Towards the application of the BFF canonical quantization to the Einstein-Maxwell Dilaton-Axion theory

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Abstract. We present the lines along which the canonical quantization of dynamical systems with curved phase space introduced by I.A. Batalin, E.S. Fradkin and T.E. Fradkina is applied to the 4D Einstein–Maxwell Dilaton–Axion theory. The spherically symmetric case with radial fields is considered. The Lagrangian density of the theory in the Einstein frame is written as an expression with first order in time derivatives of the fields. The phase space is curved due to the nontrivial interaction of the dilaton with the axion and the electromagnetic fields.

1. Introduction

The canonical quantization method of systems with curved phase space created by I.A. Batalin, E.S. Fradkin and T.E. Fradkina (BFF) \cite{1}–\cite{3} consists of doubling the dimensionality of the original phase space by introducing a conjugate canonical momentum to each original phase variable. Further, these variables are subjected to special second class constraints in such a way that the formal exclusion of the new canonical momenta reduces the system back to the original phase space. It turns out that the new phase space is flat and its quantization proceeds along the lines of \cite{4}–\cite{7}. Here we shall apply this method to the bosonic sector of the four–dimensional effective field theory of the heterotic string at tree level, better known as Einstein–Maxwell Dilaton–Axion (EMDA) theory.

After presenting a brief outline of the generalized canonical quantization method for dynamical systems in Sec 2, we consider the action of the 4D EMDA theory, perform the Arnowitt–Desser–Misner (ADM) decomposition of the metric and write the Lagrangian density of the matter sector as an expression with first order in time derivatives of the fields in Sec. 3. We further consider the spherically symmetric anzats and obtain a Lagrangian density which defines a curved phase space and possesses two irreducible first class constraints. We continue by canonically quantizing the resulting EMDA system along the lines of \cite{4}–\cite{7} in Sec. 4.

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order to achieve this aim, a suitable generalization of the method has been performed. The
details of the application of the quantization method can be consulted in [8].

2. Outline of the BFF method
Let there be a dynamical system described by the original Hamiltonian $H_0 = H_0(\Gamma)$ given as
a function of $2N$ bosonic phase variables $\Gamma^A$, $A = 1, 2, \ldots, 2N$, of certain manifold $\mathcal{M}$. If we
assume that the dynamical system is unconstrained, its Lagrangian can be written as follows [9]

$$L = a_A(\Gamma)\Gamma^A - H_0(\Gamma).$$  (1)

The Euler–Lagrange equations for the Lagrangian (1) are given by the following relations

$$\omega_{AB}(\Gamma)\dot{\Gamma}^B = \partial_A H_0(\Gamma),$$  (2)

where $\omega_{AB}(\Gamma) = \partial_A a_B(\Gamma) - \partial_B a_A(\Gamma)$ and $\partial_A \equiv \partial/\partial \Gamma^A$.

The nondegenerate tensor $\omega_{AB}(\Gamma)$ defines on the phase space manifold the covariant
components of the symplectic metric. The Poisson bracket for any two functions $X(\Gamma)$ and
$Y(\Gamma)$ of the phase space variables $\Gamma^A$ is defined as follows

$$\{X, Y\} = \partial_A X \omega^{AB} \partial_B Y.$$  (3)

The reper fields with contravariant $h^A{}^a(\Gamma)$ and covariant $h^a{}^A(\Gamma)$ components are introduced as

$$\omega^{AB}(\Gamma) = h^A{}^a(\Gamma)\omega^{ab}(0) h^b{}^B(\Gamma), \quad \omega^{AB}(\Gamma) = h^a{}^A(\Gamma)\omega^{(0)}_{ab} h^b{}^B(\Gamma),$$  (4)

where $\omega^{ab}(0)$ and $\omega^{(0)}_{ab}$ define a constant symplectic metric and its inverse in the following form

$$\omega^{(0)}_{ab} = \left(\begin{array}{cc} 0 & I_N \\ -I_N & 0 \end{array}\right),$$  (5)

where $I_N$ is a $N \times N$ unit matrix. The covariant derivatives in the phase space are defined as

$$\nabla_C V_A \equiv \partial_C V_A - \Delta^D_C A \Delta_V D, \quad \nabla_C V^A \equiv \partial_C V^A + \Delta^A_C D \Delta_V D,$$

where $\Delta^D_C A \equiv h^D{}^a \partial_C h^a{}^A$. (6)

The commutator of the covariant derivatives is given by the following relations

$$[\nabla_A, \nabla_B] = -\Lambda^C_{AB} \nabla_C,$$  (7)

where $\Lambda^C_{AB} \equiv \Delta^C_{AB} - \Delta^C_{BA}$.

The equations of motion written in terms of the Poisson brackets (3) are derived from the action

$$S = \int \left[ \Gamma^A \mathcal{L}_{AB}(\Gamma)\dot{\Gamma}^B - H_0(\Gamma) \right] dt,$$  (8)

where $\mathcal{L}_{AB}(\Gamma) \equiv \int \omega_A(\alpha \Gamma) \alpha d\alpha$.

