M-ELLIPTICITY OF FREDHOLM PSEUDO-DIFFERENTIAL OPERATORS ON $L^p(\mathbb{R}^n)$ AND GÅRDING’S INEQUALITY

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Abstract. In this paper, we study the $M$-ellipticity of Fredholm pseudo-differential operators associated with weighted symbols on $L^p(\mathbb{R}^n)$, $1 < p < \infty$. We also prove the Gårding’s inequality for $M$-elliptic operators and the hybrid class of pseudo-differential operators, namely $SGM$-elliptic operators.

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1. Introduction

A general symbolic calculus and corresponding definition of global ellipticity for partial differential operators and pseudo-differential operators on noncompact manifolds represent a relevant issue of the modern Mathematical Analysis. In the case of the Euclidean space $\mathbb{R}^n$, one can refer to the pseudo-differential operators corresponding to the Hörmander symbol class, $S^m_{\rho,\delta}$, $m \in \mathbb{R}$, $0 \leq \rho, \delta \leq 1$, [12, 19]. The other is given by the so-called SG classes. In [7], they are also called pseudo-differential operators with exit behavior. SG pseudo-differential operators and related topics can be found in [3, 5, 6, 13, 14] and the references therein. In general to prove $L^p$ boundedness of pseudo-differential operators associated with the Hörmander symbols, $S^0_{1,0}$, the key ingredient is the Mikjlin-Hörmander theorem about Fourier multipliers. But unfortunately, $L^p$-boundedness theorem for $p \neq 2$ does not hold for operators in $S^0_{\rho,0}$, for $0 < \rho < 1$. Taylor in [15] introduced a suitable symbol subclass, $M^m_{\rho,0}$, of $S^m_{\rho,0}$ giving a continuous operator for every $1 < p < \infty$ and $0 < \rho \leq 1$. Further to this, Garello and Morando introduced the subclass $M^m_{\rho,\Lambda}$ of $S^m_{\rho,\Lambda}$, which are just weighted version of the ones introduced by Taylor in [8, 9, 10] and developed the symbolic calculus for the associated pseudo-differential operators with many applications to study the regularity of multi-quasi-elliptic operators. Many studies on various properties of $M$-elliptic pseudo-differential operators can be found in [8, 18, 2, 4, 10] and [17]. But till now not much has been done on the $M$-ellipticity of Fredholm pseudo-differential operators. In this paper, we prove that Fredholm pseudo differential operators with weighted symbol are $M$-elliptic pseudo-differential operators on $\mathbb{R}^n$. Finally we prove the Gårding’s Inequality for $M$-elliptic operators. Let us start with a few historical notes and basic definitions.
A positive function $\Lambda \in C^\infty(\mathbb{R}^n)$ is said to be a weight function with polynomial growth if there exists positive constants $\mu_0, \mu_1, C_0$ and $C_1$ be such that $\mu_0 \leq \mu_1$ and $C_0 \leq C_1$, for which we have following conditions:

$$C_0(1 + |\xi|)^{\mu_0} \leq \Lambda(\xi) \leq C_1(1 + |\xi|)^{\mu_1}, \quad \xi \in \mathbb{R}^n.$$  

Also, we assume that there exists a real constant $\mu$ such that $\mu \geq \mu_1$ and for all multi-indices $\alpha, \gamma \in \mathbb{Z}^n$ with $\gamma_j \in \{0, 1\}, j = 1, 2, \ldots, n$, we can find a positive constant say $C_{\alpha,\gamma}$ such that

$$|\xi^\gamma (\partial^\alpha + \gamma \Lambda)(\xi)| \leq C_{\alpha,\gamma} \Lambda(\xi)^{1 - \frac{1}{\mu}|\alpha|}, \quad \xi \in \mathbb{R}^n. \quad \text{(1.1)}$$

Let us take $m \in \mathbb{R}$ and let $\rho \in (0, \frac{1}{\mu}]$. Then we say a function $\sigma \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ is in $S_{\rho,\Lambda}^m$ class if for all multi-indices $\alpha$ and $\beta$ we can find a positive constant $C_{\alpha,\beta}$ such that

$$|(\partial_\xi^\alpha \partial_\rho^\beta \sigma)(x, \xi)| \leq C_{\alpha,\beta} \Lambda(\xi)^{m-\rho|\beta|}, \quad x, \xi \in \mathbb{R}^n.$$  

We call such $\sigma$ be a symbol of order $m$ and type $\rho$ with weight $\Lambda$. Also, we say a symbol $\sigma \in M_{\rho,\Lambda}^m$ if for all multi-indices $\gamma$ with $\gamma_j \in \{0, 1\}, j = 1, 2, \ldots, n$, we have

$$\xi^\gamma (\partial_\xi^\gamma \sigma)(x, \xi) \in S_{\rho,\Lambda}^m.$$  

Let $\sigma \in S_{\rho,\Lambda}^m$. Define pseudo-differential operator $T_\sigma$ associate with symbol $\sigma$ by

$$(T_\sigma \varphi)(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix \cdot \xi} \sigma(x, \xi) \hat{\varphi}(\xi) \, d\xi, \quad x \in \mathbb{R}^n,$$

where $\varphi$ is in Schwartz space $S$ and

$$\hat{\varphi}(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ix \cdot \xi} \varphi(x) \, dx, \quad \xi \in \mathbb{R}^n.$$

Now we give a brief introduction to the properties of $M$-elliptic pseudo-differential operators. For this, we will start by defining a class of symbols $S_{\rho,\Lambda}^m$, where $m, \rho \in \mathbb{R}$ and $\rho \in (0, \frac{1}{\mu}]$. We say a symbol $\sigma \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ is in $S_{\rho,\Lambda}^m$ if for all multi-indices $\alpha$ we can find a positive constant $C_\alpha$ such that

$$|(\partial_\xi^\alpha \partial_\rho \sigma)(x, \xi)| \leq C_\alpha \Lambda(\xi)^{m-\rho|\alpha|}, \quad x, \xi \in \mathbb{R}^n.$$  

Also, we say a symbol $\sigma \in M_{\rho,\Lambda}^m$ if for all multi-indices $\gamma$ with $\gamma_j \in \{0, 1\}, j = 1, 2, \ldots, n$, we have

$$\xi^\gamma (\partial_\xi^\gamma \sigma)(x, \xi) \in S_{\rho,\Lambda}^m.$$  

Let $\sigma \in S_{\rho,\Lambda}^m$. Define pseudo-differential operator $T_\sigma$ associated with symbol $\sigma$ by

$$(T_\sigma \varphi)(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix \cdot \xi} \sigma(x, \xi) \hat{\varphi}(\xi) \, d\xi, \quad x \in \mathbb{R}^n, \quad \text{where} \quad \varphi \text{ is in Schwartz space } S \quad \text{and}$$

$$\hat{\varphi}(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ix \cdot \xi} \varphi(x) \, dx, \quad \xi \in \mathbb{R}^n.$$  

Note that it can be easily shown that $T_\sigma : S \rightarrow S$ is a continuous linear mapping. The following results can be found in [6].

**Theorem 1.1.** Let $\sigma \in M_{\rho,\Lambda}^m$ and $\tau \in M_{\rho,\Lambda}^\mu$. Then $T_\sigma T_\tau = T_\lambda$, where $\lambda \in M_{\rho,\Lambda}^{m+\mu}$ and

$$\lambda \sim \sum_{\mu} \frac{(-i)^{|\mu|}}{\mu!} (\partial_\xi^\mu \sigma)(\partial_\rho^\mu \tau).$$
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Here the asymptotic expansion means that for every positive integer $M$, there exists a positive integer $N$ such that

$$
\lambda - \sum_{|\mu|<N} \frac{(-i)^{|\mu|}}{\mu!} (\partial^\mu_x \sigma)(\partial^\mu_x \tau) \in M^m_{\rho, \Lambda} - \rho M.
$$

**Theorem 1.2.** Let $\sigma \in M^m_{\rho, \Lambda}$. Then the formal adjoint $T^*_\sigma$ of $T_\sigma$ is the pseudo-differential operator $T_\tau$, where $\tau \in M^m_{\rho, \Lambda}$ and

$$
\tau \sim \sum_{|\mu|<N} \frac{(-i)^{|\mu|}}{\mu!} \partial^\mu_x \sigma \partial^\mu_x \tau.
$$

Here the asymptotic expansion means that for every positive integer $M$, there exists a positive integer $N$ such that

$$
\tau - \sum_{|\mu|<N} \frac{(-i)^{|\mu|}}{\mu!} \partial^\mu_x \sigma \partial^\mu_x \tau \in M^m_{\rho, \Lambda} - \rho M.
$$

Now using formal adjoint, we can extend the definition of a pseudo-differential operator from the Schwartz space $S$ to the space $S'$ of all tempered distributions. For this, let $\sigma \in M^m_{\rho, \Lambda}$. Then for all $u \in S'$, we define $T_\sigma u : S \to \mathbb{C}$ such that

$$
(T_\sigma u)(\varphi) = u(T^*_\sigma \varphi). \quad (1.3)
$$

It is easy to check that $T_\sigma$ maps $S'$ into $S'$ continuously. In fact, we have the following theorem.

**Theorem 1.3.** Let $\sigma \in M^0_{\rho, \Lambda}$. Then $T_\sigma : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ is a bounded linear operator for $1 < p < \infty$.

Proof of this theorem can be found in from Theorem 1.4 in [18].

