Witten - Hawking wormhole and the propagation of light

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May 28, 2009

Abstract

The null geodesic congruence for the Lorentzian version of Hawking’s wormhole is studied, in spherical Rindler coordinates. One finds that the wormhole throat expands exponentially and the "flare-out" condition is satisfied.

A time reversal is equivalent with an inversion applied to the radial coordinate. The stress energy is mostly located near wormhole’s throat and the anisotropic fluid is comoving with the spherical distribution of accelerating observers. Far from the throat (the light cone in Cartesian coordinates) the negative energy density acquires an expression similar with the Casimir energy density between two perfectly reflecting plates.

We conjecture that the light propagation follows the throat of a pre-existing expanding wormhole (a "light dragging" phenomenon).

Keywords: expanding throat, null geodesic congruence, light dragging, spatial inversion.

1 Introduction

Lorentzian wormholes were often considered as topological objects ("handles" in spacetime which join two distant regions of the same universe or "bridges" linking different spacetimes [1]. Both of them give rise to multiply-connected universes with non-trivial topology and are hypersurfaces which connect two asymptotically flat spacetimes.

As Hochberg and Visser [1] have remarked, it is important to identify a local property to define a wormhole which is operative near its throat through the expansion of the null geodesic congruence propagating orthogonally to a closed two dimensional spatial hypersurface, generalizing in this way the Morris-Thorne analysis [2]. In their view, the above-mentioned hypersurface will be the wormhole throat provided the expansion \( \Theta \) of the null congruence vanishes.
on that hypersurface and the rate of change of the expansion along the null direction $\lambda$ is such that $d\Theta/d\lambda \geq 0$ (the "flare-out" condition, valid even for dynamic wormholes). In other words, the throat is a surface of minimal area w.r.t. deformations in the null direction $\lambda$, i.e. anti-trapped surface.

The "flare-out" condition at or near the throat violates the null energy condition which results directly from the Raychaudhuri equation [3]. Guendelman et al. [4] showed recently that the Einstein-Rosen "bridge" metric [5] is not a solution of the vacuum Einstein's equations but a lightlike brane source at $r = 2m$ is required (note that the Schwarzschild horizon at $r = 2m$ is a null two-surface). They observed that the brane is a natural gravitational source for transversable wormholes with the throat located at the horizon. This source is endowed with surface tension as an additional degree of freedom and is the place where the two copies of the exterior Schwarzschild spacetime are matched.

We use in this paper the dimensionally reduced Witten bubble spacetime [6] to obtain the Lorentzian version of the Hawking wormhole [11] which is conformal to the Rindler metric written in spherical coordinates [7]. By means of the Visser prescription [13] we argue that the Hawking wormhole throat expands and coincides with the null radial geodesic. We conclude from here that the light propagation follows the wormhole throat. In other words, light is "dragged" by the preexisting expanding throat to propagate. We find further that the stress tensor needed on the r.h.s. of Einstein’s equations for the Hawking wormhole to be a solution corresponds to an anisotropic fluid with negative energy density and radial pressure but positive transversal pressures.

From now on we shall use the geometrical units $\hbar = c = G = 1$.

## 2 The Witten expanding bubble

Let us consider the Witten bubble spacetime [6]

$$ds^2 = -g^2 r^2 dt^2 + (1 - \frac{4b^2}{r^2})^{-1} dr^2 + r^2 \cosh^2 gt \, d\Omega^2 + (1 - \frac{4b^2}{r^2}) \, d\chi^2$$

with $2b \leq r < \infty$, $b$ and $g$ - constants, $d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2$ is the unit two sphere and $\chi$ - the coordinate on the compactified fifth dimension. By means of the geometry (2.1) Witten has studied the decaying process (an expanding bubble) of the ground state of the original Kaluza-Klein geometry which, although stable classically, is unstable against a semiclassical barrier penetration [8].

We are now interested in the 4-dimensional subspace $\chi = \text{const.}$ of (2.1):

$$ds^2 = -g^2 r^2 dt^2 + (1 - \frac{4b^2}{r^2})^{-1} dr^2 + r^2 \cosh^2 gt \, d\Omega^2.$$  

The metric (2.2) is the ordinary Minkowski space provided $r >> 2b$, but written in spherical Rindler coordinates [9] (a spherical distribution of uniformly
accelerated observers, with the rest-system acceleration \( g \), uses this type of hyperbolically expanding coordinates. The singularity at \( r = 2b \) is only a coordinate singularity, as can be seen from the isotropical form of (2.2), with the help of a new radial coordinate \( \rho \)

\[
r = \rho + \frac{b^2}{\rho},
\]

with \( \rho \geq b > 0 \). Therefore, the spacetime (2.2) appears now as

\[
ds^2 = \left(1 + \frac{b^2}{\rho^2}\right)^2 \left(-g^2 \rho^2 dt^2 + d\rho^2 + \rho^2 \cosh^2 gt \ d\Omega^2\right).
\]

