IN VariantS OF KnOTS AND 3-MANIfoldS FrOM QUANTum GROUPOİD S

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AbStrACT. We use the categories of representations of finite dimensional quantum groupoids (weak Hopf algebras) to construct ribbon and modular categories that give rise to invariants of knots and 3-manifolds.

Contents

1. Introduction 1
2. Quantum groupoids 2
3. Examples of quantum groupoids 4
4. Representation category of a quantum groupoid 7
5. Quasitriangular quantum groupoids 10
6. The Drinfeld double for quantum groupoids 13
7. Ribbon quantum groupoids 20
8. Towards modular categories 21
9. C∗-quantum groupoids and unitary modular categories 23
10. Appendix 27
References 30

1. Introduction

In [RT2] a general method of constructing invariants of 3-manifolds from modular Hopf algebras was introduced. After appearance of [RT3] it became clear that the technique of Hopf algebras can be replaced by a more general technique of monoidal categories. An appropriate class of categories – modular categories – was introduced in [T1]. In addition to quantum groups, such categories also arise from skein categories of tangles and, as it was observed by A. Ocneanu, from certain bimodule categories of type II1 subfactors.

The goal of this paper is to study the representation categories of quantum groupoids and to give in this way a new construction of modular categories. This extends the construction of modular categories from modular Hopf algebras and in particular from quantum groups at roots of unity.

By quantum groupoids, we understand weak Hopf algebras introduced in [DNSz], [BSz1], [N]. These objects generalize Hopf algebras, usual finite groupoid algebras...
and their duals (cf. [NV1]). We use the term “quantum groupoid” rather than “weak Hopf algebra”.

It was shown in [NV2], [NV3] that quantum groupoids and their coideal subalgebras are closely related to $\text{II}_1$-subfactors. Every finite index and finite depth $\text{II}_1$-subfactor gives rise to a pair consisting of a $C^*$-quantum groupoid and its left coideal subalgebra, and vice versa. It was also explained in [NV3] how to express the known subfactor invariants such as bimodule categories and principal graphs in terms of the associated quantum groupoids. In particular, the bimodule categories arising from a finite index and finite depth $\text{II}_1$ subfactors are equivalent to the unitary representation categories of the corresponding $C^*$-quantum groupoids ([NV3], 5.8).

Thus, it is natural to study categories of representations of quantum groupoids and to extend concepts known for Hopf algebras to this setting. We show that the representation category $\text{Rep}(H)$ of a quantum groupoid $H$ is a monoidal category with duality. We introduce quasitriangular, ribbon, and modular quantum groupoids for which $\text{Rep}(H)$ is, respectively, braided, ribbon, and modular. The notion of factorizability is extended from the Hopf algebra case and used to construct modular categories. We define the Drinfeld double $D(H)$ of a quantum groupoid $H$ and show that it is a factorizable quasitriangular quantum groupoid. For a $C^*$-quantum groupoid $H$, we similarly study the unitary representation category $U\text{Rep}(H)$.

It should be mentioned that the category $U\text{Rep}(H)$ for a $C^*$-quantum groupoid $H$ was previously introduced by G. Böhm and K. Szlachányi in [BSz2]; they also introduced the notion of an $R$-matrix and the Drinfeld double for $C^*$-quantum groupoids, see [BSz1].

Our main theorem (Theorem 9.8) reads: If $H$ is a connected $C^*$-quantum groupoid, then the category $U\text{Rep}(D(H))$ of unitary representations of $D(H)$ is a unitary modular category.

Thus, any finite index and finite depth $\text{II}_1$-subfactor yields a unitary modular category as follows: consider the associated connected $C^*$-quantum groupoid $H$, then the category $U\text{Rep}(D(H))$ is a unitary modular category. We conjecture that this construction is equivalent to the one due to A. Ocneanu (see [EK]). The key role in the proof of the main theorem is played by the following Lemma 8.2: If $H$ is a connected, ribbon and factorizable quantum groupoid with a Haar measure over an algebraically closed field, then $\text{Rep}(H)$ is a modular category.

The organization of the paper is clear from the table of contents.

2. QUANTUM GROUPOIDS

In this section we recall basic properties of quantum groupoids. Most of the material presented here can be found in [BNSz] and [NV1], see also the survey [NV4].

Throughout this paper we use Sweedler’s notation for comultiplication, writing $\Delta(b) = b_{(1)} \otimes b_{(2)}$. Let $k$ be a field.

**Definition 2.1.** A (finite) quantum groupoid over $k$ is a finite dimensional $k$-vector space $H$ with the structures of an associative algebra $(H, m, 1)$ with multiplication $m : H \otimes_k H \to H$ and unit $1 \in H$ and a coassociative coalgebra $(H, \Delta, \varepsilon)$ with comultiplication $\Delta : H \to H \otimes_k H$ and counit $\varepsilon : H \to k$ such that:
(i) The comultiplication $\Delta$ is a (not necessarily unit-preserving) homomorphism of algebras such that

$$\Delta(1) = (\Delta(1) \otimes 1)(1 \otimes \Delta(1)) = (1 \otimes \Delta(1))(\Delta(1) \otimes 1),$$

(ii) The counit is a $k$-linear map satisfying the identity:

$$\varepsilon(fgh) = \varepsilon(fg_1)\varepsilon(g_2h) = \varepsilon(fg_2)\varepsilon(g_1h),$$

for all $f, g, h \in H$.

(iii) There is an algebra and coalgebra anti-homomorphism $S : H \to H$, called an antipode, such that, for all $h \in H$,

$$m(id \otimes S)\Delta(h) = (\varepsilon \otimes id)((\Delta(1)h \otimes 1)),$$

$$m(S \otimes id)\Delta(h) = (id \otimes \varepsilon)((1 \otimes h)\Delta(1)).$$

A quantum groupoid is a Hopf algebra if and only if one of the following conditions holds: (i) the comultiplication is unit-preserving or (ii) if and only if the counit is a homomorphism of algebras.

A morphism of quantum groupoids is a map between them which is both an algebra and a coalgebra morphism preserving unit and counit and commuting with the antipode. The image of such a morphism is clearly a quantum groupoid. The tensor product of two quantum groupoids is defined in an obvious way.

The set of axioms of Definition 2.1 is self-dual. This allows to define a natural quantum groupoid structure on the dual vector space $\widehat{H} = \text{Hom}_k(H, k)$ by “reversing the arrows”:

$$\langle h, \phi\psi \rangle = \langle \Delta(h), \phi \otimes \psi \rangle,$$

$$\langle g \otimes h, \hat{\Delta}(\phi) \rangle = \langle gh, \phi \rangle,$$

$$\langle h, \hat{S}(\phi) \rangle = \langle S(h), \phi \rangle,$$

for all $\phi, \psi \in \widehat{H}$, $g, h \in H$. The unit $\hat{1} \in \widehat{H}$ is $\varepsilon$ and counit $\hat{\varepsilon}$ is $\phi \mapsto \langle \phi, 1 \rangle$.

The linear endomorphisms of $H$ defined by

$$h \mapsto m(id \otimes S)\Delta(h), \quad h \mapsto m(S \otimes id)\Delta(h)$$

are called the target and source counital maps and denoted $\varepsilon_t$ and $\varepsilon_s$, respectively. From axioms (3) and (6),

$$\varepsilon_t(h) = (\varepsilon \otimes id)((\Delta(1)h \otimes 1)), \quad \varepsilon_s(h) = (id \otimes \varepsilon)((1 \otimes h)\Delta(1)).$$

In the Hopf algebra case $\varepsilon_t(h) = \varepsilon_s(h) = \varepsilon(h)1$.

We have $S \circ \varepsilon_s = \varepsilon_t \circ S$ and $\varepsilon_s \circ S = S \circ \varepsilon_t$. The images of these maps $\varepsilon_t$ and $\varepsilon_s$

$$H_t = \varepsilon_t(H) = \{h \in H \mid \Delta(h) = \Delta(1)(h \otimes 1)\},$$

$$H_s = \varepsilon_s(H) = \{h \in H \mid \Delta(h) = (1 \otimes h)\Delta(1)\}$$

are subalgebras of $H$, called the target (resp. source) counital subalgebras. They play the role of ground algebras for $H$. They commute with each other and

$$H_t = \{\phi \otimes id)\Delta(1) \mid \phi \in \widehat{H}\}, \quad H_s = \{(id \otimes \phi)\Delta(1) \mid \phi \in \widehat{H}\},$$

i.e., $H_t$ (resp. $H_s$) is generated by the right (resp. left) tensorands of $\Delta(1)$. The restriction of $S$ defines an algebra anti-isomorphism between $H_t$ and $H_s$. Any non-zero morphism $H \to K$ of quantum groupoids preserves counital subalgebras, i.e., $H_t \cong K_t$ and $H_s \cong K_s$. 
In what follows we will use the Sweedler arrows, writing for all \( h \in H, \phi \in \hat{H} \):
\[
(12) \quad h \mapsto \phi = \phi(1) h(2), \quad \phi \mapsto h = (h, \phi(1)) \phi(2)
\]
for all \( h \in H, \phi \in \hat{H} \). Then the map \( z \mapsto (z \mapsto \varepsilon) \) is an algebra isomorphism between \( H_t \) and \( \hat{H}_s \). Similarly, the map \( y \mapsto (\varepsilon \mapsto y) \) is an algebra isomorphism between \( H_s \) and \( \hat{H}_t \) (cf. [BSN3], 2.6). Thus, the counital subalgebras of \( \hat{H} \) are canonically anti-isomorphic to those of \( H \).

A quantum groupoid \( H \) is called connected if \( H_s \cap Z(H) = k \), or, equivalently, \( H_t \cap Z(H) = k \), where \( Z(H) \) denotes the center of \( H \) (cf. [N], 3.11, [BSN3], 2.4).

Let us recall that a \( k \)-algebra \( A \) is separable if the multiplication epimorphism \( m : A \otimes_k A \to A \) has a right inverse as an \( A - A \) bimodule homomorphism. This is equivalent to the existence of a separability element \( e \in A \otimes_k A \) such that \( m(e) = 1 \) and \( (a \otimes 1)e = e(1 \otimes a), \ (1 \otimes a)e = e(a \otimes 1) \) for all \( a \in A \).

The counital subalgebras \( H_t \) and \( H_s \) are separable, with separability elements \( e_t = (S \otimes \text{id}) \Delta(1) \) and \( e_s = (\text{id} \otimes S) \Delta(1) \), respectively.

Observe that the adjoint actions of \( 1 \in H \) give rise to non-trivial maps \( H \to H \):
\[
(13) \quad h \mapsto 1(1) h S(1(2)) = Ad^l_1(h), \quad h \mapsto S(1(1)) h 1(2) = Ad^r_1(h), \quad h \in H.
\]

**Lemma 2.2.** The map \( Ad^l_1 \) is a linear projection from \( H \) onto \( C_H(H_s) \), the centralizer of \( H_s \), i.e., \( (Ad^l_1)^2 = Ad^l_1 \). The map \( Ad^r_1 \) is a linear projection from \( H \) onto \( C_H(H_t) \), the centralizer of \( H_t \), i.e., \( (Ad^r_1)^2 = Ad^r_1 \).

**Proof.** Since \( 1(1) \otimes S(1(2)) \) is a separability element of \( H_s \), \( Ad^l_1(h) \) commutes with \( H_s \). The assertion about \( Ad^r_1 \) follows similarly.

**Remark 2.3.** The opposite algebra \( H^{op} \) is also a quantum groupoid with the same coalgebra structure and the antipode \( S^{-1} \). Indeed,
\[
S^{-1}(h(2)) h(1) = S^{-1}(\varepsilon_s(h)) S^{-1}(1(1)) \varepsilon(h(1(2))) = S^{-1}(h(1(1)) h(1(2))) = \varepsilon^op(h),
\]
\[
h(2) S^{-1}(h(1)) = S^{-1}(\varepsilon_t(h)) = \varepsilon(1(1)) h S^{-1}(1(2)) = \varepsilon(1(1)) h S^{-1}(1(2)) = \varepsilon^op(h),
\]
\[
S^{-1}(h(3)) h(2) S^{-1}(h(1)) = S^{-1}(h(1) S(h(2)) h(3)) = S^{-1}(h).
\]
Similarly, the co-opposite coalgebra \( H^{cop} \) (with the same algebra structure as \( H \) and the opposite coalgebra structure, and the antipode \( S^{-1} \) and \( H^{op/cop} \) (with both opposite algebra and coalgebra structures, and the antipode \( S \)) are quantum groupoids.

3. **Examples of quantum groupoids**

**Groupoid algebras and their duals ([NV1], 2.1.4).** As group algebras and their duals give the simplest examples of Hopf algebras, groupoid algebras and their duals provide simple examples of quantum groupoids.

Let \( G \) be a finite groupoid (a category with finitely many morphisms, such that each morphism is invertible). Then the groupoid algebra \( kG \) (generated by morphisms \( g \in G \) with the product of two morphisms being equal to their composition if the latter is defined and 0 otherwise) is a quantum groupoid via:
\[
(14) \quad \Delta(g) = g \otimes g, \quad \varepsilon(g) = 1, \quad S(g) = g^{-1}, \quad g \in G.
\]
The counital subalgebras of \( kG \) are equal to each other and coincide with the abelian algebra spanned by the identity morphisms: \( (kG)_e = (kG)_a = \text{span}\{gg^{-1} \mid g \in G\}. \)

The target and source counital maps are induced by the operations of taking the target (resp. source) object of a morphism:

\[
\varepsilon_t(g) = gg^{-1} = \text{id}_{\text{target}(g)} \quad \text{and} \quad \varepsilon_s(g) = g^{-1}g = \text{id}_{\text{source}(g)}.
\]

The dual quantum groupoid \( \hat{kG} \) is isomorphic to the algebra of functions on \( G \), i.e., it is generated by idempotents \( p_g, g \in G \) such that \( p_gp_h = \delta_{g,h}p_g \), with

\[
\Delta(p_g) = \sum_{uv=g} p_u \otimes p_v, \quad \varepsilon(p_g) = \delta_{g,gg^{-1}}, \quad S(p_g) = p_{g^{-1}}.
\]

The target (resp. source) counital subalgebra is precisely the algebra of functions constant on each set of morphisms of \( G \) having the same target (resp. source). The target and source maps are

\[
\varepsilon_t(p_g) = \sum_{\mathit{v}v^{-1}=g} p_v \quad \text{and} \quad \varepsilon_s(p_g) = \sum_{v^{-1}v=g} p_v.
\]

**Definition 3.1.** We call a quantum groupoid *semisimple* if its underlying algebra is semisimple.

In contrast to finite dimensional semisimple Hopf algebras, the antipode in a finite dimensional quantum groupoid is not necessarily involutive, see Section 9.

Groupoid algebras and their duals give examples of commutative and cocommutative semisimple quantum groupoids.

