On Separating Points by Lines

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Abstract
Given a set $P$ of $n$ points in the plane, its separability is the minimum number of lines needed to separate all its pairs of points from each other, denoted by $\text{sep}(P)$. We show that the minimum number of lines needed to separate $n$ points, picked randomly (and uniformly) in the unit square, is $\tilde{\Theta}(n^{2/3})$, where $\tilde{\Theta}$ hides polylogarithmic factors. In addition, we provide a fast $O(\log(\text{sep}(P)))$-approximation algorithm for computing the separability of a given point set in the plane. Finally, we point out the connection between separability and partitions.

1 Introduction

For a set $P$ of $n$ points in $\mathbb{R}^d$, a set $L$ of hyperplanes separates $P$, if for any pair of points of $x, y \in P$, there is a hyperplane in $L$ that intersects the interior of the segment $xy$ (which also does not contain $x$ or $y$). In $\mathbb{R}^2$, $L$ is a set of lines. The separability of $P$, denoted by $S_n = \text{sep}(P)$, is the size of the smallest set of lines that separates $P$. The separability of a point set captures how grid-like the point set is. In particular, the separability of the $\sqrt{n} \times \sqrt{n}$ grid is $2\sqrt{n} - 2$, while for $n$ points in convex position the separability is $\lceil n/2 \rceil$ (and this is the worst case assuming general position).
In this paper, we investigate the separability of a point set. For a collection of \( n \) points \( P \) chosen uniformly at random from the unit square, where \( S_n = \text{sep}(P) \) is now a random variable, we prove that with high probability \( S_n = O(n^{2/3}) \) and \( S_n = \Omega(n^{2/3} \log \log n / \log n) \). This bound also extends to higher dimensions. When \( P \subset \mathbb{R}^d \), we show that \( S_n = \Omega(n^{2/(d+1)} \log \log n / \log n) \) with high probability. We also give an efficient randomized approximation algorithm for computing the minimum number of separating lines in the plane. For an input set \( P \subset \mathbb{R}^2 \) of size \( n \) with \( S_n = \text{sep}(P) \), the algorithm returns a collection of lines separating \( P \) of size \( O(S_n \log S_n) \) and runs in expected time \( O(n^{2/3}S_n^{2/3} \log \log n) \).

**Grid versus random points.** There is a striking similarity between the behavior of random point sets and uniform grid point sets. For example, the convex hull of a set of \( n \) random points inside a triangle has \( O(\log n) \) vertices in expectation, and the same bound holds for the convex hull of the \( \sqrt{n} \times \sqrt{n} \) grid points when clipped to a triangle. There are many other examples of this surprising similarity in behavior (see [19] and references therein). Another striking example of this similarity is in the number of layers of the convex hull—it is \( O(n^{2/3}) \) for \( n \) random points [11], and the same bound holds for a grid of \( n \) points [21].

**Previous work.** Freimer et al. [17] showed that computing the separability of a given point set is \( \text{NP-Complete} \), and studied an extension of the problem to polygons in the plane. Nandy et al. [26] studied the problem of separating segments. Călinescu et al. [7] gave a two approximation when restricting the problem to separation via axis-parallel lines. Other work on this and related problems includes [13,16].

**Motivation.** Separating and breaking point sets, usually into clusters, is a fundamental task in computer science, needed for divide and conquer algorithms. It is thus natural to ask what can be done if restricted to lines, and if one can do the partition in a global fashion (i.e., if the partition is done locally only to the current subproblem, this results in a *binary space partition* (BSP)). Specifically, we have the following connections:

(A) **Geometric hitting set.** The separability problem reduces to a geometric hitting set problem. In recent years there was a lot of work on speeding up approximation algorithms for such problems, and it is a natural question to ask what can be done in this specific case. See [2,5] and references therein.

(B) **Polynomial partition.** For divide and conquer algorithms for lines, the classical tool to use is cuttings [8], and for points there are partitions [25]. More recently, the polynomial ham-sandwich theorem was used to partition point sets—see [4] and references therein for some recent work. This yields partitions that have stronger properties than the partitions of Matoušek [25] in some cases, but are (in many cases) algorithmically less convenient to use. It is thus natural to ask what else can be done when only using hyperplanes (or lines in \( \mathbb{R}^2 \)).

(C) **Extracting features.** Recently, there was increased interest in *autoencoders* in machine learning—here, one is interested in finding a representation of the data of a set of features, where the number of features is significantly smaller than the ambient dimension, see [23]. Thus, the separability problem can be interpreted as finding a minimum number of linear features, such that all data points are
distinguishable. The problem is usually of interest in higher dimensions, but even in constant dimension it is already challenging.

1.1 Our Results

1.1.1 Low Separability Implies Partitions

In Sect. 2 we define the problem formally, and show how low separability implies partitions (see Definition 2.6) in two and three dimensions with almost optimal parameters. Specifically, if a point set \( P \) in \( \mathbb{R}^d \) for \( d \in \{2, 3\} \) has separability \( O(n^{1/d}) \), then for any \( r > 0 \), \( P \) can be partitioned into \( O(r) \) sets, where each set is of size \( \leq n/r \).

Furthermore, for each set in the partition we can find a simplex that contains (at least) the points in the set. Lemma 2.7 shows that for \( d = 2 \), any line intersects roughly \( O(\sqrt{r}) \) triangles containing these point sets. In three dimensions, the guarantee is that any plane intersects (roughly) \( O(r^{2/3}) \) simplices that contains these sets (Lemma 2.8). Surprisingly, in the three-dimensional case, any line intersects (roughly) \( O(r^{1/3}) \) such simplices, and it is not known how to construct partitions in three dimensions that have this property in the general case (when using only planes—the polynomial method yields partitions that have this property).

1.1.2 Separability of a Random Point Set

Let \( P \) be a set of \( n \) points picked uniformly at random from the unit square \([0, 1]^2\). In Sect. 3 we study the random variable \( S_n = \text{sep}(P) \). An initial guess might be that \( \mathbb{E}[S_n] = \Theta(\sqrt{n}) \) since random points in the unit square may look like grid points. This is not the situation here—surprisingly, in Theorem 3.14 we show that \( S_n = O(n^{2/3}) \), and \( S_n = \Omega(n^{2/3} \log \log n) \) (both of these bounds hold with high probability). For \( d \geq 2 \), we prove that \( \mathbb{E}[S_{n,d}] = O(n^{2/(d+1)}) \) and \( S_{n,d} = \Omega(n^{2/(d+1)} \log \log n) \), with high probability, where the \( \Omega \) and \( O \) notations hide constants that depend polynomially on \( d \). See Corollary 3.15.

What is going on? Consider the closest pair of points in \( P \)—the distance between this pair of points is in expectation roughly \( 1/n \). Indeed, there are \( \binom{n}{2} \) pairs of points, and the probability of a specific pair of them to be in distance \( \leq 1/n \) from each other is \( \pi/n^2 \) (ignoring boring and minor boundary issues). By linearity of expectation, the expected number of pairs to be in distance \( \leq 1/n \) from each other is \( \binom{n}{2} \pi/n^2 \geq 1 \). Of course, the closest pair distance in the grid \( \{(i/\sqrt{n}, j/\sqrt{n}) \mid 1 \leq i, j \leq \sqrt{n}\} \) is \( 1/\sqrt{n} \). Thus there is a dichotomy between a random collection of points and points from a grid.

It turns out that the situation is similar in separating random points by lines—there are, in expectation, roughly \( n^{2/3} \) pairs of points in \( P \) that are in distance \( \leq 1/n^{2/3} \) from each other. Namely, there are many pairs of close points in \( P \), and a line can

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1 To see this, fix a point \( p \in P \). The probability that another point \( q \in P \) is at distance \( \leq 1/n \) from \( p \) is equal to the probability that \( q \) falls in the disk \( D \) of radius \( 1/n \) centered in \( p \). It follows that the event occurs with probability equal to \( \text{area}(D) = \pi/n^2 \).
separate only few of these pairs (this of course requires a proof). Thus, implying the lower bound. The upper bound follows readily by using a grid with cells of side length \(1/n^{2/3}\), and then separating every bad (i.e., short) pair on its own.

**What is not going on.** It is natural to think that maybe there is a convex subset of \(P\) of size \(\Theta(n^{2/3})\). Since separating \(k\) points in convex position requires \([k/2]\) lines, this would readily imply the lower bound. However it is known that for \(n\) random points, the size of the largest subset of points in convex position is \(\Theta(n^{1/3})\) with high probability [1].

Similarly, one might try to blame the number of convex layers, which is indeed \(\Theta(n^{2/3})\) for random points [11]. The similarity in the bounds seems to be a coincidence, since it is easy to construct examples of \(n\) points with \(\Omega(n)\) convex layers that can be separated with \(O(\sqrt{n})\) lines. See Fig. 1.

**A sketch of the proof of** \(S_n = \Omega(n^{2/3} \log \log n / \log n)\). While the proof of the upper bound \(S_n = O(n^{2/3})\) is easy, the lower bound is harder and requires some work:

(A) We setup the problem as a balls in bins problem by dividing the unit square into an \(n^{2/3} \times n^{2/3}\) grid. By revisiting balls and bins, and using Talagrand’s inequality, we prove that the expected number of grid cells containing exactly two points is \(\Theta(n^{2/3})\) (see Corollary 3.11), and this random variable is strongly concentrated around its expectation with high probability (the high probability interval is of width \(O(n^{1/3} \log^{1/2} n)\)).

