ON THE CHOQUET-BRUHAT–YORK–FRIEDRICH
FORMULATION OF THE EINSTEIN-EULER EQUATIONS

MARCELO M. DISCONZI AND VAMSI P. PINGALI

Abstract. Short-time existence for the Einstein-Euler and the vacuum Einstein equations is proven using a Friedrich inspired formulation due to Choquet-Bruhat and York, where the system is cast into a symmetric hyperbolic form and the Riemann tensor is treated as one of the fundamental unknowns of the problem. The reduced system of Choquet-Bruhat and York, along with the preservation of the gauge, is shown to imply the full Einstein equations.

1. Introduction

In the vast amount of literature that exists on the Cauchy problem of General Relativity (GR)\(^1\), the formulation in terms of a first order symmetric hyperbolic system (FOSH) has recently attracted significant attention (see e.g. [YB, Fri1, Fri2, FN, FR] and references therein.). Here, we focus on the Choquet-Bruhat and York [YB] formulation of the Einstein-Euler system in terms of the Lagrangian\(^2\) description of the fluid flow, which itself was adapted from an earlier formulation by Friedrich [Fri2].

In [YB] the authors wrote a system of equations in terms of the Riemann tensor (as opposed to the Weyl tensor used in [Fri2]), and chose a gauge that reduced this system to a FOSH. It turned out that this system had physical characteristics (in contrast with Friedrich’s one), i.e. the assumption that the speed of sound in the fluid is less than that of light was crucial to prove the hyperbolicity of the equations. This is important, for example, because it gives a natural breakdown criterion for the problem of long-time

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\(^1\)A complete or extensive account of all references is beyond the scope of this paper, whose length we tried to keep short. We refer the interested reader to the monographs [B1, R] and the survey papers [CGP, FR]. A long, although far from complete, review of the literature of the Cauchy problem for the Einstein-Euler system specifically, is given in [D], while a thorough and up-to-date treatment of relativistic fluids can be found in [RZ]. Further discussion on relativistic fluids, including problems such as the inclusion of viscosity, long-time existence, and other fluid-matter models, can be found in [D2, RS, S, S2] and references therein.

\(^2\)Intuitively, the picture is like this. We can think of two ways to study the flow of a river: one could float downstream on a boat, or one could sit on the bank and observe the flow. The former (Lagrangian description) corresponds to tracking the position of every particle and the latter (Eulerian description) to observing the velocity vector field. Both descriptions are useful in the study of relativistic and non-relativistic fluids.
existence. However, the task of proving that the gauge is preserved and the
original Einstein-Euler system is satisfied was not carried out in [YB]. In
this article, we complete the proof of short-time existence for the Einstein-
Euler system à la Choquet-Bruhat and York [YB]. In what follows, we shall
restrict ourselves to barotropic fluids.

In mentioning Choquet-Bruhat and York’s construction of a FOSH, it is
worth recalling the general strategy for solving Einstein’s equations. The
Einstein equations do not form an “honest” system of evolution equations,
in the sense that some of the equations are constraint equations. Such a
difficulty is a consequence of the diffeomorphism invariance, or gauge free-
dom, enjoyed by the system. To circumvent this problem one considers a
different set of equations, generally referred to as the “reduced system” con-
taining only genuine evolution equations that can be solved using standard
techniques. This system is chosen so as to correspond to the original Ein-
stein’s equations modulo the constraints (which have to be solved separately
in order to produce a full set of initial data for the evolution problem, see
below). This task can be accomplished by a suitable choice of gauge. A
solution to the original system is then obtained by showing that the gauge
conditions are in fact satisfied on the time-interval where a solution to the
reduced equations has been shown to exist, provided they are satisfied ini-
tially, i.e., at time $t = 0$. This is done by deriving a suitable system of
evolution equations for the gauge and using uniqueness.

We remark that the result here obtained is not, in itself, new. Short-time
existence for the Einstein-Euler system had been proven earlier by Choquet-
Bruhat [B2], and subsequently by Lichnerowicz [Li1, Li2]. The novelty in the
approach initiated by Friedrich [Fri2] is the use of the Lagrangian descrip-
tion of the fluid. This sheds new light in the problem of the so-called “fluid
body” modeling certain stellar dynamics, where one attempts to solve the
free-boundary problem that arises from considering the the system formed
by the Einstein equations coupled to the Euler equations within a bounded
region, with vacuum Einstein’s equations holding on the complement. Re-
cent existence results for this problem have been obtained by Brauer and
Karp [BK1, BK2].

2. Summary of results

In the study of the Cauchy problem in GR, one is usually given a RIemann-
ian smooth 3-fold $(\Sigma, h_0)$, a symmetric 2-tensor $K$, and other initial data cor-
responding to the matter fields. This initial data is required to satisfy certain
constraint equations, which are derived from the Gauss-Codazzi-Mainardi
equations and the Einstein equations, and ensure that $(\Sigma, h_0)$ embeds iso-
metrically, with $K$ as its second fundamental form, into the space-time that
is eventually obtained as a solution of the full Einstein system. This pre-
scription of data is usually facilitated by means of the conformal method of
solving the constraints. The aim then is to find an Einsteinian development,
i.e. a Lorentzian 4-fold \((M, g) = (\Sigma \times [0, T], g)\), containing matter fields satisfying Einstein’s equations and obeying the initial conditions on the matter fields.

Naturally, upon writing \(\Sigma \times [0, T]\) we are relying on a particular choice of diffeomorphism to parametrize the would-be\(^3\) “time coordinate” \(t \in [0, T]\). Although the existence of a solution to the Einstein-Euler system can be stated in a more invariant fashion, here it is convenient to write explicitly \(\Sigma \times [0, T]\) in order to follow the similar statements of \([YB]\), on which this work is largely based, and also to facilitate the identification of the spaces where solutions live in.

From a PDE perspective, using a standard 3+1 coordinate decomposition where the vectors \(\frac{\partial}{\partial x^i}, i = 1, 2, 3\) are space-like and \(\frac{\partial}{\partial x^0}\) is time-like, the constraint equations read \([B1]\)

\[
\text{Ricc}_{\mu 0} - \frac{1}{2} R g_{\mu 0} = T_{\mu 0} \text{ on } \Sigma,
\]

where \(T\) is the stress-energy tensor of the matter fields. It is easy to see that the constraints do not form a system of second order evolution equations. In particular, initial data for the full Einstein system ought to satisfy (2.1), and thus cannot be given arbitrarily. From these considerations, it is seen that, while the construction of initial data satisfying the constraint equations is doubtless a crucial aspect of the investigations surrounding Einstein’s equations, it can be considered apart from the evolution problem. Thus, in what follows, it is assumed throughout that the fields in a given initial data set always satisfy the constraint equations. We comment further on the initial conditions in section 3.

