On badly approximable subspaces

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1. INTRODUCTION

In a remarkable paper [2] devoted to the dynamical approach to Diophantine approximations on submanifolds D. Kleinbock among other results observed the following phenomenon. If an affine subspace of Euclidean space contains a vector which is "not very well approximable" by rational vectors, then almost all vectors in this affine subspace are "not very well approximable". A similar result holds for manifolds and was generalized in different directions (see [3], [4] and [9]). In [5] N. Moshchevitin gave a simple proof of the result by Kleinbock. In the present paper we prove a general result about badly approximable subspaces of a certain linear subspace of $\mathbb{R}^d$. The result by Moshchevitin from [5] is a particular case of our main theorem.

1.1. Badly approximable matrices and linear subspaces. Let

(1)

$$\Theta = \begin{pmatrix}
\theta_{1,1} & \ldots & \theta_{1,m} \\
\vdots & \ddots & \vdots \\
\theta_{n,1} & \ldots & \theta_{n,m}
\end{pmatrix}$$

be a real matrix and $\psi(t)$ be a real valued function decreasing to zero as $t \to \infty$. Matrix $\Theta$ is called $\psi$-badly approximable if for any integer vector $x = (x_1, \ldots, x_m) \in \mathbb{Z}^m$ one has

$$\max_{1 \leq j \leq n} ||\theta_{j,1}x_1 + \ldots \theta_{j,m}x_m|| \geq \psi(|x|).$$

Here $||\xi|| = \min_{x \in \mathbb{Z}} |\xi - x|$ is the distance from real $\xi$ to the nearest integer and $|\xi| = \max_{1 \leq i \leq m} |\xi_i|$ stands for the sup-norm of the vector $\xi = (\xi_1, \ldots, \xi_m) \in \mathbb{R}^m$. The irrationality measure function $\psi_\Theta(t)$ associated with matrix $\Theta$ is defined as

(2)

$$\psi_\Theta(t) = \min_{x \in \mathbb{Z}^m} \max_{0 < |x| \leq t} \max_{1 \leq j \leq n} ||\theta_{j,1}x_1 + \ldots \theta_{j,m}x_m||, \quad t \geq 1.$$ 

Then matrix $\Theta$ is $\psi$-badly approximable if and only if

$$\psi_\Theta(t) \geq \psi(t), \quad \forall t \geq 1.$$ 

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We should note that for any positive $\varepsilon$ almost all matrices $\Theta$ are $\psi$-badly approximable (with respect to Lebesgue $(m \times n)$-dimensional measure) with $\psi(t) = \gamma t^{-m/n-\varepsilon}$, for some $\gamma > 0$. Moreover the set

$$\{ \Theta : \Theta \text{ is } \gamma t^{-m/n}-\text{badly approximable for some } \gamma > 0 \}$$

is of zero measure and has full Hausdorff dimension (see [6, 7]).

However, it will be more convenient for us to consider a little bit different irrationality measure function.

Let $d = m + n$ and $\mathbb{R}^d$ be $d$-dimensional Euclidean space with coordinates

$$z = (x, y) = (x_1, \ldots, x_m, y_1, \ldots, y_n).$$

For a proper linear subspace $\mathfrak{B} \subset \mathbb{R}^d$ we consider the function

$$\psi_{\mathfrak{B}}(t) = \min_{z \in \mathbb{Z}^d, 1 \leq |z| < t} \text{dist}(z, \mathfrak{B}),$$

where $\text{dist}(A, B)$ denotes the Euclidean distance between the sets $A$ and $B$.

We define a proper linear subspace $\mathfrak{B} \subset \mathbb{R}^d$ to be $\psi$-badly approximable if

$$\psi_{\mathfrak{B}}(t) \geq \psi(t), \quad \forall t \geq 1.$$

Now, for the matrix $\Theta$ we consider the linear subspace

$$\mathcal{L}_\Theta = \{ z = (x_1, \ldots, x_m, y_1, \ldots, y_n) \in \mathbb{R}^{m+n} : y = \Theta x \}.$$

Here we should note that the projection $z = (x_1, \ldots, x_m, y_1, \ldots, y_n) \mapsto (x_1, \ldots, x_m)$ gives a bijection between $\mathcal{L}_\Theta$ and coordinate subspace $\{y_1 = \ldots = y_n = 0\}$. It is clear that

$$\kappa_1 \psi_\Theta(\kappa_2 t) \leq \psi_{\mathcal{L}_\Theta}(t) \leq \kappa_3 \psi_\Theta(\kappa_4 t)$$

with some positive $\kappa_j$ depending on $\Theta$.

