Abstract. In this paper, we construct a new class of modules for the Schrödinger algebra $\mathfrak{S}$, called quasi-Whittaker module. Different from [24], the quasi-Whittaker module is not induced by the Borel subalgebra of the Schrödinger algebra related with the triangular decomposition, but its Heisenberg subalgebra $\mathcal{H}$. We prove that, for a simple $\mathfrak{S}$-module $V$, $V$ is a quasi-Whittaker module if and only if $V$ is a locally finite $\mathcal{H}$-module; Furthermore, we classify the simple quasi-Whittaker modules by the elements with the action similar to the center elements in $U(\mathfrak{S})$ and their quasi-Whittaker vectors. Finally, we characterize arbitrary quasi-Whittaker modules.

Keywords: Schrödinger algebra; locally finite modules; quasi-Whittaker modules; Simple modules

Mathematics Subject Classification (2010): 17B10, 17B65, 17B68.

§1. Introduction

In [10], B. Kostant defined a class of modules for a finite-dimensional complex semisimple Lie algebra. He called these modules as Whittaker modules because of their connections with the Whittaker equations that arise in the study of the corresponding representations of the associated Lie group. The traditional definition of Whittaker modules is closely tied to the triangular decomposition of a Lie algebra. Results for the complex semisimple Lie algebras have been extended to quantum groups for $U_q(\mathfrak{g})$ [19], and $U_q(\mathfrak{sl}_2)$ [17], Virasoro algebra [8] [18], Schrödinger-Witt algebra [25], Heisenberg algebra and affine Kac-Moody algebra [4] [9], Heisenberg-Virasoro algebra [5] [14], Weyl algebra [2] and some other infinite dimensional Lie algebras [20] [21]. Specially, in [3], the author proved that all the simple $\mathfrak{sl}_2$-modules fall into three families: highest (lowest) weight modules, Whittaker modules and a third family obtained by localization; in [4], the author studied a new class of modules, which

\footnotesize{\begin{center}1\end{center}}

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was called imaginary Whittaker module over the non-twisted affine Lie algebra induced by its parabolic subalgebras. Inspired by recent activities on Whittaker modules over various algebras, the authors in [1] described some general framework for the study of Lie algebra modules locally finite over a subalgebra. In [16], in order to classify the simple modules over a family of finite dimensional solvable Lie algebras, the authors studied and classified the simple Virasoro modules which are locally finite over a positive part.

The Schrödinger algebra is the Lie algebra of Schrödinger group, which is the symmetry group of the free particle Schrödinger equation. The Schrödinger algebra plays an important role in mathematical physics and its applications [7]. Let $\mathfrak{G}$ denote the Schrödinger algebra. Then $\mathfrak{G} = \text{span}_\mathbb{C}\{e, h, f, p, q, z\}$ with the following Lie brackets:

\begin{align}
[h, e] &= 2e, \quad [h, f] = -2f, \quad [e, f] = h, \\
[h, p] &= p, \quad [h, q] = -q, \quad [p, q] = z, \\
[e, q] &= p, \quad [p, f] = -q, \quad [f, q] = 0, \\
[e, p] &= 0, \quad [z, \mathfrak{G}] = 0.
\end{align}

From this we see that $\mathfrak{G}$ contains two subalgebras: the Heisenberg subalgebra $\mathfrak{H} = \text{span}_\mathbb{C}\{p, q, z\}$ and $\mathfrak{sl}_2 = \text{span}_\mathbb{C}\{e, h, f\}$. The Schrödinger algebra $\mathfrak{G}$ can be viewed as a semidirect product $\mathfrak{G} = \mathfrak{H} \ltimes \mathfrak{sl}_2$. $\mathfrak{G}$ has a triangular decomposition

$$\mathfrak{G} = \mathfrak{G}_+ \oplus \mathfrak{G}_0 \oplus \mathfrak{G}_-,$$

where $\mathfrak{G}_+ = \text{span}_\mathbb{C}\{e, p\}$, $\mathfrak{G}_0 = \text{span}_\mathbb{C}\{h, z\}$ and $\mathfrak{G}_- = \text{span}_\mathbb{C}\{f, q\}$.

The representation theory of Schrödinger algebra $\mathfrak{G}$ has attracted many authors’ attention. For example, using the technique of singular vectors, a classification of the irreducible lowest weight representations of the Schrödinger algebra is given in [7]; a classification of simple weight modules with finite dimensional weight spaces in [6]; And a classification of finite-dimensional indecomposable modules in [22, 23]. All the simple weight modules were classified in [13]. The irreducible representations of conformal Galilei algebras related with Schrödinger algebra in $l$-spatial dimension was studied in [12]. In special, in [24], the authors studied the Whittaker modules induced by the triangular decomposition (1.2) of $\mathfrak{G}$, simple Whittaker modules and related Whittaker vectors were determined.

In order to classify the simple modules over a family of finite dimensional solvable Lie algebras, in this paper, we generalize the Whittaker module for Schrödinger algebra $\mathfrak{G}$. Different from the method of Ref. [24], this type of modules are neither induced by its Borel subalgebra, nor by its parabolic subalgebra, but its Heisenberg subalgebra $\mathfrak{H}$. We call this kind of new modules as the quasi-Whittaker modules. We prove that, for a simple $\mathfrak{G}$-module, $V$ is a quasi-Whittaker module if and only if $V$ is a locally finite $\mathfrak{H}$-module. Using the elements with the action similar to the center elements in $U(\mathfrak{G})$ and their quasi-Whittaker vectors, we classify the simple quasi-Whittaker modules for Schrödinger algebra.
Without the help of triangular decomposition of the Schrödinger algebra, the methods and technique of dealing with the quasi-Whittaker module studied in this paper will be interesting.

The paper is organized as follows. In section 2, we define the quasi-Whittaker module induced by its Heisenberg subalgebra \( H \), give some notations and formulae and show that any simple quasi-Whittaker module is equivalence to simple the locally finite \( H \)-module. In section 3, we characterize the quasi-Whittaker vectors of the quasi-Whittaker module. In section 4, we classify the simple quasi-Whittaker modules for the Schrödinger algebra; In section 5, we describe arbitrary quasi-Whittaker modules with generating quasi-Whittaker vectors.

