Jump Law of Co-State in Optimal Control for State-Dependent Switched Systems and Applications

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Abstract—This paper presents the jump law of co-states in optimal control for state-dependent switched systems. The number of switches and the switching modes are assumed to be known a priori. A proposed jump law is rigorously derived by theoretical analysis. An algorithm is then proposed to solve optimal control for state-dependent hybrid systems. Through numerical simulations, we further show that the proposed approach is more efficient than existing methods in solving optimal control for state-dependent switched systems.

I. INTRODUCTION

The optimal control of hybrid systems has been widely studied over the past few decades. In practice, many dynamical systems have hybrid characteristics, such as fermentation processes [1], aerospace systems [2], robots [3], [4], as well as social sciences [5] and natural systems [6]. Generally, hybrid systems are dynamical systems that exhibit interactions between continuous and discrete dynamics [7]. Switched systems are a particular class of hybrid dynamical systems composed of a family of continuous or discrete time subsystems and a law governing the transition between these subsystems. The studies of hybrid switched systems have focused on their stability analysis [8], the optimal control of their switching modes [9], [10], multi-mode switching [11], as well as on the special class of piece-wise affine systems [12] and algorithms for solving such systems [13]–[15], etc.

Switched systems may be classified into state-dependent and time-dependent switched systems. In a state-dependent switched system, the continuous state space is partitioned into a finite number of regions by several switching interfaces. In each region, the system has continuous dynamics. When the system trajectory hits a switching interface, the system dynamic “switches” and the system state either jumps (i.e., impulse effects) or continues to evolve in a continuous fashion (i.e., no state jumps) at the switching point even though the trajectory may no longer be differentiable at that point. In this paper, we will focus on switched systems without state jumps. The scenario when a state-dependent switched system experiences multiple switches (i.e., Zeno behavior) at the interface is out of the scope of this paper.

In this paper, we consider state-dependent switched systems whose state remains continuous at the switching interface and only crosses the switching interface once. Our aim is to minimize (or maximize) a continuous (or discontinuous) performance index with Bolza-type form for a fixed initial position, a given terminal time, and a-priori given switching interface.

The following is a compilation of previous work on the limited-scope problem. Reference [16] formulated a general Mayer-type optimization problem and studied its dynamics, optimal control, the notion of well-behaved solutions, and necessary conditions for optimality. The article also provided geometric intuition of the jump conditions of the co-state, namely, \( a\lambda(\tau^-) + b\lambda(\tau^+) + c\text{grad}(g(x(\tau))) = 0 \), where \( \lambda \) is the co-state, \( \text{grad}(g(x(\tau))) \) is the gradient of the switching interface at the switching state, \( a, b, c \) are constants, and \( \tau \) is the switching time instant. Reference [13] formulated a similar problem and employed classical variational and needle variation techniques to prove a set of necessary conditions for such systems, which showed the optimality condition related to the co-state satisfying \( \lambda(t_s^-) = \lambda(t_s^+) + \Delta_x m|_{t=t_s} \), where \( t_s \) is the switching time instant, \( m(x, t) = 0 \) is the switching interface, and \( p \) is a constant. The authors then proposed a gradient descent algorithm to solve the optimal control problem indirectly. Both papers provided a good qualitative analysis of the co-state, resulting in the optimality condition for the co-state. However, the full characterization of the jump law for the co-state (e.g., the exact expression at the switching interface) and its applications have never been thoroughly examined to the best of our knowledge. Therefore, in this work, we explore the jump law of the co-state based on the results in [13] and show how we can use such information to efficiently solve optimal control for hybrid systems. Our contributions include the following:

- We present jump laws of the co-state for the optimal control of general state-dependent switched systems, in both the case of time-invariant switching interface and time-varying switching interface. We also provide rigorous theoretical analysis to prove the correctness of the jump laws.
- We then propose a new algorithm for solving optimal control of hybrid systems to illustrate the applicability of our theory.
- Finally, we compare the efficiency of our approach with existing algorithms.

