HEINTZE-KARCHER INEQUALITY FOR SETS OF FINITE PERIMETER ON SPHERE

XUWEN ZHANG

Abstract. In this paper, we study the level sets of the distance function from the boundary of a set of finite perimeter in the space form \((S^{n+1}, g_{S^{n+1}})\). We find certain good subsets of these level sets, on which the distance function is differentiable. Moreover, by exploring the distance function intrinsically on \((S^{n+1}, g_{S^{n+1}})\), we will show that these good subsets are close to the level sets in some sense. By embedding \((S^{n+1}, g_{S^{n+1}})\) into \((\mathbb{R}^{n+2}, \text{euc})\), we can prove the \(C^{1,1}\)-rectifiability of the level sets. As a by-product, we prove a Heintze-Karcher inequality for sets of finite perimeter on sphere. On the other hand, we will show that the stationary points of the isoperimetric problem on \((S^{n+1}, g_{S^{n+1}})\) possess certain good properties as those in the Euclidean space.

Keywords: Sets of finite perimeter in Riemannian manifold, Level sets of distance function, Rectifiability, Heintze-Karcher inequality, Isoperimetric problem.

1. Introduction

The concept of set of finite perimeter (Caccioppoli set) was first introduced by R. Caccioppoli and then developed by L. Cesari and E. De Giorgi. The theory was developed fastly in the last century and was studied by many mathematicians. At the very first beginning, this theory is mainly set up in the Euclidean space, and then in the late 20th century, mathematicians started to extend this theory to metric measure space, Riemannian manifold, etc, see for example [Amb01], [Vol10], [AGM15],[GP15].

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Recently, M. G. Delgadino and F. Maggi studied the sets of finite perimeter in Euclidean space by exploring the superlevel sets of the distance function from the boundary of the set of finite perimeter. In [DM19], they proved that the superlevel sets are also sets of finite perimeter. Moreover, the boundaries of these superlevel sets turn out to be $C^{1,1}$-rectifiable. With the rectifiability, they followed S. Montiel and A. Ros’ approach to the A.D. Alexandrov theorem ([MR91]), and they proved a Alexandrov type theorem for Euclidean Isoperimetric problem among sets of finite perimeter. Meanwhile, they used the rectifiability and followed S. Brendle’s idea ([Bre13]) to prove the Heintze-Karcher inequality for sets of finite perimeter in the Euclidean space.

There are other approaches to the rectifiability of the level sets of the distance function from a closed set in the Euclidean space. H. Federer studied the sets with positive reach in [Fed59; Fed69], M. Santilli extended the results for sets with positive reach to arbitrary closed sets in Euclidean space. They pursued the retifiability by studying the approximate differential of the distance function.

As for the results on Riemannian manifold, by studying the Hamilton-Jacobi equations, [MM02] shows that the distance function $d_K$ on $(M,g)$ from arbitrary closed set $K$ is $C^1$ in $M \setminus (K \cup \text{Sing})$, where Sing denotes the singular set of $d_K$, namely, the set of points where $d_K$ fails to be differentiable. Moreover, Sing is proved to be $C^2$-rectifiable. To the author’s knowledge, the best rectifiability result for the level sets of the distance function from arbitrary closed set on a Riemannian manifold is [RZ12, Theorem 5.7], where they have shown that $A_r = S_r(F) \setminus \text{Crit}(d_F)$ is a Lipschitz manifold and is dense in $S_r(F)$, here $F \neq \emptyset$ is any closed set in a connected, complete, Riemannian manifold $(M,g)$, $S_r(F) = \{x \in M : \text{dist}_g(x,F) = r\}$ denotes the level set of the distance function from $F$, $\text{Crit}(d_F)$ is the set of all critical points of $d_F$ (see [RZ12, Definition 5.1]). In this paper, we improve the rectifiability result to class $C^{1,1}$, when $(M,g)$ is taken to be $\left(\mathbb{S}^{n+1}, g_{\mathbb{S}^{n+1}}\right)$.

The main purpose of this paper is to study the level sets of the distance function from the boundary of a set of finite perimeter defined on the sphere $\left(\mathbb{S}^{n+1}, g_{\mathbb{S}^{n+1}}\right)$. The main idea is as follows. If we restrict ourselves to the sphere, we can first find some good subsets of the level sets of the distance function, precisely, we will show that the distance function is differentiable on these sets. To further explore these sets, we embed the sphere $\left(\mathbb{S}^{n+1}, g_{\mathbb{S}^{n+1}}\right)$ into the Euclidean space $\mathbb{R}^{n+2}$ and then we can make full use of this embedding to do some analysis in the Euclidean space. For a set of finite perimeter in the Riemannian manifold, Volkmann showed that it’s equivalent to study the rectifiability in the ambient space when we embed the Riemannian manifold into some Euclidean space by Nash embedding ([Vol10]). Thus we turn to study the countably $n$-rectifiable sets in $\mathbb{R}^{n+2}$. Based on some analysis in $\mathbb{R}^{n+2}$, we finally arrive at the $C^{1,1}$-rectifiability of the level sets. Consequently, we derive the Heintze-Karcher inequality. On the other hand, by virtue of the embeddedness, we will show that for the sets of finite perimeter on $\left(\mathbb{S}^{n+1}, g_{\mathbb{S}^{n+1}}\right)$ which are stationary points for the isoperimetric problem among sets of finite perimeter, possess certain geometry properties as those in the Euclidean space (see e.g., [DM19, Lemma 5]).

1.1. Main results. Our main result is the following rectifiability result, here $\Gamma_s^t$ are subsets of the level sets with some good properties, whose precise definition can be found in Section 3. $N(y)$ is the unit normal of $\Gamma_s^t$ at $y$, whose existence is proved to be valid for every $y \in \Gamma_s^t$, and $\Gamma_s^+$ is taken to be the union of all $\Gamma_s^t$, i.e., $\Gamma_s^+ = \bigcup_{t \geq s} \Gamma_s^t$.

Theorem 1.1. For $0 < s < t < \frac{n}{2}$,

1. $\Gamma_s^t$ can be filled with a countable union of compact sets $\{U_j\}$, each $U_j$ can be locally written as a graph of some $C^{1,1}$-function, and $N$ is tangentially differentiable along $\Gamma_s^t$ for $\mathcal{H}^n$-a.e. $y \in \Gamma_s^t$. Moreover, the principal curvatures of $\Gamma_s^t$ are bounded from below by...
we collect some background material from Section 4, by virtue of this, we prove the Heintze-Karcher inequality (Theorem 4.1). In Section 5, as a special case, we study the sets of finite perimeter which are stationary points of the isoperimetric problem on \((S^{n+1}, g_{S^{n+1}})\).

1.2. Organization of the paper. In Section 2 we collect some background material from geometric measure theory. In Section 3 we study the level sets of the distance function to the boundary of a set of finite perimeter. In Section 4 we prove the main rectifiability result (Theorem 1.1), by virtue of this, we prove the Heintze-Karcher inequality (Theorem 4.1). In Section 5, as a special case, we study the sets of finite perimeter which are stationary points of the isoperimetric problem on \((S^{n+1}, g_{S^{n+1}})\).

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2. Notations and preliminaries

In this section, we collect some preliminaries from the theory of sets of finite perimeter in the Riemannian manifold, rectifiable set and rectifiable varifold in the Euclidean space. For the sets of finite perimeter in the Riemannian manifold, we refer to [Vol10] for more details.

Let \((M, g)\) be a complete \((n+1)\)-dimensional smooth Riemannian manifold, \(\text{div}_g\) denotes the divergence operator on \((M, g)\), \(xy\) denotes a geodesic segment on \((M, g)\) joining \(x\) and \(y\), \(\text{vol}\) denotes the volume measure of \((M, g)\), \(B_r(p)\) denotes the geodesic ball on \(M\) centered at \(p\) with radius \(r\) and \(\Gamma_{s}^{t}(TM)\) denotes the tangent vector field on \(M\) with compact support. Let \((M, d_g)\) denote the induced metric space, i.e., for \(a, b \in M\),

\[d_g(a, b) := \inf\{L_g(\gamma) : \gamma \text{ is a piecewise } C^1\text{-path joining } a \text{ and } b\}.

By Nash embedding theorem, there exists a smooth embedding \(f : M \to \mathbb{R}^N\) for some positive integer \(N\) such that \((M, g)\) is isometrically embedded into \((\mathbb{R}^N, g_{\text{euc}})\), where \(g_{\text{euc}}\) denotes the canonical Euclidean metric on \(\mathbb{R}^N\).

2.1. Hausdorff measure. In this paper, we start from the sets of finite perimeter defined intrinsically on \((S^{n+1}, g_{S^{n+1}})\) by the Hausdorff measure \(\mathcal{H}^s_g\), and then we embed \((S^{n+1}, g_{S^{n+1}})\) into \((\mathbb{R}^{n+2}, g_{\text{euc}})\), and hence we have to consider the Hausdorff measure \(\mathcal{H}^s\) on the ambient Euclidean space \(\mathbb{R}^{n+2}\) as well.

Precisely, for a Riemannian manifold \((M^{n+1}, g)\), let \(\mathcal{H}^s_g\) denote the \(s\)-dimensional Hausdorff measure defined on the metric space \((M^{n+1}, d_g)\) (see [Vol10, Definition 2.15]), on \((M, g)\) the Riemannian volume measure coincide with the \((n+1)\)-dimensional Hausdorff measure (see [Vol10, Theorem 2.17, Corollary 2.18]), i.e.,

\[\text{vol} = \mathcal{H}^{n+1}_g. \quad (2.1)\]
On the other hand, the Hausdorff measure defined on the Euclidean space has been well-studied, we refer to [Sim83; Mag12] for a detailed account. Let \( \mathcal{H} \) denote the \( s \)-dimensional Hausdorff measure on \( (\mathbb{R}^N, g_{\text{eucl}}) \), where \( (M^{n+1}, g) \) is isometrically embedded into \( (\mathbb{R}^N, g_{\text{eucl}}) \), then the Riemannian volume measure of \( (M^{n+1}, g) \) agrees with the Hausdorff measure \( \mathcal{H}^{n+1} \) of \( \mathbb{R}^N \) by [Sim83, Chapter 2, 8.6(2)], i.e.,

\[
\text{vol} = \mathcal{H}^{n+1}. \tag{2.2}
\]

In this paper we always use \( \mathcal{H}^n \) to denote the Hausdorff measure defined intrinsically on the Riemannian manifold and \( \mathcal{H}^n \) the Hausdorff measure defined on the ambient Euclidean space.

2.2. Rectifiable varifold. For basic concepts of rectifiable set and rectifiable varifold, we refer to [Sim83; De 08; Mag12] for a detailed account.

