Approximate Spielman-Teng theorems for random matrices with heavy-tailed entries: a combinatorial view

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Abstract

This paper makes two contributions to the areas of anti-concentration and non-asymptotic random matrix theory. First, we study the counting problem in inverse Littlewood-Offord theory for general random variables: for random variables $\xi_1, \ldots, \xi_n$ which are i.i.d. copies of a random variable $\xi$ (satisfying some mild hypotheses), how many integer vectors $a := (a_1, \ldots, a_n)$ in a prescribed box have large $\rho(a) := \sup_{x \in \mathbb{R}} \Pr(\sum_{i=1}^n \xi_i a_i \in B(x,1))$? Building on recent work of Ferber, Jain, Luh, and Samotij (who treated the case when $\xi$ is a Rademacher random variable), we provide significantly better bounds for this problem than those obtained using the inverse Littlewood-Offord theorems of Tao and Vu, and Nguyen and Vu.

Next, we study the non-asymptotic behavior of the least singular value $s_n(M_n)$ of a random $n \times n$ square matrix $M_n$ with i.i.d. entries, which are only assumed to be centered with variance 1. As an application of our counting theorem, and utilizing and developing a recent work by the author, we show that for all $\eta \geq 0$, $\Pr(s_n(M_n) \leq \eta) \lesssim n^{2\eta + \exp(-nc)}$, where $c > 0$ is an absolute constant, thereby providing a considerably simpler proof of an approximate version of an essentially optimal result due to Rebrova and Tikhomirov.

1 Introduction

1.1 The counting problem in inverse Littlewood-Offord theory

In its simplest form, the so-called Littlewood-Offord problem, first raised by Littlewood and Offord in [12] asks the following question. Let $a := (a_1, \ldots, a_n) \in (\mathbb{Z} \setminus \{0\})^n$ and let $\epsilon_1, \ldots, \epsilon_n$ be independent and identically distributed (i.i.d.) Rademacher random variables, i.e., each $\epsilon_i$ independently takes values $\pm 1$ with probability $1/2$ each. Estimate the largest atom probability $\rho(a)$, which is defined by

$$\rho(a) := \sup_{x \in \mathbb{Z}} \Pr(\epsilon_1 a_1 + \cdots + \epsilon_n a_n = x).$$

Littlewood and Offord showed that $\rho(a) = O(n^{-1/2} \log n)$. Soon after, Erdős [1] gave an elegant combinatorial proof of the refinement $\rho(a) \leq \binom{n}{\lfloor n/2 \rfloor} / 2^n = O(n^{-1/2})$, which is tight, as is readily seen by taking $a$ to be the all ones vector. These classic results of Littlewood-Offord and Erdős generated a lot of activity around this problem in various directions: higher-dimensional generalizations e.g. [9, 10]; better upper bounds on $\rho(a)$ given additional hypotheses on $a$ e.g. [2, 6, 24]; and obtaining similar results with the Rademacher distribution replaced by more general distributions e.g. [3, 6].

A new view was brought to the Littlewood-Offord problem by Tao and Vu [29, 27] who, guided by inverse theorems from additive combinatorics, tried to find the underlying reason why $\rho(a)$ could be large. They used deep Freiman-type results from additive combinatorics to show that, roughly speaking, the only reason for a vector $a$ to have $\rho(a)$ only polynomially small is that most

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coordinates of \( \mathbf{a} \) belong to a generalized arithmetic progression (GAP) of ‘small rank’ and ‘small volume’. Their results were subsequently sharpened by Nguyen and Vu [16], who proved an ‘optimal inverse Littlewood–Offord theorem’. We refer the reader to the survey [17] and the textbook [28] for complete definitions and statements, and much more on both forward and inverse Littlewood-Offord theory.

Recently, motivated by applications, especially those in random matrix theory, the following counting variant of the inverse Littlewood–Offord problem was isolated in work [5] of the author along with Ferber, Luh, and Samotij: for how many vectors \( \mathbf{a} \) in a given collection \( \mathcal{A} \subseteq \mathbb{Z}^n \) is the largest atom probability \( \rho(\mathbf{a}) \) greater than some prescribed value? The utility of such results is that they enable various union bound arguments, as one can control the number of terms in the relevant union/sum. In fact, the inverse Littlewood-Offord theorems are typically used precisely through such counting corollaries [17]. One of the main contributions of [5] was to show that one may obtain useful bounds for the counting variant of the inverse Littlewood-Offord problem directly, without providing a precise structural characterization like Tao-Vu. Not only does this approach make certain arguments considerably simpler, it also provides better quantitative bounds for the counting problem, since it is not hampered by losses coming from the black-box application of various theorems from additive combinatorics. In [5, 4, 7], this work was utilized to provide quantitative improvements for several problems in combinatorial random matrix theory.

A natural question left open by this line of work is whether one can adapt the strategy of [5] to study random matrices whose entries have ‘continuous’ distributions as well. We note that the inverse Littlewood-Offord theorems in [16, 27] are indeed applicable to these more general settings. However, since the proofs in [5] proceed by viewing (bounded) integer-valued random variables as random variables valued in \( \mathbb{F}_p \) for sufficiently large \( p \), it is not immediately clear whether the combinatorial techniques there can be extended. As our first main result (Theorem 1.4), we show that the combinatorial arguments of [5] can be combined with (the dual of) the Fourier-analytic arguments in [27, 16] to prove a counting result for very general distributions.

**Definition 1.1** (Lévy concentration function). Let \( z \) be an arbitrary random variable, and let \( \mathbf{v} := (v_1, \ldots, v_n) \in \mathbb{R}^n \). We define the Lévy concentration function of \( \mathbf{v} \) at radius \( r \) with respect to \( z \) by

\[
\rho_{r,z}(\mathbf{v}) := \sup_{x \in \mathbb{R}} \Pr(v_1 z_1 + \cdots + v_n z_n \in B(x, r)),
\]

where \( z_1, \ldots, z_n \) are independent copies of \( z \).

**Remark 1.2.** In particular, note that \( \rho_{r,z}(1) = \sup_{x \in \mathbb{R}} \Pr(z \in B(x, r)) \). We will use this notation repeatedly.

The next definition isolates the class of random variables we will be concerned with.

**Definition 1.3.** We say that a random variable \( z \) is C-good if

\[
\Pr(C^{-1} \leq |z_1 - z_2| \leq C) \geq C^{-1},
\]

(1)

where \( z_1 \) and \( z_2 \) denote independent copies of \( z \). The smallest \( C \geq 1 \) with respect to which \( z \) is C-good will be denoted by \( C_z \).

We can now state our counting theorem.

**Theorem 1.4.** Let \( z \) be a C_\mathcal{C}_z-good random variable. For \( \rho \in (0, 1) \) (possibly depending on \( n \)), let

\[
V_{\rho} := \{ \mathbf{v} \in \mathbb{Z}^n : \rho_{1,z}(\mathbf{v}) \geq \rho \}.
\]
There exists a constant $C_{1.4} \geq 1$, depending only on $C_z$, for which the following holds. Let $n, s, k \in \mathbb{N}$ with $k \leq \sqrt{s} \leq s \leq n/\log n$. If $\rho \geq C_{1.4} \max \{e^{-s/k}, s^{-k/4}\}$ and $p$ is an odd prime such that $2^{n/s} \geq p \geq C_{1.4} p^{-1}$, then
\[
|\varphi_p(V_{\rho})| \leq \left( \frac{5np}{s} \right)^s + \left( \frac{C_{1.4} p^{-1}}{\sqrt{s/k}} \right)^n,
\]
where $\varphi_p$ denotes the natural map from $\mathbb{Z}^n \to \mathbb{F}_p^n$.

The inverse Littlewood-Offord theorems may be used to deduce similar statements, provided we further assume that $\rho \geq n^{-C}$ for some constant $C > 0$. The freedom of taking $\rho$ to be much smaller allows us to use Theorem 1.4 to prove approximate Spielman-Teng theorems for random matrices, as we discuss in the next subsection.

1.2 The least singular value of i.i.d. heavy-tailed matrices

Recall that the singular values of an $n \times n$ matrix $M_n$, which we will denote by $s_k(M_n)$ for $k \in [n]$, are the eigenvalues of $\sqrt{M_n^T M_n}$ arranged in non-decreasing order. Here, we will be interested in the non-limiting or non-asymptotic behavior of the smallest singular value of a random matrix. This problem has a rich history, and plays a crucial role in diverse areas of mathematics (see, e.g., the surveys [17, 23, 30] and the books [26, 28] for a detailed account of the development of the subject).

Following [13, 20], it was shown in the landmark work of Rudelson and Vershynin [21] that for an $n \times n$ random matrix $M_n$ with i.i.d. centered subgaussian entries,
\[
\Pr (s_n(M_n) \leq \varepsilon n^{-1/2}) \leq C_{\varepsilon} + c^n, \tag{2}
\]
where $C > 0$ and $c \in (0, 1)$ are constants depending only on the subgaussian moment of the entries of $M_n$; this essentially confirmed a conjecture of Spielman and Teng [25], and is best possible up to the constants $C, c$.

The approach of Rudelson-Vershynin crucially used the fact that random matrices with subgaussian entries have well-controlled operator norm with very high probability. As such control is provably not available if the entries of $M_n$ do not have finite fourth moment [11], it remained open for a while whether a result similar to Equation (2) could be obtained for random matrices whose entries have heavier tails. This problem was resolved in recent work of Rebrova and Tikhomirov [18], who showed that Equation (2) holds even if the entries of $M_n$ are only assumed to be centered i.i.d. random variables with finite non-zero variance. The proof of Rebrova-Tikhomirov combines the general geometric framework of Rudelson-Vershynin with an ingenious, but very sophisticated, discretization of random ellipsoids.

Here, as our second main result, and as an application of Theorem 1.4, we provide a significantly simpler proof of an approximate version of the result of Rebrova-Tikhomirov; this proof builds upon a recent approach to controlling the least singular value of random matrices developed by the author [7].

Theorem 1.5. Let $\xi$ be a random variable with mean 0 and variance 1, and let $v_\xi \in (0, 1), u_\xi \in (0, 1)$ be such that $\rho_{u_\xi, v_\xi}(1) \leq u_\xi$. Let $M_n$ denote an $n \times n$ matrix, each of whose entries is an independent copy of $\xi$. Then, for any $\eta \geq 2^{-n^{0.0001}}$,
\[
\Pr (s_n(M_n) \leq \eta) \leq C_{1.5} n^2 \eta,
\]
where $C_{1.5} \geq 1$ is a constant depending only on $u_\xi$ and $v_\xi$. 
**Discussion and future work:** As stated, Theorem 1.5 is strictly weaker than the one in [18]. However, the reader will note that our statement and proof are essentially robust with respect to various modifications: for instance, (i) we only require that all the entries are independent, and that the entries in row $i \in [n]$ be i.i.d. copies of the centered random variables $\{\xi_i\}_{i \in [n]}$, where we have uniform control (over $i \in [n]$) on $u_{\xi_i}, v_{\xi_i}, \text{Var}(\xi_i)$; (ii) we can add an arbitrary matrix of operator norm at most $n^{0.9}$ to $M_n$; (iii) we can add an arbitrary matrix which has at most $n^{0.9}$ rows not identically zero to $M_n$ — for some of these (minor) variants, our result is possibly stronger than existing results in the literature (see also the work of Livshyts [14] which extends the result of [18] in several directions). More interestingly, we hope that a combination of the ideas in this work along with those in [4, 31] can be used to prove an analog of Theorem 1.5 for symmetric matrices. Moreover, in upcoming work [8], we will further develop the ideas here to prove results like Theorem 1.5 in the important setting of smoothed analysis.

