INVASION ENTIRE SOLUTIONS IN A TIME PERIODIC LOTKA-VOLTERRA COMPETITION SYSTEM WITH DIFFUSION

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Abstract. This paper is concerned with invasion entire solutions of a monostable time periodic Lotka-Volterra competition-diffusion system. We first give the asymptotic behaviors of time periodic traveling wave solutions at infinity by a dynamical approach coupled with the two-sided Laplace transform. According to these asymptotic behaviors, we then obtain some key estimates which are crucial for the construction of an appropriate pair of sub-super solutions. Finally, using the sub-super solutions method and comparison principle, we establish the existence of invasion entire solutions which behave as two periodic traveling fronts with different speeds propagating from both sides of x-axis. In other words, we formulate a new invasion way of the superior species to the inferior one in a time periodic environment.

1. Introduction. In this paper, we consider the following time periodic Lotka-Volterra competition-diffusion system

\[
\begin{align*}
    u_t &= u_{xx} + u(r_1(t) - a_1(t)u - b_1(t)v), \\
    v_t &= d v_{xx} + v(r_2(t) - a_2(t)u - b_2(t)v),
\end{align*}
\]

where \( u = u(t,x) \) and \( v = v(t,x) \) denote the densities of two competing species at time \( t \in \mathbb{R}^+ \) and \( x \in \mathbb{R} \), \( d \in (0,1] \) denotes the relatively diffusive coefficient of the two species, \( r_i(t), a_i(t) \) and \( b_i(t) \) are \( T \)-periodic continuous functions, \( a_i(\cdot) \) and \( b_i(\cdot) \) are positive in \( [0,T] \), and \( \overline{\tau} := \frac{1}{T} \int_0^T r_i(t) dt > 0 \), where \( i = 1,2 \). Systems like (1.1) arise in interactive populations which live in a fluctuating environment, for instance, physical environmental conditions such as temperature and humidity and the availability of food, water and other resources usually vary in time with seasonal or daily variations [45]. Time periodic traveling waves of (1.1) are solutions with the form

\[
\begin{pmatrix}
    u(t,x) \\
    v(t,x)
\end{pmatrix} = \begin{pmatrix}
    X(t,x - ct) \\
    Y(t,x - ct)
\end{pmatrix}
\]

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satisfying
\[
\begin{pmatrix}
X(t+T, z) \\
Y(t+T, z)
\end{pmatrix} = \begin{pmatrix}
X(t, z) \\
Y(t, z)
\end{pmatrix}
\]
and
\[
\begin{pmatrix}
X(t, \pm \infty) \\
Y(t, \pm \infty)
\end{pmatrix} := \lim_{z \to \pm \infty} \begin{pmatrix}
X(t, z) \\
Y(t, z)
\end{pmatrix} = \begin{pmatrix}
u^+(t) \\
v^-(t)
\end{pmatrix},
\]
where \(c \in \mathbb{R}\) is the wave speed, \(z = x - ct\) is the co-moving frame coordinate, and \((u^\pm(t), v^\pm(t))\) are periodic solutions of the corresponding kinetic system
\[
\begin{align*}
\frac{du}{dt} &= u(r_1(t) - a_1(t)u - b_1(t)v), \\
\frac{dv}{dt} &= v(r_2(t) - a_2(t)u - b_2(t)v).
\end{align*}
\]
which implies that (1.2) has only three nonnegative T-periodic solutions \((0, 0)\), \((p(t), 0)\) and \((0, q(t))\), with \((p(t), 0)\) globally stable and \((0, q(t))\) unstable in the positive quadrant \(\mathbb{R}^2_+\) = \{\(u, v\) | \(u \geq 0, v \geq 0\}\}, where \(p(t)\) and \(q(t)\) are given by

\[
\begin{align*}
\begin{cases}
p(t) = \frac{\rho_0 e^{\int_0^t r_1(s)ds}}{1 + \rho_0 \int_0^t e^{\int_0^s r_1(r)dr} a_1(s)ds}, \\
q(t) = \frac{\rho_0 e^{\int_0^t r_2(s)ds}}{1 + \rho_0 \int_0^t e^{\int_0^s r_2(r)dr} b_2(s)ds},
\end{cases}
\end{align*}
\]

For system (1.1), the time periodic traveling wave solution \((X(t, z), Y(t, z))\) connecting \((0, q(t))\) and \((p(t), 0)\) actually satisfies

\[
\begin{align*}
X_t &= X_{zz} + c X_z + X(r_1(t) - a_1(t)X - b_1(t)Y), \\
Y_t &= dY_{zz} + c Y_z + Y(r_2(t) - a_2(t)X - b_2(t)Y), \\
(X(t, z), Y(t, z)) &= (X(t + T, z), Y(t + T, z)), \\
\lim_{z \to -\infty} (X, Y) &= (0, q(t)), \quad \lim_{z \to +\infty} (X, Y) = (p(t), 0).
\end{align*}
\]

In the past few years, there were a few works devoted to the study of this issue. In particular, Zhao and Ruan [43] established the existence, uniqueness and stability of time periodic traveling waves under the monostable assumption (1.3). In 2014, the authors extended the results to a class of more general time-periodic advection-reaction-diffusion systems in [44]. In addition, Bao and Wang [3] obtained the existence and stability of time periodic traveling waves for the bistable case. Very recently, Bao et al. [2] further studied the existence, non-existence and asymptotic stability of bistable time-periodic traveling curved fronts in two-dimensional spatial space.

In our present paper, we shall consider the invasion entire solutions of system (1.1), that is, an entire solution \((u(t, x), v(t, x))\) satisfying (1.1) as well as the following conditions

\[
\begin{align*}
\lim_{t \to -\infty} \{|u(t, x)| + |v(t, x)| - q(t)|\} &= 0 \text{ locally in } x \in \mathbb{R}, \\
\lim_{t \to +\infty} \{|u(t, x) - p(t)| + |v(t, x)|\} &= 0 \text{ locally in } x \in \mathbb{R}.
\end{align*}
\]

For future reference, we denote a vector by \(u = (u_1, \ldots, u_n)\), where \(u_i\) stands for the \(i\)th component of \(u\). Let \(I, \Gamma \subset \mathbb{R}\) be two (possibly unbounded) intervals and \(M \subset \mathbb{R}^n\). Denote by \(C(I \times \Gamma, M)\) the space of continuous functions \(u : I \times \Gamma \to M\). \(C_b(I \times \Gamma, M)\) the space of functions \(u \in C(I \times \Gamma, M)\) with \(|u|_{\infty} < \infty\), \(C^{k,l}(I \times \Gamma, M)\) the space of functions \(u \in C(I \times \Gamma, M)\) such that \(u(\cdot, x)\) is \(k\)-time continuously differentiable and \(u(t, \cdot)\) is \(l\)-time continuously differentiable, \(C^{k,l}_b(I \times \Gamma, M)\) the space of functions \(u \in C^{k,l}(I \times \Gamma, M)\) such that all partial derivatives of \(u\) are uniformly bounded. Throughout the paper, we always assume that

(A1): \(r_i, a_i, b_i \in C^0([0,\infty), \mathbb{R})\) for some \(0 < \theta < 1\), \(r_i(t + T) = r_i(t), a_i(t + T) = a_i(t), b_i(t + T) = b_i(t), i = 1, 2\),

(A2): \(\tau > 0, a_i(t) > 0, b_i(t) > 0\) for all \(t \in [0, T]\). Moreover, \(\tau > \max_{t \in [0, T]} (\frac{b_1}{b_2})\tau_2\) and \(\tau \leq \min_{t \in [0, T]} (\frac{\theta}{b_1})\tau_2\),

(A3): \(a_1(t)p(t) - b_1(t)q(t) \geq a_2(t)p(t) - b_2(t)q(t) \geq 0\) for all \(t \in [0, T]\).
Now let \( u^*(t, x) = \frac{u(t, x)}{p(t)} \) and \( v^*(t, x) = \frac{q(t) - v(t, x)}{q(t)} \), then (1.1) becomes (omitting * for simplicity)

\[
\begin{align*}
& \frac{du}{dt} = u_{xx} + a_1 p u [1 - N_1(t) - u + N_1(t) v], \\
& \frac{dv}{dt} = dv_{xx} + b_2 q [1 - v] [N_2(t) u - v], \\
\end{align*}
\]

(1.5)

where \( N_1(t) = \frac{b_1(t) q(t)}{a_1(t) p(t)} \leq 1 \) and \( N_2(t) = \frac{a_2(t) p(t)}{b_2(t) q(t)} \geq 1 \) for all \( t \in \mathbb{R} \). It is easy to see that

\[
(P(t, z), Q(t, z)) := \left( \frac{X(t, z)}{p(t)}, \frac{q(t) - Y(t, z)}{q(t)} \right)
\]

is a periodic traveling wave solution of (1.5) connecting \((0, 0)\) and \((1, 1)\), that is

\[
\begin{align*}
&P_t = P_{zz} + c P_z + a_1 p [1 - N_1(t) - P + N_1(t) Q], \\
&Q_t = d Q_{zz} + c Q_z + b_2 q [1 - Q] [N_2(t) P - Q], \\
&(P(t, z), Q(t, z)) = (P(t + T, z), Q(t + T, z)), \\
&\lim_{z \to -\infty} (P, Q) = (0, 0), \quad \lim_{z \to +\infty} (P, Q) = (1, 1).
\end{align*}
\]

(1.6)

Clearly, if \((P(t, z), Q(t, z)) = (P(t, x - ct), Q(t, x - ct))\) is a periodic traveling wave solution of (1.5) with speed \( c \), then \((\tilde{P}(t, z), \tilde{Q}(t, z)) := (P(t, -x - ct), Q(t, -x - ct))\) is a periodic traveling wave solution of (1.5) as well, with speed \( \tilde{c} := -c > 0 \) and satisfies

\[
\lim_{z \to -\infty} (\tilde{P}, \tilde{Q}) = (1, 1) \quad \text{and} \quad \lim_{z \to +\infty} (\tilde{P}, \tilde{Q}) = (0, 0).
\]

Under assumptions (A1)–(A3), for any \( c \leq c^* \), (1.5) admits a time periodic traveling wave solution \((P, Q) \in C^{1,2}_b(\mathbb{R} \times \mathbb{R}, \mathbb{R}^2)\) with \((P_z, Q_z) > (0, 0)\) and \((P, Q) \leq (1, 1)\) for all \((t, z) \in \mathbb{R} \times \mathbb{R}\), where

\[
c^* = -2 \sqrt{\frac{a_1 p - b_1 q}{a_1 p + b_1 q}}
\]

is the maximal wave speed (see [43]). In particular, the authors in [43] obtained the exact exponential decay rate of solutions of (1.6) as \( z \to -\infty \) in establishing the uniqueness of the periodic traveling wave solution. Next we will consider the cooperative system (1.5) to obtain invasion entire solutions of system (1.1). In order to employ the basic idea developed in [29, 13, 27] to establish the existence of such entire solutions, we essentially need some estimates which are concerned with the asymptotic behavior of the periodic traveling wave solution. One of the main difficulties arises in obtaining the exact exponential decay rate of the periodic traveling wave as it tends to its limiting state. In the autonomous case, the asymptotic behavior is usually obtained by investigating the linearized equations at the equilibrium points (see e.g. [38, 18]), which can not be applied to system (1.1) since the presence of time dependent nonlinearities. Inspired by [43], we employ the two-sided Laplace transform method to obtain the exact exponential decay rate, which is essentially based on some a priori exponential decay estimates of the periodic traveling wave tails as \( z \to +\infty \). In particular, unlike the a priori exponential estimates as \( z \to -\infty \) characterized by the principle eigenvalue of the linear periodic eigenvalue problem associated with the linearized system at the unstable limiting state (see [43, Lemma 3.3]), the exponential estimates as \( z \to +\infty \) can only be characterized by a small perturbation of the corresponding principle eigenvalue (see ‘\( \lambda_{c^*}^\pm \)’ in Lemma 2.2). Fortunately, this small perturbation can be declined small enough such that it imposes no influence on the Laurent development of the resolvent near the isolated principle eigenvalue.
The rest of this paper is organized as follows. In Section 2, we study the exact exponential decay rate of a periodic traveling wave solution of (1.5) as it approaches its stable limiting state. We then establish some key and useful estimates in Section 3. In Section 4, we establish the existence and qualitative properties of entire solutions by a comparing argument.

2. Asymptotic behavior of periodic traveling wave fronts. In this section we shall study the asymptotic behavior of time periodic traveling waves of (1.5).

Denote
\[ \kappa = \frac{a_1p - b_1q}{a_1p - b_1q}, \quad \phi(t) = e^{\int_0^t (a_1(s)p(s) - b_1(s)q(s))ds - \kappa t}, \quad \lambda_c^+ = \frac{-c - \sqrt{c^2 - 4\kappa}}{2} \]
if \( c \leq c^* = -2\sqrt{\kappa} \), and
\[
\begin{align*}
\phi_d(t) & = \phi_d(0)e^{-\int_0^t (b_2(s)q(s) + \rho_0)ds + \int_0^t e^{-\int_0^s (b_2(\tau)q(\tau) + \rho_0)d\tau}a_2(s)p(s)\phi(s)ds}, \\
\phi_d'(0) & = (1 - e^{-\int_0^t (b_2(t)q(t) + \rho_0)dt}) - \int_0^t e^{-\int_0^s (b_2(\tau)q(\tau) + \rho_0)d\tau}a_2(s)p(s)\phi(s)ds,
\end{align*}
\]
where \( \rho_0 = \kappa + (1 - d)(\lambda_c^+)^2 \). For completeness, we first state the following asymptotic behavior of solutions of system (1.6) as it approaches its unstable limiting state (see [43, Theorem 3.8]).