Let $\sigma \in M^m_{\rho, \Lambda}$, where $m \in \mathbb{R}$. Then $\sigma$ is said to be $M$-elliptic if there exists positive constants $C$ and $R$ such that

$$
|\sigma(x, \xi)| \geq CA(\xi)^m, \quad |\xi| \geq R.
$$

**Theorem 1.4.** Let $\sigma \in M^m_{\rho, \Lambda}$ be $M$-elliptic. Then there exists a symbol $\tau \in M^{-m}_{\rho, \Lambda}$ such that

$$
T_\tau T_\sigma = I + R
$$
and

$$
T_\sigma T_\tau = I + S,
$$

where $R$ and $S$ are pseudo-differential operators with symbol in $\cap_{k \in \mathbb{R}} M^k_{\rho, \Lambda}$.

The pseudo-differential operator $T_\tau$ in Theorem 1.4 is known as parametrix of the $M$-elliptic pseudo-differential operator $T_\sigma$.

To make this paper self contained we recall here the definition of the Sobolev spaces.

For $m \in \mathbb{R}$ and $1 < p < \infty$, we define the Sobolev space $H^m_{\Lambda} \cap L^p(\mathbb{R}^n)$ by

$$
H^m_{\Lambda} \cap L^p(\mathbb{R}^n) = \{ u \in S' : J_{-m} u \in L^p(\mathbb{R}^n) \}.
$$

Then $H^m_{\Lambda} \cap L^p(\mathbb{R}^n)$ is a Banach space in which the norm $\| \cdot \|_{m,p,\Lambda}$ is given by

$$
\| u \|_{m,p,\Lambda} = \| J_{-m} u \|_{L^p(\mathbb{R}^n)}, \quad u \in H^m_{\Lambda} \cap L^p(\mathbb{R}^n).
$$

Note that $H^0_{\Lambda} \cap L^p(\mathbb{R}^n) = L^p(\mathbb{R}^n)$. 

All the above results and definitions for SG-M elliptic case can be found in [1].

The main aim of this paper is to prove the ellipticity of fredholm pseudo-differential operators associated with weighted symbols and to prove the Gårding’s Inequality for M-elliptic operators. In Section 2, first, we prove that Fredholm pseudo-differential operators are M-elliptic and then show that ellipticity of fredholm pseudo-differential operators for the hybrid case. In Section 3, we prove the Gårding’s inequality for both M-elliptic operators and SG M-elliptic operators.

2. M-Ellipticity of Fredholm Pseudo-Differential Operators

In this section, our aim is to show that Fredholm pseudo-differential operator is M-elliptic on $L^p(\mathbb{R}^n)$. For this, we need some technical preparations which are done in [5] and [19].

**Definition 2.1.** Let $\lambda > 0, \tau \geq 0$ and $x_0, \xi_0 \in \mathbb{R}^n$. For $1 < p < \infty$, we define the operator $R_{\lambda, \tau} (x_0, \xi_0) : L^p (\mathbb{R}^n) \to L^p (\mathbb{R}^n)$, by

$$
(R_{\lambda, \tau} (x_0, \xi_0) u) (x) = \lambda^{\tau n/p} e^{i\lambda x \cdot \xi} u \left( \lambda^\tau (x - x_0) \right), \quad x \in \mathbb{R}^n,
$$

for all $u \in L^p (\mathbb{R}^n)$.

**Proposition 2.2.** The operator $R_{\lambda, \tau} (x_0, \xi_0) : L^p (\mathbb{R}^n) \to L^p (\mathbb{R}^n)$ is a surjective isometry and the inverse is given by

$$
(R_{\lambda, \tau} (x_0, \xi_0)^{-1} u) (x) = \lambda^{-\tau n/p} e^{-i\lambda (x_0 + \lambda^{-\tau} x) \cdot \xi_0} u \left( x_0 + \lambda^{-\tau} x \right), \quad x \in \mathbb{R}^n,
$$

for all $u \in L^p (\mathbb{R}^n)$.

**Proposition 2.3.** Let $1 < p < \infty$ and $\tau > 0$. Then, for all $u \in L^p (\mathbb{R}^n)$ and $v \in L^q (\mathbb{R}^n)$, where $\frac{1}{p} + \frac{1}{q} = 1$,

$$(R_{\lambda, \tau} (x_0, \xi_0) u, v) \to 0$$

as $\lambda \to \infty$.

The following theorem is one of the main theorems of this paper.

**Theorem 2.4.** Let $\sigma \in M^0_{\rho, \Lambda}$ be such that $T_{\sigma} : L^p (\mathbb{R}^n) \to L^p (\mathbb{R}^n)$ is a Fredholm operator for $1 < p < \infty$. Then $\sigma$ is M-elliptic.

To prove this theorem, we need following lemma.

**Lemma 2.5.** Let $\sigma \in M^m_{\rho, \Lambda}$, where $m \in (-\infty, \infty)$. Then

$$
R_{\lambda, \tau} (x_0, \xi_0)^{-1} T_{\sigma} R_{\lambda, \tau} (x_0, \xi_0) = T_{\sigma_{\lambda, \tau}}, \quad (2.1)
$$

where

$$
\sigma_{\lambda, \tau} (x, \eta) = \sigma \left( x_0 + \lambda^{-\tau} x, \lambda \xi_0 + \lambda^{\tau} \eta \right), \quad x, \eta \in \mathbb{R}^n. \quad (2.2)
$$

Moreover, if $\sigma \in M^0_{\rho, \Lambda}$, $\lambda \geq 1, 0 \leq \tau \leq \rho \mu_0 / (1 + \rho \mu_0)$ and $\xi_0 \neq 0$, then for all multi-indices $\alpha$ and $\beta$, there exists a positive constant $C_{\beta}$ such that

$$
\left| \left( \partial^\alpha_{x} \partial^\beta_{\eta} \sigma_{\lambda, \tau} \right) (x, \eta) \right| \leq C_{\beta} p_{\alpha, \beta}(\sigma) \frac{(\Lambda(\eta))^{\beta|\beta|}}{|\xi_0|^{\mu_0|\beta|}} \lambda^{-\tau |\alpha|} \lambda^{-(\rho \mu_0 + (1 + \rho \mu_0) \tau)|\beta|}, \quad x, \eta \in \mathbb{R}^n. \quad (2.3)
$$

Here $p_{\alpha, \beta}$ denotes the corresponding norm in $M^0_{\rho, \Lambda}$.
Thus, we get \( (\eta) \).

Now by substituting \( z = \lambda^\tau (y - x_0) \), the above expression takes the form
\[
\lambda^{-\tau \sigma} (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i(x - y) \cdot \sigma} (x_0 + \lambda^{-\tau} x, \xi) e^{i\lambda y \xi_0} u (\lambda^\tau (y - x_0)) dy d\xi.
\]

Now by substituting \( \eta = \lambda^{-\tau} (\xi - \xi_0) \), the above expression takes the form
\[
(2\pi)^{-n} \int_{\mathbb{R}^n} e^{i(x - z) \cdot \eta} (x_0 + \lambda^{-\tau} x, \lambda \xi_0 + \lambda^\tau \eta) u (z) dz d\eta.
\]

Thus, we get (2.1) and (2.2), as asserted. Now, using (2.2), the chain rule and Peetre’s inequality,
\[
\left| \left( \partial^\sigma_x \partial^\beta_\eta \sigma_{\lambda, \tau} \right) (x, \eta) \right| \leq \left| \left( \partial^\sigma_x \partial^\beta_\eta \sigma \right) (x_0 + \lambda^{-\tau} x, \lambda \xi_0 + \lambda^\tau \eta) \right| \lambda^{-\tau |\alpha|} \lambda^{|\beta|} \\
\leq p_{\alpha, \beta}(\sigma) (\Lambda (\lambda \xi_0 + \lambda^\tau \eta))^{-|\beta|} \lambda^{-\tau |\alpha|} \lambda^{|\beta|} \\
\leq C_{\beta} P_{\alpha, \beta}(\sigma) (\lambda \xi_0 + \lambda^\tau \eta)^{-|\beta|} \lambda^{-\tau |\alpha|} \lambda^{|\beta|} \\
\leq C_{\beta} P_{\alpha, \beta}(\sigma) (\xi_0)^{-|\beta|} \lambda^{-\tau |\alpha|} \lambda^{|\beta|} \\
\leq C_{\beta} P_{\alpha, \beta}(\sigma) (\xi_0)^{-|\beta|} (\Lambda (\eta))^{\rho |\beta|} \lambda^{-(\rho_\alpha (1 + \rho_\mu \tau) |\beta| \lambda^{-|\alpha|}} \\
\leq C_{\beta} P_{\alpha, \beta}(\sigma) (\xi_0)^{-|\beta|} (\Lambda (\eta))^{\rho |\beta|} \lambda^{-(\rho_\alpha (1 + \rho_\mu \tau) |\beta| \lambda^{-|\alpha|}}
\]
which completes proof of Equation (2.3).

