An asymptotic existence theorem for plane curves with prescribed singularities

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Introduction

Let \( d, m_1, \ldots, m_r \) be \((r + 1)\) positive integers. Denote by \( V(d; m_1, \ldots, m_r) \) the variety of irreducible (complex) plane curves of degree \( d \) having exactly \( r \) ordinary singularities of multiplicities \( m_1, \ldots, m_r \). In most cases, it is still an open problem to know whether this variety is empty or not.

In this paper, we will concentrate on the case where the \( r \) singularities can be taken in a general position. Precisely, let \( (P_1, \ldots, P_r) \) be a general \( r \)-tuple of point in \((\mathbb{P}^2)^r\). Denote by \( E \) the linear system of plane curves of degree \( d \) passing through the points \( P_i \) \((1 \leq i \leq r)\) with multiplicity at least \( m_i \). The expected dimension of \( E \) is \( \max(-1; d(d+3)/2 - \sum m_i(m_i+1)/2) \).

**Theorem 1** Given a positive integer \( m \), there exists an integer \( d'(m) \) such that, if \( m_i \leq m \) for \( 1 \leq i \leq r \) and \( d \geq d'(m) \), then :

The system \( E \) has the expected dimension \( e \) and, if \( e \geq 0 \), then a general curve in \( E \) is irreducible, smooth away from the \( P_i \), and has an ordinary singularity of multiplicity \( m_i \) at each point \( P_i \).

As a consequence, \( V(d; m_1, \ldots, m_r) \) is not empty.

The importance of this result comes from the fact that it is still valid when the expected dimension is small (which happens when the number \( r \) of points is high) ; even say, when \( e \) is zero. In this case, the curve is isolated in \( E \), and Bertini’s theorem can not be used.

Recent existence results have been proved by Greuel, Lossen and Shustin in the case of ordinary singularities, ([6], section 3.3) ; or even for general singularities [5]. But in all these statements the dimension of the system \( E \) must be, at least, quadratic in the degree \( d \). Notice, however, that the method of [5] together with the vanishing result of Alexander-Hirschowitz cited below (see [1]) would easily give theorem 1 as soon as \( e \geq d + 1 \) (see also section 4 for such considerations).
As for zero dimensional systems, a previous theorem had been proved by the author [12] for \( m_i \leq 3 \) and \( d \geq 317 \).

An explicit value for \( d'(m) \) has been computed by the author: one may take \( d'(m) = 2((m + 2)38)^{2m-1} \). According to a theorem of Alexander and Hirschowitz [1], it is already known that there exists a bound \( d(m) \) for the degree, above which \( E \) has the expected dimension. In theorem 1 \( d'(m) \) is slightly greater than \( d(m) \) and is expressed in terms of it. It is now possible to follow the proof of [1] and give an explicit bound for \( d(m) \) (let us recall that [1] holds for any projective variety, since our bound only holds for \( \mathbb{P}^2 \)). With this approach, it seems that the doubly exponential growth for the explicit value of \( d(m) \) is unavoidable.

However this bound is far from being sharp. In fact, according to a conjecture of Hirschowitz, if the \( m_i \) are in decreasing order and if \( d \) is greater than \( m_1 + m_2 + m_3 \), then the system \( E \) should have the expected dimension and contain an irreducible and smooth curve away from the \( P_i \) (except in the well-known case \( (d; m_1, \ldots, m_r) \neq (3n; n, \ldots, n) \) with \( r = 9 \)). Thus, the conjectural bound for \( d'(m) \) is \( 3m + 1 \).

To its length, the computation of this explicit value is not described here. The author places it at the reader’s disposal.

Theorem 1 is also interesting in view of recent results on the varieties \( V(d; m_1, \ldots, m_r) \). Recall that the first variety of this type, \( V = V(d; 2, \ldots, 2) \) was studied by Severi [14]. He proved that \( V \) is not empty and smooth if and only if \( r \leq (d-1)(d-2)/2 \). In addition \( r \leq d(d+3)/6 \) we also know that the nodes can be taken in generic position except in the case \( d = 6, r = 9, m_1 = \cdots = m_9 = 2 \) (case of an isolated double cubic) (see [2] and [16]). In 1985, Harris [8] completed this work, proving that \( V(d; 2, \ldots, 2) \) is always irreducible.

The questions of irreducibility and smoothness of general varieties of curves with prescribed singularities have been treated in many papers. Let us mention recent results for general singularities [15] or for nodal curves on general surfaces in \( \mathbb{P}^3 \) [4].

However in the case considered here, i.e. plane curves with ordinary singularities, A. Bruno announced that, \( V(d; m_1, \ldots, m_r) \) is irreducible, smooth and has the expected codimension assuming that it is not empty and that the singularities can be taken in generic position (conference in Toledo, September 98). This is exactly what is proved in theorem 1.

**Strategy of the proof**

The proof of theorem 1 is based on a lemma proved by the author in [11] (see also [3] for a first –not differential– approach of this lemma). This result, which we called “Geometric Horace Lemma”, is inspired by the Horace method of Hirschowitz (see, for example, [1]). But, while the usual Horace method can only be used to compute the dimension of linear systems like \( E \),
the geometrical lemma also yields conclusions about the irreducibility and smoothness of the curves in $E$.

The principle of the Geometric Horace Lemma is the following: Let us choose an irreducible and smooth plane curve $C$. Let us specialize some of the $r$ points on $C$. Denote by $y = (Q_1, \ldots, Q_r)$ this special point of $(\mathbb{P}^2)^r$ and by $x$ the generic point of $(\mathbb{P}^2)^r$. Two linear systems may be considered: $E_x = E$ when the points are in generic position and $E_y$ when they are in special position. The specialization from $x$ to $y$ is done in such a way, that $C$ is a base component of the system $E_y$. Thus a curve in $E_y$ is the union of $C$ and of a residual curve.

Under some assumptions, detailed in 2.1, if the generic residual curve is geometrically irreducible, smooth, and has ordinary singularities, then the general curve in $E_x$ also satisfies these properties.

An important point must be mentioned: if we do not specialize enough points on $C$, then $C$ is not a base component of $E_y$ and the method fails. But, if we specialize too many points, then the dimension of the linear system grows: $\dim E_y > \dim E_x$. This phenomenon is controlled with the help of differential conditions. It means that we have to consider some sub-systems of curves bound to pass through infinitely near points.

Here is the main point of the proof: by specializing too many points on the curve $C$, it is possible to make the dimension of $E$ grow considerably; i.e. grow as high as the degree $d$. Then, assuming that some vanishing property holds true, the residual system is base point free, and Bertini’s theorem can be used. As a consequence, a general residual curve is smooth, irreducible, and the intersection variety described above is irreducible.

