Universal relation between Green’s functions in random matrix theory

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Abstract

We prove that in random matrix theory there exists a universal relation between the one-point Green’s function $G$ and the connected two-point Green’s function $G_c$ given by

$$N^2 G_c(z, w) = \frac{\partial^2}{\partial z \partial w} \log \left( \frac{G(z) - G(w)}{z - w} \right) + \text{irrelevant factorized terms}.$$ 

This relation is universal in the sense that it does not depend on the probability distribution of the random matrices for a broad class of distributions, even though $G$ is known to depend on the probability distribution in detail. The universality discussed here represents a different statement than the universality we discovered a couple of
years ago, which states that $a^2G_c(az, aw)$ is independent of the probability distribution, where $a$ denotes the width of the spectrum and depends sensitively on the probability distribution. It is shown that the universality proved here also holds for the more general problem of a Hamiltonian consisting of the sum of a deterministic term and a random term analyzed perturbatively by Brézin, Hikami, and Zee.
1 Introduction

The theory of large random matrices [5, 6, 7, 27] has been extensively developed over the years. Recently, in a series of papers [1, 2, 3, 10, 9, 30], we, and with the collaboration of J. D’Anna and of S. Hikami, have studied the correlation between the density of eigenvalues of large random matrices. For physical applications, one may imagine under some circumstances representing the Hamiltonian of a disordered system by a large hermitean random matrix.

We will try to make this paper as self-contained as possible and so we will begin with the necessary definitions. Denote by $\varphi$ an $N$ by $N$ hermitean matrix taken from the probability distribution

$$P(\varphi) = \frac{1}{Z} e^{-N \text{tr} V(\varphi)}$$

(1)

with $Z$ fixed by $\int d\varphi P(\varphi) = 1$. Here $V(\varphi)$ is an arbitrary polynomial of $\varphi$ and the large $N$ limit is understood. A hermitean matrix $\varphi$ is picked with the probability $P(\varphi)$ and we imagine that its eigenvalues are found. When this procedure is repeated a large number of times, we can define a density of eigenvalues, averaged over the matrices in an ensemble defined by the distribution (1). Similarly, the density-density correlation function can also be defined and calculated.

Before we continue with the necessary definitions, let us explain the philosophy underlying our work. We have devoted considerable effort to dis-
covering and elucidating universality properties satisfied by the correlation function, and so it behooves us to say a few words about our motivations. We say a quantity is universal if it does not depend on the probability distribution $P$, or for the specific class of $P$ examined here, on $V$.

While there are notable exceptions of course, much of the literature on random matrix theory we are aware of is concerned with only the Gaussian distribution, namely when $V(\varphi)$ is quadratic in $\varphi$. While this assumption may be plausible, it is important to understand to what extent the results obtained are general. Given a Gaussian distribution, essentially everything can be calculated. In particular, in the orthogonal polynomial approach, the relevant polynomials are just the Hermite polynomials. When asked why they restrict themselves to the Gaussian distribution, many workers in this subject would simply assert that it “suffices.” Others take the attitude that one does what is possible, and indeed if one were to go beyond random matrix theory to study actual disordered systems, it is difficult to treat non-Gaussian disorder.

What we have been trying to do is to see, at least within the confines of random matrix theory, which results are indeed universal and which results are specific to the Gaussian case. This search for universal quantities is made all the more important by the fact that the density of eigenvalues is in fact known to be not universal in the sense used here, that is, the density of eigenvalues depends sensitively on $V$, as will be mentioned below.
We would like to distinguish the universality discussed here from the “short distance” universality quoted in the literature. By short distance, we mean that we look at the correlation function on scales larger than, but comparable to, the spacing between the eigenvalues. On this scale, the physics is essentially controlled by level repulsion, leading to a more or less locally uniform spacing between eigenvalues. Thus, we expect the correlation function to be universal when suitably scaled by the local density (see for example equation (2.19) of [1]). The universality which we have discussed in our work and which we will discuss here is on distance scale large compared with the spacing between the eigenvalues and is thus less evident.

Universality in critical phenomenon is of course well understood by now, essentially because of the diverging correlation length near a second order phase transition. The subject has been placed on a solid foundation by the renormalization group approach. In [1] we suggested that a renormalization group inspired argument may also be operative here, although the details are still not completely clear, particularly since the density of eigenvalues is not universal. Perhaps, using the language of renormalization group, we may suppose that a “relevant operator” is involved.

Now that we have stated some of our motivations let us continue with our definitions. We introduce the Green’s function (or resolvent)

\[
G(z) = \langle \frac{1}{N} tr \frac{1}{z - \varphi} \rangle \equiv \int d\varphi P(\varphi) \frac{1}{N} tr \frac{1}{z - \varphi}
\]

The bracket will henceforth denote averaging with respect to \( P \): for any \( O(\varphi) \)
we write \(< O(\varphi) = \int d\varphi P(\varphi)O(\varphi).\)

The density of eigenvalues of the random matrix \(\varphi\) is then given by \(\rho(\mu) = \langle \frac{1}{N} tr \delta(\mu - \varphi) \rangle = -\frac{1}{\pi} \text{Im} G(\mu + i\epsilon).\) The limit \(N\) tending to infinity is always understood. Notice that in this paper, as in our earlier work, we choose the factors of \(N\) in our various definitions such that the interval over which \(\rho(\mu)\) is non-zero is finite (\(i.e.,\) of order \(N^0\)) in the large \(N\) limit.

