THE LOOP REPRESENTATION IN GAUGE THEORIES AND QUANTUM GRAVITY*

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ABSTRACT

We review the application of the loop representation to gauge theories and general relativity. The emphasis lies on exhibiting the loop calculus techniques, and their application to the canonical quantization. We discuss the role that knot theory and loop coordinates play in the determination of nondegenerate quantum states of the gravitational field.

1. Introduction

Since the early seventies, gauge theories appeared as the fundamental tools to describe particle interactions. After some important perturbative results such as the unification of the weak and electromagnetic forces and the proof of the renormalizability of Yang Mills theories, the treatment of the strong interactions in terms of gauge fields required the development of nonperturbative techniques. In that sense, various attempts $^{1-5}$ were made to describe gauge theories in terms of extended objects as Wilson loops and holonomies.

The loop representation $^{6-7}$ is a quantum hamiltonian representation of gauge theories in terms of loops. The aim of the loop representation, in the context of Yang Mills theories is to avoid the redundancy introduced by gauge symmetries allowing to work directly in the space of physical states. However, we shall see that the loop formalism goes far beyond a simple gauge invariant description, in fact it is the natural geometrical framework to treat gauge theories and quantum gravity in terms of their fundamental physical excitations.

The introduction by Ashtekar $^8$ of a new set of variables that cast general relativity in the same language as gauge theories allowed to apply loop techniques as a natural nonperturbative description of the Einstein’s theory. Being the new variables the basis of a canonical approach to quantum gravity, the loop representation appeared $^{9,10}$ as the most appealing application of the loop techniques to this problem. In particular, it was soon discovered a deep relationship between the physical states in the loop representation and the notions of the Knot Theory.

The organization of these lectures is as follows. In section 2 we introduce the holonomies and discuss their connection with loops. We define the group of loops

* To appear in the proceedings of the IV th. Mexican Workshop on Particles and Fields (Merida, Yucatan, 25-29 october 1993), World Scientific, Singapore
and the properties of the differential generators of the group are studied. In section 3, the loop representation of gauge theories is introduced, the role of Wilson loops is pointed out, and the formalism is applied to the study of the abelian and non abelian gauge theories. In section 4 the canonical formalism of general relativity in terms of the traditional and new variables is discussed. In section 5 the loop representation of general relativity is introduced. The constraints are realized in the loop space, and the mathematical tools required to deal with the constraints and their solutions are studied. Then, these tools are applied to the determination of a nondegenerate family of physical states of quantum gravity and its description in terms of knot invariants. Finally in section 6 we conclude with some final remarks and a general discussion of some open issues of the quantization program of gauge theories and general relativity in terms of loops.

2. Holonomies and the Group of Loops

2.1 Holonomies

All the known fundamental forces in nature may be described in terms of locally invariant gauge theories. Connections and the associated concept of parallel transport play a fundamental role in this kind of theories allowing to compare fields in neighboring points in an invariant form. In fact, let us consider fields $\psi^i(x)$ whose dynamics is invariant under local transformations

$$\psi^i(x) \rightarrow U^{ij}(x)\psi^j(x)$$  \hspace{1cm} (1)

where $U^{ij}$ are the elements of some representation of a Lie group $G$. In order to compare fields at different points $\psi^i(x+\epsilon)$ and $\psi^i(x)$ we need to introduce a notion of parallel transport that allows to compare fields in the same local frame of reference.

$$\delta \psi^i = \psi^i(x+\epsilon) - \psi^i_{\parallel}(x,\epsilon)$$  \hspace{1cm} (2)

where

$$\psi^i_{\parallel}(x,\epsilon) = V^{ij}(x,\epsilon)\psi^j(x) = (\delta^{ij} - i\epsilon^\mu A^{ij}_\mu)\psi^j(x)$$  \hspace{1cm} (3)

The matrix $V^{ij}$ is the linear transformation belonging to $G$ that relates the components of the original field at $x$ and the parallel transported field at $x+\epsilon$. Being $V$ an element of the group near to the identity it may be expressed in terms of the connection $A^{ij}_\mu = A^A_{\mu}T^{A\ ij}$ where $T^A$ is a basis of generators of the algebra. For instance if $G$ is $SU(2)$ then the $T^A, A = 1, 2, 3$ are proportional to the Pauli matrices.

Given an open curve $P_y$, one can parallel transport $\psi$ along $P$. The parallel transported field at the end point $y$ will be given by:

$$\psi_{\parallel}(x,P^y) = \lim_{N\to\infty} \prod_{h=0}^{N-1} (1 - iA_{\mu}(x_h)\Delta x_{h+1}^\mu)\psi(x)$$  \hspace{1cm} (4)
\[ \lim_{N \to \infty} (1 - iA_\mu(x_N)\Delta x_{N+1}^\mu) \ldots (1 - iA_\mu(x)\Delta x_1^\mu)\psi(x) \]

and it is usually written in terms of the path ordered exponential

\[ \psi_{||}(x, P^y_x) = \mathcal{P} \exp[-i \int_x^y A_\mu(z) dz^\mu] \psi(x) \] (5)

Under a gauge transformation

\[ A_\mu \to A_\mu(x) = U(x)A_\mu U^{-1}(x) - iU(x)\partial_\mu U^{-1}(x) \] (6)

and the path ordered exponential transforms

\[ \mathcal{P} \exp[-i \int_x^y A_\mu dz^\mu] \to U(y)\mathcal{P} \exp[-i \int_x^y A_\mu dz^\mu] U^{-1}(x) \] (7)

and therefore \( \psi_{||}(x, P^y_x) \) transforms under local transformations at \( y \).

If \( p \) is a closed curve with origin at some basepoint \( x_0 \), the path ordered exponential connects the original field with the field parallel transported along \( p \). In this case the path ordered exponential may be written as

\[ H_A(p) = \mathcal{P} \exp[-i \int_p A_\mu dy^\mu] \] (8)

and it transforms as

\[ H_A(p) \to U(x_0)H_A(p)U^{-1}(x_0) \] (9)

It is not always possible to describe a gauge theory in terms of a connection defined over all the base manifold. When there is not an unique chart covering all the space, the parallel transport along a curve will not be given by Eq.(8). The mathematical structures which describe the general case are fiber bundles with a connection. In mathematics, the parallel transport along a closed curve \( H(p_o) \) is usually called the holonomy, while in particle physics it is known as the Wu-Yang phase-factor.

Curvature will be related with the failure of a field to return to its original value when parallel transported along a small curve. For infinitesimal closed curves basepointed at \( o \), holonomies and curvatures have the same information. The knowledge of the holonomy for any closed curve with basepoint \( o \) allows to reconstruct the connection at any point of the base manifold. This property, together with its invariance under the set of gauge transformations which act trivially at the basepoint allow to use the holonomies to encode all the information of a gauge theory.

### 2.2 The group of loops

Holonomies may be defined intrinsically without any reference to connections. In fact, they can be viewed as representations from a group structure defined in terms of equivalence classes of closed curves onto a Lie group \( G \). Each equivalence class of curves is called a loop and the group structure defined by them is called the group of loops.
The group of loops is the basic underlying structure to all the formulations of gauge theories in terms of holonomies, in particular wave functions in the loop representation depend on the elements of the group of loops.

Let us consider, piecewise smooth, parameterized curves in a manifold $M$.

$$p : [0,1] \rightarrow M$$ (10)

Two parameterized curves $p_1$ and $p_2$ such that $p_1(1) = p_2(0)$ may be composed as follows:

$$p_1 \circ p_2(s) = \begin{cases} p_1(2s), & \text{for } s \in [0,1/2], \\ p_2(2(s - 1/2)), & \text{for } s \in [1/2,1]. \end{cases}$$ (11)

We shall be also interested in the curve with the opposite orientation.

$$\bar{p}(s) = p(1 - s)$$ (12)

Let us now consider closed curves $l, m, \ldots$ such that they start and end at the same point $o$. We denote by $L_o$ the set of all these closed curves. Loops will be equivalence classes of curves belonging to $L_o$. The rationale for this equivalence relation, is to identify all closed curves leading to the same holonomy. Two curves $l, m \in L_o$ are equivalent

$$l \sim m$$

iff

$$H_A(l) = H_A(m)$$ (13)

for every bundle $P(M,G)$ and connection $A$. Loops are identified with the equivalence classes of curves under this relation.

There are several equivalent definitions of a loop we give here an alternative definition. One starts by identifying curves equivalent to the null curve $i(s) = o$ for all $s$. A close curve $l$ is a ”tree”\textsuperscript{11} or ”thin”\textsuperscript{12} if there exists a homotopy of $l$ to the null curve in which the image of the homotopy is included in the image of $l$.

Examples of ”trees” are given in Fig.[1]

\textbf{Figure 1:} Trees or thin curves. These curves do not enclose any area
It is obvious that the holonomy for any of these curves is the identity no matter what is the connection or the gauge theory.

Two closed curves \( l, m \in L_o \) are equivalent \( l \sim m \) iff \( l \circ \bar{m} \) is thin.

Obviously two curves differing by an orientation preserving reparametrization are equivalent.

In Figure 2 we show two equivalent curves. Again loops are identified with the corresponding equivalence classes.

**Figure 2:** Two equivalent curves, \( l = p_1 \circ p_2 \) and \( m = p_1 \circ q \circ \bar{q} \circ p_2 \)

It may be immediately shown that the composition between loops is well defined and is again a loop. In other words, if we denote by \( \alpha = [l] \) and \( \beta = [m] \) the equivalence classes of curves that respectively contain \( l \) and \( m \) then \( \alpha \circ \beta = [l \circ m] \)

**Figure 3:** The product of two loops is given by their composition

The inverse of a loop \( \alpha = [l] \) is the loop \( \alpha^{-1} = [l] \) in fact

\[
\alpha \circ \alpha^{-1} = \iota
\]  

(14)

here \( \iota \) is the set of curves ("trees" or "thin" curves) equivalent to the null curve

We will denote by \( L_o \) the set of loops basepointed at \( o \). This set forms a non abelian group called the group of loops.
Before concluding this section, it is convenient to introduce a notion of continuity in loop space. We shall say that a loop $\alpha$ is in a neighborhood $U_{r}(\beta)$ of a loop $\beta$, if there exists at least two parameterized curves $a(s) \in \alpha$ and $b(s) \in \beta$ such that $a(s) \in U_{r}(b(s))$ with the usual curve topology of the manifold.

**Figure 4:** The inverse of a loop

**Figure 5:** Two close loops

It is possible to introduce an equivalence relation for open paths similar to the one introduced among closed curves. Given two curves $p_{x}^{\sigma}$ and $q_{x}^{\sigma}$ we shall say that $p$ and $q$ are equivalent iff $p_{x}^{\sigma} \circ q_{x}^{\sigma}$ is a "tree". We shall denote the corresponding class of equivalence by $\alpha_{x}^{\sigma}$, we shall denote by $\alpha_{x}^{e}$ the path with the opposite orientation.

**Figure 6:** Two equivalent paths
2.3 Differential operators on loop dependent functions

2.3.1 The loop derivative

In this section we are going to introduce the natural differential operators in the loop space. Due to the group structure of loop space, the differential operators are related with the infinitesimal generators of the group of loops. Although the explicit introduction of the differential operators will be made in a coordinate chart, we will show that their definition do not depend on the particular chart chosen, and they transform as tensors * under coordinate transformations.

Given \( \psi(\gamma) \) a continuous, complex value function of \( \mathcal{L}_o \). We are going to consider its variation under the action of an infinitesimal loop belonging to \( \mathcal{L}_o \). Let \( \delta \gamma(\pi, \delta u, \delta v) \) be the loop.

\[
\delta \gamma(\pi, \delta u, \delta v) = \pi_o^x \delta u \delta v \delta u \delta v \pi_o^x
\]

obtained by going first from the origin to the point \( x \) then following the loop \( \delta \gamma \) defined in a local chart by the curve going along \( u \) from \( x^a \) to \( x^a + \epsilon_1 u^a \) then going from \( x^a + \epsilon_1 u^a \) to \( x^a + \epsilon_1 u^a + \epsilon_2 v^a \) along \( v \), then going along \(-u\) to \( x^a + \epsilon_2 v^a \) and finally going back to \( x \) along \(-v\) as shown in the next figure.

For a given \( \pi \) and \( \gamma \) the function \( \psi(\delta \gamma \circ \gamma) \) only depends on the vectors \( \delta u \) and \( \delta v \). We will assume that the function \( \psi \) is differentiable and that it is possible to consider the following expansion

\[
\psi(\delta \gamma \circ \gamma) = \psi(\gamma) + \epsilon_1 u^a A_a(\pi_o^x) \psi(\gamma) + \epsilon_2 v^b B_b(\pi_o^x) \psi(\gamma) + \frac{1}{2} \epsilon_1 \epsilon_2 (u^a v^b + u^b v^a) S_{ab}(\pi_o^x) \psi(\gamma) + \frac{1}{2} \epsilon_1 \epsilon_2 (u^a v^b - u^b v^a) \Delta_{ab}(\pi_o^x) \psi(\gamma)
\]

where, \( A, B, S \) and \( \Delta \) are differential operators. It may be easily seen that if \( \epsilon_1 \) or \( \epsilon_2 \) vanish or \( u \) is colinear with \( v \), \( \delta \gamma \) is a tree and therefore all the terms except the

* A more rigorous and intrinsic mathematical treatment of the loop derivative has been recently given by J.N.Tavares\textsuperscript{13}.
first vanish. This means that the linear and symmetric terms vanish,

\[ A_a(\pi^x_0)\psi(\gamma) = 0, \quad B_a(\pi^x_0)\psi(\gamma) = 0, \quad S_{ab}(\pi^x_0)\psi(\gamma) = 0. \] (17)

All the information about the deformation of the loop is contained in the antisymmetric operator \( \Delta_{ab}(\pi) \) which is called the loop derivative 14.

\[ \psi(\delta\gamma \circ \gamma) = (1 + \frac{1}{2}\sigma^{ab}\Delta_{ab}(\pi^x_0))\psi(\gamma) \] (18)

with \( \sigma^{ab} = 2u[a_1 b_1]_1 \epsilon_1 \epsilon_2 \).

