LEFT COUNITAL HOPF ALGEBRA STRUCTURES ON FREE COMMUTATIVE NIJENHUIS ALGEBRAS

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ABSTRACT. Motivated by the Hopf algebra structures established on free commutative Rota-Baxter algebras, we explore Hopf algebra related structures on free commutative Nijenhuis algebras. Applying a cocycle condition, we first prove that a free commutative Nijenhuis algebra on a left counital bialgebra (in the sense that the right-sided counicity needs not hold) can be enriched to a left counital bialgebra. We then establish a general result that a connected graded left counital bialgebra is a left counital right antipode Hopf algebra in the sense that the antipode is also only right-sided. We finally apply this result to show that the left counital bialgebra on a free commutative Nijenhuis algebra on a connected left counital bialgebra is connected and graded, hence is a left counital right antipode Hopf algebra.

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1. INTRODUCTION

The concept of a Nijenhuis operator first appeared in the Lie algebra context from the important notion of a Nijenhuis torsion, from the study of pseudo-complex manifolds of Nijenhuis [31] in the 1950s. This concept is also related to the well-known concepts of Schouten-Nijenhuis bracket, the Frölicher-Nijenhuis bracket [17] and the Nijenhuis-Richardson bracket. Furthermore, Nijenhuis operators on Lie algebras play an important role in the study of integrability of non-linear evolution equations [8], and appeared in the contexts of Poisson-Nijenhuis manifolds [23], the classical Yang-Baxter equation [18, 19] and the Poisson structure [29]. More recently, Nijenhuis operators have been studied for $n$-Lie algebras, integrable systems, Hom-Lie algebra and integrable hierarchies [8, 28, 52, 57].
A Nijenhuis operator on an associative algebra $R$ is a linear endomorphism $P : R \rightarrow R$ satisfying the Nijenhuis equation:

$$P(x)P(y) = P(P(x)y) + P(xP(y)) - P^2(xy) \quad \text{for all } x, y \in R.$$  

The Nijenhuis operator on an associative algebra was introduced by Carinena et. al. [5] to study quantum bi-Hamiltonian systems. In [38], Nijenhuis operators are constructed by analogy with Poisson-Nijenhuis geometry, from relative Rota-Baxter operators. In [24, 25], relations with dendriform type algebras were studied.

The Nijenhuis operator is in close analogue with the Rota-Baxter operator of weight $\lambda$ (where $\lambda$ is a constant), the latter being defined by the Rota-Baxter equation

$$P(x)P(y) = P(P(x)y) + P(xP(y)) + \lambda P(xy) \quad \text{for all } x, y \in R.$$  

Originated from the probability study of G. Baxter [3], the Rota-Baxter operator was studied by Cartier and Rota [6, 34] and is closely related to the operator form of the classical Yang-Baxter equation [2, 36]. Its intensive study in the last two decades found many applications in mathematics and physics, most notably the work of Connes and Kreimer on renormalization of quantum field theory [7, 12, 13]. See [21] for further details and references.

Theoretic developments of Nijenhuis algebras have been in parallel to those of Rota-Baxter algebras. For example, free commutative and noncommutative Nijenhuis algebras were constructed in [24, 25] following the construction of free commutative and noncommutative Rota-Baxter algebras [22, 11]. In [41], the Nijenhuis operator, as one of the Rota-Baxter type operators, is studied in the context of Rota’s problem [35] on classification of operator identities.

Motivated by the close relationship between divided power Hopf algebra and shuffle product Hopf algebra with Rota-Baxter algebras, Hopf algebra structures have been found on free commutative Rota-Baxter algebras for quite a few years [1, 10]. Recently, Hopf algebra structures on free (non-commutative) Rota-Baxter algebras have been obtained [10] by a one-cocycle property in analogous to the Connes-Kreimer Hopf algebra of rooted trees [7]. Thus it is natural to explore a Hopf algebra structure on free Nijenhuis algebras. The purpose of this paper is to pursue this for free commutative Nijenhuis algebras. As it turns out, this approach does not give a bona fide Hopf algebra, only one with a left-sided counit and right-sided antipode. Interestingly, related structures have appeared in the study of quantum groups [33] (tracing back to [20]) and combinatorics [13, 16]. In both cases, the term one-side Hopf algebra was used though in the former case only one-sided antipode is optional while in the latter case both a one-sided antipode and the opposite side unit are optional. To distinguish from these notions, we will use the terms left counital bialgebra and left counital Hopf algebra for our constructions. See [4, 27, 39] for some other variations of Hopf algebras with relaxed conditions.

The layout of the paper is as follows. In Section 2 we recall the construction of free commutative Nijenhuis algebras by a generalization of the shuffle product and discuss a combinatorial identity from a special case. In Section 3 we equip the free commutative Nijenhuis algebras with a left counital bialgebra. In Section 4, we first extend the classical result that a connected graded bialgebra is a Hopf algebra to the left unital context. We then show that the above-mentioned left counital bialgebra structure on free commutative Nijenhuis algebras is graded and connected, and thus yields a left counital Hopf algebra.

**Convention.** In this paper, all algebras are taken to be unitary commutative over a unitary commutative ring $k$. Also linear maps and tensor products are taken over $k$. 


2. Free commutative Nijenhuis algebras

In this section, we recall the notion of Nijenhuis algebras and the construction of free commutative Nijenhuis algebras by the right-shift shuffle product \([14]\). We then consider some special cases of free commutative Nijenhuis algebras.

2.1. Free commutative Nijenhuis algebras on a commutative algebra.

**Definition 2.1.** A Nijenhuis algebra is an associative algebra \(R\) equipped with a linear operator \(P\), called Nijenhuis operator, satisfying the Nijenhuis equation in Eq. (1). A homomorphism from a Nijenhuis algebra \((R, P)\) to a Nijenhuis algebra \((S, Q)\) is an algebra homomorphism \(f : R \to S\) such that \(fP = Qf\).

We recall the concept and construction of the free commutative Nijenhuis algebra on a commutative algebra.

**Definition 2.2.** Let \(A\) be a commutative algebra. A free commutative Nijenhuis algebra on \(A\) is a commutative Nijenhuis algebra \(F_N(A)\) with a Nijenhuis operator \(N\), and an algebra homomorphism \(j_A : A \to F_N(A)\) such that, for any commutative Nijenhuis algebra \((R, P)\) and any algebra homomorphism \(f : A \to R\), there is a unique Nijenhuis algebra homomorphism \(\tilde{f} : F_N(A) \to R\) such that \(f = \tilde{f} \circ j_A\), that is, the following diagram

\[
\begin{array}{ccc}
A & \xrightarrow{j_A} & F_N(A) \\
\downarrow{f} & & \downarrow{\tilde{f}} \\
R & & R
\end{array}
\]

commutes.

For a given unital commutative algebra \(A\) with unit \(1_A\), we will give a Nijenhuis algebra structure on the \(k\)-module

\[
\overline{\Pi}(A) := \bigoplus_{n \geq 1} A^{\otimes n}.
\]

Here \(A^{\otimes n}\) is the \(n\)-th tensor power of \(A\).

We first define the right-shift operator \(P_r\) on \(\overline{\Pi}(A)\) by

\[
P_r : \overline{\Pi}(A) \to \overline{\Pi}(A), \quad P_r(a) = 1_A \otimes a \quad \text{for all } a \in A^{\otimes n}, n \geq 1.
\]

We then define a multiplication on \(\overline{\Pi}(A)\) as follows.

