Abstract. A local search algorithm operating on an instance of a Boolean constraint satisfaction problem (in particular, $k$-SAT) can be viewed as a stochastic process traversing successive adjacent states in an “energy landscape” defined by the problem instance on the $n$-dimensional Boolean hypercube. We investigate analytically the worst-case topography of such landscapes in the context of satisfiable $k$-SAT via a random ensemble of satisfiable “$k$-regular” linear equations modulo 2.

We show that for each fixed $k = 3, 4, \ldots$, the typical $k$-SAT energy landscape induced by an instance drawn from the ensemble has a set of $2^{\Omega(n)}$ local energy minima, each separated by an unconditional $\Omega(n)$ energy barrier from each of the $O(1)$ ground states, that is, solution states with zero energy. The main technical aspect of the analysis is that a random $k$-regular 0/1 matrix constitutes a strong boundary expander with almost full GF(2)-linear rank, a property which also enables us to prove a $2^{\Omega(n)}$ lower bound for the expected number of steps required by the focused random walk heuristic to solve typical instances drawn from the ensemble. These results paint a grim picture of the worst-case topography of $k$-SAT for local search, and constitute apparently the first rigorous analysis of the growth of energy barriers in a random ensemble of $k$-SAT landscapes as the number of variables $n$ is increased.

1. Introduction

1.1. Background and Motivation. Stochastic local search algorithms \cite{2,55} have in practice proven to be surprisingly efficient in solving instances of difficult constraint satisfaction problems (see \cite{10,93} for recent examples). Yet the basic analytical principles underlying the success or failure of local search heuristics are far from being understood.

The objective of the present work is to shed new analytical light into the combinatorial phenomena that can occur in “energy landscapes” \cite{85} governing the operation of most local search algorithms used in practice. Indeed, the difficulty in analyzing even the most elementary heuristics largely stems from the fact that the energy landscapes induced by the problem instances do not easily yield to combinatorial analysis.

To set the stage, certainly among the most well-understood settings for constraint satisfaction problems is a system of linear equations $Ax \equiv b \pmod{2}$ over $n$ variables $x_1, x_2, \ldots, x_n$ assuming 0/1 values (that is, “XORSAT”). The following example will provide to be illustrative.

\begin{equation}
\begin{bmatrix}
0 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 \\
1 & 1 & 0 & 1 \\
1 & 1 & 1 & 0 \\
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
\end{bmatrix}
\equiv
\begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
\end{bmatrix}
\pmod{2}.
\end{equation}

A local search algorithm can now be viewed as a stochastic process that traverses a sequence of adjacent states in the energy landscape associated with the problem instance. For a linear system $Ax \equiv b \pmod{2}$, the states of the landscape consist of the $2^n$ possible assignments $s = (s_1, s_2, \ldots, s_n)$ of 0/1 values to the variables $x_1, x_2, \ldots, x_n$. Any two states are adjacent if they differ in the value of exactly one variable; the distance between two states is the number of variables having different values in the two states. Associated with each state $s$ is an energy $E(s)$ equal to the number of equations violated by the assignment $x = s$. For example, with lines indicating adjacency and energy indicated by subscripts, the landscape associated with $\begin{equation}$ is depicted below.
The “simple” setting of linear equations is motivated because it provides direct insight into landscape phenomena in less tractable settings, in particular, in the context of the \textbf{k-satisfiability problem (k-SAT)} \cite{31}. Indeed, a linear equation with \( k \) variables is logically equivalent to a conjunction of \( 2^k - 1 \) SAT clauses of length \( k \) that exclude the 0/1 assignments violating the equation. Furthermore, assuming that energy in SAT is defined as the number of violated clauses, the landscape of the SAT encoding of \( Ax \equiv b \pmod{2} \) is \textit{identical} to the linear landscape. Thus, any landscape phenomenon that occurs in the context of linear equations also occurs in SAT.

In the present work we seek to understand what an energy landscape “can look like” to local search heuristics, in the worst case. The two standard heuristics that occur in most local search algorithms are: (a) \textbf{energy bias}—the algorithm prefers (in probability) moving into adjacent states with lower energy over those with higher energy; and (b) \textbf{focusing}—the algorithm prefers moving into adjacent states such that the move affects the constraints that are violated in the current state.

Exerting an energy bias does not always guide a search towards a solution, as can be immediately seen from \cite{12}. To study the worst-case extent of this phenomenon, we consider two standard combinatorial measures of “ruggedness” in a landscape: (a) the \textbf{local minimum} states, that is, the states with positive energy whose adjacent states all have strictly higher energy, and (b) the global \textbf{energy barrier} separating a state \( s \) from a state \( t \), that is, the minimum increase in energy over \( E(s) \) required by \textit{any} walk from \( s \) to \( t \) consisting of successive adjacent states. Of special interest are the barriers separating local minima from \textbf{ground states}, that is, the zero-energy solution states. For example, in \cite{31} the local minimum states are 1110, 1101, 1011, and 0111, each separated by a barrier of \( 3 - 1 = 2 \) from the unique ground state 0000.

From the perspective of the focusing heuristic, a benchmark algorithm is the \textbf{focused random walk} \cite{31} (in each step, select uniformly at random one violated constraint, and flip the value of one variable selected uniformly at random among the variables occurring in the constraint). Also focusing can perform poorly, as can be seen by considering the transition probabilities in \cite{11} and \cite{2} for the focused random walk.

The subsequent analysis paints a grim picture of the worst-case topography that heuristics face already in the “simple” case of \( k \)-regular linear equations, and hence, in the case of \( k \)-SAT. The present results constitute apparently the first rigorous topographical analysis of the energy landscapes induced by a nontrivial random ensemble. (See \S 1.3 for a discussion of related work.)

### 1.2. Statement of Results

Throughout this work we assume that \( k = 3, 4, \ldots \) is fixed. In particular, any asymptotic notation \( O(\cdot), \Omega(\cdot), \omega(\cdot) \) always refers to the parameter \( n \) growing without bound and \( k \) remaining fixed. Furthermore, the constants hidden by the asymptotic notation in general depend on the fixed parameters, such as \( k \) and \( \epsilon \) in Theorem 1.

An \( n \times n \) matrix with 0/1 entries is \textbf{\( k \)-regular} if every row and every column has exactly \( k \) nonzero entries. For a given \( n \), a \textbf{random \( k \)-regular matrix} refers to a \( k \)-regular \( n \times n \) matrix selected uniformly at random from the set of all such matrices. Similarly, a \textbf{random \( k \)-regular landscape} refers to the energy landscape associated with a system \( Ax \equiv 0 \pmod{2} \), where \( A \) is a random \( k \)-regular matrix.

**Theorem 1** (Energy barriers and local minima). For each fixed \( k = 3, 4, \ldots \) and \( \epsilon > 0 \) it holds that a random \( k \)-regular landscape has with probability at least \( 1 - \epsilon \) the following three properties:

\( \text{(i)} \) the number of ground states is \( O(1) \);

\( \text{(ii)} \) any two distinct ground states have distance \( \Omega(n) \) and are separated by an \( \Omega(n) \) energy barrier from each other;

\( \text{(iii)} \) there exists a set of \( 2^{\Omega(n)} \) local minima such that each local minimum is separated by an \( \Omega(n) \) energy barrier from every ground state.

Thus, an energy landscape can be very uneven indeed. Furthermore, Theorem 1 leaves no possibility for “trivial” barriers caused by large local fluctuations of energy. Indeed, because each variable occurs in \( k = O(1) \) equations, it follows that moving from one state into an adjacent state...
Theorem 2 (Lower bound for focused random walk). For each fixed \( k = 6, 7, \ldots \) and \( \epsilon > 0 \) it holds that the system \( Ax \equiv 0 \pmod{2} \) defined by a random \( k \)-regular matrix \( A \) has with probability at least \( 1 - \epsilon \) the property that the focused random walk requires \( 2^{\Omega(n)} \) expected steps to arrive at a ground state when started from an initial state selected uniformly at random.

