Brane-World Black Hole Solutions via a Confining Potential

M. Heydari-Fard 1,* H. Razmi 2† and H. R. Sepangi 1‡

Department of Physics, Shahid Beheshti University, Evin, Tehran 19839, Iran
Department of Physics, The University of Qom, Qom 37185-359, Iran

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Abstract

Using a confining potential, we consider spherically symmetric vacuum (static black hole) solutions in a brane-world scenario. Working with a constant curvature bulk, two interesting cases/solutions are studied. A Schwarzschild-de Sitter black hole solution similar to the standard solution in the presence of a cosmological constant is obtained which confirms the idea that an extra term in the field equations on the brane can play the role of a positive cosmological constant and may be used to account for the accelerated expansion of the universe. The other solution is one in which we can have a proper potential to explain the galaxy rotation curves without assuming the existence of dark matter and without working with new modified theories (modified Newtonian dynamics).

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1 Introduction

In recent years, models with extra dimensions have been proposed in which the standard fields are confined to a four-dimensional (4D) world viewed as a hypersurface (the brane) embedded in a higher dimensional space-time (the bulk) through which only gravity can propagate. The most well-known model in the context of brane-world theory is that proposed by Randall and Sundrum (RS). In the so-called RS1 model [1], they proposed a mechanism to solve the hierarchy problem with two branes, while in the RSII model [2], they considered a single brane with a positive tension, where 4D Newtonian gravity is recovered at low energies even if the extra dimension is not compact. This mechanism provides an alternative to compactification of extra dimensions.

The cosmological evolution of such a brane universe has been extensively investigated and effects such as a quadratic density term in the Friedmann equations have been found [3, 4]. This term arises from the imposition of the Israel junction conditions which is a relationship between the extrinsic curvature and energy-momentum tensor of the brane and that results from the singular behavior in the energy-momentum tensor. There has been concerns expressed over applying such junction conditions in that they may not be unique. Indeed, other forms of junction conditions exist, so that different conditions may lead to different physical results [5]. Furthermore, these conditions cannot be used when more than one non-compact extra dimension is involved. To avoid such concerns, an interesting higher-dimensional model was introduced where particles are trapped on a 4-dimensional hypersurface by the action of a confining potential $V$[6]. In [7], the dynamics of test particles confined to a brane by the action of such a potential at the classical and quantum levels were studied and the

*email: m.heydarifard@mail.sbu.ac.ir
†email: razmi@qom.ac.ir
‡email: hr-sepangi@sbu.ac.ir
effects of small perturbations along the extra dimensions investigated. In [8], a brane-world model was studied in which matter is confined to the brane through the action of such a potential without using any junction conditions, offering a geometrical explanation for the accelerated expansion of the universe. Another work in which localization of matter on the brane is again realized by means of a confining potential is the study of a brane scenario in which the $m$-dimensional bulk is endowed with a Gauss-Bonnet (GB) term [9]. It was shown that in the presence of the GB term, the universe accelerates faster than brane models without the GB term. The behavior of an anisotropic brane-world with Bianchi type I and V geometry in a similar vain was studied in [10].

In brane theories the covariant Einstein equations are derived by projecting the bulk equations onto the brane. This was first done by Shiromizu, Maeda and Sasaki (SMS) [11] where the Gauss-Codazzi equations together with Israel junction conditions were used to obtain the Einstein field equations on the 3-brane. The field equations on the brane is different from the Einstein equations in the standard model. An essential modification appears at high energies in the form of a new source term in the effective Einstein equation, which is quadratic in the brane energy-momentum tensor. Another modification arises whenever the bulk has a Weyl-curvature with non-vanishing projection onto the brane. This is known as the electric part of the bulk Weyl tensor. In this context, it is natural to study solutions corresponding to compact sources on the brane such as stars and black holes. Gravitational collapse on the brane has been studied by many authors [13]-[20]. In [21], the authors obtain an exact black hole solution of the effective Einstein equation on the brane under the condition that the bulk has non zero Weyl curvature and the brane space-time satisfies the null energy condition. The solution is given by the usual Reissner-Nordstrom (RN) metric where the charge parameter is thought of as a tidal charge arising from the projection of the Weyl curvature of the bulk onto the brane. The RN metric has thus been interpreted as describing a black hole on a brane where the electric charge’s role is taken over by the tidal charge and it can be thought of as the analogue of the Schwarzschild solution on the brane. The tidal charge like the RN electric charge would generate $1/r^2$ term in the potential while the high energy modification to the Newtonian potential cannot be any stronger than $1/r^3$ [2, 22]. The cause for this disagreement is the presence of tidal charge which is the measure of the bulk Weyl curvature. The main drawback of the solution is that we do not know the corresponding bulk solution. It is however agreed that RN metric is a good approximation to a black hole on the brane near the horizon [23].