§2. Preliminaries

First of all, we give the basic definitions for this paper.

**Definition 2.1** Let \( \phi : H \to \mathbb{C} \) be any Lie algebra homomorphism. Let \( V \) be a \( S \)-module.

(i) A nonzero vector \( v \in V \) is called a quasi-Whittaker vector of type \( \phi \) if \( xv = \phi(x)v \) for all \( x \in H \).

(ii) \( V \) is called a quasi-Whittaker module for \( S \) of type \( \phi \) if \( V \) contains a cyclic quasi-Whittaker vector \( v \) of type \( \phi \).

**Remark 2.2** From the definition, it is easy to see that \( \phi(z) = 0 \).

**Lemma 2.3** If \( \phi \) is a zero homomorphism, then \( V \) is a simple quasi-Whittaker module of type \( \phi \) if and only if it is a simple \( \mathfrak{sl}_2 \)-module.

**Proof.** By definition, the necessary condition is clear, thus we just prove the sufficient condition.

Let \( w \) be the cyclic quasi-Whittaker vector of \( V \), then \( pw = qw = 0 \). Let \( W = \{ v \in V | pv = qv = 0 \} \). Since

\[
p(e^i h^j f^k) = q(e^i h^j f^k) = 0.
\]

\( W \) is a submodule of \( V \). Moreover, the actions of \( p \) and \( q \) commutes, thus \( W \) is nonzero. By the simplicity of \( V \), we know \( W = V \). Therefore \( pV = qV = 0 \). It is clear that \( zV = 0 \). This implies that \( V \) is a simple \( \mathfrak{sl}_2 \)-module. \( \square \)

**Definition 2.4** Let \( \phi : H \to \mathbb{C} \) be any nonzero Lie algebra homomorphism. Define a one-dimensional \( H \)-module \( C_\phi = Cw \) by \( pw = \phi(p)w, qw = \phi(q)w \). Then we get an induced module

\[
M_\phi = \mathcal{U}(S) \otimes_{\mathcal{U}(H)} C_\phi,
\]

which is called the universal quasi-Whittaker module of type \( \phi \) in the sense that for any quasi-Whittaker module \( V \) with quasi-Whittaker vector \( v \) of type \( \phi \), there is a unique surjective homomorphism \( f : M_\phi \to V \) such that \( uw \mapsto uv, \forall u \in \mathcal{U}(\mathfrak{sl}_2) \).
The following notations will be used to describe bases for $\mathcal{U}(\mathfrak{g})$ and for quasi-Whittaker modules. Fix $\phi : \mathcal{H} \to \mathbb{C}$ be any nonzero Lie algebra homomorphism, define the following elements in $\mathcal{U}(\mathfrak{g})$:

\[
X = \delta_{\phi(q),0}e + \delta_{\phi(p),0}f,
\]
\[
C = \phi(p)^2f - \phi(q)^2e - \phi(p)\phi(q)h,
\]
\[
P_+ = \delta_{\phi(p),0}p + \delta_{\phi(q),0}q,
\]
\[
P_- = \delta_{\phi(q),0}p + \delta_{\phi(p),0}q.
\]

**Remark 2.5** It is easy to see that $X, h, C$ forms a basis for $\mathfrak{sl}_2$ when $\phi$ is nonzero with $\phi(p)\phi(q) = 0$.

**Definition 2.6** Let $M_\phi$ be the universal quasi-Whittaker module with cyclic quasi-Whittaker vector $w$. For $\xi \in \mathbb{C}$, define a submodule $W_{\phi,\xi}$ as

\[
W_{\phi,\xi} = \mathcal{U}(\mathfrak{g})(C - \xi)w.
\]

And let

\[
L_{\phi,\xi} = M_\phi/W_{\phi,\xi}.
\]

Denote " − " the canonical projection from $M_\phi$ to $L_{\phi,\xi}$.

Recall that a locally finite module for a Lie algebra is defined as follow.

**Definition 2.7** Let $L$ be a Lie algebra. An $L$-module is called locally finite if any nonzero element $v$ in $V$ is contained in a finite dimensional submodule.

From the following lemma, we can identify any simple $\mathfrak{g}$-module on which $\mathcal{H}$ act locally finite as a quasi-Whittaker module.

**Theorem 2.8** Let $V$ be a simple $\mathfrak{g}$-module, then the following two conditions are equivalent.

(i) $V$ is a locally finite $\mathcal{H}$-module.

(ii) $V$ is a quasi-Whittaker module for $\mathfrak{g}$.

**Proof.** (ii) $\Rightarrow$ (i). This is clear by definitions.

(i) $\Rightarrow$ (ii). Take any nonzero $v_0 \in V$, then there exists a submodule $V_1 \subseteq V$ such that $V_1$ is a finite dimensional $\mathcal{H}$-module containing $v_0$. Note that $\mathcal{H}$ is solvable, following from Lie’s theorem, $p$ and $q$ have common eigenvector $v$ in $V_1$, and $zv = [p, q]v = 0$. Since $V$ is simple as $\mathfrak{g}$-module, we see that $V = \mathcal{U}(\mathfrak{g})v$ and $zV = 0$. By definition, $V$ is a quasi-Whittaker module for $\mathfrak{g}$. □

By (1.1), we can deduce by induction the following identities which will be used later.
Lemma 2.9 For any $m, n, k \in \mathbb{N}$, we have

$$p^n h^m = \sum_{i=0}^{m} \binom{m}{i} (-1)^i n^i h^{m-i} p^n,$$

(2.2)

$$q^n h^m = \sum_{i=0}^{m} \binom{m}{i} n^i h^{m-i} q^n,$$

(2.3)

$$pf^k = f^k p - kf^{k-1} q,$$

(2.4)

$$qe^k = e^k q - ke^{k-1} p.$$  

(2.5)

§3. Quasi-Whittaker Vectors in $M_\phi$ and $L_{\phi, \xi}$

In the following sections, we will discuss the reducibility of the quasi-Whittaker modules for the Schrödinger algebra. There are four conditions according to the choice of $\phi$ in Definition 2.1 that is, $\phi(p) = \phi(q) = 0$; $\phi(p) = 0$ while $\phi(q) \neq 0$; $\phi(p) \neq 0$ while $\phi(q) = 0$; $\phi(p) \neq 0$ and $\phi(q) \neq 0$. The simplest condition of the four cases is $\phi(p) = \phi(q) = 0$. In this case, from Lemma 2.3, any a $\mathfrak{sl}_2$-module is a quasi-Whittaker module. While the reducibility of $\mathfrak{sl}_2$-module was completely determined by R. Block in [3, 15]. Thus, in this paper, we do not deal with this condition.

