Gravitational lens under perturbations: Symmetry of perturbing potentials with invariant caustics

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doi:10.1111/j.1365-2966.2004.08581.x

ABSTRACT
When the gravitational lensing potential can be approximated by that of a circularly symmetric system affected by weak perturbations, it is found that the shape of the resulting (tangential) caustics is entirely specified by the local azimuthal behaviour of the affecting perturbations. This provides a common mathematical groundwork for understanding problems such as the close-wide ($d \leftrightarrow d^{-1}$) separation degeneracy of binary lens microlensing lightcurves and the shear-ellipticity degeneracy of quadruple image lens modelling.

Key words: gravitational lensing – methods: analytical

1 INTRODUCTION
The recent announcement by Bond et al. (2004) of the most convincing planetary microlensing event (Mao & Paczyński 1991) to date is one of the greatest success stories of the nearly two-decade old promise of microlensing (Paczyński 1986). However, despite such successes, the study of microlensing is still hampered by various hurdles, one of the most persistent being the problem of degenerate lightcurves (e.g., Dominik 1999a; Afonso et al. 2000) – that is, several distinct physical systems can be used to model observed lightcurves.

For the purpose of formal discussion, the degeneracy of microlensing lightcurves may be categorized into two separate but related phenomena. The first, which may be termed as ‘external’ degeneracy, is caused by imperfect observations yielding highly correlated measurements of the parameters that control the observed lightcurves. For instance, in most highly blended events, without extremely good coverage of certain wing portions of the lightcurve, one only expects to measure the ratio of the time-scale to the peak magnification (or the product of the time-scale and the source flux) but not the parameters individually (c.f., Gould 1996). On the other hand, the second type of degeneracy, which will henceforth be referred to as ‘internal’ degeneracy, is due to the fact that the number of independent lightcurve-controlling parameters is smaller than the number of parameters that are used to describe the underlying lens systems. In other words, even with perfectly specified lightcurves (consequently, perfectly determined lightcurve parameters), the underlying lens system may not be uniquely recovered. This internal degeneracy of microlensing lightcurves can be further divided into two broad classes; the ‘intrinsic’ degeneracy among magnification structures from different systems, and different lens systems tracing identical or intrinsically degenerate paths in the given magnification structure (hereafter the ‘extrinsic’ degeneracy). The most widely acknowledged degeneracy of microlensing lightcurves, that is, the ‘time-scale degeneracy’ relating to the mass of the lens, the distances to the source and the lens, and the relative motions between the observer, the source, and the lens, is an example of the extrinsic degeneracy. Other examples of the extrinsic degeneracy are the discrete parallax degeneracies (e.g., Smith, Mao, & Paczyński 2003; Gould 2004). However, it is the intrinsic degeneracy that is of the most theoretical interest as it can be studied through the analysis of the lens equation as a whole without specified reference to the path of the system through the magnification structure or the observational conditions. Furthermore, understanding of the intrinsic degeneracy can greatly facilitate the identification of various extrinsic and external degeneracies. The most obvious example of an intrinsic degeneracy is the azimuthal symmetry as well as the radial similarity of the lensing magnification of the point mass lens. The well-known result of the ‘inverse-square-root’ behaviour of magnification ‘inside’ fold caustics (e.g., Gaudi & Petters 2002) is a less obvious example of a local intrinsic degeneracy.

It is notable that the intrinsic degeneracy is intimately related to the symmetry of the lens equation. In general, if (the potential of) the system possesses a certain symmetry, one can expect that there should exist an intrinsic degeneracy of magnification related to it. Note that the magnification is basically the second derivative of the potential. This also means that the study of the intrinsic degeneracy beyond the incidental case studies can be greatly systematized by recognizing the associated symmetry structure of the lens system. However, beyond

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the obvious example of the circularly symmetric lens, studies of degeneracies rooted in symmetries have been minimal, at best. This paper is one of the first attempts to understand certain intrinsic degeneracies of microlensing by searching for a symmetry in the lens system. It is found that this approach proves itself by providing a unified scheme of understanding the problem of a certain well-known microlensing degeneracy and a seemingly remote problem of lens modelling of a quadrupolar images.

The primary focus of this manuscript is a point mass lens (and a more general circularly symmetric lens in later sections) that is under the external influence of certain perturbations. It should be noted not only that the symmetry in such systems can be analysed with relative ease, but also that they can be applied to various realistic systems. In Section 2, the basic properties of the point mass lens are recapitulated, and in Section 3, a perturbative approach to gravitational lensing is developed. In particular, in section 4.1, the perturbative approach is used to derive an approximate expression for caustics and critical curves. Following this, in Section 4.2, it is shown that there exists a certain set of perturbations that yields invariant caustics, which may be seen as the main finding of this monograph. This finding is applied to interpret one of the well-known microlensing degeneracy problems in the next Section 5. The restrictions imposed on the studied lens system are subsequently relaxed in Section 6. For example, the discussion is extended from a point mass lens under perturbations to a general circularly symmetric lens system under perturbations in Section 6.1 and from external perturbations to perturbations associated with the system itself in Section 6.3. These two generalizations lay the basis for a study of the shear and ellipticity in lens modelling later in the same section.

In this paper, the lens equation formulated in terms of complex numbers is extensively used. In Appendix A, most of the mathematical terminologies regarding complex analysis found in this manuscript are summarized, and in Appendix B, the basic gravitational lens theory is redeveloped using complex-number notation. The treatments given there are minimal and deliberately casual. More in-depth treatments of complex analysis including rigorous definitions and proofs can be found in any standard text of complex analysis (e.g., Ahlfors 1979) or of mathematical methods (e.g., Arfken & Weber, 2000). Interested readers may also consult references regarding potential theory in a plane.

2 SCHWARZSCHILD LENS

When the lens system can be approximated by a Schwarzschild metric, one can write the lens equation in complex notation (see Appendix B) as

\[ \theta_s = \left( \frac{2R_{Sch}}{D_{rel}} \right)^{1/2}, \]  

(2.1)

which relates the apparent angular position of the lensed image \( z \) to the (would-be) angular position of the source \( \zeta \) in the absence of the lens. Note that, throughout this paper, the complex conjugate is represented with an overline (or upper bar) so that \( \bar{z} \) is the complex conjugate of \( z \). Here, the lens is located at the coordinate origin, and all the angular measurements are made in units of the angular Einstein ring radius, \( \theta_E \).

where \( R_{Sch} \equiv 2Gmc^2 \) is Schwarzschild radius of the lens mass \( m \), and \( D_{rel} \equiv D_{LS}^{-1}D_L D_S \) is the relative parallax distance, and \( D_S \) and \( D_L \) are the distances to the source and to the lens respectively, while \( D_{LS} \) is the distance from the source to the lens.

It is relatively straightforward, if not entirely trivial, to solve the lens equation (2.1) to find the image positions \( z \) for a given source position \( \zeta \). However, one can discover a few interesting properties of the lens equation (2.1) before actually solving it. For example, from the symmetry of the equation, one can immediately find that, if \( z_1 \) is an image position for a given source position, \( \bar{z}_2 = -\bar{z}_1 \) should also be an image position for the same source position (also note that \( \bar{z}_2^{-1} = -z_1 \)). In fact, it is easy to show that equation (2.1) always allows only two images for any given source position \( \zeta \) unless \( \bar{\zeta} = 0 \), in which case the equation (2.1) is reduced to the equation of a unit circle, \( |z| = 1 \). That is, the image becomes a ring (which is also naturally expected from the intrinsic circular symmetry of the system) with its radius given by equation (2.2). This ringed image is sometimes referred to as ‘Einstein ring.’

One important quantitative measure in gravitational lensing is the magnification factor of lensed images. In a purely mathematical sense, the lens equation defines a mapping from the image position to the source position, both of which are vectors in two dimensional spaces. The local – differential – behaviour of this mapping therefore can be studied from its Jacobian, which is basically a linear transformation that approximates the lens mapping locally. It follows naturally that the Jacobian determinant of any given lens mapping is the inverse magnification factor of the lensed image in the limit of a point source. If the lens mapping is given by the lens equation (2.1), one can easily show that the associated Jacobian determinant is that

\[ J_s = \partial_n \zeta \partial_n \bar{\zeta} - \partial_n \bar{\zeta} \partial_n \zeta = |\partial_n \zeta|^2 - |\partial_n \bar{\zeta}|^2 = 1 - \frac{1}{|z|^4}. \]  

(2.3)

Here, \( \partial_n \equiv (\partial / \partial \zeta) \) is simplified notation for the partial derivative operator. Equation (2.3) gives the inverse magnification for a single image. If one wants to find the total magnification, accounting for all images for a given source position, one needs to find the sum of the inverse absolute values of the Jacobian determinants corresponding to all the images. It is also notable that the Jacobian determinant given by

1 If \( D_{LS} = D_S - D_L \), then \( D_{rel}^{-1} = D_L^{-1} - D_S^{-1}. \)
3 SLOWLY-VARYING SMALL NULL-CONVERGENT PERTURBATION

Let us consider the situation in which a point mass lens is subject to some null-convergent – i.e., the continuous surface lens mass density is zero except finite number of isolated mass points – perturbations. Then, the perturbation part of the lensing potential \( \psi \) satisfies the two-dimensional Laplace equation \( \nabla^2 \psi = 0 \). Since the real and the imaginary parts of any complex analytic function are harmonic, there exists a complex analytic function \( \psi_0(z) \) whose real part is the same as \( \psi \) and whose imaginary part is a solution of first-order partial differential equations derived from the Cauchy-Riemann condition (see Appendix A). Since \( 2\psi(x, y) = \psi_0(z) + \bar{\psi}_0(\bar{z}) \) and \( 2\partial_0 \psi = \partial_x \psi + i\partial_y \psi \) (see Appendix A), where \( z = x + iy \), the resulting lens equation can be written in complex notation as

\[
\zeta = \zeta_i - \left[ \frac{e f(\bar{z})}{\bar{z}} + c \right],
\]

(3.1)

using a complex analytic function \( e f(z) = \psi(z) - \bar{c} \), where \( c \) is a complex constant (without the loss of generality, \( c \) may be taken as real). Note that here and throughout this paper, the use of primed symbols is exclusively reserved either for the complex total derivative of an analytic function or for the ordinary derivative of a real-valued single-real-variable function, with respect to their argument, while the argument will be dropped wherever there is little ambiguity.

If \( |e f(z)| \ll 1 \) for \( |z - z_0| \ll 1 \), where \( z_0 \) is the image position of a point mass lens satisfying \( \zeta + c = z_0 - z_0^{-1} \), one can find the image displacement \( \delta z = z - z_0 \) caused by the small perturbation of \( e f(\bar{z}) \) for a given source position, by a series expansion of equation (3.1) up to first order of both \( \delta z \) and \( e \),

\[
\delta z = \delta \zeta_0 \delta z + \partial_0(\zeta_i \delta z) - e \bar{f}(z_0) = \delta z + \frac{\delta z}{z_0} - e \bar{f}_0,
\]

(3.2)

where \( \bar{f}_0 = \bar{f}(z_0) = \bar{f}(\bar{z}_0) \). If \( \mathcal{J}_{\lambda,0} = 1 - |z_0|^{-4} \neq 0 \) (i.e., \( |z_0| \neq 1 \)), equation (3.2) is invertible for \( \delta z \). Setting \( \delta \zeta = 0 \), this inversion leads to

\[
\delta z = \frac{e}{\mathcal{J}_{\lambda,0}} \left( \frac{f_0 - \bar{f}_0}{\bar{z}_0} \right).
\]

(3.3)

That is, the new image forms where the perturbation is counterbalanced by the image of the local linear mapping that approximates the (unperturbed) lens mapping.

The Jacobian determinant of the lens mapping \( \mathcal{J}_{\lambda,0} \) is

\[
\mathcal{J} = 1 - \frac{1}{|z|^2} + \epsilon \left( \frac{f^*}{z^*} + \frac{f^*}{z} \right) - e^2 f^* f = \mathcal{J}_s + \epsilon \left( \frac{z^2 f^* + \bar{z}^2 \bar{f}^*}{|z|^2} \right) - e^2 |f|^2,
\]

(3.4)

where \( \mathcal{J}_s \) is the part of Jacobian determinant that maintains the same form \( \mathcal{J}_{\lambda,0} \) as the point mass lens. The change of the value of the Jacobian determinant with the perturbation is therefore understood as the sum of two contributions (up to the linear order); one due to the direct additional contribution from the perturbation \( (\epsilon\text{-term in eq.}\ (3.4)) \) and the other due to the change of value of \( \mathcal{J}_s \) at the new image position

\[
\delta(\mathcal{J}) = \partial_0(\mathcal{J}_{\lambda,0}) \delta z + \partial_0(\mathcal{J}_s) \delta z = \frac{2}{|z_0|^2} \left( \frac{\delta z}{z_0} + \frac{\delta z}{\bar{z}_0} \right).
\]

(3.5)

When the small-perturbation solution (eq. 3.3) is valid, one may further substitute the solution for \( \delta z \):

\[
\delta(\mathcal{J}) = \frac{2\epsilon}{(1 + |z_0|^2)|z_0|^2} \left( \frac{f_0}{z_0} + \frac{\bar{f}_0}{\bar{z}_0} \right) = \frac{2\epsilon}{|z_0|^2} \left( \frac{z_0 f_0 + \bar{z}_0 \bar{f}_0}{1 + |z_0|^2} \right).
\]

(3.6)

For this case, the final first order change of the Jacobian determinant with respect to the point mass lens case is given by the sum of the two terms

\[
\delta \mathcal{J} = \mathcal{J} - \mathcal{J}_{\lambda,0} = \frac{2\epsilon}{|z_0|^2} \left( \frac{z_0^2 f_0 + \bar{z}_0^2 \bar{f}_0}{2} + \frac{z_0 f_0 + \bar{z}_0 \bar{f}_0}{1 + |z_0|^2} \right),
\]

(3.7)

where \( z_0 \) is the position of the unperturbed image. In addition, if \( |\delta \mathcal{J}| \ll |\mathcal{J}_{\lambda,0}| = |1 - |z_0||^{-4} \), then the first order change of the magnification can be found by

\[
\delta A = p \frac{\delta \mathcal{J}}{\mathcal{J}_{\lambda,0}} = \frac{2peng}{2 - (|z_0|^2 + |z_0||z_0|^{-4})} \left( \frac{z_0^2 f_0 + \bar{z}_0^2 \bar{f}_0}{2} + \frac{z_0 f_0 + \bar{z}_0 \bar{f}_0}{1 + |z_0|^2} \right),
\]

(3.8)

where \( p \) is the parity of the image. Provided that the perturbation does not change the parity of the image, \( p = 1 \) if \( |z_0| > 1 \) and \( p = -1 \) if \( |z_0| < 1 \). Note that the effect from equation (3.8) is most likely to be negligible in practice. This is because \( \delta A \sim \epsilon \), and furthermore, for both of the most relevant regimes of equation (3.8), \( z_0^{-1}(\epsilon^{-1} \delta A) \to 0 \) as \( z_0 \to 0 \), and \( z_0(\epsilon^{-1} \delta A) \to 0 \) as \( z_0^{-1} \to 0 \).
3.1 caustic and critical curve

As \( |z_0| \to 1 \) and consequently \( J_{c,0} \to 0 \), the perturbative solution (eq. 3.3) grows and eventually diverges. That is, the linear perturbation becomes invalid or incomplete when the image for the point mass lens approaches the Einstein ring (the critical curve). This is because the local linear mapping (eq. 3.2) that approximates the lens mapping of the point mass lens becomes projective at the critical point. This implies that the general perturbative solutions at an arbitrary critical point require consideration of higher order effects. However, for the limited case in which the direction of the source displacement coincides with the projective axis, the solution can be obtained from the linear effect alone. Moreover, for the point mass lens, the caustic point is multiply degenerate, and therefore, for source positions near the lens position (= the caustic point), one can select a valid base point for the series expansion among any of the critical points on the Einstein ring that result in the source displacement along the projective axis.

