Stability of Infinite-dimensional Sampled-data Systems with Unbounded Control Operators and Perturbations*

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Abstract

We analyze the robustness of the exponential stability of infinite-dimensional sampled-data systems with unbounded control operators. The unbounded perturbations we consider are the so-called Desch-Schappacher perturbations, which arise, e.g., from the boundary perturbations of systems described by partial differential equations. As the main result, we show that the exponential stability of the sampled-data system is preserved under all Desch-Schappacher perturbations sufficiently small in a certain sense.

Keywords: Desch-Schappacher perturbations, infinite-dimensional systems, sampled-data systems, stability

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1. Introduction

Consider the following sampled-data system with state space $X$ and input space $U$ (both Banach spaces):

$$\begin{cases}
\dot{x}(t) = (A + D)x(t) + Bu(t), & t \geq 0; \quad x(0) = x^0 \in X \\
u(t) = Fx(k\tau), & k\tau \leq t < (k + 1)\tau, \; k = 0, 1, 2, \ldots,
\end{cases} \tag{1}$$

where $\tau > 0$ is the sampling period, $A$ is the generator of a strongly continuous semigroup $(T(t))_{t \geq 0}$ on $X$, and the feedback operator $F$ is a bounded linear operator from $X$ to $U$. The control operator $B$ is linear but unbounded in the sense that it maps $U$ into a larger space than $X$. More precisely, we employ the extrapolation space $X_1$, which is the completion of $X$ with respect to an appropriate norm. The perturbation $D$ is also linear but unbounded, which maps $X$ into $X_1$. In particular, we are interested in the so-called Desch-Schappacher perturbations. See Section 2 for the details of these notions. If $B$ and $D$ map boundedly into $X$, then they are called bounded; otherwise they are called unbounded. In this paper, we study the exponential stability of the perturbed sampled-data system (1).

A fundamental issue to be considered before discussing the stability of the perturbed system is the robustness of the generation of strongly continuous semigroups: Given the generator $A$ of a strongly continuous semigroup, for which perturbation $D$ does the (properly-defined) sum $A + D$ again become a generator? It is well known that an affirmative answer has been given to this question for some classes of perturbations such as relatively $A$-bounded perturbations \cite[Chapter XIII]{1}, $\alpha$-relative perturbations \cite[Section III.2]{2}, Miyadera-Voigt perturbations \cite[3, 4]{3, 4}, \cite[Section III.3.c]{2}, and Desch-Schappacher perturbations \cite[5]{5}, \cite[Section III.3.a]{2}. In this direction, Weiss \cite[6]{6} and Staffans \cite[Chapter 7]{7} have presented generation
results on $A + D$ for a class of perturbations $D$ factorized as $D = D_1D_2$ with an admissible control operator $D_1$ and an admissible observation operator $D_2$. The idea of factorizing perturbations has been also used in [8] in order to develop robustness results on sectoriality and maximal $L^p$-regularity under more general perturbations with small norm. The robustness analysis of exponential stability for infinite-dimensional continuous-time systems has been performed under several classes of unbounded perturbations in [9–12]. In the discrete-time setting, time-varying perturbations have been considered for the exponential stability of infinite-dimensional time-varying systems in [13, 14]. Strong stability and polynomial stability are much weaker notions of stability than exponential stability, and perturbations preserving these nonexponential classes of stability have been investigated, e.g., in [15–18].

For sampled-data systems, the robustness of stability with respect to sampling and perturbations have been studied. Robustness with respect to sampling means that the closed-loop stability is preserved when an idealized sample-and-hold process is applied to a stabilizing continuous-time controller. For infinite-dimensional systems, this type of robustness has been discussed in [19–22]. Robustness analysis with respect to perturbations has been also developed for infinite-dimensional sampled-data systems in [23, 24]. Bounded control operators and unbounded perturbations have been considered in [23], while unbounded control operators and bounded nonlinear perturbations have been treated in [24, Lemma 4.5].

In this paper, we continue and expand the robustness analysis developed in [23]. Desch-Schappacher perturbations, in which we are interested here, have properties different from relatively $A$-bounded perturbations and Miyadera-Voigt perturbations considered in [23]. In fact, Desch-Schappacher perturbations map
the state space $X$ into the extrapolation space $X_{-1}$, whereas the perturbations studied in [23] map a subspace of $X$, e.g., the domain of $A$, into $X$. The class of Desch-Schappacher perturbations is not a subset of the classes of perturbations considered in [23] and vice versa. Operators associated with boundary conditions of partial differential equations are often represented by using $X_{-1}$. Therefore, Desch-Schappacher perturbations appear when boundary conditions of partial differential equations are subject to perturbations, as shown in Section 4. Moreover, this class of perturbations is also used for dynamical population equations and delay differential equations; see, for example, [25, 26] and [2, Section VI.6]. Robustness of controllability under Desch-Schappacher perturbations has been studied in [27, 28].

Our aim is to prove that the exponential stability of sampled-data systems with unbounded control operators is preserved under sufficiently small Desch-Schappacher perturbations. The technical difficulties of this study come from the combination of the unbounded control operator and perturbation. In particular, we have to be careful about the operator $S_D(\tau)$ on $U$ defined by

$$S_D(\tau)u := \int_0^\tau T_D(s)Bu\,ds, \quad u \in U,$$

where $(T_D(t))_{t \geq 0}$ is the semigroup generated by the sum $A+D$, since $S_D(\tau)$ contains both the control operator $B$ and the perturbation $D$. The arguments used in [23, 24] rely on the assumption that either $B$ or $D$ is bounded. To treat the case where $B$ and $D$ are both unbounded, we develop an alternative approach which employs the fact that the resolvent of $A$ is extended to a bounded operator from $X_{-1}$ to $X$ and hence absorbs the unboundedness of $B$ and $D$.