Next we will discuss the quasi-Whittaker module for $\mathfrak{g}$ according to the remaining three conditions. When $\phi(p)\phi(q) = 0$, we take a basis for $M_\phi$ as $\{X^i h^j C^k w | i, j, k \in \mathbb{Z}_+\}$, while, when $\phi(p)\phi(q) \neq 0$, we take it as $\{h^i f^j C^k w\}$, where $w$ is a cyclic quasi-Whittaker vector of $M_\phi$.  

Lemma 3.1 For any $k \in \mathbb{Z}$, we have

$$(p - \phi(p)) C^k w = 0,$$

$$(q - \phi(q)) C^k w = 0.$$  

Thus, any vector in $\mathbb{C}[C]w$ is a quasi-Whittaker vector of type $\phi$.

Proof. The lemma follows from a direct computation. □

From (2.2)-(2.5), we know that

Lemma 3.2 Suppose $\phi(p)\phi(q) = 0$, then

$$ (P_+ - \phi(P_+))X^i = X^i P_+ - iX^{i-1} P_-,$$

(3.1)

$$ (P_- - \phi(P_-))h^m = \sum_{i=1}^{m} \binom{m}{i} (-1)^{(i\phi(p),0+1)i} h^{m-i} P_-.$$  

(3.2)
Lemma 3.3 Assume $\phi(p)\phi(q) = 0$. Suppose

$$x = \sum_{i=0}^{n} X^i \sum_{j=0}^{m} h^j a_{ij}(C)w \in M_\phi, a_{ij}(C) \in \mathbb{C}[C].$$

Then

$$(P_+ - \phi(P_+))^n x$$

$$= (-1)^n n! \sum_{j=0}^{m} P^n h^j a_{nj}(C)w$$

$$= (-1)^n n! (\phi(p) + \phi(q))^n \sum_{j=0}^{m} h^j b_j(C)w.$$ 

Proof. By (2.2)-(2.5), we see that

$$(P_+ - \phi(P_+))h^k C^i w = 0, \quad (3.3)$$

$$(P_+ - \phi(P_+))C^k w = 0. \quad (3.4)$$

Hence,

$$(P_+ - \phi(P_+))^n x = (P_+ - \phi(P_+))^{n-1} \sum_{i=0}^{n} P_+ X^i \sum_{j=0}^{m} h^j a_{ij}(C)w$$

$$= (P_+ - \phi(P_+))^{n-1} \sum_{i=0}^{n} (X^i P_+ - i X^{i-1} P_-) \sum_{j=0}^{m} h^j a_{ij}(C)w$$

$$= -(P_+ - \phi(P_+))^{n-1} \sum_{i=1}^{n} i X^{i-1} P_- \sum_{j=0}^{m} h^j a_{ij}(C)w$$

$$= (P_+ - \phi(P_+))^{n-2} \sum_{i=1}^{n} i(i - 1) X^{i-2} P_2 \sum_{j=0}^{m} h^j a_{ij}(C)w$$

$$= (-1)^n n! \sum_{j=0}^{m} P^2 h^j a_{nj}(C)w$$

$$= (-1)^n n! \sum_{j=0}^{m} \sum_{s=0}^{j} \binom{j}{s} h^{j-s} P^n a_{nj}(C)w$$

$$= (-1)^n n! (\phi(p) + \phi(q))^n \sum_{j=0}^{m} h^j b_j(C)w.$$ 

□

For later use, we also need the following lemmas.
Lemma 3.4 Assume that $\phi(p)\phi(q) = 0$.

$$(P_+ - \phi(P_-))^s(h^t b(C)w) = \begin{cases} 
0, & \text{if } s > t; \\
(\phi(p) + \phi(q))^t(-1)^{(\delta_{(p,q)}+1)t}t! b(C)w, & \text{if } s = t.
\end{cases}$$

where $b(C) \in \mathbb{C}[C]$.

**Proof.** We prove the second identity firstly. If $t = 1$, we have

$$
(P_+ - \phi(P_-))^2(h^t b(C)w) = (P_+ - \phi(P_-))((-1)^{\delta_{(p,q)}+1} P_-b(C)w) = (-1)^{\delta_{(p,q)}+1}\phi(P_-)(P_+ - \phi(P_-))(b(C)w) = 0.
$$

Hence, $(P_+ - \phi(P_-))^s(h^t b(C)w) = 0$, $\forall s > 1$. Assume that the identity holds for $t \leq k$, then for $t = k + 1$, we have

$$
(P_+ - \phi(P_-))^{k+2}(h^{k+1} b(C)w)
$$

$$
= (P_+ - \phi(P_-))^{k+1}\left(\sum_{i=1}^{k+1} \binom{k+1}{i}(-1)^{\delta_{(p,q)}+1}i h^{k+1-i} P_-b(C)w\right)
$$

$$
= (P_+ - \phi(P_-))^{k+1}(\phi(P_-)\sum_{i=1}^{k+1} \binom{k+1}{i}(-1)^{\delta_{(p,q)}+1}i h^{k+1-i} b(C)w)
$$

$$
= 0.
$$

Now we turn to prove the first identity by induction. It is easy to check that

$$(P_+ - \phi(P_-))(hb(C)w) = P_- b(C)w = \phi(P_-)b(C)w.
$$

Suppose the identity holds for $t \leq k$, then for $t = k + 1$, we have

$$
(P_+ - \phi(P_-))^{k+1}(h^{k+1} b(C)w)
$$

$$
= (P_+ - \phi(P_-))^{k}\left(\sum_{i=1}^{k+1} \binom{k+1}{i}(-1)^{\delta_{(p,q)}+1}i h^{k+1-i} P_-b(C)w\right)
$$

