A numerical method of Fourier transform based on hyperfunction theory

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Abstract

In this paper, we propose a numerical method of Fourier transform based on hyperfunction theory. In the proposed method, we compute analytic functions called the defining functions, which give the desired Fourier transform as a hyperfunction, and then obtain the Fourier transform by the analytic continuation of the defining functions onto the real axis. Numerical examples show the efficiency of the proposed method compared to the previous methods.

1 Introduction

Fourier transform

\[ \mathcal{F}[f](\xi) = \int_{-\infty}^{+\infty} f(x)e^{-2\pi i \xi x}dx \]  

(1)

is familiar in science and engineering. However, it is difficult to compute a Fourier transform by conventional numerical integration formulas such as the DE rule [10], especially, if the integrand \( f(x) \) decays slowly as \( x \to \pm \infty \). We propose a numerical method for computing Fourier transforms efficiently based on hyperfunction theory [8].

Hyperfunction theory is a theory of generalized functions based on complex analysis. Roughly speaking, a hyperfunction \( f(x) \) is the difference of the boundary values of an analytic function \( F(z) \), that is,

\[ f(x) = F(x + i0) - F(x - i0), \]

where \( F(z) \) is called a defining function of the hyperfunction \( f(x) \). For example, the Dirac delta function is defined by

\[
\delta(x) = -\frac{1}{2\pi i} \left( \frac{1}{x + i0} - \frac{1}{x - i0} \right),
\]

which coincides with an elementary definition of the delta function

\[
\delta(x) = \lim_{\epsilon \to 0^+} \frac{1}{\pi} \frac{\epsilon}{x^2 + \epsilon^2}
\]

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In hyperfunction theory, the Fourier transform of a function \(f(x)\) is defined by
\[
\mathcal{F}[f](\xi) = \int_{-\infty}^{0} f(x)e^{-2\pi i(\xi + i0)x}dx + \int_{0}^{\infty} f(x)e^{-2\pi i(\xi - i0)x}dx = \mathfrak{F}_+(\xi + i0) - \mathfrak{F}_-(\xi - i0), \tag{2}
\]
where
\[
\mathfrak{F}_+(\zeta) = \int_{-\infty}^{0} f(x)e^{-2\pi i\zeta x}dx \quad (\text{Im } \zeta > 0), \tag{3}
\]
\[
\mathfrak{F}_-(\zeta) = -\int_{0}^{\infty} f(x)e^{-2\pi i\zeta x}dx \quad (\text{Im } \zeta < 0). \tag{4}
\]
It means that the Fourier transform \(\mathcal{F}[f](\xi)\) is the hyperfunction whose defining function is
\[
\mathfrak{F}(\zeta) = \begin{cases} 
\mathfrak{F}_+(\zeta) & (\text{Im } \zeta > 0) \\
\mathfrak{F}_-(\zeta) & (\text{Im } \zeta < 0). \end{cases} \tag{5}
\]
In our method, we compute these defining functions \(F_{\pm}(\zeta)\) in \(\mathbb{C} \setminus \mathbb{R}\) instead of the integral on the right-hand side of (1), and we obtain the Fourier transform by (2) using the analytic continuation of \(\mathfrak{F}_{\pm}(\zeta)\) onto \(\mathbb{R}\). It is easy to compute the integrals on the right-hand sides of (3) and (4) because their integrands decay exponentially as \(x \to \pm \infty\). The analytic continuation of \(F_{\pm}(\zeta)\) is done using the continued fraction expansions.

