A BIVARIATE GENERATING FUNCTION FOR ZETA VALUES
AND RELATED SUPERCONGRUENCES

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Abstract. By using the Wilf-Zeilberger method, we prove a novel finite combinatorial
identity related to a bivariate generating function for \(\zeta(2 + r + 2s)\) (an extension of a
Bailey-Borwein-Bradley Apéry-like formula for even zeta values). Such identity is then
applied to show several supercongruences.

1. Introduction

The bivariate formula
\[
\sum_{k=1}^{\infty} \frac{k}{k^4 - a^2 k^2 - b^4} = \frac{1}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}(5k^2 - a^2)}{k^{2k} k} \prod_{j=1}^{k-1}(j^2 - a^2)^2 + 4b^4 \prod_{j=1}^{k}(j^4 - a^2 j^2 - b^4)
\]

(1)

has been first conjectured by H. Cohen and then proved independently by T. Rivoal [11,
Theorem 1.1] and D. M. Bradley [3, Theorem 1] by reducing it to the finite combinatorial identity
\[
\sum_{k=1}^{n} \frac{(2k)}{k} \frac{(5k^2 - a^2) \prod_{j=1}^{k-1}(n^2 - j^2)(n^2 + j^2 - a^2)}{\prod_{j=1}^{k}(n^2 + (n - j)^2 - a^2)(n^2 + (n + j)^2 - a^2)} = \frac{2}{n^2 - a^2}
\]

(2)

and by Kh. and T. Hessami Pilehrood [6, Theorem 1] by applying the Wilf-Zeilberger theory. Since the left-hand side of (1) can be written as the generating function of \(\zeta(3 + 2r + 4s)\),
\[
\sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \zeta(3 + 2r + 4s)a^{2r}b^{4s},
\]

it follows that, by extracting the coefficients for \((r, s) = (0, 0)\) and \((r, s) = (1, 0)\), we obtain the Apéry-like identities
\[
\zeta(3) = \frac{5}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^3 \binom{2k}{k}} \quad \text{and} \quad \zeta(5) = \frac{1}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{\binom{2k}{k}} \left( \frac{4}{k^5} - \frac{5H_{k-1}(2)}{k^3} \right)
\]

(3)

where \(H_n(s) = \sum_{j=1}^{n} \frac{1}{j^s}\) is the harmonic sum of weight \(s\). For more details about Apéry-like series see also [2, 4, 6, 5].

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Here we consider a similar bivariate formula
\[ \sum_{k=1}^{\infty} \frac{1}{k^2 - ak - b^2} = \sum_{k=1}^{\infty} \frac{(3k - a)}{k(2k)} \cdot \frac{\prod_{j=1}^{k-1} (j^2 - a^2 - 4b^2)}{\prod_{j=1}^{k} (j^2 - aj - b^2)} \]

where the left-hand side is the generating function of \( \zeta(2 + r + 2s) \),
\[ \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \binom{r + s}{r} \zeta(2 + r + 2s)a^rb^s. \]

For \( a = 0 \), (4) yields a formula due to D. H. Bailey, J. M. Borwein, and D. M. Bradley,
\[ \sum_{s=0}^{\infty} \zeta(2 + 2s)b^{2s} = \sum_{k=1}^{\infty} \frac{1}{k^2 - b^2} = 3 \sum_{k=1}^{\infty} \frac{1}{k(2k)} \cdot \frac{\prod_{j=1}^{k-1} (j^2 - 4b^2)}{\prod_{j=1}^{k} (j^2 - b^2)} \]

which appeared in [2, Theorem 1.1]. Moreover, for \((r, s) = (1, 0)\) and \((r, s) = (0, 1)\), we get the Apery-like identities
\[ \zeta(3) = \sum_{k=1}^{\infty} \frac{1}{(2k)} \left( \frac{2}{k^3} + \frac{3H_{k-1}(1)}{k^2} \right) \quad \text{and} \quad \zeta(4) = 3 \sum_{k=1}^{\infty} \frac{1}{(2k)} \left( \frac{1}{k^4} - \frac{3H_{k-1}(2)}{k^2} \right). \]

Replacing \( a \) by \( 2a \) and then letting \( x^2 = a^2 + b^2 \) in (4) we find the equivalent identity
\[ \sum_{k=1}^{\infty} \frac{1}{(k - a)^2 - x^2} = \sum_{k=1}^{\infty} \frac{(3k - 2a)}{k(2k)} \cdot \frac{\prod_{j=1}^{k-1} (j^2 - 4x^2)}{\prod_{j=1}^{k} ((j - a) - x^2)} \]

which has been proved by Kh. and T. Hessami Pilehrood [7 (24)].

