Critical Parameters in Particle Swarm Optimisation

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Abstract

Particle swarm optimisation is a metaheuristic algorithm which finds reasonable solutions in a wide range of applied problems if suitable parameters are used. We study the properties of the algorithm in the framework of random dynamical systems which, due to the quasi-linear swarm dynamics, yields analytical results for the stability properties of the particles. Such considerations predict a relationship between the parameters of the algorithm that marks the edge between convergent and divergent behaviours. Comparison with simulations indicates that the algorithm performs best near this margin of instability.

1 PSO Introduction

Particle Swarm Optimisation (PSO, [1]) is a metaheuristic algorithm which is widely used to solve search and optimisation tasks. It employs a number of particles as a swarm of potential solutions. Each particle shares knowledge about the current overall best solution and also retains a memory of the best solution it has encountered itself previously. Otherwise the particles, after random initialisation, obey a linear dynamics of the following form

\[
\begin{align*}
    \mathbf{v}_{i,t+1} &= \omega \mathbf{v}_{i,t} + \alpha_1 (\mathbf{p}_i - \mathbf{x}_{i,t}) + \alpha_2 (\mathbf{g} - \mathbf{x}_{i,t}) \\
    \mathbf{x}_{i,t+1} &= \mathbf{x}_{i,t} + \mathbf{v}_{i,t+1}
\end{align*}
\]

Here \(\mathbf{x}_{i,t}\) and \(\mathbf{v}_{i,t}\), \(i = 1, \ldots, N, \ t = 0, 1, 2, \ldots\), represent, respectively, the \(d\)-dimensional position in the search space and the velocity vector of the \(i\)-th particle in the swarm at time \(t\). The velocity update contains an inertial term parameterised by \(\omega\) and includes attractive forces towards the personal best location \(\mathbf{p}_i\) and towards the globally best location \(\mathbf{g}\), which are parameterised by \(\alpha_1\) and \(\alpha_2\), respectively. The symbols \(\mathbf{R}_1\) and \(\mathbf{R}_2\) denote diagonal matrices whose non-zero entries are uniformly distributed in the unit interval. The number of particles \(N\) is quite low in most applications, usually amounting to a few dozens.

In order to function as an optimiser, the algorithm uses a nonnegative cost function \(F : \mathbb{R}^d \to \mathbb{R}\), where without loss of generality \(F(\mathbf{x}^*) = 0\) is assumed at an optimal solution \(\mathbf{x}^*\). In many problems, where PSO is applied, there are also states with near-zero costs can be considered as good solutions. The cost function is evaluated for the state of each particle at each time step. If \(F(\mathbf{x}_{i,t})\) is better than \(F(\mathbf{p}_i)\), then the personal best \(\mathbf{p}_i\) is replaced by \(\mathbf{x}_{i,t}\). Similarly, if one of the particles arrives at a state with a cost less than \(F(\mathbf{g})\), then \(\mathbf{g}\) is replaced in all particles by the position of the particle that has discovered the new solution. If its velocity is non-zero, a particle will depart from the current best location, but it may still have a chance to return guided by the force terms in the dynamics.

Numerous modifications and variants have been proposed since the algorithm’s inception [1] and it continues to enjoy widespread usage. Ref. [2] groups around 700 PSO papers into 26 discernible application areas. Google Scholar reveals over 150,000 results for “Particle Swarm Optimisation” in total and 24,000 for the year 2014.

In the next section we will report observations from a simulation of a particle swarm and move on to a standard matrix formulation of the swarm dynamics in order to describe some of the existing
analytical work on PSO. In Sect. 3 we will argue for a formulation of PSO as a random dynamical system which will enable us to derive a novel exact characterisation of the dynamics of one-particle system, which will then be generalised towards the more realistic case of a multi-particle swarm. In Sect. 4 we will compare the theoretical predictions with simulations on a representative set of benchmark functions. Finally, in Sect. 5 we will discuss the assumption we have made in the theoretical solution in Sect. 3 and address the applicability of our results to other metaheuristic algorithms and to practical optimisation problems.

2 Swarm dynamics

2.1 Empirical properties

The success of the algorithm in locating good solutions depends on the dynamics of the particles in the state space of the problem. In contrast to many evolution strategies, it is not straightforward to interpret the particle swarm as following a landscape defined by the cost function. Unless the current best positions \( p \) or \( g \) change, the particles do not interact with each other and follow an intrinsic dynamics that does not even indirectly obtain any gradient information.

