Distribution of the zeros of the Riemann zeta function in longer intervals

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Abstract

In this paper, we extend the result of Fujii on the second moment of $S(t + h) - S(t)$ to longer range of $h$ under the Riemann Hypothesis and an quantitative form of the Twin Prime Conjecture.

1 Introduction

Throughout this article, we shall assume the Riemann Hypothesis RH of the Riemann zeta function $\zeta(s)$. Let

$$S(t) = \frac{1}{\pi} \arg \zeta\left(\frac{1}{2} + it\right),$$

$$F(x, T) = \sum_{0 < \gamma, \gamma' \leq T} x^{i(\gamma - \gamma')} w(\gamma - \gamma')$$

with $w(u) = \frac{4}{4 + u^2}$.

Here, $\gamma$ and $\gamma'$ run over the imaginary parts of the non-trivial zeros of $\zeta(s)$. In [4], Fujii applied Goldston’s result [6] to obtain, under RH

$$\int_0^T (S(t + h) - S(t))^2 \, dt = \frac{T}{\pi} \left[ \int_0^{h \log (T/2\pi)} \frac{1 - \cos \alpha}{\alpha} \, d\alpha ight]$$

$$+ \int_1^\infty \frac{F(\alpha)}{\alpha^2} \left( 1 - \cos (\alpha h \log \frac{T}{2\pi}) \right) \, d\alpha + o(T)$$

where

$$F(\alpha) = F(\alpha, T) := \left( \frac{T}{2\pi \log \frac{T}{2\pi}} \right)^{-1} F\left( \frac{T}{2\pi}, T \right),$$

and $0 < h = o(1)$.

To extend the above result to a longer range $0 < h = O(1)$, one needs to improve the error term. The first step in this direction was accomplished in the author’s [2] which improves the error term of the second moment of $S(t)$ to $O(T/\log^2 T)$ under an quantitative form of the Twin Prime Conjecture TPC (see next section). We shall use the more precise estimates in [2] and [3] to prove
Theorem 1.1. Assume RH and TPC. Fix a large number $A$.

\[
T \int_0^T (S(t + h) - S(t))^2 \, dt = \frac{T}{\pi^2} \int_0^{hL} \frac{1 - \cos \alpha}{\alpha^2} \, d\alpha + \frac{T}{\pi^2} \int_1^\infty \frac{F(\alpha) - F_h(\alpha)}{\alpha^2} \, d\alpha \\
+ \frac{T}{\pi^2} \left[ \log \log 2 + C_0 - \sum_{m=2}^{\infty} \sum_{p} \frac{1}{mp^m} \right] (1 - \cos (h \log 2)) - \frac{T}{\pi^2} \int_0^{h \log 2} \frac{1 - \cos \alpha}{\alpha^2} \, d\alpha \\
- \frac{T}{\pi^2} \int_2^\infty \frac{r(u) \sin (h \log u)}{u} \, du + \frac{T}{\pi^2} \sum_{m=2}^{\infty} \sum_{p} \frac{1 - \cos (hm \log p)}{m^2 p^m} \\
+ \frac{T}{\pi^2} \sin (hL) + \frac{T}{\pi^2 L^2} \frac{h^2 (20 + 3h^2)}{4(4 + h^2)^2} - \frac{3T}{2\pi^2 L^2} \int_1^\infty \frac{F(\alpha) - F_h(\alpha)}{\alpha^4} \, d\alpha + O\left( \frac{T}{L^2} \right)
\]

for $0 < h \leq A$. The implicit constant in the error term may depend on $A$. $F_h(\alpha)$ and $r(u)$ are given by (1) and (6) respectively. $C_0$ is Euler’s constant.

This prompts us to study

\[
\int_1^\infty \frac{F(\alpha) - F_h(\alpha)}{\alpha^2} \, d\alpha \quad \text{and} \quad \int_1^\infty \frac{F(\alpha) - F_h(\alpha)}{\alpha^4} \, d\alpha.
\]

We have

Theorem 1.2. Assume RH. For $T$ sufficiently large,

\[
0 \leq \int_1^\infty \frac{F(\alpha) - F_h(\alpha)}{\alpha^2} \, d\alpha < 9,
\]

and

\[
0 \leq \int_1^\infty \frac{F(\alpha) - F_h(\alpha)}{\alpha^4} \, d\alpha < 6.
\]

The author would like to thank Prof. Daniel Goldston for suggesting this problem. Here and throughout this paper, $p$ will denote a prime number. $\Lambda(n)$ is von Mangoldt’s lambda function. Also, we have $L = \log \frac{T}{2\pi}$.

2 Preparations

We shall use a strong quantitative form of the Twin Prime Conjecture (abbreviated as TPC): For any $\epsilon > 0$,

\[
\sum_{n=1}^{N} \Lambda(n) \Lambda(n + d) = \mathcal{G}(d)N + O(N^{1/2 + \epsilon}) \quad \text{uniformly in} \quad |d| \leq N.
\]

\[
\mathcal{G}(d) = 2 \prod_{p>2} \left(1 - \frac{1}{(p-1)p}\right) \prod_{p|d, p>2} \frac{p-1}{p-2} \quad \text{if} \quad d \text{ is even}, \quad \text{and} \quad \mathcal{G}(d) = 0 \quad \text{if} \quad d \text{ is odd}.
\]
Also, we need to generalize $F(x, T)$ to

$$F_h(x, T) = \sum_{0 < \gamma, \gamma' \leq T} \cos \left( (\gamma - \gamma' - h) \log x \right) w(\gamma - \gamma' - h).$$  \hspace{1cm} (1)

