MAXIMUM-ENTROPY INFERENCE AND INVERSE CONTINUITY OF THE NUMERICAL RANGE

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Abstract. We prove that the continuity of the maximum-entropy inference which refers to two quantum observables is equivalent to the strong continuity of vectors in the pre-image of the numerical range. A new condition for the continuity of the maximum-entropy inference, depending on analytic eigenfunctions, follows as a corollary. The condition implies that discontinuities are rare for two observables. A second corollary is that the continuity of the Max-Ent inference method is independent of the prior.

Key Words: maximum-entropy inference, continuity, numerical range, strong continuity, stability, strong stability.

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1. Introduction

The maximum-entropy inference method, based on the maximum-entropy principle by Boltzmann in statistical mechanics, is a standard technique to infer a quantum state from partial information both theoretically [15, 14, 31, 2] and in praxis [6]. Recently we have studied the continuity of the maximum-entropy inference [36, 34, 35, 27] when information is provided in terms of expected values of quantum observables. Discontinuity points of the maximum-entropy inference exist on the boundary of the set of expected values while the analogous maximum-entropy inference of probability distributions is continuous. The discontinuities have been discussed in condensed matter physics [8] as signatures of quantum phase transitions.

We have studied the continuity of the maximum-entropy inference using information topology [34] and convex geometry [35, 27]. Concerning the case of two observables, exact bounds for the number of discontinuity points have been derived [27] in dimensions $d = 2, 3, 4, 5$.
based on the analysis of pre-images of the numerical range [19] in operator theory.

Here we continue the numerical range approach by proving in Thm. 1 that the continuity of the maximum-entropy inference is equivalent to the strong continuity of vectors in the pre-image of the numerical range. Essential arguments are the stability of two-dimensional convex bodies [24] and the strong stability of the quantum state space [30]. The continuity theory of the numerical range [10, 20] has a lot ready to explore. We present one topic in detail which is a continuity condition depending on analytic eigenfunctions [20]. This result makes a continuity analysis of the maximum-entropy inference possible using algebraic and numerical methods.

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2. Overview

We consider the full matrix algebra $M_d$, $d \in \mathbb{N}$, of $d \times d$-matrices with complex entries and we denote by $1_d$ and $0_d$ the identity and zero elements, respectively. The real vector space of self-adjoint matrices will be denoted by $M_d^h = \{a \in M_d \mid a^* = a\}$ and we endow it with the scalar product $\langle a, b \rangle := \text{tr}(ab)$, $a, b \in M_d^h$, which makes $M_d^h$ a Euclidean space. We denote the Euclidean norm of $a \in M_d^h$ by $\|a\|_2 := \sqrt{\langle a, a \rangle}$.

The state space [1] of $M_d$ is

$$\mathcal{M}_d := \{\rho \in M_d \mid \rho \succeq 0, \text{tr}(\rho) = 1\} \subset M_d^h$$

where $a \succeq 0$ means that the matrix $a \in M_d$ is positive semi-definite.

We use physics language, see for example [4], Sec. 5.1, by calling elements of $\mathcal{M}_d$ states and self-adjoint matrices observables. The scalar product $\langle \rho, a \rangle$ of a state $\rho \in \mathcal{M}_d$ and an observable $a \in M_d^h$ has the physical interpretation of the expected value of $a$ when the system is in the state $\rho$.

Two observables in $M_d^h$ are equivalently provided by a matrix $A$ in $M_d$ in terms of its real part $\Re(A) := \frac{1}{2}(A + A^*)$ and imaginary part $\Im(A) := \frac{1}{2i}(A - A^*)$. By global assumptions we choose any natural number $d \in \mathbb{N}$ and any matrix $A \in M_d$. Adopting the identification of $\mathbb{C} \cong \mathbb{R}^2$, the real linear map

$$E_A : M_d^h \to \mathbb{C}, \quad a \mapsto \text{tr}(aA)$$
S. Weis

sends a state \( \rho \in \mathcal{M}_d \) to the pair \( \mathbb{E}_A(\rho) = (\langle \rho, \Re(A) \rangle, \langle \rho, \Im(A) \rangle) \) of expected value of \( \Re(A) \) and \( \Im(A) \).

The domain of the maximum-entropy inference which refers to the observables \( \Re(A) \) and \( \Im(A) \) is the compact convex set

\[
L_A := \{ \mathbb{E}_A(\rho) \mid \rho \in \mathcal{M}_d \} \subset \mathbb{R}^2
\]

which we call *convex support* \([36, 35, 27]\) by its name in probability theory \([3]\). The *von Neumann entropy* of a state \( \rho \in \mathcal{M}_d \) is defined by

\[
S_p(\rho) := -\text{tr}(\rho \cdot \log \rho)
\]

and the maximum-entropy inference is the map

\[
\rho^*_A : L_A \to \mathcal{M}_d, \quad \alpha \mapsto \arg \max \{ S(\rho) \mid \rho \in \mathcal{M}_d, \mathbb{E}_A(\rho) = \alpha \}.
\]

See \([15, 14]\) for more information about \( \rho^*_A \). Our analysis will be based on \([35]\), Thm. 4.9, which affirms that for all \( \alpha \in L_A \)

\[
(2.1) \quad \rho^*_A \text{ is continuous at } \alpha \text{ if, and only if, } \mathbb{E}_A|_{\mathcal{M}_d} \text{ is open at } \rho^*_A(\alpha).
\]

A different continuity problem \([10, 20]\) is that of pre-images of the *numerical range* \( W(A) \) which is the image of the quadratic form

\[
f_A : \mathbb{C}S^d \to \mathbb{C}, \quad x \mapsto x^* Ax
\]

defined on the unit sphere \( \mathbb{C}S^d \) of \( \mathbb{C}^d \). The multi-valued inverse \( f_A^{-1} : W(A) \to \mathbb{C}S^d \) is *strongly continuous* at \( \alpha \in W(A) \) if for all \( x \in f_A^{-1}(\alpha) \) the map \( f_A \) is open at \( x \). The numerical range belongs to the theory of operator theory, see for example \([18, 13]\), the statement of the Toeplitz-Hausdorff theorem is that \( W(A) \) is convex \([21]\) and \( W(A) = L_A \) holds, see for example \([5]\), Thm. 3. A main result of this article is as follows.