This definition may be extended to functions depending of open paths \( \psi(\chi^y_0) \) recalling that open paths do not see trees by definition. Therefore

\[ \psi(\delta\gamma \circ \chi^y_0) = (1 + \frac{1}{2}\sigma^{ab}\Delta_{ab}(\pi^x_0))\psi(\chi^y_0) \] (19)

The loop derivative transforms as an antisymmetric tensor under local coordinate transformations. In fact, it is immediate to see that the transformed path \( \tilde{\delta}\gamma \) will be equivalent at first order in \( \epsilon_1 \epsilon_2 \) to the path defined by the transformed vector \( \tilde{\delta}u \) and \( \tilde{\delta}v \) and therefore by quotient law \( \Delta_{ab} \) behaves as a tensor

The loop derivatives are noncommutative operators. Their commutation relations can be computed\(^{14} \) from the properties of the group of loops in the following way. Let \( \delta\gamma_1 \) and \( \delta\gamma_2 \) be two infinitesimal loops given by:

\[ \delta\gamma_1 = \pi^x_o \delta u \delta v \delta \bar{u} \delta \bar{v} \pi^x_o \] (20)

and

\[ \delta\gamma_2 = \chi^y_o \delta w \delta \bar{t} \delta \bar{w} \delta \bar{t} \chi^y_o \] (21)

with area elements

\[ \sigma_1^{ab} = 2\epsilon_1 \epsilon_2 u[a b], \quad \text{and} \quad \sigma_2^{ab} = 2\epsilon_3 \epsilon_4 w[a b] \] (22)

then it follows from the definition of the loop derivative that

\[
(1 + \frac{1}{2}\sigma_1^{ab}\Delta_{ab}(\pi^x_0))(1 + \frac{1}{2}\sigma_2^{cd}\Delta_{cd}(\chi^y_0))(1 - \frac{1}{2}\sigma_1^{ab}\Delta_{ab}(\pi^x_0))(1 - \frac{1}{2}\sigma_2^{cd}\Delta_{cd}(\chi^y_0))\psi(\gamma) \\
= 1 + \frac{1}{2}\sigma_1^{ab}\sigma_2^{cd}[(\Delta_{ab}(\pi^x_0)), \Delta_{cd}(\chi^y_0)]\psi(\gamma) = \psi(\delta\gamma_1 \circ \delta\gamma_2 \circ \delta\gamma_1 \circ \delta\gamma_2 \circ \gamma)
\] (23)

Now by introducing the open path

\[ \chi^y_o = \delta\gamma_1 \circ \chi^y_o \] (24)

one gets

\[ \delta\gamma_2' \equiv \delta\gamma_1 \circ \delta\gamma_2 \circ \delta\gamma_1 = \chi^y_o \delta w \delta \bar{t} \delta \bar{w} \delta \bar{t} \chi^y_o \] (25)

and therefore

\[
\psi(\delta\gamma_1 \circ \delta\gamma_2 \circ \delta\gamma_1 \circ \delta\gamma_2 \circ \gamma) = \psi(\delta\gamma_2' \circ \delta\gamma_2 \circ \gamma)
= (1 + \frac{1}{2}\sigma_2^{ab}\Delta_{ab}(\chi^y_o))(1 - \frac{1}{2}\sigma_2^{ab}\Delta_{ab}(\chi^y_o))\psi(\gamma)
\] (26)
Recalling the definition of the loop derivative of an open path we get

$$\psi(\delta \gamma_1 \circ \delta \gamma_2) = 1 + \frac{1}{2} \sigma^{ab}_{1} \sigma^{cd}_{2} \Delta^{ab}_{cd}(\gamma)$$

where $\Delta^{ab}_{cd}(\gamma)$ represents the action of the first loop derivative on the path dependence of the second. Therefore

$$[\Delta^{ab}_1, \Delta^{cd}_2] = \Delta^{ab}_1 \Delta^{cd}_2$$

The commutation relations of the loop derivatives may be written in a more familiar way as a linear combination of elements of the algebra. In fact

$$[\Delta^{ab}_1, \Delta^{cd}_2] = \lim_{\epsilon \to 0} \frac{1}{\epsilon} [\Delta^{ab}_1(\delta \gamma_1), \Delta^{cd}_2(\chi)]$$

However these expressions are only formally analogous with the commutation relations of a Lie group, because as it may be easily seen the group of loops is not a Lie group.

The loop derivatives are not independent. In fact, they are related by a set of identities associated with the Bianchi identities for the field strength in the usual Yang Mills theories. In order to write these relations, it is necessary to introduce a new differential operator, the end point derivative or Mandelstam derivative that acts on function of open paths. Given a function of an open path $\pi^T$, a local chart at the point $x$ and a vector $v^a$ in that chart, the end point derivative is defined by considering the change of the function when the path is extended from $x$ to $x + \epsilon v$ by the straight path $\delta v$

$$\psi(\pi^T \circ \delta v) = (1 + \epsilon v^a D_a) \psi(\pi^T)$$

**Figure 8:** The action of the end point derivative

If one performs a local coordinate transformation one can easily seen that the transformed path is approximated at first order by the extension along the transformed tangent vector at $x$ and therefore $D_a$ transforms as a one form. The Bianchi identities may now be derived by considering a tree defined by an open path $\pi^T$ and three vectors $u, v, w$ given by:
\[ l = \pi_0^x \delta v \delta w \delta \mu \delta \nu \pi_x^o \circ \pi_0^x \delta v \delta w \delta \mu \delta \nu \pi_x^o \circ \pi_0^x \delta v \delta w \delta \mu \delta \nu \pi_x^o \circ \pi_0^x \delta v \delta w \delta \mu \delta \nu \pi_x^o \circ \pi_0^x \delta v \delta w \delta \mu \delta \nu \pi_x^o \]

(31)

**Figure 9:** The tree associated with the Bianchi identities

Then

\[ \psi(i \circ \gamma) \equiv \psi(\gamma) = (1 + \epsilon_2 \epsilon_3 v^a w^b \Delta_{ab}(\pi_0^x)) (1 + \epsilon_1 \epsilon_3 u^c w^d \Delta_{cd}(\pi_0^x)) (1 + \epsilon_1 \epsilon_2 v^e w^f \Delta_{ef}(\pi_0^x)) \]

(32)

where we have denoted the extended path by \( \pi_0^{x+\epsilon v} \). Now collecting the terms of first order in \( \epsilon \) and applying the definition of the Mandelstam derivative, we get

\[ D_a \Delta_{bc}(\pi_0^x) + D_b \Delta_{ca}(\pi_0^x) + D_c \Delta_{ab}(\pi_0^x) = 0 \]

(33)

The commutation relations and the Bianchi identities are the basic tools of the loop calculus.

We conclude this section with the integral form of the commutation relations. Let us consider the loop dependent operator \( U(\eta) \), acting on the space of loop functions, defined by

\[ U(\eta) \psi(\gamma) = \psi(\eta \circ \gamma) \]

(34)

This operator verifies

\[ U(\eta_1) U(\eta_2) = U(\eta_1 \circ \eta_2) \]

(35)
and

$$U(\eta^{-1}) = U^{-1}(\eta)$$  \hspace{1cm} (36)$$

We want to compute the loop derivative evaluated for the deformed path $\eta \circ \pi_o^x$, then

$$(1 + \frac{1}{2} \sigma^{nb} \Delta_{ab}(\eta \circ \pi_o^x)) \psi(\gamma) = \psi[\eta \circ \delta\gamma(\pi, \delta u, \delta v) \circ \eta^{-1} \circ \gamma] = U(\eta)U(\delta\gamma(\pi, \delta u, \delta v))U(\eta^{-1}) \psi(\gamma)$$

$$= U(\eta)(1 + \frac{1}{2} \sigma^{nb} \Delta_{ab}(\pi_o^x))U(\eta^{-1}) \psi(\gamma)$$  \hspace{1cm} (37)$$

that implies

$$\Delta_{ab}(\eta \circ \pi_o^x) = U(\eta)\Delta_{ab}(\pi_o^x)U^{-1}(\eta)$$  \hspace{1cm} (38)$$

This expression gives the transformation law of the loop derivative under a finite deformation.

2.3.2 The connection derivative

The loop derivative is the basic building block of any finite loop $^{14,15}$, however, it is convenient to introduce a second differential operator whose properties are related with those of the connection in a gauge theory. Let us consider a continuous function $f(x)$ with support in the points of a local chart $U$, such that, to each point in the chart, it associates a path $\pi_o^x$, the origin $o$ not necessarily belonging to $U$. Given a continuous function of loop $\psi(\gamma)$ and a vector $u$ at $x$, we are going to consider the deformation of the loop $\gamma$ induced by

$$\delta\gamma = \pi_o^x \circ \delta u \circ \pi_o^{x+\epsilon u}$$  \hspace{1cm} (39)$$

where $\pi_{x+\epsilon u}$ is the path associated to the point $x + \epsilon u$ by the function $f$
We will say that the connection derivative of $\psi(\gamma)$ exists if $\psi(\delta \gamma \circ \gamma)$ may be expanded for any $x \in U$ by

$$\psi(\delta \gamma \circ \gamma) = (1 + \epsilon^a \delta_a(x)) \psi(\gamma)$$  \hspace{1cm} (40)

Notice that once the function $f(x)$ is given the connection derivative $\delta_a$ only depends on $x$.

Connection derivatives and loop derivatives are related by a relation similar to the one satisfied by the potential and field strength in a gauge theory. Let us consider the identity in loop space, given by

$$\delta \gamma = \pi_o \delta u \delta v \delta \bar{u} \delta \bar{v} \pi_x^o = \pi_o \delta u \pi_x^o \circ \pi_o \delta v \pi_x^o \circ \pi_o \delta \bar{u} \pi_x^o \circ \pi_o \delta \bar{v} \pi_x^o$$  \hspace{1cm} (41)

and shown in the next figure

\textbf{Figure 11:} The geometrical relation between the loop derivative and the connection derivative

This geometrical relation implies the following identity between differential operators

$$(1 + \epsilon_1 \epsilon_2 u^a v^b \Delta_{ab}(\pi_o^x)) \psi(\gamma) = (1 + \epsilon_1 u^a \delta_a(x))(1 + \epsilon_2 v^b \delta_b(x + \epsilon_1 u))$$

$$= (1 - \epsilon_1 u^a \delta_a(x + \epsilon_1 u + \epsilon_2 v))(1 - \epsilon_2 v^b \delta_b(x + \epsilon_2 v)) \psi(\gamma)$$  \hspace{1cm} (42)

and keeping terms linear in $\epsilon_1, \epsilon_2$ we get

$$\Delta_{ab}(\pi_o^x) = \partial_a \delta_b(x) - \partial_b \delta_a(x) + [\delta_a(x), \delta_b(x)]$$  \hspace{1cm} (43)

equation that reminds to the usual relation between fields and connections.

One may wonder what happens with the gauge dependence of the connection in the language of loops. We shall see that it is related with the prescription given by $f(x)$. To see this, let us consider two different prescriptions

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\[ \pi_o^x = f(x) \text{ and } \chi_o^x = g(x) \]  

(44)

Figure 12: The effect of a change of prescription in the definition of the connection derivative

Then

\[ \chi_o^x \delta u \chi_o^x = \chi_o^x \circ \pi_o^x \circ \pi_o^x \delta u \pi_o^x \circ \pi_o^x \circ \chi_o^x \]  

(45)

and introducing the point dependent operator \( U(x) \) constructed from the deformation operator \( U(\eta) \) by \( U(x) = U(\chi_o^x \circ \pi_o^x) \) we get

\[
(1 + \epsilon u^a \delta_a^a(x)) \psi(\gamma) = U(x)(1 + \epsilon u^a \delta_a^a(x))U^{-1}(x) \psi(\gamma)
\]  

(46)

and, from here, we get the relation between the connection derivatives evaluated for two different prescriptions

\[
\delta_a^a(x) = U(x)\delta_a^a(x)U^{-1}(x) + U(x)\partial_aU^{-1}(x).
\]  

(47)

In an analogous way we get

\[
\Delta_{ab}(\chi_o^x) = U(x)\Delta_{ab}(\pi_o^x)U^{-1}(x).
\]  

(48)

Notice that the properties of the group of loops allowed us to recover the complete set of kinematical relations of any gauge theory written in terms of the differential operators without any reference to a particular Lie group.

2.4 Gauge theories and representations of the group of loops

Classical gauge theories arise as representations (homomorphisms) of the group of loops onto same gauge group \( G \). Let \( H(\gamma) \) be such a mapping \( H(\gamma) \in G \) and

\[
H(\gamma_1)H(\gamma_2) = H(\gamma_1 \circ \gamma_2)
\]  

(49)

Let us assume that the representation is loop differentiable and that the gauge group is \( SU(N) \). Then, we may compute

\[
(1 + \epsilon u^a \delta_a^a(x))H(\gamma) = H(\pi_o^x \delta u \pi_o^x \circ \gamma) = H(\pi_o^x \delta u \pi_o^x \circ \gamma)H(\gamma)
\]  

(50)
Since $H$ is a continuous differentiable representation and $\pi_0^x \delta u \pi_{x+\epsilon u}^x$ is near to the identity with the topology of loops

\[ H(\pi_0^x \delta u \pi_{x+\epsilon u}^x) = (1 + i\epsilon u^a A_a(x)) \] (51)

with $A_a(x) = A^B_a(x) T^B$ belonging to the algebra of $SU(N)$. Thus

\[ \delta_a(x) H(\gamma) = i A_a(x) H(\gamma) \] (52)

and analogously

\[ \Delta_{ab}(\pi_0^x) H(\gamma) = i F_{ab}(x) H(\gamma) \] (53)

with $F$ belonging to the algebra. Now from each relation already derived for the operators it holds a similar relation for the elements of the algebra, fields and potentials. For instance, from Eq.(43) it follows that

\[ F_{ab}(x) = \partial_a A_b(x) - \partial_b A_a(x) + i [A_a(x), A_b(x)] \] (54)

and using Eq.[48] we get the transformation law of $F$ under a change in the prescription of the path $\pi \rightarrow \pi'$.

\[ F'_{ab}(x) = H(x) F_{ab}(x) H^{-1}(x) \] (55)

with

\[ H(x) = H(\pi_{\pi'}^x \circ \pi_0^x) \] (56)

The usual expression of the holonomy in terms of the connection may be derived from the definition of the connection derivatives and the geometrical construction shown in the next figure.

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**Figure 13:** Graphical construction of the holonomy in terms of the connection

\[ \gamma = \lim_{n \to \infty} \delta \gamma_1 \ldots \delta \gamma_n \] (57)

with
\[ \delta \gamma_i = \pi^i \Delta x_i \pi_{x, i+1} \Delta x_i \]  

and therefore

\[ U(\gamma) = \mathcal{P} \exp \int_{\gamma} dy \delta_a(y) \]  

and noticing that

\[ U(\alpha)H(\gamma) = H(\alpha \circ \gamma) = H(\alpha)H(\gamma) = \mathcal{P} \exp i \int_\alpha dy A_a(y) H(\gamma) \]

we recover the usual expression for the holonomy in terms of the connection \( A_a \). Notice that if the loop \( \alpha \) is not contained in a local chart with an unique prescription \( f(x) \), the holonomy does not take this simple form\(^{16}\). Thus all the kinematics of a gauge theory is contained in the representation of the group of loops in the gauge group under consideration. This representation is nothing but the holonomy of the corresponding gauge theory. If the representation is not loop differentiable the holonomy does not correspond to any connection and in that case we shall obtain "generalized" holonomies.

### 3. The Loop Representation

In this section we will treat the problem of the quantization of gauge theories. Our main objective is to introduce a quantum representation of Hamiltonian gauge theories in terms of loops. The use of loops for a gauge invariant description of Yang Mills theories may be traced back to the Mandelstam\(^1\) quantization without potentials. In 1974 Yang\(^{17}\) noticed the important role of the holonomies for a complete description of gauge theories.

Since the last seventies several non perturative attempts to treat Yang Mills theories in terms of loops were made. The investigation of the equation of motion for loop functionals was initiated by Polyakov\(^3\), Nambu\(^4\), Gervais and Neveu\(^18\) and further developed by many others. Makeenko and Migdal\(^2\) considered Wilson loop averages, wrote down the corresponding equations and studied the large \( N \) limit.

