Disorder Effects in Superconducting Multiple Loop Quantum Interferometers

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A theoretical study is presented on a number \( N \) of resistively shunted Josephson junctions connected in parallel as a disordered 1D array by superconducting wiring in such a manner that there are \( N - 1 \) individual SQUID loops with arbitrary shape formed. Under a constant current bias \( I > I_c \), and irrespective of the degree of the disorder, but depending on the strength of magnetic field \( B \), all junctions in the array oscillate at the same frequency \( \nu_B \). Computer simulations of the full nonlinear dynamics of a disordered junction array reveal: (i) the frequency \( \nu_B \) is not a periodic function of \( B \), (ii) in the overdamped junction regime \( \nu_B \) displays a sharp global minimum around \( B = 0 \). For zero inductive coupling the problem becomes equivalent to a virtual single junction model.

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So-called weak links, or Josephson junctions, are the basic active elements of superconductor quantum electronics. A key feature of a weak link between two superconductors, 1 and 2, is the property that there can flow a dissipationless macroscopic supercurrent \( I_s(\varphi) \) due to the tunneling of Cooper pairs with charge 2\( e \). This supercurrent depends on the gauge invariant phase difference \( \varphi = \Theta_1 - \Theta_2 + \frac{2\pi}{\hbar} \int_{S} \mathbf{A} \cdot d\mathbf{s} \), \( \mathbf{A} \) of the macroscopic BCS pairing wavefunctions on either side of the weak link. Josephson junctions made with modern fabrication techniques often have a sandwich type layered geometry, with a thin non superconducting tunneling barrier in the middle between two thick superconducting electrodes. In recent time also other types of weak links, for example of the bi-crystal type, became important in high-temperature superconductors.

For an ideal S-I-S junction the supercurrent is connected to the phase difference \( \varphi \) across the tunneling barrier by \( I_s(\varphi) = I_c \sin \varphi \). It is important to realize that the supercurrent \( I_s \) flows stationary provided it does not exceed a characteristic critical current \( I_c \), the so-called Josephson critical current, which determines the maximum dissipationless critical current that can flow across a tunneling barrier. In general, \( I_c \) depends on the material properties of the junction, on temperature \( T \), and on magnetic field \( B = \text{rot} \mathbf{A} \). Applying to a Josephson junction a bias current \( I \) with a constant strength \( I > I_c \), there appears a rapidly oscillating voltage signal \( V(t) \) across the junction, which determines the rate of change of the time dependent phase difference \( \varphi(t) \) according to

\[
\hbar \frac{d}{dt} \varphi(t) = 2eV(t)
\]

This is the fundamental non stationary Josephson relation which governs the physics of weak superconductivity. So, for \( I > I_c \) there flows, besides the dissipationless supercurrent \( I_s \), also a dissipative normal current \( I_n \) in the junction, whose physical origin is the transfer of single (unpaired) electrons.

Within the range of validity of the RCSJ model, the dissipative current may be described with sufficient accuracy as a superposition of an ohmic current, characterized by a parallel ohmic shunt resistance \( R \), and a displacement current, which is characterized by a paral-
SQUID loop, where $\alpha$ is the angle between the normal vector of the orientated area element and the magnetic field vector $\mathbf{B}$, as depicted schematically in Fig.(1a). The total magnetic field, $\mathbf{B} = \mathbf{B}^{(1)} + \mathbf{B}^{(2)}$, is then a superposition of the primary external magnetic field $\mathbf{B}^{(1)}$, which generates the flux $\Phi^{(1)} = \langle \mathbf{B}^{(1)} \cdot \mathbf{a} \rangle$, one wants to detect, and a secondary magnetic field $\mathbf{B}^{(2)}$ that results, for example, from the inductance $L$ (or other impedance effects) in the circuit. The screening current $I_{sc}$ circulating in the SQUID loop leads to a total flux $\Phi = \Phi^{(1)} + \Phi^{(2)}$. Dependent on the secondary flux term, $\Phi^{(2)} = -L \frac{dI_{sc}}{dt}$, there exists an optimal size $|a_L|$ for any SQUID loop $\mathbf{B}$. A dimensionless measure for the inductance of such a loop is $\beta_L = \frac{L}{\Phi_0}$. Note also that a two junction SQUID cannot be directly employed as a detector of absolute strength of external magnetic field. This is because the voltage response function $\langle V_{xy} \rangle$ of the SQUID, i.e. the time average of the rapidly oscillating voltage signal $V_{xy}(t)$ across the nodes $x$ and $y$ of the circuit, is a periodic function of the strength of external magnetic field, see Fig.(1a).

