Off-critical Luttinger Junctions

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We investigate Luttinger junctions of quantum wires away from criticality. The one-body scattering matrix, corresponding to the off-critical boundary conditions at the junction, admits in general antibound and/or bound states. Their contribution to the theory is fixed by causality. The presence/absence of bound states determines the existence of two different regimes with inequivalent physical properties. A scattering matrix without bound states defines an isolated equilibrium system. Bound states instead drive the system away from equilibrium, giving rise to non-trivial incoming or outgoing energy flows in the junction. We derive in both regimes and in explicit form the electromagnetic conductance tensor, pointing out the different impact of bound and antibound states.

I. INTRODUCTION

The past two decades have shown a constantly growing interest\textsuperscript{1,22} in the physics of quantum wire junctions. The quantum nature of the transport properties of these devices is fairly well described by the Tomonaga-Luttinger model\textsuperscript{23} on graphs of the type shown in FIG. 1. The edges \( \{E_i : i = 1,\ldots,n\} \) of such a star graph \( \Gamma \) are modeling the wires, whereas the vertex represents the junction, which can be treated\textsuperscript{19} as a point-like defect (impurity) in the Luttinger liquid. The defect is implemented by a nontrivial one-body scattering matrix \( S(k) \), \( k \) being the momentum. At criticality the boundary conditions at the vertex are scale invariant and the scattering matrix is a constant \( n \times n \) matrix \( S \). Various authors\textsuperscript{10,18,20,23,32} have investigated in this regime the physical properties of the system. In particular, they derived the conductance tensor \( G \), getting the following simple expression

\[
G_{ij} = G_{\text{line}}(\delta_{ij} - S_{ij}), \tag{1}
\]

\begin{figure}[h]
\centering
\includegraphics[width=0.3\textwidth]{star_graph}
\caption{A star graph \( \Gamma \) with scattering matrix \( S(k) \) at the vertex.}
\end{figure}

the coefficient \( G_{\text{line}} \) being the conductance of a single wire without junctions, which depends on the parameters (see eq. (43) below) of the Luttinger liquid.

In the present paper we pursue further the study of off-critical Luttinger junctions started in Ref. 22. Our main goal here is to extend the analysis to the case when \( S(k) \) admits bound states and to derive the off-critical generalization (see eq. (44) below) of the formula (1). This is not a purely theoretical problem, because realistic quantum wire junctions are usually noncritical. In order to reproduce this situation as close as possible, in what follows we will keep the theory critical in the bulk of \( \Gamma \), allowing for breaking of scale invariance only at the vertex. In this regime the scattering matrix is no longer constant, but depends on the momentum \( k \). The continuation of \( S(k) \) in the complex \( k \)-plane is a meromorphic function with poles of the type \( \eta = i\eta \) with real \( \eta \neq 0 \). It turns out that the behavior of the theory depends in a crucial way on the sign of \( \eta \). In absence of bound states (all \( \eta < 0 \)), the model has\textsuperscript{22} a unitary time evolution respecting time-translation invariance. Accordingly, the energy is conserved. The situation radically changes when \( S(k) \) admits bound states (some \( \eta > 0 \)). Each of them generates in the spectrum of the theory a kind of damped harmonic oscillator. These oscillators lead to a breakdown of time-translation invariance. The energy of the system is no longer conserved, which signals a nontrivial energy flow through the boundary (the vertex of \( \Gamma \)). The essential point here is that the relative non-equilibrium state is fixed by the fundamental physical principle of causality (local commutativity). Physically, the off-critical boundary conditions with \( \eta > 0 \) generate a specific boundary interaction with the environment. One possibility to implement such interaction in realistic quantum wire junctions might be\textsuperscript{2,10,15} an external magnetic field crossing the junction.

