Spectrum completion and inverse Sturm–Liouville problems

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1 | INTRODUCTION

Let $q \in L_2(0, \pi)$ be real valued. Consider the Sturm–Liouville equation

$$-y'' + q(x)y = \lambda y, \quad x \in (0, \pi),$$

where $\lambda \in \mathbb{C}$. In this work, we explore the following surprising possibility. Suppose, there are given several first eigenvalues of a corresponding regular Sturm–Liouville problem, and the potential $q(x)$ is unknown, we compute an arbitrarily large number of subsequent eigenvalues with a uniform absolute accuracy.

We show that a very limited number of the eigenvalues may be sufficient for computing hundreds of subsequent eigenvalues with a remarkable accuracy. Of course, it cannot be a question of calculating the eigenvalues based on their asymptotics, because such a reduced number of known eigenvalues is definitely not enough for obtaining the asymptotics, nor is it possible to talk about recovering the potential from several eigenvalues of one spectrum.

Our approach is based on completely different ideas. In Kravchenko et al., special representations for solutions of (1.1) and for their derivatives in the form of so-called Neumann series of Bessel functions (NSBF) were obtained, possessing...
certain important properties. First of all, the remainders of the series admit estimates independent of \( \rho = \sqrt{\lambda} \) for all real \( \rho \) or \( \rho \) belonging to a strip \( |\text{Im}\rho| < C \), where \( C > 0 \). Simply put, the truncated series approximate equally well the exact solutions and their first derivatives independently of the largeness of \( |\text{Re}\rho| \). This is extremely useful when considering direct and inverse spectral problems because it allows one to operate on large intervals of \( \rho \) and consequently of \( \lambda \). Second, the knowledge of the very first coefficient of the series is sufficient for recovering the potential \( q(x) \). These unique features of the NSBF representations were used for solving direct Sturm–Liouville problems\(^1\)–\(^3\) and for solving inverse Sturm–Liouville problems\(^4\)–\(^9\). In the present work, we show that the NSBF representations allow one to complete the spectrum of the Sturm–Liouville problem and how this spectrum completion is used for solving the two-spectra inverse Sturm–Liouville problem.

We develop the spectrum completion technique for the Dirichlet–Dirichlet spectrum, that is the eigenvalues \( \{\nu_n\}_{n=1}^{\infty} \) of (1.1) subject to the boundary conditions

\[
y(0) = y(\pi) = 0, \quad (1.2)
\]

for the Dirichlet–Neumann spectrum, that is the eigenvalues \( \{\lambda_n\}_{n=0}^{\infty} \) of (1.1) subject to the boundary conditions

\[
y(0) = y'(\pi) = 0, \quad (1.3)
\]

as well as for the Sturm–Liouville problem with the boundary conditions

\[
y'(0) - hy(0) = y'(\pi) + H y(\pi) = 0,
\]

where \( h \) and \( H \) are unknown real constants. We complete the spectrum without knowing the values of these constants.

Moreover, often, especially when solving inverse Sturm–Liouville problems numerically, the authors are forced to assume that besides the boundary conditions the important quantity

\[
\omega := \frac{1}{2} \int_0^\pi q(t)dt
\]

is known. It appears in the asymptotics of the eigenvalues (see, e.g., (2.1) and (2.3)) and is used in different steps of most existing algorithms. To extract this parameter from the asymptotics of the eigenvalues, a considerable number of the eigenvalues are required. Our approach allows us to compute the parameter \( \omega \) from very few eigenvalues (see Section 3.2 below) due to its close relation to the first coefficient of the NSBF representation for the derivative of the solution.

As an application of the spectrum completion technique, we develop a new method for solving the classical inverse Sturm–Liouville problem consisting in numerical recovering the potential \( q(x) \) from a finite set of eigenvalues from two spectra.

Several methods have been proposed for numerical solution of inverse Sturm–Liouville problems (see previous works\(^5\)–\(^21\)). However, usually they require the knowledge of additional parameters like, for example, the parameter \( \omega \).

The method proposed in the present work reduces the problem to a system of linear algebraic equations for finding the first coefficient of the NSBF representation for the solution. The accuracy of the method relies on the spectrum completion technique. To the difference from the numerical methods developed earlier on the base of the NSBF representations (see previous studies\(^4\)–\(^9\)), here the system of linear algebraic equations is obtained without using the Gelfand–Levitan integral equation but a relation between the eigenfunctions normalized at the opposite endpoints. The method is simple in its numerical realization, accurate, and fast.