So, if the subspace $\mathfrak{B} = \mathcal{L}_\Theta$ is of the form (4), then the property of $\mathfrak{B}$ to be $\psi$-badly approximable means that

$$\psi_{\Theta} \geq \kappa' \psi(\kappa'' t)$$

with certain positive $\kappa'$, $\kappa''$. Similarly $\Theta$ is $\psi$-badly approximable matrix means that

$$\psi_{\mathcal{L}_\Theta} \geq \kappa^* \psi(\kappa^{**} t)$$

with certain positive $\kappa^*$, $\kappa^{**}$. 
1.2. **Diophantine exponent.** For a real matrix $\Theta$ we define the **Diophantine exponent** as

$$\omega(\Theta) = \sup \{ \tau : \liminf_{t \to \infty} t^\tau \psi_\Theta(t) < +\infty \}.$$ 

From the discussion of previous subsection it is clear that if $B = L_\Theta$ then

$$\omega(B) := \sup \{ \gamma : \liminf_{t \to \infty} t^\gamma \psi_B(t) < \infty \} = \omega(\Theta).$$

If $B = L_\Theta$, so the equality $\omega = \omega(B)$ means that $\forall \tau < \omega(B)$ there exists infinitely many $z = (x_1, \ldots, x_m, y_1, \ldots, y_n) \in \mathbb{R}^d$ such that

$$\max_{1 \leq j \leq n} |\theta_{j,1}x_1 + \ldots + \theta_{j,m}x_m - y_j| < \left( \max_{1 \leq i \leq m} |x_i| \right)^{-\tau},$$

or

$$\omega(B) = \inf \{ \rho : \Theta \text{ is } t^{-\rho}-badly \text{ approximable} \}.$$ 

1.3. **Parametrization of subspaces.** Our results deal with the following situation. Given a linear $a$-dimensional subspace $\mathfrak{A} \subset \mathbb{R}^d$ we want to get a statement about *almost all* $c$-dimensional linear subspaces $\mathfrak{C} \subset \mathfrak{A}$. So for a fixed $\mathfrak{A}$ we should give a parametrization of all linear $c$-dimensional subspaces of $\mathfrak{A}$.

First of all we choose Euclidean coordinates

$$w = (u, v), \ u = (u_1, \ldots, u_a), \ v = (v_1, \ldots, v_{d-a})$$

in $\mathbb{R}^d$ in such a way that the subspace $\mathfrak{A}$ is defined by

$$v_1 = \ldots = v_{d-a} = 0.$$ 

We shall use these coordinates everywhere in our proofs. In these coordinates the lattice of integer points in $\mathbb{R}^d$ will be denoted as $\Lambda$. It is clear that its fundamental volume is equal to one.

Given a $c$-dimensional linear subspace $\mathfrak{C} \subset \mathfrak{A}$. There exists a coordinate subspace

$$\mathbb{R}^c = \{ u_{i_1} = \ldots = u_{i_{a-c}} = 0 \}$$

of $\mathfrak{A}$ such that the projection $\mathfrak{C} \to \mathbb{R}^c$ is bijective. So we restrict ourselves with the case when

$$\mathfrak{C} = \mathfrak{C}(\Theta) = \{ u = (x, y) : x \in \mathbb{R}^c, y \in \mathbb{R}^{a-c} \text{ such that } y = \Theta x \}$$

where $\Theta = (\theta_{ij})$ is a $(a - c) \times c$-matrix with real entries, as it was done for example in [1]. If $\mathfrak{C}$ satisfies (5) for the corresponding matrix $\Theta$ we write $\Theta = \Theta(\mathfrak{C})$.