We assume that we are given the aforementioned type of initial data. This data is then converted into the type we need for solving the FOSH. In what follows, for the fluid case, \(p \geq 0\) indicates pressure, \(\mu(p) > 0\) is the density as a function of \(p\), \(v\) is the initial 3-velocity of the fluid on \(\Sigma\), and \(u\) is the 4-velocity field on \(M\). We prove our results in the Sobolev spaces \(H^s\). In what follows, \(\Sigma, h_0, p\) denote the quantities just described, and repeated indices are summed over. We also assume the reader is familiar with the terminology and the Cauchy problem in GR and the initial conditions for the Einstein-Euler system.

Our first result is on the vacuum Einstein equations, i.e. \(\text{Ricc} = 0\):

**Theorem 2.1.** Let \(s > \frac{3}{2} + 2\), and let \((\Sigma, h_0, K)\) be an initial data set for the vacuum Einstein equations, with \(h_0 \in H^{s+1}(\Sigma)\), \(K_0 \in H^s(\Sigma)\), and \(\Sigma\) compact. Then, there exists an Einsteinian development \(M = (\Sigma \times [0, T], g)\) satisfying \(\text{Ricc}(g) = 0\). The metric \(g\) thus obtained is in \(C^0([0, T], H^{s+1}(\Sigma)) \cap C^1([0, T], H^s(\Sigma)) \cap C^2([0, T], H^{s-1}(\Sigma))\).

We have assumed \(\Sigma\) to be compact for simplicity. This can be relaxed provided suitable asymptotic conditions on the initial data are imposed.

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\(^3\)As there is no natural notion of a time coordinate in GR.
While asymptotically flat initial data is the standard choice for the vacuum case, existence under similar conditions becomes technically challenging for the case of the Einstein-Euler system; see [BK1, BK2]. Notice, also, that by stating our existence theorem on the closed interval \([0, T]\), we are not taking the maximal Cauchy development of the initial data.

For perfect fluids, the energy-momentum tensor is \(T = (\mu + p)u \otimes u + pg\). The Einstein-Euler system for a perfect fluid is

\[
\begin{align*}
\text{Ricc}_{\mu\nu} - \frac{1}{2} Rg_{\mu\nu} &= T_{\mu\nu} \\
(\mu + p)u^\alpha \partial_\alpha u^\beta + (u^\alpha u^\beta + g^{\alpha\beta})\partial_\alpha p &= 0 \\
(\mu + p)\partial_\alpha u^\alpha + u^\alpha \partial_\alpha \mu &= 0 \\
u^\alpha u_\alpha &= -1
\end{align*}
\]

(2.2)

Notice that we have chosen units such that \(\frac{8\pi G}{c^2} = 1\). Also note that the first equation maybe written as \(\text{Ricc}_{\mu\nu} = \rho_{\mu\nu}\) where \(\rho_{\mu\nu} = T_{\mu\nu} - \frac{\text{Tr}(T)}{2} g_{\mu\nu}\), where \(\text{Tr}\) denotes the trace.

The Einstein-Euler system also exists for a short period of time:

**Theorem 2.2.** Let \(s > \frac{3}{2} + 2\), and let \((\Sigma, h_0, K, p_0, v)\) be an initial data set for the Einstein-Euler system, where \(\Sigma\) is compact, \(h_0\) is in \(H^{s+1}(\Sigma)\), \(K\) is in \(H^s(\Sigma)\), \(p_0\) is in \(H^s(\Sigma)\), and \(v\) is in \(H^s(\Sigma)\). Fix a smooth invertible function \(\mu : [0, \infty) \rightarrow (0, \infty)\) with \(\mu' \geq 1\). Then there exists an Einstein development \(M = (\Sigma \times [0, T], g)\) satisfying the Einstein-Euler system. The metric \(g\) thus obtained is in \(C^0([0, T], H^{s+1}(\Sigma)) \cap C^1([0, T], H^s(\Sigma)) \cap C^2([0, T], H^{s-1}(\Sigma))\), the four-velocity \(u \in C^0([0, T], H^s(\Sigma)) \cap C^1([0, T], H^{s-1}(\Sigma))\), and the pressure \(p \in C^0([0, T], H^s(\Sigma)) \cap C^1([0, T], H^{s-1}(\Sigma))\). They obey the initial conditions, hence in particular the orthogonal projection of \(u\) onto \(T\Sigma\) is \(v\).

**Remark 2.3.** The condition \(\mu' \geq 1\) guarantees that the speed of sound is at most that of light. The Einstein-Euler system will be a FOSH only as long as this condition holds. This is one advantage of having only physical characteristics, as mentioned in the introduction.

**Remark 2.4.** Following the usual arguments relying on the finite-propagation speed property of FOSH systems, in the proof of theorems 2.1 and 2.2 we shall work solely on a single coordinate chart and use uniqueness of solutions of FOSH systems in the following way: Let \(V \subset \bar{U} \subset U \subset \Sigma\) be relatively compact open sets having smooth boundary (with \(V\) and \(\bar{U}\) contained in a coordinate chart). The domain of a solution to a FOSH system with initial data in \(\bar{U}\) contains \(V \times [0, T_V]\) for some \(T_V\) such that \(V \times [0, T_V]\) in the domain of influence of \(\bar{U}\). Moreover, if \(V \cap W\) is not empty for a relatively compact, open, smooth \(W\) then by uniqueness of solutions the solutions on \(V \times [0, \min T_V, T_W]\) and \(W \times [0, \min T_V, T_W]\) coincide. This way we get a unique solution on \(U \times [0, T_U]\) for some \(T_U > 0\). A solution on the whole of
Σ × [0, T], for some T > 0, is then obtained by a standard gluing procedure.

