We construct a low-scale seesaw model to generate the masses of active neutrinos based on $S_4$ flavor symmetry supplemented by the $Z_2 \times Z_3 \times Z_4 \times Z_{14} \times U(1)_L$ group, capable of reproducing the low energy Standard model (SM) fermion flavor data. The masses of the SM fermions and the fermionic mixings parameters are generated from a Froggatt-Nielsen mechanism after the spontaneous breaking of the $S_4 \times Z_2 \times Z_3 \times Z_4 \times Z_{14} \times U(1)_L$ group. The obtained values for the physical observables of the quark and lepton sectors are in good agreement with the most recent experimental data. The leptonic Dirac CP violating phase $\delta_{CP}$ is predicted to be $259.579^\circ$ and the predictions for the absolute neutrino masses in the model can also saturate the recent constraints.
I. INTRODUCTION

Despite its great success, the SM still has serious drawbacks such as the lack of mechanisms that explain the smallness of neutrino masses, the large hierarchy of charged fermion masses, the fermionic mixing angles, the leptonic CP violation, etc. Another puzzle of the SM is that it does not explain why there are three generations of fermions. This puzzle can be addressed in the 3-3-1 models \cite{1}. Hence, the neutrino masses and lepton mixings can be regarded as one of the most important evidence of physics beyond the SM. Among the possible extensions of the SM, discrete symmetries associated with the SM extensions are an useful tool to explain the observed pattern of SM fermion masses and mixing angles. According to the neutrino oscillation experimental data \cite{2}, the best fit values of neutrino mass squared differences and the leptonic mixing angles are

\[
\sin^2(\theta_{12}) = 0.307 \pm 0.013, \quad \sin^2(\theta_{13}) = (2.18 \pm 0.07) \times 10^{-2},
\]

\[
\sin^2(\theta_{23}) = 0.536^{+0.023}_{-0.028} \quad \text{(Inverted order)},
\]

\[
\sin^2(\theta_{23}) = 0.512^{+0.019}_{-0.022} \quad \text{(Normal order, octant I)},
\]

\[
\sin^2(\theta_{23}) = 0.542^{+0.019}_{-0.022} \quad \text{(Normal order, octant II)},
\]

\[
\Delta m^2_{21} = (7.53 \pm 0.18) \times 10^{-5}\text{eV}^2,
\]

\[
\Delta m^2_{32} = (-2.53 \pm 0.05) \times 10^{-3}\text{eV}^2 \quad (S = 1.2) \quad \text{(Inverted order)},
\]

\[
\Delta m^2_{32} = (2.444 \pm 0.034) \times 10^{-3}\text{eV}^2 \quad \text{(Normal order)}.
\]

The large leptonic mixing angles given in Eq. (1) are completely different from the quark mixing ones defined by the Cabibbo - Kobayashi - Maskawa (CKM) matrix \cite{3, 4} and this has stimulated works on flavor symmetries.

One of the most simplest possibilities to understand small non-zero neutrino masses is probably the seesaw mechanism, including type I, II, III and/or their combinations which has been briefly reviewed in Ref. \cite{5}. However, in these scenarios, the scale of the masses of the right-handed neutrinos should be very high that cannot be reached in the near future. In the inverse-and linear seesaw mechanism \cite{6–28} the small neutrino masses arise as a result of new physics at TeV scale which may be probed at the LHC experiments. In such low-scale models, both renormalizable and non-renormalizable interactions are included, which can explain the fermion masses and mixings. In the basis \((\nu, N, S)\), the neutrino mass matrix can be presented in the form of a \(3 \times 3\) block matrix where each element is a submatrix. Depending
on the position of the zero elements in the mass matrix, active neutrinos can receive masses through inverse or/and linear seesaw mechanisms that all impose some elements of the mass matrix to be zero or very small and none of them are forbidden by the SM symmetry, however, such terms can be avoided by introducing additional flavor symmetries.

In this paper we propose the possibility of predicting fermion masses and mixing angles in the framework of the low-scale seesaw mechanism with $S_4$ flavor symmetry. $S_4$ is the permutation group of four objects, which is also the symmetry group of a cube. It has 24 elements divided into 5 conjugacy classes, with $1, 1', 2, 3, 3'$ as its 5 irreducible representations. We will work in the basis in which $3, 3'$ are real representations whereas $2$ is complex. For the Clebsch-Gordan coefficients of $S_4$ group one can see, for instance, in the Ref. [29].

The content of this paper goes as follows. In Sec. II we present the necessary elements of the linear seesaw model under the $S_4$ symmetry as well as introduce the necessary Higgs fields responsible for fermion masses and mixings. Section III deals with quark masses and mixings and Section IV is devoted to lepton masses and mixings. We conclude in Section V.

II. THE MODEL

We consider a three Higgs doublet model with several gauge singlet scalars, where the SM gauge symmetry is supplemented by the $S_4 \times Z_2 \times Z_3 \times Z_4 \times Z_{14} \times U(1)_L$ group. In this work, three left-handed leptons $\psi_L$ and three right-handed neutrinos $\nu_R$ as well as extra neural leptons $N_L, N_R, S_L, S_R$ are each put in one $S_4$ triplet while the first right-handed charged leptons $l_{1R}$ and the last two right-handed charged leptons $l_{2,3R}$ transform as $\underline{1}$ and $\underline{2}$ under $S_4$ symmetry, respectively. For the quark sectors, all the families $q_{1L}, u_{1R}, d_{1R}$ are put in $\underline{1}'$ and $q_{2L}, q_{3L}, u_{2R}, u_{3R}, d_{2R}, d_{3R}$ transform as $\underline{1}$ under $S_4$. The particle spectrum of our model and their assignments under the $SU(2)_L \times U(1)_L \times S_4 \times Z_2 \times Z_3 \times Z_4 \times Z_{14}$ group is summarized in Tables I and III where the numbered subscripts on fields in order define components of their $S_4$ multiplet representations as well as the quantum numbers corresponding to other groups of the model. We use the $S_4$ discrete group since it is the smallest non Abelian discrete group having irreducible triplet and doublet representations. The discrete group $S_4$ is crucial to get a predictive fermion sector consistent with the low
energy fermion flavor data. Extra symmetries $Z_2, Z_3, Z_4$ and $Z_{14}$ are additional introduced in order to get the desired structure of the fermion mass matrices that will be discussed in detail in Sec IV.

