ZERO LEVEL PERTURBATION OF A CERTAIN THIRD-ORDER LINEAR SOLVABLE ODE WITH AN IRREGULAR SINGULARITY AT THE ORIGIN OF POINCARÉ RANK 1

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Abstract. We study an irregular singularity of Poincaré rank 1 at the origin of a certain third-order linear solvable homogeneous ODE. We perturb the equation by introducing a small parameter \( \varepsilon \in (\mathbb{R}_+, 0) \) \( (\varepsilon < 1) \), which causes the splitting of the irregular singularity into two finite Fuchsian singularities. We show that when the solutions of the perturbed equation contain logarithmic terms, the Stokes matrices of the initial equation are limits of the part of the monodromy matrices around the finite resonant Fuchsian singularities of the perturbed equation.

Key words: Third-order solvable complex linear ordinary differential equation, Stokes phenomenon, Irregular singularity, Monodromy matrices, Regular singularity, Limit

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1. Introduction

We consider a linear complex ordinary differential equation

\[ L y = 0, \]  

where \( L \) is a third-order linear differential operator of the form

\[ L = L_3 \circ L_2 \circ L_1 \]  

with

\[ L_1 = \partial - \frac{1}{x^2}, \quad L_2 = \partial - \frac{\nu - 2}{x} \frac{2}{x^2}, \quad L_3 = \partial - \frac{\nu - 4}{x}, \quad \partial = \frac{d}{dx} \]  

and \( \nu \in \mathbb{C} \). The equation (1.1) with (1.2)–(1.3) (in short, the initial equation) is a third-order solvable differential equation, in the sense that its differential Galois group is a solvable linear algebraic group. We consider the equation (1.1) over \( \mathbb{CP}^1 \), where it has an irregular singular point at the origin of Poincaré rank 1 and a regular singular point (if \( \nu \neq 0 \), see Remark 3.10) at \( x = \infty \). We associate with the initial equation its analytic invariants at the irregular singularity - the Stokes matrices and the formal monodromy.

On the other hand, following [4, 5, 8, 12, 16, 17], we can always regard the irregular singularity at the origin as a result of confluence of two Fuchsian singularities. Namely,
introducing a (small) parameter \( \varepsilon \in (\mathbb{R}_+,0) \), we consider a perturbation of the initial equation

\begin{equation}
(1.4) \quad L(\varepsilon)y = 0,
\end{equation}

where \( L(\varepsilon) \) is again a third-order differential operator of the form

\begin{equation}
(1.5) \quad L(\varepsilon) = L_3(\varepsilon) \circ L_2(\varepsilon) \circ L_1(\varepsilon).
\end{equation}

The first-order differential operators \( L_j(\varepsilon) \), \( j = 1, 2, 3 \) are defined as follows,

\begin{align*}
L_1(\varepsilon) &= \partial - \frac{1}{2\sqrt{\varepsilon}} \left( \frac{1}{x - \sqrt{\varepsilon}} - \frac{1}{x + \sqrt{\varepsilon}} \right), \\
L_2(\varepsilon) &= \partial - \left( \frac{\nu - 2}{2} + \frac{1}{\sqrt{\varepsilon}} \right) \frac{1}{x - \sqrt{\varepsilon}} - \left( \frac{\nu - 2}{2} - \frac{1}{\sqrt{\varepsilon}} \right) \frac{1}{x + \sqrt{\varepsilon}}, \\
L_3(\varepsilon) &= \partial - \frac{\nu - 4}{2} \left( \frac{1}{x - \sqrt{\varepsilon}} + \frac{1}{x + \sqrt{\varepsilon}} \right), \\
L_j(0) &= L_j, j = 1, 2, 3.
\end{align*}

The equation (1.4) with (1.5)-(1.6) (in short, the perturbed equation) is a third-order Fuchsian equation. It has over \( \mathbb{CP}^1 \) three Fuchsian singularities: \( x = -\sqrt{\varepsilon}, \sqrt{\varepsilon}, \infty \). Following [16, 17], through this article, we denote the finite regular singularities by \( x_L = -\sqrt{\varepsilon} \) and \( x_R = \sqrt{\varepsilon} \) for \( \sqrt{\varepsilon} \in \mathbb{R}_+ \). In this article we consider the perturbed equation together with its monodromy matrices around the finite Fuchsian singularities \( x_L \) and \( x_R \).

The main result of this article is, that in the presence of logarithmic terms in the solutions of the perturbed equation, the Stokes matrices of the initial equation are limits of the parts of the monodromy matrices around the resonant singular points \( x_j, j = R, L \) of the perturbed equation. More precisely, it turns out that exactly these parts of the monodromy matrices, which govern the presence of logarithmic terms tend to the Stokes matrices when \( \sqrt{\varepsilon} \) tends to zero.

The point \( x = \infty \) is a regular singularity for both initial and perturbed equations (except the case when \( \nu = 0 \), see Remark [3,16] and Remark [11,3]). Furthermore the exponents \( \rho_1^\infty = 0, \rho_2^\infty = 1 - \nu, \rho_3^\infty = 2 - \nu \) at \( x = \infty \) are the same for the both equations. The perturbation, defined above, splits the irregular singularity \( x = 0 \) of Poincaré rank 1 of the initial equation into two finite Fuchsian singular points \( x_R = \sqrt{\varepsilon} \) and \( x_L = -\sqrt{\varepsilon} \) of the perturbed equation, but does not change the infinity point. There exists a more generic perturbation, which not only splits the irregular singularity, but also perturbs the exponents \( \rho_j^\infty, j = 1, 2, 3 \) at \( x = \infty \). Generally, if the coefficients \( a_j(x), j = 1, 2, 3 \) of the initial equation (2.9) are given by

\begin{equation}
a_j(x) = \frac{\alpha_j}{x} + \frac{\beta_j}{x^2}, \quad \alpha_j, \beta_j \in \mathbb{C} \quad \text{such that} \quad (\beta_1, \beta_2, \beta_3) \neq (0, 0, 0),
\end{equation}

then the corresponding perturbed coefficients \( a_j(x, \varepsilon) \) would be

\begin{equation}
a_j(x, \varepsilon) = \left( \frac{\alpha_j(\varepsilon)}{2} + \frac{\beta_j(\varepsilon)}{2\sqrt{\varepsilon}} \right) \frac{1}{x - \sqrt{\varepsilon}} + \left( \frac{\alpha_j(\varepsilon)}{2} - \frac{\beta_j(\varepsilon)}{2\sqrt{\varepsilon}} \right) \frac{1}{x + \sqrt{\varepsilon}},
\end{equation}

\begin{equation}
a_j(x, 0) = a_j(x),
\end{equation}

where \( \alpha_j(\varepsilon) \) and \( \beta_j(\varepsilon) \) are polynomials in \( \varepsilon \) such that \( \alpha_j(0) = \alpha_j \) and \( \beta_j(0) = \beta_j \). We call such a perturbation a \( N \)-th level perturbation if all these polynomials are of degree \( N \). Then the \( N \)-th level perturbation has the exponents \( \rho_1^\infty = -A_1(\varepsilon), \rho_2^\infty = 1 - \nu - \).
Recently several authors (see below) have studied the irregular point at the origin of a first-order linear system of differential equations. In order to understand the Stokes phenomenon they perturb the system, introducing a small parameter, which causes the splitting of the irregular singularity into two finite Fuchsian singularities. Instead of considering first-order linear system, we study a higher-order scalar equation. To make our discussion simpler, we investigate a solvable (reducible) higher-order equation. With the present article we begin a research on the nature of the newly introduced perturbation. We start with the most simple example, namely, with the zero level perturbation of a certain third-order initial equation. The main goal of this paper is to show that there exists a connection, by a limit $\sqrt{\varepsilon} \to 0$, between the analytic invariants at the origin (the Stokes matrices) of the initial equation and the analytic invariants around the finite resonant Fuchsian singularities (the monodromy matrices) of the perturbed equation. In general the monodromy matrices of the perturbed equation decompose into convergent and divergent parts (see Theorem [4.9]). Consider for every fixed $\nu \in \mathbb{R}$ the particular sequence 

$$\sqrt{\varepsilon} = \sqrt{\varepsilon_n}$$

for $1/\sqrt{\varepsilon_n} = \nu + 2n$. It turns out that along this sequence (which in fact defines the so called logarithmic resonant cases) the divergent parts of the monodromy matrices stay constant. Thus along this very particular sequence of values of $\varepsilon$ the monodromy matrices converge. As a consequence of the decomposition theorem of Lambert and Rousseau (Proposition 4.31 in [16]), a theorem of Klimeš (Theorem 32. in [15]) and our decomposition theorem (Theorem [4.9]) we point which part of the monodromy matrices are the so called unfolded Stokes matrices. This result allows us to connect by a limit $\sqrt{\varepsilon} \to 0$ the pointed part of the monodromy matrices and the Stokes matrices of the initial equation (see Theorem [5.5]).

Problems of this kind have been considered in the already classical works of Glutsyuk [4], Ramis [21], Zhang [33], Duval [3], Schäfke [25] as well as in recent ones: Glutsyuk [5], Lambert and Rousseau [8, 16, 17, 18], Slavyanov and Lay [27], Klimeš [12, 13, 14, 15], Remy [24]. Our work is closer to the works of Glutsyuk [4, 5] and Lambert and Rousseau [16, 17] where the authors introduce a small parameter that splits the irregular point at the origin of a linear system (not a scalar equation) into two finite Fuchsian singularities. Then they study the confluence of the connection matrices and the monodromy matrices of the perturbed system to the Stokes matrices of the original system. The work of Glutsyuk treats the confluence on sectors in the parameter $\varepsilon$-space, on which the regular singularities are non-resonant. He shows, that, generically, the limit $\varepsilon \to 0$ of no product of monodromy matrices gives the Stokes matrices (Theorem 4.6 in [5]). The approach of Lambert, Rousseau and Hurtubise [8, 16, 17] of “mixed bases” allows to treat the resonant values of $\varepsilon$ and may be used in studying of a scalar equation too. However, the calculation of the monodromy matrices of a linear system (not a scalar equation), relevant to the same fundamental matrix solution with the resonant values of $\varepsilon$ requires more complicated computations. We also note that the approach in [16] allows to make limits of monodromy along any particular sequence $\varepsilon = \varepsilon_n$, with $1/\varepsilon_n = 1/\varepsilon_0 + 2n$, such as in Duval [3]. It should be of use for studying in case with $\varepsilon \in \mathbb{C}$.

Our approach to the above results is different. We fully exploit the resonance and the appearance of logarithmic terms in the mixed basis of solutions of the perturbed equation in order to calculate the monodromy matrices. This approach is assisted by the representation of this basis in terms of iterated integrals depending only on the solutions of the equations.
\( L_j(\varepsilon) u = 0 \). As a result we can easy determine the coefficients in front of the present logarithmic terms of the solutions of the perturbed equation just as the residues at the singular points of the functions under integration. Moreover, our approach allows us to point which part of the monodromy matrices take the part of the so called unfolded Stokes matrices. We believe that the developed technique could be useful in case of a general higher-order solvable scalar equation. It could be also applied in case of a linear system, whose fundamental matrix solutions are represented in the form of appropriate integrals.

The motivation for the study of the exactly this initial equation comes from the investigation of the integrability of the Painlevé equations. In particular, when \( \nu = 1/2 \) our initial equation appears as the second normal variational equation of the Hamiltonian system \( \mathcal{H}_{IV}(y, p, t, a, b) \) corresponding to the fourth Painlevé equation

\[
\ddot{y} = \frac{1}{2y} (\dot{y})^2 + \frac{3}{2} y^3 + 4t y^2 + 2(t^2 - a) y + \frac{b}{y}
\]

along a particular solution \( y = p = 0, a = 1, b = 0 \) [28]. In the same paper [28] we have proved that the connected component \( G^0 \) of the unit element of the differential Galois group of the initial equation with \( \nu = 1/2 \) is not Abelian using Stokes matrices. Such solvable differential equations whose differential Galois group is a solvable algebraic group can be found in our previous work on the Painlevé V (again 1 irregular point at origin of Poincaré rank 1) [29] and on the Painlevé VI equation (Fuchsian differential equations) [7, 30].

This article is organized as follows. In the next section we build global fundamental “mixed” matrices of the initial and the perturbed equations, with respect to which we are going to determine the corresponding analytic invariants. In section 3 we explicitly compute the formal monodromy and the Stokes matrices of the initial equation. In section 4 we explicitly calculate the monodromy matrices around the finite Fuchsian singularities of the perturbed equation during a resonance. In section 5 we establish the main results of this paper.

2. Global solutions

In this section we will introduce the global fundamental matrices of the initial and the perturbed equations, with respect to which we are going to determine the corresponding analytic invariants. As we have announced in the introduction we are going to use fundamental matrices different from the usual form.