The two-point Green’s function is defined by

\[
G(z, w) \equiv \left\langle \frac{1}{N} tr \frac{1}{z - \varphi} \frac{1}{w - \varphi} \right\rangle
\]

In the large \(N\) limit, \(G(z, w) \to G(z)G(w)\) and thus it is customary to define the connected Green’s function defined by \(G_c(z, w) \equiv G(z, w) - G(z)G(w),\) a quantity of order \(1/N^2.\) (Thus, we will be dealing with a quantity often ignored in discussions of large \(N\) expansion.) The connected correlation between the density of eigenvalues is then given by

\[
\rho_c(\mu, \nu) = \left\langle \frac{1}{N} tr \delta(\mu - \varphi) \frac{1}{N} tr \delta(\nu - \varphi) \right\rangle_c
\]

\[
= (-1/4\pi^2)(G_c(++) + G_c(--) - G_c(+-) - G_c(-+))
\]

with the obvious notation \(G_c(\pm, \pm) \equiv G_c(\mu \pm i\epsilon, \nu \pm i\delta)\) (signs uncorrelated).

Almost twenty years ago Brézin, Itzykson, Parisi, and Zuber [28] calculated the one-point Green’s function and found that, as might be expected, it depends on \(V\) in a complicated way. Purely for the sake of completeness, let us record that for \(V(\varphi) = \sum_{k=1}^{p} \frac{1}{2k} g_k \varphi^{2k}\) (we take \(V\) to be an even polynomial
for simplicity) the Green’s function has the form

\[ G(z) = \frac{1}{2}[V'(z) - P(z)\sqrt{z^2 - a^2}] \]  

(5)

where the polynomial

\[ P(z) = \frac{1}{2} \sum_{k=1}^{p} g_k \sum_{n=0}^{k-1} \frac{(2n)!}{(n!)^2 (\frac{a^2}{4})^n} z^{2k-2n-2} \]  

(6)

The endpoint \( a \) of the spectrum is determined by

\[ \frac{1}{2} \sum_{k=1}^{p} g_k \frac{(2k)!}{(k!)^2 (\frac{a^2}{4})^k} = 1 \]  

(7)

The density of eigenvalues is given by

\[ \rho(\mu) = \frac{1}{\pi} P(\mu)\sqrt{a^2 - \mu^2} \]  

(8)

In particular, in the simplest case, with the Gaussian distribution defined by \( V(\varphi) = m^2 \varphi^2/2 \), the one-point Green’s function is given by

\[ G_0(z) = \frac{1}{2}[z - \sqrt{z^2 - 4}] \]  

(9)

and thus the density obeys Wigner’s celebrated semi-circle law

\[ \rho(\mu) = \frac{2}{\pi a^2} \sqrt{a^2 - \mu^2}. \]  

(10)

We will emphatically not need any of these explicit formulas in what follows. We simply want to emphasize to the reader that, not surprisingly, \( G(z) \) and \( \rho(\mu) \) both depend on the potential \( V \) in detail. We say that the one-point Green’s function and the density are not universal.
In [1] we calculated the correlation function $\rho_c(\mu, \nu)$ using the method of orthogonal polynomials and again as might be expected found that it depended in detail on $V$. We will not display the full expression here but simply note that the resulting $\rho_c(\mu, \nu)$ oscillates wildly as a function of $\mu$ and $\nu$ on scales of $1/N$. These oscillations are entirely expected since between $\mu$ and $\nu$ finitely separated (that is, with $\mu - \nu \sim O(1)$) there are in general $O(N)$ eigenvalues. Thus, it is natural to smooth $\rho_c(\mu, \nu)$ by integrating over intervals $\delta \mu$ and $\delta \nu$, large compared to $O(N^{-1})$ but small compared to $O(N^0)$, centered around $\mu$ and $\nu$ respectively. Upon smoothing, the full expression for the correlation function simplified enormously and we obtained

$$\rho_c^\text{smooth}(\mu, \nu) = \frac{-1}{2N^2\pi^2} \frac{1}{(\mu - \nu)^2} \frac{(a^2 - \mu \nu)}{[(a^2 - \mu^2) (a^2 - \nu^2)]^{1/2}}.$$  \tag{11}

We found that, remarkably enough, the smoothed correlation function depended on the polynomial $V$ only through the single quantity $a$, the width of the spectrum. In other words, if we introduce the obvious scaling variables $x = \mu/a$ and $y = \nu/a$ then the correlation function (henceforth we will drop the superscript “smooth”) is equal to

$$\rho_c(\mu, \nu) = \frac{-1}{2N^2\pi^2} \frac{1}{a^2} f(x, y)$$  \tag{12}

with the universal function

$$f(x, y) = \frac{1}{(x-y)^2} \frac{(1-xy)}{[(1-x^2)(1-y^2)]^{1/2}}.$$  \tag{13}

This universality has since been derived by other authors [24, 25, 26] using alternative methods, and verified numerically [21]. It is perhaps useful to
remark here that, while there are notable exceptions of course, much of the literature on random matrix theory, as far as we know, is devoted to the Gaussian case \cite{12,13}. The whole point of our work is that it is possible to go beyond the Gaussian distribution.

