Remarks on the formulation of the cosmological constant/dark energy problems

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(Dated: August 12, 2013)

Associated with the cosmic acceleration are the old and new cosmological constant problems, recently put into the more general context of the dark energy problem. In broad terms, the old problem is related to an unexpected order of magnitude of this component while the new problem is related to this magnitude being of the same order of the matter energy density during the present epoch of cosmic evolution. Current plans to measure the equation of state or density parameters certainly constitute an important approach; however, as we discuss, this approach is faced with serious feasibility challenges and is limited in the type of conclusive answers it could provide. Therefore, is it really too early to seek actively for new tests and approaches to these problems? In view of the difficulty of this endeavor, we argue in this work that a good place to start is by questioning some of the assumptions underlying the formulation of these problems and finding new ways to put this questioning to the test. First, we calculate how much fine tuning the cosmic coincidence problem represents. Next, we discuss the potential of some cosmological probes such as weak gravitational lensing to identify novel tests to probe dark energy questions and assumptions and provide an example of consistency tests. Then, motivated by some theorems in General Relativity, we discuss if the full identification of the cosmological constant with vacuum energy is unquestionable. We discuss some implications of the simplest solution for the principles of General Relativity. Also, we point out the relevance of experiments at the interface of astrophysics and quantum field theory, such as the Casimir effect in gravitational and cosmological contexts. We conclude that challenging some of the assumptions underlying the cosmological constant problems and putting them to the test may prove useful and necessary to make progress on these questions.

PACS numbers: 98.80.Jk,04.20.Cv

I. INTRODUCTION

Several different types of astrophysical observations, e.g., have established the evidence that the expansion of the universe entered a phase of acceleration. Associated with this acceleration is a cosmological constant, or another dark energy component, that contributes significantly to the total energy density of the universe.
Cosmic acceleration and dark energy constitute one of the most important and challenging of current problems in cosmology and other areas of physics. There are many comprehensive reviews of the cosmological constant or dark energy, including the observational evidence for it and the problems associated with it, and we refer the reader to some of them [8, 9, 10, 11, 12, 13, 14, 15].

Three questions related to the cosmic acceleration are encountered in the published literature, two of which are found in different formulations or expressions. These are:

i) What is causing the cosmic acceleration? Is it a cosmological constant or a dynamical dark energy component [16]? Is this associated with the stress-energy momentum sector or with the curvature sector of the Einstein field equations (EFE)? Is this an indication of new physics at cosmological scales?

ii) The old cosmological constant problem: If the acceleration is caused by vacuum energy, then why is the value measured from astrophysics so small compared to values obtained from quantum field theory calculations (this is the puzzling smallness formulation, see for example [12])? Another formulation is: Why do all the contributions to the effective cosmological constant term cancel each other up to a very large number of decimal places (this is the fantastic cancelation formulation, see for example [8])? We contrast the two formulations in the next section.

iii) The new cosmological constant problem: Why is the acceleration happening during the present epoch of the cosmic evolution? (Any earlier would have prevented structures from forming in the universe.) This is also formulated as the cosmic coincidence: i.e. why is the dark energy density of the same order of magnitude as the matter density during the present time?

In this paper, we argue that questioning the formulation of these problems and challenging the underlying assumptions may prove useful and necessary in order to make progress on the questions. We start by discussing the cosmic coincidence problem and then make some clarifications using some fraction calculations. Then, we calculate some prospective constraints on the cosmological parameters of dark energy/cosmological constant and discuss the inferred possible answers. Constraining the equation of state parameters is certainly an important approach to pursue, however, as we show, the level of precision required is very challenging, and yet the approach is limited in the kind of answers it could provide. This points out the need for new approaches to dark energy/cosmological constant problems. In particular, questioning the assumptions underlying these questions may be found useful in order to look for new types of tests or approaches to these problems. We provide an example where cosmological probes of the cosmic expansion and the growth rate of large-scale structure-formation can be used in order to distinguish between cosmic acceleration due to dark energy and cosmic acceleration due to some modification to General Relativity at
cosmological scales. Next, we question the full identification of the cosmological constant with vacuum energy in light of some theorems on the most general curvature tensor in the Einstein Field equations. Then, we discuss how a simple intrinsic constant curvature of spacetime would fit within important interpretations of General Relativity’s principles. Finally, we point out the relevance of experiments at the interface of particle physics and astrophysics, such as the Casimir effect in astrophysical and cosmological contexts. A discussion and conclusion are provided in the last section.

II. PRELIMINARIES

A. Notation

We recall here only some preliminary equations and definitions necessary for the clarity of the paper. We refer the reader to review papers, see e.g. [8, 9, 10, 11, 12, 13, 14] and text books, see e.g. [17, 18, 19, 20]. The Einstein Field Equations (EFE) with a cosmological constant $\Lambda$ read

$$G_{\alpha\beta} + \Lambda g_{\alpha\beta} = \kappa T_{\alpha\beta}$$

where $\kappa \equiv 8\pi G$ and

$$G_{\alpha\beta} \equiv R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R$$

is the Einstein tensor, $R_{\alpha\beta}$, $R$ and $g_{\alpha\beta}$ are the Ricci tensor, the Ricci curvature scalar and the metric tensor respectively.

For a perfect fluid, the energy-momentum tensor is given by

$$T_{\alpha\beta} = (\rho + p) u_{\alpha} u_{\beta} + p g_{\alpha\beta}$$

where $u^\alpha$ is the four velocity vector and $\rho$ and $p$ are the energy-density and isotropic pressure relative to $u^\alpha$. With global isotropy, when $T_{\alpha\beta} = 0$, the EFE (1) admit de Sitter space (for $\Lambda > 0$) as a solution, for which a line element is given by (A5) in the appendix.

Motivated by current observations, e.g. [1, 2, 3, 4, 5, 6, 7], and for the sake of simplicity, let us consider the concordance spatially flat universe with a positive cosmological constant $\Lambda$. The EFE (1) with a dust source can be solved explicitly to give

$$a(t) = \left(\frac{3C}{\Lambda}\right)^{1/3} \left[ \sinh\left(\frac{\sqrt{3\Lambda}}{2} t\right) \right]^{2/3}$$

(4)
where $C \equiv \frac{8\pi\rho a^3}{3}$ is a constant; and the spacetime line element is given by

$$ds^2 = -dt^2 + a^2(t)(dr^2 + r^2d\Omega^2). \quad (5)$$

where $d\Omega^2 = (d\theta^2 + \sin^2 \theta d\phi^2)$. At early stages, the universe is matter dominated and $a(t) = \left(\frac{2G}{3\pi}t^2\right)^{1/3}$. At late stages, the universe is dominated by the cosmological constant and enters a de Sitter phase with $a(t) = \left(\frac{3C}{4\Lambda}\right)^{1/3} \exp \sqrt{\Lambda/3}t$.

We plot the curvature evolution and profile of these spacetimes in the appendix.

Now, to introduce the vacuum energy density, let us consider a scalar field with Lagrangian density

$$L_\phi = -\frac{1}{2}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi - V(\phi) \quad (6)$$

and energy momentum tensor

$$T^\phi_{\alpha\beta} = \partial_\alpha\phi\partial_\beta\phi - \frac{1}{2}(g^{\delta\gamma}\partial_\delta\phi\partial_\gamma\phi)g_{\alpha\beta} - V(\phi)g_{\alpha\beta} \quad (7)$$

The lowest energy density of the field configuration is when the kinetic (or gradient) term vanishes and the potential is at the minimum $V(\phi_{\text{min}})$. The energy momentum \cite{8, 13} reduces to

$$T^\text{vac}_{\alpha\beta} = -V(\phi_{\text{min}})g_{\alpha\beta} = -\rho_\text{vac}g_{\alpha\beta} \quad (8)$$

This form of $T^\text{vac}_{\alpha\beta}$ is also the only Lorentz-invariant form for the vacuum energy.

B. The old cosmological constant problem

The common identification of the cosmological constant with the vacuum energy density is based on the mathematical equivalence of the vacuum energy momentum tensor \cite{8} and the cosmological constant term on the LHS of the EFE \cite{11}. Also, writing a lagrangian density that includes gravitational terms and terms from quantum field theory leads to the temptation to combine some of these terms; however, one should bear in mind that we don’t have such a unified theory yet.