**Quantum transformation groupoids.** It is known that any group action on a set (i.e., on a commutative algebra of functions) gives rise to a groupoid \( \mathcal{R} \). Extending this construction, we associate a quantum groupoid with any action of a Hopf algebra on a separable algebra (“finite quantum space”). Namely, let \( H \) be a Hopf algebra and \( B \) be a separable (and, therefore, finite dimensional and semisimple \( \mathcal{R} \)) algebra with right \( H \)-action \( b \otimes h \mapsto b \cdot h \), where \( b \in B, h \in H \). Then \( B^{\text{op}} \), the algebra opposite to \( B \), becomes a left \( H \)-module via \( h \otimes a \mapsto h \cdot a = a \cdot S_H(h) \).

One can form a *double crossed product algebra* \( B^\text{op} \bowtie H \bowtie B \) with underlying vector space \( B^\text{op} \otimes H \otimes B \) and multiplication

\[
(a \otimes h \otimes b)(a' \otimes h' \otimes b') = (h_{(1)} \cdot a')a \otimes h_{(2)} h'_{(1)} \otimes (b \cdot h'_{(2)})b',
\]

for all \( a, a' \in B^\text{op}, b, b' \in B, \) and \( h, h' \in H \).

Assume that \( k \) is algebraically closed and let \( e \) be a separability element of \( B \) (note that \( e \) is an idempotent when considered in \( B^\text{op} \otimes B \)). Let \( \omega \in \hat{B} \) be uniquely determined by \( (\omega \otimes \text{id})e = (\text{id} \otimes \omega)e = 1 \). One can check that \( \omega \) is the trace of the left regular representation of \( B \)

\[
\omega((h \cdot a)b) = \omega(a \cdot h), \quad e^{(1)} \otimes (h \cdot e^{(2)}) = (e^{(1)} \cdot h) \otimes e^{(2)},
\]

where \( a \in B^\text{op}, b \in B, \) and \( e = e^{(1)} \otimes e^{(2)} \).

The structure of a quantum groupoid on \( B^\text{op} \bowtie H \bowtie B \) is given by

\[
\Delta(a \otimes h \otimes b) = (a \otimes h_{(1)} \otimes e^{(1)}) \otimes ((h_{(2)} \cdot e^{(2)}) \otimes h_{(3)} \otimes b),
\]

\[
\varepsilon(a \otimes h \otimes b) = \omega(a(h \cdot b)) = \omega(a(b \cdot S_H(h))),
\]

\[
S(a \otimes h \otimes b) = b \otimes S_H(h) \otimes a.
\]
Quantum groupoids $B^{op} \otimes B$ (BSz2, 5.2). Let $k$ be algebraically closed and let $B$ be a separable algebra over $k$, $e = e^{(1)} \otimes e^{(2)} \in B^{op} \otimes B$ be the symmetric separability idempotent of $B$, and $\omega$ be as in the previous example. The map $\pi : x \mapsto e^{(1)}xe^{(2)}$ defines a linear projection from $B$ to $Z(B)$. Let $q$ be an invertible element of $B$ such that $\pi(q) = 1$, then the following operations define a structure of quantum groupoid $H_q$ on $B^{op} \otimes B$:

$$\Delta(b \otimes c) = (b \otimes e^{(1)}q^{-1})(e^{(2)} \otimes c),$$

$$\epsilon(b \otimes c) = \omega(qbc),$$

$$S(b \otimes c) = q^{-1}cq \otimes b,$$

for all $b, c \in B$. The target and source counital subalgebras of $H_q$ are $B^{op} \otimes 1$ and $1 \otimes B$. The square of the antipode is a conjugation by $g_q = q \otimes q$. Since $H_q$ with different $q$ are non-isomorphic, this example shows that there can be uncountably many non-isomorphic semisimple quantum groupoids with the same underlying algebra (for noncommutative $B$).

This example can be also explained in terms of twisting of quantum groupoids (see EN, NV3).

Quantum groupoids from subfactors. The initial motivation for studying quantum groupoids in NV1, NV2 was their connection with depth 2 von Neumann subfactors. This connection was first mentioned in (BNSz, BSz1, BSz2, NSzW). It was shown in NV2 that quantum groupoids naturally arise as non-commutative symmetries of subfactors, namely if $N \subset M \subset M_1 \subset M_2 \subset \ldots$ is the Jones tower constructed from a finite index, depth 2 inclusion $N \subset M$ of $\Pi_1$ factors, then $H = M' \cap M_2$ has a canonical structure of a quantum groupoid acting outerly on $M_1$ such that $M = M_1^H$ and $M_2 = M_1 > \triangleright H$. Furthermore $H = N' \cap M_1$ is a quantum groupoid dual to $H$.

In NV3 this result was extended to arbitrary finite depth, via a Galois correspondence and it was shown in (NV3, 4) that any inclusion of type $\Pi_1$ von Neumann factors with finite index and depth (GHJ, 4.1) gives rise to a quantum groupoid and its coideal subalgebra.

We refer the reader to the survey NV4 (Sections 8,9) and to the Appendix of NV3 for the explanation of how quantum groupoids can be constructed from subfactors.

Temperley-Lieb algebras. We describe quantum groupoids arising from type $A_n$ subfactors, whose underlying algebras are Temperley-Lieb algebras (GHJ, 2.1).

Let $k = \mathbb{C}$, $\lambda^{-1} = 4 \cos^2 \frac{\pi}{n+3}$ ($n \geq 2$), and $e_1, e_2, \ldots$ be a sequence of idempotents satisfying, for all $i$ and $j$, the relations

$$e_i e_{i+1} e_i = \lambda e_i,$$

$$e_i e_j = e_j e_i, \quad \text{if } |i - j| \geq 2.$$  

Let $A_{k,l}$ be the algebra generated by $1, e_k, e_{k+1}, \ldots e_l$ ($k \leq l$), $\sigma$ be the algebra anti-automorphism of $H = A_{1,2n-1}$ determined by $\sigma(e_i) = e_{2n-i}$ and $P_k \in A_{2n-k,2n-1} \otimes A_{1,k}$ be the image of the separability idempotent of $A_{1,k}$ under $\sigma \otimes \text{id}$.

We denote by $\tau$ the non-degenerate Markov trace (GHJ, 2.1) on $H$ and by $w$ the index of the restriction of $\tau$ on $A_{n+1,2n-1} \subset H$, i.e., the unique central element
in $A_{n+1,2n-1}$ such that $\tau(w \cdot)$ is equal to the trace of the left regular representation of $A_{n+1,2n-1}$ (see [W]).

Then the following operations define a quantum groupoid structure on $H$:

$$
\Delta(yz) = (z \otimes y)P_{n-1}, \quad y \in A_{n+1,2n-1}, \quad z \in A_{1,n-1}
$$

$$
\Delta(e_n) = (1 \otimes w)P_n(1 \otimes w^{-1}),
$$

$$
S(h) = w^{-1}\sigma(h)w,
$$

$$
\varepsilon(h) = \lambda^{-n}\tau(hfw), \quad h \in A,
$$

where in the last line

$$
f = \lambda^{n(n-1)/2}(e_ne_{n-1} \cdots e_1)(e_{n+1}e_n \cdots e_2) \cdots (e_{2n-1}e_{2n-2} \cdots e_n)
$$

is the Jones projection corresponding to the $n$-step basic construction.

The source and target counital subalgebras of $H = A_{1,2n-1}$ are $H_s = A_{n+1,2n-1}$ and $H_t = A_{1,n-1}$. The example corresponding to $n = 2$ is a quantum groupoid of dimension 13 with antipode of infinite order (cf. [NV2], 7.3).

4. REPRESENTATION CATEGORY OF A QUANTUM GROUPOID

Throughout this paper we refer to [T2] for definitions related to categories.

For a quantum groupoid $H$ let $\text{Rep}(H)$ be the category of representations of $H$, whose objects are finite dimensional left $H$-modules and whose morphisms are $H$-linear homomorphisms. We shall show that $\text{Rep}(H)$ has a natural structure of a monoidal category with duality.

For objects $V,W$ of $\text{Rep}(H)$ set

$$
V \otimes W = \{x \in V \otimes_k W \mid x = \Delta(1) \cdot x \subseteq V \otimes_k W\},
$$

with the obvious action of $H$ via the comultiplication $\Delta$ (here $\otimes_k$ denotes the usual tensor product of vector spaces). Note that $\Delta(1)$ is an idempotent and therefore $V \otimes W = \Delta(1) \cdot (V \otimes_k W)$. The tensor product of morphisms is the restriction of usual tensor product of homomorphisms. The standard associativity isomorphisms $(U \otimes V) \otimes W \to U \otimes (V \otimes W)$ are functorial and satisfy the pentagon condition, since $\Delta$ is coassociative. We will suppress these isomorphisms and write simply $U \otimes V \otimes W$.

The target counital subalgebra $H_t \subset H$ has an $H$-module structure given by $h \cdot z = \varepsilon_t(hz)$, where $h \in H$, $z \in H_t$.

**Lemma 4.1.** $H_t$ is the unit object of $\text{Rep}(H)$.

**Proof.** Define a $k$-linear homomorphism $l_V : H_t \otimes V \to V$ by

$$
l_V(1(1) \cdot z \otimes 1(2) \cdot v) = z \cdot v, \quad z \in H_t, \quad v \in V.
$$

This map is $H$-linear, since

$$
l_V(h \cdot (1(1) \cdot z \otimes 1(2) \cdot v)) = l_V(h(1) \cdot z \otimes h(2) \cdot v)
$$

$$
= \varepsilon_t(h(1)z)h(2) \cdot v = hz \cdot v
$$

$$
= h \cdot l_V(1(1) \cdot z \otimes 1(2) \cdot v),
$$

for all $h \in H$. The inverse map $l_V^{-1} : V \to H_t \otimes V$ is given by

$$
l_V^{-1}(v) = S(1(1)) \otimes (1(2) \cdot v) = (1(1) \cdot 1) \otimes (1(2) \cdot v).
$$
The collection \( \{ \mathcal{I}_V \}_V \) gives a natural equivalence between the functor \( H_t \otimes ( \cdot ) \) and the identity functor. Indeed, for any \( H \)-linear homomorphism \( f : V \to U \) we have:

\[
\mathcal{I}_V \circ (\mathrm{id} \otimes f) (1(1) \cdot z \otimes 1(2) \cdot v) = \mathcal{I}_V (1(1) \cdot z \otimes 1(2) \cdot f(v)) = z \cdot f(v) = f(z \cdot v) = f \circ \mathcal{I}_V (1(1) \cdot z \otimes 1(2) \cdot v).
\]

Similarly, the \( k \)-linear homomorphism \( r_V : V \otimes H_t \to V \) defined by
\[
r_V (1(1) \cdot v \otimes 1(2) \cdot z) = S(z) \cdot v, \quad z \in H_t, \ v \in V,
\]
has the inverse \( r_V^{-1}(v) = 1(1) \cdot v \otimes 1(2) \) and satisfies the necessary properties.

Finally, we can check the triangle axiom \( \mathcal{I}_V \otimes l_W = r_V \otimes \mathcal{id}_W : V \otimes H_t \otimes W \to V \otimes W \) for all objects \( V, W \) of \( \text{Rep}(H) \). For \( v \in V, \ w \in W \) we have:

\[
(\mathcal{I}_V \otimes l_W)(1(1) \cdot v \otimes 1(2) \cdot z \otimes 1(1) \cdot w) = 1(1) \cdot v \otimes 1(2) \cdot z \cdot w = 1(1) S(z) \cdot v \otimes 1(2) \cdot w = (r_V \otimes \mathcal{id}_W)(1(1) \cdot v \otimes 1(1) \cdot z \otimes 1(2) \cdot w),
\]

therefore, \( \mathcal{I}_V \otimes l_W = r_V \otimes \mathcal{id}_W \).

Using the antipode \( S \) of \( H \), we can provide \( \text{Rep}(H) \) with a duality. For any object \( V \) of \( \text{Rep}(H) \), define the action of \( H \) on \( V^* = \text{Hom}_k(V, k) \) by

\[
(h \cdot \phi)(v) = \phi(S(h) \cdot v),
\]

where \( h \in H, v \in V, \phi \in V^* \). For any morphism \( f : V \to W \), let \( f^* : W^* \to V^* \) be the morphism dual to \( f \) (see [T2], I.1.8).

For any \( V \) in \( \text{Rep}(H) \), we define the duality morphisms

\[
d_V : V^* \otimes V \to H_t, \quad b_V : H_t \to V \otimes V^*
\]
as follows. For \( \sum_j \phi^j \otimes v_j \in V^* \otimes V \), set

\[
d_V (\sum_j \phi^j \otimes v_j) = \sum_j \phi^j (1(1) \cdot v_j) 1(2).
\]

Let \( \{ f_i \}_i \) and \( \{ \xi_i \}_i \) be bases of \( V \) and \( V^* \) respectively, dual to each other. The element \( \sum_i f_i \otimes \xi_i \) does not depend on choice of these bases; moreover, for all \( v \in V, \phi \in V^* \) one has \( \phi = \sum_i \phi(f_i) \xi_i \) and \( v = \sum_i f_i \xi_i(v) \). Set

\[
b_V(z) = z \cdot (\sum_i f_i \otimes \xi_i).
\]

**Proposition 4.2.** The category \( \text{Rep}(H) \) is a monoidal category with duality.

**Proof.** We know already that \( \text{Rep}(H) \) is monoidal, it remains to prove that \( d_V \) and \( b_V \) are \( H \)-linear and satisfy the identities

\[
(\mathcal{id}_V \otimes d_V)(b_V \otimes \mathcal{id}_V) = \mathcal{id}_V, \quad (d_V \otimes \mathcal{id}_V)(\mathcal{id}_V \otimes b_V) = \mathcal{id}_V.
\]
Take $\sum_j \phi^j \otimes v_j \in V^* \otimes V$, $z \in H_I$, $h \in H$. Using the axioms of a quantum groupoid, we have

$$h \cdot d_V(\sum_j \phi^j \otimes v_j) = \sum_j \phi^j(1_{(1)} \cdot v)\varepsilon_I(h1_{(2)})$$

$$= \sum_j \phi^j(\varepsilon_s(1_{(1)}h) \cdot v_j)1_{(2)}$$

$$= \sum_j \phi^j(S(h(1))1_{(3)}h(2) \cdot v_j)1_{(2)}$$

$$= \sum_j (h(1) \cdot \phi^j)(1_{(1)} \cdot (h(2) \cdot v_j))1_{(2)}$$

$$= \sum_j d_V(h(1) \cdot \phi^j \otimes h(2) \cdot v_j)$$

$$= d_V(h \cdot \sum_j \phi^j \otimes v_j),$$

therefore, $d_V$ is $H$-linear. To check the $H$-linearity of $b_V$ we have to show that $h \cdot b_V(z) = b_V(h \cdot z)$, i.e., that

$$\sum_i h(1)z \cdot f_i \otimes h(2) \cdot \xi^i = \sum_i 1_{(1)}\varepsilon_I(hz) \cdot f_i \otimes 1_{(2)} \cdot \xi^i.$$

Since both sides of the above equality are elements of $V \otimes_k V^*$, evaluating the second factor on $v \in V$, we get the equivalent condition

$$h(1)zS(h(2)) \cdot v = 1_{(1)}\varepsilon_I(hz)S(1_{(2)}) \cdot v,$$

which is easy to check. Thus, $b_V$ is $H$-linear.

