Gravitational field of a pit and maximum mass defect

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Abstract

A general relativistic solution, composed of a Zel’dovich-Letelier interior made of radial strings matched through a spherical thin shell at radius $r_0$ to an exterior Schwarzschild solution with mass $m$, is presented. It is the Zel’dovich-Letelier-Schwarzschild star. When the radius $r_0$ of the star is shrunk to its own gravitational radius $2m$, $r_0 = 2m$, the solutions that appear have very interesting properties. There are solutions with $m = 0$ and $r_0 = 0$ that further obey $\frac{2m}{r_0} = 1$. So, these solutions have a horizon, but they are not exactly black holes, they are quasiblack holes, though untypical ones. Moreover, the proper mass $m_p$ of the interior is nonzero and made of one string. Hence, a Minkowski exterior space hides an interior with matter in a pit. These are the pit solutions. These pits thus show a maximum mass defect. There are two classes of pit solutions, the first encloses a finite string and the second a semi-infinite one. So these pits are really stringpits, which can be seen as Wheeler’s bags of gold albeit totally squashed bags. There is also another class, which is a stringy compact star at the $\frac{2m}{r_0} = 1$ limit with $m$ finite. It is a typical quasiblack hole and it also shows maximum mass defect. A generic analysis is presented that shows that pit solutions with $\frac{2m}{r_0} = 1$ and $m = 0$ can exist displaying a maximum mass defect. The Zel’dovich-Letelier-Schwarzschild star at the $r_0 = 2m$ limit is actually an instance of the generic case.
I. INTRODUCTION

There is the question of whether, in general relativity, there are static configurations with some matter interior solution coupled to an exterior vacuum solution, for which three conditions are satisfied altogether: The configuration has zero radius \( r_0 \), \( r_0 = 0 \), it also has zero ADM spacetime mass \( m \), \( m = 0 \), and notwithstanding \( r_0 \) and \( m \) obey \( \frac{2m}{r_0} \) finite. We will answer this question in the positive. To do that we analyze a generic case, a generic star say, composed of a generic interior spherical symmetric static configuration with a Schwarzschild exterior. Then, as a nontrivial example, we specifically develop and present the Zel’dovich-Letelier-Schwarzschild star, which is a solution that matches the Zel’dovich-Letelier interior at some junction radius \( r_0 \) to the exterior Schwarzschild solution. We send \( r_0 \) to the gravitational radius of the star \( 2m \), \( r_0 \rightarrow 2m \), i.e., we take the highly compact star limit, or the quasiblack hole limit, of the configuration. Solutions with \( m = 0 \), \( r_0 = 0 \), and \( \frac{2m}{r_0} = 1 \), so \( \frac{2m}{r_0} \) finite, do indeed appear. They also have a proper mass \( m_p \) different from zero. Thus, clearly these solutions have maximum mass defects. Since \( m = 0 \) the exterior spacetime is Minkowski. But since \( m_p \) is not zero, such solutions enclose some matter in one form or another in a spacetime pit, thus being pit solutions. These pits are untypical quasiblack holes. In the Zel’dovich-Letelier-Schwarzschild limiting star there are two classes of pit solutions, the first encloses a finite string and the second a semi-infinite one. Hence, in this example the pits can be described as stringpits. These stringpits resemble Wheeler’s bags of gold but totally squashed. There is also another class in the Zel’dovich-Letelier-Schwarzschild limiting star, which is a stringy compact star at the \( \frac{2m}{r_0} = 1 \) limit with \( m \) finite.

Some comments are necessary. (i) The Zel’dovich-Letelier solution was discussed as an interior general relativistic solution by Zel’dovich \[1\] in connection to compact stars masses, and by Letelier \[2\] as clouds of strings. It then appeared in the context of global monopoles \[3\], of nonlinear electrodynamics \[4\], and as a rediscovery \[5\]. The equation of state for the Zel’dovich-Letelier matter is the same as used in regular black holes \[6\]. We make use of the junction formalism \[7\] to match a Zel’dovich-Letelier interior to a Schwarzschild exterior and obtain the Zel’dovich-Letelier-Schwarzschild star. (ii) Objects for which the surface radius \( r_0 \) of the matter, e.g., a star, is at its own gravitational radius, \( r_0 = 2m \), are highly compact stars called quasiblack holes \[8\]. Quasiblack holes are on the verge of becoming black holes,
have special properties, and are parents of both black holes and null naked singularities [9–11]. (iii) The answer to the question of whether in general relativity, zero mass \( m \) at a point \( r = 0 \) can have nonetheless a quotient \( \frac{2m}{r} \) finite is known to be yes in a dynamical setting. Indeed, inhomogeneous dust spherical collapse in Lemaître-Tolman-Bondi models can produce an \( m = 0, r = 0 \), naked null singularity. At the critical moment the density has a \( \frac{1}{r^2} \) profile, and \( \frac{2m(r)}{r} \) is finite, indeed \( \frac{2m(r)}{r} = 1 \) at the center [12–14]. Also, Choptuik collapse of a scalar field [15] allows for a critical case, which divides expansion of the scalar field back to infinity from collapse of the scalar field to a black hole, where a zero mass and zero radius black hole, with \( \frac{2m}{r} = 1 \), finite, forms, which can also be interpreted as the formation of a naked null zero mass singularity. (iv) Mass defects with maximum values have appeared in specific non-stationary models [16] where one can have matter with an infinite amount of interior proper mass, but the exterior spacetime has zero ADM mass [17]. This is the maximum possible gravitational mass defect. (v) Bag of gold solutions appeared in [18], see also [19], and are exemplified by a closed FLRW universe glued to the other side of a Schwarzschild black hole through an Einstein-Rosen bridge with a knot at the junction closing the bag.

