ON THE THERMODYNAMICS OF SIMPLE NON-ISENTEROPIC PERFECT FLUIDS IN GENERAL RELATIVITY*

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ABSTRACT

We examine the consistency of the thermodynamics of irrotational and non-isentropic perfect fluids complying with matter conservation by looking at the integrability conditions of the Gibbs-Duhem relation. We show that the latter is always integrable for fluids of the following types: (a) static, (b) isentropic (admits a barotropic equation of state), (c) the source of a spacetime for which $r \geq 2$, where $r$ is the dimension of the orbit of the isometry group. This consistency scheme is tested also in two large classes of known exact solutions for which $r < 2$, in general: perfect fluid Szekeres solutions (classes I and II). In none of these cases, the Gibbs-Duhem relation is integrable, in general, though specific particular cases of Szekeres class II (all complying with $r < 2$) are identified for which the integrability of this relation can be achieved. We show that Szekeres class I solutions satisfy the integrability conditions only in two trivial cases, namely the spherically symmetric limiting case and the Friedman-Roberson-Walker (FRW) cosmology. Explicit forms of the state variables and equations of state linking them are given explicitly and discussed in relation to the FRW limits of the solutions. We show that fixing free parameters in these solutions by a formal identification with FRW parameters leads, in all cases examined, to unphysical temperature evolution laws, quite unrelated to those of their FRW limiting cosmologies.

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1. Introduction

Numerous exact solutions of Einstein field equations with a perfect fluid source exist in the literature which do not admit a barotropic equation of state $p = p(\rho)$, where $p$ and $\rho$ are the pressure and matter-energy density. The thermodynamic properties of these simple non-isentropic fluids have not been properly examined, as there is a sort of consensus which regards as unphysical all exact solutions with such fluid sources. However, there are arguments supporting the idea that barotropic equations of state are too restrictive\(^1\), and so it is surprising to find so few references in the literature\(^2,3,4,5,6\) studying the properties of these fluids. There are two related levels at which this study can be posed: (1) the consistency of the thermodynamic equations with the field equations; (2) the physical relevance of the state variables and equations of state linking them. Obviously, if point (1) is not satisfied, point (2) cannot be even addressed, also, it is quite possible to find exact solutions complying with (1) but not with (2), that is, unphysical fluids whose thermodynamics is formally correct.

Coll and Ferrando\(^3\) have formally examined and solved the question behind point (1) above. They derived rigorously the necessary and sufficient conditions for the integrability of the Gibbs-Duhem relation for perfect fluid sources complying with matter conservation. These conditions become a criterion to verify if a perfect fluid source of a given exact solution admits what Coll and Ferrando denote a “thermodynamic scheme”. However, these authors did not go beyond point (1) above (the admissibility of their consistency criterion). The purpose of this paper is to expand, continue and complement their work by applying their criterion (re-phrased in a more intuitive and practical form) to irrotational and non-isentropic fluids, and then to deal with point (2) mentioned above, that is, to look at the physical nature of the resulting equations of state in the cases when the thermodynamic equations are mathematically consistent. The contents of this paper are described below.

We present in section 2 a summary of the equations of the thermodynamics of a general relativistic perfect fluid, together with the conditions for admissibility of a thermodynamic scheme as derived by Coll and Ferrando. An immediate result is the fact that a thermodynamic scheme is always admissible in the following three cases: (1) the fluid is
static; (2) the isometry group of spacetime has orbits of dimension \( r \geq 2 \); (3) isentropic fluid, admitting a barotropic equation of state \( p = p(\rho) \). Section 3 is concerned with the admissibility of a thermodynamic scheme for irrotational, non-isentropic perfect fluids. For these fluids, we investigate the conditions derived by Coll and Ferrando in terms of differential forms expanded in a coordinate basis adapted to a comoving frame. The conditions of section 3 are applied in sections 4 and 5 to the irrotational perfect fluid generalization of Szekeres solutions (class II\(_{6-12}\) and the parabolic case of class I\(_{7,8,11,13,14}\)), with vanishing 4-acceleration but non-vanishing shear. These classes of solutions admit (in general) no isometries and are considered inhomogeneous and anisotropic generalizations of FRW cosmologies. An interesting result from sections 4-5 is the fact that none of these solutions admit a thermodynamic scheme in general, that is, with unrestricted values of the free parameters characterizing them. We show that for parabolic Szekeres class I solutions the Gibbs-Duhem relation is not integrable for \( r < 2 \). However, under suitable restrictions of their parameters, specific particular cases of Szekeres II solutions are found to admit a thermodynamic scheme when \( r < 2 \). For the particular cases of the solutions of sections 4-5 found to be compatible with the thermodynamic scheme, we derive in section 6 explicit (though not unique) expressions of all state variables: \( \rho \) and \( p \), particle number density \( n \), specific entropy \( S \) and temperature \( T \), as well as two-parameter equations of state linking them. The latter turn out to be difficult to interpret as there is no clue on how to fix the time dependent free parameters of the solutions. We follow a common strategy which consists in formally identifying these parameters with the FRW scale factor and suitable state variables. The resulting equations of state are totally unfamiliar and (for most cases) unphysical. In particular, their corresponding temperature evolution laws are unrelated to those of their FRW limit. Conclusions are presented and summarized in section 7.

We show in appendix A that the formulation of the thermodynamic scheme provided by Coll and Ferrando is equivalent to that given in this paper. In appendix B we show that parabolic Szekeres class I solutions do not admit a Killing vector of the form suggested by Szafron in his original paper\(^7,^{11}\).

