Some classes of generating functions for generalized Hermite- and Chebyshev-type polynomials: Analysis of Euler’s formula

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Abstract

The aim of this paper is to construct generating functions for new families of special polynomials including the Appel polynomials, the Hermite-Kampé de Fériet polynomials, the Milne-Thomson type polynomials, parametric kinds of Apostol type numbers and polynomials. Using Euler’s formula, relations among special functions, Hermite-type polynomials, the Chebyshev polynomials and the Dickson polynomials are given. Using generating functions and their functional equations, various formulas and identities are given. With help of computational formula for new families of special polynomials, some of their numerical values are given. Using hypergeometric series, trigonometric functions and the Euler’s formula, some applications related to Hermite-type polynomials are presented. Finally, further remarks, observations and comments about generating functions for new families of special polynomials are given.

Keywords: Appel polynomials, Apostol-Bernoulli type polynomials, Apostol-Euler type polynomials, Hermite-type polynomials, Parametric kinds of Apostol-kind polynomials, Chebyshev polynomials, Dickson polynomials, Milne-Thomson type polynomials, Generating function, Functional equation, Special functions.

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1 Introduction

The Euler’s formula yields an important connection among analysis, trigonometry and special functions. This formula also gives relations between trigonometric functions and exponential functions. Because sine and cosine functions are written as sums of the exponential functions. Motivation of this paper is to construct generating functions for new families of polynomials with the help of the Euler’s formula. By using these generating functions and their functional equations, new formulas, identities, recurrence relations and properties of these polynomials, which are the Appel polynomials, Apostol-type polynomials, Hermite-type polynomials, the Chebyshev polynomials, the Dickson polynomials, Milne-Thomson type polynomials, are given. Trigonometric functions, the Euler’s formula and generating functions have applications in many different areas, which are mainly mathematics, statistics, physics, engineering and other sciences. Therefore, it can be stated that the results of this article may be used and applied in these related areas.
Notations and definitions of this paper are presented as follows:

Let $\mathbb{N}, \mathbb{Z}, \mathbb{R}$, and $\mathbb{C}$ denote the set of positive integers, the set of integers, the set of real numbers, and the set of complex numbers, respectively, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $\lambda, v \in \mathbb{C}$. $(\alpha)_v$, denotes the Pochhammer symbol, is defined by

\[
(\alpha)_v = \frac{\Gamma (\alpha + v)}{\Gamma (\alpha)} = \left\{ \begin{array}{ll}
\frac{\alpha (\alpha + 1) (\alpha + 2) \ldots (\alpha + v - 1)}{1} & v = n, \alpha \in \mathbb{C} \\
\frac{\alpha (\alpha - 1)(\alpha - 2)\ldots(\alpha - v + 1)}{v!} & v = 0, \alpha \in \mathbb{C} \setminus \{0\}
\end{array} \right.
\]

where $\Gamma (\alpha)$ denotes the Euler gamma function.

\[
(\alpha)_v = (-1)^v (\alpha)_v.
\]

Let $w = x + iy = (x, y)$ and \(\overline{w} = x - iy = (x, -y)\), where $x = \text{Re} \{w\}, \ y = \text{Im} \{w\}$ and

\[
i^2 = -1.
\]

Here, $\ln w$ takes its principal value such that

\[
\ln(w) := \ln|w| + i \arg(w),
\]

with $|w| > 0, -\pi < \arg(w) < \pi$. In addition,

\[
\exp(t) = e^t.
\]

The Euler’s formula, well-known mathematical formula in complex analysis, is given by

\[
\exp(iz) = \cos(z) + i \sin(z).
\]

This formula gives the fundamental relationship between the trigonometric functions and the complex exponential function (cf. [7], [34], [33]).

The following generating functions for well-known numbers and polynomials are needed in order give main results of this paper.

Generating function for the Apostol-Bernoulli polynomials $B^{(k)}_n(x; \lambda)$ of order $k$ is given by

\[
F_{AB} (t, x; \lambda, k) = \left( \frac{t}{\lambda \exp(t) - 1} \right)^k \exp(xt) = \sum_{n=0}^{\infty} B^{(k)}_n(x; \lambda) \frac{t^n}{n!}, \quad (1)
\]
where \(|t| < 2\pi\), when \(\lambda = 1\); \(|t| < |\log \lambda|\) when \(\lambda \neq 1\). Using (1), we have
\[
B_n^{(k)}(x) = B_n^{(k)}(x; 1)
\]
and
\[
B_n^{(k)}(\lambda) = B_n^{(k)}(0; \lambda),
\]
where \(B_n^{(k)}(x)\) and \(B_n^{(k)}(\lambda)\) denote the Bernoulli polynomials of order \(k\) and the Apostol-Bernoulli numbers of order \(k\), respectively (cf. [35], [37]).

Generating function for the Apostol-Euler polynomials \(E_n^{(k)}(x; \lambda)\) of order \(k\) is given by
\[
F_{AE}(t, x; \lambda, k) = \left(\frac{2}{\lambda \exp(t) + 1}\right)^k \exp(xt) = \sum_{n=0}^{\infty} E_n^{(k)}(x; \lambda) \frac{t^n}{n!},
\]
where \(|t| < |\log(-\lambda)|\). Using (2), we have
\[
E_n^{(k)}(x) = E_n^{(k)}(x; 1)
\]
and
\[
E_n^{(k)}(\lambda) = E_n^{(k)}(0; \lambda),
\]
where \(E_n^{(k)}(x)\) and \(E_n^{(k)}(\lambda)\) denote the Euler polynomials of order \(k\) and the Apostol-Euler numbers of order \(k\), respectively (cf. [35], [37]).

Generating functions for the polynomials \(C_n(x, y)\) and \(S_n(x, y)\) are defined as follows, respectively
\[
F_C(t, x, y) = \exp(xt) \cos(yt) = \sum_{n=0}^{\infty} C_n(x, y) \frac{t^n}{n!},
\]
and
\[
F_S(t, x, y) = \exp(xt) \sin(yt) = \sum_{n=0}^{\infty} S_n(x, y) \frac{t^n}{n!}
\]
(cf. [19], [20], [23], [24], [25], [37]).

By using equations (3) and (4), we have
\[
C_n(x, y) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n}{2k} x^{n-2k} y^{2k}
\]
and
\[
S_n(x, y) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} (-1)^k \binom{n}{2k+1} x^{n-2k-1} y^{2k+1}
\]
(cf. [19], [20], [23], [24], [25], [37]).
Generating functions for the Chebyshev polynomials of the first and second kinds are given as follows, respectively

\[
\frac{1 - xt}{1 - 2xt + t^2} = \sum_{n=0}^{\infty} T_n(x) t^n
\]  

(7)

and

\[
\frac{1}{1 - 2xt + t^2} = \sum_{n=0}^{\infty} U_n(x) t^n
\]  

(8)

(cf. [1], [4], [7], [15], [30]).

By using (7) and (8), the following well-known relations between the polynomials \(T_n(x)\) and \(U_n(x)\) are given

\[ T_n(x) = U_n(x) - xU_{n-1}(x) \]  

(9)

and

\[ T_{n+1}(x) = xT_n(x) - (1 - x^2)U_{n-1}(x). \]  

(10)

By using (7) and (8), the well-known computational formulas for the Chebyshev polynomials of the first and second kinds are given as follows, respectively

\[ T_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} (x^2 - 1)^k x^{n-2k} \]  

(11)

and

\[ U_{n-1}(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k + 1} (x^2 - 1)^k x^{n-2k-1} \]  

(12)

(cf. [1], [4], [7], [15], [30]).

Generating functions for the Dickson polynomials of the first and second kinds are given as follows, respectively

\[
\frac{1 - 2xt}{1 - xt + \alpha t^2} = \sum_{n=0}^{\infty} D_n(x, \alpha) t^n
\]  

(13)

and

\[
\frac{1}{1 - xt + \alpha t^2} = \sum_{n=0}^{\infty} \mathcal{E}_n(x, \alpha) t^n
\]  

(14)

(cf. [14], [21], [28]). The polynomials \(D_n(x, \alpha)\) and \(\mathcal{E}_n(x, \alpha)\) are of degree \(n\) in \(x\) with real parameter \(\alpha\).

By using (13) and (14), the following well-known relation between the polynomials \(D_n(x, \alpha)\) and \(\mathcal{E}_n(x, \alpha)\) is given

\[ D_n(x, \alpha) = \mathcal{E}_n(x, \alpha) - 2x\mathcal{E}_{n-1}(x, \alpha). \]
Substituting $\alpha = 1$ into (13) and (14), we have the following relations, respectively:

$$D_n(x, 1) = 2T_n\left(\frac{x}{2}\right),$$

and

$$E_n(x, 1) = U_n\left(\frac{x}{2}\right).$$

(cf. [14]).

Generating functions for the Milne-Thomson type polynomials is given by

$$R(t, x, y, z; a, b, v) = (b + f(t, a)) z^{\exp(tx + yh(t, v))} = \sum_{n=0}^{\infty} y_6(n; x, y, z; a, b, v) \frac{t^n}{n!},$$

(17)

where $f(t, a)$ is a number of family of analytic functions or meromorphic functions, $b(t, v)$ any analytic function, $a, b \in \mathbb{R}$ and $v \in \mathbb{N}$ (cf. [32]).

Note that there is one generating function for each value of $a$, $b$ and $v$.

