The spectral gap of graphs and Steklov eigenvalues on surfaces

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Abstract

Using expander graphs, we construct a sequence \( \{\Omega_N\}_{N \in \mathbb{N}} \) of smooth compact surfaces with boundary of perimeter \( N \), and with the first non-zero Steklov eigenvalue \( \sigma_1(\Omega_N) \) uniformly bounded away from zero. This answers a question which was raised in [9]. The genus of \( \Omega_N \) grows linearly with \( N \), this is the optimal growth rate.

1 Introduction

Let \( \Omega \) be a compact, connected, orientable smooth Riemannian surface with boundary \( \Sigma = \partial \Omega \). The Steklov eigenvalue problem on \( \Omega \) is

\[
\Delta f = 0 \text{ in } \Omega, \quad \partial_{\nu} f = \sigma f \text{ on } \Sigma,
\]

where \( \Delta \) is the Laplace–Beltrami operator on \( \Omega \) and \( \partial_{\nu} \) denotes the outward normal derivative along the boundary \( \Sigma \). The Steklov spectrum of \( \Omega \) is denoted by

\[
0 = \sigma_0 < \sigma_1(\Omega) \leq \sigma_2(\Omega) \leq \ldots \nrightarrow \infty,
\]

where each eigenvalue is repeated according to its multiplicity. In [9], the second author and I. Polterovich asked the following question:

Is there a sequence \( \{\Omega_N\} \) of surfaces with boundary such that \( \sigma_1(\Omega_N)L(\partial\Omega_N) \to \infty \) as \( N \to \infty \)?

The goal of this paper is to give a positive answer to this question.

Theorem 1. There exist a sequence \( \{\Omega_N\}_{N \in \mathbb{N}} \) of compact surfaces with boundary and a constant \( C > 0 \) such that for each \( N \in \mathbb{N} \), genus(\( \Omega_N \)) = 1 + \( N \), and

\[
\sigma_1(\Omega_N)L(\partial\Omega_N) \geq CN.
\]

For each \( \gamma \in \mathbb{N} \), consider the class \( S_\gamma \) of all smooth compact surfaces of genus \( \gamma \) with non-empty boundary, and define

\[
\sigma^*(\gamma) = \sup_{\Omega \in S_\gamma} \sigma_1(\Omega)L(\partial\Omega).
\]

Few results on \( \sigma^*(\gamma) \) are known. G. Kokarev proved [12] that for each genus \( \gamma \),

\[
\sigma^*(\gamma) \leq 8\pi(\gamma + 1).
\]
See also [6] for a similar bound involving the higher Steklov eigenvalues \( \sigma_k(\Omega) \), and [9] for a bound involving the number of boundary components. In [8], A. Fraser and R. Schoen proved that \( \sigma^*(0) = 4\pi \). In this case, the supremum is attained in the limit by a sequence of surfaces with their number of boundary components tending to infinity. It follows from [1] that the growth of \( \sigma^*(\gamma) \) is sublinear.

**Corollary 2.** There exists a constant \( C > 0 \) such that for each \( \gamma \in \mathbb{N} \), \( \sigma^*(\gamma) \geq C\gamma \).

In the construction of \( \{\Omega_N\} \) that we propose, the number of boundary components also tends to infinity. It would be interesting to know if this condition is necessary.

**Remark 3.** The problem of constructing closed surfaces \( M \) with large normalized first non-zero eigenvalue \( \lambda_1(M)\text{Area}(M) \) has been considered by several authors. See for instance [4, 2, 3]. Our proofs uses methods which are related to those of [7].

### Plan of the paper

In Section \( \underline{2} \) we present the construction of a surface \( \Omega_\Gamma \) which is obtained from a regular graph \( \Gamma = (V, E) \) by sewing copies of a fundamental piece following the pattern of the graph \( \Gamma \). In Section \( \underline{3} \) we introduce the spectrum of the graph \( \Gamma \) and state a comparison result (Theorem \( \underline{7} \)) between \( \lambda_1(\Gamma) \) and \( \sigma_1(\Omega_\Gamma) \). This is then used, in conjunction with expander graphs, to prove Theorem \( \underline{1} \). In Section \( \underline{4} \) we present the comparison argument leading to the proof of Theorem \( \underline{1} \).

