RELATIVE $K_0$ AND RELATIVE CYCLE CLASS MAP

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Abstract. We study relative $K_0$ of exact categories and triangulated categories. As an application, we construct a cycle class map from Chow groups with modulus to relative $K_0$.

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0. Introduction

0.1. Let $X$ be a separated regular scheme of finite type over a field. Then, to every integral closed subscheme $V$ of $X$, we can assign the cycle class $\text{cyc}(V)$ in the Grothendieck group $K_0(X)$ of algebraic vector bundles on $X$. Grothendieck has shown that the cycle classes induce surjective group homomorphisms from the Chow groups to subquotients of $K_0(X)$

\begin{equation}
\text{cyc} : \text{CH}_k(X) \to F_k K_0(X) / F_{k-1} K_0(X),
\end{equation}

for all $k \geq 0$, cf. [SGA6, Exp 0, App. Ch II]. Here, $F_s$ is the coniveau filtration; $F_k K_0(X)$ is generated by perfect complexes whose supports are of dimension $\leq k$.

The current paper constructs a relative version of the cycle class map (0.1). Let $D$ be an effective Cartier divisor on $X$. We are interested in the relative $K_0$-group $K_0(X, D)$, which is defined to be the $\pi_0$ of the homotopy fiber of the canonical map $K(X) \to K(D)$ between $K$-theory spectra. As a cycle theoretical invariant, we use the Chow group with modulus $\text{CH}_* (X | D)$ defined by Binda-Saito [BS14]: It is the group generated by cycles on $X$ which do not meet $D$ divided by “rational equivalence”. Here is the main theorem, which generalizes (0.1).

Theorem 0.1. Let $X$ be a separated regular scheme of finite type over a field and $D$ an affine effective Cartier divisor on $X$. Then there exist surjective group homomorphisms

\begin{equation}
\text{cyc} : \text{CH}_k(X | D) \to F_k K_0(X, D) / F_{k-1} K_0(X, D)
\end{equation}

for all $k \geq 0$, where $F_s$ is the “coniveau filtration” (Definition 3.3). Furthermore, if $D$ has an affine open neighborhood in $X$, then $F_{\dim X} K_0(X, D) = K_0(X, D)$.

In [BK17], Binda-Krishna constructed a cycle class map for zero cycles with modulus, namely a map from $\text{CH}_0(X | D)$ to $K_0(X | D)$, for modulus pairs $(X, D)$ with $X$ smooth quasi-projective over a perfect field. They have also shown that the cycle class map is injective if $X$ is affine and the base field is algebraically closed. Also, Binda [Bi17] constructed a cycle class map for higher zero cycles with modulus using a slightly different (stronger) modulus condition.
0.2. We explain what the cycle classes are. Let $X$ be a separated regular scheme and $V$ an integral closed subscheme of $X$. Then the coherent sheaf $\mathcal{O}_V$ is a perfect complex of $X$, i.e. quasi-isomorphic to a bounded complex $E_*$ of algebraic vector bundles on $X$, and the cycle class is given by $\text{cyc}(V) := \sum (-1)^i [E_i] \in K_0(X)$. Here, it is more natural to consider the group $K_0^{\text{perf}}(X)$ generated by perfect complexes of $X$ with the relation $[P] = [P'] + [P'']$ for each exact triangle $P' \to P \to P'' \to P'[1]$. It follows from [SGA6, Exp 1, 6.4] that the canonical map

$$(0.3) \quad K_0(X) \xrightarrow{\cong} K_0^{\text{perf}}(X)$$

is an isomorphism. Under this isomorphism, the cycle class $\text{cyc}(V)$ is just the class of the perfect complex $\mathcal{O}_V$ in $K_0^{\text{perf}}(X)$.

Now, suppose we are given an affine closed subscheme $D$ of $X$. Then the relative $K_0$-group $K_0(X, D)$ is generated by pairs $(E, E')$ of algebraic vector bundles on $X$ together with isomorphisms $E|_D \cong E'|_D$ along $D$ (Theorem 0.6). As in the absolute case, perfect complexes are more appropriate to construct cycle classes. We define $K_0^{\text{perf}}(X, D)$ to be the group generated by pairs $(P, P')$ of perfect complexes of $X$ together with isomorphisms $L^*_{\text{cyc}} P \cong L^*_{\text{cyc}} P'$ ($\iota : D \to X$) in the derived category of $D$ with suitable relations (Definition 0.3). Then we show that the canonical map

$$(0.4) \quad K_0(X, D) \cong K_0^{\text{perf}}(X, D)$$

is an isomorphism (Theorem 0.8). This is a generalization of (0.3).

Let $V$ be an integral closed subscheme of $X$ which does not meet $D$. Then $\mathcal{O}_V$ is a perfect complex of $X$ and $L^*_{\text{cyc}} \mathcal{O}_V \simeq 0$. Hence, the pair $(\mathcal{O}_V, 0)$ gives an element of $K_0^{\text{perf}}(X, D)$, which we denote by $\text{cyc}(V)$. When $X$ is of finite type over a field and $D$ is a Cartier divisor, we show that $\text{cyc}$ kills rational equivalence and get Theorem 0.1.

0.3. General results and notation. The hardest part of the above argument is the proof of the isomorphism (0.4). This isomorphism holds more generally for a certain type of exact functors between small exact categories. Actually, in large part of this paper, we discuss relative $K$-theory of exact categories and triangulated categories in general. In this subsection, we introduce some notation and explain the results.