**Theorem 1.** For all \( \alpha \in L_A \) the maximum-entropy inference \( \rho^*_A \) is continuous at \( \alpha \) if, and only if, \( f_A^{-1} \) is strongly continuous at \( \alpha \).

We point out our surprise about Thm. 1 in Rem. 1 by showing that the analysis of the continuity of \( \rho^*_A \) and of the strong continuity of \( f_A^{-1} \) are quite opposite endeavors. The proof of Thm. 1 is completed in Sec. 5 as a corollary of (2.1) and of an analysis of \( \mathbb{E}_A|_{\mathcal{M}_d} \). Based on preliminaries in Sec. 3, we prove in Sec. 4 that the strong continuity of \( f_A^{-1} \) implies the openness of \( \mathbb{E}_A|_{\mathcal{M}_d} \) where we use the *stability* of two-dimensional convex bodies \([24]\). A converse statement in Sec. 5 relies on the *strong stability* of the quantum state space \([30]\).

Corollaries of Thm. 1 and of results in \([10]\) are derived in Sec. 6. They include a continuity condition for \( \rho^*_A \) in terms of eigenfunctions which are analytic curves. This continuity condition implies that the map \( \rho^*_A \) has at most finitely many points of discontinuity and that the set of matrices \( A \) where \( \rho^*_A \) is discontinuous is nowhere dense in \( \mathcal{M}_d \).
Sec. 7 derives another corollary from Secs. 4 and 5 by proving that the continuity of the quantum MaxEnt inference method [2] is independent of the prior. This continuity problem was studied in [33, 35], the independence of the prior was addressed in [35], Remark 5.9.

Finally, we remark that our methods do not extend directly to more than two observables \( u_1, \ldots, u_r \in M_d^h \) because the corresponding convex support \( \{ \langle \rho, u \rangle_{i=1}^r \mid \rho \in \mathcal{M}_d \} \) is in general not stable as was observed for \( r = d = 3 \) in [8], Exa. 4, and proved in [27], Exa. 5.2. Furthermore the joint numerical range \( \{ (x^* u_i x)_{i=1}^r \mid x \in \mathbb{C}S^d \} \) is in general not convex [21] for \( r \geq 4 \) (for \( r = 3 \) it is convex if \( d \geq 3 \)).

3. Preliminaries

We recall several basic concepts and we connect the domains of the functions \( f_A \) and \( E_A|_{\mathcal{M}_d} \) topologically by passing from the unit sphere \( \mathbb{C}S^d \) to the quotient of pure states.

Let us recall convex geometric notions and introduce notations. The state space is a convex body that is a compact convex set of a finite-dimensional real normed vector space. We assume that all convex sets in this article are subsets of a finite-dimensional real normed vector space. A face of a convex set \( C \) is a convex subset \( F \subset C \) which contains every segment \( [x, y] := \{(1 - \lambda)x + \lambda y \mid \lambda \in [0, 1] \} \), \( x, y \in C \), whenever a point of the open segment \( [x, y] := \{(1 - \lambda)x + \lambda y \mid \lambda \in (0, 1) \} \) belongs to \( F \). An extremal point is a face of dimension zero and a facet is a face of dimension \( \dim(C) - 1 \).

The extremal points of the state space \( \mathcal{M}_d \), \( d \in \mathbb{N} \), are called pure states in physics [4, 23] and it is well-known, see for example (4.2) in [1], that the set of pure states equals the set of rank-one density matrices which we denote by

\[
(3.1) \quad \mathcal{M}^1_d := \{ \rho \in \mathcal{M}_d \mid \text{rank}(\rho) = 1 \}.
\]

Given two vectors \( x, y \in \mathbb{C}^d \) we denote by \( x^* y := \overline{x_1 y_1} + \cdots + \overline{x_d y_d} \) their inner product and by \( |x| := \sqrt{x^* x} \) the Euclidean norm. The symbol \( x y^* : \mathbb{C}^d \to \mathbb{C}^d \) denotes the linear map \( (x y^*)(z) := (y^* z)x \) defined for \( z \in \mathbb{C}^d \). The rank-one density matrices are the orthogonal projections onto one-dimensional subspaces of \( \mathbb{C}^d \). So \( \mathcal{M}^1_d \cong \mathbb{CP}^{d-1} = \mathbb{C}S^d/\mathbb{C}S^1 \) is a projective space. We denote the quotient map

\[
(3.2) \quad \beta : \mathbb{C}S^d \to \mathcal{M}^1_d, \quad x \mapsto xx^*.
\]

Its fibers are isomorphic to the circle \( \mathbb{C}S^1 \). For \( d = 2 \) the famous Hopf fibration is obtained, see [16, 4] for more details.

Let us remark on the topology of \( \beta \). We denote the square root of a positive semi-definite matrix \( a \in M_d \) by \( \sqrt{a} \), that is \( \sqrt{a} \geq 0 \) and
$(\sqrt{a})^2 = a$. For arbitrary $a \in M_d$ we set $|a| := \sqrt{a^*a}$. The trace norm of $a \in M_d$ is $\|a\|_1 := \text{tr}|a|$, the trace distance between $\rho, \sigma \in M_d$ is

$$D(\rho, \sigma) := \frac{1}{2}\|\rho - \sigma\|_1$$

and the fidelity is

$$F(\rho, \sigma) := \|\sqrt{\rho} \sqrt{\sigma}\|_1 = \text{tr} \sqrt{\rho \sigma} \sqrt{\rho}.$$

The fidelity is symmetric in the two arguments and for all $\rho, \sigma \in M_d$ we have $0 \leq F(\rho, \sigma) \leq 1$ where the upper bound is achieved if and only if $\rho = \sigma$. The inequalities

$$1 - F(\rho, \sigma) \leq D(\rho, \sigma) \leq \sqrt{1 - F(\rho, \sigma)^2}$$

hold for all $\rho, \sigma \in M_d$, see for example [23, 11]. For pure states $xx^*$ and $yy^*$ with $x, y \in CS^d$ we have $F(xx^*, yy^*) = |x^*y|$.

A function $\gamma : X \to Y$ between topological spaces $X, Y$ is open at $x \in X$ if $\gamma$ maps neighborhoods of $x$ to neighborhoods of $\gamma(x)$. Thereby, a neighborhood of a point is any subset of the topological space in question which contains an open set about the given point. The function $\gamma$ is open on a subset of $X$ if $\gamma$ is open at each point of the subset. The function $\gamma$ is open if $\gamma$ is open on $X$.

