Hyperbolic Numbers and the Dirac Spinor

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Abstract

A representation of the Lorentz group is given in terms of $4 \times 4$ matrices defined over the hyperbolic number system. The transformation properties of the corresponding four component spinor are studied, and shown to be equivalent to the transformation properties of the complex Dirac spinor. As an application, we show that there exists an algebra of automorphisms of the complex Dirac spinor that leaves the transformation properties of its eight real components invariant under any given Lorentz transformation. Interestingly, the representation of the Lorentz algebra presented here is naturally embedded in the Lie algebra of a group isomorphic to $SO(3,3;\mathbb{R})$ instead of the conformal group $SO(2,4;\mathbb{R})$. 
1 Introduction

This article is motivated by the simple observation that the transformation properties of the eight real components of a complex Dirac spinor under a Lorentz transformation may be alternatively formulated without any explicit reference to complex-valued quantities. This is accomplished by constructing a representation of the Lorentz group in terms of $4 \times 4$ matrices defined over the hyperbolic number system $\mathbb{H}$—$\mathbb{I}$–$\mathbb{L}$–$\mathbb{E}$. After studying how this new representation is related to the familiar complex one, we establish an automorphism symmetry of the complex Dirac spinor. We also discuss natural embeddings of this new representation into a maximal Lie algebra, which turns out to be isomorphic to the algebra of generators of $\text{SO}(3,3; \mathbb{R})$, and thus distinct from the conformal group $\text{SO}(2,4; \mathbb{R})$.

To begin, we revisit the familiar Lie algebra of the Lorentz group $\text{O}(1,3; \mathbb{R})$.

2 The Lorentz Algebra

2.1 A Complex Representation

Under Lorentz transformations, the complex Dirac 4-spinor $\Psi_C$ transforms as follows $\mathbb{I}$:

$$
\Psi_C \rightarrow \left( e^{\frac{i}{2} \sigma \cdot (\theta - i \phi)} \begin{pmatrix} 0 & 0 \\ 0 & e^{\frac{i}{2} \sigma \cdot (\theta + i \phi)} \end{pmatrix} \right) \cdot \Psi_C, \tag{1}
$$

where $\sigma = (\sigma_x, \sigma_y, \sigma_z)$ represents the well known Pauli spin matrices:

$$
\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2}
$$

The three real parameters $\theta = (\theta_1, \theta_2, \theta_3)$ correspond to the generators for spatial rotations, while $\phi = (\phi_1, \phi_2, \phi_3)$ represents Lorentz boosts along each of the coordinate axes. There are thus six real numbers parameterizing a given element in the Lorentz group.

Let us now introduce the six matrices $E_i$ and $F_i$, $i = 1, 2, 3$, by writing

$$
E_1 = \frac{1}{2} \begin{pmatrix} \sigma_x & 0 \\ 0 & -\sigma_x \end{pmatrix}, \quad E_2 = -\frac{1}{2} \begin{pmatrix} \sigma_y & 0 \\ 0 & \sigma_y \end{pmatrix}, \quad E_3 = \frac{1}{2} \begin{pmatrix} \sigma_z & 0 \\ 0 & -\sigma_z \end{pmatrix},
$$

$$
F_1 = \frac{1}{2} \begin{pmatrix} \sigma_x & 0 \\ 0 & \sigma_x \end{pmatrix}, \quad F_2 = \frac{1}{2} \begin{pmatrix} \sigma_y & 0 \\ 0 & -\sigma_y \end{pmatrix}, \quad F_3 = \frac{1}{2} \begin{pmatrix} \sigma_z & 0 \\ 0 & \sigma_z \end{pmatrix}. \tag{3}
$$

Then the Lorentz transformation $\mathbb{I}$ may be written as follows:

$$
\Psi_C \rightarrow \exp (\phi_1 E_1 - \theta_2 E_2 + \phi_3 E_3 + \theta_1 F_1 + \phi_2 F_2 + \theta_3 F_3) \cdot \Psi_C. \tag{4}
$$

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We remark that the transformation (4) has the form $\Psi_C \rightarrow U \cdot \Psi_C$, where $U$ may be thought of as an element of the (fifteen dimensional) conformal group $SU(2,2;\mathbb{C})$. The Lorentz symmetry is therefore a six dimensional subgroup of the conformal group.

