Matrix group actions on product of spheres and Zimmer’s program

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Abstract

Let $\text{SL}_n(\mathbb{Z})$ be the special linear group over integers and $M^r = S^{r_1} \times S^{r_2}, T^{r_1} \times S^{r_2}$, or $T^{r_0} \times S^{r_1} \times S^{r_2}$, products of spheres and tori. We prove that any group action of $\text{SL}_n(\mathbb{Z})$ on $M^r$ by diffeomorphisms or piecewise linear homeomorphisms is trivial if $r < n - 1$. This confirms a conjecture on Zimmer’s program for these manifolds.

1 Introduction

The special linear group $\text{SL}_n(\mathbb{Z})$ acts on the Euclidean space $\mathbb{R}^n$ by linear transformations. There is an induced action on the sphere $S^{n-1} (\subseteq \mathbb{R}^n)$ by $x \mapsto Ax/\|x\|$ for $x \in S^{n-1}$ and $A \in \text{SL}_n(\mathbb{Z})$. It is believed that this action is minimal in the following sense.

**Conjecture 1.1** Any action of a finite-index subgroup of $\text{SL}_n(\mathbb{Z})$ ($n \geq 3$) on a compact $r$-manifold by diffeomorphisms factors through a finite group action if $r < n - 1$.

This conjecture was formulated by Farb and Shalen [9], which is related to Zimmer’s program [29]. Conjecture 1.1 could be a special case of a general conjecture in Zimmer’s program, in which the special linear group is replaced by an arbitrary irreducible lattice $\Gamma$ in a semisimple Lie group $G$ of $\mathbb{R}$-rank at least 2, and the integer $n$ is replaced by a suitable integer $h(G)$. Some forms of Conjecture 1.1 has been discussed by Weinberger [25]. These conjectures
are part of a program to generalize the Margulis Superrigidity Theorem to a nonlinear context.

In general, it is difficult to prove this conjecture. Usually, we have to assume either that the group action preserves additional geometric structures or that the lattices themselves are special. The following is an incomplete list of some results in this direction. For more details of Zimmer’s program and related topics, see survey articles of Zimmer and Morris [31], Fisler [10] and Labourie [16].

When \( r = 1 \) and \( M = S^1 \), Ghys [13] and Burger-Monod [6] show that every \( C^1 \) action of a lattice with \( \mathbb{R} \)-rank \( \geq 2 \) on \( S^1 \) factors through a finite group action. Witte [23] proves that Conjecture 1.1 is true for an arithmetic lattice \( \Gamma \) with \( \mathbb{Q} \)-rank(\( \Gamma \)) \( \geq 2 \). He actually proves that the topological version is true as well.

When \( r = 2 \), the group action is smooth real-analytic and \( M \) is a compact surface other than the torus or Klein bottle, Farb and Shalen [9] prove that Conjecture 1.1 is true for \( n \geq 5 \) (more generally for 2-big lattices). When the group action is smooth real-analytic and volume-preserving, they also show that this result could be extended to all compact surfaces.

Polterovich (see Corollary 1.1.D of [20]) proves that if \( n \geq 3 \), then any action by \( \text{SL}(n,\mathbb{Z}) \) on a closed surface other than the sphere \( S^2 \) and the torus \( T^2 \) by area preserving diffeomorphisms factors through a finite group action. When \( r = 2 \) and the group action is by area preserving diffeomorphisms, Franks and Handel [12] prove that Conjecture 1.1 is true for an almost simple group containing a subgroup isomorphic to the three-dimensional integer Heisenberg group (e.g., any finite-index subgroup of \( \text{SL}_n(\mathbb{Z}) \) for \( n \geq 3 \)). When \( M \) is a compact surface with nonempty boundary, Ye [27] proves Conjecture 1.1 is true for \( \text{SL}_n(\mathbb{Z}) \) (\( n \geq 5 \)). The conjecture is still open for general smooth actions on closed 2-dimensional manifolds.

Bridson and Vogtmann [2] prove Conjecture 1.1 for \( \text{SL}_n(\mathbb{Z}) \) and \( M = S^r \), the sphere of dimension \( r \). Weinberger [24] shows that when \( r < n \), any group action of \( \text{SL}_n(\mathbb{Z}) \) on the lower dimensional torus \( T^r \) by diffeomorphisms is trivial, i.e., Conjecture 1.1 holds for the group \( \text{SL}_n(\mathbb{Z}) \) itself and \( M = T^r \). The aim of this article is to provide more manifolds satisfying Conjecture 1.1.

Our first result is to prove Conjecture 1.1 for manifolds with few Betti numbers.

**Theorem 1.2** Let \( X \) be an \( r \)-dimensional (\( r \geq 1 \)) orientable manifold with
Betti numbers $b_i(X; \mathbb{Z}/2)$. Suppose that $\sum_{i=0}^r b_i(X; \mathbb{Z}/2) \leq 4$. When $r < n-1$, any group action of $\text{SL}_n(\mathbb{Z})$ on $X$ is trivial if the group action is by

(i) diffeomorphisms; or

(ii) piecewise linear (PL) homeomorphisms; or

(iii) homeomorphisms and $r \leq 4$.

When the group action is by diffeomorphisms (piecewise linear homeomorphisms, resp.), we always assume that $X$ is smooth (piecewise linear, resp.). The proof of Theorem 1.2 is to do induction on the dimension $r$ by studying the action of involutions on $X$. The assumptions of various actions are to make sure that the fixed point set of an orientation-preserving involution is a manifold and is of even codimension.

By Theorem 1.2 Conjecture 1.1 holds for actions of $\text{SL}_n(\mathbb{Z})$ on manifolds $X$ of the following type:

- $X = S^r$, the sphere, or $\mathbb{R}^r$, the Euclidean space. This is first proved by Bridson and Vogtmann [2].
- $X = S^{r_1} \times S^{r_2}$, the product of two spheres.
- $X = \mathbb{R}^{r_1} \times S^{r_2}$, the product of a sphere and an Euclidean space.
- $X = \mathbb{R}^{r_0} \times S^{r_1} \times S^{r_2}$, the product of two spheres and an Euclidean space.

We study matrix group actions on manifolds with few Betti numbers by homeomorphisms. For such topological actions, we have the following.

**Theorem 1.3** Let $X$ be an $r$-dimensional ($r \geq 1$) homology manifold over $\mathbb{Z}/p$ ($p$ a prime) with Betti numbers $b_i(X; \mathbb{Z}/p)$. Suppose that $\sum_{i=0}^r b_i(X; \mathbb{Z}/p) \leq 4$. We have the following.

(i) Assume $p = 2$. When $r < n-3$ if $n$ is odd, or $r < n-4$ if $n$ is even, any group action of $\text{SL}_n(\mathbb{Z})$ on $X$ by homeomorphisms is trivial.