The RN solution can be matched to the interior solution corresponding to a constant density brane-world star. A second exterior solution, which also matches a constant density interior, has been derived in [25]. Non-singular black hole solutions in the brane-world model have been considered in [24], by relaxing the condition of the zero scalar curvature but retaining the null energy condition. It has also been shown that the vacuum field equations on the brane reduce to a system of two ordinary differential equations, which describe all the geometric properties of the vacuum as functions of the dark pressure and dark radiation terms [26].

In this paper, following the model introduced in [8], we consider an $m$-dimensional bulk space without imposing the $Z_2$ symmetry. As mentioned above, to localize the matter on the brane, a confining potential is used rather than a delta-function in the energy-momentum tensor. The vacuum field equations on the brane are modified by the $Q_{\mu\nu}$ term which is a geometrical quantity. We obtain exact solutions of the vacuum field equations on the brane for two interesting cases. The first solution can be used to explain the galaxy rotation curves without assuming the existence of dark matter and without working with new modified theories [27] (modified Newtonian dynamics) and the second solution represents a black hole in an asymptotically de Sitter space. Clearly, this work differs from the model introduced in [28, 30] in them no mechanism for the confinement of matter on the brane is introduced.
2 Geometrical considerations

In this section we present a brief review of the model proposed in [7, 8]. Consider the background manifold $V_4$ isometrically embedded in a pseudo-Riemannian manifold $V_m$ by the map $\mathcal{Y} : V_4 \rightarrow V_m$ such that

\[ G_{AB} \mathcal{Y}^A_{\mu} \mathcal{Y}^B_{\nu} = \bar{g}_{\mu\nu}, \quad G_{AB} \mathcal{Y}^A_{\mu} N^B_a = 0, \quad G_{AB} N^A_a N^B_b = \bar{g}_{ab} = \pm 1, \]  

where $G_{AB}$ is the metric of the bulk (brane) space $V_m (V_4)$ in arbitrary coordinates, $\{ \mathcal{Y}^A \}$ is the basis of the bulk (brane) coordinates, and $N^A_a$ are $(m - 4)$ normal unit vectors, orthogonal to the brane. Perturbation of $\bar{V}_4$ in a sufficiently small neighborhood of the brane along an arbitrary transverse direction $\xi$ is given by

\[ Z^A(x^\mu, \xi^a) = \mathcal{Y}^A + (L_\xi \mathcal{Y})^A, \]  

where $L_\xi$ represents the Lie derivative and $\xi^a (a = 1, 2, ..., m - 4)$ is a small parameter along $N^A_a$ that parameterizes the extra noncompact dimensions. By choosing $\xi$ orthogonal to the brane, we ensure gauge independency [7] and have perturbations of the embedding along a single orthogonal extra direction $\bar{N}_a$ giving local coordinates of the perturbed brane as

\[ Z^A_{\mu}(x^\nu, \xi^a) = \mathcal{Y}^A_{\mu} + \xi^a \bar{N}^A_{a,\mu}(x^\nu). \]  

In a similar manner, one can find that since the vectors $\bar{N}^A_a$ depend only on the local coordinates $x^\mu$, they do not propagate along the extra dimensions. The above assumptions lead to the embedding equations of the perturbed geometry

\[ g_{\mu\nu} = G_{AB} Z^A_{\mu} Z^B_{\nu}, \quad g_{\mu a} = G_{AB} Z^A_{\mu} N^B_a, \quad g_{AB} N^A_a N^B_b = g_{ab}. \]  