Using the isomorphisms $l_V$ and $r_V$ identifying $H_I \otimes V$, $V \otimes H_I$, and $V$, for all $v \in V$ and $\phi \in V^*$ we have:

$$(id_V \otimes d_V)(b_V \otimes id_V)(v) = (id_V \otimes d_V)(b_V(1_{(1)} \cdot 1) \otimes 1_{(2)} \cdot v)$$

$$= (id_V \otimes d_V)(b_V(1_{(2)})) \otimes S^{-1}(1_{(1)}) \cdot v)$$

$$= \sum_i (id_V \otimes d_V)(1_{(2)} \cdot f_i \otimes 1_{(3)} \cdot \xi^i \otimes S^{-1}(1_{(1)}) \cdot v)$$

$$= \sum_i 1_{(2)} \cdot f_i \otimes (1_{(3)} \cdot \xi^i)(1'_{(1)}S^{-1}(1_{(1)}) \cdot v)1'_{(2)}$$

$$= 1_{(2)}S(1_{(3)})1'_{(1)}S^{-1}(1_{(1)}) \cdot v \otimes 1'_{(2)} = v,$$

$$(d_V \otimes id_V \cdot )(id_V \otimes b_V)(\phi) = (d_V \otimes id_V \cdot )(1_{(1)} \cdot \phi \otimes b_V(1_{(2)}))$$

$$= \sum_i (d_V \otimes id_V \cdot )(1_{(1)} \cdot \phi \otimes 1_{(2)} \cdot f_i \otimes 1_{(3)} \cdot \xi^i)$$

$$= \sum_i (1_{(1)} \cdot \phi)(1'_{(1)}1_{(2)} \cdot f_i)1'_{(2)} \otimes 1_{(3)} \cdot \xi^i$$

$$= 1'_{(2)} \otimes 1_{(3)}1_{(1)}S(1'_{(1)}1_{(2)}) \cdot \phi = \phi,$$

which completes the proof. \qed

**Remark 4.3.** Similarly to the construction of $\text{Rep}(H)$, one can construct a category of right $H$-modules, in which $H_s$ plays the role of the unit object.
5. QUASITRIANGULAR QUANTUM GROUPOIDS

Definition 5.1. A quasitriangular quantum groupoid is a pair \((H, \mathcal{R})\) where \(H\) is a quantum groupoid and \(\mathcal{R} \in \Delta^{op}(1)(H \otimes_k H)\Delta(1)\) satisfying the following conditions:

\[
\Delta^{op}(h)\mathcal{R} = \mathcal{R}\Delta(h),
\]
for all \(h \in H\), where \(\Delta^{op}\) denotes the comultiplication opposite to \(\Delta\),

\[
(id \otimes \Delta)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{12}, \quad (\Delta \otimes id)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{23},
\]
where \(\mathcal{R}_{12} = \mathcal{R} \otimes 1, \mathcal{R}_{23} = 1 \otimes \mathcal{R}\), etc. as usual, and such that there exists \(\mathcal{R} \in \Delta(1)(H \otimes_k H)\Delta^{op}(1)\) with

\[
\mathcal{R}\mathcal{R} = \Delta^{op}(1), \quad \mathcal{R}\mathcal{R} = \Delta(1).
\]

Note that \(\mathcal{R}\) is uniquely determined by \(\mathcal{R}\): if \(\mathcal{R}\) and \(\mathcal{R}'\) are two elements of \(\Delta(1)(H \otimes_k H)\Delta^{op}(1)\) satisfying the previous equation, then

\[
\mathcal{R} = \mathcal{R}\Delta^{op}(1) = \mathcal{R}\mathcal{R}\mathcal{R}' = \Delta(1)\mathcal{R}' = \mathcal{R}'.
\]

For any two objects \(V\) and \(W\) of \(\text{Rep}(H)\) define \(c_{V,W} : V \otimes W \to W \otimes V\) as the action of \(\mathcal{R}_{21}\):

\[
c_{V,W}(x) = \mathcal{R}^{(2)}h^{(2)} \cdot x^{(2)} \otimes \mathcal{R}^{(1)}h^{(1)} \cdot x^{(1)},
\]
where \(x = x^{(1)} \otimes x^{(2)} \in V \otimes W, \mathcal{R} = \mathcal{R}^{(1)} \otimes \mathcal{R}^{(2)} \in \Delta^{op}(1)(H \otimes_k H)\Delta(1)\).

Proposition 5.2. The family of homomorphisms \(\{c_{V,W}\}_{V,W}\) defines a braiding in \(\text{Rep}(H)\). Conversely, if \(\text{Rep}(H)\) is braided, then there exists \(\mathcal{R} \in \Delta^{op}(1)(H \otimes_k H)\Delta(1)\), satisfying the properties of Definition 5.1, and inducing the given braiding.

Proof. Note that \(c_{V,W}\) is well-defined, since \(\mathcal{R}_{21} = \Delta(1)\mathcal{R}_{21}\). To prove the \(H\)-linearity of \(c_{V,W}\) we observe that

\[
c_{V,W}(h \cdot x) = \mathcal{R}^{(2)}h^{(2)} \cdot x^{(2)} \otimes \mathcal{R}^{(1)}h^{(1)} \cdot x^{(1)} = h^{(1)}\mathcal{R}^{(2)} \cdot x^{(2)} \otimes h^{(2)}\mathcal{R}^{(1)} \cdot x^{(1)} = h \cdot (c_{V,W}(x)).
\]

The inverse of \(c_{V,W}\) is given by

\[
c_{V,W}^{-1}(y) = \mathcal{R}^{(1)} \cdot y^{(2)} \otimes \mathcal{R}^{(2)} \cdot y^{(1)},
\]
where \(y = y^{(1)} \otimes y^{(2)} \in W \otimes V, \mathcal{R} = \mathcal{R}^{(1)} \otimes \mathcal{R}^{(2)}\). Therefore, \(c_{V,W}\) is an isomorphism. Finally, one can verify that the braiding identities

\[
(id_V \otimes c_{U,W})(c_{U,V} \otimes id_W) = c_{U,V} \otimes id_W, \quad (c_{U,W} \otimes id_U)(id_U \otimes c_{V,W}) = c_{U \otimes V,W}
\]
are equivalent to the relations of Definition 5.1, exactly in the same way as in the case of Hopf algebras (see, for instance, ([13], XI, 2.3.1)).

Lemma 5.3. Let \((H, \mathcal{R})\) be a quasitriangular quantum groupoid. Then for all \(y \in H, z \in H\) the following six identities hold:

\[
\begin{align*}
(1 \otimes z)\mathcal{R} &= \mathcal{R}(z \otimes 1), & (y \otimes 1)\mathcal{R} &= \mathcal{R}(1 \otimes y), \\
(z \otimes 1)\mathcal{R} &= (1 \otimes S(z))\mathcal{R}, & (1 \otimes y)\mathcal{R} &= (S(y) \otimes 1)\mathcal{R}, \\
\mathcal{R}(y \otimes 1) &= \mathcal{R}(1 \otimes S(y)), & \mathcal{R}(1 \otimes z) &= \mathcal{R}(S(z) \otimes 1).
\end{align*}
\]
Invariants of Knots and 3-Manifolds from Quantum Groupoids

Proof. Since we have $\Delta^{op}(1)R = R = R\Delta(1)$, the first line is a consequence of the relation $\Delta(yz) = (z \otimes y)\Delta(1)$, the second line follows from $(1 \otimes z)\Delta(1) = (S(z) \otimes 1)\Delta(1)$ and $(y \otimes 1)\Delta(1) = (1 \otimes S(y))\Delta(1)$. The last two identities are proven similarly. \qed

Proposition 5.4. Let $(H, R)$ be a quasitriangular quantum groupoid. Then $R$ satisfies the quantum Yang-Baxter equation:

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}. \tag{5.1}$$

Proof. It follows from the first two relations of Definition 5.1, that

$$R_{12}R_{13}R_{23} = (id \otimes \Delta^{op})(R)R_{23} = R_{23}(id \otimes \Delta)(R) = R_{23}R_{13}R_{12}. \tag{5.2}$$

Remark 5.5. Let us define two $k$-linear maps $R_1, R_2 : \hat{H} \to H$ by

$$R_1(\phi) = (id \otimes \phi)(R), \quad R_2(\phi) = (\phi \otimes id)(R), \quad \text{for } \phi \in \hat{H}. \tag{5.3}$$

Then the condition $(id \otimes \Delta)(R) = R_{13}R_{12}$ is equivalent to $R_1$ being a coalgebra homomorphism and algebra anti-homomorphism and the condition $(\Delta \otimes id)(R) = R_{13}R_{23}$ is equivalent to $R_2$ being an algebra homomorphism and coalgebra anti-homomorphism. In other words, $R_1$ and $R_2$ are homomorphisms of quantum groupoids $\hat{H} \to H^{op}$ and $H \to H^{op}$, respectively.

Proposition 5.6. For any quasitriangular quantum groupoid $(H, R)$, we have:

$$\begin{align*}
(\varepsilon_s \otimes id)(R) &= \Delta(1), \\
(id \otimes \varepsilon_s)(R) &= (S \otimes id)\Delta^{op}(1), \tag{5.4}
(\varepsilon_t \otimes id)(R) &= \Delta^{op}(1), \\
(id \otimes \varepsilon_t)(R) &= (S \otimes id)\Delta(1), \tag{5.5}
(S \otimes id)(R) &= (id \otimes S^{-1})(R) = \bar{R}, \quad (S \otimes S)(R) = R. \tag{5.6}
\end{align*}$$

Proof. First, using the same argument as in [14], XI, 2.1.1, we can show that $(\varepsilon \otimes id)(R) = (id \otimes \varepsilon)(R) = 1$. Next, using Lemma 5.3, we obtain

$$\begin{align*}
(\varepsilon_s \otimes id)(R) &= 1_{(1)}\varepsilon(\bar{R}(1^{(1)})1_{(2)}) \otimes \bar{R}(2) = \Delta(1), \\
(id \otimes \varepsilon_s)(R) &= \bar{R}(1^{(1)}) \otimes 1_{(1)}\varepsilon(\bar{R}(2)^{(2)})1_{(2)} = \Delta^{op}(1), \\
(S \otimes id)(R) &= S(1_{(2)})\bar{R}(1^{(1)}) \otimes 1_{(1)}\varepsilon(\bar{R}(2)^{(2)})1_{(2)} = (S \otimes id)\Delta^{op}(1), \\
(\varepsilon_t \otimes id)(R) &= \varepsilon(1_{(1)}\bar{R}(1^{(1)})1_{(2)}) \otimes \bar{R}(2) = \varepsilon(\bar{R}(1^{(1)})1_{(2)}) \otimes \bar{R}(2)1_{(1)} = \Delta^{op}(1), \\
(id \otimes \varepsilon_t)(R) &= \bar{R}(1^{(1)}) \otimes \varepsilon(\bar{R}(2)^{(2)})1_{(2)} = \varepsilon(\bar{R}(1^{(1)})1_{(2)}) \otimes \bar{R}(2)1_{(1)} = \Delta(1).
\end{align*}$$

Let $m$ denote multiplication $H \otimes_k H \to H$ in $H$. Set $m_{12} = m \otimes id : H^{\otimes 3} \to H^{\otimes 2}$ and $m_{23} = id \otimes m : H^{\otimes 3} \to H^{\otimes 2}$. It follows from the above relations that

$$\begin{align*}
(m_{12}(S \otimes id \otimes id))(&\Delta \otimes id)(R) = (\varepsilon_s \otimes id)(R) = \Delta(1), \\
(m_{12}((id \otimes S \otimes id))(&\Delta \otimes id)(R)) = (\varepsilon_t \otimes id)(R) = \Delta^{op}(1), \\
(m_{23}(id \otimes S^{-1})(&\Delta^{op} \otimes id)(R)) = (id \otimes S^{-1}\varepsilon_s)(R) = \Delta(1), \\
(m_{23}(id \otimes S^{-1} \otimes id)(&\Delta^{op} \otimes id)(R)) = (id \otimes S^{-1}\varepsilon_t)(R) = \Delta^{op}(1).
\end{align*}$$
On the other hand, Definition 5.1 implies that
\[ m_{12}((S \otimes \id)(\Delta \otimes \id)(R)) = \]
\[ = m_{12}((S \otimes \id)(\Delta \otimes \id)(R_{13}R_{23})) = (S \otimes \id)(R)R, \]
\[ m_{12}((\id \otimes S \otimes \id)(\Delta \otimes \id)(R)) = \]
\[ = m_{12}((\id \otimes S \otimes \id)(\Delta \otimes \id)(R)) = \]
\[ m_{23}((\id \otimes S \otimes \id)(\Delta \otimes \id)(R)) = \]
\[ = m_{23}((\id \otimes S \otimes \id)(\Delta \otimes \id)(R)) = \]
\[ \therefore (S \otimes \id)(R) = (\id \otimes S^{-1})(R) = R \text{ and } (S \otimes S)(R) = R. \]

**Proposition 5.7.** Let \((H, R)\) be a quasitriangular quantum groupoid. Then \(S^2(h) = uhu^{-1}\) for all \(h \in H\), where \(u = S(R^{(2)})\) is an invertible element of \(H\) such that
\[
\Delta(u) = \bar{R}R_{21}(u \otimes u).
\]
Likewise, \(S^{-2}(h) = vhv^{-1}\), where \(v = S(u) = R^{(1)}S(R^{(2)}), \) and
\[
v^{-1} = S^2(R^{(1)})R^{(2)}, \quad \Delta(v) = \bar{R}\bar{R}_{21}(v \otimes v).
\]

**Proof.** Note that \(S(R^{(2)})yR^{(1)} = S(y)u\) for all \(y \in H_s\), by Lemma 5.3. Hence, we have
\[
S(h^{(2)})uh^{(1)} = S(h^{(2)})S(R^{(2)})R^{(1)}h^{(1)} = S(R^{(2)}h^{(2)})R^{(1)}h^{(1)}
\]
\[
= S(h^{(1)}R^{(2)})h^{(2)}R^{(1)} = S(R^{(2)})\varepsilon_s(h)R^{(1)}
\]
\[
= S(\varepsilon_s(h))u,
\]
for all \(h \in H\). Therefore, using the axioms of a quantum groupoid, we get
\[
uh = S(1^{(2)})u1^{(1)}h = S(\varepsilon_1(h^{(2)})uh^{(1)}
\]
\[
= S(h^{(2)}S(h^{(3)})uh^{(1)} = S^2(h^{(3)})S(h^{(2)})uh^{(1)}
\]
\[
= S^2(h^{(2)})S(\varepsilon_s(h^{(1)}))u = S(\varepsilon_s(h^{(1)}))S(h^{(2)})u = S^2(h)u.
\]
The remaining part of the proof follows the lines of (M, 2.1.8). The results for \(v\) can be obtained by applying the results for \(u\) to the quasitriangular quantum groupoid \((H^{op/cop}, \bar{R})\).