A straightforward extension of the standard two junction SQUID is sketched in Fig.(1b). This is a 1D array of $N$ adjacent Josephson junctions connected in parallel. In particular, the area elements of the $N-1$ SQUID loops formed in this manner are all equal, e.g. $a_n = a_L$ for all $n$. The voltage response signal $\langle V_{xy} \rangle$ vs. strength $|\mathbf{B}^{(1)}|$ of external magnetic field of this array has the same period than a standard two junction SQUID with loop area $|a_L|$, see Fig.(1b).

A more general quantum interference device is obtained when the area elements $a_n$ of the $N-1$ loops in the array differ in size and, possibly, in orientation, as depicted schematically in Fig.(1c). If the sizes $|a_n|$ of the orientated area elements $a_n$ of the individual superconducting loops are chosen in a disordered fashion the voltage response function $\langle V_{xy} \rangle$ vs. $|\mathbf{B}^{(1)}|$ becomes nonperiodic, see Fig.(1c). Taking into account inductive couplings among the currents in the circuit, the maximum loop size in the disordered array should coincide with the corresponding optimal loop size of a standard two junction SQUID, i.e. max$|a_n| = |a_L|$. The voltage response signal $\langle V_{xy} \rangle$ vs. strength of magnetic field of a disordered junction array is, under a suitable $dc$ current bias $I$, indeed a unique function of $|\mathbf{B}|$ around its narrow global minimum at $|\mathbf{B}| = 0$. This suggests that it should be possible, e.g. by measuring control current(s) flowing through the wires of a set of suitably orientated compensation coil(s), to reconstruct absolute strength, orientation and even the phase of an incident magnetic field signal, i.e. to determine the full vector $\mathbf{B}^{(1)}(t)$.

The $n$-th Josephson junction in the array has, within the range of validity of the RCSJ model, optional individual junction parameters $R_n$, $C_n$ and $I_c, n$. The corresponding current $I_n$ flowing through the $n$-th Josephson junction is, according to Eq.(2), determined by the gauge invariant phase difference $\varphi_n(t)$ across that junction. The total current $I$ flowing through the nodes $x$ and $y$, respectively, of the circuit is then obtained from Kirchhoff’s rule as the phase sensitive superposition of the individual junction currents $I_n$:

$$I = \sum_{n=0}^{N-1} I_c \sin \varphi_n(t) + \left( \frac{\hbar C_n}{2e} \partial_t^2 + \frac{\hbar}{2e R_n} \partial_t \right) \varphi_n(t)$$

Note that the gauge invariant phase differences $\varphi_n$ of adjacent Josephson junctions in the array are not independent, but are connected to each other by the condition of flux quantization:

$$\varphi_n - \varphi_{n-1} = \frac{2\pi}{\Phi_0} (\mathbf{B} \cdot a_n) \mod 2\pi$$

Here $|a_n|$ is the area of the superconducting loop connecting adjacent Josephson junctions numbered as $n$ and $n-1$, respectively, and $\mathbf{B}$ denotes the magnetic field.

![Fig. 1. Voltage response $\langle V_{xy} \rangle$ in units of $I_c R$ vs. external flux $\Phi^{(1)}$ through largest area element $a_L$ of interferometer for bias current $I = 1.1 N I_c$ and inductance $\beta_L = 0$: a) symmetrical SQUID ($N = 2$), b) periodic 1D array ($N = 11$), c) disordered 1D array ($N = 18$), but with same total area as b).](image)
thrending the orientated area element $a_n$ of this loop. Note that Eq.$(6)$ applies quite generally, provided the superconducting material, out of which the connecting loops are made, is thick compared to the magnetic penetration depth $\lambda$. In this case there exists a path inside the wire connecting, say, junction $n$ with its neighbor junction $n-1$, on which the superfluid velocity field $v_s$ becomes negligible small. So, $\hbar \nabla \Theta = \frac{2e}{c} A$ along this path. Since all junctions in the array are connected in parallel, the rapidly oscillating voltage $V_n(t)$ at the electrodes of a particular Josephson junction, numbered as $n$ in the array, is related to the signal $V_{xy}(t)$ between the nodes $x$ and $y$ of the circuit by

$$V_{xy}(t) = V_n(t) + \int_{x \rightarrow y} ds \cdot E(t) \tag{7}$$

By Faraday’s law the electric field $E$ along an integration path $x \rightarrow y$, that starts at node $x$, traverses the tunneling barrier of the $n$-th Josephson junction just once, and then terminates at node $y$, is directly connected to the time derivative of the flux threading the area elements of the 1D array. Once the signal $V_0(t) = \frac{h}{2e} \partial t \varphi_0(t)$ is known the other voltage signals $V_n(t)$ across the electrodes of the $n$-th junction follow from

$$V_n(t) - V_{n-1}(t) = \frac{1}{c} \partial t \left( B(t) \cdot a_n \right) \tag{8}$$