This paper is organized as follows. In Section 2, we present preliminaries on extrapolation spaces, Desch-Schappacher perturbations, and the solution of the
abstract evolution equation (1). Section 3 contains our main result and its proof. Section 4 is devoted to an application to the boundary control of a heated rod with a boundary perturbation, which is extended to a diagonal system in a Banach framework.

**Notation.** Let \( \mathbb{Z}_+ \) and \( \mathbb{R}_+ \) denote the set of nonnegative integers and the set of nonnegative real numbers, respectively. Let \( X \) and \( Y \) be Banach spaces. The space of all bounded linear operators from \( X \) to \( Y \) is denoted by \( \mathcal{L}(X,Y) \). We set \( \mathcal{L}(X) \) := \( \mathcal{L}(X,X) \). For a linear operator \( A \) defined in \( X \) and mapping to \( Y \), the domain of \( A \) is denote by \( \text{dom}(A) \). We denote by \( \varrho(A) \) the resolvent set of a linear operator \( A : \text{dom}(A) \subset X \to X \). Let \( J \) be an interval of \( \mathbb{R} \). We denote by \( C(J,X) \) and \( C^1(J,X) \) the space of all continuous functions \( f : J \to X \) and the space of all continuously differential functions \( f : J \to X \), respectively. For \( 1 \leq p < \infty \), we denote by \( L^p(J,X) \) the space of all measurable functions \( f : J \to X \) such that \( \int_J \| f(t) \|_X^p \, dt < \infty \). We write \( L^p(J) \) for \( L^p(J,C) \).

2. Preliminaries

2.1. Extrapolation spaces

Let \( A \) be the generator of a strongly continuous semigroup \( (T(t))_{t \geq 0} \) on a Banach space \( X \). We denote by \( \| \cdot \| \) the norm on \( X \) and introduce a new norm

\[
\| x \|_{A^{-1}} := \| (\lambda I - A)^{-1} x \|, \quad x \in X
\]

for some \( \lambda \in \varrho(A) \). The completion of \( X \) with respect to this norm is called the **extrapolation space** associated with \( A \) (or \( (T(t))_{t \geq 0} \)) and is denoted by \( X^{A^{-1}} \). Different choices of \( \lambda \) lead to equivalent norms, and hence \( X^{A^{-1}} \) is unique up to
isomorphism. For simplicity, we shall denote the space $X_{-1}^A$ associated with $A$ by $X_{-1}$.

For $t \geq 0$, let $T_{-1}(t)$ be the continuous extension of the operator $T(t)$ to the extrapolation $X_{-1}$. Then the operators $(T_{-1}(t))_{t \geq 0}$ form a strongly continuous semigroup on $X_{-1}$. The domain of the generator $A_{-1}$ of $(T_{-1}(t))_{t \geq 0}$ is given by $\text{dom}(A_{-1}) = X$, and $A_{-1}$ is the unique continuous extension of $A : \text{dom}(A) \to X$ to $L(X, X_{-1})$. Moreover, $\varrho(A) = \varrho(A_{-1})$ holds. See Section II.5 in [2] and Section 2.10 in [29] for more details on the extrapolation space.

2.2. Desch-Shappacher perturbations

Let $X$ be a Banach space and take $t_0 > 0$. We denote by $X_{t_0}$ the space of all functions on $[0, t_0]$ into $L(X)$ that are continuous for the strong operator topology. Then $X_{t_0}$ is a Banach space with the norm

$$
\|Q\|_\infty := \sup_{t \in [0, t_0]} \|Q(t)\|_{L(X)}, \quad Q \in X_{t_0}.
$$

Let $(T(t))_{t \geq 0}$ be a strongly continuous semigroup on $X$. For $\Psi \in L(X, X_{-1})$, we define the abstract Volterra operator $V_{\Psi}$ on $X_{t_0}$:

$$(V_{\Psi}Q)(t)x := \int_0^t T_{-1}(t-s)\Psi Q(s)x ds, \quad x \in X, \ t \in [0, t_0], \ Q \in X_{t_0}.$$

This integral converges in $X_{-1}$. In general, we only have $(V_{\Psi}Q)(t) \in L(X, X_{-1})$ for $0 \leq t \leq t_0$. Hence the Volterra operator $V_{\Psi}$ may not be bounded on $X_{t_0}$. The class of Desch-Shappacher perturbations $S^{\text{DS}}_{t_0}$ is defined by

$$
S^{\text{DS}}_{t_0} := \{\Psi \in L(X, X_{-1}) : V_{\Psi} \in L(X_{t_0}), \ \|V_{\Psi}\|_{L(X_{t_0})} < 1\}.
$$

We review some important properties of Desch-Shappacher perturbations. The following theorem states that the generation of strongly continuous semigroups
is preserved under Desch-Shappacher perturbations. For the proof, see Theorem III.3.1 and Corollary III.3.2 in [2].

**Theorem 2.1.** Let $A$ be the generator of a strongly continuous semigroup $(T(t))_{t \geq 0}$ on a Banach space $X$. If $D \in S_{t_0}^{DS}$ for some $t_0 > 0$, then the operator $A_D$ defined by

$$A_Dx := (A_{-1} + D)x, \quad x \in \text{dom}(A_D) := \{x \in X : (A_{-1} + D)x \in X\} \quad (2)$$

generates a strongly continuous semigroup $(T_D(t))_{t \geq 0}$ on $X$. The semigroup $(T_D(t))_{t \geq 0}$ satisfies the following variation of constants formula:

$$T_D(t)x = T(t)x + \int_0^t T_{-1}(t-s)D T_D(s)x ds \quad (3)$$

for all $x \in X$ and all $t \geq 0$.