III. QUARK MASSES AND MIXINGS

The quarks content and the corresponding scalar fields of the model, under the $[SU(2)_L, U(1)_L, S_4, Z_2, Z_3, Z_4, Z_{14}]$, is given in Table. The quark Yukawa terms invariant under the symmetries of the model under consideration take the form:

$$\mathcal{L}^{(q)}_Y = y_{11}^{(u)} q_{1L} \tilde{H} u_{1R} \frac{\chi}{\Lambda^6} + y_{12}^{(u)} q_{1L} \tilde{H}' u_{2R} \frac{\chi^5}{\Lambda^5} + y_{13}^{(u)} q_{1L} \tilde{H}' u_{3R} \frac{\chi^3}{\Lambda^3} + y_{21}^{(u)} \tilde{q}_{2L} \tilde{H}' u_{1R} \frac{\chi^5}{\Lambda^5} + y_{22}^{(u)} \tilde{q}_{2L} \tilde{H} u_{2R} \frac{\chi^4}{\Lambda^4} + y_{23}^{(u)} \tilde{q}_{2L} \tilde{H} u_{3R} \frac{\chi^2}{\Lambda^2} + y_{31}^{(u)} \tilde{q}_{3L} \tilde{H} u_{1R} \frac{\chi^3}{\Lambda^3} + y_{32}^{(u)} \tilde{q}_{3L} \tilde{H} u_{2R} \frac{\chi^2}{\Lambda^2} + y_{33}^{(u)} \tilde{q}_{3L} \tilde{H} u_{3R} + y_{11}^{(d)} \tilde{q}_{1L} \tilde{H}' d_{1R} \frac{\chi}{\Lambda^6} + y_{12}^{(d)} \tilde{q}_{1L} \tilde{H}' d_{2R} \frac{\chi^5}{\Lambda^5} + y_{13}^{(d)} \tilde{q}_{1L} \tilde{H}' d_{3R} \frac{\chi^3}{\Lambda^3} + y_{21}^{(d)} \tilde{q}_{2L} \tilde{H} d_{1R} \frac{\chi^5}{\Lambda^5} + y_{22}^{(d)} \tilde{q}_{2L} \tilde{H} d_{2R} \frac{\chi^4}{\Lambda^4} + y_{23}^{(d)} \tilde{q}_{2L} \tilde{H} d_{3R} \frac{\chi^2}{\Lambda^2} + y_{31}^{(d)} \tilde{q}_{3L} \tilde{H} d_{1R} \frac{\chi^3}{\Lambda^3} + y_{32}^{(d)} \tilde{q}_{3L} \tilde{H} d_{2R} \frac{\chi^2}{\Lambda^2} + y_{33}^{(d)} \tilde{q}_{3L} \tilde{H} d_{3R} \frac{\chi}{\Lambda^6} + H.c.$$

(2)

Table I: $SU(2)_L \times U(1)_L \times S_4 \times Z_2 \times Z_3 \times Z_4 \times Z_{14}$ assignments for quarks and scalars.

Note that the lightest of the physical neutral scalars states of $H, H', H''$ is the SM-like 125 GeV Higgs boson discovered at the LHC. As indicated by Eq. (2), the top quark mass mainly arises from the renormalizable quark Yukawa term involving $H$. Thus the SM-like 125 GeV Higgs predominantly arises from the CP even neutral part of $H$. Furthermore, in view of the large amount of free and uncorrelated parameters of the low energy scalar
potential of the model, there is a lot of freedom to adjust the required pattern of scalar masses, thus allowing to safely assume that the remaining scalars are heavy and outside the LHC reach. In addition, the loop effects of the heavy scalars contributing to precision observables can be suppressed by making an appropriate choice of the free parameters in the scalar potential. These adjustments do not affect the physical observables in the quark and lepton sectors, which are determined mainly by the Yukawa couplings.

Assuming that the $SU(2)$ Higgs doublets $H$, $H'$, $H''$ do acquire vacuum expectation values (VEVs) at the electroweak symmetry breaking scale $v = 246$ GeV and the gauge singlet scalar $\chi$ gets a VEV of the order of $\lambda \Lambda$, with $\lambda = 0.225$ - one of the Wolfenstein parameters and $\Lambda$ - the model cutoff, we find that the SM quark mass matrices are given by:

$$
M_U = \begin{pmatrix}
    a_{11}^{(u)} \lambda^6 & a_{12}^{(u)} \lambda^5 & a_{13}^{(u)} \lambda^3 \\
    a_{12}^{(u)} \lambda^5 & a_{22}^{(u)} \lambda^2 & a_{23}^{(u)} \\
    a_{13}^{(u)} \lambda^3 & a_{23}^{(u)} \lambda^2 & a_{33}^{(u)}
\end{pmatrix} \frac{v}{\sqrt{2}},
$$

$$
M_D = \begin{pmatrix}
    a_{11}^{(d)} \lambda^7 & 0 & 0 \\
    0 & a_{22}^{(d)} \lambda^5 & a_{23}^{(d)} \lambda^5 \\
    0 & a_{32}^{(d)} \lambda^3 & a_{33}^{(d)} \lambda^3
\end{pmatrix} \frac{v}{\sqrt{2}},
$$

