GROUPOID SEMIDIRECT PRODUCT FELL BUNDLES I — ACTIONS BY ISOMORPHISMS

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Abstract. Given an action of a groupoid by isomorphisms on a Fell bundle (over another groupoid), we form a semidirect-product Fell bundle, and prove that its $C^*$-algebra is isomorphic to a crossed product.

1. Introduction

Groupoid $C^*$-algebras are a natural generalization of transformation group $C^*$-algebras first studied by Effros and Hahn back in the 1960s [EH67]. Since then the construction of $C^*$-algebras from groupoids has gone through myriad generalizations including groupoid crossed products modeled on crossed-product $C^*$-algebras and many others that are all subsumed by Fell-bundle $C^*$-algebras. In [KMQW10, KMQW13], Muhly and the last three named authors consider actions of a group $G$ on a Fell bundle $p : A \to H$ by automorphisms and analyze the consequences for the group semidirect-product $A \rtimes G \to H \rtimes G$. When the group action is specialized, the authors deduce an equivalence theorem [KMQW10, Theorem 3.1] which unifies many imprimitivity theorems under the context of Fell bundles and their semidirect products.

As suggested in [KMQW10], we want to extend some of the results there to actions by groupoids. In this note, we take up this task. We let a groupoid $G$ act on another groupoid $H$ by groupoid isomorphisms. Then we can build a semidirect-product groupoid $S(H,G)$. (We have purposely avoided the usual group notation for semidirect products, $H \rtimes G$, as this coincides with the usual notation for a transformation groupoid.) Given a Fell bundle $p : A \to H$ and an action of $G$ on $A$ by isomorphisms covering the given action on $H$, we can form a Fell bundle $q : S(A,G) \to S(H,G)$. Our main result is that the corresponding Fell-bundle $C^*$-algebra $C^*(S(H,G), S(A,G))$ is isomorphic to a groupoid crossed product $C^*(H,A) \rtimes_\alpha G$ for an action $\alpha$. This leaves off the task of extending the equivalence theorem of [KMQW10] from group to groupoid actions, which we accomplish in the forthcoming article [HKQW21].

We begin in Section 2 by recording our conventions for locally compact groupoids, groupoid actions on spaces, action groupoids, upper-semicontinuous Banach bundles, Fell bundles $p : A \to G$ over a groupoid $G$, and pull-back Fell bundles.

In Section 3 we record our conventions for bundles $(T,p,B)$, morphisms of bundles, and pull-backs. In Section 4 we record our conventions for an action of a groupoid $G$ on a bundle $p : T \to B$, and in Section 5 we apply this to actions on Fell bundles over groupoids.

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In Section 6 we develop a notion of a semidirect product $S(H, G)$ of groupoids $H$ and $G$ when $G$ acts on $H$ by isomorphisms. We then promote this to semidirect-product Fell bundles where $G$ acts on a Fell bundle $p : A \to H$ by isomorphisms.

Finally, in Section 7 we prove that if $G$ acts on a Fell bundle $p : A \to H$ by isomorphisms, then, under a mild assumption involving Haar systems, the $C^*$-algebra $C^*(S(H, G), S(A, G))$ of the semidirect-product Fell bundle is isomorphic to the crossed product by an appropriate action of $G$ on the Fell-bundle algebra $C^*(H, A)$.

A alternate version of Theorem 7.3 could be derived from the unpublished preprint [BM16] of Buss and Meyer. Their “iterated-crossed-product decomposition” result (Theorem 6.1 in their paper) would exhibit $C^*(S(H, G), S(A, G))$ as the $C^*$-algebra of an abstract Fell bundle over $G$. Because of the general nature of their result, considerable work would be required to show that their Fell bundle $C^*$-algebra is isomorphic to the crossed product by an appropriate action of $G$ on the Fell-bundle algebra $C^*(H, A)$.

We also mention the work of Duwenig and Li involving Zappa-Szép products for Fell bundles over étale groupoids [DL20]. Following Brownlowe et al, they allow for two groupoids to act on each other in a way that generalizes our semidirect product. They then consider how the Zappa-Szép product of groupoids lifts to an associated product for Fell bundles over an étale groupoid. They thus recover and generalize our result, as long as one considers the significantly restrictive case of étale groupoids. In view of the present article, there is promise that the Duwenig-Li construction generalizes to non-étale groupoids, but we do not pursue this here.

2. Preliminaries

Throughout, $G$ and $H$ will be a second countable, locally compact Hausdorff groupoids with Haar systems $\lambda_G = \{ \lambda^G_u \}_{u \in G(0)}$ and $\lambda_H = \{ \lambda^H_v \}_{v \in H(0)}$, respectively. In fact, when not obviously contradicted by context, the term “groupoid” means a locally compact Hausdorff groupoid with open range and source maps.

2.1. Open Maps. Since open maps often play a key role in the theory, it is useful to have a criterion for establishing that a given map is open. The following is called “Fell’s Criterion” in [Will19 Proposition 1.1] and comes from [FD88 Proposition II.13.2].

**Lemma 2.1** (Fell’s Criterion). Suppose that $f : X \to Y$ is a surjection. Then $f$ is open if and only if given a net $\{ y_j \}_{j \in J}$ converging to $f(x)$ in $Y$, then there is a subnet $\{ y_j \}_{j \in J}$ and a net $\{ x_j \}_{j \in J}$ such that $x_j \to x$ and $f(x_j) = y_j$.

We often use Fell’s Criterion to “lift” nets as follows. Suppose that $f : X \to Y$ is an open surjection and $y_i \to f(x)$. Then we can pass to a subnet $\{ y_i \}$ and find $x_j \to x$ with $f(x_j) = y_i$. Typically, we dispense with the distracting subnet notation and replace the original net with the subnet. We summarize this by saying
that “we can replace \( \{ y_i \} \) with a subnet, relabel, and assume that are \( x_i \to x \) with \( f(x_i) = y_i \).

Nets and subnets are discussed briefly at the end of [Wil19 §1.1].

2.2. Groupoid Actions. Groupoid actions are treated in detail in [Wil19 §2.1]. In particular, in order for \( G \) to act on the left of a space \( T \), we need a continuous moment map \( \rho_T : T \to G^{(0)} \). Then we require a a continuous map from \( G \ast T = \{ (x,t) \in G \times T : s_G(x) = \rho_T(t) \} \) to \( T \) such that the usual axioms hold—see [Wil19 Definition 2.1]. This means that \( \rho_T(t) \cdot t = t \) for all \( t \in T \), \( \rho_T(x \cdot t) = r_G(x) \), and \( x \cdot (y \cdot t) = (xy) \cdot t \) whenever \( s_G(y) = \rho_T(t) \) and \( (x,y) \in G^{(2)} \). In the literature, it is often assumed that the moment map, \( \rho_T \), is open as well as continuous. Here we will only make that assumption when necessary. Except when it would be confusing otherwise, it is common to delete the subscripts from expressions such as \( s_G(x) = \rho_T(t) \) and simply write \( s(x) = \rho(t) \) assuming that the meaning is clear from context. Right actions are defined similarly except that the moment map is usually denoted by \( \sigma_T \) and then \( x \cdot t \) is defined when \( r_G(x) = \sigma_T(t) \).

Notation. We employ the following standard notation for a \( G \)-space with moment map \( \rho_T : T \to G^{(0)} \), \( N \subset G \), and \( S \subset T \):

(a) \( N \ast S = \{ (x,t) \in N \times S : s(x) = \rho(t) \} \);
(b) \( N \cdot S = \{ x \cdot t : (x,t) \in N \ast S \} \);
(c) \( S \ast T = \{ (t,u) \in S \times T : u \in G \cdot t \} \).

Definition 2.2. Let a locally compact groupoid \( G \) act on the left of a locally compact Hausdorff space \( T \). Then the action groupoid, \( G \rtimes T \), is the space \( G \ast T = \{ (x,t) \in G \times T : s(x) = \rho(t) \} \) with operations given by

\[
(x,t)(y,s) = (xy,s) \quad \text{if } t = y \cdot s \quad \text{and} \quad (x,t)^{-1} = (x^{-1}, x \cdot t).
\]

Notice that the range and source maps are given by

\[
r(x,t) = (r(x), x \cdot t) \quad \text{and} \quad s(x,t) = (s(x), t).
\]

Then the unit space is

\[
(G \rtimes T)^{(0)} = \{ (u,t) \in G^{(0)} \times T : u = \rho(t) \}.
\]

Since the coordinate projection

\[
\pi_2 : (G \rtimes T)^{(0)} \to T
\]

is a homeomorphism, it is common practice to identify \( (G \rtimes T)^{(0)} \) with \( T \). The action groupoid \( G \rtimes T \) has open range and source maps when \( G \) does.

Remark 2.3. [Action Groupoids in the Literature] In the literature, the formulation of an action groupoids—also called transformation groupoids—is not consistent. If \( T \) is a left \( G \)-space, then in [Wil19 Definition 2.5] the action groupoid \( A(G,T) \) is the set \( \{(t,x,t') \in T \times G \times T : t = x \cdot t' \} \) of triples equipped with the relative product topology and with the natural groupoid operations: \( (t,x,t')(t',y,t'') = (t,xy,t'') \) and \( (t,x,t')^{-1} = (t',x^{-1},t) \). Using triples makes the definition both natural and transparent. It is also redundant. As a result, it is common practice to denote elements of \( A(G,T) \) more compactly as pairs in \( G \times T \). Our definition of \( G \rtimes T \) above is pulled back from the homeomorphism \( (x,t) \mapsto (x \cdot t, x,t) \) of \( G \ast T \) onto \( A(G,T) \). However, one can also consider the set \( \{(x,t) : r(x) = \rho(t) \} \) and pull-back a groupoid structure via the map \( (x,t) \mapsto (x,t,x^{-1} \cdot t) \). This groupoid structure
has the advantage that if $G$ is a group, then we arrive at the usual formulas for transformation group algebras. The convention here has the advantage that it is the same as used in [KMQW13, Appendix A.1]. However, it differs, for example, from that used in [Wil09] or [ADR00]. Of course, there are similar considerations for right actions.

2.3. Banach Bundles.

**Definition 2.4.** An (upper-semicontinuous) Banach bundle over a space $X$ is a topological space $A$ together with a continuous open surjection $p : A \to X$ and Banach space structures on the fibres $A(x) = p^{-1}(x)$ such that the following axioms hold.

1. $a \mapsto \|a\|$ is upper-semicontinuous from $A$ to $\mathbb{R}^+$.
2. $(a, b) \mapsto a + b$ is continuous from $A \times A = \{(a, b) : p(a) = p(b)\}$ to $A$.
3. $(\lambda, a) \mapsto \lambda a$ is continuous from $\mathbb{C} \times A$ to $A$.
4. If $\{a_i\}$ is a net in $A$ such that $p(a_i) \to x$ and $\|a_i\| \to 0$, then $a_i \to 0_x$ where $0_x$ is the zero element in $A(x)$.

We use $A(x)$ to denote the fibre over $x$—rather than $A_x$, say—to emphasize that the fibre comes equipped with its own fixed Banach space structure.

In addition each fibre $A(x)$ is a $C^*$-algebra and $(a, b) \mapsto ab$ and $a \mapsto a^*$ are continuous on $A \times A$ and $A$, respectively, then we call $A$ a (upper-semicontinuous) $C^*$-bundle. If axiom (B1) is replaced by “$a \mapsto \|a\|$ is continuous”, then we call $A$ a continuous Banach bundle or a continuous $C^*$-bundle. Normally, we drop the adjective “upper-semicontinuous” and add “continuous” only when specializing to that case.

