Exact solution of the $A^{(1)}_{n-1}$ trigonometric vertex model with non-diagonal open boundaries

Wen-Li Yang$^{a,b}$ and Yao-Zhong Zhang$^b$

$^a$ Institute of Modern Physics, Northwest University Xian 710069, P.R. China
$^b$ Department of Mathematics, The University of Queensland, Brisbane 4072, Australia
E-mail: wenli@ maths.uq.edu.au, yzz@ maths.uq.edu.au

Abstract

The $A^{(1)}_{n-1}$ trigonometric vertex model with generic non-diagonal boundaries is studied. The double-row transfer matrix of the model is diagonalized by algebraic Bethe ansatz method in terms of the intertwiner and the corresponding face-vertex relation. The eigenvalues and the corresponding Bethe ansatz equations are obtained.

PACS: 03.65.Fd; 05.30.-d; 05.50.+q
Keywords: Algebraic Bethe ansatz; Integrable lattice model; Open boundary conditions.
1 Introduction

Two-dimensional integrable models have traditionally been solved by imposing periodic boundary conditions. For such bulk systems, the quantum Yang-Baxter equation (QYBE)

\[ R_{12}(u_1 - u_2)R_{13}(u_1 - u_3)R_{23}(u_2 - u_3) = R_{23}(u_2 - u_3)R_{13}(u_1 - u_3)R_{12}(u_1 - u_2), \]  

leads to families of commuting row-to-row transfer matrices which may be diagonalized by the quantum inverse scattering method (QISM) (or algebraic Bethe ansatz) [1].

Not all boundary conditions are compatible with integrability in the bulk. The bulk integrability is only preserved when one imposes certain boundary conditions. In [2], Sklyanin developed the boundary QISM, which may be used to described integrable systems on a finite interval with independent boundary conditions at each end. This boundary QISM uses the new algebraic structure, the reflection equation (RE) algebra. The solutions to the RE and its dual are called boundary K-matrices which in turn give rise to boundary conditions compatible with the integrability of the bulk model [2]-[4].

The boundary QISM has been applied to diagonalize the double-row transfer matrices of various integrable models with non-trivial boundary conditions mostly corresponding to the diagonal K-matrices. However, the problem of diagonalizing the double-row transfer matrix for general non-diagonal K-matrices has been long-standing for trigonometric integrable models. To our knowledge, the only exception is the spin-1/2 XXZ (or $A^{(1)}_1$ ) model with non-diagonal K-matrices which was solved recently by fusion hierarchy of the transfer matrix with the anisotropy value being the roots of unity [5, 6], the algebraic Bethe ansatz [7, 8] and the coordinate Bethe ansatz [9]. The fundamental difficulty is that the usual highest-weight state which is the pseudo-vacuum (or reference state) for the models with periodic boundary condition or boundary conditions specified by diagonal K-matrices is no longer the pseudo-vacuum on which the Bethe ansatz analysis is based.

In a very recent work [10], we constructed a class of non-diagonal solutions to the RE for the trigonometric $A^{(1)}_{n-1}$ vertex model by the intertwiner-matrix approach. The non-diagonal K-matrices we found can be expressed in terms of the intertwiner-matrices and diagonal face-type K-matrices. In the present paper, we solve the trigonometric $A^{(1)}_{n-1}$ vertex models with boundary conditions given by the non-diagonal K-matrices in [10]. We construct the pseudo-vacuum and diagonalize the corresponding double-row transfer matrix by the generalized QISM developed in [11].
This paper is organized as follows. In section 2, we introduce our notation and some basic ingredients. In section 3, we introduce the intertwiner-matrix which satisfies the face-vertex correspondence relation between the two R-matrices $R(u)$ and $W(u)$. Through the magic intertwiner vectors, in section 4, we transform the model from the original vertex picture into its “face” picture. After succeeding in constructing the pseudo-vacuum state, we apply the algebraic Bethe ansatz method to diagonalize the transfer matrices of the boundary model. Section 5 is for conclusions. Some detailed technical calculations are given in Appendices A-C.

2 $A_{n-1}^{(1)}$ trigonometric vertex model and integrable boundary conditions

Let us fix a positive integer $n$ ($n \geq 2$) and a generic complex number $\eta$, and $R(u) \in \text{End}(C^n \otimes C^n)$ be the trigonometric solution to the $A_{n-1}^{(1)}$ type QYBE given by [12, 13, 14]

$$R(u) = \sum_{\alpha=1}^{n} R_{\alpha\alpha}^{\alpha\beta}(u) E_{\alpha\alpha} \otimes E_{\alpha\alpha} + \sum_{\alpha \neq \beta} \left\{ R_{\alpha\beta}^{\alpha\beta}(u) E_{\alpha\alpha} \otimes E_{\beta\beta} + R_{\beta\alpha}^{\alpha\beta}(u) E_{\beta\alpha} \otimes E_{\alpha\beta} \right\}, \quad (2.1)$$

where $E_{ij}$ is the matrix with elements $(E_{ij})_{kl} = \delta_{jk}\delta_{il}$. The coefficient functions are

$$R_{\alpha\beta}^{\alpha\beta}(u) = \begin{cases} \frac{\sin(u)}{\sin(u+\eta)} e^{-i\eta}, & \alpha > \beta, \\ 1, & \alpha = \beta, \\ \frac{\sin(u)}{\sin(u+\eta)} e^{i\eta}, & \alpha < \beta, \end{cases}, \quad (2.2)$$

$$R_{\beta\alpha}^{\alpha\beta}(u) = \begin{cases} \frac{\sin(u)}{\sin(u+\eta)} e^{iu}, & \alpha > \beta, \\ 1, & \alpha = \beta, \\ \frac{\sin(u)}{\sin(u+\eta)} e^{-iu}, & \alpha < \beta, \end{cases}. \quad (2.3)$$

One can check that the R-matrix satisfies the following unitarity, crossing-unitarity and quasi-classical relations:

**Unitarity:** $R_{12}(u)R_{21}(-u) = \text{id}, \quad (2.4)$

**Crossing-unitarity:** $R_{12}^{t\alpha\beta}(u)M_2^{-1}R_{21}^{\alpha\beta}(-u-n\eta)M_2 = \frac{\sin(u)\sin(u+n\eta)}{\sin(u+\eta)\sin(u+n\eta-\eta)} \text{id}. \quad (2.5)$

**Quasi-classical property:** $R_{12}(u)|_{\eta \to 0} = \text{id}. \quad (2.6)$

Here $R_{21}(u) = P_{12}R_{12}(u)P_{12}$ with $P_{12}$ being the usual permutation operator and $t_i$ denotes the transposition in the $i$-th space, and $\eta$ is the so-called crossing parameter. The crossing
matrix $M$ is a diagonal $n \times n$ matrix with elements

$$M_{\alpha\beta} = M_\alpha \delta_{\alpha\beta}, \quad M_\alpha = e^{-2i\alpha \eta}, \quad \alpha = 1, \ldots, n.$$  \hspace{1cm} (2.7)

Here and below we adopt the standard notation: for any matrix $A \in \text{End}(\mathbb{C}^n)$, $A_j$ is an embedding operator in the tensor space $\mathbb{C}^n \otimes \mathbb{C}^n \otimes \cdots$, which acts as $A$ on the $j$-th space and as an identity on the other factor spaces; $R_{ij}(u)$ is an embedding operator of R-matrix in the tensor space, which acts as an identity on the factor spaces except for the $i$-th and $j$-th ones.

One introduces the “row-to-row” monodromy matrix $T(u)$, which is an $n \times n$ matrix with elements being operators acting on $(\mathbb{C}^n)^{\otimes N}$

$$T(u) = R_{01}(u + z_1)R_{02}(u + z_2) \cdots R_{0N}(u + z_N).$$ \hspace{1cm} (2.8)

Here $\{z_i | i = 1, \ldots, N\}$ are arbitrary free complex parameters which are usually called inhomogeneous parameters. With the help of the QYBE (1.1), one can show that $T(u)$ satisfies the so-called “RLL” relation

$$R_{12}(u - v)T_1(u)T_2(v) = T_2(v)T_1(u)R_{12}(u - v).$$ \hspace{1cm} (2.9)

Integrable open chains can be constructed as follows [2]. Let us introduce the K-matrix $K^-(u)$ which gives rise to an integrable boundary condition on the right boundary. $K^-(u)$ satisfies the RE

$$R_{12}(u_1 - u_2)K^-_1(u_1)R_{21}(u_1 + u_2)K^-_2(u_2) = K^-_2(u_2)R_{12}(u_1 + u_2)K^-_1(u_1)R_{21}(u_1 - u_2).$$ \hspace{1cm} (2.10)

For models with open boundaries, instead of the standard “row-to-row” monodromy matrix $T(u)$ (2.8), one needs the “double-row” monodromy matrix $T(u)$

$$T(u) = T(u)K^-(u)T^{-1}(-u).$$ \hspace{1cm} (2.11)

Using (2.9) and (2.10), one can prove that $T(u)$ satisfies

$$R_{12}(u_1 - u_2)T_1(u_1)R_{21}(u_1 + u_2)T_2(u_2) = T_2(u_2)R_{12}(u_1 + u_2)T_1(u_1)R_{21}(u_1 - u_2).$$ \hspace{1cm} (2.12)

In order to construct the double-row transfer matrices, besides the RE, one needs another K-matrix $K^+(u)$ which gives integrable boundary condition on the left boundary. The K-matrix $K^+(u)$ satisfies the dual RE [2, 3, 10]

$$R_{12}(u_2 - u_1)K^+_1(u_1)M_1^{-1}R_{21}(-u_1 - u_2 - n\eta)M_1 K^+_2(u_2) = M_1 K^+_2(u_2)R_{12}(-u_1 - u_2 - n\eta)M_1^{-1} K^+_1(u_1)R_{21}(u_2 - u_1).$$ \hspace{1cm} (2.13)
Different integrable boundary conditions are described by different solutions $K^{-}(u)$ ($K^{+}(u)$) to the (dual) RE [2, 4]. In this paper, we consider the non-diagonal solutions $K^\pm(u)$ obtained in [10], which are respectively given by

$$
K^{-}(u)_t = \sum_{i=1}^{n} k_i(u) \phi_{\lambda,-\epsilon_i}(u) \tilde{\phi}_{\lambda,-\epsilon_i}(-u),
$$

$$
K^{+}(u)_t = \sum_{i=1}^{n} \tilde{k}_i(u) \phi_{\lambda',-\epsilon_i}(-u) \tilde{\phi}_{\lambda',-\epsilon_i}(u).
$$