2.2.1. Rectifiable set. A Borel set \( M \subset \mathbb{R}^{n+2} \) is a locally \( \mathcal{H}^n \)-rectifiable set if \( M \) can be covered, up to a \( \mathcal{H}^n \)-negligible set, by countably many Lipschitz images of \( \mathbb{R}^n \) to \( \mathbb{R}^{n+2} \), and if \( \mathcal{H}^n \cap M \) is locally finite on \( \mathbb{R}^{n+2} \). \( M \) is called \( \mathcal{H}^n \)-rectifiable if in addition, \( \mathcal{H}^n(M) < \infty \); \( M \) is said to be normalized, if \( M = \text{spt}(\mathcal{H}^n \cap M) \). In this paper, we always assume that a rectifiable set is normalized.

2.2.2. Rectifiable varifold. Let \( M \) be a countably \( n \)-rectifiable, \( \mathcal{H}^n \)-measurable set in \( \mathbb{R}^{n+2} \), let \( \theta \) be a positive locally \( \mathcal{H}^n \)-integrable function on \( M \). A rectifiable \( n \)-varifold is denoted by \( \text{var}(M, \theta) \), and is defined to be the equivalent class of all pairs \( (\tilde{M}, \tilde{\theta}) \), where \( \tilde{M} \) is countably \( n \)-rectifiable with \( \mathcal{H}^n(\tilde{M} \setminus \tilde{M}) = 0 \), and \( \tilde{\theta} = \theta \) for \( \mathcal{H}^n \)-a.e. on \( M \cap \tilde{M} \). Associated to \( V = \text{var}(M, \theta) \), the weight measure of \( V \), denoted by \( \mu_V \), and is defined by

\[
\mu_V := \mathcal{H}^n \cap \theta
\]

where we adopt the convention that \( \theta \equiv 0 \) on \( \mathbb{R}^{n+2} \setminus M \).

**Definition 2.1.** Let \( V = \text{var}(M, \theta) \) be a rectifiable \( n \)-varifold in the open set \( \Omega \subset \mathbb{R}^{n+2} \), we say that \( V \) has generalized mean curvature vector \( \mathbf{H}_M \) if

\[
\int_M \text{div}_M X d\mu_V = -\int_M \langle X, \mathbf{H}_M \rangle d\mu_V, \quad \forall X \in C^1_c(\Omega; \mathbb{R}^{n+2}). \tag{2.3}
\]

2.3. Sets of finite perimeter in a Riemannian manifold. Definitions and properties of sets of finite perimeter in a Riemannian manifold needed in the sequel are:

i. **(BV functions)** Let \( \Omega \subset M \) be an open set and \( f \in L^1(\Omega) \), then \( f \) is said to have bounded variation if

\[
||\nabla_g f||(\Omega) := \sup \left\{ \int_{\Omega} f \text{div}_g X d\mathcal{H}^n_g : \chi \in \Gamma_c^1(T\Omega), |\chi|_g \leq 1 \right\} < \infty.
\]

ii. **(Sets of finite perimeter)** A \( \mathcal{H}^n \)-measurable set \( E \subset M \) is said to be a set of finite perimeter in \( \Omega \) if

\[
P_g(E; \Omega) := ||\nabla_g \chi_E||(\Omega) < \infty.
\]

\( P_g(E; \Omega) \) is called the perimeter of \( E \) in \( \Omega \).

iii. **(Reduced boundary)** For a set of finite perimeter \( E \subset \Omega \), the structure theorem holds (see [Vol10, Theorem 2.36]), i.e., there exists a Radon measure \( \mu_{E,g} \) on \( \Omega \) and a \( \mu_{E,g} \)-measurable vector field \( \nu_{E,g} : \Omega \to T\Omega \) with \( |\mu_{E,g}|_g = 1 \) for \( \mu_{E,g} \)-a.e. such that

\[
\int_E \text{div}_g X d\mathcal{H}^n_g = -\int_{\Omega} g(X, \nu_{E,g}) d\mu_{E,g}, \quad \forall X \in \Gamma_c^1(T\Omega).
\]
The reduced boundary $\partial^* E \cap \Omega$ of $E$ in $\Omega$ is then defined by (see [Vol10, Definition 2.47]):

$$\partial^* E \cap \Omega := \{ x \in \Omega : |\nu_{E,g}|_g = 1 \}.$$  

iv. (Support) For a set of finite perimeter $E$ in $\Omega$, we can assume that $E \subset M$ is a Borel set (c.f., [Vol10, Definition 2.35, Proposition 2.45], [Mag12, Proposition 12.19]). Moreover, we can further assume that $\text{spt} \mu_{E,g} = \partial E$, where $\text{spt} \mu_{E,g}$ is characterized by

$$\text{spt} \mu_{E,g} = \left\{ x \in M : 0 < |E \cap B_r(x)|_g < |B_r(x)|_g \quad \forall r > 0 \right\}. \quad (2.4)$$

v. (Rectifiability) Let $E \subset M^{n+1}$ be a set of finite perimeter, $(M,g)$ is isometrically embedded into $(\mathbb{R}^n, g_{euc})$ by $f$, then (see [Vol10, Theorem 4.16]):

(a) $\mu := \mu_{E,g}$ is a rectifiable $n$-varifold in $\mathbb{R}^N$.

(b) Set $\Sigma = f(\partial^* E)$, then $\theta^n(\mu, x) = 1$ for every $x \in \Sigma$. In particular, $\Sigma$ is a countably $n$-rectifiable set in $\mathbb{R}^N$ and $\mu = \mathcal{H}^n \upharpoonright \Sigma$.

vi. (Gauss-Green formula) By Combining (iii) with [Vol10, Lemma 4.17] we have: for a set of finite perimeter $E$ in $\Omega \subset (M^{n+1}, g) \hookrightarrow (\mathbb{R}^n, g_{euc})$, $\mu_{E,g} = \mathcal{H}^n \upharpoonright \partial^* E$, and the Gauss-Green formula holds, i.e., for any $X \in \Gamma^1_c(\Omega)$,

$$\int_E \text{div}_g X d\mathcal{H}^{n+1}_g = -\int_{\partial^* E} g(X, \nu_{E,g}) d\mathcal{H}^n_g, \quad (2.5)$$

In the following, we study the sets of finite perimeter on $(\mathbb{S}^{n+1}, g_{\mathbb{S}^{n+1}})$. Since $(\mathbb{S}^{n+1}, g_{\mathbb{S}^{n+1}})$ can be isometrically embedded into $(\mathbb{R}^{n+2}, g_{euc})$, we can identify $(\mathbb{S}^{n+1}, g_{\mathbb{S}^{n+1}})$ with the unit sphere in $\mathbb{R}^{n+2}$, and it suffice to study the countably $n$-rectifiable set $\Sigma = f(\partial^* E)$ in $\mathbb{R}^{n+2}$.

2.4. Area formula and Coarea formula. Now we list some important material in geometric measure theory which will be needed later.

1. (Area formula for $k$-rectifiable set) If $A$ is a $\mathcal{H}^k$-rectifiable set and $f : \mathbb{R}^n \to \mathbb{R}^m$ is a Lipschitz map with $1 \leq k \leq m$, then (see [Mag12, Theorem 11.6], [Sim83, (12.4)])

$$\int_{\mathbb{R}^m} \mathcal{H}^0(A \cap \{ f = y \}) d\mathcal{H}^k(y) = \int_A J^A f(x) d\mathcal{H}^k(x), \quad (2.6)$$

where $\{ f = y \} = \{ x \in \mathbb{R}^n : f(x) = y \}$, $J^A f(x)$ is the Jacobian of $f$ with respect to $A$ at $x$, which exists for $\mathcal{H}^k$-a.e. $x \in A$.

2. (Coarea formula for $k$-rectifiable set) If $A$ is a $\mathcal{H}^k$-rectifiable set and $f : \mathbb{R}^n \to \mathbb{R}^m$ is a Lipschitz map with $k \geq m$, then (see [Sim83, (12.6)])

$$\int_{\mathbb{R}^m} \mathcal{H}^{k-m}(A \cap f^{-1}(y)) d\mathcal{H}^m(y) = \int_A J^A f(x) d\mathcal{H}^k(x). \quad (2.7)$$

3. (Coarea formula on Riemannian manifold) If $u : (M^{n+1}, g) \to \mathbb{R}^1$ is a Lipschitz function and $A \subset M$ is open, then: $t \in \mathbb{R}^1 \mapsto P_g (\{ u > t \}; A)$ is a Borel function on $M$ with

$$\int_A |\nabla u|_g = \int_{\mathbb{R}^1} P_g (\{ u > t \}; A) dt. \quad (2.8)$$

When $M = \mathbb{R}^{n+1}$, this is exactly [Mag12, Theorem 13.1], for the lack of precise reference when $(M^{n+1}, g)$ is a Riemannian manifold, we sketch the proof.
Sketch of proof. First notice that the Layer-cake representation (see [Mag12, Remark 13.6]) is valid on the measure space \((M^{n+1}, \mathcal{H}^{n+1}_g, \nu)\), i.e., for \(u \in L^1(M)\), \(u \geq 0\) and \(v \in L^\infty(M)\),

\[
\int_M u(x)v(x)d\mathcal{H}^{n+1}_g = \int_0^\infty dt \int_{\{u > t\}} v(x)d\mathcal{H}^{n+1}_g.
\]

(2.9)

Indeed, for any \(x \in M\), we have

\[
u(x) = \int_0^\infty \chi_{\{u > t\}}(x)dt,
\]

then the Fubini’s Theorem for measure space [EG15, Theorem 1.22] gives:

\[
\int_{M = \{u \geq 0\}} u(x)v(x)d\mathcal{H}^{n+1}_g(x) = \int_{\{u \geq 0\}} v(x) \int_0^\infty \chi_{\{u > t\}}(x)dt = \int_0^\infty dt \int_{\{u > t\}} v(x)d\mathcal{H}^{n+1}_g.
\]

Hence, one can readily follow the proof of [Mag12, Theorem 13.1] by using the Layer-cake representation for \((M^{n+1}, \mathcal{H}^{n+1}_g)\) and noticing that the perimeter of a set of finite perimeter \(E \subset M\) is defined by

\[
P_g (\{u > t\}; A) = \sup \left\{ \int_{\{u > t\}} \text{div}_g Td\mathcal{H}^{n+1}_g : T \in \Gamma^1_c(TA), |T|_g \leq 1 \right\}.
\]

\(\Box\)

2.5. Geometry of \((S^{n+1}, g_{S^{n+1}})\). Here we list some well-known facts about the space form \((S^{n+1}, g_{S^{n+1}})\).

(1) \((S^{n+1}, g_{S^{n+1}})\) is a smooth complete compact Riemannian manifold without boundary with sectional curvature identically 1.

(2) The injective radius of \(S^{n+1}\) is \(\pi\), i.e., \(\text{inj}(S^{n+1}) = \pi\).

(3) The only geodesics on \(S^{n+1}\) are great circles.