Again, in the same spirit of what has been done for (4), our proof of the identity (4) is reduced to show the following novel finite identity
\[ \sum_{k=1}^{n} \frac{\binom{2k}{k}}{3k - 2n + a} \cdot \prod_{j=1}^{k-1} \frac{(j - n)(j - n + a)}{j^2 - a^2} = \frac{2}{n - a}. \]

In [15, Theorem 4.2] the author established that for any prime \( p > 5 \),
\[ \sum_{k=1}^{p-1} \frac{\binom{2k}{k}}{k^2} \equiv -\frac{8H_{p-1}(1)}{3} \pmod{p^4}, \]
\[ \sum_{k=1}^{p-1} \frac{(-1)^k}{k} \binom{2k}{k} \equiv \frac{4}{5} \left( \frac{H_{p-1}(1)}{p} + 2pH_{p-1}(3) \right) \pmod{p^4}. \]
Thanks to the finite identities (2) and (6), we managed to improve congruence (7) and to show several other congruences. The main results are as follows: for any prime \( p > 5 \),
\[
\sum_{k=1}^{p-1} \frac{1}{k^3} \left( \begin{array}{c} 2k \\ k \end{array} \right) \equiv -\frac{2H_{p-1}(1)}{p^2} \pmod{p^2},
\]
\[
\sum_{k=1}^{p-1} \frac{H_k(2)}{k} \left( \begin{array}{c} 2k \\ k \end{array} \right) \equiv \frac{2H_{p-1}(1)}{3p^2} \pmod{p^2}.
\]
These congruences are known modulo \( p \) (see [8, Theorem 2]) and they confirm modulo \( p^2 \) the following conjecture by Z.-W. Sun: for each prime \( p > 7 \),
\[
\sum_{k=1}^{p-1} \frac{1}{k^3} \left( \begin{array}{c} 2k \\ k \end{array} \right) \equiv -\frac{2H_{p-1}(1)}{p^2} - \frac{13H_{p-1}(3)}{27} \pmod{p^3},
\]
\[
\sum_{k=1}^{p-1} \frac{H_k(2)}{k} \left( \begin{array}{c} 2k \\ k \end{array} \right) \equiv \frac{2H_{p-1}(1)}{3p^2} - \frac{38H_{p-1}(3)}{81} \pmod{p^3}.
\]
the first one appeared in [13, Conjecture 1.1] and the second one in [14, Conjecture 5.1].

2. Preliminaries concerning multiple harmonic sums

We define the multiple harmonic sum as
\[
H_n(s_1, \ldots, s_r) = \sum_{1 \leq k_1 < k_2 < \cdots < k_r \leq n} \frac{1}{k_1^{s_1} k_2^{s_2} \cdots k_r^{s_r}}
\]
where \( n \geq r > 0 \) and each \( s_i \) is a positive integer. The sum \( s_1 + s_2 + \cdots + s_r \) is the weight of the multiple sum. Furthermore, by \( \{s_1, s_2, \ldots, s_j\}^m \) we denote the sequence of length \( mj \) with \( m \) repetitions of \( (s_1, s_2, \ldots, s_j) \).

By [12, Theorem 5.1]), for any prime \( p > s + 2 \) we have
\[
H_{p-1}(s) \equiv \begin{cases} 
\frac{s(s+1)}{2(s+2)} p B_{p-s-2} \pmod{p^3} & \text{if } s \text{ is odd,} \\
\frac{s}{s+1} p B_{p-s-1} \pmod{p^2} & \text{if } s \text{ is even.}
\end{cases}
\]
where \( B_n \) be the \( n \)-th Bernoulli number.

