Ehlers as EM duality in the double copy

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Abstract

Given a 4D vacuum solution to gravity with an isometry direction, it is well known that the gravity equations of motion are identical to those of a 3D $\sigma$-model with two scalars parametrising the target space geometry $H^2$. Thus, any transformation by $SL(2,\mathbb{R})$ - the isometry group of $H^2$ - is a symmetry for the action, thereby providing a mechanism to generate new Ricci-flat solutions in 4D. Here we clarify recent work on electromagnetic duality in the context of the classical double copy and present it in the most natural way, namely as a transformation in the $H^2$ target geometry. In particular, we will show how the electric and magnetic Maxwell fluxes of the double copy formalism are related to derivatives of the scalars, thereby providing an equivalence between Ehlers transformations and EM duality.
1 Introduction

The classical double copy is an intriguing connection between gravity and gauge theory \cite{1}, which has been motivated from a relationship between perturbative scattering amplitudes in gauge theory and gravity \cite{2,3,4}. In its simplest form, the central observation is that solutions to Einstein gravity can be mapped to solutions of Maxwell’s equations through a Kerr-Schild (KS) decomposition of the spacetime. Interestingly, in contrast to Kaluza-Klein dimensional reduction, the KS ansatz maintains dimensionality. More concretely, one considers the spacetime metric

\[ g_{\mu\nu} = \eta_{\mu\nu} + \phi k_\mu k_\nu, \tag{1.1} \]

where \( \eta_{\mu\nu} \) denotes the metric of flat spacetime, \( \phi \) is a scalar and \( k \) is a null vector, \( k_\mu k^\mu = 0 \), satisfying the geodesic equation \( k^\rho \partial_\rho k_\mu = 0 \). The Maxwell gauge field \( A \) arises from the identification \( A = \phi k \). See \cite{6,23} for related work in this direction.

In this double copy formalism the Schwarzschild solution corresponds to a Maxwell field with an electric charge \cite{1}, while the Taub-NUT solution possesses a magnetic charge \cite{24}. Subsequently, the single copy of the Eguchi-Hanson instanton has been shown to map to a self-dual Maxwell field \cite{25}. With both electric and magnetic charges present, this raises the question whether there is a gravity analogue of electromagnetic (EM) duality, namely a rotation of the field strength \( F = dA \) into \( *_4 F \) that honours the Maxwell equations of motion. This was answered in the affirmative in two recent papers. In the first a complex transformation in the gravity is mapped to a complexified BMS supertranslation \cite{26}, while in the second \cite{27} a class of real transformations due to Ehlers \cite{28} are exploited.

The purpose of this note is to clarify comments in the latter paper. As we explain in the following section, the magic of Ehlers transformations is that given 4D gravity with a \( U(1) \) isometry direction, the equations of motion are identical to a 3D \( \sigma \)-model with a target space \( H^2 \). Being maximally symmetric, the hyperbolic space \( H^2 \) possesses an isometry group \( SL(2, \mathbb{R}) \) that rotates the scalars, but importantly leaves the 3D effective action, and therefore the equations of motion, invariant. Ultimately, two of the symmetries can be removed by rescaling the Killing vector of the \( U(1) \) isometry direction and exploiting a gauge freedom arising from shifts in one of the scalars, so this leaves a \( U(1) \) symmetry, which when recast in terms of the double copy is precisely EM duality.

Concretely, we rewrite the KS ansatz in the description appropriate for an Ehlers transformation. In the process, we show that the electric and magnetic Maxwell field strengths are related to the derivatives of the scalars of the 3D \( \sigma \)-model, thereby providing a succinct way to understand observations made in \cite{27} without recourse to transformations at the level of particular solutions. This map between the scalars parametrising \( H^2 \) and the Maxwell fluxes allows us to define electric and magnetic Maxwell charges, which transform accordingly.