(ii) Assume $p$ is odd. When $r < n-2$ if $n$ is even, or $r < n-3$ if $n$ is odd, any group action of $\text{SL}_n(\mathbb{Z})$ on $X$ by homeomorphisms is trivial.
(iii) When $p = 2$ and $r < n - 3$, any group action of $\text{SL}_n(\mathbb{Z}/2)$ on $X$ by homeomorphisms is trivial. When $p$ is odd and $r < 2n - 4$, any group action of $\text{SL}_n(\mathbb{Z}/p)$ on $X$ by homeomorphisms is trivial.

We now study the inheritance of Conjecture 1.1 for covering spaces. We obtain the following result.

**Theorem 1.4** Let $p : M' \to M$ be a universal covering of connected manifolds with the finitely generated abelian group $\mathbb{Z}^k$ as the deck transformation group. Suppose that

(i) any group action of $\text{SL}_n(\mathbb{Z})$ on $M'$ by homeomorphisms (resp. diffeomorphisms, PL homeomorphisms) is trivial;

(ii) $k \leq n - 1$.

Then any group action of $\text{SL}_n(\mathbb{Z})$ on $M$ by homeomorphisms (resp. diffeomorphisms, PL homeomorphisms) is trivial.

The proof of Theorem 1.4 is to lift the action on $M$ to the covering space $M'$, using cohomology of groups.

The linear group action of $\text{SL}_n(\mathbb{Z})$ on the Euclidean space $\mathbb{R}^n$ induces an action on the $n$-dimensional torus $T^n = \mathbb{R}^n/\mathbb{Z}^n$. Weinberger [24] shows that when $r < n$, any group action of $\text{SL}_n(\mathbb{Z})$ on the lower dimensional torus $T^r$ by diffeomorphisms is trivial, i.e., Conjecture 1.1 holds for the group $\text{SL}_n(\mathbb{Z})$ itself and $M = T^r$. As an application of Theorem 1.4, we obtain a topological version of Weinberger’s result, as a special case of the following (note that Weinberger [24] also outlines a proof of the topological version).

**Corollary 1.5** We have the following.

(i) When $r < n - 1$, any group action of $\text{SL}_n(\mathbb{Z})$ on the $r$-dimensional torus $T^r$ by homeomorphisms is trivial.

(ii) Let $M = T^{r_1} \times S^{r_2}$ be the product of a torus and a sphere of dimension $r = r_1 + r_2$, or $M = T^{r_0} \times S^{r_1} \times S^{r_2}$ be the product of a torus and a sphere of dimension $r = r_0 + r_1 + r_2$. When $r < n - 1$, any group action of $\text{SL}_n(\mathbb{Z})$ on $M$ by diffeomorphisms or PL homeomorphisms is trivial.
(iii) Let $M = S^2 \times S^2$, or $M = T^{r_1} \times S^{r_2}$ be the product of a torus and a sphere of dimension $r = r_1 + r_2 \leq 4$ (e.g. $M = T^2 \times S^2, S^1 \times S^3$). When $r < n - 1$, any group action of $\text{SL}_n(\mathbb{Z})$ on $M$ by homeomorphisms is trivial.

Corollary 1.5 shows that Conjecture 1.1 holds for the group $\text{SL}_n(\mathbb{Z})$ itself and $M = T^{r_1} \times S^{r_2}$, the product of a torus, or $M = T^{r_0} \times S^{r_1} \times S^{r_2}$, the product of a torus and two spheres.

The article is organized as follows. In Section 2, some basic facts about homology manifolds over sheaves are introduced. In Section 3 and Section 4, Theorem 1.2 and Theorem 1.3 are proved. In the last section, we prove Theorem 1.4 and Corollary 1.5.

## 2 Homology manifolds

In this section, we introduce some basic notions on generalized manifolds, following Bredon’s book [5]. Let $L = \mathbb{Z}$ or $\mathbb{Z}/p$. All homology groups in this section are Borel-Moore homology with compact supports and coefficients in a sheaf $\mathcal{A}$ of modules over a principle ideal domain $L$. The homology groups of $X$ are denoted by $H_c^*(X; \mathcal{A})$ and the Alexander-Spanier cohomology groups (with coefficients in $L$ and compact supports) are denoted by $H^*_c(X; L)$. We define $\dim_L X = \min\{n \mid H^{n+1}_c(U; L) = 0 \text{ for all open } U \subset X\}$. If $L = \mathbb{Z}/p$, we write $\dim_p X$ for $\dim_L X$. For integer $k \geq 0$, let $\mathcal{O}_k$ denote the sheaf associated to the pre-sheaf $U \mapsto H_c^k(X, X \setminus U; L)$.

**Definition 2.1** An $n$-dimensional homology manifold over $L$ (denoted $n$-hm$_L$) is a locally compact Hausdorff space $X$ with $\dim_L X < +\infty$, and $\mathcal{O}_k(X; L) = 0$ for $k \neq n$ and $\mathcal{O}_n(X; L)$ is locally constant with stalks isomorphic to $L$. The sheaf $\mathcal{O}_n$ is called the orientation sheaf.

There is a similar notion of cohomology manifold over $L$, denoted $n$-cm$_L$ (cf. [5], p.373). In this article, we assume that the homology manifolds are second countable. The following result is generally called the Local Smith Theorem (cf. [5] Theorem 20.1, Prop 20.2, pp. 409-410).

**Lemma 2.2** Let $p$ be a prime and $L = \mathbb{Z}/p$. The fixed point set of any action of $\mathbb{Z}/p$ on an $n$-hm$_L$ is the disjoint union of (open and closed) components each of which is a $r$-hm$_L$ with $r \leq n$. If $p$ is odd then each component of the fixed point set has even codimension.
In order to prove Theorem 1.2, we need several lemmas. The $i^{th}$ modulo $p$ Betti number $b_i(X; \mathbb{Z}/p)$ of $X$ is defined as the dimension of vector space $H_i^c(X; \mathbb{Z}/p)$. If the prime $p$ is clear from the context, we simply write $b_i(X)$ instead of $b_i(X; \mathbb{Z}/p)$. If $\dim_p X = n$, we say that $X$ satisfies the modulo $p$ Poincaré duality if $b_i(X; \mathbb{Z}/p) = b_{n-i}(X; \mathbb{Z}/p)$ for all $i \geq 0$. Note that homology manifolds satisfy Poincaré duality between Borel-Moore homology and sheaf cohomology ([5], Theorem 9.2, p.329), i.e. if $X$ is an $n$-hm then

$$ H_{c}^{n-k}(X; \mathcal{O}_n) \cong H_{k}^{c}(X; L). $$

The following lemma says that an elementary $p$-group acting freely on a manifold, whose Betti numbers satisfy $\sum_{i=0}^{n} b_i(X) \leq 4$, is of rank at most 2 (cf. [17], Lemma 3.1).

**Lemma 2.3** Suppose that $X$ is a space with $\dim_p X = n$, $X$ satisfies modulo $p$ Poincaré duality, and the sum of Betti numbers $\sum_{i=0}^{n} b_i(X) \leq 4$. If $G$ is an elementary $p$-group of rank $k$ acting freely on $X$, then $k \leq 2$.