**Definition 0.2.** Let $F : \mathcal{A} \to \mathcal{B}$ be a functor of categories. We define a category $\text{Rel}(F)$:

- Objects are triples $(P, \alpha, Q)$ with $P, Q \in \mathcal{A}$ and $\alpha : F(P) \cong F(Q)$ an isomorphism in $\mathcal{B}$.
- Morphisms from $(P, \alpha, Q)$ to $(P', \alpha', Q')$ are pairs $(f, g)$ of morphisms $f : P \to P'$ and $g : Q \to Q'$ in $\mathcal{A}$ which make the diagram

\[
\begin{array}{ccc}
F(P) & \xrightarrow{F(f)} & F(P') \\
\downarrow^{\alpha} & & \downarrow^{\alpha'} \\
F(Q) & \xrightarrow{F(g)} & F(Q')
\end{array}
\]

commutative.

When $F : \mathcal{A} \to \mathcal{B}$ is an exact functor between exact categories, we call a sequence

$$(P', \alpha', Q') \xrightarrow{(f, g)} (P, \alpha, Q) \xrightarrow{(f', g')} (P'', \alpha'', Q'')$$

in $\text{Rel}(F)$ exact if $P' \xrightarrow{f} P \xrightarrow{f'} P''$ and $Q' \xrightarrow{g} Q \xrightarrow{g'} Q''$ are exact sequences in $\mathcal{A}$. Under this definition, $\text{Rel}(F)$ is an exact category.

Suppose that $F : \mathcal{A} \to \mathcal{B}$ is a triangulated functor between triangulated categories. We denote by $[1]$ the shift functor of $\mathcal{A}$ or $\mathcal{B}$, and we define an endofunctor of $\text{Rel}(F)$ by $(P, \alpha, Q)[1] := (P[1], \alpha[1], Q[1])$. We call a sequence

$$(P_1, \alpha_1, Q_1) \xrightarrow{(f_1, g_1)} (P_2, \alpha_2, Q_2) \xrightarrow{(f_2, g_2)} (P_3, \alpha_3, Q_3) \xrightarrow{(f_3, g_3)} (P_1, \alpha_1, Q_1)[1]$$


in \( \text{Rel}(F) \) an exact triangle if

\[
\begin{array}{cccccc}
P_1 & \xrightarrow{f_1} & P_2 & \xrightarrow{f_2} & P_3 & \xrightarrow{f_3} & P_1[1] \\
Q_1 & \xrightarrow{g_1} & Q_2 & \xrightarrow{g_2} & Q_3 & \xrightarrow{g_3} & Q_1[1]
\end{array}
\]

are exact triangles in \( A \). Under this definition, \( \text{Rel}(F) \) is an additive category which satisfies TR1 and TR2, but may not satisfy TR3 nor TR4.

**Definition 0.3.** Let \( F : A \to B \) be an exact functor between small exact categories (resp. triangulated functor between small triangulated categories). We define \( K_0(F) \) to be the group with the generators \([X]\), one for each \( X \in \text{Rel}(F) \), and with the following relations:

(a) For each exact sequence \( X' \to X \to X'' \) (resp. exact triangle \( X' \to X \to X'' \to X'[1] \)) in \( \text{Rel}(F) \),

\[
[X] = [X'] + [X''].
\]

(b) For each pair \( (P, \alpha, Q) \), \( (Q, \beta, R) \) in \( \text{Rel}(F) \),

\[
[(P, \alpha, Q)] + [(Q, \beta, R)] = [(P, \beta \alpha, R)].
\]

**Definition 0.4.** An exact category is split exact if every exact sequence is split exact.

**Definition 0.5.** An additive functor \( F : A \to B \) between additive category is cofinal if for every \( B \in B \) there exists \( B' \in B \) and \( A \in A \) such that \( F(A) \cong B \oplus B' \).

For a small exact category \( A \), we denote by \( K(A) \) Quillen’s \( K \)-space, i.e. \( K(A) := \Omega BQA \). Here is a reinterpretation of Heller’s result in [He65].

**Theorem 0.6** (Heller). Let \( F : A \to B \) be an exact functor between small exact categories. Suppose that \( B \) is split exact and that \( F \) is cofinal. Then there exists a natural isomorphism of groups

\[
K_0(F) \cong \pi_0 \text{hofib}(K(A) \xrightarrow{F} K(B)).
\]

We discuss this result in §1

**Definition 0.7.** Let \( A \) be an additive category. We use the following notation:

(i) \( \text{Ch}^b(A) \) is the category of bounded chain complexes in \( A \).

(ii) \( K^b(A) \) is the bounded homotopy category, i.e. the same object with \( \text{Ch}^b(A) \) and morphisms up to homotopy. We regard \( K^b(A) \) as a triangulated category in the standard way (cf. [Ve77]).

Here is the general assertion of the isomorphism (0.4).

**Theorem 0.8.** Let \( A \) is a small exact category which is closed under the kernels of surjections. Let \( B \) be a small split exact category and \( F : A \to B \) an exact functor. We define \( K^{b,0}(A) \) to be the full subcategory of \( K^b(A) \) consisting of acyclic complexes. Then \( F \) induces a triangulated functor

\[
D(F) : K^b(A)/K^{b,0}(A) \to K^b(B)
\]

and the canonical map

\[
K_0(F) \xrightarrow{\sim} K_0(D(F))
\]

is an isomorphism.

