r \quad \text{as} \quad f_r < H_r \quad \text{the above equation yields} \quad \frac{z_r}{x_r} < H_r - a_r
\]  

(18)

For an essentially positive matrix all the off diagonal elements cannot be zero and consequently for any vector \( x > 0 \), \( \frac{z_r}{x_r} > 0 \). Therefore (18) becomes \( 0 < H_r - a_r \).

It is clear from equations (17) and (19) that \( \Omega(x) \) is a positive valued function. Equation (17) can also be used to derive estimates for the function \( \Omega(x) = f_r(Y(x)) \). With the abbreviation used before,

\[
\theta_r = f_r \left( \frac{z_r}{x_r} \right)
\]  

(20)

The above inequalities are equivalent to \( \theta_r = \text{max}[f_r, H_r] \).

(21)

\( \theta_r \) is the maximum of two continuous functions and is therefore continuous. By definition, \( f_r, H_r \) are either less than or equal to \( M \). Therefore from equation (21) \( \theta_r \leq M \). Then \( \Omega(x) \) has property of \( c \) in equation (7) of decreasing sequences. From the definition of \( \theta_r, \Omega(x), z \) we get

\[
\theta_r = \frac{z_r}{\Omega(x)} + a_r
\]  

(22)

Equation (20) implies \( \theta_r > f_r \).

(23)

From equations (22), (23) \( \frac{z_r}{\Omega(x)} + a_r > f_r \), or \( \frac{z_r}{\Omega(x)} + a_r \geq \frac{z_r}{x_r} + a_r \); therefore \( \Omega(x) \leq x_r \). Thus \( \Omega(x) \) has property b in equation (7) of decreasing sequences.

Equation (22) is equivalent to \( \Omega(x) = \frac{z_r}{\theta_r - a_r} \).

(24)

As stated before for an essentially positive matrix, \( z_r \) is positive, and therefore it is obvious from (22) that \( \theta_r > a_r \), and thus the denominator in equation (24) is positive. Therefore, by the established continuity of \( \theta_r \), \( \Omega(x) \) it is also continuous. Thus \( \Omega(x) \) has property of \( a \) (equation (7)) of decreasing sequences. From equation (21) \( \theta_r = M(x) \) is possible only if \( max(f_r, H_r) = M(x) \). As \( H_r < M(x) \) by assumption unless \( f_r = M(x) \), so in any case \( \theta_r = (M(x)) \), \( f_r = M(x) \).

Thus \( \Omega(x) \) has property d in equation (12) of decreasing sequences.

Finally suppose that \( \Omega(x) = x_r \) for \( r=1, 2, 3, \ldots, n \), and thus \( \theta(x) = x \). From equation (20) \( \theta_r \geq H_r \) as \( \theta_r = f_r \), we get \( \theta_r \geq H_r \). The last inequality is equivalent to \( max \ f_r = M(x) \geq H_r \).

Since \( M(x) \geq H_r \) by definition, we have \( H_r = H \). By assumption this is possible only if \( m(x)=M(x) \). Hence \( \Omega(x) \) has property of \( e \) in equation (7) of decreasing sequences. This completes the proof of theorem 2.

**Theorem 3**

Let \( h_r \) \( (r=1, 2, 3, \ldots, n) \) denotes a continuous bounded function mapping

\[
\sum_{r} x_r \quad \text{such that} \quad m(x) \leq h_r \leq \lambda(x)
\]  

(25)

and equality may hold on either side of equation (24) only if \( m(x)=M(x)=\lambda(x) \). \( x=x \). \( \sum_{r} \) has been introduced in equation (11). We further define

\[
w_r(x) = x_r
\]  

(26)

If \( f_r \leq h_r \), and
\[ \frac{z_r}{h_r - a_r}, \]  \tag{27} \]

If \( f_r > h_r \), then \( \theta_r = \frac{z_r}{a_r}, \) as \( f_r > h_r \), the above equation yields \( \sum_{r} > h_r - a_r, \) \tag{28}

For an essentially positive matrix, all the off-diagonal elements cannot be zero and accordingly for any other vector \( x > 0 \), therefore equation (28) becomes \( 0 > h_r - a_r \).