Now, if one considers that a geometrical cosmological constant term is an integral part of the EFE then the old cosmological constant problem can be expressed as (see for example \cite{8, 13}): Why do all the contributions from vacuum energy density fantastically cancel with the geometrical $\Lambda$ term? This formulation of the problem seems to be less often recalled in some of the recent literature. To see this quantitatively, one can combine equations \cite{11} and \cite{8} to write

$$\frac{\Lambda_{\text{effective}}}{8\pi G} = \frac{\Lambda}{8\pi G} + \rho_\text{vac} \quad (9)$$
where from effective quantum field theory

\[ \rho_{\text{vac}} = \frac{1}{2} \sum \bar{h}\omega. \]  

(10)

Now, if one considers only quantum field fluctuations cut off at particle energies of 100 GeV (this means, we consider only the well known physics of the standard model), one can write (with \( \bar{h} = c = 1 \)) \( \rho_{\text{vac}} \sim (100 \text{GeV})^4 \). However, from astrophysical observations \( (\Lambda_{\text{effective}}/8\pi G) \) is found to be comparable to the critical density, i.e. \( \sim 10^{-48} \text{GeV}^4 \) which means that the two terms in the RHS of (10) must cancel to 56 decimal places.

On the other hand, from the identification of the cosmological constant with the vacuum energy, the old \( \Lambda \) problem becomes: Why is the vacuum energy measured from astrophysics \( (\sim 10^{-48} \text{GeV}^4) \) so small compared to the value of the vacuum energy estimated from quantum field theory \( (\sim (100 \text{GeV})^4) \)? Of course, the situation is made even worse by taking the GUT or Planck scales.

This full identification may bring some limitations of its own to the old cosmological constant problem and is questioned later in this work.

### III. HOW MUCH FINE-TUNING DOES THE COSMIC COINCIDENCE REPRESENT? (THE NEW "PROBLEM")

The cosmic coincidence problem is usually stated as why is the cosmic acceleration recent in the cosmic history of the universe (why “now”? It is also discussed in terms of a fine tuning problem: i.e. why is the matter energy density of the same order of magnitude as the dark energy density at the present epoch? Related to this question is the fact that if the dark energy density was much larger then what is measured it would have dominated over the matter energy density much earlier and prevented structures from forming in the universe. It is important to discuss the cosmic coincidence because it is an argument that is used to motivate the search for dynamical components in order to be able to explain the coincidence. We will try to quantify some of these statements here.

The matter energy density is related to the redshift by

\[ \rho_m(z) = (1 + z)^3 \rho_m^{\text{today}}(z = 0), \]  

(11)

and in a \( \Lambda \)CDM universe

\[ t(a) = \int_0^a \frac{da'}{a'H(a')} = \int_0^a \frac{da'}{a'H_0\sqrt{\Omega_\Lambda + \Omega_m a'^{-3}}}, \]  

(12)

where \( a = \frac{1}{1+z} \) is normalized to 1 today.
Now, from the concordance model $\Omega_m \approx 0.3$ and $\Omega_\Lambda \approx 0.7$ today and it follows that

$$\rho_\Lambda = \frac{\Omega_\Lambda}{\Omega_m} \rho_m^{\text{today}} \approx 2.33 \rho_m^{\text{today}}.$$ (13)

At the transition from deceleration to acceleration the dark energy density is half the matter energy density and

$$1 + z_{\text{trans}} = \left( \frac{2\Omega_\Lambda}{\Omega_m} \right)^{1/3} \approx 1.67$$ (14)

(see for a first measurement of this \[1\]: $1 + z_{\text{trans}} = 1.46 \pm 0.13$).

It is a clarifying exercise to evaluate the age of the universe $t$ at this transition. Using (12) gives a transition age of $\approx 7.14$ billion years (we use $H_0 = 72 \text{ km/s/Mpc}$ \[3\]), showing that the transition is not that recent in the time history of the universe. Moreover, $\rho_m$ and $\rho_\Lambda$ have been within the same order of magnitude starting roughly from $1 + z \approx 2.85$ (with $\rho_m \approx 9.94 \rho_\Lambda$) and that corresponds to an age of the universe of $\approx 3.38$ billion years. i.e. $\rho_m$ and $\rho_\Lambda$ have been within the same order of magnitude for at least the last 10.32 billions years of the time history of the universe.

In order to evaluate how much fine-tuning the cosmic coincidence represents, one could evaluate some informative fractions using the transition time or the period during which dark energy has been of the same order of magnitude as the matter energy density, compared to the whole history of the universe. These fractions provide the percent chances for an observer randomly put in the time history of the universe to have the matter energy density and the dark energy density being of the same order of magnitude. We will compare energy densities using a logarithmic scale of time, so the results will be similar to those using a logarithmic redshift scale.

On a logarithmic scale, things can appear very significant on plots; however, the fractions are actually of a few percent:

$$\frac{\ln t_{\text{today}} - \ln t_{\text{transition}}}{\ln t_{\text{today}} - \ln t_i} \approx 2\%,$$ (15)

$$\frac{\ln t_{\text{today}} - \ln t_{\text{same-order}}}{\ln t_{\text{today}} - \ln t_i} \approx 3\%$$ (16)

and

$$\frac{\ln t_{\text{today}} - \ln t_{\text{same-order}}}{\ln t_{\text{today}} - \ln t_{\text{Planck}}} \approx 1\%$$ (17)

We used here $t_i = 1$ sec as we considered observational constraints from the Big Bang nucleosynthesis and CMB results to trace back in time the universe up to 1 second at the $e^+e^-$ annihilation. We also use the Planck time.
\( t_{\text{Planck}} = 10^{-44} \text{ sec} \) and find that the fraction is \( \approx 1\% \). (Of course on a linear scale of time the coincidence is insignificant:

\[
\frac{t_{\text{today}} - t_{\text{transition}}}{t_{\text{today}} - t_i} \approx 48\% \quad (18)
\]

and

\[
\frac{t_{\text{today}} - t_{\text{same-order}}}{t_{\text{today}} - t_i} \approx 75\% . \quad (19)
\]

In summary, even on a logarithmic scale the fractions remain of the order of a few percent. This is significant but not totally unusual in physics and astrophysics.

Consequently, while it remains perhaps a motivated problem to seek models that could naturally explain some of these small fractions, this fine tuning should not constitute a barrier to reject models that address successfully the other cosmological constant problems.

**IV. OBSERVATIONAL CONSTRAINTS ON THE EQUATION OF STATE PARAMETERS AND THE COSMOLOGICAL CONSTANT/DARK ENERGY QUESTIONS**

An important approach to the cosmological constant questions is to parameterize the dark energy using its equation of state, and then to use different cosmological data sets in order to determine these parameters. One can thus infer answers to the cosmological constant/dark energy questions from the values found for these parameters. As we will delineate further, some results will be more conclusive than others. It is essential to combine multiple complementary cosmological probes and techniques in order to break degeneracies between the cosmological parameters including the dark energy parameter space. A powerful combination for constraining the dark energy parameters is the Cosmic Microwave Background Radiation (CMB) plus distance measurements from the type Ia Supernovae (SNIe) and Weak Gravitational Lensing (WL) also called cosmic shear, e.g. see \[50, 51, 52\]. These probes provide orthogonal constraints that, when combined, reduce significantly the uncertainties on the individual parameters. Also, constraints from clusters of galaxies and baryons oscillations were shown to be good probes of the dark energy equation of state, see e.g. \[21, 22\]. Interestingly, WL allows one to do tomography by separating the source-galaxies into redshift bins in order to obtain further improvements on the parameter constraints, see e.g. \[23, 24, 25\]. In this analysis, we consider future constraints from the CMB+SNIe+WL combination plus WL tomography with various numbers of redshift bins.
TABLE I: Cosmological model. We use fiducial values for the parameters from recent results from the WMAP (3-years data results) + Large scale Structure + Supernova data as listed in table II with the respective references (we use for the equation of state the cosmological constant parameter values). We assume a spatially flat universe with $\Omega_m + \Omega_\Lambda = 1$. This fixes $\Omega_m$ and $H_0$ as functions of the basic parameters. We do not include massive neutrinos, or primordial isocurvature perturbations.