For \(a = a_1 \otimes \cdots \otimes a_m \in A^{\otimes m}\) and \(b = b_1 \otimes \cdots \otimes b_n \in A^{\otimes n}\), denote \(a' = a_2 \otimes \cdots \otimes a_m\) if \(m \geq 2\) and \(b' = b_2 \otimes \cdots \otimes b_n\) if \(n \geq 2\), so that \(a = a_1 \otimes a'\) and \(b = b_1 \otimes b'\). Define a multiplication \(\circ_r\) on \(\overline{\Pi}(A)\) by the following recursion.

\[
a \circ_r b = \begin{cases} 
    a_1 b_1, & m = n = 1, \\
    a_1 b_1 \otimes b', & m = 1, n \geq 2, \\
    a_1 b_1 \otimes a', & m \geq 2, n = 1, \\
    a_1 b_1 \otimes (a' \circ_r (1 \otimes b') + (1 \otimes a') \circ_r b' - 1 \otimes (a' \circ_r b')), & m, n \geq 2.
\end{cases}
\]

Alternatively, \(\circ_r\) can be defined by

\[
a \circ_r b = a_1 b_1 \otimes (a' \otimes_r b')
\]

where \(\otimes_r\) is the right-shift shuffle product defined in \([14]\) whose precise definition we will not recall since it is not needed in the sequel.
Let  
\[ j_A : A \to \mathcal{P}(A), \ a \mapsto a, \]
be the natural embedding. Then  
\[ j_A(ab) = ab = a \circ_r b = j_A(a) \circ_r j_A(b), \quad \text{for all} \ a, b \in A. \]
So \( j_A \) is an algebra homomorphism. Then we have

**Theorem 2.3.** [14] Let \( A \) be a commutative algebra with unit \( 1_A \). Let \( \mathcal{P}(A), \circ_r, \) and \( j_A \) be defined as above. Then

(a) The triple \((\mathcal{P}(A), \circ_r, P_r)\) is a commutative Nijenhuis algebra;
(b) The quadruple \((\mathcal{P}(A), \circ_r, P_r, j_A)\) is the free commutative Nijenhuis algebra on \( A \).

### 2.2. Special cases. Let \( X \) be a nonempty set and let \( A = k[X] \). Then the Nijenhuis algebra \( \mathcal{P}(k[X]) \) is the free commutative Nijenhuis algebra on \( X \) defined by the usual universal property.

Now let \( A = k \). Then we obtain the free commutative Nijenhuis algebra

\[ \mathcal{P}(k) = \bigoplus_{n \geq 0} k^{\otimes(n+1)}. \]

We denote the unit \( 1_k \) of \( k \) by 1 for abbreviation. Since the tensor product is over \( k \), by bilinearity, we have \( k^{\otimes(n+1)} = k1^{\otimes(n+1)} \) for all \( n \geq 0 \). Then

\[ \mathcal{P}(k) = \bigoplus_{n \geq 0} k1^{\otimes(n+1)}. \]

Thus \( \mathcal{P}(k) \) is a free \( k \)-module on the basis \( 1^{\otimes n}, n \geq 1 \).

**Proposition 2.4.** For any \( m, n \in \mathbb{N} \),

\[ 1^{\otimes(m+1)} \circ_r 1^{\otimes(n+1)} = 1^{\otimes(m+n+1)}. \]

**Proof.** We prove Eq. (5) by induction on \( m + n \geq 0 \). For \( m + n = 0 \), we get \( m = n = 0 \). Then the definition of \( \circ_r \) in Eq. (3) yields \( 1 \circ_r 1 = 1 \). For \( k \geq 0 \), assume that Eq. (5) has been proved for \( m + n \geq k \). Consider the case of \( m + n = k + 1 \). Then either \( m \geq 1 \) or \( n \geq 1 \). If one of \( m \) or \( n \) is zero, then by Eq. (2), we have \( 1^{\otimes(m+1)} \circ_r 1^{\otimes(n+1)} = 1^{\otimes(m+n+1)}. \) If none of \( m \) or \( n \) is zero, then by Eq. (2) and the induction hypothesis, we have

\[
1^{\otimes(m+1)} \circ_r 1^{\otimes(n+1)} = 1 \otimes \left( 1^{\otimes m} \circ_r 1^{\otimes(n+1)} + 1^{\otimes(m+1)} \circ_r 1^{\otimes n} - 1 \otimes (1^{\otimes m} \circ_r 1^{\otimes n}) \right) \\
= 1 \otimes \left( 1^{\otimes(m+n)} + 1^{\otimes(m+n)} - 1 \otimes 1^{\otimes(m+n-1)} \right) \\
= 1^{\otimes(m+n+1)}. \]

This completes the induction. \( \square \)

Note that the definition of the Nijenhuis operator in Eq. (1) can be regarded as formally taking \( \lambda = -P \) in the definition of Rota-Baxter operator in Eq. (4). In fact, the construction of commutative Nijenhuis algebras is similar to that of free commutative Rota-Baxter algebras \((\mathcal{P}_k(A) := \mathcal{P}(A), P_A, \circ_\lambda)\) on \( A \) of weight \( \lambda \in k \) [11, Chapter 3, Theorem 3.2.1]. Let \( A = k \). Then \((\mathcal{P}_k(k), P_k, \circ_\lambda)\) is the free commutative Rota-Baxter algebra on \( k \) of weight \( \lambda \). When \( A = k \), the module structure of \( \mathcal{P}(A) \) is

\[ \mathcal{P}(k) = \bigoplus_{n \geq 0} k1^{\otimes(n+1)}. \]
and the multiplication \( \cdot \) is given by

\[
\left(1 \otimes (m+1) \right) \cdot \left(1 \otimes (n+1) \right) = \sum_{k=0}^{\max(m,n)} \binom{m + n - k}{m} k! \left(1 \otimes (m+n+1-k) \right)
\]

for all \( m, n \geq 0 \).

Since the Rota-Baxter operator can be transformed to the Nijenhuis operator by formally replacing \( \lambda \) by \(-P\), doing so and hence \( \lambda^{k} \) by \((-1)^{k}P^{k} \) in the above equation, we can expect to obtain the corresponding formula for the free Nijenhuis algebra \( \mathcal{N} \), namely,

\[
\left(1 \otimes (m+1) \right) \cdot \left(1 \otimes (n+1) \right) = \sum_{k=0}^{\min(m,n)} (-1)^{k} \binom{m + n - k}{m} k! \left(1 \otimes (m+n+1-k) \right)
\]

since \( P_{r}(1 \otimes \epsilon) = 1 \otimes (\epsilon + 1) \). Comparing with Eq. (7) suggests the identity

\[
\sum_{k=0}^{\min(m,n)} (-1)^{k} \binom{m + n - k}{m} k! = 1.
\]

This is in fact a combinatorial identity which is related to the probability distribution in quantum field theory. See [23] where the formula, under the assumption \( m \geq n \) without loss of generality, is proved by identifying the coefficient of \( x^{m} \) in the series expansion of the product

\[
(1 - x)^{n}(1 - x)^{-n-1} = (1 - x)^{-1} = \sum_{k=0}^{\infty} x^{k}.
\]

The formula can also be proved by enumerating stuffles [24].