If we set $N^A_a = \delta^A_a$, the metric of the bulk space can be written in the following matrix form

\[ G_{AB} = \begin{pmatrix} g_{\mu\nu} + A_{\mu c} A^c_{\nu} & A_{\mu a} \\ A_{\nu b} & g_{ab} \end{pmatrix}, \]  

where

\[ g_{\mu\nu} = \bar{g}_{\mu\nu} - 2 \xi^a \bar{K}_{\mu a} + \xi^b \bar{g}^{a\beta} \bar{K}_{\mu a \beta} \bar{K}_{\nu b}, \]  

is the metric of the perturbed brane, so that

\[ \bar{K}_{\mu a} = -G_{AB} \mathcal{Y}^A_{\mu} N^B_{a,\nu}, \]  

represents the extrinsic curvature of the original brane (second fundamental form). We use the notation $A_{\mu c} = \xi^d A_{\mu c d}$, where

\[ A_{\mu c d} = G_{AB} N^A_{d,\mu} N^B_c = \bar{A}_{\mu c d}, \]  

represents the twisting vector fields (the normal fundamental form). Any fixed $\xi^a$ signifies a new perturbed geometry, enabling us to define an extrinsic curvature similar to the original one by

\[ \bar{K}_{\mu a} = -G_{AB} Z^A_{\mu} N^B_{a,\nu} = K_{\mu a} - \xi^b (\bar{K}_{\mu \gamma a} \bar{K}^\gamma_{\nu b} + A_{\mu c a} A^c_{b \nu}). \]  

Note that definitions (5) and (9) require

\[ \bar{K}_{\mu a} = -\frac{1}{2} \frac{\partial G_{\mu a}}{\partial \xi^a}. \]  

In geometric language, the presence of gauge fields $A_{\mu a}$ tilts the embedded family of sub-manifolds with respect to the normal vector $N^A_a$. According to our construction, the original brane is orthogonal
to the normal vector $N^A$. However, equation (4) shows that this is not true for the deformed geometry. Let us change the embedding coordinates and set

$$X^A_{\mu} = Z^A_{\mu} - g^{ab} N^A_a A^b_{\mu}. \quad (11)$$

The coordinates $X^A$ describe a new family of embedded manifolds whose members are always orthogonal to $N^A$. In this coordinates the embedding equations of the perturbed brane is similar to the original one, described by equation (1), so that $Y^A$ is replaced by $X^A$. This new embedding of the local coordinates are suitable for obtaining induced Einstein field equations on the brane. The extrinsic curvature of a perturbed brane then becomes

$$K_{\mu\nu} = -G_{AB} X^A_{\mu} N^B_a = \frac{1}{2} \frac{\partial g_{\mu\nu}}{\partial \xi^a}, \quad (12)$$

which is the generalized York’s relation and shows how the extrinsic curvature propagates as a result of the propagation of the metric in the direction of extra dimensions. The components of the Riemann tensor of the bulk written in the embedding vielbein $\{X^A_{\mu}, N^A_a\}$, lead to the Gauss-Codazzi equations

$$R_{\alpha\beta\gamma\delta} = 2g^{ab} K_{\alpha^a} K_{\beta^b} - R_{ABCD} X^A_{\mu} X^B_{\nu} X^C_{\lambda} X^D_{\sigma}, \quad (13)$$

and

$$2K_{\alpha(\gamma} = 2g^{ab} A_{\alpha^a} K_{\beta^b} + R_{ABCD} X^A_{\alpha} N^B_a X^C_{\gamma} N^D_b, \quad (14)$$

where $R_{ABCD}$ and $R_{\alpha\beta\gamma\delta}$ are the Riemann tensors for the bulk and the perturbed brane respectively. Contracting the Gauss equation (13) on $\alpha$ and $\gamma$, we find

$$R_{\mu\nu} = (K_{\mu\alpha} K_{\nu}^{\alpha^c} - K_{\mu} K_{\nu}^{\mu^c}) + R_{AB} X^A_{\mu} X^B_{\nu} - g^{ab} R_{ABCD} N^A_a X^B_{\mu} X^C_{\nu} N^D_b. \quad (15)$$

A further contraction gives the Ricci scalar

$$R = (K_{\mu\alpha} K_{\nu}^{\alpha^c} - K_{\mu} K_{\nu}^{\mu^c}) + R - 2g^{ab} R_{AB} N^A_a N^B_b + g^{ab} g^{cd} R_{ABCD} N^A_a N^B_b X^C_{\gamma} N^D_{\delta}. \quad (16)$$

