Analytical models for quark stars

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Abstract

We find two new classes of exact solutions to the Einstein-Maxwell system of equations. The matter content satisfies a linear equation of state consistent with quark matter; a particular form of one of the gravitational potentials is specified to generate solutions. The exact solutions can be written in terms of elementary functions, and these can be related to quark matter in the presence of an electromagnetic field. The first class of solutions generalises the Mak and Harko model. The second class of solutions does not admit any singularities in the matter and gravitational potentials at the centre.

Key words: Einstein-Maxwell system; exact solutions; charged quark stars.

1 Introduction

The existence of quark stars in hydrostatic equilibrium was first suggested by Itoh in a seminal treatment. The study of strange stars consisting of quark matter has stimulated much interest in the last few decades since this could represent the most energetically favourable state of baryon matter. Matter consisting of u, d and s quarks may be the absolute ground state of matter at zero pressure and temperature as suggested by Bodmer. It is expected that strange stars form during the collapse of the core of a massive star after a supernova explosion. In regions of low temperatures and sufficiently high densities hadrons are crushed into quark matter with color superconducting phases which occur in the dense cores of neutron stars as remarked by Alford.

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Consequently the core of a proto-neutron star or neutron star provides the appropriate environment for ordinary matter to convert to strange quark matter. Another possibility is that a rapidly spinning dense star can accrete sufficient mass to undergo a phase transition to form a strange star.

As the physics of ultrahigh densities for quark matter is not well understood, researchers restrict their attention to the phenomenological MIT bag model. In the bag model, the strange matter equation of state has a simple linear form given by

\[ p = \frac{1}{3}(\rho - 4B) \] (1)

where \( \rho \) is the energy density, \( p \) is the isotropic pressure and \( B \) is the bag constant. The quark confinement is determined by the vacuum pressure \( B \) in the bag model that equilibrates the pressure of quarks thereby stabilising the system. Studies of particular compact astronomical objects indicate that they could be strange stars with the quark matter. A candidate for a strange star may have been observed using the deep Chandra LETG+HRC-S observations; Drake et al. suggested that the X-ray source RXJ1856.5-3754 may be such an object. Sotani et al. have used observational data on gravitational waves to obtain the equation of state for quark matter. Harko and Cheng considered collapsing strange matter in spherically symmetric fields. Yilmaz and Baysal studied charged strange matter in rotating fields. The role of anisotropy, with the linear equation of state \( (1) \), was pursed by Mak and Harko and Sharma and Maharaj who demonstrated exact analytical solutions.

In a recent treatment Mak and Harko found a charged strange quark star under the assumption of spherical symmetry and the existence of a conformal Killing vector. In this paper we consider the Einstein-Maxwell system of equations with the linear equation of state \( (1) \) and apply them to strange stars. The existence of a conformal symmetry is not an assumption that we make. We demonstrate that exact analytical solutions to the field equations are possible that contain the Mak-Harko model. In Section 2, we rewrite the Einstein-Maxwell field equations for the static spherically line element as an equivalent set of differential equations utilising a transformation due to Durgapal and Bannerji. We then obtain a new set of differential equations with the assistance of the bag equation of state \( (1) \) for strange matter. On specifying an explicit form for one of the gravitational potentials, we obtain a first order differential equation in the remaining potential in Section 3. In Section 4 we find a new class of exact solutions to the Einstein-Maxwell system. The model of Mak and Harko is regained as a special case. In Section 5 we present a second class of exact solutions that satisfy the Einstein-Maxwell system. This category of solutions has the desirable feature of not admitting singularities at the centre.
2 Spherically symmetric spacetimes

Since our intention is to study relativistic stellar objects it seems reasonable, on physical grounds, to assume that spacetime is static and spherically symmetric. This is clearly consistent with models utilised to study physical processes in compact objects. The metric of a static spherically symmetric spacetime in curvature coordinates reads

\[ ds^2 = -e^{2\nu(r)}dt^2 + e^{2\lambda(r)}dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \]  

(2)

where \( \nu(r) \) and \( \lambda(r) \) are the two arbitrary functions. For charged perfect fluids the Einstein-Maxwell system of field equations are obtained as

\[ \frac{1}{r^2}(1 - e^{-2\lambda}) + \frac{2\nu'}{r}e^{-2\lambda} = \rho + \frac{1}{2}E^2 \]  

(3a)

\[ -\frac{1}{r^2}(1 - e^{-2\lambda}) + \frac{2\nu'}{r}e^{-2\lambda} = p - \frac{1}{2}E^2 \]  

