Moduli Space of Global Symmetry in $N = 1$ Supersymmetric Theories and the Quasi-Nambu-Goldstone Bosons

Muneto Nitta

Department of Physics, Graduate School of Science, Osaka University, Toyonaka, Osaka 560-0043, Japan and
Theory Division, Institute of Particle and Nuclear Studies, KEK, Tsukuba, Ibaraki 305-0801, Japan

Abstract

We derive the moduli space for the global symmetry in $N = 1$ supersymmetric theories. We show, at the generic points, that it coincides with the space of quasi-Nambu-Goldstone (QNG) bosons, which appear besides the ordinary Nambu-Goldstone (NG) bosons when the global symmetry $G$ breaks down spontaneously to its subgroup $H$ with preserving $N = 1$ supersymmetry. At the singular points, most of the NG bosons change to the QNG bosons and the unbroken global symmetry is enhanced. The $G$-orbits parametrized by the NG bosons are the fibre at the moduli space and the singular points correspond to the point where the $H$-orbit (in the $G$-orbit) shrinks. We also show that the low-energy effective Lagrangian is an arbitrary function of the moduli parameters.

* muneto.nitta@kek.jp, nitta@phys.wani.osaka-u.ac.jp.
1 Introduction

In this paper, we investigate the relation between two elements: one is the moduli space of global symmetry in the supersymmetric (gauge) theories; the other is the low-energy effective Lagrangian described by the supersymmetric nonlinear sigma model with the Kähler target manifold parametrized by the chiral Nambu-Goldstone (NG) superfields. The moduli space of gauge symmetry is well understood in terms of the Kähler quotient space \([1]\) or the algebraic variety \([2, 3]\). The zeros of the scalar potential is made by the F-flat condition and the D-flat condition. Since the gauge symmetry \(G\) is enhanced to its complexification \(G^C\), it is easy to deal with. However, in the case of global symmetry \(G\), although the F-term symmetry is enhanced to its complexification by the analyticity of the superpotential, the D-term symmetry is not enhanced to its complexification, since it includes both chiral and anti-chiral superfields. Therefore, the moduli space of global symmetry is obtained by the set of F-flat points divided by the symmetry \(G\), but not its complexification, since there is no D-flat condition. Since it is not well understood, we investigate it in this paper. The F-term zeros is just the \(G^C\)-orbit of the vacuum. In the case of gauge symmetry, since the D-flat points constitute the one \(G\)-orbit in the F-flat points, the moduli space of gauge symmetry is parametrized by \(G^C\)-invariant polynomials. In the case of global symmetry, the \(G\)-invariant, but not \(G^C\)-invariant, polynomials parametrize the moduli space.

On the other hand, the F-term zeros is well known in the context of the low-energy effective Lagrangian. When global symmetry breaks down spontaneously to its subgroup, there appear quasi Nambu-Goldstone (QNG) bosons besides the ordinary Nambu-Goldstone (NG) bosons as massless bosons. They constitute massless NG chiral superfields with QNG fermions (their fermion partners). After integrating out the massive mode, the low-energy effective Lagrangian is obtained as a supersymmetric nonlinear sigma model with the Kähler target manifold \([1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14]\). (For a review see \([15, 16, 26]\).) Since the target space is the F-term zeros, it is a Kähler coset manifold, where \(G^C\) acts transitively, but not isometrically, and \(G\) acts isometrically, but not transitively. The general Kähler potential with \(G\) symmetry has been obtained by Bando, Kuramoto, Maskawa and
Uehara (BKMU) in Ref. [3]. It has not been known up to recently that low energy
theorems exist [13, 14], since the Kähler potential includes an arbitrary function of
$G$-invariants. However, since it has not been known how many variables are included
in it, in general, except for few examples [9], we determine the number of variables
in the arbitrary function of the effective Kähler potential. The number of QNG
bosons changes, even if we consider one theory, since there exist supersymmetric
vacuum alignment [17, 9]. Although it is known that there must exist at least one
QNG boson [3, 7, 9], the minimum number of QNG bosons has not been known,
except for few examples [9]. We determine the range of the number of QNG bosons
and show that the minimum number of QNG bosons coincides with the number of
variables in the arbitrary function and the dimension of the moduli space of global
symmetry.

We investigate the moduli space of global symmetry in two ways: one involves
algebraic-geometrical methods, such as the algebraic variety; the other involves the
differential-geometrical or group-theoretical methods, such as the Kähler coset man-
ifolds. The former is used mainly to calculate the dimension of the moduli space;
the latter is used to investigate more complicated structures of the moduli space.

This paper is organized as follows. In the next section, we discuss the general
aspects of moduli space in both the gauge and global symmetry. We found, in the
case of global symmetry, that the moduli space is the quotient space, the set of F-
flat points divided by the symmetry. Since the set of F-flat points is the $G_C$-orbit,
known as the Kähler coset manifold, we review only its basic aspects.

In Sec. 3, we investigate the detailed structure of moduli space in the algebraic-
geometrical and the differential-geometrical ways, and show the relation to the low-
energy effective Lagrangian describing the behavior of the NG and QNG bosons.
In Sec. 3.1, we show that the moduli space of global symmetry is parametrized by
moduli parameters, which are $G$-invariants, but not $G_C$-invariants. We decompose
the moduli space into several regions with isomorphic unbroken symmetries. Sec. 3.2
is devoted to an investigation of the low-energy effective Lagrangian. We show that
it can be written as an arbitrary function of the moduli parameters, and that it is
equivalent to the known Kähler potential [3, 9] by identifying the moduli parameters
constrained by the F-term and the representative of Kähler coset manifold. (Thus
the number of variables is the dimension of moduli space.)

In Sec. 3.3, we investigate the structure of the Kähler coset manifold in detail. The main results of this paper are obtained in this subsection. We generalize the observation by Kotcheff and Shore [8, 9] that the non-compact directions belonging to the same $H$-irreducible sector are not independent. It is shown that the number of $G$-invariants agrees with the number of $H$-irreducible sectors of the mixed-type complex broken generators, but in general it does not coincide with the dimension of moduli space. We also show that the dimension of each region of moduli space is the number of $H$-singlet sectors. We prove that, in a generic region, all $H$-irreducible sectors become singlet, and that their number is just the dimension of moduli space. We also show that the singular points in moduli space correspond to the points where the $H$-orbit shrinks in the target manifold. Sec. 3.4 is devoted to a calculation of the dimension of moduli space and the number of QNG bosons.

In Sec. 4, we give some examples and demonstrate the theorems obtained in Sec. 3. We investigate $O(N)$ and $SU(N)$ with fundamental (and anti-fundamental) matters and $SU(N)$ ($N = 2, 3$) with adjoint matters. In Sec. 5, we make some comments concerning the gauging of global symmetry. Sec. 6 is devoted to conclusions.

2 Moduli space

In this paper, we assume that the vacua are transformed by some symmetry $G$. Suppose that there is a gauge and global symmetry, $G = G_{gauge} \times G_{global}$. Although it does not have to be direct product, for simplicity we suppose it here. The fundamental field $\vec{\phi}$ is in some representation space, $V = C^N$ of $G$. The scalar potential is

$$V(\phi, \phi^*) = \frac{1}{2}(\vec{\phi}^T T_A \vec{\phi})^2 + |W'|^2,$$  (2.1)

where $T_A$ is the generator of the gauge group. The first term comes from the $D$-term of the gauge symmetry and the second term from the F-term. The D-flat condition is necessary and sufficient condition that the length of the vector $\vec{\phi}$, $|\vec{\phi}|$, is minimum [18, 19]. Since the moduli parameters are the freedom that remains after bringing the fields $\vec{\phi}$ to some constant configuration by using all of the gauge and
the global symmetry, the moduli space is written as \[\mathcal{M} = \{\vec{\phi} \in \mathbb{C}^N \mid \mathcal{V}(\phi, \phi^*) = 0\}/G\]

\[= \{\vec{\phi} \mid W'' = 0, |\vec{\phi}|^2 = \text{min.}\}/G\]

\[= \{\vec{\phi} \mid W'' = 0\}/G_{\text{gauge}} \times G_{\text{global}}, \quad (2.2)\]

where we have used the fact (see [26])

\[\{\cdot \mid \text{D-flat cond.}\}/G_{\text{gauge}} = \{\cdot \}/G_{\text{gauge}}^{\text{C}}, \quad (2.3)\]

which is a result of the Higgs mechanism. The D-flat condition corresponds to \(G_{\text{gauge}}^{\text{C}}/G_{\text{gauge}}\), which is also equivalent to taking the Wess-Zumino gauge [26, 2].

Here, we consider the two extreme cases. First of all, consider the case without global symmetry, \(G = G_{\text{gauge}}\). In this case, the moduli space is

\[\mathcal{M} = \{\vec{\phi} \mid W'' = 0\}/G_{\text{gauge}}^{\text{C}}. \quad (2.4)\]

This is known as the Kähler quotient [1], and can be understood as the algebraic variety [2, 3]. It is parametrized by the \(G_{\text{gauge}}^{\text{C}}\)-invariant polynomials in \(V\), which are elements of the ring of the invariant polynomials, \(A_{\text{gauge}}^{\text{C}}[V]\). However for the case where the zeros of the potential can be transformed by the symmetries, it becomes trivial, \(\mathcal{M} \simeq \{1\}\), since there is no global symmetry. \footnote{There is another definition of the moduli space. Sometimes the moduli space is defined without the global symmetry since the quotient by the global symmetry is not easy. In such a definition, the NG bosons are included in the moduli space.}

We thus consider another extreme case, where there is no gauge symmetry, \(G = G_{\text{global}}\). In this paper we investigate this case in detail. (If there is a gauge symmetry, we consider them as global symmetry for a while, and then gauge them after finding the moduli space. This point of view is discussed in Sec. 5.) In this case, the moduli space is

\[\mathcal{M} = \{\vec{\phi} \mid W'' = 0\}/G. \quad (2.5)\]

\footnote{In the another definition of the moduli space, Eq. (2.4) is obtained as the moduli space when there is also global symmetry or not supposing that the vacua should be transformed by the symmetry. See Ref. [2, 3].}
The target manifold $M$ is defined as the zeros of F-term potential.

The zeros of the F-term potential \( M = \{ \vec{\phi} | W' = 0 \} \), is obtained when all $G^C$-invariant polynomials are fixed (see Fig. 1). Since we consider the case when any vacuum is transformed by some (complexified) symmetry, $G^C$ acts on $M$ transitively, so that $M$ can be written in complex coset space as

\[ M \simeq G^C / \hat{H}, \quad (2.6) \]

where $\hat{H}$ is the complex isotropy group at the vacuum, $\vec{v} = < \vec{\phi} >$, namely

\[ \hat{H}_v = \{ g \in G^C | g \cdot \vec{v} = \vec{v} \}. \quad (2.7) \]

(We have omitted $v$ in Eq. (2.6), since $G^C$ is transitive and each $\hat{H}_v$ is isomorphic.) Here, $\hat{H}$ includes $H^C$, $\hat{H} \supset H^C$, but need not agree with it \([5,19]\). At special points in $M$, it can be decomposed as

\[ \hat{H} = H^C \oplus \mathcal{B}, \quad (2.8) \]

where $\mathcal{B}$ is the nilpotent Lie algebra, which is written in the non-Hermitian step generators, namely, the lower-half triangle matrices in the suitable basis \([3]\) (see Sec. 4.2).

We always omit the subscript $v$ on $H$ at such points. When $\mathcal{B}$ is absent, $\hat{H}$ is called reductive. $\mathcal{B}$ is determined completely by the representations to which the vacuum

---

3 Since we are using the effective Lagrangian approach, we discuss the superpotential $W$ with quantum corrections, if any.
vectors belong. $M$ is the target space of the sigma model parametrized by the NG and QNG bosons [1, 2, 3, 4, 5, 6, 7, 8, 9, 13, 14]. The moduli space is written as

$$\mathcal{M} = (G^C/\hat{H})/G .$$

(2.9)

This is a quotient space, but not a Kähler quotient space (for a review of quotient space, see Ref. [1]). Naively, the moduli space is parametrized by the QNG bosons, since the NG bosons correspond to compact directions generated by the compact isometry group $G$.