(4) For \(x, z \in (S^{n+1}, g_{S^{n+1}})\), when \(\text{dist}_g(x, z) < \pi\), there exists a unique minimizing geodesic joining \(x\) and \(z\). In particular, if \(y \not\in \overline{xz}\), then \(\text{dist}_g(x, z) < \text{dist}_g(x, y) + \text{dist}_g(y, z)\).

3. Level sets of distance function

In this section, we explore the level sets of the distance function from the boundary of a set of finite perimeter intrinsically on \((S^{n+1}, g_{S^{n+1}})\).

Let \(\Omega\) be a set of finite perimeter in \((S^{n+1}, g_{S^{n+1}})\), \(\partial \Omega\) is its topological boundary. Let \(u : S^{n+1} \to \mathbb{R}^1\) be the distance function to \(\partial \Omega\), which is defined on the unit sphere in \(\mathbb{R}^{n+2}\) and is given by: \(u(y) = \text{dist}_g(y, \partial \Omega)\) for \(y \in S^{n+1}\). Let \(\zeta\) be the point projection of \(y\) to \(\partial \Omega\), namely, \(\text{dist}_g(y, \zeta(y)) = u(y)\).

First we need the following Lemma for \(u\) and \(\zeta\).

Lemma 3.1. Let \(\Omega\) be a set of finite perimeter in \((S^{n+1}, g_{S^{n+1}})\), then the following statements hold:

(1) \(u\) is a Lipschitz function on \(\Omega\) with Lipschitz constant at most 1, i.e., for any \(x, y \in \Omega\),

\[
|u(y) - u(x)| \leq \text{dist}_g(x, y).
\]

(2) For \(0 < s < t < \pi\), \(\zeta\) is continuous on \(\Gamma^t_s\), where \(\Gamma^t_s\) is defined in Proposition 3.1.
Proof. Since \( \partial \Omega \) is a closed, bounded set in \((S^{n+1}, g_{S^{n+1}})\), it is compact by Hopf-Rinow Theorem, and hence we can take \( a \in \partial \Omega \) such that \( u(x) = \text{dist}_g(a, x) \). Without loss of generality, assume that \( u(y) \geq u(x) \), then by the triangle inequality,

\[
|u(y) - u(x)| = u(y) - u(x) \leq \text{dist}_g(a, y) - \text{dist}_g(a, x) \leq \text{dist}_g(x, y).
\]

This completes the proof of (1).

For (2), otherwise, there exists \( \epsilon > 0 \) and a sequence of points \( y_1, y_2, y_3, \ldots \in \Gamma_s' \), converges to \( y \in \Gamma_s' \) such that \( \text{dist}_g(\zeta(y_i), \zeta(y_j)) \geq \epsilon \) for \( i = 1, 2, \ldots \). Then,

\[
\text{dist}_g(\zeta(y_i), y) = u(y_i) = s,
\]

\[
\text{dist}_g(\zeta(y_i), y) \leq \text{dist}_g(\zeta(y_i), y_i) + \text{dist}_g(y_i, y) = s + \text{dist}_g(y_i, y) < s + \epsilon.
\]

Thus, all the points \( \{\zeta(y_i)\} \) lie in \( \partial \Omega \cap B_{s+\epsilon}(y) \), which is a bounded subset of the compact set \( \partial \Omega \), and hence by passing to a subsequence, we can assume that \( \{\zeta(y_i)\} \) converges to some point \( x \in \partial \Omega \). But then,

\[
\lim_{i \to \infty} u(y_i) = \lim_{i \to \infty} \text{dist}_g(\zeta(y_i), y_i) = \text{dist}_g(x, y),
\]

which implies that \( x = \zeta(y) \), a contradiction to the fact that

\[
\text{dist}_g(x, \zeta(y)) = \lim_{i \to \infty} \text{dist}_g(\zeta(y), \zeta(y_i)) \geq \epsilon.
\]

Here when we conclude \( x = \zeta(y) \), we use the fact that for \( y \in \Gamma_s' \), \( y \) admits a unique point projection to \( \partial \Omega \). This property is included in Proposition 3.1(1), whose proof does not depend on the continuity of \( \zeta \) on \( \Gamma_s' \).

\[\square\]

Remark 3.1. When \( \Omega \) is contained in a Euclidean space, similar results are included in [Fed69, 4.8(1), (4)]

Next, we study some good subsets of the level sets of the distance function \( u \), roughly speaking, the distance function is differentiable on these sets. Such good sets are well studied in [DM19] when \( \Omega \) is a closed set in the Euclidean space. In the following, we list some good properties of these sets. In order to generalize these properties to sphere, we shall use the completeness and the injective radius of \((S^{n+1}, g_{S^{n+1}})\) and the fact that the only geodesics on the unit sphere are great circles as well.

Proposition 3.1. Let \( \Omega \subset (S^{n+1}, g_{S^{n+1}}) \) be an open set of finite perimeter, for \( 0 < s < t < \pi \), set:

\[
\Gamma_s^t := \{ y \in \partial \Omega_s : y \in \overline{xz} \text{ for some } x \in \partial \Omega, z \in \partial \Omega_t \text{ with } \text{dist}_g(y, z) = t - s \},
\]

\[
\Gamma_s^+ := \cup_{t > 0} \Gamma_s^t.
\]

Then,

(1) \( y \in \Gamma_s^t \) admits unique \( x \in \partial \Omega \) and \( z \in \partial \Omega_t \). In particular, \( y \in \Gamma_s^t \) has a unique point projection onto \( \partial \Omega \).

(2) For \( s < t_1 < t_2 < \pi \), \( \Gamma_s^{t_2} \subset \Gamma_s^{t_1} \). In particular, \( \Gamma_s^+ = \lim_{t \to s^+} \Gamma_s^t \).

(3) \( \Gamma_s^t \) is a compact set in \( S^{n+1} \) if it is not empty.

(4) for \( y \in \Gamma_s^t \), \( \Gamma_s^t \) is bounded by two tangent geodesic balls at \( y \), i.e.,

\[
\begin{align*}
\{ B_{t-s}(z) \subset \Omega_s \subset S^{n+1} \setminus B_s(x), \\
\{ y \} = \partial B_{t-s}(z) \cap \partial B_s(x).
\end{align*}
\]

(5) \( u \) is differentiable at \( y \in \Gamma_s^t \).
Proof. For any \( y \in \Gamma_s^t \), by definition, there exists \( x \in \partial \Omega, \ z \in \partial \Omega_t \) such that \( y \in \overline{zx} \) and \( \text{dist}_g(x, y) = s \).

**Claim:** \( \text{dist}_g(x, y) = s, \text{dist}_g(x, z) = t \).

Indeed, since \( y \in \partial \Omega_s \), we have: there exists \( x' \in \partial \Omega \) such that \( \text{dist}_g(x', y) = s \). If \( x' \neq x \), then by the triangle inequality and using the fact that \( s < t < \pi = \text{inj}(\mathbb{S}^{n+1}) \) and \( y \notin \overline{xz} \), we have

\[
\text{dist}_g(x', z) < \text{dist}_g(x', y) + \text{dist}_g(y, z) = t - s + s = t,
\]

which contradicts to \( z \in \partial \Omega_t \). This shows that \( \text{dist}_g(x, y) = s \), it follows that \( \text{dist}_g(x, z) = t \) since \( \overline{zx} \) is a geodesic segment, and this proves the claim.

(1) If there exists \( x' \in \partial \Omega, z' \in \partial \Omega_t \) and \( x' \neq x \) such that \( y \in \overline{x'z'} \) and \( \text{dist}_g(y, z') = t - s \), then by claim, \( \text{dist}_g(x', y) = t \). By the triangle inequality and \( y \notin \overline{x'z} \) again, we have

\[
\text{dist}_g(x', z) < \text{dist}_g(x', y) + \text{dist}_g(y, z) = t - s + s = t,
\]

which contradicts to \( z \in \partial \Omega_t \), this shows \( x' = x \).

Similarly, we can prove that \( z' = z \) by the triangle inequality, thus proof of (1) is complete.

(2) For any \( y \in \Gamma_s^t \), there exists \( x \in \partial \Omega, z \in \partial \Omega_t \) such that \( y \in \overline{xz} \) and \( \text{dist}_g(y, z) = t_2 - s \). Since \( \overline{xz} \) is a geodesic segment, we can choose \( z_1 \in y\overline{zz} \) with \( \text{dist}_g(z_1, z_2) = t_2 - t_1 \). We will prove that \( y \in \Gamma_s^{t_1} \) and the corresponding points are exactly \( x \) and \( z_1 \).

First we prove that \( z_1 \in \partial \Omega_{t_1} \). By claim, \( u(z_1) \leq \text{dist}_g(x, z_1) = \text{dist}_g(x, z_2) - \text{dist}_g(z_1, z_2) = t_2 - (t_2 - t_1) = t_1 \). Next we prove that \( u(z_1) \geq \text{dist}_g(x, z_1) \), if not, \( u(z_1) < \text{dist}_g(x, z_1) = t_1 \), and hence \( \zeta(z_1) \neq x \), which implies \( z_1 \notin \zeta(z_1)\overline{zz} \). By triangle inequality we have

\[
u(z_2) \leq \text{dist}_g(\zeta(z_1), z_2) < \text{dist}_g(\zeta(z_1), z_1) + \text{dist}_g(z_1, z_2) = u(z_1) + (t_2 - t_1) < t_1 + (t_2 - t_1) = t_2,
\]

which contradicts to the fact that \( z_2 \in \partial \Omega_{t_2} \). Hence \( u(z_1) = t_1 \) and \( z_1 \in \Omega_{t_1} \).

Since \( \overline{xz} \) is a geodesic segment and \( z_1 \in \overline{xz} \), we have: \( \overline{xz} \) is also a geodesic segment and \( \text{dist}_g(y, z_1) = \text{dist}_g(y, z_2) - \text{dist}_g(z_1, z_2) = (t_2 - s) - (t_2 - t_1) = t_1 - s \). This shows that \( y \in \Gamma_s^{t_1} \), and hence for any \( s < t_1 < t_2 < \pi \), we have: \( \Gamma_s^{t_2} \subseteq \Gamma_s^{t_1} \). By inclusion, it is apparent that \( \Gamma_s^t = \lim_{t \to \pi^+} \Gamma_s^t \).

(3) It suffices to prove that \( \Gamma_s^t \) is a closed set in \( \mathbb{S}^{n+1} \), i.e., if a sequence of points \( \{y_i \in \Gamma_s^t \}_{i=1}^{\infty} \) converges to \( y \), we will prove that \( y \in \Gamma_s^t \).

By definition of \( \Gamma_s^t \), for each \( y_i \), there exists corresponding points \( x_i \in \partial \Omega, z_i \in \partial \Omega_t \). Since \( \zeta \) is continuous on \( \Gamma_s^t \) by Lemma 3.1, we have: \( \{x_i\}_{i=1}^{\infty} \) is a Cauchy sequence in \( \partial \Omega \). Notice that \( \partial \Omega \) is closed, hence \( \{x_i\}_{i=1}^{\infty} \) converges to some \( x \in \partial \Omega \). Similarly, \( \{z_i\}_{i=1}^{\infty} \) converges to some \( z \in \partial \Omega_t \).