2. The thermodynamic scheme.
Consider the perfect fluid momentum-energy tensor

\[ T^{ab} = (\rho + p)u^a u^b + p g^{ab} \]  \hspace{1cm} (1)

where \( \rho, p \) and \( u^a \) are the matter-energy density, pressure and 4-velocity. This tensor satisfies the conservation law \( T^{ab} ;_b = 0 \) which implies the contracted Bianchi identities

\[ \dot{\rho} + (\rho + p)\Theta = 0 \]  \hspace{1cm} (2a)

\[ h^b_a p, _b + (\rho + p)\dot{u}_a = 0 \]  \hspace{1cm} (2b)

where \( \Theta = u^a_i, \dot{u}_a = u_{a,b} u^b \) and \( h^b_a = \delta^b_a + u_a u^b \) are respectively the expansion, 4-acceleration and projection tensor and \( \dot{\rho} = u^a \rho, _a \). The thermodynamics of (1) is essentially contained in the matter conservation law, the condition of vanishing entropy production and the Gibbs-Duhem relation. The first two are given by

\[ (n u^a) ;_a = 0 \]  \hspace{1cm} (3a)

\[ (n S u^a) ;_a = 0 \]  \hspace{1cm} (3b)

where \( n \) is the particle number density and \( S \) is the specific entropy. Condition (3a) inserted in (3b) leads to \( u^a S, _a = \dot{S} = 0 \), so that \( S \) is conserved along the fluid lines. The Gibbs-Duhem relation can be given as the 1-form

\[ \omega = dS = \frac{1}{T} \left[ d \left( \frac{\rho}{n} \right) + pd \left( \frac{1}{n} \right) \right] \]  \hspace{1cm} (4)

where \( T \) is the temperature and \( d \) denotes the exterior derivative. The necessary and sufficient conditions for the integrability of (4)

\[ \omega \wedge d\omega = 0 \]  \hspace{1cm} necessary  \hspace{1cm} (5)
subjected to fulfilment of the conservation laws (2) and (3), are the conditions which Coll and Ferrando denote admissibility of a “thermodynamic scheme”. Using the 1-form $T\omega$ instead of $\omega$, these authors demonstrated that a perfect fluid source admits a thermodynamic scheme iff there exists a scalar function $F = F(\rho, p)$ satisfying

$$\Theta = \dot{F} \tag{6}$$

the necessary and sufficient conditions for (6) to hold is, in turn, the fulfilment of the constraint

$$(\dot{p}d\dot{\rho} - \dot{\rho}d\dot{p}) \wedge dp \wedge d\rho = 0 \tag{7}$$

As immediate results from (6) and (7), it is easy to show that the thermodynamic scheme is always admissible in three important cases: (1) static fluids; (2) isometry groups of spacetime have orbits of dimension $r \geq 2$ and (3) isentropic fluids. In the first case, $\dot{\rho} = \dot{p} = 0$, and so (7) trivially holds, while in the second case there are always local coordinates in which all state variables, and in particular the set $(p, \rho, \dot{p}, \dot{\rho})$ are functions of only two coordinates, say $(t, x)$. Since any one of the 1-forms associated with these variables can be expanded as $dp = p_\rho dt + p_x dx$, etc, the wedge product in (7) vanishes. The third case is characterized by the existence of a barotropic equation of state $p = p(\rho)$, implying $dp = p_\rho d\rho$ and $\dot{p} = p_\rho \dot{\rho}$. Hence, the wedge product in (7) also vanishes. Since a barotropic equation of state can be shown to be equivalent to $dS = 0$ and $\dot{S} = 0$, the specific entropy in fluids compatible with such an equation of state is a global constant (isentropic fluids). For more general, non-isentropic fluids, this quantity is a different constant for different observers comoving with the fluid. Also, a relation between $p$ and $\rho$ (and between any other pair of state variables) necessarily involves a third state variable (two-parameter equations of state).

From these results, it is clear that perfect fluids which might not comply with the thermodynamic scheme are non-isentropic and sources of exact solutions with weaker spacetime symmetries: with isometry groups of dimension $r < 2$. Since exact solutions of this type
are difficult to obtain, in general, we will deal in the following sections with the important particular case of irrotational perfect fluids.

3. Thermodynamics of a non-isentropic irrotational perfect fluid.

Consider an irrotational perfect fluid (vanishing vorticity tensor, $\omega_{ab} = 0$). Since the 4-velocity is hypersurface orthogonal, there are local comoving coordinates $(t, x^i)$, such that the metric, 4-velocity, 4-acceleration and projection tensor are given by

$$ds^2 = -N^2 dt^2 + g_{ij} dx^i dx^j \quad (8a)$$

$$u^a = N^{-1} \delta^a_t \quad \dot{u}_a = (\log N)_a \delta^a_i \quad (8b)$$

$$h_{ab} = g_{ij} \delta^i_a \delta^j_b \quad (8c)$$

where all the metric coefficients $N$ and $g_{ij}$ are (in general) functions of all the coordinates $(t, x^i)$. In this representation, $\dot{X} = (1/N) X_t$ for all scalar functions and the Bianchi identities and conservation laws (2) and (3) become

$$\rho, t + (\rho + p)(\log \sqrt{\Delta}), t = 0 \quad (9a)$$

$$p, i + (\rho + p)(\log N), i = 0 \quad (9b)$$

$$n = \frac{f(x^i)}{\sqrt{\Delta}} \quad (9c)$$

$$S = S(x^i) \quad (9d)$$

where $\Delta \equiv \det(g_{ij})$ and $f(x^i)$ appearing in (9c) is an arbitrary function denoting the conserved particle number distribution. Consider the coordinate basis of 1-forms $(dt, dx^i)$ associated with the comoving frame (8); the Gibbs-Duhem relation reads
\[ \omega = S_i dx^i = \frac{1}{T} \left[ \left( \frac{\rho}{n} \right)_i + p \left( \frac{1}{n} \right)_i \right] dx^i \]  \hspace{1cm} (10)

where the \( t \) component of \( \omega \) in this coordinate basis vanishes due to (9d). A sufficient integrability condition of (10) is given by

\[ d \omega = W_{ti} \frac{dt \wedge dx^i}{nT} + W_{ij} \frac{dx^i \wedge dx^j}{nT} = 0 \]  \hspace{1cm} (11)

\[ W_{ti} = \frac{p_{[i,n,t]} - n^2 T_{[i,t} S_{i]} - n \left( \rho + p \right) \left( \frac{n}{T} \nabla_t n - \frac{1}{n} \dot{u}_t n \right)}{n} - \left( \rho, i \frac{T_t}{T} + p, i \frac{n_i}{n} \right) \]

\[ W_{ij} = \frac{p_{[i,n,j]} + n^2 T_{[i,j} S_{j]} - T_{[i,j} \rho_{j]} - \left( \rho + p \right) \left( \frac{T_{[i,j} + T \dot{u}_{[i}}}{T} \right) \frac{n_{j]}}{n} }{n} \]

where square brackets denote antisymmetrization on the corresponding indices. The necessary and sufficient condition (5) is given by

\[ d \omega \wedge \omega = X_{ijk} \frac{dx^i \wedge dx^j \wedge dx^k}{n^3 T^2} + X_{tij} \frac{dt \wedge dx^i \wedge dx^j}{n^3 T^2} = 0 \]  \hspace{1cm} (12)

\[ X_{ijk} = -\rho_{[i,p,j} n_{k]} \]

\[ X_{tij} = \rho_{[i,p,i,j]} \]

