Transverse momentum spectra of strange hadrons within extensive and nonextensive statistics

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Received 13 February 2020, revised 14 April 2020
Accepted for publication 7 May 2020
Published 15 May 2020

Abstract
Using generic (non)extensive statistics, in which the underlying system likely autonomously manifests its extensive and nonextensive statistical nature, we extract various fit parameters from the CMS experiment and compare these to the corresponding results obtained from Tsallis and Boltzmann statistics. The present study is designed to indicate the possible variations between the three types of statistical approaches and characterizes their dependence on collision energy, multiplicity, and size of the system of interest. We analyze the transverse momentum spectra $p_T$ of the strange hadrons $K^0_s$, $\Lambda$, and $\Xi^-$ produced in $\text{Pb} + \text{Pb}$ collisions, at $\sqrt{s_{NN}} = 2.76$ TeV, in $p + \text{Pb}$ collisions, at $\sqrt{s_{NN}} = 5.02$ TeV, and in $p + p$ collisions, at $\sqrt{s_{NN}} = 7$ TeV. From the comparison of the resulting fit parameters; temperature $T$, volume $V$, and nonextensvie parameter $d$, with calculations based on Tsallis and Boltzmann statistics, remarkable differences between the three types of statistics are determined besides a strong dependence on size and type of the colliding system. We conclude that the produced particles with large masses and large strange quantum numbers likely freeze out earlier than the ones with smaller masses and less strange quantum numbers. This conclusion seems not depending on the type of the particle or the collision types but apparently manifesting transitions from chemical (larger temperature) to the kinetic freezeouts (lower temperature). For the first universality (equivalent) class $c \sim 1$, the decrease in the second equivalent class $d$ with increasing energy and collision’s centrality highlights that the system departs from nonextensivity (non-equilibrium) and apparently approaches extensivity (equilibrium) indicating that the Boltzmann statistics becomes the proper statistical approach in describing that system. Last but not least, we present analytical expressions for the energy dependence of the various fit parameters.

Keywords: nonextensive thermodynamical consistency, boltzmann and Fermi–Dirac statistics, high-energy collisions and particle production

(Some figures may appear in colour only in the online journal)

1. Introduction
The high-energy experiments are designed to study the strongly interacting matter, at high temperatures and densities [1]. The deconfinement of colliding hadrons into quark-gluon plasma (QGP), which then rapidly expands and cools down [2], is conjectured to be created at such extreme collisions [3–6]. There are different signatures for the formation of QGP. The enhancement of the strangeness number [7, 8] and of the transverse momentum $p_T$ spectra of strange particles [9] are well-known examples. The importance of the latter is the ability to determine the freezeout parameters; the
temperature and the chemical potential in different statistical approaches [10–16]. These statistical models are able to describe the experimental results in a wide range of energies [6].

So far, various phenomena have been successfully described by extensive (Boltzmann) [6, 17, 18], nonextensive Tsallis [18–2018–20], and generic (non)extensive statistical approach [21–26]. Focusing the discussion on the transverse momentum spectra $p_T$, there are different Tsallis approaches [27–32] which have been utilized in characterizing the heavy-ion collisions [33–35]. Also, the $p_T$ spectra of well-identified particles in $p + p$ collisions at the energies of the Relativistic Heavy Ion Collider (RHIC) and the Large Hadron Collider (LHC) have been excellently fitted to Tsallis statistics [36–38]. Furthermore, the Tsallis approaches are used in analyzing $p_T$ of charged and strange hadrons [39–41, 31, 32, 42]. On the other hand, the generic (non)extensive approach, which was introduced in ref. [23] and utilized in estimating the particle production in a wide range of energies [24–26] was also utilized to analyze the $p_T$ spectra of charged particles measured in different types of collisions, at RHIC and LHC energies [16, 43]. This motivates the present work, in which $p_T$ of strange hadrons are fitted to extensive (Boltzmann) and nonextensive approaches [Tsallis and generic (non)extensive], at various energies. The outputs include the freezeout parameters; temperature and chemical potential besides the corresponding nonextensivity parameters, themselves. The latter is universal, i.e. both Boltzmann and Tsallis approaches are very special cases within the generic(non)extensive approach. It allows us to estimate the degree of nonextensivity of the underlying system. Accordingly, we are furnished with a trustful reliable tool enabling us to judge about formation of QGP, as this is likely accompanied by critical phenomena and thus likely has a nonextensive statistical nature.

The present script focuses on strange particles as they are suitable for the manifestation of the nonextensive statistical nature of the particle production and, on the other hand, can be compared with the conclusions drawn in [44, 45]. They could also come up with an essential contribution to the conventional dependence of the temperature on the strange quantum numbers, which likely strengthens the signature proposed for the formation of QGP.