\footnote{See the recent review \cite{5} and references therein for a wider perspective on this.}
2 Review of Ehlers

Here we follow the treatment described in appendix of [29]. Consider a 4D spacetime with a Killing vector $\partial_t$, which we will assume is in the temporal direction. The most general metric consistent with this $U(1)$ symmetry is

$$ds^4 = -V(dt + A)^2 + V^{-1}\gamma_{mn}dx^m dx^n,$$

(2.1)

where $V$ is a scalar and $A$ is a vector on the transverse 3D space with metric $\gamma_{mn}$, $m, n = 1, 2, 3$. We have rescaled the internal space judiciously so as to arrive later in Einstein frame in 3D. Now, let us demand that this is a vacuum solution to Einstein gravity, so that it satisfies the equation

$$R_{\mu\nu} = 0.$$

(2.2)

The joy of this set-up is that the equation mixing the temporal and spatial directions reduces to

$$d(V^2 *_3 F) = 0,$$

(2.3)

where $F$ is the field strength corresponding to the vector field, $F = dA$. Now comes the magic. Locally, one can replace the above equation with

$$V^2 *_3 F = d\omega,$$

(2.4)

where we have taken the opportunity to introduce a second scalar. The fact that we can do this is essentially down to dimensionality: in 3D vectors are dual to scalars. Gathering the remaining equations of motion together, it can be shown that the equations of motion follow from varying the following 3D action

$$\mathcal{L} = \sqrt{\gamma}(R - \frac{1}{2} \partial_m V \partial^m V + \partial_m \omega \partial^m \omega V^2).$$

(2.5)

From the action it is evident that there is a hyperbolic target space. Being maximally symmetric, it permits 3 Killing directions. To make these symmetries manifest, it is best to switch to the complex scalar

$$\tau = \omega + iV,$$

(2.6)

which allows us to rewrite the metric on the hyperbolic space as

$$ds^2(H^2) = \frac{dV^2 + d\omega^2}{V^2} = \frac{d\tau d\bar{\tau}}{\text{Im}(\tau)^2}.$$

(2.7)

It is now an easy task to confirm that the 2D metric, and thus the 3D action, is invariant under the $SL(2, \mathbb{R})$ transformation

$$\tau \rightarrow \tau' = \frac{a\tau + b}{c\tau + d}, \quad ad - bc = 1, \quad a, b, c, d, \in \mathbb{R}.$$ 

(2.8)
We believe that this is the simplest and most elegant way to present the class of transformations we attribute to Ehlers [28].

Just so we are all on the same page, some comments are in order. First, the $SL(2,\mathbb{R})$ clearly rotates the scalars in the action, but does not affect the 3D Ricci scalar. For this reason, the 3D space with metric $\gamma_{mn}$ is indeed invariant. Second, although we appear to have three free parameters, the freedom to rescale the Killing vector by a constant and the freedom to shift $\omega$ by a constant removes two of these parameters. In effect, if one is interested in generating new inequivalent solutions in 3D, one has only one parameter to play with. Interestingly, as explicitly highlighted in [29], the same $SL(2,\mathbb{R})$ symmetry is at the heart of Lunin & Maldacena’s TsT transformations [30], and there one finds only one parameter, in line with expectations.

Thus, the most general transformation up to redefinitions may be expressed as

$$
\begin{pmatrix}
  a & b \\
  c & d
\end{pmatrix} =
\begin{pmatrix}
  \cos \beta & \sin \beta \\
  -\sin \beta & \cos \beta
\end{pmatrix}.
$$

We explicitly check in the appendix that the most general $SL(2,\mathbb{R})$ transformation applied to the Schwarzschild solution leads to the Taub-NUT solution, i.e. in addition to the mass, only one additional charge is generated.

Returning to the above transformation (2.9), we can now comment on some special cases. The choice $\beta = \frac{\pi}{4}$ generates the pure NUT space, while $\beta = \frac{\pi}{2}$ executes the Buchdahl reciprocal transformation [31]. In contrast to [27], there is no need to rescale to the Schwarzschild metric or treat the Buchdahl transformation separately: everything naturally fits into $SL(2,\mathbb{R})$.

3 Kerr-Schild

Now comes the crux of this paper. To fully understand the Ehlers transformation in terms of the double copy, one should start with the KS ansatz and identify the scalar $V$ and vector field $A$ in terms of $\phi$ and the null vector $k$. The only problem is that nowhere in the KS ansatz is a Killing direction specified, so we will have to put one in by hand. Luckily for us, for stationary spacetimes, the most general null vector $k$ can be decomposed as

$$k = dt + \tilde{k},$$

where $\tilde{k}$ is a spatial vector with unit norm $\tilde{k}_m \tilde{k}^m = 1$. Once this is done, one can easily identify the electric and magnetic part of the Maxwell field,

$$F_{\text{elec}} = d\phi \wedge dt, \quad F_{\text{mag}} = d(\phi \tilde{k}),$$

$^2$The reader is welcome to compare with sections 3 and 5 of the recent paper [27], where the underlying simplicity (beauty) of the transformation is far from evident. For full comparison, it may be helpful to note that $\tau = i\sigma$.

