Characterizing the Zeta Distribution via Continuous Mixtures

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Abstract

We offer two novel characterizations of the Zeta distribution: first, as tractable continuous mixtures of Negative Binomial distributions (with fixed shape parameter, \( r > 0 \)), and second, as a tractable continuous mixture of Poisson distributions. In both the Negative Binomial case for \( r \in [1, \infty) \) and the Poisson case, the resulting Zeta distributions are identifiable because each mixture can be associated with a unique mixing distribution. In the Negative Binomial case for \( r \in (0, 1) \), the mixing distributions are quasi-distributions (for which the quasi-probability density function assumes some negative values).

Keywords: Zeta distribution; Negative Binomial distribution; Poisson distribution; continuous mixtures; identifiability.

1 Introduction

As part of an investigation of heavy-tailed discrete distributions in insurance and actuarial science (see Dai, Huang, Powers, and Xu, 2021), the authors derived two new characterizations of the Zeta distribution, which form the basis for the present article. Within the insurance context, Zeta random variables sometimes are employed to model heavy-tailed loss frequencies (i.e., counts of

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1By “heavy-tailed”, we mean a random variable \( X \sim f_X(x) \) for which \( E_X [X^\alpha] \to \infty \) for some \( \alpha \in (0, \infty) \).
event occurrences, claim submissions, indemnity payments, etc.; see, e.g., Doray and Arsenault, 2002). However, their use is even more common in other fields of study, in which they serve as empirical models for a wide range of discrete processes. Examples include: the number of word occurrences in a text; various measures of communication and influence, such as numbers of telephone calls, emails, and website hits; and physical intensities, such as numbers of earthquakes and solar flares within specified discrete categories (see, e.g., Newman, 2005). The Zeta distribution also plays important roles in analytic number theory, especially with regard to the distribution of prime numbers and the Riemann Hypothesis (see Lin and Hu, 2001; and Aoyama and Nakamura, 2012).

The novel formulations of the Zeta distribution presented in this article are likely to be of interest to researchers in various fields (as indicated above), especially those exploring social or physical mechanisms leading to heavy-tailed behavior. The first characterization, provided in Section 2, shows that Zeta random variables can be expressed as continuous mixtures of Negative Binomial counts with a fixed shape parameter, \( r > 0 \). This is accomplished via tractable mixing distributions that are well behaved for \( r \in [1, \infty) \), but consist of quasi-distributions (for which the quasi-probability density function assumes some negative values) for \( r \in (0, 1) \). The second characterization, given in Section 3, converts the mixtures of Negative Binomial counts from Section 2 into a tractable continuous mixture of Poisson counts by first expressing each Negative Binomial component as a continuous mixture of Poisson components. In both the Negative Binomial case for \( r \in [1, \infty) \) and the Poisson case, the resulting Zeta distributions are identifiable because the mixing distributions are unique. In Section 4, we conclude with some final observations.

## 2 Zeta as a Mixture of Negative Binomial Counts

In this section, we will show how Zeta random variables can be constructed as continuous mixtures of Negative Binomial counts with a fixed shape parameter, \( r \). To this end, let \( X|s \sim \text{Zeta}(s) \) have probability mass function (PMF) \( f_{X|s}(x) = \frac{(x+1)^{-s}}{\zeta(s)} \) for \( x \in \{0, 1, 2, \ldots\} \) and \( s \in (1, \infty) \), where \( \zeta(s) = \sum_{x=0}^{\infty}(x+1)^{-s} \) denotes the Riemann zeta function, and let \( X|r, p \sim \).
Negative Binomial \((r, p)\) have PMF \(f_{X \mid r,p} (x \mid r, p) = \frac{\Gamma(r+x)}{\Gamma(r)\Gamma(x+1)}p^r (1 - p)^x\) for \(x \in \{0, 1, 2, \ldots\}\), fixed \(r \in (0, \infty)\), and \(p \in (0, 1)\). Our basic objective is to identify a mixing random variable, \(p \mid r, s \sim f_{p \mid r,s} (p)\) for \(p \in (0, 1)\), such that

\[
f_{X \mid s} (x) = f_{X \mid r,p} (x) \wedge f_{p \mid r,s} (p).
\]

Since the form of the mixing probability density function (PDF), \(f_{p \mid r,s} (p)\), differs by the domain of the Negative Binomial shape parameter, \(r\), we must consider the cases of \(r = 1\), \(r \in (1, \infty)\), and \(r \in (0, 1)\), respectively, in the following three subsections.