In \cite{3} we developed a diagrammatic approach to calculating the connected two-point Green’s function $G_c(z,w)$. We will describe this diagrammatic approach in detail in the next section. Here we will simply outline our result from \cite{3}. (To read the rest of this paper, it is not necessary to have read \cite{3} first.) Using a diagrammatic approach, we find that $G_c(z,w)$ is given by an infinite set of Feynman diagrams. We were able to calculate and sum this infinite set only for the Gaussian case and obtained a relatively simple expression for $G_c(z,w)$ (see equation (2.11) in \cite{3}). Later we recognized\cite{9} that this expression may be written in the elegantly compact form

$$N^2G_{0c}(z,w) = \frac{\partial^2}{\partial z \partial w} \log\left(\frac{G_0(z) - G_0(w)}{z - w}\right)$$

The subscript “0” indicates that the quantities in this equation are all calculated with the Gaussian distribution. Taking the absorptive part of this according to \cite{4} we obtain the smoothed correlation function given in \cite{12}. We explained in \cite{3} that the diagrammatic method “automatically” gives the smoothed correlation function. This is because in the diagrammatic method we calculate the Green’s function $G_c(z,w)$, by first letting $N$ go to infinity to pick out an appropriate set of diagrams, and then letting $z$ and $w$ approach the real axis to extract the correlation function from the absorptive part of
$G_c(z, w)$ according to (4). With $N$ going to infinity, the discrete set of poles of $G_c(z, w)$ on the real axis merges into a cut. This is equivalent to the smoothing procedure employed in [1].

In [3], in contrast to the work we did in [1], we were unable to calculate $G_c(z, w)$ for a general $V$. Indeed, the task of summing up the infinite sets of graphs generated by the interaction terms in $V$ appeared to us at the time enormously complicated and perhaps even hopeless. It is simple enough, however, to summarize the remarkable universality discovered in [1]. The universality expressed in (12) and (13) can be stated in terms of $G_c(z, w)$ by saying that for a general $V$ we have

$$N^2 G_c(z, w) = \frac{\partial^2}{\partial z \partial w} \log \left( \frac{G_0(z/a) - G_0(w/a)}{z - w} \right)$$  \hspace{1cm} (15)$$

It is far from obvious how such a relation can be derived diagrammatically. Here $a$ has a complicated dependence on $V$ as indicated by (7). Furthermore, the Gaussian Green’s function $G_0(z)$ appears on the right hand side. The Gaussian Green’s function $G_0(z)$, in contrast to the Green’s function $G(z)$ appropriate to the general $V$, is not a “natural” object to appear in a calculation of $G_c(z, w)$. Yet, according to the orthogonal polynomial analysis of [1], this relation must be true!

To appreciate how complicated a diagrammatic calculation of $G_c(z, w)$ can get, the reader is invited to look at [9] where together with Hikami we attempted this calculation. We had to restrict ourselves to the $g\varphi^4$ case, and even so, we were able to obtain, after a long and rather involved calculation,
the correlation function $G_c(z, w)$ to only first order in $g$. This calculation, however, was instructive. It turned out that the numerous terms in our final expression for the two-point Green’s function $G_c(z, w)$ can be grouped together in precisely such a way that the rather complicated final expression can be written in terms of the one-point Green’s function $G(z)$. This observation is highly non-trivial in that, as we can see from (5), (6), (7), the Green’s function $G(z)$ to first order in $g$ is already not particularly simple.

Brézin, Hikami, and Zee\cite{9} found that, to first order in $g$,

$$N^2 G_c(z, w) = \frac{\partial^2}{\partial z \partial w} \log\left(\frac{G(z) - G(w)}{z - w}\right) (1 + 4gG(z)G(w))^{-1} + O(g^2) \quad (16)$$

Note that the factor of $(1 + 4gG(z)G(w))$ to this order in $g$ contributes to $G_c(z, w)$ only a factorized term like $h(z)h(w)$ for some function $h$. These factorized terms would not contribute to the connected correlation function $\rho_c(\mu, \nu)$.

It was thus tempting to conjecture that for an arbitrary $V(\varphi)$ the connected two-point Green’s function can be written as

$$N^2 G_c(z, w) = \frac{\partial^2}{\partial z \partial w} \log\left(\frac{G(z) - G(w)}{z - w}\right) + \text{irrelevant factorized terms} \quad (17)$$

For the Gaussian case, the two expressions in (15) and (17) are manifestly the same. For a general $V$, however, the equality of these two expressions is far from evident.

In a recent paper \cite{8} we adopted a slightly different philosophy: instead of calculating $G(z)$ and $G_c(z, w)$ in terms of $V$ and then trying to express
$G_c(z, w)$ in terms of $G(z)$ and $G(w)$ by eliminating the dependence on $V$, we attempted to calculate $G_c(z, w)$ directly in terms of $G(z)$ and $G(w)$, appealing to $V$ only for the general structure of the Feynman diagrams. This shift in philosophy is reminiscent of the dispersion approach in particle physics in the 1950’s: instead of trying to calculate various physical quantities in terms of an underlying Lagrangian, particle physicists of that era attempted to relate various physical quantities to each other. In this paper, we will exploit this philosophy to prove the conjecture in (17). In line with this philosophy, we will keep the amount of explicit calculation to a minimum. Instead, we will organize the relevant diagrams in such a way as to obtain structural relations between different Green’s functions.

2 Diagrams

Let us review the diagrammatic approach discussed in [3]. We may regard the distribution (1) as defining a $(0 + 0)$-dimensional field theory. In this context the Feynman diagram approach consists of nothing more than expanding $G(z)$ in inverse powers of $z$ and doing the integrals in (3):

$$G(z) = \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} \langle \frac{1}{N} tr \varphi^n \rangle$$

In doing the integral over $\varphi$ implied by $< ..... >$ we split off the quadratic part $V(\varphi)$ and treat the rest of $V$ perturbatively. As explained in [3], it is useful to borrow the terminology of large $N$ quantum chromodynamics [23] from the particle physics literature, and speak of quark and gluon lines. See
figure (1) for a graphical representation. (It is of course not necessary to use this language, and readers not familiar with this language can simply think of the diagrams as representing the different terms one encounters in doing the integral in (8).) The bare quark propagator simply comes from the explicit factors of \( z \) in (18) and is represented by a single line and given by \( 1/z \). The quadratic term in \( V(\varphi) \) determines the bare gluon propagator, represented by double lines, and proportional to

\[
\langle \varphi^i_j \varphi^k_l \rangle = \delta^i_k \delta^j_l \frac{1}{Nm^2}
\]

(19)

where \( m^2 \) is defined by the quadratic part of \( V(\varphi) = m^2 \varphi^2/2 + \ldots \). The non-Gaussian terms in \( V(\varphi) \) describe the interactions between gluons.