Note that $F_0(x, T) = F(x, T)$. From \( \text{(3)} \),

$$F_h(x, T) = \frac{T}{2\pi} \left[ \frac{4 \cos (h \log x)}{4 + h^2} \log x - \frac{8h \sin (h \log x)}{(4 + h^2)^2} \right] + O\left( \frac{T}{x^{1/2-\epsilon}} \right)$$

for $1 \leq x \leq \frac{T}{\log T}$ under RH, and

$$F_h(x, T) = \frac{T}{2\pi} \left[ \frac{4 \cos (h \log x)}{4 + h^2} \log x - \frac{8h \sin (h \log x)}{(4 + h^2)^2} \right] + O(1)$$

for $\frac{T}{\log^2 T} \leq x \leq T$ under RH and TPC. Define

$$F_h(\alpha) = F_h(\alpha, T) := \left( \frac{TL}{2\pi} \right)^{-1} F_h \left( \frac{T}{2\pi} \alpha, T \right)$$

for $\alpha \geq 0$ and $F_h(-\alpha) = F_h(\alpha)$. So, \( \text{(2)} \) and \( \text{(3)} \) can be summarized as

$$F_h(\alpha) = \begin{cases} 
\frac{4 \cos (h L \alpha)}{4 + h^2} \alpha - \frac{8h \sin (h L \alpha)}{(4 + h^2)^2} L^{-1} \\
+ \left( \frac{T}{2\pi} \right)^{-2} \left[ (\log \frac{T}{2\pi})^2 - 2 \log \frac{T}{2\pi} \right] + O(x \log x) + O\left( \frac{T}{x^{1/2-\epsilon}} \right) 
\end{cases}$$

for $0 \leq \alpha \leq 1 - \frac{3 \log \log T}{\log T}$,

$$F_h(\alpha) = \begin{cases} 
\frac{4 \cos (h L \alpha)}{4 + h^2} \alpha + O(T^{-1/2-\epsilon} L^{-1}) 
\end{cases}$$

for $1 - \frac{3 \log \log T}{\log T} \leq \alpha \leq 1$.  \hspace{1cm} (4)

Recall from \( \text{(5)} \),

$$k(u) = \begin{cases} 
\left( \frac{1}{2u^2} - \frac{\pi^2}{2} \cot \left( \frac{\pi^2 u}{2} \right) \right)^2, & \text{if } |u| \leq \frac{1}{2\pi}, \\
\frac{1}{2u^2}, & \text{if } |u| > \frac{1}{2\pi},
\end{cases}$$

and $\hat{k}(u)$ denotes the Fourier transform of $k(u)$. One can easily check that $k''(u)$ is bounded, piecewise continuous and $\ll u^{-4}$ when $u > \frac{1}{2\pi}$. Also,

$$k(0) = 0, k \left( \frac{1}{2\pi} \right) = \pi^2; k'(0) = 0, k' \left( \frac{1}{2\pi} \right) = 3\pi^2 - 4\pi^3.$$  \hspace{1cm} (5)

We need the following lemmas.

**Lemma 2.1.** Let $x = (T/2\pi)^{\beta}$ with $\beta > 0$,

$$\sum_{0 < \gamma, \gamma' \leq T} \hat{k}((\gamma - \gamma') \log x) = \frac{TL}{4\pi^2 \beta} \int_{-\infty}^{\infty} F(\alpha) k \left( \frac{\alpha}{2\pi \beta} \right) d\alpha + \frac{\pi^2 T}{16L} \frac{F(\beta)}{\beta^2}$$

$$- \frac{T}{64\pi^4 L \beta^3} \int_{-\infty}^{\infty} F(\alpha) k'' \left( \frac{\alpha}{2\pi \beta} \right) d\alpha.$$
Proof: This is essentially Lemma 2.6 of [2].

**Lemma 2.2.** Let \( x = (T/2\pi)^\beta \) with \( \beta > 0 \),

\[
\sum_{0 < \gamma, \gamma' \leq T} k((\gamma - \gamma' - h) \log x) = \frac{TL}{4\pi^2 \beta} \int_{-\infty}^{\infty} F_h(\alpha)k\left(\frac{\alpha}{2\pi \beta}\right) d\alpha + \frac{\pi^2 T F_h(\beta)}{16L \beta^2} - \frac{T}{64\pi^4 L \beta^3} \int_{-\infty}^{\infty} F_h(\alpha)k''\left(\frac{\alpha}{2\pi \beta}\right) d\alpha.
\]

Proof: This is just very similar to Lemma 2.1 above. We use the fact that \( k(u) \) is even.

**Lemma 2.3.**

\[
\int_{0}^{\beta^-} \sin (hL\alpha)k''\left(\frac{\alpha}{2\pi \beta}\right) d\alpha = 2(\pi^6 - 4\pi^4)\beta \sin (hL\beta) - 4\pi^4 hL\beta^2 \cos (hL\beta)
\]

\[
- (2\pi hL\beta)^2 \int_{0}^{\beta} \sin (hL\alpha)k\left(\frac{\alpha}{2\pi \beta}\right) d\alpha.
\]

Proof: Use integration by parts twice and (5).

**Lemma 2.4.**

\[
\int_{0}^{\beta^-} \alpha \cos (hL\alpha)k''\left(\frac{\alpha}{2\pi \beta}\right) d\alpha
\]

\[
= 2(\pi^6 - 6\pi^4)\beta^2 \cos (hL\beta) + 4\pi^4 hL\beta^3 \sin (hL\beta)
\]

\[
- 8\pi^2 hL\beta^2 \int_{0}^{\beta} \sin (hL\alpha)k\left(\frac{\alpha}{2\pi \beta}\right) d\alpha - (2\pi hL\beta)^2 \int_{0}^{\beta} \cos (hL\alpha)k\left(\frac{\alpha}{2\pi \beta}\right) d\alpha.
\]

Proof: Use integration by parts twice and (5) again.