To be precise, we have to choose an embedding of \( A \) into an abelian category. The terms “kernels of surjections” and “acyclic complexes” are understood in this abelian category. The result says that \( K_0(D(F)) \) does not depend on the choice of embedding. We give the proof in §2.
0.4. Question. Let $X$ be a smooth quasi-projective scheme over a field. Then Grothendieck’s Riemann-Roch type formula implies that the cycle class map (0.1) induces an isomorphism
\[(0.5) \quad \text{cyc: } CH_k(X) \otimes \mathbb{Q} \xrightarrow{\sim} F_kK_0(X) \otimes \mathbb{Q}/F_{k-1}K_0(X) \otimes \mathbb{Q}.
\]

Analogously, we may hope that the relative cycle map (0.2) is an isomorphism rationally. In [IK17], we constructed Chern classes
\[(0.6) \quad c_i: K_n(X, D) \to H^n_{\text{Nis}}(X, z^i(-| - \times X D, \bullet))
\]
for any modulus pair $(X, D)$ with $X \setminus D$ smooth. Here, $z^i(X|D, \bullet)$ is the cycle complex which defines the Chow group with modulus, i.e. $CH^i(X|D, n) = H_n(z^i(X|D, \bullet))$. One of the failure of the theory of cycles with modulus is that it does not satisfy Nisnevich descent nor Zariski descent. That is to say,
\[CH^i(X|D, n) \neq H^n_{\text{Nis}}(X, z^i(-| - \times X D, \bullet))\]
in general, though we do not know whether it fails in case $n = 0$ and $D$ is affine. To get an analogous result with (0.5), we may have to answer the following question.

*Does the cycle class map (0.2) factor through $H^0_{\text{Nis}}(X, z^i(-| - \times X D, \bullet))$? Or, does the Chern class map (0.6) in case $n = 0$ and $D$ is affine factor through $CH^i(X|D)$?*

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1. Relative $K_0$ of exact categories

1.1. Basic properties. In this section, we discuss relative $K_0$ of exact categories (Definition 0.3). Here, we collect some basic properties, whose proof is immediate from the definition.

**Lemma 1.1.** Let $F: \mathcal{A} \to \mathcal{B}$ be an exact functor between small exact categories. Then:

1. If $X, Y \in \text{Rel}(F)$ are isomorphic, then $[X] = [Y]$.
2. If $\gamma: P \xrightarrow{\sim} Q$ is an isomorphism in $\mathcal{A}$, then $[(P, F(\gamma), Q)] = 0$.
3. $[(P, \alpha, Q)] + [(Q, \alpha^{-1}, P)] = 0$.
4. Every element of $K_0(F)$ has the form $[X]$ for some $X \in \text{Rel}(F)$.

**Definition 1.2.** Let $F: \mathcal{A} \to \mathcal{B}$ be an exact functor between small exact categories. Let $(P, \alpha, Q), (P', \alpha', Q') \in \text{Rel}(F)$. We write

\[(P', \alpha', Q') \leftrightarrow (P, \alpha, Q)\]

if there exist $N \in \mathcal{A}$ and a commutator $\gamma$ in $\text{Aut}(F(Q))$ which fit into an exact sequence

\[(P', \alpha', Q') \xrightarrow{(\alpha, \gamma, Q)} (P, \alpha, Q) \xrightarrow{(N, 1, N)} \]

**Lemma 1.3.** Let $F: \mathcal{A} \to \mathcal{B}$ be an exact functor between small exact categories and $X, X' \in \text{Rel}(F)$. If $X' \leftrightarrow X$, then $[X'] = [X]$ in $K_0(F)$.

**Remark 1.4.** In fact, if $B$ is split exact, then the converse holds: All relations of $K_0(F)$ are generated by $\leftrightarrow$ (and $\oplus$).

1.2. Elementary transformations. Let $\mathcal{C}$ be an additive category. Let $P, Q \in \mathcal{C}$. Suppose that $P$ and $Q$ have the forms $P = P_1 \oplus P_2$ and $Q = Q_1 \oplus Q_2$. Then a homomorphism from $P$ to $Q$ can be expressed by a matrix

\[
\begin{pmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{pmatrix}
\]

where $a_{ij}$ is a morphism $P_j \to Q_i$ in $\mathcal{C}$. 
**Definition 1.5.** An endomorphism $\alpha$ of $P \in \mathcal{C}$ is an *elementary transformation* if there exists an embedding $\mathcal{C} \hookrightarrow \mathcal{C}'$ of additive categories and $\alpha$ is isomorphic to an endomorphism of $P_1 \oplus P_2$ of the form

$$
\begin{pmatrix}
1 & a \\
0 & 1
\end{pmatrix}
$$

for some $P_1, P_2 \in \mathcal{C}$ and $a : P_2 \to P_1$. We denote by $E(P)$ the subgroup of $\text{Aut}(P)$ generated by elementary transformations.

**Lemma 1.6.** Let $P \in \mathcal{C}$ and $\alpha \in E(P)$. Then

$$
\alpha \oplus 1 : P \oplus P \to P \oplus P
$$

is a commutator of $\text{Aut}(P \oplus P)$.

**Proof.** We may assume that $\alpha$ is an elementary transformation, i.e. $\exists \beta : P \xrightarrow{\cong} P_1 \oplus P_2$ and

$$
\beta \alpha \beta^{-1} = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}
$$

for some $a : P_2 \to P_1$. Then

$$
\begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
$$

is a commutator of $P_1 \oplus P_2 \oplus P_2$. Indeed, this is equal to

$$
\begin{pmatrix} 1 & 0 & a \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
$$

in $\text{Aut}(P_1 \oplus P_2 \oplus P_2)$. This implies that $\alpha \oplus 1 : P \oplus P \to P \oplus P$ is a commutator of $\text{Aut}(P \oplus P)$.