One can easily see that if $D_0 \in S_{t_0}^{DS}$, then $D = cD_0$ with $0 \leq c \leq 1$ also satisfies $D \in S_{t_0}^{DS}$. Moreover, the next theorem is helpful to verify the property $D \in S_{t_0}^{DS}$ in concrete examples; see Corollary III.3.4 of [2] for the proof.

**Theorem 2.2.** Let $(T(t))_{t \geq 0}$ be a strongly continuous semigroup on a Banach space $X$ and let $D \in \mathcal{L}(X, X_{-1})$. If there exist $t_1 > 0$ and $p \in [1, \infty)$ such that

$$\int_0^{t_1} T_{-1}(t-s)Df(s) ds \in X$$

for all functions $f \in L^p([0, t_1], X)$, then $D \in S_{t_0}^{DS}$ for some $t_0 > 0$.

The following result holds for the resolvent of the sum $A_D$; see (iv) of the proof of Theorem III.3.1 in [2].

**Lemma 2.3.** Let $A$ be the generator of a strongly continuous semigroup $(T(t))_{t \geq 0}$ on a Banach space $X$. If $D \in S_{t_0}^{DS}$ for some $t_0 > 0$, then there exists $\lambda^* > 0$ such
that for all $\lambda \geq \lambda^*$, one has $\lambda \in \mathcal{Q}(A) \cap \mathcal{Q}(A_D)$.

\[
\| (\lambda I - A_{-1})^{-1} D \|_{\mathcal{L}(X)} < 1,
\]

and

\[
(\lambda I - A_D)^{-1} = (I - (\lambda I - A_{-1})^{-1} D)^{-1} (\lambda I - A)^{-1}.
\]

Let $t_0 > 0$ and take $D \in S_{t_0}^{DS}$. By Lemma 2.3, the norms on $X$

\[
\| x \|_A = \| (\lambda I - A_{-1})^{-1} x \|, \quad \| x \|_{A_D} = \| (\lambda I - A_D)^{-1} x \|
\]

are equivalent for sufficiently large $\lambda > 0$. This implies that the extrapolation space $X_{-1}^{A_D}$ associated with $A_D$ is isomorphic to $X_{-1}$. In what follows, we identify $X_{-1}^{A_D}$ with $X_{-1}$. For $t \geq 0$, let $(T_D)_t$ be the continuous extension of the operator $T_D(t)$ to the extrapolation space $X_{-1}$. The generator $(A_D)_{-1}$ of the semigroup $((T_D)_t)_{t \geq 0}$ is the unique continuous extension of $A_D : \text{dom}(A_D) \to X$ to $\mathcal{L}(X, X_{-1})$. Since $A_{-1} + D$ also belongs to $\mathcal{L}(X, X_{-1})$, it follows from the definition (2) of $A_D$ that

\[
(A_D)_{-1} x = (A_{-1} + D) x \quad \forall x \in X. \tag{4}
\]

2.3. Solution of abstract evolution equation (1)

Let $B \in \mathcal{L}(U, X_{-1})$, and define the operator $S(t)$ on $U$ by

\[
S(t)u := \int_0^t T_{-1}(s)Buds, \quad u \in U, \ t \geq 0. \tag{5}
\]

For the analysis of the abstract evolution equation (1), we recall the results developed in Lemma 2.2 and its proof of [20], where $X$ and $U$ are Hilbert spaces. The idea of the proof is to use the following standard fact on strongly continuous semigroups:

\[
A_{-1} S(t)u = (T_{-1}(t) - I)Bu
\]
for \( u \in U \) and \( t \geq 0 \). Notice that \( u \) does not depend on the time \( t \). Take \( \lambda \in \mathcal{Q}(A) = \mathcal{Q}(A_{-1}) \) arbitrarily. Then

\[
S(t)u = (I - T(t))(\lambda I - A_{-1})^{-1}Bu + \lambda \int_0^t T(s)(\lambda I - A_{-1})^{-1}Bu \, ds \tag{6}
\]

for all \( u \in U \) and all \( t \geq 0 \). In (5), \((\lambda I - A_{-1})^{-1}B \in \mathcal{L}(U, X)\), that is, the resolvent \((\lambda I - A_{-1})^{-1}\) smoothens and absorbs the unboundedness of \( B \). The above observation is also applicable to the case where \( X \) and \( U \) are Banach spaces, and one can obtain the next lemma.

**Lemma 2.4.** Let \( X \) and \( U \) be Banach spaces and let \((T(t))_{t \geq 0}\) be a strongly continuous semigroup on \( X \). For all \( B \in \mathcal{L}(U, X_{-1}) \) and all \( t \geq 0 \), the operator \( S(t) \) defined by (5) belongs to \( \mathcal{L}(U, X) \). Moreover, \( \lim_{t \downarrow 0} \|S(t)u\| = 0 \) for every \( u \in U \).

