The crossover from Edwards-Wilkinson (s = 0) to KPZ (s > 0) type growth is studied for the BCSOS model. We calculate the exact numerical values for the $k = 0$ and $2\pi/N$ massgap for $N \leq 18$ using the master equation. We predict the structure of the crossover scaling function and confirm numerically that $m_0 \simeq 4(\pi/N)^2[1 + 3u^2(s)N/(2\pi^2)]^{0.5}$ and $m_1 \simeq 2(\pi/N)^2[1 + u^2(s)N/\pi^2]^{0.5}$, with $u(1) = 1.03596967$. KPZ type growth is equivalent to a phase transition in meso-scopic metallic rings where attractive interactions destroy the persistent current; and to endpoints of facet-ridges in equilibrium crystal shapes.
A large amount of theoretical effort has been devoted in recent years to establish and classify the scaling properties of dynamic processes such as those at growing interfaces. Several dynamic universality classes have emerged. One of them is the so-called KPZ universality class, named after the non-linear Langevin equation studied by Kardar, Parisi, and Zhang [1]. Older examples are Edwards-Wilkinson (EW) growth [2] and directed percolation [3]. They are distinguished by the value of the dynamic critical exponent \( z \); characteristic lengths and times scale as \( t \sim l^z \). Most of the evidence is numerical in nature, in particular from Monte Carlo (MC) studies [2], supplemented by a few exactly soluble models [4] and field theoretical renormalization studies [1,2].

Our understanding of dynamic scale invariance is still nowhere near that of equilibrium critical phenomena. In particular the comparison with two dimensional (2D) critical phenomena is relevant. 1D dynamic processes and 2D equilibrium statistical mechanics are very much alike in the master equation and transfer matrix formulation. Time becomes the second spatial coordinate. Most 2D equilibrium critical phenomena obey conformal invariance [5]. They all belong to the same conformal “dynamic” universality class \( (z = 1; \text{rotational invariance}) \). The ground state properties of 1D quantum systems are related as well. Luttinger liquids (relativistic fermions) are also part of the conformal \( z = 1 \) dynamic universality class [6]. Non-relativistic fermions have a dynamic exponent \( z = 2 \). Anisotropic scaling, with \( z \neq 1 \), occurs in 2D critical phenomena at e.g. Pokrovksy-Talapov (PT) transitions \( (z = 2) \) and the elusive Lifshitz points [6]. One prospect is that concepts similar to conformal invariance might apply to dynamic universality classes in general.

In this letter we present a detailed numerical study of the finite size scaling (FSS) crossover function from EW \( (z = 2) \) to KPZ \( (z = 1.5) \) type growth in the body-centered solid-on-solid (BCSOS) growth model. It is important to know the details of the crossover scaling between dynamic universality classes. The following simple example illustrates this.

Consider the 1D free fermion model. The fermions are non-relativistic when
the chemical potential coincides with the bottom of the energy band $E(k) \simeq k^2$. The momenta are spaced as $\Delta k = 2\pi/N$. So the massgap scales as $m \simeq (2\pi/N)^2$ and $z_0 = 2$ because $l_t \sim m^{-1}$. The fermions are relativistic for $k_F > 0$. The massgap scales as $m \simeq 4\pi k_F/N$, and $z_1 = 1$. The crossover scaling function $m = (k_F + 2\pi/N)^2 - k_F^2 \simeq 4\pi k_F/N + (2\pi/N)^2$ has two important properties. The crossover exponent $y_c$ determines the scaling of $dm/dk_F \sim N^{-z_0+y_c}$ at $k_F = 0$. $y_c$ is equal to the change in the dynamic exponent, $y_c = z_0 - z_1 = 1$. The FSS amplitude of the massgap is not a constant in the metal phase, but a universal number multiplied with the rapidity $2k_F$. The metal phase belongs to the conformal $z = 1$ dynamic universality class. The rapidity represents the lattice anisotropy in 2D critical phenomena language. The crossover operator is the stress tensor which lies at the heart of conformal field theory. We expect that crossover scaling functions between dynamic universality classes have the same generic asymptotic form

$$m \simeq A/N^{z_0} + Bu(s)/N^{z_1}, \quad (1)$$

with $A$ and $B$ universal constants and $u(s)$ the scaling field associated with the crossover operator.

