Weighted graph algorithms with Python

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Abstract

Python implementation of selected weighted graph algorithms is presented. The minimal graph interface is defined together with several classes implementing this interface. Graph nodes can be any hashable Python objects. Directed edges are instances of the Edge class. Graphs are instances of the Graph class. It is based on the adjacency-list representation, but with fast lookup of nodes and neighbors (dict-of-dict structure). Other implementations of this class are also possible.

In this work, many algorithms are implemented using a unified approach. There are separate classes and modules devoted to different algorithms. Three algorithms for finding a minimum spanning tree are implemented: the Borůvka’s algorithm, the Prim’s algorithm (three implementations), and the Kruskal’s algorithm. Three algorithms for solving the single-source shortest path problem are implemented: the dag shortest path algorithm, the Bellman-Ford algorithm, and the Dijkstra’s algorithm (two implementations). Two algorithms for solving all-pairs shortest path problem are implemented: the Floyd-Warshall algorithm and the Johnson’s algorithm.

All algorithms were tested by means of the unittest module, the Python unit testing framework. Additional computer experiments were done in order to compare real and theoretical computational complexity. The source code is available from the public GitHub repository.

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I. INTRODUCTION

Algorithms are at the heart of computer science. They are expressed as a finite sequence of operations with clearly defined input and output. Algorithms can be expressed by natural languages, pseudocode, flowcharts or programming languages. Pseudocode is often used in textbooks and scientific publications because it is compact and augmented with natural language description. However, in depth understanding of algorithms is almost impossible without computer experiments where algorithms are implemented by means of a programming language.

Our aim is to show that the Python programming language [1] can be used to implement algorithms with simplicity and elegance. On the other hand, the code is almost as readable as pseudocode because the Python syntax is very clear. That is why Python has become one of the most popular teaching languages in colleges and universities [2], and it is used in scientific research [3]. Python implementation of some algorithms from computational group theory was shown in Ref. [4]. In this paper we are interested in graph theory and weighted graph algorithms. Other algorithms will be discussed elsewhere. The source code of our programs is available from the public GitHub repository [5]. We strongly support a movement toward open source scientific software [6].

Graphs can be used to model many types of relations and processes in physical, biological, social, and information systems [7]. Many practical problems can be represented by graphs and this can be the first step to finding the solution of the problem. Sometimes the solution can be known for a long time because graph theory was born in 1736. In this year Leonard Euler solved the famous Seven Bridges of Königsberg (Królewic in Polish) problem.

We examined several Python graph packages from The Python Package Index [8] in order to checked different approaches. Some graph libraries written in other programming languages were checked briefly [9], [10], [11], [12], [13].

- Package NetworkX 1.9 by Aric A. Hagberg, Daniel A. Schult, and Pieter J. Swart [14]. A library for the creation, manipulation, and study of the structure, dynamics, and functions of complex networks. Basic classes are Graph, DiGraph, MultiGraph, and MultiDiGraph. All graph classes allow any hashable object as a node. Arbitrary edge attributes can be associated with an edge. The dictionary of dictionaries data structure is used to store graphs. NetworkX is integrated into Sage. [15].
• Package *python-igraph* 0.7.0 \[16\]. *igraph* is the network analysis package with versions for R, Python, and C/C++.

• Package *python-graph* 1.8.2 by Pedro Matiello \[17\]. A library for working with graphs in Python. Edges are standard Python tuples, weights or labels are kept separately. There are different classes for directed graphs, undirected graphs, and hypergraphs.

• Package *graph* 0.4 by Robert Dick and Kosta Gaitanis \[18\]. Directed and undirected graph data structures and algorithms.

None of the available implementations satisfy our needs. Usually high computational efficiency leads to the unreadable code. On the other hand, C/C++ syntax or Java syntax are not very close to the pseudocode used for learning algorithms. That is why we develop and advocate our approach. We note that presented Python implementations may be not as fast as C/C++ or Java counterparts but they scale with the input size according to the theory. The presented algorithms are well known and that is why we used references to many Wikipedia pages.

The paper is organized as follows. In Section II basic definitions from graph theory are given. In Section III the graph interface is presented. In Sections IV, V, and VI the following algorithms are shown: for finding a minimum spanning tree, for solving the single-source shortest path problem, for solving all-pair shortest path problem. Conclusions are contained in Section VII.

II. DEFINITIONS

Definitions of graphs vary and that is why we will present our choice which is very common \[19\]. We will not consider multigraphs with loops and multiple edges.