In 1980 a loop based\(^6\) hamiltonian approach to quantum electromagnetism was proposed and generalized\(^7\) in 1986 to include the Yang Mills theory. This hamiltonian formulation was given in terms of the traces of the holonomies (the Wilson loops) and their temporal loop derivatives as the fundamental objects. They replace the information furnished by the vector potential and the electric field operator, respectively. These gauge invariant operators verify a close algebra and may be realized on a linear space of loop dependent functions.

As we shall see, this approach has many appealing features. In first place it allows to do away with the first class constraints of the gauge theories (the Gauss law). In second place the formalism only involves gauge invariant objects. This makes the formalism specially well suited to study "white" objects as mesons and barions in Q.C.D. because the wave function will only depend on the paths associated with
the physical excitations. Finally, all the gauge invariant operators have a simple geometrical meaning when realized in the loop space.

3.1 Systems with constraints

Here we want to recall very briefly some of the main features of the systems with first class constraints in the sense of Dirac. Let us consider a hamiltonian system described by a set of canonical variables $q_i$ and momentum $p_i$ with Poisson bracket relations:

$$\{q_i, p_j\} = \delta_{ij} \quad (61)$$

We shall say that the system is constrained if the canonical variables obey a set of relations $\Phi_m(q_i, p_j) = 0$. A constraint $\Phi_k$ will be called of first class if its Poisson brackets with the other constraints is a combination of the constraints.

$$\{\Phi_k, \Phi_j\} = C_{kj}^l \Phi_l \quad (62)$$

for any $j$. Other constraints will be called second class. We shall here consider only constrained system with first class constraints. In that case all the constraints satisfy Eq.(62). The effect of having constraints is to restrict the time evolution of the system to a surface $\bar{\Gamma}$ in the phase space $\Gamma$ called the ”constraint surface”. The dynamical trajectories in $\bar{\Gamma}$ are not uniquely defined. There is an infinite family of trajectories which are physically equivalent. Two trajectories belonging to the same family are gauge equivalent. This ambiguity is due to the fact that the extension of the physical quantities from $\bar{\Gamma}$ to $\Gamma$ is not unique. For instance if $H$ is an extension, so is

$$H' = H + \lambda^j \Phi_j \quad (63)$$

where $\lambda^j$ is any smooth function on $\Gamma$. This in turns introduces an ambiguity in the dynamical evolution of the physical states in $\bar{\Gamma}$. In fact after a small amount of time, two equivalent dynamical trajectories which started from the same initial conditions will differ by terms proportional to the commutators of the dynamical variables with the constraints. In that sense, any first class constraint may be viewed as the generator of some of the gauge symmetries of the theory. Any dynamical variable with vanishing Poisson Brackets with the constraints on the constraint surface $\Gamma$ will be called an observable. These are the gauge invariant quantities of the system. To quantize a system with first class constraints one usually follows a program developed by Dirac in the sixties. One considers as states, wave functions $\psi(q)$ on the configuration space and represent the operators $\hat{q}$ as multiplicative operators

$$\hat{q}\psi(q) = q\psi(q) \quad (64)$$

and
\[ \hat{p}\psi(q) = -i\hbar \frac{\partial \psi}{\partial q} \] (65)

in order to have commutators proportional to the Poisson Brackets. Now we need to promote the classical constraints to operators

\[ \Phi_m(p, q) \rightarrow \Phi_m(\hat{p}, \hat{q}) \] (66)

in general this step involves a factor ordering choice and, in the case of fields a regularization is also required. The physical state space is defined by \( \psi_F(q) \):

\[ \Phi_m(\hat{p}, \hat{q})\psi_F(q) = 0 \] (67)

The idea is to use the space of states \( \psi_F \) as the relevant space in physics. However an important consistency requirement must hold

\[ [\Phi_m(\hat{p}, \hat{q}), \Phi_n(\hat{p}, \hat{q})] \psi_F(q) = 0 \] (68)

for all \( m \) and \( n \). At the classical level we know that the corresponding Poisson bracket is a linear combination of the constraints, but due to ordering and regularization problems this condition may fail at the quantum level.

In some cases as general relativity this program is incomplete and need to be complemented. In first place the program does not provide guidelines for introducing an appropriate inner product for general diffeomorphism invariant theories. Secondly, when the configuration space is not a trivial manifold (is not diffeomorphic to \( \mathbb{R}^n \)), one needs to work with an overcomplete set of configuration observables. In other words there are some relations between the configuration variables that will be promoted to operators at the quantum level. Here, I will not enter into the first problem because the issue of the inner product in quantum gravity will not be discussed. Concerning the second problem, we shall discuss with some detail this issue after the introduction of the natural configuration variables of the gauge theories.

Let us now discuss as an example the canonical formulation and quantization of the Maxwell field. The action is

\[ S = -\frac{1}{4} \int d^4x F_{\mu\nu}(x) F^{\mu\nu}(x) \] (69)

and the configuration variables are \( A_a \) and \( A_0 \). The canonical momentum

\[ \pi^0 = \frac{\delta S}{\delta \dot{A}_0} = 0, \pi^a = \frac{\delta S}{\delta \dot{A}_a} = F^{a\alpha} = E^a \] (70)

The vanishing of \( \pi^0 \) is a primary constraint. The corresponding hamiltonian density is

\[ \mathcal{H}_0 = \frac{1}{2}(E^a E^a + B^a B^a) - A_0(\partial_a E^a) \] (71)

We can now extend the hamiltonian to include the primary constraint
\[ H' = H_0 + \lambda_0 \pi^0 \]  

(72)

and insure the conservation of the primary constraint \( \pi^0 = 0 \)

\[ \dot{\pi}^0 = -\frac{\delta H'}{\delta A_0} = \partial_a E^a = 0 \]  

(73)

Thus the preservation of the primary constraint implies a new constraint which is in turn conserved. These constraints are first class

\[ \{ \pi^0(x, t), \pi^0(y, t) \} = \{ \partial_a E^a(x, t), \partial_b E^b(y, t) \} = 0 \]  

(74)

Let us now quantize this field. One represents quantum states as functionals of the potentials \( \psi[\vec{A}, A_0] \), and introduces the representation of the canonical Poisson algebra in which \( \hat{A}_a \) and \( \hat{A}_0 \) are multiplicative operators and

\[ \dot{\psi}^0[\vec{A}, A_0] = -i \frac{\delta \psi[\vec{A}, A_0]}{\delta A_0}, \quad \dot{E}^a[\vec{A}, A_0] = -i \frac{\delta \psi[\vec{A}, A_0]}{\delta A_a} \]  

(75)

Now promoting the constraints to quantum equations we notice that the primary constraint implies

\[ \frac{\delta \psi[\vec{A}, A_0]}{\delta A_0} = 0 \rightarrow \psi = \psi[A_a] \]  

(76)

while the meaning of the second constraint may be understood from

\[ (1 + i \int d^3 x \Lambda(x) \partial_a E^a(x)) \psi[A_a] = \psi[A_a + \Lambda_a] \]  

(77)

Thus we see that the Gauss law constraint acts as generator of infinitesimal gauge transformation of the potentials. The physical states \( \psi_F[A] \) are annihilated by the Gauss law constraint and therefore they are gauge invariant

### 3.2 Wilson loops

As we mentioned in the previous sections holonomies may be a good starting point for treating Yang Mills theories in terms of a basis of gauge invariant states. In fact let us consider the trace of the holonomy

\[ W_A(\gamma) = Tr[\mathcal{P} \exp i \int_\gamma dy_a A_a(y)] \]  

(78)

which is a gauge invariant quantity known as the Wilson loop functional. Wilson loops are restricted by a set of identities known as the Mandelstam identities and for compact gauge groups they contain all the gauge invariant information of the theory. For non compact groups, as \( SO(2, 1) \), even though holonomies carry all the gauge invariant information, some of this information is lost while taking the trace. However one can show that the Wilson loops allow even is this case to recover all the gauge invariant information up to a measure zero set of connections\(^9\).
We shall first discuss the Mandelstam identities for gauge groups that admit fundamental representations in terms of $N \times N$ matrices. These identities are usually classified in first kind and second kind. The Mandelstam identity of first kind is due to the cyclic property of traces

$$W(\gamma_1 \circ \gamma_2) = W(\gamma_2 \circ \gamma_1)$$ \hspace{1cm} (79)

The general identity of second kind ensures that $W(\gamma)$ is a trace of an $N \times N$ matrix. Depending on the particular gauge group under consideration other identities of second kind may arise. Let us first discuss the general identity. Notice that one cannot define a totally antisymmetric nonvanishing object with $N + 1$ indices in $N$ dimensions.

$$\delta_{[j_1}^{i_1} \delta_{j_2}^{i_2} \ldots \delta_{j_{N+1}}^{i_{N+1}}} = 0 \hspace{1cm} (80)$$

Now contract this with the $N \times N$ holonomies $H(\gamma_1)^{j_1}_{i_1} \ldots H(\gamma_{N+1})^{j_{N+1}}_{i_{N+1}}$, one gets a sum of products of traces of products of holonomies. For instance, for the $U(1)$ case, $N = 1$ and

$$W(\gamma_1)W(\gamma_2) - W(\gamma_1 \circ \gamma_2) = 0 \hspace{1cm} (81)$$

The Mandelstam identity for $N \times N$ matrices may be simply written in terms of the following objects defined by the recurrence relation

$$(n + 1)M_{n+1}(\gamma_1, \gamma_2 \ldots \gamma_{n+1}) = W(\gamma_{n+1})M_n(\gamma_1 \ldots \gamma_n)$$

$$- M_n(\gamma_1 \circ \gamma_{n+1}, \gamma_2 \ldots \gamma_n) - M_n(\gamma_1, \ldots \gamma_n \circ \gamma_{n+1}) \hspace{1cm} (82)$$

$$M_1(\gamma) = W(\gamma)$$

Any $N \times N$ matrix group leads to Wilson loops satisfying

$$M_{N+1}(\gamma_1 \ldots \gamma_{N+1}) = 0 \hspace{1cm} (83)$$

This is the general identity of second kind satisfied by any $N$ dimensional representation of a group $G$. For instance for $2 \times 2$ matrices this identity allows to expand the product of three traces in terms of two.

$$W(\gamma_1)W(\gamma_2)W(\gamma_3) = W(\gamma_1 \circ \gamma_2)W(\gamma_3) + W(\gamma_2 \circ \gamma_3)W(\gamma_2)$$

$$+ W(\gamma_3 \circ \gamma_1)W(\gamma_2) - W(\gamma_1 \circ \gamma_2 \circ \gamma_3) - W(\gamma_1 \circ \gamma_3 \circ \gamma_2) \hspace{1cm} (84)$$

It is obvious from Eq.(82) that if $W(\gamma)$ verifies the $N^{th}$ order Mandelstam identity, higher order identities are automatically satisfied. One may take the recurrence relation for $n = N$ and obtain the value of the Wilson loop evaluated for the identity loop $\iota, W(\iota) = N$

Further identities appear for special groups$^7$ for instance for $N \times N$ matrices with unit determinant one can prove the following identity
which allows to express products of \(N\) Wilson loops in terms of \(N-1\). For example for any special \(2 \times 2\) matrix groups
\[
M_2(\gamma_1,\gamma_2) = M_2(\gamma_1 \circ \gamma_2^{-1}, \iota) \tag{86}
\]
As
\[
M_2(\gamma_1,\gamma_2) = \frac{1}{2}(W(\gamma_1)W(\gamma_2) - W(\gamma_1 \circ \gamma_2)) \tag{87}
\]
and
\[
M_2(\gamma, \iota) = \frac{1}{2}W(\gamma) \tag{88}
\]
one has
\[
W(\gamma_1)W(\gamma_2) = W(\gamma_1 \circ \gamma_2) + W(\gamma_1 \circ \gamma_2^{-1}) \tag{89}
\]
which is the second kind identity for an \(SU(2)\) or \(SL(2,R)\) gauge theory. One can easily check that this identity implies the general identity (84) for \(2 \times 2\) matrices.

One can show that in the case of an unitary group
\[
W(\gamma) = W^*(\gamma^{-1}) \tag{90}
\]
and
\[
|W(\gamma)| \leq N \tag{91}
\]

As we have already mentioned in the case of compact gauge groups all the gauge invariant information present in the holonomy may be reconstructed from the Wilson loops. As holonomies embody all the information about connections, Wilson loops will be taken as fundamental variables since it will be possible to reconstruct all the gauge invariant information of the theory from them. Giles \(^{20}\) proved the first of such reconstruction theorems for the \(U(N)\) case. He proved that given a function \(W(\gamma)\) satisfying the Mandelstam constraint of first and second kind then it is possible to construct an explicit set of \(N \times N\) matrices \(H(\gamma)\) defined modulo similarity transformations, such that their traces are \(W(\gamma)\).

### 3.3 The Loop Representation of the Maxwell Theory

We now consider a change of representation in the quantum Maxwell gauge theory. The loop representation of the Maxwell theory was first introduced in 1980 in the covariant formalism\(^6\). The Hamiltonian formalism was discussed in Refs.\(^{21,22}\). The idea is to introduce a basis of states labeled by loops \(|\gamma\rangle\) whose inner product with the connection states is given by

\[M_N(\gamma_1 \circ \gamma, \gamma_2 \circ \gamma, \ldots \gamma_N \circ \gamma) = M_N(\gamma_1, \gamma_2, \ldots \gamma_N) \tag{85}\]
\[ < A | \gamma > = W(\gamma) = \exp[ie \int_\gamma dy^a A_a(y)] \] (92)

The loop functional \( W(\gamma) \) is the Wilson loop for the abelian \( U(1) \) case. The second kind Mandelstam identity insure that

\[ W(\gamma_1 \circ \gamma_2) = W(\gamma_1) W(\gamma_2) \] (93)

and therefore the abelian holonomy \( W(\gamma) \) vanish for elements of the form

\[ \kappa = \gamma_1 \circ \gamma_2 \circ \gamma_1^{-1} \circ \gamma_2^{-1} \] (94)

We shall call \( \kappa \) a commutator. Let us consider products of elements of this type \( \kappa_1 \circ \ldots \circ \kappa_m \). They form a group that we shall call the commutator group \( K_o \). One can show that \( K_o \) is a normal subgroup of \( L_o \), that is, given any \( \kappa \in K_o \) and \( \gamma \in L_o \)

\[ \gamma \circ \kappa \circ \gamma^{-1} \in K_o \] (95)

and therefore one may define the quotient group

\[ L_A = L_o / K_o \] (96)

In \( L_A \) any element of \( K_o \) has been identified with the identity and therefore

\[ \gamma_1 \circ \gamma_2 = \gamma_2 \circ \gamma_1, \] (97)

\( L_A \) is the abelian group of loops The wave functions of an abelian theory in the loop representation will be defined on \( L_A \).

