Hilbert schemes on planar curve singularities are generalized affine Springer fibers

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Abstract
In this paper, we show that Hilbert schemes of planar curve singularities can be interpreted as generalized affine Springer fibers for $GL_n$. This leads to a construction of a rational Cherednik algebra action on their homology, which we compute in examples. This work was inspired in part by a construction in three-dimensional $\mathcal{N} = 4$ gauge theory.

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1. Introduction

Let $\hat{\mathcal{C}} = \text{Spec} \frac{\mathbb{C}[x,t]}{f}$ be the germ of a complex plane curve singularity. In this paper, we investigate a relationship between the Hilbert scheme of points on $\hat{\mathcal{C}}$ and certain generalized affine Springer fibers in the sense of [13].

The Hilbert schemes of points on singular curves have been objects of intense study due to their connections to a wide range of topics including knot theory [14, 28], representation theory [11, 14, 20, 27, 30], and curve counting [31, 32]. Affine Springer fibers, and their various generalizations, have also seen a wide range of study in combinatorics [16], geometry [6, 23], number theory [6, 40] and representation theory [30, 33].

Our motivation comes mainly from recent predictions [10, 29] relating HOMFLY-PT knot homology to a construction in three-dimensional $\mathcal{N} = 4$ supersymmetric gauge theory and the resulting statements for their Coulomb branches, introduced to mathematicians by Braverman-Finkelberg-Nakajima [2]. The representation theory of rational Cherednik algebras as it relates to knot theory [11] and these Coulomb branches [21] also played a strong motivational role.

We now describe our approach in some detail. Interpreting torsion-free modules of $R := \frac{\mathbb{C}[x,t]}{f}$ as lattices in the total ring of fractions Frac($R$), one can identify affine Springer fibers for $GL_n$ with compactified Picard schemes of singular planar curves [22, 24], which was a starting point for Ngô’s proof of the fundamental lemma.
When the polynomial $f$ is irreducible, the affine Springer fibers for $SL_n$ can be identified with compactified Jacobians of the singularity. The compactified Jacobians have been related to the representation theory of rational Cherednik algebras by means of affine Springer theory and a perverse filtration \cite{30} which, thanks to \cite{24, 26}, we know comes from the Hilbert schemes of points via an Abel-Jacobi map.

We take the relation between affine Springer theory of $GL_n$ and Hilbert schemes slightly further on the Hilbert scheme side by interpreting ideals of $R := \mathbb{C}[x, t]/f$ as lattices in the total ring of fractions $\text{Frac}(R)$. These also have a realization as generalized affine Springer fibers in the sense of \cite{13}. In particular, our first main result is the following.

**Theorem 1.1** (Theorem 3.3). Let $\hat{C} := \text{Spec } R$ be a germ of a plane curve singularity and write $R = \mathbb{C}[x, t]/f$. If $f$ has $x$-degree $n$ then there is a generalized $\text{Ad} \oplus V$-affine Springer fiber $M_v \subset \text{Gr}_{GL_n}$ so that there is an isomorphism of (ind-)varieties

$$\varphi : M_v \to \text{Hilb}^\bullet(\hat{C}).$$

It was shown in \cite{21} that the three-dimensional $N = 4$ Coulomb branch algebra, as mathematically defined in \cite{2} by a convolution algebra construction modeled on the affine Grassmannian, for $G = GL_n$ and $N = \text{Ad} \oplus V$, where $\text{Ad}$ is the adjoint representation and $V$ is the fundamental representation, is isomorphic to the spherical rational Cherednik algebra of $GL_n$. See Theorem 4.7.

Using ideas from \cite{17} we find an action of this spherical rational Cherednik algebra on the equivariant Borel-Moore homology of $\text{Hilb}^\bullet(\hat{C})$ as a type of “generalized affine Springer theory.” This fits well with the results of \cite{11, 14, 28, 30}.

For the case where the singularity $\hat{C} = \hat{C}_{n, k}$ is quasihomogeneous and given by $f = x^n - t^k$, we find an action with parameter $m = \frac{k}{n} \hbar$ on the equivariant BM homology with respect to the stabilizer $\mathbb{C}^\times \subset \mathbb{C}^\times_{\text{rot}} \times \mathbb{C}^\times_{\text{dil}}$ of a specific element $v \in N_\mathbb{C}$, realizing an expectation of \cite{28}. When $\gcd(n, k) = 1$, $\text{Hilb}^\bullet(\hat{C}_{n, k})$ has isolated $\mathbb{C}^\times$ fixed points and we can take the analysis quite far. We compute the action in the basis of fixed points by means of an “abelianization procedure” akin to \cite{1, 4, 9} in some cases and more generally are able to prove the following.

**Theorem 1.2** (Theorem 5.4). When $\gcd(n, k) = 1$, we have

$$H^\bullet_{\mathbb{C}^\times}(\text{Hilb}^\bullet(\hat{C}_{n, k})) \simeq eL_{-k/n}(\text{triv})$$

as modules for the spherical rational Cherednik algebra of $\mathfrak{gl}_n$.

**Remark 1.3.** For the case of $(2, 2\ell + 1)$ torus knots we show that this is the case directly; see Appendix B for details. For the remaining cases the direct analysis becomes cumbersome, so we combine earlier work of \cite{20, 30, 35} to conclude the result. It is however remarkable that our approach is, in principle, amenable to completely explicit computation, when compared with e.g. \cite{30}. We also note that Theorem 1.2 is compatible with the earlier results and conjectures of \cite{14, 28, 30, 33} relating modules for the spherical rational Cherednik and $\text{Hilb}^\bullet(\hat{C}_{n, k})$.

The links of these quasihomogeneous singularities correspond to (positive) $(n, k)$ torus links, and it has been known for a while that the representations constructed above are closely connected with corresponding “lowest $a$-degree parts” of the HOMFLY-PT homologies of these links. In particular, our approach with \cite{14, 28} quite directly shows the fact that the rational Cherednik algebra of $GL_n$ acts on these link homologies. This is the subject of Section 5.
Remark 1.4 (For the physically minded reader). As is clear from the introduction, we were inspired in part by the physics of three-dimensional $\mathcal{N} = 4$ gauge theory [10] and its relationship to a recent construction of the triply graded HOMFLY-PT homology [29], whereby the various $a$-degrees are realized within a certain category of matrix factorizations.

In the upcoming (companion) work [10], the construction of [29] is interpreted as a computation in the $B$-twist of $U(n)$ gauge theory with hypermultiplets transforming in the representation $T^*R$ for $R = \mathrm{Ad} \oplus V$. For the $a$-degree $\ell$ component, one computes the supersymmetric Hilbert space of the theory in the presence of a Wilson line in the representation $\bigwedge^\ell V$ subject to a certain boundary condition whose parameters specify the link in question.

The three-dimensional mirror of this construction is a computation in the $A$-twist of the same theory. Again, one computes the supersymmetric Hilbert space of the theory but now in the presence of a vortex line and subject to a different boundary condition. The parameters of this boundary condition translate to the eigenvalues of one of the adjoint fields. For algebraic links, this computation can be reformulated algebraically and one finds that the supersymmetric Hilbert space associated to the lowest $a$-degree component of HOMFLY-PT homology can be computed as the homology of the generalized affine Springer fibers we discuss below.

In the general context of three-dimensional $\mathcal{N} = 4$ theories, the supersymmetric Hilbert spaces associated to boundary conditions and the action of the quantized Coulomb branch on them appeared previously in [4] and [3], and we make their geometric action rigorous via the BFN presentation in Section 4. In many cases of interest, we can realize the action of the Coulomb branch using an "abelianization" procedure, c.f. [1, 3, 36].

A generalization of these Hilbert spaces, and the local operators that act upon them, that includes ($1/2$-BPS) vortex line operators appeared briefly in [4] and was the central aim of [9]. The results of the current paper have a straightforward generalization to higher $a$-degrees: namely, there is a generalization of the construction in Section 3 to the incidence varieties of [28]. The homologies of these incidence varieties (supersymmetric Hilbert spaces in the presence of the above boundary conditions and vortex lines) are naturally endowed with actions of convolution algebras (the algebra of local operators bound to the vortex line) generalizing the Coulomb branch construction of BFN. Some features of this generalization will be discussed in [10]. Understanding the module structure of these homologies is a direction for future work.

Remark 1.5. Most of our results, including the computations with fixed-point localization, make sense over other algebraically closed fields, in particular $\mathbb{F}_q$ with $\mathbb{Q}_l$-coefficients in cohomology. But since it makes life easier, and the results of [2] are also written in the language of algebraic geometry over $\mathbb{C}$, we have decided to work over $\mathbb{C}$ throughout. This also makes the comparison to link homology clearer.

The paper is organized as follows. In Section 2 we recall the necessary definitions of generalized affine Springer fibers $M_v$. In Section 3 we identify the generalized affine Springer fiber (for the datum $(GL_n, \mathrm{Ad} \oplus V)$) isomorphic to $\mathrm{Hilb}^*(\tilde{C})$ for $\tilde{C}$ any germ of a plane curve singularity. In Section 4 we define a convolution action of the quantized Coulomb branches of $\mathbb{C}$ on the equivariant (Borel-Moore) homology of the generalized affine Springer fibers $M_v$, specializing in particular to the action of the spherical rational Cherednik algebra on the equivariant homology of the Hilbert schemes. The proof that the convolution really defines an action is relegated to Appendix A. In Section 5 we discuss the quasihomogeneous singularities $\tilde{C}_{n,k}$ related to $(n, k)$ torus links and show how they relate to rational Cherednik algebra representations. In Appendix B we discuss $(2, k)$ torus links in detail.
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2. Generalized affine Springer theory

This section is written in more generality than is needed for our main results. Let $G/\mathbb{C}$ be reductive, $\mathfrak{g} = \text{Lie}(G)$, and $N$ be an algebraic representation of $G$. Let $K = \mathbb{C}((t))$ and $\mathcal{O} = \mathbb{C}[[t]]$. Let $\text{Gr}_G$ be the affine Grassmannian of $G$. On the level of $\mathbb{C}$-points, $\text{Gr}_G(\mathbb{C}) = G(K)/G(\mathcal{O})$.

**Definition 2.1.** Let $v \in N(K)$. Define the generalized affine Springer fiber (GASF) associated to $v$ as the reduced ind-scheme whose closed points are $M_v(\mathbb{C}) := \{g \in G(K)|g^{-1}.v \in N(\mathcal{O})\}/G(\mathcal{O})$.

**Remark 2.2.** Note that the definition of $M_v$ also depends on $G, N$. Since we will only be working with $G = GL_n, N = \text{Ad} \oplus V$, we mostly omit these from the notation.

**Remark 2.3.** The “classical” affine Springer fibers are the case where $N = \text{Ad}$. In [13] an even more general class of “generalized affine Springer fibers” coming from data similar to a Hessenberg variety are considered. Note that both our GASF and those of [13] are different from the Kottwitz-Viehmann varieties, which are group versions of affine Springer fibers.

In [30, 33, 34], an action of the (degenerate) double affine Hecke algebra of $SL_n$ was constructed on the equivariant (K-)homology of certain (usual) affine Springer fibers using the convolution algebra technique (see e.g. [2]).

Just as affine Springer fibers are a source of affine Springer representations of affine Weyl groups and Cherednik algebras, generalized affine Springer fibers can be used to construct representations of certain convolution algebras associated to the datum $(G, N)$ as defined in [2]. These are the “quantized Coulomb branches” of three-dimensional $\mathcal{N} = 4$ field theories, or “BFN algebras”.

In particular, in [17], the convolution algebra technique was extended to any Coulomb branch algebra. The authors of loc. cit. were kind enough to share their preliminary results on the topic with us, and we expand upon these results in Section 4 (which focuses on the $N = \text{Ad} \oplus V$ case) and in Appendix A.

**Remark 2.4.** In analogy with [38], we expect there to be a “global” Springer theory defined on certain generalized Hitchin spaces (quasimaps’ spaces) at least for $N$ with good invariant-theoretic properties. This direction will be pursued in future work.
3. Hilbert schemes of points on curve singularities

Let $\hat{C} := \text{Spec } R$ be the germ of a plane curve singularity and write $R = \mathbb{C}[\![x, t]\!] / f$.

**Definition 3.1.** The *Hilbert scheme of $m$ points on $\hat{C}$* is defined as the reduced scheme

$$\hat{C}^{[m]} := \text{Hilb}^m(\hat{C}) := \{ \text{colength } m \text{ ideals in } R \}.$$ 

**Remark 3.2.** In particular, the reduced scheme

$$\text{Hilb}^\bullet(\hat{C}) := \bigsqcup_{m \geq 0} \text{Hilb}^m(\hat{C})$$

is naturally the moduli space of finite length subschemes on $\hat{C}$, see e.g. [20] for more references on the moduli problem and its solution.

We now state and prove our main theorem.

**Theorem 3.3.** For any $\hat{C}$, there is a generalized $Ad \oplus V$-affine Springer fiber $M_v \subset \text{Gr}_G$ so that there is an isomorphism of schemes

$$\varphi : M_v \rightarrow \text{Hilb}^\bullet(\hat{C})$$

**Proof.** Note that we can interpret $\hat{C}$ and $\hat{C}^{[m]}$ as follows. By Weierstrass preparation, we can assume $f(x, t)$ is a degree $n$ polynomial in $x$. Then we may write as $\mathbb{C}[\![t]\!] = \mathcal{O}$-modules that

$$\mathbb{C}[\![x, t]\!] / f = \langle 1, x, \ldots, x^{n-1} \rangle_\mathcal{O}, \quad (3.1)$$

where $\langle S \rangle_\mathcal{O}$ denotes the free $\mathcal{O}$-module generated by a set $S$.

Taking the total ring of fractions of $R$, we see that as $\mathbb{C}(t) = \mathcal{K}$-vector spaces $\text{Frac}(R) \cong (\mathcal{K}^n)^*$ ($\mathcal{K}$-linear dual of $\mathcal{K}^n$) as follows. If $f$ is squarefree so that $\hat{C}$ is reduced, $\text{Frac}(R) \cong \prod_{i=1}^d F_i$, where $d$ is the number of irreducible factors over $\mathcal{K}$ of $f$ and $F_i$ are finite extensions of $\mathcal{K}$ so that $\sum [F_i : \mathcal{K}] = n$.