Proposition 1. Assume (A1)-(A3) hold, and \((P(t, z), Q(t, z)) \in C^{1,2}_b(\mathbb{R} \times \mathbb{R}, \mathbb{R}^2)\) and \( c \) solve (1.6). Then
\[ \lim_{z \to -\infty} \frac{P(t, z)}{k_1 |z|^{1+\frac{\kappa}{c}} e^{\lambda^+ z}}\phi_d(t) = 1, \quad \lim_{z \to -\infty} \frac{Q(t, z)}{k_1 |z|^{1+\frac{\kappa}{c}} e^{\lambda^+ z}}\phi_d(t) = 1 \quad \text{uniformly in } t \in \mathbb{R}, \]
and
\[ \lim_{z \to -\infty} \frac{P_z(t, z)}{k_1 |z|^{1+\frac{\kappa}{c}} e^{\lambda^+ z}}\phi_d(t) = \lambda_c^+, \quad \lim_{z \to -\infty} \frac{Q_z(t, z)}{k_1 |z|^{1+\frac{\kappa}{c}} e^{\lambda^+ z}}\phi_d(t) = \lambda_c^+ \quad \text{uniformly in } t \in \mathbb{R}, \]
where \( k_1 > 0 \) is a constant, \( l = 0 \) if \( c < c^* \) and \( l = 1 \) if \( c = c^* \).

In order to characterize the asymptotic behavior of time periodic traveling waves as \( z \to \pm \infty \), we now list a useful lemma of the Harnack inequalities for cooperative parabolic system, which was given in [9] (see also [43, 3]).

Lemma 2.1. Let
\[ L_k := \sum_{i,j=1}^n a^i_{k,j}(t, x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^n b^i_k(t, x)\frac{\partial}{\partial x_i} - \frac{\partial}{\partial t} \quad (k = 1, 2, \ldots, l) \]
be uniformly parabolic in an open domain \((\tau, M) \times \Omega \subset \mathbb{R} \times \mathbb{R}^n\), that is, there is \( \alpha_0 > 0 \) such that \( a^i_{k,j}(t, x)\xi_i\xi_j \geq \alpha_0 \sum_{i=1}^n \xi_i^2 \) for any \( n \)-tuples of real numbers \((\xi_1, \xi_2, \cdots, \xi_n)\), where \( -\infty < \tau < M \leq +\infty \) and \( \omega \) is open and bounded. Suppose that \( a^i_{k,j}, b^i_k \in C((\tau, M) \times \Omega, \mathbb{R}) \) and
\[ \sup_{(\tau, M) \times \Omega} \left( |a^i_{k,j}(t, x)| + |b^i_k(t, x)| \right) \leq \beta_0 \text{ for some } \beta_0 > 0. \]
Assume that \( w = (w_1, w_2, \cdots, w_l) \in C([\tau, M) \times \overline{\Omega}, \mathbb{R}^l) \cap C^{1,2}((\tau, M) \times \Omega, \mathbb{R}^l) \) satisfies:
\[
\sum_{s=1}^l c^{k,s}(t, x)w_s + L_kw_k \leq 0, \quad (t, x) \in (\tau, M) \times \Omega, \quad k = 1, 2, \cdots, l, \quad (2.1)
\]
where $c^{k,s} \in C((\tau, M) \times \Omega, \mathbb{R})$ and $c^{k,s} \geq 0$ if $k \neq s$, and
\[
\sup_{(t, x) \in ([\tau, M]) \times \Omega} |c^{k,s}(t, x)| \leq \gamma_0 \quad (k, s = 1, 2, \ldots, l) \text{ for some } \gamma_0 > 0.
\]

Let $D$ and $U$ be domains in $\mathbb{R}^n$ such that $D \subset U$, $\text{dist}((\overline{D}, \partial U)) \geq \rho$ and $|D| > \varepsilon$ for certain positive constants $\rho$ and $\varepsilon > 0$. Let $\theta$ be a positive constant with $\tau + 2\theta < M$. Then there exist positive constants $\rho$, $\gamma_1$ and $\gamma_2$, determined by $\alpha_0$, $\beta_0$, $\gamma_0$, $\rho$, $\varepsilon$, $\epsilon$, $n$, $\theta$ and $\text{diam} \Omega$, such that
\[
\inf_{(\tau + 2\theta, \tau + 4\theta) \times D} \inf \left\{ \frac{1}{2}, \frac{1}{4} \right\} \text{ for } t \geq 2\theta.
\]

Here $(w_0^+) = \max\{w_0, 0\}$, $(w_0^-) = \max\{-w_0, 0\}$ and $\partial_p((\tau, \tau + 4\theta) \times U) = \tau \times U \cup [\tau, \tau + 4\theta) \times \partial U$. Moreover, if all inequalities in (2.1) are replaced by equalities, then the conclusion holds with $p = +\infty$, and with $\omega_1$, $\omega_2$ independent of $\epsilon$.

Let
\[
(U(t, z), V(t, z)) = \left( \frac{p(t) - X(t, z)}{p(t)}, \frac{Y(t, z)}{q(t)} \right),
\]
then we have
\[
\left\{ \begin{array}{l}
U_t = U_{zz} + cU_z + g(t, U, V), \\
V_t = dV_{zz} + cV_z + h(t, U, V), \\
(U(t, z), V(t, z)) = (U(t + T, z), V(t + T, z)), \\
\lim_{z \to -\infty} (U, V) = (1, 1), \lim_{z \to +\infty} (U, V) = (0, 0),
\end{array} \right.
\]
(2.2)

where
\[
\left\{ \begin{array}{l}
g(t, u, v) = -(1 - u)[a_1(t)p(t)u - b_1(t)q(t)v], \\
h(t, u, v) = -v[a_2(t)p(t)(1 - u) - b_2(t)q(t)(1 - v)].
\end{array} \right.
\]

For any $c \leq c^* = -2\sqrt{\kappa}$, denote
\[
\kappa_0 = -h_0(t, 0, 0) = a_2p - b_2q, \quad \lambda_c^+ = -c - \sqrt{c^2 + 4d\kappa_0}, \quad \psi(t) = e^{\int_0^t h_0(s, 0, 0)ds + \kappa_0 t},
\]
and
\[
\kappa_1 = -\bar{g}_0(t, 0, 0) = a_1p, \quad \lambda_c^- = -c - \sqrt{c^2 + 4\kappa_1}, \quad \bar{\psi}(t) = e^{\int_0^t \bar{g}_0(s, 0, 0)ds + \kappa_1 t}.
\]

To be specific and convenient, we give an additional assumption on the periodic coefficients.

(A4): $a_1(t)p(t) < 5 b_1(t)q(t)$ for all $t \in [0, T]$. 

Remark 1. It follows from (A4) and (A3) that $b_1(t)q(t) < a_1(t)p(t) < 5 b_1(t)q(t)$ for all $t \in [0, T]$, that is, assumption (A4) is compatible with (A3). It should be emphasized here that (A4) is a technique assumption that ensures $\lambda_c^+ > \lambda_c$ for any $c \leq c^*$, which is essential in obtaining the exact exponential decay rate of $u$ in our present work. Indeed, (A3) yields that $\kappa_0 \leq \kappa < \kappa_1$, then a direct computation shows that
and bounded, Lemma 2.1 then implies that there is some \( N > 0 \) and \( K > 0 \) such that

\[
\|z\|_{\infty} \leq (\sqrt{1 + \frac{4\kappa_0}{c^2}} + 1) = \frac{2}{\sqrt{1 + \frac{4\kappa_1}{c^2}} + 1} = \min_{0 \leq c' \leq c^*} \frac{2}{\sqrt{1 + \frac{4\kappa_1}{c^2}} + 1}.
\]

Thus, \( 2(\lambda_0 - \lambda_0^-) = \frac{c + \sqrt{c^2 + 4d}\kappa_0}{d} - (c + \sqrt{c^2 + 4\kappa_1}) \),

\[
\frac{4\kappa_0}{\sqrt{c^2 + 4d}\kappa_0 - c} - \frac{4\kappa_1}{\sqrt{c^2 + 4\kappa_1} - c} < 0 \text{ for any } c \leq c^*.
\]

Remark 2. We also remark here that when \( d = 1 \), condition (A4) can be deleted since \( \lambda_0^- > \lambda_0 \) holds certainly under condition (A3).

Noting that \( g(t, 0, 0) = h(t, 0, 0) = g(t, 1, 1) = h(t, 1, 1) = 0 \), system (2.2) can be written as

\[
\begin{align*}
U_t &= U_{zz} + cU_z + U \int_0^1 g_u(t, \tau U, \tau V) d\tau + V \int_0^1 g_v(t, \tau U, \tau V) d\tau, \\
V_t &= dV_{zz} + cV_z + U \int_0^1 h_u(t, \tau U, \tau V) d\tau + V \int_0^1 h_v(t, \tau U, \tau V) d\tau.
\end{align*}
\]

Let \( D = (z - \frac{1}{4}, z + \frac{1}{4}), U = (z - \frac{1}{2}, z + \frac{1}{2}), \Omega = (z - 1, z + 1) \) with \( z \in \mathbb{R} \), \( \tau = 0 \), and \( \theta = T \). Since \( U(\cdot, z) \) and \( V(\cdot, z) \) are periodic of \( t \), and \( U, V \) are both positive and bounded, Lemma 2.1 then implies that there is some \( N > 0 \) such that

\[
(U(t, z), V(t, z)) \leq N(U(t', z), V(t', z)) \text{ for all } z \in \mathbb{R}, t, t' \in \mathbb{R}.
\]

We now state an essential lemma for system (2.2) on the exponential decay estimates of the periodic traveling wave tails as \( z \to +\infty \) by using the method similar to [2].

Lemma 2.2. Assume (A1)-(A4) hold. Let \( (U(t, z), V(t, z)) \in C^1(\mathbb{R} \times \mathbb{R}, \mathbb{R}^2) \) be a solution of (2.2). Then for any \( 0 < \epsilon < \min\left\{ \frac{\kappa_0}{2c}, \frac{\kappa_1}{2c} \right\} \), there exist some constants \( K_i > 0 \) \((i = 1, 2, 3, 4)\) and \( \sigma < \lambda \) such that

\[
K_i e^{\sigma z} \leq U(t, z) \leq K_i e^{\lambda^+ z} \text{ for any } (t, z) \in \mathbb{R} \times [0, +\infty) \quad (2.5)
\]

and

\[
K_4 e^{\lambda^+ z} \leq V(t, z) \leq K_4 e^{\lambda^- z} \text{ for any } (t, z) \in \mathbb{R} \times [0, +\infty),
\]

where \( \lambda^\pm \in \mathbb{R} \) and \( C^+ = \max_{[0,T]} a_2(t)p(t) \) and \( C^- = \max_{[0,T]} b_2(t)q(t) \).

Proof. According to definitions of \( \psi(t) \), \( U(t, z), V(t, z) \), and \( \tilde{U}(t, z), \tilde{V}(t, z) \) are \( T \)-periodic in \( t \) for any \( z \in \mathbb{R} \). Let

\[
\hat{u}(z) = \int_0^T U(t, z) \frac{d\psi(t)}{\psi(t)} dt, \quad \hat{v}(z) = \int_0^T V(t, z) \frac{d\psi(t)}{\psi(t)} dt \text{ for any } z \in \mathbb{R},
\]
then a direct calculation yields that
\[
\begin{aligned}
\begin{cases}
a_{zz} + c\hat{u}_z - \kappa_1 \hat{u} + \int_0^T a_2(t)p(t)V(t,z)dt > 0, \\
\hat{V}_{zz} + c\hat{V}_z - \kappa_0 \hat{V} + \int_0^T \frac{a_2(t)p(t)V(t,z) - b_2(t)q(t)V^2(t,z)}{\psi(t)}dt = 0.
\end{cases}
\end{aligned}
\]
(2.7)

Since
\[
\lim_{z \to +\infty} (U(t,z), V(t,z)) = (0,0)
\]
for any 0 < \epsilon < \min\{1, \frac{\kappa_0}{c}\}, we can choose constant 0 > \kappa > 1 such that (0,0) < (U(t,z), V(t,z)) ≤ (\epsilon, \epsilon) for any (t,z) ∈ [0, T] × [M, +∞).

We first show (2.6). Let
\[
V^+(z) = \rho e^{\lambda^z_{c,\epsilon}}
\]
with \lambda^z_{c,\epsilon} = \frac{-c - \sqrt{c^2 + 4d(\kappa_0 - C^+\epsilon)}}{2d} < 0,
then \ V^+(z) \ is a solution of the linear equation
\[
d\hat{v}_{zz} + c\hat{v}_z - \kappa_0 \hat{v} + c^+ \epsilon \hat{v} = 0.
\]
(2.8)

Since \hat{v} is bounded, we can choose \rho > 0 large enough such that \hat{v}(M) ≤ \rho e^{\lambda^z_{c,\epsilon}}M.

In addition, we obtain from the second equation of (2.7) that
\[
\begin{aligned}
0 &= d\hat{v}_{zz} + c\hat{v}_z - \kappa_0 \hat{v} + \int_0^T \frac{a_2(t)p(t)V(t,z) - b_2(t)q(t)V^2(t,z)}{\psi(t)}dt \\
&\leq d\hat{v}_{zz} + c\hat{v}_z - \kappa_0 \hat{v} + \int_0^T \frac{a_2(t)p(t)V(t,z) - b_2(t)q(t)V^2(t,z)}{\psi(t)}dt \\
&\leq d\hat{v}_{zz} + c\hat{v}_z - \kappa_0 \hat{v} + C^+ \epsilon \hat{v}
\end{aligned}
\]
for any \ z \in [M, +\infty), which implies that \hat{v}(z) is a subsolution of (2.8) in [M, +\infty).