(B) We prove a high-probability counterpart to the (famous) birthday paradox—while throwing \(O(n^{1/3})\) balls into \(O(n^{2/3})\) bins, one would expect a constant number of collisions. Lemma 3.13 shows that this number is \(O(\log n / \log \log n)\) with high probability. This implies that, with high probability, a line can intersect at most \(O(\log n / \log \log n)\) cells that contains two points or more. While these results (here and in part (A)) are not difficult if one knows the machinery, surprisingly, we were unable to find a reference to them in the literature—thus we include the details.

(C) We then argue that there are only \(O(n^{8/3})\) combinatorially different lines as far as the grid is concerned. Combining (A) and (B) above then readily implies the result—see Theorem 3.14.

### 1.1.3 A Fast Algorithm for Approximating the Separability

For a given set \(P\) of \(n\) points in the plane, in Sect. 4 we present an output-sensitive reweighting algorithm for approximating the separability, with running time that depends on the size of the optimal solution. The improved running time follows by implicitly storing the set of \(\approx n^2\) candidate separating lines the solution can use. This requires using duality [18, Chap. 25] and range searching data-structures to implicitly maintain the set of separating lines (and their weights). For a given set of \(n\) points in the plane, the resulting algorithm computes a separating set of size \(O(\sigma \log \sigma)\), in time \(O(n^{2/3} \sigma^{2/3} \log O(1) n)\), where \(\sigma\) is the separability of the given point set, see Theorem 4.11. Even in the worst case scenario when \(\sigma = \Theta(n)\), the running time is \(\tilde{O}(n^{7/3})\) (where \(\tilde{O}\) hides polylogarithmic factors) which is a significant speedup over the “naive” \(O(n^3)\) time algorithm.
2 Problem Definition and an Application

Definition A set of $n$ points $P$ in the plane is in general position if no three of them are on a common line.

Definition 2.1 A set of lines $L$ separates a set of points $P$, if for every pair $p, q \in P$, $p$ and $q$ are on different sides of some line $\ell \in L$.

Definition 2.2 For a set $P$ of $n$ points in the plane, its separability, denoted by $\text{sep}(P)$, is the size of the smallest set of lines that separates $P$.

Remark 2.3 (A) Assuming no three points are colinear, one might relax the definition, and allow points to be on the separating lines. Given such a separating set of lines $L$
of size \(m\), one can generate a set of lines of size at most 3\(m\) that properly separates all the pairs of points. Indeed, for each line \(\ell\), replace it by two lines that are parallel copies close to it. In addition, add an arbitrary line that properly separates the at most two points that might be on \(\ell\) (by the general position assumption, no line can contain three points of \(P\)).

(B) For a point \(x \in P\) and a separating set of lines \(L\), there is a unique facet of the arrangement \(\mathcal{A}(L)\) that the only point of \(P\) it contains is \(x\). Since an arrangement of \(m\) hyperplanes in \(\mathbb{R}^d\) has \(O(m^d)\) faces of all dimensions, it follows that \(\text{sep}(P) = \Omega(\frac{1}{d})\).

(C) For the grid point set \(P \equiv \frac{n_1}{d} \times \cdots \times \frac{n_1}{d}\) we have that the separability is \(\leq dn_1^{1/d}\)—indeed, use the natural axis-parallel hyperplanes separating layers of the grid.

(D) Consider a set \(P\) of \(n\) points spread on a strictly convex curve \(\gamma\) in \(\mathbb{R}^d\) (i.e., \(\gamma\) is a convex curve that lies in some two-dimensional plane). Any hyperplane, that does not contain \(\gamma\), intersects \(\gamma\) in two points. It follows, that to separate the \(n\) points, we need \(n - 1\) break points along the curve. Hence, \(\text{sep}(P) \geq (n - 1)/2\) in this case.

An upper bound. The following is an easy consequence of the results of Steiger and Zhao [28] (and is probably implied by earlier work).

**Corollary 2.4** Let \(X, Y\) be two point sets in the plane that are separated by a line, and furthermore, there are no three collinear points in \(X \cup Y\). Then, for any choice of integers \(x, y\), such that \(1 \leq x < |X|, 1 \leq y < |Y|\), there exists a line \(\ell\) such that:

(a) \(\ell\) does not contain any point of \(X \cup Y\),

(b) \(\ell\) splits \(X\) into two sets of size \(x\) and \(|X| - x\), respectively, and

(c) \(\ell\) splits \(Y\) into two sets of size \(y\) and \(|Y| - y\), respectively.

**Lemma 2.5** Let \(P\) be a set of points in \(\mathbb{R}^d\) such that no three of them are on a common line. Then, \(\text{sep}(P) \leq \lceil \frac{n}{2} \rceil\).

**Proof** If \(d > 2\), we project \(P\) into a randomly rotated two-dimensional plane. Almost surely no three points in the projected point sets are colinear. In particular, a partition of the projected points by \(m\) lines can be lifted back, in the natural way, to a set of \(m\) hyperplanes separating the point set. Thus, from this point on, we assume the points of \(P\) are in the plane.

The splitting algorithm works as follows. Split \(P\) into two sets \(P_L\) and \(P_R\) of sizes \(\lceil n/2 \rceil\) and \(\lfloor n/2 \rfloor\), respectively, by a vertical line. In the \(i\)th iteration of the algorithm, if \(|P_R| \geq 3\), then by Corollary 2.4, there exists a line \(\ell_i\) that splits \(P_L\) and \(P_R\) each into two sets, such that \(P_R\) (resp. \(P_L\)) gets split into one set with two points, and another set with \(|P_R| - 2\) (resp. \(|P_L| - 2\)) points. We remove these four points from \(P_R\) and \(P_L\), and split these two pairs of points by another line \(\ell_i^\prime\).

Note, that this algorithm preserves the invariant that \(|P_L| \geq |P_R|\) (and these sizes differ by at most one). If after the last iteration we are left with \(P_L\) ad \(P_R\) having sizes 3 and 2 respectively, then we split the set with three elements into a set with 2 and a

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\(^2\) The constant depends polynomially on \(d\).
single element, and then split the two pairs by a single line. The case that $PL\text{ ad } PR$ are both size 2 can be handled by a single splitting line, as is the case that $PL$ has two points, and $PR$ is a singleton.

The number of cutting lines used is $\lceil n/2 \rceil$ as an easy case analysis based on the value of $n \mod 4$ shows. $\square$

### 2.1 Application: Partition via Separability in Two and Three Dimensions

**Definition 2.6** For a set $P$ of $n$ points in $\mathbb{R}^d$, and a parameter $r > 0$, an $r$-partition [25] is a partition of $P$ into $t = O(r)$ disjoint sets $P_1, \ldots, P_t$, with associated simplices $\Delta_1, \ldots, \Delta_t$, such that:

(i) $\forall i: P_i \subseteq \Delta_i$,
(ii) $\forall i: |P_i| \leq n/r$, and
(iii) any hyperplane $h$ intersects at most $f(r) = O(r^{1-1/d})$ simplices of $\Delta_1, \ldots, \Delta_t$.

It is not hard to see that such a partition exists for the grid point set. It is quite surprising that such a partition exists in the general case. The construction is due to Matoušek [25], and it is somewhat involved. Here, we show that if a point set has low separability, then one can easily construct a partition.

**Lemma 2.7** Let $P$ be a set of $n$ points in the plane with $\text{sep}(P) = O(\sqrt{n})$ and $r > 0$ an integer parameter. One can compute a triangulation of the plane with $O(r \log^2 r)$ triangles, such that each triangle contains $\leq n/r$ points of $P$, and any line intersects at most $O(\sqrt{r \log^2 r})$ triangles.

**Proof** Let $L$ be a set of lines that separates $P$ and realizes $\text{sep}(P)$. Let $m = \text{sep}(P) = |L|$. Consider a random sample $R$ of size $O(\rho \log \rho)$ from $L$, where $\rho = \alpha \sqrt{r}$ and $\alpha$ is a sufficiently large constant.

Consider a face $f$ of $A(R)$—it is a convex polygon with $\rho' = O(\rho \log \rho)$ sides. We triangulate it by connecting consecutive even vertices (i.e., every other vertex as we travel along the boundary of $f$), and repeat this process until the face is fully triangulated. It is easy to verify that any line can intersect at most $O(\log \rho') = O(\log \rho)$ triangles in this triangulation of the face. Repeating this triangulation for all faces of $A(L)$ results in a triangulation of the plane. Let $T$ be the resulting set of triangles. Clearly, any line intersects at most $O(\rho \log^2 r)$ triangles of $T$.

The $\epsilon$-net theorem [22] states that for a sample $R \subseteq L$ of size $O(\rho \log \rho)$, any triangle $\Delta$ that intersects more than $m/\rho$ lines of $L$ must also intersect at least one line of $R$. Furthermore this property holds for all ranges with at least some constant probability. Since all triangles of $T$ were ultimately constructed from the set $R$, it follows from the $\epsilon$-net theorem that any triangle $\Delta$ of $T$ intersects at most $m/\rho$ lines of $L$ in its interior. By assumption, there is a constant $c''$ such that $m \leq c'' \sqrt{n}$. Therefore the arrangement of $L$ restricted to $\Delta$ can have at most $c'(m/\rho)^2 \leq c'(c'' \sqrt{n}/\alpha \sqrt{r})^2 \leq n/r$ faces (including edges on the boundary of $\Delta$), for some constant $c'$, and for a sufficiently large constant $\alpha$. This also bounds the number of points of $P$ in $\Delta$, thus establishing the claim. $\square$
Lemma 2.8 Let \( P \) be a set of \( n \) points in \( \mathbb{R}^3 \) with \( \text{sep}(P) = O(n^{1/3}) \) and \( r > 0 \) an integer parameter. One can compute a triangulation with \( O(r \log^2 r) \) simplices, such that each simplex contains \( \leq n/r \) points of \( P \), any plane intersects at most \( O(r^{2/3} \log^2 r) \) simplices, and any line intersects at most \( O(r^{1/3} \log^2 r) \) simplices.

