Non-holonomic tomography I: The Born rule as a connection between experiments

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In the context of quantum tomography, we recently introduced a quantity called a partial determinant [1]. PDs (partial determinants) are explicit functions of the collected data which are sensitive to the presence of state-preparation-and-measurement (SPAM) correlated errors. As such, PDs bypass the need to estimate state-preparation or measurement parameters individually. In the present work, we suggest a theoretical perspective for the PD. We show that the PD is a holonomy and that the notions of state, measurement, and tomography can be generalized to non-holonomic constraints. To illustrate and clarify these abstract concepts, direct analogies are made to parallel transport, thermodynamics, and gauge field theory. This paper is the first of a two part series where the second paper [2] is about scalable generalizations of the PD in multiqubit systems, with possible applications for debugging a quantum computer.

I. INTRODUCTION

In quantum computing, a recent problem has been learning how to estimate quantum gates while taking into account that there are small but significant errors in the states prepared and measurements made to probe such gates, so called SPAM errors [3]. Several works have come out to solve this, [3–5], all of which speak to the notion of a “self-consistent tomography.” These works also make an important common assumption: that the uncontrollable fluctuations in the SPAM are not correlated. So in [1] the obvious question was asked: what if the states and measurements made were actually correlated with each other?

Even though this question can be asked for classical systems, this is an especially interesting question for quantum systems. Standard quantum theory tells us that reality articulates itself as discrete events. The probabilities of these events are further understood to be the product of two things: a state and a set of possible outcomes. More precisely, the Born rule in its modern form tells us that the distribution of these events is the inner product of a density operator and a POVM. This is what makes a quantum theory distinct from a classical one as it allows for fundamental randomness because a state is no longer an outcome in itself: state and outcome become statistically independent quantities, consistent with all possible state and measurement settings. However, one can still define average state and average measurement parameters locally over the space of device settings. A simple but subtle example of such locally defined quantities can be found in thermodynamics — the caloric, “Q”, and potential energy, “W”, represented by inexact heat and work forms which sum to changes in the energy, \( dU = dQ + dW \), which is globally defined over the thermodynamic state space. A more standard example can be found in quantum electrodynamics — the electron kinetic momentum, \(-iD_{\mu}\), and the photon vector potential, \(A_{\mu}\), which sum to the canonical momentum, \(-i\partial_{\mu} = -iD_{\mu} + A_{\mu}\), globally defined over position space.

In order to illustrate these analogies explicitly, we will consider a toy analogy to quantum tomography with SPAM errors. This toy model replaces the state and measurement with single parameters, which can be correlated. We demonstrate precisely how the toy analog of the partial determinant from [1] has the same structure as \(\oint dQ\) from thermodynamics or \(\oint A \cdot dx\) from QED. Such “loop” integrals are generally called holonomies and the forms they integrate can be referred to as non-holonomic constraints. Finally, we translate these results to actual quantum tomography, completing the perspective of non-holonomic quantum tomography.

II. STATE-PREPARATION, MEASUREMENT, THEIR CORRELATION, AND DATA

A. The Born Rule and Tomography

One could say that the Born rule was originally, since the 1920s, used exclusively to predict the distributions of events from states and observables. Standard textbook treatments will denote the Born rule by

\[
P(s | \psi) = | \langle s | \psi \rangle |^2,
\]

thus introducing the notions of state and measurement outcome. Statistical observables [6] are then calculated from classical probability theory and typical expressions like
appear, introducing the notions of a classically mixed state and a quantum observable. Since distinct quantum systems can interact, the notion of an ancilla can be introduced and measurements can be generalized from an orthonormal basis to a positive operator valued measure (POVM).

In more recent years, the Born rule has found a different application in so called quantum state tomography [7, 8], where states are concluded from the distribution of measured events and various known POVMs. After this, it was quickly recognized that the Born rule could just as well be used for so called detector tomography [9, 10], where POVM elements are concluded from the distribution of events and known states. It had even been demonstrated that one could perform state tomography through unknown POVMs from other known states with a technique similar to applying the Born rule twice, bypassing the need to parameterize unknown POVMs [11].

Any application of the Born rule where both state preparation and measurement are unknown [3–5] we will henceforth refer to as SPAM tomography. The central feature which makes SPAM tomography distinct from other tomographies is the presence of gauge degrees of freedom. In this case, state and measurement parameters are explicitly inseparable because the Born rule cannot uniquely determine them individually from the statistics alone. Work has been done to recover unique estimates for individual state and measurement parameters [5] under further assumptions. Of course, such work also makes the implicit assumption that there are no correlated SPAM errors.

\[
\langle s \rangle = \sum_s s P(s) = \sum_{s, \psi} s P(s|\psi) P(\psi) = \text{Tr} \left( \sum_\psi P(\psi) \langle \psi | \psi \rangle \right) \left( \sum_s s |s \rangle \langle s | \right) = \text{Tr} \rho \Sigma = \langle \Sigma \rangle \tag{1}
\]