**Lemma 1.** The map $\beta : CS^d \to M^1_d$ is continuous and open.

**Proof:** The right-hand side inequality in (3.3) shows $D^2 \leq (1 - F^2) \leq 2(1 - F)$. So for $x, y \in CS^d$ we have

$$D(xx^*, yy^*)^2 \leq |x|^2 + |y|^2 - 2|x^*y| \leq |x|^2 + |y|^2 - 2\Re(x^*y) = |x - y|^2,$$

whence $\beta$ is Lipschitz-continuous with the global constant one.

The left-hand side inequality in (3.3) implies for all $x, y \in CS^d$ such that $x^*y \in \mathbb{R}$ and such that $x^*y \geq 0$ the inequality of

$$|x - y|^2 = 2(1 - |x^*y|) = 2(1 - F(xx^*, yy^*)) \leq 2D(xx^*, yy^*).$$

This proves that the ball in $CS^d$ of (Hilbert space) radius $\epsilon > 0$ about $x \in CS^d$, mapped through $\beta$, contains the ball in $M^1_d$ of (trace distance) radius $\frac{1}{2}\epsilon^2$ about $xx^*$. Hence $\beta$ is open. \hfill $\Box$

Turning to the convex support and to the numerical range we notice that for all $x \in CS^d$

$$f_A(x) = x^*Ax = \text{tr}(xx^*A) = E_A(xx^*) = E_A(\circ \beta(x)).$$

**Lemma 2.** For all $x \in CS^d$ the following statements are equivalent.

1. $f_A$ is open at $x$,
2. $E_A|_{M^1_d}$ is open at $xx^*$. 

(3.4)
Proof: Since $f_{\tilde{A}} = E_{\tilde{A}} \circ \beta$ holds by (3.4) we can use Lemma 1 to prove (1) $\iff$ (2). The implication (1) $\implies$ (2) follows from the continuity of $\beta$. The implication (2) $\implies$ (1) follows from the openness of $\beta$. □

Let us finally point out that the continuity of $\rho^*_A$ and the strong continuity of $f^{-1}_{\tilde{A}}$ are very far from each other under several aspects. We recall that the relative interior of a convex set $C$ is the interior of $C$ in the topology of the affine hull of $C$ and the relative boundary of $C$ is the complement of the relative interior in the closure of $C$.

Remark 1. (1) Studying the continuity of $\rho^*_A$ requires by (2.1) to check the openness of $E_{\tilde{A}}$ restricted to the state space $\mathcal{M}_d$ while studying the strong continuity of $f^{-1}_{\tilde{A}}$ requires by Lemma 2 to check the openness of $E_{\tilde{A}}$ restricted to the extremal points $\mathcal{M}_1^d$ of $\mathcal{M}_d$.

(2) Lemma 5.8 in [35] shows that for all $\alpha \in L_A$ the state $\rho^*_A(\alpha)$ lies in the relative interior of the fiber $F := E_{\tilde{A}}|_{\mathcal{M}_d}(\alpha)$. So (2.1) shows that the continuity analysis of $\rho^*_A$ at $\alpha$ amounts to check the openness of $E_{\tilde{A}}|_{\mathcal{M}_d}$ at a relative interior point of $F$. On the other hand the continuity analysis of $f^{-1}_{\tilde{A}}$ at $\alpha$ requires to check the openness of $E_{\tilde{A}}|_{\mathcal{M}_d}$ at some extremal points of $F$. In fact, we have seen in the previous paragraph that the pure states $\mathcal{M}_1^d \cap F$ have to be checked. They are extremal points of $\mathcal{M}_d$ and a fortiori they are extremal points of $F$.

(3) While for all $\alpha \in L_A$ the maximum-entropy state $\rho^*_A(\alpha)$ uniquely maximizes the von Neumann entropy on the fiber $E_{\tilde{A}}|_{\mathcal{M}_d}(\alpha)$, the pre-image $f^{-1}_{\tilde{A}}(\alpha)$ is precisely the set of minimizers on that fiber where the von Neumann entropy is zero, see for example [32], Sec. A.2.

4. Strong continuity implies continuity

We prove that the strong continuity of $f^{-1}_{\tilde{A}}$ implies the continuity of the maximum entropy inference $\rho^*_A$. The main argument is the stability of two-dimensional convex bodies. A convex body $C$ is stable if the mid-point map

$$C \times C \rightarrow C, \quad (x, y) \mapsto \frac{1}{2}(x + y)$$

is open [24, 9].

We recall two facts about stable convex bodies. First, if $C$ is a stable convex body then for any integer $n \geq 2$ and for $(\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$ such
that $\lambda_i \geq 0$ for $i = 1, \ldots, n$ and $\lambda_1 + \cdots + \lambda_n = 1$ the map
\begin{equation}
\begin{aligned}
C \times \cdots \times C & \rightarrow C, \\
(x_1, \ldots, x_n) & \mapsto \lambda_1 x_1 + \cdots + \lambda_n x_n
\end{aligned}
\end{equation}
is open. The proof that (4.1) is open is given for $n = 2$ in [9], Prop. 1.1, and the case of $n \geq 3$ follows by induction.

Second, every two-dimensional convex body is stable. This follows from Thm. 2.3 in [24] which says that a convex body $C$ of any finite dimension $l \in \mathbb{N}$ is stable if, and only if, for each $k = 0, \ldots, l$ the $k$-skeleton, that is the union of all faces of $C$ of dimension at most $k$, is closed. It is well-known that the $(l-2)$-, the $(l-1)$- and the $l$-skeletons of $C$ are always closed, see [12] for a detailed proof.

The Euclidean ball of radius $\epsilon > 0$ about $\rho \in \mathcal{M}_d$ within a subset $C \subset \mathcal{M}_d$ will be denoted by $B_\epsilon(\rho, C) := \{ \sigma \in C \mid \| \rho - \sigma \|_2 \leq \epsilon \}$.

**Theorem 2.** Let $\alpha$ be an extremal point of $L_A$. If $f_A^{-1}$ is strongly continuous at $\alpha$ then $E_A|_{\mathcal{M}_d}$ is open on $E_A|_{\mathcal{M}_d}(\alpha)$.

