TWISTED COMPONENT SUMS OF VECTOR-VALUED MODULAR FORMS

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Abstract. We construct isomorphisms between spaces of vector-valued modular forms for the dual Weil representation and certain spaces of scalar-valued modular forms in the case that the underlying finite quadratic module $A$ has order $p$ or $2p$, where $p$ is an odd prime. The isomorphisms are given by twisted sums of the components of vector-valued modular forms. Our results generalize work of Bruinier and Bundschuh to the case that the components $F_\gamma$ of the vector-valued modular form are antisymmetric in the sense that $F_\gamma = -F_{-\gamma}$ for all $\gamma \in A$. As an application, we compute restrictions of Doi-Naganuma lifts of odd weight to components of Hirzebruch-Zagier curves.

1. Introduction

In the study of theta lifts (such as Maass lifts and Borcherds products) it is convenient to work with vector-valued modular forms for the dual Weil representation $\rho^*$, associated to a finite quadratic module $(A,Q)$. Therefore, it is useful to understand the precise relationship between vector-valued modular forms for $\rho^*$ and scalar-valued modular forms for congruence subgroups.

For example, in some cases there are isomorphisms between spaces of vector-valued and scalar-valued modular forms. In [2], Bruinier and Bundschuh showed that if $\rho^*$ for congruence subgroups satisfies $\rho^* \mid \mod \chi$ of vector-valued modular forms by a Dirichlet character $\chi$ of odd prime order, then modular forms $\rho^*$ of weight $k \in \mathbb{Z}$ with $k \equiv \text{sig}(A,Q)/2 \pmod{2}$ can be identified with certain modular forms of weight $k$ for $\Gamma_0(p)$ and Nebentypus $\chi_p = (\frac{p}{\rho})$. The isomorphism is given by the component sum

$$\varphi \left( \sum_{\gamma \in A} F_\gamma(\tau)e_\gamma \right) = \sum_{\gamma \in A} F_\gamma(p\tau)$$

of the vector-valued modular form $F(\tau) = \sum_{\gamma \in A} F_\gamma(\tau)e_\gamma$, where $e_\gamma$ denotes the standard basis of the group algebra $\mathbb{C}[A]$. Using similar ideas, Y. Zhang constructed isomorphisms between spaces of vector-valued and scalar-valued modular forms for certain classes of finite quadratic modules which do not necessarily have odd prime order (see [10, 11]).

The condition $k \equiv \text{sig}(A,Q)/2 \pmod{2}$ turns out to be crucial for the aforementioned result of Bruinier and Bundschuh, since otherwise the components of any modular form for $\rho^*$ satisfy $F_{-\gamma} = -F_\gamma$, and hence cancel out in pairs in the sum. To obtain a non-zero map in any weight, we twist the component sums of vector-valued modular forms by a Dirichlet character $\chi \mod p$ with $\chi(-1) = (-1)^{k+\text{sig}(A,Q)/2}$. Suppose that $|A| = p$ with an odd prime $p$. We define the twisted component sum of a modular form $F(\tau) = \sum_{\gamma \in A} F_\gamma(\tau)e_\gamma$ for $\rho^*$ by

$$\varphi_\chi \left( \sum_{\gamma \in A} F_\gamma(\tau)e_\gamma \right) = \sum_{\gamma \in A} \chi(\gamma)F_\gamma(p\tau),$$

where we fix an identification of $A$ with $\mathbb{Z}/p\mathbb{Z}$ to define $\chi(\gamma)$ for $\gamma \in A$. The assumptions on $k$ and $\chi$ imply that $\varphi_\chi$ is not trivially zero. Moreover, we have the following result.

Proposition 1. The map $\varphi_\chi$ defines an injective homomorphism from the space of modular forms of weight $k$ for $\rho^*$ to the space of scalar-valued modular forms of weight $k$ for $\Gamma_0(p^2)$ with Nebentypus $\chi \otimes \chi_p$.

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For the proof we refer to Proposition 3 below. It is immediate from the construction that the \( n \)-th Fourier coefficient of \( \varphi_\chi(F) \) vanishes unless \( n \in p(\mathbb{Z} - Q(\gamma)) \) for some \( \gamma \in A \setminus \{0\} \). However, \( \varphi_\chi \) is in general not surjective onto the subspace defined by this vanishing condition. We characterize the image of \( \varphi_\chi \) in terms of the Atkin-Lehner involution in Proposition 5 below.

We construct an analogous map \( \varphi_\chi \) in the case that \( |A| = 2p \) is twice an odd prime \( p \); see Proposition 9; in this case, \( (A, Q) \) must have odd signature and all modular forms are of half-integral weight.

As an application, we compute restrictions of Doi-Naganuma lifts of odd weight to components of Hirzebruch-Zagier curves \( T_\ell \) of prime index \( \ell \). Let \( K = \mathbb{Q}(\sqrt{p}) \) with a prime \( p \equiv 1 \pmod{4} \) and let \( \mathcal{O}_K \) be its ring of integers. Recall that the Doi-Naganuma lift maps a vector-valued cusp form \( F \) to scalar-valued modular forms makes it easier to compute the restriction of \( \Phi_F \) to \( (\mathcal{O}_K, -N_{K/Q}) \) to a component of the Hirzebruch-Zagier curve \( T_\ell \) of prime index \( \ell \) is given by the Shimura lift of the vector-valued cusp form of weight \( k + 1/2 \) for the dual Weil representation of the lattice \( (\mathbb{Z} - \ell x^2) \) obtained by the so-called theta contraction of \( F \) as defined in [6], i.e. we have the commutative diagram

\[
\begin{array}{ccc}
S_k(-N_{K/Q}) & \xrightarrow{\text{Doi-Naganuma}} & S_k(\text{SL}_2(\mathcal{O}_K)) \\
\text{theta contr.} & & \text{restr. to } T_\ell \\
\downarrow & & \downarrow \\
S_{k+1/2}((-2\ell)) & \xrightarrow{\text{Shimura}} & S_{2k}(\Gamma_0(\ell)) \\
\end{array}
\]

For a proof, see Lemma 10. We show that, on the level of the corresponding scalar-valued modular forms, the theta contraction basically becomes a multiplication by the Jacobi theta function. In this way, passing to scalar-valued modular forms makes it easier to compute the restriction of \( \Phi_F \) to a component of \( T_\ell \). To illustrate the result, we consider the case \( \ell = p \) in the introduction.

**Proposition 2.** Let \( \chi \) be a Dirichlet character mod \( p \) with \( \chi(-1) = (-1)^k \). We have the following commutative diagram:

\[
\begin{array}{ccc}
S_k(-N_{K/Q}) & \xrightarrow{\varphi_\chi} & S_k(\Gamma_0(p^2), \chi \otimes \chi_p) \\
\text{theta contr.} & & \text{G \mapsto } (G(4p\tau) \cdot \vartheta(\tau))|U_p \\
\downarrow & & \downarrow \\
S_{k+1/2}((-2\ell)) & \xrightarrow{\varphi_\chi} & S_{k+1/2}(\Gamma_0(p^2), \chi) \\
\end{array}
\]

where \( \vartheta(\tau) = \sum_{n \geq 2} q^{n^2} \) is the Jacobi theta function and \( U_p \) is the usual Hecke operator acting on Fourier expansions by \( (\sum_n c(n)q^n)|U_p = \sum_n c(pm)q^n \).

We refer to Proposition 11 for the general statement and its proof. We also give two numerical examples illustrating the use of the above proposition in Section 4.

The work is organized as follows. We start with preliminaries about modular forms for the Weil representation associated to a finite quadratic module. In Section 3, we investigate twisted component sums of vector-valued modular forms and obtain isomorphisms between spaces of vector-valued and scalar-valued modular forms in the case that the underlying finite quadratic module has order \( p \) or \( 2p \), with an odd prime \( p \). Finally, in Section 4, we explain how these isomorphisms can be used to compute restrictions of Doi-Naganuma lifts of odd weight to components of Hirzebruch-Zagier curves.