Previous studies related to our paper are as follows. Toda and Ono proposed a method of computing Fourier integrals
\[
I = \int_{0}^{\infty} f(x) \cos(2\pi \xi x + \theta_0) dx, \tag{6}
\]
where \(\xi > 0\) and \(\theta_0\) is a constant, by evaluating the limit
\[
I = \lim_{n \to \infty} \int_{0}^{\infty} f(x) \cos(2\pi \xi x + \theta_0) \exp(-2^{-n}x)dx
\]
using the DE rule and the Richardson extrapolation \cite{TodaOno}. Sugihara improved Toda and Ono’s method and proposed a method by evaluating the limit
\[
I = \lim_{n \to \infty} \int_{0}^{\infty} f(x) \cos(2\pi \xi x + \theta_0) \exp(-2^{-n}x^2)dx,
\]
using the DE rule and the Richardson extrapolation \cite{Sugihara}. Ooura and Mori proposed a DE-type numerical integration rule for oscillatory integrals with unique technique \cite{OouraMori}. They apply the variable transformation
\[
x = \frac{1}{2\pi h} \varphi \left( u + \frac{h}{2} - \frac{h\theta_0}{2} \right),
\]
where
\[
\varphi(u) = \frac{u}{1 - \exp(-2\pi \sinh u)}
\]
and $h$ is a positive constant, to the integral \( \mathcal{I} \) and evaluate the transformed integral by the trapezoidal rule with mesh $h$, that is,

\[
\mathcal{I} \simeq \frac{1}{2\pi} \sum_{k = -N_2}^{N_1} f \left( \frac{1}{2\pi} h \varphi \left( \frac{k h + h \theta_0}{\pi} \right) \right) \cos \left( \frac{\pi}{h} \varphi \left( \frac{k h + h \theta_0}{2} \right) + \theta_0 \right) \varphi' \left( \frac{k h + h \theta_0}{2} \right),
\]

where $N_1$ and $N_2$ are so small positive integers that the transformed integrand is very small at $k = N_1$ and $-N_2$. In the method, we can truncate the infinite sum of the trapezoidal rule with a small number of terms $N_1 + N_2 + 1$ because the sampling points rapidly approach the zeros of the integrand on the positive side of the $u$-axis, while the conventional DE rule are designed so that we can truncate the trapezoidal rule with a small number of the sampling points by making the transformed integrand decays double exponentially.

As applications of hyperfunction theory to numerical analysis, Mori gave a theoretical analysis of numerical integration formulas based on hyperfunction theory and showed that Gauss-type integration formulas are obtained by approximating complex integrals which are defined as hyperfunction integrals \([5]\). The author proposed a numerical integration method based on hyperfunction theory \([6]\). In the paper, they obtain desired integrals by evaluating complex integral which defines them as hyperfunction integrals, and they shows the proposed method is efficient especially for integrals with strong end-point singularities.

The contents of the paper are as follows. In Section 2, we give a brief review of hyperfunction theory. In Section 3, we propose a numerical method based on hyperfunction theory for computing Fourier transforms. In Section 4, we give some numerical examples which show the effectiveness of the proposed method compared to the previous methods. In Section 5, we give concluding remarks and refer to problems for future study.

\section{Hyperfunction theory and numerical Fourier transform}

\subsection{Hyperfunctions}

We give a brief review of hyperfunction theory. For the detail of hyperfunctions, see \([2]\).

Let $I$ be an open interval on the real axis $\mathbb{R}$ and $D$ be a complex neighborhood of $I$, that is, a complex domain which includes $I$ as a closed subset. Let $\mathcal{O}(D \setminus I)$ and $\mathcal{O}(D)$ be the sets of all the holomorphic functions in $D \setminus I$ and $D$ respectively. The set $\mathcal{O}(D \setminus I)$ forms a complex linear space with the addition $F + G$ ($F, G \in \mathcal{O}(D \setminus I)$) and the scalar product $cF$ ($F \in \mathcal{O}(D \setminus I)$, $c \in \mathbb{C}$) respectively defined by

\[
(F + G)(z) := F(z) + G(z), \quad (cF)(z) := cF(z) \quad (z \in D \setminus I),
\]

and the set $\mathcal{O}(D)$ can be regarded as a linear subspace of $\mathcal{O}(D)$.