To make all this strategy work, we still have to check the vanishing property referred to above. Roughly speaking, it means that the residual system has the expected dimension. To prove this, we make use of the following vanishing result of Alexander and Hirschowitz [1]:

Given an integer $m$, there exists an integer $a(m)$, and for $a \ge a(m)$, there exists another bound $d_0(a,m)$ such that, if $C$ is the generic curve of degree $a$, if $d \ge d_0(a,m)$ and if the points $P_i$ are either generic in $\mathbb{P}^2$ or generic on $C$ (not too many of them) then the system $E$ has the expected dimension.

In view of this result, the last choices are made: As for the curve $C$, we choose the generic curve of degree $a(m)$; and we only consider systems of curves of degree higher than $d(m) = d_0(a(m), m)$ (in fact, the final value $d'(m)$ is greater than $d(m)$, as appears in theorem 2).

**Contents**

The article is organized as follows: In the first part, notations and definitions are set. In particular, we describe the universal variety which parameterizes the curves we are studying.
In the second and third sections, we restate respectively the Geometric Horace Lemma, and the vanishing theorem of Alexander and Hirschowitz. These are the two main tools in the proof of theorem 1.

The fourth section is devoted to the study of lineal systems of “high” dimension, (precisely, a dimension greater than the degree $d$). In particular, when the $r$ points are in a good position, so that the vanishing lemma can be used, we show that theorem 1 is true for these systems.

In the last section, theorem 1 is proved.

1 Curves on rational surfaces

In the introduction, the situation has been described on the plane. Actually, most of the proofs will be done on the plane blown-up along the $r$ points $P_1, \ldots, P_r$.

In this section, we shall describe the family of rational surfaces obtained by blowing up a family of $r$ disjoint sections in $\mathbb{P}^2$, and the families of curves on these surfaces. We shall also set up most of the notations used in the article.

1.1 Families of rational surfaces and relative divisors

Let $r$ be a positive integer, and $X \subset (\mathbb{P}^2)^r$ be the open subset of $r$-tuples of distinct points. The morphism $\mathbb{P}^2_X = \mathbb{P}^2 \times X \longrightarrow X$ is naturally endowed with $r$ sections :
\[
\gamma_i : \quad (P_1, \ldots, P_r) \quad \longrightarrow \quad (P_i, (P_1, \ldots, P_r))
\]

Let $\Gamma_i$ be the image of $\gamma_i$; $\Gamma = \bigcup_{i=1}^r \Gamma_i$ is a nonsingular variety of $\mathbb{P}^2_X$. Blowing up $\mathbb{P}^2_X$ along $\Gamma$ produces a family of rational surfaces, parameterized by $X : S_X \longrightarrow X$. Let $\pi$ denote the composed morphism $S_X \longrightarrow X$. At any point $x = (P_1, \ldots, P_r)$ of $X$, the fiber of $\pi$ will be denoted by $S_x$. This surface $S_x$ is simply the projective plane blown up along the $r$ points $P_1, \ldots, P_r$.

Let us keep in mind that a relative effective Cartier divisor of $S_X$ on $X$ is simply an ideal sheaf $\mathcal{I}_D$ on $S_X$, locally principal, and not a zero-divisor in any fiber of $\pi$ (see [7]). These ideal sheaves are flat on $X$.

Examples : 1) Consider a line $L$ on $\mathbb{P}^2$, $L \times X \subset \mathbb{P}^2 \times X$ the trivial family of lines above $X$, and $H_X = b^{-1}(L \times X)$ the total transform of $L \times X$ in $S_X$. The ideal sheaf $\mathcal{I}_{H_X}$ is a relative effective Cartier divisor of $S_X$ on $X$.

2) Consider now $E_{i,X}$, the exceptional divisor obtained by blowing up the irreducible smooth variety $\Gamma_i$; the ideal sheaf $\mathcal{I}_{E_{i,X}}$ is also a relative effective Cartier divisor of $S_X$ on $X$. 

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1.2 Intersection pairing and linear systems

For any \( x \in X \), the Picard group of the surface \( S_x \) is endowed with the usual base : \([H_x], [-E_{1,x}], \ldots, [-E_{r,x}]\). The relative Cartier divisors being flat on \( X \), these bases satisfy the following property:

**Proposition 1.1** Let \( D \) be a relative Cartier divisor of \( S_X \rightarrow X \). For any point \( x \in X \), let \([D_x]\) be the class of the sheaf \( O_{S_x}(D) \) expressed in the base defined above ; then \([D_x]\) does not depend on \( x \).

The canonical divisor of Pic \( S_x \) is \( \omega_x = (-3;(-1)^r) \) (the notation \((-1)^r\) means that the integer \((-1)\) is repeated \( r \) times ; this convention will be kept in the sequel). The intersection pairing of Pic \( S_x \) is as follows : \((d;m_1,\ldots,m_r),(c;n_1,\ldots,n_r) = dc - m_1n_1 - \ldots - m_rn_r\).

Let \( \mathcal{d} = (d;m_1,\ldots,m_r) \in \text{Pic } S_x \). The sheaf \( O(dH_x - \sum_{i=1}^{r} m_iE_{i,x}) \) will be denoted by \( \mathcal{O}(\mathcal{d}) \), and the complete linear system of \( \mathcal{O}(\mathcal{d}) \) (i.e. the projective space \( \mathbb{P}(H^0(S_x, \mathcal{O}(\mathcal{d}))) \)) will be denoted by \( \mathcal{L}_x(\mathcal{d}) \).

The Riemann-Roch theorem for surfaces allows us to compute the Euler-Poincaré characteristic of \( \mathcal{O}(\mathcal{d}) : \chi(\mathcal{d}) = \frac{(d+1)(d+2)}{2} - \sum_{i=1}^{r} \frac{m_i(m_i+1)}{2} \). One may also compute the arithmetical genus of the eventual sections of \( \mathcal{O}(\mathcal{d}) : g(\mathcal{d}) = \frac{(d-1)(d-2)}{2} - \sum_{i=1}^{r} \frac{m_i(m_i-1)}{2} \).