In the context of our work, where we do allow for correlated SPAM errors, a crucial point must be made concerning our use of the \( \langle \rangle \) notation. On the leftmost side of Equation (1), \( \langle \rangle \) refers to the expectation value of a random variable, \( s \). On the rightmost side of Equation (1), \( \langle \rangle \) loses this meaning as it does not refer to the expectation value of an operator, \( \Sigma \), but rather an inner product of \( \Sigma \) with the density operator. In both cases, the distribution of quantum events is completely attributed to the state and this assumption is perhaps further obscured by Dirac’s bra-ket notation. For our purposes in SPAM tomography, we will not use \( \langle \rangle \) in this way, beyond Equation (1). Rather, \( \langle \rangle \) will refer to an expectation value where states and observables are themselves considered random variables. Specifically, if \( \rho \) is a density operator representing the state and \( E \) is a POVM element representing a possible outcome, then one must understand that

\[
f = \langle \text{Tr} \rho E \rangle \tag{2}
\]

where \( f \) is an estimate of the probability (obtained from the frequency of the measured outcome) and \( \langle \rangle \) is the
average over the ensemble of trials. The measured frequency is SPAM correlated if
\[ \langle \text{Tr} \rho E \rangle \neq \text{Tr} \langle \rho | E \rangle. \] (3)
We will examine such correlations in a much simpler context in the next section but details and examples may also be found in [1].

B. A Toy Example

The problem of whether states and measurements are correlated is fundamentally interesting because states and observables are not individually accessible in principle by experiment alone. In other words, the quantities of the right hand side of Equation (3) cannot be measured without arbitrarily well characterized devices. Nevertheless, it was demonstrated in [1] that there is still a way to detect such correlations using properties of the data alone, bypassing the need to estimate state and observable parameters separately. The basic essence of that result can be illustrated by the following toy problem:

\[ \begin{align*}
\text{FIG. 2. On the left is a device which prepares various signals on demand depending on which button, } a \in \{1, \ldots, N\}, \text{ is pressed. On the right is a device which blinks to indicate a signal with a certain property depending on which setting, } i \in \{1, \ldots, M\}, \text{ a dial is turned to.} \\
\end{align*} \]

Consider a device with various settings, \( a \), each of which prepare a different signal on demand by the press of a button. Consider also a detector with various settings, \( i \), each of which detect a particular property of the signal indicated by the blink of a light. Now suppose it is suspected that each setting of the preparation device actually produces the same signal, only that each setting produces the signal with varying probabilities of success, \( p_a \). Suppose further that the detection device is expected to simply indicate the presence of the signal, only that each setting can register a signal with varying probabilities of success, \( w_i \). We then imagine that \( p_a \) and \( w_i \) are actually unknown and that we are only able to change settings and record whether the light blinks or not.

Let us indicate by \( f_a^i \) the measured frequency with which the light blinks when the devices are set to \( (a,i) \). If one can assume that the performance of the devices and their settings are uncorrelated, then one can simply identify (after many runs of the experiment)
\[ f_a^i = p_a w_i. \] (4)
However, relaxing this assumption to allow for the possibility of correlations, one must be more careful about the quantities defined so to make the more general identification that
\[ f_a^i = \langle pw \rangle_a^i \] (5)
where we have introduced notation \( \langle pw \rangle_a^i \) to represent the average over the ensemble of trials for the pair of settings \( (a,i) \). The subtlety here is that the devices can still be represented by single parameters, \( p \) and \( w \), only now these parameters are to be understood as random variables which fluctuate depending on the setting \( (a,i) \).

The presence of SPAM correlation is simply when the frequencies, \( \langle pw \rangle_a^i \) (which are what we have access to) are such that
\[ \langle pw \rangle_a^i \neq \langle p \rangle_a \langle w \rangle^i. \] (6)
It would seem that to identify such a circumstance one would have to measure \( \langle p \rangle_a \) and \( \langle w \rangle^i \) individually. However, such measurements would require devices which are already well characterized, unlike the devices we have. What we would like to do is detect if such correlations are present between our devices given only our humble, imperfect, uncalibrated devices.

C. Gauge Degrees of Freedom

In such a situation, one must acknowledge that there will always be so called gauge degrees of freedom. If one was given the promise that a pair of device parameters were in fact SPAM uncorrelated, then there would still be a one-parameter family of possible values for the average state parameter and average detector parameter. Specifically, for a possible pair of values \( (\langle p \rangle_a, \langle w \rangle^i) \) such that \( \langle pw \rangle = \langle p \rangle \langle w \rangle \), the pair \( (g(\langle p \rangle), g^{-1}(\langle w \rangle)) \) is just as possible. If the devices are SPAM uncorrelated over a range of settings \( a \in \{1, \ldots, N\} \) and \( i \in \{1, \ldots, M\} \), then the set of possible average values continue to define exactly one gauge parameter. This is perhaps best illustrated by observing \( \langle pw \rangle_a^i = \langle p \rangle_a \langle w \rangle^i \) as a matrix equation,
\[ \begin{bmatrix} \langle pw \rangle_1^1 & \ldots & \langle pw \rangle_1^M \\ \vdots & \ddots & \vdots \\ \langle pw \rangle_N^1 & \ldots & \langle pw \rangle_N^M \end{bmatrix} = \begin{bmatrix} \langle p \rangle_1 \\ \vdots \\ \langle p \rangle_N \end{bmatrix} \begin{bmatrix} \langle w \rangle^1 & \ldots & \langle w \rangle^M \end{bmatrix}, \] (7)
so that if \( \left( [\langle p \rangle_1 \ldots \langle p \rangle_N]^T, [\langle w \rangle^1 \ldots \langle w \rangle^M] \right) \) is possible, then so is \( g \left( [\langle p \rangle_1 \ldots \langle p \rangle_N]^T, g^{-1} \left( [\langle w \rangle^1 \ldots \langle w \rangle^M] \right) \right) \).