The main technical hurdle in establishing Theorems 1 and 2 is the following result, which we also expect to be of independent interest (see [13] in particular due to its role in establishing the existence of strong \( k \)-regular boundary expanders with almost full linear rank.

Theorem 3. The expected size of the kernel of a random \( k \)-regular matrix over GF(2) is \( O(1) \).

A matrix \( A \) is a \((k, \omega, \eta)\)-boundary expander if (a) the number of nonzero entries in every column is at most \( k \), and (b) for all \( w = 1, 2, \ldots, \lceil \omega \rceil \), every submatrix consisting of \( w \) columns of \( A \) has at least \( \lfloor \eta w \rfloor \) rows containing exactly one nonzero value. The following theorem is well known (cf. [53] Theorem 4.16(2)).

Theorem 4. For each fixed \( k = 3, 4, \ldots \) and \( \delta > 0 \) there exists a \( \beta > 0 \) such that a random \( k \)-regular matrix is a \( (k, \beta n, k - 2 - \delta)\)-boundary expander with probability \( 1 - o(1) \).

[[ N.B. A proof of Theorem 4 is provided in Appendix A. ]]

Applying Markov’s inequality to Theorem 3 and combining with Theorem 4, it follows that for each fixed \( k = 3, 4, \ldots, \delta > 0 \), and \( \epsilon > 0 \) there exist constants \( d > 0 \) and \( \beta > 0 \) such that with probability at least \( 1 - \epsilon \) a random \( k \)-regular matrix both (a) has a kernel of size at most \( 2^d \) and (b) is a \( (k, \beta n, k - 2 - \delta)\)-boundary expander. This provides the technical foundation for Theorems 1 and 2.

1.3. Connections and Related Work. Random ensembles of constraint satisfaction problems such as “random \( k\)-XORSAT” [24, 25, 70] and “random \( k\)-SAT” [24, 25, 70] have received extensive attention both from the computer science and the statistical physics communities [3, 33, 41, 52, 59, 68, 71]. In particular, the random \( k\)-XORSAT ensemble is by now well-understood as regards rigorous analysis of the transition phenomena as the ratio \( \alpha \) of the number of equations to variables is increased [26, 27, 32, 69], and a similar rigorous foundation is emerging for random \( k\)-SAT [4, 5, 11, 65], where the corresponding control parameter \( \alpha \) is the ratio of the number of clauses to variables. The present work differs from these studies by (a) considering an essentially different random ensemble, and (b) focusing on the topography of the complete energy landscape, whereas most of the recent effort, e.g. [5, 65, 66, 70, 77], in studies of random \( k\)-XORSAT and random \( k\)-SAT has gone to investigating “only” the distance distribution between the ground states akin to Theorem 1(ii). (An exception is [72], where it is shown that in the limit \( n \to \infty \) the energy barriers in random \( k\)-XORSAT between nearby ground states are bounded from below by \(-C \log(\alpha_d - \alpha)\) for some constant \( C > 0 \) as the control parameter \( \alpha \) approaches the dynamical transition point \( \alpha_d \) [69].) The growth of energy barriers and local minima as a function of the system size \( n \) has apparently not been rigorously investigated in random ensembles until the present work.

The structure of energy landscapes associated with local search algorithms and spin-glass models of statistical physics [18, 67] have been the focus of many empirical and quasi-rigorous statistical-physics studies, e.g. [13, 20, 28, 37, 38, 96], however, rigorous results are more scarce. In this connection at least one result exists, namely in [75] it is shown that a deterministic \( 3\)-regular matrix family based on a triangular lattice has an associated landscape with local minima separated by an
\(\Omega(\log(n))\) barrier from the ground state; a benchmark study of SAT-solvers using 3-SAT instances derived from this family is carried out in [57]. A general survey of combinatorial landscapes in various contexts is [85].

From the perspective of computer science and statistical physics, the “random satisfiable \(k\)-regular XORSAT” (“ferromagnetic \(k\)-spin model with Ising spins and fixed connectivity \(k\)”) ensemble studied in the present work has apparently been the focus of only relatively few studies, despite the fact that the study of random \(k\)-regular matrices (equivalently, random \(k\)-regular bipartite graphs with a fixed bipartition) has a long history in mathematics [17, 99]. To the best of our knowledge, from a computational / statistical physics perspective the few works addressing the present ensemble are [73], where an analysis of the correlation times of the Glauber dynamics on a corresponding spin-glass model is carried out, and [47], where clausal encodings for the \(k = 3\) case are used to empirically benchmark SAT-solvers; further experiments for the \(k > 3\) case are reported in [56]. Statistical physics studies on analogous fixed-connectivity models include [39, 40, 74, 88].

From a mathematical perspective it is immediate that the analysis of \(k\)-regular matrices over GF(2) is closely related to the study of low-density parity-check codes (LDPC codes) [42, 87] in coding theory. In coding-theoretic language, Theorem 3 states that the expected total number of codewords in a linear code defined by a parity-check matrix drawn from the \(k\)-regular matrix ensemble is \(O(1)\) (indicating that such codes have very limited applicability from a coding-theoretic perspective). From a methodological perspective, however, the tools used to analyze the average weight distribution of the codewords in standard LDPC code ensembles are analogous to the tools used to prove Theorem 3 (cf. [12, 21, 29, 60, 80, 83], the main difference being that we want to bound the expected total number of codewords rather than the number of codewords with a specific relative weight, necessitating uniform upper bounds that enable summation over all the weights \(w = 0, 1, \ldots, n\).

Theorems 1 and 2 are apparently the first results where expansion is employed in lower bound results aimed at understanding local search, despite the fact that expansion is a basic tool in numerous lower bound constructions in, e.g., proof complexity [7, 15, 25, 97], where many constructions are based on clausal encodings of linear equations. In particular, the probabilistic full-rank boundary expander constructions in [8] apparently provide an analogue of Theorem 1 in the special case \(k = 3\); however, this is not immediate due to lack of regularity. In the converse direction, the present Theorem 3 and Theorem 4 imply (by stripping dependent rows and columns) the existence of full-rank boundary expanders for every \(k \geq 3\), thereby providing partial progress to the open lower bound questions in [55]. An interesting technical contrast to the present lower bound results is that the upper bound for the focused random walk in [6] also relies on typical expansion properties of random 3-SAT instances. A recent survey of expansion and its applications is [53].

A large number of stochastic local search algorithms for the \(k\)-SAT problem are based on variations and combinations of the energy bias and focusing heuristics. Arguably the two central algorithm families in this respect are (a) algorithms in the “WalkSAT family” [62, 91] (e.g. [54, 81, 92, 95]), and (b) algorithms based on variations of the Metropolis dynamics [63] (e.g. [10, 23, 58, 93]). (The recent survey propagation algorithm [19, 68, 75] for random \(k\)-SAT also employs local search, but only as a postprocessing step after a “global” form of belief propagation [11, 61].)

Only relatively few rigorous upper and lower bound results are known for the running time of local search algorithms for \(k\)-SAT. For the focused random walk with restarts, it is known [92] that a satisfying assignment in any satisfiable instance of \(k\)-SAT is found in \(O(n(2 - 2/k)^n)\) expected steps, \(k \geq 3\). In [6] it is shown that the focused random walk finds a satisfying assignment in \(O(n)\) steps with high probability for a typical instance drawn from the random 3-SAT ensemble for \(\alpha \leq 1.63\). An exponential upper bound improving upon the trivial \(O(2^n)\) is derived in [51] for a “cautious” randomized greedy approach. In terms of lower bounds, families of crafted instances whose solution requires an expected exponential number of steps of the focused random walk are known; see [6] and [82, §11.5.6]. Explicit families of instances forcing exponential expected running
times for certain randomized greedy heuristics are constructed in [51]. Quasi-rigorous statistical physics studies considering local search heuristics include [13, 96].

