Synthesis of Control and State Observer for Weakly Nonlinear Systems Based on the Pseudo-Linearization Technique

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Abstract—In this paper an approach for the construction of nonlinear output tracking control on a finite time interval for a class of weakly nonlinear systems with state-dependent coefficients is considered. The proposed method of control synthesis consists of two main stages. On the first stage, a nonlinear state feedback regulator is constructed using a previously proposed control algorithm based on the State-Dependent Riccati Equation (SDRE). On the second stage, the problem of full-order observer construction is formulated and then is reduced to the differential game problem. The form of its solution is obtained with the help of the guaranteed (minimax) control principle, which allows to find the best observer coefficients with respect to a given functional considering the worst-case uncertainty realization. The form of the obtained equations made it possible to use the algorithm from the first stage to determine the observer matrix. The proposed approach is characterized by the nonapplicability of the estimation and control separation principle used for linear systems, since the matrix of observer coefficients turned out to be dependent on the feedback coefficients matrix. The use of numerical-analytical procedures for determination of observer and feedback coefficients matrices significantly reduces the computational complexity of the control algorithm.

Keywords: tracking problem, nonlinear control, state-dependent Riccati equation, minimax control

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1. INTRODUCTION

Currently, many approaches are used to solve nonlinear control problems. One promising approach is a technique based on solving matrix State-Dependent Riccati Equations (SDREs) (see reviews [1, 2]). The essence of this approach is to represent the original system in a pseudo-linear form, in which the system and the control matrices depend on the state. The control law has a standard form for a linear-quadratic problem, however, for its implementation during the control process, it is necessary to numerically find a solution to the corresponding SDRE at each time point, which may run into computational limitations. In [3], an SDRE-based approach was proposed for solving the finite-horizon optimal tracking problem for weakly nonlinear systems with fully measurable state vector. Since finding the exact solution in the general case seems to be a computationally time-consuming task, a numerical-analytical algorithm for approximate synthesis has been proposed, which significantly reduces the computational complexity compared to the standard SDRE technique. However, to apply the approach from [3], it is necessary to construct an appropriate observer, which allows one to obtain estimates of the unknown states of the system. In this paper the underlined problem is reduced to the problem of control synthesis under uncertainty, for the solution of which the minimax principle from [4–6] is used.

1. THE CONTROL DESIGN

Let us consider the following control system

\begin{equation}
\dot{x} = A(x, \mu)x + B(x, \mu)u, \quad y = Cx, x(t_0) = x^0,
\end{equation}

\begin{align*}
A(x, \mu) &= A_0 + \mu A_1(x), \\
B(x, \mu) &= B_0 + \mu B_1(x),
\end{align*}

\begin{align*}
x &\in X \subset \mathbb{R}^n, \quad y \in Y \subset \mathbb{R}^m, \quad u \in \mathbb{R}^r, \quad t \in [t_0, t_1], \quad 0 < \mu \leq \mu_0,
\end{align*}

\textbf{824}
where \( x, y \) and \( u \) are the state, output, and control vectors respectively, \( \mu_0 \) is some given sufficiently small positive constant, \( A_0, B_0 \) and \( C \) are known constant matrices, \( A_i(x) \in \mathbb{R}^{n \times n}, B_i(x) \in \mathbb{R}^{m \times n} \) are known matrices with elements that are sufficiently smooth and bounded by the argument \( x \). \( X \) and \( Y \) are some bounded sets. It’s assumed that for any continuous control \( u(t) \) the trajectories of the closed loop system (1) exist, are unique and belong to \( X \) on \([t_0, t_1]\).

Let the reference behavior of system (1) be described by solving a differential equation

\[
\dot{x}_r = A_r(x_r, \mu)x_r, \quad y_r = Cx_r, \quad x_r(t_0) = x^0_r,
\]

\[
A_r(x_r, \mu) = A_{r,0} + A_{r,1}(x_r), \quad x_r \in X, y_r \in Y, \quad t \in [t_0, t_1],
\]

where \( x_r \) and \( y_r \) are the desired (reference) trajectory of the system and its output, \( A_{r,0} \) is the known constant matrix, and \( A_{r,1}(x_r) \) is the known matrix with elements that are sufficiently smooth and bounded by the argument \( x_r \). The initial states \( x^0_r \) and \( x^0 \) are generally assumed to be unknown. If \( x^0_r \) is known, then the entire reference trajectory is actually known in advance. For example, the use of a special master device, the initial state of which can be selected, can correspond to this case.