| Symbol       | Description                                               | Fiducial value | Probe-1 | Probe-2 | Probe-3 |
|--------------|-----------------------------------------------------------|----------------|---------|---------|---------|
| $\Omega_m h^2$ | physical matter density                                   | 0.1259         | WL      | CMB     | SNIe    |
| $\Omega_\Lambda$ | fraction of the critical density in a dark energy component | 0.745          | WL      | CMB     | SNIe    |
| $w_0, w_1$ (or $w_a$) | equation of state parameters                           | -1, 0          | WL      | CMB     | SNIe    |
| $\sigma_8^{\text{lin}}$ | amplitude of linear fluctuations                        | 0.712          | WL      | CMB     |         |
| $n_s(k_0 = 0.05h$/Mpc) | spectral index of the primordial scalar power spectrum at $k_0$ | 0.946          | WL      | CMB     |         |
| $\alpha_s$ | running of the primordial scalar power spectrum         | -0.06          | WL      | CMB     |         |
| $z_p$ | the characteristic redshift of source galaxies for lensing | 0.76, 1.12     | WL      |         |         |
| $\zeta_s$ | absolute calibration parameter for WL power spectrum [42, 43] | 0.0            | WL      |         |         |
| $\zeta_r$ | relative calibration parameter for WL power spectrum [42, 43] | 0.0            | WL      |         |         |
| $\Omega_b h^2$ | physical baryon density                                  | 0.0448         | CMB     |         |         |
| $\tau$ | optical depth to reionization                           | 0.08           | CMB     |         |         |
| $T/S$ | scalar-tensor fluctuation ratio                         | 0.2            | CMB     |         |         |

A. methodology

In this analysis, we consider a cosmological model with 13 parameters that take the fiducial values in table I based on recent 3-years results from the Wilkinson Microwave Anisotropy Probe (WMAP) [2, 40, 41] combined with large scale structure and supernova data as given in table II below along with the respective references (except for the dark energy equation of state, where we use the cosmological constant parameter values). We consider in this analysis, the two standard parameterizations for the dark energy equation of state $w = P/\rho$ given by

- $w(z) = w_0 + w_1 z$ if $z < 1$ and $w(z) = w_0 + w_1$ otherwise, e.g. [1, 35], and
- $w(a) = w_0 + w_a (1 - a)$, e.g. [36, 37].

The dark energy density as a function of redshift is thus given by

$$Q(z) \equiv \frac{\rho_{de}(z)}{\rho_{de}(0)} = \exp 3 \int_0^z \frac{1 + w(z')}{{1 + z'}} dz'.$$  (20)
TABLE II: Data sets used in [41] to derive the constraints on equation of state parameter $w$ using CMB, large scale structure and supernova data (table 9 in [41]). See also the NASA’s data center for Cosmic Microwave Background (CMB) research, LAMBDA at [http://lambda.gsfc.nasa.gov/]

| data set/experiment         | references                  |
|-----------------------------|-----------------------------|
| WMAP 3-years results        | (Spergel, et al., 2006) [41]|
| 2dF Galaxy Redshift Survey | (Cole, et al., 2005) [26]   |
| BOOMERanG + ACBAR           | (Montroy, et al., 2005, Kuo, et. al., 2004) [28, 29]|
| CBI + VSA                   | (Readhead, et al., 2004, Dickinson, et. al., 2004) [30, 31]|
| Sloan Digital Sky Survey    | (Tegmark, et al., 2004; Eisenstein, et. al., 2005) [27, 32]|
| Supernova "Gold Sample"     | (Riess, et al., 2004) [34]  |
| Supernova Legacy Survey (SNLS) | (Astier, et al., 2006) [33] |

Despite its simplicity, statistical inference theory using Fisher matrices was proven to be a very efficient tool to calculate constraints that will be obtained from future planned or proposed experiments. As an example of its efficiency, one could look at the good predictions on the parameter uncertainties forecasted, for instance, by [23, 38] for WMAP and compare them to the real WMAP results, obtained years later [2]. We describe here only the basic ideas of the methods that we use in the current paper but provide references [45] where the full detail can be found.

We use the standard approach to calculate the statistical error on a given parameter $p^\alpha$ by using the combination:

$$
\sigma^2(p^\alpha) \approx \left[(F_{CMB} + F_{WL} + F_{SNe} + \Pi)^{-1}\right]^{\alpha\alpha},
$$

where $F_{CMB}$, $F_{WL}$ and $F_{SNe}$ are the Fisher matrices from CMB, weak lensing, and supernovae respectively, and $\Pi$ is the prior matrix. As discussed in some detail in [23, 25], we calculate $F_{WL}$ using

$$
F_{\alpha\beta} = \sum_{\ell=\ell_{min}}^{\ell_{max}} \frac{1}{(\Delta P_\ell)^2} \frac{\partial P_\ell}{\partial p^\alpha} \frac{\partial P_\ell}{\partial p^\beta};
$$

where the lensing convergence power spectrum is given by [46, 47, 48]:

$$
P_\ell^{\kappa} = \frac{9}{4} H_0^4 \Omega_m^2 \int_0^{\chi_{max}} \frac{g^2(\chi)}{a^2(\chi)} P_{3D} \left( \frac{l}{\sin K(\chi)}, \chi \right) d\chi.
$$

where $P_{3D}$ is the 3D nonlinear power spectrum of the matter density fluctuation, $\delta$; $a(\chi)$ is the scale factor; and $\sin K \chi = K^{-1/2} \sin(K^{1/2} \chi)$ is the comoving angular diameter distance to $\chi$ (for the spatially flat universe used in this
analysis, this reduces to $\chi$). The weighting function $g(\chi)$ is the source-averaged distance ratio given by

$$g(\chi) = \int_{\chi}^{\chi_H} n(\chi') \frac{\sin K(\chi' - \chi)}{\sin K(\chi')} d\chi',$$

where $n(\chi(z))$ is the source redshift distribution normalized by $\int dz n(z) = 1$. The uncertainty in the observed lensing spectrum is given by:

$$\Delta P_\kappa(\ell) = \sqrt{\frac{2}{(2\ell + 1) f_{\text{sky}}}} \left( P_\kappa(\ell) + \frac{\langle \gamma_{int}^2 \rangle}{\bar{n}} \right),$$

where $f_{\text{sky}} = \Theta^2 \pi / 129600$ is the fraction of the sky covered by the gravitational lensing survey of dimension $\Theta$ in degrees, and $\langle \gamma_{int}^2 \rangle^{1/2}$ is the intrinsic ellipticity of galaxies. For this analysis, we consider a lensing survey with 10% sky coverage, a median redshift of 1, an average galaxy number density of $\bar{n} = 30 \, \text{gal/arcmin}^2$, and intrinsic ellipticities $\langle \gamma_{int}^2 \rangle^{1/2} = 0.4$. We calculate the constraints on the parameters using five tomographic bins for this survey. We also consider a deeper survey (space-based like) with $f_{\text{sky}} = 0.10$, a median redshift of roughly 1.5, $\bar{n} = 100 \, \text{gal/arcmin}^2$, and $\langle \gamma_{int}^2 \rangle^{1/2} \approx 0.25$. We calculate constraints from 10 tomographic bins using this survey. For both surveys, we use $\ell_{\text{max}} = 3000$ to keep the assumption of a Gaussian shear field valid.

Weak lensing has been recognized as a very powerful probe of dark energy parameters, however, several systematic effects have been identified so far, see \textsuperscript{50} and references therein for an overview. In this analysis, we included the effect of the shear calibration bias \textsuperscript{43, 49, 53, 54, 55, 56} on our results by marginalizing over its parameters. Because of this bias, the shear is systematically over or under-estimated by a multiplicative factor, and results in an overall rescaling of the shear power spectrum. Following the parameterization discussed in \textsuperscript{42}, we used the absolute power calibration parameter $\zeta_s$ and the relative calibration parameter $\zeta_r$ between two redshift bins. Another systematic effect that we considered in the analysis is the incomplete knowledge of the source redshift distribution \textsuperscript{44, 57}. A remedy to this poor knowledge of the redshift distribution using spectroscopic redshift has been explored recently in \textsuperscript{44}. In the present analysis we marginalize over the redshift bias by including the characteristic redshift of the distribution as a systematic parameter $z_s$ and we assume a reasonable prior of 0.05 on this parameter.