$^3$This rescaling can be viewed as yet another $SL(2,\mathbb{R})$ transformation where $d = 1/a$. To make comparison with the Taub-NUT geometry presented in section 3 of [27], note that $\sin^2 \beta = \frac{\gamma^2}{1+\gamma^2}$. 

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where we have opted to use differential forms. Translated into the language of the Ehlers transformation, one finds

\[ V = (1 - \phi), \quad A' = \frac{\phi}{1 - \phi} \tilde{k}, \]

\[ \gamma_{mn} dx^m dx^n = (1 - \phi) dx^2 + \phi \tilde{k}^2, \quad (3.3) \]

where we have added a prime to the Ehlers vector field to distinguish it from the Maxwell gauge field. With this mapping, it is easy to identify the electric Maxwell flux in terms of the derivative of the scalar \( V \):

\[ F_{\text{elec}} = -dV \wedge dt. \quad (3.4) \]

The magnetic Maxwell flux requires a little more work, but in the end takes a simple form. Using the condition \( \tilde{k}^n \partial_n \tilde{k}_m = 0 \), it is a straightforward calculation to show that

\[ V^2 \ast_3 F' = \ast_3 F_{\text{mag}}, \quad (3.5) \]

where the Hodge duality on the LHS is w. r. t. the metric \( \gamma_{mn} \), whereas on the RHS the metric is \( \delta_{mn} \). Since the Maxwell field is assumed to live in flat spacetime, this is in line with expectations. Now, returning to the key point in the Ehlers transformation, where the vector is replaced by a scalar, we can write

\[ \ast_3 F_{\text{mag}} = d\omega. \quad (3.6) \]

Now, everything is clear. Given a KS ansatz, the electric and magnetic fluxes of the Maxwell field of the double copy are mapped to the derivative of the scalars in the Ehlers transformation. The latter are transformed by an \( SL(2, \mathbb{R}) \) transformation, which up to redundancies is a \( U(1) \) symmetry, and this corresponds to an EM duality transformation in the Maxwell field. This provides another perspective on the results announced in the recent paper [27], but the presentation here is succinct and not at the level of specific solutions.

It is worth noting that above we have assumed a KS ansatz, but it turns out that the above relations (3.4) and (3.6) are robust. In the appendix we show that if one replaces a single KS ansatz with the double KS ansatz, so that one can describe the transformation from Schwarzschild to Taub-NUT, then the same relations hold. Therefore, provided one is careful about the asymptotics and ensures that \( V \to 1 \) at infinity, it is possible to define electric and magnetic Maxwell charges in the usual manner:

\[ Q_e = \frac{1}{4\pi} \int_{S^2} \ast_3 dV, \quad Q_m = \frac{1}{4\pi} \int_{S^2} \ast_3 d\omega. \quad (3.7) \]

Let us return the example considered in [27], to which we will apply our general one-parameter rotation (2.9), which we have argued is the most general \( SL(2, \mathbb{R}) \) up to redefinitions. The data describing the Schwarzschild solution is

\[ V = \left(1 - \frac{2m}{r}\right), \quad (3.8) \]
and \( \omega \) is a constant, so there is no vector field \( A' \) in the gravity. Translated into the double copy gauge field, the Schwarzschild solution has only an electric flux. In this case we have

\[
Q_e = 2m, \quad Q_m = 0. \tag{3.9}
\]

Performing the \( SL(2, \mathbb{R}) \) transformation, we generate new scalars and from there we read off the transformed charges,

\[
Q'_e = 2m \cos 2\beta, \quad Q'_m = -2m \sin 2\beta. \tag{3.10}
\]

As a further simple example, it is easy to convince oneself that the Buchdahl reciprocal transformation flips the sign of electric charge.

## 4 Discussion

In this note, we have married the (double) KS ansatz with the natural 4D to 3D dimensional reduction ansatz of Ehlers, which has allowed us to identify the Maxwell field strengths of the double copy formalism directly in terms of the scalars parametrising a a hyperbolic coset geometry in 3D. As we have argued, this can be done for generic spacetime geometries and it is the rotation of the scalars under a \( U(1) \subset SL(2, \mathbb{R}) \) that is mapped to EM duality in the Maxwell fluxes of the double copy formalism. We believe our work clarifies and generalises to pretty generic spacetimes admitting a KS form, the findings presented earlier in [27]. We leave the task of fleshing out examples to the interested reader.

