Some analysis on a fractional differential equation involving a noncontinuous right-hand side

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Abstract By developing new techniques we establish local existence and uniqueness theorems for an initial value problem involving a non-linear equation in the sense of Riemann-Liouville fractional derivative in the case that the nonlinear function on the right hand side of the equation is not continuous on \([0, T] \times \mathbb{R}\).

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1. Introduction and Motivation

The birth of fractional calculus dates back to the early days of differential calculus. When compared with differential calculus, fractional calculus has not made progress so much in a reasonable amount of time because of the lack of researchers studied on it during that time. However, the interest in this area has increased considerably for the last three decades and has lead the area of fractional calculus to grown rapidly by the help of the results, techniques, methods used in the ordinary differential calculus. Of course, a substantial part of the interest has been on the subject of initial-value problems (IVPs) and boundary-value problems (BVPs) for the differential equations with fractional derivatives such as Riemann-Liouville (R-L), Caputo, Caputo-Fabrizio, Grünwald-Letkinov etc. There have been many researches for the existence and uniqueness of solutions for these IVPs and BVPs (see for example [1], [4], [6], [9], [10]-[13], [17]-[22]).
In this paper, the following initial value problem for a differential equation with R-L fractional derivative which previously discussed in [12], [19], [22] is considered:

\[
\begin{cases}
D^a u(x) = f(x, u(x)), \\
u(0) = u_0,
\end{cases}
\]

where \(0 < a < 1\) (is valid throughout the paper), \(u_0 \neq 0\) is a real number, \(f\) will be specified later and, \(D^a\) represents R-L fractional derivative of order \(a\), which is defined by combining the ordinary derivative and R-L fractional integral \(I^a\) as follows:

\[
D^a u(x) := \frac{d}{dx} \left[ x I^{1-a} u \right] \quad \text{with} \quad I^a u(x) := \frac{1}{\Gamma(a)} \int_0^x \frac{u(\xi)}{(x - \xi)^{1-a}} d\xi.
\]

Here \(\Gamma(.)\) is the well-known Gamma function.

IVP (1.1) with a continuous right-hand side was first discussed by [12] and it is claimed that the continuous solution on the interval \([0, T]\) of the problem exists. However, Zhang [22] showed by an example that the initial condition \(u(0) = u_0\) (except \(u_0 = 0\)) is not suitable for investigating the existence of continuous solutions of IVP (1.1) when \(f\) is continuous on \([0, T] \times \mathbb{R}\). Correspondingly, Şan [19] considered the problem (1.1) with \(f\) satisfying the following conditions:

\[
\begin{align*}
(1.2) \quad & f(x, t)\text{ and } x^a f(x, t)\text{ are continuous on } (0, T] \times \mathbb{R} \text{ and } [0, T] \times \mathbb{R}, \\
(1.3) \quad & x^a f(x, u_0) \big|_{x=0} = u_0/\Gamma(1-a),
\end{align*}
\]

where (1.3) is a necessary condition for the existence of the continuous solution for (1.1) (See [19]). There, he gave a partial answer for the existence of continuous solutions of IVP (1.1). However, in a discussion with Manuel D. Ortigueira said (see also [15], [16]) that (1.1) represents a system and the initial condition must be independent of tools we analyze it. From this point of view, it would be more accurate to discuss the nonexistence of a continuous solution of (1.1) instead of querying the suitability of the initial condition. In fact, if there were a continuous solution \(u\) of (1.1) when \(f\) is continuous on \([0, T] \times \mathbb{R}\), then from Proposition 2.4 in [4] it would be shown as

\[
u(x) = \frac{1}{\Gamma(a)} \int_0^x \frac{f(\xi, u(\xi))}{(x - \xi)^{1-a}} d\xi, \quad x \in [0, T],
\]

since \(u \in C([0, T])\) and \(D^a u \in C([0, T])\). From here, by the continuity of \(f(x, u(x))\) on \([0, T]\) one obtains...
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$$0 \neq u_0 = \lim_{x \to 0^+} u(x) = \frac{1}{\Gamma(a)} \int_0^1 \lim_{x \to 0^+} x^a f(xt, u(xt)) \frac{x^{a-1}}{(1 - t)^{1-a}} d\xi = 0,$$

which is a contradiction. This implies that IVP (1.1) with a continuous right-hand side can not have a continuous solution \( u(x) \) with \( u(0) = u_0 \neq 0 \).