It is clear from equation (26) and (28) that \( \omega_r(x) \) is a negative valued function. Equation (26) can also be used to derive estimates for the function, \( \theta_r = \theta_r(f_r(y(x))). \) With the abbreviation used before \( \theta_r = f \), if:

\[ f_r \leq h_{r+1} \text{ or } f_r > h_r. \]  \tag{29}

The above inequalities are equivalent to:

\[ \theta_r = \min[f_r(h_r)]. \]  \tag{30}

\( \theta_r \) is the minimum of two continuous functions and is thus continuous. By definition, \( f_r, h_r \) are either greater than or equal to \( m \). Therefore, from equation (30) \( \theta_r \geq m \). Thus \( \omega_r(x) \) has property of \( c \) in equation (11) of increasing sequences. From the definition of \( \omega_r \), \( \omega_r(x)\gamma_r \), we get:

\[ \omega_r(x) = \frac{z_r}{\omega_r(x)}, \]  \tag{31} \]

Equation (29) implies

\[ \theta_r \leq f_r. \]  \tag{32}

From equation (31), (32), \( \frac{z_r}{\omega_r(x)} + a_r \leq f_r, \) or \( \frac{z_r}{\omega_r(x)} + a_r \leq z_r \), therefore \( \omega_r(x) \geq x_r. \)

Thus \( \omega_r(x) \) has property \( b \) in equation (11) of increasing sequences. Equation (31) is equivalent to:

\[ \omega_r(x) = \frac{z_r}{\theta_r - a_r}. \]  \tag{33}

As stated before, for an essentially positive matrix, \( z_r \) is positive, and hence it is obvious from (31) that \( \theta < a_r \); and thus the denominator in equation (33) is negative. Accordingly by the established continuity of \( \theta_r(x), \omega_r(x) \) it is also continuous. Thus \( \omega_r(x) \) has the property of \( f \) of equation (11) of increasing sequences. From equation (30) \( \theta_r, m(x) \) is possible only if \( \min[f_r(h_r)] = m(x) \). As \( h_r > m(x) \) by assumption unless \( f_r(x) = m(x) \), so in any case \( \theta_r(x) = m(x); f_r(x) = m(x) \). Thus \( f_r \) has property \( i \) in equation (11) of increasing sequence. Finally suppose that \( \omega_r(x) = x_r \) for \( r = 1, 2, 3, \ldots, n \), and thus \( \theta_r = f_r(x) \). From equation (29) \( \theta_r \leq h_r \), as \( \theta_r = f_r \), we get \( f_r \geq h_r \).

The last inequality is equivalent to \( \min[f_r(h_r)] = m(x) \leq h_r. \) Since \( m(x) = h_r \) by definition, we have \( m(x) = h_r \). By assumption this is possible only if \( m(x) = \infty \). Hence \( \omega(x) \) has property of \( e \) in equation (11) of increasing sequences. This completes the proof of theorem 3.

The Requirements of Functions \( H_r(x) \) and \( h_r(x) \)

The functions \( H_r(x) \) and \( h_r(x) \) can be selected in many ways. The following are a few of the possible choices:

\[
h_r(x) = \frac{1}{2}[(M(x) + m(x)) + \mu(x)]; \quad H_r(x) = \frac{1}{2}[(M'(x) + m(x))] = \mu'(x) \]  where \( r \in \mathbb{N} \) (N = 1, 2, 3, \ldots, n) and \( M(x) = \min(m(x), M_1) \) and \( M_1 \) is an upper estimate of the eigenvalue \( \lambda_r \), e.g., \( M = M(x) \) or \( M_1 = M(x) \). In [8], \( m(x) \) is defined as \( \min_{i \in \mathbb{N}} \rho_i(x) \) and \( M(x) \) is defined as \( \max_{i \in \mathbb{N}} \rho_i(x) \). However in Tedia S [19], \( m(x) = \mu(x) \), an \( \lambda(x) = \rho(x) \). While in Collatz [6], \( m(x) = \mu(x) \) and \( M(x) = \max_{i \in \mathbb{N}} \mu(x) \).