Let us define the following cost function

\[
I(u) = \frac{1}{2} \int_{t_0}^{t_1} e^T(t_i)Fe(t_i) + \frac{1}{2} \int_{t_0}^{t_1} (\dot{e}^TQ(y, y_r, \mu)e + u^TRu) dt \rightarrow \min_u.
\]

where \( Q(y, y_r, \mu) \geq 0, Q_0 > 0, R > 0, F > 0 \) are given symmetric matrices for \( y, y_r \in Y, 0 < \mu \leq \mu_0 \). Hereinafter, the signs \( >0 (\geq 0) \) denote the positive definiteness (semi-definiteness) of the corresponding matrix. It is necessary to find such a continuous output control \( u(y, \mu, t) \) that provides an approximate solution to problem (1), (2).

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It is known that the original problem (1), (2) can be represented [7] in the form

\[
\dot{\tilde{x}} = \tilde{A}(\tilde{x}, \mu)\tilde{x} + \tilde{B}(x, \mu) u, \quad \tilde{x}(t_0) = \tilde{x}^0,
\]

\[
\tilde{I}(u) = \frac{1}{2} \int_{t_0}^{t_1} \dot{\tilde{x}}^T(t_i)\tilde{F}\dot{\tilde{x}}(t_i) + \frac{1}{2} \int_{t_0}^{t_1} (\dot{\tilde{x}}^TQ(\tilde{y}, \mu)\dot{\tilde{x}} + u^TRu) dt \rightarrow \min_u,
\]

where \( \tilde{x} = \begin{bmatrix} x \\ x_r \end{bmatrix} \in \mathbb{R}^{2n}, \quad \tilde{y} = \begin{bmatrix} y \\ y_r \end{bmatrix} \in \mathbb{R}^{2n} \) are extended state and output vectors,

\[
\tilde{A}(\tilde{x}, \mu) = \begin{bmatrix} A(x, \mu) & 0 \\ 0 & A_r(x_r, \mu) \end{bmatrix} \in \mathbb{R}^{2n \times 2n}, \quad \tilde{B}(x, \mu) = \begin{bmatrix} B(x, \mu) \\ 0 \end{bmatrix} \in \mathbb{R}^{2n \times m},
\]

\[
\tilde{Q}(\tilde{y}, \mu) = \begin{bmatrix} C^TQ(\tilde{y}, \mu)C & -C^TQ(\tilde{y}, \mu)C \\ -C^TQ(\tilde{y}, \mu)C & C^TQ(\tilde{y}, \mu)C \end{bmatrix} \in \mathbb{R}^{2n \times 2n} \geq 0,
\]

\[
\tilde{F} = \begin{bmatrix} C^TFC & -C^TFC \\ -C^TFC & C^TFC \end{bmatrix} \in \mathbb{R}^{2n \times 2n} \geq 0,
\]

zero blocks in the matrices \( \tilde{A}(\tilde{x}, \mu) \) and \( \tilde{B}(x, \mu) \) are the zero matrices of the corresponding dimensions.

Let’s define the following equations

\[
\tilde{A}(\tilde{x}, \mu) = \tilde{A}_0 + \mu \tilde{A}_1, \quad \tilde{B}(x, \mu) = \tilde{B}_0 + \mu \tilde{B}_1(x), \quad \tilde{Q}(\tilde{y}, \mu) = \tilde{Q}_0 + \mu \tilde{Q}_1(\tilde{y}),
\]

\[
\tilde{A}_0 = \begin{bmatrix} A_0 & 0 \\ 0 & A_{r,0} \end{bmatrix}, \quad \tilde{A}_1 = \begin{bmatrix} A(x) & 0 \\ 0 & A_{r,1}(x_r) \end{bmatrix}, \quad \tilde{B}_0 = \begin{bmatrix} B_0 \\ 0 \end{bmatrix}, \quad \tilde{B}_1(x) = \begin{bmatrix} B_1(x) \\ 0 \end{bmatrix},
\]

\[
\tilde{Q}_0 = \begin{bmatrix} C^TQ_0C & -C^TQ_0C \\ -C^TQ_0C & C^TQ_0C \end{bmatrix}, \quad \tilde{Q}_1(\tilde{y}) = \begin{bmatrix} C^TQ_1(\tilde{y})C & -C^TQ_1(\tilde{y})C \\ -C^TQ_1(\tilde{y})C & C^TQ_1(\tilde{y})C \end{bmatrix}.
\]
Let’s define also the following conditions

