Conservation laws and symmetries of a generalized Kawahara equation

Maria Luz Gandarias\textsuperscript{1,a)}, Maria Rosa\textsuperscript{1,b)}, Elena Recio\textsuperscript{1,c)} and Stephen Anco\textsuperscript{2,d)}

\textsuperscript{1}Departamento de Matematicas, Universidad de Cadiz, Spain
\textsuperscript{2}Department of Mathematics and Statistics, Brock University, Canada

\textsuperscript{a)}Corresponding author: marialuz.gandarias@uca.es
\textsuperscript{b)}maria.rosa@uca.es
\textsuperscript{c)}elena.recio@uca.es
\textsuperscript{d)}sanco@brocku.ca

Abstract. The generalized Kawahara equation \( u_t = a(t)u_{xxxx} + b(t)u_{xxx} + c(t)f(u)u_x \) appears in many physical applications. A complete classification of low-order conservation laws and point symmetries is obtained for this equation, which includes as a special case the usual Kawahara equation \( u_t = \alpha uu_x + \beta u^2u_x + \gamma u_{xxx} + \mu u_{xxxx} \). A general connection between conservation laws and symmetries for the generalized Kawahara equation is derived through the Hamiltonian structure of this equation and its relationship to Noether’s theorem using a potential formulation.

Introduction

Dispersive wave equations arise in many areas of applied mathematics and physics. One very important equation is the Korteweg-de Vries (KdV) equation \( u_t = \alpha uu_x + \gamma u_{xxx} \) which models shallow water waves and is a bi-Hamiltonian integrable system. The dispersion is caused by the \( u_{xxx} \) term, while the other term \( uu_x \) describes nonlinear advection. A related equation is the modified KdV equation \( u_t = \beta u^2u_x + \gamma u_{xxx} \) which has the same dispersion but a stronger nonlinearity. It is also a bi-Hamiltonian integrable system and models acoustic waves in anharmonic lattices \cite{13} and Alfvén waves in collision-free plasmas \cite{8}.

An interesting nonlinear wave equation that exhibits weaker dispersion than occurs for the KdV and modified KdV equations is the Kawahara equation

\[ u_t = \alpha uu_x + \beta u^2u_x + \gamma u_{xxx} + \mu u_{xxxx}. \] (1)

This is a fifth-order dispersive nonlinear wave equation. It has a Hamiltonian structure, but in contrast to the KdV equation, it is not an integrable system.

The Kawahara equation has several important physical applications. It models plasma waves \cite{9,13} and capillary-gravity water waves \cite{7}. Kawahara \cite{10} studied this type of equation numerically and observed that it possesses both oscillatory and monotone solitary wave solutions. Other basic aspects of the Kawahara equation are its symmetry structure and its set of conservation laws. Symmetries and conservation laws of particular cases of the Kawahara equation have been studied previously \cite{11,6}, but a complete classification of remains open.

In the present paper, we consider a generalization of the Kawahara equation given by

\[ u_t = a(t)u_{xxxx} + b(t)u_{xxx} + c(t)f(u)u_x \] (2)

where \( f(u) \) is an arbitrary non-constant function, and \( a(t), b(t), c(t) \) are arbitrary non-zero functions (which are allowed to be constant). This generalization is able to model nonlinear weakly dispersive phenomena that involve a time-dependent coefficient. The Kawahara equation is a special case given by \( a = \mu, b = \gamma, c = 1, f = \alpha u + \beta u^2 \).

Our main results will be to provide a complete classification of all point symmetries and all low-order conservation laws admitted by the generalized Kawahara equation (2) in the case when \( f \) is a nonlinear function. We will also
discuss the Hamiltonian structure of this equation, which provides a connection between the conservation laws and the symmetries.

Note that, under a point transformation on \( t \), all of the coefficients \( a(t), b(t), c(t) \) in equation (2) can be multiplied by an arbitrary function of \( t \). Consequently, hereafter we will assume

\[
a(t) = 1, \quad b(t) \neq 0, \quad c(t) \neq 0, \quad f'(u) \neq 0.
\]

**Symmetries**

Symmetries are a basic structure as they can be used to find invariant solutions and yield transformations that map the set of solutions \( u(t, x) \) into itself. A general discussion of symmetries and their applications to differential equations can be found in Refs. [12, 5].