This paper is organized as follows: In Section II, we formulate the problem and propose the jump law of the co-state when the switching interface is time-invariant. We then extend our approach to the scenario of the time-varying switching interface in Section III. Next, we show how the...
jump laws of co-state can actually be used to solve the
optimal control problem of state-dependent switching sys-
tems more efficiently by comparing with existing algorithm
numerically in Section IV. Finally, we conclude our article
in Section V and envision our future work in this direction.

II. PROBLEM DESCRIPTION

We consider an optimal control problem exhibiting dis-
continuities in the system dynamics and in its performance
index. Specifically, let
\[
\dot{x} = f(x,u,t),
\]
where \(x \in \mathbb{R}^n, u \in \mathbb{R}^m, g(x) = 0\) is the switching interface,
the initial condition is \(x(t_0) = x_0, g(x_0) < 0\), and the desired
final condition is \(x(t_f) = x_f, g(x_f) > 0\) at a fixed final time
\(t_f\). Our objective is to minimize the performance index:
\[
J = \phi(x(t_f)) + \int_{t_0}^{t_f} L(x(t),u(t),t)\,dt,
\]
where
\[
L(x(t),u(t),t) = \begin{cases} 
L_1(x(u),t), & \text{if } g(x) < 0 \\
L_2(x(u),t), & \text{if } g(x) > 0
\end{cases}
\]
and \(\phi(x(t_f))\) is the terminal cost. Note that \(J\) is a function
of the switching time instant \(\tau\). Given a switching time instant
\(\tau\), this optimal control problem may be regarded as a two-
phase fixed final time optimal control problem.

Remark 1: In this paper, the switching interface is
assumed to be differentiable. The gradient of a scalar function,
g(x), is defined as a column vector \(m(x) = \frac{\partial g(x)}{\partial x}\).

Remark 2: We make the physical restriction that the state
variables must be continuous at the interface, i.e., \(x(\tau-) = \tau = x(\tau)\). There are no jumps in the system itself, and
therefore no impulsive causes.

Remark 3: The theory can be extended to systems with
multiple switching interfaces.

We define the Hamiltonian of the system as usual:
\(H(x,\lambda,u,t) = L(x,u,t) + \lambda^\top f(x,u,t)\). For a time-invariant
system, the Hamiltonian is known to be constant [17]. In the
following, we will prove that the Hamiltonian is continuous
with respect to time at the switching interface for two-modes
systems. The derivation uses standard variational principles
and follows [11].

Theorem 1: The Hamiltonian \(H\) for the two-modes
problem (1)-(2) is continuous as a function of time when the state
crosses the interface.

Proof: Consider the following cost function:
\[
J(\tau) = \phi(x(\tau)) + \int_{t_0}^{t_f} L(x(t),u(t),t)\,dt
\]
\[
= \phi(x(\tau)) + \int_{t_0}^{t_f} L_1(x(t),u(t),t)\,dt + \int_{t_0}^{t_f} L_2(x(t),u(t),t)\,dt
\]
\[
= \phi(x(\tau)) + \int_{t_0}^{t_f} L(x(t),u(t),t)\,dt
\]

The variation in \(J\) due to variations in the control \(u(t)\) and
the cross-over time \(\tau\) is
\[
\delta J = \left[ \frac{\partial \phi}{\partial x} \delta x - \lambda^\top \delta x_2 \right]_{t_f}^{t_0} + \left[ \lambda_2 - \lambda_1 + \nu \frac{\partial g}{\partial x} \right] \delta x + (H_1(x_1(\tau),u_1(\tau),\tau) - H_2(x_2(\tau),u_2(\tau),\tau)) \delta \tau
\]
\[
+ \int_{t_0}^{t_f} \left[ \frac{\partial H_1(x_1,u_1,t)}{\partial x_1} + \lambda_1^\top \frac{\partial H_1}{\partial u_1} \right] \delta x_1 + \frac{\partial H_2}{\partial u_2} \delta u_2 \right] \,dt
\]
\[
\quad + \int_{t_f}^{t_0} \left[ \frac{\partial H_2(x_2,u_2,t)}{\partial x_2} + \lambda_2^\top \frac{\partial H_2}{\partial u_2} \right] \delta x_2 + \frac{\partial H_1}{\partial u_1} \delta u_1 \right] \,dt
\]
Note that \(\delta x_1,2(\tau+\delta \tau) = \delta x_1,2(\tau) + \dot{x}_1,2\delta \tau\). Since the initial
state is fixed, \(\delta x(t_0) = 0\). The remaining parts of (4) hold...
for arbitrary $\lambda$. Choosing $\lambda$ such that