\textbf{Proof of Theorem 2.4} Since \( T_\sigma \) is a Fredholm operator, so by Theorem 20.5 in [19], we can find a non-zero bounded linear operator \( S \) on \( L^p (\mathbb{R}^n) \) and a compact operator \( K \) on \( L^p (\mathbb{R}^n) \) such that
\[
ST_\sigma = I + K.
\]

Let \( M \) be the set of all points \( \xi \) in \( \mathbb{R}^n \) such that there exists a point \( x \) in \( \mathbb{R}^n \) for which
\[
|\sigma (x, \xi)| \leq \frac{1}{2||S||}.
\]

Now, if \( M \) is bounded, then there exists a positive number \( R \) such that
\[
|\xi| < R, \quad \xi \in M.
\]

Thus, for each point \( \xi \in \mathbb{R}^n \) with \( |\xi| \geq R \), we get, for all \( x \in \mathbb{R}^n \),
\[
|\sigma (x, \xi)| \geq \frac{1}{2||S||},
\]
and this implies that that \( \sigma \) is \( M \)-elliptic. So, suppose that \( M \) is not bounded. Then there exists a sequence \( \{(x_k, \xi_k)\} \) in \( \mathbb{R}^n \times \mathbb{R}^n \) such that
\[
|\xi_k| \to \infty
\]
as \( k \to \infty \) and
\[
|\sigma (x_k, \xi_k)| \leq \frac{1}{2||S||}, \quad k = 1, 2, \ldots
\]
Thus, there exists a subsequence of \( \{(x_k, \xi_k)\} \), again denoted by \( \{(x_k, \xi_k)\} \), such that
\[
\sigma (x_k, \xi_k) \to \sigma_\infty
\]
For some complex number $\sigma_\infty$ as $k \to \infty$. Therefore

$$|\sigma_\infty| \leq \frac{1}{2\|S\|}$$ (2.4)

For $k = 1, 2, \ldots$, let $\lambda_k = |\xi_k|$. Then, by Lemma 2.5, we have

$$R_{\lambda_k, \tau} \left( x_k, \frac{\xi_k}{|\xi_k|} \right)^{-1} T_\sigma R_{\lambda_k, \tau} \left( x_k, \frac{\xi_k}{|\xi_k|} \right) = T_{\sigma_{\lambda_k, \tau}},$$

where

$$\sigma_{\lambda_k, \tau}(x, \eta) = \sigma \left( x_k + \lambda_k^{-1} x, \lambda_k \eta \right), \quad x, \eta \in \mathbb{R}^n.$$

Let $\alpha$ and $\beta$ be arbitrary multi-indices. Then, by Equation (2.3), there exists a positive constant $C_\beta$ such that

$$\left| \left( \partial^\alpha_x \partial^\beta_\eta \sigma_{\lambda_k, \tau} \right) (x, \eta) \right| \leq C_{\beta \rho} \sigma(\Lambda(\eta))^{\rho + |\beta|} \lambda_k^{-|\alpha|} \lambda_k^{-\rho - (1 + \rho_0) \tau |\beta|}, \quad x, \eta \in \mathbb{R}^n.$$ (2.5)

For $k = 1, 2, \ldots$, we define $\sigma_\infty^k$ by

$$\sigma_\infty^k = \sigma_{\lambda_k, \tau}(0, 0) = \sigma \left( x_k, \xi_k \right).$$

Then, by using Theorem 7.3 in [19] and the estimate (2.5), we get

$$\left| \sigma_{\lambda_k, \tau}(x, \eta) - \sigma_\infty^k \right| = \left| \sigma_{\lambda_k, \tau}(x, \eta) - \sigma_{\lambda_k, \tau}(0, 0) \right|$$

$$= \left| \sum_{|\gamma + \mu| = 1} x^\gamma \eta^\mu \int_0^1 \left( \partial^\alpha_x \partial^\beta_\eta \sigma_{\lambda_k, \tau} \right) (\theta x, \theta \eta) d\theta \right|$$

$$\leq \sum_{|\gamma + \mu| = 1} |x|^{|\gamma|} |\eta|^{|\mu|} \int_0^1 C_{\mu \rho} \sigma(\Lambda(\eta))^{\rho + |\mu|} \lambda_k^{-|\gamma|} \lambda_k^{-\rho - (1 + \rho_0) \tau |\mu|} d\theta \to 0$$ (2.6)

uniformly for $(x, \eta)$ on every compact subset $K$ of $\mathbb{R}^n \times \mathbb{R}^n$ as $k \to \infty$. Let $u \in S$. Then

$$\left( T_{\sigma_{\lambda_k, \tau}} u \right)(x) - \sigma_\infty^k u(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix \cdot \eta} \left( \sigma_{\lambda_k, \tau}(x, \eta) - \sigma_\infty^k \right) \hat{u}(\eta) d\eta$$

for all $x \in \mathbb{R}^n$. By (2.6), the assumption that $\sigma \in M_{\rho, A}^0$ and Lebesgue’s dominated convergence theorem,

$$\left( T_{\sigma_{\lambda_k, \tau}} u \right)(x) \to \sigma_\infty^k u(x)$$

for all $x \in \mathbb{R}^n$ as $k \to \infty$. Moreover, for all $l \in \mathbb{N}$, using (2.5) and an integration by parts, we can find a positive constant $C_\mu$ for each $\mu$ with $|\mu| \leq 2l$ such that

$$\left| \langle x \rangle^{2l} \left( T_{\sigma_{\lambda_k, \tau}} u \right)(x) \right|$$

$$= \left| \langle x \rangle^{2l} (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix \cdot \eta} \sigma_{\lambda_k, \tau}(x, \eta) \hat{u}(\eta) d\eta \right|$$

$$= \left| (2\pi)^{-n/2} \int_{\mathbb{R}^n} \left( \left( I - \Delta \xi \right)^l e^{ix \cdot \xi} \sigma_{\lambda_k, \tau}(x, \xi) \hat{u}(\xi) d\xi \right) \right|$$

$$\leq (2\pi)^{-n/2} \int_{\mathbb{R}^n} \left| \sum_{|\mu| \leq 2l} \frac{1}{\mu!} \left( P^\mu(D) \hat{u} \right)(\xi) C_\mu (\Lambda(\xi))^{|\mu|} d\xi \right|$$

$$\leq (2\pi)^{-n/2} \int_{\mathbb{R}^n} \sum_{|\mu| \leq 2l} \frac{C_\mu'}{\mu!} |\hat{u}(\xi)| |\Lambda(\xi)|^{2l-|\mu|} |\xi|^{|\mu|} d\xi$$
2.4

Now, if \( 2lp > n \), then \( \langle x \rangle^{-2lp} \in L^1(\mathbb{R}^n) \). So, there exists a positive constant \( C_1 \) such that

\[
\left| \left( T_{\sigma_{\lambda_k}, \tau} - \sigma_k^\infty \right) u(x) \right|^p \leq C_1 \langle x \rangle^{-2lp}, \quad x \in \mathbb{R}^n.
\]

Thus,

\[
T_{\sigma_{\lambda_k}, \tau} u \rightarrow \sigma_\infty u
\]

in \( L^p(\mathbb{R}^n) \) as \( k \rightarrow \infty \). Let \( u \) be a nonzero function in \( L^p(\mathbb{R}^n) \). Since \( R_{\lambda_k, \tau} \left( x_k, \frac{\xi_k}{|\xi_k|} \right) \) is an isometry, it follows that

\[
0 < \| u \|_p^p = \left\| R_{\lambda_k, \tau} \left( x_k, \frac{\xi_k}{|\xi_k|} \right) u \right\|_p^p
= \left\| (ST_\sigma - K) R_{\lambda_k, \tau} \left( x_k, \frac{\xi_k}{|\xi_k|} \right) u \right\|_p^p
\leq \left\| ST_\sigma R_{\lambda_k, \tau} \left( x_k, \frac{\xi_k}{|\xi_k|} \right) u \right\|_p^p + \left\| KR_{\lambda_k, \tau} \left( x_k, \frac{\xi_k}{|\xi_k|} \right) u \right\|_p^p
\leq \left\| R_{\lambda_k, \tau} \left( x_k, \frac{\xi_k}{|\xi_k|} \right)^{-1} T_\sigma R_{\lambda_k, \tau} \left( x_k, \frac{\xi_k}{|\xi_k|} \right) u \right\|_p^p \| S \| + \left\| KR_{\lambda_k, \tau} \left( x_k, \frac{\xi_k}{|\xi_k|} \right) u \right\|_p^p.
\]

(2.7)

Now, using the fact that \( K \) is a compact operator and Proposition 2.3, it follows that

\[
\left\| KR_{\lambda_k, \tau} \left( x_k, \frac{\xi_k}{|\xi_k|} \right) u \right\|_p^p \rightarrow 0
\]

as \( k \rightarrow \infty \). Then, by Equation (2.7),

\[
\| u \|_p^p \leq \| S \| \| \sigma_\infty \| \| u \|_p^p.
\]

(2.8)

Thus, Equation (2.4) and Equation (2.8) give the contradiction that

\[
\frac{1}{\| S \|} \leq |\sigma_\infty| \leq \frac{1}{2\| S \|},
\]

which completes the proof of the theorem.

The preceding theorem can be generalized to the following theorem.

**Theorem 2.6.** Let \( \sigma \in M_{\rho, \Lambda}^m \), where \( m \in (-\infty, \infty) \), and let \( T_\sigma : H_{\Lambda}^{s,p} \rightarrow H_{\Lambda}^{s-m,p} \) be a Fredholm operator for some \( s \in (-\infty, \infty) \). Then \( T_\sigma \) is an \( M \)-elliptic operator.

**Proof** By Theorem 1.6 in [18], the operators \( T_\sigma : H^{s,p} \rightarrow H_{\Lambda}^{s-m,p}, J_{-s} : H_{\Lambda}^{s,p} \rightarrow L^p(\mathbb{R}^n), J_s : L^p(\mathbb{R}^n) \rightarrow H_{\Lambda}^{s,p}, J_{s-m} : H_{\Lambda}^{s-m,p} \rightarrow L^p(\mathbb{R}^n) \) and \( J_{s-m} : L^p(\mathbb{R}^n) \rightarrow H_{\Lambda}^{s-m,p} \) are bounded linear operators. Here \( J_m \) is a pseudo-differential operator with symbol \( \sigma_m \in M_{\rho, \Lambda}^{-m} \), where

\[
\sigma_m(\xi) = (\Lambda(\xi))^{-m}, \quad \xi \in \mathbb{R}^n.
\]

Let

\[
J_{s-m} T_\sigma J_s = T_\tau.
\]

(2.9)

Then

\[
T_\tau : L^p(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n),
\]
where \( \tau \in M_{\rho,\Lambda}^{0} \). Since \( J_{s} \) is bijective, it follows that \( J_{s} \) is Fredholm and \( M \)-elliptic for all \( s \in (-\infty, \infty) \). So, by Theorem 2.4, \( T_{\tau} \) is \( M \)-elliptic. By Equation (2.9), the fact that \( J_{s}, s \in \mathbb{R} \), is bijective and product of two \( M \)-elliptic operators is again an \( M \)-elliptic operator, we get \( T_{\tau} \) is \( M \)-elliptic. \( \square \)

The following theorem is a simple consequence of Theorem 2.4 and Theorem 2.6, which proves the \( M \)-ellipticity of the Fredholm SG pseudo-differential operators on \( L^{p}(\mathbb{R}^{n}) \). Details about the Sobolev spaces, \( H_{\Lambda}^{s_{1},s_{2},p} \), can be found in [1].