Since \lim_{z \to +\infty} \hat{v}(z) = \lim_{z \to +\infty} V^+(z) = 0, the maximum principle then yields that
\hat{v}(z) ≤ \rho e^{\lambda^z_{c,\epsilon}}
for any \ z \in [M, +\infty). Hence, there exists constant \rho' ≥ \rho such that
\hat{v}(z) ≤ \rho' e^{\lambda^z_{c,\epsilon}}
for any \ z \in [0, +\infty). Furthermore, we have \inf_{[0,T]} V(t,z) ≤ \frac{C_1}{\rho'} e^{\lambda^z_{c,\epsilon}}
for any \ z \in [0, +\infty) with some constant \ C_1 > 0, which combining the Harnack inequalities (2.4) shows that there exists \ K_4 > 0 such that \ V(t,z) ≤ K_4 e^{\lambda^z_{c,\epsilon}}
for any \ (t,z) \in \mathbb{R} \times [0, +\infty). Similarly, let \ V^-(z) = \eta e^{\lambda^-_{c,\epsilon}}
where \lambda^-_{c,\epsilon} = \frac{-c - \sqrt{c^2 + 4d(\kappa_0 + \eta C^-\epsilon)}}{2d}
< 0 and \ V^-(M) ≥ \eta e^{\lambda^-_{c,\epsilon}}M \ for \ some \ \eta > 0. Then \ V^-(z) \ satisfies
\[
d\hat{v}_{zz} + c\hat{v}_z - \kappa_0 \hat{v} - C^- \epsilon \hat{v} = 0.
\]
(2.9)

On the other hand, by the second equation of (2.7), we have
\[
\begin{aligned}
0 &= d\hat{v}_{zz} + c\hat{v}_z - \kappa_0 \hat{v} + \int_0^T a_2(t)p(t)V(t,z)dt - b_2(t)q(t)V^2(t,z)dt \\
&\geq d\hat{v}_{zz} + c\hat{v}_z - \kappa_0 \hat{v} - \int_0^T \frac{b_2(t)q(t)V^2(t,z)}{\psi(t)}dt \\
&\geq d\hat{v}_{zz} + c\hat{v}_z - \kappa_0 \hat{v} - C^- \epsilon \hat{v}
\end{aligned}
\]
for any \ z \in [M, +\infty), thus \hat{v}(z) is a supersolution of (2.9) in [M, +\infty). Since
\[
\lim_{z \to +\infty} \hat{v}(z) = \lim_{z \to +\infty} V^-(z) = 0,
\]

it follows from the maximum principle that \hat{v}(z) ≥
for any \( z \in [M, +\infty) \). A similar argument yields that there exists \( K_3 > 0 \) such that \( V(t, z) \geq K_3 e^{\lambda_{c,z}} \) for any \((t, z) \in \mathbb{R} \times [0, +\infty)\).

We now prove \((2.5)\). Note that \((A4)\) implies \( \lambda_c < \lambda_{c, \epsilon} < \lambda_{c, \epsilon}^+ < 0 \), it then follows from the definition of \( \lambda_c \) that \( L := -(\lambda_{c, \epsilon}^+)^2 - c\lambda_{c, \epsilon}^+ + \kappa_1 > 0 \). In view of \( V(t, z) \leq K_4 e^{\lambda_{c,z}^+} \) for any \((t, z) \in \mathbb{R} \times [0, +\infty)\), the Harnack inequalities \((2.4)\) imply that there exist positive constants \( M_1 \) and \( M_2 \) such that

\[
\int_0^T \frac{b_1(t)q(t)V(t, z)}{\psi(t)} dt \leq T \max_{[0,T]} \frac{b_1(t)q(t)}{\psi(t)} K_4 e^{\lambda_{c,z}^+} \leq M_1 e^{\lambda_{c,z}^+},
\]

\[
\int_0^T \frac{a_1(t)p(t)U^2(t, z)}{\psi(t)} dt \leq \max_{[0,T]} [a_1(t)p(t)\tilde{\psi}(t)] \int_0^T \frac{U^2(t, z)}{\tilde{\psi}^2(t)} dt \leq M_2(\hat{u}(z))^2
\]

for any \( z \in [0, +\infty) \). Moreover, we know from \( \lim_{z \to +\infty} \hat{u}(z) = 0 \) that there exists a constant \( M' > 0 \) such that \( M_2\hat{u}(z) \leq \frac{1}{2} \min\{L, \kappa_1\} \) for any \( z \in [M', +\infty) \). The first equation of \((2.7)\) indicates that

\[
0 = \hat{u}_{zz} + cu_z - \kappa_1 \hat{u} + \int_0^T \frac{b_1(t)q(t)V(t, z)}{\psi(t)} dt \\
+ \int_0^T \frac{a_1(t)p(t)U^2(t, z) - b_1(t)q(t)U(t, z)V(t, z)}{\psi(t)} dt \\
\leq \hat{u}_{zz} + cu_z - \kappa_1 \hat{u} + \int_0^T \frac{b_1(t)q(t)V(t, z)}{\psi(t)} dt + \int_0^T \frac{a_1(t)p(t)U^2(t, z)}{\psi(t)} dt \\
\leq \hat{u}_{zz} + cu_z - \kappa_1 \hat{u} + M_1 e^{\lambda_{c,z}^+} + M_2(\hat{u}(z))^2 \\
\leq \hat{u}_{zz} + cu_z - \kappa_1 \hat{u} + M_1 e^{\lambda_{c,z}^+} + \frac{L}{2} \hat{u}
\]

for any \( z \in [M', +\infty) \), that is, \( \hat{u} \) is a subsolution of equation

\[
-u_{zz} - cu_z + \kappa_1 u - \frac{L}{2} u - M_1 e^{\lambda_{c,z}^+} = 0, \quad z \in [M', +\infty). \tag{2.10}
\]

Let \( U^+(z) = \delta e^{\lambda_{c,z}^+} \) with \( \delta \geq \frac{2M_1}{L} \) large enough such that \( \hat{u}(M') \leq \delta e^{\lambda_{c,z}^+}, \) then

\[
-u_{zz} - cu_z + \kappa_1 u - \frac{L}{2} u - M_1 e^{\lambda_{c,z}^+} = 0, \quad z \in [M', +\infty).
\]

Let \( U^+(z) = \delta e^{\lambda_{c,z}^+} \) with \( \delta \geq \frac{2M_1}{L} \) large enough such that \( \hat{u}(M') \leq \delta e^{\lambda_{c,z}^+}, \) then

\[
-u_{zz} - cu_z + \kappa_1 u - \frac{L}{2} u - M_1 e^{\lambda_{c,z}^+} = 0, \quad z \in [M', +\infty).
\]

Let \( U^+(z) = \delta e^{\lambda_{c,z}^+} \) with \( \delta \geq \frac{2M_1}{L} \) large enough such that \( \hat{u}(M') \leq \delta e^{\lambda_{c,z}^+}, \) then

\[
-u_{zz} - cu_z + \kappa_1 u - \frac{L}{2} u - M_1 e^{\lambda_{c,z}^+} = 0, \quad z \in [M', +\infty).
\]

Let \( U^+(z) = \delta e^{\lambda_{c,z}^+} \) with \( \delta \geq \frac{2M_1}{L} \) large enough such that \( \hat{u}(M') \leq \delta e^{\lambda_{c,z}^+}, \) then

\[
-u_{zz} - cu_z + \kappa_1 u - \frac{L}{2} u - M_1 e^{\lambda_{c,z}^+} = 0, \quad z \in [M', +\infty).
\]
for any $z \in [0, +\infty)$, and we then have
\[
0 = \dot{u}_{zz} + c\dot{u}_z - \kappa_1 \dot{u} + \int_0^T \frac{b_1(t)q(t)V(t,z)}{\psi(t)} dt + \int_0^T a_1(t)p(t)U^2(t,z) - b_1(t)q(t)U(t,z)V(t,z) dt \\
\geq \dot{u}_{zz} + c\dot{u}_z - \kappa_1 \dot{u} - \int_0^T \frac{b_1(t)q(t)U(t,z)V(t,z)}{\psi(t)} dt \\
\geq \dot{u}_{zz} + c\dot{u}_z - \kappa_1 \dot{u} - N_3 e^{\lambda^+_c z} \dot{u}
\]
for any $z \in [0, +\infty)$, that is, $\dot{u}(z)$ is a supersolution of the equation
\[
U_{zz} + cU_z - \kappa_1 U - N_3 e^{\lambda^+_c z} U = 0, \quad z \in [0, +\infty). \tag{2.11}
\]
By the definition of $\lambda_c$, we have $\sigma^2 + c\sigma - \kappa_1 > 0$ for $\sigma < \lambda_c$. Let $M'' \geq M'$ such that $e^{\lambda^+_{c'} M''} \leq \frac{e^{\lambda^+_c z} - \delta c}{N_3}$. Taking $\dot{u}^-(z) = \delta c \sigma z$ with $\delta > 0$ large enough satisfying $\dot{\check{u}}(M'') \geq \delta c \sigma M''$, it then follows that
\[
U_{zz}^+ + cU_z^+ - \kappa_1 U^+ - N_3 e^{\lambda^+_c z} U^+ = (\sigma^2 + c\sigma - \kappa_1 - N_3 e^{\lambda^+_c z}) \delta c \sigma z \\
\geq (\sigma^2 + c\sigma - \kappa_1 - N_3 e^{\lambda^+_c z} M'') \delta c \sigma z \geq 0
\]
for any $z \in [M'', +\infty)$, that is, $U^-(z)$ is a subsolution of (2.11). Note that $\lim_{z \to +\infty} \dot{u}(z) = \lim_{z \to +\infty} U^-(z) = 0$, again the maximum principle yields that $\dot{u}(z) \geq U^-(z)$ for any $z \in [M'', +\infty)$. Thus $\delta c \sigma z \leq \dot{u}(z) \leq \delta e^{\lambda^+_c z}$ for all $z \in [M'', +\infty)$, then (2.5) follows from the same argument as (2.6). The proof is complete. □

Remark 3. The definitions of $\lambda^\pm_{c'}$ indicate that there exist $\varepsilon^\pm = \varepsilon^\pm(c) > 0$ such that for any $c \leq \wedge c$ and $0 < \varepsilon < \min\{1, \frac{c}{\lambda^+_c}\}$, there holds $\lambda^\pm_{c'} = \lambda^\pm c + \varepsilon^\pm c$ with $\varepsilon^\pm(c) \to 0^+$ as $c \to 0^+$. Actually, we know from the proof of Lemma 2.2 that the exponential decay as $z \to +\infty$ can only be estimated by the perturbation $\lambda^\pm_{c'}$ rather than $\lambda^\pm_{c}$, since there is no such exponential type sub-super solutions that equipped with $\lambda^\pm_{c}$ as the exponential decay rate.

Lemma 2.3. Suppose (A1)-(A4) hold, let $(U(t,z), V(t,z)) \in C_b^{1,2}(\mathbb{R} \times \mathbb{R}, \mathbb{R}^2)$ be a solution of (2.2). Then there exist $C_1 > 0$ and $C_2 > 0$ such that
\[
|U(t,z)| + |U_z(t,z)| + |U_{zz}(t,z)| \leq C_1 e^{\lambda^+_c z} \quad \text{for any } (t,z) \in R \times [0, +\infty), \\
|V(t,z)| + |V_z(t,z)| + |V_{zz}(t,z)| \leq C_2 e^{\lambda^+_c z} \quad \text{for any } (t,z) \in R \times [0, +\infty),
\]
where $\lambda^\pm_{c'}$ are defined as in Lemma 2.2.

Proof. The proof is similar to [43, Proposition 3.4], using the interior parabolic estimates and Lemma 2.1, so we omit the details here. □
Let $Y = L^2_T \times L^2_T$, where $L^2_T := \{ \int_0^T |h(t + s)|^2 ds < \infty, \ h(t + T) = h(t) \}$ is equipped with the norm $\|h\|_{L^2_T} = (\int_0^T |h(s)|^2 ds)^{1/2}$, and $H^1_T = \{ h \in L^2_T, \ \sup_{t \in \mathbb{R}} \int_0^T |h'(t + s)|^2 ds < \infty \}$. Define $\mathcal{A} : D(\mathcal{A}) \subset Y \rightarrow Y$ as

$$\mathcal{A} = \left( \begin{array}{cc} \frac{1}{\pi} (\partial_t - h_v(t, 0, 0)) & I \\ 0 & -\frac{c}{\pi} \end{array} \right).$$

(2.12)

It is easy to see that $\mathcal{A}$ is closed and densely defined in $D(\mathcal{A}) = H^1_T \times L^2_T$. Now let $w = v_z$, then the $v$-equation of (2.2) can be written as a first order system

$$\frac{d}{dz} \begin{pmatrix} v \\ w \end{pmatrix} = \mathcal{A} \begin{pmatrix} v \\ w \end{pmatrix} + \left( \begin{array}{c} 0 \\ \frac{1}{\pi} [h_v(t, 0, 0)v - h(t, u, v)] \end{array} \right).$$

Similar to [43, Lemma 3.5], we have that $\lambda^c_\infty \in \sigma(\mathcal{A})$ (the spectrum of $\mathcal{A}$) and

$$\ker(\lambda^c_\infty I - \mathcal{A})^n = \ker(\lambda^c_\infty I - \mathcal{A}) = \text{span} \left\{ \begin{pmatrix} \psi(t) \\ \lambda^c_\infty \psi(t) \end{pmatrix} \right\} \text{ for } n = 2, 3, \ldots,$$

which implies that $\lambda^c_\infty$ is a simple pole of $(\lambda I - \mathcal{A})^{-1}$ (see [26, Remark A.2.3]). Moreover, a similar argument as [43, Proposition 3.6] shows that there exists $\epsilon' > 0$ such that $\Theta_{\epsilon'} \cap \sigma(\mathcal{A}) = \{ \lambda^c_\infty \}$, where $\Theta_{\epsilon'} = \{ \lambda \in \mathbb{C} | \lambda^c_\infty - \epsilon' \leq \Re \lambda \leq \lambda^c_\infty + \epsilon' \}$ is the vertical strip containing the vertical line $\Re \lambda = \lambda^c_\infty$. Thus, $\lambda^c_\infty$ is the only singular point of $\lambda I - \mathcal{A}$ in $\Theta_{\epsilon'}$. Then by [26], we have

$$(\lambda I - \mathcal{A})^{-1} = \sum_{n=0}^{\infty} (-1)^n (\lambda - \lambda^c_\infty)^n S^{n+1} + \frac{P}{\lambda - \lambda^c_\infty} + \sum_{n=1}^{\infty} (\lambda - \lambda^c_\infty)^{n+1} D^n, \quad (2.13)$$

where $P = \frac{1}{2\pi i} \int_\Gamma (\lambda I - \mathcal{A})^{-1} d\lambda$ is the spectral projection with $\Gamma : |\lambda - \lambda^c_\infty| < \epsilon'$ for some small $\epsilon' > 0$ and

$$S = \frac{1}{2\pi i} \int_\Gamma (\lambda I - \mathcal{A})^{-1} \lambda - \lambda^c_\infty d\lambda = \lim_{\lambda \rightarrow \lambda^c_\infty} (I - P)(\lambda I - \mathcal{A})^{-1},$$

