Repulsive Casimir force at zero and finite temperature

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\textbf{Abstract.} We study the zero and finite temperature Casimir force acting on a perfectly conducting piston with arbitrary cross section moving inside a closed cylinder with infinitely permeable walls. We show that at any temperature, the Casimir force always tends to move the piston away from the walls and toward its equilibrium position. In the case of a rectangular piston, exact expressions for the Casimir force are derived. In the high-temperature regime, we show that the leading term of the Casimir force is linear in temperature and therefore the Casimir force has a classical limit. Due to duality, all these results also hold for an infinitely permeable piston moving inside a closed cylinder with perfectly conducting walls.
1. Introduction

It is well known that vacuum fluctuations of electromagnetic fields in the presence of boundaries give rise to the Casimir force, which was shown to be attractive when the boundary consists of a pair of perfectly conducting parallel plates [1]. Since the seminal work of Casimir, many studies have been done on the Casimir effect where different quantum fields and geometric configurations have been considered. Nevertheless, 60 years after its discovery, the Casimir effect is still under active research. By definition, the calculation of Casimir energy involves an infinite sum that needs to be regularized. However, there is still no consensus on the regularization procedures, which often lead to inconsistent results. Even for a geometric configuration as simple as a rectangular cavity, despite a zeta regularization technique or a dimensional regularization method giving finite results for the Casimir force acting on a wall [2, 3], some authors consider these regularization methods which renormalize all surface divergence terms to zero as being not physical [4]. Nonetheless, Geyer et al [5] have recently developed a subtraction scheme to obtain a physically consistent Casimir force acting on a wall of a perfectly conducting rectangular cavity from the point of view of thermodynamics. Another approach to this problem was considered by Fulling et al [6]. In 2004, Cavalcanti [7] proposed an alternative to this problem by adding a piston in the rectangular cavity. He showed that for a two-dimensional rectangular piston, the Casimir force acting on the piston due to fluctuations of a scalar field with Dirichlet boundary conditions is finite without renormalization and can be computed exactly. Since then, the Casimir piston has attracted considerable interest [8]–[21]. It has been shown that at any dimension and temperature, for either scalar or electromagnetic field, if the boundary conditions assumed on all the walls are the same and an ideal condition, then the Casimir force is always an attractive force, which tends to pull the piston to the nearer wall. As the length scale shrinks to the nano range, this attractive Casimir force would create undesirable effects on microelectromechanical and nanoelectromechanical devices known as...
stiction [22, 23], which would limit the functionality of the devices. As a result, there is an impelling need to search for scenarios that would lead to the repulsive Casimir force. In the case of infinite parallel plates, Boyer [24] has shown that the Casimir force acting on the plates is repulsive if one of the plates is perfectly conducting and the other is infinitely permeable. The thermal correction of the Casimir force for this configuration was considered in [25, 26], and it was proved that the Casimir force remains repulsive at any finite temperature. Another generalization of Boyer’s work was considered in [27], where it was proved that if the plates are made of dielectric materials with nontrivial magnetic permeability, then for a large range of parameters, the Casimir force is repulsive. Similar results were obtained in [28]. For the piston geometry, Barton [11] showed that the Casimir force acting on a piston made of weakly dielectric materials can become repulsive when the plate separation is sufficiently large. In this paper, we generalize the original setup of Boyer [24] to a piston moving freely inside a closed cylinder, where the piston is perfectly conducting and the walls of the cylinder are infinitely permeable (see figure 1). This setup has been suggested by Fulling et al [29] but they only considered the zero-temperature Casimir force for rectangular piston and exact explicit formulae of the Casimir force were not given. In the present paper, we allow the piston to have arbitrary cross section and it is shown that the Casimir force is free of surface divergence even without renormalization. We prove rigorously that at any temperature, the Casimir force is always repulsive tending to restore the piston to its equilibrium position, and the magnitude of the Casimir force decreases as the piston moves toward the equilibrium position. In the high-temperature regime, we show that the Casimir forces due to the blackbody radiation from the two regions separated by the piston cancel each other, and the leading order of the Casimir force is linear in temperature. This implies the existence of a classical limit for the Casimir force. Due to the duality between electric field and magnetic field in (3 + 1) dimension, all the results in this paper also hold for an infinitely permeable piston moving freely in a closed perfectly conducting cylinder.

Figure 1. A rectangular piston moving freely inside a rectangular cylinder dividing the cylinder into Regions I and II.
In this paper, we use the units where \( h = c = k_B = 1 \) in sections 2 and 3. In sections 4 and 5, we use SI units.

2. Cut-off-dependent Casimir energy

In this section, we first discuss the eigenfrequencies of the electromagnetic field for each region divided by the piston. We then give the expressions for the Casimir energy, which was computed by the exponential cut-off method in appendix A.

2.1. Modes of the electromagnetic field

For a cylinder \([0, L] \times \Omega\), where the boundary surfaces \([0] \times \Omega\) and \(\partial \Omega\) are infinitely permeable but the boundary surface \([L] \times \Omega\) is perfectly conducting, the eigenfrequencies for the electromagnetic field can be written as follows. The boundary conditions are equivalent to the conditions \(n \cdot E = 0\) and \(n \times B = 0\) on the surfaces \([0] \times \Omega\) and \([0, L] \times \partial \Omega\), and \(n \times E = 0\) and \(n \cdot B = 0\) on the surface \([L] \times \Omega\). Here, \(n\) is the unit normal vector to the surface, \(E\) and \(B\) are the electric field and magnetic field, respectively. The fields can be expressed in terms of the vector potential \(A\) by \(E = -\partial A/\partial t\) and \(B = \nabla \times A\). Imposing the transversality condition \(\text{div } A = 0\), Maxwell’s equations are equivalent to \((\partial^2/\partial t^2 - \Delta)A = 0\). One can then check that for the TE modes (i.e. modes with \(E_1 = 0\)), a set of independent solutions for \(A\) is given by \(A_1 = 0\),

\[
A_2 = -\frac{\pi (k + (1/2))x_1}{L} \frac{1}{\omega_{D,j}} \frac{\partial \phi_j}{\partial x_3} e^{i\omega_{TE,j}x},
A_3 = \cos \frac{\pi (k + (1/2))x_1}{L} \frac{1}{\omega_{D,j}} \frac{\partial \phi_j}{\partial x_2} e^{i\omega_{TE,j}x}.
\]

