EXISTENCE OF GROUND STATE SOLUTIONS FOR A CLASS OF QUASILINEAR SCHRÖDINGER EQUATIONS WITH GENERAL CRITICAL NONLINEARITY

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Abstract. In this paper, we study the following quasilinear Schrödinger equation
\[-\Delta u + V(x)u - \Delta(u^2)u = g(u), \quad x \in \mathbb{R}^N,\]
where $N > 4$, $2^* = \frac{2N}{N-2}$, $V : \mathbb{R}^N \to \mathbb{R}$ satisfies suitable assumptions. Unlike $g \in C^1(\mathbb{R}, \mathbb{R})$, we only need to assume that $g \in C(\mathbb{R}, \mathbb{R})$. By using a change of variable, we obtain the existence of ground state solutions with general critical growth. Our results extend some known results.

1. Introduction. This article is concerned with the following quasilinear Schrödinger equation
\[-\Delta u + V(x)u - \Delta(u^2)u = g(u), \quad x \in \mathbb{R}^N,\]
where $N > 4$ and $2^* = \frac{2N}{N-2}$. It is well known that it is a hot problem in nonlinear analysis to study the existence of solitary wave solutions for the following quasilinear Schrödinger equation
\[i\partial_t z = -\Delta z + W(x)z - k(x, |z|) - \Delta l(|z|^2)'(|z|^2)z\]
where $z : \mathbb{R} \times \mathbb{R}^N \to \mathbb{C}$, $W : \mathbb{R}^N \to \mathbb{R}$ is a given potential, $l : \mathbb{R} \to \mathbb{R}$ and $k : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ are suitable functions. For various types of $l$, the quasilinear equation

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of the form (1.1) has been derived from models of several physical phenomenon. In particular, \( I(s) = s \) was used for the superfluid film equation in fluid mechanics by Kurihara [21]. For more physical background, we can refer to [2, 28] and references therein.

Set \( z(t, x) = \exp(-iEt)u(x) \), where \( E \in \mathbb{R} \) and \( u \) is a real function, (1.2) can be reduce to the corresponding equation of elliptic type (see [10]):

\[
- \Delta u + V(x)u - \Delta l(u^2)l'(u^2)u = h(x, u) \quad x \in \mathbb{R}^N.
\]

(1.3)

If we take \( g^2(u) = 1 + \frac{|(f^2(u))'|^2}{2} \), then (1.3) turns into quasilinear elliptic equations (see [32])

\[
- \text{div}(g^2(u)\nabla u) + g(u)g'(u)|\nabla u|^2 + V(x)u = h(x, u), \quad x \in \mathbb{R}^N.
\]

(1.4)

For (1.4), there are many papers (see [32, 14, 13, 4, 5]) studying the existence of positive solutions. But in our mind, there are few papers to study the existence of ground state solutions for this problem. If we set \( g(s) = \sqrt{1 + 2s^2} \) and \( h(x, u) = h(u) \) in (1.4), then (1.4) reduces to the Equ. (1.1). In [31], Poppenberge-Schmitt-Wang consider an eigenvalue problem for the following equation

\[
- \Delta u + V(x)u - \Delta l(u^2)u = \kappa |u|^{p-1}u, \quad u > 0, \quad x \in \mathbb{R}^N,
\]

(1.5)

where \( V(x) \in L^1_{\text{loc}}(\mathbb{R}^N), \) \( \inf_{x \in \mathbb{R}^N} V(x) > 0 \) and \( \kappa > 0 \) is a parameter. By using constrained variational method, they prove the existence of a positive solution of problem (1.5) in \( H^1(\mathbb{R}^N) \) for certain \( \kappa > 0 \) with some further conditions on \( V(x) \) and \( N = 1 \). Moreover, for \( p \geq 3 \), a positive solution was established for any \( \kappa > 0 \) by the Mountain Pass Theorem. If \( N \geq 2 \) and \( 1 < p < 2^* - 1 \), in [31], the authors obtain the existence of a nontrivial nonnegative solution in \( H^1(\mathbb{R}^N) \) for certain \( \kappa > 0 \). Furthermore, for \( N = 3 \) and \( 3 \leq p < 5 \), the authors [31] show that problem (1.5) has a nontrivial nonnegative solution in \( H^1(\mathbb{R}^N) \) for all \( \kappa > 0 \). Based on the work in [31], many papers focused on subcritical case, see [24, 23, 29, 30]. As the argument in [22], \( 2^* \) behaves like a critical exponent for (1.5), and they gave an open question on whether there existence results for \( p = 2^* \). For this open question, many papers studied the existence of solutions with critical exponent \( 2^* \).

For more specific works, please see [15, 25, 26]. In particular, in [15], do Ó et al. considered the following quasi-linear Schrödinger equations

\[
- \Delta u + V(x)u - \Delta (u^2)u = |u|^{q-1}u + |u|^{p-1}u \quad \text{in} \, \mathbb{R}^N,
\]

(1.6)

where \( 3 < q < p \leq 2^* - 1 \) and \( 2^* = \frac{2N}{N-2} \) is the critical exponent. They also assume that \( V(x) \) satisfying the following conditions

\begin{itemize}
  \item[(V_1)] the function \( V : \mathbb{R}^N \rightarrow \mathbb{R} \) is continuous and uniformly positive, that is, there exists a constant \( V_0 > 0 \) such that \( 0 < V_0 \leq V(x) \) for all \( x \in \mathbb{R}^N \);
  \item[(V_2)] there exists a constant \( V_{\infty} \) such that \( \lim_{|x| \to +\infty} V(x) = V_{\infty} \) and \( V(x) \leq V_{\infty} \) for all \( x \in \mathbb{R}^N \),
\end{itemize}

where the last inequality is strict on subset of positive measure in \( \mathbb{R}^N \);

\begin{itemize}
  \item[(V_2')] the function \( V \) is periodic in each variable of \( x_1, \ldots, x_N \).
\end{itemize}

For \( 3 < q < 2^* - 1 \) and \( p = 2^* - 1 \), a positive classic solution of (1.6) in \( H^1(\mathbb{R}^N) \) with \( (V_1)-(V_2) \) was obtained in [15] by introducing the changing of variable \( v = f^{-1}(u) \) (see [22, 9, 16]) and transforming (1.6) into a semilinear equation. For
more semilinear problems, we refer to [35, 39, 8] and so on. Following this work, in [33], Silva and Vieira studied the following quasilinear Schrödinger equation
\[-\Delta u + V(x)u - \Delta(u^2)u = K(x)|u|^{2^{*}_N-2}u + g(x, u), \quad u > 0, \quad x \in \mathbb{R}^N.\]

By assuming that $V(x)$, $K(x)$ and $g(x, u)$ are asymptotically periodic and satisfy some further conditions, the authors [33] established the existence of nontrivial solution by the concentration-compactness principle and a comparison argument. Specifically speaking, the authors used a change of variable to reformulate the problem obtaining a semilinear problem which has an associated functional well-defined in $H^1(\mathbb{R}^N)$ and satisfies the geometric hypotheses of the Mountain Pass Theorem. They considered the functional associated with the modified problem and used a version of the Mountain Pass Theorem without compactness condition to get a Cerami sequence associated with the minimax level. Moreover, this sequence and a technical results are applied to get a nontrivial critical point of the functional associated with the periodic problem. Furthermore, there are many papers forcing on the existence and concentration of positive ground and bound state solutions for (1.1) with critical growth. For the results of this type, we refer to [18, 36, 38].

Very recently, in [6] the first author, Tang and Cheng considered the problem (1.1) with constant potential and subcritical nonlinearity, which was proved by the critical point theorem developed by [19]. Based on this work, the first author, Tang and Cheng [7] considered the problem (1.1) with critical nonlinearity and established the existence of positive ground state solutions via Pohožaev manifold.

It is a natural problem if the existence of positive ground state of (1.1) with general critical nonlinearity? To our knowledge, the existence of the ground solutions to Eq. (1.1) with general critical nonlinearity term has not ever been considered by variational methods. In this paper, we will give an affirmative answer for the above question. The underling idea for proving our results is motivated by the method used in [11, 27, 1, 34]. As before mentioned papers, we also use a change of variable to reformulate the problem obtaining a semilinear equation. It is worth pointing out that there are many difficulties in treating this class of quasilinear Schrödinger equation in $\mathbb{R}^N$. To this end, we need to overcome those difficulties: (I) it is the lack of compactness; (II) the estimate of mountain pass level.

Now, let us recall some basic notions. Let
\[H^1(\mathbb{R}^N) = \{u \in L^2(\mathbb{R}^N) : \nabla u \in L^2(\mathbb{R}^N)\}\]
with the norm
\[\|u\| = \left(\int_{\mathbb{R}^N} (|\nabla u|^2 + u^2)\right)^{\frac{1}{2}},\]
and we denote by $L^p(\mathbb{R}^N)$ the usual Lebesgue space with norms $\|u\|_p = (\int_{\mathbb{R}^N} |u|^p)^{\frac{1}{p}}$, where $1 \leq p < \infty$. The embedding $H^1(\mathbb{R}^N) \hookrightarrow L^s(\mathbb{R}^N)$ is continuous for $s \in [2, 2^*_N]$.

Next, we consider the problem (1.1) with general critical nonlinearity. Before stating our results, we need to give some certain conditions on $g$.

*(g1)* $g \in C(\mathbb{R}, \mathbb{R})$, $g(t) = 0$ for all $t \leq 0$ and $\lim_{t \to 0^+} \frac{g(t)}{t} = 0$;

*(g2)* $\lim_{t \to +\infty} \frac{g(t)}{t^{2^{*}_N}} = 1$;

*(g3)* there exist $\beta > 0$ and $\max \left\{ \frac{2(N+4)}{N-2}, 4 \right\} < q < 2^{*}_N$ such that
\[g(t) \geq t^{2^{*}_N-1} + \beta t^{q-1} \quad \text{for all} \ t \geq 0;\]
Firstly, we give the first results by assuming that $V(x)$ is constant potential.

**Theorem 1.1.** Assume that $(g_1) - (g_3)$ and $V(x) \equiv 1$ are satisfied. Then problem (1.1) has a nontrivial ground state solution.

Secondly, we give the second result in this paper. Before stating our result, assume that $g$ satisfied some certain conditions $(g_1) - (g_3)$ and $V(x)$ is not a constant.

$(V_1)$ $V \in C^1(\mathbb{R}^N, \mathbb{R})$ and there exists a constant $V_0 > 0$ such that $\inf_{x \in \mathbb{R}^N} V(x) \geq V_0$;

$(V_2)$ $V(x) \leq V_\infty := \lim_{|x| \to +\infty} V(x)$ for all $x \in \mathbb{R}^N$;

$(V_3)$ $\|\max \{\nabla V(x), 0\}\|_{L^\infty(\mathbb{R}^N)} < 2S_*$, where $S_*$ is the best constant of the embedding $D^{1,2}(\mathbb{R}^N) \hookrightarrow L^2(\mathbb{R}^N)$.

Now, we are ready to state the second result of this case as follows.

**Theorem 1.2.** Assume that $(g_1) - (g_3)$ and $(V_1) - (V_3)$ are satisfied. Then problem (1.1) has a nontrivial ground state solution.

**Remark 1.3.** Note that Zhang and Zou [40] considered the critical case for Berestycki-Lions theorem of the Schrödinger equation $-\Delta u + V(x)u = g(u)$. They obtained positive ground state solutions when $g$ satisfies $(g_1) - (g_3)$ and $(g_4) |g'(t)| \leq C(1 + |t|^{2^* - 2})$, for $t > 0$ and some $C > 0$.

But in the present paper, $(g_4)$ is removed.

**Remark 1.4.** There are indeed functions which satisfy $(V_1) - (V_3)$. An example is given by:

$$V(x) = \begin{cases} V_\infty - 1 - \cos \left(2 \left( \frac{C_N}{2S_*} \right)^{1/N} \pi x \right), & \text{if } |x| \leq \frac{1}{2} \left( \frac{2S_*}{C_N} \right)^{1/N}, \\ V_\infty, & \text{if } |x| > \frac{1}{2} \left( \frac{2S_*}{C_N} \right)^{1/N}, \end{cases}$$

where $N > 4$, $C_N$ is the surface area of the $N$-dimensional unit ball and $S_*$ is the best constant of the embedding $D^{1,2}(\mathbb{R}^N) \hookrightarrow L^2(\mathbb{R}^N)$ and $V_\infty > 2$ is a positive constant.

The remainder of this paper is organized as follows. In section 2, we list some useful propositions and lemmas, which play an important role in proving our results. The proofs of Theorem 1.1 and Theorem 1.2 are given in Section 3 and Section 4, respectively.

In this paper, $\int_{\mathbb{R}^N} \bullet$ denotes $\int_{\mathbb{R}^N} \bullet \, dx$ and $C$ denotes the different constants.

