Flavored surface defects in $4d \mathcal{N} = 1$ SCFTs

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Abstract

We discuss supersymmetric surface defects in compactifications of six dimensional minimal conformal matter of type $SU(3)$ and $SO(8)$ to four dimensions. The relevant field theories in four dimensions are $\mathcal{N} = 1$ quiver gauge theories with $SU(3)$ and $SU(4)$ gauge groups respectively. The defects are engineered by giving space-time dependent vacuum expectation values to baryonic operators. We find evidence that in the case of $SU(3)$ minimal conformal matter the defects carry $SU(2)$ flavor symmetry which is not a symmetry of the four dimensional model. The simplest case of a model in this class is $SU(3)$ SQCD with nine flavors and thus the results suggest that this admits natural surface defects with $SU(2)$ flavor symmetry. We analyze the defects using the superconformal index and derive analytic difference operators introducing the defects into the index computation. The duality properties of the four dimensional theories imply that the index of the models is a kernel function for such difference operators. In turn, checking the kernel property constitutes an independent check of the dualities and the dictionary between six dimensional compactifications and four dimensional models.
1. Introduction

Surface defects are interesting non-local observables in quantum field theories and they have received some attention in recent years. There are various ways to introduce such defects into a $d$ dimensional model. For example, one can try and couple the degrees of freedom of a $d$ dimensional CFT to a two dimensional CFT, or define the defect by specifying boundary conditions of the $d$ dimensional theory supported on a two dimensional surface. Yet another way to introduce surface defects is by studying flows triggered by vacuum expectation values with a non trivial space-time profile. Such flow can lead to an IR CFT with some of the degrees of freedom localized to submanifolds of the $d$ dimensional spacetime where the vacuum expectation value has special properties.

Combining the latter approach with supersymmetry can lead to quantitative tools to study the defects. In this brief note we will study defects in a simple class of $\mathcal{N} = 1$ supersymmetric field theories in four dimensions. This class of theories can be engineered as compactification of six dimensional minimal conformal matter of type $SU(3)$ and $SO(8)$ \cite{1}\cite{2} on a Riemann surface with punctures. The theories in four dimensions obtained in such compactifications were identified in \cite{4} as certain gauge theories with $SU(3)$ and $SU(4)$ gauge groups respectively. We will observe a simple manifestation of an interesting phenomenon. In the class of theories obtained from minimal conformal matter of type $SU(3)$ there are supersymmetric surface defects that carry degrees of freedom charged under symmetry which is not a symmetry of the theory in the bulk. We will argue for this by engineering the defects with the RG flow construction starting from a theory which has extra symmetry. Turning on constant vacuum expectation values some of the extra symmetry is explicitly broken and the rest does not appear in the IR fixed point as the fields charged under it become massive. However, when we will turn on space-time dependent vacuum expectation values, some remnants of the massive fields will survive on two dimensional subspace where the vacuum expectation value vanishes. This will thus produce interesting defects with additional symmetry which is naively surprising from the point of view of the bulk model. An interesting question, which we leave for future research, is to understand whether our defects can be engineered in other manner, say by coupling the four dimensional models to two dimensional theories as in \cite{5}\cite{6}.

\footnote{The minimal conformal matter is described on the tensor branch as YM with single simple gauge group factor and single tensor multiplet with no matter. This set of models is a subset of the so called non-higgsable cluster theories \cite{3} in six dimensions.}
The fact that the theories we will consider have a geometric interpretation will have a mathematical implication. An interesting case here is when the six dimensional model we start with has an effective description as a five dimensional gauge theory when compactified on a circle. The theories in four dimensions, in addition to the symmetry of the six dimensional model preserved in the compactification, also have factors of global symmetry associated to the punctures and being a subgroup of the five dimensional gauge symmetry. The theories that we will study are of this type. In particular they have a description in five dimensions as $SU(3)$ Chern-Simons model for the (twisted) compactification of $SU(3)$ minimal conformal matter, and $SU(4)$ Chern-Simons model for the (twisted) compactification of the $SO(8)$ minimal conformal matter [7] (see discussion in [4]). We will compute the supersymmetric index of the theories in presence of surface defects. It is given by certain analytic difference operators acting on the index of the theory without the defect. The difference operator one obtains is a Hamiltonian of a relativistic quantum mechanical model which might be associated to the five dimensional gauge theory [8][9]. In the well studied case of class $S$ [10][11] the relevant model is Ruijsenaars-Schneider integrable system [12] as observed by Nekrasov in [8] long time ago and obtained in the context of index computations in [13]. Other examples are the van Diejen model [14] for the E-string [15] and some more intricate systems for class $S_k$ models [16][17][18]. We will identify the relevant quantum mechanical systems in the two classes of theories we study. We verify various properties such Hamiltonians have to satisfy following the conjectured map between compactifications and four dimensional theories of [4]. This constitutes additional check of the conjectures.