The present paper is organized as follows. The various types of the statistical approaches are given in section 2. The particle spectra within extensive Boltzmann and generic (non) extensive statistics are presented in section 2.1 and 2.2, respectively. The results on $p_T$ spectra of strange hadrons measured in different types of collisions are elaborated in section 3. The dependence of the resulting fit parameters on the collision’s centralities, the strange particle number, and the type of the strange particles are analyzed in section 3. The dependence of the various fit parameters on the collision energies is discussed in section 3.2. Section 4 is devoted to the conclusions.

2. Statistical Approaches

2.1. Boltzmann Statistics

We start with the total number of particles in an statistical ensemble as expressed within Boltzmann statistics [46],

$$N = \frac{g V}{(2\pi)^3} \int_0^\infty \frac{dp^3}{\exp \left( \frac{E - \mu}{T_B} \right)},$$

(1)

where $T_B$ is the Boltzmann temperature. The momentum distribution can then be deduced as [46–48]

$$E \frac{d^3N}{dp^3} = \frac{g V E}{(2\pi)^3} \exp \left( \frac{\mu - E}{T_B} \right).$$

(2)

At mid-rapidity $y = 0$ and $\mu \approx 0$, the momentum distribution reads

$$\frac{1}{2\pi p_T} \frac{d^2N}{dp_T dy} \bigg|_{y=0} = \frac{g V m_T}{(2\pi)^2} \exp \left( - \frac{m_T}{T_B} \right).$$

(3)

where $E = m_T \cosh y$. The transverse mass $m_T$ is given as $\sqrt{p_T^2 + m^2}$. At mid-rapidity ($y = 0$) but $\mu = 0$, the momentum distribution can be given as

$$\frac{1}{2\pi p_T} \frac{d^2N}{dp_T dy} \bigg|_{y=0} = \frac{g V m_T}{(2\pi)^2} \exp \left( - \frac{m_T}{T_B} \right).$$

(4)

2.2. Generic (non)extensive Statistics

For quantum gas with fermion and boson constituents, the partition function within the generic axiomatic-(non)extensive approach reads [25]

$$\ln Z_{\text{TB}}(T) = \pm V \sum_{i=1}^{N_{\text{TD}}} \ln \left[ 1 \pm \varepsilon_{c,d,r}(x_i) \right],$$

(5)

where $\pm$ refer to fermions (subscript $F$) and bosons (subscript $B$), respectively. The distribution function $\varepsilon_{c,d,r}(x_i)$ is given as [49, 50]

$$\varepsilon_{c,d,r}(x) = \exp \left[ \frac{-d}{1 - c} \left( W_k \left( \frac{(1 - \frac{x}{r})^d}{r} \right) - W_k[B] \right) \right],$$

(6)

where $W_k$ is the Lambert W-function which has real solutions at $k = 0$ with $d \geq 0$ and at $k = 1$ with $d < 0$,

$$B = \frac{(1 - c)r}{1 - (1 - c)r} \exp \left[ \frac{(1 - c)r}{1 - (1 - c)r} \right].$$

(7)

with $r = [1 - c + d]^{-1}$ and the universality (equivalent) class $(c, d)$ does not only define the entropy and throughout the extensive and nonextensive statistical nature of the underlying system, but as shown in equation (6) it determines the correspondent distribution function. Thus, the total
number of particles can be determined from as

\[ N_i = \pm V \frac{d^3N}{8\pi^3 g_i} \int_0^\infty \frac{e_{c,d,r}(x_i)}{1 + W_0\left[B\left(1 - \frac{x_i}{T}\right)^2\right]} d^3p, \]

where \( x_i = (\mu - E_i)/T, i \) runs over \( K^0_S, \Lambda, \) and \( \Xi^- \), with \( g_{K^0_S} = 1 \) and \( g_\Lambda = g_{\Xi^-} = 2 \) are degeneracy factors. The corresponding momentum distribution for strange hadrons is

\[
\frac{1}{2\pi} \frac{d^2N}{p_T dp_T} \bigg|_{y=0} = \pm g_i V T m_{T_i} \frac{e_{c,d,r}(x_i)}{8\pi^3} \times \frac{e_{c,d,r}\left(\frac{m_{T_i}}{T}\right) W_0\left[B\left(1 - \frac{\mu - m_{T_i}}{T}\right)^2\right]}{(1 - c)\left[1 \pm e_{c,d,r}\left(\frac{m_{T_i}}{T}\right)\right] (rT - \mu + m_{T_i}) \left[1 + W_0\left[B\left(1 - \frac{\mu - m_{T_i}}{T}\right)^2\right]\right]}.
\]

Given as

\[
\frac{1}{2\pi} \frac{d^2N}{d^3p} = \pm g_i V E_i T \frac{e_{c,d,r}(x_i)}{8\pi^3} \times \frac{e_{c,d,r}\left(\frac{m_{T_i}}{T}\right) W_0\left[B\left(1 - \frac{\mu - m_{T_i}}{T}\right)^2\right]}{(1 - c)\left[1 \pm e_{c,d,r}\left(\frac{m_{T_i}}{T}\right)\right] (rT - \mu + E_i) \left[1 + W_0\left[B\left(1 - \frac{\mu - m_{T_i}}{T}\right)^2\right]\right]},
\]

where \( e_{c,d,r}(x_i) \) is defined in equation (6).