where

$$
\begin{align*}
    a_{11}^{(u)} &\simeq 1.89391 + 0.404032i, & a_{12}^{(u)} = a_{21}^{(u)} &\simeq -1.42926 - 0.00898659i, \\
    a_{13}^{(u)} &\simeq 0.704581 + 0.284696i, & a_{22}^{(u)} &\simeq 1.34823 - 0.00203271i, \\
    a_{23}^{(u)} &\simeq -0.0703718 + 0.0148338i, & a_{33}^{(u)} &\simeq 0.989285 - 0.000056837i, \\
    a_{11}^{(d)} &\simeq 0.564554, & a_{22}^{(d)} &\simeq 0.534463, \\
    a_{23}^{(d)} &\simeq 1.08071, & a_{33}^{(d)} &\simeq 1.42119,
\end{align*}
$$

are $O(1)$ dimensionless couplings. The values of the $O(1)$ dimensionless couplings given above allows to successfully reproduce the experimental values of the quark mass spectrum, CKM parameters and Jarlskog invariant. As indicated by Table II our model is consistent with the low energy quark flavor data. Note that we use the $M_Z$-scale experimental values of the quark masses given by Ref. [30] (which are similar to those in [31]). The experimental values of the CKM parameters are taken from Ref. [32].

With the aim to study the sensitivity of the obtained values for the SM quark masses under variations around the best-fit values (maximum variation around the 20% of their best fit values), we show in Figs. 1 and 2 the correlations between the first and second as well as between third and second generation SM quark masses. We have found that
| Observable | Model value | Experimental value |
|------------|-------------|---------------------|
| $m_u (MeV)$ | 1.11        | $1.45^{+0.56}_{-0.45}$ |
| $m_c (MeV)$ | 639         | 635 ± 86            |
| $m_t (GeV)$ | 172.3       | 172.1 ± 0.6 ± 0.9   |
| $m_d (MeV)$ | 2.9         | $2.9^{+0.5}_{-0.4}$ |
| $m_s (MeV)$ | 57.7        | $57.7^{+16.8}_{-15.7}$ |
| $m_b (GeV)$ | 2.82        | $2.82^{+0.09}_{-0.04}$ |
| $\sin \theta^{(q)}_{12}$ | 0.225       | 0.225               |
| $\sin \theta^{(q)}_{23}$ | 0.0421      | 0.0421              |
| $\sin \theta^{(q)}_{13}$ | 0.00365     | 0.00365             |
| $J$ | $3.18 \times 10^{-5}$ | $(3.18 \pm 0.15) \times 10^{-5}$ |

Table II: Model and experimental values of the quark masses and CKM parameters.

Figure 1: Correlations between the first and second generation SM quark masses. The horizontal and vertical lines are the minimum and maximum values of the second and first generation quark masses, respectively, inside the $3\sigma$ experimentally allowed range.

such variations yield values for the SM quark masses inside the $3\sigma$ experimentally allowed range, with the exception of the top and bottom quark masses where the majority of points are outside the $3\sigma$ range. Consequently the quark sector model parameters feature some moderate amount of fine tuning. We have numerically checked that the up and down type
Figure 2: Correlations between the second and third generation SM quark masses. The horizontal and vertical lines are the minimum and maximum values of the third and second generation quark masses, respectively, inside the $3\sigma$ experimentally allowed range.

Quark sector parameters have to be varied in range around the 3% and 4% of their best fit values, respectively, in order to obtain all SM quark masses inside their $3\sigma$ experimentally allowed range.

IV. LEPTON MASSES AND MIXINGS

The lepton fields and the corresponding scalars in lepton sectors, under the $[SU(2)_L, U(1)_L, S_4, Z_2, Z_3, Z_4]$, is given in Table III. The lepton Yukawa terms invariant

| $\psi_L$ | $l_{1R}$ | $l_{2,3R}$ | $\nu_{R}$ | $N_L$ | $N_R$ | $S_L$ | $S_R$ | $\phi$ | $\varphi$ | $\xi$ | $\rho$ |
|----------|---------|-----------|----------|-------|-------|-------|-------|-------|-------|-------|-------|
| $SU(2)_L$ | 2 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $U(1)_L$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 |
| $S_4$ | 3 | 1 | 2 | 3 | 3 | 3 | 3 | 3 | 1 | 1 |
| $Z_2$ | 1 | 1 | 1 | 1 | 1 | -1 | 1 | -1 | -1 | -1 | -1 |
| $Z_3$ | 1 | 1 | 1 | $\omega$ | $\omega^2$ | 1 | $\omega^2$ | 1 | $\omega$ | $\omega$ | 1 |
| $Z_4$ | \textit{i} | \textit{i} | \textit{i} | 1 | -\textit{i} | \textit{i} | -\textit{i} | \textit{i} | 1 | -1 | -1 |

Table III: $SU(2)_L \times U(1)_L \times S_4 \times Z_2 \times Z_3 \times Z_4$ assignments for leptons and scalars.
under the symmetries of the model are:

\[-\mathcal{L}_t = \frac{h_1}{\Lambda} (\bar{\psi}_L \phi) H_{1R} + \frac{h_2}{\Lambda} (\bar{\psi}_L \phi) (H_{1} H_{2}) + \frac{h_3}{\Lambda} (\bar{\psi}_L \phi) (H'_{1} H_{2})
\]

\[
+ x_1 (\bar{\psi}_L N_{R}) i \bar{H} + x_2 (\bar{\psi}_L S_{R}) i \bar{H} + \frac{y_1}{\Lambda} (N_{L} \nu_{R})_1 \xi \rho + \frac{y_2}{\Lambda} (S_{L} \nu_{R})_1 \xi \rho
\]

\[
+ z_1 (S_{L} N_{R})_1 \xi \dagger + z_2 (S_{L} N_{R})_3 \phi \dagger + t_1 (N_{L} S_{R})_1 \xi \dagger + t_2 (N_{L} S_{R})_3 \phi \dagger
\]

\[
+ w_1 (N_{L} N_{R})_1 \xi \dagger + w_2 (N_{L} N_{R})_3 \phi \dagger + w_3 (S_{L} S_{R})_1 \xi \dagger + w_4 (S_{L} S_{R})_3 \phi \dagger + H.c.
\] (4)