**Proof.** When the space $X$ is assumed to be compact, it is already proved in [17]. Actually, the statement holds for any general space, as follows. If $\sum_{i=0}^{n} b_i(X) = 1$, the space $X$ is acyclic; hence, $k = 0$ (cf. [5], Corollary 19.8, p.144). If $\sum_{i=0}^{n} b_i(X) = 2$, the space $X$ is a modulo $p$ cohomology $n$-sphere. A classical result of Smith [21] says that $k \leq 1$. If $\sum_{i=0}^{n} b_i(X) = 4$, then $X$ has the modulo $p$ Betti numbers of a product of two spheres and $k \leq 2$ by a result of Heller [15] (Theorem 2). If $\sum_{i=0}^{n} b_i(X) = 3$, similar techniques in [15] prove that $k \leq 2$. For example, substitution $k = r = 3$ into (3.3) of [15] leads to a contradiction. ■

The following result relates the Betti numbers, Euler characteristics of the fixed point set and those of the whole space (cf. Floyd [11]).

**Lemma 2.4** Let $p$ be a prime number and $G = \mathbb{Z}/p$ act on a cohomological manifold $X$ with the fixed point set $F$. Then

$$ \chi(F) \equiv \chi(G) \mod p, $$

and

$$ \sum b_i(F) \leq \sum b_i(X). $$
There is a relation between dimensions of fixed point set and the whole space as follows (cf. Borel [7], Theorem 4.3, p182).

**Lemma 2.5** Let $G$ be an elementary $p$-group operating on a first countable cohomology $n$-manifold $X \mod p$. Let $x \in X$ be a fixed point of $G$ on $X$ and let $n(H)$ be the cohomology dimension mod $p$ of the component of $x$ in the fixed point set of a subgroup $H$ of $G$. If $r = n(G)$, we have

$$n - r = \sum_H (n(H) - r)$$

where $H$ runs through the subgroups of $G$ of index $p$.

The following lemma is a key step in the inductive proof of Theorem 1.2 (cf. Bredon [3], Theorem 7.1).

**Lemma 2.6** Let $G$ be a group of order 2 operating effectively on an $n$-cm over $\mathbb{Z}$, with nonempty fixed points. Let $F_0$ be a connected component of the fixed point set of $G$, and $r = \dim_2 F_0$. Then $n - r$ is even (respectively odd) if and only if $G$ preserves (respectively reverses) the local orientation around some point of $F_0$.

### 3 Smooth and piecewise linear actions

In order to prove Theorem 1.2 we need several technical lemmas.

**Lemma 3.1** Let $S_n$ be the permutation group of $n$ letters and $k \geq 2$ be an integer. For any $n \leq 4$ and any group homomorphism $f : \text{SL}_k(\mathbb{Z}/2) \ltimes (\mathbb{Z}/2)^k \to S_4$, where $\text{SL}_k(\mathbb{Z}/2)$ acts on $(\mathbb{Z}/2)^k$ by matrix multiplications, there is an element $\sigma \in \text{SL}_k(\mathbb{Z}/2) \ltimes (\mathbb{Z}/2)^k$ of order two such that the fixed point set $\text{Fix}(f(\sigma)) \neq \emptyset$.

**Proof.** When $k \geq 3$, the group $\text{SL}_k(\mathbb{Z}/2)$ is perfect. Therefore, $\text{Im } f$ consist only the identity element and the claim is trivial. It is only need to prove the case when $k = 2$. For the group $\text{SL}_k(\mathbb{Z}/2)$, let

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$
Note that \( A^2 = B^2 = I \). When \( n = 3 \), \( f((A, (0, 0))) \) has order at most 2, for which we take \( \sigma = (A, (0, 0)) \). When \( n = 4 \), if \( f((A, (0, 0))) \) (or \( f(B, (0, 0))) \) is not a product of disjoint transpositions, we take \( \sigma = (A, (0, 0)) \) (or \( (B, (0, 0))) \). We suppose that both \( f((A, (0, 0))) \) and \( f((B, (0, 0))) \) are products of two disjoint transpositions when \( n = 4 \) or that both \( f((A, (0, 0))) \) and \( f((B, (0, 0))) \) are transpositions when \( n = 2 \). Then \( f((AB, (0, 0))) = \text{id} \in S_n \), since \( AB \) is of order 3 while a 3-cycle in \( S_4 \) is not a product of such transpositions. Take \( \sigma = (I_2, (1, 0)) = (AB, (0, 0))(I, (0, 1))(B^{-1}A^{-1}, (0, 0))(I_2, (0, 1)) \).

It is direct that \( f(\sigma) = \text{id} \in S_n \) for \( n = 2, 4 \).

For \( 1 \leq i \neq j \leq n \), let \( e_{ij}(1) \) denote the matrix with 1s along the diagonal and 1 in the \((i, j)\)-th position and zeros elsewhere. Let \( D < \text{SL}_n(\mathbb{Z}/p) \) be the subgroup generated by \( \{e_{ij}(1), i = 2, \ldots, n\} \).

**Lemma 3.2** Let \( n \geq 3 \). The subgroup \( D \) is isomorphic to \( (\mathbb{Z}/p)^{n-1} \) and any two nontrivial elements in \( \text{SL}_n(\mathbb{Z}/p) \) are conjugate.

**Proof.** It is obvious that \( e_{1i}(1), i = 2, \ldots, n \), commutes with each other and thus \( D \cong (\mathbb{Z}/p)^{n-1} \). Let \( A = \prod_{i=2}^{n} e_{1i}(x_i) \in D \) be any nontrivial element, where \( (x_2, x_3, \ldots, x_n) \in (\mathbb{Z}/p)^{n-1} \) is a nonzero vector. After conjugating by a permutation matrix, we may assume \( x_2 \neq 0 \) (for example, when \( x_3 \neq 0 \), the \((1, 2)\)-th entry of \( P_{23}AP_{23}^{-1} \) is nonzero. Here

\[
P_{23} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \\ \end{pmatrix}
\]

). If \( x_2 \neq 0 \), we have

\[
(\prod_{i=3}^{n} e_{2i}(x_2^{-1}x_i))^{-1} \cdot A \cdot \prod_{i=3}^{n} e_{2i}(x_2^{-1}x_i) = e_{12}(x_2).
\]

For any nonzero element \( x \in \mathbb{Z}/p \), \( e_{23}(-x_2^{-1}x)e_{12}(x_2)e_{12}(x_2^{-1}x) \) has \( x \) as its \((1, 3)\)-th entry. After permuting with \( P_{23} \), the \((1, 2)\)-entry of

\[
P_{23}e_{23}(-x_2^{-1}x)e_{12}(x_2)e_{12}(x_2^{-1}x)P_{23}^{-1}
\]
is $x$. Therefore, any nontrivial element in $D$ is conjugate to $e_{12}(1)$, by taking $x = 1$. ■

We prove Theorem 1.2 by induction on the dimension $r$. As a induction step, we prove the following result first.