Surface roughness is characterized by the scaling of the height-height correlation function, $G(r, t) = \langle [h(r_0 + r) - h(r_0)]^2 \rangle \simeq b^{2\alpha} G(b^{-1}r, b^z t^{-1})$. In 1D the stationary state has typically a finite correlation length beyond which the steps are disordered, and the probability to go up or down along the interface becomes random. At large length scales $G(r)$ behaves like the root mean-square displacement of a random walk, $G(r) \sim r$. So typically $\alpha = 0.5$. Stationary states with interesting correlations like those in 2D driven diffusive systems [7] have not been found (yet) in 1D growth dynamics. $z$ characterizes the rate at which the stationary state is approached. A disturbance with characteristic size $\xi$ spreads-out as $\xi \sim t^z$.

In the BCSOS model nearest neighbour columns must differ in height by one unit, $dh = \pm 1$. This describes a body-centered type stacking of molecules, or a zig-zag type stacking of bricks. We represent the surface configuration in terms of
the steps, $S_n^z = \pm 1$. The dynamic rule is as follows. Select one of the columns at random; if $S_n^z = -1$ and $S_{n+1}^z = 1$ (a local valley) a brick is adsorbed with probability $p$; if $S_n^z = 1$ and $S_{n+1}^z = -1$ (a local hill top) evaporation occurs with probability $q$. $P(\{S_n^z\})$ is the probability of micro state $\{S_n^z\}$. $|\Psi\rangle_\tau = \sum P(\{S_n^z\})|\{S_n^z\}\rangle$ is the state vector. The time evolution operator $T$ of the discrete-time master equation $|\Psi\rangle_{\tau + \Delta} = T|\Psi\rangle_{\tau}$ has the following structure

$$T = 1 - \frac{1}{4} \epsilon N^{-1} \sum_n [1 - \lambda S_n^z S_{n+1}^z - \frac{1}{2} (S_n^+ S_{n+1}^- + S_n^- S_{n+1}^+)$$

$$- \frac{1}{2} s (S_n^+ S_{n+1}^- - S_n^- S_{n+1}^+) + \mu S_n^z]$$

with $\Delta = 1/N$, $s = (p - q)/(p + q)$, and periodic boundary conditions, $S_{n+N}^z = S_n^z$. Without loss of generality time can be rescaled to $\epsilon = p + q = 1$. The parameters $\lambda = 1$ and $\mu = 0$ are introduced for later convenience. Stochastic processes preserve probability. So the state where all microstates have the same probability, $|D\rangle = \sum |\{S_n^z\}\rangle$, is the left eigenvector for the largest eigenvalue, $\lambda_0 = 1$.

The BCSOS model has more symmetry than generic KPZ growth models: $T(p, q) = T(q, p)^\dagger$. The left eigenvectors $\langle L |$ of $T(p, q)$ are identical to the right eigenvectors of $T(q, p)$. $T(q, p)$ and $T(p, q)$ are equivalent by particle-hole symmetry. Therefore $|R\rangle = \prod S_n^z |L\rangle$, and $|D\rangle$ is both the left and the right eigenvector for $\lambda_0$. The stationary state is completely disordered; $\alpha = 1/2$ for all $s$. The BC-SOS model is used widely to study KPZ type growth [2]. This special symmetry casts some doubts on its role as a generic KPZ growth model. This becomes more apparent in the crossover scaling behaviour.

It is important to point out how this model fits into the general picture sketched above. $T$ has the generic form, $T = \exp(-\mathcal{H})$. $\mathcal{H}$ is non-Hermitian, except at $s = 0$ where the model reduces to the spin-1/2 XXZ chain [6]. The EW point at $s = 0$ maps exactly onto the ferromagnetic Heisenberg point, $\lambda = 1$. In the KPZ equation EW growth corresponds to the point where the non-linear term vanishes and the dynamics reduces to a stochastic diffusion equation. Indeed, the exact solution of
the XXZ chain at \( \lambda = 1 \) is diffusive. The equation of motion for the spin-spin correlation closes and is a deterministic diffusion equation \[8\]. We checked that also the equations of motion for the \( n \)-point spin correlation functions close within themselves and are diffusion type equations.

