Stable Husbands

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Abstract. Suppose \( n \) boys and \( n \) girls rank each other at random. We show that any particular girl has at least \( \left( \frac{1}{2} - \epsilon \right) \ln n \) and at most \( \left( 1 + \epsilon \right) \ln n \) different husbands in the set of all Gale/Shapley stable matchings defined by these rankings, with probability approaching 1 as \( n \to \infty \), if \( \epsilon \) is any positive constant. The proof emphasizes general methods that appear to be useful for the analysis of many other combinatorial algorithms.

1. Introduction. This is a tale of \( n \) girls and \( n \) boys who play a game called “stable matching,” invented by Gale and Shapley [3]. Each player ranks each player of the opposite sex according to preference; thus, there are \( n \) permutations of the set of boys, representing the preferences of the individual girls, and there are \( n \) permutations of the set of girls, representing the preferences of the individual boys. The object of the game is for the boys and girls to match up so as to obtain \( n \) marriages that are stable, in the sense that no girl and boy prefer each other to their current partners.

For example, suppose Alice, Brigitte, Cindy, and Debra play with Wilfred, Xavier, Yuri, and Zeke; and suppose their preference rankings are as follows, from favorite to least desired:

|                  | Alice likes | Wilfred likes |
|------------------|-------------|--------------|
| Brigitte likes   | Y > X > Z > W | A > B > D > C |
| Cindy likes      | W > Y > X > Z | C > A > D > B |
| Debra likes      | X > W > Z > Y | B > D > A > C |

The matching \((AW, BX, CY, DZ)\) is unstable, because for example \( A \) prefers \( Z \) to \( W \) and at the same time \( Z \) prefers \( A \) to \( D \). But the matching \((AZ, BW, CX, DY)\) is stable; most of the players are matched with a person other than their first choice, but the objects of their affections don’t want to change. The given preferences also admit another stable matching, namely \((AY, BW, CX, DZ)\). In this example only two matchings are stable.

The stable husbands of a girl are the boys she can be married to in at least one stable matching. Thus, Alice’s stable husbands in the example are Yuri and Zeke. Brigitte has only one stable husband, namely Wilfred; she likes Xavier better, but he can’t stand her.

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2. **An algorithm.** Gale and Shapley [3] gave a procedure to find a stable matching, given any set of preferences; McVitie and Wilson [10] extended the method so that all stable matchings would be found. More recently, Gusfield [5] exploited the interesting lattice structure of stable matchings to construct an elegant algorithm that simultaneously determines the stable husbands of all girls in $O(n^2)$ steps. For our purposes in the present paper it suffices to consider a simplified variant of these procedures, which finds the stable husbands of just one given girl $G$.

The basic idea is to maintain partial matchings in which each boy who currently has a partner is paired with his best possible choice among all stable matchings of a certain class. One of the boys who doesn’t have a current partner is temporarily called $P$; he will propose to one of the girls, and she will then decide whether to accept or to reject his proposal (at least for the time being). The role of $P$ passes from boy to boy according to the following simple rules:

**A0.** Initially all boys and girls are unpaired.

**A1.** If at least one boy has no current partner, let $P$ be one such boy. Otherwise all boys and girls are already paired, and we have a stable matching; output $G$’s partner $S$ as one of her stable husbands, then remove the pair $GS$ from the current matching and let $P = S$. (Henceforth we will consider only stable matchings in which $G$ is not married to $S$.)

**A2.** If $P$ has already proposed to all the girls, terminate the algorithm. Otherwise let $H$ be the girl $P$ likes best among all those he hasn’t approached so far; $P$ now proposes to $H$.

**A3.** If girl $H$ has already been proposed to by a boy she prefers to $P$, she rejects $P$’s proposal. Otherwise, she accepts $P$, and they become paired in the current matching; her previous partner (if any) now assumes the role of $P$. If she had no previous partner, the algorithm continues at step A1, otherwise it continues at A2. ☺️
For example, suppose we run this algorithm on the preference rankings given in the introduction, always choosing the alphabetically least boy when there is a choice in step A1. Let the special girl $G$ be Alice. Then the following events occur:

| Step | Current matching | $P$  | $H$  | Actions        |
|------|------------------|------|------|----------------|
| A1   | $W$              |      |      |                |
| A2   | $W$              |      |      |                |
| A1   | $AW$             |      |      |                |
| A2   | $AW$             |      |      |                |
| A1   | $AW, CX$         |      |      |                |
| A2   | $AW, CX$         |      |      |                |
| A1   | $AW, BY, CX$     |      |      |                |
| A2   | $AW, BY, CX$     |      |      |                |
| A2   | $AZ, BY, CX$     |      |      |                |
| A2   | $AZ, BW, CX$     |      |      |                |
| A1   | $AZ, BW, CX, DY$ |      |      |                |
| A2   | $(AZ), BW, CX, DY$ |      |      |                |
| A2   | $(AZ), BW, CX, DY$ |      |      |                |
| A2   | $(AZ), BW, CX, DZ$ |      |      |                |
| A1   | $AY, BW, CX, DZ$ |      |      |                |
| A2   | $(AY), BW, CX, DZ$ |      |      |                |
| A2   | $(AY), BW, CY, DZ$ |      |      |                |
| A2   | $(AY), BW, CY, DX$ |      |      |                |