2 | PRELIMINARIES

Let us recall the asymptotics of the Dirichlet–Dirichlet and Dirichlet–Neumann eigenvalues of Equation (1.1). The square roots of the eigenvalues of the Sturm–Liouville problem (1.1) and (1.2) satisfy the asymptotic relation (see, e.g., Yurko\(^22\), p. 18)

\[
\mu_n = \sqrt{\nu_n} = n + \frac{\omega}{\pi n} + \frac{\kappa_n}{n},
\]

where \( \{\kappa_n\} \in \ell_2 \) and

\[
\omega = \frac{1}{2} \int_0^\pi q(t)dt.
\]
The square roots of the eigenvalues of the Sturm–Liouville problem (1.1) and (1.3) satisfy the asymptotic relation (see, e.g., Yurko²², p. 18)

\[ \rho_n = \sqrt{\lambda_n} = n + \frac{1}{2} + \frac{\alpha}{\pi n} + \frac{\gamma_n}{n}, \quad \{\gamma_n\} \in \ell_2. \]  

(2.3)

By \( \varphi(\rho, x) \) and \( S(\rho, x) \), we denote the solutions of the equation

\[ -y''(x) + q(x)y(x) = \rho^2 y(x), \quad x \in (0, \pi) \]  

(2.4)

satisfying the initial conditions

\[ \varphi(\rho, 0) = 1, \quad \varphi'(\rho, 0) = h, \]
\[ S(\rho, 0) = 0, \quad S'(\rho, 0) = 1, \]  

(2.5)

where \( h \) is some (complex) constant. Here, \( \rho = \sqrt{\lambda}, \Im \rho \geq 0 \). The main tool used in the present work is the series representations obtained in Kravchenko et al¹ for the solutions of (2.4) and their derivatives.

**Theorem 2.1** (Kravchenko et al.¹). The solutions \( \varphi(\rho, x) \) and \( S(\rho, x) \) and their derivatives with respect to \( x \) admit the following series representations:

\[ \varphi(\rho, x) = \cos(\rho x) + \sum_{n=0}^{\infty} (-1)^ng_n(x)j_{2n}(\rho x), \]  

(2.6)

\[ S(\rho, x) = \frac{\sin(\rho x)}{\rho} + \frac{1}{\rho} \sum_{n=0}^{\infty} (-1)^ns_n(x)j_{2n+1}(\rho x), \]  

(2.7)

\[ \varphi'(\rho, x) = -\rho \sin(\rho x) + \left( h + \frac{1}{2} \int_0^x q(t) \, dt \right) \cos(\rho x) + \sum_{n=0}^{\infty} (-1)^n\gamma_n(x)j_{2n}(\rho x), \]  

(2.8)

\[ S'(\rho, x) = \cos(\rho x) + \frac{1}{2\rho} \left( \int_0^x q(t) \, dt \right) \sin(\rho x) + \frac{1}{\rho} \sum_{n=0}^{\infty} (-1)^n\sigma_n(x)j_{2n+1}(\rho x), \]  

(2.9)

where \( j_k(z) \) stands for the spherical Bessel function of order \( k \) (see, e.g., Abramovitz and Stegun²³). The coefficients \( g_n(x) \), \( s_n(x) \), \( \gamma_n(x) \) and \( \sigma_n(x) \) can be calculated following a simple recurrent integration procedure (see Kravchenko et al¹ or Kravchenko⁶, Sect. 9.4), starting with

\[ g_0(x) = \varphi(0, x) - 1, \quad s_0(x) = 3 \left( \frac{S(0, x)}{x} - 1 \right), \]  

(2.10)

\[ \gamma_0(x) = g'_0(x) - h - \frac{1}{2} \int_0^x q(t) \, dt, \quad \sigma_0(x) = \frac{s_0(x)}{x} + s'_0(x) - \frac{3}{2} \int_0^x q(t) \, dt. \]

For every \( \rho \in \mathbb{C} \), all the series converge pointwise. For every \( x \in [0, \pi] \), the series converge uniformly on any compact set of the complex plane of the variable \( \rho \), and the remainders of their partial sums admit estimates independent of \( \Re \rho \).

This last feature of the series representations (the independence of \( \Re \rho \) of the estimates for the remainders) is a direct consequence of the fact that the representations are obtained by expanding the integral kernels of the transmutation operators (for their theory we refer to previous works²⁴–²⁶) into Fourier–Legendre series (see Kravchenko et al¹ and Kravchenko⁶, Sect. 9.4). It is of crucial importance for what follows. In particular, it means that for \( S_N(\rho, x) := \frac{\sin(\rho x)}{\rho} + \frac{1}{\rho} \sum_{n=0}^{N} (-1)^n s_n(x)j_{2n+1}(\rho x) \) (and analogously for \( S'_N(\rho, x) \)) the estimate holds

\[ |S(\rho, x) - S_N(\rho, x)| < \epsilon_N(x), \]  

(2.11)

for all \( \rho \in \mathbb{R} \), where \( \epsilon_N(x) \) is a positive function tending to zero when \( N \to \infty \). That is, the approximate solution \( S_N(\rho, x) \) approximates the exact one equally well for small and for large values of \( \rho \). This is especially convenient when considering direct and inverse spectral problems. Moreover, for a fixed \( z \), the numbers \( j_k(z) \) rapidly decrease as \( k \to \infty \); see, for example, Abramovitz and Stegun²³, (9.1.62). Hence, the convergence rate of the series for any fixed \( \rho \) is, in fact, exponential.