In the case where $S_4$ is spontaneously broken down to \{identity\} by the VEVs alignment

\[
\langle \phi_1 \rangle = v, \langle \phi_2 \rangle = v e^{i \alpha}, \langle \phi_3 \rangle = v e^{i \beta} \quad \text{and} \quad \langle H \rangle = v_h, \langle H' \rangle = v'_h
\]

within the following expansions

\[
\phi_i = \langle \phi_i \rangle + \phi_i \dagger \quad (i = 1, 2, 3),
\] (5)

we get the lepton flavor changing interactions as follows

\[
-\mathcal{L}_{lep} \subset \frac{h_1 v}{\Lambda} (\bar{\nu}_{1L} H^+ + \bar{\nu}_{1L} H^0) l_{1R} + \frac{h_2 v}{\Lambda} (\bar{\nu}_{1L} H^+ + \bar{\nu}_{1L} H^0) l_{2R}
\]

\[
+ \frac{h_3 v}{\Lambda} (\bar{\nu}_{1L} H^+ + \bar{\nu}_{1L} H^0) l_{3R} + \frac{h_2 v}{\Lambda} (\bar{\nu}_{1L} H^+ + \bar{\nu}_{1L} H^0) l_{1R} + H.c.
\] (6)

From (6), it follows that, in the model under consideration, the usual Yukawa couplings are associated with the factor $\frac{v}{\Lambda}$ and the lepton flavor changing decays consist of the contribution of three Feynman diagrams as in Fig. 3.

The current experimental data on lepton flavor changing decays read \[32\]: $\text{Br}(\mu^{-} \rightarrow e^{-} \gamma) < 4.2 \times 10^{-13}$, $\text{Br}(\tau^{-} \rightarrow e^{-} \gamma) < 3.3 \times 10^{-8}$ and $\text{Br}(\tau^{-} \rightarrow \mu^{-} \gamma) < 4.4 \times 10^{-8}$. The partial decay

\[
\Gamma(l_i \rightarrow l_j \gamma) = \frac{(m_i^2 - m_j^2)^3}{16\pi m_i^3} \left( |C_L|^2 + |C_R|^2 \right),
\] (7)

where the above form factors $C_L$ and $C_R$ are determined from the process amplitude \[36, 37\]

\[
\mathcal{M} = 2(p_i \cdot \epsilon) [C_L \bar{u}_j(p_j) P_L u_i(p_i) + C_R \bar{u}_j(p_j) P_R u_i(p_i)]
\]

\[
- (m_i C_L + m_j C_L) \bar{u}_j(p_j) \epsilon P_L u_i(p_i) - (m_i C_R + m_j C_R) \bar{u}_j(p_j) \epsilon P_R u_i(p_j).
\] (8)

For the case $m_i \gg m_j$, we get

\[
\text{Br}(l_i \rightarrow l_j \gamma) = \frac{12\pi^2}{G_F^2} \left( |D_L|^2 + |D_R|^2 \right) \text{Br}(l_i \rightarrow l_j \bar{\nu}_j \nu_i),
\] (9)
where $G_F = g^2/(4\sqrt{2}m_W^2)$. In the model under consideration, one has $^{[37, 38]}$

$$D_L \propto \frac{v}{M_H \Lambda} \mathcal{O}(m_j/m_i), \quad D_R \propto \frac{v}{M_H \Lambda}$$

(10)

where $M_H$ is the mass scale of the heavy scalars (which provide the dominant contributions to the LFV decays) running in the internal lines of the loop. For further details on the form factors $D_{L,R}$, the reader is referred to Refs. $^{[36-39]}$.

Combining (9) and (10), we see that the lepton flavor changing processes in this model are suppressed by the factor $\frac{v}{G_F M_H \Lambda}$ associated with the above mentioned small Yukawa couplings and the large mass scale of the heavy scalars running in the internal lines of the loop.

Let us turn into lepton mass issue. From (11), the lepton mass terms read

$$-\mathcal{L}^\text{mass}_{cl} = \frac{v_1}{\Lambda} h_1 v_h \bar{l}_1 l_1 + \frac{v_1}{\Lambda} (h_2 v_h + h_3 v'_h) \bar{l}_1 l_2 + \frac{v_1}{\Lambda} (h_2 v_h - h_3 v'_h) \bar{l}_1 l_3
+ \frac{v_2}{\Lambda} h_1 v_h \bar{l}_2 l_2 + \frac{v_2}{\Lambda} (h_2 v_h + h_3 v'_h) \omega \bar{l}_2 l_2 + \frac{v_2}{\Lambda} (h_2 v_h - h_3 v'_h) \omega^2 \bar{l}_2 l_3
+ \frac{v_3}{\Lambda} h_1 v_h \bar{l}_3 l_3 + \frac{v_3}{\Lambda} (h_2 v_h + h_3 v'_h) \omega^2 \bar{l}_3 l_2 + \frac{v_3}{\Lambda} (h_2 v_h - h_3 v'_h) \omega \bar{l}_3 l_3 + \text{H.c.}
\equiv (\bar{l}_1, \bar{l}_2, \bar{l}_3) M_l (l_1, l_2, l_3)^T + \text{H.c.},$$