**Proof:** If $\alpha \in L_A$ is an extremal point then the fiber $F := E_A|_{\mathcal{M}_d}(\alpha)$ is a face of the state space $\mathcal{M}_d$, so all extremal points of $F$ are pure states or, equivalently by (3.1), they belong to the set of rank-one states $\mathcal{M}_d^1$.

Hence we can write an arbitrary point $\rho \in F$ in the form
$$\rho = \lambda_1 \rho_1 + \cdots + \lambda_n \rho_n$$
where $\rho_i \in \mathcal{M}_d^1$, $\lambda_i \geq 0$ for $i = 1, \ldots, n$ and $\lambda_1 + \cdots + \lambda_n = 1$. Let $x_i \in \mathbb{C}S_d$ such that $\rho_i = x_i x_i^*$ and choose a neighborhood $N_i \subset \mathcal{M}_d^1$ of $\rho_i$ in $\mathcal{M}_d^1$. By the continuity of $\beta : \mathbb{C}S_d \rightarrow \mathcal{M}_d^1$, $x \mapsto xx^*$ (see Lemma 1) the pre-image $N_i' := \beta^{-1}(N_i)$ is a neighborhood of $x_i$ in $\mathbb{C}S_d$. The assumption that $f_A^{-1}$ is strongly continuous at $\alpha$ proves that $f_A(N_i')$ is a neighborhood of $\alpha$. Hence $E_A(N_i) = f_A(N_i')$ is a neighborhood of $\alpha$.

Now let $N \subset \mathcal{M}_d$ be an arbitrary neighborhood of $\rho$ in $\mathcal{M}_d$ and choose neighborhoods $N_i \subset \mathcal{M}_d^1$ about $\rho_i$ in $\mathcal{M}_d^1$ such that $\lambda_1 N_1 + \cdots + \lambda_n N_n \subset N$. It suffices to consider a Euclidean ball $B_\epsilon(\rho, \mathcal{M}_d) \subset N$ of radius $\epsilon > 0$ about $\rho$ and to use the Euclidean balls $N_i = B_\epsilon(\rho_i, \mathcal{M}_d^1)$ about $\rho_i$, $i = 1, \ldots, n$. Then
$$\lambda_1 \mathbb{C} E_A(N_1) + \cdots + \lambda_n \mathbb{C} E_A(N_n) \subset E_A(N).$$
As is shown in the first paragraph of this proof, each set $E_A(N_i)$ is a neighborhood of $\alpha$. The convex body $L_A$ is stable since $\dim(L_A) \leq 2$. Hence (4.1) shows that $E_A(N)$ contains a neighborhood of $\alpha$.

The linear map $E_A|_{\mathcal{M}_d}$ is open on the fiber $E_A|_{\mathcal{M}_d}(\alpha)$ of $\alpha \in L_A$ if $\alpha$ is a relative interior point of $L_A$ or a relative interior point of a facet of
$L_A$. For a proof see [35], Sec. 4.3, or [27], Sec. 3 and Coro. 4.4. Since \(\dim(L_A) \leq 2\) we deduce from Thm. 2 the following.

**Corollary 1.** If \(f_A^{-1}\) is strongly continuous at \(\alpha \in L_A\) then \(E_A|\mathcal{M}_d^{-1}\) is open on the fiber \(E_A|\mathcal{M}_d^{-1}(\alpha)\).

Now (2.1) proves continuity of the maximum-entropy inference \(\rho_A^*\).

**Corollary 2.** If \(f_A^{-1}\) is strongly continuous at \(\alpha \in L_A\) then \(\rho_A^*\) is continuous at \(\alpha\).

### 5. Continuity implies strong continuity

We prove that the continuity of \(\rho_A^*\) implies the strong continuity of \(f_A^{-1}\). This is the harder part compared to converse direction because we have to deduce the openness of \(E_A\) restricted to pure states \(\mathcal{M}_d^1 \subset \mathcal{M}_d\) from the openness of \(E_A|\mathcal{M}_d\). A major argument will be a corollary of the **strong stability** of the state space \(\mathcal{M}_d\).

It is well-known that the state space \(\mathcal{M}_d\) is stable. Indeed, Lemma 3 in [29] proves that the map

\[
(5.1) \quad \mathcal{M}_d \times \mathcal{M}_d \times [0,1] \to \mathcal{M}_d, \quad (\rho, \sigma, \lambda) \mapsto (1-\lambda)\rho + \lambda\sigma
\]

is open which is equivalent to the stability of \(\mathcal{M}_d\) by Prop. 1.1 in [9].

To make an openness statement about \(E_A|\mathcal{M}_d^1\) we have to restrict the left-hand side of (5.1) from \(\mathcal{M}_d\) to \(\mathcal{M}_d^1\) while keeping the right-hand side. This restriction is indeed possible. The cost is the non-finiteness of the ensemble, see Rem. 1 in [30]. The corresponding property of \(\mathcal{M}_d\) is called **strong stability** which, by definition, means that for all \(k = 1, \ldots, d\) the barycenter map from the discrete probability measures on \(\{\rho \in \mathcal{M}_d \mid \text{rank}(\rho) \leq k\}\) to \(\mathcal{M}_d\) is open, see [30], Thm. 1.

Lemma 4 in [30] serves our purposes: Let \(\{\pi_i, \rho_i\}_{i \in \mathbb{N}}\) be a countable ensemble, that is \(\rho_i \in \mathcal{M}_d^1, \pi_i \geq 0\) for all \(i \in \mathbb{N}\) and \(\sum_{i=1}^{\infty} \pi_i = 1\). For an arbitrary sequence \(\{\rho^n\} \subset \mathcal{M}_d\) converging to the average \(\sum_{i=1}^{\infty} \pi_i \rho_i\) there exists a sequence \(\{\{\pi^n_i, \rho^n_i\}_{i \in \mathbb{N}}\}_{n \in \mathbb{N}}\) of countable ensembles such that

\[
(5.2) \quad \forall n \quad \pi^n_1 \rho^n_1 + \pi^n_2 \rho^n_2 + \cdots = \rho^n, \\
(\forall i) \quad \lim_{n \to \infty} \pi^n_i = \pi_i \quad \text{and} \quad \pi_i > 0 \implies \lim_{n \to \infty} \rho^n_i = \rho_i.
\]

We use an immediate corollary of (5.2) which is as follows.