**Theorem 3.3** Let $X$ be a homology manifold over $\mathbb{Z}/2$ with Betti numbers $b_i(X; \mathbb{Z}/2)$ of dimension $r \leq 2$. Suppose that $\Sigma_{i=0}^r b_i(X; \mathbb{Z}/2) \leq 4$. When $r < n - 1$, any group action of $\text{SL}_n(\mathbb{Z})$ ($n \geq 3$) on $X$ by homeomorphisms is trivial.

**Proof.** Since $r \leq 2$, $X$ is a topological manifold (cf. [5], 16.32, p.388). We prove the theorem by induction on $r$. When $r = 1$, $X$ is the disjoint union of several copies of $S^1$ and $\mathbb{R}^1$. Since $\text{SL}_n(\mathbb{Z})$ ($n \geq 3$) is perfect, the group action preserves each component. It is already known that $\text{SL}_n(\mathbb{Z})$ ($n \geq 3$) can only act trivially on $S^1$ or $\mathbb{R}^1$ (cf. Witte [23]).

Suppose $r = 2$. Let $D = \{\text{diag}(\pm 1, \pm 1, \ldots, \pm 1) < \text{SL}_n(\mathbb{Z})\} \cong (\mathbb{Z}/2)^{n-1}$. Let $A_i = \text{diag}(-1, \ldots, -1, \ldots, 1)$, where the second $-1$ is in the $(i+1, i+1)$-th position. By Lemma 2.3, the subgroup $A = \langle A_1, A_2, A_3 \rangle$ can’t act freely on $X$. There are several cases to consider.

**Case (1)** $\text{Fix}(A_1A_2A_3) = \emptyset$.

By Lemma 2.3, $\text{Fix}(A_1) \neq \emptyset$, since all other nontrivial elements in $A$ except $A_1A_2A_3$ are conjugate.

**Case (1.1)** $\dim_2 \text{Fix}(A_1) = 2$.

By the invariance of domain, $\text{Fix}(A_1) = X$. Let $N$ be the normal subgroup generated by $A_1$. Then $\text{SL}_n(\mathbb{Z})/N \cong \text{SL}_n(\mathbb{Z}/2)$ (cf. Ye [26], Proposition 3.4). This means that the group action of $\text{SL}_n(\mathbb{Z})$ factors through $\text{SL}_n(\mathbb{Z}/2)$. Let $e_{12}(1) \in \text{SL}_n(\mathbb{Z}/2)$. Since the subgroup $D$ generated by $\{e_{1i}(1), i = 2, \ldots, n\}$ is isomorphic to $(\mathbb{Z}/2)^{n-1}$. Using Lemma 2.3 once again, $A$ can’t act freely on $X$ (note that $n \geq 4$ when $r = 2$). Since any two nonzero elements in $D$ are conjugate in $\text{SL}_n(\mathbb{Z}/2)$ (cf. Lemma 3.2), the fixed point set $\text{Fix}(e_{12}(1)) \neq \emptyset$. We consider two cases.

**Case (1.1.1)** $\dim_2 \text{Fix}(e_{12}(1)) = 2$. 

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By the invariance of domain, we have $\text{Fix}(e_{12}(1)) = X$. This means that $e_{12}(1)$ acts trivially on $X$. Note that $\text{SL}_n(\mathbb{Z}/2)$ is a simple group for $n \geq 3$ and thus $e_{12}(1)$ normally generates the whole group $\text{SL}_n(\mathbb{Z}/2)$. Therefore, the group action of $\text{SL}_n(\mathbb{Z}/2)$ on $X$ is trivial.

**Case (1.1.2) $\dim_2 \text{Fix}(e_{12}(1)) < 2$.**

Since $\text{SL}_n(\mathbb{Z})$ ($n \geq 3$) is perfect, the group action on $X$ is orientation-preserving. By Lemma 2.6 $\dim_p \text{Fix}(e_{12}(1)) = 0$. Thus, $r = 2$ and $n \geq 4$. This means that $\text{Fix}(e_{12}(1))$ is a discrete set consisting of at most 4 points, considering Lemma 2.4. Note that the centralizer $C_{\text{SL}_n(\mathbb{Z})}(e_{12}(1))$ leaves $\text{Fix}(e_{12}(1))$ invariant and $C_{\text{SL}_n(\mathbb{Z})}(e_{12}(1))$ contains a copy of $\text{SL}_{n-2}(\mathbb{Z}/2) \ltimes (\mathbb{Z}/2)^{n-2}$, where $\text{SL}_{n-2}(\mathbb{Z}/2)$ acts on $(\mathbb{Z}/2)^{n-2}$ by matrix multiplications. Since the permutation group of $\text{Fix}(e_{12}(1))$ is at most $S_4$, there is an element $b \in \text{SL}_{n-2}(\mathbb{Z}/2) \ltimes (\mathbb{Z}/2)^{n-2}$ of order 2 such that $\text{Fix}(b) \cap \text{Fix}(e_{12}(1)) \neq \emptyset$, according to Lemma 3.1. If $\dim_p \text{Fix}(b) = 2$, a similar argument as Case (1.1.1) shows that the group action of $\text{SL}_n(\mathbb{Z})$ is trivial. Otherwise, $\dim_p \text{Fix}(b) = 0$ by 2.6. Let $B = \langle e_{12}(1), b \rangle \cong (\mathbb{Z}/2)^2$. Note that $\text{Fix}(B) \neq \emptyset$. Write $m = \dim_2(\text{Fix}(B))$ and $n(H) = \dim_2(\text{Fix}(H))$ for each non-trivial cyclic subgroup $H < B$. By Lemma 2.5

$$r - m = \sum_H (n(H) - m).$$

The only nonzero term in the right hand side is $n(\langle e_{12}(1)b \rangle) = \dim_p \text{Fix}(e_{12}(1)b) = 2$. A similar argument as (1.1.1) shows that the group action of $\text{SL}_n(\mathbb{Z})$ is trivial.

**Case (1.2) $\dim_2 \text{Fix}(A_1) < 2$.**

By Lemma 2.6 $\dim_2 \text{Fix}(A_1) = 0$. Considering Lemma 2.4, $\text{Fix}(A_1)$ is a discrete set consisting of at most 4 points. The centralizer $C_{\text{SL}_n(\mathbb{Z})}(A_1)$ leaves $\text{Fix}(A_1)$ invariant. The action of $C_{\text{SL}_n(\mathbb{Z})}(A_1)$ restricts to be an action of $(\mathbb{Z}/2)^3$ generated by $A_2, A_3$ and

$$a = \begin{pmatrix} 1 \\ -1 \\ I_{n-2} \end{pmatrix}.$$ 

By Lemma 2.3, a nontrivial element $c \in (\mathbb{Z}/2)^3$ acts trivially on $\text{Fix}(A_1)$. If $\dim_2 \text{Fix}(c) = 2$, we can continue the proof as Case (1.1). Suppose
\[ \dim_2 \text{Fix}(c) = 0. \] Let \( A = \langle A_1, c \rangle. \) Write \( m = \dim_2(\text{Fix}(A)) \) and \( n(H) = \dim_2(\text{Fix}(H)) \) for each non-trivial cyclic subgroup \( H < A. \) By Lemma 2.5

\[ r - m = \sum_H (n(H) - m), \]

where \( H \) runs through the nontrivial cyclic subgroups of \( G. \) Note that \( r = 2 \) and \( m = 0. \) We see that \( \dim_2 \text{Fix}(A_1c) = 2. \) By the invariance of domain, \( A_1c \) acts trivially on \( X. \) Therefore, the group action of \( \text{SL}_n(\mathbb{Z}) \) factors through \( \text{SL}_n(\mathbb{Z}/2). \) A similar argument as in Case (1.1) shows that the group action of \( \text{SL}_n(\mathbb{Z}) \) is trivial.