In order to calculate the supernova Fisher matrix, $F_{SNe}$, we use (see, e.g. \textsuperscript{58, 59})

$$F_{\alpha\beta} = \sum_{i=1}^{N} \frac{1}{\sigma m(D_L, i)^2} \frac{\partial D_{L,i}}{\partial p^\alpha} \frac{\partial D_{L,i}}{\partial p^\beta},$$

where $D_L \equiv H_0 d_L / c$ is the dimensionless luminosity distance to a supernova, given in a spatially flat model by

$$D_L(z) = (1 + z) \int_0^z \frac{1}{\sqrt{(1 - \Omega_A)(1 + z')^3 + \Omega_A \Omega(z')}} dz',$$
where $Q(z)$ is as defined in Eq. (20). We recall that the SN Ia apparent magnitude as a function of redshift is given by $m(z) = 5 \log_{10}(D_L(z)) + \mathcal{M}$, where $\mathcal{M}$ depends as the absolute magnitude of type Ia supernovae as well as on the Hubble parameter $H_0$. As usual, we treat $\mathcal{M}$ as a nuisance parameter. We use two sets of 2000 SNe Ia uniformly distributed with $z_{\text{max}} = 0.8$ and $z_{\text{max}} = 1.5$. It is important to briefly note here that there are systematic uncertainties associated with supernova searches: these include luminosity evolution, gravitational lensing and dust; see, e.g. and references therein. In order to partly include the effect of these systematics and the effect of the supernova peculiar velocity uncertainty [62], we follow [60, 61] and use the following quadrature for the effective magnitude uncertainty

$$\sigma_{\text{eff}}^2 = \sigma_m^2 + \left( \frac{5\sigma_v}{\ln(10)c} \right)^2 + N_{\text{(per bin)}}(\delta_m^2)$$

(28)

where, $\sigma_v = 500\text{km/sec}$ is the peculiar velocity, and $\delta_m$ is a floor uncertainty in each bin [60, 61]. The quadrature relation (28) assures that there is an uncertainty floor set by the systematic limit $\delta_m$ so that the overall uncertainty per bin cannot be reduced to arbitrarily low values by adding more supernovae.

Finally, for the CMB and lensing tomography, we use the generalized form of the Fisher matrix above (e.g. see [23]) as

$$F_{\alpha\beta} = \sum_{\ell_{\text{min}}}^{\ell_{\text{max}}} (\ell + 1/2)f_{\text{sky}}\text{Tr}\left(C_\ell^{-1} \frac{\partial C_\ell}{\partial p^\alpha} C_\ell^{-1} \frac{\partial C_\ell}{\partial p^\beta} \right),$$

(29)

where $C_\ell$ is the covariance matrix of the multipole moments of the observables $C^{XY}_{\ell} = C^{XY}_{\ell} + N^{XY}_{\ell}$ with $N^{XY}_{\ell}$ being the power spectrum of the noise in the measurement. Here $XY$ takes the values $\kappa\kappa$ for lensing tomography spectra and cross-spectra, and $TT, TE,$ and $EE$ for CMB spectra. We consider constraints from 1-year data from the Planck satellite. Our findings are discussed in the next section.

**B. Summary of results and implications for the cosmological constant/dark energy questions**

Our results, summarized in table I show that when WL with multiple-bins tomography is added to the CMB+SN combination, the uncertainties on the equation of state parameter $w_0$ reduce by roughly a factor of 3 and the uncertainties on $w_1$ and $w_a$ reduce by roughly factors of 3 or 4 at least. Also, we note that the uncertainties on the set $\{w_0, w_1\}$ are different from those of the set $\{w_0, w_a\}$ showing that the uncertainties are parameterization dependent. The deeper lensing survey, which allows one to implement 10-bins tomography, provides an additional factor of 2 improvement on $w_1$ and $w_a$ compared with the 5-bins tomography results. We find that in order to bring the uncertainties on both parameters to the order of several percent, very ambitious surveys are required. Current observations
TABLE III: Dark energy parameter constraints from various combinations of CMB, Weak Gravitational Lensing and Supernova future surveys. The constraints are (1σ uncertainties) on the two dark energy parameterizations given in section IV A. The uncertainties are calculated using combinations of 1-year data from Planck, 2000 uniformly distributed supernovae with $z_{max} = 0.8, 1.5$, a lensing survey with 5-bins tomography, and a very deep lensing survey with 10-bins tomography. The results are presented for the dark energy parameters $\{w_0, w_1\}$ and $\{w_0, w_a\}$. The systematic effects discussed in section IV A are included in the calculations.

| Simulated Experiment/Survey | Parameterization-1 $\sigma(w_0)$ | $\sigma(w_1)$ | $\sigma(w_0)$ | $\sigma(w_a)$ |
|-----------------------------|----------------------------------|---------------|---------------|---------------|
| 1-year data from PLANCK + 2000 SN with $z_{max} = 0.8$ | 0.11 | 0.26 | 0.13 | 0.47 |
| 1-year data from PLANCK + 2000 SN with $z_{max} = 0.8$ + WL survey with 5-bins tomography, $z_{med} = 1.0$ and $f_{sky} = 0.10$ | 0.039 | 0.092 | 0.033 | 0.12 |
| 1-year data from PLANCK + 2000 SN with $z_{max} = 1.5$ | 0.08 | 0.20 | 0.10 | 0.35 |
| 1-year data from PLANCK + 2000 SN with $z_{max} = 1.5$ + WL deep survey with 10-bins tomography, $z_{med} = 1.5$ and $f_{sky} = 0.10$ | 0.022 | 0.043 | 0.024 | 0.057 |

are only able to constrain, with some significance, models with a constant equation of state, i.e. one single parameter with no redshift dependence. For example, the WMAP team recently combined constraints from currently available CMB, large scale structure, and supernova data sets (see table II) and obtained $w = -0.926^{+0.051}_{-0.057}$ (table 9 in [41]).

Our table III shows the 1-sigma uncertainties on the equations of state parameters, and in order to consider the 2 and 3-sigma constraints, one has to multiply the values found by factors of two and three respectively. Therefore, one can see that even when ambitious surveys are considered the remaining uncertainty is still too large to constrain significantly multiple-parameters dark energy models. However, it will be possible to exclude some proposed models with significant deviations from the cosmological constant parameters, namely $w_0 = -1$, and $w_1 = 0$. These include for example trackers models [63] and some SUGRA inspired models [64] with for example $w_0 = -0.8$ and $w_1 = 0.3$.

The most decisive answer will be if the data can show conclusively that dark energy is not a cosmological constant. A very suggestive but less decisive answer will be to show that the dark energy parameters are those of a cosmological constant to a very high level of precision (a few percent, perhaps). But of course, the degeneracy in this case will remain and other tests, beyond the equation of state approach, will be needed. In this case, other tests are also
necessary because the problems of the cosmological constant are just re-affirmed. Finally, an important question that need to be addressed in all cases is whether the equation of state obtained is a true or forced equation of state as we explore further below. We discuss in the next sections some possible directions of such tests and illustrate one promising test using cosmological probes.

V. RECONSIDERING AND TESTING SOME OF THE ASSUMPTIONS ABOUT THE OLD
COSMOLOGICAL CONSTANT AND DARK ENERGY PROBLEMS

In this section we propose some examples on how one could question some of the assumptions made about the cosmological constant/dark energy problems and, in some cases, how to put this questioning to the test.

A. Cosmological tests beyond the equation of state approach: The expansion history versus the growth rate of large scale structure

The approach of the equation of state is certainly an important one. However, an important question that remains after some dark energy parameters are obtained from analyzing observational data is as follows. Is this an effective equation of state of some dark energy component in the Einstein’s equations, or is this just a forced equation of state obtained from fitting dark energy models on the top of some modified gravity at cosmological scales? New tests are necessary in order to address this question.

Indeed, of great importance are innovative ways of using current and future astrophysical observations that could distinguish between acceleration models beyond the equation of state. In other words, for the same degenerate effective equation of state, these novel tests could distinguish between dynamical dark energy models, a geometrical cosmological constant, and acceleration due to some modification to the gravity sector, and thus will allow one to test some of the important basic assumptions. Some cosmological probes such as weak gravitational lensing (for reviews, see [50, 51, 52, 56] and references therein) and clusters of galaxies (see for example [21, 22] and references therein) are very rich tools and very promising for identifying this type of test because they provide more than one way to constrain dark energy or cosmic acceleration. Both probes can capture the effect of dark energy on the expansion history and also its effect on the growth rate of large-scale structure (the rate at which clusters and super clusters of galaxies form over the history of the universe). Interestingly, this can be used to identify consistency checks to test the dark energy beyond the equation of state degeneracy. In particular, this could allow one to test dark energy
models based on new particles and fields versus cosmic acceleration due to some modification in the curvature sector of the EFE as suggested in some recent studies, see e.g. [65, 74, 75], recent review [15] and references therein. Most importantly, these kinds of consistency checks could be used to test some of the assumptions discussed in this paper, namely on the origin of the intrinsic constant curvature of spacetime. Another test beyond the equation of state and based on the dark energy potential was discussed in [76].