Considered in this paper equations are very particular cases of a more general third-order solvable ODE

\[
L y = 0, \quad L = L_3 \circ L_2 \circ L_1,
\]

where \( L_j, 1 \leq j \leq 3 \) are first-order differential operators of the form

\[
L_j = \partial + a_j(x) \quad \text{with} \quad a_j(x) \in \mathbb{C}(x), \quad 1 \leq j \leq 3.
\]

To introduce the fundamental matrices we at first reduce a scalar solvable equation (2.8)-(2.9) to a certain special linear system. We will call such a system, a system associated with the given scalar equation. Denote \( \tilde{L}_j = L_3 \circ \cdots \circ L_j, 1 \leq j \leq 3, \) as \( \tilde{L}_1 = L \), and by \( z_j(x), 1 \leq j \leq 3 \) a solution of the equation \( \tilde{L}_j u = 0 \). The function \( z_1(x) = y(x) \) is a solution of the equation \( L y = 0 \). Then we have
**Theorem 2.1.** The function \( y(x) \) is a solution of the third-order equation (2.8)-(2.9) if and only if the vector \( Y(x) = (y(x), L_1y(x), L_2 \circ L_1y(x))^T \) solves the system (2.10)
\[
Y'(x) = A(x) Y(x)
\]
with
\[
(2.11) \\
A(x) = \begin{pmatrix} -a_1(x) & 1 & 0 \\ 0 & -a_2(x) & 1 \\ 0 & 0 & -a_3(x) \end{pmatrix}.
\]

**Proof.** The proof is straightforward after the observation that \( z_2(x) = L_1y(x) \) and \( z_3(x) = L_2 \circ L_1y(x) \).

**Definition 2.2.** A \( 3 \times 3 \) matrix \( \Phi(x) \) is called a fundamental matrix of the scalar third-order equation (2.8)-(2.9) if it is a fundamental matrix of a system, associated with the same scalar equation.

It turns out that both initial and perturbed equations have a global fundamental matrix, whose elements are expressed in terms of iterated integrals depending only on the solutions of the equations \( L_j u = 0 \), \( 1 \leq j \leq 3 \). Denote by \( \Phi(x, \cdot) \), where the second argument is either 0 or \( \varepsilon \), the fundamental matrix of the initial or the perturbed equation respectively.

**Theorem 2.3.** Both initial and perturbed equations admit a global fundamental matrix \( \Phi(x, \cdot) \) of the form
\[
(2.12) \quad \Phi(x, \cdot) = \begin{pmatrix} \Phi_1(x, \cdot) & \Phi_{12}(x, \cdot) & \Phi_{13}(x, \cdot) \\ 0 & \Phi_2(x, \cdot) & \Phi_{23}(x, \cdot) \\ 0 & 0 & \Phi_3(x, \cdot) \end{pmatrix},
\]
where the diagonal elements \( \Phi_j(x, \cdot), j = 1, 2, 3 \) are the solutions of the equations \( L_j(\cdot)u = 0 \), with \( L_j(\varepsilon) \) and \( L_j(0) = L_j \) given by (1.6) and (1.3) respectively. The off-diagonal elements are defined as follows,
\[
(2.13) \quad \Phi_{12}(x, \cdot) = \Phi_1(x, \cdot) \int_{\Gamma_1(x, \cdot)} \frac{\Phi_2(t_1, \cdot)}{\Phi_1(t_1, \cdot)} dt_1,
\]
\[
\Phi_{23}(x, \cdot) = \Phi_2(x, \cdot) \int_{\Gamma_2(x, \cdot)} \frac{\Phi_3(t_2, \cdot)}{\Phi_2(t_2, \cdot)} dt_2,
\]
\[
\Phi_{13}(x, \cdot) = \Phi_1(x, \cdot) \int_{\Gamma_1(x, \cdot)} \frac{\Phi_2(t_1, \cdot)}{\Phi_1(t_1, \cdot)} \left( \int_{\Gamma_2(t_2, \cdot)} \frac{\Phi_3(t_2, \cdot)}{\Phi_2(t_2, \cdot)} dt_2 \right) dt_1.
\]
The paths of integration \( \Gamma_j(x, \varepsilon) \) and \( \Gamma_j(x, 0) \) are taken from the same base point \( x \) in such a way that \( \Gamma_j(x, \varepsilon) \to \Gamma_j(x, 0) \) as \( \varepsilon \to 0 \in \mathbb{R} \), and the matrices \( \Phi(x, \cdot) \) are fundamental matrix solutions of the initial and the perturbed equations respectively.

**Proof.** To prove the statement, we have to check that the so defined matrix \( \Phi(x, \cdot) \) is such that \( \Phi'(x, \cdot) = A(x) \Phi(x, \cdot) \), and \( \det \Phi(x, \cdot) \neq 0 \) outside of the singular points of the system (2.10)-(2.11).

The first condition is checked directly. Note that \( \det \Phi(x, \cdot) = \Phi_1(x, \cdot) \Phi_2(x, \cdot) \Phi_3(x, \cdot) \). But for every function \( \Phi_j(x, \cdot) \) we have that \( \Phi_j(x, \cdot) \neq 0 \) outside of the singular points of the equation \( L_j u = 0 \) respectively, since it is a fundamental solution of this equation. Now the second condition follows directly from the observations that the singular points of the system (2.10)-(2.11) and of the initial and perturbed equations coincide.

As a direct corollary of Theorem 2.3 we obtain the following proposition...
Proposition 2.4. Both initial and perturbed equations possess a global fundamental set of solutions of the form

\[
\Phi_1(x, \cdot), \Phi_1(x, \cdot) \int_{\Gamma_1(x, \cdot)} \frac{\Phi_2(t_1, \cdot)}{\Phi_1(t_1, \cdot)} dt_1,
\]

\[
\Phi_1(x, \cdot) \int_{\Gamma_2(x, \cdot)} \frac{\Phi_2(t_1, \cdot)}{\Phi_1(t_1, \cdot)} \left( \int_{\Gamma_2(t_1, \cdot)} \frac{\Phi_3(t_2, \cdot)}{\Phi_2(t_2, \cdot)} dt_2 \right) dt_1.
\]

3. The analytic invariants of the initial equation

In this section we will introduce and compute by hand the formal monodromy and the Stokes matrices at the origin of the initial equation. In this paper we are going to use the summability theory (applied to ordinary differential equations) to calculate the Stokes matrices.

All singular directions and sectors are defined on the Riemann surface of the natural logarithm. Consider the initial equation together with the formal fundamental matrix at the origin in the form (3.17), given by the next proposition.

Proposition 3.1. The initial equation admits a unique formal fundamental matrix \( \hat{\Phi}(x, 0) \) at the origin in the form

\[
(3.14) \quad \hat{\Phi}(x, 0) = \hat{H}(x) x^\Lambda \exp \left( -\frac{Q}{x} \right),
\]

where

\[
(3.15) \quad Q = \text{diag}(1, 2, 0), \quad \Lambda = \text{diag}(0, \nu - 2, \nu - 4)
\]

and

\[
(3.16) \quad \hat{H}(x) = \begin{pmatrix}
1 & x^2 \hat{\varphi}(x) & \frac{x^4 \hat{\psi}(x)}{2} \\
0 & 1 & -\frac{x^2}{2} \\
0 & 0 & 1
\end{pmatrix}.
\]

The elements \( \hat{\varphi}(x) \) and \( \hat{\psi}(x) \) of the matrix \( \hat{H}(x) \) are defined as follows,

1. If \( \nu \) is a non-positive integer, then

\[
(3.17) \quad \hat{\psi}(x) = 1 + \nu x + \nu (\nu + 1) x^2 + \nu (\nu + 1) (\nu + 2) x^3 + \cdots + (-1)^\nu (-\nu)! x^{-\nu},
\]

\[
\hat{\varphi}(x) = 1 - \nu x + \nu (\nu + 1) x^2 - \nu (\nu + 1) (\nu + 2) x^3 + \cdots + (-\nu)! x^{-\nu}
\]

are analytic at the origin functions.

2. Otherwise, \( \hat{\psi}(x) \) and \( \hat{\varphi}(x) \) are represented in terms of divergent series

\[
(3.18) \quad \hat{\psi}(x) = 1 + \nu x + \nu (\nu + 1) x^2 + \nu (\nu + 1) (\nu + 2) x^3 + \cdots,
\]

\[
\hat{\varphi}(x) = 1 - \nu x + \nu (\nu + 1) x^2 - \nu (\nu + 1) (\nu + 2) x^3 + \cdots
\]

Proof. Choosing \( \hat{\Phi}_1(x, 0) = e^{-1/x} \), \( \hat{\Phi}_2(x, 0) = x^{\nu-2} e^{-2/x} \), \( \hat{\Phi}_3(x, 0) = x^{\nu-4} \) we obtain \( \hat{\Phi}_{23}(x, 0) = -x^{\nu-2}/2 \) where \( \Gamma_2(x, 0) \) in (2.13) is a path from 0– to \( x \), approaching 0 in the direction \( \mathbb{R}_- \). Next, looking for \( \hat{\Phi}_{12}(x, 0) \) and \( \hat{\Phi}_{13}(x, 0) \) in the form \( \hat{\Phi}_{12}(x, 0) =
\[ x^\nu e^{-2/x} \bar{\varphi}(x) \] and \[ \Phi_{13}(x, 0) = x^\nu \hat{\psi}(x)/2 \] we find that \( \hat{\varphi}(x) \) and \( \hat{\psi}(x) \) satisfy the following first-order equations

\[ x^2 \hat{\psi}' + (\nu x - 1) \hat{\psi} = -1, \quad x^2 \hat{\varphi}' + (\nu x + 1) \hat{\varphi} = 1. \]  

For almost all values of the parameter \( \nu \in \mathbb{C} \) equations (3.19) admit particular solutions in terms of divergent series. Only in the exceptional cases when \( \nu = 0, -1, -2, -3, \ldots \) the functions

\[ \hat{\psi}(x) = 1 + \nu x + \nu(\nu + 1)x^2 + \nu(\nu + 1)(\nu + 2)x^3 + \cdots + (-1)^\nu(\nu)!x^{-\nu}, \]
\[ \hat{\varphi}(x) = 1 - \nu x + \nu(\nu + 1)x^2 - \nu (\nu + 1) (\nu + 2)x^3 + \cdots + (-\nu)!x^{-\nu} \]

are particular solutions of equations (3.19), analytic at the origin.

Let \( \nu \neq 0, -1, -2, -3, \ldots \) Then the divergent power series

\[ \hat{\psi}(x) = 1 + \nu x + \nu(\nu + 1)x^2 + \nu(\nu + 1)(\nu + 2)x^3 + \cdots, \]
\[ \hat{\varphi}(x) = 1 - \nu x + \nu(\nu + 1)x^2 - \nu (\nu + 1) (\nu + 2)x^3 + \cdots \]

are particular solutions of the equations (3.19). Now it is easy to write down the formal fundamental matrix \( \hat{\Phi}(x, 0) \) in the wished form.

This completes the proof. \( \square \)

Now we can make the formal monodromy matrix \( \hat{M} \) explicit.

**Proposition 3.2.** The formal monodromy matrix \( \hat{M} \) relative to the formal solution (3.14) is defined by

\[ \hat{\Phi}(x, e^{2\pi i}, 0) = \hat{\Phi}(x, 0)\hat{M}, \]

where

\[ \hat{M} = e^{2\pi i \Lambda} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{2\pi i \nu} & 0 \\ 0 & 0 & e^{2\pi i \nu} \end{pmatrix}. \]  

The formal monodromy \( \hat{M} \) is a formal analytic invariant of the initial equation.

**Definition 3.3.** 1. A sector is to be a set of the form

\[ S = S(\theta, \alpha, \rho) = \{ x = r e^{i\delta} | 0 < r < \rho, \theta - \alpha/2 < \delta < \theta + \alpha/2 \}, \]

where \( \theta \) is an arbitrary real number (the bisector of \( S \)), \( \alpha \) is a positive real (the opening of \( S \)) and \( \rho \) is either a positive number or \( +\infty \) (the radius of \( S \)).

2. A closed sector is a set of the form

\[ \bar{S} = \bar{S}(\theta, \alpha, \rho) = \{ x = r e^{i\delta} | 0 < r \leq \rho, \theta - \alpha/2 \leq \delta \leq \theta + \alpha/2 \}, \]

with \( \theta \) and \( \alpha \) as before, but where \( \rho \) is a positive real number (never equal to \( +\infty \)).

From Proposition 3.1 it follows that the initial equation has singular directions, relative only to the divergent series \( \hat{\psi}(x) \) and \( \hat{\varphi}(x) \). Then the restriction of the definition of the singular directions [20] in general gives us the following

**Definition 3.4.** For the divergent series \( \hat{\psi}(x) \) we define the singular direction (anti-Stokes direction) \( \theta_1, 0 \leq \theta_1 < 2\pi \) of the initial equation as the bisector of the sector where \( Re((-q_1 + q_2)/x) = Re(-1/x) < 0 \). Then \( \theta_1 = 0 \). In the same manner, the singular direction \( \theta_2, 0 \leq \theta_2 < 2\pi \) relative to the divergent series \( \hat{\varphi}(x) \) is the bisector of the sector where \( Re((-q_1 + q_2)/x) = Re(1/x) < 0 \). Then \( \theta_2 = \pi \).
Our next step is to make explicit an actual fundamental matrix $\Phi(x,0)$ at the origin, associate with the above formal fundamental matrix $\hat{\Phi}(x,0)$ via the next fundamental theorem

**Theorem 3.5.** (Hukuhara-Turrittin-Martinet-Ramis [19, 20]) In the formal fundamental matrix $\hat{\Phi}(x,0)$ of the initial equation defined by Hukuhara-Turrittin the entries of the matrix $\hat{H}(x)$ are 1-summable in every non-singular direction $\theta$. If we denote by $H(x)$ the 1-sum of $\hat{H}(x)$ along $\theta$ obtained from $\hat{H}(x)$ by a Borel-Laplace transform, then $\Phi(x,0) = H(x)x^\Lambda \exp(-Q/x)$ is an actual fundamental matrix of the initial equation.

Let us briefly recall some definitions and facts needed to build 1-sum (Borel sum) of the matrix $\hat{H}(x)$, following the works of Balser [1], Ramis [22, 23], van der Put and Singer [31].

