SPECTRAL FLOW INSIDE ESSENTIAL SPECTRUM II:
RESONANCE SET AND ITS STRUCTURE

NURULLA AZAMOV

Abstract. This paper is a continuation of the study of spectral flow inside essential spectrum initiated in [Az]. Given a point λ outside the essential spectrum of a self-adjoint operator $H_0$, the resonance set, $\mathcal{R}(\lambda)$, is an analytic variety which consists of self-adjoint relatively compact perturbations $H_0 + V$ of $H_0$, for which λ is an eigenvalue. One may ask for criteria for the vector $V$ to be tangent to the resonance set. Such criteria were given in [Az2].

In this paper, we study similar criteria for the case of λ inside the essential spectrum of $H_0$. For the case $\lambda \in \sigma_{ess}(H_0)$ the resonance set is defined in terms of the well-known limiting absorption principle. Among the results of this paper is that the resonance set contains plenty of straight lines, moreover, given any regular relatively compact perturbation $V$ there exists a finite rank self-adjoint operator $\tilde{V}$, such that the straight line $H_0 + \mathcal{R}(V - \tilde{V})$ belongs to the resonance set.

Another result of this paper is that inside the essential spectrum there exist plenty of transversal to the resonance set perturbations $V$ which have order $\geq 2$, in contrast to what happens outside the essential spectrum, [Az2].

1. Introduction

The importance of study of the spectrum of self-adjoint operators is well-known. It is equally important to study how the spectrum changes when a self-adjoint operator, $H_0$, undergoes a perturbation. Assuming that the essential spectrum, $\sigma_{ess}(H_0)$, stays stable, one important question is to study the net number of eigenvalues of $H_0$ which cross a given point $\lambda \notin \sigma_{ess}(H_0)$ as $H_0$ gets perturbed to $H_1$ via a continuous deformation $H_r$, $r \in [0, 1]$. The resulting number is called the spectral flow. There are different approaches to the study of spectral flow among which are an intersection number [APS], a total Fredholm index [Ph], an axiomatic approach [RoSa] and a recent total resonance index [Az2].

Another approach closely related to the approach via the intersection number is as follows. Let $\mathcal{A}$ be a real affine space of relatively compact self-adjoint perturbations of $H_0$ which leave the essential spectrum unchanged, and consider the set, $\mathcal{R}(\lambda)$, of all operators from $\mathcal{A}$ for which $\lambda$ is an eigenvalue. The set $\mathcal{R}(\lambda)$ is an analytic variety, which we call the resonance set. It can be shown that for $\lambda \notin \sigma_{ess}(H_0)$ the variety $\mathcal{R}(\lambda)$ has co-dimension 1. As such, this variety divides the affine space $\mathcal{A}$ near a point $H_0 \in \mathcal{A}$ into two or more parts, which we will call resonance cells or simply cells. Assuming for simplicity that $\lambda$ is a simple eigenvalue of $H_0$, there will be only two parts, let them be $\mathcal{R}_+$ and $\mathcal{R}_-$. The $\mathcal{R}_+$ consists of those points in $\mathcal{A}$ which have an eigenvalue slightly larger than $\lambda$, and a similar interpretation applies to $\mathcal{R}_-$. In these terms, if $H_0 \in \mathcal{R}_-$ and $H_1 \in \mathcal{R}_+$, then one should expect the
spectral flow of a norm continuous path connecting $H_0$ to $H_1$ through $\lambda$ to be $+1$. A smooth path connecting $H_0$ and $H_1$ can intersect the resonance set at many points. Thus, the following question arises: let $H_0 \in \mathcal{A}$ and let $V$ be a self-adjoint operator from the real vector space $\mathcal{A}_0 := \mathcal{A} - H_0$. If $H_0$ is perturbed in the direction $V$, where the eigenvalue $\lambda$ of $H_0$ will move: to $\mathcal{R}_+$ or $\mathcal{R}_-$? What if $H_0$ is perturbed in the direction $-V$? Intuitively it is obvious that if $V$ is tangent to the resonance set, then $H_0$ may get perturbed into the same cell, resulting in zero contribution to the spectral flow through $\lambda$. This occurs if the direction $V$ is tangent to the resonance set at $H_0$.

Thus, we conclude that it is interesting to find criteria for the tangency of a direction $V$ at a point $H_0$ of the resonance set. For a point $\lambda$ outside the essential spectrum this program was carried out in [AZ2]. The aim of this paper is to do the same for $\lambda$ inside the essential spectrum. In this case the resonance set $\mathcal{R}(\lambda)$ ought to be defined in terms of the limiting absorption principle. We assume that there is a fixed – rigging – operator $F$ which is $|H_0|^{1/2}$-compact, and using it we define $\mathcal{R}(\lambda)$ as the set of operators $H \in \mathcal{A}$ for which the norm limit $T_{\lambda+i0}(H)$ does not exist. If $\lambda$ lies outside the essential spectrum then this definition coincides with the one defined above via the eigenvalue equation. But inside the essential spectrum the character of the set $T_{\lambda+i0}(H)$ changes as it takes into account not only pure point spectrum but also singularly continuous spectrum too.