Let \( D \in S_{t_0}^{DS} \) for some \( t_0 > 0 \). To solve the abstract evolution equation (1), we define a function \( x \) recursively by

\[
\begin{cases}
  x(0) = x^0, \\
  x(k\tau + t) = T_D(t)x(k\tau) + S_D(t)Fx(k\tau), \quad t \in (0, \tau], \; k \in \mathbb{Z}_+,
\end{cases}
\tag{7}
\]

where

\[
S_D(t)u := \int_0^t (T_D)_{-1}(s)Bu \, ds, \quad u \in U, \; t \geq 0.
\tag{8}
\]

By Lemma 2.4, \( S_D(\tau) \in \mathcal{L}(U, X) \). This implies \( x(k\tau) \in X \), and therefore \( Fx(k\tau) \) is well defined for all \( k \in \mathbb{Z}_+ \). Since

\[
S_D(t_2) - S_D(t_1) = T_D(t_1)S_D(t_2 - t_1)
\]

for every \( t_2 > t_1 \geq 0 \), it follows from Lemma 2.4 that for all \( u \in U \), the function

\[
\xi_{D,u} : \mathbb{R}_+ \to X
\]

\[
: t \mapsto S_D(t)u
\]
is continuous on $\mathbb{R}_+$ with respect to $X$. Hence the function $x$ defined by (7) satisfies $x \in C(\mathbb{R}_+, X)$. Moreover, applying standard results in the semigroup theory to the extended semigroup $((T_{D_t})_{t \geq 0})$, we see that the function $x$ given in (7) satisfies $x|_{[k\tau, (k+1)\tau]} \in C^1([k\tau, (k+1)\tau], X_{-1})$

and the differential equation interpreted in $X_{-1}$

$$\dot{x}(t) = (A_{-1} + D)x(t) + BF(k\tau)$$

for all $t \in (k\tau, (k+1)\tau)$ and $k \in \mathbb{Z}_+$ with initial condition $x(0) = x^0$. Clearly, a function with these properties is unique. Therefore, we say that the function $x$ defined by (7) is the solution of the abstract evolution equation (1).

We conclude this preliminary section with the definition of the exponential stability of the sampled-data system (1).

**Definition 2.5 (Exponential stability).** The sampled-data system (1) with $D \in S_{t_0}$ for some $t_0 > 0$ is exponentially stable with decay rate greater than $\omega \geq 0$ if there exist constants $M \geq 1$ and $\tilde{\omega} > \omega$ such that the solution $x$ given by (7) satisfies

$$\|x(t)\| \leq Me^{-\tilde{\omega}t}\|x^0\| \quad \forall x^0 \in X, \forall t \geq 0.$$

### 3. Robustness analysis of exponential stability

The following theorem is the main result of this paper.

**Theorem 3.1.** Let $X$ and $U$ be Banach spaces and let $A$ be the generator of a strongly continuous semigroup $(T(t))_{t \geq 0}$ on $X$. Suppose that $B \in \mathcal{L}(U, X_{-1})$, $F \in \mathcal{L}(X, U)$, and $D_0 \in S_{t_0}$ for some $t_0 > 0$. If the nominal sampled-data system (1) with $D = 0$ is exponentially stable with decay rate greater than $\omega \geq 0$, then there
exists $c^* > 0$ such that for every $c \in [0, c^*]$, the perturbed sampled-data system (1) with $D = cD_0$ is also exponentially stable with decay rate greater than $\omega$.

In the remainder of this section, we give the proof of Theorem 3.1. Let $D \in S_{t_0}^{DS}$ for some $t_0 > 0$. For $t \geq 0$, define the operators $\Delta(t), \Delta_D(t) \in L(X)$ by

$$\Delta(t) := T(t) + S(t)F, \quad \Delta_D(t) := T_D(t) + S_D(t)F.$$  

Then the solution $x$ defined by (7) satisfies

$$x((k + 1)\tau) = \Delta_D(\tau)x(k\tau) \quad \forall k \in \mathbb{Z}_+. $$

We call $\Delta_D(\tau)$ the closed-loop operator of the discretized system.

Recall that an operator $\Delta \in L(X)$ is said to be power stable if there exist constants $M \geq 1$ and $\theta \in (0, 1)$ such that $\|\Delta^k\|_{L(X)} \leq M\theta^k$ for all $k \in \mathbb{Z}_+$. Lemma 3.2 below connects the exponential stability of the sampled-data system to the power stability of the closed-loop operator $\Delta_D(\tau)$. This result can be obtained by a slightly modification of the proof of Proposition 2.1 in [30].

**Lemma 3.2.** Let $X$ and $U$ be Banach spaces and let $A$ be the generator of a strongly continuous semigroup $(T(t))_{t \geq 0}$ on $X$. Suppose that $B \in L(U, X_{-1})$, $F \in L(X, U)$, and $D \in S^{DS}_{t_0}$ for some $t_0 > 0$. For any $\tau > 0$, the sampled-data system (1) is exponentially stable with decay rate greater than $\omega \geq 0$ if and only if $e^{\omega\tau}\Delta_D(\tau)$ is power stable.

From the result on the stability radius of a discrete-time system developed in Corollary 4.5 of [13], we obtain the following simple result.

**Lemma 3.3.** Let $X$ be a Banach space and let $\kappa > 0$. Suppose that $\Delta_1 \in L(X)$. If $\kappa\Delta_1$ is power stable, then there exists $\varepsilon > 0$ such that $\kappa\Delta_2$ is also power stable for every $\Delta_2 \in L(X)$ satisfying $\|\Delta_2 - \Delta_1\|_{L(X)} < \varepsilon$. 

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By Lemmas 3.2 and 3.3, it is enough to show that \( \| \Delta_D(\tau) - \Delta(\tau) \|_{L(X)} \) is sufficiently small. In what follows, we investigate the difference of the closed-loop operators, \( \Delta_D(\tau) - \Delta(\tau) \), by splitting it into two parts:

\[
\Delta_D(\tau) - \Delta(\tau) = (T_D(\tau) - T(\tau)) + (S_D(\tau) - S(\tau))F.
\]

First we study the difference of the semigroups, \( T_D(\tau) - T(\tau) \).

