Kinetic dominance and psi series in the Hamilton-Jacobi formulation of inflaton models

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Abstract

Single-field inflaton models in the kinetic dominance period admit formal solutions given by generalized asymptotic expansions called psi series. We present a method for computing psi series for the Hubble parameter as a function of the inflaton field in the Hamilton-Jacobi formulation of inflaton models. Similar psi series for the scale factor, the conformal time and the Hubble radius are also derived. They are applied to determine the value of the inflaton field when the inflation period starts and to estimate the contribution of the kinetic dominance period to calculate the duration of inflation. These psi series are also used to obtain explicit two-term truncated psi series near the singularity for the potentials of the Mukhanov-Sasaki equation for curvature and tensor perturbations. The method is illustrated with wide families of inflaton models determined by potential functions combining polynomial and exponential functions as well as with generalized Starobinsky models.

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I. INTRODUCTION

The theory of inflationary cosmology provides a framework to study the early universe \([1-3]\) which solves several central problems of the hot Big Bang model. In the present work we consider single-field inflaton models formulated in terms of a time dependent real field \(\phi(t)\) in a spatially flat universe \([4-7]\). For an homogeneous spatially flat Friedman-Lemaitre-Robertson-Walker (FLRW) spacetime with scale factor \(a(t)\), these models are described by the nonlinear ordinary second order differential equation

\[
\ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi}(\phi) = 0,
\]  

(1)

where \(H = \dot{a}/a\) is the Hubble parameter which is related to the inflaton field by

\[
H^2 = \frac{1}{3m_{Pl}^2} \left( \frac{1}{2} \dot{\phi}^2 + V(\phi) \right).
\]  

(2)

Here \(V = V(\phi)\) is a given potential function, \(m_{Pl} = \sqrt{\hbar c/8\pi G}\) is the Planck mass and dots indicate derivatives with respect to the cosmic time \(t\).

We concentrate on the kinetic dominance (KD) period \([8,10]\) of inflaton models, when the kinetic energy of the inflaton field dominates over its potential energy.

\[
\dot{\phi}^2 \gg V(\phi).
\]  

(3)

It is a non-inflationary or pre-inflationary stage that is followed by a short fast-roll inflation phase \([11]\) and afterwards by the traditional slow-roll inflation stage \([5,12-19]\). Recently, Handley et al \([9,10,20,22]\) have shown the relevance of the KD period \((3)\) for setting initial conditions. In fact, as they prove in \([9]\), under mild conditions on the potential \(V\), all solutions (except perhaps a single one) evolve from a KD region. Our study is devoted to the asymptotic series solutions of the inflaton equations for the KD period and their applications.

The solutions of the equation \(1\) manifest generically branch point singularities of logarithmic type. This is the same type of singularities exhibited by the solutions of the Lorentz system \([23]\) and their presence is associated to the so-called psi series \([24]\) asymptotic solutions of ordinary differential equations. Alternative different psi series containing terms with irrational or even complex exponents were found in the Hénon-Heiles system \([25]\), the rigid body problem, the Toda lattice equation, the Duffing oscillator \([26]\) and the fractional...
Ginzburg-Landau equation [27]. Logarithmic psi series solutions of the inflaton equations (1) have been also considered, see for instance [8, 9, 11]. Recently [10] a general method has been formulated for computing psi series expansions for the solutions of the equations (1) and the generalization of (2) for FLRW spacetimes with curvature. The method formulates the inflaton equations as a four-dimensional first-order system of ordinary differential equations and determines solutions as series expansions involving powers of $t - t^*$ and $\log(t - t^*)$. These series are termed logolinear series in [10].

In the present work we propose an alternative method for determining formal asymptotic solutions of (1)-(2) expressed as psi series. It uses the Hamilton-Jacobi formalism of the inflaton models [28], [29] and [30], in which the independent variable is the inflaton field instead of the cosmic time. We generate psi series for solutions of the Hamilton-Jacobi equations for the Hubble function as functions of the rescaled inflaton field $\varphi \equiv \sqrt{\frac{3}{2}} \phi/m_{Pl}$. We apply our method to the following classes of models:

1. 
   
   \[ v(\varphi) = \sum_{n=0}^{N} v_n(\varphi)e^{-n\varphi}, \]  
   
   where $v_n(\varphi)$ are polynomials in $\varphi$ and $N$ is a non-negative integer.

2. Models with generalized Starobinsky potential functions

   \[ v(\varphi) = \sum_{n=0}^{N} v_n e^{-\alpha n\varphi}, \]  

   where $\alpha$ is any irrational number and $v_n$ are constant coefficients.

In (4) and (5) $v(\varphi)$ stands for the rescaled potential $v(\varphi) = 3V(\phi)/m_{Pl}^2$.

The paper is organized as follows. In section II we briefly introduce the Hamilton-Jacobi formalism of inflation models. Section III describes our method for determining psi series for the inflaton models with potential functions (4) and (5). For the case (4) we determine a one-parameter family of logarithmic psi series solutions in the variable $u := e^{-\varphi}$ with polynomial coefficients depending on $\varphi$, which could be termed expolinear series. In particular, the models with polynomial potentials ($N = 0$) exhibit several interesting symmetry properties which are analyzed in detail. As illustrative explicit examples we apply the method to the quadratic potential $v(\varphi) = m^2 \varphi^2$ and to the Higgs potential $v(\varphi) = g^2(\varphi^2 - \lambda^2)^2$. For the models with potential functions (5) we characterize a one-parameter family of psi
series solutions of non-logarithmic type in the variable $u := e^{-\varphi}$ with coefficients which are polynomials in $e^{-\alpha \varphi}$. Furthermore, a simple limit operation shows that the results also apply to rational exponents $\alpha$. In particular, we apply the method to the Starobinsky potential

$$v(\varphi) = \lambda \left( 1 - e^{-\alpha \varphi} \right)^2,$$

and check that our results with $\alpha = \pm \frac{2}{3}$ coincide with the results in [10] for the corresponding inflaton model with potential

$$V(\phi) = \Lambda^2 \left( 1 - e^{-\sqrt{\frac{2}{3}} \phi} \right)^2. \quad (6)$$

At the end of section III we discuss how to derive from our psi series in the inflaton field $\varphi$ the logolinear series involving powers of $t - t^*$ and $\log(t - t^*)$.

Finally, section IV presents several applications of the psi series obtained in the previous section to calculate analytical approximations of several relevant quantities of inflation models and to compare them with the corresponding numerical approximations. Thus, we use the psi series to determine the value of the inflaton field at the initial moment of the inflation period. We also provide a formula for the amount of inflation, which includes the contribution of the part of the KD period which overlaps the inflation region. Finally, we consider the potentials of the Mukanov-Sasaki equation near the singularity for both curvature and tensor perturbations. It is known [11] that as functions of the conformal time the dominant term of these potentials coincides with the critical central singular attractive potential allowing the fall to the center of a quantum particle. Then, we use the psi series previously obtained to provide an explicit two-term truncated psi series approximation to these potentials.

II. HAMILTON-JACOBI FORMULATION OF INFATON MODELS

From equations (1)-(2) it follows that

$$\dot{H} = -\frac{1}{2m_{pl}^2} \dot{\phi}^2,$$  \quad (7)

As a consequence the Hubble parameter $H$ is a positive monotonically decreasing function of $t$. This property implies that for smooth and positive potential functions $V$, the solutions $\phi(t)$ of (1) with arbitrary finite initial data do not have singularities forward in the cosmic
time $t$. Nevertheless, the function $H(t)$ increases without bound backwards in time, so that $H(t)$ and $\phi(t)$ may develop singularities.

The presence of singularities backwards in time can be expected from the following argument: If the KD condition (3) holds then we may neglect $V$ and $V_\phi$ in the inflaton equations and from (1) we have

$$\ddot{\phi} + \sqrt{\frac{3}{2} \frac{1}{m_{Pl}}} |\dot{\phi}| \dot{\phi} \sim 0. \quad (8)$$

Thus we obtain two families of approximate solutions

$$\phi \sim \pm \sqrt{\frac{2}{3}} \frac{m_{Pl}}{t - t^*} \log(t - t^*) \mp \phi_p \quad \text{as} \quad t \to (t^*)^+, \quad (9)$$

where $t^*$ and $\phi_p$ are arbitrary constants. The corresponding asymptotic form of the Hubble parameters is

$$H \sim \frac{1}{3(t - t^*)} \quad \text{as} \quad t \to (t^*)^+. \quad (10)$$

These approximate solutions of the inflaton equations are the dominant terms of the psi series expansions that we will consider below.

**A. The Hamilton-Jacobi equations**

We use the rescaled variables

$$\varphi = \sqrt{\frac{3}{2} \frac{\phi}{m_{Pl}}}, \quad v(\varphi) = \frac{3}{m_{Pl}^2} V(\phi), \quad h = 3H, \quad (11)$$

and rewrite equations (1) and (2) as

$$\ddot{\varphi} + h \dot{\varphi} + \frac{1}{2} v'(\varphi) = 0, \quad (12)$$

and

$$h^2 = \varphi^2 + v(\varphi), \quad (13)$$

respectively.

