Hitchin Systems on Hyperelliptic Curves

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Abstract—We describe a class of spectral curves and find explicit formulas for the Darboux coordinates of Hitchin systems corresponding to classical simple groups on hyperelliptic curves. We consider in detail the systems with rank 2 groups on genus 2 curves.

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1. INTRODUCTION

In the vast literature on Hitchin systems there are not so many works devoted to the fundamental question of separation of variables for them. In this connection, we can only mention the papers [7, 11, 9]. The first one treats in detail the system on a genus 2 curve with SL(2) as a gauge group, and in the other two some general approaches to the problem are discussed. Separation of variables becomes effective when one uses the family of spectral curves [5, 22, 2]. The general properties of spectral curves depending on a gauge group were formulated in the pioneering work by Hitchin [10].

A description of spectral curves and separation of variables for hyperelliptic Hitchin systems and gauge groups of the series \(A_l, B_l,\) and \(C_l\) was given in [20] (see also [17–19]). These are the classical series for which the spectral curve is nonsingular and the Hamiltonians can be found explicitly in terms of separation variables. It is observed in [20] that the \(D_l\) series is specific in this respect. For the group SO(4), which is a simplest representative of this series, the separation of variables is carried out in [3]. We reproduce these results in the present paper with the corresponding references. As new results, we prove here the smoothness of the differentials of the angle coordinates on the normalization of the spectral curve in the case when it is singular (this is just the case of \(D_l\)) and show that in the case of spectral curves with holomorphic involution (i.e., for all classical simple groups except SL(\(n\), \(n > 2\)) the differentials of angles given by separation of variables coincide with the basis holomorphic Prym differentials on the spectral curve (or its normalization). The rank 1 and 2 cases, i.e., the groups SL(2), SO(4), Sp(4), and SO(5), are considered in detail (in Section 5 we discuss the relation between our results in the SL(2) case and the corresponding results in [7]). Observe that the description of the class of spectral curves as given in [20] relies on the Lax representation of Hitchin systems, which was proposed and investigated in [12] (see [13–16] for other gauge groups). Here, we derive this description directly from the properties of Higgs fields.

In Section 2 we introduce Hitchin systems following the lines of his pioneering work [10], and give a description of spectral curves for the systems on hyperelliptic curves, for all classical root systems.

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In Section 3 we give an alternative definition of Hitchin systems in terms of separation of variables. We find Darboux coordinates and investigate analytic properties of the differentials of angle-type variables.

In Section 4 we consider the rank 1 and 2 examples, i.e., the systems with gauge groups SL(2), SO(4), Sp(4), and SO(5). We present detailed characteristics of their spectral curves, namely, the genus, number of sheets, number of branch points, gluing scheme, action of the involution, list of the basis holomorphic differentials (Prym differentials), and the form of the Prym map. In the cases of SL(2) and SO(4) we find the action–angle variables.

In Section 5 we discuss the relation between our results in the case of SL(2) and the corresponding results in [7].

We are glad to devote this paper to A. G. Sergeev, our wonderful colleague and a school and university friend of one of us.

2. HITCHIN SYSTEMS ON HYPERELLIPTIC CURVES

2.1. Hitchin systems. We define Hitchin systems following the lines of [10].

Let $\Sigma$ be a compact genus $g$ Riemann surface with a conformal structure, $G$ be a complex semisimple Lie group, $\mathfrak{g} = \mathfrak{Lie}(G)$, and $P_0$ be a smooth principal $G$-bundle on $\Sigma$.

By a holomorphic structure on $P_0$ we mean a connection of type $(0, 1)$, i.e., a differential operator on the sheaf of sections of the bundle $P_0$. Locally, the operator is given as $\overline{\partial} + \omega$, where $\omega \in \Omega^{0,1}(\Sigma, \mathfrak{g})$ and a gluing function $\gamma$ acts on $\omega$ by the gauge transformation $\omega \rightarrow \gamma \omega \gamma^{-1} - (\overline{\gamma} \gamma)^{-1}$.

Let $\mathcal{A}$ be the space of semistable [10] holomorphic structures on $P_0$ and $\mathcal{G}$ be the group of smooth global gauge transformations. The quotient $\mathcal{N} = \mathcal{A}/\mathcal{G}$ is called the moduli space of stable holomorphic structures on $P_0$. Further on, we will consider $\mathcal{N}$ as a configuration space of a Hitchin system. Points in $\mathcal{N}$ are principal holomorphic $G$-bundles on $\Sigma$, which are denoted below by $P$. The dimension of $\mathcal{N}$ is $\dim \mathcal{N} = (g - 1) \dim \mathfrak{g}$.

By definition, the phase space of a Hitchin system is $T^*\mathcal{N}$. According to the Codaira–Spenser theory, $T_P(\mathcal{N}) \simeq H^1(\Sigma, \text{Ad } P)$. By Serre duality

$$T^*_P(\mathcal{N}) \simeq H^0(\Sigma, \text{Ad } P \otimes \mathcal{K}),$$

where $\mathcal{K}$ is a canonical class of $\Sigma$ and $\text{Ad } P \otimes \mathcal{K}$ is a holomorphic vector bundle with fiber $\mathfrak{g} \otimes \mathbb{C}$. We denote the points of $T^*\mathcal{N}$ by $(P, \Phi)$, where $P \in \mathcal{N}$ and $\Phi \in H^0(\Sigma, \text{Ad } P \otimes \mathcal{K})$. The sections of the sheaf $T^*\mathcal{N}$ are called Higgs fields.