The transverse momentum distribution in terms of rapidity \( y \) and transverse mass \( m_{T_i} = \sqrt{p_T^2 + m_i^2} \) can be expressed as

\[
\frac{1}{2\pi} \frac{d^2N}{d^3p} = \pm g_i V T m_{T_i} \frac{e_{c,d,r}(x_i)}{8\pi^3} \times \frac{e_{c,d,r}\left(\frac{m_{T_i}}{T}\right) W_0\left[B\left(1 - \frac{\mu - m_{T_i} \cosh y}{T}\right)^2\right]}{(1 - c)\left[1 \pm e_{c,d,r}\left(\frac{m_{T_i} \cosh y}{T}\right)\right] (rT + m_{T_i} \cosh y - \mu) \left[1 + W_0\left[B\left(1 - \frac{\mu - m_{T_i} \cosh y}{T}\right)^2\right]\right]}.
\]

At mid-rapidity and vanishing chemical potential,

\[
\frac{1}{8\pi^3} \frac{d^2N}{d^3p} = \pm g_i V T m_{T_i} \frac{e_{c,d,r}\left(\frac{m_{T_i}}{T}\right) W_0\left[B\left(1 - \frac{\mu}{T}\right)^2\right]}{(1 - c)\left[1 \pm e_{c,d,r}\left(\frac{m_{T_i}}{T}\right)\right] (rT + m_{T_i}) \left[1 + W_0\left[B\left(1 - \frac{\mu}{T}\right)^2\right]\right]}.
\]

Now, we can derive the transverse momentum distribution

\[
\frac{1}{2\pi} \frac{d^3N}{p_T dp_T} \bigg|_{y=0} = \pm g_i V T m_{T_i} \frac{e_{c,d,r}\left(\frac{m_{T_i}}{T}\right) W_0\left[B\left(1 - \frac{(\mu - m_{T_i})}{T}\right)^2\right]}{(1 - c)\left[1 \pm e_{c,d,r}\left(\frac{m_{T_i}}{T}\right)\right] (rT - \mu + m_{T_i}) \left[1 + W_0\left[B\left(1 - \frac{(\mu - m_{T_i})}{T}\right)^2\right]\right]},
\]

At mid-rapidity \( (y = 0) \) and non-vanishing chemical potential, we get

In the section that follows, we introduce results on the transverse momentum distributions of the strange hadrons \( K^0_S, \Lambda, \) and \( \Xi^- \) measured in different high-energy experiments, at various energies [34, 51, 52] within Boltzmann and generic (non)extensive statistics.

3. Results and discussion

3.1. Statistical fits to \( p_T \) spectra

The transverse momentum spectra of the strange hadrons \( K^0_S, \Lambda, \) and \( \Xi^- \) produced in Pb + Pb collisions, at \( \sqrt{s_{NN}} = 2.76 \) TeV,
Figure 1. In different multiplicity intervals, the transverse momentum distributions of the strange hadrons \( K^0_s \), \( \Lambda \), and \( \Xi^- \) produced in Pb + Pb collisions, at \( \sqrt{s_{NN}} = 2.76 \) TeV (symbols) [52] are fitted to generic (non)extensive statistical approaches (solid curves), equation (12).

Figure 2. Similar to figure 1, but in p + Pb collisions, at \( \sqrt{s_{NN}} = 5.02 \) TeV.

Figure 3. The same as in figure 1, but in p + p collisions, at \( \sqrt{s_{NN}} = 7 \) TeV.