### 2.1 The Case of \(r = 1\) (Geometric Distribution)

If the shape parameter equals 1, then the Negative Binomial \((r, p)\) distribution simplifies to Geometric \((p)\), with PMF \(f_{X \mid r=1,p} (x) = p^x (1 - p)\). Rewriting (1) as

\[
f_{X \mid s} (x) = f_{X \mid r=1,p} (x) \wedge f_{p \mid r=1,s} (p)
\]

yields

\[
\int_0^1 p^x (1 - p) f_{p \mid r=1,s} (p) \, dp = \frac{(x + 1)^{-s}}{\zeta(s)}
\]

\[
\Rightarrow \int_0^1 (p^x - p^{x+1}) f_{p \mid r=1,s} (p) \, dp = \frac{(x + 1)^{-s}}{\zeta(s)}
\]

\[
\Rightarrow E_{p \mid r=1,s} [p^{x+1}] = E_{p \mid r=1,s} [p^x] - \frac{(x + 1)^{-s}}{\zeta(s)},
\]

from which it follows that

\[
E_{p \mid r=1,s} [p^x] = 1 - \frac{1}{\zeta(s)} \sum_{i=0}^{x-1} (i + 1)^{-s}.
\]

\footnote{The Zeta \((s)\) and Negative Binomial \((r, p)\) distributions often are defined on the sample space \(x \in \{1, 2, 3, \ldots\}\) rather than \(x \in \{0, 1, 2, \ldots\}\). However, we have chosen the latter characterization both because it matches the sample space of the Poisson \((\lambda)\) distribution and because it is the more commonly used formulation in insurance applications (where it is convenient for loss frequencies to admit the possibility of \(x = 0\)).}
The system (2) then can be used to express the moment-generating function of \( p | r = 1, s \) as

\[
E_{p | r = 1, s} [e^{tp}] = 1 + \frac{t E_{p | r = 1, s} [p]}{1!} + \frac{t^2 E_{p | r = 1, s} [p^2]}{2!} + \frac{t^3 E_{p | r = 1, s} [p^3]}{3!} + \ldots
\]

\[
= 1 + \frac{t}{1!} \left( 1 - \frac{1}{\zeta (s)} \right) + \frac{t^2}{2!} \left( 1 - \frac{1 + 2^{-s}}{\zeta (s)} \right) + \frac{t^3}{3!} \left( 1 - \frac{1 + 2^{-s} + 3^{-s}}{\zeta (s)} \right) + \ldots
\]

\[
= e^t - \frac{1}{\zeta (s)} \sum_{n=1}^{\infty} \frac{t^n}{n!} \left( \sum_{i=0}^{n-1} (i + 1)^{-s} \right)
\]

\[
= e^t - \sum_{n=1}^{\infty} \frac{t^n}{n!} H_{n,s},
\]

where \( H_{n,s} = \sum_{i=0}^{n-1} (i + 1)^{-s} \) is the \( n \)th generalized harmonic number. Alternatively, (3) may be written as

\[
\frac{1}{\zeta (s)} \sum_{n=0}^{\infty} \frac{t^n}{n!} \left( \sum_{i=n}^{\infty} (i + 1)^{-s} \right)
\]

\[
= \sum_{n=0}^{\infty} \frac{t^n}{n!} \frac{\zeta (s, n+1)}{\zeta (s)},
\]

where \( \zeta (s, n+1) = \sum_{i=n}^{\infty} (i + 1)^{-s} \) is the Hurwitz zeta function of order \( n+1 \).