At this point, it is sufficient to note that the matrices $E_i$ and $F_i$ defined in (3) satisfy the following commutation relations:

\[
\begin{align*}
[E_1, E_2] &= E_3 & [F_1, F_2] &= -E_3 & [E_1, F_2] &= F_3 & [F_1, E_2] &= F_3 \\
[E_2, E_3] &= E_1 & [F_2, F_3] &= -E_1 & [E_2, F_3] &= F_1 & [F_2, E_3] &= F_1 \\
[E_3, E_1] &= -E_2 & [F_3, F_1] &= E_2 & [E_3, F_1] &= -F_2 & [F_3, E_1] &= -F_2
\end{align*}
\]

All other commutators vanish. Abstractly, these relations define the Lie algebra of the Lorentz group $O(1,3;\mathbb{R})$, and the matrices $E_i$ and $F_i$ defined by (3) correspond to a complex representation of this algebra.

### 2.2 A Hyperbolic Representation

Our goal in this section is to present an explicit representation of the Lorentz algebra (5) in terms of $4 \times 4$ matrices defined over the hyperbolic number system. This number system will be briefly discussed next.

#### 2.2.1 The Hyperbolic Number System

We consider numbers of the form

\[ x + jy, \]

where $x$ and $y$ are real numbers, and $j$ is a commuting element satisfying the relation

\[ j^2 = 1. \]

The number system generated by this simple algebra has a long history [1]–[13], and is known as the ‘hyperbolic number system’. The symbol $\mathbb{D}$ will be used to denote the hyperbolic number system, where ‘$D$’ stands for ‘double’ [13].

In this article, we exploit very basic arithmetical properties of this algebra. For example, addition, subtraction, and multiplication are defined in the obvious way:

\[
(x_1 + jy_1) \pm (x_2 + jy_2) = (x_1 \pm x_2) + j(y_1 \pm y_2),
\]

\[
(x_1 + jy_1) \cdot (x_2 + jy_2) = (x_1x_2 + y_1y_2) + j(x_1y_2 + y_1x_2).
\]

Moreover, given any hyperbolic number $w = x + jy$, we define the ‘$\mathbb{D}$-conjugate of $w$’, written $\overline{w}$, to be

\[ \overline{w} = x - jy. \]
It is easy to check the following; for any \( w_1, w_2 \in \mathbb{D} \), we have

\[
\overline{w_1 + w_2} = \overline{w_1} + \overline{w_2},
\]

\[
\overline{w_1 \cdot w_2} = \overline{w_1} \cdot \overline{w_2}.
\]

We also have the identity

\[
\overline{w} \cdot w = x^2 - y^2
\]

for any hyperbolic number \( w = x + jy \). Thus \( \overline{w} \cdot w \) is always real, although unlike the complex number system, it may take negative values.

At this point, it is convenient to define the ‘modulus squared’ of \( w \), written \(|w|^2\), as

\[
|w|^2 = \overline{w} \cdot w.
\]

A nice consequence of these definitions is that for any hyperbolic numbers \( w_1, w_2 \in \mathbb{D} \), we have

\[
|w_1 \cdot w_2|^2 = |w_1|^2 \cdot |w_2|^2.
\]

Now observe that if \(|w|^2\) doesn’t vanish, the quantity

\[
w^{-1} = \frac{1}{|w|^2} \cdot \overline{w}
\]

is a well-defined unique inverse for \( w \). So \( w \in \mathbb{D} \) fails to have an inverse if and only if \(|w|^2 = x^2 - y^2 = 0\). The hyperbolic number system is therefore a non-division algebra.