Since \( u \) is continuous on \( \Omega \), we have

\[
u(y) = \lim_{i \to \infty} u(y_i) = s,
\]

this shows that \( y \in \partial \Omega_s \). Also, by claim,

\[
\text{dist}_g(x, y) = \lim_{i \to \infty} \text{dist}_g(x_i, y_i) = \lim_{i \to \infty} s = s.
\]

Similarly, \( \text{dist}_g(y, z) = \lim_{i \to \infty} \text{dist}_g(y_i, z_i) = t - s \).

By triangle inequality,

\[
t = u(z) \leq \text{dist}_g(x, z) \leq \text{dist}_g(x, y) + \text{dist}_g(y, z) = t,
\]

this implies that \( \overline{xz} \) must be a geodesic segment which passes through \( y \), since \( t < \text{inj}(\mathbb{S}^{n+1}) \), and hence there exists a unique minimizing geodesic joining \( x \) and \( z \) whose length is \( t \). Thus, \( y \in \Gamma_s^t \), which implies that \( \Gamma_s^t \) is closed and hence compact.
Proposition 3.2. If $\Omega \subset (S^{n+1}, g_{S^{n+1}})$ is an open set of finite perimeter, then the super-level set $\Omega_s := \{ y \in \Omega : u(y) > s \}$ is a set with finite volume and perimeter with $\mathcal{H}^n(\partial \Omega_s \setminus \Gamma^+_s) = 0$ for a.e. $s > 0$, where $\Gamma^+_s := \cup_{t > s} \Gamma^+_t$ and $\Gamma^+_s := \{ y \in \partial \Omega_s : y \in \overline{\mathbb{B}}_x \text{ for some } x \in \partial \Omega_s, z \in \partial \Omega_t \text{ with } \operatorname{dist}_g(x, y) = s \}$.

Proof. Since $(S^{n+1}, g_{S^{n+1}})$ is a compact manifold, it is apparent that $\Omega_s \subseteq \Omega$ is of finite volume. By the Coarea formula on Riemannian manifold (2.8), we have
\[
\int_0^\infty P_g(\Omega_s)ds = \int_{\Omega_s} |\nabla u|_g dH^{n+1}_g = |\Omega_s|_g < |\Omega|_g < \infty.
\]
Hence
\[
P_g(\Omega_s) < \infty, \quad \text{for a.e. } s > 0,
\]
this shows that $\Omega_s$ is a set of finite perimeter for a.e. $s > 0$.

By Proposition 3.1(2)(5), we see that $\Gamma^+_s$ is indeed the set of all regular points of the distance function $u$ in $\partial \Omega_s$, then [RZ12, Theorem 5.7] shows that $\mathcal{H}^n(\partial \Omega_s \setminus \Gamma^+_s) = 0$.

Remark 3.2. When we consider $\Omega \subset (S^{n+1}, g_{S^{n+1}}) \hookrightarrow (\mathbb{R}^{n+2}, g_{\text{euc}})$, we have: $\Omega$ and $\Omega_s$ are relatively open subsets of $S^{n+1}$. Moreover, by combining the definitions of Hausdorff measure in $(S^{n+1}, g_{S^{n+1}})$ with the Hausdorff measure in $(\mathbb{R}^{n+2}, g_{\text{euc}})$, we have: In $\mathbb{R}^{n+2}$,
\[
\mathcal{H}^n(\partial \Omega_s \setminus \Gamma^+_s) = 0.
\]

4. $C^{1,1}$-Rectifiability of $\Gamma^+_s$

In this section, we will pursue the $C^{1,1}$-rectifiability of these level sets. To this end, we shall embed $(S^{n+1}, g_{S^{n+1}})$ into $(\mathbb{R}^{n+2}, g_{\text{euc}})$.

In order to further explore $\Gamma^+_s$, we will use the following fact that on $S^{n+1}$, moving along a great circle and the tangent vector at some point of this great circle can be explicitly expressed in the ambient Euclidean space $\mathbb{R}^{n+2}$.

Lemma 4.1. For $y \in \Gamma^+_s$, there exists corresponding points $x \in \partial \Omega, z \in \partial \Omega_t$ such that $y \in \overline{\mathbb{B}}_x$, let $N(y) := \nabla u(y)$, whose existence is valid by Proposition 3.1(5), then
\[
\begin{align*}
(1) \quad & N(y) = \frac{x + y}{\sin s} + \frac{y}{\tan \frac{t}{2}}, \\
(2) \quad & z = y + \frac{t}{\cos t} \cdot N(y) = \cos t \cdot y + \sin t \cdot N(y).
\end{align*}
\]

Proof. These are well-known facts and one can check by a direct computation.

Now we further explore the sets $\Gamma^+_s$; we will see that $N$ is tangentially differentiable $\mathcal{H}^n$-a.e. on $\Gamma^+_s$ and $\Gamma^+_s$ is $C^{1,1}$-rectifiable. In particular, we will generalize [Lemma 7][DM19] from Euclidean space to $(S^{n+1}, g_{S^{n+1}})$.

Proof of Theorem 1.1. First we estimate $|N(y) \cdot (y' - y)|$ in the Euclidean space $\mathbb{R}^{n+2}$ for any $y, y' \in \Gamma^+_s$ satisfying $\operatorname{dist}_g(y', y) \leq \frac{\pi}{2}$. Throughout the proof, $|\cdot|$ will denote the Euclidean norm in $\mathbb{R}^{n+2}$, $\cdot$ will denote the Euclidean inner product in $\mathbb{R}^{n+2}$, $\nabla$ will denote the gradient in Euclidean space.

Assume that $y$ admits $x \in \partial \Omega, z \in \partial \Omega_t$ as Proposition 3.1(1), on $S^{n+1}$, by the hinge version of Toponogov’s theorem, the cosine theorem in Euclidean space and Lemma 4.1(2), we have
\[
\operatorname{dist}_g^2(x, y') \leq \operatorname{dist}_g^2(x, y) + \operatorname{dist}_g^2(y, y') - 2(-s N(y)) \cdot \left[\operatorname{dist}_g(y, y') \left(\frac{y' + y}{\sin(\operatorname{dist}_g(y, y'))} - \frac{y}{\tan(\frac{\operatorname{dist}_g(y, y')}{2})}\right)\right].
\]
notice that \( \text{dist}_g(x, y') \geq s, \text{dist}_g(x, y) = s, N(y) \cdot y = 0 \), and hence we have

\[
-2s \frac{\text{dist}_g(y, y')}{\sin(\text{dist}_g(y, y'))} N(y) \cdot (y' - y) \leq \text{dist}_g^2(y, y'),
\]

since \( \text{dist}_g(y, y') \leq \frac{\pi}{2} \), we deduce that

\[
N(y) \cdot (y' - y) \geq -\frac{1}{2s} \sin(\text{dist}_g(y, y')) \text{dist}_g(y, y'). \tag{4.1}
\]

Same computation for \( y, y', z \) holds, i.e.,

\[
\text{dist}_g^2(z, y') \leq \text{dist}_g^2(y, z) + \text{dist}_g^2(y, y') - 2 ((t - s)(N(y))) \cdot \left[ \text{dist}_g(y, y') \left( \frac{y' + y}{\sin(\text{dist}_g(y, y'))} - \frac{y}{\tan(\text{dist}_g(y', y))} \right) \right],
\]

notice that \( \text{dist}_g(y', z) \geq (t - s), \text{dist}_g(y, z) = (t - s), N(y) \cdot y = 0, \text{dist}_g(y, y') \leq \frac{\pi}{2} \), we deduce

\[
N(y) \cdot (y' - y) \leq -\frac{1}{2(t - s)} \sin(\text{dist}_g(y, y')) \text{dist}_g(y, y'). \tag{4.2}
\]

By (4.1) and (4.2) we see that

\[
|N(y) \cdot (y' - y)| \leq \max \left\{ \frac{1}{2s}, \frac{1}{2(t - s)} \right\} \sin(\text{dist}_g(y, y')) \text{dist}_g(y, y'). \tag{4.3}
\]

By Lemma 4.1(1), \( x = \zeta(y) \) and Lemma 3.1(2), we see that \( N \) is continuous on \( \Gamma^t_s \).

Observe that

\[
\limsup_{\delta \to 0^+} \left\{ \frac{|u(y') - u(y) - N(y) \cdot (y' - y)|}{|y' - y|} : 0 < |y' - y| \leq \delta, y', y \in \Gamma^t_s \right\}
\]

\[
\leq \limsup_{\delta \to 0^+} \left\{ \max \left\{ \frac{1}{2(t - s)}, \frac{1}{2s} \right\} \sin(\text{dist}_g(y, y')) \cdot \text{dist}_g(y, y') : 0 < |y' - y| \leq \delta, y', y \in \Gamma^t_s \right\}
\]

\[
= 0, \tag{4.4}
\]

where in the inequality we use the fact that \( u(y') = u(y) = s \) and (4.3), in the equality we use the fact that as \( \delta \to 0^+ \), \( \text{dist}_g(y, y') \to |y' - y| \) and also \( \sin(\text{dist}_g(y, y')) \to |y' - y| \).

Now, for \((u, N) \in C^0(\Gamma^t_s; \mathbb{R} \times \mathbb{R}^{n+2})\), since (4.4) holds, by \( C^1 \)-Whitney’s extension theorem (see for example [Mag12, Section 15.2]), there exists \( \phi \in C^1(\mathbb{R}^{n+2}) \) such that \((\phi, \nabla \phi) = (u, N)\) on \( \Gamma^t_s \).