Substituting $y = 0$ into (17), we have the Appell polynomials which are defined by

$$(b + f(t, a))^2 \exp(tx) = \sum_{n=0}^{\infty} y_6(n; x, 0, z; a, b, v) \frac{t^n}{n!},$$

where $(b + f(t, a))^2 = \sum_{n=0}^{\infty} a_n^2(a, b)t^n$ is a formal power series and

$$y_6(n; x, 0, z; a, b, v) = y_6(n; x, z; a, b).$$

Setting $x = y = 0$ into (17), we obtain generating functions for special numbers of order $z$:

$$R_1(t, z; a, b) = (b + f(t, a))^2 = \sum_{n=0}^{\infty} y_6^{(z)}(n; a, b) \frac{t^n}{n!},$$

(18)

Therefore, we have

$$y_6(n; 0, 0, z; a, b, v) = y_6^{(z)}(n; a, b).$$

For instance, substituting $b = 0$ and $f(t, a) = \frac{t}{a \exp(t) - 1}$ into (17), we have

$$y_6^{(z)}(n; a, 0, v) = B_n^{(z)}(a).$$

Generating function for the Hermite-Kampé de Fériet (or Gould-Hopper) polynomials, $H_n^{(j)}(x, y)$ is given by

$$F_H(t, x, y, j) = \exp(xt + yt^j) = \sum_{n=0}^{\infty} H_n^{(j)}(x, y) \frac{t^n}{n!},$$

(19)
where for \( j \in \mathbb{N} \) with \( j \geq 2 \),
\[
H^{(j)}_n(x, y) = \sum_{s=0}^{\left\lfloor \frac{n}{j} \right\rfloor} \frac{x^n - jsy^s}{(n - js)!s!}
\]
(cf. [3], [11], [16], [36]). It is well-known that the polynomials \( H^{(j)}_n(x, y) \) are a solution of generalized heat equation.

Generating function for generalized Hermite-Kampè de Fériet polynomials is given by
\[
F_R(t, \vec{u}, r) = \exp \left( \sum_{j=1}^r u_j t^j \right) = \sum_{n=0}^\infty H_n(\vec{u}, r) \frac{t^n}{n!}, \tag{20}
\]
where for \( \vec{u} = (u_1, u_2, \ldots, u_r) \) and
\[
H_n(\vec{u}, r) = \sum_{\Pi_m(n \mid r)} n! \prod_{j=1}^r u_j^{m_j} \prod_{j=1}^r m_j!
\]
such that
\[
m = \sum_{j=1}^r m_j,
\]
and
\[
n = \sum_{j=1}^r jm_j
\]
the sum (21) runs over all restricted partitions \( \Pi_m(n \mid r) \) (containing at most \( r \) sizes) of the integer \( n \), \( m \) denoting the number of parts of the partition and \( m_j \) the number of parts of size \( j \) (cf. see for detail [3], [11], [12]).

Using equation (20), an explicit formula for the polynomials \( H_n(\vec{u}, r) \) is given by
\[
H_n(\vec{u}, r) = n! \sum_{j=0}^{\left\lfloor \frac{n}{r} \right\rfloor} u_j^{n-rj} \frac{\left( \vec{u}, r-1 \right)}{(n-rj)!}
\]
where \( [x] \) denote the largest integer \( \leq x \). (cf. [11], [12]).

2 Generating functions for new families of Hermite-type polynomials and their computation formulas

In this section, we define generating functions for families of Hermite-type polynomials. We give some identities and computation formulas for these polynomials and their generating functions.
Let

$$G(t, w, \vec{u}, r) = \exp\left( wt + \sum_{j=1}^{r} u_j t^j \right) = \sum_{n=0}^{\infty} K(n; w, \vec{u}, r) \frac{t^n}{n!}, \quad (22)$$

where \(r\)-tuples \(\vec{u} = (u_1, u_2, \ldots, u_r)\), \(w = x + iy = (x, y)\), \(u_1, u_2, \ldots, u_r, x, y \in \mathbb{R}\).

By combining equation (22) with the Euler’s formula, we obtain

$$G(t, w, \vec{u}, r) = \exp\left( xt + \sum_{j=1}^{r} u_j t^j \right) (\cos (yt) + i \sin (yt)) = \sum_{n=0}^{\infty} K(n; w, \vec{u}, r) \frac{t^n}{n!}. \quad (23)$$

In order to give an explicit formula for the polynomials \(K(n; w, \vec{u}, r)\), we give following decompositions of equation (23)

$$K_1(t, x, y, \vec{u}, r) = \Re (G(t, w, \vec{u}, r)) = \exp\left( xt + \sum_{j=1}^{r} u_j t^j \right) \cos (yt) \quad (24)$$

and

$$K_2(t, x, y, \vec{u}, r) = \Im (G(t, w, \vec{u}, r)) = \exp\left( xt + \sum_{j=1}^{r} u_j t^j \right) \sin (yt) \quad (25)$$

Therefore, by using (24) and (25), we get the following decompositions for the polynomials \(K(n; w, \vec{u}, r)\):

$$K(n; w, \vec{u}, r) = K_1(n; x, y, \vec{u}, r) + i K_2(n; x, y, \vec{u}, r). \quad (26)$$

**Lemma 1**

Let \(\vec{x} = (x + u_1, u_2, u_3, \ldots, u_r)\) and \(\vec{u} = (u_1, u_2, u_3, \ldots, u_r)\). Then we have

$$k_1(n; x, y, \vec{u}, r) = \sum_{j=0}^{n} \frac{(-1)^j}{2^j} \binom{n}{2j} y^{2j} H_{n-2j}(\vec{x}, r). \quad (27)$$

**Proof.** Combining (24) with (20), we obtain the following functional equation:

$$K_1(t, x, y, \vec{u}, r) = \cos (yt) F_R(t, \vec{x}, r)$$

where \(\vec{x} = (x + u_1, u_2, u_3, \ldots, u_r)\) and \(\vec{u} = (u_1, u_2, u_3, \ldots, u_r)\). By using above functional equation, we get

$$\sum_{n=0}^{\infty} k_1(n; x, y, \vec{u}, r) \frac{t^n}{n!} = \sum_{n=0}^{\infty} (-1)^n y^{2n} \frac{t^{2n}}{(2n)!} \sum_{n=0}^{\infty} H_n(\vec{x}, r) \frac{t^n}{n!}.$$
Therefore
\[
\sum_{n=0}^{\infty} k_1(n; x, y, \overrightarrow{u}, r) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \sum_{j=0}^{\left[\frac{n}{2}\right]} (-1)^j \frac{n}{2j} y^{2j} H_{n-2j} (\overrightarrow{z}, r) \frac{t^n}{n!}.
\]

Comparing the coefficients of \( \frac{t^n}{n!} \) on both sides of the above equation, we arrive at the desired result. ■

By using (27), we compute a few values of the polynomials \( k_1(n; x, y, \overrightarrow{u}, r) \) as follows:

For \( r = 2, \overrightarrow{u} = (u_1, u_2) \) and \( \overrightarrow{z} = (x + u_1, u_2) \), we have
\[
\begin{align*}
    k_1(0; x, y, \overrightarrow{u}, 2) &= 1, \\
    k_1(1; x, y, \overrightarrow{u}, 2) &= x + u_1, \\
    k_1(2; x, y, \overrightarrow{u}, 2) &= (x + u_1)^2 + 2u_2 - y^2, \\
    k_1(3; x, y, \overrightarrow{u}, 2) &= (x + u_1)^3 + 6(x + u_1)u_2 - 3y^2(x + u_1).
\end{align*}
\]

For \( r = 3, \overrightarrow{u} = (u_1, u_2, u_3) \) and \( \overrightarrow{z} = (x + u_1, u_2, u_3) \), we have
\[
\begin{align*}
    k_1(0; x, y, \overrightarrow{u}, 3) &= 1, \\
    k_1(1; x, y, \overrightarrow{u}, 3) &= x + u_1, \\
    k_1(2; x, y, \overrightarrow{u}, 3) &= (x + u_1)^2 + 2u_2 - y^2, \\
    k_1(3; x, y, \overrightarrow{u}, 3) &= (x + u_1)^3 + 6(x + u_1)u_2 + 6u_3 - 3y^2(x + u_1).
\end{align*}
\]

Lemma 2 Let \( \overrightarrow{z} = (x + u_1, u_2, u_3, \ldots, u_r) \) and \( \overrightarrow{u} = (u_1, u_2, u_3, \ldots, u_r) \). Then we have
\[
k_2(n; x, y, \overrightarrow{u}, r) = \sum_{j=0}^{\left[\frac{n}{2}\right]} (-1)^j \frac{n}{2j + 1} y^{2j+1} H_{n-1-2j} (\overrightarrow{z}, r).
\]

Proof. Combining (25) with (20), we get the following functional equation:
\[
K_2(t, x, y, \overrightarrow{u}, r) = \sin (yt) F_R(t, \overrightarrow{z}, r)
\]
where \( \overrightarrow{z} = (x + u_1, u_2, u_3, \ldots, u_r) \) and \( \overrightarrow{u} = (u_1, u_2, u_3, \ldots, u_r) \). By using above functional equation, we have
\[
\sum_{n=0}^{\infty} k_2(n; x, y, \overrightarrow{u}, r) \frac{t^n}{n!} = \sum_{n=0}^{\infty} (-1)^n y^{2n+1} \frac{t^{2n+1}}{(2n+1)!} \sum_{n=0}^{\infty} H_n (\overrightarrow{z}, r) \frac{t^n}{n!}.
\]

Therefore
\[
\sum_{n=0}^{\infty} k_2(n; x, y, \overrightarrow{u}, r) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \sum_{j=0}^{\left[\frac{n}{2}\right]} (-1)^j \frac{n}{2j + 1} y^{2j+1} H_{n-1-2j} (\overrightarrow{z}, r) \frac{t^n}{n!}.
\]
Comparing the coefficients of $\frac{x^n}{n!}$ on both sides of the above equation, we arrive at the desired result.