### 2.2.2 The Hyperbolic Unitary Groups

Suppose \( H \) is an \( n \times n \) matrix defined over \( \mathbb{D} \). Then \( H^\dagger \) will denote the \( n \times n \) matrix which is obtained by transposing \( H \), and then conjugating each of the entries: \( H^\dagger = \overline{H}^T \). We say \( H \) is Hermitian with respect to \( \mathbb{D} \) if \( H^\dagger = H \), and anti-Hermitian if \( H^\dagger = -H \).

Note that if \( H \) is an \( n \times n \) Hermitian matrix over \( \mathbb{D} \), then \( U = e^{iH} \) has the property

\[
U^\dagger \cdot U = U \cdot U^\dagger = 1.
\]

The set of all \( n \times n \) matrices over \( \mathbb{D} \) satisfying the constraint \((17)\) forms a group, which we will denote as \( U(n, \mathbb{D}) \), and call the ‘unitary group of \( n \times n \) matrices over \( \mathbb{D} \)’, or ‘hyperbolic unitary group’. The ‘special unitary’ subgroup \( SU(n, \mathbb{D}) \) will be defined as all elements \( U \in U(n, \mathbb{D}) \) satisfying the additional constraint

\[
\det U = 1.
\]
Note that the hyperbolic unitary groups we have defined above may be isomorphic to well known non-compact groups that are usually defined over the complex number field. For example, the special unitary hyperbolic group SU(2, D) is isomorphic to the complex group SU(1,1; C) by virtue of the identification

\[
\left( \begin{array}{cc}
 a_1 + ia_2 & b_1 + ib_2 \\
 b_1 - ib_2 & a_1 - ia_2 
\end{array} \right) \leftrightarrow \left( \begin{array}{cc}
 a_1 + jb_1 & -a_2 + jb_2 \\
 a_2 + jb_2 & a_1 - jb_1 
\end{array} \right),
\]

where the four real parameters \(a_1, a_2, b_1\) and \(b_2\) satisfy the constraint \(a_1^2 + a_2^2 - b_1^2 - b_2^2 = 1\).

Of course, one also has the isomorphism \(SU(2, D) \equiv SL(2; R)\) given by the group isomorphism

\[
\left( \begin{array}{cc}
 a_1 + jb_1 & -a_2 + jb_2 \\
 a_2 + jb_2 & a_1 - jb_1 
\end{array} \right) \leftrightarrow \left( \begin{array}{cc}
 a_1 + b_1 & -a_2 + b_2 \\
 a_2 + b_2 & a_1 - b_1 
\end{array} \right),
\]

where the real parameters \(a_1, a_2, b_1\) and \(b_2\) satisfy the constraint \(a_1^2 + a_2^2 - b_1^2 - b_2^2 = 1\) as before. Note that this correspondence was obtained by mapping the variable \(j\) to +1. Alternatively, we could have constructed an alternative isomorphism by mapping \(j\) to −1.

Actually, this example suggests that we might be able to identify the special unitary groups SU\((n; D)\) with the special linear groups SL\((n; R)\). An isomorphism was established for \(n = 2\), but what can we say about \(n > 2\)? One approach is to consider what happens near the identity. In this case, one may construct the Lie algebra for SU\((n; D)\), which is generated by \(n^2 - 1\) traceless anti-Hermitian \(n \times n\) matrices over \(D\). Any element sufficiently close to the identity is therefore obtained by exponentiating a unique real linear combination of these generators. We then map such elements into SL\((n; R)\) by mapping the variable \(j\) to +1. The generators are now real, traceless \(n \times n\) matrices, and so form the basis of the Lie algebra for SL\((n; R)\). Thus, the groups SU\((n; D)\) and SL\((n; R)\) possess isomorphic Lie algebras.