Conditions (12) are entirely equivalent to (7) provided by Coll and Ferrando. One can obtain the latter form the former simply by using (2) and (3) (in their forms (9)). However, (11) and (12) are more intuitive than (6) and (7), as they directly incorporate state variables such as \( n, S \) and \( T \), and their relations with \( \rho \) and \( p \). Condition (12) is also more practical than (7), as it is easier to use \( n \) and \( S \) from (9c) and (9d) than to compute the set \( (\rho, \dot{\rho}, p, \dot{p}) \) in exact solutions in which these quantities can be quite cumbersome. The sufficient condition (11), not examined by Coll and Ferrando, is also helpful, since its fulfilment guarantees that (7) (or (12)) holds. As shown in the following sections, it is
easier to test in some cases the admissibility of a thermodynamic scheme directly from (10) and (11). However, one must have a solution of Einstein’s equations providing \( \rho \) and \( p \) in terms of the metric functions in order to verify the admissibility of the thermodynamic scheme. Thus we will look at this scheme in known exact solutions which are particular cases of (8), and to do so we suggest the following procedure: (a) solve the conditions (12) and substitute the solution into (10), thus identifying possible (non-unique) forms for \( S \) and \( T \); insert the obtained forms of \( T \) and \( n \) into (11) in order to verify if further restrictions follow from the sufficient conditions. If these conditions hold, the equations of state linking the state variables \((\rho, p, n, S, T)\) (together with their functional relation with respect to the metric functions) follow directly from integrating it. We shown in the following sections, for various known exact solutions with \( r < 2 \) and with non-isentropic fluid sources, that the Gibbs-Duhem relation might not be integrable and if it is integrable, the resulting definitions of \( S \) and \( T \) might not be unique.

4. Geodesic perfect fluid: the Szekeres class II solutions.

Consider the geodesic case \( \dot{u}_i = 0 \). In the comoving frame (8), (9) leads to \( N, i = 0 \) and \( p = p(t) \). Without loss of generality, the metric in the comoving frame is (8a) with \( N = 1 \), so that \( u^a = \delta^a_t \) and all convective derivatives are simply derivatives with respect to the time coordinate (i.e. \( \dot{X} = X, t \) for all functions \( X \)). The remaining conservation laws (9a), (9c) and (9d) have the same forms as given by these equations. The components of the 1-form associated with the Gibbs-Duhem relation (10) becomes

\[
S, i = \frac{1}{T} \left( \frac{\rho + p}{n} \right), i
\]

while the integrability conditions \( d\omega = 0 \) and \( \omega \wedge d\omega = 0 \) become the particular cases of (11) and (12) given by

\[
W_{ti} = \frac{T, t}{T} \left( \frac{\rho + p}{n} \right), i - p, t \left( \frac{1}{n} \right), i = 0
\]

\[
W_{ij} = T, i \rho, j - \frac{\rho + p}{n} T, i n, j = T, i S, j = 0
\]
\[ X_{tij} = 0 \Rightarrow n_{[i,j]} = 0 \]  

(15)

The consistency between (13), its integrability conditions (14)-(15) and the field equations will be tested on two important classes of known exact solutions with an irrotational and geodesic perfect fluid source: the perfect fluid generalization of Szekeres solution\(^6-14\). These solutions are divided in two subclasses: I and II. This section is devoted to the class II solutions, while class I solutions are investigated in the next section.

The Szekeres class II solutions have been extensively studied as inhomogeneous generalizations of FRW cosmologies. A usual representation of the spatial metric is given by

\[
g_{ij}dx^idx^j = R^2 [B + P]^2 dx^2 + \frac{R^2(dy^2 + dz^2)}{[1 + \frac{k}{4}(y^2 + z^2)]^2} \]  

(16a)

\[
P = \frac{1}{2} \left( \frac{(y^2 + z^2)U + V_1 y + V_2 z + V}{1 + \frac{k}{4}(y^2 + z^2)} \right) \]  

(16b)

where \( R(t), V(x), V_1(x), V_2(x) \) and \( U(x) \) are arbitrary functions and \( k = 0, \pm 1 \). The function \( B = B(x,t) \) must satisfy the following constraint

\[
\dot{B} + \frac{3\dot{R}}{R} \dot{B} - \frac{k(B + \frac{1}{2}V) + U}{R^2} = 0 \]  

(16c)

which follows from the field equations. The state variables \( p = p(t), \rho \) and \( n \) corresponding to the metric (16a) are

\[
p = -\frac{2\dot{R}\ddot{R} + \dot{R}^2 + k}{R^2} \]  

(17a)

\[
\rho = \frac{2R\dot{R}\dot{B} + 3(B + P)\dot{R}^2 + k(B + 3P - V) - 2U}{R^2(B + P)} \]  

(17b)

\[
n = \frac{f}{R^3(B + P)} \]  

(17c)

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where (9c) has been used in order to arrive to (17c). The integrability condition (15) reduces to a 3-dimensional “vector product” of the type $\nabla n \times \nabla \rho = 0$, whose general solution has the form

$$\rho = a(t)n^b(t) + c(t)$$

(18)

where $(a, b, c)$ are arbitrary functions. It is evident that equation (18) does not hold in general, that is for arbitrary values of the free functions $(R, B, U, V, V_1, V_2)$ characterizing the metric (16). However, it is easy to find particular cases of the latter complying with these equations and not admitting isometries. Given the function $B_0(t)$ and the constant $b_0$, the choice of free parameters

$$B = B_0(t)V, \quad f = P, \quad U = b_0V,$$

(19a)

leads to a solution of (15) in the form (18) with

$$b(t) = 1$$

(19b)

$$a(t) = \frac{R[-2R\dot{R}\dot{B}_0 + 2b_0 + k(1 + 2B_0)]}{B_0}$$

(19c)

$$c(t) = \frac{3(\dot{R}^2 + k)}{R^2} - \frac{a(t)}{R^3}$$

(19d)

where $B_0(t)$ is restricted by (16c), and so must satisfy

$$\ddot{B}_0 + 3\frac{\dot{R}}{R} \dot{B}_0 - \frac{2kB_0 + k + 2b_0}{2R^2} = 0$$

(20)

However, as shown by equations (18)-(19), $B_0$ must also comply with $a(t) \neq 0$, or else, $\rho$ would loose its dependence on $(x, y, z)$ and would become the matter-energy density of a FRW spacetime. In order to identify $T$ and $S$, we insert (19a)-(19d) into (13) yielding

$$T(t) = c(t)R^3B_0 = [3R(\dot{R}^2 + k) - a(t)]B_0$$

(21a)
where $S_0$ is an arbitrary additive constant. Regarding the sufficient condition (14): the part (14b) holds by virtue of $T = T(t)$, while the part (14a) becomes $(c(t) + p)\dot{T} - T\dot{p} = 0$, a condition which is identically satisfied if (20) holds, this can be verified by inserting (21a) and (17a) into (14a).

