Parameterized Extension Complexity of Independent Set and Related Problems

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Abstract

Let $G$ be a graph on $n$ vertices and $\text{STAB}_k(G)$ be the convex hull of characteristic vectors of its independent sets of size at most $k$. We study extension complexity of $\text{STAB}_k(G)$ with respect to a fixed parameter $k$ (analogously to, e.g., parameterized computational complexity of problems). We show that for graphs $G$ from a class of bounded expansion it holds that $\text{xc}(\text{STAB}_k(G)) \leq O(f(k) \cdot n)$ where the function $f$ depends only on the class. This result can be extended in a simple way to a wide range of similarly defined graph polytopes. In case of general graphs we show that there is no function $f$ such that, for all values of the parameter $k$ and for all graphs on $n$ vertices, the extension complexity of $\text{STAB}_k(G)$ is at most $f(k) \cdot n^{O(1)}$. While such results are not surprising since it is known that optimizing over $\text{STAB}_k(G)$ is $\text{FPT}$ for graphs of bounded expansion and $\text{W}[1]$-hard in general, they are also not trivial and in both cases stronger than the corresponding computational complexity results.

Keywords: extension complexity, fixed-parameter polynomial extension, independent set polytope, bounded expansion

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1. Introduction

Polyhedral (aka LP) formulations of combinatorial problems belong to the basic toolbox of combinatorial optimization. In a nutshell, a set of feasible solutions of some problem is suitably encoded by a set of vectors, whose convex hull forms a polytope over which one can then optimize using established tools. A polytope \(Q\) is said to be an extended formulation or extension of a polytope \(P\) if \(P\) is a projection of \(Q\). Measuring the size of a polytope by the minimum number of inequalities required to describe it, one can define the extension complexity of a polytope to be the size of the smallest extension of the polytope. This notion has a rich history in combinatorial optimization where by adding extra variables one can sometimes obtain significantly smaller polytopes. For some recent survey on extended formulations in the context of combinatorial optimization and integer programming see [8, 13, 20, 21].

Since linear (or indeed convex) optimization of a polytope \(P\) can instead be indirectly done by optimizing over an extended formulation of \(P\), this concept provides a powerful model for solving many combinatorial problems. Various Linear Program (LP) solvers exist today that perform quite well in practice and it is desirable if a problem can be modeled as a small-sized polytope over which one can use an existing LP solver for linear optimization. However, in recent years super-polynomial lower bound on the extension complexity of polytopes associated with many combinatorial problems have been established. These bounds have been generalized to various settings, such as convex extended formulations, approximation algorithms, etc. These results are too numerous for a comprehensive listing, but we refer the interested readers to some of the landmark papers in this regard [4, 7, 10, 19].

Many of the recent lower bounds on the extension complexity of various combinatorial polytopes mimic the computational complexity of the underlying problem. For example, it is known that the extension complexities of polytopes related to various NP-hard problems are super-polynomial [1, 4, 10, 18]. One satisfying feature of these lower bounds is that they are independent of traditional complexity-theoretic assumptions such as \(P \neq NP\). Though, there also exist polytopes corresponding to polynomial time solvable optimization problems whose extension complexity is super-polynomial. In particular, the perfect matching polytope was shown to have super-polynomial extension complexity by Rothvoß [19]. Hence even if the extension complexity of a problem mimics its computational complexity, lower and upper bounds on the former do not follow from the corresponding computational complexity bounds and constitute nontrivial new results of independent interest.

One can naturally ask the related questions in the realm of parameterized complexity theory. In this rapidly grown field each problem instance comes additionally equipped with an integer parameter, and the “efficient” class denoted by FPT (fixed-parameter tractable) is the one of problems solvable, for every fixed value of the parameter, in polynomial time of degree independent of the parameter. See Section 2 for details.

Similarly as parameterized complexity provides a finer resolution of algorithm-
mic tractability of problems, parameterized extension complexity can provide a finer resolution of extension complexities of polytopes of the problems. We similarly say that a polytope has an \textit{FPT extension} if it has an extension which is, for every fixed value of the parameter, of polynomial size with degree independent of the parameter. Again, see Section 2 for details.

We follow this direction of research with a case study of the \textit{independent-set polytope} of a graph, naturally parameterized by the solution size. We confirm that the extension complexity of the independent-set polytope indeed mimics the parameterized computational complexity of the underlying independent set problem—a finding which is again not implied by the parameterized complexity status of this problem and which is actually a lot stronger than previous related complexity knowledge. Precisely, we prove:

- that the independent-set polytope cannot have an FPT extension for all graphs, independently of any computational-complexity assumptions (Section 3), but

- linear-sized FPT extensions of the independent-set polytope do exist on every graph class of bounded expansion (Section 4).

Seeing the latter result, one may naturally think whether analogous results hold for other similar problems. For example, one may consider the polytope of (induced) subgraphs isomorphic to a given graph $F$, parameterized by the size of $F$. Or, more generally, polytopes defined by solutions of non-local problems, such as the polytope of dominating sets of a certain size. While ad-hoc adaptions of our technique to such problems are surely possible, we prefer to give a “metatheorem”—a generic solution aimed at all problems defined in a certain framework.

Namely, we further formulate and prove the following generalizations:

- there is a natural way to assign a definition of a polytope to every graph problem expressible in FO logic, and these polytopes have linear-sized FPT extensions on every graph class of bounded expansion when parameterized by the size of the formula expressing the problem (Section 5),

- for a restricted fragment of FO graph problems, near-linear-sized FPT extensions of the polytopes exist even on so called nowhere dense graph classes (Section 6).

We conclude the paper with some further thoughts and suggestions in Section 7.

2. Preliminaries

We follow standard terminology of graph theory and consider finite simple undirected graphs. We refer to the vertex and edge sets of a graph $G$ as $V(G)$ and $E(G)$, respectively. An \textit{independent set} $X$ of vertices of a graph is such that no two elements of $X$ are adjacent. By a \textit{cut} in a graph $G$ we mean an edge
cut, that is, an inclusion-wise minimal set of edges \( C \subseteq E(G) \) such that \( G \setminus C \) has more connected components than \( G \).

For fundamental concepts of parameterized complexity we refer the readers, e.g., to the monograph [9]. Here we just very briefly recall the needed notions.

Considering a problem \( P \) with input of the form \((x, k) \in \Sigma^* \times \mathbb{N}\) (where \( k \) is a parameter), we say that \( A \) is fixed-parameter tractable (shortly FPT) if there is an algorithm solving \( A \) in time \( f(k) \cdot n^{O(1)} \) where \( f \) is an arbitrary computable function. In the (parameterized) \( k \)-independent set problem the input is \((G, k)\) where \( G \) is a graph and \( k \in \mathbb{N} \), and the question is whether \( G \) has an independent set of size at least \( k \).

There is no known FPT algorithm for the \( k \)-independent set problem in general and, in fact, the theory of parameterized complexity [9] defines complexity classes \( W[t] \), \( t \geq 1 \), such that the \( k \)-independent set problem is complete for \( W[1] \). Problems that are \( W[1] \)-hard do not admit an FPT algorithm unless the Exponential Time Hypothesis fails.

