SHARP CRITERIA OF LIOUVILLE TYPE FOR SOME NONLINEAR SYSTEMS

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Abstract. In this paper, we establish the sharp criteria for the nonexistence of positive solutions to the Hardy-Littlewood-Sobolev (HLS) system of nonlinear equations and the corresponding nonlinear differential systems of Lane-Emden type. These nonexistence results, known as Liouville theorems, are fundamental in PDE theory and applications. A special iteration scheme, a new shooting method and some Pohozaev identities in integral form as well as in differential form are created. Combining these new techniques with some observations and some critical asymptotic analysis, we establish the sharp criteria of Liouville type for our systems of nonlinear equations. Similar results are also derived for the system of Wolff type of integral equations and the system of γ-Laplace equations. A dichotomy description in terms of existence and nonexistence for solutions with finite energy is also obtained.

1. Introduction. In this paper, we establish sharp criteria for existence and nonexistence of positive solutions of certain types to the Hardy-Littlewood-Sobolev (HLS) system of nonlinear equations

\[ \begin{aligned}
   u(x) &= \int_{\mathbb{R}^n} \frac{v^q(y)dy}{|x-y|^{n-\alpha}}, \quad u, v > 0 \text{ in } \mathbb{R}^n, \\
v(x) &= \int_{\mathbb{R}^n} \frac{u^p(y)dy}{|x-y|^{n-\alpha}}, \quad p, q > 0
\end{aligned} \]  

and the corresponding nonlinear differential systems of Lane-Emden type of equations

\[ \begin{aligned}
   (-\Delta)^k u &= v^q, \quad u, v > 0 \text{ in } \mathbb{R}^n, \\
   (-\Delta)^k v &= u^p, \quad p, q > 0
\end{aligned} \]  

Here \( n \geq 3 \), and \( 0 < \alpha, 2k < n \). These systems are the ‘blow up’ equations for a large class of systems of nonlinear equations arising from geometric analysis, fluid dynamics, and other physical sciences. The nonexistence of positive solutions

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for systems of ‘blow up’ type like (1.1) and (1.2), known as Liouville theorem, is useful in deriving existence, a priori estimate, regularity and asymptotic analysis of solutions. Another important topic is the study of the Wolff type system of nonlinear equations:

\[
\begin{align*}
&\begin{cases}
  u(x) = W_{\beta,\gamma}(v^q)(x), & u > 0 \text{ in } \mathbb{R}^n, \\
  v(x) = W_{\beta,\gamma}(u^p)(x), & v > 0 \text{ in } \mathbb{R}^n,
\end{cases} \\
&\begin{cases}
  -\Delta u(x) = v^q(x), & u > 0 \text{ in } \mathbb{R}^n, \\
  -\Delta v(x) = u^p(x), & v > 0 \text{ in } \mathbb{R}^n,
\end{cases}
\end{align*}
\]

(1.3)

and the corresponding system of $\gamma$-Laplace equations

\[
\begin{align*}
&\begin{cases}
  -\Delta_{\gamma} u(x) = v^q(x), & u > 0 \text{ in } \mathbb{R}^n, \\
  -\Delta_{\gamma} v(x) = u^p(x), & v > 0 \text{ in } \mathbb{R}^n,
\end{cases}
\end{align*}
\]

(1.4)

where $n \geq 3$, $p, q > 0$, $\beta > 0$, $\gamma > 1$ and $0 < \beta \gamma < n$.

Recall a Liouville-type theorem for the Lane-Emden equation

\[
-\Delta u = u^p, \quad u > 0 \text{ in } \mathbb{R}^n \quad (n \geq 3) \tag{1.5}
\]

obtained by Caffarelli, Gidas and Spruck (cf. [1] and [17]): if $p \in \left[1, \frac{n+2}{n-2}\right]$, then (1.5) has no positive classical solution. When $p \geq \frac{n+2}{n-2}$, (1.5) has positive classical solution. Namely, the right end point $\frac{n+2}{n-2}$ is a sharp criterion distinguishing the existence and the nonexistence. Numbers like this, separating the existence and the nonexistence, are called the critical exponents.

The following theorem shows some results about the existence and nonexistence of positive solutions for (1.2):

**Theorem 1.1.** Assume $k \in \left[1, \frac{n}{2}\right]$ is an integer.

1. The $2k$-order equation

\[
(-\Delta)^k u(x) = u^p(x), \quad u > 0 \text{ in } \mathbb{R}^n \tag{1.6}
\]

has positive solutions if and only if $p \geq \frac{n+2k}{n-2k}$.

2. The $2k$-order system (1.2) has a pair of positive solutions $(u, v)$ if

\[
\frac{1}{p+1} + \frac{1}{q+1} \leq \frac{n-2k}{n}.
\]

**Remark 1.1.** For $k = 1$, Part 2 of Theorem 1.1, together with the nonexistence result of Souplet [51], implies that Eq. (1.2) with $n \leq 4$ has a pair of solutions if and only if $\frac{1}{p+1} + \frac{1}{q+1} \leq \frac{n-2}{n}$. When $n \geq 5$, the nonexistence of positive classical solutions to (1.2), known as the Lane-Emden conjecture, is still open. In the critical and the supercritical cases, i.e. $\frac{1}{p+1} + \frac{1}{q+1} \leq \frac{n-2k}{n}$, the existence results of positive solutions of (1.2) was obtained in [49] with $k = 1$, [15] with $k = 2$, and [38] with $k \in \left[1, \frac{n}{2}\right]$.

Here, we consider the HLS system of nonlinear equations (1.1) and its scalar case:

\[
u(x) = \int_{\mathbb{R}^n} \frac{u^p(y)dy}{|x-y|^{n-\alpha}}, \quad u > 0 \text{ in } \mathbb{R}^n \tag{1.7}
\]

Such equations are related to the study of the best constant of Hardy–Littlewood–Soblev (HLS) inequality. Lieb [35] classified all the extremal solutions of (1.7), and thus obtained the best constant in the HLS inequalities. He posed the classification of all the solutions of (1.7) as an open problem.

The corresponding PDE is the semilinear equation involving a fractional order differential operator

\[
(-\Delta)^{\alpha/2} u = u^{(n+\alpha)/(n-\alpha)}, \quad u > 0 \text{ in } \mathbb{R}^n \tag{1.8}
\]
The classification of the solutions of (1.8) with $\alpha = 2$ has provided an important ingredient in the study of the prescribing scalar curvature problem. It is also essential in deriving priori estimates in many related nonlinear elliptic equations. In [16], Gidas, Ni, and Nirenberg proved that all the positive solutions with reasonable behavior at infinity, namely

$$u(x) = O\left(\frac{1}{|x|^{n-2}}\right)$$

are radially symmetric about some point. Caffarelli, Gidas, and Spruck removed the decay condition (1.9) and obtained the same result (cf. [1]). Then Chen and Li [6], and Li [29] simplified their proofs. Later, Chang, Yang and Lin also considered some higher order equations (cf. [4], [36]). Wei and Xu [53] generalized this result to the solutions of more general equation (1.8) with $\alpha$ being any even numbers between 0 and $n$. Chen, Li, and Ou [12] and Li [32] solved the open problem as stated for the integral equation (1.7) or the corresponding PDE (1.8) respectively. The unique class of solutions assume the form

$$u(x) = c\left(\frac{t}{t^2 + |x - x_0|^2}\right)^{\frac{n-\alpha}{2}}$$

with $c, t > 0$ and $x_0 \in R^n$.

Chen, Li and Ou [11] introduced the method of moving planes in integral forms to study the symmetry of the solutions for the HLS system (1.1). Jin, Li and Hang thoroughly discussed the regularity of the solutions of (1.1) (cf. [20] and [22]). They found the optimal integrability intervals and established the smoothness for the integrable solutions. Based on the results, [28] gave the asymptotic behavior of the integrable solutions when $|x| \to 0$ and $|x| \to \infty$. Some Liouville type results can be seen in [3] and [7].

A significance of the work in [2] is on the fractional order differential operator, which is associated with integral equations and other nonlocal equations. Recently, the fractional Laplacians were applied extensively to describe various physical and finance phenomena, such as anomalous diffusion, turbulence and water waves, molecular dynamics, relativistic quantum mechanics, and stable Levy process. The equivalence between integral equations (IEs) and PDEs was studied in [9] and [12], which provides a technique in studying the PDEs: one can use the corresponding integral equations to investigate the global properties for those solutions.

A positive solution $u$ of (1.7) is called a finite energy solution, if $u \in L^{p+1}(R^n)$. Similarly, positive solutions $u, v$ are called finite energy solutions of (1.1), if $u \in L^{p+1}(R^n)$, $v \in L^{q+1}(R^n)$. Now, we point out the relation between the critical conditions and the existence of finite energy solutions of (1.7) and (1.1).

**Theorem 1.2.** (1) HLS equation (1.7) has positive solutions in $L^{p+1}(R^n)$ if and only if

$$p = \frac{n + \alpha}{n - \alpha}. \quad (1.11)$$

(2) System (1.1) has a pair of positive solutions $(u, v)$ in $L^{p+1}(R^n) \times L^{q+1}(R^n)$ if and only if

$$\frac{1}{p+1} + \frac{1}{q+1} = \frac{n-\alpha}{n}. \quad (1.12)$$
Corollary 1.3. Let $k \in [1, n/2)$ be an integer.

(1) Assume $p > 1$. The $2k$-order PDE (1.6) has positive solutions in $L^{p+1}(\mathbb{R}^n)$ if and only if

$$p = \frac{n + 2k}{n - 2k}. \quad (1.13)$$

(2) Assume $pq > 1$. System (1.2) has a pair of positive solutions $(u, v)$ in $L^{p+1}(\mathbb{R}^n) \times L^{q+1}(\mathbb{R}^n)$ if and only if

$$\frac{1}{p+1} + \frac{1}{q+1} = \frac{n - 2k}{n}. \quad (1.14)$$

Remark 1.2. (1) In the subcritical case $p < \frac{n+\alpha}{n-\alpha}$, Theorem 3 in [10] shows that (1.7) has no locally finite energy solution by using the method of moving planes and the Kelvin transformation. For system (1.1), it is a very interesting and also extremely challenging open problem to show the nonexistence of positive classical solutions in the subcritical case $1 \frac{1}{p+1} + 1 \frac{1}{q+1} > \frac{n-\alpha}{n}$. This open problem is usually called the HLS conjecture (cf. [3] and [7]). The Lane-Emden conjecture is the special case (i.e. the case of $\alpha = 2$). A possible approach we propose consists two major components. One is to find the weakest integrability requirement to derive the nonexistence. Another is to improve the integrability estimate of the solutions to meet the requirement obtained in the previous step. Our result provides such an integrability condition to guarantee the nonexistence.

(2) In the supercritical case $p > \frac{n+\alpha}{n-\alpha}$ with $\alpha = 2$, Li, Ni and Serrin proved that the semilinear Lane-Emden equation (1.5) has positive solutions (cf. [33], [41] and [44]). Our result shows that those solutions have infinite energy, namely, $\|u\|_{p+1} = \|v\|_{q+1} = \infty$.

(3) In the critical case $1 \frac{1}{p+1} + 1 \frac{1}{q+1} = \frac{n-\alpha}{n}$, Theorem 1.2 shows that the HLS integral system (1.1) has the finite energy solutions. It is very interesting to know whether all positive classical solutions $u, v$ have finite energy ($\|u\|_{p+1}, \|v\|_{q+1} < \infty$).

The following $\gamma$-Laplace equation is also studied in this paper

$$-\Delta_\gamma u(x) := -\text{div}(|\nabla u|^{\gamma-2} \nabla u) = u^p(x), \quad x \in \mathbb{R}^n. \quad (1.15)$$

Theorem 1.4. The $\gamma$-Laplace equation (1.15) has positive classical solutions with $\int_{\mathbb{R}^n} |\nabla u|^\gamma dx < \infty$ if and only if $p = \frac{n\gamma}{n-\gamma} - 1$.

Serrin and Zou [50] proved that (1.15) has a classical solution $u$ if and only if $p \geq \frac{n\gamma}{n-\gamma} - 1$.

To study the $\gamma$-Laplace equations, we introduce the Wolff potential of a positive locally integrable function $f$

$$W_{\beta,\gamma}(f)(x) := \int_0^\infty \left( \int_{B_t(x)} f(y)dy \right)^{\frac{\gamma}{\beta - \gamma}} \frac{dt}{t}. \quad (1.16)$$

The integral equations involving the Wolff potential

$$u(x) = W_{\beta,\gamma}(u^p)(x) \quad (1.16)$$

and

$$u(x) = c(x) W_{\beta,\gamma}(u^p)(x) \quad (1.17)$$
are related with the study of many nonlinear problems. The Wolff potentials are
helpful to understand the nonlinear PDEs such as the \(\gamma\)-Laplace equation and the
\(k\)-Hessian equation (cf. [25], [26], [27] and [45]). According to [24], if \(\inf_{R^n} u = 0\),
there exists \(C > 0\) such that the positive solution \(u\) of (1.15) satisfies
\[
\frac{1}{C} W_{1,\gamma}(u^p)(x) \leq u(x) \leq C W_{1,\gamma}(u^p)(x), \quad x \in R^n.
\]
Thus, \(u\) solves (1.17) for some double bounded \(c(x)\). Here, a function \(c(x)\) is called
double bounded, if there exist positive constants \(c\) and \(C\) such that \(c \leq c(x) \leq C\) for
all \(x \in R^n\). For the coupling system (1.3)
\[
\begin{aligned}
&\{ u(x) = W_{\beta,\gamma}(v^q)(x) \\
&v(x) = W_{\beta,\gamma}(u^p)(x),
\end{aligned}
\]
Chen and Li [8] proved the radial symmetry for the integrable solutions. Afterward,
Ma, Chen and Li [39] used the regularity lifting lemmas to obtain the optimal
integrability and the Lipschitz continuity. Based on these results, [52] obtained the
decay rates of the integrable solutions when \(|x| \to \infty\).

For several types of equations, we derive various critical exponents for which
there exists a scaling transformation \(u_\nu(x) = \mu^{\alpha}\nu(x)\) \((v_\nu(x) = \mu^{\gamma}\nu(\mu x))\) to keep
both the equations and the related energy functionals invariant.

(1) For the HLS equation (1.7) and the energy functional \(\|u\|_{p+1}\), the critical
exponent is \(p = \frac{n+\alpha}{n-\gamma}\).

(2) For the Wolff equation (1.16) and the energy functional \(\|u\|_{p+\gamma-1}\), the critical
exponent is \(p = \frac{n+\gamma}{n-\gamma} (\gamma - 1)\); for the \(\gamma\)-Laplace equation (1.15) and the energy
functional \(\|u\|_{p+\gamma-1}\), the critical exponent is \(p = \frac{n+\gamma}{n-\gamma} (\gamma - 1)\).

(3) For (1.16) and the energy functional \(\|u\|_{p+1}\), the critical exponent is \(p = \frac{n+\gamma}{n-\gamma} - 1\); for (1.15) and the energy functional \(\|u\|_{p+1}\), the critical exponent is
\(p = \frac{n}{n-\gamma} - 1\).

(4) For the HLS system (1.1) and the energy functional \((\|u\|_{p+1}, \|v\|_{q+1})\), the
critical condition is \(\frac{1}{p+1} + \frac{1}{q+1} = \frac{n-\alpha}{n}\).

(5) For the Wolff system (1.3) and the energy functional \((\|u\|_{p+\gamma-1}, \|v\|_{q+\gamma-1})\),
the critical condition is \(\frac{1}{p+\gamma-1} + \frac{1}{q+\gamma-1} = \frac{n-\alpha}{n(\gamma-1)}\); for the \(\gamma\)-Laplace system (1.4)
and the energy functional \((\|u\|_{p+\gamma-1}, \|v\|_{q+\gamma-1})\), the critical condition is
\(\frac{1}{p+\gamma-1} + \frac{1}{q+\gamma-1} = \frac{n-\alpha}{n(\gamma-1)}\).

(6) For (1.3) and the energy functional \((\|u\|_{p+1}, \|v\|_{q+1})\), the critical condition
is \(p = q = \frac{n-\gamma}{n-\gamma} - 1\) or \(\gamma = 2\); for (1.4) and the energy functional \((\|u\|_{p+1}, \|v\|_{q+1})\),
the critical condition is \(p = q = \frac{n-\gamma}{n-\gamma} - 1\) or \(\gamma = 2\).

**Remark 1.3.** Here an interesting observation is that the critical exponent \(\frac{n+\gamma}{n-\gamma} (\gamma - 1)\)
is different from the divided exponent \(\frac{n\gamma}{n-\gamma} - 1\) in Theorem 1.4 except \(\gamma = 2\).
In fact, those critical numbers are in the different finite energy functions classes
\(L^{p+\gamma-1}(R^n)\) and \(L^{p+1}(R^n)\), respectively.

Next we study the HLS, \(\gamma\)-Laplace, and the Wolff systems with double bounded
coefficients. Surprisingly, we found the necessary and sufficient conditions on the
exponent for the existence in all three systems. It is very interesting to derive
various integrability and asymptotic estimates for the solutions of systems with
double bounded coefficients. These estimates are key ingredients for the Lane-
Emden conjecture. Also our result shows that to establish the nonexistence one has
to use certain special properties of the coefficients \(c_1(x) = c_2(x) = 1\).
The following estimate for the solutions of (1.2) with \( k = 1 \) is essential: there exists \( C > 0 \), such that for all \( R > 0 \) and \( r \in (1, r_0], s \in (1, s_0]\),
\[
\int_{B_R(0)} u^r(x)dx \leq CR^{n-r_0}, \quad \int_{B_R(0)} v^s(x)dx \leq CR^{n-s_0}. \tag{1.18}
\]
Here \( \alpha_0 := \frac{2(q+1)}{pq-1}, \beta_0 := \frac{2(p+1)}{pq-1} \). It is true for \( r_0 = p, s_0 = q \) (cf. [48]). We expect to find the largest \( r_0 \) and \( s_0 \) such that (1.18) holds. This is helpful to well understand the nonexistence mentioned in the Lane-Emden conjecture. Similar estimates for the general systems (1.1) and (1.2) are also very interesting open problems.

**Theorem 1.5.** (1) The equation
\[
u(x) = c(x) \int_{\mathbb{R}^n} \frac{u^p(y)dy}{|x-y|^{n-\alpha}}, \tag{1.19}
\]
has positive solutions for some double bounded \( c(x) \) if and only if \( p > \frac{n}{n-\alpha} \).

