Stochastic analysis of surface roughness

M. Waechter\textsuperscript{1,2}*, F. Riess\textsuperscript{1} H. Kantz\textsuperscript{2} J. Peinke\textsuperscript{1}\dagger

\textsuperscript{1} Carl von Ossietzky University, Physics Department - D-26111 Oldenburg, Germany
\textsuperscript{2} Max Planck Institute for Physics of Complex Systems - D-01187 Dresden, Germany

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Abstract

For the characterization of surface height profiles we present a new stochastic approach which is based on the theory of Markov processes. With this analysis we achieve a characterization of the complexity of the surface roughness by means of a Fokker-Planck or Langevin equation, providing the complete stochastic information of multiscale joint probabilities. The method was applied to different road surface profiles which were measured with high resolution. Evidence of Markov properties is shown. Estimations for the parameters of the Fokker-Planck equation are based on pure, parameter free data analysis.

1 Introduction

The complexity of rough surfaces is subject of a large variety of investigations in different fields of science \textsuperscript{[1,2,3,4,5-8,10-12]}. Physical and chemical properties of surfaces and interfaces are to a significant degree determined by their topographic structure. As one example, the influence of surface roughness on boundary layer flows is discussed in turbulence research (cf. \textsuperscript{[5,6]}) and in the atmospheric (cf. \textsuperscript{[7,8]}) and oceanographic sciences (cf. \textsuperscript{[9]}). A comprehensive characterization of the topography is of vital interest for deposited, polished or otherwise processed surfaces. Therefore, to give a second example, current roughness analysis methods find practical application for the characterization of polished surfaces \textsuperscript{[10-12]}. In the context of industrial and engineering applications common roughness measures are based on standardized procedures as the maximum height difference $R_z$ and the mean value of absolute heights $R_a$ \textsuperscript{[13]}. These rather simple measures clearly cannot completely characterize the complexity of roughness. This is also confirmed by the existence of a large amount of additional measures, each describing a very special feature of a surface.

In the physical and other sciences a common local measure of roughness is the rms surface width $w_r(x) = \langle (h(x) - \bar{h})^2 \rangle_r^{1/2}$, where $h(x)$ is the measured

\*E-mail: matthias.waechter@uni-oldenburg.de
\daggerE-mail: peinke@uni-oldenburg.de
height at point $\tilde{x}$, $\langle \cdot \rangle_r$ denotes the average over an interval of length $r$ around the selected point $x$, and $\bar{h}$ the mean value of $h(\tilde{x})$ in that interval. Advantages of $w_r(x)$ are a scale dependent definition as well as clear physical and stochastical meanings. Among the techniques used to characterize scale dependent surface roughness probably the most prominent ones are the concepts of self-affinity and multi-affinity, where the multi-affine $f(\alpha)$ spectrum has been regarded as the most complete characterization of a surface \[14,15,2,3\]. From a stochastic point of view we want to point out two important properties of multi-affinity: (a) the ensemble average $\langle w_r \rangle$ must obey a scaling law $\langle w_r^\alpha \rangle \sim r^{\xi_\alpha}$, and (b) the statistics of $w_r(x)$ are investigated on distinct length scales, thus possible correlations between $w_r(x)$ and $w_{r'}(x)$ on different scales $r, r'$ are not examined.

The method we are proposing in this contribution is based on stochastic processes which should grasp the scale dependency of surface roughness in a most general way. No scaling feature is explicitly required, and especially the correlations between different scales $r$ and $r'$ are investigated. To this end we present a systematic procedure how the explicit form of a stochastic process for the $r$-evolution of a roughness measure similar to $w_r(x)$ can be extracted directly from the measured surface topography. This stochastic approach has turned out to be a promising tool also for other systems with scale dependent complexity like turbulence \[10,17\] and financial data \[18\]. In this letter we focus on the complexity of rough surfaces. As a specialized example we have picked out the applied problem of characterizing road surfaces. These are an essential component of current transportation and thus represent a class of non-idealized and widely used surfaces. We claim that the applicability of our method to this class of surfaces indicates its general relevance for improved surface characterization.

