Concentration-compactness principle for
mountain pass problems

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Abstract
In the paper we show that critical sequences associated with the
mountain pass level for semilinear elliptic problems on \( \mathbb{R}^N \) converge
when the non-linearity is subcritical, superlinear and satisfies the
penalty condition \( F_\infty(s) < F(x, s) \). The proof suggests a concen-
tration compactness framework for minimax problems, similar to that
of P.-L.Lions for constrained minima.

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1 Introduction
In this paper we prove an existence result for the classical semilinear elliptic
problem on \( \mathbb{R}^N \):

\[
-\Delta u + u = f(x, u), \quad u \in H^1(\mathbb{R}^N).
\]  

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The classical existence proof for the analogous Dirichlet problem on bounded domain, based on the mountain pass lemma of [1], fails in the case of $\mathbb{R}^N$, since the Palais-Smale condition does not anymore follow from compactness of Sobolev imbeddings and a concentration compactness argument is needed. There are numerous publications where the concentration compactness is used in minimax problems, including the problem considered here (a representative bibliography on the subject can be found in the books of Chabrowski [4] and Willem [14]), but given that in problems on $\mathbb{R}^N$ the $(PS)_c$ condition fails, typically, for every $c$ that is a linear combination, with positive integer coefficients, of critical values, the Palais-Smale condition has been proved only with severe restrictions on the nonlinearity $f(x,s)$.

We consider here a set of conditions on the functional, similar to the concentration compactness framework as set by P.-L.Lions ([6],[7],[8],[9]), where conditions for existence of minima can be formulated as the following prototype assumptions: a) the functionals are continuous, b) critical sequences are bounded (achieved by regarding constrained minima), c) constrained minimal values are subadditive with respect to the parameter of constraint level, and d) the functionals are invariant relative to transformations causing the loss of compactness or their asymptotic values (with respect to the unbounded sequences of the transformations) satisfy a penalty condition.

In the present paper we consider a functional $G$ on a Hilbert space $H$ with an asymptotic (with respect to unbounded sequence of transformations responsible for loss of compactness) value $G_\infty$. Let $\Phi$ be an appropriate set of mappings from a metric space $X$ into $H$ with fixed values on a $X_0 \subset X$, and let $\rho := \sup G(\varphi(X_0)) < c := \inf_{\varphi \in \Phi} \sup G(\varphi(X))$. We regard the following heuristic conditions, whose formal counterparts for the functional associated with (1.1) will are given in Section 2.

d') $G \in C^1(H)$ (in the semilinear elliptic case, subcritical growth of $f$);
b') critical sequences at the level $c$ are bounded (in the semilinear elliptic case follows from an assumption of superlinearity for $f$);

c') all critical points of $G_\infty$ with critical values in $(\rho, c]$ have the Morse index greater or equal to the one associated with the minimax (a weaker version: for every critical point $w$ of $G_\infty$ such that $\rho < G_\infty(w) \leq c$ there is a sequence of paths $\varphi_k$ such that $d(w, \varphi_k(X)) \to 0$ and $\sup_{x \in X} G_\infty(\varphi_k(x)) \to G(w)$) - in the semilinear elliptic case with a mountain pass, the sufficient condition is $s \mapsto f_\infty(s)/|s|$ monotone increasing; and

d') invariance or penalty condition ($f(x,s) = f(s)$ or $F(x,s) > F_\infty(s)$)
where \( F(x, t) = \int_0^t f(x, s)ds \).

Existence of critical points is proved by verifying (PS)\(_c\) for a single value \( c \), namely the one given by the mountain pass statement. Sharp estimates of \( c \) are based on the global compactness theorem by I. Schindler and the author ([2]), which is a functional-analytic generalization of earlier "multi-bump" weak convergence lemmas (Struwe, [13]; Lions, [10]; Cao and Peng [3].