In this paper we consider the generic $G^C$-orbit which has the maximal dimension. (Generalization to the singular $G^C$-orbits, which has fewer dimensions, is straightforward.) The number $N_\Phi$ of the massless NG chiral multiplets parametrizing $M$ is (see Fig. 1, see also for example [3])

$$N_\Phi = \dim_G(G^C/\hat{H}) = \dim_G V - N(G^C) ,$$

(2.10)

where $N(G^C)$ is the number of $G^C$-invariant polynomials, namely,

$$N(G^C) \overset{\text{def}}{=} \dim_G A^{G^C}[V] .$$

(2.11)

There are two types of the chiral NG multiplets [5, 6]: One is called the pure type (or non-doubled type), including two NG bosons in the scalar component. Another is called the mixed type (or doubled type), including the QNG boson besides the ordinary NG boson. The mixed types correspond to Hermitian generators, whereas the pure types correspond to non-Hermitian generators. (We can always obtain ordinary Hermitian generators by suitable complex linear combinations of the pure-type broken generators and the Borel-type unbroken generators $B$ in the complex unbroken algebra $\hat{H}$.) $N_\Phi$ can be written as

$$N_\Phi = N_M + N_P ,$$

(2.12)

where $N_M$ and $N_P$ are the number of the mixed-type and the pure-type multiplets.

In general, $N_M$ and $N_P$ can change at various points in target space $M$, with the total numbers being conserved. This is because there is a vacuum alignment [17, 9].

When the vacuum $\vec{v}$ is transformed by $g_0 \in G^C$ to $\vec{v}' = g_0 \vec{v}$, the complex isotropy

\footnote{We use symbols ‘dim’ for a real dimension and ‘dim$C$’ for a complex dimension.}
The large circle indicates the group $G$. The small circles denote the complex subgroups $\hat{H}$ and $\hat{H}'$. $\hat{H}'$ is the transform of $\hat{H}$ by $g_0$. The real subgroups $H$ or $H'$ are defined as intersections of $G$ and $\hat{H}$ or $\hat{H}'$. $K$ is the image of $H$ by the $g_0$ transformation. In general $H'$ is a subset of $K$.

group $\hat{H}$ is transformed to $g_0\hat{H}g_0^{-1}$. They are isomorphic to each other. On the other hand, the real isotropy group $H_v$ are not isomorphic to each other, since they are obtained by the equation

$$H_v = \hat{H}_v \cap G.$$  

(2.13)

Namely, the real isotropy $H$ at $v$ is transformed to $K = g_0Hg_0^{-1}$, but it is no longer included in $G$ (see Fig. 2). Since the NG bosons parametrize the compact coset manifold $G/H_v$, their number changes at each point. This means that $N_M$ and $N_P$ change. In fact, the Hermiticity of the broken generators changes at each point. It is shown in Ref. [14] how different compact cosets are embedded in the full manifold $M$. It is also shown that NG bosons are actually coupled to global $G$-currents there.

The number of the QNG bosons $N_Q (= N_M)$ is

$$N_Q = \dim(G^C/\hat{H}) - \dim(G/H_v).$$  

(2.14)

This number depends on the vacuum $v$ in the target space, since the dimension of $H_v$ also depends on. Its range is shown in Sec. 3.4.

It has been shown in Refs. [18, 19] that there is a point such that $\hat{H}$ is reductive, namely $\hat{H} = H^C$, $\dim B = 0$ and $N_P = 0$, when the $G^C$-orbit is a closed set. We call such a point a symmetric point [13, 14]. When all of the NG bosons belong to

\footnote{If the $G^C$-orbit is not closed, we call the point where the unbroken symmetry is the largest}
the mixed type, the realization is called the maximal realization.

## 3 Moduli space of global symmetry

### 3.1 Algebraic geometry of invariants

As the quotient space divided by $G^\mathbb{C}$ is parametrized by $G^\mathbb{C}$-invariant polynomials in $A^G[V]$, the quotient space divided by $G$ is parametrized by $G$-invariant polynomials in $A^G[V]$. Here, $A^G[V]$ is the ring of the $G$-invariant (but not $G^\mathbb{C}$-invariant) polynomials. We thus seek $G$-invariants composed by fundamental fields. We denote the fundamental fields which belong to unitary representations $(\rho_i, V_i)$ as $\vec{\phi}_i$. Their transformation laws under $G^\mathbb{C}$ are

$$\vec{\phi}_i \xrightarrow{g} \rho_i(g)\vec{\phi}_i, \quad \vec{\phi}_i \xrightarrow{g} \vec{\phi}_i^\dagger \rho_i(g)^\dagger, \quad g \in G^\mathbb{C},$$

where $\rho_i(g)$ are unitary matrices when $g \in G$, whereas they are not unitary matrices when $g \in G^\mathbb{C}$. Thus the moduli parameters can be chosen as

$$\theta_i = \vec{\phi}_i^\dagger \vec{\phi}_i \in \mathbb{R}, \quad \theta_{ij} = \vec{\phi}_i^\dagger \vec{\phi}_j \in \mathbb{C}.\quad (3.2)$$

They are $G$-invariant, since we use the unitary representation. Note that they are not $G^\mathbb{C}$-invariant, since $\rho_i(g)^\dagger \rho_i(g) \neq 1$ for $g \in G^\mathbb{C}$. The second invariants are possible when $\vec{\phi}_i$ and $\vec{\phi}_j$ are in the same representation of $G$. Instead of complex numbers, we call real combinations $\theta_{ij} + \theta_{ji}^\dagger, -i(\theta_{ij} - \theta_{ji})$ moduli parameters in the rest of this paper. The moduli parameters can be considered as being coordinates of the moduli space. We define the number of the $G$-invariant polynomials as $N(G)$, where

$$N(G) \overset{\text{def}}{=} \dim A^G[V],$$

and we count them in the real dimension. Since the values of the moduli parameters are constant on each $G$-orbit, the moduli parameters can be considered to be a map from $G$-orbits to the moduli space $\mathcal{M}$, namely

$$\pi : M \to \mathcal{M} = M/G.$$  \quad (3.4)
This kind of map is called an orbit map. (The orbit map in the type of $V/G$ has been discussed in Ref. [20].) Conversely, each $G$-orbit is obtained by the inverse map $\pi^{-1}(p)$ from each point $p$ in the moduli space. A $G$-orbit is parametrized by $\text{NG}$ bosons and is a coset space, $G/H_v$. In the generic region of the moduli space, a $G$-orbit has the maximal dimension,

$$\dim(G/H_g) = \dim M - N_g(G),$$

where the index $g$ denotes the generic points in the moduli space. At the singular points where the orbit shrinks, $\dim(G/H)$ takes smaller values than Eq. (3.5).

In general, the moduli parameters cannot take all values in $\mathbb{R}^{\text{NG}}$. The moduli space $\mathcal{M}$ is a subset of $\mathbb{R}^{\text{NG}}$, $\mathcal{M} \subset \mathbb{R}^{\text{NG}}$, characterized by some inequalities between the moduli parameters such as

$$r_C(\theta_i, \theta_{i^*j}) \geq 0, \ C = 1, 2, \cdots,$$

and the moduli space can be written as

$$\mathcal{M} = \{(\theta_i, \theta_{i^*j}) \in \mathbb{R}^{\text{NG}} | r_C(\theta_i, \theta_{i^*j}) \geq 0\}.$$

Although we do not give a general expression for these relations, we give some examples in the later section.

We decompose the moduli space $\mathcal{M}$ to some regions $\mathcal{M}_R$ as

$$\mathcal{M} = \bigcup_R \mathcal{M}_R.$$  

Here, the label $R$ runs over I, II, $\cdots$. Each region is defined so that the unbroken symmetry at any point is isomorphic to each other by the $G^C$ transformation, namely $H_p = g_0 H_q g_0^{-1}, g_0 \in G^C$ for any two points $p, q \in \mathcal{M}_R$. Note that they are not isomorphic to each other by $G$-action. We denote the conjugacy class of the real isotropy groups in region $R$ as

$$H_{(R)} = \{H_p | p \in \mathcal{M}_R\}.$$  

In general, the dimensions of the various regions of the moduli space are different. (We describe a way to calculate them in Sec. 3.3.) Thus the dimension of the moduli space is

$$\dim \mathcal{M} = \sup_R \dim \mathcal{M}_R.$$

9
The dimension of the moduli space is given in Sec. 3.4.

3.2 Effective Lagrangian

The leading term of the effective Lagrangian is a nonlinear sigma model whose target manifold is a Kähler manifold, and can be written by a Kähler potential as

\[ \mathcal{L}_{\text{eff.}} = \int d^4\theta K(\Phi, \Phi^\dagger). \] (3.11)

The most general \( G \)-invariant Kähler potential is written by an arbitrary function of the moduli parameters as

\[ K = f(\theta_i, \theta^*_j). \] (3.12)

Since the function \( f \) is a real function, the second type of variables appear in the form \( \theta_{i,j} + \theta_{j,i}^* \) and \(-i(\theta_{i,j} - \theta_{j,i}^*)\).

Since the target space \( M \) is a single \( G^C \)-orbit, any point in \( M \) is obtained by a \( G^C \)-action from the vacuum \( \vec{v} \). Hence, there is a remarkable relation between the fundamental fields \( \vec{\phi}_i \) and the vacuum \( \vec{v}_i \) as

\[ \vec{\phi}_i | F = \xi \vec{v}_i. \] (3.13)

Here, the subscript \( F \) means solutions of F-term constraints (corresponding to fixing the \( G^C \)-invariants) and \( \xi \) is the representative of the complex coset \( G^C/\hat{H} \), written as

\[ \xi = \exp(i\Phi^R Z_R), \] (3.14)

where \( Z_R \in G^C - \hat{H} \) are the complex broken generators and \( \Phi^R \) are the NG chiral superfields, whose scalar component parametrize the Kähler coset manifold, \( G^C/\hat{H} \).

From Eq. (3.13), the moduli parameters can be written as

\[ \theta_i = \frac{\vec{\phi}_i}{\vec{\phi}_i | F} = \vec{v}_i \xi^\dagger \xi \vec{v}_i, \quad \theta_{i,j} = \frac{\vec{\phi}_i \vec{\phi}_j}{\vec{\phi}_i | F} = \vec{v}_i \xi^\dagger \xi \vec{v}_j, \] (3.15)

where \( \vec{v}_i \) satisfy the F-term constraints: \( W'(\vec{v}_i) = 0 \). We thus find that this Kähler potential Eq. (3.12) is just the BKMU’s A-type Lagrangian \( \mathcal{L}_{\text{eff.}} \) (see also Ref. \( \mathcal{L}_{\text{eff.}} \)).
If we choose the arbitrary function $f$ linear, $f(\theta_i, \theta_j, \cdots) = a\theta_i + b\theta_j + \cdots$, the space where fundamental fields $\vec{\phi}_i$ live is a flat linear space. On the other hand, if we use variables of type $\theta_{i+j}$ or choose the arbitrary function $f$ nonlinear, the space where fundamental fields $\vec{\phi}_i$ live is no longer flat linear space.

It was not known how many variables the arbitrary function of effective Kähler potential contains. In our formalism, it is clear how many variables it contains; we show in a later section that the number coincides with the minimum number of QNG bosons and the dimension of moduli space.

It is known that the QNG bosons correspond to the non-compact directions of the target manifold \[8, 9, 13, 14\]. This is true even at points where the number of QNG bosons changes \[14\]. Since the symmetry of the theory is compact group $G$, but not $G^C$, it cannot control the non-compact directions. This is why the QNG bosons bring arbitrariness to the Kähler potential of the effective Lagrangian \[8, 9, 13, 14\]. In this paper it will become clear that the target space is the fibre bundle on the moduli space with the $G$-orbits as the fibre. The arbitrary function can be interpreted as the freedom to change the size of the $G$-orbit at each point of the moduli space, since the derivatives of the arbitrary function is related to the decay constants of the NG bosons which parametrize the $G$-orbit \[8, 9, 13, 14\].