**Theorem 2.7.** Let \( \sigma \in M_{\rho,\Lambda}^{m_{1},m_{2}} \), where \( m_{1}, m_{2} \in (-\infty, \infty) \) and let

\[ T_{\sigma} : H_{\Lambda}^{s_{1},s_{2},p} \to H_{\Lambda}^{s_{1}-m_{1},s_{2}-m_{2},p} \]

is a Fredholm operator for some \( s_{1}, s_{2} \in (-\infty, \infty) \). Then \( T_{\sigma} \) is a SG \( M \)-elliptic operator.

### 3. Gårding’s Inequality for \( M \)-Elliptic Operators

We begin with a definition.

**Definition 3.1.** Let \( \sigma \in M_{\rho,\Lambda}^{m} \), where \( m \in \mathbb{R} \). Then \( \sigma \) is said to be strongly \( M \)-elliptic if there exist positive constants \( C \) and \( R \) for which

\[ \text{Re}(\sigma(x, \xi)) \geq C \Lambda(\xi)^{m}, \quad |\xi| \geq R. \]

**Theorem 3.2.** (Gårding’s Inequality for \( M \)-Elliptic Operators)

Let \( \sigma \in M_{\rho,\Lambda}^{m} \), where \( m \in \mathbb{R} \), be strongly \( M \)-elliptic symbol. Then we can find positive constants \( C' \) and \( C_{s} \) for every real number \( s \geq \frac{m}{2} \) such that

\[ \text{Re}(T_{\sigma} \varphi, \varphi) \geq C' \| \varphi \|_{m,2,\Lambda}^{2} - C_{s} \| \varphi \|_{m-s,2,\Lambda}^{2}, \quad \varphi \in \mathcal{S}. \]

To prove this theorem, we need the following three lemmas.

**Lemma 3.3.** Let \( \sigma \in S_{\rho,\Lambda}^{0} \) and \( F \) be a \( C^{\infty} \)-function on the complex plane \( \mathbb{C} \). Then \( F \circ \sigma \in S_{\rho,\Lambda}^{0} \).

**Proof** We need to prove that for any two multi-indices \( \alpha \) and \( \beta \), we can find a positive constant \( C_{\alpha,\beta} \) such that

\[ |(\partial_{x}^{\alpha} \partial_{\xi}^{\beta} (F \circ \sigma))(x, \xi)| \leq C_{\alpha,\beta} \Lambda(\xi)^{-|\beta|}, \quad x, \xi \in \mathbb{R}^{n}. \]  \( \text{(3.1)} \)

Let \( \alpha \) and \( \beta \) be two multi-indices such that \( |\alpha + \beta| = 0 \), i.e., \( |\alpha| = 0 \) and \( |\beta| = 0 \). Since \( \sigma \in S_{\rho,\Lambda}^{0} \), so we can find a positive constant \( C \) such that

\[ |\sigma(x, \xi)| \leq C, \quad x, \xi \in \mathbb{R}^{n}. \]

Thus, \( F \circ \sigma \) is a bounded and \( C^{\infty} \) function on \( \mathbb{R}^{n} \times \mathbb{R}^{n} \). So we can find another positive constant \( C_{0} \) such that

\[ |(F \circ \sigma)(x, \xi)| \leq C_{0}, \quad x, \xi \in \mathbb{R}^{n}. \]

Hence Equation (3.1) is true for all multi-indices \( \alpha \) and \( \beta \) with \( |\alpha + \beta| = 0 \). Now, suppose that Equation (3.1) is true for all \( C^{\infty} \) functions \( F \) on \( \mathbb{C} \), \( \sigma \in S_{\rho,\Lambda}^{0} \) and multi-indices \( \alpha \) and \( \beta \) with \( |\alpha + \beta| \leq k \). Let \( \alpha \) and \( \beta \) be multi-indices with \( |\alpha + \beta| = k + 1 \). We first suppose that

\[ \partial_{x}^{\alpha} \partial_{\xi}^{\beta} = \partial_{x}^{\alpha} \partial_{\xi}^{\gamma} \partial_{\xi_{j}}, \quad j = 1, 2, ..., n. \]

Then, by chain rule,

\[ (\partial_{x}^{\alpha} \partial_{\xi}^{\beta} (F \circ \sigma))(x, \xi) = \partial_{x}^{\alpha} \partial_{\xi}^{\gamma} \{(F_{1} \circ \sigma) \partial_{\xi_{j}} \sigma + (F_{2} \circ \sigma) \partial_{\xi_{j}} \sigma\}(x, \xi) \]
Let a positive constant \( C \) be a function on the complex plane \( \mathbb{C} \) and for some multi-index \( \gamma \), we can find a positive constant \( C \) such that
\[
|\partial^\alpha_\xi (F \circ \sigma)(x, \xi)| \leq C, \quad x, \xi \in \mathbb{R}^n.
\]
Similarly, we can find another positive constant \( C'_{\alpha, \gamma, j} \) such that
\[
|\partial^\alpha_\xi (F \circ \sigma)(x, \xi)| \leq C'_{\alpha, \gamma, j} |\Lambda(\xi)|^{\rho(|\gamma| - |\beta| + 1)}, \quad x, \xi \in \mathbb{R}^n.
\]
Hence
\[
|\partial^\alpha_\xi (F \circ \sigma)(x, \xi)| \leq (C_{\alpha, \gamma, j} + C'_{\alpha, \gamma, j}) |\Lambda(\xi)|^{\rho(|\beta|)}, \quad x, \xi \in \mathbb{R}^n.
\]
Now, we assume that
\[
\partial^\alpha_\xi (F \circ \sigma) = \partial^\alpha_\xi \partial^\beta_\xi (F \circ \sigma)
\]
for some multi-index \( \gamma \) with \(|\gamma + \beta| = k\), and for some \( j = 1, 2, \ldots, n \). Then, as before, we can find a positive constant \( C''_{\alpha, \gamma, j} \) such that
\[
|\partial^\alpha_\xi (F \circ \sigma)(x, \xi)| \leq C''_{\alpha, \gamma, j} |\Lambda(\xi)|^{\rho(|\beta|)}, \quad x, \xi \in \mathbb{R}^n.
\]
So, by the principle of mathematical induction, Equation (3.1) follows. \( \square \)

**Lemma 3.4.** Let \( \sigma \in M^0_{p, \Lambda} \) and \( F \) be a \( C^\infty \)-function on the complex plane \( \mathbb{C} \). Then \( F \circ \sigma \in M^0_{p, \Lambda} \).

**Proof** Let \( S = \{ \gamma \in \mathbb{Z}^n_0 : \gamma_j \in \{0, 1\}, j = 1, 2, \ldots, n; n \in \mathbb{N} \} \), where \( \mathbb{Z}_0 = \mathbb{Z}_+ \cup \{0\} \).

Then, we need to show that, for all \( \gamma \in S \), we have
\[
\xi^\gamma \partial^\alpha_\xi (F \circ \sigma)(x, \xi) \in S^0_{p, \Lambda},
\]
i.e., for all multi-indices \( \alpha, \beta \), we can find a positive constant \( C_{\alpha, \beta} \) such that
\[
|\partial^\alpha_\xi (\xi^\gamma \partial^\beta_\xi (F \circ \sigma)(x, \xi))| \leq C_{\alpha, \beta} |\Lambda(\xi)|^{\rho(|\beta|)}, \quad \forall \gamma \in S, \quad x, \xi \in \mathbb{R}^n.
\] (3.2)

Let \(|\alpha + \beta| = 0\). This implies \(|\alpha| = 0\) and \(|\beta| = 0\). Then we need to show that there exists a positive constant \( C_{00} \) such that
\[
|\xi^\gamma \partial^\alpha_\xi (F \circ \sigma)(x, \xi)| \leq C_{00}, \quad \forall \gamma \in S, \quad x, \xi \in \mathbb{R}^n.
\] (3.3)

Let \(|\gamma| = 0\). Then by Lemma 3.3, we can find a positive constant \( C_{00} \) such that
\[
|F \circ \sigma(x, \xi)| \leq C_{00}, \quad x, \xi \in \mathbb{R}^n.
\]
Now, suppose that Equation (3.3) is true for all $C^\infty$ functions $F$ on $\mathbb{C}$, $\sigma \in M^0_{\rho, \Lambda}$ and multi-indices $\gamma$ with $|\gamma| \leq k$. Let $\gamma$ be a multi-index with $|\gamma| = k + 1$. We suppose that $\partial_x^j = \partial_{\xi^j}$, for some $j = 1, 2, \ldots, n$ and multi-index $\gamma_0$ with $|\gamma_0| = k$. Then by chain rule,

$$\xi^\gamma \partial_\xi^\gamma (F \circ \sigma)(x, \xi) = \xi^\gamma \partial_\xi^\gamma \{(F_1 \circ \sigma) \partial_{\xi_j} \sigma + (F_2 \circ \sigma) \partial_{\xi_j} \sigma\}(x, \xi)$$

for all $x$ and $\xi$ in $\mathbb{R}^n$, where $F_1$ and $F_2$ are the partial derivatives of $F$ with respect to the first and second variables respectively. Now, by Leibniz’s formula,

$$\partial_\xi^\gamma (F \circ \sigma)(x, \xi) = \partial_\xi^\gamma (F \circ \sigma)(x, \xi) \cdot \partial_{\xi_j} \sigma(x, \xi)$$

