On the stability of the tangent bundle of Fano manifolds

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March 19, 2022

Mathematics Subject Classification (1991): 14J45, 14J60, 32L07.

Introduction

A smooth variety $X$ over the field of complex numbers $\mathbb{C}$ is called Fano if its anticanonical divisor $-K_X$ is ample. Stability (in the sense of Mumford and Takemoto) with respect to $-K_X$ of the tangent bundle $T_X$ can be considered as an algebraic analogue to the existence of a Kähler-Einstein metric on $X$, since the result of Kobayashi [Ko] and Lübke [Lü] shows that the existence of a Kähler-Einstein metric implies the stability of the tangent bundle. But the converse is not true, e.g. $\mathbb{P}^2$ blown up in two points has stable tangent bundle, which do not admit a Kähler-Einstein metric, cf. [Ma].

By Tian’s solution of Calabi’s conjecture for Del-Pezzo surfaces [Ti] and by [Fa] we have a complete picture in dimension 2: If $X$ is a Del-Pezzo surface, then $X$ has stable tangent bundle $T_X$, unless $X$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, or $\mathbb{P}^2$ blown-up in a point. In both cases the relative tangent bundle $T_X/\mathbb{P}^1$ of a canonical projection to $\mathbb{P}^1$ is a destabilising subsheaf of $T_X$.

If the dimension of $X$ is $\geq 3$, then the existence of a Kähler-Einstein metric remains an open question.

In this article, our main results are as follows:

**Theorem 1** Let $X$ be a Fano 3-fold with $b_2 \geq 2$. Assume that the tangent bundle $T_X$ of $X$ is not stable.

Then the relative tangent sheaf $T_{X/Y}$ of a contraction $f : X \rightarrow Y$ of an extremal face on $X$ is a destabilising subsheaf of $T_X$.

**Theorem 2** From the 87 deformation classes of Fano 3-folds with $b_2 \geq 2$, cf. [M-M 1, M-M 2] the members of

- 68 deformation classes have stable tangent bundle,
- 12 deformation classes have semistable (but not stable) tangent bundle and
- 7 deformation classes have unstable tangent bundle.

For a detailed description of the deformation classes whose members have semistable or unstable tangent bundle see theorem 3.1 below.

*This article contains results from the author’s dissertation which was prepared at the graduate program "Complex Manifolds" at the university of Bayreuth. The author wants to express his gratitude to his advisors Prof. T. Peternell and Prof. M. Schneider.*
Our main tool in proving theorem 3 is Mori theory. By Mori theory we understand the results and techniques concerning the cone of curves on a manifold $X$ whose canonical divisor $K_X$ is not numerically effective.

The proofs of theorem 1 and 2 use the classification of Fano-3-folds. If dim $X \geq 4$, then the problem of the stability of the tangent bundle seems hopeless, if one wants to use classification. But one may expect that theorem 1 holds in any dimension.

1 Preliminaries

A smooth connected variety $X$ over the field of complex numbers $\mathbb{C}$ is called simply a manifold. All manifolds are assumed to be projective, unless otherwise stated. $K_X$ denotes the canonical divisor of a normal variety $X$.

Assume $X$ smooth and set $n = \dim X$. Let $H$ be an ample line bundle on $X$. If $F$ is a torsion free coherent sheaf on $X$ we define $\mu(F)$ to be $c_1(F).H^{n-1}/\text{rk}(F)$. We call $F$ semistable (resp. stable) if for all proper subsheaves $F'$ of $F$ with $0 \leq \text{rk}(F') \leq \text{rk}(F)$ we have $\mu(F') \leq \mu(F)$ (resp. $\mu(F') < \mu(F)$).

Let $X$ be a normal variety of dimension $n$. We use the following notation:

$$N^1(X) := \{ \text{Cartier divisors on } X \} / \equiv \otimes \mathbb{R}$$
$$N_1(X) := \{ \text{1-cycles on } X \} / \equiv \otimes \mathbb{R}$$
$$\overline{NE}(X) := \text{the closure of the convex cone generated by effective 1-cycles in } N_1(X).$$

Here the symbol $\equiv$ means numerical equivalence and the symbol $\sim$ will denoted linear equivalence.