**Remark 2.5.** The regularity hypotheses in the theorems along with Sobolev embedding imply that the metric is \( C^3 \). This would appear to be superfluous because Einstein’s equations involve only two derivatives of the metric. However, the reduced system involves the derivatives of the curvature. Hence the need for additional regularity. In fact, in whatever follows, we shall need to take two derivatives of the curvature. We are so allowed because \( h_0 \) belongs to \( H^{s+1} \) and thus has four (weak) derivatives since \( s > \frac{3}{2} + 2 \).

As already mentioned, Einstein’s equations (in vacuum and coupled to matter) are diffeomorphism invariant and hence an appropriate gauge has to be chosen in order to solve them. Traditionally, harmonic coordinates were employed to convert the Einstein equations into a second order hyperbolic system. However, in the Lagrangian framework, Choquet-Bruhat and York chose the so-called Cattaneo-Ferrarese (CF) gauge consisting of Lagrangian observers following the fluid flow. In other words, a choice of a local orthonormal frame \( \{ e_\alpha \}_{\alpha=0}^3 \) such that \( e_0 = u = \partial_x^0 \), and the remaining \( \{ e_i \}_{i=1}^3 \) are Fermi propagated, i.e., \( \omega^j_{0i} = 0 \) for \( i, j = 1, 2, 3 \), where \( \omega \) denotes the connection coefficients of the Levi-Civita connection (or Ricci rotation coefficients) with respect to \( \{ e_\alpha \}_{\alpha=0}^3 \). It is also assumed that local coordinates \( x^\alpha \) have been chosen such that \( e_0 = u = \partial_x^0 \) and \( \partial_x^a \) gives a basis for the tangent space of \( \Sigma \) within the selected coordinate chart. For notational convenience, we shall denote \( f_{ij} = \omega^j_{0i} \) and \( \partial_i = e_i, i = 1, 2, 3 \). From the above, we can write

\[
\partial_i \equiv e_i = A^j_i \left( \partial_x^j - b_i \partial_x^0 \right),
\]

for \( i = 1, 2, 3 \), and a certain invertible matrix \( A \) and a one form \( b \). Note that \( \partial_x^a \) should not be confused with a coordinate basis (which are denoted by \( \partial_x^\alpha \)). Notice, also, that \( \omega \) satisfies (see also lemma 4.2)

\[
\omega^k_{ij} = -\omega^j_{ik} \quad \text{and} \quad \omega^0_{ij} = \omega^j_{i0},
\]

for \( i, j, k = 1, 2, 3 \). In this gauge, the Einstein equations were re-written to form a reduced FOSH [YB].

**Convention 2.6.** From now on, Latin indices run from 1 to 3 and Greek indices from 0 to 3.

Following [YB], all the symbols appearing henceforth are to be treated as “abstract” — for example, \( R_{\alpha \beta \mu \nu} \) is not known, a priori, to be the Riemann tensor of a metric; except, however, for those quantities determined at \( t = 0 \), in which case they do have their usual meaning. The strategy is to write evolution equations for these “abstract” quantities, identify them as FOSH systems, and use uniqueness to conclude that indeed these “abstract” symbols correspond to the “correct” geometric objects. For the sake of brevity, we use the symbol \( \nabla \) or a semicolon to mean “covariant derivative”
with the “correct” connection coefficients in the chosen frame i.e. \( \omega_{0i}^0 = 0, \omega_{0a}^0 = Y_a, \omega_{0ab}^0 = X_{ab} = \omega_{ab}^0 \). We denote spatial covariant derivatives and curvatures with a \( \tilde{\omega} \) and \( \tilde{R} \) respectively. Square brackets enclosing two letters \( A_{[a,b]} = A_{ab} - A_{ba} \) indicates antisymmetrisation, except that we leave out a conventional factor of \( \frac{1}{2} \), whereas angular ones enclosing three letters separated by commas (\( A_{<a,b,c>} \)) indicates cyclic summation \( A_{ab} + A_{bc} + A_{ca} \). This notation is not standard but is useful in this context.

In the case of vacuum, there are no fluid flow lines. Hence we may impose the additional gauge choice \( Y_i = 0 \). The reduced system of equations as written in [YB] is

\[
\begin{align*}
\partial_0 a_{ij} & = -a_j^k X_{ik} \\
\partial_0 b_i & = 0 \\
R_{0i,j}^0 & = \partial_0 \omega_{ij}^0 + X_{j}^k \omega_{ik}^0 \\
R_{h0i0} & = -\partial_0 X_{hi} - X_{j}^i X_{ji} \\
\nabla_0 R_{hklm} & = -\nabla_k R_{hlm0} + \nabla_l R_{hkm0} \\
\nabla_0 R_{0hklm} & = \nabla^l R_{lthklm}
\end{align*}
\]

where \( a = A^{-1} \). In the above, and in what follows, we adopt the following notation: underbars “\( \_ \)” are used to denote empty slots in the order of the indices when one raises or lowers an index. For instance, in \( R_{0hij} \) the two first \( \_ \)’s on the top and the \( \_ \) on the third entry on the bottom tell us that the upper index \( i \) was obtained by raising the third lower index from \( R_{0hij} \).

Although this notation is not completely standard, it is similar to the one used in [YB], which we tried to follow.

In the perfect fluid case, let \( F = \int \frac{dp}{\mu(p) + p}, \rho_{00} = \frac{1}{2}(3p + \mu), \rho_{i0} = 0, \) and \( \rho_{ij} = \delta_{ij} \frac{1}{2}(\mu - p) \). The reduced system is

\[
\begin{align*}
\partial_0 a_{ij} & = -a_j^k X_{ik} \\
\partial_0 b_i & = -a_i^k Y_h \\
\partial_0 \omega_{ij}^0 + X_j^k \omega_{ik}^0 + Y^i X_{hj} - Y_j X_{hi} & = R_{0i,j}^0 \\
\partial_0 X_{hi} + X_{h}^j X_{ji} - Y_h Y_i - \nabla_i X_{hi} - X_{hi} & = -R_{h00i} \\
\mu^l \partial_0 Y_h - \nabla_l X_{hl} - Y^i (X_{li} - X_{il}) & = \mu^l (Y_h \partial_0 F - X_{h}^i \partial_l F) \\
& + \partial_h \mu^l \partial_0 F = 0 \\
\nabla_0 R_{hklm} & = -\nabla_k R_{h0lm} + \nabla_l R_{h0km} \\
\nabla_0 R_{0hlm} & = \nabla^l R_{lthklm} + \nabla_\mu \rho_{lm} - \nabla_\lambda \rho_{\mu h} \\
\partial_0 \mu & = -(\mu + p) X_{hi}^i
\end{align*}
\]
In both vacuum and fluid cases, the constraints are obtained from the splitting of the Riemann tensor of \((M, g)\) into that of \(\Sigma\) and the second fundamental form of \(\Sigma\) inside \(M\) (in other words, from the Gauss-Codazzi-Mainardi equations) and Einstein’s equations, upon restriction to \(\Sigma = \{t = 0\}\). This enables us to solve for temporal derivatives of all the quantities at \(t = 0\). Substituting these expressions in the so-called quasi-constraints\(^4\) which we write below, gives us the actual constraints (see also [YB]). The initial connection is the Levi-Civita one. The other quasi-constraints are