We end this section by equipping \( \mathcal{N} \) with a natural Hopf algebra structure. Denote \( u_{n} := 1 \otimes (n+1) \), \( n \geq 0 \). Then by Eq. (8),

\[
u_{n} \cdot u_{m} = u_{m+n}, \text{ for all } m, n \geq 0.
\]

Thus \((\mathcal{N}, \cdot)\) is the free commutative algebra (that is, the polynomial algebra \( k[u_{1}] \)) generated by \( u_{1}(= 1 \otimes \epsilon) \). Using the universal property of free commutative algebras, we find that \( \mathcal{N} \) has a unique bialgebra structure with the comultiplication \( \Delta(u_{n}) = \sum_{i=0}^{n} \binom{n}{i} u_{i} \otimes u_{n-i} \) and the counit \( \varepsilon(u_{0}) = 1 \), and \( \varepsilon(u_{n}) = 0 \), \( n \geq 1 \). Define a linear map \( u : k \rightarrow \mathcal{N}, c \mapsto c \) for all \( c \in k \). Thus we have

**Theorem 2.5.**

(a) The bialgebra \((\mathcal{N}, \cdot, u, \Delta, \varepsilon)\) with the unique antipode

\[
S : \mathcal{N} \rightarrow \mathcal{N}, u_{n} \mapsto (-1)^{n}u_{n}, n \geq 0,
\]

is the binomial Hopf algebra.

(b) The free commutative Nijenhuis algebra \( \mathcal{N} \) is a Hopf algebra.

We will consider its generalization to other free Nijenhuis algebras in the next section.
3. The left counital bialgebra structure on free commutative Nijenhuis algebras

In this section, we will equip a free commutative Nijenhuis algebra $\overrightarrow{\Pi}(A)$ with a left counital Hopf algebra structure, when the generating algebra $A$ is a left counital bialgebra. Let $A := (A, m_A, \mu_A, \Delta_A, \varepsilon_A)$ be a left counital bialgebra. We first construct a comultiplication on the free commutative Nijenhuis algebra $\overrightarrow{\Pi}(A) := (\overrightarrow{\Pi}(A), \cdot, P_r)$ by applying a one-cocycle property. Then we give the construction of a left counit on $\overrightarrow{\Pi}(A)$. The left counital Hopf algebra property of $\overrightarrow{\Pi}(A)$ will be considered in the next section.

3.1. Comultiplication by cocycle condition. First we give the construction of left counital coalgebra on $\overrightarrow{\Pi}(A)$ by giving the comultiplication and the left counit on free commutative Nijenhuis algebras. The coassociativity and left counicity will be proved later.

First we introduce the notions of an operated bialgebra and a cocycle bialgebra with a left counit.

Definition 3.1. (a) A left counital coalgebra is a triple $(H, \Delta, \varepsilon)$, where the comultiplication $\Delta : H \to H \otimes H$ satisfies the coassociativity and the counit $\varepsilon : H \to k$ satisfies the left counicity: $(\varepsilon \otimes \text{id})\Delta = \beta_\ell$, where $\beta_\ell : H \to k \otimes H, u \mapsto 1 \otimes u$ for all $u \in H$;

(b) A left counital operated bialgebra is a sextuple $(H, m, \mu, \Delta, \varepsilon, P)$, where the triple $(H, \Delta, \varepsilon)$ is a left counital coalgebra, and the pair $(H, P)$ is an operated algebra, that is, an algebra $H$ with a linear operator $P : H \to H$;

(c) A left counital cocycle bialgebra is a left operated bialgebra $(H, m, \mu, \Delta, \varepsilon, P)$ satisfying the one-cocycle property\footnote{The cocycle condition $\Delta P = P \otimes 1 + (\text{id} \otimes P)\Delta$, which is use to construct the Hopf algebra structure on free Rota-Baxter algebras \cite{H}, does not work for free Nijenhuis algebras.}:

\begin{equation}
\Delta P = (\text{id} \otimes P)\Delta.
\end{equation}

Any bialgebra is a left counital bialgebra. As a simple example of left counital bialgebra which is not a bialgebra, consider $k[x]$ with the coproduct $\Delta(u) = 1 \otimes u$ for all $u \in k[x]$ and the usual counit $\varepsilon : k[x] \to k$ defined by $\varepsilon(x) = 0, \varepsilon(1) = 1$. Then it is easy to check all the conditions of a bialgebra except the right counicity: $(\text{id} \otimes \varepsilon)\Delta = \beta_r$, where $\beta_r : k[x] \to k[x] \otimes k, u \mapsto u \otimes 1$. Since

\begin{equation}
(\text{id} \otimes \varepsilon)\Delta(x) = (\text{id} \otimes \varepsilon)(1 \otimes x) = 1 \otimes 0 = 0 \neq x \otimes 1,
\end{equation}

the right counicity does not hold.

Note that the tensor product $\overrightarrow{\Pi}(A) \otimes \overrightarrow{\Pi}(A)$ is also an associative algebra whose multiplication will be denoted by $\bullet$.

Now we give the definition of the comultiplicaiton $\Delta_T : \overrightarrow{\Pi}(A) \to \overrightarrow{\Pi}(A) \otimes \overrightarrow{\Pi}(A)$. For every pure tensor $a := a_1 \otimes a_2 \otimes \cdots \otimes a_n \in A^{\otimes n}, n \geq 1$, we define $\Delta_T$ by induction on $n \geq 1$. For $n = 1$, that is, $a = a_1 \in A$, we define $\Delta_T(a) := \Delta_A(a_1)$ to be the coproduct $\Delta_A$ on $A$. Assume that $\Delta_T$ has been defined for $n \geq 1$. Consider $a \in A^{\otimes(n+1)}$. Then $a = a_1 \otimes a'$ with $a' := a_2 \otimes \cdots \otimes a_n \in A^{\otimes n}$. By Eq. (\ref{eq:tensor_product}), we have

\begin{equation}
a_1 \otimes a' = a_1 \circ_r (1_A \otimes a') = a_1 \circ_r P_r(a').
\end{equation}

We first define

\begin{equation}
\Delta_T(1_A \otimes a') = (\text{id} \otimes P_r)\Delta_T(a').
\end{equation}
We then define
\[
\Delta_T(a_1 \otimes a') = \Delta_A(a_1) \bullet \Delta_T(1_A \otimes a') = \Delta_A(a_1) \bullet (\text{id} \otimes P_r)\Delta_T(a').
\]
Here since $a'$ is in $A^{\otimes n}$, $\Delta_T(a')$ in Eq. (11) is well-defined by the induction hypothesis. Thus $\Delta_T(a)$ is well-defined. The definition of $\Delta_T$ also can be rewritten as follows.
\[
\Delta_T(a) = \Delta_A(a_1) \bullet \left((\text{id} \otimes P_r)\left(\Delta_A(a_2) \bullet \left((\text{id} \otimes P_r)(\cdots \bullet (\text{id} \otimes P_r)\Delta_A(a_n))\right)\right)\right).
\]

**Example 3.1.** Note that $k$-algebra $k$ has a natural bialgebra structure, where the comultiplication $\Delta_k$ and the counit $\varepsilon_k$ are defined by
\[
\Delta_k : k \rightarrow k \otimes k, \ c \mapsto c \otimes 1, \quad \text{and} \quad \varepsilon_k : k \rightarrow k, c \mapsto c. \quad \text{for all} \quad c \in k.
\]
Then $k$ becomes a left counital bialgebra. Then by Eq. (5), we get
\[
\overline{\Pi}(k) = \bigoplus_{n \geq 0} k1^{\otimes(n+1)}.
\]
For every $c_i \in k$, $1 \leq i \leq 3$, by the definition of $\Delta_T$ in Eq. (13),
\[
\Delta_T(c_1 \otimes c_2 \otimes c_3) = \Delta_k(c_1) \bullet \left((\text{id} \otimes P_r)(\Delta_k(c_2) \bullet \left((\text{id} \otimes P_r)\Delta_k(c_3)\right))\right)
\]
\[
= \Delta_k(c_1) \bullet \left((\text{id} \otimes P_r)(c_2 \otimes 1) \bullet \left(c_3 \otimes 1^{\otimes 2}\right)\right)
\]
\[
= \Delta_k(c_1) \bullet \left((\text{id} \otimes P_r)(c_2c_3 \otimes 1^{\otimes 2})\right) \quad \text{(by Eq. (3))}
\]
\[
= (c_1 \otimes 1) \bullet (c_2c_3 \otimes 1^{\otimes 3}) \quad \text{(by Eq. (2))}
\]
\[
= c_1c_2c_3 \otimes 1^{\otimes 3}.
\]

More generally, let $u_n := 1^{\otimes(n+1)}$ for all $n \geq 0$. Then
\[
\Delta_T(u_0) = \Delta_k(1) = 1^{\otimes 2} = 1 \otimes u_0,
\]
By Eq. (11), we obtain
\[
\Delta_T(u_n) = (\text{id} \otimes P_r)\Delta_T(u_{n-1}) = 1 \otimes u_n.
\]
Thus $\Delta_T$ is different from the coproduct defined in Theorem 2.5.