2. **Modular forms for the Weil representation**

A **finite quadratic module** \((A, Q)\) consists of a finite abelian group \( A \) and a nondegenerate \( \mathbb{Q}/\mathbb{Z} \)-valued quadratic form \( Q \) on it. The **signature** of \((A, Q)\) is the number \( \text{sig}(A, Q) \in \mathbb{Z}/8\mathbb{Z} \) defined through the Gauss
sum of $\mathcal{A}$ by
\begin{equation}
\varepsilon(\text{sign}(A,Q)/8) = \frac{1}{\sqrt{|A|}} \sum_{\gamma \in A} \varepsilon(Q(\gamma)),
\end{equation}
where $\varepsilon(x) = e^{2\pi ix}$. By Milgram’s formula ([7], appendix 4), this is also the signature mod 8 of any even lattice which induces $(A,Q)$ as its discriminant form.

Let $\mathbb{C}[A]$ be the group algebra of $A$ with basis $e_{\alpha}$, $\gamma \in A$, and let $\text{Mp}_2(\mathbb{Z})$ be the integral metaplectic group, consisting of pairs $\left( \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \pm \sqrt{ct+d} \right)$ with $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$. The dual Weil representation $\rho^\ast$ is a unitary representation of $\text{Mp}_2(\mathbb{Z})$ on $\mathbb{C}[A]$ which is defined on the generators $S = \left( \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \sqrt{\gamma} \right)$ and $T = \left( \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, 1 \right)$ by
\begin{equation}
\rho^\ast(T)e_{\gamma} = \varepsilon(-Q(\gamma))e_{\gamma}, \quad \rho^\ast(S)e_{\gamma} = \frac{\varepsilon(\text{sign}(A,Q)/8)}{\sqrt{|A|}} \sum_{\beta \in A} \varepsilon((\beta,\gamma))e_{\beta},
\end{equation}
where $\langle \beta,\gamma \rangle = Q(\beta + \gamma) - Q(\beta) - Q(\gamma)$ is the bilinear form associated to $Q$. We also write $\rho^\ast_A$ if we want to emphasize the dependence on $A$, or $\rho^\ast_{A,p}$ if $A$ is the discriminant form of an even lattice $\Lambda$.

A function $F : \mathbb{H} \rightarrow \mathbb{C}[A]$ is called a weakly holomorphic modular form of weight $k \in \frac{1}{2}\mathbb{Z}$ for $\rho^\ast$ if it is holomorphic on $\mathbb{H}$, if it satisfies
\begin{equation}
F \left( \begin{pmatrix} a\tau + b \\ c\tau + d \end{pmatrix} \right) = (ct+d)^k \rho^\ast \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \sqrt{ct+d} \right) F(\tau)
\end{equation}
for all $\left( \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \sqrt{ct+d} \right) \in \text{Mp}_2(\mathbb{Z})$, and if it is meromorphic at $\infty$, which means that it has a Fourier expansion of the form
\begin{equation}
F(\tau) = \sum_{\gamma \in A} \sum_{n \in \mathbb{Z} - Q(\gamma)} c(n,\gamma)q^n e_{\gamma},
\end{equation}
with coefficients $c(n,\gamma) \in \mathbb{C}$ and $q = e^{2\pi i \tau}$. Following [2], we will denote the space of all these functions by $A_k(\rho^\ast)$ (instead of the more common $M^!_k(\rho^\ast)$). We let $M_k(\rho^\ast)$ and $S_k(\rho^\ast)$ be the subspaces of holomorphic modular forms and cusp forms, respectively. If $A$ is the discriminant form of an even lattice $\Lambda$, then we also write $A_k(Q)$ or $A_k(S)$ for $A_k(\rho^\ast)$, where $Q$ is the quadratic form on $\Lambda$ and $S$ is the Gram matrix of $Q$ with respect to some basis of $\Lambda$.

The element $Z = (-1,i) = S^2$ acts by $\rho^\ast(Z)e_{\gamma} = (-1)^{\text{sign}(A,Q)/2} e_{-\gamma}$, which implies that $A_k(\rho^\ast) = 0$ if $k + \text{sign}(A,Q)/2$ is not integral, and that the components of any weakly holomorphic modular form $F = \sum_{\gamma \in A} F_\gamma e_{\gamma} \in A_k(\rho^\ast)$ satisfy
\begin{equation}
F_\gamma = (-1)^{k + \text{sign}(A,Q)/2} F_{-\gamma}
\end{equation}
for all $\gamma \in A$. Therefore we refer to $k$ as a symmetric or antisymmetric weight if $k + \text{sign}(A,Q)/2$ is respectively even or odd.

3. Vector-valued and scalar-valued modular forms

In this section, we give isomorphisms between spaces $A_k(\rho^\ast)$ of vector-valued modular forms for $\rho^\ast$ and scalar-valued modular forms for $\Gamma_0(p^2)$ and $\Gamma_0(4p^2)$ in the cases $|A| = p$ and $|A| = 2p$ with an odd prime $p$, for both symmetric and antisymmetric weights $k \in \frac{1}{2}\mathbb{Z}$.

3.1. Finite quadratic modules of order $p$. Suppose that $|A| = p$ is an odd prime. Then $A \cong \mathbb{Z}/p\mathbb{Z}$ with $Q(\gamma) = \alpha \gamma^2/p$ for some $\alpha \in \mathbb{Z}$ with $p \nmid \alpha$. We put $\varepsilon = \chi_p(\alpha) = \left( \begin{pmatrix} \alpha \\ p \end{pmatrix} \right)$, and for odd $d \in \mathbb{Z}$ we let
\begin{equation}
\varepsilon_d = \begin{cases} 1, & p \equiv 1 \pmod{4}, \\
\iota, & p \equiv 3 \pmod{4}. \end{cases}
\end{equation}
The evaluation of the quadratic Gauss sum $\sum_{n(p)^d} \varepsilon(\alpha n^2/p) = \chi_p(\alpha)\varepsilon_p\sqrt{p}$ and Milgram’s formula (1) show that $\varepsilon\varepsilon_p = \varepsilon(\text{sign}(A,Q)/8)$. Thus the signature $\text{sign}(A,Q) \in \mathbb{Z}/8\mathbb{Z}$ depends on $p$ and $\varepsilon$ as shown in the following
table:

| $p \pmod{4}$ | 1 | 3 |
|-------------|---|---|
| $\epsilon = +1$ | 0 | 2 |
| $\epsilon = -1$ | 4 | 6 |

In particular, $\text{sig}(A, Q)$ is even. Hence we can assume that $k$ is an integer since otherwise $A_k(\rho^*) = 0$.

Let $\chi$ be a Dirichlet character mod $p$ and let $A_k(p^2, \chi \otimes \chi_p)$ be the space of scalar-valued weakly holomorphic modular forms of weight $k$ for $\Gamma_0(p^2)$ with character $\chi \otimes \chi_p$. We assume that

$$\chi(-1) = (-1)^{k+\text{sig}(A, Q)/2}$$

so that otherwise $A_k(p^2, \chi \otimes \chi_p) = 0$. We define the subspace

$$A_k^s(p^2, \chi \otimes \chi_p) = \left\{ \sum_{n \in \mathbb{Z}, n \gg -\infty} c(n)q^n \in A_k(p^2, \chi \otimes \chi_p) : \begin{cases} c(n) = 0 & \text{unless } n \in p(\mathbb{Z} - Q(\gamma)) \\ \text{for some } \gamma \in (\mathbb{Z}/p\mathbb{Z})^* \end{cases} \right\}.$$

The condition in the brackets can be restated by saying that $c(n) = 0$ unless $\chi_p(-n) = \epsilon$. Hence we also call it the $\epsilon$-condition. Note that, in contrast to the definition of the $\epsilon$-condition in [2], we also require that $c(n) = 0$ if $p \mid n$.

We define the twisted component sum of a vector-valued modular form $F(\tau) = \sum_{\gamma(p)} F_\gamma(\tau) \epsilon_\gamma \in A_k(\rho^*)$ by

$$\varphi_\epsilon \left( \sum_{\gamma(p)} F_\gamma(\tau) \epsilon_\gamma \right) = \sum_{\gamma(p)^*} \chi(\gamma) F_\gamma(p\tau).$$

The parity condition (2) ensures that $\varphi_\epsilon(F)$ is not trivially identically zero. Again, also for symmetric weight $k$ and with $\chi$ the trivial character mod $p$, our twisted component sum differs from the component sum $\sum_{\gamma(p)} F_\gamma(p\tau)$ considered in [2] since we omit the zero component $F_0(p\tau)$ in the sum. For antisymmetric weight $k$ we have $F_0 = 0$.