The paper is organized as follows. In the next section...
we briefly review the Tomonaga-Luttinger (TL) model on a star graph with off-critical boundary conditions at the vertex. The associated scattering matrix $S(k)$ and its analytic properties are described in detail. The symmetry content of the model is also analyzed here. In section III we examine the impact of the analytic structure of $S(k)$ on the physical properties of the theory. The generalization of the conductance formula (1) away from criticality is discussed in detail, confirming the difference of the conductance formula (1) away from criticality and some boundary conditions for $x = 0$. Following a standard QFT procedure, the latter are fixed by requiring that the operator $K = -\partial_x^2$ on $\Gamma$ is self-adjoint. This problem has been intensively investigated in the recent mathematical literature\cite{35–37}, where the subject goes under the name of “quantum graphs”. From these studies one infers that $K$ is self-adjoint on $\Gamma$ if and only if the field $\varphi$ satisfies the boundary condition\cite{35–37}

$$\sum_{j=1}^{n} [\Delta(\mathbb{I} - \mathbb{U})_{ij} \varphi(t, 0, j) - i(\mathbb{I} + \mathbb{U})_{ij} (\partial_x \varphi)(t, 0, j)] = 0,$$

where $\mathbb{U}$ is any unitary matrix and $\lambda > 0$ is a parameter with dimension of mass, characterizing the breaking of scale invariance.

Eq. (7) generalizes to the graph $\Gamma$ the familiar mixed (Robin) boundary condition on the half-line $\mathbb{R}_+$. The matrices $\mathbb{U} = \mathbb{I}$ and $\mathbb{U} = -\mathbb{I}$ define the Neumann and Dirichlet boundary conditions respectively. The physical interpretation of (7) in the context of bosonization was discussed in Ref. \cite{21}. At criticality \cite{7} describes the splitting of the electric current \cite{31} at the junction $38$.

\section{Luttinger Liquid with Off-Critical Boundary Conditions on $\Gamma$}

A. Bulk theory and boundary conditions

The dynamics of the Luttinger liquid in the bulk is defined by Lagrangian density

$$\mathcal{L} = iv_1^\alpha(\partial_t - v_F \partial_x)\psi_1 + iv_2^\alpha(\partial_t + v_F \partial_x)\psi_2 - g_+(\psi_1^* \psi_1 + \psi_2^* \psi_2)^2 - g_-(\psi_1^* \psi_1 - \psi_2^* \psi_2)^2. \quad (2)$$

Here $\{\psi_i(t, x, i) : \alpha = 1, 2\}$ are complex fields, depending on the time $t$ and the position $(x, i)$, where $x > 0$ is the distance from the vertex and $i = 1, ..., n$ labels the edge, as shown in FIG. \cite{1}. Finally, $v_F$ is the Fermi velocity and $g_\pm \in \mathbb{R}$ are the coupling constants\cite{34}. The equations of motion following from (2) imply the conservation law

$$\partial_t \psi_1(t, x, i) = v_F \partial_x \psi_2(t, x, i) = 0, \quad (3)$$

where

$$\psi_1(t, x, i) = (\psi^*_1 \psi_1 + \psi^*_2 \psi_2)(t, x, i), \quad (4)$$

$$j(t, x, i) = (\psi_1^* \psi_1 - \psi_2^* \psi_2)(t, x, i), \quad (5)$$

are the charge density and electric current respectively. Our main task below is to derive the relative conductance. For this purpose we have to complete first the description of the dynamics, specifying the interaction at the vertex of $\Gamma$. In other words, we must fix the boundary conditions on $\psi_\alpha$ at $x = 0$. This is a very delicate point because it strongly interferes with the solution of the model. It is useful to recall in this respect that the model \cite{2} is exactly solvable on the line $\mathbb{R}$ via bosonization. In order to preserve this nice feature on $\Gamma$, it is more convenient to formulate the boundary conditions directly in bosonic terms. We will show now that this strategy, which works\cite{20–22} nicely at criticality, can be extended to off-critical boundary conditions as well.

The basic ingredient for solving the model \cite{2} via bosonization is the scalar field $\varphi$ satisfying

$$\left(\partial_x^2 - \partial_y^2\right) \varphi(t, x, i) = 0, \quad x > 0 \quad (6)$$

and has a simple physical interpretation: the diagonal element $S_{ii}(k)$ represents the reflection amplitude from the defect on the edge $E_i$, whereas $S_{ij}(k)$ with $i \neq j$ equals the transmission amplitude from $E_i$ to $E_j$.

The matrix $S(k)$ has a number of remarkable features. It is unitary $S(k)^* = S(k)^{-1}$ by construction and satisfies $S(k)^* = S(-k)$, known as Hermitian analyticity. Notice also that $S(\lambda) = \mathbb{U}$, showing that the boundary condition \cite{7} is fixed actually by the value of scattering matrix at the scale $\lambda$.