Table 1. For \( c = 0.99988 \) and \( \mu = 0 \), the qualities of the generic (non)extensive statistical fits for the transverse momentum distributions (\( \chi^2 \)) are determined for different multiplicity intervals in Pb + Pb, p + Pb and p + p collisions, at \( \sqrt{s_{NN}} = 2.76, 5.02, 7 \) TeV.

| \( \sqrt{s_{NN}} \) [TeV] | 299 | 253 | 210 | 168 | 130 | 92  | 58  | 21  | \( K^0_s \) | \( \Lambda \) | \( \Xi^- \) |
|--------------------------|-----|-----|-----|-----|-----|-----|-----|-----|--------|-------|-------|
| 2.76                     | 0.6858 | 0.8577 | 0.5994 | 1.256 | 1.206 | 1.174 | 1.651 | 2.387 | K^0_s  | \Lambda | \Xi^-  |
|                          | 0.9626 | 1.689 | 1.676 | 0.7011 | 1.111 | 2.379 | 2.141 | 3.364 | \Lambda | \Xi^-  |
|                          | 3.517 | 2.4  | 2.47  | 2.385 | 3.409 | 3.186 | 2.982 | 2.197 | \Xi^-  |
|                          | 280  | 236  | 195  | 159  | 125  | 89   | 57   | 21   | \Xi^-  |
| 5.02                     | 2.13  | 2.467 | 1.616 | 1.33  | 1.002 | 1.68  | 1.797 | 2.763 | K^0_s  | \Lambda | \Xi^-  |
|                          | 2.182 | 2.188 | 2.507 | 1.231 | 1.353 | 0.2748 | 1.119 | 1.767 | \Lambda | \Xi^-  |
|                          | 4.987 | 5.061 | 3.644 | 2.234 | 3.392 | 1.753 | 1.712 | 2.355 | \Xi^-  |
|                          | 158  | 135  | 111  | 79   | 50   | 14   |      |      | \Xi^-  |
| 7                        | 2.301 | 1.835 | 2.325 | 2.289 | 2.853 | 4.548 |      |      | K^0_s  | \Lambda | \Xi^-  |
|                          | 2.004 | 1.267 | 2.293 | 1.917 | 1.865 | 3.125 |      |      | \Lambda | \Xi^-  |
|                          | 2.899 | 1.546 | 1.643 | 2.888 | 1.758 | 1.96  |      |      | \Xi^-  |
collisions, at $\sqrt{s_{NN}} = 5.02\text{ TeV}$, and $pp$ collisions, at $\sqrt{s_{NN}} = 0.2, 0.9, 7\text{ TeV}$ [52] in different multiplicity intervals are fitted using generic (non)extensive statistics and shown in figures 1, 2, and 3, respectively. The goodness of the statistical fits are listed in Table 1.

Figure 1 presents the transverse momentum $p_T$ spectra of (a) $K_s^0$, (b) $\Lambda$, and (c) $\Xi^-$, produced in $Pb + Pb$ collision, at $\sqrt{s_{NN}} = 2.76\text{ TeV}$. The symbols refer to the experimental data of the CMS experiment [52], which are divided into multiplicity intervals $N_{\text{track}}$ in the mid-rapidity range $|y| < 1.0$. The corresponding averaged multiplicity are $\langle N_{\text{track}} \rangle = 21, 58, 92, 130, 168, 210, 253,$ and $299$ [53]. Our calculations which are based on generic (non)extensive statistics are represented by solid curves. Here, we are interested on the smallest $p_T$ region. It is found that for all multiplicity intervals our calculations for $\Lambda$ and $\Xi^-$ hadrons agree well with the experimental data of Pb + Pb collisions, at $\sqrt{s_{NN}} = 2.76\text{ TeV}$. For $K_s^0$, the agreement becomes worse, specially with decreasing multiplicity, i.e. moving towards peripherality. Also the goodness of $K_s^0$ is less than that of other strange particles, especially at large $p_T$.

Figure 2 depicts the transverse momentum distribution $p_T$ of (a) $K_s^0$, (b) $\Lambda$, and (c) $\Xi^-$, produced in $p + Pb$ collisions, at $\sqrt{s_{NN}} = 5.02\text{ TeV}$ in different multiplicity intervals. In the mid-rapidity range $|y| < 1.0$, the experimental data (symbols) are divided into different multiplicity intervals $N_{\text{track}}$. The solid curves are the results calculated using generic (non) extensive statistics. We notice that for all multiplicity intervals our results for all hadrons are in a good agreement.

Figure 4. The resulting fit parameters $T$ and $V$ deduced within generic (non)extensive statistics from the strange hadrons $K_s^0$, $\Lambda$, and $\Xi^-$ produced in $Pb + Pb$ collisions as functions of multiplicity intervals, at $\sqrt{s_{NN}} = 2.76\text{ TeV}$, are compared with Boltzmann and Tsallis statistics [54].
with the experimental data of p + Pb collisions, at $\sqrt{s_{NN}} = 5.02$ TeV.

For different multiplicity intervals, figure 3 presents the transverse momentum distribution $p_T$ spectra of (a) $K^0_S$, (b) $\Lambda$, and (c) $\Xi^-$ produced in p + p collisions, at $\sqrt{s_{NN}} = 7$ TeV. The CMS results (symbols) divided into $N_{\text{trk}}^\text{offline} = 14, 50, 79, 111, 135$ and 158 \cite{52, 53} in the mid-rapidity range $|y| < 1.0$ are compared with our calculations (curves). We find that for all multiplicity intervals and for all hadrons our calculations are in good agreement with the results measured in the p + p collisions, at $\sqrt{s_{NN}} = 7$ TeV.

