SU(3) flavor symmetry breaking in large $N_c$ excited hyperons

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Abstract

The $1/N_c$ expansion method for studying the mass spectrum of excited baryons is shortly reviewed together with applications to mixed symmetric states. The $[\mathbf{70}, \ell^+]$ multiplet, belonging to the $N = 2$ band, is reanalyzed, with emphasis on hyperons and the SU(3) symmetry breaking operators entering the mass formula to first order. An important result is that the hierarchy of masses as a function of strangeness is correctly reproduced for all multiplets. Predictions for unknown excited hyperons to SU(6) $\times$ O(3) multiplets are made.

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I. INTRODUCTION

Understanding the baryon resonances and their group theory classification is an essential and current topic in hadronic physics. It is well known that the number of observed resonances is smaller than the number of excited baryons predicted by the quark model. The number of "missing" resonances is much larger in the strange sector. The question is whether or not the missing hyperons with strangeness $S = -1, -2, -3$ are due to lack of experimental data or due to models based on SU(3) symmetry breaking. Experimentally, the hyperons are difficult to produce. In particular for $S = -2$ hyperons, kaon-nucleon or $\Sigma$-hyperon induced reactions are required and the planned kaon beams at Thomas Jefferson National Acceleration Facility (JLAB) and the Japan Proton Accelerator Research Complex (J-PARK) are expected to improve the situation [1].

Here we discuss a theoretical approach attempting to make an SU(3) classification of excited baryons in the framework of the $1/N_c$ expansion method, where $N_c$ is the number of colors. This method, proposed by 't Hooft [2] and applied to baryons by Witten [3], is a powerful tool to study baryon spectroscopy. The underlying symmetry is SU$(2N_f)$ which results from the discovery that, for $N_f$ flavors, the ground state baryons display an exact contracted SU$(2N_f)$ spin-flavor symmetry in the large $N_c$ limit of QCD [4, 5]. The Skyrme model, the strong coupling theory [6] and the static quark model share a common symmetry with QCD baryons in the large $N_c$ limit [7].

The $1/N_c$ expansion method has been applied with great success to the ground state baryons, described by the symmetric representation 56 of SU$(6)$ [5, 8–12]. At $N_c \to \infty$ the ground state baryons are degenerate. At large, but finite $N_c$, the mass splitting starts at order $1/N_c$ as first observed in Ref. [7]. For a review regarding the ground state see, for example, Ref. [13].

The extension of the $1/N_c$ expansion method to excited states requires the symmetry group SU$(2N_f) \times O(3)$ [14], in order to introduce orbital excitations. The practice shows that the experimentally observed resonances can approximately be classified as SU$(2N_f) \times O(3)$ multiplets, grouped into excitation bands, $N = 1, 2, 3, \ldots$, each band containing a number of SU$(6) \times O(3)$ multiplets, as in quark models. In addition, lattice QCD studies have shown that the number of each spin and flavor states in the lowest energy bands is in agreement with the expectations based on a weakly broken SU$(6) \times O(3)$ symmetry [15],
used in quark models and in the treatment of excited states in large $N_c$ QCD. Presently the lattice QCD report errors bars on the baryon masses larger than the next order corrections in the mass formula of the $1/N_c$ expansion \cite{16}.

Some symmetric multiplets of $SU(6) \times O(3)$, in particular $[56, 2^+]$ and $[56, 4^+]$, containing two and four units of orbital excitations, were analyzed by analogy to the ground state in Refs. \cite{17} and \cite{18}, respectively. In this case the splitting starts at order $1/N_c$ as well.

For mixed symmetric states the situation is more intricate. Two approaches have been proposed so far. The first one is based on the Hartree approximation and describes the $N_c$ quark system as a ground state symmetric core of $N_c - 1$ quarks and an excited quark \cite{19}. This implies the split of $SU(2N_f)$ generators into two parts, one acting on the core and the other on the excited quark. Naturally, the number of generators entering the mass formula becomes larger, hence the applicability of the method beyond the $N = 1$ band becomes more problematic \cite{20}.

The second procedure, where the Pauli principle is implemented to all $N_c$ identical quarks has been proposed in Refs. \cite{21, 22}. There is no physical reason to separate the excited quark from the rest of the system. The method can straightforwardly be applied to all excitation bands $N$. It requires the knowledge of the matrix elements of all the $SU(2N_f)$ generators acting on mixed symmetric states described by the partition $(N_c - 1, 1)$. In both cases the mass splitting starts at order $N_v^0$. The latest achievements for the ground state and the current status of large $N_c$ excited baryons can be found in Ref. \cite{23}.