**Remark 4.** While this paper was in the final stage of its preparation, we learned that Mikhail Karpukhin also has developed a method for construction surfaces with large normalized Steklov eigenvalue \( \sigma_1L \). His approach is different, and his work will appear in [14].

## 2 Constructing manifolds from graphs

Let \( \Gamma \) be a finite connected regular graph of degree \( k \). The set of vertices of \( \Gamma \) is denoted \( V = V(\Gamma) \), the set of edges is denoted \( E = E(\Gamma) \). The number of vertices of \( \Gamma \) is \(|V(\Gamma)|\). We will construct a Riemannian surface \( \Omega_\Gamma \) modelled on the graphs \( \Gamma \) from a fixed orientable Riemannian surface \( M_0 \) which we call the fundamental piece (See Figure \( \underline{1} \)) and which is assumed to satisfy the following hypotheses:

1. The boundary of \( M_0 \) has \( k + 1 \) components \( \Sigma_0, B_1, \ldots, B_k \).
2. Each of the boundary component is a geodesic curve of length 1.
3. The component \( \Sigma_0 \) has a neighbourhood which is isometric to the cylinder \( C_0 = \Sigma_0 \times [0, 1] \subset M_0 \), and \( \Sigma_0 \) corresponds to \( \{0\} \times \Sigma \).

The manifold \( \Omega_\Gamma \) is obtained by sewing copies of the fundamental piece \( M_0 \) following the pattern of the graph \( \Gamma \); to each vertex \( v \in V \), there corresponds a isometric copy \( M_v \) of \( M_0 \). The edges emanating from a vertex \( v \in V(\Gamma) \) are labelled \( e_1(v), \ldots, e_k(v) \). The corresponding boundary components \( B_1, \ldots, B_k \) are identified along these edges: if \( v \sim w \) then there are \( 1 \leq i, j \leq k \) such that \( e_i(v) = e_j(w) \) and the boundary component \( B_i \) of \( M_v \) is identified to the boundary component \( B_j \) of \( M_w \). The manifold \( \Omega_\Gamma \) has one boundary component \( \Sigma_v \) for each vertex \( v \in V(\Gamma) \), each of them being isometric to \( \Sigma_0 \) with corresponding cylindrical neighbourhood \( M_v \supset C_v = [0, 1] \times \Sigma_0 \).

The following lemma shows that the genus of \( \Omega_\Gamma \) grows linearly with the number of vertices of the graph \( \Gamma \).
Lemma 5. The genus of the surface $\Omega_\Gamma$ is

$$\gamma(\Omega_\Gamma) = 1 + \left( \gamma(M_0) + \frac{k}{2} - 1 \right) |V(\Gamma)|,$$

where $\gamma(M_0)$ is the genus of the fundamental piece $M_0$, and $|V(\Gamma)|$ is the number of vertices of $\Gamma$.

Remark 6. Because the number of vertices of odd degree is always even, $1 + (\gamma(M_0) + \frac{k}{2} - 1)|V(\Gamma)|$ is an integer.

Proof of Lemma 5 The genus $\gamma$ and the Euler–Poincaré characteristic $\chi$ of a smooth compact orientable surface with $b$ boundary components are related by the formula

$$\chi = 2 - 2\gamma - b.$$

Let $K : \Omega_\Gamma \to \mathbb{R}$ be the Gauss curvature. Since the boundary curves $\Sigma_0, B_1, \cdots, B_k$ are geodesics, it follows from the Gauss–Bonnet formula hat

$$\chi(\Omega_\Gamma) = \frac{1}{2\pi} \int_{\Omega_\Gamma} K = \frac{1}{2\pi} |V(\Gamma)| \int_{M_0} K = \chi(M_0)|V(\Gamma)| = (1 - 2\gamma(M_0) - k)|V|,$$

where we have used that the number of boundary components of $M_0$ is $k + 1$. It follows that the genus of $\Omega_\Gamma$ is

$$\gamma(\Omega_\Gamma) = \frac{1}{2}(2 - \chi(\Omega_\Gamma) - |V|) = 1 + \left( \gamma(M_0) + \frac{k}{2} - 1 \right) |V|.$$