Suppose that the lens equation (3.1) is series-expanded at \( z_0 = e^{\phi} \). Then, with \( \delta z = (\delta r + i \delta \phi)e^{\phi} \), the linearized lens equation becomes

\[
\zeta = (\delta r + i \delta \phi)e^{\phi} + \frac{(\delta r - i \delta \phi)e^{-i\phi}}{e^{-2i\phi}} - e\bar{f}(e^{-i\phi}) = 2\delta re^{\phi} - e\bar{f}(e^{-i\phi}),
\]

(3.9)

where \( \zeta = \delta \zeta \) because \( z_0 - z^{-1}_0 = 0 \). Since \( \delta r \in \mathbb{R} \), the linearized lens equation (3.9) has solutions if \( e^{-i\phi}[\zeta + e\bar{f}(e^{-i\phi})] \in \mathbb{R} \) or equivalently \( \phi \) is the solution of

\[
R(\phi) = \zeta - e^{2i\phi} \bar{\zeta} + e \left[ \bar{f}(e^{-i\phi}) - e^{2i\phi} f(e^{i\phi}) \right] = 0.
\]

(3.10)

In general, equation (3.10) allows multiple solutions in \([0, 2\pi])\). In principle, for any given expansion base point \( z_0 = e^{\phi} \), one can recover multiple solutions of \( \delta z \) corresponding to each of the solutions of equation (3.10) if higher order effects are considered.

From equations (3.4) and (3.5), the Jacobian determinant corresponding to equation (3.9) is

\[
J = \frac{2}{|z_0|^2} \left[ \frac{\delta z}{\delta z_0} + \frac{\delta \bar{z}}{\delta \bar{z}_0} + \frac{2}{\bar{z}_0} \bar{z}_0 f_0^* \right] = 4 \delta r + e \left[ c^{2i\phi} f(e^{i\phi}) + e^{-2i\phi} f(e^{-i\phi}) \right] = 2e^{-i\phi} \left[ \zeta - \zeta_2(\phi) \right],
\]

(3.11)

where

\[
\zeta_2(\phi) = -\frac{e}{2} \left[ c^{2i\phi} f(e^{i\phi}) + e^{-2i\phi} f(e^{-i\phi}) + 2e^{-i\phi} \bar{f}(e^{i\phi}) \right].
\]

(3.12)

Here, \( J_{c,0} - 1 - |z_0|^2 = 0 \) so that \( J = \delta J \), and also \( J \in \mathbb{R} \) if \( \phi \) is the solution of equation (3.10). Note that \( J \ll 1 \) for any null-convergent lens system so that equation (3.11) is only valid if \( |\zeta - \zeta_2(\phi)| \ll 1/2 \). The total magnification for the given source position can be obtained by adding the inverse of all Jacobian determinants corresponding to each solution of equation (3.10) in \([0, 2\pi])\):

\[
A(\zeta) = \sum_{\phi \in [0, 2\pi]} \frac{1}{2|\zeta - \zeta_2(\phi)|}.
\]

(3.13)

Now note that if there exists \( \phi \in [0, 2\pi) \) such that \( \zeta_2(\phi) = \zeta \), then \( J(\zeta) = 0 \) and therefore \( A(\zeta) \) diverges. In other words, \( |\zeta_2(\phi)| \in [0, 2\pi) \) defines the (linear approximation of) the caustics – that is, \( \zeta_2(\phi) \) is the parametric form of the (linear approximation of) the caustics. The (parameter for) cusp points formed along the caustics can be found by solving \( d\zeta_2 / d\phi = 0 \) for the parameter \( \phi \). Since

\[
\frac{d\zeta_2}{d\phi} = -\frac{e}{2} \left[ 3i c^{2i\phi} f(e^{i\phi}) + i e^{4i\phi} f'(e^{i\phi}) - i e^{-2i\phi} f(e^{-i\phi}) - i e^{-4i\phi} f(e^{-i\phi}) - 2i e^{-i\phi} \bar{f}(e^{i\phi}) \right]
\]

\[
= \frac{e^{2i\phi}}{2i} \left[ \left( 3 c^{2i\phi} f(e^{i\phi}) + e^{3i\phi} f'(e^{i\phi}) \right) - \left( 3 c^{-2i\phi} \bar{f}(e^{-i\phi}) + e^{-3i\phi} \bar{f}'(e^{-i\phi}) \right) \right]
\]

\[
= e^{2i\phi} \mathfrak{I} \left[ 3 c^{2i\phi} f(e^{i\phi}) + e^{3i\phi} f'(e^{i\phi}) \right]
\]

(3.14)

where \( \mathfrak{I}[z] : \mathbb{C} \to \mathbb{R} \) is the imaginary part of \( z \); the condition for the parameter \( \phi \) to define a cusp point is that \( 3 c^{2i\phi} f'(e^{i\phi}) + e^{3i\phi} f''(e^{i\phi}) \in \mathbb{R} \).

The expression for the corresponding (linear approximation of the) critical curve can be easily found by setting \( J = 0 \) in equation (3.11), that is,

\[
\delta r = e \chi_1(\phi) = \frac{e}{4} \left[ c^{2i\phi} f(e^{i\phi}) + e^{-2i\phi} \bar{f}(e^{i\phi}) \right].
\]

(3.15)

Strictly speaking, this only defines the local linear approximation of the critical curve (near \( z_0 = e^{i\phi} \)). However, one may simply consider \( (1 + \delta r) e^{i\phi} \) as a global expression for the critical curve parametrized by \( \phi \),

\[
z_2(\phi) = e^{i\phi} [1 + e \chi_1(\phi)] = e^{i\phi} - \frac{e}{4} \left[ c^{2i\phi} f(e^{i\phi}) + e^{-2i\phi} \bar{f}(e^{i\phi}) \right].
\]

(3.16)

Then, the image of \( z_2(\phi) \) under the linearized lens mapping

\[
\zeta(\phi) = 2e^{i\phi} e \chi_1(\phi) - e \bar{f}(e^{-i\phi}),
\]

(3.17)

indeed recovers the previous derived expression (3.2) for the caustic.
4 Inverse symmetry of perturbation potential

4.1 Caustic invariant perturbation pair

If \( f(z) \) can be represented by a complex monomial \( az^n \) near the unit circle \( |z| = 1 \) and \( |ea| \ll 1 \), its caustic can be found from equation \( 3.12 \) by

\[
-\frac{2}{\varepsilon} \delta \psi_4(\phi) = nae^{\delta \psi(z) - (n+1)\delta \psi} + n\delta e^{\delta \psi(z) - (n-1)\delta \psi} + 2\delta e^{\delta \psi} = nae^{\delta \psi(z)} + (n+2)a\delta e^{\delta \psi(z)}.
\]

(4.1)

The resulting expression is a linear combination of two complex exponentials. Motivated by the relationship between the two exponents and the scalar coefficients of each exponential, let us consider the substitution \( m = -(n+2) \). Then,

\[
nae^{\delta \psi(z)} + (n+2)a\delta e^{\delta \psi(z)} = (m+2)(-\alpha)e^{\delta \psi(z)} + m(-\overline{\delta})e^{\delta \psi(z)}.
\]

(4.2)

that is, equation \( 4.2 \) is invariant under the exchange of \( n \leftrightarrow -(n+2) \) accompanied by \( e \leftrightarrow -\overline{e} \). In other words, up to the linear approximation, the shapes of the caustics due to the perturbations \( f(z) = az^n \) and \( f(z) = -\overline{a}z^{-(n+2)} \) are identical. Furthermore, the expression \( 3.12 \) is linear to terms involving \( f \), and therefore, the superposition principle implies that the caustic resulting from the polynomial perturbation \( f(z) = \sum_n (-\overline{a}_n)z^{-(n+2)} \) is the same as the one from another polynomial perturbation \( f_0(z) = \sum_n a_nz^n \). Moreover, one can easily show that equation \( 4.2 \) is also the same for both cases, and therefore equation \( 4.2 \) implies that this caustic invariance actually extends to the correspondence of the magnification associated with any source position in the vicinity of the caustics.

In fact, the pair of “linear caustic invariant perturbations” (hereafter, LCIP) is also connected by the symmetry between the corresponding lensing potentials. To see this, let us consider two complex potentials with inverse symmetry: \( \psi'(z) = \overline{\psi}(z^{-1}) \). If \( \psi'(z) \) has a convergent Laurent series expression in a domain containing the neighbourhood of the unit circle – since \( \psi'(z) \) is an analytic function of \( z \) for any null-convergent perturbation, this condition is basically the restriction to the absence of poles in the neighbourhood of the unit circle –, then the perturbations due to these potentials are

\[
\psi'(z) = \sum_n b_{m,n}z^n \rightarrow f(z) = \sum_n \overline{b}_{m,n}z^{-n} = \sum_n a_nz^n,
\]

(4.3)

\[
\psi'(z) = \frac{1}{z} \rightarrow f(z) = \psi''(z) = \sum_n (-m-b_m/z^n) = \sum_m (\overline{a}_m/z^{-m+1}).
\]

(4.4)

so that they are indeed an LCIP pair. In terms of the real potentials, the symmetry between the LCIP pair is more straightforward. First, a general (real) solution of the two-dimensional Laplace equation – allowing singularities at either the origin or infinity – in plane polar coordinates (e.g., Courant & Hilbert, 1962) is, by harmonic expansion,

\[
\psi' = a_0 \ln r + b_0 + \sum_{n=2}^{\infty} \left( a_n r^n + \frac{a_n}{r^n} \right) \cos n\phi + \left( b_n r^n + \frac{b_n}{r^n} \right) \sin n\phi
\]

\[
= a_0 \ln r^2 + b_0 + \frac{1}{2} \sum_{n=1}^{\infty} \left( a_n r^n + \frac{a_n}{r^n} \right) \left( e^{in\phi} + e^{-in\phi} \right) + i \left( b_n r^n + \frac{b_n}{r^n} \right) \left( e^{-in\phi} - e^{in\phi} \right)
\]

\[
= a_0 \ln z^2 + b_0 + \frac{1}{2} \sum_{n=1}^{\infty} \left( a_nz^n + \frac{a_n}{z^n} + b_nz^n + \frac{b_n}{z^n} \right) \left( e^{-in\phi} - e^{in\phi} \right)
\]

(4.5)

where all the constant coefficients, \( a_n \),'s and \( b_n \)'s are real. The corresponding complex potential is easily found as

\[
\psi' \equiv \psi_{\alpha} = a_0 \ln z + 2b_0 + \sum_{n=1}^{\infty} \left( c_n z^n + c_{-n} z^{-n} \right)
\]

(4.6)

and therefore, the complex potential of its LCIP pair is

\[
\psi_{\alpha} = a_0 \ln z + 2b_0 + \sum_{n=1}^{\infty} \left( c_n z^n - c_{-n} z^{-n} \right),
\]

(4.7)

where \( c_n = a_n + ib_n \). Finally, the corresponding real potential of the LCIP pair is

\[
\psi'' = \psi' + \psi' \equiv \psi'' = \psi_{\beta} = a_0 \ln r + b_0 + \sum_{n=1}^{\infty} \left( \frac{a_n + ib_n}{r^n} + \frac{a_n - ib_n}{r^{-n}} \right) \cos n\phi + \frac{b_n + b_n}{r^n} \sin n\phi
\]

(4.8)

In other words, the perturbing parts of the real potentials of the LCIP pair are related to each other by an inverse symmetry with respect to the unit circle (i.e., Einstein ring): that is, \( \partial \psi''(r, \phi) = \partial \psi'(r^{-1}, \phi) \) in plane polar coordinates \( (r, \phi) \), or \( \partial \psi'' = \partial \psi'(r/r^2) \) in vector notation.
Harmonic expansion of the potential, in addition, demonstrates that the presence of an LCIP pair is related to the intrinsic symmetry of two dimensional harmonic function, that is, the azimuthal structure is invariant under the radial distance inversion.

4.2 caustic invariance and magnification correspondence

Following the discussion in the previous section, the natural question arises whether the magnification correspondence between the pair of potentials with the inverse symmetry extends to sources lying far from the caustic. For the zeroth order, the answer is affirmative because the magnification of sources lying far from the caustic is just small a perturbation on the point mass lens case. However, to examine whether the correspondence actually extends to the first order perturbations, one needs to consider the approximations of equations 3.2, 3.7, and 3.8 that provide one with the lowest order non-trivial effects due to perturbations for these cases.