Each girl is paired with the boy who has made her the best offer so far, except that the special girl Alice never has a partner in step A2 after the first stable matching has been found. The notation ‘$(AZ)$’ in this chart means that Alice has no current partner, but that her best proposal so far has come from Zeke. In one place where ‘$(AY)$’ appears, A rejects X even though she is currently unattached, because she prefers Y to X and she will not lower her previous standards.

To prove that this algorithm finds all stable husbands of $G$, let us suppose for convenience that exactly one proposal is made per unit of time, so that the $t$th execution of step A2 takes place at time $t$. Denote by $M_t$ the set of all stable matchings such that $G$ is not married to any of the $S$’s already output before time $t$. Then $M_0$ is the set of all stable matchings, and $t \leq t'$ implies that $M_t \supseteq M_{t'}$. The correctness of the algorithm relies on
the following crucial fact:

If girl $H$ rejects a suitor $R$ between time $t$ and $t + 1$,
then the pair $HR$ is not part of any stable matching in $M_{t+1}$.

$\text{(\ast)}$

The proof is by induction on $t$. If (\ast) fails for the first time at $t$, suppose $H$ rejects $R$ in step A3 because she prefers $Q$. (Either $Q = P$ and $R$ is her previous best partner, or $R = P$ and $Q$ is her previous best.) Then we are assuming that $HR$ is part of a stable matching in $M_{t+1}$, and in this matching the stability condition tells us that boy $Q$ must be paired with some girl $J$ he prefers to $H$. But Quentin must then have proposed to Jane before he proposed to Helen, so he must have been rejected by $J$ at some time $t' < t$. Therefore, by (\ast) and induction, $JQ$ is not part of a stable matching from $M_{t'}$, contradicting the fact that $M_{t'} \supseteq M_{t+1}$. Rejection of $R$ by $H$ must therefore have occurred not in step A3 but in step A1; in other words, we must have $H = G$ and $R = S$, a stable husband. But then $M_{t+1}$ does not include $GS$, by definition. Therefore (\ast) must be true.

The matchings found in step A1 must be stable. For if some girl $H$ prefers boy $U$ to her current partner, she has not yet been proposed to by $U$; hence he prefers his current mate. In fact, (\ast) tells us that the stable matchings found in A1 are characterized by the property that each boy has his best choice among all matchings in $M_t$. Thus, when the algorithm outputs $S$, each boy has his best choice among all stable matchings such that $G$ is paired with $S$.

The algorithm terminates when some boy has been rejected by all the girls. According to (\ast), this happens at time $t$ when $M_t = \emptyset$, i.e., when all the stable husbands $S$ of $G$ have been output.
3. **A random model.** We wish to show that the algorithm just stated will produce at least \( c \ln n \) outputs with probability approaching 1, if \( c \) is any given constant \(< \frac{1}{2} \), assuming that the \( n!2^n \) possible preference sequences are selected uniformly at random.

The basic idea will be to use the principle of *late binding*, which also has been called the principle of “conservation of ignorance” or “deferred decisions”—*le principe d’ajournement des décisions* in [7]. Instead of fixing the preference sequences in advance, we simply let them unfold to whatever extent the algorithm needs them as it runs. Thus, whenever a boy is asked to propose, he proposes to a random girl chosen uniformly from among the girls he hasn’t tried yet. Whenever a girl receives her \( k \)th proposal, she accepts it with probability \( \frac{1}{k} \). A stochastic process with these characteristics is equivalent to the original algorithm running on random preference sequences, because it has the same transition probabilities between states.

We can also simplify the algorithm further by assuming that each proposal is uniformly random, as if each boy has “amnesia” [7] and cannot remember any of the girls he has previously asked. If it turns out that he has just repeated himself, we will say that he has just made a *redundant proposal*; such proposals are always rejected. The algorithm now reduces to a fairly simple stochastic process, which uses the following data structures:

- \( A_1, \ldots, A_n = \text{sets representing the girls proposed to so far by boys 1 to } n. \)
- \( l = \text{number of boys who have played the role of proposer.} \)
- \( p = \text{the boy who is currently proposing.} \)
- \( h = \text{the girl who is currently being proposed to.} \)
- \( x_1, \ldots, x_n = \text{boys who made the best offer so far to girls 1 to } n, \text{ or zero if the girl has received no offer.} \)
- \( k_1, \ldots, k_n = \text{number of proposals received by girls 1 to } n. \)