**Organization:** The rest of this paper is organized as follows. In Section 2, we prove Theorem 1.4; in Section 3, we outline the proof of Theorem 1.5 in the subgaussian case as a warm-up; in Section 4, we discuss and prove the main new ingredient needed to go from the subgaussian case to the heavy-tailed case (this is our replacement for the covering of random ellipsoids introduced in [18]); and finally, in Section 5, we complete the proof of Theorem 1.5.

**Notation:** Throughout the paper, we will omit floors and ceilings when they make no essential difference. For convenience, we will also say ‘let $p = x$ be a prime’, to mean that $p$ is a prime between $x$ and $2x$; again, this makes no difference to our arguments. As is standard, we will use $[n]$ to denote the discrete interval $\{1, \ldots, n\}$. We will also use the asymptotic notation $\lesssim, \gtrsim, \ll, \gg$ to denote $O(\cdot), \Omega(\cdot), o(\cdot), \omega(\cdot)$ respectively. For a matrix $M$, we will use $\|M\|$ to denote its standard $\ell^2 \to \ell^2$ operator norm. All logarithms are natural unless noted otherwise.

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## 2 Proof of Theorem 1.4

We begin with the following definition from [27] which will appear in our upper bound on the Lévy concentration function.

**Definition 2.1.** Let $z$ be an arbitrary random variable. For any $w \in \mathbb{R}$, we define

$$\|w\|^2_z := \mathbb{E}\|w(z_1 - z_2)\|^2_{\mathbb{R}/\mathbb{Z}},$$

where $z_1, z_2$ denote i.i.d. copies of $z$ and $\| \cdot \|_{\mathbb{R}/\mathbb{Z}}$ denotes the distance to the nearest integer.

Note that $\| \cdot \|_z$ is not a norm in the strict sense, since it does not satisfy homogeneity. However, it does satisfy the triangle inequality, and it is invariant under negation, which will be sufficient for us.

The next proposition, which provides a ‘Fourier-bound’ on the Lévy concentration function, appears in [27] and will be the starting point of the proof of Theorem 1.4. For the reader’s convenience, we include a complete proof here.

**Proposition 2.2.** Let $v := (v_1, \ldots, v_n) \in \mathbb{R}^n$ and let $z$ be an arbitrary random variable. Then,

$$\rho_{r,z}(v) \leq e^{\pi r^2} \int_{\mathbb{R}} \exp \left( - \sum_{i=1}^n \|v_i\|_{z}^2 / 2 - \pi \xi^2 \right) d\xi.$$
Proof. Let \( e(x) := \exp(2\pi ix) \). Then, for any \( x \in \mathbb{R} \), we have
\[
\Pr \left( \sum_{i=1}^{n} z_i v_i \in B(x, r) \right) = \Pr \left( -\sum_{i=1}^{n} z_i v_i - x \geq -r^2 \right) \\
\leq \exp(\pi r^2) \mathbb{E} \left[ \exp \left( -\pi \sum_{i=1}^{n} z_i v_i - x \right) \right] \\
= \exp(\pi r^2) \int_{\mathbb{R}} \mathbb{E} \left[ e \left( \sum_{i=1}^{n} z_i v_i \cdot \xi \right) \right] e(-x\xi) \exp(-\pi\xi^2) d\xi \\
\leq \exp(\pi r^2) \int_{\mathbb{R}} \mathbb{E} \left[ e \left( \sum_{i=1}^{n} z_i v_i \cdot \xi \right) \right] \exp(-\pi\xi^2) d\xi \\
= \exp(\pi r^2) \prod_{i=1}^{n} \mathbb{E} \left[ e(z_i v_i \xi) \right] \exp(-\pi\xi^2) d\xi,
\]
where the third line uses the standard Fourier identity
\[
\exp(-\pi|a|^2) = \int_{\mathbb{R}} e(a\xi) \exp(-\pi\xi^2) d\xi.
\]

Next, we note that
\[
|\mathbb{E} e(z_i v_i \xi)| \leq \frac{1}{2} + \frac{1}{2} |\mathbb{E} e(z_i v_i \xi)|^2 \\
= \frac{1}{2} + \frac{1}{2} \mathbb{E} e((z_i - z'_i) v_i \xi) \\
= \frac{1}{2} + \frac{1}{2} \mathbb{E} \cos(2\pi(z_i - z'_i) v_i \xi) \\
\leq \frac{1}{2} + \frac{1}{2} \mathbb{E} \left( 1 - \||z_i - z'_i||_{\mathbb{R}/\mathbb{Z}}^2 \right) \\
\leq \frac{1}{2} - \frac{1}{2} \|v_i \xi\|_{\mathbb{R}/\mathbb{Z}}^2 \\
\leq \exp(-\|v_i \xi\|_{\mathbb{R}/\mathbb{Z}}^2).
\]

Substituting this into the previous expression yields the desired conclusion.

We now proceed to the proof of Theorem 1.4, which consists of six steps. The first three steps are modelled after [16], whereas the last three steps are modelled after Halász’s proof of his anti-concentration inequality [6].

Step 1: Extracting a large sublevel set. For each integer \( 1 \leq m \leq M \), where \( M := s/k \), we define
\[
S_m := \left\{ \xi \in \mathbb{R} : \sum_{i=1}^{n} \|v_i \xi\|_{\mathbb{R}/\mathbb{Z}}^2 + \xi^2 \leq m \right\}.
\]
Since
\[
\int_{\mathbb{R}} \exp \left( -\sum_{i=1}^{n} \|v_i \xi\|_{\mathbb{R}/\mathbb{Z}}^2 / 2 - \pi\xi^2 \right) d\xi \lesssim \sum_{1 \leq m \leq M} \mu(S_m) \exp(-m/2) + \exp(-M),
\]
we have
\[
\mu(S_m) \lesssim \exp(-m/2) + \exp(-M).
\]
it follows from Proposition 2.2 that
\[ \rho_{1,z}(v) \lesssim \sum_{1 \leq m \leq M} \mu(S_m) \exp(-m/2) + \exp(-M). \]

In particular, since it is assumed that \( \rho_{1,z}(v) \geq C_{1.4} \exp(-s/k) = C_{1.4} \exp(-M) \), it follows that for sufficiently large \( C_{1.4} \geq 1 \),
\[ \rho_{1,z}(v) \lesssim \sum_{1 \leq m \leq M} \mu(S_m) \exp(-m/2) = \sum_{1 \leq m \leq M} \mu(S_m) \exp(-m/4) \exp(-m/4) \lesssim \sum_{1 \leq m \leq M} \mu(S_m) \exp(-m/4)c_m, \]
where
\[ c_m := \frac{e^{-m/4}}{\sum_{m=1}^{M} e^{-m/4}}. \]

Note that in the last line, we have used the fact that \( \sum_{m=1}^{\infty} e^{-m/4} = O(1) \). Therefore, by averaging with respect to the probability measure \( \{c_m\}_{m=1}^{M} \), it follows that there must exist some non-zero integer \( m_0 \in [1, M] \) for which
\[ \mu(S_{m_0}) \gtrsim \rho_{1,z}(v) \exp(m_0/4). \]

**Step 2: Eliminating the \( z \)-norm.** From here on, all implicit constants will be allowed to depend on \( C_z \). Since \( S_{m_0} \subset B(0, \sqrt{m_0}) \), it follows (by averaging) that there must exist some \( B(x, 1/16C_z) \subset B(0, \sqrt{m_0}) \) for which
\[ \mu(S_{m_0} \cap B(x, 1/16C_z)) \gtrsim \rho \exp(m_0/4)m_0^{-1/2} \gtrsim \rho \exp(m_0/8). \]

Moreover, for \( \xi_1, \xi_2 \in B(x, 1/16C_z) \cap S_{m_0} \), we have that
\[ \bullet \xi_1 - \xi_2 \in B(0, 1/8C_z), \text{ and} \]
\[ \bullet \sum_{i=1}^{n} \|v_i(\xi_1 - \xi_2)\|_z^2 \leq \sum_{i=1}^{n} (\|v_i\xi_1\|_z + \|v_i\xi_2\|_z)^2 \leq 2 \sum_{i=1}^{n} (\|v_i\xi_1\|^2 + \|v_i\xi_2\|^2) \leq 4m_0. \]

Since for any \( A \subseteq \mathbb{R} \), \( \mu(A - A) \geq \mu(A) \), it follows that setting
\[ T_{m_0} := \left\{ \xi \in B(0, 1/8C_z) : \sum_{i=1}^{n} \|v_i\xi\|_z^2 \leq 4m_0 \right\}, \]
we have that
\[ \mu(T_{m_0}) \gtrsim \rho \exp(m_0/8). \]

Next, let \( y := z_1 - z_2 \), where \( z_1, z_2 \) are i.i.d. copies of \( z \). Since
\[ E_y \int \sum_{i=1}^{n} \|v_i y_0 \xi\|_{\mathbb{R}/Z}^2 1_{T_{m_0}}(\xi) d\xi \leq 4m_0 \mu(T_{m_0}), \]
it follows that there exists some \( y_0 \in \mathbb{R} \) satisfying \( C_z^{-1} \leq |y_0| \leq C_z \) such that
\[ \int \sum_{i=1}^{n} \|v_i y_0 \xi\|_{\mathbb{R}/Z}^2 1_{T_{m_0}}(\xi) d\xi \leq 4m_0 \mu(T_{m_0}) \Pr \left( C_z^{-1} \leq |y| \leq C_z \right)^{-1} \leq 4C_z m_0 \mu(T_{m_0}), \]
where the final inequality follows from Equation (1). Hence, by Markov’s inequality,

\[
\mu \left( \left\{ \xi \in T_{m_0} : \sum_{i=1}^n \|v_i y_0 \xi\|_{R/Z}^2 \leq 8C_z m_0 \right\} \right) \geq \frac{\mu(T_{m_0})}{2} \geq \rho \exp(m_0/8).
\]

Since \( T_{m_0} \subset B(0, 1/8C_z) \), this shows that

\[
\mu \left( \left\{ \xi \in B(0, 1/8C_z) : \sum_{i=1}^n \|v_i y_0 \xi\|_{R/Z}^2 \leq 8C_z m_0 \right\} \right) \geq \rho \exp(m_0/8).
\]

Finally, after replacing \( \xi \) by \( y_0 \xi \), and noting that the change of measure factor lies in \([C_z^{-1}, C_z]\), it follows that

\[
T'_{m_0} := \left\{ \xi \in B(0, 1/8) : \sum_{i=1}^n \|v_i \xi\|_{R/Z}^2 \leq 8C_z m_0 \right\}
\]

satisfies

\[
\mu(T'_{m_0}) \geq \rho \exp(m_0/8).
\]

**Step 3: Discretization of \( \xi \).** For \( p \) a prime as in the statement of the theorem, let

\[
B_0 := \{ r/p : r \in \mathbb{Z}, -p/8 \leq r \leq p/8 \}.
\]