More detailed estimates for the series remainders depending on the regularity of the potential can be found in Kravchenko et al.¹
Note that formulas (2.10) indicate that the potential \( q(x) \) can be recovered from the first coefficients of the series (2.6) or (2.7). We have

\[
q(x) = \frac{g''_0(x)}{g_0(x) + 1}
\]  

(2.12)

and

\[
q(x) = \frac{(xs_0(x))''}{xs_0(x) + 3x}.
\]  

(2.13)

Note that the square roots of the Dirichlet–Dirichlet eigenvalues coincide with zeros of the function \( S(\mu, \pi) \):

\[ S(\mu_n, \pi) = 0, \quad n = 1, 2, \ldots \]

while the square roots of the Dirichlet–Neumann eigenvalues coincide with zeros of the function \( S'(\rho, \pi) \):

\[ S'(\rho_n, \pi) = 0, \quad n = 0, 1, \ldots \]

3 | SPECTRUM COMPLETION

3.1 | Dirichlet–Dirichlet spectrum

Given several first Dirichlet–Dirichlet eigenvalues \( \{ \mu^2_n \}_{n=1}^{N_1} \), let us use them to calculate the first coefficients \( s_0(\pi), s_1(\pi), \ldots, s_N(\pi) \), where \( N \leq N_1 + 1 \). First, it is convenient to consider the shifted potential

\[ \tilde{q}(x) := q(x) - \mu_1^2. \]

That is, instead of the problem (1.1) and (1.2), we consider the problem

\[
-y''(x) + \tilde{q}(x)y(x) = \rho^2 y(x), \quad x \in (0, \pi),
\]  

(3.1)

\[
y(0) = y(\pi) = 0.
\]  

(3.2)

Obviously, its eigenfunctions do not change while the eigenvalues are shifted:

\[ \mu_1^2 = 0, \quad \mu_2^2 = \mu_1^2, \quad \mu_3^2 = \mu_2^2 - \mu_1^2, \quad \mu_4^2 = \mu_3^2 - \mu_1^2, \ldots \]

The solution of (3.1) satisfying the initial conditions (2.5) we denote as \( \tilde{S}(\rho, x) \). Its NSBF representation has the form

\[
\tilde{S}(\rho, x) = \frac{\sin(\rho x)}{\rho} + \sum_{n=0}^{\infty} \left( -1 \right)^n \tilde{s}_n(x) J_{2n+1}(\rho x),
\]  

(3.3)

where the coefficients \( \tilde{s}_n(x) \), in general, do not coincide with the coefficients \( s_n(x) \); however, it is clear that

\[ S(\rho, x) = \tilde{S} \left( \sqrt{\rho^2 + \mu_1^2}, x \right). \]

Note that

\[ \tilde{S}(0, \pi) = 0, \]  

(3.4)

since zero is a Dirichlet–Dirichlet eigenvalue of \( \tilde{q}(x) \).

On the other hand, from (3.3), we have

\[ \tilde{S}(0, x) = x + \frac{x \tilde{s}_0(x)}{3}, \]  