Here \(k \in \mathbb{N} \cup \{0\}\). For \(j = 1, 2, 3, \ldots\), \(\phi_j(x_2, x_3)\) is a nonzero eigenfunction of the Laplace operator \(-\partial^2/\partial x_2^2 - \partial^2/\partial x_3^2\) on \(\Omega\) with eigenvalue \(\omega_{D,j}^2 > 0\) and with Dirichlet boundary conditions, i.e. \(\phi_j|_{\partial \Omega} = 0\). The eigenfrequency \(\omega_{TE,j,k}\) is given by

\[
\omega_{TE,j,k}^2 = \left(\frac{\pi (k + (1/2))}{L}\right)^2 + \omega_{D,j}^2. \tag{1}
\]

For TM modes (i.e. modes with \(B_1 = 0\)), a set of independent solutions for \(A\) is given by

\[
A_1 = \sin \frac{\pi (k + (1/2))x_1}{L} \psi_j(x_2, x_3) e^{i\omega_{TM,j,x}},
A_2 = \cos \frac{\pi (k + (1/2))x_1}{L} \frac{\omega_{N,j}}{\omega_{N,j}^2} \frac{\partial \psi_j}{\partial x_2} e^{i\omega_{TM,j,x}} ,
A_3 = \frac{\pi (k + (1/2))x_1}{L} \cos \frac{\omega_{N,j}}{\omega_{N,j}^2} \frac{\partial \psi_j}{\partial x_2} e^{i\omega_{TM,j,x}}.
\]

Here \(k \in \mathbb{N}\). For \(j = 1, 2, 3, \ldots\), \(\psi_j(x_2, x_3)\) is a nonconstant eigenfunction of the Laplace operator on \(\Omega\) with eigenvalue \(\omega_{N,j}^2 > 0\) and with Neumann boundary conditions, i.e. \(\frac{\partial \psi_j}{\partial n}|_{\partial \Omega} = 0\). \(\omega_{TM,j,k}\) satisfies an equation analogous to (1).
In the case the piston has rectangular cross section, i.e. \( \Omega = [0, L_2] \times [0, L_3] \), we have explicitly

\[
\omega_{D,j}^2 = \left( \frac{\pi j_2}{L_2} \right)^2 + \left( \frac{\pi j_3}{L_3} \right)^2, \quad j = (j_2, j_3) \in \mathbb{N}^2,
\]

\[
\omega_{N,j}^2 = \left( \frac{\pi j_2}{L_2} \right)^2 + \left( \frac{\pi j_3}{L_3} \right)^2, \quad j = (j_2, j_3) \in \mathbb{N}^2 \setminus \{0\}.
\]

### 2.2. Cut-off-dependent Casimir energy

The Casimir energy of the piston system, denoted by \( E_{\text{Cas}}^{\text{Piston}}(a; L_1) \), is a sum of the Casimir energies of Regions I and II, and the Casimir energy of the region outside the cylinder, i.e.

\[
E_{\text{Cas}}^{\text{Piston}}(a; L_1) = E_{\text{Cas}}^{\text{Cylinder}}(a) + E_{\text{Cas}}^{\text{Cylinder}}(L_1 - a) + E_{\text{Cas}}^{\text{Out}}.
\]

Since the Casimir energy of the exterior region \( E_{\text{Cas}}^{\text{Out}} \) does not contribute to the Casimir force acting on the piston, we need not compute it here. By definition, the finite-temperature Casimir energy of the cylinder \( E_{\text{Cas}}^{\text{Cylinder}}(L) \) can be written in terms of the partition function \( Z \) associated with a canonical ensemble in the following way:

\[
E_{\text{Cas}}^{\text{Cylinder}}(L) = -T \log Z = -T \log \prod \frac{e^{-\omega_2/2T}}{1 - e^{-\omega/T}} = \frac{1}{2} \sum \omega + T \sum \log \left(1 - e^{-\omega/T}\right).
\]

Here \( \omega \) runs through the set of eigenfrequencies. The first summation on the right-hand side of (3) is divergent. Therefore, we define a cut-off-dependent Casimir energy \( E_{\text{Cas}}^{\text{Cylinder}}(L) \) by

\[
E_{\text{Cas}}^{\text{Cylinder}}(L) = \frac{1}{2} \sum \omega e^{-\gamma \omega} + T \sum \log \left(1 - e^{-\omega/T}\right).
\]

According to the previous subsection, we can decompose this into the sum over TE modes \( E_{\text{Cas,TE}}^{\text{Cylinder}}(L) \) and the sum over TM modes \( E_{\text{Cas,TM}}^{\text{Cylinder}}(L) \). In the appendix, we use the zeta function and heat kernel techniques to compute the Casimir energies \( E_{\text{Cas,TE}}^{\text{Cylinder}}(L) \) and \( E_{\text{Cas,TM}}^{\text{Cylinder}}(L) \) due to TE and TM modes, respectively. We find that at zero temperature \( T = 0 \), the contribution to the cut-off-dependent Casimir energy of the cylinder \([0, L] \times \Omega\) from the TE modes is given by

\[
E_{\text{Cas,TE}}^{\text{Cylinder}, T=0}(L) = \frac{6L}{\pi} c_{D,0} \lambda^{-4} + \frac{L}{\sqrt{\pi}} c_{D,1} \lambda^{-3} + \frac{L}{2\pi} c_{D,2} \lambda^{-2} + \frac{L}{8\pi} \left(2 \log \lambda + \gamma + 2 - 2 \log 2\right) c_{D,4}
\]

\[
- \frac{L}{8\pi} \text{FP}_{s=1} \{ (\Gamma(s) \xi_{D,s} \zeta(s)) - \frac{1}{2\pi} \sum_{k=1}^{\infty} (-1)^k \sum_{j=1}^{\infty} \frac{\omega_{D,j}k}{k} K_1(2kL\omega_{D,j}) \}.
\]

Here \( c_{D,j}, j = 0, 1, 2, \ldots \) are the heat kernel coefficients of the Laplace operator with Dirichlet boundary coefficients on \( \Omega \). \( \gamma \) is the Euler constant, \( K_\nu(z) \) is the modified Bessel function and \( \text{FP}_{s=1}(h(s)) \) denotes the finite part of the function \( h(s) \) at \( s = s_0 \). Note that the first five terms of (5) which contain the \( \lambda \to 0^+ \) divergent parts of the Casimir energy depend on \( L \) linearly. The last term tends to 0 as \( L \to \infty \).
For the finite-temperature Casimir energy, we find that the cut-off-dependent Casimir energy \((4)\) of the cylinder \([0, L] \times \Omega\) due to TE modes is given by

\[
E^{\text{cylinder}}_{\text{Cas,TE}}(L) = \frac{6L}{\pi} c_{D,0} \lambda^{-4} + \frac{L}{\sqrt{\pi}} c_{D,1} \lambda^{-3} + \frac{L}{2\pi} c_{D,2} \lambda^{-2} + \frac{L}{8\pi} \left(2 \log \lambda + 2 + \gamma - 2 \log 2\right) c_{D,4}
\]

\[
- \frac{L}{8\pi} \left(\sum_{l=1}^{\infty} \sum_{j=1}^{\infty} \frac{\omega_{D,j}}{l} K_1 \left(\frac{l \omega_{D,j}}{T}\right) \right) + \frac{T}{2} \sum_{l=-\infty}^{\infty} \sum_{j=1}^{\infty} \log(1 + e^{-2l}\sqrt{(2\pi T)^2 + \omega_{D,j}^2})
\]