For \( y \in \Gamma^t_s \), we know that \( N(y) \neq 0 \) by Lemma 4.1(1). Let \( \{e_1, \ldots, e_{n+2}\} \) be the coordinate of \( \mathbb{R}^{n+2} \), up to a rotation, we can assume that \( y = (0, \ldots, 0, 1, 0) = \nu_{S^{n+1}}(y), N(y) = (0, \ldots, 0, 1) \), here \( \nu_{S^{n+1}}(y) \) denotes the outer unit normal of \( S^{n+1} \) in \( \mathbb{R}^{n+2} \). Since \( \Gamma^t_s \subseteq \phi^{-1}(s) \cap S^{n+1} \), consider the following system

\[
\begin{align*}
\{ \\
&f_1(x_1, \ldots, x_{n+2}) = x_1^2 + \ldots + x_{n+2}^2 = 1, \\
&f_2(x_1, \ldots, x_{n+2}) = \phi(y) = s.
\end{align*}
\]

Notice that \( N(y) = (0, \ldots, 0, 1), \nu_{S^{n+1}}(y) = (0, \ldots, 0, 1, 0) \), and hence we have

\[
\begin{align*}
\partial_{e_{n+1}} f_1(y) &= 1, \partial_{e_{n+2}} f_1(y) = 0, \\
\partial_{e_{n+1}} f_2(y) &= 0, \partial_{e_{n+2}} f_2(y) = 1.
\end{align*}
\]

Set \( F : \mathbb{R}^n \times \mathbb{R}^2 \to \mathbb{R}^2 \) by \( F(x', x_{n+1}, x_{n+2}) = (f_1(x', x_{n+1}, x_{n+2}), f_2(x', x_{n+1}, x_{n+2})) \), then by the \( C^1 \)-Implicit function theorem, there exists an open set \( U \subset \mathbb{R}^n \) and a \( C^1 \) map \( g \in C^1(U; \mathbb{R}^2) \) such that \( \Gamma^t_s = (x', g(x')) \) near \( y \), i.e., \( \Gamma^t_s \) lies in the \( C^1 \)-image of \( G : U \subset \mathbb{R}^n \to \mathbb{R}^{n+2} \), given by \( G(x') = (x', g(x')) \). In particular, this shows the \( H^n \)-rectifiability of \( \Gamma^t_s \). Precisely, one can check
the rectifiability by using the definition in [Mag12, (10.4)] and noticing that the preimage of a Borel set of $G$ is still a Borel set in $\mathbb{R}^n$ since $G$ is a continuous function.

(1) Let $C(N, \rho) := \{z + hN : z \in N^\perp, |z| < \rho, |h| < \rho\}$ be the open cylinder at the origin with axis along $N \in TS^{n+1}$, radius $\rho$ and height $2\rho$ in $\mathbb{R}^{n+2}$. By the fact that at any $y \in \Gamma_s^t$, \(y = \partial B_{-\rho}(x) \cap \partial B_{\rho}(x), \nu_{\mathbb{S}^{n+1}}(y) = y\) and $\Gamma_s^t$ is $\mathcal{H}^n$-rectifiable, we have: $\Gamma_s^t$ admits an approximate tangent plane at $\mathcal{H}^n$-a.e. of this points and this plane is then exactly span $\{N(y), \nu_{\mathbb{S}^{n+1}}(y)\}^\perp$, which is a $n$-dimensional affine plane in $\mathbb{R}^{n+2}$, i.e.,

$$T_y \Gamma_s^t = \text{span} \{N(y), y\}^\perp \quad \text{for } \mathcal{H}^n\text{-a.e. } y \in \Gamma_s^t.$$

By [Mag12, Theorem 10.2], this implies

$$\lim_{\rho \to 0^+} \frac{\mathcal{H}^n(\Gamma_s^t \cap (y + C(N(y), \rho)))}{\omega_n \rho^n} = 1, \quad \text{for } \mathcal{H}^n\text{-a.e. } y \in \Gamma_s^t,$$

here $\omega_n$ denotes the volume of $n$-dimensional unit ball in $\mathbb{R}^{n+2}$.

For a sequence $\{\rho_j\}_j$ such that $\rho_j \to 0$ as $j \to \infty$, set:

$$f_j(y) := \frac{\mathcal{H}^n(\Gamma_s^t \cap (y + C(N(y), \rho_j)))}{\omega_n \rho_j^n},$$

then $f_j \to 0$ for $\mathcal{H}^n$-a.e. $y \in \Gamma_s^t$. By Egoroff’s theorem and [EG15, Lemma 1.1], there exists a compact set $U_1 \subset \Gamma_s^t$ such that $f_j \to 0$ uniformly on $U_1$ and $\mathcal{H}^n(\Gamma_s^t \setminus U_1) < \frac{1}{4}\mathcal{H}^n(\Gamma_s^t)$. For $\Gamma_s^t \setminus U_1$, we can use Egoroff’s theorem again to find a compact set $U_2 \subset \Gamma_s^t \setminus U_1$ such that $f_j \to 0$ uniformly on $U_2$ and $\mathcal{H}^n(\Gamma_s^t \setminus (U_1 \cup U_2)) < \frac{1}{2}\mathcal{H}^n(\Gamma_s^t)$. We can repeat above argument to obtain a sequence of compact sets $\{U_j\}_j^\infty$ such that $\mathcal{H}^n(\Gamma_s^t \setminus (\cup_j^\infty U_j)) = 0$ with $f_j \to 0$ uniformly on each $U_j$, namely,

$$\mu_j^t(\rho) := \sup_{y \in U_j} \left| 1 - \frac{\mathcal{H}^n(\Gamma_s^t \cap (y + C(N(y), \rho)))}{\omega_n \rho^n} \right| \to 0 \text{ as } \rho \to 0^+. \quad (4.5)$$

This shows that $\Gamma_s^t$ can be filled with a countable union of compact sets in the $\mathcal{H}^n$-sense.

Fix a $y \in \Gamma_s^t$, we know that $\Gamma_s^t$ is a $C^1$-graph over a disk of radius $\rho_y$ in a neighborhood of $y$, combining with the construction of $U_j$ and (4.5), we have: up to a subdivision of $U_j$ and relabeling, we can assume that for each $U_j$ and for any $y \in U_j$, there exists

$$\rho_j > 0, \psi_j \in C^1(N(y)^\perp), \psi_j(0) = 0, \nabla \psi_j(0) = 0, |\nabla \psi_j|_{C^0(N(y)^\perp)} \leq 1 \quad (4.6)$$

such that: let $U_j^\prime$ denote the projection of $U_j$ on $N(y)^\perp \cap \{|z| < \rho_j\}$, then

$$U_j \cap (y + C(N(y), \rho_j)) = \Gamma_s^t \cap (y + C(N(y), \rho_j)) = \{z + \psi_j(z)N(y) : z \in U_j^\prime\}, \quad (4.7)$$

here $\rho_j, \psi_j$ depend on the choice of $y \in U_j$.

Now $\Gamma_s^t$ is written as a $C^1$-graph locally at every $y \in U_j$ and we have (4.3), (4.5), we can follow directly the proof of [DM19, (3.16)] to find that for any $y_1, y_2 \in U_j$,

$$|N(y_1) - N(y_2)| \leq C_j |y_1 - y_2|, \quad \text{for all } y_1, y_2 \in U_j, \quad (4.8)$$

this shows that $N$ is a Lipschitz map on each $U_j$, by [Mag12, Theorem 11.4] and $\mathcal{H}^n(\Gamma_s^t \setminus (\cup_j^\infty U_j)) = 0$, we see that $N$ is tangentially differentiable along $\Gamma_s^t$ for $\mathcal{H}^n$-a.e. and it suffice to explore $N$ on each $U_j$ by virtue of [Mag12, Proposition 10.5].

By (4.3), (4.8) on each $U_j$, we can use the Whitney-Glaser extension theorem (see for example [Le 09]) to see that there exists $\phi \in C^{1,1}(\mathbb{R}^{n+2})$ such that $(u, N) = (\phi, \nabla \phi)$ on $U_j$. Then, by
the $C^{1,1}$-Implicit function theorem, for each $y \in U_j$, there exists $\psi_j \in C^{1,1}(N(y)\perp)$ satisfying (4.6), (4.7). In particular, this shows the $C^{1,1}$-rectifiability of $\Gamma_s^t$.

Thus for a fixed $y \in U_j$, we have a natural Lipschitz extension from $U_j \cap (y + C(N(y), \rho_j))$ to the whole cyclinder $y + C(N(y), \rho_j)$, denoted by $N_s$ and is given by:

$$N_s(y + z + hN(y)) = \frac{(-\nabla \psi_j(z), 1)}{1 + |\psi_j(z)|^2}, \quad \forall z \in N(y)\perp, |z| < \rho_j.$$ In the following computations, we follow the notations in [DM19, Section 2.1].

Set $\Psi_j(z) := y + z + \psi_j(z)N(y)$ for $|z| < \rho_j$, by [DM19, (2-5)], we have: for $H^n$-a.e. $y' \in U_j$ and for any $\tau \in T_{y'}U_j$,

$$\left(\nabla^{U_j} N\right) |_{y'} [\tau] = \nabla (N_s \circ \Psi_j) |_{\Psi_j^{-1}(y')}[e],$$

where $e = (\nabla \Psi_j)^{-1}(y')[\tau] \in \mathbb{R}^n$.

If $\psi_j \in C^2(N(y)\perp)$, then for any $z \in N(y)\perp$,

$$\nabla (N_s \circ \Psi_j)_z[e] = \lim_{t \to 0^+} \frac{N_s(\Psi_j(z + te)) - N_s(\Psi_j(z))}{t},$$

by direct computation,

$$\nabla (N_s \circ \Psi_j)_z[e] = -S_j(\Psi_j(z))[\tau], \quad (4.9)$$

where $S_j$ denotes the shape operator with respect to the graph of $\psi_j$, and here we use the following observation: For simplicity, we write $y_z := \Psi_j(z)$, let $\{\tau_1(y_z), \ldots, \tau_n(y_z), N(y_z), \nu_{\mathbb{S}^{n+1}}(y_z) = (y_z)\}$ denotes an orthonormal basis of $T_{y_z}\mathbb{R}^{n+2}$, where $\{\tau_1(y_z), \ldots, \tau_n(y_z)\}$ is an orthonormal basis of $T_{y_z}U_j$, then

$$\nabla_{\tau_i} N(y_z) \cdot \nu_{\mathbb{S}^{n+1}}(y_z) = -N(y_z) \cdot \nabla_{\tau_i} \nu_{\mathbb{S}^{n+1}}(y_z) = -N(y_z) \cdot \nabla_{\tau_i} (y_z) = -N(y_z) \cdot \tau_i(y_z) = 0,$$

$$\nabla_{\tau_i} N \cdot N = 0,$$

since $N \cdot N = 1$.

Recall that $\Gamma_s^t$ is trapped between two mutually tangent geodesic balls on $\mathbb{S}^{n+1}$ with radius $s$ and $t - s$ by Proposition 3.1(4), and hence the principal curvatures of the graph of $\psi_j$ is bounded from below by $-\cot s$ and above by $\cot (t - s)$ when they exist, i.e., for $H^n$-a.e. $y \in \Gamma_s^t$,

$$-\cot s \leq (k_i^s)_i(y) \leq (k_i^s)_i(y) \leq \cot (t - s). \quad (4.10)$$

Since $\psi_j \in C^{1,1}(N(y)\perp)$, again by [Mag12, Theorem 11.4], above argument holds for $H^n$-a.e. $y \in U_j$, which completes the proof of (1).