**Definition 5.8.** The element \(u\) defined in Proposition 5.7 is called the Drinfeld element of \(H\).

**Corollary 5.9.** The element \(uv = vu\) is central and
\[
\Delta(uv) = (\bar{R}\bar{R}_{21})^2(uv \otimes uv).
\]
The element \(uv^{-1} = vu^{-1}\) is group-like and \(S^4(h) = uv^{-1}hvu^{-1}\) for all \(h \in H\).

**Proof.** See (M, 2.1.9).
Proposition 5.10. Given a quasitriangular quantum groupoid \((H, R)\), consider a linear map \(F : \hat{H} \to H\) given by
\[
(30) \quad F(\phi) = (\phi \otimes \text{id})(R_{21}R), \quad \phi \in \hat{H}.
\]
Then the image of \(F\) lies in \(C_H(H_s)\), the centralizer of \(H_s\).

Proof. Take \(y \in H_s\). Then we have
\[
\phi(R^{(2)}R^{(1)})R^{(1)}R^{(2)}y = \phi(R^{(2)}yR^{(1)})R^{(1)}R^{(2)} = \phi((R^{(2)}R^{(1)})yR^{(1)}R^{(2)}).
\]
Therefore \(F(\phi) \in C_H(H_s)\), as required. \(\square\)

Definition 5.11 (cf. [M], 2.1.12). A quasitriangular quantum groupoid is factorizable if the map \(F : \hat{H} \to C_H(H_s)\) from Proposition 5.10 is surjective.

The factorizability means that \(R\) is as non-trivial as possible, in contrast to trigonal quantum groupoids, for which \(\bar{R} = R_{21}\) and \(F(\hat{H}) = H_t\).

Corollary 5.12. If \(H\) is factorizable, then the restriction of \(F\) to the subspace \(W_s = \{\phi \in \hat{H} \mid \phi = \phi \circ \text{Ad}^r\}\) is a linear isomorphism onto \(C_H(H_s)\).

Proof. From the observation that \(F(\phi) = F(\phi \circ \text{Ad}^r)\) we have that the restriction of \(F\) to \(W_s\) is a linear map onto \(C_H(H_s)\). On the other hand, Lemma 2.2 allows to identify \(W_s\) with the dual vector space to \(C_H(H_t)\), from where \(\dim W_s = \dim C_H(H_s)\) and the result follows. \(\square\)

6. The Drinfeld double for quantum groupoids

Let \(H\) be a finite quantum groupoid. We define the Drinfeld double \(D(H)\) of \(H\) as follows. Consider on the vector space \(\hat{H}^{op} \otimes_k H\) a multiplication given by
\[
(31) \quad (\phi \otimes h)(\psi \otimes g) = \psi(2)\phi \otimes h(2)g(S(h_{(1)}),\psi_{(1)})h_{(3)},\psi_{(3)}),
\]
where \(\phi, \psi \in \hat{H}^{op}\) and \(h, g \in H\). We verify below that the linear span \(J\) of the elements
\[
(32) \quad \phi \otimes zh \quad (\varepsilon \leftarrow z)\phi \otimes h, \quad z \in H_t,
\]
\[
(33) \quad \phi \otimes yh \quad (y \leftarrow \varepsilon)\phi \otimes h, \quad y \in H_s,
\]
is a two-sided ideal in \(\hat{H}^{op} \otimes_k H\). Let \(D(H)\) be the factor-algebra \((\hat{H}^{op} \otimes_k H)/J\) and let \([\phi \otimes h]\) denote the class of \(\phi \otimes h\) in \(D(H)\).

Definition and Theorem 6.1. \(D(H)\) is a quantum groupoid with unit \([\varepsilon \otimes 1]\), and comultiplication, counit, and antipode given by
\[
(34) \quad \Delta([\phi \otimes h]) = [\phi_{(1)} \otimes h_{(1)}] \otimes [\phi_{(2)} \otimes h_{(2)}],
\]
\[
(35) \quad \varepsilon([\phi \otimes h]) = (\varepsilon(h),\phi),
\]
\[
(36) \quad S([\phi \otimes h]) = [S^{-1}(\phi_{(2)}) \otimes S(h_{(2)})]h_{(1)},\phi_{(1)}\langle S(h_{(3)}),\phi_{(3)}\rangle.
\]

In the case where \(H\) is a Hopf algebra, this definition and is due to Drinfeld [D].
Proof. Associativity of multiplication in $\tilde{H}^op \otimes_k H$ and hence in $D(H)$ can be verified exactly as in (M, 7.1.1). Let us check that $J$ is an ideal. We have:

\[
(\phi \otimes h)((\varepsilon \leftarrow z)\psi \otimes g) = \psi(2)\phi \otimes h(2)g(S(h(1)), (\varepsilon \leftarrow z)\psi(1)h(3), \psi(3)) = \psi(2)\phi \otimes h(1)g(zS(h(2)), \varepsilon \psi(1)h(4), \psi(3)) = \psi(2)\phi \otimes h(2)zg(S(h(1)), \psi(1)h(3), \psi(3)) = (\phi \otimes h)(\psi \otimes zg),
\]

\[
(\psi \otimes zg)(\phi \otimes h) = \phi(2)\psi \otimes g(2)h(S(\varepsilon(1)), \phi(1)g(3), \phi(3)) = \langle S, \phi(2)\phi(3)\psi \otimes g(2)h(S(\varepsilon(1)), \phi(1)g(3), \phi(3)) = \phi(2)(\varepsilon \leftarrow z)\psi \otimes g(2)h(S(\varepsilon(1)), \phi(1)g(3), \phi(3)) = ((\varepsilon \leftarrow z)\psi \otimes g)(\phi \otimes h),
\]

where $z \in H_1$ and we used the identity $zS(h(1)) \otimes h(2) = S(h(1)) \otimes h(2)z$. Similarly, one checks that

\[
(\psi \otimes yg)(\phi \otimes h) = ((y \rightarrow \varepsilon)\psi \otimes g)(\phi \otimes h),
\]

\[
(\phi \otimes h)(\psi \otimes yg) = (\phi \otimes h)((y \rightarrow \varepsilon)\psi \otimes g),
\]

therefore for all $x \in J$ we have $(\phi \otimes h)x = x(\phi \otimes h) = 0$, so $J$ is an ideal. We also compute

\[
[\varepsilon \otimes 1]([\phi \otimes h]) = [\phi(2) \otimes 1(2)h(S(1(1)), \phi(1)1(3), \phi(3)) = [\phi(2) \otimes (S(1(1)), \phi(1)1(2), \phi(3))h] = [\varepsilon(\phi(1))S(\varepsilon(\phi(3))\phi(2) \otimes h] = [\phi \otimes h],
\]

and similarly $[\phi \otimes h][\varepsilon \otimes 1] = [\phi \otimes h]$, so that $[\varepsilon \otimes 1]$ is a unit.

Now let us verify that the structure maps $\Delta$, $\varepsilon$, and $S$ are well-defined on $D(H)$. We have, using properties of a quantum groupoid and its counital subalgebras:

\[
\Delta([\phi \otimes zh]) = [\phi(1) \otimes zh(1)] \otimes [\phi(2) \otimes h(2)] = [(\varepsilon \leftarrow z)\phi(1) \otimes h(1)] \otimes [\phi(2) \otimes h(2)] = \Delta([(\varepsilon \leftarrow z, \varepsilon(1))\varepsilon(2)\phi \otimes h]),
\]

\[
\varepsilon([(\varepsilon \leftarrow z, \varepsilon(1))\varepsilon(2)\phi \otimes h]) = \langle z, \varepsilon(1)\rangle \langle \varepsilon t(h), \varepsilon(2)\phi \rangle = \langle z, \varepsilon(1)\rangle \langle 1(1)\varepsilon t(h), \varepsilon(2)\rangle \langle 1(2), \phi \rangle = \langle zz\varepsilon t(h)1(1), \varepsilon \rangle \langle 1(2), \phi \rangle = \langle zz\varepsilon t(h), \phi \rangle = \varepsilon([\phi \otimes zh]),
\]

\[
S([(\varepsilon \leftarrow z)\phi \otimes h]) = [(\varepsilon \leftarrow z)S^{-1}(\varepsilon(3)\phi(2)) \otimes S(h(2))] = (h(1), \varepsilon(2)\phi(1))S(h(3), \varepsilon(4)\phi(3)) = [S^{-1}(\phi(2)) \otimes S(h(2))] = [S^{-1}(\phi(2)) \otimes S(h(3))] = [S^{-1}(\phi(2)) \otimes S(h(3))] = [S^{-1}(\phi(2)) \otimes S(h(3))] = S([\phi \otimes zh])
\]

for all $h \in H, \phi \in \tilde{H}$, $z \in H_1$. Next, we need to check the axioms of a quantum groupoid. Coassociativity and multiplicativity of $\Delta$ are established as in (M,
7.1.1), since the computations given there do not use the unitality of multiplication and comultiplication. For the counit property, we have:

\[
(\varepsilon \otimes \text{id})\Delta([\phi \otimes h]) = \langle \varepsilon_t(h_{(1)}), \phi_{(1)} \rangle \langle \phi_{(2)} \otimes h_{(2)} \rangle \\
= \langle \varepsilon_t(h_{(1)}), \varepsilon_{(1)} \rangle \langle \varepsilon_{(2)} \phi \otimes h_{(2)} \rangle \\
= [\phi \otimes \varepsilon_t(h_{(1)}) h_{(2)}] = [\phi \otimes h],
\]

\[
(\text{id} \otimes \varepsilon)\Delta([\phi \otimes h]) = [\phi_{(1)} \otimes h_{(1)}] \langle \varepsilon_t(h_{(2)}), \phi_{(2)} \rangle \\
= [\phi_{(1)} \otimes \varepsilon_t(1_{(1)} h_{(2)}) \langle 1_{(2)}, \varepsilon_{(\phi_{(2)})} \rangle \\
= [S^{-1}(\varepsilon_t(\phi_{(2)})) \phi_{(1)} \otimes h] = [\phi \otimes h],
\]

where we used the amalgamation property \([\phi \otimes z h] = [\varepsilon \leftarrow z) \phi \otimes h], \quad z \in H_t,\]
following from (52). Now we verify the remaining axioms of a quantum groupoid. For all \(h, g, f \in H\) and \(\phi, \psi, \theta \in \hat{H}\) we compute

\[
\varepsilon(\phi \otimes h)[\psi \otimes g][\theta \otimes f]) = \\
\varepsilon(\theta_{(2)} \psi_{(2)} \phi_{(1)} g_{(3)} g_{(4)} f) \langle S(h_{(1)}), \psi_{(1)} \rangle \\
\langle S(h_{(2)} g_{(1)}), \theta_{(1)} \rangle \langle g_{(4)} g_{(3)}, \theta_{(3)} \rangle \langle h_{(5)}, \psi_{(3)} \rangle \\
= \langle h_{(3)} g_{(2)}, \varepsilon_t(\theta_{(2)} \psi_{(2)} \phi_{(1)}) \rangle \langle f, \varepsilon_{(2)} \rangle \langle S(h_{(1)}), \psi_{(1)} \rangle \\
\langle h_{(2)} g_{(1)}, S(\theta_{(1)}) \rangle \langle g_{(4)} g_{(3)}, \theta_{(3)} \rangle \langle h_{(5)}, \psi_{(3)} \rangle \\
= \langle h_{(2)} g, S(\theta_{(1)}) \varepsilon_t(\theta_{(2)} \psi_{(2)} \phi_{(1)}) \theta_{(3)} \rangle \langle f, \varepsilon_{(2)} \rangle \langle S(h_{(1)}), \psi_{(1)} \rangle \langle h_{(3)}, \psi_{(3)} \rangle \\
= \langle h_{(2)} g, \varepsilon_t(\psi_{(2)} \phi) \varepsilon_s(\varepsilon_{(1)} \theta) \rangle \langle f, \varepsilon_{(2)} \rangle \langle S(h_{(1)}), \psi_{(1)} \rangle \langle h_{(3)}, \psi_{(3)} \rangle \\
\langle f, \varepsilon_{(2)} \rangle \langle S(h_{(1)}), \psi_{(1)} \rangle \langle h_{(3)}, \varepsilon_t(1_{(1)} \psi_{(3)}) \rangle \\
= \langle h_{(2)} g, \varepsilon_t(\psi_{(2)} \phi) \rangle \langle g_{(2)}, S(\theta_{(1)}) \varepsilon_t(\theta_{(2)} \psi_{(4)}) \theta_{(3)} \rangle \\
\langle f, \varepsilon_{(2)} \rangle \langle S(h_{(1)}), \psi_{(1)} \rangle \langle h_{(3)}, \varepsilon_t(1_{(1)} \psi_{(3)}) \rangle \\
= \langle h_{(2)} g, \varepsilon_t(\psi_{(2)} \phi) \rangle \langle g_{(2)}, S(\theta_{(1)}) \varepsilon_t(\theta_{(2)} \psi_{(4)}) \theta_{(3)} \rangle \\
\langle g_{(3)} f, \varepsilon_t(\theta_{(2)} \psi_{(4)}) \rangle \langle S(g_{(2)}), \theta_{(1)} \rangle \langle g_{(4)}, \theta_{(3)} \rangle \\
= \varepsilon(\phi \otimes h)[\psi_{(2)} \otimes g_{(2)}] \varepsilon(\psi_{(1)} \otimes g_{(1)})[\theta \otimes f],
\]

\[
\varepsilon(\phi \otimes h)[\psi_{(1)} \otimes g_{(1)}][\theta \otimes f] = \\
= \langle h_{(2)} g_{(4)}, \varepsilon_t(\psi_{(3)}) \rangle \langle S(h_{(1)}), \psi_{(2)} \rangle \langle h_{(3)}, \psi_{(4)} \rangle \\
\langle g_{(2)} f, \varepsilon_t(\theta_{(2)} \psi_{(1)}) \rangle \langle S(g_{(1)}), \theta_{(1)} \rangle \langle g_{(3)}, \theta_{(3)} \rangle \\
= \langle h_{(2)} g_{(2)}, \varepsilon_t(\psi_{(3)} \phi) \rangle \langle g_{(1)}, S(\theta_{(1)}) \varepsilon_t(\theta_{(2)} \psi_{(1)}) \theta_{(3)} \rangle \\
\langle f, \varepsilon_{(2)} \rangle \langle S(h_{(1)}), \psi_{(2)} \rangle \langle h_{(3)}, \psi_{(4)} \rangle \\
= \langle h_{(2)} g_{(2)}, \varepsilon_t(\psi_{(3)} \phi) \rangle \langle g_{(1)}, \varepsilon_t(1_{(1)} \theta) \rangle \\
\langle f, \varepsilon_{(2)} \rangle \langle S(h_{(1)}), S(\varepsilon_t(1_{(1)} \psi_{(1)}) \rangle \langle h_{(3)}, \psi_{(4)} \rangle \\
= \langle h_{(2)} g, \varepsilon_t(\psi_{(2)} \phi) \varepsilon_s(\varepsilon_{(1)} \theta) \rangle \langle f, \varepsilon_{(2)} \rangle \langle S(h_{(1)}), \psi_{(1)} \rangle \langle h_{(3)}, \psi_{(3)} \rangle \\
= \varepsilon(\phi \otimes h)[\psi \otimes g][\theta \otimes f]),
\]
which is axiom (2). For axiom (1) we have:

\[(\Delta([\varepsilon \otimes 1]) \otimes [\varepsilon \otimes 1])([\varepsilon \otimes 1] \otimes \Delta([\varepsilon \otimes 1])) = \]
\[= [\varepsilon(1) \otimes 1(1)] \otimes [\varepsilon(2) \otimes 1(2)] [\varepsilon(1) \otimes 1'(1)] \otimes [\varepsilon(2) \otimes 1'(2)] \]
\[= [\varepsilon(1) \otimes 1(1)] \otimes [\varepsilon(2) \otimes 1(2)] [\varepsilon(1) \otimes 1'(1)] \otimes [\varepsilon(2) \otimes 1'(2)] \]
\[= [\varepsilon(1) \otimes 1(1)] \otimes [\varepsilon(2) \otimes 1(2)] \otimes [\varepsilon(3) \otimes 1(3)]. \]

where we used the axioms of a quantum groupoid and the definition of \( J \).