Proof We follow the proof of Lemma 2.7. Let \( L \) be a set of planes that separates \( P \) of size \( m = O(n^{1/3}) \). Let \( R \) be a random sample from \( L \) of size \( O(\rho \log \rho) \), where \( \rho = ar^{1/3} \), where \( a \) is sufficiently large constant. For a face \( f \) of \( \mathcal{A}(R) \), which is a convex polytope (or convex polyhedra, if it is unbounded), we decompose it into simplices using the Dobkin–Kirkpatrick hierarchy [14]. If the face has \( t \) vertices, the resulting decomposition has \( O(t) \) simplices, and furthermore, any line intersects at most \( O(\log t) \) such simplices. Let \( T \) be the resulting set of simplices when applying this decomposition for all the faces of \( \mathcal{A}(R) \).

As before, by the \( \varepsilon \)-net theorem, a simplex \( \triangle \in T \) intersects at most \( m/\rho \) planes of \( L \). For this reason, the arrangement of \( \mathcal{A}(L) \) when restricted to \( \triangle \), can have at most \( c((m/\rho)^3) \leq n/r \) facets, which in turn bounds the number of points of \( P \) inside such a simplex by \( n/r \).

Any line intersects \(|R| - 1\) faces of \( R \), and as such at most \( O(|R| \log \rho) = O(r^{1/3} \log^2 r) \) simplices of \( T \). For any plane \( h \), the total number of vertices that belong to facets of \( \mathcal{A}(R) \) that intersects \( h \) is \( O(|R|^2) \) by the zone theorem [27]. Since a face is decomposed into a number of simplices that is proportional to its complexity, it follows that \( h \) intersects at most \( O(r^{2/3} \log^2 r) \) simplices. \( \square \)

3 Separating Random Points by Lines

Here we consider the separability of a set \( P \) of \( n \) points picked uniformly, randomly and independently in the unit square, and the random variable \( S_n = \text{sep}(P) \), which is the separability of \( P \).

3.1 The Upper Bound

Let \( G \) be the uniform grid that partitions the unit square into \( N \times N \) cells, where \( N = n^{2/3} \). This grid is defined by \( 2(N - 1) \) lines, and the area of each grid cell is \( p = 1/N^2 = (1/n^{2/3})^2 = 1/n^{4/3} \). A grid collision is when two points \( x, y \in P \) belong to the same cell of \( G \), and in such a case \( x \) and \( y \) collide.

Lemma 3.1 Let \( Z \) be the number of pairs of points of \( P \) that collide in the grid \( G \) (i.e., \( Z \) is a random variable). Then for \( n \) sufficiently large, we have \( n^{2/3}/3 \leq \mathbb{E}[Z] \leq n^{2/3}/2 \).

Proof Let \( P = \{x_1, \ldots, x_n\} \), where the exact location of each point in this set is yet to be determined. The probability for two points \( x_i \) and \( x_j \) to collide, that is to fall into the same cell in the grid, is \( p = 1/N^2 \) — indeed, first throw in the point \( x_i \), and the desired probability is the probability of \( x_j \) to fall into the cell that contains \( x_i \). By linearity of expectation, the expected number of colliding pairs is \( \mathbb{E}[Z] = \binom{n}{2} p \leq n^2/(2n^{4/3}) = n^{2/3}/2 \).
For the lower bound, observe that $\mathbb{E}[Z] = \binom{n}{2} p = \frac{n(n-1)}{2N^2} \geq \frac{n^2}{3n^{2/3}} = \frac{n^{2/3}}{3},$ for $n \geq 3$.

**Lemma 3.2** $\mathbb{E}[S_n] = O(n^{2/3}).$

**Proof** Let $L$ be the set of $2(n^{2/3} - 1)$ separating lines used in creating $G$. By Lemma 3.1, the expected number of pairs of points of $P$ colliding is $O(n^{2/3})$. For each such colliding pair, we add to $L$ a line that separates this pair (thus $|L|$ is a random variable). At the end of this process all points of $P$ are separated, see Fig. 2. As claimed, we have $\mathbb{E}[S_n] \leq \mathbb{E}[|L|] = O(n^{2/3} + \mathbb{E}[Z]) = O(n^{2/3})$. □

### 3.2 A Detour to Balls into Bins

The problem at hand is related to the problem of balls and bins. Here, given $n$ balls, one throws them into $m$ bins, where $m \geq n$. A ball that falls into a bin with $i$ or more balls is $i$-heavy. Let $B_{\geq i}$ be the number of $i$-heavy balls. It turns out that a strong concentration on $B_{\geq i}$ follows readily from Talagrand’s inequality. While this is probably already known, we were unable to find it in the literature, and we provide a self-contained proof here for the sake of completeness.

#### 3.2.1 The Expectation of $B_{\geq i}$

**Lemma 3.3** For $i > 1$, consider throwing $n$ balls into $m$ bins, where $m \geq 3n$. Then,

$$e^{-2}F_i \leq \mathbb{E}[B_{\geq i}] \leq 6e^{i-1}F_i,$$

where $F_i = n\left(\frac{n}{im}\right)^{i-1}$, and $B_{\geq i}$ is the number of $i$-heavy balls. In addition, the expected number of pairs of $i$-heavy balls that are colliding, denoted by $\beta_i$, is $\beta_i \leq c_i n(n/m)^{i-1}$, where $c_i = O(i(e/i)^{i-1})$.

**Proof** Let $p = 1/m$. A specific ball falls into a bin with exactly $i$ balls if there are $i-1$ balls (of the remaining $n-1$ balls) that fall into the same bin. Hence the probability for that is $y_i = p^{i-1}(1-p)^{n-i}\binom{n-1}{i-1}$. We have that a specific ball is $i$-heavy with probability $p^{i-1}(1-p)^{n-i}\binom{n-1}{i-1}$. □
\[
\alpha = \sum_{j=i}^{n} \gamma_j = \sum_{j=i-1}^{n-1} \left( \frac{n-1}{j} \right) p^j (1-p)^{n-j-1} \leq \sum_{j=i-1}^{n-1} \left( \frac{e(n-1)}{jm} \right)^j
\]

\[
\leq (n/m)^{i-1} \sum_{j=i-1}^{n-1} \left( \frac{e}{j} \right)^j
\]

\[
\leq (n/m)^{i-1} 2.1 \left( \frac{e}{i-1} \right)^{i-1} \leq (n/m)^{i-1} 2.1 e^{\frac{e^i}{i}} \leq 6 e^{i-1} \left( \frac{n}{im} \right)^{i-1},
\]
as \((n/i)^i \leq \binom{n}{i} \leq (en/i)^i, i > 1\), and the summation behaves like a geometric series dominated by its first term (see Tedium 3.4 for additional details). Since \((1-p)^{n-j-1} \geq (1-1/m)^{m-1} \geq 1/e\), we have

\[
\alpha \geq \frac{1}{e} \sum_{j=i-1}^{n-1} \left( \frac{n-1}{j} \right) \frac{n-1}{n} \cdot \frac{n}{(i-1)m} \geq \frac{1}{e^2} \left( \frac{n}{im} \right)^{i-1}.
\]

We conclude that \(E[B_{\geq i}] = n\alpha = \Theta(n(n/m)^{i-1})\).

If a ball is in a bin with exactly \(j\) balls, for \(j \geq i\), then it collides directly with \(j-1\) other \(i\)-heavy balls. Thus, the expected number of collisions that a specific ball has with \(i\)-heavy balls is in expectation \(\sum_{j=i}^{n}(j-1)\gamma_j = \sum_{j=i-1}^{n-1} j \gamma_{j+1}\). Summing over all balls, and arguing as above, we have that the expected overall number of such collisions is

\[
\beta_i \leq n \sum_{j=i}^{n-1} j \gamma_{j+1} = n \sum_{j=i-1}^{n} j \left( \frac{n-1}{j} \right) p^j (1-p)^{n-j-1} = O\left( \frac{en}{im} \right)^{i-1}
\]

(note, that this counts every collision twice). \(\square\)

**Tedium 3.4** Let \(f(i) = (e/i)^i, g(i) = \sum_{j=i}^{\infty} f(j)\), and \(h(i) = g(i)/f(i)\). Observe that \(h\) is clearly monotone decreasing after \(i > 3\), and numerical calculations show that \(h(1) \approx 2.05291, h(2) \approx 1.54938, h(3) \approx 1.36421,\) and \(h(4) \approx 1.27041\). Thus \(h(i) \leq 2.1\) for all \(i \geq 1\), as desired.

### 3.2.2 Concentration of \(B_{\geq i}\)

**Talagrand’s inequality and certifiable functions.** Let \(f(x)\) be a real-valued function over some product probability space \(\Omega = \Omega_1 \times \cdots \times \Omega_n\). The function \(f\) is \(r\)-**certifiable**, if for every \(x \in \Omega\), there exists a set of indices \(J(x) \subseteq \{1, \ldots, n\}\), such that

\(\text{A)} \quad |J(x)| \leq rf(x), \text{ and} \)

\(\text{B)} \quad \text{if } y \in \Omega \text{ agrees with } x \text{ on the coordinates in } J(x), \text{ then } f(y) \geq f(x). \)

The function \(f\) is \(c\)-**Lipschitz** if for two values \(x, y \in \Omega\) that agree on all coordinates except one, we have that \(|f(x) - f(y)| \leq c\). For a real valued random variable \(f\), Springer
its median, denoted by \( v(f) \), is the infimum value \( v \), such that \( \mathbb{P}[f < v] \leq 1/2 \) and \( \mathbb{P}[f > v] \leq 1/2 \).