A. Graphs

A *(simple)* graph is an ordered pair $G = (V, E)$, where $V$ is a finite set of nodes (vertices, points) and $E$ is a finite set of edges (lines, arcs, links). An edge is an ordered pair of different nodes from $V$, $(s, t)$, where $s$ is the source node and $t$ is the target node. This is a *directed edge* and in that case $G$ is called a *directed graph*. 
An edge can be defined as a 2-element subset of $V$, $\{s, t\} = \{t, s\}$, and then it is called an undirected edge. The nodes $s$ and $t$ are called the ends of the edge (endpoints) and they are adjacent to one another. The edge connects or joins the two endpoints. A graph with undirected edges is called an undirected graph. In our approach, an undirected edge corresponds to the set of two directed edges $\{(s, t), (t, s)\}$ and the representative is usually $(s, t)$ with $s < t$.

The order of a graph $G = (V, E)$ is the number of nodes $|V|$. The degree of a node in an undirected graph is the number of edges that connect to it.

A graph $G' = (V', E')$ is a subgraph of a graph $G = (V, E)$ if $V'$ is a subset of $V$ and $E'$ is a subset of $E$.

**B. Graphs with weights**

A graph structure can be extended by assigning a number (weight) $w(s, t)$ to each edge $(s, t)$ of the graph. Weights can represent lengths, costs or capacities. In that case a graph is a weighted graph.

**C. Paths and cycles**

A path $P$ from $s$ to $t$ in a graph $G = (V, E)$ is a sequence of nodes from $V$, $(v_0, v_1, \ldots, v_n)$, where $v_0 = s$, $v_n = t$, and $(v_{i-1}, v_i) \ (i = 1, \ldots, n)$ are edges from $E$. The length of the path $P$ is $n$. A simple path is a path with distinct nodes. The weight (cost or length) of the path in a weighted graph is the sum of the weights of the corresponding edges.

A cycle is a path $C$ starting and ending at the same node, $v_0 = v_n$. A simple cycle is a cycle with no repetitions of nodes allowed, other than the repetition of the starting and ending node.

A directed path (cycle) is a path (cycle) where the corresponding edges are directed. In our implementation, a path is a list of nodes. A graph is connected if for every pair of nodes $s$ and $t$, there is a path from $s$ to $t$. 
D. Trees

A (free) tree is a connected undirected graph $T$ with no cycles. A forest is a disjoint union of trees. A spanning tree of a connected, undirected graph $G$ is a tree $T$ that includes all nodes of $G$ and is a subgraph of $G$ [20]. Spanning trees are important because they construct a sparse subgraph that tells a lot about the original graph. Also some hard problems can be solved approximately by using spanning trees (e.g. traveling salesman problem).

A rooted tree is a tree $T$ where one node is designated the root. In that case, the edges can be oriented towards or away from the root. In our implementation, a rooted tree is kept as a dictionary, where keys are nodes and values are parent nodes. The parent node of the root is None. A forest of rooted trees can be kept in a dictionary with many roots.

A shortest-path tree rooted at node $s$ is a spanning tree $T$ of $G$, such that the path distance from root $s$ to any other node $t$ in $T$ is the shortest path distance from $s$ to $t$ in $G$.

III. INTERFACE FOR GRAPHS

According to the definitions from Section II, graphs are composed of nodes and edges. In our implementation nodes can be any hashable object that can be sorted. Usually they are integer or string.

Edges are instances of the Edge class (edges module) and they are directed, hashable, and comparable. Any edge has the starting node (edge.source), the ending node (edge.target), and the weight (edge.weight). The default weight is one. The edge with the opposite direction is equal to ~edge. This is very useful for combinatorial maps used to represent planar graphs [21].