(2) First we prove that $H^n(\partial \Omega_{s+r}) \leq \{[\cot r + \cot (t - s)]|\sin r|\} H^n(\Gamma_s^t)$ for $r \in [-s, 0]$, and $H^n(\partial \Omega_{s+r}) \leq \{[\cot r + \cot s] |\sin r|\} H^n(\Gamma_s^t)$ for $r \in (0, t - s]$. Indeed, for $r \in [-s, t - s]$, we consider the mapping $f_r : \Gamma_s^t \to \partial \Omega_{s+r}$, defined by $f_r(y) = \cos ry + \sin r N(y)$. By definition of $\Gamma_s^t$ and Lemma 4.1(2), we see that $f_r(y) \in \partial \Omega_{s+r}$ and $f_r$ is surjective since for any $z \in \partial \Omega_{s+r}$, there exists some $x \in \partial \Omega$ such that $\text{dist}_{\partial} (x, z) = s + r$, and hence there exists $y \in \Gamma_s^t$ such that $y \in x$, this means $z = f_r(y)$ for some $y \in \Gamma_s^t$ and hence $f_r$ is surjective.

Then, $H^n(\partial \Omega_{s+r}) = H^n(f_r(\Gamma_s^t)) \leq \int_{f_r(\Gamma_s^t)} H^0(f_r^{-1}(z)) dH^n(z),$ by Area formula (2.6), we have

$$H^n(\partial \Omega_{s+r}) \leq \int_{f_r(\Gamma_s^t)} H^0(f_r^{-1}(z)) dH^n(z) = \int_{\Gamma_s^t} J_f^r f_r(y) dH^n(y). \quad (4.11)$$
A direct computation gives:

\[ J_{\Gamma_i}^s f_r(y) = \prod_{i=1}^n \left[ \cos r - \sin r (\kappa_i^r)^s \right]. \]

Now we consider the case \( 0 < r \leq (t - s) \leq \frac{\pi}{2} \), the case \(-\frac{\pi}{2} < -s \leq r < 0 \) follows similarly.

Since \( 0 < r \leq (t - s) < \frac{\pi}{2} \), by (4.10) we have

\[ J_{\Gamma_i}^s f_r(y) = \prod_{i=1}^n \left[ \cos r - \sin r (\kappa_i^r)^s \right] \leq \{ \cot r + \cot s \sin r \}^n. \]

Plugging into (4.11) to see that

\[ \mathcal{H}^n(\partial \Omega_{s+r}) \leq \int_{\Gamma_i^s} \{ \cot r + \cot s \sin r \}^n d\mathcal{H}^n \leq \{ \cot r + \cot s \sin r \}^n \mathcal{H}^n(\Gamma_i^t). \]  

By Coarea formula (2.7) for \( A = \Omega \setminus \Omega^* \), \( f = u \), \( k = n + 1 \), \( m = 1 \), and notice that the volume measure of \( S^{n+1} \) agrees with the Hausdorff measure \( \mathcal{H}^{n+1} \) of \( \mathbb{R}^{n+2} \) by (2.2), we have

\[ |\Omega \setminus \Omega^*|_g = \int_{\Omega \setminus \Omega^*} |\nabla u| d\mathcal{H}^{n+1} = \int_0^\infty \mathcal{H}^n((\Omega \setminus \Omega^*) \cap \partial \Omega_s) = \int_0^\infty \mathcal{H}^n(\partial \Omega_s \setminus \Gamma_i^r). \]

Again by Coarea formula (2.7), \( |\Omega_s|_g = \int_s^\infty \mathcal{H}^n(\partial \Omega_s) dt \), thus for a.e. \( s > 0 \),

\[ \mathcal{H}^n(\partial \Omega_s) = \lim_{\epsilon \to 0} \frac{|\Omega_s|_g - |\Omega_{s+\epsilon}|_g}{\epsilon} = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_0^\epsilon \mathcal{H}^n(\partial \Omega_{s+r})dr. \]

Combining with (4.12), we see that

\[ \frac{1}{\epsilon} \int_0^\epsilon \mathcal{H}^n(\partial \Omega_{s+r})dr \leq \frac{1}{\epsilon} \int_0^\epsilon \{ \cot r + \cot s \sin r \}^n \mathcal{H}^n(\Gamma_i^t)dr \]

\[ \leq \frac{1}{\epsilon} \int_0^\epsilon [1 + \sin \epsilon \cot s]^n \mathcal{H}^n(\Gamma_i^+)dr \]

\[ = [1 + \sin \epsilon \cot s]^n \mathcal{H}^n(\Gamma_i^+). \]

Notice that \( \Gamma_i^s \subset \partial \Omega_s \), so we have

\[ \mathcal{H}^n(\Gamma_i^+) \leq \mathcal{H}^n(\partial \Omega_s) \leq \lim_{\epsilon \to 0} [1 + \sin \epsilon \cot s]^n \mathcal{H}^n(\Gamma_i^+) = \mathcal{H}^n(\Gamma_i^+), \]

thus \( \mathcal{H}^n(\partial \Omega_s) = \mathcal{H}^n(\Gamma_i^+) \) for a.e. \( s > 0 \) and it follows that

\[ |\Omega \Delta \Omega^*|_g = \int_0^\infty \mathcal{H}^n(\partial \Omega_s \setminus \Gamma_i^+) = 0, \]

this proves (2).

(3) For \( r \in (0, s) \), consider the mapping \( g_r : \Gamma_i^s \rightarrow \Gamma_i^{s-r} \), defined by \( g_r(y) = \cos ry - \sin rN(y) \), for \( y \in \Gamma_i^s \). We readily see that \( g_r \) is a bijection between \( \Gamma_i^s \) and \( \Gamma_i^{s-r} \) by Proposition 3.1(2), Lemma 4.1(2). Then, if \( N \) is tangential differentiable at \( y \) along \( \Gamma_i^t \), by definition of \( \Gamma_i^t \), \( N \) is tangential differentiable at \( g_r(y) \) along \( \Gamma_i^{s-r} \).

Indeed, by a simple geometric relation on sphere, we have

\[ g_r(y) + \tan \frac{r}{2} N(g_r(y)) = y - \tan \frac{r}{2} N(y), \]  

which implies

\[ N(g_r(y)) = - \left[ (\cos r - 1) y - \left( \sin r - \tan \frac{r}{2} \right) N(y) \right] \cdot \frac{1}{\tan \frac{r}{2}} = \sin ry + \cos rN(y). \]
Thus
\[
\cos r N(y) = -\sin ry + N(g_r(y)). \tag{4.14}
\]
For any \( \tau \in T_y \Gamma_s^+ \subset T_y S^{n+1} \subset \mathbb{R}^{n+2} \), by chain rule we have
\[
\cos r \left( \nabla_{\Gamma_s^+} N \right)_y[\tau] = -\sin r \tau + \left( \nabla_{\Gamma_s^+} N \right)_{g_r(y)} \left[ \cos r \tau - \sin r \left( \nabla_{\Gamma_s^+} N \right)_y[\tau] \right],
\]
take \( \tau = \tau_i(y) \) to be the eigenvectors of the shape operators \( S_j \) in (4.9), we obtain
\[
-\cos r(\kappa^i_s)_i(y)\tau_i(y) = -\sin r\tau_i(y) + \left( \nabla_{\Gamma_s^+} N \right)_{g_r(y)} \left[ \cos r\tau_i(y) + \sin r(\kappa^i_s)_i(y)\tau_i(y) \right]
\]
\[
= -\sin r\tau_i(y) + \left( \cos r + \sin r(\kappa^i_s)_i(y) \right) \left( \nabla_{\Gamma_s^+} N \right)_{g_r(y)}[\tau_i(y)],
\]
from this we have
\[
-\tau_i(y) \cdot \left( \nabla_{\Gamma_s^+} N \right)_{g_r(y)}[\tau_i(y)] = \frac{-\sin r + \cos r(\kappa^i_s)_i(y)}{\cos r + \sin r(\kappa^i_s)_i(y)}.
\]
Hence \( \{\tau_i(y)\}_{i=1}^n \) is an orthonormal basis for \( T_{g_r(y)} \Gamma_s^{r-r} \), and the eigenvalues of \( \nabla_{\Gamma_s^+} N(g_r(y)) \) are given by:
\[
(\kappa^i_{s-r})_i (g_r(y)) = \frac{-\sin r + \cos r(\kappa^i_s)_i(y)}{\cos r + \sin r(\kappa^i_s)_i(y)}, \tag{4.15}
\]
which completes the proof of (3).
\[\square\]

**Remark 4.1.** We point out that in the previous argument, \( \mathcal{H}^n \) is the \( n \)-dimensional Hausdorff measure in \( \mathbb{R}^{n+2} \), and we restrict ourselves to \( S^{n+1} \), by the definitions of \( \mathcal{H}^n \) on \( (S^{n+1}, g_{S^{n+1}}) \) and \( \mathcal{H}^n \) on \( \mathbb{R}^{n+2} \), we see that Theorem 1.1 remains true if we replace \( \mathcal{H}^n \) by \( \mathcal{H}^n \).

Next we list some properties of \( \Gamma_s^+ \), thus extend [DM19, Lemma 7] to the sets of finite perimeter in \((S^{n+1}, g_{S^{n+1}})\).

**Proposition 4.1.** If \( \Omega \) is an open set of finite perimeter in \( S^{n+1} \), then the super level set \( \Omega_s = \{ y \in \Omega : u(y) > s \} \) is an open set of finite perimeter with \( \mathcal{H}^n(\partial \Omega_s \setminus \Gamma_s^+) = 0 \) for a.e. \( 0 < s < \frac{
}{2} \). Also,
\begin{enumerate}
\item For a.e. \( s > 0 \), \( \Gamma_s^+ \) can be filled with countably many graphs of \( C^{1,1} \)-functions from \( \mathbb{R}^n \) to \( \mathbb{R}^{n+2} \). In particular, this shows the \( C^{1,1} \)-rectifiability of \( \Gamma_s^+ \).
\item For a.e. \( s > 0 \), the principal curvatures \( (\kappa_s)_i \) of \( \Gamma_s^+ \) are defined \( \mathcal{H}^n \)-a.e. on \( \Gamma_s^+ \) by setting \( (\kappa_s)_i = (\kappa^t_s)_i \) on \( \Gamma^t_s \) for each \( t > s \). In particular, we can define the mean curvature and the length of the second fundamental form of \( \partial \Omega_s \) with respect to \( \nu_{\Omega_s} \) at \( \mathcal{H}^n \)-a.e. points of \( \Gamma^+ \) as follows:
\[
H_{\Omega_s} = \sum_{i=1}^n (\kappa_s)_i, \quad |A_{\Omega_s}|^2 = \sum_{i=1}^n (\kappa_s)_i^2.
\]
\item For every \( r < s < t \), the map \( g_r : \Gamma^t_s \to \Gamma^t_{s-r}, \) given by \( g_r(y) = \cos ry - \sin rN(y) \) for \( y \in \Gamma^t_s \), is a Lipschitz bijection from \( \Gamma^t_s \) to \( \Gamma^t_{s-r} \), with
\[
J_{\Gamma^t_s} g_r(y) = \prod_{i=1}^n \left[ \cos r + \sin r(\kappa^i_s)_i \right], \quad (\kappa_{s-r})_i (g_r(y)) = \frac{-\sin r + \cos r(\kappa_s)_i(y)}{\cos r + \sin r(\kappa_s)_i(y)} \tag{4.16}
\]
for \( \mathcal{H}^n \)-a.e. \( y \in \Gamma^t_s \).
\end{enumerate}
(4) For every \( x \in g_s(\Gamma^+_s) \subset \partial \Omega \), the limit

\[
\kappa_i(x) = \lim_{r \to s^-} (\kappa_{s-r})_i(x)
\]  

exists by monotonicity.