### 3 Field equations on the brane

We consider the total action for space-time $(M, G_{AB})$ with boundary $(\Sigma, g_{\mu\nu})$ as

$$S = \frac{1}{2\alpha^*} \int_M d^n x \sqrt{-g(R - 2\Lambda^{(b)})} + \int_\Sigma d^4 x \sqrt{-g(\mathcal{L}_{\text{surface}} + \mathcal{L}_m)}. \quad (17)$$

Variation of the total action gives the Einstein equations in the bulk space as

$$G_{AB}^{(b)} + \Lambda^{(b)} G_{AB} = \alpha^* S_{AB}, \quad (18)$$

where

$$S_{AB} = T_{AB} + \frac{1}{2} \mathcal{V} G_{AB}, \quad (19)$$

here $\alpha^* = \frac{1}{M_*^2}$ ($M_*$ is the fundamental scale of energy in the bulk space), $\Lambda^{(b)}$ is the cosmological constant of the bulk and $T_{AB} = -2 \frac{\delta S_m}{\delta g^{AB}} + g_{AB} \mathcal{L}_m$ is the energy-momentum tensor of the matter confined to the brane through the action of the confining potential $\mathcal{V}$. We require $\mathcal{V}$ to satisfy three general conditions: firstly, it has a deep minimum on the non-perturbed brane, secondly, depends only on extra coordinates and thirdly, the gauge group representing the subgroup of the isometry group of the bulk space is preserved by it [7].
Using the Einstein equations (18), we obtain the scalar curvature

\[ G_{\mu\nu} = G_{AB}A^{\mu}_{a}A^{\nu}_{b} + Q_{\mu\nu} + g^{ab}R_{AB}N^{A}_{a}N^{B}_{b}g_{\mu\nu} - g^{ab}R_{ABCD}N^{A}_{a}A^{B}_{b}N^{C}_{c}N^{D}_{d}, \]  

(20)

where

\[ Q_{\mu\nu} = -g^{ab}\left(K^{\gamma}_{\mu\nu}K^{\nu}_{\gamma b} - K^{a}_{\mu\nu}K^{\nu}_{\gamma b}\right) + \frac{1}{2}\left(K^{a\beta\alpha}K^{\alpha\beta\gamma} - K^{a}_{\mu\nu}K^{\nu}_{\gamma b}\right)g_{\mu\nu}. \]

(21)

As can be seen from the definition of \( Q_{\mu\nu} \), it is an independently conserved quantity, that is \( Q^{\mu\nu}_{\nu} = 0 \) \[28\]. Using the decomposition of the Riemann tensor into the Weyl curvature, the Ricci tensor and the scalar curvature

\[ R_{ABCD} = C_{ABCD} - \frac{2}{(m-2)}\left(G_{B[D}R_{C]|A - G_{A[D}R_{C]|B}\right) - \frac{2}{(m-1)(m-2)}R(G_{A[D}G_{C]|B), \]

(22)

we obtain the 4D Einstein equations as

\[ G_{\mu\nu} = G_{AB}A^{\mu}_{a}A^{\nu}_{b} + Q_{\mu\nu} - \varepsilon_{\mu\nu} + \frac{m-3}{(m-2)}g^{ab}R_{AB}A^{A}_{a}N^{B}_{b}g_{\mu\nu} - \frac{m-4}{(m-2)}R_{AB}A^{A}_{a}N^{B}_{b} + \frac{m-4}{(m-1)(m-2)}R_{\mu\nu}, \]

(23)

where \( \varepsilon_{\mu\nu} = g^{ab}C_{ABCD}N^{A}_{a}A^{B}_{b}N^{D}_{d} \) is the electric part of the Weyl tensor of the bulk space \( C_{ABCD} \).

Using the Einstein equations (18), we obtain

\[ R_{AB} = -\frac{\alpha^{*}}{(m-2)}G_{AB}S + \frac{2}{(m-2)}\Lambda^{(b)}G_{AB} + \alpha^{*}S_{AB}, \]

(24)

and

\[ R = -\frac{2}{m-2}(\alpha^{*}S - m\Lambda^{(b)}). \]

(25)

Substituting \( R_{AB} \) and \( R \) from the above into equation (23), we obtain

\[ G_{\mu\nu} = Q_{\mu\nu} - \varepsilon_{\mu\nu} + \frac{(m-3)}{(m-2)}\alpha^{*}g^{ab}S_{ab}g_{\mu\nu} + \frac{2\alpha^{*}}{(m-2)}S_{\mu\nu} - \frac{(m-4)(m-3)}{(m-1)(m-2)}\alpha^{*}S_{\mu\nu} + \frac{(m-7)}{(m-1)}\Lambda^{(b)}g_{\mu\nu}. \]

