Abstract

We present improved algorithms for short cycle decomposition of a graph. Short cycle decompositions were introduced in the recent work of Chu et al. [CGP+18], and were used to make progress on several questions in graph sparsification.

For all constants $\delta \in (0, 1]$, we give an $O(mn^\delta)$ time algorithm that, given a graph $G$, partitions its edges into cycles of length $O(\log n)^{1/\delta}$, with $O(n)$ extra edges not in any cycle. This gives the first subquadratic, in fact almost linear time, algorithm achieving polylogarithmic cycle lengths. We also give an $m \cdot \exp(O(\sqrt{\log n}))$ time algorithm that partitions the edges of a graph into cycles of length $\exp(O(\sqrt{\log n \log \log n}))$, with $O(n)$ extra edges not in any cycle. This improves on the short cycle decomposition algorithms given in Chu et al. [CGP+18] in terms of all parameters, and is significantly simpler.

As a result, we obtain faster algorithms and improved guarantees for several problems in graph sparsification – construction of resistance sparsifiers, graphical spectral sketches, degree preserving sparsifiers, and approximating the effective resistances of all edges.
1 Introduction

Graph sparsification is the problem of approximating a graph $G$ by a sparse graph $H$, while preserving some key properties of the graph. Several notions of graph sparsification have been studied. For instance, graph spanners introduced by Chew \[Che89\] approximately preserve distances, and cut-sparse-integers introduced by Benczur and Karger \[BK96\] approximately preserve the sizes of all cuts.

The notion of spectral sparsification defined by Spielman and Teng \[ST11, ST04\] approximately preserves the Laplacian quadratic form of the graph. To define a spectral sparsifier, we recall the definition of the Laplacian of a graph. For an undirected, weighted graph $G = (V, E_G, w_G)$, with $n$ vertices and $m$ edges, the Laplacian of $G$, $L_G$, is the unique symmetric $n \times n$ matrix such that for all $x \in \mathbb{R}^n$, we have

$$x^T L_G x = \sum_{(u,v) \in E_G} w_G(u,v)(x_u - x_v)^2.$$  

For $\varepsilon < 1$, a graph $H$ is said to be an $\varepsilon$-sparsifier for $G$ if we have

$$\forall x \in \mathbb{R}^n, \quad (1 - \varepsilon)x^T L_G x \leq x^T L_H x \leq (1 + \varepsilon)x^T L_G x.$$  

Considering $x$ as indicator vectors of a cut shows that a spectral sparsifier is also a cut sparsifier.

Spectral sparsifiers have found numerous applications in graph algorithms – they are a crucial component of several fast solvers for Laplacian linear systems (this was the main objective of Spielman and Teng \[ST04, ST14, KMP14, KMP11\]). Additionally, they are the only graph theoretic primitive in some of them \[PS14, KLP+16\], such as faster cut and flow algorithms \[She13, She09, CKM+11\], sampling random spanning trees \[DKP+17\], estimating determinants \[DPPR17\], etc.

Spectral sparsification is widely considered to be well understood: Following a sequence of works \[ST11, SS11\], Batson, Spielman, and Srivastava \[BSS12, BSS09\] showed how to construct graph sparsifiers with $O(n\varepsilon^{-2})$ edges. We also know that this bound is tight, both for graphs \[BSS12\], and even for arbitrary data-structures that can answer the sizes of all cuts up to $(1 \pm \varepsilon)$ factors \[CKST17\].

However, a sequence of recent works \[ACK+16, DKW15, JS18, CKP+17\] opened up several interesting new directions and open questions in spectral sparsification:

1. Building on the work of Andoni et al. \[ACK+16\], Jambulapati and Sidford \[JS18\], showed how to construct data structures (spectral sketches) with $\tilde{O}(n\varepsilon^{-1})$ space that can estimate the quadratic form $x^T L_G x$ for a fixed unknown vector $x \in \mathbb{R}^n$ with high probability, even though any data-structure that can answer queries even for all $x \in \{\pm 1\}^n$ needs $\Omega(n\varepsilon^{-2})$ space \[ACK+16\]. Do there exist graphs with $\tilde{O}(n^{e-1})$ edges that are spectral sketches?

2. Dinitz et al. \[DKW15\] showed that for expander graphs, there exist resistance sparsifiers with $\tilde{O}(n\varepsilon^{-1})$ edges. Resistance sparsifiers preserve the effective resistance

$^{1}$The effective resistance between $u, v$ is the potential difference between $u, v$ if the graph is considered an electrical network with edge $e$ with weight $w_e$ as a resistor with resistance $1/w_e$, and a unit current is sent from $u$ to $v$.
A key component of their algorithms is a novel decomposition of graphs – a short-cycle decomposition – into short edge-disjoint cycles and a few extra edges.

**Definition 1.1** (Short Cycle Decomposition [CGP+18]). A \((\hat{k}, L)\)-short cycle decomposition of an unweighted undirected graph \(G\) decomposes \(G\) into several edge-disjoint cycles, each of length at most \(L\), and at most \(\hat{k}\) edges not in these cycles.

In addition to resolving the above two open problems, Chu et al. also show that short cycle decompositions can be used for constructing spectral sparsifiers that preserve degrees, sparsifying Eulerian directed graphs (directed graphs with all vertices having in-degree equal to out-degree), and faster estimation of effective resistances.

The existence of \((2n, 2\log n)\)-short cycle decompositions follows from a simple observation that every graph with minimum degree 3 must have a cycle of length \(2\log n\), which can be found by a simple breadth-first search. If a graph has more than \(2n\) edges, iteratively removing vertices of degree at most 2 must leave a graph with min-degree 3, and hence the graph contains a short cycle. Removing this cycle and repeating gives a simple \(O(mn)\) time algorithm for producing a \((2n, 2\log n)\)-short cycle decomposition of a graph \(G\). It is \textsc{NaiveCycleDecomposition} [CGP+18, Algorithm 11].

In order to give nearly-linear time algorithms for their applications, Chu et al. [CGP+18] give an algorithm \textsc{ShortCycleDecomposition} [CGP+18, Algorithm 15] that runs in time \(m \cdot \exp(O(\log n)^{3/4}))\), and returns a \((n \exp(O(\log n \log \log n)^{3/4}), \exp(O(\log n)^{3/4}))\)-short cycle decomposition of the graph (see [CGP+18, Theorem 3.11]).

### 1.1 Our Contributions

Our main result is a new algorithm for short cycle decomposition, which improves over the algorithms in the work of Chu et al. [CGP+18] in terms of all parameters, is faster, and considerably simpler.

**Theorem 1.2.** For all integers \(c \geq 1\), we give an algorithm that, given a graph \(G\) with \(n\) vertices and \(m\) edges, runs in time \(O\left(\frac{mn}{\epsilon} \cdot 500^c\right)\), and returns a \((O(n), O(\log n)^c)\)-short cycle decomposition of \(G\) with high probability.

As immediate consequences the above theorem, we obtain improvements on several of the results from [CGP+18]. Throughout, we let \(G\) be a graph with \(n\) vertices and \(m\) edges, and assume that the algorithms mentioned below are run using our algorithm \textsc{ShortCycleDecomp} (Algorithm 5) as its \textsc{CycleDecomposition} algorithm.

We obtain improved degree-preserving sparsifiers by plugging in Theorem 1.2 in [CGP+18, Theorem 4.1].

**Theorem 1.3** (Degree-Preserving Sparsification). For any integer \(c \geq 1\), algorithm \textsc{DegreePreservingSparsify} from [CGP+18] returns a graph \(H\) with at most \(n\epsilon^{-2} \cdot (O(\log n)^{c+1})\) edges such that with high probability all vertices have the same weighted degrees in \(G\) and \(H\), \(H\) is an \(\epsilon\)-spectral sparsifier of \(G\). The algorithm runs in time \(\tilde{O}(500^c \cdot m \cdot n^{1-\epsilon})\).

Combining Theorem 1.2 with [CGP+18, Theorem 6.1] gives an improved construction of graph-theoretical spectral sketches and resistance sparsifiers.
Theorem 1.4. For any integer $c \geq 1$, algorithm SpectralSketch from $[\text{CGP}^+18]$, given an undirected weighted graph $G$ and parameter $\varepsilon$ as inputs, runs in time $\tilde{O}(500^c \cdot m \cdot n^{\frac{1}{c+1}})$, and returns a graph $H$ with $\tilde{O}(n\varepsilon^{-1}) \cdot (O(\log n))^{3c+1}$ edges such that with high probability

1. $H$ is an $\varepsilon$-spectral sketch for $G$, i.e., for any fixed vector $x$, with high probability $x^\top L_H x = (1 \pm \varepsilon)x^\top L_G x$.