We organize the paper as follows. In section two we will consider defects in theories obtained by compactification of the minimal conformal matter of type $SU(3)$. We will first discuss the basic physical considerations related to the defects and review the essentials of the field theories in four dimensions obtained in this compactification. We will observe that the defect might carry some symmetry. Next we will discuss the index in presence of the defects, and observe that indeed the defects carry an $SU(2)$ symmetry, and will derive the quantum mechanical model associated to this construction. In section three we will consider defects in theories obtained by compactifications of minimal conformal matter of type $SO(8)$. In Appendix we summarize some technicalities.
2. Case of the $SU(3)$ minimal conformal matter

Let us first discuss the construction of the defect in the models we are going to consider from a simple perspective. The theories we consider are constructed by combining two $\mathcal{N} = 1$ superconformal CFTs, $T_1$ and $T_2$, which have a factor of $SU(3)$ flavor symmetry each. We combine the models by coupling them to a triplet of bifundamental fields $Q_i$, a trifundamental, by gauging the two $SU(3)$ symmetries. In our constructions the only non anomalous flavor symmetry the $Q_i$ are charged under is an $SU(3)_f$ rotating the three fields, and the superconformal R symmetry is $2/3$. We then consider giving a vacuum expectation value to the baryon,

$$\det Q_1 = \epsilon_{lmk}\epsilon^{ijn}(Q_1)_j^l(Q_1)_i^k(Q_1)_m^n. \quad (2.1)$$

We can choose baryons built from $Q_2$ and $Q_3$ with equivalent results. This vacuum expectation value breaks explicitly the $SU(3)_f$ symmetry to $SU(2)_f$. The two gauge symmetries are Higgsed down to a diagonal combination with the $Q_1$ and $Q_2$ fields transforming in the adjoint (plus singlet) of the diagonal $SU(3)$ gauge symmetry and acquiring a mass term $Q_1 Q_2$. Note that if there would have been a $U(1)$ symmetry under which $Q_i$ are charged a mass term would not be generated and the theory would have the $SU(2)_f$ symmetry in the IR. In the IR the theory one obtains is just the two models $T_1$ and $T_2$ combined by gauging a diagonal combination of the two $SU(3)$ symmetries, for illustration see Figure one. Note that although we did not break the $SU(2)_f$ symmetry, nothing in the IR is charged under it.

*Fig. 1:* Two theories $T_1$ and $T_2$ combined by gauging two $SU(3)$ symmetries connected by three bifundamental chiral fields $Q_i$. We assume that there is no $U(1)$ symmetry under which $Q_i$ are charged. We give vacuum expectation value to a baryon $\det Q_1$. The theory in the IR is the two theories $T_2$ and $T_1$ glued by gauging diagonal combination of two $SU(3)$ symmetries. If we give a vacuum expectation value to derivative of $\det Q_1$ we obtain surface defect in the theory on the right with some of the states charged under $SU(2)_f$ localized to the defect.
Next we consider the same construction but we give a space time dependent vacuum expectation value. The logic is detailed in [13]. We turn on a constant expectation value for $\partial \det Q_1$ where $\partial$ is a derivative in some plane in four dimensions. For example, the plane is parametrized by complex coordinate $z$ and we take a holomorphic derivative. Away from the locus $z = 0$ we have a non vanishing vacuum expectation value and flow to the same theory as above. However at $z = 0$ the vacuum expectation value vanishes and we might have some additional degrees of freedom localize on the two dimensional surface orthogonal to the complex plane parametrized by $z$. In particular there is no reason to expect that there is no remnant of the fields charged under the $SU(2)_f$ symmetry not broken by the vacuum expectation value. We will indeed see explicitly in the index computation below that this is the case and the defects carry degrees of freedom charged under the symmetry $SU(2)_f$ in the IR.

Before turning to the index computation let us briefly review the construction of the theories $T_1$ and $T_2$ [4]. We refer the reader for details to this reference. The claim is that the theories in four dimensions corresponding to (twisted) compactifications of the minimal conformal matter of type $SU(3)$ on a general Riemann surface are constructed from the two simple blocks depicted in Figure two.

![Figure 2](image)

**Fig. 2:** On the left we have theory corresponding to compactification on sphere with two maximal punctures with $SU(3)$ symmetry and two punctures with no symmetry. The former denoted by circles and latter by a cross and referred to as empty punctures. Dashed lines correspond to bidundamental chiral fields for which a baryonic superpotential is turned on. The squares correspond to $SU(3)$ global symmetry. On the right there is a trifundamental. It does not correspond to a compactification by itself, however gluing it to a theory by gauging $SU(3)$ corresponds to removing an empty puncture and adding a maximal puncture. From these blocks any theory in the class discussed here can be constructed. Note that both models here are equivalent in the IR but it is important to distinguish them to write a precise dictionary between compactifications and four dimensional models.
Using these blocks we can construct theories corresponding to any surface. The construction proceeds iteratively by building bigger theories by gluing two smaller ones at a maximal puncture. The gluing is done by gauging with $\mathcal{N} = 1$ vector multiplet the diagonal $SU(3)$ symmetry associated to the punctures. In Figure three two important examples are depicted. These are related by RG flow we have discussed. In the geometric picture the vacuum expectation value we consider removes a maximal puncture and exchanges it with empty puncture. We note that gluing theories corresponding to general surfaces all the $U(1)$ symmetries are broken, either by the superpotentials, anomalies, or both. In particular we are in general in the setup discussed above where we have theories obtained by gluing smaller pieces with a trifundamental so it is not charged under any $U(1)$. The theories have large conformal manifolds on which all the symmetries are broken. Theories with maximal punctures reside on same conformal manifolds as theories with only empty punctures such that every maximal puncture is traded with three empty punctures. For example, the theory with four maximal punctures is the same as one with twelve empty ones. A general theory is then built from trifundamentals and baryonic superpotentials. The superconformal R symmetry of all the chiral fields is the free one.

Fig. 3: On the top we combine a theory with two maximal punctures and two empty ones with two trifundamentals resulting in a sphere with four maximal punctures from which any surface with even number of punctures can be constructed. The fact that we can get only even number of punctures is as the punctures carry a $\mathbb{Z}_2$ valued twist. On the bottom we give vacuum expectation value to one of the baryons built from a field with no superpotential and obtain a theory with three maximal and one empty puncture.

