ON IDEALS DEFINING IRREDUCIBLE REPRESENTATIONS OF REDUCTIVE $p$–ADIC GROUPS

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Abstract. Let $G$ be a reductive $p$–adic group. Assume that $L \subset G$ is an open–compact subgroup, and $\mathcal{H}_L$ is the Hecke algebra of $L$–biinvariant complex functions on $G$. It is a well–known and standard result about how to prove existence of a complex smooth irreducible $G$–module out of a maximal left ideal $I \subset \mathcal{H}_L$. Using theory about Bernstein center we make this construction explicit. This leads us to some very interesting questions.

1. Introduction

Let $k$ be a non–Archimedean local field. Let $\mathcal{O} \subset k$ be its ring of integers, and let $\varpi$ be a generator of the maximal ideal in $\mathcal{O}$. Let $q$ be the number of elements in the residue field $\mathcal{O}/\varpi \mathcal{O}$. Assume that is $G$ is a Zariski connected reductive group defined over $k$. By abuse of notation, we write $G$ for the group of $k$–points. Similarly we do for algebraic subgroups defined over $k$. We fix a Haar measure $dy$ on $G$, and consider $\mathcal{H} = C_c^\infty(G)$ as an associative algebra under the convolution defined with respect to $dy$. For any open compact subgroup $L \subset G$, we let $\mathcal{H}_L$ be the subalgebra of $\mathcal{H}$ consisting of all $L$–biinvariant functions in $\mathcal{H}$. This is an associative algebra with identity $\epsilon_L$ (see (2-1)).

In this paper maximal ideals are assumed to be proper. We will be concerned with the following simple and well–known result: Let $W$ be a (finite–dimensional) irreducible unital $\mathcal{H}_L$–module, then there exists a unique up to an isomorphism irreducible smooth $G$–module $V$ such that $V^L$ is isomorphic to $W$ as a $\mathcal{H}_L$–modules (see [5], Proposition 2.10 c)). All subsequent proofs of this result that we are able to find are not very constructive. In ([11], Theorem 3-9) we give a very explicit construction of the representation $V$ once we fix a maximal left ideal in $\mathcal{H}_L$ such that $W \simeq \mathcal{H}_L/I$ (see elementary Lemma 2-2 about relation between different possibilities for left ideal $I$; $I$ is uniquely determined by $W$ if $\mathcal{H}_L$ is commutative).

So, let us fix $L \subset G$ an open compact subgroup, and let $I \subset \mathcal{H}_L$ be a maximal left ideal. Following ([11], Theorem 3-9) (see Lemma 2-3), we put

$$J_{I,L} \overset{\text{def}}{=} \text{sum of all proper left ideals in } \mathcal{H} \star \epsilon_L \text{ which contain } \mathcal{H} \star I.$$  

Then, $J_{I,L}$ is a unique maximal left ideal in $\mathcal{H} \star \epsilon_L$ which contains $I$. The corresponding irreducible smooth $G$–module $V(I,L)$ (see Lemma 2-3) has space of $L$–invariants isomorphic to $\mathcal{H}_L/I$ as $\mathcal{H}_L$–modules. It is observed in ([11], Theorem 3-9) (see Lemma 2-3(v)) that a

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smooth $G$–module
\[ W(I, L) \overset{\text{def}}{=} \mathcal{H} \star \epsilon_L / \mathcal{H} \star I \]

has a unique maximal proper subrepresentation, and the corresponding quotient is $\mathcal{V}(I, L)$. The canonical projection $W(I, L)^L \longrightarrow V(I, L)^L$ is an isomorphism of $\mathcal{H}_L$–modules.

The goal of this note is to understand the module $W(I, L)$. We start with an important case. Assume that $L$ has an Iwahori factorization (see [4], 1.2). Then $W(I, L)$ being generated by $W(I, L)^L$ has property that every submodule has a non–zero vector invariant under $L$ by a rather deep result ([1], 3.9, see also [14], Corollaire VI.9.4 for the proof, and [4], 1.2). This immediately implies the result:

\[ W(I, L) = \mathcal{V}(I, L). \]

Hence, we have proved the following proposition:

**Proposition 1-1.** Let $L \subset G$ be an open compact subgroup having an Iwahori factorization, and $I \subset \mathcal{H}_L$ be a maximal left ideal. Then, $W(I, L) = \mathcal{V}(I, L)$, and, consequently, $J_{I,L} = \mathcal{H} \star I$.

This is a particular case of the following general theorem which is proved using techniques from ([1], [3]). We remark that a hyperspecial maximal compact subgroup does not posses an Iwahori factorization.

**Theorem 1-2.** Let $L \subset G$ be an open compact, and $I \subset \mathcal{H}_L$ be a maximal left ideal. Then, $W(I, L)$ is an admissible smooth $G$–module of finite length, all of its irreducible subquotients have the same infinitesimal character (see [3] for definition or Section 3 in this paper), it has a unique maximal proper subrepresentation, and the corresponding quotient is unique up to an isomorphism irreducible smooth $G$–module which space of $L$–invariants is isomorphic to $\mathcal{H}_L/I$ as a $\mathcal{H}_L$–module.

In Section 2 we consider elementary theory of Hecke algebras and its ideals. In Section 3 we recall from [3] some results about the Bernstein center needed in the proof of Theorem 1-2. We also prove a few more results. Finally, we prove Theorem 1-2 in Section 4. Now, we discuss some of the consequences.

Let $V$ be an irreducible smooth $G$–module. For each open compact subgroup $L \subset G$ such that $V^L \neq 0$ we fix a maximal left ideal in $\mathcal{H}_L$, say $I_{V,L}$, in the equivalence class determined by $\mathcal{H}_L$–module $V^L$ (see Lemma 2-2).

