One Sided Lipschitz Evolution Inclusions in Banach Spaces

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Abstract: Using the notion of limit solution, we study multivalued perturbations of m-dissipative differential inclusions with nonlocal initial conditions. These solutions enable us to work in general Banach spaces, in particular $L^1$. The commonly used Lipschitz condition on the right-hand side is weakened to a one-sided Lipschitz one. No compactness assumptions are required. We consider the case of an arbitrary one-sided Lipschitz condition and the case of a negative one-sided Lipschitz constant. Illustrative examples, which can be modifications of real models, are provided.

Keywords: one sided Lipschitz; limit solutions; nonlocal problems

MSC: 34C25; 34A60; 49J21

1. Introduction and Preliminaries

The goal of this paper is to prove the existence of limit and integral solutions for a class of nonlinear evolution inclusions with nonlocal initial conditions of the form:

\[
\begin{align*}
\dot{x}(t) & \in A x(t) + f(t), \quad t \in (t_0, T) \\
\int_{t_0}^{T} f(t, x(t)) dt & \in (t_0, T) \\
x(t_0) & = g(x(\cdot)) \in D(A).
\end{align*}
\]

Here, $A : D(A) \subset E \rightarrow E$ is an m–dissipative operator, $E$ is a real Banach space with norm $\| \cdot \|$, $f : I \times E \rightarrow E$ a multifunction with nonempty, closed convex and bounded values, $I = [t_0, T]$, and $g : C(I; E) \rightarrow \overline{D(A)}$ is a given function. The existence problem of solutions for (1) has been very actively studied in the last years, due to the applications in mechanics, physics, chemistry, biology, and so forth. We refer the reader to [1], where a more general form of the system (1) (system with time lag) is comprehensively studied. In that book, many examples of real systems are provided and, moreover, in the beginning the authors explain why one has to study systems like (1). We refer also to [2], where reaction–diffusion systems are studied. Both works are devoted to multivalued as well as single valued perturbations of m-dissipative operators.

The existence of solutions can be obtained under some compactness type conditions (see, e.g., [3–5] and the references therein). Usually, the dual space $E^*$ is uniformly convex. The existence of solutions can also be obtained under dissipative type assumptions, such as $F$ Lipschitz continuous. We recall the papers [6] where $E^*$ is uniformly convex and [7] in general Banach spaces. In [8], the Lipschitz assumption on $F$ is relaxed to a one-sided Lipschitz condition and $F$ is assumed to be an almost upper hemicontinuous multifunction.
with nonempty convex weakly compact values, when the state space has a uniformly convex dual.

To study nonlocal problems, one needs maybe implicit fixed point theorems. We refer the reader to [9–11] for some recent results in that field.

In the present paper, we prove the existence of limit and integral solutions for (1) in general Banach spaces. We impose that the multifunction \( F \) satisfies a one-sided Lipschitz condition, which is weaker than the commonly used Lipschitz one. First, we consider \( F \) to be a one-sided Lipschitz with respect to a positive Lipschitz function. For the function \( g(\cdot) \) we assume the same growth condition as in [7,12], which is weaker than the one from [6]. Then, we consider the case when \( F \) is a one-sided Lipschitz with respect to a negative constant. Remark that there exist multifunctions, which are not Lipschitz but are one-sided Lipschitz with a negative constant. To obtain the existence results, we provide some qualitative properties as a continuous dependence of the solutions on the initial conditions for the corresponding local problems. We also consider the system (1) with periodic nonlocal initial conditions in the particular cases when \( A \) is linear and when \( F \) is autonomous. In the last case, we prove the existence of zeros of \( A + F \). Illustrative examples are given.

We recall some well-known facts in the theory of m-dissipative systems. We refer the reader to [1,2,13–16] for the definitions and notations used here.

We define the solutions of (1) by the help of the problem,

\[
\dot{x}(t) = Ax(t) + f(t), \quad x(t_0) = x_0 \in \overline{D(A)}, \quad (2)
\]

where \( f(\cdot) \) is a Bochner integrable function. We say that a continuous function \( x : [t_0, T] \to \overline{D(A)} \) is an integral solution of (2) on \([t_0, T]\) if \( x(t_0) = x_0 \) and for every \( u \in D(A), v \in Au \) and \( t_0 \leq \tau < t \leq T \) the following inequality holds:

\[
|x(t) - u| \leq |x(\tau) - u| + \int_{\tau}^{t} |x(s) - u, f(s) + v|_+ ds
\]

(see, e.g., Definition 3.5.1 in [16]). The function \( f \) in (2) will be called pseudoderivative of \( x(\cdot) \) with respect to \( A \). To stress the dependence on \( x \), we will write \( f_x(\cdot) \), since \( A \) is given and fixed along this paper. We denoted by \(|\cdot, \cdot|_+\) the right directional derivative of the norm, that is, \(|x, y|_+ = \lim_{h \to 0^+} h^{-1}(|x + hy| - |x|)\) for \( x, y \in E \) (see, e.g., [16] for definition and properties).

Note that, for each \( x_0 \in \overline{D(A)} \), the Cauchy problem (2) has a unique integral solution on \([t_0, T]\). Moreover, if \( u(\cdot) \) and \( v(\cdot) \) are integral solutions of (2) with \( u(t_0) = u_0 \) and \( v(t_0) = v_0 \), then

\[
|u(t) - v(t)| \leq |u_0 - v_0| + \int_{t_0}^{t} |u(s) - v(s), f_u(s) - f_v(s)|_+ ds,
\]

for every \( t \in [t_0, T] \) (see, e.g., [16]). Let us consider the local evolution system,

\[
\begin{cases}
\dot{x}(t) = Ax(t) + F(t, x(t)), \quad t \in (t_0, T) \\
x(t_0) = x_0 \in \overline{D(A)}.
\end{cases}
\]

We say that the function \( x : I \to \overline{D(A)} \) is an integral solution of (4) on \( I \) when it is an integral solution of (2) such that \( f_x(t) \in F(t, x(t)) \) for a.a. \( t \in I \), where \( f_x(\cdot) \) is its pseudoderivative.

The function \( x : I \to \overline{D(A)} \) is said to be an integral solution of (1) on \( I \) if it is an integral solution of (4) on \( I \) and \( x(t_0) = g(x(\cdot)) \).
The multifunction $F : I \times E \Rightarrow E$ is called continuous if it is continuous with respect to the Hausdorff distance. We recall that the Hausdorff distance between the bounded sets $B$ and $C$ is defined by:

$$D_H(B, C) = \max\{e(B, C), e(C, B)\},$$

where $e(B, C)$ is the excess of $B$ to $C$, defined by $e(B, C) = \sup_{x \in B} \text{dist}(x, C)$. The multifunction $F(\cdot, \cdot)$ is called almost continuous if for every $\varepsilon > 0$ there exists a compact set $I_\varepsilon \subset I$ with Lebesgue measure $\text{meas}(I \setminus I_\varepsilon) \leq \varepsilon$ such that $F|_{I_\varepsilon \times X}$ is continuous.