The version of Talagrand’s inequality we need is the following.

**Theorem 3.5 (\cite{15, Thm. 11.3})** Let \( f : \Omega \rightarrow \mathbb{R} \) be a \( c \)-Lipschitz and \( r \)-certifiable function, for some constants \( r \) and \( c \), with its median being \( v = v(f) \). Then, for all \( t > 0 \), we have \( \mathbb{P}[|f - v| > t] \leq 4 \exp(-t^2/4c^2r(v + t)) \).

**Lemma 3.6** Consider throwing \( n \) balls into \( m \) bins, where \( m \geq 3n \). Furthermore, let \( i \) be a small constant integer, \( B_{\geq i} \) be the number of balls that are contained in bins with \( i \) or more balls, and let \( v_i = v(B_{\geq i}) \). In addition, assume that \( v_i \geq 16i^2c \ln n \), where \( c \) is some arbitrary constant. Then \( \mathbb{P}[|B_{\geq i} - v_i| \geq 4i\sqrt{c v_i \ln n}] \leq 1/n^c \). Furthermore, for some constant \( c' \), we have \( |v_i - \mathbb{E}[B_{\geq i}]| \leq c'\sqrt{v_i} \), and thus

\[
\mathbb{P}[|B_{\geq i} - \mathbb{E}[B_{\geq i}]| \geq c'\sqrt{v_i}] \leq 4i\sqrt{c v_i \ln n} \leq 1/n^c.
\]

**Proof** Observe that \( B_{\geq i} \) is 1-certifiable—indeed, the certificate is the list of indices of all the balls that are contained in bins with \( i \) or more balls. The variable \( B_{\geq i} \) is also \( i \)-Lipschitz. Changing the location of a single ball, can make one bin that contains \( i \) balls, into a bin that contains \( i - 1 \) balls, thus decreasing \( B_{\geq i} \) by \( i \). Applying Theorem 3.5 (Talagrand’s inequality), with \( t = 4i\sqrt{c v_i \ln n} \) we have

\[
\mathbb{P}[|B_{\geq i} - v_i| > t] \leq 4 \exp\left(-\frac{t^2}{4i^2(v_i + t)}\right) \leq \frac{1}{n^c},
\]

assuming \( t \leq v_i \).

The estimate on the distance of \( v_i \) and \( \mathbb{E}[B_{\geq i}] \) follows by estimating the expectation, by breaking the real line into intervals of length \( O(i\sqrt{v_i}) \), and using the exponential decay of the probability in each such interval as we get away from \( v_i \), as implied by the above.

Formally, we have that

\[
|v_i - \mathbb{E}[B_{\geq i}]| = |\mathbb{E}[v_i - B_{\geq i}]| \leq \mathbb{E}[|v_i - B_{\geq i}|]
\leq \sum_{k \geq 0} (k + 1)i\sqrt{v_i} \cdot \mathbb{P}[|B_{\geq i} - v_i| > ki\sqrt{v_i}].
\]

Now applying Theorem 3.5 and using the assumption \( v_i \) is sufficiently large,

\[
|v_i - \mathbb{E}[B_{\geq i}]| \leq \sum_{k \geq 0} (k + 1)i\sqrt{v_i} \cdot \mathbb{P}[|B_{\geq i} - v_i| > ki\sqrt{v_i}]
\leq \sum_{k \geq 0} (k + 1)i\sqrt{v_i} \cdot 4 \exp\left(-\frac{k^2i^2v_i}{4i^2(v_i + ki\sqrt{v_i})}\right)
\leq \sum_{k \geq 0} (k + 1)i\sqrt{v_i} \cdot 4 \exp(-k^2/4(1 + k)) = O(i\sqrt{v_i}).
\]

The final inequality is readily implied by combining the two earlier statements. \( \square \)
In the following, we need Chernoff’s inequality, which we state explicitly for completeness [15].

**Theorem 3.7** Let $X_1, \ldots, X_n$ be $n$ independent random variables where $\mathbb{P}[X_i = 1] = p_i$, and $\mathbb{P}[X_i = 0] = 1 - p_i$. And let $X = \sum_{i=1}^{b} X_i$, and $\mu = \mathbb{E}[X] = \sum_i p_i$. For any $\delta \geq 0$, we have

$$\mathbb{P}[X > (1 + \delta)\mu] < \begin{cases} \exp\left(-\mu\delta^2/4\right), & 2e - 1 \geq \delta \geq 0, \\ 2^{-\mu(1+\delta)}, & \delta \geq 2e - 1, \\ \exp\left(-\mu\delta/2\ln \delta\right), & \delta \geq e^2. \end{cases}$$

Similarly, we have $\mathbb{P}[X < (1 - \delta)\mu] < \exp(-\mu\delta^2/2)$.

### 3.2.3 Not Too Many Shared Birthdays

The **birthday paradox** states that if one throws $n$ balls (i.e., birthday dates of $n$ people) into $m = \Theta(n^2)$ bins (i.e., days of the year), then the number of bins containing two or more balls is non-zero with constant probability. The following proves that the number of such bins cannot be too large.

**Lemma 3.8** Consider throwing $n$ balls into $m = cn^2$ bins, where $c$ is some constant. Then, with high probability, the total number of bins that contain two or more balls is $O(\log n/\log \log n)$.

**Proof** Partition the set $B$ of $n$ balls into two sets $C$ and $D$, each of size $n/2$. Let $Y$ be the number of bins that contains balls of $C$—clearly, $Y \leq n/2$. As such, the probability of a ball of $D$ to fall into a bin with a ball of $C$, is $\alpha = Y/m \leq (n/2)/m = 1/(2cn)$. Now the expected number of bins that contains balls from both $C$ and $D$ is $|D|\alpha = (n/2)\alpha \leq 1/4c$. By Chernoff’s inequality, this quantity is smaller than $T = O(\log n/\log \log n)$, with high probability.³

This approach allows us to count the number of bins that contain balls from both $C$ and $D$. However, to count the number of bins that contain two or more balls, we need to count those bins which may only contain balls from $C$ (or from $D$). To overcome this, we repeat the above experiment, generating new partitions $(C_1, D_1), \ldots, (C_M, D_M)$, as above, such that any pair $x, y \in B$ appears in a constant fraction of these partitions on different sides. This is easy to do—match the balls of $B$ in pairs. To generate the $i$th partition, the algorithms goes over the pairs in the matching $(x, y)$, and puts $x$ in $C_i$ and $y$ in $D_i$, with probability half, and otherwise it assigns $x$ to $D_i$ and $y$ to $C_i$. Observe that $|C_i| = |D_i| = n/2$.

Repeating this $M = c_2 \log n$ times, guarantees with high probability, that any two balls $x, y \in B$ appear in opposing sides of at least one of these partitions (two points that are an edge in the matching are in different sides in all partitions). Furthermore,

³ By Theorem 3.7 with $\mu = 1/4c$, and $\delta = c_1 \log n/\log \log n$, where $c_1$ is a sufficiently large constant.
by Chernoff’s inequality, each pair appears in at least \( m = \Omega(\log n) \) pairs, with high probability.\(^4\)

Consequently, there are at most \( \beta = O(M \log n / \log \log n) \) heavy bins with balls that belong to different sides of some partition. Each such heavy bin gets counted at least \( m \) times, thus implying that the number of heavy bins is at most \( \beta/m = O(\log n / \log \log n) \).

The following is not required for the proof of the main result, and we include it since it might be of independent interest. Note that the next lemma bounds the number of balls colliding, while Lemma 3.8 bounded the number of bins.

**Lemma 3.9** Consider throwing \( n \) balls into \( m = cn^2 \) bins, where \( c \) is some constant. Then, with high probability, the total number of colliding pairs of balls is \( O(\log n / \log \log n) \).

**Proof** Let \( i \) be a sufficiently large constant (in particular \( i \geq e/c \)). By Lemma 3.3, the expected number of collisions of pairs of \( i \)-heavy balls is \( \beta_i = O(n i / (im)^{i-1}) = O\left(\frac{n}{i^{i-1}}\right) = O(1/i^{i-3}) \). By Markov’s inequality, the probability that are any collisions involving \( i \)-heavy balls is at most \( \beta_i \). As for collisions of pairs that are not \( i \)-heavy, by Lemma 3.8, with high probability there are at most \( O(\log n / \log \log n) \) bins that contain between 2 and \( i-1 \) balls, and each such bin contributes at most \((\binom{i-1}{2})\) colliding pairs. We conclude, that with high probability, the total number of collisions is \( O(i^2 \log n / \log \log n) = O(\log n / \log \log n) \), as claimed. \( \square \)

**Remark 3.10** Somewhat disappointingly, the upper bound \( O(\log n / \log \log n) \) on the number of colliding balls, in Lemma 3.9, is tight if the probability of success is required to be \( \geq 1 - 1/n^\tau \), where \( \tau \) is some constant. To see that, consider the partition \((C, D)\) from the proof of Lemma 3.8. With high probability, the number of bins containing balls of \( C \) is \( \geq n/4 \)—this follows by similar to, but easier, argument to the one used in the proof of Lemma 3.6. As such, the probability of a ball of \( D \) to collide with a ball of \( C \) is at least \( p = 1/(4cn) \). Thus, the probability that exactly \( i \) such collisions happen is at least \( \zeta = \binom{n/2}{i} p^i (1-p)^{n/2-i} \geq \frac{\left(\frac{n/2}{i}\right)^i}{\left(\frac{4c}{i}\right)^i} = \frac{1}{(8ci)^\tau} \). If we require the last quantity to be larger than \( 1/n^\tau \), then we have \( n^\tau \geq (8ci)^i \iff \tau \ln n \geq i \ln(8ci) \), which holds for \( i = \Theta(\log n / \log \log n) \), as \( \tau \) and \( c \) are constants. Namely, for the specified value of \( i \), we have that \( i \) collisions happen with probability \( \geq \zeta \geq 1/n^\tau \).