2. Variational framework and some preliminary lemmas. In this section, we give some useful lemmas. Let

$$E = \left\{ u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)u^2 < +\infty \right\},$$

endowed with the norm

$$\|u\|_E = \left( \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2) \right)^{\frac{1}{2}}.$$

By the conditions $(V_1)$ and $(V_2)$, it is easy to check that $\| \cdot \|_E$ is equivalent to the norm $\| \cdot \|$. In (1.1), we can deduce formally that the Euler-Lagrange functional
associated with the equation (1.1) is
\[ J(u) = \frac{1}{2} \int_{\mathbb{R}^N} [(1 + 2u^2)|\nabla u|^2 + V(x)u^2] - \int_{\mathbb{R}^N} G(u). \]

For (1.1), due to the appearance of the nonlocal term \( \int_{\mathbb{R}^N} v^2|\nabla u|^2 \), \( J \) may be not well defined. To overcome this difficulty, we apply an argument developed by Liu et al. [22] and Colin and Jeanjean [9]. We make the change of variables by \( v = f^{-1}(u) \), where \( f \) is defined by
\[ f'(t) = \frac{1}{(1 + 2f^2(t))^{\frac{1}{2}}} \text{ on } [0, \infty) \text{ and } f(t) = -f(-t) \text{ on } (-\infty, 0], \]
and then equation (1.1) in form can be transformed into
\[ I(v) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla v|^2 + V(x)f^2(v)) - \int_{\mathbb{R}^N} G(f(v)), \quad x \in \mathbb{R}^N. \tag{2.7} \]
It is easy to check to that \( I \in C^1 \). We also know that if \( v \) is a critical point of the functional \( I \), then \( u = f(v) \) is a critical point of the functional \( I \), i.e. \( u = f(v) \) is a solution of problem (1.1).

Next, let us recall some properties of the change of variables \( f : \mathbb{R} \to \mathbb{R} \), which are proved in [22, 9, 16] as follows:

**Lemma 2.1** ([22, 9, 16]). The function \( f(t) \) and its derivative satisfy the following properties:

1. \( f \) is uniquely defined, \( C^\infty \) and invertible;
2. \( |f'(t)| \leq 1 \) for all \( t \in \mathbb{R} \);
3. \( |f(t)| \leq |t| \) for all \( t \in \mathbb{R} \);
4. \( f(t)/t \to 1 \) as \( t \to 0 \);
5. \( f(t)/\sqrt{t} \to 2^\frac{1}{4} \) as \( t \to +\infty \);
6. \( f(t)/2 \leq tf'(t) \leq f(t) \) for all \( t > 0 \);
7. \( f^2(t)/2 \leq tf(t)f'(t) \leq f^2(t) \) for all \( t \in \mathbb{R} \);
8. \( |f(t)| \leq 2^{1/4}|t|^{1/2} \) for all \( t \in \mathbb{R} \);
9. there exists a positive constant \( C \) such that
   \[ |f(t)| \geq \begin{cases} \frac{C|t|}{\sqrt{t}}, & \text{if } |t| \leq 1, \\ \frac{C|t|^{1/2}}{t}, & \text{if } |t| \geq 1; \end{cases} \]
10. for each \( \alpha > 0 \), there exists a positive constant \( C(\alpha) \) such that
    \[ |f(\alpha t)|^2 \leq C(\alpha)|f(t)|^2; \]
11. \( |f(t)||f'(t)| \leq \frac{1}{\sqrt{2}}. \]

Next, we need to recall a critical point theory, which is given as follows:

**Theorem 2.2** ([19]). Let \( (X, \| \cdot \|) \) be a Banach space and \( I \subset \mathbb{R}_+ \) an interval. Consider the following family of \( C^1 \)-functionals on \( X \):
\[ I_\lambda(v) = A(v) - \lambda B(v), \quad \lambda \in I \]
with \( B \) nonnegative and either \( A(v) \to +\infty \) or \( B(v) \to +\infty \) as \( \|v\| \to \infty \). We assume there are two points \( v_1, v_2 \) in \( X \) such that
\[ c_\lambda = \inf_{\gamma \in \Gamma_\lambda} \max_{t \in [0,1]} I_\lambda(\gamma(t)) > \max\{I_\lambda(v_1), I_\lambda(v_2)\} \text{ for all } \lambda \in I \]
where
\[ \Gamma_\lambda = \{ \gamma \in C([0,1], X) : \gamma(0) = v_1, \gamma(1) = v_2 \}. \]
Then for almost every $\lambda \in \mathbb{I}$ there is a sequence $\{v_n\} \subset X$ such that

(i) $\{v_n\}$ is bounded,

(ii) $\mathcal{L}_\lambda(v_n) \to c_\lambda$,

(iii) $\mathcal{L}_\lambda'(v_n) \to 0$ in the dual $X^{-1}$ of $X$.

Moreover, the map $\lambda \mapsto c_\lambda$ is non-increasing and continuous from the left.

**Lemma 2.3** (see Lemma 1.21 in [37]). Let $\{v_n\}$ be a bounded sequence in $H^1(\mathbb{R}^N)$ such that

$$\lim_{n \to \infty} \sup_{y \in \mathbb{R}^N} \int_{B_R(y)} |v_n|^r = 0,$$

for some $R > 0$. Then $v_n \to 0$ in $L^r(\mathbb{R}^N)$ for $2 < r < 2^*$.

**Lemma 2.4** (see Lemma 8.9 in [37]). If $v_n \to v$ in $D^{1,2}(\mathbb{R}^N)$ and $v \in L^\infty_{loc}(\mathbb{R}^N)$, then

$$|v_n|^{2^* - 2}v_n - |v_n - v|^{2^* - 2}(v_n - v) \to |v|^{2^* - 2}v \text{ in } (D^{1,2}(\mathbb{R}^N))'.$$

Next, we introduce the following lemma, which was proved in [3].

**Lemma 2.5** ([3]). Let $P$ and $Q : \mathbb{R} \to \mathbb{R}$ be a continuous functions satisfying

$$\lim_{t \to \pm \infty} \frac{P(t)}{Q(t)} = 0.$$

Let $\{v_n\}$, $v$ and $w$ be measurable function from $\mathbb{R}^N$ to $\mathbb{R}$, with $w$ bounded, such that

$$\sup_{n \in \mathbb{N}} \int_{\mathbb{R}^N} |Q(v_n(x))w| < +\infty, \quad P(v_n(x)) \to v(x) \text{ a.e. in } \mathbb{R}^N.$$

Then $|(P(v_n) - v)w|_{L^1(\mathcal{B})} \to 0$, for any bounded Borel set $\mathcal{B}$. Moreover, if we have also

$$\lim_{t \to 0} \frac{P(t)}{Q(t)} = 0,$$

and

$$\lim_{n \to \infty} \sup_{x \in \mathbb{R}^N} |v_n(x)| = 0,$$

then $|(P(v_n) - v)w|_{L^1(\mathbb{R}^N)} \to 0$.

By the Brezis-Lieb Lemma in [37], we can prove the follow lemma.

**Lemma 2.6** (see Lemma 2.1 in [41]). Assume that $h \in C(\mathbb{R}^N \times \mathbb{R})$ and there exists a constant $C_0 > 0$ such that

$$\lim_{s \to 0} \frac{h(x, s)}{s} \leq C_0 \quad \text{and} \quad \lim_{s \to \infty} \frac{h(x, s)}{s^{2^* - 1}} = 0, \quad \text{uniformly in } x \in \mathbb{R}^N.$$

Let $\{v_n\} \subset H^1(\mathbb{R}^N)$ be a bounded sequence and $v \in H^1(\mathbb{R}^N)$ with $v_n \to v$ in $H^1(\mathbb{R}^N)$. Then

$$\lim_{n \to \infty} \left[ \int_{\mathbb{R}^N} H(x, v_n) - \int_{\mathbb{R}^N} H(x, v) - \int_{\mathbb{R}^N} H(x, v_n - v) \right] = 0,$$

where $H(x, s) = \int_0^s h(x, t)dt$.

By a standard argument in [37], we can obtain the following Pohozaev type identity.

**Lemma 2.7.** If $v \in H^1(\mathbb{R}^N)$ be a solution of (1.1), then $v$ satisfies

$$\frac{N - 2}{2} \int_{\mathbb{R}^N} |\nabla v|^2 + \frac{1}{2} \int_{\mathbb{R}^N} (\nabla V(x) \cdot x) f^2(v) + \frac{N}{2} \int_{\mathbb{R}^N} V(x) f^2(v) = \lambda N \int_{\mathbb{R}^N} G(f(v)),$$
3. Proof of Theorem 1.1. In this section, we want to give a proof of Theorem 1.1. At first, if we choose $V(x) \equiv 1$ in Lemma 2.7, then we can get the following version of Pohozaev identity.

Lemma 3.1. If $v \in H^1(\mathbb{R}^N)$ be a solution of (1.1), then $v$ satisfies

$$\frac{N-2}{2} \int_{\mathbb{R}^N} |\nabla v|^2 + \frac{N}{2} \int_{\mathbb{R}^N} f^2(v) = \lambda N \int_{\mathbb{R}^N} G(f(v)).$$

Let $I = [\frac{1}{2}, 1]$. We define the following energy functional

$$\Phi_\lambda(v) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla v|^2 + f^2(v)) - \lambda \int_{\mathbb{R}^N} G(f(v)),$$

where $\lambda \in [\frac{1}{2}, 1]$. Moreover, let

$$A(v) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla v|^2 + f^2(v))$$

and

$$B(v) = \int_{\mathbb{R}^N} G(f(v)).$$

By Sobolev inequality and Lemma 2.1-(9), we can get

$$\int_{\mathbb{R}^N} (|\nabla v|^2 + v^2) \leq \int_{\mathbb{R}^N} |\nabla v|^2 + \int_{|v|>1} |v|^2 + \int_{|v|\leq 1} v^2 \leq C \int_{\mathbb{R}^N} (|\nabla v|^2 + f^2(v)).$$

In fact, for any $v \in H^1(\mathbb{R}^N)$, by Sobolve inequity and Lemma 1.1, we have that

$$\int_{|v|>1} |v|^2 \leq \int_{|v|>1} |v|^{2^*} \leq \left( \int_{\mathbb{R}^N} |\nabla v|^2 \right)^{2^*/2} \leq \left( \int_{\mathbb{R}^N} |\nabla v|^2 \right)^{2^*} \int_{\mathbb{R}^N} |\nabla v|^2 = C \int_{\mathbb{R}^N} |\nabla v|^2$$

and

$$\int_{|v|\leq 1} |v|^2 \leq \int_{|v|\leq 1} f^2(v) \leq \int_{\mathbb{R}^N} f^2(v).$$

Thus

$$\int_{\mathbb{R}^N} (|\nabla v|^2 + v^2) \leq \int_{\mathbb{R}^N} |\nabla v|^2 + \int_{|v|>1} |v|^2 + \int_{|v|\leq 1} v^2 \leq C \int_{\mathbb{R}^N} (|\nabla v|^2 + f^2(v)). \quad (3.8)$$

Letting $\|v\| \to +\infty$, then $A(v) \to +\infty$. By (g1), we know that $B(v) \geq 0$.

Lemma 3.2. Assume that (g1)-(g3) are satisfied. Then there holds:

(i) there exists $v \in H^1(\mathbb{R}^N) \setminus \{0\}$ such that $\Phi_\lambda(v) < 0$ for all $\lambda \in [\frac{1}{2}, 1]$;

(ii) $c_\lambda = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \Phi_\lambda(\gamma(t)) > \max\{\Phi_\lambda(0), \Phi_\lambda(v)\}$ for all $\lambda \in [\frac{1}{2}, 1]$, where

$$\Gamma = \{ \gamma \in C([0,1], H^1(\mathbb{R}^N)) : \gamma(0) = 0, \; \gamma(1) = v \};$$

(iii) for any $v \in H^1(\mathbb{R}^N) \setminus \{0\}$, there exists a constant $C > 0$ such that $c_\lambda \leq C$ for all $\lambda \in [\frac{1}{2}, 1]$.

Proof. (i) Let $v \in H^1(\mathbb{R}^N) \setminus \{0\}$ be fixed. For any $\lambda \in [\frac{1}{2}, 1]$, one has

$$\Phi_\lambda(v) \leq \Phi_{1/2}(v) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla v|^2 + f^2(v)) - \frac{1}{2} \int_{\mathbb{R}^N} G(f(v)).$$
By (g₃), Lemma 2.1-(3) and (8), for any φ ∈ H¹(ℝᴺ) such that φ ≥ 0 and φ ≠ 0, one has

\[ \Phi_λ(φ) \leq \frac{λ^2}{2} \int_{ℝᴺ} (|∇φ|^2 + φ^2) - \frac{N - 2}{4N} \int_{ℝᴺ} |f(tφ)|^{2^*} - \frac{β}{2q} \int_{ℝᴺ} |f(tφ)|^q \]
\[ \leq \frac{λ^2}{2} \int_{ℝᴺ} (|∇φ|^2 + φ^2) - \frac{(N - 2) · 2^{2^*}}{4N} α^2 \int_{(tφ ≥ 1)} |φ|^2^* \]
\[ - \frac{β · 2^{q/4}}{2q} α^{q/2} \int_{(tφ ≥ 1)} |φ|^{q/2}. \]

It follows that \( \Phi_λ(tφ) \to −∞ \) as \( t \to +∞ \). Thus there exists a \( t_0 > 0 \) such that \( \Phi_λ(t_0 φ) < 0 \). Thus taking \( v = t_0 φ \), we have \( \Phi_λ(v) < 0 \) for all \( λ \in \left[ \frac{1}{2}, 1 \right] \).