In figures 1, 2, and 3, we focus on the lowest $p_T$, where both types of statistical approaches are conjectured to work well. There is another overall observation to be mentioned now that the resulting fit parameter $T$ greatly differs from one type of statistical approach to another. On the other hand, the resulting fit parameters deduced from generic (non)extensive statistical fits for the transverse momentum distributions in different multiplicity intervals from Pb + Pb, p + Pb and p + p are depicted in figures 4–7. The qualities of the statistical fits, at $c = 0.99988$, $\mu = 0$, and $\sqrt{s_{NN}} = 2.76, 5.02, 7$ TeV, are summarized in Table 1.

The present study is designed to determine the possible variations between the three types of statistical approaches and to characterize their dependence on energy, multiplicity, and size of the collisions. Using generic (non)extensive statistical approach, we have extracted various fit parameters and then compared these to the corresponding results obtained from fits to Tsallis and Boltzmann statistics \cite{54}.

For the sake of simplicity, we assumed that the geometry of the fireball is spherical, so that the volume reads $V = 4/3\pi R^3$, where $R$ defines the dimension of the interacting system. On the other hand, the volume can be related to the normalization of the statistical distribution function,
equation (6), which is utilized in describing the particle spectra [55] or/and yields [6], section 2.

Figure 4 shows the fit parameters $T$ (left) and $V$ (right) as obtained for the strange hadrons $K^0_S$, $\Lambda$, and $\Xi^-$ as functions of $\langle N_{\text{track}} \rangle$ in (a) Pb + Pb, at $\sqrt{s_{NN}} = 2.76$ TeV, in (b) p + Pb, at $\sqrt{s_{NN}} = 5.02$ TeV, and in (c) p + p collisions, at $\sqrt{s_{NN}} = 7$ TeV. The symbols refers to the statistical approaches as in the previous figures.

Figure 6. The same as in figure 4, but here from p + p collisions, at $\sqrt{s_{NN}} = 7$ TeV.

Figure 7. The (non)extensivity parameter $d$ as deduced from the statistical fits for the strange particles $K^0_S$, $\Lambda$, and $\Xi^-$ in dependence on $\langle N_{\text{track}} \rangle$ in (a) Pb + Pb, at $\sqrt{s_{NN}} = 2.76$ TeV, in (b) p + Pb, at $\sqrt{s_{NN}} = 5.02$ TeV, and in (c) p + p collisions, at $\sqrt{s_{NN}} = 7$ TeV. The symbols refers to the statistical approaches as in the previous figures.
We notice that the value of the temperature gained from the generic 
statistics, and Boltzmann statistics (dashed curve), equation (4).

The same as in figure 8, but here for the most central \((p+p)\) collisions in the CMS experiment (symbols) [34], at \(\sqrt{s_{\text{NN}}} = 0.9\) TeV. The generic (non)extensive, equation (12), and Boltzmann statistics, equation (3), are illustrated by solid and dashed curves, respectively.

The comparison with corresponding results deduced from Boltzmann and Tsallis statistics is also depicted, from which a few remarks are to be highlighted. First, \(T_0\) obtained from the generic (non)extensive statistics agrees well with [44, 45] but apparently disagrees with ref. [54]. This would mean that the produced particles with large masses and large strange quantum numbers likely freeze out, chemically, earlier than the ones with smaller masses and less strange quantum numbers [1]. Second, \(T_0\) deduced from Boltzmann statistics is greater than the one obtained from Tsallis statistics, \(T_{\text{T}}\), which in turn is larger than the temperature gained from the generic (non)extensive statistics, \(T_{\text{G}}\). This conclusion that \(T_0 > T_{\text{T}} > T_{\text{G}}\) isn’t depending on the type of particle or collision. Third, this result would be understood due to the different types of statistics. As highlighted, both Boltzmann and Tsallis statistics are very special cases within generic (non)extensive statistics, i.e. for Boltzmann \((c, d) = (1, 1)\) while for Tsallis \((c, d) = (q, 0)\) [26]. They apparently manifest transitions from chemical (larger temperature) to the kinetic freezeouts (lower temperature).

The volume extracted from generic (non)extensive statistics and the volumes determined from Tsallis and Boltzmann statistics [54] increase with the increase in \(\langle N_{\text{track}}\rangle\) for all systems. The volumes deduced from Tsallis and Boltzmann statistics are close to each other. Both are greater than the volume obtained from generic (non)extensive statistics except for \(K^0_s\) volume obtained from generic (non)extensive statistics is the greatest one. Furthermore, we conclude that the volume in Pb + Pb collisions is greater than the one in p + p collisions, which in turn is larger than in p + p collisions. Both conclusions are not depending on the type of the produced particle.