2. $H$ is an $\varepsilon$-resistance sparsifier for $G$. In fact, for any fixed vector $x$, with high probability, $x^\top L^+_H x = (1 \pm \varepsilon)x^\top L^+_G x$. 

Following the proof of Theorem 3.8 in Chu et al. $[\text{CGP}^+18]$ while applying our Theorem 1.2 gives an improved algorithm for estimating the effective resistances between all pairs.

Theorem 1.5. Given an undirected graph $G$ with $n$ vertices, $m$ edges, and any $t$ vertex pairs and error $\varepsilon > 0$, we can with high probability compute $\varepsilon$-approximations to the effective resistances between all $t$ of these pairs in time $\tilde{O}(m + (n + t)\varepsilon^{-1.5}) \exp(O(\log n \log \log n))$.

In contrast, Theorem 3.8 from Chu et al. $[\text{CGP}^+18]$ has a running time of $\tilde{O}(m + (n + t)\varepsilon^{-1.5}) \exp(O(\log n)^{3/4})$.

Finally, plugging in Theorem 1.2 in $[\text{CGP}^+18]$. Theorem 5.1] allows us to give an algorithm for sparsifying Eulerian directed graphs. Note that while this result is worse than the $\tilde{O}(m)$ algorithm given by Cohen et al. $[\text{CKP}^+17]$, this gives the first almost-linear time algorithm (for $c = \sqrt{\log n}$) for sparsifying Eulerian directed graphs that does not require expander decompositions.

Theorem 1.6. For any integer $c \geq 1$, algorithm EulerianSparsify from $[\text{CGP}^+18]$, given an Eulerian directed graph $\vec{G}$ with poly bounded edge weights as input, runs in time $\tilde{O}(500^c \cdot m \cdot n^{\frac{1}{c+1}})$, and returns an Eulerian directed graph $\vec{H}$ with at most $n\varepsilon^{-2} \cdot (O(\log n))^{3c+1}$ edges such that with high probability

$$\left\| L^+_{\vec{G}} \left( L^+_{\vec{G}} - L^+_{\vec{H}} \right) L^+_{\vec{G}} \right\|_2 \leq \varepsilon.$$ 

Comparison to the work of Chu et al. $[\text{CGP}^+18]$. Setting $c = 1$ in Theorem 1.2 gives an algorithm that finds an $(O(n), O(\log n))$-short cycle in time $O(m\sqrt{n})$ (in comparison with $O(mn)$ time for such a decomposition in $[\text{CGP}^+18]$). Setting $c = 1/\delta - 1$, for $\delta \in (0, 1/2]$ gives an $O(mn^\delta)$ time algorithm that finds an $(O(n), O(\log n)^{\delta-1})$-short cycle decomposition. On the other hand, the approach of Chu et al. $[\text{CGP}^+18]$ can only achieve subquadratic time if their cycles are length at least $\exp(\sqrt{\log n \log \log n})$ (see paragraph below for discussion). Setting $c = \sqrt{\log n}$ in Theorem 1.2 we obtain an algorithm that runs in $m \cdot \exp \left( O(\sqrt{\log n}) \right)$ time, and finds a $(O(n), \exp(O(\sqrt{\log n \log \log n})))$-short cycle decomposition of the graph. This beats the algorithms from Chu et al. in terms of all parameters: runtime, cycle length, and extra edges. Note that these improvements carry over immediately to all applications.

---

2 $L^+$ denotes the Moore-Penrose pseudoinverse of $L$. If the eigendecomposition of $L$ is $\sum_i \lambda_i v_i v_i^\top$, we have $L^+ = \sum_i \lambda_i > 0 \frac{v_i v_i^\top}{\lambda_i}$

3 For a directed graph $\vec{G}$, its directed Laplacian, $L_{\vec{G}}$, can be defined as

$$L_{\vec{G}}(u, v) := \begin{cases} \text{out-degree of } u & \text{if } u = v, \\ -(\text{weight of edge } v \rightarrow u) & \text{if } u \neq v \text{ and } v \rightarrow u \text{ is an edge.} \end{cases}$$
Our algorithm additionally is considerably simpler than that of \cite{CGP+18}. The algorithm in \cite{CGP+18} requires a strong version of an expander decomposition instead of the more standard expander decomposition algorithm of Spielman and Teng \cite{ST11}. Instead of each piece of the decomposition being contained in expanders, they need for each piece itself to be an expander. This requires a stronger expander decomposition which is given in work of Nanongkai and Saranurak \cite{NS17}. This immediately gives an overhead of $\exp\left(O(\sqrt{\log n \log \log n})\right)$ on the lengths of cycles produced by their algorithm, even if the recursion depth is set to some small integer constant. Our algorithm bypasses this by using only low diameter decomposition \cite{Bar96}, which allows us to generate cycles of length $\left(O(\log n)\right)^c$ for any constant $c \geq 1$.

**Summary of approach.** Our basic approach is to first partition our graph $G$ into connected components $A_1, A_2, \ldots, A_k$, each with diameter $O(\log n)$ via a low diameter decomposition. Now, we contract each component $A_i$ to a single vertex, and recursively find a short cycle decomposition on the contracted graph. Afterwards, we attempt to lift each cycle on the contracted graph back to the original graph so that they are still edge disjoint. The main obstacle is that it may be impossible to even lift two edge disjoint cycles on the contracted graph back to the original graph so that they remain edge disjoint. This was essentially the reason that Chu et al. \cite{CGP+18} required a strong expander decomposition in their algorithm. To get around this, one of our key ideas is that it suffices to work with vertex disjoint cycles, instead of edge disjoint cycles. This works well for many reasons. First, vertex disjoint cycles on the contracted graph can be easily lifted back to the original graph to stay vertex disjoint. Additionally, for a graph of bounded maximum degree, it is always possible to extract many short vertex disjoint cycles. Therefore, we do our recursive procedure to instead find vertex disjoint cycles. Our recursion structure is similar to the work of Alon et al. on constructing low-stretch spanning trees of a graph \cite{AKPW95}.

## 2 Preliminaries

Throughout we work with undirected unweighted multigraphs, allowing for multiple edges and self-loops. We say that self-loops add degree 2 to the vertex it is attached to.

In this work, we often work with vertex disjoint cycles instead of edge disjoint cycles.

**Definition 2.1.** A vertex disjoint short cycle decomposition is a short cycle decomposition where no two of the cycles share a vertex.

For a graph $G$, let $V(G)$ and $E(G)$ denote the edge and vertex sets of $G$. For a subgraph $S \subseteq G$, define $V(S)$ to be the set of vertices of $S$, and $E(S)$ the set of edges. Generally when the graph $G$ is clear from context, we let $n$ and $m$ denote $|V(G)|$ and $|E(G)|$ respectively.

For a graph $G$, let $\Delta \overset{\text{def}}{=} \Delta(G)$ denote the maximum degree of the graph $G$.

For a subgraph $G' \subseteq G$ (possibly with $V(G') \neq V(G)$), let the (strong) diameter of $G'$ be the maximum distance between two vertices in $V(G')$ using only the edges in $E(G')$. 


For disjoint subsets of vertices $A_1, A_2, \ldots, A_k$ of a graph, let $E(A_1, \ldots, A_k)$ denote the set of edges in with endpoints in different $A_i$.

### 2.1 Contraction

In this section, we discuss contraction of components in a graph, which plays a major role in our algorithms.

Let $G$ be a graph with $n$ vertices and $m$ edges, and let $A_1, A_2, \ldots, A_k$ be a partition of its vertices into disjoint components. Define the contraction of the components $A_1, A_2, \ldots, A_k$ to be the following graph, which we call $H$. $H$ has $k$ vertices numbered $1, 2, \ldots, k$, where vertex $i$ corresponds to component $A_i$ in $G$. Now, for each edge $uv \in E(G)$, if $u \in A_{u'}$ and $v \in A_{v'}$, add edge $u'v'$ to $H$. There is a clear bijection between the edges of $G$ and the edges of $H$, hence $H$ has $m$ edges too.

Define the edge injection $f : E(H) \rightarrow E(G)$ to be the function naturally obtained from the bijection described above.

### 3 Reduction to Sparse, Approximately Regular Graphs

In this section, we demonstrate that it suffices to provide algorithms for graphs $G$ which are both sparse (i.e. $m = O(n)$) and have bounded degree ($\Delta(G) = O(1)$).

**Lemma 3.1.** **GraphReduce** (Algorithm [1]) runs in $O(m + n)$ time and takes any graph $G$ with $n$ vertices and $m \geq n$ edges, and returns a graph $H$, such that:

1. $H$ has at most $2n$ vertices and exactly $m$ edges.
2. $\Delta(H) \leq \left\lceil \frac{2m}{n} \right\rceil$.
3. A $(\hat{k}, L)$-short cycle decomposition of $H$ can be mapped in $O(m + n)$ time to a $(\hat{k}, L)$-short cycle decomposition of $G$.