In order to discuss inflaton models in the Hamilton-Jacobi formalism we consider a reduced space of initial conditions $(\varphi, \dot{\varphi}) \in \mathbb{R}^2$ for (12) and (13) such that

$$\varphi \geq \varphi_0, \quad \dot{\varphi} < 0, \quad (14)$$
where $\varphi_0$ will be assumed to be a fixed value of $\varphi$ such that the potential $v$ and its first derivative $v'$ are smooth and strictly positive for $\varphi \geq \varphi_0$. The map $(\varphi, \dot{\varphi}) \mapsto (\varphi, h)$ enables us to describe the dynamics of (12)-(13) on the subset
\[ R = \{(\varphi, h) \in \mathbb{R}^2 : \varphi \geq \varphi_0, \quad \sqrt{v(\varphi)} < h < +\infty\}, \tag{15} \]
of the $(\varphi, h)$ plane. The Hamilton-Jacobi formulation of the equations (1)- (2) is given by the couple of equations
\[ h'(\varphi)^2 = h(\varphi)^2 - v(\varphi), \tag{16} \]
and
\[ \dot{\varphi} = -h'(\varphi), \tag{17} \]
Here primes denote derivatives with respect to $\varphi$ and the Hubble function $h$ is assumed to be the positive root
\[ h = \left(\varphi^2 + v(\varphi)\right)^{1/2}. \tag{18} \]
The set $R$ plays the role of the phase space of the formalism. Each solution $h = h(\varphi)$ of (16) determines a corresponding implicit solution $\varphi(t)$ of (17) given by
\[ t = \cdots \int_{\varphi(0)}^{\varphi(t)} \frac{d\varphi}{h'(\varphi)}. \tag{19} \]

From the physical point of view, the early universe is assumed to emerge from a state with energy density $3m_{Pl}^2 H$ of the same order of the Planck density $m_{Pl}^4$. Below that density the classical inflationary description of the universe is not valid. Hence the only physical constraint required for the initial data of a classical inflationary universe is that its energy density $3m_{Pl}^2 H$ should not be larger than $m_{Pl}^4$ or, equivalently, $h \leq m_{Pl}^2$, which in Planck units ($G = c = \hbar = 1$) means
\[ h < h_p := \frac{1}{8\pi} \approx 0.0398. \tag{20} \]

B. Inflation and kinetic dominance regions

The inflation period of the universe evolution is characterized by an accelerated universe expansion $\ddot{a} > 0$. From the identity
\[ \frac{\ddot{a}}{a} = \frac{1}{3m_{Pl}^2} \left(V(\phi) - \dot{\phi}^2\right), \tag{21} \]
it follows that this period is determined by the constraint
\[ \dot{\phi}^2 < V(\phi). \] (22)

Then it follows at once that the inflation region (22) in \( R \) is characterized by
\[ \sqrt{v} < h < \sqrt{\frac{3v}{2}}. \] (23)

For a successful solution to the horizon and flatness cosmological problems it is required
that the amount of inflation during the period of inflation
\[ N = \int_{\phi_{\text{in}}}^{\phi_{\text{end}}} \frac{H}{\dot{\phi}} d\phi = \frac{1}{3} \int_{\phi_{\text{in}}}^{\phi_{\text{end}}} \frac{h}{h'} d\varphi, \] (24)
should be close to \( N \sim 60 \) [6, 7, 31, 32]. Thus, given a solution \( h = h(\varphi) \) of (16) it is important to determine the values \( \varphi_{\text{in}}, \varphi_{\text{end}} \) for which inflation starts and ends, respectively. Due to (23) both values satisfy
\[ h(\varphi) = \sqrt{\frac{3v(\varphi)}{2}}. \] (25)

The approximate solutions (9) of (1) correspond to approximate solutions of (16) of the form
\[ h \sim e^{\pm \varphi} b^{-1} \text{ as } \varphi \to \pm \infty, \] (26)

where \( b \) is an arbitrary strictly positive parameter. Due to the symmetry \((v(\varphi), h(\varphi), \varphi(t)) \to (v(-\varphi), h(-\varphi), -\varphi(t))\) of the equations (16)-(17), without loss of generality we restrict our analysis to series expansions of solutions of (16) such that
\[ h \sim \frac{e^{\varphi}}{b} \text{ as } \varphi \to +\infty. \] (27)

Solutions \( h = h(\varphi) \) of (16) which have the asymptotic form (27) emerge from the KD region and blow up at a finite time \( t = t^* \) given by
\[ t^* = -\int_{\varphi(0)}^{\infty} \frac{d\varphi}{h'(\varphi)}. \] (28)
FIG. 1. Reduced hubble parameter for the quadratic model \( v(\varphi) = m^2 \varphi^2 \). The black line shows the rescaled numerical solution \( \frac{h}{m} = \frac{h(\varphi)}{m} \) of (16) such that \( h(60) = 10^3 \) m. The region between blue and red dotted lines is the inflation region.

FIG. 2. The left figure shows the solution of Figure 1 entering the inflation region in a KD regime and approaching the slow-roll regime (blue dotted line). The right figure shows the same solution leaving the inflation region.

III. PSI SERIES IN THE HAMILTON-JACOBI FORMULATION

In this section we determine a one-parameter family of psi series formal solutions of the equation (16) for the models with polynomial-exponential potentials (4) and Starobinsky potentials (5).
A. Psi series for polynomial-exponential potentials

Let us consider the differential equation (16) for a potential $v(\varphi)$ of the family (4). We look for psi series solutions of the form

$$h(\varphi) = \frac{e^{\varphi}}{b} + \sum_{n=1}^{\infty} h_n(\varphi) e^{-(n-1)\varphi},$$

(29)

where the coefficients $h_n(\varphi)$ are polynomial functions of $\varphi$ and $b$ is a non-zero positive real parameter.

By substituting (29) into (16), we obtain

$$\frac{2}{b} \sum_{n=1}^{\infty} (h'_n - nh_n) e^{-(n-2)\varphi} + \sum_{n=0}^{N} v_n(\varphi) e^{-n\varphi} =$$

$$- \sum_{n=2}^{\infty} \left[ \sum_{j+k=n} (h'_j - (j-1)h_j)(h'_k - (k-1)h_k) - h_jh_k \right] e^{-(n-2)\varphi}.$$  

(30)

Then, identifying the coefficients of $e^{\varphi}$ in (30) we have that $h_1$ must satisfy the equation

$$h'_1 - h_1 = 0,$$

whose polynomial solution is

$$h_1(\varphi) = 0.$$  

(31)

From (30) and using (31) it follows that identifying the coefficients of $e^{-n\varphi}$ for $n = 0, 1, \ldots, N$ implies the recursion relation

$$h'_{n+2} - (n+2)h_{n+2} = -\frac{b}{2} \left[ v_n + \sum_{j+k=n+2, j,k \geq 2} (h'_j - (j-1)h_j)(h'_k - (k-1)h_k) - h_jh_k \right].$$  

(32)

Furthermore, identifying the coefficients of $e^{-n\varphi}$ for $n > N$ leads to the recursion relation

$$h'_{n+2} - (n+2)h_{n+2} = -\frac{b}{2} \sum_{j+k=n+2, j,k \geq 2} (h'_j - (j-1)h_j)(h'_k - (k-1)h_k) - h_jh_k.$$  

(33)

The equations (32) and (33) are nonhomogeneous linear ordinary differential equations with constant coefficients for $h_{n+2}$. The nonhomogeneous terms depend on the coefficients $h_j$ with $j = 2, \ldots, n$ and on the polynomial coefficients $v_n(\varphi)$ with $n = 0, \ldots, N$ of the potential function (4). Therefore, it follows that the coefficients $h_n$ of the series (29) can be recursively determined as polynomials in $\varphi$. The recursion relations can be formally solved.
and provide us with the polynomial solutions

\[ h_{n+2}(\varphi) = \frac{b}{2} e^{(n+2)\varphi} \int_{\varphi}^{\infty} e^{-(n+2)s} v_n(s) \]

\[ + \frac{b}{2} e^{(n+2)\varphi} \int_{\varphi}^{\infty} e^{-(n+2)s} \sum_{j+k=n+2, j,k \geq 2} ((h'_j(s) - (j-1)h_j(s))(h'_k(s) - (k-1)h_k(s)) - h_j(s)h_k(s)) ds, \]

for \( n = 0, 1, \ldots, N \) and

\[ h_{n+2}(\varphi) = \]

\[ \frac{b}{2} e^{(n+2)\varphi} \sum_{j+k=n+2, j,k \geq 2} \int_{\varphi}^{\infty} e^{-(n+2)s} ((h'_j(s) - (j-1)h_j(s))(h'_k(s) - (k-1)h_k(s)) - h_j(s)h_k(s)) ds, \]

for \( n > N \).