**Lemma 3.4.** Let \( A \) be the generator of a strongly continuous semigroup \( (T(t))_{t \geq 0} \) on a Banach space \( X \). Let \( 0 \leq c \leq 1 \) and \( D_0 \in S^{\text{pss}}_{t_0} \) for some \( t_0 > 0 \), and define \( D := cD_0 \). Then the strongly continuous semigroup \( (T_D(t))_{t \geq 0} \) generated by the operator \( A_D \) given in Theorem 2.7 satisfies the following two properties for all \( \tau > 0 \):

1. \( \sup_{c \| 0 \leq t \leq \tau} \| T_D(t) \|_{L(X)} : 0 \leq c \leq 1, 0 \leq t \leq \tau < \infty \);
2. \( \lim_{c \downarrow 0} \sup_{0 \leq t \leq \tau} \| T_D(t) - T(t) \|_{L(X)} = 0 \).

**Proof.** Take \( \tau > 0 \). By the strong continuity of \( (T(t))_{t \geq 0} \), there exists \( M \geq 1 \) such that \( \| T(t) \| \leq M \) for all \( t \in [0, \tau] \). Let \( q := \| V_D \|_{L(X_{t_0})} < 1 \). Then \( \| V_D \|_{L(X_{t_0})} = cq \) for \( D = cD_0 \) with \( 0 \leq c \leq 1 \). Since \( T_D(t) = [(I - V_D)^{-1}T](t) \) for \( t \in [0, t_0] \) by the variation of constants formula (3), we obtain

\[
T_D - T = (I - V_D)^{-1} - I = (I - V_D)^{-1}V_DT \text{ on } [0, t_0].
\]

Hence

\[
\| T_D(t) - T(t) \|_{L(X)} \leq \frac{Mcq}{1 - cq} =: q_0 \quad \forall t \in [0, t_0].
\]

Let \( n \in \mathbb{N} \) satisfy \( nt_0 \leq \tau < (n + 1)t_0 \). Suppose that for \( m \in \mathbb{N} \) with \( m \leq n \), \( q_{m-1} > 0 \) satisfies

\[
\| T_D((m - 1)t_0 + t) - T((m - 1)t_0 + t) \|_{L(X)} \leq q_{m-1} \quad \forall t \in [0, t_0].
\]
Since
\[ \| T_D(t) \|_{\mathcal{L}(X)} \leq q_0 + M \quad \forall t \in [0, t_0], \]
we obtain
\[ \| T_D(mt_0 + t) - T(mt_0 + t) \|_{\mathcal{L}(X)} \]
\[ \leq \| T_D(t)T_D(mt_0) - T_D(t)T(mt_0) \|_{\mathcal{L}(X)} + \| T_D(t)T(mt_0) - T(mt_0 + t) \|_{\mathcal{L}(X)} \]
\[ \leq \| T_D(t) \|_{\mathcal{L}(X)} \| T_D(mt_0) - T(mt_0) \|_{\mathcal{L}(X)} + \| T_D(t) - T(t) \|_{\mathcal{L}(X)} \| T(mt_0) \|_{\mathcal{L}(X)} \]
\[ \leq (q_0 + M)q_{m-1} + Mq_0 \quad \forall t \in [0, t_0]. \]

Therefore, if we set \( q_m := (q_0 + M)q_{m-1} + Mq_0 \) for \( m \in \mathbb{N} \), then \( q_{m-1} \leq q_m \) and
\[ \| T_D(t) - T(t) \|_{\mathcal{L}(X)} \leq q_n \quad \forall t \in [0, \tau]. \]

Since
\[ \max_{0 \leq c \leq 1} q_0 \leq \frac{Mq}{1 - q}, \quad \lim_{c \downarrow 0} q_0 = 0, \]
it follows that \( \sup \{ \| T_D(t) \|_{\mathcal{L}(X)} : 0 \leq c \leq 1, 0 \leq t \leq \tau \} < \infty \) and
\[ \sup_{0 \leq t \leq \tau} \| T_D(t) - T(t) \|_{\mathcal{L}(X)} \to 0 \]
as \( c \downarrow 0. \)

Using Lemma 3.4, we next estimate \( \| S_D(\tau) - S(\tau) \|_{\mathcal{L}(U, X)} \).

**Lemma 3.5.** Let \( X \) and \( U \) be Banach spaces and let \( A \) be the generator of a strongly continuous semigroup \((T(t))_{t \geq 0}\) on \( X \). Suppose that \( B \in \mathcal{L}(U, X_{-1}) \) and \( F \in \mathcal{L}(X, U) \). Let \( 0 \leq c \leq 1 \) and \( D_0 \in S_{\text{DS}}^{cD_0} \) for some \( t_0 > 0 \), and define \( D := cD_0 \). Then the operator \( S_D(t) \in \mathcal{L}(U, X) \) defined by (8) satisfies the following two properties for all \( \tau > 0 \):

\[ 13 \]
1. \( \sup \{ \| S_D(t) \|_{\mathcal{L}(U,X)} : 0 \leq c \leq 1, \ 0 \leq t \leq \tau \} < \infty \);  

2. \( \lim_{c \downarrow 0} \sup_{0 \leq t \leq \tau} \| S_D(t) - S(t) \|_{\mathcal{L}(U,X)} = 0. \)

**Proof.** 1. Lemma 2.3 shows that there exists \( \lambda \in \mathfrak{g}(A) = \mathfrak{g}(A^{-1}) \) such that

\[
\alpha := \| (\lambda I - A^{-1})^{-1} D_0 \|_{\mathcal{L}(X)} < 1.
\]

For \( D := cD_0 \) with \( 0 \leq c \leq 1 \), we obtain

\[
\| (\lambda I - A^{-1})^{-1} D \|_{\mathcal{L}(X)} = c\alpha \leq \alpha.
\]  

(9)

From (4), we have that for all \( x \in X \),

\[
(\lambda I - (A_D)^{-1})x = (\lambda I - A^{-1} - D)x
\]

\[
= (\lambda I - A^{-1})(I - (\lambda I - A^{-1})^{-1}D)x.
\]