From the perspective of local search algorithms for k-SAT, the present Theorem 2 apparently provides the first example of a nontrivial random ensemble with exponential lower bounds on the expected running time for the focused random walk. Furthermore, the energy barriers and local minima demonstrated in Theorem 1(iii) constitute a step towards rigorous lower bounds for more complex heuristics relying on a combination of energy bias and focusing. In this regard the subsequent proof of Theorem 2 actually provides a meager first step—for large enough \( k \) it is immediate from (20) that a comparably “small” energy bias is insufficient to overcome the systematic drift away from ground states caused by focusing and expansion.

As regards energy bias heuristics alone, the convergence properties of nonfocused variants of the Metropolis dynamics (simulated annealing [23, 58] in particular) have been extensively analyzed; see [1, 22, 30, 48, 49, 90] and the references therein. However, these analyses typically adopt a worst-case setting necessitating that a ground state is found with significant probability from every possible initial state. To arrive at a rigorous analysis of the typical behavior from a random initial state akin to Theorem 2, a study of the landscape structure beyond the properties in Theorem 1 is apparently required. In particular, the structure of the attraction basins (see [37]) of the local minima in Theorem 1(iii) in relation to the attraction basins of the ground states need to be better understood.

1.4. Organization. The remainder of this work is organized as follows. The conventions and mathematical preliminaries are reviewed in §2. Theorem 3 is proved in §3. Theorems 1 and 2 are proved in §4.

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2. Preliminaries

2.1. Conventions. A vector always refers to an \( n \)-dimensional column vector with elements in the finite field \( \text{GF}(2) = \{0, 1\} \). All arithmetic on vectors is over \( \text{GF}(2) \). For \( j = 1, 2, \ldots, n \), denote by \( e_j \) the standard basis vector with the \( j \)th element equal to 1 and all other elements equal to 0. A state is a synonym for vector when landscapes are discussed. The weight \( W(v) \) of a vector \( v \) is the number of nonzero elements. In accordance with the definitions in §1, the energy of a state \( s \) with respect to the system \( Ax \equiv 0 \ (\text{mod} \ 2) \) is defined by \( E(s) = W(As) \). The distance between states \( s \) and \( t \) is \( D(s,t) = W(s + t) \). A state \( s \) is a local minimum if \( E(s) > 0 \) and \( E(s + e_j) > E(s) \) holds for all \( j = 1, 2, \ldots, n \).

2.2. Asymptotics. All logarithms are to the natural base \( \exp(1) = \sum_{k=0}^{\infty} 1/k! \). We recall a variant [89] of Stirling’s formula, valid for all positive integers \( n \),

\[
(3) \quad \frac{1}{\sqrt{2\pi}} \exp\left( n \log(n) - n + \frac{\log(n)}{2} + \frac{1}{12n + 1} \right) \leq n! \leq \frac{1}{\sqrt{2\pi}} \exp\left( n \log(n) - n + \frac{\log(n)}{2} + \frac{1}{12n} \right).
\]

For \( 0 < \lambda < 1 \), define the entropy function \( H(\lambda) = -\lambda \log(\lambda) - (1 - \lambda) \log(1 - \lambda) \). From [83] we have the following upper bounds for the binomial coefficients, valid for all integers \( n, k \geq 3 \) and \( w = 1, 2, \ldots, n - 1 \):

\[
(4) \quad \binom{n}{w} \leq \exp\left( nH\left(\frac{w}{n}\right)\right), \quad \binom{kn}{kw}^{-1} \leq \sqrt{k} \exp\left( -(k-1)nH\left(\frac{w}{n}\right) + \frac{1}{6kw}\right).
\]

In what follows we require asymptotic approximations for coefficients of large powers of certain polynomials. For a polynomial \( P(z) \), denote by \( [z^N]\{P(z)\} \) the coefficient of the term \( z^N \) in \( P(z) \).
For example, \( [z^2] \{1 + 6z^2 + z^4 \} = 6 \) and \( [z] \{1 + 3z^2 \} = 0 \). The following theorem is a well-known “local limit analogue” of the central limit theorem in probability theory; see Chap. IX.

**Theorem 5** (Local limit law for coefficients of a polynomial power). Let \( P(z) \) be a polynomial of degree \( d \geq 1 \) with a positive constant term and positive coefficients such that the greatest common divisor of the degrees of the nonzero terms of \( P(z) \) is 1, let

\[
\mu = \frac{P'(1)}{P(1)}, \quad \sigma^2 = \frac{P''(1)}{P(1)} + \mu - \mu^2, \quad \sigma > 0,
\]

and let \( 0 < \delta < 2/3 \). Then, for all large enough \( n \), it holds uniformly for all integers of the form \( N = mn + \nu \) with \( |\nu| \leq n^\delta \) that

\[
[z^N] \{ P(z)^n \} = \frac{1}{\sqrt{2\pi n\sigma}} P(1)^n \exp\left( -\frac{\nu^2}{2\sigma^2 n} \right) \left( 1 + o(1) \right).
\]

[[ N.B. A proof of Theorem 5 is provided in Appendix B. ]]

2.3. The Configuration Model for \( k \)-Regular Matrices. For integers \( n \geq k \geq 3 \), let \( X \) and \( Y \) be two \( kn \)-element sets of points, both of which are partitioned into \( n \) cells of \( k \) points each. A \((k,n)\)-configuration is a bijection \( \gamma : X \to Y \). Denote by \( I \) the set of cells in \( X \) and by \( J \) the set of cells in \( Y \). Associated with a \((k,n)\)-configuration \( \gamma \) there is a \( n \times n \) integer matrix \( A = (a_{IJ}) \) defined for all \( I \in I \) and \( J \in J \) by \( a_{IJ} = |\{ i \in I : \gamma(i) \in J \}| \). We clearly have \( \sum_J a_{IJ} = k \) for all \( I \in I \) and \( \sum_I a_{IJ} = k \) for all \( J \in J \). A configuration is simple if \( A \) is a 0/1 matrix.

The following theorem is due to Békésy, Békésy, and Komlós and O’Neil; early related results are due to Erdős and Kaplansky and Read.

**Theorem 6.** A random \((k,n)\)-configuration is simple with probability \( \exp(- (k-1)^2/2) + o(1) \).

It is well known that any given \( k \)-regular matrix is obtained from exactly \( (k!)^{2n} \) simple \((k,n)\)-configurations, enabling one to access the uniform distribution on the set of all \( k \)-regular \( n \times n \) matrices via the uniform distribution on the set of all simple \((k,n)\)-configurations.

Also considerable extensions of Theorem 6 are known, see [17, 45, 46, 99].

3. Expected Size of The Kernel

We proceed with the proof of Theorem 3.

**Proof.** By linearity of expectation, we can express the expected size of the kernel as a sum of expectations of 0/1 indicator variables, one indicator for each of the \( 2^n \) vectors. The expectation of each indicator is equal to the probability of the corresponding vector occurring in the kernel. By symmetry, for each weight \( w = 0, 1, \ldots, n \), all the \( \binom{n}{w} \) vectors of weight \( w \) have equal probability of occurring in the kernel. Denote by \( P_k(n,w) \) the probability that a given vector \( x \) of weight \( w \) occurs in the kernel.

We proceed to derive an upper bound for \( P_k(n,w) \) using the configuration model. We have that \( x \) occurs in the kernel of \( A \) if and only if the columns of \( A \) corresponding to the \( w \) nonzero coordinates of \( x \) form a submatrix with an even number of nonzero entries in every row. Let \( e_i = 0, 2, \ldots, 2[k/2] \) be the number of nonzero entries in row \( i \) of this submatrix. Because \( A \) is \( k \)-regular, \( \sum_{i=1}^n e_i = kw \). The number of simple \((k,n)\)-configurations that induce an \( A \) meeting a given nonnegative even composition \( e_1 + e_2 + \ldots + e_n = kw \) is at most \( (kw)! \cdot (k(n-w))! \cdot \prod_{i=1}^n \binom{k}{e_i} \).