A hyperfunction $f = f(x)$ on the interval $I$ is an element (an equivalence
class) of the quotient subspace
\[ B(I) := O(D \setminus I)/O(D). \]
If a hyperfunction \( f \) is an equivalence class of a function \( F \in O(D \setminus I) \), we call \( F \) a defining function of \( f \), which is denoted by
\[ f = [F] \quad \text{or} \quad f(x) = [F(z)]. \]
We also denote a hyperfunction \( f = [F] \) by
\[ f = [F_+, F_-] \quad \text{or} \quad f(x) = [F_+(z), F_-(z)], \]
where \( F_\pm(z) \) is the restriction of \( F(z) \) on \( D_\pm := \{ z \in D \mid \pm \text{Im} z > 0 \} \), that is,
\[ F(z) = \begin{cases} 
F_+(z) & \text{Im} z > 0 \\
F_-(z) & \text{Im} z < 0.
\end{cases} \]
We often use the representation
\[ f(x) = F(x + i0) - F(x - i0) = F_+(x + i0) - F_-(x - i0). \]
If the limit
\[ \lim_{\epsilon \to 0^+} \{ F(x + i\epsilon) - F(x - i\epsilon) \} \]
exists for \( x \in I \), we define the value of the hyperfunction \( f = [F] \) at the point \( x \) by this limit, and we do not define the value of \( f \) if the limit does not exist.

We should remark that there is an ambiguity of the defining function \( F(z) \) of a hyperfunction \( f(x) \in B(I) \) up to a function belonging to \( O(D) \). We mean that, if \( F(z) \in O(D \setminus I) \) defines a hyperfunction \( f(x) \in B(I) \), a function \( F(z) + \varphi(z) \) with \( \varphi \in O(D) \) also belongs to \( O(D \setminus I) \) and defines the same hyperfunction \( f(x) \in B(I) \), that is,
\[ f(x) = [F(z)] = [F(z) + \varphi(z)]. \]

The addition \( (f + g)(x) \) of two hyperfunctions \( f(x) = [F(z)] \) and \( g(x) = [G(z)] \) are defined by
\[ (f + g)(x) := [F(z) + G(z)], \]
and the multiplication \( cf(x) \) of a hyperfunction \( f(x) = [F(z)] \) by a complex constant \( c \) is defined by
\[ cf(x) := [cF(z)]. \]

\footnote{As shown here, the behavior of the defining function \( F(z) \) in the vicinity of the interval \( I \) is crucial for the hyperfunction \( f = [F] \), and we do not need to cling to a particular choice of a complex neighborhood \( D \) of \( I \). Therefore, we should define the space of hyperfunctions on an interval \( I \) by the inductive limit
\[ B(I) := \lim_{D \supset I} O(D \setminus I)/O(D). \]
However, we do not go into the exact definition of \( B(I) \) any more, and we do not need to do so.}
The space $B(I)$ of hyperfunctions on an interval $I$ forms a complex linear space by the definitions of the addition and the scalar product. In addition, the multiplication $\varphi(x)f(x)$ of a hyperfunction $f(x) = [F(z)]$ by a real analytic function $\varphi(x)$, that is, a real valued function $\varphi(x)$ on an interval $I$ which can be extended to a complex function $\varphi(z)$ holomorphic on a complex neighborhood $D$ of $I$, is defined by

$$\varphi(x)f(x) := [\varphi(z)F(z)].$$

The above definitions are well-defined, that is, the definitions are not dependent on the choice of the defining functions of hyperfunctions. For example, if $f(x) = [F(z)] = [\tilde{F}(z)]$ and $g(x) = [G(z)] = [\tilde{G}(z)]$, we have

$$f(x) + g(x) = [F(z) + G(z)] = [\tilde{F}(z) + \tilde{G}(z)].$$

The derivatives of a hyperfunction $f(x) = [F(z)]$ are defined by

$$f'(x) := [F'(z)],$$

$$f^{(n)}(x) := [F^{(n)}(z)] \quad (n = 1, 2, \ldots).$$

Therefore, hyperfunctions are infinitely differentiable because holomorphic functions are infinitely differentiable. The derivatives of a hyperfunction are also well-defined in the above sense.