$D = (\mathcal{A} - \lambda^c_\infty I)P$. Since $\lambda^c_\infty$ is a simple pole of $(\lambda I - \mathcal{A})^{-1}$, [26, Proposition A.2.2] then implies that $R(P) = \ker(\lambda^c_\infty I - \mathcal{A})$ for any $c \leq \epsilon'$, hence $D^n = 0$ for all $n \in N^+$ and (2.13) becomes

$$(\lambda I - \mathcal{A})^{-1} = \sum_{n=0}^{\infty} (-1)^n (\lambda - \lambda^c_\infty)^n S^{n+1} + \frac{P}{\lambda - \lambda^c_\infty}. \quad (2.14)$$

The formula (2.14) is therefore the Laurent series of $(\lambda I - \mathcal{A})^{-1}$ near $\lambda = \lambda^c_\infty$, and the projection $P$ is the residue of $(\lambda I - \mathcal{A})^{-1}$ at $\lambda = \lambda^c_\infty$. On the other hand, if we let $\lambda = \mu + i\eta \in \rho(\mathcal{A})$ with $\mu, \ \eta \in \mathbb{R}$, denote by

$$S = \left\{ \begin{pmatrix} 0 \\ j \end{pmatrix} \mid j \in L^2_T \right\} \subset Y$$

and $(\lambda I - \mathcal{A})^{-1}$ the restriction of $(\lambda I - \mathcal{A})^{-1}$ to $S$, similar to [43, Remark 3.7], there exist positive constants $C$ and $\varrho$ such that for any $\mu \in [\lambda^c_\infty - \epsilon', \lambda^c_\infty + \epsilon']$, there holds

$$\| (\lambda I - \mathcal{A})^{-1} \| \leq \frac{C}{|\eta|} \text{ for } |\eta| \geq \varrho. \quad (2.15)$$

Now we state the main results of this section as follows.
Theorem 2.4. Assume (A1)-(A4) hold. Let \((u(t, z), v(t, z))\) be a solution of (2.2). Then for any \(c \leq c^*\), we have
\[
\lim_{z \to +\infty} \frac{u(t, z)}{k_2 e^{\lambda_c z} \tilde{\phi}(t)} = 1, \quad \lim_{z \to +\infty} \frac{v(t, z)}{k_2 e^{\lambda_c z} \tilde{\psi}(t)} = 1, \quad \text{uniformly in } t \in \mathbb{R}
\] (2.16)
and
\[
\lim_{z \to +\infty} \frac{u_z(t, z)}{k_2 e^{\lambda_c z} \tilde{\phi}(t)} = \lambda_c^-, \quad \lim_{z \to +\infty} \frac{v_z(t, z)}{k_2 e^{\lambda_c z} \tilde{\psi}(t)} = \lambda_c^-, \quad \text{uniformly in } t \in \mathbb{R},
\] (2.17)
where \(k_2 > 0\) is some constant and
\[
\begin{aligned}
\tilde{\phi}(t) &= \tilde{\phi}(0) e^{\int_t^0 (\rho + g_u(s, 0, 0)) \, ds} + \int_t^T e^{\int_0^s (\rho + g_u(\tau, 0, 0)) \, d\tau} g_v(s, 0, 0) \psi(s) \, ds, \\
\tilde{\phi}(0) &= \left(1 - e^{\int_0^T (\rho + g_u(\tau, 0, 0)) \, d\tau} g_v(s, 0, 0) \psi(s) \, ds \right)^{-1} \int_0^T e^{\int_0^\tau (\rho + g_u(\tau, 0, 0)) \, d\tau} g_v(s, 0, 0) \psi(s) \, ds
\end{aligned}
\] (2.18)
with \(\rho = (\lambda_c^-)^2 + c \lambda_c^-\).

Proof. The proof is divided into two steps.

Step I. We prove that there exists \(k_2 > 0\) such that
\[
\lim_{z \to +\infty} \frac{v(t, z)}{k_2 e^{\lambda_c z} \tilde{\psi}(t)} = \lambda_c^-, \quad \text{uniformly in } t \in \mathbb{R}.
\]
Now we introduce an auxiliary function
\[
\chi \in C^2_0(\mathbb{R}, \mathbb{R})
\]
and set \(w = v_z, \tilde{v} = \chi v, \tilde{w} = (\chi v)_z\). A direct calculation yields that
\[
\tilde{w}_z - \frac{c}{d} \tilde{w} - \frac{1}{d} \tilde{v}_t = -\frac{1}{d} h_v(t, 0, 0) \tilde{v} + \frac{1}{d} \chi [h_v(t, 0, 0) v - h(t, u, v)] + \chi'' v + 2 \chi' v + \frac{c}{d} \chi' v.
\] (2.19)
Let
\[
\bar{g}(t, z) = \frac{1}{d} \chi [h_v(t, 0, 0) v - h(t, u, v)] + \chi'' v + 2 \chi' v + \frac{c}{d} \chi' v,
\]
then we can rewrite (2.19) as a first order system
\[
\frac{d}{dz} \begin{pmatrix} \tilde{v} \\ \tilde{w} \end{pmatrix} = \mathcal{A} \begin{pmatrix} \tilde{v} \\ \tilde{w} \end{pmatrix} + \begin{pmatrix} 0 \\ \bar{g}(t, z) \end{pmatrix}.
\] (2.20)
Taking \(0 < \epsilon' < \min \{-\frac{\lambda_c^-, \lambda_c^- - \lambda_c}{}\}\) sufficiently small such that \(\Theta_{\epsilon'} \cap \sigma(\mathcal{A}) = \{\lambda_c^-\}\). For this fixed \(\epsilon' > 0\), it follows from Remark 3 that there exists \(0 < \epsilon < \min \{1, \frac{\epsilon}{2}\}\) small enough such that \(\pm (\lambda_c^+ - \lambda_c^-) = \epsilon \pm \epsilon\) < \(\frac{1}{2} \epsilon'\). Due to Lemma 2.3, we have \(\sup_{t \in \mathbb{R}} |\tilde{v}| + |\tilde{w}| + |\tilde{v}_z| + |\tilde{w}_z| \leq O(e^{\lambda_c^+, z})\) as \(z \to +\infty\). Thus for any \(Re \lambda \in (\lambda_c^+, \lambda_c^- + \epsilon')\), there holds \(\{e^{-\lambda_c^- \tilde{v}} e^{-\lambda_c^+ \tilde{w}}\} \in W^{1,1}(\mathbb{R}, Y) \cap W^{1,\infty}(\mathbb{R}, Y)\). We now take the two-sided Laplace transform of (2.20) with respect to \(z\) and obtain that
\[
\begin{pmatrix} \int_{\mathbb{R}} e^{-\lambda x} \tilde{v}(\cdot, s) \, ds \\ \int_{\mathbb{R}} e^{-\lambda x} \tilde{w}(\cdot, s) \, ds \end{pmatrix} = \mathcal{F}(\lambda) := (\lambda I - \mathcal{A})^{-1} \begin{pmatrix} 0 \\ \int_{\mathbb{R}} e^{-\lambda x} \bar{g}(\cdot, s) \, ds \end{pmatrix},
\] (2.21)
where \(\lambda_c^- < Re \lambda \leq \lambda_c^- + \epsilon'\). It follows from the expression of \(\bar{g}\) and Lemma 2.3 that
\[
\sup_{t \in \mathbb{R}} |\bar{g}| + |\bar{g}_z| \leq O(e^{\lambda_c^+, z})\) as \(z \to +\infty\). Hence \(\int_{\mathbb{R}} e^{-\lambda x} \bar{g}(\cdot, s) \, ds\) and \(\int_{\mathbb{R}} e^{-\lambda x} \bar{g}_z(\cdot, s) \, ds\)
are analytic for \( \lambda \) with \( \text{Re}\lambda \in (\lambda_{c, \varepsilon}^+ - 3\varepsilon', 0) \). Let \( \lambda = \mu + i\eta \), then 
\[
\int_{\mathbb{R}} e^{-\mu g}(\cdot, s) ds = \int_{\mathbb{R}} e^{-i\eta g}(\cdot, s) ds = \hat{f}_\mu(\eta),
\]
where \( \hat{f}_\mu \) is the Fourier transform of \( f_\mu(s) := e^{-\mu g}(\cdot, s) \). It is easy to see that \( f_\mu(s) \in W^{1,1}(\mathbb{R}, L_2^\mu) \cap W^{1,\infty}(\mathbb{R}, L_2^\mu) \) for any fixed \( \mu \in [\lambda_{c, \varepsilon}, -\frac{3}{2} \varepsilon', -\frac{1}{2} \varepsilon'] \). Particularly, \( \|e^{-\mu g}\|_{W^{1,1}(\mathbb{R}, L_2^\mu)} \) is uniformly bounded in \( \mu \in [\lambda_{c, \varepsilon}^+, -\frac{3}{2} \varepsilon', -\frac{1}{2} \varepsilon'] \), hence there exist positive constants \( C_1 \) and \( \varrho_0 \) such that 
\[
\|\hat{f}(\eta)\|_{L_2^\mu} = \|\int_{\mathbb{R}} e^{-\lambda \mu g}(\cdot, s) ds\|_{L_2^\mu} \leq \frac{C_1}{|\eta|} \text{ for any } |\eta| \geq \varrho_0 \text{ whenever } \mu \in [\lambda_{c, \varepsilon}^+, -\frac{5}{2} \varepsilon', -\frac{1}{2} \varepsilon'].
\]
Inequality (2.15) then yields that there exist \( C_2 > 0 \) and \( \varrho > 0 \) such that 
\[
\|(\lambda I - A)^{-1} G(\lambda)\|_Y \leq \frac{C_2}{|\eta|} \text{ for any } |\eta| \geq \varrho, \tag{2.22}
\]
whenever \( \mu \in [\lambda_{c}^-, \lambda_{c}^- + \frac{1}{2} \varepsilon'] \). By the inverse Laplace transform we get that
\[
\left( \begin{array}{c}
\bar{v}(\cdot, z) \\
\bar{w}(\cdot, z)
\end{array} \right) = \frac{1}{2\pi i} \int_{\mu-i\infty}^{\mu+i\infty} e^{\lambda z}(\lambda I - A)^{-1} G(\lambda) d\lambda.
\]
Since \((\bar{v}(\cdot, z), \bar{w}(\cdot, z)) = (v(\cdot, z), w(\cdot, z))\) for \( z \geq 0 \), it follows that 
\[
\left( \begin{array}{c}
v(\cdot, z) \\
w(\cdot, z)
\end{array} \right) = \frac{1}{2\pi i} \int_{\mu-i\infty}^{\mu+i\infty} e^{\lambda z}(\lambda I - A)^{-1} G(\lambda) d\lambda \text{ for any } z \geq 0. \tag{2.23}
\]
Let \( \bar{\mu} = \lambda_{c}^- - \varepsilon' \), then \( \lambda_{c}^- \) is the only pole of \( \mathcal{F}(\lambda) \) in \( \text{Re}\lambda \in (\bar{\mu}, \lambda_{c, \varepsilon}^+) \). In view of (2.22), we have
\[
\lim_{|\eta| \to \infty} \int_{\lambda_{c}^- - \varepsilon'}^{\mu} \left\| e^{(\tau + i\eta) z} \right\|_{Y} \left\| (\tau + i\eta) I - A \right\|^{-1} G(\tau + i\eta) \right\|_{Y} d\tau = 0 \text{ for any } z \geq 0.
\]
Therefore, the path of integral in (2.23) can be shifted to \( \text{Re}\lambda = \bar{\mu} \) such that 
\[
\left( \begin{array}{c}
v(\cdot, z) \\
w(\cdot, z)
\end{array} \right) = \frac{1}{2\pi i} \int_{\lambda_{c}^- - \varepsilon'}^{\lambda_{c}^- + i\infty} e^{\lambda z}(\lambda I - A)^{-1} G(\lambda) d\lambda + \operatorname{Res}(e^{\lambda z} \mathcal{F}(\lambda), \lambda_{c}^-), \quad z \geq 0, \tag{2.24}
\]
where \( \operatorname{Res}(g, \lambda_0) := \frac{1}{2\pi i} \int_{|\lambda - \lambda_0| < \varepsilon'} g(\lambda) d\lambda \) denotes the residue of \( g \) at \( \lambda_0 \) with \( \varepsilon' > 0 \) sufficiently small. Furthermore, with the aid of 
\[
(\lambda I - A)^{-1} G(\lambda) = \sum_{n=0}^{\infty} (-1)^n (\lambda - \lambda_{c}^-)^n S^{n+1} G(\lambda) + \frac{PG(\lambda_{c}^-)}{\lambda - \lambda_{c}^-} - \frac{P[G(\lambda_{c}^-) - G(\lambda)]}{\lambda - \lambda_{c}^-}
\]
for \( |\lambda - \lambda_{c}^-| < \varepsilon' \),
\[
PG \subset \ker(\lambda_{c}^- I - A) = \text{span} \left\{ \left( \begin{array}{c}
\psi(t) \\
\lambda_{c}^- \psi(t)
\end{array} \right) \right\}
\]
and \( G(\lambda) \) is analytic in \( \Re \lambda \in (\lambda^+_{c,\varepsilon} - 3\varepsilon', 0) \), using the residue theorem, we obtain that
\[
\begin{align*}
\left( \begin{array}{c} v(t, z) \\ w(t, z) \end{array} \right) &= \frac{e^{(\lambda^+_{c,\varepsilon} - \varepsilon')z}}{2\pi} \int_{-\infty}^{+\infty} e^{in\lambda t} (\lambda^+_{c,\varepsilon} - \varepsilon' + i\eta) I - A)^{-1} G(\lambda^+_{c,\varepsilon} - \varepsilon' + i\eta) d\eta \\
&\quad + k_2 e^{\lambda^+_{c,\varepsilon}z} \left( \begin{array}{c} \psi(t) \\ \lambda^+_{c,\varepsilon} \psi(t) \end{array} \right), \quad z \geq 0,
\end{align*}
\]
(2.25)
where \( k_2 \geq 0 \) is a constant. Let \( \zeta(t, z) = v(t, z) - k_2 e^{\lambda^+_{c,\varepsilon}z} \psi(t) \) for all \((t, z) \in \mathbb{R} \times \mathbb{R}^+\).