Notice that satisfying the integrability conditions (14)-(15) leads to a non-unique form for $T$, since the function $R$ is still arbitrary. Equation (20) is a second order inhomogeneous ordinary differential equation, having (for every choice of $R$) two linearly independent solutions. The possible forms for $T$, characterized by solutions of (20), as well as equations of state linking $\rho, n$ an $p$ are discussed in section 6.

5. Parabolic Szekeres class I solutions.

A particular case of these solutions was originally examined by Szafron$^{11}$ and generalized by Bona et al.$^{13}$, and in the framework of a two-fluid interpretation, by Sussman$^{14}$. The spatial metric is given in spherical coordinates $x^i = (r, \theta, \phi)$ as

$$g_{i,j}dx^idx^j = \frac{(Y' + \nu'Y)^2}{A^2}dr^2 + e^{2\nu}Y^2(d\theta^2 + \sin^2 \theta d\phi^2)$$ (22a)

where a prime denotes the derivative with respect to $r$ and the functions $Y = Y(t, r)$ and $\nu = \nu(r, \theta, \phi)$ are given by

$$Y = (2M)^{1/3}[v + Qw]^{2/3}$$ (22b)

$$e^{-\nu} = 1 - \sin^2 \frac{\theta}{2}[1 - A^2 - B^2 - C^2] + \sin \theta[B \cos \phi + C \sin \phi]$$ (22c)

with $A(r), B(r), C(r), M(r)$, and $Q(r)$ being arbitrary functions. The pressure $p = p(t)$ is determined by the otherwise arbitrary functions $v = v(t)$ and $w = w(t)$ as

$$p = -\frac{4}{3} \frac{\ddot{v}}{v} = -\frac{4}{3} \frac{\ddot{w}}{w}$$ (23a)
while the remaining state variables $\rho = \rho(t, x^i)$ and $n = n(t, x^i)$ take the form

$$\rho = \frac{4\dot{\Psi} \dot{\Omega}}{3\Psi \Omega}, \quad n = \frac{f}{\Psi \Omega}$$  \(23b\)

with

$$\Psi = v + wQ \quad \Omega = v\Gamma + w\Delta \quad \Gamma = M' + 3M'\nu' \quad \Delta = \Gamma Q + 2MQ'$$  \(23c\)

In obtaining the expression for $n$ in (23b) we have redefined the arbitrary function $f(x^i)$ as $f \to 3Af/(2\sin \theta e^{2\nu})$.

As with the class II solutions discussed previously, there are no isometries in general, for arbitrary values of the free parameters. This fact is proven in Appendix B, thus correcting the erroneous result given by Szafron\textsuperscript{7,11}, who reported in his original paper that a one dimensional isometry group necessarily exists in these solutions. Special cases of higher symmetry are obtained by specifying the free parameters, in particular a FRW limit follows if $Y$ becomes separable as $Y = Y_1(t)Y_2(r)$.

The necessary and sufficient condition (15) for admittance of a thermodynamic scheme, with $\rho$ and $n$ given by (23b) and after lengthy algebraic manipulations, becomes ($x^i = r, \theta, \phi$)

$$\rho_{[i}^{\ n, j]} = \frac{4(v\dot{w} - w\dot{v})}{3\Psi^2 \Omega^3} \left[ A_1 \delta^1_{ij} f_{[i} f_{j]} + A_2 \delta^1_{ij} \nu'_{[i} f_{j]} + A_3 \nu'_{[i} f_{j]} \right] = 0$$  \(24a\)

where $\delta^1_{ij}$ represents the Kronecker symbol and

$$A_1 = \Psi \dot{\Psi} \left[ \left( \frac{\Delta}{\Gamma} \right)' \Gamma^2 + 6M^2 Q'\nu'' \right] + \Omega \dot{\Omega} Q'$$  \(24b\)

$$A_2 = -6fM \{ \Psi \dot{\Psi} [\Gamma Q' + (MQ')'] + MQ^2 (w \Psi') \}$$  \(24c\)

$$A_3 = -6M^2 Q' \Psi \dot{\Psi}$$  \(24d\)

It follows from (24a) that the thermodynamic scheme conditions are identically satisfied if $v\dot{w} - w\dot{v} = 0$. However, this relationship implies that $w \propto v$ and, according to (22b), the
metric function $Y(t, r)$ becomes separable. As mentioned above, this case corresponds to the limiting FRW cosmology. We therefore demand that $v \dot{w} - w \dot{v} \neq 0$ and, consequently, the expression in squared brackets in (24a) must vanish, i.e.,

$$\begin{align*}
(A_1 + A_3 \nu'')f,\eta + (A_2 - A_3 f')\nu'_{,\eta} &= 0 \\
A_3 (\nu'_{,\theta} f,\phi - \nu'_{,\phi} f,\theta) &= 0
\end{align*} \tag{25}$$

with $\eta = \theta, \phi$. Equation (25b), with $A_3 \neq 0$, can always be satisfied by fixing the angular dependence of the arbitrary function $f$. Even if we demand the fulfillment of the simplifying restriction $f' = 0$, (25b) still allows a solution which fixes the angular dependence of $f$ and imposes a condition on the radial dependence of $\nu$. In fact, using (22c) and $f' = 0$ it follows that (25b) holds if $B = C$, $B' \neq 0$, and $AA' = (B_0 - 2B)B'$ where $B_0$ is an arbitrary constant. Equation (25a) contains arbitrary functions of the time as well as spatial coordinates which can be separated by inserting (23c). Then we obtain

$$v \dot{v}(K_1 + K_2 \Gamma^2) + (vw) (K_1 Q + K_2 Q \Gamma + K_3) + w \dot{w} (K_1 Q^2 + K_2 \Delta^2 + 2K_3 Q) = 0 \tag{26a}$$

where

$$\begin{align*}
K_1 &= \left(\frac{\Delta}{\Gamma}\right)' \Gamma^2 f,\eta - 6M\{f[\Gamma Q' + (MQ')'] - MQ' f'\} \nu',\eta \tag{26b} \\
K_2 &= Q' f,\eta \tag{26c} \\
K_3 &= -6M^2 Q^2 f \nu',\eta \tag{26d}
\end{align*}$$