\[
\begin{align*}
\nabla_{(h}\left[R_{i,j}\right)\lambda\mu &= 0 \\
\nabla_h R_{h0\lambda\mu} &= \nabla_{\lambda}\rho_{\mu 0} - \nabla_{\mu}\rho_{\lambda 0} \\
R_{hkJ} &= \tilde{R}_{hkJ} + X_{ki} X_{jh} - X_{jk} X_{ih} \\
- R_{kh0j} &= \tilde{\nabla}_{[k} X_{h]j} - Y_{j} X_{[k,h]}
\end{align*}
\]

(2.5)

3. Initial data

The initial data required for the reduced system is defined on \(\Sigma \times \{0\}\):

- A field of coframes \(a^i_j\) and of covectors \(b_i\) creating a metric \(h_0\) on \(\Sigma\) via \(h_0^{j\lambda} = a^j_i a^{i\lambda} - b^j b^\lambda\) which is assumed to be positive definite. The initial metric on the manifold \(M\) is \(g(t = 0) = -\left(\theta^0\right)^2 + \sum \left(\theta^i\right)^2\) where \(\theta^i = a^i_j dx^j\) and \(\theta^0 = dx^0 + b_i dx^i\).
- Fields \(\omega^{kj}, X_{ij}\), and \(Y_i\) (which is assumed to be zero in the vacuum case). These are supposed to define the connection coefficients of the Levi-Civita connection of \(g\) initially (with \(f_{ij} = 0\)).
- Tensor components \(R_{ijkl}, R_{0ijl},\) and \(R_{00ij}\) that define the Riemann curvature tensor initially.
- In the case of the perfect fluid, we also need \(\mu(p) > 0\) obeying \(\mu' \geq 1\), and \(p \geq 0\).

In addition, the Einstein equations are imposed on this initial data at \(t = 0\) in order to derive the relation between the usual Eulerian initial data of the \(3 + 1\) decomposition — which is given in theorems 2.2 and 2.1 — and the initial data needed for the FOSH systems, as we explain below. A detailed account of the correspondence between initial data sets for the Einstein-Euler system and those of reduced equation in Lagrangian coordinates can be obtained by an argument similar to that of [D].

Given a Riemannian 3-fold \((\Sigma, h_0)\), choose local coordinates \(\tilde{x}^i\) on it. Embed it into \(M = \Sigma \times \mathbb{R}\) as \(\Sigma \times \{0\}\). Let \(\tilde{e}_i\) be an orthonormal frame on \(\Sigma\), and let \(\tilde{\theta}^i\) be the dual coframe. Then \(h_0 = \sum \tilde{\theta}^i \otimes \tilde{\theta}^i\). In the vacuum case, we may simply define the initial Lorentz metric on \(\Sigma \times \{0\}\) as \(g = -(dx^0)^2 + h_0\). This corresponds to \(b_i\) being zero initially.

In the case of a perfect fluid, define a metric \(g = h_0 + v_i dx^0 \tilde{\theta}^i - (dx^0)^2\) on \(TM\) restricted to \(\Sigma \times \{0\}\) with \(v_i\) being the components of the dual (with respect to \(h_0\)) of \(v\). This is a Lorentzian metric, with \(e_0 = \partial_{x^0}\) being a unit

\(^4\)This procedure is necessary because \(\partial_t\) contains \(\partial_{x^0}\).
timelike vector projecting to $v$, and restricting to $h_0$ on $\Sigma \times \{0\}$. Complete $e_0$ to an orthonormal basis $e_\alpha$. This gives us $a_i^j$ and $b_i$ lying in $H^s(\Sigma)$. Calculations similar to the ones in [D] maybe used to define the remaining fields, such as $\omega, X$, on $\Sigma \times \{0\}$.

The above reasoning combined with the fact that both (reduced) systems (2.3) and (2.4) are quasilinear FOSH having initial data in (at least) $H^{s-1}$ implies that both systems have solutions in $C^0([0,T_1],H^{s-1}(U)) \cap C^1([0,T_1],H^{s-2}(U))$, where $U$ is some local chart as described in remark 2.4. The mismatch between the regularity of the initial data and that of the solution is then corrected by a bootstrap argument as in [D] following the results of [FM].

Thus we have a solution to both systems with $a_i^j \in C^0([0,T_1],H^{s+1}(U)) \cap C^1([0,T_1],H^s(U)) \cap C^0([0,T_1],H^{s-1}(U))$, $b_i, \omega^k_{ij}, X_{ij}, Y_i, p$ in $C^0([0,T_1],H^s(U)) \cap C^1([0,T_1],H^{s-1}(U))$, and $R_{\alpha\beta\mu\nu}$ in $C^0([0,T_1],H^{s-1}(U)) \cap C^1([0,T_1],H^{s-2}(U))$.

4. Proofs

We prove that the constraints and the gauge are preserved. This implies that if the Einstein equations are satisfied initially, then they are satisfied in the future. We accomplish these steps by proving that the relevant quantities satisfy FOSH systems with zero as their unique solution. Note that by definition, $\partial_0 = \frac{\partial}{\partial x^0}$ and $\partial_i = A^j_i(\frac{\partial}{\partial x^j} - b_i^j \frac{\partial}{\partial x^0})$. We also note that if a linear symmetry of the $R_{\alpha\beta\mu\nu}$ is satisfied initially, then its spatial derivatives are zero. Since the temporal derivatives are related to the spatial ones by the evolution equations (which are imposed on the variables at $t = 0$), we see that $\partial_i$ applied to such a symmetry also yields zero.