### 2.2.3 A Hyperbolic Representation

As promised, we will give an explicit representation of the Lorentz algebra (5) in terms of matrices defined over \(D\). First, we define three \(2 \times 2\) matrices \(\tau = (\tau_1, \tau_2, \tau_3)\) by writing

\[
\tau_1 = \begin{pmatrix}
 0 & 1 \\
 1 & 0
\end{pmatrix}, \quad \tau_2 = \begin{pmatrix}
 0 & -j \\
 j & 0
\end{pmatrix}, \quad \tau_3 = \begin{pmatrix}
 1 & 0 \\
 0 & -1
\end{pmatrix}.
\]

\(\text{To show that any } 2 \times 2 \text{ matrix } U \in SU(2, D) \text{ has the form given in eqn (19), we use the facts } U^\dagger = U^{-1}, \text{ and } \det U = 1.\)
These matrices satisfy the following commutation relations:

\[ [\tau_1, \tau_2] = 2 j \tau_3 \quad [\tau_2, \tau_3] = 2 j \tau_1 \quad [\tau_3, \tau_1] = -2 j \tau_2 \]  

Now define the matrices \( \tilde{E}_i \) and \( \tilde{F}_i \), \( i = 1, 2, 3 \), by setting

\[
\tilde{E}_i = \frac{j}{2} \begin{pmatrix} \tau_i & 0 \\ 0 & -\tau_i \end{pmatrix}, \quad \tilde{F}_i = \frac{1}{2} \begin{pmatrix} 0 & \tau_i \\ -\tau_i & 0 \end{pmatrix}, \quad i = 1, 2, 3.
\]

The \( 4 \times 4 \) matrices \( \tilde{E}_i \) and \( \tilde{F}_i \) defined above are anti-Hermitian with respect to \( D \), and satisfy the Lorentz algebra \((5)\) after making the substitutions \( E_i \rightarrow \tilde{E}_i \) and \( F_i \rightarrow \tilde{F}_i \), \( i = 1, 2, 3 \). We may therefore introduce a 4-component ‘hyperbolic’ spinor \( \Psi_D \in D^4 \) transforming as follows under Lorentz transformations:

\[
\Psi_D \rightarrow \exp \left( \phi_1 \tilde{E}_1 - \theta_2 \tilde{E}_2 + \phi_3 \tilde{E}_3 + \theta_1 \tilde{F}_1 + \phi_2 \tilde{F}_2 + \theta_3 \tilde{F}_3 \right) \cdot \Psi_D,
\]

which is evidently the analogue of transformation \((4)\). Note that the transformation \((24)\) has the form \( \Psi_D \rightarrow U \cdot \Psi_D \), where \( U \in SU(4, D) \), since the generators \( \tilde{E}_i \) and \( \tilde{F}_i \) are traceless and anti-Hermitian with respect to \( D \). Thus the Lorentz group is a subgroup of the hyperbolic special unitary group \( SU(4, D) \).

In the next section, we discuss a relation between the complex Dirac spinor \( \Psi_C \), and the 4-component hyperbolic spinor \( \Psi_D \) defined above.

### 3 Equivalences between Spinor Transformations

#### 3.1 An Equivalence

Consider an infinitesimal Lorentz transformation of the complex Dirac spinor,

\[
\Psi_C \rightarrow \exp \left( \phi_1 E_1 - \theta_2 E_2 + \phi_3 E_3 + \theta_1 F_1 + \phi_2 F_2 + \theta_3 F_3 \right) \cdot \Psi_C,
\]

where

\[
\Psi_C = \begin{pmatrix} x_1 + iy_1 \\ x_2 + iy_2 \\ x_3 + iy_3 \\ x_4 + iy_4 \end{pmatrix},
\]

and \( E_i, F_i \) are specified by \((3)\). The eight variables \( x_i \) and \( y_i \), \( i = 1, 2, 3, 4 \), are taken to be real. Now consider the corresponding infinitesimal Lorentz transformation of the hyperbolic spinor \( \Psi_D \),

\[
\Psi_D \rightarrow \exp \left( \phi_1 \tilde{E}_1 - \theta_2 \tilde{E}_2 + \phi_3 \tilde{E}_3 + \theta_1 \tilde{F}_1 + \phi_2 \tilde{F}_2 + \theta_3 \tilde{F}_3 \right) \cdot \Psi_D,
\]
where

\[ \Psi_D = \begin{pmatrix} a_1 + jb_1 \\ a_2 + jb_2 \\ a_3 + jb_3 \\ a_4 + jb_4 \end{pmatrix}. \quad (28) \]

The matrices \( \tilde{E}_i, \tilde{F}_i \) are given by (23), and the eight variables \( a_i \) and \( b_i, \ i = 1, 2, 3, 4, \) are real-valued.