In this study, by using Leray-Schauder alternative (since Schauder fixed point theorem is not applicable, see Remark 1) we first prove the existence of continuous solutions of IVP (1.1) under conditions (1.2)- (1.3). However, as seen in the sequel, this theorem does not inform us about the existence interval of the solutions. In the literature, Mustafa and Baleanu [2] applied a method and Leray-Schauder alternative to obtain a better estimate for the existence interval of the continuous solutions of a problem they considered. Such a method can not be applied to IVP (1.1). Here, performing a new technique and using Schauder’s fixed point theorem we obtain a Peano-type existence theorem for IVP (1.1) with a class of the functions \( f \) satisfying (1.2), which explicitly shows the existence interval of solutions.

Moreover, we show the existence and uniqueness of the continuous solutions of (1.1) when the function \( f \) satisfies conditions (1.2)-(1.3) and a Nagumo-type condition which is, in fact, same with a Lipschitz-type condition used in Theorem 3.5 in [4] and Theorem 3.8 in [18]. Nagumo-type uniqueness results for fractional differential equations exist in the literature (see, for example, [6], [9], [13], [20]) and some of these results were proved by following the way developed by Diaz in [5] before. To prove the uniqueness result for (1.1), we here develop a novel technique combining with a fractional mean value theorem for functions \( u \in C([0, T]) \) satisfying \( D^a u \in C((0, T]) \) and \( x^a D^a u \in C((0, T]) \). Mean value theorems in the R-L sense or Caputo sense for functions satisfying certain conditions were previously obtained by Diethelm [6] and Odibat [14].

2. PRELIMINARIES AND MAIN RESULTS

Before proceeding to investigate problem (1.1), we give some notes which may be required for the main results. At first, we define the equivalence of solutions of problem (1.1) in the following result given in [19]:

**Lemma 1.** Under conditions (1.2)-(1.3), the continuous solutions of (1.1) are also the solutions of the integral equation (1.4), vice versa.
Now let us define the operator \( M : C([0, T]) \to C([0, T]) \) associated with integral equation (1.4) as follows:

\[
M u(x) := \frac{1}{\Gamma(a)} \int_0^x \frac{f(\xi, u(\xi))}{(x - \xi)^{1-a}} d\xi, \quad x \in [0, T].
\]  

(2.1)

Since the fixed points of operator \( M \) coincide with the solutions of integral equation (1.4), our aim here is to find out these fixed points of operator \( M \) by using the following fixed point theorems [3],[7],[8]:

**Theorem 1.** (Schauder’s fixed-point theorem) Let \( X \) be a real Banach space, \( B \subset X \) nonempty closed bounded and convex, \( M : B \to B \) compact. Then \( M \) has a fixed point.

**Remark 1.** In applications, it is usually too difficult or impossible to establish a set \( B \) so that \( M \) takes \( B \) back into \( B \) (see, for example, Remark 3.7 in [18]). Therefore, to overcome this difficulty, it will be available to consider maps \( M \) that map the whole space \( X \) into \( X \). The following result is intimately associated with what we stated above:

**Theorem 2.** (Leray-Schauder alternative). Let \( X \) be normed linear space and \( M : X \to X \) be a completely continuous (compact) operator. Then, either there exists \( u \in X \) such that \( u = M u \) or the set

\[
\mathcal{E}(M) := \{ u \in X : u = \mu M(u) \text{ for a certain } \mu \in (0, 1) \}
\]  

(2.2)

is unbounded.

The compactness of operator \( M \) was previously obtained in Theorem 2.5 in [19], therefore it will be sufficient to show the remaining conditions of the fixed point theorems given above to be satisfied. The first existence theorem for problem (1.1) is as follows:

**Theorem 3.** Let conditions (1.2) and (1.3) be satisfied. Then, there exists at least one continuous solution \( u \in C([0, T]) \) of problem (1.1).