### 3.3 Geometry of Kähler coset manifolds

In this subsection, we derive the moduli space by investigating the Kähler coset manifolds in detail. The target space of broken global symmetry, the Kähler coset manifold $M$, is a non-compact and non-homogeneous manifold. The compact isome-

\[8\] BMKU found three types of Kähler potential \[3\]. In Eq. (3.12), we have constructed the Kähler potential to be strictly invariant under $G$-transformations. If we require a Kähler potential that is not strictly $G$-invariant, but $G$-invariant up to a Kähler transformation, there may exist a Kähler potential corresponding to the BMKU’s B-type Kähler potential. Since it has been known that the B-type appears in a pure realization \[13\] (the case when there exist only pure-type multiplets and there is no QNG boson), we think that the B-type can appear when there is any pure-type multiplet. We do not know if it can appear when there exists the vacuum alignment and some pure-type multiplets turn to mixed-type multiplets. We do not obtain the C-type Kähler potential in our method.
try group $G$ connects points in compact directions. Two points apart to non-compact
directions are connected by $G^C$, but not $G$. Since the moduli space $\mathcal{M}$ is defined by
$M/G$, only non-compact directions remain in $\mathcal{M}$. The target manifold $M$ is spanned
by the broken generators, which can be decomposed as a direct sum of $H$-irreducible
sectors. In this subsection, we show that each irreducible sector comprising mixed-
type generators corresponds to an independent non-compact direction.

First of all, consider the vacuum $\vec{v}$ belonging to the $R$-th region of the moduli
space $\mathcal{M}_R$, namely $\pi(\vec{v}) \in \mathcal{M}_R$, and transform it to another vacuum, $\vec{v}' = g_0 \vec{v}$, where
$g_0$ is the element of $G^C$. To derive the moduli space, we need the $G^C$-transformation
modulo $G$-transformation. The element of $g_0 \in G^C$ is divided into

$$g_0 = \exp(i\theta^R Z_R|_M) \cdot \exp(i\theta^R Z_R|_P) \cdot \hat{h},$$

where $Z_R|_M$ are mixed-type generators and thus Hermitian; $Z_R|_P$ are pure-type
generators, and thus non-Hermitian; and $\hat{h}$ is an element of $\hat{H}_v$. The last two components transform $\vec{v}$ to points in the same $G$-orbit, since the last element does not move $\vec{v}$ and the second term can be absorbed by the some local $\hat{H}$ transformation as $\exp(i\theta^R Z_R|_P) \cdot \zeta = \exp(i \alpha^i X_i)$, where $\alpha^i$ are real and $X_i$ are some broken Hermitian generators. (Here, $\alpha^i$ can be obtained by using the Baker-Campbell-Hausdorff formula. See Ref. [L4].) Therefore, we can omit them to obtain the moduli space.

Since mixed-type generators are Hermitian, they also belong to $G - \mathcal{H}$ and are di-
vided into $n_R$ $H$-irreducible sectors, since $h(G - \mathcal{H}) h^{-1} \subset G - \mathcal{H}$. Let the number of broken generators in the $i$-th sector be $m_i$ ($i = 1, \cdots, n_R$), then the complex broken generators are divided into

$$G^C - \hat{\mathcal{H}} = \{\{Z_1^{(1)}, \cdots, Z_{m_1}^{(1)}\}_M, \cdots, \{Z_1^{(n_R)}, \cdots, Z_{m_n}^{(n_R)}\}_M\}||P\text{-type},$$

and each sector is transformed by $h \in H_v$, as

$$h Z_R^{(i)} h^{-1} = Z_S^{(i)} \rho_i(h) S_R,$$

where $\rho_i(h) S_R$ are $m_i$-by-$m_i$ representation matrices. For later convenience, we write these sectors as

$$m_{1M} \oplus \cdots \oplus m_{nM} \oplus m_{1P} \oplus \cdots,$$

Kotcheff and Shore discussed the special case of the maximal realization at the symmetric
point [9] without any discussion about the moduli space.

\[12\]
where the index M (P) denotes the mixed- (pure-) type sectors, and all components are $m_i$-dimensional irreducible representations of the unbroken symmetry, $H_v$.

The independent transformations to non-compact directions, first element of Eq. (3.16), are at most

$$
\vec{v}' = \exp(i\theta^{R^i} Z^{(i)} R) \cdot \vec{v} ,
$$

(3.20)

where all parameters $\theta^{R^i}$ are pure imaginary. This does not yet represent the independent non-compact directions. Since the new vacuum $\vec{v}'$ do not preserve the unbroken symmetry $H_v$ at $\vec{v}$, it is transformed by $H_v$. The $H_v$-orbit of the $\vec{v}'$ is

$$
h \cdot \vec{v}' = [h \exp(i\theta^{R^i} Z^{(i)} R) h^{-1}] h \vec{v}
= \exp[i\theta^{R^i} (h Z^{(i)} R h^{-1})] \vec{v}
= \exp(i\theta^{R^i} \rho_i(h) Z^{(i)} S) \vec{v}
= \exp(i\theta^{R^i} Z^{(i)} R) \cdot \vec{v} ,
$$

(3.21)

where we have used Eq. (3.20) and defined

$$
\theta^{R^i} \overset{\text{def}}{=} \rho_{i} (h) R^{i} S^{i}.
$$

(3.22)

Since we have assumed that $Z^{(i)} R$ belong to the same $H_v$-irreducible sector, $H_v$ acts transitively on the space of vacuum of the form of Eq. (3.20).

In the generic region, each $G$-orbit has the maximal dimension. Thus independent non-compact directions are parametrized by $G$-invariants. Therefore, the number of independent non-compact directions equals to the number of $G$-invariants, $N_g(G)$. Since this is also calculated as the number of $H$-irreducible sectors of mixed-types, $n_g$, we obtain a theorem concerning the number of independent non-compact directions in the generic region,

**theorem 1.** $N_g(G) = n_g$.

(3.23)

Note that the left-hand side is an algebraic geometrical quantity, whereas the right-hand side is a group-theoretical quantity.

Although the theorem is valid in the generic region, it seems to also be valid in any region as

**conjecture.** $N_R(G) = n_R$.

(3.24)
Here, the label $R$ denotes the regions of the moduli space defined in Eq. (3.8). Although we do not know any proof of this conjecture, we show that this is correct in many examples in the next section. We have stated that $N_R(G)$ changes where some of the complex moduli parameters become real. In the last example in the next section, this indeed occurs, and $n_R$ also changes accordingly; we thus believe that the conjecture is true in any region.

In general, non-compact directions do not commute, namely $g_0g_1 \neq g_1g_0$, even if they are independent. There are commuting directions and non-commuting directions.

First of all, let us discuss the case when the generators in different sectors commute. We transform $\vec{v}$ to $\vec{v}' = g_0\vec{v}$ by $g_0 = \exp(i\theta^R Z^{(i)}_R)$ in an $i$-th $H_{v}$-sector. We take $\theta^R$ to be pure imaginary, $\theta^R = i\tilde{\theta}^R$ ($\tilde{\theta}^R \in \mathbb{R}$). In this case, only one linear combination, $\tilde{\theta}^R Z^{(i)}_R$, of the transformed $i$-th broken generators, $g_0\{Z^{(i)}_1, \ldots, Z^{(i)}_{m_\nu}\}g_0^{-1}$ remains Hermitian, and the rest change to non-Hermitian, and therefore pure-type, generators. The other sectors remain unchanged (since we consider the case where the generator of $g_0$ commutes with them). Vacuum alignment can occur only in the $i$-th sector. There are $m_i$ NG and QNG bosons at the first vacuum $\vec{v}$, whereas there are the $2m_i - 1$ NG bosons and one QNG boson at the transformed vacuum $\vec{v}'$, as far as the massless bosons in the $i$-th sector are concerned. (Note that in a $H_{v}$-singlet sector with $m_i = 1$, the number of NG and QNG bosons does not change, and vacuum alignment does not occur.)

In the case when some sectors do not commute, since the transformation by the $i$-th sector induces a transformation in the other sectors not commuting with the $i$-th sector, the generators of such sectors change to non-Hermitian, thus pure-type, generators. Thus the vacuum alignments occur not only in the $i$-th sector, but also in the other sectors not commuting with the $i$-th sector.

When the vacuum $\vec{v}'$ is rotated from the symmetric points by all non-commuting mixed-type sectors with pure imaginary parameters, it belongs to the generic region $\mathcal{M}_g$ of the moduli space. The unbroken symmetry $H_{v'}$ becomes the minimum in the whole moduli space. In such a region, all of the $H_{v'}$-sectors are singlets, since the vacuum alignment to smaller unbroken symmetry cannot occur, by definition.
We can calculate the dimension of each region of the moduli space. Let us assume that the vacuum $\vec{v}$ belongs to the $R$-th region $M_R$. If we move the vacuum by the $H_v$-singlet sectors of the mixed-type broken generators, vacuum alignment does not occur. By using the $i$-th sector of $m_i = 1$, we transform the vacuum $\vec{v}$ to $\vec{v}' = g_0 \vec{v}$, where $g_0 = \exp(i\theta Z^{(i)})$. Since $Z^{(i)}$ is $H$-singlet, $H_{v'} = g_0 H_v g_0^{-1} = H_v$. Therefore, there is no vacuum alignment and the vacuum stays in the same region $M_R$.

If there are several $H_{(R)}$-singlets at a point in the $R$-th region, there are the same number of directions which bring the vacuum to another vacuum in that region of the moduli space, $M_R$. We thus obtain a theorem concerning the dimension of the $R$-th region of the moduli space,

\[ \text{theorem 2.} \quad \dim M_R = n_R(1_M), \]

where we have defined the number of $H_{(R)}$-singlet sectors in the $R$-th region $M_R$ as $n_R(1_M)$, and $1_M$ means the $H_{(R)}$-singlet of the mixed type. The number of $H_{(R)}$-singlets is less than the number of all mixed-type sectors $n_R$,

\[ n_R(1_M) \leq n_R. \]

The inequality is saturated in the generic region $M_g$ of moduli space, since all of the $H_{(g)}$-sectors of the mixed-type become singlets there, as discussed before. We thus obtain the following corollary:

\[ \text{corollary 1.} \quad \dim M_g = n_g(1_M) = n_g, \]

where the index $g$ means the generic region. Since the dimension of the generic region is the largest in all moduli space, it is also the dimension of the moduli space itself from Eq. (3.10).

We now comment on the geometry. The target space $M$ can be considered to be fibre bundle \cite{footnote}. The base space is the moduli space $M$, the fibre is the $G$-orbit, $G/H_v$, and the structure group is $G$. The projection is the orbit map $\pi$ in Eq. (3.4). The $G$-orbit shrinks somewhere on the moduli space. Consider some region $M_R$ of

---

\footnote{Exactly speaking, not the full space $M$, but the space on each region of the moduli space, $\pi^{-1}(M_R)$, can be considered as fibre bundle, since the fibre at each region has different dimension.}
moduli space and its boundary $\partial \mathcal{M}_R$ (if it exists). The boundary is another region $\mathcal{M}_{\partial R}$. In general, the unbroken symmetry, $H_{(\partial R)}$, in the boundary is larger than $H_{(R)}$ of the bulk:

$$H_{(\partial R)} \supset H_{(R)}. \quad (3.28)$$

$H_{(\partial R)}$ is broken down to $H_{(R)}$ in the bulk and constitutes the $H_{(\partial R)}$-orbit, $H_{(\partial R)}/H_{(R)}$, in the $G$-orbit. Conversely, it shrinks in the boundary $\mathcal{M}_{\partial R}$ and the unbroken symmetry, $H_{(R)}$, is enhanced to $H_{(\partial R)}$. This means that most of the NG bosons change to QNG bosons there. Namely, the NG-QNG change occurs. (See for example, Fig. 5.)