### 3.3 How Many Collisions are There, Anyway?

It is useful to think about the point set \( P \) as being generated by throwing \( n = n \) balls into \( m = N^2 = n^{4/3} \) bins – here every grid cell is a bin. Lemmas 3.3 and 3.6 together imply the following.

\(^4\) Observe that a pair \( x, y \) is separated by a partition with probability at least half. Let \( X \) be the number of partitions separating this pair. The expected number of partitions separating this pair is \( \mu = E[X] = M/2 = O(\log n) \). Setting \( \delta = 1/2 \), we have by Theorem 3.7 that \( P[X \leq (1-\delta)\mu] \leq \exp(-\mu\delta^2/2) \leq 1/\mu^{\Omega(1)} \), by making \( c_2 \) sufficiently large. It follows that with high probability, the pair is separated in at least \((1-\delta)\mu = (c_2/2) \log n \) partitions.
Corollary 3.11 When throwing \( n \) balls into \( n^{4/3} \) bins, we have, with high probability, that \( B_{\geq 2} = \Theta(n^{2/3}) \) and \( B_{\geq 3} = \Theta(n^{1/3}) \), where \( B_{\geq i} \) is the number of balls that are in bins with \( i \) or more balls.

Lemma 3.12 Let \( P \) be a set of \( n \) random point picked uniformly in the unit square. Let \( Z \) be the number of active grid cells—namely, the number of grid cells that contain two or more points of \( P \). We have, with high probability, that \( Z = \Theta(n^{2/3}) \).

Proof We interpret this as throwing \( n \) balls into \( n^{4/3} \) bins (a grid cell is a bin). The number of balls colliding with exactly one other ball is \( B_{\geq 2} - B_{\geq 3} \). Therefore the number of bins containing exactly two balls is \((B_{\geq 2} - B_{\geq 3})/2\). By Corollary 3.11, we have \( Z \geq (B_{\geq 2} - B_{\geq 3})/2 = \Theta(n^{2/3}) \).

3.3.1 A Single Line Cannot be Involved in Too Many Active Cells

Lemma 3.13 Let \( S \) be a given set of \( n \) grid cells. A cell of \( S \) is active if it contains two or more points of \( P \). Let \( Y \) be the number of cells of \( S \) that are active. We have that \( Y = O(\log n / \log \log n) \), with high probability (i.e., \( \geq 1 - 1/n^{\Omega(1)} \)).

Proof For any \( i \), let \( X_i \) be the indicator variable that is one if the \( i \)th point of \( P \) falls into a cell of \( S \), and let \( Y = \sum_{i=1}^{n} X_i \). The probability of a point \( x \in P \) to fall into a cell of \( S \) is at most \( p' = p2N = 2/n \). Hence \( \mu = E[Y] = np' \leq 2n/N = 2n^{1/3} \). By Chernoff’s inequality (Theorem 3.7), we have that

\[
\mathbb{P}[Y \geq 3n^{1/3}] \leq \mathbb{P}[Y \geq (1 + 1/2)\mu] \leq \exp\left(-\frac{\mu (1/2)^2}{4}\right) \leq \exp\left(-\frac{n^{1/3}}{8}\right).
\]

From this point on, we assume that \( Y \leq 3n^{1/3} \). Thus, we are throwing at most \( 3n^{1/3} \) balls into \( 2N = 2n^{2/3} \) bins. By Lemma 3.8, with high probability, there are at most \( O(\log n / \log \log n) \) bins with two or more balls.

3.3.2 The Result

Theorem 3.14 Let \( P \) be a set of \( n \) points picked uniformly and randomly from the unit square. Let \( S_n \) be the separability of \( P \)—the minimum number of lines separating \( P \). Then, with high probability we have \( S_n = \Omega(n^{2/3} \log \log n / \log n) \) and \( S_n = O(n^{2/3}) \).

Proof We remind the reader that \( G \) is the grid partitioning the unit square into \( N \times N \) cells, where \( N = n^{2/3} \). For a line \( \ell \) that avoids the vertices of \( G \), consider the set of grid cells that it intersects, formally \( B(\ell) = \{ \Box \mid \Box \in G \text{ and } \Box \cap \ell \neq \emptyset \} \). Since \( \ell \) intersects \( \leq N - 1 \) horizontal and \( \leq N - 1 \) vertical lines of the grid inside the unit square, it follows that \( |B(\ell)| \leq 2N - 1 \). Fix an arbitrary ordering of the cells of \( G \), and add cells according to this ordering to \( B(\ell) \) until this set is of size \( 2N \). The resulting set, \( \text{sgn}(\ell) \) is the signature of \( \ell \).

Let \( \mathcal{L} \) be a set of representative lines. Specifically, among all lines with the same signature, pick one of them to be in \( \mathcal{L} \). It is easy to verify that \( |\mathcal{L}| = O(N^4) = O(n^{8/3}) \).
We are now ready for the proof itself. Consider the randomly generated point set \( P \). We consider two points to be separated if they belong to different grid cells. It remains to separate points that collide in the grid (i.e., belong to the same grid cell). So consider a minimal separating set of lines \( L \). A line in \( L \) intersects \( \leq 2N \) cells of the grid, and by Lemma 3.13, with high probability, its signature contains at most \( T = O(\log n / \log \log n) \) active grid cells. Namely, each such line can at best only separate pairs that belong to these active cells.

However, Lemma 3.12 implies that, with high probability, the number of active grid cells is at least \( n^{2/3}/c' \), where \( c' \) is some constant. It follows that any set of lines that separates all the pairs of points that collide, must be of size \( \geq (n^{2/3}/c')/T \), with high probability.

As for the upper bound, it follows from the argument of Lemma 3.2. The number of points that need separation from other points (after we use the grid lines) is \( B_{\geq 2} \). However, \( B_{\geq 2} = \Theta(n^{2/3}) \) with high probability, by Corollary 3.11. Now use three lines to separate each point from all the other points of \( P \). This implies the upper bound. \( \square \)

### 3.4 Extensions

#### 3.4.1 Other Domains

Both the lower bound and upper bound for \( S_n \) in Theorem 3.14 hold if \( n \) points are sampled uniformly at random within any given convex region \( C \). Indeed, John’s theorem [18] implies that after appropriate scaling, there is a disk \( D \) such that \( D \subseteq C \subseteq 2D \). In particular, we can inscribe in \( D \) a large square \( S \), see Fig. 3. This square contains a constant fraction of the area of \( C \), since

\[
\text{area}(S) = (2/\pi) \cdot \text{area}(D) = (1/2\pi) \cdot \text{area}(2D) \geq (1/8) \cdot \text{area}(C).
\]

By sampling \( n \) points uniformly at random from \( C \), a Chernoff bound (Theorem 3.7) implies that with high probability, \( \Theta(n) \) points will fall inside \( S \). This implies that we can focus on separating pairs of points which fall inside the square \( S \). In particular, the number of lines needed to separate all pairs of points inside \( C \) is at least the number of lines needed to separate all pairs of points inside \( S \). The lower bound of Theorem 3.14 follows.

As for the upper bound on \( S_n \), the argument is similar. Given the disk \( 2D \supseteq C \), whose area is a constant factor larger than the area of \( C \), we can circumscribe a square \( S' \) around the disk \( 2D \). An identical argument as above implies that the area of \( S' \) is at most a constant factor larger than the area of \( C \). Thus we can consider sampling \( n \) points from this square \( S' \), where \( \Theta(n) \) of them land in \( C \) with high probability. It follows that the number of lines needed to separate pairs of points inside \( C \) is at most the number lines needed to separate pairs of points in \( S' \).
3.4.2 Higher Dimensions

Corollary 3.15 Let $P$ be a set of $n$ points picked uniformly and randomly from the unit cube $[0, 1]^d$. With high probability, the minimum number of hyperplanes separating $P$ is $\Omega(n^{2/(d+1)} \log \log n / \log n)$. Similarly, in expectation, one can separate $P$ using $O(\frac{d n^{2/(d+1)}}{\log n})$ hyperplanes.

Proof One can easily extend the two-dimensional analysis to higher dimensions. We quickly sketch the calculations without going into the low level details, which follows readily by retracing the same argumentation.

In the following $f \approx g$ means that $f = \tilde{O}(g)$. We now consider the unit cube $[0, 1]^d$. As before, we partition it into $N^d$ grid cells, in the natural way, where the value of $N$ is to be determined shortly. Let $G$ denote the resulting grid. Any hyperplane intersects at most $H \approx N^{d-1}$ grid cells. We would like to guarantee that there are $\approx O(1)$ cells that contain two and more points, for a fixed hyperplane $h$. By the birthday paradox, this means that we should have at most $\sqrt{H}$ random points falling into the $H$ cells associated with $h$, if we want a constant number of collisions. Since the probability of a point to fall into a grid cell that $h$ intersects is $H/N^d$, we get that

$$\sqrt{H} \approx nH/N^d \implies H \approx (N^d n)^2 \implies N^{d-1} \approx N^{2d}/n^2 \implies n^2 \approx N^{d+1} \implies N = n^{2/(d+1)}.$$

The overall number of grid cells that contain two or more points is

$$\binom{n}{2}/N^d \approx n^2/N^d = n^{2-2d/(d+1)} = n^{2/(d+1)}.$$

Finally, with high probability, a hyperplane can intersects only $\approx O(1)$ active grid cells, which means that the number of hyperplanes needed to separate $n$ random points is $\approx n^{2/(d+1)}/O(1)$.
The lower bound follows by plugging in the above sketch, into the detailed analysis of the two-dimensional case.