Now we introduce the new conjugate canonical momenta $\Pi_A$. For any two functions of the
$\Gamma^A$ and $\Pi_A$, $X(\Gamma^A, \Pi_A), Y(\Gamma^A, \Pi_A)$, we define the following flat Poisson bracket

$$\{X(\Gamma^A, \Pi_A), Y(\Gamma^A, \Pi_A)\}' \equiv \partial_A X \partial^A Y - \partial_A Y \partial^A X,$$  (9)

where $\partial^A \equiv \partial/\partial \Pi_A$.

Let us consider the action

$$S' = \int \left[ \Pi_A \dot{\Gamma}^A - H_0(\Gamma) - \Theta_A(\Gamma, \Pi)\lambda^A \right] dt,$$  (10)
where $\Theta_A(\Gamma, \Pi) \equiv \Pi_A + \overline{\omega}_{AB}(\Gamma) \Gamma^B$ and $\lambda^A$ are Lagrange multipliers.

The flat Poisson bracket for any two of them is given by the following relation

$$\{\Theta_A, \Theta_B\} = \omega_{AB}(\Gamma).$$  \hspace{1cm} (11)

The action (10) represents a system with Hamiltonian $H_0(\Gamma)$ independent of the momenta $\Pi_A$ subjected to special second class constraints $\Theta_A(\Gamma, \Pi)$. The equations of motion that follow from it are equivalent to those derived from (8) after the formal exclusion of the momenta $\Pi_A$.

Since the dynamical system we shall consider later on possesses a set of original irreducible first class constraints $T'_a(\Gamma)$, $a = 1, 2, \ldots, m'$ the expressions (8) and (10) are correspondingly modified as follows [3]:

$$S = \int \left[ \Gamma^A \overline{\omega}_{AB}(\Gamma) \dot{\Gamma}^B - H_0(\Gamma) - T'_a(\Gamma) \lambda^a \right] dt,$$

$$S' = \int \left[ \Pi_A \dot{\Gamma}^A - H_0(\Gamma) - \Theta_A(\Gamma, \Pi) \lambda^A - T'_a(\Gamma) \lambda^a \right] dt.$$  \hspace{1cm} (12)

Now we can perform the canonical quantization of the system. The variables $\Gamma^A$ and $\Pi_A$ are promoted to operators and must satisfy the following equal time commutation relations

$$[\Gamma^A, \Pi_B] = i\hbar \delta^A B.$$  \hspace{1cm} (14)

The second class constraints $\Theta_A(\Gamma, \Pi)$ are converted into Abelian first class by introducing the new bosonic operators $\Phi_a$, $a = 1, 2, \ldots, 2N$ [4]–[7] with equal time commutation relations

$$[\Phi_a, \Phi_b] = -i\hbar \omega_{ab}^{(0)}.$$  \hspace{1cm} (15)

The effective Abelian constraints are defined by the commutations relations

$$[T_A(\Gamma, \Pi, \Phi), T_B(\Gamma, \Pi, \Phi)] = 0, \quad \text{with} \quad T_A(\Gamma, \Pi, 0) = \Theta_A(\Gamma, \Pi).$$  \hspace{1cm} (16)

We seek for the solution in the following form

$$T_A(\Gamma, \Pi, \Phi) = \Theta_A(\Gamma, \Pi) + K_A(\Gamma, \Phi), \quad K_A(\Gamma, 0) = 0.$$  \hspace{1cm} (17)

Substituting this expression into (16) we get the following equation for $K_A(\Gamma, \Phi)$:

$$\nabla_A K_B - \nabla_B K_A + \Lambda^C_{AB} K_C - (i\hbar)^{-1} [K_A, K_B] = \omega_{AB}(\Gamma).$$  \hspace{1cm} (18)

It is worth noticing that the solution of (18) in the zero curvature case $\Lambda^C_{AB} = 0$ is given by

$$K_A = \exp \left( \Phi_a \partial^a \right) K_A(\Gamma, \varphi) |_{\varphi = 0}, \quad \tilde{K}_A = \varphi_a h^a A(\Gamma),$$  \hspace{1cm} (19)

where $\varphi_a$ are classical variables and $\partial^a \equiv \partial / \partial \varphi_a$.