The analysis presented here can clearly be generalised to other settings. Indeed, it is interesting to consider the inclusion of the cosmological constant and what happens when Einstein gravity is coupled to higher-form fields. In the case of the former, it is known that the \( SL(2, \mathbb{R}) \) group is broken by a cosmological constant [32], while for the former, there are numerous examples of supergravity theories that can be truncated to scalar sectors and similar symmetries arise. We hope to return to these in future work.

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### A General \( SL(2, \mathbb{R}) \) transformation

In this section we comment on the general \( SL(2, \mathbb{R}) \) transformation applied to Schwarzschild with a goal to convince ourselves that of the three unconstrained \( SL(2, \mathbb{R}) \) parameters,
only one is relevant after various redefinitions. Recall the most general form of a \(SL(2, \mathbb{R})\) transformation is given by (2.8). For the Schwarzschild solution we have

\[
V = 1 - \frac{2M}{r}, \quad \omega = 0,
\]  

so under the \(SL(2, \mathbb{R})\) transformation we get following expressions for \(V'\) and \(\omega'\),

\[
V' = \frac{r(r - 2M)}{c^2(r - 2M)^2 + d^2r^2}, \quad \omega' = \frac{ac(r - 2M)^2 + bdr^2}{c^2(r - 2M)^2 + d^2r^2}.
\]  

Using (2.4) the two form \(F'\) takes following form

\[
F' = 4dcM\text{vol}(S^2).
\]  

Finally the 4D metric can be written as

\[
ds^2 = -\frac{r(r - 2M)}{c^2(r - 2M)^2 + d^2r^2} (dt + 4Mdc \cos \theta d\varphi)^2 + \frac{c^2(r - 2M)^2 + d^2r^2}{r(r - 2M)} ds_3^2
\]  

where we have defined,

\[
ds_3^2 = dr^2 + r(r - 2M)d\theta^2 + r(r - 2M) \sin^2 \theta d\varphi^2
\]  

In above metric only two of independent parameters of \(SL(2, \mathbb{R})\) appear. Indeed one can show that one of these parameters can be also eliminated by a shift and a rescaling scaling of both the radial and time coordinates. If we define two positive parameters \(r_\pm\) by

\[
r_+ = \frac{2Mc^2}{\sqrt{c^2 + d^2}}, \quad r_- = \frac{2Md^2}{\sqrt{c^2 + d^2}}
\]  

After a shift and scaling

\[
r \to \frac{r + r_+}{\sqrt{c^2 + d^2}}, \quad t \to \frac{t}{\sqrt{c^2 + d^2}}
\]  

metric (A.4) take following form

\[
ds^2 = -f(r) (dt - 2\sqrt{r_+r_-} \cos \theta d\varphi)^2 + \frac{dr^2}{f(r)} + (r^2 + r_+r_-)d\Omega_2^2
\]  

where function \(f\) is defined by

\[
f(r) = \frac{(r + r_+)(r - r_-)}{r^2 + r_+r_-}
\]  

This is the metric of the Taub-NUT space time.
In this section we provide some details to support the identity (3.6). We work with a double KS ansatz for greater generality. Consider the double KS ansatz:

$$g_{\mu\nu} = \eta_{\mu\nu} + \phi k_{\mu}k_{\nu} + \psi l_{\mu}l_{\nu},$$  \hspace{1cm} (B.1)

where $k_\mu$ and $l_\mu$ satisfy following equations,

$$\eta^{\mu\nu}k_{\mu}k_{\nu} = g^{\mu\nu}k_{\mu}k_{\nu} = 0, \quad \eta^{\mu\nu}l_{\mu}l_{\nu} = g^{\mu\nu}l_{\mu}l_{\nu} = 0,$$

$$\eta^{\mu\nu}k_{\mu}l_{\nu} = g^{\mu\nu}k_{\mu}l_{\nu} = 0, \quad k_\mu \partial^{\mu}k_{\nu} = 0, \quad l_\mu \partial^{\mu}l_{\nu} = 0 \hspace{1cm} (B.2)$$

