On Coloring Random Subgraphs of a Fixed Graph

Igor Shinkar
igors@berkeley.edu
UC Berkeley

December 14, 2016

Abstract

Given a graph $G$ we study the chromatic number of a random graph $G_{1/2}$ obtained from $G$ by removing each edge of $G$ independently with probability $1/2$. Studying $\chi(G_{1/2})$ has been suggested by Bukh [Buk], who asked whether $\mathbb{E}[\chi(G_{1/2})] \geq \Omega(\chi(G)/\log(\chi(G)))$ holds for all graphs $G$. In this paper we prove several results related this problem. Denoting the chromatic number of $G$ by $k = \chi(G)$ we prove the following results.

1. For all $d \leq k^{1/3}$ it holds that $\Pr[\chi(G_{1/2}) \leq d] < \exp\left(-\Omega\left(\frac{k(k-d^3)}{d^3}\right)\right)$. In particular, $\Pr[G_{1/2} \text{ is bipartite}] < \exp\left(-\Omega\left(k^2\right)\right)$. When $G$ is the $k$-clique, this bound is tight up to a constant in $\Omega(\cdot)$.

2. For all $\varepsilon > 0$ it holds that $\Pr[\chi(G_{1/2}) \leq (1-\varepsilon)\sqrt{k}] < \exp\left(-\Omega\left(\varepsilon^2\sqrt{k}\right)\right)$.

We also prove that for graphs $G$ with $\chi(G) = k$ and $\alpha(G) \leq O(n/k)$, it holds that $\mathbb{E}[\chi(G_{1/2})] \geq \Omega(k/\log(k))$.

1 Introduction

For a given graph $G$ let $G_p$ be a random subgraph of $G$ obtained from it by removing each edge of $G$ independently with probability $1 - p$. In this paper we study the chromatic number of $G_{1/2}$ for an arbitrary graph $G$ whose chromatic number is equal to some parameter $k$. Clearly, since $G_{1/2}$ is a subgraph of $G$, it holds that $\chi(G_{1/2}) \leq \chi(G)$. If $G$ is the $k$-clique, then this is the well studied Erdős-Rényi random graph model [ER60], where is it known that $\chi(G_{1/2}) = \Theta\left(\frac{k}{\log(k)}\right)$ with high probability (see, e.g. [Bol01]). By monotonicity we also see that if $\chi(G) = k$ and $G$ contains a $k$-clique, then with high probability $\chi(G_{1/2}) \geq \Omega\left(\frac{k}{\log(k)}\right)$. It is not difficult to come up with an example of a graph $G$ for which $\chi(G_{1/2}) = \chi(G)$ with high probability. (For instance, let $G$ be the complete $k$-partite graph with poly($k$) vertices in each part.) Given the foregoing examples, it is natural to ask whether for every graph $G$
if $\chi(G)$ is large, then $\chi(G_{1/2})$ is also large with high probability. Studying $\chi(G_{1/2})$ has been suggested by Bukh [Buk], who asked the following question.

Is there a constant $c > 0$ such that $\mathbb{E}[\chi(G_{1/2})] > c \cdot \frac{\chi(G)}{\log \chi(G)}$ for all $G$?

Recently there has been some work generalizing the classical result on random graphs, asking about properties of random subgraphs of a fixed graph, or of graphs satisfying certain properties [BCvdH+05a, BCvdH+05b, FK13, KLS15]. For example, there have been several results studying the emergence of a giant component in $G_p$ when $G$ is an expander graph [KS13, ABS04, FKM04]. In this work we study a problem of a similar flavor, namely, trying to relate the chromatic random of $G_p$ to the chromatic number of a random graph in the Erdős-Rényi random model.

In a slightly different context, this problem is also motivated by a recent work of Bennett et al. [BRS16], who asked about the computational complexity of $\mathcal{NP}$-complete problems, whose inputs come from a certain semi-random model. In particular, they showed that many natural $\mathcal{NP}$-complete problems, such as finding the chromatic number of a graph, or deciding whether a graph contains a Hamiltonian path, remain $\mathcal{NP}$-hard even in the seemingly relaxed situation, where the inputs to the problem come from random subgraphs of worst case instances. In particular, they proved that if $\chi(G) = k$, then $\Pr[\chi(G_{1/2}) < d] < \text{poly}\left(\frac{1}{k-d^2}\right)$ for all $d < k^{1/3}$.

