**ERDŐS-RADO CLASSES**

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**Abstract.** We amalgamate two generalizations of Ramsey’s Theorem–Ramsey classes and the Erdős-Rado Theorem–into the notion of a combinatorial Erdős-Rado class. These classes are closely related to Erdős-Rado classes, which are those from which we can build generalized indiscernibles and blueprints in nonelementary classes, especially Abstract Elementary Classes. We give several examples and some applications.

1. **Introduction**

The motivation for this paper is to amalgamate two distinct generalizations of the classic Ramsey’s Theorem. Ramsey’s Theorem \([\text{Ram30}]\) says that, fixing finite \(n\) and \(c\) in advance, one can find large, finite homogeneous subsets for colorings of \(n\)-sized sets with \(c\) colors, as long as the set originally colored was big enough. In the well-known arrow notation\(^1\), this can be stated as follows.

**Fact 1.1 (Ramsey).** For any finite \(k, n, c\), there is finite \(R\) such that

\[
R \rightarrow (k)^n_c
\]

There are two ways for this to be generalized. The first is to coloring other classes of structures. An important observation is that coloring subsets of a given set is the same as coloring increasing tuples of that length according to some fixed linear order, so Ramsey’s Theorem can be seen as a result about coloring linear orders and finding homogeneous copies of linear orders within it. A Ramsey class \(K_0\) is a class of finite structures where a variant of Ramsey’s Theorem holds: given finite \(k < \omega\) and \(A, B \in K_0\), there is some \(C \in K_0\) such that any coloring of the copies of \(B\) appearing in \(C\) by \(k\) colors gives rise to a copy of \(A\) in \(C\) that is homogeneous for this coloring. This is written as

\[
C \rightarrow (A)^B_k
\]

Independently, Nešetřil and Rödl \([\text{NR77}]\) and Abramson and Harrington \([\text{AH78}]\) showed that the class of finite, linearly ordered \(\tau\)-structures is a Ramsey class when \(\tau\) is a finite relational language. Since then the theory of Ramsey classes has become a productive area connecting combinatorics, dynamics, and model theory (the connection to model theory is partially explained below; a nice survey on Ramsey classes is Bodirsky \([\text{Bod15}]\)).

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\(^1\)The notation

\[
\alpha \rightarrow (\beta)^r_c
\]

means that for any coloring \(c : [\alpha]^r \rightarrow \gamma\), there is \(X \subseteq \alpha\) of type \(\beta\) such that \(c[X]^r\) is a single element; such an \(X\) is called homogeneous. Hajnal and Larson \([\text{HL10}\text{, p. 130}]\) point out “[t]here are cases in mathematical history when a well-chosen notation can enormously enhance the development of a branch of mathematics and a case in point is the ordinary partition symbol.”
In another direction, one might want to remove the restriction ‘finite’ in the statement of Ramsey’s Theorem. Allowing the arity of the coloring (the upper exponent in the arrow relation) to be infinite would make positive results contradict the axiom of choice (see [EHMR84, Theorem 12.1]), so we focus on finite arity colorings. Ramsey’s Theorem can be easily generalized to \( \omega \rightarrow (\omega)^n_c \) for all finite \( n, c \). Moving to infinitely many colors and uncountable homogeneous sets, Erdős and Rado [ER56] proved the following (and, unlike most bounds in finite Ramsey theory, the left-hand cardinal is known to be optimal).

**Fact 1.2** (Erdős-Rado). For any finite \( n \) and infinite \( \kappa \),

\[
\exists_{\kappa-1}(\kappa^+) \to (\kappa^+)_\kappa^n
\]

This has been generalized in many directions, including unbalanced and polarized partition relations. Excellent surveys can be found in Erdős, Hajnal, Máté, and Rado [EHMR84] and Hajnal and Larson [HL10].

We give a general framework for generalizations of the Erdős-Rado Theorem along the lines of Ramsey classes, appropriately called combinatorial Erdős-Rado classes (Definition 3.6, see later in this introduction for a discussion of Erdős-Rado classes). Roughly, a class \( K \) is a combinatorial Erdős-Rado class if it satisfies enough instances of \( \lambda \to (\kappa^+)_{\kappa}^n \), where this means any coloring of \( n \)-tuples from any \( \lambda \)-big structure in \( K \) with \( \kappa \)-many colors has a homogeneous substructure that is \( \kappa^+ \)-big (Section 3 makes these notions of ‘big’ and ‘homogeneous’ precise). Note that we require that all \( n \)-tuples are colored, rather than coloring copies of a single structure as in Ramsey classes. Many partition relations of this sort (positive and negative) already exist in the literature, and we collect the most relevant and place them in this framework in Section 3.1.

Our main interest in these results comes from model theory, specifically building generalized indiscernibles in nonelementary classes. An (order) indiscernible sequence indexed by a linear order \( I \) is a sequence \( \{a_i : i \in I\} \) in a structure \( M \) where the information (specifically, the type) about the elements \( a_{i_1}, \ldots, a_{i_n} \) computed in the structure \( M \) only depends on the ordering of the indices \( i_1, \ldots, i_n \). Generalized indiscernibles replace the linear orders with some other index class: trees, functions spaces, etc. Generalized indiscernibles (and the related notion of generalized blueprints) appear in Shelah [She90], and we recount the definitions in Section 2.

In elementary classes (those axiomatizable in first-order logic), indiscernibles exist because of Ramsey’s Theorem and compactness. Moving to more complicated index classes \( K \), the combinatorics necessary to build generalized indiscernibles from \( K \) are exactly the same as requiring that \( K \) be the directed colimits of a Ramsey class \( K_0 \) (see [Sco12, Theorem 4.31]). In both of these constructions, restriction to finite structures is sufficient to build indiscernibles because the compactness theorem reduces satisfiability to satisfiability of finite sets.

The study of nonelementary classes typically focuses on those axiomatizable in nice logics beyond first-order and, slightly more broadly, on Abstract Elementary Classes. Abstract Elementary Classes (introduced by Shelah [She87]) give an axiomatic framework for a class of structures \( K \) and a strong substructure notion \( \prec_K \) meant to encompass a wide variety of nonelementary classes. A key feature of nonelementary classes is that they lack the structure that the compactness theorem endows on elementary classes. Indeed, Lindström’s Theorem [Lind69] says that no logic stronger than first-order can satisfy the classical (countable) compactness theorem and the downward Löwenheim-Skolem property. In practice, stronger logics tend to fail compactness (the cofinality quantifier logics \( L(Q^\omega_{\omega}) \) are a notable exception [She75, Bonf]). Thus, different methods are necessary to build indiscernibles in Abstract Elementary Classes.

For order indiscernibles, this method comes by way of Morley’s Omitting Types Theorem [Mor65] using the Erdős-Rado Theorem mentioned above (an exposition appears in [Bal09, Chapter 4 and Appendix A]). For generalized indiscernibles, the generalization of the Erdős-Rado Theorem to combinatorial Erdős-Rado classes described above gives the desired tools. We call \( K \) an
Erdős-Rado class if we can build $K$-indiscernibles in any Abstract Elementary Class (Definition 4.1 and Theorem 4.6). Generalized indiscernibles have occasionally seen use in nonelementary classes (for instance, [GS86], [She09, Chapter V.F], [BST12]). The use of structural partition relations allows us to present a unified framework for generating generalized indiscernibles in nonelementary classes. This allows us to generalize Morley’s result as Generalized Morley’s Omitting Types Theorem 4.2. There is also some work in this direction in Shelah [She], and we compare them in Remark 4.8.

We also make explicit category theoretic formulations of (generalized) Ehrenfeucht-Mostowski models and indiscernible collapse. This is motivated by a statement of Morley’s Omitting Types Theorem by Makkai and Paré in their work on accessible categories [MP89, Theorem 3.4.1]. Essentially, generalized blueprints correspond to nice functors, and we prove a converse to this as well (Theorem 5.6).

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1.1. Outline. Section 2 gives the necessary preliminaries on abstract classes of structures, types, and generalized indiscernibles and blueprints. We also include a description in Section 2.3 of the examples we will consider in this paper. Section 3 gives the definition of combinatorial Erdős-Rado classes and of the structural partition relation that defines them. Section 4 gives several known (and a few new) examples and counterexamples of these classes. Section 5 defines Erdős-Rado classes and proves the main link between the two notions, Generalized Morley’s Omitting Types Theorem 4.2. Section 5 describes several extensions and partial converses to this result, including the category theoretic perspective on blueprints. Section 6 gives three applications of this technology: stability spectra of tame AECs, indiscernible collapse in nonelementary classes, and the interpretability order.

Note that the definition of Erdős-Rado classes (Definition 4.1) does not actually depend on that of combinatorial Erdős-Rado classes (Definition 3.6) or any of Section 3. However, we give the combinatorial definitions first, as they provide the largest class of examples of Erdős-Rado classes.

1.2. Conventions. Throughout the paper, we deal with different classes of structures, normally referred to by $K$ in some font with some decoration. To aid the reader, we observe the following convention:

- the script or calligraphic $K$-typeset as $K$—will be used as the domain or index class that we wish to build generalized indiscernibles from. They typically have few assumptions of model-theoretic structure on them. Erdős-Rado classes will be of this type, and the class of linear orders form the prototypical example.
- the bold $K$-typeset as $K$—will be used as the target class that we wish to build generalized indiscernibles in. They will typically be well-structured in some model-theoretic sense. Elementary classes and Abstract Elementary Classes form the prototypical examples.

We also observe two important conventions with respect to types that might be missed by the model-theoretically inclined reader that skips the Preliminaries Section (see Definition 2.2): (1) Since we never deal with types over some parameter set, we omit the domain of types throughout. For example, we write $tp_K(a; I)$ for the $K$-type of $a$ over the empty set computed in $I$, rather than $tp_K(a/0; I)$; and

\footnote{Indiscernibles over a set of parameters can be recovered by adding those parameters to the language. The exception to this rule is Section 6.1, which deals with a specific application to Abstract Elementary Classes, and uses types over parameter sets and other techniques.}
Specifically, we use the notion of Galois types (also called orbital types) used in the study of $K$ or sometimes 'be the same as quantifier-free types. Note that we typically drop a ny adjective and use 'type' Abstract Elementary Classes (and originated in [She87b]). However, in most cases, this will (either in axiomatization of $K$ the symbol with the ambient class. Although we use the 'index class' notation

2. Preliminaries

2.1. Classes of structures and types. We want to have a very general framework for classes of structures in a common language along with a distinguished substructure relation. Although much more general than we need, we can use the notion of an abstract class (this formalization is originally due to Grossberg). Additionally, we expect our Erdős-Rado classes to have orderings (similar to, e.g., [Bod15, Proposition 2.2] for Ramsey classes), so we introduce the notion of an ordered abstract class. An alternative would be to consider equivalence classes of types in the Stone space after modding out by permutation of the indices, but requiring an ordering seems simpler.

Note that the examples presented tend to be universal classes (and mostly relational), in which case the type is determined by the quantifier-free type in $\mathcal{L}_{\omega,\omega}$. However, we offer a more general framework because it adds little technical difficulty and offers the possibility to wider applicability. For instance, well-founded trees (Example 2.12, 3.16) are not a universal class.

Definition 2.1.

1. $(K, \leq_K)$ is an abstract class iff there a language $\tau = \tau(K)$ such that each $M \in K$ is a $\tau$-structure, $\leq_K$ is a partial order contained in $\subseteq_\tau$, and membership in $K$ and $\leq_K$ both respect isomorphism. We often refer to the class simply as $K$.
   a. In an abstract class $(K, \leq_K)$, a $K$-embedding is $\tau$-embedding (an injection that preserves and reflects atomic sentences) $f : M \to N$ between elements of $K$ such that $f(M) \leq_K N$.
   b. $(K, \leq_K)$ is an ordered abstract class iff it is an abstract class with a distinguished binary relation $<_I$ in $\tau(K)$ such that $<_I$ is a total order of $I$ for every $I \in K$.

We will also use the types of elements. Most of the classes we consider will not be elementary (either in axiomatization of $K$ or ordering $\leq_K$), so syntactic types give way to semantic notions. Specifically, we use the notion of Galois types (also called orbital types) used in the study of Abstract Elementary Classes (and originated in [She87b]). However, in most cases, this will be the same as quantifier-free types. Note that we typically drop any adjective and use ‘type’ or sometimes ‘$K$-type’ to refer to the following semantic definition, although we will decorate the symbol with the ambient class. Although we use the ‘index class’ notation $K$ throughout Definition 2.2, we will use these ideas for both index classes and target classes.

Definition 2.2. Let $K$ be an abstract class.

1. Given $I_1, I_2 \in K$ and $a_1 \in I_1$, $a_2 \in I_2$, we say that $a_1$ and $a_2$ have the same $K$-type iff there are $J_1, \ldots, J_n; I_1^*, \ldots, I_{n+1}^* \in K$, $b_\ell \in I_1^*$, and $K$-embeddings $f_\ell : I_{\ell+1}^* \to J_\ell$ such that
   a. $J_1^* = I_1$, $I_{n+1}^* = I_2$, $a_1 = b_1$, and $a_2 = b_{n+1};$
   b. $I_1^* \leq_K J_2^*$; and
   c. $b_\ell = f_\ell(b_{\ell+1}).$

\[ I_1 = I_1^* \quad f_1 \quad I_2 \quad \ldots \quad f_{n-1} \quad I_n \quad f_n \quad I_2 = I_{n+1}^*. \]
We write $tp_\mathcal{K}(a; I)$ to be the equivalence class\(^3\) of all tuples in all structures that have the same type as $a$. Thus, ‘$tp_\mathcal{K}(a_1; I_1) = tp_\mathcal{K}(a_2; I_2)$’ has the same meaning as ‘$a_1$ and $a_2$ have the same type.’

(2) $S_\mathcal{K} := \{tp_\mathcal{K}(a; I) \mid a \in I \in \mathcal{K}\}$ is the Stone space or space of types.

(3) If $\mathcal{K}$ is an ordered abstract class, then $S_\mathcal{K}^{oc}$ is the subset of $S_\mathcal{K}$ whose realizations are in increasing order, namely,

$$S_\mathcal{K}^{oc} := \{tp_\mathcal{K}(a; I) \mid a \in I \in \mathcal{K} \text{ and } a_1 < \cdots < a_n\}$$

(4) Adding a superscript $n < \omega$ to either $S_\mathcal{K}$ or $S_\mathcal{K}^{oc}$ restricts to looking at types of $n$-tuples.

(5) Let $p \in S_\mathcal{K}$ be $tp_\mathcal{K}(i_1, \ldots, i_n; I)$ and $s \subseteq n$ be the set of $k_1 < \cdots < k_m$ for $m = |s|$. Then $p^s := tp_\mathcal{K}(i_{k_1}, \ldots, i_{k_m}; I) \in S_\mathcal{K}^m$.

(6) If we have an ordered abstract class decorated with a superscript $\mathcal{K}^s$, then we often use this superscript in place of the whole class in this notation, e.g., $S_{\mathcal{K}^{s-}}$ rather than $S_{\mathcal{K}^{s-\omega}}$.

(7) For a language $\tau$, we use $\mathcal{K}^\tau$ to be the abstract class of all $\tau$-structures with $\tau$-substructure.

Understanding this notation is key to the rest of the paper, so we unravel these notions in some examples.

First, consider $\mathcal{K}^\tau$. $\mathcal{K}^\tau$-embeddings are injections that preserve and reflect the $\tau$-structure. Then $S_\tau$ refers to all $\mathcal{K}^\tau$-types in this class, and these types turn out to be exactly quantifier-free types in the language.

**Proposition 2.3.** Fix a language $\tau$. Then

1. if $M_0, M_1 \in \mathcal{K}^\tau$ are structures and $a_\ell \in M_\ell$ such that

$$tp_{qf}(a_0; M_0) = tp_{qf}(a_1; M_1)$$

then the $\mathcal{K}^\tau$-types are also equal, in particular witnessed by $\mathcal{K}^\tau$-embeddings $f_\ell : M_\ell \to N$ such that $f_\ell(a_0) = f_1(a_1)$; and

2. $\mathcal{K}^\tau$ has amalgamation.

Note that although we say ‘$\mathcal{K}^\tau$-types are quantifier-free types’, the statement

$$tp_{\tau}(a; M) = tp_{qf}(a; M)$$

is technically false: the left-hand side is a (rank-initial subset of) equivalence class of pairs, while the right hand side is a $|\tau| + \aleph_0$-sized collection of quantifier-free formulas.

**Proof:** For the first item, write down the first order (and quantifier free) theory that is the union of the atomic diagram of $M_0$ and $M_1$ using the same constants for $a_0$ and $a_1$. By compactness, if this is unsatisfiable, then it witnessed by $\phi_1$ satisfied in $M_\ell$ such that

$$\models \phi_0 \rightarrow \neg \phi_1$$

By Craig’s Interpolation Theorem, there is $\psi$ in the common language of $\phi_0$ and $\phi_1$—which is $\tau$ and the constant symbols for $a_\ell$—that interpolates this. But $\psi$ is part of each atomic diagram, so we cannot have $\psi \rightarrow \neg \phi_1$. Thus, this union of atomic diagrams is satisfiable, and that model witnesses the type equality as described.