We next give the construction of the left counit $\varepsilon_T$ on $\overline{\Pi}(A)$ by using the left counit $\varepsilon_A$ of $A$. For every pure tensor $a = a_1 \otimes a_2 \otimes \cdots \otimes a_n \in A^{\otimes n}$ with $n \geq 1$, we define
\[
\varepsilon_T : \overline{\Pi}(A) \rightarrow k, \ a \mapsto \varepsilon_A(a_1)\varepsilon_A(a_2)\cdots\varepsilon_A(a_n).
\]

**Lemma 3.2.** For any pure tensors $a \in A^{\otimes m}$ and $b \in A^{\otimes n}$ with $m, n \geq 1$, we have
\[
(id \otimes \Delta_T)(id \otimes P_r)(a \otimes b) = (id \otimes id \otimes P_r)(id \otimes \Delta_T)(a \otimes b)
\]
and
\[
(\Delta_T \otimes id)(id \otimes P_r)(a \otimes b) = (id \otimes id \otimes P_r)(\Delta_T \otimes id)(a \otimes b).
\]
Proof. We first prove that Eq. (16) holds for any pure tensors $a \in A^{q_m}$ and $b \in A^{q_m}$. By Eq. (14), we obtain

$$
(id \otimes \Delta_T)(id \otimes P_r)(a \otimes b) = (id \otimes \Delta_T P_r)(a \otimes b) = a \otimes (\Delta_T P_r(b)) = a \otimes ((id \otimes P_r)\Delta_T(b)) = (id \otimes id \otimes P_r)(id \otimes \Delta_T)(a \otimes b).
$$

Eq. (17) follows from

$$
(\Delta_T \otimes id)(id \otimes P_r)(a \otimes b) = (\Delta_T \otimes P_r)(a \otimes b) = \Delta_T(a) \otimes P_r(b) = (id \otimes id \otimes P_r)(\Delta_T \otimes id)(a \otimes b).
$$

$\Box$

3.2. The compatibilities of $\Delta_T$ and $\varepsilon_T$. We now prove that $\Delta_T$ and $\varepsilon_T$ are algebra homomorphisms.

**Proposition 3.3.** The comultiplication $\Delta_T : \tilde{\Pi}(A) \rightarrow \tilde{\Pi}(A) \otimes \tilde{\Pi}(A)$ is an algebra homomorphism.

Proof. To prove that $\Delta_T$ is an algebra homomorphism, we only need to verify

$$
\Delta_T(a \circ \otimes b) = \Delta_T(a) \bullet \Delta_T(b)
$$

for any pure tensors $a := a_1 \otimes \cdots \otimes a_m \in A^{q_m}$ and $b := b_1 \otimes \cdots \otimes b_n \in A^{q_m}, m, n \geq 1$. We do this by applying the induction on $m + n \geq 2$. If $m + n = 2$, then $m = n = 1$. So $a, b$ are in $A$ and Eq. (18) follows from the assumption that $\Delta_A : A \rightarrow A \otimes A$ is an algebra homomorphism.

Let $k \geq 1$. Assume that Eq. (18) has been proved for the case of $m + n \leq k$. Consider the case of $m + n = k + 1$. Then either $m \geq 2$ or $n \geq 2$. We first verify the case when $m \geq 2$ and $n \geq 2$. Write $a = a_1 \otimes a'$ with $a' = a_2 \otimes \cdots \otimes a_m \in A^{q(m-1)}$ and $b = b_1 \otimes b'$ with $b' = b_2 \otimes \cdots \otimes b_n \in A^{q(n-1)}$. Then the left hand side of Eq. (18) is

$$
\Delta_T(a \circ \otimes b) = \Delta_T((a_1 \otimes a') \circ (b_1 \otimes b'))
$$

$$
= \Delta_T((a_1 \circ_r P_r(a')) \circ_r (b_1 \circ_r P_r(b')))
$$

$$
= \Delta_T((a_1 b_1) \circ_r (P_r(a') \circ_r P_r(b'))) \quad \text{(by the commutativity of $\circ_r$)}
$$

$$
= \Delta_T(a_1 b_1) \circ_r (P_r(a' - P_r(a') \circ_r b') + P_r(a') \circ_r b' - P_r(a' \circ_r b')) \quad \text{(by Eq. (1))}
$$

$$
= \Delta_A(a_1 b_1) \bullet ((id \otimes P_r)\Delta_T(a' \circ_r (1_A \otimes b')) + (id \otimes P_r)\Delta_T((1_A \otimes a') \circ_r b'))
$$

$$
- (id \otimes P_r)((id \otimes P_r)\Delta_T(a' \circ_r b')) \quad \text{(by Eqs. (17) and (12))}
$$

$$
= \Delta_A(a_1 b_1) \bullet ((id \otimes P_r)\Delta_T(a' \circ_r (1_A \otimes b')) + (id \otimes P_r)\Delta_T(1_A \otimes a') \bullet \Delta_T(b'))
$$

$$
- (id \otimes P_r)((id \otimes P_r)(\Delta_T(a') \bullet \Delta_T(b')))) \quad \text{(by the induction hypothesis)}
$$
\[
\Delta_T(a) \bullet \Delta_T(b) = \Delta_T(a_1 \otimes a') \bullet \Delta_T(b_1 \otimes b')
\]
\[
= \left( \Delta_T(a_1) \bullet (\text{id} \otimes P_r)\Delta_T(a') \right) \bullet \left( (\Delta_T(a_1) \bullet (\text{id} \otimes P_r)\Delta_T(b') \right)
\]
\[
= \Delta_T(a_1 b_1) \bullet \left( (\text{id} \otimes P_r)\Delta_T(a') \bullet (\text{id} \otimes P_r)\Delta_T(b') \right)
\]
\[
= \Delta_T(a_1 b_1) \bullet \left( (\text{id} \otimes P_r)(\Delta_T(a') \bullet \Delta_T(b')) \right)
\]
\[
+ (\text{id} \otimes P_r)(\Delta_T(a') \bullet \Delta_T(b'))
\]
\[
= \Delta_T(a_1 b_1) \bullet \left( (\text{id} \otimes P_r)(\Delta_T(a') \bullet \Delta_T(b')) \right)
\]
(by Eq. (13)).

The right hand side of Eq. (18) is
\[
\Delta_T(a) \bullet \Delta_T(b) = \Delta_T(a_1 b_1) \bullet \Delta_T(b_1 b')
\]
\[
= \Delta_T(a_1 b_1) \bullet \left( (\text{id} \otimes P_r)\Delta_T(a') \bullet (\text{id} \otimes P_r)\Delta_T(b') \right)
\]
(by the commutativity of \bullet).