Simple graphs (directed and undirected) are instances of the Graph class (graphs module). Multigraphs (directed and undirected) are instances of the MultiGraph class (multigraphs module) and they will not be discussed here. Let us show some properties of graphs that are listed in Table I. There are methods to report some numbers (nodes, edges, degrees). There are iterators over nodes and edges. There are also some logical functions.

```python
>>> from edges import Edge
>>> from graphs import Graph
>>> G = Graph(n=3, directed=False)
```
# n is for compatibility with other implementations.

```python
>>> G.is_directed()
False
```

```python
>>> G.add_edge(Edge('A', 'B', 5))
```

```python
>>> G.add_edge(Edge('A', 'C', 7))
```

# Nodes are added by default.

```python
>>> print G.v(), G.e()           # numbers of nodes and edges
3 2
```

```python
>>> list(G.iternodes())
['A', 'C', 'B']                  # random order of nodes
```

```python
>>> sorted((G.degree(v) for v in G.iternodes()), reverse=True)
[2, 1, 1]                        # the degree sequence
```

```python
>>> list(v for v in G.iternodes() if G.degree(v) == 1)
['C', 'B']                       # leafs
```

```python
>>> sum(edge.weight for edge in G.iteredges())
12                               # the graph (tree) weight
```

# Typical usage of an algorithm Foo from the module foo.

```python
>>> from foo import Foo
```

```python
>>> algorithm = Foo(G)            # initialization
```

```python
>>> algorithm.run()               # calculations
```

```python
>>> print algorithm.result        # results can be more...
```

Note that in the case of undirected graphs, edge and ~edge are two representatives of the same undirected edge. The method iteredges returns a representative with the ordering edge.source < edge.target.

Graph algorithms are implemented using a unified approach. There is a separate class for every algorithm. Different implementations of the same algorithm are grouped in one module. In the __init__ method, main variables and data structures are initialized. The name space of a class instance is used to access the variables and that is why interfaces of the auxiliary methods can be very short.
TABLE I. Interface for graphs; $G$ is a graph, $s$ and $t$ are nodes.

| Method name                  | Short description                                      |
|------------------------------|--------------------------------------------------------|
| Graph(n)                     | return an empty undirected graph                       |
| Graph(n, directed=True)      | return an empty directed graph                         |
| G.is_directed()              | return True if $G$ is a directed graph                  |
| G.v()                        | return the number of nodes                             |
| G.e()                        | return the number of edges                              |
| G.add_node(s)                | add $s$ to $G$                                          |
| G.del_node(s)                | remove $s$ form $G$                                    |
| G.has_node(s)                | return True if $s$ is in $G$                           |
| G.add_edge(edge)             | add a new edge to $G$                                  |
| G.del_edge(edge)             | remove the edge form $G$                               |
| G.has_edge(edge)             | return True if the edge is in $G$                      |
| G.weight(edge)               | return the edge weight or zero                         |
| G.iternodes()                | generate nodes on demand                               |
| G.iteredges()                | generate edges on demand                               |
| G.iteroutedges(s)            | generate outedges on demand                            |
| G.iterinedges(s)             | generate inedges on demand                             |
| G.degree(s)                  | return the degree of $s$ ($G$ undirected)              |
| G.indegree(s)                | return the indegree of $s$                            |
| G.outdegree(s)               | return the outdegree of $s                             |

IV. MINIMUM SPANNING TREE

Let us assume that $G = (V, E)$ is a connected, undirected, weighted graph, and $T$ is a spanning tree of $G$. We can assign a weight to the spanning tree $T$ by computing the sum of the weights of the edges in $T$. A minimum spanning tree (MST) is a spanning tree with the weight less than or equal to the weight of every other spanning tree [22]. In general, MST is not unique. There is only one MST if each edge has a distinct weight (a proof by contradiction).
Minimum spanning trees have many practical applications: the design of networks, taxonomy, cluster analysis, and others [22]. We would like to present three classical algorithms for finding MST that run in polynomial time.

A. Borůvka’s algorithm

The Borůvka’s algorithm works for a connected graph whose edges have distinct weights what implies the unique MST. In the beginning, the cheapest edge from each node to another in the graph is found, without regard to already added edges. Then joining these groupings continues in this way until MST is completed [23]. Components of MST are tracked using a disjoint-set data structure. The algorithm runs in $O(E \log V)$ time.

Our implementation of the Borůvka’s algorithm works also for disconnected graphs. In that case a forest of minimum spanning trees is created. What is more, repeated edge weights are allowed because our edge comparison use also nodes, when weights are equal.

```python
from edges import Edge
from unionfind import UnionFind

class BoruvkaMST:
    """Boruvka’s algorithm for finding MST."""

    def __init__(self, graph):
        """The algorithm initialization."""
        self.graph = graph
        self.mst = graph.__class__(graph.v()) # MST as a graph
        self.uf = UnionFind()

    def run(self):
        """Executable pseudocode."""
        for node in self.graph.iternodes():
            self.uf.create(node)
        forest = set(node for node in self.graph.iternodes())
        dummy_edge = Edge(None, None, float("inf"))
```
new_len = len(forest)
old_len = new_len + 1

while old_len > new_len:
    old_len = new_len
    min_edges = dict(((node, dummy_edge)
                     for node in forest))