$$
\lambda_1(t) = -\left(\frac{\partial H_1(x_1, \lambda_1, u_1, t)}{\partial x_1}\right)^T, \quad t_0 < t < \tau
$$

$$
\lambda_2(t) = -\left(\frac{\partial H_2(x_2, \lambda_2, u_2, t)}{\partial x_2}\right)^T, \quad \tau < t < t_f
$$

(16)

Suppose the initial state $x_0$, the initial time $t_0$, and the final time $t_f$ are given.

Adding (8) to (9) leads to

obtain

$$
\Delta \lambda^T \langle f \rangle \left(\frac{\partial g(x)}{\partial x}|_{x^o}\right) = -\left(\Delta L + (\lambda^T)\Delta f \right) \left(\frac{\partial g(x)}{\partial x}|_{x^o}\right).
$$

(11)

The optimality condition derived in (5) shows that under the optimal control, $\Delta \lambda \in (T_x g(x))^\perp$ (i.e., $\Delta \lambda$ is parallel to the gradient of the interface).

**Proposition 1:**

$$
\Delta \lambda^T \langle f \rangle \left(\frac{\partial g(x)}{\partial x}|_{x^o}\right) = \langle \frac{\partial g(x)}{\partial x}|_{x^o}\rangle \langle f \rangle \Delta \lambda.
$$

(12)

**Proof:** Denote $m = \langle \frac{\partial g(x)}{\partial x}|_{x^o}\rangle$. First, we show that

$$
\Delta \lambda^T \langle f \rangle m = m^T \langle f \rangle \Delta \lambda.
$$

Since $m$ is parallel to $\Delta \lambda$, we can write them respectively as $m = t_1 v$, $\Delta \lambda = t_2 v$, where $v$ is a direction vector, $t_1$, $t_2$ are scalars. Then $\Delta \lambda^T \langle f \rangle m = t_1 t_2^\top \langle f \rangle t_2 v = t_1 t_2 v^\top \langle f \rangle v$.

This change of order of vector multiplication in (12) allows us to separate $\Delta \lambda$ because $m^T \langle f \rangle$ is a scalar. Thus, substituting the right-hand side of (12) into the left-hand side of (11), we have the following equation:

$$
\left(\frac{\partial g(x)}{\partial x}|_{x^o}\right)^T \langle f \rangle \Delta \lambda = -\left(\Delta L + (\lambda^T)\Delta f \right) \left(\frac{\partial g(x)}{\partial x}|_{x^o}\right),
$$

which gives the relation

$$
\Delta \lambda = -\frac{(\Delta L + (\lambda^T)\Delta f) m}{m^T \langle f \rangle}.
$$

(13)

This relation specifies the quantitative behavior of the co-states at the interface. It has also been obtained directly from the singular (generalized function) interpretation of the Euler-Lagrange equation proposed in [18]–[20].

### III. TIME-VARYING SWITCHING INTERFACE

In this section, we present the jump law of co-state $\Delta \lambda$ under the time-varying (TV) switching interface.