It is now straightforward to check that the infinitesimal transformations (25) and (27) induce equivalent transformations of the eight real components of the corresponding spinors (\( \Psi_C \) and \( \Psi_D \)) if we make the following identifications\( \dagger \):

\[
\begin{align*}
    a_1 & \leftrightarrow \frac{1}{\sqrt{2}}(y_1 + y_3) \\
    b_1 & \leftrightarrow \frac{1}{\sqrt{2}}(y_1 - y_3) \\
    a_2 & \leftrightarrow \frac{1}{\sqrt{2}}(y_2 + y_4) \\
    b_2 & \leftrightarrow \frac{1}{\sqrt{2}}(y_2 - y_4) \\
    a_3 & \leftrightarrow \frac{1}{\sqrt{2}}(x_1 + x_3) \\
    b_3 & \leftrightarrow \frac{1}{\sqrt{2}}(x_1 - x_3) \\
    a_4 & \leftrightarrow \frac{1}{\sqrt{2}}(x_2 + x_4) \\
    b_4 & \leftrightarrow \frac{1}{\sqrt{2}}(x_2 - x_4)
\end{align*}
\]

(29)

In particular, we have the identification

\[
(I) \quad \Psi_C = \begin{pmatrix} x_1 + iy_1 \\ x_2 + iy_2 \\ x_3 + iy_3 \\ x_4 + iy_4 \end{pmatrix} \leftrightarrow \Psi_D = \frac{1}{\sqrt{2}} \begin{pmatrix} (y_1 + y_3) + j(y_1 - y_3) \\ (y_2 + y_4) + j(y_2 - y_4) \\ (x_1 + x_3) + j(x_1 - x_3) \\ (x_2 + x_4) + j(x_2 - x_4) \end{pmatrix}, \quad (30)
\]

which establishes an exact equivalence between a complex Lorentz transformation [Eqn(24)] acting on the Dirac 4-spinor \( \Psi_C \), and the corresponding Lorentz transformation [Eqn(24)] acting on a hyperbolic 4-spinor \( \Psi_D \).

It turns out that the equivalence specified by the identification (30) is not unique. There are additional identifications that render the complex and hyperbolic Lorentz transformations equivalent, and we list three more below:

(II) \[
\begin{pmatrix} x_1 + iy_1 \\ x_2 + iy_2 \\ x_3 + iy_3 \\ x_4 + iy_4 \end{pmatrix} \leftrightarrow \frac{1}{\sqrt{2}} \begin{pmatrix} -(y_2 + y_4) + j(y_2 - y_4) \\ (y_1 + y_3) - j(y_1 - y_3) \\ (x_2 + x_4) - j(x_2 - x_4) \\ -(x_1 + x_3) + j(x_1 - x_3) \end{pmatrix}, \quad (31)
\]

(III) \[
\begin{pmatrix} x_1 + iy_1 \\ x_2 + iy_2 \\ x_3 + iy_3 \\ x_4 + iy_4 \end{pmatrix} \leftrightarrow \frac{1}{\sqrt{2}} \begin{pmatrix} -(x_1 + x_3) - j(x_1 - x_3) \\ -(x_2 + x_4) - j(x_2 - x_4) \\ (y_1 + y_3) + j(y_1 - y_3) \\ (y_2 + y_4) + j(y_2 - y_4) \end{pmatrix}, \quad (32)
\]