With this we are ready to study the supersymmetric index and in particular the defects.
2.1. The defect and the index

Turning on a vacuum expectation value to an operator implies that we break the symmetry of the model in such a way that in IR the operator has zero charges. In the computation of the superconformal index [19] this implies that we define fugacities in such a way that the weight of the operator in the computation is one. Doing so produces a pole divergence of the index and the claim [13] is that the index of the theory in the IR is the residue of the pole.

In what follows we will compute such residues for the relevant poles. We use standard index notations detailed in [20] with the different relevant functions defined in the Appendix.

Let us denote index of some theory by $I(y)$ with $y$ standing for $(y_1, y_2, y_3)$ such that $\prod_{i=1}^{3} y_i = 1$ being the fugacities for the maximal torus of $SU(3)$ flavor symmetry of one of the maximal punctures. The index depends on fugacities for all the symmetries associated to punctures but we leave those implicit in the definition. The index of a theory obtained by gluing a trifundamental field is by usual rules of index computations given by,

$$I' = \left(\frac{q}{p}\right)^2 \left(\frac{p}{q}\right)^2 \frac{1}{6} \int \frac{dy_1}{2\pi iy_1} \int \frac{dy_2}{2\pi iy_2} \frac{\prod_{i,j,l=1}^{3} \Gamma_e((qp)^{\frac{1}{2}} b_{ij} y^j z_l^{-1})}{\prod_{i<j} \Gamma_e((y^i/y^j)^{\pm 1})} I(y).$$

(2.2)

The integration contours are around unit circle and we assume that parameters satisfy $|q|, |p| < 1$ and $|b_i| = |z_i| = 1$. We will soon take the latter to have more general values but then the contours should be properly deformed. Here the denominator comes from the vectors and the numerator from the trifundamental field. The fugacities $b$ and $z$ are for the two $SU(3)$ symmetries of the trifundamental chiral field which become global symmetries of the new theory. We next want to close one of these two maximal punctures with $SU(3)$ symmetries by giving an expectation value to a baryon. By general considerations we have detailed the resulting theory should be the same as the model we started with plus a defect if the vacuum expectation value has a non trivial profile. The vacuum expectation value corresponds to a pole in the index in the $SU(3)$ fugacities and we will study poles in $b_i$. The poles in $b_i$ of $I'$ occur as when we vary the value of $b_i$ the poles in the integrated variables $z$ can pinch the integration contour [13] (see also [21][16][17][15][18] for similar computations). We turn to the analysis of such pinchings.

The integrand has poles in $y^1$ such that the following are inside the unit circle,

$$y^1 = (qp)^{\frac{1}{2}} (y^2)^{-1} z_{j_3}^{-1} b_{k_3} q^{l_3} p^{n_3},$$

(2.3)
and the following are outside of the unit circle,

\[ y^1 = (qp)^{-\frac{1}{3}} z_{j_1} b_{k_1}^{-1} q^{-l_1} p^{-n_1}. \]  

(2.4)

Similarly we have poles in \( y^2 \) outside the unit contour,

\[ y^2 = (qp)^{-\frac{1}{3}} z_{j_2} b_{k_2}^{-1} q^{-l_2} p^{-n_2}. \]  

(2.5)

The numbers \( l_i, n_j, k_l \) are non-negative integers. From here if we want both integration contours to be pinched we have to satisfy in particular,

\[ 1 = q^{1+\sum_{h=1}^{3} l_h} p^{1+\sum_{y=1}^{3} n_y} \prod_{r=1}^{3} z_{j_r}^{-1} \prod_{c=1}^{3} b_{k_c}^{-1}. \]  

(2.6)

This is the weight of a derivative of a baryon. Setting it to 1 thus gives it vacuum expectation value. We want to consider general class of poles which are independent of \( z \) and then the indices \( j_r \) need to run over the three different values. Without loss we can take \( z_{j_r} \) as \( z_r \). When we will evaluate the residue we will need to sum over all the choices give by permutations of the three \( z_r \), which will contribute an overall factor in the computation.

We will ignore overall factors which do not depend on flavor fugacities in what follows as they will not be essential to the claims. To obtain an interesting pole let us consider taking all \( b_k \) to be the same, say \( b_1 \). Then we obtain that we have pinchings when,

\[ b_1 = q^{-\frac{1}{3}(1+\sum_{h=1}^{3} l_h)} p^{-\frac{1}{3}(1+\sum_{y=1}^{3} n_y)}. \]  

(2.7)

This corresponds to vacuum expectation value to \( \det Q_1 \). The poles are thus classified by two integers \( \sum_{h=1}^{3} l_h = L \) and \( \sum_{y=1}^{3} n_y = M \). These correspond to giving vacuum expectation values of the form \( \partial^L_1 \partial^M_2 \det Q_1 \). The two derivatives are the rotations in two orthogonal complex planes. The physical interpretation of the IR fixed point is of original theory, index of which is \( I(y) \), with two surface defects, each wrapping one of the equators of \( \mathbb{S}^3 \) and the \( \mathbb{S}^1 \).

Here we refer to the fact that the index can be thought as supersymmetric partition function on \( \mathbb{S}^3 \times \mathbb{S}^1 \). One of the defects is determined by \( L \) and another by \( M \), see [13].