The series (4) and (5) clearly converge for all \( t \in (0, \infty) \) because both \( \frac{H_{n,s}}{\zeta (s)} \) and \( \frac{\zeta (s, n+1)}{\zeta (s)} \) are bounded above as \( n \to \infty \) for all \( s \in (1, \infty) \). Although these series are not reducible to simpler expressions, it is possible to write the PDF \( f_{p | r = 1, s} (p) \) analytically, as shown in the following result.

\textbf{Theorem 1:} If \( X | s \sim \text{Zeta} (s) \) and \( X | r = 1, p \sim \text{Negative Binomial} (r = 1, p) \), then there exists a unique mixing random variable, \( p | r = 1, s \), with PDF

\[
f_{p | r = 1, s} (p) = \frac{(- \ln (p))^{s-1}}{\zeta (s) \Gamma (s) (1 - p)}
\]

for \( p \in (0, 1) \), such that

\[
f_{X | s} (x) = f_{X | r = 1, p} (x) \land f_{p | r = 1, s} (p).
\]

\textbf{Proof:} First, consider

\[
\int_{0}^{1} f_{X | r = 1, p} (x) f_{p | r = 1, s} (p) \, dp = \int_{0}^{1} p^x (1 - p) \frac{(- \ln (p))^{s-1}}{\zeta (s) \Gamma (s) (1 - p)} \, dp
\]
\[ \zeta(s) \Gamma(s) \int_0^1 p^x (-\ln(p))^{s-1} dp. \] (6)

Then, using the substitution \( y = -\ln(p) \) in the above integral, (6) can be rewritten as

\[ \frac{1}{\zeta(s) \Gamma(s)} \int_0^\infty (e^{-y})^2 y^{s-1} (-e^{-y}) \, dy \]
\[ = \frac{1}{\zeta(s) \Gamma(s)} \int_0^\infty (e^{-y})^{x+1} y^{s-1} \, dy \]
\[ = (x+1)^{-s} \int_0^\infty \frac{(x+1)^s y^{s-1} e^{-(x+1)y}}{\Gamma(s)} \, dy \]
\[ = (x+1)^{-s} \frac{\zeta(s)}{\zeta(s)}. \]

The uniqueness of \( f_{p|r=1,s}(p) \) — and equivalently, the identifiability of \( f_{X|s}(x) \) — follows from the identifiability of Negative Binomial mixtures with fixed shape parameter \( r \) (see Theorem 2.1 of Sapatinas, 1995). ■

One fairly obvious, yet interesting, aspect of the above result is that it provides a natural connection between two of the simplest convergent series in mathematical analysis: the infinite geometric series,

\[ S_G = 1 + \gamma^{-1} + \gamma^{-2} + \gamma^{-3} + \ldots = \frac{\gamma}{\gamma - 1}, \]

with \( \gamma \in (1, \infty) \); and the zeta function,

\[ S_Z = 1 + 2^{-s} + 3^{-s} + 4^{-s} \ldots = \zeta(s), \]

with \( s \in (1, \infty) \). Letting \( \tau_G(n) \) and \( \tau_Z(n) \) denote the \( n^{th} \) terms of these two series, respectively (for \( n \in \{1, 2, 3, \ldots\} \)), and treating \( \gamma \) as a random variable defined by the transformation \( \gamma = \frac{1}{p} \), it follows from Theorem 1 that

\[ f_{\gamma|s}(\gamma) = \frac{(\ln(\gamma))^{s-1}}{\zeta(s) \Gamma(s) \gamma (\gamma - 1)} \]