The number of irreducible smooth $G$–modules with the same infinitesimal character is finite (see [3] or Section 3). Therefore, for sufficiently small open compact subgroup $L_0 \subset G$, all of them have a non–zero vector invariant under $L$ where $L$ is any open–compact subgroup of $G$ contained in $L_0$. So, we obtain

**Corollary 1-3.** Let $V$ be an irreducible smooth $G$–module. Then, there exists an open compact subgroup $L_0 \subset G$ depending on $V$ such that for any open compact subgroup $L \subset L_0$, we have $W(I_{V,L}, L) = V$.

Finally, we state the following corollary:
Corollary 1-4. Let $V$ be an irreducible smooth $G$–module. If the infinitesimal character of $V$ is in general position (see Definition 3-1), then $\mathcal{W}(I_{V,L}, L) = V$ for all $L$ such that $V^L \neq 0$.

This follows immediately from Theorem 1-2 again since $\mathcal{W}(I,L)$ is irreducible, and $V$ is the only irreducible $G$–module with the infinitesimal character of $V$ (see Definition 3-1). We remark that the set of infinitesimal characters is an affine variety (\cite{3} or Section 3 here), and there exists Zariski open subset of that variety consisting of infinitesimal characters in general position (see Lemma 3-2).

We note that $\mathcal{W}(I,L)$ is not always irreducible. Before we state the result, we introduce some notation. Let $A_0$ be a maximal $k$–split torus of $G$. Let $P_0$ be a minimal $k$–parabolic subgroup of $G$ corresponding to some choice of positive roots of $A_0$ in $G$. Let $U_0$ be the unipotent radical of $P_0$. Let $K$ be a compatible hyperspecial maximal compact subgroup. We have the following result:

Proposition 1-5. Assume that $V$ is a $K$–spherical irreducible smooth $G$–module with its data in the Langlands classification of irreducible representations supported on $P_0$, i.e., there exists an unramified character $\chi$ of the Levi subgroup $M_0 = Z_G(A_0)$ of $P_0$ satisfying usual positiveness conditions such that $V$ is a unique irreducible quotient of the parabolically induced representation $\text{Ind}^G_{P_0}(\chi)$ (see for example \cite{14}, Theorem VII.4.2). Then,

$$\mathcal{W}(I_{V,K}, K) \simeq \text{Ind}^G_{P_0}(\chi).$$

Proof. Since $\text{Ind}^G_{P_0}(\chi)^K \simeq V^K$ as a $\mathcal{H}_K$–module generates whole $\text{Ind}^G_{P_0}(\chi)$, we see that $\text{Ind}^G_{P_0}(\chi)$ is a quotient of $\mathcal{W}(I_{V,K}, K)$. Next, by Theorem 1-2, $\mathcal{W}(I_{V,K}, K)$ has finite length, all of its irreducible subquotients are among those of $\text{Ind}^G_{P_0}(\chi)$. Now, the explicit computation of the Jacquet module of $\mathcal{W}(I_{V,K}, K)$ with respect to $P_0$ (see Proposition 3-9) completes the proof. \qed

The induced representation $\text{Ind}^G_{P_0}(\chi)$ can have a large number of irreducible subquotients. For example, one can see that from the Zelevinsky classification for $GL(n,k)$ (see \cite{16}). When $G = GL(2,k)$ and $V$ is the trivial representation, the induced representation $\text{Ind}^G_{P_0}(\chi)$ has length two; Steinberg representation of $GL(2,k)$ is a unique subrepresentation.

I would like to thank Gordan Savin for explaining to me the proof of Proposition 1-5 when $G$ is split and adjoint using methods of Iwahori Hecke algebras and Kazhdan–Lusztig classification (see \cite{9}). I would like to thank the referee for useful comments on the content of the paper.

2. Elementary Properties of Hecke Algebras and Ideals

Let $k$ be a non–Archimedean local field. Let $\mathcal{O} \subset k$ be its ring of integers, and let $\varpi$ be a generator of the maximal ideal in $\mathcal{O}$. Let $q$ be the number of elements in the residue field $\mathcal{O}/\varpi \mathcal{O}$. Assume that is $G$ is a Zariski connected reductive group defined over $k$. By abuse of notation, we write $G$ for the group of $k$–points. Similarly we do for subgroups defined
over $k$. We fix the Haar measure $dy$ on $G$ and let $\mathcal{H} = C^\infty_c(G)$ considered as an associative algebra under the convolution:

$$f \ast g(x) = \int_G f(xy^{-1})g(y)dy.$$  

For any open compact subgroup $L \subset G$, we let $\mathcal{H}_L$ be the subalgebra of $\mathcal{H}$ consisting of all $L$–biinvariant functions in $\mathcal{H}$. This is an associative algebra with identity

\begin{equation}
\epsilon_L \overset{\text{def}}{=} \frac{1}{\text{vol}(L)}1_L,
\end{equation}

where $1_L$ is the characteristic function of $L$ in $G$. We have

$$\mathcal{H} = \bigcup_L \mathcal{H}_L,$$

where $L$ ranges over all open compact subgroups of $G$ (or just over a basis of neighborhoods of 1). By a standard definition ([5], 2.5), $\mathcal{H}$–module $V$ is non–degenerate if for any $v \in V$ there exists an open compact subgroup $L \subset G$ such that $\epsilon_L v = v$. The space $\epsilon_L V$ is an unital module for $\mathcal{H}$. A non–degenerate $\mathcal{H}$–module gives rise to a unique smooth $G$–module such that

$$x.(f.v) \overset{\text{def}}{=} (l_x f).v, \quad f \in \mathcal{H}(G), \quad v \in V,$$

where $l_x$ is the left translation $l_x f(y) = f(x^{-1}y)$. We have

$$V^L = \epsilon_L V,$$

for all open–compact subgroups $L \subset G$.