(11)
where the mass matrix for charged leptons is given by:

$$M_l = \frac{v}{\Lambda} \begin{pmatrix}
    h_1 v_h & h_2 v_h + h_3 v'_h & h_2 v_h - h_3 v'_h \\
    h_1 v_h e^{i\alpha} (h_2 v_h + h_3 v'_h) e^{i\omega^2} & (h_2 v_h - h_3 v'_h) e^{i\omega} \\
    h_1 v_h e^{i\beta} (h_2 v_h + h_3 v'_h) e^{i\beta \omega} & (h_2 v_h - h_3 v'_h) e^{i\beta \omega^2}
\end{pmatrix}. \quad (12)$$

This matrix can be diagonalized as,

$$U_L^T M_l U_R = \frac{\sqrt{3} v}{\Lambda} \text{diag}(h_1 v_h, h_2 v_h - h_3 v'_h, h_2 v_h + h_3 v'_h) \equiv \text{diag}(m_e, m_\mu, m_\tau), \quad (13)$$

where

$$U_L = \frac{1}{\sqrt{3}} \begin{pmatrix}
1 & 0 & 0 \\
0 & e^{i\alpha} & 0 \\
0 & 0 & e^{i\beta}
\end{pmatrix}, \quad U_R = 1, \quad (14)$$

$$m_e = \frac{\sqrt{3} v}{\Lambda} h_1 v_h, \quad m_\mu, m_\tau = \frac{\sqrt{3} v}{\Lambda} (h_2 v_h \pm h_3 v'_h), \quad (15)$$

where $\omega = e^{i2\pi/3}$ is the cube root of unity.

The best fit values for the masses of charged-leptons are given in Ref. $[2]$: $m_e \simeq 0.51099 \text{ MeV}$, $m_\mu \simeq 105.65837 \text{ MeV}$, $m_\tau \simeq 1776.86 \text{ MeV}$. Then, we find the relations $\frac{h_3}{h_2} \simeq \frac{v_\mu}{v_h}$, $\frac{h_2}{h_1} \simeq 10^3$.

We also assume that in the neutrino sector, the $S_4$ discrete group is spontaneously broken down to the Klein four group $K$ by the VEV alignment $\langle \varphi \rangle = (0, v_\varphi, 0)$ of $\varphi$ and the VEVs of $\xi, \rho$ as $\langle \xi \rangle = v_\xi$, $\langle \rho \rangle = v_\rho$. In this case, the neutrino mass matrices become

$$m_{\nu N} = x_1 v_h I \equiv a_1 I, \quad M_{\nu S} = x_2 v_h I \equiv a_2 I, \quad (16)$$

$$m'_{\nu N} = \frac{y_1 v_\xi v_\rho}{\Lambda} I \equiv b_1 I, \quad M'_{\nu S} = \frac{y_2 v_\xi v_\rho}{\Lambda} I \equiv b_2 I, \quad (17)$$

$$M'_{NS} = \begin{pmatrix}
z_1 v_\xi & 0 & z_2 v_\rho \\
0 & z_1 v_\xi & 0 \\
z_2 v_\rho & z_1 v_\xi
\end{pmatrix} \equiv \begin{pmatrix}c_1 & 0 & c_2 \\
0 & c_1 & 0 \\
c_2 & 0 & c_1
\end{pmatrix}, \quad (18)$$

$$M_{NS} = \begin{pmatrix}
t_1 v_\xi & 0 & t_2 v_\rho \\
0 & t_1 v_\xi & 0 \\
t_2 v_\rho & t_1 v_\xi
\end{pmatrix} \equiv \begin{pmatrix}d_1 & 0 & d_2 \\
0 & d_1 & 0 \\
d_2 & 0 & d_1
\end{pmatrix}, \quad (19)$$

$$M_{NN} = \begin{pmatrix}
w_1 v_\xi & 0 & w_2 v_\rho \\
0 & w_1 v_\xi & 0 \\
w_2 v_\rho & w_1 v_\xi
\end{pmatrix} \equiv \begin{pmatrix}g_1 & 0 & g_2 \\
0 & g_1 & 0 \\
g_2 & 0 & g_1
\end{pmatrix}, \quad (20)$$
\[ M_{SS} = \begin{pmatrix} w_3 v_\xi & 0 & w_4 v_\varphi \\ 0 & w_3 v_\xi & 0 \\ w_4 v_\varphi & 0 & w_3 v_\xi \end{pmatrix} \equiv \begin{pmatrix} g_3 & 0 & g_4 \\ 0 & g_3 & 0 \\ g_4 & 0 & g_3 \end{pmatrix}. \quad (21) \]

Let us note that the matrices given by Eqs. (16) - (21) are all symmetric and \( m_{\nu N} \), \( M_{L S} \), \( M'_{NS} \), \( M_{NS} \), \( M_{NN} \), \( M_{SS} \) are respectively generated from the renormalizable Yukawa interactions \( x_1(\bar{\psi}_L N_R)_1 \tilde{H} \), \( x_2(\bar{\psi}_L S_R)_1 \tilde{H} \), \( \{ z_1(\bar{S}_L N_R)_1 \xi^\dagger, z_2(\bar{S}_L N_R)_3 \varphi^\dagger \} \), \( \{ t_1(\bar{N}_L S_R)_1 \xi^\dagger, t_2(\bar{N}_L S_R)_3 \varphi^\dagger \} \), \( \{ w_1(\bar{N}_L N_R)_1 \xi^\dagger, w_2(\bar{N}_L N_R)_3 \varphi^\dagger \} \), \( \{ w_3(\bar{S}_L S_R)_1 \xi^\dagger, w_4(\bar{S}_L S_R)_3 \varphi^\dagger \} \), whereas \( m'_{\nu N} \) and \( M'_{\nu S} \) arise from the non-renormalizable Yukawa interactions \( \frac{w}{\Lambda}(\bar{N}_L \nu_R)_1 \xi \rho \) and \( \frac{w}{\Lambda}(\bar{S}_L \nu_R)_1 \xi \rho \), respectively.