Here $\{k_i(u)|i=1,\ldots,n\}$ and $\{\tilde{k}_i(u)|i=1,\ldots,n\}$ are

$$
k_j(u) = \begin{cases} 
1, & 1 \leq j \leq l, \\
\frac{\sin(\xi-u)}{\sin(\xi+u)} e^{-2i\epsilon_j}, & l+1 \leq j \leq n,
\end{cases}
$$

$$
\tilde{k}_j(u) = \begin{cases} 
e^{-2i(j'\eta)}, & 1 \leq j \leq l', \\
\frac{\sin(\xi+u+\frac{\eta}{2})}{\sin(\xi-u-\frac{\eta}{2})} e^{2i(u+\frac{\eta}{2})}, & l'+1 \leq j \leq n,
\end{cases}
$$

where $l$ and $l'$ are positive integers such that $1 \leq l \leq n$, $1 \leq l' \leq n$. In (2.14) and (2.15), $\phi$, $\tilde{\phi}$ and $\tilde{\phi}$ are intertwiners which will be specified in section 3. The K-matrix $K^{-}(u)$ (resp. $K^{+}(u)$) depends on a *discrete* parameter $l$ (resp. $l'$) and continuous parameters $\xi$, $\{\lambda_i\}$ (resp. $\tilde{\xi}$, $\{\lambda'_i\}$) and $\rho$ (whose dependence is through the definition of the intertwiner-matrix below). Here and throughout we use the convention: associated with the boundary parameters $\{\lambda_i\}$ (resp. $\{\lambda'_i\}$) let us introduce a vector $\lambda = \sum_{i=1}^{n} \lambda_i \epsilon_i$ (resp. $\lambda' = \sum_{i=1}^{n} \lambda'_i \epsilon_i$), where $\{\epsilon_i|i=1,\ldots,n\}$ is the orthonormal basis of the vector space $C^n$ such that $\langle \epsilon_i, \epsilon_j \rangle = \delta_{ij}$.

Some remarks are in order. Our bulk R-matrix is different from those used in [15, 16, 17] (see (2.2)). For $n=3$ case, after a similarity transformation by a spectral-independent diagonal matrix, our K-matrix $K^{-}(u)$ has the same number of the boundary parameters as that of [17] and one more boundary parameter than that of [15, 16]. For generic $n$ ($n>3$), our K-matrix $K^{-}(u)$ has many more boundary parameters than that given in [15].

Let us emphasize that a further restriction

$$
\lambda' + \sum_{k=1}^{N} \epsilon_{j_k} = \lambda,
$$

where $\{j_k|k=1,\ldots,N\}$ are positive integers such that $2 \leq j_k \leq n$, is necessary for the the application of the algebraic Bethe ansatz method in section 4. Hereafter, we shall consider only the above case.
The double-row transfer matrix of the inhomogeneous model associated with the R-matrix (2.1)-(2.3) with open boundary specified by the K-matrices $K^\pm(u)$ (2.14)-(2.18) is given by

$$\tau(u) = \text{tr}(K^+(u) T(u)).$$

(2.19)

The commutativity of the transfer matrices

$$[\tau(u), \tau(v)] = 0,$$

(2.20)
follows as a consequence of (1.1), (2.4)-(2.5) and (2.12)-(2.13). This ensures the integrability of the inhomogeneous model with open boundary. The aim of this paper is to find the common eigenvectors and the corresponding eigenvalues of the transfer matrix (2.19) with generic non-diagonal K-matrices $K^\pm(u)$ given by (2.14)-(2.18).

3 Intertwining vectors and the associated face-vertex correspondence relations

For a vector $m \in \mathbb{C}^n$, set

$$m_i = \langle m, \epsilon_i \rangle, \quad |m| = \sum_{k=1}^{n} m_k, \quad i = 1, \ldots, n.$$  

(3.1)

Let us introduce $n$ intertwining vectors (intertwiners) $\{\phi_{m,m-\epsilon_j}(u) | j = 1, \ldots, n\}$. Each $\phi_{m,m-\epsilon_j}(u)$ is an $n$-component column vector whose $\alpha$-th elements are $\{\phi^{(\alpha)}_{m,m-\epsilon_j}(u)\}$. The $n$ intertwiners form an $n \times n$ matrix (in which $j$ and $\alpha$ stand for the column and the row indices respectively), called the intertwiner-matrix, with the non-vanishing matrix elements being

$$
\begin{pmatrix}
    e^{i \eta f_1(m)} & e^{i \eta f_2(m)} & \cdots & e^{i \eta F_n(m) + \rho e^{2iu}} \\
    e^{i \eta f_1(m)} & e^{i \eta f_2(m)} & \cdots & \cdots \\
    \vdots & \vdots & \ddots & \vdots \\
    e^{i \eta f_1(m)} & \cdots & e^{i \eta f_{n-1}(m)} & e^{i \eta f_n(m)}
\end{pmatrix}.
$$

Here $\rho$ is a complex constant with regard to $u$ and $m$, and $\{f_i(m) | i = 1, \ldots, n\}$ and $\{F_i(m) | i = 1, \ldots, n\}$ are linear functions of $m$:

$$f_i(m) = \sum_{k=1}^{i-1} m_k - m_i - \frac{1}{2}|m|, \quad i = 1, \ldots, n.$$  

(3.3)
\[ F_i(m) = \sum_{k=1}^{i} m_k - \frac{1}{2} |m|, \quad i = 1, \ldots, n - 1, \tag{3.4} \]
\[ F_n(m) = -\frac{3}{2} |m|. \tag{3.5} \]

From the above intertwiner-matrix, one may derive the following face-vertex correspondence relation \[10\]:

\[ R_{12}(u_1 - u_2) \phi_{m,m-\epsilon_i}(u_1) \otimes \phi_{m-\epsilon_i,m-\epsilon_j}(u_2) = \sum_{k,l} W^{kl}_{ij}(u_1 - u_2) \phi_{m-\epsilon_i,m-\epsilon_j-\epsilon_k}(u_1) \otimes \phi_{m,m-\epsilon_l}(u_2). \tag{3.6} \]

Here the non-vanishing elements of \{W(u)_{kl}^{ij}\} are

\[ W_{jj}^{jj}(u) = 1, \quad W_{jk}^{jk}(u) = \frac{\sin(u)}{\sin(u + \eta)}, \quad \text{for } j \neq k, \tag{3.7} \]
\[ W_{jk}^{kj}(u) = \begin{cases} 
\frac{\sin(\eta)}{\sin(u + \eta)} e^{iu}, & j > k, \\
\frac{\sin(\eta)}{\sin(u + \eta)} e^{-iu}, & j < k,
\end{cases} \quad \text{for } j \neq k. \tag{3.8} \]

Associated with \{W(u)_{kl}^{ij}\}, one may introduce “face” type R-matrix \( W(u) \)

\[ W(u) = \sum_{i,j,k,l} W^{kl}_{ij}(u) E_{ki} \otimes E_{lj}. \tag{3.9} \]

Some remarks are in order. The “face” type R-matrix \( W(u) \) does not depend on the face type parameter \( m \), in contrast to the \( \mathbb{Z}_n \) elliptic case \[18, 19\]. It follows that \( W(u) \) and \( R(u) \) satisfy the same QYBE, i.e. \( W(u) \) obeys the usual (vertex type) QYBE rather than the dynamical one \[20, 11\].

Noting that

\[ \sum_{i=1}^{n} f_i(m) = \sum_{i=1}^{n} F_i(m) = \sum_{k=1}^{n} \frac{n - 2(k + 1)}{2} m_k, \tag{3.10} \]
on one can show that the determinant of the intertwiner matrix \[8.2\] is

\[ \det \left( \phi\alpha_{m,m-\epsilon_j}(u) \right) = e^{i\eta \sum_{k=1}^{n} \frac{n-2(k+1)}{2} m_k} (1 - (-1)^n e^{2iu+\rho}). \tag{3.11} \]

For a generic \( \rho \in \mathbb{C} \) this determinant is non-vanishing and thus the inverse of \[8.2\] exists. This fact allows us to introduce other types of intertwiners \( \tilde{\phi} \) and \( \tilde{\phi} \) satisfying the following orthogonality conditions:

\[ \sum_{\alpha} \tilde{\phi}\alpha_{m,m-\epsilon_i}(u) \phi\alpha_{m,m-\epsilon_j}(u) = \delta_{ij}, \tag{3.12} \]
\[ \sum_{\alpha} \tilde{\phi}\alpha_{m+\epsilon_i,m}(u) \phi\alpha_{m+\epsilon_j,m}(u) = \delta_{ij}. \tag{3.13} \]
From these conditions we derive the “completeness” relations:

\[ \sum_k \tilde{\phi}_{m,\epsilon_k}(u) \phi_{m,\epsilon_k}(u) = \delta_{\alpha\beta}, \quad (3.14) \]

\[ \sum_k \tilde{\phi}_{m+\epsilon_k,\epsilon_k}(u) \phi_{m+\epsilon_k,\epsilon_k}(u) = \delta_{\alpha\beta}. \quad (3.15) \]