For the second item, nothing in the above proof use that $a_\ell$ was finite. So given $M_0$ that is a $\tau$-substructure of $M_1$ and $M_2$, this means

$$tp_{qf}(M_0; M_1) = tp_{qf}(M_0; M_2)$$

The same argument above gives an amalgam of these models. \(\dagger\)

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\(^3\)This is a proper class, but we can use Scott’s trick (see [Jec02 p. 65]) or some other method to only deal with sets.
Note that $\mathbb{K}^\tau$ might fail to have the joint embedding property. In fact, this property is equivalent to $\mathbb{K}^\tau$ having a unique 0-type, and is satisfied whenever $\mathbb{K}^\tau$ has no constants (or 0-ary functions).

As a second example, consider the class $\mathcal{K}^{2-\text{or}}$ (which will be used in Example 2.5). This is defined more generally in Example 2.8, but this case consists of two disjoint linear orders $(I_0, I_1)$ where everything in $I_0$ is less than everything in $I_1$. Then $S_{2-\text{or}}$ is the collection of all $\mathbb{K}^{2-\text{or}}$-types. In particular, an increasing tuple $a$ from a model $(I_0, I_1)$ in $\mathbb{K}^{2-\text{or}}$ can be written as $a_0, a_1$, where each $a_k$ is an increasing tuple (possibly empty) from $I_k$. A proof similar to (and simpler than) Proposition 2.3 shows that $\mathbb{K}^{2-\text{or}}$-types are also quantifier free types. Thus, an element $p \in S_{\mathbb{K}^{inc,n}}$ is determined by some number $k \leq n$ that indicates that the tuple is an increasing $k$-tuple from the first linear order, followed by an increasing $(n-k)$-tuple from the second part.

A similar statement is true for $\mathcal{K}^{\chi-\text{or}}$, although a type $p \in S_{\mathbb{K}^{inc,n}}$ is determined by a partition of $n$ into $\chi$-many pieces, most of which are empty when $\chi$ is infinite.

2.2. Generalized indiscernibles and blueprints. The following generalizes the normal theory of blueprints and Ehrenfeucht-Mostowski models begun in [EM56]. These generalized notions appear in [She90] Section VII.2.

**Definition 2.4.** Let $\mathcal{K}$ be an ordered abstract class.

1. A blueprint $\Phi$ proper for $\mathcal{K}$ is a function $\Phi : S^{inc,n}_{\mathcal{K}} \to S_{\tau}$ for some $\tau = \tau(\Phi)$ that satisfies the following coherence conditions:
   a. The free variables of $\Phi(p)$ are the free variables of $p$; and
   b. Given variables $s \subseteq n$ and $p \in S^{inc,n}_{\mathcal{K}}$, we have that
      \[ \Phi(p^s) = \Phi(p)^s \]
      $\mathcal{K}$ is the collection of all blueprints proper for $\mathcal{K}$.
      $\mathcal{K}$ is the collection of all blueprints proper for $\mathcal{K}$ such that $|\tau(\Phi)| \leq \kappa$.

2. Let $I \in \mathcal{K}$ and $\Phi$ be a blueprint proper for $\mathcal{K}$. Then, we can build a $\tau(\Phi)$-structure $EM(I, \Phi)$ such that, for all $i_1 < \cdots < i_n \in I$, we have that
   \[ tp_\tau(i_1, \ldots, i_n; EM(I, \Phi)) = \Phi(tp_\tau(i_1, \ldots, i_n; I)) \]
   and that every element of $EM(I, \Phi)$ is a $\tau(\Phi)$-term of a sequence from $I$.
   If $\tau \subseteq \tau(\Phi)$, then $EM_\tau(I, \Phi) := EM(I, \Phi) \upharpoonright \tau$.

3. Given an abstract class $\mathcal{K}$ of $\tau$-structures and a blueprint $\Phi$ with $\tau \subseteq \tau(\Phi)$, we say that $\Phi$ is proper for $(\mathcal{K}, \mathcal{K})$ if it is proper for $\mathcal{K}$ and, for any $I \in \mathcal{K}$, $EM_\tau(I, \Phi) \in \mathcal{K}$.
   $\mathcal{K}[\mathcal{K}]$ is the collection of all blueprints proper for $(\mathcal{K}, \mathcal{K})$.
   $\mathcal{K}[\mathcal{K}]$ is the collection of all blueprints proper for $(\mathcal{K}, \mathcal{K})$ such that $|\tau(\Phi)| \leq \kappa$.

4. Given an abstract class $\mathcal{K} = (\mathcal{K}, \mathcal{K})$ and a blueprint $\Phi$ with $\tau(\mathcal{K}) \subseteq \tau(\Phi)$, we say that $\Phi$ is proper for $(\mathcal{K}, \mathcal{K})$ if it is proper for $(\mathcal{K}, \mathcal{K})$ and, for any $I \leq J \in \mathcal{K}$, $EM_\tau(\mathcal{K})(I, \Phi) \leq EM_\tau(\mathcal{K})(J, \Phi)$.
   $\mathcal{K}[\mathcal{K}]$ is the collection of all blueprints proper for $(\mathcal{K}, \mathcal{K})$.
   $\mathcal{K}[\mathcal{K}]$ is the collection of all blueprints proper for $(\mathcal{K}, \mathcal{K})$ such that $|\tau(\Phi)| \leq \kappa$.

Being proper for $\mathcal{K}$ is the same as being proper for $(\mathcal{K}, \mathcal{K}[\tau(\Phi)])$. Note that if $\Phi \in \mathcal{K}[\mathcal{K}]$, then the blueprint $\Phi$ actually maps $S^{inc,n}_{\mathcal{K}} \to S_{\mathcal{K}}$. An observant reader might complain that the description in Definition 2.4 uniquely describes a model, but is short on proving it’s existence. However, the existence of such a model follows from standard arguments about EM models, see, e.g., [Mar02] Section 5.2. Our formalism has $I$ be the generating set for $EM(I, \Phi)$ (and later indiscernibles), rather than passing to a skeleton.

From a category-theoretic perspective, a blueprint $\Phi \in \mathcal{K}[\mathcal{K}]$ induces a functor $\Phi : \mathcal{K} \to \mathcal{K}$ that is faithful, preserves directed colimits, and induces a natural transformation between the
‘underlying set’ functor of each concrete category. We return to this perspective in Section 5.2 and derive a converse Theorem 5.2 of the Generalized Morley’s Omitting Types Theorem 4.2.

Example 2.5.

(1) These definitions generalize the standard notions of blueprints and Ehrenfeucht-Mostowski models when $K$ is the class of linear orders.

(2) Consider a bidimensional theory like the theory $T$ of a predicate $P$ that is infinite and coindfinite. Each model is determined by two infinite cardinals, the size of $P$ and the size of its complement. Using standard Ehrenfeucht-Mostowski models, one could only get blueprints that either vary one dimension and not the other or make the dimensions the same.

However, there is a generalized blueprint $\Phi \in \Upsilon_2^{\omega-or}[\text{Mod}(T)]$ for the class of two disjoint linear orders that takes $(I, J)$ to the model $M_{(I, J)}$ with universe $(I + \omega) \cup (J + \omega)$ and predicate $P_{M_{(I, J)}}(I, J) := I + \omega$. Thus, every model of $T$ is isomorphic to $E_{M_{(I, J)}}((I, J, \Phi))$ for some $I$ and $J$.

Using generalized blueprints, we can build models with generalized indiscernibles (see Theorem 4.6 for this in action).

Definition 2.6. Let $K$ an ordered abstract class and $\mathbb{K}$ be an abstract class. Then, given $I \in K$ and $M \in \mathbb{K}$, a collection $\{a_i \in \mathbb{K} \mid i \in I\}$ is a $K$-indiscernible sequence iff for every $i_1, \ldots, i_n; j_1, \ldots, j_n \in I$, if

$$tp_K(i_1, \ldots, i_n; I) = tp_K(j_1, \ldots, j_n; I)$$

then

$$tp_K(a_{i_1}, \ldots, a_{i_n}; M) = tp_K(a_{j_1}, \ldots, a_{j_n}; M)$$

An important fact to keep in mind is that, in nonelementary classes, not every collection of indiscernibles can be turned into a blueprint; [Bal09, Example 18.9] provides such an example. This is in contrast to first-order, where every infinite set of indiscernibles can be stretched (see [TZ12, Lemma 5.1.3]).

2.3. Our examples. There will be several examples that we will develop here and in Section 3.1. Here, we define the relevant classes and note the syntactic characterization of their types (normally quantifier-free). Section 3.1 explains how these classes fit within the framework of Erdős-Rado classes. In each case, the strong substructure relation is just substructure for the appropriate language unless noted otherwise.

Example 2.7 (Linear orders). $K^{or}$ is the class of linear orders in the language with a single binary relation $\prec$. This is an ordered abstract class and is universal, so $K^{or}$-type is simply quantifier-free type. This is our prototypical Erdős-Rado class.

Example 2.8 ($\chi$ disjoint linear orders). $K^{\chi-or}$ is the class of $\chi$ disjoint linear orders. In order to make this an ordered abstract class, we say $I \in K^{\chi-or}$ consists of disjoint sets $\{I_i\}_{i<\chi}$ and a total ordering $\prec$ such that $i < j < \chi$ implies that $I_i \ll I_j$ ($X \ll Y$ means that every element of $X$ is below every element of $Y$). Note that if $\chi$ is infinite, then this is not an elementary class.

4A more mathematically complex example of a bidimensional theory is the theory $Th(\oplus \mathbb{Z}(p^\infty))$ of the direct sum of countably many copies of the Prüfer $p$-group. The same analysis applies there.

5$\Upsilon^{\omega-or}$ is $\Upsilon^{\omega-or}$, and this class is consists of two disjoint linear orders; see Example 2.8.
Example 2.9 (χ-colored linear orders). We set $K^{\chi-color}$ to be a particular class of colored linear orders. $(I, <, P_\beta)_{\beta<\chi} \in K^{\chi-color}$ consists of a well-ordering $(I, <)$ such that $P_\beta = \{i \in I : i$ is the $(\chi \cdot \gamma + \beta)$th element of $I$ for some $\gamma\}$.

Example 2.10 (Trees of height $n < \omega$). Fix the language $\tau_{n-tr} = (P_n, <, \cdot, \wedge)_{n<\omega}$. Then $K^{n-tr}$ consists of all $\tau_{n-tr}$-structures $I$ such that

- $(I, <)$ is a tree of height $n$;
- $P_n$ are all vertices on level $n$;
- $<$ is a total order of $I$ coming from a lexicographic ordering of the tree; and
- $\wedge$ is the meet operation on this tree.

Then $K^{n-tr}$-type is just quantifier-free type in this language.

Example 2.11 (Trees of height $\omega$). $K^{\omega-tr}$ are the trees of height $\omega$ formalized in the language $\tau_{\omega-tr} = \cup_{n<\omega} \tau_{n-tr}$.

Of course, these tree examples can be continued on past height $\omega$, but we know of no results (positive or negative) on these classes in terms of the Erdős-Rado notions.

Example 2.12 (Well-founded trees). $K^{wf-tr}$ are the well-founded trees formalized in the language $\tau_{\omega-tr}$; recall a tree is well-founded iff it contains no infinite branch. Given any ordinal $\alpha$, we can build a well-founded tree whose nodes are decreasing sequences of ordinals starting with $\alpha$.

Going the other way, if $T$ is a well-founded tree, then we can relabel the nodes with ordinals such that each path is a decreasing sequence. Note that this relabeling can be done in many different ways and it probably disagrees with the lexicographical ordering on successors in $\tau_{\omega-tr}$.

Example 2.13 (Convexly-ordered equivalence relations). A convexly ordered equivalence relation is $(I, <, E)$, where $E$ is an equivalence relation on $I$, $<$ is a total order, and

$$\forall x, y, z \in I (xEz \land x < y < z \rightarrow xEy)$$

$K^{coe}$ is the collection of all such structures. These are similar to the class $K^{\chi-or}$ except the $\chi$ is allowed to vary. However, the type of, e.g., singletons in different equivalence classes is the same. This will make finding type homogeneous sets for colorings more difficult.

Example 2.14 (n-multi-linear orders). A $n$-multi-linear order is $(I, <_1, \ldots, <_n)$ where each $<_i$ is a linear order of $I$. $K^{n-mlo}$ is the class of these. We take $<_1$ as the distinguished linear order to view this as an ordered abstract class.

Example 2.15 (Ordered graphs). $K^{og}$ consists of the class of all ordered graphs.

Example 2.16 (Colored hypergraphs). Fix $k \leq \omega$ and a cardinal $\sigma$. $K^{(k, \sigma)-hg}$ consists of all $(I, \sigma; <, F, \alpha)_{\alpha<\sigma}$ where $<$ is a linear ordering and $F : |I|^<k \rightarrow \sigma$ is a function. If $\sigma = 2$, then one can think of $K^{(\sigma, 2)-hg}$ as the collection of all hypergraphs with all edge arities $< k$.

3. Structural Partition Relations and Combinatorial Erdős-Rado Classes

We will formulate a version of the normal partition relation for classes other than linear orders in Definition 3.5. This will encapsulate the idea that any coloring of $n$-tuples from a large structure will have a large substructure that behaves the same way to this coloring. First, we consider an example that indicates some of the difficulties and the need for new concepts, namely bigness notions (Definition 3.2) and type-homogeneity (Definition 3.4).

Example 3.1. Let $I = (I_0, I_1) \in K^{2-color}$ (recall Example 2.8). Define a coloring $c : |I|^2 \rightarrow 2$ based on whether the two elements are in the same partition: given $i, j \in (I_0, I_1)$, set

$$c(\{i, j\}) = \begin{cases} 0 & i \in I_0 \iff j \in I_1 \\ 1 & \text{otherwise} \end{cases}$$
Then any $I^* \subseteq \bar{I}$ that contains at least one element from one partition and two from the other will not be homogeneous for this coloring no matter what $\bar{I}$ is. Alternatively, any $I^* \subseteq \bar{I}$ that contains elements from just one partition will always be homogeneous.

This example exposes two issues.

• First, we could take $\bar{I}^*$ to be $(\emptyset, I_1)$, which is homogeneous for this coloring. However, taking one of the partitions to be empty goes against the point of working in $\mathcal{K}^{2-\text{or}}$. So we will attach to these classes a notion of size (or bigness) that takes the structure of the class into account.

• Second, we colored the pairs using information about their type. This meant that we could place restrictions on the structure of any homogeneous subset. To allow for big homogeneous sets we will allow for the ‘single color’ to depend on the type of tuple.

For the first issue, we define abstractly what it means to be a bigness notion. The only requirements are a monotonicity condition and some weak degree of universality. For each class from Subsection 2.3, we make its associated bigness notion explicit in Subsection 3.1. Many natural bigness notions correspond to a degrees of universality after removing the order, but other cases are more complex (see Examples 3.12 or 3.16).

Definition 3.2. Let $\mathcal{K}$ be an abstract class. A bigness notion $\textbf{big}$ for $\mathcal{K}$ is a class $\{K_{\mu}^{\textbf{big}} \subseteq \mathcal{K} \mid \mu \in \text{Card}\}$ such that

1. each $K_{\mu}^{\textbf{big}}$ is nonempty;
2. if $\mu_1 \leq \mu_2$ and $M \leq_{\mathcal{K}} N$, then $M \in K_{\mu_2}^{\textbf{big}}$ implies that $N \in K_{\mu_2}^{\textbf{big}}$; and
3. if $M \in K_{\mu_0}^{\textbf{big}}$, then every type in $S_{\mathcal{K}}$ is realized in $M$.

We write ‘$M \in \mathcal{K}$ is $\mu$-big’ for ‘$M \in K_{\mu}^{\textbf{big}}$’. Also, we will typically only have one one bigness notion for a given class, so we will omit it.

Note that the omission of $\textbf{big}$ will lead to some nonstandard notation, e.g., $K_{\mu}^{\chi-\text{or}}$ are the $\mu$-big elements of $K_{\chi-\text{or}}$ according to the bigness notion given in Example 3.8 rather than all elements of $K_{\chi-\text{or}}$ whose universe has cardinality $\mu$.

Remark 3.3. Note that the existence of a bigness notion for $\mathcal{K}$ (or just a model satisfying the conclusion of Definition 3.2 (3)) implies that there is a unique 0-type realized by any model in $\mathcal{K}$. This is because each model realizes a single 0-type, and so every model must realize the 0-type realized in any $\aleph_0$-big model.

In most examples below, there are no constants, so the 0-type does not contain information about any ‘prime structure.’ However, the semantic definition of types used here (recall Definition 2.2) means that the uniqueness of the 0-type is equivalent to a natural notion of connectedness of the category $\mathcal{K}$; this connectedness is a reflection of the completeness of a first-order theory in elementary classes.