(26)

On the other hand, again from equation (18), the trace of the Codazzi equation (14) gives the “gravi-vector equation”

\[ K^{\delta}_{\alpha\gamma\delta} - K_{\alpha\gamma} - A_{\alpha\gamma}K^{b} + A_{\alpha b\delta}K^{\gamma\delta} + B_{\alpha\gamma} = \frac{3(m-4)}{m-2}\alpha^{*}S_{\alpha\gamma}, \]

(27)

where

\[ B_{\alpha\gamma} = g^{mn}C_{ABCD}N^{A}_{m}A^{B}_{n}C_{\gamma\delta}N^{D}_{\delta}. \]

(28)

Finally, the “gravi-scalar equation” is obtained from the contraction of (15), (23) and using equation (18)

\[ \alpha^{*}\left[\frac{m-5}{m-1}S - g^{mn}S_{mn}\right]g_{ab} = \frac{m-2}{6}(Q + R + W)g_{ab} - \frac{4}{m-1}\Lambda^{(b)}g_{ab}, \]

(29)

where

\[ W = g^{ab}g^{mn}C_{ABCD}N^{A}_{m}N^{B}_{n}N^{C}_{a}N^{D}_{b}. \]

(30)

Equations (26)-(30) represent the projections of the Einstein field equations on the brane-brane, bulk-brane, and bulk-bulk directions.
As was mentioned in the introduction, localization of matter on the brane is realized in this model by the action of a confining potential. This can simply be realized by

$$\alpha \tau_{\mu \nu} = \frac{2\alpha^*}{(m - 2)} T_{\mu \nu}, \quad T_{\mu a} = 0, \quad T_{ab} = 0,$$

where $\alpha$ is the scale of energy on the brane. Now, the induced Einstein field equations on the original brane can be written as

$$G_{\mu \nu} = \alpha \tau_{\mu \nu} - \frac{(m - 4)(m - 3)}{2(m - 1)} \alpha \tau g_{\mu \nu} - \Lambda g_{\mu \nu} + Q_{\mu \nu} - E_{\mu \nu},$$

where $\Lambda = \frac{(m - 7)}{(m - 1)} \Lambda^{(b)}$ and $Q_{\mu \nu}$ is a completely geometrical quantity.

A brief discussion on the energy-momentum conservation on the brane would be in order here. The contracted Bianchi identities in the bulk space $G_{AB}^{(b)} = 0$, using equation (18), imply

$$\left( T^{AB} + \frac{1}{2} \mathcal{V} G^{AB} \right)_{;A} = 0.$$

Since the potential $\mathcal{V}$ has a minimum on the brane, the above conservation equation reduces to

$$\tau_{\mu ; \mu} = 0.$$

As we mentioned before, $Q_{\mu \nu}$ is an independently conserved quantity which according to [28] may be considered as an energy-momentum tensor of a dark energy fluid representing the x-matter, the more common phrase being “x-Cold-Dark Matter” (xCDM). This matter has the most general form of the equation of state which is characterized by the following conditions [31]: violation of the strong energy condition at the present epoch for $\omega_x < -1/3$ where $p_x = \omega_x \rho_x$, local stability i.e. $c_s^2 = \delta p_x/\delta \rho_x \geq 0$ and preservation of causality i.e. $c_s \leq 1$. Ultimately, we have three different types of ‘matter’ on the right hand side of equation (32), namely, ordinary confined conserved matter represented by $\tau_{\mu \nu}$, the Weyl matter represented by $Q_{\mu \nu}$, and the effective equations derived in the previous section are given by

$$G_{\mu \nu} = -\Lambda g_{\mu \nu} + Q_{\mu \nu} - E_{\mu \nu},$$

where $E_{\mu \nu}$ is a symmetric and traceless tensor due to the Weyl symmetries and is constrained by the conservation equations

$$E_{\mu \nu ; \nu} = 0,$$

obtained as a result of the Bianchi identities. Equations (35) and (36) determine the system of vacuum field equations on the brane. Restricting our analysis to a constant curvature bulk ($E_{\mu \nu} = 0$) and neglecting the effect of the cosmological constant, the vacuum field equation (35) reduce to

