Zeroes of holomorphic functions with almost–periodic modulus

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Abstract

We give necessary and sufficient conditions for a divisor in a tube domain to be the divisor of a holomorphic function with almost–periodic modulus.

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Zero distribution for various classes of holomorphic almost–periodic functions in a strip was studied by many authors (cf. [1], [4], [7], [8], [9], [10], [17]). The notion of almost–periodic discrete set appeared in [9] and [17] in connection with these investigations. Its generalization to several complex variables was the notion of almost–periodic divisor, introduced by L. I. Ronkin (cf. [14]) and studied in his works and works of his disciples (cf. [5], [6], [15]). But these notions are not sufficient for a complete description of zero sets of holomorphic almost–periodic functions (cf. [18]): in addition, one needs some topological characteristic, namely, Chern class of the special (generated by an almost–periodic set or a divisor) line bundle over Bohr’s compact set (cf. [2], [3]). On the other hand, the class of zero sets of holomorphic functions with almost–periodic modulus in a strip is just the class of almost–periodic discrete sets (cf. [4]). That’s why it is natural to obtain a description of zeroes of holomorphic functions with the almost–periodic modulus for several complex variables without using topological terms. This problem is just solved in our paper.

By $T_S$ denote a tube set $\{z = x + iy : x \in \mathbb{R}^m, y \in S\}$, where the base $S$ is a subset of $\mathbb{R}^m$.

Definition 1. A continuous function $f$ on $T_S$ is called almost–periodic, if for each sequence $\{f(z + h_n)\}_{h_n \in \mathbb{R}^m}$ of shifts there exists a uniformly convergent on $T_S$ subsequence.

In particular, for $S = \{0\}$ we obtain the definition of an almost–periodic function on $\mathbb{R}^m$.

It follows easily that any almost–periodic function on a tube set with a compact base is bounded.

\footnote{This definition is equivalent to another one that makes use of the notion of an $\varepsilon$–almost period; for $m = 1$ see, for example, [12], the extension to $m > 1$ is trivial.}
**Definition 2.** Let $\Omega$ be a domain in $\mathbb{R}^m$. A continuous function $f$ on $T_\Omega$ is called almost-periodic, if its restriction to every tube set $T_K$ with compact base $K \subset \Omega$ is an almost-periodic function on $T_K$.

**Definition 3** (cf. [14], for distributions from $\mathcal{D}'(\mathbb{R})$ cf. also [14]). A distribution $F(z) \in \mathcal{D}'(T_\Omega)$ is called almost-periodic, if for any test-function $\varphi(z) \in \mathcal{D}(T_\Omega)$ the function $\langle F(z), \varphi(z-t) \rangle$ is an almost-periodic function in $t \in \mathbb{R}^m$.

The next assertion is valid.

**Theorem 1** (cf. [14]). A distribution $F \in \mathcal{D}'(T_\Omega)$ is almost-periodic if and only if for each sequence $\{h^n \} \subset \mathbb{R}^m$ there exists a subsequence $\{\tilde{h}^n \}$ such that the sequence of the distributions $F(z + \tilde{h}^n)$ converges uniformly on the sets $\{\kappa(z-t) : t \in \mathbb{R}^m, \kappa \in \mathcal{K}\}$, where $\mathcal{K}$ is any compact family in $\mathcal{D}(T_\Omega)$.

**Definition 4** (cf. [14]) The mean value (in the variable $x \in \mathbb{R}^m$) of an almost-periodic distribution $F$ is the distribution $c_F(y) \otimes dx$ with $c_F(y) \in \mathcal{D}'(\Omega)$ and the Lebesgue measure $dx$ on $\mathbb{R}^m$, defined for a test-functions $\varphi \in \mathcal{D}(T_\Omega)$ by the equality

$$\langle c_F(y) \otimes dx, \varphi(z) \rangle = \lim_{N \to \infty} (2N)^{-m} \int_{\max_j |t_j| < N} \langle F(z), \varphi(z-t) \rangle dt,$$

where $t = (t_1, \ldots, t_m) \in \mathbb{R}^m$.