**Proposition 3.** If $F \in A_k(\rho^*)$, then $\varphi_\epsilon(F) \in A_k^s(p^2, \chi \otimes \chi_p)$. Furthermore, $\varphi_\epsilon$ is injective.

**Proof.** Let $M = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \Gamma_0(p^2)$ and $N = \left( \begin{smallmatrix} a & bp \\ c/p & d \end{smallmatrix} \right)$. We write

$$\varphi_\epsilon(F)(M\tau) = \sum_{\gamma(p)^*} \chi(\gamma) F_\gamma(p \cdot M \tau) = \sum_{\gamma(p)^*} \chi(\gamma) F_\gamma(N(p\tau)).$$

By [1], Theorem 5.2, we have $\rho^*(N)\epsilon_\gamma = \chi_p(d)\epsilon_{d\gamma}$, hence

$$F_\gamma(N(p\tau)) = \chi_p(d)(ct + d)^k F_{d-1, \gamma}(p\tau),$$

where $d^{-1}$ denotes an inverse of $d \pmod{p}$. Thus we find

$$\varphi_\epsilon(F)(M\tau) = \chi_p(d)(ct + d)^k \sum_{\gamma(p)^*} \chi(\gamma) F_{d-1, \gamma}(p\tau) = \chi_p(d)(ct + d)^k \varphi_\epsilon(F)(\tau).$$

It is clear that $\varphi_\epsilon(F)$ is holomorphic on $\mathbb{H}$ and meromorphic at the cusps (since $F_\gamma|_k M$ is a linear combination of components of $F$), and that it satisfies the $\epsilon$-condition, so $\varphi_\epsilon(F) \in A_k^s(p^2, \chi \otimes \chi_p)$.

Now suppose that $\varphi_\epsilon(F) = 0$ for some $F \in A_k(\rho^*)$. Since the components $F_\gamma, F_\beta$ for $\beta \neq \pm \gamma$ are supported on disjoint index sets, $\varphi_\epsilon(F) = 0$ implies that $F_\gamma = 0$ for all $\gamma \neq 0$. But then the zero component of $F$ satisfies

$$F_0|_k S = \frac{\epsilon(\text{sig}(A, Q)/8)}{\sqrt{p}} \sum_{\gamma(p)} F_\gamma = \frac{\epsilon(\text{sig}(A, Q)/8)}{\sqrt{p}} F_0,$$

and applying $|_k S$ a second time, we find $(-1)^k F_0 = \frac{\epsilon(\text{sig}(A, Q)/4)}{p} F_0$, hence $F_0 = 0$. We have shown that $F = 0$, so $\varphi_\epsilon$ is injective. \qed
The map \( \varphi_\chi \) is in general not surjective. Its image can be described in terms of the behaviour of certain twists of \( G \) under the Atkin-Lehner involution, which we explain now.

We can split \( G(\tau) = \sum_{n \gg -\infty} G(n)q^n \in A_k^\prime(p^2, \chi \otimes \chi_p) \) into components

\[
G(\tau) = \sum_{\gamma(p)^*} G_\gamma(\tau), \quad G_\gamma(\tau) = \frac{1}{2} \sum_{n \in \mathbb{Z} - Q(\gamma)} c(n)q^n.
\]

We define the **component-wise twist** of \( G \) by a Dirichlet character \( \psi \) mod \( p \) by

\[
G_\psi(\tau) = \sum_{\gamma(p)^*} \psi(\gamma)G_\gamma(\tau).
\]

Note that the component-wise twist differs from the usual twist \( \sum_{n \gg -\infty} \psi(n)c(n)q^n \) of a modular form.

**Lemma 4.** Let \( G \in A_k^\prime(p^2, \chi \otimes \chi_p) \) and let \( \psi \) be a Dirichlet character mod \( p \). Then \( G_\psi \in A_k^\prime(p^2, \psi \otimes \chi \otimes \chi_p) \).

**Proof.** We can write

\[
G_\psi = \frac{1}{2p} \sum_{\gamma(p)^*} \psi(\gamma) \sum_{j(p)} G\bigg|_k \left( \begin{array}{cc} 1 & j/p \\ 0 & 1 \end{array} \right) e(jQ(\gamma)).
\]

Let \( M = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma_0(p^2) \). We compute

\[
\left( \begin{array}{cc} 1 & j/p \\ 0 & 1 \end{array} \right) \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \left( \begin{array}{cc} 1 & -d^2j/p \\ 0 & 1 \end{array} \right) = \left( \begin{array}{cc} a + cj/p & b + (1 - ad)j/p - d^2cj^2/p^2 \\ c & d - d^2cj/p \end{array} \right) \in \Gamma_0(p^2).
\]

Note that the \( d \)-entry of this matrix equals \( d \) mod \( p \). Hence we obtain

\[
G_\psi |_k M = \chi_\chi_p(d) \frac{1}{2p} \sum_{\gamma(p)^*} \psi(\gamma) \sum_{j(p)} G\bigg|_k \left( \begin{array}{cc} 1 & d^2j/p \\ 0 & 1 \end{array} \right) e(jQ(\gamma)).
\]

Replacing \( d^2j \) by \( j \) and then \( \gamma \) by \( d\gamma \) gives a factor \( \psi(d) \) and completes the proof. \( \square \)

We let \( W_{p^2} = \left( \begin{array}{cc} 0 & -1 \\ p^2 & 0 \end{array} \right) \) be the Atkin-Lehner (or Fricke) involution. It maps \( A_k(p^2, \chi \otimes \chi_p) \) to \( A_k(p^2, \chi \otimes \chi_p) \), but it does in general not respect the \( \epsilon \)-condition. We say that \( G \in A_k^\prime(p^2, \chi \otimes \chi_p) \) satisfies the **Atkin-Lehner condition** if the twists of \( G \) by all Dirichlet characters \( \psi \) mod \( p \) with \( \psi \neq \overline{\chi} \) satisfy

\[
(3) \quad G_\psi |_{W_{p^2}} = \psi(2\alpha) \frac{g(\psi|\chi)}{\sqrt{p}} e(\text{sign}(A, Q)/8) G_{\overline{\chi}p^2},
\]

where \( g(\psi|\chi) = \sum_{n(p)^*} \psi|\chi(n) e(n/p) \) is a Gauss sum. We let \( A_k^{AL}(p^2, \chi \otimes \chi_p) \) be the subspace of \( A_k^\prime(p^2, \chi \otimes \chi_p) \) consisting of all forms satisfying the Atkin-Lehner condition.

**Proposition 5.** The linear map

\[
\varphi_\chi : A_k(p^*) \to A_k^{AL}(p^2, \chi \otimes \chi_p), \quad F(\tau) = \sum_{\gamma(p)^*} F_\gamma(\tau) \epsilon_\gamma \mapsto \sum_{\gamma(p)^*} \chi_\gamma F_\gamma(\tau\gamma),
\]

is an isomorphism. The inverse map is given by

\[
\varphi_\chi^{-1} : G(\tau) = \sum_{\gamma(p)^*} G_\gamma(\tau) \mapsto \sum_{\gamma(p)^*} \overline{\chi}\gamma G_\gamma(\tau/p) \epsilon_\gamma + G_0(\tau/p) \epsilon_0,
\]

where \( G_0 \) is defined by

\[
G_0(\tau) = \frac{\sqrt{p}}{p-1} e(-\text{sign}(A, Q)/8) \left( G_{\overline{\chi}|_W(p^*)} \right)(\tau) + \frac{1}{p-1} G_{\tau}.
\]
Let \( \psi \) by \( \mathrm{sig}(A,Q) \) for all \( \gamma \). This shows that \( F \in \mathbb{Z} \). Furthermore, it is easy to check that the third line agree for all Dirichlet characters \( \psi \). For \( \gamma = 1 \) if \( \psi = \bar{\chi} \) and \( \delta_{\psi,\bar{\chi}} = 0 \) otherwise. We compute