A collection of road surface data was measured which served previously as an empirical basis for the prediction of vibrational stress on bicycle riders \[19\]. It is common to describe the quality of a road surface by a power law fit to the power spectrum of the height profile \[20,21,22\]. This method is not appropriate especially for wavelengths below 0.3 m and for non-Gaussian height distributions. Furthermore, the power spectrum characterizes only the $r$-dependence of one moment of the two-point correlations \[23\]. While some improvements to this method have been proposed \[24\] the characterization of road surfaces still remains incomplete \[25,26\]. For the improvement of road surface characterization multifractality and multi-affinity seem not to be appropriate tools because scaling is no constant feature of road surfaces.

In the remainder of this letter we first introduce our method, the determination of a Fokker-Planck equation for the evolution of conditional probability density functions (pdf) directly from experimental data. Next, we present a typical data set, show evidence of its Markov properties, and estimate the coefficients of the Fokker-Planck equation. At last, we evaluate the precision of the estimated coefficients by numerical reconstruction of conditional and unconditional pdfs.
2 Method

It is one common procedure to characterize the complexity of a rough surface by the statistics of the height increment [27]

\[ h_r(x) := h(x + r/2) - h(x - r/2) \]  

(1)
depending on the length scale \( r \), as marked in fig. 1. Other scale dependent roughness measures can, for example, be found in [15, 3]. Here we use the height increment \( h_r \) because it is also directly linked to vehicle vibrations induced by the road surface if \( r \) is the wheelbase. Another argument for the use of \( h_r \) is that its moments are connected with spatial correlation functions, but it should be pointed out that our method presented in the following could be easily generalized to any scale dependent measure, e.g. the above-mentioned \( w_r(x) \). As a new ansatz, \( h_r \) will be regarded here as a stochastic variable in \( r \). Without loss of generality we consider the process as being directed from larger to smaller scales. Our interest is the investigation how surface roughness is linked between different length scales.

Complete information about the stochastic process would be available by the knowledge of all possible \( n \)-point, or more precisely \( n \)-scale, probability density functions (pdf) \( p(h_1, r_1; h_2, r_2; \ldots; h_n, r_n) \) describing the probability of finding simultaneously the increments \( h_1 \) on the scale \( r_1 \), \( h_2 \) on the scale \( r_2 \), and so forth up to \( h_n \) on the scale \( r_n \). Here we use the notation \( h_i(x) = h_{r_i}(x) \), see [1]. Without loss of generality we take \( r_1 < r_2 < \ldots < r_n \). As a first question one has to ask for a suitable simplification. In any case the \( n \)-scale joint pdf can be expressed by multiconditional pdf

\[
p(h_1, r_1; \ldots; h_n, r_n) = p(h_1, r_1 | h_2, r_2; \ldots; h_n, r_n) \cdot p(h_2, r_2 | h_3, r_3; \ldots; h_n, r_n) \\
\ldots \cdot p(h_{n-1}, r_{n-1} | h_n, r_n) \cdot p(h_n, r_n) 
\]  

(2)

where \( p(h_i, r_i | h_j, r_j) \) denotes a conditional probability, which is defined as the probability of finding the increment \( h_i \) on the scale \( r_i \) under the condition that simultaneously, i.e. at the same location \( x \), on a larger scale \( r_j \) the value \( h_j \) was found. An important simplification arises if

\[
p(h_i, r_i | h_{i+1}, r_{i+1}; \ldots; h_n, r_n) = p(h_i, r_i | h_{i+1}, r_{i+1}) .
\]  

(3)

