AUTOMORPHISMS OF GENERIC GRADIENT VECTOR FIELDS
WITH PRESCRIBED FINITE SYMMETRIES

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Abstract. Let $M$ be a compact and connected smooth manifold endowed with a
smooth action of a finite group $\Gamma$, and let $f$ be a $\Gamma$-invariant Morse function on $M$. We
prove that the space of $\Gamma$-invariant Riemannian metrics on $M$ contains a residual subset
$\text{Met}_f$ with the following property. Let $g \in \text{Met}_f$ and let $\nabla^g f$ be the gradient vector
field of $f$ with respect to $g$. For any diffeomorphism $\phi \in \text{Diff}(M)$ preserving $\nabla^g f$ there
exists some $t \in \mathbb{R}$ and some $\gamma \in \Gamma$ such that for every $x \in M$ we have $\phi(x) = \gamma \Phi^x_t(x)$,
where $\Phi^x_t$ is the time-$t$ flow of the vector field $\nabla^g f$.

Contents

1. Introduction 1
2. Proof of Theorem 1.1 for dim $M = 1$ 5
3. Equivariant Sternberg’s linearisation theorem for families 9
4. The space of metrics $\text{Met}_0$ 11
5. The spheres $S_g(p)$, the distributions $A_g(p)$, and the sets $F_g(p)$ 14
6. The space of metrics $\text{Met}_{1,K}$ 16
7. Proof of Theorem 1.1 for dim $M > 1$ 20
Appendix A. Change of the gradient flow as the metric varies 22
Appendix B. Glossary 24
References 25

1. Introduction

Define the automorphism group of a vector field on a smooth manifold to be the group
of diffeomorphisms of the manifold preserving the vector field. A natural question is how
small the automorphism group of a vector field can be. Suppose that we only consider
vector fields which are invariant under a fixed finite group action on the manifold. In
this situation, the automorphism group of the vector field always includes the action of
the finite group and the flow of the vector field. Our main result implies that if the
manifold is compact and connected then the set of invariant gradient vector fields whose

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automorphism group contains nothing more than this is residual in the set of all invariant gradient vector fields. (Recall that a residual subset is a countable intersection of dense open subsets. Since the space of smooth invariant gradient vector fields is Baire, any of its residual subsets is dense.)

Let us explain in more concrete terms our motivation and main result.

Take $M$ to be a smooth ($\in C^\infty$) manifold, and denote by $\mathfrak{X}(M)$ the vector space of smooth vector fields on $M$, endowed with the $C^\infty$ topology. Denote the automorphism group of a vector field $X \in \mathfrak{X}(M)$ by

$$\text{Aut}(X) = \{ \phi \in \text{Diff}(M) \mid \phi_*X = X \}.$$ 

If $X \neq 0$ then $\text{Aut}(X)$ contains a central subgroup isomorphic to $\mathbb{R}$, namely the flow generated by $X$. Denote by $\text{Aut}(X)/\mathbb{R}$ the quotient of $\text{Aut}(X)$ by this subgroup.

Suppose that $M$ is endowed with a smooth and effective action of a finite group $\Gamma$. Let $\mathfrak{X}(M)^\Gamma \subset \mathfrak{X}(M)$ be the space of $\Gamma$-invariant vector fields on $M$. F.J. Turiel and A. Viruel proved recently in [22] that there exists some $X \in \mathfrak{X}(M)^\Gamma$ such that $\text{Aut}(X)/\mathbb{R} \cong \Gamma$. The vector field $X$ is given explicitly in [22] as a gradient vector field for a carefully constructed Morse function and a suitable Riemannian metric. One may wonder, in view of that result, whether the set of $X \in \mathfrak{X}(M)^\Gamma$ satisfying $\text{Aut}(X)/\mathbb{R} \cong \Gamma$ is generic in some sense, at least if one restricts to some particular family of vector fields (such as gradient vector fields, for example). Here we give an affirmative answer to this question assuming $M$ is compact and connected.

Suppose, for the rest of the paper, that $M$ is compact and connected. For any function $f \in C^\infty(M)$ and any Riemannian metric $g$ on $M$ we denote by $\nabla^g f \in \mathfrak{X}(M)$ the gradient of $f$ with respect to $g$, defined by the condition that $g(\nabla^g f, v) = df(v)$ for every $v \in TM$. The following is our main result.

**Theorem 1.1.** Let $\text{Met} \subset C^\infty(M, S^2TM)$ denote the space of Riemannian metrics on $M$, with the $C^\infty$ topology, and let $\text{Met}^\Gamma \subset \text{Met}$ denote the space of $\Gamma$-invariant metrics. Let $f$ be a $\Gamma$-invariant Morse function on $M$. There exists a residual subset $\text{Met}_f \subset \text{Met}^\Gamma$ such that any $g \in \text{Met}_f$ satisfies $\text{Aut}(\nabla^g f)/\mathbb{R} \cong \Gamma$.

By a result of Wasserman [24, Lemma 4.8], the space of $\Gamma$-invariant Morse functions on $M$ is open and dense within the space of all $\Gamma$-invariant smooth functions on $M$ (for openness see the comments before Lemma 4.8 in [24]). Combining this result with Theorem 1.1 it follows that if $\mathfrak{G}(M)^\Gamma \subset \mathfrak{X}(M)^\Gamma$ denotes the set of $\Gamma$-invariant gradient vector fields, then the set of $X \in \mathfrak{G}(M)^\Gamma$ satisfying $\text{Aut}(X)/\mathbb{R} \cong \Gamma$ contains a residual subset in $\mathfrak{G}(M)^\Gamma$.

Probably Theorem 1.1 can be proved as well for most proper $\Gamma$-invariant Morse functions on open manifolds endowed with an smooth effective action of a finite group. However, there are exceptions: if $M = \mathbb{R}^n$ with $n > 1$, and $\Gamma$ is a finite group acting linearly on $M$ preserving the standard euclidean norm, then $f : M \to \mathbb{R}$, $f(v) = ||v||^2$, is a $\Gamma$-invariant Morse function, and for any $\Gamma$-invariant Riemannian metric $g$ on $M$, $\text{Aut}(\nabla^g f)/\mathbb{R}$ is bigger than $\Gamma$. This follows from a result of Sternberg, see Theorem 3.1 below.

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1. See e.g. [9, Chap. 2, Theorem 4.4] and the comments afterwards.
For the case \( \dim M = 1 \) (i.e., when \( M \) is the circle) we prove a stronger form of Theorem 1.1 where residual is replaced by open and dense, see Theorem 2.1. It is not inconceivable that this can be done in all dimensions: while the author has not managed to do so, he does not have any reason to suspect that it might be false. In fact, a theorem of Palis and Yoccoz answering an analogous question (in the non-equivariant setting), with the set of gradient vector fields \( \mathcal{X}(M) \) replaced by a certain set of diffeomorphisms may suggest that it is true. To be precise, let \( \mathcal{A}_1(M) \subset \text{Diff}(M) \) be the (open) set of Axiom A diffeomorphisms satisfying the transversality condition and having a sink or a source. Palis and Yoccoz prove in [14] that the set of diffeomorphisms with the smallest possible centralizer contains a \( C^\infty \) open and dense subset of \( \mathcal{A}_1(M) \). Note that the set \( \mathcal{A}_1(M) \) includes Morse–Smale diffeomorphisms, which are analogues for diffeomorphisms of gradient vector fields. However, if one considers the same question for the entire diffeomorphism group endowed with the \( C^1 \) topology then openness never holds, see [4].

Before explaining the main ideas in the proof of Theorem 1.1, let us discuss some related problems. A natural question is whether the property proved in Theorem 1.1 is true when replacing the space of invariant gradient vector fields by the entire space of invariant vector fields.

**Problem A.** Does the set of \( \mathcal{X} \in \mathcal{X}(M)^\Gamma \) satisfying \( \text{Aut}(\mathcal{X})/\mathbb{R} \simeq \Gamma \) contain a residual subset of \( \mathcal{X}(M)^\Gamma \)?

Problem A includes Theorem 1.1 as a particular case, since \( \mathcal{G}(M)^\Gamma \) is open in \( \mathcal{X}(M)^\Gamma \). Note that the case \( \Gamma = \{1\} \) of Problem A (or even Theorem 1.1) is far from being trivial.

Define the centralizer \( Z(\mathcal{X}) \) of a vector field \( \mathcal{X} \) on \( M \) to be the group of diffeomorphisms of \( M \) that send orbits of \( \mathcal{X} \) onto orbits of \( \mathcal{X} \). For any \( \mathcal{X} \), \( \text{Aut}(\mathcal{X}) \) is a subgroup of \( Z(\mathcal{X}) \), and one may try to explore analogues of Theorem 1.1 and Problem A for \( Z(\mathcal{X}) \). However, even the right question to ask is not clear in this situation. P.R. Sad [16] studied the case \( \Gamma = \{1\} \). His main result is that for a compact \( M \) there is an open and dense subset \( \mathcal{A}' \) of the set of Morse–Smale vector fields \( \mathcal{A} \subset \mathcal{X}(M) \) such that for any \( \mathcal{X} \in \mathcal{A}' \) there is a neighborhood \( V \subset \text{Diff}(M) \) of the identity with the property that any \( \phi \in V \cap Z(\mathcal{X}) \) preserves the orbits of \( \mathcal{X} \). Unfortunately the restriction to a neighborhood of the identity in \( \text{Diff}(M) \) can not be removed, as Sad shows with an example.

It is natural to consider analogues of the previous problems replacing vector fields by diffeomorphisms. Define the automorphism group of a diffeomorphism \( \phi \in \text{Diff}(M) \) to be its centralizer, i.e., \( \text{Aut}(\phi) = \{ \psi \in \text{Diff}(M) \mid \phi \psi = \psi \phi \} \). Then \( \langle \phi \rangle = \{ \phi^k \mid k \in \mathbb{Z} \} \) is a central subgroup of \( \text{Aut}(\phi) \). Let

\[
\text{Diff}^\Gamma(M) = \{ \phi \in \text{Diff}(M) \mid \phi \text{ commutes with the action of } \Gamma \}.
\]

**Problem B.** Does the set of \( \phi \in \text{Diff}^\Gamma(M) \) such that \( \text{Aut}(\phi)/\langle \phi \rangle \simeq \Gamma \) contain a residual subset of \( \text{Diff}^\Gamma(M) \)?

Of course, a positive answer to Problem B does not imply a positive answer to Problem A, since a diffeomorphism \( \phi \) such that \( \text{Aut}(\phi)/\langle \phi \rangle = \Gamma \) can not possibly belong to the flow of a vector field (for otherwise \( \text{Aut}(\phi) \) should contain a subgroup isomorphic to \( \mathbb{R} \)).

One may consider restricted versions of Problem B involving particular diffeomorphisms, for example, equivariant Morse–Smale diffeomorphisms [8]. These are very particular diffeomorphisms, but Problem B is already substantially nontrivial for them (even in the case \( \Gamma = \{1\} \), see below).
Problems A and B admit variations in which the regularity of the vector fields or the diffeomorphisms is relaxed from $C^\infty$ to $C^r$ for finite $r$. One can also consider stronger questions replacing residual by open and dense or weaker ones replacing residual by dense.

The case $\Gamma = \{1\}$ of Problem B is a famous question of Smale. It appeared for the first time in [17, Part IV, Problem (1.1)], in more elaborate form in [18], and it was included in his list of 18 problems for the present century [19]. It was solved for Morse–Smale $C^1$-diffeomorphisms by Togawa [21] and very recently for arbitrary $C^1$-diffeomorphisms by C. Bonatti, S. Crovisier, A. Wilkinson in [2] (see the survey [3] for further references). The analogous problem for higher regularity diffeomorphisms is open at present, although there are by now plenty of partial results: see e.g. [11] for the case of the circle, [14] for elements in the set $A_1(M)$ defined above, and [15] for Anosov diffeomorphisms of tori.

Theorem 1.1 may be compared to similar results for other types of tensors. For example, it has been proved in [12] that on a compact manifold the set of metrics of fixed signature with trivial isometry group is open and dense in the space of all such metrics (see also [7] for an infinitesimal version of this with the compactness condition removed).

1.1. Main ideas of the proof. To prove Theorem 1.1 we treat separately the cases $\dim M = 1$ and $\dim M > 1$. The case $\dim M = 1$ is addressed in Section 2 using rather ad hoc methods. An interesting ingredient is an invariant of vector fields which, when nonzero, distinguishes changes of orientation, and which plays an important role in the classification up to diffeomorphisms of vector fields on $S^1$ with nondegenerate zeroes.