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\[ 2 \] We parametrize the sphere as \( \sum_{j=1}^{4} x_j^2 = 1 \). The two rotations are in the planes \((x_1, x_2)\) and \((x_3, x_4)\). The defects wrap either \( x_1 = x_2 = 0 \) or \( x_3 = x_4 = 0 \).
It is straightforward to compute the residues for general $L$ and $M$. Here we will do so in two simplest cases. Let us take $L$ and $M$ to be zero. This corresponds to a vacuum expectation value for the baryon. This implies,

$$y^i = z_i,$$  \tag{2.8}

The integrand becomes,

$$\prod_{i,j} \Gamma_e(z_i/z_j) \prod_{i,j} \Gamma_e((qp)\frac{d}{b_2}z_i/z_j) \Gamma_e((qp)^\frac{d}{b_2}(b_2)^{-1}z_i/z_j) \mathcal{I}(z_i) \propto \Gamma_e(1)^3 \mathcal{I}(y).$$ \tag{2.9}

Note that $\Gamma_e(z \to 1) \to \infty$ as $1/(1 - z)$. We have thus third order singularity which can be thought as three simple poles colliding. Two of them are absorbed by the two contour integrals and the third one is the pole in $b$ we are after. We have,

$$\text{Res}_{b_1 \to (qp)^{-1/3}} \mathcal{I}' \propto \mathcal{I}(z_i).$$ \tag{2.10}

Note that although we only broke explicitly $b_1$ all fields charged under $b_2$ acquired mass and decoupled in IR. This happens as in (2.9) we observe that,

$$\Gamma_e((qp)^\frac{d}{b_2}z_i/z_j) \Gamma_e((qp)^\frac{d}{b_2}(b_2)^{-1}z_i/z_j) = 1,$$ \tag{2.11}

with the two $\Gamma_e$ corresponding to the index of $Q_2$ and $Q_3$ which is consistent with having a mass term. This will change when we turn on vacuum expectation value for derivatives of the operator.

Let us consider then the case with $L = 1$ and $M = 0$. We can satisfy this by taking one of the $l_i = 1$ and rest zero. Different choices will differ by permutations of $z_i$ and we will need to sum over all of them eventually. Let us take $l_1 = 1$ and all others zero. We thus have,

$$b_1 = (qp)^{-\frac{d}{b_2}}q^{-\frac{d}{b_2}}, \quad y^1 = z_1q^{-\frac{d}{b_2}}, \quad y^2 = z_2q^{\frac{d}{b_2}}, \quad y^3 = z_3q^{\frac{d}{b_2}}.$$  \tag{2.12}

We have then for the trifundamental matter,

$$\Gamma_e(q^{-1}z_1/z_i)\Gamma_e(z_2/z_i)\Gamma_e(z_3/z_i)\Gamma_e((qp)^\frac{d}{b_2}q^{-\frac{d}{b_2}}b_2z_1/z_i) \Gamma_e((qp)^\frac{d}{b_2}q^{\frac{d}{b_2}}b_2z_2/z_i)$$

$$\Gamma_e((qp)^\frac{d}{b_2}q^{\frac{d}{b_2}}b_2z_3/z_i)\Gamma_e((qp)^\frac{d}{b_2}q^{-\frac{d}{b_2}}b_2^{-1}z_1/z_i)\Gamma_e((qp)^\frac{d}{b_2}q^{\frac{d}{b_2}}b_2^{-1}z_2/z_i)\Gamma_e((qp)^\frac{d}{b_2}q^{\frac{d}{b_2}}b_2^{-1}z_3/z_i).$$  \tag{2.13}
Introducing the contribution of the vector and $\mathcal{I}(y)$ the residue evaluates, ignoring as usual overall factors, to

$$\frac{\theta_p((p/q)^{\frac{1}{3}} b_2/z_3) \theta_p((p/q)^{\frac{1}{3}} b_2 z_3/z_2)}{\theta_p(z_2/z_1) \theta_p(z_3/z_1)} \mathcal{I}(z_1 q^{-\frac{2}{3}}, z_2 q^{\frac{1}{3}}, z_3 q^{\frac{1}{3}}).$$

(2.14)

All in all summing over different choices of $l_i$ we have that,

$$\text{Res}_{b_1 \to (qp)^{-\frac{2}{3}} q^{-\frac{1}{3}}} \mathcal{I} \propto O_{q}^{b_2(qp)^{-\frac{1}{3}} q^{-\frac{1}{3}}} (z) \cdot \mathcal{I}(z).$$

(2.15)

Here we define,

$$O_{q}^{Y}(z) \cdot \mathcal{I}(z) = \frac{\theta_p((p/q)^{\frac{1}{3}} Y(z_2/z_3)^{\pm 1})}{\theta_p(z_2/z_1) \theta_p(z_3/z_1)} \mathcal{I}(z_1 q^{-\frac{2}{3}}, z_2 q^{\frac{1}{3}}, z_3 q^{\frac{1}{3}}) +$$

$$\frac{\theta_p((p/q)^{\frac{1}{3}} Y(z_1/z_3)^{\pm 1})}{\theta_p(z_1/z_2) \theta_p(z_3/z_2)} \mathcal{I}(z_1 q^{\frac{1}{3}}, z_2 q^{-\frac{2}{3}}, z_3 q^{\frac{1}{3}}) +$$

$$\frac{\theta_p((p/q)^{\frac{1}{3}} Y(z_2/z_1)^{\pm 1})}{\theta_p(z_2/z_3) \theta_p(z_1/z_3)} \mathcal{I}(z_1 q^{\frac{1}{3}}, z_2 q^{-\frac{2}{3}}, z_3 q^{-\frac{2}{3}}).$$

(2.16)

This is the difference operator introducing a surface defect into the index computations. Note that the operator now depends on the $SU(2)_f$ commutant of $b_1$ in $SU(3)$ fugacity for which is $Y$. When we turn on vacuum expectation values for a derivative of a baryon operator some of the states localized on the defect are charged under this $SU(2)_f$ symmetry though there are no bulk states charged under the symmetry. The expression (2.15) gives the index of the theory in the IR in presence of the defect and thus it explicitly shows that the defect carries the $SU(2)_f$ flavor symmetry. Each defect will have a factor of $SU(2)_f$ associated to it.