1.1 Our results

The main result in this paper gives an upper bound on the probability that $\chi(G_{1/2})$ is very small compared to $\chi(G)$.

**Theorem 1.1.** Let $G = (V, E)$ be a graph with $\chi(G) = k$.

1. For all $d \leq k^{1/3}$ it holds that $\Pr[\chi(G_{1/2}) \leq d] < \exp\left(-\Omega\left(\frac{k(k-d^3)}{d^3}\right)\right)$.

2. For all $\varepsilon > 0$ it holds that $\Pr[\chi(G_{1/2}) \leq (1 - \varepsilon)\sqrt{k}] < \exp\left(-\Omega\left(\varepsilon^2\sqrt{k}\right)\right)$.

**Remark.** For $d = 2$ in Item 1 we get that $\Pr[G_{1/2} \text{ is bipartite}] < \exp(-\Omega(k^2))$. When $G$ is the $k$-clique this bound is best possible up to a constant in the $\Omega(\cdot)$.

Next, we study $\chi(G_{1/2})$ for a special (rather large) class of graphs. Note that if $G$ is an $n$-vertex graph with $\chi(G) = k$, then $G$ contains an independent set of size $n/k$. However, in many cases the maximal independent set of $G$ is within a multiplicative constant factor of $n/k$, i.e., $\alpha(G) \leq C \cdot \frac{n}{k}$ for some $C > 1$ that is not too large. For example, the random graph models $G(n, p)$ and $G(n, d)$ satisfy this property with high probability for all $p > \frac{1}{n}$ and $d \geq 2$ (see, e.g., [Bol01]).

**Theorem 1.2.** Let $G = (V, E)$ be a graph with $\alpha(G) \leq C \cdot \frac{n}{k}$ for some $C > 1$. Then

$$\Pr\left[\alpha(G_{1/2}) \geq \frac{4C \log(k)}{k} n\right] > 1 - 2^{-\frac{4C \log(k)}{k} n}.$$
In particular,
\[ \mathbb{E}[\alpha(G_{1/2})] \leq \frac{k}{8C\log(k)}. \]

The following corollary follows immediately from Theorem 1.2.

**Corollary 1.3.** Let \( G = (V, E) \) be a graph with \( \chi(G) = k \), and suppose that \( G \) contains a subgraph \( G' = (V', E') \) with \( V' \subseteq V \) such that \( \alpha(G') \leq C|V'|/k \). Then,
\[ \mathbb{E}[\chi(G_{1/2})] \geq \frac{k}{8C\log(k)}. \]

Next, we discuss a related graph parameter, called the Hadwiger number of a graph. Hadwiger number of a graph \( G \), denoted by \( h(G) \), is the maximal \( t \in \mathbb{N} \) such that \( G \) contains \( K_t \) as a minor. Hadwiger’s conjecture states that \( h(G) \geq \chi(G) \) for all graphs \( G \). While the conjecture is open for general graphs, the inequality \( h(G) \geq \chi(G) \) is known to hold for a random graph \( G(n, 1/2) \) with high probability. Mader [Mad68] proved an approximate version of the conjecture, namely that \( h(G) \geq \Omega \left( \frac{\chi(G)}{\log(k)} \right) \) for all graphs \( G \).

Kostochka [Kos84] improved Mader’s result, and showed that \( h(G) \geq \Omega \left( \frac{\chi(G)}{\sqrt{\log(k)}} \right) \) for all graphs \( G \). Motivated by this line of research Adrian Vetta [Vet16] asked the following question. What is \( \min \{ \mathbb{E}[h(G_{1/2})] : G \text{ such that } \chi(G) = k \} \)? We note that a tight answer to this question follows almost immediately from [Kos84].