**B0.** Let \( A_j = \emptyset, x_j = 0, \text{ and } k_j = 0, \text{ for } 1 \leq j \leq n; \text{ also let } l = 0. \)

**B1.** If \( l < n \), increase \( l \) by 1 and let \( p = l \). Otherwise output \( x_g \), where \( g \) is the number of the special girl \( G; \text{ and let } p = x_g. \)

**B2.** Let \( h \) be a random number, uniformly chosen between 1 and \( n. \) (We say that boy \( p \) has proposed to girl \( h. \)) If \( h \in A_p \) (i.e., if \( p \)’s proposal is redundant), repeat this step. Otherwise replace \( A_p \) by \( A_p \cup \{h\} \) and go on to step B3.

**B3.** Increase \( k_h \) by one. With probability \( 1 - 1/k_h \), return to B2 (we say that girl \( h \) rejects the proposal). Otherwise interchange \( p \leftrightarrow x_h \) (she accepts the proposal and her former partner will have to propose to somebody else). If the new value of \( p \) is zero, or if \( h = g \) and at least one output has already occurred, go back to step B1; otherwise continue with step B2. ☺

Algorithm B faithfully models the previous Algorithm A on random input, except for the redundant proposals. Notice that the new algorithm never terminates; step A2 stops
when $P$ has nobody left to propose to, but step B2 keeps making redundant proposals ad infinitum when $A_p = \{1, \ldots, n\}$. The details of Algorithm B aren’t extremely simple, but we will see that certain aspects of its probable behavior are fairly easy to analyze, in part because it never terminates.

4. Probabilistic preliminaries. Algorithm B can be regarded as a branching process, an infinite tree with nodes at levels $t = 1, 2, 3, \ldots$ corresponding to the $t$th time step B2 is performed. Every node $\alpha$ in this tree corresponds to a unique path from the root, representing one of the possible behaviors of the algorithm up to time $t$. This path determines the values of the data structures $(A_1, \ldots, A_n, l, p_1, h, x_1, \ldots, x_n, k_1, \ldots, k_n)$ at node $\alpha$.

Every node $\alpha$ has $2^n$ children $\alpha^a_1, \alpha^r_1, \ldots, \alpha^a_n, \alpha^r_n$, where $\alpha^a_h$ and $\alpha^r_h$ represent the nodes following a proposal that has been accepted or rejected by $h$. If $h \in A_p$, the transition probability from $\alpha$ to $\alpha^a_h$ is 0 and the transition probability from $\alpha$ to $\alpha^r_h$ is $1/n$; this case corresponds to a redundant proposal, which is always rejected. If $h \notin A_p$, the transition probability from $\alpha$ to $\alpha^a_h$ is $1/(k_h + 1)n$ and the transition probability from $\alpha$ to $\alpha^r_h$ is $k_h/(k_h + 1)n$, where $k_h$ is the data value that becomes $k_h + 1$ in step B3.

The probability $\Pr(\alpha)$ of node $\alpha$ is the product of the transition probabilities on the path from the root to $\alpha$; this is the probability that Algorithm B will take the computational path represented by $\alpha$. Since the transition probabilities from each node to its children sum to 1, the sum of $\Pr(\alpha)$ for all nodes $\alpha$ on a given level $t$ is 1.

We say that an event occurs at node $\alpha$ with local probability $\rho$ if $\rho$ is the conditional probability of the event given that the algorithm reaches $\alpha$. Thus, for example, the local probability that a proposal is accepted at $\alpha$ is

$$\frac{1}{n} \sum_{h \notin A_p} \frac{1}{k_h + 1},$$

the sum of the transition probabilities in which the event occurs.

An event at $\alpha$ that depends only on the transition probabilities from $\alpha$ to its children will be called an immediate event. More general events may involve a sequence of node transitions in the subtree below node $\alpha$; all such events have local probabilities at $\alpha$ as defined above. For example, we might speak of the local probability that at most five consecutive rejections immediately follow node $\alpha$. Local probabilities at $\alpha$ are equivalent to unconditional probabilities in the branching process represented by the subtree whose root is $\alpha$.

