Statistics of the gravitational force in various dimensions of space: from Gaussian to Lévy laws

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To be included later

Abstract. We discuss the distribution of the gravitational force created by a Poissonian distribution of field sources (stars, galaxies,...) in different dimensions of space \( d \). In \( d = 3 \), when the particle number \( N \to +\infty \), it is given by a Lévy law called the Holtsmark distribution. It presents an algebraic tail for large fluctuations due to the contribution of the nearest neighbor. In \( d = 2 \), for large but finite values of \( N \), it is given by a marginal Gaussian distribution intermediate between Gaussian and Lévy laws. It presents a Gaussian core and an algebraic tail. In \( d = 1 \), it is exactly given by the Bernouilli distribution (for any particle number \( N \)) which becomes Gaussian for \( N \gg 1 \). Therefore, the dimension \( d = 2 \) is critical regarding the statistics of the gravitational force. We generalize these results for inhomogeneous systems with arbitrary power-law density profile and arbitrary power-law force in a \( d \)-dimensional universe.

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1 Introduction

In this paper, we study the statistics of the gravitational force arising from a random distribution of field sources (stars, galaxies,...) in arbitrary dimensions of space \( d \). This systematic study has interest both in astrophysics and probability theory. In addition, the distribution of the gravitational force presents numerous analogies with other physical systems such as Coulombian plasmas, 2D point vortices, dislocation systems etc. Many results have already been obtained by Chandrasekhar [1] for the gravitational force in \( d = 3 \) dimensions. In view of the fundamental nature of this problem and its potential applications in various areas of physics and astrophysics, it is important to formulate the mathematical problem at a general level and study how the results are affected by the dimension of space.

If we consider stellar systems such as globular clusters or elliptical galaxies, the problem is clearly three-dimensional and the gravitational force between two stars scales like \( 1/r^2 \). The structure of self-gravitating isothermal and polytropic spheres has been discussed by Emden [2] and Chandrasekhar [3] and the thermodynamics of stellar systems has been initiated by Antonov [4] and Lynden-Bell & Wood [5], and developed by several authors since then (see the reviews of Padmanabhan [6] and Chavanis [7]). On the other hand, the statistics of the gravitational force produced by a random distribution of stars has been studied by Chandrasekhar [1] by analogy with the work of Holtsmark [8] on the distribution of the electrostatic field in a plasma composed of simple ions. In a series of papers, Chandrasekhar & von Neumann [9,10,11,12,13] pursued this work in order to obtain, from a fully stochastic theory, an expression of the diffusion coefficient of stars in a cluster and understand the origin of the logarithmic divergence at large scales arising in the kinetic theory of stellar systems (see Kandrup [14] for a review). Numerical experiments have been conducted by Ahmad & Cohen [15] and more recently by Del Popolo [16] to test the predictions of this theory and take into account finite size effects. The initial theory was developed in the case of stars but the same methods can also be used in cosmology assuming that the field sources are galaxies rather than stars [17].

The standard results of Chandrasekhar [1] are valid for the gravitational force in three dimensions. However, it is important to note that some astrophysical systems have symmetries that lead to an effective gravitational interaction of lower dimensionality.

For example, some authors have considered the gravitational interaction between infinitely elongated cylindrical filaments. In that case, the force between two filaments scales like \( 1/r \) corresponding to the gravitational interaction in two dimensions. The structure of polytropic and isothermal cylinders has been studied by Ostriker [18] and the thermodynamics of gravitating rods has been developed by Katz & Lynden-Bell [19]. Aly & Perez [20] and Sire & Chavanis [21]. Polytropic and isothermal cylinders may have useful applications in the study of gaseous filaments, spiral arms and rings. Indeed, Schneider & Elmegreen [22] have shown that dark clouds have elongated or filamentary shapes. On the other hand, in some theoretical models, the spiral arms of the Galaxy are con-
sidered to be self-gravitating cylinders of infinite length \[23,24\]. Finally, gaseous rings occur in a variety of astronomical contexts (Saturn’s ring, rings in spiral galaxies,...) \[25,26\] and infinite cylinders provide the first term of a natural series expansion in which one may develop the theory of the equilibrium of such rings.

On the other hand, some authors have considered the gravitational interaction between plane-parallel sheets. In that case, the force between two sheets is independent on the distance, corresponding to the gravitational force in one dimension. The isothermal and polytropic distributions of such configurations have been determined by Spitzer \[27\] and Camm \[28\] and systematically studied by Harrison & Lake \[29\] and Ibañez & Sigalotti \[30\]. On the other hand, their thermodynamics has been worked out by Katz & Lecar \[31\] and Sire & Chavanis \[32\] in the mean field approximation valid for \(N \to \infty\). Interestingly, in the one dimensional case, Rybicki \[32\] has shown that the statistical equilibrium state can be calculated analytically for any \(N\). Isothermal sheets can have application in the study of galactic disks, collapsing clouds, pancakes in cosmology and Laplacian disk cosmogony. Indeed, in rotating disk systems, such as spiral and SO galaxies, the gas, dust and stars tend to be distributed in a symmetrical fashion about an equatorial plane. Camm \[28\] showed that the sheet model is a useful model for stellar motion in a direction perpendicular to the disk of a highly flattened galaxy. On the other hand, star-forming clouds generally collapse to a flattened (sheet), and sometimes filamentary (cylinders), configuration before fragmenting \[32\]. Sheet-like structures may also form by interstellar shocks or cloud collisions. Indeed, there seems to be strong evidence that some regions of post-shocked clouds are left near quasi-hydrostatic equilibrium plates (pancakes) at scales of galaxy formation \[33\] or at scales of stellar formation in the Galaxy \[34\]. Finally, plane-symmetric distributions of matter occur in the Saturn ring system and in the Laplacian disk cosmogony \[35\].

Apart from these various physical applications, it is interesting to investigate at a more academic level how the laws of physics, and particularly the laws of gravity, depend on the dimension of space \(d\). There is indeed a long tradition of works in that direction \[36\] starting from a seminal paper of Ehrenfest \[37\]. For example, in Ref. \[38\], we have studied how the structure of relativistic white dwarf stars would be modified in universes with lower or higher dimensions and we found that the dimensions \(d = 2\) and \(d = 4\) which surround the dimension \(d = 3\) of our universe are critical in some respect: white dwarf stars have a maximum radius in \(d = 2\), a maximum mass in \(d = 3\) and they become unstable for \(d \geq 4\). We have also found that the dimensions \(d = 2\) and \(d = 10\) are special for classical isothermal spheres \[39\] and that the dimensions \(d = 2\) and \(d = 9.96404372\ldots\) are special for self-gravitating radiation in general relativity \[39\]. This type of analysis can shed new light on the anthropic principle and explain why the dimension of our universe is particular. This is a further motivation, in addition to the physical examples mentioned above, to study gravity in \(d\) dimensions. Extra dimensions at the microscale also appear in theories of grand unification and black holes, an idea originating from Kaluza-Klein theory.

On the numerical point of view, there has been considerable interest over the years in the behaviour of one dimensional gravitational systems, essentially by reason of the simplicity of these models and their relatively cheapness for numerical study. Numerical simulations are more easily carried out in 1D than in 3D and many early numerical works have considered one-dimensional self-gravitating systems (OGS) to study (i) the process of violent relaxation (ii) the collisional evolution of the system and its relaxation to thermal equilibrium (iii) ergodicity for gravitational systems. The OGS is indeed the simplest model for studying \(N\)-body gravitational interactions even if it is not expected to capture all the features of 3D interactions. Therefore, studying gravity in one and two dimensions can be of interest to interprete numerical simulations. We refer to Yawn & Miller \[40\] for further references on this important topic.

The study of the distribution of the gravitational force in \(d\) dimensions is also important in statistical physics and probability theory \[41,42,43,44,45,46\] because it is an interesting example of a sum of random variables where the Central Limit Theorem (CLT) may or may not apply depending on the dimension of space. In particular, we show in this paper that the dimension \(d = 2\) is critical for the statistics of the gravitational field. In \(d = 3\), the variance of the gravitational force produced by one star diverges algebraically so that the distribution of the total force is a particular Lévy law called the Holtsmark distribution\[47\]. It presents an algebraic tail which is essentially due to the contribution of the nearest neighbor. In \(d = 1\), the variance of the gravitational force produced by one star is finite so that, by application of the CLT, the distribution of the total force is Gaussian (for finite \(N\) it is exactly given by the Bernoulli distribution). In \(d = 2\), the variance of the gravitational force produced by one star diverges logarithmically so that the distribution of the total force is a marginal Gaussian distribution intermediate between Gaussian and Lévy laws. It has a Gaussian core as if the CLT were applicable (but the variance diverges logarithmically with \(N\)) and an algebraic tail produced by the nearest neighbor as for a Lévy law. Therefore, by changing the dimension of space, we can pass from Gaussian (\(d = 1\)) to Lévy (\(d = 3\)) laws with an interesting limit case (\(d = 2\)). This transition has not been reported before in the context of gravitational dynamics and we think that it deserves a particular discussion.

Finally, the systematic study of the distribution of the gravitational force in various dimensions of space is interesting in view of the different analogies with other physical systems. In \(d = 3\), we have already mentioned the analogies between the statistics of the gravitational force creation between Gaussian and Lévy laws. It has a Gaussian core as if the CLT were applicable (but the variance diverges logarithmically with \(N\)) and an algebraic tail produced by the nearest neighbor as for a Lévy law. Therefore, by changing the dimension of space, we can pass from Gaussian (\(d = 1\)) to Lévy (\(d = 3\)) laws with an interesting limit case (\(d = 2\)). This transition has not been reported before in the context of gravitational dynamics and we think that it deserves a particular discussion.

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1. It is interesting to note that Chandrasekhar (1943) \[1\] did not mention the connection between the Holtsmark distribution and Lévy laws. At that time, the work of Lévy (1937) \[47\] was essentially known among mathematicians and had not diffused yet in the physical and astrophysical communities.
ated by stars in a galaxy and the statistics of the electrostatic force created by a plasma composed of simple ions. In $d = 2$, the statistics of the gravitational force is similar to the statistics of the velocity field created by a random distribution of point vortices studied by Chavanis & Sire \cite{40,41,42} (see also \cite{43,44,45,46,47,48,49}), to the statistics of the force created by dislocations \cite{50} or to the statistics of the electrostatic field in a two-dimensional plasma \cite{51}. In $d = 1$, we are not aware of any particular analogy except with one dimensional plasmas.

This paper is organized as follows. In Sec. 2 we recall the main lines of the calculation of the distribution of the gravitational force in $d = 3$ dimensions leading the the Holtsmark \cite{31} distribution and discuss the main properties of this distribution. In Sec. 3 we determine the distribution of the gravitational force in $d = 2$ dimensions by adapting the results of Chavanis & Sire \cite{40} for point vortices to the present context. In Sec. 4 we determine the distribution of the gravitational force in $d = 1$ dimension. Finally, in Sec. 5 we generalize the results of this paper to the case of an inhomogeneous distribution of particles with arbitrary power-law density profile (or fractal distribution) and arbitrary power-law force in a $d$-dimensional universe. In Appendix A we give the distribution of the force created by the nearest neighbor in arbitrary dimension $d$. For $d \geq 2$, this expression provides a good approximation of the tail of the true distribution. Concerning the notations employed in this paper, we shall use the expression of the gravitational force in $d$ dimensions obtained from the Poisson equation written as $\Delta \Phi = S_d G \rho$ where $S_d$ is the surface of a unit sphere in $d$ dimensions and $\rho$ is the density distribution. Therefore, the force (by unit of mass) created at the origin $O$ by a single star located in $r$ is $G m r / r^3$ in $d = 3$, $G m r / r^2$ in $d = 2$ and $G m \text{sgn}(x)$ in $d = 1$. With this convention, the gravitational constant $G$ depends on the dimension of space (it will be denoted $G_d$ in case of ambiguity). We shall also call the particles giving rise to the gravitational force “stars” although they can be filaments, sheets or other objects.

2 The Holtsmark distribution in $d = 3$

The statistics of the gravitational force in $d = 3$ dimensions was first studied by Chandrasekhar \cite{11} by analogy with the statistics of the electrostatic force in a plasma studied by Holtsmark \cite{31}. Chandrasekhar computed the probability density $W(F)$ that a test star experiences a random force per unity mass $F$. He assumed that there are no correlation between the positions of the stars (Poisson distribution) and that the medium is infinite and homogeneous.\footnote{As is well-known, an infinite and homogeneous distribution of stars is not a steady state of a self-gravitating system. Thus, Chandrasekhar made a sort of “Jeans’ swindle” \cite{32}. However, an infinite and homogeneous distribution of masses is justified in cosmology because the expansion of the universe has an effect similar (in the comoving frame) to that of a neutralizing background in plasma physics \cite{17}.} The case of a finite uniform density distribution has been considered by Ahmad & Cohen \cite{15}: in that case, the distribution of force $W_N(F)$ depends on the total number of stars $N$. On the other hand, the case of a system of non-uniform density has been treated by Kandrup \cite{14,59}. His study demonstrates that the basic results are rather independent upon the density profile. Chandrasekhar & von Neumann have used their stochastic model to determine the speed of fluctuations $T(F)$ \cite{10}, the diffusion coefficient of stars \cite{10} (a calculation completed by Kandrup \cite{14}) and the spatial \cite{11,12} and temporal \cite{13} correlations of the gravitational field.