**Theorem 1.4.** Let \( G = (V, E) \) be an \( n \)-vertex graph with \( \chi(G) = k \). Then,
\[ \Pr \left[ h(G) \geq \Omega \left( \frac{k}{\sqrt{\log(k)}} \right) \right] \geq 1 - \exp \left( -\Omega(k^2) \right). \]

In particular, \( \mathbb{E}[h(G_{1/2})] \geq \Omega \left( \frac{k}{\sqrt{\log(k)}} \right) \).

**Remark.** By the result of [BCE80] when \( G \) is the \( k \)-clique we have \( \mathbb{E}[h(G_{1/2})] \leq O \left( \frac{k}{\sqrt{\log(k)}} \right) \), and so the bound in Theorem 1.4 result is tight up to a multiplicative constant.

2 Preliminaries

Let \( G = (V, E) \) be an undirected graph. An independent set in \( G \) is a subset of the vertices that spans no edges. The independence number of \( G \), denoted by \( \alpha(G) \), is the largest size of an independent set in \( G \). A vertex coloring of \( G \) is an assignment of colors to \( V \) such that no two adjacent vertices have the same color. The chromatic number of \( G \), denoted by \( \chi(G) \), is the smallest number of colors required to color \( G \). Note that in any vertex coloring of \( G \)
each color class forms an independent set, and hence $\alpha(G) \geq n/\chi(G)$. Hadwiger number of a graph $G$, denoted by $h(G)$, is the maximal $t \in \mathbb{N}$ such that $G$ contains $K_t$ as a minor.

For a subset of the vertices $A \subseteq V$ let $E(A)$ be the set of edges spanned by $A$, i.e., $E(A) = \{(u, v) \in E : u, v \in A\}$. For two disjoint subsets of the vertices $A, B \subseteq V$ define $\text{cut}(A, B) = \{(u, v) \in E : u \in A, v \in B\}$ to be the set of edges with one endpoint in $A$ and one endpoint in $B$.

We will need the following easy claim saying that the number of edges in a graph is at least quadratic in its chromatic number.

**Claim 2.1.** Let $G = (V, E)$ be a graph with chromatic number $\chi(G) = k$. Then $|E| \geq \binom{k}{2}$.

**Proof.** Let $V = C_1 \cup \cdots \cup C_k$ be a partition of the vertices of $G$ into $k$ color classes. Note that there must be at least one edge between every two color classes, as otherwise, if there are no edges between $C_i$ and $C_j$, then $C_i \cup C_j$ is an independent set, and we can color them with the same color, which implies that $\chi(G) \leq k - 1$. Therefore $|E| \geq \binom{k}{2}$. \hfill $\square$

We will also need a result from extremal set theory about $r$-wise $t$-intersecting families. In order to explain the result we will need some notation. Let $X$ be a finite set, and let $P(X) = \{F : F \subseteq X\}$ be the collection of all subsets of $X$. A family of sets $\mathcal{F} \subseteq P(X)$ is said to be $r$-wise $t$-intersecting if for every $F_1, \ldots, F_t \in \mathcal{F}$ it holds that $|F_1 \cap F_2 \cap \cdots \cap F_t| \geq t$.

In the proof of Theorem 1.1 we will use the following theorem due to Frankl [Fra87].

**Theorem 2.2** (Claim 9.2 [Fra87]). Let $\mathcal{F} \subseteq P(X)$ be a $3$-wise $t$-intersecting family. Then $|\mathcal{F}| < \left(\frac{\sqrt{t} - 1}{2}\right)^t \cdot 2^n$.

### 3 Probability that $\chi(G_{1/2})$ is Small

In this section we prove Theorem 1.1. We start with the proof of Item 1 for the case of $d = 2$, as we think it is a bit cleaner than the proof for general $d$.

