Generalized Measures of Edge Fault Tolerance in \((n, k)\)-star Graphs

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Abstract: This paper considers a kind of generalized measure \(\lambda^{(h)}\) of fault tolerance in the \((n, k)\)-star graph \(S_{n,k}\) for \(2 \leq k \leq n - 1\) and \(0 \leq h \leq n - k\), and determines \(\lambda^{(h)}(S_{n,k}) = \min\{(n - h - 1)(h + 1), (n - k + 1)(k - 1)\}\), which implies that at least \(\min\{(n - k + 1)(k - 1), (n - h - 1)(h + 1)\}\) edges of \(S_{n,k}\) have to remove to get a disconnected graph that contains no vertices of degree less than \(h\). This result shows that the \((n, k)\)-star graph is robust when it is used to model the topological structure of a large-scale parallel processing system.

Keywords: Combinatorics, fault-tolerant analysis, \((n, k)\)-star graphs, edge-connectivity, \(h\)-super edge-connectivity

I. Introduction

It is well known that interconnection networks play an important role in parallel computing/communication systems. An interconnection network can be modeled by a graph \(G = (V, E)\), where \(V\) is the set of processors and \(E\) is the set of communication links in the network. The connectivity \(\lambda(G)\) of a graph \(G\) is an important measurement for fault-tolerance of the network, and the larger \(\lambda(G)\) is, the more reliable the network is.

A subset of vertices \(R\) of a connected graph \(G\) is called an edge-cut if \(G - R\) is disconnected. The edge connectivity \(\lambda(G)\) of \(G\) is defined as the minimum cardinality over all edge-cuts of \(G\). Because \(\lambda\) has many shortcomings, one proposes the concept of the \(h\)-super edge connectivity of \(G\), which can measure fault tolerance of an interconnection network more accurately than the classical connectivity \(\lambda\).

A subset of vertices \(R\) of a connected graph \(G\) is called an \(h\)-super edge-cut, or \(h\)-edge-cut for short, if \(G - R\) is disconnected and has the minimum degree at least \(h\). The \(h\)-super edge-connectivity of \(G\), denoted by \(\lambda^{(h)}(G)\), is defined as the minimum cardinality over all \(h\)-edge-cuts of \(G\). It is clear that, if \(\lambda^{(h)}(G)\) exists, then

\[
\lambda(G) = \lambda^{(0)}(G) \leq \lambda^{(1)}(G) \leq \lambda^{(2)}(G) \leq \cdots \leq \lambda^{(h-1)}(G) \leq \lambda^{(h)}(G).
\]

For any graph \(G\) and integer \(h\), determining \(\lambda^{(h)}(G)\) is quite difficult. In fact, the existence of \(\lambda^{(h)}(G)\) is an open problem so far when \(h \geq 1\). Some results have been obtained on \(\lambda^{(h)}(G)\) for particular classes of graphs and small \(h\)'s (see Section 16.7 in [5]).

This paper is concerned about \(\lambda^{(h)}\) for the \((n, k)\)-star graph \(S_{n,k}\). For the \(h\)-super connectivity, several authors have done some work. For \(k = n - 1\), \(S_{n,n-1}\) is isomorphic to a star graph \(S_n\). Akers and Krishnamurthy [1] determined \(\lambda(S_n) = n - 1\) for \(n \geq 2\) and \(\lambda^{(1)}(S_n) = 2n - 4\) for \(n \geq 3\). In this paper, we show the following result.

**Theorem:** If \(2 \leq k \leq n - 1\) and \(0 \leq h \leq n - k\), then
This result implies that at least $\min\{(n-k+1)(k-1), (n-h-1)(h+1)\}$ edges of $S_{n,k}$ have to remove to get a disconnected graph that contains no vertices of degree less than $h$. The proof of this result is in Section 3. In Section 2, we recall the structure of $S_{n,k}$ and some lemmas used in our proofs.

II. Definitions and Lemmas

For integers $n$ and $k$ with $1 \leq k \leq n-1$, let $I_n = \{1, 2, \ldots, n\}$ and $P(n, k) = \{p_1p_2 \ldots p_k : p_i \in I_n, p_i \neq p_j, 1 \leq i \neq j \leq k\}$, the set of $k$-permutations on $I_n$. Clearly, $|P(n, k)| = n!/(n-k)!$.

**Definition 2.1:** The $(n, k)$-star graph $S_{n,k}$ is a graph with vertex-set $P(n, k)$. The adjacency is defined as follows: a vertex $p = p_1p_2 \ldots p_k$ is adjacent to a vertex

(a) $p_1p_2 \ldots p_{i-1}p_1p_i+1 \ldots p_k$, where $2 \leq i \leq k$ (swap $p_i$ with $p_1$).