(2) The HLS system
\[
\begin{cases}
  u(x) = c_1(x) \int_{\mathbb{R}^n} \frac{v^q(y)dy}{|x-y|^{n-\alpha}} \\
v(x) = c_2(x) \int_{\mathbb{R}^n} \frac{u^p(y)dy}{|x-y|^{n-\alpha}}.
\end{cases} \tag{1.20}
\]
has positive solutions \( u, v \) for some double bounded \( c_1(x) \) and \( c_2(x) \), if and only if \( pq > 1 \) and \( \max\left\{ \frac{\alpha(p+1)}{pq-1}, \frac{\alpha(q+1)}{pq-1} \right\} < n-\alpha \).

**Corollary 1.6.** Let \( k \in [1, n/2) \) be an integer.

(1) Assume \( p > 1 \). The 2k-order PDE
\[
(-\Delta)^k u(x) = c(x) u^p(x), \quad x \in \mathbb{R}^n, \tag{1.21}
\]
has positive solutions for some double bounded \( c(x) \) if and only if \( p > \frac{n}{n-2k} \).

(2) Assume \( pq > 1 \). The system
\[
\begin{cases}
  (-\Delta)^k u(x) = c_1(x) v^q(x) \\
  (-\Delta)^k v(x) = c_2(x) u^p(x).
\end{cases} \tag{1.22}
\]
has positive solutions \( u, v \) for some double bounded \( c_1(x) \) and \( c_2(x) \), if and only if \( \max\left\{ \frac{2k(p+1)}{pq-1}, \frac{2k(q+1)}{pq-1} \right\} < n-2k \).

**Theorem 1.7.** (1) The equation (1.17) has positive solutions for some double bounded \( c(x) \), if and only if \( p > \frac{n(q-1)}{n-\gamma} \).

(2) The system
\[
\begin{cases}
  u(x) = c_1(x) W_{\beta, \gamma}(v^q)(x) \\
v(x) = c_2(x) W_{\beta, \gamma}(u^p)(x).
\end{cases} \tag{1.23}
\]
has positive solutions \( u, v \) for some double bounded \( c_1(x) \) and \( c_2(x) \), if and only if \( pq > (\gamma - 1)^2 \) and \( \max\left\{ \frac{\gamma(p+q-1)}{pq-(\gamma - 1)^2}, \frac{\gamma(q+1-1)}{pq-(\gamma - 1)^2} \right\} < -\frac{n-\beta_1}{\gamma-1} \).

**Corollary 1.8.** (1) If \( p > \frac{n(q-1)}{n-\gamma} \), then
\[
- \Delta_\gamma u(x) = c(x) u^p(x), \quad x \in \mathbb{R}^n \tag{1.24}
\]
has positive solutions for some double bounded \( c(x) \). If \( 0 < p \leq \frac{n(q-1)}{n-\gamma} \), then for any double bounded \( c(x) \), (1.24) has no positive solution satisfying \( \inf_{\mathbb{R}^n} u = 0 \).
(2) If \( pq > (\gamma - 1)^2 \) and \( \max\{\frac{\gamma (q+\gamma -1)}{pq-(\gamma-1)^2}, \frac{\gamma (p+\gamma -1)}{pq-(\gamma-1)^2}\} < \frac{n-\gamma}{\gamma-1} \), then there exist positive solutions \( u, v \) of the \( \gamma \)-Laplace system

\[
\begin{align*}
-\Delta_{\gamma} u(x) &= c_1(x)v^p(x), \quad x \in \mathbb{R}^n, \\
-\Delta_{\gamma} v(x) &= c_1(x)u^p(x), \quad x \in \mathbb{R}^n
\end{align*}
\]  

(1.25)

for some double bounded \( c_1(x) \) and \( c_2(x) \). On the contrary, for any double bounded functions \( c_1(x) \) and \( c_2(x) \), if one of the following conditions holds

(i) \( 0 < pq \leq (\gamma - 1)^2 \);

(ii) \( pq > (\gamma - 1)^2 \) and \( \max\{\frac{\gamma (q+\gamma -1)}{pq-(\gamma-1)^2}, \frac{\gamma (p+\gamma -1)}{pq-(\gamma-1)^2}\} \geq \frac{n-\gamma}{\gamma-1} \).

Then (1.25) has no positive solutions \( u, v \) satisfying \( \inf_{\mathbb{R}^n} u = \inf_{\mathbb{R}^n} v = 0 \).

**Remark 1.4.** Comparing with Theorem 1.2-Theorem 1.4, we obtain, from Theorem 1.5-Corollary 1.8, other divisional conditions on the existence of the positive solutions of the equations and systems with ratio coefficients \( c(x), c_1(x) \) and \( c_2(x) \). These divisional conditions are called the *secondary critical conditions*. They are the natural generalization of the Serrin exponents. The secondary critical conditions are more relaxed than those in Theorem 1.2 -Theorem 1.4 because the solutions classes of the equations and systems with ratio coefficients are larger than that in the case of \( c(x) \equiv \text{Constant} \).

In the proofs of Theorem 1.5-Corollary 1.8, we apply a special iteration scheme and some critical asymptotic analysis to establish the existence and the nonexistence, and hence obtain the sharp criteria.

The contents of this paper are as follows. In Section 2, we prove Theorem 1.5 (1), Corollary 1.6 (1), Theorem 1.7 (1) and Corollary 1.8 (1). Theorem 1.5 (2), Corollary 1.6 (2), Theorem 1.7 (2) and Corollary 1.8 (2) are proved in Section 3. In Section 4.2, we prove (1) of Theorem 1.2, which covers (1) of Corollary 1.3. The proof of of Theorem 1.4 is given in Section 4.3. In Section 5, we give the proofs of (2) of Theorem 1.2 and (2) of Corollary 1.3. The argument on Theorem 1.1 is given in Sections 6, and we give a more general existence result (see §6.3).

2. Equations with variable coefficients.

2.1. HLS integral equation. In this subsection, we give a relation between the exponents and the existence of positive solutions. First we can find a result in [47] on the semilinear Lane-Emden equations

\[- \Delta x(u(x) = c(x)u^p(x), \quad x \in \mathbb{R}^n.\]  

(2.1)

**Theorem 2.1.** Let \( p \geq 1 \). Then (2.1) has a positive radial solution of the form

\[ u(x) = \frac{1}{(1 + |x|^2)^\theta}, \]  

(2.2)

for some double bounded \( c(x) \), if and only if \( p > \frac{n}{n-2} \). Here \( \theta \in \{\frac{1}{p-1}, \frac{n-2}{2}\} \).

Similarly, we also have an analogous result for the integral equation (1.19).

**Theorem 2.2.** The HLS integral equation (1.19) has a positive solution for some double bounded \( c(x) \), if and only if

\[ p > \frac{n}{n-\alpha}. \]  

(2.3)
Proof. When $|x| \leq 2R$ for some $R > 0$, $u(x)$ in (2.2) is proportional to $\int_{B_R} |x-y|^{\alpha-n} u^p(y)dy$. Thus, we only consider the case of $|x| > 2R$.

**Step 1.** Inserting (2.2) into the right hand side of (1.19), we can find some double bounded function $c(x)$ such that as $|x| > 2R$ for some $R > 0$,

$$
\int_{B_R} \frac{u^p(y)dy}{|x-y|^{n-\alpha}} = \frac{c(x)}{(1 + |x|^2)^{(n-\alpha)/2}} \int_{B_R(0)} \frac{dy}{(1 + |y|^2)^{\rho \theta}}
$$

$$
\quad + \frac{c(x)}{(1 + |x|^2)^{\rho \theta}} \int_{B_{2R}(x)} \frac{dy}{|x-y|^{n-\alpha}} + c(x) \int_{B_R(0) \setminus B_{2R}(x)} \frac{dy}{|x-y|^{n-\alpha}|y|^{2\rho \theta}}.
$$

If $p > \frac{n}{n-\alpha}$, we take $2\theta = \frac{\alpha}{p-1}$ and hence $\alpha < 2\rho \theta < n$. Then,

$$
\int_{B_R} \frac{u^p(y)dy}{|x-y|^{n-\alpha}} = \frac{c(x)}{(1 + |x|^2)^{(n-\alpha)/2}} + \frac{c(x)}{(1 + |x|^2)^{\rho \theta-\alpha/2}}
$$

$$
\quad + \frac{c(x)}{|x|^{n-\alpha}} \int_{R} r^{n-2\rho \theta} dr + c(x) \int_{2|x|}^{\infty} r^{n-(n-\alpha+2\rho \theta)} dr
$$

$$
\quad = \frac{c(x)}{(1 + |x|^2)^{\rho \theta-\alpha/2}} = c(x)u(x)
$$

for some double bounded function $c(x)$. This result shows that (1.19) has the slowly decaying radial solution as (2.2).

On the other hand, we can also find a fast decaying solution. Now, take $2\theta = n - \alpha$, then $2\rho \theta > n$ as long as $p > \frac{n}{n-\alpha}$. Thus,

$$
\int_{B_R} \frac{u^p(y)dy}{|x-y|^{n-\alpha}} = \frac{c(x)}{(1 + |x|^2)^{(n-\alpha)/2}} + \frac{c(x)}{(1 + |x|^2)^{\rho \theta-\alpha/2}}
$$

$$
\quad + \frac{c(x)}{|x|^{n-\alpha}} \int_{R} r^{n-2\rho \theta} dr + c(x) \int_{2|x|}^{\infty} r^{n-(n-\alpha+2\rho \theta)} dr
$$

$$
\quad = \frac{c(x)}{(1 + |x|^2)^{(n-\alpha)/2}} = c(x)u(x)
$$

for some double bounded function $c(x)$.

**Step 2.** We prove that (1.19) has no positive solution when $0 < p < \frac{n}{n-\alpha}$.

Suppose $u$ is a positive solution, then it follows a contradiction. In fact, when $|x| > R$ with $R > 0$, $|x-y| \leq 2|x|$ for $y \in B_R(0)$. In addition, $\int_{B_R(0)} u^p(y)dy \geq c$. Thus,

$$
u(x) \geq c |x|^{\alpha-n} \int_{B_R(0)} u^p(y)dy \geq \frac{c}{|x|^{a_0}}, \text{ for } |x| > R.
$$

Here $a_0 = n - \alpha$. Using this estimate, for $|x| > R$ we also get

$$
u(x) \geq c \int_{B_{j_0/2}(x)} \frac{|y|^{-p a_j}dy}{|x-y|^{n-\alpha}} \geq \frac{c}{|x|^{p a_{j_0} - \alpha}} := \frac{c}{|x|^{a_1}}.
$$

By induction, we can obtain $u(x) \geq \frac{c}{|x|^{a_j}}$, for $|x| > R$, where $a_j = p a_{j-1} - \alpha$, $j = 0, 1, \cdots$. We claim that there exists $j_0$ such that

$$a_{j_0} < 0. \quad (2.4)
$$

Once it is verified, then $u(x) \geq c \int_{R^n \setminus B_R(0)} \frac{|y|^{-p a_{j_0}dy}{|x-y|^{n-\alpha}} = \infty$. The contradiction appears.
In fact,
\[ a_j = pa_j - \alpha = p(pa_{j-1} - \alpha) - \alpha = \cdots \]
\[ = p_j a_0 - \alpha(p^{j-1} + p^{j-2} + \cdots + p + 1). \]
When \( p = 1 \), \( a_j = a_0 - \alpha j \). Thus, \( a_j \) are either positive or negative for some suitably large \( j_0 \). When \( p \in (0, 1) \), let \( j \to \infty \). Then \( a_j = p^j a_0 - \alpha \frac{1-p^j}{1-p} \to -\frac{\alpha}{p-1} < 0 \). This implies \( a_j \) are negative for some \( j_0 \). When \( p \in (1, \frac{n}{n-\alpha}) \), \( a_j = p^j (n-\alpha) - \alpha \frac{p^{j-1}}{p-1} = (n-\alpha - \frac{\alpha}{p-1})p^j + \frac{\alpha}{p-1} \).
By \( p < \frac{n}{n-\alpha} \), there holds \( n - \frac{\alpha}{p-1} < 0 \). So, we can find a suitably large \( j_0 \) such that \( a_{j_0} < 0 \). Thus, (2.4) is verified.

**Step 3.** We prove that (1.19) has no positive solution when \( p = \frac{n}{n-\alpha} \).

Otherwise, \( u \) is a positive solution. For \( R > 0 \), denote \( B_R(0) \) by \( B \). From (1.19) it follows that
\[ u(x) \geq \frac{1}{(R + |x|)^{n-\alpha}} \int_B u^p(y)dy. \] (2.5)

Thus, taking \( p \) powers of (2.5) and integrating on \( B \), we have
\[ \int_B u^p(x)dx \geq \int_B \frac{dx}{(R + |x|)^n} (\int_B u^p(y)dy)^p \geq c(\int_B u^p(y)dy)^p. \] (2.6)

Here \( c \) is independent of \( R \). Letting \( R \to \infty \), we see \( u \in L^p(R^n) \).

Taking \( p \) powers of (2.5) and integrating on \( A_R := B_{2R}(0) \setminus B_R(0) \), we get
\[ \int_{A_R} u^p(x)dx \geq \int_{A_R} \frac{dx}{(R + |x|)^n} (\int_B u^p(y)dy)^p \geq c(\int_B u^p(y)dy)^p. \]

Letting \( R \to \infty \), and noting \( u \in L^p(R^n) \), we obtain \( \int_{\mathbb{R}^n} u^p(y)dy = 0 \), which contradicts with \( u > 0 \). \( \square \)

**Corollary 2.3.** Assume \( k \in [1, n/2) \) and \( p > 1 \). The higher order semilinear PDE (1.21) has a positive solution for some double bounded \( c(x) \), if and only if \( p > \frac{n}{n-2k} \).

**Proof.** If \( u > 0 \) solves the integral equation (1.19) with \( \alpha = 2k \), it is easy to see that \( u \) also solves the higher order semilinear PDE (1.21). On the contrary, if \( p > 1 \) and \( u \) solves (1.21), [37] proved \((-\Delta)^k u > 0 \) for \( i = 1, 2, \ldots, k-1 \). Similar to the argument in [12], (1.21) is equivalent to (1.19) with \( \alpha = 2k \). Therefore, if \( p > 1 \), Theorem 2.2 shows that (1.21) has positive solutions for some double bounded function \( c(x) \), if and only if \( p > \frac{n}{n-2k} \). \( \square \)

### 2.2. Integral equation involving the Wolff potential.

**Theorem 2.4.** The Wolff integral equation (1.17) has a positive solution for some double bounded \( c(x) \), if and only if \( p > \frac{n(1-\gamma)}{n-\beta} \).

**Proof.** Step 1. Existence.

Inserting (2.2) into \( W_{\beta,\gamma}(u^p)(x) \), we obtain
\[ W_{\beta,\gamma}(u^p)(x) = (\int_0^{\frac{R}{2}} + \int_{\frac{R}{2}}^{\infty}) \int_{B_r(x)} \frac{dy}{(1 + |y|^2)^\beta} t^{\beta-\gamma-n} \frac{dt}{t^{\gamma+n}} := I_1 + I_2. \]

When \( |x| \leq R \) for some \( R > 0 \), then \( u \) is proportional to \( W_{\beta,\gamma}(u^p) \). So we also only consider suitably large \( |x| \). Clearly,
\[ I_1 = \int_0^{\frac{R}{2}} \frac{\int_{B_r(x)} (1 + |y|^2)^{-\beta} dy}{t^{\beta-\gamma-n}} \frac{dt}{t^{\gamma+n}} = c(1 + |x|^2)^{-\frac{\beta}{\gamma+n}} \int_0^{\frac{R}{2}} \frac{t^{\frac{\beta}{\gamma+n}} dt}{t^{\gamma+n}} = c(1 + |x|^2)^{-\frac{\beta}{\gamma+n}}. \]
Take the slow rate $2\theta = \frac{\beta \gamma}{p-\gamma+1}$. There holds $\beta \gamma < 2p\theta < n$ by $p > \frac{n(n-1)}{n-\beta \gamma}$. So

$$I_2 = c \int_{|x|/2}^{\infty} \left( \int_{B(x)} \frac{1 + |y|^2}{t^n-\beta \gamma} \right) \frac{1}{n-\beta \gamma} \frac{dt}{t} = c(x) \int_{|x|/2}^{\infty} \left( \int_{B(x)} \frac{1}{t^n-\beta \gamma} \right) \frac{1}{n-\beta \gamma} \frac{dt}{t} = c(x)(1 + |x|^2)^{\frac{\beta \gamma - 2p\theta}{2(n-\beta \gamma)}}.$$

Thus, $I_1 + I_2 = c(x)u(x)$ for some double bounded $c(x)$.

Similarly, we also find a fast decaying solution. In fact, taking $2\theta = \frac{n-\beta \gamma}{n-\beta \gamma}$, we also have $2p\theta > n$ from $p > \frac{n(n-1)}{n-\beta \gamma}$, and hence

$$I_2 = \int_{|x|/2}^{\infty} \left( \int_{B(x) \cap B(0)} (1 + |y|^2)^{-p\theta} dy + \int_{B(x) \setminus B(0)} (1 + |y|^2)^{-p\theta} dy \right) \frac{1}{n-\beta \gamma} \frac{dt}{t} = c(x)(1 + |x|^2)^{-\frac{n-\beta \gamma}{2(n-\beta \gamma)}}.$$

There also holds $I_1 + I_2 = c(x)(1 + |x|^2)^{-\frac{n-\beta \gamma}{2(n-\beta \gamma)}} = c(x)u(x)$.

**Step 2.** Nonexistence.

**Substep 2.1.** Let

$$0 < p < \frac{n(n-1)}{n-\beta \gamma}. \quad (2.7)$$

Suppose that $u$ solves (1.17), then

$$u(x) \geq c \int_{|x|}^{\infty} \frac{dt}{t^{\frac{\beta \gamma}{n-\beta \gamma}} = c} \frac{1}{|x|^{a_0}}, \quad (2.8)$$

since $\int_{B(0)} u^p(y)dy \geq c$, where $a_0 = \frac{n-\beta \gamma}{n-1}$. By this estimate, we have

$$u(x) \geq c \int_{|x|}^{\infty} \left( \int_{B(x_0)} |y|^{-p\theta} dy \right) \frac{1}{t^{\frac{\beta \gamma}{n-\beta \gamma}}} \frac{dt}{t} \geq c \int_{|x|}^{\infty} \left( t^{\beta \gamma - p\theta} \right) \frac{1}{t^{\frac{\beta \gamma}{n-\beta \gamma}}} \frac{dt}{t}. \quad (2.9)$$

When $\frac{p}{\gamma-1} \in (0, \frac{\beta \gamma}{n-\beta \gamma})$, we have $\beta \gamma - p\theta > 0$. Eq. (2.9) implies $u(x) = \infty$. It is impossible.