b) For full matrices, a reasonable choice for \( H_r(x) \) and \( h_r(x) \) are derived from the arithmetic mean of the \( f'_r(x); H_r(x) = \sigma(x) = \frac{1}{n} \sum_{i=1}^{n} f'_i(x); \) and \( h_r(x) = \sigma(x) = \frac{1}{n} \sum_{i=1}^{n} f'_i(x) \) where \( f'_i(x) = \min(f_i(x), M_1) \).

c) Another simple choice for \( h_r(x); H_r(x) \) is a weighted arithmetic mean of \( f'_i(x) \):

\[
h_r(x) = H_r(x) = \frac{\sum_{i=1}^{n} f'_i(x)(s_i x_i)}{||x||} = u(x), \quad u(x) \] can also be defined in the following mean:

\[
v(x) = \frac{\sum_{i=1}^{n} b_i x_i}{||x||}, \quad \text{Where } b_i = \sum_{i=1}^{n} a_i. \]  \tag{34}

New Alternating Iteration Method

A step of the alternating iteration sequence method consists in modifying a single component \( x_r \) of \( x \). As a result \( z_r, f_r, ||x||, u(x) \) will have to be calculated at each iteration. Calculating \( z_r, ||x||, u(x) \) from their definition in equations (1), (5) will be referred as recalculating. A considerable reduction of calculation can be accomplished if instead of recalculating these terms are merely updated according to the following steps:

\[ ||x^{(i+1)}|| = ||x^{(i)}|| + (x^{(i+1)} - x^{(i)}) \quad \text{and} \quad z_r^{(i+1)} = z_r^{(i)} + a_r(x^{(i+1)} - x^{(i)}) \]  \( i = 1, 2, 3, \ldots, n \)

\[ u(x^{(i+1)}) = u(x^{(i)}) + b_i(x^{(i+1)} - x^{(i)}). \]  \tag{35}

These steps will be referred to as the updating iteration. The updating equations can be obtained easily from equations 1 through 5. To prevent the accumulation of round off errors after a number of iterations, the variables will have to be recalculated instead of updating. If we are working in a double precision, our previous experiences indicate that it is more than sufficient to recalculate after every twenty five iterations.

Over-Relaxation Method

From various choices for functions \( H_r(x), h_r(x) \) and \( u(x) \) as indicated in equation (34) seems to give a rapid convergence at least for full matrices. The purpose of this section is to present a variant of equation (35) by introducing the over-relaxation technique. We consider the following equation

\[ h_r(x) = (1 - \gamma) f_r(x) + \gamma u(x) \]  \tag{36}

As it well known, for several suitable values for \( \gamma \), is the over-relaxation factor, and \( 1 \leq \gamma \leq 2 \). Equation (36) may be useful in case of banded matrices. The over-relaxation method contains the following cases:
• $\gamma = 1$ for simultaneous over-relaxation method, and
• $1 < \gamma < 2$ for over-relaxation method.

Error vector in all methods the quantity $\Delta(x) = M(x) - m(x)$ as indicated in (4) is used as a measure of accuracy.

Discussion

Before we go any further, the following issues should be understood. Are both eigenvalues and eigenvectors required to be calculated, or are eigenvalues by itself enough? Is only the largest eigenvalue of interest? Does the matrix have special characteristics such as real symmetric, essentially positive, and so on? If all eigenvalues and eigenvectors are required then this new cannot be used;

If a matrix $A$ is essentially positive and the positive eigenvector $(x_1)$ and the corresponding eigenvalue $(\lambda_1)$ are of particular interest, then the new method can be used. Each step of the numerical method requires $n^2 + O(n)$ computations, if the parameters are chosen for the best rate of convergence. It is possible to assume that in half the steps practically no computations are needed, resulting thereby in $\frac{n^2}{2} + O(n)$ computations for each iteration. As previously stated, after some iteration the variables will have to be recalculated instead of updating. Recalculations need $n^2 + O(n)$ additional computations. If the computations are performed in double precision, recalculations will not have to be performed so often. As a result, recalculations do not increase the total number of computation significantly.