We note that the coordinate transformation

\[
\bar{x} = \rho \cosh g \tau \sin \theta \cos \phi, \quad \bar{y} = \rho \cosh g \tau \sin \theta \sin \phi, \quad \bar{z} = \rho \cosh g \tau \cos \theta, \quad \bar{t} = \rho \sinh g \tau
\]

changes the previous metric into

\[
ds^2 = \left(1 + \frac{b^2}{\bar{x}^2}\right)^2 \eta_{cd} \ d\bar{x}^c \ d\bar{x}^d,
\]

which is conformally flat (\( \eta_{cd} = \text{diag}(-1, +1, +1, +1) \) and the latin indices run from 0 to 3). In addition, we have \( \rho^2 = \eta_{ab} \bar{x}^a \bar{x}^b \geq b^2 \). From now on we take the constant \( b \) to be of the order of the Planck length.

Let us observe that the geometry (2.4) becomes flat provided \( \rho >> b \) (the conformal factor tends to unity) or provided \( \rho << b \), when the first term in the conformal factor may be neglected. Therefore, (2.4) represents the Lorentzian version of the euclidean Hawking wormhole \[10\] \[11\] \[12\].

It is interesting to treat the region \( \rho < b \) by means of the coordinate transformation

\[
\bar{\rho} = \frac{b^2}{\rho}.
\]

Inserting (2.7) in (2.4), we conclude that the metric (2.4) is invariant under the inversion (2.7)

\[
ds^2 = \left(1 + \frac{b^2}{\bar{\rho}^2}\right)^2 \left(-g^2 \bar{\rho}^2 dt^2 + d\bar{\rho}^2 + \bar{\rho}^2 \cosh^2 g \bar{\tau} d\Omega^2\right).
\]

When \( \rho \) varies from \( b \) to infinity, \( \bar{\rho} \in (0, b] \) and (2.8) becomes flat for \( \bar{\rho} << b \). Therefore, we may say that the \( \rho < b \) region is the image of \( \rho > b \) region obtained by inversion.

3 The energy momentum tensor

It is well known that the Hawking wormhole is not a solution of the vacuum Einstein equations. M. Visser \[13\], by surgically grafting two Schwarzschild space-time together, reached the conclusion that a boundary layer (the Schwarzschild
wormhole throat) contains a nonzero stress energy and separates two asymptotically flat regions. Using the "junction conditions" formalism, he found that the boundary layer must concentrate "exotic stress energy" which violates the weak energy condition. We saw before that, according to Guendelman et al. [4], the "Einstein - Rosen bridge" solution does not satisfy vacuum Einstein’s equations at the wormhole throat and we need a surface stress tensor on it.

We instead ask here for an energy momentum tensor on the r.h.s. of Einstein’s equations such that the spacetime (2.4) to be a solution. Using MAPLE - GRTensor Programmes, we found that the nonzero components of the Einstein tensor

\[ G_{ab} \equiv R_{ab} - \frac{1}{2} g_{ab} R = \kappa T_{ab} \]  

are given by

\[ G^t_t = G^\theta_\theta = G^\phi_\phi = -\frac{1}{3} G^\rho_\rho = \frac{4b^2 \rho^4}{(\rho^2 + b^2)^4} \]  

and \[ R^t_t = \kappa T^t_t = 0, \] where \( \kappa \) is Einstein’s constant. We see that the fluid is anisotropic and is comoving with the spherical Rindler observers (\( T^a_b \) is diagonal).

We have, using the obvious interpretation

\[ \epsilon = -T^t_t = -\frac{4b^2 \rho^4}{\kappa(\rho^2 + b^2)^4} \]  

for the energy density and

\[ T^\rho_\rho = p_\rho = -3p_\theta = -3p_\phi = \frac{-12b^2 \rho^4}{\kappa(\rho^2 + b^2)^4} \]  

for the radial and the transverse pressures, respectively. Even though the fluid is anisotropic, with pressures not only of different values but also of different signs, we could define a mean value

\[ p_m = \frac{p_\rho + p_\theta + p_\phi}{3} = \frac{\epsilon}{3}, \]  

as for a null fluid.