By (9), the operator \( I - (\lambda I - A^{-1})^{-1}D \) is invertible in \( \mathcal{L}(X) \), and we obtain

\[
\| (I - (\lambda I - A^{-1})^{-1}D)^{-1} \|_{\mathcal{L}(X)} \leq \frac{1}{1 - \alpha}.
\]

Hence, \( \lambda I - (A_D)^{-1} \) is invertible in \( \mathcal{L}(X, X^{-1}) \),

\[
(\lambda I - (A_D)^{-1})^{-1} = (I - (\lambda I - A^{-1})^{-1}D)^{-1}(\lambda I - A^{-1})^{-1},
\]  

(10)

and

\[
\| (\lambda I - (A_D)^{-1})^{-1}B \|_{\mathcal{L}(U,X)} \leq \frac{\| (\lambda I - A^{-1})^{-1}B \|_{\mathcal{L}(U,X)}}{1 - \alpha}.
\]  

(11)

Since \( (A_D)^{-1} \) is the generator of \( ((T_D)^{-1}(t))_{t \geq 0} \), we obtain

\[
(A_D)^{-1}S_D(t)u = ((T_D)^{-1}(t) - I)Bu \quad \forall u \in U, \ \forall t \geq 0.
\]  

(12)
For all \( u \in U \) and all \( t \geq 0 \), (12) yields
\[
(\lambda I - (A_D)_{-1})S_D(t)u = (I - (T_D)_{-1}(t))Bu + \lambda \int_0^t (T_D)_{-1}(s)Bu \, ds
\]
and therefore
\[
S_D(t)u = (I - T_D(t))(\lambda I - (A_D)_{-1})^{-1}Bu + \lambda \int_0^t T_D(s)(\lambda I - (A_D)_{-1})^{-1}Bu \, ds. \tag{13}
\]
Combining this with (11) and Lemma 3.4 yields the first assertion.

2. Substituting \( D = 0 \) into (13), we have that for all \( u \in U \) and all \( t \geq 0 \),
\[
S(t)u = (I - T(t))(\lambda I - A_{-1})^{-1}Bu + \lambda \int_0^t T(s)(\lambda I - A_{-1})^{-1}Bu \, ds.
\]
Hence
\[
S_D(t)u - S(t)u = (I - T_D(t))(\lambda I - (A_D)_{-1})^{-1}Bu - (I - T(t))(\lambda I - A_{-1})^{-1}Bu
\]
\[
+ \lambda \int_0^t T_D(s)(\lambda I - (A_D)_{-1})^{-1}Bu - T(s)(\lambda I - A_{-1})^{-1}Bu \, ds \tag{14}
\]
for all \( u \in U \) and all \( t \geq 0 \). Moreover,
\[
\|T_D(t)(\lambda I - (A_D)_{-1})^{-1}B - T(t)(\lambda I - A_{-1})^{-1}B\|_{L(U,X)}
\]
\[
\leq \|T_D(t) - T(t)\|_{L(X)}\|(\lambda I - (A_D)_{-1})^{-1}B\|_{L(U,X)}
\]
\[
+ \|T(t)\|_{L(X)}\|(\lambda I - (A_D)_{-1})^{-1}B - (\lambda I - A_{-1})^{-1}B\|_{L(U,X)} \tag{15}
\]
for all \( t \geq 0 \). By (10),
\[
(\lambda I - (A_D)_{-1})^{-1} - (\lambda I - A_{-1})^{-1}
\]
\[
= (\lambda I - A_{-1})^{-1}D(I - (\lambda I - A_{-1})^{-1}D)^{-1}((\lambda I - A_{-1})^{-1}.
\]
Since (9) yields
\[
\|(\lambda I - A_{-1})^{-1}D(I - (\lambda I - A_{-1})D)^{-1}\|_{L(X)} \leq \frac{c\alpha}{1 - c\alpha},
\]
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it follows that
\[
\| (\lambda I - (A_D)_{-1})^{-1} B - (\lambda I - A_{-1})^{-1} B \|_{L(U, X)} \leq \frac{c\alpha}{1 - c\alpha} \| (\lambda I - A_{-1})^{-1} B \|_{L(U, X)}. \tag{16}
\]

Take \( \varepsilon > 0 \) and \( \tau > 0 \). Combining Lemma 3.4 with the estimates (11), (15), and (16), we see that there exists \( c^* > 0 \) such that for all \( t \in [0, \tau] \) and all \( D = cD_0 \) with \( 0 \leq c \leq c^* \),
\[
\| T_D(t)(\lambda I - (A_D)_{-1})^{-1} B - T(t)(\lambda I - A_{-1})^{-1} B \|_{L(U, X)} < \varepsilon. \tag{17}
\]
This implies that if \( u \in U \) satisfies \( \| u \|_U \leq 1 \), then for all \( t \in [0, \tau] \) and all \( D = cD_0 \) with \( 0 \leq c \leq c^* \),
\[
\left\| \int_0^t \left( T_D(s)(\lambda I - (A_D)_{-1})^{-1} Bu - T(s)(\lambda I - A_{-1})^{-1} Bu \right) ds \right\| < \tau \varepsilon. \tag{18}
\]
Applying (17) and (18) to (14), we have that for all \( t \in [0, \tau] \) and all \( D = cD_0 \) with \( 0 \leq c \leq c^* \),
\[
\| S_D(t) - S(t) \|_{L(U, X)} < (2 + |\lambda|\tau)\varepsilon.
\]
Since \( \varepsilon \) was arbitrary, the second assertion follows. \( \square \)

We are now ready to prove the main theorem.