The present work considers as an example the mixed symmetric $[70, \ell^+]$ multiplet in the spirit of the procedure of Refs. \cite{21, 22}. This multiplet has already been analyzed in Ref. \cite{24} by using the 2014 version of the Review of Particle Properties (PDG2014) \cite{25}. We use the same formalism as in Ref. \cite{24} but propose a new assignment to the $\Lambda(2110)5/2^{++}$ resonance. Here we suggest that it belongs to the quartet $^4\Lambda[70, 2^+]_{5/2}^+$ instead of the $^2\Lambda[70, 2^+]_{5/2}^+$ doublet. In addition, for its experimental mass we use the average value of the 2016 Review of Particle Properties (PDG2016) \cite{26} instead of the mass found by Zhang et al. \cite{27}. This cures the previous anomaly that in some sectors the hyperon $\Lambda$ appears with a smaller mass than the nucleon partner \cite{24}. As a benefit, predictions for a few unknown hyperons are made.

In Sec. II we recall the mass formula of the $1/N_c$ expansion and in Sec. III we shortly
review the applications of the method to $N = 1, 2, 3$ and $4$ excitation bands. The matrix elements of the SU(3) flavor symmetry breaking operators $B_i$ for the mixed symmetric $[70, \ell^+]$ multiplet of the $N = 2$ band are presented in Sec. IV. The spectrum of $[70, \ell^+]$ is reanalyzed in Sec. V. The last section is devoted to conclusions.

II. MASS OPERATOR

The general form of the mass operator, where the SU(3) symmetry is broken, has the following form

$$M = \sum_i c_i O_i + \sum_i d_i B_i.$$  \hfill (1)

The rotational invariant operators $O_i$ are defined as the scalar products

$$O_i = \frac{1}{N_c-1} O_{\ell}^{(k)} \cdot O_{SF}^{(k)},$$  \hfill (2)

where $O_{\ell}^{(k)}$ is a $k$-rank tensor in SO(3) and $O_{SF}^{(k)}$ a $k$-rank tensor in SU(2)-spin, but invariant in SU($N_f$). For the ground state one has $k = 0$. The excited states also require $k = 1$ and $k = 2$ terms. The $k = 1$ tensor components are the generators $L^i$ of SO(3). In a spherical basis the components of the $k = 2$ tensor operator of SO(3) $(i, j = -1, 0, 1)$ read (see Appendix)

$$L^{(2)ij} = \frac{1}{2} \{ L^i, L^j \} - \frac{1}{3} (-)^i \delta_{i,-j} \vec{L} \cdot \vec{L}.$$  \hfill (3)

The operators $O_{SF}^{(k)}$ are constructed from the SU($N_f$) generators, $S^i$, $T^a$ and $G^{ia}$ obeying the su($2N_f$) algebra

$$[S^i, S^j] = i \varepsilon^{ijk} S^k, \quad [T^a, T^b] = i f^{abc} T^c, \quad [S^i, T^a] = 0,$$

$$[S^i, G^{ja}] = i \varepsilon^{ijk} G^{ka}, \quad [T^a, G^{jb}] = i f^{abc} G^{jc},$$

$$[G^{ia}, G^{jb}] = \frac{i}{4} \delta^{ij} f^{abc} T^c + \frac{i}{2} \varepsilon^{ijk} \left( \frac{1}{N_f \delta^{ab}} S^k + d^{abc} G^{kc} \right),$$  \hfill (4)

In the symmetric core + excited quark procedure each SU($2N_f$) generator is split into two parts

$$S^i = S^i_c + s^i, \quad T^a = T^a_c + t^a, \quad G^{ia} = G^{ia}_c + g^{ia},$$  \hfill (5)

where the operators carrying a lower index $c$ act on a symmetric ground state core and $s^i$, $t^a$ and $g^{ia}$ act on the excited quark. The procedure has the algebraical advantage that it reduces
the problem of the knowledge of the matrix elements of the SU(2N_f) generators acting on
a system described by a mixed symmetric representation of SU(2N_f) to the knowledge of
the matrix elements of S^i_c, T^a_c and G^{ia}_c, acting on symmetric states of partition \([N_c - 1]\),
which are simpler to find than the matrix elements of the SU(2N_f) generators for \([N_c - 1, 1]\)
mixed symmetric states. Then the operator reduction rules for the ground state \([10]\) may
be used for the core operators. However, the number of terms to be included in operators
describing observables remains usually very large as compared to experimental data, so that
the method cannot easily be applied to mixed symmetric highly excited baryons. It should
be remembered that the spin-orbit operator \(O_2\) of symmetric multiplets is defined in terms
of angular momentum \(L^i\) components acting on the whole system as in Ref. \([17]\) and is order
\(O(1/N_c^2)\)
\[
O_2 = \frac{1}{N_c} L \cdot S,
\]
while for mixed symmetric multiplets it is defined as a single-particle operator \([19]\)
\[
O_2 = \ell \cdot s = \sum_{i=1}^{N_c} \ell(i) \cdot s(i),
\]
the matrix elements of which are order \(O(N_c^0)\). The reason to mention \(O_2\) is that although its
collection to the mass is generally small, like in quark models, here it plays an important
role in proving the compatibility between the meson-nucleon scattering picture and the quark
model-type picture, legitimating in this way the extension of the \(1/N_c\) expansion to excited
states of mixed symmetry \([28]\).