**Definition 1.1**  
(1) A curve $C$ on $X$ is called extremal if

(a) $(K_X.C) < 0$,
(b) given $u, v \in \overline{NE}(X)$ then $u, v \in \mathbb{R}_+[C]$ if $u + v \in \mathbb{R}_+[C]$.

If $C$ is an extremal curve on $X$, then the set $R = \mathbb{R}_+[C]$ is called an extremal ray on $X$.

2) Let $H$ be a nef Cartier divisor on $X$. The set $F := H^\perp \cap \overline{NE} \setminus \{0\}$ is called an extremal face if $F$ is entirely contained in the set $\{ z \in N_1(X) \mid (K_X.z) < 0 \}$.

**Theorem 1.2 (Cone theorem (Mori, Kawamata, Kollár [Mo, KMM]))**

Assume that $X$ has only canonical singularities. Fix an ample divisor $L$. Then for any $\varepsilon > 0$, there exist extremal curves $\ell_1, \ldots, \ell_r$ such that

$$\overline{NE}(X) = \sum_{i=1}^r \mathbb{R}_+[\ell_i] + \overline{NE}_\varepsilon(X).$$

Here $\overline{NE}_\varepsilon(X) := \{ z \in \overline{NE}(X) \mid (K_X.z) > -\varepsilon(L.z) \}$.

**Theorem 1.3 (Contraction theorem (Shokurov, Kawamata [KMM]))**

Let $F$ be an extremal face of $\overline{NE}(X)$. Assume that $X$ has only canonical singularities. Then there exists a morphism $\varphi = \text{cont}_F : X \to Y$ onto a normal projective variety $Y$, such that: For any irreducible curve $C$ on $X$ the image $\varphi(C)$ is a point if and only if $[C] \in F$. 


2 Fano varieties with $b_2 = 1$

Fano 3-folds with $b_2 = 1$ are classified by Iskovskih [I 1, I 2]. There are 18 classes of Fano 3-folds with $b_2 = 1$ up to deformation.

Remark 2.1 Let $X$ be a Fano manifold of dimension $n$. By a criterion for stability [Ho], the tangent bundle $T_X$ of $X$ is stable with respect to $(-K_X)$, if one of the following equivalent conditions is fulfilled:

(A$_i$) $H^0(X, \Omega^i \otimes L^{-1}) = 0$ for all $L \in \text{Pic}(X)$ with $L.(-K_X)^{n-1} \geq -\frac{1}{n}(-K_X)^n$.

(B$_i$) $H^0(X, \wedge^i T_X \otimes L^{-1}) = 0$ for all $L \in \text{Pic}(X)$ with $L.(-K_X)^{n-1} \geq \frac{1}{n}(-K_X)^n$.

Stability is granted when all conditions (A$_i$) or (B$_{n-i}$), $1 \leq i \leq n - 1$, hold.

From now on assume that $b_2(X) = 1$. Let $L$ be the ample generator of $\text{Pic}(X) \cong \mathbb{Z}$. Then we have that $-K_X = rL$ with $1 \leq r \leq n + 1$, where the integer $r$ is called the index of $X$. By the Kobayashi-Ochiai characterisation of projective space and hyperquadrics [Ko Oc], we have:

$r = n + 1 \iff X \cong \mathbb{P}^n$ and $r = n \iff X \cong Q_n \subset \mathbb{P}^{n+1}$

Remark 2.2 If $X$ is $\mathbb{P}^n$ or $Q_n$, then on may verify the conditions (A$_i$) directly.