To handle this gauge degree of freedom, it is useful to define the following notion: The collected data, \( \langle pw \rangle_a^i \), for a pair of devices is effectively (SPAM) uncorrelated if Equation (7) exists—that is, if the experimentally accessible left-hand side can be expressed as in the right-hand side for some \( \left( [\langle p \rangle_a]^T, [\langle w \rangle^i] \right) \). Considered as a matrix, \( D = \left( [\langle pw \rangle_a^i] \right) \), one should recognize that this definition is
equivalent to an upper bound on the rank, \( \text{rank}(D) \leq 1 \). Such a bound on the rank can be further quantified by considering the determinant of every 2 × 2 submatrix of the data, so called (2 × 2) minors. Specifically, every such minor must be zero if the data is effectively uncorrelated. One should recognize that such conditions are properties of the data collected by just our humble devices alone.

Having mentioned some standard notions from linear algebra, there is an alternative set of notions which support the same analysis. These notions are also more geometric in their perspective, which one might have suspected to exist from the association of gauge. The technique which accompanies these notions further has an obvious tomographic interpretation. As a final statement of this prelude, the alternative technique we are referring to is also what generalizes to actual quantum tomography.

\[ D = \begin{bmatrix} \langle pw \rangle & \langle qw \rangle \\ \langle qw \rangle & \langleqv \rangle \end{bmatrix} = \begin{bmatrix} \langle p \rangle & \langle q \rangle \\ \langle q \rangle & \langle v \rangle \end{bmatrix} \]  

simultaneously. As observed earlier, such data is globally (effectively) uncorrelated if and only if \( \det D = 0 \). Assuming \( \langle pw \rangle \langle qw \rangle \neq 0 \), the det \( D = 0 \) condition is equivalent to

\[ \Delta(D) \equiv \frac{\langle pw \rangle \langle qw \rangle}{\langle pw \rangle \langle qw \rangle} = 1 \]  

and it is this quantity which generalizes to the full quantum problem. Since \( \Delta \) is only a function of data, it is manifestly gauge invariant. \( \Delta \) is called a partial determinant because of the analogy to the above problem and because it is not generally a single number, but rather a matrix of reduced size (\( d^2 \times d^2 \) for \( d \)-dimensional Hilbert spaces.)

Restating the (toy) result, \( D \) is globally (effectively) uncorrelated if and only if \( \Delta(D) = 1 \).

The reader may be familiar with a proof of this using the language of standard linear algebra[1][14] (considering \( D \) as an operator and considering its null space, etc.) However to emphasize the perspective, we include here a more tomographic proof: The “only if” can be proved by simple substitution. For the “if” direction, one first remembers that they can always choose \( \langle p \rangle \) and \( \langle w \rangle \) such that \( \langle pw \rangle = \langle p \rangle \langle w \rangle \). Having chosen \( \langle p \rangle \) and \( \langle w \rangle \), one may then fix \( \langle q \rangle = \langle qw \rangle / \langle w \rangle \) and \( \langle v \rangle = \langle pv \rangle / \langle p \rangle \). Notice that this fixing of \( \langle q \rangle \) and \( \langle v \rangle \) is analogous to state and detector tomography. Finally, if \( \Delta(D) = \langle pq \rangle \langle qw \rangle / \langle pw \rangle \langle w \rangle = 1 \) then \( \langle qv \rangle = \langle pq \rangle \langle qw \rangle / \langle pw \rangle = \langle q \rangle \langle w \rangle \), which finishes the proof.