In the abelian case the loop derivatives satisfy

\[ [\Delta_{ab} (\pi^x_o), \Delta_{cd} (\chi^y_o)] = 0 \] (98)

\( \forall \pi \) and \( \chi \) and therefore \( \Delta_{ab} (\pi^x_o) [\Delta_{cd} (\chi^y_o)] = 0 \) which implies that the loop derivatives

\[ \Delta_{ab} (\pi^x_o) = \Delta_{ab} (x) \] (99)

are point dependent functions. Now, it is trivial to show that the Bianchi identity takes the form

\[ \Delta_{[ab,c]} (x) = 0 \] (100)

and may be written in terms of ordinary derivatives.

Let us now show how the loop representation may be derived in the case of the electromagnetic theory. One starts by considering the non canonical algebra of a complete set of gauge invariant operators. In this case, we consider the gauge invariant holonomy

\[ \hat{H}(\gamma) = \exp[ie \int_\gamma A_a(y) dy^a] \] (101)
and the conjugate electric field $E^a(x)$. They obey the commutation relations

$$[\hat{E}^a(x), \hat{H}(\gamma)] = e \int_\gamma \delta(x - y) dy^a \hat{H}(\gamma) \equiv eX^{ax}(\gamma) \hat{H}(\gamma) \tag{102}$$

These operators act on a state space of abelian loops $\psi(\gamma)$ that may be expressed in terms of the transform

$$\psi(\gamma) = \int d_\mu [A] < \gamma | A > < A | \psi > = \int d_\mu [A] \psi(A) \exp[-ie \oint_\gamma A_a dy^a] \tag{103}$$

This transform was first introduced in 1980 in the context of the covariant formalism of quantum electromagnetism$^6$ and it is well defined in the abelian case.

Now, to realize this gauge invariant operators we may follow two different approaches. We may compute the action of the operators on the connection representation and deduce the action in the loop representation by making use of the loop transform, or we may introduce a quantum representation of these operators directly in the loop space. Even though, in general, very little is known about integration in the space of connections, the transform may be well defined in the $U(1)$ case.

Following any of these methods it is immediate to deduce the explicit action of the fundamental gauge invariant operators.

$$\hat{H}(\gamma_0)\psi(\gamma) = \psi(\gamma_0^{-1} \circ \gamma)$$

$$\hat{E}^a(x)\psi(\gamma) = +e \int_\gamma \delta(x - y) dy^a \psi(\gamma) \tag{104}$$

The physical meaning of an abelian loop may be deduced from here, in fact

$$E^a(x) \mid \gamma > = e \int_\gamma \delta(x - y) dy^a \mid \gamma > \tag{105}$$

which implies that $\mid \gamma >$ is an eigenstate of the electric field. The corresponding eigenvalue is different from zero if $x$ is on $\gamma$. Thus $\gamma$ represents a confined line of electric flux.

The action of any other gauge invariant operator may be deduced from Eqs.(104) and (105). For instance the magnetic part of the hamiltonian operator

$$\hat{B} = \frac{1}{4} \int d^3 x \hat{F}_{ij}(x,t) \hat{F}_{ij}(x,t) \tag{106}$$

may be obtained recalling that

$$\Delta_{ij}(x) \hat{H}(\gamma) = ie \hat{F}_{ij}(x) \hat{H}(\gamma) \tag{107}$$

and therefore

$$\hat{B}\psi(\gamma) = -\frac{i}{4\pi} \int d^3 x \Delta_{ij}(x) \Delta_{ij}(x) H(\alpha) \mid_{\alpha = 0} \psi(\gamma) = -\frac{i}{4\pi} \int d^3 x \Delta_{ij}(x) \Delta_{ij}(x) \psi(\gamma) \tag{108}$$
Thus, the Hamiltonian eigenvalue equation takes the form

\[-\frac{1}{4} \int d^3x \Delta_{ij}(x) \Delta_{ij}(x) + \frac{e^2}{2} l(\gamma) \psi(\gamma) = \epsilon \psi(\gamma) \] (109)

where \(l(\gamma)\) is given by

\[l(\gamma) = \int_\gamma dy^a \int_\gamma dy'^a \delta^3(y - y')\] (110)

This quantity called the quadratic length is singular and needs to be regularized. One usually introduces a regularization of the \(\delta\) function, for instance

\[f_\epsilon(x - y) = (\pi \epsilon)^{-3/2} \exp\left[\frac{-(x - y)^2}{\epsilon}\right].\] (111)

This kind of regularization is also required for the loop representation of the nonabelian gauge theories and quantum gravity. In the abelian case one can show that the Hamiltonian eigenvalue equation may be solved and the vacuum and the \(n\) photon states determined. For instance, the vacuum may be written in the form

\[\psi_0(\gamma) = \exp -\frac{e^2}{2} \int_\gamma dy^a \int_\gamma dy'^a D_1(y - y')\] (112)

where \(D_1\) is the homogeneous symmetric propagator of free electromagnetism

\[D_1(y - y') = \frac{1}{(2\pi)^3} \int \frac{d^3q}{|q|} \exp -iq(y - y')\] (113)

A complete discussion of the solutions and the inner product is out of the scope of these notes. However, it is important to remark that the usual Fock space structure of the abelian theory may be completely recovered. It is also possible to introduce an extension of the loop representation with a natural inner product, free of this kind of singularities.

Before finishing the study of the abelian case it is important to notice that in the loop representation the Gauss law is automatically satisfied due to the gauge invariance of the inner product given by Eq(92). This property may be explicitly checked by computing

\[\partial_a E^a(x) \psi(\gamma) = e \int_\gamma dy^a \partial_a \delta(x - y) \psi(\gamma) = 0\] (114)

and therefore, the first class constraint associated with the gauge invariance is automatically satisfied.

### 3.4 The \(SU(2)\) Yang Mills Theory

Let us consider the nonabelian \(SU(2)\) case. As in the Maxwell theory we start by considering a change of representation with gauge invariant inner product

\[\langle A | \gamma > = \text{Tr}[\mathcal{P} \exp i \int_\gamma A_a dy^a]\] (115)
in this case the Wilson loop functional satisfies

\[ W(\gamma_1)W(\gamma_2) = W(\gamma_1 \circ \gamma_2) + W(\gamma_1 \circ \gamma_2^{-1}) \] (116)

Of course, as in the U(1) case the Wilson loop of SU(2) do not separate any two loops. Two loops \( \gamma \) and \( \gamma' \) will belong to the same equivalence class if they lead to the same Wilson loop functional for all the SU(2) connections. In the U(1) the quotient group was the abelian loop group. In this case we shall proceed in a different way, we shall not pass to the quotient, and instead we shall impose Mandelstam constraints on the loop dependent wave functions. To quantize the Yang Mills theory, we start by considering the loop dependent algebra of gauge invariant operators.

\[ T^0(\gamma) = Tr[\hat{H}_A(\gamma)] = W_A(\gamma) \]
\[ T^a(x, \gamma) = Tr[\hat{H}_A(\gamma^a_x \circ \gamma^b_x) \hat{E}^a(x)] \] (117)

The first operator is the Wilson loop functional, the second invariant operator includes the conjugate electric field \( \hat{E}^a(x) \) and the holonomy \( H_A(\gamma_x) \) basepointed at \( x \). They satisfy the non canonical loop dependent algebra \( \mathfrak{g} \) given by.

\[ [T^0(\gamma), T^0(\gamma')] = 0 \] (118)
\[ [T^0(\gamma), T^a(x, \gamma')] = -\frac{1}{2} X^{a\gamma}(\gamma)[T^0(\gamma_x^a \circ \gamma_x^b) - T^0(\gamma_x^a \circ \gamma_x^b)] \] (119)
\[ [T^a(x, \gamma), T^b(y, \gamma')] = X^{a\gamma}(\gamma)[T^b(y, \gamma_x^b \circ \gamma_x^b) - T^0(\gamma)T^b(y, \gamma') + T^0(\gamma)T^b(y, \gamma')] - X^{b\gamma}(\gamma)[T^a(x, \gamma_x^b \circ \gamma_x^b) - T^0(\gamma)T^a(x, \gamma)] \] (120)

This algebra may be considered as the non abelian version of the algebra (104). There is, however an important difference with the abelian case, in fact in the general non abelian case these operators do not form a complete set of gauge invariant operators. The complete set includes products of any number of electric field variables, and satisfy a more general algebra\(^9\).

As before the gauge invariant operators act naturally on a state space of loop dependent functions \( \psi(\gamma) \) that may be formally expressed as the loop transform of the usual connection dependent wave function

\[ \psi(\gamma) = \int d_\mu[A]|\psi[A]Tr[\mathcal{P} \exp -i \int A_\alpha dy^\alpha] \] (121)

The existence of this transform in the SU(2) case has been studied by Ashtekar and Isham \(^{24}\). They have shown that there is a measure in a extension \( \mathcal{A}/\mathcal{G} \) of the space of gauge equivalent classes of connections.

The Mandelstam identities for the Wilson loops \( T^0(\gamma) \) induce via the loop transform some identities on the SU(2) wave functions. They are given by:

\[ \psi(\gamma_1 \circ \gamma_2) = \psi(\gamma_2 \circ \gamma_1) \]
\[ \psi(\gamma) = \psi(\gamma^{-1}) \]
\[ \psi(\gamma_1 \circ \gamma_2 \circ \gamma_3) + \psi(\gamma_1 \circ \gamma_2 \circ \gamma_3^{-1}) = \psi(\gamma_2 \circ \gamma_1 \circ \gamma_3) + \psi(\gamma_2 \circ \gamma_1 \circ \gamma_3^{-1}) \] (122)

The action of any gauge invariant operator may be deduced from its algebra or directly by means of the loop transform. For instance, the \( T^0 \) and \( T^a \) operators act as follows

\[ T^0(\gamma')\psi(\gamma) = \psi(\gamma' \circ \gamma) + \psi(\gamma'^{-1} \circ \gamma) \]

\[ T^a(x, \gamma')\psi(\gamma) = \frac{1}{2} \int_\gamma dy^a \delta(x - y)[\psi(\gamma_y^y \circ \gamma_x^x) - \psi(\gamma_y^{y'} \circ \gamma_x^{y'}^{-1} \circ \gamma_y^y)] \] (123)

Thus, we see that the \( T^a \) operator inserts the loop \( \gamma' \) with both orientations at the point \( x \) of the loop \( \gamma \). If \( x \) does not belong to \( \gamma \) the second member vanishes due to the distributional prefactor.

I conclude this section by writing the \( SU(2) \) Yang Mills hamiltonian in the loop representation. The explicit implementation may be found in [7]. It is given by

\[ \hat{H}\psi(\gamma) = \left[ -\frac{1}{2g^2} \int d^3x \Delta_{ij}(\pi^i_o)\Delta_{ij}(\pi^j_o) + \frac{1}{4}g^2l(\gamma) \right]\psi(\gamma) \]

\[ + \frac{1}{2g^2} \int_\gamma dy^a dy'^a \delta^3(y - y')\psi(\gamma_y^y \circ \gamma_x^{y'} \circ \gamma_y^{y'}^{-1}) = \epsilon\psi(\gamma) \] (124)

where \( l(\gamma) \) is given by Eq.(110). The product of loops in the argument of \( \psi \) in the last term of the left hand side of Eq.(124) must be interpreted as follows. If no double points are present \( \gamma_y^{y'} \) (premultiplied by the \( \delta \) function) coincides with \( \gamma \) and \( \gamma_y^{y'} = \iota \). When a double point (an intersecting point) is present, the loop breaks into two pieces and one of these pieces is rerouted.

This hamiltonian is singular in the continuum and need to be regularized and renormalized. A nonperturbative renormalization of this equation is not known. However, the corresponding eigenvalue equation has been extensively studied in the lattice in different approximations leading to results for the energy density, gluon mass spectrum and other observables which coincides with the obtained with more standard methods. The loop computational methods, mainly based in geometrical operations with loops seem to be more efficient when compared with other hamiltonian methods in the lattice.

4. Canonical Formulation

One of the greatest scientific challenges of our times is to unite the two fundamental theories of modern physics, quantum field theory and general relativity. These two theories together describe the fundamental forces of nature from distances less than \( 10^{-15} \text{cms} \) up to the astronomical distances. Each of them has been extraordinary successful in describing the physical phenomena in its domain. They are however strikingly different. Each of them, works independently of the other and requires two different frameworks with different mathematical methods and physical principles.

For many years this almost absolute division between both theories also included the corresponding scientific communities. The weakness of the gravitational force
allowed to examine the subatomic world simply neglecting quantization. On the other side gravitation was relevant at astronomical scales where quantum physics didn’t seem to play any role. This situation is becoming to change in the last years and there is an increasing interplay between both fields. From one side, particle physicist were led to the description of the weak, electromagnetic and strong forces in terms of the minimal $SU(3) \times SU(2) \times U(1)$ theory. The unification of these forces occurs around $10^{-28}$ cm which is very close to the Planck length of $10^{-33}$ cm where the quantum gravitational effects are expected to become dominant.

Thus, there is now a general consensus between particle physicist in the necessity of including gravitation. In the search of a renormalizable theory for gravitation, supergravity was introduced, however the renormalizability fails at three loops. A more radical revision of the quantum field theory was suggested by string theory, the theory now involves non local extended objects but it is unitary, it seem to be finite at the perturbation level, anomaly free and includes spin 2 particles among its excitations. However, there still remain some important problems as its nonuniqueness in 4 dimensions and the divergence of the sum of the perturbative expansion.

On the other side, general relativity physicist were also convinced of the necessity of including quantum mechanics in the theory for different reasons. In first place the singularity theorems of Penrose and Hawking prove that a large class of initial data for gravity plus matter evolve into singular solutions involving infinite curvatures. This kind of phenomena are typical of a classical theory going beyond its limits of validity. Furthermore, in general relativity there is not a fixed background geometry, space-time is a dynamical, physical, entity like particles or fields. Thus, a quantum theory of gravity necessary involves a quantum description of space-time at short distances.

This theory must necessary be nonperturbative, in fact, any perturbative approach assume that the smooth continuum picture holds for arbitrary small distances and that the space time may be approximated by a fixed background space with small fluctuations and leads to unrenormalizable divergences in the perturbative expansion.

There is however an important number of difficulties that a quantum theory of gravity needs to overcome. The first difficulty is related with diffeomorphism invariance and the lack of observables, then there is a number of questions related with the nature of time in a totally covariant theory. There are also problems related with the measurement theory and the axioms of quantum mechanics in absence of a background space-time. Our approach will be very conservative, we will study the canonical quantization of pure general relativity. The use of a new set of canonical variables, the Ashtekar variables, will allow us to apply the loop techniques already developed for Yang Mills to the general relativity case.

4.1 The A.D.M. canonical formulation of general relativity

Here we briefly recall the basic ideas of the standard hamiltonian approach of general relativity due to Arnowitt, Deser and Misner. General relativity is usually described in terms of the space time metric $g_{ab}$. The action is given by:
\[ S = \int d^4x \sqrt{-g} R(g_{ab}) \] (125)

where \( g \) is the determinant of \( g \) and \( R \) the scalar curvature. The equations of motion are obtained by varying the action with respect to \( g_{ab} \). They are

\[ \frac{\delta S}{\delta g_{ab}} = R_{ab} - \frac{1}{2} g_{ab} R = 0 \] (126)

In principle one has ten equations, one for each component of \( g \), but due to the general diffeomorphism invariance, the system is redundant and not all the equations are independent.