$$G_{\mu \nu} = Q_{\mu \nu},$$

where $Q_{\mu \nu}$ is a completely geometrical quantity given by

$$Q_{\mu \nu} = (K_{\mu \nu}^2 - K_{\mu} K_{\nu}^\alpha K_{\nu}^\alpha) + \frac{1}{2} \left( K_{\alpha \beta} K_{\alpha \beta} - K^2 \right) g_{\mu \nu}.$$
For the following choice of the static spherically symmetric metric on the brane
\[ ds^2 = -e^{\mu(r)} dt^2 + e^{\nu(r)} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2), \]  
the gravitational field equations are given by
\[ G^0_0 = e^{-\nu} \left( 1 - r\nu' - e^{\nu} \right), \] (40)
\[ G^1_1 = e^{-\nu} \left( 1 + r\mu' - e^{\nu} \right), \] (41)
\[ G^2_2 = G^3_3 = e^{-\nu} \frac{r}{4} \left( 2\mu' - 2\nu' - \mu' \nu r + 2\mu'' r + \mu'' r \right), \] (42)

where a prime represents differentiation with respect to \( r \). The York’s relation
\[ K_{\mu\nu a} = -\frac{1}{2} \frac{\partial g_{\mu\nu}}{\partial \xi^a}, \] (43)
shows that in a diagonal metric, \( K_{\mu\nu a} \) are diagonal. The Codazzi equations (14) with the assumption of \( E_{\mu\nu} = 0 \) take the form
\[ K_{\alpha\delta a;\gamma} - K_{\alpha\gamma a;\delta} = 0, \] (44)
where by separating the spatial components reduce to
\[ K_{\mu\nu a,\sigma} - K_{\nu\rho a} \Gamma^\rho_{\mu\sigma} = K_{\mu\sigma a,\nu} - K_{\sigma\rho a} \Gamma^\rho_{\mu
u}, \] (45)

\[ K_{00a,1} - \left( \frac{\mu'}{2} \right) K_{00a} = -\left( \frac{\mu' e^{\mu}}{2e^{\nu}} \right) K_{11a}, \] (46)
\[ K_{22a,1} - \left( \frac{1}{r} \right) K_{22a} = (r e^{-\nu}) K_{11a}. \] (47)

The first equation gives \( K_{00a,\sigma} = K_{11a,\sigma} = K_{22a,\sigma} = K_{33a,\sigma} = 0 \) for \( \sigma = 0,3 \). Repeating the same procedure for \( \sigma = 2 \), we obtain \( K_{00a,\sigma} = K_{11a,\sigma} = K_{22a,\sigma} = 0 \). This shows that \( K_{11a} \) depends only on the variable \( r \). Assuming \( K_{11a} = \alpha_a e^{\nu(r)} \) and using equations (46) and (47), one finds
\[ K_{00a}(r) = -\alpha_a e^{\mu(r)} + c_a e^{\mu(r)/2}, \] (48)
\[ K_{22a}(r) = \alpha_a r^2 + \beta_a r. \] (49)

Taking \( \mu, \nu = 3 \) in the first equation we obtain
\[ K_{33a,1} - \left( \frac{1}{r} \right) K_{33a} = (e^{-\nu} r \sin^2 \theta) K_{11a} = \alpha_a r \sin^2 \theta, \] (50)
\[ K_{33a,2} - (\cot \theta) K_{33a} = (\sin \theta \cos \theta) K_{22a}. \] (51)

Using equations (49), (50) and (51), \( k_{33a} \) is given by
\[ K_{33a}(r; \theta) = \alpha_a r^2 \sin^2 \theta + r \beta_a \sin^2 \theta + r c_{1a} \sin \theta. \] (52)
If we assume that the constants are equal, that is $\alpha_a = \alpha$, $\beta_a = \beta$ and $c_a = c$, $c_{1a} = c_1$, use of equation (38) leads to the components of $Q_{\mu \nu}$

\[
Q_{00} = -\frac{g_{00}}{r^2} \left( 3\alpha^2 r^2 + 4\alpha\beta r + \beta^2 + \frac{c_1}{\sin \theta} (2\alpha r + \beta) \right),
\]