In this way we have proved that the differential equation (16) for potentials \( v(\varphi) \) of the form (4) admits a (formal) one-parameter family of psi series solutions of the form

\[ h(\varphi) = \frac{e^\varphi}{b} + \sum_{n=2}^{\infty} h_n(\varphi)e^{-(n-1)\varphi}. \]

The first few equations (32) for \( N \geq 2 \) are

\[ h'_2 - 2h_2 = -\frac{b}{2} v_0, \]

\[ h'_3 - 3h_3 = -\frac{b}{2} v_1, \]

\[ h'_4 - 4h_4 = -\frac{b}{2} [v_2 + (h'_2)^2 - 2h_2h'_2]. \]
The corresponding solutions \((34)\) are

\[
\begin{align*}
  h_2(\varphi) &= \frac{b}{2} e^{2\varphi} \int_{\varphi}^{\infty} v_0(s)e^{-2s}ds, \\
  h_3(\varphi) &= \frac{b}{2} e^{3\varphi} \int_{\varphi}^{\infty} v_1(s)e^{-3s}ds, \\
  h_4(\varphi) &= \frac{b}{2} e^{4\varphi} \int_{\varphi}^{\infty} v_2(s)e^{-4s}ds + \frac{b^3}{8} e^{4\varphi} \int_{\varphi}^{\infty} v_0(s)^2 e^{-4s}ds \\
  &\quad - \frac{b^3}{4} e^{4\varphi} \int_{\varphi}^{\infty} e^{-2s} v_0(s_1) \left( \int_{s_1}^{\infty} e^{-2s_2} v_0(s_2)ds_2 \right) ds_1.
\end{align*}
\]

\(38\)

**B. Polynomial potentials**

In the polynomial case of \((4)\) \(v(\varphi) = v_0(\varphi)\) with \(v_0(\varphi)\) being a polynomial of degree \(d\), the family of psi series \((36)\) reduces to the form

\[
h(\varphi) = e^{\varphi} b + \sum_{n=1}^{\infty} b^{2n-1} \gamma_n(\varphi) e^{-(2n-1)\varphi},
\]

\(39\)

where \(\gamma_1\) is the unique polynomial of degree \(d\) which satisfies the equation

\[
\gamma_1' - 2\gamma_1 = -\frac{1}{2} v_0,
\]

\(40\)

and the coefficients \(\gamma_n(\varphi) \text{ (} n \geq 2 \text{)}\) are polynomials of degree \(nd - 1\), independent of the parameter \(b\), which can be recursively determined by

\[
\gamma_{n+1}' - 2(n+1)\gamma_{n+1}
\]

\[
= -\frac{1}{2} \sum_{j+k=n+1, j,k \geq 1} \left( (\gamma_j' - (2j-1)\gamma_j)(\gamma_k' - (2k-1)\gamma_k) - \gamma_j\gamma_k \right).
\]

\(41\)

Indeed, if we set \(n = 2m - 1 \text{ (} m \geq 1 \text{)}\) in \((33)\) we have that

\[
h'_{2m+1} - (2m+1)h_{2m+1}
\]

\[
= \frac{b}{2} \sum_{j+k=2m+1, j,k \geq 2} \left( (h_j' - (j-1)h_j)(h_k' - (k-1)h_k) - h_jh_k \right).
\]
Now, taking into account (31) and applying induction in \( m \) it is clear that
\[
h_{2m+1} \equiv 0 \quad \text{for all} \quad m \geq 0. \tag{42}
\]

In order to make explicit the dependence of \( h(\varphi) \) on the arbitrary parameter \( b \), we introduce the functions \( \gamma_n(\varphi) := h_{2n}(\varphi)/b^{2n-1} \). Thus, for \( n = 0 \) equation (32) reduces to (40) and (33) becomes the recursion relation (41). We notice that the coefficients \( \gamma_n(\varphi) \) are independent of \( b \). The first few equations (41) are
\[
\begin{align*}
\gamma_2' - 4\gamma_2 &= -\frac{1}{2}[(\gamma_1')^2 - 2\gamma_1\gamma_1'], \\
\gamma_3' - 6\gamma_3 &= -[\gamma_1'\gamma_2 - \gamma_1\gamma_2' - 3\gamma_1\gamma_2 + 2\gamma_1\gamma_2], \\
\gamma_4' - 8\gamma_4 &= -\frac{1}{2}[2\gamma_1'\gamma_3 + (\gamma_2')^2 - 2\gamma_1\gamma_3' - 6\gamma_2\gamma_2' - 10\gamma_1\gamma_3 + 8\gamma_1\gamma_3 + 8\gamma_2^2].
\end{align*}
\]

In particular, (40) and (43) imply the following explicit relations for general polynomial potentials \( v_0(\varphi) \)
\[
\begin{align*}
\gamma_1(\varphi) &= \frac{1}{2}e^{2\varphi} \int_{\varphi}^{\infty} e^{-2s} v_0(s) ds, \tag{46} \\
\gamma_2(\varphi) &= -\frac{1}{4}e^{4\varphi} \int_{\varphi}^{\infty} e^{-2s} v_0(s) \left( \int_{s_1}^{\infty} e^{-2s_2} v_0(s_2) ds_2 \right) ds_1 \\
&\quad + \frac{1}{8}e^{4\varphi} \int_{\varphi}^{\infty} e^{-4s} v_0(s_1^2) ds_1. \tag{47}
\end{align*}
\]

From (40) we have that \( \gamma_1 \) is a polynomial of the same degree \( d \) as \( v_0(\varphi) \). Consequently the right-hand side of (43) is a polynomial of degree \( 2d - 1 \) and so is \( \gamma_2 \). Now, since the coefficient of \( \gamma_n\gamma_1 \) in the right-hand side of (41) is given by \(-2(n-1)\), which is non-zero for \( n \geq 2 \), then using induction in \( n \) it follows that for \( n \geq 2 \) the coefficient \( \gamma_n(\varphi) \) is a polynomial in \( \varphi \) of degree \( nd - 1 \).

Next we discuss two illustrative examples.
1. The quadratic potential

For the quadratic potential

\[ v(\phi) = m^2 \phi^2, \]  

we have an expansion for \( h \) of the form (39) with \( \gamma_1 \) satisfying (40)

\[ \gamma_1' - 2\gamma_1 = -\frac{1}{2} m^2 \phi^2. \]

Hence we obtain

\[ \gamma_1(\phi) = \frac{m^2}{8} (1 + 2\phi + 2\phi^2), \]  

so that the first two terms of the psi series of \( h \) are

\[ h(\phi) = \frac{e^\phi}{b} + \frac{b m^2}{8} (1 + 2\phi + 2\phi^2)e^{-\phi} + \cdots. \]  

The coefficients \( \gamma_n (n \geq 2) \) in (39) are determined by (41). For example we obtain

\[ \gamma_2(\phi) = -\frac{m^4}{1024} \left( 5 + 20\phi + 40\phi^2 + 32\phi^3 \right), \]

\[ \gamma_3(\phi) = \frac{m^6}{1990656} \left( 703 + 4218\phi + 12654\phi^2 + 18504\phi^3 + 7344\phi^4 - 5184\phi^5 \right). \]

2. The Higgs potential

The KD period for the inflaton model with a Higgs potential

\[ v(\phi) = g^2 (\phi^2 - \lambda^2)^2, \]  

has been studied in [11], where approximate expressions for the inflaton field and the Hubble parameter as functions of \( t \) have been obtained. According to our general result for models with polynomial potentials, the Higgs model admits a psi series of the form (39). In
particular, one easily finds that the first coefficients of this psi-series are given by

\[ \gamma_1(\varphi) = \frac{g^2}{8} \left[ 3 + 6\varphi + 6\varphi^2 + 4\varphi^3 + 2\varphi^4 - 2\lambda^2(1 + 2\varphi + 2\varphi^2) + 2\lambda^4 \right], \tag{52} \]

\[ \gamma_2(\varphi) = \frac{g^4}{16384} \left[ \lambda^6(1024\varphi + 768) + \lambda^4 \left( -3072\varphi^3 - 4864\varphi^2 - 3968\varphi - 1760 \right) + \lambda^2 \left( 3072\varphi^5 + 7424\varphi^4 + 9408\varphi^2 + 4704\varphi + 1176 \right) \right. \]

\[ \left. - (1024\varphi^7 + 3328\varphi^6 + 6528\varphi^5 + 8928\varphi^4 + 8928\varphi^3 + 6696\varphi^2 + 3348\varphi + 837) \right]. \tag{53} \]

C. Starobinsky’s potentials

We now consider the class of potentials of the form (5). It includes as a particular case the Starobinsky model \[1, 33, 34\]

\[ v(\varphi) = \lambda(1 - e^{-\alpha\varphi})^2. \tag{54} \]

It can be proved \[30\] that the existence of approximate solutions (27) of Eq (16) is only possible if

\[ N\alpha > -2, \tag{55} \]

so that we will henceforth assume that (55) is satisfied.