4.1. Vacuum. Firstly, we see as to why the vacuum Einstein equations are implied by the preservation of the constraints and the gauge:

**Lemma 4.1.** If the gauge and the constraints are preserved, then $\text{Ricc}_{\alpha\beta}$ is zero in the future, if so initially.

**Proof.** Notice that $d\text{scal} = 2\text{divRicc}$, where $\text{scal}$ is the scalar curvature and $\text{div}$ means divergence. The given system implies that (assuming the constraints and the gauge) $\text{Ricc}_{\alpha\beta;\gamma} = \text{Ricc}_{\alpha\gamma;\beta}$. Contracting $\alpha$ and $\beta$, we see that $d\text{scal} = 0$ i.e. $\text{scal} = 0$ because it is so, initially. Hence

$$\text{Ricc}_{0i;0} = \text{Ricc}_{00;i}$$

(4.1)

$$\text{Ricc}_{00;0} = \sum_{i=1}^{3} \text{Ricc}_{0i;i}$$

The leading matrix ($M_0$) for the equations above is

$$\begin{pmatrix}
1 & -B_1 & -B_2 & -B_3 \\
-B_1 & 1 & 0 & 0 \\
-B_2 & 0 & 1 & 0 \\
-B_3 & 0 & 0 & 1
\end{pmatrix}$$
where $B_i = -A^j_i b_j$. It is positive definite (see lemma 11 in [YB]).

\[
\begin{align*}
\text{Ricc}_{ij;0} &= \text{Ricc}_{0i;j} \\
\text{Ricc}_{a;0} &= \text{Ricc}^a_{ij} 
\end{align*}
\] (4.2)

Equations (4.1) and (4.2) form a FOSH system (the leading matrix of equation 4.2 also has positive eigenvalues by a similar argument as for equation (4.1)). Hence $\text{Ricc}_{\alpha\beta} = 0$. Note that we treated $\text{Ricc}_{\alpha\beta}$ and $\text{Ricc}_{\beta\alpha}$ as distinct variables.

□

Now, we prove that the constraints and the gauge are preserved i.e., among other things $\partial_a$ forms an orthonormal frame for a metric whose Levi-Civita connection’s components are $\omega^0_{ab} = X_{ab}$, $\omega^\beta_0 = 0$, $\omega^i_j$, and whose Riemann curvature tensor is $R_{\alpha\beta\mu\nu}$. For further use, we prove some symmetries of $R_{\alpha\beta\mu\nu}$:

**Lemma 4.2.** The following relations are satisfied for some time:

\[
\begin{align*}
R_{\alpha\beta\mu\nu} &= -R_{\alpha\beta\nu\mu} \\
R_{\alpha\beta\mu\nu} &= -R_{\beta\alpha\mu\nu} \\
\omega^p_{ij} &= -\omega^p_{ji}
\end{align*}
\]

**Proof.**

\[
\begin{align*}
\partial_0 (R_{hk0j} + R_{kh0j}) &= -\nabla_k R_{0h0j} + \nabla_h R_{0k0j} + \nabla_k R_{00hj} - \nabla_h R_{0k0j} \\
&= 0
\end{align*}
\]

Hence $R_{hk0j} = -R_{kh0j}$ (since it is so, initially). Similarly, $R_{ijkl} = -R_{ijkl}$. We also have

\[
\begin{align*}
\partial_0 (R_{ijkl} + R_{ijlk}) &= -\nabla_j (R_{0ikl} + R_{0lik}) + \nabla_i (R_{0jkl} + R_{0jlk}) \\
\partial_0 (R_{0ikl} + R_{0lik}) &= \nabla^p (R_{pikl} + R_{pilk})
\end{align*}
\]

The system above is FOSH (having zero as its unique solution). Indeed, the leading matrix is

\[
\begin{pmatrix}
1 & 0 & 0 & B_2 & -B_1 & 0 \\
0 & 1 & 0 & B_3 & 0 & -B_1 \\
0 & 0 & 1 & 0 & B_3 & -B_2 \\
B_2 & B_3 & 0 & 1 & 0 & 0 \\
-B_1 & 0 & B_3 & 0 & 1 & 0 \\
0 & -B_1 & -B_2 & 0 & 0 & 1
\end{pmatrix}
\]

Its eigenvalues are $1, 1 \pm \sqrt{B_1^2 + B_2^2 + B_3^2}$ with multiplicity 2. They are positive for some time (by the assumptions on $a$ and $b$). This means that $R_{ijkl} = -R_{ijlk}$ and $R_{0ikl} = -R_{0ilk}$. Using these symmetries of $R_{\alpha\beta\mu\nu}$, we see that $\partial_0 (\omega^k_{ij} + \omega^i_{jk}) = X^l (\omega^l_{ij} + \omega^l_{ki})$. Hence $\omega^k_{ij} = -\omega^j_{ik}$. □
By explicit calculation, we see that
\[ \partial_0 \partial_t - \partial_0 \partial_0 = -X^a \partial_a \]
and that
\[ \partial_{i,j} = (2f_{ij} + X_{i,j}) \partial_0 + c_{i,j}^p \partial_p \]
where \( f \) and \( c \) satisfy \( f_{ij} = 0 \) and \( c_{ij}^p = \omega_{i,j}^p \) when \( t = 0 \). The evolution equations for \( f_{ij} \) and \( v_{ij}^p = c_{ij}^p - \omega_{i,j}^p \) are obtained by differentiating the above equation and using the main evolution system.