(3.5)
where we take into account that \( j_1(z) \sim \frac{z}{\lambda}, z \to 0 \) and more generally,
\[
\hat{j}_n(z) \sim \frac{z^n}{(2n+1)!}, z \to 0.
\]
Substituting \( x = \pi \) into (3.5) and taking into account (3.4), we obtain
\[
\hat{s}_0(\pi) = -3. \tag{3.6}
\]
Several subsequent coefficients \( \hat{s}_n(\pi), n = 1, \ldots, N \) can be found from the equations
\[
\hat{S}_N(\hat{\mu}_k, \pi) = 0, \quad k = 2, 3, \ldots, N_1, \tag{3.7}
\]
where
\[
\hat{S}_N(\rho, x) = \frac{\sin(\rho x)}{\rho} + \frac{1}{\rho} \sum_{n=0}^{N} (-1)^n \hat{s}_n(x) j_{2n+1}(\rho x). \tag{3.8}
\]
From (3.7), we obtain the system of linear algebraic equations for the coefficients \( \hat{s}_n(\pi) \):
\[
\sum_{n=0}^{N} (-1)^n \hat{s}_n(\pi) j_{2n+1}(\hat{\mu}_k, \pi) = -\sin(\hat{\mu}_k, \pi), \quad k = 2, 3, \ldots, N_1.
\]
Taking into account (3.6), we obtain
\[
\sum_{n=1}^{N} (-1)^n \hat{s}_n(\pi) j_{2n+1}(\hat{\mu}_k, \pi) = 3j_1(\hat{\mu}_k, \pi) - \sin(\hat{\mu}_k, \pi), \quad k = 2, 3, \ldots, N_1. \tag{3.9}
\]
Solving this system of equations, we compute \( \hat{s}_1(\pi), \ldots, \hat{s}_N(\pi) \). This gives us the possibility to compute an arbitrary number of the Dirichlet–Dirichlet eigenvalues of the problem (3.1) and (3.2) and consequently of the original problem (1.1) and (1.2). Indeed, the function
\[
\hat{S}_N(\rho, \pi) = \frac{\sin(\rho \pi)}{\rho} + \frac{1}{\rho} \sum_{n=0}^{N} (-1)^n \hat{s}_n(\pi) j_{2n+1}(\rho \pi) \tag{3.10}
\]
approximates the solution \( \hat{S}(\rho, \pi) \) at \( x = \pi \) for any value of \( \rho \). Moreover, for \( \rho \in \mathbb{R} \), we have the estimate
\[
| \hat{S}(\rho, \pi) - \hat{S}_N(\rho, \pi) | < \varepsilon_N,
\]
where \( \varepsilon_N \) is independent of \( \rho \). With the aid of complex analysis tools, the following theorem is proved.

**Theorem 3.1.** For any \( \varepsilon > 0 \), there exists such \( N \in \mathbb{N} \) that all zeros of the function \( \hat{S}(\rho, \pi) \) are approximated by corresponding zeros of the function \( \hat{S}_N(\rho, \pi) \) with errors uniformly bounded by \( \varepsilon \), and \( \hat{S}_N(\rho, \pi) \) has no other zeros.

*Proof.* The proof of this statement is completely analogous to the proof of Proposition 7.1 in Kravchenko and Torba\(^{27}\) and consists in the use of properties of characteristic functions of regular Sturm–Liouville problems and application of the Rouché theorem.

Thus, zeros of the function \( \hat{S}_N(\rho, \pi) \) give us approximate numbers \( \hat{\mu}_k \) for \( k = N_1 + 1, \ldots \).

**Example 1.** Consider the potential \( q(x) = e^x \) (first Paine’s test; see Pryce\(^{28}\)). Then \( \mu_1^2 \approx 4.8966937996 \) (here and below, we used the Matslise package\(^{29}\) to compute the “exact” eigenvalues). In Figure 1, the absolute and relative errors of \( \mu_k \) computed for \( k = 6, \ldots, 300 \) are presented. Here, five Dirichlet–Dirichlet eigenvalues were given \( (N_1 = 5) \), and we present the “completed” part of the Dirichlet–Dirichlet spectrum computed with \( N = 4 \) (so that four coefficients \( \hat{s}_n(\pi) \) in (3.10) are computed from (3.9) and together with the coefficient (3.6) they are used to compute the eigenvalues by finding zeros of the function \( \hat{S}_N(\rho, \pi) \)). It is worth mentioning that when dealing with the truncated systems of linear algebraic equations we do not seek to work with the square systems. In computations, a least-squares solution of an overdetermined system (provided by Matlab, which we used in this work) gives very satisfactory results.
FIGURE 1 Absolute and relative errors of $\mu_k$, $k = 6, \ldots, 300$ computed from five eigenvalues given for the potential $q(x) = e^x$.
[Colour figure can be viewed at wileyonlinelibrary.com]

FIGURE 2 Numbers $c_k$ defined by (3.11) computed for Example 1, $k = 3, \ldots, 10$.
[Colour figure can be viewed at wileyonlinelibrary.com]

Of course, five eigenvalues are not enough to obtain from (2.1) a meaningful value of the parameter $\omega$, so it is not clear how one could complete the spectrum in another way. Even the knowledge of 10 eigenvalues still does not give the possibility to find $\omega$ with a reasonable accuracy. Figure 2 shows the numbers

$$c_k = \pi k(\mu_k - k) - \omega, \quad k = 3, \ldots, 10,$$

which according to (2.1) represent an $\ell_2$-convergent sequence. The value of $c_{10}$ is approximately $-0.06$, which still differs from zero considerably. The attempt to compute $\omega$ by minimizing the $\ell_2$-norm of the sequence $\{c_k\}_{k=1}^{10}$ leads to a large error.