For arbitrary time functions, (26a) yields a system of three algebraic equations for the spatial coefficients. It then follows that this system allows a solution iff

$$M^2 Q^2 K_2 = M^2 Q^3 f,\eta = 0 \tag{27}$$

If $M = 0$ the spatial metric vanishes. Moreover, for $Q' = 0$ the function $Y(t, r)$ in (22b) becomes separable and the spacetime is that of the FRW cosmological models. Therefore,
the only non trivial solution of (27) is \( f, \eta = 0 \). Hence, according to (26b) and (26d), \( \nu'_{, \eta} = 0 \), i.e. \( A, B, \) and \( C \) in (22c) must be constants and the metric (22a), after a suitable coordinate transformation, becomes spherically symmetric. Another possibility for solving (26a) is to require a linear dependence between the time functions, that is \( (vw)' \propto v\dot{v} \) and \( w\dot{w} \propto v\dot{v} \). However, it can easily be shown that this case leads to \( w \propto v \) and, therefore, we obtain the FRW limiting case. The analysis presented above shows that the Szekeres class I solutions admit a thermodynamic scheme in two special cases only: the spherically symmetric limiting case and the FRW cosmologies.

We will now consider the spherically symmetric case which follows from (22c) through the conditions \( B = C = 0, A^2 = 1 \), so that \( e^\nu = 1 \). The Gibbs–Duhem relation (13) yields

\[
\alpha(t) \left( \frac{M'}{f} \right)' + 2\beta(t) \left( \frac{MQ'}{f} \right)' + \gamma(t) \left( \frac{(MQ^2)'}{f} \right)' = TS'
\]

(28a)

where

\[
\alpha(t) = -\frac{4}{3} v^2 \left( \frac{\dot{v}}{v} \right), \quad \beta(t) = -\frac{4}{3} v^2 \left( \frac{\dot{w}}{w} \right), \quad \gamma(t) = -\frac{4}{3} w^2 \left( \frac{\dot{w}}{w} \right).
\]

(28b)

In order to identify the temperature and entropy from (28a), we must impose certain functional dependence either between the time functions or the radial coefficients. We first consider the former case and require that the following relationships hold: \( \beta(t) = \kappa_1 \alpha(t) \) and \( \gamma(t) = \kappa_2 \alpha(t) \) where \( \kappa_1 \) and \( \kappa_2 \) are constants. Inserting (28b) into these constraints, we obtain a set of differential equations which may easily be integrated and yield

\[
\dot{w} = \kappa_1 \dot{v} + \epsilon_1 v \quad \dot{v} = \frac{\kappa_1}{\kappa_2} \dot{w} + \epsilon_2 w
\]

(29)

where \( \epsilon_1 \) and \( \epsilon_2 \) are constants of integration. Introducing the value of \( \dot{v} \) into the equation for \( \dot{w} \) and vice versa, and differentiating the resulting equations with respect to \( t \), we get

\[
(k_2 - k_1^2) \ddot{v} - k_1 (\epsilon_1 + \epsilon_2 k_2) \ddot{w} - \epsilon_1 \epsilon_2 \dot{v} = 0
\]

(30a)

\[
(k_2 - k_1^2) \ddot{w} - k_1 (\epsilon_1 + \epsilon_2 k_2) \ddot{v} - \epsilon_1 \epsilon_2 \dot{w} = 0
\]

(30b)
Since \( v \) and \( w \) are related by \( \frac{\dot{v}}{v} = \frac{\dot{w}}{w} \) [cf. (23a)], (30a) and (30b) implies that \( \dot{v}/v = \dot{w}/w \), a condition that leads to the FRW limiting case. Consequently, the functional dependence must be imposed on the radial functions contained in (28a). The constraints

\[
\left[ \frac{(MQ)^{'}}{f} \right]^{''} = \lambda_1 \left[ \frac{M'}{f} \right]^{''} \quad \text{and} \quad \left[ \frac{(MQ^2)^{'}}{f} \right]^{''} = \lambda_2 \left[ \frac{M'}{f} \right]^{''}
\]  

(31)

with \( \lambda_1 \) and \( \lambda_2 \) being constants, can be integrated twice yielding two algebraic equations which imply a relationship between the arbitrary functions \( M \) and \( Q \), namely

\[
Q = \frac{\lambda_2 M + \tau_2 F + \sigma_2}{\lambda_1 M + \tau_1 F + \sigma_1}
\]

(32)

where \( \tau_1, \tau_2, \sigma_1, \) and \( \sigma_2 \) are arbitrary constants and \( F = \int f(r)dr \).

The identification of \( T \) and \( S \) follows by inserting (31) into (28a), thus we obtain

\[
T(t) = -\frac{4}{3} \left[ v^2 \left( \frac{\dot{v}}{v} \right) \right]^{'} + 2\lambda_1 v^2 \left( \frac{\dot{w}}{v} \right) \right]^{'} + \lambda_2 w^2 \left( \frac{\dot{w}}{w} \right)^{'}
\]

(33a)

\[
S = \frac{M'}{f} + S_0
\]

(33b)

where \( S_0 \) is an additive constant. We see from (33a) that the temperature is determined by the arbitrary functions \( v(t) \) and \( w(t) \) together with the arbitrary constants \( \lambda_1 \) and \( \lambda_2 \) which must be appropriately specified for any given \( v(t) \) and \( w(t) \) in order to ensure the positiveness of the temperature.

6. Equations of state and FRW limits.

Szekeres class II solutions.