**Proof.** Consider the **GraphReduce** (Algorithm [1]). It can be implemented to run in $O(m + n)$ time.

Clearly when the algorithm ends, all vertices in $H$ have degree at most $\frac{2m}{n}$, as $\deg(v) - (t[v] - 1)D \leq D$. Also notice that the number of vertices in $H$ is

$$\sum_v t[v] = \sum_v \left\lfloor \frac{\deg(v)}{D} \right\rfloor \leq n + \sum_v \frac{\deg(v)}{D} = n + \frac{2m}{D} \leq n + \frac{2m}{2m/n} = 2n.$$ 

So $H$ has at most $2n$ vertices as desired. There is a natural mapping from the vertices and edges of $H$ to the vertices and edges of $G$ respectively. This mapping allows us to map each cycle in $H$ to a circuit in $G$ with identical length. Note that this circuit might now visit vertices more than once, but we can split a circuit into cycles with the same total length in time linear in the length of the cycle. This allows us to efficiently map a $(\hat{k}, L)$-short cycle decomposition of $H$ to a $(\hat{k}, L)$-short cycle decomposition of $G$. When we split the vertices, any edge is visited exactly twice, so the algorithm takes $O(m + n)$ time. \qed
Algorithm 1 GraphReduce, takes a graph $G$ with $n$ vertices, $m \geq n$ edges. Returns a graph $H$ with mapping of vertices from $H$ to $G$ where $\Delta(H) \leq \lceil \frac{2m}{n} \rceil$.

1: \[ D \leftarrow \lceil \frac{2m}{n} \rceil \]
2: Initialize $H$ to be the same as graph $G$
3: for vertex $v' \in V(G)$ do
4: \quad Let $v$ be the corresponding vertex in $V(H)$
5: \quad $t \leftarrow \left\lceil \frac{\text{deg}(v)}{D} \right\rceil$
6: \quad Let the neighbors of $v$ in $H$ be $u_1, u_2, \ldots, u_{\text{deg}(v)}$
7: \quad Delete $v$ from $H$
8: \quad for $i = 1$ to $t$ do
9: \quad \quad Add a new vertex to $H$ connected to $u_{1+(i-1)D}, u_{2+(i-1)D}, \ldots, u_{\min(\text{deg}(v), iD)}$
10: return $H$ and the vertex mapping (which vertices of $H$ come from which vertices in $G$)

Algorithm 2 Algorithm $B$, takes a graph $G$ with $n$ vertices and $m$ edges and outputs a $(20n, L(n))$-short cycle decomposition

1: Let $C \leftarrow \emptyset$ (the set of cycles we’ve found).
2: while $G$ has more than $20n$ edges do
3: \quad Let $G'$ be any subgraph of $G$ with exactly $20n$ edges.
4: \quad Let $H \leftarrow \text{GraphReduce}(G')$.
5: \quad Add isolated vertices to $H$ until $H$ has exactly $2n$ vertices.
6: \quad Let $C'$ be the set of cycles on $H$ we get when we run Algorithm $A$ on $H$.
7: \quad Let $C''$ be the set of cycles on $G$ corresponding to $C'$.
8: \quad Delete the edges of cycles in $C''$ from $G$.
9: \quad Let $C \leftarrow C \cup C''$.
10: return $C$.

We are now ready to state our main reduction result.

**Lemma 3.2.** Assume that an Algorithm $A$ takes as input graphs $G$ with $n$ vertices, $m = 10n$ edges, and maximum degree $\Delta$, and returns vertex disjoint cycles of length $L(n)$ containing at least $\Omega\left(\frac{n}{\Delta} \right)$ total vertices, in time $T(n)$.

Then, we can construct another Algorithm $B$ that takes as input a graph $G$ with $n$ vertices and $m$ edges, and outputs a $(20n, L(n))$-short cycle decomposition and runs in time $O\left(\frac{mT(n)}{n}\right)$.

**Proof.** Algorithm $B$ operates as described in Algorithm 2.

Note that because each $G'$ has $20n$ edges and $n$ vertices, we know that $\Delta(H) \leq 40$. Additionally, graph $H$ will have exactly $2n$ vertices and $20n$ edges. Therefore, by the conditions on Algorithm $A$ stated, we know that the cycles of $C'$ contain at least $\Omega\left(\frac{n}{10} \right) = \Omega(n)$ edges of $H$. Therefore, the cycles of $C''$ also contain at least $\Omega(n)$ edges. When the algorithm terminates, $G$ has less than $20n$ edges remaining, so it is clear that Algorithm $B$ returns an $(20n, L(n))$-short cycle decomposition. As each iteration of Algorithm $B$ removes cycles containing at least $\Omega(n)$ edges, it repeats at most $O\left(\frac{m}{n}\right)$ times, for a total runtime of $O\left(\frac{mT(n)}{n}\right)$ as Algorithm $A$ takes $T(n)$ time and all other
processing takes $O(n)$ time per iteration.

The above reduction allows us to work with constant degree graphs. Our main algorithms **ImprovedShortCycle** (Algorithm 3) and **ShortCycleDecomp** (Algorithm 5) both satisfy the conditions of Lemma 3.2 and assume that the input graph satisfies $m = 10n$.

## 4 Improved Naive Cycle Decomposition

In this section we present an improvement on **NaiveShortCycle** by giving an algorithm (Algorithm 3) that when given a graph $G$ with $n$ vertices and $m = 10n$ vertices, returns vertex disjoint cycles of length $O(\log n)$ containing at least $\frac{m}{10\Delta}$ vertices in total. It runs in time $O(m\sqrt{n})$.

### 4.1 High Level Overview

First, we give a high-level description of the algorithm. Assume the algorithm takes a graph $G$ with $n$ vertices, $m$ edges, and bounded maximum degree, which we can assume by the reduction given in Section 3. First, if the graph is $O(1)$ size, we can run a simple algorithm **NaiveShortCycle** (Lemma 4.1). Also, we can assume that $G$ has diameter $O(\log n)$, as we can perform a low diameter decomposition on it (Theorem 4.2) and operate on the low diameter components.

Assuming that $G$ has diameter $O(\log n)$, let $S$ be a spanning tree of $G$ with diameter $O(\log n)$, which we find by running a breadth first search. We split the graph $G$ into $\frac{\sqrt{m}}{4}$ connected components of all approximately the same size that are each also diameter $O(\log n)$. Because $G$ has bounded maximum degree, we can essentially do this by splitting the components along the spanning tree $S$ (see Lemma 4.4). Let the resulting components be $A_1, A_2, \ldots, A_k$. Now, contract the components to get a graph $H$ with $\frac{\sqrt{m}}{4}$ vertices and $m$ edges. As $G$ had bounded maximum degree, we know that $H$ has maximum degree $O(\sqrt{m})$. It is easy to check by the pigeonhole principle that $H$ has $\Omega(\sqrt{m})$ vertex disjoint cycles of length 1 and 2, because there are at most $\frac{1}{2} \left(\frac{\sqrt{m}}{4}\right)^2 = \frac{m}{32}$ possible distinct edges in $H$. Finally, given these vertex disjoint cycles of length 1 and 2 on $H$, we can recover cycles of length $O(\log n)$ in the graph $G$ by using the edges in $S$ (Lemma 4.3).

Before stating the algorithm, we state several of the subalgorithms which we have just described.

We defer the proofs to the appendix.

**Lemma 4.1.** **NaiveShortCycle** (Algorithm 6) takes a graph $G$ with $n$ vertices, $m$ edges, and maximum degree $\Delta$, and outputs vertex disjoint cycles of length at most $2\log n$ containing at least $\frac{m - 2n}{\Delta}$ total vertices. It runs in $O(m + n^2)$ time.

We will need low-diameter decompositions. They were introduced by Bartal [Bar96]. We use the following version from the work of Miller et al. [MPX13].

**Theorem 4.2** (Theorem 1.2 from [MPX13]). There is an algorithm **LowDiamDecomp**$(G, \beta)$ that takes a graph with $G$ and a parameter $\beta$ and with high probability returns a set $R$ of edges of $G$ of size $3m$ such that all connected components of $G \backslash R$ have diameter $O(\beta^{-1} \log n)$. The algorithm **LowDiamDecomp** runs in time $O(m)$.

Additionally, we need a simple routine to essentially “pull back” a short cycle decomposition on a contracted graph back to the base graph.
Algorithm 3 \textsc{ImprovedShortCycle}, takes a graph $G$ with $n$ vertices, $m = 10n$ vertices, and maximum degree $\Delta$. Returns a vertex disjoint cycles containing at least $\frac{m}{10\Delta}$ vertices.