The main ingredient in the case $\dim M > 1$, common to other papers addressing similar problems, is a theorem of Sternberg [20] on linearisation of vector fields near sinks and sources, assuming there are no resonances. The use of this result in this kind of problems goes back to work of Kopell [11], and appears in papers of Anderson [1] and Palis and Yoccoz [14] among others. To apply this theorem in our situation we need to generalize it to the equivariant setting under the presence of finite symmetries. This poses some difficulties. For example, in the equivariant case we can not suppose that the eigenvalues of the linearisation of a generic vector field at fixed points are all different: high multiplicities can not be avoided; in particular, the centraliser of the linearisation is not necessarily abelian (both Anderson [1] and Palis–Yoccoz [14] restrict themselves to the case in which the eigenvalues are different). This is relevant for example when extending the version of Sternberg’s theorem for families proved by Anderson to the equivariant setting (see Section 3 for details on this).

We close this subsection with a more concrete description of the proof of the case $\dim M > 1$. Suppose a $\Gamma$-invariant Morse function $f$ has been chosen. The set of metrics $\text{Met}_f$ is defined as the intersection of a set of invariant metrics, $\text{Met}_0$, and a countable sequence of subsets $\{\text{Met}_1, K\}_{K \in \mathbb{N}}$. Each of these sets is open and dense in $\text{Met}^\Gamma$.

The metrics $g \in \text{Met}_0$, defined in Subsection 4.3, have two properties: (1) the eigenvalues of the differential of $V^g f$ at each critical point are as much different among themselves as they can be (in particular, the collection of eigenvalues at two critical points coincide if and only if the two points belong to the same $\Gamma$-orbit), and (2) there are no resonances among eigenvalues at any critical point. The second property allows us to use Sternberg’s theorem on linearisation on neighborhoods of sinks and sources, and a theorem of
Kopell which limits enormously the automorphisms of the gradient vector field restricted to (un)stable manifolds of sinks/sources.

The metrics \( g \in \text{Met}_{1,K} \), defined in Subsection 6, have the following property. Suppose that \( p \) is a sink and \( W^s_g(p) \) is the stable manifold of \( p \) for \( \nabla^g f \), and that \( q \) is a source and \( W^u_g(q) \) is its unstable manifold for \( \nabla^g f \). If \( W^s_g(p) \cap W^u_g(q) \) is nonempty, then any automorphism of \( \nabla^g f|_{W^s_g(p)} \) whose derivative at \( p \) is at distance \( \leq K \) from the identity and which matches on \( W^s_g(p) \cap W^u_g(q) \) with an automorphism of \( \nabla^g f|_{W^u_g(q)} \) is at distance \( < K^{-1} \) from an automorphism coming from the action of \( \Gamma \) and the flow of \( \nabla^g f \).

After defining these sets of metrics, in Section 7 we prove Theorem 1.1 for manifolds of dimension greater than one, showing that if \( \Gamma \) is a finite group \( \Gamma \) acts smoothly and effectively on \( S^1 \), then \( \text{Aut}(\nabla^g f)/\Gamma \simeq \Gamma \).

The paper concludes with two appendices. The first one gives the proof of a technical result on the variation of the gradient flow of \( \nabla^g f \) with respect to variations of \( g \), and the second one contains a glossary of the notation used to address the case \( \dim M > 1 \) (Section 8 and the next ones).

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2. Proof of Theorem 1.1 for \( \dim M = 1 \)

In this section we prove a strengthening of the case \( \dim M = 1 \) of Theorem 1.1. More concretely, in Subsection 2.2 below we prove the following.

**Theorem 2.1.** Suppose that a finite group \( \Gamma \) acts smoothly and effectively on \( S^1 \). Let \( f \) be a \( \Gamma \)-equivariant Morse function on \( S^1 \). Let \( \text{Met}^\Gamma \) denote the set of \( \Gamma \)-invariant Riemannian metrics on \( S^1 \), endowed with the \( C^\infty \) topology. There exists a dense and open subset \( \text{Met}_f \subset \text{Met}^\Gamma \) such that for every \( g \in \text{Met}_f \) we have \( \text{Aut}(\nabla^g f)/\Gamma \simeq \Gamma \).

2.1. Classifying nondegenerate vector fields on the circle. To prove Theorem 2.1 we will need, in the case when \( \Gamma \) is generated by a rotation, an invariant of nondegenerate vector fields on the circle that detects change of orientations. This invariant is one of the ingredients of the classification of nondegenerate vector fields on the circle up to orientation preserving diffeomorphism. Detailed expositions of this classification (in the broader context of vector fields with zeroes of finite order) have appeared in [6, 10]. Here we briefly explain the main ideas of this result, focusing on the definition of the invariant, both for completeness and to set the notation for later use.

For any \( t \in \mathbb{R} \) and vector field \( \mathcal{X} \) we denote by \( \Phi^\mathcal{X}_t \in \text{Aut}(\mathcal{X}) \) the flow of \( \mathcal{X} \) at time \( t \).

We first consider the local classification of vector fields with a nondegenerate zero. For any nonzero real number \( \lambda \) we denote by \( \mathcal{F}_\lambda \) the set of germs of vector fields on a neighborhood of 0 in \( \mathbb{R} \) of the form \( h \partial_x \), where \( h(0) = 0 \) and \( h'(0) = \lambda \) and \( x \) is the standard coordinate in \( \mathbb{R} \). Let \( \mathcal{G} \) denote the group of germs of diffeomorphisms of neighborhoods of 0 in \( \mathbb{R} \). For any \( \mathcal{X} \in \mathcal{F}_\lambda \) we denote by \( \text{Aut}(\mathcal{X}) \) the group of all \( \phi \in \mathcal{G} \) such that \( \phi_* \mathcal{X} = \mathcal{X} \). For example, \( \Phi^\mathcal{X}_t \in \text{Aut}(\mathcal{X}) \) for every \( t \). The proof of the next lemma follows from a straightforward computation and Cauchy’s theorem on ODE’s.

**Lemma 2.2.** Let \( \lambda, \mu \) be nonzero real numbers.

\[ \text{Aut}(\mathcal{F}_\lambda) \simeq \mathbb{R} \]
Given \( X \in \mathcal{F}_\lambda \) and \( Y \in \mathcal{F}_\mu \) there exists some \( \phi \in \mathcal{G} \) satisfying \( \phi \ast X = Y \) if and only if \( \lambda = \mu \).

For any \( \lambda \) and \( X \in \mathcal{F}_\lambda \) the map \( D : \text{Aut}(X) \to \mathbb{R}^* \) sending \( \phi \) to \( \phi'(0) \) is an isomorphism of groups. Furthermore, \( D \Phi_t^X = e^{\lambda t} \).

We mention in passing that to prove the case \( \dim M > 1 \) of Theorem 1.1 we will need to extend the previous lemma to higher dimensions, in a way equivalent with respect to finite group actions. This extension will be based on non-equivariant higher dimensional analogues of statements (1) and (2), which are respectively a theorem of Sternberg (see [20] and Theorem 3.1 below) and a theorem of Kopell (see [11] and Subsection 4.3 below). Both results are substantially deeper than Lemma 2.2, and in particular they require a condition of non-resonance which is trivial in the one dimensional case.

We next explain the classification of nondegenerate vector fields on the circle. We identify \( S^1 = \mathbb{R}/2\pi\mathbb{Z} \), so vector fields on \( S^1 \) can be written as \( X = h \partial_x \) where \( h \) is a \( 2\pi \)-periodic smooth function. We say that \( X \) is nondegenerate if \( h(y) = 0 \) implies \( h'(y) \neq 0 \) (\( h'(y) \) can be identified with the derivative of \( X \) at \( y \in h^{-1}(0) \)). An immediate consequence is that \( h \) contains finitely many zeroes in \( [0, 2\pi) \). Another consequence is that \( h \) changes sign when crossing any zero of \( h \), and this implies that \( h^{-1}(0) \) contains an even number of elements in \( [0, 2\pi) \). To classify nondegenerate vector fields on the circle we will associate to them the number of zeroes, their derivatives at the zeroes (up to cyclic order), and a global invariant denoted by \( \chi \).

To define \( \chi \) suppose first of all that \( h \) has no zeroes. Denoting by \( \Phi_t^X \) the flow of \( X \) seen as a vector field on \( \mathbb{R} \), there is a unique real number \( t \) such that \( \Phi_t^X(y) = y + 2\pi \) for every \( y \in \mathbb{R} \). Then we set

\[
\chi(X) := t.
\]

Now suppose that \( h \) vanishes somewhere, and write its zeroes contained in \( [0, 2\pi) \) as

\[
0 \leq z_1 < z_2 < \cdots < z_{2r} < 2\pi.
\]

We extend this finite list to an infinite sequence by setting \( z_{i+2r} = z_i \) for every integer \( i \). Below, we implicitly consider similar periodic extensions for all objects that we are going to associate to the zeroes \( z_i \). By (2) in Lemma 2.2, for every \( i \) there exists a connected neighborhood \( U_i \) of \( z_i \), disjoint from \( z_{i-1} \) and \( z_{i+1} \), and a unique smooth involution \( \sigma_i : U_i \to U_i \) such that

\[
\sigma_i(z_i) = z_i, \quad \sigma_i'(z_i) = -1, \quad (\sigma_i)_* X = X.
\]

Choose for every \( i \) some \( t_i^+ > z_i \) contained in \( U_i \) and define \( t_i^- = \sigma_i(t_i^+) \). Then we have \( t_i^+, t_{i+1}^- \in (z_i, z_{i+1}) \), so there is a unique real number \( \rho_i \) such that

\[
t_{i+1}^- = \Phi_{\rho_i}^X(t_i^+).
\]

Note that \( \rho_i \) has the same sign as \( h'(z_i) \). Now we define

\[
\chi(X) := \sum_{i=1}^{2r} \rho_i.
\]
Lemma 2.3. The number $\chi(X)$ only depends on $X$, and not on the choices of $t_i^\pm$. Furthermore, endowing the set of generic vector fields with the $C^\infty$ topology the map $X \mapsto \chi(X)$ is continuous.

Proof. We first prove that $\chi(X)$ does not depend on the choices of $t_i^\pm$. If for any $i$ we replace $t_i^\pm$ by $(t_i')^\pm$, then the requirement that $(t_i')^- = \sigma_i((t_i')^+)$ implies that $(t_i')^\pm = \Phi_{\pm \delta}(t_i^\pm)$ for some $\delta$, so $\rho_i$ gets replaced by $\rho_i - \delta$ and $\rho_{i-1}$ gets replaced by $\rho_{i-1} + \delta$, and hence (2) remains unchanged.

To prove that $\chi(X)$ depends continuously on $X$ we first observe that any other vector field sufficiently close to $X$ is also generic and has vanishing locus close to that of $X$. Hence, once we have fixed the intervals $U_i$ and points $t_i^\pm$ above, there is a neighborhood $V$ of $X$ in the space of all vector fields on the circle such that if $Y \in V$ and we write $Y = k \partial_x$ then $k^{-1}(0) \subset \bigcup U_i$, each $U_i$ contains a unique zero $w_i$ of $k$, and $w_i < t_i^-$. So it suffices to prove that given $\delta > 0$, choosing $V$ small enough, the involution $\sigma_i^Y$ satisfying (1) with $X$ resp. $z_i$ replaced by $Y$ resp. $w_i$ has the property that $\sigma_i^Y(t_i^+)$ is well defined and at distance $< \delta$ from $\sigma_i(t_i^+)$.

The previous property will follow if we prove that $\sigma_i$ depends continuously on $X$. This is a local question, so let us assume that $X$ is a vector field defined on an open interval $0 \in I \subset \mathbb{R}$ with $X = g \partial_x$ and satisfying $g^{-1}(0) = \{0\}$ and $g'(0) \neq 0$; by Lemma 2.2 there is an open interval $0 \in J \subset \mathbb{R}$ and a smooth embedding $\phi : J \rightarrow I$ such that $\phi_*(\lambda x \partial_x) = X$ for some nonzero real $\lambda$. It is easy to check that both $\phi$ and $\lambda$ depend continuously on $X$. Take $U = \phi(J \cap -J)$. The map $\sigma : U \rightarrow U$ defined as $\sigma(x) = \phi(-\phi^{-1}(x))$ is a smooth involution of $U$ and it satisfies $\sigma_\lambda X = X$. By the previous observations it is clear that $\sigma$ depends continuously on $X$. \hfill $\Box$

This is the classification theorem of nondegenerate vector fields on $S^1$:

Theorem 2.4. Given two vector fields $X$ and $Y$ on the circle, there exists an orientation preserving diffeomorphism $\phi \in \text{Diff}^+(S^1)$ satisfying $\phi_* X = Y$ if and only if $X$ and $Y$ have the same number of zeroes, the collection of derivatives at the zeroes of $X$ and $Y$, travelling along $S^1$ counterclockwise, coincide up to a cyclic permutation, and $\chi(X) = \chi(Y)$.