(3b)

\[ e^{-2\lambda} \left( \nu'' + \nu'^2 + \frac{\nu'}{r} - \nu'\lambda' - \frac{\lambda'}{r} \right) = p + \frac{1}{2}E^2 \]  

(3c)

\[ \sigma = \frac{1}{r^2}e^{-\lambda}(r^2E)' \]  

(3d)

for the line element (2). We are utilising units in which the coupling constant \( 8\pi G/c^4 = 1 \) and the speed of light \( c = 1 \). The energy density \( \rho \) and the pressure \( p \) are measured relative to the comoving fluid 4-velocity \( u^a = e^{-\nu}\delta^a_0 \) and primes means a derivative with respect to the radial coordinate \( r \). In the system (3) the quantities \( E \) and \( \sigma \) are the electric field intensity and the proper charge density respectively.

It is convenient at this point to introduce the transformation

\[ A^2y^2(x) = e^{2\nu(r)}, \quad Z(x) = e^{-2\lambda(r)}, \quad x = Cr^2 \]  

(4)

where \( A \) and \( C \) are arbitrary constants. With this transformation, the system (3) has the equivalent form

\[ \frac{1 - Z}{x} - 2\dot{Z} = \frac{\rho}{C} + \frac{E^2}{2C} \]  

(5a)

\[ 4Z\frac{\ddot{y}}{y} + \frac{Z - 1}{x} = \frac{p}{C} - \frac{E^2}{2C} \]  

(5b)

\[ 4Z x^2 \ddot{y} + 2\dot{Z} x^2 \dot{y} + \left( \dot{Z} x - Z + 1 - \frac{E^2 x}{C} \right) y = 0 \]  

(5c)

\[ \frac{\sigma^2}{C} = \frac{4Z}{x}(xE + E)^2 \]  

(5d)

where dots denote differentiation with respect to the variable \( x \). Note that equation (5c) is the condition of pressure isotropy. We can replace the system of field equations
(5), including the bag equation of state (1), by the system

\[
\begin{align*}
\rho & = 3p + 4B \\
\frac{p}{C} & = Z\frac{\dot{y}}{y} - \frac{1}{2}\ddot{Z} - \frac{B}{C} \\
\frac{E^2}{2C} & = \frac{1-Z}{x} - 3Z\frac{\dot{y}}{y} - \frac{1}{2}\ddot{Z} - \frac{B}{C} \\
0 & = 4Zx^2\ddot{y} + (6xZ + 2x^2\dddot{Z})\dot{y} + \left[2x\left(\frac{\dot{Z}}{C} + \frac{B}{C}\right) + Z - 1\right]y \\
\sigma & = 2\sqrt{\frac{CZ}{x}}(E + x\dot{E})
\end{align*}
\]

The system of equations (6) governs the gravitational behaviour of a charged quark star.

### 3 Integration procedure

We describe one possible integration procedure that leads to an exact solution of the Einstein-Maxwell system (6). Note that other procedures are possible; our approach has the advantage of producing a first order equation that has solution in terms of elementary functions. We observe from (6a) that \(\rho\) and \(p\) are related. Therefore in the system (6) there are five independent variables (\(Z, y, p\) or \(\rho, E, \sigma\)) and only four independent equations. We have freedom to choose only one of the quantities involved. In our approach we specify \(y(x)\) on physical grounds. A number of choices for the gravitational potential \(y(x)\) are possible; clearly we should choose a form that is likely to lead to a physically reasonable solution. To make the above set of equations tractable, we choose the metric function in the particular form

\[
y(x) = (a + x^m)^n
\]

where \(a, m\) and \(n\) are constants. The form chosen ensures that the metric function \(y\) is continuous and well-behaved in the interior of the star for the wide range of values of parameters \(m\) and \(n\). The function \(y\) yields a finite value at the centre of the star. This is a very desirable feature for the model on physical grounds. It is interesting to observe that many of the solutions found previously do not share this feature.

Substitution of (7) into (6d) leads to the first order equation

\[
\dot{Z} + \frac{a^2 + 2a[1 + mn(2m + 1)]x^m + [2mn(2mn + 1) + 1]x^{2m}}{2x(a + (1 + mn)x^m)}Z
\]

\[
= \frac{(1 - \frac{2B}{C}x)(a + x^m)}{2x[a + (1 + mn)x^m]}
\]
This first order equation is linear so that it can be integrated in principle. The complicated rational coefficient of $Z$ can be simplified using partial fractions and we obtain

$$
\dot{Z} + \left[ \frac{1}{2x} + \frac{2m(n-1)x^{m-1}}{(a+x^m)} + \frac{m[4(1+mn)-3n|x|^{m-1}}{2[a+(1+mn)x^m]} \right] Z \\
= \frac{(1-\frac{2B}{C}x)(a+x^m)}{2x[a+(1+mn)x^m]}
$$

Note that we have essentially reduced the solution of the field equations (6) to integrating (8). Once the potential $Z$ in (8) is found the remaining relevant quantities $\rho$, $p$ and $E$ then follow from (6a), (6b) and (6c) respectively. It is possible to find exact solutions to the Einstein-Maxwell field equations with the linear equation of state for different values of $m$ and $n$ in (8). We illustrate this with two simple examples in terms of elementary functions. Other exact solutions are possible but the form of the solution becomes more complicated and could involve special functions.