The paper is organized as follows. In Sec. II we lay down the spacetime basic features and arrive naturally at the concept of spacetime pits, i.e., solutions with \( m = 0 \) and \( \frac{2m}{r_0} \) finite which yield maximum mass defects. In Sect. III we display the Zel’dovich-Letelier solution and make a proper matching to find the Zel’dovich-Letelier-Schwarzschild star. In Sec. IV we take the quasiblack hole limit \( r_0 \to 2m \) of the Zel’dovich-Letelier-Schwarzschild star and find three classes of highly compact objects. The first two classes are pit solutions, more precisely, stringpit solutions, one enclosing a finite string, the other a semi-infinite string, both with \( m = 0 \), \( \frac{2m}{r_0} = 1 \), and maximum mass defects. The third class is a stringy compact star, more precisely, a string star with maximum compactness, i.e., \( \frac{2m}{r_0} = 1 \), finite \( m \), and also maximum mass defect. In Sec. V we conclude summarizing also the results in a table.
II. BASIC FEATURES OF SPACETIMES WITH A PIT AND MAXIMAL MASS
DEFECT

A. Spacetime generics

A general static spherical symmetric spacetime with spacetime coordinates \((t, r, \theta, \phi)\) has a line element that can be written in the form
\[
\text{\(ds^2 = \left(1 - \frac{2m(r)}{r}\right)c^2 dt^2 + \frac{dr^2}{1 - \frac{2m(r)}{r}} + r^2 d\Omega^2\)},
\]
where \(m(r)\) and \(\psi(r)\) are functions of \(r\), and \(d\Omega^2\) is the line element on the unit sphere,
\[
d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2.
\]
Assume that for \(r \leq r_0\), for some radius \(r_0\), there is a fluid with energy-momentum tensor \(T^{ab}\) given by
\[
T^{ab} = \text{diag}(-\rho, p_r, p_t, p_t),
\]
where \(\rho\) is the fluid’s energy density, \(p_r\) its radial pressure, and \(p_t\) its tangential pressure, all functions of \(r\).
Then Einstein equation of general relativity \(G_{ab} = 8\pi T_{ab}\), where \(G_{ab}\) is the Einstein tensor, and we are putting the constant of gravitation and the velocity of light to unity, yield
\[
m(r) = 4\pi \int_0^r dr' r'^2 \rho(r')
\]
and
\[
\psi(r) = 4\pi \int_0^r dr' r'^2 \frac{\rho(r') + p_r(r')}{1 - \frac{2m(r')}{r'}}.
\]
There is yet another equation involving the tangential pressure \(p_t\) that we do not need right now. In the model that we are going to use for the interior one has \(\rho(r') + p_r(r') = 0\) so that \(\psi(r) = 0\) throughout. So, in this case the line element that we start with in the \((t, r, \theta, \phi)\) coordinates reduces to
\[
\text{\(ds^2 = \left(1 - \frac{2m(r)}{r}\right)c^2 dt^2 + \frac{dr^2}{1 - \frac{2m(r)}{r}} + r^2 d\Omega^2\)},
\]
where
\[
m(r) = 4\pi \int_0^r dr' r'^2 \rho(r'),
\]
is now the only metric function, usually called the mass function, and defined for \(r \leq r_0\).

Other functions of interest here are the proper mass \(m_p(r)\), the proper distance \(l_p(r)\) from the center to any \(r \leq r_0\), the area \(A(r)\) of a constant \(r\) sphere, and the proper volume \(V_p(r)\).

The proper mass \(m_p(r)\) is defined as
\[
m_p(r) = 4\pi \int_0^r dr' \frac{r'^2 \rho(r')}{\sqrt{1 - \frac{2m(r')}{r'}}},
\]
the proper distance \(l_p(r)\) from the center to any \(r\) is defined as
\[
l_p(r) = \int_0^r \frac{dr'}{\sqrt{1 - \frac{2m(r')}{r'}}},
\]
the area \(A(r)\) of a constant \(r\) sphere is
\[
A(r) = 4\pi r^2,
\]
and the proper volume is defined by

\[ V_p(r) = 4\pi \int_0^r dr' \frac{r'^2}{\sqrt{1 - \frac{2m(r')}{r'}}}. \]  

These functions at the boundary \( r_0 \) become specific important quantities and we put \( m \equiv m(r_0), m_p \equiv m_p(r_0), A \equiv A(r_0), l_p \equiv l_p(r_0), \) and \( V_p \equiv V_p(r_0). \)

At \( r_0 \) there is a boundary that can be smooth or have a shell. If there is a shell it can have zero or nonzero proper mass and zero or nonzero pressure.

For \( r \geq r_0 \) we assume that the solution is vacuum and so it is the Schwarzschild solution, \( m(r) = M \) constant, where \( M \) is the spacetime mass energy also called the ADM mass. In general \( M \) and \( m \) are different. Here we work with the case \( M = m \) as we will see it is the case in the matching of the Zel’dovich-Letelier interior solution to the exterior Schwarzschild solution. So \( m \) is the ADM mass of the spacetime.

**B. Features of pit spacetimes with \( m = 0, r_0 = 0, \frac{2m}{r_0} = 1, \) and maximum mass defect**

We put \( M = m \), i.e., there is no contribution to the exterior spacetime ADM mass from the boundary at \( r_0 \), and stick to call it \( m \). Considering the mass function \( m(r) \) appearing in Eq. (1), we assume that \( 1 - \frac{2m(r)}{r} \) is uniformly bounded and write \( 1 - \frac{2m(r)}{r} \geq 0 \), i.e.,

\[ \frac{2m(r)}{r} \leq 1. \]  

(7)