**Corollary 3.** Let \(\rho \in \mathcal{M}_d^1\) be a pure state and let \(N \subset \mathcal{M}_d^1\) be a neighborhood of \(\rho\) in \(\mathcal{M}_d^1\). Then for every state \(\sigma \in \mathcal{M}_d\) and \(\lambda \in (0,1]\) the set \((1-\lambda)N + \lambda\mathcal{M}_d\) is a neighborhood of \((1-\lambda)\rho + \lambda\sigma\) in \(\mathcal{M}_d\).
Based on two chapters of the numerical range theory, recall the equality of convex support and numerical range $L_A = W(A)$ from Sec. 2, the next lemma provides an extremal point argument. Firstly, Thm. 2 in [10] states that for all $\alpha \in L_A$ and $x \in \mathbb{C}S^d$ such that $\alpha = f_A(x)$ and for any neighborhood $U$ of $x$ in $\mathbb{C}S^d$ there is a constant $\delta > 0$ such that $\delta L_A + (1 - \delta)\alpha \subset f_A(U)$ holds. Secondly, Lemma 3.2 in [20] proves that for $r > 0$ and $x \in \mathbb{C}S^d$ the set $f_A(\{y \in \mathbb{C}S^d \mid |y - x| < r\})$ is convex.

**Lemma 3.** Let $\rho \in \mathcal{M}_d$ and let $N \subset \mathcal{M}_d$ be a neighborhood of $\rho$ in $\mathcal{M}_d$. There exists a neighborhood $\tilde{N} \subset N$ of $\rho$ in $\mathcal{M}_d$ such that $E_A(\tilde{N})$ is convex. If $E_A(\tilde{N})$ contains all extremal points of $L_A$ in a neighborhood of $E_A(\rho)$ in $L_A$ then $E_A(\tilde{N})$ is a neighborhood of $E_A(\rho)$ in $L_A$.

**Proof:** We switch from pure states to vectors using the quotient map (3.2) denoted $\beta : \mathbb{C}S^d \rightarrow \mathcal{M}_d$, $x \mapsto xx^*$. The continuity of $\beta$, see Lemma 1, shows that $N' := \beta^{-1}(N) \subset \mathbb{C}S^d$ is a neighborhood of any point in $\beta^{-1}(\rho)$. Let $x$ be such a point. Lemma 3.2 in [20], cited above, shows that for some neighborhood $N'' \subset N'$ of $x$ the image $f_A(N'')$ is convex. The openness of $\beta$, see Lemma 1, shows that $\tilde{N} := \beta(N'')$ is a neighborhood of $\rho$ in $\mathcal{M}_d$. Moreover, the image $E_A(\tilde{N}) = f_A(N'')$ is convex and contains $\alpha := E_A(\rho) = f_A(x)$.

It remains to prove that $f_A(N'')$ is a neighborhood of $\alpha$ if $f_A(N'')$ contains all extremal points of $L_A$ sufficiently close to $\alpha$. If $\alpha$ is a relative interior point of $L_A$ or a point in the relative interior of a facet of $L_A$, then by Thm. 2 in [10], cited above, $f_A(N'')$ is a neighborhood of $\alpha$ in $L_A$. Therefore we can assume that $L_A$ has real dimension two and that $\alpha$ is an extremal point of $L_A$. In that case, since $f_A(N'')$ is convex, it suffices to show that $f_A(N'')$ contains a neighborhood of $\alpha$ in the boundary $\partial L_A$ of $L_A$. We consider a disk $D := \{\alpha' \in \mathbb{C} \mid |\alpha' - \alpha| < \epsilon\}$ of radius $\epsilon > 0$ about $\alpha$ and the semi-arc $r_\epsilon$ of $D \cap \partial L_A$ to the right of $\alpha$ that is $r_\epsilon$ includes $\alpha$ and the points of $\partial L_A$ following $\alpha$ in clockwise direction. If $r_\epsilon$ is a segment for some $\epsilon$ then by Thm. 2 in [10] $f_A(N'')$ contains a neighborhood of $\alpha$ in $r_\epsilon$. Otherwise, given $\epsilon > 0$, the semi-arc $r_\epsilon$ has a sequence of extremal points of $L_A$ which converges to $\alpha$. By assumptions, $f_A(N'')$ includes all extremal points of $L_A$ which are sufficiently close to $\alpha$. Therefore, since it is convex, $f_A(N'')$ contains the segments between those extremal points and hence a neighborhood of $\alpha$ in $r_\epsilon$. Together with the analogous statement about the other semi-arc we have shown that $f_A(N'')$ contains a neighborhood of $\alpha$ in $\partial L_A$ which completes the proof.

We are ready to prove a main result of this paper.
Theorem 3. Let \( \alpha \in L_A \). If \( E_A|_{\mathcal{M}_d}^{-1}(\alpha) \) is open at a relative interior point of \( E_A|_{\mathcal{M}_d}^{-1}(\alpha) \) then \( f_A^{-1} \) is strongly continuous at \( \alpha \).

Proof: Let \( \rho \) be a relative interior point of the fiber \( F := E_A|_{\mathcal{M}_d}^{-1}(\alpha) \) such that \( E_A|_{\mathcal{M}_d} \) is open at \( \rho \). We have to prove that \( f_A^{-1} \) is open at every point of \( f_A^{-1}(\alpha) \). By Lemma 2, \( (2) \implies (1) \), it suffices to prove that for all pure states \( \sigma \in F \) the map \( E_A|_{\mathcal{M}_d} \) is open at \( \sigma \).