Thus by induction hypothesis and using the fact that $\sigma \in M^0_{\rho, \Lambda}$, we can find positive constants $C_{\delta_0}$ and $C_{\gamma_0, \delta_0}$ such that

$$|\xi^\gamma \partial_\xi^\gamma ((F_1 \circ \sigma)(x, \xi) \cdot \partial_{\xi_j} \sigma(x, \xi))|$$

$$\leq \sum_{\delta_0 \leq \gamma_0} \left( \frac{\delta_0}{\gamma_0} \right) C_{\delta_0} \cdot C_{\gamma_0, \delta_0}.$$

So, we have

$$|\xi^\gamma \partial_\xi^\gamma ((F_1 \circ \sigma)(x, \xi) \cdot \partial_{\xi_j} \sigma(x, \xi))| \leq C_{00}^\prime, \quad x, \xi \in \mathbb{R}^n,$$

where

$$C_{00}^\prime = \sum_{\delta_0 \leq \gamma_0} \left( \frac{\delta_0}{\gamma_0} \right) C_{\delta_0} \cdot C_{\gamma_0, \delta_0}.$$

Similarly, we can find a positive constant $C_{00}''$ such that

$$|\xi^\gamma \partial_\xi^\gamma ((F_2 \circ \sigma)(x, \xi) \cdot \partial_{\xi_j} \sigma(x, \xi))| \leq C_{00}''^\prime, \quad x, \xi \in \mathbb{R}^n.$$

Hence

$$|\xi^\gamma \partial_\xi^\gamma (F \circ \sigma)(x, \xi)| \leq (C_{00}^\prime + C_{00}''^\prime), \quad x, \xi \in \mathbb{R}^n.$$

Thus, by the principle of mathematical induction, Equation (3.3) follows.
Now, suppose that Equation (3.4) is true for all $C^\infty$ functions $F$ on $\mathbb{C}$, $\sigma \in M^0_{\rho,\Lambda}$ and multi-indices $\gamma$ with $|\gamma| \leq k$. Let $\gamma$ be a multi-index with $|\gamma| = k + 1$. We suppose that $\partial^\alpha_\xi = \partial^\alpha_\xi \partial_{\xi_i}$, for some $i = 1, 2, \ldots, n$ and multi-index $\gamma_0$ with $|\gamma_0| = k$. Then, by chain rule,

\[
\partial^\alpha_\xi \partial^\beta_\xi (\xi^\gamma \partial^\gamma_\xi (F \circ \sigma)(x, \xi)) = \partial^\alpha_\xi \partial^\beta_\xi (\xi^\gamma \partial^\gamma_\xi \partial_{\xi_i} (F \circ \sigma)(x, \xi)) = \partial^\alpha_\xi \partial^\beta_\xi (\xi^\gamma \partial^{\gamma_0}_{\xi_i} ((F_1 \circ \sigma) \partial_{\xi_i} \sigma + (F_2 \circ \sigma) \partial_{\xi_i} \sigma)(x, \xi)).
\]

Thus, by Leibniz's formula,

\[
\partial^\alpha_\xi \partial^\beta_\xi (\xi^\gamma \partial^{\gamma_0}_{\xi_i} ((F_1 \circ \sigma) \cdot \partial_{\xi_i} \sigma)(x, \xi))
\]

\[
= \partial^\alpha_\xi \partial^\beta_\xi \left( \xi^\gamma \left( \sum_{\delta_0 \leq \gamma_0} \left( \frac{\delta_0}{\gamma_0} \right) \partial^0_\xi (F_1 \circ \sigma)(x, \xi) \cdot \partial^{\gamma_0-\delta_0}_{\xi_i} (\partial_{\xi_i} \sigma(x, \xi)) \right) \right)
\]

\[
= \partial^\alpha_\xi \partial^\beta_\xi \left( \sum_{\delta_0 \leq \gamma_0} \left( \frac{\delta_0}{\gamma_0} \right) \partial^{\gamma_0}_{\xi_i} (F_1 \circ \sigma)(x, \xi) \cdot (\xi^{\gamma_0-\delta_0+1} \partial^{\gamma_0-\delta_0}_{\xi_i} (\partial^{\gamma_0-\delta_0+1}_{\xi_i} \sigma(x, \xi))) \right)
\]

\[
+ \partial^\alpha_\xi \partial^\beta_\xi \left( \sum_{\delta_0 \leq \gamma_0} \left( \frac{\delta_0}{\gamma_0} \right) \partial^{\gamma_0}_{\xi_i} (F_1 \circ \sigma)(x, \xi) \cdot \partial_{\xi_i} \left( (\xi^{\gamma_0-\delta_0+1} \partial^{\gamma_0-\delta_0+1}_{\xi_i} \sigma(x, \xi)) \right) \right).
\]

First, consider

\[
\partial^\alpha_\xi \partial^\beta_\xi \left( \sum_{\delta_0 \leq \gamma_0} \left( \frac{\delta_0}{\gamma_0} \right) \partial^{\gamma_0}_{\xi_i} (F_1 \circ \sigma)(x, \xi) \cdot (\xi^{\gamma_0-\delta_0+1} \partial^{\gamma_0-\delta_0+1}_{\xi_i} \sigma(x, \xi)) \right)
\]

\[
= \sum_{\delta_0 \leq \gamma_0} \sum_{\rho_0 \leq \delta_0} \sum_{\beta_0 \leq \alpha} \left( \frac{\delta_0}{\gamma_0} \right) \left( \frac{\rho_0}{\beta_0} \right) \left( \frac{\beta_0}{\alpha} \right) \partial^\alpha_\xi \partial^{\rho_0+1}_{\xi_i} (\xi^{\delta_0} \partial^{\delta_0}_{\xi_i} (F_1 \circ \sigma)(x, \xi)) \cdot \partial^{\rho_0-\delta_0}_{\xi_i} \partial^{\rho_0-\delta_0}_{\xi_i} \partial^{\gamma_0-\delta_0+1}_{\xi_i} \sigma(x, \xi)) \right).
\]

Thus by induction hypothesis and using the fact that $\sigma \in M^0_{\rho,\Lambda}$, we can find positive constants $C_{p,\rho_0}$ and $C_{p,\rho_0,\alpha,\beta_0}$ such that

\[
\left| \partial^\alpha_\xi \partial^\beta_\xi \left( \sum_{\delta_0 \leq \gamma_0} \left( \frac{\delta_0}{\gamma_0} \right) \partial^{\gamma_0}_{\xi_i} (F_1 \circ \sigma)(x, \xi) \cdot (\xi^{\gamma_0-\delta_0+1} \partial^{\gamma_0-\delta_0+1}_{\xi_i} \sigma(x, \xi)) \right) \right|
\]

\[
\leq \sum_{\delta_0 \leq \gamma_0} \sum_{\rho_0 \leq \delta_0} \sum_{\beta_0 \leq \alpha} \left( \frac{\delta_0}{\gamma_0} \right) \left( \frac{\rho_0}{\beta_0} \right) \left( \frac{\beta_0}{\alpha} \right) C_{p,\rho_0} (\Lambda(\xi))^{-\rho_0+1} \cdot C_{p,\rho_0,\alpha,\beta_0} (\Lambda(\xi))^{-\rho_0-\rho_0+1} \cdot C_{p,\rho_0,\alpha,\beta_0} \Lambda(\xi)^{-\rho_0-\rho_0+1} \cdot C_{p,\rho_0,\alpha,\beta_0} \Lambda(\xi)^{-\rho_0-\rho_0+1}.
\]

where

\[
C_{p,\alpha,\beta} = \sum_{\delta_0 \leq \gamma_0} \sum_{\rho_0 \leq \delta_0} \sum_{\beta_0 \leq \alpha} \left( \frac{\delta_0}{\gamma_0} \right) \left( \frac{\rho_0}{\beta_0} \right) \left( \frac{\beta_0}{\alpha} \right) C_{p,\rho_0} C_{p,\rho_0,\alpha,\beta_0}.
\]
Similarly, we can find a positive constant $C'_{\alpha,\beta}$ such that

$$\left| \partial_x^\alpha \partial_{\xi}^\beta \left( \sum_{\delta_0 \leq \gamma_0} \left( \xi^{\delta_0} \partial_{\xi}^{\delta_0} (F_1 \circ \sigma)(x, \xi) \right) \cdot \partial_{\xi}(\xi^{\gamma_0-\delta_0+1} \partial_{\xi}^{\gamma_0-\delta_0+1} \sigma(x, \xi)) \right) \right| \leq C'_{\alpha,\beta} (\Lambda(\xi))^{-\rho|\beta|}, \quad x, \xi \in \mathbb{R}^n.$$  

Hence

$$|\partial_x^\alpha \partial_{\xi}^\beta \left( (F_1 \circ \sigma)(x, \xi) \right)| \leq (C'_{\alpha,\beta} + C''_{\alpha,\beta}) (\Lambda(\xi))^{-\rho|\beta|}, \quad x, \xi \in \mathbb{R}^n.$$

Similarly, we can find a positive constant $C''_{\alpha,\beta}$ such that

$$|\partial_x^\alpha \partial_{\xi}^\beta \left( (F_2 \circ \sigma)(x, \xi) \right)| \leq C''_{\alpha,\beta} (\Lambda(\xi))^{-\rho|\beta|}, \quad x, \xi \in \mathbb{R}^n.$$

Hence

$$|\partial_x^\alpha \partial_{\xi}^\beta \left( (F \circ \sigma)(x, \xi) \right)| \leq (C'_{\alpha,\beta} + C''_{\alpha,\beta}) (\Lambda(\xi))^{-\rho|\beta|}, \quad x, \xi \in \mathbb{R}^n.$$

Now, we assume that

$$\partial_x^\alpha \partial_{\xi}^\beta = \partial_x^\gamma \partial_{\xi}^\gamma$$

for some multi-index $\gamma$ with $|\gamma + \beta| = k$ and for some $j = 1, 2, ..., n$. Then, as before, we can find a positive constant $C'''_{\alpha,\beta}$ such that

$$|\partial_x^\alpha \partial_{\xi}^\beta (F \circ \sigma)(x, \xi)| \leq C'''_{\alpha,\beta} \Lambda(\xi)^{-\rho|\beta|}, \quad x, \xi \in \mathbb{R}^n.$$

So, by the principle of mathematical induction, Equation (3.4) follows.