If $f$ has a repeated factor, we take $R \cong \prod_{i=1}^d \mathcal{O}_i$ where each $\mathcal{O}_i$ is some finite ring extension of $\mathcal{O}$ which is torsion-free over $\mathcal{O}$. Since $\mathcal{O}$ is a domain, $\text{Frac}(\mathcal{O}_i) \cong \mathcal{O}_i \otimes_\mathcal{O} \mathcal{K}$. In particular, $\text{Frac}(R) \cong (\mathcal{K}^n)^*$.

There is a natural injection $R \hookrightarrow \text{Frac}(R)$, and we choose an isomorphism $\phi^* : \text{Frac}(R) \cong (\mathcal{K}^n)^*$ identifying $R$ with $(\mathcal{O}^n)^*$ and $1 \in R$ with the vector $e_n^* = (0, 0, \ldots, 1)$ in $(\mathcal{K}^n)^*$. We may moreover choose $\phi^*$ so that in the costandard basis of $(\mathcal{K}^n)^*$, $x$ has the form

$$\hat{\gamma} = \begin{pmatrix}
a_{n-1} & a_{n-2} & \cdots & a_1 & a_0 \\
1 & 0 & \cdots & 0 & 0 \\
0 & 1 & \ddots & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & 1 & 0
\end{pmatrix}$$
so that \( \{e_k^\gamma = e_k^\gamma x^{-k} \}_{k=0}^{n-1} \) is an \( \mathcal{O} \)-basis of \( (\mathcal{O}^n)^* \). Recall that a matrix of the above form is called the \textit{companion matrix} of the polynomial \( x^n - a_{n-1}x^{n-1} - \cdots - a_1x - a_0 \).

By definition, \( \mathcal{O} \)-lattices in \( (K^n)^* \) stable under \( x \) are identified with (nonzero) fractional \( \mathcal{R} \)-ideals. The variety of nonzero ideals of finite codimension in \( \mathcal{R} \) is then identified with fractional ideals in \( \text{Frac}(\mathcal{R}) \) contained in \( \mathcal{R} \). Indeed, note that the condition of being a lattice implies that tensoring \( \Lambda \) with \( K \) and projecting to each factor of \( K^* \) is a surjective map, hence the corresponding ideal is of finite codimension. Under \( \phi^* \), we get

\[
\text{Hilb}^* (\widehat{C}) \cong X := \{ \Lambda \subset (\mathcal{O}^n)^* | \Lambda \gamma \subset \Lambda \}.
\]

Now for any \( \Lambda \), there is an element \( g \in G(K) \) so that \( \Lambda = (\mathcal{O}^n)^* g^{-1} \). It is well defined up to the stabilizer of \( (\mathcal{O}^n)^* \), which is \( G(\mathcal{O}) \). If \( \Lambda \subset (\mathcal{O}^n)^* \) and \( \Lambda \gamma \subset \Lambda \), we have

1. \( g^{-1} \in G(K) \cap \mathfrak{gl}_n(\mathcal{O}) \), because \( (\mathcal{O}^n)^* g^{-1} = \Lambda \subset (\mathcal{O}^n)^* \), and
2. \( g^{-1} \gamma g \in \text{Ad}(\mathcal{O}) \), because \( (\mathcal{O}^n)^* g^{-1} \gamma g = \Lambda \gamma g \subset \Lambda g = (\mathcal{O}^n)^* \) and the stabilizer of \( (\mathcal{O}^n)^* \) is \( \mathfrak{gl}_n(\mathcal{O}) = \text{Ad}(\mathcal{O}) \).

If \( e_1 \) denotes the standard basis in \( K^n \), the first point implies that \( g^{-1} e_1 \) belongs to \( \mathcal{O}^n \).

Let \( v := (\gamma, e_1) \) and consider the map

\[
\Lambda \mapsto [g]
\]

from \( X \) to the scheme

\[
M_v = \{ [g] \in \text{Gr}_G | g^{-1} \gamma g \in \text{Ad}(\mathcal{O}), g^{-1} e_1 \in \mathcal{O}^n \}.
\]

We will construct an inverse to this map. Given any \( [g] \in M_v \), we have

1. \( g^{-1} \in G(K) \cap \mathfrak{gl}_n(\mathcal{O}) \), because \( g^{-1} e_1 \in \mathcal{O}^n \), \( g^{-1} \gamma g \in \text{Ad}(\mathcal{O}) \) and \( g^{-1} e_k = (g^{-1} \gamma g)^{k-1} g^{-1} e_1 \), and
2. \( (\mathcal{O}^n)^* g^{-1} \gamma \subset (\mathcal{O}^n)^* g^{-1} \), because \( g^{-1} \gamma g \in \text{Ad}(\mathcal{O}) \).

The first point implies that \( \Lambda = (\mathcal{O}^n)^* g^{-1} \subset (\mathcal{O}^n)^* \) and the second implies \( \Lambda \) is closed under the action of \( \gamma \), i.e. \( \Lambda \subset X \). As these constructions are inverse to each other, we have \( X \cong M_v \).

Finally, composing with the isomorphism to \( \text{Hilb}^* (\widehat{C}) \) we get that

\[
\text{Hilb}^* (\widehat{C}) \cong M_v.
\]

By Definition \ref{def:generalized_ad affine Springer fiber}, the space \( M_v \) is the generalized \( \text{Ad} \oplus V \)-affine Springer fiber for \( v = (\gamma, e_1) \).

**Remark 3.4.** Given \([g] \in \text{Gr}_{GL_n} \), we could have instead considered \( \Lambda^{op} = g\mathcal{O}^n \) which satisfies \( \mathcal{O}^n \subset \Lambda \) and \( \gamma \Lambda \subset \Lambda \), so that we also get an isomorphism

\[
M_v \cong X^{op} := \{ \mathcal{O}^n \subset \Lambda^{op} | \gamma \Lambda^{op} \subset \Lambda^{op} \}.
\]

An equivalent, perhaps preferred, description of \( \text{Hilb}^* (\widehat{C}) \) is as lattices \( \Lambda \subset \mathcal{O}^n \) such that \( \gamma \Lambda \subset \Lambda \). Following the above proof one finds an isomorphism to the generalized \( \text{Ad} \oplus V^* \)-affine Springer fiber \( M'_w \) for the vector \( w = (\widetilde{\gamma}, e_n^\gamma) \in \text{Ad}(\mathcal{O}) \oplus (\mathcal{O}^n)^* \), c.f. \cite{40}, where \( \widetilde{\gamma} \) is obtained from \( \gamma \) by transposing along the anti-diagonal.
The induced isomorphism $M_v \cong M_w'$ comes from the following. As $F = \text{Frac}(R)$ is a finite étale extension of $K$, there is a natural modified trace $tr : F \times F \to K$, given by $tr(xy/d)$, where $d$ is a generator of the different ideal of $R$ in $\mathcal{O}$. The normalization is chosen so that if we define $I^r = \{ f \in F | tr(fI) \subset \mathcal{O} \}$, then $R^r = R$ (see [39] for details). By our chosen isomorphisms $\phi, \phi^*$ we then see that the trace pairing becomes the natural pairing $K_n \times (K_n)^* \to K$. In particular, the map $M_w' \to M_v$ corresponding to the above involution is given by $(g, g^{-1} \hat{\gamma} g, e_1^*) \mapsto \langle g, g^{-1} \hat{\gamma} g, g^{-1} e_1 \rangle$, sending lattices contained in the standard lattice to those containing the standard lattice.

Remark 3.5. Note that the proof doesn’t assume $\hat{C}$ to be reduced. In particular, this suggests us to define the “compactified Picard variety” $\text{Pic}(\hat{C})$ as the classical affine Springer fiber for $GL_n$, although it is usually not considered in the literature. For example, when $\gamma$ is the regular nilpotent matrix, we recover Hilbert schemes of points on the nonreduced curve $\{ x^n = 0 \}$.

Remark 3.6. Note that by $\text{Hilb}^\bullet(\hat{C}) \cong \{ \Lambda \subset (\mathcal{O}^n)^* | \Lambda \gamma \subset \Lambda \}$ we may identify $\text{Hilb}^\bullet(\hat{C})$ as the intersection

$$\text{Sp}_\gamma \cap \text{Gr}_{GL_n}^-$$

where $\text{Sp}_\gamma$ is the “usual” ($N = \text{Ad}$) affine Springer fiber of $\gamma$ and $\text{Gr}_{GL_n}^-$ is the negative part of the affine Grassmannian

$$\text{Gr}_{GL_n}^- := \{ \mathcal{O}^n \subset \Lambda \subset K^n \}$$

not to be confused with the “negative Grassmannian” which is a distantely related object of intense research. See also [19, Remark 4.24].

Remark 3.7. Using the decomposition of $\text{Gr}_{GL_n}$ by $\pi_1(GL_n) = \mathbb{Z}$ we find that $M_v$ can be expressed as

$$M_v = \bigsqcup_{m \leq 0} M_v^m,$$

where $M_v^m$ is the component of $M_v$ inside the degree $m$ part of $\text{Gr}_G$. Indeed, we have $M_v^0 = \text{Hilb}^{[m]}(\hat{C})$. Thus $M_v$ is a (infinite) disjoint union of projective varieties, because the Hilbert schemes are projective [13].

3.1. Links and torus actions

If $f(x, t)$ is a polynomial, we may interpret $\hat{C}$ as the germ of the curve $C = \{ f = 0 \} \subset \mathbb{C}^2$. In this case, the intersection of $C$ with a small three-sphere centered at the origin yields a compact one-manifold

$$\mathcal{L} := \text{Link}_0(C) \hookrightarrow S^3.$$

By work of Oblomkov-Rasmussen-Shende and others (see [25] and references therein) it is known that topologically, the Hilbert schemes of $\hat{C}$ are controlled by the HOMFLY-PT homology of the corresponding link $\mathcal{L}$.

Consider $f$ of the form $f = x^u - t^k$ for $u, k \geq 0$. The special form of $f$ in this case means that the singularity is \textit{quasihomogeneous}, so there is a straightforward $\mathbb{C}^\times$ action on $M_{(u, k)} := M_v$.
coming from scaling $x$ and $t$. As has been noted by various authors, we thus get an extra torus action on the Hilbert schemes. This is more nontrivial on the generalized affine Springer fiber side.

Namely, let $1 \rightarrow G \rightarrow \tilde{G} \rightarrow G_F \rightarrow 1$ be an extension of algebraic groups over $\mathbb{C}$ and let $\tilde{G}_K^\circ$ be the preimage in $\tilde{G}_K$ of $G_F, \circ$. With our definition of $M_v$, we always have an action of the stabilizer of $v$ in $\tilde{G}_K^\circ \times \mathbb{C}^\times_{rot}$ on $M_v$ (see the next section). Let $G = GL_n, G_F = \mathbb{C}^\times_{dil}, \tilde{G} = GL_n \times \mathbb{C}^\times_{dil}$, where $\mathbb{C}^\times_{dil}$ acts by dilating the Ad-part in $\text{Ad} \oplus V$. This action is considered in [30] in the case of usual affine Springer fibers, where $\mathbb{C}^\times_{rot}, \mathbb{C}^\times_{dil}$ are denoted $\mathbb{G}^\times_m, \mathbb{C}^\times_m$. For $v = (\gamma, e_1)$ corresponding to $f = x^n - t^k$ as in Theorem [3.3] the stabilizer is given as follows. It is worth noting that we use different conventions from the usual (physical) conventions used for $\mathbb{C}^\times_{rot}$ in some of the literature [2, 5]. In particular, we do not include the overall scaling of $N$ by weight $\frac{1}{2}$ in addition to scaling $t$. These conventions are those used by Webster, see e.g. [36].

**Lemma 3.8.** We have

$$L_v := \text{Stab}_{\tilde{G}_K^\circ \times \mathbb{C}^\times_{rot}} (v) \cong \mathbb{C}^\times.$$  

**Proof.** Consider acting with $(g, \mu, \lambda) \in \tilde{G}_K^\circ \times \mathbb{C}^\times_{rot}$ on $v = (\gamma, e_1)$ for $v$ corresponding to $f = x^n - t^k$. Here $\mu$ denotes the flavor part of $\tilde{g} = (g, \mu) \in \tilde{G}_K^\circ$. Preserving the determinant of $\gamma$ imposes the equation

$$\mu^n \lambda^k = 1.$$  

Preserving $e_1$ then says that the first column of $g$ is $e_1$, thus the first column of $g^{-1}$ is also $e_1$. From this, we find that the first column of $g\gamma g^{-1}$ is the second column of $g\mu$, so we need this column of $g$ to be $\mu^{-1} e_2$ for $g$ to preserve the first column of $g\gamma g^{-1}$. This process continues column-by-column so we must have

$$g = \text{diag}(1, \mu^{-1}, \ldots, \mu^{1-n}).$$  

In particular, the stabilizer is the image of the cocharacter $\mathbb{C}^\times \rightarrow \tilde{G}_K^\circ \times \mathbb{C}^\times_{rot}$ given by

$$\nu \mapsto (\text{diag}(1, \nu^k, \ldots, \nu^{(n-1)k}, \nu^{-k}, \nu^n)).$$  

□

**Remark 3.9.** In general, for nonhomogeneous $\gamma$, it’s always the case that the stabilizer is trivial by a similar argument. On the other hand, the same proof shows that $\gamma$ for the curve $\{x^n = 0\}$ has stabilizer $(\mathbb{C}^\times)^2$ given by $\text{diag}(1, \mu^{-1}, \ldots, \mu^{1-n}, \mu, \lambda)$.

**Remark 3.10.** Since $\text{Sp}_\gamma$ has a $T \times \mathbb{C}^\times_{rot}$-action in the non-coprime/multiple component case and $\text{Gr}_{\tilde{G}_L}$ is a stable subset for this action, we also get a large torus action on $\text{Hilb}^\bullet(\tilde{C})$. This has not been considered in the literature and seems harder to describe from the point of view of the Hilbert scheme.