An extra complication for \(N_f = 3\) (u, d, s quarks) is that the effects of the SU(3)
flavor symmetry breaking are comparable to \(1/N_c\) corrections. The second term in the
mass formula \([II]\) is designed to introduce the symmetry breaking. The operators \(B_i\) break
the SU(3) flavor symmetry and are defined to have zero expectation values for nonstrange
baryons. The SU(3) flavor symmetry breaking is implemented at order \(O(\epsilon)\) where \(\epsilon \sim 0.3\)
is a measure of the SU(3) flavor symmetry breaking by the strange quark mass \([12]\). Thus \(\epsilon\)
and \(1/N_c\) at \(N_c = 3\) are of similar size and both corrections have to be included. Corrections
of order \(\epsilon/N_c\) are neglected.

In the context of our approach, where the baryon is treated as a system of \(N_c\) quarks
irrespective of its spin-flavor symmetry, the SU(3) breaking operators are defined as
\[
B_1 = n_s,
\]
where \( n_s \) is the number of strange quarks and

\[
B_2 = \frac{1}{N_c} (L^i G^{i\bar{s}} - \frac{1}{2\sqrt{3}} L \cdot S),
\]

(9)

\[
B_3 = \frac{1}{N_c} (S^a G^{a\bar{s}} - \frac{1}{2\sqrt{3}} S \cdot S),
\]

(10)

where the angular momentum operator \( L^i \), the spin operator \( S^i \) and the component 8 of the spin-flavor operator \( G^{i\bar{s}} \) act on the entire system of \( N_c \) quarks.

Then, in Eq. (1) the coefficients \( c_i \) encode the quark dynamics and \( d_i \) measure the SU(3) breaking. They are determined from a numerical fit to data. An example, containing the commonly used \( O_i \) and \( B_i \) operators together with the coefficients \( c_i \) and \( d_i \) can be found in Table I below.

III. STATUS OF EXCITED HYPERONS IN THE \( 1/N_c \) EXPANSION

Here we briefly recall some important achievements in the study of baryons spectra for the \( N = 1, 2, 3 \) and 4 bands.

A. \( N = 1 \) band

The \( N = 1 \) band has been the most studied so far. It is the best known experimentally and it contains only one SU(6) × O(3) multiplet, the \([70, 1^-]\). The first application of the \( 1/N_c \) expansion was a phenomenological analysis of strong decays of resonances with one unit of orbital excitation \([29]\). There were no operators to distinguish the strange quark from \( u \) and \( d \), but the decay of some hyperons was considered via an explicit SU(3)-flavor breaking.

In the symmetric core + excited quark procedure, the \( N_f = 3 \) case has been thoroughly studied by Goity et al. \([30]\) where 11 SU(3) exact flavor symmetry operators and 4 first order SU(3)-flavor symmetry breaking operators operators were included. Two of them, proportional to the generators \( t^8 \) and \( T^8 \), thus giving a measure of the strangeness, bring significant contributions, the other two bring small contributions. The fit was made to 19 empirical quantities (17 masses and 2 mixing angles) associated to three- and four-star resonances. Predictions were made for unknown hyperons having strangeness \( S = -1, -2 \) and - 3. The masses of \( \Lambda(1405) \) and \( \Lambda(1520) \) were well reproduced, but this was due to the
simplicity of the wave function in the symmetric core + excited quark procedure where the part corresponding to \( S_c = 1 \) is missing \cite{23}. In addition one should note the absence of the pure flavor operator \( t \cdot T_c \) coupling the core flavor operator \( T_c \) to the excited quark flavor operator \( t \).

A much smaller number of operators was needed for the \([70, 1^-]\) multiplet in the approach of Refs. \cite{21, 22}. There were seven exact SU(3)-flavor symmetry, one SU(3)-flavor symmetry breaking representing the total strangeness and one isospin breaking operator. This approach, based on an exact wave function, accommodates a slightly heavier \( \Lambda(1405) \) at \( 1421 \pm 14 \) MeV. However, both procedures predict too large a mass (of 1790 MeV in Ref. \cite{31}) for the three-star puzzling \( \Xi(1690) \) resonance, a situation similar to quark models \cite{32}. The Skyrme model gives a lower mass and possibly a more natural interpretation of \( \Xi(1690) \) \cite{33}.

We note that in both approaches the \( \Lambda - N \) splitting is similar, around 150 MeV for octets. In decuplets the \( \Sigma - \Delta \) splitting is about 130 MeV in Ref. \cite{30} and about 170 MeV in Ref. \cite{31} where a different choice of \( B_i \) operators has been made, as implied by arguments given in the Introduction.

\section*{B. \( N = 2 \) band}

The \( N = 2 \) band has the following multiplets \([56', 0^+], [56, 2^+], [70, 0^+], [70, 2^+]\) and \([20, 1^+]\). The observed resonances are usually assigned to the symmetric \([56]\) or the mixed symmetric \([70]\) SU(6) multiplets. The antisymmetric SU(6) \( \times \) O(3) multiplet \([20, 1^+]\) has been ignored so far, on the basis that it does not have a real counterpart.