\(^2\)The factor of \( 1/\sqrt{2} \) is arbitrary, and introduced for later convenience.
and

\[
(IV) \quad \begin{pmatrix}
  x_1 + iy_1 \\
  x_2 + iy_2 \\
  x_3 + iy_3 \\
  x_4 + iy_4
\end{pmatrix} \leftrightarrow \frac{1}{\sqrt{2}} \begin{pmatrix}
  -(x_2 + x_4) + j(x_2 - x_4) \\
  (x_1 + x_3) - j(x_1 - x_3) \\
  -(y_2 + y_4) + j(y_2 - y_4) \\
  (y_1 + y_3) - j(y_1 - y_3)
\end{pmatrix}. \tag{33}
\]

Four more identifications may be obtained by a simple ‘reflection’ procedure; simply multiply each hyperbolic spinor appearing in identifications (I),(II),(III) and (IV) above by the variable \( j \). This has the effect of interchanging the ‘real’ and ‘imaginary’ parts of each component in the spinor. We have thus enumerated a total of eight distinct identifications, and an open question is whether there are additional (linearly independent) identifications that can be made. We leave this question for future work.

3.2 Parity

Under parity, the Dirac 4-spinor \( \Psi_C \) transforms as follows \[14\]:

\[
\Psi_C \rightarrow \begin{pmatrix}
  0 & 1_{2 \times 2} \\
  1_{2 \times 2} & 0
\end{pmatrix} \cdot \Psi_C, \tag{34}
\]

or, in terms of the eight real components \( x_i \) and \( y_i \), \( i = 1, 2, 3, 4 \), of the Dirac 4-spinor \( \Psi_C \) specified by (26), we have

\[
\begin{align*}
  x_1 & \rightarrow x_3 \\
  x_2 & \rightarrow x_4 \\
  x_3 & \rightarrow x_1 \\
  x_4 & \rightarrow x_2 \\
  y_1 & \rightarrow y_3 \\
  y_2 & \rightarrow y_4 \\
  y_3 & \rightarrow y_1 \\
  y_4 & \rightarrow y_2
\end{align*} \tag{35}
\]

According to the identifications (I),(II),(III) and (IV) of Section 3.1, a parity transformation on \( \Psi_C \) corresponds to \( D \)-conjugation of each component of \( \Psi_D \). Thus, \( \Psi_D \rightarrow \Psi_D^* \) under parity for these identifications. The ‘reflected’ forms of these identifications induces the transformation \( \Psi_D \rightarrow j\Psi_D^* \) under parity. Thus the mathematical operation of \( D \)-conjugation is closely related to the parity symmetry operation.

4 An Automorphism Algebra of the Dirac Spinor

The existence of distinct equivalences between the transformation properties of complex (or Dirac) and hyperbolic spinors permits one to construct automorphisms of the complex Dirac spinor that leave the transformation properties of its eight real components intact under Lorentz transformations.

\[\Psi_D^* \text{ denotes taking the } D\text{-conjugate of each component in } \Psi_D.\]
In order to investigate the algebra underlying the set of all possible automorphisms, it is convenient to change our current basis to the so-called ‘standard representation’ of the Lorentz group [14]. The Dirac 4-spinor $\Psi^{SR}_D$ in the standard representation is related to the original 4-spinor $\Psi_C$ according to the relation

$$\Psi^{SR}_C = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \cdot \Psi_C.$$  \hspace{1cm} (36)

The identifications (I)-(IV) stated in Section 3.1 are now equivalent to the following identifications:

\[ (I)' \quad \Psi_D = \begin{pmatrix} a_1 + jb_1 \\ a_2 + jb_2 \\ a_3 + jb_3 \\ a_4 + jb_4 \end{pmatrix} \leftrightarrow \Psi^{SR}_C = \begin{pmatrix} a_3 + ia_1 \\ a_4 + ia_2 \\ b_3 + ib_1 \\ b_4 + ib_2 \end{pmatrix} \]  \hspace{1cm} (37)