Returning back to graph structure, we shall deal with the concept of treewidth of a graph. Given a graph \( G \), a tree-decomposition of \( G \) is an ordered pair \((T, W)\), where \( T \) is a tree and \( W = \{W_x \subseteq V(G) \mid x \in V(T)\} \) is a collection of bags (vertex sets of \( G \)), such that the following hold:

1. \( \bigcup_{x \in V(T)} W_x = V(G) \);
2. for every edge \( e = uv \) in \( G \), there exists \( x \in V(T) \) such that \( u, v \in W_x \);
3. for each \( u \in V(G) \), the set \( \{x \in V(T) \mid u \in W_x\} \) induces a subtree of \( T \).

The width of this tree-decomposition is \( \max_{x \in V(T)} |W_x| - 1 \), and the treewidth \( tw(G) \) of \( G \) is the smallest width of a tree-decomposition of \( G \).

It is worth to note that computing an optimal tree-decomposition of a graph \( G \) is linear-time FPT with the parameter \( tw(G) \) [3].

2.1. Fixed-parameter extension complexity

The size of a polytope \( P \), denoted by \( \text{size}(P) \), is defined to be the number of facets of \( P \), which is the minimum number of inequalities needed to describe \( P \) if it is full-dimensional. A polytope \( Q \) is called an extension of a polytope \( P \) if \( P \) can be obtained as a linear projection of \( Q \). As a shorthand we will say that in this case \( Q \) is an EF of \( P \). As noted in the Introduction, the following is a useful notion:

**Definition 2.1** (Extension complexity). The extension complexity of a polytope \( P \), denoted by \( \text{xc}(P) \), is defined to be the size of the smallest extension. More precisely,

\[
\text{xc}(P) := \min_{Q \text{ an EF of } P} \text{size}(Q).
\]

In the context of fixed-parameter extension complexity, we deal with families of polytopes \( \mathcal{P}_n \) where \( n \in \mathbb{N} \), and a parameter \( k \). For example, for the independent set problem parameterized by a nonnegative integer \( k \), the family \( \mathcal{P}_n \) could be the family of \( k \)-independent set polytopes (cf. Subsection 2.2).
a family of $n$-vertex graphs. The prime question is whether there exists a computable function $f$ such that $\text{xc}(P) \leq f(k) \cdot n^{O(1)}$ for all $k, n$ and all $P \in \mathcal{P}_n$. As a shorthand we will say in the affirmative case that the collection of families $\{\mathcal{P}_n : n \in \mathbb{N}\}$ has FPT extension complexity (in a natural analogy with the aforementioned FPT complexity class—it also readily follows that problems with FPT extension complexity can be solved in FPT time if the extension can be efficiently constructed).

Buchanan in a recent article [5] studied the fixed-parameter extension complexity of the $k$-vertex cover problem, and proved that for any graph $G$ with $n$ vertices, the $k$-vertex cover polytope has extension complexity at most $O(c^k n)$ for some constant $c < 2$. Hence this is a nontrivial example of a polytope class with FPT extension complexity. Buchanan also raised the question whether the $k$-independent set polytope (Definition 2.4) admits an FPT extension. We answer this in the negative in Theorem 3.6. Note that our negative answer does not rely on any complexity theoretical assumptions (such as $\text{FPT} \neq \text{W}[1]$).

We also look at the positive side of the $k$-independent set problem. It is known that this problem admits an FPT algorithm (w.r.t. $k$) on quite rich restricted graph classes, e.g., on classes of bounded expansion [15] (see Subsection 2.3 for the definition). While this finding, in general, does not imply anything about the extension complexity of the $k$-independent set polytope, we manage to apply the tools of [15] in our setting, and confirm—in Theorem 4.3—FPT extension complexity of the $k$-independent set polytope on any graph class of bounded expansion. We also study a meta-generalization of this result (to all FO-definable problems) in Section 5 and partly generalize our result to nowhere dense graph classes in Section 6.

In the course of proving aforementioned Theorems 3.6 and 4.3 we are going to use the following established results on the topic of extension complexity.

**Theorem 2.2** (Balas [2]). Let $P_1, P_2, \ldots, P_s$ be polytopes and let $P := \text{conv}(\bigcup_{i=1}^s P_i)$. Then, $\text{xc}(P) \leq s + \sum_{i=1}^s \text{xc}(P_i)$.

For a graph $G$, a cut vector is a 0/1 vector of length $|E(G)|$ whose coordinates correspond to whether an edge of $G$ is in a cut $C \subseteq E(G)$ or not. The cut polytope is then the convex hull of all the cut vectors of $G$. Our negative result relies on the following lower bound.

**Theorem 2.3** (Fiorini et al. [10]). The extension complexity of the cut polytope of the complete graph $K_n$ on $n$ vertices is $2^{\Omega(n)}$.

**2.2. The $k$-independent set polytope**

Let $G$ be a graph on $n$ vertices. Every subset of vertices of $G$ can be encoded as a characteristic vector of length $n$. That is, for a subset $S \subseteq V$, define the characteristic vector $\chi_S$ as follows:

$$\chi^S_v = \begin{cases} 1 & \text{if } v \in S \\ 0 & \text{otherwise} \end{cases}$$
Definition 2.4 (Independent set polytope). The $k$-independent set polytope of $G$, denoted by $\text{STAB}^k(G)$, is defined to be the convex hull of the characteristic vectors of every independent set of size at most $k$. That is,

$$\text{STAB}^k(G) = \text{conv} \left\{ \chi^S \in \{0,1\}^n \mid S \subseteq V \text{ is an independent set of } G; |S| \leq k \right\}.$$  

In case that $k = n$ we simply speak about the independent set polytope of $G$; $\text{STAB}(G)$.

Alternatively, one could define the polytope $\text{STAB}_k(G)$ to be the convex hull of all independent sets of size exactly equal to $k$. That is,

$$\text{STAB}_k(G) = \text{conv} \left\{ \chi^S \in \{0,1\}^n \mid S \subseteq V \text{ is an independent set of } G; |S| = k \right\}.$$  

To simplify our situation, we note the following:

Lemma 2.5. $\text{xc}(\text{STAB}_k(G)) \leq \text{xc}(\text{STAB}^k(G)) \leq k + \sum_{i=0}^k \text{xc}(\text{STAB}_i(G))$.

Proof. Clearly, $\text{STAB}_k(G)$ is a face of $\text{STAB}^k(G)$. Therefore, $\text{xc}(\text{STAB}_k(G)) \leq \text{xc}(\text{STAB}^k(G))$.

On the other hand, $\text{STAB}^k(G) = \text{conv}(\bigcup_{i=1}^k \text{STAB}_i(G))$, and therefore $\text{xc}(\text{STAB}^k(G)) \leq k + \sum_{i=0}^k \text{xc}(\text{STAB}_i(G))$ by Theorem 2.2. □

We would like to remark that the above Lemma 2.5 shows that any bounds (lower or upper) that are valid for $\text{xc}(\text{STAB}_k(G))$ are also asymptotically valid for $\text{xc}(\text{STAB}^k(G))$. Therefore, the notation $\text{STAB}_k(G)$ can be freely used to mean either of the polytopes without affecting any lower or upper bounds asymptotically.

We shall also use the following result.

Theorem 2.6 (Buchanan and Butenko [6]). The extension complexity of a graph’s independent set polytope is $O(2^{tw_n})$, where $n$ is the number of vertices and $tw$ denotes its treewidth.