Let $\chi_d$ be a homogeneous invariant polynomial on $\mathfrak{g}$ of degree $d$. For every $P \in \mathcal{N}$, it defines a map $\chi_d(P): H^0(\Sigma, \text{Ad } P \otimes \mathcal{K}) \rightarrow H^0(\Sigma, \mathcal{K}^d)$. Let $\Phi$ stand for a Higgs field; then we can define $\chi_d(P, \Phi) = (\chi_d(P))(\Phi(P))$. So $\chi_d(P, \Phi) \in H^0(\Sigma, \mathcal{K}^d)$. Thus, to each point $(P, \Phi)$ of the phase space we have assigned an element of $H^0(\Sigma, \mathcal{K}^d)$. Suppose $\{\Omega^d_j\}$ is a basis in $H^0(\Sigma, \mathcal{K}^d)$; then $\chi_d(P, \Phi) = \sum_j H_{d,j}(P, \Phi)\Omega^d_j$, where $H_{d,j}(P, \Phi)$ is a scalar function on $T^*\mathcal{N}$. For any $j$ and $d$ the function $H_{d,j}(P, \Phi)$ is called a Hitchin Hamiltonian.

**Theorem 2.1** [10]. The Hitchin Hamiltonians $\{H_{d,j}\}$ Poisson commute on $T^*\mathcal{N}$.

2.2. Spectral curves of hyperelliptic Hitchin systems. Here, for convenience, we define the same Hamiltonians in another way. We fix a holomorphic differential $\omega$ on $\Sigma$ and divide the holomorphic sections of $\text{Ad } P \otimes \mathcal{K}$ by it. Thus, we obtain meromorphic sections of the bundle $\text{Ad } P$ with a divisor of poles $-D$, where $D = (\omega)$ is the divisor of zeros of $\omega$. In this way, $\chi_d(P)$ will be a map from $H^0(\Sigma, \text{Ad } P(-D))$ to $\mathcal{O}(\Sigma, -dD)$. We will also consider the basis $\Omega^d_j$ as a basis of $\mathcal{O}(\Sigma, -dD)$. 

Let $\Sigma$ be a hyperelliptic (in particular, nonsingular) curve defined by the equation

$$y^2 = P_{2g+1}(x), \quad \text{where} \quad P_{2g+1}(x) = x^{2g+1} + \sum_{i=0}^{2g} a_i x^i. \quad (2.1)$$

We choose $\omega = dx/y$; hence $D = 2(g - 1) \cdot \infty$. The following lemma enables us to find a basis of the space $\mathcal{O}(\Sigma, -dD)$.

**Lemma 2.1** [20]. The functions $1, x, \ldots, x^{d(g-1)}$ and $y, xy, \ldots, yx^{(d-1)(g-1)-2}$ form a basis of $\mathcal{O}(-dD)$, where $D = 2(g - 1) \cdot \infty$.

The spectral curve of a Higgs field $\Phi$ is defined by the relation

$$\det \left( \lambda - \frac{\Phi}{\omega} \right) = 0.$$ 

Locally, the value of $\Phi(P)$ is a $\mathfrak{g}$-valued function on $\Sigma$. Denote its value at a point $(x, y) \in \Sigma$ by $\Phi(P, x, y)$. For a fixed $P$ we consider the equation of the spectral curve as a relation between $\lambda, x,$ and $y$. Letting $P$ run through the moduli space $\mathcal{N}$, we obtain a family of spectral curves. In different local trivializations, the evaluations of $\Phi$ are related by the action of the group $\text{Ad} G$, so the equation is well defined.

Let $d_1, \ldots, d_l, l = \text{rank} \mathfrak{g}$, be a set of degrees of basis invariant polynomials of the Lie algebra $\mathfrak{g}$. For brevity, we will denote the basis invariant $\chi_{d_i}$ by $\chi_i$ and the Hamiltonian $H_{d_{i,j}}$ by $H_{i,j}$. The meromorphic functions on $\Sigma$ of the form $p_i = \chi_i(\Phi/\omega)$, where $\chi_i, i = 1, \ldots, l$, are the basis invariant polynomials, will be referred to as basis spectral invariants (because they are invariant under Hitchin flows). The integer $d_i = \deg \chi_i$ is called the degree of the basis spectral invariant $p_i$.

Consider a classical Lie algebra $\mathfrak{g}$ in the standard representation. The equation of the spectral curve has the form

$$R(x, y, \lambda) = \lambda^n + \sum_{i=1}^{l} r_i(x, y) \lambda^{n-d_i} = 0, \quad (2.2)$$

where $n$ is the dimension of the standard (or vector in another terminology) representation of $\mathfrak{g}$ and $r_i, i = 1, \ldots, l$, are meromorphic functions on $\Sigma$. Thus, for a Lie algebra $\mathfrak{g}$ of type $A_l$ we have $n = l + 1$ and $d_i = i + 1$; for the type $B_l$ we have $n = 2l + 1$ and $d_i = 2i$; for the type $C_l$ we have $n = 2l$ and $d_i = 2i$; and $r_i = p_i$ are the basis invariants for all $i = 1, \ldots, l$. The case of Lie algebras of type $D_l$ is exceptional in this respect; namely, the coefficient $r_l$ of degree $2l$ in (2.2) is a square of the basis invariant of degree $l$ (which is nothing but the Pfaffian of $\Phi/\omega$). Looking ahead, we note that this is the reason why the equations for Hamiltonians in the method of separation of variables are nonlinear for Lie algebras of type $D_l$.

A basis spectral invariant $r_i$ can be expanded over the basis from Lemma 2.1 for $d = d_i$ as follows:

$$r_i(x, y) = \sum_{k=0}^{d_i(g-1)} H_{ik}^{(0)} x^k + \sum_{s=0}^{(d_i-1)(g-1)-2} H_{is}^{(1)} y x^s, \quad (2.3)$$

where $H_{ik}^{(0)}$ and $H_{is}^{(1)}$ are independent Hamiltonians of the Hitchin system.