To obtain an upper bound for the total number of simple \((k,n)\)-configurations that induce an \( A \) with \( x \) in the kernel, let

\[
E_k(z) = \sum_{j=0}^{[k/2]} \binom{k}{2j} z^{2j} = \left(\frac{1+z}{2}\right)^k + \frac{(1-z)^k}{2}, \quad B_k(n,w) = [z^{kw}]\{E_k(z)^n\}.
\]
Now observe that the total number of simple \((k,n)-\text{configurations}\) that induce an \(A\) with \(x\) in the kernel is at most \((kw)! \cdot (k(n - w))! \cdot B_k(n, w)\), where \(B_k(n, w)\) in effect sums the product \(\prod_{i=1}^{n} \binom{k}{e_i}\) over all the eligible compositions \(e_1 + e_2 + \ldots + e_n = kw\). By Theorem 3 for all large enough \(n\) there are at least \(\rho \cdot (kn)!\) simple \((k,n)-\text{configurations}\), where \(\rho\) is any positive constant less than \(\exp(-k^2/2)\). We thus have the upper bound \(P_k(n, w) \leq \rho^{-1} \frac{1}{(kn)!} B_k(n, w)\).

Taking the sum of \(P_k(n, w)\) over all vectors of weight \(w\) and all weights \(w = 0, 1, \ldots, n\), we have that the expected size of the kernel of a random \(k\)-regular matrix of size \(n \times n\) is at most \(\rho^{-1} S_k(n)\), where

\[
S_k(n) = \sum_{w=0}^{n} \binom{n}{w} \binom{kn}{kw}^{-1} B_k(n, w).
\]

The rest of this section provides an asymptotic analysis establishing that \(S_k(n) = O(1)\).

**Theorem 7.** \(S_k(n) = 2 + o(1)\) if \(k\) is odd and \(S_k(n) = 4 + o(1)\) if \(k\) is even.

**Proof.** Partition the sum (7) into the following intervals:

- \(0 \leq w < n/(2k)\), (left extreme deviation)
- \(n/(2k) \leq w < (n - n^{3/5})/2\), (left large deviation)
- \((n - n^{3/5})/2 \leq w \leq (n + n^{3/5})/2\), (central region)
- \((n + n^{3/5})/2 < w \leq n(1 - 1/(2k))\), (right large deviation)
- \(n(1 - 1/(2k)) < w \leq n\), (right extreme deviation)

Observe that \(B_k(n, w) = 0\) if \(kw\) is odd. Furthermore, if \(k\) is even, we have \(B_k(n, w) = B_k(n, n - w)\) by symmetry of the binomial coefficients, implying that left and right regions are identical if \(k\) is even. If \(k\) is odd, then \(E_k\) has degree \(k - 1\), implying that \(B_k(n, w) = 0\) for all \(w > (k - 1)n/k\) and that the sum is zero in the right extreme region.

**Claim 8.** The sum in the central region is \(1 + o(1)\) if \(k\) is odd and \(2 + o(1)\) if \(k\) is even.

**Proof.** Using Theorem 3 we first derive Gaussian approximations to the terms \(\binom{n}{w}\), \(\binom{kn}{kw}\), and \(B_k(n, w)\) in the central region. To this end, let \(\delta = 3/5\). From the binomial theorem it follows that \(\binom{an}{aw} = [z^{aw}]\{1 + z\}^{an}\) for nonnegative integers \(a, n, w\). Setting \(P(z) = (1 + z)^a\), we have \(\mu = a/2\) and \(\sigma = \sqrt{a}/2\) in Theorem 3. We obtain that

\[
\binom{an}{aw} = \frac{1}{\sqrt{2\pi}na} 2^{an+1} \exp\left(-\frac{a(2w - n)^2}{2n}\right)(1 + o(1))
\]

uniformly for all integers \(w\) in the central region \((n - n^{3/5})/2 \leq w \leq (n + n^{3/5})/2\). To approximate \(B_k(n, w)\), let \(P(z) = E_k(\sqrt{z})\) and observe that \(P(z)\) is a polynomial meeting the requirements of Theorem 3 with \(\mu = k/4\) and \(\sigma = \sqrt{k}/4\). We obtain

\[
B_k(n, w) = \begin{cases} 
\frac{2\sqrt{k}}{\sqrt{\pi n}} 2^{(k-1)n} \exp\left(-\frac{k(2w - n)^2}{2n}\right)(1 + o(1)) & \text{if } kw \text{ is even,} \\
0 & \text{if } kw \text{ is odd,}
\end{cases}
\]

uniformly for all integers \(w\) in the central region. From (9) and (10) we have,

\[
\binom{n}{w} \binom{kn}{kw}^{-1} B_k(n, w) = \begin{cases} 
\frac{2\sqrt{k}}{\sqrt{\pi n}} \exp\left(-\frac{(2w - n)^2}{2n}\right)(1 + o(1)) & \text{if } kw \text{ is even,} \\
0 & \text{if } kw \text{ is odd.}
\end{cases}
\]
Thus, for \( w \) in the central region,
\[
\sum_{w} \binom{n}{w} \binom{kn}{kw}^{-1} B_k(n, w) = (1 + o(1)) \frac{2\sqrt{2}}{\sqrt{\pi n}} \sum_{w} \exp \left( - \frac{(2w - n)^2}{2n} \right) \]
\[
= (1 + o(1)) \frac{2\sqrt{2}}{\sqrt{\pi n}} \sum_{t} \frac{1}{\sqrt{n}} \exp \left( -2 \left( \frac{t}{\sqrt{n}} \right)^2 \right) \]
\[
\rightarrow \begin{cases} 
\frac{2\sqrt{2}}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp(-2s^2) \, ds = 2, \text{ if } k \text{ is even}, \\
\frac{2\sqrt{2}}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp(-2s^2) \, ds = 1, \text{ if } k \text{ is odd},
\end{cases}
\]
where the second equality follows from the change of variables \( t = w - n/2 \), and the limit as \( n \to \infty \) follows from the observation that for \( w \) in the central region, \( t/\sqrt{n} \) ranges over \(-n^{1/10}/2 \leq t \leq n^{1/10}/2\); the halving when \( k \) is odd is due to the terms associated with odd \( w \) being zero if \( k \) is odd.

Claim 9. The sum in the left and right large deviation regions is \( o(1) \).

Proof. First we use an approximate variant of the saddle point method (see e.g. [38 Chap. VIII]) to derive an upper bound for \( B_k(n, w) \). By Cauchy’s coefficient formula,
\[
B_k(n, w) = \frac{1}{2\pi i} \oint \frac{E_k(z)^n}{\xi^{kw+1}} \, dz,
\]
where the integration contour can be taken to be a positively oriented circle of radius \( \xi > 0 \) centered at the origin of the complex plane. Because \( E_k(z) \) is a polynomial with positive coefficients, the integrand assumes its maximum modulus on the contour at \( z = \xi \). Consequently, letting \( \lambda = w/n \),
\[
B_k(n, w) \leq \frac{1}{2\pi i} \oint \frac{E_k(\xi)^n}{\xi^{kw+1}} \, dz = \frac{E_k(\xi)^n}{\xi^{kw}} = \left( \frac{E_k(\xi)}{\xi^{k\lambda}} \right)^n.
\]
As an approximation to a saddle point contour, let \( \xi = (\lambda/(1 - \lambda))^{(k-1)/k} \) and observe that
\[
\exp(-(k - 1)H(\lambda)) = \frac{\xi^{\lambda_k}}{(1 + \xi^{k/(k-1)})^{k-1}}.
\]
Combining (11), (11) and (12) we have
\[
\left( \binom{n}{w} \binom{kn}{kw} \right)^{-1} \sum_{w} B_k(n, w) \leq \sqrt{k} \left( \frac{E_k(\xi)}{(1 + \xi^{k/(k-1)})^{k-1}} \right)^n \exp \left( \frac{1}{6kw} \right).
\]
Let \( \tau^{k-1} = \xi \).