**A. Problem formulation**

Consider the following general switched system with a time-varying switching interface $g(x, t) = 0$:

$$
\dot{x} = \begin{cases} 
    f_1(x, u, t), & \text{if } g(x, t) < 0 \\
    f_2(x, u, t), & \text{if } g(x, t) > 0
\end{cases}
$$

(15)

and stage cost

$$
L = \begin{cases} 
    L_1(x, u, t), & \text{if } g(x, t) < 0 \\
    L_2(x, u, t), & \text{if } g(x, t) > 0
\end{cases}
$$

(16)

Suppose the initial state $x_0$, the initial time $t_0$, and the final time $t_f$ are given.
B. Theoretical analysis

We change this system into a time-invariant (TIV) system by conducting a variable augmentation. Define the new augmented variable $\tilde{x} = \begin{bmatrix} x \\ t \end{bmatrix}$. Then the original system (15) is turned into the following time-invariant system:

$$\dot{\tilde{x}} = \begin{bmatrix} f(\tilde{x}, u) \\ 1 \end{bmatrix} = F(\tilde{x}, u).$$

(17)

The stage cost is $\bar{L}(x, u, t) = L(\tilde{x}, u)$. The new Hamiltonian is then $H(\tilde{x}, u, \lambda) = \bar{L}(\tilde{x}, u) + \lambda^\top F(\tilde{x}, u)$. In this case, the optimal control condition gives us

$$\frac{\partial H}{\partial u} = 0 \Rightarrow \frac{\partial \bar{L}}{\partial u} + \lambda^\top \frac{\partial F}{\partial u} = 0$$

$$\Rightarrow \frac{\partial L}{\partial u} + [\lambda^\top, \lambda] \left[ \frac{\partial f}{\partial u} \right] = \frac{\partial L}{\partial u} + \lambda^\top \frac{\partial f}{\partial u} = 0.$$

The Euler-Lagrangian equation gives us

$$\dot{\lambda} = \left[ \frac{\partial H}{\partial x} \right]^\top, \quad \lambda_t = \left( \frac{\partial L}{\partial t} + \lambda^\top \frac{\partial f}{\partial t} \right).$$

(18)

That is

$$\dot{\lambda} = -\left( \frac{\partial H}{\partial x} \right)^\top, \quad \lambda_t = -\left( \frac{\partial L}{\partial t} + \lambda^\top \frac{\partial f}{\partial t} \right).$$

With this augmented scheme and using the jump law in (14), we obtain the following law:

$$\Delta \lambda = -\frac{(\Delta L + \langle \lambda \rangle \Delta f)m}{m^\top f + \mu},$$

(19)

where the notation $\langle \cdot \rangle$ means the average, $\mu = \frac{\partial g(x, t, u)}{\partial x} |_{x_\ast}$, $m = \frac{\partial g(x, t, u)}{\partial x} |_{x_\ast}$.

Examples to verify our proposed jump law of (14) and (19) can be found in [21].

IV. APPLICATION SCENARIO

In this section, we demonstrate the effectiveness of the two jump laws of co-state (i.e., (14) and (19)) in solving optimal control problems for state-dependent switching systems. Combined with the Matlab toolbox “bvp4c”, we can solve such problems efficiently and precisely.

A. Proposed algorithm (GEL)

Suppose we have $x \in \mathbb{R}^n$. The TIV switching interface is $g(x) = 0$ and the switching time is $\tau$. To avoid confusion, for the following sections, we claim that $f_-(x, u, t)$ and $f_+(x, u, t)$ represent the system dynamics before switching and after switching respectively. Similarly, $L_-(x, u, t)$ and $L_+(x, u, t)$ represent the stage cost before switching and after switching respectively. The initial state is $x_0$, the final time is $t_f$, the final state is $x_f$. Then we have the following set of constraints for the TIV switching case:

$$\dot{x}_-(t) = f_-(x, u), \quad \dot{x}_+(t) = -\frac{\partial H_-}{\partial x} - \frac{\partial H_-}{\partial u} = 0 \quad (20)$$

and some boundary conditions $x_+(\tau) = x_-(\tau), x_-(t_0) = x_0, x_+(t_f) = x_f, g(x_+(\tau)) = 0$. In the above problem, $x_+(t), x_-(t), \lambda_+(t), \lambda_-(t), u_+(t), u_-(t)$, $\tau$ are unknown. So there are $6n+1$ unknowns and $6n+1$ constraints. Eliminating $u$, if we know the jump law equation (14) or (19), we can solve the original problem by solving $4n$ differential equations (related to $x$ and $\lambda, x \in \mathbb{R}^n, \lambda \in \mathbb{R}^m$) and one unknown $\tau$ with $4n + 1$ boundary constraints.