We use §§13–14 of [FD88, Chap. II] as a good reference for continuous Banach bundles. For the general case, see [MW08a, Appendix A], and for the $C^*$-case, [Wil07, Appendix C]. A primary motivation for working with general Banach bundles rather than continuous ones is that a $C_0(X)$-algebra $A$ is always $C_0(X)$-isomorphic to $\Gamma_0(X, A)$ for an appropriate (upper-semicontinuous) Banach bundle $A$ [Wil07, Theorem C.26].

2.4. Fell Bundles. For Fell bundles we will refer to [MW08a].

**Definition 2.5 (MW08a, Definition 1.1)).** Suppose that $p : A \to G$ is a separable Banach bundle over a second countable locally compact Hausdorff groupoid $G$. Let $A^{(2)} = \{(a, b) \in A \times A : (p(a), p(b)) \in G^{(2)}\}$. Then we call $A$ a Fell bundle if there is a continuous, bilinear, associative multiplication map $(a, b) \mapsto ab$ from $A^{(2)} \to A$ and a continuous involution $a \mapsto a^*$ from $A \to A$ such that

1. $p(ab) = p(a)p(b),$
2. $p(a^*) = p(a)^{-1},$
3. $(ab)^* = b^*a^*,$
4. for each $u \in G^{(0)}$, the Banach space fibre $A(u)$ is a $C^*$-algebra with respect to the structure induced by the multiplication and involution on $A,$ and
5. for each $x \in G$, the Banach space fibre $A(x)$ is an $A(r(x)) = A(s(x))$-imprimitive bimodule when equipped with the actions determined by multiplication and the inner products

\[ A(r(x))\langle a, b \rangle = ab^* \quad \text{and} \quad \langle a, b \rangle A(s(x)) = a^*b. \]
If \( p : A \to H \) is a Fell bundle over a locally compact groupoid (and we tacitly assume that our groupoids are Hausdorff), then to ease the notational burden, for \( a \in A \), we will write \( r(a) \) in place of \( r_H(p(a)) \) and similarly for \( s(a) \).

**Remark 2.6 (Pull-Back Fell Bundles).** Suppose that \( p : A \to G \) is a Fell bundle and that \( \phi : H \to G \) is a groupoid homomorphism. Then the pull-back,

\[
\phi^* A = \{ (h, a) \in H \times A : \phi(h) = p(a) \}
\]

is a Fell bundle over \( H \) with respect to \((h, a)(h', a') = (hh', aa') \) and \((h, a)^* = (h^{-1}, a^*)\).

**2.5. Fell Bundle Maps.** If \( p : A \to H \) and \( p' : B \to G \) are Fell bundles. Then a **Fell-bundle map** \( \phi : A \to B \) is a bundle map

\[
\begin{array}{cc}
A & \longrightarrow & B \\
\phi & \downarrow & \\
H & \longrightarrow & G \\
p & \downarrow & p'
\end{array}
\]

such that the induced map \( \tilde{\phi} \) is a groupoid homomorphism and such that \( \phi \) preserves multiplication and involution. That is, \((a, a') \in A^{(2)} \) implies \((\phi(a), \phi(a')) \in B^{(2)} \) and then \( \phi(ab) = \phi(a)\phi(b) \), and of course, we also require \( \phi(a^*) = \phi(a)^* \). Note that if \( v \in H^{(0)} \), then \( \phi \) induces a **-homomorphism** of \( A(v) \) to \( B(\tilde{\phi}(v)) \). Then for all \( a \in A \), we have

\[
\|\phi(a)\|^2 = \|\phi(a^*a)\| \leq \|a^*a\| = \|a\|^2,
\]

and \( \phi \) must be norm decreasing. Therefore if \( \phi \) is a **Fell-bundle isomorphism**—that is, if \( \tilde{\phi} \) is a groupoid isomorphism and \( \phi \) is a homeomorphism—then \( \phi \) is isometric.

**2.6. Fell Bundle \( C^* \)-Algebras.** If \( p : A \to G \) is a Fell bundle and \( G \) has a Haar system, then \( \Gamma_c(G, A) \) has a natural **-algebra structure**:

\[
f \ast g(x) = \int_G f(y) g(y^{-1} x) \, d\mu_G^x(y) \quad \text{and} \quad f^*(x) = f(x^{-1})^*.\]

The corresponding **Fell-bundle \( C^* \)-algebra** is the completion \( C^*(G, A) \) with respect to the universal norm on \( \Gamma_c(G, A) \) induced by suitably bounded **-representations** of \( \Gamma_c(G, A) \) on Hilbert space as in [MW08a §1].

**2.7. Groupoid Crossed Products.** In our main theorem, we will want to exhibit a Fell-bundle \( C^* \)-algebra as the \( C^* \)-algebra of a groupoid crossed product. We will refer to [MW08a] for the basic theory, although Goehle’s thesis [Goe99] is very good resource. We recall the basics here.

**Definition 2.7.** Suppose that \( G \) is a locally compact groupoid and that \( A \) is a \( C_0(G^{(0)}) \)-algebra such that \( A = \Gamma_0(G^{(0)}, A) \) for a \( C^* \)-bundle over \( G^{(0)} \). An **action** \( \alpha \) of \( G \) on \( A \) by **-isomorphisms** is a family \( \{ \alpha_x \}_{x \in G} \) such that

1. for each \( x \in G \), \( \alpha_x : A(s(x)) \to A(r(x)) \) is an isomorphism,
2. for all \((x, y) \in G^{(2)}\), \( \alpha_{xy} = \alpha_x \circ \alpha_y \), and
3. \( x \cdot a = \alpha_x(a) \) defines a continuous action of \( G \) on \( A \).

The triple \((A, G, \alpha)\) is called a **groupoid dynamical system**.
Classically, the groupoid crossed product $A \rtimes_\alpha G$ is built out of the sections $\Gamma_c(G, r^*A)$. In the sequel, we will use the observation from [MW08a, §2] that we can realize $A \rtimes_\alpha G$ as the Fell-bundle $C^*$-algebra $C^*(G, B)$ where $B = r^*A = \{(a, x) \in A \times G : p(a) = r(x)\}$ with multiplication given by $(a, x)(b, y) = (a\alpha_x(b), xy)$ and involution by $(a, x)^* = (\alpha^{-1}_x(a), x^{-1})$.

3. Bundles and Actions

Here we will use the term bundle for a triple $(T, p, B)$ consisting of a continuous open surjection $p : T \to B$. We call $T$ the total space and $B$ the base space of the bundle. We are using a definition of "bundle" that is suitable for the context in which we work. In [Hus94], a bundle $p : T \to B$ is merely a continuous map. We add the requirements that $p$ be open and surjective, but we do not require local triviality in any form.

If $T$ is a left $G$-space, then we call a bundle $(T, p, B)$ a $G$-bundle if $p(t) = p(t')$ if and only if $t$ and $t'$ are in the same orbit. Alternatively, $p : T \to B$ is a $G$-bundle if and only if there is a homeomorphism $\bar{p}$ of $B$ with the orbit space $G \setminus T$ such that the diagram

$$
\begin{array}{ccc}
T & \xrightarrow{p} & B \\
\downarrow{q} & & \downarrow{\bar{p}} \\
G \setminus T & \xrightarrow{\rho_T} & B
\end{array}
$$

commutes.

Let $(T, p, B)$ and $(Y, q, C)$ be $G$-bundles. Then a $G$-bundle morphism $(f, g) : (T, p, B) \to (Y, q, C)$ is a pair of continuous maps such that $f$ is $G$-equivariant and such that the diagram

$$
\begin{array}{ccc}
T & \xrightarrow{f} & Y \\
\downarrow{p} & & \downarrow{q} \\
B & \xrightarrow{g} & C
\end{array}
$$

commutes. If $B = C$ and $g = \text{id}_B$, we say $f : T \to Y$ is a $G$-bundle morphism over $B$. In any event, we say that $(f, g)$ is an isomorphism if $f$ is a homeomorphism.

If $T$ and $U$ are $G$-spaces, then the fibred product $T *_r U = \{(t, u) : \rho_T(t) = \rho_U(u)\}$ admits a diagonal $G$-action: $x \cdot (t, u) = (x \cdot t, x \cdot u)$ with $\rho(t, u) = \rho_T(t) = \rho_U(u)$.

Lemma 3.1 (Diagonal Actions). If $T$ and $U$ are $G$-spaces, then the moment map for the diagonal action of $G$ on $T *_r U$ is open if both $\rho_T$ and $\rho_U$ are.

Proof. We use Fell’s Criterion (Lemma 2.1): suppose $v_i \to \rho_{T*U}(t, u)$. Since $\rho_T$ is open, we can pass to subnet, relabel, and assume that there are $t_i \to t$ with $\rho_T(t_i) = v_i$. Since $\rho_U$ is also open, we can pass to another subnet, relabel, and assume that there are $u_i \to u$ such that $\rho_U(u_i) = v_i$. But then $(t_i, u_i) \to (t, u)$. □
If \((T, p, B)\) is a bundle and \(f : C \to B\) is continuous, then we can form the pull-back \(f^*T = C_f *_p T = \{(c, t) : f(c) = p(t)\}\) so that the diagram

\[
\begin{array}{ccc}
\pi_1 & \to & T \\
\downarrow & & \downarrow p \\
C & \to & B
\end{array}
\]

commutes.

**Lemma 3.2** (Pull-Backs). Let \((T, p, B)\) be a bundle and \(f : C \to B\) a continuous map. Then \((f^*T, \pi_1, C)\) is a bundle where \(\pi_1\) is the projection onto the first factor. If \(T\) is a \(G\)-bundle, then so is \(f^*T\) where the \(G\)-action is given by \(x \cdot (c, t) = (c, x \cdot t)\). Furthermore, \(f^*T\) has an open moment map whenever \(T\) does.

**Proof.** To see that \(\pi_1\) is open, we use Fell's Criterion. Suppose that \(c_i \to \pi_1(c, t)\). Then \(f(c_i) \to f(c) = p(t)\). Since \(p\) is open, we can pass to subnet, relabel, and assume that there are \(t_i \to t\) in \(T\) with \(p(t_i) = f(c_i)\). But then \((c_i, t_i) \to (c, t)\) in \(f^*(T)\). Hence the pull-back is always a bundle over \(C\).

Now suppose that \(T\) is a \(G\)-bundle. If \((c, t)\) and \((c, t')\) both belong to \(f^*(T)\), then \(p(t) = f(c) = p(t')\) and there is \(x \in G\) such that \(t' = x \cdot t\). Then \((c, t') = x \cdot (c, t)\). Since the converse is clear, \(\pi_1 : f^*T \to C\) is a \(G\)-bundle.

Now suppose that \(\rho_T : T \to G^{(0)}\) is open. To see that \(\rho_T \cdot f^*\) is open, we suppose that \(v_i \to v = r(c, t) = \rho_T(t)\). Since \(\rho_T\) is open, after passing to a subnet and relabeling, we can assume that there are \(t_i \to t\) with \(\rho_T(t_i) = v_i\). Then \(p(t_i) \to p(t) = f(c)\) in \(B\). Since we are now assuming that \(f\) is open, we can pass to a subnet, relabel, and assume that there are \(c_i \to c\) in \(C\) such that \(f(c_i) = p(t_i)\). Then \((c_i, t_i) \to (c, t)\) as required. \(\square\)

Note that if \((T, p, B)\) is a \(G\)-bundle and \(f : C \to B\) is a continuous map, then \((\pi_2, f) : (f^*T, \pi_1, C) \to (T, p, B)\) is bundle morphism.