Let us summarize now the analytic properties of $S(k)$ needed in what follows. We denote by $U$ the unitary matrix diagonalizing $U$ and parametrize

$$U_d = U^{-1} \mathbb{U} \quad (9)$$

as follows

$$U_d = \text{diag}(e^{2i\alpha_1}, e^{2i\alpha_2}, ..., e^{2i\alpha_n}), \quad \alpha_i \in \mathbb{R}. \quad (10)$$

Using (5), one easily verifies that $\mathbb{U}$ diagonalizes also $S(k)$.
for any \( k \) and that
\[
S_d(k) = U^{-1} S(k) U = \text{diag} \left( \frac{k + i \eta_1}{k - i \eta_1}, \frac{k + i \eta_2}{k - i \eta_2}, \ldots, \frac{k + i \eta_n}{k - i \eta_n} \right),
\]
where
\[
\eta_i = \lambda \tan(\alpha_i), \quad -\frac{\pi}{2} \leq \alpha_i \leq \frac{\pi}{2}.
\]
We conclude therefore that \( S(k) \) is a meromorphic function with \textit{simple} poles located on the imaginary axis and different from 0. In what follows we denote by \( \mathcal{P} = \{ i \eta : \eta \neq 0 \} \) the set of \textit{distinct} poles of \( S(k) \): the subset \( \mathcal{P}_+ = \{ i \eta : \eta > 0 \} \) in the upper half-plane corresponds to \textit{bound} states whereas \( \mathcal{P}_- = \{ i \eta : \eta < 0 \} \) in the lower half-plane gives raise to \textit{antibound} states. We will make use below also of the \textit{residue} matrix defined by
\[
R^{(\eta)}_{ij} = \frac{1}{i \eta} \lim_{k \to i \eta} (k - i \eta) S_{ij}(k), \quad i \eta \in \mathcal{P}.
\]
We recall that in the above parametrization the angular variables \( \alpha_i \) characterize the departure from criticality, the critical points corresponding to the values \( \alpha_i = 0, \pm \pi/2 \). The classification and the stability properties of these critical points have been extensively studied in the literature. As already stated in the introduction, our goal here is to go beyond and explore the theory away from criticality.

The basic steps in the construction of the field \( \varphi \), which satisfies the wave equation (6) and the off-critical boundary condition (7), are given in Appendix A. The subtlety point is the contribution of the bound states of \( S(k) \), which need a separate treatment where local commutativity turns out to be essential. The dual field \( \tilde{\varphi} \) is defined by
\[
\partial_t \tilde{\varphi}(t, x, i) = - \partial_x \varphi(t, x, i),
\]
\[
\partial_t \varphi(t, x, i) = - \partial_x \tilde{\varphi}(t, x, i),
\]
and the solution of the Luttinger model on \( \Gamma \) can be expressed in terms of the pair \( \{ \varphi, \tilde{\varphi} \} \). For the details we refer to Refs. 20–22, recalling here only the solution
\[
\psi_1(t, x, i) \sim e^{i \sqrt{\frac{\pi \kappa v_F + 2g_+}{\pi \kappa v_F + 2g_-}} \partial_t \varphi(ut, x, i)},
\]
\[
\psi_2(t, x, i) \sim e^{i \sqrt{\frac{\pi \kappa v_F + 2g_-}{\pi \kappa v_F + 2g_+}} \partial_t \tilde{\varphi}(ut, x, i)},
\]
and the expression of the electric current (5)
\[
j(t, x, i) = - \frac{v}{\nu_F \zeta + \sqrt{\pi}} \partial_x \varphi(ut, x, i),
\]
needed in the derivation of the conductance. In (12) \( \cdot \cdot \cdot \) denotes the normal product, whereas
\[
\zeta_\pm = \sqrt{|\kappa|} \left( \frac{\pi \kappa v_F + 2g_+}{\pi \kappa v_F + 2g_-} \right) \pm \frac{i}{\sqrt{\pi}}
\]
\[
v = \frac{\sqrt{(\pi \kappa v_F + 2g_-)(\pi \kappa v_F + 2g_+)}}{\pi |\kappa|}
\]
\( \kappa \) being the \textit{statistical parameter}. The conventional fermionic Luttinger liquid is obtained for \( \kappa = 1 \). For \( \kappa \neq 1 \) one has \textit{anyonic} Luttinger liquids. Notice also that the above solution of the Luttinger model is meaningful for coupling constants satisfying
\[
2g_\pm > -\pi \kappa v_F.
\]