Our proofs will often be based on a technique of probability estimation that can conveniently be called the principle of negligible perturbation. The idea will be to change the transition probabilities between certain nodes, obtaining a “perturbed” probability distribution $\Pr'$ on which it is relatively easy to compute the probability of some given
event. Let \( t \) be a level of the tree, and let \( E \) be the set of all nodes at level \( t \) such that the given event is true. Let \( C \) be the set of all nodes \( \alpha \) at level \( t \) whose probability has been perturbed somewhere along the path from the root to \( \alpha \); thus, \( \Pr(\alpha) = \Pr'(\alpha) \) for all \( \alpha \not\in C \). Summing over all \( \alpha \not\in C \) and taking complements tells us that \( \Pr(C) = \Pr'(C) \). If \( \Pr(C) \) is small, then the perturbation will have a negligible effect on the probability of \( E \), because

\[
\left| \Pr(E) - \Pr'(E) \right| = \left| \sum_{\alpha \in E} \Pr(\alpha) - \sum_{\alpha \in E} \Pr'(\alpha) \right|
\]

\[
= \left| \sum_{\alpha \in E \cap C} (\Pr(\alpha) - \Pr'(\alpha)) \right|
\]

\[
\leq \sum_{\alpha \in C} |\Pr(\alpha) - \Pr'(\alpha)|
\]

\[
\leq \sum_{\alpha \in C} |\Pr(\alpha)| + \sum_{\alpha \in C} |\Pr'(\alpha)| = 2\Pr(C) .
\]

Expected values can be estimated in a similar way.

(The principle of negligible perturbation seems almost absurdly simple, but we will see that it simplifies our analyses in surprisingly nontrivial ways. The idea is similar in spirit to Laplace’s method [9] of asymptotic analysis, where integrals are estimated by changing the integrand in unimportant portions of the domain. Another kindred method is Wilkinson’s well-known technique of “backward error analysis” [13], in which numerical errors are conveniently studied by assuming that exact answers have been obtained from approximate data; the actual situation, in which approximate answers are calculated from exact data, is more difficult to handle directly.)

Many of the proofs below are based on estimates of the tails of probability distributions, using the following fundamental inequalities that we shall call the tail inequalities: Let

\[
P(z) = p_0 + p_1 z + p_2 z^2 + \cdots = E(z^X)
\]

be the probability generating function (pgf) for a random variable \( X \) that takes nonnegative integer values. Then

\[
\Pr(X \leq r) \leq x^{-r}P(x) \quad \text{for } 0 < x \leq 1 ;
\]

\[
\Pr(X \geq r) \leq x^{-r}P(x) \quad \text{for } x \geq 1 .
\]

The proof is easy, since we have \( p_k \leq x^{-r}p_k x^k \) when \( 0 < x \leq 1 \) and \( k \leq r \), and also when \( x \geq 1 \) and \( k \geq r \). In spite of this easy proof, the tail inequalities lead to quite effective bounds because we can often choose \( x \) to make \( x^{-r}P(x) \) small.

(The history of these elementary inequalities takes us back to the early days of probability theory. Bienaymé [1] and Chebyshev [12] observed that \( \Pr((X - \mu)^2 \geq \ldots) \)
r) \leq E((X - \mu)^2)/r \text{ for all } r > 0. \text{ Kolmogorov} [8] \text{ went further and remarked that } \\
\Pr(X \geq r) \leq E(f(X))/s \text{ for any nonnegative function } f(X), \text{ provided that } E(f(X)) \text{ exists and } f(x) \geq s > 0 \text{ for all } x \geq r. \text{ In particular} [8, \text{equation 4.3.2}], \text{ we get the second tail inequality when } f(x) = e^{cx} \text{ and } c \geq 0. \text{ Chernoff} [2] \text{ pointed out the wide applicability of such estimates.)}

5. Probabilistic lemmas. Consider the behavior of Algorithm B as } n \to \infty. \text{ We will say that an event occurs almost surely, or ‘a.s.’, if the probability that it doesn’t happen is } o(1), \text{ i.e., if the probability of nonoccurrence approaches zero as } n \to \infty. \text{ We will also say that an event occurs quite surely, or ‘q.s.’, if the probability that it doesn’t happen is superpolynomially small, i.e., } O(n^{-K}) \text{ for all fixed } K. \text{ If } p(n) \text{ is any polynomial function, the sum of } O(p(n)) \text{ superpolynomially small probabilities is superpolynomially small; hence if } m = O(p(n)) \text{ and if the events } E_1, \ldots, E_m \text{ individually happen q.s., the combined event ‘} E_1 \text{ and } \ldots \text{ and } E_m \text{’ also happens q.s.}

Let } N = \lfloor n^{1+\delta} \rfloor, \text{ where } 0 < \delta < \frac{1}{2} \text{ is a constant. Throughout this section we shall consider only the first } N \text{ proposals made by Algorithm B. Thus, probabilities of events are measured by summing } \Pr(\alpha) \text{ over all nodes } \alpha \text{ at time } N + 1 \text{ such that the event occurs as the algorithm follows the path to } \alpha.