1: Initialize $C$ as empty ($C$ is the set of cycles we’ve found).
2: \textbf{if} $n \leq 100$ \textbf{then return} \textsc{NaiveShortCycle}(G).
3: \textbf{if} $C$ contains at least $\frac{m}{10\Delta}$ vertices \textbf{then return} $(G, C)$.
4: Define $R \leftarrow \textsc{LowDiamDecomp}(G, \frac{1}{12})$.
5: Let $H_1, H_2, \ldots, H_k$ be the connected components of $E(G) \setminus R$.
6: \textbf{for} $1 \leq i \leq k$ \textbf{do} let $C_i \leftarrow \textsc{OneRoundShortCycle}(H_i)$, where $C_i$ is a set of cycles.
7: \textbf{for} $1 \leq i \leq k$ \textbf{do} $C \leftarrow C \cup C_i$, delete vertices from $C_i$ from $G$.
8: Return to line 3.

**Lemma 4.3.** The algorithm \textsc{PullUp}($H', C', \{K_i\}_{i=1}^{V(H')}, \{S_i\}_{i=1}^{V(H')}, f$) (Algorithm 7) takes the following inputs:

1. $H'$ – a graph.
2. $C'$ – a set of vertex disjoint cycles on the vertices of $H'$.
3. $\{K_i\}_{i=1}^{V(H')}$ – A partition of the vertices of a graph $G$, where $K_i$ corresponds to vertex $i$ of graph $H'$.
4. $\{S_i\}_{i=1}^{V(H')}$ – Each $S_i$ is a spanning tree on the vertices in $K_i$.
5. $f$ – This is a function $f : E(H') \rightarrow E(G)$ such that for an edge $uv \in E(H')$, we have that $f(uv) \in E(K_u, K_v)$.

It returns a set $C$ of cycles on the vertices of $G$ such that

1. The cycles in $C$ are vertex disjoint.
2. The cycles in $C$ cover at least as many vertices as those in $C'$ did.
3. The length in $C$ have maximum length at most $O(\max_i \text{diam}(S_i))$ times the longest cycle in $C'$.

It runs in time $O(n)$.

Finally, we need an algorithm that splits a tree into smaller subtrees. We will use this in order to split connected components in a graph into smaller connected components that are all approximately equal sized.

**Lemma 4.4.** Let $T$ be a tree with $n$ vertices maximum degree $D$. Assume the vertices are labelled with nonnegative integers between 0 and $X$ (denote labels as $c_v$ for $v \in V(T)$). Then the algorithm \textsc{TreeSplit}($n, t, T, X, \{c_v\}_{v \in V(T)}$) (Algorithm 8) is an $O(n)$ time algorithm that when given $T$ and a positive integer $t \leq \sum_v c_v$, splits the tree into connected subgraphs, each of which has sum of labels between $t$ and $Dt + X$.

Now we proceed to the algorithm, which we split into two parts. The main part is \textsc{OneRoundShortCycle}, which finds many vertex disjoint cycles on a graph with diameter $O(\log n)$.

We start by analyzing the runtime and guarantees of Algorithm 4.
Algorithm 4 OneRoundShortCycle, takes a graph $G$ with $n$ vertices, $m$ edges, maximum degree $\Delta$, and diameter $O(\log n)$. Returns vertex disjoint cycles of length $O(\log n)$ containing at least $\frac{m-5n}{10\Delta \sqrt{m}}$ vertices

1: Let $T$ be a spanning tree of diameter $O(\log n)$ of $G$.
2: for $v \in V(G)$ do let $c_v \leftarrow \deg v$.
3: Let $K \leftarrow \text{Treesplit}(n, 4\sqrt{m}, T, \Delta, \{c_v\}_{v \in V(G)})$.
4: Let $K = \{K_1, K_2, \ldots, K_{|K|}\}$.
5: Initialize graph $H$ on $|K|$ vertices as empty. Each vertex $i$ will correspond to $K_i$. 
6: for $1 \leq i \leq |K|$ do 
7: Let $S_i$ be a spanning tree of $K_i$ of diameter $O(\log n)$.
8: Let $H$ be the graph obtained by contracting the components $K_1, K_2, \ldots, K_{|K|}$.
9: Remove the edges in $H$ corresponding to the edges in the trees $S_1, \ldots, S_{|K|}$.
10: Let $f : E(H) \to E(G)$ be the corresponding edge injection (see Subsection 2.1).
11: Initialize $C'$ as empty (set of vertex disjoint cycles on vertices of $H$).
12: for $1 \leq i \leq |K|$ do 
13: for $i < j \leq |K|$ do 
14: if $H$ has edge $ij$ at least twice and none of $i, j$ used in $C'$ yet then add cycle $ij$ of length 2 to $C'$.
15: for $1 \leq i \leq |K|$ do 
16: if $i$ not used in $C'$ and $i$ has a self-loop in $H$ then add the self-loop $i$ to $C'$.
17: return PullUp($H, C', K, \{S_i\}_{i=1}^{K}, f$).

Lemma 4.5. When given a graph $G$ with $n$ vertices, $m$ edges, and maximum degree $\Delta$, Algorithm 4 returns vertex disjoint cycles of length $O(\log n)$ containing at least $\frac{\max(0, m-5n)}{10\Delta \sqrt{m}}$ vertices. It runs in time $O(m)$.

Proof. First, we claim that $|K| \leq \frac{1}{4}\sqrt{m}$. This follows from the guarantees of Treesplit (Lemma 4.4) used on line 3. In order to apply Lemma 4.4 we first must check that $\sum_v c_v \geq 4\sqrt{m}$. This is clear though, as $\sum_v c_v = 2m \geq 4\sqrt{m}$. By Lemma 4.4 we know that the sum of labels in each $K_i$ is at least $4\sqrt{m}$, while the sum of labels over all $K_i$ is at most $\sum_v c_v = 2m$. Therefore, $|K| \cdot 4\sqrt{m} \leq 2m$, so $|K| \leq \frac{1}{2}\sqrt{m}$.

Also, by the guarantees of Treesplit (Lemma 4.4) and the construction of graph $H$, we know that every vertex of $H$ has degree at most $\Delta \cdot 4\sqrt{m} + \Delta \leq 5\Delta \sqrt{m}$. So $\Delta(H) \leq 5\Delta \sqrt{m}$.

Next, we show that OneRoundShortCycle indeed satisfies its guarantee of removing cycles of length $O(\log n)$ containing at least $\frac{\max(0, m-5n)}{10\Delta \sqrt{m}}$ vertices. Throughout we assume that $m \geq 5n$, or else the claim is obvious. It is clear that $C'$ (as defined in OneRoundShortCycle) must be a maximal collection of vertex disjoint cycles of length 1 and 2. In other words, it cannot be enlarged only by adding in new 1 and 2-cycles. Then, we compute the number of edges touching at least some vertex of $C'$. If $C'$ involves $t$ vertices, at most $\Delta(H)t \leq 5\Delta \sqrt{mt}$ edges touch some vertex involved in $C'$. By the pigeonhole principle, any graph with at most $\frac{1}{2}\sqrt{m}$ vertices and at least $\frac{m}{2}$ edges must have either a 1 or 2-cycle. Therefore, by maximality, we have that

$$5\Delta \sqrt{mt} \geq \frac{m}{2} - n,$$

so we have that $t \geq \frac{m - 2n}{10\Delta \sqrt{m}}$. 

9
as desired. Combining the previous discussion with the guarantees of PullUp (Lemma 4.3) shows that \text{OneRoundShortCycle} successfully removes cycles of length \(O(\log n)\) of total length at least \(\frac{m - 5n}{10\Delta\sqrt{n}}\). It is clear that \text{OneRoundShortCycle} runs in time \(O(m)\).

Now we proceed to analyzing Algorithm 3.

**Lemma 4.6.** When given a graph \(G\) with \(n\) vertices, \(m = 10n\) edges, and maximum degree \(\Delta\), Algorithm 3 with high probability returns vertex disjoint cycles of length \(O(\log n)\) containing at least \(\frac{m}{10\Delta}\) vertices. It runs in \(O(m\sqrt{n})\) time.