The particle dynamics depends on the parameterisation of the Eq. 1. To obtain the best result one needs to select parameter settings that achieve a balance between the particles exploiting the knowledge of good known locations and exploring regions of the problem space that have not been visited before. Parameter values often need to be experimentally determined, and poor selection may result in premature convergence of the swarm to poor local minima or in a divergence of the particles towards regions that are irrelevant for the problem.

Empirically we can execute PSO against a variety of problem functions with a range of \( \omega \) and \( \alpha \) values. Typically the algorithm shows performance of the form depicted in Fig. 1. The best solutions found show a curved relationship between \( \omega \) and \( \alpha \approx \alpha_1 + \alpha_2 \), with \( \omega \approx 1 \) at small \( \alpha \), and \( \alpha \gtrapprox 4 \) at small \( \omega \). Large values of both \( \alpha \) and \( \omega \) are found to cause the particles to diverge leading to results far from optimality, while at small values for both parameters the particles converge to a nearby solution which sometimes is acceptable. For other cost functions similar relationships are observed in numerical tests (see Sect. 4) unless no good solutions found due to problem complexity or run time limits, see Sect. 5. For simple cost functions, such as a single well potential, there are also parameter combinations with small \( \omega \) and small \( \alpha \) will usually lead to good results. The choice of \( \alpha_1 \) and \( \alpha_2 \) at constant \( \alpha \) may have an effect for some cost functions, but does not seem to have a big effect in most cases.

2.2 Matrix formulation

In order to analyse the behaviour of the algorithm it is convenient to use a matrix formulation by inserting the velocity explicitly in the second equation (1).

\[
z_{t+1} = Mz_t + \alpha_1 R_1 (p, p)^\top + \alpha_2 R_2 (g, g)^\top
\]

with \( z = (v, x)^\top \) and

\[
M = \begin{pmatrix}
\omega I_d & -\alpha_1 R_1 - \alpha_2 R_2 \\
\omega I_d & I_d - \alpha_1 R_1 - \alpha_2 R_2
\end{pmatrix},
\]

where \( I_d \) is the unit matrix in \( d \) dimensions. Note that the two occurrence of \( R_1 \) in Eq. 3 refer to the same realisation of the random variable. Similarly, the two \( R_2 \)'s are the same realisation, but different from \( R_1 \). Since the second and third term on the right in Eq. 2 are constant most of the time, the analysis of the algorithm can focus on the properties of the matrix \( M \). In spite of its wide applicability, PSO has not been subject to deeper theoretical study, which may be due to the multiplicative noise in the simple quasi-linear, quasi-decoupled dynamics. In previous studies the effect of the noise has largely been ignored.

2.3 Analytical results

An early exploration of the PSO dynamics [4] considered a single particle in a one-dimension space where the personal and global best locations were taken to be the same. The random components were
replaced by their averages such that apart from random initialisation the algorithm was deterministic. Varying the parameters was shown to result in a range of periodic motions and divergent behaviour for the case of $\alpha_1 + \alpha_2 \geq 4$. The addition of the random vectors was seen as beneficial as it adds noise to the deterministic search.

Control of velocity, not requiring the enforcement of an arbitrary maximum value as in Ref. [4], is derived in an analytical manner by [5]. Here eigenvalues derived from the dynamic matrix of a simplified version of the PSO algorithm are used to imply various search behaviours. Thus, again the $\alpha_1 + \alpha_2 \geq 4$ case is expected to diverge. For $\alpha_1 + \alpha_2 < 4$ various cyclic and quasi-cyclic motions are shown to exist for a non-random version of the algorithm.

In Ref. [6] again a single particle was considered in a one dimensional problem space, using a deterministic version of PSO, setting $R_1 = R_2 = 0.5$. The eigenvalues of the system were determined as functions of $\omega$ and a combined $\alpha$, which leads to three conditions: The particle is shown to converge when $\omega < 1$, $\alpha > 0$ and $2\omega - \alpha + 2 > 0$. Harmonic oscillations occur for $\omega^2 + \alpha^2 - 2\omega\alpha - 2\omega - 2\alpha + 1 < 0$ and a zigzag motion is expected if $\omega < 0$ and $\omega - \alpha + 1 < 0$. As with the preceding papers the discussion of the random numbers in the algorithm views them purely as enhancing the search capabilities by adding a drunken walk to the particle motions. Their replacement by expectation values was thus believed to simplify the analysis with no loss of generality.

