A triangular field of rational numbers related to Stirling numbers and Hyperbolic functions

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Abstract
A triangular field of rational numbers is characterized, with relations to Stirling numbers 2\textsuperscript{nd} kind, Hyperbolic functions, and centered Binomial distribution. A Generating function is given.

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Summary
Preliminary definitions

Let \( f(x) \) be a smooth function over some domain \( D \). Define the operator \([r]\) as follows:

\[
f^{[r]}(x) := \left[x \cdot \frac{d}{dx}\right]^r f(x) \quad r \in \mathbb{N}_0
\]

(1)

Please notice that this definition uses square brackets, differing from the familiar notation \( f^{(r)}(x) \), which involves round brackets and indicates the \( r \)th derivative of \( f(x) \).

Define the elementary functions \( g, h \):

\[
g(x) := \frac{1}{2} (x + x^{-1}) \quad h(x) := \frac{1}{2} (x - x^{-1})
\]

(2)

From (1) it follows immediately

\[
g^{[1]}(x) = h(x) \\
h^{[1]}(x) = g(x)
\]

(3)

Zeros of \( g \) and \( h \) are

\[
g(i) = 0 \quad h(1) = 0 \quad g(-i) = 0 \quad h(-1) = 0
\]

(4)

Here, \( i \) denotes the imaginary unit.

Based on (2), define the functions

\[
G_{s,j} : \mathbb{C} \setminus \{0, 1, -1, i, -i\} \rightarrow \mathbb{C}
\]

\[
G_{s,j} := g^{s-j} \cdot h^j \quad s \geq 0 \quad j \in \mathbb{Z}
\]

(5)

The domain of \( G_{s,j} \), in general, excludes \( \{0, 1, -1, i, -i\} \), because, if \( j < 0 \) the zeros of \( h \) cause poles of \( G_{s,j} \), and so do the zeros of \( g \) in case \( j > s \). For each given \( s \geq 0 \), the infinite set of functions

\[
G_s = \{ \ldots, G_{s,-2}, G_{s,-1}, G_{s,0}, G_{s,1}, G_{s,2}, \ldots \} \quad s \geq 0
\]

(6)

are linearly independent. The reason is, that for each \( j \in \mathbb{Z} \), \( G_{s,j} \) has unique orders of zeros/poles. For example, by definition (5), \( x = 1 \) is a zero of order \( j \) if \( j > 0 \), and a pole of order \( j \) if \( j < 0 \). Therefore, no finite linear combination of other elements in \( G_s \) can reproduce the same order \( j \) of the zero/pole at \( x = 1 \).

The triangular array \( A \)

Application of the \([r]\) operator with \( r = 1 \) to \( G_{s,j} \) as defined in (5) yields

\[
G^{[1]}_{s,j} = j \cdot G_{s,j-1} + (s-j) \cdot G_{s,j+1} \quad s \geq 0 \quad j \in \mathbb{Z}
\]

(7)
This suggests the expansion
\[
G_{s,0}^{[r]} = \sum_{j \in \mathbb{Z}} A_{s,r,j} \cdot G_{s,j} \quad s, r \geq 0
\]  
(8)
Applying \([r = 1]\) to (8) leads to a recurrence relation for the coefficients:

Here are the steps in detail:
\[
G_{s,0}^{[r+1]} = \sum_{j \in \mathbb{Z}} A_{s,r,j} \cdot G_{s,j}^{[1]}
\]
\[
= \sum_{j \in \mathbb{Z}} A_{s,r,j} \cdot j \cdot G_{s,j-1} + \sum_{j \in \mathbb{Z}} A_{s,r,j} \cdot (s - j) \cdot G_{s,j+1}
\]
\[
= \sum_{j \in \mathbb{Z}} A_{s,r,j+1} \cdot (j + 1) \cdot G_{s,j} + \sum_{j \in \mathbb{Z}} A_{s,r,j-1} \cdot (s - (j - 1)) \cdot G_{s,j}
\]
(9)

(10a) follows, if one compares this with (8) after having replaced \(r \rightarrow r + 1\), and when observing linear independence of (6).