The important point, as originally stressed by 't Hooft [23], is that this double-line formalism provides an efficient way of counting the powers of \( N \) to be associated with each diagram: each vertex counts for one power of \( N \), each gluon propagators counts for \( N^{-1} \), and each closed loop counts for \( N \).

The bare quark propagator \( 1/z \) is changed by the interaction to the dressed quark propagator \( G(z) \). The gluon propagator is dressed by gluon interaction, but note that it is not dressed by quark loops. This is clear from the definition of our problem. Another way of saying this is to note that the one-point Green’s function may be represented, by using the replica trick, as

\[
G(z) = \lim_{n \to 0} \int D\psi^\dagger D\psi D\varphi P(\varphi)\psi_1^\dagger \psi_1 e^{-\sum_{\alpha=1}^n \psi_\alpha (z-\varphi) \psi_\alpha}
\]

(20)

The replica index \( \alpha \) runs from 1 to \( n \). Note that in this language the \( \psi \)'s
represent the quark fields and \( \varphi \) the gluon fields. The interaction between gluon and quarks are given by \( \psi_\alpha^\dagger \varphi \psi_\alpha \). (Color indices are suppressed here.)

The interaction of the gluons with each other is determined by the non-Gaussian part of \( P(\varphi) \). Since internal quark loops are proportional to the number of replicas \( n \), they vanish in the \( n \to 0 \) limit.

Similarly, we can treat the two point Green’s function by expanding

\[
G_c(z, w) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{z^{n+1} w^{m+1}} \left\langle \frac{1}{N} tr \varphi^n \frac{1}{N} tr \varphi^m \right\rangle
\]

(21)

The implied integration over \( \varphi \) then generates the Feynman diagrams for \( G_c(z, w) \). We can also use the replica trick to represent \( G_c(z, w) \). Clearly, we would have to introduce two quark fields \( \psi_z \) and \( \psi_w \): the variables \( z \) and \( w \) act like a flavor label. Thus, the correlation function describes two quarks, “carrying” \( z \) and \( w \) respectively, interacting by emitting and absorbing gluons (which have complicated interactions amongst themselves.) What we are doing here may be considered as a “baby version” of quantum chromodynamics.

As a warm up exercise and to gain some familiarity with what is going on, we will first consider the Gaussian case. The derivation given here is simpler than the one given in \[3\] and in its essence was given in one of our earlier papers \[15\]. With the benefit of hindsight, we start by taking out two partial derivatives:

\[
G_c(z, w) \equiv \left\langle \frac{1}{N} tr \frac{1}{z - \varphi} \frac{1}{N} tr \frac{1}{w - \varphi} \right\rangle_c
\]
\[
\frac{\partial}{\partial z} \frac{\partial}{\partial w} \left( \frac{1}{N} \text{tr} \log(z - \varphi) \frac{1}{N} \text{tr} \log(w - \varphi) \right) \tag{22}
\]

Expanding the logarithms, we find
\[
G_c(z, w) = \frac{\partial}{\partial z} \frac{\partial}{\partial w} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{z^n w^k} \left( \frac{1}{N_n} \text{tr} \varphi^n \frac{1}{N_k} \text{tr} \varphi^k \right) \tag{23}
\]

This is represented by the “wheel” graph of Fig (2) where the quark propagator on the inner rim carries $z$ and the one on the outer rim carries $w$. For the moment we ignore quark self-energy and vertex corrections: every gluon emitted on the inner rim is absorbed on the outer rim, and vice versa. Thus we may set $n = k$ in (23).

Since we are working with a Gaussian distribution we can immediately evaluate $\langle \text{tr} \varphi^n \text{tr} \varphi^n \rangle_c = n$. (Here with no loss of generality we have scaled $m^2$ to unity.) Graphically the factor of $n$ corresponds to the fact that with the inner rim held fixed, we may rotate the outer rim by $n$ different “clicks” and leave the diagram invariant. It is this factor of $n$ which produces the logarithmic function when we evaluate the sum in (23) to obtain
\[
N^2 G(z, w)_c = -\frac{\partial}{\partial z} \frac{\partial}{\partial w} \log(1 - \frac{1}{zw}) \tag{24}
\]

This expression does not yet have the form in (17). Next we have to include self-energy and vertex corrections. Instead of doing this let us leave this expression as it is for the moment and turn our attention to a “scattering” formalism discussed in 17.
3 Some formalism

Let us go back to the expansion of $G_e(z, w)$ in (21). Note that there is an extra power of $1/z$ and $1/w$ compared to the powers of $\varphi$. Thus, in the wheel diagram of figure (2), one of the quark propagators on the inner rim, and one on the outer rim, should actually be represented by $1/z^2$ and $1/w^2$ respectively: they each consists of two quark propagators. We represent this fact graphically by two dots, one on the inner rim, and one on the outer rim, as shown in figure (3). There are two possibilities: the two dots are on the same “sector” of the wheel, as shown in figure (3a), or the two dots are on different “sectors”, as shown in figure (3b).