**Example 2.1** (spectral curve for $\mathfrak{g} = \mathfrak{sl}(2)$). There is only one basis invariant $p_2 = r_2$ of degree 2,

$$r_2(x, y) = \sum_{k=0}^{2(g-1)} H_{k}^{(0)} x^k + \sum_{s=0}^{g-3} H_{s}^{(1)} y x^s \quad (2.4)$$

(in particular, for $g = 2$ the second sum is absent). The equation of the spectral curve has the form

$$\lambda^2 + r_2(x, y) = 0. \quad (2.5)$$
Example 2.2. The spectral curve for $\mathfrak{g} = \mathfrak{so}(4)$ is defined by the equation

$$R(\lambda, x, y, H) = \lambda^4 + \lambda^2 p + q^2 = 0,$$

(2.6)

where $p$ and $q$ are the basis spectral invariants of degree 2.

For genus 2 these examples will be considered in Section 4 in detail.

3. SEPARATION OF VARIABLES FOR HYPERELLIPTIC HITCHIN SYSTEMS

3.1. Separation variables. Symplectic and Poisson structures. Denote the number of degrees of freedom of a Hitchin system by $N$. It is known [10] that $N = (g - 1)\dim \mathfrak{g}$ provided $\mathfrak{g}$ is a simple Lie algebra.

The spectral curve (2.2) is completely defined by a set of values of independent Hamiltonians, and hence by $N$ points the spectral curve passes through. Denote these points by $(x_i, y_i, \lambda_i)$, $i = 1, \ldots, N$, where $y^2 = P_{2g+1}(x) = x_i, y_i$, and $\lambda_i$ are related by equations (2.2) and (2.3) for each $i$. The variables $x_i$ and $\lambda_i$ are called separation variables.

Define a 2-form

$$\sigma = \sum_{i=1}^{N} d\lambda_i \wedge \frac{dx_i}{y_i}.$$  

(3.1)

From [12, Theorem 4.3], one can deduce that the form $\sigma$ gives a symplectic structure identical to that for a Hitchin system (see [20] for details).

Remark 3.1. To restore a Hitchin system from the above data, one should view the points $(x_i, y_i, \lambda_i)$ as poles of an eigenfunction of the Lax operator of a Hitchin system and use the inverse spectral problem following the lines of [12]. Certainly, relation (3.1) is nothing but a reformulation of [12, Theorem 4.3] for the case of a hyperelliptic base curve. In particular, $dx_i/y_i$ in (3.1) is nothing but the differential $dx/y$ fixed in our definition of Hamiltonians (and of the symplectic structure in [12] as well). A similar expression for the symplectic form also appears in [4] for elliptic curves.

To summarize, the phase space of a Hitchin system on a hyperelliptic curve $\Sigma$ is constituted by unordered sets of triples of the form

$$\{(x_i, y_i, \lambda_i) \mid i = 1, \ldots, N\},$$

with the symplectic structure given by (3.1).

The corresponding Poisson structure is given by the relation

$$\{\lambda_i, x_j\} = \delta_{ij} y_i.$$  

(3.2)

3.2. The form of Hamiltonians in separation variables. We find the Hamiltonians using the fact that the spectral curve passes through the points $(x_i, y_i, \lambda_i)$. It can be expressed as a system of equations $R(x_i, y_i, \lambda_i, H) = 0$, $i = 1, \ldots, N$, with the unknowns

$$H = \left\{ H_{jk}^{(0)}, H_{js}^{(1)} \mid j = 1, \ldots, l, \ k = 0, 1, \ldots, (2d_j - 1)(g - 1), \ s = 0, 1, \ldots, (d_j - 1)(g - 1) - 2 \right\}.$$  

We will call these equations the separation relations, as is common in the method of separation of variables. This system is linear for Lie algebras of the types $A_l, B_l, C_l$, which follows from relations (2.2) and (2.3), so the Hamiltonians can be explicitly found in terms of the separation variables by Kramers’s rule.

Example 3.1. For $\mathfrak{g} = \mathfrak{sl}(2)$ the system of equations has the following form:

$$\lambda_i^2 + \sum_{k=0}^{2(g-1)} H_{k}^{(0)} x_i^k + \sum_{s=0}^{g-3} H_{s}^{(1)} y_i x_i^s = 0, \quad i = 1, \ldots, 3(g - 1).$$  

(3.3)
Thus, $H_k^{(0)} = D_k^{(0)}/D$ and $H_s^{(1)} = D_s^{(1)}/D$, where

\[
D = \begin{vmatrix}
1 & \ldots & x_1^{2(g-1)} & y_1 & \ldots & y_1x_1^{g-3} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & \ldots & x_3^{2(g-1)} & y_3(g-1) & \ldots & y_3(g-1)x_3^{g-3}
\end{vmatrix},
\]

$D_k^{(0)}$ is obtained by replacing $x_i^k$ with $-\lambda_i^2$ in the $k$th column of the determinant $D$, and $D_s^{(1)}$ is obtained by replacing $y_ix_i^k$ with $-\lambda_i^2$ in the $(2g-1+k)$th column of $D$.

In the case of $D_n$ series the system of separation relations is quadratic in Hamiltonians. In [3], it was shown that the system is unsolvable in radicals except for the case of $n = 2$ (which corresponds to the Lie algebra $\mathfrak{g} = \mathfrak{so}(4)$). The case $n = 2$ reduces to solving an algebraic equation of degree 4 with one unknown (see Section 4). Note that the isomorphism $\mathfrak{so}(4) \cong \mathfrak{sl}(2) \times \mathfrak{sl}(2)$ does not simplify the problem, since it is an outer isomorphism not preserving the spectral curve.