**Corollary 1.7.** Let $F : \mathcal{A} \to \mathcal{B}$ be an exact functor between small exact categories. Let $(P, \alpha, Q) \in \text{Rel}(F)$ and $\gamma \in E(F(Q))$. Then

$$
[(P, \alpha, Q)] = [(P, \gamma \alpha, Q)]
$$

in $K_0(F)$.

**Proof.** According to Lemma 1.6,

$$(P, \alpha, Q) \leftrightarrow (P \oplus Q, \alpha \oplus 1, Q \oplus Q) \leftrightarrow (P, \gamma \alpha, Q).$$

Hence, $[(P, \alpha, Q)] = [(P, \gamma \alpha, Q)]$ by Lemma 1.3.

### 1.3. Heller’s theorem

Let $\mathcal{A}$ be a small exact category. We consider Quillen’s $K$-space $K(\mathcal{A}) := \Omega BQ(A)$, and denote its $n$-th homotopy group by $K_n(\mathcal{A})$.

In [Ne98], Nenashev provides generators and relations for $K_1(\mathcal{A})$; the generators are double exact sequences. In case $\mathcal{A}$ is split exact (Definition 0.4), the generators and relations become simpler and the resulting group coincides with the one considered by Heller in [He65].

**Lemma 1.8.** Let $\mathcal{A}$ be a small exact category. We define $K_{1\text{He}}(\mathcal{A})$ to be the abelian group with the generators $[h]$, one for each automorphism $h \in \text{Aut}(P)$, and with the following relations:

(a) For a commutative diagram

$$
\begin{array}{c}
P' & \xrightarrow{f} & P & \xrightarrow{g} & P'' \\
\cong & & \cong & & \cong \\
P' & \xrightarrow{h'} & P & \xrightarrow{h} & P''
\end{array}
$$

with exact rows, $[h] = [h'] + [h'']$.

(b) $[h_2 \circ h_1] = [h_2] + [h_1]$ for $h_1, h_2 \in \text{Aut}(P)$.

If $\mathcal{A}$ is split exact, then there exists a natural isomorphism

$$
K_{1\text{He}}(\mathcal{A}) \xrightarrow{\cong} K_1(\mathcal{A}).
$$
Let $F: \mathcal{A} \to \mathcal{B}$ be an exact functor between small exact categories. We suppose that $F$ is cofinal (Definition 0.4). Then every element of $K^\text{He}_1(\mathcal{B})$ is represented by $g \in \text{Aut}(F(P))$ for some $P \in \mathcal{A}$. According to [He65, Proposition 4.2], the class $[(P, g, P)]$ in $K_0(F)$ does not depend on the representative and gives a group homomorphism

$$\delta: K^\text{He}_1(\mathcal{B}) \to K_0(F).$$

We define a map $\iota: K_0(F) \to K_0(\mathcal{A})$ by sending $[(P, \alpha, Q)]$ to $[P] - [Q]$.

**Proposition 1.9** (Heller). Let $F: \mathcal{A} \to \mathcal{B}$ be an exact functor between small exact categories. Suppose that $\mathcal{B}$ is split exact and that $T$ is cofinal. Then the sequence

$$K_1(\mathcal{A}) \xrightarrow{\delta} K_0(F) \xrightarrow{\iota} K_0(\mathcal{A})$$

is exact.

**Proof.** Heller showed the sequence $K^\text{He}_1(\mathcal{A}) \xrightarrow{\delta} K^\text{He}_1(\mathcal{B}) \xrightarrow{\iota} K_0(\mathcal{A})$ is exact [He65, Proposition 5.2]. Now, $K^\text{He}_1(\mathcal{B}) = K_1(\mathcal{B})$ and the map $K^\text{He}_1(\mathcal{A}) \to K^\text{He}_1(\mathcal{B})$ factors through $K_1(\mathcal{A})$. This proves the exactness at $K_1(\mathcal{B})$. \qed

Now we can prove Theorem 0.6. We restate the statement here.

**Theorem 1.10.** Let $F: \mathcal{A} \to \mathcal{B}$ be an exact functor between small exact categories. Suppose that $\mathcal{B}$ is split exact and that $F$ is cofinal. Then there exists a natural isomorphism of groups

$$K_0(F) \cong \pi_0 \text{hofib}(K(\mathcal{A}) \xrightarrow{F} K(\mathcal{B})).$$

**Proof.** In [GG87], Gillet and Grayson have constructed a simplicial set $G\mathcal{A}$ such that its geometric realization is naturally homotopy equivalent to $K(\mathcal{A})$. The 0-simplices of $G\mathcal{A}$ are pairs $(P, Q)$ of objects in $\mathcal{A}$. The 1-simplices of $G\mathcal{A}$ are pairs of exact sequences of the forms

$$P_0 \to P_1 \to P_0, \quad Q_0 \to Q_1 \to P_{01}.$$

The face maps $G\mathcal{A}_1 \to G\mathcal{A}_0$ send the above to $(P_0, Q_0)$ and $(P_1, Q_1)$ respectively.