We shall consider the energy is mostly located on the throat of the wormhole (2.4) which is in accordance with a negative energy density (\( \epsilon < 0 \)) on the throat. From (3.3) it is clear that the case \( \rho >> b \) leads to an expression for \( \epsilon \) which does not depend on the Newton constant \( G \)

\[ \epsilon = -\frac{4b^2}{8\pi G \rho^4} = -\frac{\hbar c}{2\pi \rho^4} \]  

where \( b \) has been replaced with the Planck length \( \sqrt{\hbar c/\kappa} \). In other words, \( \epsilon \) has a purely quantum origin in this regime, with an expression similar to the Casimir energy density between two perfectly conducting parallel plates.
4 The null geodesic expansion

Hochberg and Visser [1] consider that there is a natural local geometric characterization of the existence and location of a wormhole throat. They argued that the throat is an extremal hypersurface of minimal area (minimal anti-trapped surface) with respect to deformations in the null direction $\lambda$ - the parameter on the null geodesic (the tangent vector is $k^a = dx^a/d\lambda$). In addition, the two-dimensional spatial hypersurface to which the null geodesic congruence is orthogonal will be the wormhole throat provided the scalar expansion $\Theta = 0$ and $d\Theta/d\lambda > 0$ on that surface (the "flare-out" condition generalized for a dynamic wormhole).

According to the previous authors’ prescription, we take the null geodesic congruence orthogonal to a compact two-dimensional surface embedded in the spacetime. We introduce a future-directed outgoing null 4-vector $k^a_+$, a future-directed ingoing null 4-vector $k^a_-$ and a spatial orthogonal projection tensor $\gamma^{ab}$ which observes the following relations

$$k^a_+ k^a_- = k^a_- k^a_+ = 0, \quad k^a_- k^a_- = k^a_+ k^a_+ = -1$$

$$k^a_+ \gamma^{ab} = 0, \quad \gamma^{a}_{\ b} \gamma^{bc} = \gamma^{ac}$$

(4.1)

The full spacetime metric may be decomposed in terms of the previous vectors (see also [1])

$$g_{ab} = \gamma_{ab} - k_{-a} k_{+b} - k_{+a} k_{-b}$$

(4.2)

We note that the spacelike hypersurface must have two sides and + and - are just two labels for the two null direction.

Defining the purely spatial tensors $v_{ab}^\pm = \gamma^c_a \gamma^d_b \nabla_c k_{\pm d}$, we have [1]

$$\Theta_{\pm} = \gamma^{ab} v_{ab}^\pm = \gamma^{ab} \nabla_a k_{\pm b},$$

(4.3)

where the expansions $\Theta_{\pm}$ represents the trace of $v_{ab}^\pm$ and $\gamma_{ab} \gamma^{ab} = 2$.

The Raychaudhuri equation for the congruence of null geodesics can be written as

$$\dot{\Theta}_{\pm} + 2(\sigma_{\pm}^2 - \Omega_{\pm}^2) + \frac{1}{2} \Theta_{\pm}^2 = -R_{ab} k^a_{\pm} k^b_{\pm}$$

(4.4)

where $\dot{\Theta} = d\Theta/d\lambda = k^a \nabla_a \Theta$, $2\sigma_{\pm}^2 = \sigma_{\pm}^{\pm ab} \sigma_{\pm ab}$, $2\Omega_{\pm}^2 = \Omega_{\pm ab} \Omega_{\pm ab}$ and the shear tensor $\sigma_{ab}$ and the vorticity tensor $\Omega_{ab}$ are given, respectively, by

$$\sigma_{ab}^\pm = \frac{1}{2}(v_{ab}^\pm + v_{ba}^\pm) - \frac{1}{2} \Theta_{\pm} \gamma^{ab}$$

(4.5)

and

$$\Omega_{ab}^\pm = \frac{1}{2}(v_{ab}^\pm - v_{ba}^\pm).$$

(4.6)

Let us find now the expansion $\Theta_{\pm}$ for our wormhole (2.4) and, from here, the location of the throat, namely the surface where $\Theta_{\pm} = 0$. By use of the Eqs.
(4.1) and (4.2), it is an easy task to obtain

\[ k^+ = \left( \frac{1}{\omega^2}, -\frac{g\rho}{\omega^2}, 0, 0 \right), \quad k^- = \left( \frac{1}{2g^2\rho^2}, \frac{1}{2g\rho}, 0, 0 \right), \]

\[ \gamma_{ab} = (0, 0, \omega^2 \rho^2 \cosh^2 gt, \omega^2 \rho^2 \cosh^2 gt \sin^2 \theta), \]

with \( \omega = 1 + \left( b^2/\rho^2 \right) \) and the components of the 4-vectors are in the order \( t, \rho, \theta, \phi \).

