More chiral lattice fermions

Werner Kerler

Institut für Physik, Humboldt-Universität, D-10115 Berlin, Germany

Instead of the Ginsparg-Wilson relation only generalized chiral symmetry is required. The resulting much larger class of Dirac operators for massless fermions is investigated and a general construction for them is given. It is also shown that the new class still leads properly to Weyl fermions and to chiral gauge theories.

1. INTRODUCTION

The condition for generalized chiral symmetry of Lüscher has the general form
\[ \gamma_5 D + D \gamma_5 V = 0, \quad V^\dagger = V^{-1} = \gamma_5 V \gamma_5. \] (1)

We only impose together with \( \gamma_5 \)-hermiticity
\[ D^\dagger = \gamma_5 D \gamma_5 \] (2)
on D \mathbb{R}. With (2) we get from (1)
\[ D + D^\dagger V = 0, \quad D^\dagger + DV^\dagger = 0, \] (3)
which implies \([V, D] = 0\) and \(DD^\dagger = D^\dagger D\). This is accounted for by requiring \(D\) to be of form
\[ D = F(V). \] (4)

The simplest special case is that of the operators satisfying the Ginsparg-Wilson (GW) relation \(\{\gamma_5, D\} = \rho^{-1} D \gamma_5 D\mathbb{R}\), for which one has
\[ D = F(V) = \rho (1 - V). \] (5)

A further special case is that of the operators of Fujikawa satisfying \(\{\gamma_5, D\} = 2D(\gamma_5 a_0 D)^{2k+1}\) with \(k = 0, 1, 2, \ldots\), for which we find the form
\[ D = F(V) = a_0^{-1} \left(\frac{1}{2} (1 - V)(-V)^k\right)^{1/(2k+1)}. \] (6)

2. SPECTRAL REPRESENTATION

The spectral representation of \(D = F(V)\) in terms of that of \(V\) on the finite lattice is
\[ D = f(1)(P_1^{(+)} + P_1^{(-)}) + f(-1)(P_2^{(+)} + P_2^{(-)}) \]

where the orthogonal projections satisfy
\[ \gamma_5 P_j^{(\pm)} = P_j^{(\pm)} \gamma_5 = \pm P_j^{(\pm)}, \quad \gamma_5 P_k^{(I)} = P_k^{(II)} \gamma_5. \] (8)

For the functions \(f(v)\) from (4) and (5) one gets
\[ f(v) + f(v)^* v = 0, \quad f(v)^* = f(v^*), \] (9)
so that in particular \(f(1) = 0\) and \(f(-1)\) real. In addition to determining the form of \(D\), the functions \(f(v)\) describe the location of its spectrum.

According to the projection properties we have
\[ \text{Tr}(\gamma_5 P_1^{(\pm)}) = \pm N_\pm(1), \quad \text{Tr}(\gamma_5 P_2^{(\pm)}) = \pm N_\pm(-1), \quad \text{Tr}(\gamma_5 P_k^{(I)}) = \text{Tr}(\gamma_5 P_k^{(II)}) = 0, \] (10)
where \(N_+(\pm 1)\) and \(N_-(\pm 1)\) are the dimensions of the right-handed and the left-handed eigenspaces of \(V\) for eigenvalues \(\pm 1\), respectively. Evaluating the expressions,
\[ \lim_{\zeta \to 0} \text{Tr}(\gamma_5 \frac{\zeta}{D - \zeta I}), \quad \lim_{\zeta \to 0} \text{Tr}(\gamma_5 \frac{D}{D - \zeta I}), \] (11)

which sum to zero, it follows that in any case
\[ N_+(1) - N_-(1) + N_+( -1) - N_-( -1) = 0 \] (12)
and, in addition, that to allow for a nonvanishing index of \(D\), which is given by the first expression in (4), one must require
\[ f(-1) \neq 0. \] (13)

With this in place corresponds to the sum rule found by Chiu in the GW case.
Inserting the spectral representation of $V$ into $\text{Tr}(\gamma_5 V)$ and using (10) and (12) we generally obtain for the index of $D$

$$N_+(1) - N_-(1) = \frac{1}{2} \text{Tr}(\gamma_5 V),$$

which obviously involves only $V$ for the whole new class of operators $D$. Previous results in the GW case and in the overlap formalism can be recognized as special cases of (14).

3. CONSTRUCTION OF $D$

It follows from (11) that $f$ has the form

$$f(e^{i\varphi}) = e^{i(\varphi - \pi)/2} g(\varphi), \quad g(\varphi) \text{ real},$$

$$g(\varphi) = -g(-\varphi), \quad g(\varphi + 2\pi) = -g(\varphi).$$

Thus $D = F(V)$ can be obtained by determining functions $g(\varphi)$ with the above properties.