### 3.4 Dimension of moduli space

In this subsection, we collect the obtained results concerning to the dimension of the moduli space. From theorem 1 (Eq. (3.23)) and Eqs. (3.27) and (3.5), we obtain the following corollary for the dimension of the moduli space:

**corollary 2.** \[ \dim \mathcal{M} = \dim \mathcal{M}_g = n_g(1_M) = n_g \]

\[ = N_g(G) = \dim(G^C/\hat{H}) - \dim(G/H_{(g)}). \quad (3.29) \]

As shown in Sec. 3.2, this is also the number of variables in the arbitrary function in the effective Kähler potential. It can also be stated as the number of decay constants of NG bosons.\footnote{This phenomenon has been essentially discussed by Hull et al. in the third paper of Refs. [21].}

\footnote{The authors of [9] have stated that this coincides with rank $G - \text{rank } H$. (See Appendix B in the second reference in Ref [9].) But this is not correct in general. It is correct only in their example of the chiral symmetry breaking where rank $H$ is unchanged. (It is also true in the pure realization, which is the case when there is only the pure-type multiplets, and therefore $G^C/\hat{H} \simeq G/H$ [1]. In such cases, although there is no vacuum vector and they have no linear model origin [3, 5, 6], there is no $G$-invariant namely $N(G) = 0$, which coincides with rank $G - \text{rank } H = 0$. It is also true when some Cartan generators are broken from the pure realization [6].) To include other cases, even if we would modified it to rank $G - \text{rank } H_{(g)}$, it is correct only in the case where there is only one vacuum vector transforming in the irreducible representation (besides above cases). See examples in the next section.}
We have shown that the number of QNG bosons changes even if we consider one theory. It has been known that there must be at least one QNG boson in any low-energy theory with a fundamental theory origin \[6, 7, 9\]. However the minimum number of QNG bosons was not known. In this paper, we have found it. The range of the number of QNG bosons, \(N_Q\), is calculated from Eqs. (2.14), (3.29) and (2.8) as

\[
\dim(G^C/\hat{H}) - \dim(G/H(g)) \leq N_Q \leq \dim(G^C/\hat{H}) - \dim(G/H),
\]

\[
\dim(M) \leq N_Q \leq \dim(G/H) - \dim(B),
\]

(3.30)

The left-hand inequality is saturated at the generic points, where the QNG bosons are tangent vectors of the moduli space as

\[
\pi_*(\text{QNG}) = T_p \mathcal{M}, \quad p \in \mathcal{M}_g,
\]

(3.31)

where \(\pi_*\) is a differential map of orbit map (3.4); the right inequality, however, is saturated at the most singular points in the moduli space corresponding to the symmetric points in the target space. (For closed \(G^C\)-orbits, there is no Borel algebra \((B = 0)\) and the complex isotropy \(\hat{H}\) is reductive at symmetric points. The maximal realization occurs and there are equal numbers of QNG and NG bosons.)

4 Examples

In this section, we give some examples, demonstrate the theorems and corollaries obtained in the last section and confirm that the conjecture is correct. Although we treat only the group \(O(N)\) and \(SU(N)\), the extension to another group is straightforward. The first several subsections are devoted to fundamental representations, and the last two subsections to adjoint representations. Rapid readers should read only Sections 4.1, 4.3 and 4.4.

4.1 Example of \(\dim \mathcal{M} = 1\) (closed set)

Example 1) \(O(N)\) with \(\vec{\phi} \in \mathbb{N}\)

Since this is the simplest example, we investigate this in detail. The physical result, such as low-energy theorems, are considered in Ref. [14].
First of all, we define the generators of the group $O(N)$ as
\[(T_{ij})^k_l = \frac{1}{k} (\delta_i^k \delta_{jl} - \delta_j^k \delta_{il}). \quad (4.1)\]

If we consider only the real group $G = O(N)$, the fundamental fields $\vec{\phi}$ live in the real space $\mathbf{R}^N$, since it is a real representation. However, we need the complex extension of $G$, $G^C$, and they live in the complexified space $\mathbf{C}^N$. There is only one $G^C$-singlet, $\vec{\phi}^2$, namely $N(G^C) = 1$. We consider a $G$-invariant superpotential,
\[W = g\phi_0 (\vec{\phi}^2 - f^2), \quad (4.2)\]
where $\phi_0$ is a singlet of $G$ and a non-dynamical auxiliary field. Here, $f$ is a real nonzero constant.\(^{13}\) Actually, the superpotential $W$ is $G^C$-invariant, since it includes only chiral multiplets. If we take the limit $g \to \infty$ and decouple $\phi_0$, we obtain the F-term constraint,
\[\vec{\phi}^2 - f^2 = 0. \quad (4.3)\]
(We have obtained it from the equation of motion of $\phi_0$.) The obtained space is the $G^C$-orbit with the complex dimension $N_\Phi = \dim_{\mathbf{C}} V - N(G^C) = 2N - 1$. It is the target space of the effective nonlinear sigma model.

The vacuum $\vec{v} = < \vec{\phi} >$ at the target space can be transformed to
\[\vec{v} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ v \end{pmatrix} \quad (4.4)\]
by the $G^C$-action, where $v$ is a real positive constant equal to $f$: $v = f$. We call this point a symmetric point and this region is called region I. The unbroken symmetry

\(^{13}\) There are four kinds of $G^C$-orbits. They are classified by the value of the $G^C$-invariant polynomial $\vec{\phi}^2$, namely $f$. We now consider the case when $f$ is real and nonzero as shown in Fig. 3. There is another orbit that $f$ is pure imaginary and nonzero. Other two types are cases that $f$ is zero. One is the case when $\vec{v}$ itself is zero. Another is the case when $\vec{v} \neq 0$. Only the last one is an open set. We do not consider such orbits, since it has no real submanifold corresponding to the NG manifold without QNG bosons.
is $H = O(N - 1)$ and the real broken generators are

$$X_i = T_{Ni} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \in \mathcal{G} - \mathcal{H} \quad (i = 1, \ldots, N - 1). \quad (4.5)$$

Since the NG bosons are generated by these generators, the real target space is $G/H = O(N)/O(N - 1)$ embedded in the full target space $M$. The complexification does not change the situation and the complex broken and unbroken generators coincide with those of real generators:

$$Z_R = X_i \in \mathcal{G}^C - \mathcal{H} , \quad K_M = H_a \in \mathcal{H} . \quad (4.6)$$

Since they are Hermitian, the maximal realization occurs, namely $M \simeq G^C/\hat{H} = O(N)^C/O(N - 1)^C$. Thus the numbers of NG and QNG bosons are both $N - 1$. Note that the broken generators belong to a single representation $N - 1$ of $H = O(N - 1)$. We denote this as $(N - 1)_M$, where subscript $M$ denotes a mixed type (see the first line of Table 1). Thus the number of $H_{(1)}$-irreducible sectors of the mixed-type generators in the region I is $n_1 = 1$. Since there is no $H_{(1)}$-singlet sector, the dimension of region I is $\dim \mathcal{M}_I = 0$ from theorem 2. Since there is only one $G$-invariant $|\tilde{\phi}|^2$, we have also verified the conjecture in region I as $N_I(G) = n_1 = 1$.

We transform the vacuum to another one, which belong to region II, by $G^C$-action, as

$$\vec{v} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \xrightarrow{g_0 \in G^C} \vec{v}' = g_0 \vec{v} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \overset{\text{def}}{=} \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} , \quad (4.7)$$

where we have put $g_0$ as

$$g_0 = \exp(i\theta X_{N-1})$$

$$= \begin{pmatrix} 1 & 0 \\ \cos \theta & -\sin \theta \\ 0 & \sin \theta \cos \theta \end{pmatrix} \overset{\text{def}}{=} \begin{pmatrix} 1 & 0 \\ \cosh \theta & -i \sinh \theta \\ 0 & i \sinh \theta \cosh \theta \end{pmatrix}. \quad (4.8)$$
Since the other broken generators belong to the same $H_{(1)}$ representation, they do not induce the independent non-compact directions, as shown in Sec. 3.3. The real unbroken symmetry is $H' = O(N - 2)$ and the real broken generators are $X_i$ in Eq. (4.5) and

$$X_i' = T_{N-1,i'} = \begin{pmatrix} 0 & 0 \\ i & 0 \\ 0 & 0 \end{pmatrix} \quad (i' = 1, \cdots, N - 2). \quad (4.9)$$

The real target manifold is $G/H' = O(N)/O(N - 2)$, generated by $X_i$ and $X_i'$, which has more dimensions than $G/H$ (see Fig. 4).

The complex broken and unbroken generators at the non-symmetric points are obtained as

$$Z_{R}' = g_0 Z_R g_0^{-1} = \begin{cases} g_0 X_i g_0^{-1} = \frac{2}{\nu} X_i' + \frac{\beta}{\nu} X_i : \text{Pure-type} & \in \mathcal{G}^C - \hat{\mathcal{H}}' \ , \quad (4.10) \\ g_0 X_{N-1} g_0^{-1} = X_{N-1} : \text{Mixed-type} & \end{cases}$$

$$K_{M}' = g_0 K_M g_0^{-1} = \begin{cases} g_0 X_i' g_0^{-1} = \frac{\beta}{\nu} X_i' - \frac{\alpha}{\nu} X_i \in \tilde{\mathcal{B}}' & \in \hat{\mathcal{H}}' \ \ , \quad (4.11) \\ g_0 H_a' g_0^{-1} = H_a' \in \mathcal{H}' & \end{cases}$$

We write these as

$$\mathcal{G}^C - \hat{\mathcal{H}} = \begin{pmatrix} 0 & P \\ P & 0 \end{pmatrix} \quad , \quad \hat{\mathcal{H}} = \begin{pmatrix} \hat{\mathcal{H}}^C & \tilde{\mathcal{B}}' \\ \tilde{\mathcal{B}}' & 0 \end{pmatrix} \ , \quad (4.12)$$

where $P$ and $M$ denote the pure- and mixed-type generators and $\tilde{\mathcal{B}}'$ represents the non-Hermitian, but not Borel, generators. The unbroken generators also include Hermitian and non-Hermitian generators. The Hermitian generators coincide with real symmetry $\mathcal{H}'$, whereas non-Hermitian one concerns the newly emerged NG bosons. (It is shown in Ref. [14] how different compact manifolds are embedded
Figure 3: target space of the $O(N)$ model 1

Figure 4: target space of the $O(N)$ model 2
in the full manifold.) At the non-symmetric point, only one of complex broken generators, $X_{N-1}$, is Hermitian, and thus a mixed-type generator, on the other hand, since the others are non-Hermitian, and they are the pure-type generators. Thus, the numbers of the NG chiral multiplets are $N_M = 1$ and $N_P = N - 2$. The emergence of pure-type generators is the result of QNG-NG change (see Fig. 5).

There are $2N - 3$ NG bosons and only one QNG boson. The mixed one belongs to a single representation $1_M$ of $H' = O(N - 2)$, and, the others belong to $(N - 2)_P$ (see Table 1). Since the number of the singlet is $n_{II}(1_M) = 1$, the dimension of region II is $\dim \mathcal{M}_{II} = 1$ from theorem 2. Since region II is the generic region, of course, theorem 1 is true: $N_{II}(G) = n_{II} = 1$.

We have shown that there are two kinds of vacua in this model. One is a symmetric point in region I; the other is a non-symmetric point in region II. In the target manifold $M$, there are the same vacua generated by $G$, corresponding to NG directions. Therefore, it is useful to see distinct vacua that we define moduli space as a quotient space divided by the symmetry $G : \mathcal{M} = M/G$. We define the moduli parameter as

$$\theta_1 \overset{\text{def}}{=} |\vec{\phi}|_F = |\xi \vec{v}|, \quad (4.13)$$

where $F$ denotes the F-term constraint Eq. (4.13) and $\xi$ is the representative of the complex coset, $G^C/\dot{H}$. This parameter has a minimum as shown in Figs. 3 and 4. The moduli space of this model can be written as

$$\mathcal{M} = \{\theta_1 \in \mathbb{R}|\theta_1 \geq f\}. \quad (4.14)$$

---

14 In examples we define the square root of first types of Eq. (4.2) as the moduli parameters.
This is a closed set and has two phases corresponding to the symmetric point (region I) and the non-symmetric point (region II) as, can be seen in Fig. 6.

The Kähler potential of the low-energy effective Lagrangian describing the behavior of the NG and QNG bosons is written by using moduli parameter as

\[ K = f(\theta_1^2) = f(\vec{\phi}^\dagger \vec{\phi})|_{\vec{\phi}^2 = f^2} = f(\vec{v}^\dagger \xi^\dagger \xi \vec{v}), \]

with a constraint \( \vec{v}^2 = f^2 \). The physical consequences, such as the low-energy theorems of the scattering amplitudes of the NG and QNG bosons, are discussed in Ref. [13, 14].