Turning to homogeneity, the key observation from Example 3.1 was that the types of tuples are extra information that can be used to define a coloring. In the class of linear orders, there is only one increasing type of an $n$-tuple, so this issue doesn’t arise. In the general case, we can always use the type as information to color a tuple, so we want homogeneity to mean that the type is the only information that can be used to determine the color of a tuple.

Definition 3.4. Let $\mathcal{K}$ be an ordered abstract class, $I \in \mathcal{K}$, and $c : [I]^n \rightarrow \kappa$. We say that $I_0 \leq_{\mathcal{K}} I$ is type-homogeneous for $c$ iff the color of a tuple from $I$ is determined by the $K$-type of it listed in increasing order; that is, there is a function $c^* : S_{\mathcal{K}}^{\text{nc},n} \rightarrow \kappa$ such that, for any distinct $i_1 < \cdots < i_n \in I_0$, we have that

$$c(\{i_1, \ldots, i_n\}) = c^*(tp_{\mathcal{K}}(i_1, \ldots, i_n; I))$$
In Example 3.1, the entire set \( \bar{I} \) is type-homogeneous for the given coloring.

With these new concepts in hand, we can define the structural partition relation.

**Definition 3.5.** Let \( K \) be an ordered abstract class with a bigness notion \( \text{big} \). Given cardinal \( \mu, \lambda, \alpha, \kappa \), we write

\[
(\lambda) \xrightarrow{\kappa}{\text{big}} (\mu)^\alpha_\kappa
\]

to mean that given any \( \lambda \)-\( \text{big} \) \( I \in K \) and coloring \( c : [I]^\alpha \to \kappa \), there is a \( \mu \)-\( \text{big} \) \( I_0 \leq_K I \) from \( K \) that is type-homogeneous for \( c \).

If \( K \) is one of our examples with an associated bigness notion and is denoted \( K^x \), then we simply write

\[
(\lambda) \xrightarrow{x}{(\mu)^\alpha_\kappa}
\]

for \( (\lambda) \xrightarrow{\kappa}{\text{big}} (\mu)^\alpha_\kappa \).

Since the associated bigness notion for \( K^x \) is simply cardinality, \( (\lambda) \xrightarrow{\text{or}} (\mu)^\alpha_\kappa \) is the normal partition relation. In particular, positive instances of the structural partition relation are guaranteed by the Erdős-Rado Theorem, which states that \( \beth_{\alpha-1}(\kappa) \xrightarrow{\alpha} (\kappa^+)_{\kappa}^\alpha \) for every cardinal \( \kappa \) and every \( n < \omega \). We only consider this relation with \( \alpha \) finite.

Polarized partition relations (see [EHMR84, Section III.8.7]) are similar to \( \chi \xrightarrow{\text{or}} \), but typically specify (in our language) the type of the tuple to be considered (and so are more like the Ramsey class-style partition relations, although we use them in Remark 3.10).

We have focused on the difference between the Ramsey-style structural partition relation (that fixes the type of the tuples colored) and the Erdős-Rado-style structural partition relation (that colors all tuples of a fixed length). These differ on a case-by-case basis, but the referee points out that if there are finitely many \( n \)-types in the class, then enough Ramsey structural can be strung together to get a type homogeneous coloring. That is, if \( K \) is a Ramsey class, then for each \( n, k < \omega \) and \( A \in K \), there is a \( C^* \in K \) such that any coloring of \( n \)-tuples of \( C^* \) with \( k \) colors has a type-homogeneous copy of \( A \). We also use a version of this argument in Remark 3.10.

We will list several further positive instances of structural partition relations (new and old) in Subsection 3.1.

From the structural partition relation, we can define combinatorial Erdős-Rado classes as those that satisfy structural partition relations for all inputs on the right side.

**Definition 3.6.** Let \( K \) be an ordered abstract class with a bigness notion \( \text{big} \). We say that \( K \) is a combinatorial Erdős-Rado class iff there is some function \( F : \text{Card} \times \omega \to \text{Card} \) such that, for every \( \kappa < \mu \) and \( n < \omega \), we have that

\[
(F(\mu, n)) \xrightarrow{\kappa}{\text{big}} (\mu)^n_\kappa
\]

We refer to the function \( F \) as a witness.

**3.1. Examples and some counter-examples.** We show that the examples introduced in Section 2.3 are combinatorial Erdős-Rado classes or mention results indicating they are not. In most cases, no claim of the optimality of the witnessing functions is made. While interesting from a combinatorial perspective, any reduction of the bounds on the order of ‘finitely many power set operations’ will not affect the witnesses for these classes being Erdős-Rado via an application of the Generalized Morley’s Omitting Types Theorem 1.2. Note that none of these results were originally stated in the notation of Definition 3.5 (especially since that notation was originated for this paper); however, we have translated those results into this language to illustrate our notions.
Example 3.7 (Linear orders). In $K^\text{or}$, the canonical bigness notion is just cardinality, so $I \in K^\text{or}_\mu$ iff $|I| \geq \mu$. The classic Erdős-Rado [ER56] theorem states that, for all $n < \omega$ and $\kappa$,
\[
\exists_{n-1}(\kappa^+) \overset{\text{or}}{\rightarrow} (\kappa^+)^\kappa_n
\]
Thus, $K^\text{or}$ is a combinatorial Erdős-Rado class witnessed by $(\kappa^+, n) \mapsto \exists_{n-1}(\kappa^+)$ and $(\delta, n) \mapsto \exists_{n-1}(\delta^+)$ for limit $\delta$.

Note that the classic results on the Sierpinski coloring [aS33] show that dense linear orders do not form a combinatorial Erdős-Rado class. One could work to develop an infinite version of Ramsey degree, but we defer that for later.

Example 3.8 (\(\chi\)-disjoint linear orders). As discussed in the context of Example 3.1, the canonical bigness notion for $K^{\chi\text{-or}}$ says that $I$ is $\mu$-big iff every piece has size at least $\mu$. Proposition 3.4 (in which the heavy lifting is done by Shelah [She90] Appendix, Theorem 2.7], see Remark 3.11) proves, for all $n < \omega$ and $\chi \leq \kappa$,
\[
\exists_{n(n+2)}(\kappa^+) \overset{\chi\text{-or}}{\rightarrow} (\kappa^+)^\kappa_n
\]
Thus, $K^{\chi\text{-or}}$ is a combinatorial Erdős-Rado class witnessed by $(\kappa^+, n) \mapsto \exists_{n(n+2)}(\kappa^+)$ (and so the threshold for limit $\kappa$ are the same as for $\kappa^+$).

Proposition 3.9. For all $n < \omega$ and $\chi \leq \kappa$,
\[
\exists_{n(n+1)} (\exists_{n-1}(\kappa^+)) \overset{\chi\text{-or}}{\rightarrow} (\kappa^+)^\kappa_n
\]
As alluded to above, essentially proves this proposition except that Shelah’s result had a lower cardinal on the left-hand side and assumed each piece to be well-ordered (rather than just linearly ordered); thanks again to the referee for catching this mistake. We raise the cardinal on the left-hand side to make up for this lack of well-ordering. The author believes that the theorem could probably be proven with Shelah’s original bound (and similarly for the trees in Example 3.13), but since it doesn’t affect the final bound for being an Erdős-Rado Class, we don’t pursue that here.

Proof: Let $I = (I_i; <_I)_{i < \chi} \in K^{\chi\text{-or}}$ be $\lambda$-big for $\lambda = \exists_{n(n+1)} (\exists_{n-1}(\kappa^+))$ and fix a coloring
\[
c : \left( \bigcup_{i < \chi} I_i \right) \rightarrow \kappa
\]
WLOG, each $I_i$ has size exactly $\mu$. In order to invoke Shelah’s result, the $I_i$’s must be well-ordered, so fix an ordering $<_i$ of $\bigcup_{i < \chi} I_i$ that well-orders each piece and satisfies $i < j$ implies $I_i <^*_i I_j$.

Now we can apply Shelah’s result [She90] Appendix, Theorem 2.7] to the structure $I^* = (I_i; <^*_i)_{i < \chi}$ with the coloring $c$ to get, for each $i < \chi$, $J_i \subseteq I_i$ of size $\exists_{n-1}(\kappa^+)$ such that $(J_i, <^*_i)_{i < \chi}$ is type-homogeneous for the coloring $c$. However, the $(J_i, <^*_i)_{i < \chi}$-type of a tuple is not the same as the $(J_i, <)_i <_{\chi}$-type of a tuple, and we need the type-homogeneity to be in the context of the latter.

To solve this, for each $i < \chi$, color each $n$-sized set from $J_i$ with the $(J_i, <^*_i)$-type of the set listed in increasing $<_j$-order; this is essentially a permutation of $n$ elements, so there are at most

\[^2\text{For a more accessible reference, see [Dob22] After Question 1.2].}\]

\[^3\text{By inspection, Proposition 3.9 gives a stronger result, but we use this weakening for easier comparison with other structural partition relations since}\]
\[
\exists_{n(n+1)} (\exists_{n-1}(\kappa^+)) \leq \exists_{n(n+1)} (\exists_n(\kappa^+)) = \exists_{n(n+2)}(\kappa^+)
\]
Proposition 3.12. \( \kappa \) is\n
Thus, \( \hat{\mathcal{K}} \) is an \( \kappa \)-large type-homogeneous

Remark 3.10. Shelah [She90] Historical Remarks, A.2] cites Erdős, Hajnal, and Rado [EHR65] (without further specification) for the partition relation cited as [She90] Appendix, Theorem 2.7. However, the author and referee were unable to locate the actual result there. The closest is [EHR65, Corollary 17, p. 176], which states

\[
\left( \kappa^+ \right) \twoheadrightarrow \left( \kappa, \kappa \right)_{\ell + 1}^2
\]

Unwinding this notation (see [EHR65, Section 3.3]), this means that if \( X_0 \) and \( X_1 \) are of size \( \kappa^+ \) and

\[
c : X_0 \times X_1 \to 2
\]

is a coloring, then there is \( I_1 \subseteq I_\ell \) of size \( \kappa \) and a color \( k < 2 \) such that \( c \) is constant \( k \)-valued on \( X_0 \times X_1 \). This can be combined with the classic Erdős-Rado Theorem \( \square_1(\kappa^+) \to (\kappa^+)^2 \) to prove the following partition relation restricted to well-ordered elements of the class

\[
\square_1(\kappa^+) \twoheadrightarrow (2-\text{or}) (\kappa)_{\ell + 1}^2
\]

Given a coloring \( c : [I_0 \cup I_1]^2 \to 2 \), first use Erdős-Rado to find \( I_1 \subseteq I_\ell \) of size \( \kappa^+ \) such that the partition relation \( [I_0]^2 \) is constant. Then use the polarized partition relation to find \( I_1'' \subseteq I_1'' \) of size \( \kappa \) such that the coloring restricted to \( I_0'' \times I_1'' \) is constant. Since the three types of \( S^2_{\text{or}} \) are represented by \( [I_0]^2 \), \( I_0'' \times I_0'' \), and \( [I_1]^2 \), the coloring is type homogeneous.

If [EHR65] investigated cases of the polarized with more colors or greater arity, this would similarly prove the result. However, they restrict their attention to the \( 2 \times 2 \) case above with two colors. Note that we have pieced together several results that color each type to get a type-homogeneous coloring; this technique can be used when there are finitely many types of the appropriate arity.

Example 3.11 (\( \chi \)-colored linear orders). The canonical bigness notion says that \( (I, <, P_{\beta})_{\beta < \chi} \) is \( \kappa \)-big iff \( \chi \cdot \kappa \leq \text{otp}(I) \). Then \( K_{\chi-\text{color}} \) is a combinatorial Erdős-Rado class by Proposition 3.7.3.

Proposition 3.12. If \( \lambda \overset{\text{or}}{\longrightarrow} (\kappa)_{\mu^*}^n \), where \( \mu^* = \mu(\chi^n) \), then \( \lambda \overset{\chi-\text{color}}{\longrightarrow} (\kappa)_{\mu^*}^n \).

Proof: Given a coloring \( c : [\chi : \lambda]^n \to \mu \), we define an auxiliary coloring \( d : [\lambda]^n \to (\chi^\mu) \) given by \( d(\gamma_1, \ldots, \gamma_n) = \chi^\text{c} \cdot (\chi \cdot \gamma + i_1, \ldots, \chi \cdot \gamma + i_n) \). There is a \( \kappa \)-sized homogeneous \( X \subseteq \lambda \) by assumption. Enumerate an initial segment as \( \{\gamma_\alpha \mid \alpha < \kappa \} \) and define the set

\[
\chi \otimes X := \{ \chi \cdot \gamma(\chi \cdot \beta + i) + i : \chi \cdot \beta + i < \kappa \text{ and } i < \chi \}
\]

We claim that this structure (with the induced \( \chi \)-or structure) is type homogeneous for \( c \).

Suppose that \( (\chi \cdot \gamma(\chi \cdot \beta + i_1) + i_1, \ldots, \chi \cdot \gamma(\chi \cdot \beta + i_n) + i_n) \in \chi \otimes X \) have the same \( \chi \)-or type. First, this means that \( P_{\beta} \) holds of \( \chi \cdot \gamma(\chi \cdot \beta + i_1) + i_1 \) if it holds of \( \chi \cdot \gamma(\chi \cdot \beta' + i') + i' \); thus \( i_\ell = i_\ell' \) for all \( \ell \leq n \). Second, for ordinals \( \gamma \neq \gamma' \) and \( \beta, \beta' < \chi \), we have that

\[\chi \cdot \gamma + \beta < \chi \cdot \gamma' + \beta' \iff \gamma < \gamma'.\]

\[\text{Importantly, this fails if we allow } \gamma = \gamma'. \text{ This is the reason for the complicated subscript in the definition of } X.\]
Thus, we have that for each \( \ell, k < n \),
\[
\gamma(\chi \beta_{\ell+1}) < \gamma(\chi \beta_k) \iff \chi \cdot \gamma(\chi \beta_{\ell+1}) + i_\ell < \chi \cdot \gamma(\chi \beta_k) + i_k
\]
\[
\iff \chi \cdot \gamma(\chi \beta_{\ell+1}^i) + i_\ell^i < \chi \cdot \gamma(\chi \beta_k^i) + i_k^i
\]
\[
\iff \gamma(\chi \beta_{\ell+1}^i) < \gamma(\chi \beta_k^i)
\]

Thus, \((\gamma(\chi \beta_{\ell+1}), \ldots, \gamma(\chi \beta_k)) \in [\mathcal{X}]^n\) have the same \(or\)-type and \(d\) of them is the same function. Thus, we can conclude
\[
\begin{align*}
\prod (\chi \cdot \gamma(\chi \beta_{\ell+1}) + i_1, \ldots, \chi \cdot \gamma(\chi \beta_k) + i_n) &= d(\gamma(\chi \beta_{\ell+1}), \ldots, \gamma(\chi \beta_k)) (i_1, \ldots, i_n) \\
&= d(\gamma(\chi \beta_{\ell+1}^i), \ldots, \gamma(\chi \beta_k^i)) (i_1^i, \ldots, i_n^i) \\
&= d(\gamma(\chi \beta_{\ell+1}^i), \ldots, \gamma(\chi \beta_k^i)) (i_1^i, \ldots, i_n^i) \\
&= c(\chi \cdot \gamma(\chi \beta_{\ell+1}^i) + i_1^i, \ldots, \chi \cdot \gamma(\chi \beta_k^i) + i_n^i)
\end{align*}
\]

Example 3.13 (Trees of height \(n < \omega\)). The canonical bigness notion for \(\mathcal{K}^{n-tr}\) is that of splitting: \(I \in \mathcal{K}^{n-tr}\) iff every node of the tree on level \(n < \omega\) has \(\geq \mu\)-many successors. As with Example 3.8, Shelah proved a version of the partition relation for trees built on \(\leq \omega^\lambda\) (so assuming some nice well-ordering) with a rich history\(\dagger\). By applying a similar argument, Proposition 3.14 shows
\[
\Downarrow_{k(n,m)+\mu}^n (\kappa^+)^{k(n,m)+\mu} \rightarrow (\kappa^+)^{n}^m
\]
Thus, \(\mathcal{K}^{n-tr}\) is a combinatorial Erdős-Rado class witnessed by \((\kappa^+, n) \rightarrow \Downarrow_{k(n,m)}(\kappa^+)\).

Proposition 3.14. For all \(n, m < \omega\) and \(\kappa\)
\[
\Downarrow_{k(n,m)}(\Downarrow_{m-1}(\kappa^+)^{k(n,m)})^{n} \rightarrow (\kappa^+)_{\kappa}^{m}
\]
where \(k(n,m) < \omega\) is a function as in [She97, Appendix, Theorem 2.6].

Proof: The proof follows the same strategy as Proposition 3.9. First, we take an \(n\)-tree \(T\) and a coloring \(c\). Then the global order is redone (and the tree shrank) so that the tree is a copy of \(\Downarrow_{k(n,m)}(\Downarrow_{m-1}(\kappa^+)^{k(n,m)})^{n} \rightarrow (\kappa^+)_{\kappa}^{m}\). Shelah’s result [She97, Appendix, Theorem 2.6] is applied to get \(T^* \subseteq T\) that is \(\Downarrow_{m-1}(\kappa^+)^{k(n,m)}\)-large and type-homogeneous for \(c\) for types with the redone order.