\[ \partial_0 (v_{ij}^p) \partial_p = \partial_0 v_{ij} - c_{ij}^p \partial_0 \partial_p - 2 \partial_0 (f_{ij}) \partial_0 - \partial_0 (X_{i,j}) \partial_0 - (2f_{ij} + X_{i,j}) \partial_0^2 \]
\[ + R_{0\{i,j\}l} \partial_p + \omega_{ij}^p X_{i,l} \partial_p \]
\[ = \partial_{i,j} \partial_0 - X^a \partial_a \partial_j - c_{ij}^p \partial_p \partial_0 - X^a \partial_a - 2 \partial_0 (f_{ij}) \partial_0 \]
\[ - (2f_{ij} + X_{i,j}) \partial_0^2 - \partial_0 (X_{i,j}) \partial_0 + R_{0\{i,j\}l} \partial_p + \omega_{ij}^p X_{i,l} \partial_p \]
\[ = \partial_{i,j} \partial_0 - \partial_{i,j} + (X^a_{i,j}) - X^a_{i,j} \partial_a + c_{ij}^p \partial_p + (2f_{ij} + X_{i,j}) \partial_0 \]
\[ - c_{ij}^p \partial_p - X^a \partial_a - 2 \partial_0 (f_{ij}) \partial_0 - (2f_{ij} + X_{i,j}) \partial_0^2 \]
\[ - \partial_0 (X_{i,j}) \partial_0 + R_{0\{i,j\}l} \partial_p + \omega_{ij}^p X_{i,l} \partial_p \]
\[ = (\partial_{i,j} + (X^a_{i,j}) - X^a_{i,j} \partial_a + c_{ij}^p X^a + R_{0\{i,j\}l} + \omega_{ij}^p X_{i,l}) \partial_p \]
\[ + (X^a - \{2f_{ij} + X_{i,j}\}) - 2 \partial_0 (f_{ij}) - R_{0\{i,j\}l} + X_{i,l} X_{i,l} \partial_0 \]
Comparing coefficients we see that
\[ \partial_0 (v_{ij}^p) = -\partial_{i,j} + (X^a_{i,j}) - X^a_{i,j} \partial_a + c_{ij}^p X^a + R_{0\{i,j\}l} + \omega_{ij}^p X_{i,l} \]
\[ \partial_0 (2f_{ij}) = -(X^a - \{2f_{ij} + X_{i,j}\}) - R_{0\{i,j\}l} + X_{i,l} X_{i,l} \]
(4.3)
The system (4.3) is a FOSH system for \( v \) and \( f \) and hence has a unique solution. If the first Bianchi identity and the defining equation of the Riemann tensor are satisfied, then \( f = v = 0 \) is a solution, and hence as promised, \( X \) and \( \omega \) form the Levi-Civita connection of the metric defined by \( \partial_0 \).

Next, we write evolution equations for the first Bianchi identity. In what follows \( R_{\{\alpha,\beta,\mu\nu\}} = R_{\alpha\beta\mu\nu} + R_{\beta\mu\alpha\nu} + R_{\mu\alpha\beta\nu} \).

\[ \partial_0 R_{\{h,c,d\}a} = -\nabla_h R_{\{c,0,d\}a} - \nabla_d R_{\{h,0,c\}a} - \nabla_c R_{\{0,h,d\}a} + \nabla_h R_{\{d,c\}0a} \]
\[ \partial_0 R_{\{0,h,c\}a} = -\nabla^l R_{\{l,h,c\}a} - \nabla^l R_{hca0} + \nabla_0 R_{hc0a} \]
(4.4)
The above system is FOSH for the variables \( R_{\{\alpha,\beta,\mu\nu\}} \). Indeed, it is symmetric and the eigenvalues of the leading matrix are 1 and \( 1 \pm \sqrt{B_1^2 + B_2^2 + B_3^2} \) (which are positive). We will write evolution equations for the other terms in the system (4.4). This will prove that the unique solution to the above system is 0 (since it is 0 initially).

Now, we write the evolution equations for \( \nabla^b R_{hcba} - \nabla_0 R_{hc0a} \) and prove that zero is their only solution. To accomplish this, we ought to prove that the lower order terms in these equations vanish assuming that all the identities (including the Bianchi identities, the Einstein equations, \( \nabla^b R_{hcba} - \)
\( \nabla_0 R_{hc0a} = 0, f_{ij} = 0, c^p_{ij} = \omega^p_{ij} - \omega^p_{ji}, \text{ etc.} \) hold to order zero.

\[
\nabla_0 (\nabla^b R_{hc0b}) = \\
\nabla_0 (\partial^b R_{hc0b} - \omega^c_{bh} R^{cb}_{hc0} - \omega^c_{bc} R^{cb}_{ha0} - \omega^c_{ba} R^{cb}_{hc0a} - \omega^c_{b0} R^{cb}_{h0ca}) \\
= [\partial_0, \partial^b] R_{hc0b} - \partial^b \nabla_c R_{0hc0} + \partial^b \nabla_h R_{0cb0} - X_{bh} \nabla^l R^{lb}_{hc0} \\
+ \omega^c_{bh} \nabla_c R^{cb}_{0a0} - \omega^c_{bc} \nabla_a R^{cb}_{0c0} - X_{bc} \nabla^l R^{lb}_{0c0} + \omega^c_{bc} \nabla_a R^{cb}_{0c0} \\
- \omega^c_{ba} \nabla_c R^{cb}_{0ca0} + \omega^c_{ba} \nabla_c R^{cb}_{0ca0} - X_{ba} \nabla_c R^{cb}_{0ca0} + X_{ba} \nabla_c R^{cb}_{0ca0} \\
- X_{ba} \nabla_h R^{lb}_{0ca0} + (\omega^c_{ba} + X^b_{kh}) R^{lb}_{0ca0} - (\omega^c_{ba} - X^b_{kh}) R^{lb}_{0ca0} \\
\tag{4.5} \\
+ (R_{0b0c} + X_{bk} X_{kc}) R^{lb}_{0c0} (R_{0bac} - X_{bk} \omega^p_{kc}) R^{pb}_{0c0} \\
- (R^{lb}_{0cb} - X_{bk} \omega^p_{cb}) R_{0ca0} + (R_{0ba0} + X_{bk} X_{ka}) R^{lb}_{0ca0} \\
= - \partial_0 \nabla_b R^{lb}_{0c0} + \partial_0 \nabla^b R_{0cb0} - X^c_{bb} \nabla^l R^{lb}_{0c0} - \omega^c_{ba} \nabla_c R^{cb}_{0c0} \\
+ \omega^c_{bb} \nabla_c R^{cb}_{0b0} - \omega^c_{bb} \nabla_a R^{cb}_{0c0} - X_{bb} \nabla^l R^{lb}_{0c0} + \omega^c_{bb} \nabla_a R^{cb}_{0c0} \\
- \omega^c_{bb} \nabla_h R^{lb}_{0c0} + \omega^c_{bb} \nabla_c R^{cb}_{0ca0} - \omega^c_{bb} \nabla_h R^{lb}_{0ca0} + X_{ba} \nabla_c R^{cb}_{0ca0} \\
- X_{ba} \nabla_h R^{lb}_{0ca0} - \partial_0 \nabla_c R^{lb}_{0c0} + \partial^b \nabla_h R_{0cb0} + \partial^b \nabla_c R_{0b0b} \\
- \partial_0 \nabla_b R_{0b0b} \\
\]