Let \( N \subset \mathcal{M}_d \) be a neighborhood of \( \sigma \) in \( \mathcal{M}_d \). By Lemma 3 there exists a neighborhood \( N' \subset \sigma \) of \( \sigma \) such that \( E_A(N') \) is convex. It suffices to show that \( E_A(N') \) is a neighborhood of \( \alpha \) in \( L_A \). Let \( \tau \in F \) and \( \lambda \in (0, 1) \) such that \( \rho = (1 - \lambda)\sigma + \lambda\tau \). This choice is possible [26] since \( \rho \) is a relative interior point of \( F \). The strong stability of \( \mathcal{M}_d \), see Coro. 3, proves that

\[
N'' := (1 - \lambda)N' + \lambda\mathcal{M}_d
\]

is a neighborhood of \( \rho \) in \( \mathcal{M}_d \). By assumptions \( E_A|_{\mathcal{M}_d} \) is open at \( \rho \) so

\[
E_A(N'') = (1 - \lambda)E_A(N') + \lambda L_A
\]

is a neighborhood of \( \alpha \) in \( L_A \). The definition of extremal points and (5.3) show that every extremal point of \( L_A \) which lies in \( E_A(N'') \) must lie in \( E_A(N) \). So Lemma 3 proves that the convex set \( E_A(N') \) is a neighborhood of \( \alpha \) in \( L_A \) which completes the proof.

We emphasize that, in general, the openness of \( E_A|_{\mathcal{M}_d} \) at a relative boundary point of \( E_A|_{\mathcal{M}_d}^{-1}(\alpha) \) does not imply the strong continuity of \( f_A^{-1} \) at \( \alpha \in L_A \). We will study in Example 1 a matrix \( A \) of size \( d = 3 \) where \( f_A^{-1} \) is not strongly continuous at \( \alpha = 1 \) while, by [27], Thm. 8.1, the map \( E_A|_{\mathcal{M}_3} \) is open at a relative boundary point of the three-dimensional Euclidean ball \( E_A|_{\mathcal{M}_3}^{-1}(1) \). This proves that it is essential in the assumptions of Thm. 3 (and Coro. 4) that \( E_A|_{\mathcal{M}_d} \) is open at a relative interior point of \( E_A|_{\mathcal{M}_d}^{-1}(\alpha) \).

Thm. 3 and Coro. 1 prove the following.

Corollary 4. Let \( \alpha \in L_A \). Then \( E_A|_{\mathcal{M}_d} \) is open at a relative interior point of \( E_A|_{\mathcal{M}_d}^{-1}(\alpha) \) if, and only if, \( f_A^{-1} \) is strongly continuous at \( \alpha \). In this case \( E_A|_{\mathcal{M}_d} \) is open on \( E_A|_{\mathcal{M}_d}^{-1}(\alpha) \).

We are ready to prove Theorem 1.

Proof of Theorem 1: As we have recalled in Rem. 1(2) the maximum-entropy state \( \rho_A^*(\alpha) \) lies in the relative interior of \( E_A|_{\mathcal{M}_d}^{-1}(\alpha) \) for all \( \alpha \in L_A \). Therefore the claim follows from (2.1) and Coro. 4.
6. A CONTINUITY CONDITION IN TERMS OF EIGENFUNCTIONS

We follow [20] by reinterpreting—in the context of the maximum-entropy inference—a condition for the strong continuity of $f_A^{-1}$ depending on analytic eigenfunctions and implying rarity of discontinuities.

Let us introduce a minimal selection of concepts from [20]. Secondary literature should be consulted directly there. One pillar is the existence [25] of an orthogonal basis of eigenvectors $\{x_k(\theta)\}_{k=1}^d$ of the real part $\Re(e^{-i\theta} A)$ of $e^{-i\theta} A$ and of corresponding eigenvalue curves $\{\lambda_k(\theta)\}_{k=1}^d$, called eigenfunctions [20], which depend analytically on $\theta \in \mathbb{R}$. The second pillar consists of two chapters from the theory of the numerical range. Firstly [18], the boundary generating curve (Randerzeugende Kurve in German) is a plane algebraic curve $C$ in $\mathbb{C} \cong \mathbb{R}^2$ such that the numerical range $W(A)$ is the convex hull of $C$. Secondly [17], the curve $C$ equals the union over $k = 1, \ldots, d$ of curves

$$z_k(\theta) := f_A(x_k) = e^{i\theta}(\lambda_k(\theta) + i\lambda'_k(\theta)), \quad \theta \in \mathbb{R}$$

where $\lambda'_k$ is the derivative of $\lambda_k$ with respect to $\theta$. An eigenfunction $\lambda_k$ corresponds to $\alpha \in W(A)$ at $\theta \in \mathbb{R}$ if $z_k(\theta) = \alpha$. The eigenfunctions which correspond to $\alpha$ at $\theta$ split if they are not mutually identical as functions $\mathbb{R} \to \mathbb{R}$.

A special case of [20], Thm. 2.1(1), is as follows.

**Fact 1.** Let (a) $\dim(W(A)) = 2$, let $\alpha$ be an extremal point of $W(A)$ and let (b) $\theta \in \mathbb{R}$ such that some eigenfunction corresponds to $\alpha$ at $\theta$. Then $f_A^{-1}$ is strongly continuous at $\alpha$ if, and only if, the eigenfunctions corresponding to $\alpha$ at $\theta$ do not split.

We stress that the statement of Fact 1 provides for all points of the numerical range a decisive continuity condition. The theorem addresses in (a) all points $\alpha$ of $W(A)$ where the strong continuity of $f_A^{-1}$ could possibly fail because Thm. 2 in [10], cited above in Sec. 5, proves that $f_A^{-1}$ is strongly continuous at relative interior points of $W(A)$ and at relative interior points of facets. Since the union of curves $\{z_k\}_{k=1}^d$ equals the plane algebraic curve $C$ whose convex hull is the numerical range, there exists for all extremal points $\alpha$ of $W(A)$ some $\theta \in \mathbb{R}$ such that an eigenfunction corresponds to $\alpha$ at $\theta$. So the assumption (a) in Fact 1 implies the existence of a $\theta \in \mathbb{R}$ satisfying (b). To provide more intuition we recall from [20] that in this case $\alpha$ lies on the supporting line of $W(A)$ with outward pointing unit vector $-e^{i\theta}$.