(i) The trajectories of the closed loop system (1) exist, are unique and belong to $X$ on $[t_0, t_1]$ for any continuous control $u(t)$, where $X$ is some bounded set of state space, elements of $\bar{A}(\bar{x})$. $\bar{B}(x)$ are bounded, continuous, and sufficiently smooth for $x, x_0 \in X; \mu \in (0, \mu_0]$.

(ii) The triple $[\bar{A}_r, \bar{B}_r, H_r]$, where $H_r^T H_r = \bar{Q}_0$, is stabilizable and observable.

(iii) The system matrices $\bar{A}_r, \bar{A}(x), \bar{B}_r, \bar{B}(x)$ the cost function symmetric matrices $R > 0, Q_0 \geq 0, Q(y) \geq 0, F > 0$, and $\mu_0 > 0$ are such that $\bar{P} + \mu \bar{P}(\bar{x}, t) > 0$ for $x, x_0 \in X, t \in [t_0, t_1], \mu \in (0, \mu_0]$.

In [3] under conditions (i)–(iii) for a fully measurable system (the vector $x$ is exactly measured) the following control was proposed that approximately solves problem (3)

$$u(\bar{x}, \mu, t) = -K(\bar{x}, \mu, t)\bar{x} = u_0(\bar{x}) + \mu u_0(\bar{x}, t, \mu),$$

where $K(\bar{x}, \mu, t) = R^{-1}(\bar{B}_0 + \mu \bar{B}(\bar{x}))^T (\bar{P}_0 + \mu \bar{P}(\bar{x}, t)), u_0(\bar{x}) = -R^{-1}\bar{B}_0^T \bar{P}_0 \bar{x}$ is the linear part and $\mu u_0(\bar{x}, t, \mu) = -\mu R^{-1}\left[\bar{B}_0^T (\bar{x}) \bar{P}_0 + (\bar{B}_0 + \mu \bar{B}(\bar{x}))^T \bar{P}(\bar{x}, t)\right] \bar{x}$ forms a nonlinear correction. According to [3] under conditions (i)–(iii), we have the following numerical-analytical algorithm for constructing an approximate control in the finite-horizon optimal nonlinear tracking problem.

(1) Find $u_0(\bar{x})$ as $u_0(\bar{x}) = -R^{-1}\bar{B}_0^T \bar{P}_0 \bar{x}$, where $\bar{P}_0$ is a positive definite solution of the next matrix Riccati equation $\bar{P}_0\bar{A}_0 + \bar{A}_0^T \bar{P}_0 - \bar{P}_0\bar{B}_0 R^{-1} \bar{B}_0^T \bar{P}_0 + \bar{Q}_0 = 0$.

(2) Find $\bar{P}(\bar{x}, \mu, t)$ as

$$\bar{P}(\bar{x}, \mu, t) = e^{\bar{A}(\bar{x}, t_0)\mu} M \mu = e^{\bar{A}(\bar{x}, t_0)\mu} + \int_0^\infty e^{\bar{A}(\bar{x}, t_0)\mu} D_\mu e^{\bar{A}(\bar{x}, t_0)\mu} d\sigma,$$

where $D_\mu(\bar{x}_0) = \bar{P}_0\left(\bar{A}_0 - \bar{B}_0 R^{-1} \bar{B}_0^T \bar{P}_0\right) + \left(\bar{A}_0 - \bar{B}_0 R^{-1} \bar{B}_0^T \bar{P}_0\right)^T \bar{P}_0 + \bar{Q}_1, \bar{A}_{\mu>0} = \bar{A}_0 - \bar{B}_0 R^{-1} \bar{B}_0^T \bar{P}_0$ and the matrix $M$ is calculated by means of expression

$$M = \frac{1}{\mu} (\bar{P} - \bar{P}_0) - \int_0^\infty e^{\bar{A}(\bar{x}, t_0)\mu} D_\mu (\bar{x}(t))|_{t_0}^{t} e^{\bar{A}(\bar{x}, t_0)\mu} d\sigma.$$

(3) Finally, define the control $u(\bar{x}, \mu, t)$.