Let us consider a collection of $N$ stars with mass $m$ randomly distributed in a sphere of radius $R$ with a uniform density $\rho = 3N/(4\pi R^3)$ in average. The exact gravitational force by unit of mass created at the center $O$ of the domain is

$$F = \sum_{i=1}^{N} \mathbf{f}_i, \quad \mathbf{f}_i = G m \frac{\mathbf{r}_i}{r_i^3}.$$  \hspace{1cm} (1)

In each realization, we choose at random the position of the stars with a uniform distribution. Since the positions of the individual stars fluctuate from one realization to the other, the value of the total force fluctuates too and we are interested by its distribution $W(F)$. The problem then consists in determining the distribution of a sum of random variables. The distribution of the force created by one star is obtained by writing $W(F) dF = \tau(r) dr$ where $\tau(r) = 3/(4\pi R^3)$ denotes the density probability of finding the star in $r$ and $f = G m r / r^3$ according to Eq. (1). The Jacobian of the transformation $r \rightarrow f$ is readily evaluated leading to $df = 2(Gm)^{-3/2} f^{-3/2} dr$. Therefore, the distribution of the individual forces is given, for $f > Gm/R^2$, by the pure power-law

$$W(f) = \frac{1}{2} (Gm)^{3/2} \frac{3}{4\pi R^3} f^{-9/2}. \hspace{1cm} (2)$$

The variance of the force created by one star

$$\langle f^2 \rangle = \frac{3}{4\pi R^3} \int \left( \frac{Gm}{r^2} \right)^2 4\pi r^2 dr \propto \int_0^{+\infty} \frac{1}{r^2} dr \hspace{1cm} (3)$$

diverges algebraically due to the behaviour at small distances $r \rightarrow 0$ (corresponding to large forces $f \rightarrow +\infty$). Therefore, the CLT is not applicable. As we shall see, the distribution of the total force is a Lévy law known as the Holtsmark distribution since it was first determined by Holtsmark in the context of the electric field created by a gas of simple ions \cite{31}. We briefly summarize the procedure developed by Chandrasekhar \cite{11} to compute the distribution of the force. This summary is useful to compare with the results in other dimensions.

in Chandrasekhar’s study, since the distribution of the gravitational force is dominated by the contribution of the nearest neighbor, it is permissible to extend the size of the system to infinity without strong influence on the distribution of the force. As shown by Kandrup \cite{59}, only the local density of matter close to the star under consideration is important in determining the distribution of the force.
Since there are no correlation between the stars, the distribution of the gravitational force for any value of $N$ can be expressed as

$$W_N(F) = \prod_{i=1}^{N} \tau(r_i) dr_i \delta(F - \sum_{i=1}^{N} f_i),$$

(4)

where $\tau(r_i) = 3/(4\pi R^3)$ governs the probability of occurrence of the $i$-th star at position $r_i$. Now, using a method originally due to Markov, we can express the $\delta$-function appearing in Eq. (4) in terms of its Fourier transform. In that case, $W_N(F)$ becomes

$$W_N(F) = \frac{1}{(2\pi)^3} \int A_N(k) e^{-ik \cdot F} dk,$$

(5)

with

$$A_N(k) = \left( \frac{3}{4\pi R^3} \int_{|r|=0}^{R} e^{ik \cdot r} dr \right)^N,$$

(6)

where we have written $f = Gm/r^3$. We shall see that the distribution of the force is dominated by the contribution of the nearest neighbor. Therefore, we can consider the thermodynamic limit

$$N \to +\infty, \quad R \to +\infty, \quad n = \frac{3N}{4\pi R^3} = \text{const.}$$

(7)

In this limit, we obtain [1]:

$$W(F) = \frac{1}{(2\pi)^3} \int A(k) e^{-ik \cdot F} dk,$$

(8)

with

$$A(k) = e^{-n C(k)}, \quad C(k) = \int_{|r|=0}^{+\infty} (1 - e^{ik \cdot r}) dr.$$  

(9)

The integral can be calculated easily [1] leading to

$$A(k) = e^{-ak^{3/2}}, \quad a = \frac{4}{15}(2\pi Gm)^{3/2}n.$$  

(10)

Therefore, the distribution of the gravitational force is given by the Holtsmark distribution [8,1]:

$$W(F) = \frac{1}{2\pi^2 F} \int_{0}^{+\infty} e^{-ak^{3/2}} \sin(kF) dk.$$  

(11)

It has the asymptotic behaviours [1]:

$$W(F) \sim \frac{1}{3\pi^2} \left( \frac{15}{4} \right)^2 \frac{1}{(2\pi Gm)^n n^2} \quad (F \to 0),$$  

(12)

$$W(F) \sim \frac{1}{2}(Gm)^{3/2} n F^{-9/2} \quad (F \to +\infty).$$  

(13)

Therefore, the variance of the gravitational force

$$\langle F^2 \rangle \propto \int_{0}^{+\infty} \frac{dF}{F^{1/2}}$$  

(14)

diverges algebraically because of the contribution of large fields $F \gg 1$. On the other hand the average value of the force is [14]:

$$\langle F \rangle = 4\Gamma \left( \frac{1}{3} \right) \left( \frac{8\sqrt{2}}{15} \right)^{2/3} Gmn^{2/3} \simeq 8.879Gmn^{2/3}. $$  

(15)

The typical force exerted upon a test particle is thus of the magnitude $Gm/D^2$ which might be expected to arise from a few particularly nearby field stars at the interstellar distance $D \sim n^{-1/3}$. Writing this typical force as $F_0 = Gmn^{2/3}$, Eqs. (12) and (13) give the asymptotic behaviour of the Holtsmark distribution for $F \ll F_0$ and $F \gg F_0$ respectively.

It is instructive to compare the Holtsmark distribution [11] with the distribution of the force created by the nearest neighbor given by [11]:

$$W_{n.n.}(F) = \frac{1}{2}(Gm)^{3/2} n F^{-9/2} e^{-4\pi(Gm)^{3/2} n/F^{3/2}}.$$  

(16)

It has the asymptotic behaviour

$$W_{n.n.}(F) \sim \frac{1}{2} G^{3/2} m^{3/2} n F^{-9/2} \quad (F \to +\infty),$$  

(17)

which is in exact agreement with the asymptotic behaviour [13] of the Holtsmark distribution. Therefore, the highest fields are produced only by the nearest neighbor. By contrast, in the limit of weak forces, the two distributions disagree: whereas the nearest neighbor distribution vanishes exponentially, the Holtsmark distribution tends to a constant value [12]. This reflects the fact that in the case of extremely weak forces, more than one field star plays a significant role. The Holtsmark distribution is compared to the distribution of the force due to the nearest neighbor in Fig. 1 and we get a good agreement for
sufficiently large forces $F \gg F_0$. The typical force due to the nearest neighbor is $F_{n.n.} \sim Gm/D^2 \sim Gmn^{2/3}$ where $D$ is the average distance between stars. It is precisely of the same order as the average value of the force due to all the stars. More precisely, the average value of the force due to the nearest neighbor is \[ |F| = \sum_{i=1}^{N} |\langle f_{i,j} \rangle| \] and since the effective force decreases like $1/r^4$, we find a finite value. If we write \[ \langle F \rangle = N \langle f_{eff} \rangle = n \int_{0}^{+\infty} \frac{Gm}{r^2} \frac{1}{1+r^2/A^2} 4\pi r^2 dr \] we obtain \[ \langle F \rangle = 2\pi^2 \left[ \frac{4\Gamma(\frac{2}{3})}{9\pi} \left(\frac{3}{4\pi}\right)^{2/3}\right]^{1/2} Gmn^{2/3} = 5.349Gmn^{2/3}, \] which is close to the exact result \[ 5. \] On the other hand, the average value of the squared force is clearly additive since the particles are uncorrelated \[ \langle F^2 \rangle = \sum_{i,j} f_i \cdot f_j = \sum_{i=1}^{N} \langle f_i^2 \rangle + \sum_{i \neq j} \langle f_i \rangle \cdot \langle f_j \rangle = \sum_{i=1}^{N} \langle f_i^2 \rangle. \] Therefore, we find that the variance of the gravitational force is infinite \[ \langle F^2 \rangle = N \langle f^2 \rangle = n \int_{0}^{+\infty} \frac{(Gm)^2}{r^2} 4\pi r^2 dr = +\infty \] in agreement with Eq. \[ 14. \]

3 The marginal Gaussian distribution in $d = 2$

The gravitational field produced in $O$ by an infinite rod of mass per unit length $\mu$ is $f = 2G\mu x/r^2$. This corresponds to the gravitational force $f = G_2 m x/r^2$ created by a mass $m$ in two dimensions provided that we make the correspondence $G_2 = 2G\mu/m$. Now, the statistics of the gravitational force in $d = 2$ dimensions can be directly obtained from the work of Chavanis & Sire \[ 17. \] on the statistics of the velocity created by a random distribution of point vortices in 2D hydrodynamics (it suffices to make the correspondence $\gamma/(2\pi) \rightarrow Gm$ where $\gamma$ is the circulation of a point vortex). There are indeed remarkable analogies between stellar systems and 2D vortices \[ 63. \] Adapting the procedure of Chandrasekhar & von Neumann \[ 10. \] Chavanis & Sire \[ 47, 48, 49. \] have used this stochastic approach to obtain an estimate of the diffusion coefficient of point vortices when their distribution is homogeneous. This is another manifestation of the deep formal analogy between stars and galaxies. Other related works on the statistics of the velocity created by point vortices, including direct numerical simulations to test the theoretical results, have been performed in \[ 50, 51, 52, 53, 54, 55. \] Let us consider a collection of $N$ particles with mass $m$ randomly distributed in a disk of radius $R$ with a uniform density $n = N/(\pi R^2)$ in average. The force by unit of mass created at the center $O$ of the domain is

\[ F = \sum_{i=1}^{N} f_i, \quad f_i = \frac{Gm}{r_i^2} r_i. \]
The problem consists in determining the distribution of a sum of random variables. The distribution of the force created by one star is obtained by writing \( W(f) df = \tau(r) dr \) where \( \tau(r) = 1/(r R^2) \) denotes the density probability of finding the star in \( r \). Using \( df = (Gm)^{-2} f^4 dr \), we obtain, for \( f > Gm/R \), the pure power-law:

\[
W(f) = (Gm)^2 \frac{1}{\pi R^2} f^{-4}.
\]

The variance of the force created by one star

\[
\langle f^2 \rangle = \frac{1}{(2\pi)^2} \int \left( \frac{Gm}{r} \right)^2 2\pi r dr \propto \int_0^{+\infty} \frac{dr}{r} \quad (27)
\]
diverges logarithmically due to the behaviour at small and large distances (corresponding to weak \( f \to 0 \) and large \( f \to +\infty \) forces). Therefore, strictly speaking, the CLT is not applicable. However, since the divergence of the variance is weak (logarithmic) we shall see that the distribution of the total force is intermediate between Gaussian and Lévy laws. The core of the distribution is Gaussian as if the CLT were applicable (but the variance diverges logarithmically with \( N \)) while the tail is algebraic, and produced by the nearest neighbor, as for a Lévy law.