(b) $\alpha p_3p_4 \ldots p_k$, where $\alpha \in I_n \setminus \{p_i : 1 \leq i \leq k\}$ (replace $p_i$ by $\alpha$).

The vertices of type (a) are referred to as swap-neighbors of $p$ and the edges between them are referred to as swap-edges or $i$-edges. The vertices of type (b) are referred to as unswap-neighbors of $p$ and the edges between them are referred to as unswap-edges. Clearly, every vertex in $S_{n,k}$ has $k-1$ swap-neighbors and $n-k$ unswap-neighbors. Usually, if $x = p_1p_2 \ldots p_k$ is a vertex in $S_{n,k}$, we call $p_i$ the $i$-th bit for each $i \in I_k$.

The $(n, k)$-star graph $S_{n,k}$ is proposed by Chiang and Chen [4]. Some nice properties of $S_{n,k}$ are compiled by Cheng and Lipman (see Theorem 1 in [2]).

**Lemma 2.2:** $S_{n,k}$ is $(n-1)$-regular $(n-1)$-connected.

**Lemma 2.3:** For any $\alpha = p_1p_2 \ldots p_{k-1} \in P(n, k-1)$ $(k \geq 2)$, let $V_\alpha = \{p\alpha : p \in I_n \setminus \{p_i : i \in I_{k-1}\}\}$. Then the subgraph of $S_{n,k}$ induced by $V_\alpha$ is a complete graph of order $n - k + 1$, denoted by $K_{n-k+1}^\alpha$.

Let $S_{n-1,k-1}^{i,j}$ denote the subgraph of $S_{n,k}$ induced by vertices with the $i$-th bit for $2 \leq i \leq k$. The following lemma is a slight modification of the result of Chiang and Chen [4].

**Lemma 2.4:** For a fixed integer $i$ with $2 \leq i \leq k$, $S_{n,k}$ can be decomposed into $n$ subgraphs $S_{n-1,k-1}^{i,j}$ which is isomorphic to $S_{n-1,k-1}$ for each $i \in I_n$. Moreover, there are $\frac{(n-2)!}{(n-k)!}$ independent swap-edges between $S_{n-1,k-1}^{i,j}$ and $S_{n-1,k-1}^{i,j}$ for any $i, j \in I_n$ with $i \neq j$.

Since $S_{n,1} \cong K_n$, we only consider the case of $k \geq 2$ in the following discussion.

**Lemma 2.5:** If $2 \leq k \leq n-1$ and $0 \leq h \leq n - k$, then

$$\chi_n^{(h)}(S_{n,k}) = \begin{cases} (n-h-1)(h+1) & \text{for } h \leq k-2 \text{ and } h \leq \frac{n}{2} - 1, \\ (n-k+1)(k-1) & \text{otherwise.} \end{cases}$$

**Proof:** By our hypothesis of $2 \leq k \leq n-1$, for any $\alpha \in P(n, k-1)$, we can choose a subset $X \subseteq V(K_{n-k+1}^\alpha)$ such that $|X| = h + 1$. Then the subgraph of $K_{n-k+1}^\alpha$ induced by $X$ is a complete graph $K_{h+1}$. Let $R$ be the set of incident edges with and not within $X$. Since $S_{n,k}$ is $(n-1)$-regular and $K_{h+1}$ is $h$-regular, we have that $|B| = (n-h-1)(h+1)$.

Clearly, $R$ is an edge-cut of $S_{n,k}$. Let $x$ be any vertex in $S_{n,k} - X$, and $d(x)$ denote the number of edges incident with $x$ in $S_{n,k} - X$. In order to prove that $R$ is an $h$-edge-cut, we only need to show $d(x) \geq h$. Note that $X$ is contained in $S_{n-1,k-1}^i$ and edges between $S_{n-1,k-1}^i$ and $S_{n-1,k-1}^j$ are independent for any $i, j \in I_n$ with $i \neq j$.
Lemma 2.4. If \( x \) is in \( S^i_{n-1,k-1} - K^\alpha_{n-k+1} \) or is in \( S^j_{n-1,k-1} \) with \( i \neq j \), then \( d(x) \geq n - 2 \geq n - k \geq h \). For \( x \in V(K^\alpha_{n-k+1} - X) \), if exists, then \( d(x) = n - 1 - |X| = n - h - 2 \geq h \) for \( h \leq \frac{n}{2} - 1 \). Therefore, \( R \) is an \( h \)-edge-cut of \( S_{n,k} \) and so
\[
\lambda^h_s(S_{n,k}) \leq |B| = (n - h - 1)(h + 1) \quad \text{for} \quad h \leq \frac{n}{2} - 1.
\]