Thus \( \Delta_T(a \circ \ b) = \Delta_T(a) \bullet \Delta_T(b) \). The verification of the other cases, namely when one of \( m \) or \( n \) is one, is simpler. This completes the induction. \( \square \)

Now we prove that the counit \( \varepsilon_T : \text{Nil}(A) \to k \) defined in Eq. (15) is an algebra homomorphism.

**Proposition 3.4.** The counit \( \varepsilon_T \) is an algebra homomorphism.

**Proof.** It suffices to prove that
\[
\varepsilon_T(a \circ_r b) = \varepsilon_T(a)\varepsilon_T(b)
\]
for any pure tensors \( a := a_1 \otimes a_2 \otimes \cdots \otimes a_m \in A^{\otimes m} \) and \( b := b_1 \otimes b_2 \otimes \cdots \otimes b_n \in A^{\otimes n} \) with \( m, n \geq 1 \).

We proceed with induction on \( m + n \geq 2 \). If \( m = n = 1 \), then by Eqs. (9) and (15), we obtain
\[
\varepsilon_T(a \circ_r b) = \varepsilon_A(ab) = \varepsilon_A(a)\varepsilon_A(b) = \varepsilon_T(a)\varepsilon_T(b).
\]

We proceed by induction on \( m + n \geq 2 \). Assume that Eq. (19) holds for \( m + n \leq k \) and consider \( m + n = k + 1 \). Then either \( m \geq 2 \) or \( n \geq 2 \). We will only verify the case when both \( m \geq 2 \) and \( n \geq 2 \) since the other cases are simpler. By Eq. (15), we have
\[
\varepsilon_T(a_1 \otimes a') = \varepsilon_T(a_1)\varepsilon_T(a') \quad \text{or} \quad \varepsilon_T(P_r(a)) = \varepsilon_T(a)\varepsilon_T(P_r(a')).
\]

Especially, \( \varepsilon_T(P_r(a)) = \varepsilon_T(a) \) since \( \varepsilon_A(1_A) = 1 \). Then
\[
\varepsilon_T(a \circ_r b)
\]
\[
= \varepsilon_T((a_1 \otimes a') \circ_r (b_1 \otimes b'))
\]
\[
= \varepsilon_T((a_1 \circ_r P_r(a') \circ_r (b_1 \circ_r P_r(b')) \quad \text{(by Eq. (11))}
\]
\[
= \varepsilon_T((a_1 b_1) \circ_r P_r(a') \circ_r P_r(b') + P_r(a') \circ_r b' - P_r(a' \circ_r b')) \quad \text{(by Eq. (1))}
\]
\[
= \varepsilon_T((a_1 b_1) \circ_r (a' \circ_r P_r(b')) + P_r(a') \circ_r b' - P_r(a' \circ_r b') \quad \text{(by Eq. (1))}
\]
\[
= \varepsilon_T(a_1 b_1) \left( \varepsilon_T(a')\varepsilon_T(b') + \varepsilon_T(a')\varepsilon_T(b') - \varepsilon_T(a')\varepsilon_T(b') \right) \quad \text{(by Eq. (15))}
\]
\[
= \varepsilon_T(a_1 b_1) \left( \varepsilon_T(a')\varepsilon_T(b') + \varepsilon_T(a')\varepsilon_T(b') - \varepsilon_T(a')\varepsilon_T(b') \right) \quad \text{(by the induction hypothesis and Eq. (20))}
\]

(by the induction hypothesis and Eq. (20)).
\[ \varepsilon_T(a_1)\varepsilon_T(b_1)\varepsilon_T(a')\varepsilon_T(b') \]
\[ = \varepsilon_T(a_1 \otimes a')\varepsilon_T(b_1 \otimes b') \quad \text{(by Eq. (21))} \]
\[ = \varepsilon_T(a)\varepsilon_T(b). \]

3.3. The coassociativity of \( \Delta_T \) and the left counicity of \( \varepsilon_T \). We will prove that \( \Delta_T \) is coassociative and \( \varepsilon_T \) satisfies the left counicity property.

**Proposition 3.5.** The comultiplication \( \Delta_T \) is coassociative, that is,
\[ (id \otimes \Delta_T)\Delta_T = (\Delta_T \otimes id)\Delta_T. \]

**Proof.** To prove Eq. (21), we only need to verify the equation
\[ (id \otimes \Delta_T)\Delta_T(a) = (\Delta_T \otimes id)\Delta_T(a) \]
for any pure tensor \( a := a_1 \otimes a' \in A^{\otimes n} \) with \( n \geq 1 \). For this we apply the induction on \( n \geq 1 \). If \( n = 1 \), then \( a = a_1 \) is in \( A \), so Eq. (22) follows from the coassociativity of \( \Delta_A \).

Let \( k \geq 1 \). Assume that Eq. (22) holds for \( a \in A^{\otimes k} \). Consider \( a = a_1 \otimes a' \in A^{\otimes(k+1)} \). Then we have
\[ (id \otimes \Delta_T)\Delta_T(a) = (id \otimes \Delta_T)(\Delta_T(a_1) \circ_r P_r(a')) \quad \text{(by Eq. (11))} \]
\[ = (id \otimes \Delta_T)(\Delta_T(a_1) \bullet \Delta_T(P_r(a'))) \quad \text{(by Proposition. 3.3)} \]
\[ = (id \otimes \Delta_T)(a_1) \bullet (id \otimes \Delta_T)(P_r(a')) \quad \text{(by Eq. (11))} \]
\[ = (id \otimes \Delta_T)(a_1) \bullet (id \otimes \Delta_T)(P_r \Delta_T(a')) \quad \text{(by the induction hypothesis).} \]

On the other hand,
\[ (\Delta_T \otimes id)\Delta_T(a) = (\Delta_T \otimes id)(\Delta_T(a_1) \circ_r P_r(a')) \quad \text{(by Eq. (11))} \]
\[ = (\Delta_T \otimes id)(\Delta_T(a_1) \bullet \Delta_T(P_r(a'))) \quad \text{(by Proposition. 3.3)} \]
\[ = (\Delta_T \otimes id)(a_1) \bullet (\Delta_T \otimes id)(P_r(a')) \quad \text{(by Eq. (11))} \]
\[ = (\Delta_T \otimes id)(a_1) \bullet (\Delta_T \otimes id)(P_r)\Delta_T(a') \quad \text{(by Eq. (11))} \]
\[ = (\Delta_T \otimes id)(a_1) \bullet (id \otimes \Delta_T)(a') \quad \text{(by Eq. (11))}. \]

Thus \( (id \otimes \Delta_T)\Delta_T(a) = (\Delta_T \otimes id)\Delta_T(a) \). This completes the induction. \[ \square \]

**Proposition 3.6.** The \( k \)-linear map \( \varepsilon_T \) satisfies the left counicity property:
\[ (\varepsilon_T \otimes id)\Delta_T = \beta_T, \]
where \( \beta_T : \overline{\Pi}^k(A) \to k \otimes \overline{\Pi}(A), a \mapsto 1 \otimes a \) for all \( a \in A^{\otimes k}, k \geq 1 \).