Let’s us make a few comments. Since the future trajectory of the system is unknown, in the proposed algorithm the matrix $D_\mu(\bar{x}(t))$ is calculated under the assumption that $\bar{x}(t)$ and $\bar{x}(t)$ slightly differ near $t_1$, i.e. $\bar{x}(t)$ is used instead of $\bar{x}(t)$. If the reference trajectory is known in advance, then, assuming that $x(t_i) = x_i(t_i)$, we can calculate $D_\mu$, using $x(t_i) = x_i(t_i)$. The “rigidity” of these two assumptions is mitigated by the fact that the first term in (5) is significant only in the neighborhood of $t_1$ due to $Re(\bar{A}_{\mu>0}) < 0$. Note that the algorithm offers an analytical formula for an unknown state-dependent matrix $\bar{P}(\bar{x}, t)$, which significantly reduces the computational complexity of the control algorithm.

In addition, the following is true

$$\bar{P}(\bar{x}, \mu, t) = \bar{P}_0 + \mu \bar{P}(\bar{x}, t, \mu)

= e^{\bar{A}(\bar{x}, t_0)\mu} \bar{P}_0 + \mu e^{\bar{A}(\bar{x}, t_0)\mu} \bar{P}_0 e^{\bar{A}(\bar{x}, t_0)\mu}

+ \mu \int_0^\infty e^{\bar{A}(\bar{x}, t_0)\mu} D(\bar{x}(t)) e^{\bar{A}(\bar{x}, t_0)\mu} d\sigma - e^{\bar{A}(\bar{x}, t_0)\mu} \bar{P}_0 e^{\bar{A}(\bar{x}, t_0)\mu}.$$

Thus, since $e^{\bar{A}(\bar{x}, t_0)\mu} \bar{P}_0 e^{\bar{A}(\bar{x}, t_0)\mu} + \bar{P}_0 e^{\bar{A}(\bar{x}, t_0)\mu} > 0$ for $t \in [t_0, t_1]$, then the condition (iii) can always be fulfilled for sufficiently small $\mu_0$. In addition, the condition (iii) can be met, if $D(\bar{x}(t)) > 0$ for $x, x_0 \in X, t \in [t_0, t_1]$.

The proposed algorithm can be applied to the output control problem by constructing an appropriate observer that determines estimates of the unknown state coordinates at each time moment.
2. THE OBSERVER DESIGN

We define the equation of a full order observer as follows [8]
\[
\dot{\hat{x}} = \hat{A}(\hat{x}, \mu)\hat{x} + \hat{B}(x, \mu)u + \Gamma\hat{C}(\hat{x} - \hat{x}), \quad \hat{x}(0) = \hat{x}^0,
\]
(6)
where \(\hat{x}\) is an estimation vector of state \(\hat{x}\), i.e. \(\chi\) is the estimate of \(x\), and \(\chi\) is the estimate of \(x\), \(\chi, \mu, \in \Gamma \in R^n\), and \(\Gamma = \in R^{2m+2n}\) is a observer gain matrix to be determined. Subtracting (6) from the first equation of system (3), we have
\[
\dot{\hat{x}} - \dot{x} = \hat{A}(\hat{x}, \mu)\hat{x} - \hat{A}(x, \mu)x + \hat{B}(x, \mu)u - \Gamma\hat{C}(\hat{x} - \hat{x})
\]
\[
= (\hat{A}_0 - \Gamma\hat{C})(\hat{x} - \hat{x}) + \mu(\hat{A}(\hat{x})\hat{x} - \hat{A}(x)x) + (\hat{B}(x) - \hat{B}(x))u
\]

Let’s introduce the tracking error \(e_x = \hat{x} - \chi\), then the last equation can be rewritten in the form
\[
\dot{e}_x = (\hat{A}_0 - \Gamma\hat{C})e_x + \mu(\hat{A}(\hat{x})\hat{x} - \hat{A}(x)x) + (\hat{B}(x) - \hat{B}(x))u
\]

