Bayes-optimal Methods for Finding the Source of a Cascade

Anirudh Sridhar
Department of Electrical Engineering
Princeton University
Princeton, NJ
anirudhs@princeton.edu

H. Vincent Poor
Department of Electrical Engineering
Princeton University
Princeton, NJ
poor@princeton.edu

Abstract—We study the problem of estimating the source of a network cascade. The cascade starts from a single vertex at time 0 and spreads over time, but only a noisy version of the propagation is observable. The goal is then to design a stopping time and estimator that will estimate the source well while ensuring the cost of the cascade to the system is not too large. We rigorously formulate a Bayesian approach to the problem. If vertices can be labelled by vectors in Euclidean space (which is natural in geo-spatial networks), the optimal estimator is the conditional mean estimator, and we derive an explicit form for the optimal stopping time under minimal assumptions on the cascade dynamics. We study the performance of the optimal stopping time on the line graph, and show that a computationally efficient but suboptimal stopping time which compares the posterior variance on the line graph, and show that a computationally efficient but suboptimal stopping time which compares the posterior variance to a threshold has near-optimal performance. Our theoretical results are supported by simulations.

Index Terms—Network cascade, sequential estimation, optimal stopping theory, stochastic optimization

I. INTRODUCTION

Network dynamics are often unstable: the behaviors of a small subset of vertices may rapidly disseminate to the rest of the network. This type of instability, known as a network cascade, has been observed in diverse applications such as the spread of diseases in geographical networks [1]–[3], malware in a computer networks [4], [5], and fake news in social networks [6]–[8]. When such cascading failures are present in a network, it is of utmost importance to find the source as fast as possible. Unfortunately, in many cases of interest the cascade is not directly observable. For instance, if an epidemic is spreading in a geographical network and an individual falls sick, it could be a symptom of the disease or it could be due to exogenous factors (e.g. allergies). Over time, one may better distinguish between these possibilities and construct better source estimates at the cost of allowing the cascade to propagate even further. An optimal algorithm in our framework will achieve the best possible tradeoff between estimation error and system cost caused by the cascade.

In this paper we provide a Bayesian solution to this problem. If vertices can be labelled by vectors in Euclidean space, we derive an explicit form for the optimal source estimation algorithm under minimal assumptions on the cascade dynamics. We then study its performance in the line graph for a deterministic cascade, pinning down the runtime of the optimal algorithm for a certain class of Bayes priors. Though the optimal algorithm is challenging to compute even in the line graph, we show that an easily-computable algorithm enjoys near-optimal performance for a large class of system cost functions.

A. A model of network cascades with noisy observations

Let $G$ be a graph with vertex set $V$. To avoid boundary effects, we assume that $G$ has infinitely many vertices and is locally finite (i.e., all vertices have finite degree). The network cascade starts from some vertex $v \in V$ and spreads over time via the edges of the graph according to some known (random or deterministic) discrete-time process. For each $u \in V$ and for each nonnegative integer time index $t$, we let $x_u(t) \in \{0, 1\}$ denote the private state of the vertex, where $x_u(t) = 0$ if the vertex $u$ has not been affected by the cascade yet and $x_u(t) = 1$ otherwise. We assume that once $x_u(t) = 1$, the effect of the cascade on $u$ continues for all future $t$. The private states are not observable, but the system instead monitors the public states $\{y_u(t)\}_{u \in V}$, defined as

$$y_u(t) \sim \begin{cases} Q_0 & x_u(t) = 0; \\ Q_1 & x_u(t) = 1, \end{cases}$$

where $Q_0$ and $Q_1$ are two mutually absolutely continuous probability measures. We can think of $y_u(t) \sim Q_0$ being typical behavior and $y_u(t) \sim Q_1$ as anomalous behavior caused by the cascade. As a shorthand, we denote $y(t) := \{y_u(t)\}_{u \in V}$ to be the collection of all public states at time $t$. This type of model has been studied in recent literature in the context of cascade source estimation [9] and quickest detection of cascades [10]–[13].