Suppose that \( d \geq -2 \). By Serre Duality Theorem, \( h^2(S_x, \mathcal{O}(\mathcal{d})) = 0 \). The expected dimension for \( \mathcal{L}_x(\mathcal{d}) \) is then : \( \text{max}(\chi(\mathcal{d}) - 1, -1) \). (An empty system is supposed to have dimension \(-1\)). A system \( \mathcal{L}_x(\mathcal{d}) \) will be said to be regular if it has the expected dimension. More generally, a sheaf \( \mathcal{F} \) on \( S_x \) will be said to be regular if \( h^0(S_x, \mathcal{F}) = \text{max}(\chi(\mathcal{F}), 0) \) and \( h^1(S_x, \mathcal{F}) = \text{max}(-\chi(\mathcal{F}), 0) \).

1.3 Universal family of divisors

Let \( \mathcal{d} = (d;m_1,\ldots,m_r) \) be an \( r \)-tuple of integers, and let us define \( m'_1 = \text{max}(m_i, 0) \) and \( \mathcal{d}' = (d;m'_1,\ldots,m'_r) \). Let \( \Gamma_i \) be the image of the \( i^{th} \) natural section \( \gamma_i \) defined above, and let \( Z \subset \mathbb{P}^2_X \) be the scheme defined by the ideal \( \mathcal{I}_{\Gamma_1}^{m'_1} \ldots \mathcal{I}_{\Gamma_r}^{m'_r} \). The scheme \( Z \) is a flat family above \( X \) whose fibers \( Z_x \) are unions of \( r \) fat points of multiplicities \( m'_1,\ldots,m'_r \) (see section 3 for a definition of fat points).

Consider \( x = (P_1,\ldots,P_r) \in X \). The linear system \( |\mathcal{I}_{Z_x}(\mathcal{d})| \) is the system of plane curves of degree \( d \) passing through each point \( P_i \) with multiplicity at least \( m'_i \). One can easily see that this system is isomorphic to \( \mathcal{L}_x(\mathcal{d}) \).

Consider now the linear system \( \mathbb{P}(H^0(\mathcal{I}_{\mathbb{P}^2}, \mathcal{O}(\mathcal{d}))) \xrightarrow{\sim} \mathbb{P}^{d(d+3)/2} \) of plane curves of degree \( d \). One may define (here, only under a “set-theoretic” point of view, but it is endowed with a natural scheme structure) a subscheme \( F \) of \( \mathbb{P}^{d(d+3)/2} \times X \) in the following way :

\[
F = \{ (D, (P_1,\ldots,P_r)) \mid \text{D passes through each point } P_i \text{ with multiplicity at least } m'_i \}.
\]
This scheme $F$ parameterizes a canonical family of curves $\mathcal{D}' \subset \mathbb{P}^2 \times F$ : given $x = (D, (P_1, \ldots, P_r))$, the fiber $\mathcal{D}'_x$ simply is the curve $D$. As above, it is possible to blow-up the variety $\mathbb{P}^2_F = \mathbb{P}^2 \times F$ along the $r$ disjoint natural sections. Let $b_F : S_F \to \mathbb{P}^2_F$ be this blowing-up. By assumption, the divisor $\mathcal{D} = b_F^{-1}(\mathcal{D}') - \Sigma_{i=1}^r m_i E_i$ is effective : it is a relative effective Cartier divisor of the family $S_F \to F$. Moreover, $\mathcal{D}$ is a universal divisor:

**Proposition 1.2** Let $\mathbf{d} \in \mathbb{Z}^{r+1}$. The functor $\mathbf{F}_d$ from Schemes/X to Sets such that $\mathbf{F}_d(Y) = \{ D \subset S_Y, \text{relative effective divisor of class } \mathbf{d} \text{ on } Y \}$ is represented by the couple $(F, \mathcal{D})$.

(This proposition is detailed in [10] ; see also [7] and [13]).

Let $p : F \to X$ be the natural projection from $F \subset \mathbb{P}^{d(\dim + 2)} \times X$ to $X$. The fiber of $p$ over a point $x \in X$ is nothing but the linear system $\mathcal{L}_x(\mathbf{d})$. Let $x'$ be the generic point of this fiber. By definition of the universal divisor $\mathcal{D}$, the curve $\mathcal{D}_{x'}$ is the generic curve of $\mathcal{L}_x(\mathbf{d})$. It will be denoted by $\mathcal{D}_x(\mathbf{d})$.

Suppose now that a point $y \in X$ is a specialization of $x$. We will say that the curve $\mathcal{D}_x(\mathbf{d})$ specializes to the curve $\mathcal{D}_y(\mathbf{d})$ if the generic point of $p^{-1}(y) = \mathcal{L}_y(\mathbf{d})$ is a specialization of $x'$. This notion is of special importance if one expects to find properties of $\mathcal{D}_x(\mathbf{d})$ from those of $\mathcal{D}_y(\mathbf{d})$. In particular, if $\mathcal{D}_y(\mathbf{d})$ is geometrically irreducible, smooth, or has ordinary singularities, then the same holds for $\mathcal{D}_x(\mathbf{d})$.

If the dimension of $\mathcal{L}_y(\mathbf{d})$ equals the dimension of $\mathcal{L}_x(\mathbf{d})$, one can easily see that $\mathcal{D}_x(\mathbf{d})$ always specializes to $\mathcal{D}_y(\mathbf{d})$. If the dimension grows after the specialization, extra conditions are needed. In fact, it is sufficient to prove that the strata of the cohomological stratification (associated to the sheaf $\mathcal{O}_{S_X}(\mathbf{d})$) have sufficiently big codimension. This can be done with the help of differential methods (see [11], and the lemma 2.1).

## 2 The Geometric Horace Lemma

In this section, the Geometric Horace Lemma is restated and commented on. This lemma was proved by the author in [11].

Let us first give some notations and conventions : Let $y = (Q_1, \ldots, Q_r)$ be a point of $X$ (the notation $(Q_1, \ldots, Q_r)$ is slightly incorrect, since $y$ is generally not a closed point). Let $G$ be a closed integral subscheme of $\mathbb{P}^2$. We will say that the $a$ points $Q_1, \ldots, Q_a$ ($0 \leq a \leq r$) are generic and independent on $G$ if $y$ is the generic point of a subvariety $Y \subset X$ such that $Y = G^a \times V$, where $V$ is an irreducible subscheme of $(\mathbb{P}^2)^{r-a}$.

Suppose now that the $i$-th point $Q_i$ is a nonsingular point of a plane curve $C$. On the rational surface $S_y$, the intersection point of the exceptional divisor $E_i$ and the strict transform $C$ will be denoted by $Q_i^C$ (and its ideal sheaf, $\mathcal{I}_{Q_i^C}$). Let $\mathcal{O}(\mathbf{d} = (d; m_1, \ldots, m_r))$ be an invertible sheaf on $S_x$. 
Global sections of the sheaf \( I_{Q_i} \mathcal{C}(d) \) can be seen as plane curves of degree \( d \) having multiplicity at least \( m_j \) at each point \( Q_j \) and, if the multiplicity at \( Q_i \) is exactly \( m_i \) having a branch tangent to \( C \) at this point.