Next, we consider the case $\frac{p}{\gamma-1} \in (\frac{\beta \gamma}{n-\beta \gamma}, \frac{n}{n-\beta \gamma})$. Now (2.9) leads to $u(x) \geq \frac{c}{|x|^{a_1}}$, where $a_1 = \frac{p}{\gamma-1} a_0 - \frac{\beta \gamma}{\gamma-1}$. Write

$$a_j = \frac{p}{\gamma-1} a_{j-1} - \frac{\beta \gamma}{\gamma-1}, \quad j = 1, 2, \ldots. \quad (2.10)$$

We claim that there must be $j_0 > 0$ such that $a_{j_0} \leq 0$. This leads to $u(x) = \infty$, which contradicts with the fact that $u$ is a positive solution.

In fact, by (2.10) we get

$$a_j = (\frac{p}{\gamma-1})^j a_0 - \left[ 1 + \frac{p}{\gamma-1} + \cdots + \left( \frac{p}{\gamma-1} \right)^{j-1} \right] \frac{\beta \gamma}{\gamma-1}.$$ 

If $\frac{p}{\gamma-1} = 1$, then we can find a large $j_0$ such that $a_{j_0} = a_0 - j_0 \frac{\beta \gamma}{\gamma-1} \leq 0$. If $\frac{p}{\gamma-1} \in (1, \frac{n}{n-\beta \gamma})$, then using $a_0 - \frac{\beta \gamma}{p-\gamma+1} < 0$ which is implied by (2.7), we can find a large
\( j_0 \) such that
\[
a_{j_0} = \left( \frac{p}{\gamma - 1} \right)^{j_0} a_0 - \left( \frac{p}{\gamma - 1} \right)^{j_0} \frac{\beta \gamma}{p - \gamma + 1} \frac{1}{\gamma - 1}
\]
\[
= \left( \frac{p}{\gamma - 1} \right)^{j_0} \left( a_0 - \frac{\beta \gamma}{p - \gamma + 1} \frac{1}{\gamma - 1} \right) + \frac{\beta \gamma}{p - \gamma + 1} \leq 0.
\]
If \( \frac{p}{\gamma - 1} \in (0, 1) \), letting \( j \to \infty \), we get
\[
a_j = \left( \frac{p}{\gamma - 1} \right)^{j} a_0 - \frac{1 - \left( \frac{p}{\gamma - 1} \right)^{j}}{\gamma - 1} \frac{\beta \gamma}{p - \gamma + 1} \to \frac{\beta \gamma}{p - \gamma + 1} < 0.
\]
Thus, there must be \( j_0 \) such that \( a_{j_0} \leq 0 \).

**Substep 2.2.** Let \( p = \frac{\alpha (\gamma - 1)}{n - p \gamma} \). We deduce the contradiction if \( u \) is a positive solution of \((1.17)\).

For \( R > 0 \), denote \( B_R(0) \) by \( B_R \). By using the Hölder inequality, from \((1.17)\) we deduce that for any \( x \in B_R \),
\[
\int_0^R \int_{B_t(x)} u^p(y) dy dt \\
\leq \left( \int_0^R \left( \int_{B_t(x)} u^p(y) dy \right)^{\frac{1}{\gamma - 1}} \cdot t^{\frac{n - \beta \gamma}{\gamma - 1} - 1} dt \right)^{\gamma - 1} \left( \int_0^R t^{\frac{n - \beta \gamma + 1}{\gamma - 1} - 1} dt \right)^{2 - \gamma}
\]
\[
= C R^\alpha - \beta \gamma + 1 \left( \int_0^R \left( \int_{B_t(x)} u^p(y) dy \right)^{\frac{1}{\gamma - 1}} \cdot \frac{1}{t^{\frac{n - \beta \gamma}{\gamma - 1}}} dt \right)^{\gamma - 1}.
\]
Hence, exchanging the order of the integral variables, we have
\[
u(x) \geq c \left( \int_{B_R} \left( \int_{B_t(x)} u^p(y) dy \right)^{\frac{1}{\gamma - 1}} \cdot t^{\frac{n - \beta \gamma}{\gamma - 1} - 1} dt \right)^{\gamma - 1} \geq c R^\alpha - \beta \gamma + 1 \left( \int_{B_R} u^p(y) dy \right)^{\frac{1}{\gamma - 1}} \left( \int_{B_R} t^{\frac{n - \beta \gamma}{\gamma - 1} - 1} dt \right)^{\gamma - 1}
\]
\[
\geq c R^\alpha - \beta \gamma + 1 \left( \int_{B_R} u^p(y) dy \right)^{\frac{1}{\gamma - 1}} \geq c R^\alpha - \beta \gamma + 1 \left( \int_{B_R} u^p(y) dy \right)^{\frac{1}{\gamma - 1}}.
\]
Therefore, we get
\[
u(x) \geq c R^\alpha - \beta \gamma + 1 \left( \int_{B_R} u^p(y) dy \right)^{\frac{1}{\gamma - 1}}.
\]
Integrating on \( B_R/4 \) and using \( p = \frac{\alpha (\gamma - 1)}{n - p \gamma} \) again, we get
\[
\int_{B_R/4} u^p(x) dx \geq c R^{\frac{n - p \gamma}{\gamma - 1}} \int_{B_R} dx \int_{B_R/4} u^p(y) dy \cdot \frac{x}{y} \geq c \int_{B_R/4} u^p(y) dy \cdot \frac{p}{\gamma - 1}.
\]
Here \( c \) is independent of \( R \). Letting \( R \to \infty \) and noting \( p > \gamma - 1 \), we have
\[
\int_{R^n} u^p(x) dx < \infty.
\]
Integrating \((2.11)\) on \( A_R = B_R/4 \setminus B_{R/8} \) yields
\[
\int_{A_R} u^p(x) dx \geq c R^{\frac{n - p \gamma}{\gamma - 1}} \int_{A_R} dx \int_{B_R/4} u^p(y) dy \cdot \frac{p}{\gamma - 1}.
\]
By \( p = \frac{\alpha (\gamma - 1)}{n - p \gamma} \), it follows
\[
\int_{A_R} u^p(x) dx \geq c \int_{B_R/4} u^p(y) dy \cdot \frac{p}{\gamma - 1},
\]
where $c$ is independent of $R$. Letting $R \to \infty$, and noting (2.12), we obtain
\[ \int_{R^N} u^p(y) dy = 0, \]
which implies $u \equiv 0$. It is impossible.

The proof is complete. \[ \square \]

**Remark 2.1.** When $\beta = \alpha / 2$ and $\gamma = 2$, (1.17) is reduced to (1.19). Theorem 2.4
is the generalization of Theorem 2.2.

### 2.3. $\gamma$-Laplace equation.

**Theorem 2.5.** (1) If $p > \frac{n(\gamma-1)}{n-\gamma}$, then the $\gamma$-Laplace equation (1.24) has positive
solutions for some double bounded $c(x)$.

(2) If $0 < p \leq \frac{n(\gamma-1)}{n-\gamma}$, then for any double bounded function $c(x)$, (1.24) has no
any positive solution satisfying $\inf_{R^n} u = 0$.

**Proof.** (1) For $u(x) = \frac{1}{(1+|x|^m)^\theta}$ with $m = \frac{\gamma}{\gamma-1}$, we have
\[ -\Delta_\gamma u = (1-\gamma)[U_{r}]^{-2} U_{rr} - \frac{\theta-1}{r}[U_{r}]^{-2} U_{r} \]
\[ = \frac{(m\theta)^{\gamma-1} - [m(\gamma-1)\gamma-2] + n - 1 + (\gamma-1)(m-1)]}{1+m}\]
(2.13)

Let $p > \frac{n(\gamma-1)}{\gamma-1}$. Take $m\theta = \frac{\gamma}{p-\gamma+1}$, then $n > (\theta+1)\gamma$. Therefore, (2.13) implies
\[ -\Delta_\gamma u = c(r)u^{(\theta+1)(\gamma-1)/\theta} = c(r)u^p \]
for some double bounded $c(r)$. This result shows that (1.24) has a slowly decaying radial solution.

In addition, if $p = \frac{n(\gamma-1)+\gamma}{n-\gamma}$, we can find another fast decaying solution with rate
$m\theta = \frac{\gamma}{\gamma-1}$. Now, $n = (\theta+1)\gamma$ and hence (2.13) implies
\[ -\Delta_\gamma u = c(r)u^{[(\theta+1)(\gamma-1)+1]/\theta} = c(r)u^p \]
for some double bounded $c(r)$.

(2) Suppose $u$ solves (1.24) and satisfies $\inf_{R^n} u = 0$. According to Corollary 4.13 in [24], there exists $C > 0$ such that
\[ \frac{1}{C}W_{1,\gamma}(cu^p)(x) \leq u(x) \leq CW_{1,\gamma}(cu^p)(x). \]
Since $c(x)$ is double bounded, we can see that $K(x) := u(x)[W_{1,\gamma}(u^p)(x)]^{-1}$ is
also double bounded. This shows that $u$ solves $u(x) = K(x)W_{1,\gamma}(u^p)(x)$. When
\[ 0 < p \leq \frac{n(\gamma-1)}{n-\gamma}, \] Theorem 2.4 shows that this Wolff equation has no positive solution
for any double bounded function $K(x)$. Therefore, we prove the nonexistence of positive solutions to (1.24) when $0 < p \leq \frac{n(\gamma-1)}{n-\gamma}$. \[ \square \]

**Remark 2.2.** According to Theorem 2.5, if the positive solution $u$ of (2.1) satisfies
$\inf_{R^n} u = 0$, then Theorem 2.1 still holds even though we replace the condition
$p \geq 1$ by $p > 0$.

### 3. Systems with variable coefficients.

#### 3.1. HLS system.

**Theorem 3.1.** There exist positive solutions $u, v$ of the integral system (1.20) for
some double bounded functions $c_1(x)$ and $c_2(x)$, if and only if $pq > 1$ and
\[
\max\left\{ \frac{\alpha(p + 1)}{pq - 1}, \frac{\alpha(q + 1)}{pq - 1} \right\} < n - \alpha. \tag{3.1}
\]

**Proof.** **Step 1.** Sufficiency.

Set
\[
u(x) = \frac{1}{(1 + |x|^2)^{p\theta}}, \quad v(x) = \frac{1}{(1 + |x|^2)^{q\theta}}.
\tag{3.2}
\]

Similar to the argument in the proof of Theorem 2.2, we can find four pairs solutions.

(i) Take the slow rates \(2\theta_1 = \frac{\alpha(q+1)}{pq-1}, 2\theta_2 = \frac{\alpha(p+1)}{pq-1}\). Then \(pq > 1\) as well as \(3.1\) lead to \(\alpha < 2p\theta_1 < n\) and \(\alpha < 2q\theta_2 < n\). Therefore,
\[
\int_{\mathbb{R}^n} \frac{u^p(y)dy}{|x - y|^{n-\alpha}} = \frac{c_1(x)}{(1 + |x|^2)^{p\theta_1-\alpha/2}} = c_1(x)v(x),
\]
\[
\int_{\mathbb{R}^n} \frac{v^q(y)dy}{|x - y|^{n-\alpha}} = \frac{c_2(x)}{(1 + |x|^2)^{q\theta_2-\alpha/2}} = c_2(x)u(x),
\]
for some double bounded functions \(c_1(x)\) and \(c_2(x)\). This consequence shows that \(1.20\) has a pair of radial solutions \((u, v)\) as \(3.2\).

(ii) Moreover, if the stronger condition \(p, q > \frac{n}{n-\alpha}\) holds, then we can find solutions \(u, v\) with the fast decay rate \(2\theta_1 = 2\theta_2 = n - \alpha\). Now, \(2p\theta_1 > n\) and \(2q\theta_2 > n\), then
\[
\int_{\mathbb{R}^n} \frac{u^p(y)dy}{|x - y|^{n-\alpha}} = \frac{c_1(x)}{(1 + |x|^2)^{(n-\alpha)/2}} = c_1(x)v(x),
\]
\[
\int_{\mathbb{R}^n} \frac{v^q(y)dy}{|x - y|^{n-\alpha}} = \frac{c_2(x)}{(1 + |x|^2)^{(n-\alpha)/2}} = c_2(x)u(x),
\]
for some double bounded functions \(c_1(x)\) and \(c_2(x)\). Therefore, \(1.20\) has a pair of radial solutions \((u, v)\) as \(3.2\).

(iii) If another stronger condition \(pq > 1\) as well as
\[
\frac{\alpha}{n-\alpha} < p < \frac{n}{n-\alpha}, \tag{3.3}
\]
\[
\frac{(q+1)\alpha}{pq - 1} < n - \alpha, \tag{3.4}
\]
holds, we can find a pair of solutions \(u, v\). Now, \(u, v\) have two different fast decay rates.

We claim that if \(pq > 1\), the condition \(3.3\) together with \(3.4\) are stronger than \(3.1\). In fact, we first see \(p \leq q\). Otherwise, \(3.3\) implies \(q < p < \frac{n}{n-\alpha}\), which means \(q[p(n - \alpha) - \alpha] < n\). This contradicts \(3.4\). From \(3.4\) and \(p \leq q\), it follows that \(pq(n - \alpha) - n > qa \geq pa\). This leads to \(\frac{(p+1)\alpha}{pq - 1} < n - \alpha\). Combining this with \(3.4\) yields \(3.1\).

Take two fast rates \(2\theta_1 = n - \alpha\) and \(2\theta_2 = 2p\theta_1 - \alpha = \frac{p(n - \alpha) - \alpha}{n-\alpha}\). Then \(3.3\) and \(3.4\) lead to \(\alpha < 2p\theta_1 < n\) and \(2q\theta_2 > n\). Therefore,
\[
\int_{\mathbb{R}^n} \frac{u^p(y)dy}{|x - y|^{n-\alpha}} = \frac{c_1(x)}{(1 + |x|^2)^{p\theta_1-\alpha/2}} = c_1(x)v(x),
\]
\[
\int_{\mathbb{R}^n} \frac{v^q(y)dy}{|x - y|^{n-\alpha}} = \frac{c_2(x)}{(1 + |x|^2)^{(n-\alpha)/2}} = c_2(x)u(x),
\]
for another double bounded functions \(c_1(x)\) and \(c_2(x)\). Therefore, \(1.20\) has a pair of radial solutions \((u, v)\) as \(3.2\).
By the same argument above, we know that once $pq > 1$ as well as $\frac{n}{n-\alpha} < q < \frac{n}{n-\alpha}$ and $(\frac{p+1}{p-1})_a < n - \alpha$, (1.20) has a pair of radial solutions $(u, v)$ as (3.2). Now, $u, v$ decay fast by two different rates.

(iv) We can find another pair of radial solutions to (1.20). They decay with fast rates which are different from (3.2). Now, we assume

$$u(x) = \frac{1}{(1 + |x|^2)^{(n-\alpha)/2}}, \quad v(x) = \frac{\log |x|}{(1 + |x|^2)^{(n-\alpha)/2}}.$$ 

It is easy to verify that $u, v$ also solve (1.20) with some double bounded functions $c_1, c_2$.

**Note.** According to Corollary 1.3 (2) in [52], if $(u, v) \in L^{p+1}(\mathbb{R}^n) \times L^{q+1}(\mathbb{R}^n)$ where $p, q$ satisfy the critical condition \( \frac{1}{p+1} + \frac{1}{q+1} = 1 - \frac{\alpha}{n} \), then $u, v$ decay with only three rates as in (ii)-(iv).

**Step 2.** Necessity.

(i) If either $0 < pq \leq 1$ or $\max\{\frac{(p+1)\alpha}{pq-1}, \frac{(q+1)\alpha}{pq-1}\} > n - \alpha$, we prove the nonexistence.

Assume $u, v$ are positive solutions of (1.20). First, for $|x| > R$,

$$u(x) \geq c \int_{B_R(0)} \frac{dy}{|x-y|^{n-\alpha}} \geq \frac{c}{|x|^{a_0}}.$$ 

Here $a_0 = n - \alpha$. By this estimate, for $|x| > R$ there holds

$$v(x) \geq c \int_{B_{|x|/2}(x)} \frac{|y|^{-p\alpha}dy}{|x-y|^{n-\alpha}} \geq \frac{c}{|x|^{b_0}},$$

where $b_0 = pa_0 - \alpha$. This implies

$$u(x) \geq c \int_{B_{|x|/2}(x)} \frac{|y|^{-q\alpha}dy}{|x-y|^{n-\alpha}} \geq \frac{c}{|x|^{a_1}},$$

for $|x| > R$, where $a_1 = qb_0 - \alpha$. By induction, we obtain that for $|x| > R$,

$$v(x) \geq \frac{c}{|x|^{b_0}}, \quad u(x) \geq \frac{c}{|x|^{a_k}}.$$ 

Here $a_0 = n - \alpha, b_k = pa_k - \alpha$ and $a_k = qb_k - \alpha$. Therefore, we have

$$a_j = pq a_{j-1} - \alpha(q+1) = (pq)^2 a_{j-2} - \alpha(q+1)(1 + pq) = \cdots = (pq)^j a_0 - \alpha(q+1)(1 + pq + \cdots + (pq)^{j-1}).$$

**Case I.** When $pq = 1$, there holds $a_{j_0} = a_0 - \alpha(q+1)j_0 < 0$ for some large $j_0$.

**Case II.** When $0 < pq < 1$, letting $j \to \infty$, we get $a_j = (pq)^j a_0 - \alpha(q+1) \frac{1-(pq)^j}{1-pq} \to -\frac{\alpha(q+1)}{1-pq} < 0$. Therefore, we can find $j_0$ such that $a_{j_0} < 0$.

**Case III.** When $\frac{\alpha(q+1)}{pq-1} > n - \alpha$, we deduce $a_{j_0} = (pq)^j a_0 - \alpha(q+1) \frac{(pq)^j}{pq-1} = (pq)^j [a_0 - \frac{(pq)^j}{pq-1}] + \frac{\alpha(q+1)}{pq-1} < 0$ for some large $j_0$.

**Case IV.** When $\frac{\alpha(p+1)}{pq-1} > n - \alpha$, we can also find some $k_0$ such that $b_{k_0} < 0$ by an analogous argument of Case III.

These results imply $u(x) = \infty$ or $v(x) = \infty$. It is impossible. The contradiction shows the nonexistence of the positive solutions to (1.20).