We are not going to prove the previous theorem. In fact we will only use the "only if" part of it, which is rather obvious from the definitions; the proof of the "if" part is an easy exercise using Lemma 2.2. See [6, 10] for detailed proofs of a more general result.

We close this subsection with another result that will be used in the proof of Theorem 2.1. Suppose that $h : \mathbb{R} \rightarrow \mathbb{R}$ is a smooth function such that $h(0) = h(1) = 0$, and that $h$ does not vanish on the open interval $(0, 1)$. Let $X = h \partial_x$. The next lemma follows easily from Cauchy’s theorem on ODE’s.

Lemma 2.5. Any diffeomorphism $\phi : (0, 1) \rightarrow (0, 1)$ satisfying $\phi_* X = X$ is equal to $\Phi_t^X$ for some $t \in \mathbb{R}$. In particular if a diffeomorphism $\phi : (0, 1) \rightarrow (0, 1)$ satisfying $\phi_* X = X$ is the identity on an open subset of $(0, 1)$ then $\phi$ is the identity on the entire $(0, 1)$.

2.2. Proof of Theorem 2.1. Let $\Gamma$ be a finite group acting smoothly and effectively on $S^1$, and let $f : S^1 \rightarrow \mathbb{R}$ be a $\Gamma$-invariant Morse function. Let $\text{Crit}(f)$ be the set of critical points of $f$. Any $\Gamma$-invariant Riemannian metric in $\text{Met}^\Gamma$ is isometric to the round circle $\{x^2 + y^2 = r^2\}$.
for some $r > 0$, and this allows to identify the action of $\Gamma$ on $S^1$ with the action of a cyclic or a dihedral group. We treat separately the two possibilities.

2.2.1. Dihedral groups. Suppose first that $\Gamma$ is dihedral. Then $\Gamma$ contains elements that reverse the orientation. Let $p \in S^1$ be a fixed point on an orientation reversing element of $\Gamma$, and let $\Gamma_0 \subset \Gamma$ be the subgroup of the elements which act preserving the orientation. Since $[\Gamma : \Gamma_0] = 2$, we have $\Gamma p = \Gamma_0 p$. On the other hand, $p$ is necessarily a critical point of $f$, because $f$ is $\Gamma$-invariant.

Let $\text{Met}_f$ be the set of metrics $g \in \text{Met}^\Gamma$ satisfying:

$$(3) \quad \text{if } x, y \in \text{Crit}(f) \text{ and } D\nabla^g f(x) = D\nabla^g f(y) \text{ then } \Gamma x = \Gamma y.$$  

It is clear that $\text{Met}_f$ is open and dense in $\text{Met}^\Gamma$. Now suppose that $g \in \text{Met}_f$ and let $X = \nabla^g f$. To prove that $\text{Aut}(X)/\mathbb{R} \simeq \Gamma$ we consider an arbitrary $\phi \in \text{Aut}(X)$ and show that composing $\phi$ with the action of suitably chosen elements of $\Gamma$ and with the flow $\Phi^X_t$ for some $t$ we obtain the identity.

Let $\phi \in \text{Aut}(\nabla^g f)$. Composing $\phi$ with the action of some $\gamma \in \Gamma$ we may assume that $\phi$ is orientation preserving. By (3) we have $\phi(p) \in \Gamma p$. Since $\Gamma p = \Gamma_0 p$, up to composing $\phi$ with the action of some element of $\Gamma_0$ we can assume that $\phi$ preserves the orientation and fixes $p$. This implies that $\phi$ fixes all critical points of $f$.

Let us label counterclockwise the critical points of $f$ as $p_1, p_2, \ldots, p_{2r}$. By Lemma 2.2 up to composing $\phi$ with $\Phi^X_t$ for some choice of $t$ we may assume that $\phi$ is the identity on a neighborhood of $p_1$. This implies that $\phi$ is the identity on the entire circle. Indeed, by Lemma 2.5 $\phi$ is the identity on the arc from $p_1$ to $p_2$, so by Lemma 2.2 $\phi$ is the identity on a neighborhood of $p_2$. We next apply Lemma 2.5 to the arc from $p_2$ to $p_3$ and conclude that the restriction of $\phi$ to this arc is equal to the identity. An so on, until we have traveled around the entire circle.

2.2.2. Cyclic groups. Suppose that $\Gamma$ is a cyclic group. The only case in which $\Gamma$ can contain orientation reversing elements is that in which $\Gamma$ consists of two elements, the nontrivial one being an orientation reversing involution of $S^1$. This situation can be addressed with the arguments of the previous case, so let us assume here that all elements of $\Gamma$ preserve the orientation. Then we define $\text{Met}_f$ to be the set of metrics $g \in \text{Met}^\Gamma$ satisfying property (3) above and $\chi(\nabla^g f) \neq 0$.

We claim that $\text{Met}_f$ is open and dense in $\text{Met}^\Gamma$. Since the set of metrics $g \in \text{Met}^\Gamma$ satisfying property (3) is open and dense, to see that $\text{Met}_f$ is dense it suffices to observe that if for some choice of $g$ we have $\chi(\nabla^g f) = 0$ then slightly modifying $g$ away from the critical points we may force $\chi$ to take a nonzero value; furthermore, the modification of $g$ can be made $\Gamma$-invariant because $\Gamma$ is generated by a rotation (note that, in contrast, if $\Gamma$ is a dihedral group then for any $\Gamma$-invariant metric $g$ we have $\chi(\nabla^g f) = 0$). Openness of $\text{Met}_f$ follows from the second statement in Lemma 2.3.

Let $g \in \text{Met}_f$, let $X = \nabla^g f$, and let $\phi \in \text{Aut}(X)$. We claim that $\phi$ is orientation preserving. Indeed, for any orientation reversing diffeomorphism $\psi$ of $S^1$ we have $\chi(\psi^* X) = -\chi(X)$ and since $\chi(X) \neq 0$, we can not possibly have $\psi^* X = X$. Let $p$ be any critical point of $f$. By (3) we have $\phi(p) \in \Gamma p$, so up to composing $\phi$ with the action of some element of $\Gamma$ we can assume that $\phi(p) = p$. Then, since $\phi$ preserves the orientation,
it fixes all the other critical points, and the argument is concluded as in the case of dihedral groups. Hence the proof of Theorem 2.1 is now complete.

In the remainder of the paper we are going to assume that $\dim M > 1$.

3. Equivariant Sternberg’s linearisation theorem for families

The following is Sternberg’s linearisation theorem [20, Theorem 4], which extends to the smooth setting an analytic result proved by Poincaré in his thesis:

**Theorem 3.1 (Sternberg).** Let $0 \in U \subset \mathbb{R}^n$ be an open set and let $X : U \to \mathbb{R}^n$ be a smooth vector field satisfying $X(0) = 0$. Suppose that the derivative $DX(0)$ diagonalises and has (possibly complex) eigenvalues $\lambda_1, \ldots, \lambda_n$, repeated with multiplicity. Suppose that each $\lambda_i$ has negative real part, and that

$$
\lambda_i \neq \sum_{j=1}^{n} \alpha_j \lambda_j, \quad \text{for any } i, \text{ and any } \alpha_1, \ldots, \alpha_n \in \mathbb{Z}_{\geq 0} \text{ satisfying } \sum \alpha_j \geq 2.
$$

Then there exists open sets $0 \in U' \subset \mathbb{R}^n$ and $0 \in U'' \subset U$, and a diffeomorphism $\phi : U'' \to U'$, such that $D\phi(0) = \text{Id}$ and $\phi \circ X \circ \phi^{-1} = DX(0)$.

Actually [20, Theorem 4] states that $\phi$ can be chosen to be $C^k$ for every finite and big enough $k$. The fact that $\phi$ can be assumed to be $C^\infty$ follows from [11, Theorem 6].

Sternberg proved in [20] an analogous theorem for local diffeomorphisms of $\mathbb{R}^n$. Later, Anderson proved [1, §2, Lemma] a parametric version of Sternberg’s theorem for diffeomorphisms, which can be translated, using the arguments in [20, §6], into a theorem on vector fields. Before stating it, we introduce some notation. Let $D \subset \mathbb{R}^n$ be an open disk centered at 0, and let $\Delta \subset D$ be a smaller concentric disk. Let $r$ be a natural number. For any smooth map $\chi : D \to \mathbb{R}^n$ define $\|\chi\|_{\Delta, r} = \sup_{x \in \Delta} \|D^r \chi(x)\|$, where $\|D^r \chi(x)\|$ denotes the sum of the norms of all partial derivatives of $\chi$ at $x$ of degree $\leq r$. This defines a (non separated!) topology on $\text{Map}_0(D, \mathbb{R}^n)$, the set of all smooth maps $D \to \mathbb{R}^n$ fixing 0, and we denote by $\text{Map}_0(D, \mathbb{R}^n)_{\Delta, r}$ the resulting topological space. This is the analogue of Anderson’s theorem for vector fields:

**Theorem 3.2.** Let $L : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map which diagonalises with eigenvalues $\lambda_1, \ldots, \lambda_n$ satisfying (4). Assume that each $\lambda_i$ has negative real part, and that $\lambda_i \neq \lambda_j$ for $i \neq j$. There exists a neighborhood $N$ of $L$ in $\text{Map}_0(D, \mathbb{R}^n)_{\Delta, r+1}$ and a continuous map $\Phi : N \to \text{Map}_0(D, \mathbb{R}^n)_{\Delta, r}$ such that:

1. for every $X \in N$, $D\Phi(X)(0) = \text{Id}$, so $\Phi(X)$ gives a diffeomorphism $U_X \to U'_X$ between neighborhoods of 0,

2. for every $X \in N$, $\Phi(X) \circ X \circ \Phi(X)^{-1} : U'_X \to \mathbb{R}^n$ is equal to $DX(0)$.

We will need an analogue of Theorem 3.2 in an equivariant setting. However, as was mentioned in the introduction, the presence of symmetries usually forces eigenvalues to have high multiplicity, and consequently the hypothesis in Theorem 3.2 will most of the times not hold.
Now the (only) reason why Anderson assumes the eigenvalues \( \lambda_1, \ldots, \lambda_n \) to be pairwise distinct is that he needs to be able to diagonalize the linear maps close to \( L \) in a continuous way. To state this more precisely, let \( \text{GL}^*(\mathbb{R}, n) \subset \text{GL}(n, \mathbb{R}) \) denote the open and dense set of linear automorphisms of \( \mathbb{R}^n \) all of whose eigenvalues are distinct. Anderson uses the following elementary lemma.

**Lemma 3.3.** Any \( L \in \text{GL}^*(n, \mathbb{R}) \) admits a neighborhood \( U \subset \text{GL}^*(n, \mathbb{R}) \) and smooth maps \( f_1, \ldots, f_n: U \to \mathbb{C}^n \) so that for any \( L' \in U \) the vectors \( f_1(L'), \ldots, f_n(L') \) form a basis of \( \mathbb{C}^n \) with respect to which \( L' \) diagonalizes.

So to obtain an equivariant analogue of Theorem 3.2 it suffices to define some open and dense subset of the set of equivariant automorphisms of a vector space enjoying the same property as \( \text{GL}^*(n, \mathbb{R}) \). This is the purpose of the following lemma, which also proves a property on centralizers that will be used later in the paper.

Suppose that \( V \) is an \( n \)-dimensional real vector space, and that a finite group \( G \) acts linearly on \( V \). Denote the centralizer of any \( \Lambda \in \text{Aut}(V) \) by

\[
Z(\Lambda) = \{ \Lambda' \in \text{Aut}(V) \mid \Lambda \Lambda' = \Lambda' \Lambda \}.
\]

Let \( \text{Aut}_G(V) \) denote the Lie group of automorphisms of \( V \) commuting with the \( G \)-action. Define \( \text{Aut}_G^*(V) \) to be the set of all \( \Lambda \in \text{Aut}_G(V) \) such that for any \( \lambda \in \mathbb{C} \) the \( (G\)-invariant) subspace \( \text{Ker}(\Lambda - \lambda \text{Id}) \subset V \) is irreducible as a representation of \( G \). Given a basis \( a_1, \ldots, a_n \in V \otimes \mathbb{C} \) we denote by \( (a_1, \ldots, a_n) : \mathbb{C}^n \to V \otimes \mathbb{C} \) the isomorphism \( (\lambda_1, \ldots, \lambda_n) \mapsto \lambda_i a_i \).