B. Formulation as a stochastic optimization problem

Let $(\Omega, \mathcal{P}, \mathcal{F})$ be a common probability space for all random objects, and let $\{\mathcal{F}_t\}_{t=0}^\infty$ be the natural filtration formed by the public states: $\mathcal{F}_t := \sigma(y(0), \ldots, y(t))$. Any algorithm for estimating the cascade source may be represented by $(T, \hat{v})$, where $T$ is a stopping time and $\hat{v} = \{\hat{v}(t)\}_{t=0}^\infty$ is a sequence of source estimators, $\hat{v}(t)$ being $\mathcal{F}_t$-measurable. Suppose that we...
quantifying the cost of the cascade on the network functionality at time $t$ by $c(t)$. Let $d(\cdot, \cdot)$ be some distance measure between vertices of the graph (not necessarily the shortest-path metric) and let $p$ be some probability distribution over $V$, where the cascade source is sampled as $v_0 \sim p$. Then we would like to find an algorithm $(T, \hat{v})$ that achieves

$$\min_{T, \hat{v}} \mathbb{E}\left[ d(v_0, \hat{v}(T)) + \sum_{t=1}^{T} c(t) \right]. \quad (1)$$

In other words, the optimal algorithm will minimize the sum of the estimation error and the running cost on the system caused by the cascade. This formulation implies that the optimal estimator minimizes the risk defined by the distance function:

$$\hat{v}(t) = \arg \min_{v \in V} \mathbb{E}[d(v_0, v) | F_t]. \quad (2)$$

If $d$ is the shortest-path distance, the estimator is a complex function of the graph topology and the past observations. However, if vertices can be labelled by vectors in $\mathbb{R}^d$ for some $d$, the analysis simplifies considerably. Given a probability distribution over the vertices of the graph, we can now compute basic statistical quantities such as the expected value and variance of a random variable sampled from the distribution.

If we set the distance function to be the square of the $\ell_2$ norm, the optimal estimator is exactly the conditional mean estimator: $\hat{v}(t) = \mathbb{E}[v_0 | F_t]$. This estimator has a convenient martingale structure which we will heavily exploit in deriving the optimal stopping time.

We remark that in many applications, such as in power grids and in geographical networks, vertices may represent coordinates in space and thus a labelling of vertices by vectors is natural. Well-studied mathematical examples where such a labelling is natural are in lattices and random geometric graphs. Although one may always arbitrarily label the vertices of the graph by vectors, the results and methods will only be relevant in practice if the Euclidean distance between two vertices measures closeness in some sense.

C. Related work

Shah and Zaman first studied the problem of estimating the source of a diffusion in a network [15], [16]. In their setup, they assume that the system observes a noiseless snapshot of the private states at some given timestep. The goal is then to estimate the source from this snapshot.

Our work naturally falls under the growing body of work on sequential inference of cascades, which assumes access to noisy streaming data (as opposed to inference from a noiseless snapshot) generated by the variants of the model in Section I-A. Most of this literature has studied the quickest detection problem, which aims to detect with minimum delay when the cascade affects a certain number of vertices. It is assumed that the cascade begins at some unknown time [10]–[14]. The closest work to ours is by Sridhar and Poor [9], who study the source estimation problem in a non-Bayesian setting. In their formulation, the optimal algorithm achieves the minimum expected run length subject to the estimation error being at most $\alpha$. As $\alpha \to 0$, they show that matrix sequential probability ratio tests (MSPRT) achieve the minimum expected run length, up to constant multiplicative factors. While they assume general structural conditions on the graph topology, the cascade dynamics (equivalently, the evolution of the private states) are simple and deterministic. On the other hand, our results assume that the graph admits a vertex labelling by vectors but applies to a general class of cascade dynamics for which the conditional mean estimator is consistent.

D. Organization

In Section II we derive an expression for the optimal stopping time under general assumptions on the cascade dynamics. In Section III we study the theoretical and empirical performance of the optimal stopping time for a cascade propagating in the line graph. Finally, we conclude in Section V.