Lemma 10. For all \( \tau > 0 \) it holds that \( E_k(\tau^{k-1}) \leq (1 + \tau^k)^{k-1} \), with equality if and only if \( \tau = 1 \).

Proof. Recalling (6) and using the binomial theorem, the inequality \( E_k(\tau^{k-1}) \leq (1 + \tau^k)^{k-1} \) is easily seen to be equivalent to
\[
\sum_{j=0}^{\lfloor k/2 \rfloor} \binom{k}{2j} \tau^{2j(k-1)} \leq \sum_{j=0}^{k-1} \binom{k-1}{j} \tau^j.
\]
In what follows we assume that \( e \) is an even nonnegative integer; in particular, if \( e \) is used as the index of summation, then it is assumed that \( e \) runs over all even nonnegative integers. Recalling that \( \binom{k}{j} = \binom{k-1}{j} + \binom{k-1}{j-1} \) for all nonnegative integers \( k \) and \( j \), it is straightforward to check that (11) is equivalent to
\[
\sum_{e} \left( \binom{k-1}{e} \tau^{e(k-1)} + \binom{k-1}{e-1} \tau^{e(k-1)} \right) \leq \sum_{e} \left( \binom{k-1}{e} \tau^e + \binom{k-1}{e-1} \tau^{(e-1)k} \right).
\]
The $e = 0$ terms cancel in (15), so we may assume $e > 0$. To establish (15), we show that
\begin{equation}
(\frac{k - 1}{e})(\tau^{e(k - 1)} - \tau^{e(k) - 1}) + (\frac{k - 1}{e - 1})(\tau^{e(k - 1)} - \tau^{e(k) - 1}) \geq 0
\end{equation}
holds for each $e > 0$, with equality if and only if $\tau = 1$. To this end, divide both sides of (16) by \((\frac{k - 1}{e - 1})\tau^{e(k - 1)} / e\) to obtain
\begin{equation}
f(\tau) = (k - e)\tau^k - k\tau^e + e \geq 0.
\end{equation}
Now observe that for $e > 0$ we have $f(0) = e > 0$, $f(1) = 0$, and $f(\infty) = \infty$. Taking the derivative of $f$, the real zeroes of $f'(\tau) = k(k - e)\tau^{k-e-1}(\tau^e - 1)$ are $-1, 0, 1$. Thus, for $\tau > 0$ we have
\begin{equation}
f'(\tau) \geq 0, \text{ with equality if and only if } \tau = 1.
\end{equation}

We now continue the proof of Claim 9. Observe that $\lambda \in (0, 1)$ implies $\xi, \tau \in (0, \infty)$, with $\lambda = 1/2$ if and only if $\xi = \tau = 1$. Thus, for $w$ in the large deviation regions, that is, for $\lambda = w/n$ with $n^{-2/5}/2 < |\lambda - 1/2| \leq (k - 1)/(2k)$, we have $E_k(\xi)/(1 + \xi^k/(k-1))^{k-1} < 1$ in (13) by Lemma 10. Developing $E_k(\xi)/(1 + \xi^k/(k-1))^{k-1}$ into a truncated Taylor series at $\lambda = 1/2$ and evaluating at $\lambda = (1 \pm n^{-2/5})/2$, we obtain
\begin{equation}
\left(\frac{E_k(\xi)}{(1 + \xi^k/(k-1))^{k-1}}\right)^n \leq \exp \left(-\frac{(k - 1)n^{1/5}}{2k}\right)(1 + o(1))
\end{equation}
uniformly for all $w$ in the left and right large deviation regions. The claim now follows from (13) and (18) because the regions have $O(n)$ summands.

Claim 11. The sum in the left extreme deviation region is $1 + o(1)$.

Proof. For $w = 0$ the term in the sum is 1. For $w = 1, 2$ the terms are $O(n^w n^{-kw} n^{kw/2})$. Thus, in what follows we may restrict to $3 \leq w < n/(2k)$. Observe that
\begin{equation}
B_k(n, w) \leq \left(\frac{kw/2 + n - 1}{n - 1}\right)\left(\frac{k}{2}\right)^{kw/2}.
\end{equation}
Indeed, \((kw/2 + n - 1)/(n - 1)\) counts the number of integer compositions of $kw$ into $n$ even nonnegative parts, and \((k/2)^{kw/2}\) provides an upper bound for the product $\prod_{i=1}^{n} \left(\frac{k}{e_i}\right)$ associated with each composition $e_1 + e_2 + \ldots + e_n = kw$ into even nonnegative parts at most $k$.

Observe by (14) and (19) that
\begin{equation}
\log \left(\frac{n}{w}\right) \left(\frac{kn}{kw}\right)^{-1} B_k(n, w) \leq G_k(n, w) + O(1),
\end{equation}
where
\begin{equation}
G_k(n, w) = -(k - 1)nH\left(\frac{w}{n}\right) + \left(\frac{kw}{2} + n - 1\right)H\left(\frac{n - 1}{kw/2 + n - 1}\right) + \frac{kw}{2} \log \left(\frac{k}{2}\right).
\end{equation}
Differentiating twice with respect to $w$, we have
\begin{equation}
G''_k(n, w) = \frac{k(kn - 1)w + n(n - 1)(k - 2)}{(n - w)w(kw + 2n - 2)}
\end{equation}
is positive for $0 < w < n$, implying that $G_k$ is convex in the extreme deviation region. In particular, $G_k$ assumes its maximum at the boundaries of the region. Evaluating $G_k$ at $w = 3$ and $w = n/(2k)$, we find $G_k(n, w) \leq -3/2 \log(n) + O(1)$ uniformly for all $w$ in the region. The claim follows because the number of terms in the region is $O(n)$.

Combining the results for all the regions, we have that $S_k(n) = 2 + o(1)$ if $k$ is odd and $S_k(n) = 4 + o(1)$ if $k$ is even. This completes the proof of Theorem 7.
4. Topographical Properties

Throughout this section we consider the landscape associated with a system $Ax \equiv 0 \pmod{2}$, where $A$ is a $k$-regular matrix of size $n \times n$ that both (a) has a kernel of size at most $2^d$ and (b) is a $(\beta n, k - 2 - \delta)$-boundary expander, where $0 < \beta < 1/2$, $d > 0$, and $0 < \delta < 1/3$ are constants independent of $n$.

4.1. Energy Barriers and Local Minima. The intuition underlying Theorem 4 is as follows. The boundary expansion property in effect “surrounds” a ground state with a “perimeter” of radius $\lfloor \beta n \rfloor$ in the $n$-dimensional hypercube, where the energy (“wall”) at every perimeter state is at least $(k - 2 - \delta)\lfloor \beta n \rfloor,$ so any state outside the perimeter with considerably lower energy has a considerable barrier separating it from the ground state.

Let us now make this intuition formally precise and prove Theorem 4. Property (i) is immediate by assumption. To establish property (ii), let $g_1$ and $g_2$ be any two distinct ground states. Clearly, $D(g_1, g_2) = W(g_1 + g_2) > 0$ and $Ag_1 = Ag_2 = 0.$ Thus, it follows from the boundary expansion property that $D(g_1, g_2) = W(g_1 + g_2) > \beta n.$ (Indeed, we cannot have $0 = W(A(g_1 + g_2)) > (k - 2 - \delta)W(g_1 + g_2) > 0$.) Thus, any walk of successive adjacent states from $g_1$ to $g_2$ must have a “perimeter” state $p$ with $D(g_1, p) = W(g_1 + p) = \lfloor \beta n \rfloor.$ By the boundary expansion property,

$$E(p) - E(g_1) = W(Ap) - W(Ag_1) = W(Ap) = W(A(g_1 + p)) \geq (k - 2 - \delta)W(g_1 + p) = (k - 2 - \delta)\lfloor \beta n \rfloor.$$

Since $g_1$ and $g_2$ were arbitrary, we have thus established that distinct ground states are at distance $\Omega(n)$ and separated by an $\Omega(n)$ energy barrier.