Taking into account the Biot-Savart type inductive coupling among the currents flowing in the circuit prohibits further simplification. However, it follows directly from Eq.$(7)$ that one can indeed eliminate from Eq.$(8)$ all phase variables $\varphi_n(t)$ in favor of a single phase, for example $\varphi_0(t)$, provided the extremely simplifying assumption is made, that all inductive couplings vanish. In this case the problem of $N$ coupled Josephson junctions is mapped onto a virtual single Josephson junction model. With $\frac{1}{\pi} = \frac{1}{\pi} \sum_{n=0}^{N-1} \frac{1}{\pi n}$, $T_N = \frac{2e}{c}$$\pi n$ and $\phi(t) = \varphi_0(t)$ there results a scalar differential equation determining the phase difference $\varphi(t)$:

$$|S_N(B)| \sin [\phi(t) + \delta_N(B)] + T_N (RC \partial^2 + \partial_t) \phi(t) = 0 \tag{9}$$

$$J_N - \frac{2\pi}{\Phi_0} T_N \left( RC \partial^2 + \partial_t \right) (B(t) \cdot a_C) + \partial_t (B(t) \cdot a_R)$$

where we have defined $a_0 = 0$

$$S_N(B) = \frac{1}{N} \sum_{n=0}^{N-1} \frac{I_{c,n}}{I_c} \exp \left[ \frac{2\pi i}{\Phi_0} \sum_{m=0}^{n} (B \cdot a_m) \right] \tag{10}$$

$$a_R = \frac{1}{N} \sum_{n=0}^{N-1} \frac{R}{R_n} \sum_{m=0}^{n} a_m , \quad a_C = \frac{1}{N} \sum_{n=0}^{N-1} \frac{C_n}{C} \sum_{m=0}^{n} a_m$$

and $J_N = \frac{1}{N \lambda^2}$, $I_c = \frac{1}{N} \sum_{n=0}^{N-1} I_{c,n}$, $C = \frac{1}{N} \sum_{n=0}^{N-1} C_n$. The complex function $S_N(B) = |S_N(B)| e^{i\delta_N(B)}$ denotes the characteristic structure factor of the 1D Josephson junction array, as defined in Eq.$(10)$. It is an extremely responsive function of strength and orientation of magnetic field, and it is strongly affected by the choice of the individual area elements $a_m$. In general $|S_N(B)|$ is very sensitive to permutations among the $a_m$’s.

In the overdamped junction regime, $C = 0$, under conditions where a constant current $I$ is biased such that $1 \geq |S_N(B)| / J_N \equiv \sin \alpha_B$, and assuming for simplicity a homogeneous static magnetic field $B$ (and also time independent area elements $a_m$), one finds an exact solution for the phase difference $\varphi_0(t)$:

$$V_0(t) = \frac{\hbar}{2e} \partial t \varphi_0(t) = I_c R \frac{J_N - |S_N(B)|^2}{|S_N(B)| \sin (\omega_B \cdot t - \alpha_B)}$$

For a static magnetic field $B$ the voltage response function $V_{xy}$ measured between the nodes $x$ and $y$ of the circuit is equal to the $dc$-part of the rapidly oscillating voltage signal $V_0(t)$. All Josephson junctions in the 1D array oscillate at the same frequency $\omega_B = 2\pi \nu_B$, which is related to $V_{xy}$ by:

$$h_2 \nu_B = \langle V_0 \rangle = I_c R \sqrt{J_N^2 - |S_N(B)|^2} = \langle V_{xy} \rangle \tag{11}$$

Note that the oscillation frequency $\nu_B$ of such a local oscillator is even more sensitive to changes of strength or orientation of the external magnetic field $B$ than the structure factor of the array itself, since $|S_N(B)|$ enters Eq.$(11)$ quadratically.

Consider, as a special case, an ordered array, consisting of $N - 1$ identical SQUID loops, such that $(B, a_n) = \Phi = (B, a_L)$, and $I_{c,n} = I_c$ independent on the junction index $n$. Then the structure factor $S_N(B) \equiv S_N^{(\Phi)}$ becomes a simple geometrical series:

$$S_N^{(\Phi)} = \frac{\sin \left( \frac{\pi \Phi}{\Phi_0} N \right)}{N \sin \left( \frac{\pi \Phi}{\Phi_0} \right)} \exp \left[ \frac{\pi \Phi}{\Phi_0} (N - 1) \right] \tag{12}$$

In Fig.$(1b)$ one observes the usual narrowing proportional to $1 / N$ of the width of the voltage response signal $\langle V_{xy} \rangle$ around its minima. Note the periodicity property $S_N^{(\Phi + \Phi_0)} = S_N^{(\Phi)}$ for all $N \geq 2$. For $N = 2$ Eq.$(11)$ is the periodic voltage response of a symmetric two junction SQUID in the overdamped junction regime.

