Survival of the Strictest: Stable and Unstable Equilibria under Regularized Learning with Partial Information

Angeliki Giannou  
National Technical University of Athens

Emmanouil-Vasileios Vlatakis-Gkaragkounis  
Columbia University

Panayotis Mertikopoulos  
Univ. Grenoble Alpes, CNRS, Inria, Grenoble INP, LIG, 38000 Grenoble, France & Criteo AI Lab

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Abstract

In this paper, we examine the Nash equilibrium convergence properties of no-regret learning in general $N$-player games. For concreteness, we focus on the archetypal “follow the regularized leader” (FTRL) family of algorithms, and we consider the full spectrum of uncertainty that the players may encounter – from noisy, oracle-based feedback, to bandit, payoff-based information. In this general context, we establish a comprehensive equivalence between the stability of a Nash equilibrium and its support: a Nash equilibrium is stable and attracting with arbitrarily high probability if and only if it is strict (i.e., each equilibrium strategy has a unique best response). This equivalence extends existing continuous-time versions of the “folk theorem” of evolutionary game theory to a bona fide algorithmic learning setting, and it provides a clear refinement criterion for the prediction of the day-to-day behavior of no-regret learning in games.

In more detail, we address the following questions: Is there a class of Nash equilibria that consistently attract no-regret processes? Conversely, are all Nash equilibria equally likely to emerge as outcomes of a no-regret learning process? To address them in a general setting, we focus on the “follow the regularized leader” (FTRL) algorithm and we prove the following result:

$x^*$ is a strict Nash equilibrium $\iff$ $x^*$ is stochastically asymptotically stable under FTRL

Formally, we get the following precise statements for a range of specific feedback models:

**Theorem 1.** Let $x^* \in X$ be a strict Nash equilibrium of the game under study. If FTRL is run with inexact payoff vector estimates with vanishing bias and moderately increasing variance, $x^*$ is stochastically asymptotically stable.

**Theorem 2.** Let $x^*$ be a mixed Nash equilibrium of a generic game. If FTRL is run with inexact payoff vector estimates with vanishing bias and moderately increasing variance, $x^*$ is not stochastically asymptotically stable.

These results – and, in particular, the implications for the bandit case – provide a learning justification to the abundance of arguments that have been made in the refinement literature against selecting mixed Nash equilibria [3, 6] and strengthen existing results on continuous-time game dynamics [1, 2, 5], sometimes referred to as the “folk theorem” of evolutionary game theory [4].

**Keywords:** No-regret learning, Nash Equilibrium, follow the regularized leader, asymptotic stability.

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