**Lemma 3.4.** The subset \( \text{Aut}_G^*(V) \) is open and dense in \( \text{Aut}_G(V) \). Any \( \Lambda \in \text{Aut}_G^*(V) \) has a neighborhood \( U \subset \text{Aut}_G^*(V) \) with smooth maps \( f_1, \ldots, f_n: U \to \mathbb{C} \) so that for any \( \Lambda' \in U \) the vectors \( f_1(\Lambda'), \ldots, f_n(\Lambda') \) form a basis of \( V \otimes \mathbb{C} \) with respect to which \( \Lambda' \) diagonalizes, and conjugation by \( (f'_1, \ldots, f'_n)(f_1, \ldots, f_n)^{-1} \) gives an isomorphism

\[
Z(\Lambda) \overset{\sim}{\to} Z(\Lambda').
\]

**Proof.** Denote by \( \hat{G} \) the set of irreducible characters of \( G \). For any \( \xi \in \hat{G} \) let \( V_\xi \) be a \( G \)-representation with character \( \xi \). As a \( G \)-representation, we may identify \( V \) with \( \bigoplus_{\xi \in \hat{G}} V_\xi \otimes E_\xi \), where each \( E_\xi \) is a vector space with trivial \( G \)-action. By Schur’s lemma the space of \( G \)-equivariant endomorphisms of \( V \) is

\[
\text{End}_G(V) = \bigoplus_{\xi \in \hat{G}} \text{End} E_\xi.
\]

An endomorphism \( \Lambda = (\Lambda_\xi)_\xi \) (where \( \Lambda_\xi \in \text{End} E_\xi \) for each \( \xi \)) belongs to \( \text{Aut}_G(V) \) exactly when \( \prod_\xi \det \Lambda_\xi \neq 0 \), and it belongs to \( \text{Aut}_G^*(V) \) if and only if, additionally, no root of the polynomial \( \prod_\xi \det(\Lambda_\xi - x \text{Id}_{E_\xi}) \in \mathbb{R}[x] \) has multiplicity bigger than one. This condition implies that \( \Lambda_\xi \in \text{GL}^*(E_\xi) \) for each \( \xi \). Applying Lemma 3.3 to each \( \Lambda_\xi \) we deduce the existence of a neighborhood \( U \subset \text{Aut}_G^*(V) \) of \( \Lambda \) and smooth maps \( f_1, \ldots, f_n: U \to \mathbb{C} \) and \( \lambda_1, \ldots, \lambda_n : U \to \mathbb{C} \) so that for any \( \Lambda' \in U \) we have \( \Lambda'(f_j(\Lambda')) = \lambda_j(\Lambda')f_j(\Lambda') \) for every \( j \). For any \( \Lambda' \in U \) we can identify \( Z(\Lambda') \) with the subgroup of \( \text{Aut}(V) \) preserving the subspace of \( V \otimes \mathbb{C} \) spanned by \( \{ f_j(\Lambda') \mid \lambda_j(\Lambda') = \lambda \} \) for each \( \lambda \). Shrinking \( U \) if necessary we may assume that for any \( i, j \) and any \( \Lambda' \in U \) we have

\[
\lambda_i(\Lambda') = \lambda_j(\Lambda') \iff \lambda_i(\Lambda) = \lambda_j(\Lambda),
\]
so conjugation by \((f'_1, \ldots, f'_n)(f_1, \ldots, f_n)^{-1}\) gives an isomorphism \(Z(A) \xrightarrow{\sim} Z(A')\). \(\square\)

Take some \(G\)-invariant Euclidean metric on \(V\), let \(D \subset V\) be an open disk centered at \(0\), and let \(\Delta \subset D\) be a smaller concentric disk. Let \(r\) be a natural number. For any smooth map \(\chi : D \to V\) define \(\|\chi\|_{\Delta,r} = \sup_{x \in \Delta} \|D^r\chi(x)\|\) as before. This defines a topology on \(\text{Map}_{G,0}(D,V)\), the set of all \(G\)-equivariant smooth maps \(D \to V\) fixing \(0\). Let \(\text{Map}_{G,0}(D,V)_{\Delta,r}\) be the resulting topological space. Define analogously \(\text{Map}_0(D,V)_{\Delta,r}\) by dropping the equivariance condition. Combining the previous lemma with the arguments in [1 §2, Lemma] and [20 §6] we obtain the following.

**Theorem 3.5.** Let \(L \in \text{Aut}_\ast^r(V)\) have eigenvalues \(\lambda_1, \ldots, \lambda_n\) satisfying (4) and suppose that each \(\lambda_i\) has negative real part. There is a neighborhood \(N\) of \(L\) in \(\text{Map}_{G,0}(D,V)_{\Delta,r+1}\) and a continuous map
\[
\Phi : N \to \text{Map}_{G,0}(D,V)_{\Delta,r}
\]
such that:

1. for every \(\chi \in N\), \(D\Phi(\chi)(0) = \text{Id}\), so \(\Phi(\chi)\) gives a diffeomorphism \(U_\chi \to U'_\chi\) between neighborhoods of \(0\),
2. for every \(\chi \in N\), \(\Phi(\chi) \circ \chi \circ \Phi(\chi)^{-1} : U'_\chi \to V\) is equal to \(D\chi(0)\).

The only part in the statement of Theorem 3.5 that does not follow immediately is the fact that the conjugating map \(\Phi\) may be chosen to take values in \(\text{Map}_{G,0}(D,V)_{\Delta,r}\). Sternberg’s argument provides a (continuous, by Anderson) map \(\Phi_0 : N \to \text{Map}_0(D,V)_{\Delta,r}\), satisfying (i) \(D\Phi_0(\chi)(0) = \text{Id}\) and (ii) \(\Phi_0(\chi) \circ \chi \circ \Phi_0(\chi)^{-1} = D\chi(0)\) (in a neighborhood of \(0\)), but \(\Phi_0(\chi)\) is not necessarily equivariant. Now, equality (2) is equivalent to
\[
(5) \quad \Phi_0(\chi) \circ \chi = D\chi(0) \circ \Phi_0(\chi),
\]
so setting
\[
\Phi_0(\chi)(x) = \frac{1}{|G|} \sum_{g \in G} g\Phi_0(\chi)(g^{-1}x) \in V
\]
for every \(x \in D\), we have \(\Phi(\chi) \in \text{Map}_{G,0}(D,V)_{\Delta,r}\), and equation (5) immediately gives \(\Phi(\chi) \circ \chi = D\chi(0) \circ \Phi(\chi)\). Trivially we also have \(D\Phi(\chi)(0) = \text{Id}\) for every \(\chi\), and \(\Phi(\chi) : D \to V\) is \(G\)-equivariant. The map \(\Phi : N \to \text{Map}_{G,0}(D,V)_{\Delta,r}\) is continuous, because \(\Phi_0\) is, so now Theorem 3.5 is clear.

4. The space of metrics \(\text{Met}_0\)

4.1. Preliminaries. The following result is a standard consequence of the existence of linear slices for smooth compact group actions (see e.g. [5 Chap. VI, §2]).

**Lemma 4.1.** Let \(G\) be a finite group acting smoothly on a connected manifold \(X\).

1. For each subgroup \(H \subseteq G\) the fixed point set \(X^H = \{x \in X \mid H \subseteq G_x\}\) is the disjoint union of finitely many closed submanifolds of \(X\) (not necessarily of the same dimension) satisfying \(T_x(X^H) = (T_xX)^H\) for every \(x \in X^H\). In particular, either \(X^H = X\) or \(X^H\) has empty interior.
2. Assume that the action of \(G\) on \(X\) is effective. Then \(X^\text{free} = \{x \in X \mid G_x = \{1\}\}\) is open and dense in \(X\).
Let $M$ denote the space of Riemannian metrics on $M$, and let $\text{Met}^\Gamma \subset \text{Met}$ be the subset of $\Gamma$-invariant metrics.

Let $f : M \to \mathbb{R}$ be a $\Gamma$-invariant Morse function. This function will be fixed throughout the rest of the paper. If $p$ is a critical point of $f$, so that $\nabla^g f(p) = 0$, the derivative $D\nabla^g f(p)$ is a well-defined endomorphism of $T_p M$ (one may define it using a connection on $TM$, but the result will be independent of the chosen connection). The endomorphism $D\nabla^g f(p)$ is self-adjoint with respect to the Euclidean norm on $T_p M$ given by $g$, so $D\nabla^g f(p)$ diagonalizes.

Denote the index of a critical point $p$ of $f$ by $\text{Ind}_f(p)$. Let $\text{Crit}(f) \subset M$ be the set of critical points of $f$, and for any $k$ let

$$\text{Crit}_k(f) = \{ p \in \text{Crit}(f) \mid \text{Ind}_f(p) = k \}.$$ 

Define the set of sinks of $f$ to be $\mathbb{I} = \text{Crit}_n(f)$ and the set of sources to be $\mathbb{O} = \text{Crit}_0(f)$. The points in $\mathbb{I}$ (resp. $\mathbb{O}$) are the sinks (resp. sources) of the gradient vector field $\nabla^g f$ for every $g$. Denote also by $\mathbb{E} = \mathbb{I} \cup \mathbb{O}$ the collection of all local extremes of $f$.

For any $g \in \text{Met}$ and any real number $t$ let $\Phi_t^g : M \to M$ denote the flow at time $t$ of $\nabla^g f$. Define the stable and unstable manifolds of $p \in \text{Crit}(f)$ to be, respectively,

$$W^s_g(p) = \{ q \in M \mid \lim_{t \to \infty} \Phi_t^g(q) = p \}, \quad W^u_g(p) = \{ q \in M \mid \lim_{t \to -\infty} \Phi_t^g(q) = p \}.$$

For any $p \in \mathbb{E}$ and any $g \in \text{Met}^\Gamma$ let

$$L_g(p) = \{ \psi \in \text{Aut}(T_p M) \mid (D\nabla^g f(p))\psi = \psi(D\nabla^g f(p)) \} \leq L(D\nabla^g f(p)).$$

Since $\Gamma$ is finite and acts effectively on $M$, we can identify $\Gamma_p$ with a subgroup of $L_g(p)$ using (1) in Lemma 4.1 above.

4.3. The metrics in $\text{Met}_0$: generic eigenvalues at critical points. Let $\text{Met}_0 \subset \text{Met}^\Gamma$ denote set of $\Gamma$-invariant metrics $g$ satisfying the following conditions:

(C1) for any $p \in \mathbb{E}$ the eigenvalues $\lambda_1, \ldots, \lambda_n$ of the linearization $D\nabla^g f(p)$ satisfy condition (1) in Theorem 3.1.

(C2) if $p, q \in \mathbb{E}$, then the eigenvalues of $D\nabla^g f(p)$ and $D\nabla^g f(q)$ coincide if and only if $p$ and $q$ belong to the same $\Gamma$-orbit.

(C3) for any $p \in \mathbb{E}$ we have $D\nabla^g f(p) \in \text{Aut}_{\Gamma_p}^g(T_p M)$.

Condition (C1), combined with Sternberg’s Theorem 3.1 and an easy adaptation of a theorem of Kopell [11, Theorem 6] from maps to vector fields, implies that if $p \in \mathbb{I}$ then the map

$$D(p) : \text{Aut}(\nabla^g f|_{W^s_g(p)}) \to L_g(p)$$

is a $\Gamma$-equivariant bijection. But condition (C2) means that $D(p)$ fails to be surjective as soon as $\Gamma$ is not finite.
sending any $\phi \in \text{Aut}(\nabla^g f|_{W^s_g(p)})$ to $D\phi(p) \in L_g(p)$ is an isomorphism (it is clear that any such $\phi$ fixes $p$); furthermore, there is a diffeomorphism $h(p): T_pM \to W^s_g(p)$ making the following diagram commutative:

\[
\begin{array}{c}
L_g(p) \times T_pM \xrightarrow{D(p)^{-1} \times h(p)} T_pM \\
\text{Aut}(\nabla^g f|_{W^s_g(p)}) \times W^s_g(p) \xrightarrow{h(p)} W^s_g(p),
\end{array}
\]

where the horizontal arrows are the maps defining the actions.

**Remark 4.2.** Strictly speaking, Sternberg’s theorem gives a diffeomorphism between a neighborhood of $0$ in $T_pM$ and a neighborhood of $p$ in $W^s_g(p)$ which commutes the flows of $D\nabla^g f(p)$ and of $\nabla^g f$, but such diffeomorphism can be extended uniquely imposing compatibility with the flows to yield $h(p)$.