**Proof of Item 1 for $d = 2$.** Let us assume that $k > 8$, as otherwise the claim holds trivially. Let $(A, V \setminus A)$ be a partition of the vertices of $G$ into 2 parts, Define $\text{uncut}(A) = E(A) \cup E(V \setminus A)$ to be all the edges that do not belong to $\text{cut}(A, V \setminus A)$. Let $S \subseteq E$ be a random subset of the edges, where each $e \in E$ is added to $S$ independently with probability $1/2$, and let $H = (V, E \setminus S)$ so that $H$ is distributed according to $G_{1/2}$. Note that $H$ is bipartite if and only if there exists some $A \subseteq V$ such that $\text{uncut}(A) \subseteq S$.

$$\Pr[H \text{ is bipartite}] = \Pr[\exists A : \text{uncut}(A) \subseteq S]. \quad (1)$$

Let $U$ be the monotone closure of $\{\text{uncut}(A) : A \subseteq V\}$ defined as

$$U = \{S \subseteq E : \exists A \subseteq V \text{ such that } \text{uncut}(A) \subseteq S\}.$$

Using this notation we have $\Pr[H \text{ is bipartite}] = \Pr[S \in U] = \frac{|U|}{2^n}$, and hence, it is enough to show an upper bound on $|U|$. We make the following observation:
Observation 3.1. \( U \) is a 3-wise \( t \)-intersecting family for some \( t \geq \Omega(k^2) \).

Proof. Note that by monotonicity of \( U \) it is enough to show that for any \( A_1, A_2, A_3 \subseteq V \) it holds that \( |\text{uncut}(A_1) \cap \text{uncut}(A_2) \cap \text{uncut}(A_3)| \geq \Omega(k^2) \). Indeed, the three subsets \( A_1, A_2, A_3 \) partition \( V \) into 8 parts \( V_1, \ldots, V_8 \) according to whether a vertex belongs to \( A_i \) or to its complement. Denote by \( m_i \) the number of edges spanned by \( V_i \), and denote by \( k_i \) the chromatic number of \( G[V_i] \). Then, \( \sum_{i=1}^{8} k_i \geq k \), and hence for some \( i^* \) we must have \( k_{i^*} \geq k/8 \). On the other hand, by Claim 2.1 we have \( m_{i^*} \geq \left( \frac{k_{i^*}}{2} \right) \geq \left( \frac{k/8}{2} \right) \geq \Omega(k^2) \), which immediately implies that \( |\text{uncut}(A_1) \cap \text{uncut}(A_2) \cap \text{uncut}(A_3)| \geq m_{i^*} \geq \Omega(k^2) \), as required.

Therefore, by applying Theorem 2.2 we conclude that \( |U| \leq \left( \frac{\sqrt{5} - 1}{2} \right)^t \cdot 2^{|E|} \), and hence

\[
\Pr[H \text{ is bipartite}] = \Pr[S \in U] = \frac{|U|}{2^{|E|}} \leq \exp(-\Omega(k^2)),
\]
as required.

Next, we generalize the proof above and prove Item 1 of Theorem 1.1.

Proof of Item 1. Let \( A = (A_1, \ldots, A_d) \) be a partition of the vertices of \( G \) into \( d \) classes, i.e., the \( A_i \)'s are pairwise disjoint and \( V = A_1 \cup \cdots \cup A_d \). For such a partition \( A \) define \( \text{uncut}(A) = E(A_1) \cup \cdots \cup E(A_d) \subseteq E \) to be the set of the edges of \( G \) with both endpoints in some \( A_i \). Denote by \( \mathcal{P}_d = \{ A = (A_1, \ldots, A_d) \} \) the collection of all such partitions of the vertices into \( d \) classes.

Let \( H \sim G_1/2 \), and let \( S \subseteq E \) be a random subset of the edges, where each \( e \in E \) is added to \( S \) independently with probability \( 1/2 \). Note that by considering \( H = (V, E \setminus S) \) we have

\[
\Pr[\chi(H) \leq d] = \Pr[\exists A \in \mathcal{P}_d : \text{uncut}(A) \subseteq S].
\]

Let \( U \) be the monotone closure of \( \{ \text{uncut}(A) : A \in \mathcal{P}_d \} \) defined as

\[
U = \{ \text{uncut}(A) : A \in \mathcal{P}_d \} = \{ S \subseteq E : \exists A \in \mathcal{P}_d \text{ such that } \text{uncut}(A) \subseteq S \}.
\]

Then \( \Pr[\chi(H) \leq d] = \Pr[\exists A : S \supseteq \text{uncut}(A)] = \frac{|U|}{2^{|E|}} \), and hence, it is enough to show an upper bound on \( |U| \). The key step is the following claim.