Summarizing, we have developed a perspective for analyzing toy data which parallels a perspective for analyzing quantum data: Considering the settings \( (p, w) \), \( (q, w) \), \( (p, v) \), and \( (q, v) \) as individual experiments, these settings act as coordinates for the space of experiments so that one can say, for example, experiments \( (p, w) \) and \( (q, w) \) are displaced from each other by keeping the measurement setting constant. Further, each individual experiment is effectively uncorrelated because we can always choose \( \langle p \rangle \) and \( \langle w \rangle \) such that \( \langle pw \rangle = \langle p \rangle \langle w \rangle \). The freedom of that choice is a gauge degree of freedom and is further a local one because each experiment has this property. Finally, there is a connection between the gauges of each experiment because we can write equations like \( \langle pv \rangle = \langle pv \rangle / \langle w \rangle \) — that is, a choice of \( \langle w \rangle \) fixes the gauge of experiment \( (p, w) \) which consequently fixes the gauge of experiment \( (p, v) \). With this connection, the partial determinant has the interpretation of performing tomography in a loop, with a value which measures a contradiction (see Figure 3), reflecting the presence of SPAM correlation. Cast in this language, we have demonstrated that a PD is a
holonomy. We shall proceed to explain this further. At last, it is the tomographic interpretation of this holonomy which is why we refer to any analysis with PDs non-holonomic tomography.

\[ \langle p \rangle \langle w \rangle = \langle pw \rangle \]
\[ \langle q \rangle \langle w \rangle = \langle qw \rangle \]
\[ q \langle v \rangle = q \langle vq \rangle \]
\[ w_i \]
\[ w_f \]

FIG. 3. Illustration of the PD as a Holonomy: Each experiment \((p, w)\) has a local gauge degree of freedom because it is effectively SPAM uncorrelated, \(\langle p \rangle \langle w \rangle = \langle pw \rangle\). The data \(\langle pw \rangle\) further provides a connection between adjacent gauge degrees of freedom by the assumption that they share independent settings. Such a connection defines a non-holonomic constraint when \(w_f = \frac{\langle pv \rangle \langle qw \rangle}{\langle pw \rangle \langle qv \rangle} \neq w_i\). A particular \(w_i\) fixes the gauge which can either represent an arbitrary choice or some external information. The PD \(\Delta = \frac{w_f}{w_i}\) is gauge invariant.

III. HOLOMONY

Holonomy is a concept which has become quite ubiquitous in modern physics and mathematics. Applications range from geometric phases to Yang-Mills Lagrangians, all of which share the notion of a non-holonomic constraint. Perhaps the simplest physical examples of non-holonomic constraint are the thermodynamic concepts of heat and work, although thermodynamics is typically not considered in this way. The simplest mathematical example is probably parallel transport through a sphere, where a tangent vector will turn with an angle proportional to the solid-angle subtended by the loop traversed (Figure 4).

Characteristic of these non-holonomic systems are local degrees of freedom (such as heat or angle) whose differential can be integrated over contours defined within certain dimensions (such as the thermodynamic state or the point on a sphere.) However, these integrals will have non-zero values over closed contours, reflecting that these local degrees of freedom cannot be globally defined as additional dimensions like the ones which defined the contour. Such integrals are called holonomies and their non-zero values may be interpreted as a measure of contradiction or inability to integrate the local degree of freedom to a global coordinate.

The technical notion of heat as a holonomy is not standard and so an elaboration is in order. This will allow us to draw an analogy from which the perspective of non-holonomic tomography will be more explicit. Using the language of gauges in such a non-standard way, it will also be appropriate to relate these notions to their more familiar application in gauge field theory. After having established these connections (no pun intended) we will then rewrite non-holonomic tomography in this field theoretic language. For completeness, we include a section on the actual quantum analogue of the toy problem to make all the respective technical aspects clear.

A. Analogy: Thermodynamics

For a thermodynamic system such as an ideal piston, the notion of an adiabatic process can be defined but cannot be extended to a notion of heat as a quantity. This is because heat can be transferred (into other forms of energy) over closed loops in state space (see Figure 5.) This transfer of heat is the holonomy and the integrals \(\int_{\gamma} dQ\) are the connection. Put another way, the connection \(\int_{\gamma} dQ\) can be thought of as a change in some quantity (like caloric), \(\Delta Q\), but only locally because one can have nonzero changes in the heat upon a return to the same state.

The coefficient of response is the temperature which can depend on other degrees of freedom within the state space, such as volume:

\[ T(S, V) = \frac{\partial U}{\partial S} \bigg|_V \] (11)

However, the notions of energy and entropy do exist as globally defined state variables and heat can be thought of as the energetic response generated by changes in entropy,

\[ dQ = TdS. \] (10)
Q = constant
∮dQ ≠ 0
dQ = 0
dQ ≠ 0
V
S

FIG. 5. Left: Holonomic constraints can be written globally and therefore used as coordinates. Middle: Non-holonomic constraints are only local and cannot define coordinates. The dashed lines are supposed to convey that a notion of “transverse” is still present but the distance between the layers of constraint can be correlated with coordinates along the layers. Right: Non-holonomic constraints thus give rise to holonomies or non-zero integrals over closed contours.

This extra dependence on other degrees of freedom is what makes dQ non-holonomic, non-integrable, or inexact (words which are synonymous in this context.) For such a temperature that depends on volume, one could say that the energy transfer generated by a fixed displacement in entropy is correlated with the volume.

FIG. 6. Our state and measurement devices, now with continuous settings!