With this assumption we can make some general concrete remarks. Define \( \varepsilon \) as any positive number, that can be as small as we want, and \( \chi(r) \) a function of \( r \) always greater than zero, such that \( 1 - \frac{2m(r)}{r} = \varepsilon \chi(r) \). Take the maximum value of \( \chi(r) \) as \( \chi_{\text{max}} \) and its minimum value as \( \chi_{\text{min}} \). Then, since Eq. (7) holds, the integral of Eq. (2) converges, and taking the integrals up to the boundary \( r_0 \) in Eqs. (2) and (3) leads to

\[ \frac{m}{\sqrt{\varepsilon \chi_{\text{max}}}} \leq m_p \leq \frac{m}{\sqrt{\varepsilon \chi_{\text{min}}}}. \]  

(8)

Take the quasiblack hole limit, i.e., \( r_0 \to 2m \), or \( \frac{2m}{r_0} \to 1 \) from below, so that one also has \( \varepsilon \to 0 \). This is a configuration made of some material with boundary radius at its own gravitational radius \( 2m \), it is a configuration on the verge of becoming a black hole. Suppose that \( m_p \) remains finite on this limit. Then, since \( \varepsilon \to 0 \), one has mandatorily from Eq. (8)
that \( m \) goes to zero and so, since \( r_0 = 2m \) in this limit, one also has \( r_0 \to 0 \). Thus, one has an object that has zero mass energy \( m \), and zero radius \( r_0 \), with \( \frac{2m}{r_0} = 1 \), and also has finite nonzero proper mass \( m_p \). In addition defining the mass defect \( \Delta m \) as \( \Delta m = m_p - m = m_p \), we see that this object has a maximum mass defect \( \Delta m \) given by \( \Delta m = m_p \). In brief, such an object has

\[
\begin{align*}
m &= 0, \\
r_0 &= 0, \\
\frac{2m}{r_0} &= 1, \\
m_p &= \text{finite}, \\
\Delta m &= m_p.
\end{align*}
\]  

(9)

This is an amazing object. It has zero ADM mass, zero area radius, and although \( m = 0 \), the ratio \( \frac{2m}{r_0} \) is not zero, is actually one. In addition, it has finite proper mass and maximum mass defect. We call this structure a pit, as it stores a nonzero proper mass in a zero ADM spacetime with zero area radius. As \( \frac{2m}{r_0} = 1 \), the pit is indeed a quasiblack hole, although an untypical one. A specific realization of this general analysis is through the Zel’dovich-Letelier-Schwarzschild star that we will display next and where there are three possible classes, two of them being stringpits, each one with distinct and rather interesting features, and the other being a stringy compact star at the quasiblack hole state.

III. THE ZEL’DOVICH-LETELIER-SCHWARZSCHILD STAR AND ITS LIMITS

A. The interior, the shell junction, the exterior, and the

Zel’dovich-Letelier-Schwarzschild star

1. The Zel’dovich-Letelier interior

Let us be concrete. To simplify let us choose an equation of state of the form \( p_r = -\rho \). Then, inside for \( r \leq r_0 \), we have that indeed \( \psi(r) = 0 \) and the only function that matters is \( m(r) \) that appears in Eqs. (1) and (2). The conservation law \( T^{ab}_{\;\;;b} = 0 \) with \( a = r \) gives \( p_t = p_r + \frac{2}{3} \rho \), and using the equation of state \( p_r = -\rho \) one gets \( p_t = -\rho - \frac{2}{3} \rho \). Following Zel’dovich [1] and Letelier [2], see also others [3–5], we put \( \rho = \frac{b}{8\pi r^2} \) where \( b \) is a positive constant. So the full general relativistic solution using Einstein equation is

\[
\rho = \frac{b}{8\pi r^2},
\]  

(10)

\[
p_r = -\frac{b}{8\pi r^2},
\]  

(11)
\( p_t = 0. \) \hspace{1cm} (12)

Since \( p_t = 0 \), the source is string dust, strings in the radial direction up to \( r_0 \), see also [6] for this type of matter. Putting the energy-density expression Eq. (10) into Eq. (2), one obtains \( m(r) = \frac{b}{2} r \), i.e., \( \frac{2m(r)}{r} = b \), and the line element, Eq. (11), becomes

\[
ds^2 = -(1-b) \, dt^2 + \frac{dr^2}{1-b} + r^2 d\Omega^2. \hspace{1cm} (13)
\]

This metric yields a spacetime that has a spherical conic deficit. Indeed, redefining \( \tilde{t} = \sqrt{1-b} t \) and \( \tilde{r} = \frac{r}{\sqrt{1-b}} \) one gets the conical form of the metric, namely, \( ds^2 = -d\tilde{t}^2 + d\tilde{r}^2 + \tilde{r}^2 (1-b) \, d\Omega^2 \), clearly, the inside metric is a deficit angle metric. Returning to the main functions of a static spherical symmetric spacetime, Eqs. (2)-(6), we can put them in the case of the Zel’dovich-Letelier spacetime in the form,

\[
m(r) = \frac{1}{2} br, \hspace{1cm} (14)
\]

\[
m_p(r) = \frac{m(r)}{\sqrt{1-b}}, \hspace{1cm} (15)
\]

\[
l_p(r) = \frac{r}{\sqrt{1-b}} = \frac{2m_p(r)}{b}, \hspace{1cm} (16)
\]

\[
A(r) = 4\pi r^2, \hspace{1cm} (17)
\]

\[
V_p(r) = \frac{4\pi r^3}{3\sqrt{1-b}} = \frac{32m(r)^3}{3b^3\sqrt{1-b}} = \frac{32m_p(r)^3(1-b)}{3b^3}. \hspace{1cm} (18)
\]

It is assumed that \( b \leq 1 \), so that the metric in Eq. (13) is static, and in addition it is assumed that the parameter \( b \) is positive, so that the mass function \( m(r) \) in Eq. (14) is positive, i.e., we put

\[
0 < b \leq 1. \hspace{1cm} (19)
\]

Eqs. (13)-(18) with condition (19) characterize the interior spacetime defined for \( r \leq r_0 \).

