Explosive percolations on the Bethe Lattice

Huiseung Chae, Soon-Hyung Yook, and Yup Kim

Department of Physics and Research Institute for Basic Sciences, Kyung Hee University, Seoul 130-701, Korea
(Dated: December 21, 2013)

Based on the self-consistent equations of the order parameter $P_\infty$ and the mean cluster size $S$, we develop a novel self-consistent simulation (SCS) method for arbitrary percolation on the Bethe lattice (infinite homogeneous Cayley tree). By applying SCS to the well-known percolation models, random bond percolation and bootstrap percolation, we obtain prototype functions for continuous and discontinuous phase transitions. By comparing the key functions obtained from SCSs for the Achlioptas processes (APs) with a product rule and a sum rule to the prototype functions, we show that the percolation transition of AP models on the Bethe lattice is continuous regardless of details of growth rules.

PACS numbers: 64.60.ah, 64.60.De, 05.70.Fh, 64.60.Bd

Since Achlioptas et al. [1] suggested an explosive percolation model, there have been intensive studies on the explosive percolations (EP) [2–12]. Achlioptas process (AP) was originally argued to show the discontinuous phase transition on the complete graph (CG) by suppressing growth of large clusters [1]. Subsequent studies on variants of EP models on networks and lattices also argued to show the discontinuous transition [2–5]. In contrast Riordan and Warnke [13] analytically showed that the phase transition in AP model [1] on CG is continuous by use of the arbitrary connectivity of CG. Furthermore several studies also showed that the transition of variants of EP models on CG is continuous [10–12]. However, EP on CG was still reported to undergo discontinuous transition depending on the detail of cluster growth rule [14, 15]. Therefore the transition nature of EP on CG is not still clear. Since the dimensionality of CG is infinite, physics on CG must satisfy the mean-field theory. In this sense the mean-field theory of explosive percolation is not still clearly understood.

The Bethe lattice (infinite homogeneous Cayley tree) is physically a very important substrate or medium on which the mean-field theories for various physical models become exact [16]. The analytic treatments of magnetic models [17], percolation [16, 18] and localization [16] on the Bethe lattice give important insights to the subsequent developments of the corresponding research fields. One of theoretical merits of the Bethe lattice is that one can setup the exact self-consistent equations (SCEs). In this letter, by use of the exact SCEs we develop a novel self-consistent simulation (SCS) method for arbitrary percolation process on the Bethe lattice. From SCS method, we precisely calculate the order parameter $P_\infty$ and the average size $S$ of finite clusters of the AP models with a product rule or a sum rule. The obtained $P_\infty$ and $S$ can clarify the transition nature of AP models in the infinite dimension exactly. Furthermore unlike AP on CG, the bond connections on the Bethe lattice are purely local. Since there have been some papers that EP models on lattices with local bond connections show the discontinuous transition [2, 4, 9], it is physically important to study EP models on the Bethe lattice or in the mean-field level with local connections. As we shall see, the transition of AP on the Bethe lattice is continuous regardless of the sum rule or the product rule. This result physically means that AP models with local connections in the mean-field level shows the continuous transition irrelevant to the details of growth rules. In this sense the study on the Bethe lattice should give new physical insights to the mean-field theory of AP.

From the structure of the Bethe lattice, SCEs for macroscopically measurable physical quantity can be easily setup [16]. Let’s first setup SCEs for arbitrary percolation on the Bethe lattice. To setup SCE on the Bethe lattice in Fig. 1, one starts with a center site (or origin O) having $z$ bonds. Then let’s think about a part of the Bethe lattice with $n$ generations from O, which have to-
tial \( N = 1 + z(k^n - 1)/(k - 1) \) sites with \( k = z - 1 \). To make complete Bethe lattice, one should add an infinite branch to each of \( zk^{n-1} \) edge sites. To find the probability that \( O \) belongs to the infinite cluster, we need to know the occupation probability \( p \) of a bond (or a site) and the probability \( R \) with which an edge site does not connected to the infinity through the infinite branch connected to it or the branch connected to the edge site has only finite clusters. Then the order parameter \( P_\infty \) of an arbitrary percolation, which is defined by the probability of \( O \) to belong to the infinite cluster, is a function of \( n \), \( p \) and \( R \) as \( P_\infty(p, R, n) \). Then SCE for \( P_\infty \) is