Let \( p > 5 \) be a prime, then by [15, Theorem 2.1],
\[
H_{p-1}(2) \equiv -\frac{2H_{p-1}(1)}{p} - \frac{pH_{p-1}(3)}{3} \pmod{p^4}.
\]
Moreover, by [8, Lemma 3],
\[
H_{p-1}(1, 2) \equiv -\frac{3H_{p-1}(1)}{p^2} - \frac{5H_{p-1}(3)}{12} \pmod{p^3}.
\]
and by \[16\] Proposition 3.7 and \[9\] Theorem 4.5
\[ H_{p-1}(1, 1, 2) \equiv -\frac{11H_{p-1}(3)}{12p} \pmod{p^2}, \quad H_{p-1}(1, 1, 1, 2) \equiv -\frac{5H_{p-1}(3)}{6p^2} \pmod{p}. \]

Finally, by \[16\] Theorem 3.2,
\[ H_{p-1}(2, 2) = \frac{H_{p-1}(3)}{3p}, \quad H_{p-1}(1, 3) = \frac{3H_{p-1}(3)}{4p} \pmod{p^2} \]
and by \[16\] Theorem 3.5,
\[ H_{p-1}(2, 1, 2) \equiv 0, \quad H_{p-1}(1, 2, 2) \equiv \frac{5H_{p-1}(3)}{4p^2}, \quad H_{p-1}(1, 1, 3) \equiv -\frac{5H_{p-1}(3)}{12p^2} \pmod{p}. \]

3. Proofs of the generating function \[\text{(4)}\] and the related combinatorial identity \[\text{(6)}\]

By partial fraction decomposition with respect to $b^2$, we get
\[ \frac{\prod_{j=1}^{k-1} (j^2 - a_n^2 - 4b^2)}{\prod_{j=1}^{k} (j^2 - aj - b^2)} = \sum_{n=1}^{k} \frac{C_{n,k}(a)}{n^2 - an - b^2} \]
where
\[ C_{n,k}(a) = \frac{\prod_{j=1}^{k-1} (j^2 - (a - 2n)^2)}{\prod_{j=1, j\neq n}^{k} (j - n)(j + n - a)}. \]

Hence, by inverting the summations order, the identity \[\text{(4)}\] can be written as
\[ \sum_{n=1}^{\infty} \frac{1}{n^2 - an - b^2} = \sum_{k=1}^{\infty} \frac{(3k - a)}{k(\frac{2k}{k})} \sum_{n=1}^{k} \frac{C_{n,k}(a)}{n^2 - an - b^2} = \sum_{n=1}^{\infty} \frac{1}{n^2 - an - b^2} \sum_{k=n}^{\infty} \frac{(3k - a)C_{n,k}(a)}{k(\frac{2k}{k})}. \]

It follows that \[\text{(4)}\] holds as soon as
\[ 1 = \sum_{k=n}^{\infty} \frac{(3k - a)}{k(\frac{2k}{k})} \sum_{n=1}^{k} \frac{C_{n,k}(a)}{k(\frac{2k}{k})}. \]

Taking the same approach given in \[\text{(11)}\] for the proof of \[\text{(4)}\], the above formula is equivalent to this finite combinatorial identity
\[ \sum_{k=1}^{n} \frac{\binom{2k}{k}(3k - a)}{k(\frac{2k}{k})} = \frac{2}{a - n}. \]

Both identities \[\text{(12)}\] and \[\text{(13)}\] are consequences of the next theorem after setting $z = 2n - a$.

**Theorem 1.** For any positive integer $n$,
\[ \sum_{k=1}^{n} \frac{\binom{2k}{k}(3k - 2n + z)}{k(\frac{2k}{k})} = \frac{2}{n - z}. \]
and

$$\sum_{k=n}^{\infty} \frac{(3k - 2n + z)}{k\binom{2k}{k}} \cdot \frac{\prod_{j=1}^{k-1} (j^2 - z^2)}{\prod_{j=1, j \neq n}^{k} (j - n)(j - n + z)} = 1. \tag{15}$$

**Proof.** Let

$$F(n, k) = \binom{2k}{k} (3k - 2n + z) \frac{\prod_{j=0}^{k-1} (j - n)(j - n + z)}{\prod_{j=1}^{k} (j^2 - z^2)},$$

and

$$G(n, k) = \frac{k(k^2 - z^2)F(n, k)}{(2n - 3k - z)(n + 1 - k)(n + 1 - k - z)}.$$

Then $(F, G)$ is a Wilf-Zeilberger pair, or WZ pair, which means that they satisfy the relation

$$F(n + 1, k) - F(n, k) = G(n, k + 1) - G(n, k).$$

In order to prove (14), it suffices to prove that $S_n := \sum_{k=1}^{n} F(n, k) = 2n$. Now $S_1 = F(1, 1) = 2$. Moreover,

$$S_{n+1} - S_n = \sum_{k=1}^{n+1} F(n + 1, k) - \sum_{k=1}^{n+1} F(n, k) = G(n, n + 2) - G(n, 1) = 2$$

because $F(n, n + 1) = G(n, n + 2) = 0$ and $G(n, 1) = -2$.