Note that the theta functions appearing in the numerator in the difference operator above have natural interpretation as elliptic genus of $N = (2, 0)$ Fermi multiplet and the theta functions in denominator as the elliptic genus of a chiral field, see [22]. It will be interesting to find a two dimensional CFT which when coupled to the four dimensional model is equivalent to our defect. We leave this for future work.

We can repeat the procedure with other values of $M$ and $L$ and derive operators corresponding to other surface defects. Let us just mention that the operator corresponding to $M = 1$ and $L$ being zero is the same as above with $q$ and $p$ exchanged, and we will denote it by $O_{p}^{Y}$. The two operators introduce same type of defects but on different two dimensional locus.
2.2. Checks of the compactifications using the operator

Using this result we can subject the map between the compactifications and four
dimensional models suggested in [4] to numerous checks. As the index is independent on
the couplings, it needs to be the same in all duality frames. In particular if the theory
Corresponds to compactification with several maximal punctures, acting with the difference
operator on fugacities of different symmetries, and in any order, should produce the same
result. The latter implies that all the operators have to commute. We claim that,

\[ [O^X_q, O^Z_q] = 0. \] (2.17)

This is a non trivial fact. This will follow from the following theta function identity,

\[
\frac{\theta_p(\frac{z_{21}}{2})\theta_p(\frac{z_{23}}{2})\theta_p(\frac{z_{32}}{2})}{\theta_p(\frac{z_{21}}{2})\theta_p(\frac{z_{23}}{2})} + \frac{\theta_p(\frac{z_{21}}{2})\theta_p(\frac{z_{32}}{2})\theta_p(\frac{z_{31}}{2})}{\theta_p(\frac{z_{21}}{2})\theta_p(\frac{z_{23}}{2})} = \frac{\theta_p(\frac{z_{23}}{2})\theta_p(\frac{z_{31}}{2})\theta_p(\frac{z_{32}}{2})}{\theta_p(\frac{z_{21}}{2})\theta_p(\frac{z_{23}}{2})} + \frac{\theta_p(\frac{z_{21}}{2})\theta_p(\frac{z_{32}}{2})\theta_p(\frac{z_{31}}{2})}{\theta_p(\frac{z_{21}}{2})\theta_p(\frac{z_{23}}{2})}. \] (2.18)

All the operators, for different values of \( M \) and \( L \), should commute and in particular,

\[ [O^X_q, O^Z_q] = 0. \] (2.19)

This follows from elementary identities, for example \( \theta_p(pz) = \theta_p(1/z) \) and \( \theta_p(1/z) = -1/z\theta_p(z) \). We can act with the operators on our basic building block, the trifundamental, and it should not matter on which fugacity we apply the operator. In mathematical jargon
the index of the trifundamental is a kernel function, see for example [23], of the difference
operators,

\[
O^Y_p(z) \cdot \prod_{i,j,l=1}^3 \Gamma_e((qp)^{\frac{1}{2}}z_{i}^{-1}y_{j}^{-1}b_{l}^{-1}) = \]

\[
O^Y_p(y) \cdot \prod_{i,j,l=1}^3 \Gamma_e((qp)^{\frac{1}{2}}z_{i}^{-1}y_{j}^{-1}b_{l}^{-1}) = \] (2.20)

\[
O^Y_p(b) \cdot \prod_{i,j,l=1}^3 \Gamma_e((qp)^{\frac{1}{2}}z_{i}^{-1}y_{j}^{-1}b_{l}^{-1}).
\]
Note that $\prod_i z_i = \prod_j y_j = \prod_l b_l = 1$. This equality reduces to an identity of sum of products of theta functions. We have not proven this but checked it using Mathematica in expansion in the fugacities $p$ and $q$.

The operator is self-adjoint under the vector multiplet measure,

$$
\oint \frac{dz_1}{2\pi i z_1} \oint \frac{dz_2}{2\pi i z_2} \prod_{i<j} \frac{1}{\Gamma_e((z_i/z_j)^{\pm 1})} f(z^{-1}) [O_q^Y(z) \cdot h(z)] = 
\oint \frac{dz_1}{2\pi i z_1} \oint \frac{dz_2}{2\pi i z_2} \prod_{i<j} \frac{1}{\Gamma_e((z_i/z_j)^{\pm 1})} [O_q^Y(z^{-1}) \cdot f(z^{-1})] h(z).
$$

(2.21)

This implies that if we prove the kernel function property the action of the operator will be independent of the choice of the maximal puncture for any theory as we will be able to pull the operator through the integrals. The self-adjointness can be easily shown with the assumptions that the functions do not have poles in some strip around the unit circle. For example, let us look at one of the three terms in the integrand, 

$$
\oint \frac{dz_1}{2\pi i z_1} \oint \frac{dz_2}{2\pi i z_2} \prod_{i<j} \frac{1}{\Gamma_e((z_i/z_j)^{\pm 1})} f(z^{-1}) \frac{\theta_p(p^{\pm Y}(z_2/z_3)^{\pm 1})}{\theta_p(z_2/z_1) \theta_p(z_3/z_1)} g(z_1q^{-\frac{2}{3}}, z_2q^\frac{1}{3}, z_3q^\frac{2}{3}).
$$

(2.22)