**Lemma 2.1** Let \( d = (d_1, \ldots, d_r) \) be an \( r \)-tuple of positive integers, and \( x = (P_1, \ldots, P_r), y = (Q_1, \ldots, Q_r) \) be two points of \( X \) such that \( x \) specializes to \( y \). Let \( C \) be a plane curve, and \( \tilde{C} := C_y \) its strict transform on \( S_y \). Assume that \( \tilde{C} \) is geometrically irreducible and smooth, of class \( c \in \text{Pic } S_y \) and of genus \( g(c) \).

**Dimension and specialization** Suppose that:

1) \( -\alpha := \chi(O_{\tilde{C}}(d_i)) = d_i c + 1 - g(c) \leq 0 \); 
2) At the point \( y, g(c) \) points are generic and independent on \( C \).
3) If \( \alpha \geq 1 \), there exist \( \alpha + 1 \) integers \( i_1, \ldots, i_{\alpha+1} \) such that:
   - \( P_{i_1}, \ldots, P_{i_{\alpha+1}} \) are generic and independent in the plane, and
   - \( Q_{i_1}, \ldots, Q_{i_{\alpha+1}} \) are generic and independent on \( C \).
4) If \( \alpha = 0, H^0(S, O(d-c)) \) has the expected dimension : \( \chi(d-c) = \chi(d) \).
   - If \( \alpha \geq 1, H^0(S, I_{Q_{i_1}} \cup \cdots \cup Q_{i_{\alpha+1}} C)(d-c) \) has the expected dimension : 
     \( \chi(d-c) - \alpha - 1 = \chi(d) - 1 \).

Then \( L_x(d) \) is regular, and, if \( \chi(d) > 0 \), \( D_x(d) \) specializes to \( D_y(d) \).

**Irreducibility** Suppose, moreover, that \( \chi(d) > 0 \) and:

5) the system \( L_x(d) \) is empty.
6) \( D_y(d-c) \) is geometrically irreducible.

Then \( D_x(d) \) is geometrically irreducible.

**Smoothness** Suppose finally that:

7) If \( \alpha = 0 \), \( y \) is normal and the closure of \( x \); 
8) \( D_y(d-c) \) meets \( \tilde{C} \) transversally ;
9) \( D_y(d-c) \cap \tilde{C} \) is irreducible (not: geometrically irreducible) ;
10) \( D_y(d-c) \) is a smooth curve,

then \( D_x(d) \) is smooth.

The system \( L_y(d-c) \), and the curve \( D_y(d-c) \) are respectively called residual system and residual curve. In fact, \( D_y(d) \) is the union of \( \tilde{C} \) and \( D_y(d-c) \). The curve \( \tilde{C} \) being well-known, this lemma allows one to get information on \( D_x(d) \), from the properties of \( D_y(d-c) \) and the relation between \( D_y(d-c) \) and \( \tilde{C} \).

Condition 9) is certainly best described in an example : assume that \( \dim L_y(d) = 0 \), and denote by \( Y \) the adherence of \( y \) in \( X \). There is only one curve in the system \( L_z(d) \) for every closed point \( z \) in an open subset of \( Y \). Then the intersection of \( C \) and the residual curve in \( L_z(d-c) \) makes, as \( z \) varies, a covering of degree \( (d-c) \) over the open subset of \( Y \). The condition 9) means that this “intersection variety” is irreducible ; or in other words, that the monodromy group of this covering acts transitively on the intersection points of \( \tilde{C} \) and the residual curve \( D_y(d-c) \).
A case of special interest is the case where the number $\alpha$ of $1$ is positive. Roughly speaking, this situation arises when one specializes too many points on the curve $C$. Let us suppose that $\chi(d) > 0$. Considering the exact sequence

$$0 \to \mathcal{O}_S(d - \mathcal{C}) \to \mathcal{O}_S(d) \to \mathcal{O}_{\tilde{C}}(d) \to 0,$$

we find that $\chi(d - \mathcal{C}) = \chi(d) + \alpha$. Since $\mathcal{I}_Q_{\mathcal{C}_{i_1} \cup \ldots \cup \mathcal{C}_{i_{n+1}}}(d - \mathcal{C})$ is regular, $\mathcal{L}_y(d - \mathcal{C})$ also is regular and $\dim \mathcal{L}_y(d) = \dim \mathcal{L}_y(d - \mathcal{C}) = \dim \mathcal{L}_x(d) + \alpha$. Thus, the dimension has grown by $\alpha$.

**Remark 2.2** As regards to the ordinary singularities of the plane projection of $D_x(d)$, the Geometric Horace Lemma yields no conclusion. But, if $D_y(d) = D_y(d - \mathcal{C}) \cup \mathcal{C}$ meets the exceptional divisor $E_i$ in $m_i$ distinct points, $D_x(d)$ also possesses this property. Hence, its projection has only ordinary singularities of the expected multiplicity.

This is in particular the case if $C$ has ordinary singularities and $\mathcal{L}_y(d - \mathcal{C})$ is base point free (Bertini’s theorem applied to $E_i$ and the restricted system $\mathcal{L}_y(d - \mathcal{C})|_{E_i}$).

### 3 An asymptotic vanishing theorem

In order to use the Geometric Horace Lemma, we have to check that some linear system is regular (condition 4 of 2.1). This will be done with the help of an asymptotic vanishing theorem of Alexander and Hirschowitz. In this section, we restate this result and the adequate definitions. As a corollary, we write down precisely the vanishing lemma used in the proof of theorem 1.

Here, opposed to the other sections, all the work is done on the plane, without blowing it up. It is a natural choice when dealing with the dimension of a linear system, without consideration of the smoothness of its sections.

Let us first recall some definitions: As usual, a *fat point* of support $P \in \mathbb{P}^2$ is a subscheme $P^m$ of $\mathbb{P}^2$ defined by the ideal $\mathcal{I}_P^m$; the integer $m$ is called the *multiplicity* of $P$. If $Z$ is a zero dimensional subscheme of $\mathbb{P}^2$, the *degree* of $Z$, denoted by $\deg Z$, is the length of the ring $\mathcal{O}_Z$. As an example, $\deg P^m = m(m + 1)/2$.