B. Examples

We propose an iterative algorithm (called GEL) to solve the optimal control of such systems efficiently leveraging the (co-state) jump laws in (14) and (19). The main idea is to construct the following function:

$$F(\tau) = \lambda_+ - \lambda_- - G(\lambda_+, \lambda_-, \tau, x_-, x_+),$$

(22)

where $G(\lambda_+, \lambda_-, \tau, x_-, x_+)$ is the right-hand side of (14) or (19). By using MATLAB “bvp4c”, we can solve the optimal control problem of this switching system as two boundary value problems with multiple boundary conditions. Then we find $\tau$ by Newton’s Method (a root-founding process), which is

$$\tau_{k+1} = \tau_k - \alpha \frac{F(\tau_k)}{F'(\tau_k)},$$

(23)

where $\alpha$ is the step size. The stopping criterion is then $|\tau_{k+1} - \tau_k| < tol$ and $|F(\tau_k) - 0| < tol$. For this derivative $\frac{dF}{d\tau} = \frac{F(x + \delta x) - F(x)}{\delta \tau}$, a perturbation scheme is used to obtain $F'(\tau + \delta \tau)$. We summarize the proposed approach in Algorithm 1.

**Algorithm 1** Proposed algorithm (named GEL) for solving hybrid system optimal control problem numerically

```plaintext
Initialize $\tau_0 = 0.5$, tolerance $tol = 0.0001$, $\delta \tau = 0.1$
for $k = 0$ : iter do
    Solve two-point boundary value problem (20), (21), using bvp4c
    Calculate $F(\tau_k)$
    if $|F(\tau_k) - 0| < tol$ then
        break
    else
        Add a small perturbation $\tau'_k = \tau_k + \delta \tau$ and calculate $F(\tau'_k)$
        Calculate $F'(\tau_k) = \frac{F(\tau'_k) - F(\tau_k)}{\delta \tau}$
        Newton Update: $\tau_{k+1} = \tau_k - \frac{F(\tau_k)}{F'(\tau_k)}$
        if $|\tau_{k+1} - \tau_k| < tol$ then
            break
        end if
    end if
end for
```

As the authors know, there is no direct way to solve this problem. Thus we use the ICLOCS2 [22] to brute force search the switching time $\tau$ and switching states, and regard the hybrid system into two systems. In this
way, each sub-problem is an optimal control for a continuous system with some known boundary conditions. This indirect use of ICLOCS2 toolbox is very time-consuming. Think of a second-order system with a switching interface \( x_1 + x_2 = 0 \). We need to use two for-loops to search the switching time \( \tau \) and switching state \( x_1(\tau) \). Hence, we only use it as a benchmark. In [13], the authors proposed an efficient gradient-descent-based algorithm named HMPMAS for solving hybrid switching systems, which is proven much more efficient than the algorithm in [14]. We compare our algorithm with the algorithm in [13] on two different systems (both are from the paper [13]). More examples can be found in [21]. Given a problem, we compare the time required by different algorithms to generate a solution with the same level of numerical precision. All the computations are performed using MATLAB 2020b on a personal computer with i7-8665U CPU and 16GB RAM. Each experiment is run 5 times and the average CPU time is reported for responsible comparison. The switching time instant \( \tau_{\text{ICLOCS2}} \) is obtained from the toolbox ICLOCS2. Formally, we use the absolute error of switching time instant \( \tau \) obtained by compared algorithms and the benchmark ICLOCS2 as a measure, which is given by

\[
\text{Error} = |\tau - \tau_{\text{ICLOCS2}}|.
\]  