**Proof.** We only need to verify
\[ (\varepsilon_T \otimes id)\Delta_T(a) = \beta_T(a) \]
for any pure tensor \( a \in A^{\otimes k} \) with \( k \geq 1 \) for which we apply the induction on \( k \geq 1 \). If \( k = 1 \), then \( a \in A \), and so Eq. (24) follows from the left counicity of \( \varepsilon_A \).
Let $k \geq 1$. Assume that Eq. (24) has been proved for $a \in A^{\otimes k}$. Consider $a := a_1 \otimes a' \in A^{\otimes (k+1)}$. Then

$$
(\varepsilon_T \otimes \text{id})\Delta_T(a) = (\varepsilon_T \otimes \text{id})\Delta_T(a_1 \otimes a')
= (\varepsilon_T \otimes \text{id})\Delta_T(a_1 \circ_r P_r(a')) \quad \text{(by Eq. (10))}
= (\varepsilon_T \otimes \text{id})(\Delta_A(a_1) \bullet \Delta_T(P_r(a'))) \quad \text{(by Proposition 3.3)}
= (\varepsilon_T \otimes \text{id})\Delta_A(a_1)(\varepsilon_T \otimes \text{id})(\Delta_T(P_r(a'))) \quad \text{(by Proposition 3.4)}
= (\varepsilon_T \otimes \text{id})\Delta_A(a_1)(\varepsilon_T \otimes \text{id})(\Delta_T(a')) \quad \text{(by Eq. (11))}
= (\varepsilon_T \otimes \text{id})\Delta_A(a_1)(\varepsilon_T \otimes \text{id})\Delta_T(a')
= (\varepsilon_T \otimes \text{id})\Delta_A(a_1)(\varepsilon_T \otimes \text{id})\Delta_T(a')
\quad \text{(by the definition of }\beta_r)\text{)
= \beta_r(a_1)\beta_r(P_r(a')) \quad \text{(by the definition of }\beta_r)\text{)
= \beta_r(a_1 \circ_r P_r(a')) \quad \text{(by }\beta_r\text{ being an algebra isomorphism)}
= \beta_r(a).
$$

This completes the proof of Eq. (24). \hfill \Box

Since $\varepsilon_T$ is an extension of $\varepsilon_A$, when $\varepsilon_A$ does not satisfy the right counicity, nor will $\varepsilon_T$. Assume that $\varepsilon_A$ satisfies the right counicity, that is $A$ is a bialgebra. One can check that $\varepsilon_T$, as defined is left counital, but is not right counital. For example, take $a := a_1 \otimes a_2 \in A^{\otimes 2}$. Define the linear map $\beta_r : \Pi^T(A) \to \Pi(A) \otimes k, a \mapsto a \otimes 1$. Then

$$
(\text{id} \otimes \varepsilon_T)\Delta_T(a) = (\text{id} \otimes \varepsilon_T)(a_1 \otimes a_2)
= (\text{id} \otimes \varepsilon_T)\Delta_T(a_1 \circ_r P_r(a_2)) \quad \text{(by Eq. (10))}
= (\text{id} \otimes \varepsilon_T)(\Delta_A(a_1) \bullet \Delta_T(P_r(a_2))) \quad \text{(by Proposition 3.3)}
= (\text{id} \otimes \varepsilon_T)\Delta_A(a_1)(\text{id} \otimes \varepsilon_T)(\Delta_T(P_r(a_2))) \quad \text{(by Proposition 3.4)}
= (\text{id} \otimes \varepsilon_T)\Delta_A(a_1)(\text{id} \otimes \varepsilon_T)(\text{id} \otimes P_r)\Delta_A(a_2) \quad \text{(by Eq. (11))}
= (\text{id} \otimes \varepsilon_A)\Delta_A(a_1)(\text{id} \otimes \varepsilon_A)\Delta_A(a_2) \quad \text{(by Eq. (15))}
= \beta_r(a_1)\beta_r(a_2) \quad \text{(by the right counicity of }\varepsilon_A)\text{)
= \beta_r(a_1 \circ_r a_2) \quad \text{(by }\beta_r\text{ being an algebra isomorphism)}
= \beta_r(a_1 a_2)
\neq \beta_r(a).
$$

Define

$$
\mu_T : k \to \overrightarrow{\Pi}(A), \ c \mapsto c1_A, c \in k.
$$

Then $\mu_T$ is a unit for $(\overrightarrow{\Pi}(A), \circ_r)$. Now let us put all the pieces together to give the main result of this section.

**Theorem 3.7.** The six-tuple $(\overrightarrow{\Pi}(A), \circ_r, \mu_T, \Delta_T, \varepsilon_T, P_r)$ is a left counital cocycle bialgebra.

**Proof.** First according to Theorem 2.3, the triple $(\Pi(A), \circ_r, \mu_T, P_r)$ is a Nijenhuis algebra. Applying Proposition 3.3 and Proposition 3.6, the triple $(\Pi(A), \Delta_T, \varepsilon_T)$ is a left counital coalgebra. Then by Proposition 3.3 and Proposition 3.4, the quintuple $(\Pi(A), \circ_r, \mu_T, \Delta_T, \varepsilon_T)$ is a left counital bialgebra satisfying the one-cocycle property in Eq. (2), and hence the six-tuple $(\Pi(A), \circ_r, \mu_T, \Delta_T, \varepsilon_T, P_r)$ is a left counital cocycle bialgebra. \hfill \Box
4. The Left Counital Hopf Algebra Structure on Free Commutative Nijenhuis Algebras

In this section, we provide a left counital Hopf algebra structure on the free commutative Nijenhuis algebra $\mathcal{N}(A)$.

**Definition 4.1.** (a) A left counital bialgebra $(H, m, \mu, \Delta, \varepsilon)$ is called a **graded left counital bialgebra** if there are $k$-submodules $H^{(n)}$, $n \geq 0$, of $H$ such that

\[
H = \bigoplus_{n \geq 0} H^{(n)}, \quad H^{(p)} H^{(q)} \subseteq H^{(p+q)}, \quad \Delta(H^{(n)}) \subseteq (H^{(0)} \otimes H^{(n)}) \oplus \bigoplus_{p+q=n, p>0, q>0} H^{(p)} \otimes H^{(q)}
\]

(b) A graded left counital bialgebra is called **connected** if $H^{(0)} = \text{im} \mu (= k)$ and

\[
\ker \varepsilon = \bigoplus_{n \geq 1} H^{(n)}.
\]

Let $A$ be a $k$-algebra. Let $C$ be a left counital $k$-coalgebra. Denote $R := \text{Hom}(C, A)$. For $f, g \in R$, we can still define the **convolution product** of $f$ and $g$ by

\[
f \ast g := m_A(f \otimes g)\Delta_C.
\]

**Lemma 4.2.** Let $A := (A, m_A, \mu_A)$ be a $k$-algebra. Let $C := (C, \Delta_C, \varepsilon_C)$ be a left counital $k$-algebra. Let $e := \mu_A \varepsilon_C$. Then $e$ is a left unit of $k$-algebra $\text{Hom}(C, A)$ under the convolution product $\ast$.

**Proof.** Define a linear map $\alpha_\ell : k \otimes A \to A$, $k \otimes a \mapsto ka$ for all $k \in k$ and $a \in A$, and a linear map $\beta_\ell : C \to k \otimes C$, $c \mapsto 1_k \otimes c$ for all $c \in C$. Then $\alpha_\ell$ and $\beta_\ell$ are isomorphism. Since $\mu_A$ is the unit of $A$ and $\varepsilon_C$ is the left counit of $C$, we have $m_A(\mu_A \otimes \text{id}_A) = \alpha_\ell$ and $(\varepsilon_C \otimes \text{id}_C)\Delta_C = \beta_\ell$. For every $f \in \text{Hom}(C, A)$, we have

\[
e \ast f = m_A(\mu_A \varepsilon_C \otimes f)\Delta_C = m_A(\mu_A \otimes \text{id}_A)(\text{id}_k \otimes f)(\varepsilon_C \otimes \text{id}_C)\Delta_C = \alpha_\ell(\text{id}_k \otimes f)\beta_\ell = f.
\]

As usual, by the idempotency of $e$, the surjectivity of $e$ and the injectivity of $\mu$, we obtain

**Lemma 4.3.** Let $k$ be a field. Let $H := (H, m, \mu, \Delta, \varepsilon)$ be a connected graded left counital $k$-bialgebra. Then

\[
H = \text{im} \mu \oplus \ker \varepsilon.
\]

**Definition 4.4.** Let $H := (H, m, \mu, \Delta, \varepsilon)$ be a left counital $k$-bialgebra. Let $e := \mu \varepsilon$.