Similarly, as in Figure 3, we know what it means to keep the “state device setting” constant so that we may coordinate (p, w) & (p, v) or (q, w) & (q, v) as being in the same layer. We even have the notion of an “average state parameter change” generated by an “iso-measurement-ic” process because we can write

\[ \langle q \rangle = \frac{\langle qw \rangle}{\langle pw \rangle}(p) \quad \text{or} \quad \langle q \rangle = \frac{\langle qv \rangle}{\langle pv \rangle}(p). \] (12)

Further, such an “average state parameter change” may not be holonomic because one could have

\[ \frac{\langle qw \rangle}{\langle pw \rangle} \neq \frac{\langle qv \rangle}{\langle pv \rangle}. \] (13)

so that the response in the “average state parameter” with respect to changes in the “state device setting” is a function of “measurement device setting.” Importantly, the isomorphism from the ideal piston to SPAM tomography is algebraically exponential — that is, for example,

\[ \frac{\langle qw \rangle}{\langle pw \rangle} \sim \exp \int \bar{d}Q. \] (14)

Indeed, this analogy can be made even more exact (see Table 1 and Figures 6 and 7.) Returning to our toy devices, suppose instead that the state and observable settings could be dialed continuously and call these external parameters a & i respectively. Assuming that a & i are the only controls, then the data \( \langle pw \rangle \) is a well defined function over the space of (a, i). We can also define responses in the data with respect to these parameters:

\[ \chi = \frac{\partial}{\partial a} \log \langle pw \rangle \quad \text{and} \quad \xi = -\frac{\partial}{\partial i} \log \langle pw \rangle. \] (15)

These responses provide equations of state which we may
then attribute to notions of non-holonomic average state parameter & average measurement parameter changes,
\[ d\log\langle p \rangle = \chi(a, i) da \quad \text{and} \quad d\log\langle w \rangle = -\xi(a, i) di, \]
which are related to the original data:
\[ d\log\langle pw \rangle = d\log\langle p \rangle + d\log\langle w \rangle. \]  

The exponential maps between the finite and the infinitesimal processes may now be written explicitly:
\[ \Delta = \frac{\langle pw \rangle \langle qw \rangle}{\langle pw \rangle \langle qw \rangle} = \exp\left( \int_{-1}^{1} \chi(a, w) da \right) \quad \text{and} \quad \frac{\langle pw \rangle}{\langle pw \rangle} = \exp\left( \int_{-1}^{1} \xi(p, i) di \right). \]

Finally, we have for the partial determinant
\[ \Delta = \frac{\langle pw \rangle \langle qw \rangle}{\langle pw \rangle \langle qw \rangle} = \exp\left( \int d\log\langle p \rangle \right) = \exp\left( -\int d\log\langle w \rangle \right) = \exp\left( \int \Gamma dadi \right) \]

where the integrals are counterclockwise in Figure 7 and
\[ \Gamma = \frac{\partial \chi}{\partial i} = -\frac{\partial \xi}{\partial a} = -\frac{\partial^2 \log\langle pw \rangle}{\partial a \partial i} \]
is a kind of correlation density.

![FIG. 7. An “S-V” diagram for toy SPAM tomography. Ratios between horizontally adjacent data can be interpreted as “iso-states-ic” processes and vertical ratios as “iso-observables-ic”. These processes are non-holonomic and so demote the notions of “average state” and “average observable” from physical coordinates to a gauge degree of freedom.

When considering this treatment for the response of quantum data to continuous device settings, \( p \) and \( w \) become \( d^2 \times d^2 \) matrix quantities, \( P \) and \( W \) such that
\[ D = PW, \]
representing minimally complete tomography experiments for a \( d \)-dimensional Hilbert space, as will be explained in section IIIC. As such, the inexact forms in Equation 17 should be replaced with the forms \( (d(P))^{-1}(P) \) or \( (W)^{-1}d(W) \). These forms may be recog-
γ arbitrarily to $\frac{\delta \rho}{\delta \gamma} = 0$. In which case we can write the wavefunction with a classical approximation,

$$\Psi[\gamma, A] \propto e^{i q f_i dx A}$$

where it is understood now that γ can be fixed arbitrarily. We do not bother with the normalization constant or the external phase here because we wish only to illustrate the dependence of the wavefunction on A which we can now imagine is being probed through γ, which can be externally controlled.

The quantity

$$W_\gamma = e^{i q f_i dx A}$$

is called a Wilson line. Also important is the Wilson loop

$$W_\gamma = \text{Tr} \left( e^{i q f_i dx A} \right)$$

where a trace has been introduced to include non-abelian gauge fields where there are several As, one for each generator of the gauge group. The general wavefunction, Equation (21), is often referred to as the “quantum expectation value” of the Wilson loop in this context. Normally, the application of the Wilson loop is to determine the dynamics of γ from a theory of the gauge field. However, our purpose for the Wilson loop is to represent how the gauge field could be probed by an externally fixed γ. (See Figure 1)

When we consider partial determinants in section III C, the analogous quantity will be just the closed Wilson line, i.e. a Wilson loop without the trace. Aside from the difference between a single number and a matrix, an important distinction is that closed Wilson line actually depend on the initial/final point from which γ is drawn, while Wilson loops do not. However, the dependence is simple and only such that the closed Wilson line is gauge covariant instead of invariant

$$W_\gamma \rightarrow U(\gamma_1)W_\gamma U^{-1}(\gamma_0)$$

where $\gamma_1 = \gamma_0$ for a closed contour. Although this does not have any significance in gauge field theories, it is significant for a theory of SPAM correlations.