\[
(G_{\psi} | W_{p^2}) (\tau) = p^k (p^2 \tau)^{-k} G_\psi \left( -\frac{1}{p^2 \tau} \right)
\]

\[
= \sum_{\gamma(p)^*} \psi(\gamma) \chi(\gamma) (p\tau)^{-k} F_\gamma \left( -\frac{1}{p\tau} \right)
\]

\[
= \sum_{\gamma(p)^*} \psi(\gamma) \chi(\gamma) \frac{e(\mathrm{sig}(A,Q)/8)}{\sqrt{p}} \sum_{\beta(p)^*} e((\beta,\gamma)) F_\beta(p\tau)
\]

\[
= \frac{e(\mathrm{sig}(A,Q)/8)}{\sqrt{p}} \left( \sum_{\gamma(p)^*} \psi(\gamma) \chi(\gamma) e(2\alpha \beta \gamma/p) \right) F_\beta(p\tau) + \delta_{\psi,\bar{\chi}}(p-1) F_0(p\tau)
\]

\[
= \frac{e(\mathrm{sig}(A,Q)/8)}{\sqrt{p}} \left( \psi(\gamma) (2\alpha) g(\psi(\chi) G_{\psi,\bar{\chi}}^2(\tau) + \delta_{\psi,\bar{\chi}}(p-1) F_0(p\tau)) \right).
\]

This shows that \( G \) satisfies the Atkin-Lehner condition, i.e., \( G \in A_k^{\mathrm{AL}}(p^2, \chi \otimes \chi_p) \).

Conversely, if \( G \in A_k^{\mathrm{AL}}(p^2, \chi \otimes \chi_p) \) and if \( G_0 \) is defined as in the proposition, then we can reverse the above computation (with \( F_0(p\tau) = G_0(\tau) \) and \( F_\gamma(p\tau) = \chi(\gamma) G_\gamma(\tau) \) for \( \gamma \neq 0 \)) to see that the second and third line agree for all Dirichlet characters \( \psi \) mod \( p \). By character orthogonality, we obtain that

\[
F_\gamma | k S = \frac{e(\mathrm{sig}(A,Q)/8)}{\sqrt{p}} \sum_{\beta(p)^*} e((\beta,\gamma)) F_\beta
\]

for all \( \gamma \neq 0 \). A short computation shows that this equation also implies

\[
F_0 | k S = \frac{e(\mathrm{sig}(A,Q)/8)}{\sqrt{p}} \sum_{\beta(p)^*} F_\beta.
\]

Furthermore, it is easy to check that \( F_\gamma \) transforms correctly under \( T \). We find that \( \varphi \chi^{-1}(G) \in A_k(\rho^*) \). Since \( \varphi \chi \circ \varphi \chi^{-1} = \text{id} \) and \( \varphi \chi \) is injective, \( \varphi \chi \) is an isomorphism. \( \square \)

3.2. **Finite quadratic modules of order** \( 2p \). Suppose that \( |A| = 2p \) with an odd prime \( p \). Then \( A \cong \mathbb{Z}/2p\mathbb{Z} \) with the quadratic form

\[
Q(p\gamma_1 + \gamma_2) = \delta \gamma_1^2 / 4 + \alpha \gamma_2^2 / p
\]

for \( \gamma_1 \in \mathbb{Z}/2\mathbb{Z} \) and \( \gamma_2 \in \mathbb{Z}/p\mathbb{Z} \), where \( \delta \in \{ \pm 1 \} \) and \( \alpha \in \mathbb{Z} \) with \( p \div \alpha \). Set \( \epsilon = \chi_p(\alpha) \). Using the quadratic Gauss sum and Milgram’s formula we obtain \( (1 + \delta i) \epsilon_p = \sqrt{2} e(\mathrm{sig}(A,Q)/8) \), so the signature \( \mathrm{sig}(A,Q) \) in \( \mathbb{Z}/8\mathbb{Z} \) is given in terms of \( p, \epsilon \) and \( \delta \) as follows:

\[
\begin{array}{c||c|c|c}
 p \pmod{4} & 1 & 3 \\
 \hline
 \epsilon = +1 & \delta & \delta + 2 \\
 \epsilon = -1 & \delta + 4 & \delta + 6 \\
\end{array}
\]

Now \( \mathrm{sig}(A,Q) \) is odd. Hence we can assume that \( k \) is half-integral since otherwise \( A_k(\rho^*) = 0 \).

Let us briefly recall the definition of modular forms of half-integral weight. The theta multiplier is given by

\[
\nu_\varphi(M) = \left( \frac{c}{d} \right)^{1/2} \varepsilon_d^{-1}, \quad \varepsilon_d = \left( \frac{2}{d} \right) e((1-d)/8) = \begin{cases} 1, & d \equiv 1(4), \\ i, & d \equiv 3(4), \end{cases}
\]

for \( M = \left( \frac{a}{b} \right) \in \Gamma_0(4) \). A function \( G : \mathbb{H} \rightarrow \mathbb{C} \) is called a weakly holomorphic modular form of weight \( k \in \frac{1}{2} + \mathbb{Z} \) for \( \Gamma_0(N) \) with \( 4 \mid N \) and character \( \chi \mod N \) if it is holomorphic on \( \mathbb{H} \) and meromorphic at the
By quadratic reciprocity we have
\[ G(M) = \chi(M)\nu(M)^{\pm 1}(c\tau + d)^k G(\tau) \]
for \( M = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma_0(N) \), where the sign in \( \nu(M)^{\pm 1} \) is chosen such that \( \nu(-1)^{\pm 1} \chi(-1)i^{2k} = 1 \). We denote the space of all these functions by \( A_k(N, \chi) \).

We let \( \chi \) be a character mod \( p \) and we again assume that
\[ \chi(-1) = (-1)^{k + \text{sign}(A, Q)/2}. \]
We consider the space
\[ A_k^e(16p^2, \chi) = \left\{ \sum_{n = -\infty}^\infty c(n)q^n \in A_k(16p^2, \chi) : \begin{array}{l} c(n) = 0 \text{ unless } n \in 4p(\mathbb{Z} - Q(\gamma)) \\ \text{for some } \gamma \in (\mathbb{Z}/2p\mathbb{Z}) \setminus \{0, p\} \end{array} \right\} \]
and we let \( A_k^e(4p^2, \chi) = A_k^e(16p^2, \chi) \cap A_k(4p^2, \chi) \).

We define the twisted component sum of \( F(\tau) = \sum_{\gamma(2p)} F_\gamma(\tau) \in A_k(\rho^*) \) by
\[ \varphi_\chi \left( \sum_{\gamma(2p)} F_\gamma(\tau)e_\gamma \right) = \sum_{\gamma(2p)} \chi(\gamma)F_\gamma(4p\tau). \]
Note that, since \( \chi \) is a character mod \( p \), we discard the components \( F_0 \) and \( F_p \) in the twisted component sum. If \( k \) is an antisymmetric weight, then \( F_0 = F_p = 0 \) is automatic. This map was already suggested in [3], p. 70, in the context of Jacobi forms.

**Proposition 6.** If \( F \in A_k(\rho^*) \), then \( \varphi_\chi(F) \in A_k^e(4p^2, \chi) \). Furthermore, \( \varphi_\chi \) is injective.

**Proof.** We first show that that \( \varphi_\chi(F) \) transforms correctly under \( \Gamma_0(16p^2) \). For \( M = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma_0(16p^2) \) we let \( N = \left( \begin{array}{cc} c & 4p^b \\ e & 4p^d \end{array} \right) \). Then we compute
\[ \varphi_\chi(F)(M\tau) = \sum_{\gamma(2p)} \chi(\gamma)F_\gamma(4p \cdot M\tau) = \sum_{\gamma(2p)} \chi(\gamma)F_\gamma(N(4p\tau)). \]
Using [1], Theorem 5.2, we obtain
\[ F_\gamma(N(4p\tau)) = \left( \frac{c/4p}{d} \right) \left( \frac{d}{2p} \right) \nu((1 - d)\delta/8)(c\tau + d)^k F_{d - 1, \gamma}(4p\tau). \]
We compute
\[ \left( \frac{c/4p}{d} \right) \left( \frac{d}{2p} \right) \nu((1 - d)\delta/8) = \begin{cases} 1, & \text{if } p \equiv 1 \pmod{4} \text{ and } \delta \equiv 1 \pmod{4}, \\ -1, & \text{if } d \equiv 3 \pmod{4}. \end{cases} \]
By quadratic reciprocity we have \( \left( \frac{p}{d} \right) \left( \frac{d}{p} \right) = 1 \) if \( p \equiv 1 \pmod{4} \) and
\[ \left( \frac{p}{d} \right) \left( \frac{d}{p} \right) = \begin{cases} 1, & \text{if } d \equiv 1 \pmod{4}, \\ -1, & \text{if } d \equiv 3 \pmod{4}. \end{cases} \]
if \( p \equiv 3 \pmod{4} \). This gives the stated transformation behaviour under \( \Gamma_0(16p^2) \).