\[
A_{s,r+1,j} = A_{s,r,j-1} \cdot (s - (j - 1)) + A_{s,r,j+1} \cdot (j + 1)
\]
\[
A_{s,0,0} = 1
\]
\[
A_{s,r,j} = 0 \quad \text{if not} \quad 0 \leq j \leq r
\]
(10b) / (10c) make \(A_{s,r,j}\) a Triangular array. We can therefore write (8) as a finite sum:
\[
G_{s,0}^{[r]} = \sum_{j=0}^{r} A_{s,r,j} \cdot G_{s,j} \quad s, r \geq 0
\]  
(11)

The coefficients in (11) add up to \(j + (s - j) = s\). By induction, this is readily extended to the identity
\[
\sum_{j=0}^{r} A_{s,r,j} = s^r \quad s, r \geq 0
\]  
(12)

Table 1 displays a few elements from the tip of \(A_{s,r,j}\), as computed from (10a). Here, the \((s)_k\) denote Falling factorials. The elements of \(A\) are integer functions of the size parameter \(s \geq 0\). In particular, elements in the \(j = 0\) column can be expressed in terms of \(cosh\) (15). At the same time, for any given \(r \geq 0\), the \(r\)th item in the \(j = 0\) column equals the \(r\)th moment of a centered Binomial distribution of size \(s\) (20). Coefficients of \((s)_k\) in the \(j = 0\) column match the triangle of numbers given in [Int], when selecting only even values of \(r\).
| $j$ | 0 | 1 | 2 | 3 | 4 |
|-----|---|---|---|---|---|
| 0   | 1 |   |   |   |   |
| 1   | $s$ |   |   |   |   |
| 2   | 0 | $(s)_2$ |   |   |   |
| 3   | $3(s)_2 + s$ | 0 | $(s)_3$ |   |   |
| 4   | $3(s)_2 + s$ | 0 | $6(s)_3 + 4(s)_2$ | 0 | $(s)_4$ |
| 5   | 0 | $15(s)_3 + 15(s)_2 + s$ | 0 | $10(s)_4 + 10(s)_3$ | 0 |
| 6   | $15(s)_3 + 15(s)_2 + s$ | 0 | $45(s)_4 + 75(s)_3 + 16(s)_2$ | 0 | ... |
| 7   | 0 | $105(s)_4 + 210(s)_3 + 63(s)_2 + s$ | ... |   |   |
| 8   | $105(s)_4 + 210(s)_3 + 63(s)_2 + s$ | ... |   |   |   |
| 9   | 0 |   |   |   |   |

Table 1: Apex of $A_{s,r,j}$

**Relation to Hyperbolic functions**

Exactly the same table is generated if one replaces $g / h$ from (2) by $\cosh / \sinh$, and, at the same time, replaces the $[r]$ operator from (1) by the $r$th derivative ($r$):

$$
g \rightarrow \cosh

h \rightarrow \sinh

[r] \rightarrow (r)

$$

With these replacements (11) becomes

$$
\left( \cosh(t)^s \right)^{(r)} = \sum_{j=0}^{r} A_{s,r,j} \cdot \cosh(t)^{s-j} \sinh(t)^j

s, r \geq 0

$$

and therefore, setting the formal parameter $t = 0$:

$$
A_{s,r,0} = \left[ \left( \cosh(t)^s \right)^{(r)} \right]_{t=0}

s, r \geq 0

$$

**Relation to centered Binomial distribution**

$G_{s,0}$ as defined in (5), if written as the Laurent series

$$
G_{s,0}(x) = \sum_{j \in \mathbb{Z}} b_{s,j} \cdot x^j

s \geq 0

$$

generates a centered Binomial distribution. For if we define the set of events

$$
\mathcal{V}_s := \{2k - s \mid 0 \leq k \leq s\} \quad s \geq 0

$$
each event $v \in V_s$ is associated with $b_{s,v}$ from (16), which can be written as

$$b_{s,v} = \frac{1}{2s} \left( \frac{s}{v+s} \right) \quad v \in V_s, \ s \geq 0$$

(18)

According to e.g. [Bul79] the Moment generating function $M_s(t)$ is defined by the Expected value of $e^{t \cdot v}$

$$M_s(t) = \sum_{k=0}^{s} \frac{1}{2^s} \binom{s}{k} e^{t \cdot (2k-s)} = \cosh(t)^s \quad s \geq 0$$

(19)

