Towards to Analysis in $\mathbb{R}^{pq}$

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Abstract

We are looking for a possible development of analysis in indefinite space $\mathbb{R}^{pq}$ from their group of Möbius automorphisms.

1 Introduction

Applications of Möbius transformations in Clifford analysis have attracted serious consideration recently [1, 5, 6]. The goal of the paper is to make a first step towards analysis in $\mathbb{R}^{pq}$ based on the scheme for analytic function theory described in [3] (see Section 3). In Section 4 we describe the structure of the group of Möbius transformations for positive unit sphere in $\mathbb{R}^{pq}$. Our result allows to construct a function theory in such spheres from the described scheme (to be done elsewhere).

2 Preliminaries

Let $\mathbb{R}^{pq}$ be a real $n$-dimensional vector space, where $n = p + q$ with a fixed frame $e_1, e_2, \ldots, e_p, e_{p+1}, \ldots, e_n$ and with the nondegenerate bilinear form $B(\cdot, \cdot)$ of signature $(p, q)$, which is diagonal in the frame $e_i$, i.e.:

$$B(e_i, e_j) = \epsilon_i \delta_{ij}, \quad \text{where} \quad \epsilon_i = \begin{cases} -1, & i = 1, \ldots, p \\ 1, & i = p + 1, \ldots, n \end{cases}$$

and $\delta_{ij}$ is the Kronecker delta. Particularly, the usual Euclidean space $\mathbb{R}^n$ is $\mathbb{R}^{0n}$. Let $\text{Cl}(p, q)$ be the real Clifford algebra generated by $1, e_j, 1 \leq j \leq n$ and the relations

$$e_i e_j + e_j e_i = -2B(e_i, e_j).$$

We put $e_0 = 1$ also. Then there is the natural embedding $i : \mathbb{R}^{pq} \rightarrow \text{Cl}(p, q)$. We identify $\mathbb{R}^{pq}$ with its image under $i$ and call its elements vectors. There are two linear anti-automorphisms $\ast$ (reversion) and $-\ast$ (main anti-automorphisms) and automorphism $\prime$ of $\text{Cl}(p, q)$ defined on its basis $A_\nu = e_{j_1} e_{j_2} \cdots e_{j_r}, 1 \leq j_1 < \cdots < j_r \leq n$ by the rule:

$$(A_\nu)^\ast = (-1)^{\frac{r(r-1)}{2}} A_\nu, \quad \bar{A}_\nu = (-1)^{\frac{r(r+1)}{2}} A_\nu, \quad A'_\nu = (-1)^r A_\nu.$$

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In particular, for vectors, \( \bar{x} = x' = -x \) and \( x^* = x \).

It is easy to see that \( x y = y x = 1 \) for any \( x \in \mathbb{R}^{pq} \) such that \( B(x, x) \neq 0 \) and \( y = x \|x\|^{-2} \), which is the Kelvin inverse of \( x \). Finite products of invertible vectors are invertible in \( \text{Cl}(p, q) \) and form the Clifford group \( \Gamma(p, q) \). Elements \( a \in \Gamma(p, q) \), such that \( a \overline{a} = \pm 1 \) form the Pin\((p, q)\) group—the double cover of the group of orthogonal rotations \( \text{O}(p, q) \). We also consider [\text{II} \S 5.2] \( T(p, q) \) to be the set of all products of vectors in \( \mathbb{R}^{pq} \).

Let \( (a, b, c, d) \) be a quadruple from \( T(p, q) \) with the properties:

1. \( (ad^* - bc^*) \in \mathbb{R} \setminus 0 \);
2. \( a^*b, c^*d, ac^*, bd^* \) are vectors.

Then [\text{II} Theorem 5.2.3] \( 2 \times 2 \)-matrixes \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) form the group \( \Gamma(p + 1, q + 1) \) under the usual matrix multiplication. It has a representation \( \pi_\mathbb{R}^{pq} \) by transformations of \( \mathbb{R}^{pq} \) given by:

\[
\pi_\mathbb{R}^{pq}\left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) : x \mapsto (ax + b)(cx + d)^{-1},
\]

(2.1)

which form the Möbius (or the conformal) group of \( \mathbb{R}^{pq} \). Here \( \mathbb{R}^{pq} \) the compactification of \( \mathbb{R}^{pq} \) by the “necessary number of points” (which form the light cone) at infinity (see [\text{II} \S 5.1]).