C. Symmetry content

The bulk theory, defined by the Lagrangian (2), is invariant under time reversal and global \textit{U}(1) gauge transformations. Attempting to lift these symmetries to theory on whole star graph \( \Gamma \), one gets some restrictions on the boundary conditions (7). Time reversal symmetry implies that \( U \) must be symmetric,

\[
U^t = U.
\]

Concerning the conservation of the \textit{U}(1) (electric) charge
\[
Q = \sum_{i=1}^{n} \int_{0}^{\infty} dx j(t, x, i)
\]
on \( \Gamma \), one knows already from classical electrodynamics that the Kirchhoff’s rule
\[
\sum_{i=1}^{n} j(t, 0, i) = 0
\]
must be satisfied at the vertex of \( \Gamma \). Using (18), one can verify that (23) holds if and only if
\[
\sum_{i=1}^{n} S_{ij}(k) = 1, \quad \forall j = 1, \ldots, n,
\]
or, equivalently
\[
\sum_{i=1}^{n} U_{ij} = 1, \quad \forall j = 1, \ldots, n.
\]
Combining (19) and (24), one gets
\[
\sum_{i=1}^{n} R^{(\eta)}_{ij} = 0, \quad \forall i \eta \in \mathcal{P},
\]
which will be essential below for checking the Kirchhoff’s rule for the conductance away from criticality.

In what follows we assume that both (21) and (25) hold. The case with broken time reversal has been analyzed recently in Ref. 31.

III. BASIC FEATURES OF THE MODEL AWAY FROM CRITICALITY

A. Impact of the analytic structure of \( S(k) \)

The analytic properties of the scattering matrix \( S(k) \) deeply influence the physics of the Luttinger liquid on \( \Gamma \).
The simplest observable, one can investigate in order to illustrate this fact, is the electric current $j$. More precisely, it is enough to study the relative two-point function, which is fixed up to a real parameter $\tau$ by causality (local commutativity). Postponing the discussion of the physical meaning of $\tau$ to the end of this subsection, we consider first the current-current correlator following from the definition \(\mathbf{15}\). In the Fock representation of the field $\varphi$, defined by equations \(\mathbf{A1}, \mathbf{A2}\) and \(\mathbf{A9}\) in the Appendix \(\mathbf{A}\), one finds

$$
\langle j(t_1, x_1, i_1)j(t_2, x_2, i_2) \rangle = \frac{e^2}{(2\pi \zeta + i\eta)^2} \left[ D_{t_1,t_2}(vt_{12}, x_1, x_2) + A_{t_1,t_2}(vt_{12}, x_1, x_2) + B_{t_1,t_2}(vt_{12}, x_1, x_2, \tau) \right],
$$
where

$$
D_{t_1,t_2}(t, x_1, x_2) = -\delta_{t_1,t_2} \left[ d^2(t-x_{12}) + d^2(t-x_{12}) + \eta \left[ d(t+x_{12}) - d(t-x_{12}) \right] \right] - \eta^2 \left[ w_-(\eta(t-x_{12}) + w_+ (\eta(t+x_{12})) \right],
$$
$$
A_{t_1,t_2}(t, x_1, x_2) = \sum_{i_1,i_2 \in P_-} R_{i_1,i_2}^{(\eta)} \left[ d^2(t+x_{12}) + d^2(t-x_{12}) + \eta \left[ d(t+x_{12}) - d(t-x_{12}) \right] \right],
$$
$$
B_{t_1,t_2}(t, t_2, x_1, x_2; \tau) = \sum_{i_1,i_2 \in P_+} R_{i_1,i_2}^{(\eta)} \left[ d^2(t_{12}+x_{12}) + d^2(t-x_{12}) + \eta \left[ d(t_{12}+x_{12}) - d(t_{12}-x_{12}) \right] \right] - \eta^2 \left[ w_+(\eta(t_{12}-x_{12}) + w_-(\eta(t_{12}+x_{12})) + 2\eta^2 e^{-\eta x_{12}} \left[ \cosh(\eta(\tau_{12}-2\tau)) - \sinh(\eta t_{12}) \right] \right].
$$