Note that if $F(z)$ is an almost-periodic function on $T_\Omega$, then $c_F(y)$ is a continuous function on $\Omega$. Further, if $F(z)$ is an almost-periodic complex measure on $T_\Omega$, then $c_F(y)$ is a complex measure on $\Omega$ as well, and $c_F(y) \otimes dx$ is the weak limit of the measures $F(tx + iy)dx dy$ as $|t| \to \infty$ (cf. [14]).

By $\mathcal{H}(G)$ denote the space of holomorphic functions on the domain $G \subset \mathbb{C}^m$ with respect to the topology of the uniform convergence on compact subsets of $G$.

The following assertion is true.

**Theorem 2** (cf. [14]). If a function $f \in \mathcal{H}(T_\Omega)$ is almost-periodic, then $\log |f|$ is an almost-periodic distribution on $T_\Omega$.

The main part of the proof of this theorem is the following lemma.

**Lemma 1** (cf. [14]). If $f_n \in \mathcal{H}(G)$, $n = 1, 2, \ldots$, and $f_n \to f \neq 0$ in the space $\mathcal{H}(G)$, then $\log |f_n| \to \log |f|$ in the space $\mathcal{D}'(G)$.

Now let

$$(i/\pi) \partial \bar{\partial} \log |f| = (2/\pi) \sum_{j,k=1}^m \frac{\partial^2 \log |f|}{\partial z_j \partial \bar{z}_k} (i/2) dz_j \wedge d\bar{z}_k. \quad (1)$$

be the current of integration over the divisor $d_f$ of the function $f(z) \in \mathcal{H}(G)$, $z = (z_1, \ldots, z_m)$. In the case $m = 1$ this current corresponds to the discrete measure with integer masses equal to the multiplicities of the zeroes of the function $f$. 


Note that all the coefficients of the current \(1\) are complex measures on \(G\), and the "diagonal" coefficients \(\frac{\partial^2 \log |f|}{\partial z_j \partial \overline{z}_j}\) are positive measures (cf. \([11]\)).

Clearly, the differentiation keeps the almost periodicity of distributions. Therefore, it follows from Theorem \([2]\) that all the coefficients of the current \(1\) are almost–periodic distributions for any holomorphic almost–periodic function on \(T_\Omega\). If we replace \(f\) by another holomorphic function on \(T_\Omega\) with the same divisor, then the coefficients of the current \(1\) do not change. Hence an almost–periodicity of all the coefficients does not imply almost periodicity of the function \(f\) itself.

**Definition 5** (cf. \([5]\), \([6]\)). The divisor \(d\) of a function \(f \in \mathcal{H}(T_\Omega)\) is called almost–periodic, if all the coefficients of the current \(1\) are almost–periodic distributions.

Note that in \([14]\) a divisor \(d\) was called almost–periodic, if the measure \(\sum_{j=1}^{m} \frac{\partial^2 \log |f|}{\partial z_j \partial \overline{z}_j}\), was almost–periodic on \(T_\Omega\). But that definition is equivalent to the given above (cf. \([6]\)).

There exist almost periodic divisors which cannot be generated by holomorphic almost periodic functions. For example, let \(g(w)\) be an entire function on \(\mathbb{C}\) with simple zeroes at the points of the standard integer–valued lattice, and let \(d[\lambda, \mu]\), \(\lambda, \mu \in \mathbb{R}^m\) be the divisor of the function \(g(\langle z, \lambda \rangle + i\langle z, \mu \rangle)\). This divisor is periodic for vectors \(\lambda, \mu\) that are linearly dependent over \(\mathbb{Q}\) or linearly independent over \(\mathbb{R}\) (with the periods \(\|\lambda\|^2 - (\lambda, \mu)\mu\) and \(\|\lambda\|^2 - (\lambda, \mu)\mu\)). Then \(d[\lambda, \mu]\) is almost periodic for \(\lambda, \mu\) linearly independent over \(\mathbb{Q}\) and linearly dependent over \(\mathbb{R}\) (for \(m = 1\) cf. \([18]\); since a real linearly transform in \(\mathbb{C}^m\) keeps almost–periodicity, the case \(m > 1\) follows as well). Besides, the divisor \(d[\lambda, \mu]\) for any linearly independent over \(\mathbb{Q}\) vectors \(\lambda, \mu\) is the divisor of no holomorphic almost periodic function (in the case \(m = 1\), i.e., irrational \(\lambda/\mu\) cf. \([18]\), for \(m > 1\) cf. \([14]\)). A complete description of the divisors of holomorphic almost–periodic functions is contained in the following theorem.