The expressions for the TM mode contributions are similar with TE replaced by TM and \(D\) replaced by \(N\), where now \(c_{N,j}  \), \(j = 1, 2, 3, \ldots\) are the heat kernel coefficients of the Laplace operator on \(\Omega\) with Neumann boundary coefficients, deleting the zero eigenvalue corresponding to constant functions. Analogous to \((5)\), the first six terms of \((6)\) depend on \(L\) linearly and the last term tends to 0 as \(L \to \infty\).

In general, the heat kernel coefficients \(c_{D/N,0}\), \(c_{D/N,1}\) and \(c_{D/N,2}\) are known to be given by \([30]\)

\[
c_{D/N,0} = \frac{A(\Omega)}{4\pi}, \quad c_{D/N,1} = \frac{s(\partial \Omega)}{8\sqrt{\pi}}, \quad c_{D/N,2} = \chi(\partial \Omega) - \delta_{D/N},
\]

where \(A(\Omega)\) is the area of \(\Omega\), \(s(\partial \Omega)\) is the arc length of the boundary of \(\Omega\) and

\[
\chi(\partial \Omega) = \sum_{i} \frac{1}{24} \left(\frac{\pi}{\alpha_i} - \frac{\alpha_i}{\pi}\right) + \sum_{j} \frac{1}{12\pi} \int_{\gamma_j} \kappa(\gamma_j) d\gamma_j
\]

with \(\alpha_i\) being the interior angle of each sharp corner of \(\partial \Omega\) and \(\kappa(\gamma_j)\) the curvature of each smooth section of \(\partial \Omega\). For \(l \geq 3\), \(c_{D/N,1}\) can be expressed as integrals of functions that depend on the extrinsic and intrinsic curvatures of the boundary \(\partial \Omega\) and the boundary conditions imposed. In the special case where \(\Omega\) is a rectangle \([0, L_2] \times [0, L_3]\), we have explicitly

\[
c_{D/N,0} = \frac{L_2 L_3}{4\pi}, \quad c_{D/N,1} = \frac{L_2 + L_3}{4\sqrt{\pi}}, \quad c_{D/N,2} = \frac{1}{4} - \delta_{D/N}
\]

and \(c_{D/N,1} = 0\) for all \(l \geq 3\). Here \(\delta_D = 0\) and \(\delta_N = 1\) since we exclude the zero mode from the Neumann spectrum.

### 3. The Casimir force acting on the piston

As shown by \((5)\) or \((6)\), the \(\lambda \to 0^+\) divergent terms of the Casimir energy depend linearly on \(L\). This implies that the \(\lambda \to 0^+\) divergent term of the Casimir energy of the piston system \((2)\) does not depend on the piston position \(a\). Therefore, these divergences do not contribute to the Casimir force acting on the piston, which implies that the Casimir force acting on the piston is finite even without renormalization.
3.1. Finite-temperature Casimir force and its classical limit

From (6), we find that the Casimir force is given by

\[ F_{\text{Cas}}(a; L_1) = -\frac{\partial}{\partial a} E_{\text{piston}}(a; L_1) = F_{\text{Cas}}^\infty(a) - F_{\text{Cas}}^\infty(L_1 - a), \tag{7} \]

where \( F_{\text{Cas}}^\infty(a) \) is the limit of the Casimir force when \( L_1 \to \infty \) given by

\[ F_{\text{Cas}}^\infty(a) = T \sum_{l=-\infty}^{\infty} \sum_{\omega_{D,j},\omega_{N,j}} \frac{\sqrt{(2\pi l T)^2 + \omega^2}}{e^{2\omega T} + 1}. \tag{8} \]

Since this is obviously a positive decreasing function of \( a \), (7) shows that the Casimir force acting on the piston has positive sign when \( a < L_1/2 \) and has negative sign when \( a > L_1/2 \). In other words, the Casimir force always tends to restore the piston to the equilibrium position \( a = L_1/2 \). Moreover, the magnitude of the Casimir force decreases as the piston moves toward its equilibrium position.

Note that (8) is an exact expression for the Casimir force at any finite temperature. With the knowledge of the eigenvalues of the Laplace operator with Dirichlet and Neumann boundary conditions on the surface \( \Omega \), one can compute the Casimir force by this formula to any degree of accuracy. In particular, for a rectangular piston with cross section \([0, L_2] \times [0, L_3] \), we have

\[ F_{\text{Cas}}^\infty(a; L_2, L_3) = 2\pi T \sum_{l=-\infty}^{\infty} \sum_{k_1=1}^{\infty} \sum_{k_3=1}^{\infty} \frac{\sqrt{(2lT)^2 + (k_2/L_2)^2} + (k_3/L_3)^2}{e^{2\omega T} + 1} \]

\[ + \pi T \sum_{l=-\infty}^{\infty} \sum_{j=2,3} \sum_{k_j=1}^{\infty} \frac{\sqrt{(2lT)^2 + (k_j/L_j)^2}}{e^{2\omega T} + 1}. \]

Equation (8) shows that when \( T \to \infty \), the Casimir force is dominated by a term linear in \( T \) corresponding to those terms with \( l = 0 \). The remaining terms decay exponentially as \( T \to \infty \). Note that to restore the constants \( \hbar, c \) and \( k_0 \) into the expression for the Casimir force, we need to replace \( T \) by \( k_0 T/\hbar c \) everywhere and multiply the overall expression by \( \hbar c \). Therefore the high-temperature expansion of the Casimir force is the same as the small-\( \hbar \) expansion of the Casimir force. A term with order \( T^j \) will be accompanied with the term \( \hbar^{1-j} \). The leading term of the Casimir force (8) is linear in \( T \) implying that this leading term is independent of \( \hbar \), and the remaining terms go to zero if we formally let \( \hbar \) go to 0. As a result, we find that the Casimir force has a classical limit (or high-temperature limit) given by

\[ F_{\text{Cas}}^{\text{classical}}(a; L_1) = T \sum_{\omega_{D,j},\omega_{N,j}} \left\{ \frac{\omega}{e^{2\omega T} + 1} - \frac{\omega}{e^{2(L_1-a)\omega} + 1} \right\}. \tag{9} \]