We perform change of variables with $z_1 \rightarrow z_1 q^{\frac{2}{3}}$ and $z_2 \rightarrow z_2 q^{-\frac{1}{3}}$. Then the term becomes,

$$
\oint \frac{dz_1}{2\pi i z_1} \oint \frac{dz_2}{2\pi i z_2} \prod_{i<j} \frac{1}{\Gamma_e((z_i/z_3)^{\pm 1})} \frac{1}{\Gamma_e(qz_1/z_2) \Gamma_e(qz_1/z_3) \Gamma_e(q^{-1}z_2/z_1) \Gamma_e(q^{-1}z_3/z_1)} f(z_1^{-1}q^{-\frac{2}{3}}, z_2^{-1}q^{\frac{1}{3}}, z_3^{-1}q^{\frac{2}{3}}) \frac{\theta_p(p^{\pm Y}(z_2/z_3)^{\pm 1})}{\theta_p(q^{-1}z_2/z_1) \theta_p(q^{-1}z_3/z_1)} g(z).
$$

(2.23)

We now use that $\Gamma_e(qz) = \theta_p(z) \Gamma_e(z)$ to write the above,

$$
\oint \frac{dz_1}{2\pi i z_1} \oint \frac{dz_2}{2\pi i z_2} \prod_{i<j} \frac{1}{\Gamma_e((z_i/z_j)^{\pm 1})} f(z_1^{-1}q^{-\frac{2}{3}}, z_2^{-1}q^{\frac{1}{3}}, z_3^{-1}q^{\frac{2}{3}}) \frac{\theta_p(p^{\pm Y}(z_2/z_3)^{\pm 1})}{\theta_p(z_1/z_2) \theta_p(z_1/z_3)} g(z).
$$

(2.24)

This shows the self-adjointness.

We thus find that the properties of the defects are consistent with the conjectured map between compactifications and four dimensional theories of [4] and this is a non trivial check of that suggestion.
3. Case of $SO(8)$ minimal conformal matter

We repeat the analysis of the previous section for the (twisted) compactifications of the minimal conformal matter of type $SO(8)$. The analysis is very similar but the details are a bit different. We refer again to [4] for details. First the general set of models is obtained by gluing together theories with $SU(4)$ gauge groups. The basic building block is a pair of bifundamental fields of $SU(4)$ which we will denote by $\tilde{Q}_i$. The pair $\tilde{Q}_i$ is rotated by $SU(2)_f$, has R charge half, and is not charged under any $U(1)$ symmetry. Here we consider turning on vacuum expectation values to $\text{det}\tilde{Q}_1$ and derivatives of this. The $SU(2)_f$ is broken by the vacuum expectation value. The chiral field $\tilde{Q}_2$ in the IR acquires a mass and decouples. Thus in this case we will obtain surface defects which do not carry any flavor symmetry, see Figure four for illustration.

![Diagram](image)

**Fig. 4:** Two theories $T_1$ and $T_2$ combined by gauging two $SU(4)$ symmetries connected by two bifundamental chiral fields $\tilde{Q}_i$. We assume that there is no $U(1)$ symmetry under which $\tilde{Q}_i$ are charged. We give vacuum expectation value to a baryon $\text{det}\tilde{Q}_1$. The theory in the IR is the two theories glued by gauging diagonal combination of two $SU(4)$ symmetries. The vacuum expectation value breaks $SU(2)_f$ symmetry rotating the two $Q_i$.

Before turning to the derivation of the quantum mechanical model introducing the defects to the index computation we review the basic properties of the map between the compactification of the minimal conformal matter of type $SO(8)$ and the four dimensional gauge theories derived in [4]. Here the basic building block is $\tilde{Q}_i$ with a baryonic superpotential preserving the two $SU(4)$ symmetries. It corresponds to a sphere with two maximal punctures carrying $SU(4)$ symmetry, and two basic punctures which carry no symmetry. The bifundamental with no baryonic superpotential does not correspond by itself to a compactification but when glued to another theory by gauging one of the $SU(4)$ symmetries it adds a new type of puncture carrying $SU(2)$ symmetry, which we will denote as $\tilde{SU}(2)$.

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3 For some curiosities in defining punctures in this set of models see [4].
puncture, and removes one basic puncture. See Figure five for illustration. As in the previous case the generic models have large conformal manifolds on which all the symmetries are broken. The statement is that theories with a maximal punctures reside on the same conformal manifold as the theory with the puncture traded by two $\tilde{SU}(2)$ punctures or six basic punctures. The $\tilde{SU}(2)$ punctures can be traded by three basic punctures.

![Image](image_url) **Fig. 5:** On the left we have theory corresponding to compactification on sphere with two maximal punctures with $SU(4)$ symmetry and two punctures with no symmetry. The former denoted by circles and latter by a cross and referred to as basic punctures. Dashed lines correspond to bidundamental chiral fields for which a baryonic superpotential is turned on. The colored squares correspond to $SU(4)$ global symmetry. On the right there is a pair of bifundamentals. It does not correspond to a compactification by itself, however gluing it to a theory by gauging $SU(4)$ corresponds to removing a basic puncture and adding an $\tilde{SU}(2)$ puncture. From these blocks any theory can be constructed.

In this class of theories we can construct a three punctured sphere with three maximal punctures by combining two bifundamentals with no baryon superpotential and one with the superpotential, see Figure six. Note that here naively the model has $SU(4) \times SU(4) \times SU(2) \times SU(2)$ symmetry but the conjecture of [4] is that somewhere on the conformal manifold the symmetry enhances to $SU(4) \times SU(4) \times SU(4)$. From this block theories corresponding to any Riemann surface can be constructed. The flows we consider giving vacuum expectation values to the baryons close $\tilde{SU}(2)$ puncture to the basic puncture.

We now are ready to discuss the defects in the index computation.