This paper is a natural continuation of the study of the essential spectrum initiated in [AZ] and [AZ2]. The reader should consult sections 2 and 3 of [AZ], which also has a detailed index, for all the relevant definitions which are omitted here. For more motivation for this work one can also consult the upcoming paper [AD].

As an example, among the results of this paper is the following, see Theorem [AZ3]. Assuming that for a semi-regular point $\lambda$ the limit $T_{\lambda+i0}(H_0)$ does not exist. Are there perturbations $V$ such that none of the limits $T_{\lambda+i0}(H_0 + rV), r \in \mathbb{R}$, exist? As it turns out, there are plenty such perturbations $V$, namely, any regular direction $V$ has a finite rank perturbation with this property.

2. Tangency Properties of Directions in the Case of $\lambda \in \sigma_{ess}$

Suppose, as usual, that a self-adjoint operator $H_0$ acts on a rigged Hilbert space $(\mathcal{H}, F)$. One has to assume some sort of compatibility between $H_0$ and $F$, which makes things work. To this end, we assume that $F$ is $|H_0|^{1/2}$-compact. Let $\lambda$ be a point inside the essential spectrum $\sigma_{ess}$ of $H_0$ such that the norm limit $T_{\lambda+i0}(H_0)$ of the sandwiched resolvent

$$T_{\lambda+i0}(H_0) = FR_{\lambda+i0}(H_0)F^*$$

exists. As usual, let $V$ be a self-adjoint operator from the real Banach space $\mathcal{A}_0 = F^*B_{sa}(\mathcal{K})F$ and let $H = H_0 + V$. Elements of $\mathcal{A}_0$ we call directions. The set of directions $V$ for which $T_{\lambda+i0}(H_0 + V)$ does not exist we call the resonance set and denote it $\mathcal{R}(\lambda)$. The set $\mathcal{R}(\lambda)$ is the essential spectrum case analogue of the set of operators for which $\lambda$ is an eigenvalue. It is not difficult to show that the set $\mathcal{R}(\lambda)$ is an analytic variety, in the sense that its intersection with any finite dimensional subspace of $\mathcal{A}_0$ is the set of zeros of a finite system of real analytic functions. For $\lambda$ outside the essential spectrum, the resonance set has co-dimension one, see [AZ2]. However, inside the essential spectrum the co-dimension is usually larger than one.
The co-dimension of the resonance set is closely related to the question of path-independence of singular SSF, or in other terms to the question of exactness of the infinitesimal singular spectral shift 1-form.

Since the resonance set is a smooth variety, it makes sense to ask whether a given direction $V$ is tangent to the resonance set at a given point $H_0$. In [Az2] it was found that for $\lambda$ outside the essential spectrum a direction $V$ is tangent if and only if the equation

$$[(1 - rR_\lambda(H_r)V)]^2 \varphi = 0.$$ 

has a non-zero solution for some, and therefore, for any non-resonant value of $r$. Here we discuss similar results for $\lambda$ inside $\sigma_{ess}$.

One of the results of [Az2] asserts that for a point $\lambda$ outside the essential spectrum $\sigma_{ess}$ of $H_0$ the order of a regular direction $V$ is equal to the order of tangency of $V$ to the resonance set $\Re(\lambda)$. In this paper we show that inside the essential spectrum a tangent to order $k$ direction has order at least $k$, but that the reverse in general is not true, in contrast to the case of $\lambda \notin \sigma_{ess}$. The method of proof used in [Az2] does not work inside the essential spectrum, since it relies on the eigenvalue equation $H_\lambda \varphi(s) = \lambda \varphi(s)$ for an analytic path of operators $H_\lambda$, which is not available inside $\sigma_{ess}$. However, the eigenvalue equation can be rewritten as

$$(1 + (s - r)R_\lambda(H_r)V)\varphi(s) = 0.$$ 

This equation can be adapted for $\lambda$ inside the essential spectrum by writing

$$(1 + (s - r)T_{\lambda+0}(H_r)J)u(s) = 0.$$ 

It turns out that this little trick allows to overcome difficulties.

Recall that a real number $\lambda$ is called semi-regular or essentially regular, if for some operator $H$ from $A$ the limit $T_{\lambda+0}(H)$ exists. Let $\lambda$ be a semi-regular point. Let $H(s)$ be an analytic path in $A(F)$ which consists of resonant at $\lambda$ operators. Recall that “$H$ is resonant at $\lambda$” means $T_{\lambda+0}(H)$ does not exist, that is, $H \notin \Re(\lambda)$. Let $H_0$ be an operator regular at $\lambda$, and $H(s) - H_0 = V(s) = F^* J(s) F$. Since $H_0$ is regular at $\lambda$, the limit $T_{\lambda+0}(H_0)$ is defined, and since $H(s)$ is $\lambda$-resonant, there exists a smooth path of vectors $u(s)$ such that

$$1 + T_{\lambda+0}(H_0)J(s) u(s) = 0. \quad (1)$$

Now let’s take a straight line $H_s = H_0 + sV$ which intersects the curve $H(s)$ at $H(r_\lambda)$.