for \( \gamma \in (1, \infty) \) and

\[ \frac{\tau_Z(n)}{S_Z} = \int_0^1 p^{n-1} (1 - p) f_{p|r=1,s}(p) \, dp \]
\[
\int_{0}^{1} p^{n-1} (1 - p) \frac{(- \ln (p))^{s-1}}{\zeta (s) \Gamma (s) (1 - p)} dp \\
= - \int_{0}^{1} \left( \frac{1}{\gamma} \right)^{n-1} \left( 1 - \frac{1}{\gamma} \right) \frac{(- \ln (1/\gamma))^{s-1}}{\zeta (s) \Gamma (s) (1 - 1/\gamma)} \gamma^{-2} d\gamma \\
= \int_{1}^{\infty} \frac{\gamma^{-(n-1)}}{\gamma - 1} \frac{(\ln (\gamma))^{s-1}}{\zeta (s) \Gamma (s) \gamma (\gamma - 1)} d\gamma \\
= \int_{1}^{\infty} \frac{\tau_G (n)}{S_G} f_{\gamma|s} (\gamma) d\gamma \\
= E_{\gamma|s} \left[ \frac{\tau_G (n)}{S_G} \right].
\]

2.2 The Case of Fixed \( r \in (1, \infty) \)

When the fixed Negative Binomial shape parameter differs from 1, the mixing PDF, \( f_{p|r,s} (p) \), generally becomes more complex, but remains reasonably tractable. The following result addresses the case of \( r \in (1, \infty) \).

**Theorem 2:** If \( X|s \sim \text{Zeta} (s) \) and \( X|r > 1, p \sim \text{Negative Binomial} (r > 1, p) \), then there exists a unique mixing random variable, \( p|r > 1, s \), with PDF

\[
f_{p|r>1,s} (p) = \frac{r - 1}{\zeta (s) \Gamma (s) (1 - p)} \int_{p}^{1} (\omega - p)^{r-2} (\ln (\omega))^{s-1} d\omega
\]

for \( p \in (0, 1) \), such that

\[ f_X|s (x) = f_{X|r>1,p} (x) \wedge f_{p|r>1,s} (p). \]

**Proof:** Consider

\[
\int_{0}^{1} f_{X|r>1,p} (x) f_{p|r>1,s} (p) dp \\
= \int_{0}^{1} \frac{\Gamma (r + x)}{\Gamma (r) \Gamma (x + 1)} p^{x} (1 - p)^{r} \left[ \frac{r - 1}{\zeta (s) \Gamma (s) (1 - p)} \int_{p}^{1} (\omega - p)^{r-2} (\ln (\omega))^{s-1} \omega^{x-1} d\omega \right] dp \\
= \frac{1}{\zeta (s) \Gamma (r - 1) \Gamma (x + 1)} \int_{0}^{1} \frac{p^{x}}{\Gamma (s)} \left[ \int_{p}^{1} (\omega - p)^{r-2} (\ln (\omega))^{s-1} \omega^{x-1} d\omega \right] dp. \tag{7}
\]

Using the substitution \( t = - \ln (\omega) \) in the inside integral, followed by \( y = - \ln (p) \) in the outside.
integral, (7) can be rewritten as

\[
\frac{1}{\zeta(s) \Gamma(r-1) \Gamma(x+1)} \int_0^\infty \frac{(e^{-y})^x}{\Gamma(s)} \left[ \int_y^\infty \frac{(e^{-t} - e^{-y})^{r-2} t^{s-1}}{(e^{-t})^{r-1}} \left(-e^{-t}\right) dt \right] \left(-e^{-y}\right) dy
\]

\[= \frac{1}{\zeta(s) \Gamma(r-1) \Gamma(x+1)} \int_0^\infty \int_0^y \frac{1}{\Gamma(s)} e^{-(x+1)y} (1 - e^{-y})^{r-2} t^{s-1} dt dy. \tag{8}\]

Now interchange the order of integration, and let \(\xi = y - t\) in the new inside integral, so (8) becomes

\[
\frac{1}{\zeta(s) \Gamma(r-1) \Gamma(x+1)} \int_0^\infty \int_0^\infty \frac{1}{\Gamma(s)} e^{-(x+1)(\xi+t)} (1 - e^{-\xi})^{r-2} t^{s-1} d\xi dt
\]

\[= \frac{1}{\zeta(s) \Gamma(r-1) \Gamma(x+1)} \int_0^\infty \frac{1}{\Gamma(s)} t^{s-1} e^{-(x+1)t} dt \int_0^\infty e^{-(x+1)\xi} (1 - e^{-\xi})^{r-2} d\xi
\]

\[= \frac{(x+1)^{-s}}{\zeta(s) \Gamma(r-1) \Gamma(x+1)} \int_0^\infty \frac{1}{\Gamma(s)} \frac{1}{\Gamma(r)} \frac{1}{\Gamma(x+1)} \int_0^\infty e^{-(x+1)\xi} (1 - e^{-\xi})^{r-2} d\xi.
\]