II. DERIVING THE BAYES-OPTIMAL SOLUTION

**Theorem 1.** Assume that the estimator $\hat{v}(t) := \mathbb{E}[v_0 | F_t]$ is consistent; that is, $\hat{v}(t) \to v_0$ as $t \to \infty$ almost surely, and assume that the prior has finite variance. Let $T_k$ be the set of stopping times such that $T \geq s$ a.s. for $T \in T_k$. Define, for any stopping time $T \in T_s$, the random variable

$$f_s(T) := \|\hat{v}(T) - \hat{v}(s)\|^2_2 - \sum_{t=s}^{T-1} c(t+1). \quad (3)$$

Then the optimal stopping time that solves (1) is

$$T_{opt} := \min \left\{ s \geq 0 : \sup_{T \in T_s} \mathbb{E}[f_s(T) | F_s] = 0 \right\}. \quad (4)$$

We can interpret the optimal stopping time as follows. The quantity $\mathbb{E}\left[\|\hat{v}(T) - \hat{v}(s)\|^2_2 | F_s\right]$ is the expected amount of information gained about $v_0$ at time $T$, conditioned on current information. On the other hand, $\sum_{t=s}^{T-1} c(t+1)$ is running system cost during this time. If, for every $T_s$, the information gained is less than the sampling cost, then it is not worth it to take even a single extra step. Conversely, if there is some $T \in T_s$ where the information gained is greater than the running cost until that point, then it is worth it to keep sampling.

The proof of the theorem relies on a result from optimal stopping theory, which we briefly review. Let $\{Y_t\}_t$ be an adapted sequence of stochastic rewards, so that $Y_t$ is $F_t$-measurable. Let $T$ be the set of stopping times. The goal is to find a stopping time $T \in T$ that achieves $\mathbb{E}[Y_T]$. For any integer $k \geq 0$, define

$$\gamma_k := \text{ess sup}_{T \in T_k} \mathbb{E}[Y_T | F_k].$$

Informally, $\gamma_k$ is the maximum expected reward possible, given the information at time $k$. The following result gives a closed-form expression for the optimal stopping time in terms of $\gamma_k$. We omit the proof.

**Theorem 2** ([3.7 in [17]]. Suppose that $\mathbb{E}\left[\sup_k Y_k^+\right] < \infty$. Then the stopping time

$$T_{opt} := \min\{k \geq 0 : Y_k = \gamma_k\}$$
is optimal, in the sense that it achieves $\sup_{t \in T} E[Y_T]$.

We will now use this result to prove Theorem 1.

**Proof of Theorem 1.** We begin by reformulating (1) to be in the optimal stopping framework. By orthogonality of martingale increments and consistency of the conditional mean estimator, we can write $E\left[\left\|\hat{v}(k) - v_0\right\|^2_2\right]$ as

$$E\left[\left\|\hat{v}(0) - v_0\right\|^2_2\right] - \sum_{t=0}^{k-1} E\left[\left\|\hat{v}(t+1) - \hat{v}(t)\right\|^2_2\right].$$

Thus we can rewrite the objective of (1) as

$$E\left[\left\|\hat{v}(0) - v_0\right\|^2_2\right] - \sum_{t=0}^{T-1} \left(\left\|\hat{v}(t+1) - \hat{v}(t)\right\|^2_2 + \sum_{c(t)}^T\right) .$$

It follows that optimal stopping time for (1) also achieves

$$\sup_{t \in T} E\left[\sum_{s=0}^{T-1} \left(\left\|\hat{v}(t+1) - \hat{v}(t)\right\|^2_2 - c(t+1)\right)\right].$$

To apply Theorem 2 it suffices to check that $E\left[\sup_{k} \sum_{t=0}^{k-1} \left\|\hat{v}(t+1) - \hat{v}(t)\right\|^2_2\right] < \infty$, which follows from finiteness of $E\left[\left\|v_0 - \hat{v}(0)\right\|^2_2\right]$ and orthogonality of martingale increments. Applying Theorem 2 we see that

$$T_{opt} := \min \left\{ s \geq 0 : \sum_{t=0}^{s-1} \left(\left\|\hat{v}(t+1) - \hat{v}(t)\right\|^2_2 - c(t+1)\right) \right\} = \sup_{T \in T_s} \left\{ \sum_{t=0}^{T-1} \left(\left\|\hat{v}(t+1) - \hat{v}(t)\right\|^2_2 - c(t+1)\right) \right\} .$$

Rearranging, the condition in the stopping time is

$$\sup_{T \in T_s} \left\{ \sum_{t=0}^{T-1} \left(\left\|\hat{v}(t+1) - \hat{v}(t)\right\|^2_2 - c(t+1)\right) \right\} = 0 .$$

The next result follows immediately from the form of $T_{opt}$.