**Case (2)** \( \text{Fix}(A_1A_2A_3) \neq \emptyset. \)

If \( \text{Fix}(A_1) \neq \emptyset, \) we may do a similar argument as case (1) to conclude that the action of \( \text{SL}_n(\mathbb{Z}) \) is trivial. We suppose \( \text{Fix}(A_1) = \emptyset. \) There are several cases to consider.

**Case (2.1)** \( \dim_2 \text{Fix}(A_1A_2A_3) = 2. \)

By the invariance of domain, the action of \( A_1A_2A_3 \) is trivial. If \( n > 4, \) the group action of \( \text{SL}_n(\mathbb{Z}) \) factors through \( \text{SL}_n(\mathbb{Z}/2). \) (cf. Ye [26], Proposition 3.4). We could finish the proof by a similar argument as Case (1.1). If \( n = 4, \) the group action of \( \text{SL}_n(\mathbb{Z}) \) factors through \( \text{PSL}_n(\mathbb{Z}), \) the projective linear group. Let

\[ \sigma = \begin{pmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{pmatrix} \in \text{SL}_4(\mathbb{Z}). \]

Since the subgroup \( B = \langle A_1, A_2, \sigma \rangle \cong (\mathbb{Z}/2)^3 < \text{PSL}_n(\mathbb{Z}), \) we have that the fixed point set \( \text{Fix}(\sigma), \text{Fix}(A_1\sigma), \text{Fix}(A_1A_2\sigma) \) or \( \text{Fix}(A_2\sigma) \) is not empty. Without loss of generality, assume that \( \text{Fix}(\sigma) \neq \emptyset. \)

**Case (2.1.1)** \( \dim_2 \text{Fix}(\sigma) = 2. \) By the invariance of domain, \( \sigma \) acts trivially on \( X. \) Note that \( \sigma \) normally generated the whole group \( \text{SL}_n(\mathbb{Z}). \) Thus, the group action of \( \text{SL}_n(\mathbb{Z}) \) is trivial.

**Case (2.1.2)** \( \dim_2 \text{Fix}(\sigma) < 2. \)
By Lemma 2.6, dim$_2$ Fix($\sigma$) = 0. Since Fix($A_1$) = Fix($A_2$) = Fix($A_1A_2$) = $\emptyset$ and $\langle A_1, A_2 \rangle \cong (\mathbb{Z}/2)^2$ acts freely on Fix($\sigma$), the set Fix($\sigma$) consists of 4 points. Moreover, $A_1$, $A_2$ and $A_1A_2$ acts on Fix($\sigma$) by products of two disjoint transpositions. Let

$$\sigma_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \in \text{SL}_4(\mathbb{Z}/2).$$

Since $\sigma_1 \sigma = \sigma_1 \sigma_1$, $\sigma_1$ preserves Fix($\sigma$). If Fix($\sigma_1$)$\cap$Fix($\sigma$) $\neq \emptyset$, then dim$_2$ Fix($\sigma_1\sigma_1$) = 2 by Lemma 2.5. A similar argument as Case (2.1.1) shows that the group action of $\text{SL}_n(\mathbb{Z})$ is trivial. If Fix($\sigma_1$)$\cap$Fix($\sigma$) $\neq \emptyset$, then $\sigma_1$ acts on Fix($\sigma$) identically as one of $\{A_1, A_2, A_1A_2\}$. Without loss of generality, assume that $\sigma_1|_{\text{Fix}(\sigma)} = A_1|_{\text{Fix}(\sigma)}$. By Lemma 2.5, dim$_2$ Fix($A_1^{-1}\sigma_1$) = 2 and a similar argument as Case (2.1.1) shows that the group action of $\text{SL}_n(\mathbb{Z})$ is trivial.

**Case (2.2)** dim$_2$ Fix($A_1A_2A_3$) < 2.

By Lemma 2.6, dim$_p$ Fix($A_1A_2A_3$) = 0. Then the $\text{SL}_4(\mathbb{Z})$ lying in the upper left corner of $\text{SL}_n(\mathbb{Z})$ preserves Fix($A_1A_2A_3$). Since Fix($A_1A_2A_3$) consists at most 4 points and $\text{SL}_4(\mathbb{Z}) = [\text{SL}_4(\mathbb{Z}), \text{SL}_4(\mathbb{Z})]$, the group action of $\text{SL}_4(\mathbb{Z})$ on Fix($A_1A_2A_3$) is trivial. This is a contradiction to the fact that Fix($A_1$) = $\emptyset$. The proof is finished.

We now start to prove Theorem 1.2. Note that for locally finite CW-complexes, the Borel-Moore homology coincides with singular homology. The strategy of the proof is similar to that of Theorem 3.3. We need care about the fixed point sets and the sizes of matrix groups.

**Proof of Theorem 1.2** We prove the theorem by induction on $r$. By Theorem 3.3 we may suppose that the statement holds for $r \leq k - 1$. We now consider the general case for $r = k$ and $k \geq 3$. Let $A_i = \text{diag}(-1, \ldots, -1, \ldots, 1)$, where the second $-1$ is in the $(i + 1, i + 1)$-th position. By Lemma 2.3, the subgroup $\langle A_1, A_2, A_3 \rangle$ can’t act freely on $X$. There are several cases to consider.

**Case (i)** Fix($A_1A_2A_3$) = $\emptyset$.

We have Fix($A_1$) $\neq \emptyset$, since all nontrivial element except $A_1A_2A_3$ in $\langle A_1, A_2, A_3 \rangle$ are conjugate.
Case (i.1) \( \dim_2 \text{Fix}(A_1) = k. \)

The group action of \( \text{SL}_n(\mathbb{Z}) \) factors through \( \text{SL}_n(\mathbb{Z}/2) \). Similar argument as in Case (1.1) of the proof of Theorem 3.3 shows that the group action of \( \text{SL}_n(\mathbb{Z}/2) \) is trivial.