Analogous to a Wilson line, one can define a tomography line:

$$\Delta(\gamma, \tau) = \exp \int_\gamma \tau.$$

which represents a specific type tomography, where the gauge parameter of experiment $\gamma_1$ is concluded from the gauge parameter of experiment $\gamma_0$ through the data, represented by the connection $\tau$, along changes in the device parameters, represented by the contour $\gamma$. The tomographic connection, $\tau$, is not uniquely determined by the data but is nonetheless intimately related to the interpretation of the gauge at each experiment along $\gamma$. Formally this is represented by the tomography lines being equivalent by a local gauge transformation

$$\Delta(\gamma, \tau + dq) = e^{g(\gamma_1) - g(\gamma_0)} \Delta(\gamma, \tau) = e^{g(\gamma_1)} \Delta(\gamma, \tau) e^{-g(\gamma_0)}$$

where the effect of the transformation is only to relabel the initial and final gauge parameters.

Returning to our toy devices, suppose that the gauge at each experiment is represented by an average state parameter (one could call this fixing the state gauge.) Then for $da = 0$, $\Delta$ would be the identity, while along the $a$-direction

$$\Delta(\gamma, d \log \langle p \rangle) = \exp \left( \int_\gamma \chi da \right)$$

would represent iso-measurement-ic tomography. Similarly, if the gauge at each experiment is represented by an average measurement parameter (let’s call this measurement gauge), then

$$\Delta(\gamma, -d \log \langle w \rangle) = \exp \left( \int_\gamma \xi di \right)$$

would represent iso-state-ic tomography. Most importantly, these tomographies are equivalent to each other modulo a local gauge transformation:

$$\Delta(\gamma, -d \log \langle w \rangle) = \Delta(\gamma, d \log \langle p \rangle - d \log \langle pw \rangle)$$

$$= \exp \left( - \int_\gamma d \log \langle pw \rangle \right) \Delta(\gamma, d \log \langle p \rangle)$$

$$= \frac{\langle pw \rangle(\gamma_0)}{\langle pw \rangle(\gamma_1)} \Delta(\gamma, d \log \langle p \rangle).$$

In the electromagnetism analogy, these are the equivalent of Landau gauges (see Figure 10)

C. Non-Holonomic Quantum Tomography and Non-Abelian Lattice Gauge

Having hopefully made the perspective of non-holonomic tomography clear through these analogies for the toy problem, some discussion about the actual quantum problem is due. The quantum problem is the same as the toy problem except that we assume the state and measurement devices are parameterized by Hermitian operators (a density operator and a POVM element, respectively) over a $d$-dimensional Hilbert space. In particular, this means that the devices are to be modeled by $d^2$ random variables each. If all the device parameters were uncorrelated, then one could write these operators as

$$\rho_\mu = \frac{1}{d} \rho_\mu^\mu w_\mu^i$$

where the $\{ w_\mu^i \}_{\mu=0}^{d^2-1}$ is some operator basis of Hermitian operators, $\{ \sigma_\mu^i \}$ is its reciprocal basis, and a sum over
repeated indices is implied. If $\sigma_0 = 1$ and the other $\sigma_\mu$ are traceless, then $p_0^0$ and $w_0^i$ are identical to the single device parameters of the toy problem.

The measured frequencies, a.k.a. “the data”, are now given by

$$f_a^i = \langle \text{Tr}_\rho E \rangle_{a}^{i} = (p^\mu w_\mu)_a^i. \quad (35)$$

which is equivalent to saying that the rank is bounded above by $\text{rank}(D) \leq d^2$. To define a partial determinant, the simplest way is to consider $M = N = 2d^2$ and partition the data into $4d^2 \times d^2$ corners,

$$F = \begin{bmatrix} A & B \\ C & D \end{bmatrix}. \quad (37)$$

The partial determinant is

$$\Delta(F) = D^{-1}CB^{-1}A \quad (38)$$

which is significant because of the result $F$ is globally uncorrelated if and only if $\Delta(F) = 1$.

Specifically, $\Delta$ parameterizes $d^4$ degrees of correlation. However, because of gauge covariance (Equation 26), only $d^2$ of these are gauge invariant parameters.