For its small-\( a \) behavior, we derive in appendix A that

\[ F_{\text{Cas}}^{\text{classical}}(a; L_1) = \frac{3\zeta_R(3)}{16\pi a^3} A(\Omega) T + O(a^0). \]

This shows that at high temperature, when the plate separation \( a \) is small, the Casimir force is dominated by the term

\[ F_{\text{Cas}}^{\text{classical}}(a; L_1) \sim \frac{3\zeta_R(3)}{16\pi a^3} A(\Omega) T = \frac{3}{4} \frac{\zeta_R(3)}{4\pi a^3} A(\Omega) T. \]
Interestingly, this term does not depend on the geometry of the cross section, but depends only on the area of the cross section. Moreover, it is equal to \(-\frac{3}{4}\) times the classical term of the Casimir force acting on a pair of perfectly conducting parallel plates. In fact, one can verify that if \(\tilde{F}_{\text{Cas}}(a; L_1)\) is the Casimir force when both the piston and the surrounding walls are perfectly conducting or infinitely permeable, then the Casimir force \(F_{\text{Cas}}(a; L_1)\) when the piston is perfectly conducting but the surrounding walls are infinitely permeable is related to \(\tilde{F}_{\text{Cas}}(a; L_1)\) by

\[
F_{\text{Cas}}(a; L_1) = 2\tilde{F}_{\text{Cas}}(2a; 2L_1) - \tilde{F}_{\text{Cas}}(a; L_1).
\]

In particular, when the piston has rectangular cross section, we derive from the results in \([20]\) the following alternative explicit formula for the classical term of the Casimir force:

\[
F_{\text{Cas}}^{\text{classical}}(a; L_1, L_2, L_3) = T \left\{ \frac{3\zeta_R(3)}{16\pi a^3} \frac{L_2 L_3}{L_1} + \frac{\zeta_R(3)}{8\pi} \frac{L_3}{L_2} - \frac{\pi}{24L_3} - \frac{1}{L_2} \sum_{k_2=1}^{\infty} \sum_{k_1=1}^{\infty} k_2 \right. \\
\times K_1 \left( \frac{2\pi k_2 k_3 L_3}{L_2} \right) + \frac{\pi L_2 L_3}{a^3} \sum_{k_1=1}^{\infty} \sum_{(k_2, k_3) \in \mathbb{Z} \backslash \{0\}} \left( k_1 + \frac{1}{2} \right)^2 \\
\times K_0 \left( \frac{\pi (2k_1 + 1)}{a} \sqrt{(k_2 L_2)^2 + (k_3 L_3)^2} \right) \right\} - (a \leftrightarrow L_1 - a).
\]

We would like to remark that comparing (A.2) and (A.3) in appendix A, we can deduce that in the high-temperature regime, the leading behavior of the Casimir force contribution from Region I alone is given by (see appendix A)

\[
\frac{T}{\pi} \sum_{l=1}^{\infty} \sum_{j=1}^{\infty} \omega_{D,j} \frac{1}{l} K_l \left( \frac{l \omega_{D,j}}{T} \right) + (D \leftrightarrow N) = \frac{2\pi^3}{45} c_{0,D} T^4 + \frac{\zeta_R(3)}{\sqrt{\pi}} c_{1,D} T^3 \\
+ \frac{\pi}{6} c_{2,D} T^2 + O(T) + (D \leftrightarrow N).
\]

This implies that as expected, the leading term of the Casimir force from Region I alone is the blackbody radiation term

\[
\frac{\pi^2}{45} A(\Omega) T^4.
\]

However, since the blackbody radiation from Region II would contribute a force of the same magnitude but opposite direction, the effect of the blackbody radiation on the piston cannot be observed. In fact, there is also a term proportional to \(T^2\) of magnitude

\[
\frac{\pi}{6} (2\chi - 1) T^2,
\]

which has been canceled out in the final Casimir force acting on the piston.

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3.2. Zero-temperature Casimir force

In the zero-temperature limit, the Casimir force can be obtained from (5). When \( L_1 \to \infty \), it is given by

\[
F_{\text{Cas}}^{\infty, T=0}(a) = -\frac{1}{2\pi a} \sum_{k=1}^{\infty} (-1)^k \sum_{\omega_{d,j}, \omega_{N,j}} \frac{\omega}{k} K_1(2k\omega) - \frac{1}{\pi} \sum_{k=1}^{\infty} (-1)^k \sum_{\omega_{d,j}, \omega_{N,j}} \omega^2 K_0(2k\omega).
\]

Using the same method as we derive (10), one finds that as \( a \to 0^+ \), the leading behavior of the zero-temperature Casimir force acting on the piston is given by

\[
F_{\text{Cas}}^{\infty, T=0}(a) = \frac{7\pi^2}{960} \frac{c_{0,D} + c_{0,N}}{a^4} + \frac{3\zeta(3)}{16\sqrt{\pi}} \frac{c_{1,D} + c_{1,N}}{a^3} + \frac{\pi}{48} \frac{c_{2,D} + c_{2,N}}{a^2} + O(a^0)
\]

\[
= \frac{7\pi^2}{1920a^4} A(\Omega) + \frac{\pi}{48} \frac{2\chi - 1}{a^2} + O(a^0).
\]

In particular, for small plate separation, the leading term of the Casimir force is

\[
F_{\text{Cas}}^{\infty, T=0} \approx \frac{7\pi^2}{8240a^4} A(\Omega),
\]

which is \(-\frac{7}{8}\) times the Casimir force acting on a pair of perfectly conducting infinite parallel plates, consistent with the result of Boyer [24] for parallel plates. Notice that this leading term only depends on the area but not the geometry of \( \Omega \).