We show in this contribution that the iterated use of these random factors $R_1$ and $R_2$ in fact adds a further level of complexity to the dynamics of the swarm which affects the behaviour of the algorithm in a non-trivial way. In Ref. [7] these factors were given some consideration. Regions of convergence and divergence separated by a curved line were predicted. This line separating these regions (an equation for which is given in Ref. [8]) fails to include some parameter settings that lead to convergent swarms. Our analytical solution of the stability problem for the swarm dynamics explains why parameter settings derived from the deterministic approaches are not in line with experiences from practical tests. For this purpose we will now formulate the PSO algorithm as a random dynamical system and present an analytical solution for the swarm dynamics in a simplified but representative case.

Figure 1: Typical PSO performance as a function of its $\omega$ and $\alpha$ parameters. Here a 25 particle swarm was run for pairs of $\omega$ and $\alpha$ values ($\alpha_1 = \alpha_2 = \alpha/2$). Cost function here was the $d = 10$ non-continuous rotated Rastrigin function [3]. Each parameter pair was repeated 25 times and the minimal costs after 2000 iterations were averaged.
3 Critical swarm conditions for a single particle

3.1 PSO as a random dynamical system

As in Refs. [1, 2] the dynamics of the particle swarm will be studied here as well in the single-particle case. This can be justified because the particles interact only via the global best position such that, while $g$ is unchanged, single particles exhibit qualitatively the same dynamics as in the swarm. For the one-particle case we have necessarily $p = g$, such that shift invariance allows us to set both to zero, which leads us to the following is given by the stochastic-map formulation of the PSO dynamics (2).

\[ z_{t+1} = Mz_t \] (4)

Extending earlier approaches we will explicitly consider the randomness of the dynamics, i.e. instead of averages over $R_1$ and $R_2$ we consider a random dynamical system with dynamical matrices $M$ chosen from the set

\[ \mathcal{M}_{\alpha, \omega} = \left\{ \left( \begin{array}{cc} \omega I_d & -\alpha R \\ \omega I_d & I_d - \alpha R \end{array} \right) : R_{ij} = 0 \text{ for } i \neq j \text{ and } R_{ii} \in [0, 1] \right\} \] (5)

with $R$ being in both rows the same realisation of a random diagonal matrix that combines the effects of $R_1$ and $R_2$ [1]. The parameter $\alpha$ is the sum $\alpha_1 + \alpha_2$ with $\alpha_1, \alpha_2 \geq 0$ and $\alpha > 0$. As the diagonal elements of $R_1$ and $R_2$ are uniformly distributed in $[0, 1]$, the distribution of the random variable $R_{ii} = \frac{\alpha_1}{\alpha} R_1_{ii} + \frac{\alpha_2}{\alpha} R_2_{ii}$ in Eq. (4) is given by a convolution of two uniform random variables, namely

\[ P_{\alpha_1, \alpha_2}(r) = \begin{cases} \frac{\alpha r}{\max\{\alpha_1, \alpha_2\}} & \text{if } 0 \leq r \leq \min\{\frac{\alpha_1}{\alpha}, \frac{\alpha_2}{\alpha}\} \\ \frac{\max\{\alpha_1, \alpha_2\}}{\alpha r} & \text{if } \min\{\frac{\alpha_1}{\alpha}, \frac{\alpha_2}{\alpha}\} < r \leq \max\{\frac{\alpha_1}{\alpha}, \frac{\alpha_2}{\alpha}\} \\ \frac{\alpha r}{\max\{\alpha_1, \alpha_2\}} & \text{if } \max\{\frac{\alpha_1}{\alpha}, \frac{\alpha_2}{\alpha}\} < r \leq 1 \end{cases} \] (6)

if the variable $r \in [0, 1]$ and $P_{\alpha_1, \alpha_2}(r) = 0$ otherwise. $P_{\alpha_1, \alpha_2}(r)$ has a tent shape for $\alpha_1 = \alpha_2$ and a box shape in the limits of either $\alpha_1 \rightarrow 0$ or $\alpha_2 \rightarrow 0$. The case $\alpha_1 = \alpha_2 = 0$, where the swarm does not obtain information about the fitness function, will not be considered here.