Remark 2.3 Let $X$ be a Fano manifold with $b_2 = 1$ and $L$ the ample generator of $\text{Pic}(X)$. Then we have:

1. If the index $r$ of $X$ is 1, then the conditions (A$_i$) are fulfilled (cf. [Re , Theorem 3]).

2. $H^0(X, \Omega^i \otimes L^m) = 0$ for $m \leq 0$. In particular the condition (A$_1$) is fulfilled in any case.

3. If the index $r$ of $X$ is $\leq n$, then the condition (A$_{n-1}$) is fulfilled.

Proof. 1),2) Since $1 \leq i \leq n - 1$, we have $i \frac{1}{n} < 1$. If $m < 0$, then we have $H^0(X, \Omega^i \otimes L^m) = 0$ by Kodaira-Nakano vanishing theorem. Since $-K_X$ is ample, it follows by the Kodaira vanishing theorem that $h^0(\mathcal{O}_X) = h^0(\mathcal{O}_X) = h^n(-i(\mathcal{O}_X)) = 0$.

3) Since the condition (A$_{n-1}$) is equivalent to (B$_1$), it suffices to show that $H^0(X, T_X \otimes L^{-m}) = 0$, for $m \geq 1$. But this is a consequence of [Wa, Theorem 1], because the index of $X$ is different from $n + 1$. □

Corollary 2.4 Let $X$ be a Fano 3-fold with $b_2 = 1$. Then the tangent bundle of $X$ is stable.

3 Fano 3-folds with $b_2 \geq 2$

The proof of theorem 3.1 and theorem 3.2 is a by-product of the proof of the following:

Theorem 3.1 Let $X$ be a Fano 3-fold.

i) The members of the deformation classes in the following list have semistable tangent bundle.

1) the blow-up of $\mathbb{P}^3$ with center a line.

2) the blow-up of $Y$ with center two exceptional fibers $\ell$ and $\ell'$ of the blow-up $\Phi : Y \to \mathbb{P}^3$ such that $\ell$ and $\ell'$ lie on the same irreducible component of the exceptional set of $\Phi$. Here $\Phi : Y \to \mathbb{P}^3$ is the blow-up of $\mathbb{P}^3$ with center two disjoint lines in $\mathbb{P}^3$. 

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(3) the product of a Del-Pezzo surface (i.e Fano 2-fold) with \( \mathbb{P}^1 \).

ii) The members of the deformation classes in the following list have unstable tangent bundle.

(1) \( V_7 \), that is, the \( \mathbb{P}^1 \)-bundle \( \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(1)) \) over \( \mathbb{P}^2 \).
(2) the blow-up of the Veronese cone \( W_4 \subset \mathbb{P}^6 \) with center the vertex, that is \( \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(2)) \) over \( \mathbb{P}^2 \).
(3) the blow-up of \( V_7 \) with center a line on the exceptional divisor \( D \simeq \mathbb{P}^2 \) of the blow-up \( V_7 \to \mathbb{P}^3 \).
(4) the blow-up of \( V_7 \) with center the strict transform of a line passing through the center of the blow-up \( V_7 \to \mathbb{P}^3 \).
(5) the blow-up of the cone over a smooth quadric surface in \( \mathbb{P}^3 \) with center the vertex, that is, the \( \mathbb{P}^1 \)-bundle \( \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(1,1)) \) over \( \mathbb{P}^1 \times \mathbb{P}^1 \).
(6) the blow-up of \( \mathbb{P}^1 \times \mathbb{F}_1 \) with center \( \{ t \} \times \epsilon \), where \( t \in \mathbb{P}^1 \) and \( \epsilon \) is an exceptional curve of the first kind on \( \mathbb{F}_1 \).
(7) the blow-up of \( \widetilde{\mathbb{P}}(L) \) with center two exceptional lines of the blow-up \( \widetilde{\mathbb{P}}(L) \to \mathbb{P}^3 \). Here \( \widetilde{\mathbb{P}}(L) \to \mathbb{P}^3 \) is the blow-up of \( \mathbb{P}^3 \) with center a line in \( \mathbb{P}^3 \).

iii) If \( X \) is not contained in a deformation class listed as above, then \( T_X \) is stable with respect to the anticanonical divisor \( -K_X \).