\[
P_\infty = P_\infty(p, R, n) = P_\infty(p, R, n')
\]

for any combination of \( \{n, n'\} \). For the random or normal site percolation Eq. 1 with the combination \( \{n = 1, n' = 2\} \) gives \( P = 1 - p + pR^b \), which analytically reproduces the mean-field properties of percolation transition. If the probability that a cluster including \( O \) with \( s \) sites and \( t \) edge sites within the \( n \)-generation tree is \( P(p, s, t, n, R, S_b) \), then

\[
P_\infty(p, R, n) = 1 - \sum_t R^t \sum_s P(p, s, t, n, R, S_b),
\]

where \( S_b \) is the average size of finite clusters connected to an edge site of the \( n \)-generation tree in Fig. 4 SCE for the average size \( S \) of the finite clusters including \( O \) can also be written as

\[
S = S(p, R, S_b, n) = S(p, R, s, n').
\]

If one cannot calculate \( P(p, s, t, n, R, S_b) \) analytically or if one need to know \( R \) and \( S_b \) apriori to occupy a new bond (or site), one should estimate \( P(p, s, t, n, R, S_b) \) indirectly to solve SCEs 11 and 3. One of the indirect methods is a simulation method. In this letter we develop a simulation method to solve SCEs, which we call the self-consistent simulation (SCS). In SCS, \( P(p, s, t, n, R, S_b) \) is estimated by the relation \( P(p, s, t, n, R, S_b) = N(p, s, t, n, R, S_b)/N_{\text{cluster}} \), where \( N(p, s, t, n, R, S_b) \) is the number of clusters including \( O \) with \( s \) sites and \( t \) edge sites within the \( n \)-generation tree occurred in the simulation runs. Of course \( N_{\text{cluster}} \) is the total number of clusters which includes \( O \) within the \( n \)-generation tree occurred in the same simulation runs. In the simulation both \( P(p, s, t, n, R, S_b) \) and \( P(p, s, t, n', R, S_b) \) are estimated simultaneously using the Bethe lattice with the center \( n \)-generation tree if \( n > n' \). Since we don’t know \( R(p) \) and \( S_b(p) \) apriori, the iteration processes are needed in SCS. From initially guessed values for \( R(p) \) and \( S_b(p) \), the final or saturated values of \( R(p) \) and \( S_b(p) \) are obtained by the iteration of unit simulation process. The unit simulation process consists of

the following two steps I) and II). I) By use of the simulation runs based on the given values \( R(p) \) and \( S_b(p) \)

\[
P(p, s, t, n, R, S_b)\) and \( P(p, s, t, n', R, S_b) \) are estimated. II) From the estimated \( P(p, s, t, n, R, S_b) \), new \( R(p) \) and \( S_b(p) \) are calculated by utilizing SCEs 11 and 3. In the unit simulation process to get the new \( R(p) \) and \( S_b(p) \), the quantities like \( P(p, s, t, n, R, S_b) \) are estimated by averaging over at least \( 10^3 \) simulation runs. Such unit process is repeated until \( R(p) \) and \( S_b(p) \) reach the saturation values. Using the saturated values of \( R(p) \) and \( S_b(p) \), \( P_\infty \) and \( S \) are estimated from Eqs. 2 and 4.

![FIG. 2: (color online) SCS results of \( f_p(A) \) and \( P_\infty \) for random bond percolation (RBP), bootstrap site percolation (BSP) and AP on the Bethe lattice with \( z = 4 \). (a) \( f_p(A) \) of RBP for \( p = 0.31333(< p_c) \), \( p = 0.33333(\approx p_c = 1/3) \), \( p = 0.34333(> p_c) \) and \( p = 0.35333(> p_c) \). (b) Plot of \( P_\infty \) for RBP against \( p \). The line denote the analytic result. (c) \( f_p(A) \) of BSP for \( p = 0.87888(< p_c) \), \( p = 0.88888(\approx p_c = 8/9) \), \( p = 0.89388(> p_c) \) and \( p = 0.89888(> p_c) \). (d) Plot of \( P_\infty \) for BSP against \( p \). The line denote the analytic result. Inset shows the evolution of results of iteration processes. The arrow \( \rightarrow \) denotes the direction from the earlier iteration process to the later iteration process. (e) \( f_p(A) \) of AP with a product rule (PR) for \( p = 0.60575(< p_c) \), \( p = 0.61775(\approx p_c) \), \( p = 0.61775(> p_c) \) and \( p = 0.62575(> p_c) \). (f) Plot of \( P_\infty \) for AP with PR against \( p \). Inset shows the evolution of results of iteration processes. The arrow \( \rightarrow \) means the same thing as in (d).