The category of all smooth $G$–modules can be identified with the category of all non–degenerate $\mathcal{H}$–modules. In particular, an $\mathcal{H}$–module is irreducible if and only if its is irreducible smooth $G$–module. By a theorem of Jacquet ([14], Theorem VI.2.2), every irreducible smooth $G$–module $V$ is admissible i.e., the space $V^L$ is a finite dimensional complex vector space for all open–compact subgroups of $G$. For irreducible $V$, if $V^L \neq 0$, then it is an irreducible $\mathcal{H}_L$–module (see [5], Proposition 2.10). Moreover, by the same reference, let $L \subset G$ be an open–compact subgroup, and assume that $V_i, i = 1, 2$, are irreducible smooth $G$–modules such that $V_i^L \neq 0, i = 1, 2$. Then, $V_1$ is equivalent to $V_2$ as a $G$–module if and only if $V_1^L$ is equivalent to $V_2^L$ as a $\mathcal{H}_L$–module.

Assume that $W$ is an irreducible unital $\mathcal{H}_L$–module. Let $w \in W$ be a non–zero vector in $W$. Then

$$I \overset{\text{def}}{=} I_w \overset{\text{def}}{=} \text{Ann}_{\mathcal{H}_L}(w)$$

is a maximal left ideal in $\mathcal{H}_L$. Obviously, we have

$$\mathcal{H}_L/I \simeq W$$

as $\mathcal{H}_L$–modules. Different choice of a vector $w'$ result in a existence of $f, g \in \mathcal{H}_L$ such that $w' = f.w$, and $w = g.w'$. Put $I' = I_{w'}$. Then, we have the following:

$$\begin{cases}
I \ast g \subset I', \quad I' \ast f \subset I, \\
\epsilon_L - g \ast f \equiv 0 \pmod{I}, \quad \epsilon_L - f \ast g \equiv 0 \pmod{I'}.
\end{cases}$$
We have following lemma:

**Lemma 2-2.** Let \( L \subset G \) be an open compact subgroup. We define that two maximal left ideals \( I \) and \( I' \) in \( \mathcal{H}_L \) are equivalent if and only if there exist \( f, g \in \mathcal{H}_L \) such that (2) holds. Then, maximal left ideals \( I \) and \( I' \) are equivalent if and only if the corresponding \( \mathcal{H}_L \)-modules are isomorphic:

\[
\mathcal{H}_L/I \simeq \mathcal{H}_L/I'.
\]
Furthermore, the classes of equivalence of irreducible unital \( \mathcal{H}_L \)-modules are parametrized by the equivalence classes of maximal left ideals in \( \mathcal{H}_L \).

**Proof.** We leave details to the reader. \( \square \)

We continue with the following simple result (see [3], Theorem 3-9) which makes ([5], Proposition 2.10 c)) more explicit.

**Lemma 2-3.** Let \( L \subset G \) be an open–compact subgroup. Then, for each maximal left ideal \( I \subset \mathcal{H}_L \), there exists a unique left ideal \( J' \) of \( \mathcal{H} \) such that the following three conditions hold:

(i) \( J' \subset \mathcal{H} \star \epsilon_L \);
(ii) \( I \subset J' \), or equivalently \( \mathcal{H} \star I \subset J \);
(iii) \( \mathcal{H} \star \epsilon_L / J' \) is irreducible.

The left ideal \( J' \) is a unique maximal left–ideal, denoted by

\[
J_I = J_{I,L},
\]

in \( \mathcal{H} \star \epsilon_L \) which contains \( I \). It is a sum of all proper left ideals in \( \mathcal{H} \star \epsilon_L \) which contain \( I \). Moreover, \( \epsilon_L \star J_{I,L} = I \).

(iv) Regarding

\[
\mathcal{V}(I, L) \overset{\text{def}}{=} \mathcal{H} \star \epsilon_L / J_{I,L}
\]
as a smooth \( G \)-module (see above), we have that its space of \( L \)-invariants is isomorphic to (the irreducible module) \( \mathcal{H}_L/I \) as a \( \mathcal{H}_L \)-module. Up to isomorphism, \( \mathcal{V}(I, L) \) is unique irreducible smooth \( G \)-module with this property (see [3], Proposition 2.10).

(v) The smooth \( G \)-module

\[
\mathcal{W}(I, L) \overset{\text{def}}{=} \mathcal{H} \star \epsilon_L / \mathcal{H} \star I
\]
has a unique maximal proper subrepresentation, and the corresponding quotient is \( \mathcal{V}(I, L) \). The canonical projection \( \mathcal{W}(I, L)^L \to \mathcal{V}(I, L)^L \) is an isomorphism of \( \mathcal{H}_L \)-modules. The module \( \mathcal{W}(I, L) \) is generated by \( \epsilon_L + \mathcal{H} \star I \).

3. **Bernstein center**

In this section we describe Bernstein center and its action on smooth representations of \( G \). We follow ([3], Section 2). We continue with assumption from the first paragraph of the introduction.

We fix a minimal parabolic subgroup \( P_0 \), its Levi decomposition \( P_0 = M_0 U_0 \), and, as usual related to these choices, we fix a set of standard parabolic subgroups \( P \) \( M \) such that \( M_0 \subset M \), \( P = MP_0 \). We call
$M$ a standard Levi subgroup. (See the text immediately before the statement of Proposition 1-5.)