Therefore, if $\mu_{s,r}$ denotes the $r^{th}$ moment of (18) and $M_s^{(r)}$ denotes the $r^{th}$ derivative of $M_s(t)$ with respect to the formal parameter $t$, we have

$$\mu_{s,r} = M_s^{(r)}(0) = \left[ \left( \cosh(t)^s \right)^{(r)} \right]_{t=0} \quad s, r \geq 0$$

(20)

On the other hand, by construction of $G_{s,0}^{[r]}$, we have the identity

$$\mu_{s,r} = G_{s,0}^{[r]}(1) = \sum_{j=0}^{r} A_{s,r,j} \cdot G_{s,j}(1) = A_{s,r,0} \quad s, r \geq 0$$

(21)

Comparing (20) and (21) again gives (15).

**The condensed form $B$**

In order to exclude the intermediate zero elements in table 1, we derive from $A$ the “condensed” form

$$B_{s,r,n} : = A_{s,r,r-2n} \quad n \in \mathbb{Z} \quad s, r \geq 0$$

(22)

Transcribing (10a), (10b) / (10c), and (12) in terms of (22) gives:

$$B_{s,r+1,n} = B_{s,r,n} \cdot (s - (r - 2n))$$
$$+ B_{s,r,n-1} \cdot (r - 2(n - 1)) \quad s, r \geq 0$$

(23a)

$$B_{s,0,0} = 1$$

(23b)

$$B_{s,r,n} = 0 \quad \text{IF NOT } 0 \leq n \leq \left\lfloor \frac{r}{2} \right\rfloor$$

(23c)

$$\sum_{n=0}^{\left\lfloor \frac{r}{2} \right\rfloor} B_{s,r,n} = s^r \quad s, r \geq 0$$

(24)
The triangular field $\varphi$

Expansion in terms of falling factorials

Consider the transformation

$$B_{s,r,n} = \sum_{0 \leq k \leq j \leq n} (s)_{r-n-j} \cdot (r)_{2n+k} \cdot \varphi_{n,j,k} \quad s, r \geq 0 \quad n \in \mathbb{Z}$$  \hspace{1cm} (25)$$

If one inserts (25) into (23a), while applying appropriate identities to match factorial powers of $s$ and $r$ on both sides, one gets the recurrence relation

$$(2n + k) \cdot \varphi_{n,j,k} = (n - (j - 1)) \cdot \varphi_{n-1,j,k-1}$$

$$+ (k + 1) \cdot \varphi_{n-1,j,k+1}$$

$$+ \varphi_{n-1,j,k} \quad 0 \leq k \leq j \leq n \quad (26a)$$

$$\varphi_{0,0,0} = 1 \quad (26b)$$

$$\varphi_{n,j,k} = 0 \quad \text{IF NOT} \quad 0 \leq k \leq j \leq n \quad (26c)$$

Seed values (26b) / (26c), which are counterparts of conditions (23b) / (23c), make $\varphi$ a triangular array in 3 dimensions. For given $n \geq 0$, $\varphi_{n,j,k}$ can be regarded as a lower triangular matrix of dimension $n + 1$. Indices $j$ and $k$ denote row and column indices respectively. The matrix associated with $n = 0$ reduces to the scalar 1.

Special solutions

Some special solutions of the triple (26b) / (26c) / (26a) are easily verified:

$$\varphi_{n,k,k} = \frac{1}{2^n (n-k)! \cdot k! \cdot 3^k} \quad n \geq k \geq 0 \quad (27a)$$

$$\varphi_{n,n-1,0} = \frac{1}{(2n)!} \quad \varphi_{n,n,1} = \frac{1}{(2n + 1)!} \quad n \geq 1 \quad (27b)$$

$$\varphi_{n,n,0} = \delta_{n,0} \quad n \in \mathbb{Z} \quad (27c)$$

From (27a) we have in particular

$$\varphi_{n,0,0} = \frac{1}{2^n n!} \quad \varphi_{n,n,n} = \frac{1}{6^n n!} \quad n \geq 0 \quad (28)$$
Relation to Stirling numbers

Inserting (25) into (24) gives, after rearrangement
\[
\sum_{i=0}^{r} (s)_{r-i} \sum_{j \geq 0} (r)_{j} \left[ \sum_{n \geq 0} \varphi_{n,i-n,j-2n} \right] = s^{r} \quad s, r \geq 0
\] (29)