Now imagine cutting open the quark propagators at the two dots. This converts the wheel diagrams into two sets of scattering diagrams as shown in figure (4). Note that it is necessary to include the crossed ladders in figure (4b). (Incidentally, the necessity of including the crossed ladders came to us as a bit of a surprise in carrying out this calculation using the formalism of [3] but it is made completely clear by the present formalism.)

4 Some formal relationships

We have illustrated the discussion in the two preceding paragraphs with diagrams appropriate to the Gaussian case, but this discussion applies im-
mediately to the general case. Let us define the “scattering amplitude”

\[
N \langle \left( \frac{1}{z - \varphi} \right)_i^j \left( \frac{1}{w - \varphi} \right)_m^n \rangle_c = N \langle \left( \frac{1}{z - \varphi} \right)_i^j \left( \frac{1}{w - \varphi} \right)_m^n \rangle - \delta_i^j G(z) \delta_m^n G(w) \equiv \delta_i^j \delta_m^n A + \delta_i^j \delta_m^n B
\]  

(25)

The two “scalar” scattering amplitudes \( A \) and \( B \) depend on \( z \) and \( w \) of course and correspond diagrammatically to the sets of graphs shown in figure (5a, b, c). We see from the flow of the color indices that \( A \) and \( B \) correspond to the two ways of cutting the wheel diagram, that is, to figure (4a) and figure (4b) respectively.

For a general \( V \) it is complicated to calculate the amplitudes \( A \) and \( B \) directly by perturbation theory. For example, \( A \) is given by the infinite set of graphs in figure (5a) for the \( g\varphi^4 \) theory. However, we will see that by a judicious arrangement of our calculation, we can avoid doing the explicit calculation that we had to work hard to carry out in [9] simply to obtain a result to first order in \( g \).

First, we note that by contracting (25) with \( \delta_i^n \delta_j^m \) we have

\[
N^2 A + B = N^2 \left[ \frac{G(z) - G(w)}{w - z} - G(z)G(w) \right]
\]  

(26)

We can check easily by looking at a few graphs that \( A \) is of order \( N^0 \) while \( B \) is of order \( 1/N \). Thus, we can drop \( B \) in this equation and determine \( A \) (to leading order in \( N \) of course) in terms of the one-point Green’s function \( G(z) \).
Next, by contracting (25) with $\delta_j^i \delta_m^n$ we find that the connected two-point Green’s function is given by

$$N^2 G_c(z, w) \equiv \left\langle \frac{1}{z - \varphi} tr \frac{1}{w - \varphi} \right\rangle_c = A + NB$$

(27)

Note that in this equation $A$ and $B$ both contribute to the same order in $N$. Thus, to determine $G_c$ we still have to know $A$ and $B$. It would seem that we would have to work to obtain $B$, as we did in our previous papers. In the next section, we will see how we can avoid calculating $B$.

5 General $V$

We are now ready to tackle the full problem of determining $G_c(z, w)$ for a general $V$. First, it is useful to define a two-quark irreducible scattering amplitude $\Gamma(z, w)$ consisting of those graphs that do not fall into two disconnected pieces upon cutting the two separate quark propagators, as shown in figure (6). Then the scattering function $A$ is evidently given by

$$A = \frac{1}{z^2 w^2} (\Gamma + \frac{1}{zw} \Gamma + \frac{1}{zw} \Gamma \frac{1}{zw} \Gamma + ....) = \frac{1}{z^2 w^2} \Gamma$$

(28)

Note that this expression merely relates $A$ to $\Gamma(z, w)$, which at this stage would appear to be an exceedingly complicated object to calculate directly.

Let us now start the computation of $G_c(z, w)$ with (21) which we repeat here for convenience:

$$G_c(z, w) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{z^{n+1} w^{m+1}} \left\langle \frac{1}{N} tr \varphi^n \frac{1}{N} tr \varphi^m \right\rangle$$

(29)
Again, let us proceed by first ignoring vertex and self energy corrections. But in the general case we can no longer simply set \( n \) equal to \( m \), as we did in the Gaussian case. We see, however, that we can express all the wheel diagrams representing (21) in terms of the (unknown) amplitude \( \Gamma(z, w) \) as indicated in figure (7).

We have yet to put on the two dots, one on the inner rim, one on the outer rim, as explained above. We see that now there are a number of possibilities. We can put the dot on an “exposed” quark line, or we can put the dot on a quark line hidden inside a \( \Gamma \). These two possibilities are illustrated in figure (8).

For obvious reasons, we now find it useful to use an alternative notation in which we replace \( z \) and \( w \) by \( z_1 \) and \( z_2 \) respectively and define \( \partial_a \equiv \frac{\partial}{\partial z_a} \). When we put the dot on an exposed quark line, we replace \( 1/z_a \) by \( 1/z_a^2 = -\partial_a(1/z_a) \). When we put the dot on a “hidden” quark line, we in effect replaced \( \Gamma(z_1, z_2) \) by \( -\partial_a \Gamma(z_1, z_2) \). Incidentally, in the Gaussian case there is no “hidden” quark line: all quark lines are exposed by definition. The differential operator \( \partial_1 \partial_2 \) associated with putting on the dots is precisely what relates (21) to (23).