We further construct from the initial first class constraints $T'_i(\Gamma)$, $i = 1, 2, \ldots, m'$ of our system, the quantities $\tilde{T}'_i(\Gamma, \Phi)$ that commute with the effective Abelian constraints $T_A(\Gamma, \Pi, \Phi)$ [3]:

$$[\tilde{T}'_i(\Gamma, \Phi), T_A(\Gamma, \Pi, \Phi)] = 0, \quad \text{with} \quad \tilde{T}'_i(\Gamma, 0) = T'_i(\Gamma) \quad \text{and} \quad i = 1, 2, \ldots, m'.$$  \hspace{1cm} (20)

After substituting (17) into (20) we obtain the following equation for $\tilde{T}'_i(\Gamma, \Phi)$:

$$\partial_A \tilde{T}'_i(\Gamma, \Phi) = (i\hbar)^{-1} \left[ K_A(\Gamma, \Phi), \tilde{T}'_i(\Gamma, \Phi) \right].$$  \hspace{1cm} (21)
The solution of the equation (21) in the special case $\Lambda_{AB}^C = 0$ is given by the relation [1]–[2]:

$$\tilde{T}_i'(\Gamma, \Phi) = \exp \left( \Phi \partial^\phi \right) \tilde{T}_i(\Gamma, \varphi) |_{\varphi = 0}, \quad \text{where} \quad \tilde{T}_i(\Gamma, \varphi) = T_i' \left( \Gamma(x = 0) \right)$$

(22)

and the functions $\Gamma^A(x)$ are solutions of the differential equation

$$\frac{d\Gamma^A(x)}{dx} = \varphi_a \omega_{a(0)}^b h^A_{\ b}(\Gamma), \quad \Gamma^A(x = 1) = \Gamma^A.$$  

(23)

In order to construct the fermion generating operator $\Omega$, we introduce a pair (coordinate, conjugate momentum) $(\lambda, \pi)$ together with a pair of ghosts $(C, \overline{C})$ and antighosts $(P, \overline{P})$ for every irreducible first class constraint, where $\lambda$ is an active Lagrange multiplier. The new introduced variables possess the following statistics $(\varepsilon)$ and ghost number $(gh)$ [3]:

$$\varepsilon(\lambda) = \varepsilon(\pi) = 1 + \varepsilon(C) = 1 + \varepsilon(\overline{C}) = 1 + \varepsilon(P) = 1 + \varepsilon(\overline{P})$$

$$gh(C) = -gh(\overline{C}) = gh(P) = -gh(\overline{P}) = 1.$$  

(24)

Only the following supercommutators of the above introduced operators are different from zero:

$$\lbrack \lambda, \pi \rbrack = \lbrack C, \overline{P} \rbrack = \lbrack P, \overline{C} \rbrack = i\hbar I.$$  

(25)

In the case when initial second class constraints are absent, the fermion generating operator $\Omega$, which defines the quantum physical states of the system under consideration, is given by

$$\Omega = \Omega'(\Gamma, \Phi, C', \overline{P}) + T_A(\Gamma, \Pi, \Phi)C^A + \pi_i' P^{\ni} + \pi_A {P}^A,$$

(26)

where the Fermi operator $\Omega'$ obeys the following equations

$$\lbrack \Omega', \Omega \rbrack = \lbrack \Omega', T_A \rbrack = 0, \quad gh(\Omega') = 1, \quad \text{with} \quad \Omega'(\Gamma, \Phi, C', 0) = \tilde{T}_i'(\Gamma, \Phi)C^{\ni}.$$  

(27)

By virtue of equations (27), the operator (26) turns out to be nilpotent: $[\Omega, \Omega] = 0$.

3. The four–dimensional EMDA system

The 4D EMDA system describes gravity coupled to the dilaton $\phi$, the axion $\kappa$ and just one non–vanishing $U(1)$ vector field $A_\mu$ [10]. In the Einstein frame the action reads

$$S = \frac{1}{16\pi} \int d^4 x (4g)^{1/2} \left\{-R + 2 \partial_\mu \phi \partial^\mu \phi + \frac{1}{2} e^{4\phi} \partial_\mu \kappa \partial^\mu \kappa - e^{-2\phi} F_{\mu\nu} F^{\mu\nu} - \kappa F_{\mu\nu} \tilde{F}^{\mu\nu} \right\},$$

(28)

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $\tilde{F}^{\mu\nu} = \frac{i}{2} E^{\mu\nu\lambda\sigma} F_{\lambda\sigma}$ and $E^{\mu\nu\lambda\sigma} = \sqrt{|g|} \varepsilon^{\mu\nu\lambda\sigma}$.

By using the ADM decomposition of the metric tensor [11]–[14] it adopts the form

$$I = \frac{1}{4} \int dt d\tau \mathcal{L}_{\text{tot}} = \frac{1}{4} \int dtdr \left\{-\pi_\mu \dot{\pi}_\lambda + \pi_\phi \dot{\phi} + e^{4\phi} p_\kappa \dot{\kappa} + e^{-2\phi} \mathcal{E}^r A_r + \right.$$  

$$\left. N \chi \left( e^{-\phi} - 2\phi \right) \left( \frac{1}{8} \left| \pi_\mu \right|^2 - 2\pi_\mu \pi_\lambda \right) + 2 e^{4\phi} \left( 2\lambda'' + 3\lambda' + \right) \right.$$  