Claim 3.2. \( U \) is a 3-wise \( t \)-intersecting family for \( t = \left\lfloor \frac{k(k-d)^3}{2d^3} \right\rfloor \).

We postpone the proof of the claim, and show how it implies the theorem. By applying Theorem 2.2 we conclude that \( |U| \leq \left( \frac{\sqrt{5} - 1}{2} \right)^t \cdot 2^{|E|} \), and hence

\[
\Pr[\chi(H) \leq d] = \Pr[\exists A : S \supseteq \text{uncut}(A)] = \frac{|U|}{2^{|E|}} \leq \left( \frac{\sqrt{5} - 1}{2} \right)^t \leq 0.79 \frac{k(k-d)^3}{2d^3},
\]
as required.

We now return to the proof of Claim 3.2.
Proof. Note that since $U$ is a monotone closure of $\{\text{uncut}(A) : A \in \mathcal{P}_d\}$ it is enough to show that $\{\text{uncut}(A) : A \in \mathcal{P}_d\}$ is a 3-wise $t$-intersecting family. Indeed, let $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathcal{P}_d$ be three partitions, and let $t = |\text{uncut}(A) \cap \text{uncut}(B) \cap \text{uncut}(C)|$ be the size of the intersection. Let $V_1, \ldots, V_{d^3}$ be the partition $V$ according to the partitions $\mathcal{A}, \mathcal{B},$ and $\mathcal{C}$, and for each $i \in \{1, \ldots, d^3\}$ let $t_i = |E[V_i]|$ and $k_i = \chi(G[V_i])$. Note that (1) $t = \sum_{i=1}^{d^3} t_i$ and (2) $k \leq \sum_{i=1}^{d^3} k_i$ since we color each $V_i$ with $k_i$ new colors then we will obtain a legal coloring of $G$. Therefore,

$$t \overset{(1)}{=} \sum_{i=1}^{d^3} t_i \overset{(*)}{=} \sum_{i=1}^{d^3} \left(\frac{k_i}{2}\right) \overset{(**)}{=} d^3 \left(\frac{1}{d^3} \sum_{i=1}^{d^3} k_i\right) \overset{(2)}{=} d^3 \left(\frac{k}{2}\right) = \frac{k(k-d^3)}{2d^3},$$

where $(*)$ is by Claim 2.1, and $(**)$ is by Jensen’s inequality, using the fact that $x \mapsto \frac{x(x-1)}{2}$ is a convex function. This completes the proof of Claim 3.2. \hfill \Box

We now turn to the proof of Item 2 of Theorem 1.1. We start with the following easy observation, also made by Bukh [Buk].

**Observation 3.3.** If $\chi(G) = k$, then $\mathbb{E}[\chi(G)_{1/2}] \geq \sqrt{k}$.

**Proof.** Let $G = (V, E)$. Let $H = (V, E_H)$ be a subgraph of $G$ and let $\overline{H} = (V, E \setminus E_H)$ be the complement of $H$ in $G$. Note that $\chi(H) \cdot \chi(\overline{H}) \geq k$. Indeed, if $c_H : V \rightarrow [\chi(H)]$ is a coloring of $H$ and $c_H : V \rightarrow [\chi(\overline{H})]$ is a coloring of $\overline{H}$, then we can construct a coloring $c_G$ of $G$ with at most $\chi(H) \cdot \chi(\overline{H})$ colors by letting $c_G(v) = (c_H(v), c(\overline{H})(v))$. To see that $c_G$ is indeed a legal coloring note that every edge $(u, v)$ in $G$ belongs to either $H$ or $\overline{H}$, and hence $c_G(u)$ differs from $c_G(v)$ is at least one of the coordinates. This implies that $\chi(H) + \chi(\overline{H}) \geq \sqrt{\chi(H) \cdot \chi(\overline{H})} \geq \sqrt{k}$ for all subgraphs $H \subseteq G$. Taking expectation of the inequality above, and recalling that each of $H$ and $\overline{H}$ are distributed like $G_{1/2}$ we get that