The XXZ model is equivalent to a 1D spinless interacting fermion problem \[6\]. Growth, \( s \), introduces a preferred hopping direction. It creates a persistent current around the ring, similar to those in metallic rings in perpendicular magnetic fields. Recent research is focused on coherence effects in the presence of elastic scattering from random impurities \[9\]. Eq.(2) does not include impurities. It describes a phase transition due to interactions. Weak attraction between the fermions, small \( \lambda \), only renormalizes the scaling indices due to the marginal operator in the chiral Luttinger liquid. At \( \lambda > 1 \) the fermions coalesce into a macroscopic bound state (the ferroelectric ground state of the XXZ chain). This destroys the persistent current, because in eq.(2) only single fermions can hop. (In metallic rings the macroscopic droplets of fermions might still be able to move). The transition takes place at the KPZ line, at \( \lambda = 1 \). At this transition the fermion excitations do not behave metallic, \( z = 1 \), but are described by the unconventional dynamic exponent \( z = 1.5 \). It will be interesting to investigate how elastic impurities modify this.

The spin-1/2 XXZ chain is related to the 6-vertex model for 2D equilibrium (anti-) ferroelectrics \[10\]. The equivalence becomes exact when in the dynamic rule the columns are chosen sequentially instead of randomly, first all even and then all odd columns \[8\]. The modified rule reduces to the old one in the time continuum limit, \( p \to 0 \) and \( q \to 0 \) with \( s \) constant. \( \mu \) and \( s \) are the two components of the electric field.

The 6-vertex model is equivalent to the BCSOS model for equilibrium crystal surfaces. The anti-ferroelectric side of its phase diagram describes surface roughening transitions \[10\]. The ferroelectric side is less well known. It describes the crystal shape close to a facet ridge \[11,12\]. \( \lambda \) plays the role of inverse temperature. At \( \lambda > 1 \) two facets meet at a facet ridge; see Fig. 1. The growth parameter \( s \) tilts
the surface in the time-like direction, the direction along the facet ridge. The chemical potential $\mu$ tilts the surface in the spatial direction, the direction $\perp$ to the facet ridge. The interactions in this direction are anti-ferromagnetic and favour facetting. The KPZ points are the endpoints of the facet ridge where the two facets become separated by two PT transition lines with rough rounded surface in-between; see Fig. 1. The KPZ line represents the change in location with temperature of these facet ridge endpoints. The facet ridge shortens with temperature until it vanishes at the EW critical point. At PT transition the surface rounds smoothly. At facet ridge endpoints it changes discontinuously, because at KPZ points the surface has a finite slope in the time-like direction equal to the growth rate $v_g = s/4$. The dynamic exponent $z = 1.5$ implies unusual anisotropic scaling behaviour. There is one caveat to this identification of KPZ growth with facet ridge endpoints. More general growth models such as the RSOS model lack the self-adjointness property of eq.(2). Moreover, at the facet ridge endpoints in the BCSOS model the transfer matrix happens to be stochastic; we see no reason for this to be true in general. It will be interesting to investigate how these aspects affect the scaling properties.

The eigenvalues of the transfer matrix can be classified according to wave numbers $k$, reflecting translational invariance. The largest eigenvalue, $\lambda_0 = 1$, is located in the $k = 0$ sector. Define two massgaps: $m_0 = \lambda_0(0) - \lambda_1(0)$ and $m_1 = \lambda_0(0) - \lambda_0(1)$. $\lambda_1(0)$ is the next largest eigenvalue in the $k = 0$ sector, and $\lambda_0(1)$ the largest eigenvalue in the $k = 2\pi/N$ sector. We calculated the exact numerical values of $m_0$ and $m_1$ for even system sizes $N \leq 18$. The dynamic exponent follows from the FSS of the massgaps, $m \sim N^{-z}$. The spatial exponent follows independently from the interface width, $W \sim N^\alpha$, in the stationary state (the right eigenvector of $\lambda_0(0)$).