**Definition 3.1** Let $P \in \mathbb{P}^2$ be a nonsingular point of a plane curve $C$, and $i, m$ be two integers such that $0 \leq i \leq m - 1$.

The *$i$-th residue point* supported by $P$, of multiplicity $m$, with respect to $C$, is the scheme defined by the ideal $\mathcal{I}_P^{m-1}(C) \cap (\mathcal{I}_C + \mathcal{I}_P^m)$. It is denoted by $D_{C_1}^{i}(P^m)$ or $D^i(P^m)$ if no confusion can arise. A residue of type $D^{m-1}(P^m)$ is called a simple residue ([4], 2.2.).
Proposition 3.5 ([1], 2.3) Let $C$ be an irreducible and smooth plane curve of degree $a$ and genus $g = (a-1)(a-2)/2$, and $d$ be an integer greater than $a$. Let $Z = Z_0 \cup P_1^{m_1} \cup \cdots \cup P_\beta^{m_\beta}$ be a zero dimensional subscheme of $\mathbb{P}^2$ such that $P_1, \ldots, P_\beta$ are generic and independent points of $\mathbb{P}^2$. Denote
also by \(Q_1, \ldots, Q_s\), \(\beta\) generic and independent points of \(C\). Suppose that 
\(\chi(I_Z(d)) \leq 0\). If:

i) \(H^1(\mathbb{P}^2, I_{Q_{m1} \cup \ldots \cup Q_{m\beta}}(d - a)) = 0\);

ii) \(\beta = da + 1 - g - \deg(Z \cap C)\);

iii) \(I_{(Z_0 \cup C) \cup Q_1 \cup \ldots \cup Q_{\beta}}(d)\) is a regular invertible sheaf of \(C\);

iv) \(H^0(\mathbb{P}^2, I_{Z_0 \cup D^{m1 - 1}(Q_{m1} \cup \ldots \cup D^{m\beta - 1}(Q_{m\beta}))(d - a)) = 0\);

then \(H^0(\mathbb{P}^2, I_Z(d)) = 0\), as expected.

**Corollary 3.6** Let \(m, a\) be two positive integers, and \(C\) be the generic plane curve of degree \(a\). Denote by \(x = (O_1, \ldots, O_t, P_1, \ldots, P_r)\) the generic point of \(C^t \times (\mathbb{P}^2)^r\), and consider an integer \(\alpha\) such that \(0 \leq \alpha \leq t\). Let \(\mathcal{d} = (d; n_1, \ldots, n_t, m_1, \ldots, m_r) \in \text{Pic} \ S_x\) such that \(n_i \leq m - 1\), \(m_i \leq m\), and suppose that \(\chi(\mathcal{d}) - \alpha = 0\). If

i) \(a \geq \max(a(m), 4m)\);

ii) \(d \geq \max(d_0(a, m) + a, 2am)\);

iii) \(da + 1 - g - n_1 - \ldots - n_t - \alpha \geq 0\);

then \(I_{O_{1} \cup \ldots \cup O_{\mathcal{d}}}^C(\mathcal{d})\) is regular.

**Proof:** Let \(Y_0 = D^1{(O_{m1}^{n+1})} \cup \ldots \cup D^1{(O_{m1}^{n+1})} \cup O_{\alpha+1}^n \cup \ldots \cup O_{\alpha}^{m_a}\), and 
\(Y = Y_0 \cup P_1^{m_1} \cup \ldots \cup P_r^{m_r}\). Clearly, \(\chi(Y(d)) = \chi(\mathcal{d}) - \alpha = 0\), therefore 
\(I_{O_{1} \cup \ldots \cup O_{\mathcal{d}}}^C(\mathcal{d})\) is regular if and only if \(H^0(\mathbb{P}^2, Y(d)) = 0\).

\(\bullet\) Let us first prove that there exists a non negative integer \(s\) such that:

\[
\beta := da + 1 - g - \alpha - \sum_{i=1}^{t} n_i - \sum_{j=1}^{s} m_j \in [0; m - 1]
\]

\[
s + \beta \leq r
\]

Since \(0 < m_i \leq m\), it will be enough to show that:

\[
d a + 1 - g - \alpha - \sum_{i=1}^{t} n_i - \sum_{j=1}^{s} m_j \leq m - 1
\]

\[
\iff (m_j \leq m) \quad da + 1 - g - \alpha - \sum_{i=1}^{t} n_i - \sum_{j=1}^{s} m_j \leq m - 1
\]

\[
\iff (m_j, n_i \leq m) \quad da + 1 - g - \alpha - \sum_{i=1}^{t} n_i - \sum_{j=1}^{s} m_j \leq m - 1
\]

\[
\iff (m_j, n_i \leq m) \quad da + 1 - g - \alpha - \sum_{i=1}^{t} n_i - \sum_{j=1}^{s} m_j \leq m - 1
\]

\[
\iff (m > 0) \quad da + 1 - g - \alpha - \sum_{i=1}^{t} n_i - \sum_{j=1}^{s} m_j \leq m - 1
\]

\[
\iff (d + 1 \geq \alpha + (m - 1)) \quad 1 - (\alpha + (m - 1)) \leq 0
\]

\[
\iff (a \geq 4m) \quad 1 - (\alpha + (m - 1)) \leq 0
\]

\[
\iff \quad -7m^2 - 4m + 1 \leq 0, \quad \text{which is true.}
\]
• Consider \(Q_1, \ldots, Q_{s+\beta}\), \((s + \beta)\) generic and independent points on \(C\). Denote by \(Z_0\) and \(Z\) the schemes

\[
Z_0 := Y_0 \cup Q_1^{m_1} \cup \cdots \cup Q_s^{m_s} \cup P_{s+\beta+1}^{m_{s+\beta+1}} \cup \cdots \cup P_r^{m_r}
\]

\[
Z := Z_0 \cup P_{s+1}^{m_{s+1}} \cup \cdots \cup P_{s+\beta}^{m_{s+\beta}}.
\]

By the Semicontinuity Theorem, if \(H^0(\mathbb{P}^2, \mathcal{I}_Z(d)) = 0\), then \(H^0(\mathbb{P}^2, \mathcal{I}_Y(d)) = 0\) as expected.

To prove that \(H^0(\mathcal{I}_Z(d))\) is equal to zero, we make use of proposition 3.5. Let us specialize the points \(P_{s+1}, \ldots, P_{s+\beta}\) to the points \(Q_{s+1}, \ldots, Q_{s+\beta}\). The following relation, which bound the number of generic points on \(C\), will be useful:

\[
(t + s + \beta) \geq 2a^2 - \frac{a^2}{2m}
\]

This inequality comes from (2), which yields \(\sum_{i=1}^t n_i + \sum_{j=1}^s m_j + \beta \geq da + 1 - g\). Since \(n_i, m_j \leq m\) and \(d \geq 2am\) one gets \(m(t + s + \beta) \geq 2a^2m - a^2/2 + 3a/2\).