**Theorem 2.1.** If a direction $V$ is tangent to the $\lambda$-resonance set then $u'(r_\lambda)$ is a resonance vector of order 2 and, in particular, the order of $V$ is at least 2.

**Proof.** By assumption there exists a $\lambda$-resonant curve $H(s)$ of operators such that $V$ is tangent to the curve at the point $H(r_\lambda)$. There is a straight line $H_s = H_0 + sV$ so that $H_1 = H(r_\lambda), V = H'(r_\lambda)$, and $H_0$ is regular at $\lambda$. Let $H(s) - H_0 = F^* J(s) F$. Since $H(s)$ is resonant at $\lambda$ for all $s$, there exists a smooth path of vectors $u(s)$ with $u(r_\lambda) \neq 0$ such that $u'(r_\lambda)$ holds. We differentiate this equation and take $s = r_\lambda$:

$$1 + T_{\lambda+0}(H_0)J(r_\lambda) u'(r_\lambda) = -T_{\lambda+0}(H_0)J'(r_\lambda) u(r_\lambda).$$
Since $H(r_\lambda) = H_0 + V$ is resonant at $\lambda$ and $J'(r_\lambda) = V$, the right hand side of the last display equals $u(r_\lambda)$. Indeed,

$$-T_{\lambda+i0}(H_0)J'(r_\lambda)u(r_\lambda) = -T_{\lambda+i0}(H_0)Vu(r_\lambda) = u(r_\lambda),$$

where the last equality is a resonance equation of order 1. Hence, applying the operator in the square brackets to both sides again gives

$$[1 + T_{\lambda+i0}(H_0)J(r_\lambda)]^2 u'(r_\lambda) = 0.$$ 

Therefore, $u'(r_\lambda)$ is a resonance vector of order 2 and thus the direction $V = F^*J'(r_\lambda)F$ has order at least 2 at $H(r_\lambda)$.

In the proof of the following theorem, for a rigging $F$, we will use the equivalence

$$F^*JF = 0 \iff J = 0.$$ 

**Theorem 2.2.** If a direction $V$ is tangent to the resonance set to order $k$ then the vectors $u'(r_\lambda), \ldots, u^{(k-1)}(r_\lambda)$ are resonance vectors of order, respectively, $2, \ldots, k$, and, in particular, the order of $V$ is at least $k$.

**Proof.** By premise, there exists a resonant path $H(s)$, such that $H(r_\lambda)$ is the operator at which $V$ is tangent to the resonance set and

$$H'(r_\lambda) = V, \quad H''(r_\lambda) = 0, \ldots, \quad H^{(k-1)}(r_\lambda) = 0.$$ 

Choose a straight line $H_s = H_0 + sV$ so that $H_1 = H(r_\lambda)$, and $\lambda$ is regular at $H_0$. We proceed by induction on $k$. The induction base is the previous theorem. Assume the claim for $k - 1$. Since the path $H(s)$ is resonant, there exists an analytic path $u(s)$ in $K$ such that (1) holds. Differentiating this equality $k - 1$ times and replacing $s = r_\lambda$ gives, taking into account (3) and (2).

$$[1 + T_{\lambda+i0}(H_0)J(r_\lambda)]^{k-1} u^{(k-1)}(r_\lambda) = -(k-1)T_{\lambda+i0}(H_0)J'(r_\lambda)u^{(k-2)}(r_\lambda).$$

Here we note that $J'(r_\lambda) = J$ where $J$ is from $V = F^*JF$. The operator $T_{\lambda+i0}(H_0)J$ preserves the order and the operator $[1 + T_{\lambda+i0}(H_0)J(r_\lambda)]$ decreases it. Thus, since by induction assumption the vector $u^{(k-2)}(r_\lambda)$ has order $k - 1$, it follows from the last equality that the vector $u^{(k-1)}(r_\lambda)$ has order $k$. In particular, the direction $V$ has order at least $k$.

Further, similar to $\text{Az2}$, one can show that

$$A_{\lambda+i0}(r_\lambda)u^{(k-1)}(r_\lambda) = (k-1)u^{(k-2)}(r_\lambda).$$

Proof of this equality is this: in the penultimate display replace $H_0$ by $H_s$ with complex variable $s$ and then take the integral of the both sides along a small contour which encloses $r_\lambda$.

Thus, tangent directions have order at least 2. However, inside the essential spectrum there are plenty of transversal directions which also have order at least 2. We will call such directions collar directions.