(29)

$$- \kappa^2 \left( e^{4\phi} - e^{2(\mu - \lambda)} \right) + \mathcal{E}^r \left( \frac{3}{8} e^{2\phi} \phi \mu + \frac{p_\phi^2}{2} e^{4\phi} - \mu + \frac{\kappa^2}{2} e^{4\phi} \right) - \frac{(\mathcal{E}^r)^2}{8} e^{-2\phi} = 0 \right.$$
Now we proceed as in [15]–[16] by imposing the coordinate condition \( r = e^\lambda \) and solving for \( \pi_\lambda \) the equation which results after putting the second constraint (the one multiplied by \( N^1 \)) equal to zero. Then the Lagrangian of the action (29) is modified as follows [8]

\[
\mathcal{L}_{tot} = -\pi_\mu \dot{\lambda} + \pi_\phi \dot{\phi} + e^{4\phi} \pi_\alpha \dot{\kappa} + e^{-2\phi} \dot{\varepsilon} \mathcal{A}_r + N^1 e^{-\mu} \left[ \frac{1}{8 \pi^2} \pi_\mu \frac{r}{4 \pi^2} \pi_\mu \right. \\
\left. + e^{4\phi} \pi_\alpha \dot{\kappa} + e^{-2\phi} \dot{\varepsilon} \mathcal{A}_r \right] + 2 - 4r \mu - 2e^{2\mu} + \frac{p_\phi^2}{8r^2} + 2r^2 \phi'^2 + e^{4\phi} \frac{p_\alpha^2}{2r^2} + e^{4\phi} \frac{\kappa'^2}{2} - e^{2(\mu - \phi)} \left( \frac{\varepsilon'^2}{8r^2} \right) - A_0 e^{-2\phi} \left( \varepsilon'^r - 2\phi' \varepsilon^r \right). \tag{30}
\]

We see that (30) has eight variables spanning a curved phase space and two first class constraints. Now we can apply the BFF canonical quantization method to this model.

4. Canonical quantization of the EMDA system

We rename the field variables of (30) as follows \( p_\phi = \Gamma^1, \varepsilon' = \Gamma^2, p_\kappa = \Gamma^3, \pi_\mu = \Gamma^4, \phi = \Gamma^5, A_r = \Gamma^6, \kappa = \Gamma^7, \mu = \Gamma^8. \) Then, the “canonical one–form” of (30) adopts the form

\[
\pi_\phi \dot{\phi} + e^{4\phi} p_\kappa \dot{\kappa} + e^{-2\phi} \dot{\varepsilon} \mathcal{A}_r - \pi_\mu \mu = a_A(\Gamma) \wedge A, \quad A = 1, 2, \ldots, 8; \tag{31}
\]

where the functions \( a_A \) read \( a_1 = -\frac{1}{2} \Gamma^5, \quad a_2 = -\frac{1}{2} e^{-2\Gamma^5} \Gamma^6, \quad a_3 = -\frac{1}{2} e^{-2\Gamma^5} \Gamma^7, \quad a_4 = \frac{1}{2} \Gamma^8, \quad a_5 = \frac{1}{2} \left( \Gamma^2 - 4 e^{4\Gamma^5} \Gamma^3 \Gamma^7 + 2 e^{-2\Gamma^5} \Gamma^2 \Gamma^6 \right), \quad a_6 = \frac{1}{2} e^{-2\Gamma^5} \Gamma^2, \quad a_7 = \frac{1}{2} e^{4\Gamma^5} \Gamma^3, \quad \text{and} \quad a_8 = -\frac{1}{2} \Gamma^4. \)

The equal time Poisson bracket for \( X(\Gamma(r, t), \Gamma'(r', t)) \) and \( Y(\Gamma(r, t), \Gamma'(r', t)) \) are defined as

\[
\{ X(\Gamma, \Gamma') , Y(\Gamma, \Gamma') \} = \int d\Gamma'' \left[ \frac{\partial}{\partial \Gamma'(r'', t)} X(\Gamma, \Gamma') \right] \frac{\partial}{\partial \Gamma' (r'', t)} Y(\Gamma, \Gamma'), \tag{32}
\]

where the prime over the \( \Gamma \) denotes derivative with respect to the spatial variable.

There are two irreducible constraints in the action (30), namely

\[
T_1'(\Gamma, \Gamma') = e^{-\Gamma^5} \left\{ \left( \frac{\Gamma^2}{8 \pi^2} \right) - 2r \Gamma^4 \left( \Gamma^2 - 4 \Gamma^4 \Gamma^8 + \Gamma^4 \Gamma^5 + 2 e^{-2\Gamma^5} \Gamma^3 \Gamma^7 \right) + 2 - 4r \Gamma^8 - 2e^{2\Gamma^5} + 2r^2 \left( \Gamma^5 \right)^2 + \right. \\
\left. \frac{(\Gamma^1)^2}{8r^2} + e^{4\Gamma^5} \frac{(\Gamma^3)^2}{2r^2} + e^{4\Gamma^5} \frac{\Gamma^2 \left( \Gamma^2 \right)^2}{2} - e^{2((\Gamma^8 - \Gamma^5))} \left( \frac{\Gamma^2}{8r^2} \right) \right\}, \quad T_2'(\Gamma, \Gamma') = e^{-2\Gamma^5} \left( 2 \Gamma^5 - \Gamma^2 \right). \tag{33}
\]