Proof. Instead of presenting here the long proof of theorem 3.1, we will treat some special cases and examples. For the proof of theorem 3.1, we will refer the reader to \([St]\). The plan of the proof is as follows:

Step 1. Vanishing results for \( H^0(X, T_X \otimes L^{-1}) \) and \( H^0(X, \Omega_X^1 \otimes L^{-1}) \).

Step 2. Direct check of stability for the list of families with \( b_2 = 2 \).

Step 3. Reduction of the cases with \( b_2 \geq 3 \) to those studied at Step 2, or to lower dimensional vanishing statements.

Products of Fano manifolds

Let \( Y_1, Y_2 \) be two Fano manifolds of dimension \( n_1 \) and \( n_2 \) respectively. Then \( X = Y_1 \times Y_2 \) is a Fano manifold of dimension \( n = n_1 + n_2 \). By an easy computation, one gets \( \mu(T_X) = \mu(T_{Y_1}) = \mu(T_{Y_2}) \). It is a well known fact that a vector bundle \( E_1 \oplus E_2 \) is semistable if and only if \( E_1 \) and \( E_2 \) are semistable vector bundles with \( \mu(E_1) = \mu(E_2) \). Thus, we have proved:

Proposition 3.2 Let \( Y_1, Y_2 \) be two Fano manifolds with semistable tangent bundle.

Then the Fano manifold \( X = Y_1 \times Y_2 \) has semistable tangent bundle.

Corollary 3.3 Let \( X \) be isomorphic to a product of a Del-Pezzo surface with \( \mathbb{P}^1 \).

Then \( T_X \) is semistable.

Proof. By \([Fa]\) the tangent bundle of a Del-Pezzo surface is a semistable vector bundle.

Example 3.4 Let \( X \) be the blow-up of \( \widetilde{\mathbb{P}}^3(L) \) with center two exceptional lines of the blow-up \( \widetilde{\mathbb{P}}^3(L) \to \mathbb{P}^3 \). Here \( \widetilde{\mathbb{P}}^3(L) \) is the blow-up of \( \mathbb{P}^3 \) with center a line \( L \). We will show that \( X \) has unstable tangent bundle.

Consider the following diagram
Let \( g := f_{1,2} \circ f_{1,2} \circ f_1 \). Since \( g^* \Omega^1_{\mathbb{P}^1} \simeq \mathcal{O}_X(-2H_1 + 2D_{f_1,1}) \), it follows that \( g^* \Omega^1_{\mathbb{P}^1} \subset \Omega^1_X \) is a \((-K_X)\)-destabilising subsheaf, with \( \mu(g^* \Omega^1_{\mathbb{P}^1}) = -14 > -\frac{46}{3} = \mu(\Omega^1_X) \).

Before we go to the next example we shall collect some auxiliary lemmas.

**Lemma 3.5** Let \( S \) be a smooth projective surface, \( f : X \to S \) a conic bundle and \( \Delta \subset S \) the discriminant locus. Then we have an exact sequence

\[
0 \to f^* \Omega^1_S \xrightarrow{\delta} \Omega^1_X \to \Omega^1_{X/S} \to 0
\]

and \( \Omega^1_{X/S} \simeq \mathcal{I}_\Gamma \otimes \omega_X \otimes f^* \omega^{-1}_S \), where \( \Gamma \) is a closed Cohen-Macaulay subscheme of \( X \) of pure dimension 1 with \( f(\Gamma) = \Delta \). The restriction \( f|_{\Gamma \setminus f^{-1}(\Delta_{\text{sing}})} : \Gamma \setminus f^{-1}(\Delta_{\text{sing}}) \to \Delta_{\text{reg}} \) is an isomorphism and \( \Gamma \cap X_s = (X_s)_{\text{red}} \) for all \( s \in \Delta_{\text{sing}} \).