### 4.2 Example of \( \dim \mathcal{M} = 1 \) (open set)

Example 2) \( SU(N) \) with \( \vec{\phi} \in \mathbb{N} \)

In the last example, the maximal realization occurs at the symmetric point, and \( \hat{H} \) does not include the Borel-type algebra. This example contains it. (See Ref. [10].)

Since there is only the field in the complex representation \( \vec{\phi} \in \mathbb{N} \) (no conjugate representation), we cannot compose the \( G^C \) singlet in the superpotential.\[15\] Since

\[ \text{dim}(SU(N)/SU(N-1)) = 2N-1 \]

the constraints have at least two dimensions.
\[ N(G_C) = 0, \text{ the complex dimension of the target space is } N_\Phi = \dim C V - N(G_C) = N. \]

As in Example 1, the vacuum \( \vec{v} = < \vec{\phi} > \) can be transformed to

\[
\vec{v} = \begin{pmatrix}
0 \\
\vdots \\
0 \\
v
\end{pmatrix}
\]

by the \( G_C \)-action, where \( v \) is a real positive arbitrary constant. In this case, there is no other \( G_C \)-orbit, except for the orbit \( \vec{\phi} = 0 \).

The real unbroken symmetry is \( H = SU(N - 1) \) and the number of the NG bosons is \( 2N - 1 \). (The number of the QNG bosons should be one.) We define \( 2(N - 1) \) complex (non-Hermitian) generators,

\[
X_i^- = \begin{pmatrix}
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0
\end{pmatrix}, \quad X_i^+ = \begin{pmatrix}
0 & 0 \\
1 & 0 \\
0 & 0 \\
0 & 0
\end{pmatrix},
\]

where \( i = 1, \cdots, N - 1 \). The complex unbroken symmetry is

\[
\hat{\mathcal{H}} = \begin{pmatrix}
\mathcal{H}^C & 0 \\
\mathcal{B} & 0
\end{pmatrix},
\]

where \( N - 1 \) generators \( \mathcal{B} = \{ X_i^- \} \) is the Borel generators. Therefore, the complex unbroken generator \( \hat{H} \) is not reductive, even at the symmetric point in this case. The complex broken generators are

\[
G_C - \hat{\mathcal{H}} = \begin{pmatrix}
M & \cdots & 0 \\
\vdots & \ddots & P \\
0 & \cdots & M
\end{pmatrix}
\]

They comprise \( N - 1 \) non-Hermitian pure-type generators \( \{ X_i^+ \} \) (denoted P) and one Hermitian mixed-type generator (denoted M), which is a diagonal one: \( N_P = N - 1 \).
Figure 7: moduli space of $SU(N)$ with $N$

Table 2: phase of $SU(N)$ with $N$

| $R$ | $H_{(R)}$ | $N_M$ | $N_P$ | $NG$ | QNG | $H_{(R)}$-sector | dim $\mathcal{M}_R$ | $n_R$ | $N_R(G)$ |
|-----|----------|-------|-------|------|------|------------------|----------------|------|---------|
| I   | $SU(N-1)$ | 1     | $N-1$ | $2N-1$ | 1    | $(N-1)_P \oplus 1_M$ | 1             | 1    | 1       |

and $N_M = 1$. Thus, as noted above, there are $2N-1$ NG bosons and one QNG boson (see Table 2).

In this example, the $G^C$-orbit is an open set. We transform the vacuum as

$$\vec{v} \rightarrow \vec{v}' = g_0 \vec{v},$$

where $g_0 = \exp(i \theta X^{\text{diag}})$. Here, $X^{\text{diag}} \sim \text{diag}((1, \ldots, 1, -(N-1)))$, denoted $M$ in (4.19). The angle is taken, to be pure imaginary, $\theta = i \tilde{\theta}$. If we take the limit $\tilde{\theta} \rightarrow \infty$, it reaches $\vec{\phi} = 0$. Since the origin is omitted (it is another orbit), the orbit is an open set.

If we define the moduli parameter as

$$\theta_1 \overset{\text{def}}{=} |\vec{\phi}| = |\xi \vec{v}|,$$

the moduli space is (see Fig. 7)

$$\mathcal{M} = \{ \theta_1 \in \mathbb{R} | \theta_1 > 0 \}. \tag{4.22}$$

The moduli space is also an open set.

There is only one region in this model. We can find that fact soon. The dimension of the $G^C$-orbit coincides with the dimension $V$, since there is no $G^C$-invariant. Since the origin is always omitted, it should be an open set, and there is only one phase.
4.3 Example of $\dim \mathcal{M} = 2$

Example 3) $SU(N)$ with $\vec{\phi} \in \mathbf{N}$, $\vec{\tilde{\phi}} \in \bar{\mathbf{N}}$

This example is seen in Ref. [10]. The last two examples contain only one representation vector. In such cases, the moduli space has one dimension, since the vector belongs to an irreducible representation. In this example, we introduce two representation fields. One belongs to a fundamental representation and the other to an anti-fundamental representation, transforming as

$$
\vec{\phi} \rightarrow g \cdot \vec{\phi}, \quad \vec{\tilde{\phi}}^T \rightarrow \vec{\tilde{\phi}}^T \cdot g^{-1}.
$$

(4.23)

Since there is one $G^C$-invariant, $\vec{\phi} \cdot \vec{\tilde{\phi}}$, $N(G^C) = 1$. We construct the $G$-invariant superpotential with a non-dynamical singlet $\phi_0$,

$$
W = g\phi_0 (\vec{\tilde{\phi}} \cdot \vec{\phi} - f^2),
$$

(4.24)

which is also $G^C$-invariant. As stated in Example 1, we obtain the F-term constraint, $\vec{\phi} \cdot \vec{\phi} - f^2 = 0$. The complex dimension of the target space is $N_{\phi} = \text{dim}_C V - N(G^C) = 2N - 1$.

There exist two $G$-singlets: $N(G) = 2$,

$$
\vec{\phi}^\dagger \vec{\phi}, \quad \vec{\tilde{\phi}}^\dagger \vec{\tilde{\phi}} \in \mathbf{R}.
$$

(4.25)

Thus as seen in later, the minimum number of the QNG and the dimension of the moduli space should be two from corollary 2 (Eq. (3.29)).

The vacuum $\vec{v} = \langle \vec{\phi} \rangle$, $\vec{\tilde{v}} = \langle \vec{\tilde{\phi}} \rangle$ can be transformed by $G^C$ to

$$
\vec{v} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \vec{\tilde{v}}^T = (0, \cdots, 0, \vec{v}),
$$

(4.26)

---

16 Actually there exist two kind of the anti-fundamental representation. One is given here, whereas the other is transformed as $\vec{\phi} \rightarrow \vec{\tilde{\phi}} \cdot g^\dagger$. Although both coincide in the transformation of $G$, they are distinct in the transformation of $G^C$. We do not use this one, because we cannot construct $G^C$-invariants.
where
\[ v\tilde{v} = f^2, \quad v, \tilde{v} \in \mathbb{R} > 0 \ (\text{or} \ < 0). \quad (4.27) \]

Since the case when the product \( v\tilde{v} \) is negative corresponds to another \( G^C \)-orbit, we omit such a case, as denoted in Example 1. We call this point a symmetric point, where \( \tilde{v} \propto \tilde{\tilde{v}} \). It belongs to region I in the moduli space. The breaking pattern is \( G = SU(N) \rightarrow H(I) = SU(N - 1) \). Since complexification does not change the situation, it is a maximal realization. Thus the target space is \( G^C/\hat{H} = SL(N, \mathbb{C})/SL(N - 1, \mathbb{C}) \), and there are \( N - 1 \) NG bosons and \( N - 1 \) QNG bosons.

We can transform the vacuum to the non-symmetric point belonging to region II, where \( \tilde{v} \not\propto \tilde{\tilde{v}} \). We choose \( g_0 \in G^C \) as
\[
g_0 = \exp(i\theta X_{N-1}^-) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad g_0^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & \alpha/\tilde{v} \end{pmatrix}, \quad (4.28)
\]
where \( X_{N-1}^- \) is defined in Eq. (4.17) and we have put \( e^\theta = -\alpha/\tilde{v} > 0 \ (\alpha \in \mathbb{R} < 0) \).

The transformation by \( g_0 \) is
\[
\tilde{v} \rightarrow \tilde{v}' = g_0\tilde{v} = \tilde{v}, \quad \tilde{\tilde{v}} \rightarrow \tilde{\tilde{v}}' = \tilde{\tilde{v}} g_0^{-1} = (0, \cdots, \alpha, \tilde{v}). \quad (4.29)
\]
Note that this transformation does not change \( \tilde{v} \). The breaking pattern is \( G = SU(N) \rightarrow H(II) = SU(N - 2) \). The number of NG bosons is \( 4N - 4 \) and the number of QNG bosons is \( 2 \). Although we can show these counting schemes in a generator level as in Eqs. (4.10) and (4.11) in Example 1, we do not repeat it, since it is straightforward. The results are given in Table 3.

Since the length of the vectors are
\[
|\tilde{v}'|^2 = |\tilde{v}|^2 = v^2, \quad |\tilde{\tilde{v}}|^2 = |\tilde{\tilde{v}}'|^2 = \tilde{v}^2 + \alpha^2, \quad (4.30)
\]
the \( \tilde{v} \)-fixing action induced by the \( X_{N-1}^- \) decreases the length of \( \tilde{v} \) as \( |\tilde{v}'|^2 \geq \tilde{v}^2 \). There is another independent transformation induced by \( X_{N-1}^+ \), defined in Eq. (4.17).

This time it fixes \( \tilde{\tilde{v}} \) and decreases the length of \( \tilde{\tilde{v}} : |\tilde{\tilde{v}}'|^2 \geq v^2 \)

If we define moduli parameters as
\[
\theta_1 \stackrel{\text{def}}{=} |\tilde{\phi}|_F = |\xi \tilde{v}|, \quad \theta_2 \stackrel{\text{def}}{=} |\tilde{\tilde{\phi}}|_F = |(\xi^{-1})^T \tilde{v}|, \quad (4.31)
\]

27
the moduli space is (see Fig. 8)

\[ M = \{ (\theta_1, \theta_2) \in \mathbb{R}^2 | \theta_1 \theta_2 \geq f^2 \} \, , \quad (4.32) \]

from the argument above. This is a closed set.

Since the generator \( iX^{\text{diag}} \sim \text{diag}(1, \cdots, 1, -(N-1)) \) belongs to a singlet of \( H_{(1)} \), the dimension of region I of the moduli space is \( \dim M_I = 1 \) from theorem 2. Since a transformation by \( iX^{\text{diag}} \) mixes only broken generators, it does not change the structure of the \( H \)-sectors of the broken generators, and vacuum alignment does not occur. It changes \( \theta_1 \) and \( \theta_2 \) while preserving \( v \bar{v} = f^2 \); also, the orbit of this transformation in the moduli space is just a hyperbola, as shown in Fig. 8. It is the boundary of the generic region (region II). The two generators \( X^{+}_{N-1} \) and \( X^{-}_{N-1} \) bring the vacuum in region I to another region (region II). Note that they do not commute with the generator \( iX^{\text{diag}} \).

The Kähler potential contains two \( G \)-invariant variables in the arbitrary function
as
\[
K = f(\theta_1^2, \theta_2^2) = f(\vec{\phi}^\dagger \vec{\phi}, \vec{\phi}^\dagger \vec{\phi})|_{\vec{\phi} = f^2} = f(\vec{v}^\dagger \xi \vec{\xi} \vec{v}, \vec{v}^\dagger (\vec{\xi}^\dagger \vec{\xi})^{-1} \vec{v}) ,
\]
(4.33)
with a constraint \(\vec{v} \cdot \vec{v} = f^2\).

### 4.4 Example of \(\dim M = 1\) with adjoint representation

Until the last section, we have investigated only the fundamental representation. This section and the next section are devoted to the adjoint representation.