Figure 7 presents the universality (equivalent) class \(d\) as deduced for the statistical fits of the strange hadrons \(K^0_s, \Lambda, \Xi^-\) in dependence on \(\langle N_{\text{track}}\rangle\) in different types of collisions; (a) \(\text{Pb} + \text{Pb}\), at \(\sqrt{s_{\text{NN}}} = 2.76\) TeV, (b) \(\text{p} + \text{Pb}\), at \(\sqrt{s_{\text{NN}}} = 5.02\) TeV, and (c) \(\text{p} + \text{p}\) collisions, at \(\sqrt{s_{\text{NN}}} = 7\) TeV. We notice that the value of \(d\) slightly increases with increasing \(\langle N_{\text{track}}\rangle\) in all types of collisions. Also, \(d\) decreases with the increase in both particle masses and strange quantum numbers. The observation that \(d\) decreases with \(\langle N_{\text{track}}\rangle\) apparently manifests that the statistical nature of the system approaches extensivity, especially that the second equivalent class \(c\) is taken very close to unity, \(c = 0.99988\), indicating that the Boltzmann statistics becomes the proper statistical approach in describing that system.

In the next section, we study the possible correlations between the resulting fit parameters, especially with the collision energies.
3.2. Correlations between resulting fit parameters

In this section, we propose expressions relating the dependence of the resulting fit parameters on the collision energies. To this end, we analyse, for instance, the transverse momentum distribution for the same strange hadrons measured in most central collisions, at different energies. Figure 8 presents the transverse momentum $p_T$ spectra of (a) $K^0_s$, (b) $\Lambda$, and (c) $\Xi^-$ produced in most central $p + p$ collisions in the mid-rapidity range $|y| < 1.0$, at $\sqrt{s_{NN}} = 0.2$ TeV. The symbols refer to the experimental results [51].

Our calculations using generic (non)extensive and Boltzmann statistics are depicted as solid and dashed curves, respectively. Here, we focus on the smallest $p_T$ region, where $c = 0.99988$, and $\mu \approx 25$ MeV. We find that our calculations for $\Lambda$ and $\Xi^-$ agree well with the experimental $p + p$ results. Figure 9 shows the same as figure 8, but here at $\sqrt{s_{NN}} = 0.9$ TeV and $|y| < 1.0$. The symbols represent the CMS results [34]. We also find that our calculations for $\Lambda$ and $\Xi^-$ agree well with $p + p$ collisions, at $c = 0.99988$ and $\mu = 0$. Our calculations using both extensive and generic (non)extensive statistics seem to agree well with the experimental data. We can extract various fit parameters from both types of statistics, at a wide range of energies, as shown in figures 10–12.

Figure 10 depicts the dependence of the fit parameter $T$ on the collision energies $\sqrt{s_{NN}}$ for the strange hadrons $K^0_s$, $\Lambda$, and $\Xi^-$ as obtained from generic (non)extensive and Boltzmann statistics. For all studied strange hadrons, we find that $T_G$, the temperature, determined from generic (non)extensive fits decreases with the increase in $\sqrt{s_{NN}}$, i.e. increasing collision’s centrality, while $T_B$ obtained from the Boltzmann fits increases with the increase in $\sqrt{s_{NN}}$. Also, we find that $T_G$ decreases with the increase in both particle masses and strange quantum numbers, while $T_B$ shows an opposite dependence. On the other hand, it is worthy highlighting that the obtained $T_G$ agrees well with previous studies [44, 45], in which it was concluded that the chemical freezeout of such particles with large masses and large strange quantum numbers takes place earlier than the one for light particles with small strange quantum numbers. [1].

The Boltzmann temperature $T_B$ and volume are deduced from the statistical analysis of $p_T$ spectra, at 200 and 900 GeV. The dependence of $T_B$ on the collision energy seems to
vary with the type of statistics but independent on the system size of the collision. To summarize, we find that $T_B$ is to be related to the chemical freezeout temperature, while $T_G$ to the kinetic freezeout temperature. This is based on phenomenological point-of-view, where the temperature obtained from generic (non)extensive statistics can be related to the kinetic freezeout temperature, while the temperature obtained from Boltzmann extensive statistics looks similar to the chemical freezeout temperature.