Moreover, even the knowledge of the exact value of the parameter $\omega$ leads to less accurate results in comparison with the spectrum completion technique based on solving the system (3.9) and finding zeros of the function $\tilde{S}_N(\rho, \pi)$. In Figure 3, we compare the absolute error of the numbers $\mu_k$ approximated by the asymptotic relation $\mu_k \approx k + \frac{\omega}{\pi k}$ (where the exact value of $\omega$ is used) with the absolute error of the values of $\mu_k$ obtained with the aid of the spectrum completion technique. Even for $k = 40$ the asymptotic approximation gives a much less accurate result than the spectrum completion technique: $4.6 \cdot 10^{-5}$ against $6.8 \cdot 10^{-7}$.

A similar situation is observed in other examples.

Example 2. Consider the potential $q(x) = \frac{1}{(x+0.1)^2}$ (second Paine’s test; see Pryce). In Figure 4, we make the same comparison: The absolute error of the numbers $\mu_k$ approximated by the asymptotic relation $\mu_k \approx k + \frac{\omega}{\pi k}$ with the exact
Even when the parameter \( \omega \) is known, the spectrum completion technique often gives more accurate results than the use of the asymptotic relation \( \mu_k \approx k + \frac{\omega}{\pi k} \). Here, we compare the accuracy of the “asymptotic” \( \mu_k, k = 11, \ldots, 40 \) versus those computed by the spectrum completion technique from ten eigenvalues given [Colour figure can be viewed at wileyonlinelibrary.com]

For the potential from Example 2, we compare the accuracy of the “asymptotic” \( \mu_k, k = 11, \ldots, 40 \) versus those computed by the spectrum completion technique from ten eigenvalues given [Colour figure can be viewed at wileyonlinelibrary.com]

value of \( \omega \) being used versus the absolute error of the values of \( \mu_k \) obtained with the aid of the spectrum completion technique.

### 3.2 | Dirichlet–Neumann spectrum

Given several first Dirichlet–Neumann eigenvalues \( \{\rho_n^2\}_{n=0}^{N_2} \), let us use them to calculate the first coefficients \( \sigma_0(\pi), \sigma_1(\pi), \ldots, \sigma_N(\pi) \), where \( N \leq N_2 - 1 \). Similarly to the case of the Dirichlet–Dirichlet spectrum, it is convenient to consider the shifted potential

\[
\hat{q}(x) := q(x) - \rho_0^2.
\]

That is, instead of the problem (1.1) and (1.3), we consider the problem

\[
- \gamma''(x) + \hat{q}(x)\gamma(x) = \rho^2\gamma(x), \quad x \in (0, \pi),
\]

\[
y(0) = y'(\pi) = 0.
\]

Again the eigenfunctions do not change, and the eigenvalues are shifted:

\[
\hat{\rho}_0^2 = 0, \quad \hat{\rho}_1^2 = \rho_1^2 - \rho_0^2, \quad \hat{\rho}_2^2 = \rho_2^2 - \rho_0^2, \quad \ldots
\]

The solution of (3.12) satisfying the initial conditions (2.5) we denote as \( \tilde{S}(\rho, x) \). We have the relation
\[ \hat{S}(\rho, x) = \mathcal{S}(\sqrt{\rho^2 + \rho_0^2 - \mu^2}, x). \]

We are interested in the derivative of \( \hat{S}(\rho, x) \). It has the NSBF representation

\[ \hat{S}'(\rho, x) = \cos(\rho x) + \frac{1}{2\rho} \left( \int_0^x \dot{q}(t) \, dt \right) \sin(\rho x) + \frac{1}{\rho} \sum_{n=0}^{\infty} (-1)^n \hat{\sigma}_n(x) j_{2n+1}(\rho x). \]

Note that

\[ \hat{S}'(0, x) = 1 + \frac{\dot{\sigma}_0(x)}{\rho} \int_0^x \dot{q}(t) \, dt + \frac{\dot{\sigma}_0(x)}{3}. \]

Since zero is an eigenvalue of the problem (3.12) and (3.13), we have \( \hat{S}'(0, x) = 0 \) and thus

\[ 1 + \pi \dot{\omega} + \frac{\pi}{3} \dot{\sigma}_0(x) = 0, \]

where

\[ \dot{\omega} := \frac{1}{2} \int_0^x \dot{q}(t) \, dt = \omega - \frac{\pi \rho^2}{2}. \]

Thus,

\[ \dot{\omega} = -\frac{\dot{\sigma}_0(x)}{3} - \frac{1}{\pi}. \tag{3.14} \]