FSS approximants for $z$ are shown in Fig. 2 at various values of $s$. $z(N, N+2) = \frac{\ln(m_1(N)/m_1(N + 2))}{\ln((N + 2)/N)}$. These numbers are accurate to better than 12 decimal places, unlike results from e.g. MC simulations. Therefore it is possible to take into account the leading corrections to scaling. We do this by constructing
a cascade of $1/N$ extrapolations for the $z(N, N+2)$ at successive values of $N$. At $s = 1$ the convergence is very stable, and leads to $z = 1.50 \pm 0.05$. This is the same level of accuracy as obtained by MC simulations, but with substantially less effort and much smaller system sizes.

The entire line $0 < s \leq 1$ belongs to the KPZ universality class. The dynamic exponent must be equal to $z = 1.5$ for all $s \neq 0$, but the convergence is only good for $s \approx 1$. For $s \to 0$, the $z(N, N+2)$ move towards the EW value $z = 2$ at small $N$ before they bend-down towards the KPZ value $z = 1.5$. This reflects the crossover scaling behaviour from the nearby EW point.

The crossover scaling function eq.(1) does not apply to the BCSOS model, due to the special symmetry discussed above, which implies that the massgaps are even functions of $s$. Their first derivative vanishes. Eq.(1) is likely to apply to the more generic growth models, but this needs still to be tested. We propose the modified form for the BCSOS model:

$$m = \frac{A}{N^2} [1 + Bu^2(s) N]^{0.5}. \quad (3)$$

Also in this expression $y_s$ is equal to the change in the dynamic exponent, and the amplitude at $s \neq 0$ is equal to a universal number multiplied with $u(s)$.

The crossover scaling exponent follows from the second derivatives of the massgaps with respect to $s$, $m'' \sim N^{-z+2y_s}$. We find $y_s = 0.50 \pm 0.05$; consistent with $y_s = 0.5$ from power counting at the EW point in the KPZ equation.

At $s = 0$ the $k = 2\pi/N$ mass gap is exactly equal to $m_1(0) = 2\sin^2(\pi/N)$ for all $N$. Eq.(2) reduces to the ferromagnetic Heisenberg model. Spin rotations leave $\mathcal{T}$ invariant, but mix-up the sectors of different magnetization (slopes of the surface). Since $\lambda_0(k)$ is non-degenerate in each sector it follows that $m_1$ is independent of the tilt of the surface. The equation for $m_1(0)$ is trivially true in one particle sector with surface tilt $1 - 1/N$.

The $k = 0$ massgap at $s = 0$ is more complex. We find numerically: $m_0(0) = 4\sin^2(\pi/N)(1 + \frac{2}{\pi}\sin(\pi/N) + (0.306 \pm 0.003)\sin^2(\pi/N) + ...)$. The exact result for
$m_1(0)$ suggests that $\sin(\pi/N)$ is the natural FSS parameter. Only FSS corrections
with integer powers of $\sin(\pi/N)$ are to be expected, because all n-point correlation
functions at $s=0$ obey diffusion type equations. We find no numerical evidence
otherwise. We expect amplitudes to be simple combinations of $\pi$ and integers. The
leading correction to scaling amplitude of $m_0(0)$, $0.635 \pm 0.005$, is close enough to
$2/\pi$ to guess this to be the exact value. Similarly, the second derivatives of the
massgaps at $s=0$ scale numerically as: $m''_1(0) = \frac{2}{\pi} \sin(\pi/N) - \sin^2(\pi/N) + (0.25 \pm 0.02) \sin^3(\pi/N) + ...$ and $m''_0(0) = \frac{6}{\pi} \sin(\pi/N) - (1.06 \pm 0.01) \sin^2(\pi/N) + ...$. These
results yield the amplitudes in eq.(3):

$$m_1(s, N) = 2 \sin^2(\pi/N)[1 + \frac{1}{\pi} u(s)^2/ \sin(\pi/N)]^{0.5}$$
$$m_0(s, N) = 4 \sin^2(\pi/N)[1 + \frac{3}{2\pi} u(s)^2/ \sin(\pi/N)]^{0.5}$$ (4)

At $s=1$ the leading amplitude of the $k = 2\pi/N$ massgap is known exactly from
the Bethe ansatz solution, $m_1 \simeq 1.1689666(\pi/N)^{\frac{3}{2}}$ [4]. We use this in our scaling
analysis at $s = 1$: $m_1(1) = 1.1689666 \sin^{\frac{3}{2}}(\pi/N)[1 + (0.531 \pm 0.002) \sin(\pi/N) + ...]$. This gives information about the scaling field $u(s)$. The renormalization is
small. The amplitude predicted by eq.(4) assuming $u = s$ is only 7% too small.
Our knowledge of the non-linear scaling field can be summarized as $u(s) = s(1 + 0.0359697s^2 f(s^2))$, with $f(s^2)$ an unknown function, except for $f(1) = 1$.