In this paper, we explore an example to demonstrate that tests that go beyond the equation of state degeneracy are possible. An important point for this test is that cosmic acceleration affects cosmology in two ways: 1) It affects the expansion history of the universe by speeding it up, 2) It affects the growth rate of large scale structure in the universe by suppressing it. The idea explored is that, for dark energy models, these two effects must be consistent one with another because their respective functions are mathematically related by General Relativity equations. The presence of significant inconsistencies between the expansion Hubble function and the growth rate function could be the signature of some modified gravity at cosmological scales as we will demonstrate.

In order to illustrate how the test works, we will need to use a viable modified gravity model. We choose a model proposed by Dvali, Gabadadze and Porrati (DGP) [65], where the cosmic acceleration is due to the effect of an extra large dimension modifying gravity at cosmological scales. This DGP model is motivated by higher dimensional physics and is not ruled out by current astrophysical observations [66, 69, 79]. We have no particular interest in the phenomenology and precise testing of the viability of this model. We are only interested to use it as an example in order to illustrate the test considered. We refer the interested reader to some studies dedicated to the DGP model phenomenology [69, 70, 71, 81].

We provide here a very brief description of this model but again refer the reader to [65, 67] for a full description. The action for this five-dimensional theory is [65, 67]

\[
S^{(5)} = \frac{1}{2} M_{(5)}^3 \int d^4x \sqrt{-g^{(5)}} R^{(5)} + \frac{1}{2} M_{(4)}^2 \int d^4x \sqrt{-g^{(4)}} R^{(4)} + S_{\text{matter}},
\]

(30)

where the subscripts 4 and 5 denote quantities on the brane and in the bulk, respectively; \(M_{(5)}\) is the five dimensional reduced Planck mass; \(M_{(4)} = 2.4 \times 10^{18}\)GeV is the four dimensional effective reduced Planck mass; \(R\) and \(g\) are the Ricci scalar and the determinant of the metric, respectively. The first and second terms on the right hand side describe the bulk and the brane, respectively, while \(S_{\text{matter}}\) is the action for matter confined to the brane. The two different prefactors \(M_{(5)}^3/2\) and \(M_{(4)}^2/2\) in front of the bulk and brane actions give rise to a characteristic length scale \(r_c = M_{(4)}^2/2M_{(5)}^3\). If \(M_{(5)}\) is much less than \(M_{(4)}\), then the brane terms in the action above will dominate over the bulk terms on scales much smaller than \(r_c\), and gravity will appear four dimensional. On scales larger than \(r_c\),
the full five dimensional physics will be recovered, and the gravitational force law will revert to its five dimensional \(1/r^3\) form. This is usually discussed in terms of gravity leakage into an extra dimension. Ref. [66] shows that tuning \(M(5)\) to about \(10 - 100\) MeV, implying \(r_c \sim H_0^{-1}\), is consistent with cosmological data. They have also been discussed in [79]. Low redshift cosmology in DGP brane worlds was studied in [67, 77]. Following [67], one could define the effective energy density \(\rho_{rc} \equiv \frac{3}{(32\pi G r_c^2)}\) so that Friedmann’s first equation becomes

\[
H_{\text{DGP}}^2 + \frac{k}{a^2} = \frac{8\pi G}{3} \left( \sqrt{\rho + \rho_{rc}} + \sqrt{\rho_{rc}} \right)^2.
\]

(31)

where \(\Omega_{rc} \equiv \frac{1}{4} r_c^{-2} H_0^{-2}\). We focus here on a flat universe \((k = 0)\) containing only nonrelativistic matter, such as baryons and cold dark matter, in which \(\Omega_{rc} = (1 - \Omega_m)^2\). In this model, the gravitational “leakage” into the fifth dimension, on large length scales, becomes a substitute for dark energy.

Now, for the standard general relativistic (GR) cosmological model, FLRW, with zero spatial curvature \((k=0)\), and a dark energy component \((DE)\), the expansion history is expressed by the Hubble function and is given by

\[
H_{GR+DE}(z) = H_0 \sqrt{\Omega_m (1 + z)^3 + (1 - \Omega_m) Q(z)}.
\]

(32)

And the growth rate of large scale structure \(G(a=1/(1+z))\) is given by integration of the differential equation, \[72, 73\],

\[
G''_{GR+DE} + \frac{3}{2} \frac{w(a)}{1 + X(a)} \left[ G'_{GR+DE} + \frac{3(1 - w(a)) G_{GR+DE}}{2(1 + X(a)) a^2} \right] = 0,
\]

(33)

where \(\dot{\gamma} \equiv d/da, G = D/a\) is the normalized growth rate, \(Q(a)\) as is given by equation 20 and

\[
X(a) = \frac{\Omega_m}{(1 - \Omega_m) a^3 Q(a)}.
\]

(34)

On the other part, for the spatially flat DGP model, the expansion Hubble function is given by

\[
H_{DGP}(z) = H_0 \left( \frac{1}{2} (1 - \Omega_m) + \sqrt{\frac{1}{4} (1 - \Omega_m)^2 + \Omega_m (1 + z)^3} \right).
\]

(35)

and the suppression of the growth rate function is given by, \[79, 80\]

\[
\dddot{\delta}_{DGP} + 2H_{DGP} \ddot{\delta}_{DGP} - 4\pi G \rho (1 + \frac{1}{3\beta}) \delta_{DGP} = 0
\]

(36)

where

\[
\beta = 1 - 2r_c H_{DGP} \left( 1 + \frac{\dot{H}_{DGP}}{3H_{DGP}^2} \right)
\]

(37)

We illustrate in Fig[11] Hubble expansion functions and growth rate functions for dark energy and DGP models including the cosmological constant model, \(\Lambda\)CDM. The figure shows that compared to a \(\Lambda\)CDM model with the same
FIG. 1: Supernova Hubble diagrams (Left) and Growth rate functions for dark energy and DGP models. Note that the $\Lambda$CDM model (red solid line) and the $\Omega_m = 0.20$ DGP model (blue dotted) have degenerate Hubble diagrams, but different growth rates. The degeneracy of the Hubble diagrams is even stronger for SUGRA (green dashed) and $\Omega_m = 0.27$ DGP (black double dotted) models. Interestingly, the growth rate in the $\Omega_m = 0.27$ DGP model is suppressed with respect to that in the $\Lambda$CDM model, which has the same $\Omega_m$.

matter density, a DGP model has a distinct suppression of the growth rate. Also, Fig1, displays how the degenerate models of Fig1 show distinct growth rate functions.

The basic idea for the test explored is that equations (32) and (33) must be mathematically consistent one with another via General Relativity. Similarly, equations (35) and (36) must be consistent one with another via DGP theory. A consistency cross-check of these two functions constitute a test for the fundamental underlying theory.

In order to apply the test discussed below, we assume the availability of a sample of $N_{SNe} = 2000$ type Ia supernova, evenly distributed in redshift between $z_{\text{Min}} = 0$ and $z_{\text{Max}} = 1.7$, with a magnitude uncertainty per supernova of $\sigma_m = 0.2$. As discussed in section IV A, we also include systematic effects using the quadrature (28).

We also assume the availability of measurements of the growth rate in twenty bins evenly spaced in the scale factor $a$, between $a_{\text{Min}} = 0.25$ and $a_{\text{Max}} = 1$ and an uncertainty of $\sigma_{G(a)}/G(a) = 0.02$. In order to break the usual degeneracy between $\Omega_m$ and the equation of state parameters, we add to the SN Ia data and to the growth rate measurements a constraint from the CMB shift parameter. The CMB shift parameter for a spatially flat universe is given by

$$R = \frac{1}{2} \left[ \frac{\Omega_m^{1/2} \int_{0}^{z_{\text{CMB}}} \frac{dz}{H(z)H_0(z)} - \Omega_m^{1/2} \int_{0}^{z_{\text{CMB}}} \frac{dz}{H(z)H_0(z)}}{\Omega_\Lambda^{1/2} \int_{0}^{z_{\text{CMB}}} \frac{dz}{H(z)H_0(z)}} \right],$$

where $z_{\text{CMB}} = 1089$ is the redshift of the surface of last scattering, given in [2]. Ref. 82 reports that the value of $R$ from current data is $R_{\text{obs}} = 1.716 \pm 0.062$.