**Definition 3.6.** A formal power series $\hat{f}(x) = \sum_{n=0}^{\infty} f_n x^n$ is said to be of Gevrey order 1 if there exist two positive constants $C,A > 0$ such that $|f_n| \leq CA^n n!$ for every $n \in \mathbb{N}$.

We denote by $C[[x]]_1$ the set of all power series of Gevrey order 1.

**Definition 3.7.** The formal Borel transform $\hat{B}_1$ of order 1 of a formal power series $\hat{f}(x) = \sum_{n=0}^{\infty} f_n x^n$ is called the formal series $$(\hat{B}_1 \hat{f})(\zeta) = \sum_{n=0}^{\infty} \frac{f_n}{n!} \zeta^n.$$ If $\hat{f} \in C[[x]]_1$ then its formal Borel transform $(\hat{B}_1 \hat{f})$ of order 1 converges in a neighborhood of the origin $\zeta = 0$ with a sum $f(\zeta)$.

The inverse operator of the Borel transform is the Laplace transform.

**Definition 3.8.** Let $f(\zeta)$ be analytic and of exponential size at most 1 at $\infty$, i.e. $|f(\zeta)| \leq A \exp(B|\zeta|)$, $\zeta \in \theta$ along any direction $\theta$ from 0 to $+\infty e^{i\theta}$. Then the integral $$(L_\theta f)(x) = \int_0^{+\infty e^{i\theta}} f(\zeta) \exp\left(-\frac{\zeta}{x}\right) d\left(\frac{\zeta}{x}\right)$$ is said to be the Laplace complex transform $L_\theta$ of order 1 in the direction $\theta$ of $f$.

**Definition 3.9.** The formal power series $\hat{f}(x) = \sum_{n=0}^{\infty} f_n x^n$ is 1-summable (or Borel summable) in the direction $\theta$ if there exist an open sector $V$ bisected by $\theta$ whose opening is $> \pi$ and a holomorphic function $f(x)$ on $V$ such that for every non-negative integer $N$, $$\left|f(x) - \sum_{n=0}^{N-1} f_n x^n\right| \leq C_{V_1} A_{V_1}^N N! |x|^N$$ on every closed subsector $\tilde{V}_1$ of $V$ with constants $C_{V_1}, A_{V_1} > 0$ depending only on $V_1$. The function $f(x)$ is called the 1-sum (or Borel sum) of $\hat{f}(x)$ in the direction $\theta$.

If $\hat{f}(x)$ is 1-summable in all but a finite number of directions, we will say that it is 1-summable.

One useful criterion for a Gevrey series of order 1 to be 1-summable is given in terms of Borel and Laplace transforms.
Proposition 3.10. \((\ref{31})\) Let \(\hat{f} \in \mathbb{C}[[x]]_1\) and let \(\theta\) be a direction. The following are equivalent:
1. \(\hat{f}\) is 1-summable in the direction \(\theta\).
2. The convergent power series \((\hat{B}_1\hat{f})(\zeta)\) has an analytic continuation \(h\) in a full sector \(\{\zeta \in \mathbb{C} | 0 < |\zeta| < \infty, \arg(\zeta) - \theta < \epsilon\}\). In addition, this analytic continuation has exponential growth of order \(\leq 1\) at \(\infty\) on this sector, i.e. \(| h(\zeta) | \leq A \exp(B|\zeta|)\). In this case \(f = \hat{L}_\theta(h)\) is its 1-sum.

Applying the above theory to the divergent power series \(\hat{\varphi}(x)\) and \(\hat{\psi}(x)\) we obtain their 1-sums (Borel sums).

Lemma 3.11. Let \(\nu \neq 0, -1, -2, -3, \ldots\). Then for any directions \(\theta_1 \neq 0\) and \(\theta_2 \neq \pi\) from 0 to \(+\infty e^{i\theta_k}\), \(k = 1, 2\) the functions

\[
\psi_{\theta_1}(x) = x^{-1} \int_0^{+\infty e^{i\theta_1}} (1 - \zeta)^{-\nu} e^{-\frac{x}{\zeta}} d\zeta, \\
\varphi_{\theta_2}(x) = x^{-1} \int_0^{+\infty e^{i\theta_2}} (1 + \xi)^{-\nu} e^{-\frac{x}{\xi}} d\xi
\]

define the 1-sum (Borel sum) of the divergent series \(\hat{\psi}(x)\) and \(\hat{\varphi}(x)\), respectively, in such directions.

Proof. Let us represent the divergent series \(\hat{\psi}(x)\) and \(\hat{\varphi}(x)\) as

\[
\hat{\psi}(x) = \sum_{n=0}^{\infty} (\nu)^{(n)} x^n, \quad \hat{\varphi}(x) = \sum_{n=0}^{\infty} (-1)^n (\nu)^{(n)} x^n,
\]

where \((\nu)^{(n)}\) for \(n = 0, 1, 2, \ldots\) is the rising factorial

\[
(\nu)^{(n)} = \nu (\nu + 1) (\nu + 2) \ldots (\nu + n - 1), \quad (\nu)^{(0)} = 1.
\]

Let \(|\nu| \leq 1\). Then

\[
| (\nu)^{(n)} | \leq |\nu| (|\nu| + 1) (|\nu| + 2) \ldots (|\nu| + n - 1) \leq n!.
\]

Let \(|\nu| > 1\). Then using the fact \(1 < |\nu| + 1\), we have the following estimates

\[
|\nu| < |\nu| + 1, \quad |\nu| + 1 < |\nu| + 1 + 1 < 2(|\nu| + 1), \quad |\nu| + 2 < 3(|\nu| + 1), \ldots
\]

\[
|\nu| + n - 1 < n(|\nu| + 1).
\]

Then

\[
| (\nu)^{(n)} | < (|\nu| + 1)^n n!.
\]

We have the same estimates for \((-1)^n (\nu)^{(n)}\). Therefore the series \(\hat{\psi}(x)\) and \(\hat{\varphi}(x)\) are of Gevrey order 1 with constants \(C = A = 1\) if \(|\nu| \leq 1\) and \(C = 1, A = |\nu| + 1\) if \(|\nu| > 1\) (see Definition \(3.6\)).

As a result their formal Borel transforms (Definition \(3.7\))

\[
(\hat{B}_1\hat{\psi})(\zeta) = \sum_{n=0}^{\infty} \frac{(\nu)^{(n)}}{n!} \zeta^n = (1 - \zeta)^{-\nu},
\]

\[
(\hat{B}_1\hat{\varphi})(\zeta) = \sum_{n=0}^{\infty} \frac{(-1)^n (\nu)^{(n)}}{n!} \zeta^n = (1 + \xi)^{-\nu}
\]

are analytic functions near the origin in the Borel planes.
Then for any directions $\theta_1 \neq 0$ and $\theta_2 \neq \pi$ from 0 to $+\infty e^{i\theta_k}$, $k = 1, 2$, the associate Laplace transforms (Definition 3.8)

$$\psi_{\theta_1}(x) = \int_0^{+ \infty e^{i\theta_1}} (1 - \zeta)^{-\nu} \exp \left( -\frac{\zeta}{x} \right) d \left( \frac{\zeta}{x} \right),$$

$$\varphi_{\theta_2}(x) = \int_0^{+ \infty e^{i\theta_2}} (1 + \xi)^{-\nu} \exp \left( -\frac{\xi}{x} \right) d \left( \frac{\xi}{x} \right)$$

define the corresponding 1-sum of the series $\hat{\psi}(x)$ and $\hat{\varphi}(x)$ respectively in such directions (Proposition 3.10).

This completes the proof. \(\square\)

**Remark 3.12.** Moving the direction $\theta_1$ (resp. $\theta_2$) continuously the corresponding Borel sums $\psi_{\theta_1}(x)$ (resp. $\varphi_{\theta_2}(x)$) stick each other analytically and define an function $\hat{\psi}(x)$ (resp. $\hat{\varphi}(x)$) on a sector of opening $3\pi$, $-\pi/2 < \arg x < 5\pi/2$ (resp. $-3\pi/2 < \arg x < 3\pi/2$). In these sectors the multivalued functions $\hat{\psi}(x)$ and $\hat{\varphi}(x)$ define the Borel sums of the series $\hat{\psi}(x)$ and $\hat{\varphi}(x)$, respectively. In every non-singular direction $\theta$ the multivalued functions $\phi(x)$ and $\varphi(x)$ have one value $\psi_0(x)$ and $\varphi_0(x)$, respectively. Near the singular direction $\theta = 0$ the function $\hat{\psi}(x)$ has two different values: $\psi_0^+(x) = \psi_{0+\epsilon}(x)$ and $\psi_0^-(x) = \psi_{0-\epsilon}(x)$, where $\epsilon > 0$ is a small number. Similarly, near the singular direction $\theta = \pi$ the function $\hat{\varphi}(x)\) has two different values: $\varphi_+^+(x) = \varphi_{\pi+\epsilon}(x)$ and $\varphi_+^-(x) = \varphi_{\pi-\epsilon}(x)$.

Replacing the elements $\varphi(x)$ and $\psi(x)$ of the matrix $H(x)$ in (3.16) by their sums (classical when $\nu$ is a non-positive integer and Borel otherwise), we obtain an actual function $F(x)$. It together with the actual function $F(x) = x^\Lambda \exp(-Q/x)$ form an actual fundamental matrix of the initial equation at the origin.

**Proposition 3.13.** (1) Assume that $\nu$ is a non-positive integer. Then the initial equation possesses an unique actual fundamental matrix $\Phi(x, 0)$ at the origin in the form

$$\Phi(x, 0) = H(x) F(x),$$

where $H(x)$ and $F(x)$ are analytic at the origin function, defined by (3.16), whose elements are the analytic functions (3.17). The matrix $F(x)$ is the branch of $x^\Lambda \exp(-Q/x)$ for $\arg x$.

(2) Assume that $\nu$ is not a non-positive integer. Then for every non-singular direction $\theta$ the initial equation possesses an unique actual fundamental matrix $\Phi_\theta(x, 0)$ at the origin in the form

$$\Phi_\theta(x, 0) = H_\theta(x) F_\theta(x),$$

where $H_\theta(x)$ is the Borel sum of the matrix $\hat{H}(x)$ in this direction and $F_\theta(x)$ is the branch of $x^\Lambda e^{-Q/x}$ for $\arg x = \theta$. In particular, $\Phi_{\theta+2\pi}(x, 0) = \Phi_{\theta}(x, 0) M$.

For the singular direction $\theta = 0$ the initial equation has two actual fundamental matrices

$$\Phi_0^\pm(x, 0) = \Phi_{0\pm\epsilon}(x, 0),$$

where the matrices $\Phi_{0\pm\epsilon}(x, 0)$ are given by (3.22) for a small number $\epsilon > 0$.

For the singular direction $\theta = \pi$ the initial equation again has two actual fundamental matrices

$$\Phi_\pi^\pm(x, 0) = \Phi_{\pi\pm\epsilon}(x, 0),$$
where the matrices $\Phi_{\pi \pm \epsilon}(x, 0)$ are again given by (3.22).

Moreover, the matrix $\Phi(x, 0)$ defined by (3.22)-(3.23)-(3.24) and the fundamental matrix $\Phi(x, 0)$ introduced by (2.12)-(2.13) define the same actual fundamental matrix at the origin of the initial equation.

Proof. From Theorem 3.5, Proposition 3.1, Lemma 3.11 and Remark 3.12 it follows that the matrices $\Phi(x, 0)$ defined by (3.22)-(3.23)-(3.24) are the unique actual fundamental matrices at the origin, associated with the pointed formal fundamental matrix $\hat{\Phi}(x, 0)$.

Therefore, we have only to show that these matrices and the fundamental matrix $\Phi(x, 0)$, introduced by (2.12) and (2.13) define the same actual fundamental matrix solution at the origin of the initial equation.

Let us represent $\Phi_{12}(x, 0)$ from (2.13), in the following form

$$
\Phi_{12}(x, 0) = \Phi_1(x, 0) \int_0^x \frac{\Phi_2(t, 0)}{\Phi_1(t, 0)} dt = e^{-1/x} \int_0^x t^{\nu-2} e^{-1/t} dt =
$$

where $\Gamma_1(x, 0)$ in (2.13) is a path from 0+ to $x$, approaching 0 in the direction $\mathbb{R}_+$. Then, introducing a new variable $\xi$ via

$$
\frac{1}{t} + \frac{1}{x} = -\frac{\xi}{x}
$$

we obtain

$$
\Phi_{12}(x, 0) = x^{\nu-1} e^{-2/x} \int_0^{+\infty} (1 + \xi)^{\nu} e^{-\xi/x} d\xi.
$$

In the same manner, we can represent $\Phi_{13}(x, 0)$ from (2.13) as

$$
\Phi_{13}(x, 0) = \Phi_1(x, 0) \int_{\Gamma_2(x, 0)} \frac{\Phi_{23}(t, 0)}{\Phi_1(t, 0)} dt = -\frac{1}{2} \int_0^x t^{\nu-2} e^{t^{-\frac{1}{\nu}}-\frac{1}{2}} dt,
$$

where the path of integration is the path $\Gamma_2(x, 0)$ from 0− to $x$, used in the definition of $\Phi_{23}(x, 0)$. Again, by introducing a new variable $\zeta$ by

$$
\frac{1}{t} - \frac{1}{x} = -\frac{\zeta}{x},
$$

we obtain the function

$$
\Phi_{13}(x, 0) = \frac{x^{\nu-1}}{2} \int_0^{-\infty} (1 - \zeta)^{\nu} e^{-\frac{\zeta}{x}} d\zeta.
$$

Analytic continuations of the so built $\Phi_{12}(x, 0)$ and $\Phi_{13}(x, 0)$ on $x$-plane yield analytic functions

$$
(\Phi_{12}(x, 0))_\theta = x^{\nu} e^{-2/x} \varphi_\theta(x), \quad (\Phi_{13}(x, 0))_\theta = \frac{x^{\nu}}{2} \psi_\theta(x)
$$
on every non-singular direction $\theta$.