By using (28), we compute a few values of the polynomials $k_2 (n; x, y, \overrightarrow{v}, r)$ as follows:

For $r = 2$, $\overrightarrow{v} = (u_1, u_2)$ and $\overrightarrow{x} = (x + u_1, u_2)$, we have

\[
\begin{align*}
    k_2 (0; x, y, \overrightarrow{v}, 2) &= 0, \\
    k_2 (1; x, y, \overrightarrow{v}, 2) &= y, \\
    k_2 (2; x, y, \overrightarrow{v}, 2) &= 2y (x + u_1), \\
    k_2 (3; x, y, \overrightarrow{v}, 2) &= 3y (x + u_1)^2 + 6yu_2 - y^3.
\end{align*}
\]

For $r = 3$, $\overrightarrow{v} = (u_1, u_2, u_3)$ and $\overrightarrow{x} = (x + u_1, u_2, u_3)$, we have

\[
\begin{align*}
    k_2 (0; x, y, \overrightarrow{v}, 3) &= 0, \\
    k_2 (1; x, y, \overrightarrow{v}, 3) &= y, \\
    k_2 (2; x, y, \overrightarrow{v}, 3) &= 2y (x + u_1), \\
    k_2 (3; x, y, \overrightarrow{v}, 3) &= 3y (x + u_1)^2 + 6yu_2 - y^3.
\end{align*}
\]

Combining Lemma 1 and Lemma 2 with (26), we obtain an explicit formula for the polynomials $\mathcal{K} (n; w, \overrightarrow{v}, r)$ by the following theorem:

**Theorem 3** Let $\overrightarrow{x} = (x + u_1, u_2, u_3, \ldots, u_r)$ and $\overrightarrow{v} = (u_1, u_2, u_3, \ldots, u_r)$. Then we have

\[
\mathcal{K} (n; w, \overrightarrow{v}, r) = \sum_{j=0}^{[\frac{n}{2}]} (-1)^j \binom{n}{2j} y^{2j} H_{n-2j} (\overrightarrow{x}, r) + i \sum_{j=0}^{[\frac{n-1}{2}]} (-1)^j \binom{n}{2j+1} y^{2j+1} H_{n-1-2j} (\overrightarrow{x}, r).
\]

(29)

By using (29), we compute a few values of the polynomials $\mathcal{K} (n; w, \overrightarrow{v}, r)$ as follows:

For $r = 2$, $\overrightarrow{v} = (u_1, u_2)$ and $\overrightarrow{x} = (x + u_1, u_2)$, we have

\[
\begin{align*}
    \mathcal{K} (0; w, \overrightarrow{v}, 2) &= 1, \\
    \mathcal{K} (1; w, \overrightarrow{v}, 2) &= x + u_1 + iy, \\
    \mathcal{K} (2; w, \overrightarrow{v}, 2) &= (x + u_1)^2 + 2u_2 - y^2 + 2iy (x + u_1), \\
    \mathcal{K} (3; w, \overrightarrow{v}, 2) &= (x + u_1)^3 + 6 (x + u_1) u_2 - 3y^2 (x + u_1) + i \left( 3y (x + u_1)^2 + 6yu_2 - y^3 \right).
\end{align*}
\]

For $r = 3$, $\overrightarrow{v} = (u_1, u_2, u_3)$ and $\overrightarrow{x} = (x + u_1, u_2, u_3)$, we have

\[
\begin{align*}
    \mathcal{K} (0; w, \overrightarrow{v}, 3) &= 1, \\
    \mathcal{K} (1; w, \overrightarrow{v}, 3) &= x + u_1 + iy, \\
    \mathcal{K} (2; w, \overrightarrow{v}, 3) &= (x + u_1)^2 + 2u_2 - y^2 + 2iy (x + u_1), \\
    \mathcal{K} (3; w, \overrightarrow{v}, 3) &= (x + u_1)^3 + 6 (x + u_1) u_2 + 6u_3 - 3y^2 (x + u_1) + i \left( 3y (x + u_1)^2 + 6yu_2 - y^3 \right).
\end{align*}
\]
Theorem 4 Let \( \vec{u} = (u_1, u_2, u_3, \ldots, u_r) \). Then we have

\[
k_1(n; x, y, \vec{u}, r) = \sum_{j=0}^{n} \binom{n}{j} C_j(x, y) H_{n-j}(\vec{u}, r) .
\] (30)

**Proof.** By using (3), (20) and (24), we obtain the following functional equation:

\[
K_1(t, x, y, \vec{u}, r) = F_C(t, x, y) F_R(t, \vec{u}, r) .
\]

By using the above functional equation, we get

\[
\sum_{n=0}^{\infty} k_1(n; x, y, \vec{u}, r) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \sum_{j=0}^{n} \binom{n}{j} C_j(x, y) H_{n-j}(\vec{u}, r) \frac{t^n}{n!} .
\]

Therefore

\[
\sum_{n=0}^{\infty} k_1(n; x, y, \vec{u}, r) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \sum_{j=0}^{n} \binom{n}{j} C_j(x, y) H_{n-j}(\vec{u}, r) \frac{t^n}{n!} .
\]

Comparing the coefficients of \( \frac{t^n}{n!} \) on both sides of the above equation, we arrive at the desired result. \( \blacksquare \)

Theorem 5 Let \( \vec{u} = (u_1, u_2, u_3, \ldots, u_r) \). Then we have

\[
k_2(n; x, y, \vec{u}, r) = \sum_{j=0}^{n} \binom{n}{j} S_j(x, y) H_{n-j}(\vec{u}, r) .
\] (31)

**Proof.** By using (4), (20) and (25), we derive the following functional equation:

\[
K_2(t, x, y, \vec{u}, r) = F_S(t, x, y) F_R(t, \vec{u}, r) .
\]

By using above functional equation, we get

\[
\sum_{n=0}^{\infty} k_2(n; x, y, \vec{u}, r) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \sum_{j=0}^{n} \binom{n}{j} S_j(x, y) H_{n-j}(\vec{u}, r) \frac{t^n}{n!} .
\]

Therefore

\[
\sum_{n=0}^{\infty} k_2(n; x, y, \vec{u}, r) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \sum_{j=0}^{n} \binom{n}{j} S_j(x, y) H_{n-j}(\vec{u}, r) \frac{t^n}{n!} .
\]

Comparing the coefficients of \( \frac{t^n}{n!} \) on both sides of the above equation, we arrive at the desired result. \( \blacksquare \)

Combining (30) and (31) with (26), we obtain an explicit formula for the polynomials \( \mathcal{K}(n; w, \vec{u}, r) \) by the following corollary:

**Corollary 6** Let \( \vec{u} = (u_1, u_2, u_3, \ldots, u_r) \). Then we have

\[
\mathcal{K}(n; w, \vec{u}, r) = \sum_{j=0}^{n} \binom{n}{j} H_{n-j}(\vec{u}, r) (C_j(x, y) + i S_j(x, y)) .
\]
3 Generating functions for Hermite-based $r$-parametric Milne-Thomson-type polynomials

By the aid of generating functions in (17) and (22), we construct the following generating functions for Hermite-based $r$-parametric Milne-Thomson-type polynomials:

\[ M_1 (t, w, \vec{u}, r, a, b) = (b + f(t, a))^\n \mathcal{G} (t, w, \vec{u}, r) \]
\[ = \sum_{n=0}^{\infty} h(n, w, z; \vec{u}, r, a, b) \frac{t^n}{n!}, \]

\[ M_2 (t, w, \vec{u}, r, a, b) = (b + f(t, a))^\n (\mathcal{G} (t, w, \vec{u}, r) + \mathcal{G} (t, \overline{w}, \vec{u}, r)) \]
\[ = \sum_{n=0}^{\infty} h_1(n, w, z; \vec{u}, r, a, b) \frac{t^n}{n!}, \]

and

\[ M_3 (t, w, \vec{u}, r, a, b) = (b + f(t, a))^\n (\mathcal{G} (t, w, \vec{u}, r) - \mathcal{G} (t, \overline{w}, \vec{u}, r)) \]
\[ = \sum_{n=0}^{\infty} h_2(n, w, z; \vec{u}, r, a, b) \frac{t^n}{n!}, \]

where $a, b, z \in \mathbb{R}$, $r$-tuples $\vec{u} = (u_1, u_2, \ldots, u_r)$, $w = x + iy$ and $\overline{w} = x - iy$; the function $f(t, a)$ denotes analytic or meromorphic function.