Outside the essential spectrum a direction has order $\geq 2$ iff it is tangent to the resonance set. Therefore, a linear combination of two directions of order $\geq 2$ is also a direction of order $\geq 2$. It is reasonable to believe that there should be an algebraic proof of this fact which hopefully would work inside essential spectrum as well.
3. Reduction of direction

Here we will adapt the results of [Az2 §7.1] to the case where a spectral point $\lambda$ belongs to $\sigma_{ess}$. The fact that $\lambda \in \sigma_{ess}$ creates some difficulties which we aim to overcome here.

So, let $\lambda \in \sigma_{ess}$ be a semi-regular point for a self-adjoint $H_0 \in \mathcal{A}$. Let $V = F^* J F$ be a regular direction with property $S$, see [Az §13.3] for the definition of the latter. The property $S$ can be equivalently characterised in many ways [Az Proposition 13.3.1], one of them is $Q_{\lambda - i0} J P_{\lambda + i0} = J P_{\lambda + i0}$. In particular, a point $\lambda$ has the property $S$ if and only if the bounded operator $J P_{\lambda + i0}$ on $\mathcal{K}$ is self-adjoint and thus can be treated as a direction.

Before proceeding further, we note that the property $S$ is a generic property. Suffices to say that all directions of order 1, all positive directions and all directions without the property $S$, in case $\lambda \notin \sigma_{ess}$ possess this property. In fact, it is not easy to present examples of directions without the property $S$, though this is mainly due to the fact that it is not easy to present examples of points of order $> 1$.

In this section we will use the following notation. $H_0$ is semi-regular at $\lambda$, $V = F^* J F$ is a regular direction and

$$\tilde{V} := F^* \tilde{J} F, \quad \text{where} \quad \tilde{J} := J P_{\lambda + i0}(H_0, V).$$

We often use notation $P_+ := P_{\lambda + i0}(H_0, V)$ and $T_+ := T_{\lambda + i0}(H_r)$, where $r$ is some regular point of the choice which should be clear from the context. We consider $J \mapsto \tilde{J}$ as a map on regular elements of $\mathcal{A}_0$. As usual $H_r = H_0 + r V$, and $\tilde{H}_r = H_0 + r \tilde{V}$.

The following theorem is the analogue of [Az2 Theorem 7.1.1]. It is not much surprising since it is unlikely for a direction at a semi-regular point not to be regular, but alas in mathematics we should take care of all possibilities even if they are extremely unlikely.

**Theorem 3.1.** Let $V$ be a regular direction with property $S$ at a semi-regular point $\lambda$ for $H_0$, then the direction $\tilde{V}$ is also regular.

**Proof.** By definition, that $V$ is regular means that $T_{\lambda + i0}(H_r)$ exists for some $r \in \mathbb{R}$, and we have to show the same for $T_{\lambda + i0}(H_r)$. The second resolvent identity applied to $H_r = H_0 + r (V - V)$ gives

$$T_z(\tilde{H}_r) = T_z(H_r + r F^*(\tilde{J} - J) F) = \left[ 1 + r T_z(H_r)(\tilde{J} - J) \right]^{-1} T_z(H_r).$$

Thus, $T_{\lambda + i0}(H_r)$ exists iff the operator $1 + r T_{\lambda + i0}(H_r)(\tilde{J} - J)$ is invertible. Assume the contrary, that is (as the operator is Fredholm with zero index), for some non-zero $\varphi$

$$[1 + r T_{\lambda + i0}(H_r)(\tilde{J} - J)] \varphi = 0.$$

Since $P_+$ and $T_{\lambda + i0}(H_r) J$ commute, applying $P_+$ to both sides of the equation above gives $P_+ \varphi = 0$. This gives $\tilde{J} \varphi = 0$, which combined with the last display gives

$$[1 - r T_{\lambda + i0}(H_r) J] \varphi = 0.$$

This implies that $\varphi \in \text{im} P_+$, and thus, $\varphi = P_+ \varphi = 0$. $\square$

**Theorem 3.2.** Let $V = F^* J F$ be a regular direction with property $S$ at a semi-regular point $\lambda$ for $H_0$, then for any non-resonance $s$ (w.r.t. $\lambda + i0$)

$$T_{\lambda + i0}(\tilde{H}_s) J P_+ = T_{\lambda + i0}(H_s) J P_+.$$
Proof. Given Theorem 3.1 the proof follows verbatim that of [Az2, Theorem 7.1.2] with some notational changes. Still, we give it here for reader’s convenience. We let \( A_+(s) = T_{\lambda+i0}(H_s)J \). By the second resolvent identity, we have

\[
(E) := T_{\lambda+i0}(\tilde{H}_s)JP_+ = \left[1 - sT_{\lambda+i0}(H_s)(J - \tilde{J})\right]^{-1}T_{\lambda+i0}(H_s)JP_+
\]

\[
= \left[1 - sA_+(s)(1 - P_+)\right]^{-1}A_+(s)P_+.
\]