The multiplet \([56', 0^+]\) describes states with a radial excitation, in particular the Roper resonance. It was the first to be studied in the large \( N_c \) limit \cite{34}, by using a simplified mass formula of the Gürsey-Radicati type. The analysis was free of any assumption regarding the wave function except its symmetry in SU(6). Strong decay widths were calculated as well.

The analysis of the \([56, 2^+]\) baryon masses has first been performed in Ref. \cite{17}. It has been reconsidered in Ref. \cite{18} with nearly identical results and the analysis has been extended to the higher multiplet \([56, 4^+]\) of the \( N = 4 \) band in the same paper.

The \([70, 0^+]\) and \([70, 2^+]\) baryon masses were first analyzed in Ref. \cite{35} for \( N_f = 2 \) and extended in Ref. \cite{20} to \( N_f = 3 \), both studies being performed within the symmetric core
+ excited quark procedure \[19\]. The \([70, \ell^+]\) (\(\ell = 0, 2\)) multiplets were revisited \[36\] within the approach of Ref. \[21\] where the Pauli principle was fully taken into account.

In Refs. \[35\] and \[36\] Regge-type trajectories have been drawn for the most dominant coefficient in the mass formula, \(c_1\) and \(c_2^2\) respectively, and somewhat conflicting results have been obtained. The trajectories were derived as a function of the band number \(N = 0, 1, 2, 3\) and 4. While in Ref. \[35\] a single trajectory has been obtained (note that large \(N_c\) results for the \(N = 3\) band were not available yet), in Ref. \[36\] two distinct, nearly parallel, Regge trajectories have been obtained, the lower one for symmetric \([56]\)-plets and the higher one for mixed symmetric \([70]\)-plets.

In Ref. \[24\] a combined analysis of the \([56, 2^+]\) and \([70, \ell^+]\) multiplets of the \(N = 2\) band has been made. An important aspect was that the same set of linearly independent operators in the mass formula has been used which was not the case before. Distinct Regge trajectories resulted again. The data were from PDG2014 which sometimes gives more precise values for the resonance masses with smaller error bars than before.

C. \(N = 3\) and 4 bands

The \(N = 3\) band contains eight SU(6) \(\times\) O(3) multiplets \[37\]. Those belonging to the mixed symmetric \([70, \ell^-]\) multiplets (\(\ell = 1,2,3\)) were studied in Ref. \[38\]. They were all nonstrange baryons. It is premature to perform an extended \(1/N_c\) analysis to the \(N = 3\) band, due to lack of experimental data.

The \(N = 4\) band has 17 SU(6) \(\times\) O(3) multiplets \[39\] from which only the the lowest, the \([56, 4^+]\) multiplet, has been analyzed in the \(1/N_c\) expansion method \[18\]. Being described by a symmetric representation of SU(6) it is technically simple, as mentioned in the Introduction. Despite the lack of data for highly excited hyperons, tentative predictions have been made in Ref. \[18\] by including only \(B_1\) and a single experimentally known hyperon, the \(\Lambda(2350)9/2^{+}\).

IV. MATRIX ELEMENTS OF \(B_i\) OPERATORS FOR \([70, \ell^+]\)

Here we are concerned with the \([70, \ell^+]\) multiplet. The matrix elements of \(O_i\) for \([70, \ell^+]\), as a function of \(N_c\), were derived in Ref. \[36\]. Note that in the case of mixed symmetric
states the matrix elements of $O_6$ are $O(N_c^0)$, in contrast to the symmetric case where they are $O(N_c^{-1})$, and nonvanishing only for octets, while for the symmetric case they are nonvanishing for decuplets. Thus, at large $N_c$ the splitting starts at order $O(N_c^0)$ for mixed symmetric states due both to $O_2$ and $O_6$.

The SU(3) flavor breaking operators $B_i$ were chosen to have identical definitions for mixed symmetric multiplets [24] to those for symmetric multiplets [17]. The expectation value of $B_1$ is

$$B_1 = n_s$$

(11)

where $n_s$ is the number of strange quarks in a baryon. The diagonal matrix elements of $B_2$ and $B_3$ for $[70, \ell^+]$ at arbitrary $N_c$ were first calculated in Ref. [24] where they were exhibited in Table IV. For practical purposes we do not reproduce that table. At $N_c = 3$ we have summarized those results by two simple analytic formulas. The diagonal matrix elements of $B_2$ take the following form

$$B_2 = -n_s \langle L \cdot S \rangle_{6\sqrt{3}},$$

(12)

where $\langle L \cdot S \rangle$ is the expectation value of the spin-orbit operator acting on the whole system. Thus the contribution of $B_2$ is positive or negative depending on the sign of $\langle L \cdot S \rangle$. The diagonal matrix elements of $B_3$ take the simple analytic form

$$B_3 = -n_s \frac{S(S+1)}{6\sqrt{3}},$$

(13)

where $S$ is the total spin. The contribution of $B_3$ is always negative, otherwise vanishing for nonstrange baryons. These formulas can be applied to $^2S_J$, $^4S_J$, $^210_J$ and $^211/2$ baryons of the $[70, \ell^+]$ multiplet.