These are first class constraints as they should be [18]–[22] (see [13] as well) and their Poisson brackets satisfy the following relations:

\[
\{ T_1'(\Gamma(r, t), \Gamma'(r, t)), T_2'(\Gamma(r', t), \Gamma'(r', t)) \} = \frac{\Gamma^1 (r, t)}{4r} \left[ \frac{\partial}{\partial r} \delta(r' - r) - \delta(r - r') \right]. \tag{34}
\]

\[
\{ T_1'(\Gamma(r, t), \Gamma'(r', t)), T_2'(\Gamma(r', t), \Gamma'(r', t)) \} = \{ T_2'(\Gamma(r, t), \Gamma'(r, t)), T_1'(\Gamma(r', t), \Gamma'(r', t)) \} = 0. \tag{35}
\]
The next step is to determine the reper field \( h^a A \) in such a way that the curvature \( \Lambda_{AB}^C = 0 \). This is a very crucial point because it simplifies enormously the forthcoming calculations. One such solution is given by the following quantities \( h^1 (r,t) = h^5 (r,t) = h^6 (r,t) = e^{-2G^5} \), \( h^7 (r,t) = e^{4G^5} \), \( h^8 (r,t) = 4e^{4G^5} \).

Now we introduce new bosonic fields \( \Pi_A (r,t) \), which we promote into operators along with the variables \( \Gamma^A (r,t) \). We also define the nonzero commutators as follows:

\[
[\Gamma^A (r,t), \Pi_B (r', t)] = i\hbar \delta^A_B \delta (r - r').
\]  

(36)

This is a slightly modified (generalized) version of the relation (14) since it involves the Dirac \( \delta \)-function which is not present in its original definition.

The special second class constraints \( \Theta_A (\Gamma, \Pi) \), defined in (2), are given by the expressions

\[
\begin{align*}
\Theta_1 (r,t) &= \Pi_1 (r,t) + \frac{1}{2} \Gamma^5 (r,t), \\
\Theta_2 (r,t) &= \Pi_2 (r,t) + \frac{1 - e^{-2G^5(r,t)}}{4 \Gamma^5 (r,t)^2} \Gamma^6 (r,t), \\
\Theta_3 (r,t) &= \Pi_3 (r,t) + \frac{1 + e^{4G^5(r,t)} ( -1 + 4 \Gamma^5 (r,t) )}{16 \Gamma^5 (r,t)^2} \Gamma^7 (r,t), \\
\Theta_4 (r,t) &= \Pi_4 (r,t) - \frac{1}{2} \Gamma^8 (r,t), \\
\Theta_5 (r,t) &= \Pi_5 (r,t) - \frac{1}{2} \Gamma^1 (r,t) - \frac{1 - e^{-2G^5(r,t)}}{2 \Gamma^5 (r,t)^3} \Gamma^2 (r,t) \Gamma^6 (r,t) - \\
& \quad \frac{1 - e^{4G^5(r,t)} (1 - 4 \Gamma^5 (r,t) + 8 \Gamma^5 (r,t)^2)}{8 \Gamma^5 (r,t)^3} \Gamma^3 (r,t) \Gamma^7 (r,t), \\
\Theta_6 (r,t) &= \Pi_6 (r,t) + \frac{1 - e^{-2G^5(r,t)}}{4 \Gamma^5 (r,t)^2} \Gamma^2 (r,t), \\
\Theta_7 (r,t) &= \Pi_7 (r,t) + \frac{1 - e^{4G^5(r,t)} (1 - 4 \Gamma^5 (r,t) + 16 \Gamma^5 (r,t)^2)}{16 \Gamma^5 (r,t)^2} \Gamma^3 (r,t), \\
\Theta_8 (r,t) &= \Pi_8 (r,t) + \frac{1}{2} \Gamma^4 (r,t).
\end{align*}
\]  

(37)

These quantities satisfy the following commutation relations

\[
[\Theta_A (r,t), \Theta_B (r', t)] = i\hbar \omega_{AB} (r,t) \delta (r - r')
\]  

(38)

as well as the following flat Poisson bracket relations

\[
\{\Theta_A (r,t), \Theta_B (r', t)\}' = \int dr' \left[ \partial_C \Theta_A (r,t) \partial^C \Theta_B (r', t) - \partial_C \Theta_B (r', t) \partial^C \Theta_A (r,t) \right] = \omega_{AB} (r,t) \delta (r - r'),
\]  

(39)
where $\partial_C = \partial / \partial \Gamma^C(r'', t)$ and $\partial^C = \partial / \partial \Pi^C(r'', t)$. Further, the second class constraints $\Theta_A$ are converted into Abelian first class constraints by introducing the bosonic fields $\Phi_a$, $a = 1, 2, \ldots, 8$ which satisfy the following equal time commutation relations

$$ [\Phi_a(r, t), \Phi_b(r', t)] = -i \hbar \omega^{(0)}_{ab} \delta(r - r'). \quad (40) $$

