An Alternative Proof of Hesselholt’s Conjecture on Galois Cohomology of Witt Vectors of Algebraic Integers

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Abstract

Let \( K \) be a complete discrete valuation field of characteristic zero with residue field \( k_K \) of characteristic \( p > 0 \). Let \( L/K \) be a finite Galois extension with Galois group \( G = \text{Gal}(L/K) \) and suppose that the induced extension of residue fields \( k_L/k_K \) is separable. Let \( \mathcal{W}_n(\cdot) \) denote the ring of \( p \)-typical Witt vectors of length \( n \). Hesselholt conjectured that the pro-abelian group \( \{H^1(G, \mathcal{W}_n(O_L))\}_{n \geq 1} \) is isomorphic to zero. Hogadi and Pisolkar have recently provided a proof of this conjecture. In this paper, we provide an alternative proof of Hesselholt’s conjecture which is simpler in several respects.

1 Literature Review

Let \( K \) be a complete discrete valuation field of characteristic zero with residue field \( k_K \) of characteristic \( p > 0 \). Let \( L/K \) be a finite Galois extension with Galois group \( G = \text{Gal}(L/K) \) and suppose that the induced extension of residue fields \( k_L/k_K \) is separable. Let \( \mathcal{W}_n(\cdot) \) denote the ring of \( p \)-typical Witt vectors of length \( n \). In Hesselholt’s paper [1] it is conjectured that the pro-abelian group \( \{H^1(G, \mathcal{W}_n(O_L))\}_{n \geq 1} \) is isomorphic to zero, and the conjecture is reduced to the case where \( L/K \) is a totally ramified cyclic extension of degree \( p \). Let \( \sigma \) be a generator of \( G \) and let \( t := v_L(\sigma(p_L) - \pi_L) - 1 \) denote the ramification break (see [2] Chapter V, §3) in the ramification filtration of \( G \). Recall that \( t \) does not depend on the choice of generator \( \sigma \).

Hesselholt shows his conjecture holds for extensions with \( t > \frac{p-1}{p} \). Hogadi and Pisolkar have recently provided a proof of the conjecture for all Galois extensions (see [3]). In this paper, we provide an alternative proof of Hesselholt’s conjecture which is simpler in several respects. First let us recall some lemmas from [1]:

**Lemma 1.1.** For all \( a \in O_L \), \( v_K(\text{tr}(a)) \geq \frac{v(a) + t(p-1)}{p} \).

**Proof.** We know \( a \in \mathbb{p}_L^{v_L(a)} \), so from [2] Chapter V, §3, Lemma 4], we have \( \text{tr}(a) = \pi_K^{(1+1)(p-1)+v_L(a)} b \) for some \( b \in O_K \). Now taking \( K \)-valuations gives the desired result. \( \square \)

**Lemma 1.2.** For all \( a \in O_L \), \( v_K(\text{tr}(a^p) - \text{tr}(a)^p) = v_K(p) + v_L(a) \).

**Proof.** This follows by expanding \( \text{tr}(a^p) - \text{tr}(a)^p \) using the multinomial formula and grouping the resulting expression into summands with distinct valuations. See the proof of [1] Lemma 2.2] for the details. \( \square \)

Next, we provide an alternative elementary proof of [1] Lemma 2.4:

**Lemma 1.3.** Suppose that \( a \in O_L^{\text{tr}=0} \) represents a non-zero class in \( \frac{O_L^{\text{tr}=0}}{(\sigma-1)O_L} \). Then \( v_L(a) \leq t - 1 \).

**Proof.** For each \( 0 \leq \mu \leq p-1 \), define \( x_\mu = \prod_{0 \leq i < \mu} \sigma^i(p_L) \). It is clear \( v_L(x_\mu) = \mu \). Suppose

\[
a_0x_0 + a_1x_1 + \cdots + a_{p-1}x_{p-1} = 0
\]

for some \( a_0, a_1, \ldots, a_{p-1} \in K \). The summands on the left have distinct \( L \)-valuations modulo \( p \) and thus distinct \( L \)-valuations, implying each summand must be zero by the non-archimedean property. Hence the \( x_\mu \) are linearly independent over \( K \) and thus span \( L \) over \( K \). Now recall \( \frac{\ker(\text{tr})}{(\sigma-1)L} = H^1(G, L) = 0 \) (see [2] Chapter VIII, §4] and [2] Chapter X, §1, Proposition 1]). Hence \( O_L^{\text{tr}=0} \subseteq (\sigma-1)L \) so we can write