Note that \( \zeta(t, z) \) is \( T \)-periodic in \( t \), then (2.25) and (2.22) imply that there exists \( C_3 > 0 \) such that
\[
\left( \int_{z-\frac{T}{2}}^{z+\frac{T}{2}} \left( |\zeta_z(\tau, s)|^2 + |\zeta_z(\tau, s)|^2 + |\zeta(\tau, s)|^2 \right) d\tau d\sigma \right)^{\frac{1}{2}} \leq C_3 e^{(\lambda^+_{c,\varepsilon} - \varepsilon' )z}
\]
for any \( z \geq 0 \). Sobolev embedding theorem then implies that
\[
\sup_{t \in [0, T]} |\zeta(t, z)| \leq C_5 e^{(\lambda^+_{c,\varepsilon} - \varepsilon' )z}
\]
for all \( z \geq 0 \), where \( C_5 > 0 \) is constant. Noting that \( \lambda^+_{c,\varepsilon} = \lambda^+_{c} - \varepsilon > \lambda^+_{c} - \varepsilon' \), then (2.6) yields that \( k_2 > 0 \), and thus
\[
\lim_{z \to +\infty} \frac{v(t, z)}{k_2 e^{\lambda^+_{c,\varepsilon}z} \psi(t)} = 1 \quad \text{uniformly in } t \in \mathbb{R}.
\]

Now set \( \tilde{\zeta}(t, z) = v_z(t, z) - k_2 \lambda^+_{c,\varepsilon} e^{\lambda^+_{c,\varepsilon}z} \psi(t) \) for any \((t, z) \in \mathbb{R} \times \mathbb{R}^+\). We know from (2.22) that there exists \( C_6 > 0 \) such that
\[
\left( \int_{z-\frac{T}{2}}^{z+\frac{T}{2}} \left( |\zeta_z(\tau, s)|^2 + |\zeta_z(\tau, s)|^2 + |\zeta(\tau, s)|^2 \right) d\tau d\sigma \right)^{\frac{1}{2}} \leq C_6 e^{(\lambda^+_{c,\varepsilon} - \varepsilon' )z}
\]
for any \( z \geq 0 \). Noting that for any \((t, z) \in \mathbb{R} \times \mathbb{R}^+\), \( \tilde{\zeta} \) satisfies
\[
[hu(t, u, v)u_z + hu(t, u, v)v_z - hv(t, 0, 0)v_z] + hv(t, 0, 0)\tilde{\zeta} + d\tilde{\xi}_zz + c\tilde{\xi}_z - \tilde{\xi}_t = 0
\]
and \([hu(t, u, v)u_z + hu(t, u, v)v_z - hv(t, 0, 0)v_z] = O(e^{2\lambda^+_{c,\varepsilon}z}) \) as \( z \to +\infty \) by Lemma 2.3. Through the same argument as above, we know that there exists \( C_7 > 0 \) such that
\[
\sup_{t \in [0, T]} |\tilde{\zeta}(t, z)| \leq C_7 e^{(\lambda^+_{c,\varepsilon} - \varepsilon')z}
\]
for any \( z \geq 0 \), and hence
\[
\lim_{z \to +\infty} \frac{v_z(t, z)}{k_2 e^{\lambda^+_{c,\varepsilon}z} \psi(t)} = \lambda^+_{c,\varepsilon} \quad \text{uniformly in } t \in \mathbb{R}.
\]

**Step II.** We study the asymptotic behavior of \( u \). Let \( \tilde{\rho} = \rho + g_u(t, 0, 0) \), where \( \rho = (\lambda^+_{c})^2 + c\lambda^+_{c} \), then \( \tilde{\rho} = \rho - \kappa_1 < 0 \). Hence the equation
\[
g_u(t, 0, 0)\omega(t) + [(\lambda^+_{c})^2 + c\lambda^+_{c}]w - \omega_t = 0
\]
has a unique positive periodic solution \( \tilde{\phi}(t) \) given by (2.18). A direct calculation shows that \( \omega(t, z) := k_2 e^{\lambda^+_{c}z} \tilde{\phi}(t) \) satisfies
\[
g_u(t, 0, 0)\omega + g_e(t, 0, 0)k_2 e^{\lambda^+_{c}z} \psi(t) + \omega_{zz} + cw_z - \omega_t = 0.
\]
Now let
\[\xi(t, z) = \frac{u(t, z) - k_2 e^{\lambda^- z} \tilde{\phi}(t)}{\psi(t)}, \quad \eta(t, z) = \frac{v(t, z) - k_2 e^{\lambda^+ z} \psi(t)}{\psi(t)}, \quad (t, z) \in \mathbb{R} \times \mathbb{R}^+.
\]
Then \(\xi(t, z)\) satisfies \(R(t, z) - \kappa_1 \xi + \xi_{zz} + c \xi_z - \xi_t = 0\) for any \(z \geq 0\), where
\[R(t, z) = [g(t, u, v) - g_u(t(0, 0)v - g_v(t(0, 0))\tilde{\phi}^{-1} + g_v(t(0, 0))\eta].
\]
We know from Step 1 that \(\sup_{t \in \mathbb{R}} |\eta(t, z)| = O(e^{(\lambda^- - \epsilon') z})\) as \(z \to +\infty\). In addition, we know from Lemma 2.3 that \(\sup_{t \in \mathbb{R}} |g(t, u, v) - g_u(t(0, 0)v - g_v(t(0, 0))| \leq K_M e^{(\lambda^- - \epsilon') z}\)
for all \((t, z) \in \mathbb{R} \times [M, +\infty)\). Next we show that \(\sup_{t \in \mathbb{R}} |\xi(t, z)| = O(e^{\lambda^- z})\) as \(z \to +\infty\).

In view of Lemma 2.2, we have \(\sup_{t \in \mathbb{R}} |\xi(t, z)| = O(e^{\lambda^- z})\) as \(z \to +\infty\). Notice that \(\lambda^- - \epsilon' > \lambda_0\), then \(Q = (\lambda^- - \epsilon')^2 + \epsilon(\lambda^- - \epsilon') - \kappa_1 < 0\). It is easy to verify that \(+K e^{(\lambda^- - \epsilon') z}\) satisfy respectively
\[R(t, z) - \kappa_1 \omega + \omega_z + \omega_2 - \omega_t \leq (\geq) 0 \quad \text{for all} \ z \geq M,
\]
wherever \(K \geq \frac{K_M}{|Q|}\). Since \(|\xi(t, z)|\) is bounded in \((t, z) \in \mathbb{R} \times \mathbb{R}^+\), then there exists \(K_Q \geq \frac{K_M}{|Q|}\) such that \(|\xi(t, M)| \leq K_Q e^{(\lambda^- - \epsilon') M}\) for all \(t \in \mathbb{R}\), hence
\[-K_Q e^{(\lambda^- - \epsilon') z} \leq \xi(t, z) \leq K_Q e^{(\lambda^- - \epsilon') z} \quad \text{for all} \ (t, z) \in \mathbb{R} \times [M, +\infty). \quad (2.26)
\]
Indeed, set \(\omega^+(t, z) = \pm K_Q e^{(\lambda^- - \epsilon') z} - \xi(t, z)\) for all \((t, z) \in \mathbb{R} \times [M, +\infty)\), then we have
\[\omega^-_z + \omega^-_2 - \omega^-_t - \kappa_1 \omega^-_t \leq 0, \quad \omega^-_z + \omega^-_2 - \omega^-_t - \kappa_1 \omega^-_t \geq 0. \quad (2.27)
\]
Since \(\omega^+(t, z)\) is \(T^-\) periodic in \(t\), it is sufficient to show that \(\omega^+(t, z) \geq 0\) for \((t, z) \in (0, 2T) \times [M, +\infty)\), while the similar argument holds for \(\omega^-(t, z) \leq 0\). Assume to the contrary that
\[\inf_{(t,z) \in (0,2T) \times [M, +\infty)} \omega^-(t,z) < 0, \quad \text{since} \quad \lim_{z \to +\infty} \sup_{t \in [0,2T]} \omega^+(t,z) = 0,
\]
it follows that there exists \((t^*, z^*) \in (0, 2T) \times [M, +\infty)\) such that \(\omega^+(t^*, z^*) = 0\) and hence \([\omega^+_z + \omega^+_2 - \omega^+_t - \kappa_1 \omega^+_t]_{(t^*, z^*)} > 0\), which contradicts to (2.27). Hence (2.26) implies that \(\sup_{t \in \mathbb{R}} |\xi(t, z)| = O(e^{\lambda^- z})\) as \(z \to +\infty\). Thus, we know from the definition of \(\xi(t, z)\) that
\[\lim_{z \to +\infty} \frac{u(t, z)}{k_2 e^{\lambda^- z} \tilde{\phi}(t)} = 1 \quad \text{uniformly in} \ t \in \mathbb{R}.
\]
The argument for \(u_k(t, z)\) is similar and we only give a brief sketch here. Let
\[\tilde{\xi}(t, z) = \frac{u_z(t, z) - k_2 \lambda^- e^{\lambda^- z} \tilde{\phi}(t)}{\psi(t)}, \quad \tilde{\eta}(t, z) = \frac{v_z(t, z) - k_2 \lambda^- e^{\lambda^- z} \psi(t)}{\psi(t)} \quad \text{for} \ (t, z) \in \mathbb{R} \times \mathbb{R}^+,
\]
then
\[\tilde{R}(t, z) - \kappa_1 \tilde{\xi} + \tilde{\xi}_{zz} + c \tilde{\xi}_z - \tilde{\xi}_t = 0 \quad \text{for all} \ z \geq 0.
\]
with
\[ \tilde{R}(t, z) = [(g_u(t, u, v) - g_u(t, 0, 0)) u_z + (g_v(t, u, v) - g_v(t, 0, 0)) v_z]\tilde{\psi}^{-1} + g_v(t, 0, 0)\tilde{y}. \]

The same argument as above implies that \( \sup_{c \in \mathbb{R}} |\tilde{\xi}(t, z)| = o(e^{\lambda^{-}_c z}) \) as \( z \to +\infty \), and then
\[ \lim_{z \to +\infty} \frac{u_z(t, z)}{k_2e^{\lambda^-_c z}\tilde{\phi}(t)} = \lambda^-_c \text{ uniformly in } t \in \mathbb{R}. \]

Now we complete all the proof. \( \square \)

The following is a direct result of Theorem 2.4.

**Corollary 1.** Assume (A1)-(A4) hold. Let \((P(t, z), Q(t, z)) \in C^1_{b, \theta}(\mathbb{R} \times \mathbb{R}, \mathbb{R}^2)\) be a solution of (1.6). Then
\[ \lim_{z \to +\infty} \frac{1 - P(t, z)}{k_2e^{\lambda^-_c z}\tilde{\phi}(t)} = 1, \quad \lim_{z \to +\infty} \frac{1 - Q(t, z)}{k_2e^{\lambda^-_c z}\tilde{\psi}(t)} = 1 \text{ uniformly in } t \in \mathbb{R}, \quad c \leq c^*, \]
and
\[ \lim_{z \to +\infty} \frac{P_z(t, z)}{k_2e^{\lambda^-_c z}\tilde{\phi}(t)} = -\lambda^-_c, \quad \lim_{z \to +\infty} \frac{Q_z(t, z)}{k_2e^{\lambda^-_c z}\tilde{\psi}(t)} = -\lambda^-_c \text{ uniformly in } t \in \mathbb{R}, \quad c \leq c^*, \]
for some constant \( k_2 > 0 \).

**Remark 4.** For the autonomous system
\[
\begin{cases}
    u_t = u_{xx} + u(1 - u - k_1 v), & (t, x) \in \mathbb{R} \times \mathbb{R}, \\
    v_t = dv_{xx} + av(1 - k_2 u - v), & (t, x) \in \mathbb{R} \times \mathbb{R},
\end{cases}
\]
where \( d, a, k_1, k_2 \) are positive constants. If we further assume that \( 1 - k_1 \geq a(k_2 - 1) > 0 \) and \( \frac{1}{5} < k_1 < 1 \), then the nonlinearity is monostable and (A3) and (A4) hold for (2.28). The traveling wave solution \((\phi(z), \psi(z)) (z = x - ct)\) of (2.28) connecting \((0, 1)\) and \((1, 0)\) satisfies
\[
\begin{align*}
    0 &= \phi'' + c\phi' + (1 - \phi - k_1\psi), \\
    0 &= d\psi'' + c\psi' + a(1 - k_2\phi - \psi), \\
    \lim_{z \to -\infty} (\phi, \psi) &= (0, 1), \quad \lim_{z \to +\infty} (\phi, \psi) = (1, 0).
\end{align*}
\]

Then Proposition 1 yields that
\[ \phi(z) = \alpha_1|z|^l e^{\lambda^+_c z} + h.o.t, \quad 1 - \psi(z) = \beta_1|z|^l e^{\lambda^+_c z} + h.o.t \quad \text{as } z \to -\infty, \]
where \( l = 0 \) if \( c < c^* \) and \( l = 1 \) if \( c = c^* \), and by Corollary 1, we have
\[ 1 - \phi(z) = \alpha_2 e^{\lambda^-_c z} + h.o.t, \quad \psi(z) = \beta_2 e^{\lambda^-_c z} + h.o.t \quad \text{as } z \to +\infty, \]
for all \( c \leq c^* \), where \( c^* = -2\sqrt{1 - k_1}, \ h.o.t \) denotes the higher-order terms, \( \alpha_i, \beta_i \ (i = 1, 2) \) are positive constants, \( \lambda^+_c = \frac{-c - \sqrt{c^2 + 4da(k_2 - 1)}}{2a} > 0 \) and \( \lambda^-_c = \frac{-c - \sqrt{c^2 + 4da(k_2 - 1)}}{2a} < 0 \) are roots of linear eigenvalue equations \( \lambda^2 + c\lambda + (1 - k_1) = 0 \) and \( d\lambda^2 + c\lambda + a(k_2 - 1) = 0 \), respectively. These results are consistent with those in Morita and Tachibana [29].
3. Key estimates. In this section, we give some crucial estimates which are helpful for the construction of sub-super solutions. Throughout this section, we always assume that (A1)-(A4) hold. In view of Proposition 1 and Corollary 1, the following lemma holds obviously.