**Proof of Theorem 3.7** Since the nominal sampled-data system (1) with \( D = 0 \) is exponentially stable with decay rate greater than \( \omega \geq 0 \) by assumption, it follows from Lemma 3.2 that \( e^{\omega t} \Delta(t) \) is power stable. By Lemma 3.3 there exists \( \varepsilon > 0 \) such that \( \| \Delta_D(t) - \Delta(t) \|_{L(X)} < \varepsilon \) implies the power stability of \( e^{\omega t} \Delta_D(t) \). From Lemmas 3.4 and 3.5 we find \( c^* > 0 \) satisfying \( \| \Delta_D(t) - \Delta(t) \|_{L(X)} < \varepsilon \) for all perturbations in the form \( D = cD_0 \) with \( c \in [0, c^*] \). Finally, using Lemma 3.2 again, we have that for all \( c \in [0, c^*] \), the perturbed sampled-data system (1) with \( D = cD_0 \) is exponentially stable with decay rate greater than \( \omega \). \( \square \)
4. Examples

4.1. Heat equation with boundary perturbation

Consider a metal rod of length one. Let \( z(\xi, t) \) be the temperature of the rod at position \( \xi \in [0, 1] \) and at time \( t \geq 0 \). We control the temperature of the rod by means of a heat flux acting at one boundary \( \xi = 1 \), and the input is generated by a sampled-data state-feedback controller with sampling period \( \tau > 0 \). Moreover, a certain perturbation effect appears at the other boundary \( \xi = 0 \). The system is described by the following partial differential equation:

\[
\begin{align*}
\frac{\partial z}{\partial t}(\xi, t) &= \frac{\partial^2 z}{\partial \xi^2}(\xi, t), \quad 0 \leq \xi \leq 1, \ t \geq 0 \\
z(\xi, 0) &= z^0(\xi), \quad 0 \leq \xi \leq 1 \\
\frac{\partial z}{\partial \xi}(1, t) &= Fz(\cdot, t), \quad k\tau \leq t < (k + 1)\tau, \ k \in \mathbb{Z}^+ \\
\frac{\partial z}{\partial \xi}(0, t) &= Hz(\cdot, t), \quad t \geq 0,
\end{align*}
\]

where \( F \) and \( H \) are functionals on the space of all functions mapping \([0, 1]\) into \( \mathbb{R} \).

First, we transform the heat equation (19) into the abstract evolution equation (1). Let the state space \( X \) and the input space \( U \) be \( X = L^2(0, 1) \) and \( U = \mathbb{C} \). We assume that \( F, H \in \mathcal{L}(X, \mathbb{C}) \). The generator \( A \) is given by

\[ Af = f'' \]

with domain

\[ \text{dom}(A) = \{ f \in L^2(0, 1) : f, f' \text{ are absolutely continuous,} \}
\]

\[ f'' \in L^2(0, 1), \text{ and } f'(0) = f'(1) = 0 \].

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Let \((T(t))_{t \geq 0}\) be the strongly continuous semigroup on \(X\) generated by \(A\). The control operator \(B\) and the perturbation \(D\) are given by

\[
Bu = \delta_1 u, \quad u \in \mathbb{C}
\]

\[
Df = -\delta_0 Hf, \quad f \in X,
\]

where \(\delta_\xi\) is the delta function supported at the point \(\xi \in [0, 1]\); see [31] and [29, Chap. 10] for the derivation of these operators. Since \(\delta_1\) and \(\delta_0\) belong to the extrapolation space \(X_{-1}\), it follows that \(B \in \mathcal{L}(\mathbb{C}, X_{-1})\) and \(D \in \mathcal{L}(X, X_{-1})\).

By the technique explained in Remark 10.1.4 of [29], the perturbation \(D\) is written as

\[
Df = (Hf)(I - A_{-1})\eta \quad \forall f \in X,
\]

where

\[
\eta(\xi) := -\frac{(e^\xi + e^2e^{-\xi})}{e^2 - 1}, \quad 0 \leq \xi \leq 1.
\]

Using this representation of the perturbation \(D\), we see that the perturbed operator \(A_D\) given in Theorem 2.1 satisfies

\[
\text{dom}(A_D) = \{ f \in L^2(0, 1) : f, f' \text{ are absolutely continuous,} \quad f'' \in L^2(0, 1), \quad f'(0) = Hf, \quad \text{and} \quad f'(1) = 0 \}
\]

and \(A_Df = f''\) for all \(f \in \text{dom}(A_D)\).

We shall show that \(D\) is a Desch-Shappacher perturbation. Define \(f_n \in X\) by

\[
f_0(\xi) := 1, \quad f_n(\xi) := \sqrt{2} \cos(n\pi\xi)
\]

for \(\xi \in [0, 1]\) and \(n \in \mathbb{N}\). Then \((f_n)_{n \in \mathbb{N}}\) is an orthonormal basis of \(X\). The generator \(A\) may be written as

\[
Af = -\sum_{n=0}^{\infty} n^2 \pi^2 (f_n, f_n)_{L^2} f_n \quad \forall f \in \text{dom}(A)
\]

(20)
and
\[ \text{dom}(A) = \left\{ f \in L^2(0, 1) : \sum_{n=0}^{\infty} n^4 \pi^4 \langle f, f_n \rangle_{L^2}^2 < \infty \right\}, \]
where \( \langle \cdot, \cdot \rangle_{L^2} \) is an inner product in \( L^2(0, 1) \) defined by
\[ \langle f, g \rangle_{L^2} := \int_0^1 f(\xi) \overline{g(\xi)} d\xi, \quad f, g \in L^2(0, 1). \]

We refer the reader to, e.g., Example 3.2.15 of [32] for this representation of \( A \). A standard application of the Carleson measure criterion developed in [31, 33] yields that \( \delta_0 \) is a finite-time \( L^2 \)-admissible input element for the semigroup \( (T(t))_{t \geq 0} \), i.e., there exists \( t_1 > 0 \) such that
\[ \int_0^{t_1} T^{-1}(t_1 - s) \delta_0 w(s) ds \in X \quad \forall w \in L^2(0, t_1). \]
Hence
\[ \int_0^{t_1} T^{-1}(t_1 - s) D \phi(s) ds \in X \quad \forall \phi \in L^2([0, t_1], X). \]

By Theorem 2.2 \( D \in S_{DS} \) for some \( t_0 > 0 \).