(ii) By (g₁)-(g₂), we have that for any \( ε > 0 \) there exists \( C_ε > 0 \) such that

\[ |G(t)| ≤ εt^2 + C_ε|t|^{2^*}, \text{ for all } t ∈ ℝ. \]

From (3.8) and Lemma 2.1-(3), we have

\[ \Phi_λ(v) ≥ \frac{1}{2} \int_{ℝᴺ} (|∇v|^2 + f^2(v)) - λ \int_{ℝᴺ} \left( ε|f(v)|^2 + C_ε|f(v)|^{2^*} \right) \]
\[ ≥ \frac{1}{2} \int_{ℝᴺ} |∇v|^2 + \left( \frac{1}{2} - λε \right) \int_{ℝᴺ} |f(v)|^2 - C_ε \int_{ℝᴺ} |v|^2 \]
\[ ≥ \min \left\{ \frac{1}{2}, \left( \frac{1}{2} - λε \right) C \right\} \int_{ℝᴺ} (|∇v|^2 + v^2) - C_ε \int_{ℝᴺ} |v|^2 \]
\[ ≥ \min \left\{ \frac{1}{2}, \left( \frac{1}{2} - λε \right) C \right\} \|v\|^2 - C_ε · C\|v\|^{2^*}, \]

where \( ε \) is small enough such that \( \frac{1}{2} > λε \). Since \( 2^* > 2 \), we deduce that \( \Phi_λ \) has a strict local minimum at \( 0 \) and hence \( c_λ > 0 \).

(iii) For any \( v ∈ H¹(ℝᴺ) \setminus \{0\} \), \( c_λ ≤ \max_{t>0} \Phi_λ(v_t) ≤ \max_{t>0} \Phi_{1/2}(v_t) \) for \( λ ∈ \left[ \frac{1}{2}, 1 \right] \). Thus we can choose \( C ≥ \max_{t>0} \Phi_{1/2}(v_t) ≥ 0 \) such that \( c_λ ≤ C \). This completes the proof. \( \square \)

By Theorem 2.1, it is easy to know that for a.e. \( λ ∈ \left[ \frac{1}{2}, 1 \right] \), there exists a bounded sequence \( \{v_n\} ⊂ H¹(ℝᴺ) \) such that \( \Phi_λ(v_n) → c_λ \) and \( \Phi'_λ(v_n) → 0 \), which is called (PS)-sequence.

Next, we shall estimate \( c_λ \). Let \( η ∈ C_0^∞(ℝᴺ, [0, 1]) \) be a cut-off function such that \( η ≡ 1 \) on \( B_1(0) \) and \( η ≡ 0 \) on \( ℝᴺ \setminus B_2(0) \), where \( B_1(0) \) denotes the ball in \( ℝᴺ \) of center at origin and radius 1. Given \( ε > 0 \), we consider the function \( U_ε : ℝᴺ → ℝ \) defined by

\[ U_ε(x) = C(N) \frac{ε^{(N-2)/4}}{(ε + |x|^2)^{(N-2)/2}}, \]

where

\[ C(N) = [N(N - 2)](N-2)/4. \]

Moreover, \( U_ε \) satisfies the following equations

\[ \left\{ \begin{array}{ll}
-Δu = u^{2^*-1}, & \text{in } ℝᴺ, \\
u ∈ D^{1,2}(ℝᴺ), & u(x) > 0, \text{ in } ℝᴺ.
\end{array} \right. \]

We know that \( \{U_ε\}_{ε>0} \) is a family of functions on which the infimum, that defines the best constant, \( S \), for the embedding \( D^{1,2} → L^2(ℝᴺ) \), is attained. Let us define

\[ w_ε = η(x)U_ε(x)^{1/2} \]
Lemma 3.3 ([18, 36]). If \( \varepsilon \to 0 \), then \( v_\varepsilon \) satisfies the following useful estimates:

\[
\int_{\mathbb{R}^N} |\nabla v_\varepsilon|^2 \leq S + O(\varepsilon^{N/2}),
\]

and

\[
\int_{\mathbb{R}^N} |v_\varepsilon|^2 \leq O(\varepsilon^{N/2}),
\]

and

\[
\int_{\mathbb{R}^N} |v_\varepsilon|^p = \begin{cases} O(\varepsilon^{(p(N-2))/2}), & \text{if } p \in (1, 2^*), \\ O(\varepsilon^{N/4} |\log \varepsilon|), & \text{if } p = 2^*, \\ O(\varepsilon^{N/2 - p(N-2)/2}), & \text{if } p \in (2^*, 2^*). \end{cases}
\]

Lemma 3.4. For any \( \lambda \in [\frac{1}{2}, 1] \), there holds

\[
c_\lambda < \frac{1}{2N} \frac{S^{N/2}}{\lambda^{N-2}}.
\]

Proof. Defined the energy functional \( J_\lambda : X \to \mathbb{R} \) by

\[
J_\lambda(u) := \Phi_\lambda(f^{-1}(u)) = \frac{1}{2} \int_{\mathbb{R}^N} (1 + 2u^2)|\nabla u|^2 + \frac{1}{2} \int_{\mathbb{R}^N} u^2 - \lambda \int_{\mathbb{R}^N} G(u),
\]

where \( X := \{ u \in H^1(\mathbb{R}^N) \} \) and \( \int_{\mathbb{R}^N} |\nabla u|^2 < +\infty \). It suffices to prove that there exists \( 0 \neq \vartheta \in X \) such that

\[
\sup_{t \geq 0} J_\lambda(t \vartheta) < \frac{1}{2N} \frac{S^{N/2}}{\lambda^{N-2}}.
\]

In fact, since \( J_\lambda(f^{-1}(t \vartheta)) = -\infty \) as \( t \to +\infty \), there exists some \( t_0 > 0 \) such that \( J_\lambda(f^{-1}(t_0 \vartheta)) < 0 \). Define \( \gamma^*(t) := f^{-1}(tt_0 \vartheta) \). By the mountain pass value, one has

\[
c_\lambda := \inf_{\gamma \in \mathcal{C}} \sup_{t \geq 0} \Phi_\lambda(\gamma(t)) \leq \sup_{t \geq 0} \Phi_\lambda(f^{-1}(t_0 \vartheta)) \leq \sup_{t \geq 0} J_\lambda(t_0 \vartheta).
\]

By Lemma 3.3, we know that \( v_\varepsilon \in X \). Now, by Lemma 3.3 and (g3), we have that

\[
J_\lambda(tv_\varepsilon) = \frac{t^2}{2} \int_{\mathbb{R}^N} (|\nabla v_\varepsilon|^2 + v_\varepsilon^2) + \frac{t^4}{4} \int_{\mathbb{R}^N} |\nabla v_\varepsilon|^2 - \lambda \int_{\mathbb{R}^N} H(tv_\varepsilon)
\leq \frac{t^2}{2} \int_{\mathbb{R}^N} (|\nabla v_\varepsilon|^2 + v_\varepsilon^2) + \frac{t^4}{4} \int_{\mathbb{R}^N} |\nabla v_\varepsilon|^2 - \lambda \frac{t^{2^*}}{2^{2^*}} - \lambda \frac{\beta t^q}{q} \int_{\mathbb{R}^N} v_\varepsilon^q
=: \mathcal{H}_\lambda(t).
\]

Note that \( \lim_{t \to 0} \mathcal{H}_\lambda(t) = -\infty \) and \( \mathcal{H}_\lambda(t) > 0 \) as \( t \to 0 \). Thus there exist some \( t_\varepsilon > 0 \) such that \( \mathcal{H}_\lambda(t_\varepsilon) = \sup_{t \geq 0} \mathcal{H}_\lambda(t) \). On the one hand, from

\[
0 = \mathcal{H}_\lambda'(t_\varepsilon) = t_\varepsilon \left( \int_{\mathbb{R}^N} (|\nabla v_\varepsilon|^2 + v_\varepsilon^2) + t_\varepsilon^2 \int_{\mathbb{R}^N} |\nabla v_\varepsilon|^2 - \lambda t_\varepsilon^{2^*} - \lambda \frac{\beta t^q}{q} \int_{\mathbb{R}^N} v_\varepsilon^q \right),
\]

we conclude that

\[
\int_{\mathbb{R}^N} (|\nabla v_\varepsilon|^2 + v_\varepsilon^2) + t_\varepsilon^2 \int_{\mathbb{R}^N} |\nabla v_\varepsilon|^2 = \lambda t_\varepsilon^{2^*} + \lambda \frac{\beta t^q}{q} \int_{\mathbb{R}^N} v_\varepsilon^q \geq \frac{1}{2} t_\varepsilon^{2^*}, \quad (3.12)
\]
which implies that $t_\varepsilon$ is bounded from above by some $T_1 > 0$. On the other hand, by (3.12), one has
\[
\int_{\mathbb{R}^N} |\nabla v^2_\varepsilon|^2 \leq \lambda t^*_{22} - 4 + \lambda^2 t^q_{-4} \int_{\mathbb{R}^N} v^q_\varepsilon \leq t^*_{22} - 4 + \beta t^q_{-4} \int_{\mathbb{R}^N} v^q_\varepsilon.
\]
Letting $\varepsilon \to 0$, by (3.9), (3.10), (3.11) and $4 < q < 22^*$, we have $t^*_{22} - 4 + \frac{\beta}{2}$. That is, we get a lower bound for $t_\varepsilon$ independent of $\varepsilon$. Next, we estimate $\mathcal{H}_\lambda(t)$. Note that the function
\[
\mathcal{K}_\lambda(t) := \frac{t^4}{4} \int_{\mathbb{R}^N} |\nabla v^2_\varepsilon|^2 - \lambda t^*_{22} - 4
\]
attains its unique global maximum at $t_1 = (\lambda^{-1} \int_{\mathbb{R}^N} |\nabla v^2_\varepsilon|^2)^{\frac{1}{22^*}}$. Thus by $(g_3)$ and Lemma 3.3, we get
\[
\sup_{t \geq 0} \mathcal{J}_\lambda(t v_\varepsilon) \\
\leq \mathcal{H}_\lambda(t v_\varepsilon) \\
= \mathcal{K}_\lambda(t) + \frac{t^2}{2} \int_{\mathbb{R}^N} (|\nabla v^2_\varepsilon| + v^2_\varepsilon) - \lambda \frac{\beta q}{q} \int_{\mathbb{R}^N} v^q_\varepsilon \\
\leq \left( \frac{1}{4} - \frac{1}{22^*} \right) \lambda^{-\frac{N-2}{4}} \left( \int_{\mathbb{R}^N} |\nabla v^2_\varepsilon|^2 \right)^{\frac{22^* - 1}{22^*}} + C \int_{\mathbb{R}^N} \left( |\nabla v^2_\varepsilon| + v^2_\varepsilon \right) - C \int_{\mathbb{R}^N} v^q_\varepsilon \\
\leq \frac{1}{2N} \lambda^{-\frac{N-2}{4}} \left( S + O(\varepsilon^{\frac{N-2}{4}}) \right)^{\frac{22^* - 1}{22^*}} + O(\varepsilon^{\frac{N-2}{4}}) - O(\varepsilon^{\frac{N-2}{4}}) - O(\varepsilon^{\frac{N-2}{4}}) \\
< \frac{1}{2N} \lambda^{-\frac{N-2}{4}} S^{N/2}
\]
for $\varepsilon > 0$ small enough. This completes the proof. \qed

Remark 3.5. For any $\lambda \in [1/2, 1]$, if $\{v_n\} \subset H^1(\mathbb{R}^N)$ is bounded and for $c_\lambda > 0$
\[
\Phi_\lambda(v_n) \to c_\lambda, \quad \Phi'_\lambda(v_n) \to 0,
\]
then $v_n \geq 0$ in $H^1(\mathbb{R}^N)$. In fact, it is easy to check that $v_n \geq 0$ in $H^1(\mathbb{R}^N)$ from
\[
\langle \Phi'_\lambda(v_n), v_n \rangle = o_n(1), \text{ where } v_n^- = \min\{v_n, 0\}.
\]

Remark 3.6. Note that if $(V_1)$-$V_2$ and $(g_1)$-$g_3)$, then there exists $g > 0$ independent of $\lambda \in [1/2, 1]$ such that any nontrivial critical point $v_\lambda$ of $\Phi_\lambda$ satisfied $\|v_\lambda\| \geq g > 0$. In fact, by $(g_1)$-$g_3)$, it follows that for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that
\[
g(t) \leq \varepsilon t^2 + C_\varepsilon \|t\|^{22^*}
\]
for all $t \in \mathbb{R}$. By $\langle \Phi'_\lambda(v_\lambda), v_\lambda \rangle = 0$, $\lambda \leq 1$, the Sobolev embedding, $(V_1)$-$V_2$ and (3.8), one has
\[
\|v_\lambda\|^2 \leq \frac{C}{2} \int_{\mathbb{R}^N} |\nabla v_\lambda|^2 + V(x) f^2(v_\lambda) \\
\leq C \int_{\mathbb{R}^N} |\nabla v_\lambda|^2 + V(x) f(v_\lambda) f'(v_\lambda) v_\lambda \\
= C \lambda \int_{\mathbb{R}^N} g(f(v_\lambda)) f'(v_\lambda) v_\lambda \\
\leq \varepsilon C \|v_\lambda\|^2 + C_\varepsilon C \|v_\lambda\|^{22^*}.\]
Choosing $\varepsilon > 0$ sufficiently small and by using $v_\lambda \neq 0$, we know that there exists $\varrho > 0$ such that $\|v_\lambda\| \geq \varrho > 0$.