Following the method previously exposed and considering the thermodynamic limit

\[
N \to +\infty, \quad R \to +\infty, \quad n = \frac{N}{\pi R^2} = \text{const.} \quad (28)
\]
we obtain

\[
W(F) = \frac{1}{(2\pi)^2} \int A(k) e^{-akF} dk, \quad (29)
\]

with

\[
A(k) = e^{-nC(k)}, \quad C(k) = \int_0^R (1 - e^{ikr}) dr, \quad (30)
\]

where we have written \( f = Gm r/r^2 \). Note that we cannot let \( R \to +\infty \) in the last integral since it diverges logarithmically for large \( r \). Still, the procedure is well-defined mathematically if we view \( (26) \) as an equivalent of \( W_{\text{N}}(F) \) for large \( N \), not a true limit. The integral in \( (30) \) can be calculated explicitly \(47\) leading to

\[
A(k) = e^{-ak^2 \ln \left( \frac{Gm}{R} \right)}, \quad a = \frac{1}{4} \frac{Gm}{\pi}. \quad (31)
\]

For \( F < F_{\text{crit}}(N) \) where \( F_{\text{crit}}(N) \) is defined by Eq. \( (37) \), we need to consider large values of \( k \) in Eq. \( (31) \) and we can neglect the contribution of \( k \) in the logarithm, writing \( A(k) \sim e^{-ak^2 \ln N} \). Therefore, we get a Gaussian distribution

\[
W(F) = \frac{1}{n(Gm)^2 \pi^2} e^{-\frac{F^2}{n(Gm)^2 \pi \ln N}} \quad (F < F_{\text{crit}}(N)), \quad (32)
\]
as if the CLT were applicable. However, if we were to extend this distribution for all values of \( F \), we see that the variance of this distribution

\[
\langle F^2 \rangle = \frac{n(Gm)^2 \pi \ln N}{4} \quad (33)
\]
diverges logarithmically with \( N \) due to cooperative effects. On the other hand, the average value of the force is

\[
\langle F \rangle = \left( \frac{1}{4} \frac{nG^2 m^2 \pi^2 \ln N}{4} \right)^{1/2}. \quad (34)
\]

For \( F > F_{\text{crit}}(N) \), we need to consider small values of \( k \) in Eq. \( (31) \) and its contribution in the logarithm becomes crucial, so that \( A(k) \sim e^{2ak^2 \ln k} \). In that case, we find after some calculation \(47\) that

\[
W(F) = n(Gm)^2 F^{-4} \quad (F > F_{\text{crit}}(N)). \quad (35)
\]

Therefore, the distribution of the gravitational force in \( d = 2 \) has an algebraic tail as for a Lévy law. The variance of the gravitational force

\[
\langle F^2 \rangle \propto \int_0^{+\infty} \frac{dF}{F} \quad (36)
\]
diverges logarithmically due to the contribution of large field strengths. Comparing Eqs. \( (32) \) and \( (35) \), we find that the typical force where the two regimes (Gaussian core and algebraic tail) connect each other is

\[
F_{\text{crit}}(N) \sim (nG^2 m^2 \pi \ln N)^{1/2} \ln^{1/2}(\ln N). \quad (37)
\]

For \( N \to +\infty \), \( F_{\text{crit}}(N) \to +\infty \) so, strictly speaking, the algebraic tail is rejected to infinity and the limit distribution \( W(F) \) is Gaussian. However, for large but finite values of \( N \), the convergence to the limit distribution is so slow that the algebraic tail is always visible in practice. The contribution of the high field tail to the average value of the force is

\[
\langle F \rangle = 2\pi n(Gm)^2 \int_{F_{\text{crit}}}^{+\infty} \frac{dF}{F^2} = \left( \frac{4\pi nG^2 m^2}{\ln N \ln(\ln N)} \right)^{1/2}, \quad (38)
\]

which is smaller than the contribution \( (35) \) due to the core of the distribution.

It is instructive to compare the marginal Gaussian distribution with the distribution of the force created by the nearest neighbor given by \(47\):

\[
W_{\text{n.n.}}(F) = n(Gm)^2 F^{-4} e^{-\frac{aF^2}{2}}. \quad (39)
\]

It has the asymptotic behaviour

\[
W_{\text{n.n.}}(F) \sim n(Gm)^2 F^{-4} \quad (F \to +\infty), \quad (40)
\]

which is in exact agreement with the asymptotic behaviour \( (35) \) of the marginal Gaussian distribution. Therefore, the highest fields are produced only by the nearest
neighbor as for a Lévy law. However, the Gaussian distribution of the core is created by all the particles so that the distribution \( p(x) \) does not provide a good approximation of the distribution for intermediate values of the force. The marginal Gaussian distribution is compared to the distribution of the force due to the nearest neighbor in Fig. 2 and we get a good agreement only in the tail of the distribution. The typical force due to the nearest neighbor is \( F_{n,n} \sim Gm/D \sim Gmn^{1/2} \) where \( D \sim n^{-1/2} \) is the average distance between stars. More precisely, the average value of the force due to the nearest neighbor is

\[
\langle F \rangle_{n.n.} = \pi Gmn^{1/2}. \tag{41}
\]

It is less than the average value of the force \( \langle F \rangle_{n} \) due to all the stars because of the \( \ln N \) factor arising from cooperative effects. However, apart from this logarithmic term, they are of the same order of magnitude. This means that the force created by the nearest neighbor is of the same order as the force due to all the other particles (up to a logarithmic correction). This is another manifestation of the fact that we lie at the frontier between Gaussian and Lévy laws.

The present approach shows that only stars close to the star under consideration determine the fluctuations of the gravitational field (for large forces). In fact, by adapting the calculations of Agekyan in \( d = 2 \), it is possible to show that the “effective” force created by a star at distance \( r \) from the star under consideration is given in good approximation by \( \langle F \rangle = n \int_{0}^{r_{max}} \frac{1}{1 + x/A} \, dx \),

\[
f_{\text{eff}} = \frac{Gm}{r} \left( 1 + \frac{r}{A} \right), \tag{42}
\]

where

\[
A = (16n \ln N)^{-1/2} \tag{43}
\]

is of the order of the interparticle distance \( D \). For weak separations, one has \( f_{\text{eff}} \rightarrow Gm/r \) but for large separations \( r \gg D \), the effects of individual stars compensate each other and the resulting force is reduced by a factor \( (r/D)^{4/2} \). The average value of the modulus of the force can be obtained by summing the modulus of the effective force writing

\[
\langle F \rangle = N \langle f_{\text{eff}} \rangle = n \int_{0}^{r_{max}} \frac{1}{1 + x/A} \, 2\pi r \, dr. \tag{44}
\]

This yields

\[
\langle F \rangle = \left( \frac{1}{16} n G^2 m^2 \pi^2 \ln N \right)^{1/2}, \tag{45}
\]

which is comparable to the exact result \( \langle F \rangle \approx 2.5 \langle F \rangle_1 \). On the other hand, the variance of the gravitational force is infinite

\[
\langle F^2 \rangle = N \langle f^2 \rangle = n \int_{0}^{r_{max}} \frac{Gm}{r} \left( 1 + \frac{r}{A} \right) ^2 \, 2\pi r \, dr = +\infty \tag{46}
\]

in agreement with Eq. \( \langle F \rangle \rangle_0 \).

---

4 The Gaussian distribution in \( d = 1 \)

The gravitational field produced in \( O \) by an infinite sheet of mass per unit surface \( \mu \) is \( f = 2\pi G\mu \text{sgn}(x) \). This corresponds to the gravitational force \( f = G_{1m} \text{sgn}(x) \) created by a mass \( m \) in one dimension provided that we make the correspondence \( G_{1} = 2\pi G\mu/m \). As we have mentioned in the Introduction, the one dimensional self-gravitating system (OGS) has been suggested as a model for the motion of stars perpendicular to the plane of highly flattened disk galaxies and it has been extensively studied in numerical simulations for its computational ease.

The statistics of the gravitational force in \( d = 1 \) dimension created by a Poissonian distribution of stars is relatively straightforward. Let us consider a collection of \( N \) particles with mass \( m \) randomly distributed in an interval \([-L, L] \) with a uniform density \( n = N/(2L) \) in average. The force by unit of mass created at the center \( O \) of the domain is

\[
F = \sum_{i=1}^{N} f_i, \quad f_i = Gm \text{sgn}(x_i), \tag{47}
\]

where \( \text{sgn}(x) = +1 \) if \( x > 0 \) and \( \text{sgn}(x) = -1 \) if \( x < 0 \). The total force in \( O \) can be written

\[
F = Gm(N_+ - N_-), \tag{48}
\]

where \( N_+ \) is the number of stars in the interval \( 0 < x \leq L \) and \( N_- \) is the number of stars in the interval \(-L \leq x < 0 \). Since \( N_+ + N_- = N \), we can rewrite Eq. \( \langle F \rangle \rangle_0 \) as

\[
F = Gm(2N_+ - N). \tag{49}
\]

We note that, in one dimension, the gravitational force takes only discrete values. On the other hand, according to Eq. \( \langle F \rangle \rangle_0 \), the probability of the fluctuation \( F \) is equal...
to the probability of having $N_+$ stars in the interval $0 < x \leq L$. Since the $N$ particles are uniformly distributed in the domain of size $2L$, the probability that a star is in the interval $0 < x \leq L$ is $p = L/(2L) = 1/2$ and the probability that a star is in the interval $-L \leq x < 0$ is $q = 1 - p = L/(2L) = 1/2$. Therefore, the probability to have $N_+$ stars in the interval $0 < x \leq L$ is given by the Bernouilli distribution:

$$W_N(N_+) = \frac{N!}{N_![(N - N_+)!]} \left(\frac{1}{2}\right)^N.$$  \hspace{1cm} (50)

The first two moments of this distribution are $\langle N_+ \rangle = N/2$ and $\langle (N_+ - \langle N_+ \rangle)^2 \rangle = N/4$. Using Eq. (50), the distribution of the gravitational force in $d = 1$ is exactly (i.e. for any $N$) given by the Bernouilli distribution

$$W_N(F) = \frac{N!}{\left(\frac{N}{2} + \frac{F}{2Gm}\right)!(\frac{N}{2} - \frac{F}{2Gm})!} \left(\frac{1}{2}\right)^N,$$  \hspace{1cm} (51)

with $\langle F \rangle = 0$, $\langle F^2 \rangle = NG^2m^2$.  \hspace{1cm} (52)

This last result can be obtained without computation since the variance of the force created by one particle is finite and given by

$$\langle f^2 \rangle = G^2m^2.$$  \hspace{1cm} (53)

Since the particles are uncorrelated, and since $\langle f \rangle = 0$, we have $\langle F^2 \rangle = \sum_i \sum_j (f_i f_j) = \sum_{i=1}^{N} (f_i^2) + \sum_{i \neq j} (f_i f_j) = N \langle f^2 \rangle = NG^2m^2$.

For $N \gg 1$, the CLT applies and we get the Gaussian distribution

$$W(F) = \frac{1}{\sqrt{2\pi NG^2m^2}} e^{-F^2/(2NG^2m^2)}.$$  \hspace{1cm} (54)

In the limit $N \to +\infty$, the natural scaled variable is $F/\sqrt{N}$. The thermodynamic limit $N \to +\infty$, $L \to +\infty$ with $N/L$ fixed is not valid in $d = 1$. The result (51) can also be obtained from the Bernouilli distribution (50) which becomes Gaussian in the limit of large numbers

$$W(N_+) \simeq \left(\frac{2}{N\pi}\right)^{1/2} e^{-\frac{1}{8}(N_+ - N/2)^2}.$$  \hspace{1cm} (55)

The comparison between the Bernouilli distribution and the Gaussian distribution is shown in Fig. 3. Note that in Eqs. (51) and (55) the particle number $N_+$ and the gravitational force $F$ are treated as continuous variables so that the normalization conditions are $\int_{-\infty}^{+\infty} W_N(F) dF = 1$ and $\int_{-\infty}^{+\infty} W(N_+) dN_+ = 1$ while in Eqs. (51) and (55) the particle number $N_+$ and the gravitational force $F/Gm$ are discrete variables so that the normalization conditions are $\sum_{N_+ = 0}^{\infty} W_N(F) = 1$ and $\sum_{N_+ = 0}^{\infty} W_N(N_+) = 1$.

Therefore, the relations between the discrete and the continuous distributions are $W_N(N_+) = W(N_+) dN_+$ with $dN_+ \simeq 1$ (so that $W_N(N_+) = W(N_+) dN_+$) and $W_N(F) = W(F) dF$ with $dF = 2GmdN_+ \simeq 2Gm$ (so that $W_N(F) = 2GmW(F)$).

**Fig. 3.** The Gaussian distribution $W_\infty(F) = 2GmW(F)$ in $d = 1$. It is compared with the exact Bernouilli distribution $W_N(F)$ valid for any $N$. We have set $F_0 = Gm$ and taken $N = 1000$. In the preceding calculations, we have assumed that the distribution of stars is spatially homogeneous and we have focused on the force at the center of the domain. Since the force only depends on the number of stars in the left and right intervals, and not on their precise distribution, the above results remain valid for any symmetrical distribution of the stars. If the distribution is not symmetric with respect to the point under consideration, we just have to compute the probability $p$ of finding a star in the right interval and use the general Bernouilli formula. To be specific, consider an arbitrary distribution of stars with numerical density $n(x')$ in the interval $[L_{min}, L_{max}]$. We are interested in the distribution of the gravitational force at $x$. The probability that a star is in the right interval $[x, L_{max}]$ is

$$p(x) = \frac{1}{N} \int_{x}^{L_{max}} n(x') dx',$$  \hspace{1cm} (56)

and its probability to be in the left interval $L_{min}, x]$ is $q(x) = 1 - p(x)$. For example, if the stars are uniformly distributed in the interval $[-L, L]$, we have $p(x) = \frac{1}{2}(1 - \frac{x}{L})$ and $q(x) = \frac{1}{2}(1 + \frac{x}{L})$. Now, the probability to have $N_+$ stars in the right interval is given by the Bernouilli distribution:

$$W_N(N_+, x) = \frac{N!}{N_![(N - N_+)!]} p(x)^{N_+} q(x)^{N - N_+}.$$  \hspace{1cm} (57)

The first two moments of this distribution are $\langle N_+ \rangle = Np$ and $\langle (N_+ - \langle N_+ \rangle)^2 \rangle = Npq$. Using Eq. (59), the exact distribution of the gravitational force in $x$ is

$$W_N(F, x) = \frac{N! p(x)^{\frac{F}{2Gm}} q(x)^{\frac{F}{2Gm}}}{\left(\frac{F}{2Gm}\right)!(\frac{F}{2Gm})!}.$$  \hspace{1cm} (58)

with

$$\langle F \rangle = NGm(2p - 1),$$  \hspace{1cm} (59)

$$\langle (F - \langle F \rangle)^2 \rangle = 4NG^2m^2pq.$$  \hspace{1cm} (60)
The results (65)-(69) can be obtained without computation. The distribution of the gravitational force created by one star is $W(f) = 0$ if $f \neq \pm Gm$, $W(f) = p(x)$ if $f = Gm$ and $W(f) = 1 - p(x) = q(x)$ if $f = -Gm$. The average value and the variance of the force created by one star are given by

$$\langle f \rangle = pGm + (1 - p)(-Gm) = (2p - 1)Gm,$$

$$\langle f^2 \rangle = p(Gm)^2 + (1 - p)(-Gm)^2 = G^2 m^2. \quad (61)$$

Now, $\langle F \rangle = \sum_{i=1}^{N} \langle f_i \rangle = N \langle f \rangle$ and, since the particles are uncorrelated, $\langle F^2 \rangle = \sum_{i,j} \langle f_i f_j \rangle = \sum_{i=1}^{N} \langle f_i^2 \rangle + \sum_{i \neq j} \langle f_i f_j \rangle = N \langle f^2 \rangle + N(N-1)\langle f \rangle^2$. Combining the previous results, we immediately obtain Eqs. (60) and (61).