\[
a = b_1(\sigma - 1)x_1 + b_2(\sigma - 1)x_2 + \cdots + b_{p-1}(\sigma - 1)x_{p-1}
\]
for some \(b_1, b_2, \ldots, b_{p-1} \in K\). It is clear from the definition of \(x_\mu\) that \(\pi L \sigma(x_\mu) = x_\mu \sigma^\mu(\pi L)\) for each \(1 \leq \mu \leq p - 1\) so that \(v_L((\sigma - 1)x_\mu) = v_L((\sigma^{p-1}x_\mu) \cdot x_\mu) = t + \mu\), implying the summands on the right have distinct \(L\)-valuations modulo \(p\), and thus distinct \(L\)-valuations. Since \(\alpha \notin (\sigma - 1)O_L\) by hypothesis, we must have \(b_\mu \notin O_K\) for some \(\mu\) so that \(v_L(b_\mu(\sigma - 1)x_\mu) \leq -p + t + \mu' \leq -p + t + (p - 1)\) for this \(\mu'\). Hence by the non-archimedean property, we conclude \(v_L(a) \leq t - 1\), as required.

**Lemma 1.4.** Let \(m \geq 1\) be an integer and suppose that the map

\[ R_m^n : H^1(G, \mathbb{W}_{m+n}(O_L)) \to H^1(G, \mathbb{W}_n(O_L)) \]

is equal to zero, for \(n = 1\). Then the same is true for all \(n \geq 1\).

**Proof.** This follows from the long exact sequence of cohomology. See the proof of [1, Lemma 1.1] for the details.

## 2 Proof of Hesselholt’s Conjecture

Recall for each \(n \geq 0\), we have the Witt polynomial

\[ W_n(X_0, X_1, \ldots, X_n) = X_0^{p^n} + pX_1^{p^{n-1}} + \cdots + p^nX_n = \sum_{i=0}^{n} p^i X_i^{p^{n-i}} \]

Fix any \(m \geq 0\). Let

\[ \sum_{i=0}^{p-1} (X_{i,0}, X_{i,1}, \ldots, X_{i,m}) = (z_0, z_1, \ldots, z_m) \]

where on the left we have a sum of Witt vectors. Then we know each \(z_n\) is a polynomial in \(\mathbb{Z}[\{X_{i,j}\}_{0 \leq i \leq p-1, 0 \leq j \leq n}]\) with no constant term (see [2, Chapter II, §6, Theorem 6]). By construction of Witt vector addition (see [2, Chapter II, §6, Theorem 7]) we have

\[ \sum_{i=0}^{p-1} W_n(X_{i,0}, X_{i,1}, \ldots, X_{i,n}) = W_n(z_0, z_1, \ldots, z_n) \]

for each \(0 \leq n \leq m\). Now using the expression for the Witt polynomial \(W_n\) and dividing through by \(p^n\) yields

\[ f_n + \sum_{i=0}^{p-1} X_{i,n} - z_n = 0 \quad (1) \]

where

\[ f_n = \frac{1}{p^n} \left( \sum_{i=0}^{p-1} X_{i,0}^{p^n} - z_0^{p^n} \right) + \frac{1}{p^{n-1}} \left( \sum_{i=0}^{p-1} X_{i,1}^{p^{n-1}} - z_1^{p^{n-1}} \right) + \cdots + \frac{1}{p} \left( \sum_{i=0}^{p-1} X_{i,n-1}^{p} - z_{n-1}^{p} \right) \quad (2) \]

Now for any \(1 \leq n \leq m\), we may add and subtract \(\frac{1}{p}(-f_{n-1})^p\) to obtain

\[ f_n = g_{n-2} + \frac{1}{p} \left( \sum_{i=0}^{p-1} X_{i,n-1}^{p} - z_{n-1}^{p} - (-f_{n-1})^p \right) \quad (3) \]

where

\[ g_{n-2} = \frac{1}{p^n} \left( \sum_{i=0}^{p-1} X_{i,0}^{p^n} - z_0^{p^n} \right) + \cdots + \frac{1}{p^2} \left( \sum_{i=0}^{p-1} X_{i,n-2}^{p^2} - z_{n-2}^{p^2} \right) + \frac{1}{p}(-f_{n-1})^p \quad (4) \]

**Lemma 2.1.** Suppose \((a_0, a_1, \ldots, a_m) \in \mathbb{W}_{m+1}(O_L)\). Then

\[ v_L(g_{n-2}|_{X_{i,j} = \sigma^i(a_j)}) \geq p^2 \cdot \min\{v_L(a_j): 0 \leq j \leq n - 2\} \]