Substituting \(q = e^{-\xi}\) into the above integral then yields

\[
\frac{(x+1)^{-s}}{\zeta(s) \Gamma(r-1) \Gamma(x+1)} \frac{1}{\Gamma(x+1)} \int_0^1 q^{x+1} (1 - q)^{r-2} \left(-\frac{1}{q}\right) dq
\]

\[= \frac{(x+1)^{-s}}{\zeta(s) \Gamma(r-1) \Gamma(x+1)} \int_0^1 q^x (1 - q)^{r-2} dq
\]

\[= \frac{(x+1)^{-s}}{\zeta(s) \Gamma(r-1) \Gamma(x+1)} \int_0^1 q^x (1 - q)^{r-2} dq
\]

\[= \frac{(x+1)^{-s}}{\zeta(s) \Gamma(r-1) \Gamma(x+1)} \int_0^1 q^x (1 - q)^{r-2} dq
\]

The uniqueness of \(f_{p|r>1,s}(p)\) follows in the same way as the uniqueness of \(f_{p|r=1,s}(p)\) in the proof of Theorem 1. ■

It is worth noting that, for the special case of \(r = 2\), the PDF \(f_{p|r>1,s}(p)\) simplifies to

\[f_{p|r=2,s}(p) = \frac{1}{\zeta(s) \Gamma(s) \Gamma(1-p)^2} \int_p^1 \frac{(-\ln(\omega))^{s-1}}{\omega} d\omega.
\]
an analytic form quite similar to $f_{p|\tau=1,s}(p)$.

### 2.3 The Case of Fixed $r \in (0, 1)$

For fixed $r \in (0, 1)$, the analysis is similar to that for $r \in (1, \infty)$, with one major difference: the mixing distribution has a quasi-PDF, $f_{p|r<1,s}(p)$, that assumes some negative values. We provide the details in the following result.

**Theorem 3:** If $X|s \sim \text{Zeta}(s)$ and $X|r<1, p \sim \text{Negative Binomial}(r<1, p)$, then there exists a mixing quasi-random variable, $p|r<1, s$, with quasi-PDF

$$f_{p|r<1,s}(p) = \frac{1}{\zeta(s) \Gamma(s)(1-p)^r} \left[ (s-1) \int_p^1 (\omega - p)^{r-1} (-\ln(\omega))^{s-2} \frac{d\omega}{\omega^r} ight]$$

$$+ (r-1) \int_p^1 (\omega - p)^{r-1} (-\ln(\omega))^{s-1} \frac{d\omega}{\omega^r}$$

for $p \in (0, 1)$, such that

$$f_{X|s}(x) = f_{X|r<1,p}(x) \wedge f_{p|r<1,s}(p).$$

**Proof:** After writing

$$\int_0^1 f_{X|r<1,p}(x) f_{p|r<1,s}(p) \, dp$$

$$= \int_0^1 \frac{\Gamma(r+x)}{\Gamma(r) \Gamma(x+1)} p^x (1-p)^r \frac{1}{\zeta(s) \Gamma(s)(1-p)^r} \left[ (s-1) \int_p^1 (\omega - p)^{r-1} (-\ln(\omega))^{s-2} \frac{d\omega}{\omega^r} ight]$$

$$+ (r-1) \int_p^1 (\omega - p)^{r-1} (-\ln(\omega))^{s-1} \frac{d\omega}{\omega^r} \, dp$$

$$= \frac{1}{\zeta(s) \Gamma(r) \Gamma(x+1)} \int_0^1 p^x \left[ \frac{1}{\Gamma(s-1)} \int_p^1 (\omega - p)^{r-1} (-\ln(\omega))^{s-2} \frac{d\omega}{\omega^r} \right].$$
\begin{equation}
\frac{(r-1)}{\Gamma(s)} \int_{p}^{1} \frac{(\omega - p)^{r-1}( - \ln (\omega))^{s-1}}{\omega^r} d\omega \right] dp,
\end{equation}

the proof that (10) equals \( \frac{(x+1)^{-s}}{\zeta(s)} \) is entirely analogous to the proof that (7) equals \( \frac{(x+1)^{-s}}{\zeta(s)} \) for Theorem 2.