Similarly, for any source $p \in \mathbb{O}$ the analogous map $\text{Aut}(\nabla^g pf|_{W^s_g(p)}) \to L_g(p)$ is an isomorphism and there is a diffeomorphism $T_pM \to W^u_g(p)$ which is equivariant in the obvious sense, analogous to the case of sinks.

Condition (C2) implies that for any $\phi \in \text{Aut}(\nabla^g f)$ and any $p \in \mathbb{E}$ we have $\phi(p) = \gamma p$ for some $\gamma \in \Gamma$. Of course a priori $\gamma$ may depend on $p$, but in the course of proving Theorem 1.1 we will deduce that for $g$ belonging to a residual subset of $\text{Met}_0$ and any $\phi \in \text{Aut}(\nabla^g f)$, there exists some $\gamma$ such that $\phi(p) = \gamma p$ for each $p \in \mathbb{E}$.

By Lemma 3.4 $\text{Met}_0$ is open and dense in $\text{Met}^\Gamma$. Moreover, combining (C3) with Lemma 3.4 and Theorem 3.5 (together with the obvious analogue of Remark 4.2) we deduce the following result.

**Lemma 4.3.** Any $g \in \text{Met}_0$ has a neighborhood $\mathcal{U} \subset \text{Met}_0$ such that for any $p \in \mathbb{E}$ the following holds. Let $V_p = T_pM$. Endow the space of maps $\text{Map}(V_p, M)$ with the weak (compact-open) $C^\infty$-topology [9, Chap 2, §1]. For any $g' \in \mathcal{U}$ there is a linear vector field $\mathcal{X}_{g'}(p): V_p \to V_p$ depending continuously on $g'$ and a $\Gamma_p$-equivariant embedding $h_{g'}(p): V_p \to M$ depending also continuously on $g'$ with the following properties.

1. $Z(\mathcal{X}_{g'}(p)) = Z(\mathcal{X}_g(p)) = L_g(p)$ for every $g' \in \mathcal{U}$.
2. $h_{g'}(p)$ identifies $\mathcal{X}_{g'}(p)$ with the restriction of $\nabla^{g'} f$ to $h_{g'}(p)(V_p)$; hence,

$$h_{g'}(p)(V_p) = W^s_{g'}(p) \quad \text{if} \quad p \in \mathbb{I}$$

and

$$h_{g'}(p)(V_p) = W^u_{g'}(p) \quad \text{if} \quad p \in \mathbb{O}.$$

Note that we do not claim that the derivative of $h_{g'}(p)$ at $p$ is the identity: in fact in general this will not be the case (otherwise we could not pretend to have the identifications $Z(\mathcal{X}_{g'}(p)) = Z(\mathcal{X}_g(p))$).
5. The spheres $S_g(p)$, the distributions $A_g(p)$, and the sets $F_g(p)$

Recall that we assume $\dim M > 1$.

5.1. The spheres $S_g(p)$. For any $g \in \text{Met}$ and any sink $p \in \mathbb{I}$ we denote by $\sim$ the equivalence relation in $W^s_g(p)$ that identifies two points whenever they belong to the same integral curve of $\nabla^g f$. We then define

$$S_g(p) = (W^s_g(p) \setminus \{p\})/\sim.$$

Let $\epsilon > 0$ be a real number and let $\Sigma \subset M$ be the $g$-geodesic sphere of radius $\epsilon > 0$ and center $p$. If $\epsilon$ is small enough (which we assume), then $\Sigma$ is a submanifold of $M$ diffeomorphic to $S^{n-1}$ and every equivalence class in $S_g(p)$ contains a unique representative in $\Sigma$. Hence, composing the inclusion $\Sigma \hookrightarrow W^s_g(p) \setminus \{p\}$ with the projection

$$\pi_p : W^s_g(p) \setminus \{p\} \to S_g(p)$$

gives a bijection $\Sigma \simeq S_g(q)$. This allows us to transport the smooth structure on $\Sigma$ to a smooth structure (in particular, a topology) on $S_g(p)$, independent of $\epsilon$.

If $g \in \text{Met}_0$ then the action of $L_g(p)$ on $S_g(p)$ defined via the identification $[\mathcal{O}]$ is smooth, and so is the natural action of $\Gamma_p$ on $S_g(p)$ (recall that $\text{Met}_0 \subset \text{Met}^{\Gamma}$).

For any $p \in \mathbb{I}$ and $q \in \mathbb{O}$ let

$$\Omega_g(p, q) := \pi_p(W^s_g(p) \cap W^u_g(q)) \subset S_g(p).$$

Since $W^u_g(q)$ is open in $M$, $\Omega_g(p, q)$ is open in $S_g(p)$.

Similarly, if $q$ is a source we define

$$S_g(q) = (W^u_g(q) \setminus \{q\})/\sim$$

and we denote by

$$\pi_q : W^u_g(q) \setminus \{q\} \to S_g(q)$$

the projection. If $p$ is a sink, then we define

$$\Omega_g(q, p) = \pi_q(W^s_g(p) \cap W^u_g(q)),$$

which is an open subset of $S_g(q)$.

For convenience, if $p, q \in \mathbb{I}$ or $p, q \in \mathbb{O}$ we define $\Omega_g(p, q) = \emptyset$.

Since the fibers of the restrictions of $\pi_p$ and $\pi_q$ on $W^s_g(p) \cap W^u_g(q)$ are the same, there are natural bijections

$$\sigma_{g}^{p,q} : \Omega_g(p, q) \to \Omega_g(q, p), \quad \sigma_{g}^{q,p} = (\sigma_{g}^{p,q})^{-1},$$

which are easily seen to be diffeomorphisms.

5.2. The singular distributions $A_g(q)$. Assume through the remainder of this section that $g \in \text{Met}_0$. We will consider, for every $p \in \mathbb{E}$, the diagonal action of $L_g(p)$ on $S_g(p)^k$ for some natural number $k$. If $z = (z_1, \ldots, z_k) \in S_g(p)^k$ and $\psi \in L_g(p)$ we denote

$$\psi z = (\psi z_1, \ldots, \psi z_k).$$

Similarly, we will consider the diagonal extension of the maps $\sigma_{g}^{p,q}$:

$$\sigma_{g}^{p,q} : \Omega_g(p, q)^k \to \Omega_g(q, p)^k, \quad \sigma_{g}^{p,q} z = (\sigma_{g}^{p,q} z_1, \ldots, \sigma_{g}^{p,q} z_k).$$
We are going to use below without explicit notice analogous diagonal extensions of maps to Cartesian products.

Let \( n = \dim M \). Define

\[
(8) \quad r := \left\lfloor \frac{2n^2}{n-1} + 1 \right\rfloor.
\]

The choice of this number will be justified in the proof of Lemma 5.3

For any \( q \in E \) we denote by \( \mathcal{A}_q(q) \subset T(S_g(q)^r) \) the subspace consisting of all tangent vectors given by the infinitesimal action of the Lie algebra of \( L_g(q) \). This gives, for any \( z \in S_g(q)^r \), a linear subspace \( \mathcal{A}_q(q)(z) \subset T_z S_g(q)^r \) whose dimension may vary with \( z \) (hence, one can think of \( \mathcal{A}_q(q) \) as a singular distribution). In concrete terms,

\[
\mathcal{A}_q(q)(z) = \{ y_{g,\sigma}(z) \mid \sigma \in \text{Lie} L_g(q) \},
\]

where for any \( \sigma \in \text{Lie} L_g(q) \) we denote by \( y_{g,\sigma} \) the vector field on \( S_g(q)^r \) given by the infinitesimal action of \( \sigma \).

5.3. The subset \( F_g(q) \subset S_g(q)^r \). We next want to identify a dense open subset of \( S_g(q)^r \) on which the action of \( L_g(q) \) has the smallest possible isotropy subgroup, and on which \( \mathcal{A}_q(q) \) restricts to a vector subbundle of \( T(S_g(q)^r) \). We remark that, since \( L_g(q) \) is an infinite group, in this situation we can not use (2) in Lemma 4.1. Let

\[
\mathcal{X}_q := D\nabla^g f(p) \in \text{Lie} L_g(q).
\]

Note that \( e^{t\mathcal{X}_q(q)} \) corresponds, via the isomorphism \( D(q) \) in (6), to the flow \( \Phi_t^q \), so \( e^{t\mathcal{X}_q(q)} \) acts trivially on \( S_g(q) \) and hence on \( S_g(q)^r \).

Let us denote \( V = T_q M \). Then \( X := \mathcal{X}_q(q) \) is a diagonalizable endomorphism of \( V \). Denote its eigenvalues by \( \lambda_1, \ldots, \lambda_k \). Let \( V_j \subseteq V \) be the subspace consisting of eigenvectors with eigenvalue \( \lambda_j \). We have a decomposition \( V = V_1 \oplus \cdots \oplus V_k \) with respect to which we may define projections \( \pi_j : V \to V_j \). Let us say that a collection of vectors \( w_1, \ldots, w_s \in V_j \) is thick if \( s > d_j = \dim V_j \) and for any \( 1 \leq i_1 < i_2 < \cdots < i_{d_j} \leq s \) the vectors \( w_{i_1}, \ldots, w_{i_{d_j}} \) are linearly independent. Finally, we say that a collection of vectors \( v_1, \ldots, v_s \in V \) is thick if for any \( j \) the projections \( \pi_j(v_1), \ldots, \pi_j(v_s) \) form a thick collection of vectors in \( V_j \). Let \( G = L_g(q) \).

**Lemma 5.1.** Suppose that \( v_1, \ldots, v_s \) is a thick collection of vectors, and that for some \( g \in G \) there exist real numbers \( t_1, \ldots, t_s \) satisfying \( gv_j = e^{t_j X} v_j \) for every \( j \). Then \( g = e^{t X} \) for some real number \( t \).

**Proof.** Consider first the case \( k = 1 \), so that \( X \) is a homothecy. Write \( v_{n+1} = a_1 v_1 + \cdots + a_n v_n \). The thickness condition implies that \( a_i \neq 0 \) for every \( i \). By assumption we have \( g v_i = \lambda_i v_i \) for some real numbers \( \lambda_1, \ldots, \lambda_s \). In particular,

\[
\lambda_{n+1}(a_1 v_1 + \cdots + a_n v_n) = \lambda_1 a_1 v_1 + \cdots + \lambda_n a_n v_n.
\]

Taking into account that \( v_1, \ldots, v_n \) is a basis and equating coefficients we deduce that \( \lambda_{n+1} = \lambda_1 = \cdots = \lambda_n \). So the case \( k = 1 \) is proved. The case \( k > 1 \) follows from applying the previous arguments to each \( V_j \), using the fact that every \( g \in G \) preserves \( V_j \). \( \square \)

Let \( S(V) \) denote the set of orbits of \( H = \{ e^{t X} \mid t \in \mathbb{R} \} \) acting on \( V \setminus \{0\} \). \( H \) is a central subgroup of \( G \), and the action of \( G \) on \( V \) induces an action of \( G/H \) on \( S(V) \).
Let $F \subset S(V)^r$ denote the set of tuples $(x_1, \ldots, x_r)$ such that, writing $x_i = Hx'_i$ with $x'_i \in V$ for each $i$, the vectors $x'_1, \ldots, x'_r$ form a thick collection (this is independent of the choice of representatives $x'_i$).

**Lemma 5.2.**

(1) $F$ is a dense an open subset of $S(V)^r$;

(2) the restricted action of $G/H$ on $F$ is free.

**Proof.** For (1) note that $r > n$, so the set $F'$ of thick $r$-tuples in $V^r$ can be identified with the complementary of finitely many proper subvarieties (those corresponding to the possible linear relations among projections to each summand $V_j$ of subsets of the tuple, given by the vanishing of suitable determinants). Hence $F'$ is a dense open subset of $V^r$, which implies that $F \subset S(V)^r$ is open and dense. (2) follows from Lemma 5.1. □

Assume that $q$ is a sink. Choose a diffeomorphism $h : V \to W^s_g(q)$ making commutative the diagram (7) with $p$ replaced by $q$. Then $h$ induces a diffeomorphism $S(V) \to S_g(q)$, which can be extended linearly to $S(V)^r \to S_g(q)^r$. Let

$$F_g(q) \subset S_g(q)^r$$

be the image of $F$ under the previous diffeomorphism. The set $F_g(q)$ is independent of the choice of $h$. Indeed, two different choices of $h$ differ by precomposition with an element of $G$, and the action of $G$ on $S(V)^r$ preserves $F$. If instead $q$ is a source, consider the same definition with $W^s_g(q)$ replaced by $W^u_g(q)$.

Lemma 5.2 and an obvious estimate imply:

**Lemma 5.3.**

(1) If $z \in F_g(q)$ and $\psi \in L_g(q)$ satisfies $\psi z = z$ then $\psi = e^{tX_g(q)}$ for some $t \in \mathbb{R}$.