3 Comparing eigenvalues on graphs to Steklov eigenvalues

Our main reference for spectral theory on graphs is [5]. The space $\ell^2(V(\Gamma)) = \{x : V(\Gamma) \to \mathbb{R}\}$ is equipped with the norm defined by

$$\|x\|^2 = \sum_{v \in V(\Gamma)} x(v)^2.$$
The discrete Laplacian $\Delta_\Gamma$ acts on $\ell^2(V(\Gamma))$ and is defined by the quadratic form

$$q_{\Gamma}(x) = \sum_{v \sim w} (x(v) - x(w))^2,$$

(2)

where the symbol $v \sim w$ means that the vertices $v$ and $w$ of $\Gamma$ are adjacent, and the sum appearing in (2) is over all non-oriented edges of $\Gamma$. The discrete Laplacian $\Delta_\Gamma$ has a finite non-negative spectrum which we denote by

$$0 = \lambda_0 < \lambda_1(\Gamma) \leq \lambda_2(\Gamma) \leq \ldots \leq \lambda_{|V|-1}(\Gamma),$$

where each eigenvalue is repeated according to its multiplicity. The first non zero eigenvalue admits the following variational characterization:

$$\lambda_1(\Gamma) = \min \left\{ \frac{q_{\Gamma}(x)}{\|x\|^2} \bigg| x : V(\Gamma) \to \mathbb{R}, \sum_{v \in V(\Gamma)} x(v) = 0 \right\}.$$  

(3)

In order to compare $\lambda_1(\Gamma)$ to the first non-zero Steklov eigenvalue of $\Omega_\Gamma$, the following variational characterization will be used:

$$\sigma_1(\Omega_\Gamma) = \inf \left\{ \int_{\Omega_\Gamma} |\nabla f|^2 \bigg| f \in C^\infty(\Omega_\Gamma), \int_{\partial\Omega_\Gamma} f = 0, \int_{\partial\Omega_\Gamma} f^2 = 1 \right\}.$$  

The main result of this paper will follow from the following estimate.

**Theorem 7.** There exist constants $\alpha, \beta > 0$ depending only on the fundamental piece $M_0$ such that

$$\alpha \leq \frac{\sigma_1(\Omega_\Gamma)}{\lambda_1(\Gamma)} \leq \beta.$$

The proof of Theorem 7 will be presented in Section 4.

### 3.1 Expander Graphs and the Proof of Theorem 1

To prove Theorem 1 we will use expander graphs, through one of their many characterizations.

**Definition 8.** A sequence of $k$-regular graphs $\{\Gamma_N\}_{N \in \mathbb{N}}$ is called a family of expander graphs if

$$\lim_{N \to \infty} |V(\Gamma_N)| = +\infty$$

and $\lambda_1(\Gamma_N)$ is uniformly bounded below by a positive constant.

See [10] for a survey of their properties and applications. Consider a fundamental piece $M_0$ of genus 0, with 5 boundary components, that is with $k = 4$. Let $\{\Gamma_N\}$ be a family of 4-regular expander graphs such that the number of vertices $|V(\Gamma_N)| = N$. The existence of this family of expander graphs follows from the classical probabilistic method [14]. It follows from Lemma 5 that the genus of $\Omega_{\Gamma_N}$ is

$$\gamma(\Omega_{\Gamma_N}) = 1 + N.$$

By definition, there is a constant $c > 0$ such that $\lambda_1(\Gamma_N) \geq c$ for each $N \in \mathbb{N}$. Since the boundary of $\Omega_{\Gamma_N}$ has $N$ boundary components of length 1, Theorem 7 leads to

$$\sigma_1(\Omega_{\Gamma_N}) L(\partial\Omega_{\Gamma_N}) \geq \alpha N \lambda_1(\Gamma_N) \geq c\alpha N.$$