Example 4) \(SU(2)\) with \(\vec{\phi} \in \text{adj.} = 3\)

First of all, we consider the simplest example \(SU(2)\) with adjoint matter. The fundamental fields are
\[
\phi = \phi^A T_A \ (A = 1, 2, 3) \ , \ \phi^A \in \mathbb{C} ,
\]
(4.34)
where \(T_A\) are related to the Pauli matrices as \(T_A = \frac{1}{2} \sigma^A\). They are traceless,
\[
\text{tr} \phi = 0.
\]
(4.35)

The transformation by \(G^C\) is
\[
\phi \to g \phi g^{-1} , \ \phi^\dagger \to (g^{-1})^\dagger \phi g^\dagger , \ g \in G^C .
\]
(4.36)

There is a matrix identity called the Cayley-Hamilton theorem,
\[
A^2 - (\text{tr} A)A + (\text{det} A)I_2 = 0 ,
\]
(4.37)
where \(A\) is any two-by-two matrix and \(I_2\) is a two-by-two unit matrix. By putting \(\phi\) in \(A\), we obtain the identity,
\[
\phi^2 = - (\text{det} \phi) I_2 .
\]
(4.38)

From Eq. (4.33) and this equation, we can find that there is only one \(G^C\)-invariant, \(\text{tr} \phi^2\). By choosing a suitable superpotential, we obtain a F-term constraint,
\[
\text{tr} \phi^2 - f^2 = 0 .
\]
(4.39)
Since $N(G^C) = 1$, the complex dimension of the target space is $N_\Phi = \dim_C V - N(G^C) = 2$. From Eqs. (4.38) and (4.39), we obtain
\begin{align*}
\text{tr} \phi^2 &= -2 \det \phi = f^2, \quad \text{(4.40)} \\
\phi^2 &= -(\det \phi)I_2 = \frac{1}{2}f^2I_2. \quad \text{(4.41)}
\end{align*}

To find the independent $G$-invariant, we put $\phi^\dagger \phi$ in $A$ in Eq. (4.37):
\begin{align*}
(\phi^\dagger \phi)^2 &= \text{tr} (\phi^\dagger \phi)\phi^\dagger \phi - \det(\phi^\dagger \phi)I_2 \\
&= \text{tr} (\phi^\dagger \phi)\phi^\dagger \phi - \frac{1}{4}f^4I_2, \quad \text{(4.42)}
\end{align*}
where we used Eq. (4.40). We can find that there is only one $G$-singlet: $N(G) = 1$,
\begin{equation}
\text{tr} (\phi^\dagger \phi) \in \mathbb{R}, \quad \text{(4.43)}
\end{equation}
since from Eqs. (4.41) and (4.42), other $G$-invariants,
\begin{align*}
\text{tr} (\phi^\dagger \phi^2) &= \frac{1}{2}f^2\text{tr} \phi^\dagger = 0, \quad \text{(4.44)} \\
\text{tr} (\phi^\dagger \phi^4) &= \frac{1}{2}f^4, \quad \text{(4.45)} \\
\text{tr} ((\phi^\dagger \phi)^2) &= (\text{tr} (\phi^\dagger \phi))^2 - \frac{1}{2}f^4 \quad \text{(4.46)}
\end{align*}
are not independent.

By using the $G^C$ transformation, any vacuum can be transformed to the symmetric point in region I,
\begin{align*}
v &= \langle \phi \rangle = \frac{f}{\sqrt{2}}\sigma_3 = \frac{f}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \text{(4.47)} \\
\text{tr} v^2 &= f^2. \quad \text{(4.48)}
\end{align*}
The unbroken symmetry is $H_{(I)} = U(1)$ and the broken generators are $G - H_{(I)} = \{\sigma_1, \sigma_2\}$. The maximal realization occurs at the symmetric point: $G^C/\hat{H} = SL(2, \mathbb{C})/GL(1, \mathbb{C})$ and $N_M = 2$, $N_P = 0$. $G^C - \hat{H}$ belongs to a single $H_{(I)}$-representation $2_M$. Since $n_I = 1$ agrees with $N_I(G) = 1$, the conjecture can be verified in region I. Since there is no singlet, the dimension of region I is $\dim M_I = 0$ from theorem 2.

The non-symmetric (generic) points belonging to region II are written in
\begin{equation}
v \xrightarrow{g_0} v' = g_0 v g_0^{-1} = a\sigma_3 + b\sigma_+ + c\sigma_- = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}, \quad a \in \mathbb{R}, \quad b, c \in \mathbb{C}. \quad \text{(4.49)}
\end{equation}
### Table 4: phases of $SU(2)$ with 3

| $R$ | $H_{(R)}$ | $N_M$ | $N_P$ | NG | QNG | $H_{(R)}$-sector | $\dim \mathcal{M}_R$ | $n_R$ | $N_R(G)$ |
|-----|-----------|-------|-------|-----|-----|-------------------|---------------------|------|--------|
| I   | $U(1)$    | 2     | 0     | 2   | 2   | $2_M$             | 0                   | 1    | 1      |
| II  | $\{1\}$  | 1     | 1     | 3   | 1   | $1_P \oplus 1_M$  | 1                   | 1    | 1      |

The parameter $a$ is real, since any Hermitian matrix can be diagonalized by some unitary matrix $g \in G$. There is a relation, $\text{tr} v'^2 = -2 \det v' = 2(a^2 + bc) = f^2$. The real unbroken symmetry is trivial $\{1\}$, whereas there is complex unbroken symmetry $(a\ b\ c\ -a) \in \hat{H}$. Thus in the complex broken generators there is one pure-type generator and the other is the mixed-type generator: $N_M = 1$, $N_P = 1$. Since both of them belong to singlet representations of $H_{(II)}$, the dimension of region II is $\dim \mathcal{M}_{(II)} = 1$ from theorem 2. Since region II is the generic region, theorem 1 is

$n_{II} = N_{II}(G) = 1$.

There exist two phases, the symmetric region and non-symmetric region. The moduli space is equivalent to that of Example 1, since there exists isomorphism $SU(2)/U(1) \simeq O(3)/O(2) \simeq S^2$. Their complexifications are also equivalent.

The effective Kähler potential can be written as

$$K = f \left( \text{tr} (\phi^\dagger \phi) \right) \big|_{\phi \phi^* = f^2} = f \left( \text{tr} ((\xi^\dagger \xi)^{-1} v^\dagger (\xi^\dagger \xi) v) \right),$$

(4.50)

with the constraint $\text{tr} v^2 = f^2$, where we have used the fact $\phi \big|_F = \xi v \xi^{-1}$ and its conjugate.

### 4.5 Example of $\dim \mathcal{M} = 4$ with adjoint representation

Example 5) $SU(3)$ with $\tilde{\phi} \in \text{adj.} = 8$

We now consider the $SU(3)$ case. The fundamental fields are

$$\phi = \phi^A T_A \ (A = 1, \ldots, 8) \ , \ \phi^A \in \mathbb{C},$$

(4.51)

where $T_A$ are related to the Gell-Mann matrices as $T_A = \frac{1}{2} \lambda^A$. They are traceless: $\text{tr} \phi = 0$. The transformation by $G^C$ is same as Eq (4.36).

\[17\] If we put $v = X + iY$, where $X$ and $Y$ are Hermitian matrices, $\text{tr} v^2 = \text{tr} (X^2 - Y^2) + 2\text{tr} (XY) = f^2$. By unitary transformation, $X$ can be diagonalized: $X \simeq \sigma_3$ and $Y \simeq \{\sigma_1, \sigma_2\}$.  

31
The Cayley-Hamilton theorem is
\[ A^3 - \frac{3}{2} (\text{tr} A) A^2 + \frac{1}{2} (\text{tr} A)^2 A + (\det A) I_3 = 0, \] (4.52)
where \( A \) is any three-by-three matrix and \( I_3 \) is a three-by-three unit matrix. By substituting \( A = \phi \) in Eq. (4.52), we obtain
\[ \phi^3 = -(\det \phi) I_3. \] (4.53)
From this equation, we can find that there are two \( G_C \)-invariants, \( \text{tr} \phi^2 \) and \( \text{tr} \phi^3 = -3 \det \phi, (N(G_C) = 2) \). We can obtain F-term constraints from a suitable superpotential as
\[ \text{tr} \phi^2 = \frac{1}{2} \delta_{AB} \phi^A \phi^B = f^2, \] (4.54)
\[ \text{tr} \phi^3 = -3 \det \phi = d_{ABC} \phi^A \phi^B \phi^C = g^3, \] (4.55)
where \( f \) and \( g \) are real parameters and \( d_{ABC} = \text{tr} (\{\lambda_A, \lambda_B\} \lambda_C) \). The complex dimension of the target space is \( N_\Phi = \dim_{\mathbb{C}} V - N(G_C) = 8 - 2 = 6 \).
To investigate \( G \)-invariants, we put \( A = \phi^\dagger \phi \) in Eq. (4.52),
\begin{align*}
(\phi^\dagger \phi)^3 &= \frac{3}{2} \text{tr} (\phi^\dagger \phi)(\phi^\dagger \phi)^2 - \frac{1}{2} (\text{tr} \phi^\dagger \phi)^2 \phi^\dagger \phi - \det(\phi^\dagger \phi) I_3 \\
&= \frac{3}{2} \text{tr} (\phi^\dagger \phi)(\phi^\dagger \phi)^2 - \frac{1}{2} (\text{tr} \phi^\dagger \phi)^2 \phi^\dagger \phi + \frac{1}{9} g^6 I_3 ,
\end{align*} (4.56)
where we have used Eq. (4.55).
From Eqs. (4.53), (4.55) and (4.56), we find that there are four independent \( G \)-invariants: \( N(G) = 4 \),
\[ \text{tr} (\phi^\dagger \phi) = \frac{1}{2} \delta_{AB} \phi^A \phi^B , \text{ tr} (\phi^\dagger \phi)^2 \in \mathbb{R} , \]
\[ \text{tr} (\phi^\dagger \phi^2) = d_{ABC} \phi^A \phi^B \phi^C \in \mathbb{C} , \] (4.57)
since the other \( G \)-invariants
\begin{align*}
\text{tr} (\phi^\dagger \phi^2) &= \text{tr} (\phi^\dagger \phi)^2 - \frac{3}{2} (\text{tr} \phi^\dagger \phi)^2 + \frac{3}{2} \text{tr} (\phi^\dagger \phi) , \\
\text{tr} (\phi^\dagger \phi^3) &= -(\det \phi) \text{tr} \phi^\dagger = 0 , \\
\text{tr} (\phi^\dagger \phi^2 \phi) &= -(\det \phi) \text{tr} \phi^\dagger \phi^2 = \frac{1}{3} f^2 g^3 , \\
\text{tr} ((\phi^\dagger \phi)^2 \phi) &= \frac{3}{2} \text{tr} (\phi^\dagger \phi) \text{tr} (\phi^\dagger \phi^2) ,
\end{align*} (4.58) (4.59) (4.60) (4.61)
Table 5: $G^C$-orbits of $SU(3)$ with 8

| orbit | area | $v$ | $\mathcal{H}_{(s)}$ | $H_{(s)}$ | $N_\Phi$ |
|-------|------|-----|---------------------|-----------|---------|
| (i)   | $a = b = 0$ | $v = 0$ | $\{\lambda_1, \cdots, \lambda_8\}$ | $SU(3)$ | 0       |
| (ii)  | $a = 0, b \neq 0$ | $\text{diag.}(\frac{b}{\sqrt{3}}, \frac{b}{\sqrt{3}}, -\frac{2}{\sqrt{3}}b)$ | $\{\lambda_1, \lambda_2, \lambda_3, \lambda_8\}$ | $SU(2) \times U(1)$ | 4       |
| (iii) | $a = \sqrt{3}b, b \neq 0$ | $\text{diag.}(\frac{4}{\sqrt{3}}b, -\frac{2}{\sqrt{3}}b, -\frac{2}{\sqrt{3}}b)$ | $\{\lambda_3, \lambda_6, \lambda_7, \lambda_8\}$ | $SU(2) \times U(1)$ | 4       |
| (iv)  | $a = -\sqrt{3}b, b \neq 0$ | $\text{diag.}(-\frac{2}{\sqrt{3}}b, \frac{4}{\sqrt{3}}b, -\frac{2}{\sqrt{3}}b)$ | $\{\lambda_3, \lambda_4, \lambda_5, \lambda_8\}$ | $SU(2) \times U(1)$ | 4       |
| (v)   | generic | Eq. (4.64) | $\{\lambda_3, \lambda_8\}$ | $U(1)^2$ | 6       |

$$\text{tr} \left( \phi^\dagger \phi^4 \right) = -\left( \text{det} \phi \right) \text{tr} \left( \phi^\dagger \phi \right) = \frac{1}{3} g^3 \text{tr} \left( \phi^\dagger \phi \right), \quad (4.62)$$

$$\text{tr} \left( (\phi^\dagger \phi)^3 \right) = \frac{3}{2} \text{tr} \left( \phi^\dagger \phi \right) \text{tr} \left( (\phi^\dagger \phi)^2 \right) - \frac{1}{2} \left( \text{tr} \left( \phi^\dagger \phi \right) \right)^2 \text{tr} \left( \phi^\dagger \phi \right) + \frac{1}{3} g^6, \quad (4.63)$$

are all not independent.