**Corollary 1.** Let $r$ be a positive integer. Recall the definition of $f_s$ from (3) and define the stopping times

$$T_r := \min \left\{ s \geq 0 : f_s(r) \leq 0 \right\} .$$

$$T_r := \min \left\{ s \geq 0 : E\left[\left\|v_0 - \hat{v}(s)\right\|^2_2 \mid F_s\right] \leq \sum_{c(s)}^T \right\} .$$

Then $T_r \leq T_{opt} \leq T_r$ almost surely.

In practice, computing $T_{opt}$ may be computationally intensive if standard methods such as Monte Carlo sampling are used to estimate $E[\|\hat{v}(T) - \hat{v}(s)\|^2_2 \mid F_s]$. The stopping time $T_r$ may be a desirable alternative to $T_{opt}$ in practice since it is easy to compute from the posterior distribution and directly gives a bound on estimation error. In the next section, we will show that $T_r$ is orderwise optimal in many cases.

### III. PERFORMANCE ANALYSIS IN THE LINE GRAPH

In this section we characterize the performance of the optimal stopping time on the infinite line graph. First, we introduce some notation and assumptions. Let $G = (V,E)$ refer to the infinite line graph, and identify the vertices of the line graph with the integers where $d(u,v) = |u - v|$ if and only if $u$ and $v$ are adjacent. For each $v \in V$, define the measure $P_v \equiv P(\cdot \mid v_0 = v)$. Fix a positive integer $n$, and let the prior be the uniform distribution over all vertices $v$ satisfying $|v| \leq n$; let $V_n$ be the set of all such vertices. For ease of analysis, we assume that $Q_0, Q_1$ are normally distributed, with $Q_0 \equiv N(0,1)$ and $Q_1 \equiv N(\mu,1)$, where $\mu \neq 0$. Let $T_{opt}^n$ and $T_v^n$ denote the stopping times assuming that $v_0$ is chosen uniformly from $V_n$. We assume the deterministic cascade dynamics of $[9]$: $x_0(t) = 1 \iff |u - v_0| \leq t$.

**Theorem 3.** Suppose that for some $c > 0, c(t+1) \geq (4 + c)f^2$ and $\sum_{t=0}^{r-1} c(t+1) \leq (1 - c)\frac{n^2}{2}$ where $r = \sqrt{\frac{\log n}{5\mu^2}}$. Then

$$\lim_{n \to \infty} \min_{v \in V_n} P_v \left(\sqrt{\frac{\log n}{5\mu^2}} \leq T_v^n \leq 24\sqrt{\frac{\log n}{\mu^2}}\right) = 1 .$$

Moreover, the same holds for $T_r^n$.

In particular, if $c(t) \sim t^k$ for some $k > 2$, the value of the objective in (1) for the stopping time $T_r^n$ is optimal up to a constant multiplicative factor. The constants that are in the constraints for the cost functions are artifacts of the proof, and we expect that these can be improved. See Figure 1b for empirical results in the line graph.

Due to the structure of the stopping times, the proof of Theorem 3 follows from characterizing the variance of the posterior. The following two lemmas show that the variance of $\pi(t)$ undergoes a sharp transition when $t$ is on the order of $\sqrt{\log n}$. This is illustrated empirically in Figure 1a.

In the subsequent lemmas, we use the following notation. For two functions $f_1(n)$ and $f_2(n)$, we say $f_1(n) \sim f_2(n)$ if the first-order terms in $n$ of $f_1(n)$ and $f_2(n)$ are equal. We analogously define $\propto_n$ and $\propto_n$.

**Lemma 1.** Suppose that $s \leq \sqrt{\frac{\log n}{\mu^2}}$. Then for any $v \in V_n$,

$$P_v \left(\frac{(v_0 - \hat{v}(s))^2}{F_n} \geq s_n \frac{n^2}{6}\right) = 1 - O \left(\sqrt{\frac{\log n}{n^{1.9}}}\right) .$$

Since the prior is uniform over $V_n$, the variance is initially on the order of $n^2$. The above lemma shows that with high probability the variance remains large for some time before decaying. On the other hand, the following lemma shows that the variance drops quickly once $t$ is large enough.