In order to make the types according to the two orders agree, we use the same coloring as in Proposition 3.9 but work level-by-level, starting at the level after the root. That is, using Erdős-Rado, we make sure that the orderings on \(T^*_n\) agree; then there are \(\kappa^+\)-many nodes on level 1, but each has \(\Downarrow_{m-1}(\kappa^+)^{k(n,m)}\)-many successors on level 2. Then the Erdős-Rado theorem is applied in turn on the remaining successor sets at each level, leaving with a \(\kappa^+\)-large, type homogeneous tree.

Example 3.15 (Trees of height \(\omega\)). We do not know if \(\mathcal{K}^{\omega-tr}\) is a combinatorial Erdős-Rado class (although we would expect the bigness notion to be splitting). However, we are still able to show that is an Erdős-Rado class (see Corollary 5.13).
Example 3.16 (Well-founded trees). We say that a well-founded tree is $\lambda$-big if it contains a copy of $\kappa^\lambda$, which is the well-founded tree that is made up of all decreasing sequences of ordinals less than $\lambda$ ordered by end extension. Then [GST] Conclusion 2.4] shows that, for every $n < \omega$ and $\kappa$,

$$\Downarrow_{1,n}(\kappa) \xrightarrow{wf-tr} (\kappa)^n_{\kappa}$$

where $\Downarrow_{1,n}(\kappa)$ is defined by:

- $\Downarrow_{1,0}(\kappa) = \lambda$ and
- $\Downarrow_{1,k+1}(\kappa) = \Downarrow_{1,k}(\kappa) + (\lambda)$

Note that the bound here is much larger than the other bounds (which are all below $\Downarrow_{\omega}(\kappa)$). For instance,

$$\Downarrow_{1,2}(\kappa) = \Downarrow_{\omega}(\kappa)^{+(\kappa)}$$

This impacts the witness for being an Erdős-Rado class (see Remark 4.4], but we do not know if the left-hand side here is a tight bound. Also, well-founded trees are closely related to scattered linear orders (those not containing a copy of $\mathbb{Q}$; see [GST] Observation 4] building on work of Hausdorff), so this result forms a counterpoint to the nonexample coming from Sierpinski colorings.

$\mathcal{K}^{wf-tr}$ is also important as an Erdős-Rado Class that is not axiomatizable in $\mathbb{L}_{\omega,\omega}$ and does not even form an Abstract Elementary Class because it is not closed under unions of chains. Worse, the class of well-founded trees could not even be the models of such a class (ignoring the note that (Proposition 3.18. Given infinite $\kappa$ and $n < \omega$, we have

$$\Downarrow_{n}(\kappa)^{+}_{\kappa} \xrightarrow{ccq} (\kappa^{+})^n_{\kappa}$$

Proof: Let $(I, <, E) \in \mathcal{K}^{ccq}$ is $\mu$-big if there are at least $\mu$-many equivalence classes, each of which is of size at least $\mu$. Proposition 3.18 below shows that $\mathcal{K}^{ccq}$ is a combinatorial Erdős-Rado class.

**Example 3.17 (Convexly-ordered equivalence relations).** The canonical bigness notion says that $(I, <, E) \in \mathcal{K}^{ccq}$ is $\mu$-big if there are at least $\mu$-many equivalence classes, each of which is of size at least $\mu$. Proposition 3.18 below shows that $\mathcal{K}^{ccq}$ is a combinatorial Erdős-Rado class.

**Proposition 3.18.** Given infinite $\kappa$ and $n < \omega$, we have

$$\Downarrow_{n}(\kappa)^{+}_{\kappa} \xrightarrow{ccq} (\kappa^{+})^n_{\kappa}$$

Proof: Let $(I, <, E) \in \mathcal{K}^{ccq}$ and color it with $c : [I]^n \to \kappa$. We will use two already established facts:

$$\Downarrow_{n}(\kappa)^{+} \xrightarrow{ccq} (\kappa^{+})^n_{\kappa}$$

Then find $\{i_\alpha \in I \mid \alpha \ll \Downarrow_{n-1}(\kappa)^+\}$ that are $E$-nonequivalent. Set $I_1 = \bigcup_{\alpha<\Downarrow_{n-1}(\kappa)^+} i_\alpha/E$ and note that $(I_1, i_\alpha/E, <)_{\alpha<\Downarrow_{n-1}(\kappa)^+} \in \mathcal{K}^{ccq}$. Then $c$ colors $[I_1]^n$, so use the result to find $I_2 \subseteq I_1$ and $c^* : \Downarrow_{n-1}(\kappa)^+ \to \kappa$ so that $(I_2, i_\alpha/E \cap I_2, <)_{\alpha<\Downarrow_{n-1}(\kappa)^+} \in \mathcal{K}^{ccq}$ is type-homogeneous for $c$ with $c^*$.

Now consider the structure $\{(i_\alpha \mid \alpha \ll \Downarrow_{n-1}(\kappa)^+), <\} \in \mathcal{K}^{ccq}$. We want to give an auxiliary coloring $d : \Downarrow_{n-1}(\kappa)^+ \to A\kappa$, where $A = \{s \subseteq n+1 \mid \sum_{i<n} s(i) = n\}$. Then

$$d = \{\alpha_1 < \cdots < \alpha_n\}$$

is the function that takes $s \in A$ to $\alpha = \{j_1, \ldots, j_n\}$ for $j_1, \ldots, j_n \in I_2$ such that, for each $k$, $s(k)$-many of the $j_i$'s come from the equivalence class of $i_{\alpha_k}$. Note that this is a well-defined coloring because $I_2$ was type-homogeneous for $c$. Then we can find $X \subseteq \Downarrow_{n-1}(\kappa)^+$ of size $\kappa^+$ and $d^* : A \to \kappa$ such that $X$ is homogeneous for $d$ with color $d^*$.

Set $I_3 = \{i \in I_2 \mid iEi_\alpha$ for some $\alpha \in X\}$, $E_* = E \upharpoonright (I_3^2)$, and $<_* = < \upharpoonright (I_3^2)$.
Claim: \((I_*, E_*, <_*) \in K^{ceq}_+\) is type-homogeneous for \(c\). Since \(|X| = \kappa^+\), \(I_*\) has \(\kappa^+\)-many equivalence classes. For each \(\alpha \in X\), \(i_\alpha/E_* = i_\alpha/E \cap I_2\) and has size at least \(\beth_{n-1}(\kappa)^+ > \kappa^+\). Thus, \((I_*, E_*, <_*)\) is \(\kappa^+\)-big.

For homogeneity, let \(j_1 <_* \cdots <_* j_n; j'_1 <_* \cdots <_* j'_n \in I_*\) have the same \(K^{ceq}\)-type. Then these tuples are each \(<\)-increasing, from \(I_2\), and each element of each tuple is equivalent to an element of \(\{i_\alpha \mid \alpha \in X\}\). Because they have the same \(K^{ceq}\)-type, there are \(\alpha_1 < \cdots < \alpha_n; \alpha'_1 < \cdots < \alpha'_n\) from \(X\) that contain these witnesses and a single map \(s \in A\) that maps \(\ell\) to

\[
[k \mid j_k E_* i_{\alpha_k}] = [k \mid j'_k E_* i_{\alpha'_k}]
\]

By the homogeneity of \(X\), we have that \(d^* = d(\{\alpha_1, \ldots, \alpha_n\}) = d(\{\alpha'_1, \ldots, \alpha'_n\})\). Thus,

\[
c(j_1, \ldots, j_n) = d(\{\alpha_1, \ldots, \alpha_n\})(s) = d^*(s) = d(\{\alpha'_1, \ldots, \alpha'_n\})(s) = c(j'_1, \ldots, j'_n)
\]

Example 3.19 \((n\text{-multi-orders})\). We can use \(K^{n-mlo}\) to point out that the choice of bigness notion is very important. If we say \((I, <_1, \ldots, <_n)\) is \(\mu\)-big when \(|I| \geq \mu\), then \(K^{n-mlo}\) is a combinatorial Erdős-Rado class simply because \(K^{\omega}\) is. However, this gives us no new information. A good bigness notion for this class should say something about the independence of the different linear orders.

Example 3.20 (Ordered graphs). Ordered graphs start to indicate that set theory begins to enter the picture. Hajnal and Komjáth [HK88, Theorem 12] (with correction at [HK92, Theorem 12]) show that it is consistent that there is a graph that never appears as a monochromatic subgraph. In particular, they start with a model of \(GCH\), add a single Cohen real, and construct an uncountable bipartite graph \(G\) such that every graph \(H\) has a coloring of pairs such that there is no type-homogeneous copy of \(G\) in \(H\). On the other hand, the next example (which subsumes this one by considering \(K^{(2,2)-hg}\)) shows that we can consistently get a combinatorial Erdős-Rado result.

Example 3.21 (Colored hypergraphs). Shelah [She89, Conclusion 4.2] proved that it is consistent that an Erdős-Rado Theorem holds for the classes \(K^{(k,\sigma)-hg}\) with \(k < \omega\). Specifically, he shows that, after an iterated forcing construction, for every well-ordered \(N \in K^{(k,\sigma)-hg}\), \(m < \omega\), and \(\kappa\), there is a \(M \in K^{(k,\sigma)-hg}\) with \(\|M\| < \beth_m(\|N\| + \sigma + \kappa)\) such that any coloring of \(|M|^m\) with \(\kappa\)-many colors contains a type-homogeneous substructure isomorphic to \(N\). For the right bigness notion and by using the techniques of Propositions 3.12 and 3.14 to remove the well-ordered assumption, this means that

\[
\beth_m(\kappa)^{(k,\sigma)-hg} \rightarrow (\kappa)^n\kappa
\]

4. Erdős-Rado Classes and the Generalized Morley’s Omitting Types Theorem

Erdős-Rado classes (Definition 4.1) are those that allow one to build generalized indiscernibles in nonelementary classes, especially those definable in terms of type omission. Since these classes are often axiomatized in stronger logics, one could formulate the modeling property of Ramsey classes in terms of these stronger logics (in fact, Shelah [She] does this, and we compare the notions in Remark 4.8). However, this is not how order indiscernibles are typically built in AECs. Instead, we continue to work with indiscernibility in a first-order (and even quantifier-free) context, but strengthen the modeling property so that type omission is preserved.
Note that there are two variants of being an Erdős-Rado class here, and a few more in Definition 5.1. The cofinal variant is the most common, and gives the sharpest applications.

Definition 4.1. Let $\mathcal{K}$ be an ordered abstract class.

1. $\mathcal{K}$ is a $(\mu, \chi, \text{big})$-Erdős-Rado class iff for every language $\tau$ of size $\leq \mu$, every $I \in \mathcal{K}^{\text{big}}$, every $\tau$-structure $M$, and every injection $f : I \to M$, there is a blueprint $\Phi \in \mathcal{T}^\mathcal{K}[\tau]$ such that
   - (a) $\tau(\Phi) = \tau$; and
   - (b) for each $p \in S^\mathcal{K}_\chi$, there are $i_1 < \cdots < i_n \in I$ realizing $p$ such that
     $$tp_\tau(f(i_1), \ldots, f(i_n); M) = \Phi(p)$$

2. $\mathcal{K}$ is a cofinally $(\mu, \chi, \text{big})$-Erdős-Rado class iff for every language $\tau$ of size $\leq \mu$, if we have, for each cardinal $\alpha < \chi$, a $\tau$-structure $M_\alpha$, an $\alpha$-big $I_\alpha \in \mathcal{K}$, and an injection $f_\alpha : I_\alpha \to M_\alpha$, then there is a blueprint $\Phi \in \mathcal{T}^\mathcal{K}[\tau]$ such that
   - (a) $\tau(\Phi) = \tau$; and
   - (b) for each $p \in S^\mathcal{K}_\chi$, there are cofinally many $\alpha < \chi$ such that there are $i_1 < \cdots < i_n \in I_\alpha$ realizing $p$ such that
     $$tp_\tau(f_\alpha(i_1), \ldots, f_\alpha(i_n); M_\alpha) = \Phi(p)$$

In either case, writing $\mathcal{K}$ is a \{cofinally \text{big}-Erdős-Rado class\} means that \textquote[there is a function $f : \text{Card} \to \text{Card}$ such that $\mathcal{K}$ is \{cofinally \{$(\mu, f(\mu), \text{big}$-Erdős-Rado for every $\mu$\}.} If \text{big} is the standard bigness notion for $\mathcal{K}$, then we omit it.

Note that (2) implies (1) by taking the constant sequences $I_\alpha = I$ and $M_\alpha = M$.

We refer to Definition 4.1.11 or Definition 4.1.21 as the Erdős-Rado condition. See Remark 1.8 for a comparison with Ramsey conditions.

The following is the main source of Erdős-Rado classes.

Theorem 4.2 (Generalized Morley’s Omitting Types Theorem). Let $\mathcal{K}$ be combinatorially Erdős-Rado witnessed by $F$. Define $f : \text{Card} \to \text{Card}$ by setting $f(\mu)$ to be the first $\kappa$ above
$$\sup_{n<\omega} 2^{\mu \cdot |S^\mathcal{K}_\chi|}$$
such that $\alpha < \kappa$ and $n < \omega$ implies $F(\alpha, n) < \kappa$.

Then $\mathcal{K}$ is cofinally Erdős-Rado witnessed by $f$.

The statement of Theorem 4.2 makes no mention of types in the statement, although it appears in the name. This connection is made in Theorem 4.6 which shows that if a blueprint is generated from (Skolemized) models that all omit some type, then any model built from the blueprint also omits the type. Morley is classically credited with the original result and gave the basic argument, but an important technical improvement is due to Chang, see [Chao8] and (E), p. 49].

Proof: Suppose that we are given $f_\alpha : I_\alpha \to M_\alpha$ for $\alpha < f(\mu)$, where $|\tau| \leq \mu$. Although $\mathcal{K}$ has a unique 0-type (recall Remark 5.3), the models $M_\alpha \in \mathcal{K}^{\tau}$ might realize distinct 0-types. However, the number of possible 0-types is $\leq 2^{|\tau|} < f(\mu)$. Thus, by passing to cofinal sequence (and using the monotonicity of bigness)\cite[$<$] we may assume that each $M_\alpha$ realizes the same 0-type, call it $p^* \in S^\mathcal{K}_0$. We also restrict ourselves to cardinal $\alpha$.

We are going to build, for $n < \omega$ and cardinals $\alpha$ so $2^\mu < \alpha < f(\mu)$

\footnote{A more detailed version of this technique is given at the end of the induction step of the construction.}
Moreover, the coherence condition implies that it is proper for

\[ \alpha > \beta \]

such that

\[ \alpha, \beta \in \text{Inc} \]

the \( p \)-type reflection required by the Erdős-Rado condition, see Definition 4.1.(2b).

For each \( n \), if \( \alpha < \beta \), then we have that

\[ \Phi_n \left( t_{\alpha}(i_1, \ldots, i_n; I^n_\alpha) \right) = t_{\alpha}(f^n_\alpha(i_1), \ldots, f^n_\alpha(i_n); M_{\beta}(\alpha)) \]

(see Definition 2.2.(5) for this notation);

for every \( \alpha < f(\mu) \) and \( \alpha < \beta \), we have that

\[ \Phi_n \left( t_{\alpha}(i_1, \ldots, i_n; I^n_\alpha) \right) = t_{\alpha}(f^n_\alpha(i_1), \ldots, f^n_\alpha(i_n); M_{\beta}(\alpha)) \]

(3) the \( \Phi_n \) are coherent in the following sense: if \( p \in S^{inc, n}_\alpha \) and \( s \subseteq n \), then

\[ \Phi_n(p)^s = \Phi_n \left( p^s \right) \]

This is enough: Set \( \Phi := \bigcup_{n<\omega} \Phi_n \). Then this is a function with domain \( S_\alpha \) and range \( S_\tau \).

Moreover, the coherence condition implies that it is proper for \( \mathcal{K} \). Now we wish to show that it has the type reflection required by the Erdős-Rado condition, see Definition 4.1.(2b).

Let \( p \in S^{inc, n}_\alpha \) and \( \alpha < f(\mu) \). \( I^n_\alpha \) is \( \mathbb{N}_\alpha \)-big, so there is \( i_1 < \cdots < i_n \in I^n_\alpha \) realizing \( p \). Then, by (1) of the construction

\[ \Phi(p) = \Phi_n \left( t_{\alpha}(i_1, \ldots, i_n; I^n_\alpha) \right) = t_{\alpha}(f^n_\alpha(i_1), \ldots, f^n_\alpha(i_n); M_{\beta}(\alpha)) \]

If \( n = 0 \), then \( I^n_{\alpha_0} \) and we are done. If \( n > 0 \), then composing the \( h \)-embeddings, we get

\[ h^*: I^n_{\alpha_0} \rightarrow I^\beta_\omega \]

such that \( f^n_{\alpha_0} = f^\beta_{\omega}(\alpha_0) \circ h^* \). Thus, \( h^*(i_1), \ldots, h^*(i_n) \in I^\beta_\omega \) realize \( p \) and

\[ \Phi(p) = t_{\beta}(f^\beta_{\omega}(\alpha_0) \circ h^*(i_1), \ldots, f^\beta_{\omega}(\alpha_0) \circ h^*(i_n); M_{\beta}(\alpha)) \]

Since \( \beta_\omega(\alpha_0^+) > \alpha_0 \), this completes the proof.