### 4. Actions on bundles

Everything in this section is surely standard, although we could not find it in the literature in the precise context we need; we include the details for convenient reference.

**Definition 4.1.** Let \(G\) be a locally compact groupoid, and let \(p : T \to B\) be a bundle. We say that \(G\ acts on the bundle \(p : T \to B\) if \(G\) acts on both the total space \(T\) and the base space \(B\) and the bundle map is equivariant. In particular, \(p\) intertwines the moment maps: \(\rho_B \circ p = \rho_T\).

**Remark 4.2.** Let \(p : T \to B\) be a bundle and \(C \subset B\) a closed subset, and put \(T|_C = p^{-1}(C)\). Then we have a restricted bundle

\[
p|_C : T|_C \to C.
\]

The only non-obvious property to check is that the restriction \(p|_C\) is open: let \(c_i \to p(t)\) in \(C\). Since \(p : T \to B\) is open, after replacing by a subnet and relabeling we can find \(t_i \in p^{-1}(c_i)\) such that \(t_i \to t\) in \(T\). Then in particular \(t_i \in p_u^{-1}(c_i)\) and \(t_i \to t\) in \(T|_C = p^{-1}(C)\).
Remark 4.3. Let $G$ act on a bundle $p : T \to B$, and for $u \in G^{(0)}$ let $B_u = \rho_B^{-1}(u)$. Then for all $x \in G$ we have a bundle isomorphism

$$
\begin{array}{ccc}
T|_{B_x(x)} & \xrightarrow{t \mapsto x \cdot t} & T|_{B_x(x)} \\
\downarrow \rho & & \downarrow \rho \\
B_x(x) & \xrightarrow{b \mapsto x \cdot b} & B_{r(x)}.
\end{array}
$$

In particular, the upper horizontal map is a homeomorphism between the respective subsets of $T$.

5. Actions by isomorphisms

In this section we adapt some of [KMQW13, Appendix A] to groupoid actions.

Definition 5.1. A groupoid bundle is a bundle $p : H \to B$ in which $B$ is a locally compact space, $H$ is a groupoid, and each fibre $H_b := p^{-1}(b)$ is a subgroupoid of $H$.

Remark 5.2. If $p : H \to B$ is a groupoid bundle, then $h \in p^{-1}(b)$ implies $r(h) = hh^{-1} \in p^{-1}(b)$. Thus $h \in H_b$ if and only if $p(r(h)) = b$. Thus if $p_0 = p|_{H^{(0)}}$, then $p_0 \circ r = p_0 \circ s$ and $H_b$ is the reduction $H|_{p_0^{-1}(b)} = p_0^{-1}(b)H p_0^{-1}(b)$ and $H$ is a bundle of groupoids as in [Wil19, Definition 1.16].

Definition 5.3. Let $H$ be a locally compact groupoid, and let $G$ act on the space $H$. We say $G$ acts by isomorphisms if the moment map $\rho_H : H \to G^{(0)}$ is a groupoid bundle and for each $x \in G$ the map

$$
t \mapsto x \cdot t : H_{s(x)} \to H_{r(x)}
$$

is a groupoid isomorphism.

Remark 5.4. In the unpublished preprint [BMT16, Definition 2.12], actions by isomorphisms are called “classical actions”.

Remark 5.5. If $G$ acts on $H$ by isomorphisms, then $H^{(0)}$ is a $G$-invariant subset. Moreover, $r_H(x \cdot h) = x \cdot r_H(h)$.

Example 5.6. Let $H = \{ (x, y, z) \in G \times G \times G : s(x) = r(y) \text{ and } z = yx \}$ be the action groupoid for the right action of $G$ on itself, with operations given by $(x, y, z)(z', y', x') = (x, yy', x')$ and $(x, y, z)^{-1} = (z, y^{-1}, x)$. Then $G$ acts on the left of $H$: $w \cdot (x, y, xy) = (wx, y, wxy)$. Let $\rho_H(x, y, z) = r(x)$. Then $H_u = \{ (x, y, xy) \in H : r(x) = u \} = G^u \ltimes G$ is easily seen to be a subgroupoid of $H$ and $\rho_H$ is a groupoid bundle. Furthermore,

$$w \cdot ((x, y, xy)(xy, z, xyz)) = w \cdot (x, yz, xyz) = (wx, yz, wxyz),$$

and on the other hand,

$$(w \cdot (x, y, xy))(w \cdot (x, y, z, xyz)) = (wx, y, wxy)(wxy, z, wxyz) = (wx, yz, wxyz).$$

Hence $G$ acts on $H$ by isomorphisms.

Definition 5.7 (cf., [KMQW10, Definition 6.3]). Suppose that $H$ has a Haar system $\lambda_H = \{ \lambda_H^v \}_{v \in H^{(0)}}$. Then we say that an action of $G$ on $H$ by isomorphisms is invariant if

$$
\int_H f(x \cdot h) \, d\lambda_H^v(h) = \int_H f(h) \, d\lambda_H^v(h)
$$

for all $f \in C_b(H)$.
for all $f \in C_c(H)$, $x \in G$, and $v \in H^{(0)}$.

**Remark 5.8.** While it might be more appropriate to describe actions satisfying (5.1) as “equivariant”, the term “invariant” is more common in the literature (see [Wil19 Remark 3.12]). In any event, if we view $H$ and $H^{(0)}$ as $G$-spaces, then the invariance of the $G$-action translates to saying that $\lambda_H$ is a “full invariant $r_H$-system” as defined in [Wil19 §3.2].

Now we promote the preceding to Fell bundles.

**Definition 5.9.** Let a locally compact groupoid $G$ act on a Fell bundle $p : \mathcal{A} \to H$ in the sense of Definition 4.1 with respective moment maps $\rho_A$ and $\rho_H$. We say that $G$ acts on $\mathcal{A}$ by isomorphisms if

(a) $G$ acts on the groupoid $H$ by isomorphisms, and

(b) for each $x \in vGu$ the map

$$a \mapsto x \cdot a : \mathcal{A}|_{H_u} \to \mathcal{A}|_{H_v}$$

is a Fell-bundle isomorphism.

Note that we have a restricted Fell bundle $p_u : \mathcal{A}|_{H_u} \to H_u$ for every $u \in G^{(0)}$.

**Example 5.10.** As in Example 5.9 let $H$ be the action groupoid for the right action of $G$ on itself. Let $p : \mathcal{A} \to G$ be a Fell bundle. Then $\phi(x, y, xy) = y$ is a homomorphism and we can form the pull-back Fell bundle $\phi^*\mathcal{A} = \{(x, y, xy, a) : y = \rho(a)\}$. Then $G$ acts on $\phi^*\mathcal{A}$ by isomorphisms where $z \cdot (x, y, xy, a) = (zx, y, zxy, a)$ (which clearly covers the action of $G$ on $H$ by isomorphisms given in Example 5.6).

6. **Semidirect products**

When $G$ and $H$ are groups and $G$ acts on $H$ via automorphisms, then we can construct their semidirect-product $S(H, G)$ containing $H$ as a normal subgroup. In this section we develop an analogue of this for groupoids.

**Definition 6.1.** Suppose that $G$ and $H$ are locally compact groupoids such that $G$ acts on $H$ by isomorphisms. Then the **semidirect-product groupoid** is the set $S(H, G) = \{(h, x) \in H \times G : \rho(h) = r(x)\}$, where we declare $(h, x)$ and $(k, y)$ to be composable if $s(h) = x \cdot r(k)$. Then we define

$$s(h, x)(k, y) = (h(x \cdot k), xy) \quad \text{and} \quad (h, x)^{-1} = (x^{-1} \cdot h^{-1}, x^{-1}).$$

If $s(h) = x \cdot r(k)$, then $s(h) = r(x)$ and $s(x) = \rho(r(k)) = \rho(k) = r(y)$. Hence $x \cdot k$ is defined, $(h, x \cdot k) \in H^{(2)}$ and $(x, y) \in G^{(2)}$. So the product in (6.1) makes sense. Then routine computations—say following [Wil19 Definition 1.2]—show that $S(H, G)$ is a groupoid with

$$s(h, x) = (x^{-1} \cdot s_H(h), s_G(x)) \quad \text{and} \quad r(h, x) = (r_H(h), r_G(x)).$$

In particular,

$$S(H, G)^{(0)} = \{(v, u) \in H^{(0)} \times G^{(0)} : \rho_H(v) = u\},$$

and the coordinate projection $\pi_1 : S(H, G)^{(0)} \to H^{(0)}$ is a homeomorphism.

**Example 6.2.** Suppose that $G$ acts on the left of a space $H$. Then $G$ acts on $H$ by isomorphisms when $H$ is viewed as a groupoid. Then the map $(h, x) \mapsto (x, x^{-1}h)$ is a groupoid isomorphism of the semidirect product $S(H, G)$ onto the action groupoid $G \rtimes H$. 

\[\]
We denote the pull-back with these operations by \( S_{a,x} \). Let us define the Fell-bundle operations by \( p \). It suffices to integrate over \( H \).

**Remark 6.3.** Although their notation is slightly different, in \([BM16]\) Buss and Meyer introduce semidirect products calling them “transformation groupoids for classical actions of \( G \) on \( H \).” If \( G \) and \( H \) have Haar systems and \( G \) acts on \( H \) by isomorphisms, then Buss and Meyer show in \([BM16]\) Theorem 5.1 that \( S(H,G) \) always has a Haar system. Since we want the additional structure of a dynamical system, we have to require that, in addition, \( H \) have an invariant Haar system. Since this makes constructing a Haar system on the semidirect product routine, we give the elementary construction that is required for our purposes.

For \( \varepsilon \)-at groupoids, Brownlowe et al. \([BPR+17]\) introduced Zappa-Szep products of groupoids, which generalize our semidirect products in that context.

**Lemma 6.4.** Suppose that \( G \) acts on \( H \) by isomorphisms and that \( H \) has an invariant Haar system \( \lambda_{H} = \{ \lambda_{H}^{v} \}_{v \in H^{\text{aut}}} \) so that

\[
\int_{H} f(x \cdot t) \, d\lambda_{H}^{v}(t) = \int_{H} f(t) \, d\lambda_{H}^{v}(t).
\]

Then we get a Haar system on the semidirect product \( S(H,G) \) by

\[
d\lambda^{(x,w)}(t, x) = d\lambda_{H}^{v}(t) \, d\lambda_{G}^{w}(x).
\]

**Proof.** It suffices to integrate over \( S(H,G) \) functions of the form \( f \otimes g \) for \( f \in C_{c}(H) \) and \( g \in C_{c}(G) \):

\[
\begin{align*}
\int_{S(H,G)} (f \otimes g)((t,x)(u,y)) \, d\lambda^{s(t,x)}(u,y) &= \int_{G} \int_{H} (f \otimes g)(t(x \cdot u), xy) \, d\lambda^{(x^{-1} \cdot s(t), s(x))}(u,y) \\
&= \int_{H} f(t(x \cdot u)) \, d\lambda_{H}^{s(t)}(u) \int_{G} g(xy) \, d\lambda^{s(x)}(y) \\
&= \int_{H} f(tu) \, d\lambda_{H}^{s(t)}(u) \int_{G} g(y) \, d\lambda^{r(x)}(y) \quad \text{(invariance)} \\
&= \int_{H} f(u) \, d\lambda_{H}^{r(t)}(u) \int_{G} g(y) \, d\lambda^{r(x)}(y) \\
&= \int_{S(H,G)} (f \otimes g)(u,y) \, d\lambda^{r(t,x)}(u,y).
\end{align*}
\]

Suppose that \( p : \mathcal{A} \to H \) is a Fell bundle and that \( G \) acts on \( \mathcal{A} \) by isomorphisms. We get a Banach bundle over \( S(H,G) \) via the pull-back \( \pi^{*} \mathcal{A} = \{ (a,h,x) \in \mathcal{A} \times S(H,G) : p(a) = h \} \). As usual, we adopt a more compact notation and write elements of this pull-back as pairs \((a,x)\) in \( \mathcal{A} \times G \) such that \( p(p(a)) = r(x) \). Then we define the Fell-bundle operations by

\[(a,x)(b,y) = (a(x \cdot b), xy) \quad \text{and} \quad (a,x)^{*} = (x^{-1} \cdot a^{*}, x^{-1}).\]

We denote the pull-back with these operations by \( S(\mathcal{A},G) \) and write \( \pi' \) for the projection \( \pi'(a,x) = (p(a), x) \) onto \( S(H,G) \). Then we can generalize the constructions in \([KMQW10]\) Section 6 as follows.