2 Existence theorem

We consider the Hilbert space \( H^1(\mathbb{R}^N), N \in \mathbb{N} \), defined as the completion of \( C_c^\infty(\mathbb{R}^N) \) with respect to the norm

\[
\|u\|^2 = \int_{\mathbb{R}^N} |\nabla u|^2 + u^2.
\]

(2.1)

In what follows the notation of norm without other specification will refer to this \( H^1 \)-norm. The space \( H^1(\mathbb{R}^N) \) is continuously imbedded into \( L^p(\mathbb{R}^N) \) for \( 2 \leq p \leq \frac{2N}{N-2} \) when \( N > 2 \) and for \( p \geq 2 \) for \( N = 1, 2 \). For convenience we set \( 2^* = \frac{2N}{N-2} \) for \( N > 2 \) and \( 2^* = \infty \) for \( N = 1, 2 \). Let \( f : \mathbb{R}^N \times \mathbb{R} \) be continuous function and let

\[
F(x, t) = \int_0^t f(x, s)ds,
\]

(2.2)

\[
g(u) = \int_{\mathbb{R}^N} F(x, u(x))dx,
\]

(2.3)

and

\[
G(u) = \frac{1}{2}\|u\|^2 - g(u).
\]

(2.4)

We assume that \( f(x, s) \to f_\infty(s) \) as \( |x| \to \infty \) and follow the definitions above to define \( F_\infty, g_\infty \) and \( G_\infty \).

Let \( \omega \subset H^1(\mathbb{R}^N) \) be a path-connected component of \( G^{-1}(\infty, -1) \) containing infinity, that is, the union of all path-connected subsets \( \omega' \subset G^{-1}(\infty, -1) \), such that, for any \( R > 0 \) every point of \( \omega' \) can be connected by an arc in \( G^{-1}(\infty, -1) \) to a point \( e \) with \( \|e\| > R \). Let

\[
\Phi = \{ \varphi \in C([0, 1]; H^1(\mathbb{R}^N)) : \varphi(0) = 0, \varphi(1) \in \omega \}.
\]

(2.5)
Theorem 2.1. Assume that:

for every $\epsilon > 0$ there exist $p_\epsilon \in (2, 2^*)$ and $C_\epsilon > 0$ such that

(A) $|f(x,s)| \leq \epsilon (|s| + |s|^{2^*-1}) + C_\epsilon |s|^{p_\epsilon - 1}$, $s \in \mathbb{R}, x \in \mathbb{R}^N$;

There exists a $\mu > 2$, such that

(B) $f(x,s) \geq \mu F(x,s)$, $s \in \mathbb{R}, x \in \mathbb{R}^N$;

(C) $s \mapsto f_\infty(s)/|s|$, $s \in \mathbb{R}$, is increasing;

(D) $F(x,s) > F_\infty(s)$, $s \in \mathbb{R} \setminus \{0\}, x \in \mathbb{R}^N$.

Then $\Phi \neq \emptyset$;

$$c := \inf_{\varphi \in \Phi} \max_{t \in [0, \infty)} G(\varphi(t)) > 0; \quad (2.6)$$

there is a sequence $u_k \in H^1(\mathbb{R}^N)$ such that $G'(u_k) \to 0$, $G(u_k) \to c$; every such sequence has a subsequence convergent in $H^1(\mathbb{R}^N)$. Consequently, $u = \lim u_k$ satisfies $G(u) = c$ and $G'(u) = 0$ (and therefore, $u$ is a solution of (1.1)).

Condition (A) is a well-known sufficient condition for $G \in C^1(H^1(\mathbb{R}^N))$.

Lemma 2.2. Let $G$ be as in (2.4). Assume conditions (A) and (C) of Theorem 2.1. Then for every $w \in H^1(\mathbb{R}^N) \setminus \{0\}$, the path $\varphi(t) = tw$, $t \in (0, \infty)$, is in $\Phi$ and the constant (2.4) is positive. If, in addition, $G'(w) = 0$, then $\max_t G_\infty(\varphi(t))$ is attained at $\varphi(1) = w$.

Proof. The first assertion of the lemma follows easily from (C) and the second is a consequence of (A) (the proof is a trivial modification of the one in [1]).

Let $w \neq 0$ satisfy $G'(w) = 0$. From (C) follows that the function $s \mapsto s^{-\frac{d}{ds}}(G_\infty(sw^{(n)}))$ is decreasing on $(0, \infty)$. Then, since

$$\frac{d}{ds} G_\infty(sw) = s\|w\|^2 - \int f_\infty(sw)w$$

$$= s\left(\|w\|^2 - \int \frac{f_\infty(sw)}{sw} w^2 dx\right),$$

4
the function \( s \mapsto \gamma(s) := G_\infty(sw) \) has at most one critical point. Since \( \gamma(0) = 0, \gamma(s) < 0 \) for \( s \) large and has positive values (because \( c > 0 \)), the critical point of \( \gamma \) is a point of maximum. Since \( G'(w) = 0, (G'(w), w) = 0 \), which is equivalent to \( \gamma'(1) = 0. \) Since \( \gamma(s) \) has a unique critical point, which is a point of maximum, \( s \mapsto G(sw) \) attains its maximum at \( s = 1. \) □

3 Global compactness

In this section we present statements from [12] concerning weak convergence that will be used in the proof.

