Ferromagnetism in the Blume-Emery-Griffiths model on finite-size Cayley tree

Wen-Jun Chen and Xiang-Mu Kong

Shandong Provincial Key Laboratory of Laser Polarization and Information Technology, Department of Physics, Qufu Normal University, Qufu 273165, China

Abstract

The ferromagnetic properties of the spin-1 BEG model on finite-size Cayley tree are investigated using the exact recursion method. The spontaneous magnetization of the system is studied in detail for different values of the reduced crystal-field interaction $D/J$, and it is found that there is an unusual behavior (anti-Curie temperature) when $D/J > 2.0$. We also obtain the Curie temperature of this finite-size system. When the system size is large enough, our results will fit well with that in the thermodynamic limit.
I. INTRODUCTION

The Cayley tree \([1]\) which is also called Bethe lattice \([2]\) was firstly investigated by Kurata et al. \([3]\). Then Domb studied the Ising model on such lattice and demonstrated that a Bethe-Peierls approximation is exact for the Bethe lattice \([4, 5]\). Over the years, the thermodynamic properties of the Ising system on this lattice have been extensively investigated \([6, 7, 8, 9]\). As an expanded Ising model, the Blume-Emery-Griffiths (BEG) model \([10, 11]\), which is characterized by bilinear and biquadratic exchange interactions and crystal-field interaction \([12, 13]\), has played an important role in the development of the theory of tricritical phenomena. This model has been studied by a variety of techniques, e.g., the generalized Bethe-Peierls approximation \([14, 15, 16]\), the effective-field theory \([17, 18, 19]\), the generalized constant-coupling approximation theory with two parameters \([20, 21]\), the exact recursion relations method \([22, 23, 24]\) and so on.

All of these systems mentioned above are studied in the case of the thermodynamic limit. As we all know that the systems studied by the methods of experiment and numerical simulation are all finite. So the research on the finite-size system is much more meaningful. It was not until recently that a new exact expression of finite-size system for the zero-field magnetization was established \([25]\). Then the corresponding exact expression in closed form for the zero-field susceptibility was given by T. Stosic et al. \([26, 27, 28]\). To our knowledge no exact calculation has been made in the field regarding the properties of the spin-1 BEG model on the finite-size Cayley tree yet.

In this paper we investigate the effect of the finite-size system on the ferromagnetic properties in detail. The expression of the magnetization for this system with different values of the reduced crystal-field interaction is derived and the Curie temperature is obtained. The results are compared with that of the case in the thermodynamic limit.

II. MODEL AND FORMULATION

At the beginning, we give a brief description of the construction of the Cayley tree. Starting from a single point 0, the central one of the graph \([29]\), we add \(q\) different points connected to the central point which may be called "the first shell". Then each point of the first shell is joined to \(q - 1\) new points. So the points of the first shell have \(q(q - 1)\)
nearest neighbors in total which form the second shell. The number of shells is also defined as generation number $n$ while $q$ as coordination number. If we continue in this way, the entire structure of the Cayley tree is formed.

The Hamiltonian of the BEG model on the Cayley tree is given by

$$H = -J \sum_{\langle ij \rangle} S_i S_j - K \sum_{\langle ij \rangle} S_i^2 S_j^2 + D \sum_i S_i^2 - h \sum_i S_i,$$

where $S_i (= \pm 1, 0)$ is the spin at site $i$, the summation $\sum_i$ runs over all the sites and $\sum_{\langle ij \rangle}$ denotes summation over all the nearest-neighbor pairs. $J$, $K$, $D$ and $h$ describe the bilinear exchange, biquadratic interaction, crystal-field (or single-ion anisotropy) interaction and external magnetic field, respectively. This Hamiltonian was originally proposed to explain the phase separation and superfluidity in $^3$He-$^4$He mixtures [10].

The partition function of the above system can be written as

$$Z = \sum_S \exp \left( -\beta H \right)$$
$$= \sum_S \exp \left[ \beta \left( J \sum_{\langle ij \rangle} S_i S_j + K \sum_{\langle ij \rangle} S_i^2 S_j^2 - D \sum_i S_i^2 + h \sum_i S_i \right) \right],$$

where $\beta = 1/k_B T$, $k_B$ is Boltzmann constant and $T$ is the absolute temperature. The summation $\sum_S$ in Eq. (2) goes over all spin configurations of the system.