We look for psi series solutions of (16) of the form

\[ h(\varphi) = \frac{e^\varphi}{b} + \sum_{n=1}^{\infty} h_n(u)e^{-(n-1)\varphi}, \tag{56} \]

where the coefficients \( h_n \) are polynomials in

\[ u := e^{-\alpha\varphi}. \tag{57} \]

If we substitute the series (56) into (16) we get

\[ \frac{2}{b} \sum_{n=1}^{\infty} (\alpha u h'_n + n h_n)e^{-(n-2)\varphi} = w(u) \]

\[ + \sum_{n=2}^{\infty} \left( \sum_{j+k=n} (\alpha u h'_j + (j - 1)h_j)(\alpha u h'_k + (k - 1)h_k) - h_j h_k \right) e^{-(n-2)\varphi}, \tag{58} \]
where
\[ w(u) := \sum_{n=0}^{N} v_n u^n. \]  

(59)

Since we assume that \( \alpha \) is an irrational number, the powers of \( u = e^{-\alpha \varphi} \) and the powers of \( e^{\alpha \varphi} \) are linearly independent functions, consequently the coefficients of \( e^{-k \varphi} \) for \( k \geq -1 \) in both sides of (58) must be equal. Then, for \( k = -1 \) we obtain \( \alpha u h_1' + h_1 = 0 \) and, since \( h_1 \) is a polynomial in \( u \), we have that
\[ h_1 \equiv 0. \]  

(60)

Using (60) for \( k = 0 \) we get
\[ \alpha u h_2' + 2h_2 = \frac{b}{2} w(u). \]  

(61)

Equation (61) has a polynomial solution of degree \( N \). Proceeding in the same way, vanishing the coefficients of \( e^{-n \varphi} \) in (58) with \( n \geq 1 \), we obtain the recurrence relation
\[ \alpha u h_{n+2}' + (n + 2)h_{n+2} = \frac{b}{2} \sum_{j+k=n+2, j,k \geq 2} \left( (\alpha u h_j' + (j-1)h_j)(\alpha u h_k' + (k-1)h_k) - h_j h_k \right). \]  

(62)

From (60), (61) and (62) we conclude that all the coefficients \( h_n(u) \) in (56) are recursively determined by (61) and (62) as polynomials in \( u \). Moreover, if we set \( n = 2m - 1 \) in (62), then from (60) and applying induction in \( m \), it follows immediately that
\[ h_{2m+1} \equiv 0, \quad m = 0, 1, 2, \ldots. \]

To make explicit the dependence of \( h(\varphi) \) on the arbitrary parameter \( b \), we introduce the functions
\[ \gamma_n(u) := \frac{h_{2n}(u)}{b^{2n-1}}. \]  

(63)

Thus, equations (61) and (62) reduce to
\[ \alpha u \gamma_1' + 2\gamma_1 = \frac{1}{2} w(u), \]  

(64)

and
\[ \alpha u \gamma_{n+1}' + 2(n + 1)\gamma_{n+1} = \frac{1}{2} \sum_{j+k=n+1, j,k \geq 1} \left( (\alpha u \gamma_j' + (2j-1)\gamma_j)(\alpha u \gamma_k' + (2k-1)\gamma_k) - \gamma_j \gamma_k \right), \]  

(65)
respectively. Here primes indicate derivatives with respect to $u$.

Applying induction with respect to $n$ in (65) we easily conclude that $\gamma_n(u)$ is a polynomial in $u$ of degree at most $nN$ (in particular $\gamma_1$ is a polynomial of degree $N$). Therefore we have proved the existence of a psi series expansion of the form

$$h(\varphi) = \frac{e^{\varphi}}{b} \left(1 + \sum_{n=1}^{\infty} b^2 n \gamma_n(e^{-\alpha \varphi}) e^{-2n \varphi}\right).$$ (66)

For example, we get the following explicit relations for $\alpha > 0$

$$\gamma_1(u) = \frac{1}{2\alpha} u^{-2/\alpha} \int_0^u s^{2/\alpha-1} w(s) ds,$$ (67)

$$\gamma_2(u) = -\frac{1}{4\alpha^2} u^{-4/\alpha} \int_0^u s^{2/\alpha-1} w(s) \left(\int_0^{s_1} s_2^{2/\alpha-1} w(s_2) ds_2\right) ds_1$$

$$+ \frac{1}{8\alpha} u^{-4/\alpha} \int_0^u s_1^{4/\alpha-1} w(s_1)^2 ds_1,$$ (68)

For $\alpha < 0$ the same expressions hold with the lower limits of the integrals substituted by $\infty$.

1. The Starobinsky model

The potential function of the Starobinsky model is given by

$$v(\varphi) = \lambda (1 - e^{-\alpha \varphi})^2, \quad \alpha > -1.$$ (69)

Then the corresponding equation (64) reduces to

$$\alpha u \gamma'_1 + 2 \gamma_1 = \frac{\lambda}{2} (1 - u)^2,$$ (70)

and the first two equations (65) are

$$\alpha u \gamma'_2 + 4 \gamma_2 = \frac{(\alpha u)^2}{2} (\gamma'_1)^2 + \alpha u \gamma_1 \gamma'_1,$$

$$\alpha u \gamma'_3 + 6 \gamma_3 = (\alpha u)^2 \gamma'_1 \gamma'_2 + 3 \alpha u \gamma'_1 \gamma_2 + \alpha u \gamma_1 \gamma'_2 + 2 \gamma_1 \gamma_2.$$
Therefore, the first polynomial coefficients of the expansion (56) turn out to be given by
\[
\gamma_1(u) = \frac{u^2}{4(\alpha + 1)} - \frac{u}{\alpha + 2} + \frac{1}{4},
\]
\[
\gamma_2(u) = \frac{\alpha u^4}{32(\alpha + 1)^2} - \frac{\alpha(2\alpha + 3)u^3}{4(\alpha + 1)(\alpha + 2)(3\alpha + 4)} + \frac{\alpha(5\alpha + 6)u^2}{16(\alpha + 1)(\alpha + 2)^2} - \frac{\alpha u}{4(\alpha + 2)(\alpha + 4)},
\]
\[
\gamma_3(u) = \frac{\alpha(4\alpha + 1)u^6}{384(\alpha + 1)^3} - \frac{\alpha(4\alpha + 5)(9\alpha^2 + 16\alpha + 4)u^5}{32(\alpha + 1)^2(\alpha + 2)(3\alpha + 4)(5\alpha + 6)}
+ \frac{\alpha (162\alpha^4 + 739\alpha^3 + 1190\alpha^2 + 776\alpha + 160)u^4}{128(\alpha + 1)^2(\alpha + 2)^2(2\alpha + 3)(3\alpha + 4)}
- \frac{\alpha (42\alpha^4 + 301\alpha^3 + 674\alpha^2 + 584\alpha + 160)u^3}{48(\alpha + 1)(\alpha + 2)^3(\alpha + 4)(3\alpha + 4)}
+ \frac{\alpha (13\alpha^2 + 58\alpha + 40)u^2}{64(\alpha + 2)^2(\alpha + 3)(\alpha + 4)} - \frac{\alpha u}{16(\alpha + 4)(\alpha + 6)}.
\]

D. Loglinear series

Once we have determined the psi series (36), (39) and (66) for \(h(\varphi)\), then from (17) psi series of loglinear type depending on the variable \(t\) [8–11] for the inflaton field \(\varphi\) and the reduced Hubble parameter \(h\) can be derived. Thus, for polynomial potentials, if we insert (39) into (17) we get
\[
\dot{\varphi} = -\frac{e^\varphi}{b} - \sum_{n=1}^{\infty} \frac{b^{2n-1}\left(\gamma'_n(\varphi) - (2n - 1)\gamma_n(\varphi)\right)}{2n-1} e^{-(2n-1)\varphi}. \tag{72}
\]
It can be easily checked that (72) admits a family of formal psi series solutions of the form
\[
\varphi(t) = -x + \sum_{n=1}^{\infty} a_n(x)(t - t^*)^{2n}, \tag{73}
\]
where
\[
x := \log \left( \frac{t - t^*}{b} \right), \tag{74}
\]
and $\alpha_n(x)$ can be recursively determined as polynomials in $x$. Next we sketch the main ideas of the proof. First, we introduce the polynomials $\theta_n(\varphi)$ and their coefficients $\theta_{n,j}$, $j = 0, \ldots, nd$ through

$$\theta_n(\varphi) := \gamma_n(\varphi) - (2n - 1)\gamma_n(\varphi) = \sum_{j=0}^{nd} \theta_{n,j}\varphi^j,$$

and then we rewrite (72) as

$$\dot{\varphi} = -\frac{e^{\varphi}}{b} - \sum_{n=1}^{\infty} \sum_{j=0}^{nd} \theta_{n,j}\varphi^j b^{2n-1} e^{-(2n-1)\varphi}. \quad (75)$$

and replace (73) into (75). For the left hand side we have

$$\dot{\varphi} = -\frac{1}{t - t^*} + \sum_{n=1}^{\infty} [\alpha_n'(x) + 2n\alpha_n(x)](t - t^*)^{2n-1}. \quad (76)$$