Case (i.2) \( \dim_2 \text{Fix}(A_1) < k. \)

When the group action of \( \text{SL}_n(\mathbb{Z}) \) is by diffeomorphisms or piecewise linear homeomorphisms, the fixed point set \( \text{Fix}(A_1) \) is a manifold. Since \( \text{SL}_n(\mathbb{Z}) \) \((n \geq 3)\) is perfect, i.e. \( \text{SL}_n(\mathbb{Z}) = [\text{SL}_n(\mathbb{Z}), \text{SL}_n(\mathbb{Z})] \), the group action is orientation-preserving. When the action is by homeomorphisms and \( r \leq 4 \), the fix point set \( \text{Fix}(A_1) \) is of codimension at least 2 by Lemma 2.6. Then \( \text{Fix}(A_1) \) is a manifold as well (cf. [5], 16.32, p388). Therefore, \( \text{Fix}(A_1) \) is of codimension at least 2 in any case by Lemma 2.6. We will use this fact several times in the proof. Considering Lemma 2.4, \( \text{Fix}(A_1) \) consists of at most 4 components. If \( \text{Fix}(A_1) \) has 4 components, then each component is an acyclic homology manifold over \( \mathbb{Z}/2 \) by considering the Betti numbers. Note that when \( k \geq 3 \), we have \( n \geq 5 \) and \( \text{SL}_{n-2}(\mathbb{Z}) \) is a perfect group. The \( \text{SL}_{n-2}(\mathbb{Z}) \) in the centralizer \( C_{\text{SL}_n(\mathbb{Z})}(A_1) \) will preserve each component. By induction, the group action of \( \text{SL}_{n-2}(\mathbb{Z}) \) on \( \text{Fix}(A_1) \) is trivial. Therefore, \( c = \text{diag}(1, 1, -1, -1, \ldots, 1) \) acts trivially on \( \text{Fix}(A_1) \). Let \( A = \langle A_1, c \rangle \cong (\mathbb{Z}/2)^2 \). Write \( m = \dim_2(\text{Fix}(A)) \) and \( n(H) = \dim_2(\text{Fix}(H)) \) for each non-trivial cyclic subgroup \( H < A \). By Lemma 2.5

\[
k - m = \sum_{H}(n(H) - m),
\]

where the sum is taken over the non-trivial cyclic subgroup of \( A \). We see that \( \dim_2 \text{Fix}(A_1C) = k \). By the invariance of domain, \( A_1C \) acts trivially on \( X \). Therefore, the group action of \( \text{SL}_n(\mathbb{Z}) \) factors through \( \text{SL}_n(\mathbb{Z}/2) \). Similar argument as Case (1.1) in the proof of Theorem 3.3 shows that the group action of \( \text{SL}_n(\mathbb{Z}/2) \) is trivial. If \( \text{Fix}(A_1) \) has 3 components, two of the components must be acyclic by noting that \( \sum b_i(F) \leq 4 \) (cf. Lemma 2.4). The remaining is the same as the proof of 4 components. If \( \text{Fix}(A_1) \) has at most 2 components, the \( \text{SL}_{n-2}(\mathbb{Z}) \) in the centralizer \( C_{\text{SL}_n(\mathbb{Z})}(A_1) \) will preserve each component. By induction, the group action of \( \text{SL}_{n-2}(\mathbb{Z}) \) on \( \text{Fix}(A_1) \) is trivial. The remaining is the same as the proof of 4 components.

Case (ii) \( \text{Fix}(A_1A_2A_3) \neq \emptyset. \)
Case (ii.1) \( \dim_2 \text{Fix}(A_1A_2A_3) = k. \)

By the invariance of domain, \( A_1A_2A_3 \) acts trivially on \( X \). The group factors through \( \text{SL}_n(\mathbb{Z}/2) \). A similar argument as in Case (1.1.1) in the proof of Theorem 3.3 shows that the group action of \( \text{SL}_n(\mathbb{Z}/2) \) is trivial.

Case (ii.2) \( \dim_2 \text{Fix}(A_1A_2A_3) < k. \)

If \( \text{Fix}(A_1) \neq \emptyset \), we may continue the proof as case (i). Suppose \( \text{Fix}(A_1) = \emptyset \). Let

\[
\sigma = \begin{pmatrix}
1 & & \\
-1 & 1 \\
& -1 & 
\end{pmatrix} \in \text{SL}_4(\mathbb{Z})
\]

lying in the upper left corner of \( \text{SL}_n(\mathbb{Z}) \). Since \( A_1A_2A_3 \) acts trivially on \( \text{Fix}(A_1A_2A_3) \), the group action of \( \text{SL}_4(\mathbb{Z}) \) factors through \( \text{PSL}_4(\mathbb{Z}) \). In \( \text{PSL}_4(\mathbb{Z}) \), the image of \( \langle A_1, A_2, \sigma \rangle \) is isomorphic to \((\mathbb{Z}/2)^3\). By Lemma 2.3, this subgroup can’t act freely on \( \text{Fix}(A_1A_2A_3) \). Thus,

\[
\text{Fix}(\sigma) \cap \text{Fix}(A_1A_2A_3) \neq \emptyset.
\]

If \( \dim_2 \text{Fix}(\sigma) = k \), the group action of \( \text{SL}_n(\mathbb{Z}) \) is trivial by a similar argument as Case (ii.1). If \( \dim_2 \text{Fix}(\sigma) < k \) and

\[
\dim_2 \text{Fix}(\sigma) \cap \text{Fix}(A_1A_2A_3) = \dim_2 \text{Fix}(A_1A_2A_3),
\]

the group action of \( \text{PSL}_4(\mathbb{Z}) \) is trivial. This is a contradiction to the fact that \( \text{Fix}(A_1) = \emptyset \). If

\[
\dim_2 \text{Fix}(\sigma) \cap \text{Fix}(A_1A_2A_3) < \dim_2 \text{Fix}(A_1A_2A_3),
\]

then \( \dim_2 \text{Fix}(A_1A_2A_3) - \dim_2 \text{Fix}(\sigma) \cap \text{Fix}(A_1A_2A_3) \) is at least 2. The group \( \text{SL}_{n-4}(\mathbb{Z}) \) lying in the lower right corner of \( \text{SL}_n(\mathbb{Z}) \) acts on \( \text{Fix}(\sigma) \cap \text{Fix}(A_1A_2A_3) \), which is codimension at least 4 in \( X \). If \( \dim_2 \text{Fix}(A_1A_2A_3) \leq 2 \), the \( \text{SL}_4(\mathbb{Z}) \) lying in the upper left corner of \( \text{SL}_n(\mathbb{Z}) \) acts trivially on \( \text{Fix}(A_1A_2A_3) \) by Theorem 3.3. This is impossible by the assumption that \( \text{Fix}(A_1) = \emptyset \). If \( \dim_2 \text{Fix}(A_1A_2A_3) \geq 3 \), we have \( k \geq 5 \) and \( n \geq 7 \). By induction, the group action of \( \text{SL}_{n-4}(\mathbb{Z}) \) on \( \text{Fix}(\sigma) \cap \text{Fix}(A_1A_2A_3) \) is trivial. This is a contradiction to the fact that \( \text{Fix}(A_1) = \emptyset \), since \( A_1 \) and \( \text{diag}(1, \ldots, -1, -1) \) are conjugate. The whole proof is finished. ■
4 Topological actions

In this section, we will study topological actions of \( SL_n(\mathbb{Z}) \) on manifolds with few Betti numbers. In order to prove Theorem 1.3, we need some lemmas.