We here show some typical examples of hyperfunctions. The Dirac delta function $\delta(x)$ is given as a hyperfunction by

$$\delta(x) = \left[ -\frac{1}{2\pi i} \frac{1}{z} \right] = -\frac{1}{2\pi i} \left( \frac{1}{x + i0} - \frac{1}{x - i0} \right),$$

and the Heaviside step function

$$Y(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}$$

is given as a hyperfunction by

$$Y(x) = \left[ -\frac{1}{2\pi i} \log(-z) \right] = -\frac{1}{2\pi i} [\log(-(x + i0)) - \log(-(x - i0))],$$

where the complex logarithmic function $\log z$ is the branch such that $\log x$ is real valued if $x$ is a real positive number. Figure 1 shows the graphs of the defining functions of the delta function $\delta(x)$ and the step function $Y(x)$. From these figures, we can understand visually that the difference between the boundary values of the defining function gives a hyperfunction.

### 3 Numerical Fourier transform

As mentioned in Section 1, the Fourier transform of a function $f(x)$ is given as the hyperfunction

$$\mathcal{F}[f](\xi) = [\mathcal{F}_+(\zeta), \mathcal{F}_-(-\zeta)] = \mathcal{F}_+(\xi + i0) - \mathcal{F}_-(\xi + i0)$$

(7)
in hyperfunction theory, where

\[ \mathfrak{F}_+(\zeta) = \int_{-\infty}^{0} f(x) e^{-2\pi i \zeta x} \, dx \quad (\text{Im} \, \zeta > 0) \]  

(8)

and

\[ \mathfrak{F}_-(\zeta) = -\int_{0}^{\infty} f(x) e^{-2\pi i \zeta x} \, dx \quad (\text{Im} \, \zeta < 0). \]  

(9)

The function \( \mathfrak{F}_+(\zeta) \) is holomorphic in the upper half plane \( \text{Im} \, \zeta > 0 \) and the function \( \mathfrak{F}_-(\zeta) \) is holomorphic in the lower half plane \( \text{Im} \, \zeta < 0 \). Then, the Fourier transform \( \mathcal{F}[f](\xi) \) is given as the hyperfunction whose defining functions are \( \mathfrak{F}_+(\zeta) \) and \( \mathfrak{F}_-(\zeta) \) by (7). We remark that it is easy to compute the functions \( \mathfrak{F}_\pm(\zeta) \) (\( \pm \text{Im} \, \zeta > 0 \)) even if the integrand on the right-hand-side of (1) is a slowly decaying oscillatory function because the integrands on the right-hand sides of (8) and (9) includes the exponentially decaying factors \( \exp(-2\pi |\text{Im} \, \zeta| x) \). Therefore we expect that we can compute the Fourier transform \( \mathcal{F}[f](\xi) \) by the following way.

1. We compute the defining functions \( \mathfrak{F}_\pm(\zeta) \) in \( \mathbb{C} \setminus \mathbb{R} \).
2. We obtain the Fourier transform \( \mathcal{F}[f](\xi) = \mathfrak{F}_+(\xi + i0) - \mathfrak{F}_-(\xi - i0) \) by the analytic continuation of \( \mathfrak{F}_\pm(\zeta) \) onto the real axis \( \mathbb{R} \).

We call the method above the “hyperfunction method”. More exactly, the hyperfunction method for numerical Fourier transforms is as follows.