(2) The restriction of $A_g(q)$ to $F$ is a vector bundle of rank $\dim G - 1 \leq n^2 - 1$.

6. THE SPACE OF METRICS $\text{Met}_{1,K}$

We recall again that $\dim M > 1$.

6.1. **Definition of $\text{Met}_{1,K}$.** Let $g \in \text{Met}_0$. Let $p \in \mathbb{E}$ and let $K$ be a natural number. Denote by $\| \cdot \|_g$ the operator norm in $\text{End} T_p M$ induced by $g$. Denote by

$$L_{g,K}(p) \subset L_g(p)$$

the subset consisting of those $\psi \in L_g(p)$ such that $\| \psi \|_g \leq K$, $\| \psi^{-1} \|_g \leq K$, and

$$\| \psi - e^{tX_g(p)} \gamma \|_g \geq K^{-1}$$

for every $t \in \mathbb{R}$ and $\gamma \in \Gamma_p$.

Clearly $L_{g,K}(p)$ is compact. Recall that the number $r$ has been defined in (8) in Subsection 5.2 above. For any $\psi \in L_g(p)$ we denote by

$$\alpha_\psi : S_g(q)^r \to S_g(q)^r$$

the map given by the action of $\psi$.

**Definition 6.1.** Let $p \in \mathbb{E}$. Define $\text{Met}_{1,K}(p)$ as the set of all metrics $g \in \text{Met}_0$ such that for any $\psi \in L_{g,K}(p)$ there exist:

1. $q, q' \in \mathbb{E}$ and $z \in \Omega_g(p, q)^r$ satisfying $\psi z \in \Omega_g(p, q')^r$, $\sigma^p_g z \in F_g(q)$, $\sigma_q^{p,q'} \psi z \in F_g(q')$,
Proof. We will use the following lemma, whose proof is postponed to the Appendix.

Lemma 6.3. 

Proof. Suppose that $g \in \text{Met}_{1,K}(p)$ is an open subset of $\text{Met}_0$.

Lemma 6.2. 

Proof. Let $\psi \in L_g(\mathbb{R}) \subset \text{Aut}(\mathbb{R})$. Choose subsets $V_1, \ldots, V_n \subset L_g(\mathbb{R})$ such that $L_g(\mathbb{R}) \subset V_1 \cup \cdots \cup V_n$ and such that, for every $j$, $(q_j, q_j', z_j, u_j)$ rules out an element of $V_j$. Choose subsets $V_j' \subset V_j$ with the property that $L_g(\mathbb{R}) \subset V_1' \cup \cdots \cup V_n'$, and such that $\overline{V_j'}$ is compact and contained in $V_j$ for each $j$.

Applying Lemma 6.2 to $g$ we deduce the existence of a neighborhood $\mathcal{U} \subset \text{Met}_0$ of $g$ and natural smooth identifications $S_g(\mathbb{R}) \simeq S_{g'}(\mathbb{R})$ for every $g' \in \mathcal{U}$ and $g \in \mathcal{E}$. Since in the remainder of the proof we only consider metrics from $\mathcal{U}$, we denote $S(\mathbb{R})$ instead of $S_g(\mathbb{R})$. We also get for every $g' \in \mathcal{U}$ natural isomorphisms of groups $L_g(\mathbb{R}) \simeq L_{g'}(\mathbb{R})$ which are compatible with both inclusions of $\Gamma_z$ in $L_g(\mathbb{R})$ and $L_{g'}(\mathbb{R})$ and with the identifications $S_g(\mathbb{R}) \simeq S_{g'}(\mathbb{R})$, and for this reason we write $L(\mathbb{R})$ instead of $L_{g'}(\mathbb{R})$. Now we may view $V_1, \ldots, V_n$ as subsets of $L(\mathbb{R})$. Shrinking $\mathcal{U}$ if necessary we may assume that $L_{g',K}(p) \subset V_1' \cup \cdots \cup V_n'$ for every $g' \in \mathcal{U}$.

The sets $F_{g'}(\mathbb{R}) \subset S(\mathbb{R})$ are independent of $g'$ and the distributions $A_{g'}(\mathbb{R})$ vary continuously with $g'$. Similarly the subsets $\Omega_{g'}(\mathbb{R}) \subset S(\mathbb{R})$ vary continuously with $g'$, meaning that, for any $z \in \Omega_{g'}(\mathbb{R})$, if $g'$ is sufficiently close to $g$ then $z \in \Omega_{g'}(\mathbb{R})$ as well. The maps $\sigma_{g'} : \Omega_{g'}(\mathbb{R}) \to \Omega_{g'}(\mathbb{R})$ also depend continuously on $g'$ in the obvious sense.

For any $g' \in \mathcal{U}$, any $q \in \mathcal{E}$, and any $z \in F(q)$ we define $P_{g'} : T_z S(q) \to A_{g'}(\mathbb{R})(z)$ to be the orthogonal projection with respect to $g'$ (recall that $A_{g'}(\mathbb{R})(z)$ is a vector subspace of $T_z S(q)$). The previous observations imply the following: for every $1 \leq j \leq \nu$ there exists a neighborhood of $g$, $U_j \subset \mathcal{U}$, such that for every $g' \in U_j$ and any $u_j' \in \overline{V_j}$, the tuple $(g_j, q_j', z_j, P_{g'}(u_j'))$ rules out $\psi'$. It then follows that $U_1 \cap \cdots \cap U_\nu \subset \text{Met}_{1,K}(p)$, and hence $\text{Met}_{1,K}(p)$ is open.

Lemma 6.3. 

Proof. We will use the following lemma, whose proof is postponed to the Appendix.

Lemma 6.4. Suppose that $g \in \text{Met}^T$, $x \in M \setminus \text{Crit}(f)$ and $y = \Phi_t(x)$ for some nonzero $t$. Suppose that the stabilizer $\Gamma_x$ is trivial. Let $v \in T_x M$ be a nonzero vector, and let...
Given any \( u \in T_y(TM) \) and any \( \Gamma \)-invariant open subset \( U \subset M \) containing \( \Phi^*_t(x) \) for some \( t' \in (0, t) \), there exists some \( g' \in C^\infty(M, S^2T^*M)^\Gamma \) supported on \( U \) such that

\[
\frac{\partial}{\partial \epsilon} D\Phi^*_{-\epsilon} g' (w) \bigg|_{\epsilon=0} = u.
\]

Fix some \( g \in \text{Met}_0 \). We assume for concreteness throughout the proof that \( p \) is a sink. The case in which \( p \) is a source follows from the same arguments (or replacing \( f \) by \(-f\)).

We claim that the set of points in \( S_g(p) \) with trivial stabilizer in \( \Gamma \) is open and dense. Indeed, on the one hand the points in \( W^s_\ast(p) \) with trivial stabilizer form an open and dense subset, thanks to (2) in Lemma 11 and the openness of \( W^s_\ast(p) \subset M \) and on the other hand the stabilizer of any \( z \in W^s_\ast(p) \) is equal to the stabilizer of the point in \( S_g(p) \) it represents, because the action of \( \Gamma \) preserves \( f \) and the restriction of \( f \) to the fibers of the projection \( W^s_\ast(p) \setminus \{p\} \to S_g(p) \) is injective.

Let \( \psi \in L_g,K(p) \). Since \( \psi \) does not belong to \( \Gamma_p \subset L_g(p) \) and since the union of the sets \( \{\Omega(p,q)\}_{q \in \bar{\Omega}} \) is an open and dense subset of \( S_g(p) \), there exist \( q,q' \in \bar{\Omega} \) (not necessarily distinct) and a point \( c \in \Omega(p,q) \) satisfying \( \psi \in \Omega(p,q') \) and \( \psi c \neq \gamma c \) for every \( \gamma \in \Gamma_p \).

By the previous claim, we can also assume that the stabilizer of \( c \) is trivial.

Choose a metric on the sphere \( S_g(p) \). For any \( y \in S_g(p) \) and any \( r > 0 \) denote by \( B_{S_r}(y) \) the closed ball in \( S_g(p) \) of radius \( r \) and center \( y \). Choose \( \epsilon > 0 \) in such a way that \( \gamma B_{S_r,c}(c) \cap \psi B_{S_r,c}(c) = \emptyset \) for every \( \gamma \in \Gamma_p \). By Lemma 11, the openness of \( W^s_\ast(p) \subset M \) and the restriction of \( f \) to the fibers of the projection \( W^s_\ast(p) \setminus \{p\} \to S_g(p) \) is injective.

Let \( \Omega_p \) be the set of all elements \( \psi' \in L_g(p) \) satisfying \( \gamma B_{S_r,c}(c) \cap \psi B_{S_r,c}(c) = \emptyset \) for every \( \gamma \in \Gamma_p \) and \( \sigma_g^{q,q'}(\psi') \in F_g(q') \) for every \( \gamma \in \Gamma_p \).

Denote the open ball in \( M \) with center \( x \) and radius \( \delta \) by \( B_\delta(x) \). Take real numbers \( a < b < f(p) \) in such a way that \( [a, b] \) does not contain any critical value of \( f \). Take \( \delta > 0 \) small enough so that \( B_\delta(p) \) (resp. \( B_\delta(q) \)) is entirely contained in \( W^s_\ast(p) \) (resp. \( W^s_\ast(q) \)) and \( w f(B_\delta(p)) > b \) (resp. \( f(B_\delta(q)) < a \)).

Pick, for each \( 1 \leq i \leq r \), points \( x_i \in B_\delta(p) \setminus \{p\} \) and \( y_i \in B_\delta(q) \setminus \{q\} \) both representing \( z_i \in S_g(p) \), and a tangent vector \( v_i \in T_x M \) projecting to \( u_{\psi i} \in T_{z_i} S_g(p) \). Define real numbers \( t_1, \ldots, t_r \) by the condition that \( y_i = \Phi^{t_i}_i(x_i) \), and let \( v_i = D\Phi^{t_i}_i(v_i) \).

Let \( U \subset M \) be an open \( \Gamma \)-invariant subset contained in \( f^{-1}((a,b)) \cap W^s_\ast(p) \) whose projection to \( S_g(p) \) contains \( z_1, \ldots, z_r \) and is disjoint from \( \psi(B_{S_r,c}(c)) \). By Lemma 6.4 one can pick a finite dimensional vector subspace

\[
G_\psi \subset C^\infty(M, S^2T^*M)^\Gamma,
\]

all of whose elements are supported in \( U \), with the property that the linear map

\[
G_\psi \ni g' \mapsto \left( \frac{\partial}{\partial \epsilon} D\Phi^*_{t_1} g' (w_1) \bigg|_{\epsilon=0}, \ldots, \frac{\partial}{\partial \epsilon} D\Phi^*_{t_r} g' (w_r) \bigg|_{\epsilon=0} \right) \in \bigoplus_{i=1}^r T_{u_i}(TM)
\]
is surjective.

Choose an open neighborhood $\mathcal{O}_\psi \subset \mathcal{O}_\psi$ of $\psi$ whose closure in $\mathcal{O}_\psi$ is compact. Since $L_{g,K}(p)$ is compact there exist $\psi_1, \ldots, \psi_s \in L_{g,K}(p)$ such that $L_{g,K}(p) \subset \mathcal{O}_{\psi_1} \cup \cdots \cup \mathcal{O}_{\psi_s}$. Denote $z_i = z_{\psi_i} \in S_g(p)\r$ and $u_i = u_{\psi_i} \in T(S_g(p)\r)$.

Let $G = \sum_i G_{\psi_i}$. Let $\mathcal{M}$ be the set of all $g' \in \text{Met}^T$ satisfying the following conditions:

1. $g' - g \in G$,
2. $\sigma^p_{g'}(z_i) \in F_{g'}(q) = F_g(q)$ for every $i$,
3. $\sigma^p_{g'}(\psi z_i) \in F_{g'}(q') = F_g(q')$ for every $i$ and every $\psi \in \overline{\mathcal{O}_\psi}$.

To explain conditions (2) and (3), note that since $g' - g \in \sum_i G_{\psi_i}$ and the elements in each $G_{\psi_i}$ are supported away from the critical points, we can canonically identify $S_g(q) = S_{g'}(q)$ and $S_g(q') = S_{g'}(q')$, and similarly $F_g(q) = F_{g'}(q)$ and $F_g(q') = F_{g'}(q')$.

Note that $\{g' - g \mid g' \in \mathcal{M}\}$ can be identified with an open subset of $G$ containing $0$, so $\mathcal{M}$ has a natural structure of (finite dimensional) smooth manifold.