To establish property (iii), we first require a large enough set of local minima. With foresight, select any constant $\gamma$ such that

$$0 < \gamma < \min\left(\frac{\beta(k - 2 - \delta)}{4}, \frac{1}{2^d(k(k - 1) + 1)}\right).$$

Because the kernel of $A$ has dimension at most $d$, by elementary linear algebra there is a linearly independent set of $n - d$ columns of $A.$ Furthermore, $A$ restricted to these columns has a linearly independent set of $n - d$ rows. By permuting the rows if necessary, we can assume that these rows occur first in $A.$ Applying Gaussian elimination to the selected $n - d$ linearly independent columns, we find $n - d$ vectors $y_1, y_2, \ldots, y_{n - d}$ with the property that $Ay_j = e_j + r_j,$ where $r_j$ is a vector with the first $n - d$ entries equal to 0, and $e_j$ is the $j$th vector in the standard basis. Observe that the vectors $y_1, y_2, \ldots, y_{n - d}$ are linearly independent.

We say that $y_j$ marks the rows that contain a 1 in $A$ in at least one of the columns containing a 1 in row $j$. In other words, denoting by $a_{pq}$ the entry of $A$ at row $p,$ column $q,$ we have that $y_j$ marks the rows $\{i : \exists q \ a_{iq} = a_{jq} = 1\}.$ Observe that because $A$ is $k$-regular, each vector $y_j$ marks at most $k(k - 1) + 1$ rows.

There are at most $2^d$ different vectors $r_j.$ Thus, because $d$ is a fixed constant independent of $n,$ there exist at least $(n - d)/2^d$ vectors $y_j$ that have identical associated vectors $r_j.$ Among these vectors, start selecting vectors one by one and marking associated rows subject to the constraint that no row is marked more than once, until no more vectors can be selected. Let $m$ be the number of vectors selected in this way. Clearly, $(n - d)/(2^d(k(k - 1) + 1)) \leq m \leq n.$ By re-indexing the vectors and permuting the rows and columns of $A$ if necessary, we can assume that the selected vectors are $y_1, y_2, \ldots, y_m.$

We now claim that every state $u$ of the form $u = \sum_{j=1}^{m} \chi_j y_j$ with $\chi_j \in \{0, 1\}$ and $\sum_{j=1}^{m} \chi_j \equiv 0 \pmod{2}$ is a local minimum. To see this, observe first that $Au = \sum_{j=1}^{m} \chi_j e_j$ and that $E(u) = \sum_{j=1}^{m} \chi_j.$ Now the marking constraint implies that if we flip the value of any one variable in $u,$ we satisfy at most one violated equation and introduce at least $k - 1$ new violated equations. Thus, $u$ is a local minimum.
We proceed to construct an auxiliary graph that we eventually use to establish the energy barriers separating certain local minima from all the ground states. For \( j = 1, 2, \ldots, m - 1 \), let \( z_j = y_j + \gamma m \). Observe that the vectors \( z_1, z_2, \ldots, z_{m-1} \) are linearly independent. Furthermore, the \( \binom{m-1}{2} \) sums of the form \( z_i + z_j \) with \( 1 \leq i < j \leq m - 1 \) clearly satisfy \( W(A(z_i + z_j)) = 2 \).

Because \( A \) is \( k \)-regular, there are at most \( \binom{k}{2} n = O(n) = O(m) \) vectors \( y \) with \( W(A y) = 2 \) and \( W(y) \leq 2/(k - 2 - \delta) < 3 \). To see this, observe that the two columns selected by any vector \( y \) with \( W(y) = 2 \) and \( W(A y) = 2 \) must have at least one row containing a 1 in both columns, and the total number of such “11”-patterns in \( A \) is \( \binom{k}{2} n \). Thus, by the expansion property \( \binom{m-1}{2} - O(m) \) of the sums \( z_i + z_j \) satisfy \( W(z_i + z_j) > \beta n \).

Form an auxiliary graph with the vertex set \( \{z_1, z_2, \ldots, z_{m-1}\} \) such that any two distinct vertices, \( z_i \) and \( z_j \), are adjacent if and only if \( W(z_i + z_j) \leq \beta n \). Because the number of edges in the auxiliary graph is \( O(m) \), for all sufficiently large \( n \) the auxiliary graph has an independent set of size \( 2^d + 1 \), which—by relabeling if necessary—can be assumed to consist of the vectors \( z_1, z_2, \ldots, z_{2d + 1} \).

We now construct the local minima meeting property (iii). Let \( g_1, g_2, \ldots, g_s \) be the ground states, \( s \leq 2^d \). Observe that any sum consisting of a subset of the linearly independent vectors \( z_1, z_2, \ldots, z_{m-1} \) is a local minimum. Furthermore, the energy of such a minimum is at most the number of sums plus one. Select any \( [\gamma n] \) of the vectors \( z_{2d+2}, z_{2d+3}, \ldots, z_{m-1} \). (Note that for all sufficiently large \( n \) this is possible due to the choice of \( \gamma \).)

Consider now any state \( u \) formed as the sum of a nonempty subset of the \( [\gamma n] \) selected vectors. The energy of \( u \) is \( E(u) = W(A u) \leq [\gamma n] + 1 \). The state \( u \) does not necessarily have extensive barriers separating it from each of the ground states. However, if the following condition holds, then \( u \) is separated by extensive barriers from the ground states. If the condition does not hold, then adding one of the (independent) vectors \( z_1, z_2, \ldots, z_{2d+1} \) to \( u \) will produce a local minimum that is separated by extensive barriers from the ground states.

Suppose that \( D(u, g_j) > \beta n / 2 \) holds for all \( j = 1, 2, \ldots, s \). Thus, for every solution state \( g_j \), any walk from \( u \) to \( g_j \) consisting of successive adjacent states must contain a “perimeter” state \( p \) at distance \( D(p, g_j) = W(p + g_j) = [\beta n / 2] \). By the boundary expansion property, the energy of \( p \) is

\[
E(p) = W(A p) = W(A p + A g_j) = W(A(p + g_j)) \geq (k - 2 - \delta) W(p + g_j) \geq \frac{(k - 2 - \delta) \beta n}{2}.
\]

In particular, the increase in energy at \( p \) compared with the energy of \( u \) is

\[
E(p) - E(u) = W(A p) - W(A u) \geq \frac{(k - 2 - \delta) \beta n}{2} - [\gamma n] - 1 > \gamma n.
\]

Thus, the energy barrier separating \( u \) from \( g_j \) is at least \( \gamma n \), assuming that \( D(u, g_j) > \beta n / 2 \) holds for all \( j = 1, 2, \ldots, s \).

Suppose that \( D(u, g_j) \leq \beta n / 2 \) holds for at least one \( j = 1, 2, \ldots, s \). Then, we claim that there exists at least one \( \ell = 1, 2, \ldots, 2^d + 1 \) such that \( D(u + z_\ell, g_j) > \beta n / 2 \) holds for all \( j = 1, 2, \ldots, s \). To reach a contradiction, suppose that this is not the case. Then, by the pigeonhole principle, there exists \( a_j = 1, 2, \ldots, s \) and \( 1 \leq \ell_1 < \ell_2 < 2^d + 1 \) such that \( D(u + z_{\ell_1}, g_j) \leq \beta n / 2 \) and \( D(u + z_{\ell_2}, g_j) \leq \beta n / 2 \). By the triangle inequality, \( D(u + z_{\ell_1}, u + z_{\ell_2}) \leq \beta n \), which by \( D(z_{\ell_1}, z_{\ell_2}) = D(u + z_{\ell_1}, u + z_{\ell_2}) \) contradicts the fact that \( z_1, z_2, \ldots, z_{2^d+1} \) form an independent set in the auxiliary graph. Therefore, there exists at least one \( \ell = 1, 2, \ldots, 2^d + 1 \) such that \( D(u + z_\ell, g_j) > \beta n / 2 \) holds for all \( j = 1, 2, \ldots, s \). Applying the argument in the previous paragraph to the state \( u + z_\ell \), we have that \( u + z_\ell \) is separated from every solution state by an energy barrier at least \( \gamma n \).