**Proof.** (1)(2)(3) are contained in the proof of Theorem 1.1, so we only prove (4).

Assume that \( y \in \Gamma^+_s \) is the corresponding point of \( x \in g_s(\Gamma^+_s) \), i.e., \( x = g_s(y) \). For \( 0 < r_1 < r_2 < s < \frac{\pi}{2} \), by (4.15) we have

\[
(k^t_{s-r_1})_i(x) - (k^t_{s-r_2})_i(x) = \frac{-\tan r_1 + (\kappa^t_s)_i(y)}{1 + \tan r_1 (\kappa^t_s)_i(y)} - \frac{-\tan r_2 + (\kappa^t_s)_i(y)}{1 + \tan r_2 (\kappa^t_s)_i(y)}
\]

\[
= \frac{(\tan r_2 - \tan r_1) \cdot (1 + (\kappa^t_s)_i(y))}{(1 + \tan r_1 (\kappa^t_s)_i(y)) \cdot (1 + \tan r_2 (\kappa^t_s)_i(y))}.
\]

Notice also that by Theorem 1.1(1), \( (\kappa^t_s)_i(y) \) is a fixed number and is bounded as in (4.10). Thus when \( r_1, r_2 \) are close enough, we see that \( (\kappa^t_{s-r})_i(x) \) is monotone decreasing, it follows that (4.17) exists. \qed

Now we are in the position to generalize the viscosity mean curvature of a set of finite perimeter which was first introduced in [DM19] from Euclidean space to \((S^{n+1}, g_{S^{n+1}})\), it is well-defined by Proposition 4.1(2)(4).

**Definition 4.1.** For a set of finite perimeter \( \Omega \) in \( S^{n+1} \), the viscosity boundary of \( \Omega \) is defined as

\[
\partial^v\Omega = \bigcup_{s>0} g_s(\Gamma^+_s)
\]

and the corresponding viscosity mean curvature of \( \Omega \) is defined by

\[
H^n_{\Omega} = \sum_{i=1}^{n} \kappa_i(x), \quad x \in \partial^v\Omega.
\]

Here \( g_s \) is given in the proof of Theorem 1.1(3), \( \Gamma^+_s \) is defined in Proposition 3.1 and \( \kappa_i \) is defined in Proposition 4.1(4).

As a by-product of the rectifiability result(Theorem 1.1), we derive the following Heintze-Karcher inequality for sets of finite perimeter on \((S^{n+1}, g_{S^{n+1}})\). This is enlightened by [Bre13, Theorem 3.5] and [DM19, Theorem 8].

**Theorem 4.1** (Heintze-Karcher inequality for sets of finite perimeter on Sphere). If \( \Omega \subset (S^{n+1}, g_{S^{n+1}}) \) is an open set of finite perimeter lies in a hemisphere which is mean convex in the viscosity sense as in Definition 4.1, then for every \( 0 < s < \frac{\pi}{2} \),

\[
\int_{s}^{\frac{\pi}{2}} \cos s \mathcal{H}^n(\partial \Omega_s) ds \leq \frac{n}{n+1} \int_{\Gamma^+_s} \frac{\cos s}{H^\Omega_s} d\mathcal{H}^n,
\]  

(4.18)

**Proof.** We define for \( 0 < s < \frac{\pi}{2} \),

\[
Q(s) = \int_{\Gamma^+_s} \frac{\cos s}{H^\Omega_s} d\mathcal{H}^n.
\]  

(4.19)
Notice that by monotonicity of $\kappa_{s-r}$, the viscosity mean convexity of $\Omega$ implies $H_{\Omega_s} > 0$ on $\Gamma_s^+$, for each $s \in (0, \frac{\pi}{2})$. With this observation, for every $s < t < \frac{\pi}{2}$, we define $Q^t : (0, t) \to (0, \infty)$ by setting
\[
Q^t(s) = \int_{\Gamma_s^+} \frac{\cos s}{H_{\Omega_s}} \mathrm{d}\mathcal{H}^n. \tag{4.20}
\]
Observe that by definition,
\[
Q(s) \geq Q^t(s) \geq Q^{s+t}(s) \quad \text{for all } t > s, \epsilon > 0, \tag{4.21}
\]
otice also that $\mathcal{H}^n(\Gamma_s^+)$ converges monotonically to $\mathcal{H}^n(\Gamma_s^+)$ as $t \to s^+$ by virtue of the inclusion Proposition 3.1 (2). This implies
\[
Q(s) = \lim_{t \to s^+} Q^t(s) = \sup_{t > s} Q^t(s) \quad \text{for all } 0 < s < \frac{\pi}{2}. \tag{4.22}
\]
For $r \in (0, s)$, by Proposition 4.1 (3), we have
\[
Q^t(s-r) - Q^t(s) = \int_{\Gamma_s^+} \frac{\cos(s-r)}{H_{\Omega_{s-r}}} \mathrm{d}\mathcal{H}^n - \int_{\Gamma_s^+} \frac{\cos s}{H_{\Omega_s}} \mathrm{d}\mathcal{H}^n
\]
\[
= \int_{\Gamma_s^+} \left\{ \frac{\cos(s-r) \prod_{i=1}^n \cos r + \sin r(\kappa_{s-r})_i}{\prod_{i=1}^n \cos r + \sin r(\kappa_{s-r})_i} - \frac{\cos s}{H_{\Omega_s}} \right\} \mathrm{d}\mathcal{H}^n, \tag{4.23}
\]
where
\[
\prod_{i=1}^n (-\sin r + \cos r(\kappa_{s-r})_i)/(\cos r + \sin r(\kappa_{s-r})_i)
\]
\[
= \prod_{i=1}^n (-\sin r + \cos r(\kappa_{s-r})_i) \prod_{i \neq 1} (\cos r + \sin r(\kappa_{s-r})_i)
\]
\[
= \left\{ \cos^n r H_{\Omega_s} + \cos^{n-1} r \sin r \left( H_{\Omega_s}^2 - |A_{\Omega_s}|^2 \right) - n \sin r \cos^{n-1} r + O(\sin^2 r) \right\} \prod_{i=1}^n (\cos r + \sin r(\kappa_{s-r})_i). \tag{4.24}
\]
Thus (4.23) reads
\[
Q^t(s-r) - Q^t(s) = \int_{\Gamma_s^+} \left\{ \frac{\cos(s-r)}{H_{\Omega_s}} \left( \prod_{i=1}^n \cos r + \sin r(\kappa_{s-r})_i \right)^2 - \frac{\cos s}{H_{\Omega_s}} \right\} \mathrm{d}\mathcal{H}^n
\]
\[
= \int_{\Gamma_s^+} \left\{ \frac{\cos(s-r) \cos^{2n-1} r \sin r \left( H_{\Omega_s}^2 - |A_{\Omega_s}|^2 \right) - n \sin r \cos^{n-1} r + O(\sin^2 r)}{H_{\Omega_s}} \right\} \mathrm{d}\mathcal{H}^n, \tag{4.25}
\]
Notice that
\[
\cos(s-r) \cos r = \cos s + \sin(s-r) \sin r,
\]
and hence we have
\[
Q^t(s-r) - Q^t(s)
\]
\[
= \int_{\Gamma_s^+} \left\{ \frac{\cos^{2n-1} r \cos s + \cos^{2n-2} r \sin s \sin r + 2 \cos^{2n-2} r \cos s \sin r H_{\Omega_s} + O(\sin^2 r)}{\cos^n r H_{\Omega_s} + \cos^{n-1} r \sin r \left( H_{\Omega_s}^2 - |A_{\Omega_s}|^2 \right) - n \sin r \cos^{n-1} r + O(\sin^2 r) \right\} - \frac{\cos s}{H_{\Omega_s}} \right\} \mathrm{d}\mathcal{H}^n,
\]
where \( O_t(\sin^2 r)/r \to 0 \) uniformly on \( \Gamma_s^t \) as \( r \to 0 \). We thus find \( Q' \) is differentiable on \((0, t)\) with

\[
(Q')'(s) = \lim_{r \to 0} \frac{Q^t(s - r) - Q^t(s)}{r} = -\int_{\Gamma_s^t} \cos s \left( 1 + \frac{|A_{\Omega_s}|^2}{H_{\Omega_s}^2} \right) d\mathcal{H}^n - \int_{\Gamma_s^t} \frac{n \cos s + H_{\Omega_s} \sin s}{H_{\Omega_s}^2} d\mathcal{H}^n. \tag{4.26}
\]

Notice that by Theorem 1.1 (1), \((\kappa_i^s)'_i \geq -\cot s\), which implies \( H_{\Omega_s} \sin s + n \cos s \geq 0 \). Also, by the Schwarz inequality we have \( H_{\Omega_s}^2 \leq n|A_{\Omega_s}|^2 \), these facts imply

\[
(Q')'(s) \leq -\frac{n + 1}{n} \int_{\Gamma_s^t} \cos s d\mathcal{H}^n. \tag{4.27}
\]

For \( 0 < s_1 < s_2 < \frac{\pi}{2} \), by (4.22), (4.21) and (4.26) respectively, we find

\[
Q(s_1) - Q(s_2) = \lim_{\epsilon \to 0^+} Q^{s_1+\epsilon}(s_1) - Q^{s_2+\epsilon}(s_2) \geq \lim_{\epsilon \to 0^+} Q^{s_2+\epsilon}(s_1) - Q^{s_2+\epsilon}(s_2) = Q^{s_2}(s_1) - Q^{s_2}(s_2)
\]

\[
\geq \frac{n + 1}{n} \int_{s_1}^{s_2} \left( \int_{\Gamma_s^t} \cos s d\mathcal{H}^n \right) ds = \frac{n + 1}{n} \int_{s_1}^{s_2} \cos s d\mathcal{H}^n(\Gamma_s^t) ds, \tag{4.28}
\]

in particular, \( Q \) is decreasing on \((0, \frac{\pi}{2})\) and \( Q' \) exists for a.e. \( s \) by monotonicity. Using area formula, by virtue of Proposition 4.1 (3), we have

\[
\mathcal{H}^n(\Gamma_s^t) = \int_{\Gamma_s^t} \prod_{i=1}^{n} \left[ \cos r + \sin r(\kappa_i^s) \right] d\mathcal{H}^n, \tag{4.29}
\]

where \( \left[ \cos r + \sin r(\kappa_i^s) \right] \to 1 \) uniformly on \( \Gamma_s^t \) as \( r \to 0 \) by virtue of the fact that \(-\cot s \leq (\kappa_i^s)_i \leq \cot (t - s)\), for each \( i \in \{1, \ldots, n\} \). In particular, this shows \( \mathcal{H}^n(\Gamma_s^t) \) is continuous on \( s \in (0, t) \), and the mean value property yields

\[
\int_{s_1}^{s_2} \mathcal{H}^n(\Gamma_s^t) ds = (s_2 - s_1) \mathcal{H}^n(\Gamma_{s_0}^t), \tag{4.30}
\]

for some \( s_0 \in (s_1, s_2) \).