In a similar way, we show (15) by considering the WZ pair given by

$$F(n, k) = \frac{(3k - 2n + z)}{k\binom{2k}{k}} \cdot \frac{\prod_{j=1}^{k-1} (j^2 - z^2)}{\prod_{j=1, j \neq n}^{k} (j - n)(j - n + z)},$$

and

$$G(n, k) = \frac{2(2k - 1)(k - n)F(n, k)}{n(2n - 3k - z)(n - z)}.$$

$$\square$$

4. More binomial identities

Here we collect a few identities, apparently new, involving the binomial coefficients $\binom{2k}{k}$ and $\binom{n+k}{k}$ which will play a crucial role in the next sections.
**Theorem 2.** For any positive integer $n$,

\[
\frac{3}{2} \sum_{k=1}^{n} \frac{1}{k} \binom{2k}{k} = \sum_{k=1}^{n} \frac{1}{k} \binom{n+k}{k} + H_n(1) \quad (16)
\]

\[
\sum_{k=1}^{n} \binom{2k}{k} \left( \frac{3H_k(1)}{2k} - \frac{1}{k^2} \right) = \sum_{k=1}^{n} \left( \frac{n+k}{k} \right) \frac{H_k(1)}{k} - H_n(2) \quad (17)
\]

\[
\sum_{k=1}^{n} \binom{2k}{k} \left( \frac{3H_k(2)}{k} - \frac{1}{2k^3} \right) = \sum_{k=1}^{n} \left( \frac{n+k}{k} \right) \frac{H_k(2)}{k} + H_n(2)H_n(1) - H_n(1, 2) \quad (18)
\]

**Proof.** Let us consider the WZ pair

\[
F(n, k) = \frac{1}{k} \binom{n+k}{k} \quad \text{and} \quad G(n, k) = \frac{k}{(n+1)^2} \binom{n+k}{k}
\]

then

\[
S_{n+1} - S_n = F(n+1, n+1) + \sum_{k=1}^{n} (G(n, k+1) - G(n, k))
\]

\[
= F(n+1, n+1) + G(n, n+1) - G(n, 1)
\]

\[
= \frac{3}{2} \binom{2(n+1)}{n+1} - \frac{1}{n+1}
\]

where $S_n := \sum_{k=1}^{n} F(n, k)$. Thus

\[
S_n = \frac{3}{2} \sum_{k=1}^{n} \frac{1}{k} \binom{2k}{k} - H_n(1)
\]

and we may conclude that (16) holds.

Now let $S_n^{(1)} := \sum_{k=1}^{n} F(n, k)H_k(1)$ then

\[
S_{n+1}^{(1)} - S_n^{(1)} = F(n+1, n+1)H_{n+1}(1)
\]

\[
+ \sum_{k=1}^{n} \left( G(n, k+1)H_k(1) - G(n, k) \left( H_{k-1}(1) + \frac{1}{k} \right) \right)
\]

\[
= F(n+1, n+1)H_{n+1}(1) + G(n, n+1)H_n(1) - \sum_{k=1}^{n} \frac{G(n, k)}{k}
\]

\[
= \binom{2(n+1)}{n+1} \left( \frac{3H_{n+1}(1)}{2(n+1)} - \frac{1}{(n+1)^2} \right) + \frac{1}{(n+1)^2}
\]

where we used $\sum_{k=1}^{n} \binom{n+k}{k} = \frac{1}{2} \binom{2(n+1)}{n+1} - 1$. Hence we find that

\[
S_n^{(1)} = \sum_{k=1}^{n} \binom{2k}{k} \left( \frac{3H_k(1)}{2k} - \frac{1}{k^2} \right) + H_n(2)
\]
which implies (17).
Let \( S^{(2)}_n := \sum_{k=1}^n F(n,k) H_k(2) \) then

\[
S^{(2)}_{n+1} - S^{(2)}_n = F(n+1,n+1) H_{n+1}(2) \\
+ \sum_{k=1}^n \left( G(n,k+1) H_k(2) - G(n,k) \left( H_{k-1}(2) + \frac{1}{k^2} \right) \right)
\]

\[
= F(n+1,n+1) H_{n+1}(2) + G(n,n+1) H_n(2) - \sum_{k=1}^n \frac{G(n,k)}{k^2}
\]

where we applied

\[
\sum_{k=1}^n \frac{G(n,k)}{k^2} = \frac{1}{(n+1)^2} \sum_{k=1}^n F(n,k) = \frac{S_n}{(n+1)^2}.
\]