The immediate choice for such $g$ is the set

$$g(\varphi) = \sum_\nu s_\nu w_\nu(t_1, t_2, \ldots)$$

with real functions $w_\nu$, and

$$s_\nu = \sin(2\nu + 1)\frac{\varphi}{2}, \quad t_\mu = \cos \mu \varphi.$$

Because of the identity $s_\nu = s_0(1 + 2 \sum_{\mu=1}^\nu t_\mu)$ and since the $t_\nu$ can be expressed by polynomials of $t_1$ this can be reduced to the simpler form

$$g(\varphi) = s_0 w_0(t_1) = \sin \frac{\varphi}{2} w(\cos \varphi).$$

Given a function $g$ with the required properties then $h(g)$ again has these properties provided that $h$ satisfies for real $x$

$$h(-x) = -h(x), \quad h(x)^* = h(x).$$

Thus instead of (18) we get the more general form

$$g(\varphi) = a^{-1} h\left( \sin \frac{\varphi}{2} w(\cos \varphi) \right),$$

including the factor $a^{-1}$ for later convenience.

To satisfy condition (18) we need $g(\pi) \neq 0$ or $h(w(-1)) \neq 0$. Therefore we have to impose

$$w(-1) \neq 0,$$

which is sufficient if $h(x)$ gets only zero for $x = 0$.

To guarantee this we require strict monotony,

$$h(y) > h(x) \quad \text{for} \quad y > x.$$

Then also the inverse function $\eta$ with

$$h(\eta(x)) = x$$

is uniquely defined, which we need below.

Inserting the obtained construction of $f$ into the spectral representation (7) we have

$$D = \frac{1}{i\alpha} V^\pm H \left( \frac{1}{2\pi} (V^\pm - V^{-1}) W \left( \frac{1}{2} (V + V^\dagger) \right) \right),$$

where the properties of the hermitian operator functions $W$ and $H$ are determined by those of the real functions $w$ and $h$, respectively.

To specify $V$ appropriately we generalize the overlap form of $V$ introducing

$$V = -D^{(\eta)} W \left( \sqrt{D_W^{(\eta)\dagger} D_W^{(\eta)}} \right)^{-1},$$

$$D_W^{(\eta)} = i E \left( \frac{1}{2} \sum_{\mu} \gamma_\mu (\nabla_\mu - \nabla_\mu^\dagger) \right) + E \left( \frac{1}{2} \sum_{\mu} \nabla_\mu \nabla_\mu^\dagger \right) + E(m \mathbb{1}),$$

where the properties of the hermitian operator function $E$ (of hermitian operators) are determined by those of the real function $\eta$ (of real $x$).

To check the continuum limit we note that for the Fourier transform of $V$ in the free case one has at the corners of the Brillouin zone $V = -1$ and at zero

$$\tilde{V} \rightarrow 1 - \frac{i}{|\eta(m)|} \tilde{E} \left( a \sum_{\mu} \gamma_\mu p_\mu \right) \text{ for } a \rightarrow 0.$$

Requiring $\tilde{W}(-1) \neq 0$, because of the monotony of $E(X)$ doublers are suppressed for $-2r < m < 0$ as usual. Since $H(E(X)) = X$, putting $\tilde{W}(1) = 2|\eta(m)|$ the correct limit of the propagator is obtained generally for the class of $D$ of form (24).

4. CHIRAL GAUGE THEORIES

The projection operators given in (13) and implicit in (14) are of the general form

$$P_\pm = \frac{1}{2} (1 \pm \gamma_5) \mathbb{1}, \quad \tilde{P}_\pm = \frac{1}{2} (1 \pm \gamma_5 V) \mathbb{1},$$

yielding
in which only $V$ is involved. The latter also holds for the difference of dimensions
\[
\text{Tr } P_+ - \text{Tr } \tilde{P}_- = \frac{1}{2} \text{Tr}(\gamma_5 V),
\]
which equals (44).