Following (3, 2.1), we call the pair $(M, \rho)$, where $M$ is a standard Levi subgroup and $\rho$ an irreducible supercuspidal representation of $M$, a cuspidal pair of $G$. Let $\Theta(G)$ be the set of all cuspidal pairs up to a conjugation by $G$. We write $[M, \rho]$ for the $G$–orbit in $\Theta(G)$ of $(M, \rho)$. A point in $\Theta(G)$ is called infinitesimal character of $G$. If we write $\theta \in \Theta(G)$ in the form $\theta = [M, \rho]$, then we say that infinitesimal character $\theta$ is determined by the cuspidal pair $(M, \rho)$.

Let $\Psi(M)$ be the group of all unramified characters if $M$. It has a natural structure of a complex algebraic torus. A connected component $\Theta$ in $\Theta(G)$ determined by the cuspidal pair $(M, \rho)$ is the image of the map $\Psi(M) \to \Theta(G)$ given by $\psi \mapsto [M, \psi \rho]$. The set $\Theta$ has a natural structure of a complex affine variety given by quotient of $\Psi(M)$ by a finite group. We denote by $\mathcal{Z}(\Theta)$ its algebra of regular functions.

We have $$\Theta(G) = \cup_{\Theta} \Theta$$ (disjoint union).

By standard theory, given $\theta \in \Theta(G)$, all parabolically induced representations $\text{Ind}_G^P(\rho)$ (normalized induction) have the same semi–simplifications in the Grothendieck group of all finite length smooth $G$–modules when cuspidal pairs $(M, \rho)$ range over $G$–orbit $\theta$, and $P = MP_0$. Different points in $\Theta(G)$ determine disjoint sets of irreducible $G$–modules (after semi–simplification).

Let $\sigma$ be an irreducible smooth $G$–module. Then, by remarks in the previous paragraph, there exists a unique $\theta \in \Theta(G)$ such that if we write $\theta = [M, \rho]$, then $\sigma$ is an irreducible subquotient of $\text{Ind}_G^P(\rho)$. We call $\theta$ the infinitesimal character of $\sigma$, and write $\theta = \inf\cdot \text{char.}(\sigma)$.

Let $\text{Irr}(G)$ be the set of equivalence classes of smooth irreducible $G$–modules. Then, the map $\text{Irr}(G) \to \Theta(G)$, $\sigma \mapsto \inf\cdot \text{char.}(\sigma)$ is finite to one; a preimage of $\theta \in \Theta(G)$ is the set of all irreducible subquotients of $\text{Ind}_G^P(\rho)$ where $\theta = [M, \rho]$, and $P = MP_0$.

**Definition 3-1.** We say that the infinitesimal character $\theta \in \Theta(G)$ is in general position if $\text{Ind}_G^P(\sigma)$ is irreducible, where $\theta = [M, \rho]$ and $P = MP_0$.

Of course, the definition is independent of the choice of the representative $(M, \rho)$.

**Lemma 3-2.** Let $\Theta \subset \Theta(G)$ be a connected component. Then, there exists Zariski open set $U \subset \Theta$ such that $\theta \in U$ is in general position.

**Proof.** Assume that $\Theta$ is the image of the map $\alpha : \Psi(M) \to \Theta(G)$ given by $\alpha(\psi) = [M, \psi \rho]$.

It is well–known that there exists Zariski open set $U' \subset \Psi(M)$ such that $\text{Ind}_P^G(\psi \rho)$ is irreducible for $\psi \in U'$. For example, this set is obtained if we take all $\psi$ such that the normalized Jacquet module with respect to $P$ has different central characters, and the long–intertwining operator is regular and an isomorphism. That kind of standard and well–known considerations can be easily extracted from (12). The proof is also contained in (14, Théorème VI.8.5)

As we recalled above, the set $\Theta$ has natural structure of a complex affine variety given by quotient of $\Psi(M)$ by a finite group. This implies that canonical regular map $\alpha : \Psi(M) \to \Theta$ is a finite regular map between affine algebraic varieties. In particular, the image of any closed
set is closed [13, Chapter 5, Section 3, Corollary]. The required open set is

\[ U = \Theta \setminus \alpha (\Psi(M) - U') \subset \alpha(U'). \]

□

Following [3, Section 2], we let

\[ \mathcal{Z}(G) = \prod_{\Theta} \mathcal{Z}(\Theta). \]

This \( C \)-algebra can be interpreted as an algebra of regular functions on the affine variety \( \Theta(G) \) with infinitely many connected components \( \Theta \). The ideal

\[ \mathcal{Z}(G)^0 \overset{\text{def}}{=} \oplus_{\Theta} \mathcal{Z}(\Theta) \]

is a proper ideal in \( \mathcal{Z}(G) \). One can easily check that \( C \)-algebra homomorphisms \( \mathcal{Z}(G) \rightarrow \mathbb{C} \) non–trivial on \( \mathcal{Z}(G)^0 \) are exactly evaluations at points in \( \Theta(G) \).

We recall from ([1], 2.13) the following result:

**Theorem 3-3.** For each smooth \( G \)-module \( V \) there exists a homomorphism \( \mathcal{Z}(G) \rightarrow \text{End}_G(V) \) of \( C \)-algebras such that the following holds:

(C-1) if \( V \) is irreducible, then the action of \( z \in \mathcal{Z}(G) \) is given by \( z = \inf\text{.char.}(V)(z)1_V \);

(C-2) we have \( \text{Hom}_G(V, V') \subset \text{Hom}_{\mathcal{Z}(G)}(V, V') \) for all smooth \( G \)-modules \( V \) and \( V' \).

The properties (i) and (ii) determine the system of \( C \)-algebra homomorphisms \( \mathcal{Z}(G) \rightarrow \text{End}_G(V) \), where \( V \) ranges over smooth \( G \)-modules, uniquely.

Next, we recall the following result ([1], Proposition 3.3):

**Theorem 3-4.** A finitely generated smooth representation \( V \) is \( \mathcal{Z}(G) \)-admissible i.e., for each open–compact subgroup \( L \subset G \) the space of \( L \)-invariants \( V^L \) is finitely generated \( \mathcal{Z}(G) \)-module.