(ii) If $\max\{\frac{(p+1)\alpha}{pq-1}, \frac{(q+1)\alpha}{pq-1}\} = n - \alpha$, we prove the nonexistence.
The idea is the same as Step 3 in the proof of Theorem 2.13. Denote \( B_R(0) \) by \( B \). First,
\[
u(x) \geq \frac{c}{(R + |x|)^{n-\alpha}} \int_B v^q(y)dy , \quad v(x) \geq \frac{c}{(R + |x|)^{n-\alpha}} \int_B u^p(y)dy.
\]
Thus,
\[
\int_B u^p(x)dx \geq \frac{c(\int_B v^q(y)dy)^p}{R^{(n-\alpha)p}} , \quad \int_B v^q(x)dx \geq \frac{c(\int_B u^p(y)dy)^q}{R^{(n-\alpha)q}}.
\]
Without loss of generality, assume \( p \leq q \). Combining two results above with \( \alpha(q+1) = n - \alpha \) yields
\[
\int_B v^q(x)dx \geq c(\int_B v^q(y)dy)^p,
\]
where \( c \) is independent of \( R \). Letting \( R \to \infty \), we get \( v \in L^q(R^n) \). On the other hand, we also obtain
\[
\int_{A_R} v^q(x)dx \geq c(\int_B v^q(y)dy)^p.
\]
Letting \( R \to \infty \) and noting \( v \in L^q(R^n) \), we see \( v \equiv 0 \). It is impossible.

Theorem 3.1 is proved. \( \square \)

**Corollary 3.2.** Let \( k \in [1, n/2] \) be an integer and \( pq > 1 \). There exist positive solutions \( u, v \) of the semilinear Lane-Emden system (1.22) for some double bounded functions \( c_1(x) \) and \( c_2(x) \), if and only if \( \max \{ \frac{2k(p+1)}{pq-1}, \frac{2k(q+1)}{pq-1} \} < n - 2k \).

**Proof.** When \( pq > 1 \), Liu, Guo and Zhang [37] proved \((-\Delta)^i u > 0 \) and \((-\Delta)^i v > 0 \). Similar to the argument in [9] we can also establish the equivalence between (1.22) and (1.20). So Corollary 3.2 is a direct corollary of Theorem 3.1 with \( \alpha = 2k \). \( \square \)

### 3.2. Wolff system.

**Theorem 3.3.** There exist positive solutions \( u, v \) of the integral system (1.23) for some double bounded functions \( c_1(x) \) and \( c_2(x) \), if and only if \( pq \gtrless (\gamma - 1)^2 \) and
\[
\max \{ \frac{\beta_\gamma(p + \gamma - 1)}{pq - (\gamma - 1)^2}, \frac{\beta_\gamma(q + \gamma - 1)}{pq - (\gamma - 1)^2} \} < \frac{n - \beta_\gamma}{\gamma - 1}.
\]

**Proof.** **Step 1.** Existence.

Insert (3.2) into \( W_{b,\gamma}(u^p) \) and \( W_{b,\gamma}(v^q) \). Similar to the argument in the proof of Theorem 2.4, we also discuss in four cases.

(i) Take the slow rates \( 2\theta_1 = \frac{\beta_\gamma(q + \gamma - 1)}{pq - (\gamma - 1)^2}, \quad 2\theta_2 = \frac{\beta_\gamma(p + \gamma - 1)}{pq - (\gamma - 1)^2} \). Then, \( pq > (\gamma - 1)^2 \) and (3.5) lead to \( \beta_\gamma < 2p\theta_1 < n \) and \( \beta_\gamma < 2q\theta_2 < n \). Therefore,
\[
W_{b,\gamma}(u^p)(x) = c_1(x)(1 + |x|^2)^{\frac{-\beta_\gamma - \beta_\gamma}{2(\gamma - 1)}} = c_1(x)v(x),
\]
\[
W_{b,\gamma}(v^q)(x) = c_2(x)(1 + |x|^2)^{\frac{-\beta_\gamma - \beta_\gamma}{2(\gamma - 1)}} = c_2(x)u(x)
\]
for some double bounded functions \( c_1(x), c_2(x) \). This implies that (1.23) has a pair of radial solutions \( (u, v) \) as (3.2).

(ii) If the stronger condition \( p, q > \frac{n(\gamma - 1)}{n - \beta_\gamma} \) holds, we take the fast rate \( 2\theta_1 = \frac{n - \beta_\gamma}{\gamma - 1} \). Then \( 2p\theta_1 > n \) and \( 2q\theta_2 > n \), and hence
\[
W_{b,\gamma}(u^p)(x) = c_1(x)(1 + |x|^2)^{\frac{-\beta_\gamma - \beta_\gamma}{2(\gamma - 1)}} = c_1(x)v(x),
\]
\[
W_{b,\gamma}(v^q)(x) = c_2(x)(1 + |x|^2)^{\frac{-\beta_\gamma - \beta_\gamma}{2(\gamma - 1)}} = c_2(x)u(x)
\]
for another double bounded functions $c_1(x), c_2(x)$. This implies that (1.23) has a pair of radial solutions $(u, v)$ as (3.2) with fast decay rates.

(iii) Similar to the argument in Theorem 3.1, if $pq > (\gamma - 1)^2$, the condition

$$\frac{\beta\gamma}{n-\beta\gamma} < p < \frac{\beta\gamma(p+\gamma-1)}{pq-(\gamma-1)^2} < \frac{n-\beta\gamma}{\gamma-1}$$

is also stronger than (3.5). Under this stronger condition, we take $2\theta_1 = \frac{n-\beta\gamma}{\gamma-1}$, $2\theta_2 = \frac{2\theta_1-\beta\gamma}{\gamma-1} = p\frac{n-\beta\gamma}{(\gamma-1)^2} - \frac{\beta\gamma}{\gamma-1}$. Therefore, $\beta\gamma < 2\theta_1 < n$ and $2\theta_2 > n$, and hence

$$W_{\beta,\gamma}(u^p)(x) = c_1(x)(1 + |x|^2)^{-\frac{2\theta_1-\beta\gamma}{2(\gamma-1)^2}} = c_1(x)v(x),$$

$$W_{\beta,\gamma}(v^q)(x) = c_2(x)(1 + |x|^2)^{-\frac{n-\beta\gamma}{2(\gamma-1)}} = c_2(x)u(x)$$

for another double bounded functions $c_1(x), c_2(x)$. This shows (1.23) has radial solutions as (3.2).

Similar to the argument above, if another stronger condition $pq > (\gamma - 1)^2$ as well as $\frac{\beta\gamma}{n-\beta\gamma} < q < \frac{\beta\gamma(p+\gamma-1)}{pq-(\gamma-1)^2} < \frac{n-\beta\gamma}{\gamma-1}$ holds, (1.23) also has radial solutions as (3.2) with two different fast rates $2\theta_2 = \frac{n-\beta\gamma}{\gamma-1}$, $2\theta_1 = q\frac{n-\beta\gamma}{(\gamma-1)^2} - \frac{\beta\gamma}{\gamma-1}$.

(iv) Eq. (1.23) also has another pair of radial solutions which also decay fast by two different rates. One decays with $\frac{n-\beta\gamma}{\gamma-1}$, and another decays with logarithmic order. Now, we assume

$$u(x) = \frac{1}{(1 + |x|^2)^{\frac{n-\beta\gamma}{2(\gamma-1)}}}, \quad v(x) = \frac{\log |x|^{\frac{1}{\gamma-1}}}{(1 + |x|^2)^{\frac{n-\beta\gamma}{2(\gamma-1)}}}.$$ 

It is easy to verify that $u, v$ solve (1.23) with some double bounded functions $c_1, c_2$.

**Step 2.** Nonexistence.

**Substep 2.1.** Suppose either $0 < pq \leq (\gamma - 1)^2$ or

$$\max\left\{\frac{\beta\gamma(p+\gamma-1)}{pq-(\gamma-1)^2}, \frac{\beta\gamma(q+\gamma-1)}{pq-(\gamma-1)^2}\right\} > \frac{n-\beta\gamma}{\gamma-1}.$$ 

Assume $u, v$ are positive solutions of (1.23). Noting $\int_{B_R(0)} v^q(y)dy \geq c$, we obtain that for $|x| > R$,

$$u(x) \geq \int_{|x|+R}^\infty \left(\int_{B_R(0)} v^q(y)dy\right)^\frac{1}{q} \frac{1}{t^n} dt \geq c \int_{|x|+R}^\infty t^{-\frac{n-\beta\gamma}{\gamma-1}} \frac{1}{t} dt \geq \frac{c}{|x|^{a_0}}.$$ 

Here $a_0 = \frac{n-\beta\gamma}{\gamma-1}$. By this estimate, for $|x| > R$, there holds

$$v(x) \geq c \int_{2|x|}^\infty \int_{B_R(0)} \frac{dy}{|y|^{\beta\gamma-n}|y-x|^{\alpha}} \frac{dt}{t} \geq c \int_{2|x|}^\infty t^{-\frac{\gamma-\beta\gamma p}{\gamma-1}} \frac{dt}{t}.$$ 

When $\beta\gamma - pq_0 \geq 0$, we see $v(x) = \infty$ for $|x| > R$. This implies the nonexistence of positive solutions of (1.23) since $R$ is an arbitrary positive number. When $\beta\gamma - pq_0 < 0$, then $v(x) \geq \frac{c}{|x|^{a_0}}$ for $|x| > R$, where $b_0 = \frac{pq_0-\beta\gamma}{\gamma-1}$. Similarly, using this estimate, we also obtain that if $\beta\gamma - q_0 \geq 0$, then $u(x) = \infty$; if $\beta\gamma - q_0 < 0$, then $u(x) \geq \frac{c}{|x|^{a_0}}$, for $|x| > R$, where $a_1 = q_0 - \beta\gamma$. For $k = 1, 2, \cdots$, write $a_k = \frac{n-\beta\gamma}{\gamma-1}$, $b_k = \frac{pq_0-\beta\gamma}{\gamma-1}$, $a_k = \frac{q_0 - \beta\gamma}{\gamma-1}$. By induction, we can obtain the following conclusions:

- (i) If $a_k < 0$, then $u(x) = \infty$. This leads to the nonexistence. If $a_k \geq 0$, then $u(x) \geq \frac{c}{|x|^{a_k}}$ implies $v(x) \geq \frac{c}{|x|^{a_k}}$.

- (ii) If $b_k < 0$, then $v(x) = \infty$. This also leads to the nonexistence. If $b_k \geq 0$, then $v(x) \geq \frac{c}{|x|^{b_k}}$ implies $u(x) \geq \frac{c}{|x|^{b_k}}$. 

In view of $a_k = \frac{q}{-1} b_{k-1} - \frac{\beta_\gamma}{\gamma - 1} a_k - \frac{\beta_\gamma}{\gamma} q + \frac{1}{\gamma - 1}$, we deduce that

$$a_j = \left( \frac{pq}{(\gamma - 1)^2} \right)^j a_0 - \frac{\beta_\gamma}{\gamma - 1} q + \frac{1}{\gamma - 1} \left[ 1 + \frac{pq}{(\gamma - 1)^2} + \cdots + \left( \frac{pq}{(\gamma - 1)^2} \right)^{j-1} \right].$$

When $\frac{pq}{(\gamma - 1)^2} = 1$, then for some large $j_0$, $a_{j_0} = a_0 - \frac{\beta_\gamma}{\gamma} q + \frac{1}{\gamma - 1} j_0 < 0$. This implies $u(x) = \infty$. When $0 < \frac{pq}{(\gamma - 1)^2} < 1$, letting $j \to \infty$, we get

$$a_j = \left( \frac{pq}{(\gamma - 1)^2} \right)^j a_0 - \frac{\beta_\gamma}{\gamma - 1} q + \frac{1}{\gamma - 1} \left[ 1 - \frac{pq}{(\gamma - 1)^2} \right] - \frac{\beta_\gamma (q + \gamma - 1)}{(\gamma - 1)^2 - pq} < 0.$$

Therefore, we can find $j_0$ such that $a_{j_0} < 0$. This implies $u(x) = \infty$. When $\frac{pq}{(\gamma - 1)^2} > 1$ and $\frac{\beta_\gamma (q + \gamma - 1)}{pq - (\gamma - 1)^2} > \frac{n - \beta_\gamma}{\gamma - 1}$, there holds $a_0 < \frac{\beta_\gamma (q + \gamma - 1)}{pq - (\gamma - 1)^2}$. We deduce that

$$a_{j_0} = \left( \frac{pq}{(\gamma - 1)^2} \right)^{j_0} a_0 - \frac{\beta_\gamma}{\gamma - 1} q + \frac{1}{\gamma - 1} \left[ 1 - \frac{pq}{(\gamma - 1)^2} \right] - \frac{\beta_\gamma (q + \gamma - 1)}{pq - (\gamma - 1)^2} < 0$$

for some large $j_0$. We also see $u(x) = \infty$. When $\frac{pq}{(\gamma - 1)^2} > 1$ and $\frac{\beta_\gamma (q + \gamma - 1)}{pq - (\gamma - 1)^2} > \frac{n - \beta_\gamma}{\gamma - 1}$, there also holds $a_0 < \frac{\beta_\gamma (q + \gamma - 1)}{pq - (\gamma - 1)^2}$. Via the same argument above by dealing with $b_k$ instead of $a_k$, we can also find some $k_0$ such that $b_{k_0} < 0$. This implies $v(x) = \infty$.

**Substep 2.2.** Suppose $pq > (\gamma - 1)^2$ and max$\{\frac{\beta_\gamma (q + \gamma - 1)}{pq - (\gamma - 1)^2}, \frac{\beta_\gamma (q + \gamma - 1)}{pq - (\gamma - 1)^2} \} = \frac{n - \beta_\gamma}{\gamma - 1}$.

First, write $H := \int_{B_k(y)} v^q dy$. By the Hölder inequality,

$$\int_0^R H dt \leq \left( \int_0^R H^\frac{1}{\gamma - 1} t^\frac{q - n}{\gamma - 1} dt \right)^{\gamma - 1} \left( \int_0^R t^\frac{n - \beta_\gamma}{\gamma - 1} dt \right)^\gamma = CR^{n - \beta_\gamma + 1} \left( \int_0^R \left( \frac{H}{t^{n - \beta_\gamma}} \right)^\frac{1}{\gamma - 1} dt \right)^{\gamma - 1}.$$

Therefore, exchanging the order of variables yields

$$u(x) \geq c \int_0^R \left( \frac{H}{t^{n - \beta_\gamma}} \right)^\frac{1}{\gamma - 1} dt \geq cR^{n - \beta_\gamma + 1} \left( \int_0^R H dt \right)^\frac{1}{\gamma - 1} \geq cR^{n - \beta_\gamma} \left( \int_{B_R/4} v^q(y) dy \right)^\frac{1}{\gamma - 1}.$$

Thus,

$$u^p(x) \geq cR^{n - \beta_\gamma} \left( \int_{B_R/4} v^q(y) dy \right)^{\frac{p}{\gamma - 1}}. \quad (3.6)$$

Similarly,

$$v^q(x) \geq cR^{-q \frac{n - \beta_\gamma}{\gamma - 1}} \left( \int_{B_R/4} u^p(y) dy \right)^{\frac{q}{\gamma - 1}}. \quad (3.7)$$

Without loss of generality, we suppose $p \leq q$. Thus,

$$\frac{\beta_\gamma (q + \gamma - 1)}{pq - (\gamma - 1)^2} = \frac{n - \beta_\gamma}{\gamma - 1}.$$

Inserting (3.6) into (3.7) yields

$$v^q(x) \geq cR^{-q \frac{n - \beta_\gamma}{\gamma - 1} - pq \frac{n - \beta_\gamma}{(\gamma - 1)^2} + \frac{n q}{\gamma - 1}} \left( \int_{B_R/4} v^q(y) dy \right)^{\frac{pq}{(\gamma - 1)^2}}. \quad (3.9)$$
Integrating on $B_{R/4}$, we get
\[
\int_{B_{R/4}} v^q(x)dx \geq cR^{-q}\frac{\gamma^q}{q} \frac{\beta^q}{\beta} (1 + \frac{p}{q}) + n\frac{(m\gamma)^q}{m} \left(\int_{B_{R/4}} v^q(y)dy\right)^{\frac{p}{m}}. \tag{3.10}
\]

We claim that the exponent of $R$ is zero. In fact, $q\beta\gamma + n(\gamma - 1) = \beta(\gamma + q - 1) + (n - \beta\gamma)(\gamma - 1)$. By (3.8), we obtain
\[q\beta\gamma + n(\gamma - 1) = [pq - (\gamma - 1)^2]^{\frac{n - \beta\gamma}{\gamma - 1}} + (\gamma - 1)^2 \frac{n - \beta\gamma}{\gamma - 1} = pq \frac{n - \beta\gamma}{\gamma - 1}.
\]

Multiplying by $(\gamma - 1)^{-1}$, we have
\[n\frac{q}{\gamma - 1} + 1 = \frac{pq}{\gamma - 1} - \frac{n - \beta\gamma}{\gamma - 1} = \frac{n - \beta\gamma}{\gamma - 1} (1 + \frac{p}{\gamma - 1}).
\]

The claim is proved.

Letting $R \to \infty$ in (3.10), we see that $v \in L^q(R^n)$ in view of $pq > (\gamma - 1)^2$.

Integrating (3.9) on $A_R := B_{R/4} \setminus B_{R/8}$ and letting $R \to \infty$, we also have $\int_{R^n} v^q(dy) = 0$. It is impossible.

Thus, we complete our proof. \qed

3.3. $\gamma$-Laplace system.

**Theorem 3.4.** (1) If $pq > (\gamma - 1)^2$ and
\[\max\left\{\frac{(\gamma(q + \gamma - 1)}{pq - (\gamma - 1)^2}, \frac{\gamma(p + \gamma - 1)}{pq - (\gamma - 1)^2}\right\} < \frac{n - \gamma}{\gamma - 1}, \tag{3.11}\]
then there exist positive solutions $u, v$ of the $\gamma$-Laplace system (1.25) for some double bounded $c_1(x)$ and $c_2(x)$.

(2) For any double bounded functions $c_1(x)$ and $c_2(x)$, if one of the following conditions holds:

(i) $0 < pq \leq (\gamma - 1)^2$;

(ii) $pq > (\gamma - 1)^2$ and
\[\max\left\{\frac{(\gamma(q + \gamma - 1)}{pq - (\gamma - 1)^2}, \frac{\gamma(p + \gamma - 1)}{pq - (\gamma - 1)^2}\right\} \geq \frac{n - \gamma}{\gamma - 1}, \tag{3.12}\]
then (1.25) has no positive solutions $u, v$ satisfying $\inf_{R^n} u = \inf_{R^n} v = 0$.

**Proof.** (1) Existence.

Let $m = \frac{n}{\gamma - 1}$. Take
\[u(x) = (1 + |x|^m)^{-\theta_1}, \quad v(x) = (1 + |x|^m)^{-\theta_2}. \tag{3.13}\]

Similar to the calculation in (2.13), we also obtain
\[-\Delta_\gamma u(x) = \frac{(\gamma\theta_1)^{-1}}{(1 + r^m)(\theta_1 + 1)(\gamma - 1)} [n + (\gamma + 1)^{\gamma - 1} r^m],

-\Delta_\gamma v(x) = \frac{(\gamma\theta_2)^{-1}}{(1 + r^m)(\theta_2 + 1)(\gamma - 1)} [n + (\gamma + 1)^{\gamma - 1} r^m].
\]

Therefore, the signs of both sides of the results above show four cases.