Comparing this with the basic relation for Stirling numbers 2\textsuperscript{nd} kind \[Wiki\]
\[
\sum_{i=0}^{r} (s)_{r-i} \left\{ \begin{array}{c} r \\ r-i \end{array} \right\} = s^{r} \quad s, r \geq 0
\] (30)
we get the identity
\[
\sum_{j \geq 0} (r)_{j} \left[ \sum_{n \geq 0} \varphi_{n,i-n,j-2n} \right] = \left\{ \begin{array}{c} r \\ r-i \end{array} \right\} \quad 0 \leq i \leq r
\] (31)

Relation to Hyperbolic functions

If one substitutes \( r = 2n \) in (25) one gets
\[
B_{s,2n,n} = \sum_{0 \leq k \leq j \leq n} (s)_{n-j} \cdot (2n)_{2n+k} \cdot \varphi_{n,j,k}
\]
\[
= (2n)_{2n} \cdot \sum_{0 \leq j \leq n} (s)_{n-j} \cdot \varphi_{n,j,0} \quad s, n \geq 0
\] (32)

And therefore, from (22) and (15):
\[
\sum_{0 \leq j \leq n} (s)_{n-j} \cdot \varphi_{n,j,0} = \frac{1}{(2n)!} \left[ \left( \cosh(t)^{s} \right)^{(2n)} \right]_{t=0} \quad s, n \geq 0
\] (33)

The adjoint form \( \tilde{\varphi} \)

It is useful to introduce the adjoint form
\[
\tilde{\varphi}_{n,\lambda,k} := \varphi_{n,n-\lambda+k,k} \quad \lambda, k \in \mathbb{Z} \quad n \geq 0
\] (34)

Rewriting (26a) / (26b) / (26c) in terms of \( \tilde{\varphi} \) gives
\[
(2n + k) \cdot \tilde{\varphi}_{n,\lambda,k} = (\lambda - (k - 1)) \cdot \tilde{\varphi}_{n,\lambda,k-1}
\]
\[
+ (k + 1) \cdot \tilde{\varphi}_{n-1,\lambda,k+1}
\]
\[
+ \tilde{\varphi}_{n-1,\lambda-1,k} \quad 0 \leq k \leq \lambda \leq n
\] (35a)
\[ \tilde{\varphi}_{0,0,0} = 1 \] (35b)
\[ \tilde{\varphi}_{n,\lambda,k} = 0 \quad \text{IF NOT } 0 \leq k \leq \lambda \leq n \] (35c)

The \( \tilde{\cdot} \) modifier in (34) can be viewed as a mapping which turns the lower triangular matrix \( \varphi_n \) into the lower triangular matrix \( \tilde{\varphi}_n \). In that sense, it is an \textit{involution}:

\[ \tilde{\tilde{\varphi}} = \varphi \quad n \geq 0 \] (36)

Table 2: Triangular field \( \varphi_{n,j,k} / \tilde{\varphi}_{n,\lambda,k} \)

Table 2 illustrates the triangular shape of \( \varphi_{n,j,k} / \tilde{\varphi}_{n,\lambda,k} \). Row and column indices, \( j \) and \( k \), run over all integers \( \mathbb{Z} \). The blue area, which extends in all directions, marks zero elements. The \textit{diagonals} of \( \varphi_{n,j,k} \) form the \textit{rows} of the adjoint field \( \tilde{\varphi}_{n,\lambda,k} \) (34), and vice versa, by involution (36), the \textit{diagonals} of \( \tilde{\varphi}_{n,\lambda,k} \) form the \textit{rows} of \( \varphi_{n,j,k} \). Special solutions (28), (27b), and (27c) have been inserted.
The Generating functions \( F_{n,\lambda} \)

Based upon the adjoint quantities (34), define the functions

\[
F_{n,\lambda}(z) := \sum_{k \in \mathbb{Z}} \tilde{\varphi}_{n,\lambda, k} \cdot z^k \quad n, \lambda \in \mathbb{Z}
\]  

(37)