**Proof.** We show that each iteration of Algorithm 3 also removes cycles of total length at least \(\Omega(\frac{m}{\Delta}\sqrt{n})\). Indeed, if the algorithm has not terminated yet (see line 3), then we have removed at most \(\frac{m}{10\Delta}\) total vertices. Therefore, the graph \(G\) will still have at least \(m - \frac{m}{10\Delta} \cdot \Delta = \frac{9}{10}m\) edges remaining. Additionally, after taking into account the \(\frac{m}{12}\) edges from using LowDiamDecomp, we see that

\[
\sum_{i=1}^{k} |E(H_i)| \geq \frac{9}{10}m - \frac{1}{12}m \geq \frac{4}{5}m.
\]

By Lemma 4.3, we get cycles of length \(O(\log n)\) covering at least

\[
\sum_{i=1}^{k} \frac{\max(0, |E(H_i)| - 5|V(H_i)|)}{10\Delta\sqrt{|E(H_i)|}} \geq \sum_{i=1}^{k} \frac{|E(H_i)| - 5|V(H_i)|}{100\Delta\sqrt{n}} \geq \frac{4}{5}m - \frac{5n}{100\Delta\sqrt{n}} \geq \frac{m}{500\Delta\sqrt{n}}
\]

total vertices. Here we have used that \(m = 10n\).

Therefore, we return to line 3 of Algorithm 3 at most \(O(\sqrt{n})\) times. Each iteration takes \(O(m)\) time, for a total runtime of \(O(m\sqrt{n})\) as desired.

**5 Main Algorithm and Analysis**

**5.1 High-Level Overview**

First, we give a high-level description of the algorithm, much of which proceeds similarly to Algorithm 3. Let \(k\) be the constant governing the recursion, i.e. every level has graphs of size approximately \(\frac{1}{k}\) the size of the previous level. Assume our input graph \(G\) has \(n\) vertices and \(m\) edges. From Theorem 4.2, we can decompose any graph into components with small diameter. Henceforth, we assume that we are operating on a low-diameter graph.

Let \(S\) be a spanning tree of diameter \(O(\log n)\) of our low diameter graph \(G\), which we can find by running a breadth first search. We partition the vertices of \(G\) into low diameter components \(B_1, B_2, \ldots, B_\ell\) that all have roughly \(k\) vertices, by partitioning along the spanning tree \(S\). Next, we contract each component \(B_i\) (excluding the edges in the spanning tree \(S\)) to get a new graph \(H_i\), which has approximately \(\frac{k}{k}\) vertices. We recursively find vertex disjoint short cycles on \(H_i\). Using the extra edges in \(S\), we can port the cycles in \(H_i\) back to \(G\) (see Lemma 4.3). We repeat this procedure until we have removed cycles containing enough vertices.

This approach requires some details to complete. First, it is not true that we can partition so that all components have approximately \(k\). For example, imagine a star graph. Any component that does not contain the central vertex will have size \(1\)! On the other hand, a tree of maximum degree \(\Delta\) and at least \(k\) vertices can always be partitioned into connected subtrees whose sizes are in
the range \([k, \Delta k]\) (we approximately do this but in a slightly more general way via the algorithm of Lemma 4.4). Therefore, we carefully maintain the maximum degree \(\Delta\) of the graph \(G\) throughout the process, and ensure that it doesn’t grow too much as we go deeper in the recursion. Additionally, note that while the graph \(H\) has \(\frac{n}{k}\) vertices, it still has about \(m\) edges. Therefore, we can’t pass down the whole graph \(H\) to lower levels of our recursion. Recall from Lemma 4.1 that the quantity \(\frac{m}{\Delta}\) approximately tells us the maximum number of vertex disjoint cycles we can extract from a graph. Therefore, if we could proportionally reduce the number of edges by a factor of \(k\) while also reducing \(\Delta\) by a factor of \(k\), we could still extract the same number of vertex disjoint cycles from that graph. Lemma 5.1 does precisely that, up to some constant loss. Therefore, we apply the algorithm of Lemma 5.1 to our contracted graph \(H\) before passing it down in the recursion.

Before continuing, we will state Lemma 5.1, whose proof we also defer to the appendix.

**Lemma 5.1.** Let \(G\) be a graph with \(n\) vertices and \(m\) edges. Let \(k\) be an integer such that \(m \geq k\). Then algorithm \(\text{Sparsify}(G, k)\) (Algorithm 10) returns a subgraph \(G' \subseteq G\) with \(n\) vertices, \(k\) edges, and \(\Delta(G') \leq \left(\frac{2k+4n}{m}\right)\Delta(G)\).

Now, we proceed to our algorithm and analysis. In Algorithm 5, let \(\hat{n}, \hat{m}\) denote the number of vertices and edges of the graph at the top level of the recursion.

We now analyze Algorithm \(\text{ShortCycleDecomp}\).

We first analyze the case in line 6, where many edges are within the components \(A_i\) (where \(|V(A_i)| < k\)).

**Lemma 5.2.** Consider running Algorithm \(\text{ShortCycleDecomp}\) on a graph \(G\) with \(n\) vertices, \(m = 10n\) edges and maximum degree \(\Delta\). In line 6 in the case that \(\sum_{i=1}^{\ell_1} |E(A_i)| \geq \frac{m}{4}\), we can extract vertex disjoint cycles of length \(O(\log n)\) containing at least \(\frac{m}{20\Delta}\) vertices in \(O(mk)\) time.

**Proof.** By Lemma 4.1 we know that in a component \(C\), we can find vertex disjoint cycles of length \(O(\log n)\) containing at least \(\frac{|E(C)| - 2|V(C)|}{\Delta}\) vertices in time \(O(|E(C)| \cdot |V(C)|)\). Recall that the components \(A_1, A_2, \ldots, A_{\ell_1}\) in line 5 of Algorithm 5 satisfy \(|V(A_i)| \leq k\). Then in total we can find cycles of length \(O(\log n)\) containing at least the following number of vertices:

\[
\sum_{i=1}^{\ell_1} \frac{|E(A_i)| - 2|V(A_i)|}{\Delta} \geq \frac{m/4 - 2n}{\Delta} \geq \frac{m}{20\Delta}
\]

after using that \(n = \frac{m}{10}\).

The total runtime is

\[
\sum_{i=1}^{\ell_1} O(|E(A_i)| \cdot k) = O(mk)
\]

as desired. 

Therefore, \(\text{ShortCycleDecomp}\) will process line 6 at most two times, because after that we would certainly have constructed cycles containing at least \(\frac{m}{20\Delta} \cdot 2 = \frac{m}{10\Delta}\) vertices by Lemma 5.2.

From now on, we assume that the condition of line 6 is false, so \(\sum_{i=1}^{\ell_1} |E(A_i)| < \frac{m}{4}\).

We now bound the number of vertices in our contracted graph \(H\) to show that the size of the graph passed down to lower levels of the recursion indeed decreases significantly.
Algorithm 5 **SHORTCYCLEDÉCOMP**, takes a graph $G$ with $n$ vertices, $m = 10n$ edges, max degree $\Delta$, depth $d$ of the recursion (starts at 0), constant $k = \frac{20m}{k}$.

Returns vertex disjoint cycles containing at least $\frac{m}{10\Delta}$ vertices.

1. if $d = c - 1$ then return **IMPROVEDSHORTCYCLE**($G$)
2. Initialize $C \leftarrow \emptyset$ (our set of cycles found so far).
3. while $C$ contains less than $\frac{m}{10\Delta}$ total vertices do
4. Let $R \leftarrow \text{LOWDIMPÉCOMP}(G, \frac{1}{12})$.
5. Let $(A_1, \ldots, A_{\ell_1}, B_1, \ldots, B_{\ell_2})$ be the connected components of $G \setminus R$, where $|V(A_i)| \leq k$ for $1 \leq i \leq \ell_1$ and $|V(B_i)| > k$ for $1 \leq i \leq \ell_2$.
6. if $\sum_{i=1}^{\ell_1} |E(A_i)| \geq \frac{m}{4}$ then $C \leftarrow C \cup \text{NAIVESHORTCYCLE}(A_i)$. Go to line 3.
7. For $1 \leq i \leq \ell_2$, let $T_i$ be a spanning tree of $B_i$ of diameter $O(\log n)$.
8. Initialize $K \leftarrow \emptyset$ (K is a set of subsets of $V(G)$).
9. for $1 \leq i \leq \ell_2$ do
10. for $v \in V(B_i)$ do set $c_v$ to be the degree of $v$ in $B_i$.
11. $K \leftarrow K \cup \text{TREE''SPLIT}(|V(B_i)|, k, T_i, \Delta, \{c_v\}_{v \in V(B_i)}).
12. Let $K = \{K_1, K_2, \ldots, K_{|K|}\}$ (where $K_i \subseteq V(G)$).
13. for $1 \leq i \leq |K|$ do
14. Let $S_i$ be a spanning tree of $K_i$ of diameter $O(\log n)$.
15. Let $H$ be the graph obtained by contracting the components $K_1, K_2, \ldots, K_{|K|}$
16. Remove the edges in $H$ corresponding to the edges in the trees $S_1, \ldots, S_{|K|}$.
17. Let $f : E(H) \to E(G)$ be the corresponding edge injection (see Subsection 2.1).
18. Add isolated vertices to $H$ until $H$ has $20n \frac{m}{k}$ vertices.
19. Let $H' \leftarrow \text{SPARSIFY}(H, \frac{20m}{k})$.
20. Let $f' : E(H') \to E(G)$ be the restriction of $f$ from $E(H)$ to $E(H')$.
21. Let $C' \leftarrow \text{SHORTCYCLEDÉCOMP}(H', d + 1, k)$.
22. Let $C \leftarrow C \cup \text{PULLUP}(H', C', K, \{S_i\}_{i=1}^{\bar{K}}, f')$.
23. for vertices $v$ part of a cycle in $C$ do delete $v$ from $G$.
24. return $C$.