Let us check conditions i to iv of 3.5:

i) \(H^1(\mathbb{P}^2, \mathcal{I}_{Q_{s+1}^{m_{s+1}} \cup \cdots \cup Q_{s+\beta}^{m_{s+\beta}}(d - a)}) = 0\) by the lemma 3.7 below.

ii) By definition of \(Z\), \(\deg Z \cap C = \alpha + \sum_{i=1}^t n_i + \sum_{j=1}^s m_j\). So that \(\beta = da + 1 - g - \deg Z \cap C\).

iii) The divisor of \(C\) defined by the ideal \(\mathcal{J} = \mathcal{I}(Z_0 \cap C) \cup Q_{s+1} \cup \cdots \cup Q_{s+\beta}\) is supported on the \(t + s + \beta\) points \(O_1, \ldots, O_t, Q_1, \ldots, Q_{s+\beta}\) which are generic and independent on \(C\). Hence, if \(t + s + \beta \geq g\), \(\mathcal{J}(d)\) is a nonspecial invertible sheaf. But, \(t + s + \beta \geq 2a^2 - a^2/(2m)\) (4) and then \(t + s + \beta \geq a^2/2 \geq g\).

iv) Let \(T = Z_0' \cup D_{s+1}^{m_{s+1}-1}(Q_{s+1}^{m_{s+1}}) \cup \cdots \cup D_{s+\beta}^{m_{s+\beta}-1}(Q_{s+\beta}^{m_{s+\beta}})\). Since \(Z_0'\) is a union of fat points (the \(D^1\) have disappeared), \(T\) is an \((m, a)\)-configuration. Let us prove that it is a \((d - a, m, a)\)-candidate.

From the definition of \(\beta\), one easily sees that \(\chi(\mathcal{I}_T(d - a)) = \chi(\mathcal{I}_Z(d)) = 0\). The second condition is: \(h^0(C, \mathcal{O}_C(d - a)) - \deg(T \cap C) \geq 0\). If \(s + 1 \leq j \leq s + \beta\), then \(\deg(D_{m_j}^{m_j-1}(Q_{m_j}^{m_j}))\) equals \(m_j\) if \(m_j > 1\) and 0 if \(m_j = 1\). So the inequality can be checked as follows:

\[
\begin{align*}
\sum_{i=1}^t n_i - \sum_{j=1}^s m_j - \sum_{j=s+1}^{s+\beta} m_j & \geq (d - a)a + 1 - g - \sum_{i=1}^t (n_i - 1) - \sum_{j=1}^s (m_j - 1) - \sum_{j=s+1}^{s+\beta} m_j \\
& = (d - a)a + 1 - g - \sum_{i=1}^t (n_i - 1) - \sum_{j=s+1}^{s+\beta} m_j \\
& \geq (m_j \leq m) - a^2 + t + s + \alpha + \beta - \sum_{j=s+1}^{s+\beta} m_j \\
& \geq (a \geq 4m) - a^2 + t + s + \alpha + \beta - m(m - 1) \\
& \geq (a \geq 4m) - a^2 + 2a^2 - a^2/2m - m(m - 1) \\
& \geq 15m^2 - 7m \geq 0.
\end{align*}
\]

Thus \(T\) is a \((d - a, m, a)\)-candidate. By assumption, \(a \geq a(m)\), and \(d - a \geq d_0(a, m)\); hence, by Proposition 3.4, \(T\) is winning and \(H^0(\mathcal{I}_T(d - a)) = 0\).

It is now allowed to apply proposition 3.5; it gives \(H^0(\mathcal{I}_Z(d)) = 0\). \(\square\)
Lemma 3.7 Under the assumptions of corollary 3.6, consider $Q_1, \ldots, Q_\beta$, $\beta$ generic points of $C$, and $m_1, \ldots, m_\beta$, $\beta$ integers bounded by $m$. If $\beta \leq (m - 1)$, then $H^1(\mathbb{P}^2, \mathcal{I}_{Q_1}^{m_1} \cup \cdots \cup \mathcal{I}_{Q_\beta}^{m_\beta} (d - a)) = 0$.

Proof: The only case we need to consider is $\beta = m - 1$ and $m_1 = \ldots = m_\beta = m$. By Xu’s theorem ([17], theorem 3), $H^1(\mathbb{P}^2, \mathcal{I}_{Q_1}^{m} \cup \cdots \cup \mathcal{I}_{Q_\beta}^{m} (d - a)) = 0$, as soon as $d - a \geq 3m$ and $(d - a + 3)^2 > (10/9) \sum_{i=1}^{\beta} (m_i + 1)^2 = (10/9)(m - 1)(m + 1)^2$. By assumption, $d - a \geq 2am - a \geq am$ and $a \geq \sqrt{2m}$, hence $(d - a + 3)^2 \geq a^2m^2 \geq 2m^4 \geq (10/9)(m - 1)(m + 1)^2$ for any positive $m$. If $a \geq 3$ then $d - a \geq 3m$. If $a = 2$ then $m$ necessary equals 1, and the lemma is clearly true. □

4 Systems of high dimension

Given a system $\mathcal{L}(d)$ of a sufficiently high dimension, Theorem 1 may be proved with the help of Bertini’s theorem. The main point consists in showing that $\mathcal{L}(d)$ is base point free; this is done here with the vanishing theorem of the preceding section and a kind of Castelnuovo-Mumford’s argument.

Proposition 4.1 Let $m$ be a positive integer, $a \geq a(m)$ (see prop. 3.4) and $C$ the generic curve of degree $a$. Let $x = (P_1, \ldots, P_r)$ be the generic point of $C^s \times (\mathbb{P}^2)^{r-s}$, $(0 \leq s \leq r)$ and $d = (d; m_1, \ldots, m_r) \in \text{Pic} \ S_x$ be such that $m_i \leq m$ $(1 \leq i \leq r)$. The class of $\tilde{C}$ in $\text{Pic} \ S_x$ is denoted by $\mathfrak{c}$, and its genus $g$.

Suppose that $d \geq d_0(a, m) + 1$ (see 3.4), $\chi(d) \geq d + 1$, and $d \mathfrak{c} + 1 - g \geq a$. Then, $\mathcal{L}_x(d)$ is base point free, $\mathcal{D}_x(d)$ is geometrically irreducible and smooth, $\mathcal{D}_x(d)$ meets $\tilde{C}$ transversally, and $\mathcal{D}_x(d) \cap \tilde{C}$ is irreducible.

