HEAT KERNELS FOR TIME-DEPENDENT NON-SYMMETRIC
STABLE-LIKE OPERATORS

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ABSTRACT. When studying non-symmetric nonlocal operators
\[ L_f(x) = \int_{\mathbb{R}^d} \left( f(x + z) - f(x) - \nabla f(x) \cdot z1_{|z| \leq 1} \right) \frac{\kappa(x, z)}{|z|^{d+\alpha}} dz, \]
where \( 0 < \alpha < 2 \) and \( \kappa(x, z) \) is a function on \( \mathbb{R}^d \times \mathbb{R}^d \) that is bounded between two positive constants, it is customary to assume that \( \kappa(x, z) \) is symmetric in \( z \). In this paper, we study heat kernel of \( L \) and derive its two-sided sharp bounds without the symmetric assumption \( \kappa(x, z) = \kappa(x, -z) \). In fact, we allow the kernel \( \kappa \) to be time-dependent and also derive gradient estimate when \( \beta \in (0 \lor (1 - \alpha), 1) \) as well as fractional derivative estimate of order \( \theta \in (0, (\alpha + \beta) \land 2) \) for the heat kernel, where \( \beta \) is the Hölder index of \( x \rightarrow \kappa(x, z) \). Moreover, when \( \alpha \in (1, 2) \), the drift perturbation with drift in Kato’s class is also considered. As an application, when \( \kappa(x, z) = \kappa(z) \) does not depend on \( x \), we show the boundedness of nonlocal Riesz’s transform:
\[ \| L^{1/2} f \|_p \asymp \| \Gamma(f) \|_p, \]
where \( \Gamma(f) := \frac{1}{2} L(f^2) - f L f \) is the carré du champ operator associated with \( L \), and \( L^{1/2} \) is the square root operator of \( L \) defined by using Bochner’s subordination. Here \( \asymp \) means that both sides are comparable up to a constant multiple.

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1. INTRODUCTION

For \( \alpha \in (0, 2) \), we consider the following nonlocal and non-symmetric operator:
\[ L^\alpha_f(x) := \int_{\mathbb{R}^d} \left( f(x + z) - f(x) - z^{(\alpha)} \cdot \nabla f(x) \right) \frac{\kappa(t, x, z)}{|z|^{d+\alpha}} dz, \]
where
\[ z^{(\alpha)} := (1_{\alpha \in (1, 2)} + 1_{\alpha \in (0, 1)} 1_{\alpha = 1}) z, \]
and \( \kappa(t, x, z) : \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R} \) satisfies that for some \( \kappa_0 > 1, \beta \in (0, 1) \) and \( \beta' \geq 0, \)
\[ \kappa_0^{-1} \leq \kappa(t, x, z) \leq \kappa_0, \quad |\kappa(t, x, z) - \kappa(t, y, z)| \leq \kappa_0 |x - y|^\beta (1 + |z|^{\beta'}), \]
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and for $\alpha = 1$, 
$$
\int_{R_0 < |z| < R_1} \frac{z\kappa(t, x, z)}{|z|^{d+1}} \, dz = 0 \quad \text{for any } 0 < R_0 < R_1 < \infty.
$$
(1.4)

Note that this condition is equivalent to
$$
\int_{|z| < r} z\kappa(t, x, z) \, dz = 0 \quad \text{for every } r > 0.
$$
(1.5)

The reason we use $z^{(\alpha)}$ instead of the more common $z 1_{\{|z|\leq 1\}}$ in the first order correction term in (1.1) together with condition (1.4) is that this is the form for general $\alpha$-stable Lévy processes on $\mathbb{R}^d$ when $\kappa(t, x, z)$ is a constant.

When $\kappa(t, x, z) = \kappa(x, z)$ is time-independent and $\kappa(x, z) = \kappa(x, -z)$, the heat kernel of $\mathcal{L}^x_t$ has been constructed in [6]. Moreover, sharp two-sided estimates, gradient estimate and fractional derivative estimate of the heat kernel are obtained [6]. The main goal of this paper is to drop the symmetric assumption in $z$ and extend it to time-dependent case at the same time. It in particular gives sharp two-sided heat kernel estimates to non-symmetric $\alpha$-stable processes whose Lévy measure is comparable to that of isotropic $\alpha$-stable process on $\mathbb{R}^d$. The study of heat kernels and their estimates is an active research area in analysis and in probability theory. We refer the reader to the Introduction of [6] for a brief history on the study of heat kernels for nonlocal operators.

To state our main results, we introduce the following notations for later use: for $\beta \geq 0$ and $\gamma \in \mathbb{R}$, 
$$
\xi(t, x) := \frac{t^{\gamma/\alpha} |x|^\beta \wedge 1}{(t^{\gamma/\alpha} + |x|)^{d+\gamma}},
$$
and for any $T \in (0, \infty]$ and $\varepsilon \in [0, T)$, 
$$
\mathbb{D}_\varepsilon^T := \{(t, s, x, y) : x, y \in \mathbb{R}^d \text{ and } s, t \geq 0 \text{ with } \varepsilon < s - t < T\}.
$$

It is well known that the transition density function $p(t, x)$ for an isotropic $\alpha$-stable process on $\mathbb{R}^d$ has the property 
$$
p(t, x) \equiv \xi(0, t, x) \quad \text{for all } t > 0 \text{ and } x \in \mathbb{R}^d.
$$

Here and below $\approx$ means that both sides are comparable up to a constant multiple.

In this paper, we show the following.

**Theorem 1.1.** Under (1.3) and (1.4), there is a unique continuous function $p^\kappa_{t, s}(x, y)$ on $\mathbb{D}_0^\alpha$ (called the fundamental solution or heat kernel of $\mathcal{L}^x_t$) satisfying 
$$
\partial_t p^\kappa_{t, s}(x, y) + \mathcal{L}^x_t p^\kappa_{t, s}(\cdot, y)(x) = 0 
$$
(1.6)

for Lebesgue almost all $t \in [0, s]$, and

(i) (Uniform continuity) For any bounded and uniformly continuous function $f(x)$,
$$
\lim_{|y| \to 0} \|P^\kappa_{t, s}f - f\|_{\infty} = 0, 
$$
(1.7)

where $P^\kappa_{t, s}f(x) := \int_{\mathbb{R}^d} p^\kappa_{t, s}(x, y)f(y) \, dy$. 


(ii) There exists some $\theta > \alpha$ so that $(t, x) \mapsto \Delta^{\theta/2} P^\kappa_{t,s} f(x)$ is bounded and continuous on $[0, s - \varepsilon] \times \mathbb{R}^d$ for every $s > \varepsilon > 0$.

Moreover, $p^\kappa_{t,s}(x,y)$ enjoys the following properties:

(iii) (Two-sided estimate) For any $T > 0$, there is a constant $c_0 = c_0(d, \alpha, \beta, \beta', \kappa_0) > 1$ such that on $\mathbb{D}_0^T$,

$$c_0^{-1} \varrho_\alpha^0(s-t,x-y) \leq p^\kappa_{t,s}(x,y) \leq c_0 \varrho_\alpha^0(s-t,x-y).$$

(iv) (Fractional derivative estimate) There exists an $\varepsilon = \varepsilon(\beta, \beta', \alpha) \in (0, 2 - \alpha)$ such that for any $\theta \in [0, \alpha + \varepsilon)$, $(t, x) \mapsto \Delta_x^{\varepsilon/2} p^\kappa_{t,s} (\cdot, y)(x)$ is continuous on $[0, s) \times \mathbb{R}^d$, and for any $T > 0$, there is a constant $c_1 > 0$ such that on $\mathbb{D}_0^T$,

$$|\Delta_x^{\varepsilon/2} p^\kappa_{t,s} (\cdot, y)(x)| \leq c_1 \varrho_{\alpha-\varepsilon}^0(s-t,x-y).$$