Substituting $w = 0$ and $\vec{u} = \vec{0}$ into (32), we have

\[ \sum_{n=0}^{\infty} h(n, 0, z; \vec{0}, r, a, b) \frac{t^n}{n!} = \sum_{n=0}^{\infty} y_6^{(z)}(n; a, b) \frac{t^n}{n!}. \]

Thus, we get

\[ h(n, 0, z; \vec{0}, r, a, b) = y_6^{(z)}(n; a, b). \]

Combining (34) and (33) with (32), we obtain an formula for the polynomials $h(n, w, z; \vec{u}, r, a, b)$ by the following theorem:

**Theorem 7** Let $\vec{u} = (u_1, u_2, u_3, \ldots, u_r)$ and $w = x + iy$. Then we have

\[ h(n, w, z; \vec{u}, r, a, b) = h_1(n, w, z; \vec{u}, r, a, b) + h_2(n, w, z; \vec{u}, r, a, b) \]
\[ \frac{1}{2}. \]

**Theorem 8** Let $\vec{u} = (u_1, u_2, u_3, \ldots, u_r)$. Then we have

\[ h_1(n, w, z; \vec{u}, r, a, b) = \sum_{j=0}^{n} \binom{n}{j} y_6^{(z)}(n - j; a, b) \mathcal{K}(j; w, \vec{u}, r) + \mathcal{K}(j; \overline{w}, \vec{u}, r). \]

(36)
Proof. By using (18), (22) and (33), we derive the following functional equation:

$$M_2 (t, w, z, \overrightarrow{u}, r, a, b) = R_1 (t, z; a, b) (G (t, w, \overrightarrow{u}, r) + G (t, w, \overrightarrow{u}, r)).$$

From the above equation, we have

$$\sum_{n=0}^{\infty} h_1 (n, w, z; \overrightarrow{u}, r, a, b) \frac{t^n}{n!} = \sum_{n=0}^{\infty} y_6^{(z)} (n; a, b) \frac{t^n}{n!} \sum_{n=0}^{\infty} K (n; w, \overrightarrow{u}, r) \frac{t^n}{n!}$$

$$+ \sum_{n=0}^{\infty} y_6^{(z)} (n; a, b) \frac{t^n}{n!} \sum_{n=0}^{\infty} K (n; w, \overrightarrow{u}, r) \frac{t^n}{n!}.$$

Therefore

$$\sum_{n=0}^{\infty} h_1 (n, w, z; \overrightarrow{u}, r, a, b) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \sum_{j=0}^{n} \left( \frac{n}{j} \right) y_6^{(z)} (n-j; a, b) (K (j; w, \overrightarrow{u}, r) - K (j; w, \overrightarrow{u}, r)) \frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation we arrive at the desired result.

Theorem 9 Let $\overrightarrow{u} = (u_1, u_2, u_3, \ldots, u_r)$ and $w = x + iy$. Then we have

$$h_2 (n, w, z; \overrightarrow{u}, r, a, b) = \sum_{j=0}^{n} \left( \frac{n}{j} \right) y_6^{(z)} (n-j; a, b) (K (j; w, \overrightarrow{u}, r) - K (j; w, \overrightarrow{u}, r)).$$

(37)

Proof. By using (17), (22) and (34), we derive the following functional equation:

$$M_3 (t, w, z, \overrightarrow{u}, r, a, b) = R_1 (t, z; a, b) (G (t, w, \overrightarrow{u}, r) - G (t, w, \overrightarrow{u}, r)).$$

From the above equation, we obtain

$$\sum_{n=0}^{\infty} h_2 (n, w, z; \overrightarrow{u}, r, a, b) \frac{t^n}{n!} = \sum_{n=0}^{\infty} y_6^{(z)} (n; a, b) \frac{t^n}{n!} \sum_{n=0}^{\infty} K (n; w, \overrightarrow{u}, r) \frac{t^n}{n!}$$

$$- \sum_{n=0}^{\infty} y_6^{(z)} (n; a, b) \frac{t^n}{n!} \sum_{n=0}^{\infty} K (n; w, \overrightarrow{u}, r) \frac{t^n}{n!}.$$

Therefore

$$\sum_{n=0}^{\infty} h_2 (n, w, z; \overrightarrow{u}, r, a, b) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \sum_{j=0}^{n} \left( \frac{n}{j} \right) y_6^{(z)} (n-j; a, b) (K (j; w, \overrightarrow{u}, r) - K (j; w, \overrightarrow{u}, r)) \frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation we arrive at the desired result.
3.1 Identities for Hermite-based $r$-parametric Milne-Thomson-type polynomials

By using (32)-(34), we give identities and relations for Milne-Thomson-type polynomials and Hermite-type polynomials including Hermite-based $r$-parametric Milne-Thomson-type polynomials.

**Theorem 10** Let $\vec{u} = (u_1, u_2, u_3, \ldots, u_r)$ and $w = x + iy$. Then we have

$$h(n, w, z; \vec{u}, r, a, b) = \sum_{j=0}^{n} \binom{n}{j} y_0^{(z)} (n-j; a, b) K(j; w, \vec{u}, r). \quad (38)$$

**Proof.** By using (18), (22) and (32), we derive the following functional equation:

$$M_1(t, w, z, \vec{u}, r, a, b) = R_1(t, z; a, b) G(t, w, \vec{u}, r).$$

From the above equation, we have

$$\sum_{n=0}^{\infty} h(n, w, z; \vec{u}, r, a, b) \frac{t^n}{n!} = \sum_{n=0}^{\infty} y_0^{(z)} (n; a, b) \frac{t^n}{n!} \sum_{n=0}^{\infty} K(n; w, \vec{u}, r) \frac{t^n}{n!}.$$

Therefore

$$\sum_{n=0}^{\infty} h(n, w, z; \vec{u}, r, a, b) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \sum_{j=0}^{n} \binom{n}{j} y_0^{(z)} (n-j; a, b) K(j; w, \vec{u}, r) \frac{t^n}{n!}.$$  

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation we arrive at the desired result. 

By using Euler’s formula, we modify (33) as follows:

$$B(t, x, y, z; \vec{u}, r, a, b) = 2 (b + f(t, a))^x \exp(xt) M_4(t, y, \vec{u}, r) \quad (39)$$

$$= \sum_{n=0}^{\infty} B_1(n, x, y, z; \vec{u}, r, a, b) \frac{t^n}{n!}.$$

where

$$M_4(t, y, \vec{u}, r) = \exp \left( \sum_{j=1}^{r} u_j t^j \right) \cos(yt) = \sum_{n=0}^{\infty} C_n(\vec{u}, y; r) \frac{t^n}{n!}. \quad (40)$$

Observe that when $r = 1$, (40) reduces to the (3). Setting $y = 0$ in (40), we have

$$H_n(\vec{u}, r) = C_n(\vec{u}, 0; r).$$

13
Theorem 11 Let \( \mathbf{u} = (u_1, u_2, u_3, \ldots, u_r) \). Then we have

\[
\mathbf{h}_1 \left( n, x, y, z; \mathbf{u}, r, a, b \right) = 2 \sum_{j=0}^{n} \binom{n}{j} y_j \left( n - j; x, z; a, b \right) C_j \left( \mathbf{u}, y; r \right) .
\]  

(41)

Proof. Combining (17), (40) and (39), we get

\[
\sum_{n=0}^{\infty} \mathbf{h}_1 \left( n, x, y, z; \mathbf{u}, r, a, b \right) \frac{t^n}{n!} = 2 \sum_{n=0}^{\infty} y_6 \left( n; x, z; a, b \right) \frac{t^n}{n!} \sum_{n=0}^{\infty} C_n \left( \mathbf{u}, y; r \right) \frac{t^n}{n!} .
\]

Therefore

\[
\sum_{n=0}^{\infty} \mathbf{h}_1 \left( n, x, y, z; \mathbf{u}, r, a, b \right) \frac{t^n}{n!} = 2 \sum_{n=0}^{\infty} y_6 \left( n - j; x, z; a, b \right) C_j \left( \mathbf{u}, y; r \right) \frac{t^n}{n!} .
\]

Comparing the coefficients of \( \frac{t^n}{n!} \) on both sides of the above equation we arrive at the desired result.

Substituting \( r = 1 \) into (41), we have

\[
\mathbf{h}_1 \left( n, x, y, z; u_1, 1, a, b \right) = 2 \sum_{j=0}^{n} \binom{n}{j} y_j \left( n - j; x, z; a, b \right) C_j \left( u_1, y; 1 \right) ,
\]

where

\[
C_j(u_1, y) = C_j(u_1, y; 1) .
\]

Substituting \( b = 0 \),

\[
f(t, a) = \frac{t}{a \exp(t) - 1}
\]

and \( \mathbf{u} = \mathbf{0} \) into (39), we have

\[
\sum_{n=0}^{\infty} \mathbf{h}_1 \left( n, x, y, z; \mathbf{0}, r, a, 0 \right) \frac{t^n}{n!} = \sum_{n=0}^{\infty} 2 \mathcal{B}_n^{(C, z)} \left( x, y; a \right) \frac{t^n}{n!} ,
\]  

(42)

where

\[
\left( \frac{t}{a \exp(t) - 1} \right)^z \exp(zt) \cos(zt) = \sum_{n=0}^{\infty} \mathcal{B}_n^{(C, z)} \left( x, y; a \right) \frac{t^n}{n!} .
\]

Comparing the coefficients of \( \frac{t^n}{n!} \) on both sides of equation (42), we get the following result:

Corollary 12

\[
\mathbf{h}_1 \left( n, x, y, z; \mathbf{0}, r, a, 0 \right) = 2 \mathcal{B}_n^{(C, z)} \left( x, y; a \right) .
\]
Remark 13 When \( a = 1 \) and \( z = 1 \), the polynomials \( h_1(n, x, y, z; \vec{0}, r, a, 0) \) reduce to following well-known polynomials:

\[
h_1(n, x, y, 1; \vec{0}, r, 1, 0) = 2B_n^{(C, 1)}(x, y; 1) = 2B_n^{(C)}(x, y),
\]

where the polynomials \( B_n^{(C)}(x, y) \) denote the cosine-Bernoulli polynomials (cf. [20]). When \( y = 0 \), the polynomials \( B_n^{(C, z)}(x, y; a) \) reduce to the Apostol-Bernoulli polynomials of order \( z \):

\[
B_n^{(z)}(x; a) = B_n^{(C, z)}(x, 0; a)
\]

(cf. [23], [37]).