With the help of (3.12)-(3.15), we obtain, from the face-vertex correspondence relation (3.6),

\[ \tilde{\phi}_{m+\epsilon_k,m}(u) \otimes \tilde{\phi}_{m+\epsilon_k,m}(u) \big) R_{12}(u_1 - u_2) (\phi_{m,\epsilon}(u_1) \otimes \phi_{m,\epsilon}(u_2) \big) \]

\[ = \sum_{i,j} W_{ij}^{kl}(u_1 - u_2) \tilde{\phi}_{m+\epsilon_i,\epsilon_j,m+\epsilon_j}(u_1) \otimes \phi_{m+\epsilon_i,\epsilon_j,m+\epsilon_j}(u_2), \quad (3.16) \]

\[ \tilde{\phi}_{m+\epsilon_k,m}(u_1) \otimes \tilde{\phi}_{m+\epsilon_k,m}(u_2) \big) R_{12}(u_1 - u_2) \]

\[ = \sum_{i,j} W_{ij}^{kl}(u_1 - u_2) \phi_{m,\epsilon_i,m+\epsilon_j}(u_1) \otimes \tilde{\phi}_{m,\epsilon_i,m+\epsilon_j}(u_2), \quad (3.17) \]

\[ \phi_{m,\epsilon}(u_1) \otimes \phi_{m,\epsilon}(u_2) \big) R_{12}(u_1 - u_2) \]

\[ = \sum_{k,j} W_{ij}^{kl}(u_1 - u_2) \phi_{m+\epsilon_k,\epsilon_j,m+\epsilon_j}(u_1) \otimes \phi_{m+\epsilon_k,\epsilon_j,m+\epsilon_j}(u_2), \quad (3.18) \]

\[ \tilde{\phi}_{m,\epsilon_k,m}(u_1) \otimes \phi_{m,\epsilon}(u_2) \big) R_{12}(u_1 - u_2) \]

\[ = \sum_{i,j} W_{ij}^{kl}(u_1 - u_2) \phi_{m,\epsilon_i,\epsilon_j,m+\epsilon_j}(u_1) \otimes \tilde{\phi}_{m,\epsilon_i,\epsilon_j,m+\epsilon_j}(u_2). \quad (3.19) \]

4 Algebraic Bethe ansatz for the $A^{(1)}_{n-1}$ trigonometric vertex model with non-diagonal open boundaries

In this section, we shall demonstrate that the intertwiners and the face-vertex correspondence relations (3.6)-(3.19) play a fundamental role in the construction of the eigenstates of the $A^{(1)}_{n-1}$ trigonometric vertex model with open boundary condition specified by the K-matrices $K^\pm(u)$ given in (2.14)-(2.18). In order to apply the algebraic Bethe ansatz method, we need to transform the fundamental exchange relation (2.12) from the vertex picture into its face picture so that we can construct the corresponding pseudo-vacuum and the associated Bethe ansatz states.
4.1 K-matrices in the “face” picture

Corresponding to the vertex type K-matrices (2.14) and (2.16), one introduces the following face type K-matrices \( \mathcal{K} \) and \( \tilde{\mathcal{K}} \), as in [21]

\[
\mathcal{K}(\lambda|u)^i_j = \sum_{\alpha, \beta} \tilde{\phi}^{(a)}_{\lambda^{-\epsilon_i+\epsilon_j},\lambda^{-\epsilon_i}}(u) K^{-(\beta)}(u)^{a\alpha}_{\beta\lambda^{-\epsilon_i}}(-u), \quad (4.1)
\]

\[
\tilde{\mathcal{K}}(\lambda'|u)^i_j = \sum_{\alpha, \beta} \phi^{(a)}_{\lambda'^{-\epsilon_j},\lambda'^{-\epsilon_j}}(-u) K^{+(\beta)}(u)^{a\alpha}_{\beta\lambda'^{-\epsilon_j}}(u). \quad (4.2)
\]

Through straightforward calculations, we find the \( \lambda(\lambda') \)-independent face type K-matrices have diagonal forms\(^1\)

\[
\mathcal{K}(\lambda|u)^i_j = \delta^i_j k(u; \xi)_i, \quad \tilde{\mathcal{K}}(\lambda'|u)^i_j = \delta^i_j \tilde{k}(u)_i, \quad i, j = 1, \ldots, n, \quad (4.3)
\]

where functions \( \{k(u; \xi)_i = k(u)_i\} \) and \( \{\tilde{k}(u)_i\} \) are respectively given by (2.16) and (2.17).

Although the K-matrices \( K^\pm(u) \) given by (2.14) and (2.15) are generally non-diagonal (in the vertex picture), after the face-vertex transformations (4.1) and (4.2), the face type counterparts \( \mathcal{K}(\lambda|u) \) and \( \tilde{\mathcal{K}}(\lambda'|u) \) become diagonal simultaneously. This fact enables us to apply the generalized algebraic Bethe ansatz method developed in [11] for SOS type integrable models to diagonalize the transfer matrix \( \tau(u) \) (2.19).

4.2 Exchange relations of double-row monodromy matrix of face type

By means of (3.14), (3.15), (4.2) and (4.3), the transfer matrix \( \tau(u) \) (2.19) can be recasted into the following face type form:

\[
\tau(u) = tr(K^+(u)T(u)) = \sum_{\mu, \nu} tr\left(K^+(u) \phi_{\lambda'-\epsilon_\mu,\lambda'-\epsilon_\mu}(u) \tilde{\phi}_{\lambda'-\epsilon_\nu,\lambda'-\epsilon_\nu}(u) T(u) \phi_{\lambda',\lambda'-\epsilon_\mu}(u) \tilde{\phi}_{\lambda'-\epsilon_\nu,\lambda'-\epsilon_\nu}(u) \right)
= \sum_{\mu, \nu} \tilde{\phi}_{\lambda'-\epsilon_\nu}(u) K^+(u) \phi_{\lambda'-\epsilon_\mu,\lambda'-\epsilon_\mu}(u) \tilde{\phi}_{\lambda'-\epsilon_\nu,\lambda'-\epsilon_\nu}(u) T(u) \phi_{\lambda',\lambda'-\epsilon_\mu}(u) \tilde{\phi}_{\lambda'-\epsilon_\nu,\lambda'-\epsilon_\nu}(u)
= \sum_{\mu, \nu} \tilde{\mathcal{K}}(\lambda'|u)^{\nu}_\mu T(\lambda'|u)^{\mu}_\nu = \sum_{\mu} \tilde{k}(u)_\mu T(\lambda'|u)^{\mu}_\nu. \quad (4.4)
\]

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\(^1\)The spectral parameter \( u \) and the boundary parameter \( \xi \) of the reduced double-row monodromy matrices constructed from \( \mathcal{K}(\lambda|u) \) will be shifted in each step of the nested Bethe ansatz procedure [11]. Therefore, it is convenient to specify the dependence on the boundary parameter \( \xi \) of \( \mathcal{K}(\lambda|u) \), in terms of \( k(u; \xi)_i \), in addition to the spectral parameter \( u \).
Here we have introduced the face type double-row monodromy matrix $\mathcal{T}(m|u)$,

$$\mathcal{T}(\lambda'|u)_{\mu|\nu} = \mathcal{T}(m|u)_{\mu|\nu} \big|_{m=\lambda'} = \bar{\phi}_{m-\epsilon_\mu+\epsilon_\nu,m-\epsilon_\mu}(u) \mathcal{T}(u)\phi_{m,m-\epsilon_\mu}(-u) \big|_{m=\lambda'}$$

$$\equiv \sum_{\alpha,\beta} \bar{\phi}_{\lambda'-\epsilon_\mu+\epsilon_\nu,\lambda'-\epsilon_\mu}(u) \mathcal{T}(u)_{\alpha}^\beta \phi_{\lambda',\lambda'-\epsilon_\mu}(-u). \quad (4.5)$$

Moreover from $\eqref{2.12}$, $\eqref{3.6}$ and $\eqref{3.15}$ we derive the following exchange relations among $\mathcal{T}(m|u)_{\mu|\nu}$ (see Appendix A for details):

$$\sum_{i_1,i_2,j_1,j_2} W^{i_0,j_0}_{i_1,j_1} (u_1 - u_2) \mathcal{T}(m + \epsilon_{j_1} + \epsilon_{i_2}|u_1)_{i_2}^{i_1} \times W^{j_1,i_2}_{j_2,i_3} (u_1 + u_2) \mathcal{T}(m + \epsilon_{j_3} + \epsilon_{i_3}|u_2)_{j_3}^{j_2}
= \sum_{i_1,i_2,j_1,j_2} \mathcal{T}(m + \epsilon_{j_1} + \epsilon_{i_0}|u_2)_{j_2}^{j_0} W^{i_0,j_1}_{i_1,j_1} (u_1 + u_2) \times \mathcal{T}(m + \epsilon_{j_2} + \epsilon_{i_2}|u_1)_{i_1}^{i_2} W^{j_2,i_2}_{j_3,i_3} (u_1 - u_2). \quad (4.6)$$

For convenience let us introduce the standard notation:

$$\mathcal{A}(m|u) = \mathcal{T}(m|u)_{1|1}, \mathcal{B}_i(m|u) = \mathcal{T}(m|u)_{i|1}, \mathcal{C}_i(m|u) = \mathcal{T}(m|u)_{1|i}, \quad i = 2, \ldots, n, \quad (4.7)$$

$$\mathcal{D}_i^j(m|u) = \mathcal{T}(m|u)_{1|j} - \delta_i^j W^{1_1|2}_j (2u) \mathcal{A}(m|u), \quad i, j = 2, \ldots, n. \quad (4.8)$$