Note that the interior solution is singular at \( r = 0 \), the density and radial pressure, given in Eqs. (10) and (11), respectively, diverge there, and so the Ricci and Riemann tensors and corresponding scalars diverge. Zel’dovich [1] deals with gravitational collapse issues, neglects \( p_r \) and dismisses this singularity problem showing that rounding up the energy-density \( \rho \) at the origin makes no difference for his final results. Letelier [2] suggests that the solution can be used as an intermediary solution between the Schwarzschild interior solution and a Schwarzschild exterior. Here, we use the solution to match it to a Schwarzschild exterior, and in taking the limit \( r_0 \rightarrow 2m \) it is found that this singularity is not naked because it is within a quasiblack hole.
2. The junction with a shell

The junction of the inside and the outside is at some \( r_0 \). At the junction \( r_0 \) we consider the metric to be of the form

\[
d s^2 = -d\tau^2 + r_0^2 d\Omega^2,
\]

where \( \tau \) is the proper time at the junction. For the outside we consider a vacuum spacetime and so from Birkhoff’s theorem it is the Schwarzschild spacetime. Since from the inside the radial pressure at \( r_0 \) is nonzero, namely, \( p_r = -\frac{b}{4\pi r_0^2} \), see Eq. (11), and from the outside \( p_r = 0 \), as we consider that the outside is vacuum, there is a clear jump in the radial pressure that has to be smoothed out by a thin spherical shell at the junction at \( r_0 \). The energy-momentum tensor of the thin shell can be found. We write the contribution to the energy-momentum tensor \( T_{ab} \) from the shell in the form \( T_{ab} = S_{ab} \delta(l-l_0) \), where \( S_{ab} \) is the intrinsic energy-momentum tensor associated to the shell, \( \delta \) is the Dirac delta function, \( l \) is the proper radial length in the neighborhood of the shell, and \( l_0 \) corresponds to the boundary at the shell. Then, following the junction formalism for general relativity, one finds \( S_{\tau\tau} = 0 \) and \( S_{\theta\theta} = S_{\phi\phi} = \frac{b}{16\pi r_0^2 \sqrt{1-b}} \). Writing \( S_{\tau\tau} \equiv \sigma \) and \( S_{\theta\theta} = S_{\phi\phi} \equiv P \), where \( \sigma \) is the energy density of the shell and \( P \) is the tangential pressure at the shell, we have thus

\[
\sigma = 0, \quad P = \frac{b}{16\pi r_0^2 \sqrt{1-b}}.
\]

Note that there is no mass for the shell as \( m_{\text{shell}} = 4\pi r_0^2 \sigma = 0 \). Note that \( P \) closes the conical deficit set in by the interior spacetime such that the exterior Schwarzschild spacetime has no conical deficit. The shell’s tangential pressure \( P \) is there to close ends, literally.

3. The exterior Schwarzschild

The outer spacetime is vacuum, therefore the exterior general relativistic metric, the metric for \( r \geq r_0 \), is Schwarzschild, i.e.,

\[
d s^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2 d\Omega^2
\]

for some ADM mass \( M \). In general \( M \) and \( m \) have different values. Here, since the shell has no mass, \( m_{\text{shell}} = 0 \), we deal with the case \( M = m \), and keep \( m \) throughout, so we put

\[
d s^2 = - \left(1 - \frac{2m}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2m}{r}} + r^2 d\Omega^2.
\]

as the exterior Schwarzschild metric.
4. The full solution: The Zel’dovich-Letelier-Schwarzschild star

The full general relativistic solution is composed of three parts. The inside with metric given by Eq. (13), the shell with metric given by Eq. (20), and the outside with metric given in Eq. (22). The main global features can be found at the junction, \( r_0 \). From Eqs. (14)-(18) they are,

\[
\begin{align*}
m &= \frac{1}{2} b r_0, \\
m_p &= \frac{m}{\sqrt{1 - b}}, \\
l_p &= \frac{r_0}{\sqrt{1 - b}} = \frac{2m_p}{b}, \\
A &= 4\pi r_0^2, \\
V_p &= \frac{4\pi r_0^3}{3\sqrt{1 - b}} = \frac{32m^3}{3b^3\sqrt{1 - b}} = \frac{32m_p^3(1 - b)}{3b^3}.
\end{align*}
\]

This is the full Zel’dovich-Letelier-Schwarzschild spacetime solution, i.e., the Zel’dovich-Letelier-Schwarzschild star.

B. The limit \( r_0 \rightarrow 2m \) of the Zel’dovich-Letelier-Schwarzschild star: Distinguishing features

We are interested in the limit in which

\[
r_0 \rightarrow 2m,
\]

i.e., the quasiblack hole limit in which an object is at its own gravitational radius \[8, 11\]. It follows from Eq. (23) that this means

\[
b \rightarrow 1.
\]

Let us take \( m_p \) in Eq. (24) as the quantity that identifies the possible different classes. To see this we put \( m_p(1 - b)^\gamma = \mu \) for some exponent \( \gamma \) and some finite renormalized proper mass \( \mu \), with \( \mu \geq 0 \). For now we leave \( b \) generic, only afterward we take the limit given in Eq. (29). Then, Eqs. (23)-(27) with \( r_0 = 2m \) of Eq. (28) give

\[
\begin{align*}
m &= \mu(1 - b)^{\frac{1}{2} - \gamma}, \\
m_p &= \mu(1 - b)^{-\gamma},
\end{align*}
\]
\[ l_p = 2\mu \frac{(1-b)^{-\gamma}}{b}, \quad (32) \]
\[ A = 16\pi \mu^2 \frac{(1-b)^{1-2\gamma}}{b^2}, \quad (33) \]
\[ V_p = \frac{32}{3} \mu^3 \frac{(1-b)^{1-3\gamma}}{b^3}, \quad (34) \]
respectively. From Eqs. (30) and (31) we see that a negative exponent \( \gamma \) gives zero mass \( m \) and zero proper mass \( m_p \) in the limit of Eq. (29), and so is of no interest as it gives nothing, and we impose \( \gamma \geq 0 \). From Eq. (30) we see that an exponent \( \gamma \) greater than \( \frac{1}{2} \) gives an infinite mass \( m \) in the limit of Eq. (29) and the spacetime with line element given in Eq. (22) is not well defined, and we impose \( \gamma \leq \frac{1}{2} \). Thus \( \gamma \) is within the range
\[ 0 \leq \gamma \leq \frac{1}{2}. \quad (35) \]