### 4 Generalised Mak-Harko model

An exact solution of (6) can be found with $m = 1/2$ and $n = 1$. In this case (7) gives the first metric function

$$
y(x) = (a + \sqrt{x})
$$

Equation (8) becomes

$$
\dot{Z} + \left[ \frac{1}{2x} + \frac{3}{2\sqrt{x}(2a+3\sqrt{x})} \right] Z = \frac{(1-\frac{2B}{C}x)(a+\sqrt{x})}{x(2a+3\sqrt{x})}
$$

which can be integrated to give the second metric function

$$
Z = \frac{3(2a+\sqrt{x}) - \frac{B}{C}x(4a+3\sqrt{x})}{3(2a+3\sqrt{x})}
$$

Hence we can generate the exact analytical model

$$
e^{2\nu} = A^2(a + \sqrt{x})^2 \quad (10a)$$

$$
e^{2\lambda} = \frac{3(2a+3\sqrt{x})}{3(2a+\sqrt{x}) - \frac{B}{C}x(4a+3\sqrt{x})} \quad (10b)$$

$$
\rho = f(x) + \frac{B(16a^3 + 47a^2\sqrt{x} + 48ax + 18x^3)}{2(a+\sqrt{x})(2a+3\sqrt{x})^2} \quad (10c)
$$

$$
3p = f(x) - \frac{B(16a^3 + 81a^2\sqrt{x} + 120ax + 54x^3)}{2(a+\sqrt{x})(2a+3\sqrt{x})^2} \quad (10d)
$$

$$
E^2 = \frac{C(-2a^2 - 2a\sqrt{x} + 3x) + Bx(a^2 + 2a\sqrt{x})}{\sqrt{x}(a+\sqrt{x})(2a+3\sqrt{x})^2} \quad (10e)
$$
which is a solution of \((6)\). For simplicity we have set

\[
f(x) = \frac{3C(6a^2 + 10a\sqrt{x} + 3x)}{2\sqrt{x}(a + \sqrt{x})(2a + 3\sqrt{x})^2}
\]

The exact model \((10)\) satisfies the Einstein-Maxwell system \((6)\). Note that if we set \(a = 0\), then the system \((10)\) becomes

\[
e^{2\nu} = D^2 r^2, \quad e^{2\lambda} = \frac{3}{1 - Br^2}, \quad \rho = \frac{1}{2r^2} + B, \quad p = \frac{1}{6r^2} - B, \quad E^2 = \frac{1}{3r^2}
\]

where we have set \(D^2 = A^2 C\). The particular solution \((11)\) was found by Mak and Harko\(^{23}\) under the assumption of spherical symmetry and the existence of a one-parameter group of conformal motions. It is interesting to observe that on substituting the values \(a = 0\) and \(B = 0\) for the constants, the solution \((10)\) becomes identical to that obtained by Misner and Zapolsky\(^{25}\). The physical features of the solutions \((11)\) were studied by Mak and Harko\(^{23}\) and shown to be consistent with the interior of a quark star with charged material. This corresponds to a single stable quark configuration with radius \(R = 9.46\) Km and mass \(M = 2.86M_\odot\); these figures are consistent with values obtained using numerical methods by other researchers\(^{26-28}\). Consequently our more general class of solutions is likely to produce charged quark models consistent with stellar evolution and observational data. We comment that our new class of solutions \((10)\) has a singularity in the charge density and mass density at the centre; the Mak and Harko\(^{23}\) model also shares this feature. The singularity in the charge density and mass density is physically acceptable since the total charge and mass remain finite. However our gravitational potentials \(e^{2\nu}\) and \(e^{2\lambda}\) remain finite at the centre which contrasts with the singularities in the metric functions of Mak and Harko when \(x = 0\).