**Proof.** For the proof, we use Leray-Schauder alternative and it is enough to show that \( \mathcal{E}(M) \) in (2.2) is bounded. For an arbitrary \( u \in \mathcal{E} \) one
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has

\begin{align*}
|u(x)| & \leq \frac{1}{\Gamma(a)} \left| \int_0^x \frac{f(\xi, u(\xi))}{(x - \xi)^{1-a}} d\xi \right| \\
& < \frac{1}{\Gamma(a)} \left| \int_0^x \frac{f(\xi, u(\xi)) - \xi^{a-\frac{u_0}{\Gamma(1-a)}} + \xi^{-a\frac{u_0}{\Gamma(1-a)}}}{(x - \xi)^{1-a}} d\xi \right| \\
& \leq M \Gamma(1-a) + \frac{|u_0|}{\Gamma(1-a)\Gamma(a)} \int_0^x \frac{1}{\xi^a (x - \xi)^{1-a}} d\xi \\
& = M \Gamma(1-a) + |u_0|,
\end{align*}

where \( M = \sup_{(x,t) \in [0,T] \times \mathbb{R}} |f(x,t)| \). Therefore, for any \( u \in \mathcal{E} \) one obtains

\[
\sup_{x \in [0,T]} |u(x)| < M \Gamma(1-a) + |u_0|,
\]

which means that \( \mathcal{E} \) is bounded. As a result of Leray-Schauder alternative, (1.1) admits at least one solution in \( C([0,T]) \).

Before we give a Peano-type existence theorem for problem (1.1), let us make some notes. To prove the theorem, Schauder fixed-point theorem is used and it is enough to show only \( \mathcal{M} : C \rightarrow C \), where \( C \) is an appropriate closed convex ball of \( C([0,T]) \), which will be constructed later. On the other hand, from (1.2) it is reasonable to set

\[
M_1 := \sup \{ |x^a f(x,t)| : x \in [0,T], t \in [u_0 - r, u_0 + r] \}
\]

for the fixed values \( r > 0 \) and \( T > 0 \). Keeping in mind the values \( r, T \) and \( M_1 \), we give the second existence result in the following:

**Theorem 4.** Let (1.2) be satisfied. Moreover, suppose that there exists a positive real number \( M_2 \) such that

\[
\left| x^a f(x,t) - \frac{u_0}{\Gamma(1-a)} \right| \leq M_2 \max \left( |x|, |t - u_0| \right)
\]

holds for all \( x \in [0,T] \) and for all \( t \in [u_0 - r, u_0 + r] \). Then, (1.1) has at least one continuous solution on \([0,T_0]\), where

\[
T_0 := \begin{cases} 
\frac{r}{M \Gamma(1-a)}, & \text{if } M \Gamma(1-a) > r, \\
T, & \text{if } M \Gamma(1-a) \leq r,
\end{cases}
\]

and \( M = \max \{ M_1, M_2 \} \).

**Proof.** Let us first construct an appropriate closed convex ball of \( C([0,T]) \) according to inequality (2.3). For this, it is assumed that

\[
\left| x^a f(x,t) - \frac{u_0}{\Gamma(1-a)} \right| \leq M_2 |x|
\]

(2.4)
is fulfilled for all $x \in [0, T]$ and for all $t \in [u_0 - r, u_0 + r]$. Depending on this inequality, set

$$B_r = \{ u \in C[0, T_0] : \sup_{x \in [0, T_0]} |u(x) - u_0| \leq r \}$$

with $MT(1 - a) < r$. Then, for any $u \in B_r$, from (2.4) one has

$$|\mathcal{M} u (x) - u_0| \leq \frac{1}{\Gamma(a)} \int_0^x \left| \left( f(\xi, u(\xi)) - \frac{u_0}{\Gamma(1-a)} \right) \right| \frac{\xi^a (x - \xi)^{1-a}}{(x - \xi)^{1-a}} d\xi$$