$$
= (P_+ - \phi(P_-))^{k}(\phi(P_-)\sum_{i=1}^{k+1} \binom{k+1}{i}(-1)^{\delta_{(p,q)}+1}i h^{k+1-i} b(C)w)
$$

$$
= (P_+ - \phi(P_-))^{k}((-1)^{\delta_{(p,q)}+1}(k+1)\phi(P_-)h^k b(C)w)
$$

$$
= \phi(P_-)^{k+1}(k+1)!(-1)^{\delta_{(p,q)}+1}(k+1)b(C)w
$$

$$
= (\phi(p) + \phi(q))^{k+1}(k+1)!(-1)^{\delta_{(p,q)}+1}(k+1)b(C)w.
$$

$\square$
Lemma 3.5 Assume that $\phi(p)\phi(q) \neq 0$. Then

$$(q - \phi(q))^{s}(h^t \sum_{i=0}^{m} f^i a_i(C)) = \begin{cases} 
0, & \text{if } s > t; \\
t!(\phi(q))^t \sum_{i=0}^{m} f^i a_i(C), & \text{if } s = t,
\end{cases}$$

where $a_i(C) \in \mathbb{C}[C], i = 1, \cdots, m$.

**Proof.** It is easy to check that $(q - \phi(q))f^i a(C)w = 0, \forall j \in \mathbb{Z}_{+}, a(C) \in \mathbb{C}[C]$. We use the induction on $t$. If $t = 1$, then

$$(q - \phi(q))(h \sum_{i=0}^{m} f^i a_i(C)w) = q \sum_{i=0}^{m} f^i a_i(C)w = \phi(q) \sum_{i=0}^{m} f^i a_i(C)w,$$

$$(q - \phi(q))^2(h \sum_{i=0}^{m} f^i a_i(C)w) = (q - \phi(q))(\phi(q) \sum_{i=0}^{m} f^i a_i(C)w) = 0.$$ 

Thus the identity holds for $t = 1$. Assume that the identity holds for $n = k$, then for $n = k + 1$, we have

$$(q - \phi(q))^{k+1}(h^{k+1} \sum_{i=0}^{m} f^i a_i(C)w) = (q - \phi(q))^k (\sum_{j=1}^{k+1} \binom{k+1}{j} h^{k+1-j} q \sum_{i=0}^{m} f^i a_i(C)w)$$

$$= (q - \phi(q))^k (\phi(q) \sum_{j=1}^{k+1} \binom{k+1}{j} h^{k+1-j} \sum_{i=0}^{m} f^i a_i(C)w)$$

$$= (q - \phi(q))^k (\phi(q)(k+1) h^k \sum_{i=0}^{m} f^i a_i(C)w)$$

$$= (k+1)! (\phi(q))^{k+1} \sum_{i=0}^{m} f^i a_i(C)w.$$

□

Lemma 3.6 Assume that $\phi(p)\phi(q) \neq 0$. Then

$$(p - \phi(p))^m(\sum_{i=0}^{m} f^i a_i(C)w) = (-1)^m m!(\phi(q))^m a_m(C)w.$$ 

**Proof.**

$$(p - \phi(p))^m(\sum_{i=0}^{m} f^i a_i(C)w) = (p - \phi(p))^{m-1}(\sum_{i=1}^{m} f^{i-1} q a_i(C)w)$$
\[(p - \phi(p))^{m-1}(-\phi(q) \sum_{i=1}^{m} i f^{i-1} a_i(C)w)\]

\[= (-1)^m m! (\phi(q))^m a_m(C)w.\]

By now, we can determine the quasi-Whittaker vectors for \(M_\phi\) and \(L_{\phi, \xi}\).

**Proposition 3.7** Let \(M_\phi\) be the universal quasi-Whittaker module generated by quasi-Whittaker vector \(w\). Suppose \(w' \in M_\phi\) is a quasi-Whittaker vector, then \(w' \in \mathbb{C}[C]w\).

**Proof.** (1) Assume that \(\phi(p)\phi(q) = 0\). Suppose

\[w' = \sum_{i=0}^{n} X^i \sum_{j=0}^{m} h^j a_{ij}(C)w\]

with \(a_{ij}(C) \in \mathbb{C}[C]\). Then by Lemma 3.3, we see that if \((P_+ - \phi(P_+))w' = 0\), then \(n = 0\), that is \(w' = \sum_{j=0}^{m} h^j a_j(C)w\). Applying Lemma 3.4 we get \(m = 0\). Thus, we have \(w' = b(C)w \in \mathbb{C}[C]w\). From Lemma 3.1 we know the proposition holds.

(2) If \(\phi(p)\phi(q) \neq 0\), we may assume that

\[w' = \sum_{i=0}^{m} h^i \sum_{j=0}^{n} f^j a_{ij}(C)w\]

with \(a_{ij}(C) \in \mathbb{C}[C]\). Then by Lemma 3.5 we see that if \((q - \phi(q))w' = 0\), then \(m = 0\), that is \(w' = \sum_{j=0}^{n} f^j a_j(C)w\). Applying Lemma 3.6 we get \(n = 0\). Thus, we have \(w' = b(C)w \in \mathbb{C}[C]w\). From Lemma 3.1 we know the proposition holds. \(\square\)

**Proposition 3.8** Let \(w\) be the cyclic quasi-Whittaker vector of \(M_\phi\), \(\bar{w} \in L_{\phi, \xi}\). If \(w' \in L_{\phi, \xi}\) is a quasi-Whittaker vector, then \(w' = cw\) for some \(c \in \mathbb{C}\).