From $\eqref{4.6}$, after some tedious calculation, we find the commutation relations among $\mathcal{A}(m|u)$, $\mathcal{D}(m|u)$ and $\mathcal{B}(m|u)$ (see Appendix B for details). Here we give those which are relevant for our purpose

$$\mathcal{A}(m|u)\mathcal{B}_j(m + \epsilon_j - \epsilon_1|v)$$

$$= \frac{\sin(u + v)\sin(u - v - \eta)}{\sin(u + v + \eta)\sin(u - v)} \mathcal{B}_j(m + \epsilon_j - \epsilon_1|v)\mathcal{A}(m + \epsilon_j - \epsilon_1|u) + \frac{\sin(\eta)\sin(2v)e^{i(u-v)}}{\sin(u - v)\sin(2v + \eta)} \mathcal{B}_j(m + \epsilon_j - \epsilon_1|u)\mathcal{A}(m + \epsilon_j - \epsilon_1|v)$$

$$- \frac{\sin(\eta)e^{i(u+v)}}{\sin(u + v + \eta)} \sum_{\alpha=2}^n \mathcal{B}_\alpha(m + \epsilon_\alpha - \epsilon_1|u)\mathcal{D}_j^\alpha(m + \epsilon_j - \epsilon_1|v), \quad (4.9)$$

$$\mathcal{D}_a^k(m|u)\mathcal{B}_j(m + \epsilon_j - \epsilon_1|v)$$

$$= \frac{\sin(u - v + \eta)\sin(u + v + 2\eta)}{\sin(u - v)\sin(u + v + \eta)}$$

$$\times \left\{ \sum_{\alpha_1,\alpha_2,\beta_1,\beta_2=2}^n W^{k\beta_2}_{\alpha_2\beta_1}(u + v + \eta)W^{\beta_1\alpha_1}_{j\alpha}(u - v) \times \mathcal{B}_{\beta_2}(m + \epsilon_k + \epsilon_\beta_2 - \epsilon_\alpha - \epsilon_1|v)\mathcal{D}_a^{\alpha_2}(m + \epsilon_j - \epsilon_1|u) \right\}$$
\[- \frac{\sin(\eta) \sin(2u + 2\eta) e^{-i(u-v)}}{\sin(u - v) \sin(2u + \eta)} \times \left\{ \sum_{\alpha,\beta = 2}^{n} W_{\alpha}^{\beta}(2u + \eta) B_{\beta}(m + \epsilon_{\alpha} - \epsilon_{1}|u) D_{\beta}^\alpha(m + \epsilon_{j} - \epsilon_{1}|v) \right\} \]

\[+ \frac{\sin(\eta) \sin(2v) \sin(2u + 2\eta) e^{-i(u+v)}}{\sin(u + v + \eta) \sin(2v + \eta) \sin(2u + \eta)} \times \left\{ \sum_{\alpha = 2}^{n} W_{j}^{\alpha}(2u + \eta) B_{\alpha}(m + \epsilon_{j} - \epsilon_{1}|u) A(m + \epsilon_{j} - \epsilon_{1}|v) \right\} , \tag{4.10} \]

\[B_{i}(m + \epsilon_{i} - \epsilon_{1}|u) B_{j}(m + \epsilon_{i} + \epsilon_{j} - 2\epsilon_{1}|v) = \sum_{\alpha,\beta = 2}^{n} W_{j}^{\alpha}(u - v) B_{\beta}(m + \epsilon_{\beta} - \epsilon_{1}|v) B_{\alpha}(m + \epsilon_{\alpha} + \epsilon_{\beta} - 2\epsilon_{1}|u). \tag{4.11} \]

### 4.3 Pseudo-Vacuum state

The algebraic Bethe ansatz requires, in addition to the relevant commutation relations (4.9)-(4.11), a pseudo-vacuum state which is the common eigenstate of the operators \(A, D_{i}^{\dagger}\) and is annihilated by the operators \(C_{i}\). In contrast to the models with diagonal \(K^{\pm}(u)\) \([2],[22]\), for models with non-diagonal \(K\)-matrices, the usual highest-weight state

\[
\left( \begin{array}{c} 1 \\ 0 \\ \vdots \\ 0 \\ \vdots \\
\end{array} \right) \otimes \cdots \otimes \left( \begin{array}{c} 1 \\ 0 \\ \vdots \\ 0 \\ \vdots \\
\end{array} \right), \tag{4.12} 
\]

is no longer the pseudo-vacuum state. However, after the face-vertex transformations (4.1) and (4.2), the face type \(K\)-matrices \(K(\lambda|u)\) and \(\tilde{K}(\lambda|u)\) simultaneously become diagonal. This suggests that one can translate the \(A_{\text{n-1}}^{(1)}\) trigonometric vertex model with non-diagonal \(K\)-matrices \([2],[4],[4]\) into the corresponding “face” type model with diagonal \(K\)-matrices \(K(\lambda|u)\) and \(\tilde{K}(\lambda|u)\) given by (4.1) and (4.2) respectively. Then one can construct the pseudo-vacuum in the “face” picture and use the generalized algebraic Bethe ansatz method \([11]\) to diagonalize the transfer matrix. Such a method has already been successfully used to diagonalize the transfer matrix of \(A_{\text{1}}^{(1)}\) trigonometric vertex model (XXZ model) with non-diagonal \(K\)-matrices \([8]\). In this paper we shall extend the construction to the generic \(A_{\text{n-1}}^{(1)}\) case.

Before introducing the pseudo-vacuum state, let us introduce a generic state in the quantum space by the column vectors of the intertwiner-matrix (3.2)

\[
|i_{1}, \ldots, i_{N}\rangle_{m_{0}}^{m} = \phi_{m_{0},m_{0}-\epsilon_{N}}^{N}(-z_{N}) \phi_{m_{0}-\epsilon_{N},m_{0}-\epsilon_{N}-\epsilon_{N-1}}^{N-1}(-z_{N-1}) \cdots
\]
Here we have introduced $T \otimes \cdot \ldots \otimes \phi \otimes \phi$ where the vectors $m_0, m \in C^n$ and $m = m_0 - \sum_{k=1}^{N} \epsilon_{ik}$, the vector $\phi^k = id \otimes id \ldots \otimes k-th$. Now let us evaluate the action of the face type monodromy matrix $T(m|u)$ (4.5) on the state (4.13). Using the definition of the vertex type double-row monodromy matrix $T(u)$ (2.11) and relations (5.14)-(5.15), we can further write the face type $T(m|u)$ in the following form

$$
T(m|u)^j_i = \tilde{\phi}_{m-\epsilon_i}^j(m(u)T(u)K(u)T^{-1}(-u)\phi_{m-\epsilon_i}^j(-u)
\equiv T(m, m_0|u)^j_i
= \sum_{\mu, \nu} \tilde{\phi}_{m-\epsilon_i}^j(m(u)T(u)\phi_{m_0-\epsilon_\mu, m_0-\epsilon_\nu}(u)\tilde{\phi}_{m_0-\epsilon_\nu, m_0-\epsilon_\nu}(u)K(u)
\times \phi_{m_0, m_0-\epsilon_\nu}^j(-u)\tilde{\phi}_{m_0, m_0-\epsilon_\nu}(-u)T^{-1}(-u)\phi_{m_0-\epsilon_i}^j(-u)
\equiv \sum \mathcal{T}(m - \epsilon_i, m_0 - \epsilon_\nu)^j_i \mathcal{K}(m_0|u)^\nu_\nu S(m, m_0|u)^\nu_\nu.
$$

Here we have introduced

$$
\mathcal{T}(m, m_0|u)^j_i = \tilde{\phi}_{m+\epsilon_i, m}(u)T(u)\phi_{m_0+\epsilon_\mu, m_0}(u),
\mathcal{S}(m, m_0|u)^\mu_i = \tilde{\phi}_{m_0-\epsilon_\mu, m_0}(u)T^{-1}(-u)\phi_{m_0-\epsilon_i}(-u),
\mathcal{K}(m_0|u)^j_i = \tilde{\phi}_{m_0+\epsilon_i, m_0-\epsilon_\nu}(u)K(u)\phi_{m_0, m_0-\epsilon_\nu}(-u).
$$

We can evaluate the action of the operator $T(m, m_0|u)^j_i$ on the state $|i_1, \ldots, i_N\rangle_{m_0}$ from the definition (2.8) and the face-vertex correspondence relation (3.6)

$$
T(m, m_0|u)^j_i |i_1, \ldots, i_N\rangle_{m_0}
= \tilde{\phi}_{m+\epsilon_i, m}(u)R_{01}(u + z_1)\phi_{m_0 - \sum_{k=2}^{N} \epsilon_{ik}, m_0 - \sum_{k=1}^{N} \epsilon_{ik}}^1(-z_1) \ldots
\times R_{0N}(u + z_N)\phi_{m_0, m_0 - \epsilon_i}(-z_N)\phi_{m_0+\epsilon_\mu, m_0}(u)
\equiv \sum_{\beta_1, i_N} \tilde{\phi}_{m+\epsilon_i, m}(u)R_{01}(u + z_1)\phi_{m_0 - \sum_{k=2}^{N} \epsilon_{ik}, m_0 - \sum_{k=1}^{N} \epsilon_{ik}}^1(-z_1) \ldots
\times R_{0N-1}(u + z_{N-1})\phi_{m_0+\epsilon_\mu, m_0 - \epsilon_i}(-z_N-1)\phi_{m_0-\epsilon_i, m_0-\epsilon_{i_N}}(-z_N-1)
\times W_{\mu_1, i_N}^\beta_{\beta_1}(u + z_N)\phi_{m_0+\epsilon_\mu, m_0+\epsilon_{i_N}}(-z_N)
\vdots
\equiv W_{\beta_1, i_N}^j_i(u + z_1)W_{\beta_2, i_N}^\beta_{\beta_1}(u + z_2) \ldots
\times W_{\mu, i_N}^\beta_{\beta_N} (u + z_N) |i_1', i_2', \ldots, i_N'\rangle_{m_0+\epsilon_\mu}.
$$

(4.18)
In the last equation the repeated indices imply summation over 1, 2, . . . n. Noting the unitarity of the R-matrix (2.4), \( T^{-1}(-u) \) can be written

\[
T_0^{-1}(-u) = R_{N0}(u - z_N) \cdots R_{10}(u - z_1). \tag{4.19}
\]

Then by the face-vertex correspondence relation (3.6) we can evaluate the action of the operator \( S(m, m_0 | u)_i^\mu \) on the state \( |i_1, \ldots, i_N\rangle_{m_0}^m \), similar to what we have done for \( T(m, m_0 | u) \):

\[
S(m, m_0 | u)_i^\mu |i_1, \ldots, i_N\rangle_{m_0}^m = W_i^{\beta_N^\mu, \alpha_i} (u - z_N) W_i^{\beta_{N-1}^\alpha_{N-1}} (u - z_{N-1}) \cdots \times W_i^{\beta_1^\alpha_1} (u - z_1) |i_1', \ldots, i_N'\rangle_{m_0 - \epsilon_i}^{m - \epsilon_i}. \tag{4.20}
\]

Here the repeated indices are summed over 1, 2, . . . n. Similarly, by the decomposition relation (4.14) and the equations (4.18), (4.20) we obtain the action of \( T(m | u)_i^\mu \) on the state \( |i_1, \ldots, i_N\rangle_{m_0}^m \):

\[
\mathcal{T}(m | u)_i^\mu |i_1, \ldots, i_N\rangle_{m_0}^m \equiv \mathcal{T}(m, m_0 | u)_i^\mu |i_1, \ldots, i_N\rangle_{m_0}^m = T(m - \epsilon_i, m_0 - \epsilon_i | u)_{\mu'}^\nu K(m_0 | u)_{\nu}^\mu S(m, m_0 | u)_i^\mu |i_1, \ldots, i_N\rangle_{m_0}^m
\]

\[
= W_{\mu', \nu}^{\beta_{N-1}^\alpha_{N-1}} (u + z_1) W_{\mu, \nu}^{\beta_{N-2}^\alpha_{N-2}} (u + z_2) \cdots \times W_{\mu, \nu}^{\beta_{N-1}^\alpha_{N-1}} (u + z_N) K(m_0 | u)_{\nu}^\mu W_{\nu, \nu}^{\beta_i^\nu_{i_1}} (u - z_i) \times W_{\nu, \nu}^{\beta_{N-1}^\alpha_{N-2}} (u - z_{N-1}) \cdots \times W_{\nu, \nu}^{\beta_i^\alpha_{i_1}} (u - z_1) |i_1'', \ldots, i_N''\rangle_{m_0 - \epsilon_i + \epsilon_i}^{m - \epsilon_i}. \tag{4.21}
\]

Here again it is understood that the repeated indices are summed over 1, 2, . . . n.