Given a choice of \( R = R(t) \), the constraint (20) provides \( B_0(t) \) (or given a choice of the latter function, \( R(t) \) can be obtained). Since \( p = p(t) \) and \( T = T(t) \), the functions \( (a(t), c(t)) \) appearing in (19) can always be expressed as functions of either one of the pair \( (p, T) \) once the constraint (20) has been solved. Thus, a generic formal equation of state for these solutions follows by re-writing (18) as
\[ \rho(p, n) = a(p)n + c(p) = \frac{a(p)}{R^3(p)\left[1 + B_0(p)(S - S_0)\right]} + c(p) = \rho(p, S) \quad (34a) \]

where \( n \) has been expressed in terms of \((p, S)\) by eliminating \( S \) from (21b) into (17c) as

\[ n(p, S) = \frac{1}{R^3(p)\left[1 + B_0(p)(S - S_0)\right]} \quad (34b) \]

Similar forms of the equation of state, in the form \( \rho(T, n) \) and \( n(T, S) \) can be obtained by expressing \( p \) in terms of \( T \). These formal equations of state are difficult to interpret, as they depend on the choice of one of the pair \((R, B_0)\), and there is no intrinsic way to select these functions other than using the formal analogy between the functions \( R \) and \( p \) in (16a) and (17a) with the scale factor and pressure of a perfect fluid FRW cosmology (usually obeying a “gamma law” equation of state \( p = (\gamma - 1)\mu \)). This analogy implies fixing \( R(t) \) from solving the FRW relation

\[ p = (\gamma - 1)\mu = \frac{3(\gamma - 1)(\dot{R}^2 + k)}{R^2} = 3(\gamma - 1) \left( \frac{R}{R_0} \right)^{-3\gamma} \quad (35) \]

where \( p \) is given by (17a) and \( R_0 \) is an integration constant. Several authors\(^6,9\)\(^\text{--}12\) have assumed this formal identification of these functions with their FRW analogues, hoping to provide a sort of “physical handle” in dealing with Szekeres class II solutions. The latter solutions, with these specific parameters, comply with suitably defined asymptotically FRW limits. However, as shown below, the thermodynamics of these fluids bear no relation with that of their FRW limits, and so this is an example of how nice geometric features do not always correspond to nice physics.

For Szekeres class II solutions, from equations (18) and (19), the FRW limit follows as \( a(t) \to 0 \), so that \( \rho \to \rho(t) = c(t) \equiv \mu \) and \( p(t) \) given by (17a) become the mass energy density and pressure of the limiting FRW cosmology. This FRW limit also implies \( S \to S_0 \) (entropy density a universal constant), and so, from (34b), we have \( n \to R^{-3} \) (the FRW form of \( n \)). The condition \( a(t) \to 0 \), together with (20) implies

\[ 2R\dot{R}B_0 - (2kB_0 + k + 2b_0) = 0 \]
\[ 2R^2B_0 + 6R\dot{R}B_0 - (2kB_0 + k + 2b_0) = 0 \]
which leads to the deSitter solution if $\dot{B}_0 \neq 0$, or FRW metrics if $\dot{B}_0 = 0$ (for $k = \pm 1$, one has $B_0 = -(k - 2b_0)/2k$, while for $k = 0$, $B_0$ is an arbitrary constant, but $b_0 = 0$). For the FRW limit, from (21a), the temperature function becomes $T = B_0 \mu R^3$, and $\mu = 3(\dot{R}^2 + k)/R^2$, so that one recovers the expected temperature law for a FRW cosmology with a “gamma law” equation of state, namely: $T \propto R^{3(1-\gamma)} \propto \mu^{1-1/\gamma}$. However, for $\dot{B}_0 \neq 0$ and $a(t) \neq 0$ the temperature law one obtains is unrelated to these values.

Consider first the “parabolic” case $k = 0$. Inserting the FRW form

$$R(t) = R_0 \left[1 + \frac{3}{2} \gamma (t - t_0)\right]^{3\gamma/2}$$

(36a)

into (20) and (21a) yields the following forms for $B_0$ and $T$

$$B_0(t) = c_1 + c_2 t_1^{1-9\gamma/2} - \frac{2b_0 R_0 t_1^{3\gamma/2}}{(3\gamma + 2)(3\gamma - 2)R_0^2}$$

(36b)

$$T(t) = 3c_1 \gamma R_0^3 t_1^{3\gamma-2} - \frac{3c_2 \gamma (9\gamma - 4) R_0^3}{2t_1} - 2b_0 R_0 t_1^{3\gamma/2}$$

(36c)

where $c_1$ and $c_2$ are integration constants of (20) and $t_1 \equiv 1 + \frac{3}{2} \gamma (t - t_0)$. Notice that $T(t)$ not only bears no relation with the FRW temperature law, but is a wholly unphysical temperature law (for $\gamma = 1$, dust, $T$ is not constant). The particular case $c_1 = c_2 = 0$ of (40c) was obtained by Tiomno and Lima, who report it as unrelated to the FRW temperature law. However, these authors erroneously claim that the FRW temperature law can be recovered (for non-FRW cases) by simply redefining the equation of state $\rho = \rho(p, n)$ so that it depends on three parameters ($n, p$ and a function of the spatial coordinates). Tiomno and Lima obtained $T$ from the sufficient integrability condition (14a) expressed as $\dot{T}/T = (\partial p/\partial \rho)_n \dot{n}/n$. Such integrability condition is only valid if the equation of state is of the two-parameter form.

Regarding the case $k = \pm 1$, the integration of (20) for $R$ given by its FRW analogous form leads to cumbersome hypergeometric functions. Hence it is easier to demonstrate that a FRW temperature law is incompatible with such forms of $R$ and with $B_0 \neq \text{const.}$ complying with (20). Assume that $T$ in (21a) takes the FRW form $T = T_{FRW} = 3b_1 R(\dot{R}^2 + k)$, where $b_1$ is an arbitrary constant, equation (21a) becomes
eliminating \( \dot{R} \) in (37) in terms of powers of \( R \) from (35), differentiating the result with respect to \( t \) and inserting the obtained forms of \( \ddot{B}_0 \) and \( \dot{B}_0 \) into (20) leads, after some algebraic manipulation to \( B_0 = -(k - 2b_0)/2k \), the relation defining the FRW limit, and so, indicating that a Szekeres class II solution with \( k = \pm 1 \) with \( R \) and \( T \) having FRW forms is necessarily a FRW cosmology. This means that selecting the free parameters of Szekeres II solutions in terms of their resemblance to FRW parameters does not lead to the right thermodynamics. Therefore, other criteria must be used in order to select these parameters.