**Proof.** We only show the statement for \(\phi(p)\phi(q) = 0\) since the proof is similar when \(\phi(p)\phi(q) \neq 0\). Note that \(\{X^i h^j \bar{w} | i, j \in \mathbb{Z}_+\}\) spans \(L_{\phi, \xi}\). We claim that this set is linearly independent and thus a basis for \(L_{\phi, \xi}\). To check this, suppose

\[0 = \sum_{i,j} a_{ij} X^i h^j \bar{w} = \sum_{i,j} a_{ij} X^i h^j \bar{w}.\]
Then \( \sum_{i,j} a_{ij} X^i h^j w \in \mathcal{U}(\mathfrak{g})(C - \xi)w \), and so

\[
\sum_{i,j} a_{ij} X^i h^j w = \sum_{i,j} \sum_{k=0}^{m} b_{ij}^k X^i h^j C^k (C - \xi)w
\]

for some \( m \in \mathbb{Z}_{>0} \) and \( b_{ij}^k \in \mathbb{C} \). This expression can be rewritten as

\[
\sum_{i,j} (a_{ij} + \xi b_{ij}^0) X^i h^j w + \sum_{i,j} \sum_{k=1}^{m} (\xi b_{ij}^k - b_{ij}^{k-1}) X^i h^j C^k w - \sum_{i,j} b_{ij}^m X^i h^j C^{m+1} w = 0.
\]

From this we conclude that \( b_{ij}^m = 0, \xi b_{ij}^k - b_{ij}^{k-1} = 0, a_{ij} + \xi b_{ij}^0 = 0, \) and thus \( a_{ij} = 0 \) for all \( i, j \).

With this fact now established, it is possible to use the same argument as in Proposition 3.7 to complete the proof. \( \square \)

§4. Simple Quasi-Whittaker modules for \( \mathfrak{g} \)

In this section we will determine all simple quasi-Whittaker modules of type \( \phi \), up to isomorphism. After this, following from Theorem 2.8 we can classify all simple \( \mathfrak{g} \)-modules on which \( \mathcal{H} \) acts locally finite.

**Theorem 4.1** Let \( V \) be a quasi-Whittaker module for \( \mathfrak{g} \), and let \( W \subseteq V \) be a nonzero submodule. Then there is a nonzero quasi-Whittaker vector \( w' \in W \).

**Proof.** Case1. If \( \phi(p)\phi(q) = 0 \), let

\[
x = \sum_{i=0}^{n} X^i \sum_{j=0}^{m} h^j a_{ij}(C)w \in W.
\]

Then by Lemma 3.3, we see that \( \sum_{j=0}^{m} h^j b_j(C)w \in W \). Applying Lemma 3.4, we deduce that \( W \) contains an element \( w' = b(C)w \neq 0 \). From Proposition 3.7, we know that \( w' \) is a quasi-Whittaker vector.

Case2. If \( \phi(p)\phi(q) \neq 0 \), let

\[
x = \sum_{i=0}^{n} h^i \sum_{j=0}^{m} f^j a_{ij}(C)w \in W.
\]

Then by Lemma 3.5, we see that \( \sum_{j=0}^{m} f^j b_j(C)w \in W \). Applying Lemma 3.6, we deduce that \( W \) contains an element \( w' = b(C)w \neq 0 \). From Proposition 3.7, we know that \( w' \) is a quasi-Whittaker vector. \( \square \)
Proposition 4.2 For any $\xi \in \mathbb{C}$, $W_{\phi, \xi}$ is a maximal submodule of $M_\phi$, hence $L_{\phi, \xi}$ is simple.

Proof. It is clear that $W_{\phi, \xi}$ is a proper submodule of $M_\phi$.

Case 1. If $\phi(p)\phi(q) = 0$, then for any $x \not\in W_{\phi, \xi}$, we see that

$$x \equiv \sum_{i,j} X^i h^j w (\mod W_{\phi, \xi}).$$

Let $x' = \sum_{i,j} X^i h^j w$, by Lemma 3.3 and Lemma 3.4 we can deduce that $w \in \mathcal{U}(\mathfrak{g})x' + W_{\phi, \xi}$. Hence, $W_{\phi, \xi}$ is a maximal submodule of $M_\phi$.

Case 2. If $\phi(p)\phi(q) \neq 0$, then for any $x \not\in W_{\phi, \xi}$, we see that

$$x \equiv \sum_{i,j} h^i f^j w (\mod W_{\phi, \xi}).$$

Let $x' = \sum_{i,j} h^i f^j w$, by Lemma 3.5 and Lemma 3.6 we can deduce that $w \in \mathcal{U}(\mathfrak{g})x' + W_{\phi, \xi}$. Hence, $W_{\phi, \xi}$ is a maximal submodule of $M_\phi$. □

Theorem 4.3 Let $\phi : \mathcal{H} \to \mathbb{C}$ be a Lie algebra homomorphism such that $\phi(p) \neq 0$, or $\phi(q) \neq 0$, and let $V$ be a simple quasi-Whittaker module of type $\phi$ for $\mathfrak{g}$. Then $V \cong L_{\phi, \xi}$ for some $\xi \in \mathbb{C}$.

Proof. Let $w_1$ be a cyclic quasi-Whittaker vector corresponding to $\phi$. From [13], we know that

$$[\mathcal{U}(\mathfrak{g}), p^2 f - q^2 e - hpq]V = 0.$$ By Schur’s lemma, we have $C_0 = p^2 f - q^2 e - hpq$ acts on $V$ as a scalar $\xi$. Then $\xi w_1 = C_0 w_1 = C w_1$. Thus $C$ acts on $w$ as by the scalar $\xi$. Now by the universal property of $M_\phi$, there exists a homomorphism $\varphi : M_\phi \to W$ with $uw \mapsto uw_1$. This is a surjective map since $W$ is generated by $w_1$. But $\varphi(W_{\phi, \xi}) = \mathcal{U}(\mathfrak{g})(C - \xi)w_1 = 0$, so we have

$$W_{\phi, \xi} \subseteq \text{Ker} \varphi \subseteq M_\phi.$$ By Proposition 4.2 $W_{\phi, \xi}$ is maximal, hence $\text{Ker} \varphi = W_{\phi, \xi}$. That is $W \cong L_{\phi, \xi}$. □

Note that if $w$ is a quasi-Whittaker vector of type $\phi$, then we have $Cw = C_0 w$. Hence, in the theorems and properties above, $C$ can be replaced by $C_0$. For the remaining part of this section, we will discuss the annihilator of quasi-Whittaker vectors for simple quasi-Whittaker modules.