In this work, we introduce the \( Z_2 \times Z_3 \times Z_4 \times Z_{14} \times U(1)_L \) symmetry\(^1\), which in addition to the \( S_4 \) symmetry to prevent some Yukawa interactions thus giving rise to the predictive textures for the neutrino sector shown in Eqs. (16) - (21). For instance, since the product of two \( S_4 \) triplets contains a \( S_4 \) triplet, the coupling \( \bar{\psi}_L N_R \) can transform under \( S_4 \times Z_2 \times Z_3 \times Z_4 \times Z_{14} \times U(1)_L \) as \( (\mathbf{3} \otimes \mathbf{3}, -1, 1, 1, 1, 0) \), which implies that in order to generate the mass matrix \( m_{\nu N} \), one needs one \( S_4 \) singlet transforming as \( (1, -1, 1, 1, 1, 0) \), in order to build an invariant under all given symmetries. For the known scalars, \( (\bar{\psi}_L N_R) \tilde{H}' \) is forbidden by the \( S_4 \) symmetry, \( (\bar{\psi}_L N_R) \tilde{H}'' \) is prevented by the \( Z_2 \) symmetry, \( (\bar{\psi}_L N_R) \chi \) is not allowed by the \( Z_2, Z_{14} \) and \( SU(2)_L \) symmetries, whereas \( (\bar{\psi}_L N_R) \xi \) is forbidden by \( Z_3 \) and \( Z_4 \) symmetries and \( (\bar{\psi}_L N_R) \rho \) is prevented by the \( Z_4 \) symmetry. Consequently, there is only one term involving the fields \( \psi_L, N_R \) and \( H \), invariant under the \( S_4 \times Z_2 \times Z_3 \times Z_4 \times Z_{14} \times U(1)_L \) symmetry, which corresponds to \( x_1(\bar{\psi}_L N_R)_1 \tilde{H} \) as in Eq. (1) that provide a simple form of \( m_{\nu N} \) as indicated by Eq. (16). The situation is similar for the remaining couplings that generate the other mass matrices given in Eqs. (16) - (21).

In the basis \((\nu, N, S)\), the full neutrino mass matrix predicted by our model takes the

\(^1\) All the lepton fields and the corresponding scalars in Table III carry the same charged \((+1)\) under \( Z_{14} \) which is not necessary to write out here.
The effective neutrino mass matrix \( M_{\text{eff}} \) in Eq. (22) can be rewritten in the form:

\[
M_{\text{eff}} = \begin{pmatrix} 0 & M_D & M_T \end{pmatrix}
\]

which is similar to the one resulting from a type-I seesaw mechanism. Then, the light active neutrino mass matrix takes the form:

\[
m_\nu = -M_D M_R^{-1} M_T = -m_{\nu N} M_{\nu S}^{-1} M_{\nu S} - M_{\nu S} M_{\nu S}^{-1} m_{\nu N}
- M_{\nu S} M_{\nu S}^{-1} M_{\nu S} + m_{\nu N} M_{\nu S} M_{\nu S}^{-1} M_{\nu S} + M_{\nu S} M_{\nu S}^{-1} M_{\nu S} M_{\nu S}^{-1} m_{\nu N}
+ M_{\nu S} M_{\nu S}^{-1} M_{\nu S} M_{\nu S}^{-1} M_{\nu S} - m_{\nu N} M_{\nu S} M_{\nu S}^{-1} M_{\nu S} M_{\nu S}^{-1} m_{\nu N}.
\]

Replacing Eqs. (16) - (21) in Eq. (24) yields the following mass matrix for light active neutrinos:

\[
m_\nu = \begin{pmatrix} A & 0 & B \\ 0 & C & 0 \\ B & 0 & A \end{pmatrix},
\]

where

\[
A = \frac{1}{2} (a_1 \alpha_1 + a_2 \alpha_2), \quad B = \frac{1}{2} (a_1 \beta_1 + a_2 \beta_2),
\]

\[
C = \frac{a_2 b_2 g_1 - a_2 b_1 c_1 - a_1 b_2 d_1}{c_1 d_1} - \frac{(a_1 d_1 - a_2 g_1)(b_1 c_1 - b_2 g_1)}{g_1^2 g_3},
\]
\[
\alpha_1 = -\frac{2b_2c_1}{c_1^2 - c_2^2} + \frac{(d_1 - d_2)[b_2(g_1 - g_2) - b_1(c_1 - c_2)]}{(g_1 - g_2)(g_3 - g_4)} + \frac{(d_1 + d_2)[b_2(g_1 + g_2) - b_1(c_1 + c_2)]}{(g_1 + g_2)(g_3 + g_4)},
\]
\[
\alpha_2 = -\frac{2b_1d_1}{d_1^2 - d_2^2} - \frac{2b_2[(c_1d_1 + c_2d_2)g_1 - (c_2d_1 + c_1d_2)g_2]}{(c_1^2 - c_2^2)(d_1^2 - d_2^2)} + \frac{b_1(c_1 - c_2)}{(g_1 - g_2)(g_3 - g_4)},
\]
\[
\beta_1 = \frac{2b_2c_2}{c_1^2 - c_2^2} + \frac{(d_1 - d_2)[b_1(c_1 - c_2) - b_2(g_1 - g_2)]}{(g_1 - g_2)(g_3 - g_4)} + \frac{(d_1 + d_2)[b_2(g_1 + g_2) - b_1(c_1 + c_2)]}{(g_1 + g_2)(g_3 + g_4)},
\]
\[
\beta_2 = \frac{2b_1d_2}{d_1^2 - d_2^2} + \frac{2b_2[(c_1d_1 + c_2d_2)g_2 - (c_2d_1 + c_1d_2)g_1]}{(c_1^2 - c_2^2)(d_1^2 - d_2^2)} - \frac{b_1(c_1 - c_2)}{(g_1 - g_2)(g_3 - g_4)},
\]
\begin{align*}
\text{with } a_{1,2}, b_{1,2}, c_{1,2}, d_{1,2} \text{ and } g_{1,2,3,4} \text{ defined in Eqs. (16) - (21). The mass matrix } m_\nu \text{ for light active neutrinos is diagonalized by the rotation matrix } U_\nu, \\
U_\nu = \begin{pmatrix}
\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\
0 & 1 & 0 \\
\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}
\end{pmatrix},
\end{align*}
\]
and the light active neutrino masses \( m_{1,2,3} \) are given by
\[
m_1 = A + B, \quad m_2 = C, \quad m_3 = A - B.
\]