Here and in what follows \(t_{12} = t_1 - t_2, \quad x_{12} = x_1 - x_2, \quad t_{12} = t_1 + t_2, \quad x_{12} = x_1 + x_2\) and

$$
d(\xi) = \frac{1}{\xi - i\epsilon}, \quad w_\pm(\xi) = e^{-\xi} \text{Ei}(\xi \pm i\epsilon),
$$

with $\epsilon > 0$ and $\text{Ei}$ the exponential integral function. By construction the functions $A_{t_1,t_2}$ and $B_{t_1,t_2}$ collect the contributions of the antibound and the bound states respectively. The main feature distinguishing these two functions is their time dependence. Notice indeed that $A_{t_1,t_2}$ depends exclusively on $t_{12}$, whereas $B_{t_1,t_2}$ depends in addition on $t_{12}$ and on the parameter $\tau$. This fact suggest the consideration of two separate cases:

(i) When bound states are absent ($P_+ = \emptyset$), the behavior of the system is quite orthodox; the function $B_{t_1,t_2}$ vanishes identically and time translations invariance is preserved. Accordingly, the energy is conserved. Because of \(\mathbf{21}\), the theory is also invariant under time reversal $t \rightarrow -t$.

$$
t \rightarrow -t.
$$

(ii) When bound states are present ($P_+ \neq \emptyset$) the situation changes drastically. Besides of $t_{12}$, the dynamics depends also on $t_{12}$, implying that the theory is no longer invariant under time translations. Therefore, the energy is not conserved. The theory depends on the parameter $\tau$ as well and is invariant under the time reversal transformation $t \rightarrow -t + 2\tau$.

As well known, the field products at coinciding points in \(\mathbf{44}\) contain divergences which must be subtracted. A natural way to fix the subtraction is to take as a reference the vacuum energy $\langle \theta(t, x) \rangle\langle \theta(t, x) \rangle$ on the line. Using conventional point splitting regularization, one gets in this way \(\mathbf{23}\)

$$
\mathcal{E}(t, x, i) = \langle \theta(t, x, i) \rangle - \langle \theta(t, x) \rangle\langle \theta(t, x) \rangle \sim \mathcal{E}(x, i) + \sum_{i_1 \in P_+} \eta^2 R_{i_1}^{(\eta)} e^{-2\eta x} \cosh(2\eta(t-\tau)),
$$

where $\mathcal{E}(x, i)$ is a time independent contribution, derived in Ref. \(\mathbf{10}\) but irrelevant for what follows. From \(\mathbf{96}\) we deduce that $\tau$ is the instance in which the vacuum energy flow inverts its direction, being outgoing for $t < \tau$ and...
incoming for \( t > \tau \). This feature is related to the fact that the theory is invariant under the operation (33), which obviously exchanges the two time intervals \((-\infty, \tau)\) and \((\tau, \infty)\). These considerations fix the physical meaning of the parameter \( \tau \), which appears in the theory when \( \mathcal{P}_+ \neq \emptyset \).

Summarizing, the behavior of off-critical Luttinger junctions is characterized by the two different regimes described in points (i) and (ii) above. In the case (i) one deals with an isolated system in equilibrium. Both energy and electric charge are conserved. The presence of bound states in the regime (ii) significantly modifies this behavior. The energy is no longer conserved because the relative boundary conditions determine a specific boundary interaction with the environment, which drives the system out of equilibrium. A remarkable feature is that the corresponding non-equilibrium state is fixed, up to the value of the parameter \( \tau \), by the basic physical requirement of causality.