Let’s note that \(\hat{x}\) and \(x\) are unknown here. Consider the inhomogeneity present at \(\mu\). Given that control (4) need to be based on an observer, it is fair that \(u = -K(\tilde{x}, \mu, t)\tilde{x}\). Then we have
\[
\hat{A}(\hat{x})\hat{x} - \hat{A}(x)x + (\hat{B}(x) - \hat{B}(x))K\tilde{x} = \hat{A}l(\tilde{x})\hat{x} - \hat{A}(x)x + (\hat{B}(x) - \hat{B}(x))K\tilde{x}
\]
\[
+ (\hat{A}(\hat{x}) + (\hat{B}(x) - \hat{B}(x))K)\hat{x} - (\hat{A}(\hat{x}) + (\hat{B}(x) - \hat{B}(x))K)\hat{x} = \hat{A}(\hat{x})\hat{x} - \hat{A}(x)x + (\hat{B}(x) - \hat{B}(x))K\hat{x}
\]
\[
+ (\hat{A}(\hat{x}) + (\hat{B}(x) - \hat{B}(x))K)e_x = (\hat{A}(\hat{x}) - \hat{B}(x)K)e_x + \hat{B}(x)e_x + (\hat{A}(\hat{x}) - \hat{B}(x)K)e_x
\]
\[
= (\hat{A}(\hat{x}) - \hat{B}(x)K)e_x + l(\tilde{x}, \chi, \mu, t)
\]

where \(l(\tilde{x}, \chi, \mu, t) = (\hat{A}(\hat{x}) - \hat{B}(x)K)e_x + l(\tilde{x}, \chi, \mu, t)\) \(e_x \in R^{2n}\) is unknown vector.

We represent \(l\) in the form \(l(\tilde{x}, \chi, \mu, t) = L(\tilde{x}, \chi, \mu)\), where \(L(\tilde{x}, \chi, \mu) \in R^{2m+2n}\) is an unknown matrix. Then the error dynamics equation has the form
\[
\dot{e}_x = (\hat{A}_0 - \Gamma\hat{C} + \mu(\hat{A}(\hat{x}) - \hat{B}(x)K)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)e_x + \hat{B}(x)
(v) $\tilde{C}^T R_c^{-1} \tilde{C} - R_c^{-1} > 0$.

By analogy with [6], where the control interval is assumed to be sufficiently large compared to the transition process, when (iv), (v) are fulfilled, the control and disturbance laws are defined as $v = R_c^{-1}C N(\tilde{\chi}, \mu, t)e_\chi$, $\nu = R_c^{-1}N(\tilde{\chi}, \mu, t)e_\chi$, where $N(\tilde{\chi}, \mu, t)$ is a positive definite solution of the next SDRE

$$\dot{N} + N \tilde{A}^T + \tilde{A}N - N \left( \tilde{C}^T R_c^{-1} \tilde{C} - R_c^{-1} \right) N + Q_c = 0, \quad N(t) = F_c.$$  \hspace{1cm} (9)

From the obtained relations it follows that

$$\Gamma(\tilde{\chi}, \mu, t) = N(\tilde{\chi}, \mu, t)\tilde{C}^T R_c^{-1},$$

$$L(\tilde{\chi}, \mu, t) = \frac{1}{\mu} N(\tilde{\chi}, \mu, t)R_c^{-1}. \hspace{1cm} (10)$$

The main difficulty here is the solution of differential SDRE (9). However, it can be noted that problem (3) formulated above and problem (7), (8) have a similar structure. Therefore, for an approximate solution of (9) in the problem of observer construction one can apply the above algorithm from [3], which was used to solve a similar SDRE for the control synthesis. To do this, we introduce the conditions

(vi) The trajectories of the closed loop system (7) on $[t_0, t_1]$ exist, are unique and $\chi, \chi_r \in X$ for any continuous $v(t)$, $\nu(t)$, matrix elements of $\tilde{A}(\tilde{\chi}, \mu, t)$ are bounded, continuous, and sufficiently smooth for $\chi, \chi_r \in X$, $\mu \in (0, \mu_0)$, $t \in [t_0, t_1]$.

(vii) The triple $\{\tilde{A}^T, \tilde{C}, H_N\}$, where $H_{N}^2 H_N = Q_{c,0}$, are stabilizable and observable.