In order to check the axioms (3),(4), let us compute the target counital map \( \varepsilon_t \). We have

\[\varepsilon_t([\phi \otimes h]) = \varepsilon([\varepsilon(1) \otimes 1(1)][\phi \otimes h])[\varepsilon(2) \otimes 1(2)] \]
\[= \langle \varepsilon(1)(2)h, \phi(2)\varepsilon(1) \rangle \langle S(1(1)), \phi(1) \rangle \langle 1(3), \phi(3) \rangle [\varepsilon(2) \otimes 1(4)] \]
\[= \langle S(1(1))1(1)\varepsilon(1)(2)h, \phi \rangle \langle 1(2), \varepsilon(1) \rangle [\varepsilon(2) \otimes 1(4)] \]
\[= \langle 1(1)\varepsilon(1)(h), \phi \rangle [\varepsilon \otimes 1(2)]. \]

Similarly one computes the source counital map:

\[\varepsilon_s([\phi \otimes h]) = [\varepsilon(1) \otimes 1(1)\varepsilon(1)(h), \phi]. \]

Using these formulas we have:

\[m([\text{id} \otimes S])\Delta([\phi \otimes h]) = \]
\[= [\phi(1) \otimes h(1)][S^{-1}(\phi(3)) \otimes h(2)] [\langle h(2), \phi(2) \rangle \langle S(h(4)), \phi(4) \rangle \]
\[= [S^{-1}(\phi(3))\phi(1) \otimes h(2)] [\langle S(h(1)), S^{-1}(\phi(5)) \rangle \]
\[= [h(3), S^{-1}(\phi(3))] [\langle h(4), \phi(2) \rangle \langle S(h(5)), \phi(6) \rangle \]
\[= [S^{-1}(\phi(3))\phi(1) \otimes h(2)] [\langle S(h(3)), S^{-1}(\phi(4)) \rangle \langle h(1), S(h(5)), \phi(4) \rangle \]
\[= [S^{-1}(\phi(3))\phi(1) \otimes 1(1)\varepsilon(1)(h) \langle 1(2), S^{-1}(\phi(2)) \rangle [\langle S(h(3)), \phi(4) \rangle \]
\[= [S^{-1}(\phi(3))\phi(1) \otimes 1(1)\varepsilon(1)(h) \langle 1(2), S^{-1}(\phi(2)) \rangle [\langle S(h(3)), \phi(4) \rangle \]
\[= [\varepsilon(1)S^{-1}(\phi(3))\phi(1) \otimes 1(1)\varepsilon(1)(h) \langle 1(2), \varepsilon(2) \rangle [\langle 1(1)\varepsilon(1)(h), \phi(3) \rangle \]
\[= [S^{-1}(\phi(3))\phi(1) \otimes 1(1)\varepsilon(1)(h), \phi(3) \rangle \]
\[= [\varepsilon \otimes 1(2)] [\langle 1(1)\varepsilon(1)(h), \phi \rangle = \varepsilon_t([\phi \otimes h])]. \]
and
\[
\begin{align*}
m(S \otimes \text{id})\Delta([\phi \otimes h]) = \\
= \{\phi(5)S^{-1}(\phi(2)) \otimes S(h(3))h(6)\}{(S^2(h(4)), \phi(4))}\{S(h(2)), \phi(6)\} \\
\langle h(1), \phi(1)\}\{S(h(5)), \phi(3)\} \\
= [S^{-1}(\varepsilon(1)\varepsilon(2)) \otimes S(h(2))h(4)]\{\varepsilon(h(3)), \varepsilon(h(3))\} \\
\langle h(1), \varepsilon(1)\varepsilon(2)\} \\
= [S^{-1}(\varepsilon(1)\varepsilon(2)) \otimes S(h(2))h(4)]\{\varepsilon(h(3)), \varepsilon(h(3))\} \\
\langle h(1), \varepsilon(1)\varepsilon(2)\} \\
= [\varepsilon(3)S^{-1}(\varepsilon(2)) \otimes \varepsilon(h(2))\} \\
\langle h(1), \varepsilon(h(2))S(\varepsilon(2))\} \\
= [\varepsilon(1) \otimes 1\} \\
\langle h, S(\varepsilon(2))\varepsilon(t(\phi)) = \varepsilon(\phi \otimes h)).
\end{align*}
\]

In the above computations we used repeatedly the amalgamation relations in \(D(H)\):
\[
[\phi \otimes zh] = [(\varepsilon \leftarrow z)\phi \otimes h] \quad (z \in H), \quad [\phi \otimes yh] = [(y \rightarrow \varepsilon)\phi \otimes h] \quad (y \in H_s)
\]
that follow from \([12]\), the axioms of a quantum groupoid and properties of the counital maps. Finally, we prove the relation which is equivalent to \(S\) being both algebra and coalgebra anti-homomorphism:
\[
\begin{align*}
m(\text{id} \otimes m)(S \otimes \text{id} \otimes S)(\text{id} \otimes \Delta)\Delta([\phi \otimes h]) =
= m(\phi \otimes \phi)\Delta([\phi \otimes h]) \\
= S([\phi(1) \otimes h(1)])\varepsilon([\phi(2) \otimes h(2)]) \\
= [S^{-1}(\phi(3)) \otimes S(h(2))]\{\varepsilon \otimes 1\} \\
\langle h(1), \phi(1)\} \langle S(h(3)), \phi(3)\} \langle 1(1), \varepsilon(1)\varepsilon(2)\} \\
= [S^{-1}(\phi(3)) \otimes S(h(2))1(2)]\{h(1), \phi(1)\} \langle S(h(3))1(1), \phi(3)\} = S([\phi \otimes h]).
\end{align*}
\]

Note that \(D(H)_t = [\varepsilon \otimes H_t]\) and \(D(H)_s = \hat{H}_s \otimes 1\).

Proposition 6.2. The Drinfeld double \(D(H)\) has a quasitriangular structure given by
\[
R = \sum_i [\xi^i \otimes 1] \otimes [\varepsilon \otimes f_i], \quad \bar{R} = \sum_j [S^{-1}(\xi_j) \otimes 1] \otimes [\varepsilon \otimes f_j]
\]
where \(\{f_i\}\) and \(\{\xi^i\}\) are dual bases in \(H\) and \(\hat{H}\).

Proof. The identities \((\text{id} \otimes \Delta)R = R_{13}R_{12}\) and \((\Delta \otimes \text{id})\bar{R} = R_{13}\bar{R}_{23}\) can be written as (identifying \([\hat{H}^{op} \otimes 1]\) with \(\hat{H}^{op}\) and \([\varepsilon \otimes H]\) with \(H\):
\[
\begin{align*}
\sum_i \xi^i(1) \otimes \xi^j(2) \otimes f_i = \sum_{ij} \xi^i \otimes \xi^j \otimes f_i f_j, \\
\sum_i \xi^i \otimes f_i(1) \otimes f_i(2) = \sum_{ij} \xi^i \xi^j \otimes f_j \otimes f_i.
\end{align*}
\]

The above equalities can be verified by evaluating both sides on an element \(h \in H\) in the third factor (resp., on \(\phi \in \hat{H}^{op}\) in the second factor), see \([M], 7.1.1\). To
show that $\mathcal{R}$ is an intertwiner between $\Delta$ and $\Delta^{\text{op}}$, we compute

$$\mathcal{R}\Delta([\phi \otimes h]) = \sum_i [\phi_{(1)} \xi^i \otimes h_{(1)}] \otimes [\phi_{(3)} \otimes f_{i(2)} h_{(2)}]$$

$$\langle S(f_{i(1)}), \phi_{(2)} \rangle \langle f_{i(3)}, \phi_{(4)} \rangle$$

$$= \sum_i [\phi_{(1)} S(\phi_{(2)}) \xi^i \phi_{(4)} \otimes h_{(1)}] \otimes [\phi_{(3)} \otimes f_i h_{(2)}]$$

$$= \sum_i [\xi^i \phi_{(2)} \otimes (1_{(1)}, \varepsilon_i(\phi_{(1)})) 1_{(2)} h_{(1)}] \otimes [\phi_{(2)} \otimes f_i h_{(2)}]$$

$$= \sum_i [\xi^i \phi_{(2)} \otimes (1_{(1)}, \varepsilon_i(\phi_{(1)})) 1_{(2)} h_{(1)}] \otimes [\varepsilon_i(\phi_{(1)}) \otimes f_i h_{(3)}]$$

$$= \sum_i [\xi^i \phi_{(2)} \otimes h_{(3)}] \otimes [\phi_{(1)} \otimes h_{(1)} S(h_{(2)}) f_i h_{(4)}]$$

$$= \sum_i [\xi^i \phi_{(2)} \otimes h_{(3)}] \otimes [\phi_{(1)} \otimes h_{(1)} S_{(2)} f_i h_{(4)}]$$

$$= \Delta^{\text{op}}([\phi \otimes h]) \mathcal{R},$$

where we used

$$\sum_i \langle a, \xi^i \rangle f_i = a \quad \text{and} \quad \sum_i \xi^i \langle f_i, \phi \rangle = \phi,$$

for all $a \in H$, $\phi \in \widehat{H}$. Finally, let us check that the element $\mathcal{R} = \sum_j [S^{-1}(\xi_j) \otimes 1] \otimes [\varepsilon \otimes f_j]$ satisfies $\mathcal{R} \mathcal{R} = \Delta(1)$ and $\mathcal{R} \mathcal{R} = \Delta^{\text{op}}(1)$. The first property is equivalent to

$$\sum_{i,j} [\xi^i S^{-1}(\xi^j) \otimes 1] \otimes [\varepsilon \otimes f_j f_i] = [(1_{(1)}, \varepsilon_{(2)}') \varepsilon_{(1)}' 1_{(1)}] \otimes [\varepsilon \otimes (1_{(1)}', \varepsilon_{(2)}) 1_{(1)}' 1_{(2)}].$$

which can be regarded as an equality in $\widehat{H}^{\text{op}} \otimes H$ :

$$\sum_{i,j} [\xi^i S^{-1}(\xi^j) \otimes f_j f_i] = (1_{(1)}, \varepsilon_{(2)}') \varepsilon_{(1)}' 1_{(1)}' 1_{(2)}.'$$

Evaluating both sides on arbitrary $\phi \in \widehat{H}$ in the second factor, we get

$$\phi_{(2)} S^{-1}(\phi_{(1)}) = (1_{(1)}, \varepsilon_{(2)}') \varepsilon_{(1)}' 1_{(1)}' 1_{(2)}.$$
The structure operations in $\hat{D}(H)$ are obtained by dualizing those in $D(H)$:

$$
\left( \sum_k h_k \otimes \phi_k \right) \left( \sum_l g_l \otimes \psi_l \right) = \sum_{k,l} h_k g_l \otimes \phi_k \psi_l,
$$

$$
1_{\hat{D}(H)} = 1_{(2)} \otimes (\varepsilon \leftarrow 1_{(1)}),
$$

$$
\Delta \left( \sum_k h_k \otimes \phi_k \right) = \sum_{i,j,k} (h_k(2) \otimes \xi^i \phi_{k(1)}(\xi^j) \otimes (S(f_i) h_{k(1)} f_j \otimes \phi_{k(2)})),
$$

$$
\varepsilon \left( \sum_k h_k \otimes \phi_k \right) = \sum_k \varepsilon(h_k) \varepsilon(\phi_k),
$$

$$
S \left( \sum_k h_k \otimes \phi_k \right) = \sum_{i,j,k} f_i S^{-1}(h_k) S(f_j) \otimes \xi^i S(\phi) \xi^j,
$$

for all $\sum_k h_k \otimes \phi_k$, $\sum_l g_l \otimes \psi_l \in \hat{D}(H)$, where $\{f_i\}$, $\{\xi^j\}$ are dual bases in $H, \hat{H}$, respectively.