Since we are considering the $\Lambda^C_{AB} = 0$ case, from (17) and (18) the first class constraints read

$$ T_A(\Gamma, \Pi, \Phi) = \Theta_A(\Gamma, \Pi, \Phi) + K_A(\Gamma, \Phi) = \Theta_A(\Gamma, \Pi) + \Phi_a h^a_A(\Gamma). \quad (41) $$

We can see from (38) and (40) that these first class constraints $T_A(\Gamma, \Pi, \Phi)$ are Abelian

$$ [T_A(r, t), T_B(r', t)] = \left[ \Theta_A(r, t) + \Phi_a(r, t) h^a_A(r, t), \Theta_B(r', t) + \Phi_b(r', t) h^b_B(r', t) \right] = 0. $$$$ (42)$$

Now we proceed to construct the operators $\tilde{T}_i'(\Gamma, \Gamma', \Phi, \Phi')$, $T_i'(\Gamma, \Gamma', 0, 0) = T_i'(\Gamma, \Gamma')$, $i = 1, 2$.

$$ \left[ \tilde{T}_i'(\Gamma, \Gamma', \Phi, \Phi'), T_i'(\Gamma, \Pi, \Phi) \right] = 0 \quad \text{and} \quad \tilde{T}_i'(\Gamma, \Gamma', 0, 0) = T_i'(\Gamma, \Gamma'), \quad i = 1, 2. \quad (43) $$

where $T_i'(\Gamma, \Gamma')$ are the first class constraints given in (33). Here we have to note that the quantities $\tilde{T}_i'$ depend not only on the $\Phi_a$ fields, but also on their space derivatives and this is because of the existence of space derivatives of the phase space variables $\Gamma^A$ in the initial first class constraints $T_i'$ and $T_2'$. By substituting the relation (41) into (43) we obtain

$$ \left[ \tilde{T}_i'(\Gamma, \Gamma', \Phi, \Phi'), \Pi_A(r', t), \Pi_A(r', t) \right] + \left[ \tilde{T}_i'(\Gamma, \Gamma', \Phi, \Phi'), \Phi_a(r', t) h^a_A(r', t) \right] = 0. \quad (44) $$

Due to the relation (36) this equation adopts the following form

$$ \frac{\delta}{\delta \Gamma^A(r'', t)} \tilde{T}_i'(\Gamma, \Gamma', \Phi, \Phi') = (i \hbar)^{-1} \left[ \Phi_a(r', t) h^a_A(r', t), \tilde{T}_i'(\Gamma, \Gamma', \Phi, \Phi') \right]. \quad (45) $$

Let us try to find a solution for equation (45) in the case when the constraint $T_2'(\Gamma, \Gamma')$ is given by (33). We shall consider the function $T_2'(\Gamma) = -e^{-2\Gamma^5} (\Gamma^2 - 2 \Gamma^2 \Gamma^5)$ (this relation comes from the expression of $T_2'(\Gamma, \Gamma')$ after ignoring the primes) and solve the equation

$$ \frac{\delta}{\delta \Gamma^A(r'', t)} \tilde{T}_2'(\Gamma, \Phi, \Phi) = (i \hbar)^{-1} \left[ \Phi_a(r', t) h^a_A(r', t), \tilde{T}_2'(\Gamma, \Phi, \Phi) \right] \quad (46) $$

with the boundary condition $\tilde{T}_2'(\Gamma, \Phi, \Phi) = T_2'(\Gamma, \Phi, \Phi)$ at $\Gamma(\Phi, \Phi) = 0$. The solution of equation (46) reads

$$ \tilde{T}_2'(\Gamma, \Phi, \Phi) = \exp \left( \int dr' \Phi_a(r', t) \frac{\partial}{\partial \varphi_a(r', t)} \right) T_2'(\Gamma(0), \Phi(0)) |_{\varphi = 0}, \quad (47) $$

where the functions $\Gamma^A(x)$ are solutions of the equations (23) and have the form $\Gamma^1(x) = \varphi_3(x - 1) + \Gamma^1$, $\Gamma^2(x) = e^{-2\varphi_1(x - 1)} [\Gamma^2 + \varphi_6 e^{2\varphi_1} (x - 1)]$, $\Gamma^3(x) = e^{\varphi_1 (x - 1)} [\varphi_3 + \varphi_7 e^{-4\varphi_1} (x - 1)]$, $\Gamma^4(x) = \varphi_8 (1 - x) + \Gamma^4$, $\Gamma^5(x) = \varphi_1 (1 - x) + \Gamma^5$, $\Gamma^6(x) = \varphi_2 (1 - x) + \Gamma^6$, $\Gamma^7(x) = \varphi_3 (1 - x) + \Gamma^7$, $\Gamma^8(x) = \varphi_4 (1 - x) + \Gamma^8$. 