In order to expand the right hand side of (75) in odd powers of $(t - t^*)$ we introduce the Bell’s polynomials [35] defined through

$$\exp \left( \sum_{n=1}^{\infty} x_n z^n \right) = \sum_{n=0}^{\infty} C_n(x_1, \ldots, x_n) z^n. \quad (77)$$

Then we have that

$$b^{2n-1} e^{-(2n-1)\varphi} = (t - t^*)^{2n-1} \sum_{m=0}^{\infty} C_m(A_{m,n}(x))(t - t^*)^{2m}. \quad (78)$$

where we are introducing the vectorial functions

$$A_{m,n}(x) := -(2n - 1)\left(\alpha_1(x), \ldots, \alpha_m(x)\right).$$

Thus, by replacing (76) and (78) into (75) and equating the coefficients of $(t - t^*)^{2n-1}$ we obtain the equations

$$\alpha_n' + (2n + 1)\alpha_n = \alpha_n - C_n(A_{n,0}) -$$

$$\text{coeff} \left[ \sum_{m=1}^{n} \sum_{j=0}^{md} \sum_{l=0}^{n-m} \theta_{m,j} C_l(A_{l,m}) \left( -x + \sum_{k=1}^{n-1} \alpha_k \tau^{2k} \right)^j, \tau^{2(n-m-l)} \right]. \quad (79)$$

The first equation (79) takes the form

$$\alpha_1' + 3\alpha_1 = -\theta_1(-x),$$
so that \( \alpha_1 \) can be determined as a polynomial of degree \( d \). Then, since (79) is an ordinary linear differential equation for \( \alpha_n \) with nonhomogeneous term depending only on \( \alpha_1, \ldots, \alpha_{n-1} \), then using induction in \( n \) it follows that the coefficients \( \alpha_n(x) \) can be recursively determined as polynomials of degree at most \( nd \). For instance, equation (79) for \( n = 2 \) is

\[
\alpha_2' + 5\alpha_2 = -\frac{\alpha_1(x)^2}{2} + \alpha_1(x)[\theta_1(-x) - \theta_1'(-x)] - \theta_2(-x).
\]

Furthermore, by substituting (73) into (39), we find an expansion

\[
h(t) = \frac{1}{t - t^*} + \sum_{n=1}^{\infty} \beta_n(x)(t - t^*)^{2n-1}, \tag{80}
\]

where \( \beta_n(x) \) are polynomials in \( x \) that can be written in terms of \( \alpha_k, \gamma_k, k = 1, \ldots, n \). For example, we have that

\[
\beta_1(x) = \alpha_1(x) + \gamma_1(-x),
\]

\[
\beta_2(x) = \alpha_2(x) + \frac{\alpha_1(x)^2}{2} + \alpha_1(x)[\gamma_1'(-x) - \gamma_1(-x)] + \gamma_2(-x).
\]

The psi series (73)-(80) are the logilinear expansions derived in [10] for inflaton models with polynomial potentials in a flat universe.

Analogously, for the Starobinsky potentials (5), replacement of (66) into (17) leads us to

\[
\dot{\phi} = -\frac{e^\varphi}{b} + \sum_{n=1}^{\infty} b^{2n-1} \left( \alpha e^{-\alpha\varphi} \gamma_n'(e^{-\alpha\varphi}) + (2n - 1) \gamma_n(e^{-\alpha\varphi}) \right) e^{-(2n-1)\varphi}. \tag{81}
\]

It can be proved that (81) admits a family of psi series solutions of the form

\[
\varphi(t) = -\log \left( \frac{t - t^*}{b} \right) + \sum_{n=1}^{\infty} \alpha_n(\sigma)(t - t^*)^{2n}, \tag{82}
\]

where

\[
\sigma := \left( \frac{t - t^*}{b} \right)^\alpha, \tag{83}
\]

and \( \alpha_n(\sigma) \) can be recursively determined as polynomials in \( \sigma \). We outline the main ideas of the proof. By introducing the polynomials \( \theta_n(u) \) and their coefficients \( \theta_{n,j}, j = 0, \ldots, nN \) through

\[
\theta_n(u) := \alpha u \gamma_n'(u) + (2n - 1) \gamma_n(u) = \sum_{j=0}^{nN} \theta_{n,j} u^j,
\]
the equation \( \varphi \) takes the form

\[
\dot{\varphi} = -\frac{e^\varphi}{b} + \sum_{n=1}^{\infty} \theta_{n,j} b^{2n-1} e^{-(2n-1+j\alpha)\varphi}.
\] (84)

The left hand side of (84) is expanded in odd powers of \((t - t^*)\) as

\[
\dot{\varphi} = -\frac{1}{t - t^*} + \sum_{n=1}^{\infty} \left( \alpha \sigma \alpha_n'(\sigma) + 2n\alpha_n(\sigma) \right) (t - t^*)^{2n-1}.
\] (85)

By using the Bell’s polynomials (77) we can also expand the terms in the series of the right hand side of (84) as

\[
b^{2n-1} e^{-(2n-1+j\alpha)\varphi} = (t - t^*)^{2n-1}\sigma^2 \sum_{m=0}^{\infty} C_m(A_{m,n,j}(\sigma))(t - t^*)^{2m},
\] (86)

where the vectorial polynomials \( A_{m,n,j} \) have been defined as

\[
A_{m,n,j}(\sigma) := -(\alpha j + 2n - 1)\left( \alpha_1(\sigma), \ldots, \alpha_n(\sigma) \right).
\]

Thus, by replacing (85) and (86) into (84), taking into account that as \( \alpha \) is an irrational number, the powers of \( \sigma \) and the powers of \((t - t^*)\) are linearly independent, then equating the coefficients of \((t - t^*)^{2n-1}\) in both sided of the equations, we obtain that \( \alpha_n \) satisfies the differential equation

\[
a\sigma \alpha_n' + (2n + 1)\alpha_n = \alpha_n - C_n(A_{n,0,0}) + \sum_{k+m=n, k \geq 1, m \geq 0} \sum_{j=0}^{kJ} \theta_{k,j} \sigma^j C_m(A_{m,k,j}(\sigma)).
\] (87)

As the right hand side of (87) depends only on \( \alpha_j, j = 1, \ldots, n - 1 \), this equation shows that the coefficients \( \alpha_n(\sigma) \) can be recursively obtained as polynomials of degree at most \( nN \). Thus, for instance we have that the two first equations (87) are

\[
\alpha \sigma \alpha_1' + 3\alpha_1 = \theta_1(\sigma),
\]

\[
\alpha \sigma \alpha_2' + 5\alpha_2 = -\frac{\alpha_1^2(\sigma)}{2} - \alpha_1(\sigma) [\alpha \sigma \theta_1'(\sigma) + \theta_1(\sigma)] + \theta_2(\sigma).
\]

Furthermore, the replacement of (82) into (86) provides us with the formal psi series for the reduced Hubble parameter

\[
h(t) = \frac{1}{t - t^*} + \sum_{n=1}^{\infty} \beta_n(\sigma)(t - t^*)^{2n-1},
\] (88)
where $\beta_n(\sigma)$ are polynomials in $\sigma$ that can be determined in terms of $\alpha_k, \gamma_k, k = 1, \ldots, n$. Thus, for example, we have that

$$\beta_1(\sigma) = \alpha_1(\sigma) + \gamma_1(\sigma),$$

$$\beta_2(\sigma) = \alpha_2(\sigma) + \frac{\alpha_1(\sigma)^2}{2} + \gamma_2(\sigma) - [\alpha \sigma \gamma_1'(\sigma) + \gamma_1(\sigma)] \alpha_1(\sigma).$$

We notice that for the Starobinsky model [69], the psi series (82)-(88) correspond to the logolinear expansions for the inflaton model determined in [10] for a flat universe, $\lambda = \Lambda^4$ and $\alpha = \pm \frac{2}{3}$.

The determination of logolinear series for the general class of polynomial exponential potentials [11] is more involved and together with the logolinear series for some generalizations of the class of Starobinsky potentials [5] will be the subject of a future work.

**IV. APPLICATIONS**

Many analytical calculations for inflation models assume the slow-roll approximation of (1)-(2)

$$H^2 \sim \frac{1}{3m_{pl}^2} V, \quad \dot{\phi} \sim -\frac{1}{3H} \frac{dV}{d\phi},$$

which in terms of the reduced Hubble function $h(\varphi)$ can be expressed as

$$h \sim \sqrt{v}, \quad h' \sim (\sqrt{v})'. \quad (90)$$

However for calculations involving contributions of the KD period it is more appropriate to use truncations of psi series.