Let

\[
R := \left\{ r \in \mathbb{Z} : -\frac{p}{8} \leq r \leq \frac{p}{8}, \mu \left( T'_{m_0} \cap \left[ \frac{r}{p}, \frac{r+1}{p} \right] \right) \geq \frac{1}{2p} \right\}
\]

By averaging, we see that

\[
|R| \geq \mu(T'_{m_0})p.
\]

Now, consider the random sets \( x + B_0 \), where \( x \in [0, 1/p] \) is a random point. Then, for each \( r \in R \), we have

\[
\Pr \left( x + \frac{r}{p} \in T'_{m_0} \right) \geq \frac{1}{2}.
\]

Therefore, by linearity of expectation,

\[
\mathbb{E}_{x \in [0, 1/p]} \left[ \left| (x + B_0) \cap T'_{m_0} \right| \right] \geq |R|,
\]

so there exists some \( x_0 \in [0, 1/p] \) for which

\[
\left| (x_0 + B_0) \cap T'_{m_0} \right| \geq \mu(T'_{m_0})p \geq \rho \exp(m_0/8)p.
\]

Let us now ‘recenter’ this shifted lattice. Note that for a fixed \( \xi_0 \in (x_0 + B_0) \cap T'_{m_0} \), we have for any \( \xi \in (x_0 + B_0) \cap T'_{m_0} \) that

\[
\sum_{i=1}^n \|v_i(\xi - \xi_0)\|_{R/Z}^2 \leq 2 \sum_{i=1}^n \left( \|v_i \xi\|_{R/Z}^2 + \|v_i \xi_0\|_{R/Z}^2 \right) \leq 32C_z m_0.
\]

Note also that \( \xi_0 - \xi \in B_1 := B_0 - B_0 = \{ r/p : r \in \mathbb{Z}, -p/4 \leq r \leq p/4 \} \). Hence, for a fixed \( \xi_0 \in (x_0 + B_0) \cap T'_{m_0} \), setting

\[
P_{m_0} := \{ \xi_0 - \xi : \xi \in (x_0 + B_0) \cap T'_{m_0} \}
\]

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gives a subset \( P_{m_0} \subset B_1 \) such that
\[
|P_{m_0}| \gtrsim \rho \exp(m_0/8)p,
\]
and for all \( \xi \in P_{m_0} \),
\[
\sum_{i=1}^{n} \|v_i \xi\|_{R/Z}^2 \leq 32Cz m_0.
\]

**Step 4: Embedding \( P_{m_0} \) into \( \mathbb{F}_p \) and the Halász trick.** Let \( V := \text{supp}(\varphi_p(v)) \). If \( |V| < s \), we proceed directly to Step 6. Otherwise, for \( I \subseteq V \) such that \( |I| \geq s \), we define the sets
\[
P'_m(I) := \left\{ r \in \mathbb{F}_p : \sum_{i \in I} \|v_i r\|_{R/Z}^2 \leq 32Cz m \right\},
\]
Note that the integrality of the \( v_i \) ensures that \( r \mapsto \frac{\|v_i r\|_{R/Z}}{p} \) is indeed well-defined as a map from \( \mathbb{F}_p \) to \([0, 1]\). Note also that, since \( P_{m_0} \subset B_1 \), the size of \( P'_m(I) \) (as a subset of \( \mathbb{F}_p \)) is at least the size of \( P_{m_0} \) (as a subset of \( \frac{1}{p} \cdot \mathbb{Z} \)) i.e. the way we have defined various objects ensures that there are no wrap-around issues. We claim that for all integers \( t \geq 1 \),
\[
tP'_m(I) \subseteq P'_{t^2 m}(I).
\]
Indeed, for \( r_1, \ldots, r_t \in P'_m(I) \subseteq \mathbb{F}_p \), we have
\[
\sum_{i \in I} \left\| v_i \frac{(r_1 + \cdots + r_t \mod p)}{p} \right\|_{R/Z}^2 = \sum_{i \in I} \left\| v_i r_1 \frac{1}{p} + \cdots + v_i r_t \frac{1}{p} \right\|_{R/Z}^2 \leq \sum_{i \in I} \left( \sum_{j=1}^{t} \left\| v_i r_j \frac{1}{p} \right\|_{R/Z} \right)^2 \leq \sum_{i \in I} t \sum_{j=1}^{t} \left\| v_i r_j \frac{1}{p} \right\|_{R/Z}^2 \leq t \sum_{j=1}^{t} \sum_{i \in I} \left\| v_i r_j / p \right\|_{R/Z}^2 \leq 32Cz t^2 m,
\]
which gives the desired inclusion.

Recall that the classical Cauchy-Davenport theorem states that every pair of nonempty \( A, B \subseteq \mathbb{F}_p \) satisfies
\[
|A + B| \geq \min\{p, |A| + |B| - 1\}.
\]
It follows that for all integers \( t \geq 1 \),
\[
|tP'_m(I)| \geq \min\{p, t|P'_m(I)| - t\}.
\]
Hence, by Equation (3), we have

$$|P'_{t}m(I)| \geq \min\{p, t|P'_{m}(I)| - t\}. \hspace{1cm} (4)$$

We also claim that $|P'_{m}(I)| < p$ as long as $m \leq |I|/15$. Indeed, since the map $\mathbb{F}_p \ni r \mapsto ar \in \mathbb{F}_p$ is bijective for every non-zero $a \in \mathbb{F}_p$, we have

$$\sum_{r \in \mathbb{F}_p} \sum_{i \in I} \|v_{i}r/p\|^{2}_{\mathbb{R}/\mathbb{Z}} = |I| \cdot \sum_{r \in \mathbb{F}_p} \|r/p\|^{2}_{\mathbb{R}/\mathbb{Z}}^{(p-1)/2} \geq |I| \cdot \sum_{r'} (r'/p)^{2} > |I| \cdot p/15.$$ 

On the other hand, from the definition of $P'_{m}(I)$,

$$\sum_{r \in \mathbb{F}_p} \sum_{i \in I} \|v_{i}r/p\|^{2}_{\mathbb{R}/\mathbb{Z}} \leq |P'_{m}(I)| \cdot m + (p - |P'_{m}(I)|) \cdot |I|.$$ 

Comparing these two bounds proves the claim. Combining this claim with Equation (4) shows that

$$|P'_{M}(I)| \gtrsim \sqrt{\frac{M}{m_0}} (|P'_{m_0}(I)| - 1) \gtrsim \sqrt{\frac{M}{m_0}} |P'_{m_0}(I)| \gtrsim \sqrt{\frac{M}{m_0}} \rho \exp(m_0/8)p \gtrsim \sqrt{M \rho \exp(m_0/16)p},$$

where the second line follows since $|P'_{m_0}(I)| \geq |P'_{m_0}| \gtrsim pp \geq C_{1.4}$ by assumption.

**Remark 2.3.** Whereas we have related the size of $P'_{m}(I)$ to the size of $P'_{t}m(I)$, [16] uses a similar computation to deduce information about the size of iterated sumsets of $\{v_1, \ldots, v_n\}$. This information is then combined with Freiman-type inverse theorems to provide structural information about $\{v_1, \ldots, v_n\}$. Thus, we see that by ‘dualizing’ the argument in [16], one is able to bypass the need for Freiman-type theorems, as far as the counting variant of the inverse Littlewood-Offord problem is concerned.

**Step 5: Passing to $R_k(v)$.** Since $\cos(2\pi x) \geq 1 - 20\|x\|^2_{\mathbb{R}/\mathbb{Z}}$ for all $x \in \mathbb{R}$, it follows that

$$P'_{M}(I) \subseteq P'_{M}(I) := \left\{ r \in \mathbb{F}_p : \sum_{i \in I} \cos(2\pi v_{i}r/p) \geq |I| - 2000C_2 M \right\}.$$ 

By considering the random variable $r \in \mathbb{F}_p \mapsto \sum_{i \in I} \cos(2\pi v_{i}r/p)$, we have for any $k \in \mathbb{N}$ that

$$|P'_{M}(I)| (|I| - 2000C_2 M)^2 \leq \sum_{r \in \mathbb{F}_p} \left| \sum_{j \in I} \cos(2\pi v_{j}r/p) \right|^2.$$
\[
\frac{1}{2^{2k}} \sum_{r \in \mathbb{F}_p} \left( \sum_{j \in I} e^{2\pi i v_j r/p} + e^{-2\pi i v_j r/p} \right)^{2k} \\
= \frac{1}{2^{2k}} \sum_{r \in \mathbb{F}_p} \sum_{\epsilon_1, \ldots, \epsilon_{2k} \in \{\pm 1\}} \sum_{j_1, \ldots, j_{2k} \in I} e^{2\pi i (\epsilon_1 v_{j_1} + \cdots + \epsilon_{2k} v_{j_{2k}}) r/p} \\
= \frac{1}{2^{2k}} \sum_{\epsilon_1, \ldots, \epsilon_{2k} \in \{\pm 1\}} \sum_{j_1, \ldots, j_{2k} \in I} \sum_{r \in \mathbb{F}_p} e^{2\pi i (\epsilon_1 v_{j_1} + \cdots + \epsilon_{2k} v_{j_{2k}}) r/p} \\
= \frac{1}{2^{2k}} \sum_{\epsilon_1, \ldots, \epsilon_{2k} \in \{\pm 1\}} \sum_{j_1, \ldots, j_{2k} \in I} p \cdot \delta_0(\epsilon_1 v_{j_1} + \cdots + \epsilon_{2k} v_{j_{2k}}), \tag{5}
\]

where the last line follows again using the integrality of \(v_1, \ldots, v_n\).

From here on, we will use the results of [5] to finish the proof. We begin with the following key definition.

**Definition 2.4.** Suppose that \(\mathbf{v} \in \mathbb{F}_p^n\) for an integer \(n\) and a prime \(p\), and let \(k \in \mathbb{N}\). For every \(\alpha \in [0, 1]\), we define \(R_k^\alpha(\mathbf{v})\) to be the number of solutions to

\[\pm v_{i_1} \pm \cdots \pm v_{i_{2k}} = 0 \mod p\]

that satisfy \(|\{i_1, \ldots, i_{2k}\}| \geq (1 + \alpha)k\).

The following elementary lemma from [5] shows that for ‘small’ \(\alpha\), \(R_k^\alpha(\mathbf{v})\) is not much smaller than \(R_k^0(\mathbf{v})\).