**Proposition 6.4.** The Drinfeld double $D(H)$ is factorizable in the sense of Definition 5.11.

**Proof.** First, observe that for any pair of dual bases $\{f_i\}$ and $\{\xi^j\}$ as above and all $g \in H$ and $\psi \in \hat{H}$ the element

$$
Q_{g \otimes \psi} = \sum_{k,l} f_k g f_l \otimes \xi^k \psi S(\xi^l) \in H \otimes_k \hat{H}^{op}
$$

belongs to $\hat{D}(H)$. Indeed,

$$
\langle Q_{g \otimes \psi}, \phi \otimes zh \rangle =
$$

$$
= \langle zh(1), \phi(1) \rangle \langle S(h(3)), \phi(3) \rangle \langle g, \phi(2) \rangle \langle h(2), \psi \rangle
$$

$$
= \langle h(1), (\phi \leftarrow z)(1) \rangle \langle S(h(3)), (\phi \leftarrow z)(2) \rangle \langle g, (\phi \leftarrow z)(2) \rangle \langle h(2), \psi \rangle
$$

$$
= \langle Q_{g \otimes \psi}, (\varepsilon \leftarrow z) \phi \otimes h \rangle,
$$

for all $h \in H$, $\phi \in \hat{H}$, $z \in H_t$ and, likewise,

$$
\langle Q_{g \otimes \psi}, \phi \otimes yh \rangle = \langle Q_{g \otimes \psi}, (y \rightarrow \varepsilon) \phi \otimes h \rangle.
$$

Next, we compute

$$
(\Delta_{g \otimes \psi} \otimes \text{id})(R_{21} R) =
$$

$$
= \sum_{i,j} Q_{g \otimes \psi}(\xi^i (2) \otimes f_i(2)) \xi^i \otimes f_j \langle S(f_i(1)), \xi^j(1) \rangle \langle f_i(3), \xi^j(3) \rangle
$$

$$
= \sum_{i,j} Q_{g \otimes \psi}(\xi^i (2) \otimes f_i) [S(\xi^j(1)) \xi^i \xi^j(3) \otimes f_j]
$$

$$
= \sum_{i,j,k,l} [S(\xi^j) \xi^k \psi S(\xi^l) \xi^i \otimes f_j f_k g f_l f_i]
$$

$$
= [\varepsilon(1)'(1), \varepsilon(2)'(2) \psi \varepsilon(1) \langle 1(2), \varepsilon(2) \rangle \otimes 1'(1) g 1(1)]
$$

$$
= [\psi \varepsilon(1)'(1), \varepsilon(2)'(2) \otimes g 1(1)],
$$
where we used the identities
\[ \sum_j S(\xi^j_{(1)}(g, \xi^j_{(2)}) \otimes \xi^j_{(3)} \otimes f_j) = \sum_{i,j} S(\xi^i) \otimes \xi^i \otimes f_i g f_j, \]
\[ \sum_{i,j} S(\xi^i) \xi^i \otimes f_i f_j = \varepsilon(1) \otimes 1(2) \otimes \varepsilon(2), \]
that follow from (38), (39), (40) and from axioms (3), (4) of a quantum groupoid. Therefore,
\[ \nu \]
is called a ribbon element of \( H \) with duality. Conversely, if \( \theta \) is a ribbon element of \( H \), then we conclude from Lemma 2.2 that the map
\[ \nu \]
is surjective, i.e., \( D(H) \) is factorizable.

7. Ribbon quantum groupoids

Definition 7.1. A ribbon quantum groupoid is a quasitriangular quantum groupoid \( H \) with an invertible central element \( \nu \in H \) such that
\[ \Delta(\nu) = \mathcal{R}_{21} \mathcal{R}(\nu \otimes \nu) \text{ and } S(\nu) = \nu. \]
The element \( \nu \) is called a ribbon element of \( H \).

For an object \( V \) of \( \text{Rep}(H) \) we define the twist \( \theta_V : V \rightarrow V \) to be the multiplication by \( \nu \):
\[ \theta_V(v) = \nu \cdot v, \quad v \in V. \]

Proposition 7.2. Let \( (H, \mathcal{R}, \nu) \) be a ribbon quantum groupoid. The family of homomorphisms \( \{\theta_V\}_V \) is a twist in the braided monoidal category \( \text{Rep}(H) \) compatible with duality. Conversely, if \( \theta_V(v) = \nu \cdot v \) with \( \nu \in H \) is a twist in \( \text{Rep}(H) \), then \( \nu \) is a ribbon element of \( H \).

Proof. Since \( \nu \) is an invertible central element of \( H \), the homomorphism \( \theta_V \) is an \( H \)-linear isomorphism. The twist identity
\[ c_{W,V} c_{V,W}(\theta_V \otimes \theta_W) = \theta_{V \otimes W} \]
follows from the properties of \( \nu \):
\[ c_{W,V} c_{V,W}(\theta_V \otimes \theta_W)(x) = \mathcal{R}_{21} \mathcal{R}(\nu \cdot x^{(1)} \otimes \nu \cdot x^{(2)}) = \Delta(\nu) \cdot x = \theta_{V \otimes W}(x), \]
for all \( x = x^{(1)} \otimes x^{(2)} \in V \otimes W \). Clearly, the identity
\[ \mathcal{R}_{21} \mathcal{R}(\nu \otimes \nu) = \Delta(\nu) \]
is equivalent to the twist property. It remains to prove that
\[ (\theta_V \otimes \text{id}_V \cdot \cdot b_V(z) = (\text{id}_V \otimes \theta_V \cdot \cdot b_V(z), \]
for all \( z \in H_t \), i.e., that
\[ \sum_i \nu z^{(1)}_i \cdot \varepsilon^i \otimes z^{(2)}_i \cdot f_i = \sum_i z^{(1)}_i \cdot \varepsilon^i \otimes \nu z^{(2)}_i \cdot f_i, \]
where \( \sum_i \varepsilon^i \otimes f_i \) is the canonical element in \( V^* \otimes V \). Evaluating the first factors of the above equality on an arbitrary \( v \in V \), we get the equivalent condition:
\[ \sum_i (\nu z^{(1)}_i \cdot \varepsilon^i)(v) z^{(2)}_i \cdot f_i = \sum_i (z^{(1)}_i \cdot \varepsilon^i)(v) \nu z^{(2)}_i \cdot f_i, \]
which reduces to $z(2)S(\nu z(1)) \cdot v = S(\nu z(2)) \cdot v$. The latter easily follows from
the centrality of $\nu = S(\nu)$ and properties of $H_i$.  \hfill \Box

**Proposition 7.3.** The category $\text{Rep}(H)$ is a ribbon category if and only if $H$ is a
ribbon quantum groupoid.

**Proof.** Follows from Propositions 4.4, 5.2, and 7.2.  \hfill \Box

For any endomorphism $f$ of an object $V$ of $\text{Rep}(H)$, we define, following ([T2], I.1.5), its quantum trace
\begin{equation}
\text{tr}_q(f) = d_V c_{V,V} \cdot (\theta_V f \otimes \text{id}_V) b_V \tag{43}
\end{equation}
with values in $\text{End}(H_i)$ and the quantum dimension of $V$ by $\text{dim}_q(V) = \text{tr}_q(\text{id}_V)$. The next lemma gives an explicit computation of $\text{tr}_q$ and $\text{dim}_q$ via the usual trace of endomorphisms.

**Proposition 7.4.** Let $(H, \mathcal{R}, \nu)$ be a ribbon quantum groupoid, $f$ be an endomorphism of an object $V$ in $\text{Rep}(H)$. Then
\begin{equation}
\text{tr}_q(f)(z) = \text{Tr}(S(1(1))uf)z1(2), \quad \text{dim}_q(V)(z) = \text{Tr}(S(1(1))uv)z1(2), \tag{44}
\end{equation}
where $\text{Tr}$ is the usual trace of endomorphisms, and $u \in H$ is the Drinfeld element.

**Proof.** Since the trace of an endomorphism $h \in \text{End}_k(H_i)$, in terms of the canonical element $\sum_i f_i \otimes \xi^i \in V \otimes_k V^*$, is $\text{Tr}(h) = \sum_i \xi^i(h(f_i))$, the definition of $\text{tr}_q$ gives:
\begin{align*}
\text{tr}_q(f)(z) &= d_V c_{V,V} \cdot (\theta_V f \otimes \text{id}_V) b_V(z) \\
&= d_V \sum_i \mathcal{R}^{(2)} z(2) \cdot \xi_i \otimes \mathcal{R}^{(1)} \nu z(1) f(f_i) \\
&= \sum_i \xi_i (\mathcal{R}^{(2)} z(2)) \cdot \xi_i(1(1) \mathcal{R}^{(1)} \nu z(1) f(f_i))1(2) \\
&= \text{Tr}(S(1(1)uv)z1(2),
\end{align*}
where we used formulas ([24] and [25]) defining $b_V$ and $d_V$.

**Corollary 7.5.** Let $k$ be algebraically closed. If $H$-module $H_i$ is irreducible (which happens exactly when $H_i \cap Z(H) = k$, i.e., when $H$ is connected ([3], 3.11, [BNSZ], 2.4), then $\text{tr}_q(f)$ and $\text{dim}_q(V)$ are scalars:
\begin{equation}
\text{tr}_q(f) = (\text{dim} H_i)^{-1} \text{Tr}(uvf), \quad \text{dim}_q(V) = (\text{dim} H_i)^{-1} \text{Tr}(uv). \tag{45}
\end{equation}

**Proof.** An endomorphism of an irreducible module is multiplication by a scalar, therefore, we must have $\text{Tr}(S(1(1)uvf)1(2)) = \text{tr}_q(f)(1) = \text{tr}_q(f)1$. Applying the counit to both sides and using that $\varepsilon(1) = \text{dim} H_i$, we get the result.  \hfill \Box

8. Towards modular categories

Let us first recall some definitions needed in this section. Let $\mathcal{V}$ be a ribbon $Ab$-category over $k$, i.e., such that all $\text{Hom}(V,W)$ are $k$-vector spaces (for all objects $V, W \in \mathcal{V}$) and both operations $\circ$ and $\otimes$ are $k$-bilinear.

An object $V \in \mathcal{V}$ is said to be simple if any endomorphism of $V$ is multiplication by an element of $k$. We say that a family $\{V_i\}_{i \in I}$ of objects of $\mathcal{V}$ dominates an object $V$ of $\mathcal{V}$ if there exists a finite set $\{V_{i(r)}\}_r$ of objects of this family (possibly,
with repetitions) and a family of morphisms \( f_r : V_{i(r)} \to V, g_r : V \to V_{i(r)} \) such that \( \text{id}_V = \sum_r f_r g_r \).

A modular category \( \mathcal{E}_s \) II.1.4 is a pair consisting of a ribbon \( Ab \)-category \( \mathcal{V} \) and a finite family \( \{V_i\}_{i \in I} \) of simple objects of \( \mathcal{V} \) satisfying four axioms:

(i) There exists \( 0 \in I \) such that \( V_0 \) is the unit object.
(ii) For any \( i \in I \), there exists \( i^* \in I \) such that \( V_i \) is isomorphic to \( V_{i^*} \).
(iii) All objects of \( \mathcal{V} \) are dominated by the family \( \{V_i\}_{i \in I} \).
(iv) The square matrix \( S = \{S_{ij}\}_{i,j \in I} = \{\text{tr}_q(c_{V_i,V_j} \circ c_{V_j,V_i})\}_{i,j \in I} \) is invertible over \( k \) (here \( \text{tr}_q \) is the quantum trace in a ribbon category defined by (43)).

If a quantum groupoid \( H \) is connected and semisimple over an algebraically closed field, modularity of \( \text{Rep}(H) \) is equivalent to \( \text{Rep}(H) \) being ribbon and such that the matrix \( S = \{S_{ij}\}_{i,j \in I} = \{\text{tr}_q(c_{V_i,V_j} \circ c_{V_j,V_i})\}_{i,j \in I} \), where \( I \) is the set of all (equivalent classes of) irreducible representations, is invertible.

**Remark 8.1.** Recall that \( h \in H \) is a left (resp. right) integral if \( x h = \varepsilon_l(x) h \) (resp. \( h x = h \varepsilon_r(x) \)) for all \( x \in H \) ([BNS], 3.24). A Haar integral is a two-sided integral \( h \) which is normalized, i.e., \( \varepsilon_l(h) = \varepsilon_r(h) = 1 \). Existence of a Haar integral in a quantum groupoid \( H \) is equivalent to \( H \) being semisimple and possessing an invertible element \( g \) such that \( S^{(2)}(x) = g x g^{-1} \) for any \( x \in H \) and \( \chi(g^{-1}) \neq 0 \) for all irreducible characters \( \chi \) of \( H \) ([BNS], 3.27).

The following lemma extends a result known for Hopf algebras ([EC], 1.1).

**Lemma 8.2.** Let \( H \) be a connected, ribbon, factorizable quantum groupoid over an algebraically closed field \( k \), and assume that \( H \) has a Haar integral. Then \( \text{Rep}(H) \) is a modular category.

**Proof.** Note that \( H \) is semisimple by Remark 8.1. We only need to prove the invertibility of the matrix formed by

\[
S_{ij} = \text{tr}_q(c_{V_i,V_j} \circ c_{V_j,V_i}) = (\dim H_i)^{-1} \text{Tr}((uv) \circ c_{V_i,V_j} \circ c_{V_j,V_i}) = (\dim H_i)^{-1}(\chi_j \otimes \chi_i)((uv \otimes uv)\mathcal{R}_{21}\mathcal{R}),
\]

where \( V_i \) are as above, \( I = \{1, \ldots, n\} \), \( \{\chi_i\} \) is a basis in the space \( C(H) \) of characters of \( H \) (we used above the formula (15) for the quantum trace).

Observe that the linear map \( F : \phi \mapsto (\phi \otimes \text{id})(\mathcal{R}_{21}\mathcal{R}) \) takes any element of the form \( \phi = \chi \leftrightarrow uv \) (i.e., \( \phi(h) = \chi(uvh) \forall h \in H \)), where \( \chi \in C(H) \), into \( Z(H) \). Indeed, for any such \( \phi \) and all \( \psi \in \mathcal{H}, h \in H \) we have, using the fact that \( u \in C_H(H) \) (this follows from Lemma 33), the properties of \( \varepsilon_s \) and \( \varepsilon^{op}_s \), the relation \( \Delta^{op}(h)\mathcal{R} = \Delta(h)\mathcal{R} \) (\( h \in H \)), and the centrality of \( \chi \):

\[
\langle F(\phi)h, \psi \rangle = \langle uv\mathcal{R}(2)\mathcal{R}'(1), \chi \rangle \langle \mathcal{R}(1)\mathcal{R}'(2)h, \psi \rangle = \langle uv\mathcal{R}(2)\mathcal{R}'(1)\varepsilon^{op}_s(h(1)), \chi \rangle \langle \mathcal{R}(1)\mathcal{R}'(2)h(2), \psi \rangle = \langle uv\mathcal{R}(2)h(2)\mathcal{R}'(1)S^{-1}(h(1)), \chi \rangle \langle \mathcal{R}(1)h(3)\mathcal{R}'(2), \psi \rangle = \langle uh(2)\mathcal{R}(2)\mathcal{R}'(1)S^{-1}(h(1)), \chi \rangle \langle h(3)\mathcal{R}(1)\mathcal{R}'(2), \psi \rangle = \langle uhS(h(1))h(2)\mathcal{R}(2)\mathcal{R}'(1), \chi \rangle \langle h(3)\mathcal{R}(1)\mathcal{R}'(2), \psi \rangle = \langle uh\mathcal{R}(2)\mathcal{R}'(1), \chi \rangle \langle h\mathcal{R}(1)\mathcal{R}'(2), \psi \rangle = \langle hF(\phi), \psi \rangle,
\]
therefore $F(\phi) \in Z(H)$. Since $H$ is factorizable, we know from Corollary 5.12 that the restriction

$$F : \{ \phi \in \hat{H} \mid \phi = \phi \circ \text{Ad}_T \} \to C(H)$$

is a linear isomorphism. Since $\chi \leftarrow uv$ belongs to the subspace on the left hand side, we have a linear isomorphism between $C(H) \leftarrow uv$ and $Z(H)$, hence, there exists an invertible matrix $T = (T_{ij})$ representing the map $F$ in the bases of $C(H)$ and $Z(H)$, i.e., such that $F(\chi_j \leftarrow uv) = \sum_i T_{ij} \varepsilon_i$. Then

$$S_{ij} = (\dim H_i)^{-1} \chi_i(uvF(\chi_j \leftarrow uv)) = (\dim H_i)^{-1} \sum_k T_{kj} \chi_i(uek)$$

$$= (\dim H_i)^{-1}(\dim V_i)\chi_i(uv)T_{ij}.$$  

Therefore, $S = DT$, where $D = \text{diag}((\dim H_i)^{-1}(\dim V_i)\chi_i(uv))$. If $g$ is an element from Remark 8.1 then $u^{-1}g$ is an invertible central element of $h$ and $\chi_i(u^{-1}) \neq 0$ for all $\chi_i$. By Corollary 8.2 $uS(u) = c$ is invertible central, therefore $\chi_i(u) = \chi_i(c)\chi_i(S(u^{-1})) \neq 0$. Hence, $\chi_i(uv) \neq 0$ for all $i$ and $D$ is invertible.