5 Nonsingular quark model

Another exact solution of \((6)\) can be found with \(m = 1\) and \(n = 2\). For this case \((7)\) gives the first metric function

\[
y(x) = (a + x)^2.
\]

Equation \((8)\) becomes

\[
\dot{Z} + \left[\frac{1}{2x} + \frac{2}{a + x} + \frac{3}{a + 3x}\right] Z = \frac{(1 - \frac{2B}{C}x)(a + x)}{2x(a + 3x)}
\]

which can be integrated to give the second metric function

\[
Z = \frac{9(35a^3 + 35a^2x + 21ax^2 + 5x^3) - \frac{2B}{C}x(105a^3 + 189a^2x + 135ax^2 + 35x^3)}{315(a + x)^2(a + 3x)}
\]
Therefore we can find the exact analytical model

\[ e^{2\nu} = A^2(a + x)^4 \]  

(13a)

\[ e^{2\lambda} = \frac{315(a + x)^2(a + 3x)}{9(35a^3 + 35a^2x + 21ax^2 + 5x^3) - \frac{2B}{C}x(105a^3 + 189a^2x + 135ax^2 + 35x^3)} \]  

(13b)

\[ \rho = g(x) + \frac{2B[3(35a^5 + 133a^4x + 246a^3x^2) + 5(254a^2x^3 + 209ax^4 + 63x^5)]}{105(a + x)^3(a + 3x)^2} \]  

(13c)

\[ 3p = g(x) - \frac{2B[3(35a^5 + 497a^4x + 1854a^3x^2) + 5(1678a^2x^3 + 1177ax^4 + 315x^5)]}{105(a + x)^3(a + 3x)^2} \]  

(13d)

\[ E^2 = \frac{4x[9C(49a^3 + 363a^2x + 339ax^2 + 105x^3) - 2B(21a^4 + 162a^3x + 816a^2x^2 + 910ax^3 + 315x^4)]}{315(a + x)^3(a + 3x)^2} \]  

(13e)

Again for simplicity we have set

\[ g(x) = \frac{6C(70a^4 + 217a^3x + 159a^2x^2 + 75ax^3 + 15x^4)}{35(a + x)^3(a + 3x)^2} \]

The exact model (13) satisfies the Einstein-Maxwell system (6) and constitutes a new family of analytical solutions for a quark star with charged material. The gravitational potentials \( e^{2\nu} \) and \( e^{2\lambda} \) in (13) have the advantage of having a simple analytic form, and they are written in terms of polynomials and rational functions. Consequently the matter variables and the electric field intensity have a simple analytic representation. The function \( e^{2\nu} \) is continuous and well behaved in the interior and finite at the centre \( x = 0 \). The function \( e^{2\lambda} \) is well behaved and has a constant value at the centre \( x = 0 \). The energy density \( \rho \) is positive throughout the interior, regular at the centre with value \( \rho_0 = 2\left(\frac{6C}{a} + B\right) \). The pressure \( p \) is regular at the centre with value \( p_0 = 2\left(\frac{2C}{a} - \frac{B}{3}\right) = \frac{1}{3}(\rho_0 - 4B) \). The electric field intensity \( E \) is continuous in the interior and vanishes at the centre. Hence the matter variables and gravitational potentials comply with usual conditions for a stellar source. The finiteness of \( e^{2\nu} \), \( e^{2\lambda} \), \( \rho \), \( p \) and \( E \) at the origin \( x = 0 \) is a very welcome feature which is absent in the previous class of solutions. Consequently the exact solutions (13) are likely to produce charged quark stars with physically acceptable interiors. A recent attempt in this direction is the strange star model of Jotania and Tikekar admitting compact configurations with mass to size ratio consistent with strange matter.
6 Conclusion

We have generated a new category of exact solutions to the Einstein-Maxwell system of equations. The linear equation of state (1) was imposed which is relevant in the description of quark stars. Two classes of exact solutions were identified by specifying the form of one of the gravitational potentials. The first class comprises the generalised Mak-Harko model as the earlier solution of Mak and Harko is regained as a special case. The second class comprises the nonsingular quark model which has finite values for both the matter and metric variables at the stellar centre. The method in this paper depends crucially on the choice (7) for the metric potential $Z$ which leads to solutions of the condition of pressure isotropy. In future work it would be desirable to seek physically reasonable solutions, with new forms of $Z$, which are consistent with the relationship (1). We also intend to study more closely the physical features of quark stars, from the exact solutions generated, and relate these to specific astronomical objects following the treatment of Jotania and Tikekar.