Let $\epsilon > 0$ be a small number and let $0 + \epsilon$ and $0 - \epsilon$ be two non-singular directions near the singular direction $\theta = 0$. Then the function $\Phi_{13}(x, 0)$ has two different values near $\theta = 0$

$$
(\Phi_{13}(x, 0))_0^+ = (\Phi_{13}(x, 0))_0^- = \frac{x^{\nu}}{2} \psi_{0\pm \epsilon}(x),
$$

where $\psi_{0\pm \epsilon}(x, 0)$ are the Borel sums of the series $\hat{\psi}(x)$, built by Lemma 3.11 and extended by Remark 3.12.
Similarly, let \( \pi + \epsilon \) and \( \pi - \epsilon \) be two non-singular directions near the singular direction \( \theta = \pi \). Then near \( \theta = \pi \) the function \( \Phi_{12}(x, 0) \) has two different values

\[
(\Phi_{12}(x, 0))_{\pi}^{\pm} = (\Phi_{12}(x, 0))_{\pi \pm \epsilon} = x^\nu e^{-2/x} \varphi_{\pi \pm \epsilon}(x).
\]

Here \( \varphi_{\pi \pm \epsilon}(x) \) are the Borel sums of the series \( \hat{\varphi}(x) \), built and extended by Lemma 3.11 and Remark 3.12.

In the same manner near \( \theta = 2\pi \) the function \( \Phi_{13}(x, 0) \) has two different values

\[
(\Phi_{13}(x, 0))_{2\pi}^{\pm} = x^\nu e^{2\pi i \nu/2} \psi_{2\pi \pm \epsilon}(x).
\]

Note that near \( \theta = 2\pi \) the actual function \( F(x) \) is transformed into the function \( F(x) \hat{M} = x^\Lambda e^{-Q/\pi} \hat{M} \).

This proves Proposition 3.13.

In what follows we define and compute the Stokes matrices of the initial equation.

Let \( \theta \) be a singular direction of the initial equation. Denote by \( \Phi_+^\theta(x, 0) \) and \( \Phi_-^\theta(x, 0) \) the actual fundamental matrix of the initial equation, defined by Proposition 3.13. Then

**Definition 3.14.** With respect to the given actual fundamental matrices the Stokes matrix \( St^\theta \in \text{GL}_3(\mathbb{C}) \) corresponding to the singular direction \( \theta \) is defined by

\[
St^\theta = (\Phi_+^\theta(x, 0))^{-1} \Phi_-^\theta(x, 0).
\]

This definition implies that the Stokes matrix measures the difference between two fundamental matrices when we turn around a singular direction in a positive sense, which is keeping with the definition of the monodromy matrices \( M_j(\epsilon), j = L, R \) of the perturbed equation (see next section).

**Theorem 3.15.** With respect to the formal fundamental matrix \( \hat{\Phi}(x, 0) \) at the origin given by (3.14) - (3.15) - (3.16) and associated to it the actual fundamental matrix at the origin given by (3.22) - (3.23) - (3.24), the initial equation has two singular directions \( \theta_1 = 0 \) and \( \theta_2 = \pi \). The corresponding Stokes matrices are given by

\[
St_0 = \begin{pmatrix}
1 & 0 & -\frac{\pi i}{1(\nu)} \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix},
\]

\[
St_\pi = \begin{pmatrix}
1 & -\frac{2\pi i e^{-\pi i \nu}}{1(\nu)} & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]

**Proof.** The direction \( \theta_1 = 0 \) is a singular direction only to the element \( \Phi_{13}(x, 0) \) of the fundamental matrix. Therefore to build the Stokes matrix \( St_0 \), corresponding to the singular direction \( \theta_1 = 0 \), we have only to compare the functions \( (\Phi_{13}(x, 0))_0^+ = x^\nu \psi_{0-}(x)/2 \) and \( (\Phi_{13}(x, 0))_0^- = x^\nu \psi_{0+}(x)/2 \). We have

\[
(\Phi_{13}(x, 0))_0^- = (\Phi_{13}(x, 0))_0^+ + \frac{x^{\nu-1}}{2} \int_\gamma (1 - \zeta)^{-\nu} e^{-\frac{x}{\zeta}} d\zeta,
\]

where \( \gamma = (0 - \epsilon) - (0 + \epsilon) \). In this case, without changing the integral, we can deform \( \gamma \) in a Hankel type path \( \gamma' \) going along the positive real axis from infinity to 1, encircles 1 in the positive direction and back to infinity in the positive sense. Then since \( \arg(1 - \zeta) = -\pi \)
when $\text{Re}(\zeta) > 1$ and $\zeta$ lies on the direction $0 + \epsilon$ the integral becomes

$$
\frac{x^{\nu-1}(e^{-\pi i \nu} - e^{\pi i \nu})}{2} \int_1^{+\infty} (\zeta - 1)^{-\nu} e^{-\frac{\zeta}{x}} d\zeta = 
$$

$$
= \frac{x^{\nu-1}}{2} e^{-1/x} (e^{-\pi i \nu} - e^{\pi i \nu}) \int_0^{+\infty} u^{-\nu} e^{-\frac{u}{x}} du = 
$$

$$
= \frac{e^{-1/x} (e^{-\pi i \nu} - e^{\pi i \nu})}{2} \int_0^{+\infty} \tau^{-\nu} e^{-\tau} d\tau = \frac{1}{2} (e^{-\pi i \nu} - e^{\pi i \nu}) \Gamma(1 - \nu) e^{-1/x} = 
$$

$$
= -\frac{\pi i}{\Gamma(\nu)} e^{-1/x},
$$

where we used that the Gamma function $\Gamma(1 - \nu)$ is related to $\Gamma(\nu)$ by (see [2])

$$
\Gamma(\nu) \Gamma(1 - \nu) = \frac{\pi}{\sin(\pi \nu)}.
$$

In the same manner, comparing the functions $(\Phi_{12}(x,0))_\pi^-$ and $(\Phi_{12}(x,0))_\pi^+$, we obtain

$$(\Phi_{12}(x,0))_\pi^- = (\Phi_{12}(x,0))_\pi^+ - \frac{2\pi i e^{-\pi i \nu}}{\Gamma(\nu)} e^{-1/x}.
$$

Then the straightforward application of the Definition 3.14 gives us the Stokes matrices at the origin of the initial equation

$$
St_\pi = \begin{pmatrix}
1 & -\frac{2\pi i e^{-\pi i \nu}}{\Gamma(\nu)} & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix},
St_0 = \begin{pmatrix}
1 & 0 & -\frac{\pi i}{\Gamma(\nu)} \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}.
$$

We note that the function $1/\Gamma(\nu)$ is an entire function with zeros at $\nu = 0, -1, -2, \ldots$. Then, by a theorem on the analytic dependence of the Stokes matrices on the parameter $\nu$ (see [3]), it follows (according to expectation), that when $\nu \in \mathbb{Z}_{\leq 0}$ we have $St_0 = St_\pi = I_3$.

This completes the proof.

In order to use the results of Hurtubuse, Lambert and Rousseau [8, 16, 17], we consider now the initial equation and its actual fundamental matrices at the origin in the ramified domain $\{x \in \mathbb{C} : -\alpha < \arg(x) < \alpha\}$, where $0 < \alpha < \pi/2$. We cover this domain by two open sectors $\Omega_1$ and $\Omega_2$

$$
\Omega_1 = \Omega_1(\alpha, \rho) = \left\{ x = r e^{i\delta} \mid 0 < r < \rho, -\alpha < \delta < \pi + \alpha \right\},
\Omega_2 = \Omega_2(\alpha, \rho) = \left\{ x = r e^{i\delta} \mid 0 < r < \rho, -(\pi + \alpha) < \delta < \alpha \right\}.
$$

Denote by $\Omega_R$ and $\Omega_L$ the connected components of the intersection $\Omega_1 \cap \Omega_2$. The radius $\rho$ of the sectors $\Omega_j$ is chosen in such a way that $x_R = \sqrt{\varepsilon}$ belongs to $\Omega_R$, and $x_L = -\sqrt{\varepsilon}$ belongs to $\Omega_L$. The lower sector $\Omega_2$ contains only the Stokes ray $i \mathbb{R}_-$, and the upper sector $\Omega_1$ contains only the Stokes ray $i \mathbb{R}_+$. Then from the sectorial normalization theorem of Sibuya [24] it follows, that over the sector $\Omega_1$ the matrices $H^+_\pi(x)$ and $H^-_\pi(x)$ represent the same analytic function, asymptotic at the origin in $\Omega_1$ to the matrix $H(x)$ in [3,16].

Denote by $H_1(x) = H^+_\pi(x) = H^-_\pi(x)$. Similarly, over the sector $\Omega_2$ the matrices $H^+_\pi(x)$ and $H^-_\pi(x)$ represent the same analytic function, asymptotic at the origin in $\Omega_2$ to the matrix $H(x)$. Denote by $H_2(x) = H^+_\pi(x) = H^-_\pi(x)$. Then the matrix

$$
\Phi_j(x) = H_j(x) F(x) \quad j = 1, 2
$$
with the corresponding branch of $F(x)$ is an actual fundamental matrix at the origin of the initial equation over the sector $\Omega_j$, $j = 1, 2$. Note that we can observe the Stokes phenomenon on $\Omega_R$ and $\Omega_L$. Let us turn around the origin in the positive sense, starting from the sector $\Omega_1$. On $\Omega_L$ we define the Stokes matrix $St_L$ as 
$$St_L = (\Phi_2(x, 0))^{-1} \Phi_1(x, 0) = St_\pi,$$
where $St_\pi$ is the Stokes matrix, corresponding to the singular direction $\theta = \pi$ and defined by Theorem 3.15. On $\Omega_R$ we define the Stokes matrix $St_R$ as 
$$(\Phi_0^+(x, 0))^{-1} \Phi_2^-(x, 0) = (\Phi_1(x, 0))^{-1} \Phi_2(x, 0) \dot{M} = St_R \dot{M} = St_0 \dot{M},$$
where $St_0$ is the Stokes matrix, corresponding to the singular direction $\theta = 0$ and defined by Theorem 3.15. Then the actual monodromy matrix $M_0$ around $x = 0$ with respect to a base point on the lower sector $\Omega_2$ and the corresponding fundamental solution there is given by 
$$M_0 = St_\pi St_0 \dot{M}.$$ 
Then the monodromy around $x = \infty$ is given by $M_\infty^{-1} = St_\pi St_0 \dot{M}$. In the last section we will make a cut between the singular points $x_L$ and $x_R$ of the perturbed equation. Then the sectors $\Omega_j$ of this section will become new sectors $\Omega_j(\varepsilon)$ such that $\Omega_j(\varepsilon)$ tend to $\Omega_j$ when $\varepsilon$ tends to zero. This limit procedure implies a limit between the monodromy matrices $M_j(\varepsilon)$ and the so called unfolded Stokes matrices $St_j(\varepsilon)$ of the perturbed equation. The latter depend analytically on $\varepsilon$ and tend to the Stokes matrices $St_j, j = L, R$ defined above (see Theorem 4.25 in [16]).

We end this section fixing the behavior of the other singularity of the initial equation.

**Remark 3.16.** Except for the case when $\nu = 0$ the point $x = \infty$ is a regular singular point for the initial equation. The characteristic exponents $\rho_i^\infty$, $i = 1, 2, 3$ at $x = \infty$ are $\rho_1^\infty = 0, \rho_2^\infty = 1 - \nu, \rho_3^\infty = 2 - \nu$.

When $\nu = 0$ the change $x = 1/t$ transforms the initial equation into equation

$$\ddot{y}(t) + 3 \dot{y}(t) + 2 \dot{y}(t) = 0, \quad dt =$$

for which the point $t = 0$ (resp. the point $x = \infty$ for the original equation) is an ordinary point.

**4. The analytic invariants of the perturbed equation**

In this section we will introduce and find the monodromy matrices of the perturbed equation, connecting it with the initial equation partially. To do this, we firstly make the global fundamental matrix $\Phi(x, \varepsilon)$ (2.12) of the perturbed equation explicit. Let us defines the paths of integrations $\Gamma_j(x, \varepsilon)$ in (2.13). Once fixing the paths of integration $\Gamma_j(x, 0)$, we immediately determine the paths $\Gamma_j(x, \varepsilon)$ as follows: the path $\Gamma_1(x, \varepsilon)$ (resp. $\Gamma_2(x, \varepsilon)$) is a positive (resp. negative) real trajectory of the vector field $(x^2 - \varepsilon) \partial_x$ from $x_R = \sqrt{\varepsilon}$ to $x$ (resp. from $x_L = -\sqrt{\varepsilon}$ to $x$). The path $\Gamma_2(t_1, \varepsilon)$, similar to the path $\Gamma_2(t_1, 0)$, is a path from $-\sqrt{\varepsilon}$ to $t_1$ in the direction $\mathbb{R}_{-}$. Then we have that $\Gamma_j(x, \varepsilon) \rightarrow \Gamma_j(x, 0)$ when $\varepsilon \rightarrow 0$. This choice of the paths implies that the parameter of perturbation $\varepsilon$ is a small real positive number, i.e. $0 < \varepsilon < 1$.