Without loss of generality, we consider a Cayley tree of $n$ generations with branch number $B = 5$ (coordination number minus one). Then, the $n$-generation branch consists of $N_n = \frac{5^{n+1} - 1}{4}$ spins, while the 0-generation being a single spin. Let $Z_n^{(+)}$, $Z_n^{(-)}$ and $Z_n^{(0)}$ be the partial partition functions of the system, with the central spin takes values $+1$, $0$ and $-1$ respectively. Based on Eq. (2), we can obtain the recursion relations [6] as

$$Z_{n+1}^{(\pm)} = e^{-\beta D \pm \beta h} \left( e^{\pm \beta J + \beta K} Z_n^{(\pm)} + Z_n^{(0)} + e^{\mp \beta J + \beta K} Z_n^{(-)} \right)^5$$

and

$$Z_{n+1}^{(0)} = \left( Z_n^{(+)} + Z_n^{(0)} + Z_n^{(-)} \right)^5.$$

The partition function Eqs. (3) and (4) can be differentiated with respect to field, thus
the recursion relations for the field derivatives of the partition function are easily written as

\[
\frac{\partial Z_{n+1}^{(\pm)}}{\partial \beta h} = \pm e^{-\beta D \pm \beta h} \left( e^{\pm \beta J + \beta K} Z_{n}^{(\pm)} + Z_{n}^{(0)} + e^{\mp \beta J + \beta K} Z_{n}^{(-)} \right)^5
\]

\[
+ 5e^{-\beta D \pm \beta h} \left( e^{\pm \beta J + \beta K} Z_{n}^{(\pm)} + Z_{n}^{(0)} + e^{\mp \beta J + \beta K} Z_{n}^{(-)} \right)^4
\]

\times \left( e^{\pm \beta J + \beta K} \frac{\partial Z_{n}^{(\pm)}}{\partial \beta h} + \frac{\partial Z_{n}^{(0)}}{\partial \beta h} + e^{\mp \beta J + \beta K} \frac{\partial Z_{n}^{(-)}}{\partial \beta h} \right),
\]

\[
\frac{\partial Z_{n+1}^{(0)}}{\partial \beta h} = 5 \left( Z_{n}^{(0)} + Z_{n}^{(0)} + Z_{n}^{(-)} \right)^4 \left( \frac{\partial Z_{n}^{(0)}}{\partial \beta h} + \frac{\partial Z_{n}^{(0)}}{\partial \beta h} + \frac{\partial Z_{n}^{(-)}}{\partial \beta h} \right). \tag{5}
\]

Starting from a single spin (0-th generation branch) we have

\[
Z_{0}^{(\pm)} = e^{-\beta D \pm \beta h}, \quad Z_{0}^{(0)} = 1,
\]

and

\[
\frac{\partial Z_{0}^{(\pm)}}{\partial \beta h} = \pm e^{-\beta D \pm \beta h}, \quad \frac{\partial Z_{0}^{(0)}}{\partial \beta h} = 0.
\]

Then the magnetization of a site in the Cayley tree can be written as

\[
\langle m \rangle_{n}^{\pm} = \frac{1}{N_n} \frac{1}{Z_{n}^{(\pm)}} \frac{\partial Z_{n}^{(\pm)}}{\partial \beta h}. \tag{6}
\]

The exact recursion relations and the magnetization expression allow us to study the thermodynamic behavior of the system in detail. In next section, we will give the numerical results of the magnetization.

III. NUMERICAL RESULTS AND DISCUSSIONS

In the following, we investigate numerically the magnetization of the BEG model on this finite-size Cayley tree. Based on the above Eqs. (1-6), taking the limit \( h \to 0 \), we make a detailed calculation to the magnetization for various strengths of the interaction \( J, K \) and \( D \) and for several system sizes \( n \).

The magnetization as a function of temperature is shown in Fig. 1 for various reduced values of crystal-field interaction \( D/J \) while \( K/J = 1, n = 12 \). The curve corresponding to \( D/J = 2.0 \) separates the curves into two different trend. In the case of \( D/J > 2.0 \), there is an anti-Curie temperature \( T_{aC} \) [13], i.e., a temperature below which the magnetization
vanishes, in addition to the Curie temperature $T_C$. As $D/J$ decreases, the anti-Curie temperature drops to lower values and the maximum magnetization becomes larger. We can see that the anti-Curie temperature vanishes in the case of $D/J = 2.0$ and the magnetization decreases steadily from its saturation value 0.58 to zero with growing temperature. In the case of $D/J < 2.0$, the maximum magnetization of curves increases and it approaches to unity at zero temperature for $D/J = 1.5$. Moreover the magnetization curve would keep the conventional shape for even smaller $D/J$ values that is not shown in this figure. An approximative value of the deduced Curie temperature $T_C = 1.4$ can also be obtained for this two-order transition.

In Fig. 2 we present the magnetization as a function of temperature for several system sizes $n = 3, 6, 9, 12$ and different values of $D/J$. It is seen that the curves exhibit a slow decay of magnetic ordering with the increase of the system sites. The larger the system size, the sharper the magnetization curve is. If the system sizes could get large enough value, the magnetization curve of finite-size system would become close to that in thermodynamic limit. The unusual shape of the figure is due to the complicated interactions including the bilinear exchange and biquadratic exchange. This result is in agreement with the system in thermodynamic limit which is studied by K. G. Chakraborty et al. [30, 31].