$$= \frac{1}{\Gamma(a)} \int_0^x \left| \left( \xi^a f(\xi, u(\xi)) - \frac{u_0}{\Gamma(1-a)} \right) \right| \frac{\xi^a (x - \xi)^{1-a}}{(x - \xi)^{1-a}} d\xi$$

$$\leq \frac{M_1}{\Gamma(a)} \int_0^x \left| \xi \right| \frac{\xi^a (x - \xi)^{1-a}}{(x - \xi)^{1-a}} d\xi$$

$$\leq M \left| x \right| \Gamma(2 - a) \leq MT_0 \Gamma(1 - a),$$

where we used the inequality $\Gamma(2 - a) < \Gamma(1 - a)$ for all $a \in (0, 1)$. From here and the definition of $T_0$, one obtains

$$\sup_{x \in [0, T_0]} |\mathcal{M} u (x) - u_0| \leq MT(1 - a)T_0 \leq r. \quad (2.5)$$

On the other hand, if

$$\left| x^a f(x, t) - \frac{u_0}{\Gamma(1-a)} \right| \leq M_2 |t - u_0| \quad (2.6)$$

holds for all $x \in [0, T]$ and for all $t \in [u_0 - r, u_0 + r]$, then set $B_r$ as

$$B_r = \{ u \in C[0, T] : \sup_{x \in [0, T]} |u(x) - u_0| \leq r \},$$

for $r \geq MT(1 - a)$. Then, using (2.6) one gets

$$|\mathcal{M} u (x) - u_0| \leq \frac{1}{\Gamma(a)} \int_0^x \left| \left( \xi^a f(\xi, u(\xi)) - \frac{u_0}{\Gamma(1-a)} \right) \right| \frac{\xi^a (x - \xi)^{1-a}}{(x - \xi)^{1-a}} d\xi$$

$$\leq \frac{M_2}{\Gamma(a)} \int_0^x \left| \xi \right| \frac{\xi^a (x - \xi)^{1-a}}{(x - \xi)^{1-a}} d\xi$$

$$\leq M \left| x \right| \Gamma(2 - a) \leq r$$

for any $u \in B_r$ and for all $x \in [0, T]$. Therefore,

$$\sup_{x \in [0, T]} |\mathcal{M} u (x) - u_0| \leq MT(1 - a) \leq r. \quad (2.7)$$

From (2.5) and (2.7) one can see $\mathcal{M}(B_r) \subset B_r$ which is the desired result.  \qed
Remark 2. There exists a considerable amount of functions satisfying inequality (2.3). Indeed, for example, if \( g(x,t) = x^a f(x,t) \) is continuously differentiable in \( x \) and \( t \), then from the mean value theorem for several variables and equivalence of the Euclidean and maximum norms on \( \mathbb{R}^2 \) one can write

\[
|x^a f(x,t) - x^a f(x,u_0)|_{x=0} = |g(x,t) - g(0,u_0)| \\
\leq \|\nabla g\| \|(x,t - u_0)\| \\
\leq \|\nabla g\| \sqrt{x^2 + (t-u_0)^2} \\
\leq \sqrt{2} \|\nabla g\| \max( |x|, |t-u_0| ),
\]

which means that \( g(x,t) \) fulfills inequality (2.3).

For obtaining the uniqueness result, we first give a mean value theorem for R-L derivative, and for its proof we follow the path used in [6] and [14].

Lemma 2. Let \( u \in C[0,T] \) with \( D^a u \in C(0,T] \) and \( x^a D^a u \in C[0,T] \) for \( 0 < a < 1 \). Then, there exists a function \( \lambda = \lambda(x), \lambda : [0,T] \rightarrow (0,x) \) such that

\[
u(x) = \Gamma(1-a)(\lambda(x))^a D^a u(\lambda(x)) \tag{2.8}\]

is satisfied.