Finally, for $N \gg 1$, the CLT applies and we get the Gaussian distribution

$$W(F) = \frac{1}{\sqrt{2\pi \langle F - \langle F \rangle \rangle^2}} e^{-\frac{(F - \langle F \rangle)^2}{2\langle F - \langle F \rangle \rangle^2}}. \quad (62)$$

The result (62) can also be obtained from the Bernoulli distribution (60) which becomes Gaussian in the limit of large numbers.

5 Inhomogeneous medium and power-law potential in $d$ dimensions

In this section, we determine the distribution of the force created by an inhomogeneous distribution of particles in $d$ dimensions. To be specific, we consider a power-law density profile $n(r) = K/r^p$ for $r \leq R$ and $n(r) = 0$ for $r > R$. We assume $0 \leq p < d$ in order to have a decreasing density distribution that is normalizable as $r \to 0$. Then, $K = (d - p) N/(S_d R^{d-p})$. The uniform profile is recovered for $p = 0$ and $K = n$. For the sake of generality, we consider a force of the form

$$f = Gm \frac{r}{r^{d+\alpha}}, \quad (64)$$

with $d + \alpha - 1 > 0$. The gravitational force is recovered for $\alpha = 0$. The distribution of the force created by one particle is obtained by writing $W(f) df = \tau(r) dr$ where $\tau(r) = (K/n) r^{-p}$. Using the transformation

$$df = (d + \alpha - 1)(Gm)^{-\frac{d}{d+\alpha}} f^{\frac{d+\alpha}{d+\alpha-1}} dr, \quad (65)$$

we obtain, for $f > Gm/R^{d+\alpha-1}$, a pure power law

$$W(f) = \frac{1}{d + \alpha - 1} \frac{d - p}{S_d R^{d-p}} (Gm)^{-\frac{d-p}{d+\alpha-1}} f^{-(d+\alpha-1)/d}, \quad (66)$$

decreasing with an exponent

$$\gamma = d + \frac{d-p}{d+\alpha-1}. \quad (67)$$

This distribution is normalizable provided that $\gamma > d$ which is equivalent to our previous assumptions. For the gravitational force ($\alpha = 0$) and for a homogeneous distribution of stars ($p = 0$), one has $\gamma = d^2/(d-1)$ for $d > 1$. We note that the variance of the force created by one star diverges algebraically for

$$2\alpha + p > 2 - d, \quad (69)$$
due to the behaviour at small distances $r \to 0$. In that case, the distribution of the (total) force is a Lévy law. For a uniform distribution ($p = 0$), the criterion (69) gives $\alpha > (2-d)/2$ and for the gravitational case ($\alpha = 0$) it gives $p > 2 - d$. For a uniform distribution and a gravitational force ($p = \alpha = 0$), we recover the condition $d > 2$. In the following, we assume that inequality (69) is fulfilled. The critical case where (69) is an equality, corresponding to a logarithmic divergence of the variance, will be treated specifically in Sec. 5.7.

These criteria can also be expressed in terms of the index $\gamma$ of the individual distribution (60). For a given dimension of space $d$, we introduce the critical exponent $\gamma_c = 2 + d$. The variance of $W(f)$ diverges algebraically (due to its behaviour for large $f$) when $\gamma < \gamma_c$. In that case, the distribution $W(F)$ is a Lévy law. For the critical case $\gamma = \gamma_c$, the variance of $W(f)$ diverges logarithmically and $W(F)$ is a marginal Gaussian distribution. For $\gamma > \gamma_c$, the variance of $W(f)$ is finite and $W(F)$ is a Gaussian distribution. In the following, we assume

$$d < \gamma < \gamma_c = 2 + d \quad (70)$$

and in Sec. 5.7 we consider the critical case $\gamma = \gamma_c$.

5.1 The distribution of the force

We wish to determine the distribution of the total force

$$F = \sum_{i=1}^{N} f_i, \quad (71)$$

created by the particles. Since there are no correlation between the particles, the distribution of the gravitational force for any value of $N$ is given by

$$W_N(F) = \prod_{i=1}^{N} \tau(r_i) dr_i \delta \left( F - \sum_{i=1}^{N} f_i \right), \quad (72)$$

where $\tau(r_i)$ governs the probability of occurrence of the $i$-th star at position $r_i$. Now, using the Markov method, we express the $\delta$-function appearing in Eq. (72) in terms of its Fourier transform

$$\delta(x) = \frac{1}{(2\pi)^d} \int e^{-ik \cdot x} dk. \quad (73)$$

Then, $W_N(F)$ can be written

$$W_N(F) = \frac{1}{(2\pi)^d} \int A_N(k) e^{-ik \cdot F} dk, \quad (74)$$
with
\[ A_N(k) = \left( \int_{|r|=0}^{R} \tau(r) e^{i k \cdot r} dr \right)^N, \]  
\[ (75) \]
where \( f \) is given by Eq. (64). Using \( \int_{|r|=0}^{R} \tau(r) dr = 1 \), and \( \tau(r) = n(r)/N \), the foregoing expression is equivalent to
\[ A_N(k) = \left( 1 - \frac{1}{N} \int_{|r|=0}^{R} (1 - e^{i k \cdot r}) n(r) dr \right)^N. \]  
\[ (76) \]
We now consider the limit \( N \to +\infty, R \to +\infty \) with \( N/R^{d-p} \) fixed. In this limit, the distribution of the force can be written
\[ W(\mathbf{F}) = \frac{1}{(2\pi)^d} \int A(k) e^{-i k \cdot \mathbf{F}} dk, \]  
\[ (77) \]
with
\[ A(k) = e^{-C(k)}, \quad C(k) = \int_{|r|=0}^{+\infty} n(r) (1 - e^{i k \cdot r}) dr. \]  
\[ (78) \]
Using the transformation \( \Theta \), we obtain
\[ C(k) = \frac{K}{d + \alpha - 1} (\Gamma m)^{\frac{d-p}{d+\alpha-1}} \times \int_{|r|=0}^{+\infty} (1 - e^{i k \cdot r}) \int_{|r|}^{\infty} \frac{d}{d+\alpha+1} df. \]  
\[ (79) \]
The characteristic function \( C(k) \) converges for \( f \to +\infty \) if \( p < d \) and \( d + \alpha - 1 > 0 \), and for \( f \to 0 \) if \( 2 \alpha + p > 2 - d \).
In other words, it converges if \( d < \gamma < \gamma_c \). Introducing a spherical system of coordinates, we get
\[ C(k) = \frac{K C_d}{d + \alpha - 1} (\Gamma m)^{\frac{d-p}{d+\alpha-1}} \int_{0}^{+\infty} df f^{d-1} \times \int_{0}^{\pi} \sin \theta d^2 (1 - \cos(k f \cos \theta)) f^{\frac{d-d(d+p)}{d+\alpha-1}}. \]  
\[ (80) \]
We have introduced the notation \( C_d = S_d/ \int_{0}^{\pi} (\sin \theta)^{d-2} d\theta \) where \( S_d = 2\pi^{d/2}/\Gamma(d/2) \) represents the surface of a unit sphere in \( d \) dimensions. Using the identity
\[ \int_{0}^{\pi} (\sin \theta)^{d-2} d\theta = \frac{\sqrt{\pi} \Gamma \left( \frac{d-1}{2} \right)}{\Gamma \left( \frac{d}{2} \right)}, \]  
\[ (81) \]
we obtain
\[ C_d = \frac{2\pi^{d-1}}{\Gamma \left( \frac{d}{2} \right)}. \]  
\[ (82) \]
Next, setting \( x = k f \) and using the identity
\[ \int_{0}^{\pi} \cos(x \cos \theta)(\sin \theta)^{d-2} d\theta \]  
\[ = \sqrt{\pi} \left( \frac{2}{x} \right)^{\frac{d}{2}-1} J_{\frac{d}{2}-1}(x) \Gamma \left( \frac{d-1}{2} \right), \]  
\[ (83) \]
we can rewrite Eq. (80) in the form
\[ C(k) = ak^{d-H}, \]  
\[ (84) \]
where
\[ H = \frac{d-p}{d+\alpha-1} = \gamma - d, \]  
\[ (85) \]
\[ a = \frac{S_d}{d+\alpha-1} (\Gamma m)^H KB, \]  
\[ (86) \]
\[ B = \int_{0}^{+\infty} \frac{dx}{x^{H+1}} \left[ 1 - \Gamma \left( \frac{d}{2} \right) \left( \frac{2}{x} \right)^{\frac{d}{2}-1} J_{\frac{d}{2}-1}(x) \right]. \]  
\[ (87) \]
The integral converges for \( x \to +\infty \) if \( H > 0 \), i.e. if \( d > p \) and \( d + \alpha - 1 > 0 \) or, equivalently, if \( \gamma > d \). On the other hand, using
\[ J_d(x) \sim_{x \to 0} \frac{1}{\Gamma(n+1)} \left( \frac{x}{2} \right)^{n} - \frac{1}{\Gamma(n+2)} \left( \frac{x}{2} \right)^{n+2}, \]  
\[ (88) \]
we see that the integral converges for \( x \to 0 \) if \( H < 2 \), i.e. if \( 2\alpha + p < 2 - d \) or, equivalently, if \( \gamma < \gamma_c \). Making an integration by parts and using the identity
\[ \frac{d}{dx} \left[ J_n(x) \right] = \frac{J_n(x) - \frac{n}{x} J_n(x)}{x^n} = -J_{n+1}(x)/x^n, \]  
\[ (89) \]
we can rewrite the function \( B \) in the form
\[ B = \frac{1}{H} \Gamma \left( \frac{d}{2} \right) 2^{\frac{d}{2}-1} \int_{0}^{+\infty} J_{\frac{d}{2}}(x) \frac{dx}{x^{H+\frac{d}{2}-1}}. \]  
\[ (90) \]
It can also be expressed in terms of Gamma functions as
\[ B = \frac{1}{H} \Gamma \left( \frac{d}{2} \right) \Gamma \left( \frac{1-H}{2} \right) \Gamma \left( \frac{d+1-H}{2} \right). \]  
\[ (91) \]
The distribution of the force is given by Eq. (77). Introducing a spherical system of coordinates, it can be rewritten
\[ W(\mathbf{F}) = \frac{C_d}{(2\pi)^d} \int_{0}^{+\infty} dk k^{d-1} \]  
\[ \times \int_{0}^{\pi} \sin \theta d^2 e^{-ak^{d-H} e^{-ikF \cos \theta}}. \]  
\[ (92) \]
Using identity (80), we obtain
\[ W(\mathbf{F}) = \frac{S_d}{(2\pi)^d} \Gamma \left( \frac{d}{2} \right) \left( \frac{2}{F} \right)^{\frac{d}{2}-1} \]  
\[ \times \int_{0}^{+\infty} e^{-ak^{d-H} k^{d} J_{\frac{d}{2}-1}(kF)} dk. \]  
\[ (93) \]
Note that the structure of the distribution only depends on the scaling exponent \( H \) which takes values in the range
\[ 0 < H < \gamma_c = 2. \]  
\[ (94) \]
The asymptotic behaviour of $W(F)$ for small $|F|$ can be obtained by expanding the Bessel function in Taylor series and integrating term by term. This yields

$$W(F) = \frac{S_d}{(2\pi)^d} F^d \left(\frac{d}{2}\right) \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \Gamma\left(\frac{2l+d}{2}\right) F^{2l}. \tag{95}$$

The asymptotic behaviour of $W(F)$ for large $|F|$ can be obtained by using a method similar to the one exposed in Sec. II.C. of [47]. Performing the changes of variables $z = kF$ and $t = -\cos \theta$ in Eq. (92), we obtain

$$W(F) = \frac{C_d}{(2\pi)^d F^d} \int_{1}^{+1} (1 - t^2)^{d/2} dt \times \int_{0}^{+\infty} e^{-a(\frac{1}{4} \pi H)} e^{izt} z^{d-1} dz. \tag{96}$$