For \( h \geq \frac{n}{3} \), we choose \( X = V(K^\alpha_{n-k+1}) \). Then \( |B| = (n - k + 1)(k - 1) \). For any \( x \) in \( S^i_{n-1,k-1} - X \) or \( S^j_{n-1,k-1} \) with \( i \neq j \), we have \( d(x) \geq n - 2 \geq n - k \geq h \). Thus, \( R \) is an \( h \)-edge-cut of \( S_{n,k} \) and so
\[
\lambda^h_s(S_{n,k}) \leq |B| = (n - k + 1)(k - 1) \quad \text{for} \quad h \geq \frac{n}{2}.
\]

The lemma follows.

Corollary 2.6: \( \lambda^h_s(S_{n,2}) = n - 1 \) for \( 0 \leq h \leq n - 2 \).

Proof: On the one hand, \( \lambda^h_s(S_{n,2}) \leq n - 1 \) by Lemma 2.5 when \( k = 2 \). On the other hand, \( \lambda^h_s(S_{n,2}) \geq \lambda(S_{n,2}) = n - 1 \) by Lemma 2.2.

The following lemma shows the relations between \( (n - h - 1)(h + 1) \) and \( (n - k + 1)(k - 1) \).

Lemma 2.7: For \( 2 \leq k \leq n - 1 \), \( 0 \leq h \leq n - k \), let
\[
\psi(h, k) = \min\{(n - h - 1)(h + 1), (n - k + 1)(k - 1)\}.
\]
If \( h \leq \frac{n}{2} - 1 \), then
\[
\psi(h, k) = \begin{cases} (n - h - 1)(h + 1) & \text{if } 0 \leq h \leq k - 2; \\ (n - k + 1)(k - 1) & \text{if } h \geq k - 1. \end{cases}
\]

Proof: Let \( f(x) = (n - x)x \), then \( \psi(h, k) = \min\{f(h + 1), f(k - 1)\} \). It can be easily checked that \( f(x) \) is a convex function on the interval \([0, n]\); the maximum value is reached at \( x = \frac{n}{2} \). Thus, \( f(x) \) is an increasing function on the interval \([0, \frac{n}{2}]\).

If \( 0 \leq h \leq k - 2 \), then \( h + 1 \leq k - 1 \). Since \( h \leq n - k, h + 1 \leq n - k + 1 \) and \( \min\{k - 1, n + k - 1\} \leq \frac{n}{2} \). Thus, when \( h \leq \frac{n}{2} - 1 \), \( f(h + 1) \leq f(k - 1) = f(n - k + 1) \), and so \( \psi(h, k) = f(h + 1) = (n - h - 1)(h + 1) \).

If \( h \geq k - 1 \), then \( k - 1 < h + 1 \leq \frac{n}{2} \), \( f(k - 1) < f(h + 1) \), so \( \psi(n, k) = f(k - 1) = (n - k + 1)(k - 1) \).

The lemma follows.

To state and prove our main results, we need some notations. Let \( R \) be a minimum \( h \)-edge-cut of \( S_{n,k} \). Clearly, \( S_{n,k} - B \) has exactly two connected components. Let \( X \) and \( Y \) be two vertex-set of two connected components of \( S_{n,k} - B \). For a fixed \( t \in I_k \setminus \{1\} \) and any \( i \in I_n \), let
\[
X_i = X \cap V(S^i_{n-1,k-1}), \\
Y_i = Y \cap V(S^i_{n-1,k-1}), \\
B_i = B \cap E(S^i_{n-1,k-1}) \quad \text{and} \\
B_{ij} = B \cap E(S^i_{n-1,k-1}, S^j_{n-1,k-1}),
\]
and let
\[
J = \{i \in I_n : X_i \neq \emptyset\}, \\
J' = \{i \in J : Y_i \neq \emptyset\} \quad \text{and} \\
T = \{i \in I_n : Y_i \neq \emptyset\}.
\]

Lemma 2.8: Let \( R \) be a minimum \( h \)-edge-cut of \( S_{n,k} \) and \( X \) be the vertex-set of a connected component of \( S_{n,k} - B \). If \( 3 \leq k \leq n - 1 \) and \( 1 \leq h \leq n - k \) then, for any \( t \in I_k \setminus \{1\} \),

(a) \( B_i \) is an \( (h - 1 \)-edge-cut of \( S^i_{n-1,k-1} \) for any \( i \in J' \).
(b) $\lambda_s^{(h)}(S_{n,k}) \geq |J| \lambda_s^{(h-1)}(S_{n-1,k-1})$.