### 3.1. The defect and the index

In this case we denote the index of general theory with a maximal puncture by $I(y)$ with the $y_i$ being fugacities for $SU(4)$. We glue to it $\tilde{Q}_i$ which has an additional $SU(4)$ parametrized by $z_i$ and an $SU(2)$ parametrized by $b$. The index of the combined model is given by the following integral,
\[ I' = (q; q)^3(p; p)^3 \frac{1}{24} \oint \frac{dy^1}{2\pi i y^1} \oint \frac{dy^2}{2\pi i y^2} \oint \frac{dy^3}{2\pi i y^3} \frac{\prod_{i,j=1}^4 \Gamma_e \left((qp)\frac{y^i b^\pm 1}{y^j z_i} \right)}{\prod_{i<j} \Gamma_e \left((y^i/y^j)^\pm 1 \right)} I(y). \]  

(3.1)

We are interested in poles of the index in \( b \), which is in closing the \( SU(2) \) puncture. We will perform the analysis of the divergences in \( b \) and discuss interesting set of poles corresponding to vacuum expectation values to derivatives of baryon \( \tilde{Q}_1 \). The integrand has poles in \( y^1 \) such that the following are inside the unit circle,

\[ y^1 = (qp)^{-\frac{1}{4}}(y^2)^{-1}(y^3)^{-1}z_j^{-1}b^\pm 1q^{l_1}p^{n_4}, \]

(3.2)

and the following are outside of the unit circle,

\[ y^1 = (qp)^{-\frac{1}{4}} z_j^{b^\pm 1}q^{-l_1}p^{-n_1}. \]

(3.3)

Similarly we have poles in \( y^2 \) and \( y^3 \) outside the unit contour,

\[ y^2 = (qp)^{-\frac{1}{4}} z_j^{b^\pm 1}q^{-l_2}p^{-n_2}, \]

\[ y^3 = (qp)^{-\frac{1}{4}} z_j^{b^\pm 1}q^{-l_3}p^{-n_3}. \]

(3.4)

From here a necessary condition if we want all the integration contours to pinch will be,

\[ 1 = q^{1+\sum_{h=1}^4 l_h p + \sum_{y=1}^4 n_y} \prod_{r=1}^4 z_j^{-1}b^x. \]

(3.5)

Fig. 6: On the top we combine a theory with two maximal punctures and two basic ones with two \( \tilde{Q}_i \) resulting in a sphere with three maximal punctures from which any surface can be constructed. On the bottom we give vacuum expectation value to one of the baryons built from a field with no superpotential and obtain a theory with two maximal, one basic puncture, and one \( \tilde{SU}(2) \) puncture. We denote the \( \tilde{SU}(2) \) puncture with colored circle.
This is the weight of a derivative of a baryon. Setting it to 1 thus gives it vacuum expectation value. As we want general poles independent of \( z \) will take without loss \( z_j = z_r \).

We will also take \( x = 4 \) which is corresponds to vacuum expectation values for derivatives of \( \text{det} Q_1 \) (note that it could be \(-4, -2, 0, 2, 4\)). The we have a pole when,

\[
b = (qp)^{-\frac{1}{4}} q^{-\frac{M}{4}} p^{-\frac{L}{4}} .
\]  
(3.6)

First we analyze the case of both \( L \) and \( M = 0 \). We have,

\[
y^i = z_i .
\]  
(3.7)

The integrand evaluates to,

\[
\frac{\prod_{j,l=1}^4 \Gamma_e((qp)^{\frac{1}{2}} z_j/z_l) \Gamma_e(z_j/z_i)}{\prod_{i<j} \Gamma_e(z_j/z_i) \Gamma_e(z^i/z^j)} \mathcal{I}(z) \rightarrow \Gamma_e(1)^4 \mathcal{I}(z) .
\]  
(3.8)

We see that we have order four divergence which gives us a pole and the residue is (as always ignoring overall factors),

\[
\text{Res}_{b \rightarrow (qp)^{-\frac{1}{4}}} \propto \mathcal{I}(z_i) .
\]  
(3.9)

Now we take \( M \) to be 1 and \( L \) to be \( = 0 \). We have taking \( l_1 = 1 \),

\[
y^1 = z_1 q^{-\frac{3}{4}} , \quad y^2 = z_2 q^\frac{1}{4} , \quad y^3 = z_3 q^\frac{1}{4} , \quad y^4 = z_4 q^\frac{1}{4} .
\]  
(3.10)

The integrand becomes,

\[
\frac{\Gamma_e(q^{-1} z_1/z_l) \Gamma_e(z_j/z_l) \Gamma_e((qp)^{\frac{1}{2}} q^{-\frac{3}{4}} z_1/z_j) \Gamma_e((qp)^{\frac{1}{2}} q^\frac{1}{4} z_j/z_l)}{\Gamma_e(q^{-1} z_1/z_l) \Gamma_e(q z_l/z_1) \Gamma_e(z_l/z_j)} \mathcal{I}(z_1 q^{-\frac{3}{4}} , z_2 q^\frac{1}{4} , z_3 q^\frac{1}{4} , z_4 q^\frac{1}{4}) .
\]  
(3.11)

Evaluating the residue we get removing overall factors,

\[
\frac{\theta_p(p^\frac{1}{4} z_3/z_2) \theta_p(p^\frac{1}{4} z_4/z_2) \theta_p(p^\frac{1}{4} z_3/z_4)}{\theta_p(z_2/z_1) \theta_p(z_3/z_1) \theta_p(z_4/z_1)} \mathcal{I}(z_1 q^{-\frac{3}{4}} , z_2 q^\frac{1}{4} , z_3 q^\frac{1}{4} , z_4 q^\frac{1}{4}) .
\]  
(3.12)