**Theorem 3** (for \(m = 1\) cf. \([2]\), for \(m > 1\) cf. \([3]\)). A holomorphic bundle over Bohr’s compactification \(K_B\) of the space \(\mathbb{R}^m\) is assigned to each almost–periodic divisor \(d\) on a tube domain \(T_\Omega\) with convex base \(\Omega\) such that:

the map \(d \mapsto c(d)\), \(c(d)\) being the first Chern class of this bundle, is a homomorphism of the semigroup of positive almost–periodic divisors on \(T_\Omega\) to the cohomology group \(H^2(K_B, \mathbb{Z})\), the kernel of this homomorphism is just the set of all divisors of holomorphic almost–periodic functions on \(T_\Omega\),

a finite family \(\lambda^j, \mu^j \in \mathbb{R}^m\) corresponds to each cohomology class \(c(d)\) such that \(c(d) = \sum_j c(d[\lambda^j, \mu^j])\),

the mapping \(W : (\lambda, \mu) \mapsto c(d[\lambda, \mu])\) is skew-symmetric and additive in variables \(\lambda, \mu \in \mathbb{R}^m\).

A description of zeroes for holomorphic functions of one variable with the almost–periodic modulus is given in the following theorem.

**Theorem 4** (cf. \([4]\); for divisors \(d[\lambda, \mu]\), \(\lambda, \mu \in \mathbb{R}\) cf. \([18]\)). A divisor \(d\) on a strip is the divisor of some holomorphic function on the strip with almost–periodic modulus if and only if \(d\) is almost–periodic.
Now consider the multidimensional case again. Note that for an almost–periodic divisor $d$ on $T_\Omega$ all the coefficients of the current (1) have mean values in $x$. The imaginary parts of these mean values, i.e., the mean values of the real measures $(2/\pi)\xi_1^{\partial^2\log |f|}/\partial z_1\partial z_2$ have the form $a_{j,k}dy\otimes dx$, $a_{j,k} \in \mathbb{R}$ (cf. [13]). By $A(d)$ denote the matrix with the entries $a_{j,k}$. In the case $d = d_f$ for an almost–periodic function $f \in \mathcal{H}(T_\Omega)$ we have $A(d) = 0$ (cf. [13]).

**Theorem 5.** A divisor $d$ on a tube domain $T_\Omega$ with convex base $\Omega$ is the divisor of a holomorphic function with almost–periodic modulus if and only if divisor $d$ is almost–periodic, and the skew-symmetric matrix $A(d)$ is zero.

To prove this theorem we need the following improvement of Theorem 2:

**Theorem 6.** A function $f \in \mathcal{H}(T_\Omega)$, $f \neq 0$, has almost–periodic modulus if and only if the distribution $\log |f| \in \mathcal{D}'(T_\Omega)$ is almost–periodic.

**Proof of Theorem 6.** Let $|f(z)|$ be an almost–periodic function on $T_\Omega$, and let $\{h^n\}$ be an arbitrary sequence from $\mathbb{R}^m$. In order to check that $\log |f| \neq 0$ is an almost–periodic distribution on $T_\Omega$, we will prove that for any continuous function $\varphi$ with compact support in $T_\Omega$, the sequence of functions

$$
\psi_n(t) = \int \log |f(z + h^n)\varphi(z-t)|dx\,dy
$$

contains a convergent, uniformly on $\mathbb{R}^m$, subsequence. We will prove this assertion by contradiction.