In the case of a rectangular piston, we have the following explicit formula for the zero-temperature Casimir force in the \( L_1 \to \infty \) limit:

\[
F_{\text{Cas}}^{\infty, T=0}(a; L_2, L_3) = \frac{7\pi^2}{1920} \frac{L_2 L_3}{a^4} - \frac{\pi}{96} \frac{1}{a^2} - \frac{\pi^2}{720} \frac{L_2}{L_3^3} - \frac{\zeta(3)}{16\pi} \frac{L_2^2}{L_3^3} - \frac{1}{2} \left[ \sum_{k=1}^{\infty} \sum_{k_1=1}^{\infty} \frac{\left( \frac{k_2}{k_3} \right)^{3/2}}{k_2 k_3} K_{3/2} \left( \frac{2\pi k_2 k_3 L_3}{L_2} \right) \right] + \frac{\pi L_2 L_3}{4a^3} \sum_{k_1=0}^{\infty} \sum_{(k_2,k_3) \in Z^2 \setminus \{0\}} \left( \frac{(k_1 + (1/2))^2}{(k_2 L_2)^2 + (k_3 L_3)^2} \right) \times \exp \left( -\frac{\pi (2k_1 + 1)}{a} \sqrt{(k_2 L_2)^2 + (k_3 L_3)^2} \right).
\]

In particular,

\[
F_{\text{Cas}}^{\infty, T=0}(a) = \frac{7\pi^2}{1920} \frac{L_2 L_3}{a^4} - \frac{\pi}{96} \frac{1}{a^2} + O(a^0),
\]

in agreement with the general result (11).

4. Numerical results and discussions

We compute the magnitude of the Casimir force when the cross section of the piston is a square of dimension 0.3 m \( \times \) 0.3 m, \( L_1 \to \infty \) and the temperature is equal to 0, 1 and 300 K, respectively. The results are compared to the contribution from the term

\[
F_{\text{Cas}}^{\parallel} = \frac{7\pi^2}{1920} \frac{L_2 L_3}{a^4},
\]

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and 15 nm on the piston when the cross section of the piston has dimension 100 nm at both zero and room temperatures. 10 nm, then the Casimir pressure on the piston is approximately equal to 1 atmospheric pressure is negligible. However, when the separation between the piston and an opposite wall shrinks to the Casimir force from the leading terms \( F_{\text{Cas}} \), whereas if \( a/L_2 \) is small but \( a T k_B/(\hbar c) \gg 1 \), the Casimir force is dominated by the term \( F_{\text{Cas}}^{\parallel} \). When \( a/L_2 \approx 1 \), there is a considerable deviation of the Casimir force from the leading terms \( F_{\text{Cas}}^{\parallel} \) and \( F_{\text{Cas}}^{\parallel, \text{classical}} \). At visible length, the Casimir force is negligible. However, when the separation between the piston and an opposite wall shrinks to 10 nm, then the Casimir pressure on the piston is approximately equal to 1 atmospheric pressure at both zero and room temperatures.

For possible applications to nanotechnology, we also compute the Casimir pressure acting on the piston when the cross section of the piston has dimension 100 nm \( \times \) 100 nm and the temperature is 0 and 300 K. The results are tabulated in tables 4 and 5. At this length scale, the Casimir force is not very much affected when the temperature changes from 0 to 300 K. We also find that the Casimir pressure is almost the same as the atmospheric pressure when \( a = 10 \text{ nm} \). Reducing \( a \) at a rate \( r \) will result in the increase of the pressure at the rate \( r^4 \). Since the Casimir force is pushing the piston outward, this will prevent the undesirable collapse of the piston to the opposite wall.

**Table 1.** The Casimir force \( F_{\text{Cas}} \) and the contribution from \( F_{\text{Cas}}^{\parallel} \) when \( L_2 = L_3 = 0.3 \text{ m} \) and \( T = 0 \text{ K} \).

| \( a \) (m) | \( F_{\text{Cas}} \) (N) | \( F_{\text{Cas}}^{\parallel} \) (N) |
|---|---|---|
| \( 10^{-8} \) | \( 1.0238 \times 10^4 \) | \( 1.0238 \times 10^4 \) |
| \( 10^{-4} \) | \( 1.0238 \times 10^{-12} \) | \( 1.0238 \times 10^{-12} \) |
| \( 0.1 \) | \( 9.0759 \times 10^{-25} \) | \( 9.0759 \times 10^{-24} \) |
| \( 0.3 \) | \( 2.6683 \times 10^{-27} \) | \( 2.6460 \times 10^{-26} \) |

**Table 2.** The Casimir force \( F_{\text{Cas}} \) and the contribution from the classical term \( F_{\text{Cas}}^{\text{classical}} \) and \( F_{\text{Cas}}^{\parallel, \text{classical}} \) when \( L_2 = L_3 = 0.3 \text{ m} \) and \( T = 1 \text{ K} \).

| \( a \) (m) | \( F_{\text{Cas}} \) (N) | \( F_{\text{Cas}}^{\text{classical}} \) (N) | \( F_{\text{Cas}}^{\parallel, \text{classical}} \) (N) |
|---|---|---|---|
| \( 10^{-8} \) | \( 1.0238 \times 10^4 \) | \( 8.9146 \times 10^{-2} \) | \( 8.9146 \times 10^{-2} \) |
| \( 10^{-4} \) | \( 1.0238 \times 10^{-12} \) | \( 8.9146 \times 10^{-14} \) | \( 8.9146 \times 10^{-14} \) |
| \( 0.1 \) | \( 8.1004 \times 10^{-23} \) | \( 8.1004 \times 10^{-23} \) | \( 8.1004 \times 10^{-23} \) |
| \( 0.3 \) | \( 5.9862 \times 10^{-25} \) | \( 5.9862 \times 10^{-25} \) | \( 3.3017 \times 10^{-24} \) |

| \( a \) (m) | \( F_{\text{Cas}} \) (N) | \( F_{\text{Cas}}^{\parallel} \) (N) | \( F_{\text{Cas}}^{\parallel, \text{classical}} \) (N) |
|---|---|---|---|
| \( 10^{-8} \) | \( 1.0238 \times 10^4 \) | \( 2.6744 \times 10 \) | \( 2.6744 \times 10 \) |
| \( 10^{-4} \) | \( 2.6744 \times 10^{-11} \) | \( 2.6744 \times 10^{-11} \) | \( 2.6744 \times 10^{-11} \) |
| \( 0.1 \) | \( 2.4301 \times 10^{-20} \) | \( 2.4301 \times 10^{-20} \) | \( 2.4301 \times 10^{-20} \) |
| \( 0.3 \) | \( 1.7959 \times 10^{-22} \) | \( 1.7959 \times 10^{-22} \) | \( 9.9051 \times 10^{-22} \) |

the classical term (9) and the term

\[
\frac{F_{\text{Cas}}^{\parallel, \text{classical}}}{L_2 L_3 T} = \frac{3 \zeta_R(3)}{16 \pi a^3} L_2 L_3 T,
\]

and are tabulated in tables 1–3. Note that when \( a/L_2 \) is small and \( a T k_B/(\hbar c) \ll 1 \), the Casimir force is dominated by the term \( F_{\text{Cas}}^{\parallel} \), whereas if \( a/L_2 \) is small but \( a T k_B/(\hbar c) \gg 1 \), the Casimir force is dominated by the term \( F_{\text{Cas}}^{\parallel, \text{classical}} \). When \( a/L_2 \approx 1 \), there is a considerable deviation of the classical term \( F_{\text{Cas}}^{\parallel, \text{classical}} \) and \( F_{\text{Cas}}^{\parallel} \). At visible length, the Casimir force is negligible. However, when the separation between the piston and an opposite wall shrinks to 10 nm, then the Casimir pressure on the piston is approximately equal to 1 atmospheric pressure at both zero and room temperatures.