Assuming $\partial_t$ is a Killing direction, we can always write

$$k = dt + \tilde{k}, \quad l = dt + \tilde{l}. \hspace{1cm} (B.3)$$

Rewriting everything in terms of the Ehlers ansatz (2.1) gives

$$V = 1 - \phi - \psi, \quad A = -V^{-1} \left( \phi \tilde{k} + \psi \tilde{l} \right) \hspace{1cm} (B.4)$$

and the 3D metric $d\mathbf{s}^2_3$ becomes

$$d\mathbf{s}^2_3 = \gamma_{mn}dx^m dx^n = V(dx_t^2 + \phi \tilde{k}^2 + \psi \tilde{l}^2) + (\phi \tilde{k} + \psi \tilde{l})^2 \hspace{1cm} (B.5)$$

To find the Hodge dual, first we need to invert the above metric. It is easy to show that the inverse metric is

$$\gamma^{mn} = (1 - \phi - \psi)^{-1} \left( \delta^{mn} - \phi \tilde{k}^m \tilde{k}^n - \psi \tilde{l}^m \tilde{l}^n \right) \hspace{1cm} (B.6)$$

where we defined $\tilde{k}^m$ and $\tilde{l}^m$ by

$$\tilde{k}^m = \delta^{mn}\tilde{k}_n, \quad \tilde{l}^m = \delta^{mn}\tilde{l}_n. \hspace{1cm} (B.7)$$

Now, using (B.3) and (B.2) we get

$$(V^2 \wedge_3 F)_p = -2\partial^m(\phi \tilde{k}^n + \psi \tilde{l}^n)\epsilon_{mpn} \hspace{1cm} (B.8)$$

which can be further rewritten as (3.6). Although we have not performed the calculation, there is nothing that suggests the same analysis will not work for a KS ansatz with three null vectors.

References

[1] R. Monteiro, D. O’Connell, and C. D. White, “Black holes and the double copy,” JHEP 1412 (2014) 056, 1410.0239.
[2] Z. Bern, J. J. M. Carrasco and H. Johansson, “New Relations for Gauge-Theory Amplitudes,” Phys. Rev. D 78, 085011 (2008) [arXiv:0805.3993 [hep-ph]].

[3] Z. Bern, J. J. M. Carrasco and H. Johansson, “Perturbative Quantum Gravity as a Double Copy of Gauge Theory,” Phys. Rev. Lett. 105, 061602 (2010) [arXiv:1004.0476 [hep-th]].

[4] Z. Bern, T. Dennen, Y. t. Huang and M. Kiermaier, “Gravity as the Square of Gauge Theory,” Phys. Rev. D 82, 065003 (2010) [arXiv:1004.0693 [hep-th]].

[5] Z. Bern, J. J. Carrasco, M. Chiodaroli, H. Johansson and R. Roiban, arXiv:1909.01358 [hep-th].

[6] S. Sabharwal and J. W. Dalhuisen, “Anti-Self-Dual Spacetimes, Gravitational Instantons and Knotted Zeros of the Weyl Tensor,” JHEP 07 (2019) 004, 1904.06030.

[7] A. Anastasiou, L. Borsten, M. J. Duff, L. J. Hughes, and S. Nagy, “Yang-Mills origin of gravitational symmetries,” Phys. Rev. Lett. 113 (2014), no. 23, 231606, 1408.4434.

[8] L. Borsten and M. J. Duff, “Gravity as the square of Yang-Mills?,” Phys. Scripta 90 (2015) 108012, 1602.08267.

[9] A. K. Ridgway and M. B. Wise, “Static Spherically Symmetric Kerr-Schild Metrics and Implications for the Classical Double Copy,” Phys. Rev. D 94, no. 4, 044023 (2016) [arXiv:1512.02243 [hep-th]].

[10] A. Luna, R. Monteiro, I. Nicholson, D. O’Connell and C. D. White, JHEP 1606, 023 (2016) doi:10.1007/JHEP06(2016)023 [arXiv:1603.05737 [hep-th]].

[11] M. Carrillo-Gonzalez, R. Penco and M. Trodden, “The classical double copy in maximally symmetric spacetimes,” JHEP 1804, 028 (2018) [arXiv:1711.01296 [hep-th]].