To demonstrate \( f_{p|r<1,s}(p) < 0 \) for some \( p \in (0,1) \) for all \( s \in (1,\infty) \), it suffices to insert \( p = 0 \) into the right-hand side of (9), revealing \( \lim_{p \to 0^+} f_{p|r<1,s}(p) = -\infty \). This implies there exists an interval \( (0, \varepsilon) \), for some \( \varepsilon > 0 \), such that \( p \in (0, \varepsilon) \implies f_{p|r<1,s}(p) < 0 \).

The reason \( f_{p|r<1,s}(p) < 0 \) in some neighborhood of zero is quite intuitive. Essentially, when \( r \in (0,1) \), the Negative Binomial PMF becomes very steep at the lower end of its sample space (in the sense that \( \sup_r \left( \frac{f_{X|r,p}(0)}{f_{X|r,p}(1)} \right) = \lim_{r \to 0^+} \left( \frac{1}{rp} \right) = \infty \)), and this steepness is aggravated for values of \( p \) close to zero. In this region of the sample space, it is impossible to construct the much flatter Zeta PMF (for which \( \inf_s \left( \frac{f_{X|s}(0)}{f_{X|s}(1)} \right) = \lim_{s \to 1^+} 2^s = 2 \)) as a convex combination of Negative Binomial PMFs. However, if one can assign negative weight to those Negative Binomial PMFs for which \( p \) is very small, then one can mitigate the impact of small \( r \) by offsetting it with negative contributions from small \( p \).

### 2.4 The Case of Random \( r \)

If the Negative Binomial shape parameter, \( r \), is not fixed, but rather is a continuous random variable on \((0,\infty)\), then it is possible to express \( X|s \sim \text{Zeta}(s) \) as a mixture

\[
f_{X|s}(x) = f_{X|r,p}(x) \wedge f_{r,p|s}(r,p),
\]

for some joint mixing PDF, \( f_{r,p|s}(r,p) \). Unfortunately, this joint PDF is not unique, and the resulting Zeta distribution therefore is not identifiable. In fact, if \( f_{r,p|s}(r,p) = f_{p|r,s}(p) f_{r|s}(r) \), where \( f_{p|r,s}(p) \) denotes the mixing PDF given by Theorems 1 or 2 above, then

\[
f_{X|s}(x) = \int_{0}^{\infty} \int_{0}^{1} f_{X|r,p}(x) f_{r,p|s}(r,p) dr dp
\]

\[
= \int_{0}^{\infty} \int_{0}^{1} f_{X|r,p}(x) f_{p|r,s}(p) f_{r|s}(r) dr dp
\]
\[ \int_{0}^{\infty} \left[ \int_{0}^{1} f_{X|r,p} (x) f_{p|r,s} (p) \, dp \right] f_{r|s} (r) \, dr \]
\[ = f_{X|s} (x) \int_{0}^{\infty} f_{r|s} (r) \, dr, \]

so that \( f_{r|s} (r) \) can be any well-defined PDF on \((0, \infty)\). Without the identifiability property, it is impossible to probe the statistical processes generating the mixed distribution of interest (in our case, \( f_{X|s} (x) \)). In particular, one cannot estimate the parameters of the mixing distribution from observations of the mixed random variable (see, e.g., Karlis and Xekalaki, 2005).