We stress in conclusion that the above results are obtained in an abstract setup, where we attempt to describe the physics of a quantum wire junction by a Luttinger liquid with specific off-critical boundary conditions at the vertex of the star graph approximating the junction. Further investigations are needed for clarifying the plausibility of these assumptions and the applicability of the results to real-life quantum wire junctions. The case (i) looks more physical since the energy is conserved and bounded from below. In our opinion however, also the regime (ii) can not be excluded a priori, if one considers the possible interactions of the junction with the environment. In order to clarify the situation, one should analyze in the above framework some physical observables, which in principle can be checked experimentally. As an example, we derive in the next subsection the electromagnetic conductance in explicit form. Another attractive possibility to apply the above results is the development of an effective description of the Luttinger liquid on complex quantum wire networks with several junctions and loops, crossed possibly by magnetic fluxes implementing the interaction with the environment. The complete field theory analysis of such networks is usually a complicated problem. One approximate way to face the problem could be to use the star product approach\(^\text{17,18}\), the “gluing” technique\(^\text{49–51}\), or transfer matrix formalism\(^\text{52}\) for deriving the effective scattering matrix \( S_{\text{eff}}(k) \) relative to the external edges of the network. \( S_{\text{eff}}(k) \) admits in general both bound and antibound states and can be used\(^\text{20}\) for constructing a simplified model with one effective off-critical junction.

### B. Off-critical conductance

In order to compute the conductance, we couple the Luttinger liquid on \( \Gamma \) to a uniform classical electric field\(^\text{53}\).\(^\text{54}\)

\[ E(t, i) = \partial_t A_x(t, i) \]

performing in (2) the substitution

\[ \partial_x \longrightarrow \partial_x + iA_x(t, i). \]  

(36)

This external field coupling deforms the bosonized version of the current according to\(^\text{20}\)

\[ j(t, x, i) \longrightarrow j(t, x, i) = \frac{v}{v_F \zeta + i\pi} \left[ \partial_x \varphi(vt, x, i) + \frac{1}{\zeta + \sqrt{\pi}} A_x(t, i) \right] \]  

(37)

The Hamiltonian encoding the interaction of \( \varphi \) and \( A_x \) is time dependent and reads\(^\text{20}\)

\[ H_{\text{int}}(t) = \frac{1}{\zeta + \sqrt{\pi}} \sum_{i=1}^{n} \int_0^\infty dx \left( \partial_x \varphi(vt, x, i)A_x(t, i) \right). \]  

(38)

We stress that now the system can exchange energy with the environment in two different ways. The first one corresponds to the external force produced by the coupling with the time dependent electric field \( E(t, i) \). The second one represents an intrinsic property in the regime (ii), being related to the presence of bound states of \( S(k) \). The conductance, derived below, keeps track of both of them.

In order to derive the conductance, one should compute the expectation value of the current \( J(t, x, i) \) in the external field \( A_x \). This expectation value is given by the following well known\(^\text{19,20}\) series expansion

\[ \langle J(t, x, i) \rangle_{A_x} = \langle J(t, x, i) \rangle - i \int_{-\infty}^{t} d\tau_1 \langle J(t, x, i), H_{\text{int}}(\tau_1) \rangle + \cdots \]

\[ + (-i)^n \int_{-\infty}^{t} d\tau_1 \int_{-\infty}^{\tau_1} d\tau_2 \cdots \int_{-\infty}^{\tau_{n-1}} d\tau_n \left[ \left[ \left[ J(t, x, i), H_{\text{int}}(\tau_1) \right], H_{\text{int}}(\tau_2) \right], \cdots, H_{\text{int}}(\tau_n) \right] \]  

(39)

in powers of \( H_{\text{int}} \). This expansion is particularly simple in our case. In fact, from (37,38) one obtains that

\[ \left[ J(t, x, i), H_{\text{int}}(\tau_1) \right], H_{\text{int}}(\tau_2) \right] = 0, \]  

(40)
where \[ \langle J(t, x, i) \rangle_{A_x} = \frac{v}{v_F \xi^+} \int_{-\infty}^{t} \left[ A_x(t, i) + n \int_{0}^{t} d\tau \int_{0}^{\infty} dy A_y(\tau, j) \langle [\partial_y \varphi(v\tau, y, j), \partial_x \varphi(vt, x, i)] \rangle \right]. \] \hfill (41)

The conductance can be extracted directly from (42). The result is

\[ \langle J(t, x, i) \rangle_{A_x} \text{ is a consequence of the conservation of the electric charge, namely of equations (24, 26).} \]

which is a consequence of the conservation of the electric charge, namely of equations (24, 26). Notice that away from criticality \[ G_{ij}(\omega, t - t_0) \] is in general complex, showing that off-critical junctions may have a nontrivial impedance.