(v) (Gradient estimate) If $\alpha \in [1, 2)$ or $\alpha + \beta > 1$ and $\beta' = 0$ in (1.3), then for any $T > 0$, there is a constant $c_2 > 0$ such that on $\mathbb{D}_0^T$,

$$|\nabla_x \log p^\kappa_{t,s} (\cdot, y)(x)| \leq c_2 (s-t)^{-1/\alpha}.$$

(vi) (Conservativeness) For every $0 < s < t$ and $x \in \mathbb{R}^d$,

$$\int_{\mathbb{R}^d} p^\kappa_{t,s}(x,y)dy = 1.$$

(vii) (C-K equation) For all $0 < t < r < s < \infty$ and $x, y \in \mathbb{R}^d$, the following Chapman-Kolmogorov equation holds:

$$\int_{\mathbb{R}^d} p^\kappa_{r,t}(x,z)p^\kappa_{r,s}(z,y)dz = p^\kappa_{t,s}(x,y).$$

(viii) (Generator) For any $f \in C^2_b(\mathbb{R}^d)$, we have

$$P^\kappa_{t,s} f(x) - f(x) = \int_t^s P^\kappa_{t,r} \mathcal{L}^\kappa_{r,t} f(x)dr = \int_t^s \mathcal{L}^\kappa_{r,s} P^\kappa_{t,r} f(x)dr.$$

Remark 1.2.  
(i) It should be noticed that in (1.9), the differentiability index $\theta$ could be greater than $\alpha$. From the proof below, one sees that if $\beta' = 0$ in (1.3), then we can take $\varepsilon = (2 - \alpha) \wedge \beta$ in (iv) of Theorem 1.1.

(ii) It follows from Theorem 1.1 that the fundamental solution $p^\kappa_{t,s}(x,y)$ uniquely determines a time-inhomogeneous Feller process $X := \{X_t^{t,x}, s \geq 0; \mathbb{P}^{s,x}, s \geq 0, x \in \mathbb{R}^d\}$ that has $p^\kappa_{t,s}(x,y)$ as its transition density function with respect to the Lebesgue measure on $\mathbb{R}^d$. Clearly, by Theorem 1.1(iii), the probability law of $X$ solves the martingale problem for $(\mathcal{L}^\kappa_{r,t}, C^\infty_c(\mathbb{R}^d))$ in the sense that for every $f \in C^\infty_c(\mathbb{R}^d)$ and $(t, x) \in [0, \infty) \times \mathbb{R}^d$,

$$M^f_s := f(X_s^{t,x}) - f(x) - \int_t^s \mathcal{L}^\kappa_{r,s} f(X_r^{t,x})dr; \quad s \geq t,$$

is an $\mathbb{P}^{t,x}$-martingale. Here $C^\infty_c(\mathbb{R}^d)$ is the space of smooth functions on $\mathbb{R}^d$ with compact support. This gives a constructive proof of the existence of solution to
the martingale problem \((\mathcal{L}_t^\kappa, C_0^\infty(\mathbb{R}^d))\). The existence and uniqueness of solutions to the martingale problem for \((\mathcal{L}_t^\kappa, C_0^\infty(\mathbb{R}^d))\) have been established in [11, Proposition 3] and [7, Theorem 4.6].

Next we consider the perturbation of \(\mathcal{L}_t^\kappa\) by a drift \(b\) belonging to some Kato’s class when \(\alpha \in (1, 2)\). First of all, we introduce the following Kato’s class as in [15].

**Definition 1.3.** For \(\alpha \in (1, 2)\), a Borel measurable function \(f : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}\) is said to be in Kato’s class \(\mathcal{K}_{\alpha, d}\) if

\[
\lim_{\varepsilon \downarrow 0} K_f^\alpha(\varepsilon) = 0,
\]

where

\[
K_f^\alpha(\varepsilon) := \sup_{(t, x) \in [0, \infty) \times \mathbb{R}^d} \int_0^\varepsilon \int_{\mathbb{R}^d} \frac{g_0^\alpha(s, x - y)|f(t + s, y)|}{s^{1/\alpha}(\varepsilon - s)^{1/\alpha}} dy ds.
\]

Here we have extended \(f\) to \(\mathbb{R}\) by setting \(f(t, \cdot) = 0\) for \(t < 0\).

**Remark 1.4.** By Hölder’s inequality, one sees that \(L_{t,loc}^q(\mathbb{R}_+; L^p(\mathbb{R}^d)) \subset \mathcal{K}_{\alpha, d}\) provided that \(\frac{d}{p} + \frac{\alpha}{q} < \alpha - 1\). In particular, \(\mathcal{K}_{\alpha, d}\) contains all bounded functions on \(\mathbb{R}_+ \times \mathbb{R}^d\).

When \(\alpha \in (1, 2)\) and \(b(t, x) \in \mathcal{K}_{\alpha, d}, \mathcal{L}_t^\kappa + b(t, x) \cdot \nabla\) can be viewed as the perturbation of \(\mathcal{L}_t^\kappa\) by \(b(t, x) \cdot \nabla\). Hence, heuristically, the fundamental solution \(p_{t,s}^{\infty}(x, y)\) of \(\mathcal{L}_t^\kappa + b(t, x) \cdot \nabla\) is related to \(p_{t,s}(x, y)\) of \(\mathcal{L}_t^\kappa\) by the following Duhamel’s formula: for any \(t < s\) and \(x, y \in \mathbb{R}^d\),

\[
p_{t,s}^{\infty}(x, y) = p_{t,s}(x, y) + \int_t^s \int_{\mathbb{R}^d} p_{r,s}(x, z)b(r, z)\nabla_z p_{r,t}(z, y) dz dr. \quad (1.14)
\]

The following result can be shown as in [2] and [15], while the uniqueness can be shown as in [4, Theorem 3.10] or [3, Theorem 1.1]. We omit the details.

**Theorem 1.5.** Let \(\alpha \in (1, 2)\) and \(b \in \mathcal{K}_{\alpha, d}\). Under \((1.3)\) and \((1.4)\), there is a unique continuous function \(p_{t,s}^{\infty}(x, y)\) on \(\mathbb{D}_0^\infty\) satisfying \((1.14)\) and that for every \(T > 0\), there is a constant \(c_0 > 1\) such that on \(\mathbb{D}_0^T\),

\[
|p_{t,s}^{\infty}(x, y)| \leq c_0 g_0^\alpha(s - t, x - y).
\]

Moreover, \(p_{t,s}^{\infty}(x, y)\) enjoys the following properties.

(i) **(Two-sided estimate)** For any \(T > 0\), there is a constant \(c_1 > 1\) such that on \(\mathbb{D}_0^T\),

\[
c_1^{-1} g_0^\alpha(s - t, x - y) \leq p_{t,s}^{\infty}(x, y) \leq c_1 g_0^\alpha(s - t, x - y).
\]

(ii) **(Conservativeness)** For every \(0 < s < t\) and \(x \in \mathbb{R}^d\),

\[
\int_{\mathbb{R}^d} p_{t,s}^{\infty}(x, y) dy = 1.
\]

(iii) **(C-K equation)** For all \(0 < t < r < s < \infty\) and \(x, y \in \mathbb{R}^d\), the following Chapman-Kolmogorov equation holds:

\[
\int_{\mathbb{R}^d} p_{t,r}^{\infty}(x, z)p_{r,s}^{\infty}(z, y) dz = p_{t,s}^{\infty}(x, y).
\]
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Remark 1.6. uniquely determines a time-inhomogeneous Feller process 0 can in fact establish the uniqueness of the martingale problem for (L

established in [11, Proposition 3] and [7, Theorem 4.6]. Using the approach in [5], one non-symmetric) Lévy process X

Theorem 1.7. Let α ∈ (0, 2) and L be defined as in (1.1) with κ(t, x, z) = κ(z). Suppose

where \( \kappa_0^{-1} \leq \kappa(z) \leq \kappa_0 \) and \( 1_{\alpha=1} \int_{|z|<r} \kappa(z)dz = 0 \) for every \( r > 0 \). (Uniformly continuity) For any bounded and uniformly continuous function \( f(x) \),

\[ \lim_{|x-s| \to 0} \| P_{t,s}^{x,b} f - f \|_{\infty} = 0. \]

(vi) (Gradient estimate) For any \( T > 0 \), there is a constant \( c_2 > 0 \) such that on \( \mathbb{D}_0^T \),

\[ |\nabla_x \log p_{t,s}^{x,b}(\cdot, y)(x)| \leq c_2(s-t)^{-\alpha}. \]

Remark 1.6. Under the condition of Theorem 1.5, the fundamental solution \( p_{t,s}^{x,b}(x, y) \) uniquely determines a time-inhomogeneous Feller process \( X := \{X_t^x; s \geq 0; \mathbb{P}^{x,s}, s \geq 0, x \in \mathbb{R}^d \} \) that has \( p_{t,s}^{x,b}(x, y) \) as its transition density function with respect to the Lebesgue measure on \( \mathbb{R}^d \). It follows from Theorem 1.5(iv) that the probability law of \( X \) solves the martingale problem for \((L_t^{x,b}, C_c^\infty(\mathbb{R}^d))\). When \( b \) is bounded, the existence and uniqueness of solutions to the martingale problem for \((L_t^{x,b}, C_c^\infty(\mathbb{R}^d))\) have been established in [11, Proposition 3] and [7, Theorem 4.6]. Using the approach in [5], one can in fact establish the uniqueness of the martingale problem for \((L_t^{x,b}, C_c^\infty(\mathbb{R}^d))\) for general \( b \in \mathbb{R}^d \).