**Example 8.3.** An example of a modular category can be constructed from *elementary* quantum groupoids classified in [NV], 3.2. A quantum groupoid $H$ is called elementary if $H \cong M_n(k)$. Then it is determined, up to an isomorphism, by one of its counital subalgebras

$$H_t \cong \oplus_\alpha M_{n_\alpha}(k),$$

$n_\alpha$, $\alpha = 1 \ldots N$ are positive integers, $n = \sum_\alpha n_\alpha^2$. From this one can see that

$$D(H) = [\hat{H}^{\text{op}} \otimes H] = [\hat{H}_s \hat{H}_t \otimes H][\varepsilon \otimes H] = H,$$

where for subsets $A \subset \hat{H}^{\text{op}}$, $b \subset H$ we set $[A \otimes b] = \{[a \otimes b] \in D(H) \mid a \in A, b \in B\}$. Hence $H$ is the Drinfeld double of itself. The $R$-matrix of $H$ is

$$R = \sum_{i,j,k,\alpha} \frac{1}{n_\alpha} E^{k\beta}_{ija} \otimes E^{\alpha \beta}_{j\alpha},$$

where $\{E^{k\beta}_{ija}\}_{i,j=1 \ldots n_\alpha}$ is a system of matrix units in $H$. Both the Drinfeld and the ribbon elements are equal to 1. Thus, all the conditions of Lemma 5.2 are satisfied, the category Rep($H$) is modular with a unique irreducible object.

**9. $C^*$-QUANTUM GROUPOIDS AND UNITARY MODULAR CATEGORIES**

**Definition 9.1.** A *-quantum groupoid is a quantum groupoid over a field $k$ with involution, whose underlying algebra $H$ is equipped with an antilinear involutive algebra anti-homomorphism $\ast : H \to H$ such that $\Delta \circ \ast = (\ast \otimes \ast)\Delta$. A *-quantum groupoid is said to be $C^*$-quantum groupoid, if $k = \mathbb{C}$ and $H$ is a finite-dimensional $C^*$-algebra, i.e., $x^*x = 0$ if and only if $x = 0$, $\forall x \in H$.

Definition 8.1 together with the uniqueness of the unit, counit and antipode imply that

$$1^* = 1, \quad \varepsilon(h^*) = \overline{\varepsilon(h)}, \quad (S \circ \ast)^2 = \text{id}$$

for all $h$ in a *-quantum groupoid $H$. It is also easy to check the relations

$$\varepsilon_t(h)^* = \varepsilon_t(S(h)^*), \quad \varepsilon_t(h)^* = \varepsilon_t(S(h)^*),$$
therefore, \( H_t \) and \( H_s \) are \(*\)-subalgebras. The dual \( \hat{H} \) is also a \(*\)-quantum groupoid with respect to the \(*\)-operation
\begin{equation}
\langle \phi^*, h \rangle = \frac{1}{\langle h, S(h)^* \rangle} \quad \text{for all } \phi \in \hat{H}, \ h \in H.
\end{equation}
The square of the antipode of a \( C^* \)-quantum groupoid is an inner automorphism, i.e., \( S^2(h) = ghg^{-1} \) for some \( g \in H \). It is easy to see that there is a unique such \( g \) satisfying the following conditions ([BNSz], 4.4):
\begin{enumerate}
\item \( \operatorname{tr}(\pi(\phi^{-1})) = \operatorname{tr}(\phi(\pi(\phi))) \neq 0 \) for all irreducible representations \( \pi_{\alpha} \) of \( H \) (here \( \operatorname{tr} \) is the usual trace on a matrix algebra);
\item \( S(g) = g^{-1} \), and
\item \( \Delta(g) = (g \otimes g)\Delta(1) = \Delta(1)(g \otimes g) \).
\end{enumerate}
This element \( g \) is called the \textit{canonical group-like element} of \( H \).

\begin{remark}
(i) Any \( C^* \)-quantum groupoid satisfies the conditions of Remark 8.1, so it always possess a Haar integral.

(ii) If \( H \) is a \( C^* \)-quantum groupoid, then its dual \( \hat{H} \) is also a \( C^* \)-quantum groupoid (see [BNSz], 4.5, [NV1], 2.3.10).
\end{remark}

Groupoid algebras and their duals give examples of commutative and cocommutative \( C^* \)-quantum groupoids if the ground field \( k = \mathbb{C} \) (in which case \( g^* = g^{-1} \) for all \( g \in G \)).

One can check that for a quasitriangular \(*\)-quantum groupoid \( \mathcal{R} = \mathcal{R}^* \).

**Proposition 9.3.** If \( H \) is a \( C^* \)-quantum groupoid, then \( D(H) \) is a quasitriangular \( C^* \)-quantum groupoid.

**Proof.** First let us show that \( \mathcal{D}(H) \), equipped with a natural involution
\[ \langle X^*, \phi \otimes h \rangle = \sum_k \langle g_k, (S\phi)^* \rangle \langle S(h)^*, \psi_k \rangle, \]
where \( X = \sum_k g_k \otimes \psi_k \in \mathcal{D}(H), h \in H, \phi \in \hat{H} \), is a \( C^* \)-subalgebra of the tensor product \( C^* \)-algebra \( H \otimes \hat{H}^{op} \). For this it suffices to show that \( X^* |_{\mathcal{H}_I} = 0 \), i.e.,
\[ \langle X^*, \phi \otimes z h \rangle = \langle X^*, (\varepsilon \leftarrow z) \phi \otimes h \rangle, \langle X^*, \phi \otimes y h \rangle = \langle X^*, (y \rightarrow \varepsilon) \phi \otimes h \rangle \]
for all \( z \in H_t, y \in H_s \). For instance, one computes:
\[ \langle X^*, \phi \otimes z h \rangle = \sum_k \langle g_k, (S\phi)^* \rangle \langle S(z)^* S(h)^*, \psi_k \rangle, \]
\[ \langle X^*, (\varepsilon \leftarrow z) \phi \otimes h \rangle = \sum_k \langle g_k, S((\varepsilon \leftarrow z) \phi)^* \rangle \langle S(h)^*, \psi_k \rangle, \]
for all \( z \in H_t \). The right-hand sides of the above equations are equal since
\[ \langle z \varepsilon(1), S(\varepsilon(2))^* \rangle = \langle S(z)^*, \varepsilon(2) \rangle \varepsilon(1). \]
Similarly one gets the other relation.

To prove that the comultiplication of \( \mathcal{D}(H) \) is a \(*\)-homomorphism we compute
\[ \Delta(X^*) = \sum_{i,j,k} (g_{k(2)}^* \otimes \xi^j \psi_{k(1)}^* \xi^i) \otimes (f_j^* g_{k(1)}^* S(f_i^*) \otimes \psi_{k(2)}), \]
\[ = \sum_{i,j,k} (g_{k(2)}^* \otimes \xi^j \psi_{k(1)}^* \xi^i) \otimes (S(f_i^*) g_{k(1)}^* f_j \otimes \psi_{k(2)}^*) = \Delta(X^*), \]
Invariants of Knots and 3-Manifolds from Quantum Groupoids

where we use that $\sum_j (\xi^j)^* \otimes S(f_j)^* = \sum_j \xi^j \otimes f_j$ for every pair of dual bases. Thus, $\tilde{D}(H)$ is a $C^*$-quantum groupoid and so is $D(H)$ (see Remark 9.2). □

In [EG], it was shown that a quasitriangular semisimple Hopf algebra is automatically ribbon with ribbon element $\nu = u^{-1}$, where $u$ is the Drinfeld element. We are able to get a similar result for $C^*$-quantum groupoids.

**Proposition 9.4.** A quasitriangular $C^*$-quantum groupoid $H$ is automatically ribbon with ribbon element $\nu = u^{-1}g = gu^{-1}$, where $u$ is the Drinfeld element from Definition 5.8 and $g$ is the canonical group-like element implementing $S^2$.

**Proof.** Since $u$ also implements $S^2$ (Proposition 5.7), $\nu = u^{-1}g$ is central, therefore $S(\nu)$ is also central. Clearly, $u$ must commute with $g$. The same Proposition gives $\Delta(u^{-1}) = R_{21}R(u^{-1} \otimes u^{-1})$, which allows us to compute

$\Delta(\nu) = \Delta(u^{-1})(g \otimes g) = R_{21}R(u^{-1}g \otimes u^{-1}g) = R_{21}R(\nu \otimes \nu)$.

Propositions 5.6 and 5.7 and the trace property imply that

$\text{tr}(\pi_\alpha(u^{-1})) = \text{tr}(\pi_\alpha(R^{(2)}S^2(R^{(1)}))) = \text{tr}(\pi_\alpha(S^3(R^{(1)}S(R^{(2)}))) = \text{tr}(\pi_\alpha(S(u^{-1}))).$

Since $u^{-1} = \nu g^{-1}$ and $\nu$ is central, the above relation means that

$\text{tr}(\pi_\alpha(\nu))\text{tr}(\pi_\alpha(g^{-1})) = \text{tr}(\pi_\alpha(S(\nu)))\text{tr}(\pi_\alpha(g)),$

and, therefore, $\text{tr}(\pi_\alpha(\nu)) = \text{tr}(\pi_\alpha(S(\nu)))$ for any irreducible representation $\pi_\alpha$, which shows that that $\nu = S(\nu)$. □

**Corollary 9.5.** For a connected ribbon $C^*$-quantum groupoid $H$ we have:

$\text{tr}_H(f) = (\dim H)^{-1} Tr_\nu(g \circ f), \quad \dim_\nu(V) = (\dim H)^{-1} Tr_\nu(g)$.

for any $f \in \text{End}(V)$, where $V$ is an $H$-module.

To define the (unitary) representation category $U\text{Rep}(H)$ of a $C^*$-quantum groupoid $H$ we consider unitary $H$-modules, i.e., $H$-modules $V$ equipped with a scalar product

$(\cdot, \cdot) : V \times V \to \mathbb{C}$ such that $(h \cdot v, w) = (v, h^* \cdot w) \forall h \in H, v, w \in V$.

The notion of a morphism in this category remains the same as in $\text{Rep}(H)$. The monoidal product of $V, W \in U\text{Rep}(H)$ is defined as follows. We construct a tensor product $V \otimes_{\mathbb{C}} W$ of Hilbert spaces and remark that the action of $\Delta(1)$ on this left $H$-module is an orthogonal projection. The image of this projection is, by definition, the monoidal product of $V, W$ in $U\text{Rep}(H)$. Clearly, this definition is compatible with the monoidal product of morphisms in $\text{Rep}(H)$.

For any $V \in U\text{Rep}(H)$, the dual space $V^*$ is naturally identified $(v \to \overline{v})$ with the conjugate Hilbert space, and under this identification we have $h \cdot \overline{v} = S(h)^* \cdot v \forall h \in H, v \in V$. In this way $V^*$ becomes a unitary $H$-module with scalar product $(\overline{v}, \overline{w}) = (w, gv)$, where $g$ is the canonical group-like element of $H$.

The unit object in $U\text{Rep}(H)$ is $H_I$ equipped with scalar product $(z, t)_{H_I} = \varepsilon(zt^*)$ (it is known [BNS], [NV] that the restriction of $\varepsilon$ to $H_I$ is a non-degenerate positive form). One can verify that the maps $l_H, r_H$ and their inverses are isometries. For example, let us show that the adjoint map for $l_H$ is exactly its inverse. We have:

$(l_H(1_{(1)} \cdot z \otimes 1_{(2)} \cdot v), w) = (z \cdot v, w) \forall z \in H_I, v, w \in V$. 

On the other hand:

\[(1_1 \cdot z \otimes 1_{(2)} \cdot v, l^{-1}_V w) = (z S(1_{(1)}) \otimes 1_{(2)} \cdot v, S(1_{(1)}) \otimes 1_{(2)} \cdot w) = \varepsilon(z S(1_{(1)}) S(1_{(1)}^*) (1_{(2)}^* 1_{(2)}^* \cdot v, w) = (z \cdot v, w).\]

Proposition 4.2 implies that URep(H) is a monoidal category with duality (see also [BNS2], Section 3).

**Remark 9.6.** a) One can check that for a quasitriangular \(*\)-quantum groupoid the braiding is an isometry in URep(H): \(c^{-1}_{V,W} = c_{V,W}^*\).

b) For a ribbon \(C^*\)-quantum groupoid \(H\), the twist is an isometry in URep(H). Indeed, the relation \(\theta_v = \theta_v^{-1}\) is equivalent to the identity \(S(u^{-1}) = u^*\), which follows from Proposition 5.6 and Remark 9.6a).

A **Hermitian ribbon category** over the field \(k\) with involution is an Ab-ribbon category over \(k\) endowed with a conjugation of morphisms \(f \mapsto \overline{f}\) satisfying natural conditions (see [T2], I.5.2):

\begin{align*}
\overline{f} &= f, \\
\overline{f + g} &= \overline{f} + \overline{g}, \\
\overline{f \cdot g} &= \overline{g} \cdot \overline{f}, \\
\overline{c_{V,W} f} &= (c_{V,W})^{-1}, \\
\overline{b_V} &= id_V \otimes \overline{b_V}.
\end{align*}

A **unitary ribbon category** is a Hermitian ribbon category over the field \(\mathbb{C}\) such that for any morphism \(f\) we have \(\text{tr}_f(\overline{f} f) \geq 0\).

In a natural way we have a conjugation of morphisms in URep(H). Namely, for any morphism \(f : V \to W\) we define \(\overline{f} : W \to V\) as \(\overline{f}(w) = \overline{f}(\overline{w})\) for any \(w \in W\). Here \(\overline{f} \in W^*, f^* : W^* \to V^*\) is the standard dual of \(f\) (see [T2], I.1.8) and \(\overline{f}(\overline{w}) \in V\).