**Proposition 3.11.** In the case $\gcd(n, k) = 1$, the action of $L_v$ on $M_v$ has isolated fixed points labeled by cocharacters $A$ of the maximal torus $T \subset GL_n$ such that

$$\langle A, \omega_1 \rangle \geq 0 \quad \langle A, \alpha_i \rangle \leq 0 \quad \sum_{i=1}^{n-1} \langle A, \alpha_i \rangle \geq -k,$$  

(3.2)
where $\omega_1$ is the fundamental weight of $GL_n$, $\alpha_i$ are the simple roots of $GL_n$ and $\langle , \rangle$ is the pairing of cocharacters and weights.

**Proof.** The action of $\nu \in L_v$ on $[g] \in M_v$ is simply $[\nu g]$, where the product of $L$ and $G(K)$ are viewed within $G(K) \rtimes \mathbb{C}^\times$. In particular, we have

$$\nu_* [g(t)] = [\nu(0, k, \ldots, (n-1) k) g(\nu^n t)]$$

where $\nu^{(m_1, \ldots, m_n)} := \text{diag}(\nu^{m_1}, \ldots, \nu^{m_n})$. Define the "orbital variety" (see the next section for motivation)

$$\tilde{V}^v := G_K \cdot v \cap N_O.$$

A point $g(t) \in \tilde{V}^v$ will not be invariant under $L$ but will require a compensating $G(O)$ transformation.

We now describe $M_v \cong \tilde{V}^v / G(O)$. By the Iwasawa decomposition of $G(K)$, we can choose to represent elements of $Gr_G$ by an upper-triangular matrix in $G(K)$ of the form $h = t^{-A} + u$, where $u$ is strictly upper triangular. Moreover, we can always use $G(O)$ to make the (non-zero) $u_{ij}$ Laurent polynomials and with no terms of degree larger than $-A_i - 1$, c.f. 23. We interpret $A$ as a cocharacter of $T \subset GL_n$.

Under the action of $\nu$, the diagonal entries of $h$ transform as $t^{-A_i} \mapsto \nu^{(i-1)k - n A_i} t^A_i$ whereas $u_{ij}(t) \mapsto \nu^{(i-1)k} u_{ij}(\nu^n t)$. We can always return the diagonal entries to $t^{-A_i}$ by means of a diagonal $G(O)$ transformation, sending $\nu^{(i-1)k} u_{ij}(\nu^n t) \mapsto \nu^{(i-1)k + n A_i} u_{ij}(\nu^n t)$. Since the non-zero entries of $u$ are (Laurent) polynomial and have degree at most $-A_i - 1$, it follows that there is no upper-triangular matrix that can send this back to $h$. For example, when $j = i+1$ we must solve the equation

$$\nu^{k+n A_i} u_{ii+1}(\nu^n t) + t^{-A_i} p_i(t) = u_{ii+1}(t)$$

for $p_i(t) \in O$. This requires $t^{A_i} (u_{ii+1}(t) - \nu^{k+n A_i} u_{ii+1}(\nu^n t))$ to belong to $O$, hence

$$u_{ii+1}(t) - \nu^{k+n A_i} u_{ii+1}(\nu^n t) = 0$$

since $u_{ii+1}$ has no terms of degree more than $-A_i - 1$. Finally, since $k$ is coprime to $n$ we conclude that $u_{ii+1}(t) = 0$. With $u_{ii+1} = 0$, it is straightforward to inductively show that $u = 0$.

For $t^A \cdot v$ to belong to $\tilde{V}^v$, for $v$ corresponding to the $(n, k)$ torus knot, requires

$$\langle A, \omega_1 \rangle \geq 0 \quad \langle A, \alpha_i \rangle \leq 0 \quad \sum_{i=1}^{n-1} \langle A, \alpha_i \rangle \geq -k.$$

\[\square\]

**Remark 3.12.** When $n$ and $k$ are not coprime it is possible to have

$$u_{ii+1}(t) - \nu^{k+n A_i} u_{ii+1}(\nu^n t) = 0$$

for $u_{ii+1}(t)$ nonzero. In these circumstances there are still fixed points but they need not be isolated. See also Remark 8310.

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1We parameterize $h$ with $t^{-A}$ as opposed to $t^A$ so that our fixed points are labeled by positive cocharacters; i.e. positive linear combinations of the usual basis vector for the cocharacter lattice of $T \subset GL_n$. 

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4. Action of the spherical RCA

In this section, we construct an action of the (spherical subalgebra of the) rational Cherednik algebra for \( \mathfrak{gl}_n \) on equivariant BM homologies of Hilbert schemes.

We first recall the construction of the BFN algebras in general. Suppose \( 1 \to G \to \tilde{G} \to G_F \to 1 \) is an extension of algebraic groups and let \( \tilde{G}_K^O \) be the preimage in \( \tilde{G}_K \) of \( G_F \). Note that

\[
\text{Gr}_G \cong \tilde{G}_K^O/\tilde{G}_O \cong (\tilde{G}_K^O \rtimes \mathbb{C}_{\text{rot}})/(\tilde{G}_O \rtimes \mathbb{C}_{\text{rot}}).
\]

Let \( N \) be an algebraic representation of \( \tilde{G} \).

Definition 4.1. Define the BFN space of \((G, N)\) as

\[
\mathcal{R}_{G,N} = \{([g], s) \in \text{Gr}_G \times N_O | g^{-1} \cdot s \in N_O \}.
\]

Remark 4.2. We naturally have \( \mathcal{R}_{G,N} \subset T_{G,N} := G_K \times G_{O,N} \cong \{([g], s) \in \text{Gr}_G \times N_O | g^{-1} \cdot s \in N_O \} \). The last isomorphism is given by the embedding \([g, s] \to ([g], g \cdot s)\), see [2, discussion on p.6]. We use these descriptions interchangeably. The vector bundle \( T_{G,N} \) is of infinite rank and has the modular interpretation

\[
T_{G,N} \cong \{(P, \sigma, s) | P \text{ is a } G \text{-torsor on the formal disk } D, \sigma : P|_D \to G|_D, s \in \Gamma(D, P \times G_N) \}.
\]

The locally closed sub-ind-scheme \( \mathcal{R}_{G,N} \) consists of those triples where \( \sigma(s) \) extends to a section over \( D \).

Theorem 4.3 (Braverman-Finkelberg-Nakajima). There is a natural convolution product on \( A_{G,N} := H^{G_O}(\mathcal{R}_{G,N}) \) and \( A^h_{G,N} := H^{G_O \rtimes \mathbb{C}_{\text{rot}}}(\mathcal{R}_{G,N}) \), making them associative algebras with unit. Moreover, \( A^h_{G,N} \) is a filtered quantization of \( A_{G,N} \), which is commutative.

Definition 4.4. We will call either of these algebras the BFN algebra or the (quantized) Coulomb branch.

Remark 4.5. The BFN algebra \( A_{G,N} \) and its quantization have natural deformations given an extension as above. Namely, the homologies \( \tilde{A}_{G,N} := H^{G_O}(\mathcal{R}_{G,N}) \) and \( \tilde{A}^h_{G,N} := H^{G_O \rtimes \mathbb{C}_{\text{rot}}}(\mathcal{R}_{G,N}) \) have the structures of algebras that deform \( A_{G,N} \) and \( A^h_{G,N} \), respectively, with \( \tilde{A}^h_{G,N} \) a filtered quantization of the commutative \( \tilde{A}_{G,N} \). See [2, Section 3(viii)] for more details. This physically corresponds to turning on complex mass parameters for the flavor group \( G_F \). In that context, one assumes that \( G_F \) is a torus.

4.1. Convolution action of Coulomb branches on GASF

Recall that we have defined the BFN space \( \mathcal{R} := \mathcal{R}_{G,N} \) of a representation \( N \). We will also consider the infinite-rank vector bundle

\[
\mathcal{T} := T_{G,N} \to \text{Gr}_G, ([g], s) \mapsto [g].
\]
Recall also that
\[ V^v := (\tilde{G}_K^O \rtimes \mathbb{C}_{rot}^\times).v \cap N_\mathcal{O}. \]
This is analogous to the orbital varieties in [7], and is also called such by [17]. Note that on the level of closed points (which is what we are concerned with, since we only work with the reduced structure), it is clear that \( V^v/(\tilde{G}_O \rtimes \mathbb{C}_{rot}^\times) = M_v. \)

We now define the convolution action of \( \tilde{A}_\hbar := \tilde{A}_\hbar G_0 N, \) following [2] and [17].

**Theorem 4.6.** Suppose the stabilizer \( L_v \) of \( v \) is contained in \( \tilde{G}_O \rtimes \mathbb{C}_{rot}^\times \). Then there is an action of \( \tilde{A}_\hbar \) on \( \mathcal{H}^L_v(M_v) \).

**Proof.** Note that there is a natural map
\[
\pi: \tilde{G}_O K \rtimes C \times \mathrm{rot} \times \mathcal{N}_O \to T \times \mathcal{N}_O \quad (4.1)
\]
given by
\[
(g, s) \mapsto ([g], g.s, s).
\]
Consider the groupoid
\[
\mathcal{P} := \{(g, s) \in \tilde{G}_K^O \rtimes \mathbb{C}_{rot}^\times \times \mathcal{N}_O | g.s \in \mathcal{N}_O \} \xrightarrow{\pi_1: (g, s) \mapsto s} \mathcal{N}_O.
\]
Note that there is another projection map \( \pi_2 \) to \( \mathcal{N}_O \) given by \((g, s) \mapsto g.s\). Then for \( \mathcal{F}_v := \omega_{V^v[-2 \dim \tilde{G}_O]} \) (which is an object in the \( \tilde{G}_O \times \mathbb{C}_{rot}^\times \)-equivariant derived category of \( \mathcal{N}_O \) we have a natural isomorphism
\[
\pi_1^* \mathcal{F}_v \cong \pi_2^* \mathcal{F}_v.
\]
Let \( L_v \) be the stabilizer of \( v \) in \( \tilde{G}_K^O \times \mathbb{C}_{rot}^\times \). Taking the equivariant cohomology of \( \mathcal{N}_O \) with coefficients in \( \mathcal{F}_v \), we get
\[
\mathcal{H}^L_v(M_v) \cong H^\ast_{\mathcal{G}_O \times \mathbb{C}_{rot}^\times}(V^v),
\]
where the left-hand side makes sense because \( L \) is compact.

By definition we have \( p^{-1}(\mathcal{R} \times \mathcal{N}_O) = \mathcal{P} \), and that \( m \circ q = \pi_2, \pi \circ j = \pi_1 \), where \( \pi : \tilde{G}_K^O \times \mathbb{C}_{rot}^\times \times \mathcal{N}_O \to \mathcal{N}_O \) is the projection.

Consider then the following diagram:
\[
\begin{array}{ccc}
\mathcal{R} \times \mathcal{N}_O & \xleftarrow{p} & \mathcal{P} \\
\downarrow{id} & & \downarrow{j} \\
\mathcal{T} \times \mathcal{N}_O & \xleftarrow{p} & \tilde{G}_K^O \times \mathbb{C}_{rot}^\times \times \mathcal{N}_O
\end{array}
\]
Here \( p: (g, s) \mapsto ([g], g.s, s) \) is as above, \( q: (g, s) \mapsto [g, s] \) is quotient by the \( \tilde{G}_O \times \mathbb{C}_{rot}^\times \)-action \( h.(g, s) = (gh^{-1}, h.s) \) and \( m \) is the multiplication map \([g, s] \mapsto g.s\).

Using the “restriction with support” map of Section [Appendix A.1](see [2, Section 3(ii)]) applied to the leftmost Cartesian square, and the map
\[
p^* \omega_T [-2 \dim \mathcal{N}_O] \boxtimes \mathcal{F}_v \cong \omega_{\tilde{G}_K^O \times \mathbb{C}_{rot}^\times} [-2 \dim \tilde{G}_O \times \mathbb{C}_{rot}^\times] \boxtimes \mathcal{F}_v
\]
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we get a map (omitting the shifts for sake of readability)

\[ p^* : H_{\tilde{G}_O \times C_{rot}^{\times}}(\mathbb{R} \times N_O, \omega_R \boxtimes F_v) = H_{\tilde{G}_O \times C_{rot}^{\times}}(\mathbb{R}) \otimes H_{\tilde{G}_O \times C_{rot}^{\times}}(\mathcal{V})^n \to H_{\tilde{G}_O \times C_{rot}^{\times}}(P, \pi_1^1 F_v). \]

Since \( F_v \) is a \( \tilde{G}_O \)-equivariant complex, we have \( \pi_1^1 F_v \cong \pi_2^1 F_v \) and since \( \pi_2 = m \circ q \), we get

\[ H_{\tilde{G}_O \times C_{rot}^{\times}}(P, \pi_1^1 F_v) = H_{\tilde{G}_O \times C_{rot}^{\times}}(q(P), m^1 F_v) \]

Finally, \( m \) is proper, so that using the adjunction \( m \circ q \to id \) we get a map

\[ (m \circ q)_* : H_{\tilde{G}_O \times C_{rot}^{\times}}(q(p^{-1}(P)), m^1 F_v) \to H_{\tilde{G}_O \times C_{rot}^{\times}}(\mathcal{V}). \]

In particular, composing gives us an “intersection pairing”

\[ * := (m \circ q)_* p^* : H_{\tilde{G}_O \times C_{rot}^{\times}}(\mathbb{R}) \times H_{L_v}(M_v) \to H_{L_v}(M_v). \]

This is clearly bilinear over \( \mathbb{Q} \). We prove the associativity in Lemma Appendix A.3 and the fact that the identity acts by 1 in Lemma Appendix A.4.

### 4.1.1. The case of Hilbert schemes

Specializing the construction of the Theorem to \( N = Ad \oplus V \), the \( L_v \)-equivariant homology of our GASF admits an action of the spherical rational Cherednik algebra of \( \mathfrak{gl}_n \), as we now describe.