Finally, as an application of heat kernel estimate obtained in Theorem 2.1 below, we have the following boundedness of nonlocal Riesz’s transform. We use \( L^{1/2} \) to denote the “square root” operator of \( L \) through Bochner’s subordination as follows. Suppose \( \kappa(t, x, z) = \kappa(z) \). Then \( L \) defined by (1.1) is the generator of a (possibly non-symmetric) Lévy process \( X_t \) with transition semigroup \( \{P_t; t \geq 0\} \). Let \( S_t \) be an independent \( \frac{1}{2} \)-subordinator. Clearly \( Y_t := X_{S_t} \) is again a Lévy process, whose generator we denote as \( L^{1/2} \). It is well-known (see [12, p.216, (32.11)]) that

\[ L^{1/2}f(x) = \frac{1}{2\Gamma(1/2)} \int_0^\infty (P_t f(x) - f(x)) t^{-3/2}dt. \] (1.15)

Theorem 1.7. Let \( \alpha \in (0, 2) \) and \( L \) be defined as in (1.1) with \( \kappa(t, x, z) = \kappa(z) \). Suppose

\[ \kappa_0^{-1} \leq \kappa(z) \leq \kappa_0 \quad \text{and} \quad 1_{\alpha=1} \int_{|z|<r} z \kappa(z)dz = 0 \quad \text{for every} \quad r > 0. \] (1.16)

Then for any \( f \in C_c^\infty(\mathbb{R}^d) \) and \( p > 2d/(d + 2\alpha) \),

\[ ||L^{1/2}f||_p \lesssim ||\Gamma(f)||^{1/2}_p, \] (1.17)

where \( \Gamma(f) := \frac{1}{2}(L(f^2) - fL f) = \int_{\mathbb{R}^d} (f(\cdot + z) - f(\cdot))^2 \kappa(z)/|z|^{d+\alpha}dz \).
Classical Riesz’s transform says that for any $p > 1$, there is a constant $c > 0$ such that (see \([14]\))
\[
\|\Delta^{1/2} f\|_p \asymp \|\nabla f\|_p.
\] (1.18)

Up to now, there are a large amount of literatures devoting to the study of various Riesz’s transformation. Here we only mention that P.A. Meyer \([9]\) used the probabilistic technique to prove (1.18), and consider the more general problem (called Meyer’s problem by now): determining whether for nice $f$,
\[
\|A^{1/2} f\|_p \asymp \|\Gamma(f)^{1/2}\|_p, \quad p \in (1, \infty),
\] (1.19)
where $A$ is an abstract symmetric Markov operator and $\Gamma(f) := \frac{1}{2}A(f^2) - fAf$ is the associated carré du champ operator (if it exists). Meyer \([10]\) established (1.19) for Ornstein-Uhlenbeck operator on Wiener space. Bakry \([1]\) later showed it holds for diffusion operators on Riemannian manifold under the condition that Ricci curvature is bounded from below. It seems to us that Theorem 1.7 is the first result on Riesz’s transform for non-symmetric and nonlocal operators. An interesting open problem is whether (1.17) holds for nonlocal operator $\mathcal{L}$ with spatial dependent kernel $\kappa(x, z)$. This is a quite challenging problem even for second order elliptic operators with variable coefficients, see, for example, \([13]\) and the references therein. We plan to investigate this problem in a future project.

The main part of this paper was reported at the IMS-China International Conference on Statistics and Probability held from June 28-July 2, 2017 at Nanning, China. At the time when we are finalizing this paper, we notice a preprint \([8]\) by Peng Jin just posted on arXiv, where heat kernels for time-independent $\mathcal{L}$ is studied using the approach from our previous work \([6]\). The results in \([8]\) overlap some of ours in Theorems 1.1 and 1.5 in the time-independent and bounded drift case.

2. Heat kernel estimates of $\mathcal{L}_t^\kappa$ with $\kappa(t, x, z) = \kappa(t, z)$

Throughout this section, we assume
\[
\kappa_0^{-1} \leq \kappa(t, z) \leq \kappa_0
\] (2.1)
and when $\alpha = 1$,
\[
\int_{R_0 < |z| < R_1} z \frac{\kappa(t, z)}{|z|^{d+1}} \, dz = 0 \quad \text{for every } 0 < R_0 < R_1 < \infty.
\] (2.2)

Let $N(dt, dz)$ be a time-inhomogenous Poisson random measure with intensity measure $\frac{\kappa(t, z)}{|z|^{d+1}} \, dz \, dt$. Define
\[
X_{t,s}^\kappa := \int_t^s \int_{\mathbb{R}^d} z \tilde{N}(dr, dz) + \int_t^s \int_{\mathbb{R}^d} (z - z^{(r)}) \frac{\kappa(r, z)}{|z|^{d+1}} \, dz \, dr,
\] (2.3)
where $\tilde{N}(dr, dz) := N(dt, dz) - \frac{dt \, dz}{|z|^{d+1}}$. By Itô’s formula, we have
\[
\mathbb{E} f(X_{t,s}^\kappa) = \mathbb{E} \int_t^s \mathcal{L}_r^\kappa f(X_{t,r}^\kappa) \, dr, \quad f \in C_0^2(\mathbb{R}^d).
\]
In particular, if we take \( f(x) = e^{ix} \), then one finds that the characteristic function of \( X^\kappa_{t,s} \) is given by
\[
\mathbb{E}e^{i\xi X^\kappa_{t,s}} = \exp \left\{ \int_{\mathbb{R}^d} \left( e^{i\xi z} - 1 - i\xi \cdot z^{(\alpha)} \right) \frac{\kappa(r, z)}{|z|^{d+\alpha}} \, dz \right\}.
\]
By the definition of \( z^{(\alpha)} \) and a change of variable, we conclude from the last display that for every \( \lambda > 0 \),
\[
\{ \lambda^{-1/\alpha} X^\kappa_{t_\lambda,t_\lambda}, s > t \} \text{ has the same distribution as } \{ X^\kappa_{t,s}, s > t \} \tag{2.4}
\]
when \( \alpha \neq 1 \), where \( \kappa(r, z) = \kappa(\lambda r, \lambda^{1/\alpha} z) \). This is the reason why we define \( z^{(\alpha)} \) in this way as \( 1.2 \). See Remark 2.8 below. The scaling property (2.4) holds for \( \alpha = 1 \) as well but under condition (2.2). By the change of variables, we can write
\[
\mathbb{E}e^{i\xi X^\kappa_{t,s}} = \exp \left( (s - t) \int_0^1 \int_{\mathbb{R}^d} \left( e^{i\xi z} - 1 - i\xi \cdot z^{(\alpha)} \right) \frac{\kappa(s - t, r, z)}{|z|^{d+\alpha}} \, dz \, dr \right). \tag{2.5}
\]
By (2.1), there is a constant \( c > 0 \) depending only on \( \kappa_0, d, \alpha \) so that
\[
|\mathbb{E}e^{i\xi X^\kappa_{t,s}}| \leq \exp \left( (s - t) \int_0^1 \int_{\mathbb{R}^d} (\cos(\xi \cdot z) - 1) \frac{\kappa(s - t, r, z)}{|z|^{d+\alpha}} \, dz \, dr \right) \leq e^{-c(s-t)|\xi|^p}.
\]
Hence, \( X^\kappa_{t,s} \) admits a continuous density \( p^\kappa_{t,s}(x) \) given by Fourier’s inverse transform
\[
p^\kappa_{t,s}(x) = \int_{\mathbb{R}^d} e^{-i\xi x} \mathbb{E}e^{i\xi X^\kappa_{t,s}} \, d\xi = \int_{\mathbb{R}^d} \mathbb{E}e^{i\xi (X^\kappa_{t,s} - x)} \, d\xi. \tag{2.6}
\]
Moreover, we also have
\[
\partial_t p^\kappa_{t,s}(x) + L^\kappa p^\kappa_{t,s}(x) = 0 \text{ for } s > t \text{ with } \lim_{s \downarrow t} p^\kappa_{t,s}(x) \, dx = \delta_0(\, dx),
\]
where the limit is taken in the weak sense.