**Lemma 2.** For $t \geq 24\sqrt{\sqrt{\frac{\log n}{\mu^2}}} \text{ and any } r \leq t, v \in V_n$,

$$P_v \left(\frac{(v_0 - \hat{v}(s))^2}{F_n} \leq 4t^2\right) \geq 1 - O \left(\frac{1}{n^{3.9}}\right) .$$

The proof of Theorem 3 follows directly from the lemmas.

**Proof of Theorem 3.** Set $r = 24\sqrt{\frac{\log n}{\mu^2}}$. Then for any $s \leq r$, we can decompose $E[(v_0 - \hat{v}(s))^2 \mid F_s]$ as

$$E[(v_0 - \hat{v}(s))^2 \mid F_s] - E[(v_0 - \hat{v}(s))^2 \mid F_s] .$$
If \( s \leq \sqrt{\frac{\log n}{5\mu^2}} \), then Lemmas 1 and 2 imply that with probability at least 1 \(- O\left(\frac{\log n}{n^{1/4}n^r}\right)\), \(\mathbb{E}[(\hat{v}(r) - \hat{v}(s))^2 \mid F_s] \geq r \cdot \frac{1}{4} n^2 \).

Using the assumption that \( \sum_{t=0}^{r-1} c(t + 1) = (1 - \epsilon) \frac{n^2}{6} \), we can upper bound \( \mathbb{P}_v\left(T^+ \leq \sqrt{\frac{\log n}{5\mu^2}}\right) \) by

\[
\mathbb{P}_v\left(\exists s \leq \sqrt{\frac{\log n}{5\mu^2}} \colon \mathbb{E}[(\hat{v}(r) - \hat{v}(s))^2 \mid F_s] \leq \sum_{t=0}^{r-1} c(t + 1)\right) \\
\leq \sum_{s=0}^{\sqrt{\frac{\log n}{5\mu^2}}} \mathbb{P}_v\left(\mathbb{E}[(\hat{v}(r) - \hat{v}(s))^2 \mid F_s] \leq (1 - \epsilon) \frac{n^2}{6}\right),
\]

which in turn is bounded by \( O\left(\frac{\log n}{n^{1/4}n^r}\right) \). In light of Corollary 1, \( T_{\text{opt}} \geq \sqrt{\frac{\log n}{\mu^2}} \) with probability tending to 1 as \( n \to \infty \). On the other hand,

\[
\mathbb{P}_v(T^+ > t) \leq \mathbb{P}_v(\mathbb{E}[(v_0 - \hat{v}(t))^2 \mid F_t] > c(t + 1)) \\
\leq \mathbb{P}_v(\mathbb{E}[(v_0 - \hat{v}(t))^2 \mid F_t] > (4 + \epsilon)t^2). \]

By Lemma 2, the above is \( O(n^{-3}) \) for \( t \geq 24\sqrt{\frac{\log n}{\mu^2}} \) and the desired result follows. \( \square \)

**IV. CHARACTERIZING THE POSTERIOR VARIANCE**

In this section we prove Lemmas 1 and 2. We begin by describing the distribution of the posterior probabilities. For any \( u, v \in V \), we have the recursion

\[
\frac{\pi_u(t)}{\pi_v(t)} = \frac{\pi_u(t-1)}{\pi_v(t-1)} \cdot \frac{d\mathbb{P}_u(y(t))}{d\mathbb{P}_v(y(t))}.
\]

Let \( \mathcal{N}_v(t) \) denote the vertices within distance \( t \) of \( u \). Then

\[
\frac{d\mathbb{P}_u(y(t))}{d\mathbb{P}_v(y(t))} = \prod_{w \in \mathcal{N}_v(t)} dQ_1(y_u(t)) \cdot \prod_{w \notin \mathcal{N}_v(t)} dQ_0(y_v(t))
\]

\[
= \prod_{w \in \mathcal{N}_v(t) \cap \mathcal{N}_u(t)} dQ_1(y_u(t)) \cdot \prod_{w \not\in \mathcal{N}_u(t) \cap \mathcal{N}_v(t)} dQ_0(y_v(t)).
\]