This completes the proof of the Equation (3.2).

\[\square\]

**Lemma 3.5.** Let $\sigma$ be a strongly $M$-elliptic symbol in $M^{2m}_{\rho, \Lambda}$, where $m \in \mathbb{R}$. Then we can find two positive constants $\eta$ and $\kappa$ such that

$$\text{Re}(\sigma(x, \xi)) \geq \eta (\Lambda(\xi))^{2m} - \kappa (\Lambda(\xi))^{2m-\rho}, \quad x, \xi \in \mathbb{R}^n.$$

**Proof** By strong ellipticity, there exist positive constants $C$ and $R$ such that

$$\text{Re}(\sigma(x, \xi)) \geq C (\Lambda(\xi))^{2m}, \quad |\xi| \geq R.$$

Since $\sigma \in M^{2m}_{\rho, \Lambda}$, we can find a positive constant $K$ such that

$$|\sigma(x, \xi)| \leq K (\Lambda(\xi))^{2m}, \quad x, \xi \in \mathbb{R}^n.$$

Therefore, if $m \geq 0$, then

$$|\text{Re}(\sigma(x, \xi))| \leq K (\Lambda(\xi))^{2m} \leq K_{2m} (1 + |\xi|)^{2m} \leq K_{2m} (1 + R)^{2m}, \quad |\xi| \leq R,$$

where

$$K_{2m} = K \cdot C_1^{2m}.$$

And if $m < 0$, then

$$|\text{Re}(\sigma(x, \xi))| \leq K (\Lambda(\xi))^{2m} \leq K, \quad |\xi| \leq R.$$

By Equation (3.5) and Equation (3.6), for given $m \in \mathbb{R}$, we can find a positive constant $M$ such that

$$\text{Re}(\sigma(x, \xi)) \geq -M, \quad |\xi| \leq R.$$

Since $\frac{\text{Re} \sigma}{(\Lambda(\xi))^{2m-\rho}}$ is continuous on the compact set $\{\xi \in \mathbb{R}^n : |\xi| \leq R\}$, so we can find a positive constant $\kappa$ such that

$$\frac{\text{Re} \sigma}{(\Lambda(\xi))^{2m-\rho}} > -\kappa, \quad |\xi| \leq R.$$
Therefore
\[ \text{Re}(\sigma(x, \xi)) + \kappa(\Lambda(\xi))^{2m-\rho} > 0, \quad |\xi| \leq R. \]

Since \( \frac{\text{Re}\sigma + \kappa(\Lambda(\xi))^{2m-\rho}}{(\Lambda(\xi))^{2m}} \) is a positive and continuous function on the compact set \( \{\xi \in \mathbb{R}^n : |\xi| \leq R\} \), so we can find another positive constant \( \delta \) such that
\[ \frac{\text{Re}\sigma + \kappa(\Lambda(\xi))^{2m-\rho}}{(\Lambda(\xi))^{2m}} \geq \delta, \quad |\xi| \leq R. \]

So, the lemma is proved if we let \( \eta = \min(C, \delta) \). \( \Box \)

**Proof of Theorem 3.2** Let \( T_\tau = J_m T_\tau J_m \), where \( J_m = T_{\sigma_m} \) and \( \sigma_m(\xi) = (\Lambda(\xi))^{-m} \). Then, using the asymptotic expansion for the product of two pseudo-differential operators in Theorem 1.2 in \([18]\),
\[ T_\sigma J_m = T_{\tau_1}, \]
where
\[ \tau_1 - (\Lambda(\cdot))^{-m}\sigma \in M_{p,\Lambda}^{m-\rho}. \quad (3.7) \]

Similarly,
\[ T_\tau = J_m T_{\tau_1} \]
and
\[ \tau - (\Lambda(\cdot))^{-m}\tau_1 \in M_{p,\Lambda}^{-\rho}. \quad (3.8) \]

Multiplying Equation (3.7) by \((\Lambda(\cdot))^{-m}\) and adding the result to Equation (3.8), we get
\[ \tau - (\Lambda(\cdot))^{-2m}\sigma \in M_{p,\Lambda}^{-\rho}. \]

Therefore
\[ \tau = (\Lambda(\cdot))^{-2m}\sigma + r, \]
where \( r \in M_{p,\Lambda}^{-\rho} \). So, by Lemma 3.5,
\[ \text{Re}\tau = (\Lambda(\cdot))^{-2m}\text{Re}\sigma + \text{Re}r \geq \eta - \kappa(\Lambda(\cdot))^{-\rho} + \text{Re}r \geq \eta - \kappa'(\Lambda(\cdot))^{-\rho}, \]
where \( \kappa' \) is another positive constant. Therefore \( \tau \) satisfies the conclusion ofLemma 3.5 with \( m = 0 \). Let us suppose for a moment that Gårding's inequality is valid for \( m = 0 \). Then we can find a positive constant \( C' \) and a positive constant \( C_s \) for every real number \( s \geq \frac{n}{2} \) such that
\[ \text{Re}(T_\sigma \varphi, \varphi) = \text{Re}(J_m T_\sigma J_m \varphi, \varphi) = \text{Re}(T_\tau J_m \varphi, J_m \varphi) \geq C' \|J_m \varphi\|^2_{\theta,2,\Lambda} - C_s \|J_m \varphi\|^2_{s,2,\Lambda} = C' \|\varphi\|^2_{m,2,\Lambda} - C_s \|\varphi\|^2_{m-s,2,\Lambda} \]
for all \( \varphi \) in \( \mathcal{S} \). Now we need only to prove Gårding’s inequality for \( m = 0 \). Let \( \sigma \in M_{p,\Lambda}^0 \). Then, by Lemma 3.5, we can find positive constants \( \eta \) and \( \kappa \) such that
\[ \text{Re}\sigma + \kappa(\Lambda(\cdot))^{-\rho} \geq \eta. \]

Let \( F \) be a \( C^\infty \) function on \( \mathbb{C} \) such that
\[ F(z) = \sqrt{\frac{\eta}{2} + z}, \quad z \in [0, \infty). \]

Let \( \tau \) be the function defined on \( \mathbb{R}^n \times \mathbb{R}^n \) by
\[ \tau(x, \xi) = F\left(2 \left(\text{Re}(\sigma(x, \xi)) + \kappa(\Lambda(\xi))^{-\rho} - \eta\right)\right), \quad x, \xi \in \mathbb{R}^n. \]
Then, by Lemma 3.4, \( \tau \in M^0_{\rho, \Lambda} \), and for all \( x \) and \( \xi \) in \( \mathbb{R}^n \),

\[
\tau(x, \xi) = \sqrt{\frac{\eta}{2} + 2 \Re \sigma(x, \xi) + 2\kappa(\Lambda(\xi))^{-\rho} - 2\eta}
\]

Using the asymptotic expansion for the formal adjoint of a pseudo-differential operator in Theorem 1.3 in [18], we get

\[
T^*_\tau = T^*_\tau^*,
\]

where \( \tau^* \in M^0_{\rho, \Lambda} \) and \( \tau - \tau^* \in M^{-\rho}_{\rho, \Lambda} \). Again, by using Theorem 1.2 in [18], we have

\[
T^*\tau T = T_\lambda,
\]

where

\[
\lambda - \tau^* \tau \in M^{-\rho}_{\rho, \Lambda}.
\]

If we let \( r_1 \) and \( r'_1 \) in \( M^{-\rho}_{\rho, \Lambda} \) be such that

\[
\tau^* = \tau + r_1
\]

and

\[
\lambda = \tau^* \tau + r'_1.
\]

Then

\[
\lambda = (\tau + r_1) \tau + r'_1 = 2 \Re \sigma + 2\kappa(\Lambda(\xi))^{-\rho} - \frac{3}{2} \eta + r_2,
\]

where

\[
r_2 = r_1 \tau + r'_1 \in M^{-\rho}_{\rho, \Lambda}.
\]

So, if we let \( r_3 = 2\kappa(\Lambda(\xi))^{-\rho} + r_2 \in M^{-\rho}_{\rho, \Lambda} \), then we get

\[
\lambda = 2 \Re \sigma - \frac{3}{2} \eta + r_3.
\]

But

\[
2 \Re \sigma = \sigma + \bar{\sigma} = \sigma + \sigma^* + r_4
\]

for some \( r_4 \) in \( M^{-\rho}_{\rho, \Lambda} \). Therefore

\[
\lambda = \sigma + \sigma^* - \frac{3}{2} \eta + r_5
\]

for some \( r_5 \) in \( M^{-\rho}_{\rho, \Lambda} \). Thus,

\[
\sigma + \sigma^* = \lambda + \frac{3}{2} \eta - r_5.
\]