**Lemma 5.3.** Consider running Algorithm 5 on a graph $G$ with $n$ vertices, $m = 10n$ edges and maximum degree $\Delta$. As defined in Algorithm 2, we have that with high probability $|K| \leq \frac{20n}{k}$. Therefore, after line 78 graph $H$ will have exactly $\frac{20n}{k}$ vertices.

**Proof.** This essentially follows from the guarantees of TREE''SPLIT (Lemma 4.4) as used in line 11. To apply Lemma 4.4, we must show that in each component $B_i$, we have $\sum_{v \in V(B_i)} c_v \geq k$. Because $B_i$ is connected, we know that

$$\sum_{v \in V(B_i)} c_v \geq 2(|V(B_i)| - 1) \geq k$$

as $|V(B_i)| > k$ by assumption. Thus, by the guarantees of TREE''SPLIT (Lemma 4.4) used on line 11 of Algorithm 5, we know that each $K_i$ will have total sum of labels at least $k$. The sum of labels over all $K_i$ is at most the sum of degrees of $G$, which equals $2m = 20n$. Therefore, $k \cdot |K| \leq 20n$, so $|K| \leq \frac{20n}{k}$.

12
Additionally, we need to show that the maximum degree of our graphs doesn’t increase significantly from one recursion level to the next.

**Lemma 5.4.** Consider running Algorithm 5 on a graph $G$ with $n$ vertices, $m = 10n$ edges and maximum degree $\Delta$. Consider graph $H'$ as defined in line 17. Then we have with high probability that $\Delta(H') \leq 110\Delta$.

**Proof.** First, we claim that each time we return to line 3 of Algorithm 5, the graph $G$ still contains at least $\frac{9}{10}m$ edges. Indeed, we return to line 3 only if we have only created cycles containing at most $\frac{m}{10\Delta} \cdot \Delta = \frac{9}{10}m$ vertices. Therefore, the number of edges $G$ still has among the remaining vertices is at least $m - \frac{m}{10\Delta} \cdot \Delta = \frac{9}{10}m$.

Next, we show that $\Delta(H') \leq (k+1)\Delta(G) = (k+1)\Delta$. This essentially follows from the guarantees of TreeSplit (Lemma 4.4). To apply Lemma 4.4, must show that in each component $B_i$, we have $\sum_{v \in V(B_i)} c_v \geq k$ because $|V(B_i)| > k$ by assumption. Thus, by the construction of $H$ in Algorithm 5, we can see that the degree of vertex $h \in H$ equals the sum of the labels of the vertices in $K_h$. By the guarantees of TreeSplit (Lemma 4.4) used on line 11 of Algorithm 5, we can see that the sum of labels of $K_h$ is at most $\Delta k + \Delta = \Delta(k+1)$.

If we are to get to line 19 then we must have skipped over line 6. Therefore, we know that

$$\sum_{i=1}^{\ell_2} |E(B_i)| \geq \frac{9}{10}m - \frac{1}{12}m - \frac{1}{4}m \geq \frac{5}{9}m.$$ 

Because we are removing the edges in $S_1, S_2, \ldots, S_{|K|}$ before contracting to get $H$, we have

$$|E(H)| \geq \sum_{i=1}^{\ell_2} |E(B_i)| - n \geq \frac{4}{9}m.$$ 

By Lemma 5.1, we know that

$$\Delta(H') \leq \frac{2 \cdot \frac{20m}{k} + 4 \cdot \frac{20m}{k}}{4m/9} \cdot \Delta(H) \leq \frac{2 \cdot \frac{20m}{k} + 4 \cdot \frac{20m}{k}}{4m/9} \cdot \Delta(k+1) \leq 110\Delta$$

as desired. \qed

Now, Lemma 5.4 allows us to bound the total number of edges processed per level in the recursion.

**Lemma 5.5.** Consider running Algorithm 5 on a graph $G$ with $n$ vertices, $m = 10n$ edges and maximum degree $\Delta$. The number of edges processed in level $\ell$ of the recursion with high probability is $O(110^\ell \cdot m)$.

**Proof.** We go by induction and show that the number of edges processed in the next level is at most 110 times the number of edges in the previous level. By the guarantees of algorithm PullUp (Lemma 4.3) and the recursive guarantees of Algorithm 5, we can see that during each iteration of the algorithm the total length of cycles in $C$ (our set of cycles) will increase by at least $\frac{|E(H')|}{10\Delta(H')} \geq \frac{20m/k}{10 \cdot 110\Delta} \geq \frac{m}{55\Delta k}$. Here we used Lemma 5.4 to show that $\Delta(H') \leq 110\Delta$. Therefore, the algorithm will return to line 3 at most $\frac{m}{55\Delta k} \cdot \frac{11}{2}k = \frac{m}{10\Delta} \cdot \frac{11}{2}k = \frac{110m}{k}$ times on the same node in the recursion tree, because after that we will have removed at least $\frac{m}{55\Delta k} \cdot \frac{11}{2}k = \frac{m}{10\Delta} \cdot \frac{11}{2}k = 110m$ vertices. As each $H'$ has $\frac{20m}{k}$ edges, the total number of edges passed to the next level is at most $\frac{20m}{k} \cdot \frac{11}{2}k = 110m$ as desired. \qed
We now are ready to state a bound on the runtime of Algorithm 5.

**Theorem 5.6.** For all integers \(c\), Algorithm 5 takes as input a graph \(G\) with \(\tilde{n}\) vertices, \(\tilde{m} = 10\tilde{n}\) edges, and maximum degree \(\Delta\), and with high probability returns vertex disjoint cycles of length \(O(\log \tilde{n})^c\) containing at least \(\tilde{m}/10\Delta\) vertices. The runtime is \(O(\tilde{m}\tilde{n}^{\frac{c+1}{c}} \cdot 500^c)\).

**Proof.** First, note that per recursion level, by Lemma 5.2, we will only perform the computation listed on line 6 at most twice. The computation up to this point takes time \(\tilde{O}(m + n)\) by Lemma 4.2.

Hence, by Lemma 5.5, we know that total number of edges processed on the \(\ell\)-th level of recursion is \(O(110^{c-1}\tilde{m})\). Therefore, the total runtime contribution from running NaiveShortCycle on small components (see Lemma 5.2) is \(O(110^{c-1}\tilde{m}k)\). By Lemma 4.6, the cost of running ImprovedShortCycle on the bottom level (which is level \(c-1\)) is at most

\[
O \left(110^{c-1}\tilde{m} \cdot \sqrt{\frac{20^{c-1}\tilde{n}}{k^{c-1}}} \right) = O \left(\frac{500^c\tilde{m}\tilde{n}^{\frac{1}{c}}}{k^{\frac{1}{c}}} \right) = O(500^c \cdot \tilde{m}k).
\]

The cost of processing the graphs (running the non-recursive steps of the algorithm) on the \(i\)-th level is \(O(110^c\tilde{m}k)\), which sums to \(O(110^c\tilde{m}k)\) in total over all levels.

Therefore, the total runtime is then

\[
O(500^c \cdot \tilde{m}k + 110^c\tilde{m}k) = O \left(\tilde{m}\tilde{n}^{\frac{1}{c}} \cdot 500^c \right)
\]

as desired.

At each level, the cycle lengths grow by a factor of \(O(\log n)\). Therefore, the total length at the end will be \(O(\log \tilde{n})^c\) as desired. \qed

We can now complete the proof of Theorem 1.2

**Proof of Theorem 1.2** Note the Theorem 5.6 implies that Algorithm 5 satisfies the constraints of Lemma 3.2. Also, Theorem 5.6 tells us that Algorithm 5 runs time \(O(500^c \cdot \tilde{m}n^{\frac{c+1}{c}}) = O(500^c \cdot n^{\frac{c+1}{c}})\) because \(\tilde{m} = 10n\).