Interestingly, for the decuplet members of the symmetric $[56, 2^+]$ multiplet the expressions of $B_2$ and $B_3$ at $N_c = 3$ given by Eqs. (12) and (13) of Ref. [24] are the same as those of Eqs. (12) and (13) shown above.

V. SPECTRUM OF $[70, \ell^+]$

Presently we use the PDG2016 [26] to reanalyze the mixed symmetric multiplet $[70, \ell^+]$ with $\ell = 0$ or 2. The values of the fitted coefficients $c_i$ and $d_i$ are exhibited in Table II together with the value of $\chi^2_{\text{dof}} = 1.80$. The results can only roughly be compared to those
presented in Table I, Fit 2 of Ref. [36], because $B_2$ and $B_3$ were missing there. Note that the factor 15 of $O_6$ has been removed here, which explains the larger value of $c_6$ now. In fact the product $c_6O_6$ matters in the mass. The value of $c_2$ is similar to that of Ref. [36]. The $1/N_c$ corrections are dominated by $O_3$ in octets and by $O_4$ in decuplets. The SU(3) flavor breaking is dominated by $B_1$ for all hyperons.

**TABLE I.** List of dominant operators and their coefficients (MeV) $c_i$ and $d_i$ from the mass formula (1) obtained in a numerical fit for the $[70, \ell^+]$ multiplet. The spin-orbit operator $O_2$ is defined by Eq. (7) for $[70, \ell^+]$.

| Operator | Coefficient (MeV) |
|----------|-------------------|
| $O_1 = N_c \mathbb{I}$ | $630 \pm 11$ |
| $O_2 = \ell \cdot s$ | $62 \pm 26$ |
| $O_3 = \frac{1}{N_c} S^i S^i$ | $95 \pm 31$ |
| $O_4 = \frac{1}{N_c} \left[ T^a T^a - \frac{1}{12} N_c (N_c + 6) \right]$ | $108 \pm 43$ |
| $O_6 = \frac{1}{N_c} L^{(2)ij} G^{ia} G^{ja}$ | $137 \pm 57$ |
| $B_1 = n_s$ | $40 \pm 33$ |
| $B_2 = \frac{1}{N_c} (L^i G^{i8} - \frac{1}{2\sqrt{3}} L^i S^i)$ | $-37 \pm 122$ |
| $B_3 = \frac{1}{N_c} (S^i G^{i8} - \frac{1}{2\sqrt{3}} S^i S^i)$ | $60 \pm 162$ |
| $\chi^2_{\text{dof}}$ | $1.80$ |

The PDG2016 as well as PDG2014 incorporate the new multichannel partial wave analysis of the Bonn-Gatchina group [40]. Accordingly the resonance $P_{13}(1900)$ has been upgraded from two to three stars with a Breit-Wigner mass of $1905 \pm 30$ MeV. The resonance $N(2000)5/2^+$ has been split into two two-star resonances, namely $N(1860)5/2^+$ and $N(2000)5/2^+$, with masses indicated in Table III. There is a new one-star resonance $N(2040)3/2^+$ observed in the decay $J/\psi \rightarrow p\bar{p}\pi^0$ [41]. There is also a new two-star resonance $N(1880)1/2^+$ observed by the Bonn-Gatchina group with a mass of $1870 \pm 35$ MeV [40].

In a previous work [36] only 11 resonances have been included in the numerical fit. Here,
as well as in Ref. \[24\], 16 resonances have been included, with a status of three, two or one star. These extra resonances are the hyperons \(\Xi(2120)^{??}\), \(\Sigma(2070)5/2^{++}\), \(\Sigma(1940)^{??}\), \(\Xi(1950)^{??}\) and \(\Sigma(2080)3/2^{++}\). For the three-star resonances we use the Breit-Wigner mass of PDG2016 except for \(\Xi(1950)^{??}\) where we take the value found in Ref. \[42\] which reduces the \(\chi^2_{\text{dof}}\) value from 1.96 to 1.80. For the spectrum, such a choice would not make much difference.

For the resonances omitted from the summary table of PDG2016 the masses and the error bars considered in the fit correspond to averages over those data taken into account in the particle listings, except for a few which favor specific experimental values cited in the headings of Table III.

The \(N(1710)1/2^{++}\) and \(\Sigma(1770)1/2^{++}\) resonances have been ignored in this fit. The theoretical argument is that their masses are too low, leading to unnatural sizes for the coefficients \(c_i\) or \(d_i\) \[43\]. Experimentally the controversial \(N(1710)1/2^{++}\) resonance has not been seen in the latest GWU analysis of Arndt et al. \[44\]. We have also ignored \(\Delta(1750)1/2^{++}\), inasmuch as, neither Arndt et al. \[44\] nor Anisovich et al. \[40\] find evidence for it.