At this point we note that

\[
\partial_0 \nabla^b R_{0hbb0} - \partial^b \nabla_c R^{cb}_{0hbb0} = v_{cb} \partial_0 R_{0hbb0} + 2 f_{cb} \nabla^l R^{lb}_{0hbb0} + (X_{cb} - X_{bc}) \nabla^l R^{lb}_{0hbb0} \\
+ (\omega^c_{cb} - \omega^c_{bc}) \partial_0 R^{cb}_{0hbb0} + \omega^c_{bb} \nabla_c R^{cb}_{0hbb0} + \omega^c_{ba} R^{cb}_{0ba0} + X_{ba} R^{cb}_{0ba0} \\
+ \partial^b (X_{ca} R_{0b0b} + \omega^c_{cb} R_{0b0b} + \omega^c_{cb} R_{0ba0} + X_{ca} R^{cb}_{0ba0}) \\
\]

Noticing that \( \partial_0 \omega^a_{ck} - \partial_0 \omega^a_{bc} = R^a_{bc0k} + \omega^a_{ck} (\omega^p_{bc} - \omega^p_{cb}) - \omega^p_{bp} \omega^p_{ck} + \omega^a_{cp} \omega^p_{bk} \) up to to the zeroeth order by assumption, we have

\[
\partial_0 \nabla^b R_{0hbb0} - \partial^b \nabla_c R^{cb}_{0hbb0} = \\
v_{cb} \partial_0 R_{0hbb0} + 2 f_{cb} \nabla^l R^{lb}_{0hbb0} + (X_{cb} - X_{bc}) \nabla^l R^{lb}_{0hbb0} \\
+ (\omega^c_{cb} - \omega^c_{bc}) \nabla_c R^{cb}_{0hbb0} - X_{ba} \partial_0 R^{cb}_{0ba0} - \omega^c_{bb} \nabla_c R^{cb}_{0hbb0} \\
+ \omega^c_{cb} \partial_0 R_{0b0b} + \omega^c_{cb} \partial^b R_{0hbb0} + \omega^c_{ba} R^{cb}_{0ba0} + \omega^c_{ba} R^{cb}_{0ba0} \\
+ (R^{lb}_{0cb} + R^{lb}_{0ba}) (-R_{0bca} + X_{bp} \omega^p_{cb} + X_{cp} \omega^p_{ba}) \\
+ R^{lb}_{0ca} (X_{bah} - \omega^p_{bp} \omega^p_{ch} + \omega^p_{cp} \omega^p_{bh} - X_{ba} X_{ch} + X_{ca} X_{bh}) \\
+ R_{0ba0} (R^{lb}_{0cb} - \omega^p_{bp} \omega^p_{cb} - \omega^p_{cp} \omega^p_{bh} - X_{ba} X_{ch} - X_{ca} X_{bh}) \\
\tag{4.6} \\
\]

Inserting (4.6) and another equation (the same one as (4.6) with \( h \) and \( c \) interchanged and the sign flipped) into equation (4.5) we see that the zeroeth and the first order terms cancel assuming all the identities hold to order zero.
The system (4.7) (along with equation (4.5)) is easily verified to be FOSH with zero as the unique solution if zero initially. We note that $\nabla_0 R_{0\alpha \beta \gamma}$ and $\nabla_0 (\partial_{(i} R_{j)\lambda \mu})$ evolve according to

$$\nabla_0 (\nabla^h R_{0\alpha \beta \gamma} ) = L_7$$

$$\nabla_0 (\nabla_0 (\partial_{(i} R_{j)\lambda \mu}) ) = L_8$$

Finally, we calculate the evolution of $B_{i,j} = R_{i,j}^{h0}_0 + \partial_k X_{i,j} - \partial_h X_{i,j} - X_{i,j}^p (\omega^p_h - \omega^p_{hk}) + X_{i,j}^{p,k} \omega^{p,k}_{hj} - X_{i,j}^{p,k} \omega^{p,k}_{jh} + X_{i,j} X_{hj}$ (i.e. the definitions of the components of the Riemann tensor)

$$\partial_0 B_{i,j} = -X_{i,j}^{p} B_{k,p} + X_{i,j}^{p} B_{k,p} - X_{i,j}^{p} B_{k,p} + X_{i,j}^{p} R_{i,j}^{h0}_{(0,k,h)}$$

$$\partial_0 W_{i,j} = -\omega^l_{i,j} R_{(i,0,k,h)}^{00} + \omega^l_{i,j} B_{k,p} + X_{i,j}^{p} W_{k,l} - X_{i,j}^{p} W_{k,l}$$

The system (4.9) is FOSH having zero as its solution. If $B$ and $W$ are zero then $R_{\alpha \beta \gamma \delta}$ is the Riemann tensor of the metric.

Proof of theorem 2.1: Equations (4.3), (4.4), (4.5), (4.7), (4.8), (4.9) form a FOSH. Using the uniqueness theory for the same, we conclude that zero is the unique solution (zero in $H^{s-2}$ is the same as zero throughout because of Sobolev embedding) if the variables are zero initially. A calculation shows that they are zero initially. Such a calculation is quite long and will not be presented here, but it is done in essentially the same fashion as in [D]. This also holds for the system in lemma (4.1). This proves that we have a solution to the vacuum Einstein equations satisfying all the conditions required by theorem 2.1.

4.2. Perfect fluids. Just as before, we write equations for the preservation of the gauge. Indeed, we show that the Levi-Civita connection corresponding to the orthonormal frame defined by $\partial_\alpha$ has components $\omega^k_i = \omega^{k,i}$, $\omega^0_i = 0$, $\omega^i_{00} = Y_i$, and $\omega^0_{ij} = X_{ij}$.

Calculations similar to the ones in lemma (4.2) show that the same lemma holds for the Einstein-Euler system as well. We assume this implicitly in
what follows. We define $S_{αβ} = \text{Ricc}_αβ - ραβ$ so that the Einstein equations are $S_{αβ} = 0$.