Therefore, combining Theorem 5.6 and Lemma 3.2 tells us that we have an algorithm that returns a \((10\tilde{n}, (O(\log \tilde{n}))^c)\)-short cycle decomposition which runs in time

\[
O \left(\frac{\tilde{m}500^c \cdot n^{\frac{c+1}{c}}}{\tilde{n}} \right) = O(500^c \cdot \tilde{m}n^{\frac{c+1}{c+1}})
\]

as desired. \qed

**References**

[ACK+16] Alexandr Andoni, Jiecao Chen, Robert Krauthgamer, Bo Qin, David P Woodruff, and Qin Zhang. On sketching quadratic forms. In *Proceedings of the 2016 ACM Conference on Innovations in Theoretical Computer Science*, pages 311–319. ACM, 2016. Available at: https://arxiv.org/abs/1511.06099.

[AKPW95] Noga Alon, Richard M. Karp, David Peleg, and Douglas West. A graph-theoretic game and its application to the \(k\)-server problem. *SIAM J. Comput.*, 24(1):78–100, 1995.

[Bar96] Y. Bartal. Probabilistic approximation of metric spaces and its algorithmic applications. In *Proceedings of 37th Conference on Foundations of Computer Science*, pages 184–193, Oct 1996.
[BK96] András A. Benczúr and David R. Karger. Approximating s-t minimum cuts in $\tilde{O}(n^2)$ time. In Proceedings of the twenty-eighth annual ACM symposium on Theory of computing, STOC ’96, pages 47–55, New York, NY, USA, 1996. ACM.

[BSS09] Joshua D. Batson, Daniel A. Spielman, and Nikhil Srivastava. Twice-ramanujan sparsifiers. In Proceedings of the Forty-first Annual ACM Symposium on Theory of Computing, STOC ’09, pages 255–262, New York, NY, USA, 2009. ACM.

[BSS12] Joshua Batson, Daniel A Spielman, and Nikhil Srivastava. Twice-Ramanujan sparsifiers. SIAM Journal on Computing, 41(6):1704–1721, 2012.

[CGP+18] T. Chu, Y. Gao, R. Peng, S. Sachdeva, S. Sawlani, and J. Wang. Graph Sparsification, Spectral Sketches, and Faster Resistance Computation, via Short Cycle Decompositions. ArXiv e-prints, May 2018. To appear at FOCS 2018.

[Che89] L. Paul Chew. There are planar graphs almost as good as the complete graph. J. Comput. System Sci., 39(2):205–219, 1989. Computational geometry.

[CKM+11] Paul Christiano, Jonathan A. Kelner, Aleksander Madry, Daniel A. Spielman, and Shang-Hua Teng. Electrical flows, laplacian systems, and faster approximation of maximum flow in undirected graphs. In Proceedings of the 43rd annual ACM symposium on Theory of computing, STOC ’11, pages 273–282, New York, NY, USA, 2011. ACM. Available at http://arxiv.org/abs/1010.2921.

[CKP+17] Michael B Cohen, Jonathan Kelner, John Peebles, Richard Peng, Anup B Rao, Aaron Sidford, and Adrian Vladu. Almost-linear-time algorithms for markov chains and new spectral primitives for directed graphs. In Proceedings of the 49th Annual ACM SIGACT Symposium on Theory of Computing, pages 410–419. ACM, 2017. Available at: https://arxiv.org/abs/1611.00755.

[CKST17] Charles Carlson, Alexandra Kolla, Nikhil Srivastava, and Luca Trevisan. Optimal lower bounds for sketching graph cuts. CoRR, abs/1712.10261, 2017. Available at: https://arxiv.org/abs/1712.10261.

[DKP+17] David Durfee, Rasmus Kyng, John Peebles, Anup B Rao, and Sushant Sachdeva. Sampling random spanning trees faster than matrix multiplication. In Proceedings of the 49th Annual ACM SIGACT Symposium on Theory of Computing, pages 730–742. ACM, 2017. Available at: https://arxiv.org/abs/1611.07451.

[DKW15] Michael Dinitz, Robert Krauthgamer, and Tal Wagner. Towards Resistance Sparsifiers. In Naveen Garg, Klaus Jansen, Anup Rao, and José D. P. Rolim, editors, Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques (APPROX/RANDOM 2015), volume 40 of Leibniz International Proceedings in Informatics (LIPIcs), pages 738–755, Dagstuhl, Germany, 2015. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik.

[DPPR17] David Durfee, John Peebles, Richard Peng, and Anup B. Rao. Determinant-preserving sparsification of SDDM matrices with applications to counting and sampling spanning trees. In FOCS, pages 926–937. IEEE Computer Society, 2017. Available at: https://arxiv.org/abs/1705.00985.

[JS18] Arun Jambulapati and Aaron Sidford. Efficient $\tilde{O}(n/\varepsilon)$ spectral sketches for the laplacian and its pseudoinverse. In SODA, pages 2487–2503. SIAM, 2018. Available at: https://arxiv.org/abs/1711.00571.

[KLP+16] Rasmus Kyng, Yin Tat Lee, Richard Peng, Sushant Sachdeva, and Daniel A Spielman. Sparsified cholesky and multigrid solvers for connection laplacians. In Proceedings of the 48th Annual ACM SIGACT Symposium on Theory of Computing, pages 842–850. ACM, 2016. Available at http://arxiv.org/abs/1512.01892.
A Omitted Proofs

In this section we give proofs for various lemmas which we omitted.

**Lemma 4.1.** \textsc{NaiveShortCycle} (Algorithm 6) takes a graph $G$ with $n$ vertices, $m$ edges, and maximum degree $\Delta$, and outputs vertex disjoint cycles of length at most $2 \log n$ containing at least $m - 2n$ total vertices. It runs in $O(m + n^2)$ time.

**Proof.** Consider the algorithm \textsc{NaiveShortCycle} described as Algorithm 6.

\begin{algorithm}
\begin{enumerate}
\item Initialize $C$ to be empty
\item repeat
\item While $G$ has a vertex $u$ of degree $\leq 2$, remove $u$ and edges incident to $u$ from $G$
\item Run BFS from an arbitrary vertex $r$ until first non-tree edge $e$ found
\item Add the cycle formed by $e$ and tree edges in $C$
\item Remove the vertices in the cycle
\item Remove all corresponding edges
\item until $G$ is empty
\item return $C$
\end{enumerate}
\end{algorithm}

After line 3 we will get a graph with minimum degree 3. Thus, after the BFS, we are guaranteed to have a tree such that no non-leaf vertex of it has less than 2 children. Thus, when we find a cycle from the non-tree edge $e$, it is guaranteed to have length at most $2 \log n$.

Note that when the algorithm ends, we have removed all the vertices. The only way we remove a vertex not in a cycle is by line 2, where that vertex has at most 2 edges incident to it when we remove it. Thus, line 2 can remove at most $2n$ edges incident in total. Thus, the total number of edges removed by removing vertices in cycles are at least $m - 2n$. Since the maximum degree of a vertex is at most $\Delta$, the number of vertices contained in the cycles must be at least $\frac{m - 2n}{\Delta}$.

The BFS run in each iteration of the loop runs in $O(n)$ time since we stop the BFS when the first non-tree edges is found. Since the loop can run at most $O(n)$ times, the time taken by the BFS over all iterations is $O(n^2)$. Removing the edges incident to the cycle vertices requires a total time $O(m + n)$ over all iterations, giving a total running time of $O(m + n^2)$.

**Lemma 4.3.** The algorithm \textsc{PullUp}($H', C', \{K_i\}_{i=1}^{\left|V(H')\right|}, \{S_i\}_{i=1}^{\left|V(H')\right|}, f$) (Algorithm 7) takes the following inputs:

1. $H'$ – a graph.
2. $C'$ – a set of vertex disjoint cycles on the vertices of $H'$.
3. $\{K_i\}_{i=1}^{\left|V(H')\right|}$ – A partition of the vertices of a graph $G$, where $K_i$ corresponds to vertex $i$ of graph $H'$.
4. $\{S_i\}_{i=1}^{\left|V(H')\right|}$ – Each $S_i$ is a spanning tree on the vertices in $K_i$.
5. $f$ – This is a function $f : E(H') \rightarrow E(G)$ such that for an edge $uv \in E(H')$, we have that $f(uv) \in E(K_u, K_v)$.
It returns a set $C$ of cycles on the vertices of $G$ such that

1. The cycles in $C$ are vertex disjoint.
2. The cycles in $C$ cover at least as many vertices as those in $C'$ did.
3. The length in $C$ have maximum length at most $O(\max_i \text{diam}(S_i))$ times the longest cycle in $C'$.

It runs in time $O(n)$.