Then, Eqs. (30) and (31) show that there are three distinct main classes: A. \( \gamma = 0 \), which yields \( m = 0 \), and \( m_p \) equal to \( \mu \), and so \( m_p \) is finite, it is a stringpit solution; B. \( 0 < \gamma < \frac{1}{2} \), which yields \( m = 0 \) and \( m_p \) infinite, it is also a stringpit solution with different properties; C. \( \gamma = \frac{1}{2} \), which yields \( m \) finite and \( m_p \) infinite, it is a stringy highly compact star, not a pit. Let us analyze these three classes in detail.

**IV. THE THREE LIMITING SOLUTIONS: TWO STRINGPITS AND A STRINGY COMPACT STAR**

**A. A finite string in a pit, i.e., a stringpit, almost detached from spacetime hanging from a point**

Here we find a finite string in a pit, a stringpit, almost detached from spacetime hanging from a point, with indeed the spacetime mass \( m = 0 \) and the proper mass \( m_p \) = finite. This is the class \( \gamma = 0 \).

We are interested in the limit in which \( r_0 \to 2m \), see Eq. (28), i.e., the quasiblack hole state. It follows from Eq. (23) that this implies \( b \to 1 \), see Eq. (29). We put \( \gamma = 0 \) in Eqs. (30)-(34) and analyze the spacetime main features. From Eq. (30) we have \( m \to 0 \), i.e., \( m = 0 \) in the limit. Equation (31) yields \( m_p = \mu \) so \( m_p \) is finite, the subcase \( m_p = 0 \) being a trivial case. From Eq. (32), the total proper length \( l_p \) remains finite, this is clear as \( l_p = 2m_p \) and \( m_p \) is finite, see Eq. (25) with \( b = 1 \). From Eq. (33), the surface area is \( A = 0 \),
and also the area radius of the boundary is $r_0 = 0$. From Eq. (34), the proper volume is zero, $V_p = 0$.

Thus, the full spacetime can be understood as follows. The inside solution is made of a one-dimensional open string, with finite length and zero volume. That the inside spacetime is a one-dimensional string can also be seen from the conical form of the inside metric, where for $b = 1$, and $\bar{r}$ finite, as it is the case, the angular part disappears leaving a one-dimensional space, i.e., a two-dimensional spacetime. This single string in the inside spacetime pit is what is left from the hedgehog continuous spherical distribution of strings in the original Zel’dovich-Letelier star solution. It is a finite string almost detached from spacetime hanging from a point. For the shell, that joins the inside to the outside, one deduces it is now a point as $r_0 = 0$. Then, from Eq. (21), since $b = 1$ and $r_0 = 0$, the tangential stresses tend to infinity, $P \to \infty$, and thus the point $r_0 = 0$ is singular, a type of singular horizon. For the outside, one has that the spacetime is Minkowski as $m = 0$. Thus, in a nutshell, a Minkowski exterior spacetime hides a finite stringpit. For a $t = \text{constant}$ and, e.g., $\theta = \frac{\pi}{2}$ space representation of the spacetime see Fig. 1, where it is clear that the packed region with matter is a pit with a string hanging in the middle of flat space.

Note five important and interesting properties of this class of stringpit solution. First, the interior mass of the Zel’dovich-Letelier-Schwarzschild star [1–7] in this limit is hidden to the outside as it does not manifest itself gravitationally to the outer space since $m = 0$. Second, it is also hidden because it is invisible since it is a quasiblack hole [8–11]. So it is invisible because of two reasons. Third, although $m = 0$, its ratio to $r_0$ is finite, indeed, $\frac{2m}{r_0} = 1$, these three features characterizing an untypical quasiblack hole. Thus the dynamical gravitational collapse setting in [12–15] for which a null naked singularity, i.e., a singular horizon, forms when $m = 0$ at $r = 0$, and $\frac{2m}{r_0} = 1$, is also established in the static case we are analyzing. Fourth, the mass defect, i.e., the proper mass minus the energy of the assembled object, is $\Delta m = m_p - m = m_p$, it is then an object with maximum mass defect, see also [16, 17]. Fifth, it is a Wheeler’s bag of gold [18, 19] but totally squashed.

For a study of the geodesics in this spacetime see Appendix A.
FIG. 1: A \( t = \text{constant} \) and \( \theta = \frac{\pi}{2} \) space representation of the spacetime given by the Zel’dovich-Letelier-Schwarzschild star with proper mass \( m_p = \mu = \text{finite} \), actually \( m_p(1 - b)\gamma = \mu \) with \( \gamma = 0 \) and \( b = 1 \), and spacetime mass \( m = 0 \). This class has \( r_0 = 2m \), and so it is a quasiblack hole, an untypical one, as it satisfies \( r_0 = 2m = 0 \). The space inside is a highly packed region of matter, composed of a pit made of one-dimensional string with finite proper length, hanged from a point with \( r_0 = 0 \), which opens up to a massless \( m = 0 \) Minkowski spacetime, i.e., a flat space here. The point \( r_0 = 0 \) yields the singular horizon of the quasiblack hole and joins the almost detached string to the rest of space. Note that although \( m = 0 \) and \( r_0 = 0 \) their ratio is finite as \( \frac{2m}{r_0} = 1 \). This object has maximum mass defect. The representation of this class of stringpit solution shows clearly that the solution is a totally squashed Wheeler’s bag of gold.