Construction: We work by induction on \( n \).

For \( n = 0 \), we must deal with the unique \( 0 \)-type \( p_0 \in S^{inc, 0}_\alpha \). Recalling the first step to ensure each \( M_\beta \) has the same \( 0 \)-type \( p^* \), we can set \( \Phi_0 := \{ (p_0, p^*) \} \) and use what we are given: \( \alpha_0(\alpha) = \alpha; I^0_\alpha = I_\alpha; \) and \( f^0_\alpha = f_\alpha \).

For \( n + 1 \), suppose we have completed the construction up to stage \( n \). Fix \( \alpha < f(\mu) \). Recall \( \alpha > 2^\mu \geq \aleph_{\tau+1} \), then \( F(\alpha, n + 1) < f(\mu) \). Consider the coloring

\[ c^{n+1}_{\alpha} : [I_{F(\alpha, n + 1)}]^{n+1}_\tau \rightarrow S^{n+1}_\tau \]
Corollary 4.3. Each of the examples of combinatorial Erdős-Rado classes in Section \[3.2\] are cofinal Erdős-Rado classes.

Remark 4.4. Whenever \( F(\mu, n) \leq 2^\omega(\mu) \), then this gives the bound \( f(\mu) = 2^{(2^\omega)^+} \) that often appears in the theory of nonelementary classes. In the case of well-founded trees, we get the bound \( 2_{(2^\omega)^+} \). These bounds can be improved by phrasing in terms of the undefinability of well-ordering of certain PC classes (this is done for specific cases in [She98, GS86]).

The following extends the normal notion of PC classes to include classes with a strong substructure relation. Note that Chang’s Presentation Theorem [Cham88] implies any \( \mathbb{L}_{\omega, \omega} \)-axiomatizable class with ‘elementary according to a fragment’ as the strong substructure is what we will call a PC pair, and Shelah’s Presentation Theorem [She77a] extends this to Abstract Elementary Classes.

Definition 4.5.

1. Let \( \mathbb{K} \) be an abstract class with \( \tau = \tau(\mathbb{K}) \). \( \mathbb{K} \) is a PC class iff there is a language \( \tau_1 \supset \tau \), a (first-order) \( \tau_1 \)-theory \( T_1 \), and a collection \( \Gamma \) of \( \tau_1 \)-types such that, for any \( \tau \)-structure \( M, M \in \mathbb{K} \) iff there is an expansion \( M_1 \) of \( M \) to \( \tau_1 \) that models \( T_1 \) and omits all types in \( \Gamma \).

2. Let \( \mathbb{K} \) be a class of \( \tau \)-structures and \( \prec \) be a partial order on \( \mathbb{K} \). \( \mathbb{K} \) is a PC pair iff there is a language \( \tau_1 \supset \tau \), a \( \tau_1 \)-theory \( T_1 \), and a collection \( \Gamma \) of \( \tau_1 \)-types such that: for any \( \tau \)-structure \( M, M \in \mathbb{K} \) iff there is an expansion \( M_1 \) of \( M \) to \( \tau_1 \) that models \( T_1 \) and omits all types in \( \Gamma \); and
for any $M, N \in K$, $M \preceq K N$ iff there are expansions $M_1$ of $M$ and $N_1$ of $N$ to $\tau_1$ that models $T_1$ and omits all types in $\Gamma$ such that $M_1 \subseteq \tau_1 N_1$

**Theorem 4.6.** Let $K$ be a cofinal Erdős-Rado class witnessed by $f$ and let $K$ be a PC pair with $\tau = \tau(K)$ and $\tau_1$ the witnessing language. Suppose that, for every $\alpha < f(\vert \tau_1 \vert)$, there is $M_\alpha \in K$; $\alpha$-big $I_\alpha \in K$; and $f_\alpha : I_\alpha \to M_\alpha$. Then, there is $\Phi \in \mathcal{T}^{[\tau_1]}_K$ such that, for every $p \in S^\mathrm{nc}_K$, there are cofinally many $\alpha < f(\vert \tau_1 \vert)$ such that there are $i_1 < \cdots < i_n \in I_\alpha$ realizing $p$ such that

$$tp_\tau (f_\alpha(i_1), \ldots, f_\alpha(i_n); M_\alpha) = \Phi(p)$$

A version of Theorem 4.6 also holds for Erdős-Rado classes (without the cofinal adjective) when there is a single embedding from a $f(\vert \tau_1 \vert)$-big member of $K$ into $M$.

**Proof:** Let $T_1$ and $\Gamma$ in the language $\tau_1$ witness that $K$ is a PC pair. By a further Skolem expansion, we can assume that $T_1$ is universal and the types of $\Gamma$ are quantifier-free. Let $f_\alpha : I_\alpha \to M_\alpha$ for $\alpha < f(\vert \tau_1 \vert)$ as in the hypothesis. Since $K$ is $(\vert \tau_1 \vert, f(\vert \tau_1 \vert))$-cofinally Erdős-Rado, we can find a blueprint $\Phi \in \mathcal{T}^{[\tau_1]}_K$ satisfying the Erdős-Rado condition, Definition 4.1.(2b).

First, we wish to show that $\Phi$ is proper for $(K, \mathbb{K})$. For membership in $K$, let $I \in K$. First, suppose that $\forall x \phi(x) \in T_1$ with $\phi(x)$ quantifier-free. If $EM(I, \Phi) \models \neg \forall x \phi(x)$, then there is $a \in EM(I, \Phi)$ witnessing this. Since $EM(I, \Phi)$ is generated by $\tau_1$-terms, there are $\tau_1$-terms $\sigma_1, \ldots, \sigma_n$, and $i_1, \ldots, i_k \in I$ such that $a = \sigma_1^\tau EM(I, \Phi)(i_1), \ldots, \sigma_n^\tau EM(I, \Phi)(i_k)$; without loss, these satisfy $i_1 < \cdots < i_k$.

Set $p = tp_{\mathbb{K}}(I, \Phi)$. By the Erdős-Rado condition, there is some $\alpha < f(p)$ and $j_1 < \cdots < j_k$ such that

$$tp_{\tau_1} (i_1, \ldots, i_k; EM(I, \Phi)) = \Phi(p) = tp_{\tau_1} (f_\alpha(j_1), \ldots, f_\alpha(j_k); M_\alpha)$$

In particular,

$$M_\alpha \models \neg \phi(\sigma_1(j_1), \ldots, \sigma_n(j))$$

But this contradicts that $M_\alpha$ models $T_1$.

The same argument shows that any quantifier-free type which is realized in $EM(I, \Phi)$ is realized cofinally many $M_\alpha$. Since all types in $\Gamma$ are quantifier-free, $EM(I, \Phi)$ must omit all of them. So $EM_r(I, \Phi) \in \mathbb{K}$.

For substructure, this follows from the definition for PC pair and the fact that $\Phi$ is proper for $(K, \mathbb{K}^{\tau_1})$.

**Remark 4.7.** Since a $\Phi$ proper for $(K, \mathbb{K})$ is a map $S^\mathrm{nc}_K \to S_K$, the blueprint $\Phi$ also determines Galois types in $\mathbb{K}$ in the following sense: if $\sigma_1, \ldots, \sigma_k$ are $\tau(\Phi)$-terms; $I, J \in \mathbb{K}$; and $i_1, \ldots, i_n \in I$ and $j_1, \ldots, j_n \in J$ are tuples such that

$$tp_{\mathbb{K}}(i_1, \ldots, i_n; \Phi) = tp_{\mathbb{K}}(j_1, \ldots, j_n; \Phi)$$

then

$$tp_{\Phi}(\sigma_1(i_1, \ldots, i_n), \ldots, \sigma_k(i_1, \ldots, i_n); EM_r(\Phi)(I, \Phi)) = tp_{\Phi}(\sigma_1(j_1, \ldots, j_n), \ldots, \sigma_k(j_1, \ldots, j_n); EM_r(\Phi)(J, \Phi))$$

This could also be proved by applying the functor $EM_r(\cdot, \Phi)$ to the diagram witnessing type equality in $\mathbb{K}$ to get a diagram proving the type equality in $\mathbb{K}$.

In particular, $I \subseteq EM_r(I, \Phi)$ is a collection of $\mathbb{K}$-indiscernibles.

**Remark 4.8.** We want to highlight the differences between the Erdős-Rado condition (Definition 4.1.(2b)) to the relevant condition in uses of Ramsey classes, such as [Sh4, Definition 1.15] or [GHS, Definition 2.12]. We rephrase the Ramsey modeling condition and the Erdős-Rado condition to highlight the comparison:
Ramsey: for each $p \in S^\text{inc}_K$ and for each $\phi(x) \in \Phi(p)$, there is $i_1 < \cdots < i_n \in I$ realizing $p$ so

$$M \models \phi(f(i_1), \ldots, f(i_n); M)$$

Erdős-Rado: for each $p \in S^\text{inc}_K$, there is $i_1 < \cdots < i_n \in I$ realizing $p$ so for each $\phi(x) \in \Phi(p)$

$$M \models \phi(f(i_1), \ldots, f(i_n); M)$$

The witnesses for Ramsey condition depend on the formula under consideration, but the witness for the Erdős-Rado condition is uniform for all formulas.

This makes Ramsey classes ill-equipped to handle type omission and nonelementary classes. These is because, after Skolemization, the generating sequence might not agree on where terms omit the types, so the blueprint is not guaranteed to omit types. Shelah [She, Definition 1.15] addresses this by introducing $L$-nice Ramsey classes (for a logic fragment $L$) that considers formulas in $L$. However, it is unclear how to get a $L$-nice Ramsey class outside of Erdős-Rado classes. He also considers the notion of a strongly Ramsey class, which is similar to our notion.

5. Further results

5.1. Reversing Generalized Morley’s Omitting Types Theorem. We would like to have a converse to the Generalized Morley’s Omitting Types Theorem 4.2 that says that all Erdős-Rado classes come from a combinatorial result. However, this seems unlikely to be true (and we discuss candidates for this in Section 5.4). The issue is that the definition of a (cofinally) Erdős-Rado class is not as tied to the relevant bigness notion, but the definition of a combinatorial Erdős-Rado class is. In particular, the definition leaves open the possibility that there is only a single witness to the Erdős-Rado condition, while combinatorial Erdős-Rado classes require a big set of witnesses to the type-homogeneity. If we strengthen this requirement, then we get a converse.

Definition 5.1. We say that $K$ is strongly $(\mu, \chi, \text{big})$-Erdős-Rado iff for all $I \in K^{\text{big}_\chi}$ and every injection $f : I \rightarrow M$ with $|\tau(M)| \leq \mu$, there is a blueprint $\Phi \in \mathcal{T}^K[\tau]$ with $\tau(\Phi) = \tau(M)$ such that for all $\alpha < \chi$ and $n < \omega$, there is an $\alpha$-big $I^n_\alpha \subseteq K I$ such that, for every $i_1 < \cdots < i_n \in I^n_\alpha$, we have

$$tp_{\tau(M)}(f(i_1), \ldots, f(i_n); M) = \Phi(tp_{\tau}(i_1, \ldots, i_n; I))$$

We define the cofinal variant and what it means to omit the $(\mu, \chi, \text{big})$-prefix as in Definition 4.7.

Theorem 5.2. Let $K$ be an ordered abstract class.

1. If $K$ is combinatorially Erdős-Rado witnessed by $F$, then $K$ is a strongly, cofinally Erdős-Rado witnessed by the function in Theorem 4.2.

2. If $K$ is strongly, cofinally $(\mu, \chi, \text{big})$-Erdős-Rado, then $K$ is strongly $(\mu, \chi, \text{big})$-Erdős-Rado.

3. If $K$ is strongly $(\mu, \chi, \text{big})$-Erdős-Rado, then, for each $n < \omega$ and $\lambda < \chi$,

$$\langle \chi \rangle \xrightarrow{\text{big}} (\lambda)_n^\mu$$

4. If $K$ is strongly Erdős-Rado witnessed by $f$, then $K$ is combinatorially Erdős-Rado witnessed by $F(\mu^+, n) = f(\mu)$.

Proof: The proof of the Generalized Morley’s Omitting Types Theorem 4.2 proves (1): the $I^n_\alpha$ built in that proof are exactly the ones needed to witness ‘strong.’ The proof of (2) is straightforward. We prove (3), which is enough to prove (4). The idea is that a potential coloring is turned into a structure, and the derived blueprint is used to figure out the colors for the large set.
Let \( \lambda < \chi \) and \( c : |I|^{\lambda} \to \mu \) be a coloring of \( I \in K^{\text{big}}_\chi \). We build this into a two-sorted structure
\[
M = (\{I, \mu; c, \alpha\}_{\alpha < \mu})
\]
We have an embedding \( f : I \to M \) given by the identity. Then \( |\tau(M)| = \mu \), so the strong Erdős-Rado property gives us a blueprint \( \Phi : S^{inc}_K \to \Sigma_\tau(M) \) as in Definition 5.3.

Claim 1: For every \( p \in S^{inc,n}_K \), there is a unique \( \alpha_p < \mu \) such that \( "\alpha(1, \ldots, \lambda_n) = \alpha_p" \in \Phi(p) \).

Take \( I^\lambda_n \leq_k I \) witnessing the strong Erdős-Rado property and find \( i_1 < \cdots < i_n \in I^\lambda_n \) realizing \( p \); such a tuple exists by the definition of a bigness notion. Then \( \Phi(p) = tp_{\tau(M)}(i_1, \ldots, i_n; M) \). This has a color, so \( \alpha_p = c(i_1, \ldots, i_n) \).
\[\uparrow\text{Claim 1}\]

Set \( c^* : S^{inc,n}_K \to \mu \) to be the function that takes \( p \) to \( \alpha_p \).

Claim 2: \( I^\lambda_n \) is type-homogeneous for \( c \) as witnessed by \( c^* \).
Straightforward.
\[\uparrow\text{Claim 2}\]

Since \( I^\lambda_n \) is \( \lambda \text{-big} \), this proves the theorem.
\[\uparrow\text{Theorem 5.3}\]

Note that this is not an exact converse because there is some slippage in the witnessing functions. However, this doesn’t affect the bounds on the Erdős-Rado class.

5.2. A category theoretic interpretation of blueprints. This section gives a category theoretic perspective on the results we’ve proven and indiscernibles in general. It requires more category theoretic background than the rest of the paper (such as [MP89] or [AR94]), but can be skipped. This theme will be explored further in [Bona].

Makkai and Paré give the following statement credited to Morley. For a logic \( \mathcal{L} \), an ‘\( \mathcal{L} \)-elementary category’ is (a category equivalent to) one where the objects are models of some fixed \( \mathcal{L} \)-theory \( T \) and arrows are \( \tau \)-homomorphisms between models that are elementary for some fragment of \( \mathcal{L} \) containing \( T \).

Fact 5.3 ([MP89] Theorem 3.4.1], \( K^{\text{er}} \) is a “minimal” large, \( \mathbb{L}_{\omega,\omega} \)-elementary category. This means that if \( K \) is a large, \( \mathbb{L}_{\omega,\omega} \)-elementary category, then there is a faithful functor \( \Phi : K^{\text{er}} \to K \) that preserves directed colimits.

This is not phrased as Morley (likely) ever wrote it, but this is the classic proof of Morley’s Omitting Types Theorem. The functor \( \Phi \) comes from the blueprint that takes \( I \in K^{\text{er}} \) to \( EM_L(I, \Phi) \in K \). With generalized indiscernibles in hand, we have a generalization.

Theorem 5.4. Erdős-Rado classes are below every large, \( \mathbb{L}_{\omega,\omega} \)-elementary category (in the sense of Fact 5.3). In particular, every \( \mathbb{L}_{\omega,\omega} \)-axiomatizable Erdős-Rado class is minimal amongst the large, \( \mathbb{L}_{\omega,\omega} \)-elementary categories.

We include a proof to make the translation more clear (and in part because Makkai and Paré do not give a proof). Recall that \( K^{\text{er}} \) is an Erdős-Rado class that is not \( \mathbb{L}_{\omega,\omega} \)-axiomatizable.

Proof: Let \( K \) be an Erdős-Rado class and \( K \) be a large, \( \mathbb{L}_{\omega,\omega} \)-elementary category. Fix \( f : \text{Card} \to \text{Card} \) witnessing that \( K \) is Erdős-Rado. By virtue of being large, there is \( M \in K \) such that \( |M| \geq f(\mu) \), where \( M \) is the size of fragment witnessing that \( K \) is a \( \mathbb{L}_{\omega,\omega} \)-elementary category. Thus by Theorem 1.6 and Chang’s Presentation Theorem, there is a blueprint \( \Phi \in \mathcal{Y}^K[K] \). Define a functor \( F : K \to K \) by, for \( I \in K \), \( F(I) = EM_L(I, \Phi) \) and, for \( f : I \to J \in K \), \( Ff \) the map that takes \( \sigma^{EM_L(I, \Phi)}(i_1, \ldots, i_n) \) for a \( \tau(\Phi) \)-term \( \sigma \) to \( \sigma^{EM_L(J, \Phi)}(f(i_1), \ldots, f(i_n)) \).
This is clearly faithful. Moreover, the EM construction commutes with directed colimits, so F preserves them.