IV. CONCLUSIONS

In this paper, using the exact recursion relations, we present an exact calculation for the spontaneous magnetization of the spin-1 BEG system on the finite-size Cayley tree. The magnetization properties are studied in detail for different values of the crystal-field interaction and system size of the Cayley tree. It is shown that the magnetization curves exhibit some unusual features including an anti-Curie temperature as the variance of the strength of the crystal-field interaction. The Curie temperature is also obtained. The curves would become close to that in the thermodynamic limit with the increase of the system size.

V. ACKNOWLEDGMENTS

This work was supported by the National Natural Science foundation of China under Grant NO. 10775088, and the Shandong Natural Science foundation under Grant NO.
One of the authors (Chen) thanks Shuxia Chen, Sai Wang and Shengxin Liu for fruitful discussions.

[1] A. Cayley, Coll. Math. Papers 3, 242 (1889).
[2] H. A. Bethe, Proc. R. Soc. A 150, 552 (1935).
[3] M. Kurata, R. Kikuchi and T. Watari, J. Chem. Phys. 21, 434 (1953).
[4] C. Domb, Adv. Phys. 9, 283 (1960).
[5] C. Domb, Adv. Phys. 9, 145 (1960); 81, 3088 (1984).
[6] T. P. Eggarter, Phys. Rev. B 9, 2989 (1974).
[7] J. von Heimburg and H Thomas, J. Phys. C 7, 3433 (1974).
[8] Y. Tanaka and N. Uryu, J. Phys. Soc. Jap. 50, 1140 (1981).
[9] S. Katsura and M. Takizawa, Prog. Theor. Phys. 51, 82 (1974).
[10] M. Blume, V. J. Emery and R. B. Griffiths, Phys. Rev. A 4, 1071 (1971).
[11] R. B. Griffiths, Phys. Rev. Lett. 24, 715 (1970).
[12] M. Blume, Phys. Rev. 141, 517 (1966).
[13] H. W. Capel, Physica 32, 966 (1966).
[14] T. Obokata and T. Oguchi, J. Phys. Soc. Jap. 25, 321 (1968).
[15] T. Iwashita and N. Uryu, J. Phys. C 12, 4007 (1979).
[16] T. Iwashita and N. Uryu, Phys. Lett. A 73, 333 (1979).
[17] K. G. Chakraborty, J. Phys. C 21, 2911 (1988).
[18] K. G. Chakraborty and J. W. Tucker, Physica 137A, 122 (1986).
[19] I. P. Fittipaldi and A. F. Siqueira, J. Magn. Magn. Mater. 54, 694 (1986).
[20] K. Takahashi and M. Tanaka, J. Phys. Soc. Jpn. 46, 1428 (1979).
[21] K. Takahashi and M. Tanaka, J. Phys. Soc. Jap. 48, 1423 (1980).
[22] C. Ekiz, E. Albayrak, M. Keskin, J. Magn. Magn. Mater. 256, 311 (2003).
[23] C. Ekiz, Phys. Status Solidi B 241, 1324 (2004).
[24] C. Ekiz, Phys. Lett. A 325, 99 (2004).
[25] R. Melin, J. C. Angles d’Auriac, P. Chandra and B. Doucou J. Phys. A 29, 5773 (1996).
[26] T. Stosic, B. D. Stosic, I. P. Fittipaldi, J. Magn. Magn. Mater. 177, 185 (1998).
[27] T. Stosic, B. D. Stosic, I. P. Fittipaldi, Physica A 320, 443 (2003).
[28] T. Stosic, B. D. Stosic, I. P. Fittipaldi, Physica A 355, 346 (2005).

[29] R. J. Baxter, Exactly Solved Models in Statistical Mechanics (Academic Press, New York, 1982).

[30] K. G. Chakraborty and J. W. Tucker Phys. Lett. 111A, 4 (1985).

[31] K. G. Chakraborty, Physica A 129, 415 (1985).
FIG. 1: The magnetization $M$ as a function of $T$ for $K/J = 1, n = 12$ and $B = 5$. The successive curves from (a) to (b) are for $D/J = 2.3, 2.1, 2.0, 1.9, 1.5$ and 1 respectively. There exhibits a two-order phase transition.

FIG. 2: The magnetization versus temperature $T$ for different $D/J$ and $n = 3, 6, 9, 12$. The successive curves from (a) to (f) are for $D/J = 2.3, 2.1, 2.0, 1.9, 1.5$ and 1 respectively. The slope of each subgraph curve steepens with the increase of the system sizes $n$. 
FIG. 1: The magnetization $M$ as a function of $T$ for $K/J = 1$, $n = 12$ and $B = 5$. The successive curves from (a) to (b) are for $D/J = 2.3, 2.1, 2.0, 1.9, 1.5$ and 1 respectively. There exhibits a two-order phase transition.
FIG. 2: The magnetization versus temperature $T$ for different $D/J$ and $n = 3, 6, 9, 12$. The successive curves from (a) to (f) are for $D/J = 2.3, 2.1, 2.0, 1.9, 1.5$ and 1 respectively. The slope of each subgraph curve steepens with the increase of the system sizes $n$. 