(i) Take the slow decay rates $m\theta_1 = \frac{\gamma(q + \gamma - 1)}{pq - (\gamma - 1)^2}$, $m\theta_2 = \frac{\gamma(p + \gamma - 1)}{pq - (\gamma - 1)^2}$. Then $pq > (\gamma - 1)^2$ and (3.11) lead to $(\theta_1 + 1)\gamma < n$ and $(\theta_2 + 1)\gamma < n$, and hence
\[-\Delta_\gamma u(x) = \frac{c_1(r)}{(1 + r^{\gamma\theta_1})(\gamma - 1)} = c_1(x)v^q(x), \quad -\Delta_\gamma v(x) = \frac{c_2(r)}{(1 + r^{\gamma\theta_2})(\gamma - 1)} = c_2(x)u^p(x). \]
This shows that (1.25) has the radial solutions as (3.13) with slow decay rates.

(ii) Moreover, if \( p = q = \frac{n(\gamma-1)+\gamma}{n-\gamma} \), then we take the fast decay rates \( m\theta_1 = m\theta_2 = \frac{n-\gamma}{\gamma-1} \). This leads to \( n = (\theta_1 + 1)\gamma = (\theta_2 + 1)\gamma \). Therefore,

\[
-\Delta \gamma u(x) = \frac{c_1(r)}{(1 + r^m)^{(\theta_1+1)(\gamma-1)+1}} = c_1(x)v^q(x),
-\Delta \gamma v(x) = \frac{c_2(r)}{(1 + r^m)^{(\theta_2+1)(\gamma-1)+1}} = c_2(x)u^p(x).
\]

This shows that (1.25) has the radial solutions as (3.13) with fast decay rates.

(iii) If \( \frac{\gamma(p+\gamma)}{pq-(\gamma-1)p} = \frac{n-\gamma}{\gamma-1} \), then we take other fast decay rates \( m\theta_1 = \frac{n-\gamma}{\gamma-1}, m\theta_2 = \frac{n-\gamma}{\gamma-1} - \frac{\gamma}{\gamma-1} \). Thus, \( n = (\theta_1 + 1)\gamma, n > (\theta_2 + 1)\gamma \). Therefore,

\[
-\Delta \gamma u(x) = \frac{c_1(r)}{(1 + r^m)^{(\theta_1+1)(\gamma-1)+1}} = c_1(x)v^q(x),
-\Delta \gamma v(x) = \frac{c_2(r)}{(1 + r^m)^{(\theta_2+1)(\gamma-1)+1}} = c_2(x)u^p(x).
\]

This shows that (1.25) has the radial solutions as (3.13) with the second fast decay rates.

Similar to the argument above, if \( \frac{\gamma(p+\gamma)}{pq-(\gamma-1)p} = \frac{n-\gamma}{\gamma-1} \) holds, (1.23) also has radial solutions as (3.13) with the third fast rates \( m\theta_2 = \frac{n-\gamma}{\gamma-1}, m\theta_1 = q\frac{n-\gamma}{\gamma-1} - \frac{\gamma}{\gamma-1} \).

(iv) Eq. (1.23) also has another pair of radial solutions which also decay fast with the different rates. One decays with \( \frac{n-\gamma}{\gamma-1} \), and another decays with logarithmic order. Now, we assume

\[
u(x) = (1 + |x|^m)^{-\frac{n-\gamma}{\gamma-1}}, \quad v(x) = (\log |x|)^{\frac{1}{\gamma-1}}(1 + |x|^m)^{-\frac{n-\gamma}{\gamma-1}}.
\]

It is easy to verify that \( u, v \) solve (1.23) with some double bounded functions \( c_1, c_2 \).

(2) **Nonexistence.**

Suppose \( u, v \) are positive solutions of (1.25) satisfying \( \inf_{R^n} u = \inf_{R^n} v = 0 \). According to Corollary 4.13 in [24], there exists \( C > 0 \) such that

\[
\frac{1}{C} W_{1,\gamma}(c_1v^q)(x) \leq u(x) \leq CW_{1,\gamma}(c_1v^q)(x),
\]

\[
\frac{1}{C} W_{1,\gamma}(c_2u^p)(x) \leq v(x) \leq CW_{1,\gamma}(c_2u^p)(x).
\]

Since \( c_1 \) and \( c_2 \) are double bounded, we can find two other double bounded functions \( K_1(x) \) and \( K_2(x) \) such that

\[
u(x) = K_1(x)W_{1,\gamma}(v^q)(x), \quad v(x) = K_2(x)W_{1,\gamma}(u^p)(x).
\]

By Theorem 3.3 with \( \beta = 1 \), we can see the nonexistence. \( \square \)

4. **Finite energy solutions: Scalar equations.** In this section, we consider the critical conditions associated with the existence of the positive solutions when the coefficient \( c(x) \equiv 1 \).
4.1. **Critical exponents and scaling invariants.** Take a scaling transformation
\( u_\mu(x) = \mu^{-\frac{2}{n-2}} u(\mu x) \). Assume \( u \) solves
\[-\Delta u = u^{\frac{n+2}{n-2}}, \quad u > 0 \text{ in } \mathbb{R}^n.\]
By a simple calculation, we have \(-\Delta u_\mu = u^{\frac{n+2}{n-2}}\) and \( \|u\|_{L^\infty} = \|u_\mu\|_{L^\infty} \). For the higher-order equation, the corresponding conformal properties are still true. Furthermore, we have the more general result.

**Theorem 4.1.** *The HLS equation (1.7) and the energy \( \|u\|_{L^p+1(\mathbb{R}^n)} \) are invariant under the scaling transformation, if and only if (1.11) holds.*

**Proof.** Take the scaling transformation \( u_\mu(x) = \mu^\sigma u(\mu x) \). Then
\[
u_\mu(x) = \mu^\sigma \int_{\mathbb{R}^n} \frac{u_\mu(y)dy}{|\mu x - y|^{n-\alpha}} = \mu^{-p\sigma + \alpha} \int_{\mathbb{R}^n} \frac{u_\mu(y)dy}{|x - y|^{n-\alpha}}.
\]
Clearly, \( u_\mu \) still solves (1.7) if and only if
\[
\sigma = \frac{\alpha}{p - 1}.
\] (4.1)

On the other hand,
\[
\int_{\mathbb{R}^n} u_\mu^{p+1}(x)dx = \int_{\mathbb{R}^n} [\mu^\sigma u(\mu x)]^{p+1}dx = \mu^{\sigma(p+1) - n} \int_{\mathbb{R}^n} u^{p+1}(z)dz.
\]
If the \( L^{p+1}(\mathbb{R}^n) \)-norm is invariant, then there holds \( \sigma = \frac{n}{p+1} \). Combining this with (4.1), we get (1.11). On the contrary, if (1.11) is true, then we can also deduce the invariance by the same calculation above.

**Theorem 4.2.** *The Wolff equation (1.16) and the energy \( \|u\|_{L^{p+\gamma-1}(\mathbb{R}^n)} \) are invariant under the scaling transformation, and only if*
\[
p = \frac{n + \beta \gamma}{n - \beta \gamma} (\gamma - 1).
\] (4.2)

*In addition, (1.16) and another energy \( \|u\|_{L^{p+1}(\mathbb{R}^n)} \) are invariant under the scaling transformation, if and only if*
\[
p = \gamma^* - 1 \quad \text{(where} \quad \gamma^* = \frac{n \gamma}{n - \beta \gamma}).
\] (4.3)

**Proof.** Take the scaling transformation \( u_\mu(x) = \mu^\sigma u(\mu x) \). Then
\[
u_\mu(x) = \mu^{\sigma + \frac{\beta \gamma - p}{\gamma - 1}} \int_0^\infty \frac{B_s(x)}{s^{n-\beta \gamma}} \frac{u_\mu^p(z)dz}{s^{\gamma - 1}} ds.
\]
Thus, \( u_\mu \) solves (1.16) if and only if
\[
\sigma = \frac{\beta \gamma}{p - \gamma + 1}.
\] (4.4)

By virtue of
\[
\int_{\mathbb{R}^n} u_\mu^{p+\gamma-1}(x)dx = \mu^{\sigma(p+\gamma-1) - n} \int_{\mathbb{R}^n} u^{p+\gamma-1}(z)dz,
\]
the \( L^{p+\gamma-1}(\mathbb{R}^n) \) norm is invariant if and only if \( \sigma = \frac{n}{p+\gamma-1} \). Combining this with (4.4), we get (4.2).
Theorem 4.3. The \( L^p+1(R^n) \) norm is invariant if and only if \( \sigma = \frac{n}{p+1} \). Combining this with (4.4),
we get (4.3).

Since the corresponding result of the \( \gamma \)-Laplace equation can not be covered by
that of the Wolff equation, we should point out the following conclusion.

**Theorem 4.4.** The \( \gamma \)-Laplace equation (1.15) and the energy \( \|u\|_{L^{p+\gamma-1}(R^n)} \) are
invariant under the scaling transformation, if and only if (4.2) with \( \beta = 1 \) holds.

In addition, (1.15) and another energy \( \|u\|_{L^{p+1}(R^n)} \) are invariant under the scaling
transformation, if and only if (4.3) with \( \beta = 1 \) holds.

**Proof.** Suppose \( u_\mu \) is a solution of (1.15). Then
\[
-\mu^{\sigma(\gamma-1)}\text{div}_x([\nabla_x u(\mu x)]^{\gamma-2}\nabla_x u(\mu x)) = \mu^{\rho\sigma} u^p(\mu x).
\]
Let \( y = \mu x \), then
\[
-\mu^{\sigma(\gamma-1)+\gamma}\text{div}_y([\nabla_y u(y)]^{\gamma-2}\nabla_y u(y)) = \mu^{\rho\sigma} u^p(y).
\]
This result shows that the equation is invariant if and only if \( \sigma = \frac{2}{p+\gamma-1} \). By the
same argument as in Theorem 4.2, the invariance of the energy is equivalent to
\( \sigma = \frac{n}{p+\gamma-1} \). Eliminating \( \sigma \) from the two formulas above yields \( p = \frac{n}{n-1}(\gamma-1) \).
The proof that (4.3) with \( \beta = 1 \) is the sufficient and necessary condition is the
same as the argument above.

### 4.2. HLS equation.

**Theorem 4.4.** Let \( u > 0 \) be a classical solution of (1.5). Assume \( u \in L^2(R^n) \).
Then \( \nabla u \in L^2(R^n) \) if and only if \( u \in L^{p+1}(R^n) \).

A classical positive solution \( u \in L^2(R^n) \) of (1.5) is called finite energy solution,
if \( u \in L^{p+1}(R^n) \) or \( \nabla u \in L^2(R^n) \).

**Proof.** Take smooth function \( \zeta(x) \) satisfying
\[
\begin{cases}
\zeta(x) = 1, & \text{for } |x| \leq 1; \\
\zeta(x) \in [0, 1], & \text{for } |x| \in [1, 2]; \\
\zeta(x) = 0, & \text{for } |x| \geq 2.
\end{cases}
\]
Define the cut-off function
\[
\zeta_R(x) = \zeta\left(\frac{x}{R}\right). \tag{4.5}
\]
Multiplying (1.5) by \( u\zeta_R^2 \) and integrating on \( D := B_{3R}(0) \), we have
\[
-\int_D u\zeta_R^2 \Delta u dx = \int_D u^{p+1} \zeta_R^2 dx.
\]
Integrating by parts, we obtain
\[
\int_D |\nabla u|^2 \zeta_R^2 dx + 2 \int_D u\zeta_R \nabla u \nabla \zeta_R dx = \int_D u^{p+1} \zeta_R^2 dx. \tag{4.6}
\]
Applying the Young inequality, we get
\[
|\int_D u\zeta_R \nabla u \nabla \zeta_R dx| \leq \delta \int_D |\nabla u|^2 \zeta_R^2 dx + C \int_D u^2 |\nabla \zeta_R|^2 dx \tag{4.7}
\]
for any $\delta \in (0,1/2)$. If $u \in L^2(R^n)$, we can find $C$ which is independent of $R$ such that
\[
\int_D u^2 |\nabla \zeta_R|^2 dx \leq C. \tag{4.8}
\]
If $u \in L^{p+1}(R^n) \cap L^2(R^n)$, then (4.6)-(4.8) imply $\int_D |\nabla u|^2 \zeta_R^2 dx \leq C$. Letting $R \to \infty$ yields $\nabla u \in L^2(R^n)$. This and $u \in L^2(R^n)$ show $R \int_{\partial D} (|\nabla u|^2 + u^2) ds \to 0$ for some $R = R_j \to \infty$. Therefore,
\[
|\int_{\partial D} u \partial_\nu u ds| \leq (\int_{\partial D} u^2 ds)^{1/2} (\int_{\partial D} |\nabla u|^2 ds)^{1/2} R^{(n-1)(1/2-1/2^*)} \to 0, \tag{4.9}
\]
when $R \to \infty$. Multiplying (1.5) by $u$ yields
\[
\int_D u^{p+1} dx = \int_D |\nabla u|^2 dx - \int_{\partial D} u \partial_\nu u ds. \tag{4.10}
\]
Letting $R \to \infty$ and using the result above, we have $\|\nabla u\|_2^2 = \|u\|_{p+1}^{p+1}$.

If $\nabla u \in L^2(R^n)$ and $u \in L^2(R^n)$, (4.9) still holds. If letting $R \to \infty$ in (4.10) and inserting (4.9) into it, we obtain $\|u\|_{p+1}^{p+1} = \|\nabla u\|_2^2$ and hence $u \in L^{p+1}(R^n)$. \qed

Next, we use the Pohozaev identity in integral forms to discuss the existence of the finite energy solutions of (1.7). A positive classical solution $u$ of (1.7) is called finite energy solution, if $u \in L^{p+1}(R^n)$.

**Theorem 4.5.** The HLS integral equation (1.7) has positive classical solution in $L^{p+1}(R^n)$ if and only if (1.11) holds.

**Proof.** If (1.11) holds, (1.7) exists a unique class of finite energy solutions
\[
u(x) = c(t^2 + |x - x_0|^2)^{(\alpha-n)/2}
\]
with $c, t > 0$ and $x_0 \in R^n$ (cf. [12] or [35]).

On the contrary, if $u \in L^{p+1}(R^n)$ solves (1.7), we claim that (1.11) is true. In fact, for any $\mu > 0$, from (1.7) it follows
\[
\int_{R^n} \frac{u^p(\mu z) dz}{|x - z|^{n-\alpha}} = \mu^\alpha \int_{R^n} \frac{u^{p-1}(\mu z) (z \cdot \nabla u) dz}{|x - z|^{n-\alpha}}.
\]
Differentiate both sides with respect to $\mu$. Then,
\[
x \cdot \nabla u(\mu x) = \alpha \mu^{\alpha-1} \int_{R^n} \frac{u^p(\mu z) dz}{|x - z|^{n-\alpha}} \mu^\alpha \int_{R^n} \frac{pu^{p-1}(\mu z) (z \cdot \nabla u) dz}{|x - z|^{n-\alpha}}.
\]
Letting $\mu = 1$ yields
\[
x \cdot \nabla u(x) = \alpha u(x) + \int_{R^n} \frac{z \cdot \nabla u^p(z) dz}{|x - z|^{n-\alpha}}. \tag{4.11}
\]
By $u \in L^{p+1}(R^n)$, we can find $R_j \to \infty$ such that
\[
R_j \int_{\partial B_{R_j}} u^{p+1} ds \to 0. \tag{4.12}
\]
Therefore, integrating by parts we get
\[
\int_{R^n} u^p (x \cdot \nabla u) dx = \frac{1}{p+1} \int_{R^n} x \cdot \nabla u^{p+1} dx = \frac{-n}{p+1} \int_{R^n} u^{p+1} dx. \tag{4.13}
\]
This implies \( u^p(x \cdot \nabla u) \in L^1(R^n) \). Using (1.7) and Fubini’s theorem, we have

\[
p \int_{R^n} u^p(z)(z \cdot \nabla u(z))dz = \int_{R^n} (z \cdot \nabla u^p(z))u(z)dz
\]

\[
= \int_{R^n} z \cdot \nabla u^p(z) \int_{R^n} \frac{u^p(x)dx}{|x-z|^{n-\alpha}}dz = \int_{R^n} u^p(x)dx \int_{R^n} \frac{z \cdot \nabla u^p(z)dz}{|x-z|^{n-\alpha}}.
\]

Multiply (4.11) by \( u^p \) and integrate on \( R^n \). Combining with the result above, we obtain

\[
\int_{R^n} u^p(x)(x \cdot \nabla u(x))dx = \alpha \int_{R^n} u^{p+1}(x)dx + p \int_{R^n} u^p(x)(x \cdot \nabla u)dx,
\]

which implies

\[(p-1) \int_{R^n} u^p(x \cdot \nabla u)dx + \alpha \int_{R^n} u^{p+1}dx = 0.\]

Inserting (4.13) into the result above, we see \((n\frac{p-1}{p+1} - \alpha) \int_{R^n} u^{p+1}dx = 0\) It leads to

\[p = \frac{n+\alpha}{n-\alpha}.\]

Theorem 4.5 is proved.

**Corollary 4.6.** Let \( k \in [1, n/2) \) be an integer and \( p > 1 \). The 2k-order Lane-Emden PDE (1.6) has positive classical solution in \( L^{p+1}(R^n) \) if and only if \( p = \frac{n+2k}{n-2k} \).

**Proof.** When \( p > 1 \), Corollary 2.3 shows that (1.6) is equivalent to the HLS equation (1.7) with \( \alpha = 2k \). According to Theorem 4.5, we have the corresponding critical conditions \( p = \frac{n+2k}{n-2k} \) for the existence of the finite energy solutions of the (1.5).

**Remark 4.1.** Theorem 4.5 shows another critical condition (1.11) for the existence of the positive solutions to (1.7). Since the finite energy solutions class of (1.7) is smaller than the positive solutions class of (1.19), the critical condition (1.11) is stronger than (2.3).

4.3. \( \gamma \)-Laplace equation. Serrin and Zou [50] proved that \( \gamma \)-Laplace equation has positive classical solutions if and only if \( p \geq \gamma^* - 1 \), where \( \gamma^* = \frac{n\gamma}{n-\gamma} \). Naturally, we conjecture that \( \gamma \)-Laplace equation has the finite energy solution if and only if \( p = \gamma^* - 1 \).

To define the finite energy solution, we first introduce the following theorem. It is a natural generalization of Theorem 4.4.