In order to introduce a canonical formalism and a notion of hamiltonian, it is necessary to split the space time into space and time, the hamiltonian will give the evolution along this time. The splitting is only formal, the covariance is not lost and this time has no physical meaning. Thus, we foliate the space-time \( (M, g_{ab}) \) of signature \(-+++\), with spacelike Cauchy surfaces \( \Sigma_t \), parameterized by a function \( t \).

The time direction \( t^a \) is such that

\[ t^a \partial_a t = 1 \] (127)

and may be decomposed into normal and tangent components to the three-surface

\[ t^a = N n^a + N^a \] (128)

where \( n^a \) is the normal to \( \Sigma_t \) and \( N^a \) is tangent to the surface, and is called the shift. The scalar \( N \) is called the lapse function.

The space-time metric \( g_{ab} \) induces a spatial metric \( q_{ab} \) on each \( \Sigma_t \)

\[ q_{ab} = g_{ab} + n_a n_b \] (129)

\( q^b_a \) can be considered a projection operator on \( \Sigma_t \). Let us now call \( X^a \) the coordinates for which \( g \) has components \( g_{ab} \), one may introduce coordinates adapted to the foliation in such a way that the foliation \( \Sigma_t \) is given by \( X^a(t, x^i) \) where \( t \) is the parameter that define \( \Sigma_t \). Now the space-time metric tensor may be easily written in the \( (t, x^i) \) coordinates

\[ ds^2 = -N^2 dt^2 + q_{ij}(dx^i + N^i dt)(dx^j + N^j dt). \] (131)

When the action is written in terms of these variables, one can notice that there are no momentum canonically conjugate to \( N \) and \( N_a \) because the Lagrangian does not contain their time derivatives.

\[ \tilde{\pi} = \frac{\delta L}{\delta N} = 0, \quad \tilde{\pi}_a = \frac{\delta L}{\delta N^a} = 0 \] (132)

while

\[ \tilde{\pi}_{ab} = \frac{\delta L}{\delta q_{ab}} = \sqrt{q} (K_{ab} - K q_{ab}) \] (133)
where $K_{ab}$ is the extrinsic curvature, defined by

$$K_{ab} = q^c_d q^d_b \nabla_c n_d$$

and $\nabla_c$ is the covariant derivative associated with $g_{ab}$.

It is easy to show that

$$K_{ab} = \frac{1}{2} \mathcal{L}_{\dot{\pi}} q_{ab}$$

and therefore, roughly speaking, it is the "time derivative" of the metric and measures how the three metric change with evolution.

Performing a Legendre transform of the original action, one can obtain the hamiltonian

$$H(\tilde{\pi}, q) = \int d^3 x [N(-\frac{q^{1/2}}{2} R + q^{-1/2}(\tilde{\pi}^{ab} \tilde{\pi}_{ab} - \frac{1}{2} \tilde{\pi}^2) - 2N^b D_a \tilde{\pi}^a)]$$

$N$ and $N^a$ are arbitrary functions and the hamiltonian turns out to be a linear combination of the constraints:

$$C_a(\tilde{\pi}, q) = 2D_b \tilde{\pi}^b_a = 0, \quad C(\tilde{\pi}, q) = -q^{1/2} R + q^{-1/2}(\tilde{\pi}^{ab} \tilde{\pi}_{ab} - \frac{1}{2} \tilde{\pi}^2) = 0$$

To see this, one can follow the Dirac method for constrained systems, $\tilde{\pi} = 0$, and $\tilde{\pi}_a = 0$ are primary constraints, their conservation in time implies that $C_a$ and $C$ are also constraints and finally one can check that they are first class.

As we have already mentioned, each first class constraint is related with some gauge invariance of the dynamical system. The general relativity constraints $C_a$ and $C$ are the generators of diffeomorphism transformation of the three surface and of the evolution from one surface to the other. For instance, if we consider the Poisson Brackets of the constraint

$$C(\bar{N}) = \int d^3 x N^a(x) C_a(\tilde{\pi}, q)$$

with any dynamical quantity $f(\tilde{\pi}, q)$, one gets

$$[f(\tilde{\pi}, q), C(N)] = \mathcal{L}_{\bar{N}} f(\tilde{\pi}, q)$$

which is the Lie derivative associated with the infinitesimal spatial diffeomorphism.

$$\bar{x}^a = x^a + N^a(x).$$

Now at the quantum level we take wavefunctionals $\psi(q^{ab})$ and represent $q^{ab}$ as a multiplicative operator and $\tilde{\pi}_{ab}$ as a functional derivative.

$$\hat{q}^{ab} \psi(q^{ab}) = q^{ab} \psi(q^{ab}) \quad \text{and} \quad \hat{\pi}_{ab} \psi(q^{ab}) = -i\hbar \frac{\delta}{\delta q^{ab}} \psi(q^{ab}).$$

Then, we need to promote the constraints to quantum operators. That implies a choice of factor ordering and a regularization. A physical requirement is that the
factor ordering should be consistent with the property of $C_a$ as generator of diffeomorphism. It is possible to find solutions of the diffeomorphism constraint, they are functions of the "geometry of the three space" in other words, functions of the orbits of the metric under diffeomorphism. Even though several examples of functionals that satisfy this requirement are known, there is not a general way to encode this information. However the biggest trouble is the search of solutions of the hamiltonian constraint. In fact the Wheeler-De Witt equation is highly non linear in the configuration variables $q_{ab}$, and so far, not a single solution of this constraint is known.

4.2 The Ashtekar new variables

Let us, now introduce the main ideas of the Ashtekar canonical formulation. The underlying idea in the new variables approach is to cast general relativity in terms of connections rather than metrics. Some authors have followed this approach starting from the Palatini form of the action based on a $SO(3,1)$ connection and a tetrad. The Palatini action depends on the tetrads $e^a_I$ and the Lorentz connection $\omega^I_{aJ}$. A tetrad is a vector basis at each point of space-time. The Lorentz index $I$ labels the vectors. The space-time metric $g_{ab}$ is constructed from the inverse tetrad $e^I_a$

$$e^I_a e^J_b = \delta^I_J$$

by

$$g_{ab} = e^I_a e^J_b \eta_{IJ}$$

where $\eta_{IJ} = \text{diag}(-++)$ is the Minkowski metric. Thus, the tetrad define the linear transformation leading from the original metric to the flat metric.

Notice that the tetrad $e^I_a$ has sixteen independent components. This is due to the fact that Eq.(143) is invariant under local Lorentz transformations. Now, as we have a Lorentz gauge invariance we may introduce a Lorentz connection $\omega^I_{aJ}$ (antisymmetric in $I$ and $J$) and define a covariant derivative

$$D_a K_I = \partial_a K_I + \omega^I_{aJ} K_J$$

and notice that this derivative annihilates the Minkowski metric. As usual, the curvature $\Omega$ associated with the connection is defined by

$$\Omega^I_{ab} = 2\partial_{[a} \omega^I_{b]} + [\omega_a, \omega_b]^I_J$$

and

$$\Omega_{ab} = [D_a, D_b]$$

transforms under local Lorentz transformations in the following way

$$\Omega'_{ab} = L \Omega_{ab} L^{-1} = L \Omega_{ab} L^T$$
and therefore \( e_J^a \Omega_{ab}^I \) is a scalar gauge invariant object.

Now we may consider the action

\[
S(e, \omega) = \int d^4x \, e e^a_I e^b_J \Omega_{ab}^{IJ}
\]

(148)

where \( e = \text{det}[e^a_I] = \sqrt{-g} \) and \( e^a_I e^b_J \Omega_{ab}^{IJ} \) the Ricci scalar associated with the spin connection. Variations of this action with respect to the connection leads to a connection related to the tetrad via

\[
D_a e^b_J = \partial_a e^b_J + \omega^J_a e^b_J - \Gamma^b_{ac} e^c_J = 0
\]

(149)

where \( \Gamma \) is a torsion free connection associated with the metric \( g \). It can be easily seen\(^{28}\) that this condition implies that

\[
\Omega_{ab}^{IJ} = e^c_I e^d_J R_{cd}^{IJ}
\]

(150)

where \( R \) is the Riemann tensor. Now varying the action with respect to the tetrad one obtains the second field equation

\[
e^c_I e^d_J - \frac{1}{2} \Omega_{cd}^{MN} e^c_M e^d_N e^e_J = 0
\]

(151)

that after multiplication by \( e_J a \) leads to the Einstein equations

\[
G_{ab} \equiv R_{ab} - \frac{1}{2} g_{ab} R = 0
\]

(152)

This formulation is well known, the question is: Does this theory in terms of connections has any advantage when written in a hamiltonian form? The answer is negative and the reason is the following\(^{29}\): The conjugate momentum to the connection \( w_a^{IJ} \) is quadratic in the tetrad vectors. Thus, the theory has new constraints. These constraints spoil the first class nature of the constraint algebra. In order to quantize this theory one needs to solve the second class constraints and express the theory in terms of new canonical variables. These variables essentially coincide with the ordinary variables of the A.D.M. geometrodynamics.

The introduction of selfdual variables allows to solve this problem. The idea is to use a complex Lorentz connection \( A_a^{IJ} \) which is selfdual in the internal indices.

\[
A_a^{IJ} = \omega_a^{IJ} - \frac{i}{2} \epsilon_{KL} \omega_a^{KL}
\]

(153)

and therefore satisfy

\[
\frac{1}{2} \epsilon_{KL} A_a^{KL} = i A_a^{IJ}
\]

(154)

The corresponding curvature is

\[
F_a^{IJ} = 2 \partial_a A_a^{IJ} + [A_a, A_b]^{IJ}
\]

(155)

which is also selfdual, \( * F_a^{IJ} = i F_a^{IJ} \).

The action is now defined by
\[ S[e, A] = \int d^4x e e_I^a e_J^b F^{IJ}_{ab} \] (156)

From here one can obtain the field equations by repeating the before mentioned calculations.

Variations of the selfdual action with respect to the connection \( A_a \) impose that the covariant derivative annihilates the tetrad. Variations with respect to the tetrad leads again to the Einstein equations. The field equations are not modified because the new part of the selfdual action
\[ T[e, A] = \int d^4x e e_I^a e_J^b \epsilon^{IJ}_{MN} \Omega_{ab}^{MN}(A) \] (157)

can be added without affecting the equation of motion. In fact \( T[e, A] \) is a pure divergence and only contribute to boundary terms.

Let us now consider the canonical formulation for this action. We again introduce a foliation \( \Sigma \) with normal vector \( n^a \) and projection \( q^b_a(e) \). Now we consider the projection of the tetrad
\[ E_I^a = q^a_b e_I^b \] (158)

and the quantities
\[ n_I = e_I^a n_a, \quad \epsilon^{IJK} = \epsilon^{IJKL} n_L, \quad \tilde{E}_I^a = \sqrt{q} E_I^a \] (159)

and
\[ N = \frac{N}{\sqrt{q}} \] (160)

finally, we recall that \( t^a = N n^a + N^a \). After a long by straightforward calculation that is explained in detail in Ref 29 one can get
\[ S = \int d^4x [-i \tilde{E}_b^a \epsilon_M^{IJ} \mathcal{L}_t A_b^{MN} - N^a F_{ab}^{MN} - i A_a^{MN} t^a D_b [\tilde{E}_J^b - \tilde{E}_b^J F^{IJ}_{ab}] + N \tilde{E}_I^a \tilde{E}_b^J F^{IJ}_{ab}] \] (161)

The action is now written in canonical form and the conjugate variables can be read off directly. The configuration variable is the selfdual connection \( A_a \). The conjugate momentum is the selfdual part of \(-i \tilde{E}_b^a \epsilon_M^{IJ}\)
\[ \tilde{\pi}^a_{MN} = \tilde{E}_{[M}^a n_{N]} - \frac{i}{2} \tilde{E}^a_I \epsilon_{MN} \] (162)

Now, in terms of the canonical variables the Lagrangian takes the form
\[ \int \Sigma d^3x Tr (-\tilde{\pi}^a \mathcal{L}_t A_a + N^a \tilde{\pi}^b F_{ab} - A.tD_a \tilde{\pi}^a - N \tilde{\pi}^a \tilde{\pi}^b F_{ab}) \] (163)

where any reference to the internal vector \( n^I \) has disappeared. As \( n_I \) is not a dynamical variable it can be gauge fixed. We fix \( n_I = (1,0,0,0) \) and therefore \( \epsilon^{IJKL} n_L = \)
\[ \epsilon^{IJK0}, \quad \text{since} \quad A^{IJ}_a \quad \text{and} \quad \tilde{\pi}^a_{IJ} \quad \text{are selfdual, they can be determined by its 0I components.} \]

Then, we may define

\[ A^{i}_a = i A^{0I}_a, \quad \tilde{E}^a_i = \tilde{\pi}^a_{0I} \] (164)

where internal indices \( i, j \) refer to the \( SO(3) \) Lie Algebra. In fact, as it is well known the selfdual Lorentz Lie Algebra is isomorphic to the \( SO(3) \) algebra. The new variables now satisfy the Poisson Bracket relations

\[ \{ A^i_a(x) E^b_j(y) \} = +i \delta^b_a \delta^i_j \delta^3(x - y) \] (165)

Now the constraints may be read off from the Lagrangian (163), and take the form

\[ \tilde{G}^i = D_a \tilde{E}^a_i \] (166)

\[ \tilde{C}_a = \tilde{E}^b_i F^i_{ab} \] (167)

\[ \tilde{C} = \epsilon^{ij} \tilde{E}^a_i \tilde{E}^b_j F^k_{ab} \] (168)

The Hamiltonian is again a linear combination of the constraints.

The constraints are respectively related with the gauge invariances of the theory, under internal \( SO(3) \) transformations, under diffeomorphism and under the evolution of \( \Sigma \) in space-time. Notice the simplification of the constraints which are polynomial and at most involve quartic powers of the phase space variables. Moreover the formalism now takes the form of a complex Yang Mills theory. In particular the first constraint is the Gauss law and therefore the physical states of quantum gravity are a subspace of the reduced phase space of a complex Yang Mills theory. This property allows to apply to gravity the loop techniques already developed for the abelian and nonabelian gauge theories.

In principle, we have a canonical formalism with \( A \) and \( \tilde{\pi} \) complex, however in the original action tetrads were real and consequently \( \tilde{E}^a_i \) and \( \tilde{\pi}^a_{IJ} \) are also real. Then, to recover the real general relativity from this canonical description one need to impose that

\[ q q^{ab} = \tilde{E}^a_i \tilde{E}^{bi} \] (169)

be real. This is a new constraint and its conservation in time induces another constraint. They are second class in the sense of Dirac and when solved they lead back to the A.D.M formulation. However, the idea is to follow an alternative procedure and use the reality conditions as a guideline in order to find the appropriated inner product after the theory has been quantized. One need to require that the real quantities in the classical theory, become selfadjoint operators under the chosen inner product.