Lemma 4.4 Fix $\phi : \mathcal{H} \to \mathbb{C}$. Define the left ideal $L$ of $\mathcal{U}(\mathfrak{g})$ by $L = \mathcal{U}(\mathfrak{g})(C_0 - \xi 1) + \mathcal{U}(\mathfrak{g})(p - \phi(p)1) + \mathcal{U}(\mathfrak{g})(q - \phi(q)1)$, and regard $V = \mathcal{U}(\mathfrak{g})/L$ as a left $\mathcal{U}(\mathfrak{g})$-module. Then $V \cong L_{\phi, \xi}$, and thus $V$ is simple.
Proof. For $u \in \mathcal{U}(\mathfrak{g})$, let $\bar{u} = u + L \in \mathcal{U}(\mathfrak{g})/L$. Then we may regard $V$ as a quasi-Whittaker module of type $\phi$ with cyclic quasi-Whittaker vector $\bar{1}$. By the universal property of $M_\phi$, there exists a homomorphism $\varphi : M_\phi \to V$ with $uw \mapsto u\bar{1}$. This is a surjective map since $\bar{1}$ is the generator of $V$. However, for any $u(C_0 - \xi)w \in W_{\phi, \xi}$, we have $\varphi(u(C_0 - \xi)w) = u(C_0 - \xi)\bar{1} = 0$. Hence,

$$W_{\phi, \xi} \subseteq \text{Ker} \varphi \subseteq M_\phi.$$ 

Since $W_{\phi, \xi}$ is maximal, it follows that $V \cong M_\phi/\text{Ker} \varphi \cong L_{\phi, \xi}$. □

Proposition 4.5 Let $V$ be a quasi-Whittaker module of type $\phi$ such that $C_0$ acts on the cyclic quasi-Whittaker vector by the scalar $\xi \in \mathbb{C}$. Then $V$ is simple. Moreover, if $w$ is a cyclic quasi-Whittaker vector for $V$, then

$$\text{Ann}_{\mathcal{U}(\mathfrak{g})}(w) = \mathcal{U}(\mathfrak{g})(C_0 - \xi 1) + \mathcal{U}(\mathfrak{g})(p - \phi(p)1) + \mathcal{U}(\mathfrak{g})(q - \phi(q)1).$$

Proof. Let $K$ denote the kernel of the natural surjective map $\mathcal{U}(\mathfrak{g}) \to V$ given by $u \mapsto uw$. Then $K$ is a proper left ideal containing

$$L = \mathcal{U}(\mathfrak{g})(C_0 - \xi 1) + \mathcal{U}(\mathfrak{g})(p - \phi(p)1) + \mathcal{U}(\mathfrak{g})(q - \phi(q)1).$$

By Lemma 4.4, $L$ is maximal, thus $K = L$ and $V \cong \mathcal{U}(\mathfrak{g})/L$ is simple. □

§5. Arbitrary Quasi-Whittaker Modules

In this section, we always assume that $V$ is any a quasi-Whittaker module of type $\phi$ with cyclic quasi-Whittaker vector $w$, similar to [18], we will describe its reducibility and its quasi-Whittaker vectors in terms of $\text{Ann}_{\mathcal{U}[C_0]}(w)$.

Lemma 5.1 Assume that $\text{Ann}_{\mathcal{U}[C_0]}(w) = (C_0 - \xi 1)^k$ for some $k > 0$, and the submodules sequence

$$V = V_0 \supseteq V_1 \supseteq \cdots \supseteq V_k = 0$$

are defined by $V_i = \mathcal{U}(\mathfrak{g})(C_0 - \xi 1)^i w$. Then (i) $V_i$ is a quasi-Whittaker module of type $\phi$, with cyclic quasi-Whittaker vector $w_i = (C_0 - \xi 1)^i w$ and (ii) $V_i/V_{i+1}$ is simple for $0 \leq i < k$. Particularly, (iii) the submodules $V_0, \ldots, V_k$ are the only submodules of $V$.

Proof. It is easy to see that $V_i$ is a quasi-Whittaker module with cyclic quasi-Whittaker vector $w_i$ by (3.2) and (3.4). Obviously, $V_i/V_{i+1}$ is a quasi-Whittaker module of type $\phi$ with quasi-Whittaker vector $\bar{w}_i$. Since $C_0$ acts by the scalar on $\bar{w}_i$, following from Proposition 4.5, we deduce that $V_i/V_{i+1}$ is simple, and thus isomorphic to $L_{\phi, \xi}$. Thus $V_0 \supseteq V_1 \supseteq \cdots \supseteq V_k$ form a composition series for $V$, and any simple subquotient of $V$ is isomorphic to $V_i/V_{i+1} \cong L_{\phi, \xi}$.

Let $M$ be any maximal submodule of $V$, then $V/M$ is a simple quasi-Whittaker module of type $\phi$ with quasi-Whittaker vector $\bar{w}$. By Theorem 4.3, we see that $C_0$ acts on $w$ by
some scalar $\kappa \in \mathbb{C}$. While, $(C_0 - \xi_1)^k$ acts as 0 on $w$, and therefore on $\bar{w}$. So we have $(\kappa - \xi)^k w \in M$. Since $w \not\in M$, we deduce that $\kappa = \xi$. It follows that $(C_0 - \xi_1)w \in M$, thus $V_1 = \mathcal{U}(\mathfrak{g})(C_0 - \xi_1)w \subseteq M$. Since $V_i$ is a maximal submodule of $M$, we get $V_1 = M$. Similarly, we can show that $V_{i+1}$ is the unique maximal submodule of $V_i$ for every $i < k$. Thus $V_i$ for $1 \leq i \leq k$ are the only submodules of $V$. \hfill \Box

\textbf{Theorem 5.2} Assume that $\text{Ann}_{\mathbb{C}[C_0]}(w) \neq 0$ and $d(C_0) = \prod_{i=1}^{k} (C_0 - \xi_i)\alpha_i$ for distinct $\xi_1, \ldots, \xi_k \in \mathbb{C}$ be the unique monic generator of $\text{Ann}_{\mathbb{C}[C_0]}(w)$ in $\mathbb{C}[C_0]$,

(i) Define $V_{\phi, \xi} = \mathcal{U}(\mathfrak{g})(C_0 - \xi_i)w, i = 1, \ldots, k$, then $V_{\phi, \xi_1}, \ldots, V_{\phi, \xi_k}$ are the only maximal submodules of $V$.