Note that Buchanan and Butenko give an explicit description of an extension of the independent set polytope, provided the corresponding tree-decomposition is given.

2.3. Sparsity and bounded expansion

A useful toolbox in our research is the theory of sparse graph classes, largely developed by Nešetřil and Ossona de Mendez. We follow their monograph [17].

We start by defining the notion of edge contraction. Given an edge $e = uv$ of a graph $G$, we let $G/e$ denote the graph obtained from $G$ by contracting the edge $e$, which amounts to deleting the endpoints of $e$, introducing a new vertex $w_e$ and making it adjacent to all vertices in the union of the neighborhoods of $u$ and $v$ (excluding $u, v$ themselves). A minor of $G$ is a graph obtained from a subgraph of $G$ by contractin zero or more edges. In a more general view, if
$H$ is isomorphic to a minor of $G$, then we call $H$ a minor of $G$ as well, and we write $H \preceq G$.

Alternatively, $H$ is a minor of $G$ if there exists a bijection $\psi \colon V(H) \to \{V_1, \ldots, V_p\}$ where $V_1, \ldots, V_p$ are pairwise disjoint subsets of $V(G)$ inducing connected subgraphs of $G$, and $uv \in E(H)$ only if there is an edge in $G$ with an endpoint in each of $\psi(u)$ and $\psi(v)$. If, moreover, it is required that each subgraph $G[V_i]$ has radius at most $d$, meaning that there exist $c_i \in V_i$ (a center) such that every vertex in $V_i$ is within distance at most $d$ from $c_i$ in $G[V_i]$; then $H$ is called a shallow minor at depth $d$ of $G$ (shortly, a $d$-shallow minor).

One of the most prominent notions of “sparsity” for graph classes is the following one:

**Definition 2.7** (Grad and bounded expansion [16]). Let $\mathcal{G}$ be a graph class. Then the greatest reduced average density of $\mathcal{G}$ with rank $d$ is defined as

$$\nabla_d(\mathcal{G}) = \sup_{H \in \mathcal{G}} \frac{|E(H)|}{|V(H)|}.$$ 

A graph class $\mathcal{G}$ has bounded expansion if there exists a function $f : \mathbb{N} \to \mathbb{R}$ (called the expansion function) such that for all $d \in \mathbb{N}$, $\nabla_d(\mathcal{G}) \leq f(d)$.

We provide a brief informal explanation of Definition 2.7. A graph to be considered “sparse” should not, in particular, contain subgraphs with relatively many edges. Since $G \nabla 0$ is the set of all subgraphs of $G$, this is captured by $2\nabla_0(G)$ being the maximum average degree over all subgraphs of $G$. However, the definition requires more; even after contracting edges up to limited depth $d$, the resulting shallow minors stay free of relatively dense subgraphs, with the maximum average degree bounded by $2\nabla_d(\mathcal{G}) \leq 2f(d)$.

For example, the class $\mathcal{P}$ of all planar graphs has bounded expansion (even with a constant expansion function). On the other hand, a class $\mathcal{Q}$ obtained from all cliques by subdividing each edge twice, although also having relatively few edges, does not have bounded expansion since $\mathcal{Q} \nabla 1$ contains all graphs.

3. Lower Bound: Paired Local-Cut Graphs

In this section we deal with a specially crafted graph for a lower-bound reduction for the $k$-independent set polytope. We use a shorthand notation $[n] = \{1, 2, \ldots, n\}$.

**Definition 3.1** (Paired local-cut graph). Given positive integers $k$ and $n$, let a Paired Local-Cut Graph, denoted by PLC($k, n$), be defined as follows:

1. We create $k \cdot 2^{\lceil \log n \rceil}$ vertices labeled with tuples $(i, S)$ for $i \in [k]$ and $S \subseteq \lceil \lfloor \log n \rfloor \rfloor$. These vertices will be called cut vertices. Then we create
Proof. Recall that the independent set vectors of $\text{STAB}^{\text{CUT}(\cdot \cdot)}$ that correspond to a given cut $(i, S_1)$ and $(i, S_2)$, where $S_1, S_2 \subseteq [\lceil \log n \rceil]$ are arbitrary. For each index pair $i, j \in [k]$, we add edges between every pair of cut nodes that have labels $(i, S_1)$ and $(i, S_2)$, where $S_1, S_2 \subseteq [\lceil \log n \rceil]$ are arbitrary.

Finally, let $u$ be a cut vertex labeled $(i, S)$ and let $v$ be a pairing vertex labeled $(j, S_1, S_2)$. If $i = j$, but $S \neq S_1$, then we add the edge $uv$. Symmetrically, if $i = j_2$ but $S \neq S_2$, then we also add the edge $uv$.

For ease of exposition we will identify vertices of $\text{PLC}(k, n)$ with their labels whenever convenient. We first state two easy claims.

**Observation 3.2.** The number of vertices of the graph $\text{PLC}(k, n)$ equals $k(k - 1) \cdot 2^{[\log n]} + k \cdot 2^{[\log n]} \leq (kn)^2$.  

**Lemma 3.3.** Let $I$ be an independent set in $\text{PLC}(k, n)$. Then, $|I| \leq k^2$. Moreover, equality holds if and only if $I$ contains exactly one cut vertex for each $1 \leq i \leq k$ and exactly one pairing vertex for each $1 \leq i \neq j \leq k$.

**Proof.** By Definition 3.1.2, the set $I$ can contain at most $k$ cut vertices—at most one vertex $(i, S)$ where $S \subseteq [\lceil \log n \rceil]$ for each $1 \leq i \leq k$. Also, $I$ can contain at most $k(k - 1) = k^2 - k$ pairing vertices—at most one vertex $(i, j, S, S')$ for each ordered pair $1 \leq i \neq j \leq k$.

In subsequent Lemma 3.4, we will relate the vertices of $\text{STAB}_{k^2}(\text{PLC}(k, n))$ with the vertices of the **polytope** $\text{CUT}(K_r)$ where $r = k \lceil \log n \rceil$, to be defined as follows.

We group the $r$ vertices of the complete graph $K_r$ into $k$ groups, each of size $\lceil \log n \rceil$, and label the vertices as $v^i_j$ where $1 \leq i \leq k$ and $1 \leq j \leq \lceil \log n \rceil$. We also order the vertices lexicographically according to their labels. A cut vector of $K_r$, corresponding to a cut $C$, is a $0/1$ vector of length $|E(K_r)| = \binom{k}{2}$ whose coordinates correspond to whether an edge of $K_r$ is in the cut $C$ or not. The edges of $K_r$ are labeled with pairs $(i_1, j_1, i_2, j_2)$ where $1 \leq i_1, i_2 \leq k : 1 \leq j_1, j_2 \leq \lceil \log n \rceil$, and $(i_1, j_1) < (i_2, j_2)$ lexicographically. So, if $z$ is a cut vector corresponding to a given cut $C \subset E_r$, then $z_{i_1, j_1, i_2, j_2} = 1$ if and only if the edge $(i_1, j_1, i_2, j_2)$ is in $C$. The polytope $\text{CUT}(K_r)$ is the convex hull of all such cut vectors.