**Lemma 2.5** (Lemma 1.6 in [5]). For all integers \(k, n\) with \(k \leq n/2\), any prime \(p\), vector \(\mathbf{v} \in \mathbb{F}_p^n\), and \(\alpha \in [0, 1]\),

\[R_k^0(\mathbf{v}) \leq R_k^\alpha(\mathbf{v}) + (40k^{1-\alpha}n^{1+\alpha})^k.\]

**Proof.** By definition, \(R_k^0(\mathbf{v})\) is equal to \(R_k^\alpha(\mathbf{v})\) plus the number of solutions to \(\pm v_{i_1} \pm v_{i_2} \cdots \pm v_{i_{2k}} = 0\) that satisfy \(|\{i_1, \ldots, i_{2k}\}| < (1 + \alpha)k\). The latter quantity is bounded from above by the number of sequences \((i_1, \ldots, i_{2k}) \in [n]^{2k}\) with at most \((1 + \alpha)k\) distinct entries times \(2^{2k}\), the number of choices for the \(\pm\) signs. Thus

\[R_k^0(\mathbf{v}) \leq R_k^\alpha(\mathbf{v}) + \binom{n}{(1 + \alpha)k}((1 + \alpha)k)^{2k} 2^{2k} \leq R_k^\alpha(\mathbf{v}) + (4e^{1+\alpha}k^{1-\alpha}n^{1+\alpha})^k,
\]

where the final inequality follows from the well-known bound \((\binom{a}{b}) \leq (ea/b)^b\). Finally, noting that \(4e^{1+\alpha} \leq 4e^2 \leq 40\) completes the proof. \(\square\)

Let \(\mathbf{v}_I\) denote the \(|I|\)-dimensional vector obtained by restricting \(\mathbf{v}\) to the coordinates corresponding to \(I\). Recognizing the right hand side of Equation (5) as

\[
\frac{pR_k^0(\mathbf{v}_I)}{2^{2k}},
\]

it follows from Equation (5) and the above lemma that for any \(k \leq \sqrt{|I|}\) and \(\alpha \in [0, 1/8]\),

\[
R_k^\alpha(\mathbf{v}_I) \gtrsim (|I| - 2000C_2 M)^{2k} 2^{2k} \rho \sqrt{|I|} - (40k^{1-\alpha}|I|^{1+\alpha})^k \\
\gtrsim |I|^{2k} 2^{2k} \rho \sqrt{|I|} - (40k^{1-\alpha}|I|^{1+\alpha})^k
\]

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\[ \geq |I|^{2k2k \rho \sqrt{M} - (40|I|^{(3/2)}+\alpha)k} \]
\[ \geq |I|^{(3/2)k} \left(2^{2k} \sqrt{|I|} \rho \sqrt{M} - (40)^k |I|^\alpha k\right) \]
\[ \geq |I|^{(3/2)k} \left(2^{2k} \sqrt{|I|} \rho \sqrt{M}\right) \]
\[ \geq |I|^{2k2k \rho \sqrt{M}}, \]

where the second line follows from the assumption that \( Mk \leq s \leq |I| \), the third line follows from the assumption that \( k \leq \sqrt{s} \leq \sqrt{|I|} \), and the fifth line follows from the assumption that \( \rho > s^{-k/4} \geq s^{-(k/2)+2\alpha k} \geq |I|^{-(k/2)+2\alpha k} \).

**Step 6: Applying the counting lemma.** Let us summarize where we stand. We have proved that for any random variable \( Z \) satisfying Equation (1), there exists an absolute constant \( C := C(C_2) > 0 \) for which the following holds. If \( V \in \mathbb{Z}^n \) satisfies \( p_{1/2}(V) := \rho \geq C_{1.4} \max\{e^{-s/k}, s^{-k/4}\} \) for some \( k \leq \sqrt{s} \leq s \leq n/\log n \) and sufficiently large \( C_{1.4} \), and if \( \alpha \in [0, 1/8] \), then either

1. \(|V| < s\) (where \( V := \text{supp}(\varphi_p(v)) \)), or
2. for all \( I \subseteq V \) with \(|I| \geq s\),

\[ R^\alpha_k(v_I) \geq \frac{|I|^{2k2k \rho \sqrt{M}}}{C}. \]

Hence, it follows that

\[ \varphi_p(V) \subseteq X_s + \bigcup_{m=s}^n Y^\alpha_{k,s,p}(m), \]

where

\[ X_s := \{ a \in \mathbb{F}_p^n : |\text{supp}(a)| < s \}, \]

and

\[ Y^\alpha_{k,s,p}(m) := \{ a \in \mathbb{F}_p^n : |\text{supp}(a)| = m \text{ and } R^\alpha_k(a_I) \geq \frac{2^{2k}|I|^{2k \rho \sqrt{M}}}{C} \forall I \subseteq \text{supp}(a) \text{ with } |I| \geq s \}. \]

We will bound the size of each of these pieces separately. For \(|X_s|\), the following simple bound suffices:

\[ |X_s| \leq \sum_{t=0}^{s-1} \left(\binom{n}{t} p^t \right) \leq s \left(\frac{n}{s} \right) p^s \leq s \left(\frac{\exp}{s} \right)^s \leq \left(\frac{5np}{s} \right)^s. \]

On the other hand, the desired bound on \( Y^\alpha_{k,s,p}(m) \) follows easily from the work in [5].

**Theorem 2.6** (Theorem 1.7 in [5]). Let \( p \) be a prime, let \( k, n \in \mathbb{N} \), \( s \in [n] \), \( t \in [p] \), and let \( \alpha \in (0, 1) \). Denoting

\[ B^\alpha_{k,s,\geq t}(n) := \{ v \in \mathbb{F}_p^n : R^\alpha_k(v_I) \geq t \frac{2^{2k} \cdot |I|^{2k}}{p} \text{ for every } I \subseteq [n] \text{ with } |I| \geq s \}, \]

we have

\[ |B^\alpha_{k,s,\geq t}(n)| \leq (\alpha t)^{s-n} p^n. \]

**Corollary 2.7.** For our choice of parameters, \(|Y^\alpha_{k,s,p}(m)| \leq \left(\frac{16C}{p\sqrt{M}} \right)^n. \)
Proof. After paying an overall factor of \( \left( \begin{array}{c} n \\ m \end{array} \right) \), it suffices to count only those \( a \in Y_{k,s,\rho}^\alpha(m) \) for which \( \text{supp}(a) = [m] \). The key point is that, by definition, for any such \( a \), we have

\[
a|_{[m]} \in B_{k,s,\geq t}^\alpha(m),
\]

for \( t = [pp\sqrt{M}/C] \). Therefore, by Theorem 2.6, it easily follows that

\[
|Y_{k,s,\rho}^\alpha(m)| \leq \left( \begin{array}{c} n \\ m \end{array} \right) (at)^s \left( \frac{p}{t} \right)^m
\]
\[
\leq 2^n t^s \left( \frac{p}{t} \right)^n
\]
\[
\leq 2^n \left( p\sqrt{M} \right)^s \left( \frac{2Cp}{pp\sqrt{M}} \right)^n
\]
\[
\leq (p\sqrt{M})^s \left( \frac{4C}{p\rho M} \right)^n
\]
\[
\leq \left( \frac{16C}{p\rho M} \right)^n,
\]

as desired. \( \square \)

From Equations (6) and (7) and Corollary 2.7, and noting that \( M = s/k \), it follows that

\[
|\varphi_p(V_\rho)| \leq \left( \frac{5np}{s} \right)^s + n \cdot \left( \frac{16C\rho^{-1}}{\sqrt{s/k}} \right)^n
\]
\[
\leq \left( \frac{5np}{s} \right)^s + \left( \frac{32C\rho^{-1}}{\sqrt{s/k}} \right)^n
\]
\[
\leq \left( \frac{5np}{s} \right)^s + \left( \frac{C_{1.4}\rho^{-1}}{\sqrt{s/k}} \right)^n,
\]

where the final inequality follows since we can take \( C_{1.4} \) larger than \( 32C \). This completes the proof of Theorem 1.4.

3 Warm-up: proof of Theorem 1.5 in the subgaussian case

In this section, we will sketch the main elements of the proof of Theorem 1.5, in the special case when the entries are further assumed to be i.i.d. subgaussian. This will allow the reader to see many ideas in the proof of the general version of Theorem 1.5 in a simpler, less technical, setting, as well as motivate and shed greater light on the key Propositions 4.1 and 4.2. We begin with some preliminaries.

Definition 3.1. A random variable \( X \) is said to be \( C \)-subgaussian if, for all \( t > 0 \),

\[
\Pr(|X| > t) \leq 2 \exp \left( -\frac{t^2}{C^2} \right).
\]

For the remainder of this section, we fix a centered subgaussian random variable \( \xi \) with variance 1. All implicit constants will be allowed to depend only on the following quantities associated to \( \xi \):

(P1) \( \tilde{C}_\xi > 0 \) such that \( \xi \) is \( \tilde{C}_\xi \)-subgaussian.
Note that for the above choice of $u_\xi, v_\xi$, and letting $\xi'$ denote an independent copy of $\xi$, we have

$$\Pr \left( |\xi - \xi'| \leq \frac{v_\xi}{2} \right) \leq \rho_{v_\xi, \xi - \xi'}(1) \leq \rho_{v_\xi, \xi}(1) \leq u_\xi.$$  

Moreover, since $\text{Var}(\xi - \xi') = \text{Var}(\xi) + \text{Var}(\xi') = 2$, it follows from Markov's inequality that

$$\Pr \left( |\xi - \xi'| \geq 4(1 - u_\xi)^{-1/2} \right) \leq \frac{1 - u_\xi}{2}.$$  

Combining these two bounds, we see that

$$\Pr \left( \frac{v_\xi}{2} \leq |\xi - \xi'| \leq 4(1 - u_\xi)^{-1/2} \right) \geq \frac{1 - u_\xi}{2}.$$  

In other words, we see that any random variable $\xi$ with variance 1 and satisfying (P2) is $C_\xi$-good (in the sense of Equation (1)), for some constant $C_\xi$ depending only on $u_\xi, v_\xi$.

A useful fact that we will need about subgaussian random variables is the following concentration inequality.

**Lemma 3.2** (see, e.g., Corollary 5.17 in [30]). There exists an absolute constant $C_{3.2} > 0$ with the following property. Let $X_1, \ldots, X_n$ be independent centered $\tilde{C}_\xi$-subgaussian random variables. Then,

$$\Pr \left( \sum_{i=1}^n X_i^2 \geq C_{3.2} \tilde{C}_\xi^2 n \right) \leq \exp(-2n).$$  

Crucially, the subgaussian concentration inequality, combined with a standard epsilon-net argument, shows that the operator norm of a random matrix with i.i.d. subgaussian entries is typically $O(\sqrt{n})$.

**Lemma 3.3** (see, e.g., Lemma 2.4 in [21]). Let $M_n$ be an $n \times n$ random matrix whose entries are i.i.d. centered $\tilde{C}_\xi$-subgaussian random variables. Then,

$$\Pr \left( \|M_n\| \geq C_{3.3} \sqrt{n} \right) \leq 2 \exp(-n),$$  

where $C_{3.3} \geq 1$ depends only on $\tilde{C}_\xi$.

In the next three subsections, we discuss the proof of the subgaussian case of Theorem 1.5, which is essentially the same as that of the Rademacher case in [7] – the only difference is that the estimates on the size of the union bound coming from [5] will be replaced by a corresponding application of Theorem 1.4 (Claim 5.7).

### 3.1 Reduction to integer vectors

Our reduction to integer vectors relies on the key notion of the Least Common Denominator (LCD) of a vector, and its connection to the Lévy concentration function, as developed in [21].

**Definition 3.4** (Least Common Denominator (LCD)). For $\gamma \in (0, 1)$ and $\alpha > 0$, define

$$\text{LCD}_{\gamma, \alpha}(a) := \inf \{ \theta > 0 : \text{dist}(\theta a, \mathbb{Z}^n) < \min\{\gamma \|\theta a\|, \alpha\} \}.$$  

Note that the requirement that the distance is smaller than $\gamma \|\theta a\|_2$ forces us to consider only non-trivial integer points as approximations of $\theta a$. 