The following is the main result of this section.

**Theorem 2.1.** Under (2.1) and (2.2), there is a constant \( c_0 > 1 \) only depending on \( \kappa_0, d, \alpha \) such that for all \( t < s \) and \( x \in \mathbb{R}^d \),
\[
c_0^{-1} G^0_{\kappa_0}(s - t, x) \leq p^\kappa_{t,s}(x) \leq c_0 G^0_{\kappa_0}(s - t, x). \tag{2.7}
\]

Notice that by (2.2) and (2.4),
\[
p^\kappa_{t,s}(x) = (s - t)^{-d/\alpha} p^\kappa_{0,1} ((s - t)^{-1/\alpha} x), \tag{2.8}
\]
where \( \tilde{\kappa}(r, z) := \kappa(s + (t - s)r, (t - s)^{1/\alpha} z) \). Thus, to prove (2.7), it suffices to show it for \( t = 0 \) and \( s = 1 \). The main point for us is to show that \( c_0 \) only depends on \( \kappa_0, d, \alpha \). Let \( \chi_1 \) and \( \chi_2 \) be the small and large jump parts of \( X^\kappa_{0,1} \) defined respectively by
\[
\chi_1 := \int_0^1 \int_{|z| < 1} z \tilde{N}(dr, dz) + \int_0^1 \int_{|z| < 1} (z - z^{(\alpha)}) \frac{\tilde{\kappa}(r, z)}{|z|^{d+\alpha}} \, dz \, dr,
\]
\[
\chi_2 := \int_0^1 \int_{|z| > 1} z \tilde{N}(dr, dz) + \int_0^1 \int_{|z| > 1} (z - z^{(\alpha)}) \frac{\tilde{\kappa}(r, z)}{|z|^{d+\alpha}} \, dz \, dr.
\]
Note that $\chi_1$ and $\chi_2$ are independent and have the characteristic functions

\[
\mathbb{E}e^{i\xi \chi_1} = \exp \left\{ \int_{|z| \leq 1} (e^{i\xi \cdot z} - 1 - i\xi \cdot z^{(a)}) \frac{\kappa(z)}{|z|^{(a)p+1}} \, dz \right\} =: e^{\Phi_1(\xi)},
\]

(2.9)

\[
\mathbb{E}e^{i\xi \chi_2} = \exp \left\{ \int_{|z| > 1} (e^{i\xi \cdot z} - 1 - i\xi \cdot z^{(a)}) \frac{\kappa(z)}{|z|^{(a)p+1}} \, dz \right\} =: e^{\Phi_2(\xi)},
\]

(2.10)

In particular, $\chi_1$ has a smooth density $p_1(x)$ with respect to the Lebesgue measure on $\mathbb{R}^d$. Since $X_{0,1}^\kappa$ is the independent sum of $\chi_1$ and $\chi_2$, we have

\[
p_{0,1}(x) = \mathbb{E}[p_1(x - \chi_2)].
\]

(2.11)

To show the two-sided estimate of $p_{0,1}(x)$, we prepare the following two lemmas.

**Lemma 2.2.** (i) For any $R > 0$, there is a $\delta > 0$ only depending on $R, \kappa_0, d, \alpha$ such that

\[
\inf_{x \in B_R} p_1(x) \geq \delta.
\]

(ii) For any integer $m \geq 1$, there is a constant $c = c(\kappa_0, d, \alpha, m) > 0$ such that

\[
p_1(x) \leq c(1 + |x|)^{-m}, \quad x \in \mathbb{R}^d.
\]

**Proof.** (i) Let $\chi_{11}$ and $\chi_{12}$ be two independent random variables with the characteristic functions

\[
\mathbb{E}e^{i\xi \chi_{11}} = \exp \left\{ \int_{|z| \leq 1} (e^{i\xi \cdot z} - 1 - i\xi \cdot z^{(a)}) \frac{\kappa(z)}{|z|^{(a)p+1}} \, dz \right\} =: e^{\Phi_{11}(\xi)},
\]

(2.12)

\[
\mathbb{E}e^{i\xi \chi_{12}} = \exp \left\{ \int_{|z| \leq 1} (e^{i\xi \cdot z} - 1 - i\xi \cdot z^{(a)}) \frac{\kappa(z)}{|z|^{(a)p+1}} \, dz \right\} =: e^{\Phi_{12}(\xi)},
\]

where $\kappa(z) := \int_0^1 \kappa(r, z) \, dr \geq \kappa_0$ by (2.1). Let $p_{11}$ and $p_{12}$ be the continuous distribution density of $\chi_{11}$ and $\chi_{12}$. Clearly, we have

\[
p_1(x) = \int_{\mathbb{R}^d} p_{11}(x - z)p_{12}(z) \, dz.
\]

(2.13)

Since $\chi_{12}$ is a truncated $\alpha$-stable random variable, it is well known that $p_{12}$ is strictly positive. On the other hand, by (2.12) we also have

\[
\mathbb{E}[\chi_{11}] \leq c(\kappa_0, p, \alpha).
\]

Hence, by (2.13), we have for any $R' > 0$,

\[
p_1(x) = \int_{\mathbb{R}^d} p_{11}(x - z)p_{12}(z) \, dz \geq \inf_{z \in B_{R'}} p_{12}(z) \int_{|z| \leq R'} p_{11}(x - z) \, dz
\]

\[
= \inf_{z \in B_{R'}} p_{12}(z) \left( 1 - \mathbb{P}(|\chi_{11} - x| > R') \right)
\]

\[
\geq \inf_{z \in B_{R'}} p_{12}(z) \left( 1 - \mathbb{E}[|\chi_{11} - x|]/R' \right),
\]

which yields (i) by choosing $R'$ large enough.
Lemma 2.3. For any $R > 2$, there is a constant $c_1 = c_1(R, \kappa_0, d, \alpha) > 0$ such that

\[ c_1^{-1} (1 + |x|)^{-d-\alpha} \leq \mathbb{P}(\chi_2 \in B_R(x)) \leq c_1 (1 + |x|)^{-d-\alpha}. \tag{2.14} \]

**Proof.** Observe that by (2.10),

\[ \mathbb{E} e^{i \xi \cdot \chi_2} = \exp \left( \int_{\mathbb{R}^d} (e^{i \xi \cdot z} - 1) \nu(dz) \right) \exp \left( -i \xi \cdot b \right), \]

where $\nu(dz) := 1_{|z| > 1} |z|^{-d-\alpha} \left( \int_0^1 \kappa(r, z) dr \right) dz$ and $b := \int_{\mathbb{R}^d} \xi \nu(dz)$. Let $\eta := \{\eta_n, n \in \mathbb{N}\}$ be a family of i.i.d. random variables in $\mathbb{R}^d$ with distribution $\nu/\lambda$, where

\[ \lambda := \nu(\mathbb{R}^d) \leq \kappa_0 \int |z|^{-d-\alpha} dz < \infty. \]