(ii) Define

$$w_j = d_j(C_0)w, \text{ where } d_j(C_0) = \prod_{i \neq j} (C_0 - \xi_i)\alpha_i, \text{ and } V_j = \mathcal{U}(\mathfrak{g})w_j.$$

Then $V_i$ is a quasi-Whittaker module of type $\phi$ with cyclic quasi-Whittaker vector $w_i$ and $V = V_1 \oplus \cdots \oplus V_k$. Furthermore, the submodules $V_1, \ldots, V_k$ are indecomposable; $V_j$ is simple if and only if $\alpha_j = 1$; and $\alpha_j$ is the composition length of $V_j$. In particular, $V$ has a composition series of length $\sum_{i=1}^{k} \alpha_i$.

\textbf{Proof.} (i) Let $M$ be a maximal submodule of $V$, then $V/M$ is simple and thus $C$ acts on $w$ by some scalar $\kappa \in \mathbb{C}$. On the other hand, $d(C)$ acts as 0 on $w$, and therefore on $\bar{w}$. Thus $d(\kappa) = 0$, which implies that $\kappa = \xi$ for some $1 \leq i \leq k$. Hence,

$$V_{\phi, \xi} = \mathcal{U}(\mathfrak{g})(C - \xi_i)w \subseteq M.$$

Since $V_{\phi, \xi}$ is the image of the maximal submodule $W_{\phi, \xi}$ under the epimorphism $M_{\phi} \to V$, $V_{\phi, \xi}$ is maximal. Thus, $M = V_{\phi, \xi}$.

(ii) Firstly, we show that: $V = V_1 + \cdots + V_k$. Since $\gcd(d_1, \ldots, d_k) = 1$, there exist polynomials $r_1(C_0), \ldots, r_k(C_0) \in \mathbb{C}[C_0]$ such that $\sum_{i=1}^{k} r_i(C_0)d_i(C_0) = 1$. Therefore,

$$w = 1w = (\sum_{i=1}^{k} r_i(C_0)d_i(C_0))w \in V_1 + \cdots + V_k.$$

To show that the sum $V = V_1 + \cdots + V_k$ is direct, note that for $i \neq j, d(C_0)$ is a factor of $d_i(C_0)d_j(C_0)$, which implies that $d_j(C_0)w_i = 0$. Following from this, we have

$$w_i = 1w_i.$$
Suppose that \( u_1w_1 + \cdots + u_kw_k = 0 \) for \( u_1, \ldots, u_k \in U(\mathcal{G}) \), then
\[
0 = r_i(C_0)d_i(C_0)\left(\sum_{j=1}^{k} u_jw_j\right) = u_ir_i(C_0)d_i(C_0)w_i = u_iw_i.
\]

Thus, the sum is direct.

To finish the proof, by Lemma 5.1 we know that the submodules \( V_1, \ldots, V_k \) are indecomposable with the stated composition length.

If we consider the annihilator of \( w \) in \( \mathbb{C}[C] \), then we can get similar result as follow:

**Theorem 5.2** Assume that \( \text{Ann}_{\mathbb{C}[C]}(w) \neq 0 \) and \( d(C) = \prod_{i=1}^{k} (C - \xi_i)^{a_i} \) for distinct \( \xi_1, \ldots, \xi_k \in \mathbb{C} \) be the unique monic generator of the ideal \( \text{Ann}_{\mathbb{C}[C]}(w) \) in \( \mathbb{C}[C] \).

(i) Define \( V_{\phi,\xi_i} = U(\mathcal{G})(C - \xi_i)w, i = 1, \ldots, k \), then \( V_{\phi,\xi_1}, \ldots, V_{\phi,\xi_k} \) are the only maximal submodules of \( V \).

(ii) Let \( n = \deg d(C) \), then \( V \) has a unique composition series (up to permutation): \( V = V_0 \supseteq V_1 \supseteq \cdots \supseteq V_n = 0 \) with \( V_i/V_{i+1} \cong L_{\phi,\xi_j} \) for some \( j = 1, \ldots, k \). And the composition factors are \( a_i \) copies of \( L_{\phi,\xi_i}, i = 1, \ldots, k \).

However, we can not decompose \( V \) as a direct sum of submodules using \( \text{Ann}_{\mathbb{C}[C]}(w) \).

**Corollary 5.3** Assume that \( \text{Ann}_{\mathbb{C}[C_0]}(w) \neq 0 \) and \( d(C_0) \) be the unique monic generator of \( \text{Ann}_{\mathbb{C}[C_0]}(w) \). Then
\[
\text{Ann}_{U(\mathcal{G})}(w) = U(\mathcal{G})d(C_0) + U(\mathcal{G})(p - \phi(p)1) + U(\mathcal{G})(q - \phi(q)1).
\]

**Proof.** We use induction on the composition length \( n \) of \( V \) (or equivalently, the degree of \( d(C_0) \)). If \( n = 1 \), then \( d(C_0) = C_0 - \xi \), thus \( V \) is simple, therefore the result is true by Proposition 4.5. Assume that \( n > 1 \), write \( d(C_0) = (C_0 - \xi)d'(C_0) \) for some \( \xi \in \mathbb{C} \) and some monic polynomial \( d'(C_0) \in \mathbb{C}[C_0] \) with positive degree. Then \( w' = (C_0 - \xi)w \neq 0 \).