Consider, for each $i \in \{1, \ldots, s\}$,

$$V_i = \{(g', \psi', b) \in \mathcal{M} \times \mathcal{O}_{\psi_i} \times T(S_g(q')\r) \mid b = D(\sigma^p_{g'} \circ \alpha_{\psi'} \circ \sigma^p_{g})(u_i)\}$$

and its subvariety

$$V'_i = \{(g', \psi', b) \in \mathcal{M} \times \mathcal{O}_{\psi_i} \times T(S_g(q')\r) \mid b = D(\sigma^p_{g'} \circ \alpha_{\psi'} \circ \sigma^p_{g})(u_i)\}.$$

Let also

$$A = \mathcal{M} \times L_g(p) \times J_g(q')|_{F_g(q')}.$$

Note that $V_i$, $V'_i$, $A$ are subvarieties of $\mathcal{M} \times L_g(p) \times T(S_g(q')\r)$.

Let $N_i = V_i \cap A \cap \{(g) \times \mathcal{O}_{\psi_i} \times T(S_g(p)\r)\}$. The definition of $G_{\psi_i}$ guarantees that $V_i$ and $A$ intersect transversely along $N_i$. Consequently, there exists a neighborhood of $N_i$,

$$N_i \subset \mathcal{M} \times \mathcal{O}_{\psi_i} \times T(S_g(q')\r),$$

such that the intersection $V_i \cap A \cap N_i$ is a smooth manifold whose dimension satisfies

$$d - \dim(V_i \cap A \cap N_i) = \min\{d + 1, (d - \dim V_i) + (d - \dim A)\},$$

where

$$d = \dim \mathcal{M} \times L_g(p) \times T(S_g(q')\r).$$

This formula is consistent with the convention that a set is empty if and only if its dimension is $-1$. Consider the projection

$$\pi_i : V_i \cap A \to \mathcal{M}.$$

Since the closure of $V'_i$ inside $V_i$ is compact, there exists a neighborhood of $g$, $M_i \subset \mathcal{M}$, with the property that $\pi^{-1}_i(M_i) \cap V'_i \subset N_i$. Hence, $\pi^{-1}_i(M_i) \cap V'_i$ is a smooth manifold. Let

$$\mathcal{M}^\text{reg}_i \subset M_i$$

be the set of regular values of $\pi_i$ restricted to $\pi^{-1}_i(M_i)$.
We claim that for every $g' \in \mathcal{M}^\text{reg}_i$ we have $\pi_i^{-1}(g') \cap \mathcal{V}_i' = \emptyset$. To prove the claim it suffices to check that $\dim \pi_i^{-1}(g') \cap \mathcal{V}_i' = -1$. Now,

$$\dim \pi_i^{-1}(g') \cap \mathcal{V}_i' = \dim(\mathcal{V}_i \cap \mathcal{A} \cap \mathcal{N}_i) - \dim \mathcal{M}$$

$$= d - \min\{d + 1, (d - \dim \mathcal{V}_i) + (d - \dim \mathcal{A})\} - \dim \mathcal{M}.$$ If $(d - \dim \mathcal{V}_i) + (d - \dim \mathcal{A}) \geq d + 1$ then this is clearly negative. So assume that instead $(d - \dim \mathcal{V}_i) + (d - \dim \mathcal{A}) < d + 1$. Since the projection of $\mathcal{V}_i$ to $\mathcal{M} \times \mathcal{O}_{\psi_i}$ is a diffeomorphism, we have

$$d - \dim \mathcal{V}_i = d - (\dim \mathcal{M} \times \mathcal{O}_{\psi_i}) = d - (\dim \mathcal{M} \times \mathcal{O}_g) = \dim T(S_g(q')^r) = 2r(n - 1).$$

On the other hand we have, using (2) in Lemma 5.3,

$$d - \dim \mathcal{A} = \dim T(S_g(q')^r) - \dim \mathcal{A}_g(q')|_{F_g(q')}$$

$$\geq 2r(n - 1) - (r(n - 1) + n^2 - 1) = r(n - 1) - n^2 + 1.$$ Combining both estimates we compute:

$$\dim \pi_i^{-1}(g') \cap \mathcal{V}_i' \leq d - 2r(n - 1) - r(n - 1) + n^2 - 1 - \dim \mathcal{M}$$

$$= \dim L_p(g) \times T(S_g(q')^r) - 3r(n - 1) + n^2 - 1$$

$$\leq n^2 + 2r(n - 1) - 3r(n - 1) + n^2 - 1$$

$$= 2n^2 - r(n - 1) - 1.$$ Our choice of $r$, see (3), implies that $2n^2 - r(n - 1) - 1 < 0$, so the claim is proved.

Finally, let

$$\mathcal{M}^\text{reg} = \mathcal{M}^\text{reg}_1 \cap \cdots \cap \mathcal{M}^\text{reg}_n.$$ We claim that $\mathcal{M}^\text{reg} \subset \text{Met}_{1,K}(p)$. Indeed, suppose that $g' \in \mathcal{M}^\text{reg}$ and let $\psi \in L_{g,K}(p)$ be any element. Then $\psi \in \mathcal{O}_{\psi_i}$ for some $i$ and we have, on the one hand,

$$z_i \in S_{g'}(p,q')^r, \quad \psi z_i \in S_{g'}(p,q')^r, \quad \sigma_{g'}^{p,q}(z_i) \in F_{g'}(q), \quad \sigma_{g'}^{p,q'}(\psi z_i) \in F_{g'}(q'),$$

and, on the other hand, the fact that $\pi_i^{-1}(g') \cap \mathcal{V}_i' = \emptyset$ implies that

$$D(\sigma_{g'}^{p,q'} \circ \alpha_{\psi} \circ \sigma_{g'}^{p,q})(u_i) \notin \mathcal{A}_{g'}(q')(\sigma_{g'}^{p,q'} \circ \alpha_{\psi}(z_i)).$$

This proves the claim.

Sard’s theorem (see e.g. [9] Chap 3, §1.3) implies that $\mathcal{M}^\text{reg}$ is residual in $\mathcal{M}$. Hence $\mathcal{M}^\text{reg}$ is dense in a neighborhood of $g \in \mathcal{M}$, so $\text{Met}_{1,K}$ is dense in a neighborhood of $g$. □

Recall that $\text{Met}_{1,K} = \bigcap_{p \in \mathcal{E}} \text{Met}_{1,K}(p)$. The preceding two lemmas imply:

**Lemma 6.5.** $\text{Met}_{1,K}$ is a dense open subset of $\text{Met}_0$.

7. **Proof of Theorem 1.1 for dim $M > 1$**

Continuing with the notation of the previous sections, let us define

$$\text{Met}_f = \text{Met}_0 \cap \bigcap_{K \in \mathbb{N}} \text{Met}_{1,K}.$$ Since each of the sets appearing in the right hand side of the equality is open and dense in $\text{Met}^\Gamma$ (see Subsection 4.3 and Lemma 6.5), $\text{Met}_f$ is a residual subset of $\text{Met}^\Gamma$. Fix
some \( g \in \text{Met}_f \) and let \( \phi \in \text{Aut}(\nabla^g f) \). We are going to check that there exists some \( \gamma \in \Gamma \) and some \( t \in \mathbb{R} \) such that

\[
\phi(x) = \Phi_t^\gamma(\gamma x)
\]

for every \( x \in M \). This will prove Theorem [11]

**Lemma 7.1.** For each \( p \in \mathbb{I} \) (resp. \( p \in \mathbb{O} \)) there exists some \( \gamma \in \Gamma \) and some \( t \in \mathbb{R} \) such that \( \phi(x) = \Phi_t^\gamma(\gamma x) \) for every \( x \in W^s_g(p) \) (resp. for every \( x \in W^u_g(p) \)).

**Proof.** Suppose that \( p \in \mathbb{I} \) (the case \( p \in \mathbb{O} \) is dealt with in the same way with the obvious modifications). By property (C2) in the definition of \( \text{Met}_0 \) (see Subsection [4.3] there exists some \( \gamma \in \Gamma \) such that \( \phi(p) = \gamma p \). Hence, up to composing \( \phi \) with the action of \( \gamma \), we can (and do) suppose that \( \phi(p) = p \).

Once we know that \( \phi \) fixes \( p \), we conclude that it restricts to a diffeomorphism of \( W^s_g(p) \) preserving \( \nabla^g f \), which we identify with an element \( \phi_p \in L_g(p) \) via the isomorphism (3.

Next, let us prove that the action of \( \phi_p \) on \( S_g(p) \) coincides with the action of some \( \gamma \in \Gamma_p \). If this is not the case, then \( \phi_p \in L_{g,K}(p) \) for some natural \( K \) (see Subsection [6.1]). Since \( g \in \text{Met}_{1,K} \), it follows that there exist sources \( q, q' \in \mathbb{O} \) and \( z \in \Omega(p, q)^r \) satisfying

\[
\phi_p z \in \Omega_g(p, q')^r, \quad \sigma^{p,q}_g z \in F_g(q), \quad \sigma^{p,q}_g \phi_p z \in F_g(q'),
\]

and a vector \( u \in A_g(q)(\sigma^{p,q}_g(z)) \) satisfying

\[
(11) \quad D(\sigma^{p,q}_g \circ \alpha_{\phi_p} \circ \sigma^{q,p}_g)(u) \notin A_g(q')(\sigma^{p,q}_g \circ \alpha_{\phi_p}(z)).
\]

By the definition of \( A_g(q) \), we may write \( u = y_{g,s}(\sigma^{p,q}_g z) \) for some \( s \in \text{Lie} L_g(q) \).

The fact that \( z \in \Omega(p, q)^r \) and \( \phi_p z \in \Omega_g(p, q')^r \) implies that \( \phi(q) = q' \), so \( \phi \) maps \( W^u_g(q) \) diffeomorphically to \( W^u_g(q') \); since \( \phi \) preserves \( \nabla^g f \), \( \phi \) induces by conjugation an isomorphism

\[
\psi : L_g(q) \to L_g(q').
\]

The corresponding map at the level of Lie algebras associates to \( s \) an element \( \psi(s) \in \text{Lie} L_g(q') \), and in fact we have

\[
D(\sigma^{p,q}_g \circ \alpha_{\phi_p} \circ \sigma^{q,p}_g)(u) = D(\sigma^{p,q}_g \circ \alpha_{\phi_p} \circ \sigma^{q,p}_g)(y_{g,s}(\sigma^{p,q}_g z)) = y_{g,\psi(s)}(\sigma^{p,q}_g(\phi_p z)).
\]

The last expression manifestly belongs to \( A_g(q')(\sigma^{p,q}_g \circ \alpha_{\phi_p}(z)) \), and this contradicts (11).

So we have proved that there is some \( \gamma \in \Gamma_p \) such that \( \gamma^{-1} \phi_p \) acts trivially on \( S_g(p) \). Now statement (1) in Lemma [5.3] implies that \( \gamma^{-1} \phi_p = e^{t \alpha_p}(p) \) for some \( t \in \mathbb{R} \), so we may write \( \phi_p = \gamma = e^{t \alpha_p} \) or, equivalently, that \( \phi_p(y) = \Phi_t^\gamma(\gamma y) \) for every \( y \in W^u_g(p) \).

For any \( p \in \mathbb{E} \) we denote \( W_g(p) := W^s_g(p) \) (resp. \( W_g(p) := W^u_g(p) \)) if \( p \in \mathbb{I} \) (resp. if \( p \in \mathbb{O} \)). Now the proof of the case \( \dim M > 1 \) in Theorem [14] is concluded as the proof of the main theorem in [22]. This is done in two steps. We know there exist \( \{t_p \in \mathbb{R} \}_{p \in \mathbb{E}} \) and \( \{\gamma_p \in \Gamma \}_{p \in \mathbb{E}} \) such that \( \phi(x) = \Phi_{t_p}^{\gamma_p}(\gamma_p x) \) for every \( p \in \mathbb{E} \) and \( x \in W_g(p) \). The first step consists in proving that if all \( \gamma_p \)’s are equal then all \( t_p \)'s are equal as well (this is [22, Lemma 5]). The second step consists on reducing the general case to the one covered by the first step. This is explained in the three paragraphs following [22, Lemma 5].
Appendix A. Change of the Gradient Flow as the Metric Varies

Recall what we want to prove.