Let $S_0 = 0$ and $S_n := \eta_1 + \ldots + \eta_n$. Let $N$ be a Poisson random variable with parameter $\lambda$, which is independent of $\eta$. It is easy to see that

\[ S_N^{(d)} = \chi_2 + b. \]

Now, by definition we have

\[ \mathbb{P}(\chi_2 \in B_R(x)) = \mathbb{P}(S_N \in B_R(x + b)) = \sum_{n=1}^{\infty} \mathbb{P}(S_n \in B_R(x + b) \mid N = n) \]

\[ = \mathbb{E} e^{-\lambda} \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\mathbb{R}^d} 1_{\sum_{j=1}^n |z_j| \in B_R(x + b)} \nu(dz_1) \cdots \nu(dz_n). \]

When $|x + b| < R + 1$, the upper bound in (2.14) for $\mathbb{P}(\chi_2 \in B_R(x))$ trivially holds. Thus we assume that $|x + b| \geq R + 1$. Notice that $\sum_{j=1}^n z_j \in B_R(x + b)$ implies that there is at least one $i$ such that $|z_i| > (|x + b| - R)/n$. Hence,

\[ \mathbb{P}(\chi_2 \in B_R(x)) \leq e^{-\lambda} \sum_{n=1}^{\infty} \frac{1}{n!} \left( \int_{\mathbb{R}^d} 1_{\sum_{j=1}^n |z_j| \in B_R(x + b)} 1_{|z_i| > (|x + b| - R)/n} \nu(dz_1) \cdots \nu(dz_n) \right). \]

Recalling $\nu(dz_i) = 1_{|z_i| > 1} |z_i|^{-d-\alpha} \left( \int_0^1 \kappa(r, z_i) dr \right) dz_i$ and by (2.1), we get

\[ \mathbb{P}(\chi_2 \in B_R(x)) \leq \frac{\kappa_0}{(|x + b| - R)^d + \alpha} e^{-\lambda} \sum_{n=1}^{\infty} \frac{1}{n!} \left( \int_{\mathbb{R}^d} 1_{\sum_{j=1}^n |z_j| \in B_R(x + b)} \nu(dz_1) \cdots \nu(dz_n) \right) \]

\[ = \frac{\kappa_0}{(|x + b| - R)^d + \alpha} e^{-\lambda} |B_R| \sum_{n=1}^{\infty} \frac{n^{d+\alpha+1} \lambda^{n-1}}{n!}. \]
\[
\leq \frac{c_1(d, \kappa_0) \kappa_0}{|x + b - R|^{d + \alpha}} |B_R| \leq \frac{c_2(d, \kappa_0, \alpha, R)}{(1 + |x|)^{d + \alpha}}.
\]

where in the second equality we used the translation invariance property of the Lebesgue measure. On the other hand, for any \( x \in \mathbb{R}^d \), since \( R > 2 \),

\[
\begin{align*}
\mathbb{P}(\chi_2 \in B_R(x)) &\geq e^{-\lambda} \int_{\mathbb{R}^d} 1_{\{|z_1| \leq B_R(x + b)\}} \nu(dz_1) \\
&\geq \frac{\kappa_0^{-1} e^{-\lambda} B_R(x + b) \cap B_1^c}{(|x| + |b| + R)^{d + \alpha}} \\
&\geq \frac{c_3(d, \kappa_0, \alpha, R)}{(1 + |x|)^{d + \alpha}}.
\end{align*}
\]

Combining the above calculations, we get the desired estimate. \( \square \)

Now we can give

**Proof of Theorem 2.7** Our proof is adapted from [16]. Let \( R > 2 \). For the lower bound, by (i) of Lemma 2.2 we have

\[
\delta := \inf_{z \in B_R} p_1(z) > 0.
\]

Hence, by (2.11) and Lemma 2.3

\[
p_{0,1}(x) = \mathbb{E}[p_1(x - \chi_2)] \geq \delta \mathbb{P}(|x - \chi_2| \leq R) \geq \delta c_1^{-1}(1 + |x|)^{-d - \alpha}.
\]

For the upper bound, by (2.11) again, we have

\[
p_{0,1}^*(x) \leq \mathbb{E}
\begin{pmatrix}
p_1(x - \chi_2) 1_{|x - \chi_2| \leq |x|/2} \\
p_1(z)
\end{pmatrix}
+ \sup_{|z| > |x|/2} p_1(z). \tag{2.15}
\]

By (ii) of Lemma 2.2 we can choose \( N \)-points \( z_1, \ldots, z_N \in B_{|x|/2} \) such that

\[
B_{|x|/2} \subset \bigcup_{j=1}^N B_{\varepsilon}(z_j) \quad \text{and} \quad \sum_{j=1}^N \sup_{z \in B_{\varepsilon}(z_j)} p_1(z) \leq c_4,
\]

where \( c_4 \) only depends on \( \varepsilon, \kappa_0, d, \alpha \). Hence, by Lemma 2.3 we have

\[
\mathbb{E}
\begin{pmatrix}
p_1(x - \chi_2) 1_{|x - \chi_2| \leq |x|/2} \\
p_1(z)
\end{pmatrix}
\leq \sum_{j=1}^N \mathbb{E}
\begin{pmatrix}
p_1(x - \chi_2) 1_{x - \chi_2 \in B_{\varepsilon}(z_j)}
\end{pmatrix}
\leq \sum_{j=1}^N \sup_{z \in B_{\varepsilon}(z_j)} p_1(z) \mathbb{P}(|x - \chi_2 - z_j| \leq \varepsilon)
\leq c_1 \sum_{j=1}^N \sup_{z \in B_{\varepsilon}(z_j)} p_1(z)(1 + |x - z_j|)^{-d - \alpha}
\leq c_1 c_4 (1 + |x|/2)^{-d - \alpha},
\]

which together with Lemma 2.2(ii) yields (2.7) for \( t = 0 \) and \( s = 1 \). The proof is complete by (2.8). \( \square \)
Lemma 2.4. Under (2.1)-(2.2), for any \( \theta \in (0, 2) \), there is a constant \( c = c(\kappa_0, d, \alpha, \theta) > 0 \) such that for every \( s > t > 0 \) and \( x, x', z \in \mathbb{R}^d \),

\[
|p_{t,s}^\kappa(x) - p_{t,s}^\kappa(x')| \leq c(((s - t)^{-1/\alpha}|x - x'|) \wedge 1) (\epsilon_0^\alpha(s - t, x) + \epsilon_0^\alpha(s - t, x')),
\]

(2.16)

and

\[
|\nabla p_{t,s}^\kappa(x)| \leq c\epsilon_0^{\alpha-1}(s - t, x),
\]

(2.17)

and

\[
|\delta^\theta_{p_{t,s}^\kappa}(s - t, x; z)| \leq c\ell^\theta(((s - t)^{-1/\alpha}|x - x'|) \wedge 1) \ell^\theta((s - t)^{-1/\alpha}z)
\]

\[
\times (\epsilon_0^\alpha(s - t, x + z) + \epsilon_0^\alpha(s - t, x) + \epsilon_0^\alpha(s - t, x' + z) + \epsilon_0^\alpha(s - t, x')),
\]

(2.19)

where \( \ell^\theta(z) := 1_{\theta \in (1,2)}(|z|^2 \wedge |z|) + 1_{\theta = 1}(|z|^2 \wedge 1) + 1_{\theta \in (0,1)}(|z| \wedge 1) \).

Using this lemma, it is easy to derive by definition (see [6] Theorem 2.4]).

Lemma 2.5. Under (2.1)-(2.2), for any \( \theta \in (0, 2) \), there is a constant \( c = c(\kappa_0, d, \alpha, \theta) > 0 \) such that for all \( s > t > 0 \) and \( x, x' \in \mathbb{R}^d \),

\[
\int_{\mathbb{R}^d} |\delta^\theta_{p_{t,s}^\kappa}(s - t, x; z)| \cdot |z|^{-d-\theta} dz \leq c\epsilon_0^{\alpha-\theta}(s - t, x),
\]

(2.20)

and

\[
\int_{\mathbb{R}^d} |\delta^\theta_{p_{t,s}^\kappa}(s - t, x; z) - \delta^\theta_{p_{t,s}^\kappa}(s - t, x'; z)| \cdot |z|^{-d-\theta} dz \leq c(((s - t)^{-1/\alpha}|x - x'|) \wedge 1) (\epsilon_0^{\alpha-\theta}(s - t, x) + \epsilon_0^{\alpha-\theta}(s - t, x')).
\]

(2.21)

Next we show the continuous dependence of \( p_{t,s}^\kappa(x) \) with respect to \( \kappa \).