The operator \( \tilde{A}_+(s) = A_+(s)(1 - P_+) \) is the holomorphic part of the Laurent expansion of \( A_+(s) \) at \( s = 0 \). So, we have for small enough \( s \)

\[
(E) = \left[1 - s\tilde{A}_+(s)\right]^{-1}A_+(s)P_+
\]

\[
= \left[1 + s\tilde{A}_+(s) + s^2\tilde{A}_+^2(s) + \ldots \right]A_+(s)P_+.
\]

Since \( [A_+(s), P_+] = 0 \) and \( \tilde{A}_+(s)P_+ = 0 \) it follows that \( (E) = A_+(s)P_+ \), as required. By analytic continuation, the equality holds for all not necessarily small \( s \). □

**Theorem 3.3.** Let \( V \) be a regular direction with property \( S \) at a semi-regular point \( \lambda \) for \( H_0 \). Then

\[
P_{\lambda+i0}(H_0, \tilde{V}) = P_{\lambda+i0}(H_0, V)
\]

and

\[
A_{\lambda+i0}(H_0, \tilde{V}) = A_{\lambda+i0}(H_0, V).
\]

Proof. Given Theorems 3.1 and 3.2 this proof follows verbatim that of [Az2, Theorem 7.1.3] with some obvious notational changes. Still, we give this proof as well. We prove the second equality, the first one is proved by the same argument. Using the definition of \( A_+ \) and Theorem 3.2 we have

\[
A_+(H_0, \tilde{V}) = \frac{1}{2\pi i} \oint_{C(0)} sT_+(\tilde{H}_s)\tilde{J} ds
\]

\[
= \frac{1}{2\pi i} \oint_{C(0)} sT_+(H_s)JP_+ ds
\]

\[
= A_+P_+
\]

\[
= A_+.
\]

The following theorem is a direct consequence of the previous theorems (see also Theorems 7.1.4 and 7.1.5 in [Az2]).

**Theorem 3.4.** Let \( V \) be a regular direction with property \( S \) at a semi-regular point \( \lambda \) for \( H_0 \). Then the resonance matrices of the directions \( V \) and \( \tilde{V} \) are equal, and therefore so are their resonance indices.

Indeed, the resonance matrix of \( V \) is \( JP_+(H_0, V) \) and Theorem 3.3 gives

\[
JP_+(H_0, V) = JP_+(H_0, V) \cdot P_+(H_0, V) = \tilde{J} \cdot P_+(H_0, \tilde{V}),
\]

where the last operator is the resonance matrix of \( \tilde{V} \). Further, the resonance index is the signature of the resonance matrix, see [Az2, Theorem 9.2.1], and thus the resonance indices of \( V \) and \( \tilde{V} \) are also equal.

Definition of plain homotopic directions is the same as [Az2, Definition 7.1.6].
Theorem 3.5. Let $V$ be a regular direction with property $S$ at a semi-regular point $\lambda$ for $H_0$. Then the directions $V$ and $\tilde{V}$ are plain homotopic.

Proof. The proof follows verbatim that of [Az2, Theorem 7.1.7] with some obvious notational changes and one more change: everywhere the statement of this form “$H - \lambda$ is invertible” should be replaced by “$T_{\lambda+\iota 0}(H)$ exists”. □

Lemma 3.6. The idempotent $P_z(H_0, rV)$ does not depend on $r \neq 0$.

Proof. By definition,

$$P_z(H_0, rV) = \frac{1}{2\pi i} \oint_{C(0)} T_z(H_{sr}) rJ \, ds.$$  

From this one can see that this idempotent does not depend on $r$ including its sign. Indeed, scaling of $r$ results in scaling of $C(0)$, which does not affect the contour integral (for large $r$ one can always choose $C(0)$ to be small enough). Replacing $r$ by $-r$ also does not change the left side:

$$P_z(H_0, -V) = \frac{1}{2\pi i} \oint_{C(0)} T_z(H_{0} - sV)(-J) \, ds$$

$$= \frac{1}{2\pi i} \oint_{C(0)} T_z(H_{0} + tV)J \, dt = P_z(H_0, V).$$

since $t = -s$ also traces the contour $-C(0)$ in the counterclockwise direction. □

At the same time, it is obvious that

$$\text{ind}_{\text{res}}(\lambda; H_0, -V) = -\text{ind}_{\text{res}}(\lambda; H_0, V).$$

Recall that for a semi-regular point $\lambda$ the resonance set

$$\mathbb{R}(\lambda) := \{ H \in \mathcal{A}: H \text{ is not regular at } \lambda \}$$

is an analytic variety of co-dimension 1 outside the essential spectrum and $> 1$ in the inside. Therefore, inside the essential spectrum we can find an intersection of the resonance set with three dimensional real affine plane so that the intersection is one-dimensional. In this 3D section we can transversally deform $V$ changing the sign of the resonance index. That is, the resonance index is not homotopically invariant for $\lambda$ inside $\sigma_{\text{ess}}$. For a semi-regular point of geometric multiplicity 1 the resonance index can have only the values $\pm 1$ or 0, according to the U-turn inequality, see [Az].