**Lemma 3.7.** Suppose that $V(x) \equiv 1$ and $q \in \left( \max \left\{ \frac{2(N+4)}{N-2}, 4 \right\}, 2^* \right)$ hold. If $\lambda \in [1/2, 1]$ is fixed and $\{v_n\} \subset H^1(\mathbb{R}^N)$ be a bounded $(PS)_{c_\lambda}$ sequence of $\Phi_\lambda$ with $c_\lambda < \frac{SN_2}{2NL_2}\omega_1$, then there exists a positive integer $l \in \mathbb{N} \cup \{0\}$, a subsequence denoted again by $\{v_n\}$ and sequences $\{\eta_n^k\} \subset \mathbb{R}^N$ and $\omega^k \in H^1(\mathbb{R}^N)$ for $1 \leq k \leq l$ such that

1. $v_n \rightharpoonup v_0$ in $H^1(\mathbb{R}^N)$ with $\Phi_\lambda'(v_0) = 0$;
2. $|\eta_n^k| \to +\infty$ and $|\eta_n^k - \eta_n^{k'}| \to +\infty$ for $k \neq k'$ and $n \to +\infty$;
3. $\omega^k \neq 0$ and $\Phi_\lambda(\omega^k) = 0$ for all $1 \leq k \leq l$;
4. $\|v_n - v_0 - \sum_{k=1}^l \omega^k(\cdot - \eta_n^k)\| \to 0$;
5. $\Phi_\lambda(v_n) \to \Phi_\lambda(v_0) + \sum_{k=1}^l \Phi_\lambda(\omega^k)$.

**Proof.** Since $\{v_n\}$ is a bounded sequence, then, up to subsequence, there exists $v_0$ such that $v_n \rightharpoonup v_0$ in $H^1(\mathbb{R}^N)$. By $H^1(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N)$ for $2 \leq r < 2^*$, we know that $v_n \to v_0$ in $L^r_{loc}$ $1 \leq r < 2^*$ and $v_n \to v_0$ a.e. in $\mathbb{R}^N$. Let $h(t) = g(t) - t^{2^*-1}$. Then we can know that

$$\Phi_\lambda(v) = \frac{1}{2} \int_{\mathbb{R}^N} \langle |\nabla v|^2 + f^2(v) \rangle - \lambda \int_{\mathbb{R}^N} H(f(v)) - \frac{1}{2^*} \int_{\mathbb{R}^N} |f(v)|^{2^*}, \tag{3.13}$$

where $H(t) = \int_0^t h(s)ds$. Hence it follows from $\Phi_\lambda'(v_n) \to 0$ that for any $\psi \in C_0^\infty(\mathbb{R}^N)$,

$$(\Phi_\lambda'(v_n) - \Phi_\lambda'(v_0), \psi) = \int_{\mathbb{R}^N} \nabla(v_n - v_0) \cdot \nabla \psi + \int_{\mathbb{R}^N} (f(v_n)f'(v_n) - f(v_0)f'(v_0))\psi$$

$$- \lambda \int_{\mathbb{R}^N} [h(f(v_n))f'(v_n) - h(f(v_0))f'(v_0)]\psi$$

$$- \lambda \int_{\mathbb{R}^N} [f(v_n)^{2^*-2}f(v_n)f'(v_n) - f(v_0)^{2^*-2}f(v_0)f'(v_0)]\psi. \tag{3.14}$$

Similar to the proof in [33], let $\Omega := \text{supp} \psi$, then we know that $v_n \to v_0$ in $L^r(\Omega)$, where $1 \leq r < 2^*$. By Lemma A.1 in [37], there exists $\omega_r(x) \in L^r(\Omega)$ such that for every $n \in \mathbb{N}$ and a.e. $x \in \Omega$,

$$|v_n(x)| \leq \omega_r(x).$$

Thus we have that

$$f(v_n)f'(v_n) - f(v_0)f'(v_0) \to 0 \text{ a.e. on } \Omega, \text{ as } n \to +\infty,$$

$$h(f(v_n))f'(v_n) - h(f(v_0))f'(v_0) \to 0 \text{ a.e. on } \Omega, \text{ as } n \to +\infty$$

and

$$|f(v_n)|^{2^*-2}f(v_n)f'(v_n) - |f(v_0)|^{2^*-2}f(v_0)f'(v_0) \to 0 \text{ a.e. on } \Omega, \text{ as } n \to +\infty.$$  

Moreover, by Lemma 2.1-(1) and (2), we have

$$|f(v_n)\psi| \leq |f(v_0)\psi| \leq |v_n\psi| \leq |\omega_2||\psi| \in L^1(\mathbb{R}^N)$$

and

$$|f(v_n)|^{2^*-2}f(v_n)\psi| \leq 2^{2^*-2} |v_n|^{2^*-2} |\psi| \leq 2^{2^*-2} |\omega_{2-1}|^{2^*-1} |\psi| \in L^1(\mathbb{R}^N).$$
Form \((g_1)\) and \((g_2)\), we have

\[
|h(f(v_n))f'(v_n)\psi| \leq \left(\varepsilon |f(v_n)| + C_\varepsilon |f(v_n)|^{2^* - 1}\right) |f'(v_n)\psi| \\
+ |f(v_n)|^{2^* - 2}f(v_n)f'(v_n)\psi| \\
\leq (C|\omega_2| |\psi| + C|\omega_2|^{-1}|\psi|^{2^* - 1}) \in L^1(\mathbb{R}^N).
\]

Thus by the Lebesgue Dominated Convergence Theorem, we have \(\Psi'_{\lambda}(v_0) = 0\). So (1) holds.

**Step 1.** Let \(v_n^1 = v_n - v_0\). Thus by Brezis-Lieb Lemma and Lemma 2.6, we have that the following hold:

(i) \(\|v_n^1\|^2 = \|v_n\|^2 - \|v_0\|^2 + o_n(1)\),
(ii) \(\|v_n^1\|^p_p = \|v_n\|^p_p - \|v_0\|^p_p + o_n(1)\),
(iii) \(\|v_n^1\|^{2^*_p} = \|v_n\|^{2^*_p} - \|v_0\|^{2^*_p} + o_n(1)\),
(iv) \(\Phi_{\lambda}(v_n) - \Phi_{\lambda}(v_0) = \Phi_{\lambda}(v_n^1) + o_n(1)\),
(v) \(\Phi_{\lambda}'(v_n^1) \to 0\) in \(H^{-1}(\mathbb{R}^N)\).

It is a standard to prove (i)-(iii) via Brezis-Lieb Lemma. Next, if we set

\[
\tilde{h}_{\lambda}(v) = \lambda h(f(v))f'(v) + \lambda |f(v)|^{2^* - 2}f(v)f'(v) - f(v)f'(v) + v \\
- \lambda 4^{\frac{1}{2^*}} |v|^{2^* - 2}v
\]

and

\[
\tilde{H}_{\lambda}(v) = \lambda H(f(v)) + \frac{\lambda}{2^*} |f(v)|^{2^*} + \frac{1}{2} |f(v)|^2 + \frac{1}{2} v^2 - \lambda \frac{4}{2^*} |v|^{2^*},
\]

then we write Eq. (1.1) in the following form

\[-\Delta v + v = \tilde{h}_{\lambda}(v) + \lambda 4^{\frac{1}{2^*}} |v|^{2^* - 2}v.
\]

Similar to the proof of Lemma 2.2 of [12], it is easy to check that

\[
\lim_{|v| \to 0} \frac{\tilde{H}_{\lambda}(v)}{v^2} = 0, \quad \lim_{|v| \to \infty} \frac{\tilde{H}_{\lambda}(v)}{v^{2^*}} = 0, \quad \lim_{|v| \to 0} \frac{\tilde{\lambda}(v)}{v} = 0, \quad \lim_{|v| \to \infty} \frac{\tilde{\lambda}(v)}{|v|^{2^* - 1}} = 0.
\]

By (3.15) and Lemma 2.6, we have

\[
\int_{\mathbb{R}^N} \tilde{H}_{\lambda}(v_n^1) = \int_{\mathbb{R}^N} \tilde{H}_{\lambda}(v_n) - \int_{\mathbb{R}^N} \tilde{H}_{\lambda}(v_0) + o_n(1).
\]

Thus it follows from \(v_n^1 \to 0\) and (iii) that

\[
\Phi_{\lambda}(v_n) - \Phi_{\lambda}(v_0) = \Phi_{\lambda}(v_n^1) + o_n(1),
\]

which implies that (iv) holds.

Next, we prove (v). By (3.15), for any \(\varepsilon > 0\), there exists a constant \(C_\varepsilon > 0\) such that

\[
|\tilde{h}_{\lambda}(v_n)| \leq \varepsilon (|v_n| + |v_n|^{2^* - 1}) + C_\varepsilon |v_n| r, \quad \text{for all } \ 2 < r < 2^*.
\]

By the similar argument of Lemma 8.1 in [37], for any \(\psi \in H^1(\mathbb{R}^N)\), we get

\[
\int_{\mathbb{R}^N} \left[\tilde{h}_{\lambda}(v_n) - \tilde{h}_{\lambda}(v_n^1) - \tilde{h}_{\lambda}(v_0)\right] \psi = o_n(1) \|\psi\|.
\]
Moreover, by Lemma 2.4, for any \( \psi \in H^1(\mathbb{R}^N) \), we have
\[
\int_{\mathbb{R}^N} \left[ |v_n|^2 - |v_0|^2 \right] \psi = o_n(1) \|\psi\|. \tag{3.19}
\]
By (3.14)-(3.19) and the fact \( v_n^1 \to 0 \) in \( H^1(\mathbb{R}^N) \), we have that
\[
\langle \Phi_\lambda(v_n^1), \psi \rangle + o_n(1) = \langle \Phi_\lambda(v_n) - \Phi_\lambda(v_0), \psi \rangle + o_n(1) = o_n(1).
\]
Thus \( \{v_n^1\} \) is a \((PS)\) sequence of \( \Phi_\lambda \).

We define
\[
\mu^1 := \limsup_{n \to \infty} \sup_{y \in \mathbb{R}^N} \int_{B_1(y)} |v_n^1|^2.
\]

**Vanishing :** If \( \mu^1 = 0 \), then by Lemma 2.3, one has
\[
v_n^1 \to 0 \quad \text{in } L^r(\mathbb{R}^N) \quad \text{for } 2 < r < 2^*.
\]

By (3.15), for any \( \varepsilon > 0 \), there exists \( C_\varepsilon > 0 \) such that
\[
\hat{H}_\lambda(v_n^1), \hat{H}_\lambda(v_n^1)v_n^1 \leq \varepsilon (|v_n^1|^2 + |v_n^1|^{2^*}) + C_\varepsilon |v_n^1|^r, \quad \text{for all} \quad 2 < r < 2^*.
\]

Thus we have
\[
\int_{\mathbb{R}^N} \hat{H}_\lambda(v_n^1) \to 0 \quad \text{and} \quad \int_{\mathbb{R}^N} \hat{H}_\lambda(v_n^1)v_n^1 \to 0, \quad \text{as } n \to +\infty.
\]

Since \( \{v_n^1\} \) is bounded in \( H^1(\mathbb{R}^N) \). Then we have
\[
\Phi_\lambda(v_n^1) + o_n(1) = \Phi_\lambda(v_n) - \Phi_\lambda(v_0) = \frac{1}{2} \|v_n^1\|^2 - \int_{\mathbb{R}^N} \hat{H}_\lambda(x, v_n^1) - \lambda \frac{4^*-1}{2^*} \|v_n^1\|^{2^*} \tag{3.23}
\]
and
\[
\|v_n^1\|^2 = \lambda 4^{\frac{1}{1-2^*}} \|v_n^1\|^{2^*} + o_n(1).
\]
Let \( \|v_n^1\|^{2^*} = \varrho + o_n(1) \), then we have that \( \|v_n^1\|^2 = \lambda 4^{\frac{1}{1-2^*}} \varrho + o_n(1) \). By the Sobolev inequality, we get
\[
o_n(1) + S \varrho^{2/2^*} = S \left( \int_{\mathbb{R}^N} |v_n^1|^2 \right)^{2/2^*} \leq \|v_n^1\|^2 = \lambda 4^{\frac{1}{1-2^*}} \varrho + o_n(1).
\]
If \( \varrho > 0 \), then we have that
\[
\varrho \geq \left( \frac{1}{\lambda 4^{\frac{1}{1-2^*}}} \right)^{N/2} S^{N/2}
\]
From Step 1, one has
\[
c_\lambda \geq \Psi_\lambda(v_n) - \Phi_\lambda(v_0) + o_n(1) = \left( \frac{1}{2} - \frac{1}{2^*} \right) \lambda 4^{\frac{1}{1-2^*}} \varrho \geq \frac{\frac{N}{2} \lambda^{\frac{N}{2^*}}}{S^{N/2}} ,
\]
which is contradiction. Thus \( \varrho = 0 \). Therefore, \( v_n \to v_0 \) in \( H^1(\mathbb{R}^N) \) and Lemma 3.7 holds with \( l = 0 \).