Since \( Q_0 \) and \( Q_1 \) are Gaussian, \( \log \frac{dQ_1}{dQ_0}(y_w(t)) = \mu_y(t) - \frac{\mu_y(t)^2}{2} \). It follows that

\[
\log \frac{d\mathbb{P}_u}{d\mathbb{P}_v}(y(t)) = \mu \left( \sum_{w \in \mathcal{N}_u(t)} y_u(t) - \sum_{w \notin \mathcal{N}_u(t)} y_v(t) \right).
\]

Putting everything together, we can see that for \( u \in V_n \),

\[
\pi_u(t) \approx \frac{1}{2n + 1} \exp \left( \mu \sum_{s=0}^{t} \sum_{w \in \mathcal{N}_u(s)} y_u(s) \right),
\]

with normalizing constant

\[
Z(t) := \frac{1}{2n + 1} \sum_{u \in V_n} \exp \left( \mu \sum_{s=0}^{t} \sum_{w \in \mathcal{N}_u(s)} y_u(s) \right).
\]

Denote \( X_u(t) := \mu \sum_{s=0}^{t} \sum_{w \in \mathcal{N}_u(s)} y_u(s) \). Under \( \mathbb{P}_v \),

\[
X_u(t) \sim N \left( \mu^2 \sum_{s=0}^{t} |\mathcal{N}_v(s) \cap \mathcal{N}_u(s)|, \mu^2 \sum_{s=0}^{t} |\mathcal{N}_u(s)| \right).
\]

As a shorthand, we write \( g_v(u) := \sum_{s=0}^{t} |\mathcal{N}_v(s) \cap \mathcal{N}_u(s)| \). In the case of the line graph, we can explicitly compute \( |\mathcal{N}_v(t)| = 2t + 1 \) and \( \sum_{s=0}^{t} |\mathcal{N}_v(s)| = (t + 1)^2 \).

**Lemma 3.** Assume that \( t \leq \sqrt{\log n} \). Then for any \( v \in V_n \),

\[
\mathbb{P}_v \left( Z(t) - e^{\frac{\mu^2}{2}}(t + 1)^2 > \frac{1}{\log n} e^{\frac{\mu^2}{2}}(t + 1)^2 \right) \leq O \left\{ \frac{1}{n} \right\}.
\]

**Proof.** Under \( \mathbb{P}_v \), using basic properties of the log-normal distribution we can write

\[
\mathbb{E}_v[Z(t)] = \frac{1}{2n + 1} \sum_{u \in V_n} \exp\left( \mu^2 g_v(u) + \frac{\mu^2}{2} (t + 1)^2 \right) = \sum_{u \in V_n} \left( e^{\mu^2 g_v(u)}(t + 1)^2 \right).
\]

\[
eq e^{\frac{\mu^2}{2}}(t + 1)^2 \left( 1 + \frac{1}{2n + 1} \sum_{u \in V_n} \left( e^{\mu^2 g_v(u)}(t + 1)^2 - 1 \right) \right).
\]
Since $g_{uv}(t) \leq (t + 1)^2$, the summation is bounded by
$\frac{(4t+1)^2(2n+1)^2}{12n^2}$, which tends to 0 as $n \to \infty$ for $t \leq \sqrt{\log n}$.

Next, to establish concentration, we bound the variance of $Z(t)$. We can write

$$\text{Var}_v(Z(t)) = \frac{1}{(2n+1)^2} \sum_{u,w \in V_n} \text{Cov}_v(e^{X_u(t)}, e^{X_w(t)}) .$$

If $d(u,v) > 2t$ then $X_u(t)$ and $X_w(t)$ are independent, so the covariance term is zero in these cases. Thus we can bound the variance by

$$\frac{1}{(2n+1)^2} \sum_{u,w \in V_n; d(u,w) \leq 2t} \text{E}_v(e^{X_u(t)}+X_w(t)) .$$

By the Cauchy-Schwartz inequality,

$$\text{E}_v\left[ e^{X_u(t)+X_w(t)} \right] \leq \text{E}_v\left[ e^{2X_u(t)} \right]^{1/2} \text{E}_v\left[ e^{2X_w(t)} \right]^{1/2} = \exp\left( \mu^2 g_{uv}(t) + \mu^2 g_{uw}(t) + 2\mu^2 (t + 1)^2 \right) \leq e^{4\mu^2(t+1)^2} .$$

Putting everything together, the bound is for $d(u,v) > 2t$.