**Lemma 6.5.** With the operations defined above, \( \pi' : S(\mathcal{A},G) \to S(H,G) \) is a Fell bundle.
Proof. We have the candidates for multiplication and involution defined above, so let’s proceed with checking Definition 2.5 with respect to these operations. It is routine to verify that the multiplication is continuous, bilinear, and associative, and the involution is continuous. Notice that $p$ respects all of the operations on $A$ and is also equivariant for the $G$ action. Moreover, $G$ acts on $A$ by isomorphisms.

For the multiplication condition, see that

$$p'(a, x)(b, y) = p'(a x b, xy)$$

$$= \left(p(a x b), xy\right)$$

$$= \left(p(a)p(x \cdot b), xy\right)$$

$$= \left(p(a)(x \cdot p(b)), xy\right)$$

$$= \left(p(a), x\right)(p(b), y)$$

$$= p'(a, x)p'(b, y).$$

For the compatibility of the projection with involution,

$$p'(a, x)^* = p'(x^{-1} \cdot a^*, x^{-1})$$

$$= \left(p(x^{-1} \cdot a^*), x^{-1}\right)$$

$$= \left(x^{-1} \cdot p(a)^*, x^{-1}\right)$$

$$= \left(p(a), x\right)^*$$

$$= p'(a, x)^*.$$

Next, we check that the involution on $S(A, G)$ is, in fact, involutive:

$$((a, x)(b, y))^* = (a x b, xy)^*$$

$$= \left(\left(xy\right)^{-1} \cdot (a x b)^*, (xy)^{-1}\right)$$

$$= \left(y^{-1} x^{-1} \cdot (x \cdot b^*) a^*, y^{-1} x^{-1}\right)$$

$$= \left((y^{-1} x^{-1} \cdot (x \cdot b^*)) \left(y^{-1} x^{-1} \cdot a^*\right), y^{-1} x^{-1}\right)$$

$$= \left(\left(y^{-1} \cdot b^*\right) \left(y^{-1} x^{-1} \cdot a^*\right), y^{-1} x^{-1}\right)$$

$$= \left((y^{-1} \cdot b^*) (y^{-1} \cdot (x^{-1} \cdot a^*)), y^{-1} x^{-1}\right)$$

$$= \left(y^{-1} \cdot b^*, y^{-1}(x^{-1} \cdot a^*, x^{-1}\right)$$

$$= \left(b, y\right)^*(a, x)^*.$$

Before turning to the final conditions, notice that the fibre over $(h, x) \in S(H, G)$ can be identified as a Banach space with $A(h)$ via $(a, x) \mapsto a$, and $A(h)$ is, by assumption, an $A(v(h)) - A(s(h))$-imprimitivity bimodule with respect to the operations inherited from $A$. Likewise, the fibre over a unit $(u, v) \in S(H, G)^{(0)}$ is identified with the $C^*$-algebra $A(u)$. The operations on $A(u, v)$ correspond to those of $A(u)$, so $A(u, v)$ is a $C^*$-algebra. Similarly, on the left-hand side the module and inner product operations on $A(h, x)$ correspond to those on $A(h)$. On the right-hand side, there is a subtlety: the module and inner product on $A(h, x)$ correspond to those on $A(h)$ provided that the $C^*$-algebra $A(s(h, x))$ is identified with $A(s(h))$. 
via the map \((a, s(x)) \mapsto x \cdot a\). More precisely,
\[ (a, x) \cdot (b, u) = (a(x \cdot b), x), \]
which, after \((b, u)\) is sent to \(x \cdot b\), corresponds to \(a \cdot c = ac\) for \(a \in A(h), c \in A(s(h))\), and
\[ (a, x)^* (b, x) = (x^{-1} \cdot (a^* b), s(x)), \]
which would be sent to the element \(a^* b \in A(s(h))\).

\[\Box\]

**Remark 6.6.** Duwenig and Li [DL20, Theorem 3.8] studied Zappa-Szép products of Fell bundles over étale groupoids, which generalize our semidirect-product Fell bundles in that context.

### 7. Crossed Products

We want to see that there is a natural groupoid dynamical system associated to a groupoid action on a Fell bundle by isomorphisms. For this, we require that \(H\) has an invariant Haar system \(\lambda_H = \{ \lambda_H^v \}_{v \in H(0)}\). Furthermore, if \(u \in G(0)\), then \(H_u = \{ h \in H : \rho(h) = u \}\) is the groupoid \(\rho^{-1}(u)H \rho^{-1}(u)\) and \(\{ \lambda_H^v \}_{v \in A(0)}\) is a Haar system for \(H_u\). We fix a Fell bundle \(p : \mathcal{A} \to H\) and an action of \(G\) by isomorphisms on \(\mathcal{A}\). Let \(\rho_H : H \to G(0)\) be the moment map. If \(\phi \in C_b(G(0))\) and \(f, g \in \Gamma_c(H, \mathcal{A})\), then let
\[ V(\phi)f(h) = \phi(\rho_H(h)) f(h). \]
If \(f, g \in \Gamma_c(H, \mathcal{A})\), then
\[ V(\phi)(f * g)(h) = \phi(\rho_H(h)) \int_H f(k)g(k^{-1}h) \, d\lambda_H^{r(h)}(k) \]
which, since \(\rho_H\) is constant on \(H_{r(h)}\) by Remark 5.2, is
\[ = \int_H f(k)\phi(\rho_H(k^{-1}h)g(k^{-1}h) \, d\lambda_H^{r(h)}(k) = f * (V(\phi)g)(h). \]

We can view \(M(C^*(H, \mathcal{A}))\) as the collection of adjointable operators on \(C^*(H, \mathcal{A})\) viewed a right Hilbert module over itself with the inner product \(\langle f, g \rangle = f^* g\) [RW98, Definition 2.48]. Since \(V(\phi)f^* = (V(\phi)f)^*\), it follows from
\[ (7.1) \]
\[ V(\phi)(f * g) = f * (V(\phi)g) \]
that \(\langle V(\phi)f, g \rangle = \langle f, V(\phi)g \rangle\). Then the usual arguments (like [Wil19, Lemma 1.48] for example) show that \(V(\phi)\) extends to a bounded adjointable operator in the multiplier algebra \(M(C^*(H, \mathcal{A}))\) which lies in the center in view of (7.1) and [Wil07, Lemma 8.3]. Thus we have proved most of the following.

**Lemma 7.1.** Suppose that \(p : \mathcal{A} \to H\) is a Fell bundle and that \(G\) acts on \(\mathcal{A}\) by isomorphisms. Then \(C^*(H, \mathcal{A})\) is a \(C_0(G(0))\)-algebra with \(C^*(H, \mathcal{A})(u) = C^*(H_u, \mathcal{A})\).

**Proof.** All that is left is to identify the fibre. But this can be done as in [Wil19, Proposition 5.37] using [SW13, Lemma 9] in place of [Wil19, Theorem 5.1]. (This identification requires the universal \(C^*\)-norm. We do not know if it holds for the reduced norm.) \[\Box\]
Since $\lambda_H$ is invariant, for each $x \in G$ we get an isomorphism
\[
\alpha_x : C^*(H_{s(x)}, A) \to C^*(H_{r(x)}, A)
\]
given by
\[
(7.2) \quad \alpha_x(f)(h) = x \cdot f(x^{-1} \cdot h) \quad \text{for } f \in \Gamma_c(H_{s(x)}, A).
\]

**Lemma 7.2.** Let $p : A \to H$ be a Fell bundle. Suppose that $G$ acts on $H$ by isomorphisms and that $\lambda_H = \{\lambda_H^u\}_{u \in H^{(0)}}$ is an invariant Haar system on $H$. Then $\alpha = \{\alpha_x\}_{x \in G}$ as in (7.2) above implements a groupoid dynamical system $(C^*(H, A), G, \alpha)$.

**Sketch of the Proof.** The only issue is continuity of the action. Since $C^*(H, A)$ is a $C_0(G^{(0)})$-algebra, it can be realized as the sections of a Banach bundle $q : \mathcal{E} \to G^{(0)}$ with fibres $E(u) = C^*(H_u, A)$. We get a $G$-action on $\mathcal{E}$ via $x \cdot e = e x$. To see that this is a continuous action, suppose that $e_i \to e$ and $x_i \to x$ with $q(e_i) = s(x_i)$. To see that $x_i \cdot e_i \to x \cdot e$, we’ll use [MW08a] Lemma A.3. If $f \in \Gamma_c(H, A)$, then we get a section $\hat{f} \in \Gamma_c(G^{(0)}, \mathcal{E})$ where $\hat{f}(u) \in \Gamma_c(H_u, A)$ is given by
\[
\hat{f}(x)(h) = f(h) \quad \text{for } h \in H_u.
\]

Fix $\epsilon > 0$. Then we can choose $f$ so that $\|\hat{f}(q(e_i)) - f\| < \epsilon$. By upper semi-continuity, we can assume $\|\hat{f}(q(e_i)) - f\| < \epsilon$ for all $i$. Since $\alpha$ is norm reducing, we also have $\|\alpha_{x_i}(\hat{f}(q(e_i))) - \alpha_x(f)\| < \epsilon$ for all $i$. Thus it will suffice to see that $\alpha_{x_i}(\hat{f}(q(e_i))) \to \alpha_x(f(q(e)))$ in the topology on $\mathcal{E}$.

Our approach is modeled on the proof of [MW08a] Lemma 4.3. Let $G \ast_s H = \{(x, h) : \rho(p(h)) = s(x) \}$, and let $\mathcal{C}_c(G \ast_s H, A)$ be the collection of continuous functions $F : G \ast H \to A$ such that $\rho(p(F(x, h))) = s(x)$ and $F$ vanishes off a compact subset of $G \ast_s H$. Then $F$ determines a function $\tilde{F}$ of $G$ into $\mathcal{E}$ where $\tilde{F}(x) \in \Gamma_c(H_{s(x)}(x), A) \subset E(s(x))$ is given by
\[
\tilde{F}(x)(h) = F(x, h).
\]

If $F$ is of the form $(x, h) \mapsto \phi(x) f(h)$ for $\phi \in \mathcal{C}_c(G)$ and $f \in \Gamma_c(H, A)$, then $\tilde{F}$ is continuous from $G$ into $\mathcal{E}$. Since sums of such functions are appropriately dense in $\mathcal{C}_c(G \ast_s H, A)$, we see that $x \mapsto \tilde{F}(x)$ is continuous for all $F \in \mathcal{C}_c(G \ast_s H, A)$.