In order to show the transformation behaviour under \( \Gamma_0(4p^2) \), it suffices to check the transformation under the matrices \( U_{4p^2,j} = \left( \begin{array}{cc} 1 & 0 \\ 4p^2j & 1 \end{array} \right) \) for \( j = 0, 1, 2, 3 \) since they represent \( \Gamma_0(16p^2) \setminus \Gamma_0(4p^2) \). We compute
\[ \varphi_\chi(F)(U_{4p^2,j}\tau) = \sum_{\gamma(2p)} \chi(\gamma)F_\gamma(4p \cdot U_{4p^2,j}\tau) = \sum_{\gamma(2p)} \chi(\gamma)F_\gamma(U_{4p,j}(4p\tau)) = \sum_{\gamma(2p)} \chi(\gamma)F_\gamma(S^{-1}T^{-pj}S(4p\tau)) = (4p^2j\tau + 1)^k \sum_{\gamma(2p)} \chi(\gamma) \sum_{\beta(2p)} e(-\beta, \gamma) e(jpQ(\beta)) \sum_{\mu(2p)} e((\mu, \beta)) F_\mu(4p\tau). \]
If we write $\gamma = p\gamma_1 + \gamma_2$ with $\gamma_1 \in \mathbb{Z}/2\mathbb{Z}$ and $\gamma_2 \in \mathbb{Z}/p\mathbb{Z}$, and similarly for $\beta$, and use that $\chi$ only depends on $\gamma_2$, we see that the sum over $\gamma_1$ vanishes unless $\beta_2 = 0$. This means that we can replace $e(jpQ(\beta))$ by 1 in the above sum. But then the sum over $\beta$ equals $2p$ if $\mu = \gamma$, and vanishes otherwise. Hence we get
\[
\varphi_\chi(F)(U_{4p^2}) = (4p^2j^2 + 1)^k \sum_{\gamma(2p)} \chi(\gamma)F_\gamma(j^2 + 1)^k\varphi_\chi(F)(\gamma).
\]

This shows that $\varphi_\chi(F)$ transforms correctly under $\Gamma_0(4p^2)$. It is easy to see that $\varphi_\chi$ is holomorphic on $\mathbb{H}$ and meromorphic at the cusps, and that it satisfies the $c$-condition.

Now suppose that $\varphi_\chi(F) = 0$ for some $F \in \mathcal{A}_k(p^*)$. By comparing the index sets on which the components $F_\gamma$ are supported, we obtain that $F_\gamma = 0$ for $\gamma \notin \{0, p\}$. Then the transformation behaviour of $F$ under $S$ implies
\[
F_0(-1/\tau) = \tau^k \frac{e(\text{sig}(A,Q)/8)}{\sqrt{2p}} (F_0(\tau) + F_p(\tau)), \quad F_p(-1/\tau) = \tau^k \frac{e(\text{sig}(A,Q)/8)}{\sqrt{2p}} (F_0(\tau) - F_p(\tau)).
\]
Applying $\tau \mapsto -1/\tau$ a second time, we get $i^{2k}F_0 = \frac{e(\text{sig}(A,Q)/4)}{p} F_0$ and $i^{2k}F_p = \frac{e(\text{sig}(A,Q)/4)}{p} F_p$, hence $F_0 = F_p = 0$. Thus $F = 0$ and $\varphi_\chi$ is injective.

We split $G = \sum_{n > -\infty} c(n)q^n \in \mathcal{A}_k(16p^2, \chi)$ into components
\[
G(\tau) = \sum_{\gamma(2p)} G_\gamma(\tau), \quad G_\gamma(\tau) = \frac{1}{2} \sum_{n \in \mathbb{Z}/(4p-\text{sig}(\gamma))} c(n)q^n,
\]
and define its component-wise twist by a Dirichlet character $\psi$ mod $p$ by
\[
G_\psi(\tau) = \sum_{\gamma(2p)} \psi(\gamma)G_\gamma(\tau).
\]

**Lemma 7.** Let $G \in \mathcal{A}_k(16p^2, \chi)$ and let $\psi$ be a Dirichlet character mod $p$. Then $G_\psi \in \mathcal{A}_k(16p^2, \psi \otimes \chi)$.

**Proof.** The proof is analogous to the proof of Lemma 4, so we leave the details to the reader. \hfill \square

In contrast to the case $|A| = p$ we need another notion to describe the image of $\varphi_\chi$. We call $\gamma \in \mathbb{Z}/2p\mathbb{Z}$ even if $4pQ(\gamma)$ is even, and odd if $4pQ(\gamma)$ is odd. The even and odd parts of $G \in \mathcal{A}_k(16p^2, \chi)$ are defined by
\[
G^\text{even}(\tau) = \sum_{\gamma\text{ even}} G_\gamma(\tau), \quad G^\text{odd}(\tau) = \sum_{\gamma\text{ odd}} G_\gamma(\tau).
\]
Note that taking the even and odd parts of $G$ commutes with component-wise twisting.

**Lemma 8.** If $G \in \mathcal{A}_k(16p^2, \chi)$ then $G^\text{even}, G^\text{odd} \in \mathcal{A}_k(16p^2, \chi)$ as well.

**Proof.** We can write
\[
G^\text{even}(\tau) = \frac{1}{2} (G(\tau) + G(\tau + 1/2)), \quad G^\text{odd}(\tau) = \frac{1}{2} (G(\tau) - G(\tau + 1/2)).
\]
For $\left(\begin{array}{cc} a & b \\ c & d \end{array}\right) \in \Gamma_0(16p^2)$ we have
\[
\left(\begin{array}{cc} 1 & 1/2 \\ 0 & 1 \end{array}\right) \left(\begin{array}{cc} a & b \\ c & d \end{array}\right) \left(\begin{array}{cc} 1 & -1/2 \\ 0 & 1 \end{array}\right) = \left(\begin{array}{cc} a + c/2 & b + (d-a)/2 - c/4 \\ c & d + c/2 \end{array}\right) \in \Gamma_0(16p^2),
\]
which easily implies $G(\tau + 1/2) \in \mathcal{A}_k(16p^2, \chi)$. This proves the lemma. \hfill \square

For $G \in \mathcal{A}_k(16p^2, \chi)$ we define the Atkin-Lehner involution
\[
G|_{16p^2} = (4p)^{-k}(-i\tau)^{-k}G(-1/16p^2\tau).
\]
Then $G|_k W_{16p^2} \in A_k(16p^2, \chi)$ and $G|_k W_{16p^2}|_k W_{16p^2} = G$. In general, the Atkin-Lehner involution does not preserve the $\epsilon$-condition. We say that $G \in A_k(16p^2, \chi)$ satisfies the **Atkin-Lehner condition** if its twists by all Dirichlet characters $\psi$ mod $p$ with $\psi \not\equiv \chi$ satisfy

$$G|_k W_{16p^2} = \psi\chi(2\alpha)\sqrt{2g(\psi\chi)} \frac{e(\text{sig}(A, Q)/8)}{\sqrt{p}} G_{\chi}^{G_{\chi}^{\text{even}}},$$

where $g(\psi\chi) = \sum_{n(p)} \psi\chi(n)e(n/p)$ is a Gauss sum. Let $A_k^{\text{AL}}(16p^2, \chi)$ be the subspace of $A_k(16p^2, \chi)$ satisfying the Atkin-Lehner condition. Note that, after applying $W_{16p^2}$ and a short calculation, the Atkin-Lehner condition also implies that

$$(G_{\chi}^{\text{even}} - G_{\chi}^{\text{odd}})|_k W_{16p^2} = \psi\chi(2\alpha)\sqrt{2g(\psi\chi)} \frac{e(\text{sig}(A, Q)/8)}{\sqrt{p}} G_{\chi}^{G_{\chi}^{\text{even}}},$$

for $\psi \not\equiv \chi$.