[12] A. Anastasiou, L. Borsten, M. J. Duff, M. J. Hughes, A. Marrani, S. Nagy, and M. Zoccali, “Twin supergravities from Yang-Mills theory squared,” Phys. Rev. D96 (2017), no. 2, 026013, 1610.07192.

[13] A. Anastasiou, L. Borsten, M. J. Duff, A. Marrani, S. Nagy, and M. Zoccali, “Are all supergravity theories Yang-Mills squared?,” 1707.03234.

[14] G. L. Cardoso, S. Nagy, and S. Nampuri, “A double copy for $\mathcal{N} = 2$ supergravity: a linearised tale told on-shell,” JHEP 10 (2016) 127, 1609.05022.

[15] G. Cardoso, S. Nagy and S. Nampuri, “Multi-centered $\mathcal{N} = 2$ BPS black holes: a double copy description,” JHEP 1704, 037 (2017) [arXiv:1611.04409 [hep-th]].

[16] L. Borsten, “On $D = 6$, $\mathcal{N} = (2,0)$ and $\mathcal{N} = (4,0)$ theories,” 1708.02573.
[17] A. Anastasiou, L. Borsten, M. J. Duff, A. Marrani, S. Nagy, and M. Zoccali, “The Mile High Magic Pyramid,” 1711.08476.

[18] A. Anastasiou, L. Borsten, M. J. Duff, S. Nagy and M. Zoccali, “Gravity as Gauge Theory Squared: A Ghost Story,” Phys. Rev. Lett. 121, no. 21, 211601 (2018) [arXiv:1807.02486 [hep-th]].

[19] M. Gurses and B. Tekin, “Classical Double Copy: Kerr-Schild-Kundt metrics from Yang-Mills Theory,” Phys. Rev. D 98, no. 12, 126017 (2018) [arXiv:1810.03411 [gr-qc]].

[20] A. Anastasiou, L. Borsten, M. J. Duff, S. Nagy, and M. Zoccali, “BRST squared,” 1807.02486.

[21] G. Lopes Cardoso, G. Inverso, S. Nagy, and S. Nampuri, “Comments on the double copy construction for gravitational theories,” in 17th Hellenic School and Workshops on Elementary Particle Physics and Gravity (CORFU2017) Corfu, Greece, September 2-28, 2017. 2018. 1803.07670.

[22] W. D. Goldberger and J. Li, “Strings, extended objects, and the classical double copy,” arXiv:1912.01650 [hep-th].

[23] K. Kim, K. Lee, R. Monteiro, I. Nicholson and D. Peinador Veiga, “The Classical Double Copy of a Point Charge,” arXiv:1912.02177 [hep-th].

[24] A. Luna, R. Monteiro, D. O’Connell and C. D. White, “The classical double copy for Taub-NUT spacetime,” Phys. Lett. B 750, 272 (2015) [arXiv:1507.01869 [hep-th]].

[25] D. S. Berman, E. Chacon, A. Luna, and C. D. White, “The self-dual classical double copy, and the Eguchi-Hanson instanton,” 1809.04063.

[26] Y. T. Huang, U. Kol and D. O’Connell, “The Double Copy of Electric-Magnetic Duality,” arXiv:1911.06318 [hep-th].

[27] R. Alawadhi, D. S. Berman, B. Spence and D. P. Veiga, “S-duality and the Double Copy,” arXiv:1911.06797 [hep-th].

[28] J. Ehlers, “Transformations of static exterior solutions of Einstein’s gravitational field equations into different solutions by means of conformal mapping,” Colloq. Int. CNRS 91, 275 (1962).

[29] I. Bakhmatov, N. S. Deger, E. T. Musaev, E. Ó Colgáin and M. M. Sheikh-Jabbari, “Trivector deformations in $d = 11$ supergravity,” JHEP 1908, 126 (2019) [arXiv:1906.09052 [hep-th]].

[30] O. Lunin and J. M. Maldacena, “Deforming field theories with U(1) x U(1) global symmetry and their gravity duals,” JHEP 0505, 033 (2005) [hep-th/0502086].
[31] H. A. Buchdahl, “Reciprocal Static Metrics and Scalar Fields in the General Theory of Relativity,” Phys. Rev. 115, 1325 (1959).

[32] A. C. Petkou, P. M. Petropoulos and K. Siampos, “Geroch group for Einstein spaces and holographic integrability,” PoS PLANCK 2015, 104 (2015) [arXiv:1512.04970 [hep-th]].