Here, \( z \) is a formal parameter. If \( F'_{n,\lambda} \) denotes the derivative of (37) with respect to \( z \), we have

\[
F'_{n,\lambda}(z) = \sum_{k \in \mathbb{Z}} k \cdot \varphi_{n, n-\lambda+k, k} \cdot z^{k-1} \quad n, \lambda \in \mathbb{Z}
\]  

(38)

If one multiplies (35a) by \( z^k \), performs summation over all \( k \in \mathbb{Z} \), matches powers of \( z \) on both sides, and replaces (37) and (38), one gets the recursive differential equation

\[
(2n - \lambda z) F_{n,\lambda}(z) + z(z+1) F'_{n,\lambda}(z) = F_{n-1,\lambda-1}(z) + F'_{n-1,\lambda}(z) \quad n \geq \lambda \geq 0
\]  

(39a)

\[
F_{0,0} = 1
\]  

(39b)

\[
F_{n,\lambda} = 0 \quad \text{IF NOT } n \geq \lambda \geq 0
\]  

(39c)

with boundary conditions (39b) and (39c) being direct consequences of the corresponding properties (35b) and (35c) of \( \tilde{\varphi} \). A distinctive special value is

\[
F_{n,\lambda}(0) = \tilde{\varphi}_{n,\lambda,0} = \varphi_{n,n-\lambda,0} \quad n, \lambda \in \mathbb{Z}
\]  

(40)

Frequency representation of integer partitions

The solution (43) of (39a) involves integer partitions in frequency representation [And98]. Let \( P_{n,\lambda} \) denote the set of integer partitions of \( n \geq 0 \) with a given number \( \lambda \geq 0 \) of parts [AS64]. Let \( \pi \in P_{n,\lambda} \) be a partition of \( n \) with \( \lambda \) parts. Then \( \pi(k) \) denotes the frequency of part \( k \geq 1 \) in \( \pi \). This implies the identities

\[
n = \sum_{k=1}^{n} \pi(k) \cdot k
\]  

(41a)

\[
\lambda = \sum_{k=1}^{n} \pi(k)
\]  

(41b)

In the degenerate case \( \lambda = 0 \) we have

\[
P_{n,0} = \begin{cases} \{ \text{empty partition} \} & \text{if } n = 0 \\ \emptyset & \text{if } n > 0 \end{cases}
\]  

(42)

(42) means, that \( n = 0 \) has exactly one partition, namely the empty partition, without parts. And on the other hand, clearly, there is no partition of \( n > 0 \) without parts.
Solution of (39a) / (39b)

The recursive differential equation (39a), including boundary condition (39b), is solved by

\[ F_{n,\lambda}(z) = \sum_{\pi \in P_{n,\lambda}} \prod_{k=1}^{n} \left( \frac{1}{\pi(k)!} \cdot f_{k}(z)^{\pi(k)} \right) \quad n \geq \lambda \geq 0 \quad (43) \]

the component functions \( f_{k}(z) \) being defined as

\[ f_{k}(z) := \frac{1}{(2k)!} \left( 1 + \frac{z}{2k + 1} \right) \quad k \geq 0 \quad (44) \]

Setting \( z = 0 \) in (43) while using (40) gives

\[ \varphi_{n,n-\lambda,0} = \sum_{\pi \in P_{n,\lambda}} \left[ \prod_{k=1}^{n} \frac{\pi(k)!}{(2k)!} \right]^{-1} \quad n \geq \lambda \geq 0 \quad (45) \]

For \( \lambda = 0 \) this reduces to (27c), and for \( \lambda = n \) to the first part of (28).

Proof that (43) solves (39a) / (39b)

The proof is divided into three sections. First, get rid of the recursiveness in (39a) by forming power series over the whole range of the indices \( \lambda \) and \( n \). This leads to the linear partial differential equation (50a) including boundary condition (50b). Second, solve (50a) / (50b). And third, focus on specific powers of \( x \) and \( y \) in the representation of solution (51) as an infinite product of infinite sums (52).