Let $GF_0$ be the set of objects in $\text{Rel}(F)$. Let $GF_1$ be the set of all pairs $(l, \gamma)$ where $l$ is an exact sequence in $\text{Rel}(F)$ of the form

$$l: \quad (P, \alpha, Q) \xrightarrow{(R, \beta, S)} (N, 1, N)$$

and $\gamma$ is a commutator of $\text{Aut}(F(S))$. We define face maps $d_1, d_2: GF_1 \to GF_0$ by $d_1((l, \gamma)) := (P, \alpha, Q)$ and $d_2((l, \gamma)) := (R, \gamma \beta, S)$, and a degeneracy map $s: GF_0 \to GF_1$ by $s(X) := (X \to X \to 0, 1)$. We define $GF$ to be the simplicial set generated by $GF_1, GF_0$. Then

$$\pi_0|GF| = K_0(F).$$

We have a natural map $GF \to G\mathcal{A}$ which sends $(P, \alpha, Q) \in GF_0$ to $(P, Q)$ and $(l, \gamma) \in GF_1$ to the underlying pair of exact sequences of $l$. Then the composite $GF \to G\mathcal{A} \to GB$ is homotopic to zero. Therefore, we obtain a natural map

$$\theta: K_0(F) \to \pi_0 \text{hofib}(K(\mathcal{A}) \to K(\mathcal{B})).$$

According to Proposition 1.9, it remains to show that $\delta: K_1(\mathcal{B}) \to K_0(F)$ followed by $\theta$ is equal to the boundary map

$$\partial: K_1(\mathcal{B}) \to \pi_0 \text{hofib}(K(\mathcal{A}) \to K(\mathcal{B})),$$

and it is straightforward. \qed

**Example 1.11.** Examples of exact functors $F: \mathcal{A} \to \mathcal{B}$ such that $\mathcal{B}$ is a split exact category and that $T$ is cofinal.

1. A base change functor $P(\mathcal{A}) \to P(\mathcal{B})$ induced from a ring homomorphism $A \to B$. Here, $P(-)$ is the category of finitely generated projective modules.
(2) A base change functor Vec(X) → Vec(Y) induced from a morphism of schemes Y → X with Y affine. Here, Vec(−) is the category of algebraic vector bundles.

2. Relative $K_0$ of triangulated categories

2.1. Basic properties. In this section, we study relative $K_0$ of triangulated categories (Definition 0.3). The same properties as in Lemma 1.1 also hold for triangulated categories. Again, the proof is immediate.

**Lemma 2.1.** Let $F : A → B$ be a triangulated functor between small triangulated categories. Then:

(i) $[0] = 0$. $[X[n]] = (-1)^n[X]$. If $X, Y ∈ \text{Rel}(F)$ are isomorphic, then $[X] = [Y]$.

(ii) If $γ : P \rightarrow Q$ is an isomorphism in $A$, then $[(P, F(γ), Q)] = 0$.

(iii) $[(P, α, Q)] + [(Q, α^{-1}, P)] = 0$.

(iv) Every element of $K_0(F)$ has the form $[X]$ for some $X ∈ \text{Rel}(F)$.

**Lemma 2.2.** Let $F : A → B$ be a triangulated functor between small triangulated categories. Let $A_0$ be a thick triangulated subcategory of $A$ such that $F(A_0) = 0$. Then $F$ factors through the Verdier quotient $A/A_0$, and there is an exact sequence of abelian groups

$$0 \rightarrow K_0(A_0) \rightarrow K_0(A, F) \rightarrow K_0(A/A_0, \overline{F}) \rightarrow 0.$$  

**Proof.** Note that $\text{Ob}(\text{Rel}(F)) = \text{Ob}(\text{Rel}(\overline{F}))$. An exact triangle in $\text{Rel}(\overline{F})$ is a sequence

$$(P_1, α_1, Q_1) \rightarrow (P_2, α_2, Q_2) \rightarrow (P_3, α_3, Q_3) \rightarrow (P_1, α_1, Q_1)[1]$$

in $\text{Rel}(\overline{F})$ such that

$$P_1 \rightarrow P_2 \rightarrow P_3 \rightarrow P_1[1]$$

$$Q_1 \rightarrow Q_2 \rightarrow Q_3 \rightarrow Q_1[1]$$

are isomorphic in $A/A_0$ to exact triangles in $A$. It follows that $K_0(\overline{F})$ is the group with the generators $[X]$, one for each $X ∈ \text{Rel}(F)$, and with the relations (a) (b) for $K_0(F)$ and

(c) For $X = (P, 0, Q)$ with $P, Q ∈ A_0$, $[X] = 0$.

This says that $K_0(\overline{F})$ is the quotient of $K_0(F)$ by the image of $K_0(A_0)$.

2.2. Comparison theorem. Let $\mathcal{A}$ be a small abelian category and $A$ a full subcategory of $\mathcal{A}$ closed under extensions and kernels of surjections. We regard $A$ as an exact category. We define $K^b(\mathcal{A})$ to be the full subcategory of $K^b(A)$ (Definition 0.7) consisting of acyclic complexes in $\mathcal{A}$, i.e. complexes in $A$ whose homology taken in $\mathcal{A}$ vanish. Let $B$ be a small split exact category and $F : A → B$ an exact functor.

For every $P ∈ K^b(\mathcal{A})$, $F(P)$ is homotopic to zero. Hence, $F$ induces a triangulated functor

$$D(F) : K^b(A)/K^b(\mathcal{A}) → K^b(B).$$

The rest of this section is devoted to the proof of Theorem 0.8. That is:

**Theorem 2.3.** The canonical map

$$K_0(F) \rightarrow K_0(D(F))$$

is an isomorphism.

The problem is how to define the inverse. In absolute case, i.e. $B = 0$, the inverse is clear, which is given by $E_∗ → \sum (-1)^i[E_i]$. We cannot imitate this directly because we keep track of homotopy equivalences on $B$. However, still a variant does work. We call it Euler characteristic and study in §2.3 and §2.4. Using this machinery, the proof of the theorem is completed in §2.5.