The generic vacua can be transformed by a $G^C$ transformation to the symmetric point,

$$v = <\phi> = a\lambda_3 + b\lambda_8 = \begin{pmatrix}
a + \frac{b}{\sqrt{3}} & 0 & 0 \\
0 & -a + \frac{b}{\sqrt{3}} & 0 \\
0 & 0 & -\frac{2}{\sqrt{3}}b
\end{pmatrix}, \quad a, b \in \mathbb{R}. \quad (4.64)$$

Since there are relations between parameters $a$ and $b$ and the value of the $G^C$-invariants, $f$ and $g$ as

$$\text{tr} v^2 = 2(a^2 + b^2) = f^2, \quad \text{tr} v^3 = 2\sqrt{3} \left( a^2 b - \frac{1}{3} b^3 \right) = g^3, \quad (4.65)$$

$a$ and $b$ are constant at the single $G^C$-orbit. Thus they also parametrize the $G^C$-orbit space $V/G^C$. There are five types of $G^C$-orbits. We list them in Table 5: (i)-(v).

In all cases, the maximal realization occurs, since the adjoint representation is a real representation. The last type contains the generic $G^C$-orbits which have the maximal dimension and the others are singular $G^C$-orbits with fewer dimensions than generic orbits. In this paper we consider the generic $G^C$-orbit. Thus, the unbroken symmetry at the symmetric point (region I) is $H_{(I)} = U(1)^2$ and the generators of it,
$H_{(I)} = \{ \lambda_3, \lambda_8 \}$ are the Cartan generators. Since it is the maximal realization, the target space is $G^C/\hat{H} = SL(3, C)/GL(1, C)^2$. From the commutation relations,

$$
\begin{align*}
[\lambda_3, \lambda_1] &\sim \lambda_2, \ [\lambda_3, \lambda_2] \sim \lambda_1, \ [\lambda_8, \lambda_1] \sim 0, \ [\lambda_8, \lambda_2] \sim 0, \\
[\lambda_3, \lambda_4] &\sim \lambda_5, \ [\lambda_3, \lambda_5] \sim \lambda_4, \ [\lambda_8, \lambda_4] \sim \lambda_5, \ [\lambda_8, \lambda_5] \sim \lambda_4, \\
[\lambda_3, \lambda_6] &\sim \lambda_7, \ [\lambda_3, \lambda_7] \sim \lambda_6, \ [\lambda_8, \lambda_6] \sim \lambda_7, \ [\lambda_8, \lambda_7] \sim \lambda_6,
\end{align*}
$$

we can find the $H_{(I)}$-representations to which the complex broken generators belong, as follows. $\lambda_1 \oplus \lambda_2$ belongs to $(2_M, 1_M)$ from the first line of Eq. (4.66) and $\lambda_4 \oplus \lambda_5$ and $\lambda_6 \oplus \lambda_7$ belong to $(2_M, 2_M)$ from the second and third lines of Eq. (4.66). Here, $(\cdot, \cdot)$ denotes the representation of the unbroken symmetry generated by $(\lambda_3, \lambda_8)$. There are three sectors: $n_I = 3$, which coincides with the number of the $G$-invariants, since at this point $\phi$ becomes Hermitian, $\phi^\dagger = \phi$, so the third type of the $G$-invariants in Eq.(4.57) becomes real and $N_I(G) = 3$. Thus the conjecture is nontrivially also true in this region of the model. Since there is no $H_{(I)}$-singlet, the dimension of region $I$ is $\dim \mathcal{M}_{(I)} = 0$ from theorem 2.

We transform the symmetric vacuum $v_{(I)}$ to another vacuum $v_{(II)}$ as

$$
v_{(I)} \xrightarrow{g_0} v_{(II)} = g_0 v_{(I)} g_0^{-1}, \ g_0 = \exp(\alpha E_{(+,0)} + \beta E_{(-,0)}) \in G^C, \quad (4.67)
$$

where $\alpha$ and $\beta$ are some real parameters and

$$
E_{(\pm,0)} = (T_1 \pm iT_2) = \frac{1}{2} \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad \frac{1}{2} \begin{pmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
$$

are broken generators in the Cartan form, whose root vectors are $(\pm, 0)$. Since the unbroken generators are transformed to

$$
\begin{align*}
\lambda_3 &\xrightarrow{g_0} g_0 \lambda_3 g_0^{-1} = a' \lambda_3 + c \lambda_+ + d \lambda_-, \ a' \in \mathbb{R}, \ c, d \in \mathbb{C} \\
\lambda_8 &\xrightarrow{g_0} g_0 \lambda_8 g_0^{-1} = \lambda_8,
\end{align*}
$$

the obtained vacuum is in the form

$$
v_{(II)} = a \lambda_3 + b \lambda_8 + c \lambda_+ + d \lambda_- = \begin{pmatrix}
da' + \frac{b}{\sqrt{3}} & c & 0 \\
d & -a' + \frac{b}{\sqrt{3}} & 0 \\
0 & 0 & -\frac{2}{\sqrt{3}} b
\end{pmatrix}. \quad (4.71)
$$
Table 6: phases of SU(3) with 8

|   | $R$ | $H_{(R)}$ | $N_M$ | $N_P$ | NG | QNG | $H_{(R)}$-sector | dim $\mathcal{M}_R$ | $n_R$ | $N_{R(G)}$ |
|---|-----|----------|------|------|----|-----|------------------|----------------|------|-----------|
| I | $U(1)^2$ | 6       | 0    | 6    | 6  | (2, 1)$_M$ $\oplus$ 2(2, 2)$_M$ | 0    | 3   | 3         |
| II| $U(1)$  | 5       | 1    | 7    | 5  | 1$_P$ $\oplus$ 22$_M$ $\oplus$ 1$_M$ | 1    | 3   | 3         |
| III| $\{1\}$ | 4       | 2    | 8    | 4  | 21$_P$ $\oplus$ 41$_M$ | 4    | 4   | 4         |

Note that $cd \in \mathbb{R}$, since $\det \phi = \left(- \frac{2}{\sqrt{3}}\right)\left(\frac{b}{\sqrt{3}}\right)^2 - a^2 - cd = -\frac{1}{3}g^3 \in \mathbb{R}$. Therefore, from a short calculation, the third type of $G$-invariant in Eq. (4.57), which is complex at general point, also remains real in region II, since $\text{tr} \left(\phi^\dagger \phi^2\right) = 2(-a' + \frac{b}{\sqrt{3}})^3 + \left(-\frac{2}{\sqrt{3}}b\right)^3 + \frac{2}{\sqrt{3}}b(|c|^2 + |d|^2 + cd) \in \mathbb{R}$. Thus, the number of $G$-invariants is $N_{II}(G) = 3$.

The unbroken symmetry in this region is $H_{(II)} = U(1)$, generated by $\lambda_8$. From Eqs. (4.69) and (4.70), $\lambda_1$ is transformed to a non-Hermitian generator and $\lambda_8$ remains Hermitian. Since there is one more complex unbroken generator in addition to the real unbroken generators, there is one pure-type complex broken generator besides five mixed-type broken generators. The $H_{(II)}$-sectors of the complex broken generators are as follows: $(\lambda_4, \lambda_5)$ and $(\lambda_6, \lambda_7)$ belong to 2$_M$, one of the complex combination of $\lambda_1, \lambda_2$ and $\lambda_3$ to 1$_M$ and another combination to 1$_P$. (The rest combination is in $H_{(II)}$ as noted above.) Thus, the number of mixed $H_{(II)}$-sectors is $n_{II} = 3$, which is in agreement with $N_{II}(G)$. Therefore, the conjecture is also non-trivially verified in region II. The results are given in the second line of Table 6.

There exists one more region as a generic region (region III). The most generic vacuum can be written as

\[
v_{(III)} = a \lambda_3 + b \lambda_8 + \sum_{A \neq 3,8} c_A \lambda_A , \quad a, b \in \mathbb{R} , \quad c_A \in \mathbb{C} . \tag{4.72}
\]

Although this vacuum breaks the symmetry $G$ completely, there exist two complex unbroken generators corresponding to $\lambda_3$ and $\lambda_8$ in the symmetric point. Thus the complex broken generators are constituted from two pure-type and six mixed-type generators, and all of them belong to singlets, since there is no unbroken symmetry, as is the third line of Table 6.
The effective Kähler potential can be written as

\[ K = f \left( \text{tr} \left( \phi^\dagger \phi \right), \text{tr} \left( \phi^\dagger \phi^2 + \phi \phi^\dagger \phi \right), -i \text{tr} \left( \phi^\dagger \phi^2 - \phi \phi^\dagger \phi \right) \right) \mid_{\text{tr} \phi^2 = f^2, \text{tr} \phi^3 = g^3} \]

\[ = f \left[ \text{tr} \left( (\xi^\dagger \xi)^{-1} v^\dagger (\xi^\dagger \xi) v \right), \text{tr} \left( (\xi^\dagger \xi)^{-1} v^\dagger (\xi^\dagger \xi) v \right)^2, \right. \]

\[ \text{tr} \left( (\xi^\dagger \xi)^{-1} v^\dagger (\xi^\dagger \xi) v^2 + (\xi^\dagger \xi) v (\xi^\dagger \xi)^{-1} v \right), \]

\[ -i \text{tr} \left( (\xi^\dagger \xi)^{-1} v^\dagger (\xi^\dagger \xi) v^2 - (\xi^\dagger \xi) v (\xi^\dagger \xi)^{-1} v \right) \right], \]

with constraints \( \text{tr} v^2 = f^2, \text{tr} v^3 = g^3 \).

In this example, since it is nontrivially verified that \( N_R(G) \) changes accordingly to \( n_R \), we believe that the conjecture is generically true.

5 Comments on gauging global symmetry

In this paper, we have mainly considered a theory which has only global symmetry. We comment in this section on a theory with gauge symmetry. If a theory has a gauge symmetry, we consider it as being global symmetry for a while, and gauge it after finding the moduli space. If the theory has global and gauge symmetry, we require partial gauging of the global symmetry, while if it has only gauge symmetry, we require full gauging of the global symmetry.

First of all, we consider the case that all of the global symmetry \( G \) is gauged. The gauging brings the D-flat condition, \( (\bar{\phi}^\dagger T_A \bar{\phi})^2 = 0 \), besides the F-flat condition. This condition can be replaced by the condition that the length \( |\bar{\phi}| \) is minimum \([18, 19]\). It chooses the one \( G \)-orbit (if it exists) from the set of F-flat points, namely the target space \( M \). (It is called the D-orbit in Ref. \([2]\).) It is known that a closed \( G^C \)-orbit has one D-flat \( G \)-orbit \([18, 13, 2]\). For example, in Example 1 the \( G^C \)-orbit is closed and there exists one D-orbit where the moduli parameter \( \theta_1 = |\bar{\phi}|^2 \) is minimum. (See Fig. 3 and Fig. 4.) They are a set of symmetric points. It has been proved that the complex unbroken symmetry \( \hat{H} \) is reductive, namely \( \hat{H} = H^C \), and that there is no Borel algebra in the D-flat orbit \([18, 19]\). Actually, the symmetric points have this property and the maximal realization occurs there. In moduli space, the D-orbit corresponds to one point. Thus, moduli space is trivial. In Example 1, it is the point labeled D in Fig. 6. In example 3, the symmetric region, where \( \hat{H} \) is reductive, has one dimension. The D-point is one of them, where the length of the

36
vector $\vec{\Phi} \overset{\text{def.}}{=} (\vec{\phi}, \vec{\tilde{\phi}})$ in a reducible representation of $G$ is minimum. The minimum of $|\vec{\Phi}|^2 = |\vec{\phi}|^2 + |\vec{\tilde{\phi}}|^2 = (\theta_1)^2 + (\theta_2)^2$ is shown in Fig. 8. The D-orbit is the fibre at D-point in the moduli space.