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The next proposition, which appears in [22], shows that vectors with large LCD have small Lévy concentration function on scales which are larger than $\Omega(1/\text{LCD})$.

**Theorem 3.5** (Theorem 3.4 in [22]). Let $\\xi$ denote a centered random variable satisfying $\rho_{v,\\xi}(1) \leq u$ for some $v \geq 0$ and $u \in (0, 1)$. Then, for every $a \in \mathbb{S}^{n-1}$, for every $\alpha > 0, \gamma \in (0, 1)$, and for $\delta \geq \frac{1}{\text{LCD}_{\alpha, \gamma}(a)}$, we have

$$\rho_{\delta v, \xi}(a) \leq \frac{C_{3.5} \delta}{\gamma \sqrt{1 - u}} + C_{3.5} \exp\left(-2(1 - u)\alpha^2\right),$$

where $C_{3.5} \geq 1$ is an absolute constant.

For the rest of this paper, we will take $\alpha := n^{1/4}$ and $\gamma := n^{-1/2}$. Moreover, since **Theorem 1.5** is trivially true for $\eta \geq n^{-3/2}$, we will henceforth assume that $2^{-n^{0.0001}} \leq \eta < n^{-3/2}$.

**Remark 3.6.** In fact, in the subgaussian case, the arguments below go through with $\gamma$ equal to a sufficiently small constant, and this choice of $\gamma$ recovers (a quantitative strengthening of) the main result of Rudelson in [20]. However, we have chosen to work with a (slightly) smaller value of $\gamma$ so as to have the same choice of parameters as in the heavy-tailed case.

We decompose the unit sphere $\mathbb{S}^{n-1}$ into $\Gamma^1(\eta) \cup \Gamma^2(\eta)$, where

$$\Gamma^1(\eta) := \{ a \in \mathbb{S}^{n-1} : \text{LCD}_{\alpha, \gamma}(a) \geq n^{3/4} \cdot \eta^{-1} \}$$

and $\Gamma^2(\eta) := \mathbb{S}^{n-1} \setminus \Gamma^1(\eta)$. Accordingly, we have

$$\Pr(s_n(M_n) \leq \eta) \leq \Pr(\exists a \in \Gamma^1(\eta) : \|M_n a\|_2 \leq \eta) + \Pr(\exists a \in \Gamma^2(\eta) : \|M_n a\|_2 \leq \eta).$$

(8)

Therefore, **Theorem 1.5** follows from the following two propositions and the union bound.

**Proposition 3.7.** $\Pr(\exists a \in \Gamma^1(\eta) : \|M_n a\|_2 \leq \eta) \leq 2C_{3.5} \left(\eta \gamma^{-1} n^{3/2} + n \exp(-c_{3.7} \sqrt{n})\right)$, where $c_{3.7} > 0$ depends only on $u_\xi, v_\xi$.

**Proposition 3.8.** $\Pr(\exists a \in \Gamma^2(\eta) : \|M_n a\|_2 \leq \eta) \leq C_{3.8} \left(e^{-n^{0.98}} + \exp(-c_{3.8} \eta)\right)$, where $C_{3.8} \geq 1$ and $c_{3.8} > 0$ are constants depending only on $u_\xi, v_\xi$.

**Remark 3.9.** For the proof of **Theorem 1.5** in the subgaussian case, discussed in this section, we will allow the constants $C_{3.8}, c_{3.8}$ to depend on $\tilde{C}_\xi$ as well. Consequently, the constant $C_{1.5}$ resulting from this simpler proof also depends on $\tilde{C}_\xi$.

The proof of **Proposition 3.7** is relatively simple, and follows from a conditioning argument developed in [13], once we observe the crucial fact (**Theorem 3.5**) that for any $a \in \Gamma^1(\eta)$,

$$\rho_{\delta v, \xi}(a) \lesssim \gamma^{-1} \delta + \exp(-2(1 - u_\xi) \sqrt{n})$$

for all $\delta \geq \eta \cdot n^{-3/4}$.
Proof of Proposition 3.7 following [13, 29]. Since $M_n^T$ and $M_n$ have the same singular values, it follows that a necessary condition for a matrix $M_n$ to satisfy the event in Proposition 3.7 is that there exists a unit vector $a' = (a_1', \ldots, a_n')$ such that $\|a'^T M_n\|_2 \leq \eta$. To every matrix $M_n$, associate such a vector $a'$ arbitrarily (if one exists) and denote it by $a'M_n$; this leads to a partition of the space of all matrices with least singular value at most $\eta$. Then, by taking a union bound, it suffices to show the following.

$$\Pr \left( \exists a \in \Gamma(\eta) : \|M_n a\|_2 \leq \eta \wedge \|a'M_n\|_\infty = |a'_n| \right) \leq \rho_{2\eta\sqrt{\pi}\xi}(y) \leq 2C_{3.5} \left( \eta \gamma^{-1} \sqrt{n} + \exp(-2(1-u\xi)\sqrt{n}) \right).$$

(9)

To this end, we expose the first $n-1$ rows $X_1, \ldots, X_{n-1}$ of $M_n$. Note that if there is some $a \in \Gamma(\eta)$ satisfying $\|M_n a\|_2 \leq \eta$, then there must exist a vector $y \in \Gamma(\eta)$, depending only on the first $n-1$ rows $X_1, \ldots, X_{n-1}$, such that

$$\left( \sum_{i=1}^{n-1} (X_i \cdot y)^2 \right)^{1/2} \leq \eta.$$

In other words, once we expose the first $n-1$ rows of the matrix, either the matrix cannot be extended to one satisfying the event in Proposition 3.7, or there is some unit vector $y \in \Gamma(\eta)$, which can be chosen after looking only at the first $n-1$ rows, and which satisfies the equation above. For the rest of the proof, we condition on the first $n-1$ rows $X_1, \ldots, X_{n-1}$ (and hence, a choice of $y$).

For any vector $w' \in S^{n-1}$ with $w'_n \neq 0$, we can write

$$X_n = \frac{1}{w'_n} \left( u - \sum_{i=1}^{n-1} w'_i X_i \right),$$

where $u := w'^T M_n$. Thus, restricted to the event $\{s_n(M_n) \leq \eta\} \wedge \{|a'_M| = |a'_n|\}$, we have

$$|X_n \cdot y| = \inf_{w' \in S^{n-1}, w'_n \neq 0} \frac{1}{|w'_n|} \left| u \cdot y - \sum_{i=1}^{n-1} w'_i X_i \cdot y \right|$$

$$\leq \frac{1}{|a'_n|} \left( \|a'^T M_n N_n\|_2 \|y\|_2 + \|a'M_n\|_2 \left( \sum_{i=1}^{n-1} (X_i \cdot y)^2 \right)^{1/2} \right)$$

$$\leq \eta \sqrt{n} \left( \|y\|_2 + \|a'M_n\|_2 \right) \leq 2\eta \sqrt{n},$$

where the second line is due to the Cauchy-Schwarz inequality and the particular choice $w' = a'M_n$. It follows that the probability in Equation (9) is bounded by

$$\rho_{2\eta\sqrt{\pi}\xi}(y) \leq 2C_{3.5} \left( \eta \gamma^{-1} \sqrt{n} + \exp(-2(1-u\xi)\sqrt{n}) \right),$$

which completes the proof. 

\qed

Remark 3.10. Note that, until now, we have not used the subgaussianity of our random variables, or even the assumption that the random variables have finite variance. The only property that has been used thus far is (P2). Hence, the discussion above carries over verbatim to the heavy-tailed case as well.
The proof of Proposition 3.8 is the content of the next two subsections. Here, we present the initial crucial step, which consists of efficiently passing from vectors on the unit sphere to integer vectors. This is the only place in the argument where we will use (via Lemma 3.3) that our entries are subgaussian.

**Proposition 3.11.** With notation as above, we have

\[
\Pr (\exists a \in \Gamma^2(\eta) : \|M_n a\|_2 \leq \eta) \leq 2e^{-n} + \\
\Pr(\exists w \in (\mathbb{Z}^n \setminus \{0\}) \cap [-2\eta^{-1}n^{3/4}, 2\eta^{-1}n^{3/4}]^n : \|M_n w\|_2 \leq \min\{4\gamma C_{3.3}\sqrt{n}\|w\|_2, 2C_{3.3}\alpha \sqrt{n}\}).
\]

**Proof.** Since by Lemma 3.3, \(\Pr(\|M_n\| \geq C_{3.3}\sqrt{n}) \leq 2\exp(-n)\), we may henceforth restrict to the complement of this event. Let \(a \in \Gamma^2(\eta)\). Then, by definition, there exists some \(0 < \theta \leq \text{LCD}_{\alpha, \gamma}(a) \leq n^{3/4}\eta^{-1}\) and some \(w \in \mathbb{Z}^n \setminus \{0\}\) such that \(\|\theta a - w\|_2 \leq \min\{\gamma \theta, \alpha\}\). Thus, if \(\|M_n a\|_2 \leq \eta\), it follows from the triangle inequality that

\[
\|M_n w\|_2 = \|M_n(w - \theta a) + M_n(\theta a)\|_2 \\
\leq \|M_n\| \cdot \|\theta a - w\|_2 + \theta \cdot \|M_n a\|_2 \\
\leq C_{3.3}\sqrt{n} \cdot \min\{\gamma \theta, \alpha\} + \theta \eta \\
\leq 2C_{3.3}\sqrt{n} \cdot \min\{\gamma \theta, \alpha\},
\]

where the last inequality follows since \(\eta \leq \gamma \sqrt{n}\) and \(\theta \eta \leq n^{3/4} \leq \sqrt{n} \alpha\). The desired conclusion now follows from the straightforward case analysis below.

**Case I:** \(\gamma \theta \leq \alpha\). In this case, \(w\) is a non-zero integer vector of norm \(\|w\|_2 = \theta(1 + \gamma)\) satisfying

\[
\|M_n w\|_2 \leq 2\gamma C_{3.3}\sqrt{n} \theta \leq \min\{4\gamma C_{3.3}\sqrt{n}\|w\|_2, 2C_{3.3}\alpha \sqrt{n}\},
\]

where the last inequality uses \(\theta \leq \|w\|_2\) and \(\gamma \theta \leq \alpha\).

**Case II:** \(\gamma \theta > \alpha\). In this case, \(w\) is a non-zero integer vector of norm \(\|w\|_2 = \theta(1 + \gamma) \geq \gamma^{-1}\alpha/2\) satisfying

\[
\|M_n w\|_2 \leq 2C_{3.3}\alpha \sqrt{n} \leq \min\{2C_{3.3}\gamma^{-1}\alpha \gamma \sqrt{n}, 2C_{3.3}\alpha \sqrt{n}\} \leq \min\{4\gamma C_{3.3}\sqrt{n}\|w\|_2, 2C_{3.3}\alpha \sqrt{n}\}.
\]

\[\square\]

**Remark 3.12.** As mentioned in the introduction, the proof of the heavy-tailed case of Theorem 1.5 is complicated precisely by the absence of an estimate such as Lemma 3.3 on the operator norm on \(M_n\). However, in Proposition 4.2, we will see how to obtain enough control on various operator norms associated to \(M_n\) in order to carry out (a version of) a rounding argument like the one above (Proposition 4.1).