In addition to the choice of putting the two dots on exposed or hidden quark lines, we also have the choice of putting the two dots in the same sector or in different sectors. (Sectors are defined as the segments of the wheel divided by the different \( \Gamma \)'s: each sector consists of one the spaces
Let us first consider putting the two dots in the same sector. Then we have the following four possibilities, corresponding to the four diagrams in figure (9): (a) both dots are on an exposed line, thus giving $\frac{\Gamma}{z_1 z_2}$, (b) the dot on the inner rim is on a hidden line, while the dot on the outer rim is on an exposed line, thus giving $-\frac{\partial_1 \Gamma}{z_1 z_2}$, (c) the previous case with inner and outer exchanged, and (d) both dots are on a hidden line, thus giving $\frac{\partial_1 \partial_2 \Gamma}{z_1 z_2}$. These four terms add up to $(\frac{\Gamma}{z_1 z_2} + -\frac{\partial_1 \Gamma}{z_1 z_2} + -\frac{\partial_2 \Gamma}{z_1 z_2} + \frac{\partial_1 \partial_2 \Gamma}{z_1 z_2}) = \partial_1 \partial_2 (\frac{\Gamma}{z_1 z_2})$.

The rest of the wheel (see figure (9)) can be filled with nothing, one $\Gamma$, two $\Gamma$’s, and so on, that is, with the series $1 + \frac{\Gamma}{z_1 z_2} + (\frac{\Gamma}{z_1 z_2})^2 + \ldots$. Putting all of this together we have the following contribution to $G_c(z_1, z_2)$:

$$\left(1 - \frac{\Gamma}{z_1 z_2}\right) \partial_1 \partial_2 (\frac{\Gamma}{z_1 z_2}) \quad (30)$$

Next we have to consider the possibilities of putting the two dots on two different sectors. The dot on the inner rim can be either on an exposed line or a hidden line, and thus we obtain a factor $(\frac{\Gamma}{z_1 z_2} - \frac{\partial_1 \Gamma}{z_1 z_2}) = -\partial_1 (\frac{\Gamma}{z_1 z_2})$. Similarly, the dot on the outer rim can be either on an exposed line or a hidden line, and we obtain the factor just given but with 1 and 2 interchanged. These two factors combine to give $\partial_1 (\frac{\Gamma}{z_1 z_2}) \partial_2 (\frac{\Gamma}{z_1 z_2})$.

The two different sectors, where the two dots are placed, divide the wheel into two segments, each of which can be filled, just as above, with nothing, one $\Gamma$, two $\Gamma$’s, and so on, that is, each segment leads to the factor $\frac{1}{1 - \frac{\Gamma}{z_1 z_2}}$. 

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Thus, we obtain in $G_c(z_1, z_2)$ the contribution

$$
(\frac{1}{1 - \frac{\Gamma}{z_1z_2}})^2 \partial_1(\frac{\Gamma}{z_1 z_2}) \partial_2(\frac{\Gamma}{z_1 z_2})
$$

(31)

Putting (30) and (31) together, we finally obtain

$$
N^2 G_c(z_1, z_2) = (\frac{1}{1 - \frac{\Gamma}{z_1 z_2}}) \partial_1 \partial_2(\frac{\Gamma}{z_1 z_2})
$$

+ $(\frac{1}{1 - \frac{\Gamma}{z_1 z_2}})^2 \partial_1(\frac{\Gamma}{z_1 z_2}) \partial_2(\frac{\Gamma}{z_1 z_2})$

(32)

which we happily recognize as just

$$
N^2 G_c(z_1, z_2) = -\partial_1 \partial_2 \log(1 - \frac{\Gamma}{z_1 z_2})
$$

(33)

Note that we could have given a shorter derivation by simply “working backwards”: we could have started with (33) and simply said that the operator $(\partial_1 \partial_2 \log)$ distributes the two dots in all the possible ways that we had enumerated, but we believe that our longer derivation just given is more transparent and easier for the reader to follow.

For $\Gamma = 1$ we recover our previous expression (24): surely we are on the right track. We are almost there but we have yet to put in the quark self energy corrections and the vertex corrections. The quark self energy corrections are easy to put in: we simply replace the bare quark propagator $1/z$ by the dressed propagator $G(z)$ appropriate for the interaction potential $V$. The vertex corrections require more thought. First, note that our usage of the term “vertex corrections” differs slightly from the standard usage. For
example, the diagram in figure (10a) has already been counted in $\Gamma$. We include in vertex corrections the diagrams in figure (10b) for example. As seen in figure (10c), including the vertex corrections we simply multiply the amplitude without vertex corrections by a factor $v(z)$.

Already, we noted in [3] that the gluons in the vertex corrections must “span” the whole amplitude [20] lest we lose factors of $N$. This is illustrated in figure (11). The remark given in [3] concerning the vertex corrections for the Gaussian case clearly generalizes to the case of an arbitrary $V$.

We are now faced with the task of calculating $v(z)$, which we calculated for the Gaussian case in [3]. Fortunately, we can avoid doing any work by noting that there is a “Ward identity”

$$v(z) = \frac{dG^{-1}}{dz}$$

(34)

The reader can easily convince himself or herself of this identity by contemplating the diagrams in figure (12). The differentiation $\frac{d}{dz}$ simply puts the vertex in all possible places with the correct counting factor. Note that the $v(z)$ given by this identity is fully dressed, that is, the quark propagators that enter in $v(z)$ are already dressed.

But now we see a remarkable cancellation of the vertex corrections because we have the foresight (or hindsight!) of arranging our calculation of $G_c$ so that it has the form $\partial_1 \partial_2(...)$ before self energy and vertex corrections
are included (see (33)). Including these corrections we obtain

\[ N^2 G_c(z_1, z_2) = v(z_1) v(z_2) ([\partial_1 \partial_2 \log(1 - \frac{\Gamma}{z_1 z_2})]|_{dressed}) \]

\[ = \partial_1 \partial_2 (\log(1 - \frac{\Gamma}{z_1 z_2})|_{dressed}) \]  \hspace{1cm} (35)

since \( (\frac{d}{dz})|_{dressed} = \frac{d}{dG^{-1}(z)} = \frac{1}{v(z)} \frac{d}{dz} \). The vertex corrections disappear!