On the other hand, letting \( r = t - s \) in (4.12), we find

\[
\mathcal{H}^n(\partial \Omega_t) \leq \{ \cot (t - s) + \cot s \sin (t - s) \}^n \mathcal{H}^n(\Gamma_s^t), \tag{4.31}
\]

and it follows that

\[
\mathcal{H}^n(\partial \Omega_{s_2}) \leq \liminf_{s \to (s_2)^-} (\cos (s_2 - s) + \cot s_2 \cdot \sin (s_2 - s))^n \mathcal{H}^n(\Gamma_s^{s_2}) \leq \liminf_{s \to (s_2)^-} \mathcal{H}^n(\Gamma_s^{s_2}). \tag{4.32}
\]

From this observation we obtain

\[
\liminf_{s_1 \to (s_2)^-} \frac{1}{s_2 - s_1} \int_{s_1}^{s_2} \cos s d\mathcal{H}^n(\Gamma_s^{s_2}) ds \geq \cos s d\mathcal{H}^n(\partial \Omega_{s_2}) \quad \text{for all } 0 < s_2 < \frac{\pi}{2}. \tag{4.33}
\]

Thus we conclude from (4.28) that

\[
-Q'(s) \geq \frac{n + 1}{n} \cos s d\mathcal{H}^n(\partial \Omega_s) \quad \text{for a.e. } s > 0. \tag{4.34}
\]

Finally, integrating this over \((s, \frac{\pi}{2})\) \(^1\) we obtain the Heintze-Karcher inequality (4.18). This completes the proof.

\(^1\)Since \( \Omega \) lies in a hemisphere, we know that \( Q(\frac{\pi}{2}) = 0 \).
5. Critical points of the isoperimetric problem for \((S^{n+1}, g_{S^{n+1}})\)

In this section, as a special case, we study the stationary points of the isoperimetric problem on \((S^{n+1}, g_{S^{n+1}})\) among sets of finite perimeter.

First we list and prove some properties of critical points of the isoperimetric problem for \((S^{n+1}, g_{S^{n+1}})\), the Euclidean case can be found in [DM19, Section 2.4].

We say that a set of finite perimeter \(E \subset (S^{n+1}, g_{S^{n+1}})\) is a critical point for the isoperimetric problem if

\[
\frac{d}{dt} \mid_{t=0} P_g(\phi_t(E)) = 0, \tag{5.1}
\]

for any one-parameter family of diffeomorphisms \(\{\phi_t\}_{t<1}\) with \(\phi_0 = Id, |\phi_t(E)|_g = |E|_g\) and \(\text{spt}(\phi_t - Id) \subset S^{n+1}\) for every small \(t\). By [Vol10, Proposition 4.10], we see that there exists a constant \(H \in \mathbb{R}\) such that

\[
\int_{S^{n+1}} \text{div}^g \phi^E X d\mu_{E,g} = H \int_{S^{n+1}} g(X, \nu_{E,g}) d\mu_{E,g}, \quad \forall X \in \Gamma^1_c(TS^{n+1}), \tag{5.2}
\]

Here \(\text{div}^g \phi^E X\) denotes the tangential divergence of \(X\) with respect to the reduced boundary \(\partial^* E\) on \((S^{n+1}, g_{S^{n+1}})\).

**Proposition 5.1.** If \(E \subset (S^{n+1}, g_{S^{n+1}})\) is a critical point for the isoperimetric problem, then up to a measure zero modification, \(E\) is an open set of finite perimeter with \(\partial E = \text{spt} \mu_{E,g}\) and

\[
\mathcal{H}_g^0(\partial E \setminus \partial^* E) = 0.
\]

Moreover,

\[
\partial^* E = f^{-1}\left( \left\{ x \in \partial E : \lim_{\rho \to 0^+} \mathcal{H}^n(B_\rho(x) \cap \partial E) = 0 \right\} \right)
\]

is locally an analytic hypersurface with constant mean curvature relatively open in \(\partial E\).

**Proof.** We embed \((S^{n+1}, g_{S^{n+1}})\) into the Euclidean space \((\mathbb{R}^{n+2}, g_{\text{euc}})\) by \(f\). First, we prove that \(f(\partial^* E)\) is a \(n\)-rectifiable varifold in \((\mathbb{R}^{n+2}, g_{\text{euc}})\) with constant generalized mean curvature \(\sqrt{1 + n^2}\). In the following, we use \(\nabla\) and \(\langle \cdot, \cdot \rangle\) to denote the gradient and the inner product in \((\mathbb{R}^{n+2}, g_{\text{euc}})\), respectively.

By Section 2 (v.) (vi.), we know that \(\mu_{E,g} = \mathcal{H}_g^n \llcorner \partial^* E, f(\mu_{E,g}) = \mathcal{H}^n \llcorner f(\partial^* E)\), since the isometrically embedding map \(f\) is just the inclusion map, we identify \(f(\partial^* E)\) with \(\partial^* E\), \(f(\mu_{E,g})\) with \(\mu_{E,g}\) and \(f_*(\nu_{E,g})\) with \(\nu_{E,g}\). In \(\mathbb{R}^{n+2}\), for any \(X \in C^1_c(\mathbb{R}^{n+2}; \mathbb{R}^{n+2})\), we have

\[
\int_{\partial^* E} \text{div} \phi^E X d\mathcal{H}^n = \int_{\partial^* E} \text{div} \phi^E (X^T + X^\perp) d\mathcal{H}^n,
\]

here \(X^T, X^\perp\) denote the tangential part and the normal part with respect to \(\partial^* E\) in \(\mathbb{R}^{n+2}\), respectively. By [Vol10, Proposition 2.51(ii)] and (5.2), we have \(\mathcal{H}^n \llcorner \partial^* E = \mathcal{H}_g^n \llcorner \partial^* E\), and

\[
\int_{\partial^* E} \text{div} \phi^E X d\mathcal{H}^n = H \int_{\partial^* E} \langle X, \nu_{E,g} \rangle d\mathcal{H}^n + \int_{\partial^* E} \text{div} \phi^E X^\perp(y) d\mathcal{H}^n(y). \tag{5.3}
\]
Let \( \{ \tau_1, \ldots, \tau_n \} \) denote the orthonormal basis for the approximate tangent space of \( \partial^* E \) at \( y \), notice that \( \nu_{S^{n+1}}(y) = y \), we have

\[
\int_{\partial^* E} \text{div} \tau_y^* E X\,d\mathcal{H}^n(y) = \sum_{i=1}^n \int_{\partial^* E} \langle \nabla \tau_i (\langle X(y), y \rangle \), \tau_i \rangle \,d\mathcal{H}^n(y) = \sum_{i=1}^n \int_{\partial^* E} \langle X(y), y \rangle \langle \nabla \tau_i y, \tau_i \rangle \,d\mathcal{H}^n(y) = \int_{\partial^* E} \langle X(y), ny \rangle \,d\mathcal{H}^n(y),
\]

where in the second equality we use the fact that \( y = \nu_{S^{n+1}}(y) \perp \tau_i(y) \) for each \( i \); in the last equality we use the fact that \( \nabla \tau_i y = \tau_i(y) \).

Back to (5.3), we have

\[
\int_{\partial^* E} \text{div} \tau_y^* E X\,d\mathcal{H}^n = \int_{\partial^* E} \langle X, H\nu_{E, g}(y) + ny \rangle \,d\mathcal{H}^n(y),
\]

set \( \tilde{\nu}(y) = \frac{\nu_{E, g}(y) + ny}{|\nu_{E, g}(y) + ny|} \), we see that

\[
\int_{\partial^* E} \text{div} \tau_y^* E X\,d\mathcal{H}^n = \sqrt{H^2 + n^2} \int_{\partial^* E} \langle X, \tilde{\nu} \rangle \,d\mathcal{H}^n. \tag{5.4}
\]

Combining with Section 2 (v.), we deduce that \( \partial^* E \) is a \( n \)-rectifiable varifold with a constant generalized mean curvature vector in \((\mathbb{R}^{n+2}, g_{\text{euc}})\).

Using the well-known monotonicity formula for \( n \)-rectifiable varifold with bounded generalized mean curvature in \((\mathbb{R}^{n+2}, g_{\text{euc}}) ([\text{Sim83}, \text{Theorem 17.6}] \), we have that for any \( x \in \mathbb{R}^{n+2} \),

\[
e^{\sqrt{H^2 + n^2}} \frac{\mathcal{H}^n(B_{\rho}(x) \cap \partial^* E)}{\rho^n} \text{ is increasing on } \rho > 0. \tag{5.5}
\]

The monotonicity formula (5.5) together with the definition of the approximate tangent space \([\text{Mag12, Theorem 10.2, (10.7)}] \) implies that

\[
\mathcal{H}^n(\text{spt} \mu_{E, g} \setminus \partial^* E) = 0,
\]

see for example, \([\text{DM19, (2.21), (2.22)}] \). Consequently, on \((S^{n+1}, g_{S^{n+1}}) \), we have

\[
\mathcal{H}^n(\text{spt} \mu_{E, g} \setminus \partial^* E) = 0. \tag{5.6}
\]

Moreover, if we restrict ourselves to \((S^{n+1}, g_{S^{n+1}}) \), we can follow the proof in [DM19, Lemma 5] to find an open set \( E_1 \subset (S^{n+1}, g_{S^{n+1}}) \) such that

\[
|(E \setminus E_1) \cup (E_1 \setminus E)|_g = 0, \quad \partial E_1 = \text{spt} \mu_{E_1, g}. \tag{5.7}
\]

Indeed, \( E_1 \) is taken to be the set of \( x \in (S^{n+1}, g_{S^{n+1}}) \) such that \( |E \cap B_{\rho}(x)|_g = |B_{\rho}(x)|_g \) for every \( \rho \) small enough. We thus find the desired set \( E_1 \) to replace \( E \).

Finally, by applying the Allard’s regularity theorem to the \( n \)-rectifiable varifold \( \partial^* E \) in \( \mathbb{R}^{n+2} \), we see that \( \text{spt} \mu_{E, g} \) is locally an analytic hypersurface with constant mean curvature, which combined with the measure zero modification (5.7) shows that \( \partial^* E \) is locally an analytic hypersurface with constant mean curvature. This completes the proof. \( \square \)
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*School of Mathematical Sciences, Xiamen University, 361005, Xiamen, P.R. China*

*Email address: xuwenzhang97@gmail.com*