This property is the defining feature of a Markov process evolving from \( r_{i+1} \) to \( r_i \). Thus for a Markov process the \( n \)-scale joint pdf factorize into \( n \) conditional pdf

\[
p(h_1, r_1; \ldots; h_n, r_n) = p(h_1, r_1 | h_2, r_2) \cdot \ldots \cdot p(h_{n-1}, r_{n-1} | h_n, r_n) \cdot p(h_n, r_n) .
\]  

(4)

This Markov property implies that the \( r \)-dependence of \( h_r \) can be regarded as a stochastic process evolving in \( r \), driven by deterministic and random forces. If additionally the included noise is Gaussian distributed, the process can be described by a Fokker-Planck equation [28]. For our height profiles it takes the
form

\[-r \frac{\partial}{\partial r} p(h_r, r|h_0, r_0) = \left\{ -\frac{\partial}{\partial h_r} D^{(1)}(h_r, r) + \frac{\partial^2}{\partial h_r^2} D^{(2)}(h_r, r) \right\} p(h_r, r|h_0, r_0) \,.
\]  

The Fokker-Planck equation then describes the evolution of the conditional probability density function from larger to smaller length scales and thus also the complete n-scale statistics. The minus sign on the left side of eq. (5) expresses this direction of the process, furthermore the factor \( r \) corresponds to a logarithmic variable \( \rho = \ln r \) which leads to simplified results in the case of scaling behaviour [29].

The term \( D^{(1)}(h_r, r) \) is commonly denoted as drift term, describing the deterministic part of the process, and \( D^{(2)}(h_r, r) \) as diffusion term being the variance of a Gaussian, \( \delta \)-correlated noise. Focus of our analysis is to show evidence of the above mentioned Markov property and to derive the drift and diffusion coefficients \( D^{(1)} \) and \( D^{(2)} \) in eq. (5) from experimental data.

3 Data

![Figure 1: Cut-out from a height profile of an irregular cobblestone road. Additionally the construction of the height increment \( h_r(x_0) = h(x_0 + r/2) - h(x_0 - r/2) \) is illustrated.](image1)

![Figure 2: Power spectral density of the data set. The arrow indicates beginning and direction of the range where drift and diffusion coefficient could be obtained and verified.](image2)

Height profiles were measured from numerous road and cycle track surfaces typical for West German bicycle traffic. Road sections were selected in such a way that stationarity is given. Profile length is typically 20 m or 19 000 samples, respectively. Longitudinal resolution was 1.04 mm and vertical error smaller than 0.5 mm. As an example we present results from a data set of an irregular cobblestone road consisting of ten profiles with a total of about 190 000 samples [30]. Figure 1 shows a short section of the data. In fig. 2 the power spectral density of the height profiles is plotted against the wavenumber. Scaling behaviour is not found at the beginning of the analysed region of length scales,
while for smaller scales \((20 < k/m^{-1} < 100)\) it appears to be present. In the following, the height increments are normalized by \(\sigma_\infty := \lim_{r \to \infty} \langle h_r^2 \rangle^{1/2}\), with \(\sigma_\infty = 6.3 \text{mm}\) for the given data set.

### 4 Markov Properties

For a Markov process the defining feature is that the \(n\)-point conditional probability distributions are equal to the single conditional probabilities, according to eq. (3). With the given amount of data points the verification of this condition is only possible for \(n = 3\) and for \(r_1 < r_2 < r_3 < 300 \text{mm}\).

![Figure 3](image)

Figure 3: a) Contour plot of single and double conditional probabilities \(p(h_1, r_1|h_2, r_2)\) (dashed lines) and \(p(h_1, r_1|h_2, r_2; h_3 = 0, r_3)\) (solid lines) for \(r_1 = 126; r_2 = 188; r_3 = 251 \text{mm}\). Contour levels differ by a factor of 10, with an additional level at \(p = 0.3\). b), c) Two one-dimensional cuts at \(h_2 \approx \pm \sigma_\infty\) with \(p(h_1, r_1|h_2, r_2)\) as dashed lines and \(p(h_1, r_1|h_2, r_2; h_3, r_3)\) as circles.