Let \( V' = U(\mathcal{G})w' \subseteq V \). Then \( V' \) is a quasi-Whittaker module with cyclic quasi-Whittaker vector \( w' \), and \( \text{Ann}_{\mathbb{C}[C_0]}(w') = \mathbb{C}[C_0]d'(C_0) \). Theorem 5.2 therefore implies that the composition length of \( V' \) is \( n - 1 \), and the induction hypothesis deduce that
\[
\text{Ann}_{\mathbb{C}[C_0]}(w') = U(\mathcal{G})d'(C_0) + U(\mathcal{G})(p - \phi(p)1) + U(\mathcal{G})(q - \phi(q)1).
\]

Let \( \tilde{w} = w + V' \in V/V' \), and note that \( \text{Ann}_{\mathbb{C}[C_0]}(\tilde{w}) = \mathbb{C}[C_0](C_0 - \xi) \).
Let \( u \in \text{Ann}_U(\mathfrak{g})(w) \). Since \( \text{Ann}_U(\mathfrak{g})(w) \subseteq \text{Ann}_U(\mathfrak{g})(\bar{w}) \), following from proposition 4.5, we have

\[
\begin{align*}
    u &= u_0(C_0 - \xi 1) + u_1(p - \phi(p)) + u_2(q - \phi(q)) \\
    &\in U(\mathfrak{g})(C - \xi 1) + U(\mathfrak{g})(p - \phi(p)) + U(\mathfrak{g})(q - \phi(q)).
\end{align*}
\]  
(5.1)

While \( u_1(p - \phi(p)) + u_2(q - \phi(q)) \in \text{Ann}_U(\mathfrak{g})(w) \), thus \( u_0(C_0 - \xi 1) \in \text{Ann}_U(\mathfrak{g})(w) \). Observe that \( 0 = u_0(C_0 - \xi 1)w = u_0 w' \), we have

\[
\begin{align*}
    u_0 &\in \text{Ann}_U(\mathfrak{g})(w') = U(\mathfrak{g})d'(C_0) + U(\mathfrak{g})(p - \phi(p) 1) + U(\mathfrak{g})(q - \phi(q) 1).
\end{align*}
\]

(5.1) implies that \( u \) has the required form. \( \square \)

**Theorem 5.4** Let \( M_\phi \) be the universal quasi-Whittaker module of type \( \phi \) with cyclic quasi-Whittaker vector \( w \). If \( U \subseteq M_\phi \) is a submodule, then \( U \cong M_\phi \). Furthermore, \( V \) is generated by a quasi-Whittaker vector of the form \( d(C)w \) for some \( d(x) \in \mathbb{C}[x] \).

To prove this theorem, we need the following lemma.

**Lemma 5.5** If \( \text{Ann}_{\mathbb{C}[C_0]}(w) = 0 \), then \( V \cong M_\phi \).

**Proof.** By the universal property of \( M_\phi \), there exists an epimorphism \( \varphi : M_\phi \to V \). It is clear that \( \text{Ker}\varphi \) is a submodule of \( M_\phi \). If \( \text{Ker}\varphi \neq 0 \), then Theorem 4.1 implies that there is a nonzero quasi-Whittaker vector \( w' \in \text{Ker}\varphi \). It follows from Proposition 3.7, \( 0 \neq w' = d(C_0)1 \otimes 1 \) and thus \( 0 \neq d(C_0) \in \text{Ann}_{\mathbb{C}[C_0]}(w) \), which is impossible. Therefore, \( \text{Ker}\varphi \) must be 0 and \( \varphi \) is an isomorphism. \( \square \)

**Proof of Theorem 5.4** Since \( W_{\phi,\xi} \) is a maximal submodule and a quasi-Whittaker module of type \( \phi \) with cyclic quasi-Whittaker vector \( (C_0 - \xi)w \), using Lemma 5.5, we obtain that \( W_{\phi,\xi} \cong M_\phi \).

By Proposition 3.7 and Theorem 4.1, we know that \( V \) contains a quasi-Whittaker vector \( d(C_0)w \). By Lemma 5.5, we have \( U(\mathfrak{g})d(C_0)w \cong M_\phi \). Since \( d(C_0) \) can be written as a product of linear factors, from the above we know that there exists a chain of universal quasi-Whittaker modules between \( U(\mathfrak{g})d(C_0)w \) and \( M_\phi \) satisfying that each quotient is simple. Thus \( V \) must be one of the submodules in the chain, thus \( V \) is a universal quasi-Whittaker module of type \( \phi \). \( \square \)

**Theorem 5.6** The set of quasi-Whittaker vectors in \( V \) is \( \mathbb{C}[C_0]w \).

**Proof.** If \( \text{Ann}_{\mathbb{C}[C_0]}(w) = 0 \), using Proposition 3.7 and Lemma 5.5, we know that the conclusion holds. If \( \text{Ann}_{\mathbb{C}[C_0]}(w) \neq 0 \), by Theorem 5.2, we obtain that \( V \) has finite composition length \( n \).
Next we induce on \( n \). If \( n = 1 \), then \( V \) is simple, using Proposition 3.8 we obtain the conclusion. Suppose that \( V \) is a module with arbitrary composition length \( n \). Let

\[
V = V_0 \supseteq V_1 \supseteq \cdots \supseteq V_n = 0
\]

be a composition series of \( V \), and assume \( V_1 \) has cyclic quasi-Whittaker vector \( w_1 = (C_0 - \xi_1)w \). Let \( w' \in V \) be a quasi-Whittaker vector. Since \( V/V_1 \) is simple, by Proposition 3.8 we obtain that the image of \( w' \) in \( V/V_1 \) is a scalar multiple of \( \bar{w} \). Therefore, in \( V \), \( w' = cw + w'' \) for some \( c \in \mathbb{C} \) and \( w'' \in V_1 \). Note that \( w'' = w' - cw \) is also a quasi-Whittaker vector. Since \( V_1 \) has composition length \( n - 1 \), by induction, we have

\[
w'' = d(C_0)w_1 = d(C_0)(C_0 - \xi_1)w \text{ for some } d(C_0) \in \mathbb{C}[C_0].
\]

Therefore, \( w' = cw + d(C_0)(C_0 - \xi_1)w \), the statement holds. \( \square \)

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