    # Finding the cheapest edges.
    for edge in self.graph.iteredges(): # O(E) time
        source = self.uf.find(edge.source)
        target = self.uf.find(edge.target)
        if source != target: # different components
            if edge < min_edges[source]:
                min_edges[source] = edge
            if edge < min_edges[target]:
                min_edges[target] = edge

    # Connecting components, total time is O(V).
    forest = set()
    for edge in min_edges.itervalues():
        if edge is dummy_edge: # a disconnected graph
            continue
        source = self.uf.find(edge.source)
        target = self.uf.find(edge.target)
        if source != target: # different components
            self.uf.union(source, target)
            forest.add(source)
            self.mst.add_edge(edge)

    # Remove duplicates, total time is O(V).
    forest = set(self.uf.find(node) for node in forest)
    new_len = len(forest)

    if new_len == 1: # a connected graph
        break
We note that the Borůvka's algorithm is well suited for parallel computation.

B. Prim's algorithm

The Prim's algorithm has the property that the edges in growing $T$ always form a single tree. The weights from $G$ can be negative. We begin with some node $s$ from $G$ and $s$ is added to the empty $T$. Then, in each iteration, we choose a minimum-weight edge $(s, t)$, joining $s$ inside $T$ to $t$ outside $T$. Then the minimum-weight edge is added to $T$. This process is repeated until MST is formed.

The performance of Prim's algorithm depends on how the priority queue is implemented. In the case of the first implementation a binary heap is used and it takes $O(E \log V)$ time.

```python
from edges import Edge
from Queue import PriorityQueue

class PrimMST:
    """Prim's algorithm for finding MST."""

def __init__(self, graph):
    """The algorithm initialization."""
    self.graph = graph
    self.distance = {node: float("inf") for node in self.graph.iternodes()}
    self.parent = {node: None for node in self.graph.iternodes()}
    self.in_queue = {node: True for node in self.graph.iternodes()}  # MST as a dict
    self.pq = PriorityQueue()

def run(self, source=None):
    """Executable pseudocode."""
    if source is None:  # get first random node
        source = self.graph.iternodes().next()
```
The second implementation is better for dense graphs ($|E| \sim |V|^2$), where the adjacency-matrix representation of graphs is often used. For-loop is executed $|V|$ times, finding the minimum takes $O(V)$. Therefore, the total time is $O(V^2)$.

```python
class PrimMatrixMST:
    """Prim's algorithm for finding MST in $O(V^2)$ time."""

def __init__(self, graph):
    """The algorithm initialization."""
    self.graph = graph
    self.distance = dict((node, float("inf")))
    for node in self.graph.iternodes()
        self.parent = dict((node, None))
        for node in self.graph.iternodes() # MST as a dict
            self.in_queue = dict((node, True))
            for node in self.graph.iternodes()
```python
def run(self, source=None):
    """Executable pseudocode."""
    if source is None:  # get first random node
        source = self.graph.iternodes().next()
    self.source = source
    self.distance[source] = 0
    for step in xrange(self.graph.v()):  # |V| times
        # Find min node in the graph, O(V) time.
        node = min((node for node in self.graph.iternodes()
                     if self.in_queue[node]), key=self.distance.get)
        self.in_queue[node] = False
        for edge in self.graph.iteroutedges(node):  # O(V) time
            if (self.in_queue[edge.target] and
                edge.weight < self.distance[edge.target]):
                self.distance[edge.target] = edge.weight
                self.parent[edge.target] = edge.source
```

C. Kruskal’s algorithm

The Kruskal’s algorithm builds the MST in forest [25], [26]. In the beginning, each node is in its own tree in forest. Then all edges are scanned in increasing weight order. If an edge connects two different trees, then the edge is added to the MST and the trees are merged. If an edge connects two nodes in the same tree, then the edge is discarded.

The presented implementation uses a priority queue in order to sort edges by weights. Components of the MST are tracked using a disjoint-set data structure (the UnionFind class). The total time is $O(E \log V)$.