Before discussing the main features of (44), we would like to mention two useful checks. First of all we observe that at criticality, where scale invariance implies that \[ S \] is constant (\( k \)-independent), the sum over the poles vanishes and (44) precisely reproduces the expression (11) from the introduction. A second highly nontrivial check is the Kirchhoff rule

\[ \sum_{i=1}^{n} G_{ij}(\omega, t - t_0) = 0, \] \hfill (45)

which is a consequence of the conservation of the electric charge, namely of equations (44) and 20.

The sum in the right hand side of (44) runs over both negative and positive poles \( P = P_- \cup P_+ \). The antibound states \( P_- \) produce damped oscillations in \( t - t_0 \). If bound states are absent (\( P_+ = \emptyset \)),

\[ \lim_{t \to \infty} G_{ij}(\omega, t - t_0) = \lim_{t, \omega \to \infty} G_{ij}(\omega, t - t_0) = G_{\text{line}} \left[ \delta_{ij} - S_{ij} \left( \frac{\omega}{v} \right) \right], \] \hfill (46)

which gives the conductance one will observe in the regime (i) long time after switching on the external field.

Finally, the bound states \( P_+ \) give origin to oscillations whose amplitude is growing exponentially with \( t - t_0 \). The oscillations in (44) provide therefore a nice experimental signature for testing the analytic structure of the scattering matrix \( S(k) \).

**IV. OUTLOOK AND CONCLUSIONS**

Off-critical Luttinger junctions are characterized by a scattering matrix \( S(k) \) which admits in general bound and antibound states. The presence/absence of bound states determines two different regimes of the theory. A scattering matrix without bound states gives raise to an isolated equilibrium system. One bound state is enough to change radically the situation. We have show in fact that each such a state generates an oscillator degree of freedom, whose contribution is fixed by local commutativity up to a common free parameter \( \tau \). These additional degrees of freedom break the invariance under time translations and drive the system out of equilibrium. Accordingly, the vacuum energy is time dependent: it decays exponentially in the interval \( (-\infty, \tau) \) and grows at the same rate in \( (\tau, \infty) \). These two intervals are related by time reversal and are characterized by a nontrivial outgoing and incoming vacuum energy flows. Recalling that the traditional way for driving a system out of equilibrium is to couple it with a “bath” of external oscillators, we have shown above that under certain conditions such oscillators can be automatically generated by boundary effects. Junctions with bound states provide therefore
an intrinsic mechanism for constructing non-equilibrium quantum systems, whose behavior is governed by purely boundary phenomena. Since this mechanism is based on the general physical requirement of causality, it extends also to systems in space dimensions greater then one.

The off-critical conductance formula (44) has various interesting properties. A first remarkable feature of (44) is the different impact of antibound and bound states, which hopefully can be tested experimentally. The result (44) turns out to be useful also at the level of effective theory for quantum wire networks with several junctions, where it applies for the effective scattering matrix $S_{\text{eff}}(k)$ associated to the external edges of the network.

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Appendix A: Effect of the bound states of $S(k)$ on the field $\varphi$

We assume below that $S(k)$ admits bound states, the case $P_+ = \emptyset$ being standard. When $P_+ \neq \emptyset$ it is natural to represent the field $\varphi$ as a linear combination

$$\varphi(t, x, i) = \varphi^{(s)}(t, x, i) + \varphi^{(b)}(t, x, i),$$

where $\varphi^{(s)}$ collects the contribution of the scattering states of $S(k)$ and $\varphi^{(b)}$, that of its bound states. The scattering component $\varphi^{(s)}$ is known from previous studies. One has

$$\varphi^{(s)}(t, x, i) = \int_{-\infty}^{\infty} \frac{dk}{2\pi \sqrt{2|k|}} \left[ a^*_i(k) e^{i(k|x-t|)} + a_i(k) e^{-i(k|x-t|)} \right],$$

where $\{a_i(k), a^*_i(k) : k \in \mathbb{R}\}$ generate the reflection-transmission algebra. $\varphi^{(s)}(t, x, i) = \varphi^{(s)}(0, x, i)$.