**Lemma 1.** Each girl q.s. receives at least } \frac{1}{2}n^\delta \text{ proposals and at most } 2n^\delta \text{ proposals (including redundant ones).}

The statement of this lemma and those below is deliberately somewhat ambiguous. One interpretation is that, if } g \text{ is any particular girl, she q.s. receives the stated number of proposals. Another interpretation is that q.s. all } n \text{ of the girls receive the stated number. The second statement is a corollary of the first, because of the nature of ‘q.s.’; therefore we can prove each lemma using the first (weak) interpretation, but we can apply each lemma by using the second (strong) interpretation.

**Proof.** Let } g \text{ be one of the girls, and let } E_k \text{ be the event that the } k\text{th proposal is to } g. \text{ This immediate event has local probability } \frac{1}{n}, \text{ because each proposal in step B2 is uniformly random. Therefore proposals to } g \text{ are like Bernoulli trials with parameter } \frac{1}{n}, \text{ and the pgf for the total number of proposals received by } g \text{ in the first } N \text{ levels is simply}

\[ P(z) = \left(\frac{n - 1 + z}{n}\right)^N. \]

Let } r = \frac{1}{2}n^\delta. \text{ By the first tail inequality, the probability that } g \text{ receives at most } r \text{ proposals is at most}

\[ \left(\frac{1}{2}\right)^r P\left(\frac{1}{2}\right) = 2^r \left(1 - \frac{1}{2n}\right)^{\lfloor 2nr \rfloor} \leq 2^r \left(1 - \frac{1}{2n}\right)^{2nr - 1} \leq 2^{r+1}e^{-r} \]

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since \(1 - x \leq e^{-x}\), and this is superpolynomially small.

Similarly, if \(r = 2n^\delta\), the second tail inequality tells us that \(g\) receives \(r\) or more proposals with probability at most

\[
2^{-r} P(2) = 2^{-r} \left(1 + \frac{1}{n}\right)^{\left\lfloor \frac{1}{2}nr \right\rfloor} \leq 2^{-r} \left(1 + \frac{1}{n}\right)^{\frac{1}{2}nr} \leq 2^{-r} e^{\frac{1}{2}r}
\]

since \(1 + x \leq e^x\), again superpolynomially small. ☺

Let us say that a boy begins a run of proposals when he becomes the proposer \(p\) in step B1 or B3; his run ends when one of his subsequent proposals is first accepted in step B3. In terms of the branching process, a run continues when a transition is from node \(\alpha\) to a “rejected” node of the form \(\alpha^\prime_h\), and it ends at a transition from \(\alpha\) to an “accepted” node of the form \(\alpha^a_h\).

**Lemma 2.** Each boy q.s. begins at most \(2n^\delta\) runs of proposals.

*Proof.* Let \(b\) be one of the boys. His first run of proposals begins just after \(l\) increases to \(b\) in step B1; his subsequent runs occur just after \(p\) is set to \(x_g = b\) in step B1 or to \(x_h = b\) in step B3. Thus, at most two of his runs begin immediately after \(p\) becomes \(b\) in step B1.

The other runs occur when \(p\) becomes \(b = x_h\) in step B3; and this can happen only if \(h\) is the girl who accepted \(b\) at the end of his previous run. Let \(E_t\) be the immediate event that the proposal at time \(t\) is to the girl who has most recently accepted \(b\), or to girl 1 if \(b\) has never yet been accepted. Then the number of runs begun by \(b\) is at most 2 plus the number of occurrences of \(E_t\); in other words, \(b\) can begin \(r\) or more runs only if \(E_t\) occurs \(r - 2\) or more times. But the local probability of \(E_t\) is \(\frac{1}{n}\), so again we have the binomial pgf

\[
P(x) = \left(\frac{n - 1 + x}{n}\right)^N
\]

for the distribution of occurrences of \(E_t\).

We now complete the proof as in Lemma 1, by setting \(r = 2n^\delta\); the probability of \(r\) or more runs is at most \(x^{2-r} P(x)\) for all \(x > 1\). And we have seen that this bound is superpolynomially small when \(x = 2\). ☺

**Lemma 3.** Each run q.s. contains at most \(n^\delta (\log n)^2\) non-redundant proposals.

*Proof.* We will prove that for any fixed time \(t, 1 \leq t \leq N\), a run starting at \(t\) q.s. has the stated property. Let \(\alpha\) be any node at level \(t\), and let \(P(\alpha, m, t)\) be the local probability that the proposals immediately following \(\alpha\) will include at least \(m\) rejected non-redundant proposals before reaching time \(N + 1\) or before the first acceptance, whichever comes first.
Then we have the recursive formulas

\[
P(\alpha, m, t) = \begin{cases} 
1, & \text{if } m = 0; \\
0, & \text{if } m > 0 \text{ and } t = N + 1; \\
\frac{\sum_{h \in A_p} P(\alpha_h, m, t + 1)}{n} + \frac{\sum_{h \notin A_p} k_h P(\alpha'_h, m - 1, t + 1)}{(k_h + 1)n}, & \text{otherwise.}
\end{cases}
\]