By combining Eqs. (17) and (28) we find that the leptonic mixing matrix takes the form:
\[
U^{\text{lep}} = U_L^\dagger U_\nu = \begin{pmatrix}
\frac{1 + e^{-i\beta}}{\sqrt{6}} & \frac{e^{-i\alpha}}{\sqrt{3}} & -\frac{1 + e^{-i\beta}}{\sqrt{6}} \\
\frac{1 - e^{-i\beta}}{\sqrt{6}} & \frac{e^{-i\alpha}}{\sqrt{3}} & -\frac{1 + e^{-i\beta}}{\sqrt{6}} \\
\frac{1 + e^{-i\beta}}{\sqrt{6}} & \frac{-e^{-i\alpha}}{\sqrt{3}} & -\frac{1 + e^{-i\beta}}{\sqrt{6}}
\end{pmatrix}.
\]

We see that all the elements of the matrix \( U^{\text{lep}} \) in Eq. (30) depend only on two parameters \( \alpha \) and \( \beta \). From experimental constraints on the elements of the lepton mixing matrix given in Ref. [33], we can find out the regions of \( \alpha \) and \( \beta \) to establish experimental constraints for lepton mixing matrix. In the standard Particle Data Group (PDG) parametrization, the leptonic mixing matrix can be parameterized in three Euler’s angles as follows:
\[
s_{13} = |U_{13}| = \frac{\sqrt{1 - \cos \beta}}{\sqrt{3}},
\]
\[
t_{23} = \frac{U_{23}}{U_{33}} = \left| \frac{1 + 2 \cos \beta}{2 + \cos \beta - \sqrt{3} \sin \beta} \right|,
\]
\[
t_{12} = \frac{U_{12}}{U_{11}} = \sqrt{\frac{1}{1 + \cos \beta}}.
\]
i.e., $s_{13}, t_{12}$ and $t_{23}$ in Eqs. (31) and (33) depend only on one parameter $\beta$. Eqs. (31) - (33) yields:

$$\beta = -\arccos(1 - 3s_{13}^2),$$

$$t_{23} = \frac{1 - 2s_{13}^2}{1 - s_{13}^2 + s_{13}\sqrt{2 - 3s_{13}^2}},$$

$$t_{12} = \frac{1}{\sqrt{2 - 3s_{13}^2}}. \quad (35)$$

The data in Particle Data Group 2018 shows that $s_{13} \in (0.145258, 0.15)$ rad so $t_{23} \in (0.806, 0.811)$ and $t_{12} \in (0.7811, 0.7192)$ rad as depicted in Figs. 4 and 5, respectively. Taking the best fit value given in Ref. 2, $s_{13} = 0.147648$ rad ($\theta_{13} = 8.45963^\circ$) we get $t_{23} = 0.808068$ ($\theta_{23} = 38.9406^\circ$) and $t_{12} = 0.718959$ ($\theta_{12} = 35.7146^\circ$) which are in good agreement with the

![Figure 4](image1.png)

**Figure 4:** $t_{23}$ as a function of $s_{13}$ with $s_{13} \in (\sqrt{0.0211}, \sqrt{0.0225})$ rad.

![Figure 5](image2.png)

**Figure 5:** $t_{12}$ as a function of $s_{13}$ with $s_{13} \in (\sqrt{0.0211}, \sqrt{0.0225})$ rad.
values of $\theta_{23}$ and $\theta_{12}$ given in Ref. [2]. On the other hand, with this best value of $\theta_{13}$, we get $\beta = -0.363663 \text{ rad} \left( \sim 339.163^\circ \right)$ and Dirac CP violation phase $\delta_{\text{CP}} = 259.579^\circ$ which is a viable value of the CP violating Dirac phase [2]. The leptonic mixing matrix in Eq. (36) takes the explicit form

$$U_{\text{lep}} = \begin{pmatrix} 0.789797 + 0.145214i & 0.57735e^{-ia} & -0.0266994 + 0.145214i \\ 0.343233 - 0.403038i & (-0.288675 + 0.5i)e^{-ia} & -0.473264 - 0.403038i \\ 0.0917147 + 0.257824i & (-0.288675 - 0.5i)e^{-ia} & -0.724782 + 0.257824i \end{pmatrix}, (36)$$

which is an unitary matrix.