For our numerical comparisons all three methods, Power, New Method, and QR methods, were tried to solve eigenvalue of the following matrices:

All three methods were used to estimate the eigenvalue of Hilbert matrices of various orders. Let $H_n$ be a Hilbert matrix of order $n$. The elements of Hilbert matrix are defined according to the following relations:

$$A_{ij} = \frac{1}{(i+j-1)} \quad 1 \leq i, j \leq n.$$

The results of the 3 methods can be seen in Tables 1-3.

- We would like to find the efficiency of the three numerical methods, when a matrix had eigenvalues of nearly the same modulus. So it was decided to pick a matrix of order $n$ that was almost cyclic ($c_n$). Consider the below mentioned matrix

$$\begin{bmatrix}
A_{1,1} & A_{1,2} \\
A_{2,1} & A_{2,2}
\end{bmatrix}.$$ 

The elements of $A_{1,1}$ and $A_{2,1}$ were defined as follows,

$$a_{1,ij} = \frac{1}{(i+j-1)}, \quad A_{2,1} \text{ is a (8, 12) matrix, and } A_{1,1} \text{ is a (12, 8) matrix.}$$

The elements of $A_{2,2}$ were defined in the following respects,

$$\begin{array}{c|c|c|c|c|c|c|c}
\text{Operations} & \Delta(x) & \log|\Delta(x)| \\
\hline
1640 & 1.423 & 0.3528 \\
4921 & 1.41 \times 10^{-1} & -1.958 \\
8200 & 1.39 \times 10^{-2} & -4.275 \\
11490 & 1.3441 \times 10^{-3} & -6.611 \\
14764 & 1.296 \times 10^{-4} & -8.949 \\
18040 & 1.25 \times 10^{-5} & -11.288 \\
21322 & 1.208 \times 10^{-6} & -13.626 \\
24600 & 1.116 \times 10^{-7} & -15.965 \\
27880 & 1.123 \times 10^{-8} & -18.304 \\
\end{array}$$

Table 1: Hilbert matrix $H_n$ new method.

$$\begin{array}{c|c|c|c|c|c|c|c}
\text{Operations} & \Delta(x) & \log|\Delta(x)| \\
\hline
17400 & 2.423 \times 10^{2} & -3.21 \\
23400 & 1.007 \times 10^{4} & -9.202 \\
26600 & 8.263 \times 10^{4} & -18.614 \\
332000 & 2.422 \times 10^{2} & -3.722 \\
442000 & 1.0096 \times 10^{4} & -9.212 \\
506000 & 8.239 \times 10^{4} & -18.615 \\
\end{array}$$

Table 2: Hilbert matrix $H_n$ power method.

$$a_{i,j} = \frac{10^{-2}}{(i+j-1)} \cdot A_{i,j} \text{ is a (12, 12) matrix, and } A_{2,1} \text{ is a (8, 8) matrix.}$$

If the elements of $A_{1,1}$ and $A_{2,1}$ were replaced by zero, then the matrix would be nearly cyclic. For comparisons, the results of those 3 methods can be seen in Tables 4-6.

- Introducing a proper shift of origin could speed up the convergence of power method [9]. So it was decided to try that kind of matrix by introducing a shift of origin would not help the speed of convergence. Such a matrix of order $n(Q_n)$ can be given by the following relations.

$$a_{i,j} = \max \left\{ \frac{n-i}{n}, \frac{n-j}{n} \right\}, \quad 1 \leq i, j \leq n \quad \text{and} \quad a_{i,j} = \frac{(n-1)}{n} \left( \frac{2(i + n - 1)}{n} \right)^{\frac{1}{n}}$$

The results of our tests are indicated in Tables 7-10 for Arnoldi.