**Lemma 2.5.**

\[
\int_{0}^{\beta^-} \alpha^2 k''\left(\frac{\alpha}{2\pi \beta}\right) d\alpha = 2(\pi^6 - 6\pi^4)\beta^2.
\]

Proof: Set \( h = 0 \) in Lemma 2.4.

**Lemma 2.6.**

\[
\int_{\beta}^{1} \frac{\cos (hL\alpha)}{\alpha^3} d\alpha = \frac{\cos (hL\beta)}{2\beta^2} - \frac{\cos (hL)_2}{2} - \frac{hL \sin (hL\beta)}{2\beta}
\]

\[
+ \frac{hL \sin (hL)}{2} - \frac{(hL)^2}{2} \int_{\beta}^{1} \frac{\cos (hL\alpha)}{\alpha} d\alpha.
\]

Proof: Use integration by parts twice.
Lemma 2.7.
\[
\int_\beta^1 \sin (hL\alpha) \frac{d\alpha}{\alpha^4} = \frac{\sin (hL/3)}{3\beta^3} - \frac{\sin (hL)}{3} + \frac{hL \cos (hL/3)}{6\beta^2} - \frac{hL \cos (hL)}{6} - \frac{(hL)^2 \sin (hL/3)}{6\beta} + \frac{(hL)^2 \sin (hL)}{6} - \frac{(hL)^3}{6} \int_\beta^1 \cos (hL\alpha) \frac{d\alpha}{\alpha}.
\]

Proof: Use integration by parts thrice.

3 \(S_4\) and \(S_5\)

We shall follow Fujii \[4\] closely. Let \(x = (T/2\pi)^{\beta}\) with \(0 < \beta < 1\). By Goldston’s explicit formula of \(S(t)\) in \[6\] under RH, Fujii got (see p. 76 & 77 of \[4\])
\[
\int_0^T (S(t + h) - S(t))^2 \, dt = S_3 + S_4 + S_5 + O\left(\frac{x \log \log x}{\log x}\right) + O(\log^3 T)
\]

where
\[
S_3 = \frac{2}{\pi^2 \log x} \sum_{0 < \gamma, \gamma' \leq T} \hat{k}((\gamma - \gamma') \log x) - \hat{k}((\gamma - \gamma' - h) \log x),
\]
\[
S_4 = \frac{T}{\pi^2} \sum_{p \leq x} \cos (h \log p) - 1 \left( f^2 \left( \frac{\log p}{\log x} \right) - 2 f \left( \frac{\log p}{\log x} \right) \right),
\]
\[
S_5 = \frac{T}{\pi^2} \sum_{m = 2}^{\infty} \sum_{p^m \leq x} \cos (hm \log p) - 1 \left( f^2 \left( \frac{m \log p}{\log x} \right) - 2 f \left( \frac{m \log p}{\log x} \right) \right).
\]

Here \(f(u) = \frac{u}{\pi} \cot \left( \frac{u}{\pi} \right)\) and \(\hat{k}(u)\) is defined as in the previous section. We note that with Euler’s constant \(C_0\),
\[
\sum_{p \leq u} \frac{1}{p} = \log \log u + C_0 + \sum_p \left( \log \left( 1 - \frac{1}{p} \right) + \frac{1}{p} \right) + r(u),
\]
\[
r(u) \ll \frac{\log u}{\sqrt{u}} \tag{6}
\]
under RH. Consider
\[
\Sigma_1 = \sum_{p \leq x} \cos (h \log p) - 1 \frac{1}{p} f^2 \left( \frac{\log p}{\log x} \right), \quad \Sigma_2 = \sum_{p \leq x} \cos (h \log p) - 1 \frac{1}{p} f \left( \frac{\log p}{\log x} \right).
\]

By partial summation,
\[
\Sigma_1 = \int_2^x \frac{\cos (h \log u) - 1}{u \log u} f^2 \left( \frac{\log u}{\log x} \right) du + \int_2^x (\cos (h \log u) - 1) f^2 \left( \frac{\log u}{\log x} \right) du \tag{\ref{eq:partial_summation}}
\]
\[
= \Sigma_{1,1} + \Sigma_{1,2},
\]
\[
\Sigma_2 = \int_2^x \frac{\cos (h \log u) - 1}{u \log u} f \left( \frac{\log u}{\log x} \right) du + \int_2^x (\cos (h \log u) - 1) f \left( \frac{\log u}{\log x} \right) du \tag{\ref{eq:partial_summation}}
\]
\[
= \Sigma_{2,1} + \Sigma_{2,2}.
\]
Then
\[ \Sigma_1 - 2\Sigma_2 = (\Sigma_{1,1} - 2\Sigma_{2,1}) + (\Sigma_{1,2} - 2\Sigma_{2,2}). \]