Figure 11 presents the fit parameter $V$, the volume, as a function of $\sqrt{s_{NN}}$ for the strange hadrons $K^0_s$, $\Lambda$, and $\Xi^-$ as fitted to generic (non)extensive and Boltzmann statistics. For all hadrons, we find that the volume deduced from both types of statistics has almost the same behaviour. On the other hand, the Boltzmann volume is found greater than the generic (non)extensive volume. Also, it is apparent that the volume deduced from both types of statistics increases with the increase in $\sqrt{s_{NN}}$ till 2.76 GeV, and decreases with the increase in $\sqrt{s_{NN}}$. This trend indicates that the dependence on the energy is only more effective at low energy. It is noticed that the value of the volume from both statistics decreases with the increase in the particle masses and strange quantum numbers. In other words, the volume of the system is small if this system contains hadrons with large mass and/or large strange quantum numbers.

Figure 12 shows the nonextensive parameter $d$ as a function of $\sqrt{s_{NN}}$ for the strange hadrons $K^0_s$, $\Lambda$, and $\Xi^-$, which have been fitted to generic (non)extensive statistics. For all strange hadrons, we find that $d$ decreases with the increase in $\sqrt{s_{NN}}$. Also, we observe that $d$ decreases with the increase in both particle masses and strange quantum numbers. The decrease in $d$ with $\sqrt{s_{NN}}$ indicates that the system starts from a non-equilibrium state, nonextensive, at low $\sqrt{s_{NN}}$, and apparently approaches an equilibrium state, extensive, at large $\sqrt{s_{NN}}$, i.e. a novel kind of a statistical transition, which can only be characterized by generic (non)extensive statistics.

### 3.2.1. Analytical expressions for the resulting fit parameters

Tables 2 and 3 and figures 10, 11, and 12 list out and depict various fit parameters. In the following, we summarize the various dependences of these thermodynamical quantities on the collision energy.

- Using both Boltzmann and generic (non)extensive statistics, the dependence of temperature on $\sqrt{s_{NN}}$ for all particles is given as

$$ T = a \sqrt{s_{NN}}^b + c \sqrt{s_{NN}}, $$

(14)

where the values of $a$, $b$, and $c$ are taken from figure 10, see Tables 2, 3.

- For $K^0_s$, the dependence of the volume $V$ on $\sqrt{s_{NN}}$ using both statistics can be expressed as

$$ V = a \sqrt{s_{NN}} + b \sqrt{s_{NN}}^2 + c \sqrt{s_{NN}}, $$

(15)

where the values of $a$, $b$, and $c$ are taken from figure 11, see Tables 2, 3.

- For $\Lambda$, the dependence of volume $V$ on $\sqrt{s_{NN}}$ using both statistics can be expressed as

$$ V = a \sqrt{s_{NN}} + b \sqrt{s_{NN}}^2 + c \sqrt{s_{NN}}^3, $$

(16)

where the values of $a$, $b$, and $c$ are taken from figure 11, see Tables 2, 3.

- For $\Xi^-$, the dependence of volume $V$ on $\sqrt{s_{NN}}$ using both statistics can be expressed as

$$ V = a \sqrt{s_{NN}} + b \sqrt{s_{NN}}^2 + c \sqrt{s_{NN}}^3 + f, $$

(17)

where the values of $a$, $b$, $c$, and $f$ are taken from figure 11, see Tables 2, 3.

- For all particles, the dependence of the equivalent class $d$ on $\sqrt{s_{NN}}$ is suggested as

$$ d = (a + \sqrt{s_{NN}}^{-b})c + f \sqrt{s_{NN}}, $$

(18)

where the values of $a$, $b$, $c$, and $f$, are shown in figure 12, see Table 3.

So far, we conclude that the fit parameters obtained depend on the collision energy and on the type of the statistical approach applied, especially when moving from extensive (Boltzmann) to nonextensive statistical approach.

### 4. Conclusions

Within generic (non)extensive statistics, we have analysed the transverse momentum distribution $p_T$ of the strange hadrons $K^0_s$, $\Lambda$, and $\Xi^-$ in different multiplicity classes measured in the CMS experiment in $\text{Pb + Pb}$ collisions, at $\sqrt{s_{NN}} = 2.76$ TeV, in $\text{p + Pb}$ collisions, at $\sqrt{s_{NN}} = 5.02$ TeV, and in $\text{p + p}$ collisions, at $\sqrt{s_{NN}} = 0.2, 0.9$, and $7$ TeV. The fit parameters deduced are compared with previous studies, in which Tsallis and Boltzmann statistics have been utilized.
Table 3. The various fit parameters obtained from generic (non)extensive statistics, equations (14)–(18).