A transversal direction $V$ of resonance index $+1$ can be transversally deformed to $-V$ with resonance index $-1$. Thus, along the way we get a transversal direction of resonance index 0. This indicates that there is a set of “invisible” directions of resonance index zero, which form a barrier for deforming $+1$ resonance index directions to $-1$ resonance index directions. The U-turn inequality shows that these zero resonance index directions should have the algebraic multiplicity at least 2. Thus, we conclude that

Proposition 3.7. Inside the essential spectrum there exist transversal directions of order $\geq 2$ (that is, collar directions).

This is in contrast to [Az2, Theorem 4.3.3] which asserts that outside the essential spectrum a regular direction has order 1 iff it is transversal.
**Theorem 3.8.** Let $V = F^* JF$ be a regular direction with property $S$ at a semi-regular point $\lambda$ for $H_0$. Let $s$ and $t$ be real numbers (such that $s + t \neq 0$) such that $sV + t\tilde{V}$ is regular. Then

$$T_{\lambda+i0}(H_0 + sV + t\tilde{V})JP_+ = T_{\lambda+i0}(H_0 + (s + t)V)JP_+.$$ 

**Proof.** Let $H_{s,t} = H_0 + sV + t\tilde{V}$. In this notation Theorem 3.2 asserts

$$T_{\lambda+i0}(H_{s,0})JP_+ = T_{\lambda+i0}(H_{0,s})JP_+.$$ 

Thus, using alternately the second resolvent identity and the last display twice we get

$$T_{\lambda+i0}(H_{s,t})JP_+ = \left[1 + tT_{\lambda+i0}(H_{s,0})JP_+\right]^{-1} T_{\lambda+i0}(H_{s,0})JP_+$$

$$= \left[1 + tT_{\lambda+i0}(H_{0,s})JP_+\right]^{-1} T_{\lambda+i0}(H_{0,s})JP_+$$

$$= T_{\lambda+i0}(H_{0,s+t})JP_+$$

$$= T_{\lambda+i0}(H_{s+t,0})JP_+. \quad \square$$

**Corollary 3.10.** (a) The resonance set contains a lot of straight lines.

(b) At any semi-regular $H_0$ there are plenty of non-regular directions. Namely, any regular direction has a finite-rank perturbation which is not regular.

Outside the essential spectrum ($\lambda \notin \sigma_{ess}$) this corollary can be proved in another simpler way, and if in addition $\lambda$ is a simple eigenvalue and $V$ has order 1 then a proof becomes particularly trivial.
Theorem 3.11. For $H_0$ semi-regular at $\lambda$, and $V$ a regular direction, we have

$$P_{\lambda+i0}(H_0, V)P_{\lambda+i0}(H_0 + V - \hat{V}, V) = P_{\lambda+i0}(H_0 + V - \hat{V}, V)P_{\lambda+i0}(H_0, V)$$

and

$$P_{\lambda+i0}(H_0, V)A_{\lambda+i0}(H_0 + V - \hat{V}, V) = A_{\lambda+i0}(H_0 + V - \hat{V}, V)P_{\lambda+i0}(H_0, V)$$

Proof. We have, using Theorem 3.8,

(4) \[2\pi i P_{\lambda+i0}(H_0 + V - \hat{V}, V) P_+ = \oint_{C(0)} T_{\lambda+i0}(H_0 + V - \hat{V} + sV) J P_+ ds\]

Thus,

$$P_{\lambda+i0}(H_0 + V - \hat{V}, V) P_+ = P_+.$$

A display in the proof of Theorem 3.8 implies that $P_+$ commutes with $T_{\lambda+i0}(H_{s,t}) J$. Therefore, the first display of this proof shows that $P_+$ commutes with $P_{\lambda+i0}(H_0 + V - \hat{V}, V)$.

Proof of other equalities is similar. \qed

Theorem 3.12. Under the premise of Theorem 3.8, for any real numbers $\alpha$ and $\beta$, such that $\alpha + \beta \neq 0$,

$$P_{\lambda+i0}(H_0, \alpha V + \beta \hat{V}) = P_{\lambda+i0}(H_0, V)$$

and,

$$A_{\lambda+i0}(H_0, \alpha V + \beta \hat{V}) = \frac{1}{\alpha + \beta} A_{\lambda+i0}(H_0, V).$$

Proof. We have, by definition,

$$P_{\lambda+i0}(H_0, \alpha V + \beta \hat{V}) = \frac{1}{2\pi i} \oint_{C} T_{\lambda+i0}(H_0 + s(\alpha V + \beta \hat{V})) J J ds.$$

We split the last integral into the sum of two integrals and calculate them separately. By Theorem 3.8 for the second summand we have

$$\frac{1}{2\pi i} \oint_{C} T_{\lambda+i0}(H_0 + s(\alpha V + \beta \hat{V})) \beta \hat{J} ds = \frac{\beta}{\alpha + \beta} P_+.$$