Next, the perturbed equation is invariant under transformation

$$(4.25) \quad \sqrt{\varepsilon} \rightarrow -\sqrt{\varepsilon}.$$ 

So, throughout this section, we assume that $1/\sqrt{\varepsilon} > 1$. 


The next readily verified Lemma simplifies the elements \( \Phi_{23}(x, \varepsilon) \) and \( \Phi_{13}(x, \varepsilon) \) of the fundamental matrix \( \Phi(x, \varepsilon) \).

**Lemma 4.1.** Let \( a, b \in \mathbb{R} \) such that \( a > 0 \) and \( b > 1 \). Then

\[
\int_{-a}^{x} \frac{(s + a)^{b-1}}{(s - a)^{b+1}} ds = -\frac{1}{2ab} \left( \frac{x + a}{x - a} \right)^{b},
\]

where the integral is taken in the direction \( \mathbb{R}_{-} \) from \(-a\) to \( x < -a \).

Theorem [2.3] in combination with Lemma [4.1] gives the explicit form of the fundamental matrix of the perturbed equation.

**Theorem 4.2.** Assume that \( 1/\sqrt{\varepsilon} > 1 \). Then the explicit form of the elements of the fundamental matrix \( \Phi(x, \varepsilon) \) is given as follows

\[
\Phi_1(x, \varepsilon) = \left( \frac{x - \sqrt{\varepsilon}}{x + \sqrt{\varepsilon}} \right)^{\frac{1}{2\varepsilon}}, \quad \Phi_2(x, \varepsilon) = (x^2 - \varepsilon)^{\frac{\nu - 2}{2}} \left( \frac{x - \sqrt{\varepsilon}}{x + \sqrt{\varepsilon}} \right)^{\frac{1}{2\varepsilon}},
\]

\[
\Phi_3(x, \varepsilon) = (x^2 - \varepsilon)^{\frac{\nu - 4}{2}},
\]

for the diagonal elements, and

\[
\Phi_{12}(x, \varepsilon) = \Phi_1(x, \varepsilon) \int_{\Gamma_1(x, \varepsilon)} \left( \frac{t - \sqrt{\varepsilon}}{t + \sqrt{\varepsilon}} \right)^{\frac{1}{2\varepsilon}} dt,
\]

\[
\Phi_{23}(x, \varepsilon) = -\frac{\Phi_2(x, \varepsilon)}{2} \left( \frac{x + \sqrt{\varepsilon}}{x - \sqrt{\varepsilon}} \right)^{\frac{1}{2\varepsilon}} = -\frac{1}{2}(x^2 - \varepsilon)^{\frac{\nu - 2}{2}},
\]

\[
\Phi_{13}(x, \varepsilon) = -\frac{\Phi_1(x, \varepsilon)}{2} \int_{\Gamma_2(x, \varepsilon)} \left( \frac{t + \sqrt{\varepsilon}}{t - \sqrt{\varepsilon}} \right)^{\frac{1}{2\varepsilon}} dt,
\]

for the off-diagonal elements. The path of integration \( \Gamma_1(x, \varepsilon) \) is a path from \( x_R = \sqrt{\varepsilon} \) to \( x \) in the direction \( \mathbb{R}_+ \). The path of integration \( \Gamma_2(x, \varepsilon) \) is a path from \( x_L = -\sqrt{\varepsilon} \) to \( x \) in the direction \( \mathbb{R}_- \).

Let us briefly introduce the monodromy matrices of the perturbed equation, following Iwasaki et al. [10]. The perturbed equation is a third-order Fuchsian equation with three regular points over \( \mathbb{C}P^1 \): two of them \( x_R = \sqrt{\varepsilon} \) and \( x_L = -\sqrt{\varepsilon} \) are finite singularities and the third one is the infinity point. Let us consider the perturbed equation over \( X = \mathbb{C}P^1 - \{ x_R, x_L, x = \infty \} \). Its fundamental matrix \( \Phi(x, \varepsilon) \) is a multi-valued analytic function on the punctured Riemann sphere \( X \). Its multivaluedness is described by the monodromy matrices. Let \( \gamma_j \in X, j = R, L \) be simple closed loops starting and ending at point \( x_0 = 0 \in X \), defined by

\[
\gamma_R(t) = \sqrt{\varepsilon} + \sqrt{\varepsilon} e^{\pi i(2t+1)}, \quad \gamma_L(t) = -\sqrt{\varepsilon} + \sqrt{\varepsilon} e^{2\pi i} t, \quad 0 \leq t \leq 1.
\]

Let the matrix \( \Phi_{\gamma_j}(x, \varepsilon) \) be the analytic continuation of the fundamental matrix \( \Phi(x, \varepsilon) \) of the perturbed equation along the loop \( \gamma_j \). The matrix \( \Phi_{\gamma_j}(x, \varepsilon) \) depends only on the homotopy class \( [\gamma_j] \) of \( \gamma_j \). Since the perturbed equation is a linear equation, the matrix \( \Phi_{\gamma_j}(x, \varepsilon) \) is also a fundamental matrix of the same equation and there is a unique invertible constant matrix \( M_{\gamma_j}(\varepsilon) \in \text{GL}_3(\mathbb{C}) \) such that

\[
\Phi_{\gamma_j}(x, \varepsilon) = \Phi(x, \varepsilon) M_{\gamma_j}(\varepsilon).
\]
Definition 4.3. The antihomomorphism mapping
\[ \pi_1(X, x_0) \longrightarrow \text{GL}(3, \mathbb{C}), \]
\[ [\gamma_j] \longrightarrow M_{\gamma_j}(\varepsilon), \]
\[ M_{\gamma_R^j \gamma_L}(\varepsilon) = M_{\gamma_L}(\varepsilon) M_{\gamma_R}(\varepsilon), \quad M_{\gamma_R^j}^{-1}(\varepsilon) = M_{\gamma_L}^{-1}(\varepsilon) \]
determines monodromy representation of the perturbed equation associated with the given fundamental matrix \[ \left[ 10 \right]. \]

The product \( (\gamma_L^j \gamma_R)^{-1} \) of the so chosen loops \( \gamma_j \) is homotopic to a simple loop \( \gamma_{\infty} \) around infinity starting and ending at point \( x_0 \). Therefore the loops \( \gamma_j, j = R, L \) generate \( \pi_1(X, x_0) \).

Definition 4.4. The images \( M_{\gamma_j}(\varepsilon) \) of the generators \( \gamma_j, j = R, L \) of \( \pi_1(X, x_0) \) are called monodromy matrices of the perturbed equation \[ \left[ 10 \right]. \]

They satisfy the following relation:
\[ M_{\gamma_L}(\varepsilon) M_{\gamma_R}(\varepsilon) = M_{\gamma_L}^{-1}(\varepsilon), \]
where the matrix \( M_{\gamma_{\infty}}(\varepsilon) \in \text{GL}(3, \mathbb{C}) \) is the image of the loop \( \gamma_{\infty} \).

Now we have to determine when the matrix \( \Phi(x, \varepsilon) \) contains logarithmic therms near the singular points \( x_j, j = R, L \). In fact, only the elements \( \Phi_{12}(x, \varepsilon) \) and \( \Phi_{13}(x, \varepsilon) \) can contain such terms. To solve this problem, we are going to apply the local theory of the scalar Fucshian equations. Let us briefly recall some aspects of this theory needed to our goal, following Golubev \[ \left[ 6 \right] \] and Iwasaki et al. \[ \left[ 10 \right] \]. For simplicity we restrict ourselves to the perturbed equation.

As a Fuchsian equation, the perturbed equation can be written down as, \[ \left[ 6 \right] \]
\[ (4.30) \quad Y'''(x) + \frac{Q_2}{(x - x_R)(x - x_L)} Y''(x) + \frac{Q_1}{(x - x_R)^2(x - x_L)^2} Y'(x) + \frac{Q_0}{(x - x_R)^3(x - x_L)^3} Y(x) = 0, \quad t = \frac{d}{dx}, \]
where \( Q_i(x), i = 0, 1, 2 \) are polynomials of degree \( 3 - i \). Denote by \( c_i(x) \) the coefficient \( Q_i(x)/(x - x_R)^{3-i}(x - x_L)^{3-i}) \), \( i = 0, 1, 2 \) of the perturbed equation. At every regular point of the perturbed equation one can consider the so called characteristic (or the indicial) equation.

Definition 4.5. \[ \left[ 10 \right] \]
1. The third order algebraic equation
\[ \rho (\rho - 1) (\rho - 2) + b_2 \rho (\rho - 1) + b_1 \rho + b_0 = 0, \]
where \( b_i = \lim_{x \rightarrow x_j} c_i(x)(x - x_j)^{3-i}, j = R, L \), is called the characteristic (or the indicial) equation of the perturbed equation at the regular singular point \( x_j, j = R, L \). Its roots \( \rho_j^i, j = R, L, i = 1, 2, 3 \) are called the characteristic exponents at the singularities \( x_j, j = R, L \).

2. The characteristic equation at the point \( t = 0 \) of the equation, obtained from the perturbed equation after the transformation \( x = 1/t \), is called the characteristic equation at the point \( x = \infty \). Its roots \( \rho_i^\infty, i = 1, 2, 3 \) are called the characteristic exponents at the regular point \( x = \infty \).

It turns out that the coefficients \( a_j(x, \varepsilon) \in \mathbb{C}(x), j = 1, 2, 3 \) \[ \left[ 1.7 \right] \] of the perturbed equation in the representation \[ \left( 1.4 \right) - \left( 1.5 \right) \] are expressed in the terms of the characteristic exponents.
Proposition 4.6. The coefficients $a_j(x, \varepsilon)$ from (1.7) of the perturbed equation are uniquely determined only by the characteristic exponents $\rho_i^j$, $j = R, L$, $i = 1, 2, 3$ at the finite singularities $x_R$ and $x_L$.

Proof. The coefficients $a_j(x, \varepsilon)$ must have the form:

$$
(4.31) \quad a_1(x, \varepsilon) = -\frac{\rho_1^R}{x-x_R} - \frac{\rho_1^L}{x-x_L}, \quad a_2(x, \varepsilon) = -\frac{\rho_2^R - 1}{x-x_R} - \frac{\rho_2^L - 1}{x-x_L},
$$

$$
\quad a_3(x, \varepsilon) = -\frac{\rho_3^R - 2}{x-x_R} - \frac{\rho_3^L - 2}{x-x_L}.
$$

According to Proposition 4.6, equations (4.31) and (1.6), the fundamental matrix $\Phi(x, \varepsilon)$ of the perturbed equation becomes

$$
\Phi(x, \varepsilon) = e^{\int a_1(x, \varepsilon) dx} e^{\int a_2(x, \varepsilon) dx} e^{\int a_3(x, \varepsilon) dx}.
$$

The coefficients $a_j(x, \varepsilon)$ are uniquely determined only by the characteristic exponents $\rho_i^j$, $j = R, L$, $i = 1, 2, 3$ at the finite singular points $x_R = \sqrt{\varepsilon}$ and $x_L = -\sqrt{\varepsilon}$. After the change $x = 1/t$ the perturbed equation becomes

$$
(4.32) \quad \ddot{y}(t) + \left(\frac{2\nu}{t} + 3\right) \dot{y}(t) + \left(\frac{\nu(\nu - 1)}{t^2} + 4\nu + 2\right) y(t) + \left(\frac{\nu(\nu - 1)}{t^2} + 2\nu\right) y(t) = 0.
$$

According to Definition 4.5(2) the characteristic equation at $x = \infty$ of the perturbed equation is just the characteristic equation at $t = 0$ of the equation (4.32). It has the form

$$
\rho \left(\rho - 1\right) \left(\rho - 2\right) + 2\nu \rho \left(\rho - 1\right) + \nu \left(\nu - 1\right) \rho = 0
$$

and $\rho_1^\infty = 0, \rho_2^\infty = 1 - \nu, \rho_3^\infty = 2 - \nu$ are its roots. As we mentioned in the introduction the characteristic exponents $\rho_i^\infty$, $i = 1, 2, 3$ at $x = \infty$ coincide with the characteristic exponents at the same point $x = \infty$ of the initial equation.