**Lemma 3.4.** For every pair of natural numbers $(k, n)$ and $r = k \lceil \log n \rceil$ it holds that $\text{CUT}(K_r)$ is a projection of $\text{STAB}_{k^2}(\text{PLC}(k, n))$.

**Proof.** Recall that the independent set vectors of $\text{STAB}_{k^2}(\text{PLC}(k, n))$ are of length $s = k(k - 1) \cdot 2^{[\log n]} + k \cdot 2^{[\log n]}$, by Observation 3.2. We describe an affine map $\pi : \mathbb{R}^s \to \mathbb{R}^{\binom{k}{2}}$ such that for every vertex $C$ of $\text{CUT}(K_r)$ there exists a vertex $I$ of $\text{STAB}_{k^2}(\text{PLC}(k, n))$ such that $\pi(I) = C$. Moreover, for every vertex
$I$ of $\text{STAB}_{k^2}(\text{PLC}(k,n))$ we show that $\pi(I)$ is a vertex of $\text{CUT}(K_r)$. Since $\pi$ is an affine map, this will complete the proof.

To make it easy to follow our arguments, we relate the coordinates of $\mathbb{R}^s$ to the vertices of $\text{PLC}(k,n)$, labeling the coordinates with tuples of the form $(i,j,S,S')$ as follows. For a coordinate corresponding to a cut vertex $(i,S)$ we label this coordinate with $(i,i,S)$. For a coordinate corresponding to a pairing vertex $(i,j,S,S')$ we label the coordinate with same $(i,j,S,S')$. Similarly, we identify the coordinates of $\mathbb{R}^{\ell}$ with the pairs of vertices of $K_r$: the coordinate corresponding to an edge between two different vertices $v_{\ell_1}^i$ and $v_{\ell_2}^j$ is to be labeled with the integer tuple $(i,\ell_1,j,\ell_2)$, assuming that $v_{\ell_1}^i < v_{\ell_2}^j$ lexicographically (that is, $i < j$ and if $i = j$ then $\ell_1 < \ell_2$). Also note that $1 \leq i,j \leq k$ and $1 \leq \ell_1,\ell_2 \leq \lfloor \log n \rfloor$.

Given a vector $y \in \mathbb{R}^s$ we define $\pi(y) := z \in \mathbb{R}^{\ell}$ where

$$z_{i_1,\ell_1,i_2,\ell_2} = \begin{cases} \sum_{S \subseteq [\lfloor \log n \rfloor]} y_{i_1,i_2,S,S} + \sum_{S' \subseteq [\lfloor \log n \rfloor]} y_{i_1,i_2,S',S'} & \text{if } i_1 = i_2, \\ \sum_{S_1 \subseteq [\lfloor \log n \rfloor]} y_{i_1,i_2,S_1,S_1} + \sum_{S_1' \subseteq [\lfloor \log n \rfloor]} y_{i_1,i_2,S_1',S_1'} & \text{if } i_1 \neq i_2. \end{cases}$$

Let $y \in \mathbb{R}^s$ be a vertex of $\text{STAB}_{k^2}(\text{PLC}(k,n))$. That is, $y$ is the characteristic vector of an independent set $I \in \mathcal{I}$. Since $I$ is of size $k^2$, for every $1 \leq i \neq j \leq k$ the following hold by Lemma 3.3:

- there is exactly one $S_t \subseteq [\lfloor \log n \rfloor]$ such that $y_{i+i, S_t, 1} = 1$, and
- there is exactly one pair $S_{ij} \subseteq [\lfloor \log n \rfloor]$ such that $y_{i+i, S_{ij}, S_{ij}} = 1$.

Furthermore, by Definition 3.3 for any pairing vertex $(i_1, i_2, S', S'')$ picked in $I$, that is $y_{i_1,i_2,S',S''} = 1$, it holds; if $i = i_1$ then $S_i = S'$, and if $i = i_2$ then $S_i = S''$. Consider the subsets $S(I), \overline{S}(I)$ of vertices of $K_r$ defined as follows:

$$S(I) := \{v^i_\ell \mid i \in [k] \land \ell \in S_i \} \quad \text{and} \quad \overline{S}(I) := \{v^i_\ell \mid i \in [k] \land \ell \notin S_i \}$$

It is routine to check that $\pi(y)$ is exactly the characteristic vector of the cut defined by $S(I), \overline{S}(I)$ because $z_{i_1,\ell_1,i_2,\ell_2} = 1$ if and only if $v_{\ell_1}^i$ and $v_{\ell_2}^j$ do not lie both in $S(I)$ or both in $\overline{S}(I)$.

On the other hand, any cut in $K_r$ of characteristic vector $z$ creates a bipartition $(S,\overline{S})$ of the vertices of $K_r$. The bipartition $(S,\overline{S})$ consequently induces bipartitions $(S_i,\overline{S}_i)$, $i = 1, \ldots, k$, within each of the $k$ groups of the vertices of $K_r$; namely $S_i := \{j \mid 1 \leq j \leq \lfloor \log n \rfloor \land v^i_j \in S\}$. Then $\{(i,S_i) \mid 1 \leq i \leq k\} \cup \{(i,j,S_i,\overline{S}_j) \mid 1 \leq i \neq j \leq k\}$ is an independent set of $\text{PLC}(k,n)$ whose size is $k^2$ and whose characteristic vector projects to $z$ under $\pi$.

Hence $\pi$ defines a projection from $\text{PLC}(k,n)$ to $\text{CUT}(K_r)$. \hfill \Box
Corollary 3.5. There exists a constant $c' > 0$ such that for $k, n \in \mathbb{N}$,

$$xc(STAB_{k^2}(PLC(k,n))) \geq n^{c'k}.$$ 

Proof. By Lemma 3.4, $STAB_{k^2}(PLC(k,n))$ is an extended formulation of $CUT(K_r)$ with $r = k \lfloor \log n \rfloor$. So any extended formulation of $STAB_{k^2}(PLC(k,n))$ is also an extended formulation of $CUT(K_r)$. By Theorem 2.3, $xc(CUT(K_r)) \geq 2^{\Omega(r)}$. Therefore, $xc(STAB_{k^2}(PLC(k,n))) \geq xc(CUT(K_r)) \geq 2^{\Omega(r)} \geq n^{c'k}$ for some constant $c' > 0$.

We can now easily finish with the main result of this section.

Theorem 3.6. There is no function $f : \mathbb{N} \to \mathbb{R}$ such that $xc(STAB_k(G)) \leq f(k) \cdot n^{O(1)}$ for all natural numbers $k$ and all graphs $G$ on $n$ vertices.

Proof. Suppose, on the contrary, that such a function $f$ does exist. That is, there is a constant $c$ such that for every pair of natural numbers $(\ell, m)$ and for all $m$-vertex graphs $G$ it holds that $xc(STAB_k(G)) \leq f(\ell) \cdot m^c$.