Now we complete the spectrum of the problem (3.12) and (3.13) and hence also the Dirichlet–Neumann spectrum of the potential \( q(x) \). For this, we consider the equations

\[ \hat{S}'_N(\hat{\sigma}_k, \pi) = 0, \quad k = 1, 2, \ldots, N_2, \]

which can be written in the form

\[ \hat{\sigma}_k \cos(\hat{\sigma}_k \pi) + \hat{\sigma}_k \sin(\hat{\sigma}_k \pi) + \sum_{n=1}^{N_2} (-1)^n \hat{\sigma}(\pi) j_{2n+1}(\hat{\sigma}_k \pi) = 0. \]

Taking into account (3.14), we write these equations in the form of the system of linear algebraic equations for the coefficients \( \hat{\sigma}_0(\pi), \ldots, \hat{\sigma}_{N_2}(\pi) \):

\[ \left( j_1(\hat{\sigma}_k \pi) - \frac{\sin(\hat{\sigma}_k \pi)}{3} \right) \hat{\sigma}_0(\pi) + \sum_{n=1}^{N_2} (-1)^n \hat{\sigma}_n(\pi) j_{2n+1}(\hat{\sigma}_k \pi) = -\hat{\sigma}_k \cos(\hat{\sigma}_k \pi) + \frac{\sin(\hat{\sigma}_k \pi)}{\pi}, \quad k = 1, \ldots, N_2. \tag{3.15} \]

Solving this system, we find \( \hat{\sigma}_0(\pi), \ldots, \hat{\sigma}_{N_2}(\pi) \) as well as the parameter \( \dot{\omega} \) (from (3.14)) and the parameter \( \omega = \dot{\omega} + \frac{\pi \rho^2}{2} \).

Having computed the coefficients \( \hat{\sigma}_0(\pi), \ldots, \hat{\sigma}_{N_2}(\pi) \), we consider the function

\[ \hat{S}'_N(\rho, \pi) = \cos(\rho \pi) + \frac{\dot{\sigma}(\rho \pi)}{\rho} + \frac{\sin(\rho \pi)}{\pi} \sum_{n=0}^{N_2} (-1)^n \hat{\sigma}_n(\pi) j_{2n+1}(\rho \pi), \tag{3.16} \]

which approximates the derivative \( \hat{S}'(\rho, \pi) \) in such a way that

\[ \left| \hat{S}'(\rho, \pi) - \hat{S}'_N(\rho, \pi) \right| < \hat{\epsilon}_N, \quad \rho \in \mathbb{R}, \]

where \( \hat{\epsilon}_N \) is a positive constant. Theorem 3.1 is valid as well if instead of the functions \( \hat{S}(\rho, \pi) \) and \( \hat{S}_N(\rho, \pi) \) one considers the functions \( \hat{S}'(\rho, \pi) \) and \( \hat{S}'_N(\rho, \pi) \). Thus, zeros of \( \hat{S}_N(\rho, \pi) \) approximate the Dirichlet–Neumann eigenvalues of the potential \( \dot{q}(x) \).
In Figure 5, the absolute and relative errors of $\rho_k$ computed for $k = 5, \ldots, 300$ are presented in the case of the potential from Example 2. Here, five Dirichlet–Neumann eigenvalues were given ($N_2 = 4$), so the “completed” part of the Dirichlet–Neumann spectrum was computed with four coefficients $\hat{\sigma}_n(\pi)$ in (3.16) computed from (3.15) and $\hat{\omega}$ from (3.14) [Colour figure can be viewed at wileyonlinelibrary.com].

Naturally the accuracy improves when a larger number of the eigenvalues are known. In Figure 6, the results are presented for the same potential but in the case when ten eigenvalues are known ($N_2 = 9$). Here, $N = 8$. The absolute error of the computed $\hat{\omega}$ was already 0.09.

### 3.3 Completion of other spectra

Suppose several eigenvalues $\rho_0^2, \rho_1^2, \ldots, \rho_{N_3}^2$ of the following Sturm–Liouville problem are given

$$-y''(x) + q(x)y(x) = \rho^2 y(x), \quad x \in (0, \pi),$$

$$y'(0) - hy(0) = 0, \quad y'(\pi) + Hy(\pi) = 0,$$

where $h$ and $H$ are unknown real constants. Again, as in the previous cases, we always can shift the eigenvalues in such a way that the first shifted eigenvalue becomes zero. So, without loss of generality, we assume that $\rho_0 = 0$. Note that

$$\varphi(0, x) = 1 + g_0(x)$$
and
\[ \varphi'(0, x) = h + \frac{1}{2} \int_0^x q(t) dt + \gamma_0(x). \]
Since zero is an eigenvalue, we have that
\[ \varphi'(0, \pi) + H\varphi(0, \pi) = 0. \]
Thus,
\[ h + H + \omega = -\gamma_0(\pi) - Hg_0(\pi), \]
where \( \omega = \frac{1}{2} \int_0^\pi q(t) dt. \)