We obtain that,

\[
\text{Res}_{b \rightarrow (qp)^{-\frac{1}{4}} q^{-\frac{1}{4}}} \propto \mathcal{O}_q(z) \cdot \mathcal{I}(z) .
\]  
(3.13)
We have,

\[ O_q(z) \cdot \mathcal{I}(z) = \frac{\theta_p(p^{\frac{q}{2}}z_3/z_2)\theta_p(p^{\frac{q}{2}}z_4/z_2)\theta_p(p^{\frac{q}{2}}z_3/z_4)}{\theta_p(z_2/z_1)\theta_p(z_3/z_1)\theta_p(z_4/z_1)}\mathcal{I}(z_{1}q^{-\frac{q}{4}}, z_{2}q^{\frac{q}{4}}, z_{3}q^{\frac{q}{4}}, z_{4}q^{\frac{q}{4}}) + \]

\[ \frac{\theta_p(p^{\frac{q}{2}}z_3/z_1)\theta_p(p^{\frac{q}{2}}z_4/z_1)\theta_p(p^{\frac{q}{2}}z_3/z_4)}{\theta_p(z_2/z_1)\theta_p(z_3/z_1)\theta_p(z_4/z_1)}\mathcal{I}(z_{1}q^{\frac{q}{4}}, z_{2}q^{-\frac{q}{4}}, z_{3}q^{\frac{q}{4}}, z_{4}q^{\frac{q}{4}}) + \]

\[ \frac{\theta_p(p^{\frac{q}{2}}z_3/z_2)\theta_p(p^{\frac{q}{2}}z_4/z_2)\theta_p(p^{\frac{q}{2}}z_3/z_4)}{\theta_p(z_2/z_2)\theta_p(z_2/z_3)\theta_p(z_2/z_4)}\mathcal{I}(z_{1}q^{\frac{q}{4}}, z_{2}q^{\frac{q}{4}}, z_{3}q^{\frac{q}{4}}, z_{4}q^{\frac{q}{4}}) + \]

\[ \frac{\theta_p(p^{\frac{q}{2}}z_3/z_1)\theta_p(p^{\frac{q}{2}}z_4/z_1)\theta_p(p^{\frac{q}{2}}z_3/z_2)}{\theta_p(z_2/z_1)\theta_p(z_2/z_2)\theta_p(z_2/z_4)}\mathcal{I}(z_{1}q^{\frac{q}{4}}, z_{2}q^{\frac{q}{4}}, z_{3}q^{\frac{q}{4}}, z_{4}q^{\frac{q}{4}}). \]

(3.14)

This is the difference operator introducing the surface defect into the index computation.

We can define more general operators taking \( M \) and \( L \) arbitrary. For \( M = 1 \) and \( L \) equal zero we get the same operator with \( q \) and \( p \) exchanged. The operators \( O_q \) and \( O_p \) commute. The operator \( O_q \) (and all the others) should be self-adjoint under vector multiplet measure,

\[ \oint \frac{dz_1}{2\pi i z_1} \oint \frac{dz_2}{2\pi i z_2} \oint \frac{dz_3}{2\pi i z_3} \prod_{i<j} \frac{1}{\Gamma_e((z_i/z_j)^{\pm 1})} f(z^{-1}) \left[ O_q(z) \cdot h(z) \right] = \]

\[ \oint \frac{dz_1}{2\pi i z_1} \oint \frac{dz_2}{2\pi i z_2} \oint \frac{dz_3}{2\pi i z_3} \prod_{i<j} \frac{1}{\Gamma_e((z_i/z_j)^{\pm 1})} \left[ O_q(z^{-1}) \cdot f(z^{-1}) \right] h(z). \]

(3.15)

The proof for the operator (3.14) is elementary as in the previous case. The index of the \( Q_i \) is a kernel function for the operator,

\[ O_p(z) \cdot \prod_{i,j=1}^{4} \Gamma_e((qp)^{\frac{1}{2}}z_{i}^{-1}b^{\pm 1}y_{j}^{-1}) = O_p(y) \cdot \prod_{i,j=1}^{4} \Gamma_e((qp)^{\frac{1}{2}}z_{i}^{-1}y_{j}^{-1}b^{\pm 1}). \]

(3.16)

Note that \( \prod_{i} z_{i} = \prod_{j} y_{j} = 1 \). As in the previous case we checked this equality in expansion in fugacities. These properties guarantee that the action of the difference operator is consistent with the conjectured map between compactifications and four dimensional theories.

Acknowledgments: We would like to thank Simon Ruijsenaars and Gabi Zafrir for comments and relevant discussions. The research was supported by the Israel Science Foundation under grant No. 1696/15 and by the I-CORE Program of the Planning and Budgeting Committee.
Appendix A. Definitions of functions

We define the theta functions and the q-Pochhammer symbol,

\[ \theta_q(z) = \prod_{j=1}^{\infty} (1 - zq^{j-1})(1 - 1/zq^j), \quad (z; q) = \prod_{j=1}^{\infty} (1 - zq^{j-1}). \]  

(A.1)

The elliptic Gamma function is,

\[ \Gamma_e(z) = \prod_{i,j=1}^{\infty} \frac{1 - \frac{q}{z}q^{j-1}p^{i-1}}{1 - zq^{i-1}p^{j-1}}. \]  

(A.2)

We omit the parameters \( q \) and \( p \) from the definitions of the elliptic Gamma function for brevity. We also use the short-hand notations,

\[ f(az \pm 1) = f(az)f(az^{-1}). \]  

(A.3)

The theta functions and the elliptic Gamma functions satisfy many identities and here are some of them,

\[ \theta_p(pz) = \theta_p(1/z) = -\frac{1}{z} \theta_p(z), \quad \Gamma_e(qz) = \theta_p(z)\Gamma_e(z), \]

\[ \Gamma_e(qp/z)\Gamma_e(z) = 1. \]  

(A.4)

See [24] for a useful reference on elliptic Gamma functions and their properties.
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