4.3 Quantum Theory
Let us now proceed to the canonical quantization of the theory. We proceed as in usual gauge theories by taking wavefunctionals of the connection $\psi[A]$ and representing the connection as a multiplicative operator and the triad as a functional derivative.

\[
\hat{A}_a^i \psi[A] = A_a^i \psi[A] \\
\hat{E}_a^i \psi[A] = \frac{\delta}{\delta A_a^i} \psi[A]
\] (170)

The form of the quantum constraints depend on the regularization and the factor ordering. We shall consider the factor ordering with the triads (or "electric fields") to the left. With this ordering the constraint algebra formally closes and it leads to the simplest form of the loop representation. They may be explicitly written

\[
\hat{G}^i(x) \psi[A] = D_a \frac{\delta}{\delta A_a^i(x)} \psi[A] \\
\hat{C}_a(x) \psi[A] = \frac{\delta}{\delta A_a^b(x)} F_{ab}^i(x) \psi[A] \\
\hat{C}(x) \psi[A] = \epsilon^{ijk} \frac{\delta}{\delta A_a^i(x)} \frac{\delta}{\delta A_b^j(x)} F_{ab}^i(x) \psi[A]
\] (171)

Let us stress that by now all these expressions are formal and need to be regularized.

5. Quantum gravity in the loop representation

5.1 The constraints of quantum gravity

As it was already discussed in the case of usual gauge theories the first motivation to introduce a loop representation is to get rid of the gauge invariance and the Gauss law constraint. Thus, in the case of quantum gravity in the loop representation, we only need to deal with the diffeomorphism and the Hamiltonian constraint.

The quantization of general relativity in the loop representation is now very similar to the $SU(2)$ Yang Mills theory. We first consider the loop dependent algebra of gauge invariant operators $T^0(\gamma), T^a(x, \gamma), \ldots, T^{a_1 \ldots a_n}(x_1, \ldots, x_n, \gamma)$ where

\[
T^{a_1 \ldots a_n}(x_1, \ldots, x_n, \gamma) = Tr[H_A(\gamma x_{z_1}) E^{a_2}(x_2) H_A(\gamma x_{z_2}) E^{a_3}(x_3) \ldots H_A(\gamma x_{z_n}) E^{a_1}(x_1)]
\] (172)

These operators satisfy the same algebra that in the Yang-Mills Mills case (up to a global factor $i$ absorbed in the connection) The Wilson loop functional $T(\gamma)$ satisfies the $SU(2)$ Mandelstam identities. As in the previous cases these operators have a natural action on loop dependent wavefunctions $\psi(\gamma)$. This action may be deduced from their algebra or with the help of the loop transform

\[
\psi(\gamma) = \int d_\mu[A] \psi[A] Tr[\mathcal{P} \exp \int_\gamma A_a dy^a]
\] (173)
Even though the existence of the loop transform in the complex $SU(2)$ case has not been proved, it is an useful tool for the realization of the gauge invariant operators in the loop space. At the end one has to check that these operators satisfy the algebra derived from the canonical quantization. Notice that as in the $SU(2)$ Yang Mills case each Mandelstam identity for $T^a(\gamma)$ induces an identity on the wavefunctions and consequently they obey Eqs(122).

The operators $T^a(\gamma)$ and $T^a(x_1, \gamma)$ have been already realized in the Yang Mills case. One may also realize $T^a,_{x_2}(x_1, x_2, \gamma)$ and from them obtain the explicit form of the constraints in the loop representation by taking appropriate limits of $\gamma$.

This construction was first proposed by Rovelli and Smolin\textsuperscript{9}. I will follow here a different approach\textsuperscript{10} that takes advantage of the group structure of the loop space and makes use of the loop derivative as the basic object in terms of which we are going to write the constraints. It has been shown that both methods lead to the same form of both the hamiltonian and the diffeomorphism constraint\textsuperscript{30}.

Let us consider the loop transform of the diffeomorphism constraint in the connection representation.

\[
\hat{C}(\vec{N})\psi[\gamma] = \int d_\mu[A] \int d^3 x N^a(x) \frac{\delta}{\delta A^a_k(x)} F_{ab}(x) \psi[A] Tr[\mathcal{P} \exp \int_{\gamma} A_a dy^a] 
\]

we now integrate by parts and we compute

\[
I_a(x, \gamma) \equiv F^i_{ab}(x) \frac{\delta}{\delta A^i_k(x)} Tr[\mathcal{P} \exp \int_{\gamma} A_a dy^a] = 
\]

\[
F^i_{ab}(x) \int_{\gamma} dy^c \delta(x - y) \delta^b_c Tr[H_A(\gamma) \tau^i H_A(\gamma)]
\]

\[
= \int_{\gamma} dy^b \delta(x - y) Tr[F_{ab}(y) H_A(\gamma)]
\]

where the $\tau^i$ are the $SU(2)$ generators, by making use of the fact that the holonomy is a representation of the group of loops, and of the definition of the loop derivative, we get:

\[
I_a(x, \gamma) = \int_{\gamma} dy^b \delta(x - y) \Delta^a_{ab}(\gamma(y)) H_A(\gamma)y)
\]

and replacing this expression in the constraint we find

\[
\hat{C}(\vec{N})\psi[\gamma] = \int_{\gamma} dy^b N^a(y) \Delta^a_{ab}(\gamma(y)) \psi[\gamma]
\]

This operator was first introduced\textsuperscript{31} in 1983 within the context of the chiral formulation of Yang Mills theory in loop space and it is the generator of infinitesimal deformations of the loop. One can prove\textsuperscript{31} by making use of the identities of the loop derivative that it satisfies the algebra of the diffeomorphism group.
Thus, the diffeomorphism

\[ x^a \rightarrow x^a + \epsilon N^a(x) \]  

will be generated by \( \hat{C}(\vec{N}) \) and we get

\[ (1 + \epsilon \hat{C}(\vec{N})) \psi[\gamma] = \psi[\gamma_\epsilon] \]  

where \( \gamma_\epsilon \) is shown in the next figure.

Figure 14: The deformed loop \( \gamma_\epsilon \) obtained by dragging along \( \vec{N} \) the loop \( \gamma \)

Therefore, making use of the diffeomorphism constraint, we get

\[ \psi[\gamma] = \psi[\gamma_\epsilon] \]  

and the wavefunction is invariant under smooth deformations of the loop and only depends on the equivalence classes of loops under diffeomorphisms. In other words the solution are knot invariants. Thus, the loop representation has allowed to solve six of the seven constraints of quantum gravity simply by considering knot dependent functions.

To obtain the Hamiltonian constraint, one may follow a similar procedure, starting from the regularized Hamiltonian in the connection representation and making use of the loop transform one is led to compute

\[ \hat{C}_c(x) W(\gamma) = \int d^3 y Z_c(x, y) e^{ijk} F_{ab}^k(x) \frac{\delta}{\delta A_a^i(x)} \frac{\delta}{\delta A_b^j(y)} \text{Tr}[H_A(\gamma)] \]  

where \( Z_c \) is a regularization of the \( \delta \) function, for instance

\[ Z_c(x, y) = \frac{1}{(\sqrt{\pi \epsilon})^3} \exp[- |x - y|^2 / \epsilon]. \]  

The action of the constraint on the Wilson loop may be expressed in terms of the loop derivative as
\[
\int_{\gamma} dy^a \int_{\gamma} dz^b \delta(x - z) Z_c(z, y) \Delta_{ab}(\gamma^y_o) \text{Tr}[H_A(\gamma^y_o) H_A(\gamma^y_o)]
\]

(184)

Here \( O \) is an arbitrarily chosen basepoint \(*\).

From here, we obtain the action of the Hamiltonian constraint on an arbitrary wave function

\[
\hat{C}_c(x) \psi(\gamma) = \int_{\gamma} dy^a \int_{\gamma} dy'^b \delta(x - z) Z_c(z, y) \Delta_{ab}(\gamma^y_o) \psi(\gamma^y_o \circ \gamma^y_o)
\]

(185)

This very compact equation, should be considered as the Wheeler-De Witt equation in the loop representation. Notice, in first place, the analogy with the \( SU(2) \) Yang Mills theory, as in that case, the argument of the wavefunction contains a rerouting of a portion of the loop and the intersections play a crucial role.

When the wavefunctions are evaluated on simple nonintersecting loops, the Hamiltonian constraint has in principle to tangents evaluated at the same point contracted with the (antisymmetric) loop derivative and therefore this term is naively zero when the regulator is removed. However some care must be taken. In fact, we have an infinite factor coming from the \( \delta(0) \) and therefore the computation need to be performed taking into account the regulator. There are two contributions, the one shown in Fig(15) where \( \gamma^y_o \) coincides with the loop \( \gamma \) while \( \gamma^y' \) vanishes, and a second contribution arising when \( y \) and \( y' \) are in the opposite order along the loop, and then, \( \gamma^y_o \) vanishes while \( \gamma^y' = \gamma^{-1} \).

One can show that these contributions lead to a term which vanishes on diffeomorphism invariant wave functions. In other words, any nonintersecting knot automatically satisfy the Hamiltonian constraint.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{loop_no_intersections}
\caption{A loop without intersections}
\end{figure}

\(*\) We have made use of the Fiertz identity: \( T^a_{ij} T^a_{kl} = \delta_{jk} \delta_{il} - \frac{1}{N} \delta_{ij} \delta_{kl} \).
One can extend this analysis and show that the only non trivial contributions arise at the intersections. In this case, one of the portions of $\gamma$ is rerouted.

**Figure 16:** The loops $\gamma$ and $\gamma_y' \circ \gamma_{yo}$.  

As we have different tangents at the intersection, this term gives a nontrivial contribution.

As in the previous cases of electromagnetism and Yang-Mills, loop equations involve a regularization, in this case the regularization breaks the diffeomorphism invariance. It is therefore necessary to check that the space of solutions is diffeomorphism invariant. This condition is equivalent to require that we have a solution no matter what was the choice of coordinates used to define the regulator.

Up to this point, we have been able to obtain the general solution of six of the seven constraints, the knot invariants. Furthermore, a particular set of solutions of all the constraints have been determined, the non intersecting knots. However, it is not clear to what extent the nonintersecting knot invariant solutions can represent interesting physics. In fact, if we naively compute the determinant of the three-metric and apply the operator $\det \hat{q}$ on any nonintersecting knot dependent wavefunction, we obtain:

$$\det[\hat{q}] \psi(K) \equiv 0$$  \hspace{1cm} (186)

and this solutions would lead to degenerate metrics. What seems even more important, the algebra of gauge invariant operators $T^0, T^a, T^{a\ldots a_n}$ is nontrivial only at intersections, if we neglect intersections there is not difference between the $U(1)$ theory and the nonabelian $SU(2)$ Yang-Mills theory (see, for instance Eqs(109) and (124)). It is, therefore, necessary to study with more care the physical space of states and include other physical solutions.

### 5.2 Mathematical Tools

Up to now we have determined the explicit form of the constraints in the loop representation and found a trivial set of solutions. Here we shall set up the mathematical framework needed to discuss the construction of the nondegenerate solutions. These techniques are also important in other problems that we are not going to treat in this
course as the existence of the loop transform, the equivalence between the connection and the loop representation, the inner product and other related problems.

5.2 Loop Coordinates

All the gauge invariant information present in a gauge field is contained in the holonomy and, as we have shown, loops may be defined in terms of the \( \gamma \). Thus, all the relevant information about loops is contained in the holonomy. Let us write the explicit expansion of the holonomy

\[
H_A(\gamma) = \mathcal{P} \exp \oint_\gamma A_a(x) dy^a = 1 + \sum_{n=1}^{\infty} \int dx_1^3 \cdots dx_n^3 A_{a_1}(x_1) \cdots A_{a_n}(x_n) X^{a_1 \cdots a_n}(x_1 \cdots x_n, \gamma) \tag{187}
\]

where the loop dependent objects \( X \) of ”rank” \( n \) are given by

\[
X^{a_1 \cdots a_n}(x_1 \cdots x_n, \gamma) = \oint_\gamma dy_n^a \oint_{y_{n-1}}^{y_n} dy_{n-1}^{a_{n-1}} \cdots \oint_{y_1}^{y_n} dy_1^{a_1} \delta(x_n - y_n) \cdots \delta(x_1 - y_1) = \oint_\gamma dy_n^a \oint_{y_{n-1}}^{y_n} dy_{n-1}^{a_{n-1}} \cdots \oint_{y_1}^{y_n} dy_1^{a_1} \delta(x_n - y_n) \cdots \delta(x_1 - y_1) \theta_\gamma(0, y_1, \ldots y_n) \tag{188}
\]

where the \( \theta_\gamma(0, y_1, \ldots y_n) \) orders the points along the curve \( \gamma \), \( \theta_\gamma(0, y_1, \ldots y_n) = 1 \) if \( 0 < y_1 < y_2 < \cdots < y_n \) along the loop.

All the relevant information about the loop is contained in the quantities \( X \). It will be convenient to introduce the notation

\[
X^{\mu_1 \cdots \mu_n}(\gamma) \equiv X^{a_1 x_1 \cdots a_n x_n}(\gamma) = X^{a_1 \cdots a_n}(x_1 \cdots x_n, \gamma) \tag{189}
\]

with \( \mu_1 = (a_1 x_1) \), and a ”generalized Einstein convention” meaning that repeated \( x_i \) coordinates are integrated over and treated as indices. The holonomy may be rewritten with this notation

\[
H_A(\gamma) = 1 + \sum_{n=1}^{\infty} A_{a_1 x_1} \cdots A_{a_n x_n} X^{a_1 x_1 \cdots a_n x_n}(\gamma) \tag{190}
\]

The \( X \) objects behave like multivector densities at the point \( x_i \) of the three manifold \( M \). The loop dependent wave functions for any gauge theory or quantum gravity are functions of the \( X \)’s

\[
\psi(\gamma) = \psi(X(\gamma)) \tag{191}
\]

The \( X \)’s are not really coordinates in the sense that they are not freely specifiable objects, in other words they are constrained quantities. They obey algebraic and differential constraints. The algebraic constraints arise from the following relations of the \( \theta_\gamma \) functions.

\[
\theta_\gamma(0, y_1) = 1
\]

38
\[ \theta_z(0,y_1,y_2) + \theta_z(0,y_2,y_1) = 1 \] (192)

\[ \theta_z(0,y_1,y_2,y_3) + \theta_z(0,y_2,y_1,y_3) + \theta_z(0,y_2,y_3,y_1) = \theta_z(0,y_2,y_3) \]

and so on, which imply

\[
X^{\mu_1} = X^{\mu_1}, \quad X^{\mu_1\mu_2} + X^{\mu_2\mu_1} = X^{\mu_1} X^{\mu_2}
\]

\[
X^{\mu_1\mu_2\mu_3} + X^{\mu_2\mu_1\mu_3} + X^{\mu_2\mu_3\mu_1} = X^{\mu_1} X^{\mu_2} X^{\mu_3}
\] (193)

And in general

\[
X^{\mu_1...\mu_k...\mu_{n+1}} = \sum_{P_\kappa} X^{P_\kappa(\mu_1...\mu_n)} = X^{\mu_1...\mu_k} X^{\mu_{k+1}...\mu_n}
\] (194)

where the sum goes over all the permutations of the \( \mu \)'s which preserve the ordering of the \( \mu_1...\mu_k \) and the \( \mu_{k+1}...\mu_n \) among themselves. The differential constraint can be readily obtained from Eq.(188) and is given by

\[
\frac{\partial}{\partial x_i} X^{a_1 x_1...a_i x_i...a_n x_n} = (\delta(x_i - x_{i-1}) - \delta(x_i - x_{i+1})) X^{a_1 x_1...a_i x_i...a_{i-1} x_{i-1} a_{i+1} x_{i+1}...x_n}
\] (195)

in this expression the point \( x_0 \) and \( x_{n+1} \) are to be understood as the basepoint of the loop.