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Figure 2. The Casimir force $F_{\text{Cas}}$ at $T = 0$ and 300 K and the contribution from $F_{\text{Cas}}^\parallel$ and $F_{\text{Cas}}^\text{classical}$ plotted as a function of $a$. 

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Table 4. The Casimir pressure \( P_{\text{Cas}} \) and the contribution from \( P_{\text{Cas}}^\parallel \) when \( L_2 = L_3 = 100 \text{ nm} \) and \( T = 0 \text{ K} \).

| \( a \) (nm) | \( P_{\text{Cas}} \) (N m\(^{-2}\)) | \( P_{\text{Cas}}^\parallel \) (N m\(^{-2}\)) |
|-------------|-------------------------------|-------------------------------|
| 1           | \( 1.1375 \times 10^9 \)       | \( 1.1376 \times 10^9 \)       |
| 10          | \( 1.1271 \times 10^5 \)       | \( 1.1367 \times 10^5 \)       |
| 50          | \( 1.3248 \times 10^2 \)       | \( 1.8202 \times 10^2 \)       |
| 100         | \( 2.4015 \times 10^5 \)       | \( 1.1376 \times 10^5 \)       |

Table 5. The Casimir pressure \( P_{\text{Cas}} \) and the contribution from the classical terms \( P_{\text{Cas}}^{\text{classical}} \) and \( P_{\text{Cas}}^{\parallel} \) when \( L_2 = L_3 = 100 \text{ nm} \) and \( T = 300 \text{ K} \).

| \( a \) (nm) | \( P_{\text{Cas}} \) (N m\(^{-2}\)) | \( P_{\text{Cas}}^{\text{classical}} \) (N m\(^{-2}\)) | \( P_{\text{Cas}}^{\parallel} \) (N m\(^{-2}\)) |
|-------------|-------------------------------|-------------------------------|-------------------------------|
| 1           | \( 1.1375 \times 10^9 \)       | \( 2.9715 \times 10^5 \)       | \( 2.9715 \times 10^5 \)       |
| 10          | \( 1.1273 \times 10^5 \)       | \( 2.9641 \times 10^5 \)       | \( 2.9715 \times 10^5 \)       |
| 50          | \( 1.4299 \times 10^2 \)       | \( 1.7344 \)                   | \( 2.3772 \)                   |
| 100         | \( 1.2061 \times 10^7 \)       | \( 5.3876 \times 10^{-2} \)    | \( 2.9715 \times 10^{-1} \)    |

Figure 3. This graph shows that the Casimir force is not an increasing function of temperature. Here \( a = 0.001 \text{ m} \), \( L_2 = L_3 = 0.01 \text{ m} \).

Figure 2 shows the behavior of the Casimir force \( F_{\text{Cas}} \) as a function of \( a \) at \( T = 0 \) and 300 K. It is compared with \( F_{\text{Cas}}^{\parallel} \) and \( F_{\text{Cas}}^{\text{classical}} \). We see that when \( a \) is in the range 10 nm–1 \( \mu \text{m} \), the Casimir force at both 0 and 300 K is dominated by the zero-temperature parallel plate term \( F_{\text{Cas}}^{\parallel} \). When \( a \) is in the range 1 \( \mu \text{m}–0.3 \text{ m} \), the force is then dominated by the classical term at \( T = 300 \text{ K} \). From the graphs, one would tend to conclude that the Casimir force is an increasing function of the temperature. However, this is not the case as shown in figure 3.
5. Conclusion

We have shown that for a perfectly conducting piston moving freely inside a cylinder with infinitely permeable walls, the Casimir force acting on the piston is a repulsive force which tends to push the piston to its equilibrium position. At zero temperature, when the separation $a$ between the piston and one of its opposite walls is small, then the magnitude of the Casimir pressure is asymptotically equal to

$$\frac{7}{8} \frac{\pi^2 \hbar c}{240 a^4},$$

which is the result obtained by Boyer [24] for a pair of infinite parallel plates, one being infinitely conducting and the other being infinitely permeable. It is $-\frac{7}{8}$ times the zero-temperature Casimir pressure acting on a pair of perfectly conducting parallel plates. However, at high temperature, the Casimir pressure is dominated by

$$\frac{3\zeta_R(3)}{4} \frac{\zeta}{4\pi a^3} k_B T$$

when $a$ is small. This is $-\frac{3}{4}$ times the Casimir pressure when both plates are perfectly conducting. It is interesting to note the change from the ratio $\frac{7}{8}$ at zero temperature to the ratio $\frac{3}{4}$ at high temperature. We have also shown that at high temperature, the Casimir force acting on the piston is dominated by a term linear in $T$ known as the classical term. Moreover, when $a$ is small, the classical term is asymptotically equal to the area of the cross section $A(\Omega)$ multiplied by pressure (12), which is of order $a^{-3}$. The correction term is of order $a^0$.

For simplicity, in this paper we have assumed that the piston is perfectly conducting, whereas the surrounding walls are infinitely permeable. We would like to consider in a future work the more general case where the piston and its surrounding walls are allowed to have different electric permittivity and magnetic permeability. It would then be interesting to determine the range of the parameters for which the Casimir force acting on the piston is repulsive. One can anticipate that this would have important applications in nanotechnology.