Lemma 2.6. Let \( \kappa \) and \( \tilde{\kappa} \) be two kernels satisfying (2.1)-(2.2) with the same constant \( \kappa_0 \). Let \( \alpha \in (0, 2) \) and \( \gamma \in [0, \alpha - 1) \) when \( \alpha \in (1, 2) \), \( \gamma \in [0, \alpha) \) if \( \alpha \in (0, 1) \). Assume that for some \( K > 0 \),

\[
|\kappa(t, z) - \tilde{\kappa}(t, z)| \leq K(|z|^\gamma + 1).
\]

(2.22)
Then for any $\theta \in (0, 2)$, there exists a constant $c = c(\kappa_0, d, \alpha, \theta) > 0$ such that

$$|\nabla^j p_{r,s}^\kappa(x) - \nabla^j p_{r,s}^\kappa(x)| \leq c K \xi_0^0(s - t, x), \quad j = 0, 1,$$

$$\int_{\mathbb{R}^d} |\delta_{q_{r,s}}^{(\theta)}(s - t, x; z) - \delta_{q_{r,s}}^{(\theta)}(s - t, x; z)| \cdot |z|^{-d-\theta} dz \leq c K \xi_0^{0-\theta}(s - t, x).$$

**Proof.** By the scaling property (2.8), it suffices to prove (2.23) and (2.24) for $s = 1$ and $t = 0$. The argument of proving (2.23) in [6, Theorem 2.5] strongly depends on the symmetry of $\kappa(z) = \kappa(-z)$. Here we provide a different proof. Noticing that by (2.5) and (2.6),

$$q_1(x) := p_{0,1}^{\lambda + (1-\lambda)\tilde{z}}(x) = \int_{\mathbb{R}^d} e^{-ix \cdot \xi} \exp \left\{ \int_{\mathbb{R}^d} (e^{i\zeta \cdot z} - 1 - i\xi \cdot z^{(a)} \frac{\lambda \zeta + (1-\lambda)\tilde{z} \cdot \tilde{z}}{\lambda^d + a} ) dz \right\} d\xi,$$

where $\kappa(z) := \int_0^1 \kappa(r, z) dr$ and $\tilde{z}(z) := \int_0^1 \tilde{z}(r, z) dr$. We claim that

$$\partial_\lambda q_1(x) = (\mathcal{L}^\lambda - \mathcal{L}^\tilde{z}) q_1(x).$$

In fact, by Fourier’s transform it suffices to show that

$$\partial_\lambda \hat{q}_1(\xi) = \hat{\mathcal{L}^\lambda} q_1(\xi) - \hat{\mathcal{L}^\tilde{z}} q_1(\xi).$$

Notice that

$$\mathcal{L}^\xi f(\xi) = \left( \int_{\mathbb{R}^d} (e^{i\zeta \cdot z} - 1 - i\xi \cdot z^{(a)} \frac{\lambda \zeta + (1-\lambda)\tilde{z} \cdot \tilde{z}}{\lambda^d + a} ) dz \right) \hat{f}(\xi)$$

and

$$\hat{q}_1(\xi) = \exp \left\{ \int_{\mathbb{R}^d} (e^{i\zeta \cdot z} - 1 - i\xi \cdot z^{(a)} \frac{\lambda \zeta + (1-\lambda)\tilde{z} \cdot \tilde{z}}{\lambda^d + a} ) dz \right\}.$$

The desired claim now follows. Now, by (2.25) and the assumption,

$$|p_{0,1}^\kappa(x) - p_{0,1}^\kappa(x)| = \int_0^1 \partial_\lambda q_1(x) d\lambda = \left| \int_0^1 \int_{\mathbb{R}^d} \delta_{q_{r,s}}^{(a)}(1, x; z) \frac{\kappa(z) - \tilde{z}(z)}{\lambda^d + a} dz d\lambda \right| \leq K \int_0^1 \int_{\mathbb{R}^d} |\delta_{q_{r,s}}^{(a)}(1, x; z)| \cdot |z|^{\gamma + 1} |z|^{d+\alpha} dz d\lambda.$$

If $\alpha \in (1, 2)$, since $\gamma \in [0, \alpha - 1)$, by definition we have

$$\int_{\mathbb{R}^d} |\delta_{q_{r,s}}^{(a)}(1, x; z)| \cdot |z|^{\gamma - d - \alpha} dz = \int_{\mathbb{R}^d} |\delta_{q_{r,s}}^{(1-a)}(1, x; z)| \cdot |z|^{\gamma - d - \alpha} dz.$$

If $\alpha = 1$, then

$$\int_{\mathbb{R}^d} |\delta_{q_{r,s}}^{(1)}(1, x; z)| \cdot |z|^{\gamma - d} dz \leq \int_{\mathbb{R}^d} |\delta_{q_{r,s}}^{(1-a)}(1, x; z)| \cdot |z|^{\gamma - d} dz + |\nabla q_1(x)| \int_{|z| < 1} |z|^{\gamma - d} dz.$$

If $\alpha \in (0, 1)$, then

$$\int_{\mathbb{R}^d} |\delta_{q_{r,s}}^{(a)}(1, x; z)| \cdot |z|^{\gamma - d - \alpha} dz = \int_{\mathbb{R}^d} |\delta_{q_{r,s}}^{(1-a)}(1, x; z)| \cdot |z|^{\gamma - d - \alpha} dz.$$
Thus, by (2.26) and (2.20), we obtain
\[ |p_k^{(C)}(x) - p_{0,1}^{(C)}(x)| \leq cK(\varepsilon_0^0(1, x) + \varepsilon_0^0(1, x)) = cK\varepsilon_0^0(1, x), \]
(2.27)
which gives (2.23) for \( j = 0 \). As for (2.23) with \( j = 1 \) and (2.24), it follows by the same argument as used in [6, Theorem 2.5].

\[ \square \]

**Remark 2.7.** The assumption (2.22) will be used to treat the more general kernel functions as in (1.3). For instance, \( \kappa(t, x, z) := \cos(x \cdot z) + 2 \) satisfies (1.3), but is not Hölder continuous in \( x \) uniformly with respect to \( z \).

**Remark 2.8.** The strong Markov process \( X_{t,s}^\kappa \) of (2.3) has infinitesimal generator
\[ \mathcal{L}_t^\kappa f(x) = \int_{\mathbb{R}^d} \left( f(x + z) - f(x) - \nabla f(x) \cdot z^{(\alpha)} \right) \frac{\kappa(t, z)}{|z|^{d+\alpha}} \, dz, \]
The reason we use \( z^{(\alpha)} \) instead of the more common \( z1_{|z|<1} \) in the first order correction term is that this is the form for general \( \alpha \)-stable Lévy processes. Suppose
\[ \tilde{X}_{t,s}^\kappa := \int_t^s \int_{\mathbb{R}^d} z\tilde{N}(dr, dz) + \int_t^s \int_{\{|z|>1\}} z \frac{\kappa(r, z)}{|z|^{d+\alpha}} \, dz \, dr, \]
(2.28)
which has infinitesimal generator
\[ \widetilde{\mathcal{L}}_t^\kappa f(x) := \int_{\mathbb{R}^d} \left( f(x + z) - f(x) - \nabla f(x) \cdot z1_{|z|<1} \right) \frac{\kappa(t, z)}{|z|^{d+\alpha}} \, dz. \]

Clearly,
\[ \tilde{X}_{t,s}^\kappa = X_{t,s}^\kappa + \int_t^s b(r) \, dr \quad \text{and} \quad \widetilde{\mathcal{L}}_t^\kappa = \mathcal{L}_t^\kappa + b(r) \cdot \nabla, \]
(2.29)
where
\[ b(r) = \begin{cases} -\int_{\{|z|<1\}} z \frac{\kappa(x, z)}{|z|^{d+\alpha}} \, dz & \text{if } \alpha < 1, \\ 0 & \text{if } \alpha = 1, \\ \int_{\{|z|>1\}} z \frac{\kappa(x, z)}{|z|^{d+\alpha}} \, dz & \text{if } \alpha > 1. \end{cases} \]
(2.30)