With respect to the above characteristic exponents $\rho_i^j$ we define the following exponent differences:

$$
\Delta_{32}^L = \rho_3^L - \rho_2^L = \frac{1}{\sqrt{\varepsilon}}, \quad \Delta_{32}^R = \rho_3^R - \rho_2^R = -\Delta_{32}^L,
$$

$$
\Delta_{21}^L = \rho_2^L - \rho_1^L = \frac{\nu}{2} + \frac{1}{2\sqrt{\varepsilon}}, \quad \Delta_{21}^R = \rho_2^R - \rho_1^R = \Delta_{21}^L + \Delta_{32}^R,
$$

$$
\Delta_{31}^L = \rho_3^L - \rho_1^L = \Delta_{31}^L, \quad \Delta_{31}^R = \rho_3^R - \rho_1^R = \Delta_{31}^R.
$$

Classically, the Fuchsian singular point $x_j$, $j = R, L$ is called a resonant Fuchsian singularity, if there is a $\Delta_{kp}^L$, $k \neq p$, $k = 2, 3$, $p = 1, 2$, which is an integer [10]. Otherwise, it is called a non-resonant Fuchsian singularity [10]. The local theory of the Fuchsian singularities says that the presence of a resonant regular singular point $x_j$, $j = R, L$ is a necessary but not a sufficient condition the fundamental matrix $\Phi(x, \varepsilon)$ to contain logarithmic terms [10]. On the other hand we have already observed that only the elements $\Phi_{12}(x, \varepsilon)$ and $\Phi_{13}(x, \varepsilon)$ of the matrix $\Phi(x, \varepsilon)$ can contain logarithmic terms. So, from here on we focus mainly on the computation of the monodromy matrices $M_j(\varepsilon)$, $j = R, L$ relevant to the so called resonant logarithmic cases (see below).
For simplicity, through the present and the next section, we call these values of the parameters $\nu$ and $\varepsilon$, for which there is a chance of the appearance of logarithmic terms near a resonant Fuchsian singularity $x_j, j = R, L$ just the resonant logarithmic cases or a logarithmic resonance. We distinguish three different types of resonant logarithmic cases. They are:

- the resonant logarithmic cases of type (B) if
  
  \[
  (B) \quad \Delta_{21}^R = \Delta_{31}^L \in \mathbb{Z}, \quad \Delta_{21}^L = \Delta_{31}^R \in \mathbb{Z}
  \]
  simultaneously;

- the resonant logarithmic cases of type (C) if
  
  \[
  (C) \quad \Delta_{21}^L = \Delta_{31}^R \in \mathbb{Z}, \quad \Delta_{21}^R = \Delta_{31}^L \notin \mathbb{Z}
  \]
  simultaneously;

- the resonant logarithmic cases of type (D) if
  
  \[
  (D) \quad \Delta_{21}^R = \Delta_{31}^L \in \mathbb{Z}, \quad \Delta_{21}^L = \Delta_{31}^R \notin \mathbb{Z}
  \]
  simultaneously.

It turns out that we can already here reduce the number of the types of the resonant logarithmic cases. Indeed, the simultaneous conditions, which define the resonant logarithmic cases of type (D), imply that

\[
\frac{\nu}{2} + \frac{1}{2\sqrt{\varepsilon}} \in \mathbb{Z} \quad \text{but} \quad \frac{\nu}{2} - \frac{1}{2\sqrt{\varepsilon}} \notin \mathbb{Z}.
\]

Compare this conditions with the form of the elements $\Phi_{12}(x, \varepsilon)$ and $\Phi_{13}(x, \varepsilon)$. We see that the element $\Phi_{12}(x, \varepsilon)$ (resp. $\Phi_{13}(x, \varepsilon)$) can contain logarithmic term near $x_R$ (resp. $x_L$) if and only if $\nu/2 + 1/2\sqrt{\varepsilon} - 1 < 0$. On the other hand the limit $\sqrt{\varepsilon} \to 0 \in \mathbb{R}_+$ is equivalent to the limit $\nu/2 + 1/2\sqrt{\varepsilon} \to +\infty$ for a fixed $\nu$. But this implies that $\nu/2 + 1/2\sqrt{\varepsilon} \in \mathbb{N}$. Therefore during a logarithmic resonance of type (D) the fundamental matrix $\Phi(x, \varepsilon)$ does not contain logarithmic terms. So, in this article we consider only the rest resonant logarithmic cases.

Now, we are in a position to describe the behavior of the fundamental matrix $\Phi(x, \varepsilon)$ at the finite singularities $x_j, j = R, L$ during a logarithmic resonance of type B and C. We have the following result.

**Proposition 4.7.** During a logarithmic resonance of type (B) and (C) the fundamental matrix $\Phi(x, \varepsilon)$ of the perturbed equation is represented near the singular points $x_j, j = R, L$, as

\[
\Phi(x, \varepsilon) = (I_L(\varepsilon) + \mathcal{O}(x-x_L))(x-x_L)^{\Lambda + \frac{1}{2\sqrt{\varepsilon}}}Q(x-x_L)^{T_L}
\]

in a neighborhood of $x_L$, which does not contain the point $x_R$, and

\[
\Phi(x, \varepsilon) = (I_R(\varepsilon) + \mathcal{O}(x-x_R))(x-x_R)^{\Lambda + \frac{1}{2\sqrt{\varepsilon}}}Q(x-x_R)^{T_R}
\]

in a neighborhood of $x_R$, which does not contain the point $x_L$. The matrices $I_j(\varepsilon) + \mathcal{O}(x-x_j)$ are analytic matrix-functions near the point $x_j, j = R, L$, respectively. The matrices $\Lambda$ and $Q$ are the matrices, associated with the initial equation and defined by (3.15),

\[
(4.34) \quad T_j = \begin{pmatrix}
0 & d_2^j & d_3^j \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]
The elements $d^j_i$, $i = 2, 3$, $j = R, L$ of the matrices $T_j$, $j = R, L$ are defined as follows,

\begin{equation}
\begin{align*}
&d^R_2 = 0, \quad d^L_2 = \text{Res} \left( \frac{(x - \sqrt{\varepsilon})^{2\nu} + \frac{2\nu}{3}}{(x + \sqrt{\varepsilon})^{2\nu} - \frac{2\nu}{3}}, \ x = x_L \right), \\
&d^R_3 = 0, \quad d^L_3 = -\frac{1}{2} \text{Res} \left( \frac{(x + \sqrt{\varepsilon})^{2\nu} + \frac{2\nu}{3}}{(x - \sqrt{\varepsilon})^{2\nu} - \frac{2\nu}{3}}, \ x = x_R \right).
\end{align*}
\end{equation}

Proof. To study the behavior of the fundamental matrix $\Phi(x, \varepsilon)$ we will use its explicit form given by Theorem 4.2. Let us firstly consider the elements $\Phi_s(x, \varepsilon)$, $s = 2, 3, 23$. We have

$$
\Phi_s(x, \varepsilon) = \begin{cases}
(x - x_L)^{m^L_s} g_{s,L}(x) & \text{near the point } x_L, \\
(x - x_R)^{m^R_s} g_{s,R} & \text{near the point } x_R,
\end{cases}
$$

where $s = 2, 3, 23$, the exponents $m^L_2 = \rho^2_2 - 1$, $m^L_3 = \rho^2_3 + m^L_{23} = \rho^2_3 - 2$, $j = L, R$. The functions $g_{s,j}(x)$ are analytic functions in $x$ in a neighborhood of the singular point $x_j$, which does not contain the other singular point. In particular, $g_{s,j}(x_j) \neq 0$ for $s = 2, 3$, and $g_{23,j}(x_j) = 0$.

Now we consider the elements $\Phi_s(x, \varepsilon)$ for $s = 1, 12, 13$. We have

$$
\Phi_1(x, \varepsilon) = \begin{cases}
(x - x_L)^{\nu^L_1} g_{1,L}(x) & \text{near the point } x_L, \\
(x - x_R)^{\nu^R_1} g_{1,R}(x) & \text{near the point } x_R.
\end{cases}
$$

For the element $\Phi_{12}(x, \varepsilon)$ we have

$$
\Phi_{12}(x, \varepsilon) = d^L_2 (x - x_L)^{\nu^L_2} \log(x - x_L) g_{1,L}(x) + (x - x_L)^{\nu^L_2 - 1} h_{2,L}(x)
$$

in a neighborhood of $x_L$, which does not contain $x_R$, and

$$
\Phi_{12}(x, \varepsilon) = (x - x_R)^{\nu^R_2 - 1} h_{2,R}(x)
$$

in a neighborhood of $x_R$, which does not contain the point $x_L$. For the element $\Phi_{13}(x, \varepsilon)$ we have

$$
\Phi_{13}(x, \varepsilon) = (x - x_L)^{\nu^L_3 - 2} h_{3,L}(x)
$$

near the point $x_L$, and

$$
\Phi_{13}(x, \varepsilon) = d^R_3 (x - x_R)^{\nu^R_3} \log(x - x_R) g_{1,R}(x) + (x - x_R)^{\nu^R_3 - 2} h_{3,R}(x)
$$

in a neighborhood of $x_R$, which does not contain the point $x_L$. The functions $g_{1,j}(x)$ and $h_{s,j}$, $j = L, R$, $s = 2, 3$ are analytic functions in $x$ in a neighborhood of the singular point $x_j$, which does not contain the other singular point. In particular, $g_{1,j}(x_j) \neq 0$, and $h_{s,j}(x_j) = 0$.

Note that the absence of logarithmic terms during a resonance implies nullity of the numbers $d^j_i$, $i = 2, 3$.

Remark 4.8. From Proposition 4.7 it follows that, the norm of the $j$-column of the fundamental matrix $\Phi(x, \varepsilon)$ has the same growth rate near the singular points as the function $\Phi_j(x, \varepsilon)$, $j = 1, 2, 3$. In particular, near $x_R$ the function $\Phi_2(x, \varepsilon)$ has the smallest growth, and the function $\Phi_3(x, \varepsilon)$ has the largest one. Near $x_L$ the function $\Phi_3(x, \varepsilon)$ has the smallest growth, and the function $\Phi_2(x, \varepsilon)$ has the largest one. The first column is determined as the unique solution (up to a constant factor) that has a mid-growth at both $x_R$ and $x_L$. □
Theorem 4.9. During a logarithmic resonance of type (B) and (C) the monodromy matrices $M_j(x, \varepsilon)$, $j = R, L$ of the perturbed equation with respect to the fundamental matrices $\Phi(x, \varepsilon)$, defined by Theorem 4.2, are given by

$$M_j(x, \varepsilon) = e^{\pi i (\Lambda + \frac{1}{x_j} Q)} e^{2\pi i T_j} e^{\pi i (\Lambda + \frac{1}{x_j} Q)}.$$  

(4.36)

Proof. The proof follows from Proposition 4.7 and Definition 4.3. Definition 4.4 together with the observation that during a resonance the matrices $e^{\pi i (\Lambda + \frac{1}{x_j} Q)}$ and $(x - x_j)T_j$ commute. Note that during the logarithmic resonance of type (B) and (C) the matrices $e^{2\pi i T_j}$ and $e^{\pi i (\Lambda + \frac{1}{x_j} Q)}$ also commute. \qed

Remark 4.10. Combining Proposition 4.7 and Theorem 4.9 we observe that the first two columns of the matrix $\Phi(x, \varepsilon)$ are eigenvectors of the monodromy operators $M_R(x, \varepsilon)$ with eigenvalues $e^{2\pi i \rho^R_1} = e^{\pi i/\sqrt{\varepsilon}}$ and $e^{2\pi i (\rho^R_1 - 1)} = e^{\pi i(\nu + 1/\sqrt{\varepsilon})}$, respectively. In the same manner the first column and the third column of the matrix $\Phi(x, \varepsilon)$ are eigenvectors of the monodromy operator $M_L(x, \varepsilon)$ with eigenvalues $e^{2\pi i \rho^L_1} = e^{-\pi i/\sqrt{\varepsilon}}$ and $e^{2\pi i (\rho^L_1 - 1)} = e^{\pi i(\nu - 1/\sqrt{\varepsilon})}$, respectively. The numbers $d^R_3$ and $d^L_2$ (when they are different from zero) block the third and the second column of $\Phi(x, \varepsilon)$ of being eigenvectors of the monodromy operators $M_R(x, \varepsilon)$ and $M_L(x, \varepsilon)$ respectively (see also [18, 33] about the confluence of the hypergeometric equation).

Combining the observations of Remark 4.8 and Remark 4.10 we see that the columns of the fundamental matrix of the perturbed equation leads to the so called “mixed basis” of solutions (see [8], Theorem 5.4 and 5.5 for more details).

In what follows we will calculate explicitly the numbers $d^R_i$, $i = 2, 3$, $j = R, L$ (4.35) and write down the corresponding monodromy matrices (4.36).

4.1. The resonant logarithmic cases of type (B). We begin our calculations with the resonant logarithmic cases of type (B).

The simultaneous conditions $\Delta^R_{21} = \Delta^L_{31} \in \mathbb{Z}$, $\Delta^L_{21} = \Delta^R_{31} \in \mathbb{Z}$ imply that

$$\nu \in \mathbb{Z}, \quad \frac{1}{\sqrt{\varepsilon}} \in \mathbb{N} \quad \text{such that} \quad \frac{\nu}{2} + \frac{1}{2\sqrt{\varepsilon}} \in \mathbb{Z}. \quad \text{(4.27)}$$

In the last section we are going to study the behavior of the monodromy matrices, obtained in the present section, when $\sqrt{\varepsilon} \to 0 \in \mathbb{R}_+$ and $1/\sqrt{\varepsilon} \in \mathbb{N}$ for fixed $\nu$. But the limit when $\sqrt{\varepsilon} \to 0 \in \mathbb{R}_+$ (for fixed $\nu$) is equivalent to the limit $\nu/2 + 1/2\sqrt{\varepsilon} \to +\infty$. So, we limit calculation to the case when $\nu/2 + 1/2\sqrt{\varepsilon} \in \mathbb{N}$. We also note that when $\nu/2 + 1/2\sqrt{\varepsilon} \in \mathbb{N}$ the integral $\Phi_{12}(x, \varepsilon)/\Phi_1(x, \varepsilon)$ in (4.21) is a convergent one.