According to (1) we have $D = \frac{1}{2}(D - \gamma_5 D \gamma_5 V)$, which with (28) becomes $D = P_+ \tilde{D} \tilde{P}_- + P_- \tilde{D} \tilde{P}_+$. With this we generally get for Weyl operators
\[
P_{\pm} D \tilde{P}_\mp = D \tilde{P}_\mp = P_{\pm} D.
\]

The projections can be represented in terms of bases with $u_j^i u_j = \delta_{ij}$, $\bar{u}_k^i \bar{u}_l = \delta_{kl}$ as
\[
P_+ = \sum_j u_j^i u_j^i, \quad \tilde{P}_- = \sum_k \bar{u}_k^i \bar{u}_k^i,
\]
which implies that one has the eigenequations
\[
P_+ u_j = u_j, \quad \tilde{P}_- \bar{u}_k = \bar{u}_k.
\]

Associating Grassmann variables $\bar{\chi}_j$ and $\chi_k$ to the degrees of freedom gives the fields
\[
\bar{\psi} = \sum_j \bar{\chi}_j u_j^i, \quad \psi = \sum_k \bar{u}_k \chi_k,
\]
and for $\text{Tr } \tilde{P}_- = \text{Tr } P_+$ for correlation functions
\[
\int \prod_l (d \bar{\chi}_l d \chi_l) e^{-\bar{\psi} D \psi} \bar{\psi}_{n(i)} \psi_{n(r_1)} \ldots \psi_{n(f)} \bar{\psi}_{n(r_f)} = \sum_{s_1, \ldots, s_f} \epsilon_{s_1 s_2 \ldots s_f} (\tilde{P}_- D^{-1} P_+)_{n(s_i) n(r_i)} \ldots
\]
\[
\ldots (\tilde{P}_- D^{-1} P_+)_{n(s_f) n(r_f)} \det M,
\]
where the matrix $M$ and its inverse are given by
\[
M_{jk} = u_j^i \bar{u}_k^i, \quad (M^{-1})_{kl} = \bar{u}_k^i D^{-1} u_l.
\]

Obviously for the derivations here generalized chiral symmetry of $D$ has been sufficient and the GW relation has not been needed. This will also hold in the following Section.

5. INVARIANCE OF DETERMINANT

We note that the transformations of interest in fermion space are closely related. For unitary $T$ with $D' = TDT^\dagger$ because of (3) we also have $V' = TVT^\dagger$ and vice versa. Then, requiring $T$ to satisfy also $[\gamma_5, T] = 0$, with $V' = TVT^\dagger$ because of (28) we also get $\tilde{P}'_+ = \tilde{T} \tilde{P}_- \tilde{T}^\dagger$ and vice versa. Therefore putting $T = \exp \mathcal{G}$, where
\[
\mathcal{G}^\dagger = -\mathcal{G}, \quad [\gamma_5, \mathcal{G}] = 0,
\]
we obtain for the related variations
\[
\delta V = [\mathcal{G}, V], \quad \delta D = [\mathcal{G}, D], \quad \delta \tilde{P}_- = [\mathcal{G}, \tilde{P}_-].
\]

Varying the second relation in (22) gives
\[
(\delta \tilde{P}_-)u_k = (1 - \tilde{P}_-) \delta \bar{u}_k.
\]

and inserting the last relation of (37) into this
\[
\tilde{P}_+ \delta \bar{u}_k = \tilde{P}_+ \mathcal{G} \bar{u}_k.
\]

Thus unfortunately only a relation for $\tilde{P}_+ \delta \bar{u}_k$ follows here while one for $\tilde{P}_- \bar{u}_k$ would be needed to evaluate the last term of (46) below.

For the gauge variation $\delta \ln \det M = \text{Tr}(M^{-1} \delta M)$ of the chiral determinant one gets
\[
\sum_{k,l} (M^{-1})_{kl} \delta M_{kl} = \text{Tr}(\tilde{P}_- D^{-1} \delta D) + \sum_k \bar{u}_k^i \delta \bar{u}_k,
\]
which requiring gauge invariance has to vanish. The first contribution, giving the gauge anomaly, turns out to involve generally only $V$,
\[
\text{Tr}(\tilde{P}_- D^{-1} \delta D) = \frac{1}{2} \text{Tr}(\mathcal{G} \gamma_5 V).
\]

REFERENCES

1. M. Lüscher, Phys. Lett. B 428 (1998) 342.
2. M. Lüscher, Nucl. Phys. B 549 (1999) 295; Nucl. Phys. B 568 (2000) 162.
3. W. Kerler, Nucl. Phys. B 646 (2002) 201.
4. P.H. Ginsparg and K.G. Wilson, Phys. Rev. D 25 (1982) 2649.
5. K. Fujikawa, Nucl. Phys. B 589 (2000) 487.
6. T.-W. Chiu, Phys. Rev. D 58 (1998) 074511.
7. P. Hasenfratz, V. Lalena, F. Niedermayer, Phys. Lett. B 427 (1998) 125.
8. R. Narayanan and H. Neuberger, Phys. Rev. Lett. 71, (1993) 3251; Nucl. Phys. B 412 (1994) 574; Nucl. Phys. B 443 (1995) 305.
9. H. Neuberger, Phys. Lett. B 417 (1998) 141; Phys. Lett. B 427 (1998) 353.