We will assume that we are interested in the positive eigenvector and the corresponding eigenvalue of the essentially positive matrix. From our trials, it is obvious that in all three cases the rate of convergence of our new method is better than or at least as fast as the power [9]. The QR [26] method converges very slowly in the last two cases, when the separation between the eigenvalues is poor. Let us consider the results of case b, when the matrix is nearly cyclic. For a cyclic matrix there are some eigenvalues of equal modulus, and so for a matrix that is “near cyclic” it is plausible to assume the separation between the modules is very poor. The new method takes about 5, 700 multiplications and divisions to reach an accuracy of 8 digits; which is about 5 times the computations of the power method and the QR method reaches an accuracy of 2 digits and 4 digits respectively. We should remember that the QR method is not specifically designed to calculate just one eigenvalue; therefore, a comparable efficiency cannot be expected. Thus from our recent experience, we can conclude that the new method shows a good speed of convergence even when the separation of the eigenvalues is poor. However, in the case of banded matrices the new method converges slowly. The new was tried on various banded matrices arising from finite difference approximation to boundary value problems of ordinary differential equations. A computer code was written specially for banded matrices, to take advantage of the large number of zero elements in a banded matrix. We will here summarize the results of our computer runs with the following (20, 20) matrix.
We will now return our attention to full matrices. Let $R_n$ be a matrix of order $n$ with pseudo-random entries. The new method and the power method were tried on each family of matrices ($R_n$, $C_n$, $H_n$).

### Table 4: Almost cyclic matrix $H_n$ power method.

| Operations | $\Delta(x)$ | Log$\Delta(x)$ |
|------------|--------------|-----------------|
| 1400       | 2.76 $\times$ 10$^{-2}$ | -3.589          |
| 2800       | 7.607 $\times$ 10$^{-1}$ | -7.187          |
| 4200       | 2.293 $\times$ 10$^{1}$  | -10.699         |
| 5600       | 6.349 $\times$ 10$^{1}$  | -14.277         |
| 7000       | 1.9264 $\times$ 10$^{1}$ | -17.774         |
| 8400       | 5.3279 $\times$ 10$^{-1}$ | -21.358         |

### Table 5: Almost cyclic matrix $C_n$ new method.

| Operations | $\Delta(x)$ | Log$\Delta(x)$ |
|------------|--------------|-----------------|
| 2400       | 3.091 $\times$ 10$^{-1}$ | -6.622          |
| 3260       | 2.089 $\times$ 10$^{1}$ | -8.308          |
| 4000       | 1.421 $\times$ 10$^{1}$ | -14.178         |
| 44680      | 9.706 $\times$ 10$^{-2}$ | -21.798         |
| 21340      | 6.349 $\times$ 10$^{-2}$ | -6.620          |
| 29340      | 4.5461 $\times$ 10$^{-2}$ | -8.30992        |
| 37300      | 3.111 $\times$ 10$^{-1}$ | -14.178         |
| 41340      | 2.1341 $\times$ 10$^{-2}$ | -21.798         |

### Table 6: Almost cyclic matrix $C_n$ QR.

| Operations | $\Delta(x)$ | Log$\Delta(x)$ |
|------------|--------------|-----------------|
| 4200       | 6.506 $\times$ 10$^{2}$ | 8.788           |
| 8788       | 1.0223 $\times$ 10$^{1}$ | 4.608           |
| 3780       | 6.196 | 1.824           |
| 4.608      | 1.065 | 0.0609         |
| 7990       | 2.1967 $\times$ 10$^{-1}$ | -1.518          |
| 1.824      | 4.985 $\times$ 10$^{-2}$ | -2.988          |
| 12190      | 1.160 $\times$ 10$^{-1}$ | -4.458          |
| 0.0609     | 2.703 $\times$ 10$^{-1}$ | -5.916          |
| 16390      | 6.294 $\times$ 10$^{-2}$ | -7.372          |