Finally, we recall the Decomposition theorem ([1], 2.10):

**Theorem 3-5.** Let \( V \) be a smooth \( G \)-module. Let \( 1_\Theta \in \mathcal{Z}(\Theta) \) be the identity for each connected component \( \Theta \). Then, \( 1_\Theta \) act on \( V \) as a projector on a \( G \)-submodule denoted by \( V_\Theta \). We have

\[ V = \oplus_{\Theta} V_\Theta \]

Moreover, for any open compact subgroup \( L \subset G \), there exists only finitely many connected components \( \Theta \) such that \( V^L_\Theta \neq 0 \).

We end this section with the following two well–known observations:

**Corollary 3-6.** Assume that \( V \) is an irreducible smooth \( G \)-module. Then, there exists a unique connected component \( \Theta \) such that \( V = V_\Theta \). Moreover, we have \( \inf\text{.char.}(V) \in \Theta \).
Proof. Since $V$ is irreducible, the first claim is obvious from Theorem 3-5. This implies that $1_\Theta$ acts as identity on $V$. In particular,

$$inf.char.(V)(1_\Theta) = 1$$

by (C-1) in Theorem 3-3. This implies the second claim. \hfill \Box

Corollary 3-7. Let $\Theta$ be a connected component. Then, the functor $V \mapsto V_\Theta$ is exact functor from the category of all smooth $G$–modules into the same category.

Proof. Consider the short exact sequence

$$0 \rightarrow V \xrightarrow{\alpha} W \xrightarrow{\beta} U \rightarrow 0$$

of smooth $G$–modules maps. By Theorem 3-3 (C-2), we have

$$(\text{the action of } 1_\Theta \text{ on } W) \circ \alpha = \alpha \circ (\text{the action of } 1_\Theta \text{ on } V),$$

and

$$(\text{the action of } 1_\Theta \text{ on } U) \circ \beta = \alpha \circ (\text{the action of } 1_\Theta \text{ on } W).$$

Since $1_\Theta$ acts as a projection, i.e., $1_\Theta^2 = 1_\Theta$, this implies that we have the following sequence of $G$–modules maps:

$$V_\Theta \xrightarrow{\alpha|_{V_\Theta}} W_\Theta \xrightarrow{\beta|_{W_\Theta}} U_\Theta.$$  

It is obvious that $\alpha|_{V_\Theta}$ is injective and that $\beta|_{W_\Theta} \circ \alpha|_{V_\Theta} = 0$. Again, since $1_\Theta$ acts as a projection, one sees that the kernel of $\beta|_{W_\Theta} \circ \alpha|_{V_\Theta}$ is isomorphic to the image of $\alpha|_{V_\Theta}$ as well as that $\beta|_{W_\Theta}$ is surjective. For example, if $\beta|_{V_\Theta}(w) = 0$, then there exists $v \in V$ such that $w = \alpha(v)$ since the original sequence is exact. Then,

$$w = 1_\Theta.w = 1_\Theta.\alpha(v) = \alpha(1_\Theta.v) = \alpha|_{V_\Theta}(1_\Theta.v).$$

\hfill \Box

4. The proof of Theorem 1-2

As we already mentioned in the introduction, it is observed in ([11], Theorem 3-9) (see Lemma 2-3 (v)) that a smooth $G$–module $W(I,L)$ has a unique maximal proper subrepresentation, and the corresponding quotient is $V(I,L)$. Moreover, the canonical projection $W(I,L) \rightarrow V(I,L)$ is an isomorphism of $\mathcal{H}_L$–modules.

It remains to prove the most difficult part: $W(I,L)$ is an admissible smooth $G$–module of finite length, and all irreducible subquotients have the same infinitesimal character. For this, we use results on Bernstein center ([3], [1]) stated in Section 3.

First, we use the Decomposition theorem (see Theorem 3-5). As a result, we obtain the decomposition as smooth $G$–modules

$$W(I,L) = \bigoplus_\Theta W(I,L)_\Theta,$$

and as $\mathcal{H}_L$–modules

$$W(I,L)^L = \bigoplus_\Theta W(I,L)_\Theta^L.$$
But as we recalled above, $\mathcal{W}(I, L)^L$ is isomorphic to $\mathcal{V}(I, L)^L$ as a $\mathcal{H}_L$–module. Hence, $\mathcal{W}(I, L)^L$ is irreducible $\mathcal{H}_L$–module. Therefore, there exists a unique connected component $\Theta$ such that

$$\mathcal{W}(I, L)^L = \mathcal{W}(I, L)_\Theta^L.$$  

Since

$$(4-1) \quad \epsilon_L + \mathcal{H} \cdot I \in \mathcal{W}(I, L)^L$$

generates $\mathcal{W}(I, L)$ as a $\mathcal{H}$–module (see Lemma 2-3 (v)), we see that

$$\mathcal{W}(I, L) = \mathcal{W}(I, L)_\Theta.$$  

Since irreducible smooth module $\mathcal{V}(I, L)$ is a quotient of $\mathcal{W}(I, L)$, we must have by Corollary 3-7

$$\mathcal{V}(I, L) = \mathcal{V}(I, L)_\Theta.$$  

This implies that the infinitesimal character of $\mathcal{V}(I, L)$, say

$$\theta = [M, \rho],$$

must belong to the connected component $\Theta$ (see Corollary 3-6). Next, by Theorem 3-3 (C-1), we must have that $\mathcal{Z}(G)$ acts as as a character $\theta$ i.e.,

$$z = \theta(z)1_{\mathcal{W}(I, L)}, \quad z \in \mathcal{Z}(G).$$

Using the isomorphism of $\mathcal{W}(I, L)^L \simeq \mathcal{V}(I, L)^L$ of $\mathcal{H}_L$–modules, we see that $\mathcal{Z}(G)$ acts as a character $\theta$ on $\mathcal{W}(I, L)^L$. But, $\mathcal{W}(I, L)$ is as a smooth $G$–module generated by a class in (4-1). Hence, $\mathcal{Z}(G)$ acts as a character $\theta$ on $\mathcal{W}(I, L)$. This implies that every irreducible subquotient of $\mathcal{W}(I, L)$ has infinitesimal character $\theta$.