**A. The inflation region**

From [20] it follows that we may estimate the value $\varphi_{class}(b)$ of the inflaton field at the beginning of the classical period as a function of $b$ by means of the equation

$$h(\varphi_{class}, b) = h_p = \frac{1}{8\pi}. \quad (91)$$

In particular, since the approximation (27) is model independent, we obtain the general estimation

$$\varphi_{class} \approx \log \left( \frac{b}{8\pi} \right). \quad (92)$$
A standard procedure \[6, 7\] to calculate the boundary values \(\varphi_{in}\) and \(\varphi_{end}\) of the inflation period proceeds as follows: we first determine \(\varphi_{end}\) by imposing \(\epsilon_V = 1\) where

\[\epsilon_V = \frac{m_{Pl}^2}{2} \left( \frac{V}{V'} \right)^2,\] (93)

is the potential slow-roll parameter. It leads to the equation

\[v'(\varphi_{end}) = \frac{2}{\sqrt{3}} v(\varphi_{end}).\] (94)

Then \(\varphi_{in}\) is calculated by using the slow-roll approximation during the whole period of inflation and by adjusting the amount of inflation

\[N \approx \frac{1}{3} \int_{\varphi_{in}}^{\varphi_{end}} \frac{\sqrt{v(\varphi)}}{v(\varphi)} d\varphi\] (95)

to \(N \sim 60\). This method leads to the same inflation period for all the solutions of the inflaton model, since (94) and (95) depend only on the potential \(v(\varphi)\). Nevertheless, in general, the inflation period depends on the solution used since each solution has a different kinetically dominated fraction of its inflation region.

In order to include the effect of the KD part of the inflation period, we may proceed as follows. We first calculate the function \(\varphi_{in}(b)\) by means of the equation (25)

\[h(\varphi_{in}, b) = \sqrt{\frac{3v(\varphi_{in})}{2}},\] (96)

where \(h\) is approximated by a truncation with a given number \(n + 1\) of terms of the psi series ((36), (39) or (66)) corresponding to the model. Then, to calculate \(N(b)\) we look for a value \(\varphi^*(b)\) such that the KD approximation holds for the interval \(\varphi \in (\varphi^*(b), \varphi_{in}(b))\), while the slow-roll approximation holds for the remaining part of the inflation period. Thus, we estimate the amount of inflation \(N(b)\) for the psi series corresponding to \(b\) by

\[N(b) \approx \frac{1}{3} \int_{\varphi_{end}}^{\varphi^*(b)} \frac{\sqrt{v(\varphi)}}{v(\varphi)} d\varphi + \frac{1}{3} \int_{\varphi^*(b)}^{\varphi_{in}(b)} \frac{h(\varphi, b)}{h(\varphi, b)} d\varphi.\] (97)

The point now is how to get an appropriate value for \(\varphi^*(b)\). To this end we observe that the KD approximation means that \(h' \sim h\) and \(h \gg \sqrt{v}\), and that it does not hold as \(h'(\varphi)\) gets close to zero. Thus, from (16) we have that the truncated psi series cannot provide a good approximation when \(h(\varphi, b)^2\) becomes small. Consequently, we take \(\varphi^{(n)}(b)\) \((\varphi^{(n)}(b) < \varphi_{in}(b))\) as the local minimum of \(h(\varphi, b)^2\) closest to \(\varphi_{in}(b)\). Numerical evidence
shows that the KD approximation is not acceptable for \( \phi \) very close to \( \phi^{(n)}(b) \), then \( \phi^*(b) \) must be strictly larger than \( \phi^{(n)}(b) \). In order to introduce this correction we use for \( \phi^*(b) \) an expression of the form

\[
\phi^*(b) = \alpha \phi^{(n)}(b) + \beta \phi_{in}(b),
\]

with two real positive parameters \( \alpha \) and \( \beta \) such that \( \alpha + \beta = 1 \). For example, we have taken \( \alpha = 0.9 \) and \( \beta = 0.1 \) in the examples of figures 3-7. Figures 4 and 7 show how the KD approximations fit accurately the numerical approximation for \( \phi \in (\phi^*(b), \phi_{in}(b)) \).

The approximation \([97]\) can be used to select the appropriate value of the parameter \( b \), and consequently the initial conditions, corresponding to a solution of \([16]\) with a previously fixed value of \( N(b) \).

Figures 3 and 4 show the application of this method to the quadratic model \( v(\phi) = m^2 \phi^2 \). We use the truncated psi series with 7 terms. It is worthy to point out that very good approximations are also obtained when fewer terms 4, 5 or 6 are used. We illustrate this fact in figure 3 (right) where we use also truncated series with 4, 5 or 6 terms. It means that the KD approximation is rather reliable.

FIG. 3. The left figure shows the lines \( \phi_{in}(b) \) (black line), \( \phi^*(b) \) (blue line) and \( \phi_{end} \) (red line) for the quadratic model \( v(\phi) = m^2 \phi^2 \). It also shows the regions in the \((m, \phi)\) plane corresponding to the KD approximation (grey region) and the SR approximation (blue region). The right figure shows the amount of inflation \( N(b) \) corresponding to the approximations with 4 (brown line), 5 (green line), 6 (red line) and 7 (black line) terms, respectively, in the truncated series \([39]\) provided by \([97]\). The lines are almost completely overlapped, which shows that a very good approximation is obtained with only a few terms of the truncated series. The blue dots indicate the values \( N = 50, 60 \) and 70.
In particular for $N(b) = 60$ we obtain $b_{60} = \frac{b_{60}}{m}$, $\hat{b}_{60} \approx 1.79104 \times 10^7$. In figure 4, we plot the numerical solution $h(\varphi)$ of (16) with initial condition $\varphi_0 = 22$, $h_0 = h(\varphi_0, b_{60})$. Numerical computation of the amount of inflation for this solution leads to $N_{num} \approx 60.86$.

![Figure 4](image-url)

**FIG. 4.** The green line shows the rescaled numerical solution $h/m = h(\varphi)/m$ of (16) with initial condition $(\varphi_0 = 22, h_0 = h(\varphi_0, b_{60}) \approx 200.79 m)$ for the quadratic model $v(\varphi) = m^2 \varphi^2$. The black line shows the KD approximation. The region between blue and red dotted lines is the inflation region. The grey dots correspond (from right to left) to $\varphi_{in}$, $\varphi^*$ and $\varphi_{end}$. We notice that the black line (KD approximation) overlaps the green line (numerical approximation) for $\varphi \in (\varphi^*, \varphi_{in})$, while the red dashed line (SR approximation) overlaps the green line for $\varphi \in (\varphi_{end}, \varphi^*)$.

We may also consider $\varphi_{in}$ as a function of $N$, and compare it with the SR approximation for $\varphi_{in}$ which derives from (95) i.e.

$$\tilde{\varphi}_{in}(N) = \sqrt{3 + 6N}. \tag{98}$$

Figure 5 shows the graphs of both approximations to $\varphi_{in}(N)$. It can be observed that the result of taking into account the effect of the KD stage leads to greater values of $\varphi_{in}(N)$.

We apply the same scheme to the Starobinsky model $v(\varphi) = \lambda(1 - e^{-\sqrt{3} \varphi})^2$, and exhibit the corresponding graphics in figures 6 and 7. We use the truncated psi series with 7 terms, although we exhibit also, in figure 6 (right) how the difference with the results corresponding
FIG. 5. The black line shows $\varphi_{in}$ as a function of $N$, where $\varphi_{in}$ is obtained from (96) and $N$ from the formula (97), which takes into account the contributions of both the KD stage and the SR stage. The blue line shows $\varphi_{in}$ as a function of $N$ as given by the SR approximation (98).

to truncated series with 4, 5 or 6 terms are almost imperceptible. In this case, a solution with $N(b) = 60$ corresponds to a value $b_{60} = \hat{b}_{60}/\sqrt{\lambda}$, $\hat{b}_{60} \approx 2649.03$ and the numerical solution with initial condition $\varphi_0 = 12$, $h_0 = h(\varphi_0, b_{60})$ gives the number of e-folds $N_{num}(b) \approx 59.5$.

FIG. 6. The left figure shows the lines $\varphi_{in}(b)$ (black line), $\varphi^*(b)$ (blue line) and $\varphi_{end}$ (red line), as well as the regions in the $(\sqrt{\lambda}b, \varphi)$ plane corresponding to the KD approximation (grey region) and the SR approximation (blue region) for the Starobinsky model $v(\varphi) = \lambda(1 - e^{-\sqrt{3} \varphi})^2$. The right figure shows the amount of inflation $N(b)$ determined by the approximations (97) with 4 (brown line), 5 (green line), 6 (red line) and 7 (black line) terms, respectively, in the truncated series (66) provided by (97). The lines are almost completely overlapped, which shows that a very good approximation is obtained with only a few terms in the truncated series. The blue dots correspond to $N = 50, 60$ and 70.
FIG. 7. The green line shows the numerical solution \( h/\sqrt{\lambda} = h(\varphi)/\sqrt{\lambda} \) of (16) with initial condition \( (\varphi_0 = 12, h_0 = h(\varphi_0, b_0) \approx 61.4435\sqrt{\lambda}) \) for the Starobinsky model \( v(\varphi) = \lambda(1 - e^{-\sqrt{3}\varphi})^2 \). The black line shows the KD approximation. The region between blue and red dotted lines is the inflation region. The grey dots correspond (from right to left) to \( \varphi_{in}, \varphi^*, \) and \( \varphi_{end} \). It has to be noticed that the black line (KD approximation) overlaps the green line (numerical approximation) for \( \varphi \in (\varphi^*, \varphi_{in}) \), while the red dashed line (SR approximation) overlaps the green line for \( \varphi \in (\varphi_{end}, \varphi^*) \).