In this expression, $t$ and $z$ are real and the domains of integration are on the real axis $-1 \leq t \leq 1$ and $z \geq 0$. Under this form, we cannot expand the exponential in power series for $F \to +\infty$ and integrate term by term because the integrals would diverge. The idea is to work in the complex plane and deform the contours of integration as indicated in Sec. II.C. of [47]. It is then possible to perform the integration on $t$ along the semi-circle $C_+$ of radius unity in the upper-half plane $\text{Im}(t) \geq 0$, and the integration on $z$ along the line such that $izt = -y$ with $y$ real $\geq 0$. We then obtain

$$W(F) = \frac{C_d}{(2\pi)^d} \text{Re} \int_{C_+} (1 - t^2)^{d/2} dt \times \int_{0}^{+\infty} e^{-a(\frac{1}{4} \pi H)} e^{-y} y^{d-1} dy. \tag{97}$$

We can now expand the exponential term in Taylor series and perform the integration on $y$ to obtain

$$W(F) = \frac{C_d}{(2\pi)^d} \sum_{l=0}^{+\infty} \text{Re} \int_{C_+} (1 - t^2)^{d/2} \frac{(-a)^l}{l!} t^{Hl+d} \times \Gamma(Hl+d). \tag{98}$$

We now need to evaluate the integral

$$I = \text{Re} \int_{C_+} \frac{(1 - t^2)^{d/2} \frac{d}{2} i^{Hl+d}}{t^{Hl+d}} \Gamma(Hl+d), \tag{99}$$

with $t = e^{i\theta}$ and $\theta$ going from $\pi$ to $0$. After straightforward calculations, we can rewrite the integral (99) in the form

$$I = -2 i^{d/2} \int_{0}^{\pi} \cos \left(\frac{1}{2} + d + Hl\right) \theta \left( Hl + \frac{d}{2} + \frac{5}{2} \right) \left( \sin \theta \right)^{d/2} d\theta. \tag{100}$$

Using the identity

$$\int_{0}^{\pi} \cos(a\theta + b)(\sin \theta)^k d\theta = \frac{\pi \Gamma\left(\frac{d-1}{2}\right)}{\Gamma\left(\frac{1+d+H}{2}\right). H_l}, \tag{101}$$

we find that

$$I = \frac{\pi \Gamma\left(\frac{d-1}{2}\right)}{\Gamma\left(\frac{1+d+H}{2}\right). H_l}. \tag{102}$$

Combining the previous results, the large $F$ expansion of the distribution of the force can be written

$$W(F) = \frac{C_d}{(2\pi)^d} \sum_{l=0}^{+\infty} \frac{(-a)^l}{l!} t^{Hl+d} \Gamma(Hl+d) \times \frac{\pi \Gamma\left(\frac{d-1}{2}\right)}{\Gamma\left(\frac{1+d+H}{2}\right). H_l}. \tag{103}$$

Now, combining Eqs. (86) and (91), we have

$$a = -\frac{\pi^{d/2}}{d + 1} (Gm)^H K \frac{1}{2^H} \Gamma\left(\frac{1}{2} + Hl\right). \tag{104}$$

Substituting this expression in Eq. (101), using Eq. (82) and the identity

$$2^{2z-1} \Gamma(z) \Gamma\left(z + \frac{1}{2}\right) = \sqrt{\pi} \Gamma(2z), \tag{105}$$

we finally obtain

$$W(F) \sim K \frac{(Gm)^H}{d + 1} \frac{1}{2^H} \Gamma\left(\frac{1}{2} + Hl\right) (F \to +\infty). \tag{106}$$

Therefore, the tail of the distribution decreases with the exponent

$$\gamma = d + H, \tag{107}$$

like for the individual distribution [46]. Furthermore, we shall see in Sec. 5.2 that the asymptotic behaviour of the distribution of the force coincides with the expression [118] derived in the nearest neighbor approximation.

The previous results can also be used for analyzing the stochastic gravitational fluctuations generated by a fractal distribution of field sources (stars or galaxies) provided that we make the correspondence

$$p = d - d_f, \tag{108}$$
where $d_f$ is the fractal dimension of the distribution in a $d$-dimensional universe [63]. We also introduce the exponent $\nu = d + \alpha - 1$ characterizing the power-law decay of the force. For the gravitational interaction $\nu = d - 1$. In terms of these quantities, the scaling exponent $H$ can be written as

$$H = \frac{d_f}{\nu}.$$

(110)

The condition [113] required to have a Lévy law is

$$0 < d_f < 2\nu.$$

(111)

Such a formalism can be useful in cosmology where observations suggest that galaxies are distributed according to a fractal law characterized by a fractal dimension $1 < d_f < 2$ (in a $d = 3$ universe) [63,64]. For this range of values, the scaling exponent satisfies $1/2 < H < 1$.

### 5.2 Nearest neighbor approximation

Let us compare these results with those obtained by making the nearest neighbor approximation (see also Appendix A). For an arbitrary inhomogeneous distribution of particles, the distribution of the nearest neighbor is obtained from the relation

$$\tau_{n.n.}(r)dr = \left(1 - \int_0^r \tau_{n.n.}(r')dr'\right)n(r)S_dr^{d-1}dr,$$

(112)

leading to

$$\tau_{n.n.}(r) = \frac{\tau_{n.n.}(r)}{S_dr^{d-1}} = n(r)e^{-\int_0^r n(x)S_dx^{d-1}dx}.$$

(113)

For a power-law density profile $n(r) = K/r^p$, we get

$$\tau_{n.n.}(r) = \frac{K}{r^p}e^{-\frac{\pi r K}{\sqrt{2}}}.$$

(114)

The distribution of the force due to the nearest neighbor is obtained from the relation $W_{n.n.}(F)dF = \tau_{n.n.}(r)dr$ with

$$F = Gm\frac{r}{r^{(d+\alpha)}}.$$

(115)

Using

$$dF = (d + \alpha - 1)(Gm)^{-d/\alpha+1}F^{-(d+\alpha)}dr,$$

(116)

we obtain

$$W_{n.n.}(F) = K\frac{(Gm)^H}{d + \alpha - 1}F^{-(d+H)}e^{-\frac{\pi FK}{\sqrt{2}}}.$$

(117)

For $F \to +\infty$, we get the asymptotic behaviour

$$W_{n.n.}(F) \sim K\frac{(Gm)^H}{d + \alpha - 1}F^{-(d+H)},$$

(118)

which coincides with the asymptotic behaviour [107] of the exact distribution. Therefore, the tail of the distribution is dominated by the contribution of the nearest neighbour. We note that the moment of order $b$ of the distribution [118] is finite for $b < H$ and its value is

$$\langle F^b \rangle_{n.n.} = \left(\frac{S_bK}{d - p}\right)^{b/H} \langle Gm \rangle^b \Gamma\left(1 - \frac{b}{H}\right).$$

(119)

### 5.3 The dimension $d = 3$

For the ordinary dimension $d = 3$, we have

$$C(k) = a k^H,$$

(120)

where

$$H = \frac{3 - p}{2 + \alpha},$$

(121)

$$a = \frac{4\pi}{2 + \alpha} (Gm)^H KB,$$

(122)

$$K = \frac{(3 - p)N}{4\pi R^3}.$$

(123)

As before, we assume that $0 \leq p < 3$, $\alpha > -2$ and $2\alpha + p > -1$ (or, equivalently, $3 < \gamma < 5$). Therefore, $0 < H < 2$. On the other hand, using

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x,$$

(124)

and $G(3/2) = \sqrt{\pi}/2$, the function $B$ defined by Eq. [87] takes the form

$$B = \int_0^{+\infty} dx \frac{x}{x^H} (x - \sin x) = \frac{1}{H(H + 1)} \int_0^{+\infty} \sin x \frac{x}{x^H} dx,$$

(125)

where we have used two integrations by parts to get the last equality. This expression can also be obtained from Eq. [90] by using the identity

$$J_{3/2}(x) = \sqrt{\frac{2}{\pi x}} \left(\frac{\sin x}{x} - \cos x\right),$$

(126)

and performing an integration by parts. The function $B$ can finally be expressed in terms of the Gamma function as

$$B = \frac{\pi}{2(\Gamma(H + 2) \sin \left(\frac{\pi H}{2}\right))}.$$

(127)

Using identity [108] and the identity

$$\Gamma(x) \Gamma(1 - x) = \frac{\pi}{\sin(\pi x)},$$

(128)

we can check that Eq. [127] is consistent with Eq. [91]. The distribution of the force is then given by

$$W(F) = \frac{1}{2\pi^2 F} \int_0^{+\infty} e^{-akH} \sin(kF) k dk,$$

(129)

where we recall that $0 < H < 2$. For $\alpha = 0$ (gravity), we recover the situation considered by Kandrup [14]. In that case, $H = (3 - p)/2$. If in addition $p = 0$ (homogeneous system), we recover the situation considered by Chandrasekhar [1]. In that case, $H = 3/2$, $K = n$, $B = 4\sqrt{2\pi}/15$ and $a = n/(2\pi Gm)^{3/2}$. We will see that certain results derived by Kandrup [14] contain some mistakes, so it is important to reconsider this situation in detail. In addition, we treat the case of a general power-law potential where $\alpha$ can be non-zero.
The general properties of the distribution \(W(F)\) have been derived by Chandrasekhar \[67\]. Although Chandrasekhar considered a uniform medium, it is important to note that his results remain valid for a power-law density profile; we just need to replace the index that appears in his analysis by \(H\). Let us rewrite the results obtained by Chandrasekhar \[67\] with the present notations. When \(F\) is given by \[67\]:

\[
F = \frac{1}{2\pi^2 H} \sum_{l=0}^{\infty} (-1)^l \Gamma \left( \frac{2l + 3}{H} \right) \frac{1}{a^{2l+3}} (2l+1)!F^{2l}.
\]

In particular,

\[
W(F) \sim \frac{1}{2\pi^2 H} \Gamma \left( \frac{3}{H} \right) \frac{1}{a^{1/2}} F, \quad (F \rightarrow 0).
\]  
(130)

Using identity (106), we can check that Eq. (130) is consistent with Eq. (95). For \(F \rightarrow +\infty\), we have the series expansion \[67\]:

\[
W(F) = \frac{1}{2\pi^2} \sum_{l=1}^{\infty} (-1)^{l+1} \frac{q^l}{l!} \Gamma(Hl + 2) \sin \left( \frac{Hl \pi}{2} \right) \frac{1}{F^{3+l}}.
\]

In particular,

\[
W(F) \sim \frac{1}{2\pi^2} a \Gamma(H + 2) \sin \left( \frac{H \pi}{2} \right) \frac{1}{F^{3+H}}, \quad (F \rightarrow +\infty).
\]  
(132)

Using Eqs. (128) and (127), we have

\[
W(F) \sim \frac{K(Gm)^H}{2 + \alpha} \frac{1}{F^{3+H}}, \quad (F \rightarrow +\infty).
\]  
(134)

This asymptotic behaviour coincides with the asymptotic behaviour of the distribution of the force \[118\] due to the nearest neighbor. Using identity (109), we can check that Eqs. (132), (134) are consistent with Eqs. (103), (104) and (107). The moments of the force are finite iff \(-3 < b < H\). For \(0 \leq b < H\), we have \(\[67\]:

\[
\langle F^b \rangle = a^{b/H} \frac{2}{\pi} (b + 1) \Gamma(b) \Gamma \left( 1 - \frac{b}{H} \right) \sin \left( \frac{b\pi}{2} \right).
\]  
(135)

In particular, for \(b = 1\), we get

\[
\langle F \rangle = a^{1/H} \frac{4}{\pi} \Gamma \left( 1 - \frac{1}{H} \right).
\]  
(136)