On the other hand, using (??), we have

\[
\sum_{n=0}^{\infty} h_1(n, x, y, z; \vec{0}, r, a, 0) \frac{t^n}{n!} = 2 \sum_{n=0}^{\infty} B_n^{(z)}(x; a) \frac{t^n}{n!} \sum_{n=0}^{\infty} (-1)^n (yt)^{2n} \frac{(2n)!}{(2n)!}.
\]

Therefore, we obtain

\[
\sum_{n=0}^{\infty} h_1(n, x, y, z; \vec{0}, r, a, 0) \frac{t^n}{n!} = 2 \sum_{n=0}^{\infty} B_n^{(z)}(x; a) \frac{t^n}{n!} \sum_{n=0}^{\infty} (-1)^n (yt)^{2n} \frac{(2n)!}{(2n)!}.
\]

Comparing the coefficients of \( \frac{t^n}{n!} \) on both sides of the above equation, we arrive at the following result:

Corollary 14 Let \( r \)-tuples \( \vec{0} = (0, 0, ..., 0) \). Then we have

\[
h_1(n, x, y, z; \vec{0}, r, a, 0) = 2 \sum_{n=0}^{\infty} B_n^{(z)}(x; a) \frac{t^n}{n!} \sum_{n=0}^{\infty} (-1)^n (yt)^{2n} \frac{(2n)!}{(2n)!}.
\]

We modify (??) as follows:

\[
\sum_{n=0}^{\infty} h_1(n, x, y, z; \vec{0}, r, -a, 0) \frac{t^n}{n!} = \frac{(-1)^z}{2^{z-1}} \sum_{n=0}^{\infty} \mathcal{E}_n^{(C, z)}(x, y; a) \frac{t^n}{n!},
\]

where \( \mathcal{E}_n^{(C, z)}(x, y; a) \) denote the two parametric kinds of Apostol-Euler polynomials of order \( z \), which are defined by

\[
\left( \frac{2}{a \exp(t) + 1} \right)^z \exp(xt) \cos(yt) = \sum_{n=0}^{\infty} \mathcal{E}_n^{(C, z)}(x, y; a) \frac{t^n}{n!},
\]

(cf. [38]).

Comparing the coefficients of \( \frac{t^n}{n!} \) on both sides of equation (43), we get the following result:
Corollary 15 Let \( r \)-tuples \( \overrightarrow{0} = (0, 0, \ldots, 0) \). Then we have
\[
\mathfrak{h}_1 \left( n, x, y, z; \overrightarrow{0}, r, -a, 0 \right) = \frac{(1)^{z} (n)^{z}}{2^{z-1}} C^{(C,z)}_{n} (x, y; a).
\] (45)

Remark 16 When \( a = 1 \) and \( z = 1 \), the polynomials \( \mathfrak{h}_1 \left( n, x, y, z; \overrightarrow{0}, r, -a, 0 \right) \) reduce to the polynomials the cosine-Euler polynomials:
\[
\mathfrak{h}_1 \left( n, x, y, z; \overrightarrow{0}, r, -1, 0 \right) = -n C^{(C,1)}_{n-1} (x, y; 1) = -n E^{(C)}_{n-1} (x, y).
\]
(cf. [20], [24]). Setting \( y = 0 \) in (44), the polynomials \( \mathcal{E}^{(C,z)}_{n} (x; y; a) \) reduce to the Apostol-Euler polynomials of order \( z \):
\[
\mathcal{E}^{(z)}_{n} (x; a) = \mathcal{E}^{(C,z)}_{n} (x, 0; a).
\]
(cf. [33], [37]).

By using (43), we have
\[
\sum_{n=0}^{\infty} \mathfrak{h}_1 \left( n, x, y, z; \overrightarrow{0}, r, -a, 0 \right) \frac{t^n}{n!} = \frac{(1)^{z} t^z}{2^{z-1}} \sum_{n=0}^{\infty} \mathcal{E}^{(z)}_{n} (x; a) \frac{t^n}{n!} \sum_{n=0}^{\infty} \frac{(1)^{n} (yt)^{2n}}{(2n)!}.
\]
Therefore
\[
\sum_{n=0}^{\infty} \mathfrak{h}_1 \left( n, x, y, z; \overrightarrow{0}, r, -a, 0 \right) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left( \frac{-1}{} \sum_{j=0}^{\left\lfloor \frac{z}{2} \right\rfloor} (-1)^{j} 2^{1-z} \binom{n}{2j} (n-2j)^{y^{2j} \mathcal{E}^{(z)}_{n-2j-z} (x; a)} \right) \frac{t^n}{n!}.
\]
Comparing the coefficients of \( \frac{t^n}{n!} \) on both sides of the above equation we arrive at the following result:

Corollary 17 Let \( r \)-tuples \( \overrightarrow{0} = (0, 0, \ldots, 0) \). Then we have
\[
\mathfrak{h}_1 \left( n, x, y, z; \overrightarrow{0}, r, -a, 0 \right) = (1)^{z} 2^{1-z} \sum_{j=0}^{\left\lfloor \frac{z}{2} \right\rfloor} (-1)^{j} \binom{n}{2j} (n-2j)^{y^{2j} \mathcal{E}^{(z)}_{n-2j-z} (x; a)}.
\]

By using Euler’s formula, we modify and unify equation (44) as follows:
\[
B_{1} \left( t, x, y, z, \overrightarrow{u}, r, a, b \right) = 2 (b + f (t, a))^{z} \exp (xt) M_{5} \left( t, y, \overrightarrow{u}, r \right)
\]
\[
= \sum_{n=0}^{\infty} \mathfrak{h}_2 \left( n, w, z; \overrightarrow{u}, r, a, b \right) \frac{t^n}{n!}.
\]
where
\[
M_{5} \left( t, y, \overrightarrow{u}, r \right) = \exp \left( \sum_{j=1}^{r} u_{j} t^{j} \right) \sin (yt) = \sum_{n=0}^{\infty} S_{n} \left( \overrightarrow{u}, y; r \right) \frac{t^n}{n!}.
\] (47)

Observe that when \( r = 1 \), (47) reduces to (41). Setting \( y = \frac{\pi}{2} \) in (47), we have
\[
H_{n} \left( \overrightarrow{u}, r \right) = S_{n} \left( \overrightarrow{u}, \frac{\pi}{2}; r \right).
\]

16
Theorem 18  Let $\vec{u} = (u_1, u_2, u_3, \ldots, u_r)$. Then we have
\[
b_2 (n, x, y, z; \vec{u}, r, a, b) = 2 \sum_{j=0}^{n} \binom{n}{j} y_6 (n - j; x, z; a, b) S_j (\vec{u}, y r) . \tag{48}
\]

Proof. Combining (17), (47) and (46), we have
\[
\sum_{n=0}^{\infty} h_2 (n, x, y, z; \vec{u}, r, a, b) \frac{t^n}{n!} = 2 \sum_{n=0}^{\infty} y_6 (n; x, z; a, b) \frac{t^n}{n!} \sum_{n=0}^{\infty} S_n (\vec{u}, y r) \frac{t^n}{n!} .
\]
Therefore
\[
\sum_{n=0}^{\infty} h_2 (n, x, y, z; \vec{u}, r, a, b) \frac{t^n}{n!} = 2 \sum_{n=0}^{\infty} \sum_{j=0}^{n} \binom{n}{j} y_6 (n - j; x, z; a, b) S_j (\vec{u}, y r) \frac{t^n}{n!} .
\]
Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation we arrive at the desired result.