Putting the second term on the right hand side above, note that the covariance term is 0 when $d(u,w) > 2s$ and when $d(u,w) \leq 2s$ it is bounded by $e^{4\mu^2(s+1)^2}$. Thus we can bound the second term of (5) by

$$e^{4\mu^2(s+1)^2} \sum_{u,w \in V_n; d(u,w) \leq 2s} |u||w| \leq n \left( s + \frac{1}{4} \right) ne^{4\mu^2(s+1)^2} .$$

For the first term in (5), we can write

$$\sum_{u \in V_n} \text{E}_n\left[ e^{X_u(s)} \right] = \sum_{u \in V_n} e^{\mu^2(s+1)^2} .$$

By Chebyshev’s inequality, it then follows that

$$\mathbb{P}_v\left( \frac{\sum_{u \in V} u^2 e^{X_u(s)}}{2n+1} \leq 1 - \frac{1}{6} e^{2(s+1)^2} n^2 \right) \leq 18e^{3\mu^2(s+1)^2} 4t + 1 .$$

Since $s \leq \sqrt{\log n}$, the probability bound is $O\left( \frac{\sqrt{\log n}}{n^{\gamma/10}} \right)$. Combining this result with Lemma 3 shows that

$$\sum_{u \in V_n} u^2 \pi_u(s) \geq n \cdot n^2 \mu^2 \text{ with probability at least } 1 - O\left( \frac{\sqrt{\log n}}{n^{\gamma/10}} \right) .$$

Next, we turn to upper bounding the second term on the right hand side in (4). We can write

$$\text{E}_v\left[ \left( \sum_{u \in V_n} e^{X_u(s)} \right)^2 \right] = \left( \sum_{u \in V_n} \text{E}_v\left[ e^{X_u(s)} \right] \right)^2 + \sum_{u,u \in V_n} \text{Cov}_v\left( e^{X_u(s)}, e^{X_w(s)} \right) .$$

To bound the second term on the right hand side above, note that the covariance term is 0 when $d(u,w) > 2s$ and when $d(u,w) \leq 2s$, it is bounded by $e^{4\mu^2(s+1)^2}$. Thus we can bound the second term of (4) by

$$e^{4\mu^2(s+1)^2} \sum_{u,u \in V_n; d(u,w) \leq 2s} u^2 \mu^2 g_{uw}(s) \leq n \left( 1 + \frac{1}{4} \right) ne^{4\mu^2(s+1)^2} .$$

In obtaining the last equality, we have used the fact that

$$\sum_{u \in V_n} u = 0 .$$

Bounding $u \leq n$ and $e^{\mu^2(s+1)^2} - 1 \leq e^{\mu^2(s+1)^2}$, we obtain the bound

$$\left( \sum_{u \in V_n} \text{E}_n\left[ e^{X_u(s)} \right] \right)^2 \leq n \left( 4s + 1 \right) e^{4\mu^2(s+1)^2} .$$

Putting everything together, it follows from Markov’s inequality and $s \leq \sqrt{\log n}$ that

$$\mathbb{P}_v\left( \frac{\sum_{u \in V_n} e^{X_u(s)}}{2n+1} > n^{2-s/4} \right) = O\left( \frac{\sqrt{\log n}}{n^{1/10}} \right) .$$