### 3.3. Darboux coordinates

We will also use the following through numbering of the Hamiltonians: $H = (\ldots, H_j, \ldots)$. Our next goal is to find the coordinates $\phi_j$ conjugate to Hamiltonians. These are coordinates in a universal covering of a generic leaf of the Lagrangian foliation $H = \text{const}$ (which is called the Hitchin foliation in the case of Hitchin systems). They can be found in a standard way by means of the technique of generating functions [1]. Finally, we obtain the following result:

\[
\phi_j = \sum_{i=1}^{N} \left( x_i, y_i, \lambda_i \right) \int \omega_j, \quad \text{where} \quad \omega_j = \frac{\partial R/\partial H_j}{\partial R/\partial \lambda_j} \frac{dx}{y}
\]  

(3.4)

(the calculation goes back to [11]; we refer to [20] for details; see also [12, Eq. (4.61)]).

The coordinates $(H_j, \phi_j)$ possess the Darboux property, which immediately follows from the method of generating functions. In the case where the equation of a spectral curve is linear in the $H$-coordinates, it also follows from the results of [2, 23]. For the root system $D_l$ these results do not work.

Next, we study the properties of abelian differentials $\omega_j$.

**Proposition 3.1.** Assume that a spectral curve has at worst simple singularities and that the projections of its ramification divisors and of its divisor of singularities do not intersect with the ramification divisor of the base curve over $\mathbb{CP}^1$. Then, away from infinity, the differentials $\omega_j$ are holomorphic at smooth points of the spectral curve. If the spectral curve is singular, then their pull-back onto the normalization results in holomorphic differentials (away from infinity).

**Proof.** If $R'_\Lambda \neq 0$, then $\omega_j$ is holomorphic because $dx/y$ is holomorphic and $\partial R/\partial H_j$ in the numerator is a polynomial in $x$, $y$, and $\lambda$. If $R'_\Lambda = 0$, i.e., the point $(x, y, \lambda)$ is a finite branch point, and, moreover, this point is nonsingular, then $R'_\Lambda \neq 0$. From the equation $R'_x dx + R'_\Lambda d\lambda = 0$ we obtain $dx/R'_x = -d\lambda/R'_x$, and the right-hand side of the relation is holomorphic. Therefore, the left-hand side is also holomorphic. In addition, by assumption, this point is not a branch point for the base curve; hence $y \neq 0$. Therefore, at a finite branch point, $\omega_j$ is holomorphic for each $j$.

In the case of a singular point, it is a simple singularity by assumption. Locally, in the neighborhood of the singularity, the equation of the curve has the form $R = g_1 \cdot g_2$. The normalization of the curve splits into two smooth branches $\eta_1$ and $\eta_2$ (Fig. 1). The first branch is given by the relations $g_1 = 0$ and $g_2 \neq 0$, and the second, by the relations $g_1 \neq 0$ and $g_2 = 0$. On the first branch, we have

\[
\omega_j = \frac{(g_1)'_{H_j} g_2 + g_1 (g_2)'_{H_j}}{(g_1)'_{H_j} g_2 + g_1 (g_2)'_{\Lambda}} \frac{dx}{y}\big|_{\eta_1} = \frac{(g_1)'_{H_j} dx}{(g_1)'_{\Lambda} dx}.
\]
The latter differential is constructed according to the same rules with respect to the equation $g_1 = 0$ as $\omega_j$ with respect to the equation $R = 0$. Therefore, by the first part of the proposition, it is holomorphic (locally, in a neighborhood of the point). Similarly, the pull-back of $\omega_j$ onto the second branch of the normalized curve is also holomorphic. In this way, we proceed in the neighborhood of the full list of the differentials $\omega_j$ is as follows:

$$\frac{x^k\lambda^{n-d_k} dx}{R'_\lambda(x,y,\lambda)y}, \quad 0 \leq k \leq d_i(g-1), \quad \text{and} \quad \frac{x^s\lambda^{n-d_i} dx}{R'_\lambda(x,y,\lambda)}, \quad 0 \leq s \leq (d_i-1)(g-1) - 2, \quad (3.5)$$

where $i = 1, \ldots, l$. For the series $D_l$ the differentials are the same for $i < l$, and for $i = l$ they are as follows:

$$\frac{x^k\eta dx}{R'_\lambda(x,y,\lambda)y}, \quad 0 \leq k \leq l(g-1), \quad \text{and} \quad \frac{x^s\eta dx}{R'_\lambda(x,y,\lambda)}, \quad 0 \leq s \leq (l-1)(g-1) - 2, \quad (3.6)$$

where $q = q(x,y)$ is the Pfaffian of the Lax operator.

2. The differentials $\omega_j$ are holomorphic at infinity. In the case of the systems $A_l$, $l \geq 2$, together with the differentials $x^p dx/y$, $p = 0, 1, \ldots, g-1$, lifted from the base, they form a basis of holomorphic differentials on the spectral curve. For the systems $A_1$, $B_1$, and $C_l$ they form a basis of holomorphic Prym differentials on the spectral curve with respect to the involution $\lambda \rightarrow -\lambda$. In the $D_l$ case they form a basis of holomorphic Prym differentials on the normalization of the spectral curve.

Proof. In the cases $A_l$, $B_l$, and $C_l$ this was shown in [20]; in particular, formulas (3.5) are obtained there. Formulas (3.6) follow from (3.4) in a similar way.