However, if the $G^C$-orbit is an open set, there is no D-flat point. In Example 2, the $G^C$-orbit is open and there is no D-flat point. Since there exist Borel subalgebra, there is no point such that $\hat{H}$ becomes reductive. Thus supersymmetry must be broken spontaneously.

It is known that supersymmetry is spontaneously broken in a gauged sigma model with only pure-type multiplets [21, 26, 15]. Only when the maximal realization can occur, supersymmetry is preserved. There is a physical explanation to this phenomenon [15]. When the massless vector superfields absorb the NG chiral superfields, if there exist pure-type multiplets, they cannot constitute massive vector multiplets and supersymmetry must be spontaneously broken.

In our method, it is sufficient to see whether the $G^C$-orbit is open or closed, instead investigating whether supersymmetry is broken or not. Thus, our method may be useful to investigate dynamical supersymmetry breaking. (For a review, see Ref. [24].)

The case of partial gauging is more complicated. In the general embedding case, the ordinary vacuum alignment problem occurs besides the supersymmetric vacuum alignment [17]. However, when the gauged group is an ideal, namely the whole symmetry is the direct product of the global symmetry and the gauge symmetry, it does not occur. Since this includes the case of the supersymmetric QCD [9, 23], it is interesting to consider this case. We leave it to future works.

6 Conclusions

The moduli space of the gauge symmetry in $N = 1$ supersymmetric theory is well understood. However, for the case of global symmetry, it was not known at all. We have investigated the moduli space of the $N = 1$ supersymmetric theory with only global symmetry. In the case of global symmetry, although the complexified group

\footnote{We assume the whole symmetry is compact group.}
is a symmetry of the superpotential, and thus the F-term scalar potential, since it contains only chiral superfields, it is not the symmetry of the D-term potential, since the D-term contains chiral and anti-chiral superfields. Thus, the moduli space of global symmetry is the quotient space of the set of F-flat points divided by the symmetry $G$.

On the other hand, it has been known that when the global symmetry $G$ spontaneously breaks down to $H$ while preserving $N = 1$ supersymmetry, the low-energy effective Lagrangian is the nonlinear sigma model with the Kähler target manifold $M = G^C/\hat{H}$ parametrized by NG and QNG bosons. The target manifold is embedded in the space of the fundamental fields. Since the target manifold $M \simeq G^C/\hat{H}$ just comprises the F-flat points, the moduli space is $\mathcal{M} = (G^C/\hat{H})/G$. An investigation in this direction requires a deep understanding of the Kähler coset manifolds which had not yet been done.

It has been known that there is a supersymmetric vacuum alignment in this type of theory. The target manifold has non-compact directions corresponding to the appearance of the QNG bosons. The vacuum degeneracy in this non-compact direction has a one-to-one correspondence with the freedom to embed the complex unbroken symmetry $\hat{H}_v$ to $G^C$. Therefore, the unbroken symmetry $H_v = \hat{H}_v \cap G$ depends on the points $\vec{v}$ in the target manifold. (We have called the symmetric point $\vec{v}$ the point with the largest real unbroken group $H_v$ and the generic point the point with the least symmetry.) The number of NG and QNG bosons changes from point to point with the total number of NG and QNG bosons being unchanged. The compact coset manifolds $G/H_v$ (G-orbit of $\vec{v}$), with various dimensions, parametrized by the NG bosons are embedded in the full target manifold $M$.

The Kähler potential of the low-energy effective Lagrangian which describes the low energy behavior of the NG and QNG bosons can be written as the arbitrary function of some moduli parameters. By identification of the fundamental fields with the F-term constraints and the Kähler coset representative, we have found that it coincides with the known Kähler potential constructed by a group-theoretical way.

We have decomposed the moduli space into some regions $\mathcal{M}_R$, such that the real unbroken symmetries at different points in the same region are isomorphic to each other by a $G^C$-transformation. We have investigated the moduli space by differential
geometric (or the group theoretical) viewpoints, such as the Kähler coset manifold, and by algebraic geometric view points such as the ring of $G$-invariant polynomials.

From the differential geometrical view points, the moduli space is obtained by the identification of $G$-orbits in the full target manifold. Thus, the target manifold $M$ is considered to be a fibre bundle with fiber $G$-orbits $G/H_v$ on the base moduli space $\mathcal{M}$. At the boundary region $R_2$ of some region $R_1$ ($\partial R_1 = R_2$), the unbroken symmetry $H_{(R_1)}$ is enhanced to $H_{(R_2)}$. This corresponds to that $H_{(R_1)}$-orbit in $G$-orbit shrinks and most of the NG bosons change to the QNG bosons there. (The NG-QNG change occurs.) The symmetric points of target space correspond to the most singular point of the moduli space, and the number of QNG bosons is maximal. On the other hand, at generic points of moduli space, the space of QNG bosons is identical to the tangent vector space of the moduli space, $T_p\mathcal{M}$, and the number of QNG bosons is minimum with agreement with the dimension of the moduli space.

The complex broken generators can be decomposed to $H_{(R)}$-irreducible sectors, since they are transformed linearly by the action of $H_{(R)}$. The number of the $H_{(R)}$-irreducible sectors is the number of independent non-compact directions. They also correspond to the directions of moduli space. It can change at each region.

From algebraic geometrical view points, the ring of $G_C$-invariant polynomials is generated by the finite $G_C$-invariants and the target manifold is obtained by fixing all of them. We have considered the generic $G_C$-orbits. On the other hand, the ring of the $G$-invariant polynomials is also generated by the finite $G$-invariant polynomial and the $G$-orbit is obtained by fixing all of them. So the moduli space $\mathcal{M}$ is parametrized by such the $G$-invariants (after fixing the $G_C$-invariants).

From the relation of these two methods, we have obtained theorem 1 (Eq. (3.23)) which states that, in the generic region, the number of the $G$-invariants coincides with the number of the $H_{(R)}$-irreducible sectors of mixed types. We have also conjectured that it is true in any region, since both quantities are equal to the number of independent non-compact directions. We indeed show that this is true in examples, especially in example 5. We have also obtained theorem 2 (Eq. (3.25)) concerning the dimension of the regions of the moduli space denoted that $\dim \mathcal{M}_R$ coincides with the number of $H_{(R)}$-singlet sectors of mixed types. From these theorems, we have obtained formulae (3.29) to calculate the dimension of the moduli space in
various ways.

We have examined the results in many examples using the method of the algebraic geometry and the differential geometry (or the group theory).

When the fields belong to the fundamental representation, it is quite easy to calculate the dimension of the moduli space by using the former method. However it is difficult to calculate it by the latter method, since we must classify the complex broken generators to the pure- and mixed-types and study their transformation properties under unbroken symmetry.

On the other hand, when the fields belong to the adjoint representation, it is quite easy to calculate the dimension of the moduli space by using the latter method, since the unbroken symmetry is just the Cartan subalgebra. However, it is quite difficult to calculate it by the former method, since we must use freely the Cayley-Hamilton theorem to reduce the number of $G$-invariants.

In this paper, we have considered only the generic $G^C$-orbits. Generalization to the other $G^C$-orbits is straightforward. By considering it, it is possible to generalize to vacua with non-transitive $G^C$ action. (In such theories, there are extra flat directions not with related to the symmetry breaking.)

We hope that our method is useful to construct the supersymmetric Wess-Zumino term \cite{22}, to satisfy anomaly matching, to consider dynamical supersymmetry breaking \cite{24}, to investigate the effective action of the branes in curved space \cite{25} and other theoretical and phenomenological subjects of the modern physics. We hope to return to these subjects in future studies.

**Acknowledgements**

We thank K. Higashijima for useful discussions, encouragement and careful reading of the manuscript. We are grateful to K. Ohta and N. Ohta for arguments during the early stage of this work. We also thank T. Yokono for some useful comments on the moduli space of the supersymmetric gauge theories and M. Goto for some comments on the orbit space.
References

[1] N. J. Hitchin, A. Karlhede, U. Lindström and M. Roček, Comm. Math. Phys. 108 (1987) 535.

[2] M. A. Luty and W. Taylor IV, Phys. Rev. D53 (1996) 3339, hep-th/9506098.

[3] G. D. Dotti and A. V. Manohar, Anomaly Matching Conditions and the Moduli Space of Supersymmetric Gauge Theories, hep-th/9710024.

[4] B. Zumino, Phys. Lett. 87B (1979) 203.

[5] M. Bando, T. Kuramoto, T. Maskawa and S. Uehara, Phys. Lett. 138B (1984) 94; Prog. Theor. Phys. 72 (1984) 313, 1207.

[6] W. Lerche, Nucl. Phys. B238 (1984) 582.

[7] W. Buchmüller and W. Lerche, Ann. Phys. 175 (1987) 159.

[8] G. M. Shore, Nucl. Phys. B320 (1989) 202; Nucl. Phys. B334 (1990) 172.

[9] A. C. Kotcheff and G. M. Shore, Int. J. Mod. Phys. A4 (1989) 4391; Nucl. Phys. B333 (1990) 701; Nucl. Phys. B336 (1990) 245.

[10] M. A. Luty, J. March-Russel and H. Murayama, Phys. Rev. D52 (1995) 1190, hep-ph/9501233.

[11] K. Itoh, T. Kugo and H. Kunitomo, Nucl. Phys. B263 (1986) 295; Prog. Theor. Phys. 75 (1986) 386.

[12] W. Buchmüller and U. Ellwanger, Phys. Lett. 166B (1985) 325.

[13] K. Higashijima, M. Nitta, K. Ohta and N. Ohta, Prog. Theor. Phys. 98 (1997) 1165, hep-th/9706219.

[14] K. Higashijima and M. Nitta, Geometry of N = 1 Supersymmetric Low Energy Theorems, KEK-TH 571, to appear.

[15] T. Kugo, Soryuusiron Kenkyuu (Kyoto) 95 (1997) C56.
[16] M. Bando, T. Kugo and K. Yamawaki, Phys. Rep. 164 (1988) 217.

[17] A. C. Kotcheff and G. M. Shore, Nucl. Phys. B301 (1988) 267.

[18] R. Gatto and G. Sartori, Phys. Lett. 118B (1982) 79; G. Girardi, P. Sorba and R. Stora, Phys. Lett. 144B (1984) 212; C. Procesi and G. W. Schwarz, Phys. Lett. 161B (1984) 117; R. Gatto and G. Sartori, Phys. Lett. 157B (1985) 389.

[19] R. Gatto and G. Sartori, Comm. Math. Phys. 109 (1987) 327.

[20] M. Abud and G. Sartori, Phys. Lett. 104B (1981) 147; Ann. Phys. 150 (1983) 307.

[21] J. Bagger and E. Witten, Phys. Lett. 118B (1982) 103; A. J. Buras and W. Slominski, Nucl. Phys. B223 (1983) 157; C. M. Hull, A. Karlhede, U. Lindström and M. Roček, Nucl. Phys. B266 (1986) 1; J. Bagger and J. Wess, Phys. Lett. 199B (1987) 243; E. J. Chun, Phys. Rev. D41 (1990) 2003.

[22] D. Nemeschansky and R. Rohm, Nucl. Phys. B249 (1985) 157; E. Cohen and C. Gómez, Nucl. Phys. B254 (1985) 235; S. Aoyama and J. W. van Holten, Nucl. Phys. B258 (1985) 18.

[23] F. Feruglio, A smooth massless limit for supersymmetric QCD, [hep-th/9802178].

[24] E. Poppitz and S. P. Trivedi, Dynamical Supersymmetry Breaking, [hep-th/9803107].

[25] M. R. Douglas, D-branes in Curved Space, [hep-th/9703050]; M. R. Douglas, A. Kato and H. Ooguri, D-brane Actions on Kähler manifolds, [hep-th/9708012].

[26] J. Wess and J. Bagger, Supersymmetry and Supergravity, Princeton Univ. Press, Princeton(1992).