So in the present paper, when we speak about almost all $c$-dimensional linear subspace $\mathfrak{C} \subset \mathfrak{A}$ we mean almost all $(a - c) \times c$-matrices with respect to Lebesgue measure in $\mathbb{R}^{(a-c)\times c}$. An alternative approach to
consider the variety of all $c$-dimensional subspaces of $\mathfrak{A}$ is connected with Grassman coordinates (see [8], Ch. 1, §5).

1.4. The main result for Diophantine exponents. Here we formulate our main results in terms of Diophantine exponents of linear subspaces of $\mathbb{R}^d$.

**Theorem 1.** Let $\mathfrak{A}$ be a $d$-dimensional linear subspace of $\mathbb{R}^d$. Suppose that $\mathfrak{B}$ is a $b$-dimensional linear subspace of $\mathfrak{A}$ with $\omega(\mathfrak{B}) = \omega$.

1) If $c < b$, then

$$\omega(\mathfrak{C}) \leq \omega + \frac{(\omega + 1) \cdot (c - b)}{d - c} = \frac{\omega \cdot (d - b) + c - b}{d - c}$$

for almost all $c$-dimensional linear subspaces $\mathfrak{C} \subset \mathfrak{A}$.

2) If $c \geq b$, then

$$\omega(\mathfrak{C}) \leq \omega + \frac{(\omega + 1) \cdot (c - b)}{a - c} = \frac{\omega \cdot (a - b) + c - b}{a - c}$$

for almost all $c$-dimensional linear subspaces $\mathfrak{C} \subset \mathfrak{A}$.

Theorem 1 means that if there exists $b$-dimensional linear subspace $\mathfrak{B} \subset \mathfrak{A}$ which doesn’t have good rational approximations then almost all $c$-dimensional subspaces $\mathfrak{C} \subset \mathfrak{A}$ don’t have good rational approximations also. Theorem 1 is a particular case of more general result (Theorem 2), which will be formulated in the next section.

1.5. An example. We do not want to give here examples of application our results for $b = 1$ as this case was completely considered in [5]. To illustrate our theorem we consider the following example.

Let us deal with 2-dimensional badly approximable subspace in $\mathbb{R}^4$. It is clear that there exists $T^{-\beta}$-badly approximable 2-dimensional subspace with $\beta = 1$.

However if we would like to find a badly approximable two-dimensional subspace in a three-dimensional subspace $\mathfrak{A} \subset \mathbb{R}^4$, dim $\mathfrak{A} = 3$, it may happen that we be able to find $T^{-\beta}$ badly approximable 2-dimensional subspace with $\beta = 2$ only. Indeed, this happens if we take $\mathfrak{A} \subset \mathbb{R}^4$ to be a completely rational 3-dimensional linear subspace.

Our Theorem 1 gives the following results. Suppose that $\mathfrak{A} \subset \mathbb{R}^4$ is a 3-dimensional linear subspace of $\mathbb{R}^4$ such that there exists a vector $\mathbf{w} = (1, w_1, w_2, w_3) \in \mathfrak{A}$ with

$$\max_{1 \leq i \leq 3} ||w_i q|| \geq \frac{c}{q^{1/\beta}}, \quad c > 0$$
so, 1-dimensional linear subspace spanned by \( w \) is \( cT^{-\frac{3}{2}} \)-badly approximable. Then almost all 2-dimensional linear subspace \( \mathcal{C} \subset \mathfrak{A} \) are \( \gamma T^{-\beta} \)-badly approximable with any \( \beta > \frac{5}{3} \) and some positive \( \gamma \).

2. General result

2.1. Functions and examples. Let \( \psi(T) \) and \( \varphi(T), T \geq 1 \) be two positive valued functions monotonically decreasing to zero as \( T \to \infty \). Here we deal with \( \psi \)-badly approximable \( b \)-dimensional linear subspace \( \mathcal{B} \subset \mathbb{R}^d \). We may suppose that

\[
\psi(T) \leq T^{-\frac{b}{d}}.
\]

For this functions we define quantities

\[
\lambda_T = \frac{\varphi^{a-c}(T)}{T^{a-c}} - \frac{\varphi^{a-c}(T+1)}{(T+1)^{a-c}} \geq 0
\]

and

\[
\mu_T = \left( \frac{T}{\psi(T)} \right)^{a-b} \cdot \max \left\{ 1, \left( \frac{\varphi(T)}{\psi(T)} \right)^{d-a} \right\}.
\]

**Theorem 2.** Let the series

\[
\sum_{T=M}^{+\infty} \lambda_T \mu_T
\]

converges. Suppose that \( \mathcal{B} \) is \( b \)-dimensional \( \psi \)-badly approximable linear subspace. Then for each \( a \)-dimensional linear subspace \( \mathcal{A} \supset \mathcal{B} \) almost all \( c \)-dimensional linear subspaces \( \mathcal{C} \subset \mathcal{A} \) are \( \varphi \)-badly approximable subspaces.