For the system $D_l$ one should check that the differentials (3.6) are holomorphic. As in the proof of [20, Proposition 3.4], $q\lambda^{-l}$ is holomorphic at infinity; hence $q \sim z^{-2(g-1)}$. Substituting the latter, together with the asymptotic expressions for the other components [20], into (3.6), we obtain the statement.

In [10] it was shown that the dimension of the Jacobian of a spectral curve in the case of $A_l$ is equal to $n^2(g-1) + 1$ ($n = l + 1$), and the dimension of its Prym variety (or Prym variety of its normalization) in the other cases is equal to $(g-1) \dim \mathfrak{g}$. It matches the number of differentials $\omega_j$ (joined with the differentials lifted from the base in the case of $A_l$, $l \geq 2$), as was essentially shown in [20, statement 3 of Proposition 2.3]. □

3.4. Action–angle coordinates. Among the Darboux coordinates, the algebro-geometric action–angle coordinates play a special role. By algebro-geometric angle coordinates we mean coordinates of the form (3.4) subject to the condition that the corresponding differentials are holomorphic and normalized. The angle coordinates are coordinates in a generic leaf of the Hitchin foliation. By action coordinates we mean the conjugate coordinates. For $\mathfrak{g} = \mathfrak{gl}(n)$, (3.4) is nothing but the Abel transform, and for $\mathfrak{g} = \mathfrak{sl}(2), \mathfrak{so}(2l), \mathfrak{so}(2l + 1), \mathfrak{sp}(2l)$ it is the Prym transform of the points in the phase space.
Consider the problem of finding the algebro-geometric angle coordinates in the case of a smooth spectral curve, i.e., for the Lie algebras $\mathfrak{sl}(l+1)$, $\mathfrak{sp}(2l)$, and $\mathfrak{so}(2l+1)$ (in the last case we take the nontrivial irreducible component of the spectral curve). According to Proposition 3.2, the differentials listed in its statement 1 form a basis of holomorphic differentials (holomorphic Prym differentials). As shown in [20], the normalization of this basis is given by a transformation of the form

$$\omega \to A\omega, \quad H \to HA^{-1},$$

where $\omega = (\omega_1, \ldots, \omega_N)^T$ and $H = (H_1, \ldots, H_N)$ (the symbol T means the transpose of the vector). At the same time, the Darboux property is preserved. A standard expression for the matrix $A^{-1}$ is as follows:

$$A^{-1} = \left(2 \int_{l_i}^{l_j} \omega_k \right)_{i,k=1,\ldots,N}. \quad (3.7)$$

Here, $l_i$, $i = 1, \ldots, N$, are the cuts on the spectral curve that connect pairs of branch points. According to [20], the spectral curve can be represented as a result of gluing $n$ copies of the base curve (sheets) along $\nu/2$ cuts (where $\nu$ is the number of branch points of the spectral curve over the base curve) connecting some pairs of branch points (for an arbitrary pairing). However, independent cycles are given by just $\nu/2 - n + 1$ cuts (denote them by $l_1, \ldots, l_{\nu/2-n+1}$). In turn, each sheet is obtained by gluing two copies of CP$^1$ along $g + 1$ cuts (in this way, we obtain $g$ independent cycles on each sheet). The total number of cycles is $\nu/2 - n + 1 + ng = \hat{g}$ (by the Riemann–Hurwitz formula). Among them, we can choose $N$ independent ones (in particular, with respect to the involution, if it exists) arbitrarily. The schemes of gluing the spectral curves from the sheets for systems of rank 2 are presented in the next section.

As follows from the above, to compute the matrix $A$ (which would solve the normalization problem), we need to know the branch points of the spectral curve over a base curve and the branch points of the base curve. The latter points are known, but finding the former requires solving the system of equations $R(\lambda, x, y) = 0$, $R'_\lambda(\lambda, x, y) = 0$, or, equivalently, $D(R) = 0$, where $D(R)$ is the discriminant of $R$ as a polynomial in $\lambda$ (these systems should be supplemented with the equation $y^2 = P_{2g+1}(x)$). In general, it is a system of algebraic equations which turns out to be too complicated to express the matrix $A$ in terms of the coordinates $(\lambda_i, x_i, y_i)$ effectively. However, for $\mathfrak{g} = \mathfrak{sl}(2)$ and $\mathfrak{g} = \mathfrak{so}(4)$ (and genus 2) it is possible to solve this problem explicitly. These examples and other examples of rank 2 Hitchin systems for classical Lie algebras will be considered in the next section.

4. SYSTEMS OF RANK 1 AND 2 ON CURVES OF GENUS 2

By the rank of a system we mean the rank of the corresponding Lie algebra. Thus, in this section we will consider Hitchin systems for the Lie algebras $\mathfrak{sl}(2)$, $\mathfrak{so}(4)$, $\mathfrak{sp}(4)$, and $\mathfrak{so}(5)$. The ranks of the corresponding bundles are 2, 4, 4, and 5, respectively.

4.1. Hitchin system for the Lie algebra $\mathfrak{sl}(2)$ on a curve of genus 2. A Hitchin system for the Lie algebra $\mathfrak{sl}(2)$ on a hyperelliptic curve of genus $g$ has $N = 3(g-1)$ independent Hamiltonians. For $g = 2$ we obtain a system with three Hamiltonians. The Lie algebra $\mathfrak{sl}(2)$ has only a second-order invariant; hence a spectral curve for this case has the form

$$R(\lambda, x, y) = \lambda^2 + r_2(x) = \lambda^2 + (H_0 + xH_1 + x^2H_2) = 0. \quad (4.1)$$

The curve is smooth, because the equations of singular points ($R'_\lambda = 0$ and $R'_x = 0$) reduce to $r_2(x) = 0$ and $(r_2(x))' = 0$ and are incompatible unless $r_2(x)$ has a multiple root.
The spectral curve is a two-sheeted branch covering of the base curve. The branch points can be obtained as solutions of the equation \( R'_\lambda = 0 \), which reduces to \( r_2(x) = 0 \). Due to the symmetry in \( y \), the number of solutions of this equation, and hence the number of branch points, is four. The genus \( \hat{g} \) of the spectral curve can be obtained by means of the Riemann–Hurwitz formula:

\[
2\hat{g} - 2 = 2(2g - 2) + 4.
\]

For \( g = 2 \) we obtain \( \hat{g} = 5 \).