Denote by \( \tilde{p}_{t,s}^\kappa(x) \) the density function of \( \tilde{X}_{t,s}^\kappa \). Then by (2.29),
\[ \tilde{p}_{t,s}^\kappa(x) = p_{t,s}^\kappa \left(x - \int_t^s b(r) \, dr\right). \]

Thus under conditions (2.1)-(2.2), one can get two-sided estimates on \( \tilde{p}_{t,s}^\kappa(x) \) from that of \( p_{t,s}^\kappa(x) \). In particular, it follows from Theorem 2.1 that for every \( \alpha \in [1, 2) \), there is a constant \( c_0 > 1 \) depending only on \( \kappa_0, d \) and \( \alpha \) so that for every \( t < s \) and \( x \in \mathbb{R}^d \),
\[ c_0^{-1} \frac{s-t}{(s-t)^{1/\alpha} + |x|^{d+\alpha}} \leq \tilde{p}_{t,s}^\kappa(x) \leq c_0 \frac{s-t}{(s-t)^{1/\alpha} + |x|^{d+\alpha}}. \]

However, the above estimates in general fails for \( \tilde{p}_{t,s}^\kappa(x) \) when \( \alpha \in (0, 1) \) as the drift \( \int_t^s b(r) \, dr \) may not be controlled by \( (s-t)^{1/\alpha} \) when \( 0 < s-t \leq 1 \).

We close this section by giving a proof to Theorem 1.7.
Proof of Theorem 1.7. First note that by (1.15) and Fubini’s theorem,
\[
\mathcal{L}^{1/2} f(x) = \frac{1}{2\Gamma(1/2)} \int_{\mathbb{R}^d} (f(x) - f(z)) \left( \int_0^\infty p_t(z) r^{-3/2} \, dt \right) \, dz,
\]
where \(p_t(z)\) is the heat kernel of \(\mathcal{L}\) and so \(P_t f(x) = \int_{\mathbb{R}^d} f(z) p_t(x-z) \, dz\). Note also that \(\mathcal{L}^{1/2}\) is a Lévy-type operator with Lévy measure \(\tilde{\nu}(dz) := \frac{1}{2\Gamma(1/2)} \int_0^\infty p_t(z) r^{-3/2} \, dr \, dz\). By Theorem 2.1, one sees that
\[
\int_0^\infty p_t(z) r^{-3/2} \, dt = \int_0^\infty t/(t^{1/\alpha} + |z|^{d+\alpha}) r^{-3/2} \, dt = |z|^{-d-\alpha/2} \int_0^\infty r^{-1/2}(t^{1/\alpha} + 1)^{d+\alpha} \, dt.
\]
Hence
\[
\tilde{\nu}(dz) = \tilde{\kappa}(z) dz/|z|^{d+\alpha/2} \text{ with } \tilde{\kappa}(z) \equiv 1,
\]
and by [17] Corollary 4.4, for any \(p > 1\),
\[
\|\mathcal{L}^{1/2} f\|_p \approx \|\Delta^{\alpha/4} f\|_p. \tag{2.31}
\]
On the other hand, by [14], it is well known that for any \(p > 2d/(d+2\alpha)\),
\[
\|\Delta^{\alpha/4} f\|_p \approx \left\| \int_{\mathbb{R}^d} \frac{(f(\cdot + z) - f(\cdot))^2}{|z|^{d+\alpha}} \, dz \right\|_p. \tag{2.32}
\]
Moreover, by (1.16), it is clear that
\[
\|\Gamma(f)^{1/2}\|_p \approx \left\| \int_{\mathbb{R}^d} \frac{(f(\cdot + z) - f(\cdot))^2}{|z|^{d+\alpha}} \, dz \right\|_p. \tag{2.33}
\]
The desired estimate (1.17) follows by combining (2.31)-(2.33).
\[\square\]

3. Proof of Theorem 1.1

In this section we consider the space and time dependent nonlocal operator \(\mathcal{L}_t^x\) defined by (1.1), with the kernel function \(\kappa(t, x, z)\) satisfying conditions (1.3)-(1.4), and give a proof for Theorem 1.1. In order to emphasize the dependence on \(x\), we also write
\[
\mathcal{L}_t^x f(x) = \mathcal{L}_t^x f(x) = \int_{\mathbb{R}^d} \delta_x^{(\alpha)}(t, x; z) \kappa(t, x, z) |z|^{-d-\alpha} \, dz.
\]
We use Levi’s construction. For fixed \(y \in \mathbb{R}^d\), let \(\mathcal{L}_t^{x(y)}\) be the freezing operator
\[
\mathcal{L}_t^{x(y)} f(x) = \int_{\mathbb{R}^d} \delta_x^{(\alpha)}(t, x; z) \kappa(t, y, z) |z|^{-d-\alpha} \, dz.
\]
Let \(p_{t,x}^{y(x)}(y) := p_{t,x}^{x(y)}(x)\) be the heat kernel of operator \(\mathcal{L}_t^{x(y)}\), i.e.,
\[
\partial_t p_{t,x}^{y(x)}(x) + \mathcal{L}_t^{x(y)} p_{t,x}^{y(x)}(x) = 0, \quad \lim_{t \to 0} p_{t,x}^{y(x)}(x) = \delta_{(0)}(x), \tag{3.1}
\]
where \(\delta_{(0)}(x)\) denotes the usual Dirac function.
Now, we want to seek the heat kernel $p_{t,s}^x(x, y)$ of $\mathcal{L}_t^x$ with the following form:

$$p_{t,s}^x(x, y) = p_{t,s}^{(y)}(x - y) + \int_t^s \int_{\mathbb{R}^d} \partial_t p_{t,r}^{(z)}(x - z)q_{r,s}(z,y)dzdr. \quad (3.2)$$

The classical Levi’s method suggests that $q_{t,s}(x, y)$ solves the following integral equation:

$$q_{t,s}(x, y) = q_{t,s}^{(0)}(x, y) + \int_t^s \int_{\mathbb{R}^d} q_{t,r}^{(0)}(x, z)q_{r,s}(z, y)dzdr, \quad (3.3)$$

where

$$q_{t,s}^{(0)}(x, y) := (\mathcal{L}_t^{\kappa(x)} - \mathcal{L}_t^{\kappa(y)})p_{t,s}^{(y)}(x - y).$$

In fact, we formally have

$$\partial_t p_{t,s}^x(x, y) = \int_t^s \int_{\mathbb{R}^d} \partial_t p_{t,r}^{(z)}(x - z)q_{r,s}(z, y)dzdr - q_{t,s}(x, y) - \mathcal{L}_t^{\kappa(x)} p_{t,s}^{(y)}(x - y)$$

$$= - \int_t^s \int_{\mathbb{R}^d} \mathcal{L}_t^{\kappa(x)} p_{t,r}^{(z)}(x - z)q_{r,s}(z, y)dzdr - \mathcal{L}_t^{\kappa(x)} p_{t,s}^{(y)}(x - y)$$

$$= - \mathcal{L}_t^{\kappa(x)} p_{t,s}^x(x, y). \quad (3.4)$$

The main point for us is to make the above calculations rigorous. First of all, by (1.3), we have for any $\eta \in (0, 1),$

$$|\kappa(t, x, z) - \kappa(t, y, z)| \leq \kappa_0|x - y|^\beta(1 + |z|^\beta\eta).$$

Thus, in the following, without loss of generality, we may assume that

$$\beta' \in (0, \alpha - 1) \text{ if } \alpha \in (1, 2) \text{ and } \beta' \in (0, \alpha) \text{ if } \alpha \in (0, 1].$$