**Theorem 4.7** (Kodera-Nakajima). For \( G = GL_n, N = Ad \oplus V \), the quantized BFN algebra \( \tilde{A}_G^{\mathbb{C}^{\times}} \) is isomorphic to the spherical rational Cherednik algebra of \( \mathfrak{gl}_n \).

**Remark 4.8.** The extended group \( \tilde{G} \) in the above theorem is simply \( G \times G_F \) where \( G_F = \mathbb{C}^{\times}_{dil} \) acts by scaling \( Ad \) with weight 1 and \( V \) with weight 0.

**Definition 4.9.** The rational Cherednik algebra of \( \mathfrak{gl}_n \) is the quotient algebra

\[ \mathcal{H}_n = \mathbb{C}[[\hbar, m]][x_1, \ldots, x_n, y_1, \ldots, y_n] \rtimes \mathbb{C}S_n \]

where \( \sim \) consists of the relations \( [x_i, x_j] = [y_i, y_j] = 0 \) for all \( i, j \), and

\[ [y_i, x_j] = \begin{cases} -\hbar + m \sum_{k \neq i} (i \, k) & \text{if } i = j, \\ -m(i \, j) & \text{if } i \neq j. \end{cases} \]

The spherical subalgebra is defined as \( e \mathcal{H}_n e \) where \( e = |\mathbb{C}_n|^{-1} \sum_{w \in \mathbb{C}_n} w \). We often refer to the spherical subalgebra simply as the spherical rational Cherednik algebra of \( \mathfrak{gl}_n \).

In the situation of Theorem 3.3 we get

**Corollary 4.10.** The spherical rational Cherednik algebra \( e \mathcal{H}_n e \) of \( \mathfrak{gl}_n \) acts on \( H_{L_v}(\text{Hilb}^n(\tilde{C})) \) where \( L_v \) is the stabilizer in \( \tilde{G}_K^{\mathbb{C}^{\times}} \times C_{rot}^{\times} \) of \( v \in \text{Ad}(K) \oplus K^n \) associated to \( \tilde{C} \) as in Theorem 3.3.
\[ \text{4.2. Comparison of the convolution action to an action by correspondences} \]

For many of our results, in particular Theorem 5.4, we will need to compare the convolution action from Theorem 4.6 to another action by correspondences. We will do this again in greater generality than needed for the rest of the paper. In particular, we make rigorous expectations from [3] and [4].

**Definition 4.11.** Define the *raviolo space/Hecke stack* for \( v \) which has \( \mathbb{C} \)-points given by

\[ \mathcal{R}^v(\mathbb{C}) = \{(s_2, g, s_1) \in \mathcal{V}^v \times \tilde{G}_\mathcal{O} \times \mathbb{C}_\text{rot}^\times \mathcal{V}^v | g.s_1 \leq s_2 \} / \tilde{G}_\mathcal{O} \times \mathbb{C}_\text{rot}^\times. \]

Here the \( \tilde{G}_\mathcal{O} \times \mathbb{C}_\text{rot}^\times \)-action is on the left of \( s_1 \) and the right of \( g \).

**Definition 4.12.** Define also

\[ \mathcal{T}^v(\mathbb{C}) = \{(s_2, g, s_1) \in \mathcal{W}^v \times \tilde{G}_\mathcal{O} \times \mathbb{C}_\text{rot}^\times \mathcal{V}^v | g.s_1 \leq s_2 \} / \tilde{G}_\mathcal{O} \times \mathbb{C}_\text{rot}^\times, \]

where \( \mathcal{W}^v := (\tilde{G}_\mathcal{O} \times \mathbb{C}_\text{rot}^\times).v \subset \mathcal{N}(\mathcal{K}). \)

Next, note that \( \mathcal{R}^v \) is a locally closed sub-ind-variety of \( \mathcal{R} \) via \( [s_2, g, s_1] \mapsto [g, s_1] \) and therefore inherits a stratification \( \mathcal{R}^v_{\leq \lambda} := \mathcal{R}^v \cap \mathcal{R}_{\leq \lambda} \) (ditto for \( \mathcal{T}^v, \mathcal{T} \)). It also has maps

\[ \varphi_1 : \mathcal{R}^v \xrightarrow{\sim} \mathcal{N}(\mathcal{O}) \]

\[ \mathcal{M}_v \xleftarrow{\varphi_1} \mathcal{R}^v \xrightarrow{\varphi_2} \mathcal{N}(\mathcal{O}) \]

where \( \varphi_1 \) is induced from the \( \tilde{G}_\mathcal{O} \times \mathbb{C}_\text{rot}^\times \)-equivariant projection map \( (s_2, g, s_1) \mapsto s_1 \), and whose restriction to \( \mathcal{R}^v_{\leq \lambda} \) is smooth, and \( \varphi_2 \) is another proper equivariant projection given by

\[ \varphi_2 : [s_2, g, s_1] \mapsto s_2, \]

whose image is naturally identified with \( \mathcal{V}^v \). The map in Equation (4.1) restricts to

\[ p : p^{-1}(\mathcal{R} \times \mathcal{V}^v) \to \mathcal{R}^v \times \mathcal{V}^v \]

\[ p : \tilde{G}_\mathcal{O} \times \mathbb{C}_\text{rot}^\times \times \mathcal{V}^v \to \mathcal{T}^v \times \mathcal{W}^v. \]

And \( q(p^{-1}(\mathcal{R} \times \mathcal{V}^v)) \cong \mathcal{R}^v \) by the right quotient. Our goal is to interpret the “push-pull” maps in equivariant cohomology of \( \mathcal{V}^v \) giving rise to the action. Note that when the stabilizer of \( v \) is trivial, we have \( p^{-1}(\mathcal{R} \times \mathcal{V}^v) \cong \mathcal{V}^v \times \mathcal{V}^v \) by \( (g, s) \mapsto (s, g.s) \).

Note that \( q_*p^* \) where \( p^* \) is defined in Theorem 4.6 defines a map

\[ H^*_{\tilde{G}_\mathcal{O} \times \mathbb{C}_\text{rot}^\times}(\mathcal{R}^v \times \mathcal{V}^v) \to H^*_{\tilde{G}_\mathcal{O} \times \mathbb{C}_\text{rot}^\times}(\mathcal{P}, \pi_1^*\mathcal{F}_v) = H^*_{\tilde{G}_\mathcal{O} \times \mathbb{C}_\text{rot}^\times}(\mathcal{R}^v). \]

Given a class \( [\mathcal{R}_{\leq \lambda}] \in \mathcal{A}^h \) and \( \alpha \in H^*_s(\mathcal{M}_v) \cong H^*_{\tilde{G}_\mathcal{O} \times \mathbb{C}_\text{rot}^\times}(\mathcal{V}^v) \) we have that \( q_*p^*([\mathcal{R}_{\leq \lambda}] \otimes \alpha) \) is identified with the restriction of the map \( q_*p^* \) to \( \mathcal{R}^v_{\leq \lambda} \times \mathcal{V}^v \). In particular, by smoothness of the maps in Eq. 4.22 and the natural inclusion \( \mathcal{R} \to \mathcal{T} \) we may use the "classical" refined pullback map as in [12] to compute \( q_*p^*([\mathcal{R}_{\leq \lambda}] \otimes \alpha) \) given good enough understanding of \( \mathcal{R}^v \) and how it sits in \( \mathcal{T}^v \). Moreover, \( m_\alpha : H^*_{\tilde{G}_\mathcal{O} \times \mathbb{C}_\text{rot}^\times}(\mathcal{R}_v) \to H^*_s(\mathcal{M}_v) \) as given as in Theorem 4.6 is identified with \( \varphi_{2*} \). In Section 4.3 we will see that it is possible to compute \( (m \circ q)_*, p^* \) using this interpretation in the abelian setting, which enormously simplifies computations.
### 4.2.1. The case of Hilbert schemes

Suppose now \( \lambda = (1, \ldots, 0) \) and we are in the setting of Theorem 3.3. Then \( H^*_{\tilde{G}_n \times S^\times} (\mathcal{R}_{\Sigma}^{\leq \lambda}) \cong H^*_{\tilde{G}} (\text{Hilb}^{\bullet, \bullet + 1}(C)) \) where \( \text{Hilb}^{\bullet, \bullet + 1}(C) \) is the flag Hilbert scheme and (after forgetting equivalence) the map \( \phi_1 = q_* \rho^* \) can be identified with the refined pullback map also denoted “\( p^* \)” in [20, Theorem 1] restricted to the punctual Hilbert scheme (the versal deformations needed in loc. cit. work locally, whereas global curves are needed for the rest of the arguments). Similarly, if \( \lambda = (-1, \ldots, 0) \), we recover the map “\( q^* \)” of loc. cit.

Let us now explain why this happens. The affine Grassmannian of \( \text{GL}_n \) is the increasing union of the projective varieties

\[
\text{Gr}_{\text{GL}_n} := \{ \Lambda \subset \mathcal{K}^n | t^d \mathcal{O}^n \subset \Lambda \subset t^{-d} \mathcal{O}^n \}.
\]

It is clear that \( M_v \) as in Theorem 3.3 corresponding to the germ of a curve \( \tilde{C} \) has \( M_v^{m} := \bigsqcup_{m=0}^n \text{Hilb}^m(\tilde{C}) \subset M_v \) contained in \( \text{Gr}_{\text{GL}_n} \) for all \( m \) and some \( d \) depending on \( m \).

Let moreover \( N_d := N(\mathcal{O})/t^d N(\mathcal{O}) \) and \( \mathcal{V}_{d}^\nu \) be the image in the quotient. Let also \( \mathcal{R}^d := \{ [g, s] \in \text{Gr}^d_G \times G(\mathcal{O})/t^d N_d | g^{-1}.s \in N_d \} \). Then \( \mathcal{R} \) is the colimit of \( \mathcal{R}^d \) for the inclusions coming from \( \text{Gr}^d \to \text{Gr}^{d+1}_G \), in particular the equivariant Borel-Moore homology is the corresponding colimit.

Choose \( d \gg 0 \) and some open neighborhood \( U \) of \( v \in N_d \). Then choosing some transversal slice \( S \) to \( \mathcal{V}_{d}^\nu \), we locally have \( \mathcal{V}_{d}^\nu \times S \cong U \). In particular, if we let \( \varphi : \mathcal{R}^d \to N_d \) be the projection, and

\[
\Sigma := \varphi^{-1}(\mathcal{V}_{d}^\nu), \quad \Sigma_U := \varphi^{-1}(U \cap \mathcal{V}_{d}^\nu)
\]

then

\[
\Sigma_U \cong (\mathcal{V}_{d}^\nu \cap U) \times M_v^d.
\]

Consider the inclusion \( \mathcal{V}_{d}^\nu \cap U \hookrightarrow U \). The map \( \varphi^{-1}(U) \to U \) is smooth, so we get a refined pullback map \[12\]

\[
\varphi^* : H^*_\tilde{G}(\mathcal{O})/t^d \mathcal{O}^\times (\mathcal{V}_{d}^\nu \cap U) \to H^*_\tilde{G}(\mathcal{O})/t^d \mathcal{O}^\times (\Sigma_U).
\]

We will in fact abuse notation and denote by \( \varphi^* \) the composition of this map and the pushforward

\[
H^*_\tilde{G}(\mathcal{O})/t^d \mathcal{O}^\times (\Sigma_U) \to H^*_\tilde{G}(\mathcal{O})/t^d \mathcal{O}^\times (\Sigma).
\]

Possibly further increasing \( d \) and throwing away some high codimension subset of \( U \) not containing \( v \), note that by Theorem 3.3 it is possible to identify \( \varphi^{-1}(U) \to U \) with the family of Hilbert schemes of 0, 1, \ldots, \( d \) points (i.e. the union thereof) since having a cyclic vector is an open condition.

Since \( N_d \) is the space of all matrices and vectors in \( \mathcal{O}/t^d \), the associated family of (germs of) planar curves is versal for large enough \( U \). By results of Shende and others (see e.g. Sections 2 and 3 of [20] for discussion and references), the associated total space is smooth.

Further restricting \( \varphi \) to \( \varphi^{-1}(U) \cap \mathcal{R}^\leq_{d \lambda} \) for the cocharacter \( \lambda = (1, \ldots, 0) \) identifies the refined intersection map \( \rho^* \) for the inclusion \( v \hookrightarrow U \) in [20, Definition 3.4] with \( \varphi_{\leq \lambda}^* \). The other case is similar.

In particular, this gives an interpretation of one of the Weyl algebras appearing in [20, 33]. The other one has to do with the Hilbert schemes of global curves and cannot be defined in our setting. Indeed, the other Weyl algebra depends on the number of components of the curve, whereas our Cherednik algebra depends on the degree of the curve.
Remark 4.13. It is remarkable to note that the convolution action works on the level of punctual Hilbert schemes directly. In [24] and [33], one of the main points is to define convolution maps for the Hilbert schemes of (locally planar) singular curves using refined intersection products, which are constructed by deforming the singularities as we saw above. The role of the deformation in our context is played by considering the infinite-dimensional ind-variety \( \mathcal{V}^w \) in place of \( M_v \). Note also that the “restriction with supports” map is a refined intersection product in the case of a regular embedding, while here we use a rather special form of the map \( p \), which is very far from anything like a regular embedding, but rather like a principal bundle.