This completes the proof of Theorem 1.
4 Proof of the comparison results

Let \( f : \Omega \rightarrow \mathbb{R} \) be a smooth function. Given a vertex \( v \in V(\Gamma) \), the function \( f_v \) is defined to be the restriction of \( f \) to the cylinder \( C_v \). On each cylinder \( C_v \), the function \( f_v \) admits a decomposition \( f_v = \overline{f}_v + \tilde{f}_v \) where

\[
\overline{f}_v(r) = \int_{\Sigma_v} f(r, x) \, dV_{\Sigma_v}(x)
\]

is the average of \( f \) on the corresponding slice of \( C_v \). It follows that for each \( r \in [0, 1] \),

\[
\int_{\Sigma_v} \tilde{f}(r, x) \, dV_{\Sigma_v}(x) = 0.
\]

The function \( \overline{f} \) is defined to be \( \overline{f}_v \) on each cylinder \( C_v \), and similarly the function \( \tilde{f} \) is defined to be \( \tilde{f}_v \) on each \( C_v \).

Let \( f \in C^\infty(\Omega \Gamma) \) be a Steklov eigenfunction corresponding to \( \sigma_1(\Omega \Gamma) \). The function \( x = x_f : V(\Gamma) \rightarrow \mathbb{R} \) is defined to be the average of \( f \) over the boundary component \( \Sigma_v \). Since \( |\Sigma_v| = 1 \) for each vertex \( v \), this is expressed by

\[
x(v) = \int_{\Sigma_v} f \, dV_{\Sigma_v} = \overline{f}_v(0).
\]

Because

\[
\sum_{v \in V(\Gamma)} x(v) = \sum_{v} \int_{\Sigma_v} f \, dV_{\Sigma_v} = \int_{\Sigma} f \, dV_{\Sigma} = 0,
\]

the function \( x \) can be used as a trial function in the variational characterization (3) of \( \lambda_1(\Gamma) \). It follows from the orthogonality of \( \overline{f} \) and \( \tilde{f} \) on the boundary \( \Sigma = \partial \Omega \Gamma \) that

\[
\int_{\partial \Omega \Gamma} f^2 \, dV_{\Sigma} = \int_{\partial \Omega \Gamma} (\overline{f} + \tilde{f})^2 \, dV_{\Sigma} = \sum_{v \in V(\Gamma)} x(v)^2 + \int_{\partial \Omega \Gamma} \tilde{f}^2 \, dV_{\Sigma} \leq \frac{1}{\lambda_1(\Gamma)} q_{\Gamma}(x) + \int_{\partial \Omega \Gamma} \tilde{f}^2 \, dV_{\Sigma}.
\]  

(4)

The two terms on the right-hand side of the previous inequality will be bounded above in terms of \( \|\nabla f\|^2_{L^2(\Omega \Gamma)} \). In order to bound \( \int_{\partial \Omega \Gamma} \tilde{f}^2 \, dV_{\Sigma} \), it will be sufficient to consider the behaviour of \( \tilde{f} \) locally on each cylinders \( C_v \). More work will be required to bound \( q_{\Gamma}(x) \).

4.1 Local estimate of smooth functions on cylindrical neighbourhoods

On the model cylinder \( C_0 = [0, 1] \times \Sigma_0 \), consider the following mixed Neumann–Steklov spectral problem:

\[
\Delta f = 0 \text{ in } (0, 1) \times \Sigma_0,
\]

\[
\partial_n f = 0 \text{ on } \{1\} \times \Sigma_0,
\]

\[
\partial_n f = \mu f \text{ on } \{0\} \times \Sigma_0.
\]