```python
from unionfind import UnionFind
from Queue import PriorityQueue

class KruskalMST:
```
def __init__(self, graph):
    """The algorithm initialization."
    self.graph = graph
    self.mst = graph.__class__(graph.v())  # MST as a graph
    self.uf = UnionFind()
    self.pq = PriorityQueue()

def run(self):
    """Executable pseudocode."""
    for node in self.graph.iternodes():  # O(V) time
        self.uf.create(node)
    for edge in self.graph.iteredges():  # O(E*\log(V)) time
        self.pq.put((edge.weight, edge))
    while not self.pq.empty():  # |E| steps
        _, edge = self.pq.get()  # O(\log(V)) time
        if (self.uf.find(edge.source) !=
            self.uf.find(edge.target)):
            self.uf.union(edge.source, edge.target)
            self.mst.add_edge(edge)

The Kruskal's algorithm is regarded as best when the edges can be sorted fast or are already sorted.

V. SINGLE-SOURCE SHORTEST PATH PROBLEM

Let us assume that $G = (V, E)$ is a weighted directed graph. The shortest path problem is the problem of finding a path between two nodes in $G$ such that the path weight is minimized \[27\]. The negative edge weights are allowed but the negative weight cycles are forbidden (in that case there is no shortest path).

The Dijkstra's algorithm and the Bellman-Ford are based on the principle of relaxation.
Approximate distances (overestimates) are gradually replaced by more accurate values until the correct distances are reached. However, the details of the relaxation process differ. Many additional algorithms may be found in Ref. [28].

A. Bellman-Ford algorithm

The Bellman-Ford algorithm computes shortest paths from a single source to all of the other nodes in a weighted graph \( G \) in which some of the edge weights are negative [29]. The algorithm relaxes all the edges and it does this \(|V| - 1\) times. At the last stage, negative cycles detection is performed and their existence is reported. The algorithm runs in \( O(V \cdot E) \) time.

class BellmanFord:

"""The Bellman–Ford algorithm for the shortest path problem."""

def __init__(self, graph):
    """The algorithm initialization."""
    if not graph.is_directed():
        raise ValueError("graph is not directed")
    self.graph = graph
    self.distance = dict(((node, float("inf"))
                          for node in self.graph.iternodes()))
    # Shortest path tree as a dictionary.
    self.parent = dict(((node, None)
                         for node in self.graph.iternodes()))

def run(self, source):
    """Executable pseudocode."""
    self.source = source
    self.distance[source] = 0
    for step in xrange(self.graph.v()-1):  # \(|V| - 1\) times
        for edge in self.graph.iteredges():  # \(O(E)\) time
            self._relax(edge)
# Check for negative cycles.
for edge in self.graph.iteredges():  # O(E) time
    if (self.distance[edge.target] > self.distance[edge.source] + edge.weight):
        raise ValueError("negative cycle")

def _relax(self, edge):
    """Edge relaxation.""
    alt = self.distance[edge.source] + edge.weight
    if self.distance[edge.target] > alt:
        self.distance[edge.target] = alt
        self.parent[edge.target] = edge.source
    return True
return False

def path(self, target):
    """Construct a path from source to target.""
    if self.source == target:
        return [self.source]
    elif self.parent[target] is None:
        raise ValueError("no path to target")
    else:
        return self.path(self.parent[target]) + [target]

B. Dijkstra’s algorithm

The Dijkstra’s algorithm solves the single-source shortest path problem for a graph \( G \) with non-negative edge weights, producing a shortest path tree \( T \) [30], [31]. It is a greedy algorithm that starts at the source node, then it grows \( T \) and spans all nodes reachable from the source. Nodes are added to \( T \) in order of distance. The relaxation process is performed on outgoing edges of minimum-weight nodes. The total time is \( O(E \log V) \).
from Queue import PriorityQueue
class Dijkstra:
    """Dijkstra's algorithm for the shortest path problem."""
    def __init__(self, graph):
        """The algorithm initialization."""
        if not graph.is_directed():
            raise ValueError("graph is not directed")
        self.graph = graph
        self.distance = dict((node, float("inf"))
            for node in self.graph.iternodes())
        self.parent = dict((node, None)
            for node in self.graph.iternodes())
        self.in_queue = dict((node, True)
            for node in self.graph.iternodes())
        self.pq = PriorityQueue()

    def run(self, source):
        """Executable pseudocode."""
        self.source = source
        self.distance[source] = 0
        for node in self.graph.iternodes():
            self.pq.put((self.distance[node], node))
        while not self.pq.empty():
            _, node = self.pq.get()
            if self.in_queue[node]:
                self.in_queue[node] = False
            else:
                continue
            for edge in self.graph.iteroutedges(node):
if self.in_queue[edge.target] and self._relax(edge):
    self.pq.put(((self.distance[edge.target],
                  edge.target))

The second implementation is better for dense graphs with the adjacency-matrix representation, the total time is $O(V^2)$.

class DijkstraMatrix:
    """Dijkstra's algorithm with $O(V^2)$ time."""

def __init__(self, graph):
    """The algorithm initialization."""
    if not graph.is_directed():
        raise ValueError("graph is not directed")
    self.graph = graph
    self.distance = dict((node, float("inf"))
                         for node in self.graph.iternodes())
    self.parent = dict((node, None)
                      for node in self.graph.iternodes())
    self.in_queue = dict((node, True)
                         for node in self.graph.iternodes())

def run(self, source):
    """Executable pseudocode."""
    self.source = source
    self.distance[source] = 0
    for step in xrange(self.graph.v()):  # /V/ times
        # Find min node, $O(V)$ time.
        node = min((node for node in self.graph.iternodes()
                     if self.in_queue[node]), key=self.distance.get)
        self.in_queue[node] = False
        for edge in self.graph.iteroutedges(node):  # $O(V)$ time
            if self.in_queue[edge.target]:

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Le us note that a time of $O(E + V \log V)$ is best possible for Dijkstra’s algorithm, if edge weights are real numbers and only binary comparisons are used. This bond is attainable using Fibonacci heaps, relaxed heaps or Vheaps [32]. Researchers are also working on adaptive algorithms which profit from graph easiness (small density, not many cycles).

C. DAG Shortest Path

Shortest paths in DAG are always defined because there are no cycles. The algorithm runs in $O(V + E)$ time because topological sorting of graph nodes is conducted [19].