**Theorem 3.1.** Let \( u_k \in H \) be a bounded sequence. Then there exists \( w^{(n)} \in H, g_k^{(n)} \in D, k, n \in \mathbb{N} \) such that for a renumbered subsequence

\[
\begin{align*}
g_k^{(1)} &= \text{id}, \quad g_k^{(n)} g_k^{(m)} \to 0 \text{ for } n \neq m, \\
w^{(n)} &= \text{w-lim} \ g_k^{(n)} u_k \\
\sum_{n \in \mathbb{N}} \|w^{(n)}\|^2 &\leq \limsup \|u_k\|^2 \\
\|u_k - \sum_{n \in \mathbb{N}} g_k^{(n)} w^{(n)}\|_D &\to 0.
\end{align*}
\]

(3.1)

(3.2)

(3.3)

(3.4)

In particular \( u \mapsto u(\cdot - y), y \in \mathbb{Z}^N, \) form a dislocation group in \( H_0^1(\mathbb{R}^N), \) and \( u_k \overset{D}{\rightharpoonup} 0 \) is equivalent to \( u_k \to 0 \) in \( L^p(\mathbb{R}^N), \) \( p \in (2, 2^*) \) (an equivalent statement is found in [3]).

The following lemma is similar to the Brezis-Lieb lemma from [2] and is a trivial modification of analogous lemma from [12].

**Lemma 3.2.** Assume that \( F \) satisfies (A) and that \( u_k \) and \( (w^{(n)}) \) are as in Theorem 3.1. Then

\[
\int F(x,u_k) \to \int F(x,w^{(1)}) + \sum_{n \geq 2} \int F_\infty(w^{(n)}).
\]

(3.5)

4 Proof of Theorem 2.1

Step 1. By Lemma 2.2 \( \Phi \neq \emptyset \) and \( c > 0. \) By (A), (2.6) and the mountain pass lemma ([1]), there is a sequence \( u_k \) such that \( G'(u_k) \to 0 \) and \( G(u_k) \to c. \)
By (B), \( u_k \) is bounded in \( H^1 \) (see, again, the argument of [1]) and we can apply Theorem 3.1, referring in what follows to the renamed subsequence.

By (3.5) and (3.3),

\[
c \geq \frac{1}{2} \sum_{n \in \mathbb{N}} \| w^{(n)} \|^2 - \int F(x, w^{(1)}) - \sum_{n \geq 2} \int F_{\infty}(w^{(n)}). \tag{4.1}
\]

From \( G'(u_k) \to 0 \) (since (A), by compactness of local imbeddings of \( H^1 \) into \( L^p \) implies weak convergence of \( g'(u_k) \) for weakly convergent \( u_k \)) follows

\[
\| w^{(1)} \|^2 = \int f(x, w^{(1)})w^{(1)} , \| w^{(n)} \|^2 = \int f_{\infty}(w^{(n)})w^{(n)} \text{ for } n \geq 2. \tag{4.2}
\]

Substituting (4.2) into (4.1) we get

\[
c \geq \int \left( \frac{1}{2} f(x, w^{(1)})w^{(1)} - F(x, w^{(1)}) \right) + \sum_{n \geq 2} \int \left( \frac{1}{2} f_{\infty}(w^{(n)})w^{(n)} - F_{\infty}(w^{(n)}) \right). \tag{4.3}
\]

Step 2. Note that

\[
\frac{1}{2} f(x, s)s - F(x, s) > 0 \quad \text{and} \quad \frac{1}{2} f_{\infty}(s)s - F_{\infty}(s) > 0, s \neq 0. \tag{4.4}
\]

The first relation follows from (B):

\[
\frac{1}{2} f(x, s)s - F(x, s) \geq \left( \frac{\mu}{2} - 1 \right) F(x, s), \tag{4.5}
\]

\( F(x, s) > F_{\infty}(s) \) by (D) and \( F_{\infty}(s) > 0 \) for \( s \neq 0 \) due to (A). The second relation follows going to the limit in (4.5) as \( |x| \to \infty \) and using positivity of \( F_{\infty}(s) \).