### 3 Zeta as a Mixture of Poisson Counts

For any choice of fixed \( r \in (0, \infty) \) and \( p \in (0, 1) \), the Negative Binomial \((r, p)\) random variable can be expressed as a continuous mixture of Poisson \((\lambda)\) counts, using a Gamma \((r, \beta)\) mixing distribution with PDF \( f_{\lambda|r,\beta=\frac{1-p}{p}} (\lambda) = \left( \frac{1-p}{p} \right)^{\frac{\lambda-1}{r}} \Gamma^{\frac{\lambda}{r}} \exp \left( - \left( \frac{1-p}{p} \right) \lambda \right) \) for \( \lambda \in (0, \infty), r \in (1, \infty), \) and \( \beta = \frac{1-p}{p} \in (0, \infty) \). This mixture may be written as

\[ f_{X|r,p} (x) = f_{X|\lambda} (x) \wedge f_{\lambda|r,\beta=\frac{1-p}{p}} (\lambda). \]

Substituting the right-hand side of the above equation into (1) yields

\[ f_{X|s} (x) = f_{X|\lambda} (x) \wedge f_{\lambda|r,\beta=\frac{1-p}{p}} (\lambda) \wedge f_{p|r,s} (p), \]

which, by the associative property of distribution mixing, is equivalent to

\[ f_{X|s} (x) = f_{X|\lambda} (x) \wedge f_{\lambda|r,\beta=\frac{1-p}{p}} (\lambda) \wedge f_{p|r,s} (p) \]
\[ = f_{X|\lambda} (x) \wedge f_{\lambda|r,s} (\lambda). \]

It thus follows that Zeta random variables can be expressed as continuous mixtures of Poisson counts with the mixing PDF,

\[ f_{\lambda|r,s} (\lambda) = f_{\lambda|r,\beta=\frac{1-p}{p}} (\lambda) \wedge f_{p|r,s} (p). \]
\begin{align*}
\int_0^1 \left( \frac{1 - p}{p} \right)^r \lambda^{r-1} \exp \left( - \left( \frac{1 - p}{p} \right) \lambda \right) f_{p|r,s}(p) \, dp.
\end{align*}

(11)

Although the right-hand side of (11) appears to depend on the Negative Binomial shape parameter \( r \), that actually is not the case. As shown by Feller (1943), any mixed-Poisson random variable must possess the identifiability property, and thus a unique mixing distribution. Consequently, \( f_{\lambda|r,s}(\lambda) \) must be invariant over \( r \), and may be expressed using the simplest form of the above integral (i.e., by inserting \( r = 1 \)). Therefore,

\begin{align*}
 f_{\lambda|r,s}(\lambda) &= f_{\lambda|s}(\lambda) \\
 &= \int_0^1 \left( \frac{1 - p}{p} \right) \exp \left( - \left( \frac{1 - p}{p} \right) \lambda \right) \frac{(- \ln (p))^{s-1}}{\zeta(s) \Gamma(s) (1 - p)} \, dp \\
 &= \frac{1}{\zeta(s) \Gamma(s)} \int_0^1 \frac{1}{p} (- \ln (p))^{s-1} \exp \left( - \left( \frac{1 - p}{p} \right) \lambda \right) \, dp.
\end{align*}

(12)

Applying the substitution \( y = \frac{1 - p}{p} \) to the above integral, (12) can be rewritten as

\begin{align*}
\frac{1}{\zeta(s) \Gamma(s)} \int_0^\infty (y + 1) (\ln (y + 1))^{s-1} \exp (- \lambda y) \left[ \frac{-1}{(y + 1)^2} \right] \, dy \\
= \frac{1}{\zeta(s) \Gamma(s)} \int_0^\infty \frac{1}{y + 1} (\ln (y + 1))^{s-1} \exp (- \lambda y) \, dy,
\end{align*}

which is not further reducible.