As \( f(0) = 1, f(1) = 0 \) and \( f(u) = 1 + O(u^2) \), by integration by parts,
\[
\Sigma_{1,2} - 2\Sigma_{2,2} = -r(2^-)(c\log(h\log 2) - 1) \left[ f^2 \left( \frac{\log 2}{\log x} \right) - 2f \left( \frac{\log 2}{\log x} \right) \right] \\
- \int_2^x r(u) \cos \left( \frac{\log u}{\log x} \right) \left[ \frac{1}{u \log x} \left( \frac{\log u}{\log x} \right) f' \left( \frac{\log u}{\log x} \right) - 2f \left( \frac{\log u}{\log x} \right) \right] \, du \\
+ h \int_2^x r(u) \left[ f^2 \left( \frac{\log u}{\log x} \right) - 2f \left( \frac{\log u}{\log x} \right) \right] \, du \\
= \left[ \log \log 2 + C_0 - \sum_{m=2}^{\infty} \sum_{p} \frac{1}{mp^m} \right] (1 - \cos(h\log 2)) \\
- h \int_2^\infty r(u) \frac{\sin(h\log u)}{u} \, du + O \left( \frac{1}{\beta^3 L^3} \right)
\]

by \( f^2 - 2f = (f - 1)^2 \) and Taylor series of \( \log(1 + x) \).

\[
\Sigma_{1,1} - 2\Sigma_{2,1} = \int_2^x \cos \left( \frac{\log u}{\log x} \right) \left[ f^2 \left( \frac{\log u}{\log x} \right) - 2f \left( \frac{\log u}{\log x} \right) \right] \, du \\
- \int_2^x 1 - \cos \left( \frac{\log u}{\log x} \right) \left[ 1 - \frac{\pi \log u}{2 \log x} \cot \left( \frac{\pi \log u}{2 \log x} \right) \right]^2 \, du + \int_2^x \frac{1 - \cos \left( \frac{\log u}{\log x} \right) \alpha}{u \log x} \, du \\
= - \int_0^1 \frac{1 - \cos (hL\alpha)}{\alpha} \left[ 1 - \frac{\pi \alpha}{2\beta} \cot \left( \frac{\pi \alpha}{2\beta} \right) \right]^2 \, d\alpha + \int_0^1 \frac{1 - \cos (hL\alpha)}{\alpha} \, d\alpha \\
- \int_0^{\log 2/L} \frac{1 - \cos (hL\alpha)}{\alpha} \, d\alpha + O \left( \frac{1}{\beta^4 L^6} \right)
\]

by substituting \( \alpha = \log u/L \). Therefore,
\[
S_4 = \frac{T}{\pi} \left[ \log \log 2 + C_0 - \sum_{m=2}^{\infty} \sum_{p} \frac{1}{mp^m} \right] (1 - \cos(h\log 2)) \\
+ \frac{T}{\pi^2} \int_0^\beta \frac{1 - \cos (hL\alpha)}{\alpha} \, d\alpha - \frac{T}{\pi^2} \int_0^\beta \frac{1 - \cos (hL\alpha)}{\alpha} \left[ 1 - \frac{\pi \alpha}{2\beta} \cot \left( \frac{\pi \alpha}{2\beta} \right) \right]^2 \, d\alpha \\
- \frac{T}{\pi^2} \int_0^{\log 2/L} \frac{1 - \cos \alpha}{\alpha} \, d\alpha - \frac{Th}{\pi^2} \int_2^\infty r(u) \frac{\sin(h\log u)}{u} \, du + O \left( \frac{1}{\beta^4 L^3} \right).
\]
\[
S_3 = \frac{T}{\pi^2} \left[ \sum_{m=2}^{\infty} \sum_{p^m \leq x} \frac{1 - \cos (hm \log p)}{m^2 p^m} \right] - \sum_{m=2}^{\infty} \sum_{p^m \leq x} \frac{1 - \cos (hm \log p)}{m^2 p^m} \left( f\left( \frac{m \log p}{\log x} \right) - 1 \right)^2
\]
\[
= \frac{T}{\pi^2} \sum_{m=2}^{\infty} \sum_{p} \frac{1 - \cos (hm \log p)}{m^2 p^m} + O\left( \frac{1}{\beta^4 L^4} \right).
\]

\section{S_3}

Applying Lemma 2.1 and Lemma 2.2, we have

\[
S_3 = \frac{T}{2\pi^4 \beta^2} \int_{-\infty}^{\infty} (F(\alpha) - F_h(\alpha)) k\left( \frac{\alpha}{2\pi \beta} \right) d\alpha + \frac{T}{8L^2} \frac{F(\beta) - F_h(\beta)}{\beta^3}
\]
\[
- \frac{T}{32\pi^6 L^2 \beta^4} \int_{-\infty}^{\infty} (F(\alpha) - F_h(\alpha)) k''\left( \frac{\alpha}{2\pi \beta} \right) d\alpha
\]
\[
= S_{3,1} + S_{3,2} - S_{3,3}
\]

From (9),

\[
F(\alpha) - F_h(\alpha) = (1 - \cos (hL\alpha))\alpha + \frac{h^2}{4 + h^2} \cos (hL\alpha)\alpha + \frac{8h \sin (hL\alpha)}{(4 + h^2)^2 \beta} + E
\]

where

\[
E = \begin{cases} 
O(T^{-(1/2-c)L^{-1}}) + O(\alpha T^{\alpha-1}), & \text{if } 0 \leq \alpha \leq 1 - \frac{3 \log \log T}{\log T}, \\
O(T^{-(1/2-c)L^{-1}}) + O(\alpha T^{\alpha-1} L^{-1}), & \text{if } 1 - \frac{3 \log \log T}{\log T} \leq \alpha \leq 1.
\end{cases}
\]

Since \(F(\alpha)\) and \(F_h(\alpha)\) are even,

\[
S_{3,1} = \frac{T}{\pi^4 \beta^2} \int_{0}^{\infty} (F(\alpha) - F_h(\alpha)) k\left( \frac{\alpha}{2\pi \beta} \right) d\alpha.
\]