According to Lemma 1, we may assume that \(k_h \leq 2n^\delta\) for \(1 \leq h \leq n\). (The validity of this assumption is discussed below.) Then it follows by induction on \(N + 1 - t\) that

\[
P(\alpha, m, t) \leq \left(\frac{2n^\delta}{2n^\delta + 1}\right)^m.
\]

If we now choose \(m = \lfloor n^\delta (\log n)^2 \rfloor\), the local probability that there are more than \(m\) non-redundant proposals in a run starting at \(\alpha\) is at most

\[
\exp\left(\frac{-m}{2n^\delta + 1}\right) = \exp\left(-\frac{1}{2}(\log n)^2 + o(1)\right).
\]

Multiplying by \(\Pr(\alpha)\) and summing over all \(\alpha\) on level \(t\) gives a total probability of at most \(\exp\left(-\frac{1}{2}(\log n)^2 + o(1)\right)\), which is superpolynomially small. \(\square\)

The previous proof uses a convenient simplification, indicated by the words ‘According to Lemma 1, we may assume that . . .’. The assumption we are making holds q.s., but it is not always true; moreover, it is a probabilistic assertion about time \(N + 1\). So we should be careful that we are not fallaciously using the future to influence probability calculations in the past. A rigorous justification can be made by appealing to the principle of negligible perturbation: We simply recompute the transition probabilities when the assumption \(k_h \leq 2n^\delta\) is invalid.

More precisely, if \(\alpha\) is any node in the branching process, we let the perturbed transition probabilities from \(\alpha\) to \(\alpha'_h\) and \(\alpha_h'\) be \(1/(k'_h + 1)n\) and \(k'_h/(k'_h + 1)n\), respectively, where

\[
k'_h = \min(k_h, 2n^\delta).
\]

The proof of Lemma 3 is valid for the perturbed branching process, using \(k'_h\) in place of \(k_h\) in the formula for \(P(\alpha, m, t)\). Thus, the proof establishes that each run in the perturbed branching process q.s. contains at most \(n^\delta(\log n)^2\) non-redundant proposals. And this same conclusion also holds q.s. in the unperturbed branching process, because the probability of its falsity can increase by at most \(2\Pr(C)\), where \(C\) is the condition that some transition probability has been perturbed between the root and level \(N + 1\). Lemma 1 tells us that \(\Pr(C)\) is superpolynomially small, because the path to a node \(\alpha\) at level \(N + 1\) involves a perturbed transition probability only if some girl in the state represented by \(\alpha\) has received more than \(2n^\delta\) proposals before time \(N + 1\).
Lemma 4. Each boy q.s. proposes to at most $2n^{2\delta}(\log n)^2$ girls.

Proof. Multiply the results of Lemmas 2 and 3. ☐

Lemma 5. Each run q.s. contains at most $n^\delta(\log n)^2$ proposals.

Proof. Let $t$ be a fixed time, $1 \leq t \leq N$, and let $\alpha$ be any node at level $t$. A proposal is rejected with local probability $\sum_{h \in A_p} 1/n + \sum_{h \not\in A_p} k_h/(k_h + 1)n$.

By the previous lemmas and the principle of negligible perturbation, we can assume that $\|A_p\| \leq 2n^{2\delta}(\log n)^2$ and $k_h \leq 2n^\delta$. Let $\rho = \|A_p\|/n$. Then the local probability of a run continuing one more step is at most

$$\rho + (1 - \rho) \frac{2n^\delta}{2n^\delta + 1} = \frac{2n^\delta + \rho}{2n^\delta + 1} \leq \frac{2n^\delta + 2n^{2\delta - 1}(\log n)^2}{2n^\delta + 1} = \rho'.$$

(If the assumptions fail and the local probability is actually greater than this number $\rho'$, we can perturb it by artificially decreasing the probability of rejection and increasing the probability of acceptance. For example, we can define the transition probabilities from $\alpha$ to $\alpha_h^a$ and $\alpha_h^r$ to be respectively $(1 - \rho')/n$ and $\rho'/n$, for $1 \leq h \leq n$. The perturbed algorithm need not behave at all like the original algorithm does; for example, a boy’s redundant proposals might be accepted with positive probability. The principle of negligible perturbations requires only that the nodes of the tree remain the same and that the transition probabilities be consistent with all assumptions of the proof.)