(a) A $k$-linear map $S$ of $H$ is called a **right antipode** for $H$ if

\[
\text{id}_H \ast S = e.
\]

(b) A left counital bialgebra with a right antipode is called a **left counital right antipode Hopf algebra** or simply a **left counital Hopf algebra**.

Since any Hopf algebra or any right Hopf algebra in the sense of [20] is a left counital (right antipode) Hopf algebra, there are plenty examples of left counital Hopf algebras.

**Proposition 4.5.** Let $k$ be a field. Let $H := (H, m, \mu, \Delta, \varepsilon)$ be a connected graded left counital $k$-bialgebra. Then for any $x \in H^{(n)}$, $n \geq 1$,

\[
\Delta(x) = 1 \otimes x + \tilde{\Delta}(x), \quad \tilde{\Delta}(x) \in \ker \varepsilon \otimes \ker \varepsilon.
\]
Proof. According to $\Delta(H^{(n)}) \subseteq (H^{(0)} \otimes H^{(n)}) \oplus \bigoplus_{p+q=n} H^{(p)} \otimes H^{(q)}$ and $H^{(0)} = \mathbf{k}$, we get

$$\Delta(x) = 1 \otimes u + \tilde{\Delta}(x),$$

where $u \in H^{(n)}$ and $\tilde{\Delta}(x) \in \bigoplus_{p+q=n} H^{(p)} \otimes H^{(q)}$. Then by Eq. (26), $\tilde{\Delta}(x) \in \ker \varepsilon \otimes \ker \varepsilon$. By the left counit property, we obtain

$$x = \beta^{-1}_\ell(\beta_\ell(x)) = \beta^{-1}_\ell(\varepsilon \otimes I)\Delta(x) = \beta^{-1}_\ell(\varepsilon \otimes I)(1 \otimes u + \tilde{\Delta}(x)) = u.$$

Thus,

$$\Delta(x) = 1 \otimes x + \tilde{\Delta}(x).$$

□

**Theorem 4.6.** Let $\mathbf{k}$ be a field. Then a connected graded left counital $\mathbf{k}$-bialgebra is a left counital Hopf algebra. Its antipode $S$ is defined by the recursion

$$S(1_H) = 1_H, \quad S(x) = - \sum_{(x)} x' S(x''), \quad x \in \ker \varepsilon,$$

using Sweedler’s notation $\tilde{\Delta}(x) = \sum_{(x)} x' \otimes x''$.

**Proof.** The proof is the same as the proof of the fact that a connected graded bialgebra is a Hopf algebra [23, 30], by checking directly that the map $S$ defined by Eq. (30) satisfies

$$(\text{id}_H \ast S)(x) = e(x)$$

for $x = 1_H$ and $x \in \ker \varepsilon$. □

**Theorem 4.7.** Let $A = \bigoplus_{n \geq 0} A^{(n)}$ be a connected graded left counital bialgebra. Then the bialgebra $\overrightarrow{\Pi}(A)$ is a left counital Hopf algebra.

**Proof.** By Theorem 4.6, it suffices to show that $\overrightarrow{\Pi}(A)$ is a connected graded left counital bialgebra. For any element $0 \neq a \in A$, we define the degree of $a$ by defining

$$\deg(a) := k, \quad \text{if } a \in A^{(k)}, \quad k \geq 0.$$

In general, for any pure tensor $0 \neq a := a_1 \otimes a_2 \otimes \cdots \otimes a_m \in A^{\otimes m}$, we define

$$\deg(a) := \deg(a_1) + \deg(a_2) + \cdots + \deg(a_m) + m - 1.$$

Then for $m \geq 2$, taking $a = a_1 \otimes a'$ with $a' := a_2 \otimes \cdots \otimes a_m$, we have

$$\deg(a) = \deg(a_1) + \deg(a') + 1.$$

Let $\mathcal{U} := \overrightarrow{\Pi}(A)$. We denote the linear span of $\{a \in \mathcal{U} \mid \deg(a) = k\}$ by $\mathcal{U}^{(k)}$. Then we get $\mathcal{U}^{(0)} = A^{(0)}$, and thus $\mathcal{U}^{(0)} = \mathbf{k}$. Furthermore, we have

$$A^{(n)} \subseteq \mathcal{U}^{(n)}, \quad \mathcal{U}^{(i)} \cap \mathcal{U}^{(j)} = \{0\}, \quad i \neq j, \quad \text{and so } \mathcal{U} = \bigoplus_{n \geq 0} \mathcal{U}^{(n)}.$$

Let $a = a_1 \otimes a' \in A^{\otimes m}$. Then by Eq. (33),

$$\deg(a) = n \Rightarrow \deg(a') = n - \deg(a_1) - 1.$$

and

$$\deg(a') = n \Rightarrow \deg(a) = \deg(a_1) + n + 1.$$

To prove that

$$\mathcal{U}^{(p)} \circ \mathcal{U}^{(q)} \subseteq \mathcal{U}^{(p+q)}, \quad \text{for all } p, q \geq 0,$$

we have

$$(\mathcal{U}^{(p)} \circ \mathcal{U}^{(q)}) \subseteq \mathcal{U}^{(p+q)}.$$
it suffices to prove
\begin{equation}
\deg(a \diamond b) = \deg(a) + \deg(b), \quad \text{for all } a \in \mathcal{U}^{(p)}, b \in \mathcal{U}^{(q)}.
\end{equation}

Use induction on \( p + q \geq 0 \). If \( p = q = 0 \), then \( \mathcal{U}^{(p)} = \mathcal{U}^{(q)} = \mathcal{U}^{(0)} = k \), and thus \( a \diamond b \in k \). Then \( \deg(a \diamond b) = 0 = \deg(a) + \deg(b) \). Assume that Eq. 37 has been proved for \( p + q \leq k \). Consider \( p + q = k + 1 \). If either \( p = 0 \) or \( q = 0 \), then \( \mathcal{U}^{(p)} = k \) or \( \mathcal{U}^{(q)} = k \). Then by the definition of \( \diamond \), either \( a \diamond b = b \) or \( a \diamond b = a \). So Eq. 37 holds. Thus we assume \( p, q \geq 1 \). Let \( a \in \mathcal{U}^{(p)} \) and \( b \in \mathcal{U}^{(q)} \). If \( a, b \in A \), then \( a \diamond b = ab \), and so
\[
\deg(a \diamond b) = \deg(ab) = \deg(a) + \deg(b).
\]

If \( a \in A \) and \( b := b_1 \otimes b' \in A^{\otimes \ell}, \ell \geq 1 \), then \( a \diamond b = ab_1 \otimes b' \). By Eq. 37, we get
\[
\deg(a \diamond b) = \deg(ab_1 \otimes b) = \deg(ab_1) + \deg(b') + \ell - 1 = \deg(a) + \deg(b_1) + \deg(b') + \ell - 1 = \deg(a) + \deg(b).
\]