**Proof.** (a) By the definition of $J$, $B_i$ is an edge-cut of $S_{n-1,k-1}^{i_1}$ for any $i \in J$. For any vertex $x$ in $S_{n-1,k-1}^{i_1} - B_i$, since $x$ has degree at least $h$ in $S_{n,k} - S$ and has exactly one neighbor outsider $S_{n-1,k-1}^{i_1}$, $x$ has degree at least $h-1$ in $S_{n,k}^{i_1} - B_i$. This fact shows that $B_i$ is an $(h-1)$-edge-cut of $S_{n-1,k-1}^{i_1}$ for any $i \in J$.

(b) By the assertion (a), we have $|B_i| \geq \lambda_s^{(h-1)}(S_{n-1,k-1})$, and so

$$\lambda_s^{(h)}(S_{n,k}) = |B| \geq \sum_{i \in J} |B_i| \geq |J| \lambda_s^{(h-1)}(S_{n-1,k-1})$$

The lemma follows.

### III. Proof of Theorem

By Lemma 2.5 and Lemma 2.7, we only need to prove that, for $2 \leq k \leq n-1$ and $0 \leq h \leq n-k$,

$$\lambda_s^{(h)}(S_{n,k}) \geq \begin{cases} (n-h-1)(h+1) & \text{for } h \leq k-2 \text{ and } h \leq \frac{n}{2} - 1, \\ (n-k+1)(k-1) & \text{otherwise.} \end{cases}$$

Let $\omega(h,k) = \max\{(n-h-1)(h+1), (n-k+1)(k-1)\}$.

We proceed by induction on $k \geq 2$ and $h \geq 0$. The inequality is true for $k = 2$ and any $h$ with $0 \leq h \leq n-2$ by Corollary 2.6. The inequality is also true for $h = 0$ and any $k$ with $2 \leq k \leq n-1$ since $\lambda_s^{(0)}(S_{n,k}) = \lambda(S_{n,k}) = n-1$. Assume the induction hypothesis for $k-1$ with $k \geq 3$ and for $h-1$ with $h \geq 1$, that is,

$$\lambda_s^{(h-1)}(S_{n-1,k-1}) \geq \begin{cases} (n-h+1)h & \text{for } h \leq k-2 \text{ and } h \leq \frac{n-1}{2}, \\ (n-k+2)(k-2) & \text{otherwise.} \end{cases}$$

Let $B$ be a minimum $h$-edge-cut of $S_{n,k}$ and $X$ be the vertex-set of a minimum connected component of $S_{n,k} - B$. By Lemma 2.5, we have

$$|B| \leq \omega(h,k).$$

Use notations defined in Section II. Choose $t \in I_k \setminus \{1\}$ such that $|J|$ is as large as possible. For each $i \in I_n$, we write $S_{n-1,k-1}^{i_1}$ for $S_{n-1,k-1}^{i_1}$ for short.

We first show $|J| = 1$. Suppose to the contrary $|J| \geq 2$. We will deduce contradictions by considering three cases depending on $|J| = 0$, $|J| = 1$ or $|J| \geq 2$.

**Case 1.** $|J| = 0$.

In this case, $X_i \neq \emptyset$ and $Y_i = \emptyset$ for each $i \in J$, that is, $J \cap T = \emptyset$. By $|J| \geq 2$ and the minimality of $X$, $|T| \geq 2$. Assume $\{i_1, i_2\} \subseteq J$ and $\{i_3, i_4\} \subseteq T$. By Lemma 2.4, there are $\binom{n-2}{(n-k)}$ independent swap-edges between $S_{n-1,k-1}^{i_1}$ (resp. $S_{n-1,k-1}^{i_2}$) and $S_{n-1,k-1}^{i_3}$ (resp. $S_{n-1,k-1}^{i_4}$), all of which are contained in $B$. Since $J \cap T = \emptyset$, we have that

$$|B| \geq 4 \binom{n-2}{(n-k)}.$$

For $k = 3$,

$$|B| \geq 4 \binom{n-2}{(n-3)} \geq 4(n-2) > 2(n-2)$$

Combining Lemma 2.5 with Lemma 2.7 yields $|B| \leq \lambda_s^{(h)}(S_{n,3}) \leq 2(n-2)$, a contradiction. For $k \geq 4$, it is easy to check that
which contradicts the inequality (1).

Case 2. \(|J'| = 1\).