**Theorem 4.7.** Assume \( u > 0 \) is a classical solution of the \( \gamma \)-Laplace equation (1.15). Assume \( u \in L^{\gamma}(R^n) \) with \( \gamma^* = \frac{n\gamma}{n-\gamma} \). Then \( \nabla u \in L^{\gamma}(R^n) \) if and only if \( u \in L^{p+1}(R^n) \). In addition, \( \|\nabla u\|_{\gamma} = \|u\|_{p+1}^{1-p}\).

A classical positive solution \( u \in L^{\gamma}(R^n) \) of (1.15) is called finite energy solution if \( u \in L^{p+1}(R^n) \) or \( \nabla u \in L^{\gamma}(R^n) \).

**Proof.** Let \( u \in L^{\gamma}(R^n) \). Take a cut-off function \( \zeta_R \) as (4.5). Using the Hölder inequality, we get

\[
\int_D u^\gamma |\nabla \zeta_R|^\gamma dx \leq \|u\|_{\gamma^*,D} \|\nabla \zeta\|_{\gamma^*,D} \leq C,
\]

where \( D = B_{2R}(0) \), and \( C > 0 \) is independent of \( R \).

1. Sufficiency: Supposing \( u \in L^{p+1}(R^n) \cap L^{\gamma}(R^n) \) solves (1.15), we claim \( \nabla u \in L^{\gamma}(R^n) \) and \( \|\nabla u\|_{\gamma^*} = \|u\|_{p+1}^{1-p} \).
By means of the Hölder inequality and (4.16), we get
\[ \int_D |\nabla u|^\gamma \zeta_R dx + \gamma \int_D |\nabla u|^{-2} (u \zeta_R^{-1}) \nabla u \nabla \zeta_R dx = \int_D u^{p+1} \zeta_R^d dx. \] (4.15)

Using the Young inequality, from (4.15) we deduce that for any \( \delta \)
\[ \int_D |\nabla u|^\gamma \zeta_R dx \leq C \int_D |\nabla u|^{-2} (u \zeta_R^{-1}) \nabla u \nabla \zeta_R dx + \int_D u^{p+1} \zeta_R^d dx \]
\[ \leq \delta \int_D |\nabla u|^\gamma \zeta_R dx + C \int_D u^\gamma \zeta_R^d dx + \int_D u^{p+1} \zeta_R^d dx. \]
Combining this result with (4.14), we see that \( \int_D |\nabla u|^\gamma \zeta_R^d dx \leq C \). Let \( R \to \infty \), then \( \nabla u \in L^\gamma (R^n) \). From this result as well as \( u \in L^{\gamma^*} (R^n) \), we deduce that for some \( R \), denoted by \( R \),
\[ R \int_{\partial D} (|\nabla u|^\gamma + u^\gamma) ds < o(1), \quad \text{as } R \to \infty. \] (4.16)

Multiplying (1.15) by \( u \) and integrating on \( D \), we have
\[ \int_D u^{p+1} dx = - \int_D u \Delta u dx + \int_D |\nabla u|^\gamma dx - \int_{\partial D} u |\nabla u|^{-2} \partial_n u ds. \] (4.17)

By means of the Hölder inequality and (4.16), we get
\[ |\int_{\partial D} u |\nabla u|^{-2} \partial_n u ds| \]
\[ \leq |R \int_{\partial D} |\nabla u|^\gamma ds|^{\gamma^*} |\int_{\partial D} u^\gamma ds|^{1/\gamma^*} |R^{\gamma^*} - \gamma^* R^{-1/\gamma^*} (R^{\gamma^*} - 1)(1/\gamma^*/1/\gamma^*)| \]
\[ < o(1) \quad \text{as } R \to \infty. \]

Letting \( R \to \infty \) in (4.17), we have
\[ \int_{R^n} |\nabla u|^\gamma dx = \int_{R^n} u^{p+1} dx. \] (4.18)

(2) Necessity. Let \( u > 0 \) solve (1.15). If \( \nabla u \in L^\gamma (R^n) \) and \( u \in L^{\gamma^*} (R^n) \), then (4.16) is still true. Using (4.16) to handle the last term of the right hand side of (4.17), we also derive (4.18) and hence \( u \in L^{p+1} (R^n) \).

Theorem 4.8. The \( \gamma \)-Laplace equation (1.15) has a classical solution satisfying \( \nabla u \in L^\gamma (R^n) \) if and only if
\[ p = \gamma^* - 1. \] (4.19)

Proof. If \( p = \gamma^* - 1 \), according to p.328 in [14], (1.15) admits a class of solutions
\[ u(x) = d[1 + D(d \frac{\pi}{\gamma^*} \frac{1}{|x|^\gamma})]^{\frac{\gamma^* - 1}{\gamma^*}}. \]
Here \( d, D \) are positive constants.

Next, we prove the sufficiency. Write \( B = B_R(0) \). Multiplying the equation with \( (x \cdot \nabla u) \) and integrating on \( B \), we obtain
\[ \int_B |\nabla u|^\gamma \nabla u (x \cdot \nabla u) dx - \int_{\partial B} |\nabla u|^\gamma \partial_n u (x \cdot \nabla u) ds = \int_B u^p (x \cdot \nabla u) dx. \]
Here \( \nu \) is the unit outward normal vector to \( \partial B \). Noting \( \nabla u \nabla (x \cdot \nabla u) = |\nabla u|^2 + \frac{1}{2} x \cdot \nabla (|\nabla u|^2) \) and \( x = |x| \nu \), we have
\[ \int_B |\nabla u|^\gamma dx + \int_B \nabla (|\nabla u|^\gamma) \cdot \frac{x}{|x|} dx - R \int_{\partial B} |\nabla u|^\gamma \partial_n u^2 ds = \int_B \frac{x \cdot \nabla u^{p+1}}{p+1} dx. \]
Integrating by parts, we get
\[
(1 - \frac{n}{\gamma}) \int_B |\nabla u|^\gamma dx + R \int_{\partial B} |\nabla u|^\gamma ds - R \int_{\partial B} |\nabla u|^{\gamma-2} |\partial_x u|^2 ds = R \int_{\partial B} u^{p+1} ds - \frac{n}{p+1} \int_B u^{p+1} dx.
\]
(4.20)

According to Theorem 4.7, \(\nabla u \in L^\gamma(R^n)\) implies \(u \in L^{p+1}(R^n)\). Therefore, we can find \(R_j \to \infty\), such that
\[
R_j \int_{\partial B_{R_j}} (u^{p+1} + |\nabla u|^\gamma) ds \to 0.
\]

Let \(R = R_j \to \infty\) in (4.20). By means of the result above, we deduce that
\[
(1 - \frac{n}{\gamma}) \int_{R^n} |\nabla u|^\gamma dx = -\frac{n}{p+1} \int_{R^n} u^{p+1} dx.
\]
Inserting (4.18) into this result yields \(p = \gamma^* - 1\).

**Remark 4.2.** (1) For the Wolff equation (1.16), we do not know whether (4.2) is the necessary and sufficient condition for the existence of positive solution in \(L^{p+\gamma-1}(R^n)\).

(2) A surprising observation is, when \(\gamma \neq 2\), the critical condition (4.19) is different from (4.2) with \(\beta = 1\). One reason is that the solution of (1.15) only solves a Wolff equation with variable coefficient instead of (1.16). Another reason is that the finite energy functions spaces \(L^{p+1}(R^n)\) and \(L^{p+\gamma-1}(R^n)\) are also different except for \(\gamma = 2\). This distinction shows that (1.11) and (4.19) are not the same class critical exponents. For \(\gamma\)-Laplace equation, besides the divisional number in Theorem 2.5, we also have two critical exponents mentioned above. The relation of them is
\[
\frac{n(\gamma - 1)}{n - \gamma} < \frac{n + \gamma}{n - \gamma} (\gamma - 1) < \frac{n \gamma}{n - \gamma} - 1
\]
as long as \(\gamma \in (1, 2)\). This is also implied by the difference of the existence spaces of positive solutions.

5. **Finite energy solutions: Systems.**

5.1. **Critical conditions and scaling invariants.**

**Theorem 5.1.** (1) Both the semilinear Lane-Emden system
\[
\begin{align*}
-\Delta u &= v^q, \\
-\Delta v &= u^p.
\end{align*}
\]
and the energy integrals \(\|u\|_{L^{p+1}(R^n)}\) and \(\|v\|_{L^{q+1}(R^n)}\) are invariant under the scaling transformations, if and only if
\[
\frac{1}{p+1} + \frac{1}{q+1} = \frac{n-2}{n}.
\]
(5.2)

(2) Both the \(\gamma\)-Laplace system (1.4) and the energy integrals \(\|u\|_{L^{p+\gamma-1}(R^n)}\) and \(\|v\|_{L^{q+\gamma-1}(R^n)}\) are invariant under the scaling transformations, if and only if
\[
\frac{1}{p+\gamma-1} + \frac{1}{q+\gamma-1} = \frac{n-\gamma}{n(\gamma-1)}.
\]
(5.3)
In addition, (1.4) and the energy integrals \( \|u\|_{L^{p+1}(\mathbb{R}^n)} \) and \( \|v\|_{L^{q+1}(\mathbb{R}^n)} \) are invariant under the scaling transformations, if and only if
\[
p = q = \frac{n\gamma}{n - \gamma} \quad \text{or} \quad \gamma = 2. \tag{5.4}
\]

(3) HLS system (1.1) and the energy integrals \( \|u\|_{L^{p+1}(\mathbb{R}^n)} \) and \( \|v\|_{L^{q+1}(\mathbb{R}^n)} \) are invariant under the scaling transformations, if and only if (1.12) holds.

(4) Wolff system (1.3) and the energy integrals \( \|u\|_{L^{p+\gamma-1}(\mathbb{R}^n)} \) and \( \|v\|_{L^{q+\gamma-1}(\mathbb{R}^n)} \) are invariant under the scaling transformations, if and only if
\[
\frac{1}{p + \gamma - 1} + \frac{1}{q + \gamma - 1} = \frac{n - \beta\gamma}{n(\gamma - 1)}. \tag{5.5}
\]

In addition, (1.3) and the energy integrals \( \|u\|_{L^{p+1}(\mathbb{R}^n)} \) and \( \|v\|_{L^{q+1}(\mathbb{R}^n)} \) are invariant under the scaling transformations, if and only if
\[
p = q = \frac{n\gamma}{n - \beta\gamma} - 1 \quad \text{or} \quad \gamma = 2. \tag{5.6}
\]

Proof. (1) Take the scaling transformations
\[
u_{\mu}(x) = \mu^{p\gamma}u(\mu x), \quad \nu_{\mu}(x) = \mu^{q\gamma}u(\mu x). \tag{5.7}
\]
Then
\[-\Delta u_{\mu}(x) = \mu^{p\gamma}2^{-\gamma+2 - q\sigma_2}u_{\mu}(x), \quad -\Delta v_{\mu} = \mu^{q\gamma}2^{-\gamma+2 - p\sigma_1}v_{\mu};
\]
and
\[
\|u_{\mu}\|_{p+1} = \mu^{\sigma_1(p+1) - n}\|u\|_{p+1}, \quad \|v_{\mu}\|_{q+1} = \mu^{\sigma_2(q+1) - n}\|v\|_{q+1}.
\]
Clearly, (5.1) is invariant if and only if \( \sigma_1 + 2 = q\sigma_2, \sigma_2 + 2 = p\sigma_1 \). Energy integrals are invariant if and only if \( \sigma_1(p+1) = n, \sigma_2(q+1) = n \). Eliminate \( \sigma_1 \) and \( \sigma_2 \). Then
\[
\frac{pq - (\beta\gamma)^2}{(p+1)(q+1)} = \frac{n}{2}. \tag{5.8}
\]
This is equivalent to (5.2).

(2) In view of (5.7), we have
\[-\Delta u_{\mu}(x) = \mu^{\sigma_1(\gamma - 1) + \gamma - q\sigma_2}u_{\mu}(x), \quad -\Delta v_{\lambda} = \mu^{\sigma_2(\gamma - 1) + \gamma - p\sigma_1}v_{\mu};
\]
and
\[
\|u_{\mu}\|_{p+\gamma - 1} = \mu^{\sigma_1(p+\gamma - 1) - n}\|u\|_{p+\gamma - 1}, \quad \|v_{\mu}\|_{q+\gamma - 1} = \mu^{\sigma_2(q+\gamma - 1) - n}\|v\|_{q+\gamma - 1}.
\]

Eq. (1.4) is invariant if and only if \( \sigma_1(1 + \gamma - q\sigma_2) = 0, \sigma_2(1 + \gamma - p\sigma_1) = 0 \). Namely,
\[
\sigma_1 = \frac{\gamma(q + \gamma - 1)}{pq - (\gamma - 1)^2}, \quad \sigma_2 = \frac{\gamma(p + \gamma - 1)}{pq - (\gamma - 1)^2}. \tag{5.8}
\]
Energy integrals \( \|u\|_{L^{p+\gamma-1}(\mathbb{R}^n)} \) and \( \|v\|_{L^{q+\gamma-1}(\mathbb{R}^n)} \) are invariant if and only if \( \sigma_1(p + \gamma - 1) - n = 0, \sigma_2(q + \gamma - 1) - n = 0 \). Eliminating \( \sigma_1 \) and \( \sigma_2 \), we obtain
\[
\frac{(p + \gamma - 1)(q + \gamma - 1)}{pq - (\gamma - 1)^2} = \frac{n}{\gamma},
\]
This is (5.3) if we notice \( (p + \gamma - 1)(q + \gamma - 1) = pq - (\gamma - 1)^2 + (\gamma - 1)((p + \gamma - 1) + (q + \gamma - 1)) \).

Similarly, \( \|u\|_{L^{p+1}(\mathbb{R}^n)} \) and \( \|v\|_{L^{q+1}(\mathbb{R}^n)} \) are invariant if and only if \( \sigma_1 = \frac{n}{p+1}, \sigma_2 = \frac{n}{q+1} \). Combining with (5.8), we get \( (q + 1)(p + \gamma - 1) = (p + 1)(q + \gamma - 1) \).

This is equivalent to \( (p - q)(\gamma - 2) = 0 \). If \( p = q \), from \( \frac{\gamma(p + \gamma - 1)}{pq - (\gamma - 1)^2} = \frac{n}{p+1} \), we can see \( \sigma_1 = \frac{n\gamma}{n - \gamma} - 1 \). Thus, (5.4) is the sufficient and necessary condition.
(3) Noting (5.7), we have
\[
v_\mu(x) = \mu^{\sigma_2 - p\sigma_1 + \alpha} \int_{\mathbb{R}^n} \frac{u_\mu^p(y)dy}{|x - y|^{n-\alpha}}, \quad u_\mu(x) = \mu^{\sigma_1 - q\sigma_2 + \alpha} \int_{\mathbb{R}^n} \frac{v_\mu^q(y)dy}{|x - y|^{n-\alpha}}.
\]
Thus, \( u_\mu, v_\mu \) still solve (1.1) if and only if \( \sigma_1 + \alpha = q\sigma_2, \sigma_2 + \alpha = p\sigma_1 \). By the same calculation in (1), energy integrals are invariant if and only if \( \sigma_1(p + 1) = \sigma_2(q + 1) = n \). Eliminating \( \sigma_1 \) and \( \sigma_2 \), we deduce (1.12).

(4) Noting (5.7), we have
\[
v_\mu(x) = \mu^{\sigma_2 + \frac{\beta_\gamma - p\sigma_1}{\gamma - 1}} \int_0^\infty \left( \frac{\int_{B_s(x)} u_\mu^p(z)dz}{s^{n-\beta\gamma}} \right)^{\frac{1}{p-1}} ds, \quad u_\mu(x) = \mu^{\sigma_1 + \frac{\beta_\gamma - q\sigma_2}{\gamma - 1}} \int_0^\infty \left( \frac{\int_{B_s(x)} v_\mu^q(z)dz}{s^{n-\beta\gamma}} \right)^{\frac{1}{q-1}} ds.
\]
Thus, \( u_\mu, v_\mu \) still solve (1.3) if and only if \( \gamma - 1)\sigma_1 + \beta_\gamma = q\sigma_2, (\gamma - 1)\sigma_2 + \beta_\gamma = p\sigma_1 \). By the same calculation in (2), energy integrals \( \|u\|_{L^{p+\gamma-1}(\mathbb{R}^n)} \) and \( \|v\|_{L^{q+\gamma-1}(\mathbb{R}^n)} \) are invariant if and only if \( \sigma_1(p + \gamma - 1) = \sigma_2(q + \gamma - 1) = n \). Eliminating \( \sigma_1 \) and \( \sigma_2 \), we deduce (5.5).

By the same argument in (2), (5.6) is another corresponding necessary and sufficient condition.

5.2. Existence and the critical conditions. Pucci, Serrin and Zou showed that (5.2) is the critical condition of the existence of the finite energy solution of (5.1). We call the positive classical solutions \( u, v \in L^2(\mathbb{R}^n) \) of (5.1) finite energy solutions, if \( (u, v) \in L^{p+1}(\mathbb{R}^n) \times L^{q+1}(\mathbb{R}^n) \). The following is the sketch.

**Theorem 5.2.** The system (5.1) has a pair of finite energy solutions \( (u, v) \) if and only if (5.2) holds.

**Proof.** Serrin and Zou [49] proved the existence if (5.2) is true. According to the Pohozaev identity established in [46] (or cf. Proposition 5.1 in [48]), there holds
\[
\left( \frac{n}{p + 1} - a_1 \right) \int_B u^{p+1}dx + \left( \frac{n}{q + 1} - a_2 \right) \int_B v^{q+1}dx
= R^n \int_{S^{n-1}} \left( \frac{u^{p+1}}{p + 1} + \frac{v^{q+1}}{q + 1} \right)ds + R^{n-1} \int_{S^{n-1}} (a_1 u \partial_r v + a_2 v \partial_r u)ds
+ R^n \int_{S^{n-1}} (\partial_r u \partial_r v - \frac{\partial u \partial v}{R^2})ds,
\]
where \( B = B_R(0), a_2, a_2 \in R \) satisfy \( a_1 + a_2 = n - 2 \). Since \( u, v \) are finite energy solutions, we know \( \nabla u, \nabla v \in L^2(\mathbb{R}^n) \) by an analogous argument of Theorem 4.4. Using Applying the Young inequality, we can find \( R = R_j \rightarrow \infty \), such that all the terms in the right hand side of (5.9) converge to zero. Thus, if we take \( a_2 = \frac{n}{q+1}, (5.2) \) can also be verified.

Next, we consider the HLS system. Since (1.1) is the Euler-Lagrange system of the extremal functions of the HLS inequality which implies \( (u, v) \in L^{p+1}(\mathbb{R}^n) \times L^{q+1}(\mathbb{R}^n) \), we naturally call such solutions (belonging to \( L^{p+1}(\mathbb{R}^n) \times L^{q+1}(\mathbb{R}^n) \)) of (1.1) as finite energy solutions.