Lemma 3.1. Let \((P(t, z), Q(t, z)) \in C^{1,2}_{b}(\mathbb{R} \times \mathbb{R}, \mathbb{R}^2)\) be solution of (1.6) with \(c \leq c^{*}\). Then there exist positive constants \(M(c), N(c), m(c), n(c), \delta_{j}(c), \gamma_{j}(c)\) \((j = 1, 2)\) such that

\[
Q(t, z) \leq M(c)P(t, z), \quad t \in \mathbb{R}, \quad z \leq 0,
\]

\[
\delta_{1}(c)P(t, z) \leq P_{2}(t, z) \leq \delta_{2}(c)P(t, z), \quad t \in \mathbb{R}, \quad z \leq 0,
\]

\[
\gamma_{1}(c)Q(t, z) \leq Q_{2}(t, z) \leq \gamma_{2}(c)Q(t, z), \quad t \in \mathbb{R}, \quad z \leq 0.
\]

\[
1 - Q(t, z) \leq N(c)(1 - P(t, z)), \quad t \in \mathbb{R}, \quad z \geq 0,
\]

\[
\delta_{1}(c)m(c)e^{\lambda_{+}^{z}z} \leq \delta_{1}(c)(1 - P(t, z)) \leq P_{2}(t, z), \quad t \in \mathbb{R}, \quad z \geq 0,
\]

\[
\gamma_{1}(c)m(c)e^{\lambda_{+}^{z}z} \leq \gamma_{1}(c)(1 - Q(t, z)) \leq Q_{2}(t, z), \quad t \in \mathbb{R}, \quad z \geq 0.
\]

In particular, for any \(0 < \varepsilon < \lambda_{+}^{z}\), there exist \(K_{\varepsilon}(c) > 0\) such that

\[
P(t, z) \leq K_{\varepsilon}(c)e^{(\lambda_{+}^{z}-\varepsilon)z}, \quad t \in \mathbb{R}, \quad z \leq 0,
\]

\[
Q(t, z) \leq K_{\varepsilon}(c)e^{(\lambda_{+}^{z}-\varepsilon)z}, \quad t \in \mathbb{R}, \quad z \leq 0.
\]

We now give some key estimates in the following two lemmas.

Lemma 3.2. Let \((P_{i}(t, z), Q_{i}(t, z)) \in C^{1,2}_{b}(\mathbb{R} \times \mathbb{R}, \mathbb{R}^2)\) \((i = 1, 2)\) be solutions of (1.6). Assume that \(c_{1} \leq c^{*}\) and \(p_{2} \leq p_{1} \leq 0\). Denote

\[
P_{1} = P_{1}(t, x + p_{1}), \quad P_{2} = P_{2}(t, -x + p_{2}), \quad Q_{1} = Q_{1}(t, x + p_{1}), \quad Q_{2} = Q_{2}(t, -x + p_{2})
\]

and

\[
H_{1}(t, x) = -2a_{1}p_{1}P_{1} + a_{1}p_{1}Q_{1} - 2a_{2}p_{2}Q_{2} + a_{1}p_{1}Q_{1} - 2a_{1}p_{1}Q_{2} - 2a_{2}p_{2}Q_{2}.
\]

Then there exist positive constants \(\alpha_{1}\) and \(K_{1}\) such that

\[
\frac{H_{1}(t, x)}{P_{1,z}(t, x + p_{1}) + P_{2,z}(t, -x + p_{2})} \leq K_{1}e^{\alpha_{1}p_{1}} \text{ for any } (t, x) \in \mathbb{R} \times \mathbb{R}.
\]

Proof. We divide \(x \in \mathbb{R}\) into four intervals.

Case A. \(p_{2} \leq x \leq 0\). Then \(x + p_{1} \leq 0\) and \(-x + p_{2} \leq 0\). By (3.1), (3.2), (3.7) and (3.8), we have

\[
H_{1}(t, x) \leq \frac{a_{1}p_{1}Q_{1}}{P_{2,z}(t, -x + p_{2})} + \frac{a_{1}p_{1}Q_{2}}{P_{2,z}(t, -x + p_{2})} + \frac{K_{e}(c_{1})e^{(\lambda_{+}^{z}-\varepsilon)(x + p_{1})}}{\delta_{1}(c_{2})P_{2}}
\]

\[
\leq \max_{t \in [0, T]}(b_{1})K_{e}(c_{1})(M(c_{2}) + 1)e^{(\lambda_{+}^{z}-\varepsilon)p_{1}}, \quad t \in \mathbb{R}.
\]

Case B. \(0 \leq x \leq -p_{1}\). Then \(x + p_{1} \leq 0\) and \(-x + p_{2} \leq 0\). Similar to case A,

\[
H_{1}(t, x) \leq \frac{a_{1}p_{1}Q_{1}}{P_{2,z}(t, -x + p_{2})} + \frac{a_{1}p_{1}Q_{2}}{P_{2,z}(t, -x + p_{2})} + \frac{K_{e}(c_{2})(M(c_{1}) + 1)}{\delta_{1}(c_{1})}e^{(\lambda_{+}^{z}-\varepsilon)p_{1}}, \quad t \in \mathbb{R}.
\]
Lemma 3.3. Let $(P_i(t,z), Q_i(t,z)) \in C^{1,2}_b(\mathbb{R} \times \mathbb{R}, \mathbb{R}^2)$ $(i = 1, 2)$ be solutions of (1.6). Assume that $c_i \leq c^*$ and $p_2 \leq p_1 \leq 0$. Denote

$$Q_1 = Q_1(t,x + p_1), \quad Q_2 = Q_2(t,-x + p_2)$$

and

$$\tilde{H}_2(t,x) = 2dQ_{1,z}Q_{2,z} + b_2qQ_1Q_2(1 - Q_1)(1 - Q_2).$$

Then there exist positive constants $\alpha_2$ and $K_2$ such that

$$\frac{\tilde{H}_2(t,x)}{(1 - Q_2)Q_{1,z} + (1 - Q_1)Q_{2,z}} \leq K_2 e^{\alpha_2 p_1}, \quad (t,x) \in \mathbb{R} \times \mathbb{R}. \quad (3.10)$$

Proof. We divide $x \in \mathbb{R}$ into four intervals.

Case A. $p_2 \leq x \leq 0$. Then $x + p_1 \leq 0$ and $-x + p_2 \leq 0$. By (3.3) and (3.8), we have

$$H_1(t,x) \leq -a_1 p N_1 P_1 P_2 + a_1 p N_1 [P_1 Q_2 (1 - Q_1) + P_2 Q_1] \leq a_1 p N_1 Q_2 (1 - Q_1) + a_1 p N_1 P_2 (1 - P_1).$$

By (3.4), (3.5), (3.7) and (3.8), we have

$$\frac{H_1(t,x)}{P_{1,z}(t,x + p_1) + P_{2,z}(t,-x + p_2)} \leq \max_{t \in [0,T]} (b_1 q) \times \left[ \frac{K_2 (c_2) e^{(\lambda_2^+ - \epsilon)(-x + p_2)} N(c_1)(1 - P_1)}{\delta_1(c_1)(1 - P_1)} + \frac{K_2 (c_2) e^{(\lambda_2^+ - \epsilon)(-x + p_2)} (1 - P_1)}{\delta_1(c_1)(1 - P_1)} \right] \leq \max_{t \in [0,T]} (b_1 q) K_2 (c_2) (N(c_1) + 1) e^{(\lambda_2^+ - \epsilon) p_1}, \quad t \in \mathbb{R}.$$

Case B. $x = 0$. Then $x + p_1 = 0$ and $-x + p_2 = 0$. Note that $N_1 \leq 1$, then

$$H_1(t,x) \leq -a_1 p N_1 P_1 P_2 + a_1 p N_1 P_1 Q_2 (1 - Q_2) \leq a_1 p N_1 P_1 (1 - P_2) + a_1 p N_1 Q_1 (1 - Q_2),$$

Case C. $x \geq -p_1$. Then $x + p_1 \geq 0$ and $-x + p_2 \leq 0$. Note that $N_1 \leq 1$ and $P_1, Q_i \leq 1$ $(i = 1, 2)$, then

$$H_1(t,x) \leq -a_1 p N_1 P_1 P_2 + a_1 p N_1 [P_1 Q_2 (1 - Q_1) + P_2 Q_1] \leq a_1 p N_1 Q_2 (1 - Q_1) + a_1 p N_1 P_2 (1 - P_1).$$

For any fixed $0 < \epsilon < \min\{\lambda_1^+, \lambda_2^+\}$, now let $\alpha_1 = \min\{\lambda_1^+, \lambda_2^+, -\epsilon\}$ and $K_1 = \max_{t \in [0,T]} (b_1 q) \max_{i,j=1,2, i \neq j} \left\{ \frac{K_2 (c_2) M(c_1) + 1}{\delta_1(c_j)} \right\}$, then (3.9) holds. \qed
4. **Entire solutions.** In this section, we establish the existence and some qualitative properties of invasion entire solutions by constructing appropriate sub-super
solutions and using the comparison principle. Let
\[
\begin{align*}
F_1(t, u, v) &= u_t - u_{xx} - f_1(t, u, v), \\
F_2(t, u, v) &= v_t - dv_{xx} - f_2(t, u, v),
\end{align*}
\]
where \( f_1(t, u, v) = a_1pu(1 - N_1(t) - u + N_1(t)v) \) and \( f_2(t, u, v) = b_2 q(1 - v)(N_2(t)u - v) \). Then \((1.5)\) can be written as
\[
\begin{align*}
F_1(t, u, v) &= 0, \\
F_2(t, u, v) &= 0.
\end{align*}
\]

**Definition 4.1.** Suppose \( s < T \leq \infty \), a pair \((U(t, x), V(t, x)) \in C^{1,2}((s, T) \times \mathbb{R}, [0, 1]^2)\) is said to be a supersolution of \((1.5)\) in \((t, x) \in (s, T) \times \mathbb{R}\), if there holds
\[
\begin{align*}
F_1(t, U, V) &\geq 0, \\
F_2(t, U, V) &\geq 0.
\end{align*}
\]
If for any \( s < T \), \((U(t, x), V(t, x))\) is a supersolution of \((1.5)\) in \((t, x) \in (s, T) \times \mathbb{R}\), then we call that \((U(t, x), V(t, x))\) is a supersolution of \((1.5)\) in \((t, x) \in (-\infty, T) \times \mathbb{R}\). The subsolution \((u(t, x), v(t, x))\) can be defined in a similar way by reversing the inequality.

**Lemma 4.2.** (i) For any \((0, 0) \leq (u_0, v_0) \leq (1, 1)\), system \((1.5)\) admits a unique classical solution \((u(t, x; u_0), v(t, x; v_0))\) with \((u(s, x; u_0), v(s, x; v_0)) = (u_0, v_0)\) which satisfies \(0, 0 \leq (u, v) \leq (1, 1)\) for all \((t, x) \in [s, +\infty) \times \mathbb{R}\).

(ii) Let \((\bar{U}, \bar{V})\) and \((\underline{u}, \underline{v})\) be supersolution and subsolution of \((1.5)\) in \((t, x) \in (s, T) \times \mathbb{R}, \text{respectively. If } (u(s, \cdot), v(s, \cdot)) \leq (\bar{U}(s, \cdot), \bar{V}(s, \cdot))\), then \((u(t, \cdot), v(t, \cdot)) \leq (\bar{U}(t, \cdot), \bar{V}(t, \cdot))\) for all \( s \leq t \leq T \).

**Proof.** The proof is similar to that of [14, Lemma 3.1] and we omit the details here.

To construct a supersolution of \((1.5)\), we first introduce an auxiliary coupled system of ordinary differential equations
\[
\begin{align*}
p_1'(t) &= -c_1 + K e^{\alpha p_1(t)}, & t < 0, \\
p_2'(t) &= -c_2 + K e^{\alpha p_1(t)}, & t < 0, \\
p_2(0) &\leq p_1(0) \leq 0, 
\end{align*}
\]
where \( c_2 \leq c_1 \leq e^*, \alpha \) and \( K \) are positive constants. Solving the equations explicitly, we obtain
\[
\begin{align*}
p_1(t) &= p_1(0) - c_1 t - \frac{1}{\alpha} \ln \left( 1 - \frac{K}{c_1} e^{\alpha p_1(0)} (1 - e^{-c_1 \alpha t}) \right) \leq 0 \ (t \leq 0), \\
p_2(t) &= p_2(0) - c_2 t - \frac{1}{\alpha} \ln \left( 1 - \frac{K}{c_1} e^{\alpha p_1(0)} (1 - e^{-c_1 \alpha t}) \right) \leq 0 \ (t \leq 0).
\end{align*}
\]
Then \( p_1(t) \) is monotone increasing, and by virtue of \( p_2'(t) - p_1'(t) = c_1 - c_2 \geq 0 \), we have \( p_2(t) \leq p_1(t) \leq 0 \) for all \( t \leq 0 \). Let
\[
\omega_1 = p_1(0) - \frac{1}{\alpha} \ln \left( 1 - \frac{K}{c_1} e^{\alpha p_1(0)} \right), \quad \omega_2 = p_2(0) - \frac{1}{\alpha} \ln \left( 1 - \frac{K}{c_1} e^{\alpha p_1(0)} \right).
\]
Then
\[
p(t) - (-c_i t + \omega_i) = -\frac{1}{\alpha} \ln \left( 1 - \frac{c_i}{1 + c_i e^{-c_i \alpha t}} \right) \quad \text{with } \zeta = \frac{K}{c_1} e^{\alpha p_1(0)}.
\]
and there is a constant $C_0 > 0$ such that
\[
0 < p_1(t) - (-c_1t + \omega_1) = p_2(t) - (-c_2t + \omega_2) \leq C_0 e^{-c_1t} \text{ for all } t \leq 0.
\]

Now we can construct a supersolution of (1.5) as follows.