With the help of the following expressions of the Christoffel symbols

\[ \Gamma_{\theta\theta}^t = \frac{\Gamma_{\phi\phi}^t}{\sin^2 \theta} = \frac{1}{g} \cosh gt \sinh gt, \quad \Gamma_{\theta\theta}^\rho = -\frac{\rho(b^2 - \rho^2) \cosh^2 gt}{\rho^2 + b^2}, \quad \Gamma_{\rho\rho}^\rho = -\frac{2b^2}{\rho(\rho^2 + b^2)} \]

we get

\[ \Theta_+ = \frac{2g}{\omega^2 \cosh gt} \left[ (1 + \frac{b^2}{\rho^2}) \sinh gt - (1 - \frac{b^2}{\rho^2}) \cosh gt \right] \]

and

\[ \Theta_- = \frac{1}{g \rho^2 \omega \cosh gt} \left[ (1 + \frac{b^2}{\rho^2}) \sinh gt + (1 - \frac{b^2}{\rho^2}) \cosh gt \right] \]

The conditions \( \Theta_\pm = 0 \) for the throats gives us

\[ \rho_+(t) = b \, e^{gt}, \quad \rho_-(t) = b \, e^{-gt}, \]

with \( \rho_\pm(0) = b \). A time reversal changes \( \rho_+(t) \) in \( \rho_-(t) \). We see that, while one throat is expanding the other is shrinking. Since the spacetime (2.4) is invariant under an inversion \( \rho \rightarrow \tilde{\rho} = b^2/\rho \), we could replace \( \rho_+(t) < b \) (\( t < 0 \)) with \( \tilde{\rho}_+(t) > b \) (\( t > 0 \)), obtaining in this way two copies of the region \([b, \infty) : \tilde{\rho}_+(t)\) for \( t < 0 \) and \( \rho_+(t) \) for \( t > 0 \). Therefore, one throat comes from infinity at \( t \rightarrow -\infty \), reaches \( \tilde{\rho}_+(t) = b \) at \( t = 0 \) and returns to infinity at \( t \rightarrow \infty \). Similar arguments may be applied for the domain \([0, b)\).

An inspection of the null radial geodesics in the geometry (2.4) yields

\[ -g^2 \rho^2 dt^2 + d\rho^2 = 0, \]

whence \( \rho_\pm(t) = \rho_0 \, e^{\pm gt} \), with \( \rho_0 = \rho(0) \). The conformal factor does not change, of course, the null trajectories, so that the previous null curves correspond to those viewed by a spherical distribution of uniformly accelerated observer in Minkowski space, with acceleration \( g \). In addition, we notice that the particular null radial geodesic with \( \rho(0) = b \) represents the expanding (contracting) wormhole throat, found before. Reverting the logic, we conjecture that the propagation of light follows the throat of a preexisting wormhole. The expanding one corresponds to the retarded radiation (\( t > 0 \)) and the contracting one - to the advanced radiation (\( t < 0 \)). In other words, our “empty” space is full of expanding - contracting throats which “drag” the light with them. Their throats
correspond, in the conformally flat metric (2.6), to the light cones \( \tilde{r} = \pm \tilde{t} \) or \( \tilde{r} = \pm \tilde{t} + b \), with \( \tilde{r} = \sqrt{\tilde{x}^2 + \tilde{y}^2 + \tilde{z}^2} \). On the grounds of the previous results, we conclude that the geometry (2.6) is more suitable for the spacetime felt by an inertial observer than the Minkowski spacetime. We have a nonvanishing stress tensor near the wormhole throat (light cone) but the components of \( T_{ab} \) tends to zero far from the light cone.

We notice that any geodesic with \( \rho_0 \neq b \) does not represents a wormhole throat. That is a consequence of the fact that the spacetime (2.4) is not invariant under \( \rho \) - translations.

Let us check now whether the minimal condition \( \dot{\Theta}_\pm \equiv d\Theta_\pm /d\lambda \geq 0 \) is fulfilled. We have

\[
\begin{align*}
k^a_+ \nabla_a \Theta_+ &= \frac{d\Theta_+}{d\lambda} \bigg|_{\Theta_+ = 0} = \frac{16b^2g^2}{\rho^2\omega^6}, \\
k^a_- \nabla_a \Theta_- &= \frac{d\Theta_-}{d\lambda} \bigg|_{\Theta_- = 0} = \frac{4b^2}{g^2\rho^6\omega^2}.
\end{align*}
\]

Hence, both \( \dot{\Theta}_\pm \) are everywhere positive and the "flare-out" condition is satisfied. That is in accordance with the negative value of the energy density \( \epsilon \) ("exotic" matter).