Since $\delta < \frac{1}{2}$, the local probability of $m$ consecutive redundant or rejected proposals is at most

$$(\rho')^m < \left(1 - \frac{1}{3n^\delta}\right)^m$$

for sufficiently large $n$. Hence we can complete the proof as in Lemma 3. ☐

Notice that the principle of negligible perturbations has made it legitimate for us, in this proof, to estimate probabilities of events that start at time $t$ by using assumptions that might fail at some future time $> t$. (Thus, $\|A_p\|$ might be $\leq 2n^{2\delta}(\log n)^2$ at the beginning of a run but not at the end.) Arguments based on a weaker principle, which would require only that the assumptions hold at time $t$, would be more complicated; we would have to argue that $\|A_p\|$ cannot grow by more than 1 at each time step, and our upper bound would be $((\rho' + m/(n(2n^\delta + 1)))^m$ instead of $(\rho')^m$.

Lemma 6. Each boy q.s. makes at most $2n^{2\delta}(\log n)^2$ proposals.

Proof. Multiply the results of Lemmas 2 and 5. ☐

Lemma 7. Each boy q.s. proposes to a given girl at most $\log n$ times.

Proof. Let $b$ be one of the boys and let $j$ be one of the girls. Perturb the process so that after $b$ makes $n$ proposals, none of his subsequent proposals has positive probability of being
made to $j$. This perturbation is negligible, because $b$ q.s. makes fewer than $n$ proposals (Lemma 6).

Furthermore, if $b$ hasn’t made $n$ proposals by time $N+1$, pretend that he continues proposing until he has done it $n$ times. This can only increase the number of proposals he makes to $j$.

The pgf for the total number of proposals by $b$ to $j$ is then

$$P(z) = \left(\frac{n-1+z}{n}\right)^n,$$

because $b$ has amnesia; each of his $n$ proposals is uniform among the girls. The probability that he has made more than $\log n$ of them to $j$ is therefore at most

$$(\ln n)^{-\log n} \left(\frac{n-1+\ln n}{n}\right)^n = \exp\left(-(-\log n)(\ln \ln n + O(\ln n))\right),$$

and this is superpolynomially small. ☺

Incidentally, we have adopted here the convention of [4, p. 435] that ‘log’ is used for logarithms in contexts where the base is immaterial, while ‘ln’ denotes the special case of natural logs.

**Lemma 8.** Each girl q.s. receives at least $\frac{1}{2} n^\delta / \log n$ non-redundant proposals.

**Proof.** A girl receives q.s. $\frac{1}{2} n^\delta$ proposals by Lemma 1, but at most $\log n$ from any one boy by Lemma 7. ☺

6. **The main theorem.** We are almost ready to show that Algorithm B a.s. produces $\Theta(\log n)$ outputs. (This final result will be “almost sure” but not “quite sure.”) But first we need to analyze the time of the first output, because steps B1 and B3 change their behavior at that time.

The first output occurs as soon as each of the $n$ girls has received at least one proposal. We can prove that this q.s. happens long before time $N = [n^{1+\delta}]$:

**Lemma 9.** Let $N_0 = [\ln n \ln \ln n]$. Each girl q.s. receives at least one proposal and at most $\ln n (\ln \ln n)^2$ proposals during the first $N_0$ steps.

**Proof.** The pgf for proposals to $g$ satisfies

$$P(x) = \left(\frac{n-1+x}{n}\right)^{N_0} \leq \exp\left((x-1) \ln n \ln \ln n + o(1)\right)$$

for all real $x$. The probability that $g$ receives no proposal is $P(0) \leq \exp\left(-\ln n \ln \ln n + o(1)\right)$; the probability that she receives $\ln n (\ln \ln n)^2$ or more is at most

$$2^{-\ln n (\ln \ln n)^2} P(2) \leq \exp\left(-(-2)(\ln n)(\ln \ln n)^2 + \ln n \ln \ln n + o(1)\right).$$

Both of these bounds are superpolynomially small. ☺
Lemma 10. Let $\epsilon$ be a positive constant. A girl who has received $m$ non-redundant proposals will accept at least $(1-\epsilon) \ln m$ of them, with probability $1-O(m^{-\epsilon^2/2})$ as $m \to \infty$. She will accept at most $(1+\epsilon) \ln m$ of them with probability $1-O(m^{-\epsilon^2/2+\epsilon^3/6})$ as $m \to \infty$. And she will accept at most $m/(\ln m)^3$ of them with probability $1-O\left(\exp\left(-m/(\ln m)^2 + 7m (\ln \ln m)/(\ln m)^3\right)\right)$ as $m \to \infty$.