From equation (3.8), the magnification change associated with the perturbation \( f(z) = \alpha e^z \) is found by

\[
\delta A_p = \frac{p \epsilon}{2 - (|c_0|^2 + |c_1|^2)} \left( n + \frac{2}{1 + |c_0|^2} \right) \left( \alpha \zeta_{0}^{n+1} \zeta_{0}^{*} - \alpha \zeta_{0}^{n+1} \zeta_{0}^{*} \right).
\]

(4.9)

On the other hand, the magnification associated with the corresponding perturbation \( f_p(z) = -\delta z^{-(n+2)} \) is

\[
\delta A_p = \frac{p \epsilon}{2 - (|c_0|^2 + |c_1|^2)} \left( n + \frac{2}{1 + |c_0|^2} \right) \left( \alpha \zeta_{0}^{n+1} \zeta_{0}^{*} + \frac{\bar{a}}{\zeta_{0}} \right).
\]

(4.10)

Next, one notes that, for a given source position, the symmetry of the point mass lens implies that the positive and negative parity images should be related by \( z_0 \leftrightarrow -z_0^{-1} \). Hence, the linear changes in magnification on alternative parity images by a pair of perturbations \( f(z) \) and \( f_p(z) \) are symmetric for even \( n \) – an odd potential – while they are antisymmetric for odd \( n \) – an even potential. In other words, the pair of perturbations \( f(z) = \alpha e^z \) and \( f_p(z) = -(1)^n \delta z^{-(n+2)} \) yields correspondent linearly perturbed magnifications for the source position far from the caustics. Using the same argument in the previous section, one may generally conclude that this pair is in fact related, in terms of potentials, with a property that \( \psi_p(r) = -\psi(-r/\zeta) \) or \( \psi_p'(z) = -\psi'(z^{-1}) \). In addition, one may also state that the magnification correspondence between the LCIP pair of odd potentials is stronger than that of even potentials.

5 EXAMPLES

5.1 planetary perturbations

Bozza (1999) showed the perturbative analysis provides an ideal method to study microlensing from planetary perturbations. Here, some well-known degeneracies of planetary microlensing are reexamined using the perturbative approach. In general, microlensing by a star with a planet can be described by the lens equation,

\[
\zeta = z - \frac{1}{\zeta} - \frac{q}{z - z_p}.
\]

(5.1)

Here, the position of star is chosen as the coordinate origin, and Einstein ring radius corresponding to the stellar mass alone as the unit of angular measurements. The projected angular location of the planet is given by \( z_p \), while \( q \) is the mass ratio of the planet to the star. From the comparison to equation (5.4), if \( 0 < q \ll 1 \) (and \( |z - z_p| \gg q \)), equation (5.1) can be considered as a point mass lens under a small null-convergent perturbation that comes from

\[
f(z) = (z - z_p)^{-1} ; \quad f'(z) = -(z - z_p)^{-2}.
\]

(5.2)

which is analytic everywhere except \( z = z_p \) where it forms a pole. Then, if \( z_p \) is sufficiently far from the circle \( |z| = 1 \), equation (5.2) provides a parametric representation for the linear approximation of the (central) caustics;

\[
\frac{2q}{\zeta} = \frac{e^{\phi i}d}{(e^{\phi i}d - z_p)} + \frac{e^{-\phi i}d}{(e^{-\phi i}d - z_p)} + \frac{2}{e^{\phi i}d - z_p} \equiv \frac{2}{e^{\phi i}d - z_p}.
\]

(5.3)

where \( \phi \in [0, 2\pi) \) is a parameter. To examine the degeneracies of planetary microlensing related to these caustics, it is helpful to further manipulate the last two terms in the equation (5.3) algebraically, that is,

\[
\frac{e^{\phi i}d}{(e^{\phi i}d - z_p)^2} = -\frac{1}{(1 - z_p e^{i\phi})^2} - \frac{1}{(1 - \bar{z_p} e^{-i\phi})^2} = \frac{1}{(1 - z_p e^{i\phi})^2} + \frac{1}{(1 - \bar{z_p} e^{-i\phi})^2} - 1.
\]

(5.4)

Then, combining equations (5.3) and (5.4) leads to a more symmetric representation of the linear approximation of the caustics;

\[
\frac{2q}{\zeta} = \frac{e^{\phi i}d}{(e^{\phi i}d - z_p)^2} + \frac{e^{-\phi i}d}{(e^{-\phi i}d - z_p)^2} - \frac{1}{(1 - z_p e^{i\phi})^2} + \frac{1}{(1 - \bar{z_p} e^{-i\phi})^2} - 1.
\]

(5.5)

Therefore, up to linear order, the shapes of the caustics due to the perturbation from the planet lying at \( z_p \) and the one at \( \bar{z_p}^{-1} \) are identical. Note that \( z_0 \) and \( \bar{z_p}^{-1} \) both have the same argument but that their norms are related inversely to each other. It is also straightforward to show
that this symmetry further implies the magnification degeneracy between these two planet positions through equation (5.11). Compare this to the degeneracy identified by Gaudi & Gould (1997) as their second class of discrete degeneracies of planetary microlensing — “whether the planet lies closer to or farther from the star than does the position of the image that it is perturbing.” Note that this degeneracy also extends to systems of any number of multiple planets since the superposition principle applies to the linear approximation.

If one compares the actual central caustics of a planetary microlens to its linear approximation in equation (5.5), one finds that deviations between them take place for smaller $q$ with $|z_q| < 1$ than $|z_q| > 1$, and therefore that the $z_q^{-1}$ degeneracy appears not to be strict for $q \gtrsim 10^2$. This is because the magnitude of the actual perturbation term $(qf \sim q)$ for $|z_q| < 1$ grows faster than the one for $|z_q| > 1$, which grows $|qf| \sim q|z_q|^{-1}$ for $|z_q| \sim 1$. To find a better approximation of the caustics for close-in ($|z_q| < 1$) planetary systems, one may need to include the second order effects of the perturbation (see Appendix A). However, if $0 < |z_q| \ll 1$, it is possible to find an alternative perturbative approach for which the leading perturbation term behaves as $\sim q|z_q|^{2}$, and therefore the linear approximation that can be used for larger $q$. Similarly, one can devise a different description of the system in which the leading perturbation term behaves as $\sim q|z_q|^{-2}$, which can be applied for wide-separation ($|z_q| \gg 1$) planetary systems. Furthermore, one can also show that these two types of perturbative description of system, in fact, lead to a certain pair of LCIP, which may be understood as a natural extension of the $z_q^{-1}$ degeneracy.

5.2 Extreme binary lens

It was Dominik (1999b) who noticed that the approximation of binary lenses with extreme separations by certain classes of perturbed point mass lens systems reveals the underlying connection between various binary lens systems. Alcubierre et al. (2002) tried explicit but rather limited calculations based on the perturbative approach to demonstrate the presence of the magnification degeneracy between two types of binary lens systems. Here, this magnification degeneracy is studied in a more general way as an archetypal example of the LCIP pair.

Let us first think of the situation in which the lens equation is given by equation (B6) with two component masses. By series-expanding the deflection term caused by one of the masses at the location of the other mass;

$$\zeta = z - \frac{q_1}{\bar{\ell}_1} - \frac{q_2}{\bar{\ell}_2} = z - \frac{q_1}{\bar{\ell}_1} + \frac{q_2}{\bar{\ell}_2} \sum_{k=0}^{\infty} \left(\frac{\bar{\ell}_1}{\bar{\ell}_2} - \frac{\bar{\ell}_1}{\bar{\ell}_2}\right)^k = z - \frac{q_1}{\bar{\ell}_1} + \frac{q_2}{\bar{\ell}_2} + \sum_{k=0}^{\infty} \frac{q_2 (\bar{\ell}_1 - \bar{\ell}_2)}{(\bar{\ell}_1 - \bar{\ell}_2)^{k+1}},$$

(5.6)

where $\bar{\ell}_1$ and $\bar{\ell}_2$ are the positions of the two mass points. The radius of the convergence for the infinite series in equation (5.6) is given by $|z - \bar{\ell}_1| < |\bar{\ell}_2 - \bar{\ell}_1|$. After some substitutions of symbols,

$$w = q_1^{-1/2}(\bar{\ell}_1 - \bar{\ell}_2); \quad \omega = \frac{1}{\sqrt{q_1}} \left(\zeta - \bar{\ell}_1 - \frac{q_2}{\bar{\ell}_2 - \bar{\ell}_1}\right) = q_1^{-1/2}(\zeta - \bar{\ell}_1) - \frac{q_2}{q_1} (\bar{\ell}_2 - \bar{\ell}_1)^{-1},$$

(5.7)

which is basically the choice of new coordinate origins $[\bar{\ell}_1$ for the image space and $\bar{\ell}_1 - q_2 (\bar{\ell}_2 - \bar{\ell}_1)^{-1}$ for the source space] and Einstein ring radius of the mass at $\bar{\ell}_1$ as the new unit of angular measurements, it is easy to see that lens equation (5.6) is a specific example of lens equation (5.1) for a point mass lens under perturbations,

$$\omega = w - \frac{1}{w} + \sum_{k=1}^{\infty} \gamma_k \, w^k,$$

(5.8)

where

$$\gamma_k = \frac{q_2 q_1^{k/2}}{(\bar{\ell}_2 - \bar{\ell}_1)^{k+2}}; \quad \gamma_0 = \frac{\gamma_0}{(\bar{\ell}_2 - \bar{\ell}_1)^{k+2}}.$$

(5.9)

Here, $\bar{\ell}_2 = q_1^{-1/2}(\bar{\ell}_1 - \bar{\ell}_1)$, that is, the lens location in this new coordinate system. In particular,

$$\gamma_0 = \frac{q_2 q_1^{1/2}}{q_1 (\bar{\ell}_2 - \bar{\ell}_1)^{-1}}; \quad \gamma_k = \frac{\gamma_0}{(\bar{\ell}_2 - \bar{\ell}_1)^{k+2}}.$$ 

(5.10)

If $|\gamma_0| \ll 1$, then, the circle $|w| = 1$ lies well within the region of the convergence of the infinite series, and also the perturbative approach for the caustics in the previous section is valid. If one considers the linear approximation alone, the perturbation series may be truncated after the second term. Note that, here, the second order effect of the leading term in the series $|\gamma_0| \sim |\bar{\ell}_2 - \bar{\ell}_1|^{-1}$ is higher order than the linear effect from the second term $|\gamma_1| \sim |\bar{\ell}_2 - \bar{\ell}_1|^{-2}$. In effect, the lens equation (5.8) describes the system as a point mass lens lying in the gravitational tidal field due to masses that are well outside of the region around its Einstein ring.

Next, let us consider the multipole expansion of the deflection terms of the lens equation (B6) at some point $z_c$;

$$\zeta = z - \frac{q_1}{\bar{\ell}_1} - \frac{q_2}{\bar{\ell}_2} = z - \frac{q_1}{\bar{\ell}_1} + \frac{q_2}{\bar{\ell}_2} \sum_{k=0}^{\infty} \left(\frac{\bar{\ell}_1}{\bar{\ell}_2} - \frac{\bar{\ell}_1}{\bar{\ell}_2}\right)^k = z - \frac{q_1}{\bar{\ell}_1} + \frac{q_2}{\bar{\ell}_2} \sum_{k=0}^{\infty} \left(\frac{\bar{\ell}_1}{\bar{\ell}_2} - \frac{\bar{\ell}_1}{\bar{\ell}_2}\right)^k.$$

(5.11)
\[
\frac{z_c}{q_1 + q_2} l_1 + \frac{q_2}{q_1 + q_2} l_2 = \frac{q_1 l_{1w} + q_2 l_{2w}}{q_1 + q_2},
\]
then the dipole term vanishes. Consequently, substituting symbols via
\[
w = (q_1 + q_2)^{-1/2}(z - z_c); \quad \omega = (q_1 + q_2)^{-1/2}(\zeta - z_c),
\]
which is the choice of the centre of the mass \(z_c\) being the coordinate origin and Einstein ring radius corresponding to the total mass of the system being the unit of angular measurements, one finds equation (5.11) to be consistent with being another example of a point mass lens under perturbations (eq. 5.1).

\[
\omega = w - \frac{1}{w} - \sum_{k=1}^{\infty} \hat{Q}_{2k+1} \tilde{w}^{-k/2},
\]
where
\[
\hat{Q}_{2k} = \frac{q_1 (l_{1w} - z_c z)^k + q_2 (l_{2w} - z_c z)^k}{(q_1 + q_2)^{k/2} z_c} = \frac{q_1 l_{1w}^k + q_2 l_{2w}^k}{q_1 + q_2}
\]
\[
= \frac{q_1 q_2^k (l_{1w} - l_{2w})^k + q_2 q_2^k (l_{2w} - l_{1w})^k}{(q_1 + q_2)^{k/2} (q_1 + q_2)^{k/2}} = \frac{q_1 (-q_2)^{k/2} + q_2^k q_2}{(q_1 + q_2)^{k+1}} (l_{1w} - l_{2w})^{k/2} = \left(\sum_{k=1}^{\infty} q_1 (-q_2)^{k/2} (q_1 + q_2)^{k/2}ight) (l_{1w} - l_{2w})^{k/2}.
\]
are basically multipole moments. Here, again \(l_{1w} = (q_1 + q_2)^{-1/2}(l_2 - z_c)\) is the lens location in the new coordinate system, but the relation to the original representation differs from the previous case because of the difference of the new coordinate system. The coefficient for the leading term (the quadrupole moment) and the following higher order moments can also be written as
\[
\hat{Q}_4 = \frac{q_1 q_2}{(q_1 + q_2)^2} (l_{1w} - l_{2w})^2; \quad \hat{Q}_4^2 = \hat{Q}_4 \sum_{k=2}^4 q_1 (-q_2)^{k/2} (l_{1w} - l_{2w})^{k} = \hat{Q}_4 \frac{q_1 (-q_2)^{k+1} (l_{1w} - l_{2w})^{k}}{(q_1 + q_2)^{k+1}}.
\]
Again, if \(|\hat{Q}_4| \ll 1\), one finds not only that the infinite series converges at \(|w| \equiv 1\) but also that the perturbation series is small enough for the linear approximation of the caustics given in equation (5.12) to be reasonably valid. Similar to the previous case, one finds that the second order effect of the leading term in the series \(|\hat{Q}_4|^2 \sim |l_{1w} - l_{2w}|^2\) is a higher order effect than the linear effect of the second term \(|\hat{Q}_4 l_{1w} - l_{2w}|^2\) and so one may truncate the perturbation series after the second term when considering of the linear approximation.