In view of Propositions 3.7, 3.8 and 3.11, it suffices to prove the following in order to complete the proof of Theorem 1.5 in the subgaussian case. Let \(V := (\mathbb{Z}^n \setminus \{0\}) \cap [-2\eta^{-1}n^{3/4}, 2\eta^{-1}n^{3/4}]^n\)

**Proposition 3.13.** \(\Pr(\exists w \in V : \|M_n w\|_2 \leq \min\{4\gamma C_{3.3}\sqrt{n}\|w\|_2, 2C_{3.3}\alpha \sqrt{n}\}) \leq C_{3.13} \exp(-c_{3.13}n)\), where \(C_{3.13} \geq 1\) and \(c_{3.13} > 0\) are constants depending only on \(u_{\xi}, v_{\xi}\), and \(\tilde{C}_{\xi}\).

We outline the proof of this proposition in the following two subsections.
3.2 Dealing with sparse integer vectors

Throughout this subsection and the next one, $p = 2^{n^{0.001}}$ is a prime. Note, in particular, that $p \gg n^{3/4}$. The proof of Proposition 3.13 proceeds in two steps. The first step is to show that the probability of the event appearing in Proposition 3.13 is small, provided we restrict ourselves only to sufficiently sparse integer vectors. Let $S := \{w \in (\mathbb{Z}^n \setminus \{0\}) \cap [-p, p]^n : |\text{supp}(w)| \leq n^{0.99}\}$.

**Lemma 3.14.** $\Pr (\exists w \in S : \|M_n w\|_2 \leq 4\gamma C_{3.3} \sqrt{n} \|w\|_2) \leq C_{3.14} \exp(-c_{3.14} n)$, where $C_{3.14} \geq 1$ is an absolute constant, and $c_{3.14} > 0$ depends only on $u_\xi, v_\xi$.

The proof of this lemma follows from a simple union bound, using the following standard estimate on the ‘invertibility with respect to a single vector’.

**Proposition 3.15** (see, e.g., Lemma 4.9 in [18]). There exists a constant $c_{3.15} \in (0, 1)$ depending only on $u_\xi, v_\xi$ from (P2) for which the following holds. Let $A := (a_{ij})$ be an $n \times m$ random matrix with i.i.d. entries, each of which is a copy of $\xi$. Then, for any $a \in \mathbb{S}^{n-1}$,

$$\Pr (\|Aa\|_2 \leq c_{3.15} \sqrt{n}) \leq (1 - c_{3.15})^n.$$ 

We omit the details of the union bound here, since later in Lemma 5.2, we will prove a more general statement.

3.3 Dealing with non-sparse integer vectors

It remains to deal with integer vectors with support of size at least $n^{0.99}$. Formally, let

$$W := \{w \in (\mathbb{Z}^n \setminus \{0\}) \cap [-\eta^{-4}, \eta^{-4}]^n : |\text{supp}(w)| \geq n^{0.99}\}.$$ 

In view of Lemma 3.14, and since $\eta \leq n^{-3/2}$, the following proposition suffices to prove Proposition 3.13.

**Proposition 3.16.** $\Pr (\exists w \in W : \|M_n w\|_2 \leq 2C_{3.3} n^{3/4}) \leq C_{3.16} \exp(-n)$, where $C_{3.16} \geq 1$ is a constant depending only on $u_\xi, v_\xi$, and $C_\xi$.

We omit the details of the proof of this proposition, since later in Proposition 5.3, we will prove a more general statement. However, let us mention here that the proof of this proposition (and that of Proposition 5.3) follows by a ‘double union bound’ based on the following simple observation. For $z \in \mathbb{Z}^n$, let $D_z$ denote the unit cube in $\mathbb{R}^n$ centered at $z$. Then, for any function $C(n)$, we have

$$\Pr (\exists w \in W : \|M_n w\|_2 \leq C(n) \sqrt{n}) \leq \sum_{z \in \mathbb{Z}^n \cap B(0, C(n) \sqrt{n})} \Pr (\exists w \in W : M_n w \in D_z) \leq \left| \mathbb{Z}^n \cap B(0, C(n) \sqrt{n}) \right| \cdot \sup_{z \in \mathbb{Z}^n \cap B(0, C(n) \sqrt{n})} \Pr (\exists w \in W : M_n w \in D_z) \leq (100C(n))^n \cdot \sup_{z \in \mathbb{Z}^n} \Pr (\exists w \in W : M_n w \in D_z),$$

where the last inequality uses a standard (loose) volumetric estimate on the number of integer points in an $n$-dimensional ball of radius $R$. The second quantity in the last equation i.e.

$$\sup_{z \in \mathbb{Z}^n} \Pr (\exists w \in W : M_n w \in D_z)$$

is reminiscent of the singularity problem for random matrices. Indeed, as we will see later in the proof of Proposition 5.3, the approach to the singularity problem for random matrices via either inverse Littlewood-Offord theory [29] or its counting variant [5] uses a union bound argument (involving the decomposition of $W$) to show that one may bound the quantity in Equation (11) by $O(n^{-cn})$ for some absolute constant $c \geq 0.44$. Hence, for $C(n) = 2C_{3.3} n^{1/4} = o(n^{0.44})$, the quantity on the right hand side of Equation (10) is $(o(1))^n$, as desired.
4 Reduction to integer vectors for random matrices with heavy-tailed entries

For a subset $I \subseteq [n]$, let $P_I : \mathbb{R}^n \to \mathbb{R}^n$ denote the orthogonal projection onto the subspace spanned by the vectors \{e_i : i \in I\}. Also, as before, let $V := (\mathbb{Z}^n \setminus \{0\}) \cap [-2\eta^{-1}n^{3/4}, 2\eta^{-1}n^{3/4}]^n$ and let $K := \{K \subseteq [n] : |K| \geq n - 4n^{0.99}\}$. The goal of this section is to prove the following analog of Proposition 3.11 in the heavy-tailed setting.

**Proposition 4.1.** With notation as above,
\[
\Pr(\exists a \in \Gamma^2(\eta) : \|M_n a\|_2 \leq \eta) \leq C_{4.1} e^{-n^{0.99}/10} + \Pr(\exists w \in V \text{ and } K \subseteq K : \|P_K M_n w\|_2 \leq 4 \min\{n^{0.41}\|w\|_2, n^{0.91}\}),
\]
where $C_{4.1} \geq 1$ is an absolute constant.

### 4.1 Norms of large projections of random matrices

The following proposition will turn out to be an appropriate substitute for controlling the operator norm in the subgaussian setting.

**Proposition 4.2.** Let $M_n := (m_{ij})$ be an $n \times n$ random matrix with i.i.d. entries, each with mean 0 and variance 1. For $\epsilon, \delta \in (0, 1/2)$ with $\delta \geq 4\epsilon$, there exists $C_{4.2}(\epsilon) \geq 1$ such that, except with probability at most $C_{4.2}(\epsilon) \exp\left(-n^{1-\epsilon}/8\right)$, the following hold.

1. There exists $I \subseteq [n]$ with $|I| \geq n - 2n^{1-\epsilon}$ such that
\[
\|P_I M_n\|_{\infty \to 2} \leq C_{4.2}(1)n^{1+\epsilon}.
\]

2. For every $J \subseteq [n]$ with $|J| = n^{1-\delta}$, there exists some $I(J) \subseteq [n]$ such that $|I(J)| \geq n - 2n^{1-\epsilon}$, and
\[
\|P_{I(J)} M_n P_J\|_{\infty \to 2} \leq C_{4.2}(1)n^{1+\epsilon-0.5\delta}.
\]

**Remark 4.3.** A statement similar to the one above, and with some common proof ideas, already appears in the work of Rebrova and Vershynin [19]. In that work, the primary interest is in obtaining optimal bounds on the restricted operator norms and consequently, the proofs are much more involved. In contrast, we do not require such optimal bounds for our application, and are therefore able to give a much shorter proof of the above proposition.

The proof of this proposition will occupy the remainder of this subsection. We begin with a simple lemma showing that, with high probability, most rows of a random matrix with i.i.d. centered entries of finite variance have small $\ell_1$ and $\ell_2$ norms.

**Lemma 4.4.** Let $A := (a_{ij})$ be an $n \times m$ random matrix with i.i.d. entries, each with mean 0 and variance 1. For $\epsilon \in (0, 1/2)$, let $I \subseteq [n]$ denote the (random) subset of coordinates such that for each $i \in I$,
\[
\left(\sum_{j=1}^m a_{ij}^2 \leq n^{2\epsilon}m\right) \land \left(\left|\sum_{j=1}^m a_{ij}\right| \leq n^\epsilon \sqrt{m}\right).
\]
Then,
\[
\Pr\left(|I^c| \geq 2n^{1-\epsilon}\right) \leq 2 \exp\left(-\frac{n^{1-\epsilon}}{4}\right).
\]
Proof. Since for each $i \in [n]$,
\[
\mathbb{E} \left[ \sum_{j=1}^{m} a_{ij}^2 \right] = \mathbb{E} \left[ \sum_{j=1}^{m} a_{ij}^2 \right] = m,
\]
it follows from Markov’s inequality that
\[
\Pr \left( \sum_{j=1}^{m} a_{ij}^2 > n^{2\epsilon} m \right) \leq n^{-2\epsilon}
\]
and
\[
\Pr \left( \sum_{j=1}^{m} a_{ij} > n^{\epsilon} \sqrt{m} \right) \leq n^{-2\epsilon}.
\]
Let $I_1 \subseteq [n]$ denote the subset of coordinates such that for each $i \in I_1$,
\[
\sum_{j=1}^{m} a_{ij}^2 \leq n^{2\epsilon} m
\]
and let $I_2 \subseteq [n]$ denote the subset of coordinates such that for each $i \in I_2$,
\[
\left| \sum_{j=1}^{m} a_{ij} \right| \leq n^{\epsilon} \sqrt{m}.
\]
Since the rows of the matrix are independent, it follows from the standard Chernoff bound that for $k \in \{1, 2\}$
\[
\Pr \left( |I_k^c| \geq n^{1-\epsilon} \right) \leq \exp \left( -\frac{n^{1-\epsilon}}{4} \right).
\]
Hence, by the union bound,
\[
|I^c| \leq |I_1^c| + |I_2^c| \leq 2n^{1-\epsilon},
\]
except with probability at most $2 \exp \left( -\frac{n^{1-\epsilon}}{4} \right)$. \qed

The next proposition controls the $\infty \to 2$ operator norm of a random matrix with i.i.d. entries, conditioned on no row having $\ell_1$ or $\ell_2$ norm which is ‘too large’, and essentially appears as Proposition 3.10 in [18]. Since our statement uses somewhat different parameters than in [18], we provide a complete proof below for the reader’s convenience.

**Proposition 4.5.** Fix $\epsilon \in (0, 1/2)$. Let $B := (b_{ij})$ be an $n \times m$ matrix such that the Euclidean norm of every row is at most $n^\epsilon \sqrt{m}$ and such that for all $i \in [n]$,
\[
\left| \sum_{j=1}^{m} b_{ij} \right| \leq n^{\epsilon} \sqrt{m}.
\]
Let $\pi_1, \ldots, \pi_n$ be independent random permutations uniformly distributed on $S_m$, and let $\tilde{B} := (\tilde{b}_{ij})$ denote the random $n \times m$ matrix whose entries are given by
\[
\tilde{b}_{ij} := b_{i, \pi(j)}.
\]
Then,
\[ \Pr \left( \| \tilde{B} \|_{\infty \to 2} \geq C_{4.5} \sqrt{mn\epsilon} \right) \leq \exp(-n), \]
where \( C_{4.5} \geq 1 \) is an absolute constant.