Finally, we prove that $\mathcal{W}(I, L)$ is an admissible smooth $G$–module of finite length. Let $L' \subset G$ be an open compact subgroup. Then, by Theorem 3-4 $\mathcal{W}(I, L)^L$ is finitely generated as $\mathcal{Z}(G)$–module. Since $\mathcal{Z}(G)$ acts on $\mathcal{W}(I, L)$ as a character $\theta$, we see that $\dim_{\mathbb{C}} \mathcal{W}(I, L)^L < \infty$. Since $L'$ is arbitrary, we obtain that $\mathcal{W}(I, L)$ is admissible. Finally, it has finite length since every finitely generated admissible $G$–module has finite length (see [8], theorem 6.3.10, or [5], Theorem 4.1 for $GL_N$). This completes the proof of Theorem 1-2.

5. Computation of Certain Jacquet modules and Proof of Proposition 1-5

We maintain the notation from the Introduction and Section 3. We recall the notion of a normalized Jacquet module. Let $P = MU$ be a standard parabolic subgroup of $G$. Let $V$ be a smooth $G$–module. Then, $\mathbb{C}$–span, say $V(U)$, of all $v - u.v, \ v \in V$ and $u \in U$, is $M$–invariant. Therefore, the quotient $V/V(U)$ is canonically smooth $M$–module. The corresponding normalized Jacquet module $r_{M,G}(V)$ is the space $V/V(U)$ under the action of $M$ given by

$$m.(v + V(U)) = \delta_P^{-1/2}(m)m.v + V(U), \quad v \in V, \ m \in M.$$  

Here $\delta_P$ is usual modular character of $P$.

Let us fix some Haar measure on $U$, for example normalized with $\int_{U \cap K} du = 1$, where $K$ is a hyperspecial maximal compact subgroup (see the text before the statement of Proposition
The space $V(U)$, can also be characterized as follows (see \[8\], 3.2): $v \in V(U)$ if and only if $\int_{L_U} u.v \, du = 0$ for some open compact subgroup $L_U \subset U$.

We begin with

**Lemma 5-1.** Let $P = MU$ be a standard parabolic subgroup of $G$. Let $f \mapsto \mapsto f_P$ be the constant term map $C^\infty_c(G) \to C^\infty_c(U \setminus G)$: $f_P(x) = \int_U f(ux) \, du$. Let $V \subset C^\infty_c(G)$ be a smooth $G$–submodule under the left translation $l$. Then, $r_{M,G}(V)$ is the image of $V$ under the constant term map, and the action of $m \in M$ is given by $\delta^{1/2}(m)l(m)$.

**Proof.** It is obvious that the restriction of the constant term map factors through $V/V(U)$. It remains to prove that $f_P$ restricted to $V$ has kernel exactly $V(U)$. So, let $f \in V$ such that

$$f_P(x) = \int_U f(ux) \, du = 0, \quad x \in G. \tag{5-2}$$

Then, by above recalled characterization of $V(U)$, we must prove that there exists an open compact subgroup $L_U \subset U$ such that

$$\int_{L_U} f(ux) \, du = \int_{L_U} f(u^{-1}x) \, du = \int_{L_U} l(u)f(x) \, du = 0, \quad x \in G. \tag{5-3}$$

Let $L \subset K$ be a normal open compact subgroup such that $f$ is right invariant under $L$. We fix a decomposition

$$K = \bigcup_{i=1}^l k_i L = \bigcup_{i=1}^l Lk_i, \quad \text{(disjoint union)}.$$ 

By Iwasawa decomposition $P = UMK$, the function $f_P$ is determined by its values on the sets $Mk_i$, $i = 1, \ldots, l$. Let $\Omega$ be the (compact) support of $f$. We may assume that $\Omega = \Omega \cdot L$. For any $i = 1, \ldots, l$, we define a compact set $\Omega_{M,i}$ as the image of $\Omega \cdot k_i^{-1} \cap P$ of the projection of $P$ onto $M \cong U \setminus P$. Now, for any $i = 1, \ldots, l$, there exists a compact subset $\Omega_{U,i} \subset U$ such that if for $u \in U$, $f(umk_i) \neq 0$, for some $m \in M$, then $u \in \Omega_{U,i}$.

But since $U$ contains arbitrarily large open compact subgroups, there exists an open compact subgroup $L_U \subset U$ such that

$$\bigcup_{i=1}^l \Omega_{U,i} \subset L_U. \tag{5-4}$$

Thus, we have

$$\text{if for } u \in U, \quad f(umk_i) \neq 0, \quad \text{for some } m \in M \text{ and } i, \text{ then } u \in L_U. \tag{5-5}$$

Now, we compute using (5-2) and (5-5)

$$\int_{L_U} f(umk_i) \, du = \int_U f(umk_i) \, du = f_P(mk_i) = 0, \tag{5-6}$$

for all $m \in M$, and all $i = 1, \ldots, l$.