Summation over \( \lambda \in \mathbb{Z} \)

Starting from (37), define the functions

\[ F_{n}(y, z) := \sum_{\lambda \in \mathbb{Z}} F_{n,\lambda}(z) \cdot y^{\lambda} \quad n \in \mathbb{Z} \quad (46) \]

Here, \( y \) and \( z \) are formal parameters. If one multiplies (39a) by \( y^{\lambda} \), performs summation over all \( \lambda \in \mathbb{Z} \), matches powers of \( y \) and \( z \) on both sides, and replaces (46), or their partial derivatives, one gets the recursive partial differential equation

\[ 2 n F_{n}(y, z) + z(z+1) \partial_{z} F_{n}(y, z) - z y \partial_{y} F_{n}(y, z) = y F_{n-1}(y, z) + \partial_{z} F_{n-1}(y, z) \quad n \geq 0 \quad (47a) \]

\[ F_{0} = 1 \quad (47b) \]
\[ F_n = 0 \quad n < 0 \quad (47c) \]

with boundary conditions (47b) and (47c) again being direct consequences of the corresponding properties (39b) and (39c). A distinctive special value here is

\[ F_n(0,0) = F_{n,0}(0) = \tilde{\varphi}_{n,0,0} = \varphi_{n,n,0} = \delta_{n,0} \quad n \in \mathbb{Z} \quad (48) \]

Here, we have made use of (40) and (27c).

**Summation over** \( n \in \mathbb{Z} \)

Next, in a quite analogous manner, define the function

\[ F(x, y, z) := \sum_{n \in \mathbb{Z}} F_n(y, z) \cdot x^n \quad (49) \]

where \( F_n \) has been declared in (46). Again, \( x, y, \) and \( z \) are formal parameters. If one multiplies (47a) by \( x^n \), performs summation over all \( n \in \mathbb{Z} \), matches powers of \( x, y, \) and \( z \) on both sides, and replaces (49), or their partial derivatives, one gets the linear partial differential equation

\[ 2 x \partial_x F(x, y, z) + z(z + 1) \partial_z F(x, y, z) - z y \partial_y F(x, y, z) = y x F(x, y, z) + x \partial_z F(x, y, z) \quad (50a) \]

\[ F(0, 0, 0) = F_0(0, 0) = 1 \quad (50b) \]

with boundary condition (50b) derived from (48).

**Solution of** (50a)

The linear partial differential equation (50a), including boundary condition (50b), is solved by

\[ F(x, y, z) = \exp \left[ y \cdot \sum_{k \geq 1} f_k(z) \cdot x^k \right] \quad (51) \]

where \( f_k \) is from (44). This is verified by insertion. Result (43) becomes clear if one writes down (51) as an infinite product of infinite sums, as illustrated in (52). Here, each row represents an infinite sum.

\[
\begin{array}{cccccccc}
1 & + & y f_1(z) x^{1 \cdot 1} & + & \frac{1}{2!} y^2 f_1^2(z) x^{1 \cdot 2} & + & \frac{1}{3!} y^3 f_1^3(z) x^{1 \cdot 3} & + & \cdots \\
1 & + & y f_2(z) x^{2 \cdot 1} & + & \frac{1}{2!} y^2 f_2^2(z) x^{2 \cdot 2} & + & \frac{1}{3!} y^3 f_2^3(z) x^{2 \cdot 3} & + & \cdots \\
1 & + & y f_3(z) x^{3 \cdot 1} & + & \frac{1}{2!} y^2 f_3^2(z) x^{3 \cdot 2} & + & \frac{1}{3!} y^3 f_3^3(z) x^{3 \cdot 3} & + & \cdots \\
\vdots & & \vdots & & \vdots & & \vdots & & \ddots \\
\end{array}
\]
(13) follows for any given pair \( n, \lambda \geq 0 \), if, upon multiplication of all rows, one combines all coefficients of \( x^n \), and at the same time, all coefficients of \( y^\lambda \).

Summary

Starting from the pair of elementary functions (2), the triangular array \( A \) (see Table 1) has been generated, whose elements are integer functions of the size parameter \( s \geq 0 \). The 3-dimensional triangular field of rational numbers \( \varphi \) has been introduced, being a result of the transformation (25), which involves the condensed form \( B \) (22). \( \varphi \) satisfies the triple of conditions (26a) / (26b) / (26c). Distinctive special solutions have been given in (27a), (27b), (27c), and (45). Relations to Stirling numbers (31) and Hyperbolic functions (33) have been described. A Generating function (37), based upon the adjoint form \( \tilde{\varphi} \) (34), has been defined, and it was shown that it can be written as (43).

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