B. A semi-infinite string in a pit, i.e., a stringpit, almost detached from spacetime hanging from a point

Here we find a semi-infinite string in a pit, a semi-infinite stringpit, almost detached from spacetime hanging from a point, with indeed the spacetime mass \( m = 0 \) and the proper mass \( m_p = \infty \). This is the class \( 0 < \gamma < \frac{1}{2} \).

We are again interested in the limit in which \( r_0 \to 2m \), see Eq. (28), i.e., the quasiblack hole state. It follows from Eq. (23) that again this implies \( b \to 1 \), see Eq. (29). We put \( 0 < \gamma < \frac{1}{2} \) in Eqs. (30)-(34) and analyze the spacetime main features. From Eq. (30) we have \( m \to 0 \), i.e., \( m = 0 \) in the limit. Equation (31) yields \( m_p = \infty \), the proper mass is infinite. From Eq. (32), the total proper length \( l_p \) is then infinite. From Eq. (33) the surface area is \( A = 0 \), and also the area radius of the boundary is \( r_0 = 0 \). From Eq. (34), the proper volume is zero, \( V_p = 0 \), for \( 0 < \gamma < \frac{1}{3} \), it is finite nonzero, \( V_p = \frac{32\mu^3}{3} \), for \( \gamma = \frac{1}{3} \), in which case it is a string or a rope with zero cross sectional area, infinite length, but finite volume,
and it is infinite, $V_p = \infty$, for $\frac{1}{3} < \gamma < \frac{1}{2}$.

Thus, the full spacetime can be understood as follows. The inside solution is made of a one-dimensional string, with infinite length, zero area, and zero, finite, or infinite volume, depending on the specific $\gamma$. That the inside spacetime is a one-dimensional string can be also seen from the conical form of the inside metric, where for $b = 1$, and $r^2(1 - b)$ tends to zero as it is the case for the range of $\gamma$ under study, the angular part disappears leaving a one-dimensional space, i.e., a two-dimensional spacetime. This packed region of matter inside, made of a lonely boundless semi-infinite string in a pit almost detached from the outer spacetime hanging from a point, is the remnant of the infinite number of strings stemming radially from $r = 0$ up to $r_0$ in a hedgehog distribution in the original Zel’dovich-Letelier star solution. For the shell, that joins the inside to the outside, one deduces it is now a point as $r_0 = 0$. Then, from Eq. (21), since $b = 1$ and $r_0 = 0$, the tangential stresses tend to infinity, $P \to \infty$, and thus the point $r_0 = 0$ is singular, a type of singular horizon. For the outside, one has that the spacetime is Minkowski as $m = 0$. Thus, in a nutshell, a Minkowski exterior space hides a semi-infinite stringpit. For a $t = \text{constant}$ and, e.g., $\theta = \frac{\pi}{2}$ space representation of the spacetime, see Fig. 2, where it is clear that the packed region with matter is a pit with a semi-infinite string hanging in the middle of flat space.

Note also five additional important and interesting properties of this class of stringpit solutions. First, the interior mass of the Zel’dovich-Letelier-Schwarzschild star in this limit is hidden to the outside as it does not manifest itself gravitationally to the outer space since $m = 0$. Second, it is also hidden because it is invisible since it is a quasiblack hole. So it is invisible because of two reasons. Third, although $m = 0$, its ratio to $r_0$ is finite, indeed, $\frac{2m}{r_0} = 1$, these three features characterizing an untypical quasiblack hole. Thus the dynamical gravitational collapse setting in for which a null naked singularity, i.e., a singular horizon, forms when $m = 0$ at $r = 0$, and $\frac{2m}{r} = 1$, is also established in the static case we are analyzing. Fourth, the mass defect, i.e., the proper mass minus the energy of the assembled object, is $\Delta m = m_p - m = m_p = \infty$, it is then an object with infinite mass defect, see also. Fifth, it is a totally squeezed Wheeler’s bag of gold if we allow the bag to have infinite length.

The study of the geodesics in this spacetime can be done along the lines sketched in the previous spacetime.
FIG. 2: A $t = \text{constant}$ and $\theta = \frac{\pi}{2}$ space representation of the spacetime given by the Zel’dovich-Letelier-Schwarzschild star with proper mass $m_p = \infty$, actually $m_p(1-b)^\gamma = \mu$ with $0 < \gamma < \frac{1}{2}$ and $b = 1$, and spacetime mass $m = 0$. This class has $r_0 = 2m$, and so it is a quasiblack hole, an untypical one, as it satisfies $r_0 = 2m = 0$. The space inside is a highly packed region of matter, composed of a pit made of one-dimensional string with infinite proper length, hanged from a point with zero area $A = 0$ and $r_0 = 0$, and, depending on the parameter $\gamma$, with zero, finite, or infinite volume, hanged from a point with $r_0 = 0$, which opens up to a massless $m = 0$ Minkowski spacetime, i.e., a flat space. The point $r_0 = 0$ yields the singular horizon of the quasiblack hole and joins the almost detached semi-infinite string to the rest of space. Note that although $m = 0$ and $r_0 = 0$ their ratio is finite as $\frac{2m}{r_0} = 1$. This object has maximum mass defect, indeed infinite mass defect. The representation of the semi-infinite stringpit solution shows that the solution is a totally squashed Wheeler’s bag of gold although infinite in this class.

C. A stringy compact star at its gravitational radius

Here we find a stringy compact star at its gravitational radius, with spacetime mass $m = \text{finite}$ and the proper mass $m_p = \infty$. This is the class $\gamma = \frac{1}{2}$.