We get the result by considering $F(y, h) = \phi(y) y \cdot f(y^{-1} \cdot h)$ for $\phi \in \mathcal{C}_c(G)$ identically one near $x$. Then $\tilde{F}(y)(h) = \alpha_y(\tilde{f}(s(y))(h))$ for $y$ near $x$.

As in [MW08a] Example 2.1], we can realize $C^*(H, A) \rtimes_\alpha G$ as the $C^*$-algebra of the Fell bundle $\mathcal{B} = r^* \mathcal{E} = \{(b, x) \in \mathcal{E} \times G : q(b) = r(x) \}$ where $C^*(H, A) \cong \Gamma_0(G^{(0)}, \mathcal{E})$ as above. The multiplication is given by $(b, x)(b', y) = (b \alpha_x(b'), xy)$ and involution by $(b, x)^* = (\alpha_x^{-1}(b), x^{-1})$. If $a \in \Gamma_c(H, A)$, then we can define $b_a \in \Gamma_c(G^{(0)}, \mathcal{E})$ by $b_a(u) = \Gamma_c(H_u, A)$ and $b_a(h) = a(h)$. If $\phi \in \mathcal{C}_c(G)$, then we get $(a \otimes \phi) \in \Gamma_c(G, \mathcal{B})$ where $(a \otimes \phi)(x) = (\phi(x) b_a(r(x)), x)$. To keep the notation from obscuring what is going on, we usually simply write $(a \otimes \phi)(x) = (\phi(x) a(\cdot), x)$. It is not hard to see that such sections span a dense subspace of $\Gamma_c(G, \mathcal{B})$ in the inductive limit topology. Note that
\[
(a \otimes \phi)(x)(a' \otimes \phi')(y) = (\phi(x) a(\cdot), x) (\phi'(y) a'(\cdot), y)
\]
\[
= (\phi(x) \phi'(y) a(\cdot) \alpha_x(a'(\cdot)), xy)
\]
\[
= (\phi(x) \phi'(y) a''(\cdot), xy),
\]
Lemma 7.2. \(\text{isomorphic to the crossed product}\)

This map is continuous in the inductive limit topology and has dense range. Hence

Suppose that \(\Gamma \in \Gamma_c(S(H,G), S(A,G))\) is determined by a function \(F_f : S(H,G) \to A\) where \(h(x,y) = (F_f(h,x), x)\) and \(\rho(f,h(x)) = r(x)\). Then we get a section \(\hat{f} \in \Gamma_c(G,B)\) given by \(\hat{f}(x) = (F_f(h,x), x)\) where \(F_f(h,x)\) is the corresponding element of \(\Gamma_c(H_r(x), A)\) as above: \(F_f(\cdot, x)(h) = F_f(h, x)\).

On the other hand, supposing \(f, g \in \Gamma_c(S(H,G), S(A,G))\), we have

\[
\hat{f} \ast \hat{g}(x) = \int_G \hat{f}(y)\hat{g}(y^{-1}x) d\lambda_G^{r(x)}(y)
\]

where \(a' \in \Gamma_c(H_r(x), A)\) is given by

\[
a'(h) = \int_H F_f(k,y) y \cdot F_g(y^{-1} \cdot (k^{-1}h), y^{-1}x) d\lambda_H^{r(h)}(k).
\]

Arguing as in \cite[Lemma 1.108]{Will07], we see that \(\hat{f} \ast \hat{g}(x) = (a''(\cdot), x)\) with \(a'' \in \Gamma_c(H_r(x), A)\) and

\[
a''(h) = \int_G \int_H F_f(k,y) y \cdot F_g(y^{-1} \cdot (k^{-1}h), y^{-1}x) d\lambda_H^{r(h)}(k) d\lambda_G^{r(x)}(y).
\]

On the other hand, supposing \(f, g \in \Gamma_c(S(H,G), S(A,G))\), we have

\[
f \ast g(h,x) = \int_G \int_H f(k,y) g((k^{-1}h, y^{-1})(h, x)) d\lambda_H^{r(h)}(k) d\lambda_G^{r(x)}(y)
\]

\[
= \int_G \int_H f(k,y) g((y^{-1} \cdot k^{-1}, y^{-1})(h, x)) d\lambda_H^{r(h)}(k) d\lambda_G^{r(x)}(y)
\]

\[
= \int_G \int_H f(k,y) g((y^{-1} \cdot (k^{-1}h), y^{-1}x)) d\lambda_H^{r(h)}(k) d\lambda_G^{r(x)}(y)
\]

\[
= \int_G \int_H (F_f(k,y))(F_g(y^{-1} \cdot (k^{-1}h), y^{-1}x), y^{-1}x) d\lambda_H^{r(h)}(k) d\lambda_G^{r(x)}(y)
\]

\[
= \int_G \int_H (F_f(k,y)) y \cdot F_g(y^{-1} \cdot (k^{-1}h), y^{-1}x, x) d\lambda_H^{r(h)}(k) d\lambda_G^{r(x)}(y).
\]

Theorem 7.3. Suppose that \(p : A \to H\) is a Fell bundle and that \(G\) acts on \(A\) by automorphisms. We suppose that the Haar system on \(H\) is \(G\)-invariant so that \(S(H,G)\) has a Haar system as in Lemma 6.4. Then \(C^*(S(H,G), S(A,G))\) is isomorphic to the crossed product \(C^*(H,A) \rtimes G\) for the dynamical system from Lemma 7.2.

By the above, the map \(f \to \hat{f}\) from \(\Gamma_c(S(H,G), S(A,G))\) to \(\Gamma_c(G,B)\) sends \(f \ast g\) to \(\hat{f} \ast \hat{g}\). A similar computation shows that it sends \(f^*\) to \((\hat{f})^*\). Thus we get a \(*\)-homomorphism \(\Phi : \Gamma_c(S(H,G), S(A,G)) \to \Gamma_c(G, B)\). It is not hard to see that this map is continuous in the inductive limit topology and has dense range. Hence
ϕ extends to a surjective homomorphism of $C^∗(S(H, G), S(\mathcal{A}, G))$ onto $C^∗(G, B) \cong C^∗(H, \mathcal{A}) \rtimes_\alpha G$.

To show that $\Phi$ is isometric, and therefore the desired isomorphism, we will proceed as follows. Given a faithful representation, $L$, of $C^∗(S(H, G), S(\mathcal{A}, G))$, we will produce a representation $\mathcal{L}$ of $C^∗(G, B)$ such that $\mathcal{L}(f) = L(\Phi(f))$. Then $\|f\| = \|\mathcal{L}(f)\| = \|\mathcal{L}(\Phi(f))\| \leq \|\Phi(f)\| \leq \|f\|$. This will suffice.

We will identify $S(H, G)^{(0)}$ with $H^{(0)}$. (Then $s(h, x) = x^{-1} \cdot s_H(h)$ and $r(h, x) = r_H(h)$.) By [M08a, Theorem 4.13], we can assume that $L$ is the integrated form of a strict representation $(\mu, H^{(0)} \ast H, \pi)$. We can assume that $\mu$ is finite and use [Wil07, Theorem I.5] to disintegrate the quasi-invariant measure $\mu$ with respect to the moment map $\rho : H^{(0)} \to G^{(0)}$. Thus off a $\mu$-null set $N(\mu)$ we obtain probability measures $\mu^u$ on $H^{(0)}$ with support in $H^{(0)}_u = (H_u)^{(0)}$ such that for every suitable Borel function $h$ on $H^{(0)}$

$$\int_{H^{(0)}} h(v) \, d\mu(v) = \int_{G^{(0)}} \int_{H^{(0)}} h(v) \, d\mu^u(v) \, d\mu_G(u)$$

where $\mu_G = \rho_* \mu$ is the push-forward of $\mu$ to $G^{(0)}$ (push-forward measures are examined in [Wil07, Lemma H.13]). We let $\Delta$ be the modular function on $S(H, G)$ for $\mu$. Since there are a number of groupoids that will support representations, we will have multiple quasi-invariant measures and their modular functions in play. Although we will always assume modular functions on our groupoids are taken to be Borel homomorphisms into the positive multiplicative reals ([Wil19, Proposition 7.6]), one of the technical difficulties will be to insure all these functions play nice with one-another. In part because modular functions are only defined almost everywhere, this will be a delicate operation culminating in Proposition [7.7] where it is crucial that the equality proved there holds everywhere.

Let $\nu_H = \mu \circ \lambda_H$ and $\nu_G = \mu_G \circ \lambda_G$. For a summary of some of the measure theory required, see [Wil19] §3.1.

Lemma 7.4. The push forward $\mu_G = \rho_* \mu$ is quasi-invariant on $G^{(0)}$. Thus if $\Delta_G$ is the corresponding modular function on $G$, we have

$$\int_{G^{(0)}} \int_G f(x^{-1}) \Delta_G(x^{-1}) \, d\mu_G^u(x) \, d\mu_G(u) = \int_{G^{(0)}} \int_G f(x) \, d\mu_G^u(x) \, d\mu_G(x)$$

for all $f \in C_c(G)$.

Proof. Suppose that $f$ is a non-negative Borel function on $G$ and that $g \in C^+(H)$ is a nowhere vanishing function such that $\lambda_H(g)(v) = 1$ for all $u \in H^{(0)}$ (see [Wil19, Lemma 3.19]). Then

$$\nu_G(f) = \int_{H^{(0)}} \int_G f(x) \, d\lambda_G^u(x) \, d\mu_G(u)$$

$$= \int_{H^{(0)}} \int_H f(x) \, d\lambda_G^u(x) \, d\mu(v)$$

$$= \int_{H^{(0)}} \int_H \int_G f(x) g(h) \, d\lambda_H^u(h) \, d\lambda_G^u(x) \, d\mu(v)$$

$$= \int_{H^{(0)}} \int_H \int_G f(x^{-1}) g(x^{-1} \cdot h^{-1}) \Delta(x^{-1} \cdot h^{-1}, x^{-1}) \, d\lambda_H^u(h) \, d\lambda_G^u(x) \, d\mu(v)$$
= \int_{H(0)} \int_G f(x^{-1}) \left( \int_H g(x^{-1} \cdot h^{-1}) \Delta(x^{-1} \cdot h^{-1}, x^{-1}) d\lambda_H^r(h) \right) d\lambda_G^{\rho(v)}(x) d\mu(v)

= \int_{H(0)} \int_G f(x^{-1}) B(x, v) d\lambda_G^{\rho(v)}(x) d\mu(v)

= \int_{G(0)} \int_{H(0)} \int_G f(x^{-1}) B(x, v) d\lambda_G^{\rho(v)}(x) d\mu^u(v) d\mu_G(u)

= \int_{G(0)} \int_{H(0)} \int_G f(x^{-1}) B(x, u) d\lambda_G^{\rho(v)}(x) d\mu^u(v) d\mu_G(u)

= \int_{G(0)} \int_G f(x^{-1}) \left( \int_{H(0)} B(x, v) d\mu^u(v) \right) d\lambda_G^v(x) d\mu_G(u)

= \int_{G(0)} \int_G f(x^{-1}) B(x) d\lambda_g^u(x) d\mu_G(u)

= \nu_{G}^{-1}(f B^*)

Since the $g$ and $\Delta$ are both strictly positive, $B$ is a Borel function taking values in $(0, \infty]$. Thus if $f$ is the characteristic function of a Borel set in $G$, $f$ is equal to zero $\nu_G$-almost everywhere and only if it is equal to zero $\nu_G^{-1}$-almost everywhere. Hence $\nu_G$ and $\nu_G^{-1}$ are equivalent measures on $G$. That is, $\mu_G$ is $G$-quasi-invariant.