Let \(\epsilon_T = 3 \log \log T/\log T\). We split the above integral into four pieces:

\[
I = \int_{0}^{\infty} = \int_{0}^{\beta} + \int_{\beta}^{1-\epsilon_T} + \int_{1-\epsilon_T}^{1} + \int_{1}^{\infty} = I_1 + I_2 + I_3 + I_4.
\]
By (9), (10) and the definition of $k(u)$,

$$I_1 = (\pi \beta)^2 \int_0^\beta \frac{1 - \cos(hL\alpha)}{\alpha} \left[1 - \frac{\pi \alpha}{2\beta} \cot \left(\frac{\pi \alpha}{2\beta}\right)\right]^2 d\alpha$$

$$+ \frac{h^2}{4 + h^2} \int_0^\beta \alpha \cos(hL\alpha)k\left(\frac{\alpha}{2\pi \beta}\right) d\alpha$$

$$+ \frac{8h}{(4 + h^2)^2L} \int_0^\beta \sin(hL\alpha)k\left(\frac{\alpha}{2\pi \beta}\right) d\alpha + O(\beta^{-1}L^{-3}),$$

$$I_2 = (\pi \beta)^2 \int_0^{1-\epsilon_T} \frac{1 - \cos(hL\alpha)}{\alpha} d\alpha + \frac{(\pi \beta)^2 h^2}{4 + h^2} \int_0^{1-\epsilon_T} \cos(hL\alpha) d\alpha$$

$$+ \frac{(\pi \beta)^2 8h}{(4 + h^2)^2L} \int_0^{1-\epsilon_T} \sin(hL\alpha) d\alpha + O(L^{-4}),$$

$$I_3 = (\pi \beta)^2 \int_{1-\epsilon_T}^1 \frac{1 - \cos(hL\alpha)}{\alpha} d\alpha + \frac{(\pi \beta)^2 h^2}{4 + h^2} \int_{1-\epsilon_T}^1 \cos(hL\alpha) d\alpha$$

$$+ \frac{(\pi \beta)^2 8h}{(4 + h^2)^2L} \int_{1-\epsilon_T}^1 \sin(hL\alpha) d\alpha + O(L^{-4}),$$

$$I_4 = (\pi \beta)^2 \int_1^\infty \frac{F(\alpha) - F_h(\alpha)}{\alpha^2} d\alpha.$$

Thus,

$$S_{3,1} = \frac{T}{\pi^2} \int_0^\beta \frac{1 - \cos(hL\alpha)}{\alpha} \left[1 - \frac{\pi \alpha}{2\beta} \cot \left(\frac{\pi \alpha}{2\beta}\right)\right]^2 d\alpha + \frac{T}{\pi^2} \int_0^1 \frac{1 - \cos(hL\alpha)}{\alpha} d\alpha$$

$$+ \frac{T}{\pi^4 \beta^2} \left[\frac{h^2}{4 + h^2} \int_0^\beta \alpha \cos(hL\alpha)k\left(\frac{\alpha}{2\pi \beta}\right) d\alpha\right]$$

$$+ \frac{8h}{(4 + h^2)^2L} \int_0^\beta \sin(hL\alpha)k\left(\frac{\alpha}{2\pi \beta}\right) d\alpha + \frac{T h^2}{\pi^2(4 + h^2)} \int_0^1 \cos(hL\alpha) d\alpha$$

$$+ \frac{8Th}{\pi^2(4 + h^2)^2L} \int_0^1 \sin(hL\alpha) d\alpha + \frac{T}{\pi^2} \int_1^\infty \frac{F(\alpha) - F_h(\alpha)}{\alpha^2} d\alpha + O(\beta^{-1}L^{-2}).$$

Apply (9) and (10) directly,

$$S_{3,2} = \frac{T}{8 L^2 \beta^2} - \frac{T \cos(hL\beta)}{2(4 + h^2)L^2 \beta^2} + \frac{T h \sin(hL\beta)}{(4 + h^2)^2 L^3 \beta^3} + O(L^{-4}).$$

Similar to the treatment of $S_{3,1}$,

$$S_{3,3} = \frac{T}{16 \pi^6 L^2 \beta^4} \int_0^\infty (F(\alpha) - F_h(\alpha))k''\left(\frac{\alpha}{2\pi \beta}\right) d\alpha$$

and we split the integral into four pieces:

$$J = \int_0^\infty = \int_0^\beta + \int_{\beta}^{1-\epsilon_T} + \int_{1-\epsilon_T}^1 + \int_1^\infty = J_1 + J_2 + J_3 + J_4.$$
By Lemma 2.6 and 2.7

\[ J_1 = 2(\pi^6 - 6\pi^4)\beta^2 \left[ 1 - \frac{4\cos(hL\beta)}{4 + h^2} \right] - \frac{16\pi^4h\sin(hL\beta)}{4 + h^2} + \frac{32\pi^4h^2 \cos(hL\beta)}{(4 + h^2)^2} \beta^2 + \frac{16(\pi^6 - 4\pi^4)h \sin(hL\beta)}{(4 + h^2)^3} \beta^2 \]
\[ + \frac{16\pi^2h^2 \beta^2 L^2}{4 + h^2} \int_0^\beta \alpha \cos(h\Lambda)k\left(\frac{\alpha}{2\pi\beta}\right)d\alpha \]
\[ + \frac{128\pi^2h}{(4 + h^2)^2} \beta^2 L \int_0^\beta \sin(h\Lambda)k\left(\frac{\alpha}{2\pi\beta}\right)d\alpha + O(\beta^{-1}L^{-2}). \]