Theorem 4.11. Assume that $\nu \in \mathbb{Z}$. Assume also that $1/\sqrt{\varepsilon} \in \mathbb{N}$ such that $\nu/2 + 1/2\sqrt{\varepsilon} \in \mathbb{N}$. Then for $d^R_i$, $i = 2, 3$, $j = R, L$ (4.35) and the corresponding monodromy matrices $M_j$, $j = R, L$ (4.36) we have

$$d^R_3 = d^L_3 = 0, \quad d^L_2 = \left(-\frac{1}{2\sqrt{\varepsilon}}\right)^{1-\nu} \frac{\Gamma\left(\frac{1}{2\sqrt{\varepsilon}} + \frac{\nu}{2}\right)}{\Gamma(\nu)\Gamma\left(\frac{1}{2\sqrt{\varepsilon}} - \frac{\nu}{2} + 1\right)},$$

$$d^R_2 = -\frac{1}{2} \left(\frac{1}{2\sqrt{\varepsilon}}\right)^{1-\nu} \frac{\Gamma\left(\frac{1}{2\sqrt{\varepsilon}} + \frac{\nu}{2}\right)}{\Gamma(\nu)\Gamma\left(\frac{1}{2\sqrt{\varepsilon}} - \frac{\nu}{2} + 1\right)}.$$
The corresponding monodromy matrices are given by
\[
M_j(ε) = \begin{pmatrix}
(-1)\nu & 2\pi i(-1)\nu d_2 & 2\pi i(-1)\nu d_3 \\
0 & (-1)\nu & 0 \\
0 & 0 & (-1)\nu
\end{pmatrix}, \quad j = R, L.
\]

Proof. According to (4.35) the numbers \(d_j^R, j = R, L\) are defined by
\[
d_j^R = \text{Res} \left( \frac{1}{x - \sqrt{ε}} \frac{1}{2\sqrt{ε} + \frac{\nu}{2} - 1}, x = x_j \right), \quad j = R, L.
\]
Since \(1/2\sqrt{ε} + \nu/2 \in \mathbb{N}\) we find that \(d_j^R = 0\). When \(\nu \in \mathbb{Z}\) and \(1/2\sqrt{ε} + \nu/2 \in \mathbb{N}\) the exponent \(1/2\sqrt{ε} - \nu/2 + 1 \in \mathbb{Z}\). Then for the number \(d_j^L\) we obtain consecutively
\[
d_j^L = \frac{1}{(1/2\sqrt{ε} - \nu/2)!} \left( \frac{d}{dx} \frac{1}{x - \sqrt{ε} \frac{1}{2\sqrt{ε} + \frac{\nu}{2} - 1}} \right)_{x=x_L} = \left( -\frac{1}{2} \right)^{1-\nu} \frac{\Gamma(1/2\sqrt{ε} + \nu/2)}{\Gamma(\nu) \Gamma(1/2\sqrt{ε} - \nu/2 + 1)},
\]
where \(\Gamma(z)\) is the classical Gamma function.

Next, we compute the numbers \(d_j^3\). Applying the formula (4.35), we see that \(d_j^3 = 0\) when \(1/2\sqrt{ε} + \nu/2 \in \mathbb{N}\). For the number \(d_j^R\) we have
\[
d_j^R = -\frac{1}{2} \left( \frac{1}{2\sqrt{ε} - \frac{\nu}{2}} \right) \left( \frac{d}{dx} \frac{1}{x - \sqrt{ε} \frac{1}{2\sqrt{ε} + \frac{\nu}{2} - 1}} \right)_{x=x_L} = \frac{1}{2} \left( \frac{1}{2\sqrt{ε}} \right)^{1-\nu} \frac{\Gamma(1/2\sqrt{ε} + \nu/2)}{\Gamma(\nu) \Gamma(1/2\sqrt{ε} - \nu/2 + 1)}.
\]

We also note that in this case the integral, including in the definition of \(Φ_{13}(x, ε)\)
\[
\int_{-\sqrt{ε}}^{x} \frac{(t + \sqrt{ε})^{1/2\sqrt{ε} + \frac{\nu}{2} - 1}}{(t - \sqrt{ε})^{1/2\sqrt{ε} - \frac{\nu}{2} + 1}} dt
\]
is a convergent one.

Applying Theorem 4.9 we see that the corresponding monodromy are given by
\[
M_j(ε) = \begin{pmatrix}
(-1)\nu & 2\pi i(-1)\nu d_2 & 2\pi i(-1)\nu d_3 \\
0 & (-1)\nu & 0 \\
0 & 0 & (-1)\nu
\end{pmatrix}, \quad j = R, L,
\]
since \(\nu\) is an integer.

This completes the proof of Theorem 4.11. □

4.2. The resonant logarithmic cases of type (C) . The conditions \(Δ_{21}^R = Δ_{31}^R \in \mathbb{Z}\) but \(Δ_{21}^L = Δ_{31}^L \notin \mathbb{Z}\) which define the resonant logarithmic case (C) imply that
\[
\frac{\nu}{2} - \frac{1}{2\sqrt{ε}} \in \mathbb{Z} \quad \text{but} \quad \nu \notin \mathbb{Z}.
\]
Similar to the resonant logarithmic cases of type B, we can restrict the calculations to the non-positive values of \(\nu/2 - 1/2\sqrt{ε}\).
Then we have

**Theorem 4.12.** Assume that \(1/2\sqrt{\epsilon} - \nu/2 \in \mathbb{N} \) but \(\nu \notin \mathbb{Z} \). Then for the numbers \(d^j_3\) and the corresponding monodromy matrices \(M_R\) and \(M_L\) of the perturbed equation we have

\[
\begin{align*}
  d^R_2 &= d^L_3 = 0, \\
  d^L_2 &= \left(-\frac{1}{2\sqrt{\epsilon}}\right)^{1-\nu} \frac{\Gamma\left(\frac{1}{2\sqrt{\epsilon}} + \frac{\nu}{2}\right)}{\Gamma\left(\nu\right) \Gamma\left(\frac{1}{2\sqrt{\epsilon}} - \frac{\nu}{2} + 1\right)}, \\
  d^R_3 &= -\frac{1}{2} \left(\frac{1}{2\sqrt{\epsilon}}\right)^{1-\nu} \frac{\Gamma\left(\frac{1}{2\sqrt{\epsilon}} + \frac{\nu}{2}\right)}{\Gamma\left(\nu\right) \Gamma\left(\frac{1}{2\sqrt{\epsilon}} - \frac{\nu}{2} + 1\right)}.
\end{align*}
\]

The monodromy matrices are given by

\[
M_R(\epsilon) = \begin{pmatrix}
  e^{\pi i \nu} & 0 & 2\pi i e^{\pi i \nu} d^R_3 \\
  0 & e^{2\pi i \nu} & 0 \\
  0 & 0 & e^{\pi i \nu}
\end{pmatrix},
M_L(\epsilon) = \begin{pmatrix}
  e^{-\pi i \nu} & 2\pi i e^{-\pi i \nu} d^L_2 & 0 \\
  0 & e^{-\pi i \nu} & 0 \\
  0 & 0 & e^{\pi i \nu}
\end{pmatrix}.
\]

**Proof.** As in the previous theorems for the numbers \(d^j_3\) we have that

\[
d^j_3 = -\frac{1}{2} \text{Res} \left( \frac{(x + \sqrt{\epsilon})^{1/2 \epsilon + \frac{\nu}{2} - 1}}{(x - \sqrt{\epsilon})^{1/2 \epsilon - \frac{\nu}{2} + 1}}, x = x_j \right).
\]

Since the exponent \(1/2\sqrt{\epsilon} + \nu/2 - 1 \notin \mathbb{Z}\), when \(\nu \notin \mathbb{Z}\) then \(d^L_3 = 0\). We also note that for sufficiently big \(1/\sqrt{\epsilon}\) the integral, including in the definition of \(\Phi_1(x, \epsilon)\)

\[
\int_{-\sqrt{\epsilon}}^{x} \frac{(t + \sqrt{\epsilon})^{1/2 \epsilon + \frac{\nu}{2} - 1}}{(t - \sqrt{\epsilon})^{1/2 \epsilon - \frac{\nu}{2} + 1}} dt
\]

is a convergent one. Next, for the number \(d^R_3\) we find

\[
d^R_3 = -\frac{1}{2} \left(\frac{1}{2\sqrt{\epsilon}}\right)^{1-\nu} \frac{\Gamma\left(\frac{1}{2\sqrt{\epsilon}} + \frac{\nu}{2}\right)}{\Gamma\left(\nu\right) \Gamma\left(\frac{1}{2\sqrt{\epsilon}} - \frac{\nu}{2} + 1\right)}.
\]

For the number \(d^L_2\) we obtain

\[
d^L_2 = \left(-\frac{1}{2\sqrt{\epsilon}}\right)^{1-\nu} \frac{\Gamma\left(\frac{1}{2\sqrt{\epsilon}} + \frac{\nu}{2}\right)}{\Gamma\left(\nu\right) \Gamma\left(\frac{1}{2\sqrt{\epsilon}} - \frac{\nu}{2} + 1\right)}.
\]

Again since the exponent \(1/2\sqrt{\epsilon} + \nu/2 - 1 \notin \mathbb{Z}\), then \(d^R_2 = 0\). This ends the calculations of the numbers \(d^j_3\).

Finally, applying (4.39) and using the connection \(1/2\sqrt{\epsilon} - \nu/2 \in \mathbb{N}\), we write down the monodromy matrices \(M_R(\epsilon)\) and \(M_L(\epsilon)\). \(\square\)

**Remark 4.13.** Similar to the non-perturbed equation when \(\nu = 0\) the point \(x = \infty\) is no longer a singular point for the perturbed equation. Indeed, when \(\nu = 0\) we set \(x = 1/t\). Then the perturbed equation becomes

\[
\dot{y}(t) + \left[-3\sqrt{\epsilon} \left(1 - \frac{1}{2\sqrt{\epsilon}}\right) \frac{1}{1 - \sqrt{\epsilon} t} + 3\sqrt{\epsilon} \left(1 + \frac{1}{2\sqrt{\epsilon}}\right) \frac{1}{1 + \sqrt{\epsilon} t}\right] \dot{y}(t) + \\
+ \left[-\frac{2\epsilon + \frac{1}{2}}{1 - \sqrt{\epsilon} t} + \frac{2\epsilon + \frac{1}{2}}{1 + \sqrt{\epsilon} t} + \frac{\epsilon(1 - 3/2\sqrt{\epsilon} + \frac{1}{2})}{(1 - \sqrt{\epsilon})^2} + \frac{\epsilon(1 + 3/2\sqrt{\epsilon} + \frac{1}{2})}{(1 + \sqrt{\epsilon})^2}\right] \dot{y}(t) = 0
\]
for which the point $t = 0$ (resp. the point $x = \infty$ for the original equation) is an ordinary point.

5. **Main results**

In this section we will connect, by a limit \( \sqrt{\varepsilon} \to 0 \), analytic invariants of the initial and the perturbed equations, computed in the previous two sections. In order to connect by a limit \( \sqrt{\varepsilon} \to 0 \) the solution of the perturbed equation with the solution of the initial equation, as well as their invariants, we consider the perturbed equation on the sectorial domains \( \Omega_1(\varepsilon) \) and \( \Omega_2(\varepsilon) \). These domains are obtained from \( \Omega_1 \) and \( \Omega_2 \) (relevant to the initial equation) by making a cut between the singular points \( x_L \) and \( x_R \) through the real axis, The point \( x_0 = 0 \) belongs to this cut. When \( \varepsilon \to 0 \) then \( \Omega_j(\varepsilon), j = 1, 2 \) tend to \( \Omega_j, j = 1, 2 \). The domains \( \Omega_1(\varepsilon) \) and \( \Omega_2(\varepsilon) \) intersect in the left \( \Omega_L(\varepsilon) \), right \( \Omega_R(\varepsilon) \) sectors and along the cut. The points \( x_j, j = L, R \) do not belong to \( \Omega_j(\varepsilon) \), but belong to their closure, respectively. Both points belong to the cut (see [16]).

In the keeping with the initial equation, we rewrite the fundamental matrix \( \Phi(x, \varepsilon) \) of the perturbed equation in the form

\[
\Phi(x, \varepsilon) = H(x, \varepsilon) F(x, \varepsilon).
\]

Here \( F(x, \varepsilon) = (x - x_L)^{\Lambda/2 + Q/2x_L} (x - x_R)^{\Lambda/2 + Q/2x_R}. \) The matrix \( H(x, \varepsilon) \) is defined as

\[
H(x, \varepsilon) = \begin{pmatrix}
1 & \frac{\Phi_1(x, \varepsilon)}{\Phi_2(x, \varepsilon)} \int_{\Gamma_1(x, \varepsilon)} \frac{\Phi_1(t, \varepsilon)}{\Phi_2(t, \varepsilon)} dt & \frac{\Phi_1(x, \varepsilon)}{\Phi_2(x, \varepsilon)} \int_{\Gamma_2(x, \varepsilon)} \frac{\Phi_3(t, \varepsilon)}{\Phi_2(t, \varepsilon)} dt \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]