**Non-Vanishing :** If \( \mu^1 > 0 \), there exists a sequence \( \{\eta_n\} \subset \mathbb{R}^N \) such that
\[
\int_{B_1(\eta_n^1)} |v_n^1|^2 \geq \frac{H_1^1}{2} > 0. \tag{3.24}
\]
Set \( w_n^1 = v_n^1(\cdot + \eta_n^1) \). Then \( \{w_n^1\} \) is bounded in \( H^1(\mathbb{R}^N) \) and we assume that \( w_n^1 \to w_0^1 \) in \( H^1(\mathbb{R}^N) \). By (3.24), we know that \( w_0^1 \neq 0 \). Moreover, \( v_n^1 \to 0 \) in \( H^1(\mathbb{R}^N) \), which shows that \( \{v_n^1\} \) is unbounded. Hence we may assume that \( |\eta_n| \to +\infty \). In addition, it is easy to check that \( \Phi_\lambda(w_0^1) = 0 \).
Step 2. Set $v_n^2 = v_n - v_0 - w_0^1(-\eta_n^1)$. Similar to the argument in Step 1, we can check that

(I) $\|v_n^2\|^2 = \|v_n\|^2 - \|v_0\|^2 - \|w_0^1\|^2 + o_n(1)$,

(II) $\|v_n^2\|^2 - \|v_0\|^2 - w_0^1\|v_n^2\|^2 + o_n(1)$,

(III) $\Phi_\lambda(v_n) - \Phi_\lambda(v_0) - \Phi_\lambda(w_0^1) = \Phi_\lambda(v_n^2) + o_n(1)$,

(IV) $\Phi_\lambda(v_n^2) \to 0$ in $H^{-1}(\mathbb{R}^N)$.

Let

$$\mu^2 = \limsup_{n \to \infty} \sup_{y \in \mathbb{R}^N} \int_{B_1(y)} |v_n^2|^2.$$ 

If vanishing occurs, then $\|v_n^2\|^2 \to 0$, that is, $v_n - v_0 - w_0^1(-\eta_n^1) \to 0$ in $H^1(\mathbb{R}^N)$. Moreover, by (III), we see that $\Phi_\lambda(v_n) = \Phi_\lambda(v_0) + \Phi_\lambda(w_0^1) + o_n(1)$ and Lemma 3.7 holds with $l = 1$.

If non-vanishing occurs, then there exists a sequence $\{\eta_n^2\} \subset \mathbb{R}^N$ and a nontrivial $w_0^2 \in H^1(\mathbb{R}^N)$ such that $w_n^2 = v_n^2(-\eta_n^2) \to w_0^2$ in $H^1(\mathbb{R}^N)$. Thus by (IV), we know that $\Phi_\lambda'(w_0^2) = 0$. Moreover, $v_n^2 \to 0$ in $H^1(\mathbb{R}^N)$, which shows that $|\eta_n^2| \to +\infty$ and $|\eta_n^2 - \eta_n^1| \to +\infty$.

Finally, by iteration, we can finish the proof.

Lemma 3.8. Suppose that $V(x) \equiv 1$ hold. Let $\lambda \in [1/2, 1]$ be fixed and $\{v_n\} \subset H^1(\mathbb{R}^N)$ be a bounded (PS)$_{c_\lambda}$ sequence of the energy functional $\Phi_\lambda$ with $c_\lambda < \frac{S^{N/2}}{2\sqrt{\lambda}}$. Then there exists a subsequence of $\{v_n\}$, still denote by $\{v_n\}$, such that $v_n \to v_\lambda \neq 0$ in $H^1(\mathbb{R}^N)$ with $\Psi_\lambda'(v_\lambda) = 0$ and $\Phi_\lambda(v_\lambda) = c_\lambda$.

Proof. By the boundedness of $\{v_n\}$, there exists $v_\lambda \in H^1(\mathbb{R}^N)$ such that $v_n \rightharpoonup v_\lambda$ in $H^1(\mathbb{R}^N)$. From Remark 3.5, we assume that $v_n \geq 0$ in $H^1(\mathbb{R}^N)$. If $v_\lambda \neq 0$, then the have finished. Otherwise, we may suppose that $v_n \to 0$ in $H^1(\mathbb{R}^N)$. Next, we will show that there exists $\delta > 0$ such that

$$\lim_{n \to \infty} \sup_{y \in \mathbb{R}^N} \int_{B_1(y)} |v_n|^2 \geq \delta > 0. \quad (3.25)$$

If (3.25) does not hold, then by Lemma 2.3, we know that $v_n \to 0$ in $L^r(\mathbb{R}^N)$ for any $r \in (2, 2^*)$. Similar to (3.22), we have

$$\int_{\mathbb{R}^N} \tilde{\mathcal{H}}_\lambda(v_n) \to 0 \quad \text{and} \quad \int_{\mathbb{R}^N} \tilde{h}_\lambda(v_n)v_n \to 0, \quad \text{as} \quad n \to +\infty.$$

By (3.13) and $\Phi_\lambda(v_n) \to c_\lambda$ and $\Phi_\lambda'(v_n) \to 0$, one has

$$c_\lambda + o_n(1) = \frac{1}{2} \|v_n\|^2 - \lambda \frac{4^{\frac{1}{2}}}{{2^*}} \|v_n\|^2 - \|v_n\|^2, \quad (3.26)$$

and

$$\|v_n\|^2 = \lambda \frac{4^{\frac{1}{2}}}{{2^*}} \|v_n\|^2 + o_n(1). \quad (3.27)$$

Assume that $\|v_n\| \to \sigma$ for some $\sigma > 0$, by Sobolev embedding, then we know that $\sigma \geq \frac{S^{N/2}}{2\sqrt{\lambda}}$, which together with (3.26) and (3.27), it follows that $c_\lambda \geq \frac{S^{N/2}}{2\sqrt{\lambda}}$. This is a contradiction. Thus we can infer that (3.25) holds. Then there exists a sequence $\{y_n\} \subset \mathbb{R}^N$ such that $|y_n| \to +\infty$ and

$$\int_{B_1(y_n)} |v_n|^2 \geq \delta > 0.$$
Set \( w_n = v_n(y_n) \). Thus we get \( \Phi_\lambda(w_n) \to c_\lambda \) and \( \Phi'_\lambda(w_n) \to 0 \). By (3.25), we deduce that \( w_n \to w_\lambda \neq 0 \) in \( H^1(\mathbb{R}^N) \) and \( \Phi'_\lambda(w_\lambda) = 0 \). It is easy to check that \( w_\lambda \geq 0 \) in \( \mathbb{R}^N \) and due to the fact \( w_\lambda \neq 0 \), we get \( w_\lambda > 0 \) in \( \mathbb{R}^N \).

**Proof of Theorem 1.1.** By Lemma 3.8, for a.e. \( \lambda \in [1/2,1] \), there exists \( v_\lambda \in H^1(\mathbb{R}^N) \) such that \( v_n \geq 0, v_n \to v_\lambda > 0 \) in \( H^1(\mathbb{R}^N) \), \( \Phi_\lambda(v_\lambda) \to c_\lambda < \frac{S^{N/2}}{2N^{N/2}} \) and \( \Phi'_\lambda(v_\lambda) \to 0 \). Using Lemma 3.7, we have that

\[
c_\lambda = \Phi_\lambda(v_\lambda) + o_n(1) = \Phi_\lambda(v_\lambda) + \sum_{k=1}^l \Phi_\lambda(\omega^k),
\]

(3.28)

\( \Phi'_\lambda(v_\lambda) = 0 \) and \( \Phi'_\lambda(\omega^k) = 0 \) for \( 1 \leq k \leq l \). It follows from Lemma 3.1 and the definition of \( \Phi_\lambda \) that

\[
\Phi_\lambda(v) = \frac{1}{N} \int_{\mathbb{R}^N} |\nabla v|^2 \geq 0.
\]

(3.29)

Thus we deduce that \( \Phi_\lambda(v_\lambda) > 0 \) and \( \Phi_\lambda(\omega^k) \geq 0 \) for \( 1 \leq k \leq l \). By (3.28), we have \( c_\lambda \geq \Phi_\lambda(v_\lambda) > 0 \). Thus there exists \( \{\lambda_n\} \subset [1/2,1] \) such that \( \lambda_n \to 1 \), \( v_{\lambda_n} \in H^1(\mathbb{R}^N) \), \( v_{\lambda_n} > 0 \), \( \Phi'_{\lambda_n}(v_{\lambda_n}) = 0 \) and \( 0 < \Phi_{\lambda_n}(v_{\lambda_n}) \leq c_{\lambda_n} < \frac{S^{N/2}}{2N^{N/2}} \). From (3.29), we get

\[
c_{1/2} \geq c_{\lambda_n} \geq \Phi_{\lambda_n}(v_{\lambda_n}) = \frac{1}{N} \int_{\mathbb{R}^N} |\nabla v_{\lambda_n}|^2 > 0.
\]

Next, by the Sobolev embedding inequality, we have that \( \int_{\mathbb{R}^N} |\nabla v_{\lambda_n}|^2 \leq C \) and \( \|v_{\lambda_n}\|_2 \leq C \) for all \( n \in \mathbb{N} \). Combine (g1)-(g3), Lemma 2.1-(8) with Lemma 3.1, we deduce that for any \( \varepsilon > 0 \), there exists \( C_{\varepsilon} > 0 \) such that

\[
\frac{N-2}{2N} \int_{\mathbb{R}^N} |\nabla v_{\lambda_n}|^2 + \frac{1}{2} \int_{\mathbb{R}^N} f^2(v_{\lambda_n}) = \lambda N \int_{\mathbb{R}^N} G(f(v_{\lambda_n})) \leq \varepsilon \|f(v_{\lambda_n})\|_2^2 + C_{\varepsilon} \|f(v_{\lambda_n})\|_{2^*}^{2^*}.
\]

(3.30)

which implies that

\[
\frac{1}{2} \int_{\mathbb{R}^N} f^2(v_{\lambda_n}) \leq \varepsilon \|f(v_{\lambda_n})\|_2^2 + C_{\varepsilon} C.
\]

Therefore, choosing \( \varepsilon < 1/2 \), we can infer that there exists \( C > 0 \) such that

\[
\int_{\mathbb{R}^N} f^2(v_{\lambda_n}) \leq C.
\]

By Sobolev inequality and Lemma 2.1-(9), we can get

\[
\int_{\mathbb{R}^N} (|\nabla v_{\lambda_n}|^2 + v_{\lambda_n}^2) \leq \int_{\mathbb{R}^N} |\nabla v_{\lambda_n}|^2 + \int_{|v_{\lambda_n}| > 1} |v_{\lambda_n}|^2 + \int_{|v_{\lambda_n}| \leq 1} v_{\lambda_n}^2 \leq C \int_{\mathbb{R}^N} (|\nabla v_{\lambda_n}|^2 + f^2(v_{\lambda_n})) \leq C.
\]

(3.30)

Thus we have that \( \{v_{\lambda_n}\} \) is bounded in \( H^1(\mathbb{R}^N) \). Next, we can assume that the limit of \( \Phi_{\lambda_n}(v_{\lambda_n}) \) exists. By Theorem 2.2, we know that \( \lambda \to c_\lambda \) is continuous from the left. Thus we get

\[
0 \leq \lim_{n \to \infty} \Phi_{\lambda_n}(v_{\lambda_n}) \leq c_1 \leq \frac{S^{N/2}}{2N}.
\]
Then by using the fact that
\[ \Phi(v_{\lambda_n}) = \Phi_{\lambda_n}(v_{\lambda_n}) + (\lambda_n - 1) \int_{\mathbb{R}^N} G(f(v_{\lambda_n})), \]
\[ \langle \Phi'(v_{\lambda_n}), \phi \rangle = \langle \Phi'_{\lambda_n}(v_{\lambda_n}), \phi \rangle + (\lambda_n - 1) \int_{\mathbb{R}^N} g(f(v_{\lambda_n}))f'(v_{\lambda_n})\phi \]
for any \( \phi \in C_0^\infty(\mathbb{R}^N) \) and \( \|v_{\lambda_n}\| \leq C \), it follows that
\[ 0 \leq \lim_{n \to \infty} \Phi(v_{\lambda_n}) \leq c_1 < \frac{S^{N/2}}{2N}. \]

By Remark 3.6, there exists \( \rho > 0 \) independent of \( \lambda_n \) such that \( \|v_{\lambda_n}\| \geq \rho \). Similar to the proof of Lemma 3.8, we can obtain the existence of a positive solution \( v_0 \) for (1.1). By Lemma 3.7, we have
\[ \Phi(0) \leq \lim_{n \to \infty} \Phi(v_{\lambda_n}) \leq c_1 < \frac{S^{N/2}}{2N}. \]

Let
\[ m = \inf \{ \Phi(v) : u \in H^1(\mathbb{R}^N), v \neq 0, \Phi'(v) = 0 \}. \]

Since \( \Phi'(v_0) = 0 \), we have \( m \leq \Phi(v_0) < \frac{S^{N/2}}{2N} \), and by Lemma 3.1, we get \( 0 \leq m < \frac{S^{N/2}}{2N} \).