Substituting $b = 0$, 
\[
f (t, a) = \frac{t}{a \exp (t) - 1}
\]
and $\vec{u} = \vec{0}$ into (16), we have the following equation:
\[
B_1 (t, x, y, z, \vec{0}, r, a, 0) = 2 F_{BS} (t, x, y; a, z) \tag{49}
\]
where the function $F_{BS} (t, x, y; a, z)$ is a generating function for the two parametric kinds of the Apostol-Bernoulli polynomials of order $z$,
\[
F_{BS} (t, x, y; a, z) = \left( \frac{t}{a \exp (t) - 1} \right)^z \exp (xt) \sin (yt) = \sum_{n=0}^{\infty} B_n^{(S, z)} (x, y; a) \frac{t^n}{n!} \tag{50}
\]
(cf. [38]). Thus, using (49), we have the following result:

Corollary 19  Let $r$-tuples $\vec{0} = (0, 0, \ldots, 0)$. Then we have
\[
b_2 (n, x, y, z; \vec{0}, r, a, 0) = 2 B_n^{(S, z)} (x, y; a) .
\]

Remark 20  When $a = 1$ and $z = 1$, the polynomials $b_2 (n, x, y, z; \vec{0}, r, a, 0)$ reduces to the polynomials $B_n^{(S, 1)} (x, y; 1) = B_n^{(S)} (x, y)$, which denote sine-Bernoulli polynomials:
\[
b_2 (n, x, y, 1; \vec{0}, r, 1, 0) = 2 B_n^{(S)} (x, y) \tag{cf. [20]}
\]
Setting $y = \frac{\pi}{2}$ in (50), we have
\[
B_n^{(z)} (x; a) = B_n^{(S, z)} \left( x, \frac{\pi}{2}; a \right) \tag{cf. [35], [37]}
\]
By using (49), we have
\[
\sum_{n=0}^{\infty} h_2 \left( n, x, y, z; \overrightarrow{0}, r, a, 0 \right) \frac{t^n}{n!} = 2 \sum_{n=0}^{\infty} E_n^{(z)} (x; a) \frac{t^n}{n!} \sum_{n=0}^{\infty} (-1)^n (yt)^{2n+1} \frac{1}{(2n+1)!}.
\]

Therefore
\[
\sum_{n=0}^{\infty} h_2 \left( n, x, y, z; \overrightarrow{0}, r, a, 0 \right) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left( 2 \sum_{j=0}^{n} (-1)^j \binom{n}{2j+1} y^{2j+1} B_{n-1-2j}^{(z)} (x; a) \right) \frac{t^n}{n!}
\]

**Corollary 21** Let \( r \)-tuples \( \overrightarrow{0} = (0, 0, \ldots, 0) \). Then we have
\[
h_2 \left( n, x, y, z; \overrightarrow{0}, r, a, 0 \right) = 2 \sum_{j=0}^{n} (-1)^j \binom{n}{2j+1} y^{2j+1} B_{n-1-2j}^{(z)} (x; a).
\]

We modify (49) as follows:
\[
\sum_{n=0}^{\infty} h_2 \left( n, x, y, z; \overrightarrow{0}, r, -a, 0 \right) \frac{t^n}{n!} = \frac{(-1)^z t^z}{2^{z-1}} \sum_{n=0}^{\infty} E_n^{(S,z)} (x, y; a) \frac{t^n}{n!}
\]

where \( E_n^{(S,z)} (x, y; a) \) denote the two parametric kinds of Apostol-Euler polynomials of order \( z \), which are defined by the following generating function:
\[
\left( \frac{2}{a \exp (t) + 1} \right)^z \exp (xt) \sin (yt) = \sum_{n=0}^{\infty} E_n^{(S,z)} (x, y; a) \frac{t^n}{n!}
\]

(cf. [38]).

By using (51), we get the following result:

**Corollary 22** Let \( r \)-tuples \( \overrightarrow{0} = (0, 0, \ldots, 0) \). Then we have
\[
h_2 \left( n, x, y, z; \overrightarrow{0}, r, -a, 0 \right) = \frac{(-1)^z (n)^z}{2^{z-1}} E_n^{(S,z)} (x, y; a).
\]

**Remark 23** When \( a = 1 \) and \( z = 1 \), the polynomials \( h_2 \left( n, x, y, z; \overrightarrow{0}, r, -a, 0 \right) \) reduce to the polynomials \( E_{n-1}^{(1)} (x, y; 1) = E_{n-1}^{(S)} (x, y) \), which denote sine-Euler polynomials:
\[
h_2 \left( n, x, y, z; \overrightarrow{0}, r, -1, 0 \right) = -n E_{n-1}^{(S)} (x, y)
\]

(cf. [20], [24]). Setting \( y = \frac{\pi}{2} \) in (52), we have
\[
E_n^{(z)} (x; a) = E_n^{(S,z)} \left( x, \frac{\pi}{2}; a \right)
\]

(cf. [39], [37]).
Using (51), we obtain

\[ \sum_{n=0}^{\infty} b_2 \left( n, x, y, z; \vec{0}, r, -a, 0 \right) \frac{t^n}{n!} = (-t)^z 2^{1-z} \sum_{n=0}^{\infty} c_n^{(z)} (x; a) \frac{t^n}{n!} \sum_{n=0}^{\infty} (-1)^n \frac{yt^{2n+1}}{(2n+1)!}. \]

Therefore

\[ \sum_{n=0}^{\infty} b_2 \left( n, x, y, z; \vec{0}, r, -a, 0 \right) \frac{t^n}{n!} = \sum_{n=0}^{\infty} (-1)^z 2^{1-z} \sum_{j=0}^{n-1} (-1)^j \left( \begin{array}{c} n \\ 2j+1 \end{array} \right) (n-1-2j) y^{2j+1} E_{n-1-z-2j} (x; a) \frac{t^n}{n!}. \]

By using (52), we obtain the following result:

**Corollary 24** Let \( r \)-tuples \( \vec{0} = (0, 0, \ldots, 0) \). Then we have

\[ b_2 \left( n, x, y, z; \vec{0}, r, -a, 0 \right) = (-1)^z 2^{1-z} \sum_{j=0}^{n-1} (-1)^j \left( \begin{array}{c} n \\ 2j+1 \end{array} \right) (n-1-2j) y^{2j+1} E_{n-1-z-2j} (x; a). \]

4 Relations among the polynomials \( K(n; w, \vec{u}, r) \), trigonometric functions and hypergeometric function

In this section, we study the following two variable polynomials

\[ N_n (w) = K \left( n; w, \vec{0}, r \right) \quad (54) \]

where \( r \)-tuples \( \vec{0} = (0, 0, \ldots, 0) \), the polynomials \( K \left( n; w, \vec{0}, r \right) \) are given in equation (22). We set \( N_n (w) = N_n ((x, y)) \). We investigate some properties of the polynomials \( N_n (w) \). We give relations among the polynomials \( N_n (w) \), trigonometric functions and hypergeometric functions. The polynomials \( N_n (w) \) are also related to other special polynomials, such as the Milne-Thomson-type polynomials and the generalized Hermite-Kampé de Fériet polynomials.

A series representation of the polynomials \( N_n (w) \) is given by

\[ G(t, w) = \sum_{n=0}^{\infty} N_n (w) \frac{t^n}{n!} = \exp (wt). \quad (55) \]

Alternative forms of the above generating functions are given as follows:

\[ G(t, w) = {}_0F_0 \left[ - ; wt \right], \quad (56) \]
\[
G(t, w) = {}_0F_0 \left[ \begin{array}{c}
- \\
\end{array} ; xt \right] \left\{ {}_0F_1 \left[ \begin{array}{c}
\frac{-y^2}{4} \\
\end{array} ; -y^2t^2 \right] + it {}_0F_1 \left[ \begin{array}{c}
\frac{-y^2}{4} \\
\end{array} ; -y^2t^2 \right] \right\},
\]
\text{(57)}
and
\[
N_m (w) = \frac{\partial^m}{\partial t^m} \left\{ {}_0F_0 \left[ \begin{array}{c}
- \\
\end{array} ; wt \right] \right\} \bigg|_{t=0},
\]
where \( {}_pF_q \left[ \begin{array}{c}
\alpha_1, \ldots, \alpha_p \\
\beta_1, \ldots, \beta_q \\
\end{array} ; z \right] \) denotes hypergeometric function, defined by
\[
{}_pF_q \left[ \begin{array}{c}
\alpha_1, \ldots, \alpha_p \\
\beta_1, \ldots, \beta_q \\
\end{array} ; z \right] = \sum_{m=0}^{\infty} \frac{\prod_{j=1}^{p} (\alpha_j)_m}{\prod_{j=1}^{q} (\beta_j)_m} \frac{z^m}{m!}.
\]
\text{(58)}
A series in \( \text{(58)} \) converges for all \( z \) if \( p < q + 1 \), and for \( |z| < 1 \) if \( p = q + 1 \) and also all \( \beta_j, (j = 1, 2, \ldots, q) \) are real or complex parameters with \( \beta_j \not\in \mathbb{N} \) (cf. \[2], \[6], \[27], \[39], \[42\]).

By using the above generating functions, we obtain the following well-known identity:
\[
N_n (w) = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} (iy)^j = (x + iy)^n.
\]
\text{(59)}
Replacing \( w \) by \( \overline{w} \), we modify \( \text{(59)} \), we have
\[
N_n (\overline{w}) = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} (-iy)^j = (x - iy)^n.
\]
\text{(60)}
Observe that
\[
N_n (w) N_n (\overline{w}) = |w|^{2n}.
\]
By using the Riemann integral, we derive some identities and formulas including the polynomials \( N_n (w) \), the Bernoulli numbers and other special polynomials.

**Theorem 25** Let \( n \in \mathbb{N} \) and \( w = x + iy \). Then we have
\[
N_{n-1} (w) = \frac{\overline{w}}{x^2 + y^2} C_n (x, y) + \frac{iw + 2y}{x^2 + y^2} S_n (x, y).
\]
\text{(61)}
**Proof.** Integrating both sides of equation \( \text{(55)} \) from 0 to \( t \) with respect to the variable \( v \), we get
\[
\sum_{n=0}^{\infty} N_n (w) \int_{0}^{t} \frac{v^n}{n!} dv = \int_{0}^{t} e^{xv} \cos (yv) dv + i \int_{0}^{t} e^{xv} \sin (yv) dv.
\]
\text{(62)}
After some elementary calculations in the above equation, then combining with (3) and (4), respectively, we obtain

\[ \sum_{n=1}^{\infty} N_{n-1}(w) \frac{t^n}{n!} = \frac{w}{x^2 + y^2} \sum_{n=1}^{\infty} C_n(x, y) \frac{t^n}{n!} + \frac{iw + 2y}{x^2 + y^2} \sum_{n=1}^{\infty} S_n(x, y) \frac{t^n}{n!}. \]

Comparing the coefficients of \( \frac{t^n}{n!} \) on both sides of the above equation, we arrive at the desired result. 