Proof: Let us first prove that $\mathcal{L}_x(d)$ is base point free: the characteristic $\chi(d)$ is greater than 1, so we just have to show that, given a point $Q \in S_x$, $h^1(S_x, \mathcal{I}_Q(d)) = 0$. Whatever the position of $Q$ is (even on an exceptional divisor), there exists a line $L$ on $\mathbb{P}^2$ such that $Q$ belongs to the strict transform $\tilde{L}$ of $L$. Let $L \in \text{Pic} \ S_x$ be the class of $\tilde{L}$; one may write $L = (1; \varepsilon_1, \ldots, \varepsilon_r)$, where $\varepsilon_i = 1$ if $P_i \in L$ and 0 otherwise. Consider the exact sequence:

$$H^1(S_x, \mathcal{O}(d - L)) \rightarrow H^1(S_x, \mathcal{I}_Q(d)) \rightarrow H^1(L, \mathcal{I}_{Q,L}(d))$$

The scheme $Z = P_1^{m_1-\varepsilon_1} \cup \cdots \cup P_r^{m_r-\varepsilon_r}$ is clearly an $(m, a)$-configuration. Since $(d - L)_d + 1 - g \geq 0$, $h^0(C, \mathcal{O}(d - 1)) \geq (d - 1)a + 1 - g \geq \deg(Z \cap C)$, hence $Z$ is a $(d - 1, m, a)$-candidate (in the extended sense of remark 3.3). Moreover, $(d - 1) \geq d_0(a, m)$; Proposition 3.4 shows that $\mathcal{L}_x(d - L)$ is a regular system. But $\chi(d - L) \geq \chi(d) - (d + 1) \geq 0$, therefore $h^1(d - L) = 0$.

Moreover, since $\tilde{L}$ is a rational curve, $|\mathcal{I}_{Q,L}(d)|$ is a regular system of degree $d \mathfrak{c} - 1$. If $d \mathfrak{c} \geq 0$, then $h^1(\mathcal{I}_{Q,L}(d)) = 0$, as expected. Otherwise, the
exact sequence $0 \rightarrow \mathcal{O}_{S_x}(d-L) \rightarrow \mathcal{O}_{S_x}(d) \rightarrow \mathcal{O}_{\tilde{L}}(d) \rightarrow 0$ shows that $\tilde{L}$ is a base component of $\mathcal{L}_x(d)$. Consider another line $L'$ of $\mathbb{P}^2$ containing none of the $r$ points $P_1, \ldots, P_r$. The preceding argument is true, with $L'$ in place of $L$, showing that the point $Q' = L \cap L'$ cannot be a base point of $\mathcal{L}_x(d)$. This is a contradiction.

Therefore, $h^1(\mathcal{O}_{S_x}(d-L)) = h^1(\mathcal{I}_{Q,L}(d)) = 0$, and the first exact sequence yields $h^1(\mathcal{I}_Q(d)) = 0$.

- Thus $\mathcal{L}_x(d)$ is base point free. Bertini’s theorem shows that $\mathcal{D}_x(d)$ is a smooth curve. Suppose it is not geometrically irreducible, and denote by $D_1, \ldots, D_l$ ($l \geq 2$) its geometrically irreducible components (over a bigger base field). Let $d_1 \in \text{Pic } S_x$ be the class of $D_i$ ($1 \leq i \leq l$).

Consider two integers $1 \leq i \neq j \leq l$. The curve $\mathcal{D}_x(d)$ being smooth, $D_i$ does not intersect $D_j$ and $d_i - d_j = 0$. Let $P$ be a point of $D_i$. The dimension of $\mathcal{L}_x(d_j)$ being positive (this system is base point free), there exists a curve $D'_j \subset \mathcal{L}_x(d_j)$ containing $P$. But $D'_j \cap D_i = 0$, so $D_i$ is a component of $D'_j$, and $\mathcal{L}_x(d_j - d_i)$ is effective. By the same argument $\mathcal{L}_x(d_i - d_j)$ is effective too, hence $d_i = d_j$ for every $i \neq j$.

Thus $d = l d_1$ and $d^2 = 0$. The equality $\chi(d) + g(d) = d^2 + 2$ yields $g(d) \leq 1 - d$. Moreover, one can easily see that $g(l d_1) = l g(d_1) - (l - 1)$. Therefore, $l g(d_1) \leq l - d$ and (since $g(d_1) \geq 0$), $l \geq d$. The only possibility is $l = d$. Then $D_1$ is the strict transform of a line such that $D_1^2 = 0$. One may suppose that $d_1 = (1; 1)$, and $d = (d; d)$. This situation never happens since, by assumption $d \geq d_0(a, m) > m$.

- We still have to prove that $\mathcal{D}_x(d)$ meets $\tilde{C}$ transversally and that $\mathcal{D}_x(d) \cap \tilde{C}$ is irreducible. The first point comes from Bertini’s theorem, applied to the curve $\tilde{C}$ and the restricted (base point free) linear system $\mathcal{L}_x(d)|_{\tilde{C}}$.

As for the second point, consider $\mathcal{G} \subset S_x \times \mathcal{L}(d)$, the universal divisor over $\mathcal{L}(d)$. Let $I = \mathcal{G} \cap (\tilde{C} \times \mathcal{L}(d))$ be the intersection variety. Since $\mathcal{L}(d)$ is base point free, the fibers of the natural projection $I \rightarrow \tilde{C}$ are projective spaces of constant dimension. Therefore, $I$ is irreducible and $\mathcal{D}_x(d) \cap \tilde{C}$, which is the generic fiber of $I \rightarrow \mathcal{L}(d)$ also.

5 Proof of Theorem 1

In this section we gather the preceding results to prove the announced theorem. Actually, the work is not made directly on the projective plane but, rather, on the plane blown up at the $r$ points. However, the theorem proved below clearly implies the statement of the introduction.