In order to demonstrate the working of the consistency check we proceed as follows: We assume that the true
cosmology is described by a DGP model and simulate the expansion and growth data using a fiducial DGP model. Then we ask what contradictions arise when the data are instead analyzed based on the assumption of a dark energy model (as mentioned earlier, we are not particularly interested in the DGP cosmology here but we want to use it as an example to illustrate the procedure.)

Because we will generate the data using the DGP model, the consistency relation from General Relativity between the expansion history and the growth rate of large scale structure will be broken. The dark energy equation of state \( w_{\text{exp}}(z) \) which best fits measurements of the expansion will not be consistent with the equation of state \( w_{\text{growth}}(z) \) which best fits measurements of the growth.

The methods and steps we use are as follows:

i) We use a fiducial DGP model with \( \Omega_m = 0.27 \) and simulate the data for the expansion and the growth rate.

   We generate supernova magnitudes, a growth rate function, and the CMB shift parameter

ii) We use a \( \chi^2 \) minimization method in order to find the best fit dark energy model to the expansion. We obtain the best fit model to the supernova magnitudes and the CMB shift parameter with a first dark energy parameter space \( \{\Omega_{de}, w_0, w_1\} \). The \( \chi^2 \) minimization method was discussed in detail in [35] and was shown to give similar results to those from Monte Carlo Markov Chain method although the \( \chi^2 \) minimization can have a better handle on degeneracies [35].

iii) We use the \( \chi^2 \) minimization in order to find the best fit dark energy model to measurements of the growth rate of large scale structure. We determine the best fit model to the growth rate function and the CMB shift parameter and obtain a second dark energy parameter space \( \{\Omega'_{de}, w'_0, w'_1\} \)

iv) Next, we use a standard Fisher matrix approach to calculate the confidence regions (or \( \chi^2 \) contours) around the two best fit dark energy models (this standard procedure is discussed in detail in references [45]).

v) Then, we compare the allowed regions in the \( \{\Omega_{de}, w_0, w_1\} \) parameter space between the two data combinations in order to look for inconsistencies.

Our results, in figure 2, show that the two allowed regions in the dark energy parameter space are significantly different. This signals the expected inconsistency between the expansion history and the growth rate of large scale structure. The source of the inconsistency is from point (i) where the data was generated using a DGP model, i.e. from our hypothesis that the true cosmology is that of a modified gravity DGP model. Thus, the inconsistency
FIG. 2: LEFT(2D): Best fit equations of state: Solid contours are for fits to SN Ia simulated data and the CMB shift parameter $R$, while dashed contours are for fits to the growth-rate simulated data and the CMB shift parameter. RIGHT(3D): Best fit Dark Energy parameter spaces ($\Omega_{de}$, $w_0$, $w_1$). The ellipsoid to the right of the 3D-figure is for fit to SN Ia simulated data and the CMB shift parameter, while the ellipsoid to the left (3D-figure) is for fits to the growth rate simulated data and the CMB shift parameter data. For both figures, the significant difference (inconsistency) between the parameter spaces found using the two combinations is due to the DGP model assumed by hypothesis and used to simulate the data. The inconsistency is thus an indication that cosmic acceleration in this case is due to modified gravity at cosmological scales rather then a Dark Energy component.

constitute an observational detection of the assumed underlying modified gravity model. And finding two significantly different equations of state implies that these are not true but forced ones.

The test is based on the comparison of measurements of the expansion history and measurements of the growth rate of large scale structure and shows that we can go beyond the equation of state analysis. We demonstrated here the working of consistency tests based on this comparison and provided a preliminary implementation in [83]. Other works that explored the same idea include [84, 85]. Future work is needed in order to make these tests more robust and generic: e.g. to consider other dark energy models (with couplings, unusual sound speeds), other modified gravity models, and comparison with systematic effects of the probes. Another interesting approach was discussed in reference [86] where the authors considered signatures of quintessence models and their extension to scalar-tensor gravity on weak gravitational lensing observables. They found that some models can let an imprint of ten percent on lensing observables. The important point from these examples and others is that cosmological observations that can probe the growth of cosmological perturbations are promising tools to learn about the acceleration of the universe beyond
the effective equation of state degeneracy.

B. The Weyl-Lovelock theorem as an argument against the full identification of the cosmological constant with vacuum energy

Based on the following theorems, one could argue against the full identification of the cosmological constant with vacuum energy. Indeed, Cartan, Weyl, and Vermeil proved different theorems showing that the only tensor of valency two, $A_{\alpha\beta}$, that is:

a) constructed from the metric tensor $g_{\alpha\beta}$ and its first two partial derivatives, $g_{\alpha\beta,\gamma}$ and $g_{\alpha\beta,\gamma\delta}$,

b) divergence free, i.e. $A^{\alpha\beta,\beta} = 0$,

c) symmetric, i.e. $A_{\alpha\beta} = A_{\beta\alpha}$,

d) linear in the second derivatives of $g_{\alpha\beta}$,

is

$$A_{\alpha\beta} = c_1 G_{\alpha\beta} + c_2 g_{\alpha\beta}$$

where $c_1$ and $c_2$ are constants and $G_{\alpha\beta}$ is the Einstein tensor. Lovelock showed that conditions c) and d) are superfluous when the spacetime dimension is 4. (Note that when $A_{\alpha\beta}$ is put in the EFE, $c_1$ is absorbed in the $\kappa$ factor.)

In Refs. and , it was first proven that the most general curvature tensor $A_{\alpha\beta}$ is a linear combination of $R_{\alpha\beta}$, $Rg_{\alpha\beta}$ and $g_{\alpha\beta}$, i.e. of the form

$$A_{\alpha\beta} = aR_{\alpha\beta} + bRg_{\alpha\beta} + cg_{\alpha\beta},$$

where $a$, $b$, and $c$ are constants. Then values $a = 1$ and $b = -\frac{1}{2}$ are derived from the divergence free condition (conservation law).

Consequently, one is tempted to take the standpoint that unless one is guided by some physical laws or measurements, setting the constant $c$ (i.e. $\Lambda$) in equation to any particular value, including zero is unjustifiable.

Imposing a priori a particular value on $\Lambda$ (for instance zero) is perhaps making the same mistake Einstein did by putting the particular value

$$\Lambda_{Einstein} = \frac{4}{9C^2}$$

(for a closed static universe) where $C$ is as defined previously, after equation.
Thus, this suggests that a geometrical constant $\Lambda$-term in the EFE (1) is part of the equations on its own right with no reference to any energy momentum tensor. This could be used as an argument not in favor of the exact identification of the geometrical cosmological constant with vacuum energy.

C. **What are the implications of the simplest solution in view of principles of General Relativity?**

As discussed in the previous sections, some of the assumptions underlying the cosmological constant problems are not unquestionable and it is important to find ways to challenge them and put them to the test.

Needless to recall, the simplest solution can arise from, first, abandoning the assumed identification of the cosmological constant with vacuum energy (based on the theorems discussed in the previous sub-section IV B), and second, putting on the side the cosmic coincidence (see section III for a discussion). The remaining question is then why the huge vacuum energy densities (see section II B) from quantum field theory estimations do not contribute to the energy budget in the universe. This question can be legitimately replaced by, how does vacuum energy contribute to the EFE? For example, is it correct to try to add contributions from vacuum energy density to the EFE using some ultra-violet cutoff energy? We discuss this point more in section V D, see also [93, 94, 95, 96].

Further, did we really exhaust all possibility of exact cancelation mechanisms for vacuum energy? [8, 9, 10, 11, 12, 13, 14].

It is perhaps worth mentioning that a rather negative vacuum energy/cosmological constant was expected within some candidates for a unified theory such as String Theory [98], as there is no attractive way to derive a stable vacuum with a positive cosmological constant [98]. So, in addition to the magnitude problems, could this sign problem be a further indication to revise the full identification above?

In this simple solution, what is measured currently is simply an intrinsic curvature of the spacetime, and the value of $\Lambda$ is just a constant measured from experiments, as is Newton’s gravitational constant $G$.