An infinitesimal point symmetry for the generalized Kawahara equation (2) is a generator

\[
X = \xi(x, t, u) \partial_x + \tau(x, t, u) \partial_t + \eta(x, t, u) \partial_u
\]

whose prolongation leaves invariant the equation (2). Every point symmetry can be expressed in an equivalent, characteristic form

\[
\dot{X} = P \partial_u, \quad P = \eta(x, t, u) - \xi(x, t, u) u_x - \tau(x, t, u) u_t
\]

which acts only on \( u \). Invariance of the equation (2) is given by the condition

\[
0 = D_t P - a(t) D_x^2 P - b(t) D_x P - c(t) D_u (f(u) P)
\]

holding for all solutions \( u(t, x) \) of equation (2). This condition (6) splits with respect to \( x \)-derivatives of \( u \), which yields a overdetermined system of equations on \( \eta(x, t, u), \xi(x, t, u), \tau(x, t, u) \), along with \( a(t), b(t), c(t), f(u) \), subject to the classification conditions (5). It is straightforward to set up and solve this determining system by Maple.

We obtain the following results. The point symmetries admitted by the generalized Kawahara equation (2) in the general case (3) are generated by

\[
\tau = 0, \quad \xi = 1, \quad \eta = 0.
\]

All special cases for which additional point symmetries are admitted consist of:

\[
\tau = a t + \beta, \quad \xi = \frac{1}{2} a x, \quad \eta = 0
\]

\[
f(u) \text{ arbitrary,} \quad b(t) = (a t + \beta)^{1/2}, \quad c(t) = \gamma (a t + \beta)^{-1/2}; \quad (8a)
\]

\[
\tau = a t + \beta, \quad \xi = \frac{1}{3} a x - \delta f_0 \int c(t) dt, \quad \eta = - (a \delta / f_3) (u + f_2)
\]

\[
f(u) = f_1 (u + f_2) f_1, \quad b(t) = (a t + \beta)^{-1/2}, \quad c(t) = \gamma (a t + \beta)^{1/2}, \quad f_1, f_3 \neq 0; \quad (9a)
\]

\[
\tau = 0, \quad \xi = \int c(t) dt, \quad \eta = -1 / f_1
\]

\[
f(u) = f_1 u + f_0, \quad b(t) \text{ arbitrary,} \quad c(t) \text{ arbitrary,} \quad f_1 \neq 0; \quad (10a)
\]

\[
\tau = d(t) / D(t)^2, \quad \xi = (1/2 - \delta + \delta \beta / D(t)) x, \quad \eta = (a \beta \delta / (\gamma^3 f_1)) x - (1/2 - \delta \beta / D(t)) (u + f_0 / f_1)
\]

\[
f(u) = f_1 u + f_0, \quad b(t) = \gamma^2 d(t)^{-1/2} D(t)^2, \quad c(t) = \gamma^3 d(t)^{d-1} D(t)^3, \quad f_1 \neq 0,
\]

\[
D(t) = a d(t)^{\delta} + \beta, \quad d'(t) = D(t)^{\gamma}, \quad \gamma, \delta \neq 0; \quad (11a)
\]

\[
\tau = 0, \quad \xi = \int c(t) dt, \quad \eta = -(1 / f_1) (u + f_2)
\]

\[
f(u) = f_1 \ln(u + f_2) + f_0, \quad b(t) \text{ arbitrary,} \quad c(t) \text{ arbitrary,} \quad f_1 \neq 0. \quad (12a)
\]
These symmetries can be interpreted by considering the forms of generators for space translations $X_{\text{space}} = \partial_x$, time translations $X_{\text{time}} = \partial_t$, shifts $X_{\text{shift}} = -\partial_x$, scalings $X_{\text{scal}} = pt\partial_t + q\partial_x + ru\partial_u$ where $p, q, r$ are the weights, time-dependent dilations $X_{\text{dil}} = q(t)\partial_t + p(t)\partial_x + r(t)u\partial_u$, and Galilean boosts $X_{\text{Gal}} = (\int v(t)\,dt)\partial_x$ where $v(t)$ is the relative speed.