The expression (36) shows that $\alpha$ is free parameter so we can choose the VEV alignment $\phi$ in the charged-lepton sector as $\langle \phi \rangle = v(1,1,e^{i\beta})$, i.e, $\alpha$ may get the value $\alpha = 0$. In this case, the leptonic mixing matrix becomes:

$$U_{\text{lep}} = \begin{pmatrix} 0.789797 + 0.145214i & 0.57735 & -0.0266994 + 0.145214i \\ 0.343233 - 0.403038i & -0.288675 + 0.5i & -0.473264 - 0.403038i \\ 0.0917147 + 0.257824i & -0.288675 - 0.5i & -0.724782 + 0.257824i \end{pmatrix}, (37)$$

or

$$|U_{\text{lep}}| = \begin{pmatrix} 0.803036 & 0.57735 & 0.147648 \\ 0.529385 & 0.57735 & 0.621625 \\ 0.273651 & 0.57735 & 0.769274 \end{pmatrix}, (38)$$

i.e, the ranges of the magnitude of the elements of the three-flavour leptonic mixing matrix is consistent with those of given in Ref. [33]. At present, the values of neutrino masses (or the absolute neutrino masses) as well as the mass ordering of neutrinos are still unknown. The result in Ref. [34] shows that $m_i \leq 0.6 \text{ eV} (i = 1,2,3)$ while the upper bound on the sum of light active neutrino masses is given by [35],

$$\sum_{i=1}^{3} m_i \leq 0.17 \text{ eV. (39)}$$

The experimental neutrino oscillation data given in Eq. (1) are compatible with two possible signs of $\Delta m_{23}^2$ which is currently unknown and correspond to two types of neutrino mass spectra.
A. Normal spectrum \((m_1 < m_2 < m_3)\)

By taking the best fit values on neutrino mass squared differences for normal spectrum, given in Ref. [2], \(\Delta m_{21}^2 = 7.53 \times 10^{-5}\text{eV}^2\) and \(\Delta m_{32}^2 = 2.444 \times 10^{-3}\text{eV}^2\), we obtain four solutions, however, they have the same absolute values of \(m_{1,2,3}\), the unique difference is the sign of them. So, here we only consider the following solution:

\[
A = 1.58114 \times 10^{-2} \Gamma, \\
B = (-0.0148662 - 12.5522C^2 + 7.93871 \times 10^{-3} \gamma) \Gamma,
\]

where

\[
\Gamma = \sqrt{2.3687 + 2 \times 10^3 C^2 + 1.26491 \sqrt{\gamma}}, \\
\gamma = \sqrt{-0.460083 + 5.92175 \times 10^3 C^2 + 2.5 \times 10^6 C^4}.
\]

In the model under consideration, \(C \equiv m_2 \in (0.001, 0.0506)\) eV is a good region of \(C\) that can reach the realistic normal neutrino mass hierarchy which is depicted in Fig. 7. In the case \(C \equiv m_2 = 0.0087\) eV, the parameters \(A, B\) and the other neutrino masses are explicitly given as \(A = 2.54105 \times 10^{-2}, B = -2.4786 \times 10^{-2}, m_1 = 6.245 \times 10^{-4}\) eV and \(m_3 = 5.01965 \times 10^{-2}\) eV which corresponds to a normal neutrino mass spectrum. The sum of all three neutrino in this case is given by \(\sum^{N}_i = \sum^{3}_{i=1} m_i = 5.9521 \times 10^{-2}\) eV lying within the cosmological bound from Planck data given in Eq. (39).

Figure 6: \(m_{1,3}\) as functions of \(m_2\) with \(m_2 \in (0.001, 0.0506)\) in the normal spectrum.
Figure 7: $\Sigma^N = \sum_{i=1}^{3} m_i^N$ as functions of $m_2$ with $m_2 \in (0.001, 0.0506)$ in the normal spectrum.

B. Inverted spectrum ($m_3 < m_1 < m_2$)

Similar to the normal spectrum, by taking the best fit values on neutrino mass squared differences for inverted spectrum, given in Ref. \[2\], $\Delta m^2_{21} = 7.53 \times 10^{-5}$eV$^2$ and $\Delta m^2_{32} = -2.53 \times 10^{-3}$eV$^2$, we get a solution as follows:

$$A = 1.5811 \times 10^{-2} \Gamma',$$
$$B = (-0.0167814 + 12.8825C^2 - 1.2882 \times 10^{-2} \gamma') \Gamma',$$

(42)

where

$$\Gamma' = \sqrt{2.6053 + 2 \times 10^3 C^2 + 2 \sqrt{\gamma'}},$$
$$\gamma' = \sqrt{0.190509 - 2.6053 \times 10^3 C^2 + x \times 10^6 C^4}.$$  

(43)

In this model, $C \equiv m_2 \in (0.051, 0.065)$ eV is a good region of $C$ that can reach the inverted neutrino mass hierarchy which is depicted in Fig. 8. In the case $C \equiv m_2 = 5.1 \times 10^{-2}$ eV, the parameters $A, B$ and the other neutrino masses are explicitly given as $A = 2.93412 \times 10^{-2}, B = 2.09151 \times 10^{-2}, m_1^I = 5.02563 \times 10^{-2}$ eV and $m_3^I = 8.42615 \times 10^{-3}$ eV which corresponds to an inverted neutrino mass spectrum. The sum of all three neutrino in this case is given by $\sum^I = \sum_{i=1}^{3} m_i^I = 0.10968$ eV which is consistent with the cosmological bound from Planck data in Eq.(39).
We have proposed a low-scale seesaw model to generate the masses for the active neutrinos based on $S_4$ flavour symmetry supplemented by the $Z_2 \times Z_3 \times Z_4 \times Z_{14} \times U(1)_L$ group, where the masses of the SM charged fermions and the fermionic mixing angles are generated from a Froggatt-Nielsen mechanism after the spontaneous breaking of the $S_4 \times Z_2 \times Z_3 \times Z_4 \times Z_{14} \times U(1)_L$ group. The obtained values for the physical observables of the quark and lepton sectors are in good agreement with the most recent experimental data. The Dirac CP violating phase $\delta_{CP}$ is predicted to be $259.579^\circ$ which is consistent with the most recent neutrino oscillation experimental data $[2]$. The predictions for the absolute neutrino masses in the model can also saturate the recent constraints.
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