The analogy with fractional-linear transformations of the complex line \( \mathbb{C} \) is useful, as well as representations of shifts \( x \mapsto x + y \), orthogonal rotations \( x \mapsto k(a)x \), dilatations \( x \mapsto \lambda x \), and the Kelvin inverse \( x \mapsto x^{-1} \) by the matrixes \( \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \), \( \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \), \( \begin{pmatrix} \lambda^{1/2} & 0 \\ 0 & \lambda^{-1/2} \end{pmatrix} \), \( \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \) respectively.

### 3 Groups of Symmetries and Analytic Function Theories

Let \( G \) be a group that acts via transformation of a closed domain \( \Omega \). Moreover, let \( G : \partial \Omega \to \partial \Omega \) and \( G \) act on \( \Omega \) and \( \partial \Omega \) transitively. Let us fix a point \( x_0 \in \Omega \) and let \( H \subset G \) be a stationary subgroup of point \( x_0 \). Then domain \( \Omega \) is naturally identified with the homogeneous space \( G/H \). And let there exist a \( H \)-invariant measure \( d\mu \) on \( \partial \Omega \).

We consider the Hilbert space \( L_2(\partial \Omega, d\mu) \). Then geometrical transformations of \( \partial \Omega \) give us the representation \( \pi \) of \( G \) in \( L_2(\partial \Omega, d\mu) \). Let \( f_0(x) \equiv 1 \) and \( F_2(\partial \Omega, d\mu) \) be the closed linear subspace of \( L_2(\partial \Omega, d\mu) \) with the properties:

1. \( f_0 \in F_2(\partial \Omega, d\mu) \);
2. \( F_2(\partial \Omega, d\mu) \) is \( G \)-invariant;
3. \( F_2(\partial \Omega, d\mu) \) is \( G \)-irreducible.

The standard wavelet transform \( W \) is defined by

\[
W : F_2(\partial \Omega, d\mu) \to L_2(G) : f(x) \mapsto \hat{f}(g) = \langle f(x), \pi(g)f_0(x) \rangle_{L_2(\partial \Omega, d\mu)}
\]

Due to the property \( |\pi(h)f_0(x)| = f_0(x) \), \( h \in H \) and identification \( \Omega \sim G/H \) it could be translated to the embedding:

\[
\tilde{W} : F_2(\partial \Omega, d\mu) \to L_2(\Omega) : f(x) \mapsto \tilde{f}(y) = \langle f(x), \pi(g)f_0(x) \rangle_{L_2(\partial \Omega, d\mu)},
\]

(3.1)
where \( y \in \Omega \) for some \( h \in H \). The imbedding \([11]\) is an abstract analog of the Cauchy integral formula. Let functions \( V_n \) be the special functions generated by the representation of \( H \). Then the decomposition of \( \hat{f}_0(y) \) by \( V_n \) gives us the Taylor series.

The Bergman kernel in our approach is given by the formula

\[
K(x, y) = c \int_G [\pi_g f_0](x) \overline{\pi_g f_0}(y) \, dg,
\]

where \( c \) is a constant.

The interpretation of complex analysis based on the given scheme could be found in [3].

4 Möbius Transformations of the Positive Unit Sphere in \( \mathbb{R}^{pq} \)

One usually says that the conformal group in \( \mathbb{R}^{pq} \), \( n > 2 \) is not so rich as the conformal group in \( \mathbb{R}^2 \). Nevertheless, the conformal covariance has many applications in Clifford analysis [1, 6]. Notably, groups of conformal mappings of unit spheres \( S^{pq} = \{ x \mid x \in \mathbb{R}^{pq}, B(x, x) = 1 \} \) onto itself are similar for all \( (p, q) \) and as sets can be parametrized by the product of \( B^{pq} := \mathbb{R}^{pq} \setminus S^{pq} \) and the group of isometries of \( S^{pq} \).