Proof. Using equality (1.3) and by the help of mean value theorem of integral calculus we have

\[
u(x) = I^a D^a u(x) = \frac{1}{\Gamma(a)} \int_0^x \frac{D^a u(\xi)}{(x-\xi)^{1-a}} d\xi \\
= \frac{1}{\Gamma(a)} \int_0^x \frac{\xi^a D^a u(\xi)}{\xi^a (x-\xi)^{1-a}} d\xi \\
= \frac{(\lambda(x))^a D^a u(\lambda(x))}{\Gamma(a)} \int_0^x \frac{1}{\xi^a (x-\xi)^{1-a}} d\xi \\
= \Gamma(1-a)(\lambda(x))^a D^a u(\lambda(x)),
\]

where \( \lambda = \lambda(x) \in (0,x) \) for all \( x \in [0,T] \).

Remark 3. In Theorem 1 in [14], the dependence of \( \lambda \) (which is taken as \( \xi \) there) on \( x \) was not clearly expressed. However, \( \lambda \) is generally is a function of \( x \). To see this, let \( u(x) = 1 + x^2 \). Then from (2.8), one has

\[
1 + x^2 = \Gamma(1-a)\lambda^a \left( \frac{\lambda^{-a}}{\Gamma(1-a)} + \frac{\lambda^{2-a}}{\Gamma(3-a)} \right) = 1 + \frac{2\lambda^2}{(1-a)(2-a)}.
\]
From here,
\[
\lambda = \sqrt{\frac{(1-a)(2-a)}{2}} x \in (0, x)
\]
is obtained, which shows that \(\lambda\) is a function of \(x\).

**Theorem 5.** Let \(0 < a < 1\) and \(T > 0\), and let conditions (1.2) and (1.3) be satisfied. Moreover, suppose that the inequality
\[
x^a |f(x, t_1) - f(x, t_2)| \leq \frac{1}{\Gamma(1-a)} |t_1 - t_2|
\]
holds for all \(x \in [0, T]\) and for all \(t_1, t_2 \in \mathbb{R}\). Then IVP (1.1) admits a unique continuous solution on \([0, T]\).

**Proof.** We first assume that IVP (1.1) has two different continuous solutions \(u_1\) and \(u_2\), and then we show, by contradiction, that this can not happen. We do this in the following way: we initially suppose \(\omega(x) \not\equiv 0\), where
\[
\omega(x) := \begin{cases} 
|u_1(x) - u_2(x)|, & x > 0 \\
0, & x = 0
\end{cases}
\]
and by contradiction we prove the contrary.

Let \(\omega(x) \not\equiv 0\). It is easily seen that \(\omega(x)\) is nonnegative for all \(x \in [0, T]\) and continuous for all these \(x\) values except \(x = 0\). For the continuity of \(\omega(x)\) at \(x = 0\), using (1.2), variable substitution \(\xi = xt\) and condition (1.4), respectively, we have
\[
0 \leq \lim_{x \to 0^+} \omega(x) = \lim_{x \to 0^+} \frac{1}{\Gamma(a)} \left| \int_0^x \frac{f(\xi, u_1(\xi)) - f(\xi, u_2(\xi))}{(x - \xi)^{1-a}} d\xi \right|
\]
\[
\leq \lim_{x \to 0^+} \frac{1}{\Gamma(a)} \left| \int_0^{xt} \frac{\left[ f(xt, u_1(x)) - f(\xi, u_2(xt)) \right]}{t^a (1-t)^{1-a}} dt \right|
\]
\[
\leq \frac{1}{\Gamma(a)} \int_0^{1} \lim_{x \to 0^+} \left| \frac{\left[ f(xt, u_1(x)) - f(\xi, u_2(xt)) \right]}{t^a (1-t)^{1-a}} \right| dt
\]
\[
= 0,
\]
which of course means that \(\lim_{x \to 0^+} \omega(x) = 0 = \omega(0)\).