**Proposition 9.** The linear map

$$\varphi_{\chi} : A_k(\rho^*) \to A_k^{\text{AL}}(16p^2, \chi), \quad F(\tau) = \sum_{\gamma(2p)} F_{\gamma}(\tau)e_{\gamma} \mapsto \sum_{\gamma(2p)} \chi(\gamma) F_{\gamma}(4p\tau),$$

is an isomorphism. The inverse map is given by

$$\varphi_{\chi}^{-1} : G(\tau) = \sum_{\gamma(2p)} G_{\gamma}(\tau) \mapsto \sum_{\gamma(2p)} \chi(\gamma) G_{\gamma}(\tau/4p)e_{\gamma} + G_0(\tau/4p)e_0 + G_p(\tau/4p)e_p,$$

where $G_0$ and $G_p$ are defined by

$$G_0(\tau) = \frac{\sqrt{p}}{\sqrt{2(p-1)}} e(-\text{sig}(A, Q)/8)i^{-k(G_{\chi}^{\text{even}}(\tau) + G_{\chi}^{\text{odd}}(\tau))}|_k W_{16p^2} + \frac{1}{p-1} G_{\chi}^{\text{even}}(\tau),$$

$$G_p(\tau) = \frac{\sqrt{p}}{\sqrt{2(p-1)}} e(-\text{sig}(A, Q)/8)i^{-k(G_{\chi}^{\text{even}}(\tau) - G_{\chi}^{\text{odd}}(\tau))}|_k W_{16p^2} + \frac{1}{p-1} G_{\chi}^{\text{odd}}(\tau).$$

**Proof.** The proof is very similar to the proof of Proposition 5, so we omit it for brevity. \hfill \Box

4. **Application: The Doi-Naganuma lift and theta contraction**

Let $K = \mathbb{Q}(\sqrt{p})$ for a prime $p \equiv 1 \pmod{4}$ and let $\mathcal{O}_K$ be its ring of integers. Let $\mathcal{O}_K^\# = (1/\sqrt{p})\mathcal{O}_K$ be the dual lattice of $\mathcal{O}_K$ with respect to the trace and consider the finite quadratic module $(\mathcal{O}_K^\#/\mathcal{O}_K, -N_{K/Q})$. It has order $p$ and signature $0 \pmod{8}$. Let $\lambda \in \mathcal{O}_K$ be any totally positive prime with norm $\ell = N_{K/Q}(\lambda)$, and let $b \in \mathbb{Z}$ be any integer with $b^2 \equiv p \pmod{4\ell}$ (which exists by quadratic reciprocity). Since $\ell$ is prime, one of $(b + \sqrt{p})/2\lambda, (b - \sqrt{p})/2\lambda$ is integral; it is then straightforward to show that $\{\lambda, (b \pm \sqrt{p})/2\lambda\}$ is a $\mathbb{Z}$-basis of $\mathcal{O}_K$ and

$$S = \begin{pmatrix} -2\ell & -b \\ -b & \frac{-b^2}{2\ell} \end{pmatrix}$$

is the Gram matrix of $-N_{K/Q}$ in that basis.

If $\ell \neq p$, then for each $r \in \mathbb{Z}/p\mathbb{Z}$ there exists a unique $a \in \mathbb{Z}/2\ell\mathbb{Z}$ with $r \equiv ab \pmod{2\ell}$. We fix a bijection $\mathcal{O}_K^\# / \mathcal{O}_K \cong \mathbb{Z}/p\mathbb{Z}$ by sending $r \in \mathbb{Z}/p\mathbb{Z}$ to the element

$$\gamma_{a,r} + \mathcal{O}_K = \frac{a \pm r/\sqrt{p}}{2\lambda} + \mathcal{O}_K \in \mathcal{O}_K^\#/\mathcal{O}_K.$$

If $\ell = p$, then we fix the bijection which identifies $a \in \mathbb{Z}/p\mathbb{Z}$ with $\gamma_a + \mathcal{O}_K = \frac{a(1 + \sqrt{p})}{2\lambda} + \mathcal{O}_K$ instead. (In this case, one can always take $b = p$ and $\lambda = \varepsilon \sqrt{p}$ if $\varepsilon$ is the fundamental unit of $\mathcal{O}_K$.)

The **theta decomposition** identifies vector-valued modular forms for the dual Weil representation attached to $(\mathcal{O}_K^\#/\mathcal{O}_K, -N_{K/Q})$ with vector-valued Jacobi forms of fractional index $p/4\ell$ for the dual Weil representation attached to the discriminant form with Gram matrix $(-2\ell)$ and a particular representation of the Heisenberg group, see e.g. [9]. By setting the Heisenberg variable of those Jacobi forms equal to
zero one obtains the **theta contraction**, a graded homomorphism between the modular forms $M_\ast(-N_{K/Q})$ and $M_{\ast+1/2}((-2\ell))$ as graded modules over the ring $M_\ast(\text{SL}_2(\mathbb{Z}))$ of scalar-valued modular forms. This was introduced by Ma [6] in order to study the quasi-pullback of Borcherds products. Explicitly in terms of Fourier coefficients, it is the map

$$\Theta : M_\ast(-N_{K/Q}) \to M_{\ast+1/2}((-2\ell)), \quad \sum_{\gamma \in O_K^\ast/O_K} \sum_{n \in \mathbb{Z}^{+N_{K/Q}}(\gamma)} c(n,\gamma)q^n e_\gamma \to \sum_{a \in \mathbb{Z}/2\mathbb{Z}} \sum_{n \in \mathbb{Z}^{+a^2/4\ell}} \tilde{c}(n,a)q^n e_a,$$

where

$$\tilde{c}(n,a) = \sum_{r \equiv a (2\ell)} c\left(n - \frac{r^2}{4lp}, \gamma_{a,r}\right).$$

Possibly the most important aspect of the theta contraction (which can be defined more generally) is that it fits into a commutative diagram involving the additive theta lift (of Oda and Rallis-Schiffmann) and restriction to Heegner divisors: letting $\Lambda$ be an even lattice of type $(2, b)$ that it fits into a commutative diagram involving the additive theta lift (of Oda and Rallis-Schiffmann) and restriction to Heegner divisors: letting $\Lambda$ be an even lattice of type $(2, b)$ such that $b^- \geq 2$ is greater than the Witt rank of $\Lambda$, and $\lambda^\perp$ the orthogonal complement of a primitive, negative-norm vector $\lambda \in \Lambda$, the natural pullback map Res for orthogonal modular forms satisfies

$$S_{\ast+1-b^-/2}(\rho_\lambda^*) \xrightarrow{\text{Theta lift}} S_k(O(\Lambda)) \xrightarrow{\Theta} S_{\ast+1-(b^- - 1)/2}(\rho_{\lambda^\perp}^*) \xrightarrow{\text{Theta lift}} S_k(O(\lambda^\perp))$$

for $k \geq 2$.

For the Doi-Naganuma lift (i.e. $b^- = 2$) this can be made very explicit. Recall that for a cusp form $F(\tau) = \sum_{\gamma \in O_K^\ast/O_K} \sum_{n \in \mathbb{Z}^{+N_{K/Q}}(\gamma)} c(n,\gamma)q^n e_\gamma \in S_k(-N_{K/Q})$ of weight $k \geq 2$, the Doi-Naganuma lift is a Hilbert cusp form $\Phi_F$ of weight $k$ for $\text{SL}_2(O_K)$ with Fourier expansion

$$\Phi_F(\tau_1, \tau_2) = \sum_{\nu, \nu' = 1}^{\infty} c(\nu\nu', \nu)n^{k-1}q^{\nu}\nu', \quad \text{where } q^{\nu, \nu'} = e^{2\pi i(\nu\tau_1 + \nu'\tau_2)}.$$

Moreover $\Phi_F$ satisfies the graded symmetry $\Phi_F(\tau_1, \tau_2) = (-1)^k\Phi_F(\tau_2, \tau_1)$.