**Theorem 5.3.** The HLS system (1.1) has the finite energy solutions if and only if (1.12) holds.
Proof. Sufficiency. Clearly, the extremal functions of the HLS inequality are the finite energy solutions. Lieb [35] obtained the existence of those extremal functions.

Necessity. The Pohozaev identity in integral forms is used here.

For any $\mu > 0$, there holds

$$ v(\mu x) = \mu^\alpha \int_{R^n} \frac{u^p(\mu z)dz}{|x - z|^{n-\alpha}}. $$

Differentiate both sides with respect to $\mu$ and let $\mu = 1$. Then,

$$ x \cdot \nabla v = \alpha v + \int_{R^n} \frac{z \cdot \nabla u^p(z)dz}{|x - z|^{n-\alpha}}. \tag{5.10} $$

Since $u, v$ are the finite energy solutions, it follows

$$ R \int_{\partial B_R} u^{p+1}(z)ds \to 0, \quad R \int_{\partial B_R} v^{q+1}(z)ds \to 0, $$

when $R = R_j \to \infty$. Thus, integrating by parts, we obtain

$$ \int_{R^n} v^q(x)(x \cdot \nabla v(x))dx = \frac{-n}{q + 1} \int_{R^n} v^{q+1}(x)dx, \tag{5.11} $$

$$ \int_{R^n} (x \cdot \nabla u^{p+1}(x))dx = -n \int_{R^n} u^{p+1}(x)dx. \tag{5.12} $$

Multiplying (5.10) by $v^q(x)$, and using (1.1), we get

$$ \int_{R^n} v^q(x)(x \cdot \nabla v(x))dx = \alpha \int_{R^n} v^{q+1}(x)dx + \int_{R^n} v^q(x)dx \int_{R^n} \frac{z \cdot \nabla u^p(z)dz}{|x - z|^{n-\alpha}} $$

$$ = \alpha \int_{R^n} v^{q+1}(x)dx + \int_{R^n} z \cdot \nabla u(z)dz \int_{R^n} \frac{v^q(x)}{|x - z|^{n-\alpha}}dx $$

$$ = \alpha \int_{R^n} v^{q+1}(x)dx + \int_{R^n} [z \cdot \nabla u^p(z)]u(z)dz $$

$$ = \alpha \int_{R^n} v^{q+1}(x)dx + \frac{p}{p + 1} \int_{R^n} z \cdot \nabla u^{p+1}(z)dz. $$

Combining this result with (5.11) and (5.12) yields

$$ -\frac{n}{q + 1} \int_{R^n} v^{q+1}dx = \alpha \int_{R^n} v^{q+1}dx - \frac{np}{p + 1} \int_{R^n} u^{p+1}dx. \tag{5.13} $$

From (1.1), it follows that

$$ \int_{R^n} v^{q+1}dx = \int_{R^n} v^q(x)dx \int_{R^n} \frac{u^p(y)dy}{|x - y|^{n-\alpha}} $$

$$ = \int_{R^n} u^p(x)dx \int_{R^n} \frac{v^q(y)dy}{|x - y|^{n-\alpha}} = \int_{R^n} u^{p+1}(x)dx. $$

Inserting this into (5.13) we get $\frac{np}{p + 1} - \frac{n}{q + 1} = \alpha$, which implies (1.12). Theorem 5.3 is proved.

Corollary 5.4. Let $k \in [1, n/2)$ be an integer and $pq > 1$. The $2k$-order system (1.2) has a pair of finite energy positive solutions $(u, v)$, then (1.14) holds.
Step 3. We claim that for \( 0 < r < a \leq \varepsilon \), this system is equivalent to the integral system (1.1) with \( \alpha = 2k \) (cf. [9]). According to Theorem 5.3, we can also derive the conclusion.

6. Infinite energy solutions.

6.1. Existence in supercritical case for bi-Laplace equations. For semilinear Lane-Emden equation (1.5) with the supercritical exponents, Li and Ni [34] obtained positive radial solutions with the slow decay rate \( u(x) = O(|x|^{-\frac{n+4}{n-4}}) \) when \( |x| \to \infty \). According to Corollary 1.3, it is not the finite energy solution.

In this subsection, we prove that there also exists an infinite energy solution for bi-Laplace equation in the supercritical case \( p > \frac{n+4}{n-4} \). This result was obtained firstly in [15] by the shooting method which was also employed in [49]. They handled a 4-order ODE and found the radial solutions. Here, we split the bi-Laplace equation into a system of 2-order ODEs. Namely,

\[
(-\Delta)^2 u = u^p, \quad u > 0 \text{ in } R^n,
\]

is equivalent to

\[
\begin{cases}
-\Delta u = v, & u > 0 \text{ in } R^n, \\
-\Delta v = u^p, & v > 0 \text{ in } R^n.
\end{cases}
\]

We search the positive solutions with radial structures. The existence can be implied by the following argument.

Theorem 6.1. Let \( p > \frac{n+4}{n-4} \). Then the following ODE system

\[
\begin{cases}
-(u'' + \frac{n-1}{r} u') = v, & u'' + \frac{n-1}{r} u' = u^p, & r > 0 \\
u'(0) = v'(0) = 0, & u(0) = 1, & v(0) = a,
\end{cases}
\]  

(6.1)

has entire solutions satisfying \( \lim_{|x| \to \infty} u(x) = \lim_{|x| \to \infty} v(x) = 0 \).

Proof. Here we use the shooting method.

We denote the solutions of (6.1) by \( u_a(r), v_a(r) \).

Step 1. By the standard contraction argument, we can see the local existence.

Step 2. We claim that for \( a \geq 4n \), there exists \( R \in (0, 1] \) such that \( u_a(r), v_a(r) > 0 \) for \( r \in [0,R) \) and \( u_a(R) = 0 \).

In fact, from (6.1) we obtain \( u'_a < 0 \) which implies \( u_a(r) \leq u_a(0) = 1 \), and

\[
u_a(r) = u_a(0) - \int_0^r \tau^{1-n} \int_0^\tau s^{n-1} u_a^p(s) ds d\tau \geq a - \frac{r^2}{2n} \geq \frac{a}{2}\]

for \( r \in [0,1] \). Therefore,

\[
u_a(r) = u_a(0) - \int_0^r \tau^{1-n} \int_0^\tau s^{n-1} v_a(s) ds d\tau \leq 1 - \frac{ar^2}{4n}.
\]

This proves that for \( a \geq 4n \), we can find \( R \in (0,1] \) such that \( u_a(r), v_a(r) > 0 \) for \( r \in (0,R) \) and \( u_a(R) = 0 \).

Step 3. We claim that for \( 0 < a < \varepsilon_0 = \frac{1}{n^2+n} \), there exists \( R \in (0,1] \), such that \( u_a(r), v_a(r) > 0 \) for \( r \in (0,R) \) and \( v_a(R) = 0 \).

In fact,

\[
u_a(r) \geq 1 - \frac{\varepsilon_0 r^2}{2n} \geq \frac{1}{2}.
\]
for \( r \in (0, 1) \). Therefore,

\[
v_a(r) < \varepsilon_0 - \frac{1}{2n} r^2.
\]

This proves that for \( a < \varepsilon_0 \), we can find \( R \in (0, 1) \) such that \( u_a(r), v_a(r) > 0 \) for \( r \in (0, R) \) and \( v_a(R) = 0 \).

**Step 4.** Let \( a = \sup S \), where

\[
S := \{ \varepsilon; \quad \text{when } a \in (0, \varepsilon), \exists R_a > 0, \text{ such that } u_a(r) > 0, v_a(r) \geq 0, \text{ for } r \in (0, R_a], v_a(R_a) = 0 \}.
\]

Clearly, \( S \neq \emptyset \) by virtue of \( \varepsilon_0 \in S \). From Step 2, it follows \( \varepsilon \leq 4n \) for \( \varepsilon \in S \). Namely, \( S \) is upper bounded, which implies the existence of \( a \).

**Step 5.** Write \( \bar{u}(r) = u_{\frac{a}{2}}(r) \) and \( \bar{v}(r) = v_{\frac{a}{2}}(r) \). We claim that \( \bar{u}(r), \bar{v}(r) > 0 \) for \( r \in (0, R) \), and hence they are entire positive solutions of (6.1).

Otherwise, there exists \( \tilde{R} > 0 \) such that \( \bar{u}(r), \bar{v}(r) > 0 \) for \( r \in (0, \tilde{R}) \) and one of the following consequences holds:

\begin{itemize}
  \item[(i)] \( \bar{u}(\tilde{R}) = 0, \bar{v}(\tilde{R}) > 0 \);
  \item[(ii)] \( \bar{v}(\tilde{R}) = 0, \bar{u}(\tilde{R}) > 0 \);
  \item[(iii)] \( \bar{u}(\tilde{R}) = 0, \bar{v}(\tilde{R}) = 0 \).
\end{itemize}

We deduce the contradictions from three consequences above.

(i) By \( C^1 \)-continuous dependence of \( u_a, v_a \) in \( a \), and the fact \( \bar{u}'(\tilde{R}) < 0 \), we see that for all \( [a - \delta, a] \) small, there exists \( R_a > 0 \) such that

\[
\bar{u}(r), \bar{v}(r) > 0, \quad \text{for } r \in (0, R_a); \quad \bar{u}(R_a) = 0, \quad \bar{v}(R_a) > 0.
\]

This contradicts with the definition of \( a \).

(ii) Similarly, for \( |a - \delta| \) small, there exists \( R_a > 0 \) such that

\[
\bar{u}(r), \bar{v}(r) > 0, \quad \text{for } r \in (0, R_a); \quad \bar{u}(R_a) > 0, \quad \bar{v}(R_a) = 0.
\]

This implies that \( a + \delta \in S \) for some \( \delta > 0 \), which contradicts with the definition of \( a \).

(iii) The consequence implies that \( u(x) = \bar{u}(|x|) \) and \( v(x) = \bar{v}(|x|) \) are solutions of the system

\[
\begin{cases}
-\Delta u = v, & -\Delta v = u^p, \text{ in } B_R, \\
u, v > 0 \text{ in } B_R, & u = v = 0 \text{ on } \partial B_R.
\end{cases}
\]

It is impossible by the Pohozaev identity proved later (cf. Theorem 6.3).

All the contradictions show that our claim is true. Thus, the entire positive solutions exist.

**Step 6.** We claim \( \lim_{r \to \infty} \bar{u}(r), \bar{v}(r) = 0 \).

Eq. (6.1) implies \( \bar{u}' < 0 \) and \( \bar{v}' < 0 \) for \( r > 0 \). So \( \bar{u} \) and \( \bar{v} \) are decreasing positive solutions, and \( \lim_{r \to \infty} \bar{u}(r), \lim_{r \to \infty} \bar{v}(r) \) exist.

If there exists \( c > 0 \) such that \( \bar{v}(r) \geq c \) for \( r > 0 \), then (6.1) shows that \( \bar{u} \) satisfies

\[
\bar{u}'' + \frac{n-1}{r} \bar{u}' \leq -c.
\]

Integrating twice yields

\[
\bar{u}(r) \leq \bar{u}(0) - \frac{cr^2}{2n}
\]

for \( r > 0 \). It is impossible since \( \bar{u} \) is a entire positive solution. This shows that \( \bar{v} \to 0 \) when \( r \to \infty \).

Similarly, \( u \) has the same property.
Remark 6.1. (1) When $k \in (2, n/2)$ is an integer, the existence of the $2k$-order PDEs (1.2) in the supercritical cases was obtained by Liu, Guo and Zhang [38]. In §6.3, the shooting method and the analysis of the target map via the degree theory are developed to be employed in dealing with more general systems.

(2) In the critical case $p = \frac{n+\alpha}{n-\alpha}$, (1.10) with $\alpha = 2k$ is a solution of (1.6). For the system (1.2), the critical condition $\frac{1}{p+1} + \frac{1}{q+1} = 1 - \frac{2}{n}$ leads to $pq > 1$. The argument in Corollary 5.4 shows the equivalence between (1.2) and the HLS system (1.1). Therefore, the existence of (1.2) is implied by the sufficiency of Theorem 5.3.

(3) In the subcritical case $p < \frac{n+\alpha}{n-\alpha}$, the nonexistence of positive solutions of (1.7) had been proved (cf. [1], [10] and [54]). On the other hand, by the equivalence between (1.6) and (1.7) (cf. [9] and [12]), we also see that (1.6) has no positive solution. As regards the nonexistence for the system (1.2) (or (1.1)), it is the Lane-Emden conjecture (or the HLS conjecture) (cf. [3] and [51]).

6.2. Nonexistence in bounded domain. In this subsection, we give the Pohozaev identity which the proof of Theorem 6.1 needs. In fact, we can give more general ones which imply nonexistence of positive solutions of the following $2k$-order PDE (1.6)

$$(-\Delta)^ku = u^p, \quad k \geq 1, \quad u > 0$$

in a bounded star-shaped domain with the Navier boundary conditions, where $p > \frac{n+2k}{n-2k}$. The argument plays an essential role to prove the existence results in $\mathbb{R}^n$.

It should be pointed out that the conclusions here are not covered by the results in [46], since the boundary conditions are different.

Proposition 6.2. Let $D \subset \mathbb{R}^n$ be a bounded domain. Assume that $u_j$ ($j = 1, 2, \cdots, k$) solve the following boundary value problem

$$\begin{cases}
-\Delta u_1 = u_2, & -\Delta u_2 = u_3, \cdots, \\
-\Delta u_{k-1} = u_k, & -\Delta u_k = u_{k+1} := u_1^p, \quad \text{in } D, \\
u_1 = u_2 = \cdots = u_k = 0, \quad \text{on } \partial D.
\end{cases}$$

(6.3)

Then for $j = 1, 2, \cdots, k$, there hold

$$\int_D u_j^{p+1} dx = \int_D u_j u_{k+2-j} dx, \quad \int_D u_j^{p+1} dx = \int_D \nabla u_j \nabla u_{k+1-j} dx. \quad (6.4)$$

Proof. Applying the boundary value condition, from (6.3) we obtain

$$\int_D u_1^{p+1} dx = -\int_D u_1 \Delta u_k dx = \int_D \nabla u_1 \nabla u_k dx$$

$$= -\int_D u_k \Delta u_1 dx = \int_D u_2 u_k dx = -\int_D u_2 \Delta u_{k-1} dx$$

$$= \int_D \nabla u_2 \nabla u_{k-1} dx = \int_D u_3 u_{k-1} dx = \cdots$$

$$= \int_D \nabla u_j \nabla u_{k+1-j} dx = \int_D u_j u_{k+2-j} dx.$$  

This result implies (6.4). \qed

Theorem 6.3. Let $D \subset \mathbb{R}^n$ be a bounded star-shaped domain. If

$$p \geq \frac{n+2k}{n-2k}, \quad (6.5)$$

then

$$\int_D u_j^{p+1} dx = \int_D u_j u_{k+2-j} dx, \quad \int_D u_j^{p+1} dx = \int_D \nabla u_j \nabla u_{k+1-j} dx.$$
then the following Navier boundary value problem has no positive radial solution in $C^{2k}(D) \cap C^{2k-1}(\overline{D})$

$$
\begin{cases}
(-\Delta)^k u = u^p & \text{in } D, \\
u = \Delta u = \cdots = \Delta^{k-1} u = 0 & \text{on } \partial D.
\end{cases}
$$

(6.6)

**Proof.** Suppose $u$ is a radial solution. Clearly, $u = u_1$ satisfies

$$
\begin{cases}
-\Delta u_1 = u_2, \\
-\Delta u_2 = u_3, \\
\cdots \\
-\Delta u_{k-1} = u_k, \\
-\Delta u_k = u_{k+1} := u_1^p, \\
u_1 = u_2 = \cdots = u_k = 0, \text{ on } \partial D.
\end{cases}
$$

By the maximum principle, from $-\Delta u_k = u_1^p > 0$ and $u_k|_{\partial D} = 0$, we see $u_k > 0$ in $D$. By the same way, we also deduce by induction that

$$u_j > 0 \text{ in } D, \quad j = 1, 2, \cdots, k. \quad (6.7)$$

Multiplying the $j$-th equation by $(x \cdot \nabla u_{k+1-j})$, we have

$$
\begin{align*}
&-\int_{\partial D} (x \cdot \nu) \partial_r u_j \partial_r u_{k+1-j} ds + \int_D \nabla u_j \nabla u_{k+1-j} dx \\
&+ \int_D x \cdot \nabla u_{jx} (u_{k+1-j})_x dx = \int_D u_{j+1} (x \cdot \nabla u_{k+1-j}) dx.
\end{align*}
$$

(6.8)

for $j = 1, 2, \cdots, k$, where $\nu$ is the unit outward normal vector on $\partial D$. Integrating by parts, we can see that

$$
\begin{align*}
\int_D x \cdot (u_{jx}, \nabla (u_{k+1-j})_{x_i} + (u_{k+1-j})_{x_i} \nabla u_{jx_i}) dx \\
= \int_D x \cdot \nabla (\nabla u_j \nabla u_{k+1-j}) dx \\
= \int_{\partial D} (x \cdot \nu) \partial_r u_j \partial_r u_{k+1-j} ds - n \int_D \nabla u_j \nabla u_{k+1-j} dx.
\end{align*}
$$

Combining the results of (6.8) with $j$ and $k+1-j$, and using the result above, we deduce that, for $j = 1, 2, 3, \cdots, k$,

$$
\begin{align*}
&-\int_{\partial D} (x \cdot \nu) \partial_r u_j \partial_r u_{k+1-j} ds + (2 - n) \int_D \nabla u_j \nabla u_{k+1-j} dx \\
&= \int_D u_{k+2-j} (x \cdot \nabla u_j) dx + \int_D u_{j+1} (x \cdot \nabla u_{k+1-j}) dx.
\end{align*}
$$

(6.9)

Integrating by parts, we also see that for $j = 2, 3, \cdots, k$,

$$
\begin{align*}
\int_D x \cdot (u_j \nabla u_{k+2-j} + u_{k+2-j} \nabla u_j) dx \\
= \int_D x \cdot \nabla (u_j u_{k+2-j}) dx = -n \int_D u_j u_{k+2-j} dx.
\end{align*}
$$

Summing $j$ from 1 to $k$ in (6.9) and using the result above, we obtain

$$
\begin{align*}
\frac{2 - n}{2} \int_D (\nabla u_1 \nabla u_k + \nabla u_2 \nabla u_{k-1} + \cdots + \nabla u_k \nabla u_1) dx \\
+ \frac{n}{2} \int_D (u_2 u_k + u_3 u_{k-1} + \cdots + u_k u_2) dx - \int_D u_{k+1} (x \cdot \nabla u_1) dx \\
= \int_{\partial D} (x \cdot \nu) (\partial_r u_1 \partial_r u_k + \partial_r u_2 \partial_r u_{k-1} + \cdots + \partial_r u_k \partial_r u_1).
\end{align*}
$$
By virtue of (6.7) and the boundary value condition, the Hopf lemma shows that
\( \partial_n u_j < 0 \) on \( \partial D \) for \( j = 1, 2, \cdots, k \). Noting \( D \) is star-shaped, we know that all
terms in the right hand side of the result above are positive. Namely,
\[
\frac{2 - n}{2} \int_D (\nabla u_1 \nabla u_k + \nabla u_2 \nabla u_{k-1} + \cdots + \nabla u_k \nabla u_1)dx \\
+ \frac{n}{2} \int_D (u_2 u_k + u_3 u_{k-1} + \cdots + u_k u_2)dx + \frac{n}{p+1} \int_D u^{p+1} dx > 0.
\]
Inserting (6.4) into (6.10), we have
\[
\frac{n}{p+1} + \frac{k(2-n)}{2} + \frac{n(k-1)}{2} > 0.
\]
This contradicts (6.5). \( \Box \)

The following result is necessary to prove Theorem 1.5 (2) (cf. [30]).