The previous identities may be solved in terms of a set of objects that are freely specifiable and behave as loop coordinates. These objects, however, not only include distributional quantities associated to the \( X(\gamma) \) but also smooth functions. An important property of the coordinates is that any multitensor density \( X^{\mu_1...\mu_n} \) that satisfies them can be put into Eq.(190) and the resulting object is a gauge covariant quantity. When restricted to \( X(\gamma) \) associated with loops, the resulting object is the holonomy. It is this property that allows to extend the loops to a more general structure. With this construction in hand, one could go further and forget loops and holonomies altogether and represent 23 a gauge theory entirely in terms of the \( X \)'s. The underlying mathematical structure of this extended representation is the “extended group of loops” which has the structure of an infinite dimensional Lie group 33.

Coming back to the problem of the determination of the physical state space of solutions of the constraints, we will need to study the action of the constraints on the \( X \)'s. The fundamental information comes from the action of the loop derivative on these objects. We give here the expressions for the loop derivatives \( \Delta_{ab}(\pi^x_\alpha) X(\gamma) \) in the particular case that \( \pi^x_0 \) is a portion of the loop \( \gamma \). This is the relevant case needed to compute the action of the constraints. They may be simply derived from the definition of the loop derivative and the \( X \) variables. They are

\[
\Delta_{ab}(\gamma^z_\alpha) X^{a_1 x_1}(\gamma) = \delta^{a_1}_{ab} \partial_b \delta(x_1 - z)
\] (196)

for the "rank" one \( X \),

\[
\Delta_{ab}(\gamma^z_\alpha) X^{a_1 x_1 a_2 x_2}(\gamma) = \delta^{a_1 a_2}_{ab} \delta(x_1 - z) \delta(x_2 - z) + X^{a_1 x_1}(\gamma^z_\alpha) \delta^{a_2}_{ab} \partial_b \delta(x_2 - z)
\]
\[
+\delta^a_{a+1}d\partial_d(\delta(x_1-z)X^{a_2a_2}(\gamma^a_z)) \tag{197}
\]
for the "rank" two \(X\), and for \(n \geq 3\)
\[
\Delta_{ab}(\gamma^a_z)X^{a_1b_1...a_nb_n}(\gamma) = \sum_{i=0}^{n-1} X^{a_1b_1...a_i}a_{i+1}(\gamma^a_z)\delta^a_{a+1}d\partial_d(\delta(x_{i+1}-z)X^{a_{i+2}b_{i+2}...b_n}(\gamma^a_z)) + \\
+\sum_{i=0}^{n-2} X^{a_1b_1...a_i}a_{i+1}(\gamma^a_z)\delta^a_{a+1}d\partial_d(\delta(x_{i+1}-z)\delta(x_{i+2}-z)X^{a_{i+3}b_{i+3}...b_n}(\gamma^a_z)) \tag{198}
\]

where
\[
\delta^d_{cd} = \frac{1}{2}(\delta^d_{ca} - \delta^d_{cb}). \tag{199}
\]

The action of the diffeomorphism generator \(C(\vec{N})\) may be simply derived from these equations, it simply corresponds to the transformation under infinitesimal diffeomorphisms
\[
x^a \rightarrow x^a + \epsilon N^a(x) \tag{200}
\]
of the multivector densities \(X^{\mu_1...\mu_n}\).

5.2.2 Knot Theory

The aim of knot theory is the study of the properties of the knots and links that one can construct in three dimensions. A central issue in knot theory is to distinguish and classify all the unequivalent knots. A powerful method for accomplishing this is the use of the link and knot invariants. (The term link is used to refer knots involving more than one connected curve). In fact, if a given knot invariant takes different values when it is evaluated on two different curves there is no deformation leading from one into the other.

Links may be described by considering their projections on a two dimensional surface.

**Figure 17:** A link with two components and a knot
One can show that it is always possible to represent a nonintersecting knot by this kind of diagrams containing only over and under crossings.

**Figure 18:** Over crossings and under crossings

Two loops $K_1$ and $K_2$ are called *ambient isotopic* $K_1 \sim K_2$ if there is a diffeomorphism of the manifold in which $K_1$ is embedded leading from $K_1$ to $K_2$. A practical procedure for verifying ambient isotopy is with the help of the Reidemeister moves shown in the next figure.

**Figure 19:** The Reidemeister moves

It can be shown that $K_1 \sim K_2$ if and only if there is a finite sequence of Reidemeister moves leading from the diagram $K_1$ into the diagram $K_2$.

Two diagrams are said *regular isotopic* if one can be obtained from the other by a finite sequence of Reidemeister moves of type II and III only. Regular isotopy is
related with the invariance of twisted bands. The type I move change the twist of the band.

A very important kind of link invariant are the link polynomials. They are polynomials in one or several variables associated to each link. Two planar diagrams corresponding to the same knot lead to the same polynomial. Some link polynomials may be defined by a set of implicit relations, known as skein relations. The Alexander-Conway Polynomial the Jones Polynomial and the HOMFLY Polynomial are examples of objects that may be defined in terms of a skein relation. For instance the Jones Polynomial is an ambient isotopy invariant defined by

\[ P(U, q) = 1 \]

\[ qP(L_+) - q^{-1}P(L_-) = (q^{1/2} - q^{-1/2})P(L_0) \]

where \( U \) is the unknotted knot and the links \( L_+, L_- \) and \( L_0 \) are identical except inside a disk containing one crossing as shown in the following figure

Figure 20: The links \( L_+, L_- \) and \( L_0 \)

As an example, let us apply the skein relations to determine the Jones Polynomial associated to the link \( L_1 \) shown in Fig(21).

Figure 21: The links \( L_1, L_2 \) and \( L_3 \).
then

\[(q^{1/2} - q^{-1/2})P(L_1) = qP(L_2) - q^{-1}P(L_3)\]  (202)

but

\[P(L_2) = P(L_3) = 1\]  (203)

and therefore

\[P(L_1) = \frac{q - q^{-1}}{q^{1/2} - q^{-1/2}} = q^{1/2} + q^{-1/2}\]  (204)

which is a Laurent polynomial in powers of \(q^{1/2}\). The reader is invited to apply the skein relations in order to derive the Jones Polynomial of the links and knots shown in Fig(17).

As we have already shown, nondegenerate solutions involve intersections. It is therefore necessary to generalize the notions of knot polynomials to the intersecting case. A standard technique for constructing knot polynomials is to start from the braid group. The braid group \(B_n\) is generated by elements \(g_i\) with \(0 < i \leq n\) that satisfy

\[g_i g_j = g_j g_i \quad |i - j| > 1\]

\[g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}\]  (205)

**Figure 22:** Grafic representation of the algebraic relations

Each element of \(B_n\) represents a braid diagram composed by lines, called strands that evolve from an initial plane to a final plane. The initial and final position of the lines in these planes must coincide up to permutations. If \(g_i\) represents an overcrossing of the lines \(i\) and \(i+1\), \(g_i^{-1}\) represents the correspondent undercrossing. Two braids are equivalent if they are smoothly deformable into each other, leaving their endpoint fixed. To proceed from braids to knots, one identifies the top and bottom ends of the braid.
The braid algebra can be enlarged to consider the case of braids with intersections. To do that, one introduces a new generator $a_i$ representing a four-valent rigid vertex.

\begin{align}
a_i g_i &= g_i a_i \\
g_i^{-1} a_{i+1} g_i &= g_{i+1} a_i g_i^{-1} \\
[g_i, a_j] &= 0, \ [a_i, a_j] = 0 \mid i - j \mid > 1 \tag{206}
\end{align}

From the matrix representations of the braid algebra one can derive skein relations for the knot polynomials. For instance the generalized Kauffman bracket polynomial $F(q, a)$ for 4-valent intersections may be obtained in this way and satisfy the skein relations.

\begin{align}
F_{L_+} &= q^{3/4} F_{L_0} \\
F_{L_-} &= q^{-3/4} F_{L_0} \\
q^{1/4} F_{L_+} - q^{-1/4} F_{L_-} &= (q^{1/2} - q^{-1/2}) F_{L_0} \tag{207} \\
F_{L_1} &= q^{1/4}(1 - a) F_{L_-} + a F_{L_0} \\
F_0 &= 1
\end{align}

where the crossings $\hat{L}_+, \hat{L}_-$ and $\hat{L}_0$ are shown in Fig(24).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig23.png}
\caption{The generators of the enlarged $B_n$ algebra}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig24.png}
\caption{The crossings $\hat{L}_+, \hat{L}_-$ and $\hat{L}_0$}
\end{figure}
Notice that this polynomial is a regular isotopy invariant, and it allows to distinguish bands differing by type I Reidemeister moves.

5.2.3 Chern Simons Theory and Knot Invariants

In the last section we have mentioned some knot theory techniques. Up to now the relation with quantum gravity is not apparent. In particular the Quantum Gravity constraints are realized on the space of loop functionals and we would like to have analytic expressions for the knot invariants in order to apply the constraints to them and see if they are quantum states of the gravitational theory.

Some years ago Atiyah$^{36}$ and Witten$^{37}$ pointed out that the three dimensional field theories with a pure Chern Simons action could be relevant for knot theory and might shed new light on the link invariants. Chern-Simons theory is a gauge theory defined in $2+1$ dimensions (or 3 euclidean dimensions) having as action.

$$S_{\text{CS}} = \frac{k}{4\pi} \int d^3x (A_a \partial_b A_c + \frac{2}{3} i A_a A_b A_c) \epsilon^{abc}$$

The real parameter $k$ is called the coupling constant of the model and the connection $A$ takes values in the algebra of a simple compact group $G$. The action is invariant under infinitesimal gauge transformation and under general coordinate transformations. The fundamental feature of this action is its metric independence. This property ensures that the quantum expectation values of gauge invariant and metric independent observables as the Wilson loop operator will be characterized by their topological properties. In the language of path integrals

$$\langle W(\gamma) \rangle = \int d\mu[A] e^{i S_{\text{CS}}} W_A(\gamma)$$

should be a knot invariant, parameterized by the coupling constant $k$. More precisely one can prove that $\langle W(\gamma) \rangle$ satisfies the skein relation of the Kauffman bracket. Witten proved this result non perturbatively, by relating the three-dimensional Chern Simons models and certain two-dimensional conformal field theories.

On the other hand, as the Chern Simons theory is a renormalizable theory, one can compute the Wilson loop expectation value perturbatively in terms of the inverse of the coupling constant. The coefficients of this expansion will be knot invariants related with the Kauffman bracket. Here, I will give this expansion up to terms of order $(1/k)^2$ for an $SU(2)$ gauge theory. A more complete treatment of the perturbative Chern Simons theory may be found in Ref.$^{41}$. Making use of Eq(209) and the expression of the holonomy in terms of the loop coordinates we get

$$\langle W(\gamma) \rangle = 2 + \sum_{i=2}^{\infty} \langle Tr[A_{a_1 x_1} ... A_{a_i x_i}] \rangle X^{a_1 x_1} ... a_i x_i (\gamma)$$

$$= 2 - \frac{4\pi}{k^2} a_1(\gamma) - \left(\frac{4\pi}{k^2}\right)^2 [\frac{2}{3} a_1^2(\gamma) - 6 a_2(\gamma)]$$

where the knot invariants $a_1(\gamma)$ and $a_2(\gamma)$ are given in terms of the Chern Simons propagator.
\[ g_{axby} = \frac{1}{4\pi^2} \epsilon_{abc} \frac{(x - y)^c}{|x - y|^3} \]  

(211)

by

\[ a_1(\gamma) = g_{ax_1x_2}X_{a_1x_1a_2x_2}^{x_1x_2}(\gamma) \]  

(212)

and

\[ a_2(\gamma) = +2h_{ax_1x_2x_3x_4}X_{a_1x_1a_2x_2a_3x_3a_4x_4}^{x_1x_2x_3x_4}(\gamma) + 2g_{ax_1x_2x_3x_4}X_{a_1x_1a_2x_2a_3x_3a_4x_4}^{x_1x_2x_3x_4} \]  

(213)

where

\[ h_{axbycz} = \int d^3w \epsilon_{def} g_{axdw}g_{byew}g_{czfw} \]  

(214)

The first knot invariant may be rewritten as follows:

\[ a_1(\gamma) = \frac{1}{4\pi} \int \gamma dy \int \gamma dz \epsilon_{abc} \frac{(y - z)^c}{|y - z|^3} \]  

(215)

and it is usually called the Gauss self-linking number. The Gauss linking number

\[ n(\gamma_1, \gamma_2) = \frac{1}{4\pi} \int_{\gamma_1} dy \int_{\gamma_2} dz \epsilon_{abc} \frac{(y - z)^c}{|y - z|^3} \]  

(216)

was obtained by C.F. Gauss and is the first result on knot theory, roughly speaking it measures how many times \( \gamma_2 \) winds around \( \gamma_1 \).

The Gauss self-linking number presents an ambiguity related with the fact that it is a coefficient of the Kauffman bracket polynomial which is a regular invariant polynomial. It depends on the limit \( \gamma_1 \to \gamma_2 = \gamma \). This means that it is framing dependent. A "framing" is a prescription to convert the loop into a ribbon. This object is not strictly speaking, a diffeomorphism invariant because the same loop may yield to different ribbons with a different value of \( a_1 \).

The second knot invariant \( a_2(\gamma) \) is related to the second coefficient of the Conway Polynomial which is an ambient isotopy invariant and therefore \( a_2 \) is diffeomorphism invariant and framing independent. The expansion in \( k \) of the Kauffman bracket may be computed at higher orders \(^{38}\) and one can show that at each order one gets some framing dependent terms (as the \( a_1^2 \) term at second order) and some diffeomorphism invariant coefficients. As we shall see in the next section they are the natural candidates to solutions of the hamiltonian constraint.

5.3 Non degenerate states of quantum gravity and knot theory

The previous mathematical detour was motivated by our interest in looking for nondegenerate solutions of the hamiltonian constraint. As we have already noticed these solutions involve intersections. Only in this case, the full topological content of
the hamiltonian constraint is displayed. Let us first analyze what kind of intersections are required in order to have a nonvanishing determinant. The expression of the square root of the determinant in the loop representation may be simply derived from its explicit form in terms of the triads

\[
(det[\dot{q}(x)]\psi(\gamma_1 \circ \gamma_2 \circ \gamma_3) = A\delta^3(x - x_{int})\epsilon_{abc}\dot{\gamma}_1^a \dot{\gamma}_2^b \dot{\gamma}_3^c
\]  

(217)

\[
[\psi(\gamma_1 \circ \gamma_3 \circ \gamma_2) + \psi(\gamma_2 \circ \gamma_1 \circ \gamma_3) + \psi(\gamma_3 \circ \gamma_1 \circ \gamma_2)]
\]

Thus we see that we need a loop with a triple self intersection and three independent tangent vectors (see Fig(25)) to get a nonvanishing contribution.