Finally, if one compares equations (5.8) and (5.14), one immediately discovers that the two become a strict LCIP pair if \(y_{k-1} = y_{k+1}\) and \(y_k = \hat{Q}_4\), in fact, suffice to define an LCIP pair from equations (5.8) and (5.14). Following the usual convention of parameter definition for the binary lens, that is, \(q = q_2/q_1 > 0\) as the mass ratio between the two components, one can rewrite the leading coefficients of the series
\[
\hat{y}_0 = \frac{q_1}{1 + q_2} d_2^2; \quad \hat{y}_1 = \frac{q_1}{1 + q_2} d_2^2;
\]
\[
\hat{Q}_4 = \frac{q_1 q_2}{1 + q_2} \frac{d_0}{1 + q_2} d_0 = \hat{Q}_4 \frac{1}{1 + q_2} d_0^2.
\]
where the subscripts ‘’ and ‘’ are used to distinguish the parameters associated with the tidal approximation (for \(|d| \gg 1\)) and the multipole expansion (for \(|d| \ll 1\)). Then, the condition for two system to be an LCIP pair becomes
\[
(1 + q_2) d_0^2 = \frac{q_1}{q_2} \frac{1 + q_2}{1 - q_2} d_0^2 = \frac{1}{1 - q_2} d_0^2 = \frac{1 + q_2}{1 - q_2} d_0^2 = \frac{1 + q_2}{1 - q_2} d_0^2.
\]
This condition can be used to find the system \((q, d_0)\) that is degenerate in the first order to the system \((q_0, d_0)\):
\[
q_0 = \frac{q_0}{1 - q_0}; \quad d_0 = \frac{1 + q_0}{d_0 (1 - q_0 + q_0^2)},
\]
or the system \((q, d_0)\) to the system \((q_0, d_0)\):
\[
q_0 = \frac{1 + 2 q_0 - (1 + 4 q_0)^{1/2}}{2 q_0} = 1 - \frac{(1 + 4 q_0)^{1/2} - 1}{2 q_0}; \quad d_0 = \frac{1}{d_0} \left(1 + 4 q_0\right)^{1/2}.
\]
Algebraically, there is a second solution for the inversion of the equation (5.21). However, the second solution is related to equation (5.22) by \(q_0 \leftrightarrow q_0^*\) and \(d_0 \leftrightarrow -d_0\), which results in the identical system because each relationship on its own corresponds to a 180° rotation of the system (with respect to the centre of mass) and the combination of the two becomes the identity transformation. Note that, if \(q_0 \ll 1\), then \(q \approx q_0 + 2 q_0^2\), \(q_0 \approx q_0 - 2 q_0^2\), and \(d_0 \approx 1 + (3/2) q_0\). That is, this LCIP pair is reduced to the \(z_0 - z_0^*\) degeneracy of planetary perturbations.
to one can show that there is either one more solution of equation (5.25) in $\mathbb{R}$ – it is essentially controlled by two parameters.

Detailed discussion regarding the analysis of the shape of the (central) caustics of extreme binary lens is available in the literature (e.g. Dominik 1999b; Bozza 2000). Here, the basic properties of these caustics are reexamined as an example of the simplest LCIP caustics, essentially controlled by two parameters.

Suppose that a point mass lens is perturbed by a null-convergent perturbation of the form of $f = -c_1 z - c_2 z^2$. From equation (4.1), the linear approximation of the resulting caustic is

$$\xi = \frac{1}{2} \left( c_1 e^{i\phi} + 3c_2 e^{-i\phi} + 2c_2 e^{i\phi} + 4c_2 e^{-2i\phi} \right) = \frac{|c_1|}{2} e^{i\phi} \left[ e^{i(\phi - \phi_1)} + 3e^{-i(\phi - \phi_1)} \right] + |c_2| e^{i\phi} \left[ e^{i(\phi - \phi_2)} + 2e^{-2i(\phi - \phi_2)} \right],$$

where $-2\phi_1$ and $-3\phi_2$ are the arguments of $c_1$ and $c_2$ respectively (i.e., $c_1 = |c_1| e^{-2i\phi_1}$ and $c_2 = |c_2| e^{-3i\phi_2}$). Up to the same order, this caustic is identical to that caused by its LCIP, $f = \bar{q} z^{-3} + \bar{q} z^{-4}$. In general, $e^{i(\phi_1 - \phi_2)}$ is not necessarily real. However, if one restricts to the approximation for extreme binary lenses, one finds that $\phi_1 = \phi_2$. That is, $c_1 = \bar{q}_0$ and $c_2/c_1 = \bar{q}_1/\bar{q}_0$ for the tidal approximation, or $c_1 = \bar{q}_4$ and $c_2/c_1 = \bar{q}_4/\bar{q}_4$, and therefore, from equations (5.16) and (5.19), both $\phi_1$ and $\phi_2$ are the argument of $d$ (i.e., $d/d\phi = e^{i\phi_1} = e^{i\phi_2}$). Then,

$$\xi e^{-i\phi} = 2|c_1| \left[ \cos^2(\phi - \phi_1) - i \sin^2(\phi - \phi_1) \right] + |c_2| \left[ 4 \cos^2(\phi - \phi_2) \cos[2(\phi - \phi_2)] - 1 - 4i \sin^2(\phi - \phi_2) \sin[2(\phi - \phi_2)] \right],$$

where $\phi_1 = \phi_2$. If $c_2 = 0$ (and the real axis is chosen such that $\phi = 0$), the expression is reduced to the one derived by Kovner (1987b).

To find the cusp points (eq. (5.24)),

$$\frac{d\xi}{d\phi} = \frac{e^{i\phi}}{2i} \left[ 3 \left( c_1 e^{-2i\phi} - c_1 e^{i\phi} \right) + 8 \left( c_2 e^{-3i\phi} - c_2 e^{3i\phi} \right) \right] = e^{i\phi} \left[ 3|c_1| \sin[2(\phi - \phi_1)] + 8|c_2| \sin[3(\phi - \phi_2)] \right] = 0.$$

Here, if one still assumes that $\phi_1 = \phi_2$, one immediately finds that a cusp forms with $\phi - \phi_i = n\pi$, where $n$ is integer. For these cusps, $\Re[\xi e^{-i\phi}] = (-1)^n \Re[c_1] + 3|c_2|$ and $\Im[\xi e^{-i\phi}] = 0$ so that they lie along the axis with the angle of $\phi_i$ and are separated by $4|c_1|$. Here, similar to $\Im[z]$, the imaginary part of $z$, $\Re[z] : \mathbb{C} \rightarrow \mathbb{R}$ is the real part of $z$ (i.e., $\Re[z] = (z + \bar{z})/2$, $\Im[z] = i(z - \bar{z})/2$, $z = \Re[z] + i\Im[z]$). In addition, one can show that there is either one more solution of equation (5.24) in $0 < \phi - \phi_i < \pi$ if $4|c_2| < |c_1|$ or two solutions in the same interval if...
4|c_2| > |c_1|, and therefore that the caustic has four \((4|c_2| < |c_1|)^2\) or six cusps \((4|c_2| > |c_1| > 0)\) unless \(c_1 = 0\). If \(c_1 = 0\) and \(c_2 \neq 0\), it becomes tricuspoid – three cusps – since the \(\phi\)-period for eq. 5.23 then becomes \(\pi\) rather than \(2\pi\). However, hexacuspoid caustics generated by equation 5.24 with \(4|c_2| > |c_1|\) are doubly-wound self-intersecting curves (Fig. 1), which cannot be the caustics of any real pure binary lens (note that certain ternary lenses may be approximated by an octupole dominated perturbation series and thus consistent with hexacuspoid caustics although \(\phi_1\) and \(\phi_2\) may not be the same any more). Hence, the tetracuspoid condition may be regarded as one of the limit for the approximation of an extreme binary lens by a perturbation series. The area enclosed by a tetracuspoid caustic can be found by

\[
S = \frac{1}{2} \int \xi d\zeta = \frac{1}{2i} \int_0^{2\pi} \xi \frac{d\zeta}{d\phi} d\phi = \left(\frac{3}{2} \pi |c_1|^2 + 4\pi|c_2|^2\right),
\]  

(5.26)

where the negative sign is caused by the orientation of the current \(\phi\)-parametrization of the caustic. With the restriction that \(|c_2|/c_1| < 1/4\), one may regard \(c_1\) as the main parameter controlling the size of the caustics while \(s = c_2/c_1\) is an asymmetry shape parameter for the caustics.

In Fig. 1 the change of the caustics shape with varying value of \(s\) at fixed \(c_1\) is represented. The dimension is intentionally omitted since it can be linearly scaled with \(c_1\). In particular, the distance between two cusps on the horizontal direction is given by \(4|c_1|\) and is independent of \(s\). For completeness, examples of the caustics with \(s > 0.25\) are also shown in the bottom of Fig. 1 although they are not applicable for the approximation of the caustics of any binary lens.

The fact that the shape of the central caustics of extreme binary lenses is controlled not by the leading coefficient of the perturbation series but by the ratio of its two first coefficients also implies that, when the wing of the lightcurve is not well observed, there may exist a continuous degeneracy of the lens model running along the (almost) constant shape parameter. While the behaviour of the highly-magnified part of lightcurve when the source is near the central caustic can constrain the structure of the caustic well enough to determine the shape parameter, the constraints on the relative size of the caustics with respect to the Einstein ring require the precise determination of the overall time-scale. Since the magnification near the centre is essentially scale-free \((\lambda \sim |\zeta|^{-1})\; c.f., \; eq. 5.14\) when the size of the caustic is sufficiently small compared to the distance between the caustic and the source, one may rescale the whole lightcurve by changing the blend fractions. Therefore, without the detailed constraints from the wing, the overall scale factor is left to be unconstrained and the model exhibits a continuous degeneracy with strongly correlated time-scale, peak magnification, blending, and the size parameter of the caustics (which is the leading coefficient of the perturbation series). One can find an archetypal demonstration of this degeneracy in the modelling of the lightcurve of MACHO 99-BLG-47 (Albrow et al. 2003). In Fig. 1 one can not only find an example of a two-fold LCIP degeneracy but also the continuous degeneracy with the constant shape parameter.

It is notable that, as \(q \to 0\), the shape parameter is mainly controlled by \(d\) alone (i.e., \(s \sim \bar{z}_p\) or \(s \sim d^{-1}\)). In fact, if \(q \ll 1\), the approximation of the caustic by equation 5.23 can be valid, and one can immediately find that the shape of the caustic is completely (up to linear order) determined by the planetary position \(z_p\) while the planetary mass ratio \(q\) becomes an overall linear scale. Moreover, Taylor-series expansion of equation 5.23, either for \(|z_p| \ll 1\):

\[
\zeta(\phi) = q \bar{z}_p + \frac{q}{2} \left[3z_p^2 e^{-i\phi} + 4z_p e^{i\phi} + 2z_p e^{i\phi} + O(|z_p|^4)\right],
\]

(5.27)

or for \(|z_p| \gg 1\):

\[
\zeta(\phi) = q \bar{z}_p + \frac{q}{2} \left[\frac{e^{i\phi}}{z_p^2} + \frac{3 e^{i\phi}}{z_p^2} + \frac{2 e^{-i\phi}}{z_p^2} + \frac{e^{-i\phi}}{z_p^2} + O(|z_p|^{-4})\right]
\]

(5.28)

approaches equation 5.23 with the shape parameter given by \(s = \bar{z}_p\) (for \(|z_p| \ll 1\)) or \(s = z_p^{-1}\) (for \(|z_p| \gg 1\)) as expected.

5.2.2 planetary caustic

Note that the condition for the series expansion 5.23 to be valid for \(|\zeta| \sim 1\) is that \(|\mu_2 - \mu_1| = (1 + q)\alpha|\phi| \gg 1\). That is, if \(q \gg 1\), then the tidal approximation can be used even if \(|\phi| \sim 1\). However, for this case, one can easily notice that \(|\gamma_0| = (1 + q^{-1})\alpha^2 \sim 1\) (eq. 5.18) so that the lensing behaviour of these systems cannot be properly described by the perturbative analysis although the subsequent higher order terms may be ignored (or viewed as perturbations on a lens with linear shear). If one truncates the series after the leading term, one finds that the system modelled by the lens equation 5.23 is a point mass lens subject to a constant external shear given by \(-\gamma_0\), which is generally referred to as a Chang-Refsdal lens after Chang &Refsdal (1979, 1984). While the lensing behaviour of Chang-Refsdal lenses is for the most part analytically tractable, its study is beyond the scope of the current monograph. Here, it is simply noted that, for planetary microlensing, provided that the planet lies sufficiently far from the Einstein ring, there exist two types of caustics; one that can be

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2 Strictly speaking, at \(4|c_2| = |c_1|\), the caustics develop a butterfly catastrophe, and therefore, it has three simple cusps plus one butterfly catastrophe.

3 Note that the area two-form can be written as \(d\zeta = d\zeta = (d\xi, d\zeta)/\sqrt{(2i)} = d(\zeta d\zeta)/(2i)\). Hence, from the fundamental theorem of multivariate calculus, the area bounded by \(\zeta\) can be found to be \((2\pi)^2\int d(\zeta d\zeta) = (2\pi)^2 \frac{\bar{z}_p^2 d\zeta}{\sqrt{(2i)}}\).

4 The negative sign follows from the definition given in Appendix B which conforms to the usual convention found in weak lensing literature – the direction of the shear to be the direction of eigenvector associated with the positive eigenvalue of \(\gamma\). Assuming \(\vec{q} \in \mathbb{R}\), ‘\(\sim \gamma_0\)’ is negative real (eq. 5.23) so that, according to this definition, the direction of the external shear due to the companion at the position of the planetary caustics is along the imaginary axis, that is, the shear runs perpendicular to the axis joining two masses. 

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approximated by equation (6.1) – the central caustic, and the other, which is basically the caustic of Chang-Refsdal lens perturbed by the higher order tidal-effect terms – the planetary caustic. In general, the lensing effects due to planetary caustics are rather smaller than those associated with the central caustics so that they usually appear as small additional signals on the main lensing event (Gould & Loeb 1992; Gaudi & Gould 1997).