At this point, if we go back to the Gaussian formula (24) we see that in the Gaussian case we have essentially finished our calculation since \( \Gamma = 1 \). We obtain

\[ N^2 G_{0c}(z, w) = -\frac{\partial}{\partial z} \frac{\partial}{\partial w} \log(1 - G_0(z) G_0(w)) \]  \hspace{1cm} (36)

Inserting the explicit form for \( G_0(z) \) given in (9) we obtain after some simple manipulations (14). This derivation is simpler than that given in (3).

It is always nice to recover the Gaussian result as a check but here we want to do the much more ambitious problem of calculating \( G_c \) for a general \( V \). To go further, we have to calculate \( \Gamma(z_1, z_2) \) and then to dress it by replacing \( 1/z_a \) by \( G(z_a) \) which we will write as \( G_a \) for short. This would have been a long involved calculation, but again we note happily that we can avoid doing it simply by noting that the bare \( \Gamma \) is related to the bare \( A \) by (28):

\[ \Gamma = \frac{z_1 z_2 A}{A + \frac{1}{z_1 z_2}} \]  \hspace{1cm} (37)

But the dressed \( A \) is given by the identity in (20)! Thus, we don’t have to
do any further work. We simply dress to obtain
\[ \Gamma_{dressed} = \left( \frac{1}{G_1 G_2} + \frac{G_1 - G_2}{z_1 - z_2} \right) \] (38)

Finally, then
\[ (z_1 z_2 - \Gamma(z_1, z_2))_{dressed} = \left( \frac{1}{G_1 G_2} - \Gamma_{dressed} \right) = -\left( \frac{z_1 - z_2}{G_1 - G_2} \right) \] (39)

And thus, we have proved our conjecture. Combining (35) and (39) we obtain our conjectured relation
\[ N^2 G_c(z, w) = \frac{\partial^2}{\partial z \partial w} \log \left( \frac{G(z) - G(w)}{z - w} \right) + \text{irrelevant factorized terms} \] (40)

Finally, in the language of the wheel diagram, it is easy to see where the irrelevant factorized terms in (40) come from. They come from diagrams which disconnect the inner rim and the outer rim of the wheel from each other and thus clearly has a factorized dependence on \( z \) and \( w \). See figure (13).

6 Deterministic plus random

In our earlier work, we have also generalized the problem outlined in the introduction of this paper to the problem of a Hamiltonian given by the sum of a deterministic term and a random term
\[ H = H_0 + \varphi \] (41)

Here \( H_0 \) is a diagonal matrix with diagonal elements \( \epsilon_i, i = 1, 2, ...N \), and \( \varphi \) a random matrix taken from the ensemble (1). For the Gaussian case, namely
with $V(\varphi) = \frac{1}{2}\varphi^2$, Pastur \cite{Pastur} has long ago determined the density of eigenvalues. Our work in \cite{our_work} went beyond Pastur’s work in that the correlation function between the density of eigenvalues in the Gaussian case was also determined. In our recent work with Hikami, \cite{Hikami} we studied this correlation function for a $g\varphi^4$ theory to first order in $g$.

This problem of “determinism plus chance” may be regarded as a generic problem in physics, and as such represents a significant generalization of Wigner’s problem. For example, consider an electron moving in a magnetic field and scattering off impurities. We note that these “deterministic plus random” problems are considerably more difficult than purely random problems. The orthogonal polynomial approach used in \cite{orthogonal} involves diagonalizing the random matrix $\varphi$ and is clearly no longer available: in \cite{non_diagonalization} we cannot diagonalize $\varphi$ without un-diagonalizing $H_0$. Thus, we do not have the analog of (15) for this problem.

Indeed, as discussed in a recent paper \cite{recent_paper}, the problem described here represents a special case of a broader class of problems involving the addition of random matrices. The deterministic Hamiltonian $H_0$ may in turn be replaced by a random Hamiltonian. Indeed, a deterministic matrix is but a special case of a random matrix.

Consider then a Hamiltonian given by

\[ H = \varphi_1 + \varphi_2 \]  

(42)
with the matrices $\varphi_{1,2}$ taken from a factorized probability distribution

$$P(\varphi_1, \varphi_2) = \frac{1}{Z} e^{-N \text{tr}[V_1(\varphi_1) + V_2(\varphi_2)]} \equiv P_1(\varphi_1) P_2(\varphi_2). \quad (43)$$

In [8] it was shown how the one-point Green’s function $G$ can be obtained for the Hamiltonian given in (42). Here we would like to solve the problem of determining the connected two-point Green’s function $G_c$.

A slightly sloppy but essentially correct argument is that given our universal relation (40) between $G_c$ and $G$ our problem is solved instantly. The desired connected two-point Green’s function is given in terms of the one-point Green’s function appropriate to the distribution in (43). This result is precisely what was conjectured in [9].

We can put this argument on a more solid footing by using the formalism discussed in [8]. (The following discussion will be sketchy and not self-contained.) In that work, it was shown that $G(z)$ may be determined in terms of $G_1(z)$ and $G_2(z)$, the Green’s functions corresponding to the distribution $P_1$ and $P_2$ respectively, according to the following procedure. First, solve the equations $G_a(B_a(z)) = z$, for $a = 1, 2$, that is, find the functional inverses of $G_a(z)$, denoted by $B_a(z)$ here. Next, define the function

$$B_{1+2}(z) = B_1(z) + B_2(z) - \frac{1}{z}. \quad (44)$$

The functional inverse of $B_{1+2}(z)$ is then the desired Green’s function $G(z)$.