We fix $A, B$ and $F$ until the end of this section.
2.3. **Euler characteristic I.** Let $C$ be a bounded complex in $B$ which is homotopic to zero. Then there exists $s: C_n \to C_{n+1}$ such that $sd + ds = 1$ and $ss = 0$, which we call a *strict split* of $C$. Then the direct sum of the maps

$$
\begin{array}{c}
C_n \\
\downarrow d \quad s \\
C_{n-1} \\
\end{array}
$$

give an isomorphism

$$
\Phi_{C,s}: \bigoplus_n C_{2n+1} \xrightarrow{\sim} \bigoplus_n C_{2n}.
$$

**Lemma 2.4.** If $s'$ is another strict split of $C$, then there exists an elementary transformation (Definition 1.5) $\gamma$ of $C_{2s}$ such that $\Phi_{C,s} = \gamma \Phi_{C,s'}$.

**Proof.** Take an embedding of $B$ into an abelian category $\mathcal{B}$. Let $Z_n \in \mathcal{B}$ be the kernel of $d: C_n \to C_{n-1}$. Then $C$ decomposes into short exact sequences

$$0 \to Z_n \xrightarrow{\epsilon} C_n \xrightarrow{\delta} Z_{n-1} \to 0,$$

and $s\epsilon: Z_{n-1} \to C_n$ or $s\delta: C_n \to Z_n$ give splits of these short exact sequences. Hence, the map

$$\phi_{s,n} := (s\delta, \delta): C_n \xrightarrow{\sim} Z_n \oplus Z_{n-1},$$

is an isomorphism with the inverse $\phi_{s,n}^{-1} = (\epsilon, s\epsilon)$.

Now, it is clear that there is an elementary transformation $\gamma_n$ of $Z_n \oplus Z_{n-1}$ such that $\phi_{s,n} = \gamma_n \phi_{s',n}$. On the other hand, $\phi_{s,n}$'s give isomorphisms

$$C_{2s+1} \xrightarrow{\phi_{s,2s+1}} Z_n \xrightarrow{\phi^{-1}_{s,2s}} C_{2s},$$

and the composite is equal to $\Phi_{s,n}$ because $s\delta = d$ and $s\epsilon \delta = ss = s$. Therefore, there exists an elementary transformation $\gamma$ such that $\Phi_{C,s} = \gamma \Phi_{C,s'}$. \(\square\)

Let $P, Q$ be bounded complexes in $\mathcal{A}$ and $\alpha$ a homotopy equivalence $F(P) \xrightarrow{\sim} F(Q)$. We apply the above construction to cone $\alpha$. Set $\Phi := \Phi_{\text{cone}, \alpha, s}$ for some strict split $s$ of cone $\alpha$. Thanks to Lemma 2.4 and Corollary 1.7, the following is well-defined.

**Definition 2.5.** We define the *Euler characteristic* of $(P, \alpha, Q)$ by

$$
\chi(P, \alpha, Q) := \left[ \left( \bigoplus_n (P_{2n} \oplus Q_{2n+1}), \Phi, \bigoplus_n (P_{2n-1} \oplus Q_{2n}) \right) \right] \in K_0(F).
$$

Here are first properties of Euler characteristic.

**Lemma 2.6.** Let $P, Q$ be bounded complexes in $\mathcal{A}$ with a homotopy equivalence $\alpha: F(P) \xrightarrow{\sim} F(Q)$.

(i) Let $P', Q' \in \text{Ch}^i(\mathcal{A})$ with isomorphisms of complexes $\gamma: F(P) \xrightarrow{\sim} F(P')$ and $\delta: F(Q) \xrightarrow{\sim} F(Q')$. If $[\{P_{n}, \gamma_n, P'_n\}] = [\{Q_{n}, \delta_n, Q'_n\}] = 0$ for all $n$, then

$$
\chi(P', \delta \alpha \gamma^{-1}, Q') = \chi(P, \alpha, Q).
$$

(ii) $\chi(P[n], \alpha[n], Q[n]) = (-1)^n \chi(P, \alpha, Q)$.

**Proof.** (i) $\gamma := (\gamma, \delta)$ gives an isomorphism of complexes cone $\alpha \xrightarrow{\sim} \text{cone}(\delta \alpha \gamma^{-1})$. Hence, $\Phi_{\text{cone}(\delta \alpha \gamma^{-1})}$ is equal to the composite

$$
\begin{array}{c}
F(P'_{2s}) \oplus F(Q'_{2s+1}) \\
\xrightarrow{(\gamma_{2s-1}^{-1}, \delta_{2s+1}^{-1})} \\
F(P_{2s}) \oplus F(Q_{2s+1}) \\
\xrightarrow{\Phi_{\text{cone}, \alpha, s}} \\
F(P_{2s-1}) \oplus F(Q_{2s}) \\
\xrightarrow{(\gamma_{2s-1}^{-1}, \delta_{2s+1}^{-1})} \\
F(P'_{2s-1}) \oplus F(Q'_{2s}).
\end{array}
$$
It follows from the assumption that
\[(P_{2s}^I \oplus Q_{2s+1}^I, (\gamma_{2s}^{-1}, \delta_{2s+1}^{-1}), \gamma_{2s+1}, P_{2s}^I \oplus Q_{2s+1}^I, (\gamma_{2s+1}^{-1}, \delta_{2s+1})) = 0,\]
and thus \(\chi(P', \delta \alpha \gamma^{-1}, Q') = \chi(P, \alpha, Q).\)

(ii) By the construction,
\[
\chi(P[1], \alpha[1], Q[1]) = [(P_{2s}^I \oplus Q_{2s+1}^I, -\Phi_{\text{cone}(-\alpha)}, P_{2s+1} \oplus Q_{2s})].
\]
By Lemma 2.1 (iii), the right hand side equal to
\[-[(P_{2s-1} \oplus Q_{2s}, \Phi_{\text{cone}(-\alpha)}, P_{2s} \oplus Q_{2s+1})] = -\chi(P, -\alpha, Q),\]
which equals to \(-\chi(P, \alpha, Q)\) by (i).