4.3. Localization to fixed points

Let us analyze the construction of Theorem 4.6 first in the case \( G = T \) is a torus. In this case, \( \mathcal{R}_T \) is a collection of (infinite rank) vector bundles over a discrete set \( \text{Gr}_T \cong X^*(T) \), of finite codimension in \( \text{Gr}_T \). Its complex points are

\[
\mathcal{R}_T(\mathbb{C}) = \{(g,s) \in \tilde{T}_C^\mathbb{C} \times \mathbb{C}_{\mathbb{C}}^\times \times N(\mathcal{O}) : g,s \in N_\mathcal{O} \}/\tilde{T}_C \times \mathbb{C}_{\mathbb{C}}^\times,
\]

and the map \( \pi_T : \mathcal{R}_T \to \text{Gr}_T \) given by forgetting \( s \). Here, the action of \( \tilde{T}_C \times \mathbb{C}_{\mathbb{C}}^\times \) is on the left of \( s \) and the right of \( g \). The map

\[
\tilde{T}_C^\mathbb{C} \times N_\mathcal{O} \to \mathcal{T}_C \times N_\mathcal{O}
\]

is simply many copies of the quotient map

\[
\mathbb{C}((t)) \to \mathbb{C}((t))/\mathbb{C}[[t]].
\]

Fix now \( G \) reductive and \( T \) a maximal torus in it. We may think of \( \mathcal{R}_T \) as an “abelianized” BFN space for \( G \), as it also admits an inclusion map \( \iota : \mathcal{R}_T \hookrightarrow \mathcal{R} \) via inclusion of \( \text{Gr}_T \hookrightarrow \text{Gr}_G \) and viewing \( N \) as a representation of \( T \) (obtained by restricting the action of \( G \) to \( T \)). The space \( \mathcal{R}_T \) has a natural convolution product and it admits a natural action of the Weyl group \( W \). By Lemma 5.10 of [2] there is an algebra homomorphism \( (\iota_R)_*: (\mathcal{A}_T^h)^w \to \mathcal{A}_T^h \) coming from the inclusion \( \iota_\mathcal{R} : \mathcal{R}_T \hookrightarrow \mathcal{R} \). We call \( \mathcal{A}_T^h \) the “abelianized” BFN algebra. This construction generalizes to the flavor deformed algebras \( (\mathcal{A}_T^h)^w \to \mathcal{A}_T^h \), where \( \mathcal{A}_T^h := \mathcal{T}_\mathcal{O}^{\mathbb{C}}/\mathbb{C}_{\mathbb{C}}^\times \mathcal{R}_T \).

Consider \( \mathcal{R}_T^w = \mathcal{R}_T \cap \mathcal{R}^w \). By definition of the generalized affine Springer fiber for \( v \), where we consider \( N \) as a representation of \( T \subset G \), we see that \( \mathcal{R}_T^w \) is the Hecke stack associated to the datum \( (T,N,v) \). Using the convolution action of Theorem 4.6 for \( (T,N) \), we get an action of \( \mathcal{A}_T^h \) on \( H_L^{l_T,v}(M_{v,T}) \) where \( L_{T,v} \) is the stabilizer of \( v \) in \( T \).

We can now try to compare the two actions.

Proposition 4.14. Suppose \( \text{Gr}_G \) has isolated fixed points under the stabilizer \( L_v \subset \mathcal{G}_K^\mathbb{C} \times \mathbb{C}_{\mathbb{C}}^\times \) of \( v \) and that \( L_v \) is contained \( \tilde{T}_C \times \mathbb{C}_{\mathbb{C}}^\times \). Then

1. \( M_{v,T} = M_{v,L_v}^L \)
2. \( (\iota_{M_v})_* : H_L^{l_v}(M_{v,L_v}) \to H_L^{l_v}(M_v) \) becomes an isomorphism after inverting countably many characters of \( L \).
3. \( (\iota_{M_v})_* \) intertwines the actions of \( \mathcal{A}_T^h(\mathcal{N}_{T,N})^w \) and \( \mathcal{A}_G^h(\mathcal{N}_{G,N})^w \).
Proof. The first assertion follows from the fact that the $L_v$-fixed points are contained in the $L_v = T$-fixed points on the affine Grassmanian, which for $T$ is topologically a discrete set of points coinciding with $\Gr T$. The second assertion is the Atiyah-Bott localization theorem.

Consider the following diagram:

\[ \begin{array}{c}
\mathcal{R} \times \mathcal{V}^v \\
\mathcal{T} \times \mathcal{V}^v \\
\mathcal{T}_T \times \mathcal{V}_T^v \\
\mathcal{R}_T \times \mathcal{V}_T^v \\
\end{array} \]

\[ \begin{array}{c}
\mathcal{P}^v \\
\mathcal{P}_T^v \\
\end{array} \]

\[ \begin{array}{c}
\mathcal{V}^v \\
\mathcal{V}_T^v \\
\end{array} \]

Here $i, j, p, q, m$ are as before, and the versions with subscript $T$ are the corresponding maps for $T \subset G$. The inclusions $\iota_T$ come from the maps $T \to G, \Gr_T \to \Gr_G$ and variations. The space $\mathcal{P}^v$ is defined as $\mathcal{P}^v := p^{-1}(\mathcal{R} \times \mathcal{V}^v)$ and $\mathcal{P}_T^v$ by replacing $G$ with $T$.

Note that the upper and lower squares on the left tower of squares are clearly cartesian. We claim that the middle one is so too. By definition the fiber product

\[ (\tilde{G}^O_K \times \mathbb{C}_{rot}^x \times \mathcal{V}^v) \times_{\mathcal{T} \times \mathcal{V}^v} (\mathcal{T}_T \times \mathcal{V}_T^v) \]

consists of $(g, v'', [t, v'])$ so that $[g, v''] = [t, v']$ and $v' = v''$. In particular, there is some $g' \in L_v$ such that $g g' = t$. But since $L_v$ is contained in $\tilde{T}^O \times \mathbb{C}_{rot}^x$, we must have $g \in \tilde{T}^O \times \mathbb{C}_{rot}^x$. So every square in the tower is cartesian. Note that this is not true without our assumptions (take for example $N = 0, v = 0$).

Let $\mathcal{F}_v = \omega_{\mathcal{V}^v}[-2 \dim \tilde{G}^O]$ and $\mathcal{F}_{T,v} = \omega_{\mathcal{V}_T^v}[-2 \dim \mathcal{T}^O]$. Let $\iota_{\mathcal{V}^v}: \mathcal{V}_T^v \to \mathcal{V}^v$. Then $\iota_{\mathcal{V}^v}^* \mathcal{F}_v = \mathcal{F}_{T,v}$. Let then

\[ r \otimes \alpha \in H_\ast \tilde{G}^O \times \mathbb{C}_{rot}^x (\mathcal{R}) \otimes H_\ast^{L_v}(M_v) \cong H_\ast \tilde{G}^O \times \tilde{G}_T \times \mathbb{C}_{rot}^x \times \mathbb{C}_{rot}^x (\mathcal{R} \times \mathcal{V}^v, \omega_{\mathcal{R}} \boxtimes \mathcal{F}_v). \]

By Lemma 5.10. of [Z], the pushforward map

\[ (\iota_{\mathcal{R}})_\ast^W : (\tilde{\mathcal{A}}^h_T)W \to \tilde{\mathcal{A}}^h \]

given by taking the $W$-invariants of the $\tilde{T}^O \times \mathbb{C}_{rot}^x$-equivariant pushforward becomes an isomorphism after localizing at countably many characters of $\tilde{T} \times \mathbb{C}_{rot}^x$. By parts (1) and (2),

\[ (\iota_{\mathcal{V}^v})_\ast : H_\ast \tilde{T}^O \times \mathbb{C}_{rot}^x (\mathcal{V}_T^v) \cong H_\ast^{L_v}(M_v^{L_v}) \to H_\ast^{L_v}(M_v) \cong H_\ast \tilde{G}^O \times \mathbb{C}_{rot}^x (\mathcal{V}^v) \]

also becomes an isomorphism after localizing at countably many characters of $L_v$.

If we define moreover

\[ \iota_{\ast} := (\iota_{\mathcal{R}})_\ast^W \otimes (\iota_{\mathcal{V}^v})_\ast \]
and work in this localization, the intertwining property we need to show becomes

\[ \iota_*(m_T \circ q_T) \circ p_T^* (((\iota_*)^{-1}(r \otimes \alpha))) = (m \circ q)_* p^*(r \otimes \alpha). \]

Define

\[ A := \omega_T[-2 \dim N_\mathcal{O}] \boxtimes F_v, A_T := \omega_T[-2 \dim N_\mathcal{O}] \boxtimes F_{T,v} \]

and

\[ B := \omega_{\tilde{T}_\mathcal{O} \times \mathbb{C}_\text{rot}^\times}[-2 \dim \tilde{G}_\mathcal{O} \times \mathbb{C}_\text{rot}^\times] \boxtimes F_v, B_T := \omega_{\tilde{T}_\mathcal{O} \times \mathbb{C}_\text{rot}^\times}[-2 \dim \tilde{T}_\mathcal{O} \times \mathbb{C}_\text{rot}^\times] \boxtimes F_{T,v} \]

The restriction with support map \( p^* \) from Theorem 4.6 and Definition Appendix A.1 is (the induced map in hypercohomology of) the composition

\[ \iota_T^! A \to \iota_T^! p_* p^* A = p_* j_T^! p^* A \to p_* j_T^! B. \]

Similarly we have

\[ \iota_T^! A_T = (\iota_T \times \iota_{V'})(\iota_T^! A) \to \iota_T^! p_T^* p_T^* A_T \to p_T^* j_T^! B_T = (\iota_T \times \iota_{V'}) \iota_T^! A \to p_T^* j_T^! B. \]

Using proper base change, we rewrite this as

\[ (\iota_T \times \iota_{V'}) \iota_T^! A \to \iota_T^! p_T^* p_T^* A_T = (\iota_T \times \iota_{V'}) \iota_T^! A \to p_T^* j_T^! B_T = (\iota_T \times \iota_{V'}) \iota_T^! A \to p_T^* j_T^! B. \]

Passing to \( \tilde{T}_\mathcal{O} \times \mathbb{C}_\text{rot}^\times \)-equivariant hypercohomology, we get that the square

\[
\begin{array}{ccc}
H^{-\ast}_{\tilde{T}_\mathcal{O} \times \mathbb{C}_\text{rot}^\times \times \tilde{G}_\mathcal{O} \times \mathbb{C}_\text{rot}^\times} (R \times V^v, i_T^! A) & \xrightarrow{p_T^*} & H^{-\ast}_{T_\mathcal{O} \times \mathbb{C}_\text{rot}^\times \times \tilde{G}_\mathcal{O} \times \mathbb{C}_\text{rot}^\times} (P^v \circ \pi_1^T, F_v) \\
(\iota_T \times (\iota_{V'})). & & (\iota_T \times (\iota_{V'})). \\
H^{-\ast}_{T_\mathcal{O} \times \mathbb{C}_\text{rot}^\times \times \tilde{T}_\mathcal{O} \times \mathbb{C}_\text{rot}^\times} (R_T^v \times V_T^v, i_T^! A_T) & \xrightarrow{p_T^*} & H^{-\ast}_{T_\mathcal{O} \times \mathbb{C}_\text{rot}^\times \times \tilde{T}_\mathcal{O} \times \mathbb{C}_\text{rot}^\times} (P_T^v \circ \pi_1^T, F_{T,v}) \\
& & (\iota_T \times (\iota_{V'})).
\end{array}
\]

commutes. Now taking \( W \)-invariants on the \( R \)-factor everywhere and passing to the localization where the left column becomes an isomorphism, we get

\[ p_T^*((\iota_*)^{-1}(r \otimes \alpha)) = (\iota_T^! p_*)^{-1} p^*(r \otimes \alpha). \]

Since the right large square is also cartesian and \( \iota_T^! \) is a closed embedding, using proper base change once more we get

\[ \iota_*(m_T \circ q_T)_* (\iota_T^! p_*)^{-1} p^*(r \otimes \alpha) = (m \circ q)_* p^*(r \otimes \alpha). \]

\[ \square \]

**Remark 4.15.** Parts (1) and (2) of the above Proposition were also obtained in [17] Theorem 5.27.

Let \( \text{Gr}_G^\lambda \) be the \( G \)-orbit of \([\lambda] \in \text{Gr}_G \) and set \( \mathcal{R}^{\leq \lambda} = \mathcal{R} \cap \pi^{-1}(\text{Gr}_G^\lambda) \), where \( \pi : \mathcal{R} \to \text{Gr}_G \) is the projection forgetting \( N_\mathcal{O} \). In what follows we will determine the action of various classes in \( \tilde{A}^9 \) by means of two-fold fixed-point localization. Recall that there are commutative subalgebras
Assume that \( m \) \( \ell \) that the map \( \tilde{T} \times C^\times_{rot} \) collectively by \( \varphi \) (for \( T \)), \( m \) (for \( G_F \)) and \( h \) (for \( C^\times_{rot} \)).

Let \([t^\lambda]\) denote the fundamental class of \( \mathcal{R}_T \cap p_T^{-1}(\text{Gr}_G^\lambda) \), often called an “abelianized monopole” \([8\text{ }11, 17]\). For \( \lambda \) dominant with \( \text{Gr}_G^\lambda \) closed we can then write the following localization formula, c.f. Proposition 6.6:

\[
[R^\leq \lambda] = \ell_* \left( \sum_{w \in W/W_\lambda} \frac{[tw^\lambda]}{e(T_{w^\lambda} G^{\leq \lambda}_{\text{Gr}})} \right), \tag{4.4}
\]

where \( W_\lambda \) is the stabilizer of \( \lambda \) in the Weyl group \( W \). The unit of the algebra \( \tilde{A}^h \) is 1 := \([R^\leq 0]\). Other generators of \( \tilde{A}^h \) can be constructed by including a \( W_\lambda \)-invariant function \( f(\varphi, m, h) \) to the numerator of this expression:

\[
[R^\leq \lambda][f] = \ell_* \left( \sum_{w \in W/W_\lambda} \frac{(w, f)(tw^\lambda)}{e(T_{w^\lambda} G^{\leq \lambda}_{\text{Gr}})} \right). \tag{4.5}
\]

These are called “dressed” monopole operators, which are known to generate \( \tilde{A}^h \) \([1, 37]\).

**Remark 4.16.** More precisely, it was shown in \([37]\) that the \([R^\leq \lambda][f]\) with minuscule \( \lambda \) and a slightly smaller collection of \( f \)'s generate \( \tilde{A}^h_{G,N} \) for any quiver gauge theory; the quiver in this case is a Jordan quiver with a framing node of rank 1.

**Remark 4.17.** The terminology “dressed monopole” has its origins in the physics literature, in our context they appear for example in \([8]\). These operators also appear as the dimensional reduction of the 4d mixed Wilson-t Hooft operators of \([18]\).