In turn, the base curve is a 2-sheeted branch covering of a Riemann sphere. Thus we can consider the spectral curve as a four-sheeted covering of the sphere. Moreover, there are eight branch points on each sheet (six points coming from gluing the base curve, and two points coming from gluing the two copies of the base curve). The Riemann surface obtained in this way is represented in Fig. 2, where circles mean Riemann spheres and each line is a gluing along a cut connecting a pair of branch points (one can interpret the lines as tubes connecting the spheres).

The spectral curve is invariant under the involution \( \lambda \to -\lambda \) (in the figure, it is the reflection in the vertical axis). The fixed points of the involution correspond to the branch points defined by the condition \( \lambda = 0 \). In other words, they correspond to the branch points of the spectral curve over the base curve (i.e., to the horizontal lines in the figure).

The basis of Prym differentials, according to Proposition 3.2, is given by 

\[
\omega_i = \left(\frac{x^i \, dx}{2y\lambda}\right), \quad i = 0, 1, 2.
\]

To calculate the matrix \( A^{-1} \) with the help of relation (3.7), we can choose cuts in two ways. One way is to choose three cuts corresponding to vertical lines. Another way is to choose two cuts corresponding to vertical lines and one cut corresponding to a horizontal line. By the definition of the Prym map, the lower limit of integration in (3.7) is chosen at one of the fixed points.

4.2. Hitchin system for the Lie algebra \( \mathfrak{so}(4) \) on a hyperelliptic curve of genus 2. In this case, the spectral curve is singular. According to [10], the Hitchin foliation is a foliation with Prym varieties of normalized spectral curves as leaves.

The Hitchin system in question has \( N = 6 \) independent Hamiltonians. The Lie algebra \( \mathfrak{so}(4) \) has two basis invariants \( p \) and \( q \). They are both second-degree invariants (one of them is the Pfaffian, and we denote it by \( q \)). The spectral curve has the form

\[
R(\lambda, x, y, H) = \lambda^4 + \lambda^2 p + q^2 = 0,
\]

where \( p = H_0 + xH_1 + x^2H_2 \) and \( q = H_3 + xH_4 + x^2H_5 \). Singular points are solutions of the following system supplemented by equation (4.2):

\[
\begin{align*}
R'_\lambda(\lambda, x, y, H) &= \lambda(4\lambda^2 + 2p) = 0, \\
R'_x(\lambda, x, y, H) &= \lambda^2 p'_x + 2q q'_x = 0.
\end{align*}
\]

In general position, the spectral curve has four singular points given by the equations \( \lambda = 0 \) and \( q = 0 \).

In the case of \( \lambda^2 = -p/2 \) and generic \( H \), the second equation of system (4.3) does not hold and, thus, we obtain branch points. By (4.2) they satisfy the following system of equations:

\[
\lambda^2 = -\frac{p}{2}, \quad q = \pm\frac{p}{2}.
\]
Due to the symmetries in $y$ and $\lambda$ we obtain 16 branch points. For a singular curve, the Riemann–Hurwitz formula gives the genus of its normalization (we continue to denote it by $\hat{g}$). For this reason we have

$$2\hat{g} - 2 = 4(2g - 2) + 16$$

for our spectral curve, which gives $\hat{g} = 13$ for $g = 2$.

The normalized curve is a four-sheeted branch covering of the base curve. If we consider it as a branch covering of a sphere, then it has eight sheets, with 10 branch points on each sheet (six points for gluing the curve of genus 2, and four points for gluing the two copies of the sheets thus obtained). The normalized curve is shown in Fig. 3.

The involution $\sigma: \lambda \to -\lambda$ corresponds to the rotation around the center of the picture through an angle of $\pi$. It has no fixed points. Although the singular points of the spectral curve are fixed, their preimages under the normalization map are permuted by the involution. In the figure, the preimages are located in the middles of the tubes corresponding to the horizontal lines, with two preimages on each. One can think of the normalization map as the identification of points on the opposite horizontal lines.

As a consequence of Propositions 3.1 and 3.2, the basis of holomorphic Prym differentials on the normalized curve is given by

$$\omega_i^{(0)} = \frac{x_i^{-1}q(x) dx}{y\lambda(4\lambda^2 + 2p(x))}, \quad i = 1, 2, 3, \quad \omega_i^{(1)} = \frac{\lambda^2 x_i^{-4} dx}{y\lambda(4\lambda^2 + 2p(x))}, \quad i = 4, 5, 6$$

(cf. (3.5), (3.6)). The cuts needed for normalizing the basis are uniquely defined and correspond to the lines marked with dots in Fig. 3.