We will use the following lemma on homological algebra.

**Lemma 2.7.** Let \(C\) be an additive category. Suppose given a commutative diagram
\[
\begin{array}{ccc}
A' & \xrightarrow{f_1} & A \\
\uparrow{d'} & & \uparrow{d} \\
B' & \xrightarrow{f_2} & B \\
\downarrow{g_1} & & \downarrow{g_2} \\
A'' & \xrightarrow{\alpha} & A'' \\
\end{array}
\]
in \(C\) such that the rows are split exact sequences and that \(d', d''\) are split epimorphisms. Let \(s', s''\) be splits of \(d', d''\). Then there exists \(\tilde{s} : B \to A\) such that \(\tilde{s} f_2 = f_1 s', g_1 \tilde{s} = s'' g_2\) and \(\gamma := d\tilde{s}\) is an elementary transformation of \(B\). In particular, \(s := \tilde{s} \gamma^{-1}\) is a split of \(d\).

**Proof.** Let \(a\) and \(b\) are splits of \(g_1\) and \(f_2\) respectively;
\[
\begin{array}{ccc}
A' & \xrightarrow{f_1} & A \\
\uparrow{s'} & & \uparrow{d} \\
B' & \xrightarrow{b} & B \\
\downarrow{g_1} & & \downarrow{g_2} \\
A'' & \xrightarrow{\alpha} & A'' \\
\end{array}
\]
We define
\[
\tilde{s} := f_1 s' b + a s'' g_2 : B \to A.
\]
Then \(\tilde{s} f_2 = f_1 s'\) and \(g_1 \tilde{s} = s'' g_2\). Set \(\gamma := d\tilde{s}\). Then \(\gamma f_2 = f_2\) and \(g_2 \gamma = g_2\), which implies that \(\gamma\) is an elementary transformation of \(B\). \(\square\)

2.4. Euler characteristic II. In this subsection, we prove that the Euler characteristic gives a group homomorphism \(\chi : K_0(\text{D}(F)) \to K_0(F)\).

**Lemma 2.8.** Suppose we are given exact sequences
\[
P' \xrightarrow{P} \xrightarrow{P'} P'' , \quad Q' \xrightarrow{Q} \xrightarrow{Q'} Q''
\]
in \(\text{Ch}^b(A)\) and homotopy equivalences \(\alpha, \alpha', \alpha''\) fitting into a commutative diagram
\[
\begin{array}{ccc}
F(P') & \xrightarrow{\alpha'} & F(P) \\
\sim & & \sim \\
F(Q') & \xrightarrow{\alpha''} & F(Q'')
\end{array}
\]
Then
\[
\chi(P, \alpha, Q) = \chi(P', \alpha', Q') + \chi(P'', \alpha'', Q'').
\]
Proof. We have a commutative diagram

\[
\begin{array}{c}
\begin{array}{ccc}
Z'_n & \to & Z_n & \to & Z''_n \\
\uparrow{\epsilon'} & & \uparrow{\epsilon} & & \uparrow{\epsilon''} \\
F(P'_{n-1}) \oplus F(Q'_n) & \to & F(P_{n-1}) \oplus F(Q_n) & \to & F(P''_{n-1}) \oplus F(Q''_n) \\
\downarrow{s'} & & \downarrow{s} & & \downarrow{s''} \\
Z'_{n-1} & \to & Z_{n-1} & \to & Z''_{n-1}
\end{array}
\end{array}
\]

with exact rows and columns, where \(Z', Z'_n, Z''_n\) are the kernels of the differentials of cone \(\alpha, \alpha', \alpha''\). We take splits \(s'\) and \(s''\) of \(\delta'\) and \(\delta''\). Let \(\tilde{s}\) be a map \(Z_{n-1} \to F(P_{n-1}) \oplus F(Q_n)\) as in Lemma 2.7, and set \(\gamma := \delta \tilde{s}, s := \tilde{s} \gamma\). Then \(\gamma\) is an elementary transformation and \(s\) is a split of \(\delta\).

Now, we have isomorphisms \(\tilde{\phi}_n, \phi'_n, \phi''_n\) fitting into a commutative diagram

\[
\begin{array}{c}
\begin{array}{ccc}
Z'_n \oplus Z'_{n-1} & \to & Z_n \oplus Z_{n-1} & \to & Z''_n \oplus Z''_{n-1} \\
\uparrow{\phi'_n=\langle \epsilon', s' \rangle} & & \uparrow{\phi_n=\langle \epsilon, \tilde{s} \rangle} & & \uparrow{\phi''_n=\langle \epsilon'', s'' \rangle} \\
F(P'_{n-1}) \oplus F(Q'_n) & \to & F(P_{n-1}) \oplus F(Q_n) & \to & F(P''_{n-1}) \oplus F(Q''_n).
\end{array}
\end{array}
\]

Set

\[
\Phi' := (\phi'_2)^{-1} \phi'_{2+1}, \quad \Phi'' := (\phi''_2)^{-1} \phi''_{2+1}, \quad \tilde{\Phi} := \tilde{\phi}_2^{-1} \tilde{\phi}_{2+1}.
\]

Then we obtain a sequence in Rel(\(F\))

\[
(P_2' \oplus Q_{2+1}', \Phi', P_{2+1}' \oplus Q_{2+1}'; \tilde{\Phi}, P_{2+1}' \oplus Q_{2+1}; \Phi'', P_{2+1}' \oplus Q_{2+1}'),
\]

and it is an exact sequence since the given exact sequences are degreewise exact.