Let $H(0) \ast_r G = \{ (v, x) \in S(H, G) : v \in H(0) \}$. Then $H(0) \ast_r G$ is a closed subgroupoid of $S(H, G)$ with unit space identified with $H(0)$. Thus $s(v, x) = x^{-1} \cdot v$ and $r(v, x) = v$. Then $(v, x)(x^{-1} \cdot v, y) = (v, xy)$. We can view $H(0) \ast_r G$ as the action groupoid for the $G$ action on $G(0)$ where we have used a different convention for writing the action groupoid as pairs—see Remark 2.3. As in Lemma 6.4, we get a Haar system $\sigma$ on $H(0) \ast_r G$ where

$$\sigma(f)(v) = \int_G f(v, x) d\lambda_G^{\rho(v)}(x).$$

**Lemma 7.5.** The measure $\mu$ is quasi-invariant when $H(0)$ is viewed as the unit space of $H(0) \ast_r G$. Thus if we write $\delta$ for its modular function on $H(0) \ast_r G$, then $\delta$ is a Borel homomorphism such that

$$\int_{H(0)} \int_G \phi(x^{-1} \cdot v, x^{-1}) \delta(x^{-1} \cdot v, x^{-1}) d\lambda_G^{\rho(v)}(x) d\mu(v)$$

$$= \int_{H(0)} \int_G \phi(v, x) d\lambda_G^{\rho(v)}(x) d\mu(v)$$

for suitable Borel functions $\phi$ on $H(0) \ast_r G$.

**Proof.** The proof is similar to that for Lemma 6.4. In particular, let $g \in C^+(H)$ be nowhere vanishing with $\lambda_H(g)(v) = 1$ for all $v \in H(0)$. Let $\nu_{H(0) \ast_r G} = \sigma \circ \nu$. Then $\nu_{H(0) \ast_r G}^{-1}(f)$

$$= \int_{H(0)} \int_G f(x^{-1} \cdot v, x^{-1}) d\lambda_G^{\rho(v)}(x) d\mu(v)$$
\[ = \int_{H^{(0)}} \int_{G} J_{H} f(x^{-1} \cdot r_{H}(h), x^{-1})g(h) \frac{d\lambda_{H}^{v}(h) d\lambda_{G}^{\rho(v)}(x)}{F(h,x)} d\mu(v) \]

\[ = \int_{H^{(0)}} \int_{G} J_{H} F(h,x) d\lambda_{H}^{v}(h) d\lambda_{G}^{\rho(v)}(x) d\mu(v) \]

\[ = \int_{H^{(0)}} \int_{G} J_{H} F(x^{-1} \cdot h^{-1}, x^{-1}) \Delta(x^{-1} \cdot h^{-1}, x^{-1}) d\lambda_{H}^{v}(h) d\lambda_{G}^{\rho(v)}(x) d\mu(v) \]

\[ = \int_{H^{(0)}} \int_{G} f(v,x) \left( \int_{H} b(x^{-1} \cdot h^{-1}) \Delta(x^{-1} \cdot h^{-1}, x^{-1}) d\lambda_{H}^{v}(h) \right) d\lambda_{G}^{\rho(v)}(x) d\mu(v) \]

\[ = \nu_{H^{(0)},G}(fB). \]

Thus \( \nu_{H^{(0)},G} \) and \( \nu_{H^{(0)},s,G} \) are equivalent and \( \nu \) is \( H^{(0)} \cdot G \)-quasi-invariant. As usual, we can then choose the modular function \( \delta \) to be a Borel homomorphism. \( \square \)

**Lemma 7.6.** The measure \( \mu \) is also quasi-invariant when \( H^{(0)} \) is viewed as the unit space of \( H \).

**Proof.** Let \( f \) be a non-negative Borel function and then use [Wil19, Proposition 3.19] to find \( b \in C^{+}(G) \) such that \( \lambda_{G}(b)(u) = 1 \) for all \( u \in G^{(0)} \) and such that \( b \) never vanishes. Then

\[ \nu_{H}(f) = \int_{H^{(0)}} \int_{H} f(h) d\lambda_{H}^{v}(h) d\mu(v) \]

\[ = \int_{H^{(0)}} \int_{G} f(h)b(x) d\lambda_{H}^{v}(h) d\lambda_{G}^{\rho(v)}(x) d\mu(v) \]

which, since \( \mu \) is \( S(H, G) \)-quasi-invariant, is

\[ = \int_{H^{(0)}} \int_{G} f(x^{-1} \cdot h^{-1})b(x^{-1}) \Delta(x^{-1} \cdot h^{-1}, x^{-1}) d\lambda_{H}^{v}(h) d\lambda_{G}^{\rho(v)}(x) d\mu(v) \]

which, since \( x^{-1} \cdot h^{-1} = (x^{-1} \cdot h)^{-1} \) and since \( \lambda_{H} \) is invariant, is

\[ = \int_{H^{(0)}} \int_{G} \left( \int_{H} f(h^{-1})b(x^{-1}) \Delta(-h^{-1}, x^{-1}) d\lambda_{H}^{v}(h) \right) d\lambda_{G}^{\rho(v)}(x) d\mu(v) \]

\[ = \int_{H^{(0)}} \int_{G} C(v,x) \delta(v,x)^{-1} d\lambda_{G}^{\rho(v)}(x) d\mu(v) \]

\[ = \int_{H^{(0)}} \int_{H} f(h^{-1})b(x) \Delta(h^{-1}, x) \delta(v,x) d\lambda_{H}^{v}(h) d\lambda_{G}^{\rho(v)}(x) d\mu(v) \]

\[ = \int_{H^{(0)}} \int_{H} f(h^{-1}) \left( \int_{G} b(x) \Delta(h^{-1}, x) \delta(v,x) d\lambda_{G}^{\rho(v)}(x) \right) d\lambda_{H}^{v}(h) d\mu(v) \]

\[ = \int_{H^{(0)}} \int_{H} f(h^{-1})D(h^{-1}) d\lambda_{H}^{v}(h) d\mu(v) = \nu_{H}^{-1}(fD). \]

The result follows as \( D \) takes values in \((0, \infty]\). \( \square \)
In view of Lemma 7.6 we can write $\Delta_H$ for the modular function on $H$ when $\mu$ is viewed as a quasi-invariant measure on $H^{(0)} \subset H$. Recall that we are using $\delta$ for the modular function on $H^{(0)} \ast_r G$ from Lemma 7.5.

**Proposition 7.7.** We can take $\Delta_H(h) = \Delta(h, \rho(h))$ and $\delta(v, x) = \Delta(v, x)$. Then for all $(h, x) \in S(H, G)$, we have

$$\Delta(h, x) = \Delta_H(x) \delta(s_H(h), x).$$

For the proof, we need the following observation.

**Lemma 7.8.** Suppose that $f$ and $\phi$ are bounded non-negative Borel functions on $H$ and $H^{(0)} \ast_r G$, respectively, and that

$$f(h) = \phi(s(h), x) \text{ for } \nu-\text{almost all } (h, x) \in S(H, G).$$

Then there is a non-negative bounded Borel function on $A$ on $H^{(0)}$ such that

$$f(h) = A(s(h)) \text{ for } \nu_H-\text{almost all } h \in H.$$

**Proof.** Let $b \in C_c^+(G)$ be such that $\lambda_G(b)(u) = 1$ for all $u \in G^{(0)}$. Then let

$$A(v) = \int_G \phi(v, x) b(x) d\lambda_G^{(v)}(x).$$

Let $g \in C_c(H)$. Then

$$\int_{H^{(0)}} \int_H (f(h) - A(s(h))) g(h) d\lambda_H^v(h) d\mu(v)$$

$$= \int_{H^{(0)}} \int_H \int_G f(h) g(h) b(x) d\lambda_G^{(v)}(x) d\lambda_H^v(h) d\mu(v)$$

$$- \int_{H^{(0)}} \int_H \phi(s(h), x) b(x) g(h) d\lambda_G^{(s(h))}(x) d\lambda_H^v(h) d\mu(v)$$

which, since $r(h) = v$ implies $\rho(s(h)) = \rho(v)$, is

$$= \int_{H^{(0)}} \int_H \int f(h) - \phi(s(h), x)) b(h) g(h) d\lambda_H^v(h) d\lambda_G^{(v)}(x) d\mu(v) = 0$$

Since $g \in C_c(H)$ is arbitrary, the result follows. \hfill \Box

**Proof of Proposition 7.7.** We have

$$\nu(f) = \int_{H^{(0)}} \int_G \left( \int_H f(h, x) d\lambda_H^v(h) \right) d\lambda_G^{(v)}(x) d\mu(v)$$

which, in view of Lemma 7.5 is

$$= \int_{H^{(0)}} \int_G \int_H f(h, x^{-1}) \cdot d\lambda_H^{v^{-1}}(h) \delta(x^{-1} \cdot v, x^{-1}) \cdot d\lambda_G^{(v)}(x) d\mu(v)$$

which, by invariance, is

$$= \int_{H^{(0)}} \int_G \int_H f(x^{-1} \cdot h, x^{-1}) \delta(x^{-1} \cdot v, x^{-1}) \cdot d\lambda_H^v(h) \cdot d\lambda_G^{(v)}(x) d\mu(v)$$

$$= \int_{H^{(0)}} \int_G \left( \int_H f(x^{-1} \cdot h, x^{-1}) \delta(x^{-1} \cdot r_H(h), x^{-1}) \cdot d\lambda_H^v(h) \cdot d\lambda_G^{(v)}(x) \right) d\lambda_H^v(h) d\mu(h)$$
which, since μ is H-quasi-invariant, is
\[
\int_{H^{(0)}} \int_{H} f(x^{-1} \cdot h^{-1}, x^{-1}) \delta(x^{-1} \cdot s_H(h), x^{-1}) d\lambda^\rho_G(x) \Delta_H(h^{-1}) d\mu(v)
\]
\[
= \int_{H^{(0)}} \int_{H} f(x^{-1} \cdot h^{-1}, x^{-1}) \Delta_H(h^{-1}) \delta(x^{-1} \cdot s_H(h), x^{-1}) d\lambda^\rho_G(x) d\mu(v).
\]
Therefore
\[
\Delta(x^{-1} \cdot h^{-1}, x^{-1}) = \Delta_H(h^{-1}) \delta(x^{-1} \cdot s_H(h), x^{-1})
\]
ν-almost everywhere. Since \( \Delta \), \( \Delta_H \), and \( \delta \) are homomorphisms, we have
\[
\Delta(h, x) = \Delta_H(h) \delta(s_H(h), x)
\]
ν-almost everywhere as claimed.

However, if \((h, x) \in S(H, G)\), then \(\rho(h) = r(x)\) and \((h, k)\) may be written as \((h, x) = (h, r(x))(s_H(h), x) = (h, \rho(h))(s_H(h), x)\). Since \(\Delta\) is a homomorphism
\[
(7.3) \quad \Delta(h, x) = \Delta(h, \rho(h))\Delta(s_H(h), x).
\]
Therefore it will suffice to see that \(\Delta_H(h) = \Delta(\rho(h), h)\) off a \(\nu_H\)-null set that \(\delta(v, x) = \Delta(v, x)\) off a \(\nu_{H^{(0)}+r,G}\)-null set.