The phase transition of random or normal percolation (RBP) on the Bethe lattice is well-known to be continuous [18]. We have applied our SCS to random bond percolation on the Bethe lattice with \( z = 4 \). The results
for $z = 4$, $n = 13$, $n' = 5$ are displayed in Figs. (a) and (b). In Fig. (a), we display the simulation results for

$$f_p(A) = P_\infty(p, 1 - A, n') - P_\infty(p, 1 - A, n),$$

(5)

where $A \equiv 1 - R$ for various $p$. From Eq. (1), $A^*$ (or $R^*$) which satisfies $f_p(A^*) = 0$ is the real physical value for a given occupation probability $p$. For $p < p_c$ there occurs only trivial solution $A^* = 0$ as shown in Fig. (a). Increasing $p$ from $p_c$, the nontrivial solution $A^* > 0$ continuously increases from zero. This continuous increase makes the order parameter $P_\infty(p)$ increase continuously as in Fig. (b). The simulation result for $P_\infty(p)$ exactly coincides with the analytic result $P_\infty = 1 - \sqrt{(1 - 3p)/4p^4}$ for $p > p_c(= 1/k = 1/3)$ as shown in Fig. (b). Therefore $f_p(A)$ which behaves like in Fig. (a) is a prototype function for the continuous transition.

Bootstrap site percolation (BSP) on the Bethe lattice [19] is analytically known to show the discontinuous transition. In BSP the order parameter $P_\infty$ is the probability for an occupied site to be a site of the infinite $m$-cluster. Here the $m$-cluster means the cluster in which every occupied site has at least $m$ occupied nearest neighbors. For $m \geq 3$ the phase transition of BSP on the Bethe lattice is discontinuous [19]. By using SCS we have obtained $f_p(A)$ and $P_\infty$ for BSP with $m = 3$ on the Bethe lattice with $z = 4$. The results of SCS for BSP with $z = 4$, $m = 3$, $n = 13$, $n' = 1$ are depicted in Figs. (c) and (d). As shown in Fig. (c), $f_p(A) = 0$ for $p < p_c$ has only the trivial solution $A^* = 0$ as in RBP. In contrast to RBP the nontrivial solution of $f_p(A) = 0$ for $p > p_c$ comes from the peculiar behavior of $f_p(A)$, which reminds us the thermodynamic instability in the thermal mean-field first order transition, which makes the sudden jump of the $A^*$ from zero at $p = p_c = 8/9$. The jump of $A^*$ causes the discontinuous increase of $P_\infty$ as shown in Fig. (d), which is exactly as the analytic result, $P_\infty = p(1 - R)^4 + 4pR(1 - R)^3$ with $R = 1$ for $p < p_c$ and $R = [1 - 3\sqrt{1 - p_c/p}]/4$ for $p > p_c$. [19]. Therefore $f_p(A)$ like in Fig. (c) is a prototype function for the discontinuous transition. Furthermore, as shown in the inset of Fig. (d), the iteration processes in SCS for the BSP with the initial value $R = 0$ drives $P_\infty(p)$ from the continuous increase to the final discontinuous jump at $p = p_c$, which is also a typical behavior of the discontinuous transition on the Bethe lattice.

We now focus the AP model. To occupy a bond in the AP model [1], two bonds $\alpha$ and $\beta$ are randomly chosen. Let $S_{\alpha 1}$ and $S_{\beta 2}(S_{\beta 1}$ and $S_{\beta 2})$ be the sizes of the two clusters which would be connected by occupying bond $\alpha$ ($\beta$). Under a product rule (PR), the bond $\alpha$ is chosen for the growth of clusters and $\beta$ is discarded if $\prod_{j=1}^z S_{\alpha j} < \prod_{j=1}^z S_{\beta j}$. Otherwise, the bond $\beta$ is chosen. AP model with a sum rule (SR) is the same as that with PR except for the change of the condition into $\sum_{j=1}^z S_{\alpha j} < \sum_{j=1}^z S_{\beta j}$. Note that arbitrary edge site is connected to an infinite cluster with the probability $A$ depicted as in Fig. 1. Therefore $S_{\alpha(\beta)j}$ is calculated as

$$S_{\alpha(\beta)j} = s_{\alpha(\beta)j} + \infty \times I_{\alpha(\beta)j} + (t_{\alpha(\beta)j} - I_{\alpha(\beta)j})S_b,$$