\[
Q_{11} = -\frac{g_{11}}{r^2} \left( 3\alpha^2 r^2 + 4\alpha\beta r + \beta^2 + \frac{c_1}{\sin \theta} (2\alpha r + \beta - cre^{-\mu/2}) \right),
\]

\[
Q_{22} = \frac{g_{22}}{r} \left( -3\alpha^2 r - 2\alpha\beta + c e^{-\mu/2} (2\alpha r + \beta) + \frac{c_1}{\sin \theta} (-2\alpha + ce^{-\mu/2}) \right),
\]

\[
Q_{33} = \frac{g_{33}}{r} \left( -3\alpha^2 r - 2\alpha\beta + ce^{-\mu/2} (2\alpha r + \beta) \right).
\] (53)

Since $G_2^2 = G_3^2$ and thus $Q_2^2 = Q_3^2$ one obtains $c_1 = 0$. The relations (53), equations (37) and (40)-(42) lead to the vacuum field equations on the brane

\[
e^{-\nu} \left( -\frac{1}{r^2} + \frac{\mu'}{r} \right) + \frac{1}{r^2} = 3\alpha^2 + \frac{4\alpha\beta}{r} + \frac{\beta^2}{r^2},
\] (54)

\[
e^{-\nu} \left( \frac{1}{r^2} + \frac{\mu'}{r} \right) + \frac{1}{r^2} = 3\alpha^2 + \frac{4\alpha\beta}{r} + \frac{\beta^2}{r^2} - 2ce^{-\mu/2} (\alpha + \frac{\beta}{r}),
\] (55)

\[
e^{-\nu} \left( \frac{\mu' - \nu'}{r} - \frac{\mu' \nu'}{2} + \mu'' + \frac{\mu'^2}{2} \right) = -6\alpha^2 - \frac{4\alpha\beta}{r} + 2ce^{-\mu/2} (2\alpha + \frac{\beta}{r}).
\] (56)

Equation (54) can immediately be integrated to give

\[
e^{-\nu(r)} = 1 - \frac{C_1}{r} - \alpha^2 r^2 - 2\alpha\beta r - \beta^2,
\] (57)

where $C_1$ is an integration constant. Substitution of $e^{-\nu(r)}$ into equation (55) leads to

\[
e^{\mu(r)} = \frac{f(r)}{4r} \left( -C_2 + 2ac \int r^{5/2} dr + 2bc \int r^{3/2} dr \right)^2,
\] (58)

where $C_2$ is an integration constant and

\[
f(r) = -r + C_1 + \alpha^2 r^3 + 2\alpha\beta r^2 + \beta^2 r.
\] (59)

Although we have considered all the necessary equations/relations (even with special choices), there are still a number of arbitrary constants which do not let us find the unique vacuum solution of the gravitational field equations on the brane because the Birkhoff theorem does not apply here [25, 26]. All the way, among various possible solutions that depend on different choices of arbitrary constants $c, \alpha, \beta$, we consider the two following interesting cases. The choice $c = 0$ and use of equation (58) result in

\[
e^{\mu(r)} = e^{-\nu(r)} = 1 - \frac{C_1}{r} - \alpha^2 r^2 - 2\alpha\beta r - \beta^2.
\] (60)

For small distances associated with the standard stellar/astrophysical scales (e.g. solar system scales), the corresponding metric for this case is the familiar Schwarzschild solution. On larger distance scales associated with galaxies, assuming a small value of $\alpha$ in order to neglecting its second order term, the model differs from Einstein theory through the following potential function

\[
\phi(r) = \frac{C_1}{2r} + \alpha \beta r + \frac{\beta^2}{2},
\] (61)
which can be used to explain the galactic rotation curves corresponding to the well-known dark matter problem without resorting to dark matter and even without assuming any new/modified mechanics (e.g., modified Newtonian dynamics [31]).