By Lemma 2.6 and 2.7

\[ J_2 + J_3 = \int_\beta ^1 \left[ \alpha - \frac{4\cos(h\Lambda)}{4 + h^2} + \frac{8h \sin(h\Lambda)}{(4 + h^2)^2} \right] \frac{24\pi^4\beta^4}{4 - \beta^4} d\alpha \]
\[ = 12\pi^4(\beta^2 - \beta^4) - \frac{16\pi^4(12 + h^2) \cos(hL\beta)}{(4 + h^2)^2} \beta^2 \]
\[ + \frac{16\pi^4(12 + h^2) \cos(hL\beta)}{(4 + h^2)^2} \beta^4 \delta^4 + \frac{16h(12 + h^2) \sin(hL\beta)}{(4 + h^2)^2} \beta^3 L \]
\[ - 64\pi^4h \sin(hL) \beta^4 + \frac{16\pi^4h^2(12 + h^2)}{(4 + h^2)^2} \beta^4 L \int_\beta ^1 \delta^4 \left( \frac{\cos(h\Lambda)}{\alpha} \right) d\alpha + O(L^{-2}). \]
\[ J_4 = 24\pi^4 \beta^4 \int_1 ^\infty \frac{F(\alpha) - F_h(\alpha)}{\alpha^4} d\alpha. \]

Thus,

\[ S_{3,3} = \frac{T}{8\beta L} - \frac{3T}{4\pi L} + \frac{T \cos(hL\beta)}{2(4 + h^2)L^2 \beta^2} + \frac{8Th \sin(hL\beta)}{\pi^2(4 + h^2)^2 \beta L} \]
\[ + \frac{Th \sin(hL\beta)}{(4 + h^2)^2 L^2} \beta^3 + \frac{T(12 + h^2) \cos(hL\beta)}{\pi^2(4 + h^2)^2 L^2} - \frac{Th(12 + h^2) \sin(hL\beta)}{\pi^2(4 + h^2)^2 L} \]
\[ - \frac{4T \sin(hL\beta)}{\pi^2(4 + h^2)^2 L^3} + \frac{Th^2}{\pi^4 \beta^2 (4 + h^2)^2} \int_0^\beta \alpha \cos(h\Lambda)k\left(\frac{\alpha}{2\pi\beta}\right)d\alpha \]
\[ + \frac{8Th}{\pi^4 \beta^2 (4 + h^2)^2 L} \int_0^\beta \sin(h\Lambda)k\left(\frac{\alpha}{2\pi\beta}\right)d\alpha \]
\[ + \frac{Th^2(12 + h^2)}{\pi^2(4 + h^2)^2} \int_1 ^1 \cos(h\Lambda) \left( \frac{\alpha}{\alpha} \right) d\alpha \]
\[ + \frac{3T}{2\pi^3 L^2} \int_1 ^\infty \frac{F(\alpha) - F_h(\alpha)}{\alpha^4} d\alpha + O(\beta^{-1}L^{-4}). \]
Finally, combining the results for \( S_{3,1}, S_{3,2} \) and \( S_{3,3} \), we have

\[
S_3 = \frac{T}{\pi^2} \int_0^\beta \frac{1 - \cos (hL \alpha)}{\alpha} \left[ 1 - \frac{\pi \alpha}{2\beta} \cot \left( \frac{\pi \alpha}{2\beta} \right) \right]^2 d\alpha + \frac{T}{\pi^2} \int_0^\beta \frac{1 - \cos (hL \alpha)}{\alpha} d\alpha + T \int_0^\beta \frac{F(\alpha) - F_h(\alpha)}{\alpha^2} d\alpha + \frac{T}{\pi^2} \int_1^\beta \frac{F(\alpha) - F_h(\alpha)}{\alpha^2} d\alpha + \frac{3T}{\pi(1 + h^2) \cos (hL)} \frac{1}{2\pi^2 L^2} \int_1^\beta \frac{F(\alpha) - F_h(\alpha)}{\alpha^4} d\alpha + O(\beta^{-1} L^{-2})
\]

as

\[
\int_\beta^1 \frac{\sin (hL \alpha)}{\alpha^2} d\alpha = -\sin (hL) + \frac{\sin (hL \beta)}{\beta} + hL \int_\beta^1 \frac{\cos (hL \alpha)}{\alpha} d\alpha.
\]

Remark: We keep some of the \( O(TL^{-2}) \) terms explicit because, with more effort, one can make the error term = \( C_1 TL^{-2} + o(TL^{-2}) \).

5 Proof of Theorem 1.1

Take \( \beta = 1/2 \). Combining the results on \( S_3, S_4 \) and \( S_5 \), we have

\[
\int_0^T (S(t + h) - S(t))^2 dt = \frac{T}{\pi^2} \int_0^1 \frac{1 - \cos (hL \alpha)}{\alpha} d\alpha + \frac{T}{\pi^2} \int_1^\infty \frac{F(\alpha) - F_h(\alpha)}{\alpha^2} d\alpha + \frac{T}{\pi^2} \left[ \log 2 + C_0 - \sum_{m=2}^{\infty} \sum_p \frac{1}{mp^m} \right] (1 - \cos (h \log 2)) + \frac{T}{\pi^2} \int_0^{h \log 2} \frac{1 - \cos \alpha}{\alpha} d\alpha
\]

\[
+ \frac{Th}{2\pi^2} \int_2^\infty \frac{r(u)}{u^2} du + \frac{T}{\pi^2} \sum_{m=2}^{\infty} \sum_{p} \frac{1 - \cos (hm \log p)}{m^2 p^m}
\]

\[
+ \frac{Th \sin (hL)}{\pi^2 (1 + h^2) L} + \frac{T}{\pi^2 L^2} \frac{h^2 (20 + 3h^2)}{4(1 + h^2)^2} - \frac{3T}{2\pi^2 L^2} \int_1^\beta \frac{F(\alpha) - F_h(\alpha)}{\alpha^4} d\alpha + O \left( \frac{T}{L^2} \right)
\]

which gives the theorem. Again, one can make the error term = \( C_1 TL^{-2} + o(TL^{-2}) \) with more effort.