Figure 3 shows both sides of eq. (4) with \(n = 3\) in a contour plot and two cuts at \(h_2 \approx \pm \sigma_\infty\). The value of \(h_3\) was chosen to be \(h_3 = 0\). We take this rather good correspondence as a strong hint for a Markov process. Markov properties were found for scale distances from about 17 mm upwards. Note that the main axis of the distribution is tilted, indicating that \(p(h_1, r_1|h_2, r_2) \neq p(h_1, r_1)\) and thus height increments on different scales are not independent.

### 5 Estimation of Drift and Diffusion Coefficients

In order to obtain the drift \((D^{(1)})\) and diffusion coefficient \((D^{(2)})\) for eq. (5) we proceed in a well defined way like it was already expressed by Kolmogorov [31], see also [28, 16]. First, the conditional moments \(M^{(k)}(h_r, r, \Delta r)\) for finite step sizes \(\Delta r\) are directly estimated from the data via moments of the conditional probabilities

\[
M^{(k)}(h_r, r, \Delta r) = \frac{r}{k! \Delta r} \int_{-\infty}^{+\infty} (\bar{h} - h_r)^k p(\bar{h}, r - \Delta r|h_r, r) \, d\bar{h}.
\]

(6)
Second, the coefficients \( D^{(k)}(h_r, r) \) are obtained from the limit of \( M^{(k)}(h_r, r, \Delta r) \) when \( \Delta r \) approaches zero

\[
D^{(k)}(h_r, r) = \lim_{\Delta r \to 0} M^{(k)}(h_r, r, \Delta r) .
\]  

Figure 4 shows estimations of the drift coefficient \( D^{(1)}(h_r, r) \) and the diffusion coefficient \( D^{(2)}(h_r, r) \) for \( r = 188 \text{ mm} \). The error bars are estimated from the errors of \( M^{(1)} \) and \( M^{(2)} \) via the number of events contributing to each value. The limit \( \Delta r \to 0 \) was performed in both cases by a linear fit to \( M^{(k)}(h_r, r, \Delta r) \) in the range \( 17 \text{ mm} \leq \Delta r \leq 29 \text{ mm} \). Both coefficients were parameterized as piecewise linear functions where the behaviour within \( |h_r| \leq 2.5\sigma_\infty \) could be derived directly from the above estimations. Outside this range increasing errors make a precise estimate more difficult. Parameterizations were chosen here to additionally obtain good results in the verification procedures (see below). Figure 4 shows that the resulting parameterizations are in good agreement with the estimations. It is easy to verify that with linear \( D^{(1)} \) and constant \( D^{(2)} \) the Fokker-Planck equation \ref{eq:fp} describes a Gaussian process, while with a parabolic \( D^{(2)} \) the distributions become non-Gaussian, also called intermittent or heavy tailed.

In our case, here, \( D^{(1)}(h_r, r) \) is characterized by the slope \(-\gamma(r)\) within \( |h_r| \leq 2.5\sigma_\infty, -4\gamma(r) \) for \( h_r < -2.5\sigma_\infty \), and \(-2\gamma(r) \) for \( h_r > 2.5\sigma_\infty \) (compare fig. 4). The dependence of \( \gamma \) on \( r \) is nontrivial with the value ranging from 0.82 for \( r = 83 \text{ mm} \) to 1.9 for \( r = 188 \text{ mm} \). \( D^{(2)}(h_r, r) \) was found to have a value \( \beta(r) \) independent of \( h_r \) within \( |h_r| \leq 2.5\sigma_\infty \). For \( h_r < -2.5\sigma_\infty \) \( D^{(2)}(h_r, r) \) was parameterized to be linear with slope \(-4\beta(r)\), for \( h_r > 2.5\sigma_\infty \) with slope \( 3\beta(r) \). The dependence of \( \beta \) on \( r \) can be approximated by \( \beta(r) = 0.0117 r/\text{mm} \).