Explicit computation shows that $[∂_0, ∂_i] = X_iα∂_α + Y_0∂_0$, and $[∂_i, ∂_j] = (2f_{ij} + X_{[i,j]})∂_0 + (v^p_{[i,j]} + ω^p_{[α,β]}ω_{β,γ}^q)∂_p$ where $f = 0 = v$ initially. If we prove that $f = 0 = v$ is preserved, then indeed the components of the Levi-Civita connection are as described above. Computations similar to the ones leading to the system (4.3) prove that

$$
∂_0v^p_{ij} = R_{[0][i,j]} + ω^p_{[k][j]}X_i[k] + Y^pX_{[i,j]}
+ (v^0_{[i,j]} + ω^p_{[α,β]}ω_{β,γ}^q)X_i[α] - ∂_iX^p_{α}
(4.10)
$$

$$
2∂_0f_{ij} = -R_{0[i,j]} + 2X_{[i,j]}X^p_{k} - v^0_{ij}Y_a - 2f_{a[i]j}X_{[i]} - 2\mathcal{\nabla}_{[i}Y_{j]}
$$

If indeed $v = f = 0$, and the first Bianchi identity holds, then both the equations in system (4.10) are satisfied provided $F_{ij} = \mathcal{\nabla}_{[i}Y_{j]} + X_{[i,j]}X^p_{d} = 0$. Let $P_t = Y_{t} + ∂_tF$. We record the following calculations for further use

$$
F_{ij} = \mathcal{\nabla}_{[i}Y_{j]} + X_{[i,j]}\partial_0F
(4.11)
$$

$$
\partial_{[i}F_{ij]} = \tilde{L}_1 + \partial_{j}F_{hk}
$$

where $\tilde{L}_i$ denote lower order terms as before. They vanish when all the identities are satisfied. The evolution of $F$ is given by

$$
μ'∂_0F_{ij} = \text{lower order} + \partial_{[i}(∂_0P_{j]})
= \text{lower order} + \partial_{[i}\partial_{j}X_{j]} - \partial_{[i}|\partial_{j}X_{j]|}
= \text{lower order} + \partial_{[i}\partial_{j}X_{j]}
(4.12)
$$

We now write the evolution equation of $R_{(α,β,γ,δ)}$ as before:

$$
\nabla_{0}R_{(h,c,d,\alpha)} = -\nabla_{h}R_{(c,0,d)}\alpha - \nabla_{d}R_{(h,0,c)}\alpha - \nabla_{c}R_{(0,h,d)}\alpha
+ \nabla_{(h,d,c)}0\alpha
(4.13)
$$

$$
\nabla_{0}R_{(h,c)} = \nabla_{l}R_{(h,c)} - \nabla_{(l}R_{c)}\alpha + \nabla_{c}S_{h\alpha} - \nabla_{h}S_{c\alpha}
$$

We wish to make sure that the Euler equation $Y_t = -∂_tF$ is satisfied. The evolution of $P_t$ is computed to be

$$
∂_0P_t = \frac{1}{μ'}\left[\mathcal{\nabla}_{j}X_{ij} - \mathcal{\nabla}_{i}X_{i} - Y^k(X_{ki} - X_{ik})\right] - X_{i}^tP_t
(4.14)
$$

$$
= \frac{1}{μ'}\left[S_{i0} + (-\text{Ricc}_{i0} + \mathcal{\nabla}_{j}X_{ij} - \mathcal{\nabla}_{i}X_{i} - Y^k(X_{ki} - X_{ik}))\right]
- X_{i}^tP_t
$$

Now, we calculate the evolution of the (quasi-)constraints (remembering that $R_{h00} = \tilde{R}_{h00} - F_{hi}$ where $\tilde{R}_{h00}$ is the “true” Riemann tensor). Let $B_{khj} = \tilde{R}_{khij}$
\[ R_{kh0j} + \partial_k X_{hj} - \partial_h X_{kj} - X_{pj}(\omega_{kh}^p - \omega_{hk}^p) + X_{kj} \omega_{hj}^p - X_{hj} \omega_{kj}^p - Y_j(X_{kh} - X_{hk}) \]

and \[ W_{hkl} = R_{hkl} - \dot{R}_{hkl} - X_{kl} X_{jl} + X_{jk} X_{hl}. \]

\[
\partial_0 B_{kh} = -X_{ij} B_{lij} + X_{k} B_{lij} + X_{l} B_{ikj} + X_{pj} R_{(0,k,h)lj} - Y_p W_{hklj} - Y_j R_{0[k,h]0} + 2 f_{kh} \partial_0 Y_j + \nu_{kh} \partial_p Y_j + Y_{[k,F_{hlj]} - Y_j F_{[k,h]} + \tilde{\nabla}_{[k,F_{hlj]]}
\]

(4.15)

\[
\partial_0 W_{hkl} = \tilde{L}_3 + \partial_j F_{hk}
\]

Remark 4.3. Notice that (4.10), (4.12), (4.13), (4.14), (4.15), and (4.16) form a FOSH. They have zero as their unique solution if the Bianchi identities, the constraints, and the Euler equations hold.

Proof of theorem 2.2: By arguments similar to those used in the proof of theorem 2.1, we obtain a solution of the Einstein-Euler system. This satisfies almost all the conditions required by theorem 2.2 except ostensibly, the regularity of \( g \), because of the regularity of \( b_i \) that is lower than desired. This however, is an artifact of our chosen coordinate system (which depends on \( v \) which in turn has lower regularity than \( h_0 \)). To see this, first we note that the \( g \) we obtained is in \( C^0([0,T], H^s(U)) \cap C^1([0,T], H^{s-1}(U)) \cap C^2([0,T], H^{s-2}(U)) \), with \( U \) some local chart (see remark 2.4). However, its restriction to \( \Sigma \times \{0\} \) is in \( H^{s+1} \). Using the ADM decomposition of lapse and shift, one may choose the initial lapse to be 1, the initial shift vector to be 0 and their time derivative appropriately so as to satisfy the wave gauge condition initially. The energy-momentum tensor will be in \( C^0([0,T], H^s(U)) \cap C^1([0,T], H^{s-1}(U)) \). The Einstein equations form a quasilinear hyperbolic system in the wave gauge. Hence we get a metric
solving the full Einstein equations with the correct regularity. This coincides our original solution due to local geometric uniqueness [B1].

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Department of Mathematics, Vanderbilt University, 1326 Stevenson Center, Nashville, TN 37240, USA.
E-mail address: marcelo.disconzi@vanderbilt.edu

Department of Mathematics, Johns Hopkins University, 404 Krieger Hall 3400 N. Charles Street, Baltimore, MD 21218, USA.
E-mail address: vpingali@math.jhu.edu