\[ (II)' \quad \Psi_D = \begin{pmatrix} a_1 + jb_1 \\ a_2 + jb_2 \\ a_3 + jb_3 \\ a_4 + jb_4 \end{pmatrix} \leftrightarrow \Psi^{SR}_C = \begin{pmatrix} -a_4 + ia_2 \\ a_3 - ia_1 \\ b_4 - ib_2 \\ -b_3 + ib_1 \end{pmatrix} \]  \hspace{1cm} (38)

\[ (III)' \quad \Psi_D = \begin{pmatrix} a_1 + jb_1 \\ a_2 + jb_2 \\ a_3 + jb_3 \\ a_4 + jb_4 \end{pmatrix} \leftrightarrow \Psi^{SR}_C = \begin{pmatrix} -a_1 + ia_3 \\ -a_2 + ia_4 \\ -b_1 + ib_3 \\ -b_2 + ib_4 \end{pmatrix} \]  \hspace{1cm} (39)

\[ (IV)' \quad \Psi_D = \begin{pmatrix} a_1 + jb_1 \\ a_2 + jb_2 \\ a_3 + jb_3 \\ a_4 + jb_4 \end{pmatrix} \leftrightarrow \Psi^{SR}_C = \begin{pmatrix} a_2 + ia_4 \\ -a_1 - ia_3 \\ -b_2 - ib_4 \\ b_1 + ib_3 \end{pmatrix}. \]  \hspace{1cm} (40)

In addition, we have four more which correspond to the ‘reflected’ form of the above identifications, and are obtained by interchanging the ‘real’ and ‘imaginary’ parts of the components of $\Psi_D$:

\[ (V)' \quad \Psi_D = \begin{pmatrix} a_1 + jb_1 \\ a_2 + jb_2 \\ a_3 + jb_3 \\ a_4 + jb_4 \end{pmatrix} \leftrightarrow \Psi^{SR}_C = \begin{pmatrix} b_3 + ib_1 \\ b_4 + ib_2 \\ a_3 + ia_1 \\ a_4 + ia_2 \end{pmatrix} \]  \hspace{1cm} (41)

\[ (VI)' \quad \Psi_D = \begin{pmatrix} a_1 + jb_1 \\ a_2 + jb_2 \\ a_3 + jb_3 \\ a_4 + jb_4 \end{pmatrix} \leftrightarrow \Psi^{SR}_C = \begin{pmatrix} -b_4 + ib_2 \\ b_3 - ib_1 \\ a_4 - ia_2 \\ -a_3 + ia_1 \end{pmatrix}. \]  \hspace{1cm} (42)
Recall what these identifications mean; namely, under any given Lorentz transformation [Eqn(24)] of $\Psi_D$, the eight real components $a_i$ and $b_i$ ($i = 1, 2, 3, 4$) transform in exactly the same way as the eight real components $a_i$ and $b_i$ that appear in the (eight) complex spinors $\Psi^{SR}_C$ listed above, after being acted on by the corresponding complex Lorentz transformation $\rho$ [Eqn(3)].

We now define an operator $\rho_{II}$ which takes the complex spinor $\Psi^{SR}_C$ in the identification (I)' above and maps it to the complex spinor $\Psi^{SR}_C$ in the identification (II)'. Thus $\rho_{II}$ is defined by

$$
(\rho_{II} \cdot \begin{pmatrix} x_1 + iy_1 \\ x_2 + iy_2 \\ x_3 + iy_3 \\ x_4 + iy_4 \end{pmatrix}) = \begin{pmatrix} -x_2 + iy_2 \\ x_1 - iy_1 \\ x_4 - iy_4 \\ -x_3 + iy_3 \end{pmatrix},
$$

for any real variables $x_i$ and $y_i$. Similarly, we may construct the operators $\rho_{III}, \rho_{IV}, \ldots, \rho_{VIII}$, whose explicit form we omit for brevity.