With $\lambda$ as above, there is a natural restriction map onto a component of the Hirzebruch-Zagier curve $T_\ell$: $\text{Res} : S_k(\text{SL}_2(O_K)) \to S_{2k}(\Gamma_0(\ell)), \quad f(\tau_1, \tau_2) \mapsto f(\lambda\tau, \lambda^\ast\tau)$.

It turns out that $\text{Res}(\Phi_F)$ equals the Shimura lift

$$\sum_{a=1}^{\infty} \sum_{n=1}^{\infty} \tilde{c}(a^2/4\ell, a)n^{k-1}q^{na}$$

of the contracted form $\Theta F = \sum_{a,n} \tilde{c}(n,a)q^n e_a \in S_{k+1/2}((-2\ell))$.

**Lemma 10.** We have the following commutative diagram:

$$\begin{align*}
S_k(-N_{K/Q}) & \xrightarrow{\text{Doi} - \text{Naganuma}} S_k(\text{SL}_2(O_K)) \\
\Theta & \downarrow \quad \text{Res} \\
S_{k+1/2}((-2\ell)) & \xrightarrow{\text{Shimura}} S_{2k}(\Gamma_0(\ell))
\end{align*}$$
Proof. Since the elements \( \nu \in \mathcal{O}_K^\# \) with \( \text{Tr}(\nu \lambda) = a \in \mathbb{N} \) are exactly those of the form \( \gamma_{a,r} = \frac{a \pm r \sqrt{\ell}}{2\lambda} \) with \( r \equiv ab \mod 2\ell \), we find

\[
\Phi_F(\lambda \tau, \lambda' \tau) = \sum_{n=1}^{\infty} \sum_{\nu \in \mathcal{O}_K} c(\nu \nu', \nu)n^{k-1}q^{n}\nu
\]

\[
= \sum_{n=1}^{\infty} \sum_{a=1}^{\infty} \sum_{r \equiv ab(2\ell)} c \left( \frac{a^2}{4\ell} - \frac{r^2}{4\ell}, \frac{a \pm r \sqrt{\ell}}{2\lambda} \right) n^{k-1}q^{n}\nu
\]

\[
= \sum_{n,a=1}^{\infty} \tilde{c}(a^2/4\ell, a)n^{k-1}q^{n}\nu,
\]

i.e. \( \text{Res}(\Phi_F)(\tau) = \Phi_F(\lambda \tau, \lambda' \tau) \) is the Shimura lift of the contracted form \( \Theta F \in S_{k+1/2}((-2\ell)) \).

In this section we observe that this relationship takes a simple form in terms of twisted component sums of \( F \) and \( \Theta F \). Recall that for a \( q \)-series \( f(\tau) = \sum_n c(n)q^n \) the Hecke operator \( U_p \) is defined by

\[
f(U_p(\tau)) = \sum_{n \equiv 0 (p)} c(n)q^n = \sum_{n \equiv 0 (p)} c(n)q^n/p.
\]

**Proposition 11.**

(i) Suppose \( \ell \neq p \). Let \( \psi_\ell \) and \( \psi_p \) be Dirichlet characters modulo \( \ell \) and \( p \) with \( \psi_\ell(-1) = \psi_p(-1) = (-1)^k \)

and let \( \chi \) be the Dirichlet character modulo \( \ell p \) defined by \( \chi(r) = \psi_\ell(r) \psi_p(r) \) for all \( r \in \mathbb{Z} \). By abuse of notation, let \( \psi_p \) denote the “character” on the cosets \( \mathcal{O}_K^\#/\mathcal{O}_K \) defined by setting \( \psi_p((a \pm r \sqrt{\ell})/2\lambda) = \psi_p(r) \) for any \( a, r \in \mathbb{Z} \) satisfying \( r \equiv ab \mod 2\ell \). Then

\[
\varphi_\psi(F)(\Theta F)(\tau) = \frac{1}{2\psi_\ell(b)} \cdot \left( \varphi_\psi(F)(4\ell \tau) \cdot \vartheta_\chi(\tau) \right) U_p
\]

where \( \vartheta_\chi(\tau) = \sum_{r \in \mathbb{Z}} \chi(r)q^{r^2} \) is the twisted Jacobi theta series.

(ii) Suppose \( \ell = p \), and let \( \psi_p \) be a Dirichlet character modulo \( p \) with \( \psi_p(-1) = (-1)^k \). By abuse of notation, define \( \psi_p \) on \( \mathcal{O}_K^\#/\mathcal{O}_K \) by \( \psi_p(a/\lambda) = \psi_p(2a), a \in \mathbb{Z}/p\mathbb{Z} \), where \( \lambda = \varepsilon \sqrt{p} \) and \( \varepsilon \) is the fundamental unit of \( \mathcal{O}_K \). Then

\[
\varphi_\psi(F)(\Theta F)(\tau) = \left( \varphi_\psi(F)(4p \tau) \cdot \vartheta(\tau) \right) U_p,
\]

where \( \vartheta(\tau) = \sum_{r \in \mathbb{Z}} q^{r^2} \).

**Proof.**

(i) Write \( f(\tau) = \varphi_\psi(F)(\tau) = \sum_{\gamma \in \mathcal{O}_K^\#/\mathcal{O}_K} \sum_{\gamma \in \mathbb{Z}^+\mathcal{O}_K/\mathcal{O}_K} \overline{\psi_p(\gamma)}c(n, \gamma)q^{rn} \). In the product

\[
f(4\ell \tau) \vartheta_\chi(\tau) = \left( \sum_{\gamma, n} \overline{\psi_p(\gamma)}c(n, \gamma)4\ell^n \right) \left( \sum_{r=-\infty}^{\infty} \chi(r)q^{r^2} \right)
\]

\[
= \sum_{\gamma, n, r} \psi_p(\gamma) \chi(r) c(n, \gamma) q^{4\ell^n + r^2},
\]

we get exponents which are divisible by \( p \) only when \( n \in \mathbb{Z} + a^2/4\ell \) and \( r \equiv \pm ab(2\ell) \) for some \( a \in \mathbb{N} \) (which is uniquely determined modulo \( 2\ell \)). In this case \( \gamma \in \gamma_{a,r} + \mathcal{O}_K \) with \( \gamma_{a,r} = \frac{a \pm r \sqrt{\ell}}{2\lambda} \) as before. Applying the \( U_p \) operator yields

\[
\left( f(4\ell \tau) \vartheta_\chi(\tau) \right) U_p = 2 \sum_{a \in \mathbb{Z} + \mathcal{O}_K} \sum_{n \in \mathbb{Z} + a^2/4\ell} \sum_{r \equiv ab(2\ell)} \overline{\psi_p(\gamma_{a,r})} \chi(r) c(n + r^2/4\ell, \gamma_{a,r}) q^{4\ell^n}
\]

\[
= 2 \sum_{a,n,r} \psi_\ell(ab)c(r^2/4\ell, \gamma_{a,r}) q^{4\ell^n}
\]

\[
= 2 \psi_\ell(b) \sum_{a, n} \psi_\ell(a) \overline{\psi_p(\gamma)} c(n, a) q^{4\ell^n}
\]

\[
= 2 \psi_\ell(b) \varphi_\psi(\Theta F)(\tau).
\]
(ii) This is proved similarly to part (i). We do not need to divide by two, since the sum over \( r \) runs through only one congruence class (namely, \( r \equiv \alpha p \pmod{2p} \)). The definition of \( \psi_p \) on \( \mathcal{O}_K^\# / \mathcal{O}_K \) is such that \( \psi_p(\gamma a) = \psi_p\left(\frac{a(1+p)}{2\lambda}\right) = \psi_p(a) \) for all \( a \in \mathbb{Z}/2p\mathbb{Z} \).