We expect that the multi-particle PSO is well represented by the simplified version for $\alpha_2 \gg \alpha_1$ or $\alpha_1 \gg \alpha_2$, the latter case being irrelevant in practice. For $\alpha_1 \approx \alpha_2$ deviations from the theory may occur because in the multi-particle case $p$ and $g$ will be different for most particles. We will discuss this as well as the effects of the switching of the dynamics at discovery of better solutions in Sect. 5.2.

3.2 Marginal stability

While the swarm does not discover any new solutions, its dynamical properties are determined by an infinite product of matrices from the set $\mathcal{M}$ [3]. Such products have been studied for several decades [9] and have found applications in physics, biology and economics. Here they provide a convenient way to explicitly model the stochasticity of the swarm dynamics such that we can claim that the performance of PSO is determined by the stability properties of the random dynamical system $\mathcal{M}$.

Since the equation (4) is linear, the analysis can be restricted to vectors on the unit sphere in the $(v, x)$ space, i.e. to unit vectors

\[ a = (x, v)^\top / \| (x, v)^\top \|, \] (7)

where $\| \cdot \|$ denotes the Euclidean norm. Unless the set of matrices shares the same eigenvectors (which is not the case here) standard stability analysis in terms of eigenvalues is not applicable. Instead we will use means from the theory of random matrix products in order to decide whether the set of matrices is stochastically contractive. The properties of the asymptotic dynamics can be described based on a double Lebesgue integral over the unit sphere $S^{2d-1}$ and the set $\mathcal{M}$ [10, 11]. As in Lyapunov exponents, the effect of the dynamics is measured in logarithmic units in order to account for multiplicative action.

\[ \lambda(\alpha, \omega) = \int d\nu_{\alpha, \omega}(a) \int dP_{\alpha, \omega}(M) \log \| Ma \| \] (8)

4
If \((a, \omega)\) is negative the algorithm will converge to \(p\) with probability 1, while for positive \(a\) arbitrarily large fluctuations are possible. While the measure for the inner integral (8) is given by Eq. 6, we have to determine the stationary distribution \(\nu\) on the unit sphere for the outer integral. It is given as the solution of the integral equation

\[
\nu_{\alpha,\omega}(a) = \int d\nu_{\alpha,\omega}(b) \int dP_{\alpha,\omega}(M) \delta(a, Mb/\|Mb\|), \quad a, b \in S^{2d-1}.
\]

The existence of the invariant measure requires the dynamics to be ergodic which is ensured if at least some of elements of \(M\) have complex eigenvalues, such as being the case for \(\omega^2 + \alpha^2/4 - \omega\alpha - 2\omega - \alpha + 1 < 0\) (see above, [6]). This condition excludes a small region in the parameters space at small values of \(\omega\), such that there we have to take all ergodic components into account. There are not more than two components which due to symmetry have the same stability properties. It depends on the parameters \(\alpha\) and \(\omega\) and differs strongly from a homogenous distribution, see Fig. 2 for a few examples in the case \(d = 1\). Critical parameters are obtained from Eq. 8 by the relation

\[
\lambda(\alpha, \omega) = 0.
\]

Solving Eq. 10 is difficult in higher dimensions, so we rely on the linearity of the system when considering the \((d = 1)\)-case as representative. The curve in Fig. 3 represents the solution of Eq. 10 for \(d = 1\) and \(\alpha = \alpha_2\). For other settings of \(\alpha_1\) and \(\alpha_2\) the distribution of the random factors has a smaller variance rendering the dynamics more stable such that the contour moves towards larger parameter values (see Fig. 4). Inside the contour \(\lambda(\alpha, \omega)\) is negative, meaning that the state will approach the origin with probability 1. Along the contour and in the outside region large state fluctuations are possible. Interesting parameter values are expected near the curve where due to a coexistence of stable and unstable dynamics (induced by different sequences of random matrices) a theoretically optimal combination of exploration and exploitation is possible. For specific problems, however, deviations from the critical curve can be expected to be beneficial.