Specializing the face-type parameters \( \{(m_0)_i\} \) to the boundary parameters \( \{\lambda_i\} \), i.e. \( m = \lambda \), in equation (4.17), then from equation (4.3) the corresponding face type boundary K-matrix \( K(\lambda | u) \) becomes diagonal. This enables us to construct the pseudo-vacuum state of the model and apply the algebraic Bethe ansatz method to diagonalize the double-row transfer matrices (2.19) later.

Now, let us construct the pseudo-vacuum state \( |\Omega\rangle \):

\[
|\Omega\rangle \equiv |\text{vac}\rangle_{\lambda}^{\lambda - N \epsilon_1} = |1, \ldots, 1\rangle_{\lambda}^{\lambda - N \epsilon_1}, \tag{4.22}
\]

where \( \lambda \) is related to the boundary parameters \( \{\lambda_i\} \) of the boundary K-matrix \( K^- (u) \) in (2.14). Then from equations (4.18) and (4.20) we find that the actions of the operators
Moreover, after a tedious calculation, we have (see Appendix C for details) the equations Noting that the diagonal form of $K(\lambda|u)$ (4.3) and the above equations, we derive

\begin{align}
T(\lambda - N\epsilon_1, \lambda|u)_{1}^{1} |\text{vac}\rangle_{\lambda}^{\lambda-N\epsilon_1} &= |\text{vac}\rangle_{\lambda + \epsilon_1}^{\lambda-N\epsilon_1 + \epsilon_1}, \quad (4.23) \\
T(\lambda - N\epsilon_1, \lambda|u)_{i}^{i} |\text{vac}\rangle_{\lambda}^{\lambda-N\epsilon_1} &= 0, \quad i = 2, \ldots, n, \quad (4.24) \\
T(\lambda - N\epsilon_1, \lambda|u)_{i}^{j} |\text{vac}\rangle_{\lambda}^{\lambda-N\epsilon_1} &= \delta_{j}^{i} \prod_{k=1}^{N} W_{j}^{j1}(u + z_k) |\text{vac}\rangle_{\lambda + \epsilon_j}^{\lambda-N\epsilon_1 + \epsilon_j}, \quad i, j = 2, \ldots, n, \quad (4.25) \\
S(\lambda - N\epsilon_1, \lambda|u)_{1}^{1} |\text{vac}\rangle_{\lambda}^{\lambda-N\epsilon_1} &= |\text{vac}\rangle_{\lambda - \epsilon_1}^{\lambda-N\epsilon_1 - \epsilon_1}, \quad (4.26) \\
S(\lambda - N\epsilon_1, \lambda|u)_{i}^{j} |\text{vac}\rangle_{\lambda}^{\lambda-N\epsilon_1} &= 0, \quad i = 2, \ldots, n, \quad (4.27) \\
S(\lambda - N\epsilon_1, \lambda|u)_{i}^{j} |\text{vac}\rangle_{\lambda}^{\lambda-N\epsilon_1} &= \delta_{j}^{i} \prod_{k=1}^{N} W_{1j}^{1j}(u - z_k) |\text{vac}\rangle_{\lambda - \epsilon_j}^{\lambda-N\epsilon_1 - \epsilon_j}, \quad i, j = 2, \ldots, n. \quad (4.28)
\end{align}

Noting that the above equations, we have (see Appendix C for details)

\begin{align}
T(\lambda - N\epsilon_1, \lambda|u)_{1}^{1} |\text{vac}\rangle_{\lambda}^{\lambda-N\epsilon_1} &= k(u; \xi)_{1} |\text{vac}\rangle_{\lambda}^{\lambda-N\epsilon_1}, \quad (4.29) \\
T(\lambda - N\epsilon_1, \lambda|u)_{i}^{i} |\text{vac}\rangle_{\lambda}^{\lambda-N\epsilon_1} &= 0, \quad i = 2, \ldots, n. \quad (4.30)
\end{align}

Moreover, after a tedious calculation, we have (see Appendix C for details)

\begin{align}
T(\lambda - N\epsilon_1, \lambda|u)_{i}^{j} |\text{vac}\rangle_{\lambda}^{\lambda-N\epsilon_1} &= \delta_{j}^{i} \left\{ W_{j}^{j1}(2u)k(u; \xi)_{1} \left( 1 - \prod_{k=1}^{N} W_{1j}^{1j}(u - z_k)W_{j}^{j1}(u + z_k) \right) \right. \\
& \quad \left. + k(u; \xi)_{j} \prod_{k=1}^{N} W_{1j}^{1j}(u - z_k)W_{j}^{j1}(u + z_k) \right\} |\text{vac}\rangle_{\lambda}^{\lambda-N\epsilon_1}, \quad i, j = 2, \ldots, n. \quad (4.31)
\end{align}

Keeping the definition of operators $A$ (4.7) and $D^{j}$ (4.8) in mind, and using the relations $(4.29)$-$(4.31)$, we find that the pseudo-vacuum state given by $(4.22)$ satisfies the following equations

\begin{align}
A(\lambda - N\epsilon_1|u) |\Omega\rangle &= k(u; \xi)_{1} |\Omega\rangle, \quad (4.32) \\
D^{j}_{j}(\lambda - N\epsilon_1|u) |\Omega\rangle &= \delta_{j}^{i} \frac{\sin(2u) e^{i\eta}}{\sin(2u + \eta)} k(u + \frac{\eta}{2}; \xi - \frac{\eta}{2})_{j} \\
& \quad \times \left\{ \prod_{k=1}^{N} \frac{\sin(u + z_k) \sin(u - z_k)}{\sin(u + z_k + \eta) \sin(u - z_k + \eta)} \right\} |\Omega\rangle.
\end{align}
\[a, j = 2, \ldots, n,\]  
\[C_i(\lambda - N\epsilon_1 |u\rangle |\Omega\rangle = 0, \ i = 2, \ldots, n,\]  
\[B_i(\lambda - N\epsilon_1 |u\rangle |\Omega\rangle \neq 0, \ i = 2, \ldots, n,\]  
\[(4.33)\]  
\[(4.34)\]  
\[(4.35)\]

as required. In deriving the equation (4.33), we have used the following equation
\[k(u; \xi)_j - k(u; \xi)_1 W_{1j}^1(2u) = \frac{\sin(2u)e^{i\eta}}{\sin(2u + \eta)} k(u + \frac{\eta}{2}; \xi - \frac{\eta}{2})_j.\]  
\[(4.36)\]

Therefore, we have constructed the pseudo-vacuum state \(|\Omega\rangle\) which is the common eigenstate of the operators \(A, D^i, i = 2, \ldots, n\), and is annihilated by the operators \(C_i, i = 2, \ldots, n\).

The operators \(B_i, i = 2, \ldots, n\), will play the role of creation operators used to generate the Bethe ansatz states.

### 4.4 Nested Bethe ansatz

Having derived the relevant commutation relations (4.10)-(4.11) and constructed the pseudo-vacuum state (4.22), we now apply the generalized algebraic Bethe ansatz method developed in [11] to solve the eigenvalue problem for the transfer matrices (2.19) of the \(A_n^{(1)}\) trigonometric vertex model with open boundary condition specified by the K-matrices \(K^\pm(u)\) given in (2.14)-(2.18).

For convenience, let us introduce a set of non-negative integers \(\{N_i| i = 1, \ldots, n - 1\}\) with \(N_1 = N\) and complex parameters \(\{v_k^{(i)}| k = 1, 2, \ldots, N_i + 1, \ i = 0, 1, \ldots, n - 2\}\). As in the usual nested Bethe ansatz method [23, 24, 22, 20, 11, 25], the parameters \(\{v_k^{(i)}\}\) will be used to specify the eigenvectors of the corresponding reduced transfer matrices. They will be constrained later by the Bethe ansatz equations. For convenience, we adopt the following convention:

\[v_k = v_k^{(0)}, \ k = 1, 2, \ldots, N.\]  
\[(4.37)\]

We will seek the common eigenvectors (i.e. the so-called Bethe states) of the transfer matrix in the form
\[|v_1, \ldots, v_N\rangle = \sum_{i_1, \ldots, i_N=2}^n F^{i_1,i_2,\ldots,i_N} B_{i_1}(\lambda' + \epsilon_{i_1} - \epsilon_1 |v_1\rangle B_{i_2}(\lambda' + \epsilon_{i_1} + \epsilon_{i_2} - 2\epsilon_1 |v_2\rangle \cdots \times B_{i_{N-1}}(\lambda' + \sum_{k=1}^{N-1} \epsilon_k - (N-1)\epsilon_1 |v_{N-1}\rangle \times B_{i_N}(\lambda' + \sum_{k=1}^N \epsilon_k - N\epsilon_1 |v_N\rangle |\Omega\rangle.\]  
\[(4.38)\]
The summing indices in the above equation should obey the following restriction:

\[ \lambda' + \sum_{k=1}^{N} \epsilon_{i_k} = \lambda, \]  