This proof works by noting that blueprints can be seen as well-behaved functors. We can actually specify the properties of these functors to obtain a converse. The one additional property that we need is that the size of EM, (I, Φ) is determined by |I| and an additional cardinal parameter representing |τ(Φ)|. The following is based on an argument developed with John Baldwin in the case K = \( K^{\omega} \).

For this, we need the following definition:

**Definition 5.5.** Fix an abstract class \( K \).

1. \( K \) is universal iff \( \leq_K \) is \( \subseteq_\tau \) and \( K \) is closed under substructure.
2. \( K \) is a universal Erdős-Rado class iff it is universal and an Erdős-Rado class.

Note that all universal classes \( K \) are \( \mathbb{L}_{\infty, \omega} \)-axiomatizable by saying no finite tuple generates a structure not in \( K \).

**Theorem 5.6.** Suppose \( K \) is a universal Erdős-Rado class and \( \mathbb{K} \) is a large, \( \mathbb{L}_{\infty, \omega} \)-elementary category. Let \( F : K \to \mathbb{K} \) be a faithful functor that preserves directed colimits such that there is a cardinal \( \mu_F \) so that \( \| F(I) \| = |I| + \mu_F \) for every \( I \in K \). Then there is a blueprint \( \Phi \in \mathcal{Y}_{\mu_F}^K[\mathbb{K}] \) such that the functor induced by \( I \in K \mapsto EM,F(I, \Phi) \) is naturally isomorphic to \( F \).

**Proof:** Let \( T \subseteq \mathbb{L}_{\infty, \omega}(\tau) \) such that \( \mathbb{K} \) is (equivalent to) Mod T. Enumerate the \( K \)-types as \( \{ p^n_i \in S^n_i \mid i < \mu_n \} \), and pick some \( I^n_i \in K \) that is generated by elements \( a^n_1, \ldots, a^n_n \) that realize \( p^n_i \). We expand each \( F(I^n_i) \) to a \( \tau^* := \tau(\mathbb{K}) \cup \{ F^n_\alpha : \alpha < \mu_F, n < \omega \} \)-structure as in Shelah’s Presentation Theorem. In fact, we only give an explicit description of the \( \{ F^n_\alpha : \alpha < \mu_F \} \) structure on the \( F(I^n_i) \): for each \( n < \omega \) and \( i < \mu_n \), define these functions so that \( \{ F^n_\alpha(a^n_1, \ldots, a^n_n) : \alpha < \mu \} \) enumerates the universe of \( F(I^n_i) \). Then define the remaining functions arbitrarily.

Since \( F \) preserves directed colimits and \( K \) is generated by the \( I^n_i \) under directed colimits, we can lift these expansions to the rest of \( F''K \). Taking \( I \) large enough, we can define a blueprint \( \Phi \in \mathcal{Y}_K[\tau^*] \). For all \( I \), the \( \tau \)-reduct of \( EM,F(I, \Phi) \) is canonically isomorphic to \( F(I) \). Thus, \( \Phi \) is as desired.

Note that this converse requires that models be of a predictable size. Specializing to linear order, we demand that \( \Phi(n) \) be the same for all \( n < \omega \). This is necessary for the formalism we’ve given where \( \tau(\Phi) \) consists of functions that can be applied to any element of \( EM(I, \Phi) \). To state the most general result, we could change this to only apply the functions of \( \tau(\Phi) \) to the skeleton \( I \). Then the different sizes of \( \Phi(n) \) could be dealt with by having different numbers of functions of different cardinalities. But this seems like a marginal gain after what would be significant technical pain. Additionally, the requirement that \( K \) be universal can be removed.

We return to this category theoretic perspective in Section 6.2 when discussing indiscernible collapse.

### 5.3. Generalized Shelah’s Omitting Types Theorem

The application of Morley’s Omitting Types Theorem to Abstract Elementary Classes is normally done through Shelah’s Presentation Theorem, which gives a type omitting characterization of these classes (see Theorem 4.5 for this argument). Moving beyond this, Shelah has proved an omitting types theorem that strengthens this and specifically applies to to Abstract Elementary Classes in that it references Galois types rather than syntactic types ([MS90] and [She93] Lemma 8.7 both use some version of this). The key addition is a reduction in the cardinal threshold for type omission at the cost
of less control over what types are omitted. The main combinatorial tool is still using the Erdős-Rado Theorem to build Ehrenfeucht-Mostowski models, so we can similarly prove a version for any Erdős-Rado class.

One nonstandard piece of notation is necessary.

**Definition 5.7.** Suppose \( K \) is an AEC, and let \( N \prec_{\mathbb{K}} M, p \in S_{\mathbb{K}}(N), \) and \( \chi \leq \|N\| \). We say \( M \) omits \( p/E \chi \) iff, for every \( c \in M \), there is some \( N_0 \prec_{\mathbb{K}} N \) of size \( \chi \) such that \( c \) does not realize \( p \restriction N_0 \).

**Theorem 5.8 (Generalized Shelah’s Omitting Types Theorem).** Let \( K \) be an Erdős-Rado class and \( \mathbb{K} \) be an Abstract Elementary Class and \( |\tau(K)| + LS(\mathbb{K}) \leq \chi \leq \lambda \) with

1. \( f(\mu) < \sum_{LS(\mathbb{K})}(\mu) \) where \( f \) witnesses that \( K \) is Erdős-Rado (for simplicity);
2. \( N_0 \prec_{\mathbb{K}} N_1 \) with \( \|N_0\| \leq \chi \) and \( \|N_1\| = \lambda \);
3. \( \Gamma_0 = \{p_0^i : i < i_0^*\} \subseteq S_{\mathbb{K}}(N_0) \); and
4. \( \Gamma_1 = \{p_1^i : i < i_1^*\} \subseteq S_{\mathbb{K}}(N_1) \) with \( i_1^* \leq \chi \).

Suppose that, for each \( \alpha < (2^\chi)^+ \), there is \( M_\alpha \in \mathbb{K} \) such that

1. \( f_0^\alpha : I_0^\alpha \to M_\alpha \) for \( I_0^\alpha \in K_{\sum(\lambda)} \);
2. \( N_1 \prec M_\alpha \);
3. \( M_\alpha \) omits \( \Gamma_0 \); and
4. \( M_\alpha \) omits \( p_1^i/E_\chi \) for each \( i < i_1^* \).

Then there is \( \Phi \in \mathcal{Y}_{\mathbb{K}}^{\mathbb{K}}(\chi) \); increasing, continuous \( \{N_q^i \in \mathbb{K}_{\leq \chi} : q \in S_{\mathbb{K}}^{inc}\} \); and increasing Galois types \( p_{1,q}^i \in S_{\mathbb{K}}(N_q^i) \) for \( q \in S_{\mathbb{K}}^{inc} \) and \( i < i_1^* \) such that

1. \( N_0 = N_0^q = EM_r(0, \Phi) \);
2. for each \( q \in S_{\mathbb{K}}^{inc} \), there is \( \mathcal{V} \in I \in K \) realizing \( q \) such that \( N_q^i \prec_{\mathbb{K}} EM_r(\mathcal{V}, \Phi) \) and \( f_q : EM_r(\mathcal{V}, \Phi) \to M_\alpha \) for some \( \alpha_q < (2^\chi)^+ \) such that \( f_q(N_q^i) \prec_{\mathbb{K}} N_1 \);
3. \( p_{1,q}^i = f_q^{-1}(p_1^i \restriction f_q(N_q^i)) \); and
4. For every \( I \in K_\omega \), \( EM_r(I, \Phi) \) omits every type in \( \Gamma_0 \) and omits any type that extends \( \{p_{1,q}^i : q \in S_{\mathbb{K}}^{inc}\} \) in the following strong sense: if \( a \in EM_r(I, \Phi) \) is in the \( \tau(\Phi) \)-closure of \( 1 \in I \), then \( a \) doesn’t realize \( H(p^i_1), q \), where \( H : EM_r(\mathcal{V}, \Phi) \cong EM_r(1, \Phi) \) is the lifting \( 1 \mapsto \mathcal{V} \).

The proof of the above adapts the proof of Shelah’s Omitting Types Theorem just as Theorem 4.2 adapts Morley’s original; see the notes by the author for a very detailed proof of (the ordinary) Shelah’s Omitting Types Theorem [Bone].

### 5.4. End-approximations.

Although, most classes are proved to be cofinally Erdős-Rado by proving a combinatorial theorem and then applying Generalized Morley’s Omitting Types Theorem 4.2, the example of \( K^{\omega-ir} \) gives a case where this doesn’t happen. Instead, the fact that they are approximated by the trees of height \( n \), each of which satisfy a combinatorial theorem, allows us to show they are a cofinal Erdős-Rado class. We give an abstract condition in Definition 5.9 below, and Proposition 5.10 shows that \( K^{\omega-ir} \) fits into this framework.

**Definition 5.9.** We say that \( K \) is end-approximated by combinatorial Erdős-Rado classes \( \{K^n \mid n < \omega \} \) such that

1. \( \tau(K^n) \subseteq \tau(K^{n+1}) \) and \( \tau(K) = \bigcup_{n < \omega} \tau(K^n) \);
2. there are coherent restriction maps

\[
\cdot \restriction n : K^{(\geq n)} \to K^n
\]

where
(a) ‘coherent’ means that for \( n > m \) and \( I \in \mathcal{K}^{\leq n} \),
\[
I \upharpoonright m = (I \upharpoonright n) \upharpoonright m
\]
(b) ‘restriction maps’ has the normal meaning (restricting a structure to a smaller language) with the important detail that any sorts in \( \tau(\mathcal{K}) \) that are not \( \tau(\mathcal{K}^n) \) are removed from the structure, so the universe might shrink (see Proposition 5.11).

(3) various structures involving \( \mathcal{K} \) are the (co)limit of the same structures on the \( \mathcal{K}^n \):
(a) if \( (I_n \in \mathcal{K}^n : n < \omega) \) is a sequence so \( (I_{n+1}) \upharpoonright n = I_n \) for all \( n < \omega \), then the
\[
\bigcup_{n<\omega} I_n
\]
is in \( \mathcal{K} \);
(b) in the above, if each \( I_n \) is \( \mu \)-big, then the union is \( \mu \)-big;
(c) if \( (p^n = tp_{\mathcal{K}^n}(a^n; I_n) \in S_{\mathcal{K}^n} \mid k_0 \leq n < \omega) \) is a sequence for some \( k_0 < \omega \) so \( a^{n+1} \in I_{n+1} \upharpoonright n \) and
\[
p^n = p^{n+1} \upharpoonright n := tp_{\mathcal{K}^{n+1}}(a^{n+1}; I_{n+1} \upharpoonright n)
\]
for all \( n \geq k_0 \), then there is a unique \( p = tp_{\mathcal{K}}(a; I) \in S_{\mathcal{K}} \) such that
\[
p^n = p \upharpoonright n := tp_k(a; I \upharpoonright n)
\]
(4) the restriction of a \( \mu \)-big model (with bigness computed in the domain) is \( \mu \)-big (in the restricted class); and
(5) if \( I \in \mathcal{K}_\alpha^n, J \in \mathcal{K}_\alpha^n \), and \( f : I \to J \upharpoonright n \) is a \( \mathcal{K}^n \)-embedding, then this can be lifted to \( \hat{I} \in \mathcal{K}^{n+1}_\alpha \) and \( \hat{f} : \hat{I} \to J \upharpoonright (n+1) \) such that \( \hat{I} \upharpoonright n = I \) and \( f \upharpoonright \hat{I} = \hat{f} \).

Note that since the language is finitary, we can always decompose the objects covered in Definition 5.9(3) as the canonical colimits of its restrictions; that is, if \( I \in \mathcal{K} \), then
\[
I = \bigcup_{n<\omega} I \upharpoonright n
\]
The lifting condition Definition 5.9(3) is the key property. Our initial motivation for this framework was to show that \( \mathcal{K}^{\omega-\text{tr}} \) is an Erdős-Rado class. After sending him a draft, Baldwin pointed us to [BS12, Section 4], where this is already shown. However, this framework can also be used to show that \( \mathcal{K}^{(\omega, \sigma)-h\#} \) is an Erdős-Rado class in Shelah’s model in [She89, Conclusion 4.2] (specifically, from the conclusion that \( \mathcal{K}^{(k, \sigma)-h\#} \) is an Erdős-Rado class for each \( k < \omega \)).

**Proposition 5.10.** \( \mathcal{K}^{\omega-\text{tr}} \) is end approximated by \( \{\mathcal{K}^{n-\text{tr}} \mid n < \omega\} \).

**Proof:** The proof is straightforward. The truncation map \( \cdot \upharpoonright n \) truncates a tree of height \( \geq n \) to its \( \leq n \) levels. Any \( p \in S_{\mathcal{K}^{n-\text{tr}}} \) specifies the max height \( k_p \) of a realization.

For condition (3), let \( I, J, f \) be as there. Build \( \hat{I} \) by specifying \( \hat{I} \upharpoonright n = I \) and, given maximal \( \eta \in I \), the successors of \( \eta \) in \( I \) are an isomorphic copy the successors of \( f(\eta) \) in \( J \). Since \( J \) is at least \( \alpha \)-splitting, so is \( \hat{I} \in \mathcal{K}^{n+1}_\alpha \), and the isomorphisms give the lift \( \hat{f} : \hat{I} \to J \upharpoonright (n+1) \).

Similarly, we can show the following.

**Proposition 5.11.** \( \mathcal{K}^{(\omega, \sigma)-h\#} \) is end approximated by \( \{\mathcal{K}^{(k, \sigma)-h\#} \mid k < \omega\} \).

**Theorem 5.12.** Let \( \mathcal{K} \) be end-approximated by combinatorial Erdős-Rado classes. Then \( \mathcal{K} \) is a cofinal Erdős-Rado class. A witnessing function \( f \) is the one taking \( \mu \) to the first cardinal above
\[
\sup_{n<\omega} 2^{\mu \mid S_{\mathcal{K}^{n-\text{tr}}} \mid}
\]
that is closed under the application of the functions witnessing that each $K^n$ is combinatorial Erdős-Rado.

The goal is to repeat the proof of the Generalized Morley’s Omitting Types Theorem except we restrict our set-up to the $n$th approximation $K^n$ in stage $n$. Then when we move to stage $n + 1$, we use the lifting condition Definition 5.9 to lift the set-up to the next level. At the advice of the referee, we repeat the proof here so that we can highlight where the different conditions of Definition 5.9 are used.

**Proof:** Suppose that we are given $f_\alpha : I_\alpha \to M_\alpha$ with $I_\alpha \in K_\alpha$ for cardinals $\alpha < f(\mu)$, where $|\tau| \leq \mu$.

We want to build

- $\Phi_n: S_{K^n} \to S_{r^n}$
- $\beta_n(\alpha) < f(\mu)$;
- $\gamma_n+1(\alpha) < f(\mu)$;
- $\alpha$-big $I^n_\alpha \in K^n_\alpha$;
- a $K^n$-embedding $k^{n+1}_\alpha : I^{n+1}_\alpha \restriction n \to I^n_{\gamma_n+1(\alpha)}$;
- a $K^n$-embedding $g^n_\alpha : I^n_\alpha \to I^{\beta_n(\alpha)}_\alpha \restriction n$; and
- $f^n_\alpha : I^n_\alpha \to M^{\beta_n(\alpha)}$ such that

1. $\beta_0(\alpha) = \alpha; I^0_\alpha = I^0 \restriction 0; f^0_\alpha = f \restriction (I^0 \restriction 0)$; and $g^0_\alpha = id_{I^0}$;
2. for each $\alpha < f(\mu)$ and $i_1 < \cdots < i_m \in I^n_\alpha$ with $m \leq n$, we have that $\Phi_n(t_{p\kappa^n}(i_1, \ldots, i_m; I^n_\alpha)) = tp_{\tau}(f^n_\alpha(i_1), \ldots, f^n_\alpha(i_m); M^{\beta_n(\alpha)})$
3. the $\Phi_n$ are coherent in the following senses:

   - (a) if $p \in S^m_{K^n}$ and $s \subseteq n$, then $\Phi_n(p)^s = \Phi_n(p^s)$
   - (b) if $m < n$ and $p \in S^m_{K^n}$ so $p \restriction m$ is defined and $m \leq \ell(p)$, then $\Phi_m(p \restriction m) = \Phi_n(p)$
4. for every $\alpha < f(\mu)$ and $n < \omega$, $\alpha \leq \beta_n(\alpha)$ and $\alpha \leq \gamma_n(\alpha)$;
5. for every $\alpha < f(\mu)$ and $n < \omega$, we have that $\beta_n(\gamma_n+1(\alpha)) = \beta_{n+1}(\alpha)$ and the following commutes

\[
\begin{array}{ccc}
I^{n+1}_\alpha \restriction n & \xrightarrow{h^{n+1}_{\alpha}} & I^n_{\gamma_n+1(\alpha)} \\
| & | & | \\
\downarrow & \downarrow & \downarrow \\
I^n_\alpha & \xrightarrow{g^{\alpha}_n} & I^{\beta_n(\alpha)}_\alpha
\end{array}
\]

This is enough: We define $\Phi : S_K \to S_\tau$ as the ‘colimit’ of the $\Phi_n$’s: given $p = tp_\kappa(a; I) \in S_K$, there is some minimal $k_p < \omega$ such that $a \in I \restriction k_p$ and $\ell(a) \leq k_p$. Then set $\Phi(p) := \Phi_{k_p}(p \restriction k_p)$

To show that this is a blueprint, we must check the coherence condition. So fix $p \in S^m_{K^n}$ and $s \subseteq n$. Then the types $p$ and $p^s$ has defining indices $k_p$ and $k_{p^s}$ (respectively) with $k_p \geq k_{p^s}$.