Assume the hypothesis of Proposition 4.14 and, moreover, that the map \( L_v \to G_F \times C^\times_{rot} \) is injective. Thus, the action of \( H^*_T \times C^\times (pt) \) factors through the action of \( H^*_L \times C^\times (pt) \) \([17]\).

A representative in \( T_C \) of a fixed point \( p \in M_v \) will generically not be exactly fixed by \( L_v \), instead requiring a compensating \( T \subset T_C \) transformation. The requirement that \( L_v \to G_F \times C^\times_{rot} \) is injective implies that there is a unique such compensating transformation, hence the action of \( H^*_T \times C^\times_{rot} (pt) \) on the fixed point class \([p]\) is uniquely determined by the action of \( H^*_T \times C^\times_{rot} (pt) \) on \([p]\). We write \( \varphi[p] = \varphi(p)[p] \). The action of \( H^*_T \times C^\times_{rot} (pt) \) is then determined by the injection \( L_v \to G_F \times C^\times_{rot} \), which imposes rank \( G_F + 1 - \text{rank } L_v \) linear relations on the \( m[p] \), \( h[p] \). This is the source of the specialization discussed earlier.

**Lemma 4.18.** Assume that \( M_v \) has isolated fixed points under the action of \( L_v \subset \tilde{T} \times C^\times_{rot} \) and that the map \( L_v \to G_F \times C^\times_{rot} \) is injective.

For a minuscule cocharacter and \( f(\varphi, m, h) \) a \( W_\lambda \)-invariant function we have

\[
[R^\leq \lambda][f][p] = \sum_{w \in W/W_\lambda} \frac{(w, f(tw^\lambda p))}{e(T_{w^\lambda} G^{\leq \lambda}_{\text{Gr}})} \big| tw^\lambda p \big|, \tag{4.6}
\]

\(^2\text{Since the } \varphi \text{ do not commute with } [t^\lambda], \text{ we take the convention that the denominator is to the right of the numerator in writing this formula.}\)
where $E_{p,\nu}$ is an excess intersection factor. The denominator in this formula should be understood as replacing $\varphi$ in the polynomials $e(T_{w,\lambda}\text{Gr}^{<\lambda}_G)$ with $\varphi(p)$.

Proof. By the previous Proposition we only need to compute this inside $H^L_{\nu}(M^L_{\nu})$ be (the inclusion of) the fundamental class of a fixed point in $M_v \subset N(O)/G(O)$. The subalgebra $H^2_T(pt) = C[\bar{t}] < A_{\bar{t}}$ acts as $\varphi_{a}|p⟩ = \varphi_a(p)|p⟩$. Since $\pi_{\nu}^{-1}(\text{Gr}_{\lambda}^{<\lambda})$ is a vector bundle over a point, using the excess intersection formula for the refined pullback $p^*$ (see Fulton [12, Section 6.3]) we have

$$[t^k]|p⟩ = (m \circ q)_*p^*([t^k] \otimes |p⟩) = e(E_{p,\lambda})|t^k⟩.$$ 

As a vector space over $C$, $E_{p,\lambda}$ can be expressed as

$$E_{p,\lambda} \simeq N(O)/(N(O) \cap t^{-\lambda}N(O)).$$

The equivariant structure of this vector space is determined by $\lambda$ and $p$; $E_{p,\lambda}$ should be thought of as a quotient of tangent spaces at $(t^k p, t^k, p) \in T_T$. A straightforward computation shows that

$$e(E_{p,\lambda}) = \prod_{\bar{\mu} \text{ s.t. } (\mu, \lambda) < 0} (\langle \bar{\mu}, \varphi(p) + m \rangle - 1).$$

where the product runs over the $\bar{G}$ weights $\bar{\mu}$ of $N$, with $\mu$ its restriction to $G$. We also use the notation that

$$[x]^r = \begin{cases} \prod_{j=0}^{r-1} (x + jh) & r > 0 \\ 1 & r = 0 \\ \prod_{j=1}^{r} (x - jh) & r < 0 \end{cases}.$$ 

It is worth noting that if $t^k$ maps $p$ outside $N(O)$ then $E_{p,\lambda}$ will necessarily have a vector that transforms trivially under $C^\times$, i.e. $e(E_{p,\lambda}) = 0$. By Eq. (4.5) the result follows. 

Remark 4.19. The above localization computations and the “abelianization procedure” appear in [1] as an embedding of the algebra $A_{\bar{t}}$ to an algebra of differential(-difference) operators on the maximal torus $T \subset G$.

5. Torus Links and the spherical RCA

In this section we turn to a conjecture of [28] concerning the relation between the homology of the Hilbert schemes of points on plane curve singularities and minimal $a$-degree HOMFLY homology of the associated link. 

Fix $G = GL_n, N = Ad \oplus V, G_F = C^\times_{di}$ and set $R = R_{G,N}, A^h = A_{G,N}^h$. We focus on the case of $v \in N(O)$ corresponding to positive $(n, k)$ torus links, which can be realized by the plane curve singularities $\hat{C}_{n,k}$ associated to $f = x^n - t^k$. Based on the relation between Hilbert schemes of points on $\hat{C}_{n,k}$ and GASF in Theorem 3.3 we see that

$$\mathcal{M}_{(n,k)} := \Hilb^\bullet(\hat{C}_{n,k}) = M_v$$
rational Cherednik algebra of Proposition 5.3. For coprime $v$ with
\[ \overset{\text{γ}}{\rightarrow} \] classes inside $C$ points are labeled by cocharacters as described in Proposition 3.11. We will label the fixed point
\[ \text{There is a similar expression for the action of } (ν, ν^k). \] This implies the relation $(nm + k) | A) = 0$ for all $A$. We explicitly solve this by replacing $m | A) = -\frac{2}{n} h | A)$. Let $\varphi, a = 1, ..., n$ be the components of $\varphi$ in the standard basis.

**Lemma 5.1.** The action of $\mathbb{C}[t]$ is given by
\[ \varphi_a | A) = (a - 1) \frac{k}{n} - A_a) | h | A) \]
and the action of $[t^{\lambda}]$ is given by
\[ [t^{\lambda}] | A) = \left( \prod_{\lambda_a < 0} \prod_{\alpha = 0}^{\lambda_a - 1} ((a - 1) \frac{k}{n} - A_a + \alpha) h \right) \] \[ \left( \prod_{\lambda_a = 0} \prod_{\beta = 0}^{\lambda_a - 1} ((a - b + 1) \frac{k}{n} - A_a + A_b + \beta) h \right) | A + \lambda) . \]

**Proof.** This is a direct application of Lemma 4.18.

Using these ingredients and equation (4.10) one can obtain an expression for the action of any $[R_{\leq \lambda})|f)$. Therefore, for $\lambda$ a minuscule cocharacter we have
\[ [R_{\leq \lambda})|1)|A) = \sum_{\lambda' \in W \cdot \lambda} \left( \prod_{\lambda' \in 0} \prod_{\alpha = 1}^{\lambda'} (\varphi_a - \alpha h) \right) \left( \prod_{\lambda' > \lambda} \prod_{\beta = 1}^{\lambda' - \lambda} (\varphi_b - \varphi_a - m - \beta h) \right) \]
\[ \left( \prod_{\gamma = 1}^{\lambda' - \lambda} (\varphi_b - \varphi_a - \gamma h) \right) \] \[ |A + \lambda') . \] (5.1)

There is a similar expression for the action of $[R_{\leq \lambda})|f) = f(\varphi, m, h)$ a $W_\lambda$-invariant function, though we will not need it in the following.

**Proposition 5.2.** Comparing to $A(iii)$, we have an identification (up to numerical factors)
\[ E_r[f] = [R_{\leq \lambda_r})|f) \] and $F_r[f] = [R_{\leq -\lambda_r})|f)$ where $\lambda_r = (1, 1, ..., 1, 0, 0, ..., 0)$ with $r$ 1’s and $f(\varphi) = f(\varphi - h)$.

Using this presentation of the algebra, the following result is straightforward.

**Proposition 5.3.** For coprime $(n, k)$, $H^C_{s}(\mathcal{M}(n,k))$ is irreducible as the module for the spherical rational Cherednik algebra of $\mathfrak{gl}_n$ at parameter $m = -\frac{2}{n} h$. 

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Proof. We show that this module is irreducible by identifying the unique singular vector, namely \([0]\). Recall that being a singular vector for the spherical rational Cherednik algebra corresponds to being in the kernel of all \(F_r[f] = [F_{\leq -\lambda}, [f]]\). First consider the kernel of \(F_n[f]\), or the classes corresponding to the cocharacter \(\lambda = (-1, -1, ..., -1)\). The choice of \(f\) is that of a \(W\) invariant polynomial \(f(\varphi, m, h)\). From the action given in (5.1), we find that

\[
F_n[1] | A) = \prod_{b=1}^{n} (\varphi - \frac{k}{n} - A_b) h | A - (1, 1, ..., 1) = \prod_{b=1}^{n} (b - 1) \frac{k}{n} - A_b) h | A - (1, 1, ..., 1) .
\]

Since \(\gcd(n, k) = 1\), the factor \((b - 1) \frac{k}{n} - A_b)\) can only vanish for \(b = 1\) and \(A_1 = 0\). It follows that the kernel of \(F_n[1]\) is exactly those classes \([A]\) with \(A_1 = 0\). Moreover, such classes are in the kernel of \(F_n[f]\) for all \(f\).

Now consider the action of \(F_{n-1}[f]\) on sums of fixed point classes with \(A_1 = 0\). Using Eq. (4.9) for we have, after a dramatic simplification following from \(A_1 = 0\),

\[
F_{n-1}[1] | [0, A_2, ..., A_n) = \left( \prod_{b=2}^{n} ((b - 2) \frac{k}{n} - A_b) h \right) | [0, A_2 - 1, ..., A_n - 1] .
\]

Again, since \(\gcd(n, k) = 1\), the factor \((b - 2) \frac{k}{n} - A_b)\) can only vanish for \(b = 2\) and \(A_2 = 0\). Therefore \([0, A_2, ..., A_n)\) is in the kernel of \(F_{n-1}[1]\) if and only if \(A_2 = 0\). Thus \(\ker F_n[1] \cap \ker F_{n-1}[1]\) only contains classes with \(A_1 = A_2 = 0\). Moreover, these classes belong to the kernel of \(F_{n-1}[f]\) for all \(f\). Continuing this process shows that \(\ker F_n[1] \cap \ker F_{n-1}[1] \cap ... \cap F_1[1] = \text{span}\{[0]\}\)

and that it also belongs to the kernel of all \(F_r[f]\). \(\square\)

Now we state and prove the main theorem of this section.

**Theorem 5.4.** For coprime \((n, k)\), \(H_\infty^\infty(M_{n,k})\) can be identified with the irreducible representation \(eL_{-\frac{k}{n}}(\text{triv})\) of the spherical rational Cherednik algebra of \(\mathfrak{gl}_n\) at parameter \(m = -\frac{k}{n} h\). That is, setting the equivariant parameter in \(H_\infty^\infty(M_{n,k})\) to 1, the quotient algebra \(eH_n e/(m + \frac{k}{n} h)\) acts.

**Proof.** From [21], or a direct computation using (5.2), it follows that for all \(n\) the operators \(X = [F_{\leq ,0}, ..., 0)] = E_1[1]\) and \(Y = [F_{\leq ,0}, ..., 0)] = F_1[1]\) generate an appropriately scaled copy of the Weyl algebra: \([X, Y] = nh\). Since we have shown that \(H^\infty_\infty(M_{n,k})\) is irreducible as a module for the spherical rational Cherednik algebra of \(\mathfrak{gl}_n\) at parameter \(m = -\frac{k}{n} h\) it follows that it must decompose as a product \(\mathbb{C}[X] \otimes M\), where \(M\) is some irreducible module for the spherical rational Cherednik algebra of \(\mathfrak{sl}_n\). Finally, noting that the spherical rational Cherednik algebra for \(\mathfrak{sl}_n\) at parameter \(m = -\frac{k}{n} h\) has a unique finite dimensional, irreducible module, it suffices to show that \(\ker Y \simeq M\) is finite dimensional.

Consider the graded Euler character of this homology, which can easily be computed from counting fixed points. Recall that the fixed points in \(M_{n,k}\) are labeled by cocharacters \(A\) as in Prop. 3.11, denote the set of such \(A\) by \(A_{n,k}\). The degree in the Hilbert scheme is given by

\[
d(A) = \sum_{a=1}^{n} A_a.
\]
and one finds

\[ \chi(M_{n,k}) = \sum_{A \in \mathfrak{A}_{n,k}} q^{d(A)} = \frac{1}{1 - q^n} \binom{n - 1 + k}{n - 1} q. \]

Noting that \( X \) changes \( q \)-degree by 1, we can determine the dimension of \( M \) by multiplying the above by \( 1 - q \), counting the \( \mathbb{C}[X] \) factor, and setting \( q = 1 \). One finds

\[ \dim_{\mathbb{C}} M = \frac{1}{n} \binom{n + k - 1}{n - 1} = \dim_{\mathbb{C}} H^*(\mathcal{J}_{n,k}), \]

where \( \mathcal{J}_{n,k} \) is the compactified Jacobian of the curve \( \hat{C}_{n,k} \).

\[ \square \]

**Remark 5.5.** It is worth noting that \( \mathfrak{A}^h \) is bi-filtered by the degree in \( \text{Gr}_G \), called “monopole number” in the physics literature, and by the action induced by scaling \( \mathbb{C}[t, \hbar]^W \) with weight 2, called “R-charge” in the physics literature. In particular, we assign the degree \( (\pm r, r + 2 \deg f) \) to \( [\mathcal{R}_{\leq \pm \lambda_r}] |f| \). The spherical rational Cherednik algebra for \( \mathfrak{g}l_n \) is also bi-filtered by total polynomial degree and by difference in degree of \( x \)'s and \( y \)'s. That the respective filtrations agree follows from [21].
Appendix A.

Appendix A.1. Restriction with supports

In this section, we define the restriction with support homomorphisms used in the definition of $p^*$ in Theorem 4.6. We follow [2].