First, since the function $|f(z)|$ is uniformly bounded on $T_K$ for every compact set $K \subset \Omega$, we may assume that the sequence of the functions $\{f(z + h^n)\}$ converges to some function $g(z)$ in the space $\mathcal{H}(T_\Omega)$. Further, since the function $|f(z)|$ is almost–periodic on $T_\Omega$, we may assume that the sequence of the functions $\{|f(z + h^n)|\}$ converges to some function $\Phi(z) \neq 0$ uniformly on each $T_K$. If the sequence (2) does not converge uniformly on $\mathbb{R}^m$, then for some $\delta > 0$ and some subsequence of $n$ there exist $t^n \in \mathbb{R}^m$ with the property

$$
|\psi_n(t^n) - \int \log |g(z)|\varphi(z-t^n)dx\,dy| \geq \delta. \tag{3}
$$

The function $|g(z)| \equiv \Phi(z)$ is almost–periodic on $T_\Omega$, hence we may assume that the same subsequence of the functions $\{|g(z + t^n)|\}$ converges uniformly on each $T_K$ to some function $\Psi(z) \neq 0$. Since the sequence of the functions $\{|f(z + h^n + t)|\}$ converges uniformly in $t \in \mathbb{R}^m$ and $z \in T_K$ to the function $|g(z + t)|$, we see that the subsequence $\{|f(z + h^n + t^n)|\}$ converges to $\Psi(z)$ uniformly on $T_K$. Also, the subsequence of the functions $\{f(z + h^n + t^n)|\}$ and $\{g(z - t^n)\}$ are bounded uniformly on compact subsets of $T_\Omega$, therefore passing to a subsequence again, we get that $f(z + h^n + t^n) \to H_1(z)$ and $g(z + t^n) \to H_2(z)$ in the space $\mathcal{H}(T_\Omega)$, and $|H_1(z)| = \Psi(z) = |H_2(z)|$. Using Lemma 1 we obtain that the corresponding subsequences of the functions $\{|f(z + h^n + t^n)|\}$ and
The last assertion contradicts (3).

On the other hand, let \( \log |f(z)| \) be an almost–periodic distribution on \( T_\Omega \), and let \( \varphi_\varepsilon(z) \) be a nonnegative, depending on \( |z| \) smooth function such that \( \varphi(z) = 0 \) for \( |z| > \varepsilon \) and \( \int_{Cm} \varphi_\varepsilon(z)dx
dy = 1 \). Evidently, the family of functions \( \{ \varphi_\varepsilon(z + iy) \}_{y| \leq C} \) is a compact set in the space \( \mathcal{D}(Cm) \) for every \( C < \infty \). Let \( K \) be a compact set in \( \Omega \) and \( \varepsilon < \text{dist}\{K, \partial \Omega\} \). Now Theorem \( \mathbb{I} \) implies that the convolution \( (\log |f| * \varphi_\varepsilon)(z) \) is an almost–periodic function on \( T_K \). Hence this convolution is bounded on \( T_K \), and the inequality \( \log |f(z)| \leq (\log |f| * \varphi_\varepsilon)(z) \) shows that \( |f(z)| \) is bounded on \( T_K \) as well.

Suppose that \( |f| \) is not an almost–periodic function on \( T_\Omega \). Then there exists a sequence of functions \( \{|f(z + h^n)|\}, h^n \in \mathbb{R}^m \), such that every its subsequence does not converge uniformly on \( T_{K'} \) for some compact set \( K' \subset \Omega \). Without loss of generality it can be assumed that the sequence of functions \( \{f(z + h^n)\} \) converges in the space \( \mathcal{H}(T_\Omega) \) to some function \( g(z) \). It is clear that \( g(z) \) is bounded on \( T_K \) for every compact set \( K \subset \Omega \). Further, by Lemma \( \mathbb{II} \) we get \( \log |f(z + h^n)| \to \log |g(z)| \) in the sense of distributions. Using Theorem \( \mathbb{I} \) and passing to a subsequence, we obtain

\[
\int (\log |f(z + h^n)| - \log |g(z)|) \varphi_\varepsilon(z - t - is)dx
dy \to 0.
\]

(4)

uniformly in \( t \in \mathbb{R}^m \) and \( s \in K' \). On the other hand, for some \( \delta > 0 \) and some subsequence of \( n \) there exist points \( z^n = x^n + iy^n \in T_K' \) such that

\[
||f(h^n + x^n + iy^n)| - |g(x^n + iy^n)|| > \delta.
\]