In the quantum case, it becomes important to pay attention to the arrangement of the settings when the data is considered in the form of Equation (37) so let us define indices:

$$F = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} D_0^0 & D_1^1 \\ D_0^1 & D_1^0 \end{bmatrix} \quad (39)$$

where the matrix elements of these corners are

$$(D_a^i)^i_\alpha = f_{\alpha d^2+\alpha+1}^{d^2+1+1}. \quad (40)$$

The corners are coordinated by $(a, i)$ and understood to be $2 \times 2$ minimally complete tomography experiments we call a square. Each minimally complete tomography experiment consists of $d^2$ states enumerated by $a$ and $d^2$ measurements enumerated by $i$ and is further associated with $d^4$ gauge degrees of freedom reflecting the fact that the data of each corner is locally (effectively) uncorrelated,

$$(D_a^i)^i_\alpha = (P_a)_\alpha^\mu (W^i)_\mu^i = (P_a)_0^\mu G_\mu^\lambda G_\lambda^{1\nu} (W^i)_\nu^i. \quad (41)$$

The corners are understood to be displaced from each other through changes in $(a, i)$ and it is useful to think of these indices as pairs of points on a continuum (see Figure 8). As such, the data matrix is conceptually reorganized as a square which has gauges at each corner (experiment) which are connected to each other by the edges over which the data define a connection.

To be effectively uncorrelated in this case means that the data can be decomposed into the form

$$\begin{pmatrix} \langle p^0_0 \rangle & \cdots & \langle p^{d^2-1}_0 \rangle \\ \vdots & \ddots & \vdots \\ \langle p^0_N \rangle & \cdots & \langle p^{d^2-1}_N \rangle \end{pmatrix} = \begin{pmatrix} \langle w_0^1 \rangle & \cdots & \langle w_{d^2-1}^1 \rangle \\ \vdots & \ddots & \vdots \\ \langle w_0^0 \rangle & \cdots & \langle w_{d^2-1}^0 \rangle \end{pmatrix}, \quad (36)$$

FIG. 8. The $d^2$ buttons enumerate a (detector) tomographically complete frame of states. The $d^2$ notches enumerate a (state) tomographically complete frame of observables. The continuous slider and continuous dial are the square coordinates which displace settings.

For simplicity, each minimally complete tomography experiment will henceforth be referred to as just an experiment. For each experiment, the Born rule, $A = PW$, can be thought of as a connection between gauge parameters, e.g. $P = AW^{-1}$ or $W^{-1} A \rightarrow P$. In other words, the data from experiments can be interpreted as defining maps. For multiple experiments sharing devices, there are degrees of choice as to how one can represent the gauge degrees of freedom for each pair of devices. These choices simultaneously correspond to the choices of how to embed the data in the maps between these experiments. Let us go over a few particularly meaningful examples.

A couple of gauges that should be familiar are what we would like to call standard gauges (Figure 9). Every arrow represents a constraint which may be interpreted as a tomography — e.g. in the right diagram of Figure 9 $\lambda \rightarrow W^{-1}$ represents the equation $W^{-1} = A^{-1}P$ which may be interpreted as a detector tomography. This gauge is in fact the gauge used in the tomographic proof of section [II D]. Also important are what we call tomographies in “Landau” gauge (see Figure 10) which have actually appeared (sections [II D] and [II A]). The reader is encouraged to stare at these 4 gauges and try to see how
they are each an equivalent representation of the same organization of information as Figure 7.

\[
\begin{array}{ccc}
W^{-1} & A & P \\
C & P & B^{-1} \\
Q & A^{-1} & V^{-1} \\
D^{-1} & W^{-1} & C \\
Q & & \end{array}
\]

FIG. 9. Tomography in “Standard” Gauge. We call them standard gauges because, considering for instance the left connection: The measurement parameters of the top-left experiment are imagined to be fixed in which case the data from this experiment can be interpreted as a standard state tomography on the top-right experiment, and from the top-right the connection does standard detector tomography on the bottom-right, etc. The choice of representing the top-left experiment’s gauge by its measurement device parameters, the top-right experiment’s gauge by its state device parameters, etc. uniquely defines how the data is to be organized as a connection in between these experiment’s gauge parameters.

\[
\begin{array}{ccc}
P & 1 & P \\
CA^{-1} & DB^{-1} & 1 \\
Q & 1 & Q \\
W^{-1} & B^{-1}A & V^{-1} \\
1 & 1 & \end{array}
\]

FIG. 10. Tomography in “Landau” Gauge. Left: iso-measurement-ic tomography, the arrangement of quantum data in state gauge, Equation (29). Right: iso-state-ic tomography, the arrangement of quantum data in measurement gauge, Equation (30). These are called Landau because they keep gauge parameters in either the state or measurement direction constant just like the vector potential for a 2-d surface in the x- or y-direction can be chosen to be zero. The left gauge is a tomography where data from two experiments (either A and C or B and D) with a common measurement device is used to infer an unknown state device (Q) from a “known” state device (P). This kind of tomography has been thought of before and already put into practice [11] (instead using a maximum likelihood method to estimate parameters rather than linear inversion, which we are considering.) As far as the authors are aware, the right gauge is a tomography yet unperformed.

All of these gauges are formally related to each other by local gauge transformations. As such, an explanation of gauge transformations on a lattice is in order (see Figure 11). Instead of considering only a square of experiments, it is conceptually more useful to think about a lattice of experiments sharing devices. Something to notice is that \(g\) is not exactly the \(G\) in Equation (31) but rather \(g\Gamma = \Gamma G\) or \(g = \Gamma G T^{-1}\).