It follows from (7.3) that
\[
\Delta_H(h)\Delta(h, \rho(h))^{-1} = \Delta(s_H(h), x)\delta(s_H(h), x)^{-1}
\]
for \(\nu\)-almost all \((h, x)\). Therefore Lemma 7.5 implies that there is a bounded Borel function \(A\) on \(H^{(0)}\) such that
\[
\Delta_H(h) = \Delta(h, \rho(h))A(s(h)) \quad \nu_H\text{-almost everywhere.}
\]

Therefore \(\Delta_{\nu_H}(h) = \Delta(h, \rho(h))\) is a modular function on \(H\) for \(\mu\) and hence must be multiplicative for \(\nu_H^{(2)}\)-almost all \((h, k) \in H^{(2)}\) by [Will9] Lemma 7.5. It follows that \(A(s(h)) = 1\) for \(\nu_H^{(2)}\)-almost all \((h, k) \in H^{(2)}\). But then for any \(g \in C_c(H)\) and an appropriate \(b \in C^+(G)\), we have
\[
\int_{H^{(0)}} \int_{H} (A(s(h)) - 1) g(h) d(\lambda_H)_c(h) d\mu(h)
\]
\[
= \int_{H^{(0)}} \int_{H} (A(s(h)) - 1) g(h) b(k) d(\lambda_H)_c(h) d\lambda^\rho_G(k) d\mu(v) = 0.
\]
Therefore \(A(s(h)) = 1\) \(\nu_H^{-1}\)-almost everywhere, and hence \(\nu_H\)-almost everywhere.

Hence we immediately have \(\Delta(h, \rho(h)) = \Delta_H(h)\) \(\nu_H\text{-almost everywhere as required.}

Now let \(f \in C_c(S(H, G))\). Then
\[
\int_{H^{(0)}} \int_{H} f(h, x) d\lambda^\rho_G(h) d\lambda^\rho_G(x) d\mu(v)
\]
\[
= \int_{H^{(0)}} \int_{H} \left( \int_{H} f(x^{-1} \cdot h^{-1}, x^{-1}) \Delta(x^{-1} \cdot s_H(h), x^{-1}) d\lambda^\rho_G(h) \right) d\lambda^\rho_G(x) d\mu(v)
\]
\[
= \int_{H^{(0)}} \int_{H} f(h^{-1}, x^{-1}) \Delta(h^{-1}, s_H(h)) d\lambda^\rho_G(x) d\mu(v).
\]
\[ \begin{aligned}
\int_{H(0)} \int_{G} \left( \int_{H} f(h^{-1}, x^{-1}) \Delta(h^{-1}, x^{-1}) \, d\lambda_{H}^{\rho}(v(h)) \right) \, d\lambda_{G}^{\rho(v)}(x) \, d\mu(v) \\
= \int_{H(0)} \int_{G} C(v, x) \delta(v, x)^{-1} \, d\lambda_{G}^{\rho(v)}(x) \, d\mu(v) \\
= \int_{H(0)} \int_{G} \int_{H} f(h^{-1}, x^{-1}) \Delta(h^{-1}, x^{-1}) \, d\lambda_{H}(h) \, d\lambda_{G}^{\rho(v)}(x) \, d\mu(v) \\
= \int_{H(0)} \int_{G} \int_{H} f(h^{-1}, x^{-1}) \Delta(h^{-1}, x^{-1}, \rho(h^{-1})) \delta(v, x)^{-1} \, d\lambda_{H}(h) \, d\lambda_{G}^{\rho(v)}(x) \, d\mu(v) \\
= \int_{H(0)} \int_{G} \int_{H} f(h^{-1}, x^{-1}) \Delta(h^{-1}, x^{-1}, \rho(h^{-1})) \delta(v, x)^{-1} \, d\lambda_{H}(h) \, d\lambda_{G}^{\rho(v)}(x) \, d\mu(v) 
\end{aligned} \]

which, since \( \Delta_{H}(h) = \Delta(h, \rho(h)) \) is a modular function for \( \mu \), is

\[ \int_{H(0)} \int_{G} \int_{H} f(h, x) \, d\lambda_{H}(h) \delta(v, x)^{-1} \, d\lambda_{G}^{\rho(v)}(x) \, d\mu(v). \]

Since \( f \in C_{c}(S(H, G)) \) is arbitrary, it follows that \( \Delta(v, x) = \delta(v, x) \) for \( \nu_{H(0), r, G} \)-almost all \( (v, x) \) as required.

This completes the proof. \( \square \)

From this point forward we will assume \( \Delta_{H}(h) = \Delta(h, \rho(h)) \) and \( \delta(v, x) = \Delta(v, x) \), and hence that \( \Delta(h, x) = \Delta_{H}(h) \delta(s(h), x) \) for all \( (h, x) \in S(H, G) \).

Since \( \mu_{G} \) is a quasi-invariant measure on \( G(0) \), whenever \( U \subset G(0) \) is \( \mu_{G} \)-conull the reduction \( G\big|_{U} = \{ x \in G : r(x) \in U \text{ and } s(x) \in U \} \) is \( \nu_{G} \)-conull (see \[Wil19, Remark 7.8\]). Muhly has called such reductions inessential.

**Proposition 7.9.** There is a \( \mu_{G} \)-conull set \( U \subset G(0) \) such that for all \( x \in G\big|_{U} \) we have \( \mu^{s(x)} = \delta(\cdot, x^{-1}) \Delta_{G}(x)(x^{-1}, \mu^{r(x)}) \). That is, for all suitable Borel functions \( \phi \) on \( H(0) \) and \( x \) in the \( \nu_{G} \)-conull reduction \( G\big|_{U} \), we have

\[ (7.4) \int_{H(0)} \phi(v) \, d\mu(x(v)) = \int_{H(0)} \phi(x^{-1} \cdot v) \delta(x^{-1} \cdot v, x^{-1}) \Delta_{G}(x) \, d\mu^{r(x)}(v). \]

**Proof.** Suppose \( \phi \in C_{c}(H(0)) \) and \( \psi \in C_{c}(G) \). Then

\[ \nu_{H(0), r, G}(\phi \otimes \psi) = \int_{H(0)} \int_{G} \phi(v) \psi(x) \, d\lambda_{G}^{\rho(v)}(x) \, d\mu(v) \]

\[ = \int_{G(0)} \int_{H(0)} \int_{G} \phi(v) \psi(x) \, d\lambda_{G}^{\rho(v)}(x) \, d\mu^{u}(v) \, d\mu_{G}(u) \]

\[ = \int_{G(0)} \int_{G} \left( \int_{H(0)} \phi(v) \, d\mu^{u}(v) \right) \psi(x) \, d\lambda_{G}^{\rho(v)}(x) \, d\mu_{G}(u) \]

\[ = \int_{G} \left( \int_{H(0)} \phi(v) \, d\mu^{r(x)}(v) \right) \psi(x) \, d\mu_{G}(x). \]

On the other hand, using Lemma 7.4,

\[ \nu_{H(0), r, G}(\phi \otimes \psi) = \int_{H(0)} \int_{G} \phi(x^{-1} \cdot v) \psi(x^{-1}) \delta(x^{-1} \cdot v, x^{-1}) \, d\lambda_{G}^{\rho(v)}(x) \, d\mu(v) \]

\[ = \int_{G(0)} \int_{G} \left( \int_{H(0)} \phi(x^{-1} \cdot v) \delta(x^{-1} \cdot v, x^{-1}) \, d\mu^{r(x)}(v) \right) \psi(x^{-1}) \, d\lambda_{G}^{\rho(v)}(x) \, d\mu_{G}(u) \]

\[ = \int_{G} \left( \int_{H(0)} \phi(x^{-1} \cdot v) \delta(x^{-1} \cdot v, x^{-1}) \, d\mu^{r(x)}(v) \right) \psi(x^{-1}) \, d\lambda_{G}^{\rho(v)}(x) \, d\mu_{G}(u). \]
which, since $\mu_G$ is quasi-invariant, is
\[
\int_{H^{(0)}} \int_G D(x^{-1}) \Delta_G(x^{-1}) \, d\lambda^w_G(x) \, d\mu_G(u)
= \int_{H^{(0)}} \int_G \left( \int_{H^{(0)}} \phi(x \cdot v) \delta(x \cdot v, x) \Delta_G(x^{-1}) \, d\mu^w(x)(v) \right) \psi(x) \, d\lambda^w_G(x) \, d\mu_G(u)
= \int_{H^{(0)}} \int_G \phi(v) \delta(v, x) \Delta_G(x^{-1}) \, d\lambda^w_G(x) \, d\mu_G(u)
\]
\[
\int_H \phi(v) \, d\mu^r(v) = \int_H \phi(v) \delta(v, x) \Delta_G(x^{-1}) \, d(x \cdot \mu^w(x))(v) \quad \text{if } x \notin N(\phi).
\]
Since $C_c(H^{(0)})$ is separable in the inductive limit topology ([Wil19 Lemma C.2]), there is a $\nu_G$-null set $N(\phi)$ such that (7.5) holds for all $\phi \in C_c(H^{(0)})$. Since $\nu_G$ and $\nu^{-1}_G$ are equivalent, we get (7.4).

Let $\Sigma = \{ x \in G : \mu^w(x) = \delta(, x^{-1}) \Delta_G(x)(x^{-1} \cdot \mu^r(x)) \}$. If $x, y \in \Sigma$ and $(y, x) \in G^{(2)}$, then
\[
\int_{H^{(0)}} \phi(v) \, d\mu^w(x)(v) = \int_{H^{(0)}} \phi(x^{-1} \cdot v) \delta(x^{-1} \cdot v, x^{-1}) \Delta_G(x) \, d\mu^r(v)
\]
which, since $r(x) = s(y)$, is
\[
\int_{H^{(0)}} \phi(x^{-1} \cdot y^{-1} \cdot v) \delta(x^{-1} \cdot y^{-1} \cdot v, x^{-1}) \delta(y^{-1} \cdot v, y^{-1}) \Delta_G(y) \Delta_G(x) \, d\mu^r(y)(v)
\]
which, since $\delta$ and $\Delta_H$ are homomorphisms, is
\[
\int_{H^{(0)}} \phi((yx)^{-1} \cdot v) \delta((yx)^{-1} \cdot v, (yx)^{-1}) \Delta_H(xy) \, d\mu^w((yx)(v)).
\]
That is, $yx \in \Sigma$. It now follows from [Wil19 Proposition D.2] that there is an inessential reduction $G|U \subset \Sigma$. \hfill \square

As usual, we let $H_u$ be the subgroupoid $\rho^{-1}(u)$ equipped with the Haar system $\lambda_{H_u} = \{ \lambda_H \}_{v \in H_u^{(0)}}$ where $H_u^{(0)} = H^{(0)} \cap H_u$.

**Lemma 7.10.** For $\mu_G$-almost all $u$, $\mu^u$ is quasi-invariant on $H_u^{(0)}$ and $\Delta_H$ restricts to a modular function for $\mu^u$ on $H_u$.

**Proof.** Let $\nu_{H_u} = \mu^u \circ \lambda_{H_u}$. Then if $f \in C_c(H)$ and $\phi \in C_c(G^{(0)})$ we have
\[
\int_{H^{(0)}} \int_H f(h) \phi(\rho(h) \, d\lambda^w_H(h) \, d\mu(v)
\]
that Lemma 7.10 holds for all $u$.