(5)

Passing to a subsequence if necessary, we may assume that \( y^n \to y^0 \in K' \), and the sequences of the functions \( \{f(z + h^n + z^n - iy^0)\} \) and \( \{g(z + z^n - iy^0)\} \) converge in the space \( \mathcal{H}(T_\Omega) \) to functions \( H_1(z) \) and \( H_2(z) \), respectively. Then Lemma \( \mathbb{II} \) implies that \( \log |f(z + h^n + z^n - iy^0)| \to \log |H_1(z)| \) and \( \log |g(z + z^n - iy^0)| \to \log |H_2(z)| \) in the space \( \mathcal{D}'(T_\Omega) \). Taking into account \( \mathbb{II} \), we obtain

\[
\int \log |H_1(z)| \varphi_\varepsilon(z - iy^0)dx
dy = \int \log |H_2(z)| \varphi_\varepsilon(z - iy^0)dxdy.
\]

Since \( \varepsilon \) is arbitrary small, we get \( |H_1(iy^0)| = |H_2(iy^0)| \). At the same time, by \( \mathbb{V} \) we have \( |H_1(iy^0)| \neq |H_2(iy^0)| \). This contradiction proves Theorem \( \mathbb{III} \).

Proof of the necessity in Theorem \( \mathbb{V} \). It follows from Theorem \( \mathbb{II} \) that every function \( f \in \mathcal{H}(T_\Omega) \) with almost–periodic modulus has an almost–periodic divisor. Further, the mean value \( c_{\log |f|}(y) \otimes dx \) of the function \( \log |f| \) is the weak limit of the measures \( \log |f(tx + iy)|dxdy \) as \( t \to \infty \) in the space \( \mathcal{D}'(T_\Omega) \), therefore for all \( j \neq k \) the mean values of the distributions

\[
3 \frac{\partial^2 \log |f|}{\partial z_j \partial z_k} = \frac{1}{4} \left( \frac{\partial^2}{\partial x_j \partial y_k} - \frac{\partial^2}{\partial x_k \partial y_j} \right) \log |f|
\]

equal

\[
\lim_{|t| \to \infty} \frac{1}{4} \left( \frac{\partial^2}{\partial x_j \partial y_k} - \frac{\partial^2}{\partial x_k \partial y_j} \right) \log |f(tx + iy)|dxdy = \frac{1}{4} \left( \frac{\partial^2}{\partial x_j \partial y_k} - \frac{\partial^2}{\partial x_k \partial y_j} \right) c_{\log |f|}(y) \otimes dx = 0.
\]
Lemma 3. The entries $\lambda, \mu \in \mathbb{R}^m$ of the matrix $A_0 = A(d[e^1, e^2])$ vanish for $(j, k) \neq (1, 2)$ or $(2, 1)$, and $a_{1, 2} = -1, a_{2, 1} = 1$.

Proof of Lemma 3. The divisor $d[\lambda, \mu]$ with $\lambda = t\mu, \mu \in \mathbb{R}^m, t \in \mathbb{R}$, is the divisor of an entire function on $\mathbb{C}^m$ with almost-periodic modulus.

The necessity of the conditions in Theorem 5 is proved.

The proof of the sufficiency makes use of the following lemmas. As above, $d[\lambda, \mu], \lambda, \mu \in \mathbb{R}^m$ is the divisor of the function $g((z, \lambda) + i(z, \mu))$, where $g(w)$ is an entire function on $\mathbb{C}$ with simple zeroes at the points of the standard integer-valued lattice.

Lemma 2. The divisor $d[\lambda, \mu]$ with $\lambda = t\mu, \lambda \in \mathbb{R}^m, t \in \mathbb{R}$, is the divisor of an entire function on $\mathbb{C}^m$ with almost-periodic modulus.

Proof of Lemma 2. After a suitable regular real linear transform we obtain the case $\mu = (1, 0, \ldots, 0)$, i.e., the case of a divisor depending only on one coordinate, therefore the assertion of our lemma is a consequence of Theorem 4.

Further, let $e^1, \ldots, e^m$ be the coordinate vectors in $\mathbb{C}^m$.