Figure 25: A loop with triple selfintersections

Now, as we know the analytic expressions of some knot invariants associated to the SU(2) Chern-Simons theory and as this knot invariants automatically satisfy the SU(2) Mandelstam identities, we can consider \(a_1(\gamma)\) and \(a_2(\gamma)\) as good candidates to wavefunctions of quantum gravity. The explicit computation of the action of the hamiltonian constraint on this wave functions involve the use of the loop derivatives of the loop coordinates and leads to the final result.

\[
\hat{C}(N)a_2(\gamma_1 \circ \gamma_2 \circ \gamma_3) = 0
\]  

(218)

This result was first derived by direct computation and was the first hint about the relation between the Kauffman bracket and quantum gravity. This relation became apparent after the following observation. Let us consider the hamiltonian constraint with a cosmological constant in the connection representation given by.

\[
\hat{C}_A(x)\psi[A] = \{\epsilon^{ijk} \frac{\delta}{\delta A^i_k(x)} \frac{\delta}{\delta A^k_j(x)} F_{ab}^i(x) - \frac{\Lambda}{6} \epsilon_{abc} \epsilon^{ijk} \frac{\delta}{\delta A^a_k(x)} \frac{\delta}{\delta A^b_j(x)} \frac{\delta}{\delta A^c_l(x)} \} \psi[A]
\]  

(219)
Then, there is a solution of the hamiltonian constraint with cosmological constant, which is also a solution of the Gauss constraint and the diffeomorphism constraint, given by the exponential of the Chern Simons action.

\[ \psi_\Lambda[A] = \exp -\frac{\Lambda}{6} \int d^3x \varepsilon^{abc} Tr[A_a \partial_b A_c + \frac{2}{3} A_a A_b A_c] \]  

(220)

This can be checked using the relation

\[ \frac{\delta}{\delta A^a_\nu(x)} \psi_\Lambda[A] = \frac{3}{\Lambda} \varepsilon^{abc} F^i_{bc}(x) \psi_\Lambda[A] \]  

(221)

Furthermore, this solution is obviously nondegenerate. Now, we know that the corresponding solution in the loop representation is given by the loop transform of \( \psi_\Lambda[A] \)

\[ \psi_\Lambda(\gamma) = \int d\mu[A] \psi_\Lambda[A] W_A(\gamma) \]  

(222)

but this loop transform is nothing but the Wilson loop average of the Chern-Simons theory and it was computed by Witten in the \( SU(2) \) case for nonintersecting knots.

Thus, in this case one can compute the transform, which turns out to be the Kauffman bracket polynomial. There are however important differences between this well known result in Chern-Simons theory and what we need here. First, in the Chern-Simons case the connection is a real element of the \( SU(2) \) algebra while here is complex and satisfies same reality conditions. Second, the result was proved for nonintersecting knots. Two questions are in order: Can this result be extended to the complex case?. Does this transform lead to a generalized Kauffman bracket in the intersecting case? The following perturbative argument suggests that both questions should be answered in the positive. Similar calculations were performed by Smolin\textsuperscript{40}, Kauffman\textsuperscript{34} and Cotta-Ramusino et al\textsuperscript{41}. Here we sketch the argument in the simplest non intersecting case. Intersections may be taken into account making use of similar techniques\textsuperscript{42}.

One starts by applying the loop derivative to the loop transform at a point \( x \) that may be taken as the base point of \( \gamma \)

\[ \sigma^{ab} \Delta_{ab}(x) \psi(\gamma) = \int d\mu[A] \psi_\Lambda[A] \sigma^{ab} \sigma^*_{ab}(x) Tr[\tau^i H_A(\gamma_x)] \]  

(223)

Now using Eq.(221) and integrating by parts, we obtain

\[ -\frac{1}{6} \int d\mu[A] \sigma^{ab} \epsilon_{abc} \int dy \delta(x - y) Tr[\tau^i H_A(\gamma_y) \tau^i H_A(\gamma_x)] e^{-\frac{\Lambda}{6} S_{CS}} \]  

(224)

This integral depends on the volume factor

\[ \sigma^{ab} \epsilon_{abc} dy \delta(x - y) \]  

(225)

which depending on the relative orientation of the two-surface \( \sigma \) and the tangent to the loop \( \gamma \) can lead to +1, -1 or 0 (times a regularization dependent factor that
can be absorbed in the definition of the cosmological constant. There are, therefore, three possibilities depending on the value of the volume.

\[ \delta \psi(\gamma) = 0 \]

\[ \delta \psi(\gamma) = \pm \frac{\Lambda}{8} \psi(\gamma) \]  

(226)

These equations may be diagrammatically interpreted in the following way

\[ \psi(\hat{L}_+) - \psi(\hat{L}_0) = \pm \frac{\Lambda}{8} \psi(\hat{L}_0) \]  

(227)

and correspond to the first two equations of the skein relations for the Kauffman Bracket polynomial. The other skein relations may be obtained by following a similar procedure. This heuristic argument may be extended to higher order in perturbation theory and is based on the assumption of the existence of the loop transform.

Therefore, we have good reasons to think that the Kauffman Bracket is a solution of the Hamiltonian constraint with cosmological constant \( \Lambda \). This property can be explicitly checked by making use of the full technology of the loop representation to verify if the expansion of the Kauffman bracket in powers of \( \Lambda \) is a solution. We need to compute

\[ \hat{C}_\Lambda \psi_\Lambda(\gamma) = (\hat{C}_0 - \frac{\Lambda}{4} \text{det}\hat{q})[2 - \frac{\Lambda}{4} a_1(\gamma) \]  

\[ + (\frac{\Lambda}{16})^2 [a_1^2 - 6a_2 + \ldots] = 0 \]  

(228)

This expression should vanish order by order in \( \Lambda \) and leads to the following set of equations

\[ \hat{C}_0 2(\gamma) = 0 \]  

(229)

\[ \text{det}\hat{q} 1(\gamma) + \frac{\Lambda}{2} \hat{C}_0 a_1(\gamma) = 0 \]  

(230)

\[ \text{det}\hat{q} a_1(\gamma) + \frac{\Lambda}{2} \hat{C}_0 a_1^2(\gamma) - \hat{C}_0 a_2(\gamma) = 0 \]  

(231)

and to similar equations for higher order. It is very easy to check that the first two equations hold while in the third the first two terms cancel among themselves and therefore

\[ \hat{C}_0 a_2(\gamma) = 0 \]  

(232)

must hold independently. Thus we recover the previously mentioned result. The second Conway coefficient \( a_2(\gamma) \) is a solution of the vacuum Hamiltonian constraint of quantum gravity.

While the first proof involved a long and complicated computation, here it takes half an hour to reobtain the same result. One can show that similar decompositions...
of the Kauffman Brackets coefficients occur at higher order in $\Lambda$. At each order some terms are framing dependent functions of the Gauss selflinking number and there appear ambient isotopy invariants $a_n(\gamma)$ related with the expansion in $\Lambda$ of the Jones polynomial. This fact, together with some evidence coming from the loop coordinate dependence of these coefficients has led us to conjecture\(^{43}\) that the Jones Polynomial could be a solution of the Quantum Gravity constraints.

To conclude we have shown that we have now at our disposal a set of mathematical techniques that allow to study the physical state space of Quantum Gravity and show a deep relation between this space and the Knot theory. This review does not have the aim of completeness There are some significant results concerning the loop representation that I am not going to discuss. I would like to mention two of them. In first place, one can define \(^{44}\) diffeomorphism invariant operators as the area of a 2-surface that carry geometric information and have discrete eigenvalues, quantized in Planck units. Secondly, it is possible to define ”semiclassical” states -the weaves- which approximate\(^{45}\) smooth classical solutions at large distances but exhibit a discrete structure at the Planck scale. Even though these states are not physical states of quantum gravity, (they don’t satisfy the constraints) they indicate the type of new structures that one could expect to found.

6 Open Issues

Here, we would like to mention some open issues related with the loop representation of gauge theories and quantum gravity. We are not going to treat some of the fundamental question of quantum gravity, as the issue of time. Instead, we would like to mention some more specific problems that will probably be under consideration in the near future.

i) Pure gauge theories

Even though the loop representation has very appealing features as the gauge invariance and its geometrical content, up to now, it has not allowed to improve other computational methods.

In the continuum we have a very compact description of the theory but, up to now, no exact solutions of the nonperturbative hamiltonian eigenvalue equation is known. In the case of renormalizable theories, these loop equations need to be renormalized, but we don’t know how to introduce a nonperturbative renormalization and therefore the strategy has been trying to solve the regularized equations and renormalize at the end. The new techniques developed in the last years, for the study of quantum gravity\(^ {24,33}\) may be useful in the Yang Mills case, in particular they seem to be well suited to introduce a rigorous defined inner product in the space of connections modulo gauge transformations.

On the lattice, the loop representation leads to a hamiltonian description of gauge theories in terms of loop clusters\(^ {46}\) that works better than other lattice hamiltonian methods. However, the standard statistical methods like Montecarlo are more effi-
cient. To apply functional methods in the loop representation we need to compute the action of the gauge theories in terms of loops. This is a non trivial problem that has been recently solved in the abelian case leading \(^47\) to an action proportional to the quadratic area associated to the world sheet defined by the evolution of the loop.

**ii) Gauge theories with matter fields**

Several matter fields, Higgs bosons, electrons, and quarks, have been introduced in this picture. The fundamental objects are open path with matter fields at the end points. A very appealing geometrical picture of the interacting fields appears, where the basic states of the theory are associated to the physical excitations in the confinement phase. For instance, in QCD the physical state space is defined in terms of loops and open paths with two or three quarks at the end points. These variables are respectively associated with the physical excitations, gluons, mesons, and baryons. On the lattice, the phase structure of QED was studied in the cluster approximation of the loop representation. In 3 + 1 dimensions the theory show a second order phase transition. The path representation is particularly useful when fermions are present because it allows to treat fermions in terms of even variables without an explosive proliferation of diagrams.

As in the pure gauge theories the introduction of an action principle in terms of loops and paths could be a good starting point for a more powerful computational treatment of these theories.

**iii) Equivalence between the connection and the loop representation**

The problem of the isomorphism between both representations is a non trivial one. The equivalence is well established on the lattice for abelian and non abelian Yang-Mills theories. However, in the continuum, even in the abelian case one needs to extend the usual loop representation to a loop coordinate or form factor representation in order to have a well defined inner product and an invertible loop transform connecting both spaces.

In 2 + 1 Gravity, Marolf has recently shown that both representations are inequivalent. In fact, the loop transform has a kernel that is dense in the connection representation, again the introduction of an extended loop representation allows to avoid this problem. A second way of solving the problem has been recently proposed by Ashtekar and Loll and is based in a modification of the loop transform by introducing a weight factor.

This problem is highly nontrivial in the 3 + 1 case. However, some partial results as the framing dependence of the loop transform of the Chern-Simons state, suggest that the connection and loop representation will be inequivalent. An extension of the group of loops and of the corresponding loop representation has recently been proposed. This extended representation seem to be free of some of the problems of the usual loop representation, but more work need to be done to confirm these
preliminary results.

iv) Algebra of the quantum constraints

It is known that the consistency of the algebra of constraints at the quantum level depends not only on the ordering of the operators but also of the regularization. Even though we know a set of solutions of all the constraints it is important to verify the consistency of the algebra. In particular it is necessary to check how the background dependent regularization of the constraints affects this consistency. Partial results in this direction has been recently obtained\textsuperscript{54}.

v) Framing dependence

The loop transform of the Chern-Simons state in the connection representation is the regular isotopic framing dependent Kauffman polynomial. Notice that the Chern-Simons state was invariant under diffeomorphism while its transform is not. That means that, probably, both connection and loop representation are not isomorphic. This nonequivalence should be expected, in fact, while the Chern-Simons state is not invariant under big gauge transformations (not connected with the identity), the states of the loop representation are invariant under general gauge transformations. Thus, the framing dependence is somehow related with the additional degrees of freedom of the big gauge transformations. If one insists in taking the loop representation as more fundamental, that would mean that we have a consistent set of framing independent diffeomorphism invariant solutions only in the case of vanishing cosmological constant. As we mentioned before, it is also possible to extend the loop representation and work in a loop coordinate representation were loops are substituted by smooth functions and all the extended knot invariants are perfectly well defined\textsuperscript{33} diffeomorphism invariants objects.

vi) Integration measures in the space of connections modulo gauge transformations

Ashtekar and Isham have recently\textsuperscript{24} introduced a notion of integration on the space of connections modulo gauge transformations of the $SU(2)$ theory in terms of the so called holonomy algebra. In principle such notion of integration would allow to introduce an inner product and a well defined loop transform in this space. When defining measures on functional spaces in quantum field theory it is generally necessary to enlarge the space of functions and include distributions. The space of smooth functions is of measure zero. However Rendall\textsuperscript{55} has recently proved that the space induced by the holonomy algebras includes an infinite set of unphysical nondistributional connections and may exclude some of the distributional ones. Thus, it seems that the original measures introduced by Ashtekar and Isham need to be improved, and that we are still far from having the physically relevant integration measures for the Yang Mills theory and quantum gravity. This is a subject under very active research, and some progress has recently\textsuperscript{56} been done.
vii) Inner product and reality conditions in loop space

A very important point in any quantization program of general relativity is to find the inner product on the space of physical states. Since the general solutions of the Gauss and vector constraints are known, it seems convenient to first introduce an inner product in the space of general knot invariant functions and then restrict the space to the solutions of the scalar constraint. The appealing point of this approach is the apparent discreteness of space of knots which could simplify significantly the problem of finding measures. However we have no control about the reality conditions in the space of diffeomorphism invariant loop dependent objects and we are very far from having a complete set of observables which commute with the gauge and vector constraints. Furthermore, when knots with self intersections are present, the space of knots is not a countable set any more.

An alternative approach could be to impose the reality constraints at the level of the solutions of the Gauss constraint only. There, the main problem is to introduce a measure in loop space. Again the extended loops may be of some help because as it occurs in the abelian case, they allow to transform loop integrals into functional integrals.

8. Acknowledgements

I am most grateful to the organizers, especially Luis Urrutia and Miguel Angel Pérez Angón for giving me the opportunity to present my views on these topics. Most of the original research presented in section 5 was done in collaboration with Jorge Pullin (Penn. State) and Bernd Brügmann (Max-Planck). Collaborating with them has been a great pleasure and has enriched a lot my views on these subjects. I would like to thank to Abhay Ashtekar, John Baez, Cayetano Di Bartolo, Ricardo Capovilla, Jorge Griego, Renate Loll, Jose Mourão, Carlo Rovelli, Lee Smolin, and Madhavan Varadarajan. To all of them I am indebted for the insights they offered on various topics. This work was supported in part by PEDECIBA (Montevideo).

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