**Theorem 26** Let \( n \in \mathbb{N} \) and \( w = x + iy \). Then we have

\[ B_{n-1}^{(-1)} = \frac{w^{1-n}}{(x^2 + y^2) n} (wC_n(x, y) + (iw + 2y) S_n(x, y)). \]  

**Proof.** Using (62) and (55), we get

\[ G(t, w) - 1 = \exp(wt) - 1. \]

Therefore

\[ \frac{\exp(wt) - 1}{w} = \frac{w}{x^2 + y^2} \sum_{n=1}^{\infty} C_n(x, y) \frac{t^n}{n!} + \frac{iw + 2y}{x^2 + y^2} \sum_{n=1}^{\infty} S_n(x, y) \frac{t^n}{n!}. \]

Combining the above equation with (1), we obtain

\[ \sum_{n=1}^{\infty} nw^{n-1} B_{n-1}^{(-1)} \frac{t^n}{n!} = \frac{w}{x^2 + y^2} \sum_{n=1}^{\infty} C_n(x, y) \frac{t^n}{n!} + \frac{iw + 2y}{x^2 + y^2} \sum_{n=1}^{\infty} S_n(x, y) \frac{t^n}{n!}. \]

Comparing the coefficients of \( \frac{t^n}{n!} \) on both sides of the above equation, we arrive at the desired result. 

Combining (61) with (63), we arrive at the following corollary:

**Corollary 27** Let \( n \in \mathbb{N} \) and \( w = x + iy \). Then we have

\[ N_{n-1}(w) = nw^{n-1} B_{n-1}^{(-1)}. \]  

By using (55), we get the following well-known identity for the numbers \( B_{n-1}^{(-1)} \):

**Corollary 28** Let \( n \in \mathbb{N}_0 \). Then we have

\[ B_n^{(-1)} = \frac{1}{n + 1}. \]  

**Remark 29** In work of Srivastava (cf. [24, Eq. (7.17)]), we have the following well-known formula including the Stirling numbers of the second kind and the Bernoulli numbers of order \(-k\):

\[ B_n^{(-k)} = \frac{1}{(n+k)} S_2(n+k,k). \]

Substituting \( k = 1 \) into the above formula, since \( S_2(n+1,1) = 1 \), we also arrive at (65).
The polynomials $K(n; w, \vec{u}, r)$ are linear combinations of the polynomials $N_n(w)$, presented by the following theorem.

**Theorem 30** Let $\vec{u} = (u_1, u_2, u_3, \ldots, u_r)$ and $w = x + iy$. Then we have

$$K(n; w, \vec{u}, r) = \sum_{j=0}^{n} \binom{n}{j} H_j(\vec{u}, r) N_{n-j}(w). \quad (66)$$

**Proof.** By using (20), (22) and (55), we derive the following functional equation:

$$G(t, w, \vec{u}, r) = G(t, w) F_R(t, \vec{u}, r). \quad (67)$$

From the above equation, we have

$$\sum_{n=0}^{\infty} K(n; w, \vec{u}, r) \frac{t^n}{n!} = \sum_{n=0}^{\infty} N_n(w) \frac{t^n}{n!} \sum_{n=0}^{\infty} H_n(\vec{u}, r) \frac{t^n}{n!}. \quad (68)$$

Therefore

$$\sum_{n=0}^{\infty} K(n; w, \vec{u}, r) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \sum_{j=0}^{n} \binom{n}{j} H_j(\vec{u}, r) N_{n-j}(w) \frac{t^n}{n!}. \quad (69)$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the desired result.

Similarly, the polynomials $h(n, w, z; \vec{0}, r, a, b)$, $h_1(n, w, z; \vec{0}, r, a, b)$, and $h_2(n, w, z; \vec{0}, r, a, b)$ are also linear combinations of the polynomials $N_n(w)$, presented as follows:

Substituting $\vec{u} = \vec{0}$ into (33), (36) and (37), after that combining the last equation with equation (54), we arrive at the following identities, respectively:

**Corollary 31**

$$h(n, w, z; \vec{0}, r, a, b) = \sum_{j=0}^{n} \binom{n}{j} y_6^{(z)}(n-j;a,b) N_j(w),$$

$$h_1(n, w, z; \vec{0}, r, a, b) = \sum_{j=0}^{n} \binom{n}{j} y_6^{(z)}(n-j;a,b) (N_j(w) + N_j(\overline{w})), \quad (70)$$

and

$$h_2(n, w, z; \vec{0}, r, a, b) = \sum_{j=0}^{n} \binom{n}{j} y_6^{(z)}(n-j;a,b) (N_j(w) - N_j(\overline{w})). \quad (71)$$
5 Relations among Hermite-type polynomials and Chebyshev-type polynomials and Dickson polynomials

In this section, we give relations among the Hermite-type polynomials, the generalized Hermite-Kampé de Fériet polynomials, and the Chebyshev polynomials. Let \( w = x + iy \). By using (59), we modify (66) as follows:

\[
K(n; x + iy, \vec{u}, r) = \sum_{j=0}^{n} \binom{n}{j} H_j(\vec{u}, r) \sum_{k=0}^{n-j} \binom{n-j}{k} x^{n-j-k} (iy)^k.
\]

Therefore, we define the following polynomials:

\[
P_1(n, x, y, \vec{u}, r) = \text{Re} \{ K(n; x + iy, \vec{u}, r) \}
\]

and

\[
P_2(n, x, y, \vec{u}, r) = \text{Im} \{ K(n; x + iy, \vec{u}, r) \}.
\]

Explicit formulas for these polynomials are given as follows:

\[
P_1(n, x, y, \vec{u}, r) = \sum_{j=0}^{n} \binom{n}{j} H_j(\vec{u}, r) \sum_{k=0}^{[n-j]} (-1)^k \binom{n-j}{2k} x^{n-j-2k} y^{2k} \tag{68}
\]

and

\[
P_2(n, x, y, \vec{u}, r) = \sum_{j=0}^{n} \binom{n}{j} H_j(\vec{u}, r) \sum_{k=0}^{[n-j-1]} (-1)^k \binom{n-j}{2k+1} x^{n-j-2k-1} y^{2k+1} \tag{69}
\]

Combining equations (68) and (69) with \(5\) and \(6\), respectively, we arrive at the following theorem:

**Theorem 32** Let \( \vec{u} = (u_1, u_2, u_3, \ldots, u_r) \). Then we have

\[
P_1(n, x, y, \vec{u}, r) = \sum_{j=0}^{n} \binom{n}{j} H_j(\vec{u}, r) C_{n-j}(x, y) \tag{70}
\]

and

\[
P_2(n, x, y, \vec{u}, r) = \sum_{j=0}^{n} \binom{n}{j} H_j(\vec{u}, r) S_{n-j}(x, y). \tag{71}
\]

Substituting \( y = \sqrt{1 - x^2} \) into (68) and (69), we obtain relations among the Chebyshev polynomials of the first kind \( T_n(x) \), the Chebyshev polynomials of the second kind \( U_n(x) \), the generalized Hermite-Kampé de Fériet polynomials \( H_n(\vec{u}, r) \) by the following theorem:
Theorem 33  Let \( \overrightarrow{u} = (u_1, u_2, u_3, \ldots, u_r) \). Then we have

\[
P_3(n, x, \overrightarrow{u}, r) = \sum_{j=0}^{n} \binom{n}{j} H_j(\overrightarrow{u}, r) T_{n-j}(x),
\]

where

\[
P_3(n, x, \overrightarrow{u}, r) = P_1 \left( n, x, \sqrt{1-x^2}, \overrightarrow{u}, r \right)
\]

and

\[
P_4(n, x, \overrightarrow{u}, r) = \sum_{j=0}^{n} \binom{n}{j} H_j(\overrightarrow{u}, r) U_{n-j-1}(x)
\]

where

\[
P_4(n, x, \overrightarrow{u}, r) = \frac{P_2 \left( n, x, \sqrt{1-x^2}, \overrightarrow{u}, r \right)}{\sqrt{1-x^2}}.
\]

Using the polynomials \( P_3(n, x, \overrightarrow{u}, r) \) and \( P_4(n, x, \overrightarrow{u}, r) \), we arrive at the following corollary:

Corollary 34  Let \( n \in \mathbb{N}_0 \). Then we have

\[
T_n(x) = C_n \left( x, \sqrt{1-x^2} \right) = \text{Re} \left\{ N_n \left( \left( x, \sqrt{1-x^2} \right) \right) \right\} \quad (72)
\]

and for \( n \geq 1 \)

\[
U_{n-1}(x) = \frac{S_n \left( x, \sqrt{1-x^2} \right)}{\sqrt{1-x^2}} = \frac{1}{\sqrt{1-x^2}} \text{Im} \left\{ N_n \left( \left( x, \sqrt{1-x^2} \right) \right) \right\}. \quad (73)
\]

By (15), (16), (72) and (73), we obtain the following result which are related to the Dickson polynomials, the polynomials \( C_n(x, y) \), \( S_n(x, y) \) and the polynomials \( N_n(w) \):

Corollary 35  Let \( n \in \mathbb{N}_0 \). Then we have

\[
D_n(2x, 1) = 2C_n \left( x, \sqrt{1-x^2} \right) = 2 \text{Re} \left\{ N_n \left( \left( x, \sqrt{1-x^2} \right) \right) \right\}.
\]

Corollary 36  Let \( n \in \mathbb{N} \). Then we have

\[
\mathcal{C}_{n-1}(2x, 1) = \frac{S_n \left( x, \sqrt{1-x^2} \right)}{\sqrt{1-x^2}} = \frac{1}{\sqrt{1-x^2}} \text{Im} \left\{ N_n \left( \left( x, \sqrt{1-x^2} \right) \right) \right\}.
\]

6  Identities and relations including Chebyshev polynomials and trigonometric polynomials

Here, we give some identities and formulas which are relations among the Chebyshev polynomials, the Dickson polynomials, the Bernoulli numbers, the Euler numbers and other special polynomials.