Now consider the characteristic function of the Sturm–Liouville problem (3.17) and (3.18). Taking into account (2.6) and (2.8), it can be written in the form
\[ \Phi(\rho) := \varphi'(\rho, \pi) + H\varphi(\rho, \pi) = -\rho \sin(\rho \pi) + (h + H + \omega) \cos(\rho \pi) + \sum_{n=0}^{\infty} (-1)^n \gamma_n(\pi) j_{2n}(\rho \pi) + H \sum_{n=0}^{\infty} (-1)^n g_n(\pi) j_{2n}(\rho \pi). \]

Denote
\[ h_n := \gamma_n(\pi) + Hg_n(\pi), \quad n = 0, 1, \ldots. \]

Then, taking into account (3.19), we obtain
\[ \varphi'(\rho, \pi) + H\varphi(\rho, \pi) = -\rho \sin(\rho \pi) + h_0 (j_0(\rho \pi) - \cos(\rho \pi)) + \sum_{n=1}^{\infty} (-1)^n h_n j_{2n}(\rho \pi). \]

Several given eigenvalues \( \rho_1^2, \ldots, \rho_{N_1}^2 \) allow us to compute several constants \( h_n, n = 0, \ldots, N \), where \( N \leq N_1 - 1 \), from the system of linear algebraic equations
\[ h_0 (j_0(\rho_k \pi) - \cos(\rho_k \pi)) + \sum_{n=1}^{N} (-1)^n h_n j_{2n}(\rho_k \pi) = \rho_k \sin(\rho_k \pi), \quad k = 1, \ldots, N_3. \]

Thus, we obtain an approximate characteristic function of the problem (3.17) and (3.18)
\[ \Phi_N(\rho) := h_0 (j_0(\rho \pi) - \cos(\rho \pi)) + \sum_{n=1}^{N} (-1)^n h_n j_{2n}(\rho \pi) - \rho \sin(\rho \pi), \]
whose zeros approximate the square roots of the eigenvalues of the problem (3.17) and (3.18).

We emphasize that to complete the spectrum of (3.17) and (3.18), we require no information neither on the potential \( q(x) \) nor on the boundary conditions (the constants \( h \) and \( H \) remain unknown). Moreover, the parameter \( \overline{\omega} := h + H + \omega \) that appears in the second term of the asymptotics for \( \rho_k \) in this problem
\[ \rho_k = k + \frac{\overline{\omega}}{\pi k} + \chi_k, \quad \{\chi_k\} \in \ell_2, \]
is computed as well, since due to (3.19), \( \overline{\omega} = -h_0 \). Numerical results are similar to those for the Dirichlet–Neumann spectrum.

4 SOLUTION OF THE INVERSE PROBLEM

Let us consider the inverse Sturm–Liouville problem consisting in recovering the potential \( q(x) \) from given several first eigenvalues of two spectra. For definiteness, we restrict ourselves to the Dirichlet–Dirichlet and Dirichlet–Neumann spectra. Thus, given \( \{\rho_n^2\}_{n=1}^{N_1} \) and \( \{\rho_n^2\}_{n=0}^{N_2} \), first eigenvalues of problem (1.1) and (1.2) and problem (1.1) and (1.3), respectively. In the first step, as it was explained in the previous section, we compute several coefficients \( \tilde{s}_n(\pi), n = 1, \ldots, N, N \leq \min\{N_1 - 1, N_2 - 1\} \) and hence obtain the function \( \tilde{S}_N(\rho, \pi) \), which approximates the characteristic function of the problem (3.1) and (3.2). In the second step, we compute several coefficients \( \tilde{\delta}_n(\pi), \ldots, \tilde{\delta}_N(\pi) \) and then complete the
spectrum of problem (1.1) and (1.3). Next, let us consider the solution \( \psi(\rho, x) \) of Equation (1.1) satisfying the initial conditions at \( x = \pi \):

\[
\psi(\rho, x) = 1, \quad \psi'(\rho, x) = 0.
\]

Analogously to the solution (2.6), the solution \( \psi(\rho, x) \) admits the series representation:

\[
\psi(\rho, x) = \cos(\rho (x - \pi)) + \sum_{n=0}^{\infty} (-1)^n \tau_n(x) j_{2n}(\rho (x - \pi)), \tag{4.1}
\]

where \( \tau_n(x) \) are corresponding coefficients, analogous to \( g_n(x) \) from (2.6). Similarly to (2.12), the equality

\[
q(x) = \frac{\tau'''(x)}{\tau_0(x) + 1} \tag{4.2}
\]

is valid.