Proof. She accepts the $k$th with probability $1/k$, so the pgf for the total number of acceptances is

$$P(z) = \left(\frac{1}{1} \right) \left(\frac{1+z}{2} \right) \ldots \left(\frac{m-1+z}{m} \right) = \left(\frac{m-1+z}{m} \right)^m = \frac{1}{\Gamma(z)m^{1-z}}.$$  

(The notation $z^w = z!/z-w)!$ for factorial powers is discussed in [4, p. 211].) The probability that she accepts fewer than $(1-\epsilon) \ln m$ is at most

$$(1-\epsilon)^{(1-\epsilon)\ln m} P(1-\epsilon) = \frac{m^{-(1-\epsilon)\ln(1-\epsilon)}}{\Gamma(1-\epsilon) m^{1-\epsilon}} \leq \frac{m^{\epsilon-\epsilon^2/2}}{\Gamma(1-\epsilon) m^{1-\epsilon}},$$

and this is $O(m^{-\epsilon^2/2})$ because $m^{1-\epsilon} = m^\epsilon + O(m^{\epsilon-1})$. (See the answer to exercise 9.44 in [4].) Similarly, she accepts more than $(1+\epsilon) \ln n$ with probability at most

$$(1+\epsilon)^{(1+\epsilon)\ln m} P(1+\epsilon) \leq \frac{m^{\epsilon-\epsilon^2/2+\epsilon^3/6}}{\Gamma(1+\epsilon) m^{1-\epsilon}} = O(m^{-\epsilon^2/2+\epsilon^3/6}).$$

The probability that she accepts more than $m_0 = m/(\ln m)^3$ of them is at most

$$m_0^{-m_0} P(m_0) = \exp\left(-m_0 \ln m_0 + \ln \Gamma(m + m_0) - \ln \Gamma(m_0) - \ln m!\right)$$

$$= \exp\left(-m_0 (\ln m - 6 \ln \ln m + O(1))\right)$$

by Stirling’s approximation.

Theorem. Assume that $n$ girls and $n$ boys have independent random preference rankings, and let $G$ be one of the girls. Let $c$ be a constant $< \frac{1}{2}$ and let $C$ be a constant $> 1$. Then $G$ a.s. has at least $c \ln n$ and at most $C \ln n$ stable husbands.

Proof. The stable husbands of $G$ are output by Algorithm A, which is equivalent to Algorithm B. The number of outputs is the number of times $g$ accepts a proposal in Algorithm B, minus the number of times she accepts a proposal before the first output.

We have shown in Lemma 8 that $g$ will q.s. receive at least $\frac{1}{2} n^\delta / \log n$ non-redundant proposals, among the first $n^{1+\delta}$ proposals made by Algorithm B, if $\delta$ is any constant between 0 and $\frac{1}{2}$. Therefore, by the first estimate of Lemma 10, she will a.s. accept at least $(1-\epsilon)\delta \ln n - O(\log \log n)$ proposals.
On the other hand, \( g \) receives at most \( n \) non-redundant proposals altogether. Therefore, by the second estimate of Lemma 10, she will a.s. accept at most \((1 + \epsilon) \ln n\) of them.

Furthermore, by Lemma 9, the first output q.s. occurs before she has received \( m = \ln n \left(\ln \ln n\right)^2 \) non-redundant proposals. Therefore (by the third estimate of Lemma 10) she will accept at most

\[
\frac{m}{(\ln m)^3} = \frac{\ln n}{\ln \ln n} \left(1 + O\left(\frac{\log \log n}{\log n}\right)\right) = o(\ln n)
\]

before the first output, with probability

\[
1 - O\left(\exp\left(-\ln n + O\left(\frac{\log n \log \log n}{\log n}\right)\right)\right) = 1 - \frac{1}{n^{1+o(1)}}.
\]

So the number of outputs will a.s. exceed \( c \ln n \) and be less than \( C \ln n \) for all large \( n \), if we choose \( \delta \) and \( \epsilon \) so that \((1 - \epsilon)\delta > c \) and \( 1 + \epsilon < C \).

7. Remarks. Inspection of the proof of the theorem shows that the conclusion holds with probability \( 1 - O(n^{-\gamma}) \), where \( \gamma \) is any constant less than both \((1 - 2c)^2/2\) and \((C - 1)^2/2 - (C - 1)^3/6\). We cannot improve this estimate to \( 1 - O(n^{-1}) \), because there is probability \( \sqrt{\ln n}/n \) that the first proposal to \( G \) will come from one of her \( \sqrt{\ln n} \) favorite boys. In such a case she can have at most \( \sqrt{\ln n} \) stable husbands, because the first stable marriage found by Algorithm A gives every girl her least preferred stable husband.

Our theorem proves that random preferences a.s. guarantee an unbounded number of stable matchings, since every stable husband is part of at least one stable matching. Can it be shown that the a.s. lower bound of stable matchings grows faster than this, say as \( \Omega(\log n)^2 \)? Pittel [11] has proved that the expected number of stable matchings is asymptotically \( e^{-1} n \ln n \). However, Pittel’s theorem does not prove that a large number of matchings will almost surely occur; constructions are known [6] where certain preference matrices give rise to at least \( 2^{n-1} \) stable matchings, and such examples may be common enough to account for the relatively high expected value.

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