The following concentration inequality will be used to establish the subgaussianity of certain random variables appearing in the proof of Proposition 4.5. It appears as Lemma 3.9 in [18], and is a direct application of Theorem 7.8 in [15].

**Lemma 4.6** (Lemma 3.9 in [18]). Let \( y := (y_1, \ldots, y_m) \) be a non-zero vector, and let \( v \in \{\pm 1\}^m \). Consider the function \( f : S_m \to \mathbb{R} \) defined by
\[ f(\pi) := \sum_{j=1}^{m} v_{\pi(j)} y_j. \]
Then, for all \( t > 0 \),
\[ \Pr (|f(\pi) - \mathbb{E}f| \geq t) \leq 2 \exp \left( -\frac{t^2}{64\|y\|_2^2} \right). \]

**Proof of Proposition 4.5.** By convexity and the union bound, it suffices to show that for any fixed \( v \in \{\pm 1\}^m \),
\[ \Pr \left( \| \tilde{B} v \|_2^2 \geq (128C_{3.2} + 2)mn^{1+2\epsilon} \right) \leq \exp(-n - n \ln 2). \]
To see this, we begin by noting that the random variables \( X_i := \langle \tilde{B} v, e_i \rangle \) are independent and
\[ X_i \sim \sum_{j=1}^{m} v_{\pi(i)} b_{ij}. \]
In particular, if \( \ell \) denotes the number of ones in \((v_1, \ldots, v_m)\), then
\[ |\mathbb{E}[X_i]| = \left| \sum_{j=1}^{m} \mathbb{E} \left[ v_{\pi(i)} \right] b_{ij} \right| = \left| \sum_{j=1}^{m} \frac{2\ell - m}{m} b_{ij} \right| \leq \left| \sum_{j=1}^{m} b_{ij} \right| \leq n^\epsilon \sqrt{m}. \]
By Lemma 4.6, for all \( t > 0 \), we have
\[ \Pr (|X_i - \mathbb{E}[X_i]| \geq t) \leq 2 \exp \left( -\frac{t^2}{64\|b_i\|_2^2} \right) \leq 2 \exp \left( -\frac{t^2}{64mn2\epsilon} \right). \]
In particular, by Definition 3.1, the random variables \( n^{-\epsilon}m^{-1/2}(X_i - \mathbb{E}[X_i]) \) are \( 8 \)-subgaussian, so that by Lemma 3.2,
\[ \Pr \left( \sum_{i=1}^{n} (X_i - \mathbb{E}[X_i])^2 \geq 64C_{3.2}mn^{1+2\epsilon} \right) \leq \exp(-2n) \leq \exp(-n - n \ln 2). \]
Finally, since
\[ \sum_{i=1}^{n} X_i^2 = \sum_{i=1}^{n} (X_i - \mathbb{E}[X_i] + \mathbb{E}[X_i])^2 \leq 2 \sum_{i=1}^{n} (X_i - \mathbb{E}[X_i])^2 + 2 \sum_{i=1}^{n} \mathbb{E}[X_i]^2 \]
Lemma 4.4 applied to the Proposition 4.2 Proposition 4.5 i.e. it suffices to show that is almost immediate.

\[
\Pr \left( \sum_{i=1}^{n} X_i^2 \geq (128C_{3.2} + 2)mn^{1+2\epsilon} \right) \leq \Pr \left( \sum_{i=1}^{n} (X_i - \mathbb{E}[X_i])^2 \geq 64C_{3.2}mn^{1+2\epsilon} \right) \leq \exp(-n - n \ln 2),
\]

which completes the proof. \(\square\)

Given the above results, Proposition 4.2 is almost immediate.

**Proof of Proposition 4.2.** 1. Let \(M_n\) be the \(n \times n\) random matrix appearing in the statement of the proposition, and let \(\mathcal{E}\) denote the ‘good’ event appearing in Lemma 4.4 i.e. \(\mathcal{E}\) is the event that there exists some \(I \subseteq [n]\) with \(|I| \geq n - 2n^{1-\epsilon}\) such that for all \(i \in I\),

\[
\left( \sum_{j=1}^{n} m_{ij}^2 \leq n^{1+2\epsilon} \right) \wedge \left( \left| \sum_{j=1}^{n} m_{ij} \right| \leq n^{(1/2)+\epsilon} \right).
\]

Since \(\Pr(\mathcal{E}^c) \leq 2\exp(-n^{1-\epsilon}/4)\) by Lemma 4.4, it suffices to show that

\[
\Pr \left( \left\{ \inf_{t \in \mathcal{T}} \| P_t M_n \|_{\infty \to 2} \geq C_{4.5}n^{1+\epsilon} \right\} \cap \mathcal{E} \right) \leq \exp(-n),
\]

where \(\mathcal{T}\) denotes the collection of subsets of \([n]\) of size at least \(n - 2n^{1-\epsilon}\). For this, note that since both the event \(\mathcal{E}\) as well as our distribution on \(n \times n\) matrices are invariant under permuting each row of \(M_n\) separately, it suffices to show the following: for each (fixed) \(n \times n\) matrix \(A_n\) for which there exists a subset \(I \subseteq [n]\) as above,

\[
\Pr \left( \| P_t A_n \|_{\infty \to 2} \geq C_{4.5}n^{1+\epsilon} \right) \leq \exp(-n),
\]

where \(\tilde{A}_n\) is the random matrix obtained by permuting each row of \(A_n\) independently and uniformly. But this follows immediately from Proposition 4.5 applied to the \(n \times n\) matrix \(P_t A_n\).

2. The proof of this part is very similar to the previous one. Let \(\mathcal{J}\) denote the collection of all subsets of \([n]\) of size \(n^{1-\delta}\) and let \(\mathcal{I}\) denote the collection of all subsets of \([n]\) of size at least \(n - 2n^{1-\epsilon}\). For \(J \in \mathcal{J}\) and \(\epsilon, \delta \in (0, 1/2)\) with \(\delta \geq 4\epsilon\), let \(\mathcal{E}_{\epsilon, \delta}(J)\) denote the event that there exists some \(I \in \mathcal{I}\) such that for all \(i \in I\),

\[
\left( \sum_{j \in J} m_{ij}^2 \leq n^{2\epsilon |J|} \right) \wedge \left( \left| \sum_{j \in J} m_{ij} \right| \leq n^{\epsilon \sqrt{|J|}} \right).
\]

We show that the desired conclusion in 2. holds with sufficiently high probability for fixed \(J \in \mathcal{J}\); the proof is completed by taking the union bound over the at most

\[
\binom{n}{n^{1-\delta}} \leq \exp(n^{1-\delta} \log n) \leq C(\epsilon) \exp(n^{1-3\epsilon})
\]

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choices for $J \subseteq \mathcal{J}$, where $C(\epsilon) \geq 1$ depends only on $\epsilon$, and the last inequality uses that $\delta \geq 4\epsilon$.

As before, by Lemma 4.4 applied to the operator $M_n P_J$ viewed as an $n \times |J|$ matrix, we see that $\Pr(\mathcal{E}_{\epsilon, \delta}(J)) \leq 2 \exp(-n^{1-\epsilon}/4)$. Therefore, it suffices to show that

$$\Pr \left( \inf_{i \in I} \|P_I M_n P_J\|_{\infty \to 2} \geq C_{4.5} n^{1+\epsilon-0.5\delta} \right) \cap \mathcal{E}_{\epsilon, \delta}(J) \leq \exp(-n).$$

But this follows by exactly the same argument (using Proposition 4.5) as above.

\[\square\]

4.2 Proof of Proposition 4.1

Proof of Proposition 4.1. Let $\epsilon = 0.01$ and $\delta = 0.2$, and let $\mathcal{G}$ denote the event appearing in the conclusion of Proposition 4.2 for this choice of $\epsilon, \delta$. Since $\Pr(\mathcal{G}^c) \leq C_{4.2}(0.01) \exp(-n^{1-\epsilon}/8)$, we may restrict ourselves to the event $\mathcal{G}$.

Let $\theta a \in \Gamma^2(\eta)$. Then, by definition, there exists some $0 < \theta \leq \text{LCD}_{\alpha, \gamma}(a) \leq n^{3/4} \eta^{-1}$ and some $w \in \mathbb{Z}^n \setminus \{0\}$ such that $\|\theta a - w\|_2 \leq \min\{\gamma \theta, \alpha\}$. Note also that $\|\theta a - w\|_\infty \leq \min\{\gamma \theta, 1\}$.

To leverage the control we have over various norms associated to the matrix $M_n$, we decompose the ‘error’ vector $\theta a - w$ into a ‘small’ part (with respect to the $\ell^\infty$-norm) and a ‘sparse’ part. Accordingly, let $v_{\text{small}} \in \mathbb{R}^n$ and let $v_{\text{small}} = \theta a - w - v_{\text{sparse}}$. Then, we have that

$$\|v_{\text{sparse}}\|_\infty \leq \min\{\gamma \theta, 1\}$$

and

$$\|v_{\text{small}}\|_\infty \leq \frac{\min\{\gamma \theta, \alpha\}}{\sqrt{\ell}}.$$  \hfill (12)

Indeed, the first inequality is immediate from $\|\theta a - w\|_\infty \leq \min\{\gamma \theta, 1\}$, whereas the second inequality follows from

$$\ell \cdot \|v_{\text{small}}\|_\infty^2 \leq \|\theta a - w\|_2^2.$$  

Let $J \subseteq [n]$ denote the support of $v_{\text{sparse}}$, so that $|J| \leq \ell$. If $|J| < \ell$, we arbitrarily extend it to a subset of size exactly $\ell$. Moreover, since we have restricted to $M_n \in \mathcal{G}$, let $I \subseteq [n]$ denote a subset of size at least $n - 2n^{1-\epsilon}$ with respect to which conclusion 1. of Proposition 4.2 holds. Then, from the triangle inequality, we have

$$\|P_I(J) P_I M_n(\theta a - w)\|_2 \leq \|P_I M_n v_{\text{small}}\|_2 + \|P_I(J) M_n P_J v_{\text{small}}\|_2$$

$$= \|P_I M_n v_{\text{small}}\|_2 + \|P_I(J) M_n P_J v_{\text{sparse}}\|_2$$

$$\leq \|P_I M_n\|_{\infty \to 2}\|v_{\text{small}}\|_\infty + \|P_I(J) M_n P_J\|_{\infty \to 2}\|v_{\text{sparse}}\|_\infty$$

$$\leq n^{0.5+\epsilon+0.5\delta} \min\{\gamma \theta, \alpha\} + n^{1+\epsilon-0.5\delta} \min\{\gamma \theta, 1\}$$

$$\leq \min\{n^{\epsilon+0.5\delta \theta}, n^{0.75+\epsilon+0.5\delta}\} + \min\{n^{0.5+\epsilon-0.5\delta \theta}, n^{1+\epsilon-0.5\delta}\}$$

$$\leq \min\{n^{0.41 \theta}, n^{0.91}\},$$

where the second line uses that $P_J v_{\text{sparse}} = v_{\text{sparse}}$; the fourth line uses Proposition 4.2 and Equation (12); the fifth line uses the parameter value $\gamma = n^{-1/2}$; and the last line uses the parameter values $\epsilon = 0.01$, $\delta = 0.2$.