This problem is related to the sloshing spectral problem. See [11, 13] for details.
Lemma 9. Let $\mu$ be the first non-zero eigenvalue of the sloshing problem $\mathcal{E}$. For any smooth function $f : \Omega \Gamma \to \mathbb{R}$,

$$
\int_{\partial \Omega} \tilde{f}^2 \, dV_{\Sigma} \leq \mu^{-1} \int_{\Omega} |\nabla f|^2. 
$$

(6)

Proof. The first non-zero eigenvalue of this problem is characterized by

$$
\mu = \inf \left\{ \int_{(0,1) \times \Sigma_0} |\nabla f|^2 \right\} : f \in C^\infty((0,1) \times \Sigma_0), \int_{(0,1) \times \Sigma_0} f \, ds = 0
$$

(7)

Since $\tilde{f}$ is orthogonal to constants on each boundary component $\Sigma_v$, it follows from (7) that

$$
\int_{\partial \Omega} \tilde{f}^2 \, dV_{\Sigma} \leq \mu^{-1} \sum_{v \in V(\Gamma)} \int_{C_v} |\nabla \tilde{f}|^2.
$$

The proof of Lemma 9 is completed by observing that the Dirichlet energy of $f_v : C_v \to \mathbb{R}$ is expressed by

$$
\int_{C_v} |\nabla f_v|^2 = \int_0^1 (f_v'(r))^2 \, dr + \int_{C_v} |\nabla \tilde{f}_v|^2.
$$

4.2 Global estimate and graph structure

Lemma 10. There exists a positive constant $C_0$ depending only on the fundamental piece $M_0$ such that the following holds for any function $f$ on $\Omega_\Gamma$:

$$
\sum_{v \sim w} (x(v) - x(w))^2 \leq C_0 \int_{\Omega_\Gamma} |\nabla f|^2.
$$

(8)

The proof of Lemma 10 is based on the following general estimate.

Lemma 11. Let $\Omega$ be a smooth compact connected Riemannian surface with boundary. Let $A$ and $B$ be two of the connected components of the boundary $\partial \Omega$, both of length 1. There exists a constant $C > 0$ depending only on $\Omega$ such that any smooth function $f \in C^\infty(\Omega)$ satisfies

$$
\left| \int_A f - \int_B f \right|^2 \leq C \int_\Omega |\nabla f|^2
$$

(9)

In fact, we will use this estimate only for harmonic functions, in which case it is also possible to prove it using a method similar to that of [7].

Proof of Lemma 11. Let $x = \int_A f, y = \int_B f$ be the average of $f$ on the two boundary components $A, B$. Let

$$
\langle f \rangle = \frac{1}{|\Omega|} \int_\Omega f,
$$

(10)
be the average of $f$ on the surface $\Omega$. Finally, set $g = f - \langle f \rangle$. Now, since $\int_\Omega g = \int_\Omega (f - \langle f \rangle) = 0$, 
\[
\int_\Omega g^2 \leq \mu^{-1} \int_\Omega |\nabla g|^2 = \mu^{-1} \int_\Omega |\nabla f|^2,
\]
where $\mu > 0$ is the first non-zero Neumann eigenvalue of $\Omega$. It follows that 
\[
\|f - \langle f \rangle\|_{H^1(\Omega)} = \|g\|_{H^1(\Omega)} = \|g\|_{L^2(\Omega)} + \|\nabla f\|_{L^2(\Omega)} \leq (\mu^{-1} + 1)\|\nabla f\|_{L^2(\Omega)},
\]
In other words, the Dirichlet energy of $f$ controls how far $f$ is from its average $\langle f \rangle$ in $H^1$-norm. This is essentially a version of the Poincaré Inequality. The restriction of $f$ to $A$ and $B$ are also close to the average $\langle f \rangle$ in $L^2$-norm. Indeed, it follows from the fact that the trace operators $\tau_A : H^1(\Omega) \to L^2(A)$ and $\tau_B : H^1(\Omega) \to L^2(B)$ are bounded that 
\[
|x - \langle f \rangle| = |\int_A (f - \langle f \rangle)| = |\int_A (\tau_A(g))| \leq \|\tau_A(g)\|_{L^2(A)} \leq \|\tau_A\|\|g\|_{H^1(\Omega)},
\]
and similarly $|y - \langle f \rangle| \leq \|\tau_B\|\|g\|_{H^1(\Omega)}$, where $\|\tau_A\|$ and $\|\tau_B\|$ are the operator norms. These two inequalities together lead to 
\[
|x - y| \leq |x - \langle f \rangle| + |y - \langle f \rangle| \leq (\|\tau_A\| + \|\tau_B\|)\|g\|_{H^1(\Omega)}.
\]
In combination with (10) this imply 
\[
|x - y|^2 \leq (\|\tau_A\| + \|\tau_B\|)^2(\mu^{-1} + 1) \int_\Omega |\nabla f|^2.
\]
One can take $C = (\|\tau_A\| + \|\tau_B\|)^2(\mu^{-1} + 1)$. The proof is completed.