**Lemma 4.3.** Let $(P_i(t, z), Q_i(t, z)) \in C^{1,2}_b(\mathbb{R} \times \mathbb{R}, \mathbb{R}^2) \ (i = 1, 2)$ be the periodic traveling wave of (1.5) with $c_2 \leq c_1 \leq c^*$. Choose $\alpha = \min\{\alpha_1, \alpha_2\}$ and $K = \max\{K_1, K_2\}$ in (4.1), where $(\alpha_1, K_1)$ and $(\alpha_2, K_2)$ are defined as in Lemmas 3.2 and 3.3, respectively. Then

\[
\begin{cases}
\overline{U}(t, x) := \min\{1, P_1(t, x + p_1(t)) + P_2(t, -x + p_2(t))\}, \\
\overline{V}(t, x) := Q_1(t, x + p_1(t)) + Q_2(t, -x + p_2(t)) \\
-\alpha p_1(t, x + p_1(t) + Q_2(t, -x + p_2(t))
\end{cases}
\]

is a supersolution of (1.5) defined in $(t, x) \in (-\infty, 0) \times \mathbb{R}$.

**Proof.** Firstly, we prove $\mathcal{F}_1(t, \overline{U}, \overline{V}) \geq 0$. Denote
\[
\begin{align*}
S_1 &= \{(t, x) | P_1(t, x + p_1(t)) + P_2(t, -x + p_2(t)) > 1\}, \\
S_2 &= \{(t, x) | P_1(t, x + p_1(t)) + P_2(t, -x + p_2(t)) \leq 1\},
\end{align*}
\]
If $(t, x) \in S_1$, then $\overline{U} \equiv 1$ and thus $\mathcal{F}_1(t, \overline{U}, \overline{V}) = b_1 q(1 - \overline{V}) \geq 0$. If $(t, x) \in S_2$, then $\overline{U} = P_1(t, x + p_1(t)) + P_2(t, -x + p_2(t))$. Moreover, we have
\[
\begin{align*}
\mathcal{F}_1(t, \overline{U}, \overline{V}) &= P_1(t, x + p_1(t)) + P_2(t, -x + p_2(t)) \\
&\quad - a_1 p_1(t, x + p_1(t)) + a_1 p_2(t, -x + p_2(t)) \\
&\quad + K e^{\alpha p_1}(P_1(t, x + p_1(t)) + P_2(t, -x + p_2(t))) \\
&\quad - a_1 p_1(t, x + p_1(t)) - a_1 p_2(t, -x + p_2(t)) \\
&\quad + K e^{\alpha p_1}((P_1(t, x + p_1(t)) + P_2(t, -x + p_2(t))) - H_1(t, x),
\end{align*}
\]
where $H_1(t, x) = -2a_1 p_1(t, x + p_1(t)) + a_1 p N_1[p_1 Q_2(1 - Q_1) + P_2 Q_1(1 - Q_2)]$. By Lemma 3.2, there hold
\[
H_1(t, x) \leq K_1 e^{\alpha p_1} \leq K e^{\alpha p_1} \text{ for any } (t, x) \in (-\infty, 0) \times \mathbb{R},
\]
and hence
\[
\mathcal{F}_1(t, \overline{U}, \overline{V}) = K e^{\alpha p_1}(P_1(t, x + p_1(t)) + P_2(t, -x + p_2(t))) - H_1(t, x) \geq 0 \quad \text{for any } (t, x) \in (-\infty, 0) \times \mathbb{R}.
\]
Then we prove that $\mathcal{F}_2(t, \overline{U}, \overline{V}) \geq 0$. Noting that
\[
\begin{align*}
\mathcal{F}_2(t, \overline{U}, \overline{V}) &= (Q_1(t, x + p_1(t) - c_1 Q_1 + d Q_1, z_1)(1 - Q_2) + (Q_2(t, x - c_2 Q_2, z_2)(1 - Q_1)) \\
&\quad - 2d Q_1 Q_2 + b_2 q(1 - (Q_1 + Q_2 - Q_1 Q_2)[N_2 U - (Q_1 + Q_2 - Q_1 Q_2)] \\
&\quad + K e^{\alpha p_1}[(1 - Q_2) Q_1, z_1 + (1 - Q_1) Q_2, z_2] \\
&\quad = b_2 q(1 - Q_1)[N_2 P_2 - Q_1)(1 - Q_2) + b_2 q(1 - Q_2)(N_2 P_2 - Q_2) - Q_1) \\
&\quad - 2d Q_1 Q_2 + b_2 q(1 - (Q_1 + Q_2 - Q_1 Q_2)[N_2 U - (Q_1 + Q_2 - Q_1 Q_2)] \\
&\quad + K e^{\alpha p_1}[(1 - Q_2) Q_1, z_1 + (1 - Q_1) Q_2, z_2]
\end{align*}
\]

$= K e^{\alpha p_1}[(1 - Q_2) Q_1, z_1 + (1 - Q_1) Q_2, z_2] - H_2(t, x),$
where \( H_2(t, x) = 2dQ_{1,z}Q_{2,z} + b_2q(1 - Q_1)(1 - Q_2)|N_2(\bar{U} - P_1 - P_2) + Q_1Q_2 \). It is easy to see that \( H_2(t, x) \leq 2dQ_{1,z}Q_{2,z} + b_2qQ_1Q_2(1 - Q_1)(1 - Q_2) = H_2(t, x) \).

Then it follows from Lemma 3.3 that
\[
\frac{\dot{H}_2(t, x)}{(1 - Q_2)Q_{1,z} + (1 - Q_1)Q_{2,z}} \leq K_2 e^{\alpha_2 p_1} \leq K e^{\alpha p_1} \text{ for any } (t, x) \in (-\infty, 0] \times \mathbb{R}.
\]

Hence
\[
\mathcal{F}_2(t, U, V) = K e^{\alpha p_1}[(1 - Q_2)Q_{1,z} + (1 - Q_1)Q_{2,z}] - H_2(t, x) \geq 0
\]
for any \((t, x) \in (-\infty, 0] \times \mathbb{R}\). The proof is complete. 

We now state our main result as follows.

**Theorem 4.4.** Assume (A1)-(A4) hold. Let \((P_1(t, z), Q_1(t, z))\) be the periodic traveling wave solution of system (1.5) with \( 0 \leq c_1 \leq -2\sqrt{\gamma} \). Then for any given constants \( \theta_1, \theta_2 \in \mathbb{R} \), there exists an entire solution \((U_{\theta_1, \theta_2}(t, x), V_{\theta_1, \theta_2}(t, x))\) of (1.5) such that \((0, 0) < (U_{\theta_1, \theta_2}, V_{\theta_1, \theta_2}) < (1, 1)\), and satisfying

\[
\lim_{t \to -\infty} \left\{ \sup_{x \geq 0} |U_{\theta_1, \theta_2}(t, x) - P_1(t, x - c_1 t + \theta_1)| + \sup_{x \leq 0} |U_{\theta_1, \theta_2}(t, x) - P_2(t, -x - c_2 t + \theta_2)| \right\} = 0
\]  

(4.4)

\[
\lim_{t \to -\infty} \left\{ \sup_{x \geq 0} |\theta_1, \theta_2(t, x) - Q_1(t, x - c_1 t + \theta_1)| + \sup_{x \leq 0} |\theta_1, \theta_2(t, x) - Q_2(t, -x - c_2 t + \theta_2)| \right\} = 0.
\]  

(4.5)

Furthermore, we have

(i) \((U_{\theta_1, \theta_2}(t + T, x), V_{\theta_1, \theta_2}(t + T, x)) = (U_{\theta_1, \theta_2}(t, x), V_{\theta_1, \theta_2}(t, x))\) for any \((t, x) \in \mathbb{R} \times \mathbb{R}\); or \((U_{\theta_1, \theta_2}(t, x), V_{\theta_1, \theta_2}(t, x))\) for any \((t, x) \in \mathbb{R} \times \mathbb{R}\); (ii) \(\lim_{t \to +\infty} \sup_{x \geq 0} \{|U_{\theta_1, \theta_2}(t, x) - 1| + |V_{\theta_1, \theta_2}(t, x) - 1|\} = 0\); (iii) \(\lim_{t \to -\infty} \sup_{x \in (x_i, x_j)} \{|U_{\theta_1, \theta_2}(t, x)| + |V_{\theta_1, \theta_2}(t, x)|\} = 0\) for any \(x_1 < x_2\); (iv) \(\lim_{t \to +\infty} \sup_{|x| \leq t_0} \{|U_{\theta_1, \theta_2}(t, x) - 1| + |V_{\theta_1, \theta_2}(t, x) - 1|\} = 0\) for any \(t_0 \in \mathbb{R}\); (v) \((U_{\theta_1, \theta_2}(t, x), V_{\theta_1, \theta_2}(t, x))\) is monotone increasing with respect to \(\theta_1\) and \(\theta_2\) for any \((t, x) \in \mathbb{R}^2\); (vi) \((U_{\theta_1, \theta_2}(t, x), V_{\theta_1, \theta_2}(t, x))\) converges to \((1, 1)\) locally in \((t, x) \in \mathbb{R}^2\) as \(\theta_1 \to +\infty\).

**Proof.** Let \(\omega_1\) and \(\omega_2\) be as in (4.2) and define

\[
\begin{align*}
(u(t, x)) &= \max\{P_1(t, x - c_1 t + \omega_1), P_2(t, -x - c_2 t + \omega_2)\}, \\
(v(t, x)) &= \max\{Q_1(t, x - c_1 t + \omega_1), Q_2(t, -x - c_2 t + \omega_2)\},
\end{align*}
\]

(4.6)

then \((u, v)\) is a subsolution of (1.5) in \((t, x) \in \mathbb{R} \times \mathbb{R}\), satisfying \((u, v) \leq (U, V)\) for any \((t, x) \in (-\infty, 0] \times \mathbb{R}\), where \((U, V)\) is defined in (4.3). Now consider the
following initial value problem

\[
\begin{align*}
\frac{\partial u^n}{\partial t} &= f_1(t, u^n, v^n), & (t, x) \in (-n, +\infty) \times \mathbb{R}, \\
\frac{\partial v^n}{\partial t} &= f_2(t, u^n, v^n), & (t, x) \in (-n, +\infty) \times \mathbb{R}, \\
u^n(-n, x) &= u_0^n(x) = u_0(-n, x), & x \in \mathbb{R}, \\
v^n(-n, x) &= v_0^n(x) = v_0(-n, x), & x \in \mathbb{R}.
\end{align*}
\] (4.7)

We know from [26] that the problem (4.7) is well posed and the (strong) maximum principle holds since all the coefficients are periodic with respect to t. By virtue of Lemmas 4.2 and 4.3, for \( x \in \mathbb{R} \), we have

\[
\begin{align*}
\left\{ (u(t, x), v(t, x)) \leq (u^n(t, x), v^n(t, x)) \leq (u^{n+1}(t, x), v^{n+1}(t, x)) \leq (1, 1), t \geq -n, \\
(u(t, x), v(t, x)) \leq (u^n(t, x), v^n(t, x)) \leq (U(t, x), V(t, x)), t \in (-n, 0].
\end{align*}
\]

Using the standard parabolic estimates and the diagonal extraction process, there exists a subsequence \( \{(u^{n_k}(t, x), v^{n_k}(t, x))\}_{k \in \mathbb{N}} \) such that \( \{(u^{n_k}(t, x), v^{n_k}(t, x))\}_{k \in \mathbb{N}} \) converges to a function \((u(t, x), v(t, x))\) locally in \((t, x) \in \mathbb{R} \times \mathbb{R}\) as \( k \to +\infty \) \((n_k \to +\infty)\). In view of \((u^n(t, x), v^n(t, x)) \leq (u^{n+1}(t, x), v^{n+1}(t, x))\) for any \( t > -n \), \((u^n(t, x), v^n(t, x))\) converges to \((u(t, x), v(t, x))\) in \( \mathbb{R}^2 \) as \( n \to +\infty \). Clearly, \((u(t, x), v(t, x))\) is an entire solution of (1.5) and satisfies

\[
\begin{align*}
\left\{ (u(t, x), v(t, x)) \leq (u(t, x), v(t, x)) \leq (U(t, x), V(t, x)), t \in (-n, 0].
\end{align*}
\]

(4.8)

Particularly, the (strong) maximum principle implies that for any \((t, x) \in \mathbb{R} \times \mathbb{R}, (0, 0) < (u(t, x), v(t, x)) < (1, 1)\).

We now prove (4.4) and (4.5). Firstly, we prove

\[
\lim_{t \to -\infty} \left\{ \sup_{z \geq 0} |u(t, x) - P_1(t, x - c_1 t + \omega_1)| + \sup_{x \leq 0} |u(t, x) - P_2(t, -x - c_2 t + \omega_2)| \right\} = 0.
\]

(4.9)

For \( x \geq 0 \), there exists \( L_1 > 0 \) such that

\[
\begin{align*}
0 &\leq u(t, x) - u(t, x) \\
&\leq u(t, x) - P_1(t, x - c_1 t + \omega_1) \\
&\leq U(t, x) - P_1(t, x - c_1 t + \omega_1) \\
&\leq P_1(t, x + p_1(t)) + P_2(t, -x + p_2(t)) - P_1(t, x - c_1 t + \omega_1) \\
&\leq K \varepsilon (c_2) e^{(\lambda_2 - \varepsilon)(x + p_2)} + \sup_{(t, x) \in [0, T] \times \mathbb{R}} |P_1(t, z)| \cdot |P_1(t) - (c_1 t + \omega_1)| \\
&\leq K \varepsilon (c_2) e^{\alpha p_1} + L_1 e^{(-c_1 \omega_1)} \to 0 \text{ as } t \to -\infty.
\end{align*}
\]

For \( x \leq 0 \), there exists \( L_2 > 0 \) such that

\[
\begin{align*}
0 &\leq u(t, x) - u(t, x) \\
&\leq u(t, x) - P_2(t, -x - c_2 t + \omega_2) \\
&\leq U(t, x) - P_2(t, -x - c_2 t + \omega_2) \\
&\leq P_1(t, x + p_1(t)) + P_2(t, -x + p_2(t)) - P_2(t, -x - c_2 t + \omega_2) \\
&\leq K \varepsilon (c_1) e^{(\lambda_1 - \varepsilon)(x + p_1)} + \sup_{(t, x) \in [0, T] \times \mathbb{R}} |P_2(t, z)| \cdot |P_2(t) - (-c_2 t + \omega_2)| \\
&\leq K \varepsilon (c_1) e^{\alpha p_1} + L_2 e^{(-c_1 \omega_1)} \to 0 \text{ as } t \to -\infty.
\end{align*}
\]
(4.9) then follows.