Thus, from equation (47) we obtain for the solution \( \tilde{T}_2 (\Gamma(r, t), \Phi(r, t)) \) of equation (46) the following expression

\[
\tilde{T}_2 (\Gamma(r, t), \Phi(r, t)) = -e^{-2 (\Phi_1 + \Gamma^5)} \left[ e^{2 \Phi_1} \left( \Gamma^2 - \Phi_6 e^{2 \Gamma^5} \right) - 2 \left( \Phi_1 + \Gamma^5 \right) e^{2 \Phi_1} \left( \Gamma^2 - \Phi_6 e^{2 \Gamma^5} \right) \right].
\]  

(48)

From the relation (48) we can write down a solution of (45) in the form

\[
\tilde{T}'_2 (\Gamma, \Gamma', \Phi, \Phi') = -e^{-2 (\Phi_1 + \Gamma^5)} \left\{ \frac{d}{dr} \left[ e^{2 \Phi_1} \left( \Gamma^2 - \Phi_6 e^{2 \Gamma^5} \right) \right] - 2 \left( \Phi_1 + \Gamma^5' \right) e^{2 \Phi_1} \left( \Gamma^2 - \Phi_6 e^{2 \Gamma^5} \right) \right\}.
\]  

(49)

Thus, the simplified expression for the equation (49) reads

\[
\tilde{T}'_2 (\Gamma, \Gamma', \Phi, \Phi') = -e^{-2 \Gamma^5} \left( \Gamma^{2'} - \Phi_6' e^{2 \Gamma^5} - 2 \Gamma^{5'} \Gamma^{2'} \right).
\]  

(50)

It is easy to verify that (50) satisfies the equation (45) in all cases and that the boundary condition \( \tilde{T}'_2 (\Gamma, \Gamma', 0, 0) = T'_2 (\Gamma, \Gamma') \) is satisfied as well.

It turns out that the same procedure can be performed when solving the equation (45) for the constraint \( \tilde{T}'_1 (\Gamma, \Gamma') \) of (33). Avoiding the intermediate computations, we just quote the solution

\[
\tilde{T}'_1 (\Gamma(r, t), \Gamma'(r, t), \Phi(r, t), \Phi'(r, t)) = \sum_{i=1}^{8} \tilde{T}'_{1i} (\Gamma(r, t), \Gamma'(r, t), \Phi(r, t), \Phi'(r, t)), \quad \text{where}
\]  

(51)

\[
\tilde{T}'_{11} (\Gamma, \Gamma', \Phi, \Phi') = \frac{1}{8r^2} \left[ \left( \Gamma^4 \right)^2 c_1 + \frac{1}{2} \Phi_8 c_1 \Phi_8 + \Gamma^4 (\Phi_8 c_1 + c_1 \Phi_8) + \frac{1}{4} \left( \Phi_8^2 c_1 + c_1 \Phi_8^2 \right) \right] -
\]

\[
\frac{1}{4r} \left[ \Gamma^4 \Gamma^{4'} c_1 + \frac{1}{4} (\Phi_8 c_1 \Phi_8 + \Phi_8 c_1 \Phi_8') + \frac{\Gamma^{4'}}{2} (\Phi_8 c_1 + c_1 \Phi_8) + \frac{\Gamma^4}{2} (\Phi_8' c_1 + c_1 \Phi_8') + \frac{1}{4} (\Phi_8' \Phi_8 c_1 + c_1 \Phi_8' \Phi_8) \right],
\]

\[
\tilde{T}'_{12} (\Gamma, \Gamma', \Phi, \Phi') = \frac{1}{4r} \left[ \left( \Gamma^4 \right)^2 c_2 + \Gamma^4 (\Phi_8 c_2 + c_2 \Phi_8) + \frac{1}{2} \left( \Phi_8 c_2 \Phi_8 + \Phi_8^2 c_2 + c_2 \Phi_8^2 \right) \right],
\]

\[
\tilde{T}'_{13} (\Gamma, \Gamma', \Phi, \Phi') = -\frac{1}{4r} \left[ \Gamma^4 c_1 + \frac{1}{2} (c_1 \Phi_8 + \Phi_8 c_1) \right] \left[ \Gamma^4 \Gamma^{5'} + (\Gamma^4 \Phi_1 - \Gamma^5 \Phi_5) - \frac{1}{2} (\Phi_1 \Phi_5 + \Phi_5 \Phi_1) \right],
\]

\[
\tilde{T}'_{14} (\Gamma, \Gamma', \Phi, \Phi') = -\frac{e^{4 \Gamma^5}}{4r} \left[ \Gamma^4 c_1 + \frac{1}{2} (c_1 \Phi_8 + \Phi_8 c_1) \right] \left[ \Gamma^3 \Gamma^{5'} + (\Gamma^5 \Phi_3 - e^{4 \Gamma^5} \Gamma^{7'} \Phi_7) - \frac{e^{4 \Gamma^5}}{2} (\Phi_7 \Phi_5 + \Phi_5 \Phi_7) \right],
\]