Next, by the definition of \( m \), we can find \( \{\nu_n\} \subset H^1(\mathbb{R}^N) \) such that \( \Phi(\nu_n) \to m \) and \( \Phi'(\nu_n) \to 0 \). From Remark 3.6, we deduce that \( \|\nu_n\| \geq \rho \), where \( \rho > 0 \) is independent of \( n \). Similar to the proof of (3.30), we deduce that \( \{\nu_n\} \) is bounded in \( H^1(\mathbb{R}^N) \). In virtue of Remark 3.5, we may assume that \( \nu_n \geq 0 \) in \( H^1(\mathbb{R}^N) \). Taking in mind that \( \|\nu_n\| \geq \rho > 0 \), we can proceed as in proof of Lemma 3.8, to show that there exists \( \{\nu_n\} \subset H^1(\mathbb{R}^N) \) such that \( \nu_n \geq 0, \nu_n \to v_0 > 0 \) in \( H^1(\mathbb{R}^N) \), \( \Phi(\nu_n) \to m \) and \( \Phi'(\nu_n) \to 0 \). By Lemma 3.7, we have that \( \Phi(v_0) \leq m \) and \( \Phi'(v_0) = 0 \). In order to complete the proof, we need to prove that \( \Phi(v_0) \geq m \). In fact, since \( \Phi'(v_0) = 0 \), we also get \( \Phi(\nu_n) \geq m \). Thus we have that \( \Phi(v_0) = m \) and \( \Phi'(v_0) = 0 \). The proof is completed. \( \square \)

4. Proof of Theorem 1.2. In this section, we want to prove Theorem 1.2, that is, we prove the existence of ground state solution for (1.1) with the assumptions that \( V(x) \) is not equality to a constant. For this case, we will assume that \( V(x) \neq V_\infty \). From (1.6), for \( \lambda \in [1/2, 1] \), we introduce the following family of functionals by
\[ \Psi_\lambda(v) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla v|^2 + V(x)f^2(v)) - \lambda \int_{\mathbb{R}^N} G(f(v)) \]
and
\[ \Psi_\infty^\lambda(v) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla v|^2 + V_\infty f^2(v)) - \lambda \int_{\mathbb{R}^N} G(f(v)) \]
where \( v \in H^1(\mathbb{R}^N) \).

Lemma 4.1. Let \( v \in E \) be a nontrivial critical point for
\[ \Psi_\lambda^\infty(v) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla v|^2 + V_\infty f^2(v)) - \int_{\mathbb{R}^N} G(f(v)) \]
Then there exists \( \gamma \in C([0,1], H^1(\mathbb{R}^N)) \) such that \( \gamma(0) = 0, \) \( \Psi^\infty(\gamma(1)) < 0, \) and \( \gamma([0,1]), v \in \gamma([0,1]) \) and
\[
\max_{t \in [0,1]} \Psi^\infty(\gamma(t)) = \Psi^\infty(v).
\]

Proof. Let \( v \in H^1(\mathbb{R}^N) \) be a nontrivial critical point of \( \Psi^\infty. \) Set
\[
v'(x) = \begin{cases} v\left(\frac{x}{t}\right), & \text{for } t > 0, \\ 0, & \text{for } t = 0.
\end{cases}
\]
By a directly calculation, it is easy to see that \( v' \) has the following properties:
\[
\begin{align*}
\int_{\mathbb{R}^N} |\nabla v'|^2 &= t^{N-2} \int_{\mathbb{R}^N} |\nabla v|^2, \\
\int_{\mathbb{R}^N} f^2(v') &= t^N \int_{\mathbb{R}^N} f^2(v), \\
\int_{\mathbb{R}^N} G(f(v')) &= t^N \int_{\mathbb{R}^N} G(f(v)).
\end{align*}
\]
By Lemma 3.1, we know
\[
\frac{N-2}{2} \int_{\mathbb{R}^N} |\nabla v|^2 + \frac{N}{2} \int_{\mathbb{R}^N} V_\infty f^2(v) = \lambda N \int_{\mathbb{R}^N} G(f(v)).
\]
and we can see that
\[
\Psi^\infty(v') = \frac{t^{N-2}}{2} \int_{\mathbb{R}^N} |\nabla v|^2 + \frac{t^N}{2} \int_{\mathbb{R}^N} V_\infty f^2(v) + \frac{N-2}{2N} t^N \int_{\mathbb{R}^N} |\nabla v|^2.
\]
Hence we can deduce that \( \max_{t>0} \Psi^\infty(v') = \Psi^\infty(v), \) \( \Psi^\infty(v') \to -\infty \) as \( t \to \infty, \) and
\[
\|v'\|_H^2 = \|\nabla v'|^2 + \|v'|^2 = t^{N-2} \|\nabla v|^2 + t^N \|v|^2 \to 0 \quad \text{as } t \to 0.
\]
Choose \( \alpha > 1 \) such that \( \Psi^\infty(v'^\alpha) < 0 \) and set \( \gamma(t) = v'^\alpha \) for \( t \in (0,1] \) and \( \gamma(0) = 0, \) we obtain the desired \( \gamma. \) This completes the proof. \( \square \)

Remark 4.2. By Theorem 1.1, we know that \( \Psi^\infty_\lambda(v) \) has a ground state solution.

Lemma 4.3. Assume that (\( g_1 \))-(\( g_3 \)) and (\( V_1 \))-(\( V_3 \)) are satisfied. Then for a.e. \( \lambda \in [1/2,1], \) \( \Psi_\lambda \) has a positive critical point.

Proof. By Lemma 3.2 and Remark 3.5, we may assume that for a.e. \( \lambda \in [1/2,1], \) there exists \( \{v_n\} \subset H^1(\mathbb{R}^N) \) such that \( v_n \geq 0 \) in \( H^1(\mathbb{R}^N), \)
\[
\Psi_\lambda(v_n) \to c_\lambda < \frac{S^{N/2}}{2N\lambda^{N/2}}
\]
and
\[
\Psi'_\lambda(v_n) \to 0.
\]
Next, we prove that \( v_\lambda \neq 0. \) In fact, if \( v_\lambda = 0, \) then by the proof of Lemma 3.8, there exists a sequence \( \{y_n\} \subset H^1(\mathbb{R}^N) \) such that \( |y_n| \to +\infty \) and \( w_n = v_n(-y_n) \to w_\lambda \neq 0 \) in \( H^1(\mathbb{R}^N). \) Moreover, by \( v_n \to 0 \) in \( H^1(\mathbb{R}^N), \) we can get that
\[
\Psi^\infty_\lambda(v_n) \to c_\lambda \text{ and } \Psi^\infty_\lambda(w_n) \to 0.
\]
Thus we have \( \Psi^\infty_\lambda(v_n) \to c_\lambda \) and \( \Psi^\infty_\lambda(w_n) \to 0. \)
Since \( w_n \to w_\lambda \neq 0 \) in \( H^1(\mathbb{R}^N), \) it follows that \( \Psi^\infty_\lambda(v_\lambda) = 0. \) In view of Lemma 3.7, we have \( \Psi^\infty_\lambda(w_\lambda) \leq c_\lambda. \) By Remark 4.2, we know that \( \Psi^\infty_\lambda \) has a ground state \( v_\lambda. \) Thus \( \Psi^\infty_\lambda(\theta_\lambda) \leq c_\lambda. \) By Lemma 4.1, we can find a path \( \gamma \in C([0,1], H^1(\mathbb{R}^N)) \) such that \( \gamma(0) = 0, \) \( \Psi^\infty_\lambda(\gamma(1)) < 0, \) \( \theta_\lambda \in \gamma([0,1]) \) and
\[
\max_{t \in [0,1]} \Psi^\infty_\lambda(\gamma(t)) = \Psi^\infty_\lambda(\theta_\lambda).
\]
Thus we have
\[ \max_{t \in [0,1]} \Psi_{\lambda}^\infty(\gamma(t)) = \Psi_{\lambda}^\infty(\theta_{\lambda}) \leq c_\lambda. \]

From (V2), \( V(x) \neq V_\infty \) and \( 0 \not\in \gamma([0,1]) \) we have that \( \Psi_{\lambda}(\gamma(t)) < \Psi_{\lambda}^\infty(\gamma(t)) \) for all \( t \in (0,1) \). Next, we choose \( v_1 = 0 \) and \( v_2 = \gamma(1) \) in Theorem 2.2. Thus by the definition of \( c_\lambda \), we deduce that
\[ c_\lambda \leq \max_{t \in [0,1]} \Psi_{\lambda}(\gamma(t)) < \max_{t \in [0,1]} \Psi_{\lambda}^\infty(\gamma(t)) \leq c_\lambda, \]
which is a contradiction. Thus \( v_\lambda \neq 0 \), and by applying the stronger maximum principle, we have \( v_\lambda > 0 \). The proof is completed.

**Lemma 4.4.** Assume that \((g_1)-(g_3)\) and \((V_1)-(V_3)\) are satisfied. Then for a.e. \( \lambda \in [1/2,1] \), if \( \{v_n\} \subset H^1(\mathbb{R}^N) \) is a bounded sequence and \( v_n \geq 0 \) in \( H^1(\mathbb{R}^N) \) and \( \Psi_{\lambda}(v_n) \to c_\lambda < \frac{S^{N/2}}{2N} \) and \( \Psi'_{\lambda}(v_n) \to 0 \), then there exists a subsequence \( \{v_n\} \), still denoted by \( \{v_n\} \), such that

(i) \( v_n \rightharpoonup v_\lambda \) in \( H^1(\mathbb{R}^N) \) and \( \Psi'_{\lambda}(v_\lambda) = 0 \),

(ii) \( \Psi_{\lambda}(v_\lambda) \leq c_\lambda \).

**Proof.** Since \( \{v_n\} \) is bounded in \( H^1(\mathbb{R}^N) \), we can assume that \( v_n \rightharpoonup v_\lambda \). By the proof of Lemma 3.7 and using (V2), we can easily prove that \( \Psi'_{\lambda}(v_\lambda) = 0 \) and \( \Psi_{\lambda}(v_\lambda) \to c_\lambda \). Thus (i) holds.

Next, by Brezis-Lieb Lemma and Lemma 2.6, we have that the following hold:

1. \( \|v_n^1\|^2_2 = \|v_\lambda\|^2_2 - \|v_0\|^2_2 + o_n(1) \),
2. \( \|v_n^1\|^2_2 = \|v_\lambda\|^2_2 - \|v_0\|^2_2 + o_n(1) \),
3. \( \Psi_{\lambda}(v_n) - \Psi_{\lambda}(v_0) = \Psi_{\lambda}^\infty(v_n) + o_n(1) \),
4. \( \Psi_{\lambda}^\infty(v_n) \to 0 \) in \( H^{-1}(\mathbb{R}^N) \).

It is a standard to prove (1) and (2) via Brezis-Lieb Lemma. Let \( h(t) = g(t) - t^{2^* - 1} \).

Then we can get the following energy functional
\[ \Psi_{\lambda}(v) = \frac{1}{2} \int_{\mathbb{R}^N} (\|\nabla v\|^2 + V(x)f^2(v)) - \lambda \int_{\mathbb{R}^N} H(f(v)) - \frac{\lambda}{2^{2^*}} \int_{\mathbb{R}^N} |f(v)|^{2^{2^*}} \quad (4.31) \]
and
\[ \Psi_{\lambda}^\infty(v) = \frac{1}{2} \int_{\mathbb{R}^N} (\|\nabla v\|^2 + V_\infty f^2(v)) - \lambda \int_{\mathbb{R}^N} H(f(v)) - \frac{\lambda}{2^{2^*}} \int_{\mathbb{R}^N} |f(v)|^{2^{2^*}}, \]
where \( H(t) = \int_0^t h(s) ds \).