At this point, we would like to discuss the following subtle point. The usual aesthetic interpretation of one of General Relativity’s principles is that the mass-energy content of the universe creates curvature of the spacetime (assuming a zero Weyl tensor as in standard FLRW cosmology). One could then ask the following question: if spacetime is to have a curvature in absence of mass-energy sources and this curvature is not due to vacuum energy then what is generating this curvature? There are two possible answers to the question: If one wants to preserve the interpretation above, then one needs to explain the source of this curvature. However, it is also correct to take the other standpoint and consider that the Einstein field equations are a set of differential equations containing a cosmological constant and
governing the laws of General Relativity, and that in absence of sources the trivial spacetime is simply de Sitter with an intrinsic curvature.

The simple solution we discussed in this section is of the latter type and is a consequence of questioning some of the assumptions usually made. To our best knowledge, this particular simple but subtle point about loosing the aesthetic interpretation above has not been discussed in literature about dark energy.

With these reconsiderations in mind, it is important to think about identifying new astrophysical tests or experiments at the interface of particle and gravitational physics, as we discuss in the next sub-section.

D. Vacuum energy and Casimir effect in gravitational and cosmological contexts

Another possible successful approach to the cosmological constant problems is to think of a situation or an experiment where the validity of the cosmological constant-vacuum energy identification can be put to the test. A geometrical cosmological constant has no quantum properties while vacuum energy has both gravitational and quantum properties. Also, is it possible to learn more on how vacuum energy contributes to the cosmological constant? Some of these questions started to be addressed in the literature as we cite further.

It is perhaps relevant at this point to recall the Casimir effect [99] which is a purely quantum field theory phenomenon (see [100] for a recent comprehensive review, and references therein.) The Casimir effect results from a change in the zero-point oscillations spectrum of a quantized field when the quantization domain is restricted or when the topology of the space is non-trivial. For example, a Casimir force appears as the result of the alteration of the vacuum energy by some boundaries. In its simplest form, predicted by Casimir [99], two neutral plane parallel conducting plates placed in a vacuum at a distance $a$ from one another will experience an attractive force $F_C = -\frac{\pi^2 \hbar c}{240 S} S$ where $S \gg a$ is the plate area. The Casimir effect has been now extensively measured with a few percent precision [100].

Cosmologically, the Casimir effect is significant when the topology of the model of the universe is non-trivial (different from an infinite Euclidean topology), see e.g. [101]. The effect has been discussed in models with non-trivial topology, notably the simple case of a closed FRW universe with a 3-torus topology, see e.g. [102]. Also, the Casimir energy has been used from compact extra dimensions [94, 95, 96] to discuss the cosmological constant problem, and with models with supersymmetric large extra dimensions to propose cancellation mechanisms for the cosmological constant problem, see e.g. [97].

In a more relevant context for our discussion, one would like to study, via the Casimir effect, the gravitational properties of the vacuum energy. For example, Refs. [103, 104] calculated correction terms to the Casimir force due
to the weak gravitational field. Such corrections represent the effect of gravitational curvature on quantum vacuum fluctuations. The authors of Ref. [104] evaluated the order of the force acting on a Casimir apparatus redshifting in a weak gravitational field and concluded that, although some issues with signal modulation need to be solved, testing such force should be feasible and within reach of present technological resources.

Now, related to our question on how the vacuum energy may fit within the EFE, it has been argued in some papers, see for example [94, 95, 96], that as the measured Casimir effect is related to vacuum energy differences, the vacuum energy may not contribute to the cosmological dynamics via some fixed cutoff energy but rather via energy differences as in Casimir energy. This Casimir energy can be produced from some compact extra dimensions [94, 95, 96] or non-trivial topology of the spacetime [101].

We could state that if this is the case, then as we have not yet detected any non-trivial topology for a wide range of models [105], this could imply the vanishing of the vacuum energy contribution at cosmological scales. On the other hand, not all non-trivial topologies have been ruled out and one could push the idea further.

Therefore, questioning and testing how vacuum energy contributes to the cosmological constant using, for example, the Casimir effect in a cosmological context may prove helpful to the dark energy questions.

VI. CONCLUDING REMARKS

We discussed different formulations of the cosmological constant/dark energy problems and some of the assumptions underlying them. We argued for the usefulness of clarifying and questioning some of these assumptions and identifying new strategies in order to put them to the test.

We used some fraction calculations in order to evaluate how much of a fine-tuning is involved in the cosmic coincidence. We found that these fractions are of the order of a few percent even in the worst cases. This is significant but not totally unusual in physics and astrophysics. Therefore, on one hand, it remains perhaps a motivated and interesting problem to seek models that could naturally explain these numbers. On the other hand, it was important to clarify that this fine tuning should not constitute a barrier that rejects a successful solution for the other cosmological constant problems.

Current and future plans are focused on constraining the equation of state of dark energy using cosmological probes. This is certainly an important approach and some progress has been made, however as we showed in section IV, constraining a variable equation of state will require very sophisticated and challenging future experiments. Furthermore, this approach is limited in the kind of decisive answers it could provide on the nature of dark energy.
Indeed, unless we are lucky enough to find a dark energy that has an equation of state significantly different from that of a cosmological constant, new kinds of tests or experiments will be necessary in order to provide conclusive answers to the dark energy problem. For example, finding that dark energy parameters are those of a cosmological constant to a few percent precision will be very suggestive but will require tests different from the equation of state in order to rule out decisively dynamical dark energy models. Now, even if we are ready to accept some high level of precision to be satisfactory (or if we reach fundamental limitations of our experiments), finding dark energy parameters that are characteristic of a cosmological constant will only confirm the cosmological constant problems with no further clues.

Further, once an equation of state is determined from cosmological observations, one is always left with the following question: Is this an effective equation of state of some dark energy component in the energy momentum tensor or is this a forced equation of state obtained by fitting dark energy models on the top of some modified model of gravity?

Therefore it is important to encourage other directions and strategies for approaching the dark energy problems. In particular, we discussed the relevance of questioning and challenging some of the assumptions underlying the formulation of the cosmological constant problems in order to look for new types of tests.

Next, we showed that comparing cosmological observations of the expansion history and cosmological observations of the growth rate of large-scale structure can distinguish between cosmic acceleration due to some dark energy models and cosmic acceleration due to some modification to gravity physics at cosmological scales. The basic idea is that the effect of cosmic acceleration on the expansion function and its effect on the growth rate function must be mathematically consistent one with another because of the underlying gravity theory (General Relativity). As shown in section V-A, the failure in the consistency relation can be used as a test to distinguish between cosmic acceleration due to dark energy models and acceleration due to modified gravity at cosmological scales. This consistency test shows the potential of some cosmological probes such as, supernova searches, gravitational lensing and clusters of galaxies to go beyond the equation of state approach in order to address the cosmological constant/dark energy questions.

Next, motivated by some theorems on the most general curvature tensor in the Einstein field equations, we argued that the identification of the cosmological constant with the vacuum energy is not unquestionable and might bring some limitations of its own because it changes the formulation of the old cosmological constant problem. Recall that as a result of this questioning, dark energy can be identified as a simple geometrical cosmological constant. In the absence of sources, the trivial spacetime is then de Sitter with an intrinsic constant curvature. However, then two questions arise. i) An important interpretation of one of principles of General Relativity is that the mass-energy content of the universe creates curvature of the spacetime. One could then ask the following question: if spacetime is
to have a curvature in the absence of mass-energy sources, and this curvature is not due to vacuum energy, then what is generating this curvature? There are two possible answers to this question: If we want to preserve the interpretation above then we do need to explain this curvature. However, it is fully correct to take the other standpoint and consider that the Einstein field equations are a set of differential equations containing a cosmological constant and governing the laws of gravity (General Relativity), and that in the absence of sources, spacetime has an intrinsic constant curvature. This last possibility requires one to sacrifice the important interpretation mentioned above. ii) The second question is why would the huge vacuum energy densities evaluated from quantum field theory calculations not contribute to the measured effective cosmological constant? As we discussed, this question could be re-addressed in the context of how the vacuum energy may contribute to the Einstein field equations. In particular, is the usual method of using a given ultraviolet cutoff energy as a source of gravity questionable? For example, other propositions have been made in literature [94, 95, 96] where vacuum energy will contribute via energy differences as experienced with the Casimir effect.