Given a pair $(k, n)$ of natural numbers consider the graph $PLC(k,n)$. By Corollary 3.5, we have that $xc(STAB_{k^2}(PLC(k,n))) \geq n^{c'k}$ for some constant $c' > 0$. On the other hand, we have $\ell = k^2$ and $m \leq (kn)^2$ by Observation 3.2, and so we derive from our assumption that $xc(STAB_{k^2}(G)) \leq f(k^2) \cdot (kn)^{2c}$. However, implied $n^{c'k} \leq f(k^2) \cdot (kn)^{2c}$ clearly cannot hold true for a sufficiently large but fixed parameter $k$ and arbitrary $n$. More precisely, we can choose $k > 2c/c'$ and $n$ such that $\log n > (\log f(k^2) + 2c \log k)/(c'k - 2c) > 0$ and obtain

$$n^{c'k} = n^{2c} \cdot n^{c'k - 2c} > n^{2c} \cdot 2^{\log f(k^2) + 2c \log k} = f(k^2) \cdot n^{2c} k^{2c},$$

a contradiction. Hence no such function $f$ exists.

We remark that the function $f$ in the previous theorem need not even be computable.

4. Upper Bound: Bounded Expansion Classes

While Theorem 3.6 asserts that FPT extensions are not possible for the $k$-independent set polytopes of all graphs, there is still a good chance to prove a positive result for restricted classes of graphs. An example of such restriction is, by a simple modification of Theorem 2.6, presented by graph classes of bounded treewidth; although, this is somehow too restrictive. We show that in the case of $k$ being a fixed parameter, one can go much further.

The underlying idea of our approach can be informally explained as follows. Imagine we can “guess”, in advance, a (short) list of well-structured subgraphs of our graph such that every possible independent set is fully contained in at least one of them. Then we can separately construct an independent set polytope for each one of the subgraphs, and make their union at the end (Theorem 2.2). This ambitious plan indeed turns out to be viable for graph classes of bounded...
expansion (Definition 2.7), and the key to the success is a combination of a powerful structural characterization of bounded expansion (Theorem 4.2) with the size bound \( k \) on the independent sets.

In order to state the desired structural characterization, we need the notion of treedepth. In this context, a rooted forest is a disjoint union of rooted trees. The height of a rooted forest is the maximum distance from one of the forest’s roots to a vertex in the same tree. The closure \( \text{clos}(F) \) of a rooted forest \( F \) is the graph with the vertex set \( \bigcup_{T \in F} V(T) \) and the edge set \( \{xy : x \text{ is an ancestor of } y \text{ in a tree of } F\} \). The treedepth \( \text{td}(G) \) of a graph \( G \) is the minimum height plus one of a rooted forest \( F \) such that \( G \subseteq \text{clos}(F) \).

The following fact, which will allow us to connect the expansion concept of this section with Theorem 2.6, is easy to establish directly from the definitions:

**Observation 4.1.** For any \( G \), the treewidth of \( G \) is at most the treedepth of \( G \) minus one.

The amazing connection between graph classes of bounded expansion and treedepth is captured by the notion of low treedepth coloring: For an integer \( d \geq 1 \), an assignment of colors to the vertices of a graph \( G \) is a low treedepth coloring of order \( d \) if, for every \( s = 1, 2, \ldots, d \), the union of any \( s \) color classes induces a subgraph of \( G \) of treedepth at most \( s \).

In particular, every low treedepth coloring of \( G \) is a proper coloring of \( G \) (but not the other way round), and the union of any two color classes induces a forest of stars. The following result is crucial:

**Theorem 4.2** (Nešetřil and Ossona de Mendez [15, 16]). If \( \mathcal{G} \) is a class of graphs of bounded expansion, then there is a function \( N_{\mathcal{G}} : \mathbb{N} \rightarrow \mathbb{N} \) (depending on the expansion function of \( \mathcal{G} \)) such that for any graph \( G \in \mathcal{G} \) and \( k \), there exists a low treedepth coloring of order \( k \) of \( G \) using at most \( N_{\mathcal{G}}(k) \) colors. This coloring can be found in linear time for a fixed \( k \).

We are now ready to state and prove the main theorem of this section.

**Theorem 4.3.** Let \( \mathcal{G} \) be any graph class of bounded expansion. Then there exists a computable function \( f : \mathbb{N} \rightarrow \mathbb{N} \), depending on the expansion function of \( \mathcal{G} \), such that

\[
\text{xc}(\text{STAB}_k(G)) \leq f(k) \cdot n
\]

holds for every integer \( n \) and every \( n \)-vertex graph \( G \in \mathcal{G} \). Moreover, an explicit extension of \( \text{STAB}_k(G) \) of size at most \( f(k) \cdot n \) can be found in linear time for fixed \( k \) and \( \mathcal{G} \).

**Proof.** Since \( \mathcal{G} \) is a graph class of bounded expansion, by Theorem 4.2 we can for any \( G \in \mathcal{G} \) and given \( k \) find an assignment \( c : V(G) \rightarrow [N_{\mathcal{G}}(k)] \) such that \( c \) is a low treedepth coloring of order \( k \). Let \( \mathcal{J}_k := \binom{[N_{\mathcal{G}}(k)]}{k} \) denote the set of \( k \)-element subsets of \([N_{\mathcal{G}}(k)]\), and let a subgraph \( G_J \subseteq G \) where \( J \in \mathcal{J}_k \), be defined as the subgraph of \( G \) induced on \( \bigcup_{J \in \mathcal{J}_k} c^{-1}(J) \) – the color classes indexed by \( J \).

Note the following two immediate facts:
a) by the definition, each \( G_J, J \in \mathcal{J}_k \), is of treedepth at most \(|J| = k\);

b) for every set \( X \subseteq V(G) \) (independent or not) of size \(|X| \leq k\), there is \( J \in \mathcal{J}_k \) such that \( X \subseteq V(G_J) \).

Consequently,

\[
\text{STAB}_k(G) = \text{conv} \left( \bigcup_{J \in \mathcal{J}_k} \text{STAB}_k(G_J) \right)
\]

and it is sufficient to bound the extension complexity of each \( \text{STAB}_k(G_J) \).

By (a) and Observation 4.1, \( \text{tw}(G_J) \leq k - 1 \) and Theorem 2.6 applies here:

\[
\text{xc}(\text{STAB}_k(G_J)) \leq O(2^k \cdot |G_J|) \leq O(2^k \cdot n) \leq c'2^k \cdot n \text{ for a suitable constant } c'.
\]

Then, by Theorem 2.2 we have

\[
\text{xc}(\text{STAB}_k(G)) \leq |\mathcal{J}_k| + \sum_{J \in \mathcal{J}_k} \text{xc}(\text{STAB}_k(G_J)) \\
\leq |\mathcal{J}_k| \cdot (1 + c'2^k \cdot n) \leq \left( \frac{N_G(k)}{k} \right)(1 + c'2^k \cdot n) \leq f(k) \cdot n.
\]

Note that this extended formulation can be constructed in linear time for fixed \( k \) since the low treedepth coloring in Theorem 4.2 can be found in linear time, the extended formulation in Theorem 2.6 is explicit using a tree-decomposition trivially derived from the definition of tree-depth, and the extended formulation of union of polytopes can be constructed in linear time from the extensions of the component polytopes \[2\].

5. Generalizing the Upper Bound

As advertised in the introduction, the positive results and the proof method from Section 4 can be generalized to the polytopes of many more graph problems than only the \( k \)-independent set polytope. In this respect one could think about other established problems like, for example, the \( k \)-clique or \( k \)-dominating set. Instead of giving a list of particular extensions, we provide here a metaresult covering a whole range of graph problems which share a common ground with the independent set problem.