The following example demonstrates the use of Fact 1.
Example 1. A discontinuity of the maximum-entropy inference \( \rho^*_A \) is known [36, 8, 27] for

\[
A := \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \oplus [1]
\]

where the direct sum denotes a block-diagonal matrix in \( M_3 \). The numerical range \( W(A) \) is the unit disk in \( \mathbb{C} \cong \mathbb{R}^2 \) and \( \alpha = 1 \) is a point of discontinuity of \( \rho^*_A \). The real part of \( e^{-i\theta}A \) is

\[
\Re(e^{-i\theta} A) = (\cos(\theta)\sigma_1 + \sin(\theta)\sigma_2) \oplus \cos(\theta), \quad \theta \in \mathbb{R}
\]

for Pauli matrices

\[
\sigma_1 := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad \sigma_2 := \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}.
\]

Instead of the eigenvectors we write down the spectral projections of \( \Re(e^{-i\theta} A) \) which are

\[
P_1(\theta) := \frac{1}{2}(1_2 + \cos(\theta)\sigma_1 + \sin(\theta)\sigma_2) \oplus 0,
\]

\[
P_2(\theta) := \frac{1}{2}(1_2 - \cos(\theta)\sigma_1 - \sin(\theta)\sigma_2) \oplus 0,
\]

\[
P_3(\theta) := 0_2 \oplus 1.
\]

The corresponding eigenfunctions are given by \( \lambda_1(\theta) = 1 \), \( \lambda_2(\theta) = -1 \) and \( \lambda_3(\theta) = \cos(\theta) \) while \( z_1(\theta) = e^{i\theta} \), \( z_2(\theta) = -e^{i\theta} \) and \( z_3(\theta) = 1 \). The eigenfunctions \( \lambda_1 \) and \( \lambda_3 \) corresponding to \( \alpha = 1 \) at \( \theta = 0 \) split so Fact 1 proves that \( f_A^{-1} \) is not strongly continuous at \( \alpha = 1 \). Thus, Theorem 1 provides another proof of the discontinuity of \( \rho^*_A \) at \( \alpha = 1 \).

Continuity conditions for \( \rho^*_A \) arise from Fact 1 (and from the discussion in the paragraph thereafter). Recalling from Sec. 2 that \( L_A = W(A) \) holds, Coro. 4 and Thm. 1 show the following.

Corollary 5. Let \( \alpha \in L_A \).

1. The map \( E_A|_{M_d} \) is open on \( E_A|_{M_d}^{-1}(\alpha) \) unless \( L_A \) has dimension two, \( \alpha \) is an extremal point of \( L_A \) and there exists \( \theta \in \mathbb{R} \) such that the eigenfunctions corresponding to \( \alpha \) at \( \theta \) split. In the latter case \( E_A|_{M_d} \) is not open at any relative interior point of \( E_A|_{M_d}^{-1}(\alpha) \) but may be open at some relative boundary points of \( E_A|_{M_d}^{-1}(\alpha) \).

2. The maximum-entropy inference \( \rho^*_A \) is continuous at \( \alpha \) unless \( L_A \) has dimension two, \( \alpha \) is an extremal point of \( L_A \) and there exists \( \theta \in \mathbb{R} \) such that the eigenfunctions corresponding to \( \alpha \) at \( \theta \) split. In the latter case \( \rho^*_A \) is discontinuous at \( \alpha \).
We follow [20] and derive two rarity statements about the discontinuity of $\rho_A^*$. Firstly, the eigenfunctions $\lambda_k$ are analytic in a neighborhood of $\mathbb{R}$ in $\mathbb{C}$ and therefore, see for example [28], Thm. 10.18, distinct eigenfunctions can only coincide at finitely many exceptional values of $\theta \in [0, 2\pi)$. Thus Coro. 5 shows the following.

**Corollary 6.** (1) For all except possibly finitely many points $\alpha$ of $L_A$ the map $E_A|_{M_d}$ is open on $E_A|_{M_d}^{-1}(\alpha)$.

(2) For all except possibly finitely many points $\alpha$ of $L_A$ the maximum-entropy inference $\rho_A^*$ is continuous at $\alpha$.

Secondly, the von Neumann-Wigner non-crossing rule [22] in the formulation of [16], Thm. 9, implies that the set of matrices $A \in M_d$ such that the vector space $\{s\Re(A) + t\Im(A) \mid s, t \in \mathbb{R}\}$ of self-adjoint matrices contains a matrix with multiple eigenvalues is nowhere dense in $M_d$. Thus Coro. 5 shows the following.

**Corollary 7.** (1) The set of matrices $A \in M_d$ such that $E_A|_{M_d}$ is not open is nowhere dense in $M_d$.

(2) The set of matrices $A \in M_d$ such that $\rho_A^*$ is discontinuous is nowhere dense in $M_d$.

7. Independence of the prior

The MaxEnt inference method allows to make an update of an estimated or assumed state of a quantum system, called prior state, if new information is provided by expected values of one or more observables [7, 2]. We prove for two observables that the continuity of the MaxEnt method does not depend on the prior state.

As we have seen in Sec. 2 the set of expected values of two observables is provided by the convex support $L_A$ which refers to a matrix $A \in M_d$. The prior state $\rho \in M_d$ is assumed to be a positive definite matrix and the MaxEnt inference with respect to $A$ and $\rho$ is defined by

$$\Psi_{A,\rho} : L_A \rightarrow M_d, \quad \alpha \mapsto \arg\min\{S(\sigma, \rho) \mid \sigma \in E_A|_{M_d}^{-1}(\alpha)\}$$

where the Umegaki relative entropy $S : M_d \times M_d \rightarrow [0, \infty]$ is an asymmetric distance which is zero only for equal arguments. By definition $S(\sigma, \rho) = \text{tr} \sigma (\log(\sigma) - \log(\rho))$ holds for states $\rho$ of maximal rank and we have $S(\sigma, 1_d/d) = \log(d) - S(\sigma)$ for all $\sigma \in M_d$ where $S(\sigma)$ is the von Neumann entropy. Notice $\Psi_{A,1_d/d} = \rho_A^*$, the MaxEnt inference for the uniform prior $\rho = 1_d/d$ is the maximum-entropy inference.

The question whether the continuity of $\Psi_{A,\rho}$ is independent of the prior $\rho$ is natural and was asked in [35], Remark 5.9. We can solve this question for two observables affirmatively.
Theorem 4. If the MaxEnt inference $\Psi_{A,\rho}$ is continuous at $\alpha \in L_A$ for a prior state $\rho$ then $\Psi_{A,\rho}$ is continuous at $\alpha$ for all prior states $\rho$.