Now, we consider the first summand

$$(E) = \frac{1}{2\pi i} \oint_{C} T_{\lambda+i0}(H_0 + s(\alpha V + \beta \hat{V})) \alpha J ds.$$
We have, for small enough $\beta$, using the fact that $A_+ \coloneqq A_{\lambda+i0}(\alpha)$ and $P_+$ commute
\[
T_{\lambda+i0}(H_0 + s(\alpha V + \beta \hat{V}))),J = \left[1 + s \beta T_{\lambda+i0}(H_0 + s\alpha V)JP_+\right]^{-1}T_{\lambda+i0}(H_0 + s\alpha V)J = \left[1 + s \beta A_{\lambda+i0}(\alpha)\right]^{-1}A_{\lambda+i0}(\alpha) \\
= \left[1 - s \beta A_+P_+ + s^2 \beta^2 A_+^2 - \ldots \right]A_+ \\
= \left[1 - P_+ + P_+ - s \beta A_+P_+ + s^2 \beta^2 A_+^2 - \ldots \right]A_+ \\
= \tilde{A}_+ + \left[1 - s \beta A_+ + s^2 \beta^2 A_+^2 - \ldots \right]A_+P_+ \\
= \tilde{A}_+ + \left[1 + s \beta A_+\right]^{-1}A_+P_+ \\
= \tilde{A}_+ + A_{\lambda+i0}(\alpha + s\beta)P_+.
\]
The function $\tilde{A}_+$ is holomorphic at $s = 0$, so its integral vanishes. Thus,
\[
(E) = \frac{\alpha}{\alpha + \beta}P_+.
\]
Combining this with the second display of the proof completes the proof of the first equality, for small enough $\beta$. For other $\beta$ the equality holds by analytic continuation. The second one is proved by the same argument. \qed

Of course, in these theorems one can replace $\lambda + i0$ by $\lambda - i0$.

3.1. The map $V \mapsto \tilde{V}$. Let $H_0$ be semi-regular at $\lambda$. To any regular direction $V$ with property $S$ we can assign the direction $\tilde{V} = F^*JF = F^*J\lambda+i0(H_0, V)F$. We summarise some properties of this map.

**Theorem 3.13.** Fix a semi-regular point $\lambda$ for $H_0 \in \mathcal{A}$. The map $V \mapsto \tilde{V}$, defined on regular directions $V$, has the following properties:

(1) The direction $\tilde{V}$ is regular.
(2) The direction $V - \tilde{V}$ is not regular.
(3) $P_{\lambda+i0}(H_0, \tilde{V}) = P_{\lambda+i0}(H_0, V)$.
(4) $A_{\lambda+i0}(H_0, \tilde{V}) = A_{\lambda+i0}(H_0, V)$.
(5) $\tilde{J} = \tilde{J}$.
(6) For any non-zero complex $r$ we have $\tilde{rJ} = r\tilde{J}$.

**Proof.** All these properties, except (5), have been proved before in Theorems 3.1, 3.9, 3.13 and Lemma 3.6. Now we prove (5):

\[
\tilde{J} = \tilde{J}P_+(H_0, \tilde{V}) = JP_+(H_0, V) \cdot P_+(H_0, V) = \tilde{J}.
\]

\qed

Question: is it true that $\tilde{(J_1 + J_2)} = \tilde{J}_1 + \tilde{J}_2$, provided $J_1 + J_2$ is regular?

**Lemma 3.14.** Let $H_0$ be resonant at $\lambda$. Let $V \in \mathcal{A}^6_0$ be a regular direction. The operator $T_{\lambda+i0}(H_0 + V)$ depends continuously on $V$ in the $\mathcal{A}_0$ norm.

**Proof.** Let $V_1 \in \mathcal{A}_0$ have a small norm. Then by the second resolvent identity

\[
T_{\lambda+i0}(H_0 + V + V_1) = T_{\lambda+i0}(H_0 + V) \left[1 + J_1T_{\lambda+i0}(H_0 + V)\right]^{-1}.
\]
Since \( J_1 \) has small norm, – by the premise, the operator in the square brackets is invertible and therefore the expression is continuous in \( J_1 \).

**Theorem 3.15.** The resonance index of a direction of order 1 is stable under small perturbations in \( A_0 \) norm.

**Proof.** Since the resonance index is the signature of the resonance matrix, it is enough to show that the resonance matrix \( J P_+ (H_0, V) \) depends continuously on \( V \) for order 1 directions and has constant rank. The resonance matrix is equal to

\[
\frac{1}{2\pi i} \oint_{C(0)} J T_{\lambda+i0} (H_0 + sV) J ds.
\]

Using compactness of the contour \( C(0) \) and Lemma 3.14, one can show that \( V \) has a neighbourhood in \( A_0 \) such that for all \( V' \) from it the operator \( T_{\lambda+i0} (H_0 + sV') \) exists for all \( s \in C(0) \) and is continuous in \( s \). Thus, the integral above is continuous in \( V \).