(6)

where $s_{\alpha(\beta)j}$ is the number of sites within the $n$-generation tree in the cluster $\alpha(\beta)j$, $I_{\alpha(\beta)j}$ is the number of edge sites and $I_{\alpha(\beta)j}$ is the number of edge sites which are connected to the infinite cluster. Of course $I_{\alpha(\beta)j}$ depends on $R$ or $A$. Therefore $P(p, s, t, n, R, S_b)(= N(p, s, t, n, R, S_b)/N_{\text{cluster}})$ depends apriori on $R$ and $S_b$ and thus the iteration is essential in SCS for AP model. For SCS of AP model, it should be careful to choose $n'(< n)$ for a given $n$, because too small $n'$ cannot convey enough information for AP and $n'$ close to $n$ can never gives physically plausible solutions for SCEs [1] and [3]. From the simulations with various $\{n, n'\}$ it is confirmed that the suitable choice of $n'$ should be in the interval $n/3 < n' < n/2$. $f_p(A)$ and $P_\infty$ for AP model with PR obtained from SCS with $z = 4$, $n = 14$, $n' = 5$ are displayed in Figs. (e) and (f). The results for AP model with SR are nearly the same as those in Figs. (e) and (f) except that $p_c$ for SR is slightly smaller than $p_c$ for PR. As can be seen from Fig. (e), $f_p(A) = 0$ has only trivial solution $A^* = 0$ for $p < p_c$ as RBP and BSP. Increasing $p$ from $p_c$, the nontrivial solution $A^* > 0$ continuously increase from zero. $A^*$ for AP increases very rapidly compared to $A^*$ for RBP as $p$ increases. Except for this rapid increase, $f_p(A)$ for AP behaves like $f_p(A)$ for BSP in Fig. 2(c). The continuous increase of $A^*$ makes the order parameter $P_\infty(p)$ for AP increase continuously as in Fig. 2(f). Moreover, as can be seen from the inset of Fig. (f) the iteration processes in SCS for AP with PR drives $P_\infty(p)$ from the discontinuous jump to the final continuous increase at $p = p_c$, contrary to those in the inset of Fig. 2(d).

![Figure 3](image-url)  

**FIG. 3:** $P_\infty$ against $\Delta p(\equiv 1/(N - 1))$ for $\{n, n'\} = \{7, 4\}, \{9, 4\}, \{11, 5\}, \{13, 5\}, \{14, 5\}$. (a) AP model with PR and (b) the model with SR (b).
Δp = 1/(N − 1), where N is the total number of sites in the n-generation Cayley tree. Therefore in SCS the lowest nonzero $P_n^1$ occurs at $p = p'_c$ very close to the true $p_c$ with $0 < (p'_c − p_c) ≤ Δp$. In Fig. 2(b), we display the dependence of $P_n^1$ on $Δp$ for SCSs of AP models with $(n, n') = \{7, 4\}, \{9, 4\}, \{11, 5\}, \{13, 5\}, \{14, 5\}$. As can be seen from Fig. 2(b), $P_n^1$ for both AP models with PR and SR decreases monotonically to zero as $Δp$ decreases to zero. This result for $P_n^1$ also supports the fact that the phase transition nature of AP on the Bethe lattice is continuous.

![Image of Fig. 4](image-url)

**FIG. 4:** $S$ on the Bethe lattice with $z = 4$. (a) Plot of $S$ for RBP against $p$. The data are obtained from SCS with $n = 13$, $n' = 5$. (b) Plot of $S$ for AP with PR against $p$. The data are obtained from SCS with $n = 14$, $n' = 5$.

From SCS based on Eq. (3), the average size $S$ of the finite clusters is obtained for RBP and AP model with PR as in Fig. 2(a). $S$ for AP model with SR shows nearly the same behavior as Fig. 2(b). It is confirmed that $S$ for RBP from SCS is nearly identical to the analytic result $S = 1 + 4pR/(1 - p - 2pR)$, where $R = 1$ for $p < p_c$ and $R = 1 + \sqrt{(4 - 3p)/4p^2 - 3/2p}$ for $p > p_c$. Even though $S$ for AP model diverges more rapidly than $S$ for RBP, $S$ for both RBP and AP diverges as $S(p) \simeq |p - p_c|^{-\gamma}$ with the susceptibility exponent $\gamma = 1.00(1)$. The result for $S$ of AP models also supports that the transition in AP models is continuous on the Bethe lattice.