Step 3. Assume that

\[
w^{(n)} \neq 0 \text{ for some } n \neq 1. \tag{4.6}
\]

Let us estimate \( c \) from above by choosing paths \( s \mapsto sw^{(n)}(\cdot - y_k) \in H^1(\mathbb{R}^N) \) with \( y_k \in \mathbb{Z}^N, |y_k| \to \infty \). Then

\[
c \leq \sup_{s \in (0, \infty)} G(sw^{(n)}(\cdot - y_k)). \tag{4.7}
\]
By taking $k \to \infty$ we have
\[ c \leq \sup_{s \in (0, \infty)} G_{\infty}(sw^{(n)}). \]  
(4.8)

By Lemma 2.2, \( \sup_{s \in (0, \infty)} G_{\infty}(sw^{(n)}) = G_{\infty}(w^{(n)}) \), and, therefore,
\[ c \leq G_{\infty}(w^{(n)}). \]  
(4.9)

Comparing this with (4.3), we see, due to (4.4), that for \( m \neq n \), \( w^{(m)} = 0 \) with necessity and therefore
\[ c = G_{\infty}(w^{(n)}). \]  
(4.10)

This is clearly false: consider a path \( s \mapsto sw^{(n)} \). Then by (D), \( \sup_{s} G(sw^{(n)}) < \sup_{s} G_{\infty}(sw^{(n)}) = G_{\infty}(w^{(n)}) = c \), which contradicts the definition of \( c \).

We conclude that the assumption (4.6) is false and \( w^{(n)} = 0 \) for all \( n \neq 1 \).

Step 4. We conclude from Step 4 and (3.4) that \( u^{k} \to w^{(1)} \) in \( L^{r} \) for any \( r \in (2, 2^{*}) \). Then from (A) follows \( g'(u^{k}) \to g'(w^{(1)}) \), and, since \( u^{k} - g'(u^{k}) \to 0 \), \( u^{k} \) is a convergent sequence in \( H^{1}(\mathbb{R}^{N}) \). We conclude that \( u^{k} \to w^{(1)} \) in \( H^{1}(\mathbb{R}^{N}) \). By continuity, \( G'(w^{(1)}) = 0 \) and \( G(w^{(1)}) = c \).

References

[1] Ambrosetti A., Rabinowitz P.H., Dual variational methods in critical point theory and applications. J. Functional Analysis 14 (1973), 349–381.

[2] Brezis, H., Lieb, E., A relation between pointwise convergence of functions and convergence of functionals, Proc. Amer. Math. Soc. 88 (1983), 486-490.

[3] Cao, D., Peng, S., A global compactness result for singular elliptic problems involving critical Sobolev exponent, Proc. Amer. Math. Soc. 131 (2003), 1857-1866.

[4] Chabrowski J., Weak convergence methods for semilinear elliptic equations. World Scientific Publishing Co., Inc., River Edge, NJ, 1999.

[5] Lieb, E., On the lowest eigenvalue of the Laplacian for the intersection of two domains. Invent. Math. 74, 441-448 (1983).
[6] Lions P.-L., The concentration-compactness principle in the calculus of variations. The locally compact case, part 1. Ann.Inst.H.Poincare, Analyse non linéaire 1, 109-1453 (1984)

[7] Lions P.-L., The concentration-compactness principle in the calculus of variations. The locally compact case, part 2. Ann.Inst.H.Poincare, Analyse non linéaire 1, 223-283 (1984)

[8] Lions P.-L., The concentration-compactness principle in the calculus of variations. The Limit Case, Revista Matematica Iberoamericana, Part 1, 1.1 145-201 (1985)

[9] Lions P.-L., The concentration-compactness principle in the calculus of variations. The Limit Case, Revista Matematica Iberoamericana, Part 2, 1.2 45-121 (1985)

[10] Lions P.-L., Solutions of Hartree-Fock equations for Coulomb systems, Comm.Math.Phys. 109, 33-97 (1987).

[11] Del Pino M., Felmer P., Least energy solutions for elliptic equations in unbounded domains, Proc. Royal Soc. Edinburgh 126A, 195-208 (1996)

[12] Schindler I., Tintarev K., An abstract version of the concentration compactness principle, Revista Mat.Complutense 15, 1-20 (2002).

[13] Struwe, M. A global compactness result for elliptic boundary value problems involving limiting nonlinearities, Math. Z. 187 (1984), 511-517.

[14] Willem M., Minimax theorems. Progress in Nonlinear Differential Equations and their Applications, 24. Birkhäuser Boston, Inc., Boston, MA, 1996.