The first step in this generalization is introducing our descriptive framework, namely the first-order logic of graphs, and the polytopes associated with graphs under a given logical formula. At this point the reader should understand that defining such a polytope for a logical formula cannot be as simple as Definition 2.3 due to necessity to handle formula arguments in full generality (as they may not be only mutually interchangeable elements of a set). However, as it will be clear in the case of the independent set polytope, our polytopes defined from logical formulas naturally form extensions of what one would call “standard” problem polytopes. In particular, an upper bound on the extension complexity of our FO polytope from Definition 2.4 applies also to such “standard” problem polytopes of particular graph problems.
5.1. FO logic and FO polytope

The first-order logic of graphs (abbreviated as FO) applies the standard language of first-order logic to a graph \( G \) viewed as a relational structure with the domain \( V(G) \) and the single binary (symmetric) relation \( E(G) \). That is, in FO we have got the standard predicate \( x = y \), a binary predicate \( \text{edge}(x, y) \), usual logical connectives \( \land, \lor, \rightarrow \), and quantifiers \( \forall x, \exists x \) over the vertex set \( V(G) \). For example, \( \phi(x, y) \equiv \exists z (\text{edge}(x, z) \land \text{edge}(y, z)) \) states that the vertices \( x, y \) have a common neighbor in \( G \).

If \( \phi \) is a formula of \( k \) free variables and \( W = (w_1, w_2, \ldots, w_k) \in V(G)^k \) is such that \( \phi(w_1, w_2, \ldots, w_k) \) holds true in \( G \), we write \( G \models \phi(w_1, w_2, \ldots, w_k) \).

Consider now the FO formula

\[
\psi(x_1, x_2, \ldots, x_k) \equiv \bigwedge_{i \neq j} (\neg \text{edge}(x_i, x_j) \land x_i \neq x_j)
\]

which is quantifier-free. It is easy to see that \( G \models \psi(w_1, w_2, \ldots, w_k) \) if and only if \( \{w_1, w_2, \ldots, w_k\} \) is an independent set of size exactly \( k \).

In another example,

\[
\delta(x_1, x_2, \ldots, x_k) \equiv \forall y \bigvee_{i=1, \ldots, k} (\text{edge}(x_i, y) \lor x_i = y)
\]

is an FO formula with one quantifier such that \( G \models \delta(w_1, \ldots, w_k) \) if and only if \( \{w_1, \ldots, w_k\} \) is a dominating set (of size \( \leq k \)). A more involved example is the following formula with two quantifiers describing a distance-2 dominating set:

\[
\delta_2(x_1, \ldots, x_k) \equiv \forall y \exists z \bigvee_{i=1, \ldots, k} [(\text{edge}(x_i, z) \lor x_i = z) \land (\text{edge}(z, y) \lor z = y)]
\]

For our purposes we will consider FO logic on graphs labeled by labels from a finite set \( \text{Lab} \). Formally, vertex labels are modelled as subsets of \( V(G) \) – for each \( a \in \text{Lab} \) there is a subset \( V_a \) of vertices having label \( a \). From FO formulas labels are accessed using unary predicates: \( L_a(v) \) is true if and only of \( v \in V_a \). As an example, if we work with graphs labeled by \( \text{Lab} = \{a, b\} \) then the formula

\[
\delta'(x_1, x_2, \ldots, x_k) \equiv \forall y L_a(y) \Rightarrow \bigvee_{i=1, \ldots, k} (\text{edge}(x_i, y) \lor x_i = y)
\]

such that \( G \models \delta'(w_1, \ldots, w_k) \) if and only if all vertices labeled by \( a \) are dominated by \( \{w_1, \ldots, w_k\} \). Apart from graphs with labeled vertices, one can also consider graphs with labeled edges. Edge labels are realized as subsets \( E_a \) of \( E(G) \) and are accessed by FO formulas using binary predicates \( \text{edge}_a(x, y) \), i.e. \( \text{edge}_a(u, v) \) is true if and only if \( \{u, v\} \in E_a \). As an example, if we work with graphs with edges labeled by \( \text{Lab} = \{a, b\} \) then the formula

\[
\delta''(x_1, x_2, \ldots, x_k) \equiv \forall y \bigvee_{i=1, \ldots, k} (\text{edge}_a(x_i, y) \lor x_i = y)
\]
such that \( G \models \delta(w_1, \ldots, w_k) \) if and only if vertices \( \{w_1, \ldots, w_k\} \) dominate all vertices of \( G \) using only edges labeled by label \( a \). Edge labels and vertex labels can be used simultaneously – this is the setting we use in Lemma 5.5.

Now we assign, to any FO formula \( \phi(x_1, x_2, \ldots, x_k) \), a graph polytope as follows. As we have already mentioned above, it has to be somehow more complicated than the independent set polytope since the order of arguments of \( \phi \) matters in general, and the same vertex may be repeated among the arguments. For an ordered \( k \)-tuple of vertices \( W = (w_1, w_2, \ldots, w_k) \in V(G)^k \) we thus define its characteristic vector \( \chi^W \) of length \( |V(G)| \) by

\[
\chi^W_{v,i} = \begin{cases} 
1 & \text{if } v = w_i, \\
0 & \text{otherwise}.
\end{cases}
\]

Note that \( \chi^W \) always satisfies \( \sum_{v \in V(G)} \chi^W_{v,i} = 1 \) for each \( i = 1, \ldots, k \), by the definition.

We can now give the following definition:

**Definition 5.1 (FO polytope).** Let \( \phi(x_1, \ldots, x_k) \) be an FO formula with \( k \) free variables. The \((first-order) \ \phi\)-polytope of \( G \), denoted by \( \text{FOP}_\phi(G) \), is defined to be the convex hull of the characteristic vectors of every \( k \)-tuple of vertices of \( G \) such that \( \phi(w_1, w_2, \ldots, w_k) \) holds true in \( G \). That is,

\[
\text{FOP}_\phi(G) = \text{conv} \left\{ \chi^W \in \{0, 1\}^n \mid W = (w_1, w_2, \ldots, w_k) \in V(G)^k, \ G \models \phi(w_1, w_2, \ldots, w_k) \right\}.
\]

The definition of an FO polytope is, at least in the case of an independent set problem, indeed very naturally related to Definition 2.4 of the independent set polytope. See the following:

**Lemma 5.2.** Let \( \iota(x_1, \ldots, x_k) \equiv \bigwedge_{i \neq j} (\neg \text{edge}(x_i, x_j) \land x_i \neq x_j) \) (the above \( k \)-independent set formula). For every graph \( G \), the \( \iota \)-polytope \( \text{FOP}_\iota(G) \) is an extension of \( \text{STAB}_k(G) \).

**Proof.** If \( G \) has \( n \) vertices then

\[
\text{STAB}_k(G) = \left\{ y \in \mathbb{R}^n \mid y_v = \sum_{i=1}^k \chi^W_{v,i}, \ \chi^W \in \text{FOP}_\iota(G) \right\}.
\]

Therefore, \( \text{STAB}_k(G) \) is a projection of \( \text{FOP}_\iota(G) \) given by the projection map described by \( y_v = \sum_{i=1}^k \chi^W_{v,i} \) for all vertices \( v \) of \( G \).