**Szekeres class I solutions.**

As in the previous case, the physical interpretation of the thermodynamic variables of Szekeres’ class I solutions presents serious difficulties. Consider, for instance, the temperature law (33a). Inserting the value

of the pressure (23a) into (33a), we obtain

\[
T(t) = p(v^2 + 2\lambda_1 vw + \lambda_2 w^2) + \frac{4}{3}(\dot{v}^2 + 2\lambda_1 \dot{v}\dot{w} + \lambda_2 \dot{w}^2) \tag{38}
\]

This temperature law already shows an unphysical behaviour as it does not reduce, in general, to a constant in the limiting case of vanishing pressure. Even in the extreme case \( \lambda_1 = \lambda_2 = 0 \), the required behaviour holds only if \( \dot{v} = const. \) This implies an additional condition on the function \( v \) which cannot be satisfied in general.

An equation of state relating \( \rho \) and \( n \) can be obtained from (23b) by inserting the constraints (31) and (32), and using the definition of the entropy as in (33b). Then

\[
\rho = \frac{4}{3}n[(S - S_0)(\dot{v}^2 + 2\lambda_1 \dot{v}\dot{w} + \lambda_2 \dot{w}^2) + 2\tau_1 \dot{v}\dot{w} + \lambda_2 \dot{w}^2] \tag{39}
\]

where the baryon number density, according to (23b) and (33b), can be expressed as

\[
n = (v + wQ)^{-1}[(S - S_0)(1 + \lambda_2 w) + \tau_2 w]^{-1} \tag{40}
\]
According to (38) the variable $t$ can always be replaced by a function of one of the pair $p$ and $T$ (or both of them). Furthermore, (32) and (33) can be used to express $Q$ as a function of the entropy $S$. Consequently, (40) may be interpreted as an equation of state of the form $n = n(p, S)$ and (39) relates $\rho$ with $p$ and $S$. The physical interpretation of these equations of state remains unclear as they explicitly depend on the choice of the arbitrary functions $v$ and $w$. Therefore, it is necessary to fix these functions by choosing a special case of Szekeres class I solutions. Consider the spherically symmetric Szafron model\textsuperscript{11,14}

\[ v(t) = \left( \frac{t}{t_0} \right)^{1/\gamma} \quad w(t) = \left( \frac{t}{t_0} \right)^{1-1/\gamma} \quad t_0 = \frac{2R_0}{3\gamma} \quad \gamma \neq 0, 2 \] (41)

where $R_0$ is the constant FRW scale factor. The function $M(r)$ remains arbitrary, and the thermodynamic pressure satisfies a “gamma law” and is given by

\[ p = (\gamma - 1)\rho = \frac{4(\gamma - 1)}{3(\gamma t)^2} \] (42)

The consistency of the equations of state and the definition of temperature can be analyzed by looking at their specific behaviour under physically reasonable assumptions. In the FRW limiting case of dust ($S \to S_0$ and $\gamma = 1$), the temperature law (38) shows the correct behaviour as $T = t_0^{-2} = \text{const.}$ while the equations of state (39) and (40) yield $\rho = 0$ and $n \propto t^{-1}$, respectively. Obviously, this result has no relation to the evolution law expected at the FRW limit ($n = \rho^{1/3}$). For radiation ($\gamma = 4/3$) the temperature evolution predicted by (38) becomes $T \propto t^{1/2}$, and the equations of state lead to $\rho \propto t^{-3}$ and $n \propto t^{-3/2}$ so that $n \propto \rho^{1/2}$; whereas the expected FRW limiting behaviour should be $T \propto t^{-1/2}$ and $n \propto \rho^{1/4}$. These examples indicate that the requirement of a thermodynamic scheme for the Szafron class I solutions leads to an unphysical behaviour of the thermodynamic variables and equations of state.

7. Conclusions.

We have verified the consistency of the thermodynamic equations (i.e. admittance of a thermodynamic scheme) for two large class of exact solutions (classes I and II Szekeres solutions) whose sources are non-isentropic and irrotational perfect fluids. This work has
aimed at improving the study of this type of solutions, as classical fluid models generalizing
FRW cosmologies, in contrast to a widespread attitude of simply disregarding them for
not admitting a barotropic equation of state.

For the particular cases of the solutions examined admitting a thermodynamic scheme,
the resulting equations of state have an elusive interpretation, as there is no blue print on
how to select the free parameters of the solutions. We have show that formally identifying
the time dependent parameter $R$ of these solutions with the FRW scale factor leads to
unphysical temperature evolution laws, totally unrelated to that of their limiting FRW
cosmology. The question of how to select these parameters in a convenient matter remains
unsolved, though the adequate theoretical framework to carry this task has been presented
in this paper.

It has been very interesting to find that these solutions are not compatible, in general,
with a thermodynamic scheme. This fact seems to disqualify these solutions as classical
fluids of physical interest. However, these exact solutions can still be useful if they are
examined under a less restrictive framework than that of the simple perfect fluid. In this
context, both class I and II Szekeres solutions have been re-interpreted as mixtures of inho-
mogeneous dust and a homogeneous perfect fluid\cite{12,14}, subjected to adiabatic interaction.
In all these cases, the thermodynamics is totally different (and much less restrictive) than
that presented here.

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Appendix A. Thermodynamic scheme conditions

In this appendix we explicitly derive the condition for the existence of a thermody-
namic scheme as given by Coll and Ferrando\cite{3}, and show its equivalence to the condition
$\omega \wedge d\omega = 0$. 

20
Consider a perfect fluid with mass–energy density \( \rho \), pressure \( p \) and baryon number density \( n \), satisfying the contracted Bianchi identities (2). If no other conditions are specified, (2) describe an open system. However, when an equation of state is given, the closure of the system is obtained. The existence of a thermodynamic scheme for the fluid is therefore closely related to the existence condition of an equation of state. To derive this condition one may use only the matter conservation law (3a). (Condition (3b) cannot be used since it involves the entropy \( S \) which can be defined only if the fluid accepts a thermodynamic scheme.)