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Appendix A. Computations of the Casimir energy and asymptotic behavior of the Casimir force

A.1. Computation of the Casimir energy

Define the following zeta functions

$$\zeta_{\Omega,D}(s) = \sum_{j=1}^{\infty} \omega_{D,j}^{-2s}, \quad \zeta_{\text{cylinder,TE}}(s) = \sum_{k=0}^{\infty} \sum_{j=1}^{\infty} \omega_{TE,j,k}^{-2s},$$

$$\zeta_{TE}(s) = \sum_{k=0}^{\infty} \sum_{j=1}^{\infty} \sum_{n=-\infty}^{\infty} \left( \omega_{TE,j,k}^2 + [2\pi T]^2 \right)^{-s}$$
and the heat kernels

\[
K_{\Omega,D}(t) = \sum_{j=1}^{\infty} e^{-t \omega_{D,j}^2}
\]

\[
K_{\text{cylinder,TE}}(t) = \sum_{k=0}^{\infty} \sum_{j=1}^{\infty} e^{-\omega_{k,j}^2 t} = \sum_{k=0}^{\infty} e^{-\pi [(k+1/2)/L]^2} K_{\Omega,D}(t).
\]

The zeta functions \( \zeta_{\Omega,N}(s) \), \( \zeta_{\text{cylinder,TM}}(s) \), \( \zeta_{\text{TM}}(s) \) and the heat kernels \( K_{\Omega,N}(t) \) and \( K_{\text{cylinder,TM}}(s) \) for the TM modes are defined analogously. It is well known that the heat kernel \( K_{\Omega,D}(t) \) has the following asymptotic expansion:

\[
K_{\Omega,D}(t) \simeq \sum_{l=0}^{M} c_{D,l} t^{(l-2)/2} + O \left( t^{(M-1)/2} \right) \quad \text{as} \quad t \to 0^+.
\]

Therefore,

\[
K_{\text{cylinder,TE}}(t) = \frac{L}{2 \sqrt{\pi} t} \left( 1 + 2 \sum_{k=1}^{\infty} (-1)^k e^{-(k^2 L^2)/t} \right) K_{\Omega,D}(t)
\]

\[
\simeq \frac{L}{2 \sqrt{\pi}} \sum_{l=0}^{M} c_{D,l} t^{(l-3)/2} + O \left( t^{(M-2)/2} \right) \quad \text{as} \quad t \to 0^+.
\]

Consequently, the function \( \Gamma(s) \zeta_{\text{cylinder,TE}}(s) \) has at most simple poles at \( s = (3-l)/2 \), \( l = 0, 1, 2, \ldots \) with residues

\[
\text{Res}_{s=(3-l)/2} \left\{ \Gamma(s) \zeta_{\text{cylinder,TE}}(s) \right\} = \frac{L}{2 \sqrt{\pi}} c_{D,l}.
\]

The \( \lambda \to 0^+ \) behavior of the \( T = 0 \) part of the cut-off-dependent Casimir energy (4) can be determined as follows:

\[
E_{\text{Cas,TE}}(L) = \frac{1}{2} \sum_{k=0}^{\infty} \sum_{j=1}^{\infty} \omega_{E,j,k} e^{-k \omega_{E,j,k}}
\]

\[
= -\frac{1}{2} \frac{\partial}{\partial \lambda} \left[ \frac{1}{2 \pi i} \int_{-i \infty}^{u+i \infty} dz \Gamma(z) \lambda^{-2} \zeta_{\text{cylinder,TE}} \left( \frac{z}{2} \right) \right]
\]

\[
\simeq \frac{6L}{\pi} c_{D,0} \lambda^{-4} + \frac{L}{\sqrt{\pi}} c_{D,1} \lambda^{-3} + \frac{L}{2 \pi} c_{D,2} \lambda^{-2}
\]

\[
+ \frac{L}{4 \pi} (\log \lambda + \gamma) c_{D,4} + \frac{1}{2} \text{FP}_{s=-1/2} \zeta_{\text{cylinder,TE}}(s).
\]

Here, \( \gamma \) is the Euler constant and \( \text{FP}_{s=-1/2} \zeta_{\text{cylinder,TE}}(s) \) is the finite part of the zeta function \( \zeta_{\text{cylinder,TE}}(s) \) at \( s = -1/2 \). Since

\[
\zeta_{\text{cylinder,TE}}(s) = \frac{1}{\Gamma(s)} \int_{0}^{\infty} t^{s-1} K_{\text{cylinder,TE}}(t) \, dt,
\]
a straightforward computation gives
\[
\frac{1}{2} \pi \zeta_{cylinder, TE}(s) = \frac{L}{8\pi} \left[ c_{D,4} \left( -2 - \gamma - 2 \log 2 \right) \right] - \frac{1}{2\pi} \sum_{k=1}^{\infty} (-1)^k \sum_{j=1}^{\infty} \frac{\omega_{D,j}}{k} K_1 \left( 2kL\omega_{D,j} \right),
\]

where \( K_\nu(z) \) is the modified Bessel function. Gathering the above results, we find that the contribution to the cut-off-dependent zero-temperature Casimir energy of the cylinder \([0, L] \times \Omega\) from the TE modes is given by
\[
E_{\text{cylinder},T=0}(L) = \frac{6L}{\pi} \zeta_{cylinder, TE}(s) = \frac{L}{8\pi} \left[ c_{D,4} \left( -2 - \gamma - 2 \log 2 \right) \right] - \frac{L}{8\pi} \sum_{k=1}^{\infty} (-1)^k \sum_{j=1}^{\infty} \frac{\omega_{D,j}}{k} K_1 \left( 2kL\omega_{D,j} \right). \tag{A.1}
\]

For the finite-temperature Casimir energy, we can use the following formulae. On the one hand, we have (see e.g. [31])
\[
-\frac{T}{2} \zeta_{TE}(0) = \frac{1}{2} \pi \zeta_{cylinder, TE}(s) - \frac{L}{4\pi} c_{D,4} (1 - \log 2) + T \sum_{k=0}^{\infty} \sum_{j=1}^{\infty} \log \left( 1 - e^{-\omega_{D,j} / T} \right). \tag{A.2}
\]

On the other hand,
\[
-\frac{T}{2} \zeta_{TE}(0) = -\frac{L}{8\pi} \left[ \gamma c_{D,4} + \pi \zeta_{cylinder, TE}(s) \right] - \frac{LT}{\pi} \sum_{l=1}^{\infty} \sum_{j=1}^{\infty} \frac{\omega_{D,j}}{l} K_1 \left( \frac{l\omega_{D,j}}{T} \right) + \frac{T}{2} \sum_{l=-\infty}^{\infty} \sum_{j=1}^{\infty} \log \left( 1 + e^{-2L\sqrt{(2\pi lT)^2 + \omega_{D,j}}} \right). \tag{A.3}
\]