Let $\omega$ be the set of normalized Prym differentials, and let $Q_1$ and $Q_2$ be a pair of points permuted by the involution. We can write the Prym map in the following way:

$$\eta(P) = \frac{1}{2} \int_{\gamma} \omega,$$

where $\gamma = 2\gamma_1 + \rho$, $\gamma_1$ is an arbitrary path from $Q_1$ to $P$, and $\rho$ is an arbitrary fixed path from $Q_2$ to $Q_1$ (hence $\rho + \gamma_1$ is a path from $Q_2$ to $P$; see Fig. 4). The ambiguity in choosing the path reduces to choosing $\gamma_1$ and leads to a change of the integral by a double lattice of periods. Thus, $\eta$ is a well-defined map to the Jacobian of the spectral curve. It remains to show that this map is skew-symmetric. For any path $\pi$ and a Prym differential $\omega$ we have $\int_{\sigma(\pi)} \omega = -\int_{\sigma(\pi)} \omega$. Indeed, both sides of this relation are equal to $\int_{\sigma(\pi)} \omega^\sigma$: the left-hand side by the invariance of the integral under a change of variables, and the right-hand side because the differential is skew-symmetric, $\omega^\sigma = -\omega$. We apply this to the proof of the relation $\eta(\sigma(P)) = -\eta(P)$. As soon as the path $\gamma_1$ from $Q_1$ to $P$ is chosen, we take the path from $Q_1$ to $\sigma(P)$ in the form $\gamma' = \sigma(\rho) + \sigma(\gamma_1)$ (Fig. 4). Then

\[\text{Fig. 3. Normalized spectral curve in the case of } \mathfrak{so}(4) \text{ and } g = 2.\]
Fig. 4. Paths on the normalized spectral curve.

$$2\eta(\sigma(P)) = \int_{2\gamma' + \rho} \omega.$$ We have

$$2\gamma' + \rho = 2\sigma(\gamma_1) + \sigma(\rho) + (\rho + \sigma(\rho)) = \sigma(\gamma) + (\rho + \sigma(\rho)).$$

By the above remark, $\int_{\rho + \sigma(\rho)} \omega = 0$; hence

$$2\eta(\sigma(P)) = \int_{\sigma(\gamma)} \omega = -\int_{\gamma} \omega = -2\eta(P).$$

It is essential that system (4.2), (4.4) is biquadratic, and therefore the branch points can be found explicitly. The Hamiltonians of the system can also be found explicitly in this case due to the following statement.

**Proposition 4.1** [3]. For $g = \mathfrak{so}(4)$ and $g = 2$, the system of separation equations reduces to one algebraic equation of degree 4 in one variable; consequently, it is solvable in radicals.

Thus, for $g = \mathfrak{so}(4)$ and $g = 2$ the problem of finding the algebro-geometric action–angle coordinates is solved as explicitly as possible, in the sense that finding the action coordinates has been reduced to the solution of a degree 4 equation, and the angle coordinates are expressed in terms of abelian integrals in the known integration limits.

### 4.3. Hitchin system for the Lie algebra $\mathfrak{sp}(4)$ on a curve of genus 2

The number of degrees of freedom of this system is 10. The Lie algebra $\mathfrak{sp}(4)$ has two basis invariants of degrees 2 and 4. Thus the equation of the spectral curve is

$$R(\lambda, x, y) = \lambda^4 + \lambda^2 r_2(x) + r_4(x) = 0,$$

where, due to (2.3),

$$r_2(x) = H_0 + xH_1 + x^2H_2,$$

$$r_4(x) = H_3 + \ldots + x^4H_7 + yH_8 + xyH_9.$$

Similar to the $\mathfrak{sl}(2)$ case, it is a smooth curve, because the system of equations for singular points has no solutions in general position. Indeed, the system is as follows:

$$\begin{cases}
R(\lambda, x, y, H) = \lambda^4 + \lambda^2 r_2(x) + r_4(x) = 0,
R'_\lambda(\lambda, x, y, H) = \lambda(4\lambda^2 + 2r_2(x)) = 0,
R'_x(\lambda, x, y, H) = \lambda^2(r_2)'_x + (r_4)'_x = 0.
\end{cases}$$

(4.6)

From the second equation we have either $\lambda = 0$ or $\lambda^2 = -r_2/2$. If $\lambda = 0$, the system takes the following form:

$$\lambda = 0, \quad r_4(x) = 0, \quad (r_4)'_x = 0.$$ 

It may have solutions only in the case when $r_4(x)$ has multiple roots, but this is not so in general position. In the case of $\lambda^2 = -r_2/2$, system (4.6) reduces to the equation $r_2(x) = \text{const}$. This is not the case in general position either.
The branch points of the spectral curve are given by the first two equations of system (4.6). For $\lambda = 0$ one has $r_4(x) = 0$, which is equivalent to an equation of degree 8 and hence gives eight branch points. For $\lambda^2 = -r_2(x)/2$ we obtain $r_2(x)^2/4 = r_4(x)$, which also leads to an eighth-degree equation, but due to the symmetry in $\lambda$ the number of branch points doubles, i.e., it is 16. The total number of branch points is 24. By the Riemann–Hurwitz formula, for the genus $\hat{g}$ of the spectral curve we obtain

$$2\hat{g} - 2 = 4(2g - 2) + 24,$$

which gives $\hat{g} = 17$ for $g = 2$.

The basis of holomorphic Prym differentials on the spectral curve is given by

$$\omega_i^{(0)} = \frac{\lambda^2 x_i \, dx}{y\lambda(4\lambda^2 + 2r_2)}, \quad i = 0, 1, 2, \quad \omega_i^{(1)} = \frac{x^{i-3} \, dx}{y\lambda(4\lambda^2 + 2r_2)}, \quad i = 3, \ldots, 7,$$

$$\omega_i^{(2)} = \frac{x^{i-8} \, dx}{\lambda(4\lambda^2 + 2r_2)}, \quad i = 8, 9.$$

The involution with respect to $\lambda$ acts as a reflection in the vertical axis in Fig. 5. To normalize the basis of differentials, one should choose $N = 10$ cuts which are not permuted by the involution; for instance, one can take the cuts corresponding to the lines marked by dots in Fig. 5. The involution has eight fixed points located on the horizontal lines on the picture. To construct the Prym map, one should choose an arbitrary fixed point of the involution. Denote it by $Q$. Then the Prym map is $\eta(P) = \int_Q^P \omega$, where $\omega$ is a set of normalized basis differentials, as above (see [6]).