Thus, if $\|M_n a\|_2 \leq \eta$, it follows from the triangle inequality that

$$\|P_I(J) P_I M_n w\|_2 = \|P_I(J) P_I M_n(\theta a) + P_I(J) P_I M_n(\theta a)\|_2$$

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Theorem 1.5.

\[ \exp(\text{Proposition 5.1}) \]

proceeds in two steps. The first step is to show that the \text{Proposition 3.7} \text{Proposition 3.8} \exp(\text{is small, provided we restrict ourselves only to applied to the matrix } P, \text{and } \eta \text{ where the fourth line follows since Lemma 5.2.})

\[ \Pr(\exists w \in V \text{ and } K \in \mathcal{K} : \|P_K M_n w\|_2 \leq 4 \min\{n^{0.41} \|w\|_2, n^{0.91}\}) \leq C_{5.1} \exp(-c_{5.1} n), \]

where \( C_{5.1} \geq 1 \) and \( c_{5.1} > 0 \) are constants depending only on \( u, v \).

The proof of this proposition is the content of the next two subsections.

5 Completing the proof of Theorem 1.5

In view of \text{Propositions 3.7 and 4.1}, it suffices to show the following in order to prove \text{Proposition 3.8}, and hence, complete the proof of \text{Theorem 1.5}.

\text{Proposition 5.1.} \Pr(\exists w \in V \text{ and } K \in \mathcal{K} : \|P_K M_n w\|_2 \leq 4 \min\{n^{0.41} \|w\|_2, n^{0.91}\}) \leq C_{5.1} \exp(-c_{5.1} n), \]

where \( C_{5.1} \geq 1 \) and \( c_{5.1} > 0 \) are constants depending only on \( u, v \).

The proof of this proposition is the content of the next two subsections.

5.1 Dealing with sparse integer vectors

Throughout this subsection and the next one, \( p = 2^{n^{0.001}} \) is a prime. Note, in particular, that \( p \gg n^{3/4} \). The proof of \text{Proposition 5.1} proceeds in two steps. The first step is to show that the probability of the event appearing in \text{Proposition 5.1} is small, provided we restrict ourselves only to sufficiently sparse integer vectors. Recall that \( S := \{w \in (\mathbb{Z}^n \setminus \{0\}) \cap [-p, p]^n : |\text{supp}(w)| \leq n^{0.99}\} \).

\text{Lemma 5.2.} \Pr(\exists w \in S \text{ and } K \in \mathcal{K} : \|P_K M_n w\|_2 \leq 4n^{0.41}\|w\|_2) \leq C_{5.2} \exp(-c_{3.15} n/4), \text{ where } C_{5.2} \geq 1 \text{ is an absolute constant.}

\text{Proof.} By taking the union bound over all the at most \( n^{n^{0.99}} \ll \exp(n^{0.991}) \) choices of \( K \in \mathcal{K} \), it suffices to show that for a fixed \( K_0 \in \mathcal{K} \),

\[ \Pr(\exists w \in S : \|P_{K_0} M_n w\|_2 \leq 4n^{0.41}\|w\|_2) \leq C \exp(-c_{3.15} n/2) \]

for some absolute constant \( C \geq 1 \). The number of vectors \( w \in S \) is at most

\[ \binom{n}{n^{0.99}} (3p)^{n^{0.99}} \ll 2^{n^{0.992}}. \]

By \text{Proposition 3.15} applied to the matrix \( P_{K_0} M_n \), for any such vector,

\[ \Pr(\|P_{K_0} M_n w\|_2 \leq c_{3.15} \sqrt{n}\|w\|_2/2) \leq \exp(-c_{3.15} n). \]

Therefore, the union bound gives the desired conclusion. \qed

5.2 Dealing with non-sparse integer vectors

It remains to deal with integer vectors with support of size at least \( n^{0.99} \). Recall that

\[ W := \{w \in (\mathbb{Z}^n \setminus \{0\}) \cap [-\eta^{-4}, \eta^{-4}^n] : |\text{supp}(w)| \geq n^{0.99}\}. \]
Note that for our choice of parameters, the natural map
\[ \varphi_p : W \to \mathbb{R}^n \]
is injective.

In view of Lemma 5.2, since \( \eta \leq n^{-3/2} \), and taking the union bound over all the at most \( n(\frac{n}{4n^{0.99}}) \ll \exp(n^{0.99}) \) choices of \( K \in \mathcal{K} \), the following proposition suffices to prove Proposition 5.1.

**Proposition 5.3.** For all \( K_0 \in \mathcal{K} \),
\[ \Pr \left( \exists w \in W : \|P_{K_0} M_n w\|_2 \leq 4n^{0.91} \right) \leq C_{5.3} \exp(-c_{5.3} n), \]
where \( C_{5.3} \geq 1 \) and \( c_{5.3} > 0 \) are constants depending only on \( u_{\xi}, v_{\xi} \).

The proof of Proposition 5.3 is accomplished by a union bound, following the strategy outlined in Equation (10). To execute this, we need the following preliminary claims.

**Claim 5.4.** There exists a constant \( C_{5.4} \geq 1 \) (depending only on \( u_{\xi}, v_{\xi} \)) such that for all \( w \in W \),
\[ \rho_{1,\xi}(w) \leq C_{5.4} n^{-0.495}. \]

The proof of this claim is a direct consequence of the following classical anti-concentration inequality due to Rogozin.

**Theorem 5.5** (see, e.g., [3]). Let \( n \in \mathbb{N} \), let \( \xi_1, \ldots, \xi_n \) be jointly independent random variables and let \( t_1, \ldots, t_n \) be some positive real numbers. Then, for any \( t \geq \max_j t_j \), we have
\[ \rho_{t, \sum_{j=1}^n \xi_j}(1) \leq C_{5.5} t \left( \sum_{j=1}^n t_j^2 \left( 1 - \rho_{t_j, \xi_j}(1) \right) \right)^{-1/2}, \]
where \( C_{5.5} \geq 1 \) is an absolute constant.

**Proof of Claim 5.4.** Let \( w \in W \). From (P2), we know that \( \rho_{v_{\xi}, \xi}(1) \leq u_{\xi} \). Therefore, for all \( j \in \text{supp}(w) \),
\[ \rho_{v_{\xi}, w_j \xi_j}(1) \leq \rho_{|w_j| v_{\xi}, w_j \xi_j}(1) \leq \rho_{v_{\xi}, \xi_j}(1) \leq u_{\xi}. \]
Hence, by Theorem 5.5,
\[ \rho_{v_{\xi}, \sum_{j=1}^n w_j \xi_j}(1) \leq \frac{C_{5.5}}{\sqrt{|\text{supp}(w)| (1 - u_{\xi})}}. \]
Since \( |\text{supp}(w)| \geq n^{0.99} \), and since \( \rho_{1, \xi}(w) \leq \max\{1, v_{\xi}^{-1}\} \rho_{v_{\xi}, \sum_{j=1}^n w_j \xi_j}(1) \), the desired conclusion follows.

**Claim 5.6.** For all \( w \in W \), \( \rho_{1, \xi}(w) \geq n^{-1/2} \eta^4/10 \).

**Proof.** The random variable \( \sum_{j=1}^n w_j \xi_j \) has mean 0 and variance at most \( n \eta^{-8} \). Therefore, by Markov’s inequality,
\[ \Pr \left( \left| \sum_{j=1}^n w_j \xi_j \right| \leq 2\sqrt{n} \eta^{-4} \right) \geq \frac{3}{4}. \]
Hence, by the pigeonhole principle, it follows that
\[ \rho_{1, \xi}(w) \geq n^{-1/2} \eta^4/10, \]
as desired.
For the next claim, let
\[ W_t := \{ w \in W : \rho_{t,\xi}(w) \in [t, 2t) \}. \]

Note that the previous two claims show that \( W_t \) is nonempty only if \( n^{-1/2} \eta^4/10 \leq t \leq C5.4 n^{-0.495} \).

**Claim 5.7.** For all \( n^{-1/2} \eta^4/10 \leq t \leq C5.4 n^{-0.495} \),
\[ |W_t| \leq C5.7 \left( \frac{C_{1.4} t^{-1}}{n^{0.45}} \right)^n, \]
where \( C5.7 \geq 1 \) is a constant depending only on \( C_{1.4} \cdot C5.4 \).

**Proof.** Fix \( s = n^{0.998} \) and \( k = n^{0.098} \). Then, \( k \leq \sqrt{s} \leq s \leq n / \log n, n^{-1/2} \eta^4 \gg \max\{e^{-s/k}, s^{-k/4}\} \), and \( 2^{n/s} \geq p \gg n^{1/2} \eta^{-4} \). Hence, for large enough \( n \), the hypotheses of **Theorem 1.4** are satisfied, so that
\[
|W_t| = |\varphi_p(W_t)| \\
\leq |\varphi_p(V_t)| \\
\leq \left( \frac{5np}{s} \right)^s + \left( \frac{C_{1.4} t^{-1}}{n^{0.45}} \right)^n \\
\leq 2 \left( \frac{C_{1.4} t^{-1}}{n^{0.45}} \right)^n,
\]
where the first line follows from the injectivity of \( \varphi_p \) on \( W \), the third line follows from **Theorem 1.4**, and the last line follows since \( t^{-1} \gg n^{0.49} \). \( \Box \)

We now have all the ingredients to prove **Proposition 5.3**.

**Proof of Proposition 5.3.** For all \( n \) sufficiently large, we have
\[
\Pr \left( \exists w \in W : \|P_{K_0} M_n w\|_2 \leq 4n^{0.91} \right) \leq \sum_{t = 0.1n^{-1/2} \eta^4}^{n^{-0.494}} \Pr \left( \exists w \in W_t : \|P_{K_0} M_n w\|_2 \leq 4n^{0.91} \right) \\
\leq \sum_{t = 0.1n^{-1/2} \eta^4}^{n^{-0.494}} (400n^{0.41})^n \sup_{z \in \mathbb{Z}^n} \Pr (\exists w \in W_t : P_{K_0} M_n w \in D_z) \\
\leq \sum_{t = 0.1n^{-1/2} \eta^4}^{n^{-0.494}} (400n^{0.41})^n |W_t| (2t)^{|K_0|} \\
\leq \sum_{t = 0.1n^{-1/2} \eta^4}^{n^{-0.494}} (400n^{0.41})^n \cdot C5.7 \left( \frac{C_{1.4} t^{-1}}{n^{0.45}} \right)^n \cdot (2t)^{n-4n^{0.99}} \\
\leq \sum_{t = 0.1n^{-1/2} \eta^4}^{n^{-0.494}} C5.7 (800C_{1.4} n^{-0.04})^n \cdot t^{-4n^{0.99}} \\
\leq C5.7 n \cdot (800C_{1.4} n^{-0.04})^n \cdot \eta^{-20n^{0.99}} \\
\leq C5.7 n \cdot (800C_{1.4} n^{-0.04})^n \cdot 2^{20n^{0.991}} \\
\leq C5.3 \exp(-C5.3 n),
\]
where the second line follows from **Equation (10)**, the fourth line follows from **Claim 5.7**, and the sixth and seventh lines follow from the assumed bounds on \( \eta \). \( \Box \)
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