In particular, symmetry (7) represents a space-translation; symmetry (8a) represents a time-translation ($\beta$) combined with a scaling ($\alpha$); symmetry (9a) represents a time-translation ($\beta$) combined with the composition of a scaling ($\alpha$), a shift ($\delta\alpha f_2/f_3$), and a Galilean boost ($-\delta\gamma f_0$) with relative speed $c(t)$; symmetry (10a) represents a shift ($1/f_1$) combined with a Galilean boost with relative speed $c(t)$; symmetry (12a) represents a shift ($f_2/f_1$) combined with a scaling and a Galilean boost with relative speed $c(t)$; symmetry (11a) represents the composition of an $x$-dependent shift and a time-dependent dilation.

**Conservation laws**

Conservation laws are of basic importance because they provide physical, conserved quantities for all solutions $u(x,t)$, and they can be used to check the accuracy of numerical solution methods. A general discussion of conservation laws and their applications to differential equations can be found in Refs. [12, 5].

A local conservation law for the generalized Kawahara equation (2) is a continuity equation

$$D_t T + D_x X = 0$$

(13)

holding for all solutions $u(x,t)$ of equation (2), where the conserved density $T$ and the spatial flux $X$ are functions of $t$, $x$, $u$, and $x$-derivatives of $u$ (with $t$-derivatives of $u$ being eliminated through the equation (2)). If $T = D_x \Theta$ and $X = -D_t \Theta$ hold for all solutions $u(x,t)$, where $\Theta$ is some function of $t$, $x$, $u$, and $x$-derivatives of $u$, then the continuity equation (13) becomes an identity. Conservation laws of this form are called locally trivial, and two conservation laws are considered to be locally equivalent if they differ by a locally trivial conservation law. The global form of a non-trivial conservation law is given by

$$\frac{d}{dt} \int_{\Omega} T\,dx = -X|_{\partial \Omega}$$

(14)

where $\Omega \subseteq \mathbb{R}$ is any fixed spatial domain.

Every local conservation law can be expressed in an equivalent, characteristic form (analogous to the evolutionary form for symmetries) [12] which is given by a divergence identity

$$D_t \tilde{T} + D_x \tilde{X} = (u_x - a(t)u_{xxxx} - b(t)u_{xxx} - c(t)f(u)u_x)Q$$

(15)

holding off of the set of solutions of the generalized Kawahara equation (2), where $\tilde{T} = T + D_x \Theta$ and $\tilde{X} = X - D_x \Theta$ are a conserved density and spatial flux that are locally equivalent to $T$ and $X$, and where $Q = E_u(\tilde{T})$

(16)

is a function of $t$, $x$, $u$, and $x$-derivatives of $u$. This function is a called a multiplier [12, 1, 5]. Here $E_u$ denotes the Euler operator with respect to $u$ [12].

For evolution equations, there is a one-to-one correspondence between non-zero multipliers and non-trivial conservation laws up to local equivalence [12, 2]. and the conservation laws of basic physical interest arise from multipliers of low order [3]

$$Q(t,x,u,u_x,u_{xx},u_{xxx},u_{xxxx}).$$

(17)

Such multipliers correspond to conserved densities of the form $T(t,x,u,u_x,u_{xx})$ modulo a trivial conserved density. A function (17) will be a multiplier if $E_u((u_x - a(t)u_{xxxx} - b(t)u_{xxx} - c(t)f(u)u_x)Q) = 0$ holds identically. This condition splits with respects to the $x$-derivatives of $u$ that do not appear in $Q$. The resulting overdetermined system consists of the adjoint of the symmetry determining equation (6)

$$0 = -D_t Q + a(t)D_x^3 P + b(t)D_x^2 P + c(t)f(u)D_x Q$$

(18)

holding for all solutions $u(x,t)$ of equation (2), plus the Helmholtz equations [2, 3]

$$Q_u = E_u(Q), \quad Q_{u_x} = -E_u^{(1)}(Q), \quad Q_{u_{xx}} = E_u^{(2)}(Q), \quad Q_{u_{xxx}} = -E_u^{(3)}(Q)$$

(19)
which are necessary and sufficient for $Q$ to have the variational form \((16)\). Here $E_r^{(1)}$, and so on, denote the higher Euler operators with respect to a variable $v^{(R)}$.