Therefore

\[
S^{(2)}_n = \sum_{k=1}^n \left( \frac{2k}{k} \right) \left( \frac{3H_k(2)}{2k} - \frac{1}{2k^3} \right) - \sum_{k=1}^n \frac{S_{k-1}}{k^2}
\]

\[
= \sum_{k=1}^n \left( \frac{2k}{k} \right) \left( \frac{3H_k(2)}{2k} - \frac{1}{2k^3} \right) - \frac{3}{2} \sum_{k=1}^n \frac{1}{k^2} \sum_{j=1}^{k-1} \frac{1}{j} \left( \frac{2j}{j} \right) + H_n(1,2)
\]

\[
= \sum_{k=1}^n \left( \frac{2k}{k} \right) \left( \frac{3H_k(2)}{2k} - \frac{1}{2k^3} \right) - \frac{3}{2} \sum_{j=1}^n \frac{1}{j} \left( \frac{2j}{j} \right) (H_n(2) - H_j(2)) + H_n(1,2)
\]

\[
= \sum_{k=1}^n \left( \frac{2k}{k} \right) \left( \frac{3H_k(2)}{k} - \frac{1}{2k^3} \right) - \frac{3H_n(2)}{2} \sum_{k=1}^n \frac{1}{k} \left( \frac{2k}{k} \right) + H_n(1,2)
\]

and the proof of (18) is complete. \( \square \)

5. Proofs of the main supercongruences

**Theorem 3.** For any prime \( p > 3 \),

\[
\sum_{k=1}^{p-1} \frac{1}{k} \left( \frac{2k}{k} \right) \equiv -\frac{8H_{p-1}(1)}{3} - \frac{5p^2H_{p-1}(3)}{3} \pmod{p^5}
\]

(19)
Moreover, for any prime \( p > 5 \),
\[
\sum_{k=1}^{p-1} \frac{1}{k^3} \binom{2k}{k} \equiv -\frac{2H_{p-1}(1)}{p^2} \pmod{p^2},
\]
\[
\sum_{k=1}^{p-1} \binom{2k}{k} \frac{H_k(2)}{k} \equiv \frac{2H_{p-1}(1)}{3p^2} \pmod{p^2}.
\]

**Proof.** We first note that
\[
\binom{p-1 + k}{k} = \frac{p}{k} \binom{p + k - 1}{k - 1} = \frac{p}{k} \prod_{j=1}^{k-1} \left( 1 + \frac{p}{j} \right) = \frac{1}{k} \sum_{j=0}^{k-1} p^{j+1} H_{k-1}(\{1\}^j).
\]
Therefore, by (19) with \( n = p - 1 \) we obtain the desired congruence (19),
\[
\sum_{k=1}^{p-1} 1 \binom{2k}{k} = \frac{2}{3} \left( H_{p-1}(1) + \sum_{j=0}^{p-2} p^{j+1} H_{p-1}(\{1\}^j, 2) \right)
\equiv \frac{2}{3} \left( H_{p-1}(1) + pH_{p-1}(2) + p^2 H_{p-1}(1, 2) \right.
\quad + p^3 H_{p-1}(1, 1, 2) + p^4 H_{p-1}(1, 1, 1, 2) \left. \right)
\equiv \frac{-8H_{p-1}(1)}{3} - \frac{5p^2 H_{p-1}(3)}{3} \pmod{p^6}.
\]

By letting \( z = 2n \) in (14) we have
\[
\sum_{k=1}^{n} \binom{2k}{k} k \frac{k}{k^2 - 4n^2} \prod_{j=1}^{k-1} \frac{j^2 - n^2}{j^2 - 4n^2} = \frac{2}{3n}.
\]