Lemma 3. The entries $a_{j, k}$ of the matrix $A_0 = A(d[e^1, e^2])$ vanish for $(j, k) \neq (1, 2)$ or $(2, 1)$.

Proof of Lemma 3. The divisor of the function $g(z_1 + iz_2)$ does not depend on variables $z_j$ with $j > 2$, hence the distributions $\Re \frac{\partial^2 \log |g(z_1 + iz_2)|}{\partial z_j \partial z_k}$ vanish for $(j, k) \neq (1, 2)$ or $(2, 1)$.

Consider the expression

$$L_z \log |g(z_1 + iz_2)|, \varphi(z_1 + t_1, z_2 + t_2), (t_1, t_2) \in \mathbb{R}^2,$$

for $L_z = \frac{2}{\pi} \Im \frac{\partial^2}{\partial z_1 \partial z_2}$ and a function $\varphi(z) \geq 0$ from the space $\mathcal{D}(\mathbb{C}^2)$. In the coordinates $\zeta_1 = z_1 + iz_2, \zeta_2 = z_1 - iz_2$, it has a form

$$\frac{1}{4} (\tilde{L}_\zeta \log |g(\zeta_1)|, \varphi((\zeta_1 + \zeta_2)/2 + t_1, (\zeta_1 - \zeta_2)/2i + t_2))$$

with

$$\tilde{L}_\zeta = \frac{2}{\pi} \Re \left( \frac{\partial^2}{\partial \zeta_1 \partial \zeta_1} - \frac{\partial^2}{\partial \zeta_2 \partial \zeta_1} + \frac{\partial^2}{\partial \zeta_1 \partial \zeta_2} - \frac{\partial^2}{\partial \zeta_2 \partial \zeta_2} \right).$$

Using the definition of $g$ and properties of the Laplace operator, we get

$$\tilde{L}_\zeta \log |g(\zeta_1)| = \frac{2}{\pi} \frac{\partial^2}{\partial \zeta_1 \partial \zeta_1} \log |g(\zeta_1)| = \sum_{q_1, q_2 \in \mathbb{Z}} \delta(\zeta_1 - q_1 - iq_2) \otimes d\xi d\eta,$$

where $\delta$ is the Dirac function on the plane, $\xi = \Re \zeta_2, \eta = \Im \zeta_2$. Therefore, (6) is equal to

$$\frac{1}{4} \sum_{q_1, q_2 \in \mathbb{Z}} \int_{\mathbb{C}} \varphi(t_1 + (q_1 + iq_2 + \xi + i\eta)/2, t_2 + (q_1 + iq_2 - \xi - i\eta)/2i) d\xi d\eta.$$

Substituting $\xi = q_1, \eta + q_2$ for $u, v$, respectively, we get

$$\frac{1}{4} \sum_{q_1, q_2 \in \mathbb{Z}} \int_{\mathbb{C}} \varphi(u/2 + iv/2 + q_1, -v/2 + iu/2 + t_2 + q_2) du dv. \quad (7)$$
Since the divisor $d_{e_1,e_2}$ has period 1 in each variable, we see that the mean value of (6) is the integral of (7) over the square $0 \leq t_1 \leq 1$, $0 \leq t_2 \leq 1$. Then it is equal to the integral
\[
\frac{1}{4} \int_{\mathbb{R}^4} \varphi(\frac{u}{2} + iv/2 + t_1, -v/2 + iu/2 + t_2)du \, dv \, dt_1 \, dt_2.
\]
Finally, substituting $u/2 + t_1 = x_1$, $v/2 = y_1$, $t_2 - v/2 = x_2$, $u/2 = y_2$, we obtain the equality
\[
\int_0^1 \int_0^1 (L_z \log |g(z_1 + iz_2)|, \varphi(z_1 + t_1, z_2 + t_2)) dt_1 \, dt_2 = \int_{\mathbb{R}^4} \varphi(x_1 + iy_1, x_2 + iy_2) dx_1 \, dy_1 \, dx_2 \, dy_2,
\]
which is the integral of $\varphi$ over the square $0 \leq x_1 \leq 1$, $0 \leq x_2 \leq 1$. Hence the mean value of the distribution $L_z \log |g(z_1 + iz_2)|$ is the Lebesgue measure in $\mathbb{C}^2$. The lemma is proved.