A second class of solutions of the system of equations (54)-(56) can be obtained by considering \( c = \beta = 0 \) and \( \alpha \neq 0 \). In this case by means of equation (58) we find

\[
e^{\mu(r)} = e^{-\nu(r)} = 1 - \frac{C_1}{r} - \alpha^2 r^2.
\]

(62)

The corresponding line element, choosing \( C_1 = 2GM = 2m \) based on proper correspondence principle in the limit of standard gravity, takes the form

\[
d s^2 = -\left(1 - \frac{2m}{r} - \alpha^2 r^2\right) dt^2 + \frac{dr^2}{\left(1 - \frac{2m}{r} - \alpha^2 r^2\right)} + r^2 \left(d\theta^2 + \sin^2 \theta d\varphi^2\right).
\]

(63)

Comparing the above result with the following line element for the black hole solution in an asymptotically de Sitter space

\[
d s^2 = -\left(1 - \frac{2m}{r} - \frac{\Lambda}{3} r^2\right) dt^2 + \frac{dr^2}{\left(1 - \frac{2m}{r} - \frac{\Lambda}{3} r^2\right)} + r^2 \left(d\theta^2 + \sin^2 \theta d\varphi^2\right),
\]

(64)

the cosmological constant is found as \( \Lambda = 3\alpha^2 \). This positive value is in agreement with present observations. Now, let us find the Kretschmann scalar of the metric (60) as

\[
R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} = \frac{4(6\alpha^4 r^6 + 12\alpha^3 \beta r^5 + 10\alpha^2 \beta^2 r^4 + 4\alpha \beta^3 r^3 + \beta^4 r^2 + 4m \beta^2 r + 12m^2)}{r^6},
\]

(65)

the singularity at \( r = 0 \) is an intrinsic singularity. In the case that \( \alpha = 0 \) we have the Schwarzschild horizon \( r = 2m \), and for \( m = 0 \) we have the de Sitter horizon \( r = \frac{1}{\alpha} \). For \( \sqrt{27m} < \frac{1}{\alpha} \) there are two horizons

\[
r_1 = \frac{2}{\sqrt{3}\alpha} \cos \frac{\Theta}{3},
\]

(66)

\[
r_2 = -\frac{1}{\sqrt{3}\alpha} \left(\cos \frac{\Theta}{3} - \sqrt{3} \sin \frac{\Theta}{3}\right),
\]

(67)

where \( \Theta \) is given by

\[
\cos \Theta = -\sqrt{27m}\alpha.
\]

(68)

If \( m = 0 \) we have \( r_2 = 0 \) and \( r_1 = \frac{1}{\alpha} \), then we call \( r_1 \) the cosmological horizon generalized when \( m \neq 0 \), and \( r_2 \) the black hole horizon generalized when \( \alpha \neq 0 \). For \( \sqrt{27m} = \frac{1}{\alpha} \), \( r_1 \) and \( r_2 \) coincide and there is only one horizon,

\[
r = \frac{1}{\sqrt{3}\alpha}.
\]

(69)

In general, the range for \( r_1 \) and \( r_2 \) is given by

\[
0 \leq r_2 \leq \frac{1}{\sqrt{3}\alpha} \leq r_1 \leq \frac{1}{\alpha}.
\]

(70)

For \( \sqrt{27m} > \frac{1}{\alpha} \) there are no horizons.

Finally, we should mention that if we had worked as in the usual brane-world models where the Israel junction condition is used to calculate the extrinsic curvature in terms of the energy-momentum tensor on the brane and its trace, that is

\[
K_{\mu\nu} = -\frac{1}{2} \alpha^2 \left(\tau_{\mu\nu} - \frac{1}{3} \tau g_{\mu\nu}\right),
\]

(71)
where $\alpha^*$ is proportional to the gravitational constant in the bulk, then by substituting equation (71) into equation (38), the vacuum field equation (35) with a constant curvature bulk would have been reduced to the vacuum field equations in the standard general relativity $G_{\mu\nu} = 0$; but, the $Q_{\mu\nu}$ term here modifies the vacuum field equations on the brane and this is because it originates from the so-called confining potential and not the usual junction condition.

5 Conclusion

In this paper, we have considered the vacuum field equations in a brane world model where the matter is confined to the brane through the action of a confining potential, rendering the use of any junction condition redundant. We have obtained the exact solutions for static black holes localized on a 3-brane in a constant curvature bulk. A particular solution of the field equations represents a Schwarzschild-de Sitter black hole in the presence of a positive cosmological constant. This shows that the extra term $Q_{\mu\nu}$ may play the role of dark energy, supporting our previous results that the accelerated expansion of the universe could be explained in a purely geometrical fashion based on the extrinsic curvature. Another interesting solution to the model considered here is one in which we can have a proper potential to explain the galaxy rotation curves without assuming the existence of dark matter and without working with new modified theories (modified Newtonian dynamics [31,32]).

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