(4.39)

where \( \lambda' \) and \( \lambda \) are the boundary parameters which satisfy the restriction (2.18). The condition (4.39) leads to

\[ |v_1, \ldots, v_N \rangle = \sum_{i_1, \ldots, i_N=2}^{n} F^{i_1,i_2,\ldots,i_N} B_{i_1}(\lambda' + \epsilon_{i_1} - \epsilon_{1}|v_1) B_{i_2}(\lambda' + \epsilon_{i_1} + \epsilon_{i_2} - 2\epsilon_{1}|v_2) \cdots \]

\[ \times B_{i_{N-1}}(\lambda' + \sum_{k=1}^{N-1} \epsilon_{i_k} - (N-1)\epsilon_{1}|v_{N-1}) \]

\[ \times B_{i_N}(\lambda - N\epsilon_{1}|v_{N}) |\Omega\rangle. \]  

(4.40)

With the help of (4.4), (4.7) and (4.8) we rewrite the transfer matrix (2.19) in terms of the operators \( A \) and \( D_i \):

\[
\tau(u) = \sum_{\nu=1}^{n} \tilde{k}(u)_\nu T(\lambda'|u)_\nu^\mu \\
= \tilde{k}(u)_1 A(\lambda'|u) + \sum_{i=2}^{n} \tilde{k}(u)_i T(\lambda'|u)_i^i \\
= \tilde{k}(u)_1 A(\lambda'|u) + \sum_{i=2}^{n} \tilde{k}(u)_i W_{1i}^{i1}(2u) A(\lambda'|u) \\
\quad + \sum_{i=2}^{n} \tilde{k}(u)_i (T(\lambda'|u)_i^i - W_{1i}^{i1}(2u) A(\lambda'|u)) \\
= \sum_{i=1}^{n} \tilde{k}(u)_i W_{1i}^{i1}(2u) A(\lambda'|u) \\
\quad + \sum_{i=2}^{n} \tilde{k}(i) (u + \frac{\eta}{2})_i (T(\lambda'|u)_i^i - W_{1i}^{i1}(2u) A(\lambda'|u)) \\
= \alpha^{(1)}(u) A(\lambda'|u) + \sum_{i=2}^{n} \tilde{k}(i) (u + \frac{\eta}{2})_i D(\lambda'|u)_i^i. \]  

(4.41)

Here we have used (4.8) and introduced the function \( \alpha^{(1)}(u) \),

\[
\alpha^{(1)}(u) = \sum_{i=1}^{n} \tilde{k}(u)_i W_{1i}^{i1}(2u), \]  

(4.42)

and the reduced K-matrix \( \tilde{K}^{(1)}(\lambda'|u) \) with the elements given by

\[
\tilde{K}^{(1)}(\lambda'|u)_i^j = \delta_i^j \tilde{k}^{(1)}(u)_i, \quad i, j = 2, \ldots, n, \]  

(4.43)

\[
\tilde{k}^{(1)}(u)_i = \tilde{k}(u - \frac{\eta}{2})_i, \quad i = 2, \ldots, n. \]  

(4.44)
Moreover we introduce a set of functions \( \{ \alpha^{(b)}(u) \} \) (including the one in (4.42)) related to the reduced K-matrices \( \tilde{K}^{(b)}(\lambda'|u) \) and the ones in (4.43) and (4.44):

\[
\begin{align*}
\tilde{K}^{(b)}(\lambda'|u)_{ij}^{(0)} &= \delta_{ij} \tilde{k}^{(b)}(u), \quad i, j = b + 1, \ldots, n, \quad b = 0, \ldots, n - 1, \\
\tilde{k}^{(b)}(u) &= k(u - \frac{\eta b}{2}), \quad i = b + 1, \ldots, n, \quad b = 0, \ldots, n - 1.
\end{align*}
\]

Carrying out the nested Bethe ansatz, we finally find that, with the coefficients \( F_{i_1 i_2 \cdots : i_N} \) in (4.40) properly chosen, the Bethe state \( |v_1, \ldots, v_N\rangle \) is the eigenstate of the transfer matrix (2.19),

\[
\tau(u) |v_1, \ldots, v_N\rangle = \Lambda(u; \xi, \{v_k\}) |v_1, \ldots, v_N\rangle,
\]

with eigenvalue given by

\[
\Lambda(u; \xi, \{v_k\}) = \alpha^{(1)}(u) k(u; \xi_1) \prod_{k=1}^{N} \frac{\sin(u + v_k) \sin(u - v_k - \eta)}{\sin(u + v_k + \eta) \sin(u - v_k)} \\
+ \frac{\sin(2u) e^{\eta \xi}}{\sin(2u + \eta)} \left\{ \prod_{k=1}^{N} \frac{\sin(u - v_k + \eta) \sin(u + v_k + 2\eta)}{\sin(u - v_k) \sin(u + v_k + \eta)} \right\} \\
\times \prod_{k=1}^{N} \frac{\sin(u + z_k) \sin(u - z_k)}{\sin(u + z_k + \eta) \sin(u - z_k + \eta)} \\
\times \Lambda^{(1)}(u + \frac{\eta}{2}; \xi - \frac{\eta}{2}, \{v_k^{(1)}\}) \right\}.
\]

The eigenvalues \( \{ \Lambda^{(j)}(u; \xi, \{v_k^{(j)}\}) \} \) \( j = 0, \ldots, n - 1 \) (with \( \Lambda(u; \xi, \{v_k\}) = \Lambda^{(0)}(u; \xi, \{v_k^{(0)}\}) \)) of the reduced transfer matrices are given by the following recurrence relations:

\[
\Lambda^{(j)}(u; \xi^{(j)}, \{v_k^{(j)}\}) = \alpha^{(j+1)}(u) k(u; \xi^{(j)}) \prod_{k=1}^{N_{j+1}} \frac{\sin(u + v_k^{(j)}) \sin(u - v_k^{(j)} - \eta)}{\sin(u + v_k^{(j)} + \eta) \sin(u - v_k^{(j)})}
\]

\( \lambda_{\tau} = \alpha^{(1)}(u) k(u; \xi_1) \prod_{k=1}^{N} \frac{\sin(u + v_k) \sin(u - v_k - \eta)}{\sin(u + v_k + \eta) \sin(u - v_k)} \\
+ \frac{\sin(2u) e^{\eta \xi}}{\sin(2u + \eta)} \right\} \prod_{k=1}^{N} \frac{\sin(u - v_k + \eta) \sin(u + v_k + 2\eta)}{\sin(u - v_k) \sin(u + v_k + \eta)} \\
\times \right\} \prod_{k=1}^{N} \frac{\sin(u + z_k) \sin(u - z_k)}{\sin(u + z_k + \eta) \sin(u - z_k + \eta)} \\
\times \Lambda^{(1)}(u + \frac{\eta}{2}; \xi - \frac{\eta}{2}, \{v_k^{(1)}\}) \right\}.
\]
satisfy the following Bethe ansatz equations:

$$\xi \text{ here we have adopted the convention:}$$

$$\{ \xi \} = \{ \xi \} - \eta \frac{1}{2}$$

The reduced boundary parameters \{ \xi \} and inhomogeneous parameters \{ z \} are given by

$$\xi^{(j+1)} = \xi^{(j)} - \eta \frac{1}{2}, \quad z^{(j+1)} = v^{(j)} + \eta \frac{1}{2}, \quad j = 0, \ldots, n - 2.$$  (4.52)

Here we have adopted the convention: \( \xi = \xi^{(0)}, \ z^{(0)} = z \). The complex parameters \{ v^{(j)} \}

satisfy the following Bethe ansatz equations:

$$\alpha^{(1)}(v_s) k(v_s; \xi) \frac{\sin(2v_s + \eta)e^{i\eta}}{\sin(2v_s + 2\eta)} \times \prod_{k \neq s, k=1}^{N_1} \frac{\sin(v_s + v_k) \sin(v_s - v_k - \eta)}{\sin(v_s + v_k + 2\eta) \sin(v_s - v_k + \eta)}$$

$$= \prod_{k=1}^{N} \frac{\sin(v_s + z_k) \sin(v_s - z_k)}{\sin(v_s + z_k + \eta) \sin(v_s - z_k + \eta)} \times \Lambda^{(1)}(v_s + \eta \frac{1}{2}; \xi - \eta \frac{1}{2}; \{ v^{(1)} \}),$$  (4.53)

$$\alpha^{(j+1)}(v^{(j)}_s) k(v^{(j)}; \xi^{(j)}) \frac{\sin(2v^{(j)} + \eta)e^{i\eta}}{\sin(2v^{(j)} + 2\eta)} \times \prod_{k \neq s, k=1}^{N_{j+1}} \frac{\sin(v^{(j)} + v^{(j)}_k) \sin(v^{(j)} - v^{(j)}_k - \eta)}{\sin(v^{(j)} + v^{(j)}_k + 2\eta) \sin(v^{(j)} - v^{(j)}_k + \eta)}$$

$$= \prod_{k=1}^{N_j} \frac{\sin(v^{(j)} + z^{(j)}_k) \sin(v^{(j)} - z^{(j)}_k)}{\sin(v^{(j)} + z^{(j)}_k + \eta) \sin(v^{(j)} - z^{(j)}_k + \eta)} \times \Lambda^{(j+1)}(v^{(j)} + \eta \frac{1}{2}; \xi^{(j)} - \eta \frac{1}{2}; \{ v^{(j+1)} \}),$$  (4.54)

$$j = 1, \ldots, n - 2,$$
5 Conclusions

We have studied the $A_{n-1}^{(1)}$ trigonometric vertex model with integrable open boundary condition described by the generic non-diagonal boundary K-matrix $K^-(u)$ given in (2.14) and its dual $K^+(u)$ given in (2.15) with restriction (2.18). In addition to the two discrete (positive integers) parameters $l$ and $l'$, the total number of the independent free boundary parameters: $\xi, \bar{\xi}, \rho$ and $\lambda_i$, $i = 1, \ldots, n$, is actually $n + 3$. Although the K-matrices given in (2.14) and (2.15) are non-diagonal in the vertex picture, they become diagonal simultaneously in the “face” picture after the face-vertex transformation given by (4.1)-(4.3). This fact enables us to successfully construct the corresponding pseudo-vacuum state $|\Omega\rangle$ (4.22) and apply the algebraic Bethe ansatz method to diagonalize the corresponding double-row transfer matrices. The eigenvalues of the transfer matrices and associated Bethe ansatz equations are given by (4.49), (4.50)-(4.52), and (4.53), (4.54). Taking the rational limit of our results with $\{\lim_{\eta \to 0}(\eta \lambda_i)\}$ being kept finite, we recover the results obtained in [26].