Because the vectors \( z_1, z_2, \ldots, z_{m-1} \) are linearly independent, we have thus established the existence of at least \( 2^\gamma m - 1 \) distinct local minima, each separated from every ground state by an energy barrier of at least \( \gamma n \). This establishes property (iii).

4.2. Lower Bound for the Focused Random Walk. The intuition underlying Theorem 2 is as follows. Consider any ground state \( g \). In any state \( s \neq g \) with \( D(s, g) \leq [\beta n] \), the boundary
expansion property implies that most equations violated by \( s \) in the system \( Ax \equiv 0 \) (mod 2) have exactly one variable that assumes different values in \( s \) and \( g \). In particular, the focused random walk is unlikely to flip this variable (there are \( k - 1 \) other choices), thereby exerting a systematic drift away from \( g \). Thus, expansion in effect induces a region of entropic repulsion around every ground state.

Let us now make this intuition formally precise and prove Theorem 2. As was demonstrated in Section 4.1, all ground states in the landscape have distance at least \( [\beta n] \). Because \( \beta < 1/2 \), it follows by standard tail bounds for the binomial distribution (see e.g. \([17, \S 1]\)) that with probability \( 1 - 2^{-\Omega(n)} \) the random initial state for the focused random walk has distance at least \( [\beta n] \) to each of the at most \( 2^d \) ground states. It thus suffices to show that the expected number of steps to reach the ground state from the perimeter of a region of repulsion is \( 2^{\Omega(n)} \). To this end, let \( g \) be any ground state, and consider any state \( s \neq g \) with \( D(s, g) = W(s + g) \leq [\beta n] \). Because \( Ag = 0 \), the number of violated equations in \( s \) is \( E(s) = W(As) = W(A(s + g)) = E(s + g) \). By \( k \)-regularity, \( E(s + g) \leq kW(s + g) \). By the expansion property and \( As = A(s + g) \), at least \((k - 2 - \delta)W(s + g)\) equations violated by \( s \) contain exactly one variable having a different value in \( s \) and \( g \). Thus, assuming that the focused random walk is in the state \( s \), the random step will increase the distance to \( g \) by 1 with probability at least

\[
(20) \quad \frac{(k - 1)(k - 2 - \delta)W(s + g)}{kE(s)} \geq \frac{(k - 1)(k - 2 - \delta)W(s + g)}{k^2W(s + g)} \geq \frac{(k - 1)(k - 2 - \delta)}{k^2},
\]

otherwise the distance to \( g \) decreases by 1. For \( k \geq 6 \) the probability \((20)\) is at least 55/108, thereby establishing a systematic drift away from \( g \) for every state \( s \neq g \) with \( D(s, g) \leq [\beta n] \). A standard analysis of the gambler’s ruin problem (see e.g. [35] Chap. XIV) with one absorbing ruin state and one reflecting barrier now establishes that, starting from a (reflecting) state \( s \) at distance \( D(s, g) \geq \beta n \) from each ground state \( g \), the expected number of steps required to reach a ground (ruin) state is \( 2^{\Omega(n)} \).

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APPENDIX

This appendix is provided only for convenience of verification of the earlier results. In particular, we stress that Theorems 4 and 5 are well known; cf. [53, Theorem 4.16(2)] and [50, 36] Chaps. VIII and IX.

APPENDIX A. PROOF OF THEOREM 4

A matrix $A$ is a $(k, \omega, \eta)$-expander if (a) the number of nonzero entries in every column is at most $k$, and (b) for all $w = 1, 2, \ldots, \omega$, every submatrix consisting of $w$ columns of $A$ has at least $\lceil \eta w \rceil$ rows containing at least one nonzero value.

Theorem 4 follows immediately by combining the following two results.

**Lemma 12.** Let $A$ be a $(k, \omega, \eta)$-expander. Then, $A$ is a $(k, \omega; 2\eta - k)$-boundary expander.

**Proof.** Consider any submatrix of $A$ consisting of $w$ of its columns, $1 \leq w \leq \omega$. Denote by $c_{\ell}$ the number of rows with exactly $\ell$ nonzero values in these columns. Because $A$ is an expander, we have $C = c_1 + c_2 + \ldots + c_w \geq \eta w$ and $C' = c_1 + 2c_2 + \ldots + wc_w \leq kw$. In particular, $c_1 \geq 2C - C' \geq (2\eta - k)w$.

**Theorem 13.** For every $\delta > 0$ there exists an $\beta > 0$ such that a random $k$-regular matrix is a $(k, \beta n, k - 1 - \delta)$-expander with probability $1 - o(1)$.

**Proof.** Select a $0 < \delta < 1$. Let $\eta = k - 1 - \delta$ and

$$
U_k(n, w) = \binom{n}{w} \binom{\lceil \eta w \rceil}{\lceil \eta w \rceil} \binom{\lceil \eta w \rceil}{\lceil \eta w \rceil - kw} \binom{\lceil \eta w \rceil}{\lceil \eta w \rceil - kw} = \binom{n}{w} \binom{\lceil \eta w \rceil}{\lceil \eta w \rceil} \binom{\lceil \eta w \rceil}{\lceil \eta w \rceil - kw} \binom{\lceil \eta w \rceil}{\lceil \eta w \rceil - kw}^{-1}.
$$

Recalling the configuration model from [28, 29] we claim that the probability that a random $(k, n)$-configuration does not define a $(k, \beta n, \eta)$-expander is bounded from above by $\sum_{w=1}^{\beta n} U_k(n, w)$. To see this, observe that for every configuration violating the expansion property, there exists a set of $w$ points in $\mathcal{J}$ and a set of $\lceil \eta w \rceil$ points in $\mathcal{I}$ such that the $kw$ points in the former set of cells are paired with points in the latter. There are $(\lceil \eta w \rceil)!/(\lceil \eta w \rceil - kw)!$ ways to pair the $kw$ points, and $(kn - kw)!$ ways to pair the remaining points.

We proceed to show that $\sum_{w=1}^{\beta n} U_k(n, w) = O(1)$ for an appropriate fixed $\beta > 0$. For $w = 1, 2, \ldots, \lfloor 2/\delta \rfloor$ we may view $U_k(n, w)$ as a rational function of two polynomials of $n$. The denominator polynomial has degree $kw$ and the numerator polynomial has degree $w + \lfloor \eta w \rfloor = \lfloor (k - \delta)w \rfloor \leq kw - 1$. Thus, $\sum_{w=1}^{\beta n} U_k(n, w) = O(1/n)$. Now let

$$
L_k(n, w) = nH\left(\frac{\eta w}{n}\right) + k\eta w H\left(\frac{1}{\eta}\right) - (k - 1)nH\left(\frac{w}{n}\right)
$$

and observe by [4] and (21) that $\log U_k(n, w) \leq L_k(n, w) + O(1)$ for all large enough $n$ and $w = \lfloor 2/\delta \rfloor + 1, \lfloor 2/\delta \rfloor + 2, \ldots, \lfloor n/(2\eta) \rfloor$. Observe that $L_k(n, 2/\delta) \leq -2\log(n) + O(1)$ and that, differentiating (22) with respect to $w$ and letting $\lambda = w/n$,

$$
L_k(n, w) = k\eta H\left(\frac{1}{\eta}\right) + \log\left(\frac{\lambda^\delta(1 - \eta\lambda)^\eta}{\eta^\eta(1 - \lambda)^{k-1}}\right).
$$

Because the log-term in (23) decreases without bound as $\lambda \to 0^+$, there exists an $\beta > 0$ such that $L_k(n, w)$ is decreasing as $w = \lfloor 2/\delta \rfloor + 1, \lfloor 2/\delta \rfloor + 2, \ldots, \lfloor \beta n \rfloor$. Thus, $\sum_{w=1}^{\beta n} U_k(n, w) < O(1/n) + \beta n/(n^2 \cdot O(1)) = O(1/n)$. The claim now follows from Theorem 6. \qed
We require first a preliminary result.