5.2. **Upper bound for existential FO**

For the subsequent arguments we recall the following weaker form\(^4\) of a recent result of Kolman et al.\(^14\):

---

\(^4\)The original result of Kolman et al. applies to Monadic Second Order logic: a logic that subsumes FO logic.
We say that an FO formula $\phi$ directly extends Theorem 4.3 to the following restrictive fragment of FO logic.

$\phi_{k,\ell}$

Again, this extended formulation can be constructed in linear time for fixed $k,\ell$. Let Lemma 5.4.

We start with two simple facts from model theory:

Proof.

$\bigcup$ induced on $\ell$ variables and $\ell$

Then there exists a computable function $f$ such that $g$ is an induced subgraph of $G$ such that $G|\cup U \leq \ell$. Let $\phi(x_1,\ldots,x_k)$ be an existential FO if it can be written as $\phi(x_1,\ldots,x_k) \equiv \exists y_1\ldots y_{\ell} \psi(x_1,\ldots,x_k,y_1,\ldots,y_{\ell})$, where $\psi$ is quantifier-free.

**Lemma 5.4.** Let $\phi(x_1,\ldots,x_k)$ be an existential FO formula with $k$ free variables and $\ell$ quantifiers. Also, let $G$ be any graph class of bounded expansion. Then there exists a computable function $f : \mathbb{N} \to \mathbb{N}$, depending on the expansion function of $G$, such that $\text{xc}(\text{FOP}_\phi(G)) \leq f(k+\ell) \cdot n$

holds for every integer $n$ and every $n$-vertex graph $G$ of treewidth $\tau$. Furthermore, this extension can be computed in linear time for fixed $k,\ell$ and $\tau$.

Using this and the decomposition provided by Theorem 4.2 we are able to directly extend Theorem 4.3 to the following restrictive fragment of FO logic. We say that an FO formula $\phi(x_1,\ldots,x_k)$ is existential FO if it can be written as $\phi(x_1,\ldots,x_k) \equiv \exists y_1\ldots y_{\ell} \psi(x_1,\ldots,x_k,y_1,\ldots,y_{\ell})$, where $\psi$ is quantifier-free.

**Theorem 5.3** (Kolman, Koutecký and Tiwary [14]). Let $\phi(x_1,\ldots,x_k)$ be an FO formula with $k$ free variables and $\ell$ quantifiers. Then there exists a computable function $g : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, such that $\text{xc}(\text{FOP}_\phi(G)) \leq g(k+\ell,\tau) \cdot n$

holds for every integer $n$ and every $n$-vertex graph $G$ of treewidth $\tau$. Furthermore, this extension can be computed in linear time for fixed $k,\ell$ and $\tau$.

Proof. We start with two simple facts from model theory:

a) If $H$ is an induced subgraph of $G$, and $H \models \phi(w_1,\ldots,w_k)$ for $w_1,\ldots,w_k \in V(H)$, then $G \models \phi(w_1,\ldots,w_k)$ (since $\phi$ is existential).

b) If $G \models \phi(w_1,\ldots,w_k)$ for any $W = \{w_1,\ldots,w_k\} \subseteq V(G)$, then there is $U \subseteq V(G)$, $|U| \leq \ell$, such that $G|W \cup U \models \phi(w_1,\ldots,w_k)$ where $G|W \cup U$ is the subgraph of $G$ induced on $W \cup U$ (since $\phi$ has at most $\ell$ quantifiers).

We can hence apply the same technique as in the proof of Theorem 4.3 – using a low treedepth coloring $c$ of $G$ now by $N_G(k+\ell)$ colors from Theorem 4.2. Again, let $J_{k+\ell} := \binom{[N_G(k+\ell)]}{k+\ell}$ denote the set of $(k+\ell)$-element subsets of $[N_G(k+\ell)]$, and let a subgraph $G_J \subseteq G$ where $J \in J_{k+\ell}$, be defined as the subgraph of $G$ induced on $\bigcup_{j \in J_{k+\ell}} c^{-1}(j)$ – the color classes of $c$ indexed by $J$. By a),b) we immediately get

$$\text{FOP}_\phi(G) = \text{conv}\left( \bigcup_{J \in J_{k+\ell}} \text{FOP}_\phi(G_J) \right).$$

From Theorems 4.2 and 5.3 (via Observation 4.1) we analogously conclude

$$\text{xc}(\text{FOP}_\phi(G)) \leq |J_{k+\ell}| + \sum_{J \in J_{k+\ell}} \text{xc}(\text{FOP}_\phi(G_J))$$

$$\leq |J_{k+\ell}| \cdot (1 + g(k+\ell,k+\ell-1) \cdot n) \leq f(k+\ell) \cdot n.$$ 

Again, this extended formulation can be constructed in linear time for fixed $k,\ell$. 

\[ \square \]
5.3. Extension towards full FO

Existential FO is a rather restricted fragment as, for example, the vertex cover or dominating set problems cannot (at least not immediately) be formulated in it. Though, using another established logical tool, explained next, we can circumvent this restriction and cover problems in full FO logic of graphs, i.e., allowing also for universal quantifiers in the problem expression. We base our approach on the exposition of an FO model checking algorithm for graphs of bounded expansion presented in [11].

For \(i, q \in \mathbb{N}, i \leq q\), we say that two \(i\)-tuples of vertices \(\bar{u}, \bar{v} \in V(G)^i\) have the same logical \(q\)-type \(^5\) (denoted by \(tp^q_i(\bar{u}) = tp^q_i(\bar{v})\)) if they satisfy the same set of FO formulas with quantifier rank at most \(q - i\). We note that even though there are infinitely many formulas of a given quantifier rank, there are only finitely many semantically different ones, and therefore there are only finitely many \(q\)-types. Let \(T^q_i\) denote the finite set of all \(q\)-types of \(i\)-tuples of vertices.

The following lemma (Lemma 8.21 in [11] adjusted to our setting) says that on graph classes of bounded expansion one can reduce the problem of determining the \(q\)-type of an \(i\)-tuple of vertices of \(G\) to evaluating certain existential FO formula on a suitable and efficiently computable labeling of \(G\).

**Lemma 5.5** ([11]). Let \(G\) be a class of graphs of bounded expansion, and let \(q \geq 0\). There exists \(r := r(q, G) \in \mathbb{N}\) and a finite set Lab\(_r\) of special labels such that the following holds for all \(1 \leq i \leq q\): there are existential first-order formulas \(\psi_t(x_1, \ldots, x_i)\) for \(t \in T^q_i\), using labels from Lab\(_r\), such that for every graph \(G \in G\) and a low treedepth coloring \(c\) of \(G\) of order \(r\) (using \(N_G(r)\) colors), it is possible to efficiently (in polynomial time for fixed \(C\) and \(q\)) label vertices and edges of \(G\) using \(c\) and labels from Lab\(_r\), to get a labeled graph \(G(c)\) such that for every tuple \(\bar{v} \in V(G)^i\) it holds

\[
G(c) \models \psi_t(v_1, \ldots, v_i) \quad \text{if and only if} \quad tp^q_i(\bar{v}) = t \text{ in } G.
\]

Now we state the final strengthening of Lemma 5.4.