It is worth to note that we use a non-affine connection for our congruence of null geodesic. Therefore, the Raychaudhuri equation (4.4) must have an extra term \(-K \Theta_\pm \) on its l.h.s., where \( K \) is obtained from [3] [15]

\[
a^b \equiv k^a_\pm \nabla_a k^b_\pm = K_\pm k^b_\pm \quad (4.14)
\]

This is the geodesic equation in a non-affine parameterization and \( K \) plays the role of the "surface gravity" [16]. From the expressions (4.7) for \( k^a_\pm \) we find the components of the accelerations \( a^b_\pm \):

\[
\begin{align*}
a^b_+ &= \left( \frac{-2g}{\omega^4}, \frac{2g^2\rho}{\omega^4}, 0, 0 \right), \\
a^b_- &= \left( \frac{-b^2}{g^3\rho^2\omega}, \frac{-b^2}{g^2\rho^3\omega}, 0, 0 \right),
\end{align*}
\]

with \( a^b_+ a^b_+ = a^b_- a^b_- = 0 \). Therefore, (4.13) and (4.15) yield

\[
K_+ = \frac{-2g}{\omega^2}, \quad K_- = \frac{-2b^2}{g^2\rho^2\omega} \quad (4.16)
\]

Using the above equations for \( K_\pm \), keeping in mind that \( \sigma^b_\pm \) and \( \Omega^b_\pm \) vanish in our spacetime (2.4) and that, with the help of the Eqs. (3.3) and (3.4), the r.h.s. of Raychaudhuri’s equations are given by

\[
\kappa T_{ab} k^a_+ k^b_+ = -\frac{16b^2g^2}{\rho^2\omega^6}, \quad \kappa T_{ab} k^a_- k^b_- = -\frac{4b^2}{g^2\rho^6\omega^2} \quad (4.17)
\]

we convince ourselves that the (±) Raychaudhuri equations are satisfied.

5 The anisotropic stress tensor

Let us show now that the energy momentum tensor \( T_{ab} \) with the components (3.3) - (3.4) can be expressed in the general form characterizing an anisotropic
fluid \[ T_{ab} = (\epsilon + p_{\perp})k_a k_b + p_{\perp} g_{ab} + (p_\rho - p_{\perp})s_a s_b \] (5.1)

where \( p_{\perp} \) is the transversal pressure and \( s_a \) is a spacelike vector in the direction of anisotropy. Taking \( p_{\perp} = p_\theta = p_\phi \), we get \( \epsilon + p_{\perp} = 0 \) and the first term on the r.h.s. of (5.1) is vanishing. With \( s_a = (0, \omega, 0, 0) \) and \( s_a s^a = 1 \), our stress tensor acquires the simple form

\[ T_{ab} = (\epsilon + p_\rho) s_a s_b - \epsilon g_{ab} \] (5.2)

It corresponds to an anisotropic fluid due to the fact that the radial and transversal pressures are different.

Let us note that the energy density on the throats, \( \rho_{\pm}(t) = b e^{\pm gt} \) can be written as

\[ \epsilon(t) = -\frac{1}{\kappa \rho_{th}^2(t)} \] (5.3)

where \( \rho_{th}(t) = 2b \cosh^2 gt \) is the time dependent radius of the throats, obtained from the metric

\[ ds_{th}^2 = 4b^2 \cosh^2 gt \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) \] (5.4)

It is invariant at time reversal and the inversion \( \rho \rightarrow b^2/\rho \).

6 Conclusions

Keeping in mind that the Witten - Hawking wormhole is not a solution of vacuum Einstein’s equations, we introduced an energy momentum tensor on the r.h.s. corresponding to an anisotropic fluid with negative energy density \( \epsilon = -p_\theta = -p_\phi \). For a radial coordinate \( \rho \) much greater than the Planck length \( (\rho >> b) \), we found that \( \epsilon \) no longer depends on the Newton’s constant \( G \), having a purely quantum origin.

We proved that the scalar expansions \( \Theta_{\pm} \) of the outgoing and, respectively, ingoing congruence of null geodesics are vanishing for \( \rho(t) = b e^{-gt} \), which suggests that the throats of the dynamic wormholes expand (shrink) exponentially, \( \rho(t) \) being identified with the null radial geodesics.

To check the Raychaudhuri equations, we kept in mind that we have not used an affine parameterization for the null congruence, adding an extra term proportional to the scalar expansion.

The stress tensor located near the throat (or near the light cone in Cartesian coordinates) is traceless, the null energy condition is violated and the "flare-out" condition is obeyed.

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