Let \( n = p > 5 \) be a prime and move the \( p \)-th term of the sum to the right-hand side,
\[
\sum_{k=1}^{p-1} \frac{1}{k} \binom{2k}{k} \frac{1}{1 - 4p^2/k^2} \prod_{j=1}^{k-1} \frac{1 - p^2/j^2}{1 - 4p^2/j^2} = \frac{2}{3p} \left( \frac{2p}{p} \prod_{j=1}^{p-1} \frac{1 - p^2/j^2}{1 - 4p^2/j^2} - 1 \right).
\]
The left-hand side modulo \( p^4 \) is congruent to
\[
\sum_{k=1}^{p-1} \frac{1}{k} \binom{2k}{k} \left( 1 + 4p^2/k^2 \right) \prod_{j=1}^{k-1} \left( 1 + \frac{3p^2}{j^2} \right) \equiv \sum_{k=1}^{p-1} \frac{1}{k} \binom{2k}{k} + p^2 \sum_{k=1}^{p-1} \binom{2k}{k} \left( \frac{1}{k^3} + \frac{3H_k(2)}{k} \right).
\]
On the other hand, by [15, Theorem 2.4],
\[
\frac{1}{2} \binom{2p}{p} \equiv 1 + 2pH_{p-1}(1) + \frac{2p^3 H_{p-1}(3)}{3} \equiv 1 - p^2 H_{p-1}(2) - \frac{p^4 H_{p-1}(4)}{2} \pmod{p^6},
\]
(21)
the right-hand side is
\[
\frac{1}{2} \binom{2p}{p} \prod_{j=1}^{p-1} \frac{1 - \frac{p^2}{j^2}}{1 - \frac{p^2}{j^2}} = \frac{1}{2} \binom{2p}{p} \prod_{j=1}^{p-1} \left( 1 + \frac{3p^2}{j^2} + \frac{12p^4}{j^4} \right)
\equiv \left( 1 - p^2 H_{p-1}(2) - \frac{p^4 H_{p-1}(4)}{2} \right)
\cdot \left( 1 + 3p^2 H_{p-1}(2) + 12p^4 H_{p-1}(4) + 9p^4 H_{p-1}(2,2) \right)
\equiv 1 + 2p^2 H_{p-1}(2) + p^4 \left( \frac{17 H_{p-1}(4)}{2} + 3 H_{p-1}(2,2) \right)
\equiv 1 + 2p^2 H_{p-1}(2) \pmod{p^5},
\]

where \(2 H_{p-1}(2,2) = (H_{p-1}(2))^2 - H_{p-1}(4) \equiv 0 \pmod{p} \). Finally, by (19),
\[
\sum_{k=1}^{p-1} \binom{2k}{k} \left( \frac{1}{k^3} + \frac{3H_k(2)}{k} \right) \equiv 8H_{p-1}(1) + 5H_{p-1}(3) + 4pH_{p-1}(2) \equiv 0 \pmod{p^2}. \tag{22}
\]

where we used (11).

By (18), with \(n = p - 1\), we have that
\[
\sum_{k=1}^{p-1} \binom{2k}{k} \left( \frac{3H_k(2)}{k} - \frac{1}{2k^3} \right) = p \sum_{k=1}^{p-1} \prod_{j=1}^{k-1} \left( 1 + \frac{p}{j} \right) \frac{H_k(2) + H_{p-1}(2)}{k^2}
+ H_{p-1}(2) H_{p-1}(1) - H_{p-1}(1,2)
\equiv p \sum_{k=1}^{p-1} \frac{H_k(2)}{k^2} - H_{p-1}(1,2)
= pH_{p-1}(2,2) + pH_{p-1}(4) - H_{p-1}(1,2)
\equiv -H_{p-1}(1,2) = \frac{3H_{p-1}(1)}{p^2} \pmod{p^2}. \tag{23}
\]

The proof is completety as soon as we combine properly congruences (22) and (23). \(\square\)

6. Finale - Two Apery-like congruences

The following congruences are related to the second series in (3) and to the first series in (5).