Acknowledgements

This work was financially supported by the Australian Research Council.

Appendix A: The exchange relation of $\mathcal{T}$

The starting point for deriving the exchange relations (4.16) among $\mathcal{T}(m|u)_\mu$ is the exchange relation (2.12). Multiplying both sides of (2.12) from the right by $\phi_{m+\epsilon_{i_3}, m}(-u_1) \otimes \phi_{m+\epsilon_{i_3}+\epsilon_{j_3}, m+\epsilon_{i_3}}(-u_2)$, and using the face-vertex correspondence relation (3.6) and the “completeness” relation (3.13), we have, for the L.H.S. of the resulting relation,

\[
\text{L.H.S.} = R_{12}(u_1 - u_2)T_1(u_1)R_{21}(u_1 + u_2) \\
\quad \times (\phi_{m+\epsilon_{i_3}, m}(-u_1) \otimes T(u_2) \phi_{m+\epsilon_{i_3}+\epsilon_{j_3}, m+\epsilon_{i_3}}(-u_2)) \\
= R_{12}(u_1 - u_2)T_1(u_1)R_{21}(u_1 + u_2)(\phi_{m+\epsilon_{i_3}, m}(-u_1) \otimes 1) \\
\quad \times (1 \otimes \{\sum_{j_2} \phi_{m+\epsilon_{i_3}+\epsilon_{j_2}, m+\epsilon_{i_3}}(u_2)\tilde{T}_{m+\epsilon_{i_3}+\epsilon_{j_2}, m+\epsilon_{i_3}}(u_2) \\
\quad \times T(u_2)\phi_{m+\epsilon_{i_3}+\epsilon_{j_3}, m+\epsilon_{i_3}}(-u_2)) \}) \\
= \sum_{j_2} R_{12}(u_1 - u_2)T_1(u_1)R_{21}(u_1 + u_2)
\]
Similarly for the R.H.S. of the resulting relation, we obtain

\[ A \phi_{m+i_3, m}(u_1) \otimes \phi_{m+i_3+\epsilon_j, m+i_3}(u_2) \] \[ \mathcal{T}(m+\epsilon_i + \epsilon_j, u_2) \] \[ = \sum_{i_2} \sum_{j_1, j_2} R_{12}(u_1 - u_2) T_1(u_1) W_{j_1}^{j_2} (u_1 + u_2) \times \phi_{m+i_3+\epsilon_j, m+i_3}(u_2) \] \[ \mathcal{T}(m+\epsilon_i + \epsilon_j, u_2) \] 

\[ \vdots \]

\[ = \sum_{i_0, j_0} \phi_{m+i_0, m}(u_1) \otimes \phi_{m+i_0+\epsilon_j, m+i_0}(u_2) \]

\[ \mathcal{T}(m+\epsilon_i + \epsilon_j, u_2) \]

\[ \times \left\{ \sum_{i_1, i_2} \sum_{j_1, j_2} T(m+\epsilon_i + \epsilon_j) \mathcal{T}(u_1 - u_2) \right\} \]

Similarly for the R.H.S. of the resulting relation, we obtain

\[ \text{R.H.S.} = \sum_{i_0, j_0} \phi_{m+i_0, m}(u_1) \otimes \phi_{m+i_0+\epsilon_j, m+i_0}(u_2) \]

\[ \mathcal{T}(m+\epsilon_i + \epsilon_j, u_2) \]

\[ \times \left\{ \sum_{i_1, i_2} \sum_{j_1, j_2} T(m+\epsilon_i + \epsilon_j) \mathcal{T}(u_1 - u_2) \right\} \]

Note that intertwiners are linearly independent, which follows from (3.11). Thus we obtain the exchange relation (4.6) by comparing (A.1) with (A.2).

**Appendix B: The relevant commutation relations**

Let us introduce

\[ D_i^j(m|u) = \mathcal{T}(m|u), \quad i, j = 2, \ldots, n. \]  \[ (B.1) \]

The starting point for deriving the commutation relations among \( A(m|u) \), \( D_i^j(m|u) \) and \( B_i(m|u) \) \( i, j = 2, \ldots, n \) is the exchange relation (4.6).

For \( i_0 = j_0 = j_3 = 1, \ i_3 = i \neq 1 \), we obtain

\[ A(m + 2\epsilon_1|v) B_j(m + \epsilon_j + \epsilon_1|u) \]

\[ = \frac{\sin(u + v) \sin(u + v + \eta)}{\sin(u + v + \eta) \sin(u + v)} B_j(m + \epsilon_j + \epsilon_1|u) A(m + \epsilon_j + \epsilon_1|v) \]

\[ - \frac{\sin(\eta) \sin(u + v) e^{-i(u-v)}}{\sin(u - v) \sin(u + v + \eta)} B_j(m + \epsilon_j + \epsilon_1|u) A(m + \epsilon_j + \epsilon_1|v) \]

\[ - \frac{\sin(\eta) e^{i(u+v)}}{\sin(u + v + \eta)} \sum_{\alpha=2}^n B_\alpha(m + \epsilon_\alpha + \epsilon_1|v) D_j^\alpha(m + \epsilon_j + \epsilon_1|u). \]  \[ (B.2) \]
The commutation relation \((4.9)\) is a simple consequence of \((B.2)\), \((4.7)\) and \((4.8)\).

For \(i_0 = k \neq 1, j_0 = 1, i_3 = i \neq 1\) and \(j_3 = j \neq 1\), we obtain

\[
D_a^k(m + \epsilon_a + \epsilon_1|u)B_j(m + \epsilon_j + \epsilon_a|v)
= \sum_{\alpha_1,\alpha_2,\beta_1,\beta_2=2}^n \frac{W_{\alpha_2\beta_2}^k(u + v)W_{\beta_1}^{\alpha_1}(u - v)}{W_{k1}^{\alpha_1}(u - v)W_{1a}^{\alpha_1}(u + v)}
\times B_{\beta_2}(m + \epsilon_k + \epsilon_{\beta_2}|v)D_{\alpha_1}^\beta(m + \epsilon_a + \epsilon_j|u)
+ \sum_{\alpha=2}^n \frac{W_{k1}^{\alpha}(u + v)W_{\alpha}^{k}(u - v)}{W_{k1}^{\alpha}(u - v)W_{1a}^{\alpha}(u + v)}A(m + \epsilon_k + \epsilon_1|v)B_{\alpha}(m + \epsilon_a + \epsilon_j|u)
- \sum_{\alpha,\beta=2}^n \frac{W_{1k}^{\alpha}(u - v)W_{k}^{\beta}(u + v)}{W_{k1}^{\alpha}(u - v)W_{1a}^{\alpha}(u + v)}B_{\alpha}(m + \epsilon_k + \epsilon_a|u)D_j^\beta(m + \epsilon_a + \epsilon_j|v)
- \frac{W_{1k}^{\alpha}(u - v)W_{k}^{1}(u + v)}{W_{k1}^{\alpha}(u - v)W_{1a}^{\alpha}(u + v)}A(m + \epsilon_k + \epsilon_1|u)B_j(m + \epsilon_a + \epsilon_j|v).
\]

In order to separate the contribution of \(A\) and \(D_j^\beta\) in the above relations, one needs to introduce the operator \(D_a^k\) as \((4.8)\) (cf. [2]). Then we can derive the commutation relations among \(D_a^k\) and \(B_j\) from \((B.3)\).

\[
D_a^k(m|u)B_j(m + \epsilon_j - \epsilon_1|v)
= \sum_{\alpha_1,\alpha_2,\beta_1,\beta_2=2}^n \frac{W_{\alpha_2\beta_2}^k(u + v)W_{\beta_1}^{\alpha_1}(u - v)}{W_{k1}^{\alpha_1}(u - v)W_{1a}^{\alpha_1}(u + v)}
\times B_{\beta_2}(m + \epsilon_k + \epsilon_{\beta_2} - \epsilon_a - \epsilon_1|v)D_{\alpha_1}^\beta(m + \epsilon_j - \epsilon_1|u)
- \sum_{\alpha,\beta=2}^n \frac{W_{1k}^{\alpha}(u - v)W_{k}^{\beta}(u + v)}{W_{k1}^{\alpha}(u - v)W_{1a}^{\alpha}(u + v)}B_{\alpha}(m + \epsilon_k + \epsilon_a - \epsilon_1|v)A(m + \epsilon_j - \epsilon_1|u)
+ \sum_{\alpha,\beta_1,\beta_2=2}^n \frac{W_{\alpha_2\beta_2}^k(u + v)W_{\beta_1}^{\alpha_1}(u - v)}{W_{k1}^{\alpha_1}(u - v)W_{1a}^{\alpha_1}(u + v)}W_{1j}^{\beta_2}(2u)
\times B_{\beta_2}(m + \epsilon_k + \epsilon_a - \epsilon_1|v)A(m + \epsilon_j - \epsilon_1|u)
- \sum_{\alpha=2}^n \frac{W_{1k}^{\alpha}(u - v)W_{k}^{\beta}(u + v)}{W_{k1}^{\alpha}(u - v)W_{1a}^{\alpha}(u + v)}W_{1j}^{\beta_2}(2v)
\times B_{\alpha}(m + \epsilon_j - \epsilon_a - \epsilon_1|v)A(m + \epsilon_k - \epsilon_1|u)
+ \sum_{\alpha=2}^n \frac{W_{1k}^{\alpha}(u + v)W_{k}^{\beta}(u - v)}{W_{k1}^{\alpha}(u - v)W_{1a}^{\alpha}(u + v)}A(m + \epsilon_k - \epsilon_a|v)B_{\alpha}(m + \epsilon_j - \epsilon_1|u)
- \frac{\sin(u - v + \eta)\sin(u + v + \eta)\sin(\eta)e^{-2iu}}{\sin(u - v)\sin(u + v)\sin(2u + \eta)}
\times \delta_{\alpha}^kA(m - \epsilon_a + \epsilon_k|u)B_j(m + \epsilon_j - \epsilon_1|v).
\]

\(21\)
We have used the following equation
\[ \frac{W^{k_{1}}_{1k}(u-v)W^{k_{1}}_{1k}(u+v)}{W^{k_{1}}_{1k}(u-v)W^{k_{1}}_{1k}(u+v)} + W^{k_{1}}_{1k}(2u) = \frac{\sin(u-v+\eta)\sin(u+v+\eta)e^{-2iu}}{\sin(u-v)\sin(u+v)\sin(2u+\eta)}, \] (B.5)

to derive the last term on the right side of the above equation. After some long tedious calculation, we finally obtain the commutation relation (4.10) from (B.5) by noting the definitions (4.7) and (4.8).