Proof: Since the function $\sigma \mapsto S(\sigma, \rho)$ is continuous for a positive definite prior state $\rho \in \mathcal{M}_d$, the continuity of $\Psi_{A,\rho}$ at $\alpha \in L_A$ is equivalent to the openness of $\mathbb{E}_A|_{\mathcal{M}_d}$ at $\Psi_{A,\rho}(\alpha)$, see [35], Thm. 4.9. In addition, Lemma 5.8 in [35] proves that $\Psi_{A,\rho}(\alpha)$ lies in the relative interior of the fiber $\mathbb{E}_A|_{\mathcal{M}_d}^{-1}(\alpha)$. Now Coro. 4 proves the claim. \qed

It is clear from Thms. 4 and 1 that for all prior states $\rho$ the MaxEnt inference $\Psi_{A,\rho}$ is continuous at a point $\alpha \in L_A$ if and only if $f_A^{-1}$ is strongly continuous at $\alpha$.

References

[1] E. M. Alfsen, F. W. Shultz (2001) State Spaces of Operator Algebras: Basic Theory, Orientations, and C*-Products, Springer-Verlag
[2] S. A. Ali, C. Cafaro, A. Giffin, C. Lupo, S. Mancini (2012) On a differential geometric viewpoint of Jaynes' MaxEnt method and its quantum extension, AIP Conf. Proc. 1443 120-128
[3] O. Barndorff-Nielsen (1978) Information and Exponential Families in Statistical Theory, John Wiley & Sons, New York
[4] I. Bengtsson, K. Życzkowski (2006) Geometry of Quantum States, Cambridge University Press
[5] S. K. Berberian, G. H. Orland (1967) On the closure of the numerical range of an operator, Proc Amer Math Soc 18(3) 499-503
[6] V. Bužek, G. Drobný, R. Derka, G. Adam, H. Wiedemann (1999) Quantum state reconstruction from incomplete data, Chaos, Solitons & Fractals 10 981–1074
[7] A. Caticha, A. Giffin (2006) Updating Probabilities, AIP Conf. Proc. 872 31–42
[8] J. Chen, Z. Ji, C.-K. Li, Y.-T. Poon, Y. Shen, N. Yu, B. Zeng, D. Zhou (2014) Principle of maximum entropy and quantum phase transitions, arXiv:1406.5046[quant-ph]
[9] A. Clausing, S. Papadopoulou (1978) Stable convex sets and extremal operators, Mathematische Annalen 231 193–203
[10] D. Corey, C. R. Johnson, R. Kirk, B. Lins, I. Spitkovsky (2013) Continuity properties of vectors realizing points in the classical field of values, Linear and Multilinear Algebra 61(10) 1329–1338
[11] C. A. Fuchs, J. Van De Graaf (1999) Cryptographic distinguishability measures for quantum-mechanical states, IEEE Trans Inf Theory 45(4) 1216–1227
[12] R. Grzaslewicz (1997) Extreme continuous function property, Acta Mathematica Hungarica 74(1–2) 93–99
[13] R. A. Horn, C. R. Johnson (2012) Matrix Analysis, 2nd Edition, Cambridge University Press
[14] R. S. Ingarden, A. Kossakowski, M. Ohya (1997) Information Dynamics and Open Systems, Kluwer Academic Publishers Group
[15] E. T. Jaynes (1957) Information theory and statistical mechanics., Phys Rev 106 620–630 and 108 171–190
[16] E. A. Jonckheere, F. Ahmad, E. Gutkin (1998) *Differential topology of numerical range*, Linear Algebra Appl 279(1–3) 227–254

[17] M. Joswig, B. Straub (1998) *On the numerical range map*, Journal of the Australian Mathematical Society 65 267–283

[18] R. Kippenhahn (1951) Über den Wertevorrat einer Matrix, Mathematische Nachrichten 6(3–4) 193–228

[19] T. Leake, B. Lins, I. M. Spitkovsky (2014) *Pre-images of boundary points of the numerical range*, Operators and Matrices 8(3) 699–724

[20] T. Leake, B. Lins, I. M. Spitkovsky (2014) *Inverse continuity on the boundary of the numerical range*, Linear and Multilinear Algebra 62 1335–1345

[21] C.-K. Li, Y.-T. Poon (2000) *Convexity of the joint numerical range*, SIAM J Matrix Anal A 21(2) 668–678

[22] J. von Neumann, E. P. Wigner (1929) Über das Verhalten von Eigenwerten bei adiabatischen Prozessen, Physikalische Zeitschrift 30 467–470

[23] M. A. Nielsen, I. L. Chuang (2000) *Quantum Computation and Quantum Information*, Cambridge University Press

[24] S. Papadopoulou (1977) On the geometry of stable compact convex sets, Math Ann 229 193–200

[25] F. Rellich (1954) *Perturbation Theory of Eigenvalue Problems*, Research in the Field of Perturbation Theory and Linear Operators, Technical Report No. 1, Courant Institute of Mathematical Sciences, New York University

[26] R. T. Rockafellar (1972) *Convex Analysis*, Princeton University Press

[27] L. Rodman, I. M. Spitkovsky, A. Szkola, S. Weis (submitted) *Continuity of the maximum-entropy inference: Convex geometry and numerical ranges approach*, arXiv:1502.02018 [math-ph]

[28] W. Rudin (1987) *Real and Complex Analysis*, 3rd ed., McGraw-Hill

[29] M. E. Shirokov (2006) *The Holevo capacity of infinite dimensional channels and the additivity problem*, Commun Math Phys 262 137–159

[30] M. E. Shirokov (2010) *Continuity of the von Neumann entropy*, Commun Math Phys 296(3) 625–654

[31] R. F. Streater (2011) *Proof of a modified Jaynes’s estimation theory*, Open Systems & Information Dynamics 18(2) 223–233

[32] A. Wehrl (1978) *General properties of entropy*, Reviews of Modern Physics 50 221–260

[33] S. Weis (2013) *Discontinuities in the maximum-entropy inference*, AIP Conf. Proc. 1553 192–199

[34] S. Weis (2014) *Information topologies on non-commutative state spaces*, Journal of Convex Analysis 21(2) 339–399

[35] S. Weis (2014) *Continuity of the maximum-entropy inference*, Communications in Mathematical Physics 330(3) 1263–1292

[36] S. Weis, A. Knauf (2012) *Entropy distance: New quantum phenomena*, J Math Phys 53(10) 102206
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