It is straightforward using Maple to set up and solve this determining system \((18)-(19)\) for $Q(t, x, u, u_t, u_{xx}, u_{xxx}, u_{xxxx})$ along with $a(t)$, $b(t)$, $c(t)$, $f(u)$, subject to the classification conditions \((3)\).

For each solution $Q$, a corresponding conserved density $T$ and spatial flux $X$ can be derived (up to local equivalence) by integration of the divergence identity \((15)\). We obtain the following results.

The multipliers and conserved densities admitted by the generalized Kawahara equation \((2)\) in the general case \((4)\) are linear combinations of

\[
Q = 1, \quad T = u; \quad Q = u, \quad T = \frac{1}{2}u^2.
\]

All special cases for which additional multipliers and conserved densities are admitted consist of:

\[
Q = u_{xxxx} + au_{xx} + \beta \int f(u) \, du, \quad T = \frac{1}{2}u_{xx}^2 - \frac{1}{2}au_x^2 + \beta(u \int f(u) \, du - \int uf(u) \, du) \tag{22a}
\]

\[
f(u) \text{ arbitrary}, \quad b(t) = \alpha, \quad c(t) = \beta; \quad (22b)
\]

\[
Q = (at + \beta)u_{xxxx} + (at + \beta)^2u_{xx} + \frac{1}{4}ax(u + f_2) + (\gamma/(f_3 + 1))(at + \beta)^4(u + f_2)f(u),
\]

\[
T = \frac{1}{2}(at + \beta)u_{xx}^2 - \frac{1}{2}(at + \beta)^2u_x^2 + \frac{1}{10}ax(u + f_2)^2 + (\gamma/(f_3 + 1))(at + \beta)^4(u + f_2)f(u) \tag{23b}
\]

\[
f(u) = f_1(u + f_2)^\gamma + f_0, \quad b(t) = (at + \beta)^\gamma, \quad c(t) = \gamma(at + \beta)^{4\gamma - 4}, \quad f_4 \neq 0; \tag{23c}
\]

The physical meaning of these conservation laws can be seen by considering their global form \((14)\).

For general $f(u)$, the three admitted conservation laws \((20), (21), (22a)\) respectively yield the conserved integrals

\[
C_1 = \int \Omega u \, dx, \quad C_2 = \int \Omega u^2 \, dx, \quad C_3 = \int \Omega \left(\frac{1}{2}u_x^2 - \frac{1}{2}au_x^2 + \beta(u \int f(u) \, du - \int uf(u) \, du)\right) \, dx \tag{26}
\]

These represent the mass, the $L^2$-norm, and the gradient-energy for solutions $u(x, t)$. When $f(u)$ has a power-law form \((21)\), the conservation law \((22a)\) yields the conserved integral

\[
C_4 = \int \Omega \left(\frac{1}{2}(at + \beta)u_{xx}^2 - \frac{1}{2}(at + \beta)^2u_x^2 + \frac{1}{10}ax(u + f_2)^3 + (\gamma/(f_3 + 1))(at + \beta)^4(u + f_2)f(u) \, du\right) \, dx \tag{27}
\]

which represents a dilatational Galilean energy for solutions $u(x, t)$. When $f(u)$ is a linear polynomial \((24b)\) and \((25b)\), the two admitted conservation laws \((24a)\) and \((25b)\) yield the respective conserved integrals

\[
C_5 = \int \Omega \left(\frac{1}{2}C(t)(f_2u^2 + 2fu) + xu\right) \, dx \tag{28}
\]
which represents a Galilean momentum, and

\[
C_6 = \int_{\Omega} \left( \frac{25}{2} \gamma f_1 (d(t)u^2_x - d(t)\phi_x^2 + \frac{1}{2} x D(t)u^2) + \frac{1}{2} \alpha x^2 u + \frac{1}{2} a_0 (\int f(t) dt) u \right) dx,
\]

which represents a generalized dilational Galilean energy-momentum, where \( C(t) = \int f(t) dt \).

These interpretations are reinforced by the Hamiltonian symmetries associated to the conserved integrals, as discussed in the next section.