**Definition 1.** The multifunction $F : I \times E \Rightarrow E$ is said to be one-sided Lipschitz (OSL) if there exists a Lebesgue integrable $L(\cdot)$ (maybe negative) and a null set $\mathcal{J} \subset I$ such that for every $u, v \in E$, every $t \in I \setminus \mathcal{J}$, every $f_u \in F(t, u)$ and every $\varepsilon > 0$ there exists $f_v \in F(t, v)$ such that:

$$|u - v, f_u - f_v|_+ < L(t)|u - v| + \varepsilon.$$

The OSL condition in the case of multifunctions was first introduced (in a more general form and with different name) in [17]. Afterwards, it was investigated in [18,19]. An interesting approach (implicit Euler method) was introduced in [20]. This approach works in finite dimensional systems (cf. [21]) and can be extended for parabolic differential inclusions in the evolution triple (cf. [22]).

The OSL condition is very elegant when the duality map is single valued. However, many partial differential equations are studied in $L^1$ and this is why, in this paper, we work in general Banach spaces, in particular in $L^1$.

We will give an example of a map, satisfying the OSL condition. To this end, we will prove some elementary propositions, since they are used in the example and we do not know if they are proved anywhere. Notice that, to our knowledge, there are no nontrivial examples of OSL maps in the existing literature, except the one given in [8] in Hilbert spaces.

**Proposition 1.** The multifunction $F(\cdot, \cdot)$ is OSL with respect to the function $L(\cdot)$ iff there exists a null set $\mathcal{J} \subset I$ such that for every $u, v \in E$ every $t \in I \setminus \mathcal{J}$, every $f_u \in F(t, u)$ and every $\varepsilon > 0$ there exists $f_v \in F(t, v)$ such that:

$$|u - v, f_u - f_v|_+ < (L(t) + \varepsilon)|u - v|.$$

**Proof.** Clearly for $u = v$, one can take $f_u = f_v$ and hence $|u - v, f_u - f_v|_+ = 0$. If $u \neq v$, then one can replace $\varepsilon$ by $\frac{|u - v|}{|u - v|}$.

We end this section by giving an example of a one-sided Lipschitz map, which is neither Lipschitz nor full Perron.

Consider the following function:

$$\mu(x) = \begin{cases} -\frac{x}{\sqrt{|x|}} & x \neq 0 \\ 0 & x = 0. \end{cases}$$

**Lemma 1.** For any real numbers $a, b$ and $h > 0$, we have that:

$$h(a, b) := |a - b + h(\mu(a) - \mu(b))| - |a - b| \leq 0.$$

Consequently, $|a - b, \mu(a) - \mu(b)|_+ \leq 0$.

**Proof.** We have to consider the following cases:

(I) If $a = b = 0$ then $h(a, b) = 0$.

(II) If $a \neq 0$ and $b = 0$, then for small $h > 0$ we have $|a + h\mu(a)| < |a|$ and hence $h(a, b) < 0$. 

(III) If \(a > b > 0\) then for small \(h > 0\) we have \(h(a, b) = -h(\sqrt{a} - \sqrt{b}) < 0\).

(IV) If \(a > 0 > b\) then for small \(h > 0\) we have \(h(a, b) = -h(\sqrt{a} + \sqrt{-b}) < 0\).

All the other cases can be devoted to one of these. \(\square\)

Let \(L^1(\Omega)\) be the set of all real valued Lebesgue integrable functions on the bounded domain \(\Omega \subset \mathbb{R}\).

**Proposition 2.** The function \(f(p)(\cdot) = \mu(p(\cdot)), p \in L^1(\Omega)\), is a one-sided Lipschitz with a constant 0.

**Proof.** Clearly, \(f(\cdot)\) maps \(L^1(\Omega)\) into itself. Let \(p, q \in L^1(\Omega)\) and fix \(h > 0\). Due to Lemma 1, for every \(x \in \Omega\), \(h(p(x), q(x)) \leq 0\). Then we have that:

\[
\|p - q + h(f(p) - f(q))\|_{L^1(\Omega)} - \|p - q\|_{L^1(\Omega)}
\]

\[
= \int_{\Omega} |p(x) - q(x) + h(f(p)(x) - f(q)(x))|dx - \int_{\Omega} |p(x) - q(x)|dx
\]

\[
= \int_{\Omega} \left[ |p(x) - q(x) + h(\mu(p(x)) - \mu(q(x)))| - |p(x) - q(x)| \right]dx
\]

\[
= \int_{\Omega} h(p(x), q(x))dx \leq 0.
\]

Divide the above inequality by \(h\), take the limit \(h \to 0^+\) and we get that \([p - q, f(p) - f(q)]_+ \leq 0\). \(\square\)

Consider now the same function \(f(\cdot)\); however, in \(L^\infty(\Omega)\).

**Proposition 3.** The function \(f(p)(\cdot) = \mu(p(\cdot)), p \in L^\infty(\Omega)\), is a one-sided Lipschitz with a constant 0.

**Proof.** Let \(p, q \in L^\infty(\Omega)\). Fix \(h > 0\). Let \((x_i)_i\) be a sequence such that:

\[
\lim_{i \to \infty} |p(x_i) - q(x_i) + h(f(p)(x_i) - f(q)(x_i))| = \|p - q + h(f(p) - f(q))\|_{L^\infty(\Omega)}.
\]

By Lemma 1,

\[
0 \geq \lim_{i \to \infty} \left[ |p(x_i) - q(x_i) + h(f(p)(x_i) - f(q)(x_i))| - |p(x_i) - q(x_i)| \right]
\]

\[
\geq \|p - q + h(f(p) - f(q))\|_{L^\infty(\Omega)} - \|p - q\|_{L^\infty(\Omega)}.
\]

Dividing by \(h\) and passing to the limit as \(h \to 0^+\) we have that:

\[
[p - q, f(p) - f(q)]_+ \leq 0
\]

also in \(L^\infty(\Omega)\). \(\square\)

Note that, if the function \(f(p)(\cdot) = \mu(p(\cdot))\) is defined on \(L^1(\Omega; \mathbb{R}^k)\) or \(L^\infty(\Omega; \mathbb{R}^k)\), it is still a one-sided Lipschitz with a constant 0.

Let us consider:

\[
f(x) = \begin{cases} 
3\mu(x) & |x| \leq 1 \\
\mu(x) - 2x & |x| > 1.
\end{cases}
\]

It is easy to see that \(f(\cdot)\) is OSL with a negative constant.