6 DISCUSSION

6.1 What about the critical curves?

So far, the discussion on the LCIP pair has been focused on the identical (up to linear approximation) caustics and the degenerate (total) magnification. A natural question to follow is whether they are indistinguishable (at least up to the linear order) in every sense. The answer is actually negative as one can easily discover the difference in the critical curves and the individual image positions. For the monomial perturbation of \( f(z) = az^n \), one finds, from equation (3.16), the corresponding critical curves to be:

\[
z_c(\phi) = e^{i\phi} \left\{ 1 - \frac{\epsilon}{2} |a| \cos[(n+1)\phi + \phi_a] \right\},
\]

where \( \phi_a \) is the argument of \( a \), that is, \( a = |a|e^{i\phi_a} \). It is straightforward to show that equation (6.1) is not invariant under the LCIP transformation of \( n \leftrightarrow -(n+2) \) and \( a \leftrightarrow -\bar{a} \) (equivalently, \( \phi_a \leftrightarrow \pi - \phi_a \)), but rather the amplitude of the cosine linear deviation term changes to \( \epsilon(n+2)|a|/2 \). In addition, while the azimuthal positions of the images for the LCIP pair are the same since equation (3.10) is identical for them, one finds that the radial positional deviations given by

\[
\delta r = \frac{e^{-i\phi}}{2} \left[ \zeta + \epsilon f(e^{-i\phi}) \right] (\in \mathbb{R})
\]

cannot be the same for the two different perturbations. However, it should be noted that the leading order descriptions of the critical curves and the image radial positions are dominated not by the first order but by the zeroth order terms, and they are still identical up to their leading zeroth order. More importantly, despite this difference, the actual observational distinction between the LCIP pair would be rather difficult.
Figure 3. Geometrical determination of the azimuthal positions and magnifications of images with a linear approximation of the caustics. For this example, the two cusps lying along the horizontal direction are positive so that the tangent to a caustic point on the bottom half points upwards and vice versa. For a given shape the caustics, if one can draw a tangent to the caustics through a given source position (represented by a dot), there exists an image towards the direction (represented by an arrow) parallel to that tangent from the image plane centre. Note that the image plane centre is not necessarily coincident with the source plane centre nor with the caustics centre. The magnification of the image is proportional to the inverse of the distance between the source and the tangent point.

since the critical curve is physically unobservable and two dominant observable characteristics of the individual images – the magnification and the azimuthal position – are indeed identical up to the linear order of the perturbations.

To find the magnification and the azimuthal position of the individual images, one must solve equation (3.10), which is a transcendental equation for \( \phi \), and thus it may not be possible to find algebraic solutions although it is straightforward to find its roots graphically. Incidentally, it is also possible to find them in a purely geometrical manner provided that one can draw its caustics (eq. 3.12). From equation (3.14), one finds that \( (d\zeta_c/d\phi) \parallel e^{i\phi} \), that is, the parameter \( \phi \) also determines the tangent direction to the caustics. This, together with the fact that equation (3.11) is real when \( \phi \) is the solution of equation (3.10), also implies that, if one can draw a tangent to the caustic through the given source position, the angle of that tangent line \( \phi \) is a root of equation (3.10). Hence, there exists an image with its azimuthal position with respect to the image plane centre being \( \phi \) and magnification given by \( (2l)^{-1} \) where \( l \) is the distance between the source and the tangent point. Examples of this procedure for the perturbation given by \( f(z) = az \) or its LCIP \( f(z) = -\bar{a}z^{-2} \) is found in Fig. 3. For definiteness, \( a \) is assumed to be real (horizontal direction) in this example, and therefore, the two cusps lying along the horizontal direction are positive. Finally, note that not all images are actually solutions of equation (3.10). However, missing images, if any, are typically higher order solutions that do not significantly contribute to the total magnification provided that the linear approximation is valid and dominant and that the source lies sufficiently close to the centre (and therefore the caustics).

6.2 What is so special about the point-mass lens?

The discussion until now has been limited to the case for which the unperturbed lens system is described by the point-mass lens equation (2.1). However, the perturbation approach outlined in Section 3 can be applied to any lens system whose solutions are known. In fact, the procedure in Section 3 can be generalized to the case for which the unperturbed lens system is given by any circularly symmetric lens system that can form an Einstein ring. Suppose that the system is described by the lens equation (3.1) with the unperturbed part given by
\[ \zeta = z \left(1 - \frac{1}{r} \frac{d \psi}{dr} \right) = \zeta \left[1 - \frac{4G}{D_\text{Ein} c^2} \frac{M(<r)}{r^2} \right], \] (6.3)

where \( r = |z| \) is the radial coordinate of image, \( \psi(r) \) is a circularly symmetric lensing potential, and \( M(<r) \) is the mass enclosed within a given angular radius \( r \). If the equation

\[ \frac{4G}{D_\text{Ein} c^2} M(<r) = r^2 \] (6.4)

possesses a positive root \( r = r_0 \), one can find the linear approximation for the tangential caustics of the system by series-expanding the lens equation at \( z_0 = r_0 e^{i \theta_0} \) and following procedure similar to Section 3.4. In particular, one finds

\[ \zeta = (2 - \beta_0) \delta r e^{i \theta} - \epsilon \tilde{f}(r_0 e^{i \theta}), \] (6.5)

\[ R(\phi) = \zeta + \epsilon \tilde{f}(r_0 e^{i \theta}) - e^{2i \phi} [\zeta + \epsilon \tilde{f}(r_0 e^{i \theta})] = 0, \] (6.6)

\[ \mathcal{F} = (2 - \beta_0) \frac{\zeta - \zeta(\phi)}{z_0}, \] (6.7)

\[ \frac{\zeta(\phi)}{r_0} = -\frac{\epsilon}{2(2 - \beta_0)} \left[ e^{i \phi} (r_0 e^{i \theta}) + e^{-i \phi} r_0 e^{i \theta} + 2 \tilde{f}(r_0 e^{i \theta}) \right], \] (6.8)

and

\[ \frac{z(\phi)}{r_0} = e^{i \phi} \frac{\epsilon}{2(2 - \beta_0)} \left[ e^{i \phi} (r_0 e^{i \theta}) + e^{-i \phi} r_0 e^{i \theta} \right]. \] (6.9)

in place of equations (6.9), (6.10), (6.11), (6.12), and (6.16). Here, \( \beta_0 = (d \ln M/d \ln r)|_{r_0} \) is the local power index for the mass \( M \) at the projected angular radius \( r = r_0 \). In other words, the shape of (the linear approximation of) the caustics is completely determined by the perturbation alone once they are scaled by the angular Einstein ring radius (note that eq. 6.4 actually defines \( r_0 \) to be the angular Einstein ring radius for the given circularly symmetric system), and the unperturbed mass distribution only affects the image magnification as a common multiplicative factor, and therefore, the magnification ratios between images are also completely independent of the unperturbed mass distribution. Moreover, it is easy to establish the same LCIP relation \( f(\zeta) = a \bar{r}(z/r_0)^{\beta} \) and \( f(\zeta) = -\bar{a} r(z/r_0)^{(\beta + 1)/2} \), or more generally, that the pair of perturbation potentials with an inverse symmetry with respect to Einstein ring radius \( r_0 \) [i.e., \( \delta \psi(\phi) \) and \( \delta \phi(\phi) = \bar{\psi}(r_0^2 r/\bar{r}) \)] has the same expression for the linear approximation of the tangential caustics.

### 6.3 Are there LCIP pairs for convergent perturbations?

As has been noted, the LCIP relation originates from a certain intrinsic symmetry of harmonic functions, that is, their azimuthal structures are invariant under the radial distance inversion. This implies that the two-fold correspondence LCIP does not exist for convergent perturbations, for which the perturbation potential is no longer harmonic. However, examination of equations (6.5) and (6.8) reveals more basic properties of the LCIP, that is, the azimuthal Fourier coefficients of the perturbation potentials at the angular Einstein ring radius are the same, which is indeed the generalization of LCIP to include convergent perturbations.

Let us think of the potential

\[ \delta \psi(r, \phi) = \sum_n \epsilon_n(r) \cos n \phi + i \eta_n(r) \sin n \phi = \frac{1}{2} \sum_n \left[ \frac{\zeta^2}{r^2} \epsilon_n(r) + \frac{\zeta^2}{z^2} \eta_n(r) \right] \] (6.10)

perturbing a circularly symmetric lens system. Here, \( \epsilon_n = \epsilon_n + i \eta_n \). Then, the lens equation is given by

\[ \zeta = z \left(1 - \frac{4G}{D_\text{Ein} c^2} \frac{M}{r^2} \right) = \delta \alpha, \] (6.11)

where

\[ \delta \alpha = 2 \partial_r \delta \psi = \sum_n \left[ \frac{r^{n+1}}{2} \frac{d}{dr} \left( \frac{\bar{\epsilon}_n}{r} \right) + \frac{1}{2} \frac{d}{dr} \left( r^n \epsilon_n \right) \right] = \frac{r_0}{2} \sum_n \left[ r_0 \frac{r^{n+1}}{n+1} \tilde{\epsilon}_n - \frac{r_0^{n-1}}{n-1} \tilde{\eta}_n \right]. \] (6.12)

\[ f^{(\phi)} = \frac{r^{n+1}}{r_0^{n+1}} \frac{d}{dr} \left( \frac{\epsilon_n}{r} \right); \quad g^{(\phi)} = -\frac{r_0^{n-2}}{r_0^{n-2}} \frac{d}{dr} \left( r^n \epsilon_n \right) \] (6.13)

Here, the calculation utilizes the fact that \( r^2 = z \bar{z} \) and consequently that \( \delta \alpha = z(2r)^{-1} = r(2z)^{-1} \). That is, for any real function \( F(r) \), one can relate the complex derivative and the ordinary derivatives via \( \delta \alpha = F^*(r) \), \( \delta \alpha = z F'(2r)^{-1} = r F'(2z)^{-1} \). To find the expression for the linear approximation of the caustics, one series-expands equation (6.11) and its Jacobian determinant at \( z_0 = r_0 e^{i \theta} \) as in Section 3.4. After some tedious but none the less straightforward algebra, one can find

\[ \mathcal{F} = 2(1 - \kappa_{\text{E}0}) \frac{\zeta - \zeta(\phi)}{z_0} \] (6.14)
in place of equations (3.11) and (3.12), where $\kappa_0$ is the (local) convergence of the unperturbed circularly symmetric lens at $r = r_0$ (note that $\beta_0 = 2\kappa_0$ at $r = r_0$), and the subscript ‘0’ indicates the value of the function at $r = r_0$ [e.g., $f_{00} = f(r_0)$ and $g_{00} = g(r_0)$]. Then,

$$f_{00} = \frac{\partial}{\partial r} \Phi(r_0) = \frac{1}{r_0} \frac{d}{dr} \Phi(r)$$

(6.16)

$$g_{00} = \frac{\partial}{\partial r} \Phi(r_0) = \frac{1}{r_0} \frac{d}{dr} \Phi(r)$$

(6.17)

so that

$$\zeta(r) = -\frac{r_0}{2} \sum_n n \left[ (n + 1)e^{-\alpha^2} \tilde{e}_n \tilde{\kappa}_n + (n - 1)e^{-\alpha^2} \tilde{e}_n \tilde{\kappa}_n \right].$$

(6.18)

or

$$\Re\left[ \frac{\zeta}{r_0} \right] = -\sum_n \left[ \frac{\tilde{e}_n(\cos n \phi \cos \phi + \sin n \phi \sin \phi) + \frac{\tilde{\kappa}_n}{r_0} (\sin n \phi \cos \phi - \cos n \phi \sin \phi)}{r_0} \right]$$

(6.19)

$$\Im\left[ \frac{\zeta}{r_0} \right] = -\sum_n \left[ \frac{\tilde{e}_n(\cos n \phi \sin \phi - \sin n \phi \cos \phi) + \frac{\tilde{\kappa}_n}{r_0} (\sin n \phi \sin \phi + \cos n \phi \cos \phi)}{r_0} \right].$$

(6.20)

Note that the expressions do not involve the radial derivatives at all. In other words, up to the linear order, the caustics are completely specified by $\tilde{e}_n, \tilde{\kappa}_n$ or the azimuthal behaviour of the potential $\Phi(r_0, \phi)$ exactly at $r = r_0$, and independent of its radial behaviour even in its immediate neighbourhood. In terms of the deflections due to the perturbations (instead of the perturbation potentials) given as in equation (6.12), this translates into the linear approximation of the caustics being a function of $f_{00}, g_{00}$ alone, and thus the two perturbations producing the same linear approximation of the caustics if $f_{00}, g_{00}$ is the same. The null-convergent LCIP pair is a special case of a pair of two perturbations given respectively by $(f_{00}, g_{00}) = (\epsilon_{00}, 0)$ and $(f_{00}, g_{00}) = (0, \epsilon_{00})$ where the $\epsilon_n$’s are (complex) constants. In terms of the potentials, they are the pair of $e_n = a_0(r/r_0)^n$ and $e_n = a_n(r_0/r)^n$ where the $a_n$’s are again some common constants.

### 6.3.1 Ellipticity, external shear, and quadrupole moment

In many applications of gravitational lensing, the lensing convergence may be elliptically symmetric, that is, $\kappa(r, \phi) = \kappa_s(a(r, \phi))$ is the function of $r$ and $\phi$ only through the combination $a = r_0^2 (\sin^2 \phi + \rho^2 \cos^2 \phi)^{1/2}$, where $\rho$ is a constant. It is straightforward to show that this yields concentric ellipses with eccentricity of $1 - \rho^2$ as iso-convergence contours. The analysis of these types of lensing systems is of great interest, but also very much more complicated than the circularly symmetric system. However, if $\rho \approx 1$, the problem can be greatly simplified by applying Fourier analysis. Let us consider the radial Taylor expansion of $\kappa(r, \phi)$ with fixed $\phi$, that is, $\kappa(r, \phi) \approx \kappa(r, \phi) + (d \kappa/dr)|_{r_0} (r - \hat{r})$. Suppose that the expansion was at $\hat{r} = r_0^2 (1 + \rho^2)^{-1/2}$. Then the convergence at $r = a\rho (\sin^2 \phi + \rho^2 \cos^2 \phi)^{1/2}$ is

$$\kappa_s(a) \approx \kappa_s(\hat{r}) \left( 1 + \frac{d \ln \kappa}{d \ln r} \right) \left( 1 - \frac{1 - \rho^2}{2} \cos 2\phi \right).$$

(6.21)

Here, by noticing that $\kappa_s(a)$ and $(d \ln \kappa/d \ln r)|_{r_0}$ do not involve any azimuthal dependence, equation (6.21) can be used to derive the quadrupole approximation (c.f., Kovner 1987b) of the elliptically symmetric convergence

$$\kappa(r, \phi) \approx \kappa_s(\hat{r}) \left( 1 + \frac{d \ln \kappa}{d \ln r} \right) \left( 1 - \frac{1 - \rho^2}{2} \cos 2\phi \right),$$

(6.22)

provided that $|1 - \rho^2| << 1$.