Similarly, Eq. 37 holds for \( a \in A^{\otimes m}, m \geq 2 \), and \( b \in A \). We now consider pure tensors \( a := a_1 \otimes a' \in A^{\otimes \ell} \) and \( b := b_1 \otimes b' \in A^{\otimes m} \) with \( \ell, m \geq 2 \). Then
\[
a \diamond b = (a_1 \otimes a') \diamond (b_1 \otimes b')
\]
\[
= (a_1 b_1) \otimes (a' \diamond (P_r(a') + P_r(b') + P_r(a') \diamond b' - P_r(a' \diamond b'))
\]
\[
= (a_1 b_1) \otimes (a' \diamond P_r(b') + P_r(a') \diamond b' - P_r(a' \diamond b')).
\]

By Eq. 37, we obtain
\[
\deg(a') = p - \deg(a_1) - 1 \quad \text{and} \quad \deg(b') = q - \deg(b_1) - 1,
\]
and then by Eq. 37,
\[
\deg(P_r(a')) = \deg(1_A \otimes a') = p - \deg(a_1) \quad \text{and} \quad \deg(P_r(b')) = \deg(1_A \otimes b') = q - \deg(b_1).
\]

By the induction hypothesis, we get
\[
\deg(a' \diamond P_r(b')) = \deg(P_r(a') \diamond b') = \deg(P_r(a' \diamond b')) = p + q - \deg(a_1) - \deg(b_1) - 1.
\]

Thus by Eq. 37,
\[
\deg(a \diamond b) = \deg(a_1 b_1) + \deg((a' \diamond P_r(b') + P_r(a') \diamond b' - P_r(a' \diamond b')) + 1
\]
\[
= \deg(a_1) + \deg(b_1) + p + q - \deg(a_1) - \deg(b_1) - 1 + 1
\]
\[
= p + q
\]
\[
= \deg(a) + \deg(b).
\]

This completes the induction.

Finally, we prove
\begin{equation}
\Delta_T(\mathcal{U}^{(n)}) \subseteq (\mathcal{U}^{(0)} \otimes \mathcal{U}^{(n)}) \oplus \left( \bigoplus_{\substack{p+q=n \atop p \geq 0, q \geq 0}} \mathcal{U}^{(p)} \otimes \mathcal{U}^{(q)} \right) \text{ for all } p, q, n \geq 0,
\end{equation}
by induction on \( n \geq 0 \). If \( n = 0 \), then \( \mathcal{U}^{(0)} = k \), and so \( \Delta_T(\mathcal{U}^{(0)}) = \Delta_A(k) = k \otimes k = \mathcal{U}^{(0)} \otimes \mathcal{U}^{(0)} \). Assume that Eq. (38) has been proved for \( n \geq 0 \). Consider \( a \in \mathcal{U}^{(n+1)} \). If \( a \in \mathcal{A}(= \bigoplus_{k \geq 0} A^{(k)}) \), then we get \( a \in A^{(n+1)} \). Since \( A \) is a connected graded left counital bialgebra, we have

\[
\Delta_T(a) = \Delta_A(a) \subseteq (A^{(0)} \otimes A^{(n+1)}) \oplus \left( \bigoplus_{p+q=n+1 \atop p>0, q>0} A^{(p)} \otimes A^{(q)} \right) \subseteq (\mathcal{U}^{(0)} \otimes \mathcal{U}^{(n+1)}) \oplus \left( \bigoplus_{p+q=n+1 \atop p>0, q>0} \mathcal{U}^{(p)} \otimes \mathcal{U}^{(q)} \right).
\]

Let \( a = a_1 \otimes a' \in A^{\otimes (\ell+1)} \) with \( \ell \geq 1 \). Then

\[
\Delta_T(a) = \Delta_T(a_1 \circ P_r(a')) \quad \text{(by Eq. (10))}
\]

\[
= \Delta_T(a_1) \bullet \Delta_T(P_r(a')) \quad \text{(by Proposition 3.3)}
\]

\[
= \Delta_T(a_1) \bullet (\text{id} \otimes P_r) \Delta_T(a') \quad \text{(by Proposition 1)}
\]

\[
= \Delta_T(a_1) \bullet (\text{id} \otimes P_r) \Delta_T(a').
\]

By Eq. (34), we get \( a' \in \mathcal{U}^{(n-\deg(a_1))} \). Then by the induction hypothesis,

\[
\Delta_T(a') \subseteq (\mathcal{U}^{(0)} \otimes \mathcal{U}^{(n-\deg(a_1))}) \oplus \left( \bigoplus_{p_1+q_1=n-\deg(a_1) \atop p_1>0, q_1>0} \mathcal{U}^{(p_1)} \otimes \mathcal{U}^{(q_1)} \right).
\]

Thus

\[
\Delta_T(a) = \Delta_A(a_1) \bullet (\text{id} \otimes P_r) \Delta_T(a')
\]

\[
\subseteq \left( \left( A^{(0)} \otimes A^{(\deg(a_1))} \right) \oplus \left( \bigoplus_{p_1+q_1=\deg(a_1) \atop p_1>0, q_1>0} A^{(p_1)} \otimes A^{(q_1)} \right) \right)
\]

\[
\bullet (\text{id} \otimes P_r) \left( \left( \mathcal{U}^{(0)} \otimes \mathcal{U}^{(n-\deg(a_1))} \oplus \left( \bigoplus_{p_2+q_2=n-\deg(a_1) \atop p_2>0, q_2>0} \mathcal{U}^{(p_2)} \otimes \mathcal{U}^{(q_2)} \right) \right) \right)
\]

\[
\subseteq \left( \left( \mathcal{U}^{(0)} \otimes \mathcal{U}^{(\deg(a_1))} \right) \oplus \left( \bigoplus_{p_1+q_1=\deg(a_1) \atop p_1>0, q_1>0} A^{(p_1)} \otimes A^{(q_1)} \right) \right)
\]

\[
\bullet \left( \left( \mathcal{U}^{(0)} \otimes \mathcal{U}^{(n-\deg(a_1)+1)} \oplus \left( \bigoplus_{p_2+q_2=n-\deg(a_1) \atop p_2>0, q_2>0} \mathcal{U}^{(p_2)} \otimes \mathcal{U}^{(q_2+1)} \right) \right) \right)
\]

\[
\subseteq \left( \left( \mathcal{U}^{(0)} \otimes \mathcal{U}^{(n+1)} \right) \oplus \left( \bigoplus_{p_2+q_2=n-\deg(a_1) \atop p_2>0, q_2>0} \mathcal{U}^{(p_2)} \otimes \mathcal{U}^{(q_2+1+\deg(a_1))} \right) \right)
\]

\[
\oplus \left( \bigoplus_{p_1+q_1=\deg(a_1) \atop p_1>0, q_1>0} \mathcal{U}^{(p_1)} \otimes \mathcal{U}^{(q_1+n-\deg(a_1)+1)} \right)
\]

\[
\bullet \left( \left( \bigoplus_{p_2+q_2=n-\deg(a_1) \atop p_2>0, q_2>0} \mathcal{U}^{(p_2)} \otimes \mathcal{U}^{(q_2+1)} \right) \right)
\]
\[ \subseteq \left( \mathcal{U}^{(0)} \otimes \mathcal{U}^{(n+1)} \right) \oplus \left( \bigoplus_{p_1+k_1=\deg(a_1)} \mathcal{U}^{(p_1+p_2)} \otimes \mathcal{U}^{(k_1+k_2+1)} \right) \]

\[ \subseteq \left( \mathcal{U}^{(0)} \otimes \mathcal{U}^{(n+1)} \right) \oplus \left( \bigoplus_{p+q=n+1 \quad p>0, q>0} \mathcal{U}^{(p)} \otimes \mathcal{U}^{(q)} \right) \quad (p := p_1 + p_2, q := q_1 + q_2 + 1), \]

as needed. This completes the proof. \(\square\)

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