The effective group actions of elementary \( p \)-group on homology manifolds with few Betti numbers was already studied by Mann [17] and Mann-Su [18]. The compact case of the following lemma was first obtained by Mann [17] (Theorem 3.2).

**Lemma 4.1** We have the following.

(i) If \( r < d - 2 \), the group \((\mathbb{Z}/2)^d\) cannot act effectively on a \( r \)-dimensional homology manifold \( X \) over \( \mathbb{Z}/2 \) with Betti numbers \( \sum_{i=0}^{r} b_i(X; \mathbb{Z}/2) \leq 4 \).

(ii) Let \( p > 2 \) be a prime. If \( r < 2d - 2 \), the group \((\mathbb{Z}/p)^d\) cannot act effectively on a \( r \)-dimensional homology manifold \( X \) over \( \mathbb{Z}/p \) with Betti numbers \( \sum_{i=0}^{r} b_i(X; \mathbb{Z}/p) \leq 4 \).

**Proof.** The case is vacuous for \( d = 1 \). For \( d = 2 \) and \( p \) is an odd prime, it is not hard to see that \((\mathbb{Z}/p)^2\) cannot act effectively on a 1-dimensional manifold \( X \) with \( \sum_{i=0}^{1} b_i(X; \mathbb{Z}/2) \leq 4 \) (note that 1-homology is actually an manifold).

We assume \( d \geq 3 \) and prove the theorem by induction. Let \( G = (\mathbb{Z}/p)^d \) be a group acting on \( X \). Choose a nontrivial element \( a \in G \) such that the fixed point set \( \text{Fix}(a) \) is maximal with respect to inclusion. Denote by \( G_0 \cong (\mathbb{Z}/p)^{d-1} \) the complement of \( \langle a \rangle \) in \( G \). By Lemma 2.3, \( \text{Fix}(a) \) is not empty. If \( \text{Fix}(a) = X \), we are done. If \( \text{Fix}(a) \neq X \), the co-dimension of \( \text{Fix}(a) \) is at least 1 for \( p = 2 \) and at least 2 for \( p \) odd by Lemma 2.2. By Lemma 2.4

\[ \sum_{i=0}^{r} b_i(\text{Fix}(a); \mathbb{Z}/p) \leq \sum_{i=0}^{r} b_i(X; \mathbb{Z}/p) \leq 4. \]

Applying induction to the action of \( G_0 \) on \( \text{Fix}(a) \), there is a nontrivial element \( b \in G_0 \) fixing \( \text{Fix}(a) \) pointwise. By the maximality of \( \text{Fix}(a) \), we have \( \text{Fix}(a) = \text{Fix}(b) \). Let \( A = \langle a, b \rangle \). For any nontrivial element \( x \in A \), we get \( \text{Fix}(a) = \text{Fix}(x) \). Write \( m = \dim_p(\text{Fix}(A)) \) and \( n(H) = \dim_p(\text{Fix}(H)) \) for each non-trivial cyclic subgroup \( H < A \). By Lemma 2.5

\[ r - m = \sum_{H} (n(H) - m), \]

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where the sum is taken over the non-trivial cyclic subgroup of \( A \). We have just proved \( n(H) = m \) for any \( H \). Thus \( r = m \). By the invariance of domain, we get \( \text{Fix}(a) = X \), which is a contradiction. ■

**Proof of Theorem 1.3.** We prove (i) first. Let \( A_i = \text{diag}(-1, \ldots, -1, \ldots, 1) \in \text{SL}_n(\mathbb{Z}) \), where the second \( -1 \) is in the \((i+1, i+1)\)-th position. The subgroup \( \langle A_i, i = 1, 2, \ldots, n-1 \rangle \cong (\mathbb{Z}/2)^{n-1} \). When \( n \) is odd, the center of \( \text{SL}_n(\mathbb{Z}) \) is trivial. By Lemma 4.1, when \( r < n - 3 \) if \( n \) is odd, or \( r < n - 4 \) if \( n \) is even, there is a noncentral order-two element in \( \text{SL}_n(\mathbb{Z}) \) acts trivially on \( X \). Therefore, the group action of \( \text{SL}_n(\mathbb{Z}/2) \) factors through \( \text{SL}_n(\mathbb{Z}/2) \).

Let \( e_{1i}(1) \) denote the matrix with 1s along the diagonal, 1 in the \((1, i)\)-th position and 0s elsewhere. We have \( \langle e_{1i}(1), i = 2, 3, \ldots, n \rangle \cong (\mathbb{Z}/2)^{n-1} \). By Lemma 4.1, there is a nontrivial element acting trivially on \( X \). Since \( \text{SL}_n(\mathbb{Z}/2) \) \((n \geq 3)\) is simple, the group action of \( \text{SL}_n(\mathbb{Z}/2) \) is trivial.

For (ii), let \( B = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \). It is directly that \( \text{SL}_n(\mathbb{Z}) \) contains \( \left\lceil \frac{n}{2} \right\rceil \) copies of \( \langle B \rangle \), which is isomorphic to \( (\mathbb{Z}/3)^{\left\lceil \frac{n}{2} \right\rceil} \). Here \( \left\lceil \frac{n}{2} \right\rceil \) is the integral part of \( \frac{n}{2} \). By Lemma 4.1, there is a nontrivial element in this \( (\mathbb{Z}/3)^{\left\lceil \frac{n}{2} \right\rceil} \) acting trivially on \( X \), when \( r < n - 2 \) if \( n \) is even, or \( r < n - 3 \) if \( n \) is odd. The nontrivial element normally generates the whole group \( \text{SL}_n(\mathbb{Z}) \). Therefore, the group action of \( \text{SL}_n(\mathbb{Z}) \) is trivial.

We now prove (iii). For \( 1 \leq i \neq j \leq n \), let \( e_{ij}(1) \) denote the matrix with 1s along the diagonal and 1 in the \((i, j)\)-th position and zeros elsewhere. Let \( D \subset \text{SL}_n(\mathbb{Z}/p) \) be the subgroup generated by \( \{e_{1i}(1), i = 2, \ldots, n\} \). It is not hard to see that \( D \) is isomorphic to \( (\mathbb{Z}/p)^{n-1} \). When \( r < n - 3 \) for \( p = 2 \) or \( r < 2n - 4 \) for odd \( p \), there is a nontrivial element \( \sigma \) in \( D \) acting trivially on \( X \). However, \( \text{PSL}_n(\mathbb{Z}/p) \) is simple and thus a noncentral element \( \sigma \) normally generates the whole group. Therefore, the group action of \( \text{SL}_n(\mathbb{Z}/p) \) is trivial. ■

5 Lifting group actions

Let \( \text{SL}_n(\mathbb{Z}) \) act on the Euclidean space \( \mathbb{R}^n \) by matrix multiplications. It induces an action on the torus \( T^n = \mathbb{R}^n/\mathbb{Z}^n \). Weinberger [24] proves that any smooth action of \( \text{SL}_n(\mathbb{Z}) \) on \( T^r \) is trivial for \( r < n \). We want to study the group action of \( \text{SL}_n(\mathbb{Z}) \) on \( T^r \) by homeomorphisms. Before proving Theorem
we prove a general result on group actions on manifolds and covering spaces.