Since for $P_+ \neq \emptyset$ the scattering states are not complete, the scattering component $\varphi^{(s)}$ is neither a canonical nor a local field. In fact, keeping in mind that $\bar{x}_{12} > 0$ and using the Cauchy integral formula one finds

$$[(\partial_t \varphi^{(s)})(0, x_1, i_1), \varphi^{(s)}(0, x_2, i_2)] = -i \delta_{i_1 i_2} \delta(x_1 - x_2) + i \sum_{i \in P_+} \eta e^{-\eta \bar{x}_{12}} R^{(\eta)}_{i_1 i_2},$$

and, at space-like separations $t_{12}^2 - x_{12}^2 < 0$,

$$[\varphi^{(s)}(t_1, x_1, i_1), \varphi^{(s)}(t_2, x_2, i_2)] = i \sum_{i \in P_+} e^{-\eta \bar{x}_{12}} \sinh(\eta t_{12}) R^{(\eta)}_{i_1 i_2}.$$

We turn now to the construction of the boundary component $\varphi^{(b)}$ which is designed in such a way that the total field $\varphi$ is both canonical and local. Let us introduce first the set $I_+ = \{ i : \eta_i > 0 \}$ labeling the different poles of $S(k)$ in the upper half plane. Then the normalizable solutions of the wave equation on $\Gamma$ corresponding to the bound states are $\{ e^{-\eta_i (x \pm t)} : i \in I_+ \}$. The main idea now is to associate with each index $i \in I_+$ a quantum oscillator. The quantum boundary degrees of freedom are described therefore by the algebra $B$ generated by $\{ b_i, b^*_i : i \in I_+ \}$, which satisfy

$$[b_i, b^*_j] = [b^*_i, b^*_j] = 0, \quad [b_i, b_j] = \delta_{i j} \delta_{ij},$$

and commute with $\{ a_i(k), a^*_i(k) \}$. Now, the field $\varphi^{(b)}$ is defined by

$$\varphi^{(b)}(t, x, i) = \frac{1}{\sqrt{2}} \sum_{j \in I_+} U_{ij} \left[ (b^*_j + b_j) e^{-\eta (x + t - \tau)} + i (b^*_j - b_j) e^{-\eta (x - t + \tau)} \right],$$

$U$ being the unitary matrix which appears in (9). By construction the field $\varphi^{(b)}$ satisfies the equation of motion (6), the boundary condition (7) and depends on the parameter $\tau \in \mathbb{R}$, whose physical meaning has been discussed in section II.D. One can also verify that the total field $\varphi$ is both canonical and local. In fact,

$$[\varphi^{(b)}(t_1, x_1, i_1), \varphi^{(b)}(t_2, x_2, i_2)] = -i \sum_{i \in P_+} e^{-\eta \bar{x}_{12}} \sinh(\eta t_{12}) R^{(\eta)}_{i_1 i_2},$$

which precisely compensates the right hand side of (A7).

The Hamiltonian $H$, generating the time evolution of $\varphi$ according to

$$[H, \varphi(t, x, i)] = -i (\partial_t \varphi)(t, x, i),$$

is given by

$$H = H^{(s)} + H^{(b)},$$

where $H^{(s)}$ and $H^{(b)}$ are the Hamiltonians associated to the scattering and bound states, respectively.
where the scattering contribution $H^{(s)}$ has the standard form

$$
H^{(s)} = \frac{1}{2} \sum_{i=1}^{n} \int_{-\infty}^{\infty} \frac{dk}{2\pi} |k| a_i^\dagger(k) a_i(k), \quad (A13)
$$

and

$$
H^{(b)} = \frac{i}{2} \sum_{j \in I_+} \eta_j (b_j^2 - b_j^*)^2. \quad (A14)
$$

We see that (A15) is $\tau$-independent. Finally, the current-current correlator $\langle \varphi(t_1, x_1, 1), \varphi(t_2, x_2, 2) \rangle$ is obtained in the Fock representation of the algebras $A$ and $B$, characterized as usual by a vacuum state $\Omega$ annihilated by $a_i(k)$ and $b_i$. In conclusion, we would like to stress once more the fundamental role played by causality (local commutativity) in the above construction. In fact, this deep physical requirement fixes (up to the parameter $\tau$) the whole structure of the field $\varphi$ and its time evolution.

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