Since

\[
(T_\lambda \varphi, \varphi) = (T^*_\tau \varphi, T^*_\tau \varphi) \geq 0, \quad \varphi \in \mathcal{S},
\]

it follows that

\[
2 \Re (T_\sigma \varphi, \varphi) = (T_\sigma \varphi, \varphi) + (T^*_\sigma \varphi, \varphi) = (T_{\sigma + \sigma^*} \varphi, \varphi)
\]

\[
= (T_\lambda \varphi, \varphi) + \frac{3}{2} \eta \| \varphi \|_{0,2,\Lambda}^2 - (T_{r_5} \varphi, \varphi)
\]

\[
\geq \eta \| \varphi \|_{0,2,\Lambda}^2 + \left\{ \frac{\eta}{2} \| \varphi \|_{0,2,\Lambda}^2 - \| T_{r_5} \varphi \|_{\frac{2}{5},2,\Lambda} \| \varphi \|_{-\frac{2}{5},2,\Lambda} \right\}, \quad \varphi \in \mathcal{S}.
\]

Then, by Theorem 1.6 in [18] and using the fact that \( r_5 \in M^{-\rho}_{\rho, \Lambda} \), we can find a positive constant \( \nu \) such that

\[
2 \Re (T_\sigma \varphi, \varphi) \geq \eta \| \varphi \|_{0,2,\Lambda}^2 + \left\{ \frac{\eta}{2} \| \varphi \|_{0,2,\Lambda}^2 - \nu \| \varphi \|_{-\frac{2}{5},2,\Lambda}^2 \right\}, \quad \varphi \in \mathcal{S}.
\]
But
\[ \nu \| \varphi \|^2_{-\frac{\rho}{2}, 2, \Lambda} = \int_{\mathbb{R}^n} \nu (\Lambda(\xi))^{-\rho} |\hat{\varphi}(\xi)|^2 d\xi = I + J, \]

where
\[ I = \int_{\nu (\Lambda(\xi))^{-\rho} \leq \frac{\eta}{2}} \nu (\Lambda(\xi))^{-\rho} |\hat{\varphi}(\xi)|^2 d\xi \]

and
\[ J = \int_{\nu (\Lambda(\xi))^{-\rho} \geq \frac{\eta}{2}} \nu (\Lambda(\xi))^{-\rho} |\hat{\varphi}(\xi)|^2 d\xi. \]

Obviously,
\[ I \leq \frac{\eta}{2} \int_{\mathbb{R}^n} |\hat{\varphi}(\xi)|^2 d\xi = \frac{\eta}{2} \| \varphi \|^2_{0, 2, \Lambda}. \]

To estimate \( J \), we note that
\[ \nu (\Lambda(\xi))^{-\rho} \geq \frac{\eta}{2} \Rightarrow \Lambda(\xi) \leq \left( \frac{2 \nu}{\eta} \right)^{\frac{1}{\rho}}. \]

So, for \( \nu (\Lambda(\xi))^{-\rho} \geq \frac{\eta}{2} \), we get, for every real number \( s \geq \frac{\rho}{2} \),
\[ \nu (\Lambda(\xi))^{-\rho} = \nu (\Lambda(\xi))^{2s - \rho} (\Lambda(\xi))^{-2s} \leq \nu \left( \frac{2 \nu}{\eta} \right)^{\frac{2s - \rho}{\rho}} (\Lambda(\xi))^{-2s}. \]

Thus for every real number \( s \geq \frac{\rho}{2} \),
\[ J \leq \nu \left( \frac{2 \nu}{\eta} \right)^{\frac{2s - \rho}{\rho}} \int_{\mathbb{R}^n} (\Lambda(\xi))^{-2s} |\hat{\varphi}(\xi)|^2 d\xi = C_s' \| \varphi \|^2_{-s, 2, \Lambda}. \]

where \( C_s' = \nu \left( \frac{2 \nu}{\eta} \right)^{\frac{2s - \rho}{\rho}} \). Therefore
\[ 2 \text{Re}(T_\sigma \varphi, \varphi) \geq \eta \| \varphi \|^2_{0, 2, \Lambda} - C_s' \| \varphi \|^2_{-s, 2, \Lambda}, \quad \varphi \in \mathcal{S}, \]

and this completes the proof of the theorem.

**Definition 3.6.** Let \( \sigma \in M^{m_1, m_2}_{\rho, \Lambda} \), where \( m_1, m_2 \in \mathbb{R} \). Then \( \sigma \) is said to be strongly \( \nu \)-elliptic if there exist positive constants \( C \) and \( R \) such that
\[ \text{Re}(\sigma(x, \xi)) \geq C \Lambda(x)^{m_2} \Lambda(\xi)^{m_1}, \quad |x|^2 + |\xi|^2 \geq R^2. \]

**Theorem 3.7. (Gårding’s Inequality For SG M-elliptic Operators)**
Let \( \sigma \in M^{2m_1, 2m_2}_{\rho, \Lambda} \), where \( m_1, m_2 \in \mathbb{R} \), be strongly \( \nu \)-elliptic symbol. Then we can find positive constants \( C' \) and \( C_{s_1, s_2} \) for every real number \( s_1 \leq \frac{\rho}{2}, s_2 \geq \frac{\rho}{2} \) such that
\[ \text{Re}(T_\sigma \varphi, \varphi) \geq C' \| \varphi \|^2_{m_1, m_2, \Lambda} - C_{s_1, s_2} \| \varphi \|^2_{m_1 - s_1, m_2 - s_2, \Lambda}, \quad \varphi \in \mathcal{S}. \]

To prove this theorem, we need the following two lemmas.

**Lemma 3.8.** Let \( \sigma \in M^{0, 0}_{\rho, \Lambda} \) and \( F \) be a \( C^\infty \)- function on the complex plane \( \mathbb{C} \). Then \( F \circ \sigma \in M^{0, 0}_{\rho, \Lambda} \).

Proof of the above lemma follows from the similar techniques as in Lemma 3.3 and Lemma 3.4.

**Lemma 3.9.** Let \( \sigma \in M^{2m_1, 2m_2}_{\rho, \Lambda} \), where \( m_1, m_2 \in \mathbb{R} \), be strongly \( \nu \)-elliptic symbol. Then we can find two positive constants \( \eta \) and \( \kappa \) such that
\[ \text{Re}(\sigma(x, \xi)) \geq \eta (\Lambda(x))^{2m_2} (\Lambda(\xi))^{2m_1} - \kappa (\Lambda(x))^{2m_2 - \rho} (\Lambda(\xi))^{2m_1 - \rho}, \quad x, \xi \in \mathbb{R}^n. \]
Proof of the above lemma follows from the similar techniques as in Lemma 3.5.

**Proof of Theorem 3.7** Let $T_\tau = J_{m_1,m_2} T_\sigma J_{m_1,m_2}$, where $J_{m_1,m_2} = T_{\sigma_{m_1,m_2}}$ and $\sigma_{m_1,m_2}(x,\xi) = (\Lambda(x))^{-m_2} (\Lambda(\xi))^{-m_1}$. Then, by Theorem 1.1, we have

$$T_\sigma J_{m_1,m_2} = T_{\tau_1},$$

where

$$\tau_1 - \sigma_{m_1,m_2}\sigma \in M_{\rho,\Lambda}^{m_1,-\rho,m_2,-\rho}. \quad \text{(3.9)}$$

Similarly,

$$T_\tau = J_{m_1,m_2} T_{\tau_1}$$

and

$$\tau - \sigma_{m_1,m_2}\tau_1 \in M_{\rho,\Lambda}^{-\rho,-\rho}. \quad \text{(3.10)}$$

Multiplying Equation (3.9) by $\sigma_{m_1,m_2}$ and adding the result to Equation (3.10), we get

$$\tau - \sigma_{m_1,m_2}^2 \sigma \in M_{\rho,\Lambda}^{-\rho,-\rho}.$$ 

Therefore

$$\tau = \sigma_{m_1,m_2}^2 \sigma + r,$$

where $r \in M_{\rho,\Lambda}^{-\rho,-\rho}$. So, by Lemma 3.9,

$$\text{Re} \tau = \sigma_{m_1,m_2}^2 \text{Re} \sigma + \text{Re} r \geq \eta - \kappa (\Lambda(x))^{-\rho} (\Lambda(\xi))^{-\rho} + \text{Re} r \geq \eta - \kappa' (\Lambda(x))^{-\rho} (\Lambda(\xi))^{-\rho},$$

where $\kappa'$ is another positive constant. Therefore $\tau$ satisfies the conclusion of Lemma 3.9 with $m_1 = 0$ and $m_2 = 0$. Let us suppose for a moment that Gårding’s inequality is valid for $m_1 = 0$ and $m_2 = 0$. Then we can find a positive constant $C'$ and a positive constant $C_{s_1,s_2}$ for every real number $s_1 \leq \frac{3}{2}$, $s_2 \geq \frac{\eta}{2}$ such that

$$\text{Re} (T_\sigma \varphi, \varphi) = \text{Re} (J_{-m_1,-m_2} T_\tau J_{-m_1,-m_2} \varphi, \varphi) = \text{Re} (T_\sigma J_{-m_1,-m_2} \varphi, J_{-m_1,-m_2} \varphi) \geq C' \|J_{-m_1,-m_2} \varphi\|_{L^2(\Sigma_0,2,\Lambda)}^2 - C_{s_1,s_2} \|J_{-m_1,-m_2} \varphi\|_{L^{s_1,-s_2,2,\Lambda}}^2 = C' \|\varphi\|_{L^{m_1,m_2,2,\Lambda}}^2 - C' \|\varphi\|_{L^{m_1,m_2,-s_2,2,\Lambda}}^2$$

for all $\varphi \in \mathcal{S}$. Now we need only to prove Gårding’s inequality for $m_1 = 0$ and $m_2 = 0$. Let $\sigma \in M_{\rho,\Lambda}^{0,0}$. Then, by Lemma 3.9, we can find positive constants $\eta$ and $\kappa$ such that

$$\text{Re} \sigma + \kappa (\Lambda(x))^{-\rho} (\Lambda(\xi))^{-\rho} \geq \eta.$$ 