We are again interested in the limit in which $r_0 \to 2m$, see Eq. (28), i.e., the quasiblack hole state. It follows from Eq. (23) that again this implies $b \to 1$, see Eq. (29). We put $\gamma = \frac{1}{2}$ in Eqs. (30)-(34) and analyze the spacetime main features. From Eq. (30) we have $m = \mu$, i.e., $m$ is finite in the limit. Equation (31) yields $m_p = \infty$, the proper mass is infinite. From Eq. (32), the total proper length $l_p$ is then infinite. From Eq. (33) the surface area $A = 16\pi m^2$ which is finite, and also the area radius of the boundary $r_0 = 2m$ is finite.
From Eq. (34), the proper volume is infinite, $V_p = \infty$.

Thus the full spacetime can be understood as follows. The inside solution is made of a bulk and all the strings from the original Zel’dovich-Letelier solution remain, but now hidden in a spacetime inside a horizon at finite nonzero area $A$ and finite $r_0$. Note that $\bar{r}_0$ is infinite in this case and therefore the inside space is three dimensional, not one-dimensional as in the previous two classes. For the shell, that joins the inside to the outside, one deduces it is a sphere with radius $r_0 = 2m$. Then, from Eq. (21), since $b = 1$, the tangential stresses tend to infinity, $P \rightarrow \infty$, and thus the horizon $r_0$ is singular. For the outside, one has that the spacetime is Schwarzschild as $m$ is finite and not zero. In brief, the solution represents a stringy compact star at the quasiblack hole state, made of strings from $r = 0$ to $r_0$, the compact star’s boundary is a quasihorizon, and the outside is Schwarzschild. This solution is not a pit. For a $t = \text{constant}$ and, e.g., $\theta = \pi/2$ space representation of the spacetime, see Fig. 3, where it clear that the packed region with matter has finite boundary area radius $r_0$ and unbound volume.

![Diagram](image)

**FIG. 3:** A $t = \text{constant}$ and $\theta = \pi/2$ space representation of the spacetime given by the Zel’dovich-Letelier-Schwarzschild star with proper mass $m_p = \infty$, actually $m_p(1 - b)^\gamma = \mu$ with $\gamma = \frac{1}{2}$ and $b = 1$, and spacetime mass $m = \text{finite}$. This class has $r_0 = 2m$, and so it is a quasiblack hole, a typical one, as it satisfies $r_0 = 2m$ with $m$ finite. The space inside is an infinite volume region of matter, composed of the strings from the original Zel’dovich-Letelier solution but now hidden in a spacetime inside a horizon at finite nonzero $r_0$ that is singular and joins the inside to the curved Schwarzschild exterior space. Note that $\frac{2m}{r_0} = 1$ as it should be for a quasihorizon. This object has maximum, actually infinite, mass defect. The representation of the stringy compact star solution shows that there is no Wheeler’s bag of gold in this class.

The five additional properties for this class of the stringy compact star solution can be put in the form. First, the interior mass of the Zel’dovich-Letelier-Schwarzschild star
is not hidden in this class to the outside as it does manifest itself gravitationally to the outer space since \( m \) is finite. Second, nonetheless it is still invisible since it is a quasiblack hole \[8–11\]. Third, here \( \frac{2m}{r_0} = 1 \) with \( m \) and \( r_0 \) finite, so the solution is a quasiblack hole, a typical one in this class. In the dynamical gravitational collapse setting \[12–15\] there are also cases for which \( \frac{2m}{r} = 1 \), with \( m \) finite, characterizing the formation of a typical black hole not a naked singularity. Fourth, the mass defect, i.e., the proper mass minus the energy of the assembled object, is \( \Delta m = m_p - m = \infty \), it is then an object with infinite mass defect, see also \[16, 17\]. Fifth, this class does not resemble Wheeler’s bag of gold \[18, 19\] at all.

The study of the geodesics in this spacetime can be done along the lines sketched in the first spacetime.

\[\text{V. CONCLUSIONS}\]

The main results of this work are the finding of two classes of stringpit solutions with unusual interesting structures and unusual interesting general relativistic gravitational fields. There is also another class, a stringy compact star solution that has standard properties. These three classes, although obtained from an appropriate limit of the Zel’dovich-Letelier-Schwarzschild star, stand on their own as separate general relativistic solutions, if one wishes to envisage them as such. In Table 1, a summary of the main features of the three classes parameterized by the exponent \( \gamma \) are displayed.

| Class          | stringpit\(_1\) | stringpit\(_2\) | stringy compact star |
|----------------|----------------|----------------|----------------------|
| \( \gamma \)   | 0              | \((0, \frac{1}{2})\) | \( \frac{1}{2} \)    |
| \( m \)        | 0              | 0              | finite               |
| \( m_p \)      | finite         | infinite       | infinite             |
| \( l_p \)      | finite         | infinite       | infinite             |
| \( A \)        | 0              | 0              | finite               |
| \( V_p \)      | 0, finite, infinite | infinite       | infinite             |

TABLE 1: The main physical features along with its values of the three different classes of solutions, i.e., the first stringpit class, the second stringpit class, and the stringy compact star class, distinguished by the values of \( \gamma \), namely, \( \gamma = 0 \), \( 0 < \gamma < \frac{1}{2} \), and \( \gamma = \frac{1}{2} \), are displayed. The physical features are the spacetime ADM mass \( m \), the interior proper mass \( m_p \), the interior proper
length $l_p$, the surface area at the junction $A$, and the interior proper volume $V_p$.