The partial contributions and the calculated total masses obtained from the fit are presented in Table III. One can see that the fit is generally good except for \(\Sigma(1880)1/2^{++}\) where the calculated mass somewhat too high. The operator \(B_2\) has a vanishing expectation value and the contribution of \(B_3\), although negative, is negligible. The mass of the \(N(1860)5/2^{++}\) seems large too, but it is within the large error bars of Ref. \[40\].

The good fit for the \(N(1880)1/2^{++}\) resonance was due to the negative contributions of \(-93\) MeV and \(-80\) MeV of the spin-orbit operator \(O_2\) and of \(O_6\) operators respectively. However its strange partners are almost degenerate because the positive contribution of \(B_1\) is accidentally cancelled out by the negative contribution of \(B_2 + B_3\).

The assignment of \(\Sigma(1940)^{??}\) and \(\Xi(1950)^{??}\) to the \(^2[70, 2^+]\) multiplet seems reasonable. Thus these resonances may have \(J^P = 3/2^+\), hopefully to be confirmed experimentally in future analyses.

Some predictions have also been made for experimentally unknown strange partners in octets and decuplets. Note that \(\Lambda\) and \(\Sigma\) are degenerate in our approach because the expectation values of \(B_2\) and \(B_3\) are identical at \(N_c = 3\), although they are different at arbitrary \(N_c\). This is not the case for the \([56, 2^+]\) multiplet. Also, the total contribution
TABLE II. Partial contribution and the total mass (MeV) predicted by the $1/N_c$ expansion with matrix elements of $O_i$ from Ref. [24] and of $B_i$ given in the text. The column Ref.[24] gives the total mass of Ref. [24]. The last two columns give the empirically known masses and status from the 2016 Review of Particles Properties [26] unless specified by (A) from [40], (L) from [45], (G1) from [46], (B) from [42], (AB) from [41], (G2) from [47].

| Part. contrib. (MeV) | Total(MeV) | Ref.[24] | Expt.(MeV) | Name, status |
|---------------------|------------|----------|------------|--------------|
| $c_1 O_1$ | 1889 | 62 | 118 | 27 | - 23 | 0 | 0 | 0 | 2073 ± 38 | 2080 ± 39 | 2060 ± 65(A) | $N(1990)7/2^+**$ |
| $c_2 O_2$ | 40 | 11 | - 22 | 2102 ± 19 | 2105 ± 19 | 2100 ± 30(L) | $\Lambda(2020)7/2^+$ |
| $c_3 O_3$ | 79 | 22 | - 43 | 2131 ± 8 | 2130 ± 8 | 2130 ± 8 | $\Xi(2120)?^*$ |
| $c_4 O_4$ | 4 | 1 | - 2 | 2097 ± 18 | 2009 ± 40 | 2110 ± 20 | $\Lambda(2110)5/2^+***$ |
| $c_5 O_5$ | 79 | 4 | - 43 | 2113 ± 41 | |
| $c_6 O_6$ | 0 | 0 | 0 | 0 | 1972 ± 29 | 1955 ± 32 | |
| $d_1 B_1$ | 0 | 0 | 0 | 0 | 1863 ± 16 | 1933 ± 7 | |
| $d_2 B_2$ | 79 | - 22 | - 43 | 1865 ± 79 | |
| $d_3 B_3$ | 0 | 0 | 0 | 0 | 1860 ± 31 | 1870 ± 0 | 0 | 0 | 0 | 1878 ± 34 | 1870 ± 35(A) | $N(1880)1/2^+**$ |
| $d_4 B_4$ | 40 | 16 | - 22 | 1863 ± 79 | |
| $d_5 B_5$ | 79 | 32 | - 43 | 1865 ± 153 | |
| $d_6 B_6$ | 0 | 0 | 0 | 0 | 1860 ± 29 | 1959 ± 29 | $120(A)$ | $N(1860)5/2^+**$ |
| $e_1 B_1$ | 0 | 4 | - 4 | 2000 ± 18 | 2031 ± 11 | 2051 ± 25(G1) | $\Sigma(2070)5/2^+**$ |
| $e_2 B_2$ | 79 | 7 | - 8 | 2038 ± 45 | |
| $e_3 B_3$ | 0 | 0 | 0 | 0 | 1900 ± 22 | 1900 ± 30(A) | $N(1900)3/2^+**$ |
| $e_4 B_4$ | 0 | 0 | 0 | 0 | 1938 ± 16 | 1933 ± 11 | 1941 ± 18 | $\Sigma(1940)?^*$ |
| $e_5 B_5$ | 0 | 79 | - 11 | - 8 | 1968 ± 7 | 1964 ± 70 | 1967 ± 7(B) | $\Xi(1950)?***$ |
| $e_6 B_6$ | 0 | 0 | 0 | 0 | 2034 ± 18 | 2024 ± 20 | 2040 ± 28(AB) | $N(2040)3/2^+**$ |
| $f_1 B_1$ | 40 | 0 | - 22 | 2052 ± 22 | 2000 ± 23 | 2100 ± 69 | $\Sigma(2080)3/2^+**$ |
| $f_2 B_2$ | 79 | 0 | - 43 | 2070 ± 46 | |
| Part. contrib. (MeV) | Total(MeV) | Ref.[24] | Expt.(MeV) | Name, status |
|----------------------|------------|----------|------------|--------------|
| $c_1O_1$ | $c_2O_2$ | $c_3O_3$ | $c_4O_4$ | $c_5O_5$ | $d_1B_1$ | $d_2B_2$ | $d_3B_3$ |
| $^2\Delta[70, 2^+]_{\frac{5}{2}^+}$ | 1889 | -21 | 24 | 134 | 0 | 0 | 0 | 0 | 2026 $\pm$ 48 | 2086 $\pm$ 37 | 1962 $\pm$ 139 | $\Delta(2000)5/2^+\ast\ast$ |
| $^2\Sigma^*[70, 2^+]_{\frac{5}{2}^+}$ | 0 | 40 | 3 | -4 | 2065 $\pm$ 52 |
| $^2\Xi^*[70, 2^+]_{\frac{5}{2}^+}$ | 0 | 79 | 7 | -8 | 2104 $\pm$ 73 |
| $^2\Delta[70, 0^+]_{\frac{1}{2}^+}$ | 1889 | 0 | 24 | 134 | 0 | 0 | 0 | 0 | 2047 $\pm$ 49 |
| $^2\Sigma^*[70, 0^+]_{\frac{1}{2}^+}$ | 0 | 40 | 0 | -4 | 2083 $\pm$ 46 | 2119 $\pm$ 25 | 1902 $\pm$ 96 | $\Sigma(1880)1/2^+\ast\ast\ast$ |
| $^2\Sigma^*[70, 0^+]_{\frac{1}{2}^+}$ | 0 | 79 | 0 | -8 | 2118 $\pm$ 53 |
| $^2\Lambda'[70, 2^+]_{\frac{5}{2}^+}$ | 1889 | 62 | 24 | -81 | 0 | 40 | 3 | -4 | 1933 $\pm$ 47 |
| $^2\Lambda'[70, 0^+]_{\frac{1}{2}^+}$ | 1889 | 0 | 24 | -81 | 0 | 40 | 0 | -4 | 1868 $\pm$ 43 | 1865 $\pm$ 19 | 1853 $\pm$ 20(G2) | $\Lambda(1810)1/2^+\ast\ast\ast\ast$ |