From the matter conservation law we define the function \( F \) by

\[
\dot{F} = \Theta = -\frac{\dot{n}}{n}
\]

and consider the one–form

\[
\Gamma = d\rho + (\rho + p)dF.
\]

If \( \Gamma \) is completely integrable, we can define the entropy \( S \) as \( dS = \Gamma/T \), where \( T \) is the integral factor which can be associated with the absolute temperature. Now we calculate the condition for the complete integrability of \( \Gamma \) which according to Frobenius’ theorem\(^\text{15}\) is equivalent to the vanishing of the three–form \( \Gamma \wedge d\Gamma \). Clearly, this condition will not be satisfied in general unless we demand certain functional dependence for \( F \). Assuming that (A2) allows a solution of the form \( F = F(\rho, p) \), it follows that

\[
dF = F_{,\rho} d\rho + F_{,p} dp \quad \text{and} \quad \dot{F} = F_{,\rho} \dot{\rho} + F_{,p} \dot{p}
\]

where \( F_{,\rho} = \partial F/\partial \rho \) and \( F_{,p} = \partial F/\partial p \). Accordingly, (A3) and the Bianchi identity (2a) may be written as

\[
\Gamma = [1 + (\rho + p)F_{,\rho}]d\rho + (\rho + p)F_{,p} dp
\]

\[
[1 + (\rho + p)F_{,\rho}] \dot{\rho} + (\rho + p)F_{,p} \dot{p} = 0
\]

respectively, and lead to

\[
\Gamma = (\rho + p)F_{,p} \left( dp - \frac{\dot{p}}{\rho} d\rho \right).
\]
It follows from the last equation that the integrability condition for $\Gamma$ is equivalent to

$$\Gamma \wedge d\Gamma = - (\rho + p) F_{\rho p} \dot{\rho}^2 (d\dot{\rho} - \dot{\rho} d\dot{\rho}) \wedge dp \wedge d\rho = 0 . \quad (A8)$$

The last expression leads to the condition (7) obtained by Coll and Ferrando.

Consider now the Gibbs–Duhem relationship as given in (4), that is $\omega = \omega(\rho, p, n, T)$. Since $\omega$ is a 1–form, the condition for its complete integrability, $\omega \wedge d\omega = 0$, will be satisfied if $\omega$ can be defined in a two–dimensional space with thermodynamic coordinates, say, $\rho$ and $p$. This means that an equation of state must exist such that $n = n(\rho, p)$ and $T = T(\rho, p)$. Consequently, the Gibbs–Duhem relationship becomes

$$\omega = \frac{1}{nT} \left\{ \left[ 1 - (\rho + p) \frac{n_\rho}{n} \right] d\rho - (\rho + p) \frac{n_{\rho p}}{n} dp \right\} \quad (A9)$$

where $n_{\rho} = \partial n / \partial p$, etc. Introducing the Bianchi identity (2a) [with $n = n(\rho, p)$] into (A9) yields

$$\omega = (\rho + p) \frac{n_{\rho p}}{n^2 T} \left( \frac{\dot{\rho}}{\rho} d\rho - dp \right) . \quad (A10)$$

It is now easy to see that the integrability condition $\omega \wedge d\omega = 0$ is equivalent up to a constant factor to (A8).

Appendix B. Parabolic Szekeres class I solutions and isometry groups

In his original paper\textsuperscript{11}, Szafron introduced the particular class of parabolic Szekeres class I solutions that has become known in the literature as the “Szafron models”. He claimed that the latter admit at least a one parameter isometry group, and provide specific forms for the components of this Killing vector. However, Szafron’s claim, stated again by Kramer et al\textsuperscript{7}, is false. We prove in this appendix that Szekeres class I solutions admit no Killing vector of the type suggested by Szafron.

In section 5 we have use spherical coordinates to investigate the Szekeres class I solutions; however, for the investigation of their isometries it is convenient to introduce the original coordinates used by Szafron\textsuperscript{11} and Bona et al.\textsuperscript{13}. The spatial metric, save changes in notation, is given as
\[ g_{ij} dx^i dx^j = P^2 \left[ \left( \frac{B^{2/3}}{P} \right)^2 dx^2 + \frac{B^{4/3}}{P^2} (dy^2 + dz^2) \right] \quad (B1) \]

where a prime denotes derivative wrt \( x \), the function \( P(x, y, z) \) is given by

\[ P = \frac{1}{2} U(x)(y^2 + z^2) + V_1(x)y + V_2(x) + V(x) \quad (B2) \]

where the functions \( U(x), V_1(x), V_2(x) \) and \( V(x) \) restricted by the condition: \( UV - V_1^2 - V_2^2 - \frac{1}{2} = 0 \), while \( B(t, x) \) is determined by the field equation

\[ \ddot{B} + \frac{3}{4} p(t) B = 0 \quad (B3) \]

which can be considered as linear second order differential equation, and so \( B \) has the generic form \( B = M(x)v(t) + Q(x)w(t) \) identifying the two arbitrary integration constants as the arbitrary functions \( M(x) \) and \( Q(x) \). The two linearly independent solutions of (B3), \( v(t) \) and \( w(t) \), are related to the pressure by

\[ p(t) = -\frac{4\ddot{v}}{3v} = -\frac{4\ddot{w}}{3w} \quad (B4) \]

Adapted to the coordinates and notation of (B1), the generic form of the Killing vector provided by Szafron is

\[ K = Y(x, y, z) \partial_y + Z(x, y, z) \partial_z \quad (B5) \]

Computing the Killing equation for the metric (B1) and vector field (B5) leads immediately to the conditions

\[ Y_x = Z_x = 0 \quad (B6) \]

\[ Y_z + Z_y = 0 \quad (B7) \]

\[ Z_z - Y_y = 0 \quad (B8) \]
Although Szafron’s result refers to a particular case of (B1), condition (B6) is sufficient to prove that this result is wrong: the components of the vector fields he provided cannot be components of a Killing vector of (B1). This is so, because Szafron’s particular case follows by giving specific forms to the functions $v(t)$ and $w(t)$ in (B4) and these functions are not involved in the calculation which leads to (B6–B9). However, even assuming $Y = Y(y, z)$ and $Z = Z(y, z)$, (B5) is not an isometry of the metric (B1). This can be proven by inserting (B2) into (B9) so that the resulting equation

$$(yY + zZ) \left( \frac{U}{P} \right)' + y \left( \frac{V_1}{P} \right)' = 0$$

implies a functional dependence between the functions $U$ and $V_1$ of the form $(V_1/P)' = G(y, z)(U/P)'$, where $G(y, z)$ is an arbitrary function of its arguments. Obviously, the latter condition cannot hold for an unrestricted form of $P$.

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