for each \(2 \leq n \leq m\).
Proof. From (1) and (2) we know \( f_n \) is a polynomial in \( \mathbb{Z}[\{X_{i,j}\}]_{0 \leq i \leq r-1, 0 \leq j \leq s-1} \) with no constant term, and each monomial of \( f_n \) has degree at least \( p \). From (1) we know \( \sum_{i=0}^{p-1} X_{i,n-1} = z_{n-1} - f_{n-1} \), implying \( \sum_{i=0}^{p-1} X_{i,n-1} \equiv z_{n-1}^p - (f_{n-1})^p \pmod{p} \), so in view of (3) we see \( g_{n-2} \) has integer coefficients. Thus from (4) we know \( g_{n-2} \) is a polynomial in \( \mathbb{Z}[\{X_{i,j}\}]_{0 \leq i \leq r-1, 0 \leq j \leq s-1} \) with no constant term, and each monomial of \( g_{n-2} \) has degree at least \( p^2 \). Hence recalling that \( v_L(\sigma^i(a_j)) = v_L(a_j) \) (see [2, Chapter II, §2, Corollary 3]), and using the properties of valuations, it is clear we have the desired inequality.

**Lemma 2.2.** Suppose \( (a_0, a_1, \ldots, a_m) \in \mathbb{W}_{m+1}(\mathcal{O}_L)^{tr=0} \). Then

\[
v_L(a_{n-1}) \geq \min\left\{ \frac{v_L(a_n) + t(p-1)}{p}, \frac{v_K(g_{n-2}|_{X_{i,j}=\sigma^i(a_j)})}{p^2} \right\}
\]

for each \( 1 \leq n \leq m \).

**Proof.** Since \( (a_0, a_1, \ldots, a_m) \in \mathbb{W}_{m+1}(\mathcal{O}_L)^{tr=0} \), by definition of the \( z_n \) we can take \( z_n = 0 \) for \( 0 \leq n \leq m \) and \( X_{i,j} = \sigma^i(a_j) \). Then from (1) we see \( -f_n = \text{tr}(a_n) \) for each \( n \), and hence (3) reduces to \( tr(a^p_{n-1}) - tr(a_{n-1})^p = -\text{tr}(a_n) - g_{n-2}|_{X_{i,j}=\sigma^i(a_j)} \). Taking \( K \)-valuations of both sides of this equation then applying Lemma 1.2 and Lemma 1.4 gives

\[
v_L(a_{n-1}) \geq \min\left\{ \frac{v_L(a_n) + t(p-1)}{p}, \frac{v_K(g_{n-2}|_{X_{i,j}=\sigma^i(a_j)})}{p^2} \right\}
\]

(5)

Since \( f_0 = 0 \) by (2), we see \( g_{-1} = 0 \) by (1). Hence taking \( n = 1 \) in (5), we see that the claim holds for \( n = 1 \). Now for the inductive step let \( N \geq 2 \) and suppose the claim holds for all \( 1 \leq n \leq N - 1 \). Then we have

\[
v_L(a_{N-1}) \geq \min\left\{ \frac{v_L(a_N) + t(p-1)}{p}, \frac{v_K(g_{N-2}|_{X_{i,j}=\sigma^i(a_j)})}{p^2} \right\}
\]

where the first inequality follows from (3), the second by Lemma 2.1 and the third by the induction hypothesis. This completes the inductive step.

By Lemma 1.4 and recalling that \( H^1(G, \mathbb{W}_{m+1}(\mathcal{O}_L)) = \frac{\mathbb{W}_{m+1}(\mathcal{O}_L)^{tr=0}}{(\sigma-1)\mathbb{W}_{m+1}(\mathcal{O}_L)} \) (see [2, Chapter VIII, §4]), the following proposition (a generalisation of [1, Proposition 2.5]) proves Hesselholt’s conjecture.

**Proposition 2.3.** The map

\[
R_m: \frac{\mathbb{W}_{m+1}(\mathcal{O}_L)^{tr=0}}{(\sigma-1)\mathbb{W}_{m+1}(\mathcal{O}_L)} \rightarrow \frac{\mathcal{O}^{tr=0}_L}{(\sigma-1)\mathcal{O}_L}
\]

is equal to zero, provided that \( p^m > t \).

**Proof.** Suppose \( (a_0, a_1, \ldots, a_m) \in \mathbb{W}_{m+1}(\mathcal{O}_L)^{tr=0} \). Note that \( v_L(a_n) > t - p^n \) implies

\[
v_L(a_{n-1}) \geq \min\left\{ \frac{v_L(a_n) + t(p-1)}{p}, \frac{v_K(g_{n-2}|_{X_{i,j}=\sigma^i(a_j)})}{p^2} \right\} > \frac{(t - p^n) + t(p-1)}{p} = t - p^{n-1}
\]

Since \( v_L(a_m) > t - p^m \) by hypothesis, we see \( v_L(a_0) > t - p^0 \) by downward induction. Thus by Lemma 1.3 we see \( a_0 \) must represent the zero class in \( \frac{\mathcal{O}^{tr=0}_L}{(\sigma-1)\mathcal{O}_L} \).
References

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