**Theorem 5.6.** Let \(\phi(x_1, \ldots, x_k)\) be an FO formula with \(k\) free variables and \(\ell\) quantifiers. Also, let \(G\) be any graph class of bounded expansion. Then there exists a computable function \(f : \mathbb{N} \to \mathbb{N}\), depending on \(G\), such that

\[
xc(FOP_{\phi}(G)) \leq f(k + \ell) \cdot n
\]

holds for every integer \(n\) and every \(n\)-vertex graph \(G \in G\). Furthermore, an explicit extension of \(FOP_{\phi}(G)\) of size at most \(f(k + \ell) \cdot n\) can be found in polynomial time for fixed \(G\) and \(k, \ell\).

**Proof.** We set \(q := k + \ell\) and \(i := k\), and first apply Lemma 5.5 to obtain the labeling \(G(c)\) of \(G\) (note that \(G(c)\) has the same underlying graph as \(G\) and so having the same bounded expansion) and the existential FO formulas \(\psi_t\). Let

\(^5\)Our logical types correspond to full types (Definition 8.17) in [11]
\( T' \subseteq T_k^q \) be the subset of those \( q \)-types which include our \( \phi \). Then, by the definition of type and Lemma 5.5, we have \( G \models \phi(\bar{v}) \) for \( \bar{v} \in V(G)^i \), if and only if \( G(c) \models \psi_t(\bar{v}) \) for some \( t \in T' \).

Hence the polytope \( \text{FOP}_\phi(G) \) is the convex hull of the union of the polytopes \( \text{FOP}_{\psi_t}(G(c)) \) for \( t \in T' \). Since the cardinality of \( T' \subseteq T_k^q \) is finite and bounded in terms of \( q = k + \ell \), and the formulas \( \psi_t \) depend only on \( t \in T_k^q \) and \( q \) for a fixed class \( \mathcal{G} \), our result now directly follows from Lemma 5.4, applied to each \( t \in T' \), via Theorem 2.2.

6. Nowhere Dense Classes

In this section we present yet another extension of Theorem 4.3, studying the \( k \)-independent set polytope (and more generally existential FO polytopes) on graph classes larger than those with bounded expansion.

A graph class \( \mathcal{G} \) is nowhere dense [17] if there is no integer \( d \) such that \( \mathcal{G} \cap \mathcal{G}^d \) contains all graphs. Every graph class of bounded expansion is nowhere dense, but the converse is not true. The \( k \)-independent-set problem, and existential FO problems in greater generality, are also known to be in FPT on every nowhere dense class [17], see also a more general result of [12]. It is natural to ask whether the same can hold for the fixed-parameter extension complexity of their polytopes. Indeed, the following similarly holds true.

**Theorem 6.1.** Let \( \phi(x_1, x_2, \ldots, x_k) \) be an existential FO formula with \( k \) free variables and \( \ell \) quantifiers. Also, let \( \mathcal{G} \) be any nowhere dense graph class. Then, for every \( \varepsilon > 0 \), there exists a computable function \( f : \mathbb{N} \to \mathbb{N} \) depending on \( \varepsilon \) and \( \mathcal{G} \), such that

\[
\text{xc}(\text{FOP}_\phi(G)) \leq f(k + \ell) \cdot n^{1+\varepsilon}
\]

holds for every integer \( n \) and every \( n \)-vertex graph \( G \in \mathcal{G} \). Furthermore, an explicit extension of \( \text{FOP}_\phi(G) \) of this size can be found in polynomial time for fixed \( \mathcal{G} \) and \( k, \ell \).

**Proof.** The approach for nowhere dense classes is nearly the same as in the proof of Lemma 5.4 by [16, 17], for a nowhere dense class \( \mathcal{G} \) and \( \varepsilon' > 0 \), \( p \in \mathbb{N} \) there exists a threshold \( N_{\varepsilon', p} \) such that each \( G \in \mathcal{G} \) with \( n = |G| \geq N_{\varepsilon', p} \) admits a low treedepth coloring of order \( p \) with at most \( N = n^{\varepsilon'} \) colors. In such case we take \( p := k + \ell \) and \( \varepsilon' := \varepsilon/(k + \ell) \).

Now, setting \( J_{k+\ell} := \binom{[N]}{k+\ell} \), we conclude as in the proof of Lemma 5.4

\[
\text{xc}(\text{FOP}_\phi(G)) \leq |J_{k+\ell}| \cdot (1 + g(k + \ell, k + \ell - 1) \cdot n)
\leq N^{k+\ell} \cdot f(k + \ell) \cdot n = f(k + \ell) \cdot (n^{\varepsilon/(k+\ell)})^{k+\ell} \cdot n
= f(k + \ell) \cdot n^{1+\varepsilon}.
\]

On the other hand, for \( n = |G| < N_{\varepsilon', p} \) we have got a finite problem which is solved by brute force.\( \square \)
However, we now cannot directly proceed towards full FO logic in the direction of Theorem 5.6 since we do not have a tool alike Lemma 5.5 available for the case of nowhere dense classes.

7. Conclusions

We have begun to study the question: to which extent FP tractability of the $k$-independent set problem on graph classes is related to the FPT extension complexity of the (corresponding) $k$-independent-set polytope? Not surprisingly, we confirm that there cannot be FPT extensions of this polytope in the class of all graphs (note, though, that our proof is absolute and does not rely on the assumption $\text{FPT} \neq \text{W}[1]$). On the other hand, the $k$-independent-set problem is linear-time FPT on graph classes of bounded expansion [15], and we construct a linear FPT extension for its polytope on such classes. This positive result then routinely carries over to all FO problems on graph classes of bounded expansion.

We now outline possible natural directions of future research in this regard.

1. The deep tractability result of [12] addresses problems in full FO logic of graphs from nowhere dense classes. This suggests that perhaps, for every FO formula $\phi$ (not only the existential ones as in Theorem 6.1), the related $\phi$-polytope can also have FPT extension complexity on nowhere dense graph classes. Though, the involved proof techniques of [12] do not seem to easily translate to the extension complexity setting.

In a broader view, one may regard the property of a problem polytope having an FPT extension complexity as a finer (case-by-case) resolution of the class FPT. For an explanation; the well-established assumption $\text{FPT} \neq \text{W}[1]$ implies that problems not in FPT do not have FPT extensions, while on the other hand the example of the matching polytope [19] suggests that there may also be FPT problems whose polytopes do not have FPT extensions. (In other words, a situation could be analogous to that of polynomial kernelization; while every problem with a polynomial kernel is FPT, many FPT problems do not admit a polynomial kernel.) We believe that this task is worth further detailed investigation.

One may try to proceed even further and ask a general question:

2. Is it true that all $\text{W}[t]$-hard problems for some $t \geq 1$ do not admit FPT extensions?

However, this question is not even easy to formulate since the polytope we associate with a problem remains a specific choice which by no means is the only choice.

Moreover, it can be argued that either possible answer to the very broad Question 2 would be a significant breakthrough in the complexity world. Say, since some FPT problems such as $k$-vertex cover do admit FPT extensions [5], an affirmative answer to 2 would imply that this problem is not $\text{W}[t]$-complete.
and so $FPT \neq W[t]$. On the other hand, if the answer to \(^2\) was no, then this would imply the existence of non-uniform FPT circuits for $W[t]$-complete problems which is considered unlikely.

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