Suppose that the lensing convergence is given by

$$\kappa(r, \phi) = \kappa_s(\phi) \left( 1 + \xi \cos [2(\phi - \phi_s)] \right).$$

(6.23)

Then, with the canonical boundary condition, the potential can be found, using Green’s function, to be

$$\psi(r, \phi) = 2 \ln r \int_0^\infty \frac{d\hat{r}}{\hat{r}} \chi(\hat{r}) \frac{d}{d\hat{r}} \Phi(\hat{r}) \ln \hat{r} + \frac{1}{2} \int_0^\infty \frac{d\hat{r}^2}{\hat{r}} \chi(\hat{r}) \cos [2(\phi - \phi_s)] \right) \right) + \int_\infty^\infty \frac{d\hat{r}}{\hat{r}} \chi(\hat{r}) \cos [2(\phi - \phi_s)] \right) \right) \right).$$

(6.24)

If one defines radial functions $\mu_0(\hat{r}) = \int_0^\infty \kappa_s\hat{r}d\hat{r}$, which is basically a scaled mass associated with the circularly symmetric part of the convergence, and $\mu_0(\hat{r}) = \int_0^\infty \kappa_s\hat{r}d\hat{r}$, then the potential (up to an addictive constant) may also be written in (assuming $\phi_s$ is constant)

$$\psi(r, \phi) = 2\phi_0 + 2 \int_\infty^\infty \frac{d\hat{r}}{\hat{r}} \frac{\mu_0(\hat{r})}{\hat{r}} \left( \frac{1}{2} \int_\infty^\infty \frac{d\hat{r}^2}{\hat{r}} \chi(\hat{r}) \cos [2(\phi - \phi_s)] \right).$$

(6.25)
where \( \mathbf{\xi} \) is some fiducial radius, \( \phi_0 \) is a constant that may be formally defined by \( \phi_0 \equiv \mu_0(x) \ln \mathbf{\xi} + \int_0^x \mu'(r) \ln r \, dr \). In addition, one can also find the deflection function

\[
\alpha = 2 \partial_\phi \psi = 2 \frac{\partial_\phi \mu}{r^2} \mathbf{\xi} + \frac{\mu}{r^2} \int_0^x \frac{\partial_\phi \mu(r)}{r} \hat{\mathbf{r}} \, e^{-2\phi_0} - \frac{2}{3} \int_0^x \frac{\partial_\phi \mu^2}{r} \, e^{-2\phi_0}.
\]

(6.26)

One can note that the potential given by equation (6.25) can be considered as a circularly symmetric potential perturbed by a small perturbing potential of the form of equation (6.10) with \( n = 2 \) provided that both of the integrals are small. Assuming that the circularly symmetric part of the potential in equation (6.25) allows to form an Einstein ring at \( r = r_0 \) [i.e., \( 2\delta_0(r_0) = r_0 \)], the linear approximation of the tangential caustics is given by

\[
\frac{\zeta(\phi)}{\rho_0} = \frac{\left( f_0^* + g_0^* \right)}{2} \left[ 3 e^{i(\phi - \phi_0)} + e^{i(\phi + \phi_0)} \right] = 2 \left( f_0^* + g_0^* \right) \left[ \cos^4(\phi - \phi_0) - i \sin^4(\phi - \phi_0) \right]
\]

(6.27)

\[
f_0^* = \int_0^{\infty} \frac{d\xi}{r_0} \left( \frac{r^2}{r_0} \right) \mu(r) \, e^{-2\phi_0} \quad \text{and} \quad g_0^* = \int_0^{\infty} \frac{d\xi}{r_0} \left( \frac{r_0}{r} \right)^2 \frac{\mu^2}{r} \, e^{-2\phi_0}.
\]

(6.28)

Here, the caustic is a tetracosipoid so that there are four images for a source position within the caustic. Equation (6.27), from a comparison to equations (6.23), (6.24), and (6.26), also implies that the cross-section of quadruple images is \((3\pi/2)(f_0^* + g_0^*)^2\). Furthermore, the lensing behaviour of the source near the lens centre is essentially described by two constants \( f_0^* \) and \( g_0^* \). In particular, when the source and the centre of the lens are perfectly aligned \( (\zeta = 0) \), there are four images associated with the caustics, and their azimuthal coordinates are given by \( \phi = \pi r/2 + \phi_0 \), where \( \rho_0 \) is an integer. It is easy to verify that equation (6.24) is real and that

\[
z = (r_0 + \delta r) e^{i\phi} = r_0 e^{i\phi} \left[ 1 + \left( -1 \right)^n \frac{(f_0^* + g_0^*)}{2(1 - \kappa_{\infty})} \right] e^{-\phi_0} \quad \text{and} \quad \mathcal{J} = \left( -1 \right)^n \left( 1 - \kappa_{\infty} \right)(f_0^* + g_0^*)
\]

(6.29)

so that the resulting images are an alternating-parity equal-magnification two-fold symmetric crucifix-form quartet with an axis ratio of \( 1 - \left| (f_0^* - g_0^*)/(1 - \kappa_{\infty}) \right| \). For more general source positions, once the caustic is drawn using equation (6.27), the azimuthal coordinates and the individual magnifications of the images can be found by following the same procedure as in Section 6.1 using equation (6.14). In addition, since the form of the perturbation here is specified, it is also possible to find the radial coordinates of the images by the equivalent of equation (6.2)

\[
\frac{d\phi}{2(1 - \kappa_{\infty})}(\zeta + r_0 \left[ e^{i(\phi + 2\phi_0)} f_0^* - e^{i(\phi - 2\phi_0)} g_0^* \right]) = \frac{\xi e^{-i\phi}}{2(1 - \kappa_{\infty})} \left( f_0^* - g_0^* \right) \cos[2(\phi - \phi_0)] + i f_0^* + g_0^* \sin[2(\phi - \phi_0)]
\]

(6.30)

for the given radial coordinate of the images \( \phi \). Algebraically, \( \phi \) can be determined from the condition that \( d\phi \) is zero, that is, \( \mathcal{J} [\xi e^{-i\phi}] + (f_0^* + g_0^*) \sin[2(\phi - \phi_0)] = 0 \). In other words, for a given source position, \( \phi \) is determined by \( f_0^* + g_0^* \), which is in accordance with the geometric construction of \( \phi \) since it is the same coefficient for the caustics (eq. 6.27).

Note the linear approximation of the caustics (eq. 6.27) is identical (after a proper rotation of coordinate) to those caused either by the null-convergent compact quadrupole perturbation with the quadrupole moment of \((f_0^* + g_0^*) \rho_0^2\) or by the null-convergent tidal perturbation by a constant external shear of \( (f_0^* + g_0^*) \) (c.f. Witt, Mao, & Schechter 1992). In fact, the examination of equation (6.26) suggests that those locally (near Einstein ring region) null-convergent perturbations can be caused by the quadrupole moment of the mass distribution, either entirely enclosed within the angular Einstein ring radius \( (f \propto \mathcal{E}^2) \), or lying completely outside of the angular Einstein ring radius \( (f \propto \mathcal{E}) \). In general, the lensing behaviour for the source near the lens centre when the convergence is given by equation (6.23) with \( \mathcal{E} \ll 1 \) is basically equivalent to that of the system with a circularly symmetric convergence \( \kappa_0 \) under the perturbation of \( f = r_0 \left( \bar{f} / f_0 \right)^3 - \bar{g}_0 \) \( (\bar{f} / f_0) \), where \( f_0^* = f_0 e^{2\phi_0} \) and \( g_0^* = g_0 e^{2\phi_0} \). This can be generalized to the elliptically symmetric system that is additionally affected by external shear and/or compact quadrupole mass moment.\(^5\) For this case, \( f_0^* = f_0 e^{2\phi_0} + \hat{Q}_4 \) and \( g_0^* = g_0 e^{2\phi_0} + \gamma_0 \), and they are not necessarily parallel to each other any more. Here, note that \( \hat{Q}_4 = \hat{Q}_4 e^{2\phi_0} \) and \( \gamma_0 = |\gamma_0| e^{2i\phi_0} \) are complex numbers. (Strictly speaking, they are complex number representations of certain symmetric 2×2 traceless tensors. The absolute value of the complex number corresponds to the positive eigenvalue of the tensor and its argument is the same as the angle between the corresponding eigenvector and the real direction.) Since equations (6.14) and (6.15) are still valid regardless of \( f_0 \) and \( g_0 \) not being parallel, one finds, in the place of equation (6.27), \(~\text{note the factor of two difference in the definition compared to eq. (6.12)~}\) that

\[
\zeta(\phi) = \frac{\rho_0 e^{i\phi}}{2} \left[ f_0^* + g_0^* \right] \left[ 3 e^{i(\phi+\phi_0)} + e^{i(\phi-\phi_0)} \right] = 2 \rho_0 e^{i\phi} \left[ f_0^* + g_0^* \right] \left[ \cos^4(\phi - \phi_0) - i \sin^4(\phi - \phi_0) \right]
\]

(6.31)

and also that

\(^5\) In much of literature, a quadrupole lens and a lens with shear are used interchangeably without distinction. Although this practice has a certain justification since both the effects due to the constant external shear and the compact quadrupole moment are associated with \( 2\phi \) terms in the azimuthal part of the potential, they are always explicitly distinguished in this paper.

\(^6\) The sign for the external shear is chosen to follow the usual convention in strong lensing literatures (e.g., Huterer et al. 2004). This choice leads the lensing shear due to the external shear term to be \( -\gamma_0 \), as defined in Appendix II.
in place of equation (6.30). Here, $f'_0 + g'_0 = |f'_0 + g'_0|e^{2i\delta}$ and $f'_0 - g'_0 = |f'_0 - g'_0|e^{2i\phi}$. As before, $\delta r \in \mathbb{R}$ for $\phi$ that is the azimuthal coordinate of the image for the given source position. For $\xi = 0$, one finds the azimuthal coordinates of images $\phi = n\pi/2 + \phi_s$ and also that $2(1 - \kappa_{s,0})\delta r = -(1)^{i}r_{0}(f'_0 - g'_0)|\cos[2(\phi - \phi_s)]$. This implies that the resulting image quartets are identical for the set of $f'_0$ and $g'_0$ if $\{f'_0 + g'_0\}$ and $\{f'_0 - g'_0\}|\cos[2(\phi - \phi_s)]$ are constant. For a more general source position $\xi \neq 0$, while one finds that the azimuthal coordinates and the magnification ratios of images are completely specified by $f'_0 + g'_0$, the projection that controls the radial coordinates depends on each azimuthal coordinate of the image, so that they are in general not common for the four images. However, if the source is sufficiently close to the centre, the projections are more or less parallel to one another, and therefore, there still exists a certain near degeneracy of the images concerning the choice of $\{f'_0, g'_0\}$, and their relative orientation in the underlying lens model.

If the convergence is given by a power-law ellipsoid

\[\kappa_i(r, \phi) = \frac{\beta}{2} \left( \frac{a_0}{r} \right)^{2-\beta} = \beta \left( \frac{a_0}{r} \right)^{2-\beta} \left( \frac{r^2 \sin^2 \phi + \cos^2 \phi}{1 + \rho^2} \right)^{(2-\beta)/2} = \beta \left( \frac{a_0}{r} \right)^{2-\beta} \left( \frac{2\rho^2}{1 + \rho^2} \right)^{(2-\beta)/2} \left( 1 - \frac{1 - \rho^2}{1 + \rho^2} \cos 2\phi \right)^{(2-\beta)/2},\]

it can be approximated by a quadrupole convergence in a form of equation (6.23) with

\[\kappa_i = \frac{\beta}{2} \left( \frac{a_0}{r} \right)^{2-\beta} ; \quad \xi = \frac{1 - \rho^2}{1 + \rho^2} \left( 1 - \frac{\beta}{2} \right).\]

Here, $0 < \beta < 2$. In addition,

\[\mu_0(r) = \int_0^r d\varphi \kappa_i(r, \varphi) = \frac{\beta}{2} \left( \frac{2\rho^2}{1 + \rho^2} \right)^{(2-\beta)/2} \int_0^r d\varphi \varphi^{2-\beta} = \frac{\beta}{2 \beta + 2} \int_0^r d\varphi \varphi^{2-\beta} = \frac{\xi \beta}{2(\beta + 2)} = \frac{1 - \rho^2}{1 + \rho^2} \frac{\beta - 2}{4 + 2 \beta}.\]

so that one finds $n_0 = 2^{1/2}a_0(1 + \rho^2)^{-1}$ by solving $2\mu_0(r_0) = r_0^2$. Furthermore, $\mu_1(r) = \xi \mu_0(r)$ since $\xi$ is now constant, and $2\kappa_{s,0} = \beta$. Note that $2\mu_0(r) = r_{0}^{2-\beta}e^\beta$ and $(d \ln \mu_0 / d \ln r) = (d \ln \mu_1 / d \ln r) = \beta$. Finally,

\[f'_0 = \int_0^r \frac{dx}{r_0^2(r_0)^{2-\beta}} = \frac{1}{2} \int_0^r d\varphi \varphi^{2-\beta} = \frac{\xi \beta}{2(\beta + 2)} = \frac{1 - \rho^2}{1 + \rho^2} \frac{\beta - 2}{4 + 2 \beta} \]

and thus

\[f'_0 + g'_0 = \frac{1 - \rho^2}{1 + \rho^2} \frac{\beta^2}{2} ; \quad f'_0 - g'_0 = \frac{1 - \rho^2}{1 + \rho^2} \frac{\beta^2}{2(2 + \beta)}.\]

Hence, for a singular isothermal ellipsoid (SIE) lens ($2\kappa_{s,0} = \beta = 1$), with an ellipticity parameter $\varepsilon = (1 - \rho^2)/(1 + \rho^2) = 1 - \rho$, one finds that $f'_0 + g'_0 = \varepsilon/3$ and $f'_0 - g'_0 = -\varepsilon/6$, and thus, the four-image cross-section is given by $(3\pi/2)(f'_0 + g'_0)^2 = (\pi/6)^2$, and the axis ratio for a crucifix-form quartet is $1 - \varepsilon(\gamma + \delta - 1)/3 = 1 - (\varepsilon/3)$ (Keeton, Kochanek, & Seljaas 1997).