**Computation of \( \mathfrak{F}_\pm(\zeta) \)** We compute the defining functions \( \mathfrak{F}_\pm(\zeta) \) in \( \mathbb{C} \setminus \mathbb{R} \) in Taylor series

\[ \mathfrak{F}_\pm(\zeta) = \sum_{n=0}^{\infty} c_n^{(\pm)} (\zeta - \zeta_0^{(\pm)})^n, \]  

(10)
where $\zeta_0^{(\pm)}$ are given imaginary numbers such that $\pm \text{Im} \zeta_0^{(\pm)} > 0$ and
\[
c_n^{(\pm)} = \frac{1}{n!} \delta_n^{(\pm)} (\zeta_0^{(\pm)}) = \frac{1}{n!} \int_0^\infty (\pm 2\pi i x)^n f(x) e^{\pm 2\pi i c_n^{(\pm)} x} dx
= \frac{1}{2\pi \eta n!} \int_0^\infty \left(\frac{1}{\eta} \right)^n f \left( -\frac{x}{2\pi \eta} \right) e^{(\zeta/\eta)x} e^{-x} dx
\]
where $\Re \eta > 0$ and $\Im \eta > 0$.

It is easier to compute the coefficients $c_n$ than the oscillatory integral on the right-hand side of (11) because the integrands on the right-hand side of (11) involve the exponentially decaying factor $e^{-2\pi |\text{Im} \zeta^2| x}$. We can compute the integrals appearing in (11) easily using conventional quadrature formula, for example, the DE rule.

**Analytic continuation of $\mathfrak{F}_\pm(\zeta)$**  
Next, we obtain the analytic continuation of $\mathfrak{F}_\pm(\zeta)$ onto the real axis $\mathbb{R}$. For this purpose, we transform the Taylor series (10) into the continued fractions
\[
\mathfrak{F}_\pm(\zeta) = \frac{a_1^{(\pm)}}{1 + a_2^{(\pm)} (\zeta - c_0^{(\pm)})}
= \frac{1 + a_n^{(\pm)} (\zeta - c_0^{(\pm)})}{1 + a_n^{(\pm)} (\zeta - c_0^{(\pm)})}
\]
We expect that we can get the analytic continuation of $\mathfrak{F}_\pm$ by the continued fraction since, in general, the convergence regions of the continued fractions are wider than the convergence disk of the Taylor series (10). The coefficients $a_n^{(\pm)}$ of the continued fractions are obtained by the quotient-difference algorithm as follows. We compute the series \( \{e_n^{(n)}| n = 0, 1, 2, \ldots; k = 0, 1, 2, \ldots \} \) and \( \{q_n^{(n)}| n = 0, 1, 2, \ldots; k = 1, 2, \ldots \} \) by

\[
e_0^{(n)} = 0, \quad q_1^{(n)} = \frac{c_1^{(\pm)} e_n^{(\pm)}}{c_n^{(\pm)}}, \quad (n = 0, 1, 2, \ldots),
\]
\[
e_k^{(n)} = q_k^{(n+1)} - q_k^{(n)} + e_k^{(n+1)}, \quad q_{k+1}^{(n+1)} = \frac{e_k^{(n+1)}}{e_k^{(n)}} q_k^{(n)}
\]

\( (n = 0, 1, 2, \ldots; k = 1, 2, \ldots) \),

and then we obtain the coefficients $a_n^{(\pm)}$ by

\[
a_1^{(\pm)} = r_0^{(\pm)}, \quad a_2^{(\pm)} = -q_2^{(0)}, \quad a_{2k+1}^{(\pm)} = -e_k^{(0)} \quad (k = 1, 2, \ldots).
\]

The numbers $e_k^{(n)}$ and $q_k^{(n)}$ are generated as shown in the tableau of Figure 2. It is known that the quotient-difference algorithm is numerically unstable. Therefore we carry out the computation of the algorithm using multiple precision arithmetics. If some of the coefficients $e_k^{(\pm)}$ are zeros, we cannot carry out the algorithm. In that case, we change $\zeta_0^{(\pm)}$ so that all the coefficients $c_n^{(\pm)}$ are not zeros.
4 Numerical examples

We here show some numerical examples which show the effectiveness of the proposed method. All the computations were carried out by using programs coded in C++ with 100 decimal digit precision working by the multiple precision arithmetic library exflib [1]. We computed the Fourier series \( \mathcal{F}[f](\xi) \) for

\[
f(x) = \begin{cases} 
(i) & 1/(1 + x^2) \\
(ii) & \tanh(\pi x) \\
(iii) & \log |x| \\
(iv) & |x|
\end{cases}
\]  

(12)

by the hyperfunction method. The exact expression of the Fourier transforms for the functions (12) are known as follows.