The amplitude of the $k = 0$ massgap at $s=1$ is not known from the Bethe Ansatz
solution [4]. The two massgaps must depend on $u(s)$ in the same way. Therefore
eq.(4) predicts the exact value, $m_0(1) \simeq 2.8633717 \sin^{\frac{3}{2}}(\pi/N)$. This agrees well
with a free numerical estimate, $m_0(1) \simeq (2.87 \pm 0.01) \sin^{\frac{3}{2}}(\pi/N)$. Moreover, the
leading corrections to scaling amplitude is very stable when we assume this value
to be exact: $m_0(1) \simeq 2.8633717 \sin^{\frac{3}{2}}(\pi/N)[1 + (0.940 \pm 0.005) \sin(\pi/N) + ...]$.

It is tempting to visualize the low lying excitations in terms of quasi-particles
with a dispersion relation $E(k) \sim k^z$. The amplitude of the massgap $m_1 \simeq AN^{-z}$
yields the slope of such a dispersion relation. In this picture, $m_0$ must have an
amplitude twice a big as $m_1$, because the lowest $k = 0$ excitations involves the creation of two quasi particles of opposite momentum. This is true at the EW point, but not for KPZ type growth where the amplitudes differ by more than a factor 2. So the large length scale behaviour of KPZ growth can not be described by a free field theory this simple in structure.

Fig. 3 shows the convergence of the ratio between the numerical values of $m_1(s)$ and those predicted by eq.(4), as function of $s$. The leading corrections to scaling are incorporated. The representation of corrections to scaling is not unique because their origins cannot be distinguished numerically. We choose to represent them as:

$$m_1 = 2 \sin^2 \left( \frac{\pi}{N} \right) \left[ 1 + \frac{1}{\pi} \frac{u^2(s)}{\sin(\pi/N)} - b_1(s) s^2 + ... \right]^{0.5}$$

with $b_1(s) \simeq \frac{1}{2} + 0.137 s^2$ from the corrections to scaling in $m_1''(0)$ and $m_1(1)$. The ratio is a sensitive test because both quantities vanish in the thermodynamic limit. The $k = 2\pi/N$ massgap ratio converges to 1 within 0.1% for all $s$. Similarly, the $k = 0$ massgap ratio converges to within 0.5%. The quality of the convergence varies slightly with $s$. This reflects the uncertainty in the scaling field $u(s)$. The shape of the curves is sensitive to the precise choice for $f(s^2)$. $f(s^2) = 1$ can be ruled out, because our numerical estimates of the amplitudes at $s = 0.6$ is slightly but significantly smaller than predicted by $f(s^2) = 1$. We investigated the simple form $f(s^2) = (1-a)s^2 - a$ for various values of $a$. $f(s^2) = s^2$, shown in Fig. 3, gives the smoothest curves, but the differences are marginal. The convergence of both ratios stays within the limits quoted above for any of these choices.

We conclude that the asymptotic crossover scaling function eq.(3) is correct. The agreement with the exact numerical results is within 7% over the entire range $0 \leq s \leq 1$ when $s$ is assumed to be a pure scaling field. The agreement improves to within 0.5% when the renormalization of the growth parameter is incorporated.

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**Figure Captions**

Fig.1: Schematic temperature evolution of a facet ridge in an equilibrium crystal. KPZ growth coincides with the facet edge endpoint (a) and EW growth with the point where the facet ridge vanishes (b).

Fig.2: Finite size scaling approximants of the dynamic exponent $z$ at various values of $s$ from the $k = 2\pi/N$ massgap.

Fig.3: The ratio between the numerical values of the $k = 2\pi/N$ massgap and those of the proposed crossover scaling function eq.(4) for system sizes $N = 10 – 18$. 
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