### 3.3 Personal best vs. global best

Due to linearity, the particle swarm update rule (1) is subject to a scaling invariance which was already used in Eq. 7. We now consider the consequences of linearity for the case where personal best and global best differ, i.e. \(p \neq g\). For an interval where \(p_i\) and \(g\) remain unchanged, the particle \(i\) with personal best \(p_i\) will behave like a particle in a swarm where together with \(x\) and \(v\), \(p_i\) is also scaled by a factor \(\kappa > 0\). The finite-time approximation of the Lyapunov exponent (see Eq. 8)

\[
\lambda(t) = \frac{1}{t} \log \langle \| (x_t, v_t) \| \rangle
\]

Figure 2: Stationary distribution \(\nu_{\alpha,\omega}(a)\) on the unit circle \((a \in [0, 2\pi])\) in the \((x, v)\) plane for a one-particle system (4) for \(\omega = 0.7\) and \(\alpha = \alpha_2 = 0.5, 1.5, 2.5, 3.5, 4.5\) (the distribution with peak near \(\pi\) is for \(\alpha = 0.5\), otherwise main peaks are highest for largest \(\alpha\)).
Figure 3: Solution of Eq. 10 representing a single particle in one dimension with a fixed best value at $g = p = 0$. The curve that has higher $\alpha$-values on the right (magenta) is for $\alpha_1 = \alpha_2$, the other curve (green) is for $\alpha = \alpha_2, \alpha_1 = 0$. Except for the regions near $\omega = \pm 1$, where numerical instabilities can occur, a simulation produces an indistinguishable curve. In the simulation we tracked the probability of a particle to either reach a small region ($10^{-6}$) near the origin or to escape beyond a radius of $10^6$ after starting from a random location on the unit circle. Along the curve both probabilities are equal.

will be changed by an amount of $1/\log \kappa$ by the scaling. Although this has no effect on the asymptotic behaviour, we will have to expect an effect on the stability of the swarm for finite times which may be relevant for practical applications. For the same parameters, the swarm will be more stable if $\kappa < 1$ and less stable for $\kappa > 1$, provided that the initial conditions are scaled in the same way. Likewise, if $||p||$ is increased, then the critical contour will move inwards, see Fig. 5. Note that in this figure, the low number of iterations lead to a few erroneous trials at parameter pairs outside the outer contour which have been omitted here. We also do not consider the behaviour near $\alpha = 0$ which is complex but irrelevant for PSO. The contour (10) can be seen as the limit $\kappa \to 0$ such that only an increase of $||p||$ is relevant for comparison with the theoretical stability result. When comparing the stability results with numerical simulations for real optimisation problems, we will need to take into account the effects caused by differences between $p$ and $g$ in a multi-particle swarm with finite runtimes.

4 Optimisation of benchmark functions

Metaheuristic algorithms are often tested in competition against benchmark functions designed to present different problem space characteristics. The 28 functions [3] contain a mix of unimodal, basic multimodal and composite functions. The domain of the functions in this test set are all defined to be $[-100, 100]^d$ where $d$ is the dimensionality of the problem. Particles were initialised within the same domain. We use 10-dimensional problems throughout. Our implementation of PSO performed no spatial or velocity clamping. In all trials a swarm of 25 particles was used. We repeated the algorithm 100 times, on each occasion allowing 200, 2000, 20000 iterations to pass before recording the best solution found by the swarm. For the competition 50000 fitness evaluation were allowed which corresponds to 2000 iterations with 25 particles. Other iteration numbers were included for comparison. This protocol was carried out for pairs of $\omega \in [-1.1, 1.1]$ and $\alpha \in [0, 5]$ This was repeated for all 28 functions. The averaged solution costs as a function of the two parameters showed curved valleys similar to that in Fig. 1 for all problems. For each function we obtain different best values along (or near) the theoretical curve (10). There appears to be no preferable location within the valley. Some individual functions yield best performance near $\omega = 1$. This is not the case near $\omega = 0$, although the global average performance over all test functions is better in the valley near $\omega = 0$ than near $\omega = 1$, see Fig [3].
At medium values of $\omega$ the difference between the analytical solutions for the cases $\alpha_1 = \alpha_2$ and $\alpha_1 = 0$ is strongest, see Fig. 4. In simulations this shows to a lesser extent, thus revealing a shortcoming of the one-particle approximation. Because in the multi-particle case, $p$ and $g$ are often different, the resulting vector will have a smaller norm than in the one-particle case, where $p = g$. The case $p \neq g$ violates the assumption of the theory the dynamics can be described based on unit vectors. While a particle far away from both $p$ and $g$ will behave as predicted from the one-particle case, at length scales smaller than $\|p - g\|$ the retractive forces will tend to be reduced such that the inertia becomes more effective and the particle is locally less stable which shows numerically in optimal parameters that are smaller than predicted.