Note that for \( \rho = \rho_k \), the solutions \( S(\rho_k, x) \) and \( \psi(\rho_k, x) \) are linearly dependent because both are eigenfunctions of problem (1.1) and (1.3). Hence, there exist such real constants \( \beta_k \neq 0 \) that

\[
S(\rho_k, x) = \beta_k \psi(\rho_k, x). \tag{4.3}
\]

Moreover, these multiplier constants can be easily calculated by recalling that \( \psi(\rho_k, \pi) = 1 \). Thus,

\[
\beta_k = S(\rho_k, \pi),
\]

and we approximate these constants with the aid of the coefficients \( s_n(x) \):

\[
\beta_k \approx \tilde{S}_N(\sqrt{\rho_k^2 + \mu_1^2}, \pi) = \frac{\sin\left(\sqrt{\rho_k^2 + \mu_1^2} \pi\right)}{\sqrt{\rho_k^2 + \mu_1^2}} + \frac{1}{N} \sum_{n=0}^{N} (-1)^n \tilde{s}_n(x) j_{2n+1}\left(\sqrt{\rho_k^2 + \mu_1^2}\right). \tag{4.4}
\]

Having computed these constants, we use Equation (4.3) for constructing a system of linear algebraic equations for the coefficients \( s_n(x) \) and \( \tau_n(x) \). Indeed, Equation (4.3) can be written in the form

\[
\frac{1}{\rho_k} \sum_{n=0}^{\infty} (-1)^n s_n(x) j_{2n+1}(\rho_k x) = \beta_k \sum_{n=0}^{\infty} (-1)^n \tau_n(x) j_{2n}(\rho_k (x - \pi)) \tag{4.4}
\]

\[
= -\frac{\sin(\rho_k x)}{\rho_k} + \beta_k \cos(\rho_k (x - \pi)).
\]

We have as many of such equations as many Dirichlet–Neumann singular numbers \( \rho_k \) are computed. For computational purposes, we choose some natural number \( N_c \)—the number of the coefficients \( s_n(x) \) and \( \tau_n(x) \) to be computed. More precisely, we choose a sufficiently dense set of points \( x_m \in (0, \pi) \) and at every \( x_m \) consider the equations

\[
\frac{1}{\rho_k} \sum_{n=0}^{N} (-1)^n s_n(x_m) j_{2n+1}(\rho_k x_m) = \beta_k \sum_{n=0}^{N} (-1)^n \tau_n(x_m) j_{2n}(\rho_k (x_m - x_m)) \tag{4.4}
\]

\[
= -\frac{\sin(\rho_k x_m)}{\rho_k} + \beta_k \cos(\rho_k (x_m - x_m)).
\]

Solving this system of equations, we find \( s_0(x_m) \) and \( \tau_0(x_m) \) and consequently \( s_0(x) \) and \( \tau_0(x) \) at a dense set of points of the interval \( (0, \pi) \). Finally, with the aid of (2.13) or (4.2), we compute \( q(x) \).
5 | NUMERICAL EXAMPLES

Consider the potential from Example 2. In Figure 7, we show the recovered potential in the case of 10 pairs of the eigenvalues given. Here, \( N = 9 \), the parameter \( \omega \) was recovered with the accuracy 0.092, and additional to the 10 Dirichlet–Neumann eigenvalues given, 90 eigenvalues were computed by the spectrum completion technique. Their use when constructing the system (4.4) is crucial, because the accuracy without these additional eigenvalues computed is considerably worse.

**Example 3.** Consider a less smooth potential

\[
q(x) = |x - 1| + 1.
\]

In Figure 8, we show the results of two spectra completion in the case of seven eigenpairs given, \( N = 6 \).

The result of the recovery of the potential in this case is presented in Figure 9.

Doubling the number of the eigenpairs given (for the same number of the coefficients used, \( N = 6 \)) delivers more accurate results shown in Figures 10 and 11.
6 | CONCLUSIONS

A simple method for completing the sequence of the eigenvalues of a regular Sturm–Liouville problem is developed, which does not require neither information on the potential nor the knowledge of the boundary conditions. The spectrum is
completed with a uniform absolute accuracy. Based on this spectrum completion technique, a direct method for solving two-spectra inverse Sturm–Liouville problems on a finite interval is developed. The main role in the proposed approach is played by the coefficients of the Neumann series of Bessel functions expansion of solutions of the Sturm–Liouville equation and of their derivatives. The given spectral data leads to an infinite system of linear algebraic equations for the coefficients, and the potential is recovered from the first coefficient alone.

The method is simple, direct, and accurate. Its performance is illustrated by numerical examples.

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CONFLICT OF INTEREST

This work does not have any conflict of interest.

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