**Theorem 4.** For any prime \(p > 3\),
\[
\sum_{k=1}^{p-1} \binom{2k}{k} \left( \frac{2}{k^2} - \frac{3H_k(1)}{k} \right) \equiv \frac{2H_{p-1}(1)}{p} + 3pH_{p-1}(3) \pmod{p^4}, \tag{24}
\]
\[
\sum_{k=1}^{p-1} (-1)^k \binom{2k}{k} \left( \frac{4}{k^4} + \frac{5H_k(2)}{k^2} \right) \equiv -H_{p-1}(4) \pmod{p^2}. \tag{25}
\]
Proof. By (17) with \( n = p - 1 \) and by (20)

\[
\sum_{k=1}^{p-1} \binom{2k}{k} \left( \frac{3H_k(1)}{2k} - \frac{1}{k^2} \right) = \sum_{k=1}^{p-1} \frac{H_k(1)}{k^2} \sum_{j=0}^{k-1} p^{j+1} H_{k-1} \{ \{1\}^j \} - H_{p-1}(2)
\]

\[
\equiv p \sum_{k=1}^{p-1} \frac{H_{k-1}(1) + \frac{1}{k^2}}{k^2} (1 + pH_{k-1}(1) + p^2 H_{k-1}(1, 1)) - H_{p-1}(2)
\]

\[
\equiv -H_{p-1}(2) + pH_{p-1}(1, 2) + pH_{p-1}(3)
\]

\[
+ 2p^2 H_{p-1}(1, 1, 2) + p^2 H_{p-1}(2, 2) + p^2 H_{p-1}(1, 3)
\]

\[
+ 3p^3 H_{p-1}(1, 1, 2) + p^3 H_{p-1}(2, 1, 2) + p^3 H_{p-1}(1, 2, 2)
\]

\[
+ p^3 H_{p-1}(1, 1, 3)
\]

\[
\equiv -\frac{H_{p-1}(1)}{p} - \frac{3pH_{p-1}(3)}{2} \pmod{p^4}
\]

where at the last step we applied the results mentioned in the preliminaries.

By comparing the coefficient of \( a^2 \) in the expansion of both sides of (2) at \( a = 0 \) we have

\[
\sum_{k=1}^{n} \binom{2k}{k} \frac{5k^2}{4n^4 + k^4} \prod_{j=1}^{k-1} \frac{n^4 - j^4}{4n^4 + j^4} \left( \frac{1}{5k^2} + \sum_{j=1}^{k-1} \frac{1}{n^2 + j^2} - 2 \sum_{j=1}^{k} \frac{2n^2 + j^2}{4n^4 + j^4} \right) = -\frac{2}{n^4}.
\]

Let \( n = p > 3 \) be a prime then move to the right-hand side the \( p \)-th term of the sum on the left. Thus, the left-hand side modulo \( p^2 \) is congruent to

\[
\sum_{k=1}^{p-1} (-1)^{k-1} \binom{2k}{k} \frac{5}{k^2} \left( \frac{1}{5k^2} + H_{k-1}(2) - 2H_k(2) \right) = \sum_{k=1}^{p-1} (-1)^{k} \binom{2k}{k} \left( \frac{4}{k^4} + \frac{5H_k(2)}{k^2} \right).
\]

The right-hand side multiplied by \( p^4 \) is

\[
-2 - \left( \frac{2p}{p} \right) \prod_{j=1}^{p-1} \frac{p^4 - j^4}{4p^4 + j^4} \left( \frac{1}{5} + p^2 \sum_{j=1}^{p-1} \frac{1}{p^2 + j^2} - 2p^2 \sum_{j=1}^{p} \frac{2p^2 + j^2}{4p^4 + j^4} \right)
\]

and it remains to verify that it is congruent to \(-p^4 H_{p-1}(4) \pmod{p^6} \). We note that

\[
\prod_{j=1}^{p-1} \frac{p^4 - j^4}{4p^4 + j^4} \equiv 1 - 5p^4 H_{p-1}(4) \pmod{p^6},
\]

\[
p^2 \sum_{j=1}^{p-1} \frac{1}{p^2 + j^2} \equiv p^2 H_{p-1}(2) - p^4 H_{p-1}(4) \pmod{p^6},
\]

\[
2p^2 \sum_{j=1}^{p} \frac{2p^2 + j^2}{4p^4 + j^4} \equiv 6 + 2p^2 H_{p-1}(2) + 4p^4 H_{p-1}(4) \pmod{p^6},
\]
Hence, by (21), (26) simplifies to
\[-2 + 2 \left( 1 - p^2 H_{p-1}(2) - \frac{p^4 H_{p-1}(4)}{2} \right) (1 + p^2 H_{p-1}(2)) \equiv -p^4 H_{p-1}(4) \pmod{p^6} \]
and the proof is finished. □

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