\[
\tilde{T}'_{15} (\Gamma, \Gamma', \Phi, \Phi') = \tilde{T}_{15} (\Gamma, \Phi), \quad \tilde{T}'_{16} (\Gamma, \Gamma', \Phi, \Phi') = -4rc_2,
\]

\[
\tilde{T}'_{17} (\Gamma, \Gamma', \Phi, \Phi') = 2r^2 c_1 \left( \Phi_1' + \Gamma^5 \right)^2, \quad \tilde{T}'_{18} (\Gamma, \Gamma', \Phi, \Phi') = \frac{r^2}{2} c_1 e^{4 (\Phi_2 + \Gamma^5)} \left( \Phi_3' + \Gamma^{7'} \right)^2,
\]

where \( c_1 = e^{-(\Phi_4 + \Gamma^8)} \) and \( c_2 = (\Phi_4' + \Gamma^8') c_1 \).
Now we are in position to calculate the fermion generating operator Ω. In order to achieve this aim, let us recall that we have the following set of irreducible first class constraints

\[
T \equiv \begin{pmatrix} \tilde{T}'_a \\ \tilde{T}'_A \end{pmatrix}, \quad a = 1, 2; \quad A = 1, 2, \ldots, 8;
\]

whose expressions are given in the equations (41), (50) and (51). We put into correspondence with this complete set of irreducible first class constraints the following ordered operator pairs

\[
\begin{pmatrix} \tilde{T}'_a \\ \tilde{T}'_A \end{pmatrix} \rightarrow \begin{pmatrix} \lambda^a(r, t), \pi'_a(r, t) \\ \lambda^A(r, t), \pi'_A(r, t) \end{pmatrix}, \quad \begin{pmatrix} C'^a(r, t), \overline{\mathcal{P}}^a(r, t) \\ C'^A(r, t), \overline{\mathcal{P}}^A(r, t) \end{pmatrix}, \quad \begin{pmatrix} \mathcal{P}^a(r, t), \overline{\mathcal{C}}^a(r, t) \\ \mathcal{P}^A(r, t), \overline{\mathcal{C}}^A(r, t) \end{pmatrix},
\]

where the active Lagrange multipliers read \( \lambda^1 = N^\perp, \lambda^2 = A_0 \). The ghost numbers and the statistics of these magnitudes read \( gh(C) = gh(\mathcal{P}) = 1, \ gh(\overline{C}) = gh(\overline{\mathcal{P}}) = -1, \ v(\lambda) = v(\pi) = 0, \ v(C) = v(\overline{C}) = v(\mathcal{P}) = v(\overline{\mathcal{P}}) = 1 \).

The only nontrivial supercommutators read

\[
[\lambda^i(r, t), \pi'_j(r', t)] = [C'^i(r, t), \overline{\mathcal{P}}^j(r', t)] = [\mathcal{P}^i(r, t), \overline{\mathcal{C}}^j(r', t)] = i\hbar \delta(r - r'), \quad i, j = a, A;
\]

while other supercommutators vanish, i.e. \([C'^i(r, t), C'^j(r', t)] = [\overline{\mathcal{P}}^i(r, t), \overline{\mathcal{C}}^j(r', t)] = 0, \) etc.

Thus, the fermion generating operator Ω adopts the form given by the expression (26)

\[
\Omega = \Omega'(\Gamma, \Gamma', \Phi, \Phi', C', \overline{\mathcal{P}}') + T_A(G, \Pi, \Phi)C^A + \pi'_a \mathcal{P}^a + \pi'_A \mathcal{P}^A, \quad a = 1, 2; \quad A = 1, 2, \ldots 8;
\]

where the the Fermi operator \( \Omega' \) obeys the following relations \( [\Omega', \Omega'] = [\Omega', T_A] = 0, \) possesses the ghost number \( gh(\Omega') = 1, \) and satisfies the boundary condition \( \Omega'(\Gamma, \Gamma', \Phi, \Phi', C', 0) = \tilde{T}'_a(\Gamma, \Gamma', \Phi, \Phi')C'^a. \)

We try to find the solution for the equation (55) in the following form (we use a Weyl basis)

\[
\Omega' = \tilde{T}'_a(\Gamma, \Phi, \Phi')C'^a + \tilde{U}^a_{bc}(\Gamma, \Phi, \Phi') \begin{pmatrix} C'^c \left( \frac{\partial}{\partial r} C'^b \right) + \left( \frac{\partial}{\partial t} C'^b \right) \end{pmatrix} + \begin{pmatrix} \frac{\partial}{\partial r} C'^c \end{pmatrix} \begin{pmatrix} C'^c \left( \frac{\partial}{\partial t} C'^b \right) \end{pmatrix} = \tilde{\mathcal{P}}'_a.
\]

The exact form of the function \( \tilde{U}^a_{bc}(\Gamma, \Phi, \Phi') \) are currently under investigation. This report constitutes the first part of our research because the explicit form of the fermion generating operator Ω and the total unitarizing Hamiltonian of the theory \( H \) are still under construction.

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