5 Discussion

5.1 Relevance of criticality

Our analytical approach predicts a locus of $\alpha$ and $\omega$ pairings that maintain the critical behaviour of the PSO swarm. Outside this line the swarm will diverge unless steps are taken to constrain it. Inside, the swarm will eventually converge to a single solution. In order to locate a solution within the search space, the swarm needs to converge at some point, so the line represents an upper bound on the exploration-exploitation mix that a swarm manifests. For parameters on the critical line, fluctuations are still arbitrary large. Therefore, subcritical parameter values can be preferable if the settling time is of the same order as the scheduled runtime of the algorithm. If, in addition, a typical length scale of the problem is known, then the finite standard deviation of the particles in the stable parameter region can be used to decide about the distance of the parameter values from the critical curve. These dynamical quantities can be approximately set, based on the theory presented here, such that a precise control of the behaviour of the algorithm is in principle possible.

The observation of the distribution of empirically optimal parameter values along the critical curve, confirms the expectation that critical or near-critical behaviour is the main reason for success of the algorithm. Critical fluctuations are a plausible tool in search problem if apart from certain smoothness assumption nothing is known about the cost landscape: The majority of excursions will exploit the smoothness of the cost function by local search, whereas the flat tails of the distribution allow the particles to escape from local minima.
Figure 5: For \( p \neq g \) we define neutral stability as the equilibrium between divergence and convergence. Convergence means here that the particle approaches the line connecting \( p \) and \( g \). Curves are for a one-dimensional problem with \( p = 0.1 \) and \( g = 0 \) scaled (see Sect. 3.3) by \( \kappa = 1 \) (outer curve) \( \kappa = 0.1 \) and \( \kappa = 0.04 \) (inner curve). Results are for 200 iterations and averaged over 100000 repetitions.

5.2 Switching dynamics at discovery of better solutions

Eq. 2 shows that the discovery of a better solution affects only the constant terms of the linear dynamics of a particle, whereas its dynamical properties are governed by the linear coefficient matrices. However, in the time step after a particle has found a new solution the corresponding force term in the dynamics is zero (see Eq. 1) such that the particle dynamics slows down compared to the theoretical solution which assumes a finite distance from the best position at all (finite) times. As this affects usually only one particle at a time and because new discoveries tend to become rarer over time, this effect will be small in the asymptotic dynamics, although it could justify the empirical optimality of parameters in the unstable region for some test cases.

The question is nevertheless, how often these changes occur. A weakly converging swarm can still produce good results if it often discovers better solutions by means of the fluctuations it performs before settling into the current best position. For cost functions that are not ‘deceptive’, i.e. where local optima tend to be near better optima, parameter values far inside the critical contour (see Fig. 3) may give good results, while in other cases more exploration is needed.

5.3 The role of personal best and global best

A numerical scan of the \((\alpha_1, \alpha_2)\) plane shows a valley of good fitness values, which, at small fixed positive \( \omega \), is roughly linear and described by the relation \( \alpha_1 + \alpha_2 = \text{const} \), i.e. only the joint parameter \( \alpha = \alpha_1 + \alpha_2 \) matters. For large \( \omega \), and accordingly small predicted optimal \( \alpha \) values, the valley is less straight. This may be because the effect of the known solutions is relatively weak, so the interaction of the two components becomes more important. In other words if the movement of the particles is mainly due to inertia, then the relation between the global and local best is non-trivial, while at low inertia the particles can adjust their \( p \) vectors quickly towards the \( g \) vector such that both terms become interchangeable.

Finally, we should mention that more particles, longer runtime as well as lower search space dimension increase the potential for exploration. They all lead to the empirically determined optimal parameters being closer to the critical curve.
6 Conclusion

PSO is a widely used optimisation scheme which is theoretically not well understood. Existing theory concentrates on a deterministic version of the algorithm which does not possess useful exploration capabilities. We have studied the algorithm by means of a product of random matrices which allows us to predict useful parameter ranges and may allow for more precise settings if a typical length scale of the problem is known. A weakness of the current approach is that it focuses on the standard PSO [1] which is known to include biases [12, 13], that are not necessarily justifiable, and to be outperformed on benchmark set and in practical applications by many of the existing PSO variants. Similar analyses are certainly possible and are expected to be carried out for some of the variants, even though the field of metaheuristic search is often portrayed as largely inert to theoretical advances. If the dynamics of particle swarms is better understood, the algorithms may become useful as efficient particle filters which have many applications beyond heuristic optimisation.

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