Some properties of the solutions in the three classes found here are: (1) For the two first classes, comprising the stringpit solutions that arise as the quasiblack hole limit of the Zel’dovich-Letelier-Schwarzschild star, in spite of having in the core a nonzero mass $m_p$, which in one class is arbitrarily large, this mass is hidden as it does not manifest itself gravitationally to the outer spacetime since $m = 0$. The third class, the stringy compact star solution, does not possess this property, the outer spacetime is Schwarzschild, it has a finite nonzero $m$. (2) The three classes of solutions are invisible to the exterior since they are quasiblack holes, and as such no particle or light emanates from them. (3) The three class of solutions obey the quasiblack hole condition $\frac{2m}{r_0} = 1$. The two classes of stringpit solutions are remarkable because they obey this condition, having in addition $m = 0$ and $r_0 = 0$, and are indeed untypical quasiblack holes. These are the static solutions akin to the naked singularities, i.e., singular horizons, that form in dynamical gravitational collapse when $m = 0$ at $r = 0$, and $\frac{2m}{r} = 1$. The class of the stringy compact star solution has finite $m$ and finite $r_0$ and are typical quasiblack holes, akin to the black holes that form in dynamical gravitational collapse with $m$ finite and some finite horizon radius $r$. (4) In the three classes, the mass defect, i.e., the proper mass $m_p$ minus the ADM mass $m$ of the assembled objects, is maximum. In the first class of stringpit solutions the mass defect is finite and maximum, and in the other two classes, i.e., the second class of stringpit solutions and the stringy compact star solution, it is a superstrong mass defect, it is infinite. Moreover, as far as the gravitational mass defect is concerned, we have obtained general results, indeed we have shown that the maximum mass defect result can be obtained without specifying any equation of state, the Zel’dovich-Letelier-Schwarzschild star is a realization of the general result. (5) For the two first classes, i.e., the two stringpit solutions, one finds Wheeler’s bags of gold albeit totally squashed. The third class, the stringy compact star solution, has no bag.
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Appendix A: Geodesics in the spacetime of case A. The finite string in a pit almost detached from spacetime hanging from a point, i.e., in the spacetime with \( m_p = \text{finite} \)

Here, we study the geodesics of the spacetime of case A. above, i.e., a finite string in a pit almost detached from spacetime hanging from a point, with \( m = 0 \) and \( m_p = \text{finite} \).

Timelike geodesics:
A radial geodesic has as one of its equations, the following equation \( \left( 1 - \frac{2m(r)}{r} \right) \dot{t} = E \), where a dot means derivative with respect to proper time \( \tau \) and \( E \) is a constant representing the energy per unit mass of the test massive particle along the geodesic. The other equation is \( \left( 1 - \frac{2m(r)}{r} \right) \dot{t}^2 - \frac{\dot{r}^2}{1 - \frac{2m(r)}{r}} = 1 \). Combining the two one gets \( \dot{r}^2 = E^2 - 1 + \frac{2m(r)}{r} \). Thus, \( d\tau = \pm \frac{dr}{\sqrt{E^2 - 1 + \frac{2m(r)}{r}}} \). For the inside \( \frac{2m(r)}{r} = b \) and so letting a test massive particle fall from some \( r \) in the inside region to the center the proper time is \( \tau = \int_{r_0}^{r} \frac{dr'}{\sqrt{E^2 - 1 + b}} \), which yields \( \tau = \frac{r}{\sqrt{E^2 - 1 + b}} \). If the particle comes from \( r_0 \) with \( E \geq 1 \) then

\[
\tau_{in} = \frac{r_0}{\sqrt{E^2 - 1 + b}}. \tag{A1}
\]

In the limiting spacetime \( b \to 1 \) and \( r_0 \to 0 \) yields \( \tau = 0 \).

Null geodesics:
For a null geodesic \( ds^2 = 0 \). So, the time between the center and some \( r \) is \( t = \int_{r_0}^{r} \frac{dr'}{\sqrt{1 - \frac{2m(r)}{r}}} \).

For the inside \( \frac{2m(r)}{r} = b \) and so the time between the center and the boundary \( r_0 \) is

\[
t_{in} = \frac{r_0}{1 - b}. \tag{A2}
\]

In the limiting spacetime, \( b \to 1 \) and \( r_0 \to 0 \), and taking into account Eqs. (14) and (15) with \( m_p \) finite, one has \( t_{in} \to \infty \) with \( \frac{1}{\sqrt{1-b}} \). The time between \( r_0 \) and some \( r_1 > r_0 \) is

\[
t_{out} = \int_{r_0}^{r_1} \frac{dr}{1 - \frac{2m(r)}{r}} = \int_{r_0}^{r_1} \frac{dr}{1 - \frac{br_0}{r}} = \int_{r_0}^{r_1} \frac{dr}{r - br_0} = r_1 - r_0 + br_0 \ln \frac{r_1 - br_0}{r_0(1-b)}, \text{ i.e.,} \]

\[
t_{out} = r_1 - r_0 + br_0 \ln \frac{r_1 - br_0}{r_0 - br_0}. \tag{A3}
\]
When \( b \to 1 \), \( t_{\text{out}} \to \infty \). The divergences of \( t_{\text{in}} \) are stronger than those of \( t_{\text{out}} \).

Redshift:

Now we analyze the redshift and blueshift of light. According to the standard formulas,

\[
\omega \sqrt{1 - \frac{2m(r)}{r}} = \omega_0, \tag{A4}
\]

\( \omega \) is the frequency measured by a local static observer at a given point \( r \), and \( \omega_0 \) is a constant.

Inside, one has \( 1 - \frac{2m(r)}{r} = 1 - b \), and outside \( 1 - \frac{2m(r)}{r} = 1 - \frac{2m}{r} \). Inside, for \( r \leq r_0 \), the frequency of light does not change as it propagates. On the other hand when it goes from \( r_0 \) to larger \( r \) it will arrive at infinity with \( \omega_{\infty} = \omega \sqrt{1 - b} \). In the limit \( b \to 1 \), we have a strong redshift. If light with finite \( \omega_{\infty} \) comes from infinity and enters the inner region, it has \( \omega = \frac{\omega_{\infty}}{\sqrt{1 - b}} \) there.

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