**Connection between conservation laws and symmetries**

The generalized Kawahara equation (2) has a Hamiltonian structure, on any fixed spatial domain \( \Omega \subseteq \mathbb{R} \), which is given by

\[
u_t = \mathcal{H}(\delta H/\delta u), \quad H = \int_{\Omega} \left( \frac{1}{2} (a(t)u_x^2 + b(t)\phi_x^2) - c(t)F(u) \right) dx, \quad F = \int \int f(\theta) du \, du = u \int f(\theta) du - \int uf \, du
\]

where the Hamiltonian operator (12) is a total \( x \)-derivative

\[
\mathcal{H} = D_x.
\]

This Hamiltonian structure yields a corresponding Lagrangian

\[
L = \frac{1}{2} (-\nu_x v_x + a(t) v_x^2) - b(t) F(v_x), \quad \nu = v_x
\]

when a potential is introduced. In potential form, the generalized Kawahara equation (2) is an Euler-Lagrange equation

\[
E_u(L) = \nu_x - a(t) u_{xxx} - b(t) F(v_x), \quad u_t = a(t) u_{xxx} - b(t) u_x - c(t) f(u) u_x.
\]

For any Hamiltonian evolution equation, there is a correspondence (12) that produces a symmetry from each admitted conservation law. This correspondence is a Hamiltonian analog of Noether’s theorem. It can be formulated for the generalized Kawahara equation (2) by the explicit relation

\[
P = \mathcal{H}(\delta C/\delta u) = D_x Q
\]

involving the characteristic function \( P \) of the symmetry generator \( \hat{X} = P \partial_u \) and the multiplier \( Q \) associated to the conserved integral \( C = \int_{\Omega} T \, dx \) given by a local conservation law (13). This correspondence is one way: every conservation law yields a symmetry. The converse holds if the symmetry has the Hamiltonian form (34), which requires that \( E_u(P) = 0 \).

The same correspondence (34) can be derived from Noether’s theorem applied to the Lagrangian for the generalized Kawahara equation in potential form (33) as follows. Every local conservation law (13) admitted by the generalized Kawahara equation arises from a multiplier \( Q \). When the multiplier is expressed in terms of the potential, \( Q \big|_{\nu = v_x} = Q^\nu \), it yields a local conservation law for the equation in potential form. From Noether’s theorem, there is one-to-one correspondence between the local conservation laws (up to equivalence) and the variational symmetries of this equation (33). This correspondence is explicitly given by \( P^\nu = Q^\nu \), where \( P^\nu \) is the characteristic function of the variational symmetry \( \hat{Y}^\nu = P^\nu \partial_u \) and \( Q^\nu \) is multiplier of the conservation law. Through the relation \( u = v_x \), the prolongation of the variational symmetry \( P^\nu = \hat{Y} + D_x P^\nu \partial_u \) yields a symmetry of the generalized Kawahara equation itself (2), as given by \( P \big|_{\nu = v_x} = D_x P^\nu \). This symmetry relation combined with \( P^\nu = Q^\nu \big|_{\nu = v_x} = Q^\nu \) is precisely the correspondence (34).

The set of Hamiltonian symmetries

\[
\hat{X}_{\text{Ham}} = D_x Q \partial_u,
\]

or equivalently the set of variational symmetries

\[
\hat{Y}_{\text{var}} = Q \big|_{\nu = v_x} \partial_v,
\]

will be a Lie subalgebra of the Lie algebra of symmetries admitted by the generalized Kawahara equation (2). From the conserved integrals (26–29), we find that \( C_2 \) produces a space-translation, \( C_3 \) produces a time-translation, \( C_4 \) produces a scaling combined with a Galilean boost, \( C_5 \) produces a shift combined with a Galilean boost, and \( C_6 \) produces a Galilean boost combined with an \( x \)-dependent shift and a time-dependent dilation.
Concluding remarks

We have presented a complete classification of all low-order conservation laws and all point symmetries admitted by the generalized Kawahara equation (2). This classification includes as a special case the usual Kawahara equation (1).

The symmetries can be used to obtain exact group-invariant solutions, while the conservation laws can be used to investigate these solutions as well as to study the initial-value problem.

We have also explained a general connection between conservation laws and symmetries for the generalized Kawahara equation through the Hamiltonian structure of this equation and its relationship to Noether’s theorem using a potential formulation.

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