Since \( V \) has order 1, this small perturbation does not generate any new resonance points inside the contour \( C(0) \), in addition to the zero resonance point, and therefore the rank of the resonance matrix is constant. Therefore, the integral above with perturbed \( V \) would be the resonance matrix of the perturbed direction, and proof is thus complete.

**3.2. The reduction \( \tilde{V} \) outside \( \sigma_{ess} \).** Outside the essential spectrum there is an explicit formula for \( P_\lambda \), see [Az2, (5.9.2)], and therefore there is such a formula for the reduction. In the case of an order 1 direction \( V \) at a simple point this formula takes very simple form:

\[
\tilde{V} = \langle \chi, V \chi \rangle^{-1} \langle V \chi, \cdot \rangle V \chi,
\]

where \( \chi \) is an eigenvector of \( H_0 \) corresponding to \( \lambda \). Since the point \( H_0 \) is simple, this eigenvector is unique up to scaling, the choice of which does not affect the formula above. Also, since \( V \) has order 1, the inner product \( \langle \chi, V \chi \rangle \) is non-zero, see [Az2].

**3.3. The case of \( V \) with no property \( S \).** If a direction \( V \) has no property \( S \), then the operators \( Q_- J P_+ \) and \( Q_+ J P_- \) are different, and so we have two candidates for a resonance matrix. This case is still work in progress, but we will present one initial result.

**Theorem 3.16.** If \( V \) is a regular direction then each of the directions \( Q_- J P_+ \) and \( Q_+ J P_- \) are regular.

**Proof.** Let \( \tilde{H}_r = H_0 + rQ_- J P_+ \). We show that if \( T_+(H_r) \) exists then so does \( T_+(\tilde{H}_r) \).

We assume, wlog, that \( r = 1 \) and write \( H = H_1 \) and \( \tilde{H} = \tilde{H}_1 \). We have by the second resolvent identity

\[
T_+(\tilde{H}) = \left[ 1 + T_+(H) (Q_- J P_+ - J) \right]^{-1} T_+(H).
\]

Thus, existence of \( T_+(\tilde{H}) \) is equivalent to invertibility of \( 1 + T_+(H) (Q_- J P_+ - J) \).

Assume it is not invertible. Then for some non-zero \( \varphi \) we have

\[
\left[ 1 + T_+(H) (Q_- J P_+ - J) \right] \varphi = 0.
\]

(A) It suffices to show that \( P_+ \varphi = 0 \). Indeed, if this holds then

\[
0 = \left[ 1 + T_+(H) (Q_- J P_+ - J) \right] \varphi = \left[ 1 - T_+(H) J \right] \varphi.
\]
This equality means that $\varphi \in \text{im } P_+$, and so $\varphi = 0$.

(B) We show that $P_+ \varphi = 0$. Apply $P_+$ to the equation

$$[1 + T_+(H)(Q - JP_+ - J)] \varphi = 0.$$ 

This gives (using standard properties of $T_+, P_\pm$ and $Q_\pm$, see [Az Section 3])

$$-P_+ \varphi = (P_+ P_- P_+ - P_+) T_+ J P_+ \varphi.$$ 

The image of $P_+ P_- P_+ - P_+$ consists of vectors of type I (see [Az Lemma 11.2.4]). Thus $P_+ \varphi$ is of type I. But $T_+ J$ preserves property of resonance vectors to be of type I. By the same [Az Lemma 11.2.4], the operator $P_+ P_- P_+ - P_+$ maps to zero any vector of type I. Hence, from the last display we have $P_+ \varphi = 0$.

For $Q_+ J P_-$ proof is the same, — since $T_+$ exists iff $T_-$ does, all we need to change is to replace $T_+$ by $T_-$. \hfill \Box

Acknowledgements. The author thanks his wife, Feruza, for financially supporting him during the work on this paper.

References

[APS] M. Atiyah, V. Patodi, I. M. Singer, *Spectral Asymmetry and Riemannian Geometry. III*, Math. Proc. Camb. Phil. Soc. 79 (1976), 71–99.

[Az] N. A. Azamov, *Spectral flow inside essential spectrum*, Dissertationes Math. 518 (2016), 1–156.

[Az2] N. A. Azamov, *Spectral flow and resonance index*, Dissertationes Math. 528 (2017), 1–91.

[AD] N. A. Azamov, T. W. Daniels, *Coupling resonances and spectral properties of the product of resolvent and perturbation*, in preparation.

[Ph] J. Phillips, *Spectral flow in type I and type II factors — a new approach*, Fields Inst. Comm. 17 (1997), 137–153.

[RoSa] J. Robbin, D. Salamon, *The spectral flow and the Maslov index*, Bull. London Math. Soc. 27 (1995), 1–33.

Independent scholar, Adelaide, SA, Australia

Email address: azamovnurulla@gmail.com