Finally, we pointed out the possible role of the Casimir effect used in gravitational and cosmological contexts for testing some of the assumptions and questions discussed. This is a purely quantum field theory phenomenon and could be used to look for clues on how the vacuum energy may fit within the Einstein field equations. Other kinds of experiments at the interface between quantum field theory and general relativistic principles have been also discussed in [106, 107, 108, 109] and might be of similar interest.

We conclude that challenging some of the assumptions underlying the formulation of the cosmological constant/dark energy problems and putting them to the test may prove useful and necessary to make progress on these questions.

Acknowledgments

The author thanks Latham Boyle, Simon DeDeo, G.F.R. Ellis, Chris Hirata, Pat McDonald, David Spergel, Paul Steinhardt, and Amol Upadhye for useful comments. The author thanks James Richardson for reading the manuscript.

This is not a review paper and we acknowledge that the list of references cited here is incomplete. We tried to provide only some examples from the literature when necessary.

Partial support from the Natural Sciences and Engineering Research Council of Canada (NSERC) and NASA Theory Award NNG04GK55G at Princeton University is acknowledged. The author acknowledges the partial support from the Hoblitzelle Foundation and a Clark award at the University of Texas at Dallas.
FIG. 3: a) Plot of the curvature invariant \( \mathcal{K} = R_{\alpha\beta}^{\gamma\delta} R_{\gamma\delta}^{\alpha\beta} \). b) Plot of the differential invariant \( DiRiem = R_{\alpha\beta}^{\gamma\delta} \gamma ;_\eta R_{\gamma\delta}^{\alpha\beta} \eta ;_\eta \).

The curvature decreases during a matter dominated universe to reach a constant curvature Lambda-dominated universe. Length units are used with \( \Lambda = 10^{-56} \text{cm}^{-2} \). We also display on the left vertical lines for \( c t_{\text{transition}} = 0.67 \times 10^{28} \text{cm} \) and \( c t_{\rho = \Lambda} = 0.90 \times 10^{28} \text{cm} \).

**APPENDIX A: EXAMPLES OF SPACETIME CURVATURE INCLUDING A COSMOLOGICAL CONSTANT**

In order to plot the evolution of spacetime curvature with a cosmological constant, we consider curvature invariants constructed from the Riemann tensor, \( R_{\alpha\beta\gamma\delta} \). These scalars allow a coordinate independent study of some geometrical features of a spacetime. They can also be linked to physical quantities via the EFE. For the special spacetimes we consider here the invariants are all related via algebraic relations [110, 111], and for the sake of simplicity we just choose here the Kretchman scalar,

\[
\mathcal{K} = R_{\alpha\beta}^{\gamma\delta} R_{\gamma\delta}^{\alpha\beta} \quad (A1)
\]

and the differential invariant

\[
DiRiem = R_{\alpha\beta}^{\gamma\delta} \gamma ;_\eta R_{\gamma\delta}^{\alpha\beta} \eta ;_\eta \quad (A2)
\]

to trace the evolution of the curvature of spacetime. For the metric (5), the invariants read

\[
\mathcal{K} = \Lambda^2 \left[ \frac{5}{3} \left( \coth \left( \frac{\sqrt{3} \Lambda}{2} t \right) \right)^4 - 2 \left( \coth \left( \frac{\sqrt{3} \Lambda}{2} t \right) \right)^2 + 3 \right] \quad (A3)
\]

and

\[
DiRiem = -9\Lambda^4 \left( \frac{\coth \left( \frac{\sqrt{3} \Lambda}{2} t \right)}{\sinh \left( \frac{\sqrt{3} \Lambda}{2} t \right)} \right)^2 \quad (A4)
\]
FIG. 4: a) LEFT: Plot of the Riemann components $R_{txtx} = R_{tyty} = R_{tztz}$. It starts with a power law decrease to reach a negative range exponential decrease during a de Sitter phase. For comparison, the no-Lambda curve shows how this component continues with a power law decrease within a positive range. Length units are used with $\Lambda = 10^{-56}\text{ cm}^{-2}$. We display vertical for $ct_{\text{transition}} = 0.67 \times 10^{28}\text{ cm}$ and $ct_{\rho_\Lambda=\rho_m} = 0.90 \times 10^{28}\text{ cm}$.

b) RIGHT: Plot of the Riemann components $R_{xyxy} = R_{xzxz} = R_{yzyz}$. The profile is similar to that of the scale factor. These components transit to an exponential increase at very large $t$. We also display $ct_{\text{transition}} = 0.67 \times 10^{28}\text{ cm}$ and $ct_{\rho_\Lambda=\rho_m} = 0.90 \times 10^{28}\text{ cm}$.

For the matter dominated universe, these are simply given by, $\mathcal{K} = \frac{80}{37} \frac{1}{t^4}$ and $DiRiem = -\frac{60}{3} \frac{1}{t^6}$. The de Sitter space has the usual line element

$$ds^2 = -dt^2 + \exp \left(2\sqrt{\frac{\Lambda}{3}}t\right) \left(dr^2 + r^2 d\Omega^2\right)$$  \hspace{1cm} (A5)

and constant curvature with $\mathcal{K} = \frac{3}{8} \Lambda^2$.

Figure 3a and Figure 3b show the profile of $\mathcal{K}$ and $DiRiem$ and how the spacetime curvature decreases during the expanding matter dominated universe to reach a constant curvature $\Lambda$-dominated universe at late times. This can also be seen from taking the limits of (A3) and (A4) at very large $t$.

The vertical lines in Figure 3a are the time at equality of dark energy density with matter energy density and the time of transition from deceleration to acceleration. The no-Lambda curves are shown for comparison.

Furthermore, in order to trace some features of the curvature lost in the squared quantities, we recourse to plotting directly the non-vanishing components of the Riemann tensor. Though coordinate dependent, these can be informative
FIG. 5: Plot of $K = R_{\alpha\beta} \gamma^\alpha R_{\gamma\delta} \alpha^\beta$ as a function of $r$. The curvature decreases as $1/r^2$ from the central mass $m$ to become dominated by the $\Lambda$ term after $r = (\frac{48m^2}{\Lambda})^{\frac{1}{3}} = .36 \times 10^{25}$ cm. Length units are used with $\Lambda = 10^{-56}$ cm$^{-2}$ and a mass $m = 0.74 \times 10^{17}$ cm.

In cartesian coordinates, these are:

$$R_{txtx} = R_{tyty} = R_{tztz} = \Lambda \left( \frac{3C}{\Lambda} \right)^{2/3} \times \left[ \left( \sinh \left( \frac{\sqrt{3}\Lambda}{2} t \right) \right)^{-2/3} - 2 \left( \sinh \left( \frac{\sqrt{3}\Lambda}{2} t \right) \right)^{4/3} \right]$$

(A6)

and

$$R_{xyxy} = R_{xzzz} = R_{yzyz} = 3^{1/3} \Lambda \left( \frac{C}{\Lambda} \right)^{4/3} \times \left[ \left( \sinh \left( \frac{\sqrt{3}\Lambda}{2} t \right) \right)^{8/3} + \left( \sinh \left( \frac{\sqrt{3}\Lambda}{2} t \right) \right)^{2/3} \right]$$

(A7)

The time evolution of the $R_{txtx}$ components is shown in Figure 5a where it starts with a power law decrease to reach a negative range exponential decrease during a de Sitter phase. For comparison, the no-$\Lambda$ curve shows how this component continues with power law decrease within a positive range.

Figure 5b shows that the profile of the $R_{xyxy}$ component is similar to that of the scale factor. This component transits to an exponential increase at very large $t$.

Also, we consider another informative example, the Schwarzschild-de Sitter spacetime with

$$ds^2 = - \left( 1 - \frac{2m}{r} - \frac{\Lambda r^2}{3} \right) dt^2 + \left( 1 - \frac{2m}{r} - \frac{\Lambda r^2}{3} \right)^{-1} dr^2 + r^2 d\Omega^2. \quad (A8)$$

At large $r$, it tends to the de Sitter space limit. The explicit de Sitter case is obtained by setting $m = 0$ while the
explicit Schwarzschild case is obtained by setting $\Lambda = 0$. Here, for (A8)

$$K = \frac{8}{3} \Lambda^2 + \frac{48m^2}{r^2}$$

(A9)

and is plotted in Figure 5 where the curvature due to the central mass $m$ decreases as a function of $r$ and is overtaken by the $\Lambda$ term at very large $r$. The Schwarzschild curve is plotted for comparison.

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