(viii) The system matrices $\tilde{A}_\theta \tilde{A}_{d,1}(\tilde{\chi}, \mu, t), \tilde{C}$, the cost function matrices $R_c > 0$, $R_0 > 0$, $Q_{c,0} \geq 0$, $Q_{c,1}(\tilde{\chi}) \geq 0$, $F_c \geq 0$ and $\mu_0 > 0$ are such that $N_0 + \mu N_1(\tilde{\chi}, \mu, t) > 0$ for $\chi, \chi_r \in X$, $t \in [t_0, t_1]$, $\mu \in (0, \mu_0]$.

Thus, in accordance with Section 1, the state control is defined firstly. Then, under conditions (iv)–(viii) the observer is constructed by means of the following numerical-analytical procedure.

1. Find $N_0$ as a positive definite solution to the next equation

$$N_0 \tilde{A}_0^T + \tilde{A}_0 N_0 = N_0 \left( \tilde{C}^T R_c^{-1} \tilde{C} - R_c^{-1} \right) N_0 + Q_{c,0} = 0.$$

2. Find $N_1(\tilde{\chi}, \mu, t)$ as

$$N_1(\tilde{\chi}, \mu, t) = e^{\tilde{A}_{N,1}(d, t)} M_N e^{\tilde{A}_{N,1}(d, 0)} + \int_{0}^{\infty} e^{\tilde{A}_{N,1} \sigma} D_N(\tilde{\chi}, \mu, t) e^{\tilde{A}_{N,1} \sigma} d\sigma,$$

where

$$M_N = \frac{1}{\mu} (F_c - N_0) - \int_{0}^{\infty} e^{\tilde{A}_{N,1} \sigma} D_N(\tilde{\chi}(t), \mu, t) \left|_{\tilde{\chi}(t) = \tilde{\chi}(t)} \right. e^{\tilde{A}_{N,1} \sigma} d\sigma,$$

$$D_N(\tilde{\chi}, \mu, t) = N_0 \tilde{A}_{d,1}^T(\tilde{\chi}, \mu, t) + \tilde{A}_{d,1}(\tilde{\chi}, \mu, t) N_0 + Q_{c,1}(\tilde{\chi}),$$

$$\tilde{A}_{N,1}(t) = \tilde{A}_0^T - \left( \tilde{C}^T R_c^{-1} \tilde{C} - R_c^{-1} \right) N_0.$$  \hspace{1cm} (10)

3. Find $\Gamma(\tilde{\chi}, \mu, t)$ by means of (10), where $N(\tilde{\chi}, \mu, t) = N_0 + \mu N_1(\tilde{\chi}, \mu, t)$.

4. Finally, find the observer (6), by setting the initial state $\tilde{\chi}^0$ in an arbitrary way.

Now we can apply the control (4), using estimate vector $\tilde{x}$ instead of the unknown state vector $\tilde{\chi}$.

**Comment 1.** In contrast to linear systems [9], in this case, the principle of separation of control synthesis and observer synthesis problems is not fully satisfied, since the observer gain matrix $I$ depends on the feedback gain matrix $K$.

**Comment 2.** If the initial state $x^0$ or $x_r^0$ is known, then for $t \geq t_0$ using the observer, one can obtain absolutely accurate estimates of the vector $x$ or $x_r$, respectively, by setting $\chi^0 = x^0$ or $\chi_r^0 = x_r^0$. If the reference path $x_r$ is known in advance (which means $\chi_r^0 = x_r^0$ according to the remark above), then, assuming the proximity of $x(t_1)$ to $x_r(t_1)$ and the proximity of $\chi(t_1)$ to $\chi_r(t_1)$, one may calculate $D_N$, using $\chi(t_1) = \chi_r(t_1) = x_r(t_1)$.
Comment 3. Like the condition (iii), the condition (viii) may be fulfilled for a sufficiently small $\mu_0$.

CONCLUSIONS

This paper devoted to the approach to constructing a nonlinear observer-based control for the finite-horizon tracking problem for a weakly nonlinear system. The construction of a state observer is based on the principle of guaranteed control. The synthesis of the control and the observer makes it necessary to consider similar matrix state-dependent Riccati equations, and the same approximate approach is used to solve them. The approach main advantage is the use of analytical expressions, which significantly reduce computational costs compared to the usual approach, when the corresponding Riccati equation is solved in the control process for each state of the system.

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CONFLICT OF INTEREST

The authors declare that they have no conflicts of interest.

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