2. Existence of Solutions

In this section, we prove the existence of solutions when the right-hand side is OSL with the non-negative Lebesgue integrable function. Notice that, in this case, the right-hand
side is a one-sided Perron. The results of [12] are not applicable, because here we study the nonlocal problem.

First, we present the hypotheses needed in this section.

(F1) $F$ is almost continuous with closed bounded values.

(F2) There exists a Lebesgue integrable function $\lambda(\cdot)$ such that $\|F(t, x)\| \leq \lambda(t)(1 + |x|)$ for a.a. $t \in I$ and every $x \in E$, where $\|F(t, x)\| = \sup_{y \in F(t, x)} |y|$.

(F3) $F$ is OSL w.r.t. a positive Lebesgue integrable function $L(\cdot)$.

(g) There exists a constant $K > 0$ such that $|g(x(\cdot)) - g(y(\cdot))| \leq K\|x(\cdot) - y(\cdot)\|_{C(I; E)}$ for every $x, y \in C(I; E)$.

**Remark 1.** We could assume that $0 \in D(A)$, and moreover, that $0 \in A0$. Indeed, let $q \in D(A)$ and replace $A$ by $B$, where $Bx := A(x + q)$. Then $B$ is $m$-dissipative and $0 \in D(B)$. Now, let $p \in B0$. Then the operator $C = B - p$ is also $m$-dissipative and we can put it in (1) instead of $A$. Clearly, $0 \in C0$.

In [12], a new concept of solution for the local problem (4) was introduced, called the limit solution.

**Definition 2.** (i) Let $\varepsilon > 0$. The continuous function $y : I \to D(A)$ is said to be an $\varepsilon$–solution of (4) on $I$ if it is a solution of:

\[
\begin{align*}
\dot{y}(t) &\in Ay(t) + F(t, y(t) + B) + B, \\
y(t_0) &= x_0,
\end{align*}
\]

and its pseudoderivative $f_y(\cdot)$ satisfies:

\[
\int_{t_0}^t \text{dist}(f_y(\cdot), F(t, y(\cdot)))dt \leq \varepsilon.
\]

(ii) The function $y(\cdot)$ is a limit solution of (4) on $I$ if there exists a sequence $(y^n(\cdot))$ of $\varepsilon_n$–solutions as $\varepsilon_n \downarrow 0^+$ such that $y(t) = \lim_{n \to \infty} y^n(t)$ uniformly on $I$.

Following [12], we define the limit solution of the nonlocal problem (1).

**Definition 3.** The continuous function $y(\cdot)$ is said to be the limit solution of (1) if it is the limit solution of (4) on $I$ with $x_0 = g(y(\cdot))$.

**Remark 2.** It is easy to see that, under (F1), (F2), for every $x_0 \in D(A)$ and every $\varepsilon > 0$ there exist a constant $M > 0$ and a Lebesgue integrable function $k(\cdot)$ such that $|x(t)| \leq M$ and $\|F(t, M\mathbb{1})\| \leq k(t)$ for every $t \in I$ and every $\varepsilon$–solution $x(\cdot)$ of (4).

First, we present the following result regarding the limit solutions of the local problem (4), which comes directly from Theorem 2 in [12].

**Proposition 4.** Assume (F1)–(F3). Let $x(\cdot)$ be a limit solution of (4) with $x(t_0) = x_0$. Then, for every $y_0 \in D(A)$ and every $\varepsilon > 0$ there exists a limit solution $y(\cdot)$ of (4) with $y(t_0) = y_0$ such that

\[
|x(t) - y(t)| \leq |x_0 - y_0| \exp \int_{t_0}^t L(s)ds + \varepsilon
\]

for any $t \in I$.

**Proof.** Let $z(\cdot)$ be a $\delta$–solution of (4) with $z(t_0) = x_0$ such that $\|x(\cdot) - z(\cdot)\|_{C(I; E)} \leq \delta$, where $\delta > 0$. Then its pseudoderivative $f_z(\cdot)$ satisfies $f_z(t) \in F(t, z(t)) + \mu(t)B$ for any $t \in I$ with $\int_t^\mu(t)dt \leq \delta.$
Due to Theorem 2 in [12] there exists a limit solution \( y(\cdot) \) of (4) with \( y(t_0) = y_0 \) such that \( |z(t) - y(t)| \leq r(t) + \delta \), where \( r(\cdot) \) is the maximal solution of \( \dot{r}(t) = L(t)r(t) + \mu(t), \; r(t_0) = |x_0 - y_0| \). Therefore,

\[
r(t) = \left( |x_0 - y_0| + \int_{t_0}^t \mu(s) \exp\left( - \int_{t_0}^s L(\tau) d\tau \right) ds \right) \exp \int_{t_0}^t L(s) ds
\]

for any \( t \in I \). Thus, \( |x(t) - y(t)| \leq \delta \left( 2 + \exp \int_I L(t) dt \right) + |x_0 - y_0| \exp \left( \int_{t_0}^t L(s) ds \right) \) for any \( t \in I \). The proof is therefore complete since \( \delta > 0 \) is arbitrary. \( \square \)

We are ready now to prove the existence of limit solutions for the nonlocal problem (1). We mention that, in Theorem 4.1 in [6], the existence of the solution of (1) was proved when \( F(t, \cdot) \) is Lipschitz and \( K + \int I L(s) ds < 1 \). We prove here the existence of limit solutions in general Banach spaces under much weaker assumptions.

**Theorem 1.** Assume (F1)--(F3) and (g). Suppose that

\[
K \exp \left( \int I L(s) ds \right) < 1. \tag{5}
\]

Then, the nonlocal problem (1) has a limit solution.

**Proof.** Let \( z(\cdot) \) be a continuous function on \( I \) and denote by \( \text{Sol}(z(\cdot)) \) the limit solutions set of (4) with \( x_0 = g(z(\cdot)) \). It follows from Proposition 4 that:

\[
D_H(\text{Sol}(y(\cdot)), \text{Sol}(z(\cdot))) \leq |g(y(\cdot)) - g(z(\cdot))| \left( \exp \int_I L(s) ds \right)
\]

\[
\leq K \| y(\cdot) - z(\cdot) \|_{C(I; E)} \left( \exp \int_I L(s) ds \right).
\]

Therefore, \( z(\cdot) =: \text{Sol}(z(\cdot)) \) is a set valued contraction and hence it admits a fixed point \( x(\cdot) \). This fixed point is a limit solution of (1). \( \square \)

**Remark 3.** If we study the evolution equation:

\[
x(t) \in Ax(t) + f(t, x(t)), \quad x(t_0) = x_0,
\]

where \( f \) is single valued, then every limit solution is also integral solution, because if \( x_n(t) \to x(t) \) uniformly, then \( f(\cdot, x_n(\cdot)) \to f(\cdot, x(\cdot)) \) a.e. in \( I \).

**Existence of Integral Solutions**

We will study the relation between the limit solutions and the integral solutions.