For \( i_{0} = j_{0} = 1, i_{3} = i \neq 1 \) and \( j_{3} = j \neq 1 \), we obtain
\[ B_{i}(m+\epsilon_{i}+\epsilon_{1}|u)B_{j}(m+\epsilon_{i}+\epsilon_{j}|v) = \sum_{\beta,\alpha=2}^{n} \frac{W^{\beta\alpha}_{i1}(u-v)W^{\lambda\beta}_{j1}(u+v)}{W^{\lambda\alpha}_{i1}(u-v)W^{\lambda\beta}_{j1}(u+v)}B_{\beta}(m+\epsilon_{\beta}+\epsilon_{1}|v)B_{\alpha}(m+\epsilon_{i}+\epsilon_{j}|u), \] (B.6)

which leads to the commutation relation (4.11).

**Appendix C: The action of \( T_{i}^{j} \) on the pseudo-vacuum state**

Carrying out the calculation similar to that leading to (4.29) and (4.30), we have
\[
\mathcal{T}(\lambda - N\epsilon_{1}, \lambda|u)_{ij}|\text{vac}\rangle_{\lambda}^{\lambda-N\epsilon_{1}}
= k(u;\xi)_{ij}T(\lambda - N\epsilon_{1} - \epsilon_{j}, \lambda - \epsilon_{j}|u)_{ij}S(\lambda - N\epsilon_{1}, \lambda|u)_{ij}|\text{vac}\rangle_{\lambda}^{\lambda-N\epsilon_{1}}
+ \delta_{ij}k(u;\xi)_{ij} \prod_{k=1}^{N} W_{ij1}(u-z_{k})W_{j1}(u+z_{k})
\times |\text{vac}\rangle_{\lambda}^{\lambda-N\epsilon_{1}}, \quad i, j = 2, \ldots, n. \quad (C.1)
\]
The first term on the right hand of the above equation is obtained as follows.

From “RLL” relation (2.9), we derive the following exchange relations
\[ T_{1}(u)R_{12}(2u)T_{2}^{-1}(-u) = T_{2}^{-1}(-u)R_{12}(2u)T_{1}(u). \] (C.2)

Multiplying both sides of the above equation from the left by \( \tilde{\phi}_{\lambda-N\epsilon_{1}+\epsilon_{i},\lambda-N\epsilon_{1}-\epsilon_{i}}(u) \otimes \tilde{\phi}_{\lambda+\epsilon_{1},\lambda}(-u) \) and from the right by \( \phi_{\lambda+\epsilon_{1},\lambda}(u) \otimes \phi_{\lambda-N\epsilon_{1},\lambda-N\epsilon_{1}-\epsilon_{i}}(-u) \), we obtain the following
exchange relation from the face-vertex correspondence relations (3.6) and (3.16)-(3.19)

\[
\sum_{\alpha=1}^{n} w_{1\alpha}^{ij}(2u) T(\lambda - N\epsilon_1 - \epsilon_i, \lambda - \epsilon_\alpha |u\rangle_\alpha^j S(\lambda - N\epsilon_1, \lambda |u\rangle_i^a)
\]

\[= \sum_{\alpha,\beta=1}^{n} w_{\alpha\beta}^{ij}(2u) S(\lambda - N\epsilon_1 + \epsilon_\alpha, \lambda + \epsilon_1 |u\rangle_\beta^j T(\lambda - N\epsilon_1, \lambda |u\rangle_i^a). \quad (C.3)\]

Acting both sides on the pseudo-vacuum state |vac⟩_\lambda^{\lambda-N\epsilon_1}, and using the equations (4.23)-(4.28), we obtain

\[
T(\lambda - N\epsilon_1 - \epsilon_i, \lambda - \epsilon_\lambda |u\rangle_i^1 S(\lambda - N\epsilon_1, \lambda |u\rangle_1^1 |vac\rangle_\lambda^{\lambda-N\epsilon_1}
\]

\[= \delta_i^j \left\{ w_{1j}^{ij}(2u) - w_{1j}^{ij}(2u) \left( \prod_{k=1}^{N} w_{1j}^{1j}(u-z_k) w_{j1}^{j1}(u+z_k) \right) \right\}
\]

\[\times |vac\rangle_\lambda^{\lambda-N\epsilon_1}. \quad (C.4)\]

The equation (4.31) is a simple consequence of the equations (C.1) and (C.4).

References

[1] V. E. Korepin, N. M. Bogoliubov and A. G. Izergin, *Quantum Inverse Scattering Method and correlation Function*, Cambridge Univ. Press, Cambridge, 1993.

[2] E. K. Sklyanin, *Boundary conditions for integrable quantum systems*, J. Phys. A 21 (1988), 2375.

[3] L. Mezincescue and R. I. Nepomechie, *Integrable open spin chains with non-symmetric R-matrices*, J. Phys. A 24 (1991), L17.

[4] S. Ghoshal and A.B. Zamolodchikov, *Boundary s-matrix and boundary state in two-dimensional integrable quantum field theory*, Int. J. Mod. Phys. A 9 (1994), 3841 hep-th/9306002.

[5] R. I. Nepomechie, *Sloving the open XXZ spin chain with non-diagonal boundary terms at roots of unity*, Nucl. Phys. B 622 (2002), 615 hep-th/0110116.

[6] R. I. Nepomechie, *Functional relations and Bethe ansatz for the XXZ chain*, J. Stat. Phys. 111 (2003), 1363 hep-th/0211001; *Bethe ansatz solution of the open XXZ chain with nondiagonal boundary terms*, J. Phys. A 37 (2004), 433 hep-th/0304092.
[7] J. Cao, H.-Q. Lin, K.-J. Shi and Y. Wang, Exact solution of XXZ spin chain with unparallel boundary fields, Nucl. Phys. B 663 (2003), 487.

[8] W.-L. Yang, Y.-Z. Zhang and M. Gould, Exact solution of the XXZ Gaudin model with generic open boundaries, Nucl. Phys. B 698 (2004), 503 [hep-th/0411048].

[9] J. de Gier and P. Pyatov, Bethe ansatz for the Temperley-Lieb loop model with open boundaries, JSTAT 03 (2004), P002 [hep-th/0312235].

[10] W.-L. Yang and Y.-Z. Zhang, Non-diagonal solutions of the reflection equation for the trigonometric $A_{n-1}^{(1)}$ vertex model, JHEP 12 (2004), 019 [hep-th/0411160].

[11] W.-L. Yang and R. Sasaki, Exact solution of $Z_n$ belavin model with open boundary condition, Nucl. Phys. B 679 (2004), 495 [hep-th/0308127].

[12] I. V. Cherednik, On a method of constructing factorized S-matrices in terms of elementary functions, Theor. Mat. Fiz 43 (1980), 117.

[13] J. H. H. Perk and C. L. Schultz, New families of commuting transfer matrices in q-state vertex models, Phys. Lett. A 84 (1981), 407.

[14] V. V. Bazhanov, R. M. Kashaev, V. V. Mangazeev and Yu. G. Stroganov, $(Z_N \times)^{n-1}$ generalization of the chiral potts model, Commun. Math. Phys. 138 (1991), 393.

[15] A. Lima-Santos, $A_{n-1}^{(1)}$ reflection K-matrices, Nucl. Phys. B 644 (2002), 568.

[16] J. Abad and M. Rios, Non-diagonal solutions to reflection equations in $su(n)$ spin chains, Phys. Lett. B 352 (1995), 92.

[17] Y. Yamada, Segre threefold and $N = 3$ reflection equation, Phys. Lett A 298 (2002), 350.

[18] A. Belavin, Dynamical symmetry of integrable quantum systems, Nucl. Phys. B 180 (1981), 189.

[19] M. Jimbo, T. Miwa and M. Okado, Solvable lattice models whose states are dominant integral weights of $A_{n-1}^{(1)}$, Lett. Math. Phys. 14 (1987), 123; Local state probabilities of solvable lattice models: $A_{n-1}^{(1)}$ family, Nucl. Phys. B 300 (1988), 74.

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[20] B. Y. Hou, R. Sasaki and W.-L. Yang, *Algebraic bethe ansatz for the elliptic quantum group $e_{r,n}(sl_n)$ and its applications*, Nucl. Phys. B 663 (2003), 467 [hep-th/0303077].

[21] W.-L. Yang and R. Sasaki, *Solution of the dual reflection equation for $A^{(1)}_{n-1}$ SOS model*, J. Math. Phys. 45 (2004), 4301 [hep-th/0308118].

[22] H. J. de Vega and A. Gonzalez-Ruiz, *Exact solution of the $SU_q(n)$ invariant quantum spin chains*, Nucl. Phys. B 417 (1994), 553 [hep-th/9309022].

[23] O. Babelon, H. J. de Vega and C. M. Viallet, *Exact solution of the $Z(N+1)_xZ(N+1)$ symmetric generalization of the XXZ model*, Nucl. Phys. B 200 (1982), 266.

[24] C. L. Schultz, *Eigenvectors of the multicomponent generalization of the six vertex model*, Physica A 122 (1983), 71.

[25] W.-L. Yang, R. Sasaki and Y.-Z. Zhang, *$Z_n$ elliptic Gaudin model with open boundaries*, JHEP 09 (2004), 046 [hep-th/0409002].

[26] W. Galleas and M. J. Martins, *Solution of the $SU(N)$ vertex model with non-diagonal open boundaries*, e-print: [nlin.SI/0407027](nlin.SI/0407027).