**Lemma 9.7.** Given a quasitriangular \(C^*\)-quantum groupoid \(H\), URep(H) is a unitary ribbon Ab-category with respect to the above conjugation of morphisms.

**Proof.** Relations (47) are obvious, (48) follows from Remarks 9.6. Let us prove relations (49). On the one hand, for all \(v, \phi \in V^*, z \in H_t\) we have, using the definitions of \(d_V, c_{V,V^*}, \theta_V\), Propositions 5.4, 5.7 and the notation \(\omega_{v,\phi}(L) = (Lv, \phi)\) for a linear operator \(L\) and two vectors \(v, \phi\) of a Hilbert space:

\[(d_V \circ c_{V,V^*}(\theta_V \otimes id_{V^*}))(v \otimes \phi), z\)]

\[= (d_V \circ c_{V,V^*}(gu^{-1} \otimes \phi), z)_{H_t} =
\]

\[= (d_V [S(R^{(2)})(\phi \otimes R^{(1)} gu^{-1} v), z]_{H_t} =
\]

\[= \varepsilon([1_{(1)} R^{(1)} gu^{-1} v, S(R^{(2)})(\phi)] 1_{(2)} (S(z^*))]
\]

\[= \omega_{v,\phi}(S(R^{(2)})(\phi))(S(z^*) R^{(1)} gu^{-1})]
\]

\[= \omega_{v,\phi}(S(R^{(2)})(\phi))(z^* gu^{-1}) = (z^* gv, \phi).\]
And, on the other hand, using the definition of \( b_V \), we compute:

\[
(b_V (v \otimes \phi), z)_{H_t} = (v \otimes \phi, \sum_i z(1)f_i \otimes S(z(2))^*\xi^i)V \otimes V^*
\]

\[
= \sum_i (v, z(1)f_i)(S(z(2))^*\xi^i, g\phi)
\]

\[
= \sum_i (z^*_i v, f_i)(\xi^i, S(z(2))g\phi)
\]

\[
= (z^*_i v, S(z(2))g\phi) = (v, zG\phi) = (z^*gv, \phi),
\]

whence the first part of (49) follows. To establish the second part, note that for all \( v \in V, \phi \in V^*, z \in H_t \) we have, using the definitions of \( b_V, c_{V^*}, \theta_V \), Propositions 5.6, 5.7 and the properties of \( \nu \):

\[
((\id_V, \otimes \theta_V^{-1})c_V^{-1,V} \circ b_V(z) \otimes \phi \otimes v)_{V^* \otimes V} = \]

\[
= ((\id_V, \otimes \theta_V^{-1})c_V^{-1,V} \sum_i z(1)f_i \otimes S(z(2))^*\xi^i, \phi \otimes v)_{V^* \otimes V}
\]

\[
= ((\id_V, \otimes \theta_V^{-1}) \sum_i S(R^{s(1)}_h^*)^*S(z(2))^*\xi^i \otimes R^{s(2)}v, z(1)f_i, \phi \otimes v)_{V^* \otimes V}
\]

\[
= \sum_i (\phi, gS(R^{s(1)}_h^*)^*S(z(2))^*\xi^i)(\nu^{-1}R^{s(2)}v, z(1)f_i, v)
\]

\[
= \sum_i (S(z(2))S(R^{s(1)}_h)g\phi, \xi^i)(\nu, z(1)^*R^{s(2)}v)
\]

\[
= (S(z(2))S(R^{s(1)}_h)\phi, \nu z(1)^*R^{s(2)}v) = (R^{s(2)}v, zS(R^{s(1)}_h)g\phi, \nu v)
\]

\[
= ((S^{-1}(R^{s(1)}_h))^*S(z)g\phi, \nu v) = (S^{-1}(u)^*S(z)g\phi, \nu v) = (\phi, S(z^*)v).
\]

On the other hand, using the definition of \( d_V \), we obtain:

\[
(d_V (z), \phi \otimes v)_{V^* \otimes V} = (z, d_V (\phi \otimes v))_{H_t} = (z, (1(1)v, \phi)(1(2))_{H_t}) = (z, (1(1)v, \phi)(\Delta(1)(1 \otimes z^*)) = (z, (\omega_{\nu, \phi}(S(z^*))) = (\phi, S(z^*)v).
\]

The condition \( \text{tr}_q(fT) = \text{Tr}(gff^*) \geq 0 \) for any morphism \( f \) follows from Remark 9.6(6) and from the positivity of \( g \).

The next proposition extends (\[RT\], 1.2).

**Theorem 9.8.** If \( H \) is a connected \( C^* \)-quantum groupoid, then \( U\text{Rep}(D(H)) \) is a unitary modular category.

**Proof.** The proof follows from Lemmas 7.7, 8.2 and Propositions 6.4, 9.3.

10. Appendix

Here we collected some results on ribbon and modular quantum groupoids which extend the corresponding facts for Hopf algebras. 1. There is a procedure analogous to (\[RT\], 3.4), that extends any quasitriangular quantum groupoid \((H, \mathcal{R}, \bar{\mathcal{R}})\) to a ribbon quantum groupoid in a canonical way. For this we need

**Lemma 10.1** (cf. (\[RT\], 3.3)). (i) A ribbon element \( \nu \) satisfies

\[
\varepsilon_1(\nu) = \varepsilon_2(\nu) = 1 \quad \text{and} \quad \nu^2 = (\nu u)^{-1},
\]

\[
\]
where \(u\) and \(v\) are the elements defined in Proposition 5.7.

(ii) If \(\nu_1\) and \(\nu_2\) are two ribbon elements of \((H, \mathcal{R})\), then \(\nu_2 = E\nu_1\), where \(E \in H\) is an invertible central element such that \(E = S(E) = E^{-1}\), \(\Delta(E) = \Delta(1)(E \otimes E)\) (i.e., \(E\) is group-like), and \(\varepsilon_s(E) = \varepsilon_s(E) = 1\).

Proof. (i) The definition of counit implies:

\[
\nu = (\text{id} \otimes \varepsilon)\Delta(\nu) = \nu \mathcal{R}^{(2)} \mathcal{R}^{(1)} \varepsilon(\mathcal{R}^{(1)} \mathcal{R}^{(2)} \nu) = \nu \mathcal{R}^{(2)} \mathcal{R}^{(1)} \varepsilon(\varepsilon_s(\mathcal{R}^{(1)}) \mathcal{R}^{(2)} \nu) = \nu \mathcal{R}^{(1)} \varepsilon(\varepsilon_s(\mathcal{R}^{(2)} \nu)) = \nu S(1_{(2)}) \varepsilon(1_{(1)} \nu) = \nu S(\varepsilon_1(\nu)),
\]

hence \(\varepsilon_1(\nu) = 1\). We used here the identity \(\varepsilon(hg) = \varepsilon(\varepsilon_s(h)g)\), \(h, g \in H\) and Lemma 5.6. Similarly, \(\varepsilon_s(\nu) = 1\). Using the antipode property, we compute

\[
1 = \varepsilon_1(\nu) = m(\text{id} \otimes S)\Delta(\nu) = \mathcal{R}^{(2)} \mathcal{R}^{(1)} S(\mathcal{R}^{(2)}) S(\mathcal{R}^{(1)}) \nu^2 = \nu S(\mathcal{R}^{(2)}) S(\mathcal{R}^{(1)}) \nu^2 = vv^2.
\]

(ii) Set \(E = \nu_1^{-1} \nu_2\). Then \(E\) is central and invertible, \(S(E) = E\), and from part (i) we conclude that \(E^2 = 1\). Next,

\[
\Delta(E) = \mathcal{R} \mathcal{R}_{21}(\nu_1^{-1} \otimes \nu_1^{-1}) \mathcal{R}_{21} \mathcal{R}(\nu_2 \otimes \nu_2) = \Delta(1)(E \otimes E).
\]

Applying the counit to both sides of the last equality, we get \(E = E\varepsilon_1(E) = E\varepsilon_s(E)\), i.e., \(\varepsilon_1(E) = \varepsilon_s(E) = 1\).

\(\square\)

**Proposition 10.2.** Let \(\hat{H} = H + Hv\) be a central extension of \(H\), consisting of formal linear combinations \(h + g\nu\) with \(h, g \in H\). Then \(\hat{H}, \mathcal{R}, \nu\) is a ribbon quantum groupoid with operations

\[
(h + g\nu)(h' + g'\nu) = (hh' + gg'(vu)^{-1}) + (hg' + gh')\nu,
\]

\[
\Delta(h + g\nu) = \Delta(h) + \Delta(g) \mathcal{R}_{21} \mathcal{R}(\nu \otimes \nu),
\]

\[
\varepsilon(h + g\nu) = \varepsilon(h) + \varepsilon(g),
\]

\[
S(h + g\nu) = S(h) + S(g)\nu.
\]

Note that \(\hat{H}\) contains \(H = \{h + 0\nu \mid g \in H\}\) as a quantum subgroupoid.

**Proof.** One verifies that \(\Delta\) is a homomorphism exactly as in [RT1]. The properties of \(\mathcal{R}\) and \(\nu\) follow directly from definitions. For the counit axiom we have, using the properties of comutal maps, Proposition 5.6 and Lemma 10.1:

\[
(\varepsilon \otimes \text{id})\Delta(h + g\nu) = h + \varepsilon(g_{(1)} \nu \varepsilon_1(\mathcal{R}^{(2)} \mathcal{R}^{(1)})) g_{(2)} \nu \mathcal{R}^{(1)} \mathcal{R}^{(2)} = h + g(2) S(\varepsilon_s(g_{(1)} \nu)) = h + g\nu,
\]

\[
(\text{id} \otimes \varepsilon)\Delta(h + g\nu) = h + \mathcal{R}^{(2)} \mathcal{R}^{(1)} g_{(1)} \nu \varepsilon_s(\mathcal{R}^{(1)} \mathcal{R}^{(2)} g_{(2)} \nu) = h + S(\varepsilon_1(g_{(2)} \nu)) g_{(1)} \nu = h + g\nu.
\]

Axioms (\(\mathbb{P}\)) and (\(\mathbb{B}\)) of Definition 2.1 can be verified by a direct computation.
Next, we observe that \( \varepsilon_t(h + gu) = \varepsilon_t(h) + \varepsilon_t(g) \) and \( \varepsilon_s(h + gu) = \varepsilon_s(h) + \varepsilon_s(g) \). The antipode axiom follows from the identity \( \nu^2v = 1 \) provided by Lemma 10.1.

\[
m(\text{id} \otimes S)\Delta(h + gu) = \varepsilon_t(h) + g(1)R^{(2)}R^{(1)}S(R^{(2)})S(R^{(1)})S(g(2))\nu^2 = \varepsilon_t(h + gu),
\]

\[
m(S \otimes \text{id})\Delta(h + gu) = \varepsilon_s(h) + S(g(1))S(R^{(1)})S(R^{(2)})R^{(1)}R^{(2)}g(2)\nu^2 = \varepsilon_s(h + gu).
\]

The anti-multiplicative properties of the antipode follow from the facts that \( S(uv) = uv \) and \( S(\nu) = \nu \).

2. Let us establish a relation between modular quantum groupoids and modular categories. A morphism \( f : V \to W \) in a ribbon Ab-category \( V \) is said to be negligible if for any morphism \( g : W \to V \) we have \( \text{tr}(fg) = 0 \). \( V \) is said to be pure if all negligible morphisms in this category are equal to zero. A purification procedure transforming any ribbon Ab-category into a pure ribbon Ab-category is described in ([12], XI.4.2); this procedure transforms hermitian ribbon Ab-categories into hermitian pure ribbon Ab-categories ([12], XI.4.3). We say that a family \( \{V_i\}, I \) of objects of \( V \) quasidominates an object \( V \) of \( V \) if there exists a finite set \( \{V_{i(r)}\}_r \) of objects of this family (possibly, with repetitions) and a family of morphisms \( f_r : V_{i(r)} \to V, g_r : V \to V_{i(r)} \) such that \( \text{id}_V - \sum_r f_r g_r \) is negligible. If \( V \) is pure, then quasidomination coincides with domination. Let \( (H, \mathcal{R}, \nu) \) be a ribbon quantum groupoid. Then an \( H \)-module \( V \) of finite \( k \)-rank is said to be negligible if \( \text{tr}_q(f) = 0 \) for any \( f \in \text{End}(V) \). If \( V \) is algebraically closed, then any irreducible \( H \)-module is a simple object of \( \text{Rep}(H) \).

**Definition 10.3.** A modular quantum groupoid consists of a ribbon quantum groupoid \( (H, \mathcal{R}, \nu) \) together with a finite family of simple \( H \)-modules of finite rank \( \{V_i\}, I \) such that:

(i) for some \( 0 \in I \), we have \( V_0 = H_1 \), the unit object of \( \text{Rep}(H) \);

(ii) for each \( i \in I \), there exists \( i^* \in I \) such that \( V_i \) is isomorphic to \( V_{i^*} \);

(iii) for any \( k, l \in I \), the tensor product \( V_k \otimes V_l \) splits as a finite direct sum of certain \( \{V_{i}\} \) (possibly with multiplicities) and a negligible \( H \)-module;

To formulate the last condition, let \( S_{i,j} = \text{tr}_q(c_{V_i, V_j} \circ c_{V_j, V_i}) \), \( i, j \in I \), where the braiding \( c_{V_i, V_j} \) was defined in 11.1 and the quantum trace \( \text{tr}_q \) in 11.4.

(iv) The square matrix \( [S_{i,j}]_{i,j \in I} \) is invertible in \( M_{|I|}(k) \).

For any modular quantum groupoid, we define a subcategory \( \mathcal{C} \) of \( \text{Rep}(H) \), as follows. The objects of \( \mathcal{C} \) are \( H \)-modules of finite rank quasidominated by \( \{V_i\}_{i \in I} \) and morphisms are \( H \)-morphisms of such modules; all the operations in \( \mathcal{C} \) are induced by the corresponding operations in \( \text{Rep}(H) \). Now taking into account the results of the previous sections and repeating the proof of ([12], XI.5.3.2), we have the first statement of the following

**Proposition 10.4.** If \( (H, \mathcal{R}, \nu, \{V_i\}, I) \) is modular, then the subcategory \( (\mathcal{C}, \{V_i\}, I) \) of \( \text{Rep}(H) \) is quasimodular in the sense of ([12], XI.4.3). Conversely, if \( (\mathcal{C}, \{V_i\}, I) \) is quasimodular, then \( (H, \mathcal{R}, \nu, \{V_i\}, I) \) is modular.

The proof of the second statement follows directly from the comparison of ([12], XI.4.3) and the above definition of a modular quantum groupoid. Purifying \( (\mathcal{C}, \{V_i\}, I) \) as in ([12], XI.4.2), we get a modular category ([12], II.1.4).
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INVARIENTS OF KNOTS AND 3-MANIFOLDS FROM QUANTUM GROUPOIDS

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