**Definition 4.** (See Definition 1.8.5 in [1]) The m-dissipative operator \( A \) is said to be of complete continuous type if for every \( a < b \) and every \( (f_n) \) in \( L^1([a, b]; X) \) and \( (x_n) \) in \( C([a, b]; X) \), with \( x_n(\cdot) \) a solution on \([a, b]\) of \( x_n(t) \in Ax_n(t) + f_n(t), \; n = 1, 2, \ldots, \lim_{n \to \infty} f_n = f \) weakly in \( L^1([a, b]; X) \) and \( \lim_{n \to \infty} x_n = x \) uniformly in \( C([a, b]; X) \), it follows that \( x \) is a solution on \([a, b]\) of

\[
x(t) \in Ax(t) + f(t).
\]

There exist several classes of m-dissipative operators \( A \) of a complete continuous type, which do not necessarily generate a compact semigroup.
Further, we need the following assumption:
(F4) \( F \) has nonempty closed convex weakly compact values.

**Theorem 2.** Let \( A \) be of complete continuous type. If (F1)–(F4) hold, then the set of limit solutions of (1) and the set of integral solutions of (1) coincide.

**Proof.** Let \( (x^n(\cdot)) \) be a sequence of \( \epsilon_n \)-solutions of (1) with \( \epsilon_n \downarrow 0 \) such that \( \lim_{n \to \infty} x^n(t) = x(t) \) uniformly on \( I \). Then, due to (F4) the sequence of the corresponding pseudoderivatives \( \{f_n(\cdot)\} \) is bounded, and for almost every \( t \in I \), they are contained in a weakly compact set. Then, passing to subsequences, \( f_n(\cdot) \to f(\cdot) \) weakly in \( L^1(I; X) \). Furthermore, \( A \) is of complete continuous type, that is, \( x(\cdot) \) is the solution of:

\[
\dot{x}(t) \in Ax(t) + f(t), \quad x(t_0) = x_0.
\]

Since \( F(\cdot, \cdot) \) is almost continuous, \( f(t) \in F(t, x(t)) \). The proof is therefore complete. \( \square \)

We now give a new existence result for the nonlocal problem (1).

**Corollary 1.** Assume the conditions of Theorem 1. Moreover, assume (F4) and that \( A \) is of complete continuous type. Then the nonlocal problem (1) has an integral solution.

We present an example, which is a modification of the one from [12], to illustrate the applicability of the abstract results obtained.

**Example 1.** Let \( \Omega \subset \mathbb{R}^n \) with \( n \geq 4 \) be a domain with smooth boundary \( \partial \Omega \). Let \( \varphi(\cdot) \) be strictly increasing and continuous with \( \varphi(0) = 0 \). We consider the following system:

\[
\begin{cases}
    u_t \in \Delta \varphi(u) + G(t, y, u) \text{ on } (0, T) \times \Omega \\
    -\frac{\partial \varphi(u)}{\partial v} \in \beta(u) \text{ on } (0, T) \times \partial \Omega
\end{cases}
\]

subject to the initial condition:

\[
\begin{align*}
\text{(a)} \quad u(0, y) &= \int_0^T l(s, u(s, y)) \, ds.
\end{align*}
\]

Here, \( u \in \mathbb{R} \), \( \frac{\partial \varphi(u)}{\partial v} \) is the outward normal derivative on \( \partial \Omega \), \( \beta(\cdot) \) is a maximal monotone graph in \( \mathbb{R} \) with \( \beta(0) \ni 0 \) and \( l : [0, T] \times \mathbb{R} \to \mathbb{R} \) is an almost continuous function. The multifunction \( G : [0, T] \times \Omega \times \mathbb{R} \rightrightarrows \mathbb{R} \) has nonempty compact values, is measurable on all variables and continuous on the third one.

Define the operator \( A : D(B) \subseteq L^1(\Omega) \rightrightarrows L^1(\Omega) \) by

\[
Au = \Delta \varphi(u), \quad \text{with}
\]

\[
D(A) = \{ u \in L^1(\Omega) ; \varphi(u) \in W^{1,1}(\Omega), \Delta \varphi(u) \in L^1(\Omega), -\frac{\partial \varphi(u)}{\partial v} \in \beta(u) \text{ on } \partial \Omega \}.
\]

The derivatives here are understood in the sense of distributions. As is shown in [14], p. 97, the operator \( A \) defined above is \( m \)-dissipative (i.e., \( -A \) is \( m \)-accretive) in \( L^1(\Omega) \).

Let

\[
F(t, x) = \{ f \in L^1(\Omega) ; f(y) \in G(t, y, x(y)) \text{ a.e. in } \Omega \},
\]

for \( (t, x) \in [0, T] \times L^1(\Omega) \), which is jointly measurable and continuous on \( x \). We also assume that there exists \( \lambda \in L^1([0, T]) \) such that \( \|F(t, x)\| \leq \lambda(t)(1 + |x|) \). Let \( x_0 = u(\cdot) \in D(A) \). Therefore, (F1) and (F2) hold true.
We suppose that there exists a Lebesgue function \( L(\cdot) \) such that for every \( x, z \in L^1(\Omega) \) and every \( f \in F(t, x) \) there exists \( g \in F(t, z) \) with:

\[
\int_{\Omega_{x-z}} (f(y) - g(y)) dy - \int_{\Omega_{x-z}} (f(y) - g(y)) dy + \int_{\Omega_{x-z}} |f(y) - g(y)| dy \leq L(t) \int_{\Omega} |x(y) - z(y)| dy.
\]

(7)

Here, \( \Omega^{+(-\beta)}_{x-z} = \{ y \in \Omega; x(y) > (\leq) z(y) \} \). It follows from the characterization of \([\cdot, \cdot]_+\) that (F3) also holds true (see, e.g., Example 1.4.3 in [23]). Furthermore, it is easy to see that \( F \) has convex weakly compact values.

We assume, in addition, that there exists a positive integrable function \( \mu(\cdot) \) with \( |l(t, u) - l(t, v)| \leq \mu(t)|u - v| \) for any \( t \in [0, T] \), \( u, v \in \mathbb{R} \). Then, (g) holds true.

The following result follows from Corollary 1:

**Theorem 3.** Suppose that \( K \int_0^T L(t) dt < 1 \), where \( K := \int_0^T u(t) dt > 0 \). Then, under the conditions above, the problem (6) has a solution. Moreover, the solution set depends continuously on \( h \).