A structure factor with a much longer period may be obtained in a parallel junction array where the orientated area elements increase in size according to a linear relation:

$$a_m = (2m - 1) a_1 \tag{13}$$

For simplicity, identical junction parameters $R_n, C_n$ and $I_{c,n}$ are assumed. Then:

$$S_N(B) = \frac{1}{N} \sum_{n=0}^{N-1} \exp \left[ 2\pi i \frac{\langle B, a_1 \rangle}{\Phi_0} n^2 \right] \tag{14}$$
The total area occupied by such a Gaussian array is $(N - 1)^2 a_1$, where $a_1$ is the smallest area element, and $a_{N-1} = (2N - 3) a_1$ is the largest area element. Compare a Gaussian array, with $N - 1$ area elements as described in Eq. (3), with a periodic array, consisting of $N_P - 1$ identical SQUID-loops with size $|a_L|$. For a useful comparison, both arrays should occupy the same total area: $(N - 1)^2 a_1 = (N_P - 1) a_{L}$. Also the largest area element in the Gaussian array should coincide with the area element of an optimal single SQUID-loop, i.e. $a_{N-1} = a_L$. Both requirements together imply for $N_P \gg 1$ that the Gaussian array has the double number of junctions compared to a corresponding periodic junction array: $N \approx 2N_P$. To determine the period of the Gaussian array consider a case where the flux threading the area of the smallest element, $a_1$, is equal to a rational multiple of half a flux quantum: $(B, a_1) = \frac{\pi}{M}$. Then the largest area element in the array, $a_{N-1}$, is threaded by a flux $\Phi_M = \left(1 - \frac{3}{2N}\right) M \Phi_0$. In this case the structure factor $S_N(B) = e^{-\langle \Phi_M \rangle}$ may be determined using a result of C.F. Gauss:

$$S_N^{(\Phi_M)} = \frac{1}{N} \sum_{n=0}^{N-1} e^{-\pi i n^2} = \frac{e^{\pi^2}}{\sqrt{N} \cdot M} \sum_{n=0}^{M-1} e^{-\pi i n^2}$$

Note the periodicity $|S_N^{(\Phi_M + 2\pi n)}| = |S_N^{(\Phi_M)}|$, with period $2\pi = (2N - 3) \Phi_0$. Remarkably, for $N = (N_1 N_2)$ being the product of two prime numbers $N_1$ and $N_2$, there holds the factorization:

$$S_N^{(\Phi_2)} = (-1)^{(N_1 - 1)(N_2 - 1)} S_{N_1}^{(\Phi_2)} S_{N_2}^{(\Phi_2)}$$

Apparently, such Gaussian junction arrays are governed by the laws of number theory (quadratic residues).

The long periodicity of the structure factor vs. flux $\Phi^{(1)}$ threading the largest area element $a_L$ of Gaussian junction arrays is also visible in the calculated voltage response function $\langle V_{xy} \rangle$, irrespective of the degree of the inductive coupling represented by the parameter $\beta_L$. This is illustrated in Fig.(2). Note the asymmetry of $\langle V_{xy} \rangle$ under $\Phi \rightarrow -\Phi$ (for a constant bias current $I$) for finite inductive coupling. As far as disorder is concerned, we also find that $\langle V_{xy} \rangle$ in Gaussian junction arrays is very responsive to adding small random fluctuations to the size distribution of the area elements, so that $\langle V_{xy} \rangle$ becomes non periodic with a pronounced antipeak only around $\Phi = 0$. As $\beta_L$ increases, the difference max$(\langle V_{xy} \rangle) - \min(\langle V_{xy} \rangle)$ decreases, and the linewidth of the global minimum $\langle V_{xy} \rangle$ around $\Phi = 0$ ceases in this case to scale proportional to $1$. Note all this applies in the overdamped junction regime. Our results for weak damping will be published elsewhere.

We hope that an experimental verification of the predicted magnetic field dependence of the voltage response function of disordered 1D parallel Josephson junction arrays will stimulate the development of new types of robust superconducting quantum interferometers, which would allow (for the first time) a technically rather simple precision measurement of absolute strength of external magnetic fields.

![Fig. 2. Voltage response $\langle V_{xy} \rangle$ in units of $I R$ vs. external flux $\Phi^{(1)}$ through largest area element $a_L$ for a Gaussian array with $N = 18$ (overdamped) junctions for bias current $I = 1.1 N I_c$ and various inductive couplings: a) $\beta_L = 0$, b) $\beta_L = 0.3$ and c) $\beta_L = 0.7$.](image)

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