On the other hand, if we start directly from Eq. (39) of Chandrasekhar \[67\] and use the identity

\[
\int_0^{+\infty} \frac{\sin x}{x^\alpha} dx = \frac{\pi}{2\Gamma(\alpha) \sin \left( \frac{\pi \alpha}{2} \right)}, \quad (0 < \alpha < 2).
\]  
(137)
Using identities (106) and (128), we can check that Eq. (147) is consistent with Eq. (121). The distribution of the force is then given by
\[ W(F) = \frac{1}{\pi} \int_0^{+\infty} e^{-a/k} \cos(kF) \, dk, \tag{148} \]
where we recall that \(0 < H < 2\). The general properties of the distribution (148) can be derived by adapting the method developed by Chandrasekhar [67] for \(d = 3\). When \(F\) is small, a convenient series expansion of \(W(F)\) is obtained by expanding \(\cos(kF)\) in Taylor series and integrating term by term. In this manner, we obtain
\[ W(F) = \frac{1}{\pi H} \sum_{l=0}^{+\infty} (-1)^l \Gamma\left(\frac{2l+1}{H}\right) \frac{1}{a} \frac{F^{2l}}{(2l)!}. \tag{149} \]
In particular,
\[ W(F) \to \frac{1}{\pi H} \Gamma\left(\frac{1}{H}\right) \frac{1}{a^{1/H}}, \quad (F \to 0). \tag{150} \]
Using identity (106), we can check that Eq. (149) is consistent with Eq. (121). To obtain the series expansion of \(W(F)\) for \(F \to +\infty\), we first set \(z = kF\) and rewrite Eq. (148) in the form
\[ W(F) = \frac{1}{\pi F} \int_0^{+\infty} e^{-a/(\pi F)} e^{iz} \, dz. \tag{151} \]
We now integrate in the complex plane along the line passing through the origin and inclined at an angle \(\pi F\) to the real axis instead of along the real axis itself. Thus, we set \(z = e^{i\pi F} y\) where \(y\) is real \(\geq 0\) in Eq. (151) and we obtain
\[ W(F) = \frac{1}{\pi F} \text{Re} \int_0^{+\infty} e^{-a/(\pi F)} e^{-i(\pi F) y} \times \exp\left\{ -e^{-i(1-\pi F) y}\right\} \, dy. \tag{152} \]
Expanding \(e^{-i(\pi F) y}\) in Taylor series, we have
\[ W(F) = \frac{1}{\pi F} \text{Re} \sum_{l=0}^{+\infty} (-i)^l \frac{a^l}{l!} e^{i\pi F} \frac{1}{F^{Hl}} \times \int_0^{+\infty} \exp\left\{ -e^{-i(1-\pi F) y}\right\} y^{Hl} \, dy. \tag{153} \]
We rotate again the line of integration by an angle \((1 - \frac{1}{\pi F}) \pi\). Thus, we set \(z = e^{-i(1-\pi F) y}\) where \(z\) is real \(\geq 0\) and we obtain
\[ W(F) = \frac{1}{\pi F} \text{Re} \sum_{l=0}^{+\infty} (-i)^l \frac{a^l}{l!} e^{i\pi F} \frac{1}{F^{Hl}} \times e^{i(Hl+1)(1-\pi F) \tau} \Gamma(Hl + 1). \tag{154} \]
Now, we verify that
\[ \text{Re} \left( (-i)^l e^{i\pi F} e^{i(Hl+1)(1-\pi F) \tau}\right) = (-1)^{l+1} \sin\left(\frac{HL\pi}{2}\right). \tag{155} \]
Hence, we obtain the asymptotic expansion for \(F \to +\infty:\)
\[ W(F) = \frac{1}{\pi} \sum_{l=1}^{+\infty} (-1)^{l+1} \frac{a^l}{l!} \Gamma(Hl + 1) \sin\left(\frac{Hl\pi}{2}\right) \frac{1}{F^{1+Hl}}. \tag{156} \]
In particular,
\[ W(F) \sim \frac{1}{\pi} a \Gamma(H + 1) \sin\left(\frac{H\pi}{2}\right) \frac{1}{F^{1+H}}, \quad (F \to +\infty). \tag{157} \]
Using Eqs. (142) and (147), this can be rewritten
\[ W(F) \sim \frac{K(Gm)^H}{\alpha} \frac{1}{F^{1+H}}, \quad (F \to +\infty). \tag{158} \]
This asymptotic behaviour coincides with the asymptotic behaviour of the distribution of the force (118) due to the nearest neighbor.

To evaluate the asymptotic expansion of \(W(F)\) given by Eq. (151) for \(F \to +\infty\), we can also integrate along the imaginary axis. Thus, we set \(z = iy\) with \(y\) real \(\geq 0\). In that case, Eq. (151) becomes
\[ W(F) = \frac{1}{\pi F} \text{Re} i \int_0^{+\infty} e^{-a/(\pi F)} e^{-y} \, dy. \tag{159} \]
Expanding the first term in the integral in Taylor series, we get
\[ W(F) = \frac{1}{\pi F} \text{Re} \int_0^{+\infty} i^{Hl+1} (-a)^l \frac{1}{l!} \Gamma(Hl + 1). \tag{160} \]
Noting that
\[ \text{Re} i^{Hl+1} = -\sin\left(\frac{HL\pi}{2}\right), \tag{161} \]
we recover Eq. (156).

From the asymptotic behaviour (158), it is clear that the moments of the force \(\langle F^b \rangle = 2 \int_0^{+\infty} W(F) F^b \, dF\) are finite iff \(-1 < b < H\). We can obtain an analytical expression for \(-1 < b < 0\). For ease of notations, we set \(\nu = -b\) with \(0 < \nu < 1\). From Eq. (148), we have
\[ \langle F^{-\nu} \rangle = \frac{2}{\pi} \int_0^{+\infty} \int_0^{+\infty} e^{-akH} F^{-\nu} \cos(kF) \, dF \, dk. \tag{162} \]
We first integrate on \(F\) using the identity
\[ \int_0^{+\infty} \cos x \, dx = \frac{\pi}{2\Gamma(\alpha) \cos\left(\frac{\pi}{2}\alpha\right)}, \quad (0 < \alpha < 1). \tag{163} \]
This yields
\[ \langle F^{-\nu} \rangle = \frac{1}{\Gamma(\nu) \cos(\nu \pi/2)} \int_0^{+\infty} e^{-akH} k^{\nu-1} \, dk. \tag{164} \]
Expressing the integral in terms of $\Gamma$-functions, we finally obtain the formula

$$
\langle F^{-\nu} \rangle = \frac{\Gamma(\nu/H)}{H(\nu) \cos(\nu\pi)} a^{\nu/H}, \quad 0 < \nu < 1.
$$

(165)

5.5 The dimension $d = 2$

For the dimension $d = 2$, we have

$$
C(k) = a k^H,
$$

(166)

where

$$
H = \frac{2 - p}{1 + \alpha},
$$

(167)

$$
a = \frac{2\pi}{1 + \alpha} (Gm)^H K B,
$$

(168)

$$
K = \frac{(2 - p)N}{2\pi R^{2-p}}.
$$

(169)

As before, we assume that $0 \leq p < 2$, $\alpha > -1$ and $2\alpha + p > 0$ (or, equivalently, $2 < \gamma < 4$). Therefore, $0 < H < 2$. The function $B$ is given by

$$
B = \int_0^{+\infty} \frac{dx}{x^{\alpha}+1} \left[ 1 - J_0(x) \right].
$$

(170)

Integrating by parts, the foregoing integral can be rewritten

$$
B = \frac{1}{H} \int_0^{+\infty} \frac{dx}{x^{\alpha}+1} J_1(x).
$$

(171)

Finally, the function $B$ can be expressed in terms of $\Gamma$-functions under the form

$$
B = \frac{1}{2\pi} \int_0^{+\infty} \frac{dx}{x^{\alpha}+1} J_1(x).
$$

(172)

in agreement with Eq. (171). The distribution of the force is then given by

$$
W(F) = \frac{1}{2\pi} \int_0^{+\infty} e^{-akH} J_0(kF)k dk,
$$

(173)

where we recall that $0 < H < 2$. When $F$ is small, a convenient series expansion of $W(F)$ is obtained by expanding $J_0(kF)$ is Taylor series and integrating term by term. This yields

$$
W(F) = \frac{1}{2\pi H} \sum_{l=0}^{+\infty} \frac{(-1)^l}{l!} \frac{F^{2l}}{4^r} \frac{1}{\alpha} \frac{\Gamma(2l+2)}{(Hl+2)}
$$

(174)

in agreement with Eq. (172). The asymptotic expansion of the distribution $W(F)$ for large $F$ can be obtained from the general method developed in Sec. 5.1 leading to

$$
W(F) = \frac{1}{2\sqrt{\pi}} \sum_{l=0}^{+\infty} \frac{(-1)^l}{l!} \frac{1}{F^{Hl+2}} \frac{\Gamma(Hl+2)}{\Gamma\left(\frac{Hl+2}{2}\right)}.
$$

(175)

Using Eqs. (168), (172) and the identity (100), we obtain the equivalent for $F \to +\infty$:

$$
W(F) \sim \frac{1}{1 + \alpha} (Gm)^H K \frac{1}{F^{H+1}}.
$$

(176)

This asymptotic behaviour coincides with the asymptotic behaviour of the distribution of the force (118) due to the nearest neighbor.

From the asymptotic behaviour (176), the moments

$$
\langle F^b \rangle = \int W(F) F^b dF
$$

of the force exist iff $-2 < b < H$. We can obtain an analytical expression for $-2 < b < -1/2$. For convenience, we set $\nu = -b$ with $1/2 < \nu < 2$. Using Eq. (173) and setting $t = kF$, we have

$$
\langle F^{-\nu} \rangle = \int_0^{+\infty} dt e^{-akH} \Gamma_{\nu-1} \int_0^{+\infty} J_0(t) \frac{1}{t^{\nu-1}} dt.
$$

(177)

Using the identity

$$
\int_0^{+\infty} J_0(t) \frac{1}{t^{\alpha}} dt = \frac{1}{2\alpha} \frac{\Gamma\left(\frac{1-\alpha}{2}\right)}{\Gamma\left(\frac{1+\alpha}{2}\right)}, \quad \left(\frac{1}{2} < \alpha < 1\right).
$$

(178)

we obtain for $1/2 < \nu < 2$:

$$
\langle F^{-\nu} \rangle = \frac{1}{2\nu-1} \frac{\Gamma\left(\frac{2\nu}{\nu}\right)}{(H\nu)^{\nu}} \frac{\Gamma\left(\frac{\nu}{H}\right)}{(2\nu-1)}.
$$

(179)

5.6 The Cauchy distribution

We note that the characteristic function (91) is linear when

$$
H = 1, \quad (\gamma = d + 1).
$$

(180)

In that case, the distribution of the force is a Cauchy law. This corresponds to $\alpha + p = -1$ independently on the dimension of space $d$. For the gravity case ($\alpha = 0$), the Cauchy law is obtained for $p = 1$, i.e. for a fractal dimension $d_f = d - 1$ (assuming $d > 1$). In $d$ dimensions, the Cauchy law has the form

$$
W(F) = \frac{a}{\pi H} \frac{\Gamma\left(\frac{1}{2} + \frac{d}{2}\right)}{\Gamma\left(\frac{a}{2H}\right)}
$$

(181)

The moments $\langle F^b \rangle$ exist for $-d < b < 1$ and they are given by

$$
\langle F^b \rangle = \frac{a^b}{\sqrt{\pi} \Gamma\left(\frac{1}{2} + \frac{d}{2}\right)} \frac{\Gamma\left(\frac{1-b}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} \frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d+2}{2}\right)}.
$$

(182)

For $d = 3$, we obtain

$$
W(F) = \frac{a}{\pi^2} \frac{1}{(a^2 + F^2)^2}.
$$

(183)

with

$$
a = \frac{\pi^2}{2} GmK.
$$

(184)

We have

$$
W(F) \to \frac{1}{\pi^2 a^3}, \quad (F \to 0),
$$

(185)
\[ W(\mathbf{F}) \sim \frac{a}{\pi^2 F^d}, \quad (F \to +\infty). \tag{186} \]

For \(-3 < b < 1\), the moments are
\[ \langle F^b \rangle = \frac{a^b (b + 1)}{\cos \left( \frac{b \pi}{2} \right)}. \tag{187} \]

Using identity \([128]\), we check that Eqs. \([187]\), \([185]\) and \([182]\) coincide.

For \(d = 2\), we obtain
\[ W(\mathbf{F}) = \frac{a}{2\pi \alpha} \left( \frac{1}{a^2 + F^2} \right)^{3/2}, \tag{188} \]
with
\[ a = \frac{2\pi}{\alpha + 1} GmK. \tag{189} \]

We have
\[ W(\mathbf{F}) \to \frac{1}{\pi a^2}, \quad (F \to 0), \tag{190} \]
\[ W(\mathbf{F}) \sim \frac{a}{\pi F^3}, \quad (F \to +\infty). \tag{191} \]

For \(-2 < b < 1\), the moments are
\[ \langle F^b \rangle = \frac{a^b}{\sqrt{\pi}} \Gamma \left( 1 - \frac{b}{2} \right) \Gamma \left( \frac{b + 2}{2} \right). \tag{192} \]

For \(d = 1\), we obtain
\[ W(F) = \frac{a}{\pi} \frac{1}{a^2 + F^2}, \tag{193} \]
with
\[ a = \frac{\pi}{\alpha} GmK. \tag{194} \]

We have
\[ W(F) \to \frac{1}{\pi a}, \quad (F \to 0), \tag{195} \]
\[ W(F) \sim \frac{a}{\pi F^2}, \quad (F \to +\infty). \tag{196} \]

For \(-1 < b < 1\), the moments are
\[ \langle F^b \rangle = \frac{a^b}{\cos (b \pi/2)}. \tag{197} \]

Using the identity \([128]\), we check that Eqs. \([197]\), \([182]\) and \([103]\) coincide.