of $B_i$ is generally of about 30 MeV which is much less than for the $[56, 2^+]$ multiplet. We did not present predictions for the $\Omega$’s in the $[70, \ell^+]$ multiplet because we thought them irrelevant at this stage of theory and experiment.

A useful remark is that the contributions of $B_2$ and $B_3$ mutually cancel out for hyperons belonging to decuplets with $\ell \neq 0$. In that case $B_1$ is enough in the mass formula, like in Ref. [36]. The contributions of $B_2$ and $B_3$ are generally small. This is due to the smallness of the coefficients $d_2$ and $d_3$ of Table I, having sizes of a similar order of magnitude to the corresponding ones from Ref. [30] obtained for the $N = 1$ band in the excited quark + ground state core method.

Presently the negative contribution of $B_3$ (see Eq. (13)) makes the hyperons masses larger than those derived in Ref. [24] and helps in restoring the correct hierarchy as a function of strangeness.

It is important to make a comparison between the present results and those of Ref. [24] where a different assignment and mass have been chosen for $\Lambda(2110)5/2^+\ast\ast\ast$. For this purpose we have included in Table II the column called Ref. [24] which gives the total masses obtained in our previous work. One can notice that presently the fit to the resonances $N(2000)5/2^+\ast\ast\ast$ and $\Sigma(2070)5/2^+\ast\ast$ slightly deteriorates, which may be a reason for the increase of $\chi^2_{dof}$ from 1.48 to 1.80. Note that all these resonances have $J = 5/2^+$.

Our suggestions for assignments of resonances in the $[70, \ell^+]$ multiplet can be compared
to those made in Ref. [48] as educated guesses. The assignment of $\Sigma(1880)1/2^{++}\ast$ as a $[70, 0^+]1/2^+$ decuplet resonance is confirmed as well as the assignment of $\Lambda(1810)1/2^{++}\ast$ as a flavor singlet. We agree with Ref. [48] regarding $\Lambda(2110)5/2^{++}\ast$ as a partner of $N(2000)5/2^{++}\ast$ in a spin quartet. We disagree with Ref. [48] that $N(1900)3/2^{++}\ast$ is a member of a spin quartet. We propose it as a partner of $\Sigma(1940)5^{++}$ and $\Xi(1950)5^{++}$ in a spin doublet.