4.4. Hitchin system for the Lie algebra $\mathfrak{so}(5)$ on a curve of genus 2. This system is locally isomorphic to the previous one [10]. It also has 10 degrees of freedom and two basis invariants of degrees 2 and 4. The spectral curve has the form

$$R(\lambda, x, y, H) = \lambda^5 + \lambda^3 r_2(x) + \lambda r_4(x) = 0.$$

It splits into two irreducible components, $\lambda = 0$ and $\lambda^2 r_2(x) + r_4(x) = 0$. The first component is trivial, and the second one has the same form as the spectral curve for $\mathfrak{sp}(4)$. Thus a local description of this system completely reduces to the $\mathfrak{sp}(4)$ case.

5. DISCUSSION

In this section, we compare our results with those of the paper [7], which pursued similar goals and, in a sense, held a record from 1998 to 2018. In [7], a specific Lax representation was used, which gives the Hitchin system in the simplest case of the gauge group $\text{SL}(2)$ and a base curve of genus 2. The arising spectral curve is different from the standard spectral curve of the Hitchin system; in particular, it differs from our spectral curve. The standard curve has genus 5 and is a
covering of the spectral curve in [7], which is hyperelliptic of genus 3. It was proved in [7] that in
the case of rank 2 and genus 2
(1) the equation of the spectral curve in [7] is a term-by-term product of the equation of the
standard spectral curve and the equation of the base curve;
(2) the Prym differentials on the standard curve are the lifts of all holomorphic differentials on
the spectral curve in [7].

Here we prove a generalization of this result for an arbitrary genus (a gauge group is still SL(2)).

**Proposition 5.1.** 1. The normalization of the curve defined by the equation obtained by term-
by-term multiplication of the equations of the spectral and base curves has genus $3(g - 1)$ (which
coincides with the dimension of the corresponding Prym variety).

2. The Prym differentials on the spectral curve can be obtained by lifting all holomorphic differ-
entials on this curve.

This actually suggests that the Prym varieties of spectral curves for SL(2)-type Hitchin systems
may be Jacobians. A classification of the curves possessing this property is given in [21]. The
example considered above does not fall under the terms of this classification, at least because the
set of singular points does not coincide with the set of fixed points of the involution: a singular
point is unique, while the branch points coincide with the zeros of the polynomial $P_{2g+1}$, as follows
from equation (5.2) below.

**Proof of Proposition 5.1.** Let us write the equation of the spectral curve in the form
$\lambda^2 - r(x, y) = 0$. The base curve is given by $y^2 = P_{2g+1}(x)$. Multiplying these equations, we
obtain
$$\lambda^2 y^2 + r(x, y)P_{2g+1}(x) = 0.$$  (5.1)

Following the lines of [7], we make a substitution $\sigma = \lambda y$:
$$\sigma^2 + r(x, y)P_{2g+1}(x) = 0.$$  (5.2)

We calculate the genus of the normalization of this curve as a branch covering over the $x$-line. The
set of its branch points is the union of the branch points of the spectral and base curves, but there is
a subtlety: the infinity is a branch point for both curves, so it becomes a simple (in general position)
singular point. The number of branch points of the spectral curve is $4g - 4$. The base curve has
$2g + 2$ branch points (including infinity; see Remark 5.1 below). Thus, the total number of branch
points of the curve (5.2) is $(4g - 4) + (2g + 2) - 2 = 6g - 4$ (see Remark 5.1). The genus $\tilde{g}$ of
the normalization is given by the Riemann–Hurwitz formula $2\tilde{g} - 2 = 2(h - 2) + 6g - 4$, where $h$ is the
genus of the base. In the case in question, $h = 0$, which implies $\tilde{g} = 3g - 3$.

Calculating the orders at infinity shows that the holomorphic differentials on the curve (5.2) are
given by the relation
$$\tilde{\omega}_j = \frac{\partial r/\partial H_j}{2\sigma} \frac{d\sigma}{\lambda}, \quad j = 1, \ldots, 3g - 3.$$

Substituting $\sigma = \lambda y$, one obtains
$$\tilde{\omega}_j = \frac{\partial r/\partial H_j}{2\lambda} \frac{dx}{y},$$
i.e., the basic Prym differentials on the standard spectral curve (according to Proposition 3.2).  \(\square\)

**Remark 5.1.** We should explain why the number of branch points of the spectral curve (5.2)
is equal to $4g - 4$, including the infinity point. Write $r$ in the form $r(x, y) = a(x) + yb(x)$, where $a$
and $b$ are polynomials of degrees $4g - 4$ and $4g - 5$, respectively. Then we represent the equation
of the spectral curve in the form $\lambda^2 + a = -yb$ and square both parts of it. Thus, we get rid of irrationalities but the equation becomes reducible:

$$(\lambda^2 + a)^2 = y^2b^2, \quad \text{or} \quad (\lambda^2 + a + yb)(\lambda^2 + a - yb) = 0,$$

and one of the irreducible components is the same as the spectral curve itself. At the finite branch points either $\lambda = 0$ or $\lambda^2 = -a$. It is easy to see that the number of branch points of the first kind is $4g - 4$, and the number of those of the second kind is $4g - 5$. As the number of all branch points must be even, we should add infinity as a branch point to one of the irreducible components. On the other hand, the irreducible components are symmetric; hence infinity is a branch point for both components.

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