Then the coherence conditions ensure that $\Phi(p)^s = \Phi_{k_p}(p \restriction k_p)^s = \Phi_{k_{p^s}}((p \restriction k_p)^s) = \Phi_{k_{p^s}}((p \restriction k_{p^s})^s \restriction k_{p^s}) = \Phi_{k_{p^s}}(p^s \restriction k_{p^s}) = \Phi(p^s)$
where we have used the equality

\[(p \upharpoonright k_p)'' \upharpoonright k_{p'} = p'' \upharpoonright k_{p'}\]

which follows from direct computation.

Now we check the Erdős-Rado condition, Definition 6.1.1. Let \( p = tp_X(a; I) \in S^{\text{inv}, n}_K \) and \( \alpha_0 < f(\mu) \). Set \( k_p \) to be the defining index with \( n \leq k_p \). Then the bigness of \( I_{\alpha_0}^n \) implies that there is \( i_1 < \cdots < i_n \in I_{\alpha_0}^n \) realizing \( p \upharpoonright k_p \). Then, by (2) of the construction, we have

\[\Phi(p) = \Phi_{k_p}(p \upharpoonright k_p) = tp_T(I_{\alpha_0}^n(i_1), \ldots, f_{k_p}(i_n); M_{\beta_n(\alpha_0)})\]

Since \( \beta_n \) is a cofinal function, this completes the proof.

**Construction:** \( n = 0 \): Like before.\n
\[n+1: \text{As before, suppose we have the construction at stage } n. \text{ Fix } \alpha < f(\mu), \text{ WLOG } 2^n < \alpha. \]

We have the \( K^n \)-embidding \( g^n_\alpha : I^n_\alpha \rightarrow I_{\beta_n(\alpha)} \upharpoonright n \). Using the lifting condition Definition 5.9.3, we can lift to \( \tilde{g}^{n+1}_\alpha : I^{n+1}_\alpha \rightarrow I_{\beta_n(\alpha)} \upharpoonright (n+1) \) with \( I^{n+1}_\alpha \in K^{n+1}_\alpha \) and \( I^{n+1}_\alpha \upharpoonright n = I^n_\alpha \).

Color \( c^{n+1}_\alpha : [I^{n+1}_{F_{\beta_n(\alpha)+1}(a,n+1)}]^{n+1} \rightarrow S^{n+1}_n \) by

\[c^{n+1}_\alpha \left( \{i_1 < \cdots < i_{n+1}\} \right) = tp_T \left( f_{\beta_n(F_{\beta_n(\alpha)+1}(a,n+1))} \circ \tilde{g}^{n+1}_\alpha((i_1, \ldots, i_{n+1}))/\emptyset; M_{\beta_n(F_{\beta_n(\alpha)+1}(a,n+1))} \right)\]

Because \( K^{n+1} \) is a combinatorial Erdős-Rado class, there are

\[\tilde{h}^{n+1}_\alpha : I^{n+1}_\alpha \rightarrow I^{n+1}_{\tilde{F}_{\beta_n(\alpha)+1}(a,n+1)} \]

\[c^{n+1}_\alpha : S^{\text{inv}, n+1}_n \rightarrow S^{\text{inv}, n+1}_n\]

such that \( c^{n+1}_\alpha \) witnesses that \( \tilde{h}^{n+1}(I^{n+1}_\alpha) \) is type homogeneous for \( c^{n+1}_\alpha \).

Then finish as in Theorem 5.12. \( \dagger \)

Thus, while we have no combinatorial partition result for \( K^{\omega-\text{tr}} \), it is an Erdős-Rado class.

**Corollary 5.13.** \( K^{\omega-\text{tr}} \) is a cofinal Erdős-Rado class witnessed by \( \mu \mapsto \Delta_\omega(2^\omega)^+ \).

**Proof:** By Theorem 5.12 applied to Proposition 5.10. \( \dagger \)

6. Applications

6.1. Unsuperstability in Abstract Elementary Classes. In countable first-order theories, strict stability can be detected by counting types at cardinals \( \lambda \) satisfying \( \lambda < \lambda^{\omega} \); if \( T \) is stable, then \( T \) is superstabile iff \( T \) is stable in some \( \lambda \) so \( \omega < \lambda < \lambda^{\omega} \) iff \( T \) is stable in all \( \lambda \) so \( \omega < \lambda < \lambda^{\omega} \). This is done by building what is called a ‘Shelah tree’ [Bal88 p. 85]. This is a way of embedding the \( \omega + 1 \)-height tree \( \tilde{S}^\omega \lambda \) into a model of \( T \) so the types of branches are differentiated over their initial segments. In the context of nonelementary classes, Baldwin and Shelah [BST12 Theorem 3.3] generalizes this to atomic classes by use of \( K^{\omega-\text{tr}} \)-indiscernibles.

Here, we generalize this to tame Abstract Elementary Classes with amalgamation. Note that we break our convention of always using types over the empty set here. In fact, we will consider types over arbitrary sets. Given \( N \in K, a \in N, \) and \( B \subseteq N \), we write

\[ gtp_G(a/B; N) \]

for the equivalence class of triples where the equivalence relation in Definition 2.2 is required to fix the parameter set \( B \) as well. Since we lack a monster model, we fix an ambient model \( N \) to give meaning to \( B \). Then \( S_G(B; N) \) is the collection of all Galois types over \( B \) as seen as a set in \( N \). We omit other basics of Abstract Elementary Classes, but the key definitions can be found in one of [GroX][Bal09][BV17].
Theorem 6.1. Let $K$ be a $<\kappa$-tame Abstract Elementary Class with amalgamation (allowing for the possibility that $\kappa < \text{LS}(K)$). One of the following holds:

1. there is $\chi < 2^{(2^{\kappa+\text{LS}(K)})^+}$ such that for all $M \in K_\chi$, $|S_K(M)| \leq \|M\|^{<\kappa}$; or
2. $K$ is Galois unstable in every $\lambda$ satisfying $\lambda^\kappa > \lambda \geq \kappa + \text{LS}(K)$.

The first case roughly corresponds to Galois superstability, while the second is not superstable. However, the necessary involvement of $\|M\|^{<\kappa}$ in the type counting (as opposed to just $\|M\|$) makes comparison with other results awkward. Here we state two corollaries that rephrase the result directly in terms of Galois stability:

Corollary 6.2.

1. Let $K$ be a $<\omega$-tame Abstract Elementary Class with amalgamation. One of the following holds:
   a. there is $\chi < 2^{(2^{\kappa+\text{LS}(K)})^+}$ such that $K$ is Galois stable in every $\lambda > \chi$; or
   b. $K$ is Galois unstable in every $\lambda$ satisfying $\lambda^\kappa > \lambda \geq \kappa + \text{LS}(K)$.
2. Let $K$ be a $<\kappa$-tame Abstract Elementary Class with amalgamation (allowing for the possibility that $\kappa < \text{LS}(K)$). One of the following holds:
   a. there is $\chi < 2^{(2^{\kappa+\text{LS}(K)})^+}$ such that $K$ is Galois stable in every $\lambda > \chi$ so $\lambda^{<\kappa} = \lambda$; or
   b. $K$ is Galois unstable in every $\lambda$ satisfying $\lambda^\kappa > \lambda \geq \kappa + \text{LS}(K)$.

The subscript $(2^{\kappa + \text{LS}(K)})^+$ can be replaced by the relevant undefinability of well-ordering number. This fits into the project summarized in [GV17]; while superstability for arbitrary Abstract Elementary Classes seems poorly behaved (exhibiting what Shelah terms ‘schizophrenia’), superstability in the context of tame Abstract Elementary Classes is much better behaved. Vasey [Vas] computes stability spectra of Abstract Elementary Classes. For tame classes with amalgamation, [Vas Corollary 4.24] uses a technical analysis of nonsplitting to show that failure of ‘$K$ is Galois stable on a tail’ implies $\chi(K) > \omega$ and [Vas Corollary 4.17] shows that for ‘most’ $\lambda$, if $\lambda < \chi(K)$ implies $K$ is Galois unstable in $\lambda$; ‘most $\lambda$’ means all sufficiently large, almost $\lambda(K)$-closed cardinals. Theorem 6.1 offers a tighter bound on when the tail of stability must start and also a better condition on where the instability must happen.

Allowing for $\kappa \leq \text{LS}(K)$, especially the case $\kappa = \omega$ is important because of the requirement that $\lambda^{<\kappa} = \lambda$ in (1) of Theorem 6.1. This allows us to recover comparisons to first-order results and [BS12].

Our proof follows [BS12], but adapts the argument to Abstract Elementary Classes. The following notion of type fragments will make our argument smoother. These are essentially the partial Galois types that allow us to specify extending or not extending small Galois types. This is motivated by the idea that small Galois types should occasionally be able to stand in for formulas in tame AECs (e.g., [Bon14, Section 3] or Vasey’s Galois Morleyization [Vas16 Definition 3.3]).

Hypothesis 6.3. In the rest of the section, we assume that $K$ is a $<\kappa$-tame Abstract Elementary class with amalgamation.

Definition 6.4.

1. Given $M \in K$, $P^*_M := \{M_0 \in K_{<\kappa} : M_0 \prec M\}$.
2. $A < \kappa$-(Galois) type fragment over $B \subseteq N \in K$ is a collection $\Sigma$ of objects of the form $\{p\}$ or $\{\neg p\}$ for $p \in S_K(A; N)$ for $A \in P_K$ such that some element realizes every $p \in \Sigma$.
3. Some $a \in M$ realizes a $<\kappa$-type fragment $\Sigma$ over $M$ if $a \models p$ for all $p \in \Sigma$ and $a \not\models p$ for all $\neg p \in \Sigma$.
4. $A < \kappa$-type fragment is satisfiable if some element realizes it.
We won't have use for unsatisfiable type fragments, so all type fragments will be assumed to be satisfiable.

We fix some important notation for this section: given \(q \in S^\kappa_\kappa(A;N)\), we write
\[
q^0 := q \\
q^1 := \neg q
\]

In the following, we will want consider the number of types of length \(<\kappa\) over the empty set.

We have
\[
S^\kappa_\omega(\emptyset) = \bigcup_{M \in S^\kappa_\omega(\emptyset)} S^\kappa_\omega(\emptyset; M)
\]
and can estimate it's size by
\[
|S^\kappa_\omega(\emptyset)| \leq \bigcup_{\lambda<\kappa} \lambda\lambda \geq \kappa
\]

The following is similar to an argument of Baldwin-Kueker-VanDieren (see [Bal09, Theorem 11.11]), generalizing first-order arguments of Morley. It will give us more than we need.

**Lemma 6.5.** Let \(N \in \mathbb{K}\) and \(A \subseteq N\) with \(\Gamma \subseteq S^\kappa_\kappa(A;N)\) of size \(|A|^\kappa\). Then there is \(B \subseteq A\) of size \(<\kappa\) with \(q \neq r \in S^\kappa_\kappa(B;N)\) such that both \(q\) and \(r\) have \(|A|^\kappa\)-many extensions to \(\Gamma\).

**Proof:** If not, then for every \(B \in \mathcal{P}_\kappa A\), there is a unique \(q_B \in S^\kappa_\kappa(B;N)\) that has many extensions to \(\Gamma\). Then every \(p \in \Gamma\) falls into one of two categories:

1. \(p \geq q_B\) for all \(B \in \mathcal{P}_\kappa A\); or
2. there is \(B \in \mathcal{P}_\kappa A\) such that \(p \nleq q_B\).

By tameness, there is at most one type of the first kind. For each \(B\), there are \(|A|^\kappa\)-many such types by the choice of \(q_B\) and there are \(|A|^\kappa\)-many such \(B\)’s. Thus, there are \(|A|^\kappa\)-many types of the second kind, which contradicts that there are \(|A|^\kappa\)-many types in \(\Gamma\). \(\dagger\)

**Lemma 6.6.** Let \(\mu \geq |S^\kappa_\kappa(\emptyset)|\). If \(M \in \mathbb{K}_{\geq \mu}\) and \(\Gamma = \langle p_\alpha \in S^\kappa_\kappa(M) : \alpha < (|M|^\kappa)^+ \rangle\) are distinct, then there is \(\langle A^i \in \mathcal{P}_\kappa M : i < \mu \rangle\) and \(q(x;X) \in S^\kappa_\kappa(\emptyset)\) such that one of the following occur:

1. for all \(j_1 < \mu\), the following set has size \((|M|^\kappa)^+\)
\[
\big\{|i < (|M|^\kappa)^+: q(x; A^{j_1}) \nleq p_i \text{ and } j_0 < j_1 \implies q(x; A^{j_0}) \nl \leq p_i\big\}
\]
2. for all \(j_1 < \mu\), the following set has size \((|M|^\kappa)^+\)
\[
\big\{|i < (|M|^\kappa)^+: q(x; A^{j_1}) \leq p_i \text{ and } j_0 < j_1 \implies q(x; A^{j_0}) \nleq p_i\big\}
\]

**Proof:** We will construct

1. a tree \(T \subseteq \mathcal{S}^\mu_2\);
2. types \(\{q_\eta(x;X) \in S^\kappa_{\kappa}(\emptyset) : \eta \in T\}\);
3. sets \(\{A^\eta \in \mathcal{P}_\kappa M : \eta \in T\}\); and
4. type fragments \([1]\)
\[
\Sigma_\eta := \{q_\eta^{(j)}(x; A^{\eta(j)}) : j < \ell(\eta)\}
\]
\[
\text{such that for each } i \leq \mu:\
\]
1. every level \(T_i\) of \(T\) is nonempty;

\(\text{Note that this type fragment is actually determined by the other objects in the construction.}\)
This is a consistent type fragment over $M$ types over $\eta$ we apply Lemma 6.5 to the elements of $\Gamma$-extending $\Sigma$. Because every type extending $r$ we have that $\Sigma_1$ extending $\eta$ elements of $\Gamma$ extending $\eta$. Our type fragments $\Sigma_\eta$ and $\eta \in \kappa$. The following lemma is the key inductive step that allows us to build our tree of types.

For $i = 0$, we apply Lemma 6.5 to $\Gamma$ to find $A^0_\eta \in \mathcal{P}_\eta M$ and $q_\eta(x; A^0_\eta) \neq r \in S_\eta(A; M)$. Then we have that $\Sigma_{(0)} = \{q_\eta(x; A^0_\eta)\}$ and $\Sigma_{(1)} = \{\neg q_\eta(x; A^0_\eta)\}$, with the latter satisfying our conditions because every type extending $r$ extends $\Sigma_{(1)}$.

For $i = j + 1$, for each $\eta \in T_i$ we follow the same strategy except we apply Lemma 6.5 to the elements of $\Gamma$-extending $\Sigma_\eta$. This gives $A^0_\eta \in \mathcal{P}_\eta M$ and $q_\eta(x; A^0_\eta) \neq r \in S_\eta(A^0_\eta)$ that satisfy the requirements of the construction.

At limit stage $i$, we note that every type $p_\eta$ (or more generally, every $p \in S_\eta(M)$) extends one of our type fragments $\Sigma_\eta$ for $\eta \in \omega$. There are $2^\kappa$ many branches at this stage, and $\omega$-many $p_\eta$’s, so there must be some $\eta \in \omega$ many of them extend $\Sigma_\eta$; then $\eta \in T_i$. Once again, we apply Lemma 6.5 to the elements of $\Gamma$-extending $\Sigma_\eta$ to define $N^0_\eta$ and $q_\eta$.

The following lemma is the key inductive step that allows us to build our tree of types.

**Lemma 6.7.** Fix $\mu > |S_\kappa(\emptyset)|$. Suppose $M \in K_{\geq \kappa}$ with $|S_\kappa(M)| > (||M|^{< \kappa})$, and let $M$ be a $\mu^+$-Galois saturated extension of $M$.