By Lemma 3, the covariance term is 0 if $d(u,w) > 2s$, otherwise $u^2 \mu^2 \leq n^2 \mu^2$. We conclude by noting that with high probability,

$$\sum_{u \in V_n} u^2 \pi_u(s)^2 \text{ is orderwise (in } n \text{) smaller than } \sum_{u \in V_n} u^2 \pi_u(s) .$$
Proof of Lemma 2. For any vertex \( v \in V \), we can write
\[
E \left[ (v_0 - \hat{v}(t))^2 \mid F_s \right] \leq \sum_{u \in V_n} (u - v)^2 \pi_u(t).
\]
Taking an expectation conditioned on \( F_s \) gives the bound
\[
\sum_{v \in V_n} \pi_v(s) \sum_{u \in V_n} (u - v)^2 \mathbb{E}_v[\pi_u(t) \mid F_s].
\]
Fix \( v \in V_n \), we will proceed by bounding the inner summation over \( u \in V_n \). Since \( \sum_{u \in V_n} \mathbb{E}_v[\pi_u(t) \mid F_s] = 1 \), we can bound
\[
\sum_{u \in V_n, d(v,u) \leq 2t} (u - v)^2 \mathbb{E}_v[\pi_u(t) \mid F_s] \leq 4t^2.
\]
It remains to bound the summation over \( u \in V_n \) such that \( d(v,u) > 2t \). To this end, define the events
\[
E_{1,v} := \{ |X_u(s) - \mu^2 g_v(u(s))| \leq 3|\mu|(s + 1)\sqrt{\log n}, \forall u \in V_n \};
\]
\[
E_{2,v} := \{ |X_u(t) - X_u(s) - \mu^2 g_v(u(t) - g_v(v(s)))| \leq 3|\mu|(s + 1)\sqrt{\log n}, \forall u \in V_n \}.
\]
Since \( X_u(s) \) and \( X_u(t) - X_u(s) \) are independent under \( \mathbb{P}_v \), the two events are \( \mathbb{P}_v \)-independent as well. We will show that \( E_{1,v} \) and \( E_{2,v} \) occur with high probability under \( \mathbb{P}_v \). By Gaussian concentration,
\[
\mathbb{P}_v \left( |X_u(s) - \mu^2 g_v(u(s))| > 3|\mu|(s + 1)\sqrt{\log n} \right) \leq \frac{2}{n^4}.
\]
Taking a union bound over \( u \in V_n \), we see that \( \mathbb{P}_v(\mathbb{E}_{1,v}^c) = O(n^{-3}) \). Using identical arguments, we can show that \( \mathbb{P}_v(\mathbb{E}_{2,v}^c \mid F_s) = O(n^{-3}) \) as well. On the event \( E_{1,v} \cap E_{2,v} \), we have that under \( \mathbb{P}_v \),
\[
Z_v(t) \geq \frac{e^{X_v(t)}}{2n + 1} \geq \frac{e^{\mu^2(t+1)^2 - 3|\mu|(t+1)\sqrt{\log n} - 3|\mu|(s+1)\sqrt{\log n}}}{2n + 1} \geq \frac{e^{\mu^2(t+1)^2 - 6|\mu|(t+1)\sqrt{\log n}}}{2n + 1}.
\]
It follows that on \( E_{1,v} \cap E_{2,v} \) and under \( \mathbb{P}_v \), if \( d(v,u) > 2t \) then
\[
\pi_u(t) \leq e^{-\mu^2(t+1)^2 + 12|\mu|(t+1)\sqrt{\log n}} \leq e^{-\frac{4}{n^2}(t+1)^2},
\]
which tends to 0 as \( n \to \infty \) since \( t \geq 24 \sqrt{\frac{\log n}{\mu^2}} \). On the other hand,
\[
\sum_{u \in V_n, d(v,u) > 2t} (u - v)^2 \mathbb{E}_v[\pi_u(t) \mathbb{1}_{E_{2,v}}] \lesssim \mathbb{P}_v(\mathbb{E}_{2,v}^c \mid F_s) n^3,
\]
which is \( O(n^{-1}) \). Thus the summation over \( u \in V_n \) with \( d(v,u) \leq 2t \) dominates, and the desired result follows.

V. Conclusion

In this work we have formulated and developed a Bayesian approach to the problem of estimating the source of a cascade, given noisy time-series observations of the network. If vertices can be labelled by vectors in Euclidean space, we use optimal stopping theory to derive the Bayes-optimal stopping time. We then studied the performance of the optimal stopping time in the line graph. Though the optimal estimator is computationally intensive, the stopping time \( T^+ \) which compares the posterior variance to a threshold is orderwise optimal. There are a number of future directions, including a rigorous study of the estimator when there is no vector labelling, and a performance analysis of other cascade dynamics and graphs.

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