Next, if we set
\[ \tilde{h}_{\lambda}(x,v) = \lambda h(f(v)) f'(v) + \lambda |f(v)|^{2^{2^*} - 2} f(v) f'(v) - V(x) f(v) f'(v) + v - \lambda 4^{\frac{1}{2^{*}}} |v|^{2^{*} - 2} v \]
and
\[ \tilde{H}_{\lambda}(x,v) = \lambda H(f(v)) + \frac{\lambda}{2^{2^*}} |f(v)|^{2^{2^*}} + \frac{1}{2} V(x) |f(v)|^2 + \frac{1}{2} v^2 - \lambda 4^{\frac{1}{2^{*}}} |v|^{2^{*}}, \]
then we write Eq. (1.1) in the following form
\[ -\Delta v + V(x)v = \tilde{h}_{\lambda}(x,v) + \lambda 4^{\frac{1}{2^{*}}} |v|^{2^{*} - 2} v \]
we get
\[ \int (|\nabla v|^2 + V(x)v^2) - \lambda \int \tilde{H}_\lambda(x,v) - \frac{\lambda}{2^*} \frac{4}{s} \int |v|^{2^*}. \] (4.32)
and
\[ \Psi_\lambda(v^n) = \frac{1}{2} \int \tilde{H}_\lambda(x,v^n) - \frac{\lambda}{2^*} \frac{4}{s} \int |v|^{2^*}. \] (4.33)

By (4.33) and similar to the proof of (3.17) in Lemma 3.7, we have that
\[ \Psi_\lambda(v^n) = \Psi_\lambda(v_1^n) + o_n(1) = \Psi_\lambda^\infty(v_1^n) + o_n(1), \] (4.36)

Thus it follows from (4.31) to (4.39) and the fact \( v_1^n \to 0 \) in \( H^1(\mathbb{R}^N) \), we have that
\[ \langle \Psi_\lambda'(v_1^n), \psi \rangle + o_n(1) = \langle \Psi_\lambda'(v_1), \psi \rangle + o_n(1) = o_n(1), \] (4.40)

which shows that
\[ \Psi_\lambda^\prime(v_1^n) = o_n(1). \] (4.41)

It follows from (4.34), (4.36), (4.40) and (4.41) that
\[ c_\lambda - \Psi_\lambda(v_1^n) = \Psi_\lambda^\infty(v_1^n) = o_n(1), \] (4.42)

which shows that
\[ \Psi_\lambda^\infty(v_1^n) = o_n(1). \] (4.43)

Thus (4) holds.

Now, we need to distinguish the following two cases:

**Case 1.** we assume that
\[ \lim_{n \to \infty} \sup_{y \in \mathbb{R}^N} \int_{B_1(y)} |v_1^n|^2 = 0. \]
By Lemma 2.3, we can know that $v_n^1 \to 0$ in $L^r(\mathbb{R}^N)$ for any $2 < r < 2^*$. Let
\[
\Psi_\lambda(v) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla v|^2 + V(x)v^2) - \frac{\lambda}{2^*} v^2 \mu_{\frac{N}{2^*}}^2 \int_{\mathbb{R}^N} |v|^2^*
\]
and
\[
\Psi_\infty(v) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla v|^2 + V_\infty v^2) - \frac{\lambda}{2^*} 4^\frac{N}{2^*} \int_{\mathbb{R}^N} |v|^2^*.
\]
Thus by (4.44) and (4.42)-(4.43), we know that
\[
c_\lambda - \Psi_\lambda(v_\lambda) = \Psi_\infty(v_n^1) + o_n(1) \tag{4.44}
\]
and
\[
\Psi_\lambda(v_n^1) = o_n(1). \tag{4.45}
\]
Thus it follows from (4.44) and (4.45) that
\[
c_\lambda - \Psi_\lambda(v_\lambda) = \frac{1}{2} \int_{\mathbb{R}^N} (V_\infty - V(x))|v_n^1|^2 + \left(\frac{1}{2} - \frac{1}{2^*}\right) \int_{\mathbb{R}^N} |v_n^1|^2^* + o_n(1) \geq 0,
\]
and so $\Psi_\lambda(v_\lambda) \leq c_\lambda$.

**Case 2.** we assume that $\limsup_{n \to \infty} \sup_{y \in \mathbb{R}^N} \int_{B_1(y)} |v_n^1|^2 \geq \delta_1$ for some $\delta_1 > 0$. Thus there exists $\{y_n^1\} \subset \mathbb{R}^N$, $|y_n^1| \to \infty$ such that
\[
\int_{B_1(y_n^1)} |v_n^1|^2 \geq \frac{\delta_1}{2}.
\]
As a consequence, we know that $v_n^1(y_n^1 + y_n^1) \to v_\lambda^1 \neq 0$ in $H^1(\mathbb{R}^N)$,
\[
c_\lambda - \Psi_\lambda(v_\lambda) = \Psi_\infty(v_n^1(y_n^1 + y_n^1)) + o_n(1) \tag{4.46}
\]
and
\[
\Psi_\infty(v_n^1(y_n^1 + y_n^1)) = o_n(1). \tag{4.47}
\]
By (4.47), we have $\Psi_\infty(v_n^1) = 0$. Now, if $c_\lambda - \Psi_\lambda(v_\lambda) < \frac{\delta_1}{2^*}$, then we can proceed as in the proof of Lemma 3.7 to obtain the thesis. Otherwise, let $v_n^2 = v_n^1(y_n^1 + y_n^1) - v_\lambda^1$. Similar to the arguments of (4.42) and (4.43), we can get the following
\[
c_\lambda - \Psi_\lambda(v_\lambda) - \Psi_\infty(v_n^1) = \Psi_\infty(v_n^2) + o_n(1), \tag{4.48}
\]
which shows that
\[
\Psi_\infty(v_n^2) = o_n(1). \tag{4.49}
\]
As Case 1 and Case 2 before, the following two cases occur
\[
\lim_{n \to \infty} \sup_{y \in \mathbb{R}^N} \int_{B_1(y)} |v_n^2|^2 = 0 \tag{4.50}
\]
and
\[
\lim_{n \to \infty} \sup_{y \in \mathbb{R}^N} \int_{B_1(y)} |v_n^1|^2 \geq \delta_2 > 0. \tag{4.51}
\]
Firstly, we assume that (4.50) holds. Thus one has
\[
c_\lambda - \Psi_\lambda(v_\lambda) - \Psi_\infty(v_n^1) \geq 0
\]
and by Lemma 3.1, we get $\Psi_\infty(v_\lambda^1) \geq 0$. Thus we know that $\Psi_\lambda(v_\lambda) \leq c_\lambda$. Secondly, we can assume that (4.51) holds. Repeating the above procedure, we can find
\{v_i^j\} \subset H^1(\mathbb{R}^N), y_i^j \in \mathbb{R}^N, |y_i^j| \to +\infty, i \in \mathbb{N} \text{ such that } v_i^j(\cdot + y_i^j) \to v_\lambda \neq 0 \text{ in } H^1(\mathbb{R}^N), \Psi_\lambda^\infty(v_\lambda) = 0,$

$$c_\lambda - \Psi_\lambda(v_\lambda) - \sum_{i=1}^j \Psi_\lambda^\infty(v_i^j) + o_n(1) = \Psi_\lambda^\infty(v_j^j + 1)$$

and

$$\Psi_\lambda^\infty(v_j^j + 1) = o_n(1),$$

where $v_j^j + 1 = v_j^j(\cdot + y_j^j) - v_\lambda$. Since $\Psi_\lambda^\infty(v_j^j + 1) = 0$, we have

$$\Psi_\lambda^\infty(v_i^j) = \frac{1}{2} \int_{\mathbb{R}^N} (V_N - V(x))|v_i^j|^2 + \left(\frac{1}{2} - \frac{1}{2^*}\right) 4^{\frac{1}{2^*}} \int_{\mathbb{R}^N} |v_i^j|^{2^*} \geq 0. \quad (4.52)$$

Next, we prove that there exists $\sigma > 0$ independent of $i$ such that

$$\int_{\mathbb{R}^N} |\nabla v_i^j|^2 \geq \sigma > 0. \quad (4.53)$$

In fact, by $\Psi_\lambda^\infty(v_i^j) = 0 \lambda \in [1/2, 1]$ and $(g_1)-(g_3)$, we deduce that for $\varepsilon > 0$, there exists $C \varepsilon > 0$ such that

$$\int_{\mathbb{R}^N} (|\nabla v_i^j|^2 + V_N |v_i^j|^2) \leq \varepsilon \int_{\mathbb{R}^N} |f(v_i^j)|^2 + C \varepsilon \int_{\mathbb{R}^N} |f(v_i^j)|^{2^*} \leq \varepsilon \int_{\mathbb{R}^N} |v_i^j|^2 + C \varepsilon \cdot 2^{N/(N-2)} \int_{\mathbb{R}^N} |v_i^j|^2 \leq \frac{\varepsilon}{V_N} \int_{\mathbb{R}^N} V_N |v_i^j|^2 + C \varepsilon \cdot 2^{N/(N-2)} \int_{\mathbb{R}^N} |v_i^j|^2.$$}

Thus we can choose $\varepsilon \in (0, V_N)$ such that

$$\int_{\mathbb{R}^N} (|\nabla v_i^j|^2 + |v_i^j|^2) \leq C \int_{\mathbb{R}^N} |v_i^j|^2. \quad (4.54)$$

By Sobolev inequality and (4.54), one has

$$\int_{\mathbb{R}^N} |\nabla v_i^j|^2 \leq \int_{\mathbb{R}^N} (|\nabla v_i^j|^2 + |v_i^j|^2) \leq C \int_{\mathbb{R}^N} |v_i^j|^2 \leq C \left( \int_{\mathbb{R}^N} |\nabla v_i^j|^2 \right)^{2^*},$$

which shows that (4.53) holds. Therefore, by (4.52), (4.53), at some $j = l$, we have

$$c_\lambda - \Psi_\lambda(v_\lambda) - \sum_{i=1}^l \Psi_\lambda(v_i^j) < \frac{S^{N/2}}{2\lambda^{N/2}}.$$}

By Lemma 3.7, the conclusion is obvious. This completes the proof. \(\square\)

**Proof of Theorem 1.2.** By Lemma 4.2, for a.e. $\lambda \in [1/2, 1]$, there exists $v_\lambda \in H^1(\mathbb{R}^N)$ such that $v_n \to v_\lambda \neq 0$ in $H^1(\mathbb{R}^N), \Phi_\lambda(v_n) \to c_\lambda < \frac{S^{N/2}}{2\lambda^{N/2}}$ and $\Phi_\lambda'(v_n) \to 0$. Using Lemma 4.4, we have $\Phi_\lambda(v_\lambda) \leq c_\lambda$ and $\Phi_\lambda'(v_\lambda) = 0$. Thus there exists $\{\lambda_n\} \subset [1/2, 1]$ such that $\lambda_n \to 1, v_\lambda_n \in H^1(\mathbb{R}^N), \Phi_\lambda'(v_\lambda_n) = 0$ and $\Phi_\lambda(v_\lambda_n) \leq c_\lambda < \frac{S^{N/2}}{2\lambda_n^{N/2}}$. Next, we prove that $\{v_\lambda_n\}$ is bounded in $H^1(\mathbb{R}^N)$. In fact, by $(V_3)$, we can deduce that there exists $\varsigma \in (0, 2)$ such that

$$|\max\{|\nabla V(x), x\}|_{L^\infty} \leq \varsigma.$$}  

(4.55)
From $Φ_{λ_n}(v_{λ_n}) ≤ c_{1/2}$, Lemma 3.1, Lemma 2.1-(8), H"older inequality, Sobolev inequality and (4.55), we deduce that

\[
\int_{\mathbb{R}^N} |\nabla v_{λ_n}|^2 = \frac{N}{2} \int_{\mathbb{R}^N} |\nabla v_{λ_n}|^2 + \frac{N}{2} \int_{\mathbb{R}^N} V(x) f^2(v_{λ_n}) \\
+ \frac{1}{2} \int_{\mathbb{R}^N} (\nabla V(x), x) f^2(v_{λ_n}) - λ_n N \int_{\mathbb{R}^N} G(f(v_{λ_n})) \\
= N Φ_{λ_n}(v_{λ_n}) + \frac{1}{2} \int_{\mathbb{R}^N} (\nabla V(x), x) f^2(v_{λ_n}) \\
≤ N c_{1/2} + \frac{1}{2} \left( \int_{\mathbb{R}^N} |(\nabla V(x), x)|^N \right)^{\frac{2}{N}} \left( \int_{\mathbb{R}^N} |f(v_{λ_n})|^2 \right)^{\frac{N-2}{N-2}} \\
≤ N c_{1/2} + \frac{1}{2} \|S_ε\| \left( \int_{\mathbb{R}^N} |v_{λ_n}|^2 \right)^{\frac{N-2}{2}} \\
≤ N c_{1/2} + \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v_{λ_n}|^2 ,
\]

which implies that there exists $C > 0$ such that $\int_{\mathbb{R}^N} |\nabla v_{λ_n}|^2 ≤ C$ for any $n ∈ \mathbb{N}$.