Noticing that by definition and (1.4),

$$q_{t,s}^{(0)}(x, y) = \int_{\mathbb{R}^d} \mathcal{L}_t^{\kappa(x)}(s - t, x - y, z) \kappa(t, x, z) - \kappa(t, y, z)|z|^{-d-\alpha}dz,$$

as in (2.27), we get by (1.3) and (2.20),

$$|q_{t,s}^{(0)}(x, y)| \leq c\bar{q}_0^\beta(s - t, x - y). \quad (3.5)$$

Hence, by Picard’s iteration, we can show (see [6, Theorem 3.1])

**Theorem 3.1.** For $n \in \mathbb{N}$, define $q_{t,s}^{(n)}(x, y)$ recursively by

$$q_{t,s}^{(n)}(x, y) := \int_t^s \int_{\mathbb{R}^d} q_{t,r}^{(0)}(x, z)q_{r,s}^{(n-1)}(z, y)dzdr. \quad (3.6)$$

Under (1.3) and (1.4), the series $q_{t,s}(x, y) := \sum_{n=0}^{\infty} q_{t,s}^{(n)}(x, y)$ is absolutely and locally uniformly convergent on $\mathbb{D}_0^{+\infty}$ and solves the integral equation (3.3). Moreover, $q_{t,s}(x, y)$ is jointly continuous in $\mathbb{D}_0^{+\infty}$, and has the following estimates: For any $T > 0$, there is a constant $c_1 = c_1(T, \kappa_0, d, \alpha, \beta) > 0$ so that on $\mathbb{D}_T^+$,

$$|q_{t,s}(x, y)| \leq c_1(\bar{q}_0^\beta + q_0^\beta)(s - t, x - y), \quad (3.7)$$
and for any $\gamma \in (0, \beta)$, there is a constant $c_2 = c_2(T, \kappa_0, d, \alpha, \beta, \gamma) > 0$ so that on $D_0^T$,

$$
|q_{t,s}(x, y) - q_{t,s}(x', y)| \leq c_2 \left| |x - x'|^{\beta - \gamma} \wedge 1 \right| \times \left( (\mathcal{E}_0^0 + \mathcal{E}_{\gamma - \beta}^\beta)(s - t, x - y) + (\mathcal{E}_0^0 + \mathcal{E}_{\gamma - \beta}^\beta)(s - t, x' - y) \right).
$$

(3.8)

Now using Lemmas 2.4 and 2.6 and as in [6] (see also [3]), we can make the calculations in (3.4) rigorous, and show that $p^k_{t,s}(x, y)$ defined by (3.2) has the properties stated in Theorem 1.1 except for (1.9) and (1.10). Note that in Theorem 1.1, property (ii) is implied by property (iv). Below we give a proof for (1.9) and (1.10), which are slight extensions of [6] Lemma 4.2.

**Proofs of (1.10) and (1.9).** We only show (1.9) since (1.10) is similar by replacing $\Delta^{1/2}$ with $\nabla$ and using (2.17) and (2.23). Let $\theta \in (0, (\alpha + \beta) \wedge 2)$. By (3.2), we have

$$
\Delta^{\theta/2} p_{t,s}^k(x, y) = \Delta^{\theta/2} p_{t,s}^{(y)}(x - y) + \int_{(s+t)/2}^s \int_{\mathbb{R}^d} \Delta^{\theta/2} p_{t,r}^{(z)}(x - z)q_{r,s}(z, y)dzdr
$$

$$
+ \int_t^{(s+t)/2} \int_{\mathbb{R}^d} \Delta^{\theta/2} p_{t,r}^{(z)}(x - z)(q_{r,s}(z, y) - q_{r,s}(x, y))dzdr
$$

$$
+ \int_t^{(s+t)/2} \left( \int_{\mathbb{R}^d} \Delta^{\theta/2} p_{t,r}^{(z)}(x - z)dz \right)q_{r,s}(x, y)dr
$$

$$
=: J_1 + J_2 + J_3 + J_4.
$$

For $J_1$, we have by (2.20)

$$I_1 \lesssim \mathcal{E}_{\alpha - \theta}^0(s - t, x - y).
$$

For $J_2$, by (2.20), (3.7) and [6] Lemma 2.1, we have

$$J_2 \lesssim \int_{(s+t)/2}^s \int_{\mathbb{R}^d} \mathcal{E}_{\alpha - \theta}^0(r - t, x - z)(\mathcal{E}_0^0 + \mathcal{E}_{\beta - \theta}^\beta)(s - r, z - y)dzdr \lesssim \mathcal{E}_{\alpha - \theta}^0(s - t, x - y).
$$

For $J_3$, we get by (2.20), (3.8) and [6] Lemma 2.1 that, for any $\gamma \in (0, (\alpha + \beta - \theta) \wedge \beta)$,

$$J_3 \lesssim \int_t^{(s+t)/2} \int_{\mathbb{R}^d} \mathcal{E}_{\alpha - \theta}^{\beta - \gamma}(r - t, x - z)(\mathcal{E}_0^0 + \mathcal{E}_{\gamma - \theta}^\beta)(s - r, z - y)dzdr
$$

$$+ \int_t^{(s+t)/2} \int_{\mathbb{R}^d} \mathcal{E}_{\alpha - \theta}^{\beta - \gamma}(r - t, x - z)(\mathcal{E}_0^0 + \mathcal{E}_{\gamma - \theta}^\beta)(s - r, x - y)dzdr
$$

$$\lesssim (\mathcal{E}_{\alpha + \beta - \theta}^0 + \mathcal{E}_{\alpha + \beta - \theta}^\beta + \mathcal{E}_{\alpha - \theta}^\beta)(s - t, x - y)
$$

$$+ \int_t^{(s+t)/2} (r - t)^{(\beta - \gamma - \theta)/\alpha}(\mathcal{E}_0^0 + \mathcal{E}_{\gamma - \theta}^\beta)(s - r, x - y)dr
$$

$$\lesssim \mathcal{E}_{\alpha - \theta}^0(s - t, x - y).
$$

For $J_4$, noticing that by (2.24),

$$\left| \int_{\mathbb{R}^d} \Delta^{\theta/2} p_{t,r}^{(z)}(x - z)dz \right| = \left| \int_{\mathbb{R}^d} (\Delta^{\theta/2} p_{t,r}^{(z)} - \Delta^{\theta/2} p_{t,r}^{(x)})(x - z)dz \right|
$$

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\[ \int_{\mathbb{R}^d} \mathcal{G}^\beta_{\alpha-\theta}(r-t, x-z)dz \lesssim (r-t)^{(\beta-\theta)/\alpha}, \]

we have by (3.7),

\[ I_4 \lesssim \int_t^{(s+t)/2} (r-t)^{(\beta-\theta)/\alpha}(\xi_0^\beta + \xi_0^\theta)(s-r, x-y)dr \lesssim \xi_0^\beta(s-t, x-y). \]

Combining the above calculations, we get (1.9). \( \Box \)

**Proof of Uniqueness.** We now show the uniqueness. Let \( \tilde{p}_{t,s}^\varkappa(x, y) \) be another continuous function on \( D_0^\infty \) satisfying (1.6)-(1.7). For \( f \in C_c(\mathbb{R}^d) \), a continuous function with compact support, let \( \tilde{P}_{t,s}^\varkappa f(x) := \int_{\mathbb{R}^d} p_{t,s}^\varkappa(x, y) f(y)dy \). By (1.6) and (1.7), one sees that \( \tilde{u}(t, x) := \tilde{P}_{t,s}^\varkappa f(x) \) solves the following equation

\[ \partial_t \tilde{u} + \mathcal{L}^\varkappa_t \tilde{u} = 0 \quad \text{with} \quad \lim_{t \uparrow s} \| \tilde{u}(t) - f \|_\infty = 0. \]

Note that by (i)-(ii) of Theorem 1.1 \( u(t, x) := P_{t,s}^\varkappa f(x) \) has the same property as that for \( \tilde{u}(t, x) \). Hence by the maximum principle (can be proved in a similar way as that for [3, Theorem 6.1]), we have \( \tilde{u}(t, x) = u(t, x) \); that is, \( P_{t,s}^\varkappa f(x) = \tilde{P}_{t,s}^\varkappa f(x) \). This implies \( p_{t,s}^\varkappa(x, y) = \tilde{p}_{t,s}^\varkappa(x, y) \). \( \Box \)

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