Let $x \in G$. Then, we can write $x = umk_i l$, where $u \in U, m \in M, l \in L$ for some $i$. Since $f$ is right invariant under $L$, we may assume that $l = 1$. Now, for $u' \in L_U$, by (5-5), we have
that \( f(u'umk) \neq 0 \) implies \( u'u \in L_U \), and consequently \( u \in L_U \). Thus, if \( u \in L_U \), then, using (5-6), we have the following:

\[
\int_{L_U} f(u'x)du' = \int_{L_U} f(u'mk)du' = 0
\]

proving (5-3). If \( u \notin L_U \), then \( f(u'umk) = 0 \) for all \( u' \in L_U \). Again, this implies

\[
\int_{L_U} f(u'x)du' = 0.
\]

The action of \( M \) on \( r_{M,G}(V) \) is given by

\[
m.f_P(x) = \delta_{P^{1/2}}(m) \int_U f(m^{-1}ux)du = \delta_{P^{1/2}}(m) \int_U f(um^{-1}x)du = \delta_{P^{1/2}}(m)l(m)f_P(x),
\]

for \( x \in G \). \( \square \)

Let \( \mathfrak{N}_0 \) be as usual the intersection of the kernels of all characters \( |\chi|_k \), where \( \chi \) ranges over all rational characters \( \chi : M_0 \to k^\times \). Here \( | \cdot |_k \) is the norm of \( k \). Since \( P_0 \) is a minimal \( k \)-parabolic subgroup of \( G \), we have (see [7])

\[
\mathfrak{N}_0 = M_0 \cap K.
\]

Lemma 5-1 implies the following result:

**Corollary 5-7.** Let \( \delta_0 \) be the modular character of \( P_0 \). Assume that \( I \) is a maximal left ideal in \( \mathcal{H}_K \). We define \( J \) to be the \( \mathbb{C} \)-span of all \( \delta_0^{1/2}(m_0)l(m_0)f_P \), where \( f \in I \), and \( m_0 \in M_0 \). Then, we have the following isomorphism \( M_0 \)-modules:

\[
r_{M_0,G}(W(I,K)) \simeq C^\infty_0(M_0/\mathfrak{N}_0)/J.
\]

**Proof.** By Lemma 5-1, we have

\[
r_{M_0,G}(H \ast \epsilon_K) = r_{M_0,G}(C^\infty_0(G,K)) = C^\infty_0(U \setminus G/K) = C^\infty_0(M_0/\mathfrak{N}_0),
\]

with the action of \( m_0 \in M_0 \) given by \( \delta_0^{1/2}(m_0)l(m_0) \). Also, its submodule \( \mathcal{H} \ast I \), generated by \( I \), satisfies

\[
r_{M_0,G}(H \ast I) = J.
\]

Now, the exactness of Jacquet modules ([8], Proposition 3.2.3) implies the claim:

\[
r_{M_0,G}(W(I,K)) \simeq r_{M_0,G}(H \ast \epsilon_K)/r_{M_0,G}(H \ast I) \simeq C^\infty_0(M_0/\mathfrak{N}_0)/J.
\]

\( \square \)

Now, we compute the Jacquet module from Corollary 5-7 using some commutative algebra via Satake isomorphism. For this, we recall some more results from [7]. The group of all unramified characters of \( M_0 \) (see beginning of Section 3) is given by

\[
\Psi(M_0) = \text{Hom}_\mathbb{Z}(M_0/\mathfrak{N}_0, \mathbb{C}^\times).
\]
It is also obvious that the group $M_0/^0M_0$ is commutative. Therefore, under the convolution normalized by $\int_{M_0} dm = 1$,

$$A \overset{\text{def}}{=} C_c^\infty(M/^0M)$$

is an associative algebra with identity $1_{M_0}$.

Since $M_0/^0M_0$ is a finitely generated free Abelian group, the algebra is finitely generated. The Weyl group

$$W = N_G(A_0)/Z_G(A_0) = N_G(A_0) \cap K/^0M_0$$

acts by conjugation on $M_0$, $^0M_0$, and $M_0/^0M_0$, and consequently on $A$.

The subalgebra of all $W$–invariants $A^W$ of $A$ is the image of the algebra $\mathcal{H}_K$ under the Satake isomorphism

$$Sf(m) = \delta_0^{-1/2}(m)f_{P_0}(m), \quad m \in M_0,$$

using the notation of Lemma 5-1. In particular, there is a one–to–one correspondence $I \leftrightarrow m_I$ between maximal (left) ideals in $\mathcal{H}_K$, and in $A^W$.

Now, we make Corollary 5-7 explicit. We have the following:

**Corollary 5-8.** Assume that $I \subset \mathcal{H}_K$ is a maximal left ideal. Put $m = m_I$. Then,

$$r_{M_0,G}(W(I, K)) \simeq A/mA$$

$$r_{M_0,G}(\mathcal{H}_K/I) \simeq A^W/mA^W,$$

where the action on the right is just the usual left translation twisted by $\delta_0^{1/2}$.

We also recall that for $\chi \in \Psi(M_0)$ defines a $\mathbb{C}$–algebra homomorphism $A \longrightarrow \mathbb{C}$ (page 148) given by

$$f \mapsto \int_M f(m)\chi(m)dm.$$

Two unramified characters define the same $\mathbb{C}$–algebra homomorphism $A^W \longrightarrow \mathbb{C}$ if and only if are $W$–conjugate. In fact, $A$ is the algebra of regular functions on complex affine variety $\Psi(M_0)$ and $A^W$ is the algebra of regular functions on the affine variety of the $W$–orbits. In particular, maximal ideals in $A^W$ are in one–to–one correspondence with $W$–orbits of unramified characters in $\Psi(M_0)$: $m \leftrightarrow O_m$. Thus, there exists a one–to–one correspondence between maximal ideals in $\mathcal{H}_K$, and $W$–orbits of unramified characters in $\Psi(M_0)$: $I \leftrightarrow O_{m_I}$. We say that $I$ (or $m_I$) is regular if $\#O_{m_I} = \#W$. This is equivalent to the fact that orbit contains $W$–regular (i.e., its stabilizer in $W$ is trivial) unramified character.