Now for the upper bound. In the grid \( G \), the volume of each grid cell is \( p = 1/N^d = 1/n^{2d/(d+1)} \). Thus the expected number of collisions happening inside the grid cells is \( \mathbb{E}[Z] = \binom{n}{2} p \leq n^2/2n^{2d/(d+1)} = O(n^{2/(d+1)}) \). We separate each such colliding pair by its own hyperplane. Note, that creating the grid \( G \), requires \( d(N-1) \) separating hyperplanes. As such, the expected number of separating hyperplanes one needs is at most \( O(dN + \mathbb{E}[Z]) = O(dn^{2/(d+1)}) \). \( \square \)

**Remark 3.16** A set of \( n \) points of the grid \( n^{1/d} \times \cdots \times n^{1/d} \) in \( \mathbb{R}^d \) requires \( \leq dn^{1/d} \) hyperplanes to separate them. Therefore the gap demonstrated in two dimensions (between the grid and random points) also holds in higher dimensions.

### 3.4.3 Allowing More Points to Collide

Here, we change the problem—we allow groups of up to \( t \) points to not be separated by the points.

**Lemma 3.17** Let \( t > 1 \) be a fixed integer constant. Consider a set \( P \) of \( n \) random points chosen uniformly and independently from \( [0,1]^2 \). In expectation there is a set \( L \) of \( O\left(n^{(t+1)/(2r+1)}\right) \) lines, such that every face of \( \mathcal{A}(L) \) contains at most \( t \) points of \( L \).

**Proof** Let \( N = n^{(t+1)/(2r+1)} \). And consider the set of lines forming the grid \( N \times N \). Let \( m = N^2 \). Consider the distribution of the points of \( P \) in the grid cells. Any grid cell that contains more than \( t \) points, is further split by introducing additional lines until every cell in the resulting arrangement contains at most \( t \) points.

To bound the number of these additional fix-up lines, recall the balls and bins interpretation. By Lemma 3.3, the number of points that falls into grid cells with \( t+1 \) or more balls is

\[
\Theta\left(n^{t+1}/m^t\right) = \Theta\left(n^{t+1}/n^{2(t+1)/(2r+1)}\right) \\
= \Theta\left((n^{1-2t/(2r+1)})^{t+1}\right) = O\left(n^{(t+1)/(2r+1)}\right).
\]

Clearly, this also provides an upper bound on the number of fix-up lines needed. \( \square \)

### 4 Approximating a Minimum Separating Set of Lines

#### 4.1 Problem Statement and a Slow Algorithm

Given a set \( P \) of \( n \) points in general position (i.e., no three points are collinear) in the plane, our goal is to approximate the minimal set of lines \( L \) separating all the pairs of points of \( P \).
4.1.1 Reduction to Hitting Set

Given a set \( P \) as above, one can restate the problem as a hitting set problem. Indeed, let \( \mathcal{C} = \{ \text{line}(p, q) \mid p, q \in P \} \) be the set of candidates, which contains all lines that pass through every pair of points of \( P \), where \( \text{line}(p, q) \) denotes the line passing through \( p \) and \( q \).

To simplify the description of the algorithm, we use an alternative definition of separation as suggested in Remark 2.3(A). Specifically, we modify Definition 2.1 by allowing points to lie on the separating lines. In particular, two points \( p \) and \( q \) are additionally considered to be separated if the line separating them passes through \( p \) or \( q \).

For each pair of points \( p, q \in P \), consider the set of all lines of \( \mathcal{C} \) that intersect this segment \( pq \):

\[
L_{pq} = \{ \ell \in \mathcal{C} \mid pq \cap \ell \neq \emptyset \}.
\]

By our modified definition, any of the lines of \( L_{pq} \) can be interpreted as separating the two points \( p \) and \( q \). Consider the set system

\[
\mathcal{F} = (\mathcal{C}, \mathcal{L}), \quad \text{where } \mathcal{L} = \{ L_{pq} \mid p, q \in P, p \neq q \}.
\]  

(4.1)

**Observation 4.1** Given a set \( L' \) of \( m \) lines that separates \( P \), there exists a subset \( L \subseteq \mathcal{C} \) of \( m \) lines, such that \( L \) separates \( P \). Indeed, translate and rotate every line of \( L' \) till it passes through two points of \( P \). Clearly, the resulting set of lines separates the points of \( P \).

**Lemma 4.2** The set system \( \mathcal{F} \) defined by (4.1) has VC dimension at most 11.

**Proof** The following argument is due to Jan Kynčl [24]. The arrangement of \( m \) lines in the plane has at most \( f = m(m + 1)/2 + 1 \) faces. As such, there are at most \( \binom{m}{2} \) distinct segments (as far as what lines they intersect). If a set \( L \) of \( m \) lines is shattered by the range space, then we must have \( 2^m \leq \binom{m}{2} \leq (m(m+1)/2+1)^2 \), and this inequality breaks for \( m = 12 \), which implies that the VC dimension is at most 11. A further improvement might be possible by more involved argument [24], but one has to be careful since the lines of \( L \) are not in general position. \( \square \)

It follows that one can compute a separating set by computing (approximately) a hitting set for the set system \( \mathcal{F} \), using known approximation algorithms for hitting sets for spaces with bounded VC dimension [18].

4.1.2 The Basic Approximation Algorithm for Hitting Set for \( \mathcal{F} \)

We next describe the standard reweighting algorithm for hitting set in our context. Such reweighting algorithms in Computational Geometry go back to the work by Chazelle and Welzl [9]. In the context of geometric set-cover/hitting-set, Clarkson [10] was the first to suggest such an algorithm. Clarkson’s algorithm was further formalized by Brönnimann and Goodrich [6]. See [18, Chap. 6] for more details.
The Algorithm. Given $\mathcal{F}$ as above, let $L_{\text{opt}}$ be the optimal solution, and let $\sigma = \text{sep}(P) = |L_{\text{opt}}|$ denote the separability of $P$. The algorithm will perform an exponential search for the value of $\sigma$. Let $k$ be the current guess for the value of $\sigma$ (at the beginning of the algorithm $k = 1$).

Initially, each line in $\ell \in \mathcal{C}$ is assigned weight $\omega(\ell) = 1$. For a subset $L \subseteq \mathcal{C}$, its weight is $\omega(L) = \sum_{\ell \in L} \omega(\ell)$. In each iteration, the algorithm samples a set of lines $R \subseteq \mathcal{C}$ of size $O(\varepsilon^{-1} \log \varepsilon^{-1})$ (where $\varepsilon = 1/(4k)$ and $k$ is the current guess for $\sigma$) picked according to their weights. By the $\varepsilon$-net theorem [22], $R$ is an $\varepsilon$-net with probability at least $1 - \varepsilon^c = 1 - 1/(4k)^c$ (for some sufficiently large constant $c$). The algorithm next checks if the sample $R$ separates $P$, and if so, it returns the sample as the desired separating set.

To this end, the algorithm builds the arrangement $\mathcal{A}(R)$ and preprocesses it for point-location queries. Next, it locates all of the faces in this arrangement that contain points of $P$. If there is a pair of points $p, q \in P$ that are in the same face, then this pair is not separated by $R$. If the weight of the lines $L_{pq}$ is at most an $\varepsilon$ fraction of the total weight of $\mathcal{C}$ (formally, $\omega(L_{pq}) \leq \varepsilon \omega(\mathcal{C})$), the algorithm doubles the weight of all the lines in $L_{pq}$. Otherwise, this iteration failed, and the algorithm continues to the next iteration.

If after $16k \log n$ iterations the algorithm did not output a solution, then the guess $k$ for $\sigma$ is too small. In which case, the algorithm doubles the value of $k$ and starts from scratch.

Lemma 4.3 Given a set $P$ of $n$ points in general position, one can return a set of separating lines $R$ of size $O(\sigma \log \sigma)$ in expected time $O(n^2 \sigma \log n + \sigma^3 \log n \log^2 \sigma)$, where $\sigma$ is the separability of $P$.

Proof For the sake of completeness, we sketch the proof of correctness of the algorithm. Assume that the guess $k$ is such that $\sigma \leq k \leq 2\sigma$.

Initially, the total weight of the $\mathcal{C}$ is $\binom{n}{2}$. In each successful iteration, the total weight increases by a factor of at most $\varepsilon$. (Assume for the time being that all iterations are successful.) If $W_i$ is the total weight of the lines of $\mathcal{C}$ in the end of the $i$th successful iteration, then $W_i \leq (1 + \varepsilon)^i n^2$. On the other hand, any successful iteration doubles the weight of at least one the lines in the optimal hitting set $L_{\text{opt}}$. For a line $\ell \in L_{\text{opt}}$, let $h(\ell)$ be the number of times its weight had been doubled. We have that $\sum_{\ell \in L_{\text{opt}}} h(\ell) \geq i$ and $W_i \geq \sum_{\ell \in L_{\text{opt}}} 2^{h(\ell)}$. Clearly, the right side is minimized when all the “hits” are distributed uniformly. That is, we have that $W_i \geq \sum_{\ell \in L_{\text{opt}}} 2^{i/\sigma} \geq \sigma 2^{i/\sigma - 1}$.