**Proposition 4.1** The group \( S_{pq} \) of conformal mappings of the open unit sphere \( S^{pq} \) onto itself represented by matrixes

\[
\begin{pmatrix}
\alpha & \beta \\
\beta' & \alpha'
\end{pmatrix}, \quad \alpha, \beta \in T(p, q), \quad \alpha\beta^* \in \mathbb{R}^{pq}, \quad \alpha\bar{\alpha} - \beta\bar{\beta} = \pm 1.
\]

(4.1)

Alternatively, let \( a \in B^{pq} \), \( b \in \Gamma(p, q) \) then the Möbius transformations of the form

\[
\phi_{(a,b)} = \begin{pmatrix} 1 & a \\ a' & 1 \end{pmatrix} \begin{pmatrix} b & 0 \\ 0 & b' \end{pmatrix} = \begin{pmatrix} b & ab' \\ d' & b' \end{pmatrix}
\]

constitute \( S_{pq} \). \( S_{pq} \) acts on \( B^{pq} \) transitively. Transformations of the form \( \phi_{(a,b)} \) constitute a subgroup isomorphic to \( O(p, q) \). The homogeneous space \( B_{pq}/O(p, q) \) is isomorphic as a set to \( B^{pq} \). Moreover:

1. \( \phi^{2}_{(a,1)} = -1 \) on \( B^{pq} \) \( (\phi^{-1}_{(a,1)} = -\phi_{(a,1)}) \).
2. \( \phi_{(a,1)}(0) = a, \phi_{(a,1)}(a) = 0. \)
3. \( \phi_{(a,1)}\phi_{(c,1)} = \phi_{(d,f)} \) where \( d = \phi_{(a,1)}(c) \) and \( f = a - c. \)

**Proof.** We are using here the notations and results of [1, § 5.1–5.2]. As any Möbius transformations \( B_{pq} \) is represented via fractional-linear transformations associated to matrixes \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \). Its characteristic property is that it preserves (up to projectivity) the unit sphere \( S^{pq} \). \( B^{pq} \) is described in the Fillmore-Springer construction [1, § 5.1] by the matrix \( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \). So we are looking for matrix \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) with the property

\[
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \bar{d} & \bar{b} \\ \bar{c} & \bar{a} \end{pmatrix} = \begin{pmatrix} 0 & r \\ r & 0 \end{pmatrix}, \quad \text{for some } r \in \mathbb{C}l(p, q).
\]

This gives us two equations

\[
bd + ac = 0, \quad d\bar{d} + c\bar{c} = b\bar{b} + a\bar{a}.
\]

(4.2)
Within different matrices, which satisfy to (4.2) and define the same transformation, there exist exactly one of the form

\[
\begin{pmatrix}
\alpha & \beta \\
\beta' & -\alpha'
\end{pmatrix}, \quad \alpha \bar{\alpha} + \beta \bar{\beta} = 1, \quad \alpha, \beta \in \text{Cl}(p, q).
\]

This condition defines the group $S_{pq}$ analogously to $SL(2, \mathbb{R})$.

Now we are looking for the stationary subgroup $S \subset S_{pq}$ of the origin 0. The simple equation

\[
a0 + b \bar{b} = 0
\]

convinces us that $S$ consists of matrices

\[
\begin{pmatrix}
a & 0 \\
0 & -a'
\end{pmatrix}, \quad a \bar{a} = 1
\]

or equivalently $S = \text{Pin}(p, q)$, which acts on the ball $\mathbb{B}^{pq}$ and the sphere $S^{pq}$ by the isometries. As well known any homogeneous space $X$ is topologically equivalent to the quotient of the symmetry group $G$ with respect to the stationary subgroup $G_0$ of a point $x_0$: $X \sim G/G_0$. In our case this means $\mathbb{B}^{pq} = B_{pq}/O(p, q)$.

Identities 4.1.1–4.1.3 could be checked by the direct calculations. □

**Remark 4.2** The above Proposition is a generalization of Lemma 2.1 from [4], which was given without proof. Related results for Euclidean spaces are considered in [11 § 6.1].

For Euclidean space one could split $\mathbb{R}^{pq} \setminus S^{pq}$ on the unit ball $\{x \mid x^2 > -1\}$ and its exterior $\{x \mid x^2 < -1\}$. This also splits group $S_{on}$ onto two subgroups. In the general $pq$ case this could not be done because sphere $S^{pq}$ is not orientable. The details to be given elsewhere [2].

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