The theorem improves that of Fujii as

\[
\begin{itemize}
  \item \( h \) is allowed to be \( O(1) \).
  \item All the terms besides first two contribute \( O(Th^2) + O(TL^{-2}) \).
  \item It is conjectured in [3] that 
    \[
    F_h(\alpha) = F(\alpha) \frac{4 \cos (hL \alpha)}{4 + h^2} + o(1) \text{ for } 1 \leq \alpha \leq A \text{ with arbitrary large } A.
    \]
\end{itemize}

(11)
Furthermore, as
\[ \sum_{p \leq x} \frac{1 - \cos (h \log p)}{p} = \int_{2}^{x} \frac{1 - \cos (h \log u)}{u \log u} du + \int_{2}^{x} 1 - \cos (h \log u) dr(u) \]
\[ = \int_{\log 2}^{\log x} \frac{1 - \cos \alpha}{\alpha} d\alpha - r(2^{-}) (1 - \cos (h \log 2)) \]
\[ - h \int_{2}^{\infty} \frac{r(u) \sin (h \log u)}{u} du + O\left( \frac{\log x}{\sqrt{x}} \right), \]

Theorem 1.1 gives
\[ \int_{0}^{T} (S(t+h) - S(t))^2 dt \]
\[ = \frac{T}{\pi^2} \int_{0}^{1} \frac{1 - \cos (h\alpha)}{\alpha^2} d\alpha + \frac{T}{\pi^2} \int_{1}^{\infty} \frac{F(\alpha) - F_h(\alpha)}{\alpha^2} d\alpha \]
\[ + \frac{T}{\pi^2} \left[ \sum_{m=1}^{x} \sum_{p^m \leq x} \frac{1 - \cos (h \log p)}{m^2 p^m} + C_i(h \log x) - \log (h \log x) - C_0 \right] \]
\[ + \frac{Th \sin (hL)}{\pi^2 (4 + h^2)L} + \frac{T}{\pi^2 L^2} \frac{h^2 (20 + 3h^2)}{4(4 + h^2)^2} - \frac{3T}{2\pi^2 L^2} \int_{1}^{\infty} \frac{F(\alpha) - F_h(\alpha)}{\alpha^4} d\alpha + O\left( \frac{T}{L^2} \right) \]

where \( C_i(x) = -\int_{x}^{\infty} \frac{\cos t}{t} dt = C_0 + \log x + \int_{0}^{x} \frac{\cos t - 1}{t} dt \) is the cosine integral. If we assume Montgomery’s conjecture on \( F(\alpha) \) and \( F_h(\alpha) \), the first two terms account for the GUE part of Berry’s formula (19) conjectured in [1] by a similar calculation as page 79 of [4]. Moreover, the third term is the non-GUE part of Berry’s formula. So, our theorem is even more precise than Berry’s formula.

### 6 Proof of Theorem 1.2

First, let us consider
\[ L(x, t) = \sum_{0 < \gamma \leq T} \frac{x^{i(\gamma - t)}}{1 + (t - \gamma)^2} \]
where the sum here is over of the imaginary parts of the non-trivial zeros of the Riemann zeta function.

**Lemma 6.1.** For all \( \alpha \) and \( h \), we have
\[ F_h(\alpha) \leq F(\alpha). \]

**Proof:** First, by partial fractions and Cauchy’s residue theorem,
\[ \int_{-\infty}^{\infty} \frac{1}{(1 + (t - a)^2)(1 + (t - b)^2)} dt = \frac{2\pi}{4 + (a - b)^2}, \]
Then
\[
0 \leq \int_{-\infty}^\infty |L(x,t) - L(x,t-h)|^2 dt
= 2 \sum_{0 < \gamma, \gamma' \leq T} x^{i(\gamma - \gamma')} \int_{-\infty}^\infty \frac{1}{(1 + (t - \gamma)^2)(1 + (t - \gamma')^2)} dt
- 2 Re \sum_{0 < \gamma, \gamma' \leq T} x^{i(\gamma - \gamma' - h)} \int_{-\infty}^\infty \frac{1}{(1 + (t - \gamma)^2)(1 + (t - h - \gamma')^2)} dt
= \pi F(x, T) - \pi F_h(x, T).
\]

Set \( x = \left( \frac{T}{2\pi} \right)^\alpha \) and divide through by \( T L^2 \pi^2 \), we have the lemma.