(i) \( \mathcal{F}[(1 + x^2)^{-1}](\xi) = \pi e^{-2\pi|\xi|} \),

(ii) \( \mathcal{F}[\tanh(\pi x)](\xi) = -i \coth(\pi \xi) \),

(iii) \( \mathcal{F}[^{\log} |x|](\xi) = -\gamma \delta(\xi) - \frac{1}{2|\xi|} \),

(iv) \( \mathcal{F}[|^{| x |}|](\xi) = -\frac{1}{2(\pi \xi)^2} \).

Figure 3 shows the absolute errors of the numerical Fourier transforms by the hyperfunction method with the center of Taylor series \( z_0^{(\pm)} = \pm i, \pm 2i, 1 \pm i \) and \(-1 \pm i\), and Table 1 shows the numbers of the evaluations of \( f(x) \) in (12) in computing the Fourier transforms of the functions \( f(x) \) by the hyperfunction method. From the figures, the hyperfunction method works well, especially, for \( f(x) = \tanh(\pi x) \) and \( |x| \).
Figure 3: The absolute errors of the numerical Fourier transforms of the functions (12) by the hyperfunction method with the center of the Taylor series in (10) taken as (1) $\zeta_0^{(\pm)} = \pm i$, (2) $\zeta_0^{(\pm)} = \pm 2i$, (3) $\zeta_0^{(\pm)} = 1 \pm i$ and $\zeta_0^{(\pm)} = -1 \pm i$.

We remark the interesting fact that we can obtain Fourier transforms by the hyperfunction method without computing oscillatory integrals. In fact, if we take $\zeta_0^{(\pm)}$ as a purely imaginary number $\xi = i\eta$, the integrals in (11) which gives the coefficients of the Taylor series of $F_{\pm}(\xi)$ become

$$c_n^{(\pm)} = \frac{1}{2\pi i\eta} \int_0^{\infty} \left( i\eta x \right)^n f\left( -\frac{x}{2\pi \eta} \right) e^{-x} dx,$$

which include no oscillatory function.

We compared the hyperfunction method with Sugihara’s method and the DE-type formula by Ooura and Mori. We computed the Fourier transform $\mathcal{F}[f](\xi)$ for $f(x)$ given in (12) with $\xi = 1$ by our method and the two previous methods. Table 2 shows the numbers of the evaluations of $f(x)$ used for numerical integration and the errors of the methods. From the table, the our method is superior to Sugihara’s method, and it is competitive with Ooura and Mori’s method in some examples. Besides, we remark that our method gives Fourier transform as a function while the two previous methods give Fourier transform as an integral or a number. We means that, in our method, we can use the same coefficients $a_n$ of the continued fraction for Fourier transforms $\mathcal{F}[f](\xi)$ with many $\xi$’s once the coefficients $a_n$ are obtained. On the other hand, in the two previous methods, we have to compute numerical integrals again every time the value $\xi$ is changed.
Table 1: The numbers of the evaluations of $f(x)$ in computing the Fourier transform of the functions $\mathcal{F}_{\pm}^{(\pm)}(\zeta)$ by the hyperfunction method.

| $\mathcal{F}_{\pm}^{(\pm)}(\zeta_0)$ | $\pm i$ | $\pm 2i$ | $1 \pm i$ | $-1 \pm i$ |
|--------------------------------------|---------|---------|---------|---------|
| (i)                                  | 1420    | 720     | 2820    | 2820    |
| (ii)                                 | 1330    | 666     | 2838    | 2838    |
| (iii)                                | 1430    | 714     | 2838    | 2838    |
| (iv)                                 | 1332    | 668     | 2646    | 2646    |

5 Concluding remarks

In this paper, we proposed a numerical method for obtaining Fourier transforms based on hyperfunction theory. In hyperfunction theory, a Fourier transform is given as a hyperfunction, the difference of the real axis of analytic functions which are called the defining functions of the hyperfunction, and the defining functions of a Fourier transform are given by integrals including the integrand of the desired Fourier transform and exponentially decaying factors. In our method, we compute the defining functions in the upper or lower half complex plane and obtain the Fourier transform by the analytic continuations of the defining functions onto the real axis. Numerical examples show that our method is effective and competitive with the previous methods.