4 Conclusions

The present article provided two novel characterizations of \( X|s \sim \text{Zeta} \,(s) \). First, we showed that these random variables can be expressed as tractable continuous mixtures of \( X|r,p \sim \text{Negative Binomial} \,(r,p) \) with fixed shape parameter \( r \); that is,

\[ f_{X|s}(x) = f_{X|r,p}(x) \wedge f_{p|r,s}(p), \]
where

\[
f_{p|r,s}(p) = \frac{1}{\zeta(s) \Gamma(s) (1-p)^r} \times \begin{cases} 
(s-1) \int_p^1 \frac{(\omega-p)^{r-1} (-\ln(\omega))^{s-2}}{\omega^r} d\omega \\
+ (r-1) \int_p^1 \frac{(\omega-p)^{r-1} (-\ln(\omega))^{s-1}}{\omega^r} d\omega \\
(-\ln(p))^{s-1} \\
(r-1) \int_p^1 \frac{(\omega-p)^{r-2} (-\ln(\omega))^{s-1}}{\omega^{r-1}} d\omega 
\end{cases} 
\]

for \( r \in (0, 1) \),

\[
(\zeta(s) \Gamma(s)) (s-1) \int_1^\infty \omega^{r-2} (-\ln(\omega))^{s-2} \exp(-\omega) d\omega 
\]

for \( r = 1 \),

\[
\int_1^\infty \omega^{r-1} (-\ln(\omega))^{s-2} \exp(-\omega) d\omega 
\]

for \( r \in (1, \infty) \).

\[ f_{p|r,s}(p) = f_{p|r=1,s}(p) \text{ and } f_{p|r,s}(p) = f_{p|r>1,s}(p) \text{ are unique PDFs, and } f_{p|r,s}(p) = f_{p|r<1,s}(p) \text{ is a quas-PDF (i.e., with some negative values). Next, based on the fact that Negative Binomial } (r, p) \text{ random variables can be constructed as mixtures of } X|\lambda \sim \text{Poisson } (\lambda), \text{ with a Gamma } (r, \beta = \frac{1-p}{p}) \text{ mixing distribution, we showed that Zeta random variables also can be expressed as a unique and tractable continuous mixture of Poisson counts; that is,}
\]

\[
f_{X|\lambda}(x) = f_{X|\lambda}(x) \wedge f_{\lambda|s}(\lambda),
\]

where

\[
f_{\lambda|s}(\lambda) = \frac{1}{\zeta(s) \Gamma(s)} \int_0^\infty \frac{1}{y+1} (\ln(y+1))^{s-1} \exp(-\lambda y) dy.
\]

The appearance of quasi-PDFs in the Negative Binomial case for \( r \in (0, 1) \) was somewhat unexpected, but – as argued in Subsection 2.3 – has a fairly intuitive explanation. Therefore, it is natural to consider whether or not other heavy-tailed discrete random variables formed as mixtures of Negative Binomial counts also involve quasi-distributions. One obvious family to consider is \( X|b \sim \text{Yule } (b) \), with PMF \( f_{X|b}(x) = \frac{b^x (b+1)^{b+1}}{\Gamma(x+2)} \) for \( x \in \{0, 1, 2, \ldots\} \) and \( b \in (0, \infty) \), which approximates \( X|s \sim \text{Zeta } (s) \) for \( b = s - 1 \). In fact, Yule \((b)\) random variables may be expressed as mixtures of Negative Binomial \((r = 1, p)\) (i.e., Geometric \((p)\)) counts, using \( f_{p|a=1,b}(p) \sim \text{Beta } (a = 1, b) \) as the mixing distribution; that is,

\[
f_{X|b}(x) = f_{X|r=1,p}(x) \wedge f_{p|a=1,b}(p).
\]

For the Yule distribution, \( \inf_b \left( \frac{f_{X|b}(0)}{f_{X|b}(s)} \right) = \lim_{b \to 0^+} (b+2) = 2 \), which is identical to the corre-
sponding result for the Zeta distribution presented at the end of Subsection 2.3. Consequently, one might anticipate that constructing Yule random variables as mixtures of Negative Binomial counts with $r \in (0, 1)$ would require quasi-distributions, as in the Zeta case. In Dai, Huang, Powers, and Xu (2021), we not only show that this is indeed true, but also provide complete results for the Yule distribution analogous to those of Theorems 1-3 and Section 3 of the present article.

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