$$\mathbb{E}[\chi(G)_{1/2}] = \frac{\mathbb{E}[\chi(H)] + \mathbb{E}[\chi(\overline{H})]}{2} \geq \sqrt{k}. \hfill \Box$$

Next, we claim that $\chi(G_{1/2})$ is concentrated around its expectation.

**Lemma 3.4.** Let $G = (V, E)$ be a graph with $\chi(G) = k$. Then for all $\varepsilon > 0$ we have $\Pr[\chi(G_{1/2}) < (1 - \varepsilon) \cdot \mathbb{E}[\chi(G_{1/2})]] < e^{-\Omega(\varepsilon^2 \mathbb{E}[\chi(G_{1/2})])}$.

**Proof.** Let $V = C_1 \cup \cdots \cup C_k$ be a partition of the vertices of $G$ into $k$ color classes, and let $H = G_{1/2}$. Let $X_1, X_2, \ldots, X_k$ be a sequence of random variables defined by $X_i = \chi(H[U \cup \{C_j\}])$, i.e. $X_i$ is the chromatic number of the subgraph of $H$ induced by $C_1 \cup \cdots \cup C_i$. In particular $\mathbb{E}[X_k] = \mathbb{E}[\chi(G)_{1/2}]$. Note that since each $C_i$ is an independent set in $G$, it is also an independent set in $H$, and hence $X_{i+1} - X_i \in \{0, 1\}$.

We now use the following large deviation result due to Alon et al. [AGGL10], which, in turn relies of the Bernstein-Kolmogorov type inequality of Freedman [Fre75].
**Proposition 3.5.** Let $X_1, \ldots, X_k$ sequence of random variables adapted to some filter $(F_i)$ such that $|X_{i+1} - X_i| \leq 1$ for all $i = 1, \ldots, k-1$. Then,

$$\Pr \left[ \left| \frac{X_k}{\mathbb{E}[X_k]} - 1 \right| > \varepsilon \right] < O(e^{-\Omega(\varepsilon^2 \mathbb{E}[X_k])})$$

By Proposition 3.5 we have

$$\Pr[\chi(G_{1/2}) < (1 - \varepsilon) \cdot \mathbb{E}[\chi(G_{1/2})]] = \Pr \left[ \frac{X_k}{\mathbb{E}[X_k]} < 1 - \varepsilon \right] < O(e^{-\Omega(\varepsilon^2 \mathbb{E}[X_k])}).$$

\[\square\]

**Proof of Item 2.** The proof follows immediately from Observation 3.3 and Lemma 3.4. Indeed, by Observation 3.3 we have $\mathbb{E}[\chi(G_{1/2})] \geq \sqrt{k}$, and by applying Lemma 3.4 we get that for all $\varepsilon > 0$ it holds that

$$\Pr[\chi(G_{1/2}) < (1 - \varepsilon) \cdot \sqrt{k}] < e^{-\Omega(\varepsilon^2 \sqrt{k})},$$

as required. \[\square\]

### 4 Proofs of Theorem 1.2 and Corollary 1.3

Theorem 1.2 has been essentially proven in [BRS16] Theorem 3. We reproduce the proof here mostly for completeness.

**Proof of Theorem 1.2.** Let $G = (V, E)$ be a graph with $\chi(G) = k$ and $\alpha(G) \leq C \cdot \frac{n}{k}$ for some $C > 1$. We will need the claim, also known as Turán’s theorem [Tur41].