**Remark 4.** If we replace the second equation of (6) by \( \phi(u(t, y)) = 0 \) and assume that \( 0 \in \varphi(0) \cap D(\varphi) \), then this example becomes a slight modification of the porous medium system considered in [1], p. 272. Of course, in that case, one has to change the domain of \( A \). Namely, we obtain the system:

\[
\begin{cases}
  u_t(t, y) \in \Delta \varphi(u) + G(t, y, u) & \text{in } I \times \Omega \\
  \varphi(u(t, y)) = 0 & \text{in } \partial \Omega \\
  u(0, y) \in \int_0^T l(t, u(t, y)) dt.
\end{cases}
\]

(8)

Here, \( \Omega \subset \mathbb{R}^n \) is an open bounded domain with smooth boundary \( \partial \Omega \), \( I = (0, T) \). Recall that \( \Delta \) denotes Laplace operator. \( A \varphi = \Delta \varphi(u) \) for each \( u \in D(A) \), where

\[
D(A) = \{ u \in L^1(\Omega), \varphi(u) \in W_0^{1,1}(\Omega), \Delta \varphi(u) \in L^1(\Omega) \}.
\]

In that case, the conclusion of Theorem 3 holds true.

3. OSL Condition with Negative Constant

In this section, we assume that the multifunction \( F \) is OSL with a negative constant, that is, \( \lambda(t) \equiv -L < 0 \). More exactly, we replace (F3) with the following hypothesis:

(F3') \( F : I \times E \Rightarrow E \) is OSL w.r.t. a negative constant, i.e., there exists \( L > 0 \) such that for every \( u, v \in \overline{D(A)} \) and every \( f_u \in F(t, u) \) there exists \( f_v \in F(t, v) \) such that:

\[
[u - v, f_u - f_v]_+ < -L[u - v] + \varepsilon.
\]

We will need the following theorem, which is a modification of ([13], Theorem 4.1).

**Theorem 4.** Let \( \omega > 0 \) and let \( A : D(A) \subset X \Rightarrow X \) be an \( m \)-dissipative operator such that \( A + \omega I \) is dissipative. If \( f, g \in L^1(t_0, T; X) \) and \( x(\cdot), y(\cdot) \) are two solutions of (2) corresponding to \( f(\cdot) \) and \( g(\cdot) \), respectively, then:

\[
|x(t) - y(t)| \leq e^{-\omega(t-s)}|x(s) - y(s)| + \int_s^t e^{-\omega(t-\tau)}|x(\tau) - y(\tau), f(\tau) - g(\tau)| d\tau
\]

(9)

for each \( t_0 \leq s < t \leq T \).

The next theorem will play a crucial role in getting the main results of this section. Let us remark that \(-L\) is not a Perron function (see [12] for the definition of a Perron function), hence we cannot directly apply the results from [12].
Theorem 5. Assume (F1), (F2) and (F3'). Let \( x_0, y_0 \in \overline{D(A)} \). If \( x(\cdot) \) is a limit solution of (4) with \( x(t_0) = x_0 \) then for every \( \epsilon > 0 \) there exists a limit solution \( y(\cdot) \) of (4) with \( y(t_0) = y_0 \) such that:
\[
|x(t) - y(t)| \leq e^{-L(t-t_0)}|x_0 - y_0| + \epsilon,
\]
for every \( t \in I \).

Proof. First, let us note that, if we consider a new operator \( B = A - LL \), where \( I \) is the identity operator, and a new multifunction \( \tilde{F}(t,x) = F(t,x) + Lx \) for any \((t,x) \in I \times X\), the problem (4) can be rewritten as:
\[
\begin{cases}
x(t) \in Bx(t) + \tilde{F}(t,x(t)) \\
x(t_0) = x_0.
\end{cases}
\]

It is easy to see that \( B \) is m-dissipative and \( \tilde{F}(t,\cdot) \) is OSL with constant 0. Hence, in what follows, we assume that \( A \) is m-dissipative such that \( A + LI \) is dissipative and \( \tilde{F}(t,\cdot) \) is OSL with constant 0.

Let \( \epsilon > 0 \) and take \( \mu := \epsilon/2 \) and \( \delta := \mu / [2(2 + T - t_0)] \). Let \( x(\cdot) \) be a limit solution of (4) with \( x(t_0) = x_0 \). Then there exists a \( \delta \)-solution \( z(\cdot) \) of (4) with \( z(t_0) = x_0 \) such that:
\[
|x(t) - z(t)| \leq \mu,
\]
for any \( t \in I \). Denote by \( f_z(\cdot) \) the pseudoderivative of \( z(\cdot) \). As is shown in the proof of Lemma 2 in [12], for \( \delta_1 = \delta/2 \) there exists a \( \delta_1 \)-solution \( v_1(\cdot) \) of (4) with \( v_1(t_0) = y_0 \) such that
\[
[z(t) - v_1(t), f_z(t) - f_{v_1}(t)]_+ < l(t),
\]
where \( l(t) \leq \text{dist}(f_z(t), F(t,z(t))) + \text{dist}(f_{v_1}(t), F(t,v_1(t))) + \delta \). We denoted by \( f_{v_1}(\cdot) \) the pseudoderivative of \( v_1(\cdot) \). By (9) we get that:
\[
|z(t) - v_1(t)| \leq e^{-L(t-t_0)}|x_0 - y_0| + \int_{t_0}^t e^{-L(t-s)}l(s)ds
\]
\[
\leq e^{-L(t-t_0)}|x_0 - y_0| + \int_{t_0}^t \text{dist}(f_z(s), F(s,z(s)))ds + \int_{t_0}^t \text{dist}(f_{v_1}(s), F(s,v_1(s)))ds + \delta(t - t_0)
\]
\[
\leq e^{-L(t-t_0)}|x_0 - y_0| + \delta + \delta_1 + \delta(t - t_0);
\]
hence,
\[
|z(t) - v_1(t))| \leq e^{-L(t-t_0)}|x_0 - y_0| + \delta(2 + T - t_0),
\]
for any \( t \in I \). By a similar way we can prove that for \( \delta_2 = \delta/2^2 \) there exists a \( \delta_2 \)-solution \( v_2(\cdot) \) of (4) with \( v_2(t_0) = y_0 \) such that:
\[
|v_1(t) - v_2(t)| \leq \delta_1(2 + T - t_0),
\]
for any \( t \in I \). Following this technique, we can construct a sequence \( (\delta_n)_n \) by \( \delta_n := \delta / 2^n \), \( \delta_n \downarrow 0 \), and a sequence \( (v_n(\cdot)) \) of \( \delta_n \)-solutions of (4) with \( v_n(t_0) = y_0 \) such that
\[
|v_{n-1}(t) - v_n(t)| \leq \delta_{n-1}(2 + T - t_0),
\]
for \( t \in I \) and \( n \geq 2 \). Clearly \( v_n(t) \to y(t) \) uniformly on \([t_0,T]\) and, using (10)–(12), we get that
\[
|x(t) - y(t)| \leq e^{-L(t-t_0)}|x_0 - y_0| + \epsilon
\]
for \( t \in [t_0,T] \). □
Now we study the nonlocal problem (1). We replace condition (g) by:

(g') There exists $\tau \in (t_0, T]$ such that for any $x, y \in C([t_0, T]; \overline{D(A)})$ one has:

$$|g(x(\cdot)) - g(y(\cdot))| \leq \|x(\cdot) - y(\cdot)\|_{C([\tau, T]; \overline{D(A)})}.$$ 

**Remark 5.** From condition (g'), it is easy to see that for any $x, \tilde{x} \in C([t_0, T]; \overline{D(A)})$ with $x(t) = \tilde{x}(t)$ for any $t \in [\tau, T]$ we have that $g(x(\cdot)) = g(\tilde{x}(\cdot))$, hence $g$ depends only on the restriction of $x$ on $[\tau, T]$.