Substituting \( y = \sqrt{1-x^2} \) into Theorem 2.9 of [19], then combining (72) and (73), we have the following result:
Corollary 37 Let $n \in \mathbb{N}$. Then we have

$$U_{n-1}(x) = 2^{1-n} \sum_{j=1}^{n} \binom{n}{j} U_{j-1}(x) T_{n-j}(x). \quad (74)$$

By using (15), (16) and (74), we also obtain the following result:

Corollary 38 Let $n \in \mathbb{N}$. Then we have

$$E_{n-1}(2x, 1) = 2^{-n} \sum_{j=1}^{n} \binom{n}{j} E_{j-1}(2x, 1) D_{n-j}(2x, 1).$$

Substituting $y = \sqrt{1-x^2}$ into Theorem 1 of [20], we have

$$E^{(C)}_n(x, \sqrt{1-x^2}) = \sum_{j=0}^{n} \binom{n}{j} C_j(x, \sqrt{1-x^2}) E_{n-j}$$

and

$$E^{(S)}_n(x, \sqrt{1-x^2}) = \sum_{j=1}^{n} \binom{n}{j} S_j(x, \sqrt{1-x^2}) E_{n-j}.$$

Combining above equations with (72) and (73), respectively, we obtain the following results:

Corollary 39 Let $n \in \mathbb{N}_0$. Then we have

$$E^{(C)}_n(x, \sqrt{1-x^2}) = \sum_{j=0}^{n} \binom{n}{j} T_j(x) E_{n-j}. \quad (75)$$

Corollary 40 Let $n \in \mathbb{N}$. Then we have

$$E^{(S)}_n(x, \sqrt{1-x^2}) = \sqrt{1-x^2} \sum_{j=1}^{n} \binom{n}{j} U_{j-1}(x) E_{n-j}. \quad (76)$$

By using (15), (16), (75) and (76), we also obtain the following result:

Corollary 41

$$E^{(C)}_n(x, \sqrt{1-x^2}) = \frac{1}{2} \sum_{j=0}^{n} \binom{n}{j} D_j(2x, 1) E_{n-j}$$

and

$$E^{(S)}_n(x, \sqrt{1-x^2}) = \sqrt{1-x^2} \sum_{j=1}^{n} \binom{n}{j} E_{j-1}(2x, 1) E_{n-j}.$$
On the other hand, substituting \( y = \sqrt{1 - x^2} \) into Theorem 6 of \[20\], we have
\[
B_n^{(C)} (x, \sqrt{1 - x^2}) = \sum_{j=0}^{n} \binom{n}{j} C_j (x, \sqrt{1 - x^2}) B_{n-j}
\]
and
\[
B_n^{(S)} (x, \sqrt{1 - x^2}) = \sum_{j=1}^{n} \binom{n}{j} S_j (x, \sqrt{1 - x^2}) B_{n-j}.
\]
Combining above equations with (72) and (73), respectively, we obtain the following results:

**Corollary 42** Let \( n \in \mathbb{N}_0 \). Then we have
\[
B_n^{(C)} (x, \sqrt{1 - x^2}) = \sum_{j=0}^{n} \binom{n}{j} T_j (x) B_{n-j}. \tag{77}
\]

**Corollary 43** Let \( n \in \mathbb{N} \). The we have
\[
B_n^{(S)} (x, \sqrt{1 - x^2}) = \sqrt{1 - x^2} \sum_{j=1}^{n} \binom{n}{j} U_{j-1} (x) B_{n-j}. \tag{78}
\]

By using (15), (16), (77) and (78) we also obtain the following result:

**Corollary 44**
\[
B_n^{(C)} (x, \sqrt{1 - x^2}) = \frac{1}{2} \sum_{j=0}^{n} \binom{n}{j} D_j (2x, 1) B_{n-j}
\]
and
\[
B_n^{(S)} (x, \sqrt{1 - x^2}) = \sqrt{1 - x^2} \sum_{j=1}^{n} \binom{n}{j} E_{j-1} (2x, 1) B_{n-j}.
\]

By applying derivative operator to (3) and (4) with respect to \( x \), then with respect to \( y \), we obtain the following partial differential equation:
\[
\frac{\partial}{\partial x} F_C (t, x, y) = \sum_{n=0}^{\infty} \frac{\partial}{\partial x} C_n (x, y) \frac{t^n}{n!} = \sum_{n=0}^{\infty} nC_{n-1} (x, y) \frac{t^n}{n!}.
\]
Comparing the coefficients of \( \frac{t^n}{n!} \) on both sides of the above equation, we have
\[
\frac{\partial}{\partial x} C_n (x, y) = nC_{n-1} (x, y).
\]
Similarly, for \( n \geq 1 \), we have
\[
\frac{\partial}{\partial x} S_n (x, y) = nS_{n-1} (x, y),
\]
26
\[
\frac{\partial}{\partial y} C_n (x, y) = -n S_{n-1} (x, y), \quad (79)
\]
\[
\frac{\partial}{\partial y} S_n (x, y) = n C_{n-1} (x, y).
\]
Substituting \( y = \sqrt{1 - x^2} \) into (79), we obtain the following well-known identity as follows:
\[
T'_n (x) = n U_{n-1} (x)
\]
(cf. [30]).

**Theorem 45** Let \( n \geq 2 \). Then we have
\[
\frac{\partial^2}{\partial x \partial y} C_n (x, y) = -n (n - 1) S_{n-2} (x, y)
\]
and
\[
\frac{\partial^2}{\partial x \partial y} S_n (x, y) = n (n - 1) C_{n-2} (x, y).
\]

**Proof.** By applying derivative operator to (3) and (4) with respect to \( x \) and \( y \), we obtain the following partial differential equations, respectively:
\[
\frac{\partial^2}{\partial x \partial y} F_C (t, x, y) = -t^2 F_S (t, x, y)
\]
and
\[
\frac{\partial^2}{\partial x \partial y} F_S (t, x, y) = t^2 F_C (t, x, y).
\]
From the above functional equations, we obtain
\[
\sum_{n=0}^{\infty} \frac{\partial^2}{\partial x \partial y} C_n (x, y) \frac{t^n}{n!} = - \sum_{n=0}^{\infty} n (n - 1) S_{n-2} (x, y) \frac{t^n}{n!}
\]
and
\[
\sum_{n=0}^{\infty} \frac{\partial^2}{\partial x \partial y} S_n (x, y) \frac{t^n}{n!} = \sum_{n=0}^{\infty} n (n - 1) C_{n-2} (x, y) \frac{t^n}{n!}.
\]
Comparing the coefficients of \( \frac{t^n}{n!} \) on both sides of the above equations, we arrive at the desired result.

**Theorem 46** Let \( n \in \mathbb{N}_0 \). Then we have
\[
C_{n+1} (x, y) = x C_n (x, y) - y S_n (x, y).
\]
(80)
Proof. By applying derivative operator to (3) with respect to $t$, we obtain the following partial differential equation:

$$\frac{\partial}{\partial t} F_C (t, x, y) = x F_C (t, x, y) - y F_S (t, x, y).$$

From the above equation, we have

$$\sum_{n=0}^{\infty} C_{n+1} (x, y) \frac{t^n}{n!} = x \sum_{n=0}^{\infty} C_n (x, y) \frac{t^n}{n!} - y \sum_{n=0}^{\infty} S_n (x, y) \frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the desired result. $\blacksquare$

Remark 47 By using (72), (73) and (80), we arrive at the equation (11).

Theorem 48 Let $n \in \mathbb{N}_0$. Then we have

$$S_{n+1} (x, y) = x S_n (x, y) + y C_n (x, y). \quad (81)$$

Proof. By applying derivative operator to (4) with respect to $t$, we obtain the following partial differential equation:

$$\frac{\partial}{\partial t} F_S (t, x, y) = x F_S (t, x, y) + y F_C (t, x, y).$$

From the above equation, we get

$$\sum_{n=0}^{\infty} S_{n+1} (x, y) \frac{t^n}{n!} = x \sum_{n=0}^{\infty} S_n (x, y) \frac{t^n}{n!} + y \sum_{n=0}^{\infty} C_n (x, y) \frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the desired result. $\blacksquare$

Remark 49 By using (72), (73) and (81), we arrive at the equation (9). On the other hand, multiplying (80) by $x$ and (81) by $y$ and then side-by-side adding, and multiplying (80) by $y$ and (81) by $x$ and then side-by-side subtracting, then after some calculation, we arrive at the equation (61).

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