1) Example 1:

\[
\begin{align*}
\text{s1:} & \quad \dot{x} = x + u \tau \\
\text{s2:} & \quad \dot{x} = -x + xu,
\end{align*}
\]

with cost function \( J = \int_0^T \frac{1}{2} u^2 dt \). \( t_0 = 0, t_f = 2, x(t_0) = 1, x(t_f) = 1 \), and a time-varying switching interface \( m(x,t) = x - \varepsilon t = 0 \). The stopping condition is \( \text{tol} = 0.0001 \). The initial values for HMPMAS algorithm are set as \( \tau_0 = 0.5, x_0 = 0.1 \) and step size \( r_k = 0.04k+1 \). Table I is the result. In the following tables, “-” means a very large number; “n/a” means unavailable. Only 5 iterations are needed to reach convergence using the proposed algorithm. We also find that the HMPMAS algorithm is very fragile to the step size \( r_k \). One has to play with the step size to find the best one to make this algorithm converge. Fig. 1 is the trajectory of \( x \) using proposed algorithm and HMPMAS. Fig. 2 is the absolute error curve of \( \tau \) using proposed algorithm and HMPMAS.

2) Example 2: We then consider a second-order system:

\[
\begin{align*}
\text{s1:} & \quad \dot{x} = \begin{bmatrix} 1.5 & 0 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u, \\
\text{s2:} & \quad \dot{x} = \begin{bmatrix} 0.5 & 0.866 \\ 0.866 & -0.5 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u,
\end{align*}
\]

The initial time, final time, and initial state are \( t_0 = 0, t_f = 2, x(t_0) = [1, 1]^T \) respectively. In this example, the terminal state is not fixed. The initial value is set as \( \tau_0 = 1.5, x_0 = [4.5, 2.5]^T \) (same as [13]), \( \text{tol} = 0.0001 \), \( r_k = 0.02k+1 \), \( \alpha = 1 \). Table II shows the result. It shows that the proposed algorithm needs less iteration and time to find an optimal value. Moreover, the proposed algorithm has higher precision compared to that in [13].

### Table I

**Algorithm Performance Comparison: Example 1**

| Methods   | \( J \)          | Time (s) | Iteration |
|-----------|------------------|----------|-----------|
| ICLOCS2   | 1.0000           | 0 hours  | -         |
| HMPMAS    | 0.9678           | 0.0032   | 57.62577  |
| Proposed (GEL) | 1.0004     | 8.1008e-04 | 45.2587  |

### Table II

**Algorithm Performance Comparison: Example 2**

| Methods   | \( \tau \)          | \( J \)          | Time (s) | Iteration |
|-----------|---------------------|------------------|----------|-----------|
| ICLOCS2   | (4.5556, 2.4444)   | 0.1130           | hours    | -         |
| HMPMAS    | (4.5456, 2.4326)   | 0.0001           | 83.1040  | 5         |
| Proposed (GEL) | (4.5562, 2.4438) | 0.1130           | 21.3526  | 5         |

In this example, one finds that despite keeping the initialization of the state \( x_0 \) on the switching interface and \( \tau \) starting from different initial values, the convergence to the switching interface is not guaranteed for the HMPMAS algorithm. However, for the proposed algorithm, we only need to search for one variable \( \tau \); thus, it converges faster.
Fig. 3 shows the optimal trajectory of $x$. Fig. 4 is the iterative curve of absolute error of switching time $\tau$ as defined in (24).

![Graph showing optimal trajectory and absolute error curve]

**V. CONCLUSION**

In this paper, we derived the jump law of co-state in optimal control for state-dependent switching systems, based on which we developed an efficient algorithm for solving optimal control problems with hybrid systems. Future work will be extended to the following aspects:

- Explore the same optimal control problem with a non-differentiable switching interface.
- Develop an efficient algorithm to solve the optimal control for state-dependent switched time-delayed systems, as formulated in [23].

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