Definition Appendix A.1. Suppose we have a Cartesian diagram of ind-varieties

\[
\begin{array}{ccc}
Y & \xleftarrow{g} & Z \\
\downarrow{j} & & \downarrow{i} \\
W & \xleftarrow{f} & X
\end{array}
\]

and let $A, B$ be (possibly unbounded) complexes of constructible sheaves on $W, X$. Then suppose we are given $\varphi \in \text{Hom}(A, f^*B) \cong \text{Hom}(f^*A, B)$. Define the morphism of complexes

\[j^!A \to j^!f_*f^*A \cong g_*i^!f^*A \to g_*i^!B\]

as the composition of the adjunction map and $\varphi$. This induces a map on hypercohomology:

\[H^*(Y, j^!A) \to H^*(Z, i^!B).\]

We will call this map “restriction with supports”.

Remark Appendix A.2. Suppose we have a cartesian diagram of varieties

\[
\begin{array}{ccc}
Z & \longrightarrow & Y \\
\downarrow & & \downarrow \\
X & \longrightarrow & W
\end{array}
\]

the first arrow is a regular embedding, let $N$ be the pullback to $Z$ of the normal bundle $N_{X/W}$. There is a specialization map

\[\sigma : H_*(Y) \to H_*(N), [V] \mapsto [C(C \cap Z)/V].\]

The usual refined intersection map/pullback with support is defined as the composition $H_*(Y) \to H_*(N) \to H_*(Z)$.

Appendix A.2. Finite-dimensional approximation

In many parts of this paper, we consider equivariant complexes on infinite-dimensional ind-varieties, in particular $R, T$ and $N_O$ and their substacks. We refer the reader to [2, Section 2] for more precise definitions in the first two cases, and in the latter case define

\[D^b_{G_O \times C^{\times}_{rot}}(N_O)\]

to be the direct limit over the finite-dimensional approximations to $N_O$ given by $N_O/t^iN_O$. The degree shifts such as $[-2 \dim N_O]$ we use, are also to be understood as in [2, Section 2].
Appendix A.3. Associativity

In this section, we prove that the convolution product defined in Theorem 4.6 is associative. We follow the proof of associativity of the convolution product of $\tilde{A}_{G,N}^\hbar := A^\hbar$ in [2, Section 3] and the outline in the preprint [17].

Lemma Appendix A.3. The convolution product defined in Theorem 4.6 is associative.

Proof. We consider the following commutative diagram, which is a ‘product’ of the upper row of (4.11) and [2, (3.2)]:

\[
\begin{array}{cccccccccccc}
R \times N_\mathcal{O} & \xleftarrow{p} & \mathcal{P} & \xrightarrow{q} & q(\mathcal{P}) & \xrightarrow{m} & N_\mathcal{O} \\
\downarrow{m \times \text{id}_R} & & \downarrow{q} & & \downarrow{m} & & \\
q(p^{-1}(R \times R)) \times N_\mathcal{O} & \xleftarrow{3} & 4 & \xrightarrow{q(\mathcal{P})} & & & \\
q \times \text{id}_{N_\mathcal{O}} & & & & & & \\
p^{-1}(R \times R) \times N_\mathcal{O} & \xleftarrow{1} & 2 & \xrightarrow{\mathcal{P}} & & & \\
p \times \text{id}_{N_\mathcal{O}} & & & & & & \\
R \times R \times N_\mathcal{O} & \xrightarrow{id_R \times p} & R \times \mathcal{P} & \xrightarrow{id_R \times q} & R \times q(\mathcal{P}) & \xrightarrow{id_R \times m} & R \times N_\mathcal{O},
\end{array}
\]  

(A.1)

where we have defined

\[ 1 = \{(g_1, g_2, s) \in \tilde{G}_K^r \times \mathbb{C}_\text{rot}^\times \times \tilde{G}_K^r \times \mathbb{C}_\text{rot}^\times \times N_\mathcal{O} \mid g_2 s, g_1 g_2 s \in N_\mathcal{O}\}, \]

and 2, 3, 4 are quotients of 1 by $1 \times \tilde{G}_\mathcal{O} \times \mathbb{C}_\text{rot}^\times$, $\tilde{G}_\mathcal{O} \times \mathbb{C}_\text{rot}^\times \times 1$, $\tilde{G}_\mathcal{O} \times \mathbb{C}_\text{rot}^\times \times \tilde{G}_\mathcal{O} \times \mathbb{C}_\text{rot}^\times$, respectively. Here $\tilde{G}_\mathcal{O} \times \mathbb{C}_\text{rot}^\times \times \tilde{G}_\mathcal{O} \times \mathbb{C}_\text{rot}^\times$ acts on 1 by

\[(h_1, h_2) \cdot (g_1, g_2, s) = (g_1 h_1^{-1}, h_1 g_2 h_2^{-1}, h_2 s) \quad \text{for } (h_1, h_2) \in \tilde{G}_\mathcal{O} \times \mathbb{C}_\text{rot}^\times \times \tilde{G}_\mathcal{O} \times \mathbb{C}_\text{rot}^\times.\]

The horizontal and vertical arrows from 1, 4 are given by

\[ (g_1, [g_2, s], s) \xleftarrow{p_1} (g_1, g_2, s) \in 1 \]  
\[ [g_1 g_2, s] \xrightarrow{p_2} \]

\[ ([g_1, g_2 s], [g_2, s]), \quad 4 \ni [g_1, [g_2, s]] \mapsto [g_1, g_2 s]. \]  

(A.2)

Arrows from 2, 3 are given by the obvious modification of above ones, as 1 → 3 etc. are fiber bundles. Also, $p_R$ is as defined in 2, i.e.

\[ (g_1, [g_2, s]) \mapsto ([g_1, g_2 s], [g_2, s]). \]

Let $\alpha \in H^{L^\nu}(M_\nu)$ and $c_1, c_2 \in \tilde{A}^\hbar$. The convolution product $c_2 \ast \alpha$ is given by applying the construction in Theorem 4.6 (i.e. induced homomorphisms in BM homology) to the bottom row
from left to right, and \(c_1 * (c_2 * \alpha)\) is then obtained by going up in the rightmost column. Similarly 
\((c_1 * c_2) * \alpha\) is given by going up the leftmost column using the construction in \([2]\) and then from left to right along the top row.

Therefore the associativity of the convolution product is the statement that the induced morphisms

\[- * ( - * -), - * ( - - *) - : \tilde{\mathcal{A}}^* \otimes \tilde{\mathcal{A}}^* \otimes H^L(M_v) \to H^L(M_v)\]

are equal. This would follow commutativity of the associated “large square” in BM homology. (It might be helpful for the reader to recall the usual diagram for associativity of an algebra action).

We will in fact prove that each square is commutative after applying BM homology.

Let us first look at the bottom left square. We can extend the square to a cube as

\[
\begin{array}{c}
\tilde{G}_K^C \times C_{rot}^\times \mathcal{R} \times N_O \\
p^{-1}(\mathcal{R} \times \mathcal{R}) \times N_O
\end{array}
\begin{array}{c}
\tilde{G}_K^C \times C_{rot}^\times \mathcal{P} \\
p \times \mathcal{T} \times \mathcal{P}
\end{array}
\begin{array}{c}
\mathcal{T} \times \mathcal{R} \times N_O \\
\mathcal{R} \times \mathcal{P}
\end{array}
\begin{array}{c}
p' \times \text{id}_{N_O} \\
p \times \text{id}_\mathcal{P}
\end{array}
\]

Arrows from spaces in the front square to those in the rear square are closed embeddings. Arrows in the rear square are as indicated, where we have defined \(P : \tilde{G}_K^C \times C_{rot}^\times \mathcal{P} \to \mathcal{T} \times \mathcal{P}\) by \((g_1, g_2, s) \mapsto ([g_1, g_2], s)\), just as the downward arrow from \([1]\) above.

The top, right, left and bottom faces of the cube are Cartesian and we have the isomorphisms

\[P^*(\omega_T \boxtimes \pi_1^1F_v) \cong \omega_{\tilde{G}_K^C \times C_{rot}^\times} \boxtimes \pi_1^1F_v (p' \times \text{id}_{N_O})^* \omega_T \boxtimes \omega_{\mathcal{R} \boxtimes \mathcal{P}} \cong \omega_{\tilde{G}_K^C \times C_{rot}^\times} \boxtimes \omega_{\mathcal{R} \boxtimes \mathcal{P}} \boxtimes F_v.\]

This gives us two pullbacks with supports

\[H^*_{\tilde{G}_K^C \times C_{rot}^\times \times \tilde{G}_K^C \times C_{rot}^\times}(\mathcal{R} \times \mathcal{P}, \omega_{\mathcal{R} \boxtimes \mathcal{P}} \boxtimes \pi_1^1F_v) \to H^*_{\tilde{G}_K^C \times C_{rot}^\times \times \tilde{G}_K^C \times C_{rot}^\times}([1] \omega_{\tilde{G}_K^C \times C_{rot}^\times} \boxtimes \pi_1^1F_v)\]

and

\[H^*_{\tilde{G}_K^C \times C_{rot}^\times \times \tilde{G}_K^C \times C_{rot}^\times}(p^{-1}(\mathcal{R} \times \mathcal{R}) \times N_O, \omega_{\mathcal{R} \boxtimes \mathcal{R} \boxtimes \mathcal{F}_v) \to H^*_{\tilde{G}_K^C \times C_{rot}^\times \times \tilde{G}_K^C \times C_{rot}^\times}([1] \omega_{\tilde{G}_K^C \times C_{rot}^\times} \boxtimes \pi_1^1F_v).\]

We claim that these are the same homomorphism. Consider \(\omega_T \boxtimes \omega_{\mathcal{T} \boxtimes \mathcal{R} \boxtimes \mathcal{N}_O}\) and consider the pull-backs of \(\omega_T\) and \(\omega_{\mathcal{R} \boxtimes \mathcal{P}} \boxtimes F_v\) separately. Let us first consider \(\omega_{\mathcal{R} \boxtimes \mathcal{P}} \boxtimes F_v\).

\[P^*(\text{id}_\mathcal{T} \times p)^*(\omega_T \boxtimes \omega_{\mathcal{R} \boxtimes \mathcal{F}_v}) \longrightarrow \omega_{\tilde{G}_K^C \times C_{rot}^\times} \boxtimes \pi_1^1F_v[2 \dim \mathcal{N}_O - 2 \dim \tilde{G}_K^C]\]

\[(p' \times \text{id}_{N_O})^*(\text{id}_{\tilde{G}_K^C \times C_{rot}^\times} \times p)^*(\omega_T \boxtimes \omega_{\mathcal{R} \boxtimes \mathcal{F}_v}) \longrightarrow \omega_{\tilde{G}_K^C \times C_{rot}^\times} \boxtimes \pi_1^1F_v[2 \dim \mathcal{N}_O - 2 \dim \tilde{G}_K^C]\]

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by following left, top arrows and bottom, right arrows in the rear square. They are the same, as both are essentially given by the homomorphism

$$p^* \omega_{\mathcal{R}} \boxtimes \mathcal{F}_v \to \pi_1^1 \mathcal{F}_v.$$ 

Next consider $$\omega_T$$. The $$T$$-component of $$(\text{id}_T \times p) \circ P = (\text{id}_{\tilde{G}_K^O \times \mathcal{C}_{rot}^x} \times p) \circ (p' \times \text{id}_R)$$ (which is $$(g_1, g_2, s) \mapsto [g_1, g_1 g_2, s]$$) factors as

$$\tilde{G}_K^O \times \mathcal{C}_{rot}^x \times \mathcal{P} \xrightarrow{\text{id}_{\tilde{G}_K^O \times \mathcal{C}_{rot}^x} \times \Pi'} \tilde{G}_K^O \times \mathcal{C}_{rot}^x \times \mathcal{N}_\mathcal{O} \xrightarrow{\text{id}_T \times m} T,$$

where $$\Pi': \mathcal{P} \to \mathcal{N}_\mathcal{O}$$ is $$(g_2, s) \mapsto g_2, s$$. So we have

$$(\text{id}_T \times p) \circ P)^* (\omega_T \boxtimes \omega_{\mathcal{R}} \boxtimes \mathcal{F}_v) \cong \omega_{\tilde{G}_K^O \times \mathcal{C}_{rot}^x} \boxtimes \pi_1^1 \mathcal{F}_v [2 \dim \mathcal{N}_\mathcal{O} - 2 \dim \tilde{G}_O].$$

The two restriction with supports homomorphisms from above constructed by going along left, top arrows and bottom, right arrows in the rear square are thus identical. This completes the proof of the commutativity of the bottom left square.

Since $$\tilde{q}: \mathcal{P} \to q(\mathcal{P})$$ is a fiber bundle with fibers $$\tilde{G}_O$$, commutativity for squares involving $$q$$ is obvious.

Let us finally consider the right bottom square. We extend it to a cube:

Arrows from the front to rear are closed embeddings. The map $$P': \tilde{G}_K^O \times \mathcal{C}_{rot}^x \times q(\mathcal{P}) \to T \times q(\mathcal{P})$$ is given by

$$(g_1, [g_2, s]) \mapsto ([g_1, g_1 g_2, s], [g_2, s]).$$

The left and right faces of the cube are cartesian, and the commutativity of the rear square in the cube is enough to conclude that the corresponding proper pushforwards give the same map.

Finally, the commutativity of the induced maps in the right top square is clear, as it involves only pushforward homomorphisms. In particular, the whole large square is commutative.

**Lemma Appendix A.4.** The class of $$[1] \in H^*_* (\tilde{G}_O \times \mathcal{C}_{rot}^x (\mathcal{R})$$ acts by the identity on $$H^*_* (M_v).$$

**Proof.** Consider the following diagram.
The vertical maps are the natural inclusions (where we include $N_{\mathcal{O}} \hookrightarrow \mathcal{R}$ as the fiber over $\text{Gr}_{\mathcal{G}}^0$). Since $[1] \otimes c$ is the pushforward of $1 \otimes c$ along the left inclusion, by proper base change, $q_* p^* ([1] \otimes c)$ is given by the pushforward along right vertical embedding $N_{\mathcal{O}} \rightarrow q(\mathcal{P})$.