**Proof.** \( f^* \Omega^1_S \to \Omega^1_X \) drops rank in codimension 2, whence the first three assertions follow from the theory of the Eagon-Northcott complex [E N]. It is also clear that \( f(\Gamma) = \Delta \).

\( S \) can be covered by affine open sets \( U_\alpha \) such that \( f^{-1}(U_\alpha) \) is isomorphic over \( U_\alpha \) to the closed subscheme of \( U_\alpha \times \mathbb{P}^2 \) given by a quadratic equation:

\[
g_\alpha := \sum_{0 \leq i \leq j \leq 2} A_{ij} X_i X_j, \quad A_{ij} \in H^0(U_\alpha, \mathcal{O}_{U_\alpha}).
\]
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Using the diagram

\[
\begin{array}{c}
0 \\ \\
\mathcal{O}_{U_\alpha \times \mathbb{P}^2}(-2)|_{f^{-1}(U_\alpha)} \quad \xrightarrow{\partial g_0 \partial g_1 \partial g_2} \quad \Omega^1_{U_\alpha \times \mathbb{P}^2}|_{f^{-1}(U_\alpha)} \quad \xrightarrow{\partial g_0 \partial g_1 \partial g_2} \quad \Omega^1_{X/S}|_{f^{-1}(U_\alpha)} \quad \xrightarrow{0} \\
\end{array}
\]

one can deduce that \( \Gamma \cap f^{-1}(U_\alpha) \) is the closed subscheme of \( U_\alpha \times \mathbb{P}^2 \) given by the equations:

\[
\frac{\partial g_0}{\partial X_0} = \frac{\partial g_1}{\partial X_1} = \frac{\partial g_2}{\partial X_2} = 0
\]

(in fact, the three equations are enough by Euler’s identity). Now the last 2 assertions are clear.

\[\square\]

**Lemma 3.6** Let \( Y \xleftarrow{\pi} X \) be the blow-up of a conic bundle \( X \xrightarrow{f} S \) with center a smooth irreducible subsection \( C \) over \( S \) (i.e. \( f|_C : C \to S \) is an embedding). Let \( L \in \text{Pic}(Y) \), such that \( \pi^* L \) is a \( f \)-ample line bundle on \( X \). Then:

(i) \( H^0(Y, T_Y \otimes L^{-1}) = 0 \), if \( \mu(L) > \mu(T_Y/S) \) and

(ii) \( H^0(Y, \Omega^1_Y \otimes L^{-1}) = 0 \).

**Proof.** Straightforward and left to the reader. \[\square\]

**Example 3.7** Let \( X \) be the blow-up of \( \mathbb{P}^3 \) with center a union of a cubic \( C \) in a plane \( S \) and a point \( p \) not in \( S \). We will prove that \( X \) has stable tangent bundle. For this we make use of the following diagram

\[
\begin{array}{c}
\mathbb{P}^3 \\
\xleftarrow{\varepsilon} \\
\xrightarrow{\zeta} \\
\mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1)) \\
\xleftarrow{f_1} \\
\xrightarrow{f_4} \\
Y_4 \\
\mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1)) \\
\xleftarrow{f_1} \\
\xrightarrow{f_4} \\
\mathbb{P}^3 \\
\mathbb{P}^2 \\
\xleftarrow{\beta} \\
\xrightarrow{\alpha} \\
\mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(2)) \\
\xleftarrow{f_2} \\
\xrightarrow{f_3} \\
X \\
\xleftarrow{f_2} \\
\xrightarrow{f_3} \\
Y_3 \\
\mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(2)) \\
\xleftarrow{f_2} \\
\xrightarrow{f_3} \\
\mathbb{P}^3 \\
\xleftarrow{\gamma} \\
\xrightarrow{\delta} \\
W_4 \\
\end{array}
\]

Define \( \zeta : V_7 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1)) \to \mathbb{P}^3 \) to be the blow-up of \( \mathbb{P}^3 \) in \( p \) and denote again by \( C \subset V_7 \) the proper transform of \( C \subset S \subset \mathbb{P}^3 \). Define also \( f_2 \) to be the elementary
transformation of $f_1$ along $C$, and $W_4$ to be the cone over the Veronese surface in $\mathbb{P}^5$ and $\gamma$
the blow-up of the vertex.