**Claim 4.1.** Let $G$ be a graph with maximal independent set of size $r$. Then for every $t > 1$ every set of vertices of size $t r$ spans at least $\frac{t^2}{2} r$ edges.\(^1\)

In our setting we have $\alpha(G) \leq \frac{C}{k} n$, and hence, every set of $\ell = \frac{4C \log(k)}{k} n$ vertices spans at least $\frac{t(t-1)}{2} \alpha(G) \geq \frac{2^2}{4} \frac{4C \log(k)^2}{k} n$ edges. Let $S \subseteq V$ be a subset of the vertices of size $|S| = \ell = \frac{4C \log(k)}{k} n$. Then the probability that $S$ is an independent set in $G_{1/2}$ is at most $2 |E[S]| \leq 2 \frac{4C \log(k)^2}{k} n$. Therefore, by taking union bound over all subsets of size $\ell$ we get that

$$\Pr[\alpha(G_{1/2}) \geq \ell] \leq \left( \frac{n}{\ell} \right) \cdot 2^{\frac{4C \log(k)^2}{k}} n \leq \left( \frac{n - e}{\ell} \right) \ell \cdot \left( \frac{n - e}{\ell} \right)^{\ell} \cdot \left( \frac{n - e}{\ell} \right)^{\ell} < 2^{-\ell}.$$ 

In particular, $\Pr[\chi(G_{1/2}) \leq \frac{k}{4C \log(k)}] \leq \Pr[\alpha(G_{1/2}) \geq \ell] < 2^{-\ell} < 1/2$. This implies that $\mathbb{E}[\chi(G_{1/2})] > \frac{k}{8C \log(k)}$, and the theorem follows. \[\square\]

\(^1\) We remark that Turán’s theorem is usually stated as follows: If an $n$ vertex graph $G$ does not contain an $(r + 1)$-clique, then the number of edges in $G$ is at most $\frac{r + 1}{2} n^2$ edges. The statement of Claim 4.1 can be easily derived by considering the complement graph.
Next we turn to proving Corollary 1.3.

**Proof of Corollary 1.3.** Let $G'$ be the subgraph as in the assumption. By Theorem 1.2 we have

$$
\mathbb{E}[\chi(G_{1/2})] > \mathbb{E}[\chi(G'_{1/2})] > \frac{k}{8C \log(k)},
$$

as required.

\[\square\]

5 Proof of Theorem 1.4

The theorem follows almost immediately from the following result of Kostochka [Kos84].

**Theorem 5.1 ([Kos84] Theorem 1).** Let $G = (V, E)$ be a graph such that $|E| \geq k \cdot |V|$. Then $h(G) \geq \Omega\left(\frac{k}{\log(k)}\right)$.

**Proof of Theorem 1.4.** Suppose that $k$ is sufficiently large (e.g., $k \geq 10$), as otherwise the theorem holds trivially. Let $G$ be an $n$ vertex graph with $\chi(G) = k$. Let $G' = (V', E')$ be a $k$-critical subgraph of $G$, i.e., $G'$ is a subgraph of $G$ such that $\chi(G') = k$ but removing any edge from $G'$ reduces its chromatic number. Then, every vertex of $G'$ has degree at least $k - 1$, and hence, $|V'| \geq k$ and $|E'| \geq \frac{k-1}{2}|V'| \geq \frac{k}{4}|V'|$. Let $H = (V', E_H) \sim G_{1/2}'$. Then, by Chernoff bound we have $\Pr[|E_H| < k \cdot |V'|/8] < \exp(-\Omega(k \cdot |V'|)) < \exp(-\Omega(k^2))$. Applying Theorem 5.1 to $H$ we get that $h(H) \geq \Omega\left(\frac{k}{\log(k)}\right)$ with probability $1 - \exp(-\Omega(k^2))$. This completes the proof of Theorem 1.4.

\[\square\]

6 Open Problems

The most obvious open problem in this context is the original question of Bukh.

**Question 6.1.** Is there a constant $c > 0$ such that $\mathbb{E}[\chi(G_{1/2})] > c \cdot \frac{\chi(G)}{\log \chi(G)}$ for all $G$?