From these, we find that the cut-off-dependent Casimir energy due to the TE modes is given by
\[
E_{\text{cas,TE}}(L) = \frac{6L}{\pi} \zeta_{cylinder, TE}(s) = \frac{L}{8\pi} \left[ c_{D,4} \left( -2 - \gamma - 2 \log 2 \right) \right] - \frac{L}{8\pi} \sum_{k=1}^{\infty} (-1)^k \sum_{j=1}^{\infty} \frac{\omega_{D,j}}{k} K_1 \left( 2kL\omega_{D,j} \right) + \frac{T}{2} \sum_{l=-\infty}^{\infty} \sum_{j=1}^{\infty} \log \left( 1 + e^{-2L\sqrt{(2\pi lT)^2 + \omega_{D,j}}} \right). \tag{A.4}
\]
A.2. The leading behavior of the classical term of the Casimir force at small plate separation

Here we compute the small-$a$ asymptotic behavior of the classical term of the Casimir force (9). We have

\[ F_{\text{Cas}}^{\text{classical}}(a; L_1) = T \sum_{\omega_{D,j}, \omega_{N,j}} \frac{\omega}{e^{2\lambda_\omega} + 1} + O(a^0) \]

\[ = -\frac{T}{\sqrt{\pi}} \sum_{k=1}^{\infty} \sum_{\omega} (-1)^k \omega^2 \int_0^\infty t^{-3/2} \exp \left(-t (ka)^2 - \frac{\omega^2}{t} \right) dt \]

\[ = T \frac{1}{\sqrt{\pi}} \int_{3-i\infty}^{3+i\infty} a^{1-2z} \Gamma(z) (z-1/2) (\zeta_R(2z-1) (1-2^{2z-2}) \times \Gamma(z) (\zeta_D(z-1) + \zeta_N(z-1)) \right) dz \]

\[ = T \left\{ \frac{3 \zeta_R(3)}{8a^3} (c_{0,D} + c_{0,N}) + \frac{\pi^{3/2}}{24a^2} (c_{1,D} + c_{1,N}) \right\} + O(a^0) \]

\[ = \frac{3 \zeta_R(3)}{16\pi a^3} A(\Omega) T + O(a^0). \]

A.3. Verification of (10)

\[ \frac{T}{\pi} \sum_{l=1}^{\infty} \sum_{j=1}^{\infty} \frac{\omega_{D,j}}{l} K_1 \left( \frac{l \omega_{D,j}}{T} \right) = \frac{1}{4\pi} \int_0^\infty \sum_{l=1}^{\infty} \sum_{j=1}^{\infty} \exp \left\{ -\frac{tl^2}{4T^2} - \frac{\omega_{D,j}^2}{l} \right\} dt \]

\[ = \frac{1}{8\pi^2} \int_{2-i\infty}^{2+i\infty} \Gamma(z+1) \zeta_R(2z+2)(2T)^{2z+2} \Gamma(z) \zeta_{\Omega,D}(z) \right) dz \]

\[ = \frac{2\pi^3}{45} c_{0,D} + \frac{\zeta_R(3)}{\sqrt{\pi}} c_{1,D} T^3 + \frac{\pi}{6} c_{2,D} T^2 + O(T). \]

Appendix B. Alternative formulae for the Casimir force acting on a rectangular piston

Here we present two exact formulae for the Casimir force when the piston has rectangular cross section. These formulae are useful when the temperature $T$ is high and the separation $a$ is small. The formulae can be used to study the behavior of the Casimir force when the combination $aT$ is small and large, respectively. They can be derived using the Chowla–Selberg formula as presented in [20].

B.1. $aT \ll 1$

\[ F_{\text{Cas}}(a; l_2, L_3) = \frac{7 \pi^2 L_2 L_3}{8 \cdot 240 a^4} - \frac{\pi}{96a^2} - \frac{L_3 T}{8\pi L_2^2} \zeta_R(3) + \frac{\pi T^2}{12} - \frac{\pi T}{24 L_3} - \frac{\pi^2 T^4 L_2 L_3}{45} \]

\[ - \frac{T}{L_2} \sum_{k_2=1}^{\infty} \sum_{k_3=1}^{\infty} K_1 \left( \frac{2\pi k_2 k_3 L_3}{L_2} \right) - \frac{\pi}{2a^2} \sum_{k_1=0}^{\infty} \exp \left( \frac{\pi(k_1+1/2)}{a T} \right) - 1 \]
\[-\frac{\pi T L_2 L_3}{a^3} \sum_{k_1=0}^{\infty} \left( k_1 + \frac{1}{2} \right)^2 \log \left( 1 - \exp \left( -\frac{\pi (k_1 + (1/2))}{a T} \right) \right) \]

\[+ \frac{\pi L_2 L_3 T}{a^3} \sum_{(k_2,k_3) \in \mathbb{Z}^2} \sum_{k_1=0}^{\infty} \left( k_1 + \frac{1}{2} \right)^2 \]

\[\times K_0 \left( 2\pi \sqrt{\left( \frac{k_1 + (1/2)}{a} \right)^2 + [2/T]^2} \right) \left( [k_2 L_2]^2 + [k_3 L_3]^2 \right). \]

### B.2. \( aT \gg 1 \)

\[F_{\text{Cas}}^\infty (a; L_2, L_3) = \frac{3\zeta_R(3)T}{16\pi a^3} L_2 L_3 \frac{T}{8\pi L_2^2} \zeta_R(3) - \frac{\pi T}{24 L_3} - \frac{T}{L_2} \sum_{k_2=1}^{\infty} \sum_{k_3=1}^{\infty} K_1 \left( \frac{2\pi k_2 k_3 L_3}{L_2} \right) \]

\[- \frac{2T^2 L_2 L_3}{a^2} \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} \frac{e^{4\pi k aT}}{\left( e^{4\pi k aT} - 1 \right)^2} - \frac{L_2 L_3 T}{2\pi a^3} \sum_{k=1}^{\infty} \frac{(-1)^k}{k^3} \frac{1}{e^{4\pi k aT} - 1} \]

\[+ \frac{4\pi T^3 L_2 L_3}{a} \sum_{l=1}^{\infty} l^2 \log \left( 1 + e^{-4\pi laT} \right) - 2\pi T^2 \sum_{l=1}^{\infty} \frac{l}{e^{4\pi laT} + 1} \]

\[+ \frac{\pi L_2 L_3 T}{a^3} \sum_{(k_2,k_3) \in \mathbb{Z}^2} \sum_{k_1=0}^{\infty} \left( k_1 + \frac{1}{2} \right)^2 \]

\[\times K_0 \left( 2\pi \sqrt{\left( \frac{k_1 + (1/2)}{a} \right)^2 + [2/T]^2} \right) \left( [k_2 L_2]^2 + [k_3 L_3]^2 \right). \]

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