Without loss of generality, assume \(J' = \{1\}\). By Lemma 2.8 (a), \(B_1\) is an \((h - 1)\)-edge-cut of \(S_{n-1,k-1}^1\). By \(|J| \geq 2\), there exists an \(i \in J - J'\) such that \(X_i = V(S_{n-1,k-1}^i)\). By the minimality of \(X\), there exists some \(j \in T - J'\) such that \(Y_j = V(S_{n-1,k-1}^j)\). By Lemma 2.4, there are \(\frac{(n-2)!}{(n-k)!}\) independent swap-edges between \(S_{n-1,k-1}^i\) and \(S_{n-1,k-1}^j\), thus \(|B_{ij}| = \frac{(n-2)!}{(n-k)!} \geq n-2\). We consider the following two cases.

If \(\lambda_s^{(h-1)}(S_{n-1,k-1}) \geq (n-h)h\), then
\[
|B| \geq |B_1| + |B_{ij}|
\geq (n-h)h + (n-2)
\geq (n-h-1)h + (n-h-1)
= (n-h-1)(h+1),
\]

If \(\lambda_s^{(h-1)}(S_{n-1,k-1}) \geq (n-k+2)(k-2)\), then
\[
|B| \geq |B_1| + |B_{ij}|
\geq (n-k+2)(k-2) + (n-2)
\geq (n-k+1)(k-2) + (n-k+1)
= (n-k+1)(k-1).
\]

Therefore, we have \(|B| > \omega(h,k)\), which contradicts the inequality (1).

Case 3. \(|J'| \geq 2\).

By Lemma 2.8 (b), if \(\lambda_s^{(h-1)}(S_{n-1,k-1}) \geq (n-h)h\) then
\[
|B| \geq |J'| \lambda_s^{(h-1)}(S_{n-1,k-1})
\geq 2(n-h)h \geq (n-h)h + (n-h)
\geq (n-h-1)(h+1),
\]

if \(\lambda_s^{(h-1)}(S_{n-1,k-1}) \geq (n-k+2)(k-2)\) then
\[
|B| \geq |J'| \lambda_s^{(h-1)}(S_{n-1,k-1})
\geq 2(n-k+2)(k-2)
\geq (n-k+2)(k-2) + (n-k+2)
\geq (n-k+1)(k-1).
\]

Therefore, we have \(|B| > \omega(h,k)\), which contradicts the inequality (1).

Thus, we have \(|J| = 1\). By the choice of \(i\), the \(i\)-th bits of all vertices in \(X\) are same for each \(i = 2, 3, \ldots, k\), and so \(X\) is a complete graph. Thus, we have that
\[
\lambda_s^{(h)}(S_{n,k}) = |B| = (n - |X|)|X|
\]

Since \(h + 1 \leq |X| \leq n - k + 1\) and \(f(x) = (n-x)x\) is a convex function on the interval \([0, n]\), we have that
\[
\lambda_s^{(h)}(S_{n,k}) = |B| = (n - |X|)|X| \geq \psi(h,k),
\]

where \(\psi(h,k)\) is defined in Lemma 2.7.

If \(h \leq \frac{n}{2} - 1\), using Lemma 2.7, we have
\[ \lambda^{(h)}_e(S_{n,k}) \geq \psi(h,k) = \begin{cases} (n-h-1)(h+1) & \text{if } 0 \leq h \leq k-2; \\ (n-k+1)(k-1) & \text{if } h \geq k-1. \end{cases} \]

If \( h \geq \frac{n}{2} \), then \( X = V(K_{n-k+1}) \). Otherwise, there exists some \( x \in V(K_{n-k+1} - X) \) such that

\[ h \leq d(x) = n - 1 - |X| \leq n - h - 2, \]

which implies \( h \leq \frac{n}{2} - 1 \), a contradiction. Therefore, we have \( |X| = n - k + 1 \), and

\[ \lambda^{(h)}_e(S_{n,k}) = |B| = (n - |X|)|X| = (n - k + 1)(k - 1) \text{ for } h \geq \frac{n}{2}. \]

By the induction principle, the theorem follows.

As we have known, when \( k = n - 1 \), \( S_{n,n-1} \) is isomorphic to the star graph \( S_n \). Akers and Krishnamurthy [1] determined \( \lambda(S_n) \) and \( \lambda^{(1)}_e(S_n) \), which can be obtained from our result by setting \( k = n - 1 \) and \( h = 0,1 \), respectively.

**Corollary:** \( \lambda(S_n) = n - 1 \) for \( n \geq 2 \) and \( \lambda^{(1)}_e(S_n) = 2n - 4 \) for \( n \geq 3 \).

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