Problems for future study are as follows.

- How should we choose the centers $\zeta_{0+}^{(\pm)}$ of the Taylor series of the defining functions $\mathcal{F}_{\pm}(\zeta)$? As shown in the numerical examples, the accuracy of our method depends on the choice of $\zeta_{0+}^{(\pm)}$, and it is a crucial problem which points $\zeta_{0+}^{(\pm)}$ are the best.

- Theoretical error estimate of our method is an important problem.

- We employed an analytic continuation by transforming the defining functions into continued fractions. However, this process is expensive because we use the quotient-difference algorithm here and we use multiple precision arithmetics due to the numerical instability of the quotient-difference algorithm. Therefore, we need a numerically stable method of analytic continuation or transformation of an analytic function into a continued fraction.

Acknowledgements

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Table 2: The numbers $N$ of the evaluations of $f(x)$ and the errors of the hyperfunction method, Sugihara’s method and Ooura-Mori’s method applied to the Fourier transforms $\mathcal{F}[f](\xi = 1)$ of the functions $f(x)$ in (12).

| $f(x)$ method | $N$ | error  |
|---------------|-----|--------|
| hyperfunction ($\zeta_0^{(\pm)} = \pm i$) | 1420 | $8.1 \times 10^{-18}$ |
| hyperfunction ($\zeta_0^{(\pm)} = \pm 2i$) | 710 | $1.0 \times 10^{-16}$ |
| (i) hyperfunction ($\zeta_0^{(\pm)} = 1 \pm i$) | 2820 | $3.1 \times 10^{-27}$ |
| Sugihara | 17808 | $8.1 \times 10^{-22}$ |
| Ooura-Mori | 1010 | $2.2 \times 10^{-28}$ |
| hyperfunction ($\zeta_0^{(\pm)} = \pm i$) | 1330 | $4.9 \times 10^{-50}$ |
| hyperfunction ($\zeta_0^{(\pm)} = \pm 2i$) | 666 | $7.4 \times 10^{-43}$ |
| (ii) hyperfunction ($\zeta_0^{(\pm)} = 1 \pm i$) | 2642 | $2.1 \times 10^{-56}$ |
| Sugihara | 17156 | $7.9 \times 10^{-21}$ |
| Ooura-Mori | 1892 | $1.5 \times 10^{-46}$ |
| hyperfunction ($\zeta_0^{(\pm)} = \pm i$) | 1430 | $4.2 \times 10^{-16}$ |
| hyperfunction ($\zeta_0^{(\pm)} = \pm 2i$) | 714 | $9.8 \times 10^{-28}$ |
| (iii) hyperfunction ($\zeta_0^{(\pm)} = 1 \pm i$) | 2838 | $2.8 \times 10^{-28}$ |
| Sugihara | 17916 | $2.0 \times 10^{-20}$ |
| Ooura-Mori | 1020 | $6.3 \times 10^{-32}$ |
| hyperfunction ($\zeta_0^{(\pm)} = \pm i$) | 1332 | $2.2 \times 10^{-85}$ |
| hyperfunction ($\zeta_0^{(\pm)} = \pm 2i$) | 668 | $7.5 \times 10^{-84}$ |
| (iv) hyperfunction ($\zeta_0^{(\pm)} = 1 \pm i$) | 2646 | $6.3 \times 10^{-83}$ |
| Sugihara | 17182 | $3.8 \times 10^{-20}$ |
| Ooura-Mori | 15198 | $1.5 \times 10^{-80}$ |

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