If this SIE lens is subject to perturbations due to the constant external shear, one finds that

\[f'_0 = \frac{\varepsilon}{12} e^{2i\delta} = \frac{\varepsilon}{12} \left[ (e^{2i\delta} + 4|\gamma_0|e^{2i\phi_0}) - |\gamma_0|e^{2i\phi_0} \right], \quad g'_0 = \frac{\varepsilon}{4} e^{2i\delta} + |\gamma_0|e^{2i\phi_0} = \frac{1}{4} \left( e^{2i\delta} + 4|\gamma_0|e^{2i\phi_0} \right).\]

By comparing this to another SIE lens system that is affected by the compact quadrupole moment, $f'_0 = (\varepsilon/12)e^{2i\delta} + |\gamma_0|e^{2i\phi_0}$ and $g'_0 = (\varepsilon/4)e^{2i\delta}$, one can conclude that the lensing behaviours of two systems are identical up to the linear approximation provided that the external shear and the compact quadrupole moment are related by $|\gamma_0| = 3|Q_4|$ and $e^{2i\phi_0 - \delta} = -1$ and the two ellipticity parameters by

\[e^{2i\phi_0 - \delta} = \frac{1}{\varepsilon} \left[ 1 + \frac{4|\gamma_0|}{e^{2i\phi_0 - \delta}} \right] ; \quad e^{2i\phi_0 - \delta} = \frac{1}{\varepsilon} \left[ 1 + \frac{4|Q_4|}{e^{2i\phi_0 - \delta}} \right].\]

In other words, the lens system that can be modelled by a SIE lens with an external shear can also be modelled by a different SIE lens with a quadrupole moment, the magnitude of which is a third of that of the external shear, provided that the system’s departure from the circular symmetry is small. This can be further generalized to systems that are subject to an external shear and a compact quadrupole moment at the same time to yield a certain degeneracy of the lens model regarding the ellipticity, the shear, and the quadrupole moment (as well as their relative orientations).

If one models an image quartet using a SIE lens with a constant external shear but no quadrupole moment, then $f'_0 + g'_0 = |f'_0 + g'_0|e^{2i\delta} = (\varepsilon/3)e^{2i\delta} + |\gamma_0|e^{2i\phi_0} = (\varepsilon/3)e^{2i\delta} + |\gamma_0|e^{2i\phi_0}$ and $f'_0 - g'_0 = |f'_0 - g'_0|e^{2i\phi} = (\varepsilon/6)e^{2i\delta} - |\gamma_0|e^{2i\phi}$. Hence, the four-image cross-section is found to be

\[\frac{3\pi}{2} |f'_0 + g'_0|^2 = \pi \left( \frac{\varepsilon}{6} + \frac{3|\gamma_0|^2}{2} + |\gamma_0| \cos [2(\phi_0 - \phi_s)] \right).\]

It is easy to see that, for given $\varepsilon$ and $|\gamma_0|$, the cross-section is largest [i.e., $\pi(\varepsilon + 3|\gamma_0|^2)/6$] if $\phi_0 = \phi_s$ while it is at minimum [$\pi(\varepsilon - 3|\gamma_0|^2)/6$] when $\phi_0 - \phi_s = \pi/2$. In addition, one can find that
\[
\left( \frac{\lambda}{3} \right)^2 - |\gamma| = |f_0^* + g_0^*| |3|f_0^* + g_0^*| + 4|f_0^* - g_0^*| \cos[2(\phi_+ - \phi_-)]
\]

That is to say, for fixed \( |f_0^* + g_0^*| \) and \( |f_0^* - g_0^*| \cos[2(\phi_+ - \phi_-)] \), which produce identical crucifix-form image quartets, there exists a degeneracy running along a hyperbolic path in \((\epsilon, |\gamma|)\) space. This may be seen as a prototype for the ‘cancellation branches’ of the ‘U’ shape degenerate path found in fig. 8 of Keeton et al. (1997). In reality, it is highly unlikely that the source is perfectly aligned with the lens centre so that the actual degeneracy becomes rather more complex. (Furthermore, as the ellipticity and/or the shear get larger, the whole perturbative approach will start to break down.) Nevertheless, the basic idea that there exists a certain set of \( f_0^* + g_0^* \) and \( f_0^* - g_0^* \) that yields similar (or nearly degenerate) image configurations (with the proper rotation of frames) is thought to be more or less valid, and this can lead to the observed degeneracy of the shear and ellipticity combination.

ACKNOWLEDGMENT

The author thanks N. Wyn Evans for pointing out possible connection of the subject to macrolens modelling. The author is grateful to A. Gould for his careful reading of the manuscript. Fig. 2 is reproduced from Albrow et al. (2002), in a modified form, by permission of the American Astronomical Society.

REFERENCES

Ahlfors L. V., 1979, Complex Analysis. McGraw-Hill, New York
Afonso C. et al., 2000, ApJ, 532, 340
Albrow M. D. et al., 2002, ApJ, 572, 1031
Arfken G. B., Weber H. J., 2000, Mathematical Methods for Physicists. Academic Press, London
Bond I. A. et al., 2004, ApJ, 606, L155
Bourassa R. R., Kantowski R., 1975, ApJ, 195, 13
Bozza V., 1999, A&A, 348, 311
Bozza V., 2000, A&A, 355, 423
Chang K.,Refsdal S., 1979, Nature, 282, 561
Chang K.,Refsdal S., 1984, A&A, 132, 168
Courant R., Hilbert D., 1962, Methods of Mathematical Physics, volume II. John Wiley & Sons Inc., New York
Dominik M., 1999a, A&A, 341, 943
Dominik M., 1999b, A&A, 349, 108
Gaudi B. S., Gould A., 1987, ApJ, 486, 85
Gaudi B. S., Petters A. O., 2002, ApJ, 574, 970
Gould A., 1996, ApJ, 470, 201
Gould A., 2004, ApJ, 606, 319
Gould A., Loeb A., 1992, ApJ, 396, 104
Huterer D., Keeton C. R., Ma C.-P., 2004, ApJ, submitted (astro-ph/0405040)
Keeton C. R., Kochanek C. S., Seljak U., 1997, ApJ, 482, 604
Kovner I., 1987a, ApJ, 312, 22
Kovner I., 1987b, ApJ, 316, 52
Mao S, Paczyński B., 1991, ApJ, 374, 37
Paczyński B., 1986, ApJ, 304, 1
Smith, M. C., Mao S., Paczyński B., 2003, MNRAS, 339, 925
Witt H. J., 1990, A&A, 236, 311
Witt H. J., Mao S., 1995, ApJ, 447, L105
Witt H. J., Mao S., Schechter P. L., 1995, ApJ, 443, 18

APPENDIX A: COMPLEX ANALYTIC FUNCTION

Let \( f = u(x,y) + iv(x,y) \) be a function of \( z = x + iy \) in the region \( R \subset \mathbb{C} \) containing \( z_0 \), where \( u \) and \( v \) are real-valued functions of real variables \( x \) and \( y \). If there exist continuous partial derivatives at \( z = z_0 \) and they satisfy a differential constraint in the form of coupled differential equations given by

\[
\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0,
\]

one can show that \( f \) is complex differentiable, that is, there exists a unique complex derivative

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at $z = z_0$. (Here and in the following appendices, the subscripted comma notation for partial derivatives is used, that is, $u_x = \partial_x u = \partial u/\partial x$.) Furthermore, if the condition given by equation (A1) is met for all points in $R$, then the function $f$ is complex differentiable arbitrarily many times (i.e., infinitely complex differentiable) at every point in $R$. In general, a complex-valued complex-variable function $f$ is referred to as an analytic function (or a holomorphic function) in $R$ if $f$ is complex differentiable at every point in $R$. Note that $f$ is analytic in $R$ if and only if it satisfies the condition given by equation (A1) in $R$, which is usually referred to as Cauchy-Riemann condition. If $f = u + iv$ is an analytic function, Cauchy-Riemann condition (A1) implies that

$$u_{xx} + u_{yy} = (u_x)_x + (u_y)_y = (v_x)_x + (v_y)_y = v_{xx} - v_{yy} = 0,$$

$$v_{xx} + v_{yy} = (v_x)_x + (v_y)_y = -(u_x)_x + (u_y)_y = -u_{xx} + u_{yy} = 0.$$  

That is, both component functions $u$ and $v$ satisfy the (two-dimensional) Laplace equation. In general, a function $F : \mathbb{R}^2 \to \mathbb{R}^2$ is referred to as a harmonic function if it satisfies Laplace equation $\nabla^2 F = (\partial^2_x + \partial^2_y)F = F_{xx} + F_{yy} = 0$. In other words, both the real and imaginary parts of any analytic function are harmonic.

If one considers a complex conjugation $\bar{z} = x - iy$ of a complex variable, the function $f = u + iv$ may be regarded as a function of $z$ and $\bar{z}$, by the relations $x = (z + \bar{z})/2$ and $y = (i(z - \bar{z})/2$. Then, formally, one may consider a partial derivative

$$f_z = u_z + iv_z = (u_x, x_2 + u_y, y_2) + i(v_x, x_2 + v_y, y_2) = \frac{(u_x + iu_y) + i(v_x + iv_y)}{2} = \frac{u_x - v_y}{2} + i\frac{u_y + v_x}{2}.$$  

That is, $f_z = 0$ if and only if $f$ is analytic. In other words, an analytic function may be casually understood as a complex function of $z$ alone. (Roughly speaking, $f$ is analytic if it can be expressed by complex constants and elementary differentiable real functions with their arguments replaced by the complex variable $z$.)

If $f$ is analytic in $R$ containing $z_0$, there exists a convergent Taylor series expression of $f$ at $z_0$

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n = f(z_0) + f'(z_0)(z - z_0) + \frac{f''(z_0)}{2}(z - z_0)^2 + \frac{f'''(z_0)}{6}(z - z_0)^3 + \cdots ,$$  

(A2)

where $f^{(n)}(z) = d^n f/dz^n$. A Taylor series (absolutely) converges within its radius of convergence. That is, there exists a positive real number $r$ such that the series in equation (A2) converges if $|z - z_0| < r$. A complex function $f$ is said to have a pole of order $n$ at $z_0$ if $\lim_{z \to z_0} (z - z_0)^n f(z)$ is unbounded for non-negative integers $m < n$ and $(z - z_0)^m f(z)$ is analytic in a region containing $z_0$ for positive integers $m \geq n$. If $z_0 \in R$ is a pole of order $n$ of a function $f$ that is analytic in $R - \{z_0\}$, then $f$ has a series expression of the form of

$$f(z) = \frac{(z - z_0)^m f(z)}{(z - z_0)^n} = (z - z_0)^{-n} \sum_{m=0}^{\infty} \frac{1}{m!} \frac{d^m}{dz^m} \left[ (z - z_0)^n f(z_0) \right]_{z_0} (z - z_0)^m = \sum_{m+n=n}^{\infty} \frac{1}{(m+n)!} \frac{d^{m+n}}{dz^{m+n}} \left[ (z - z_0)^n f(z_0) \right]_{z_0} (z - z_0)^m,$$

which converges if $|z - z_0| < r$ and $r \neq z_0$ where $r$ is the radius of convergence. Note that the series expansion of a complex function of the form of $f = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ is generally referred to as Laurent series. In other words, if the $a_n$'s are coefficients of the Laurent series expansion of $f$ at a pole of order $n$, then $a_n = 0$ for negative integers $m < -n$ and $a_n \neq 0$. If $f$ is analytic in $C$ except at isolated poles, the radius of convergence for a Taylor series at a point $z_0$ that is not a pole is the same as the distance to the nearest pole while that for a Laurent series at a pole is also given by the distance to the next nearest pole.

### Appendix B: Lens Equation with Complex Numbers

In general, the lens equation for a given lensing potential $\psi$ is given by

$$y = x - \nabla_0 \psi(x),$$  

(B1)

where $y$ is the angular position of the source and $x$ is the angular position of the image. In addition, the lensing potential is related to the mass column density per unit cross-sectional area of lens $\Sigma$ via Poisson’s equation

$$\nabla_0^2 \psi = \frac{8\pi G D_{20} \Sigma}{c^2},$$  

(B2)

where $D_{02} \equiv D_{0S} D_{S2}^{-1}$ is the reduced distance$^7$. Note that one may also rewrite equation (B2) using the surface mass density per unit solid angle $\Sigma_A = \Sigma D_{02}^2$ as in $\nabla_0^2 \psi = \frac{8\pi G D_{\text{rel}}^2}{c^2} \Sigma_A$. Here, also used is that $D_{R} D_{\text{rel}} = D_{L}^2$.