```python
from topsort import TopologicalSort

class DAGShortestPath:
    """The shortest path problem for DAG."""

    def __init__(self, graph):
        """The algorithm initialization."""
        if not graph.is_directed():
            raise ValueError("graph is not directed")
        self.graph = graph
        self.distance = dict((node, float("inf"))
        for node in self.graph.iternodes())
        self.parent = dict((node, None)
        for node in self.graph.iternodes())

    def run(self, source):
        """Executable pseudocode."""
        self.source = source
        self.distance[source] = 0
        algorithm = TopologicalSort(self.graph)
        algorithm.run()
```
VI. ALL-PAIRS SHORTEST PATH PROBLEM

Let us assume that $G = (V, E)$ is a weighted directed graph. The \textit{all-pairs shortest path problem} is the problem of finding shortest paths between every pair of nodes \[27\]. Two algorithms solving this problem are shown: the Floyd-Warshall algorithm and the Johnson’s algorithm. There is the third algorithm in our repository, which is based on matrix multiplication (the \texttt{allpairs.py} module) \[19\]. A basic version has a running time of $O(V^4)$ (the \texttt{SlowAllPairs} class), but it is improved to $O(V^3 \log V)$ (the \texttt{FasterAllPairs} class).

A. Floyd-Warshall algorithm

The Floyd-Warshall algorithm computes shortest paths in a weighted graph with positive or negative edge weights, but with no negative cycles \[33\]. The algorithm uses a method of dynamic programming. The shortest path from $s$ to $t$ without intermediate nodes has the length $w(s, t)$. The shortest paths estimates are incrementally improved using a growing set of intermediate nodes. $O(V^3)$ comparisons are needed to solve the problem. Three nested for-loops are the heart of the algorithm.

Our implementation allows the reconstruction of the path between any two endpoint nodes. The shortest-path tree for each node is calculated and $O(V^2)$ memory is used. The diagonal of the path matrix is inspected and the presence of negative numbers indicates negative cycles.