Since the transition in the AP models is continuous, the order parameter exponent $\beta$ is calculated by fitting the relation $P_\infty \simeq (p - p_c)^\beta$ to the data for $p > p_c$ but very close to $p_c$ in Fig. 2(f). Since $f_p(A)$ for $p \simeq p_c$ in Fig. 2(e) increases very slowly as $A$ increases from zero, $P_\infty$ increases very rapidly and the exponent $\beta$ is expected to be very small. From the best fit the obtained exponent is $\beta = 0.05(5)$. The data even fits very well to $P_\infty \simeq |\ln(p - p_c)|^{-\chi}$ with $\chi = 3.4(1)$. This result means that the exponent $\beta$ on the Bethe lattice is very small and nearly identical to those obtained on the complete graph [4, 5, 12].

In summary, we show that AP models on the Bethe lattice have a continuous transition from the SCSs developed to cover arbitrary percolation processes on the Bethe lattice. For this $f_p(A)$ for AP is first shown to be physically identical to that for RBP. We also shown that $P_\infty^1$ for AP model decrease to zero as $Δp = 1/(N − 1)$ goes to zero. The divergent behavior of $S$ for AP models is also shown to be the same as that for RBP with $\gamma = 1$. The exponent $\beta$ is also shown to be very small or $\beta \simeq 0.05$.

This work was supported by National Research Foundation of Korea (NRF) Grant funded by the Korean Government (MEST) (Grant Nos. 2009-0073939 and 2011-0015257).

[1] D. Achlioptas, R. M. D’Souza, and J. Spencer, Science 323, 1453 (2009).
[2] R. M. Ziff, Phys. Rev. Lett. 103, 045701 (2009) and Phys. Rev. E 82, 051105 (2010).
[3] Y. S. Cho, J. S. Kim, J. Park, B. Kahng, and D. Kim, Phys. Rev. Lett. 103, 135702 (2009).
[4] F. Radicchi and S. Fortunato, Phys. Rev. Lett. 103, 168701 (2009) and Phys. Rev. E 81, 036110 (2010) and S. Fortunato, F. Radicchi, J. Phys.: Conf. Ser. 297, 012009 (2011).
[5] E. J. Friedman and A. S. Landsberg, Phys. Rev. Lett. 103, 255701 (2009).
[6] R. M. D’Souza and M. Mitzenmacher, Phys. Rev. Lett. 104, 195702 (2009).
[7] Y. S. Cho, S.-W. Kim, J. D. Noh, B. Kahng, and D. Kim, Phys. Rev. E 82, 042102 (2010).
[8] Y. Kim, Y. K. Yun, and S.-H. Yook, Phys. Rev. E 82, 061105 (2010).
[9] W. Choi, S.-H. Yook and Y. Kim, Phys. Rev. E 84, 020102 (2010).
[10] R. A. da Costa, S. N. Dorogovtsev, A. V. Goltsev, and J. F. F. Mendes, Phys. Rev. Lett. 105, 255701 (2010).
[11] H. K. Lee, B. J. Kim, and H. Park, Phys. Rev. E 84, 020101 (2011).
[12] P. Grassberger, C. Christensen, G. Bizhani, S.-W. Son, and M. Paczuski, Phys. Rev. Lett. 106, 252701 (2011).
[13] O. Riordan and L. Warnke, Science 333, 322 (2011).
[14] Y. S. Cho and B. Kahng, Phys. Rev. Lett. 107, 275703 (2011).
[15] R. M. D’Souza and M. Mitzenmacher, Phys. Rev. Lett. 104, 195702 (2010).
[16] M. F. Thorpe, Excitations in disordered systems (N. Y. New York, Plenum Press, 1982).
[17] H. A. Bethe, Proc. R. Soc. London, Ser. A 150 (1935).
[18] D. Stauffer and A. Aharony, “Introduction to Percolation Theory” 2nd Ed. (Taylor & Francis, London and New York, 1994).
[19] J. Chalupa, P. L. Leath, and G. R. Reich, J. Phys. C 12, L31 (1979).