By choosing some axis as a real axis, one may rewrite the lens equation (B1) using complex numbers;

$$\zeta = z - \alpha(z, \bar{z}).$$  

(B3)

7 If $D_{S} = D_{0} + D_{LS}$, then $D_{L}^2 = D_{0}^2 + D_{LS}^2$. 

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where \( z = x_1 + i x_2, \ \zeta = y_1 + iy_2 \) and \( \alpha = \psi_{i1} + i \psi_{i2} \). Here \( \psi_{i1} = \psi_{x1} \) and so on. The complex deflection function \( \alpha \) is related to the lensing potential \( \psi \) by

\[
\psi_z = \psi_{x1,i1} + \psi_{x2,i2} = \frac{\psi_{i1} + i \psi_{i2}}{2} = \frac{\alpha}{2},
\]

\[
\alpha_z = \alpha_{x1,i1} + \alpha_{x2,i2} = \frac{(\psi_{i1} + i \psi_{i2}) - i(\psi_{i1} + i \psi_{i2})}{2} = \frac{(\psi_{i1} + \psi_{i2}) + i(\psi_{i1} - \psi_{i2})}{2} = \frac{1}{2} \nabla_z^2 \psi = \frac{4 \pi G D_{\nu k} \Sigma}{c^2}.
\]

Here, note that \( x_1 = (z + \bar{z})/2 \) and \( x_2 = i(z - \bar{z})/2 \). (The above equations also imply that \( 2F_z = F_{i1} + i F_{i2} = (\nabla \psi)_{i1} + i(\nabla \psi)_{i2} \) and \( 4F_{i2} = \nabla^2 \psi \) for any real valued function \( F \).) Sometimes, \( \alpha_z \) and \( \alpha_z \) are also referred to as the convergence \( \kappa \) and the shear \( \gamma \), respectively, of the lensing potential. Note the differential relation connecting the convergence and the shear, \( \kappa_{z2} = \alpha_{z2} = \gamma_z \). The convergence \( \kappa \) is always a non-negative real number, which is related to the mass column density by \( \kappa = (4 \pi G D_{\nu k} e^{-2}) \Sigma - \) some authors also define a critical mass column density \( \Sigma_c = c^2/(4 \pi G D_{\nu k}) \), and then \( \kappa = \Sigma/\Sigma_c \). On the other hand, the shear \( \gamma \) here is a complex number (i.e., it has two independent components);

\[
\gamma = \alpha_{z2} = \alpha_{x1,i1} + \alpha_{x2,i2} = \frac{(\psi_{i1} + i \psi_{i2}) + i(\psi_{i1} + i \psi_{i2})}{2} = \frac{(\psi_{i1} - \psi_{i2}) + i(\psi_{i1} + \psi_{i2})}{2} = \frac{1}{2} \nabla_z^2 \psi = \frac{4 \pi G D_{\nu k} \Sigma}{c^2}.
\]

It is notable, however, that the corresponding shear in real 2-dimensional notation does not behave as a vector under the coordinate transformation (in particular, rotation of coordinate axes). Rather it is a symmetric tensor of a rank of two with a null contraction — a real symmetric 2×2 traceless matrix, that is,

\[
\gamma = \begin{pmatrix} \Re[\gamma] & \Im[\gamma] \\ \Im[\gamma] & -\Re[\gamma] \end{pmatrix} = \begin{pmatrix} \psi_{x1} - \psi_{x2} & 2\psi_{y1} \\ 2\psi_{y2} & \psi_{x1} + \psi_{x2} \end{pmatrix} = |\gamma| \begin{pmatrix} \cos 2\phi & \sin 2\phi \\ \sin 2\phi & -\cos 2\phi \end{pmatrix},
\]

where \( 2\phi \) is the argument of \( \gamma \), i.e., \( \gamma = |\gamma| e^{i2\phi} \). The two eigenvalues of \( \gamma \) are \( \pm |\gamma| \) and its determinant is \( -|\gamma|^2 = -\gamma \bar{\gamma} \). Note that the unit eigenvector associated with the positive eigenvalue \( |\gamma| \) is \( (\cos \phi, \sin \phi)^T \) whose direction defines the direction of the shear. On the other hand, the second unit eigenvector (−\( \sin \phi, \cos \phi \))^T that is associated with the negative eigenvalue \( -|\gamma| \) is perpendicular to it. It is also notable that the lensing shear field should satisfy the differential constraint that \( \gamma_{z2} = \gamma_{x1,i1} \in \Re \), that is, \( 2\gamma_{y1,i1} = \gamma_{x1} - \gamma_{y2,i2} \), where \( \gamma_{x1} = \Re[\gamma] = |\gamma| \cos 2\phi \) and \( \gamma_{y2,i2} = \Im[\gamma] = |\gamma| \sin 2\phi \). This constraint is the direct result of the fact that the lensing deflection can be described by a scalar potential — i.e., \( \gamma_{z2} = \kappa_{z2} = 4 \nabla^2 \kappa = 8 \nabla^2 \psi \) and \( \nabla^2 \) is a scalar operator.

In the case of the null convergence \( (\kappa = 0) \), equation (B3) becomes Laplace equation so that \( \psi \) becomes a harmonic function. Then, one can find a complex analytic function \( \psi_z(z) = \psi(x_1, x_2) + i(\varphi(x_1, x_2)), \) where a function \( \varphi \) satisfies the Cauchy-Riemann condition for \( \psi_z(z) \). Then, it is easy to show that

\[
\psi_z = \psi_z + i \varphi_z = \frac{\psi_{i1} - i \psi_{i2}}{2} + \frac{i \psi_{i1} - i \psi_{i2}}{2} = \frac{1}{2} (\psi_{i1} - i \psi_{i2} - i \psi_{i2} + \psi_{i1}) = \bar{\alpha},
\]

which also indicates that, for the case of the null convergence, \( \bar{\alpha} \) becomes also a complex analytic function since it is a total complex derivative of an analytic function.

If the lensing potential is given by

\[
\psi_z = \frac{4Gm}{D_{\text{tot}} c^2} \ln(z - l),
\]

\[
\psi = \frac{\psi_z + \bar{\psi}_z}{2} = \frac{2Gm}{D_{\text{tot}} c^2} \left[ \ln(z - l) + \ln(\bar{z} - \bar{l}) \right] = \frac{4Gm}{D_{\text{tot}} c^2} \ln(|z - l|),
\]

the corresponding convergence is found to be

\[
\kappa = \frac{4 \pi Gm}{D_{\text{tot}} c^2} \delta^2(x - l) = \frac{1}{\sum_k m_k D_{\text{tot}} D_{\text{tot}}} \delta^2(x - l),
\]

where \( \delta^2(x) \) is the 2-dimensional Dirac delta function and \( l \) is the lens position in real 2-dimensional notation. In other words, \( \Sigma = m D_{\text{tot}}^{-2} \delta^2(x - l) \), so that the potential describes lensing by a point mass \( m \) in empty space lying at \( l \) (note that \( D_{\text{tot}} D_{\text{tot}} = D_l^2 \)). Moreover, since Poisson’s equation is linear in mass (density), the lensing by a finite number of multiple point masses can be described by the sum of individual potentials. Then, the deflection function is found to be

\[
\alpha = \bar{\psi} \frac{4G}{D_{\text{tot}} c^2} \sum_k \frac{m_k}{z - l_k}.
\]

With this deflection function, the lens equation (B3) is reduced to a particularly simple (dimensionless) form

\[
\zeta = \sum_k \frac{q_k}{z - l_k} ; \quad q_k = \frac{m_k}{M},
\]

provided that every angular measurement is rescaled by the angular Einstein radius (eq. (2.2)) corresponding to some mass \( M \). Here, \( m_k \) is the mass of \( k \)-th component located at the complex (angular) position \( l_{z_k} \). While it is customary that the lens equation (B6) is rescaled by the
Einstein ring radius corresponding either to the total mass of system ($\sum q_i = 1$) – usually when several component masses are comparable – or to the most massive component ($q_1 = 1$) – usually when the mass of the system is dominated by a single component –, it is equally valid to choose any mass $M$ as a unit of mass measurement as long as it is connected to the unit of angular measurement $\theta$ by $D_{oc}\theta^2 = 4GM$.

The local (differentiable) behaviour of the lens mapping can be studied from the Jacobian matrix of the lens equation, which is basically the linear transformation that approximates the lens mapping locally. For the complex lens equation (83), its Jacobian matrix is

$$J = \begin{pmatrix} \xi_z & \eta_z \\ \zeta_z & \xi_z \end{pmatrix} = \begin{pmatrix} 1 - \alpha_z - \xi_z \\ -\alpha_z - \eta_z \end{pmatrix} = \begin{pmatrix} 1 - \kappa - \gamma \\ -\gamma \end{pmatrix} - (1 - \kappa - \gamma).$$  \hspace{0.5cm} \text{(B7)}

Here, $\xi_z = 1 - \alpha_z = 1 - \kappa$ and $\zeta_z = -\alpha_z = -\gamma$. Since this matrix is Hermitian – the corresponding Jacobian matrix of the lens equation (B1) with real 2-dimensional notation is real symmetric –, it has two real eigenvalues, which are $\lambda_\pm = 1 - (\kappa \pm |\gamma|)$. Note that $\lambda_+ + \lambda_- = 2(1 - \kappa)$ and $\lambda_+ - \lambda_- = 2|\gamma| \leq 0$. Thus, $\lambda_1^2 - \lambda_2^2 = 4|\gamma|\kappa$ (1) and so $|\lambda_+| < |\lambda_-|$ if $\kappa < 1$ and vice versa. Suppose that the principal argument of $\gamma = |\gamma|e^{i\phi}$ is $2\phi \in [0, 2\pi)$. Then, one finds that $E_+ = e^{i\phi}$ and $E_- = iE_+$. are the unit eigenvectors of the Jacobian matrix, whose associated eigenvalues are $\lambda_\pm$, respectively. Note that these two eigenvectors of the Jacobian matrix are also the eigenvectors of the shear tensor (eq. (B5)). This is a natural consequence of the fact that $J = (1 - \kappa)I - \gamma$, where $J$ is the real 2-dimensional Jacobian matrix and $I$ is a $2 \times 2$ identity matrix. Finally, Jacobian determinant is invariant under the transformation of the lens equation (81) to the complex equation (83), that is,

$$d\xi \wedge d\eta = (dx_1 + idx_2) \wedge (dy_1 - idy_2) = -2i dyn \wedge dy_2 = -2i f dx_1 \wedge dx_2 = -2i f \left[ \left( \frac{dz + \bar{d}z}{2} \right) \wedge \left( \frac{dz - \bar{d}z}{2i} \right) \right] = J dz \wedge d\xi.$$

Hence,

$$J = \text{det} \; J = \lambda_+, \lambda_- = (1 - \kappa - |\gamma|)(1 - \kappa - |\gamma|) = (1 - \kappa)^2 - |\gamma|^2.$$

**APPENDIX C: SECOND ORDER PERTURBATION CAUSTIC**

With the lens mapping given by equation (3.1), the linear perturbation approach becomes inadequate or at least insufficient as the magnitude of the perturbation grows. Here, the discussion in Section 3.1 is extended to find the appropriate expression of the caustic up to second order of the perturbation.

When the lens mapping is described by equation (3.1), its Jacobian determinant is found as equation (3.4) or

$$J = 1 - \frac{1}{z^2} + \epsilon (\hat{f(i)} \overline{\hat{f(i)}}) - \epsilon^2 f(i) \overline{f(i)}. \hspace{0.5cm} \text{(C1)}$$

If one Taylor-expands equation (C1) at $z_0 = e^{i\phi}$ and keeps up to the second order terms of $\delta z = z - z_0$ and $\epsilon$,

$$J = \epsilon(g_1 + \bar{g}_1) - e^{i\phi}g_1 \bar{g}_1 + [2 + \epsilon(g_2 - 2g_1)]e^{i2\phi} \delta z - 3e^{-2i\phi} \delta z^2 - 4\delta z \bar{g}_1 - 3e^{2i\phi} \delta z^2; \hspace{0.5cm} \text{(C2)}$$

where $g_1(\phi) = e^{i\phi}f(i)$. Also used are

$$\partial_z J = 2 \frac{z}{z^2} + e \left( \frac{-2f^{(1)}}{z^2} - \frac{f^{(2)}}{z^2} \right) + O(e^2); \hspace{0.5cm} \partial_z \bar{f} = 2 \frac{z}{z^2} + e \left( \frac{-2\overline{f^{(1)}}}{z^2} + \frac{\overline{f^{(2)}}}{z^2} \right) + O(e^2) \hspace{0.5cm} \text{(C1)}$$

$$\partial^2_z J = -\frac{6}{z^4} + O(e); \hspace{0.5cm} \partial_z \partial_z J = -\frac{4}{z^2} + O(e); \hspace{0.5cm} \partial^2_z \bar{f} = -\frac{6}{z^4} + O(e).$$

Next, one inserts $\delta z = (e^{i\phi} + e^{i2\phi}) de^{i\phi}$ into equation (C1):

$$J = \epsilon(4x_1 + g_1 + \bar{g}_1) + e^i \left[ 4x_2 - g_1 \bar{g}_1 + (g_2 + \bar{g}_2)x_1 - 2(g_1 + \bar{g}_1)x_1 - 10x_1^2 \right] + O(e^2).$$

Here, $x_1$ and $x_2$ are assumed to be real-valued. To find the expression for the second order approximation of the critical curve, one solves for $J = 0 + O(e^2)$, that is,

$$x_1(\phi) = \frac{g_1 + \bar{g}_1}{4}; \hspace{0.5cm} x_2(\phi) = \frac{2x_1^2 - (g_2 + \bar{g}_2)x_1 + g_1 \bar{g}_1}{32} = \frac{(g_1 + \bar{g}_1)^2}{16} + \frac{(g_1 + \bar{g}_1)(g_2 + \bar{g}_2)}{16} + \frac{g_1 \bar{g}_1}{4}.$$

Then, the critical curve is found to be $z_c(\phi) = z_0 + \delta z = e^{i\phi}(1 + e^{i\phi}X_2)$, which, up to the linear approximation, recovers equation (3.15). The corresponding caustic can be found as the (second-order) image of $z_c(\phi)$ under the lens mapping (3.1). To do this, first, one similarly Taylor-expands equation (3.1) at $z_0 = e^{i\phi}$ and keeps up to second order terms of $\epsilon$ and $\delta z$;

$$\delta z = e^{i\phi} \delta z + e^{i2\phi} \delta z^2 - e^{i\phi} \delta z - e^{i\phi} \delta z^2 + e^{i\phi} \delta z - e^{i\phi} \delta z^2 - e^{i\phi} \delta z - e^{i\phi} \delta z^2,$$

and replaces $\delta z$ by $(e^{i\phi} + e^{i2\phi}) e^{i\phi}$ and takes up to the quadratic terms of $\epsilon$;

$$\zeta(\phi) = e^{i\phi} \left[ e^{i(2x_1 - \bar{g}_1)X_2} + e^{i(2x_2 - \bar{g}_1)x_1} \right]. \hspace{0.5cm} \text{(C3)}$$

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Figure 4. Various approximation of the central caustic of the system described by the lens equation: $\zeta = z - \bar{z} - \frac{q}{(z - \bar{z})} - \frac{1}{q^2}$ where $q = 0.1$ and $z_\ell = 0.2$. The horizontal direction is same as the real direction. The parameters are chosen such that the linear approximation of Section 5.1 deviates noticeably from the true caustic although the perturbative approach does not fail completely. The curves shown in solid lines are (i) the linear order approximation in Section 5.1, (ii) the second order approximation in Appendix C, (iii) the linear caustic of the perturbation series of Section 5.2 keeping only the quadrupole term, and (iv) the linear caustic of the perturbation series of Section 5.2 keeping up to the octupole term. Also shown in dotted lines are the ‘true’ caustic found by the method of Witt (1990). which is again reduced to equation (3.17) and consequently equation (3.12) if one keeps only up to the linear term. One thing to note regarding the second order approximation, albeit obvious, is that the expression is no longer linear in the perturbation so that the superposition principle does not hold. That is, to properly account for all the cross terms, one always needs to work with all components of the perturbation together. An example of the second order perturbation caustic is shown in Fig. 4, along with alternative approximations for the same underlying caustic.