**Algorithm 7** PullUp, takes as inputs a graph $H'$, vertex disjoint cycles $C'$ on the vertices on $H'$, a partition $\{K_i\}_{i=1}^{\lvert V(H')\rvert}$ of the vertices of $G$, a vertex disjoint union of spanning trees $\{S_i\}_{i=1}^{\lvert V(H')\rvert}$ on another graph $G$, and a function $f : E(H') \to E(G)$ which satisfies $f(uv) \in E(K_u, K_v)$. Returns vertex disjoint cycles on the vertices on $G$.

Throughout, we used indices (mod $k$) where it is obvious.

**Proof.**

1: Initialize $C$ as empty (ending set of cycles on $G$).
2: for cycle $v_1v_2 \ldots v_k \in C'$ do
   3: for $1 \leq i \leq k$ do let $a_ib_{i+1} \leftarrow f(v_iv_{i+1})$.
   4: for $1 \leq i \leq k$ do let $p_i$ be the path from $b_i \to a_i$ in tree $S_i$.
   5: $C \leftarrow C \cup b_1p_1a_1b_2p_2a_2 \ldots b_kp_ka_k$ (concatenation of paths).

return $C$

All guarantees follow very easily from the description of Algorithm 7. At a high level, note that by the definition of $H'$ and $K_i$, we know that cycles on $H'$ corresponds to “cycles” on the components $K_i$ in the graph $G$. Now, simply use the edges of the spanning trees $S_i$ to recover a cycle on $G$.

It is obvious that we cover at least as many vertices among $C$ as in $C'$. Additionally, the lengths will increase by at most a factor of the diameter of some spanning tree $S_i$ by the construction. Vertex disjointness follows trivially. The runtime follows because the only operation we need to do is find paths between vertices in a spanning tree, which is time $O(n)$.

**Lemma 4.4.** Let $T$ be a tree with $n$ vertices maximum degree $D$. Assume the vertices are labelled with nonnegative integers between $0$ and $X$ (denote labels as $c_v$ for $v \in V(T)$). Then the algorithm TreeSplit($n, t, T, X, \{c_v\}_{v \in V(T)}$) (Algorithm 3) is an $O(n)$ time algorithm that when given $T$ and a positive integer $t \leq \sum_v c_v$, splits the tree into connected subgraphs, each of which has sum of labels between $t$ and $Dt + X$.

**Proof.** Let the vertices be numbered $1, 2, \ldots, n$, and let the corresponding labels be $c_1, c_2, \ldots, c_n$. We outline the algorithm TreeSplit, which takes $n, t, T, X$, and $\{c_i\}_{i=1}^{n}$ as inputs.

In the algorithm, extra [$v$] denotes the total sum of labels still attached to the rest of the tree $v$ after processing it.

We can show the correctness of the algorithm by induction, specifically that after processing vertex $v$, that extra [$v$] $< t$ always. Consider a vertex $v$. Note that it has at most $D - 1$ children $u_1, \ldots, u_k$ (as we rooted the tree at a leaf $\ell$). By induction it is clear that after visiting all children
Algorithm 8  

TreeSplit, takes inputs \( n, t, T, X \), and \( \{c_v\}_{v \in V(T)} \) such that \( \sum_i c_i \geq t \). Runs in time \( O(n) \). Partitions \( T \) into connected subtrees so that each subtree has sum of labels between \( t \) and \( Dt + X \).

1: Root \( T \) at a leaf \( \ell \).
2: Initialize an array \( \text{extra}[1 \ldots n] \).
3: Set \( \text{extra}[v] \leftarrow c_v \) for all vertices \( v \).
4: Run a depth first search through \( T \), starting at \( \ell \).
5: Let \( v \) denote the vertex that is being currently processed.
6: for children \( u \) of \( v \) do
7:  \text{Recursively visit} \( u \).
8:  \text{extra}[v] \leftarrow \text{extra}[v] + \text{extra}[u].
9: if \( \text{extra}[v] \geq t \) then
10:  Remove the edge between \( v \) and its parent.
11:  Set \( \text{extra}[v] \leftarrow 0 \).
12: End the depth first search.
13: if \( \text{extra}[\ell] < t \) then
14:  Reconnect the component with \( \ell \) to another component.
15: Let \( C_1, C_2, \ldots, C_k \) be the connected components of the resulting forest.
16: return \((C_1, C_2, \ldots, C_k)\).

of \( v \), it will be true that

\[
\text{extra}[v] \leq c_v + \sum_i \text{extra}[u_i] \leq (D - 1)t + X.
\]

If \( t \leq \text{extra}[v] \leq (D - 1)t + X \), then we split off \( v \) from its parent (we made a new component). Otherwise, \( \text{extra}[v] < t \), verifying the induction. Finally, if the root \( \ell \) satisfies \( \text{extra}[\ell] < t \) but \( \text{extra}[\ell] \neq 0 \), then connect the component containing \( \ell \) to another one (which we know has label sum at most \( (D - 1)t + X \)). Therefore, it still holds that all components have sum of labels at most \( t + (D - 1)t + X = Dt + X \).

This shows the correctness of Algorithm 8. \(\Box\)

Lemma 5.1. Let \( G \) be a graph with \( n \) vertices and \( m \) edges. Let \( k \) be an integer such that \( m \geq k \). Then algorithm \text{Sparsify}(G, k) (Algorithm 10) returns a subgraph \( G' \subseteq G \) with \( n \) vertices, \( k \) edges, and \( \Delta(G') \leq \frac{(2k + 4n)\Delta(G)}{m} \).

Proof. Consider the following algorithms:

Algorithm 9 \text{SparsifyHelper}, takes the graph \( G \) and a spanning tree root \( r \), recursively remove the tree edge from leaves to the root with odd degree vertices.

1: for \( v \) be child of \( r \) do
2: \text{SparsifyHelper}(G, v)
3: if \( r \) has odd degree then
4: remove the edge formed by \( r \) and parent of \( r \)
Algorithm 10 Sparsify, takes a graph $G$ with $m$ vertices, $n$ edges, and max degree $\Delta$, and a target edge count $k$. Returns a subgraph with exactly $k$ edges and max degree at most $(2k + 4n)\Delta/m$.

1: while $|E(G)| \geq 2k + 2n$ do
2:   find a spanning tree of $G$ root at an arbitrary vertex $r_i$ for each connected component.
3:   for Each root $r_i$ do
4:       SparsifyHelper($G, r_i$)
5:   Perform an Eulerian Tour on each connected component, and remove every other edge starting from the first edge. Specifically, if the Eulerian tour has edges $e_1, e_2, \ldots, e_\ell$ in that order, remove edges $e_1, e_3, e_5, \ldots$
6:   Remove edges until the resulting graph has exactly $k$ edges.

First we claim that the remaining graph after SparsifyHelper will have even degree for all vertices. Since we are removing the tree edge when the degree of the vertex is odd from the bottom to the root, we are guaranteed that all vertices except the root will have an even degree. But the sum of degrees of all vertices is even, so the root will have an even degree as well. Thus, we can perform an Eulerian Tour on each connected component. After removing every other edge, every degree will be reduced by half. Note that in components with an odd number of edges, we must remove both the first and last edges in the Eulerian tour. In total, if we remove $t$ edges from the call to SparsifyHelper and that after doing this, we have $o$ components of odd size remaining.

It is easy to see that our resulting graph after deleting every other edge from the Eulerian tours on connected components has $\frac{m-t-o}{2} \geq \frac{m}{2} - n$ edges.

We claim that after $i$th round, $\frac{m}{2^i} - 2n \leq |E_i| \leq \frac{m}{2^i}$. This can be easily shown by induction. After the $(i + 1)$th round,

$$|E_{i+1}| \geq \frac{|E_i|}{2} - n \geq \frac{m/2^i - 2n}{2} - n = \frac{m}{2^{i+1}} - 2n$$

$$|E_{i+1}| \leq \frac{|E_i|}{2} \leq \frac{m/2^i}{2} = \frac{m}{2^{i+1}}$$

Also, when the algorithm ends, we have $k \leq |E| < 2k + 2n$. Assume that the algorithm needs $r$ rounds. Then, we have $k \leq \frac{m}{2^r}$ and $\frac{m}{2^r} - 2n < 2k + 2n$. So $\frac{m}{2^r} < 2k + 4n$. Thus, the number of rounds is in the range of $(\log_2 \frac{m}{2k+4n}, \log_2 \frac{m}{2k})$. Therefore, our final graph $G'$ will have

$$\Delta(G') < \frac{\Delta(G)}{2^{\log_2 \frac{m}{2k+4n}}} = \frac{(2k + 4n)\Delta(G)}{m}.$$ 

To analyze the running time, note that the $i$-th round takes time $O(\frac{m}{2^i} + n)$ time. As we are running for $O(\log n)$ rounds, our total runtime will be $\sum_{i=0}^{O(\log n)} O(\frac{m}{2^i} + n) = O(m + n \log n)$ as desired. □