B. The KD period and the Mukhanov-Sasaki equation

As it was shown in [9,11] truncations of logilinear series expansions involving powers of \( t - t^* \) and \( \log(t - t^*) \) are useful to determine the effect of the KD period on the power spectrum of the primordial curvature and tensor perturbations. These perturbations are characterized by the Mukhanov-Sasaki equation [29]

\[
\left( \frac{d^2}{d\eta^2} + k^2 - W_\alpha(\eta) \right) S_\alpha(k, \eta) = 0; \quad \alpha = R, T, \tag{99}
\]

where the potential \( W_\alpha(\eta) \) felt by the perturbations is

\[
W_\alpha(\eta) = \begin{cases} 
\frac{Z_{\eta\eta}}{Z} & \text{for curvature perturbations} \\
\frac{a_{\eta\eta}}{a} & \text{for tensor perturbations} 
\end{cases} \tag{100}
\]

Here \( Z = a\dot{\phi}/H \) and \( \eta \) is the conformal time defined, up to a constant, by \( dt = a\,d\eta \).
We now apply the psi series depending on $\varphi$ obtained above and several of their consequences to determine approximate expressions for the potentials $W_\alpha(\eta)$.

1. **Polynomial-exponential potentials**

For models with a potential $v(\varphi)$ of the form (4) we consider the two-term approximation

$$h(\varphi) \approx e^{\varphi}\left(\frac{1}{b} + h_2(\varphi)e^{-2\varphi}\right),$$

of the psi series (36). The polynomial coefficient $h_2(\varphi)$ satisfies

$$h_2' - 2 h_2 = -\frac{b}{2} v_0,$$

so that it is given by

$$h_2(\varphi) = \frac{b}{2} e^{2\varphi} \int_\varphi^{+\infty} v_0(x) e^{-2x} dx.$$  

We show below several psi series of the form

$$f(\varphi) = e^{a\varphi}\left(\beta + \sum_{n=2}^{\infty} f_n(\varphi)e^{-n\varphi}\right),$$

with polynomial coefficients $f_n(\varphi)$, which are consequences of (36) and are required for our calculations.

1. The scale factor admits the expansion

$$a(\varphi) = c e^{-\frac{\varphi}{3}} \left(1 + \sum_{n=2}^{\infty} a_n(\varphi)e^{-n\varphi}\right),$$

where $c$ is an arbitrary nonzero positive constant and

$$a_2(\varphi) = \frac{b}{3} h_2(\varphi).$$

2. The conformal time $\eta$, considered as a function of $\varphi$, satisfies the differential equation

$$\frac{d\eta}{d\varphi} = -\frac{1}{a h'},$$

and then it can be expanded as a psi series

$$\eta(\varphi) = \frac{e^{-\frac{2\varphi}{3}}}{c} \left(\frac{3}{2} b + \sum_{n=2}^{\infty} \eta_n(\varphi)e^{-n\varphi}\right),$$

with

$$\eta_2(\varphi) = b^2 e^{\frac{5}{3}\varphi} \int_\varphi^{+\infty} \left(h_2'(x) - \frac{2}{3} h_2(x)\right) e^{-\frac{5}{3}x} dx.$$
3. The Hubble radius $1/aH = 3/ah$ can be expanded as

$$\frac{1}{aH} = \frac{3b}{c} e^{-\frac{2}{3}\phi} \left(1 + \sum_{n=2}^{\infty} r_n(\varphi)e^{-n\varphi}\right),$$

(110)

where

$$r_2(\varphi) = -\frac{4b}{3} h_2(\varphi).$$

(111)

4. The function $Z = a\dot{\phi}/H$ is proportional to $z(\varphi) \equiv -a h'/h$, which has the psi series expansion

$$z(\varphi) = -c e^{-\frac{2}{3}\phi} \left(1 + \sum_{n=2}^{\infty} z_n(\varphi)e^{-n\varphi}\right),$$

(112)

where

$$z_2(\varphi) = b \left[h_2'(\varphi) - \frac{5}{3} h_2(\varphi)\right].$$

(113)

Let us now consider the potentials $W_\alpha(\eta)$ (100). From (108) we get the following approximation for $\varphi(\eta)$

$$\varphi \approx -\frac{3}{2} \log \left(\frac{2c}{3b}\eta\right).$$

(114)

Inserting this result in (108) we get an approximation with two terms for $\varphi(\eta)$ given by

$$\varphi \approx -\frac{3}{2} \log \left(\frac{2c}{3b}\eta\right) + \eta \left(\frac{3}{2} \log \left(\frac{2c}{3b}\eta\right)\right),$$

(115)

Then, using (112) we obtain the two-term approximation

$$z(\eta) \approx -c \left[\frac{2c}{3b}\eta\right]^{1/2} \left(1 + z_2(\log \eta)\eta^2\right), \quad \eta \to 0,$$

(116)

where

$$z_2(\log \eta) \equiv \left(\frac{2c}{3b}\right)^3 \left[z_2 - \frac{3}{2} \log \left(\frac{2c}{3b}\eta\right) - \frac{1}{3b} \eta z_2 - \frac{3}{2} \log \left(\frac{2c}{3b}\eta\right)\right].$$

(117)

Thus, we get that

$$W_R(\eta) = \frac{z_m}{z} \approx \frac{1}{\eta^2} \left[-\frac{1}{4} + \eta^3 \left(z''(\log \eta) + 6 z_2'(\log \eta) + 9 z_2(\log \eta)\right)\right], \quad \eta \to 0.$$  

(118)

A completely similar calculation starting from (105) and (106) leads to

$$W_T(\eta) \approx \frac{1}{\eta^2} \left[-\frac{1}{4} + \eta^3 \left(\tilde{a}_2''(\log \eta) + 6 \tilde{a}_2'(\log \eta) + 9 \tilde{a}_2(\log \eta)\right)\right], \quad \eta \to 0,$$

(119)

with

$$\tilde{a}_2(\log \eta) \equiv \left(\frac{2c}{3b}\right)^3 \left[a_2 - \frac{3}{2} \log \left(\frac{2c}{3b}\eta\right) - \frac{1}{3b} \eta a_2 - \frac{3}{2} \log \left(\frac{2c}{3b}\eta\right)\right].$$

(120)
FIG. 8. Potentials $W_R(\eta)$ (left) and $W_T(\eta)$ (right) for the quadratic model $v(\varphi) = m^2 \varphi^2$. The black lines show the numerical approximations while the blue lines show the psi series approximation (121). Here we have used the rescaled variables $m \eta$, $m^{-2} W_R$, $m^{-2} W_T$, the value $b = b_{60} = \hat{b}_{60}/m$, $\hat{b}_{60} \approx 1.79104 \times 10^7$ obtained in section IV A for the quadratic model and $c$ is such that $a(\varphi_{in}) = 1$. We have taken a value $\eta(t^*) = 0$ of the conformal time in agreement with (108).

For example, for the quadratic model (48), from (49), (106), (109), (113) and (117)-(120) we obtain

$$W_R(\eta) \approx -\frac{1}{4\eta^2} - \frac{m^2 c^3}{198 b} \eta \left[ 131 + 756 \log \left( \frac{2c}{3b} \eta \right) + 504 \log \left( \frac{2c}{3b} \eta \right)^2 \right],$$

$$W_T(\eta) \approx -\frac{1}{4\eta^2} + \frac{m^2 c^3}{64b} \eta \left[ -1 + 4 \log \left( \frac{2c}{3b} \eta \right) + 24 \log \left( \frac{2c}{3b} \eta \right)^2 \right].$$

(121)

2. Starobinsky potentials

For models with a potential $v(\varphi)$ of the form (5) we consider the psi series (66) for $h(\varphi)$ and its truncation to two terms

$$h(\varphi) \approx \frac{e^{\varphi}}{b} \left( 1 + b^2 \gamma_1 (e^{-\alpha \varphi}) e^{-2\varphi} \right),$$

(122)

where $\gamma_1$ is given by (67) for $\alpha > 0$ and by setting the lower integration index as $\infty$ for $\alpha \in \left( -\frac{2}{N}, 0 \right)$. From (66) we derive the psi series which follow.
1. The scale factor

\[ a(\varphi) = ce^{-\frac{\varphi}{3}} \left( 1 + \sum_{n=1}^{\infty} b^{2n} a_n(u) e^{-2n\varphi} \right), \]  