### 5.7 The marginal Gaussian distribution (critical case)

We have seen that the distribution of the total force is a Lévy law when condition \([69]\), or equivalently condition \([70]\), is fulfilled corresponding to \(0 < H < 2\). The critical case happens for \(2\alpha + p = 2 - d\) corresponding to
\[ H = 2, \quad (\gamma = d + 2). \tag{198} \]

In that case, the variance of the force produced by a star diverges logarithmically. For the gravity case \((\alpha = 0)\) this corresponds to \(p = 2 - d\). But, the condition \(d > 1\) is required to have a decreasing force and the condition \(d \leq 2\) must hold to have a non increasing density profile. Therefore, the only possibility is \(d = 2\) and \(p = 0\) treated in Sec. \([6]\). For other values of \(\alpha\), the dimension of space must lie in the range \(1 - \alpha < d \leq 2(1 - \alpha)\). When \(H = 2\), Eq. \([77]\) remains valid with
\[ A(k) = e^{-C(k)}, \quad C(k) = \int_{|r|=0}^{R} n(r) \left(1 - e^{ikr} \right) dr. \tag{199} \]

Note that the integral defining the characteristic function diverges logarithmically as \(R \to +\infty\). Therefore, Eq. \([199]\) with Eq. \([199]\) must be viewed as an equivalent of the distribution \(W_N(\mathbf{F})\) for large values of \(R\) or \(N\), not a true limit. Equations \([83]-[87]\) are now replaced by
\[ C(k) = ak^2, \tag{200} \]
where
\[ a = \frac{S_d}{d + \alpha - 1} (Gm)^2 KB, \tag{201} \]
\[ B = \int_{\frac{GmK}{R^{d+\alpha-1}}}^{+\infty} \frac{dx}{x^3} \left[ 1 - \Gamma \left( \frac{d}{2} \right) \left( \frac{2}{x} \right)^{d/2} J_{d/2-1}(x) \right]. \tag{202} \]

Since this integral diverges logarithmically as \(R \to +\infty\), we can replace the term in brackets by its leading order expression for \(x \to 0\) using Eq. \([83]\) and we obtain
\[ B = \frac{1}{2d} \int_{\frac{GmK}{R^{d+\alpha-1}}}^{+\infty} \frac{dx}{x} = \frac{1}{4d} \ln \left( \frac{N}{G^2 m^2 k^2} \right), \tag{203} \]
where we have used \(N \sim R^{d-p} \sim R^{2(d+\alpha-1)}\) for \(R, N \to +\infty\). Redefining
\[ \alpha = \frac{S_d}{d + \alpha - 1} (Gm)^2 K \tag{204} \]
the distribution of the force can be written
\[ W(\mathbf{F}) = \frac{S_d}{(2\pi)^d} \Gamma \left( \frac{d}{2} \right) \left( \frac{2}{F} \right)^{d/2-1} \]
\[ \times \int_{0}^{+\infty} e^{-\pi k^2 \ln \left( \frac{N}{(\alpha m + \pi F^2)^2} \right)} k^{d/2} J_{d/2-1}(kF) dk. \tag{205} \]

For \(F\) not too large (corresponding to \(k\) not too small), we can replace the logarithmic term by its leading contribution in \(N\) and we get
\[ W(\mathbf{F}) = \frac{S_d}{(2\pi)^d} \Gamma \left( \frac{d}{2} \right) \left( \frac{2}{F} \right)^{d/2-1} \]
\[ \times \int_{0}^{+\infty} e^{-\pi k^2 \ln N} k^{d/2} J_{d/2-1}(kF) dk. \tag{206} \]
In that case, the distribution of the force is Gaussian
\[
W(F) = \frac{1}{(4\pi \ln N)^{d/2}} e^{-\frac{k^2}{2 \sigma^2 N}} \quad (207)
\]
with a variance \( \langle F^2 \rangle = 2\sigma^2 \ln N \) diverging like \( \ln N \). For \( F \to +\infty \) (corresponding to \( k \to 0 \)), we can replace the logarithmic term by \(-2\ln k\) and we get
\[
W(F) = \frac{S_d}{(2\pi)^d} \int_{C^+} (1-t^2)^{-\frac{d-1}{2}} d\theta \cdot \int_0^{+\infty} e^{\frac{\alpha y}{2}} \ln \left( \frac{iy}{F} \right) e^{-y} d\beta \cdot \int_0^{+\infty} e^{-\frac{\alpha y}{2}} \ln \left( \frac{iy}{F} \right) e^{-y} d\gamma.
\]
Repeating the procedure of Sec. 5.1, we can rewrite the following integral in the complex plane along the semi-circle of unit radius in the upper half plane \( \text{Im}(t) > 0 \). Expanding the exponential term for large \( F \), we obtain
\[
W(F) = \frac{C_d}{(2\pi)^d} \int_{C^+} (1-t^2)^{-\frac{d-3}{2}} dt \int_0^{+\infty} e^{-y} dy \cdot \left[ 1 - 2\pi \frac{1}{t^2} \ln \left( \frac{iy}{F} \right) + 2\pi \ln t \frac{y^2}{t^2} F^2 + \ldots \right] dt d\gamma.
\]
Now, setting \( t = e^{i\theta} \) and integrating on \( C^+ \) from \( \theta = \pi \) to \( \theta = 0 \), we obtain the following results
\[
\text{Re} \int_{C^+} (1-t^2)^{-\frac{d-1}{2}} \frac{1}{t^d} dt = 0, \quad (211)
\]
\[
\text{Re} \int_{C^+} (1-t^2)^{-\frac{d-3}{2}} \frac{1}{t^{d+2}} dt = 0, \quad (212)
\]
\[
\text{Re} \int_{C^+} (1-t^2)^{-\frac{d-3}{2}} \frac{1}{t^{d+2}} dt = 0, \quad (213)
\]
\[
\text{Re} \int_{C^+} (1-t^2)^{-\frac{d-1}{2}} \ln t \frac{t^d}{t^{d+2}} dt = \frac{2^{d-1} d\sqrt{\pi}}{F(d+2)} \Gamma \left( \frac{d}{2} \right) \Gamma \left( \frac{d-1}{2} \right).
\]
Substituting these relations in Eq. (211) and using Eq. (201), we obtain the equivalent for \( F \to +\infty \):
\[
W(F) \sim K \frac{(Gm)^2}{d + \alpha - 1} \frac{1}{F^{d+2}}, \quad (215)
\]
which coincides with the distribution (118) due to the nearest neighbor for \( H = 2 \). We also note that the asymptotic behaviour (107) obtained for \( H < 2 \) passes to the limit \( H \to 2 \). Finally, the crossover between the two distributions (207) and (215) occurs for a typical force
\[
F_{\text{crit}}(N) \sim (4\pi \ln N)^{1/2} \ln(\ln N)^{1/2}. \quad (216)
\]
in any dimension of space \( d \).

It may be useful to study the physical dimensions \( d = 3, 2, 1 \) specifically and re-derive the previous results in a different manner. For the physical dimension \( d = 3 \), the expression (205) of the distribution of the force reduces to
\[
W(F) = \frac{1}{2\pi^2 F} \int_0^{+\infty} e^{-\pi k^2 \ln \left( \frac{Gm}{\sigma e^{2\pi N}} \right)} \sin(kF) k dk, \quad (217)
\]
with
\[
\pi = \frac{K \pi (Gm)^2}{3(2 + \alpha)}. \quad (218)
\]
This critical case corresponds to \( 2\alpha + p = -1 \) or, equivalently, \( \gamma = 5 \). In the core of the distribution, we can make the approximation
\[
W(F) = \frac{1}{2\pi^2 F} \int_0^{+\infty} e^{-\pi \ln N k^2} \sin(kF) k dk. \quad (219)
\]
This leads to the Gaussian distribution
\[
W(F) = \frac{1}{(4\pi \ln N)^{3/2}} e^{-\frac{y^2}{4\pi N}}, \quad (220)
\]
with a variance \( \langle F^2 \rangle = 6\pi \ln N \) that diverges logarithmically with \( N \). In the tail of the distribution, we can make the approximation
\[
W(F) = \frac{1}{2\pi^2 F} \int_0^{+\infty} e^{2\pi k^2} \sin(kF) k dk. \quad (221)
\]
Setting \( z = kF \), this can be rewritten
\[
W(F) = \frac{1}{2\pi^2 F^3} \int_0^{+\infty} e^{2\pi k^2} \ln(kF) e^{iz} z dz. \quad (222)
\]
In this expression, \( z \) is real and the domain of integration is on the real axis \( z \geq 0 \). Under this form, we cannot expand the exponential in power series for \( F \to +\infty \) and integrate term by term because the integrals would diverge. However, we can modify the domain of integration and work in the complex plane. Then, \( z \) is viewed as a complex variable and we can replace the domain of integration by the imaginary axis, i.e. \( z = iy \) with \( y \) real \( \geq 0 \). Thus, we have
\[
W(F) = -\frac{1}{2\pi^2 F^3} \int_0^{+\infty} e^{-2\pi y} \ln(kF) e^{-iy} dy. \quad (223)
\]
Under this form, it is possible to expand the exponential in powers of $y/F$ and integrate term by term. To leading order, we find

$$W(F) = -\frac{1}{2\pi^2 F^3} \text{Im} \int_0^{+\infty} \left[1 - 2\pi \left(\frac{y}{F}\right)^2 \times \ln\left(\frac{i y}{F}\right) + \ldots\right] e^{-y} dy. \quad (224)$$

Since only the imaginary part of the integral matters, the foregoing expression reduces to

$$W(F) \sim \frac{\pi}{2\pi F^3} \int_0^{+\infty} \ln(i) e^{-y} dy. \quad (225)$$

Then, using $\ln(i) = \ln(e^{i\pi/2}) = i\pi/2$ we finally obtain

$$W(F) \sim \frac{\pi}{2\pi F^3} \int_0^{+\infty} e^{-y} dy. \quad (226)$$

which yields

$$W(F) \sim \frac{\pi}{2\pi F^3} \Gamma(4) \sim \frac{3\pi}{F^3}. \quad (227)$$

We obtain the same result if we deform the contour of integration in Eq. (222) as indicated in Chandrasekhar [67] for the case $H < 2$. Substituting the value of $\pi$ from Eq. (223), we obtain the equivalent

$$W(F) \sim \frac{K(Gm)^2}{2 + \alpha} \frac{1}{F^3}, \quad (F \to +\infty) \quad (228)$$

which is in exact agreement with the distribution of the force [113] due to the nearest neighbor for $H = 2$ and $d = 3$. We also note that the asymptotic behaviour [13] obtained for $H < 2$ passes to the limit $H \to 2$. Finally, the crossover between the two distributions [220] and [228] occurs for a typical force

$$F_{\text{crit}}(N) \sim (4\pi \ln N)^{1/2} \ln(\ln N)^{1/2}. \quad (229)$$

For the dimension $d = 1$, the expression [205] of the distribution of the force reduces to

$$W(F) = \frac{1}{\alpha} \int_0^{+\infty} e^{-\pi k^2 \ln(\frac{N}{2\pi F \alpha})} \cos(kF) dk, \quad (230)$$

with

$$\alpha = \frac{K(Gm)^2}{2\alpha}. \quad (231)$$

This critical case corresponds to $2\alpha + p = 1$ or, equivalently, $\gamma = 3$. In the core of the distribution, we can make the approximation

$$W(F) = \frac{1}{\alpha} \int_0^{+\infty} e^{-\pi \ln k^2} \cos(kF) dk. \quad (232)$$

This leads to the Gaussian distribution

$$W(F) = \frac{1}{(4\pi \pi \ln N)^{1/2} F^{3}} e^{-\frac{\pi^2}{\pi \ln N}}. \quad (233)$$

with a variance $(\langle F^2 \rangle) = 2\pi \ln N$ that diverges logarithmically with $N$. In the tail of the distribution, we can make the approximation

$$W(F) = \frac{1}{\pi} \int_0^{+\infty} e^{\pi \kappa^2 \ln k} \cos(kF) dk. \quad (234)$$

Setting $z = kF$, this can be rewritten

$$W(F) = \frac{1}{\pi F} \int_0^{+\infty} e^{\pi \kappa^2 \ln k} \cos(kF) dz. \quad (235)$$

To determine the asymptotic behaviour of $W(F)$ for $F \to +\infty$, we deform the contour of integration and integrate on the imaginary axis, setting $z = iy$ with $y$ real $\geq 0$. Thus, we have

$$W(F) = \frac{1}{\pi F} \int_0^{+\infty} e^{-\pi \kappa^2 \ln k} \cos(kF) e^{-y} dk. \quad (236)$$

We can now expand the exponential in powers of $y/F$ and integrate term by term. To leading order, we find

$$W(F) = \frac{1}{\alpha} \int_0^{+\infty} i \left[1 - 2\pi \left(\frac{y}{F}\right)^2 \times \ln\left(\frac{i y}{F}\right) + \ldots\right] e^{-y} dy. \quad (237)$$

Since only the real part of the integral matters, the foregoing expression reduces to

$$W(F) \sim \frac{2\pi}{\pi F^3} \int_0^{+\infty} \ln(i) e^{-y} dy. \quad (238)$$

Then, using $\ln(i) = \ln(e^{i\pi/2}) = i\pi/2$ we finally obtain

$$W(F) \sim \frac{\pi}{F^3} \int_0^{+\infty} e^{-y} dy. \quad (239)$$

which yields

$$W(F) \sim \frac{\pi}{F^3} \Gamma(3) \sim \frac{2\pi}{F^3}. \quad (240)$$

We obtain the same result if we deform the contour of integration in Eq. (235) as indicated between Eqs. (151) and (157) for the case $H < 2$. Substituting the value of $\alpha$ from Eq. (231), we obtain the equivalent

$$W(F) \sim \frac{K(Gm)^2}{\alpha} \frac{1}{F^3}, \quad (F \to +\infty) \quad (241)$$

which is in exact agreement with the distribution of the force [113] due to the nearest neighbor for $H = 2$ and $d = 1$. We also note that the asymptotic behaviour [13] obtained for $H < 2$ passes to the limit $H \to 2$. Finally, the crossover between the two distributions [220] and [240] occurs for a typical force

$$F_{\text{crit}}(N) \sim (4\pi \ln N)^{1/2} \ln(\ln N)^{1/2}. \quad (242)$$

The critical case in $d = 2$ corresponding to $2\alpha + p = 0$ or, equivalently $\gamma = 4$, can be treated like in [47].
6 Conclusion

In this paper, we have studied how the statistics of the gravitational force created by a random distribution of field sources changes with the dimension of space $d$. The dimensions $d = 1$, $d = 2$ and $d = 3$ correspond respectively to plane-parallel (sheets), cylindrical (filaments) and spherical (stars) configurations. We have shown that the dimension $d = 2$ is critical as it separates Gaussian laws (for $d = 1$) from Lévy laws (for $d \geq 3$). This transition may have interesting implications for the kinetic theory of self-gravitating systems (in $d$ dimensions) since the distribution of the gravitational force is a key ingredient for the determination of the diffusion coefficient of stars [14,57]. Furthermore, even if our study has astrophysical motivations at the start, it can be of interest in probability theory to illustrate the differences between Gaussian and Lévy laws.