**Remark.** For \( b = 1 \) the result of Theorem 2 coincides with the main result form the paper [5].

To illustrate our Theorem 2 we consider the function \( \psi(T) = T^{-\beta} \log^B T \). Then we can choose \( \varphi = T^{-\gamma} \log^C T \) with

\[
\gamma > \frac{\beta \cdot (d - b) + c - b}{d - c} \quad \text{where } C \text{ and } B \text{ are arbitrary}
\]

or

\[
\gamma = \frac{\beta \cdot (d - b) + c - b}{d - c} \quad \text{and } C(d - c) - B(d - b) < -1,
\]

for the case when \( c < b \), or
\[
\gamma > \frac{\beta \cdot (a - b) + c - b}{a - c} \quad \text{where } C \text{ and } B \text{ are arbitrary}
\]

or

\[
\gamma = \frac{\beta \cdot (a - b) + c - b}{a - c} \quad \text{and } C(a - c) - B(a - b) < -1,
\]

for the case when \( c \geq b \).

For these functions \( \varphi \) and \( \psi \) the conclusion of the Theorem 2 is valid. In particular the choice of

\[
\psi(T) = t^{-\omega^{(B)} - \varepsilon}, \quad \varepsilon > 0
\]

shows that the Theorem 1 from the previous section follows from Theorem 2.

3. Proofs

3.1. Simple lemmas. Consider the sets

\[
\Omega_T = \{ \mathbf{w} = (w_1, \ldots, w_d) \in \mathbb{R}^d : |\mathbf{w}| \leq T, \inf_{b \in B} \text{dist } (\mathbf{w}, b) \leq \psi(T) \}
\]

and

\[
\Pi_T = \{ \mathbf{w} = (w_1, \ldots, w_d) \in \mathbb{R}^d : |\mathbf{w}| \leq T, \inf_{a \in A} \text{dist } (\mathbf{w}, a) \leq \varphi(T) \}.
\]

Lemma 1. Let \( N_T \) be the minimal number of points \( c_j, 1 \leq j \leq N_T \) such that

\[
\Pi_T \subset \bigcup_{j=1}^{N_T} \left( \frac{1}{2} \Omega_T + \mathbf{c}_j \right).
\]

Then \( N_T \ll \mu_T \), where \( \mu_T \) is defined in (6).

Proof. As \( \Pi_T, \Omega_T \subset E_T = \{ \mathbf{x} : |\mathbf{x}| < T \} \), by \( O \left( \left( \frac{T}{\psi(T)} \right)^{a-b} \right) \) shifted sets \( \frac{1}{2} \Omega_T + \mathbf{c}_j \) we are able to cover \( \frac{\psi(T)}{4} \) neighbourhood of \( A \cap E_T \).

So, in the case \( \varphi \leq \psi \) by \( O \left( \left( \frac{T}{\psi(T)} \right)^{a-b} \right) \) shifted sets \( \frac{1}{2} \Omega_T + \mathbf{c}_j \) we will cover the whole \( \Pi_T \). So,

\[
N_T \ll \left( \frac{T}{\psi(T)} \right)^{a-b}.
\]
and in this case lemma is proved. In the case \( \varphi > \psi \), to cover \( \varphi(T) \)-neighbourhood of \( \mathfrak{A} \cap E_T \) we need to cover \( O \left( \frac{\varphi(T)}{\psi(T)} \right)^{d-a} \) shifted copies of \( \psi(T) \)-neighbourhood of \( \mathfrak{A} \cap E_T \). So in this case

\[
N_T \ll \frac{T^{a-b} \cdot \varphi^{d-a}(T)}{\psi^{d-b}(T)}.
\]

and everything is proved. □

**Lemma 2.** Suppose that \( \mathfrak{B} \) is \( \psi \)-badly approximable. Then for any \( T > 1 \) and for any \( c \in \mathbb{R}^d \) the set \( \frac{1}{2} \times \Omega_T + c \) consists of not more than one lattice point.