Let the operator \( H \in \mathcal{L}(X, \mathbb{C}) \) be represented as \( H = cH_0 \) with \( 0 \leq c \leq 1 \) and \( H_0 \in \mathcal{L}(X, \mathbb{C}) \). Theorem 3.1 shows that if the feedback operator \( F \in \mathcal{L}(X, \mathbb{C}) \) and the sampling period \( \tau > 0 \) are chosen so that the nominal sampled-data system with \( D = 0 \) is exponentially stable with decay rate greater than \( \omega \geq 0 \), then the perturbed sampled-data system also has the same stability property for all sufficiently small \( c > 0 \).

4.2. Perturbed diagonal system

Let \( \ell^q := \{(x_n)_{n \in \mathbb{Z}_+} \subset \mathbb{C} : \sum_{n=0}^{\infty} |x_n|^q < \infty \} \) for \( q \in [1, \infty) \) and \( \ell^\infty := \{(x_n)_{n \in \mathbb{Z}_+} \subset \mathbb{C} : \sup_{n \in \mathbb{Z}_+} |x_n| < \infty \} \). From the expansion (20) of the generator \( A \), the heat equation (19) can be regarded as a diagonal system. More precisely, the heat
equation (19) may be written as the abstract evolution equation (1) with state space \( X = \ell^2 \) and input space \( U = \mathbb{C} \). The generator \( A \) is a diagonal operator on \( \ell^2 \) given by
\[
(Ax)_{n \in \mathbb{Z}_+} = (-n^2 \pi^2 x_n)_{n \in \mathbb{Z}_+}
\]
with domain
\[
\text{dom}(A) = \{ x = (x_n)_{n \in \mathbb{Z}_+} \in \ell^2 : (n^2 \pi^2 x_n)_{n \in \mathbb{Z}_+} \in \ell^2 \}.
\]

The control operator \( B \) and the perturbation \( D \) are written as
\[
Bu = bu, \quad u \in \mathbb{C}
\]
\[
Df = dHf, \quad f \in \ell^2,
\]
where \( b = (b_n)_{n \in \mathbb{Z}_+} \in \ell^\infty \) and \( d = (d_n)_{n \in \mathbb{Z}_+} \in \ell^\infty \) are given by
\[
b_0 = 1, \quad d_0 = 1, \quad b_n = (-1)^n \sqrt{2}, \quad d_n = -\sqrt{2} \quad \forall n \in \mathbb{N}
\]
and \( H \in \mathcal{L}(\ell^2, \mathbb{C}) \). Here we consider a more general case \( X = \ell^q \) with \( q \in (1, \infty) \). Let \( \kappa, \gamma > 0 \) and \( \zeta \in \mathbb{R} \). Set \( \lambda_n := -\kappa n^\gamma + \zeta \) for \( n \in \mathbb{Z}_+ \). Define a diagonal operator \( A \) on \( \ell^q \) by
\[
(Ax)_{n \in \mathbb{Z}_+} := (\lambda_n x_n)_{n \in \mathbb{Z}_+}
\]
with domain
\[
\text{dom}(A) := \{ x = (x_n)_{n \in \mathbb{Z}_+} \in \ell^q : (\lambda_n x_n)_{n \in \mathbb{Z}_+} \in \ell^q \}.
\]
Let \( b, d \in \ell^\infty \) and \( H \in \mathcal{L}(\ell^q, \mathbb{C}) \). Define
\[
Bu := bu, \quad u \in \mathbb{C}
\]
\[
Df := dHf, \quad f \in \ell^q.
\]
Applying to the diagonal system the Carleson measure criterion developed in Theorem 3.2 of [34], one can obtain the following fact: Suppose that $q \in (1, \infty)$ and $\gamma > 0$ satisfy
\[ q \geq \frac{\gamma + 1}{\gamma}. \] (21)
Define
\[ p := \frac{\gamma q}{\gamma q - 1} \leq q. \]
Then $b$ and $d$ belong to the extrapolation space $X_{-1}$. Moreover, $d$ is a finite-time $L^p$-admissible input element for the semigroup $(T(t))_{t \geq 0}$ generated by $A$, i.e., there exists $t_1 > 0$ such that
\[ \int_0^{t_1} T_{-1}(t_1 - s)dw(s)ds \in X \quad \forall w \in L^p(0, t_1). \] (22)

By Theorem 2.2 and (22), we obtain $D \in S_{t_0}^{DS}$ for some $t_0 > 0$. Thus, Theorem 3.1 can be applied to the perturbed diagonal system as in Section 4.1.

5. Conclusion

We have studied the exponential stability of infinite-dimensional sampled-data systems with unbounded control operators and Desch-Schappacher perturbations. Since the exponential stability of the sampled-data system is equivalent to the power stability of the closed-loop operator of the discretized system, we have investigated the discretized system for the stability analysis of the sampled-data system. To treat the unbounded control operator and perturbation, we have utilized the property that their products with the resolvent of the generator map boundedly into the state space. Future work involves the analysis of the strong stability of perturbed infinite-dimensional sampled-data systems.
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