**Proof of Lemma 10.** For each adjacent vertices $v \sim w$ of the graph $\Gamma$, we apply Lemma 11 to the surface $M_v \cup M_w$ with $A = \Sigma_v$ and $B = \Sigma_w$ to get 
\[
(x(v) - x(w))^2 \leq C \int_{M_v \cup M_w} |\nabla f|^2.
\]
Since the graph $\Gamma$ is regular of degree $k$, it follows that 
\[
\sum_{v \sim w} (x(v) - x(w))^2 \leq Ck \sum_{v \sim w} \int_{M_v \cup M_w} |\nabla f|^2 = Ck \int_{\Omega_\Gamma} |\nabla f|^2.
\]

**4.3 The proof of Theorem 7**

The upper bound

Let $f$ be a Steklov eigenfunction corresponding to the first non-zero Steklov eigenvalue $\sigma_1(\Omega_\Gamma)$. Combining the local estimate obtained in Lemma 9 and the global estimate of Lemma 10 with Inequality (11) leads to 
\[
\int_{\partial \Omega_\Gamma} f^2 dV_{\Sigma} \leq \left( \frac{C_0}{\lambda_1(\Gamma)} + \frac{1}{\mu} \right) \int_{\Omega_\Gamma} |\nabla f|^2,
\]
which of course can be rewritten
\[
\sigma_1(\Omega) = \frac{\int_{\Omega} |\nabla f|^2}{\int_{\partial \Omega} f^2} \geq \left( \frac{C_0}{\lambda_1(\Gamma)} + \frac{1}{\mu} \right)^{-1} = \frac{\lambda_1(\Gamma)}{\lambda_1(\Gamma)/\mu + C_0} \geq \min \left\{ \mu, \frac{1}{C_0} \right\} \lambda_1(\Gamma) + 1
\]

Now, because we are on a regular graph of degree \( k \), \( \lambda_1 \leq k \), so that
\[
\sigma_1(\Omega) \geq \frac{1}{k+1} \min \left\{ \mu, \frac{1}{C_0} \right\} \lambda_1(\Gamma),
\]

and taking \( \beta = \frac{1}{k+1} \min \left\{ \mu, \frac{1}{C_0} \right\} \), the proof of Theorem 7 is completed.

The lower bound

Let \( x : \in \ell^2(\Gamma) \) be an normalized eigenfunction corresponding to \( \lambda_1(\Gamma) \). The function \( x \) satisfy
\[
\sum_{v \in V(\Gamma)} x(v)^2 = 1 \quad \text{and} \quad \sum_{v \in V(\Gamma)} x(v) = 0.
\]

Using \( x \), a function \( f_x : \Omega \rightarrow \mathbb{R} \) is defined to be \( x(v) \) on \( \Sigma_v \) and to decay linearly to zero on the cylinder \( C_v \). The function \( f_x \) satisfy
\[
\int_{\partial \Omega} f_x = \sum_{v \in V} x(v) = 0,
\]

and can therefore be used in the variational characterization of \( \sigma_1(\Omega) \). The estimates of the Rayleigh quotient are simple and follows [7, p. 290] verbatim. We will not reproduce it here.

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