Consequently,

$$\exp \left( \frac{i}{2\sigma} - 1 + \ln \sigma \right) \leq \sigma 2^{i/\sigma - 1} \leq W_i \leq (1 + \varepsilon)^i n^2 \leq \left(1 + \frac{1}{4k}\right)^i n^2 \leq \left(1 + \frac{1}{4\sigma}\right)^i n^2 \leq \exp \left( \frac{i}{4\sigma} + 2 \ln n \right).$$
since \( k \geq \sigma \). This is equivalent to \( \frac{i}{2\sigma} - 1 + \ln \sigma \leq \frac{i}{4\sigma} + 2 \ln n \iff \frac{i}{4\sigma} \leq 2 \ln n - \ln \sigma + 1 \), which holds only for \( i \leq 8\sigma \ln n \). Namely, the algorithm must stop after this number of successful iterations. Note that the separating lines returned is a sample of size \( O(k \log k) = O(\sigma \log \sigma) \) that separates all the points of \( P \).

By the \( \varepsilon \)-net theorem, every iteration is successful with probability \( 1 - \varepsilon^c \geq 1 - 1/\sigma^c \), where the constant \( c \) is sufficiently large. As such, the number of failed iterations is tiny compared to the number of successful iterations, and we can ignore this issue.

In each iteration, the algorithm samples a set \( R \) of size \( r = O(\varepsilon^{-1} \log \varepsilon^{-1}) = O(k \log k) \). The arrangement \( \mathcal{A}(R) \) is constructed in \( O(r^2) \) time. We then perform \( n \) point location queries in \( \mathcal{A}(R) \), in \( O(\log r) = O(\log k) \) time per query. Thus, the running time for a fixed value of \( k \) is \( O((r^2 + n \log k + n^2)k \log n) = O((k^2 \log^2 k + n \log k + n^2)k \log n) \). Here, the \( O(n^2) \) term is the time it takes to scan the lines of \( C \) and update their weights. Summing this over exponentially growing values of \( k = 2^0, 2^1, \ldots \), where the final \( k \) is at most \( 2\sigma \), the total running time is bounded by the sum (where \( \lg = \log_2 \) denotes the binary logarithm):

\[
O\left( \sum_{i=0}^{\lfloor \lg(2\sigma) \rfloor} (2^i)^2 \log^2 2^i + n \log 2^i + n^2) 2^i \log n \right)
\]

\[
= O\left( \sum_{i=0}^{\lfloor \lg(2\sigma) \rfloor} i^2 2^i \log n + n^2 \log n \sum_{i=0}^{\lfloor \lg(2\sigma) \rfloor} 2^i \right)
\]

\[
= O\left( \sigma^3 \log n \log^2 \sigma + n^2 \sigma \log n \right).
\]

4.2 Faster Algorithm

4.2.1 Challenge and the Main Ideas

**Challenge.** We want to get a faster algorithm than the “naive” algorithm described above. In the above algorithm, the bottleneck is the \( O(n^2) \) term in the running time, which is the result of explicitly maintaining the set \( C \) and the weights for each line in \( C \). Note that the number of iterations the algorithm performs is pretty small, only \( O(\sigma \log n) \).

The idea is to maintain the set \( C \) and the weights implicitly. To this end, consider the given set \( P \) of \( n \) points. In the dual, the set \( P^* \) corresponds to a set of \( n \) lines (see [18, Chap. 25] for more details about duality). A line \( \ell \in C \) corresponds to an intersection point between two lines \( p^*, q^* \in P^* \)—that is, a vertex of \( \mathcal{A}(P^*) \) (and this vertex represents \( \ell \) uniquely).

Now, in the \( i \)th iteration of the (inner) algorithm, it doubles the weight of the lines that are in the set \( L_{p_i, q_i} \). In other words, the lines that intersect the segment \( s_i = p_i q_i \).

In the dual, the segment \( s_i \) is a double-wedge \( D_i = s_i^* \). The double-wedge is the region “sandwiched” between the two dual lines \( p_i^* \) and \( q_i^* \), and its interior are all the points in the plane that are above exactly one line of out of \( p_i^* \) and \( q_i^* \).
At the end of the $i$th iteration, the dual plane is partitioned into the arrangement $\mathcal{A}(\mathcal{D}_i)$, where $\mathcal{D}_i = \{D_1, \ldots, D_i\}$. A vertex $v \in \mathcal{A}(P^*)$, at the end of the $i$th iteration, has weight $2^h(v)$ where $h(v)$ is the number of double wedges of $\mathcal{D}_i$ that contains $v$.

Observe that the arrangement $\mathcal{A}(\mathcal{D}_i)$ has complexity $O(i^2)$, which is relatively small, and it can be maintained efficiently. The problem is that to implement the algorithm, one needs to be able to sample efficiently a line from $\mathcal{C}$ according to their weights. To this end, we need to maintain for each face of $\mathcal{A}(\mathcal{D}_i)$ the number of vertices of $\mathcal{A}(P^*)$ that it contains.

4.2.2 Building Blocks

We next describe data-structures for counting intersections inside a simple region, sampling a vertex from such a region, and how to maintain such a partition of the plane under insertion of double-wedges.

Counting and sampling intersections

Lemma 4.4 Let $\psi$ be a convex polygon in the plane with constant number of edges, and let $L$ be a set of $m$ lines. The number of vertices of $\mathcal{A}(L)$ that lie in $\psi$ can be computed in $O(m \log m)$ time.

Furthermore, this algorithm constructs a data-structure, using $O(m \log m)$ space, such that one can uniformly at random pick, in $O(\log m)$ time, a vertex of $\mathcal{A}(L)$ that lies in $\psi$.

Proof Conceptually, select a point on the boundary of $\psi$ and cut $\psi$ at that point. Take this (now open) polygon and straighten it into a straight line. Finally, translate and rotate the plane, so that this straightened line becomes parallel to the $x$-axis, see Fig. 4.

Furthermore, for a line $\ell \in L$ that intersects $\partial \psi$, treat the segment $s = \ell \cap \psi$ as a rubber band. In the end of this straightening process, $s$ became an interval on the $x$-axis. For two lines $\ell, \ell' \in L$ that have an intersection inside $\psi$, this results in two intervals $I, I'$, such that each interval contains exactly one endpoint of the other interval in its interior. This also holds in the other direction—two intervals that have this property corresponds to a common intersection of the original lines inside $\psi$. Counting such pairs is quite easy by sweeping the $x$-axis from left to right. We next describe this algorithm more formally in the original setup.

Assume that $L = \{\ell_1, \ldots, \ell_m\}$. The algorithm computes the intersection points of the lines of $L$ with the boundary of $\psi$, and sorts them in their counterclockwise order on the boundary of $\psi$ (starting, say, in the top left vertex of $\psi$).
The resulting order is a sequence \( p_1, \ldots, p_{m'} \), where \( m' \leq 2m \), and every point \( p_i \) has a label \( \alpha = \text{id}(p_i) \) which is the index of the line \( \ell_{\alpha} \in L \) that defines it (i.e., \( p_i \in \partial \psi \cap \ell_{\alpha} \)). Next, the algorithm scans this sequence:

- When it encounters an intersection \( p_j \) such that \( \text{id}(p_j) \) was not seen before, it inserts the line of \( p_j \) into a balanced binary search tree (BST), using the value of \( j \) for the ordering. This BST has the added feature that each internal node stores the number of elements stored in its subtree.
- When the algorithm encounters a point \( p_k \) such that the line defining it was already inserted into the BST (i.e., \( \text{id}(p_k) = \text{id}(p_j) \) for some \( j < k \)), the algorithm reports the number of lines stored in the tree between \( j \) and \( k \), which corresponds to the number of lines of \( L \) that intersects the line of \( p_k \) in \( \psi \). Next, we remove the line of \( p_k \) (stored with the key value \( j \)) from the tree.

All of these operations can be implemented in \( O(\log m) \) time, so that the overall running time is \( O(m \log m) \). Observe, that every relevant intersection is counted exactly once by this process.

To get the sampling data-structure, rerun the above algorithm using a BST with persistence. This persistence costs \( O(\log m) \) additional space per operation, since we use the path copying approach. This modification does not effect the overall running time. Thus, the resulting data-structure uses \( O(m \log m) \) space. Now, every line \( \ell \in L \), corresponds to an interval \( I_\ell = [i(\ell), i'(\ell)] \) in the BST. Furthermore, the lines intersecting \( \ell \) in \( \psi \), are stored in the BST (in the version just after \( \ell \) was deleted) in the interval \( I_\ell \).

As such, every line intersecting \( \psi \) has an associated interval, with an associated weight (i.e., the number of intersections assigned to it by the construction). To pick a random vertex, the algorithm first picks an interval according to their weights—this corresponds to a random line \( \ell \). Next, given this random line, the algorithm picks a random element stored in the \( O(\log m) \) subtrees representing the lines in \( I_\ell \). Since the algorithm used path copying, it has the exact number of lines stored in each subtree, and it is straightforward to sample a line in uniform. This second random line \( \ell' \), such that \( \ell \cap \ell' \in \psi \) is the desired random vertex.

\[ \square \]

**Sampling a trapezoid.** The algorithm maintains a collection of \( m \) trapezoids that are interior-disjoint, such that their (disjoint) union covers the plane. Furthermore, assume that each such trapezoid \( \psi \) already has the data-structure of Lemma 4.4 built for it.