Let $H_1 := f_1^* \mathcal{O}_{\mathbb{P}^2}(1)$, $H_2 := f_1^* \mathcal{O}_{\mathbb{P}^3}(1)$, $H_3 := f_2^* \mathcal{O}_{\mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(2))}(1)$ and $D_{f_1}, D_{f_2}$ the
exceptional divisors of $f_1$ resp. $f_2$. Then it follows that we have

$$D_{f_2} \sim 3H_1 - D_{f_1}, \quad H_2 - H_1 \sim H_3 - D_{f_2}$$

$$-K_X \sim 2H_1 + 2H_2 - D_{f_1} \sim H_1 + 2H_3 - D_{f_2}$$

$$\Rightarrow -K_X \sim H_2 + H_3$$

$$(a_1H_1 + a_2H_2 + a_3H_3)(-K_X)^2 = 9a_1 + 13a_2 + 19a_3, \quad (-K_X)^3 = 32.$$
If \( \mathcal{L} \subset g^*\Omega^1_{\mathbb{P}^2} \) has maximal \( \mu(\mathcal{L}) \), then we have:

\[
0 \neq H^0(X, g^*\Omega^1_{\mathbb{P}^2} \otimes \mathcal{L}^{-1}) \subset H^0(D_{f_2}, g^*\Omega^1_{\mathbb{P}^2} \otimes \mathcal{L}^{-1}|D_{f_2}) \cong H^0(C', \Omega^1_{\mathbb{P}^2}(C) \otimes O_{C'}(-a_1 - 2a_3) \otimes S^{-a_2}(O_C \oplus O_{C'}(1)))
\]

\[a_2 \leq 0 \text{ and } a_1 + a_2 + 2a_3 \leq -2.\]

(2) \( \mathcal{I}_{C''} \otimes \omega_{X/\mathbb{P}^2} \otimes \mathcal{L}^{-1} \subset O_X((3 - a_1)H_1 + (-1 - a_2)H_2 + (-1 - a_3)H_3) \)

\[\Rightarrow \begin{cases} 
(\alpha \circ f_1)_* : & a_2 + a_3 \leq -2 \text{ and } a_1 + a_2 + 3a_3 \leq -1; \\
(\beta \circ f_2)_* : & a_2 + a_3 \leq -2 \text{ and } a_1 + 4a_2 + 2a_3 \leq -2.
\end{cases}\]

If \( 0 \neq H^0(X, \mathcal{I}_{C''} \otimes O_X((3 - a_1)H_1 + (-1 - a_2)H_2 + (-1 - a_3)H_3)) \), then \( \omega_{X/\mathbb{P}^2} \otimes \mathcal{L}^{-1} \) has a global section vanishing on \( C'' \). It follows that \( \omega_{X/\mathbb{P}^2} \otimes \mathcal{L}^{-1}|D_{f_2} \) has a global section vanishing on \( C'' \). Therefore

\[0 \neq H^0(D_{f_2}, O_X((3 - a_1)H_1 + (-1 - a_2)H_2 + (-1 - a_3)H_3) \otimes O_{D_{f_2}}(-C'')) \]

\[\cong H^0(Z, \rho^*O_{C'}(1-a_1 - 2a_3) \otimes O_Z(-2 - a_2)) \text{, implies that } a_2 \leq -2 \text{ and } a_1 + a_2 + 2a_3 \leq -1.\]

Hence \( \mu(\mathcal{L}) = 9a_1 + 13a_2 + 19a_3 = 9(a_1 + a_2 + 2a_3) + (a_2 + a_3) + 3a_2 \leq -17. \) \qed

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