However, one has to keep in mind that at the same $J$ spin doublets and quartets can mix, for example for $N[70, 2^+]$ at $J = 3/2$ or $5/2$. The mixing would be due to the off-diagonal matrix elements of the spin-orbit operator $O_2$ and the tensor operator $O_6$.

The problem of assignment is not trivial. Within the $1/N_c$ expansion method Ref. [17] suggested that $\Sigma(2080)3/2^{++}\ast$ and $\Sigma(2070)5/2^{++}\ast$ could be members of two distinct decuplets in the $[56, 2^+]$ multiplet while here and in Ref. [48] they seem to be good candidates for mixed symmetric states.

VI. CONCLUSIONS

The inclusion of three SU(3) symmetry breaking operators, $B_1$, $B_2$ and $B_3$ in the mass formula of the $[70, \ell^+]$ multiplet helps to brings more insight into the SU(6) $\times$ O(3) classification of highly excited baryons when accompanied by realistic assignments. Presently it seems that the evolution of the $\Lambda - N$ or $\Sigma - N$ splitting with excitation energy in baryon multiplets described by the $1/N_c$ expansion remains an open problem.

Alternative suggestions for assignments of the known baryons should be studied and more data for excited hyperons are highly desirable. The continuing study of the presently available data and the production of new hyperons are needed for understanding the structure of baryons and disentangle between various models. At the Workshop on Physics with Neutral Kaon beam at JLAB [1] it was pointed out that a $K_L$ beam at JLAB would open new opportunities for studying excited hyperons which may help in understanding the multiplet structure of excited baryons. Similar hopes are at J-PARK.
Appendix A: The two-rank tensor of SO(3)

In this Appendix we derive the expression (3) of the two-rank tensor $L^{(2)ij}$ of SO(3) in a spherical basis. Let us denote the spherical components of the SO(3) generators by $L_i$. Then the product $L_i L_j$ can be written as

$$L_i L_j = \sum_{k=0}^{2} \sum_{\mu=-k}^{k} C_{i j \mu}^{1 \mu} T_{\mu}^{k}, \quad (A1)$$

in terms of a Clebsch-Gordan coefficient and the irreducible $k$-rank tensor $T_{\mu}^{k}$. In the anticommutator \{ $L_i, L_j$ \} only the tensors $k = 0$ and 2 survive for symmetry reasons. Then one can write

$$\frac{1}{2} \{ L_i, L_j \} = \sum_{\mu} C_{i j \mu}^{1 1} T_{\mu}^{2} + C_{i j 0}^{1 0} T_{0}^{0}. \quad (A2)$$

The second term contains the Clebsch-Gordan coefficient

$$C_{i j 0}^{1 0} = (\mathbf{1} - i) \frac{1}{\sqrt{3}} \delta_{i, j}. \quad (A3)$$

The standard definition of $T_{0}^{0}$ is (see, for example, Eq. (4.7) of Ref. [49])

$$T_{0}^{0} = -\frac{1}{\sqrt{3}} \mathbf{L} \cdot \mathbf{L}. \quad (A4)$$

Then shifting the second term of Eq (A2) from right to left we obtain the second rank tensor $L^{(2)ij}$ of SO(3) as

$$L^{(2)ij} = \sum_{\mu} C_{i j \mu}^{1 1} T_{\mu}^{2}, \quad (A5)$$

or alternatively

$$L^{(2)ij} = \frac{1}{2} \{ L_i, L_j \} - \frac{1}{3} (-)^{i} \delta_{i, j} \mathbf{L} \cdot \mathbf{L}. \quad (A6)$$

Equation (A5) can be used to calculate the matrix elements of $L^{(2)ij}$ defined as an irreducible two-rank tensor. Using the Wigner-Eckart theorem and a spherical harmonic basis one has

$$\langle \ell' m' | L^{(2)ij} | \ell m \rangle = \sum_{\mu, \mu'} C_{i j \mu}^{1 1} C_{m m'}^{\ell' \ell} \langle \ell' | T_{\mu}^{2} | \ell \rangle. \quad (A7)$$

The reduced matrix element $\langle \ell' | T_{\mu}^{2} | \ell \rangle$ can be easily calculated. The result leads to

$$\langle \ell' m' | L^{(2)ij} | \ell m \rangle = \delta_{\ell \ell'} \sqrt{\ell(\ell + 1)(2\ell - 1)(2\ell + 3)} \sum_{\mu, \mu'} C_{i j \mu}^{1 1} C_{m m'}^{\ell' \ell} \mu \mu', \quad (A8)$$
which has been used in deriving the matrix elements of $O_6$ and is consistent with Eq. (A5) of Ref. [19].

Equation (A6) indicates that the definition of $L^{(2)ij}$ from Ref. [35] contains a typographic error in the second term on the right-hand side, namely the phase $(-)^i$ is missing. Previous and present results are not affected by this inadvertence.

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