We have the following:

**Proposition 5-9.** Assume that $I \subset \mathcal{H}_K$ is a regular maximal left ideal. Let $\chi \in O_{m_I}$. Then, $\chi$ is $W$–regular, and we have an isomorphism of $M_0$–modules

$$r_{M_0,G}(W(I, K)) \simeq r_{M_0,G}(\text{Ind}^{G}_{P_0}(\chi)) \simeq \oplus_{w \in W} w(\chi).$$
Proof. The isomorphism
\[ r_{M_0,G} \left( \text{Ind}_{P_0}^G(\chi) \right) \simeq \bigoplus_{w \in W} w(\chi) \]
is well–known (see [8], Proposition 6.4.1). It remains to prove

\[ r_{M_0,G}(W(I,K)) \simeq \bigoplus_{w \in W} w(\chi). \tag{5-10} \]

First, let \( \pi \) be a unique irreducible subquotient of \( \text{Ind}_{P_0}^G(\chi) \) which is \( K \)–spherical. Then, there exists a subset \( S \subset W \) such that
\[ r_{M_0,G}(\pi) \simeq \bigoplus_{w \in S} w(\chi). \]
Since \( \chi \) is \( W \)–regular, for any other irreducible subquotient \( \pi' \) of \( \text{Ind}_{P_0}^G(\chi) \), \( w(\chi), w \in S \), does not show up in \( r_{M_0,G}(\pi') \).

Next, by above description of \( r_{M_0,G}(W(I,K)) \) (see Corollary 5-8), and general Lemma 5-11, there exists a positive integer \( L \) such that the semi–simplification of \( r_{M_0,G}(W(I,K)) \) is given by
\[ r_{M_0,G}(W(I,K)) = L \cdot \sum_{w \in W} w(\chi) \]
in the Grothendieck group of finite length admissible modules of \( M_0 \).

By Theorem 1-2, \( W(I,K) \) has finite length, all of its irreducible subquotients are among those of \( \text{Ind}_{P_0}^G(\chi) \), and \( \pi \) appears with multiplicity one in its composition series.

This implies \( L = 1 \) because \( w(\chi), w \in S \), show up only in \( r_{M_0,G}(\pi) \), and \( \pi \) appears with multiplicity one in the composition series of \( W(I,K) \).

Finally, since \( \chi \) is regular, we obtain (5-10). \( \square \)

We include the following general lemma. Before we state the lemma, we introduce some notation. Let \( X \) be a complex affine variety, and let \( \mathcal{A} \) be its algebra of regular functions. Assume that a finite group \( W \) acts on \( X \) as a group of regular (algebraic) transformations. Let \( Y \) be the affine variety of \( W \)–orbits. Then, by standard theory, its algebra of regular functions is \( \mathcal{A}^W \).

**Lemma 5-11.** Let \( x \in X \) be a point such that the orbit \( W.x \) has \#\( W \)–distinct elements (a generic orbit). Let \( m_w \) be the maximal ideal that corresponds to \( w.x \) for \( w \in W \). Then,
\[ m \overset{\text{def}}{=} m_w \cap \mathcal{A}^W \]
is independent of \( w \in W \). As a \( \mathcal{A} \)–module, \( \mathcal{A}/m \mathcal{A} \) has filtration by irreducible \( \mathcal{A} \)–modules \( \mathcal{A}/m_w, w \in W \), where their multiplicity in the composition series is independent of \( w \).

**Proof.** We use Nullstellensatz, and results about the support of finitely generated modules (see [10], Chapter X, Section 2). Put
\[ M \overset{\text{def}}{=} \mathcal{A}/m \mathcal{A}. \]
This is a finitely generated \( \mathcal{A} \)–module which by the general theory has filtration by \( \mathcal{A}/p \) where \( p \) ranges over primes in the support \( \text{supp}(M) \) of \( M \). By definition, \( p \in \text{supp}(M) \) if
and only if the localization $M_p$ satisfies $M_p \neq 0$. The radical of the annihilator of $M$ is given by the following general formula:

(5-12) \[ \text{rad} \left( \text{ann} \left( M \right) \right) = \cap_{p \in \text{supp} \left( M \right)} p. \]

It is obvious that \[ \text{ann} \left( M \right) = mA, \]
and by Nullstellensatz

(5-13) \[ \text{rad} \left( mA \right) = \cap_{w \in W} m_w. \]

This implies that we have an epimorphism of $A$-modules \[ M = A/mA \rightarrow A/\cap_{w \in W} m_w \rightarrow 0. \]

Next, by the Chinese remainder theorem, this gives the following epimorphism of $A$-modules: \[ M \rightarrow \bigoplus_{w \in W} A/m_w \rightarrow 0. \]

Thus, \[ \{m_w; \ w \in W\} \subset \text{supp} \left( M \right). \]

But (5-12) and (5-13) imply \[ \cap_{w \in W} m_w = \cap_{p \in \text{supp} \left( M \right)} p. \]

Then, for each $q \in \text{supp} \left( M \right)$, we have \[ \prod_{w \in W} m_w \subset \cap_{w \in W} m_w \subset q. \]

Then, the definition of a prime ideal implies that \[ m_w \subset q, \]
for some $w$. Since $m_w$ is a maximal ideal, we obtain $q = m_w$. Thus, we have \[ \text{supp} \left( M \right) = \{m_w; \ w \in W\}. \]

Finally, this implies that $M$ has filtration by irreducible $A$-modules $A/m_w$, $w \in W$, where their multiplicity in the composition series is independent of $w$ since $W$-permutes composition factors. \[ \square \]

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