Condition (g') is satisfied by the following remarkable function

$$g(x(\cdot)) = \sum_{i=1}^{k} a_i x(t_i),$$

where $t_0 < t_1 < \ldots < t_k \leq T$ are arbitrary, but fixed, and $a_i \in \mathbb{R}$, with $\sum_{i=1}^{k} |a_i| \leq 1$ (multipoint discrete mean condition). In this case the constant $\tau > t_0$ in (g') is $t_1$. We also mention the particular cases: periodic and antiperiodic condition, that is, $g(x(\cdot)) = \pm x(T)$. A more general case is the following:

$$g(x(\cdot)) = \int_{t_0}^{T} N(x(t))dv(t),$$

where $N : E \to E$ is a (possible nonlinear) nonexpansive operator and $v$ is a $\sigma$-finite and complete measure for which there exists $b > t_0$ such that $\text{supp } v = [b, T]$ and $\nu([b, T]) = 1$. In this case, the constant $\tau > t_0$ is $b$.

**Theorem 6.** Under (F1), (F2), (F3') and (g'), the problem (1) has a limit solution.

**Proof.** In view of Remark 5, we can consider that $g$ is defined only on $C([\tau, T]; \overline{D(A)})$.

We define the multifunction $S : C([\tau, T]; \overline{D(A)}) \Rightarrow C([\tau, T]; \overline{D(A)})$ by:

$$S(u(\cdot)) = \left\{ x_{[\tau, T]} : x(\cdot) \text{ is a limit solution of (4) with } x(t_0) = g(u(\cdot)) \right\}$$

for any $u(\cdot) \in C([\tau, T]; \overline{D(A)})$. We denoted by $x_{[\tau, T]}$ the restriction of $x(\cdot)$ on $[\tau, T]$. We will prove that $S$ is a contraction.

To this end, let $u(\cdot), v(\cdot) \in C([\tau, T]; \overline{D(A)})$. Let $x(\cdot)$ be a limit solution of (4) with $x(t_0) = g(u(\cdot))$. By Theorem 5, for any $\epsilon > 0$ there exists $y(\cdot)$ a limit solution of (4) with $y(t_0) = g(v(\cdot))$ such that:

$$|x(t) - y(t)| \leq \exp(-L(t-t_0))|g(u(\cdot)) - g(v(\cdot))| + \epsilon$$

for any $t \in [t_0, T]$. Then, using (g'), for any $t \in [\tau, T]$ we have

$$|x(t) - y(t)| \leq \exp(-L(\tau-t_0))\|u(\cdot) - v(\cdot)\|_{C([\tau, T]; \overline{D(A)})} + \epsilon.$$

It follows that:

$$D_H(S(u(\cdot)), S(v(\cdot))) \leq \exp(-L(\tau-t_0))\|u(\cdot) - v(\cdot)\|_{C([\tau, T]; \overline{D(A)})} + \epsilon,$$

hence $S$ is a set valued contraction with closed values, so it has a fixed point $u(\cdot) \in S(u(\cdot))$. Then, $u(\cdot)$ is the restriction on $[\tau, T]$ of a limit solution $\tilde{u}(\cdot)$ of (4) with $u(t_0) = g(u(\cdot))$, i.e., $x_0 = g(\tilde{u}(\cdot))$. Hence, $\tilde{u}(\cdot)$ is a limit solution of (1). □

From Theorems 2 and 6 it follows the following result:
Corollary 2. If the conditions of Theorem 6 are satisfied, A is of complete continuous type and \((F4)\) holds, then the problem \((1)\) has an integral solution.

Remark 6. In case \(F(\cdot, \cdot)\) is single valued, Theorem 6 improves Theorem 3.1 of [24] for finite interval and the problem without time lag, because the Lipshitz condition from [24] is replaced by the OSL one. In this case it is possible to prove existence of solutions on \([t_0, \infty)\); however, this is out of the scope of this paper.

Example 2. We study the system \((6)\) with initial condition:

\[
(b) \quad u(0, x) = \sum_{i=1}^{k} u(t_i, x).
\]

Here, \(t_0 < t_1 < t_2 < \cdots < t_k \leq T\). We assume that there exists a negative constant \(N\) such that the inequality \((7)\) holds true with \(L(t)\) replaced by \(N\). Thus, there exists a solution of \((6)\) thanks to Corollary 2.

Theorem 7. Assume that there exists a negative constant \(N\) such that the inequality \((7)\) holds true with \(L(t)\) replaced by \(N\). Then, under the conditions above, there exists a solution of \((6)\).

Remark 7. If we change the second Equation \((6)\) by \(q(u(t, y)) = 0\) then the system \((6)\) becomes a slight modification of the nonlinear reaction equation under nonlocal conditions \((b)\) considered in [1], p. 136.

3.1. Case of Linear A

In this subsection, we study the following periodic problem:

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + f_x(t), \quad t \in (t_0, T) \\
f_x(t) &\in F(t, x(t)) \\
x(t_0) &= x(T) \in D(A),
\end{align*}
\]

under the following assumptions:

(A) \(A : D(A) \subseteq E \to E\) is a densely defined linear operator generating a \(C_0\)-semigroup \(S(\cdot)\) and

\[x - y, A(x - y)\] \(\geq 0, \forall x, y \in D(A)\).

We denoted by \([\cdot, \cdot]_\cdot\) the left directional derivative of the norm, that is,

\[|x - y, A(x - y)|_\cdot = \lim_{h \to 0} h^{-1} \left( ||x + hy|| - ||x|| \right).\]

(F) \(F(\cdot, \cdot)\) is almost continuous and there exists a constant \(L > 0\) such that for every \(x, y \in E\) and every \(u_x \in F(t, x)\) there exists \(u_y \in F(t, y)\) with

\[|x - y, u_x - u_y|_\cdot \geq L|x - y|\.
\]

Definition 5. The continuous function \(x : [t_0, T] \to E\) is said to be a mild solution of \((13)\) on \([t_0, T]\) if \(x(t) = S(t - t_0)x(T) + \int_{t_0}^{t} S(t - \tau)f_x(\tau)d\tau\) for every \(t \in [t_0, T]\) and \(f_x(t) \in F(t, x(t))\) a.e. on \([t_0, T]\).