This type of addition laws has been discussed recently in the mathematical [17, 18, 19] and physical literature [14, 11, 16].

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Following the argument given in [8] which we won’t repeat here, we find for example the undressed $\Gamma$ is given by

$$\Gamma(z, w) = zw(1 - \frac{z^2 G_{gc1}(z) - w^2 G_{gc1}(w)}{z - w}) + \text{(1} \leftrightarrow \text{2)}$$ (45)

Here $G_{gc1}$ and $G_{gc2}$ are “gluon connected” Green’s functions defined in [8] and are related to $B_1$ and $B_2$ respectively. Following the same steps as above we find that (35) still holds. Inserting the expression for $\Gamma$ given here and using (44) we immediately find that (40) indeed holds for this more general class of problems.

### 7 Conclusion

In conclusion, we have found a remarkable universal relation between the one point Green’s function $G(z)$ and the connected two point Green’s function $G_c(z, w)$. This represents an entirely different sort of universality as the one found in [8]. There it was shown that the scaled two point Green’s function $a^2 G_c(az, aw)$, with $a$ the endpoint of the spectrum given by a complicated function of the potential $V(\varphi)$, is independent of $V$. Here it is shown that the structural relation between $G_c(z, w)$ and $G(z)$ is independent of $V$, even though $G(z)$ is known to depend on $V$ in a complicated way.

While we know that these two forms of universality must be equivalent, it is not obvious how to show this equivalence directly.

The compact form of $G_c(z, w)$ obtained here renders the universality property of $\rho_c^{\text{smooth}}(\mu, \nu)$ as $\mu$ approaches $\nu$ particularly transparent. Consider our
universal form

\[ N^2 G_c(z, w) = \frac{\partial}{\partial z} \frac{\partial}{\partial w} \log \left( \frac{G(z) - G(w)}{z - w} \right) \] (46)

If \( z \) and \( w \) approach each other on the same side of the cut of \( G \), the argument of the logarithm is a smooth function of \( z - w \), and thus would not contribute to \( \rho_{\text{smooth}}(\mu, \nu) \) a term proportional to \( 1/(\mu - \nu)^2 \) that we know from (11) must be there. On the other hand, if \( z \) and \( w \) approach each other from opposite sides of the cut, then writing \( G(\mu \pm i\epsilon) \equiv R(\mu) \pm iI(\mu) \) as \( \epsilon \) goes to zero, we have the universal singular behavior

\[ N^2 G_c(\mu + i\epsilon, \nu - i\delta) = \frac{\partial}{\partial \mu} \frac{\partial}{\partial \nu} \log \left( \frac{R(\mu) - R(\nu)}{\mu - \nu} + i \frac{I(\mu) + I(\nu)}{\mu - \nu} \right) \]

\[ \rightarrow - \frac{\partial}{\partial \mu} \frac{\partial}{\partial \nu} \log(\mu - \nu) = -\frac{1}{(\mu - \nu)^2} \] (47)

Inserting this into (4) we find the singular part of (11). The emergence of the universal behavior, in which the dependence on \( R(\mu) \) and \( I(\mu) \) drops away, is made particularly clear by (47).

We may be tempted to conjecture that the elegantly compact form of our universal relation (46) may be associated with a deeper mathematical structure.

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9 Figure Captions

Fig 1. Diagrammatic rules: (a) quark propagator, (b) gluon propagator, (c) quark gluon vertex, and (d) gluon interaction, illustrated here with a $g\varphi^4$ vertex.

Fig 2. Wheel diagram.

Fig 3. The two dots may be placed in the same sector (a) or in different sectors (b).

Fig 4. The diagrams in Fig 3. cut open at the dots. (The diagrams in the same topological class, rather than the exact correspondents, are shown.)

Fig 5. Diagrams contributing to $A$ (a) and diagrams contributing to $B$ (b) and (c). Note that two topological distinct classes of graphs contribute to $B$.

Fig 6. Typical diagrams contributing to $\Gamma$.

Fig 7. The wheel diagrams for a general potential $V$.

Fig 8. A dot may be put on an exposed line (a) or on a hidden line (b).

Fig 9. (a) The two dots are both placed on exposed lines. (b) The dot on the outer rim is placed on an exposed line, while the dot on the inner rim is placed on a hidden line. (c) The situation in (b) reversed. (d) The two dots
are both placed on hidden lines.

Fig 10. (a) Diagram not included in what we called vertex correction. (b) Diagram included in what we called vertex correction. (c) A vertex correction: the shaded portion can include many gluon lines, possibly interacting with each other.

Fig 11. (a) The gluon in the vertex correction spans the whole diagram. (b) The gluon in the vertex correction spans only part of the diagram. We see that the diagram in (b) has one less loop than the one in (a).

Fig 12. The “Ward identity” relating the vertex correction to the self-energy.

Fig 13. A diagram contributing to the “irrelevant” factorized terms in $N^2G_c(z, w)$. Note the inner rim and the outer rim are “decoupled” from each other.

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$$
\Gamma(z, w) = 1 - g(2 + \frac{1}{z^2} + \frac{1}{zw} + \frac{1}{w^2}) + g^2(9 + 10(\frac{1}{z^2} + \frac{1}{zw} + \frac{1}{w^2})

+ 3(\frac{1}{z^4} + \frac{1}{z^3w} + \frac{1}{z^2w^2} + \frac{1}{zw^3} + \frac{1}{w^4})
$$

(48)

It is a theorem that the coefficient of $\frac{1}{z^m w^n}$ depends only on $m + n$.

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