FIG. 11. Local Gauge Transformations: The vertical direction represents displacements in state \(a\) and the horizontal direction represents displacements in measurement \(i\). At each vertex (experiment) is a \(d^2 \times d^2\) matrix of gauge parameters, \(\Gamma\). At each adjacent edge (connection to the adjacent experiment) is a component of the connection, \(X = \tau(a,i), Y = \tau(a,i), Z = \tau(a,1,1), T = \tau(a,i-1)\) (see Equation (27)) The distance between lattice sites is defined by distances along continuous device settings (see Figure 8) The right lattice is a gauge transformation, \(g\), of the left lattice at just one vertex. These transformations leave the constraints represented by each connection invariant.

Having re-expressed non-holonomic tomography for quantum systems, some distinctions are in order. First, as already mentioned one should not forget that unlike in the toy model, the gauges of quantum tomography are non-abelian — particularly, the gauge does not generally commute with the connection — which results in a covariance (see Equation (26) of closed-line tomographies on the gauge at the initial/terminal experiment. Second, the gauge groups, \(GL(d^2,\mathbb{R})\), we are concerned with are actually not compact like the unitary groups of Yang-Mills theories. [10] Third, one could imagine having \(d^2\) continuous settings per device, in which case the gauge group becomes a tangent space, where the frame, \(P\), and coframe, \(W^{-1}\), are then like vierbein. Fourth, an experimentalist may not have any “sliders” but rather just have \(2d^2\) “buttons” per device in which case a metric for the distance between experiments is obscured. Finally, in the “only buttons” scenario, localizing settings to corners of a square becomes arbitrary — i.e. whether settings \([1,2,3,4]\) are to appear in the first corner or \([2,6,4,7]\) is arbitrary.

IV. CONCLUSION AND DISCUSSION

In this work, we considered non-holonomic quantum tomography as a perspective for the method of partial determinants [11]. Partial determinants are matrix quantities which analyze quantum data to detect and quantify SPAM correlations, without estimating average state-preparation or measurement parameters. We particularly focused on a toy model to illustrate that the partial determinant is in fact a holonomy, showing that one can formalize SPAM tomographies in direct analogy to thermodynamic theories and gauge field theories. A SPAM tomography is then non-holonomic if the partial determinants (i.e. tomographic holonomies) have nontrivial...
values, which can be interpreted as correlations between state and measurement parameters.

\[ \lambda = 0 \]
\[ |\mu\lambda| > 0 \]
\[ \mu = 0 \]

FIG. 12. Using a determinant to define the distance of a rank 2 matrix from the space of rank 1 matrices can be a subtle point. If \( \lambda \) and \( \mu \) are the singular values of a matrix \( M \), then \( \text{Det}(M) = \lambda \mu \) is a type of distance from the axes (which are rank 1), modulo area preserving transformations. The axes are drawn askew to emphasize that there is no notion of metric distance.

From a practical perspective, the matrix elements of a PD can be used to detect amounts of SPAM correlation. However, the way in which a PD measures distances away from a correlated model can be a little subtle because these distances are not a metric, in the standard mathematical sense. The subtlety simply reduces to the fact that the determinant of a matrix alone does not actually tell you how large its smallest singular value is (see Figure 12). Rather than think of distances away from the space of uncorrelated data, one must think in terms of inherited notions of distance from continuous device settings. The equations such as (20) can quantitatively measure correlations relative to areas in setting space.

A broader observation should also be made about device parameters and gauge dimensions. Importantly, one should notice that the only property which distinguished the toy problem from the quantum problem was a mere “speculation” about the number of degrees of freedom which parameterize the devices. In the most general scheme, an \( r \times r \) PD is a test of the ability to model the data by uncorrelated \( r \)-dimensional state and measurement vectors. For quantum probabilities, one has further interpretations for the \( r = d^2 \) dimensions reflecting that the state and measurement vectors are also operators on a \( d \)-dimensional vector space. As an example of a more general application of PDs, one could consider \( 2 \times 2 \) PDs for an uncorrelated qubit system. Such a PD would generally take a value different from the identity which can be interpreted as a measure of the inability to model the data by uncorrelated classical bit state and measurement parameters.

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[12] Of course, \( g \) must has a compact range so that the interpretation of \( (g(p), g^{-1}(w)) \) as a pair of probabilities still makes sense. However, this detail is not of concern for this paper.
[13] One can argue that such devices do not exist other than by assumption!
[14] Remember that this result is equivalent to the commonly known property that \( D \) is rank (\( \leq \)) 1 if and only if \( \text{Det}D = 0 \).
[15] Indeed, we could just as well have a discussion about general partition functions in statistical mechanics. Their dependence on reservoir parameters can be probed with the mode of the ensemble distribution. The conclusions of such a discussion would have the same essence as the previous section with only the advantage of a technically broader perspective. The logic would exactly parallel the following discussion so we will not go further than to simply acknowledge its existence.
[16] .. ignoring positivity constraints.