**Theorem 6.4.** Let \( D \subset \mathbb{R}^n \) be a bounded star-shaped domain. If
\[
\frac{1}{p+1} + \frac{1}{q+1} \leq \frac{n-2k}{n},
\]
then the following Navier boundary value problem has no positive radial solution in
\( C^{2k}(D) \cap C^{2k-1}(\partial D) \)
\[
\begin{cases}
(-\Delta)^ku = v^q, & (-\Delta)^kv = u^p \quad \text{in} \ D, \\
u = \Delta u = \cdots = \Delta^{k-1}u = 0 \quad \text{on} \ \partial D, \\
v = \Delta v = \cdots = \Delta^{k-1}v = 0 \quad \text{on} \ \partial D.
\end{cases}
\]

**Proof.** Clearly, the solutions \( u_1(=u) \) and \( v_1(=v) \) of (6.12) satisfy
\[
\begin{cases}
-\Delta u_1 = u_2, -\Delta u_2 = u_3, \cdots, -\Delta u_k = u_{k+1} := v^q, \quad \text{in} \ D, \\
-\Delta v_1 = v_2, -\Delta v_2 = v_3, \cdots, -\Delta v_k = v_{k+1} := u^p, \quad \text{in} \ D, \\
u_1 = u_2 = \cdots = u_k = 0 \quad \text{on} \ \partial D, \\
v_1 = v_2 = \cdots = v_k = 0 \quad \text{on} \ \partial D.
\end{cases}
\]
Multiply \(-\Delta u_j = u_{j+1}\) and \(-\Delta v_j = v_{j+1}\) by \((x \cdot \nabla v_{k+1-j})\) and \((x \cdot \nabla u_{k+1-j})\), respectively. Integrating by parts yields
\[
- \int_{\partial D} (x \cdot \nabla) \partial_v u_j \partial_v u_{k+1-j} ds + \int_D \nabla u_j \nabla v_{k+1-j} dx \\
+ \int_D [x \cdot \nabla (v_{k+1-j})_x]u_{jx} dx = \int_D u_{j+1}(x \cdot \nabla v_{k+1-j}) dx
\]
and
\[
- \int_{\partial D} (x \cdot \nabla) \partial_v v_j \partial_v u_{k+1-j} ds + \int_D \nabla v_j \nabla u_{k+1-j} dx \\
+ \int_D [x \cdot \nabla (u_{k+1-j})_x]v_{jx} dx = \int_D v_{j+1}(x \cdot \nabla u_{k+1-j}) dx.
\]
Adding the \((k+1-j)\)-th (6.14) and the \(j\)-th (6.15) together, we obtain
\[
- \int_{\partial D} (x \cdot \nabla) \partial_v v_1 \partial_v u_k ds + (2-n) \int_D \nabla v_1 \nabla u_k dx \\
= -\frac{n}{q+1} \int_D v_1^{q+1} dx + \int_D v_2(x \cdot \nabla u_k) dx,
\]
\[
- \int_{\partial D} (x \cdot \nu) \partial_{\nu} u_1 \partial_{\nu} v_k ds + (2 - n) \int_D \nabla u_1 \nabla v_k dx
= - \frac{n}{p+1} \int_D u_1^{p+1} dx + \int_D u_2 (x \cdot \nabla v_k) dx,
\]

and
\[
- \int_{\partial D} (x \cdot \nu) \partial_{\nu} v_j \partial_{\nu} u_{k+1-j} ds + (2 - n) \int_D \nabla v_j \nabla u_{k+1-j} dx
= \int_D [v_{j+1} (x \cdot \nabla u_{k+1-j}) + u_{k+2-j} (x \cdot \nabla v_j)] dx.
\]

Summing \( j \) from 1 to \( k \), by (6.16), (6.17) and (6.18) we deduce that
\[
\begin{align*}
&\frac{n}{p+1} \int_D \left( \frac{u_1^{p+1}}{p+1} + \frac{u_1^{q+1}}{q+1} \right) dx + n \int_D \sum_{j=2}^k v_j u_{k+2-j} \ dx \\
&\quad + (2 - n) \int_D \sum_{j=1}^k \nabla v_j \nabla u_{k+1-j} dx \\
&= \int_D (x \cdot \nu) \sum_{j=1}^k \partial_{\nu} v_j \partial_{\nu} u_{k+1-j} ds > 0.
\end{align*}
\]

Similar to Proposition 6.2, it also follows
\[
\int_D u_1^{p+1} dx = \int_D u_1^{q+1} dx = \int_D \nabla v_j \nabla u_{k+1-j} dx = \int_D v_l u_{k+2-l}
\]
for \( 1 \leq l \leq k \) and \( 2 \leq l \leq k \).

Combining two results above, we have
\[
\frac{n}{p+1} + \frac{n}{q+1} + n(k-1) + (2-n)k > 0,
\]
which contradicts (6.11).

\[\square\]

6.3. Systems with non-degenerate terms. Consider a more general system
\[
\begin{cases}
-\Delta u_i = f_i(u) \quad \text{in} \ R^n, \quad i = 1, 2, \cdots, L, \\
u_i > 0 \quad \text{in} \ R^n.
\end{cases}
\]

(6.19)

We rewrite the above system in radial coordinate as an initial value problem and choose a set of suitable initial values as the domain of our target map. With the proof of the continuity of the target map, we apply the degree theory (property 1.5.5 in [42]) to compute the index of the target map and to show that the target map is onto which guarantees that we can shoot to the desired target. We then prove that this guarantees the existence of some global positive solutions.

We only need some very mild assumptions on the function: \( F = (f_1, f_2, \cdots, f_L): R_+^L = R_+ \times R_+ \times \cdots \times R_+ \to R_+^L \), where \( R_+ = [0, \infty) \). Here, we always assume that \( F \) is continuous on \( R_+^L \) and is Lipschitz in the interior of \( R_+^L \).

Definition 6.5. We say that \( F \), or system (6.19), is non-degenerate if for any permutation \( i_1, i_2, \cdots, i_L \) of 1, 2, \cdots, L, any \( 1 \leq k < L \), and any \( u = (u_1, u_2, \cdots, u_L) \in R_+^L \) with \( u_{i_1} > 0, \cdots, u_{i_k} > 0 \), and \( u_{i_{k+1}} = \cdots = u_{i_L} = 0 \), we have \( f_{i_{k+1}}(u) + \cdots + f_{i_L}(u) > 0 \).
The main idea is to associate the existence of solutions to (6.19) with the non-existence of solutions to the Dirichlet boundary value problem of the same elliptic system on balls:

\[
\begin{aligned}
-\Delta u_i &= f_i(u) \quad \text{in } B_R(0) \subset \mathbb{R}^n, \quad i = 1, 2, \cdots, L, \\
u_i > 0 &= \text{in } B_R(0); \quad u_i = 0 \quad \text{on } \partial B_R(0).
\end{aligned}
\] (6.20)

**Theorem 6.6.** Assuming that \( F \) is non-degenerate, then system (6.19) admits a solution if the corresponding system (6.20) admits no solution for any given \( R > 0 \). Furthermore, if we assume that \( F(u) \neq 0 \) for \( u > 0 \) (we write \( u > 0 \) if all components of \( u \) are positive i.e. \( u_i > 0 \) for \( i = 1, 2, \cdots, L \)), then (6.19) admits a solution \( u(x) \) such that \( u(x) \to 0 \) uniformly as \( |x| \to \infty \).

First, if \( F(\beta) = 0 \) for some \( \beta > 0 \), then \( u = \beta \) is a trivial positive solution of (6.19) and we are done with our proof of theorem 6.6. Thus, we may assume that \( F(\beta) \neq 0 \) for \( \beta > 0 \). Later, we can see that we only need the assumption holds for \( \beta \) small. The key ingredient of this subsection is the definition and the analysis of the target map \( \psi \).

For any initial value \( \alpha = (\alpha_1, \alpha_2, \cdots, \alpha_L) \) with \( \alpha_i > 0, i = 1, 2, \cdots, L, \) we solve the following initial value problem and denote the solution as \( u(r, \alpha) \):

\[
\begin{aligned}
u''_i(r) + \frac{n+1}{r} u'_i(r) &= -f_i(u), \\
u'_i(0) &= 0; \quad u_i(0) = \alpha_i, \quad i = 1, 2, \cdots, L.
\end{aligned}
\] (6.21)

For \( \alpha > 0 \), we define the target map with \( \psi(\alpha) = u(r_0, \alpha) \) where \( r_0 \) is the smallest value of \( r \) for which \( u_i(r, \alpha) = 0 \) for some \( i \) or when there exists no such \( r \), we define \( \psi(\alpha) = \lim_{r \to \infty} u(r, \alpha) \). In the later case, one can see that \( F(\psi(\alpha)) = 0 \). This and the assumption that \( F(\beta) \neq 0 \) when \( \beta > 0 \) ensure that \( \psi(\alpha) \in \partial R^L_+ \). With the natural extension of \( \psi(\alpha) = \alpha \) for \( \alpha \in R^L_+ \), we then show that \( \psi \) is continuous from \( R^L_+ \) to \( \partial R^L_+ \).

Applying the degree theory, we show that \( \psi \) is onto from \( A_a \) to \( B_a \) where:

\[
A_a := \{ \alpha \in R^L_+: \sum_{i=1}^{L} \alpha_i = a \}, \quad B_a := \{ \alpha \in \partial R^L_+: \sum_{i=1}^{L} \alpha_i \leq a \}
\]

for any \( a > 0 \). In particular, there exists at least one \( \alpha_a \in A_a \) for every \( a > 0 \) such that \( \psi(\alpha_a) = 0 \). Shooting from the initial value \( \alpha_a \), using the fact that the system (6.20) admits no solution, we obtain a solution of (6.19). In fact, we get a solution for every \( a > 0 \). We remark that we get infinity many solutions even if our assumptions on \( F \) only hold for \( u \) small. In this case, we just employ the above method for \( a \) small.

**Lemma 6.7.** The map \( \psi : R^L_+ \to \partial R^L_+ \) is continuous.

**Proof.** For any \( \alpha \in R^L_+ \), there are three cases to be considered:

1. \( \alpha \in \partial R^L_+ \).
2. \( \alpha > 0 \), and the solution \( u(r, \alpha) \) of (6.21) with initial value \( \alpha \) touches the wall at the smallest possible value \( r_0 \) with \( u_i(r_0, \alpha) = 0 \), for some \( 1 \leq i_0 \leq L \).
3. \( \alpha > 0 \), and the solution \( u(r, \alpha) \) of (6.21) never touches the wall, or \( u_i(r, \alpha) > 0 \) for \( i = 1, 2, \cdots, L \) and \( r \in [0, \infty) \).

**Case (1).** If \( \bar{\alpha} = 0 \), then \( |\psi(\alpha) - \psi(\bar{\alpha})| = |\psi(\alpha)| \leq |\alpha| = |\alpha - \bar{\alpha}| \to 0 \) as \( \alpha \to 0 = \bar{\alpha} \).

When \( \alpha \neq 0 \), without loss of generality, we assume that \( \bar{\alpha}_1 = \cdots = \bar{\alpha}_j = 0 \) and \( \bar{\alpha}_{j+1} > 0, \cdots, \bar{\alpha}_L > 0, 1 \leq j \leq L \). We must have \( f_1(\bar{\alpha}) + \cdots + f_j(\bar{\alpha}) > 0 \) by the assumption that \( F \) is non-degenerate. Thus we may assume \( f_1(\bar{\alpha}) = c > 0 \). By
continuity, there exists $\delta_1 > 0$ such that $f_i(\alpha) \geq c/2$ if $|\alpha - \bar{\alpha}| \leq \delta_1$. Classical ODE theory shows that there exists a $\delta_2 > 0$ such that if $|\alpha - \bar{\alpha}| \leq \delta_2$ and $r < \delta_2$ then $|u(r, \alpha) - \bar{\alpha}| \leq \delta_1$ before $u(r, \alpha)$ touches the wall. One sees that as $|\alpha - \bar{\alpha}| \to 0$, $u_1(r, \alpha) = 0$ for some $r_1 \to 0$. Hence, $|\psi(\alpha) - \psi(\bar{\alpha})| \leq |u(r_1, \alpha) - \alpha| + |\alpha - \bar{\alpha}| \to 0$ as $\alpha \to \bar{\alpha}$.

**Case (2).** From the fact that $f_i \geq 0$, one derives that $u'_i(\alpha, \bar{\alpha}) < 0$. This transversality condition and the ODE stability imply that $\psi$ is continuous at $\bar{\alpha}$.

**Case (3).** In this case, we first show that $\psi(\bar{\alpha}) = 0$. Classical ODE or Elliptic theory shows that $F(\psi(\bar{\alpha})) = 0$. Hence $\psi(\bar{\alpha}) \in \partial R_1^\alpha$. By our non-degeneracy assumption, we conclude that $\psi(\bar{\alpha}) = 0$. In fact, the non-degeneracy condition implies that $F(\beta) \neq 0$ when $\beta \in \partial R_1^\alpha$ and $\beta \neq 0$. Then $u(r, \bar{\alpha})$ is positive and small for $r$ large. Continuous dependence of initial values for our ODE implies that for any $R$ large but fixed when $\alpha$ is close to $\bar{\alpha}$, then $u(r, \alpha) > 0$ for $r \in [0, R]$ and $u(R, \alpha)$ is close to $u(r, \bar{\alpha})$ and thus is small. Consequently $|\psi(\alpha)| \leq |u(R, \alpha)|$ is small. This shows that $\psi$ is continuous at $\bar{\alpha}$.

**Lemma 6.8.** For any $a > 0$, $\psi$ is an onto map from $A_a$ to $B_a$ and thus there exists at least one $\alpha_a \in A_a$ such that $\psi(\alpha_a) = 0$.

**Proof.** Recall the definition of $B_a$, then as a consequence of the non-increasing property of the solutions of (6.21), we see that $\psi$ maps $A_a \to B_a$. Let $\phi(\alpha) = \alpha + \frac{1}{L} (a - \sum_{i=1}^n \alpha_i)(1, \cdots, 1) : B_a \to A_a$, then $\phi$ is continuous with a continuous inverse $\phi^{-1}(\alpha) = \alpha - (\min_{i \in \{1, \cdots, L\}} \alpha_i)(1, \cdots, 1) : A_a \to B_a$. The map $G = \phi \circ \psi : A_a \to A_a$ is continuous and $G(\alpha) = \alpha$ on the boundary of $A_a$. Based on the Heinz-Lax-Nirenberg version of the degree theory, property 1.5.5 in page 8 of [42], we calculate that $\deg(G, A_a, \alpha) = \deg(\text{Identity}, A_a, \alpha) = 1$ for any interior point $\alpha \in A_a$. Consequently, $G$ is onto which implies that $\psi$ is also onto. This shows that there exists an $\alpha_a \in A_a$ such that $\psi(\alpha_a) = 0$ for any $a > 0$.

**Proof of Theorem 6.6.** First, we show that the solution $\bar{u}(r)$ of (6.21) with the initial value $\bar{u}(0) = \alpha_a$ never touches the wall for any $r$ and thus is defined for all $r > 0$. Suppose in the contrary that $\bar{u}_i(r) = 0$ for some $i$. Then by definition $\bar{u}(r_0) = \psi(\alpha_a) = 0$. This implies that $u(x) = \bar{u}(|x|)$ is a solution of (6.20) with $R = r_0$. This contradicts with the assumption that system (6.20) admits no solutions. Consequently, we get $\bar{u}_i(r) > 0$ for $i = 1, 2, \cdots, L$ and $r > 0$ and $\lim_{r \to \infty} \bar{u}(r) = \psi(\alpha_a) = 0$. Applying Lemma 6.7, we can see the first conclusion of Theorem 6.6. Clearly, $u(x) = \bar{u}(|x|)$ is an radially symmetric classical solution of (6.19) with $u(x) \to 0$ uniformly as $|x| \to \infty$.

**Proof of Theorem 1.1.** We define $w_i = (-\Delta)^{i-1}u$, $w_{k+i} = (-\Delta)^{i-1}v$, $i = 1, \cdots, k$. Then $w$ satisfies

$$
\begin{cases}
-\Delta w_1 = w_2, \cdots, -\Delta w_{k-1} = w_k, -\Delta w_k = w_{k+1}, \\
-\Delta w_{k+1} = w_{k+2}, \cdots, -\Delta w_{2k-1} = w_{2k}, -\Delta w_{2k} = w_{2k+1}, \\
w_i > 0 \text{ in } R^n, \quad i = 1, 2, \cdots, 2k,
\end{cases}
$$

(6.22)

with $F(w) = (w_2, \cdots, w_k, w_{k+1}, w_{k+1}, \cdots, w_{2k}, w_{2k}) : R_1^{2k} \to R_1^{2k}$ continuous. $F(w)$ is also Lipschitz for $w > 0$. It is easy to check that $F$ is non-degenerate and $F(\beta) \neq 0$ when $\beta \neq 0$. According to the proof of Theorem 6.4, (6.13) admits no radially symmetric solution in the critical and super-critical cases $\frac{1}{p+1} + \frac{1}{q+1} = \frac{n-2k}{n}$ for any $2k < n$. Thus, according to Theorem 6.6, system (6.22) admits a positive
radial solution $w_i(x) > 0$, $x \in \mathbb{R}^n$, $i = 1, \cdots, 2k$. Consequently, $u = w_1$ and $v = w_{k+1}$ solve system (1.2).

For the corresponding scalar case with $u = v$ and $p = q$, the existence of radial solutions of (1.6) follows exactly as the above proof. The nonexistence of the subcritical case is a corollary of the Liouville theorem in [54] and the equivalence result between PDEs and IEs in [9].

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