If we let $\mathcal{V}(\Psi^{SR}_C)$ denote the eight-dimensional vector space formed by all real linear combinations of complex 4-spinors, then the linear map $\rho_{II}$, for example, is an automorphism of $\mathcal{V}(\Psi^{SR}_C)$. In particular, the transformation properties of the eight real components of $\Psi^{SR}_C$ under a Lorentz transformation is identical to the transformation properties of the transformed spinor $\rho_{II}(\Psi^{SR}_C)$ under the same Lorentz transformation. One can show that the set of eight operators

$$\{1, \rho_{II}, \rho_{III}, \ldots, \rho_{VIII}\}$$

generate an eight dimensional closed algebra with respect to the real numbers. The subset $\{1, \rho_{II}, \rho_{III}, \rho_{IV}\}$, for example, generates the algebra of quaternions.

One may also consider all commutators of the seven elements $\rho_{II}, \rho_{III}, \ldots, \rho_{VIII}$. These turn out to generate a Lie algebra that is isomorphic to $\text{SU}(2) \times \text{SU}(2) \times \text{U}(1)$. The $\text{SU}(2) \times \text{SU}(2)$ part is a Lorentz symmetry. The $\text{U}(1)$ factor is intriguing.

\footnote{We assume the $E_i$’s and $F_i$’s are now in the standard representation.}
As we pointed out earlier, we have not established that the algebra generated by the eight operators \( \{1, \rho_{II}, \rho_{III}, \ldots, \rho_{VIII}\} \) is maximal; additional independent automorphism operators could exist. We leave this question for a future investigation.

5 Discussion

In this work, we constructed a representation of the six-dimensional Lorentz group in terms of \( 4 \times 4 \) generating matrices defined over the hyperbolic number system, \( \mathbb{D} \).

The transformation properties of the eight real components of the corresponding ‘hyperbolic’ 4-spinor under a Lorentz transformation was shown to be equivalent to the transformation properties of the eight real components of the familiar complex Dirac spinor, after making an appropriate identification of components. The non-uniqueness of this identification led to an automorphism algebra defined on the vector space of Dirac spinors. These automorphisms have the property of preserving the transformation properties of the eight-real components of a Dirac 4-spinor in any given Lorentz frame. Properties of this algebra were studied, although we were unable to prove that the algebra studied here was maximal.

It is interesting to note that the hyperbolic representation of the Lorentz group turns out to be a subgroup of the (fifteen dimensional) special unitary group \( SU(4,\mathbb{D}) \). A simple consequence is that \( \Psi_D^\dagger \Psi_D \) is a Lorentz invariant scalar. Moreover, after identifying \( \Psi_D \) with \( \Psi_{SR} \), as in equation (37), for example, it becomes manifest that the six-dimensional complex representation of the Lorentz group is a subgroup of \( SU(2,2;\mathbb{C}) \). This group is also fifteen dimensional, and it is tempting to assume that \( SU(4,\mathbb{D}) \) and \( SU(2,2;\mathbb{C}) \) are isomorphic. This seems to be supported by the proven correspondence \( SU(2,\mathbb{D}) \cong SU(1,1;\mathbb{C}) \).

However, from general arguments, we were able to assert that \( SU(n,\mathbb{D}) \) and \( SL(n,\mathbb{R}) \) possess isomorphic Lie algebras for \( n \geq 2 \). But we also know \( SL(4,\mathbb{R}) \cong SO(3,3;\mathbb{R}) \), and so we conclude that the Lie algebra of \( SU(4,\mathbb{D}) \) is isomorphic to the Lie algebra of \( SO(3,3;\mathbb{R}) \). But this symmetry evidently differs from the algebra of generators of the conformal group \( SU(2,2;\mathbb{C}) \), which is equivalent to the algebra for \( SO(2,4;\mathbb{R}) \). Thus \( SU(4,\mathbb{D}) \) and \( SU(2,2;\mathbb{C}) \) are inequivalent groups.

Thus, from the viewpoint of naturally embedding the Lorentz symmetry into some larger group, the hyperbolic and complex representations stand apart. We leave the physics of \( SU(4,\mathbb{D}) \) as an intriguing topic yet to be studied.
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