Theorem 2 Let $m$ be a positive integer, $x$ the generic point of $(\mathbb{P}^2)^r$ and $d = (d; m_1, \ldots, m_r) \in \text{Pic } S_x$ such that $0 < m_i \leq m$ ($1 \leq i \leq r$). With the notations of 3.4, let
\[ a = a'(m) = \max(a(m), 4m); \]
\[ d'(m) = \max(d_0(a, m) + 2a, a(2m + 1)). \]

If \( d \geq d'(m) \) then \( \mathcal{L}_x(\overline{d}) \) is regular, and if \( \dim \mathcal{L}_x(\overline{d}) \geq 0 \), the generic curve \( \mathcal{D}_x(\overline{d}) \) is geometrically irreducible, smooth, and meets each exceptional divisor \( E_i \) in \( m_i \) distinct points \( (1 \leq i \leq r) \). As a consequence, the image of \( \mathcal{D}_x(\overline{d}) \) on the plane has an ordinary singularity of the prescribed multiplicity \( m_i \) at every \( P_i \).

**Proof:** Let \( C \) be the generic curve of degree \( a \) and genus \( g \).

If \( \chi(\overline{d}) \leq 0 \), then the scheme \( Z = P_1^{m_1} \cup \ldots \cup P_r^{m_r} \) is a \((d, m, a)\)-candidate with an empty constraint part. By Proposition 3.4, \( Z \) is winning, \( \mathcal{L}_x(\overline{d}) \) is empty and the theorem is true. If \( \chi(\overline{d}) > 1 \) one may consider \( \chi(\overline{d}) - 1 \) more general points \( P_{r+1}, \ldots, P_{r+\chi(\overline{d})-1} \) of \( \mathbb{P}^2 \) and study the sub-system of curves in \( \mathcal{L}_x(\overline{d}) \) passing through these supplementary points with multiplicity at least 1. It is equivalent to study the curves in \( \mathcal{L}(d; m_1, \ldots, m_r) \) or in \( \mathcal{L}(d; m_1, \ldots, m_r, 1^{\chi(\overline{d})-1}) \). As a consequence, we can make the assumption that \( \chi(\overline{d}) = 1 \).

- There exists a positive integer \( s \leq r \) such that:

  \[
  (5) \quad \alpha = da - m_1 - \ldots - m_s + 1 - g \in [-d + a - m, -d + a - 1] \\
  (6) \quad s \geq (2da - a^2)/(2m)
  \]

  The second inequality follows from the first one: (5) together with \( m_i \leq m \)
gives \( da - ms + 1 - g \leq -d + a - 1 \) hence (since \( g \leq a^2/2 \), \( ms \geq (2da - a^2)/2 \).

As for (5), since \( 0 < m_i \leq m \), it is sufficient to show that \( da - \sum_{i=1}^s m_i + 1 - g \leq -d + a - 1 \). This is a consequence of the assumption \( \chi(\overline{d}) = 1 \):

\[
\begin{align*}
\iff_1 (m_i &\leq m, g > 0) \\
& da - (m_1 + \ldots + m_r) + 1 - g \leq -d + a - 1 \\
& da - 2 \left( \frac{m_1(m_1+1)}{2} \ldots + m_r(m_r+1) \right) \leq -d + a - 1 \\
\iff_2 (\chi(\overline{d}) = 1) \\
& da - \frac{2}{m+1} \left( \frac{d^2}{2} \ldots - \chi(\overline{d}) \right) \leq -d + a - 1 \\
& d(a+1) - \frac{d(d+3)}{(m+1)} - a + 1 \leq 0
\end{align*}
\]

which is true when \( d \geq a(2m + 1) \).

- Let \( x = (P_1, \ldots, P_r) \) denote the generic point of \( (\mathbb{P}^2)^r \), and \( y = (Q_1, \ldots, Q_s, P_{s+1}, \ldots, P_r) \) the generic point of \( C^s \times (\mathbb{P}^2)^{r-s} \). The \( r \)-tuple \( y \) is a specialization of \( x \). The class of \( \overline{C}_y \) is \( \underline{c} = (a; 1^s, 0^{r-s}) \). In order to apply the Geometric Horace Lemma we are going to check the points \( 1^0 \) to \( 10^0 \) of lemma 2.1.

Condition \( 1^0 \) is nothing but the relation (5) above. The assumption \( d \geq a(2m + 1) \) and (6) yield \( s \geq a^2 \geq g \), hence \( 2^0 \) is true; moreover (6) and \( a \geq 4m \) also give \( s \geq 4d - 2a \geq d - a + m + 1 \), hence \( 3^0 \) is true.

As for the regularity of \( \mathcal{I}_{Q_1 \cup \ldots \cup Q_s}^{C_{a+1}}(\overline{d} - \underline{c}) \), we make use of the corollary 3.6. Since \( \chi(\overline{d}) = 1 \), the exact sequence 1 (section 2) and 5 yield \( \chi(\overline{d} - \underline{c}) - \]

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\( (\alpha + 1) = 0 \). The choice of \( a'(m) \) and \( d'(m) \) gives \( a = \max(a(m), 4m) \) and \( d - a \geq \max(d_0(a, m) + a, 2am) \). Therefore, the only remaining condition of 3.6 is

\[
(d - \alpha, \alpha - 1 + 1 - g \geq 0 \\
\iff (d - a) a - \sum_{i=1}^{8}(m_i - 1) + 1 - g - (\alpha + 1) \geq 0 \\
\iff (a \leq d - a + m) -2\alpha - a^2 + s - 1 \geq 0 \\
\iff (a \geq 2(2m+1)) a^2 - 4am - 2a - 2m - 1 \geq 0 \\
\iff (a \geq 4m) 8m - 2m - 1 \geq 0
\]

which is true since \( m \geq 1 \).

\[ \bullet \] We are now left with the “irreducibility” and “smoothness” part of lemma 2.1. The sheaf \( \mathcal{O}(\gamma) \) is effective if and only if \( s \leq a(a+3)/2 \), which is not the case by (6); so \( 5^{\circ} \) is true. The 7-th point is empty since \( \alpha \geq d - a + 1 > 0 \).

Now, the residual system \( \mathcal{L}_y(d - \gamma) \) has a “high” dimension: Precisely, the exact sequence 1 shows that \( \dim \mathcal{L}_x(d - \gamma) = \alpha \in [d - a + 1, d - a + m] \). Thus, the remaining condition of the Horace lemma can be proved with the proposition 4.1. By assumption \( d - a \geq d_0(a, m) + 1 \). It is then sufficient to prove that \( (d - \gamma) \alpha - 1 - g \geq a \).

But the preceding computation shows that \( (d - \gamma) \alpha - 1 - g \geq \alpha + 1 \geq d - a + 2 \geq (d \geq 3a) a \).

\[ \bullet \] Let us turn now to the question of ordinary singularities. Recalling the remark 2.2, we only have to check that the system \( \mathcal{L}_y(d - \gamma) \) is base point free, which is the case by Proposition 4.1. \( \square \)

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