By $(\lambda, \mu)$ denote the matrix product $\prod_{j,k=1}^m \lambda_j \mu_k$ of the vectors $\lambda = (\lambda_1, \ldots, \lambda_m)$, $\mu = (\mu_1, \ldots, \mu_m) \in \mathbb{R}^m$.

**Lemma 4**. For any $\lambda, \mu \in \mathbb{R}^m$, the matrix $A(d[\lambda, \mu])$ equals the difference $(\mu, \lambda) - (\lambda, \mu)$.

**Proof of Lemma 4**. If $\lambda, \mu$ are linearly dependent over $\mathbb{R}$, then $(\mu, \lambda) - (\lambda, \mu) = 0$. On the other hand, it follows from Lemma 2 that in this case the divisor $d[\lambda, \mu]$ is the divisor of some holomorphic in $\mathbb{C}^m$ function with almost–periodic modulus. Using the proved part of Theorem 5, we have $A(d[\lambda, \mu]) = 0$.

Let $\lambda, \mu$ be linearly independent over $\mathbb{R}$. The divisor $d[\lambda, \mu]$ is the divisor $d[e_1, e_2]$ in the coordinates $\zeta = Bz$ for some real nondegenerate matrix $B$ with the first and second rows $\lambda$ and $\mu$, respectively. Note that the matrix $A(d)$ is the matrix of the mean values for the matrix $\frac{1}{2i}(D(z) - \overline{D}(z))$, where
\[
D(z) = \left( \frac{\partial^2 \log |g(z, \lambda) + i(z, \mu)|}{\partial z_j \partial \overline{z}_k} \right).
\]
$D$ being the matrix with all the entries complex conjugated to the corresponding entries of the matrix $D$. Therefore $D(z) = B' D(\zeta) B$, $B'$ being the transpose matrix to $B$, and $A(d[\lambda, \mu]) = B' A_0 B$ for the matrix $A_0$ from the previous lemma. This completes the proof of Lemma 4.

**Lemma 5**. If numbers $\alpha_j, \beta_j \in \mathbb{R}$, $j = 1, \ldots, n$, satisfy the condition $\sum_{j=1}^n \alpha_j \beta_j = 0$, then for some $\gamma_k \in \mathbb{R}$, $\nu^k \in \mathbb{R}^m$, $k = 1, \ldots, N$, we get
\[
\sum_{j=1}^n W(\alpha_j e^1, \beta_j e^2) = \sum_{k=1}^N W(\gamma_k \nu^k, \nu^k),
\]
where $W$ being the mapping from Theorem 4.
Proof of Lemma 5 The case \( n = 1 \) means that the left-hand side of (8) vanishes. For \( n > 1 \) we have

\[
W(\alpha_{n-1}e^1, \beta_{n-1}e^2) + W(\alpha_ne^1, \beta_ne^2) = W(\alpha_{n-1}e^1, \alpha_ne^1) + W(\beta_ne^2, \beta_n\alpha_{n-1}/\alpha_{n-1}e^2) \\
+ W(\alpha_ne^1 + \alpha_n\beta_{n-1}/\alpha_{n-1}e^1, \alpha_ne^1) + \beta_ne^2) + W(\alpha_{n-1}e^1, (\beta_{n-1} + \beta_n\alpha_{n-1}/\alpha_{n-1})e^2).
\]

The first three terms of the right-hand side have the form \( W(\gamma\nu, \nu) \), \( \gamma \in \mathbb{R}, \nu \in \mathbb{R}^m \). Subtracting these terms from the left-hand side of (8), we get

\[
\sum_{j=1}^{n-2} W(\alpha_je^1, \beta_je^2) + W(\alpha_{n-1}e^1, (\beta_{n-1} + \beta_n\alpha_{n-1}/\alpha_{n-1})e^2).
\]

Hence the lemma can be proved by induction over \( n \).