Notice that, when the linear, densely defined operator \(A\) is \(m\)-dissipative, the mild solutions of \((13)\) coincide with integral ones (see, e.g., Proposition 1.5 in [14]).

The main result in this subsection is the following:

Theorem 8. Under the hypotheses (A), (F), (F2) and (F4), the problem \((13)\) has a solution.
Proof. Clearly the operator $B = -A$ is $m$-dissipative. Furthermore, it is of a completely continuous type since $A$ is linear (see, e.g., [1,2,13]). We put $s = T - t$ and consider:

$$
\begin{cases}
    x(s) = Bx(s) + f_x(s), \ s \in (0, T - t_0) \\
    f_x(t) \in F(s, x(s)) \\
    x(0) = x(T - t_0) \in \overline{D(A)}.
\end{cases}
$$

Then, all the conditions of Corollary 2 hold true, hence the problem (14) has a mild solution, which is also a mild solution of (13). \qed

Notice that Theorem 8 also holds true when the boundary condition is antiperiodic, that is, $x(t_0) = -x(T)$.

It is also clear that, if we study the local problem, when the boundary condition is replaced by $x(t_0) = x_0 \in \overline{D(A)}$, the solution of the problem (13) does not necessarily exist.

3.2. Case of Autonomous $F$

Now we consider the periodic problem when $F(t, x) \equiv F(x)$, that is, the right hand side is autonomous,

$$
\begin{aligned}
    x(t) &\in Ax(t) + F(x), \ t \in [S, T] \\
    x(S) &= x(T), \ S < T.
\end{aligned}
$$

Here, $A$ is of a complete continuous type.

We assume that $F(\cdot)$ is OSL with negative constant $-L$, that is, for every $u, v \in \overline{D(A)}$, every $\varepsilon > 0$ and every $f_u \in F(u)$ there exists $f_v \in F(v)$ with $|u - v, f_u - f_v| < -L|u - v| + \varepsilon$, and continuous with convex weakly compact values. Moreover, there exist constants $a, b$ such that $|F(x)| \leq a + b|x|$ for every $x \in E$. The following result follows immediately from Corollary 2.

Proposition 5. Under the above assumptions the problem (15) has a nonempty $C(I; E)$ closed set of solutions.

Clearly, every solution of (15) can be extended to $(-\infty, +\infty)$ by periodicity, that is, it will be periodic with period $T - S$.

Theorem 9. Let $A$ generate a compact semigroup. Then under the conditions of Proposition 5 there exists $y \in \overline{D(A)}$ such that $0 \in Ay + F(y)$.

Proof. Due to Remark 1, we can assume, without loss of generality, that $0 \in D(A)$ and $0 \in A0$. Therefore, the unique solution of $\dot{y}(t) \in Ay(t)$, $y(0) = 0$ is $y(t) \equiv 0$.

We can also replace $A$ by $A - L I$, where $I$ is the identity operator and $F(x)$ by $F(x) + Lx$ for any $x \in E$. In this case $A + L I$ becomes dissipative and $F(\cdot)$ OSL with a constant $0$.

Therefore for every $\varepsilon > 0$ and every solution $y(\cdot)$ of

$$
\begin{cases}
    \dot{y}(t) \in Ay(t) + F(y(t)) \\
    y(0) = y_0 \in \overline{D(A)},
\end{cases}
$$

with pseudoderivative $f_y(\cdot)$ there exists a strongly measurable selection $g_y(t) \in F(0)$ such that: $[y(t) - 0, f_y(t) - g_y(t)]_+ < \varepsilon$. Let $\|F(0)\| := P > 0$. Therefore, $[y(t), f_y(t)]_+ < P + \varepsilon$. Due to Theorem 4,

$$
|y(t)| \leq e^{-Lt}|y(0)| + \int_0^t e^{-L(t-s)}(P + \varepsilon)ds
\leq e^{-Lt}|y(0)| + (P + \varepsilon) \frac{1 - e^{-Lt}}{L}.
$$
Clearly, there exists $T_0 > 0$ such that $e^{-lt}|y_0| \leq 1$ for $t \geq T_0$. Consequently, without loss of generality, one can consider only solutions $y(\cdot)$ of (16) for which $|y(t)| \leq 1 + \frac{\overline{F}}{L} := M$. Denote by $\text{Reach}(t, y_0)$ the reachable set of (16), that is,

$$\text{Reach}(t, y_0) = \{ v \in E; \exists \text{ solution } y(\cdot) \text{ of (16) with } y(t) = v \}.$$ 

Since $A$ generates a compact semigroup, one has that for every $y_0 \in \overline{D(A)}$ the reachable set $\text{Reach}(t, y_0)$ of $y(t) = Ay + F(y)$, $y(0) = y_0$, is compact for any $t \geq 0$. Furthermore, by Theorem 5, $\mathcal{D}_H(\text{Reach}(t, u))$, $\text{Reach}(t, v)) \leq e^{-lt}|u - v|$. Therefore, $\lim_{t \to \infty} \text{Reach}(t, MB)$ exists and it is a precompact set with closure, say $U$. Let $z(\cdot)$ be a solution to (4) with initial condition $z(0) \in U$. Clearly, $z(t) \in U$. It follows from Corollary 2 and Remark 5 that, for every $T > 0$, the problem (4) has a $T$ periodic solution.

Let $\{y_n(\cdot)\}_{n \geq 1}$ be a sequence of solutions with period, respectively, $\frac{1}{n}$. It is easy to see that this sequence is equicontinuous and due to Arzela theorem, passing to subsequences, $y_n(t)$ converges uniformly on $[0, 1]$ to a continuous function $z(\cdot)$. Clearly, $z(t) = z(0)$ on $[0, 1]$ and $z(\cdot)$ is a solution of (4) with initial condition $z_0 = z(0)$. Since $z(\cdot)$ is constant, then $z(t) = 0$. Therefore, $z(0) \in D(A)$ and $0 \in Az(0) + F(z(0))$. 

Example 3. We study the system:

$$\begin{align*}
\{ u_t & \in \Delta \varphi(u) + G(y, u) \text{ on } (0,T) \times \Omega \\
\frac{\partial \varphi(u)}{\partial v} & \in \beta(u) \text{ on } (0,T) \times \partial \Omega. 
\end{align*} \tag{17}$$

We assume that all the conditions of Example 2 hold true, and moreover, $\varphi(\cdot)$ is continuously differentiable on $\mathbb{R} \setminus \{0\}$ with $\varphi'(r) \geq Cr^{-\gamma-1}$. Here $C > 0$ and $\gamma > \max\{0, \frac{n-2}{2}\}$. In this case, $A$ generates a compact semigroup and it is of a complete continuous type (see p. 39 in [1]).