**Theorem 14** (Saddle point asymptotics for coefficients of a polynomial power). Let $P(z)$ be a polynomial of degree $d \geq 1$ with a positive constant term and positive coefficients such that the greatest common divisor of the degrees of the nonzero terms of $P(z)$ is 1, and let $\Lambda$ be any compact subinterval of the open interval $(0,d)$. Then, for all large enough $n$, it holds uniformly for all integers $N = \lambda n$ with $\lambda \in \Lambda$ that

$$[z^N]\{P(z)^n\} = \frac{1}{\sqrt{2\pi nK_\Lambda'(\xi)}} \frac{P(\xi)^n}{\xi^{\lambda n+1}}(1+o(1)), \tag{24}$$

where $K_\Lambda(z) = \log(P(z)) - \lambda \log(z)$ and $\xi = \xi(\lambda)$ is the unique positive solution of $K_\Lambda'(\xi) = 0$.

**Proof.** It follows from the assumptions on $P$ that $\xi$ exists and is unique for every $\lambda \in \Lambda$. Applying Cauchy’s coefficient formula on the circular contour $z(\theta) = \xi \exp(i\theta), -\pi \leq \theta \leq \pi$, we have

$$[z^N]\{P(z)^n\} = \frac{1}{2\pi i} \int_{\pi}^{-\pi} \frac{P(\xi \exp(i\theta))^n}{(\xi \exp(i\theta))^N+1} \xi \exp(i\theta) d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{P(\xi \exp(i\theta))^n}{(\xi \exp(i\theta))^n} d\theta.$$

By the assumptions on $P$, the modulus of $P(z)$ assumes its maximum value on the contour if and only if $z = \xi$. Thus, as $n$ increases, the neighborhood of $\xi$ produces the (exponentially) dominant contribution to the integral. In particular, letting $\theta_0 = n^{-2/5}$, we have

$$[z^N]\{P(z)^n\} = \frac{1}{2\pi}(1+o(1)) \int_{-\theta_0}^{\theta_0} \frac{P(\xi \exp(i\theta))^n}{(\xi \exp(i\theta))^n} d\theta = \frac{1}{2\pi}(1+o(1)) \int_{-\theta_0}^{\theta_0} \exp(nK_\Lambda(\xi \exp(i\theta))) d\theta.$$

Assuming that $n$ is large enough, $K_\Lambda$ is analytic on every straight line segment connecting $\xi$ to $\xi \exp(i\theta)$ with $-\theta_0 \leq \theta \leq \theta_0$. Thus, we have (see e.g. [38, §7.1]),

$$K_\Lambda(\xi \exp(i\theta)) = K_\Lambda(\xi) + K_\Lambda'(\xi)(\xi \exp(i\theta) - \xi) + K_\Lambda''(\xi)(\xi \exp(i\theta) - \xi)^2/2 +$$

$$\frac{(\xi \exp(i\theta) - \xi)^3}{2} \int_{0}^{1} (t - 1)^2 K_\Lambda'''(\xi + t(\xi \exp(i\theta) - \xi)) dt.$$

By assumption we have that $K'(\xi) = 0$. Furthermore, the last integral and $\xi$ are bounded when $\lambda \in \Lambda$. Thus, using $\exp(i\theta) = 1+i\theta+O(n^{-4/5})$ for the second-order term and $\exp(i\theta) = 1+O(n^{-2/5})$ for the coefficient of the integral, we have

$$K_\Lambda(\xi \exp(i\theta)) = K_\Lambda(\xi) - K_\Lambda''(\xi)\xi^2\theta^2/2 + O(n^{-6/5})$$

uniformly for $-\theta_0 \leq \theta \leq \theta_0$. It follows that

$$[z^N]\{P(z)^n\} = \frac{1}{2\pi}(1+o(1)) \frac{P(\xi)^n}{\xi^{\lambda n}} \int_{-\theta_0}^{\theta_0} \exp(-nK_\Lambda''(\xi)\xi^2\theta^2/2) d\theta$$

$$= \frac{1}{2\pi \sqrt{n}}(1+o(1)) \frac{P(\xi)^n}{\xi^{\lambda n}} \int_{-\theta_0 \sqrt{n}}^{\theta_0 \sqrt{n}} \exp(-K_\Lambda''(\xi)\xi^2 t^2/2) dt$$

$$= \frac{1}{2\pi \sqrt{n}}(1+o(1)) \frac{P(\xi)^n}{\xi^{\lambda n}} \int_{-\infty}^{\infty} \exp(-K_\Lambda''(\xi)\xi^2 t^2/2) dt$$

$$= \frac{1}{\sqrt{2\pi nK_\Lambda'(\xi)}} \frac{P(\xi)^n}{\xi^{\lambda n+1}}(1+o(1)),$$

where the second equality follows from the change of variables $\theta = t/\sqrt{n}$ and the last equality follows from $\int_{-\infty}^{\infty} \exp(-t^2) dt = \sqrt{\pi}$. \hfill \Box

We now proceed with the proof of Theorem 5.
Proof. Observe that $0 < \mu < d$ and let $\Lambda$ be any compact subinterval of $(0, d)$ with $\mu$ in its interior. We proceed to apply Theorem 14 with sufficient approximations to $\xi$, $K''_\lambda(\xi)$, and $P(\xi)/\xi^\lambda$ as $n$ increases and $\lambda = N/n = \mu + \nu/n$ with $|\lambda - \mu| \leq n^{\delta - 1}$. First, observe that $\lambda = \mu$ implies $\xi(\lambda) = 1$. Developing $\xi(\lambda)$ into a truncated Taylor series at $\lambda = \mu$ using the defining equality $\xi(\lambda)P'(\xi(\lambda)) = \lambda P(\xi(\lambda))$, we have, after some calculation, uniformly
\[
(25) \quad \xi(\lambda) = 1 + \xi'(\mu)(\lambda - \mu) + O\left(n^{2(\delta - 1)}\right) = 1 + \frac{\nu}{\sigma^2 n} + O\left(n^{2(\delta - 1)}\right).
\]
Some more calculation gives the uniform approximations
\[
(26) \quad K''_\lambda(\xi(\lambda)) = \frac{\lambda}{\xi(\lambda)^2} - \frac{P'(\xi(\lambda))^2}{P(\xi(\lambda))^2} + \frac{P''(\xi(\lambda))}{P(\xi(\lambda))} = \sigma^2 + O(n^{\delta - 1})
\]
and
\[
(27) \quad \frac{P(\xi(\lambda))}{\xi(\lambda)^\lambda} = P(1) - \frac{P(1)}{2\sigma^2}(\lambda - \mu)^2 + O\left(n^{3(\delta - 1)}\right) = P(1)\left(1 + \frac{-\nu^2/(2\sigma^2 n) + O(n^{3\delta - 2})}{n}\right).
\]
The approximations (25), (26), and (27) applied to (24) establish (5). \qed

Helsinki Institute for Information Technology HIIT, Department of Computer Science, University of Helsinki, P.O.Box 68, FI-00014 University of Helsinki, Finland

E-mail address: petteri.kaski@cs.helsinki.fi