If we abbreviate

\[
\Theta \chi G = \left(G(4\ell\tau) \cdot \vartheta(\chi\tau)\right)|U_p
\]

for a scalar valued modular form \( G \), then the first item of the proposition (the case \( \ell \neq p \)) can be illustrated by the diagram

\[
\begin{array}{ccc}
S_k(-N_{K/Q}) & \xrightarrow{\varphi_{\psi_p}} & S^\tau_{k}(p^2, \psi_p \otimes \chi_p) \\
\Theta & & \Theta \\
S_{k+1/2}((-2\ell)) & \xrightarrow{\varphi_{\psi_p}} & S^\tau_{k+1/2}(p^2, \psi_p)
\end{array}
\]

which commutes up to a constant factor. Note that the horizontal arrows are isomorphisms.

**Example 12.** Let \( p = 5 \). Fix the element \( \lambda = 4 + \sqrt{5} \) of norm \( \ell = 11 \), and fix \( b = 7 \). We fix the Dirichlet characters \( \psi_{11} \) and \( \psi_5 \) by specifying \( \psi_{11}(2) = e^{\pi i/5} \) and \( \psi_5(2) = i \). Up to scalar multiples there is a unique cusp form of (antisymmetric) weight 5 for the dual Weil representation attached to \((\mathcal{O}_K, -N_{K/Q})\) with \( K = \mathbb{Q}(\sqrt{5}) \), and it is

\[
F(\tau) = \left(q^{1/5} + 42q^{6/5} - 108q^{11/5} + 4q^{16/5} - 378q^{21/5} \pm \ldots\right)(\epsilon_{3/\sqrt{5}} - \epsilon_{2/\sqrt{5}}) \\
+ \left(26q^{1/5} + 39q^{9/5} - 378q^{14/5} + 140q^{19/5} + 420q^{24/5} \pm \ldots\right)(\epsilon_{4/\sqrt{5}} - \epsilon_{1/\sqrt{5}}).
\]

One can compute \( F \) using, for example, the algorithm described in [8] (compare the example of Section 7 there); and after enough coefficients have been computed, one can identify its twisted component sum in \( S_5(\Gamma_1(25)) \) using standard methods for computing scalar-valued modular forms. The Doi-Naganuma lift of \( F \) is, up to a multiple, the well-known product \( s_5 \) of theta constants for \( \mathbb{Q}(\sqrt{5}) \) constructed by Gundlach ([5]; see also the example of Section 4 of [2]). The character \( \psi_5 \) on \( \mathcal{O}_K^\# / \mathcal{O}_K \) is defined such that e.g.

\[
\psi_5\left(1/\sqrt{5} + \mathcal{O}_K\right) = \psi_5\left(1 - 7/\sqrt{5} + 2\lambda + \mathcal{O}_K\right) = \psi_5(7) = i.
\]

Therefore the twisted component sum of \( F \) by \( \psi_5 \) is the cusp form

\[
\varphi_{\psi_5}(F)(\tau) = 2q + 52iq^4 + 84q^6 + 78iq^9 - 216q^{11} - 756iq^{14} - 8q^{16} \pm \ldots \in S_5(\Gamma_0(25), \psi_5 \otimes \chi_5) = S_5(\Gamma_0(25), \psi_5).
\]

After multiplying

\[
\varphi_{\psi_5}(F)(44\tau) \cdot \vartheta(\chi(\tau)) = \left(2q^{44} + 52iq^{76} \pm \ldots\right) \left(2q + 2\zeta_{20}q^4 - 2\zeta_{20}q^9 \pm \ldots\right) \\
= 4q^{45} + 4\zeta_{20}q^{48} - 4\zeta_{20}q^{53} - 4\zeta_{20}q^{60} \pm \ldots
\]

and applying \( U_5 \) we get the series

\[
4q^9 - 4\zeta_{20}q^{12} + 4\zeta_{20}q^{16} + 4\zeta_{20}q^{25} - 104\zeta_{20}q^{36} \pm \ldots
\]

Dividing by \( 2\psi_{11}(b) = -2\zeta_{20}^4 \) yields the twisted component sum of the theta contraction \( \Theta F \):

\[
\varphi_{\psi_{11}}(\Theta F)(\tau) = 2\zeta_{20}q^9 + 2q^{12} + 2\zeta_{20}q^{16} - 2\zeta_{20}q^{25} + 52\zeta_{20}q^{36} \pm \ldots
\]

From this we can read off the Shimura lift of the underlying vector-valued modular form \( \Theta F \): the coefficient of \( q^n \) is zero if \( 11|n \), and otherwise \( \sum_{d|n} \frac{1}{2\psi_{11}(d)} (n/d)^{5/2} \cdot (d^2) \) if \( c(n) \) is the coefficient of \( q^n \) in \( \varphi_{\psi_{11}}(\Theta F)(\tau) \), so

\[
s_5(\lambda\tau, \lambda'\tau) = -q^3 + q^4 + q^5 + 10q^6 - 10q^8 - 121q^9 + 98q^{10} + 275q^{12} + 32q^{13} + 140q^{14} \pm \ldots \in S_{10}(\Gamma_0(11)).
\]
Example 13. Let $p = 13$. Fix the totally positive element $\lambda = \frac{13+3\sqrt{13}}{2}$ of norm $\ell = 13$ and fix $b = 13$. We fix an odd Dirichlet character $\psi_{13}$ mod 13 by specifying $\psi_{13}(2) = \zeta_{12} = e^{\pi i/6}$. The dual Weil representation attached to $(\mathcal{O}_K, -N_{K/Q})$, $K = \mathbb{Q}(\sqrt{13})$ admits up to scalar multiples a unique cusp form of weight 3:

$$F(\tau) = (q^{1/13} - 33q^{14/13} + 27q^{27/13} + 33q^{40/13} \pm \ldots) (\epsilon_{1/\lambda} - \epsilon_{12/\lambda})$$

$$+ (3q^{3/13} + 5q^{16/13} + 42q^{29/13} - 99q^{42/13} \pm \ldots) (\epsilon_{4/\lambda} - \epsilon_{9/\lambda})$$

$$+ (-7q^{4/13} - 3q^{17/13} - 33q^{30/13} + 96q^{43/13} \pm \ldots) (\epsilon_{2/\lambda} - \epsilon_{11/\lambda})$$

$$+ (0q^{9/13} - 22q^{22/13} + 33q^{35/13} + 15q^{48/13} \pm \ldots) (\epsilon_{3/\lambda} - \epsilon_{10/\lambda})$$

$$+ (11q^{10/13} - 12q^{23/13} + 9q^{36/13} + 50q^{49/13} \pm \ldots) (\epsilon_{6/\lambda} - \epsilon_{7/\lambda})$$

$$+ (21q^{12/13} + 14q^{25/13} - 66q^{38/13} + 9q^{51/13} \pm \ldots) (\epsilon_{5/\lambda} - \epsilon_{8/\lambda}).$$

Under the Doi-Naganuma lift, $F$ is mapped to the cusp form $\omega_3$ used by van der Geer and Zagier to compute the ring of Hilbert modular forms for $\mathcal{O}_K$ ([4], Section 10). The twisted component sum of $F$ by $\psi_{13}$ is

$$\varphi_{\psi_{13}}(F)(\tau) = 2\zeta_{12}q + 6\zeta_{12}^3q^3 - 14\zeta_{12}^4q^4 - 22q^{10} - 42\zeta_{12}^4q^{12} \pm \ldots \in S_3(\Gamma_0(169), \psi_{13} \otimes \chi_{13}).$$

With this we can compute $\omega_3(\lambda\tau, \lambda'\tau)$ as follows: multiply

$$\varphi_{\psi_{13}}(F)(52\tau) \cdot \vartheta(\tau) = 2\zeta_{12}q^{52} + 4\zeta_{12}q^{53} + 4\zeta_{12}q^{56} \pm \ldots$$

and apply the Hecke operator $U_{13}$ to obtain

$$\varphi_{\psi_{13}}(\Theta F)(\tau) = 2\zeta_{12}q^2 + 6\zeta_{12}^3q^{12} - 14\zeta_{12}^4q^{16} + 4\zeta_{12}^4q^{17} + 12\zeta_{12}^4q^{25} \pm \ldots$$

and therefore the Shimura lift

$$\omega_3(\lambda\tau, \lambda'\tau) = q^2 - 3q^4 - 6q^5 + 9q^6 - q^8 + 6q^9 + 57q^{10} \pm \ldots \in S_6(\Gamma_0(13)).$$

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