Now suppose that there exists a \(\lambda \in (0, T]\) such that \(\omega(\lambda) \not\equiv 0\), i.e. \(\omega(\lambda) > 0\). From here, by using Lemma 2 and inequality (2.9), one
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obtains

\[ 0 < \omega(\lambda) = |u_1(\lambda) - u_2(\lambda)| = \Gamma(1 - a) |\lambda^a D^a(u_1 - u_2)(\lambda_s)| = \Gamma(1 - a) |f(\lambda_s, u_1(\lambda_s)) - f(\lambda_s, u_2(\lambda_s))| \leq |u_1(\lambda_s) - u_2(\lambda_s)| = \omega(\lambda_s) \]

for some \( \lambda_s \in (0, \lambda) \). If one follows the same procedure applied above for the point \( \lambda_s \), then there exist some \( \lambda_2 \in (0, \lambda) \) such that \( 0 < \omega(\lambda_s) \leq \omega(\lambda_2) \). Continuing in the same way, a sequence \( \{\lambda_n\}_{n=1}^{\infty} \subset [0, \lambda) \) with \( \lambda_1 = \lambda \) satisfying \( \lambda_n \to 0 \) can be obtained, for which

\[ 0 < \omega(\lambda) \leq \omega(\lambda_1) \leq \omega(\lambda_2) \leq ... \leq \omega(\lambda_n) \leq ... \quad (2.10) \]

On the other hand, since \( \omega(x) \) is continuous at \( x = 0 \) and \( \lambda_n \to 0 \), one has \( \omega(\lambda_n) \to \omega(0) = 0 \), which, however, contradicts with (2.10). In that case, \( \omega(x) \equiv 0 \), namely IVP (1.1) admits a unique continuous solution. \( \square \)

It is to be observed that condition (1.2) forces us to impose the Nagumo-type condition in (2.9). Under (2.9) without equality, Delbosco and Rodino in [4] showed the uniqueness of the continuous solution of the equation in (1.1) by using Banach contraction principle. In Theorem 4 one can see the effect of the initial condition on the uniqueness. On the other hand, there are techniques and theorems (see for example Theorem 3.4 in [4] and Theorem 4.1 in [21]) enable us to take a positive fixed real number bigger than \( \frac{1}{\Gamma(1-a)} \) in (2.9) or an arbitrary positive real number instead of it so that problem (1.1) admits a unique solution under the Nagumo-type condition. However, this cannot be applied to problem (1.1), namely there does not exist a bigger number than \( \frac{1}{\Gamma(1-a)} \) or an arbitrary positive real number instead of \( \frac{1}{\Gamma(1-a)} \) to guarantee the uniqueness of the continuous solution of (1.1). The following example may clearly express the foregoing discussion.

**Example 1.** Let us take \( f_\beta(x, t) := \frac{\Gamma(\beta+1)}{\Gamma(1-a+\beta)} x^{-a} (t + k) \) where \( k = \frac{\Gamma(\beta-a+1) - \Gamma(1+\beta) \Gamma(1-a)}{\Gamma(1+\beta) \Gamma(1-a)} \), \( \beta > 0 \) and \( u_0 = 1 \) in problem (1.1). It is clear that conditions (1.2) and (1.3) are satisfied. However, inequality (2.9) does not hold for this right-hand side function, since \( \frac{\Gamma(\beta+1)}{\Gamma(1-a+\beta)} \) exists instead of \( \frac{1}{\Gamma(1-a)} \) in (2.9) and \( \frac{\Gamma(\beta+1)}{\Gamma(1-a+\beta)} > \frac{1}{\Gamma(1-a)} \) for \( \beta > 0 \) and \( a \in (0,1) \). That is to say, the solution of (1.1) may not be unique. Indeed, (1.1) has the solutions \( u(x) = cx + 1 \), where \( c \) is an arbitrary real number. Moreover, it is to be pointed out that \( \frac{\Gamma(\beta+1)}{\Gamma(1-a+\beta)} \to \frac{1}{\Gamma(1-a)} \) and
\[ f_\beta(x, t) \rightarrow f(x, t) = \frac{x^{-\alpha t}}{\Gamma(1-\alpha)} \text{ when } \beta \rightarrow 0 \text{ and that, for } f(x, t) = \frac{x^{-\alpha t}}{\Gamma(1-\alpha)}, \]

(1.1) with \( u_0 = 1 \) admits a unique solution in the form \( u(x) = 1 \).

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