Note that by the martingale argument from Lemma 3.4 a positive answer would follow from the (seemingly weaker) bound $\Pr[\chi(G_{1/2}) > \Omega(\frac{\chi(G)}{\log \chi(G)})] > \exp(-\Omega(\frac{\chi(G)}{\log \chi(G)}))$. Other than Bukh's original question, this paper raises several additional problems which we mention below.

**Question 6.2.** Is it true that every graph $G$ contains an induced subgraph $G' \subseteq G$ such that $\chi(G') \geq c \cdot \chi(G)$, and $\alpha(G') \leq C \frac{|V(G')|}{\chi(G')}$ for some absolute constants $C, c > 0$?

A positive answer to this question would immediately give a positive answer to Bukh's question using Theorem 1.2. We stress that Question 6.2 does not require any conditions on the number of vertices on $G'$, except for the obvious $|V(G')| \geq \chi(G') \cdot \alpha(G')/C$. 

8
Recall that $G(k, 1/2)$ is the random Erdős-Rényi graph model \cite{ER60} obtained from the $k$-clique where each of the clique is kept with probability $1/2$. A routine calculation shows that in this case we have $\Pr[\chi(G(k, 1/2) \leq d] < e^{-\Omega(dk^2/d\log(d))}$ for all $d < ck/\log(k)$.\footnote{For $d < k^{\frac{k}{2\log(k)}}$, and for any partition of the $k$ vertices into $d$ classes, the $k$-clique has at least $k^2/4d$ edges that belong to one of the color classes, and hence, by taking union bound over all possible $d$-colorings we have $\Pr[\chi(G(k, 1/2) \leq d] \leq d^k \cdot 2^{-k^2/4d} = 2^{-(\frac{k^2}{4d}\log(d))}$.} It seems natural to ask whether it is possible to compare $\chi(G(k, 1/2))$ with $\chi(G_{1/2})$ for an arbitrary $G$ with $\chi(G) = k$. In particular, is it true that for all graphs $G$ with $\chi(G) = k$ and for all $d < k$ if holds that $\Pr[\chi(G_{1/2}) \leq d] < \Pr[\chi(G(k, 1/2)) \leq d]$? However, it is not difficult to see that for $k = 3$ this is false by taking $G = C_n$ to be the odd length cycle of length $n$. Indeed, in this case $\Pr[\chi(G_{1/2}) \leq 2] = 1 - 2^n$, while $\Pr[\chi(G(k, 1/2)) \leq 2] = 7/8$. This example can be easily extended to any $k$ and $d = 2k/3$; we omit the details. Still, it is natural to ask whether a slightly relaxed comparison is true. We ask a slightly more general question about $G_p$ for an arbitrary $p \in (0, 1)$.

**Question 6.3.** Let $p \in (0, 1)$. Is there a constant $C > 1$ and a polynomial $\text{poly} : \mathbb{R} \to \mathbb{R}$ such that $\Pr[\chi(G_p) \leq d] \leq \text{poly}(\Pr[\chi(G(k,p)) \leq C \cdot d])$ holds for every graph $G$, and for all $d \leq k$, where $k = \chi(G)$ and $G(k,p)$ is the random Erdős-Rényi graph model?

Note that Theorem 1.1 gives a positive answer to Question 6.3 for $p = 1/2$ and $d = O(1)$.

We conclude with two more problems that we find interesting.

**Question 6.4.** Let $p \in (0, 1/2)$. Is there a constant $c > 0$ such that $\mathbb{E}[\chi(G_p)] \geq c \cdot \chi(G)^p$ for all $G$?

**Question 6.5.** Let $p \in (0, 1)$. Is there a constant $c > 0$ such that $\mathbb{E}[\chi(G_p/2)] \geq c \cdot \mathbb{E}[\chi(G_p)]$ for all $G$?

### 7 Acknowledgements

I am grateful to Huck Bennett for many helpful discussions related to this work. Huck had several crucial observations related to this work, but persistently refused to co-author the paper. I am thankful to Daniel Reichman for valuable pointers to the literature and for many discussions related to this paper. I am also thankful to Boris Bukh for his helpful comments.

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