When we move the path \( \Gamma_1(x, \varepsilon) \) analytically around the origin we obtain near the negative real axis \( \mathbb{R}_- \) two branches of the element \( h_{12}(x, \varepsilon) \) of the matrix \( H(x, \varepsilon) \). We denote them by \( h_{12}^+(x, \varepsilon) \) and \( h_{12}^-(x, \varepsilon) \). The function \( h_{12}^+(x, \varepsilon) \) is defined on the path in the direction \( \pi - \varepsilon \), and the function \( h_{12}^-(x, \varepsilon) \) is defined in the direction \( \pi + \varepsilon \). Similarly, near the positive real axis \( \mathbb{R}_+ \) we have two branches, \( h_{13}^+(x, \varepsilon) \) and \( h_{13}^-(x, \varepsilon) \), of the function \( h_{13}(x, \varepsilon) \). When \( \Gamma_1(x, \varepsilon) \) (resp. \( \Gamma_2(x, \varepsilon) \)) crosses \( \mathbb{R}_- \) (resp. \( \mathbb{R}_+ \)) we rather observe a Stokes phenomenon, that a linear monodromy. This phenomenon is described by the so called unfolded Stokes matrices. In accordance with the initial equation, we determine on the sector \( \Omega_1(x, \varepsilon) \) the fundamental matrix of the perturbed equation as

\[
\Phi_1(x, \varepsilon) = H_1(x, \varepsilon) F_1(x, \varepsilon),
\]

where

\[
H_1(x, \varepsilon) = \begin{pmatrix}
1 & h_{12}(x, \varepsilon) & h_{12}^+(x, \varepsilon) \\
0 & 1 & h_{23}(x, \varepsilon) \\
0 & 0 & 1
\end{pmatrix},
\]

and \( F_1(x, \varepsilon) \) is the branch of \( F(x, \varepsilon) \) on \( \Omega_1(\varepsilon) \). On the sector \( \Omega_2(\varepsilon) \) we define the fundamental matrix as

\[
\Phi_2(x, \varepsilon) = H_2(x, \varepsilon) F_2(x, \varepsilon),
\]

where

\[
H_2(x, \varepsilon) = \begin{pmatrix}
1 & h_{12}^+(x, \varepsilon) & h_{12}^-(x, \varepsilon) \\
0 & 1 & h_{23}(x, \varepsilon) \\
0 & 0 & 1
\end{pmatrix},
\]

as \( F_2(x, \varepsilon) = F_1(x, \varepsilon) \) on \( \Omega_L(\varepsilon) \) and \( F_2(x, \varepsilon) = F_1(x, \varepsilon) \) \( \tilde{M} \) on \( \Omega_R(\varepsilon) \). Denote by \( St_L(\varepsilon) \) and \( St_R(\varepsilon) \) the unfolded Stokes matrices with respect to the fundamental matrix \( \Phi_1(x, \varepsilon) \) on the upper sector \( \Omega_1(\varepsilon) \). Now we will describe the change of the fundamental matrix when
we turn around the origin analytically in the positive sense. We start from the sector $\Omega_1(\varepsilon)$ and the solution $\Phi_1(x, \varepsilon)$ on it. When $\Gamma_1(x, \varepsilon)$ crosses the negative real axis, we observe a Stokes phenomenon on $\Omega_L(\varepsilon)$. In particular, the unfolded Stokes matrix $\hat{St}_L(\varepsilon)$ is defined by

$$\hat{St}_L(\varepsilon) = (\Phi_2(x, \varepsilon))^{-1} \Phi_1(x, \varepsilon) \quad \text{on} \quad \Omega_L(\varepsilon)$$

If we continue circling round the origin then when $\Gamma_2(x, \varepsilon)$ crosses the positive real axis, we observe a Stokes phenomenon on $\Omega_R(\varepsilon)$. The jump of the solution $\Phi_2(x, \varepsilon)$ to the solution $\Phi_1(x, \varepsilon)$ is defined by

$$(\Phi_1(x, \varepsilon))^{-1} \Phi_2(x, \varepsilon) = St_R(\varepsilon) \hat{M} \quad \text{on} \quad \Omega_R(\varepsilon),$$

since on $\Omega_2(\varepsilon)$ we have $F_2(x, \varepsilon) = F_1(x, \varepsilon) \hat{M}$.

From Theorem 4.25 in [16] it follows that the unfolded Stokes matrices $St_L(\varepsilon)$ and $St_R(\varepsilon)$ depend analytically on the parameter of perturbation $\varepsilon$ and they converge when $\varepsilon \to 0$ to the Stokes matrices $St_L = St_\pi$ and $St_R = St_0$ of the initial equation.

In the previous section we have computed the monodromy matrix $M_R(\varepsilon)$ of the perturbed equation with respect to the fundamental solution, defined on the upper sector $\Omega_1(\varepsilon)$. With respect to the fundamental solution on the lower sector $\Omega_2(\varepsilon)$ the monodromy matrix $\hat{M}_R(\varepsilon)$ is given by

$$\hat{M}_R(\varepsilon) = \hat{M}^{-1} M_R(\varepsilon) \hat{M},$$

where $M_R(\varepsilon)$ is the monodromy matrix, defined by (4.36).

Now, we can give the connection between the monodromy matrices and the unfolded Stokes matrices. Proposition 4.31 in [16] states that the monodromy operator acting on the solution $\Phi_j(x, \varepsilon)$ decomposes into the Stokes operator multiplied, from the right, by the classical monodromy operator acting on branch of $F(x, \varepsilon)$. In [15] Theorem 32, Klimeš expresses in a remarkable way the acting of the monodromy operators on analytic extension of the solutions of the perturbed equation to the whole $\Omega_1(\varepsilon) \cup \Omega_2(\varepsilon)$ by the monodromy matrices $M_j(\varepsilon)$, unfolded Stokes matrices $St_j(\varepsilon)$ and the matrices $e^{\pi i (\Lambda+Q/\varepsilon)}$, $j = L, R$.

His formulas have been deduced provided that there is an agreement of the matrices $F(x)$ and $F(x, \varepsilon)$ on the right intersections $\Omega_R$ and $\Omega_L(\varepsilon)$. In the next proposition we reformulate (without giving a proof) his formulas, provided that the above agreement is on the left intersections (see for details and proof [15]).

**Proposition 5.1.** Let $M_j(\varepsilon)$ and $St_j(\varepsilon)$, $j = R, L$ be the monodromy matrices and the unfolded Stokes matrices of the perturbed equation with respect to the fundamental solution on the upper sector $\Omega_1(\varepsilon)$. Then on the upper sector $\Omega_1(\varepsilon)$ they satisfy the following relations

$$M_L(\varepsilon) = e^{\pi i (\Lambda + \frac{1}{\varepsilon L} Q)} St_L(\varepsilon), \quad M_R(\varepsilon) = St_R(\varepsilon) e^{\pi i (\Lambda + \frac{1}{\varepsilon R} Q)}.$$

On the lower sector $\Omega_2(\varepsilon)$ they satisfy the following relations

$$M_L(\varepsilon) = St_L(\varepsilon) e^{\pi i (\Lambda + \frac{1}{\varepsilon L} Q)}, \quad M_R(\varepsilon) = e^{\pi i (\Lambda + \frac{1}{\varepsilon R} Q)} St_R(\varepsilon).$$

Note that these relations are in concordance with the definition of the monodromy around $x = \infty$ for both equations. Indeed, from Proposition 5.1 it follows that on the lower sector $\Omega_2(\varepsilon)$

$$St_L(\varepsilon) St_R(\varepsilon) \hat{M} = M_L(\varepsilon) \hat{M}^{-1} M_R(\varepsilon) \hat{M} = M_L(\varepsilon) \hat{M}_R(\varepsilon) = M_{\infty}^{-1}(\varepsilon) = e^{2\pi i T_L} e^{2\pi i T_R} \hat{M}.$$
When $\sqrt{\varepsilon} \to 0$ the monodromy $M^{-1}(\varepsilon)$ around $x = \infty$ of the initial equation tends to $St_\pi St_0 \hat{M}$. Since the matrices $T_j$ are convergent under this limit, then the monodromy matrices $M^{-1}_\infty$ of the perturbed equation is well defined. Recall that the exponents $\rho_i^\infty$ at $x = \infty$ do not change under the perturbation. So the above phenomenon is expected. It is interesting to study if this connection remains valid under perturbation, that makes the characteristic exponents $\rho_i^\infty$ at $x = \infty$ dependent on the parameter of perturbation.

**Remark 5.2.** In fact Theorem 32 in [15] states that on the lower sector $\Omega_2(\varepsilon)$ the monodromy matrix $M_R(\varepsilon)$ is expressed as

$$
\hat{M}_R(\varepsilon) = e^{-\pi i(\Lambda + \frac{1}{2R}Q)} St_R(\varepsilon) \hat{M}.
$$

Using the relation $\hat{M}_R(\varepsilon) = \hat{M}^{-1} M_R(\varepsilon) \hat{M}$, we rewrite it as $M_R(\varepsilon) = e^{\pi i(\Lambda + Q/2R)} St_R(\varepsilon)$.

In a consequence of Proposition 5.1 and Theorem 4.9 we have the following relation.

**Proposition 5.3.** The unfolded Stokes matrices $St_j(\varepsilon)$ and the matrices $e^{2\pi iT_j}$, $j = L, R$ satisfy the following relation

$$
St_L(\varepsilon) = e^{2\pi i T_L}, \quad St_R(\varepsilon) = e^{2\pi i T_R}.
$$

**Proof.** From Proposition 5.1 we have that

$$
M_R(\varepsilon) = St_R(\varepsilon) e^{\pi i(\Lambda + \frac{1}{2R}Q)}.
$$

On other hand the monodromy matrix $M_R(\varepsilon)$, given by (4.36), is

$$
M_R(\varepsilon) = e^{2\pi i T_R} e^{\pi i(\Lambda + \frac{1}{2R}Q)}.
$$

Then comparing the both expressions for $M_R(\varepsilon)$ we obtain the relation between $St_R(\varepsilon)$ and $e^{2\pi i T_L}$. In the same manner one have the direct relation between $St_L(\varepsilon)$ and $e^{2\pi i T_R}$. □

It turns out that the matrices $T_j$, $j = L, R$ convergent when $\sqrt{\varepsilon} \to 0 \in \mathbb{R}_+$. The next preliminary lemma deal with the limits of the numbers $d_3^R$ and $d_2^L$, obtained in Theorem 4.1.4 and Theorem 4.1.12

**Lemma 5.4.** Assume that $\nu \in \mathbb{R}$ is fixed. Then the numbers $d_2^L$ and $d_3^R$ derived in Theorem 4.1.4 and Theorem 4.1.12 satisfy the following limits

$$
\lim_{1/\sqrt{\varepsilon} \to +\infty} d_3^R = -\frac{1}{2\Gamma(\nu)}, \quad \lim_{1/\sqrt{\varepsilon} \to +\infty} d_2^L = -\frac{e^{-\pi i \nu}}{\Gamma(\nu)}.
$$

**Proof.** Let us represent $d_3^R$ as

$$
d_3^R = -\frac{1}{2\Gamma(\nu)} z^{1-\nu} \frac{\Gamma(z + \frac{\nu}{2})}{\Gamma(z)} z^{\frac{\nu}{2}} \frac{\Gamma(z + \frac{1}{2})}{\Gamma(z - \frac{\nu}{2} + 1)} z^{-1 + \frac{\nu}{2}},
$$

where $z := 1/2\sqrt{\varepsilon}$. Then the statement follows from the limit (see [2] formula 1.18(5))

$$
\lim_{|z| \to \infty} \frac{\Gamma(z + \alpha)}{\Gamma(z)} z^{\alpha} = 1.
$$

In the same manner one can prove the statement for $d_2^L$. □

Note that for non-resonant values of the parameter $\sqrt{\varepsilon}$ the matrices

$$
e^{\pi i (\Lambda + \frac{1}{2} Q)} = \begin{pmatrix} e^{\pm \pi i \nu} & 0 & 0 \\ 0 & e^{\pi i (\nu - 2\pm \frac{1}{2} \nu)} & 0 \\ 0 & 0 & e^{\pi i (\nu - 4)} \end{pmatrix}$$
will be divergent when $\sqrt{\varepsilon} \to 0$. But during a logarithmic resonance of type (B) and (C) these matrices stay constant
\[ e^{\pi i (\Lambda + \frac{1}{2} Q)} = \text{diag} \left( e^{-\pi i \nu}, e^{-\pi i \nu}, e^{\pi i \nu} \right), \quad e^{\pi i (\Lambda + \frac{1}{2} R)} = \text{diag} \left( e^{\pi i \nu}, e^{3\pi i \nu}, e^{\pi i \nu} \right), \]
because of the relation $1/2\sqrt{\varepsilon} - \nu/2 \in \mathbb{N}$. So, in these cases the limits $\lim_{\sqrt{\varepsilon} \to 0} M_j(\varepsilon)$, $j = L, R$ exist. That is why taking values of $\sqrt{\varepsilon}$ for which these matrices stay constant is a good idea.

Now we can state the main result of this paper.

**Theorem 5.5.** Assume that $\nu \in \mathbb{R}$ is fixed. Assume also that $1/\sqrt{\varepsilon} - \nu \in 2\mathbb{N}$. Then
\[ e^{2\pi i T_L} \to St_\pi, \quad e^{2\pi i T_R} \to St_0, \]
when $\sqrt{\varepsilon} \to 0$.

**Proof.** From Theorem 4.25 of [16] it follows that the unfolded Stokes matrices $St_L(\varepsilon)$ and $St_R(\varepsilon)$ tend to the Stokes matrices $St_\pi$ and $St_0$ of the initial equation when $\sqrt{\varepsilon} \to 0$. Then from Proposition 5.3 and the symmetry (4.25) of the perturbed equation it follows that the matrices $e^{2\pi i T_L}$ and $e^{2\pi i T_R}$ tend to the Stokes matrices $St_\pi$ and $St_0$ when $\sqrt{\varepsilon}$ tends to 0. In particular, thanks to Lemma 5.4 we have that
\[ e^{2\pi i T_L} \to \begin{pmatrix} 1 & -2\pi i e^{-\pi i \nu} \Gamma(\nu) \varepsilon \\ 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} = St_\pi \]
and
\[ e^{2\pi i T_R} \to \begin{pmatrix} 1 & 0 & -\pi i \Gamma(\nu) \varepsilon \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = St_0, \]
when $\sqrt{\varepsilon} \to 0$. The latter confirms one more time the statement of the theorem. This end the proof. \(\square\)

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