Let $F$ be a $C^\infty$ function on $\mathcal{C}$ such that

$$F(z) = \sqrt{\frac{\eta}{2} + z}, \quad z \in [0, \infty).$$

Let $\tau$ be the function defined on $\mathbb{R}^n \times \mathbb{R}^n$ by

$$\tau(x,\xi) = F \left( 2 \left( \text{Re} \sigma(x,\xi) + \kappa (\Lambda(x))^{-\rho} (\Lambda(\xi))^{-\rho} - \eta \right) \right), \quad x,\xi \in \mathbb{R}^n.$$ 

Then, by Lemma 3.8, $\tau \in M_{\rho,\Lambda}^{0,0}$, and for all $x$ and $\xi$ in $\mathbb{R}^n$,

$$\tau(x,\xi) = \sqrt{\frac{\eta}{2} + 2 \text{Re} \sigma(x,\xi) + 2\kappa (\Lambda(x))^{-\rho} (\Lambda(\xi))^{-\rho} - 2\eta}$$

$$= \sqrt{2 \text{Re} \sigma(x,\xi) + 2\kappa (\Lambda(x))^{-\rho} (\Lambda(\xi))^{-\rho} - \frac{3}{2} \eta}.$$
Hence, by Theorem 1.2, we get

\[ T_\tau^* = T_{\tau^*}, \]

where \( \tau^* \in \mathcal{M}_{\rho}^{0,0} \) and \( \tau - \tau^* \in \mathcal{M}_{\rho}^{\rho,\rho} \). Again, by using Theorem 1.1, we have

\[ T_\tau^* T_\tau = T_\lambda, \]

where

\[ \lambda - \tau^* \tau \in \mathcal{M}_{\rho}^{-\rho,-\rho}. \]

If we let \( r_1 \) and \( r'_1 \) in \( \mathcal{M}_{\rho}^{-\rho,-\rho} \) be such that

\[ \tau^* = \tau + r_1 \]

and

\[ \lambda = \tau^* \tau + r'_1. \]

Then

\[ \lambda = (\tau + r_1) \tau + r'_1 = 2 \Re \sigma + 2 \kappa (\Lambda(x))^{-\rho} (\Lambda(\xi))^{-\rho} - \frac{3}{2} \eta + r_2, \]

where

\[ r_2 = r_1 \tau + r'_1 \in \mathcal{M}_{\rho}^{-\rho,-\rho}. \]

So, if we let \( r_3 = 2 \kappa (\Lambda(x))^{-\rho} (\Lambda(\xi))^{-\rho} + r_2 \in \mathcal{M}_{\rho}^{-\rho,-\rho} \), then we get

\[ \lambda = 2 \Re \sigma - \frac{3}{2} \eta + r_3. \]

But

\[ 2 \Re \sigma = \sigma + \sigma^* = \sigma + \sigma^* + r_4 \]

for some \( r_4 \) in \( \mathcal{M}_{\rho}^{-\rho,-\rho}. \) Therefore

\[ \lambda = \sigma + \sigma^* - \frac{3}{2} \eta + r_5 \]

for some \( r_5 \) in \( \mathcal{M}_{\rho}^{-\rho,-\rho}. \) Thus,

\[ \sigma + \sigma^* = \lambda + \frac{3}{2} \eta - r_5. \]

Since

\[ (T_\lambda \varphi, \varphi) = (T_\tau \varphi, T_\tau \varphi) \geq 0, \quad \varphi \in \mathcal{S}, \]

it follows that

\[ 2 \Re (T_\sigma \varphi, \varphi) = (T_\sigma \varphi, \varphi) + (T_\sigma^* \varphi, \varphi) = (T_{\sigma + \sigma^*} \varphi, \varphi) \]

\[ = (T_\lambda \varphi, \varphi) + \frac{3}{2} \eta \| \varphi \|^2_{0,0,2,\Lambda} - (T_\tau \varphi, \varphi) \]

\[ \geq \eta \| \varphi \|^2_{0,0,2,\Lambda} + \left\{ \eta \frac{1}{2} \| \varphi \|^2_{0,0,2,\Lambda} - \| T_{\tau^*} \varphi \|^2_{-\frac{\rho}{2},-\frac{\rho}{2},2,\Lambda} \right\}, \quad \varphi \in \mathcal{S}. \]

Then, by Theorem ?? and using the fact that \( r_5 \in \mathcal{M}_{\rho}^{-\rho,-\rho}, \) we can find a positive constant \( \nu \) such that

\[ 2 \Re (T_\sigma \varphi, \varphi) \geq \eta \| \varphi \|^2_{0,0,2,\Lambda} + \left\{ \eta \frac{1}{2} \| \varphi \|^2_{0,0,2,\Lambda} - \nu \| \varphi \|^2_{-\frac{\rho}{2},-\frac{\rho}{2},2,\Lambda} \right\}, \quad \varphi \in \mathcal{S}. \]

Since \( \frac{\eta}{2} > 0, \) so by Theorem ??, there exists a positive constant \( C_0 \) such that

\[ \| \varphi \|^2_{-\frac{\rho}{2},-\frac{\rho}{2},2,\Lambda} \leq C_0 \| \varphi \|^2_{0,0,2,\Lambda}, \quad \varphi \in \mathcal{S}. \]

Now

\[ \nu \| \varphi \|^2_{-\frac{\rho}{2},-\frac{\rho}{2},2,\Lambda} = \int_{\mathbb{R}^n} \nu (\Lambda(x))^{-\rho} |(T_{\frac{\tau}{2}} \varphi)(x)|^2 dx = I + J, \]
Since $\rho$, thus for every real number $s$ where $C_1$, we get:

$$I = \int_{\nu(\Lambda(x))^{-\rho} \leq \frac{\eta}{2C_0}} \nu(\Lambda(x))^{-\rho} |(T_{\sigma_2^{\frac{1}{2}}} \varphi)(x)|^2 \, dx$$

and

$$J = \int_{\nu(\Lambda(x))^{-\rho} \geq \frac{\eta}{2C_0}} \nu(\Lambda(x))^{-\rho} |(T_{\sigma_2^{\frac{1}{2}}} \varphi)(x)|^2 \, dx.$$  

Obviously,

$$I \leq \frac{\eta}{2C_0} \int_{\mathbb{R}^n} |(T_{\sigma_2^{\frac{1}{2}}} \varphi)(x)|^2 \, dx = \frac{\eta}{2C_0} \nu(\Lambda(x))^{-\rho} \leq \frac{\eta}{2} \nu(\Lambda(x))^{-\rho} \nu(\Lambda(x))^{-2s_2}.$$  

To estimate $J$, we note that

$$\nu(\Lambda(x))^{-\rho} \geq \frac{\eta}{2C_0} \Rightarrow \Lambda(x) \leq \left(\frac{2C_0}{\eta}\right)^{\frac{1}{\rho}}.$$  

So, for $\nu(\Lambda(x))^{-\rho} \geq \frac{\eta}{2C_0}$, we get, for every real number $s_2 \geq \frac{\rho}{2}$,

$$\nu(\Lambda(x))^{-\rho} \nu(\Lambda(x))^{2s_2-\rho} (\Lambda(x))^{-2s_2} \leq \nu(\frac{2C_0}{\eta})^{\frac{2s_2-\rho}{\rho}} (\Lambda(x))^{-2s_2}.$$  

Thus for every real number $s_2 \geq \frac{\rho}{2}$,

$$J \leq \nu \left(\frac{2C_0}{\eta}\right)^{\frac{2s_2-\rho}{\rho}} \int_{\mathbb{R}^n} (\Lambda(x))^{-2s_2} |(T_{\sigma_2^{\frac{1}{2}}} \varphi)(x)|^2 \, dx = C'_{s_2} \nu(\Lambda(x))^{-2s_2}.$$  

where $C'_{s_2} = \nu \left(\frac{2C_0}{\eta}\right)^{\frac{2s_2-\rho}{\rho}}$. Therefore

$$2 \Re(T_{\sigma_2^{\frac{1}{2}}} \varphi, \varphi) \geq \eta \nu(\Lambda(x))^{-2s_2} |(T_{\sigma_2^{\frac{1}{2}}} \varphi)(x)|^2 \, dx = C_{s_2} \nu(\Lambda(x))^{-2s_2}.$$  

Since $\frac{\rho}{2} \geq s_1$, so by Theorem 2, there exists a positive constant $C'_{s_1}$ such that

$$|\varphi|^2 \nu(\Lambda(x))^{-2s_2} \leq C'_{s_1} \nu(\Lambda(x))^{-2s_2} \nu(\Lambda(x))^{-2s_2} \varphi \in \mathcal{S}.$$  

Thus for every real number $s_1 \leq \frac{\rho}{2}$ and $s_2 \geq \frac{\rho}{2}$, we have

$$2 \Re(T_{\sigma_2^{\frac{1}{2}}} \varphi, \varphi) \geq \eta \nu(\Lambda(x))^{-2s_2} |(T_{\sigma_2^{\frac{1}{2}}} \varphi)(x)|^2 \, dx = C_{s_1, s_2} \nu(\Lambda(x))^{-2s_2} \varphi \in \mathcal{S},$$

where $C_{s_1, s_2} = C'_{s_1} C'_{s_2}$ and this completes the proof of the theorem.

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