Assuming RH, Montgomery [8] proved that, for fixed \( 0 < \beta < 1 \),
\[
\left( \frac{TL}{2\pi} \right)^{-1} \sum_{0 < \gamma, \gamma' \leq T} \left[ \sin \frac{\beta(\gamma - \gamma')L}{2} \right]^2 w(\gamma - \gamma') \sim \frac{1}{\beta} + \frac{\beta}{3}
\]  
(12)

as \( T \to \infty \). This also holds for \( \beta = 1 \) by Goldston [5]. Using (5), one can prove similarly that for fixed \( 0 < \beta \leq 1 \),
\[
\left( \frac{TL}{2\pi} \right)^{-1} \sum_{0 < \gamma, \gamma' \leq T} \left[ \sin \frac{\beta(\gamma - \gamma' - h)L}{2} \right]^2 w(\gamma - \gamma' - h) \\
\sim \frac{1}{\beta} + \frac{8\beta}{4 + h^2} \frac{2 \sin (hL\beta)}{hL\beta} - 1 - \cos (hL\beta)
\]  
(13)
as \( T \to \infty \) under RH only (similar to [5] or using Lemma 7 of [7] in the argument of Chan [3]). Note that \( 2 \sin x - 1 - \cos x \leq \frac{x^2}{6} \) by simply looking at their Taylor series. We also need the following

Lemma 6.2. For any real number \( c \),
\[
\int_{c-1}^{c+1} F_h(\alpha)(1 - |\alpha - c|) d\alpha
= \left( \frac{TL}{2\pi} \right)^{-1} \sum_{0 < \gamma, \gamma' \leq T} \cos (L(\gamma - \gamma' - h)c) \left[ \sin \frac{(\gamma - \gamma' - h)L}{2} \right]^2 w(\gamma - \gamma' - h),
\]
\[
\int_{c-1}^{c+1} F(\alpha)(1 - |\alpha - c|) d\alpha
= \left( \frac{TL}{2\pi} \right)^{-1} \sum_{0 < \gamma, \gamma' \leq T} \left( \frac{T}{2\pi} \right)^{i(\gamma - \gamma')} \left[ \sin \frac{(\gamma - \gamma')L}{2} \right]^2 w(\gamma - \gamma').
\]
Proof: The second one follows from the first one by setting $h = 0$. To prove the first one, we have, from the definition of $F_h(\alpha)$,

$$\int_{c-1}^{c+1} F_h(\alpha)(1 - |\alpha - c|)d\alpha = \int_{-1}^{1} F_h(\alpha - c)(1 - |\alpha|)d\alpha$$

$$= \sum_{0<\gamma,\gamma'\leq T} \int_{-1}^{1} \cos(L(\gamma - \gamma' - h)(\alpha - c))(1 - |\alpha|)d\alpha w(\gamma - \gamma' - h)$$

$$= \sum_{0<\gamma,\gamma'\leq T} 2 \cos(L(\gamma - \gamma' - h)c) \frac{1 - \cos(L(\gamma - \gamma' - h))}{((\gamma - \gamma' - h)L)^2} w(\gamma - \gamma' - h)$$

by integration by parts. This gives the lemma as $\cos 2x = 1 - 2 \sin^2 x$.

Note: (12) and (13) can be proved by setting $c = 0$ in Lemma 6.2 and using the asymptotic formulas for $F(\alpha)$ and $F_h(\alpha)$ like (5).

**Lemma 6.3.** Assume RH. For any $\epsilon > 0$ and $T$ sufficiently large,

$$0 \leq \int_{c}^{c+1} (F(\alpha) - F_h(\alpha))d\alpha \leq \frac{16}{3} + \epsilon$$

uniformly for any real number $c$.

Proof: From Lemma 6.1 we have $F(\alpha) - F_h(\alpha) \geq 0$. This gives the lower bound as well as

$$\frac{1}{2} \int_{c-1/2}^{c+1/2} (F(\alpha) - F_h(\alpha))d\alpha \leq \int_{c-1}^{c+1} (F(\alpha) - F_h(\alpha))(1 - |\alpha - c|)d\alpha.$$ 

So, by Lemma 6.2

$$\int_{c-1/2}^{c+1/2} (F(\alpha) - F_h(\alpha))d\alpha \leq 2 \left(\frac{T L}{2\pi}\right)^{-1} \sum_{0<\gamma,\gamma'\leq T} \left[\frac{\sin \left(\frac{\gamma-\gamma'}{2}\right)L}{\frac{(\gamma-\gamma')L}{2}}\right]^2 w(\gamma - \gamma')$$

$$+ 2 \left(\frac{T L}{2\pi}\right)^{-1} \sum_{0<\gamma,\gamma'\leq T} \left[\frac{\sin \left(\frac{\gamma-\gamma'-h}{2}\right)L}{\frac{(\gamma-\gamma'-h)L}{2}}\right]^2 w(\gamma - \gamma' - h).$$

Now, using (12) and (13) with $\beta = 1$ and $2 \sin \frac{x}{2} - 1 - \cos x \leq \frac{x^2}{6}$, the right hand side of the above inequality is

$$\leq 2 \left(\frac{4}{3} + \frac{\epsilon}{4}\right) + 2 \left(\frac{4}{3} + \frac{\epsilon}{4}\right) = \frac{16}{3} + \epsilon$$

when $T$ is sufficiently large.

We are now in the position to prove Theorem 1.2. By Lemma 6.1 and 6.3

$$0 \leq \int_{1}^{\infty} \frac{F(\alpha) - F_h(\alpha)}{\alpha^2}d\alpha \leq \sum_{c=1}^{\infty} \frac{1}{c^2} \int_{c}^{c+1} (F(\alpha) - F_h(\alpha))d\alpha < 5.4 \frac{\pi^2}{6} < 9.$$ 

Similarly,

$$0 \leq \int_{1}^{\infty} \frac{F(\alpha) - F_h(\alpha)}{\alpha^4}d\alpha \leq \sum_{c=1}^{\infty} \frac{1}{c^4} \int_{c}^{c+1} (F(\alpha) - F_h(\alpha))d\alpha < 5.4 \frac{\pi^4}{90} < 6.$$
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