Lemma 6. Let vectors \( \lambda^j, \mu^j \in \mathbb{R}^m \), \( j = 1, \ldots, n \) be such that the matrix \( \sum_1^n (\lambda^j, \mu^j) \) is symmetric. Then

\[
\sum_{j=1}^n W(\lambda^j, \mu^j) = \sum_{k=1}^N W(\gamma_k \nu^k, \nu^k)
\]

for some \( \gamma_k \in \mathbb{R}, \nu^k \in \mathbb{R}^m \), \( k = 1, \ldots, N \).

Proof of Lemma 6. The vectors \( \lambda^j, \mu^j \) are linear combinations of the vectors \( e^1, \ldots, e^m \), therefore the left-hand side of (9) has the form

\[
\sum_{1 \leq p, q \leq m} \left( \sum_{j=1}^{M(p,q)} W(\alpha_{j,p}e^p, \beta_{j,q}e^q) \right)
\]

with \( \alpha_{j,p}, \beta_{j,q} \in \mathbb{R} \). The mapping \( W \) is skew-symmetric, hence we may assume that all the terms in (10) vanish for \( p > q \), and the entries of the corresponding matrix \( \sum_{j=1}^{M(p,q)} \alpha_{j,p} \beta_{j,q} \) vanish for all \( p > q \). Since this matrix coincides with the symmetric matrix \( \sum_1^n (\lambda^j, \mu^j) \), we see that \( \sum_{j=1}^{M(p,q)} \alpha_{j,p} \beta_{j,q} = 0 \) for \( p < q \) as well. Now it follows from Lemma 5 that for \( p < q \) the sum

\[
\sum_{j=1}^{M(p,q)} W(\alpha_{j,p}e^p, \beta_{j,q}e^q)
\]

has the form of the right-hand side of (9). The terms of (10) with \( p = q \) have already the form \( W(\gamma\nu, \nu) \). The lemma is proved.

Proof of the sufficiency in Theorem 5. Let \( d \) be a divisor in \( T_{\Omega} \) such that \( A(d) = 0 \). It follows from Theorem 3 that there exist \( \lambda^j, \mu^j \in \mathbb{R}^m \), \( j = 1, \ldots, n \), such that the sum \( d + \sum_j d[\lambda^j, \mu^j] \) is the divisor of a holomorphic almost–periodic function. Now, by (13), \( A(d + \sum_1^n d[\lambda^j, \mu^j]) = 0 \). Since the mapping \( d \mapsto A(d) \) is a homomorphism,
we get $\sum_{\lambda,\mu}^n (\lambda, \mu) = \sum_{\lambda,\mu}^n A(d[\lambda, \mu]) = 0$, i.e., the matrix $\sum_{\lambda,\mu}^n (\lambda, \mu)$ is symmetric. Using Lemma 6 we get (9) for some $\gamma_k \in \mathbb{R}$, $\nu_k \in \mathbb{R}^m$, $k = 1, \ldots, N$. Therefore,

$$c(d + \sum_{\lambda,\mu}^n d[\lambda, \mu]) = c(d) + \sum_{\lambda,\mu}^n W(\gamma_k \nu_k, \nu_k)$$

$$= c(d) + \sum_{\lambda,\mu}^n W(\lambda, \mu) = c(d + \sum_{\lambda,\mu}^n d[\lambda, \mu]) = 0.$$

An application of Theorem 3 yields that there exists an almost–periodic function $F \in \mathcal{H}(T_\Omega)$ with the divisor $d + \sum_{\lambda,\mu}^N d[\gamma_k \nu^k, \nu^k]$. Using Lemma 2, we can take functions $f_k \in \mathcal{H}(T_\Omega)$ with the divisors $d[\gamma_k \nu^k, \nu^k]$ and almost–periodic modula. The function $f(z) = F(z)(\prod_{k=1}^N f_k(z))^{-1}$ is holomorphic on $T_\Omega$ and has the divisor $d$. Then Theorem 6 implies that the distributions $\log |F|$ and $\log |f_k|$, $k = 1, \ldots, N$, are almost–periodic. Hence the distribution $\log |f| = \log |F| - \sum_{\lambda,\mu}^N \log |f_k|$ is almost–periodic as well. Using Theorem 6 again, we see that the function $|f|$ is almost–periodic. This completes the proof of Theorem 5.

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