Recall that the solution $u(\cdot)$ of the system (17) is said to be steady state if it does not depend on $t$.

Therefore, it follows from Theorem 9 that the system (17) has a steady state solution.

4. Conclusions

This paper can be considered an extension of [12], where the local problem (4) is studied under conditions weaker than here (namely, one-sided Perron condition). The conditions in the latter paper are not appropriate for the nonlocal problems studied here. Notice that, in the case of Banach spaces with uniformly convex dual existence, results are proved under the OSL condition, but under weaker continuity assumptions.

The topic studied in the present paper is important and it is investigated in many previous publications. In [2,14] the author essentially uses some compactness type conditions. Notice also [25], where the theory of monotone operators in the evolution (Gelfand) triple is studied, again using compact embedding of reflexive and separable Banach space in a separable Hilbert one. Such a problem, but with many qualitative results, is studied in [22]. In [6], either compactness conditions or Lipschitz continuity in a Banach space with uniformly convex dual are used. The case of Lipschitz continuity is then studied in [7], where some conditions are also relaxed. In [3], multivalued perturbations of non autonomous m-dissipative evolution inclusions under compactness type assumptions are studied. Notice also the very good book [1], where the authors study evolution inclusions with a time lag, assuming mainly that $A$ generates a compact semigroup.

The novelties in the present article are, in general:

I. We study evolution inclusions in general Banach spaces and prove the existence of solutions under similar conditions as in the existing literature, where the Banach space is with uniformly convex dual and we improve, for example, the results from [6,7,25].
II. We also investigate the problem when the OSL constant is negative. In the case of the Lipschitz right-hand side, we assume that the multivalued perturbation is only OSL, but not Lipschitz. We partially improve the main results from [20].

III. We prove under relatively weak conditions the existence of zeros of $A + F$.

IV. We provide examples that cover some real reaction-diffusion problems.

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**References**

1. Burlică, M.; Necula, M.; Roșu, D.; Vrabie, I.I. *Delay Differential Equations Subjected to Nonlocal Initial Conditions; Monographs and Research Notes in Mathematics*, CRC Press: New York, NY, USA, 2016.

2. Bothe, D. Multivalued perturbations of m-accretive differential inclusions. *Israel J. Math.* 1998, 108, 109–138. [CrossRef]

3. Aizicovici, S.; Staicu, V. Multivalued evolution equations with nonlocal initial conditions in Banach spaces. *NoDEA Nonlinear Differ. Equ. Appl.* 2007, 14, 361–376. [CrossRef]

4. Paicu, A.; Vrabie, I.I. A class of nonlinear evolution equations subjected to nonlocal initial conditions. *Nonlinear Anal.* 2010, 72, 4091–4100. [CrossRef]

5. Vrabie, I.I. Existence in the large for nonlinear delay evolution inclusions with nonlocal initial conditions. *J. Funct. Anal.* 2012, 262, 1363–1391. [CrossRef]

6. Zhu, L.; Hiang, Q.; Li, G. Existence and asymptotic properties of solutions of nonlinear multivalued differential inclusions with nonlocal conditions. *J. Math. Anal. Appl.* 2012, 390, 523–534. [CrossRef]

7. Ahmed, R.; Donchev, T.; Lazu, A.I. Nonlocal m-dissipative evolution inclusions in general Banach spaces. *Mediterr. J. Math.* 2017, 14, 215. [CrossRef]

8. Bilal, S.; Cărjă, O.; Donchev, T.; Javaid, N.; Lazu, A.I. Nonlocal evolution inclusions under weak condition. *Adv. Differ. Equ.* 2018, 399. [CrossRef]

9. Debrath, P.; Mitrović, Z.; Cho, S.Y. Common fixed points of Kannan, Chatterjea and Reich type pairs of self-maps in a complete metric space. *Sao Paulo J. Math. Sci.* 2021, 15, 383–391. [CrossRef]

10. Damjanović, B.; Samet, B.; Vetro, C. Common fixed point theorems for multi-valued maps. *Acta Math. Sci.* 2012, 32, 818–824. [CrossRef]

11. Konwar, N.; Debrath, P. Some new contractive conditions and related fixed point theorems in intuitionistic fuzzy n-Banach spaces. *J. Intell. Fuzzy Syst.* 2018, 34, 361–372. [CrossRef]

12. Donchev, T.; Bilal, S.; Cărjă, O.; Javaid, N.; Lazu, A.I. Evolution inclusion in Banach space under dissipative conditions. *Mathematics* 2020, 8, 750. [CrossRef]

13. Barbu, V. *Nonlinear Differential Equations of Monotone Types in Banach Spaces*; Springer: New York, NY, USA, 2010. [CrossRef]

14. Bothe, D. *Nonlinear Evolutions in Banach Spaces*; Paderborn, Germany, 1999. [CrossRef]

15. Ito, K.; Kappel, F. *Evolution Equations and Approximations; World Scientific*: Singapore, 2002.

16. Lakshmikantham, V.; Leela, S. *Nonlinear Differential Equations in Abstract Spaces*; Pergamon Press: Oxford, UK, 1981.

17. Donchev, T. Functional differential inclusions with monotone right-hand side. *Nonlinear Anal.* 1991, 16, 543–552.

18. Donchev, T.; Farkhi, E. Stability and Euler Approximations of one-sided Lipschitz convex differential inclusions. *SIAM J. Control Optim.* 1996, 36, 780–796.

19. Donchev, T.; Farkhi, E.; Mordukhovich, B. Discrete approximations, relaxation, and optimization of one-sided Lipschitzian differential inclusions in Hilbert spaces. *J. Differ. Equ.* 2007, 24, 301–328. [CrossRef]

20. Beyn, W.; Rieger, J. The implicit Euler scheme for one-sided Lipschitz differential equations. *Discr. Cont. Dyn. Syst.* 2010, 14, 409–428. [CrossRef]

21. Mordukhovich, B.; Tian, Y. Implicit Euler approximation and optimization of one-sided Lipschitzian differential inclusions. *Nonlinear Anal. Optim.* 2016, 650, 165–188. [CrossRef]

22. Beyn, W.; Rieger, J. Semilinear differential inclusions with one-sided Lipschitz nonlinearities. *J. Evol. Equ.* 2018, 18, 1319–1338. [CrossRef]

23. Vrabie, I.I. *Compactness Methods for Nonlinear Evolutions*; Wiley: Harlow, UK, 1995.
24. Burlică, M.; Roșu, D. A class of nonlocal evolution equations with nonlocal initial conditions. *Proc. AMS* 2014, 142, 2445–2458. [CrossRef]

25. Hu, S.; Papageorgiou, N. *Handbook of Multivalued Analysis; Applications*, Kluwer: Dordrecht, The Nederlands, 2000; Volume II.