Let $M$ be a connected manifold and let $G$ be a subgroup the group of homeomorphisms of $M$. Suppose that $p : M' \to M$ is a universal covering of connected manifolds with deck transformation group $\pi$ and denote by $G'$ all homeomorphisms of $M'$ covering those in $G$ (cf. [4], Theorem 9.3, page 66). The group $G'$ fits into an exact sequence

$$1 \to \pi \to G' \to G \to 1.$$ (*)

When the group action is by diffeomorphisms (piecewise linear homeomorphisms, resp.), we always assume that the manifolds are smooth (piecewise linear, resp.).

**Theorem 5.1** Let $G$ be a group and $p : M' \to M$ be a universal covering of connected manifolds with an abelian group $\pi$ as the deck transformation group. Suppose that

(i) any group action of $G$ on $M'$ by homeomorphisms (resp. diffeomorphisms, PL homeomorphisms) is trivial;

(ii) for any surjective group homomorphism $f : G \to Q$, the second cohomology group $H^2(Q; \pi) = 0$, if there is an effective action of $Q$ on $M$. Here $Q$ acts on $\pi$ through the exact sequence (*) (as $G$ does).

Then any group action of $G$ on $M$ by homeomorphisms (resp. diffeomorphisms, PL homeomorphisms) is trivial.

**Proof.** Let Homeo$(M)$ be the group of homeomorphisms of $M$ and $f : G \to$ Homeo$(M)$ a group homomorphism. Denote by $Q$ the image $\text{Im } f$ and $G'$ the group of all liftings of elements in $Q$ to Homeo$(M')$. We have a short exact sequence

$$1 \to \pi \to G' \to Q \to 1.$$ 

By (ii), $H^2(Q; \pi) = 0$. Therefore, this exact sequence is split and $G' \cong \pi \rtimes Q$. The group $Q$ could act on $M'$ through $G'$. Then the group $G$ could act on $M'$ through $Q$. However, this group action is trivial by assumption (i). Therefore, $Q$ is the trivial group. This proves that any group action of $G$ on $M$ by homeomorphisms is trivial. The proof for the case of diffeomorphisms and PL homeomorphisms is similar. ■

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Lemma 5.2 Let $\pi$ be a finitely generated abelian group without 2-torsions. For any $n \geq 3$, the second cohomology group

$$H^2(\text{SL}_n(\mathbb{Z}); \pi) = 0 \text{ and } H^2(\text{SL}_n(\mathbb{Z}/k); \pi) = 0$$

for any nonzero integer $k$.

Proof. By van der Kallen [22], the second homology group $H_2(\text{SL}_n(\mathbb{Z}); \mathbb{Z}) = \mathbb{Z}/2$ when $n \geq 5$ and $H_2(\text{SL}_3(\mathbb{Z}); \mathbb{Z}) = \mathbb{Z}/2 \bigoplus \mathbb{Z}/2$. Since $\text{SL}_n(\mathbb{Z})$ is perfect, $H_1(\text{SL}_n(\mathbb{Z}); \mathbb{Z}) = 0$ for any $n \geq 3$. By universal coefficient theorem, $H^2(\text{SL}_n(\mathbb{Z}); \mathbb{Z}) = 0$ for any $n \geq 3$. Dennis and Stein proved that $H_2(\text{SL}_n(\mathbb{Z}/k)) = \mathbb{Z}/2$, for $k \equiv 0 \text{ (mod } 4)$, while $H_2(\text{SL}_n(\mathbb{Z}/k)) = 0$, otherwise (cf. [8] corollary 10.2). By universal coefficient theorem again, $H^2(\text{SL}_n(\mathbb{Z}/k); \pi) = 0$ for any $k$.

Proof of Theorem 1.4. It suffices to check that the two conditions in Theorem 5.1 are satisfied. It is obvious that (i) in Theorem 5.1 is same as (i) in Theorem 1.4 by taking $G = \text{SL}_n(\mathbb{Z})$. Since any group homomorphism $\text{SL}_n(\mathbb{Z}) \to \text{SL}_k(\mathbb{Z})$ ($k < n$) is trivial (cf. [24] or [26], Corollary 1.11), the group $\text{SL}_n(\mathbb{Z})$ can only act on $\mathbb{Z}^k$ trivially. Denote by $\pi = \mathbb{Z}^k$. Let $f : \text{SL}_n(\mathbb{Z}) \to Q$ be any surjective homomorphism. If $f$ is injective, $H^2(\text{SL}_n(\mathbb{Z}); \pi) = 0$ by Lemma 5.2. If $f$ is not injective, the congruence subgroup property [1] implies that $Q$ is a quotient of $\text{SL}_n(\mathbb{Z}/k)$ by a central subgroup $K$ for some integer $k$. From the Serre spectral sequence

$$H^p(Q; H^q(K; \pi)) \implies H^{p+q}(\text{SL}_{n+1}(\mathbb{Z}/k); \pi),$$

we have the exact sequence

$$0 \to H^1(Q; \pi) \to H^1(\text{SL}_n(\mathbb{Z}/k); \pi) \to H^0(Q; H^1(K; \pi)) \to H^2(Q; \pi) \to H^2(\text{SL}_n(\mathbb{Z}/k); \pi).$$

This implies that $H^2(Q; \pi) \cong H^1(K; \pi) = 0$, by Lemma 5.2. Therefore, condition (ii) of Theorem 5.1 is satisfied and any group action of $\text{SL}_n(\mathbb{Z})$ on $M$ is trivial. 

Proof of Corollary 1.5. Denote by $M' = \mathbb{R}^r$, $\mathbb{R}^{r_1} \times S^{r_2}$ or $\mathbb{R}^{r_0} \times S^{r_1} \times S^{r_2}$, a universal cover of $M$. Bridson and Vogtmann [2] prove that when $r < n,$
any group action of $\text{SL}_n(\mathbb{Z})$ on $\mathbb{R}^r$ by homeomorphism is trivial. Theorem 1.4 implies (i). When $r_1 + r_2 < n - 1$, any group action of $\text{SL}_n(\mathbb{Z})$ on $\mathbb{R}^{r_1} \times S^{r_2}$, or $\mathbb{R}^{r_0} \times S^{r_1} \times S^{r_2}$ is trivial by Theorem 1.2. Therefore, the statements (ii) and (iii) follows Theorem 1.4 directly.

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