| Fit parameters | $K_0$ | $\Lambda$ | $\Xi$ |
|----------------|-------|-----------|-------|
| T              | $a$   | 245.879 ± 19.79 | 206.964 ± 39.69 | 379.443 ± 161.5 |
| b              | $-0.0428$ ± 0.0017 | $-0.0315$ ± 0.0032 | $-0.138$ ± 0.0054 |
| c              | $-0.0059$ ± 0.0003 | $-0.0035$ ± 0.0004 | $0.0036$ ± 0.0016 |
| V              | $a$   | $0.0004$ ± 0.0002 | $0.0111$ ± $4.806 \times 10^{-6}$ | $(3.7378 \pm 0.0214) \times 10^{-5}$ |
| b              | $(-6.6553 \pm 2.652) \times 10^{-8}$ | $0.9945$ ± $4.922 \times 10^{-5}$ | $-1.166 \times 10^{-8}$ ± $3.35 \times 10^{-11}$ |
| c              | $(-1.83 \pm 0.4492) \times 10^{-12}$ | $-0.0105$ ± $4.58 \times 10^{-6}$ | $9.3786 \times 10^{-11}$ ± $4.868 \times 10^{-15}$ |
| f              | $a$   | $-0.006$ ± 0.0001 |
| d              | $a$   | $0.9789$ ± 0.044 | $0.3792$ ± 0.0006 | $0.2503$ ± 0.0008 |
| b              | $0.8876$ ± 0.0113 | $0.558$ ± 0.0043 | $0.4906$ ± 0.0053 |
| c              | $17.3014$ ± 0.3036 | $0.4066$ ± 0.0006 | $0.2869$ ± 0.0006 |
| f              | $-2.622 \times 10^{-6}$ ± $9.802 \times 10^{-8}$ | $(-9.3153 \pm 0.7578) \times 10^{-7}$ | $(-1.2104 \pm 0.116) \times 10^{-6}$ |
This comprehensive comparison indicates variations between the three types of statistical approaches.

We conclude that the temperature obtained from the generic (non)extensive statistics agrees well with previous studies [44, 45] but not with other ones, such as [54]. Thus, we conclude that the produced strange particles with large masses and large strange quantum numbers seem to freezeout, either chemically or kinetically, earlier than the ones with smaller masses and less strange quantum numbers [1]. As for the dependence on the type of the statistical approaches, we have obtained that the temperature deduced from Boltzmann statistics is larger than the one obtained from Tsallis statistics. The latter is in turn larger than the one determined from generic (non)extensive statistics. This conclusion is not affected by the type of particles or collisions. As an explanation, we suggest that the different types of statistics play an essential role. The different temperatures obtained would be conjectured to manifest transitions from chemical (larger temperature) to kinetic freezeouts (lower temperature). Accordingly, Boltzmann and Tsallis statistics can be related to the chemical and the kinetic freezeouts, respectively. The generic (non)extensive statistics combines both types of statistical nature, where to the universality (equivalent) class \((c, d)\) specific values are assigned. For Boltzmann \((c, d) = (1, 1)\), while for Tsallis \((c, d) = (q, 0)\). The values assigned to the universality (equivalent) classes \((c, d)\) autonomously dictates statistical nature of the studies system, i.e. within the ranges of the classes \((c, d)\) both Boltzmann and Tsallis are very special cases.

As for the volume extracted from generic (non)extensive statistics and compared with Tsallis and Boltzmann statistics, we find that the volume apart from the type of the produced particles increases with the collision’s centrality for all collision sizes. While the volume values deduced from Tsallis and Boltzmann are close to each other and both are greater than the one obtained from generic (non)extensive statistics. We found that the temperature and the volume obtained within the different statistical approaches increase with the increase in the multiplicity classes for all types of collisions.

As for the energy dependence of the various fit parameters, we first found that the temperature deduced from generic (non)extensive fits decreases with the increase in \(\sqrt{\text{NN}}\), while the Boltzmann temperature increases. The generic (non)extensive temperature decreases with the masses and and the strange quantum numbers of the particles, while the Boltzmann temperature has an opposite dependence. The earlier refers to early freezeout of particles with large masses and large strange quantum numbers. Lighter particles with smaller strange quantum numbers freezeout afterward. The system size seems not affect this conclusion. From phenomenological point-of-view, we conclude that the temperature obtained from generic (non)extensive statistics can be compared with the kinetic freezeout temperature, while the Boltzmann temperature can be related to the chemical freezeout temperature.

The nonextensive parameter \(d\) decreases with the increase in \(\sqrt{\text{NN}}\), particle masses, and strangeness contents indicating that the system moves from a non-equilibrium state, nonextensive, at low \(\sqrt{\text{NN}}\) towards an equilibrium state, extensive, at large \(\sqrt{\text{NN}}\). This novel statistical transition can only be characterized by the generic (non)extensive statistics. In other words, when utilizing either Boltzmann or Tsallis statistics, the system of interest is ad hoc biased the corresponding statistical nature. Particle masses autonomously manifest the real nature of statistics describing the particle production.

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