Thus if $f \in \Gamma_c(H, \mathcal{G}, G)$ and $\xi, \eta \in L^2(H^{(0)}, \mathcal{H}, \mu)$, then

$$
\mathcal{L}(f) \xi = \int_{H^{(0)}} \int_{G} \int_{H} (\pi(f(h,x))\xi(x^{-1} \cdot s_h(h)) \mid \eta(v)) \Delta(h,x)^{-\frac{1}{2}}
$$

$$
d\lambda_H(h) d\lambda_G^{(e)}(x) d\mu(v)
$$

$$
= \int_{H^{(0)}} \int_{G} \int_{H} (\pi(f(h,x))\xi(x^{-1} \cdot s_h(h)) \mid \eta(v)) \Delta(h,x)^{-\frac{1}{2}}
$$

$$
d\lambda_H(h) d\lambda_G^{(e)}(x) d\mu(v) d\mu_G(u)
$$

$$
= \int_{H^{(0)}} \int_{G} \int_{H} \int_{H^{(0)}} (\eta(v) \mid \eta(v)) \Delta(h,x)^{-\frac{1}{2}}
$$

$$
\lambda_H(h) d\mu(u) \lambda_G^{(e)}(x) d\mu_G(u).
$$

On the other hand,

$$
\int_{H^{(0)}} \int_{H} f(h) \phi(\rho(h)) d\lambda_H^{(e)}(h) d\mu(v)
$$

$$
= \int_{H^{(0)}} \int_{H} f(h^{-1}) \Delta_H(h^{-1}) d\lambda_H^{(e)}(h) \phi(u) d\mu(v)
$$

$$
= \int_{H^{(0)}} \int_{H} f(h^{-1}) \Delta_H(h^{-1}) d\mu(u) \phi(u) d\mu_G(u)
$$

$$
= \int_{H^{(0)}} \nu_{H_c}(f) \phi(u) d\mu_G(u).
$$

Since $\phi$ is arbitrary, there is a $\mu_G$-null set $N(f)$ such that $\nu_{H_c}(f) = \nu_{H_c}^{-1}(\Delta_H f)$ if $u \notin N(f)$. Since $C_c(H)$ is separable in the inductive limit topology [Will9], there is a $\mu_G$-null set $N$ such that $\mu_{H_c}(f) = \nu_{H_c}(\Delta_H f)$ for all $f \in C_c(H)$ provided $u \notin N$. The result follows. \hfill $\square$

Now we can finish the proof of Theorem 7.3

Proof of Theorem 7.3. We let $L$ be the integrated form of $(\mu, H^{(0)} \ast \mathcal{H}, \pi)$ as above. Thus if $f \in \Gamma_c(S(H,G), S(A,G))$ and $\xi, \eta \in L^2(H^{(0)} \ast \mathcal{H}, \mu)$, then

$$
(L(f) \xi \mid \eta) = \int_{H^{(0)}} \int_{G} \int_{H} (\pi(f(h,x))\xi(x^{-1} \cdot s_h(h)) \mid \eta(v)) \Delta(h,x)^{-\frac{1}{2}}
$$

$$
d\lambda_H(h) d\lambda_G^{(e)}(x) d\mu(v)
$$

Recall that using Proposition 7.7 we have $\Delta(h,x) = \Delta_H(h) \delta(s_h(h), x)$ for all $(h,x) \in S(H,G)$. We may as well replace $G$ by $G|_U$ and assume that Proposition 7.9 holds for all $x \in G$. We can also shrink $U$ a bit if necessary and assume that Lemma 7.10 holds for all $u \in G^{(0)}$ as well.

Let $x \in G$ and $b \in \Gamma_c(H_s^{(e)}, A)$. Then we define $\bar{\pi}_x(b) : L^2(H^{(0)}, \mu^e(v)) \rightarrow L^2(H^{(0)}, \mu^{\tau(x)})$ by

$$
(\bar{\pi}_x(b) \xi \mid \eta)
$$

$$
= \int_{H^{(0)}} \int_{H} (\pi(b(h,x))\xi(x^{-1} \cdot s_h(h)) \mid \eta(v)) \Delta(h,x)^{-\frac{1}{2}} \Delta_G(x)^{-\frac{1}{2}}
$$

$$
d\lambda_H(h) d\mu^e(v)
$$

$$
= \int_{H^{(0)}} \int_{H} (\pi(b(h,x))\xi(x^{-1} \cdot s_h(h)) \mid \eta(v)) \Delta(h,x)^{-\frac{1}{2}} \delta(s_h(h), x)^{-\frac{1}{2}}
$$
\[ \Delta_G(x)^{\frac{1}{2}} d\lambda^{\nu}_{H_{r(x)}}(h) d\mu^{r(x)}(v) \]

Then

\[ \left( \frac{\Delta_H(h)^{\frac{1}{2}} \delta(s_H(h), x)^{-\frac{1}{2}} \Delta_G(x)^{\frac{1}{2}} d\lambda^{\nu}_{H_{r(x)}}(h) d\mu^{r(x)}(v)}{\|b(h)\| \|\xi(x^{-1} \cdot s_H(h))\| \|\eta(v)\|} \right)^2 \]

which, using the Cauchy-Schwarz inequality, is

\[ \leq \left( \int_{H(0)} \|b(h)\| \|\xi(x^{-1} \cdot s_H(h))\| d\lambda^{\nu}_{H_{r(x)}}(h) d\mu^{r(x)}(v) \right)^2 \]

\[ \times \left( \int_{H(0)} \|f(h, x)\| \|\eta(v)\| d\lambda^{\nu}_{H_{r(x)}}(h) d\mu^{r(x)}(v) \right)^2 \]

which, using Lemma 7.10 is

\[ \leq \left( \int_{H(0)} \|\xi(x^{-1} \cdot v)\|^2 d\lambda^{\nu}_{H_{r(x)}}(h) d\mu^{r(x)}(v) \right)^2 \]

\[ \times \left( \int_{H(0)} \|f(h, x)\| \|\eta(v)\| d\lambda^{\nu}_{H_{r(x)}}(h) d\mu^{r(x)}(v) \right)^2 \]

\[ \leq \left( \int_{H(0)} \|\xi(x^{-1} \cdot v)\|^2 d\lambda^{\nu}_{H_{r(x)}}(h) d\mu^{r(x)}(v) \right)^2 \]

\[ \times \left( \int_{H(0)} \|b(h)\| \|\eta(v)\|^2 d\lambda^{\nu}_{H_{r(x)}}(h) d\mu^{r(x)}(v) \right)^2 \]

which, by Proposition 7.9 is

\[ = \|b\|^2 \|\xi\|^2 \|L^{2}(H(0), \mu^{r(x)})\| \|\eta\|^2 \|L^{2}(H(0), \mu^{r(x)})\|^2 \]

Therefore we can extend \( \tilde{\pi}_x \) to all of \( C^*(H_{s(x)}, A) \) and define \( \Pi : B \to \text{End}(H^{(0)} * \mathcal{H}) \) by \( \Pi(b, x) = \tilde{\pi}_x(b) \). We can view \( L^{2}(H^{(0)} * \mathcal{H}, \mu) \) as a direct integral \( L^{2}(G^{(0)} * \mathcal{K}, \mu^G) \) with fibres \( K(u) = L^{2}(H^{(0)} * \mathcal{H}, \mu^u) \) as in [MW07, Example F.19]. In particular, if \( \xi \in L^{2}(H^{(0)} * \mathcal{H}, \mu) \), then \( \xi \) defines a Borel section \( \underline{\xi} \in L^{2}(G^{(0)} * \mathcal{K}, \mu_G) \) with \( \underline{\xi}(u) = \xi \). Thus

\[ (L(f)\xi | \eta) = \int_{G^{(0)}} \int_{G} \langle \Pi(f(x))\xi(s(x)) | \eta(u) \rangle \Delta_G(x)^{-\frac{1}{2}} d\lambda^{\nu}_{G}(x) d\mu_{G}(u) \]

Therefore it will suffice to see that \( \hat{\Pi}(b, x) = (r(x), \tilde{\pi}_x(b), s(x)) \) is a Borel \( * \)-functor as in [MW08, Definition 4.5]. Then we can let \( L = (\mu_G, G^{(0)} * \mathcal{K}, \Pi) \).

Linearity is clear. To check multiplicativity, let \( f(x) = (b(\cdot), x) \) and \( g(y) = (c(\cdot), y) \) with \( (x, y) \in G^{(2)} \). Then \( f(x)g(y) = (d(\cdot)\alpha_{x}(c(\cdot)), xy) \) where \( d(\cdot) := b(\cdot)\alpha_{x}(c(\cdot)) \in \Gamma_{c}(H_{r(xy)}, A) \) is given by

\[ d(h) = \int_{H} b(k)x \cdot c(x^{-1} \cdot (k^{-1}h)) d\lambda^{\nu}_{H}(h) \] for \( h \in H_{r(x)} \).
Thus using a vector-valued version of (7.6),
\[ \bar{\pi}_x(b)(\bar{\pi}_y(c)\xi)(v) \]
\[ = \int_H \pi(b(h), x)(\bar{\pi}_y(c)\xi)(x^{-1} \cdot s(h)) \Delta(h, x)^{-\frac{1}{2}} \Delta_G(x)^{\frac{1}{2}} d\lambda_H^v(h) \]
\[ = \int_H \int_H \pi(b(h), x)\pi(c(k), y)\xi(y^{-1} \cdot s(k)) \Delta(k, y)^{-\frac{1}{2}} \Delta_G(y)^{\frac{1}{2}} \]
\[ \Delta(h, x)^{-\frac{1}{2}} \Delta_G(x)^{\frac{1}{2}} d\lambda_H^v(h) \]
which, after \( k \mapsto h^{-1}k \), is
\[ = \int_H \int_H \pi(b(h), x)\pi(c(x^{-1} \cdot (h^{-1}k)), y)\xi(y^{-1} \cdot x^{-1} \cdot s(k)) \Delta(x^{-1} \cdot (h^{-1}k), y)^{-\frac{1}{2}} \Delta_G(y)^{\frac{1}{2}} \]
\[ \Delta(h, x)^{-\frac{1}{2}} \Delta_G(x)^{\frac{1}{2}} d\lambda_H^v(h) \]
which, since \( \Delta \) and \( \Delta_G \) are homomorphism, is
\[ = \int_H \int_H \pi(b(h), x)\pi(c(x^{-1} \cdot (h^{-1}k)), y)\xi(y^{-1} \cdot x^{-1} \cdot s(k)) \Delta(k, xy)^{-\frac{1}{2}} \Delta_G(xy)^{\frac{1}{2}} d\lambda_H^v(k) d\lambda_H^v(h) \]
\[ = \int_H \left( \int_H \pi(b(h) \cdot c(x^{-1} \cdot (h^{-1}k)) \right) d\lambda_H^v(h) \right) \xi(y^{-1} \cdot x^{-1} \cdot s(k)) \Delta(k, xy)^{-\frac{1}{2}} \Delta_G(xy)^{\frac{1}{2}} d\lambda_H^v(k) \]
\[ = \int_H \pi(d(h), xy)\xi((xy)^{-1} \cdot s(h)) \Delta(k, xy)^{-\frac{1}{2}} \Delta_G(xy)^{\frac{1}{2}} d\lambda_H^v(k) \]
\[ = \bar{\pi}_{xy}(d)\xi(v). \]
Thus \( \Pi(b, x)\Pi(c, y) = \Pi((b, x)(c, y)) \) as required.

Another computation shows that \( \Pi((b, x)^*) = \Pi((b, x))^* \).

Since \( \pi \) is Borel, it is not hard to check that \( \Pi \) is as well. This completes the proof. \( \square \)

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