Note, however, that our analysis is based on several simplifying assumptions:

(i) we have assumed (for the cases $d = 3$ and $d = 2$) that the number of stars $N \to +\infty$ or equivalently that the system is infinite, i.e. we have considered the limit $N, R \to +\infty$ with fixed $n = N/V$. The distribution of the force $W_N(F)$ for a finite system must be computed numerically, using Eqs. (4)-(6). In $d = 3$ this study has been performed by Ahmad & Cohen [15]. It is shown that the convergence to the limit distribution $W(F)$ is quite rapid: for $N = 2$, the overall agreement with the $N = \infty$ case is not unreasonable, for $N = 50$ the agreement is accurate to 10% and for $N = 1000$, the agreement is excellent. In $d = 2$, the convergence of the distribution of the gravitational force to the limit distribution (Gaussian) is very slow and, as we have seen, a power-law tail develops at a typical value of the force [47] which increases logarithmically with $N$. Numerical simulations exhibiting this power-law tail are reported (for point vortices) in [30,53,55]. In $d = 1$, the exact distribution of the gravitational force has been obtained for any $N$.

(ii) we have assumed in Secs. 2-4 that the distribution of stars is spatially homogeneous. In fact, an infinite and homogeneous distribution of stars is not stable and it clusters in dense objects (galaxies or clusters of galaxies). The case of a power-law distribution of stars has been treated by Kandrup [14] in $d = 3$ and generalized to any dimension and to any power-law force in Sec. 5. In practice, since the distribution of the force is dominated by the nearest neighbor, the assumption of an infinite and homogeneous distribution is not crucial; indeed, Kandrup [57] shows that the inhomogeneous case, for a smooth density distribution $n(r)$, is still described by the Holtsmark distribution (for the fluctuating force) where the density $n$ is replaced by the local density $n(a)$ at the point under consideration (see also Appendix A of [58] in $d = 2$).

(iii) we have assumed that the positions of the stars are uncorrelated (Poisson distribution). This approximation may be correct in stellar dynamics where it is known that the two-body distribution function can be approximated by a product of two one-body distributions in a proper thermodynamic limit $N \to +\infty$ [71]. However, in that case, the one-body distribution is spatially inhomogeneous and we are led to point (ii). On the other hand, in cosmology, the system is statistically spatially homogeneous but the particles are correlated and have the tendency to form clusters. In that case, the theoretical framework developed to determine the statistics of the gravitational force must be modified. Some interesting attempts to take into account spatial correlations in the position of the particles have been made in [72,73]. Therefore, the results presented in this paper can be improved in several directions by relaxing the above assumptions. This will be considered in future works.

A last comment, suggested by the referee, may be in order. The limit distribution for a sum of random variables $F = \sum_{i=1}^{N} F_i$ is a classical problem in probability theory and there exists rigorous results and general theorems about it [41,42,43,44,45,46]. Our approach, which is based on the seminal work of Chandrasekhar [1], is consistent with these general theorems. In fact, Chandrasekhar (1943) [1] and Holtsmark (1919) [8] obtained “Lévy laws” independently from Lévy (1937) [21] and other mathematicians of that time. No reference to Lévy laws are made in the classical papers of Chandrasekhar (1943, 1948) [1,67] nor in the more recent review of Kandrup (1980) [14]. Reciprocally, the books [42,43,44,45] do not make any reference to Chandrasekhar’s work and derive the limit distributions in a more formal manner. In the present paper, we have used and extended the method introduced by Chandrasekhar [1]. One interest of this method is that it is fully explicit and amounts to the calculation of integrals. It is, however, restricted to pure power-law distributions $\tau(f) \propto f^{-\gamma}$ (with a cut-off at small $f$) while the theorems of [42,43,44,45] are more general. We have obtained $d$-dimensional generalizations of Lévy laws [see Eq. (93)] and given their main properties (usually, the problem of the sum of random variables is formulated in one dimension). On the other hand, the critical case $\gamma = \gamma_c = d + 2$ reported in Secs. 5 and 6 of the present paper (where the variance of the random variables $\langle f^2 \rangle$ diverges logarithmically) has not been treated in depth in [42,43,44,45]. It is usually argued that, in that case, the limit distribution $W(F)$ is Gaussian. This is true in a strict sense when $N \to +\infty$. However, we have shown that for large but finite $N$, the physical distribution $W(F)$ has a Gaussian core and an algebraic tail. The separation between these two behaviours is obtained for a typical value of the force $F_c(N) \propto (\ln N)^{1/2} \ln(\ln N)^{1/2}$ which diverges with $N$. Therefore, at the limit $N \to +\infty$, the power-law tail is rejected to infinity and only the Gaussian core remains. However, the convergence is slow (logarithmic) that, in practice, the power-law tail is visible. This point may have been overlooked in [42,43,44,45].

\footnote{The critical nature of the dimension $d = 2$ has also been noted in [21] regarding the gravitational collapse of isothermal systems.}

\footnote{Of course, for finite $N$ systems, correlations must be taken into account in the so-called “collisional” regime of the dynamics, as they drive the kinetic evolution of the system [17,60].}
Finally, our study of the statistics of the gravitational force created by a uniform distribution of sources in $d$ dimensions illustrates the three kinds of laws that can be read from Fig. 1.1. of the review of Bouchaud & Georges [44]: (i) For $d > d_c$, $2$, the variance of the individual forces diverges algebraically since $\gamma = d^2/(d - 1) < \gamma_c = 2 + d/\sqrt{d}$, and the distribution of the total force is a marginal Gaussian law [93] with index $H = d/(d - 1)$ [H = 3/2 in $d = 3$]. (ii) For $d < d_c$, $2$, the variance of the individual forces is finite and the distribution of the total force is Gaussian according to the CLT (in $d = 1$ it is exactly given by a Bernouilli law for all $N$). (iii) For $d = d_c = 2$ (critical case), the variance of the individual forces diverges logarithmically since $\gamma = \gamma_c = 4$ and the distribution of the total force is a Gaussian distribution. In that case, we are at the border between Gaussian and Lévy laws (see again Fig. 1.1. of [44]). More generally, our results can be expressed in terms of $\gamma$ alone (in a space of dimension $d$). For $d < \gamma < \gamma_c = d + 2$, the variance of the individual forces diverges algebraically and the distribution of the total force is a $d$-dimensional Lévy law [93] with index $H = \gamma - d$ (its tail decreases algebraically with an exponent $\gamma$ due to the nearest neighbour). For $\gamma > \gamma_c$, the variance of the individual forces is finite and the distribution of the total force is Gaussian according to the CLT. For $\gamma = \gamma_c$ (critical case), the variance of the individual forces diverges logarithmically and the distribution of the total force is a marginal Gaussian distribution. Therefore, for a fixed dimension of space $d$, the different laws correspond to $d < \gamma < \gamma_c$ (Lévy), $\gamma = \gamma_c$ (Gaussian) and $\gamma > \gamma_c$ (Gaussian). Alternatively, when we consider the gravitational force created by a homogeneous distribution of sources, $\gamma = d^2/(d - 1)$ is fixed and the different laws correspond to $d > d_c = 2$ (Lévy), $d = d_c$ (critical) and $d < 2$ (Gaussian).

A The distribution of the gravitational force created by the nearest neighbor in $d$ dimensions

In this Appendix, we determine the distribution of the gravitational force in $O$ due to the contribution of the nearest neighbor for a uniform distribution of stars. For the sake of generality, we work in a space of $d$ dimensions. The probability $\tau_{n,n.}(r) dr$ that the position of the nearest neighbor occurs between $r$ and $r + dr$ is equal to the probability that no star exist interior to $r$ times the probability that a star (any) exists in the shell between $r$ and $r + dr$. Therefore, it satisfies an equation of the form

$$\tau_{n,n.}(r) dr = \left( 1 - \int_r^\infty \tau_{n,n.}(r') dr' \right) n S_d r^{d-1} dr,$$  

where $n$ is the mean density of stars. Differentiating this expression with respect to $r$, we obtain

$$\frac{d}{dr} \left[ \frac{\tau_{n,n.}(r)}{S_d r^{d-1}} \right] = -\tau_{n,n.}(r).$$

This equation is readily integrated with the condition $\tau_{n,n.}(r) \sim S_d r^{d-1}$ for $r \to 0$ and we find

$$\tau_{n,n.}(r) = S_d r^{d-1} e^{-\frac{S_d r^d}{d}}.$$  

From this formula, we can obtain the exact expression of the “average distance” $D$ between stars:

$$D = \int_0^{+\infty} \tau_{n,n.}(r) r dr = \int_0^{+\infty} S_d r^{d-1} e^{-\frac{S_d r^d}{d}} dr$$

$$= \left( \frac{d}{S_d n} \right)^{1/d} \int_0^{+\infty} x^{1/d} e^{-x} dx = \left( \frac{d}{S_d n} \right)^{1/d} \Gamma \left( \frac{1}{d} + 1 \right).$$  

(246)

For example, $D = \left( \frac{3}{4\pi n} \right)^{1/3} \Gamma(4/3)$ in $d = 3$, $D = \frac{1}{2\sqrt{\pi}}$ in $d = 2$ and $D = \frac{1}{2\sqrt{n}}$ in $d = 1$.

The gravitational force created in $O$ by a star in $r$ in a $d$-dimensional space is given by

$$F = G m r.$$  

(247)

We note that the variance of the force created by one star

$$\langle F^2 \rangle \propto \int_0^{+\infty} \frac{1}{r^{2(d-1)}} r^{d-1} dr \propto \int_0^{+\infty} \frac{1}{r^{d-1}} dr,$$

(248)

diverges for $d \geq 2$ due to the behaviour at small distances $r \to 0$. In that case, the distribution of the (total) gravitational force is a Lévy law. For large field strengths, it is well-approximated by the distribution created by the nearest neighbor. For the case of the force created by the nearest neighbor is such that $W_{n,n.}(r) dr = \tau_{n,n.}(r) dr$ where

$$\tau_{n,n.}(r) = \frac{\tau_{n,n.}(r)}{S_d r^{d-1}} = n e^{-\frac{S_d r^d}{d}}.$$  

(249)

Using

$$dF = (d-1) (Gm)^{-d/(d-1)} F^{d/(d-1)} dr,$$

(250)

we obtain

$$W_{n,n.}(F) = n \left( \frac{G m}{(d-1) F^{d/(d-1)}} \right) \frac{S_d n (G m)^{d/(d-1)}}{d}.$$  

(251)

For $F \to +\infty$, we get

$$W_{n,n.}(F) \sim n \left( \frac{G m}{(d-1) F^{d/(d-1)}} \right).$$  

(252)

The distribution of the force decreases with an exponent $\gamma = d + \frac{1}{d - 1}$. (253)

We note that the average force $\langle F \rangle$ is always finite and its value is

$$\langle F \rangle_{n,n.} = \left( \frac{S_d}{d} \right)^{\frac{d-1}{d}} \Gamma \left( \frac{1}{d} \right) G m n \frac{d}{d-1}.$$  

(254)

It can be expressed in terms of the exact average distance between stars [240] under the form

$$\langle F \rangle_{n,n.} = \frac{1}{d-1} \Gamma \left( \frac{1}{d} \right) d G m \frac{d}{d-1}.$$  

(255)
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