```python
class FloydWarshallPaths :
    """The Floyd–Warshall algorithm with path reconstruction."""

    def __init__(self, graph):
        """The algorithm initialization."""
        if not graph.is_directed():
```
```python
raise ValueError("graph is not directed")
self.graph = graph
self.distance = dict()
self.parent = dict()
for source in self.graph.iternodes():
    self.distance[source] = dict()
    self.parent[source] = dict()
for target in self.graph.iternodes():
    self.distance[source][target] = float("inf")
    self.parent[source][target] = None
self.distance[source][source] = 0
for edge in self.graph.iteredges():
    self.distance[edge.source][edge.target] = edge.weight
    self.parent[edge.source][edge.target] = edge.source

def run(self):
    """Executable pseudocode."""
    for node in self.graph.iternodes():
        for source in self.graph.iternodes():
            for target in self.graph.iternodes():
                alt = self.distance[source][node] + \
                self.distance[node][target]
                if alt < self.distance[source][target]:
                    self.distance[source][target] = alt
                    self.parent[source][target] = \
                    self.parent[node][target]
            if any(self.distance[node][node] < 0)
                for node in self.graph.iternodes():
                    raise ValueError("negative cycle detected")

def path(self, source, target):
```
"""Path reconstruction."""

if source == target:
    return [source]
elif self.parent[source][target] is None:
    raise ValueError("no path to target")
else:
    return self.path(source, self.parent[target]) + [target]

B. Johnson’s algorithm

The Johnson’s algorithm finds the shortest paths between all pairs of nodes in a sparse
directed graph. The algorithm uses the technique of reweighting. It works by using the
Bellman-Ford algorithm to compute a transformation of the input graph that removes all
negative weights, allowing Dijkstra’s algorithm to be used on the transformed graph [34].
The time complexity of our implementation is $O(VE \log V)$, because the binary min-heap
is used in the Dijkstra’s algorithm. It is asymptotically faster than the Floyd-Warshall
algorithm if the graph is sparse.

```python
from edges import Edge
from bellmanford import BellmanFord
from dijkstra import Dijkstra

class Johnson:
    """The Johnson algorithm for the shortest path problem."""

    def __init__(self, graph):
        """The algorithm initialization."""
        if not graph.is_directed():
            raise ValueError("graph is not directed")
        self.graph = graph

    def run(self):
```
"""Executable pseudocode."""

```python
self.new_graph = self.graph.__class__(
    self.graph.v() + 1, directed=True)

for node in self.graph.iternodes():  # O(V) time
    self.new_graph.add_node(node)

for edge in self.graph.iteredges():  # O(E) time
    self.new_graph.add_edge(edge)

self.new_node = self.graph.v()
self.new_graph.add_node(self.new_node)

for node in self.graph.iternodes():  # O(V) time
    self.new_graph.add_edge(Edge(self.new_node, node, 0))

self.bf = BellmanFord(self.new_graph)
# If this step detects a negative cycle,
# the algorithm is terminated.
self.bf.run(self.new_node)  # O(V*E) time
# Edges are reweighted.

for edge in list(self.new_graph.iteredges()):  # O(E) time
    edge.weight = (edge.weight
                   + self.bf.distance[edge.source]
                   - self.bf.distance[edge.target])
    self.new_graph.del_edge(edge)
    self.new_graph.add_edge(edge)

# Remove new_node with edges.
self.new_graph.del_node(self.new_node)
# Weights are now modified!

self.distance = dict()

for source in self.graph.iternodes():
    self.distance[source] = dict()

algorithm = Dijkstra(self.new_graph)
algorithm.run(source)

for target in self.graph.iternodes():
```

\[
\text{self.distance}[\text{source}][\text{target}] = (\text{algorithm.distance}[\text{target}] - \text{self.bf.distance}[\text{source}] + \text{self.bf.distance}[\text{target}])
\]

VII. CONCLUSIONS

In this paper, we presented Python implementation of several weighted graph algorithms. The algorithms are represented by classes where graph objects are processed via proposed graph interface. The presented implementation is unique in several ways. The source code is readable like a pseudocode from textbooks or scientific articles. On the other hand, the code can be executed with efficiency established by the corresponding theory. Python’s class mechanism adds classes with minimum of new syntax and it is easy to create desired data structures (e.g., an edge, a graph, a union-find data structure) or to use objects from standard modules (e.g., queues, stacks).

The source code is available from the public GitHub repository [5]. It can be used in education, scientific research, or as a starting point for implementations in other programming languages. The number of available algorithms is growing, let us list some of them:

- Graph traversal (breadth-first search, depth-first search)
- Connectivity (connected components, strongly connected components)
- Accessibility (transitive closure)
- Topological sorting
- Cycle detection
- Testing bipartiteness
- Minimum spanning tree
- Matching (Augmenting path algorithm, Hopcroft-Karp algorithm)
- Shortest path search
• Maximum flow (Ford-Fulkerson algorithm, Edmonds-Karp algorithm)

• Graph generators

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