**Proof.** Since \( \mathfrak{B} \) is a \( \psi \)-badly approximable subspace it is clear that for any \( T \)

\[
\Omega_T \cap \Lambda = \{0\}.
\]

Consider any translation of the \( 1/2 \)-contracted set

\[
\frac{1}{2} \times \Omega_T + c, \quad c \in \mathbb{R}^d.
\]

If two different integer points \( \mathbf{w}, \mathbf{t} \) belong to the same set of this form, then

\[
0 \neq \mathbf{w} - \mathbf{t} \in \Omega_T.
\]

But this is not possible. □

**3.2. Proof of Theorem 2.** As was explained in the section 1.4 we are dealing with \( a \)-dimensional space \( \mathfrak{A} \) with coordinates

\[
x_1, \ldots, x_c, y_1, \ldots, y_{a-c}.
\]

It is clear that any \( c \)-dimensional subspace under consideration \( \mathfrak{C} \subset \mathfrak{A} \) can be defined by the system of \( a-c \) equations

\[
\begin{align*}
\theta_{1,1} x_1 + \ldots + \theta_{1,c} x_c &= y_1 \\
&\vdots \\
\theta_{a-c,1} x_1 + \theta_{a-c,c} x_c &= y_{a-c}.
\end{align*}
\]

We take \( R > 1 \) and suppose that

(8) \[ \max_{i,j} |\theta_{i,j}| \leq R. \]

Consider \( (a-1) \)-dimensional subspaces

\[
\mathfrak{C}_j = \{ (x, y) \in \mathfrak{A} : \theta_{j,1} x_1 + \ldots + \theta_{j,c} x_c = y_j \}.
\]

Then

\[
\mathfrak{C} = \bigcap_{1 \leq j \leq a-c} \mathfrak{C}_j.
\]
Let $z \in \mathbb{Z}^d$ be an integer point. Denote by $z^* = z|_A$ the orthogonal projection of $z$ onto $A$. Consider the neighbourhood

$$U_{\varphi(|z|)}(z) = \{z' \in A : \text{dist}(z', z) < \varphi(|z|)\}.$$ 

We are interested in those $z$ for which

$$A \cap U_{\varphi(|z|)}(z) \neq \emptyset.$$ 

These $z$ are close $A$ and for them $|z| \asymp |z^*|$. Consider the set of $c$-dimensional subspaces

$$L_{z^*} = \{C = C(\Theta) \subset A : \Theta \text{ satisfies } (S), \text{ dist}(C, z^*) < \varepsilon\}.$$ 

Our parametrization (see Section 1.4) allows to consider

$$\Theta(L_{z^*}) = \{\Theta : \Theta = \Theta(C), C \in L_{z^*}\}$$

as a subsets of $\mathbb{R}^{(a-c) \times c}$. The condition $\text{dist}(C, z^*) < \varepsilon$ leads to

$$\text{dist}(C_i, z^*) = \frac{\left|\theta_{i,1}z_1^* + \ldots + \theta_{i,c}z_c^* - z_{c+1}\right|}{\sqrt{\theta_{i,1}^2 + \ldots + \theta_{i,c}^2 + 1}} < \varepsilon \quad \forall i = 1, \ldots, a - c.$$ 

So

$$\Theta(L_{z^*}) \subset \hat{L}_{z^*}(R, \varepsilon) = \{\Theta \in \mathbb{R}^{(a-c) \times c} : \text{dist}(C_i, z^*) = \left|\sum_{j=1}^{c} \theta_{ij}z_j^* - z_{c+1}\right| < \sqrt{c + 1}R\varepsilon \quad \forall i = 1, \ldots, a - c\}.$$