By $(g_1)-(g_3)$ and Sobolev inequality, we know that for any $ε > 0$, there exists $C_ε > 0$ such that

\[
\frac{V_0}{2} \int_{\mathbb{R}^N} |f(v_{λ_n})|^2 ≤ \int_{\mathbb{R}^N} (|\nabla v_{λ_n}|^2 + V(x) f(v_{λ_n}) f'(v_{λ_n}) v_{λ_n}) \\
≤ \int_{\mathbb{R}^N} g(f(v_{λ_n})) f'(v_{λ_n}) v_{λ_n} \\
≤ \int_{\mathbb{R}^N} g(f(v_{λ_n})) f(v_{λ_n}) \\
≤ ε \int_{\mathbb{R}^N} |f(v_{λ_n})|^2 + C_ε \int_{\mathbb{R}^N} |f(v_{λ_n})|^{2ε} \\
≤ ε \int_{\mathbb{R}^N} |f(v_{λ_n})|^2 + C_ε 2^{N/2} \int_{\mathbb{R}^N} |v_{λ_n}|^2 \\
≤ ε \int_{\mathbb{R}^N} |f(v_{λ_n})|^2 + C_ε \left( \int_{\mathbb{R}^N} |\nabla v_{λ_n}|^2 \right)^{2ε} ,
\]

which implies that

\[
\frac{V_0}{2} \int_{\mathbb{R}^N} f^2(v_{λ_n}) ≤ ε \|f(v_{λ_n})\|^2 + C_ε C.
\]

Therefore, choosing $ε < \frac{V_0}{2}$, we can know that there exists $C > 0$ such that $\int_{\mathbb{R}^N} f^2(v_{λ_n}) ≤ C$. By Sobolev inequality and Lemma 2.1-(9), we can get

\[
\int_{\mathbb{R}^N} (|\nabla v_{λ_n}|^2 + V(x) v_{λ_n}^2) ≤ \max \{V_∞, 1\} \int_{\mathbb{R}^N} (|\nabla v_{λ_n}|^2 + v_{λ_n}^2) \\
≤ C \left( \int_{\mathbb{R}^N} |\nabla v_{λ_n}|^2 + \int_{|v_{λ_n}| > 1} |v_{λ_n}|^2 + \int_{|v_{λ_n}| ≤ 1} v_{λ_n}^2 \right) \\
≤ C \int_{\mathbb{R}^N} (|\nabla v_{λ_n}|^2 + f^2(v_{λ_n})) \\
≤ C.
\]

(4.56)
Thus there exists a constant $C > 0$ independent of $n$ such that
\[ \|v_{\lambda_n}\|^2 = \int_{\mathbb{R}^N} \left(|\nabla v_{\lambda_n}|^2 + V(x)v_{\lambda_n}^2\right) \leq C. \]

Next, we can assume that the limit of $\Phi_{\lambda_n}(v_{\lambda_n})$ exists. By Theorem 2.2, we know that $\lambda \mapsto c_\lambda$ is continuous from the left. Thus we get
\[ 0 \leq \lim_{n \to \infty} \Phi_{\lambda_n}(v_{\lambda_n}) \leq c_1 < \frac{S^{N/2}}{2N}. \]

Then by using the fact that
\[ \Phi(v_{\lambda_n}) = \Phi_{\lambda_n}(v_{\lambda_n}) + (\lambda_n - 1) \int_{\mathbb{R}^N} G(f(v_{\lambda_n})), \]
\[ \langle \Phi'(v_{\lambda_n}), \phi \rangle = \langle \Phi'_{\lambda_n}(v_{\lambda_n}), \phi \rangle + (\lambda_n - 1) \int_{\mathbb{R}^N} g(f(v_{\lambda_n}))f'(v_{\lambda_n})\phi \]
for any $\phi \in C_0^\infty(\mathbb{R}^N)$ and $\|v_{\lambda_n}\| \leq C$, it follows that
\[ 0 \leq \lim_{n \to \infty} \Phi(v_{\lambda_n}) \leq c_1 < \frac{S^{N/2}}{2N}, \]
and
\[ \lim_{n \to \infty} \Phi'(v_{\lambda_n}) = 0. \]

By Remark 3.6, there exists $\varrho > 0$ independent of $\lambda_n$ such that $\|v_{\lambda_n}\| \geq \varrho$. Similar to the proof of Lemma 4.2, we can obtain the existence of a positive solution $v_0$ for (1.1). By Lemma 4.4, we have
\[ \Phi(v_0) \leq \lim_{n \to \infty} \Phi(v_{\lambda_n}) \leq c_1 < \frac{S^{N/2}}{2N}. \]

Let
\[ m = \inf\{\Phi(v) : v \in H^1(\mathbb{R}^N), v \neq 0, \Phi'(v) = 0\}. \]

Since $\Phi'(v_0) = 0$, we have $m \leq \Phi(v_0) < \frac{S^{N/2}}{2N}$.

Next, from the definition of $m$, there exists a sequence $\{\nu_n\} \subset H^1(\mathbb{R}^N)$ such that $\Phi(\nu_n) \to m$ and $\Phi'(\nu_n) \to 0$. From Remark 3.6, we deduce that $\|\nu_n\| \geq \varrho$, where $\varrho > 0$ is independent of $n$. Similar to the proof of (3.30), we deduce that $\{\nu_n\}$ is bounded in $H^1(\mathbb{R}^N)$. Taking in mind that $\|\nu_n\| \geq \varrho > 0$, we can proceed as in proof of Lemma 4.2, to show that there exists $\{\nu_n\} \subset H^1(\mathbb{R}^N)$ such that $\nu_n \to v_0 \neq 0$ in $H^1(\mathbb{R}^N)$, $\Phi(\nu_n) \to m$ and $\Phi'(\nu_n) \to 0$. By Lemma 4.4, we have that $\Phi(\nu_0) \leq m$ and $\Phi'(\nu_0) = 0$. In order to complete the proof, we need to prove that $\Phi(\nu_0) \geq m$. In fact, since $\Phi'(\nu_0) = 0$, we also get $\Phi(\nu_0) \geq m$. Thus we have that $\Phi(\nu_0) = m$ and $\Phi'(\nu_0) = 0$. The proof is completed. \qed

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REFERENCES

[1] V. Ambrosio and G. M. Figueiredo, Ground state solutions for a fractional Schrödinger equation with critical growth, *Asymptotic Anal.*, 105 (2017), 159–191.
[2] F. G. Bass and N. N. Nasanov, Nonlinear electromagnetic-spin waves, *Phys. Rep.*, 180 (1990), 165–223.
[3] H. Berestycki and P. L. Lions, Nonlinear scalar field equations. I. Existence of a ground state, *Arch. Rational Mech. Anal.*, 82 (1983), 313–345.
[4] J. H. Chen, X. H. Tang and B. T. Cheng, Non-Nehari manifold method for a class of generalized quasilinear Schrödinger equations, *Appl. Math. Lett.*, 74 (2017), 20–26.
[5] J. H. Chen, X. H. Tang and B. T. Cheng, Ground states for a class of generalized quasilinear Schrödinger equations in $\mathbb{R}^N$, *Mediterr. J. Math.*, 14 (2017), 190.

[6] J. H. Chen, X. H. Tang and B. T. Cheng, Existence of ground state solutions for quasilinear Schrödinger equations with super-quadratic condition, *Appl. Math. Lett.*, 79 (2018), 27–33.

[7] J. H. Chen, X. H. Tang and B. T. Cheng, Ground state solutions for a class of quasilinear Schrödinger equations via Pohožaev manifold, Submitted.

[8] S. T. Chen, X. H. Tang, Improved results for Klein-Gordon-Maxwell systems with general nonlinearity, *Discrete Contin. Dyn. Syst. A.*, 38 (2018), 2333–2348.

[9] M. Colin and L. Jeanjean, Solutions for a quasilinear Schrödinger equation: a dual approach, *Nonlinear Anal.*, 56 (2004), 213–226.

[10] S. Cuccagna, On instability of excited states of the nonlinear Schrödinger equation, *Phys. D*, 238 (2009), 38–54.

[11] Y. Deng and W. Huang, Ground state solutions for a quasilinear Elliptic equations with critical growth, *Discrete Contin. Dyn. Syst. A.*, 37 (2017), 4213–4230.

[12] Y. Deng, S. Peng and J. Wang, Nodal soliton solutions for quasilinear Schrödinger equations with critical exponent, *J. Math. Phys.*, 54 (2013), 011504.

[13] Y. Deng, S. Peng and S. Yan, Positive soliton solutions for generalized quasilinear Schrödinger equations with critical growth, *J. Differential Equations*, 258 (2015), 115–147.

[14] Y. Deng, S. Peng and S. Yan, Critical exponents and solitary wave solutions for generalized quasilinear Schrödinger equations, *J. Differential Equations*, 260 (2016), 1228–1262.

[15] J. M. do Ó, O. Miyagaki and S. Soares, Soliton solutions for quasilinear Schrödinger equations with critical growth, *J. Differential Equations*, 248 (2010), 712–744.

[16] J. M. do Ó and U. Severo, Solitary waves for a class of quasilinear Schrödinger equations in dimension two, *Calc. Var. Partial Differential Equations*, 38 (2010), 275–315.

[17] D. Gilbarg and N. Trudinger, *Elliptic Partial Differential Equations of Second Order*, 2nd ed. Berlin: Springer, 1983.

[18] X. He, A. Qian and W. Zou, Existence and concentration of positive solutions for quasilinear equations with critical growth, *Nonlinearity*, 26 (2013), 3137–3168.

[19] L. Jeanjean, On the existence of bounded Palais-Smale sequences and application to a Landesman-Lazer type problem set on $\mathbb{R}^N$, *Proc. R. Soc. Edinburgh Sect A.*, 129 (1999), 787–809.

[20] L. Jeanjean and K. Tanaka, A positive solution for an asymptotically linear elliptic problem on $\mathbb{R}^N$ autonomous at infinity, *ESAIM Control Optim. Calc. Var.*, 7 (2002), 597–614.

[21] S. Kurihara, Large-amplitude quasi-solitons in superfluid films, *J. Phys. Soc. Japan*, 50 (1981), 3262–3267.

[22] J. Q. Liu, Y. Wang and Z. Q. Wang, Solutions for quasilinear Schrödinger equations, II, *J. Differential Equations*, 187 (2003), 473–493.

[23] J. Q. Liu, Y. Wang and Z. Q. Wang, Solutions for quasilinear Schrödinger equations via the Nehari method, *Comm. Partial Differential Equations*, 29 (2004), 879–901.

[24] J. Q. Liu and Z. Q. Wang, Soliton solutions for quasilinear Schrödinger equations, *Proc. Amer. Math. Soc.*, 131 (2002), 441–448.

[25] X. Q. Liu, J. Q. Liu and Z. Q. Wang, Ground states for quasilinear Schrödinger equations with critical growth, *Calc. Var. Partial Differential Equations*, 46 (2013), 641–669.

[26] X. Q. Liu, J. Q. Liu and Z. Q. Wang, Quasilinear elliptic equations with critical growth via perturbation method, *J. Differential Equations*, 254 (2013), 102–124.

[27] Z. Liu and S. Guo, On ground state solutions for the Schrödinger-Poisson equations with critical growth, *J. Math. Anal. Appl.*, 412 (2014), 435–448.

[28] V. G. Makhankov and V. K. Fedyanin, Nonlinear effects in quasi-one-dimensional models and condensed matter theory, *Phys. Rep.*, 104 (1984), 1–86.

[29] A. Moameni, Existence of soliton solutions for a quasilinear Schrödinger equation involving critical exponent in $\mathbb{R}^N$, *J. Differential Equations*, 229 (2006), 570–587.

[30] A. Moameni, On the existence of standing wave solutions to quasilinear Schrödinger equations, *Nonlinearity*, 19 (2006), 937–957.

[31] M. Poppenberg, K. Schmitt and Z. Q. Wang, On the existence of soliton solutions to quasilinear Schrödinger equation, *Calc. Var. Partial Differential Equations*, 14 (2002), 329–344.

[32] Y. Shen and Y. Wang, Soliton solutions for generalized quasilinear Schrödinger equations, *Nonlinear Anal. Theory Methods Appl.*, 80 (2013), 194–201.

[33] E. A. Silva and G. F. Vieira, Quasilinear asymptotically periodic Schrödinger equations with critical growth, *Calc. Var. Partial Differential Equations*, 39 (2010), 1–33.
[34] X. H. Tang and S. T. Chen, Ground state solutions of Nehari-Pohožaev type for Schrödinger Poisson problems with general potentials, *Discrete Contin. Dyn. Syst. A*, 37 (2017), 4973–5002.

[35] X. H. Tang, X. Y. Lin and J. S. Yu, Nontrivial solutions for Schrödinger equation with local super-quadratic conditions, *J. Dyn. Differ. Equ.*, (2018), 1–15.

[36] Y. Wang and W. Zou, Bound states to critical quasilinear Schrödinger equations, *Nonlinear Differ. Equ. Appl.*, 19 (2012), 19–47.

[37] M. Willem, *Minimax Theorems*, Birkhäuser, Berlin, 1996.

[38] H. Ye and G. Li, Concentrating solition solutions for quasilinear Schrödinger equations involving critical Sobolev exponents, *Discrete Contin. Dyn. Syst. A*, 36 (2016), 731–762.

[39] J. Zhang, W. Zhang and X. H. Tang, Ground state solutions for Hamiltonian elliptic system with inverse square potential, *Discrete Contin. Dyn. Syst. A*, 37 (2017), 4565–4583.

[40] J. Zhang and W. Zou, The critical case for a Berestycki-Lions theorem, *Sci. China Math.*, 14 (2014), 541–554.

[41] X. Zhu and D. Cao, The concentration-compactness principle in nonlinear elliptic equations, *Acta Math. Sci.*, 9 (1989), 307–328.

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