Shearlets and Optimally Sparse Approximations

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Abstract Multivariate functions are typically governed by anisotropic features such as edges in images or shock fronts in solutions of transport-dominated equations. One major goal both for the purpose of compression and for an efficient analysis is the provision of optimally sparse approximations of such functions. Recently, cartoon-like images were introduced in 2D and 3D as a suitable model class, and approximation properties were measured by considering the decay rate of the $L^2$ error of the best $N$-term approximation. Shearlet systems are to date the only representation system, which provide optimally sparse approximations of this model class in 2D as well as 3D. Even more, in contrast to all other directional representation systems, a theory for compactly supported shearlet frames was derived which moreover also satisfy this optimality benchmark. This chapter shall serve as an introduction to and a survey about sparse approximations of cartoon-like images by band-limited and also compactly supported shearlet frames as well as a reference for the state of the art of this research field.

Key words: Anisotropic features, Band-limited shearlets, Cartoon-like images, Compactly supported shearlets, Linear and nonlinear approximations, Multidimensional data, Sparse approximations
1 Introduction

Scientists face a rapidly growing deluge of data, which requires highly sophisticated methodologies for analysis and compression. Simultaneously, the data itself are becoming increasingly complex and higher dimensional. One of the most prominent features of data is singularities. This statement is justified, for instance, by the observation from neuroscientists that the human eye is most sensitive to smooth geometric areas divided by sharp edges. Intriguingly, already the step from univariate to multivariate data causes a significant change in the behavior of singularities. While one-dimensional (1D) functions can only exhibit point singularities, singularities of two-dimensional (2D) functions can already be of both point and curvilinear type. In fact, multivariate functions are typically governed by anisotropic phenomena. Think, for instance, of edges in digital images or evolving shock fronts in solutions of transport-dominated equations. These two exemplary situations also show that such phenomena occur even for both explicitly and implicitly given data.

One major goal both for the purpose of compression and for an efficient analysis is the introduction of representation systems for “good” approximation of anisotropic phenomena, more precisely, of multivariate functions governed by anisotropic features. This raises the following fundamental questions:

(P1) What is a suitable model for functions governed by anisotropic features?
(P2) How do we measure “good” approximation and what is a benchmark for optimality?
(P3) Is the step from 1D to 2D already the crucial step or how does this framework scale with increasing dimension?
(P4) Which representation system behaves optimally?

Let us now first debate these questions on a higher and more intuitive level, and later on delve into the precise mathematical formalism.

1.1 Choice of Model for Anisotropic Features

Each model design has to face the trade-off between closeness to the true situation versus sufficient simplicity to enable analysis of the model. The suggestion of a suitable model for functions governed by anisotropic features in [6] solved this problem in the following way. As a model for an image, it first of all requires the $L^2(\mathbb{R}^2)$ functions serving as a model to be supported on the unit square $[0, 1]^2$. These functions shall then consist of the minimal number of smooth parts, namely two. To avoid artificial problems with a discontinuity ending at the boundary of $[0, 1]^2$, the boundary curve of one of the smooth parts is entirely contained in $(0, 1)^2$. It now remains to decide upon the regularity of the smooth parts of the model functions and of the boundary curve, which were chosen to both be $C^2$. Thus, concluding, a possible suitable model for functions governed by anisotropic features are 2D