Note from (4.2) that \( \omega_1 = \omega_1(p_1(0)) \) and \( \omega_2 = \omega_2(p_1(0), p_2(0)) \) are defined for any \( p_2(0) \leq p_1(0) \leq 0 \). Then for any \( \theta_1, \theta_2 \in \mathbb{R} \), there exist \( p_2(0) \leq p_1(0) \leq 0 \) such that \( \omega_1 = \omega_1(p_1(0)) \) and \( \omega_2 = \omega_2(p_1(0), p_2(0)) \) satisfy \( n^* := \frac{\omega_1 - \theta_1 - \omega_2 - \theta_2}{c_1 + c_2} \in \mathbb{Z} \).

Define

\[
(U_{\theta_1, \theta_2}(t, x), V_{\theta_1, \theta_2}(t, x)) = (u(t + n^* T, x + x_0), v(t + n^* T, x + x_0))
\]

with \( x_0 = \frac{\omega_1 - \theta_1 - \omega_2 - \theta_2}{c_1 + c_2} \), then \( (U_{\theta_1, \theta_2}(t, x), V_{\theta_1, \theta_2}(t, x)) \) is an entire solution of (1.5). In view of (4.9), we can easily see (4.4) holds. A similar argument yields that (4.5) holds.

The assertions (ii)-(vi) in Theorem 4.4 are straightforward consequences of (4.8). Therefore, we only prove the assertion (i).

(i) For any \( (0, 0) \leq (u_0, v_0) \leq (1, 1) \), let \((u(t, x; u_0), v(t, x; v_0))\) be the unique classical solution of (1.5) with initial value \((u(0, x; u_0), v(0, x; v_0)) = (u_0, v_0)\), then it is easy to see that \(u^n(t, x), v^n(t, x) = (u(t + n, x; u(-n, \cdot)), v(t + n, x; v(-n, \cdot)))\).

Note that for any \((t, x) \in \mathbb{R} \times \mathbb{R}\), there is

\[
u(t + T, x) = \max \{P_1(t + T, x - c_1(t + T) + \omega_1), P_2(t + T, -x - c_2(t + T) + \omega_2)\} \geq \max \{P_1(t, x - c_1(t + T) + \omega_1), P_2(t, -x - c_2(t + T) + \omega_2)\} = u(T, x),
\]

and similarly \(v(t + T, x) \geq v(T, x)\). It follows from the uniqueness of solutions and the comparison principle that for any \((t, x) \in [-n, +\infty) \times \mathbb{R}\), there hold

\[
u^n(t + T, x) = u(t + T + n, x; u(-n, \cdot)) = u(t + n, x; u(T, x; u(-n, \cdot))) \geq u(T + n, x; u(-n, \cdot)) = u^n(t, x),
\]

and similarly \(v^n(t + T, x) \geq v^n(t, x)\). Then there holds \((u(t + T, x), v(t + T, x)) \geq (u(t, x), v(t, x))\) for any \((t, x) \in \mathbb{R} \times \mathbb{R}\). Therefore, the (strong) maximum principle further implies that \((u(t + T, x), v(t + T, x)) = (u(t, x), v(t, x))\) or \((u(t + T, x), v(t + T, x)) > (u(t, x), v(t, x))\) for any \((t, x) \in \mathbb{R} \times \mathbb{R}\). (i) then follows. This completes the proof.

\[\square\]

Remark 5. For the autonomous Lotka-Volterra competition system with random (local) and nonlocal dispersal, Morita and Tachibana [29] and Li et al. [24] established the existence of invasion entire solutions, respectively. Notice that in their papers, the following condition is needed, which may be technical:

\[\text{(C): There exists a positive number } \eta_0 \text{ such that } \frac{\phi(z)}{1 - \varphi(z)} \geq \eta_0 \text{ for } z \leq 0, \text{ where } \phi(z), \varphi(z) \text{ is the invasion traveling wave solution.}\]

In fact, according to Remark 4, when the time periodic system (1.1) degenerates into the homogeneous case, the condition (C) holds obviously under our assumptions (A1)-(A3). We point out that the following supersolution

\[
\begin{align*}
\mathbf{U}(t, x) &:= P_1(t, x + p_1(t)) + P_2(t, -x + p_2(t)) - P_1(t, x + p_1(t)) P_2(t, -x + p_2(t)), \\
\mathbf{V}(t, x) &:= Q_1(t, x + p_1(t)) + Q_2(t, -x + p_2(t)) - Q_1(t, x + p_1(t)) Q_2(t, -x + p_2(t)),
\end{align*}
\]

which has been used in [29, 24], is also applicable to our problem. In this sense, we generalize the result about entire solutions from autonomous case to periodic case.
Remark 6. By the relation between systems (1.5) and (1.1), we get that (1.1) admits an entire solution \((u(t, x), v(t, x)) := (p(t)U_{\theta_1, \theta_2}(t, x), q(t)(1 - V_{\theta_1, \theta_2}(t, x)))\). According to Theorem 4.4 (ii) and (iii), we have
\[
\lim_{t \to -\infty} \{|u(t, x)| + |v(t, x)| - q(t)| = 0 \text{ locally in } x \in \mathbb{R},
\]
\[
\lim_{t \to +\infty} \{|u(t, x) - p(t)| + |v(t, x)| = 0 \text{ uniformly in } x \in \mathbb{R},
\]
which indicates that the entire solution \((u, v)\) exhibits the extinction of the inferior species \(v\) by the superior one \(u\) invading from both sides of \(x\)-axis. In fact, this kind of entire solution describes a different type of biological invasion from one presented by traveling waves in a time periodic environment. On the other hand, we point out in particular that the continuous dependence of such an entire solution on parameters such as wave speeds and the shifted variables is important but still open. For some related works on this issue, one can see Hamel and Nadirashvili [15] for a local dispersal KPP equation, Wang et al. [40] for a delayed lattice differential equation, and Li et al. [23] for a nonlocal dispersal periodic monostable equation. We will leave such problems about our system (1.1) for a future study.

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REFERENCES

[1] N. D. Alikakos, P. W. Bates and X. Chen, Periodic traveling waves and locating oscillating patterns in multidimensional domains, Trans. Amer. Math. Soc., 351 (1999), 2777–2805.
[2] X. Bao, W. T. Li and Z. C. Wang, Time periodic traveling curved fronts in the periodic Lotka-Volterra competition-diffusion system, J. Dynam. Differential Equations, (2015), 1–36.
[3] X. Bao and Z. C. Wang, Existence and stability of time periodic traveling waves for a periodic bistable Lotka-Volterra competition system, J. Differential Equations, 255 (2013), 2402–2435.
[4] P. W. Bates and F. Chen, Periodic traveling waves for a nonlocal integro-differential model, Electron. J. Differential Equations, 1999 (1999), 1–19.
[5] H. Berestycki and F. Hamel, Front propagation in periodic excitable media, Comm. Pure Appl. Math., 55 (2002), 949–1032.
[6] Z. H. Bu, Z. C. Wang and N. W. Liu, Asymptotic behavior of pulsating fronts and entire solutions of reaction-advection-diffusion equations in periodic media, Nonlinear Anal. Real World Appl., 28 (2016), 48–71.
[7] X. Chen and J. S. Guo, Existence and uniqueness of entire solutions for a reaction-diffusion equation, J. Differential Equations, 212 (2005), 62–84.
[8] C. Conley and R. Gardner, An application of the generalized morse index to travelling wave solutions of a competitive reaction-diffusion model, Indiana Univ. Math. J., 33 (1984), 319–343.
[9] J. Foldes and P. Poláčik, On cooperative parabolic systems: Harnack inequalities and asymptotic symmetry, Discrete Cont. Dynam. Syst. Ser. A., 25 (2009), 133–157.
[10] Y. Fukao, Y. Morita and H. Ninomiya, Some entire solutions of the Allen-Cahn equation, Taiwanese J. Math., 8 (2004), 15–32.
[11] R. A. Gardner, Existence and stability of traveling wave solutions of competition models: A degree theoretic approach, J. Differential Equations, 44 (1982), 343–364.
[12] J. S. Guo and Y. Morita, Entire solutions of reaction-diffusion equations and an application to discrete diffusive equations, Discrete Contin. Dyn. Syst. Ser. A., 12 (2005), 193–212.
[13] J. S. Guo and C. H. Wu, Entire solutions for a two-component competition system in a lattice, Tohoku Math. J., 62 (2010), 17–28.
[14] F. Hamel, Qualitative properties of monostable pulsating fronts: Exponential decayed monotonicity, J. Math. Pures Appl., 89 (2008), 355–399.
[15] F. Hamel and N. Nadirashvili, Entire solutions of the KPP equation, *Comm. Pure Appl. Math.*, **52** (1999), 1255–1276.
[16] F. Hamel and N. Nadirashvili, Travelling fronts and entire solutions of the Fisher-KPP equation in $\mathbb{R}^N$, *Arch. Ration. Mech. Anal.*, **157** (2001), 91–163.
[17] Y. Hosono, Singular perturbation analysis of travelling waves of diffusive Lotka-Volterra competition models, *Numerical and Applied Mathematics, Part II (Paris 1988)*, Baltzer, Basel, (1989), 687–692.
[18] X. Hou and A. W. Leung, Traveling wave solutions for a competitive reaction-diffusion system and their asymptotics, *Nonlinear Anal. Real World Appl.*, **9** (2008), 2196–2213.
[19] Y. Kan-On, Parameter dependence of propagation speed of travelling waves for competition-diffusion equations, *SIAM J. Math. Anal.*, **26** (1995), 340–363.
[20] Y. Kan-On, Fisher wave fronts for the Lotka-Volterra competition model with diffusion, *Nonlinear Anal.*, **28** (1997), 145–164.
[21] W. T. Li, Y. J. Sun and Z. C. Wang, Entire solutions in the Fisher-KPP equation with nonlocal dispersal, *Nonlinear Anal. Real World Appl.*, **11** (2010), 2302–2313.
[22] W. T. Li, Z. C. Wang and J. Wu, Entire solutions in monostable reaction-diffusion equations with delayed nonlinearity, *J. Differential Equations*, **245** (2008), 102–129.
[23] W. T. Li, J. B. Wang and L. Zhang, Entire solutions of nonlocal dispersal equations with monostable nonlinearity in space periodic habitats, *J. Differential Equations*, **261** (2016), 2472–2501.
[24] W. T. Li, L. Zhang and G. B. Zhang, Invasion entire solutions in a competition system with nonlocal dispersal, *Discrete Contin. Dyn. Syst. Ser. A.*, **35** (2015), 1531–1560.
[25] A. Lunardi, *Analytic Semigroups and Optimal Regularity in Parabolic Problems*, Birkhäuser, Boston, 1995.
[26] G. Lv and M. Wang, Traveling wave front in diffusive and competitive Lotka-Volterra system with delays, *Nonlinear Anal. Real World Appl.*, **11** (2010), 1323–1329.
[27] Y. Morita and H. Ninomiya, Entire solutions with merging fronts to reaction-diffusion equations, *J. Dynam. Differential Equations*, **18** (2006), 841–861.
[28] Y. Morita and K. Tachibana, An entire solution to the Lotka-Volterra competition-diffusion equations, *SIAM J. Math. Anal.*, **40** (2009), 2217–2240.
[29] G. Nadin, Traveling fronts in space-time periodic media, *J. Math. Pures Appl.*, **92** (2009), 232–262.
[30] G. Nadin, Existence and uniqueness of the solution of a space-time periodic reaction-diffusion equation, *J. Differential Equations*, **249** (2010), 1288–1304.
[31] J. Nolen, M. Rudd and J. Xin, Existence of KPP fronts in spatially-temporally periodic advection and variational principle for propagation speeds, *Dyn. Partial Differ. Equ.*, **2** (2005), 1–24.
[32] W. Shen, Traveling waves in time periodic lattice differential equations, *Nonlinear Anal.*, **54** (2003), 319–339.
[33] W. J. Sheng and J. B. Wang, Entire solutions of time periodic bistable reaction-advection-diffusion equations in infinite cylinders, *J. Math. Phys.*, **56** (2015), 081501, 17 pp.
[34] Y. J. Sun, W. T. Li and Z. C. Wang, Entire solutions in nonlocal dispersal equations with bistable nonlinearity, *J. Differential Equations*, **251** (2011), 551–581.
[35] M. M. Tang and P. C. Fife, Propagating fronts for competing species equations with diffusion, *Arch. Ration. Mech. Anal.*, **73** (1980), 69–77.
[36] J. H. Vuuren, The existence of traveling plane waves in a general class of competition-diffusion systems, *SIMA J. Appl. Math.*, **55** (1995), 135–148.
[37] M. Wang and G. Lv, Entire solutions of a diffusive and competitive Lotka-Volterra type system with nonlocal delays, *Nonlinearity*, **23** (2010), 1609–1630.
[38] Z. C. Wang, W. T. Li and S. Ruan, Entire solutions in bistable reaction-diffusion equations with nonlocal delays, *Trans. Amer. Math. Soc.*, **361** (2009), 2047–2084.
[39] Z. C. Wang, W. T. Li and J. Wu, Entire solutions in delayed lattice differential equations with monostable nonlinearity, *SIAM J. Math. Anal.*, **40** (2009), 2392–2420.
[40] H. Yagisita, Backward global solutions characterizing annihilation dynamics of travelling fronts, *Publ. Res. Inst. Math. Sci.*, **39** (2003), 117–164.
[42] L. Zhang, W. T. Li and S. L. Wu, Multi-type entire solutions in a nonlocal dispersal epidemic model, *J. Dynom. Differential Equations*, 28 (2016), 189–224.

[43] G. Zhao and S. Ruan, Existence, uniqueness and asymptotic stability of time periodic traveling waves for a periodic Lotka-Volterra competition system with diffusion, *J. Math. Pures Appl.*, 95 (2011), 627–671.

[44] G. Zhao and S. Ruan, Time periodic traveling wave solutions for periodic advection-reaction-diffusion systems, *J. Differential Equations*, 257 (2014), 1078–1147.

[45] X. Q. Zhao, *Dynamical Systems in Population Biology*, Springer-Verlag, New York, 2003.

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