### Table 7: Matrix $Q_n$ power method.

| Operations | $\Delta(x)$ | Log$\Delta(x)$ |
|------------|--------------|-----------------|
| 2400       | 3.150 | 1.134           |
| 3260       | 2.089 $\times$ 10$^{1}$ | -8.472          |
| 4000       | 4.778 $\times$ 10$^{1}$ | -14.558         |
| 44680      | 3.124 $\times$ 10$^{-1}$ | -21.798         |

### Table 8: Matrix $Q_n$ new method.

\[
a_{ij} = \begin{cases} 
-2 & \text{if } 1 \leq i \leq n, \; i \neq j, \; i + 1 \leq n, \; i \\ 1 & \text{if } 1 \leq i \leq n-1, \; i + 1 = n, \; j = 0 \\
0 & \text{otherwise} \end{cases} \tag{37}
\]

The over relaxation method as described in equation (36), was tried on the previously mentioned matrix with values of $\gamma$ ranging from 1 to 1.99. The speed convergence did not show a remarkable dependence of $\gamma$. An 8 digit of accuracy was obtained in 168 iterations for $\gamma=1.73$, whereas for full matrices the new method gave a 9 digit of accuracy in 21 steps.

We will now return our attention to full matrices. Let $R_n$ be a matrix (of order $n$) with pseudo-random entries. The new method and the power method were tried on each family of matrices ($R_n$, $C_n$, $H_n$).

### Table 9: Matrix $Q_n$ new method.

| Operations | $\Delta(x)$ | Log$\Delta(x)$ |
|------------|--------------|-----------------|
| 4260       | 1.429 $\times$ 10$^{2}$ | 2.6591          |
| 5260       | 1.322 $\times$ 10$^{2}$ | -4.3429         |
| 5870       | 4.664 $\times$ 10$^{3}$ | -9.9722         |
| 41300      | 1.4294 $\times$ 10$^{4}$ | 2.6588          |
| 51300      | 1.324 $\times$ 10$^{4}$ | -4.3262         |
| 55300      | 4.688 $\times$ 10$^{4}$ | -9.9722         |

### Table 10: Matrix $Q_n$ Arnoldi method.

| Operations | $\Delta(x)$ | Log$\Delta(x)$ |
|------------|--------------|-----------------|
| 4350       | 1.439 $\times$ 10$^{3}$ | 2.5591          |
| 4500       | 1.322 $\times$ 10$^{3}$ | -6.473          |
| 5000       | 4.5778 $\times$ 10$^{2}$ | -15.558         |
| 54678      | 4.134 $\times$ 10$^{5}$ |                |

Acknowledgment

The author wishes to thank Prof. Schneider, Mathematics Department at the University of Wisconsin in Madison, for his suggestions during the writing of the author’s earlier manuscript and criticism during the writing. His criticism and suggestions yielded the development of this method. The author acknowledges indebtedness to him, and to his stimulating comments during the review of the earlier article [19]. The author would also like to thank Professor Tsuypshi Ando at Hokkaido University, for his proof of corollaries 1, 2, and 3 for its convergences, which is what led me to pursue most of the research collected herein. Also, a special thanks to him for his enthusiasm and help. Lastly, the author would like to express his gratitude to Prof. Thomas Laffey in the school of mathematical sciences, University of Dublin, for his suggestions during the writing of his earlier article [19]. The author would also like to thank Professor Tsuypshi Ando at Hokkaido University, for his proof of corollaries 1, 2, and 3 for its convergences, which is what led me to pursue most of the research collected herein. Also, a special thanks to him for his enthusiasm and help. Lastly, the author would like to express his gratitude to Prof. Thomas Laffey in the school of mathematical sciences, University of Dublin, for his suggestions during the writing of his earlier article [19].

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