6 Verification of Coefficients

Next, we want to evaluate the precision of our result. Therefore we return to eq. \ref{eq:fp}. Knowing \( D^{(1)} \) and \( D^{(2)} \) it should be possible to calculate the pdf of \( h_r \),
with the corresponding Fokker-Planck equation. Equation (5) can be integrated over $h_0$ and is then valid also for the unconditional pdf. Now the empirical pdf at $r_0 = 188$ mm is parameterized (see fig. 5) and used as initial condition for a numerical solution of the integrated form of eq. (5). For several values of $r$ the reconstructed pdf is compared to the respective empirical pdf, as shown in fig. 5. Please note that the interchange of steeper and flatter regions in the reconstructed pdf is achieved by the piecewise linear parameterization of $D^{(1)}$ and $D^{(2)}$.

![Figure 5: Numerical solution of the integrated form of Fokker-Planck equation (symbols) at different scales $r$. Solid line: empirical pdf parameterized at $r = 188$ mm, dashed lines: reconstructed pdf. Scales $r$ are 188, 158, 112, 79, 46 mm from top to bottom. Pdf are shifted in vertical direction for clarity of presentation.](image)

A second verification is the reconstruction of conditional pdf by direct numerical solution of the Fokker-Planck equation (5). An example for the scales $r_0 = 188$ mm and $r_1 = 131$ mm is shown in fig. 6. Reconstructing the conditional pdf this way is much more sensitive to deviations in $D^{(1)}$ and $D^{(2)}$. This becomes evident by the fact that the conditional pdf (and not the unconditional pdf of fig. 5) determine $D^{(1)}$ and $D^{(2)}$ (see (6) and (7)). Here also the difference to the multiscaling analysis becomes clear, which analyses higher moments $\langle h^q r \rangle = \frac{\int h^q \cdot p(h_r) \, dh_r}{\langle h^0_r \rangle}$ of $h_r$, and does not depend on conditional pdf. It is easy to show that there are many different stochastic processes which lead to the same single scale pdf $p(h_r)$.}

![Figure 6: Direct numerical solution of Fokker-Planck equation (symbols) compared to the empirical pdf at scales $r_0 = 188$, $r_1 = 131$ mm. a) Contour plot of empirical (solid lines) and reconstructed pdf (dashed lines). Contour levels are as in fig. 3 b,c) Cuts at $h_0 \approx \pm \sigma_{\infty}$. Empirical pdf are plotted as symbols.](image)
7 Conclusions

The height increment $h_r$ of surface height profiles as a stochastic variable in $r$ can be correctly described by a Fokker-Planck equation with drift and diffusion coefficients derived directly from measured data. The results of the presented example support the hypothesis that the noise term in the evolution of the stochastic variable $h_r$ in $r$ is sufficiently well described by a Gaussian, $\delta$-correlated random process.

As the Fokker-Planck equation describes the evolution of $p(h_r, r | h_0, r_0)$ and $p(h_r, r)$ with $r$, it covers also the behaviour of the moments $\langle h^n_r \rangle$ including any possible scaling behaviour. From the integrated form of eq. (5) an equation for the moments can be obtained by multiplying with $h^n_r$ and integrating over $h_r$. For $D^{(1)}$ being purely linear in $h_r$ and $D^{(2)}$ purely quadratic, multifractal scaling is obtained. We note again that, compared to scaling features, the knowledge of the Fokker-Planck equation provides more information on the complexity of the surface roughness in the sense of multi-scale joint probability density functions, eq. (2), which correspond to multipoint statistics. While to this end we do not seem to find universal laws concerning rough structures, we do achieve a comprehensive characterization of a specific surface, showing the strength and generality of this method.

At last we want to point out that the Fokker-Planck equation (5) corresponds to an equivalent Langevin equation [28]. The use of this Langevin equation in the scale variable should open the possibility to directly simulate surface profiles with given stochastic properties for different applications.

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