Composing with $m : q(\mathcal{P}) \rightarrow N_{\mathcal{O}}$, this embedding becomes the identity map on $N_{\mathcal{O}}$, so we must have $m_* q_* p^* ([1] \otimes c) = c$. \qed

Appendix B.

Appendix B.1. Generalized Affine Springer Fibers for $(2, k)$ Torus Links

In this section we discuss the example where $\hat{C} = \text{Spec} \mathbb{C}[[x,t]]_{x^2 - t^k}$, whose links are $(2, k)$ torus links. We are interested in the GASF $\mathcal{M}_{(2, k)} := M_v$ for

$$v = \begin{pmatrix} 0 & t^k \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$ 

As remarked above, under the decomposition of $\text{Gr}_{GL_2}$ by $\pi_1 (GL_2) = \mathbb{Z}$ we have

$$\mathcal{M}_{(2, k)} = \bigsqcup_{m \leq 0} \mathcal{M}_{(2, k)}^m.$$ 

Lemma Appendix B.1. For $k = 2\ell + 1$, we have

$$\mathcal{M}_{(2, 2\ell + 1)}^m = \begin{cases} \mathbb{P} \left[ \frac{|m|}{2} \right] & |m| \leq 2\ell, \\ \mathbb{P}^\ell & |m| \geq 2\ell \end{cases}$$

Proof. As described in Prop. 5.11, we can parameterize $\mathcal{M}_{(2, 2\ell + 1)}$ by $h(t)$ of the form

$$h(t) = \begin{pmatrix} t^{-A_1} & u(t) \\ 0 & t^{-A_2} \end{pmatrix},$$

such that $u$ is a Laurent polynomial with entries of degree at most $-A_1 - 1$. Only trivial $G(\mathcal{O})$ transformations fix this form and therefore the above $h$, subject to some algebraic equations on $u(t)$, parameterizes a closed, affine variety in $\mathcal{M}_{(2, 2\ell + 1)} \setminus (A_1 + A_2)$. With this in mind, we have

$$h^{-1}.v = \begin{pmatrix} -t^{A_2}u & t^{A_1 + A_2}(t^{2\ell + 1} - u^2) \\ t^{A_2 - A_1} & t^{A_2}u \end{pmatrix}, \begin{pmatrix} t^{A_1} \\ 0 \end{pmatrix}.$$ 

Belonging to $\mathcal{V}^v$ implies that $A_1, A_2 - A_1 \geq 0$ and $w := -t^{A_2}u, t^{A_1 - A_2}(t^{2\ell + 1} - u^2)$ belong to $\mathcal{O}$. 27
The latter condition implies that \( A_2 - A_1 \leq 2\ell + 1 \) and \( w \) should have no terms of degree less than \( \frac{A_2 - A_1}{2} \). Thus, this is simply a copy of \( \mathbb{C}[\frac{A_2 - A_1}{2}] \). The closure of this cell is a copy of \( \mathbb{P}[\frac{A_2 - A_1}{2}] \) containing the fixed points \( \{(A_1, A_2), (A_1 + 1, A_2 - 1), \ldots, (A_1 + \lfloor \frac{A_2 - A_1}{2} \rfloor, A_2 - \lfloor \frac{A_1 - A_2}{2} \rfloor)\} \). Therefore

\[
\mathcal{M}^m_{(2,2\ell+1)} = \begin{cases} 
\mathbb{P}[\frac{|m|}{2}] & |m| \leq 2\ell, \\
\mathbb{P}^{\ell} & |m| \geq 2\ell.
\end{cases}
\]

**Lemma Appendix B.2.** For \( k = 2\ell \), we have

\[
\mathcal{M}^m_{(2,2\ell)} = \mathbb{P}[\frac{|m|}{2}]
\]

for \( |m| \leq 2\ell \) and \( \mathcal{M}^m_{(2,2\ell)} \) is a chain of \( |m| - 2\ell + 1 \) copies of \( \mathbb{P}^{\ell} \) intersecting transversely for \( |m| \geq 2\ell \).

**Proof.** We can parameterize \( \mathcal{M}_{(2,2\ell)} \) via

\[
h^{-1}.v = \left( \begin{array}{c}
w \\
t^{A_2-A_1} - t^{A_1-A_2} (t^{2\ell} - w^2) \\
t^{A_1}
\end{array} \right),
\]

where \( w := -t^{A_2}.u \). Belonging to \( \mathcal{V}^w \) implies that \( A_1, A_2 - A_1 \geq 0 \) and \( w, t^{A_1-A_2}(t^{2\ell} - w^2) \in \mathcal{O} \).

For \( A_2 - A_1 \leq \ell \) we need \( w \) to have no terms of degree less than \( \frac{A_2 - A_1}{2} \), i.e. the above cell is \( \lfloor \frac{A_2 - 1}{2} \rfloor \) dimensional. When \( A_2 - A_1 > \ell \) there are more possibilities, i.e. when \( w \) takes the form

\[
w_\pm = \pm \ell \ell + \sum_{j = j_0}^{N_2 - 1} u_j t^j.
\]

Belonging to \( \mathcal{N}(\mathcal{O}) \) requires \( A_2 - A_1 \leq 2j_0 \) and \( A_2 - A_1 \leq j_0 + \ell \) so that the minimal solution is \( j_0 = \max\{\lfloor \frac{A_2 - A_1}{2} \rfloor, A_2 - A_1 - \ell\} \). Thus, this cell is \( C[\frac{A_2 - A_1}{2}] \) for \( \ell < A_2 - A_1 \leq 2\ell \) and \( C^{\ell} \) for \( A_2 - A_1 \geq 2\ell \). \( w_+ \) and \( w_- \) parameterize the same cell when \( A_2 - A_1 \leq 2\ell \), but are otherwise inequivalent. We again find that the closure of each of the above is a copy of projective space.

Tracking through the closures, we find that \( \mathcal{M}^m_{(2,2\ell)} \) is a copy of \( \mathbb{P}[\frac{|m|}{2}] \) for \( |m| \leq 2\ell \) and a chain of \( |m| - 2\ell + 1 \) copies of \( \mathbb{P}^{\ell} \) intersecting transversely for \( |m| \geq 2\ell \). \( \square \)

**Appendix B.2. Modules for (2, 2\ell + 1) Torus Knots**

In this section we discuss the module structure of \( H_{L^+}^*(\mathcal{M}_{(2,2\ell+1)}) \).

Recall that the spherical rational Cherednik algebra for \( GL_2 \) has generators given by an \( \mathfrak{sl}_2 \) triple \( E, F, H \) and a Weyl pair \( X, Y \) transforming in the defining representation of that \( \mathfrak{sl}_2 \). In particular, the non-zero commutation relations between these generators are those defining \( \mathfrak{sl}_2 \) and the Weyl algebra

\[
[E, F] = hH \quad [H, E] = 2hE \quad [H, F] = -2hF \quad [X, Y] = 2h,
\]

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and those describing the way \( X, Y \) transform under \( \mathfrak{sl}_2 \)
\[
[E, X] = [F, Y] = 0 \quad [H, X] = [E, Y] = hX \quad [H, Y] = -[F, X] = -hY.
\]
Denote \( W^+ = \frac{1}{2}X^2, W^0 = -\frac{1}{2}(XY + YX), W^- = -\frac{1}{2}Y^2 \), so that the \( W^\pm, W^0 \) transform in the adjoint representation of the above \( \mathfrak{sl}_2 \). There is one additional relation amongst these operators:
\[
C_2 = 2(EW^- + FW^+) + HW^0 + m(m - \hbar),
\]
where \( C_2 = 2(EF + FE) + H^2 \) is the quadratic Casimir of the \( \mathfrak{sl}_2 \) triple and \( m \) is a complex parameter.

**Theorem Appendix B.3.** The above algebra, realized as the quantized BFN algebra \( \widehat{\mathcal{A}}_{G, V}^\mathbb{C} \) for \( G = GL_2, N = \text{Ad} \oplus \mathbb{C}^2 \), acts via convolution on \( H^\mathbb{C}^\times(M_{(2, 2\ell+1)}) \) for \( m = -\frac{2\ell+1}{2} \hbar \). As a module for the spherical rational Cherednik algebra for \( \mathfrak{gl}_2 \), we have
\[
H^\mathbb{C}^\times(M_{(2, 2\ell+1)}) \simeq eL_{-(2\ell+2)/2}(\text{triv}),
\]
where \( e \) is the \( \mathfrak{S}_2 \) symmetrizer and \( L_{-(2\ell+2)/2}(\text{triv}) \) is the simple module of the rational Cherednik algebra of \( \mathfrak{gl}_2 \) (at parameter \( m = -\frac{2\ell+1}{2} \hbar \)) induced from the trivial representation of \( \mathfrak{S}_2 \).

**Proof.** First consider the monopole operator \( X := [\mathcal{R}_{(1,0)}] \). This arises from the orbits \( \text{Gr}^{(1,0)}_{GL_2} \), which form a copy of \( \mathbb{P}^1 \) parameterized by two affine charts given by
\[
\begin{pmatrix}
t & 0 \\ a_1 & 1
\end{pmatrix}
\]
with transition function \( a_2 = \frac{1}{a_1} \). There are \( G(\mathcal{O}) \) torus fixed points at the origins of these affine charts, and the coordinate \( a_1 \) (resp. \( a_2 \)) transforms with weight \( \varphi_2 - \varphi_1 \) (resp. \( \varphi_1 - \varphi_2 \)). Applying Eq. (5.1) yields
\[
X |A_1, A_2\rangle = \frac{A_2 - A_1 - 2\ell - 1}{2} |A_1, A_2 + 1\rangle + \frac{A_2 - A_1}{A_2 - A_1 - 2\ell - 1} |A_1 + 1, A_2\rangle.
\]

Similarly, there is the monopole operator \( Y := [\mathcal{R}_{(0,-1)}] \) coming from the orbits \( \text{Gr}^{(0,-1)}_{GL_2} \), which forms a copy of \( \mathbb{P}^1 \) parameterized by two affine charts
\[
\begin{pmatrix}
1 & 0 \\ a_1 t^{-1} & t^{-1}
\end{pmatrix}
\]
with transition function \( a_2 = \frac{1}{a_1 t} \). The coordinate \( a_1 \) again transforms with weight \( \varphi_1 - \varphi_2 \). We find that
\[
Y |A_1, A_2\rangle = \frac{(A_2 - A_1)(2\ell + 1 + A_2)\hbar}{A_2 - A_1 - 2\ell - 1} |A_1, A_2 - 1\rangle + \frac{A_1(2\ell + 1 - A_2 + A_1)\hbar}{A_2 - A_1 - 2\ell - 1} |A_1 - 1, A_2\rangle.
\]

There are two other monopole operators we will be interested in, namely \( E = [\mathcal{R}_{(1,1)}] \) and \( F = -[\mathcal{R}_{(-1,-1)}] \). They come from \( \text{Gr}^{(1,1)}_{GL_2} \) and \( \text{Gr}^{(-1,-1)}_{GL_2} \) respectively, both of which are single orbits. Applying Eq. (5.1) gives
\[
E |A_1, A_2\rangle = |A_1 + 1, A_2 + 1\rangle \quad F |A_1, A_2\rangle = A_1(\frac{2\ell + 1}{2} - A_2)\hbar^2 |A_1 - 1, A_2 - 1\rangle.
\]
from which it is straightforward to compute that \( H = h - \varphi_1 - \varphi_2 \) acts as
\[
H | A_1, A_2 \rangle = (A_1 + A_2 + 1 - \frac{2\ell+1}{2})h | A_1, A_2 \rangle
\]
and makes \((E, F, H)\) an \(\mathfrak{sl}_2\) triple. The quadratic Casimir \( C_2 = 2(EF + FE) + H^2 \) acts as
\[
C_2 | A_1, A_2 \rangle = \left( (A_2 - A_1 - \frac{2\ell+1}{2})^2 - 1 \right) h^2 | A_1, A_2 \rangle.
\]
It is straightforward to check that the desired relations are indeed satisfied with \( m = -\frac{2\ell+1}{2} \).

From the action of \(\mathfrak{sl}_2\), we see that the classes \([0, A_2]\) are lowest weight vectors with weights \( \nu = (A_2 + 1 - \frac{2\ell+1}{2})h \). Therefore, the homology of this GASF can be expressed as an \(\mathfrak{sl}_2\) module as
\[
H^C_\ell (\mathcal{M}(2,2\ell+1)) = \bigoplus_{A_2=0}^{2\ell+1} A_{(A_2+1-\frac{2\ell+1}{2})} h^j,
\]
where \( A_\nu \) is the \(\mathfrak{sl}_2\) Verma module generated by a lowest weight vector of weight \( \nu \). It is also worth noting that there is another presentation of the \(\mathfrak{sl}_2\) Verma module for the rational Cherednik algebra of \( R = 2 \) given by a (different) \(\mathfrak{sl}_2\)-triple \((E, F, H)\) and the Weyl pair \( X, Y \). In this presentation \( X, Y \) transform trivially under \(\mathfrak{sl}_2\) and the quadratic Casimir of the \(\mathfrak{sl}_2\)-triple is given by
\[
\tilde{C}_2 = (m - \frac{3}{2} \varepsilon)(m + \frac{1}{2} \varepsilon)
\]
with no other constraints. The above vectors can be identified (up to scaling) with \( E^N | 0, 0 \rangle \). We find that the homology of our GASF is given by
\[
H^C_\ell (\mathcal{M}(2,2\ell+1)) \cong C[X] \otimes \text{Sym}^\ell \square,
\]
where \( \text{Sym}^\ell \square \) is the \(\ell + 1\) dimensional representation of \(\mathfrak{sl}_2\). We can identify \(\text{Sym}^\ell \square\) as the cohomology of \( \mathbb{P}^\ell \), the compactified Jacobian for the \((2, 2\ell + 1)\) torus knots. This feature was predicted in [28].

\[3\] The change of variables is given by \( \tilde{E} = E - \frac{1}{4} X^2, \tilde{F} = F + \frac{1}{4} Y^2, \tilde{H} = H + \frac{1}{4} (XY + YX) \).
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