In conclusion, we make two further points that are of significance in the modelling process. Firstly, in the general solutions (10) and (13), when studying models of charged spheres, we should consider only those values of parameters for which the energy density $\rho$, the pressure $p$ and the electric field intensity $E^2$ are positive. Clearly a wide range of charged spheres, with nonsingular potentials and matter variables, are possible for relevant choices of $a, B$ and $C$. Secondly, the interior metric (2) must match to the Reissner-Nordstrom exterior spacetime

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)$$

across the boundary of the star. This generate relationships between the mass $M$, the charge $Q$, and the constants $a, B$ and $C$. Consequently the continuity of the metric coefficients across the boundary of the star is easily achieved. There are sufficient free parameters available to satisfy the necessary conditions that may arise from a particular physical model under consideration.

Acknowledgements

KK thanks the National Research Foundation and the University of KwaZulu-Natal for financial support, and also extends his appreciation to the South Eastern University of Sri Lanka for granting study leave. SDM acknowledges that this work is based upon research supported by the South African Research Chair Initiative of the Department of Science and Technology and the National Research Foundation.
References

[1] N. Itoh, *Prog. Theor. Phys.* **44**, 291 (1970).

[2] A. R. Bodmer, *Phys. Rev. D* **4**, 1601 (1971).

[3] K. S. Cheng, Z. G. Dai and T. Lu, *Int. J. Mod. Phys. D* **7**, 139 (1998).

[4] M. Alford, *Ann. Rev. Nucl. Part. Sci.* **51**, 131 (2001).

[5] A. Chodos, R. L. Jaffe, K. Johnson, C. B. Thorn and V. F. Weisskopf, *Phys. Rev. D* **9**, 3471 (1974).

[6] E. Farhi and R. L. Jaffe, *Phys. Rev. D* **30**, 2379 (1984).

[7] E. Witten, *Phys. Rev. D* **30**, 272 (1984).

[8] I. Bombaci, *Phys. Rev. C* **55**, 1587 (1997).

[9] X.-D Li, Z.-G Dai and Z.-R Wang, *Astron. Astrophys.* **303**, L1 (1995).

[10] M. Dey, I. Bombaci, J. Dey, S. Ray and B. C. Samanta, *Phys. Lett. B* **438**, 123 (1998).

[11] X.-D. Li, I. Bombaci, M. Dey, J. Dey and E. P. J. van den Heuvel, *Phys. Rev. Lett.* **83**, 3776 (1999).

[12] X.-D. Li, S. Ray, J. Dey, M. Dey and I. Bombaci, *Astrophys. J.* **527**, L51 (1999).

[13] R. X. Xu, G. J. Qiao and B. Zhang, *Astrophys. J.* **522**, L109 (1999).

[14] R. X. Xu, X. B. Xu and X. J. Wu, *Chin. Phys. Lett.* **18**, 837 (2001).

[15] J. A. Pons, F. M. Walter, J. M. Lattimer, M. Prakash, R. Neuhauser and A. Penghui, *Astrophys. J.* **564**, 981 (2002).

[16] V. V. Usov, *Phys. Rev. D* **70**, 067301 (2004).

[17] J. J. Drake, H. L. Marshall, S. Dreizler, P. E. Freeman, A. Fruscione, M. Juda, V. Kashyap, F. Nicastro, D. O. Pease, B. J. Wargelin and K. Werner, *Astrophys. J.* **572**, 996 (2002).

[18] H. Sotani, K. Kohri and T. Harada, *Phys. Rev. D* **69**, 084008 (2004).

[19] T. Harko and K. S. Cheng, *Phys. Lett. A* **266**, 249 (2000).

[20] I. Yilmaz and H. Baysal, *Int. J. Mod. Phys. D* **14**, 697 (2005).
[21] M. K. Mak and T. Harko, *Chin. J. Astron. Astrophys.* **2**, 248 (2002).

[22] R. Sharma and S. D. Maharaj, *Mon. Not. R. Astron. Soc.* **375**, 1265 (2007).

[23] M. K. Mak and T. Harko, *Int. J. Mod. Phys. D* **13**, 149 (2004).

[24] M. C. Durgapal and R. Bannerji, *Phys. Rev. D* **27**, 328 (1983).

[25] C. W. Misner and H. S. Zapolsky, *Phys. Rev. Lett.* **12**, 635 (1964).

[26] P. Haensel, J. L. Zdunik and R. Schaefer, *Astron. Astrophys.* **160**, 121 (1986).

[27] P. Haensel and J. L. Zdunik, *Nature.* **340**, 617 (1989).

[28] E. Gourgoulhon, P. Haensel, R. Livirne, E. Paluch, S. Bonazzola and J. A. Marck, *Astron. Astrophys.* **349**, 851 (1999).

[29] K. Jotania and R. Tikekar, *Int. J. Mod. Phys. D* **15**, 1175 (2006).