(123)

with \( a_n(u) \) being polynomials in \( u \). In particular,

\[ a_1(u) = \frac{1}{3} \gamma_1(u). \]  

(124)

2. The conformal time

\[ \eta(\varphi) = \frac{3 b}{2c} e^{-\frac{2\varphi}{3}} \left( 1 + \sum_{n=1}^{\infty} b^{2n} \eta_n(u) e^{-2n\varphi} \right), \]  

(125)

where \( \eta_n(u) \) are polynomials in \( u \). For example we have that

\[ \eta_1(u) = \left\{ \begin{array}{ll}
\frac{2}{3} \int_0^u \left( \gamma'_1(\mu) + \frac{2}{3\alpha\mu} \gamma_1(\mu) \right) \left( \frac{\mu}{u} \right)^{\frac{8}{3}} d\mu, & \alpha > 0, \\
-\frac{2}{3} \int_u^{\infty} \left( \gamma'_1(\mu) + \frac{2}{3\alpha\mu} \gamma_1(\mu) \right) \left( \frac{\mu}{u} \right)^{\frac{8}{3}} d\mu, & \alpha \in (-\frac{2}{N}, 0).
\end{array} \right. \]  

(126)

3. The Hubble radius

\[ \frac{1}{a H} = \frac{3 b}{c} e^{-\frac{2\varphi}{3}} \left( 1 + \sum_{n=1}^{\infty} b^{2n} r_n(u) e^{-2n\varphi} \right), \]  

(127)

where \( r_n(u) \) are polynomials in \( u \). For instance

\[ r_1(u) = -\frac{4}{3} \gamma_1(u). \]  

(128)

4. The variable \( z(\varphi) \equiv -a h'/h \)

\[ z(\varphi) = -c e^{-\frac{\varphi}{3}} \left( 1 + \sum_{n=1}^{\infty} b^{2n} z_n(u) e^{-2n\varphi} \right), \]  

(129)

with \( z_n(u) \) being polynomials in \( u \). In particular

\[ z_1(u) = -\left( \alpha u \gamma'_1(u) + \frac{5}{3} \gamma_1(u) \right). \]  

(130)

From (125) we can derive the approximation

\[ \varphi \approx -\frac{3}{2} \log \left( \frac{2c\eta}{3b} \right) + \frac{3b^2}{2} \left( \frac{2c\eta}{3b} \right)^3 \eta \left( \frac{2c\eta}{3b} \right)^{\frac{3\eta}{2}}. \]  

(131)
By replacing (131) into (129), then it is found that
\[ z(\eta) \approx -c \left( \frac{2c}{3b} \right)^{\frac{1}{2}} \eta^{\frac{1}{2}} \left[ 1 + \eta^{3} \tilde{z}_{1}(\eta^{\alpha}) \right], \]  
(132)

where
\[ \tilde{z}_{1}(\eta^{\alpha}) = \frac{8c^{3}}{27b} \left[ z_{1}\left( \left( \frac{2cn}{3b} \right)^{\frac{3\alpha}{2}} \right) - \frac{1}{2} \eta_{1} \left( \frac{2cn}{3b} \right)^{\frac{3\alpha}{2}} \right]. \]  
(133)

Similarly, from (123)-(124) and (131) we have that
\[ a(\eta) \approx c \left( \frac{2c}{3b} \right)^{\frac{1}{2}} \eta^{\frac{1}{2}} \left[ 1 + \eta^{3} \tilde{a}_{1}(\eta^{\alpha}) \right], \]  
(134)

with
\[ \tilde{a}_{1}(\eta^{\alpha}) = \frac{8c^{3}}{27b} \left[ \frac{1}{3} \gamma_{1}\left( \left( \frac{2cn}{3b} \right)^{\frac{3\alpha}{2}} \right) - \frac{1}{2} \eta_{1} \left( \frac{2cn}{3b} \right)^{\frac{3\alpha}{2}} \right]. \]  
(135)

Thus, (132) leads to
\[ W_{R}(\eta) \approx \frac{1}{\eta^{2}} \left[ -\frac{1}{4} + \eta^{3} \left( \alpha^{2} \eta^{2\alpha} \tilde{z}_{1}''(\eta^{\alpha}) + \alpha(\alpha + 6)\eta^{\alpha} \tilde{z}_{1}'(\eta^{\alpha}) + 9\tilde{z}_{1}(\eta^{\alpha}) \right) \right], \quad \eta \to 0. \]  
(136)

Analogously, from (134) one gets
\[ W_{T}(\eta) \approx \frac{1}{\eta^{2}} \left[ -\frac{1}{4} + \eta^{3} \left( \alpha^{2} \eta^{2\alpha} \tilde{a}_{1}''(\eta^{\alpha}) + \alpha(\alpha + 6)\eta^{\alpha} \tilde{a}_{1}'(\eta^{\alpha}) + 9\tilde{a}_{1}(\eta^{\alpha}) \right) \right], \quad \eta \to 0. \]  
(137)

For example, for the Starobinsky model (69), from the first equation (71), (124), (126), (130), (133) and (135)-(137), it is found that
\[ W_{R}(\eta) \approx -\frac{1}{4\eta^{2}} - \frac{\lambda c^{3}}{3b^{3}} \left[ \frac{2(\alpha + 1)(6\alpha^{2} + 14\alpha + 7)}{3\alpha + 4} \left( \frac{2cn}{3b} \right)^{3\alpha} \right. \]  
\[ \left. - \frac{2(\alpha + 2)(3\alpha^{2} + 14\alpha + 14)}{3\alpha + 8} \left( \frac{2cn}{3b} \right)^{\frac{3\alpha}{2}} + \frac{7}{2} \right], \]  
(138)

\[ W_{T}(\eta) \approx -\frac{1}{4\eta^{2}} + \frac{\lambda c^{3}}{3b^{3}} \left[ \frac{2(\alpha + 1)}{3\alpha + 4} \left( \frac{2cn}{3b} \right)^{3\alpha} \right. \]  
\[ \left. - \frac{4(\alpha + 2)}{3\alpha + 8} \left( \frac{2cn}{3b} \right)^{\frac{3\alpha}{2}} + \frac{1}{2} \right]. \]

We notice that, as it was found in [11], the common dominant term near the singularity \( \eta = 0 \) of the potentials (118), (119), (136) and (137) coincides with the critical central singular attractive potential allowing the fall to the center of a quantum particle.

Finally, we point out that we may also use our psi-series in the variable \( \varphi \) starting from the Mukhanov-Sasaki equation (99) with the potential \( W_{\alpha} \) expressed as a function of \( \varphi \). However, as it is showed in Equation (2.27) of [11], the function \( W_{\alpha}(\varphi) \) is rather involved, so that we do not find any advantage in using our psi-series in that way.
FIG. 9. Potentials $W_R(\eta)$ (left) and $W_T(\eta)$ (right) for the Starobinsky potential $v(\varphi) = \lambda(1 - e^{-\sqrt{3}\frac{\varphi}{\lambda}})^2$. The black lines show the numerical approximations while the blue lines show the psi series approximation $\overset{\text{IV}}{\text{A}}$. Here we have used the rescaled variables $\sqrt{\lambda} \eta$, $\lambda^{-1} W_R$, $\lambda^{-1} W_T$ and the value $b = \hat{b}_{60}/\sqrt{\lambda}$, $\hat{b}_{60} \approx 2649.03$ obtained in section $\overset{\text{IV}}{\text{A}}$, $c$ is taken so that $a(\varphi_{in}) = 1$ and $\eta(t^*) = 0$ in agreement with $\overset{\text{IV}}{\text{A}}$.

V. CONCLUSIONS

In this paper, we have developed a method to determine psi-series formal solutions of single field inflation models $\overset{\text{I}}{\text{I}}$ during the kinetic dominance period. The scheme is based on the Hamilton-Jacobi formalism of inflaton models $\overset{\text{I}}{\text{A}}$ and provides psi-series depending on the inflaton field. The method has been applied to models with polynomial-exponential potential functions $\overset{\text{I}}{\text{I}}$ (two particularly important examples are the quadratic potential $\overset{\text{I}}{\text{I}}$ and the Higgs potential $\overset{\text{I}}{\text{I}}$) and to models with generalized Starobinsky potential functions $\overset{\text{I}}{\text{I}}$ (including the standard Starobinsky potential $\overset{\text{I}}{\text{I}}$). We have also proved that there exist psi series near the singularity for several physically relevant quantities such as the scale factor, the conformal time and the Hubble radius. The explicit form of the first two terms of these expansions has been given.

We have found that truncations of these psi series can be used to determine the value of the inflaton field at the initial moment of the inflation period, and to include the effect of the KD period to estimate the amount of inflation. Furthermore, we have shown that psi series can be applied to determine explicit corrections depending on the inflaton field to the dominant term of the potentials of the Mukanov-Sasaki equation for both, curvature and tensor perturbations.
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