**Lemma A.1.** Suppose that $g \in \text{Met}^\Gamma$, $x \in M \setminus \text{Crit}(f)$ and $y = \Phi_t^g(x)$ for some nonzero $t$. Suppose that the stabilizer $\Gamma_x$ is trivial. Let $v \in T_xM$ be a nonzero vector, and let $w = D\Phi_t^g(x)(v)$. Given any $u \in T_y(TM)$ and any $\Gamma$-invariant open subset $U \subset M$ containing $\Phi_t^g(x)$ for some $t' \in (0, t)$, there exists some $g' \in C^\infty(M, S^2T^*M)^\Gamma$ supported on $U$ such that

$$
\frac{\partial}{\partial \epsilon} D\Phi_{t-\epsilon}^{g+\epsilon g'}(w) \bigg|_{\epsilon=0} = u.
$$

We will prove Lemma A.2 using the following weaker version of it.

**Lemma A.2.** Let $g, x, t, y$ be as in Lemma A.1. Given any $v \in T_yM$ and any $\Gamma$-invariant open subset $U \subset M$ containing $\Phi_t^g(x)$ for some $t' \in (0, t)$, there exists some $g' \in C^\infty(M, S^2T^*M)^\Gamma$ supported on $U$ such that

$$
\frac{\partial}{\partial \epsilon} \Phi_{t-\epsilon}^{g+\epsilon g'}(y) \bigg|_{\epsilon=0} = v.
$$

Before proving Lemma A.2 we prove two auxiliary lemmas.

**Lemma A.3.** Let $g \in \text{Met}^\Gamma$. Let $Y \in C^\infty(M; TM)^\Gamma$ satisfy $\text{supp} \ Y \cap \text{Crit}(f) = \emptyset$. There exists some $\delta > 0$ and a smooth map $G : (-\delta, \delta) \to \text{Met}^\Gamma$ such that $G(0) = g$ and, for every $\epsilon \in (-\delta, \delta)$, $\nabla^{G(\epsilon)} Y = \nabla^{g} Y + \epsilon Y$.

**Proof.** This is a consequence of the following elementary fact in linear algebra. Let $V$ be a finite dimensional real vector space and let $\alpha \in V^*$ be a nonzero element. Let $E \subset S^2V^*$ be the open subset of Euclidean pairings, and let $\nabla : E \to V$ be the map defined by the property that $e(\nabla(e), u) = \alpha(u)$ for every $u \in V$. Then $\nabla$ is a submersion with contractible fibers. \qed

For any vector field $X$ on $M$ we denote by $\Phi_t^X : M \to M$ the flow at time $t$ of $X$.

**Lemma A.4.** Let $X, Y \in C^\infty(M; TM)$. Suppose that $p \notin \text{supp} Y$ and that $X$ has no regular periodic integral curve. For any $t$ we have

$$
\frac{\partial}{\partial s} \Phi_t^{X+sL_XY}(p) \bigg|_{s=0} = Y(\Phi_t^X(p)).
$$

**Proof.** If $X(p) = 0$ then the formula is immediate. So suppose that $X(p) \neq 0$. Then $Z := \{\Phi_\tau^X | 0 \leq \tau \leq t\}$ is diffeomorphic to $[0, 1]$, because $X$ has no regular periodic integral curve. Take an open neighborhood $U \subset M$ of $Z$ and coordinates $x = (x_1, \ldots, x_n) : U \to \mathbb{R}^n$ with respect to which $p = (0, \ldots, 0)$ and $X = \frac{\partial}{\partial x_1}$, so that $\Phi_\tau^X(x_1, \ldots, x_n) = (x_1 + \tau, x_2, \ldots, x_n)$. Suppose that $x_\ast(L_XY|_U) = \sum a_j \frac{\partial}{\partial x_j}$ and that $x_\ast(Y|_U) = \sum b_j \frac{\partial}{\partial x_j}$. Since $Y(p) = 0$, we have $b_j(t, 0, \ldots, 0) = \int_0^t a_j(\tau, 0, \ldots, 0) \, d\tau$ for every $j$. Let $\gamma(t, s) = \gamma(t, s)$.
Let \( e_1, \ldots, e_n \) denote the canonical basis of \( \mathbb{R}^n \). We have
\[
\frac{\partial}{\partial t} \frac{\partial \gamma(t,s)}{\partial s} \bigg|_{s=0} = \frac{\partial}{\partial s} \frac{\partial \gamma(t,s)}{\partial t} \bigg|_{s=0} = \frac{\partial}{\partial s} \left( (1,0,\ldots,0) + s \sum a_j(\gamma(t,s)) e_j \right) \bigg|_{s=0} = \sum a_j(\gamma(t,0)) e_j = \sum a_j(t,0,\ldots,0) e_j.
\]
Consequently,
\[
\frac{\partial \gamma(t,s)}{\partial s} \bigg|_{s=0} = \sum \left( \int_0^t a_j(\tau,0,\ldots,0) d\tau \right) e_j = \sum b_j(t,0,\ldots,0) e_j = x_*(Y(\Phi^X_t(p))),
\]
which proves the desired formula.

Let us now prove Lemma \( \textbf{A.2} \). Fix some \( g \in \text{Met}^\Gamma \), let \( x \in M \setminus \text{Crit}(f) \), and let \( y = \Phi^g_t(x) \) for some nonzero \( t \). Suppose that the stabilizer \( \Gamma_x \) is trivial, let \( v \in T_xM \) be any element and let \( U \subset M \) be a \( \Gamma \)-invariant open subset containing \( \Phi^g_t(x) \) for some nonzero \( t \in (0,t) \). \( X = \nabla^g f \). Since \( x \notin \text{Crit}(f) \) and \( \Gamma_x \) is trivial, there exists an invariant vector field \( Y \in \mathcal{C}_\infty(M;TM)^\Gamma \) whose support is contained in \( U \setminus \text{Crit}(f) \) and which satisfies \( Y(x) = v \). By Lemma \( \textbf{A.3} \) we have
\[
\frac{\partial}{\partial s} \Phi^{X+sL_XY}_{-t} (y) \bigg|_{s=0} = Y(x).
\]
By Lemma \( \textbf{A.3} \) there exists some \( \delta > 0 \) and a family of metrics \( \{g_\epsilon\}_{\epsilon\in(-\delta,\delta)} \) satisfying \( g_0 = g \) and \( \nabla^{g_\epsilon} f = \nabla^g f + \epsilon Y \) for every \( \epsilon \in (-\delta,\delta) \). Setting \( g' = \partial g_\epsilon / \partial \epsilon |_{\epsilon=0} \) it follows that
\[
\frac{\partial}{\partial \epsilon} \Phi^{g+\epsilon g'}_{-t} (y) \bigg|_{\epsilon=0} = v.
\]
Finally we prove Lemma \( \textbf{6.4}/\textbf{A.1} \).

Let \( \pi : TM \rightarrow M \) denote the projection and let \( D\pi : T(TM) \rightarrow TM \) denote its derivative. A vector field \( X \) on \( M \) defines a vector field \( \tilde{X} \) on \( TM \) by the condition that
(14)
\[
D\Phi^X_t = \Phi^X_t
\]
for every \( t \). In particular, \( \Phi^X_{-t}(w) = v \).

We will use these properties of the map \( X \mapsto \tilde{X} \): (1) it is linear, (2) \( D\pi(\tilde{X}(u)) = X(\pi(u)) \) for every vector field \( X \) and any \( u \in TM \), (3) if \( X(a) = 0 \) for some point \( a \in M \), then the restriction of \( \tilde{X} \) to \( T_aM \) is vertical and can be identified with the linear vector field \( T_aM \rightarrow T_aM \) given by the endomorphism \( DX(a) \) of \( T_aM \), and (4) it is compatible with Lie brackets: \( [X,Y] = [\tilde{X},\tilde{Y}] \).

Let \( g \in \text{Met}^\Gamma \), let \( x \in M \setminus \text{Crit}(f) \) be a point with trivial stabilizer, and let \( y = \Phi^g_t(x) \) for some nonzero \( t \). Let \( v \in T_xM \) be nonzero and let \( w = D\Phi^g_t(x)(v) \). Finally, suppose given \( u \in T_{\gamma}(TM) \) and a \( \Gamma \)-invariant open subset \( U \subset M \) containing \( \Phi^g_t(x) \) for some \( t' \in (0,t) \).

Let \( X = \nabla^g f \). Since \( x \notin \text{Crit}(f) \) and \( \Gamma_x \) is trivial, there exists an invariant vector field \( Y_0 \in \mathcal{C}_\infty(M;TM)^\Gamma \) whose support is contained in \( U \setminus \text{Crit}(f) \) and which satisfies \( Y_0(x) = D\pi(u) \). Then \( u = Y_0(v) - u \) satisfies \( D\pi(u_1) = 0 \). Let \( L : T_xM \rightarrow T_xM \) be a linear map satisfying \( Lu = u_1 \), and let \( Y_1 \in \mathcal{C}_\infty(M;TM)^\Gamma \) have support contained in \( U \setminus \text{Crit}(f) \) and satisfy \( Y_1(x) = 0 \) and \( DY_1(x) = L \). Let \( Y = Y_0 + Y_1 \). Then \( \tilde{Y}(v) = u \).
By (14), the properties of the map $X \mapsto \tilde{X}$, and Lemma A.4 (with $M$ replaced by $TM$ in the statement), we have
\[
\frac{\partial}{\partial s} D\Phi^{-s}_{-t} LXY(w)\bigg|_{s=0} = \frac{\partial}{\partial s} \Phi^{-s}_{-t} sLXY(w)\bigg|_{s=0} = \frac{\partial}{\partial s} \tilde{\Phi}^{-s}_{-t} \tilde{X}(w)\bigg|_{s=0} = \tilde{Y}(v).
\]
By Lemma A.3 there exists some $\delta > 0$ and a smooth map $G : (-\delta, \delta) \to \text{Met}^\Gamma$ satisfying $G(0) = g$ and $\nabla^{G(\epsilon)} f = \nabla^g f + \epsilon \tilde{Y}$ for every $\epsilon \in (-\delta, \delta)$. Then $g' = G'(0)$ satisfies (13).

**Appendix B. Glossary**

Here we list in alphabetical order some of the symbols used in Section 3 and the next ones.

- $A_g(p)$ = the distribution on $S_g(p)^r$ given by the infinitesimal diagonal action of the Lie algebra of $L_g(p)$, see Subsection 5.2
- $\alpha_\psi$ = the map $S_g(p)^r \to S_g(p)^r$ given by the action of $\psi \in L_g(p)$
- $E = \mathbb{I} \cup \mathbb{O}
- f : M \to \mathbb{R} = \Gamma$-invariant Morse function on $M$, see Subsection 4.2
- $F_g(p)$ = a dense open subset of $S_g(p)^r$ on which the only elements of $L_g(p)$ with fixed points are those of the form $D\Phi^t_g(p)$, see Subsection 5.3
- $\Phi^t_g$ = the time $t$ gradient flow of $f$ w.r.t. the metric $g$
- $\Gamma$ = a finite group acting smoothly and effectively on $M$
- $\mathbb{I}$ = the critical points of $f$ of index $n = \dim M$ (sinks of $\nabla^g f$)
- $L_g(p)$ = the automorphisms of $T_p M$ commuting with $D\nabla^g f(p)$, see Subsection 4.2
  - if $p \in \mathbb{I}$ and $g \in \text{Met}_0$ then $L_g(p)$ is naturally isomorphic to $\text{Aut}(\nabla^g f|_{W^s_g(p)})$
  - (if $p \in \mathbb{O}$ then the same holds for $W^u_g(p)$), see Subsection 4.3
- $M$ = a compact connected smooth manifold of dimension at least 2
- $\text{Met}_0$ = the set of $\Gamma$-invariant metrics on $M$ defined in Subsection 4.3
- $\text{Met}_{1,K}$ = the set of $\Gamma$-invariant metrics on $M$ defined in Section 6
- $\mathbb{O}$ = the critical points of $f$ of index 0 (sources of $\nabla^g f$)
- $\Omega_g(p, q)$ = the projection to $S_g(q)$ of $W^s_g(p) \cap W^u_g(q)$ if $p \in \mathbb{I}$ and $q \in \mathbb{O}$, and the projection of $W^u_g(p) \cap W^s_g(q)$ if $p \in \mathbb{O}$ and $q \in \mathbb{I}$, see Subsection 5.1
- $S_g(p)$ = the set of nonconstant integral curves of $\nabla^g f|_{W^s_g(p)}$ (resp. $\nabla^g f|_{W^u_g(p)}$)
  - if $p \in \mathbb{I}$ (resp. $p \in \mathbb{O}$), see Subsection 5.1
- $\sigma^{p,q}_g$ = the natural isomorphism $\Omega_g(p, q) \to \Omega_g(q, p)$, see Subsection 5.1
- $W^s_g(p)$ = the stable set of $p \in \mathbb{I}$
- $W^u_g(q)$ = the unstable set of $q \in \mathbb{O}$
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