Now we consider $\theta_{ij}$ as a variables. Then equations

$$\theta_{i,1}z_1^* + \ldots + \theta_{i,c}z_c^* - z_{c+1} = 0 \quad \forall i = 1, \ldots, a - c.$$ 

define an affine $(a - c) \cdot (c - 1)$-dimensional subspace

$$\{\Theta \in \mathbb{R}^{(a-c) \times c} : \text{dist}(C_i, z^*) = 0 \quad \forall i = 1, \ldots, a - c\}.$$ 

So the set $L_{z^*}$ is a neighbourhood of this affine subspace. Now for $(a - c) \times c$-dimensional Lebesgue measure we have

$$\mu(\hat{L}_{z^*}) \ll \left(\frac{R\varepsilon}{\max_{1 \leq i \leq c}|z_i^*|}\right)^{a-c} R^c \ll \frac{R^{a+2\varepsilon} a-c}{|z|^{a-c}}.$$ 

In the last inequality we use the upper bound $|z_{c+1}^*| \ll R(|z_1^*| + \ldots + |z_c^*|)$ for all $i \in \{1, \ldots, a - c\}$ which follows form $(S)$. Now we suppose that $C \subset A$ is not a $\varphi$-badly approximable linear subspace. Then there exist infinitely many $z \in \Lambda \setminus \{0\}$ such that $\text{dist}(z, C) < \varphi(|z|)$ and then $C$ belongs to the set

$$\{C = C(\Theta) : \Theta \text{ satisfies } (S), \forall M \in \mathbb{N} \ \exists z \in \Lambda \setminus \{0\} : |z| > M \text{ such that } U_{\varphi(|z|)}(z) \cap C \neq \emptyset\} \subset$$
By (9) for all \( M \in \mathbb{N} \) for \((a - c) \times c\)-dimensional Lebesgue measure we have
\[
\mu \left( \bigcap_{M \in \mathbb{N}} \bigcup_{z \in \Lambda \setminus \{0\}} \bigcup_{|z| > M} \hat{L}_z^*(R, \varphi(|z|)) \right) \leq \sum_{z \in \Lambda \setminus \{0\}} \mu \left( \hat{L}_z^*(R, \varphi(|z|)) \right) \ll R^{a+2} \sum_{T=M}^{W} \frac{\varphi^{a-c}(T)}{|z| a-c} \leq R^{a+2} \sum_{T=M}^{+\infty} \frac{\varphi^{a-c}(T)}{T a-c} K_T
\]
where \( K_T = \#\{z \in \Lambda : |z| = T, \ Z \cap U_{\varphi(|z|)}(z) \neq \emptyset\} \). By partial summation for \( W > M \) we get
\[
\sum_{T=M}^{W} \frac{\varphi^{a-c}(T)}{T a-c} K_T = \sum_{T=M}^{W} \lambda_T \cdot \sum_{j=M}^{T} K_j - \frac{\varphi^{a-c}(W+1)}{(W+1)^{a-c}} \sum_{T=M}^{W+1} K_j < \sum_{T=M}^{W} \lambda_T \sum_{j=M}^{T} K_j \leq \sum_{T=M}^{W} \lambda_T H_T;
\]
where
\[
H_T = \#\{z \in \Lambda : |z| \leq T, \ Z \cap U_{\varphi(|z|)}(z) \neq \emptyset\}
\]
and \( \sum_{j \leq T} K_j \leq H_T \). We see that
\[
\{ z \in \Lambda : |z| \leq T, \ Z \cap U_{\varphi(|z|)}(z) \neq \emptyset \} \subset \{ z \in \Lambda : |z| \leq T, \ P_T \cap U_{\varphi(|z|)}(z) \neq \emptyset \}.
\]
So \( H_T \leq N_T \) by Lemma 2. But \( N_T \ll \mu_T \) by Lemma 1. Now
\[
\mu \left( \bigcap_{M \in \mathbb{N}} \bigcup_{z \in \Lambda \setminus \{0\}} \bigcup_{|z| > M} \hat{L}_z^*(R, \varphi(|z|)) \right) \leq \sum_{T=M}^{+\infty} \lambda_T \mu_T
\]
for any \( M \). By the condition of the Theorem 2 the series from right hand side converges. Thus, by Borel-Cantelli lemma argument the measure of the set of subspaces \( C \subset Z \) which are not \( \varphi \)-badly approximable is zero.
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