**Definition 4.5** Consider a set \( D \) of double-wedges and a trapezoid \( \psi \) such that its interior is contained in a single face of \( \mathcal{A}(D) \). For a set of lines \( L \), the number of vertices of \( \mathcal{A}(L) \) in \( \psi \) is the support of \( \psi \), and it is denoted by \( \#(\psi) \). The depth of \( \psi \) is the number of double-wedges of \( D \) that fully contain \( \psi \) in their interior. The depth of \( \psi \) is denoted by \( d(\psi) \). The mass of \( \psi \) is defined as \( m(\psi) = \#(\psi)2^{d(\psi)} \).

**Lemma 4.6** Given a (dynamic) set at most \( m \) interior-disjoint trapezoids, covering the plane, each with the associated data-structure of Lemma 4.4 and their known mass, one can sample a random vertex from \( \mathcal{A}(L) \) in \( O(\log m + \log m') \) time, where \( m' \) is the maximum size of a conflict list of such a trapezoid. Furthermore, one can update this data-structure under insertion and deletion in \( O(\log m) \) time.
**Proof**  The task at hand is to pick a vertex of $A(L)$ uniformly at random according to these weights. To this end, we construct a balanced binary search tree having the trapezoids as leafs—a trapezoid is stored together with its mass. Every internal node of this tree has the total mass of the leafs in its subtree.

Now, one can traverse down the tree randomly, starting at the root, as follows. If the current node is $u$, consider its two children $v$ and $v'$. The algorithm picks an integer number randomly and uniformly in the range $[1, 1 + m(v) + m(v')]$. If this number is in the range $[1, m(v)]$, the algorithm continues the traversal into $v$, otherwise, it continues into $v'$. Clearly, this traversal randomly and uniformly chooses a leaf of the tree (according to their mass). Once the algorithm arrived to such a leaf, it uses the data-structure of Lemma 4.4 to pick a random vertex inside the associated trapezoid.  

\\[\square\\]

### 4.2.3 Maintaining Vertex Weights Efficiently Under Insertions

Our purpose here is to present an efficient data-structure that solves the following problem.

**Problem 4.7** Given a set $L$ of $n$ lines and a parameter $k$, we would like to maintain a vertical decomposition of the plane, such that each trapezoid $\psi$ in this decomposition maintains the sampling data-structure of Lemma 4.4 for the vertices of $A(L)$. This data-structure should support insertions of up to $O(k \log n)$ double-wedges. Here, each trapezoid maintains its support, depth, and mass, see Definition 4.5.

**The basic scheme**

**Lemma 4.8** One can maintain a data-structure for Problem 4.7, over $O(k \log n)$ insertions, with total running time $O((k^3 + nk) \log^3 n)$.

**Proof**  Let $R$ be a random sample of $L$ of size $K = O(k \log n)$, where $L$ is the set of $n$ lines that are dual to the original set of points. Compute the vertical decomposition of $R$. For each trapezoid $\psi$ in this decomposition, we compute the conflict list of $\psi$ (i.e., the set of lines from $L$ intersecting the interior of $\psi$). This can be done in $O(K^2 + Kn)$ time, using standard algorithms, see [12]. Next, the algorithm computes for each trapezoid the data-structure of Lemma 4.4.

By the $\varepsilon$-net theorem, every vertical trapezoid that does not intersect a line of $R$ in its interior intersects at most $\varepsilon n$ lines of $L$ (where $\varepsilon = 1/4k$). This property holds with high probability. As such, the conflict lists that the algorithm deals with are of size $O(n/k)$.

Let $L_0 = R$. In the $i$th iteration, the $i$th double-wedge $D_i$ is inserted. To this end, the two lines $\ell_i, \ell'_i$ bounding the double wedge are inserted into the current vertical decomposition, splitting and merging trapezoids as necessary. At the end of this process we have the vertical decomposition of $L_i = L_{i-1} \cup \{\ell_i, \ell'_i\}$. This involves creating $O(K + i)$ new trapezoids, since the zone complexity of a line in $A(L_{i-1})$ is $O(K + i) = O(K)$, and $i = O(K)$. For each such trapezoid we rebuild the data-structure of Lemma 4.4, which takes overall $O((n/k) \log(n/k) \cdot K) = O(n \log^2 n)$ time. Finally, we scan all the vertical trapezoids, and update their depth count, if they are contained inside the inserted wedge. This takes (naively) $O(K^2)$ time.
Recall that we perform $O(K)$ insertions in total, and therefore the overall running time of the data-structure is $O\left(K (K^2 + n \log^2 n)\right) = O((k^3 + nk) \log^3 n)$.

**A more efficient scheme.** The overall running time of Lemma 4.8 can be further improved by using dynamic partition trees.

**Lemma 4.9** One can maintain a data-structure for Problem 4.7 with overall running time $O(nk \log^3 n + k^2 \log^{O(1)} n)$. (This running time includes $O(k \log n)$ double-wedge insertions.) Furthermore, one can sample a random vertex of $A(L)$ according to their weight in $O(\log n)$ time.

**Proof** Partition trees are used to maintain the depth of the vertical trapezoids. This maintenance step is the bottleneck in the scheme of Lemma 4.8, since the algorithm must scan all of the existing trapezoids to update their depth after each insertion of a double wedge.

A partition tree is a hierarchical partition of the point set, until each leaf has a constant number of points. Each node uses a partition (see Definition 2.6) to break its point set into subsets, and for each subset a partition tree is constructed recursively. Performing a simplex query in a partition tree is done by starting at the root, inspecting its children simplices. If such a simplex $\Delta$ lies entirely within the query, the algorithm reports the number of points inside it. Otherwise if $\Delta$ intersects the query, the algorithm recurses on that child node. Given a set of $n$ points in $\mathbb{R}^2$, Matoušek showed that one can construct a partition tree in $O(n \log n)$ time and return the number of points inside the simplex query in time $O(\sqrt{n} \log^{O(1)} n)$ [25].

For our purposes, we pick a point inside a vertical trapezoid (in the current vertical decomposition) to represent it. Overall, there are $m = O(K^2) = O(k^2 \log^2 n)$ representatives at any given time. We next build the data-structure of Matoušek [25] to dynamically maintain this point-set under insertions and deletions (each operation takes amortized $O(\log^2 m)$ time). Updating the weight of a trapezoid corresponds to two simplex queries, where we have to increase the depth count for the canonical sets reported by this range-searching query. There are $O(\sqrt{m} \log^{O(1)} m) = O(k \log^{O(1)} n)$ such canonical sets, and this is the time to perform such an update. Thus an insertion of a double wedge with respect to this partition tree takes $O(K \log^2 K + k \log^{O(1)} n)$ time. Therefore, over the $O(K)$ insertions, the algorithm requires $O(k^2 \log^{O(1)} n)$ time to maintain the weights of the vertices of $A(L)$.

Using the above, and the sampling data structure of Lemma 4.6, implies the claim.

**4.2.4 Putting Everything Together**

**Remark 4.10** (More efficient point-location) Each iteration of the algorithm needs to solve the following subproblem. Given a set of $m$ lines $L$ and $n$ points $P$, compute for each point $p \in P$ the face of $A(L)$ containing $p$. Agarwal et al. [3] describe an algorithm for this problem with running time $O\left((n + m + n^{2/3}m^{2/3}) \log n\right)$. This subroutine can be applied in each iteration of our algorithm on the $m = O(k \log k)$ lines sampled. Substituting the value for $m$ in the preceding bound, we conclude the subroutine can be completed in time $O(n \log n + k \log^2 n + n^{2/3}k^{2/3} \log^2 n)$.  

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Theorem 4.11  Given a set $P$ of $n$ points in the plane, one can compute a set of $O(\sigma \log \sigma)$ lines that separates all the points of $P$, where $\sigma$ is the minimal set of lines that separates $P$. The overall expected running time of this algorithm is $O\left(n^{2/3}\sigma^{5/3}\log^{O(1)} n\right)$.

Proof We implement the algorithm of Lemma 4.3 using the data-structure of Lemma 4.9 to maintain the vertices of the dual arrangement, and use the point-location data-structure of Remark 4.10. For a fixed value of $k$, the algorithm performs $O\left(k \log n\right)$ inner iterations, and the resulting running time is

$$O\left((n + k + n^{2/3}k^{2/3})k \log^3 n + nk\log^3 n + k^2 \log^{O(1)} n\right)$$

$$= O\left(nk \log^3 n + n^{2/3}k^{5/3}\log^{O(1)} n\right).$$

Summing the above bound over exponentially growing values of $k$, ending at $k = O(\sigma)$ (as in the proof of Lemma 4.3), the overall running time is $O\left(n\sigma \log^3 n + n^{2/3}\sigma^{5/3}\log^{O(1)} n\right)$. Observe that by Remark 2.3(B), $\sigma = \text{sep}(P) = \Omega(\sqrt{n})$ which implies that the second term is bigger than the first term. The result then follows.  

Remark 4.12  To appreciate Theorem 4.11, consider the grid-like case where $\sigma = O(\sqrt{n})$. The running time then becomes $O\left(n^{3/2}\log^{O(1)} n\right)$, which is well below quadratic time. The worst case for this algorithm is when $\sigma = \Omega(n)$ (for example, if the input points are in convex position), where the running time becomes $O\left(n^{7/3}\log^{O(1)} n\right)$.

5 Conclusion

Two interesting open problems suggested by our work are the following:

(A) Improve the lower bound of Theorem 3.14, so it matches the upper bound up to a constant.

(B) Improve the running time of the approximation algorithm of Theorem 4.11, or prove a matching lower bound in some reasonable model.

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