- There are increasing $\{M_\eta \in K_\mu : \eta < \omega\}$; types $\{q_\eta \in S_\kappa(M_{\ell(\mu)}) : \eta < \omega\}$; sets $\{A^\eta \in \mathcal{P}_\kappa M_{\ell(\mu)} : \eta < \omega\}$ such that
  1. $M_\eta \prec_M M$ for all $\eta < \omega$;
  2. each $q_\eta$ has $||M|^{< \kappa}$-many extensions to $S_\kappa(M)$ and $a_\eta$ realizes $q_\eta$;
  3. if $\eta < \mu$ and $\nu < \omega_\mu$, then $q_\eta|_{\nu} \leq q_\nu$;
  4. for every $\nu < \omega$ and $i < j < \mu$,
     $q_\eta|_{(i)} \uparrow A^\nu|_{(i)} \neq q_\eta|_{(j)} \uparrow A^\nu|_{(j)}$

**Proof:** We do this by induction. For the base case $n = 0$, we pick $M_0 \prec_M M$ and $A^i_0 \in \mathcal{P}_\kappa M_0$ arbitrarily, then use the pigeonhole principle to find $q_0 \in S_\kappa(M_0)$ with $q_0|_{\nu} \leq q_\nu$ for $\nu < \mu$.

Given stage $n$, we know that each $q_\nu$ for $\nu < \mu$ has $||M|^{< \kappa}$-many extensions to $S_\kappa(M)$ and $|M| \geq 2^\kappa$. So we can apply Lemma 6.5 to get $q \in S_\kappa(\emptyset)$ and $\{A^0_\nu \in \mathcal{P}_\kappa M : i < \mu\}$ and $\ell_\nu \in \{0, 1\}$ such that

\[
\text{for all } j_i < \mu, \text{the following has size } \geq (||M|^{< \kappa})^+:
\]

\[
\{p \in S_\kappa(M) : q_\eta \leq p \text{ and } q(x; A^0_\nu)^{1-\ell_\nu} \leq p \text{ and } j_0 < j_1 \text{ implies } q(x; A^0_\nu)^{\ell_\nu} \leq p\}
\]

Set $A^\nu|_{(i)} = A^0_\nu$.

Let $M_{n+1} \prec M$ contain $M_n$ and $\bigcup_{\mu \in \alpha^+} A_\mu$ of size $\mu$. For each $i < \mu$, set

\[
\Sigma^\nu_{i} := q_\nu \cup \{q(x; A^0_\nu)^{1-\ell_\nu}, q(x; A^0_\nu)^{\ell_\nu} : j < i\}
\]

This is a consistent type fragment over $M$ by definition that can be extended to $\nu$-many types over $M$. Since there are at most $2^\kappa$ extensions of $\Sigma^\nu_{i}$ to $S_\kappa(M_{n+1})$, the pigeonhole principle says we can extend $\Sigma^\nu_{i}$ to a type $q_\nu|_{(i)} \in S_\kappa(M_{n+1})$ that can be extended to $\nu$-many types over $M$. 

By the saturation of \( \hat{M} \), we can find \( a_\nu \in \hat{M} \) realizing \( p_\nu \) for each \( \nu \in ^{<\omega}\mu \). 

**Proof of Theorem 6.1** Suppose that (1) fails, so that for \( \alpha < (2^\kappa)^+ \), there is \( M^\alpha \in \mathbb{K}_{>\underline{\mathbb{U}}^+} \) such that \( |S_\mathbb{K}(M_\alpha)| > |M_\alpha|^{<\kappa} \); by [Bon17, Theorem 3.1], we can assume this is witnessed by the 1-ary types. Using amalgamation, we can extend \( M^\alpha \) to \( \|M^\alpha\|^{\kappa^+} \)-Galois saturated \( M^\alpha \). Apply Lemma 6.7 for each \( \alpha \) to get \( \{M^\alpha_n, q^n_\nu, A^n_\alpha, a^n_\alpha : \nu \in \kappa, n < \omega \} \) as there. Expand \( \hat{M}^\alpha \) to \( M^\alpha_n \) by

1. adding the presentation language \( \tau_1 \) to witness the \( \prec \)-relations and so there is \( b^n_\alpha \in A^n_\alpha \) that generates \( A^n_\alpha \) in a uniform way and \( M^\alpha_n \) is generated by \( \{b^n_\alpha : \nu \in n = \kappa \} \);
2. interpreting \( \mathcal{K}^{\omega^{-1}} \)-structure as follows:
   - (a) \( P_n \) is \( M^\alpha_n \);
   - (b) \( \prec \) and \( \prec \) are the tree and lexicographic orderings on the \( b^n_\alpha \) (by their indices);
3. \( R \) are the pairs \((m, a^n_\alpha)\) for \( m \in \bigcup_n^{<\omega} M^\alpha_n \) and \( \nu \in ^{<\omega}\mathbb{U}_\alpha \); and
4. \( F_n \) takes \( b^n_\alpha \) to \( a^n_\alpha \) for \( \nu \in n = \kappa \);
5. \( S \) is a four place relation so that \( S(\ldots, b^n_\alpha, b^n_\beta) \) defines the graph of a function that witnesses \( a^n_\alpha \) and \( a^n_\beta \) have the same type over \( P_\ell(\nu \cap \eta) = M^\alpha_{\nu \cap \eta} \); it does this by mapping the structure generated by \( a^n_\alpha \) and the elements of \( P_\ell(\nu \cap \eta) \) into a structure generated by \( a^n_\beta \) and the images of the domain elements in a way that sends \( a^n_\alpha \) to \( a^n_\eta \) and fixes the elements of \( P_\ell(\nu \cap \eta) \).

Then \( |\tau_1| \leq \kappa \).

Since \( \mathcal{K}^{\omega^{-1}} \) is a cofinal \( \mathcal{E}_\mathbb{R} \)-class (Corollary 5.13), we can build a blueprint \( \Phi \in \mathcal{T}^{\omega^{-1}}[\mathbb{K}] \) that is modeled off of the embedding \( \nu \in ^{<\omega}\mathbb{U}_\alpha \mapsto b^n_\alpha \). Given \( \lambda \geq \kappa \), set \( T^\lambda \) to be the tree \( <^{<\omega}\lambda \). We then read off the various structures we have built:

1. \( M^\lambda = EM(T^\lambda, \Phi) \);
2. \( M^\lambda := EM_r(T^\lambda, \Phi) \in \mathbb{K}_\lambda \);
3. \( N^\lambda := \bigcup_{n < \omega} \left( P_n^{M_n^\lambda} \upharpoonright \tau \right) <^L M^\lambda \); and
4. \( \rho \in ^{<\omega}\mu \), set \( A^\rho \) to be the structure generated by \( \rho \) as the \( A^n_\alpha \) were by \( b^n_\alpha \).

For each \( n < \omega \) and \( \rho \in n = \lambda \), define

\[ p_\rho := gtp \left( E^{M_n^\lambda}((\rho)/P_n^{M_n^\lambda} : M^\lambda) \right) \]

We finish with a series of claims:

**Claim 1:** If \( \nu \leq \eta < ^{<\omega}\mu \), then \( p_\nu \leq p_\rho \).

**Proof:** The predicate \( S \) induces a map given by \( S^{M_n^\lambda}(\cdot, \cdot, \nu, \eta) \) that witnesses this.

**Claim 2:** Given \( \rho \in ^{<\omega}\lambda \) and \( i < j < \lambda \), we have

\[ p_{\rho^{<_i}}(i) \upharpoonright A^{\rho^{<_j}}_{i}(j) \neq p_{\rho^{<_j}}(j) \upharpoonright A^{\rho^{<_j}}_{i}(j) \]

**Proof:** The domain model is generated by \( \rho^{<_i}(i) \in T^\lambda \), so the types are controlled by where the blueprint sends

\[ tp_{K^{\omega^{-1}}(\rho^{<_i}(i), \rho^{<_j}(j); T^\lambda)} \]

Using the modeling property, we can find (cofinally many) \( \alpha < (2^\kappa)^+ \) and \( \rho' \in ^{<\omega}\mu_\alpha \) and \( i' < j' < \mu_\alpha \) that realize the above type. Then

\[ gtp \left( E^{M_n^\alpha}((\rho^{<_i}(i'))/A^{\rho^{<_j}}_{i'}(i'); M_\alpha) \right) \neq gtp \left( E^{M_n^\alpha}((\rho^{<_j}(j'))/A^{\rho^{<_j}}_{i'}(i'); M_\alpha) \right) \]

and this is preserved by \( \Phi \).

**Claim 3:** \( |S_\mathbb{K}(N^\lambda)| \geq \lambda^\omega \).
Proof. For each \(\eta \in \mathcal{P}^{<\omega}_\lambda\), by Claim 1 we have the increasing \(\omega\)-chain of types \(\{p_\eta(n) : n < \omega\}\). By the \(\omega\)-compactness of AECs with amalgamation (see [Bal09, Theorem 11.1]), there is \(p_\eta \in S_\mathcal{K}(N^\lambda)\) that extends all of them. By Claim 2, they are all distinct. 

6.2. Indiscernible Collapse in Nonelementary Classes. One of the uses of generalized indiscernibles in first-order is to characterize various dividing lines via indiscernible collapse. An old result of Shelah [She90, Theorem II.2.13] says that a theory \(T\) is stable iff any order indiscernibles in a model of \(T\) are in fact set indiscernibles. Scow [Sco12, Theorem 5.11] proved that \(T\) is NIP iff any ordered graph indiscernibles in a model of \(T\) are in fact just order indiscernibles. In each of these cases, there are abstract (Ramsey) classes \(\mathcal{K}_0\) and \(\mathcal{K}\) with \(\mathcal{K}\) a reduct of \(\mathcal{K}_0\) where some property of \(T\) can be detected by whether or not there are \(\mathcal{K}_0\)-indiscernibles that are actually \(\mathcal{K}_0\)-indiscernibles (after reducting the index). Guingona, Hill, and Scow [GHS] have formalized this notion of indiscernible collapse and given several more examples.

Following this work, we can give definitions of several dividing lines in Abstract Elementary Classes making use of the fact that the determining classes are Erdős-Rado classes in addition to being Ramsey classes. Unfortunately, at this time, we don’t know of any indiscernible collapses characterizing dividing lines that start with an Erdős-Rado class (other than order indiscernibles, but this collapse result is already known for Abstract Elementary Classes). [GHS, Theorem 3.4] uses \(\mathcal{K}^{n-mlo}\) and [GHS, Theorem 4.7] uses a class of trees that doesn’t restrict the height (in a similar way that \(\mathcal{K}^{ceq}\) generalizes \(\mathcal{K}^{\chi-\text{or}}\)), but neither of these are known to be Erdős-Rado classes. [GHS, Corollary 5.9] characterizes \(NTP_2\) theories via a collapse of \(\mathcal{K}^{ceq}\)-indiscernibles, but involves notions of formulas dividing that does not easily generalize to Abstract Elementary Classes. This leads us to the following question:

Question 6.8. Is \(\mathcal{K}^{op}\) an Erdős-Rado class?

Recall that it is consistently not a combinatorial Erdős-Rado class by Example [3.20]. However, this does not rule out the possibility it is an Erdős-Rado class. A positive answer for this question would give a prospective definition for the notion of NIP for Abstract Elementary Classes.

Definition 6.9. Suppose that \(\mathcal{K}^{op}\) is an Erdős-Rado class and let \(\mathcal{K}\) be an Abstract Elementary Class with arbitrarily large models. We say that \(\mathcal{K}\) is NIP iff for every \(\Phi \in \mathcal{T}^{op}[\mathcal{K}]\), there is \(\Psi \in \mathcal{T}[\mathcal{K}]\) such that \(\Phi = \Psi \circ U\), where \(U \in \mathcal{T}^{op}[\mathcal{K}^{op}]\) forgets the graph structure and \(\Psi \circ U\) is the composition of these blueprints.

This has advantages over other prospective definitions in that no amalgamation, tameness, etc. assumption is necessary. Of course, it has the disadvantage that it needs more results to be viable. This is being explored further in [BS].

6.2.1. Category theoretic interpretation of indiscernible collapse. We can build on the category theoretic interpretation of indiscernibles from Section 5.2 to give the same gloss to indiscernible collapse in terms of injectivity conditions ([AR94, Section 4.A] gives this background). In general, if \(f : A \to B\) is a morphism, then another object \(C\) is injective with respect to \(f\) iff every \(g : A \to C\) can be lifted along \(f\) to a \(g' : B \to C\) so \(g = g' \circ f\). Then Shelah’s result [She90, Theorem II.2.13] can be rephrased as follows.

Theorem 6.10. Let \(\mathcal{K}^{set}\) be the abstract class of sets and \(U : \mathcal{K}^{op} \to \mathcal{K}^{set}\) be the functor forgetting the ordering. An elementary class \(\mathcal{K}\) is stable iff it is injective with respect to \(U\) (in the category of accessible categories whose morphisms are faithful functors preserving directed colimits).
Note that, since Ψ agrees with Φ on τ which is Erdős-Rado by Example 3.11 and Corollary 4.3. Also, they use  
\( T(\MS, \text{Theorem 5.3}) \) 
Fact 6.12

Definition 6.11.

Let \( T_0 \) and \( T_1 \) be complete first-order theories and let \( \mu \) be an infinite cardinal.

1. We say that \( T_0 \prec_1^* T_1 \) iff for all large enough, regular \( \mu \), there is a first-order theory \( T_\ast \) of size \( \leq |T_0| + |T_1| + \aleph_0 \) that interprets \( T_\ell \) via \( \phi_\ell \) such that, for every \( M_\ast \models T_\ast \), if the interpretation \( M^{[\phi_\ell]}_\ast \) of \( T_\ell \) is \( \mu \)-saturated, then the \( M^{[\phi_\ell]}_\ast \) is \( \mu \)-saturated. (Shelah)

2. We say that \( T_0 \prec_1^{* \kappa} T_1 \) iff for all large enough, regular \( \mu \), there is an \( L_{\kappa, \omega} \)-theory \( T_\ast \) of size \( \leq |T_0| + |T_1| + \aleph_0 \) that interprets \( T_\ell \) via \( \phi_\ell \) such that, for every \( M_\ast \models T_\ast \), if the interpretation \( M^{[\phi_\ell]}_\ast \) of \( T_\ell \) is \( \mu \)-saturated, then \( M^{[\phi_\ell]}_\ast \) is \( \mu \)-saturated.

So \( \prec_1^{* \kappa} \) differs from \( \prec_1^* \) in that it allows for infinitary theories to do the interpreting. In particular, the statement that \( \neg(T_0 \prec_1^{* \kappa} T_1) \) is a stronger statement than \( \neg(T_0 \prec_1^* T_1) \). In [MS], Malliaris and Shelah show several positive and negative instances of the interpretability order.

The negative instances are proved by using various Ramsey classes to build generalized blueprints that saturate \( T_1 \) without saturating \( T_0 \). When these Ramsey classes are in fact Erdős-Rado classes, the stronger negative instance can be shown. In the following statement, \( T_{DLO} \) is the theory of dense linear orders and \( T_{RG} \) is the theory of the random graph.

Fact 6.12 ([MS Theorem 5.3]). \( \neg(T_{DLO} \prec_1^* T_{RG}) \)

Theorem 6.13. For every cardinal \( \kappa \), \( \neg(T_{DLO} \prec_1^{* \kappa} T_{RG}) \).

Proof: We rely heavily on citations from [MS]. Note their \( \mathcal{K} = \mathcal{K}_{34} \) is essentially our \( \mathcal{K}^{\lambda-\text{color}} \), which is Erdős-Rado by Example 3.11] and Corollary 4.3. Also, they use GEM to emphasize that the Ehrenfeucht-Mostowski construction uses a generalized blueprint. We adopt [MS Hypothesis 5.5] with our infinitary change, so

1. \( \lambda = \lambda^{<\mu} \geq 2^\mu \).
2. \( T_\ast \) is a skolemized \( L_{\kappa, \omega} \)-theory with \( |T_\ast| \leq \lambda \) that interprets \( T_{RG} \) by \( R_{RG} \) and interprets \( T_{DLO} \) by \( <_{DLO} \).

Note that they point out that their results in this area work for uncountable languages as well.

Since \( \mathcal{K} \) is a combinatorial Erdős-Rado class, there is \( \Phi \in \Upsilon^\mathcal{K}[T_\ast] \). By [MS Corollary 5.10], we can find \( \Psi \) extending \( \Phi \) such that for every separated \( I \in \mathcal{K} \), \( EM_{RG}(I, \Psi) \) is \( \mu \)-saturated. Note that, since \( \Psi \) agrees with \( \Phi \) on \( \tau(T_\ast) \), \( \Psi \) is still in \( \Upsilon^\mathcal{K}[T_\ast] \). By [MS Claim 5.11], if \( J \) is a
separated linear order, then for any $\Phi^* \in \mathcal{Y}^\kappa(T)$, $EM_{DLO}(I, \Phi^*)$ is not $\kappa^+$-saturated. Thus, by taking $I$ separated with a $(\kappa, \kappa)$-cut, we have $EM_{RG}(I, \Psi)$ is $\mu$-saturated, but $EM_{DLO}(I, \Psi)$ is not $\kappa^+$-saturated, as desired.

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