On the other hand, \(\Phi\) is equal to \(\Phi := \phi_2^{-1} \phi_{2+1}, \phi_n := (\epsilon, s)\), modulo elementary transformations. Therefore,

\[
\chi(P, \alpha, Q) = [(P_2 \oplus Q_{2+1}, \Phi, P_{2+1} \oplus Q_{2+1})] = [(P_{2+1} \oplus Q_{2+1}, \tilde{\Phi}, P_{2+1} \oplus Q_{2+1})] = [(P_2' \oplus Q_{2+1}, \Phi', P_{2+1} \oplus Q_{2+1})] + [(P_{2+1}' \oplus Q_{2+1}, \Phi'', P_{2+1} \oplus Q_{2+1})] = \chi(P', \alpha', Q') + \chi(P'', \alpha'', Q'').
\]

\[\square\]

Lemma 2.9. Let \(P, Q\) be bounded complexes in \(A\) with a homotopy equivalence \(\alpha : F(P) \xrightarrow{\sim} F(Q)\) such that \(H_* X, H_* Y \in A\). Then we have

\[
\chi(P, \alpha, Q) = \sum_i (-1)^i [(H_i P, H_i \alpha, H_i Q)].
\]

Proof. Since \(\chi(P[1], \alpha[1], Q[1]) = -\chi(P, \alpha, Q)\) (Lemma 2.6 (ii)), we may assume that \(P_i = Q_i = 0\) for \(i < 0\). It is easy to see from our assumptions that the kernels and the images of \(d_P : P_n \to P_{n-1}\) and \(d_Q : Q_n \to Q_{n-1}\) are in \(A\). Now, we have an exact sequence

\[
(P, \alpha, Q) \xrightarrow{(\tau_{\geq n+1} P, \tau_{\geq n+1} \alpha, \tau_{\geq n+1} Q)} (P, \alpha, Q) \xrightarrow{(\tau_{\leq n} P, \tau_{\leq n} \alpha, \tau_{\leq n} Q)}.
\]

By Lemma 2.8 and by induction, we may assume that \(P_i = Q_i = 0\) for \(i \geq 2\) and that \(d_P : P_1 \to P_0\) and \(d_Q : Q_1 \to Q_0\) are admissible monomorphisms. Now, the cone of \(\alpha\) has the form

\[
F(P_1) \to F(P_0) \oplus F(Q_1) \to F(Q_0),
\]
which fits into a commutative diagram

\[
\begin{array}{ccc}
F(P) & \xrightarrow{c} & F(P_1) \\
\downarrow{e} & & \downarrow{d} \\
F(Q_1) & \xrightarrow{s} & F(P_0) \\
\downarrow{\bar{s}} & & \downarrow{\bar{a}} \\
F(Q_1) & \xrightarrow{F} & F(Q_0) \\
\end{array}
\]

with exact rows and columns. We take a split \(s''\) of \(\bar{s}\), and take \(\bar{s} : F(Q_0) \to F(P_0) \oplus F(Q_1)\) as in Lemma 2.7. Then we have a commutative diagram

\[
\begin{array}{ccc}
F(Q_1) & \xrightarrow{\simeq} & F(P_0) \oplus F(Q_1) \\
\downarrow{\bar{s}} & & \downarrow{\simeq} \\
F(Q_1) & \xrightarrow{\simeq} & F(P_0) \\
\end{array}
\]

Since the rows lift canonically to exact sequences in \(\mathcal{A}\), we have

\[
\chi(P, \alpha, Q) = [(P_0 \oplus Q_1, \hat{\phi}, P_1 \oplus Q_0)] = [(P_0, \psi, H_0(Q) \oplus P_1)].
\]

Finally, it follows from the exact sequence

\[
(P_1, 1, P_1) \xrightarrow{(P_0, \psi, H_0(Q) \oplus P_1)} (H_0(P), H_0(\alpha), H_0(Q))
\]

that

\[
[(P_0, \psi, H_0(Q) \oplus P_1)] = [(H_0(P), H_0(\alpha), H_0(Q))].
\]

\[\square\]

**Corollary 2.10.**

(i) Let \(f : P \to P'\) and \(g : Q \to Q'\) be quasi-isomorphisms of bounded complexes in \(\mathcal{A}\) with homotopy equivalences \(\alpha : F(P) \xrightarrow{\sim} F(Q)\) and \(\alpha' : F(P') \xrightarrow{\sim} F(Q')\) such that \(\alpha' F(f) = F(g) \alpha\). Then

\[
\chi(P, \alpha, Q) = \chi(P', \alpha', Q').
\]

(ii) Let \(P, Q\) be bounded complexes in \(\mathcal{A}\) with a homotopy equivalence \(\alpha : F(P) \xrightarrow{\sim} F(Q)\). Suppose that \(\alpha\) is homotopic to another homotopy equivalence \(\beta : F(P) \xrightarrow{\sim} F(Q)\). Then

\[
\chi(P, \alpha, Q) = \chi(P, \beta, Q).
\]

**Proof.** By considering the mapping cylinder, (ii) reduces to (i): Let \(C(F(P))\) be the mapping cylinder of the identity map of \(F(P)\). Then \(\alpha\) and \(\beta\) extend to a homotopy equivalence \(C(F(P)) \to F(Q)\). Since the canonical map \(F(P) \to C(F(P))\) lifts to a quasi-isomorphism in \(\mathcal{A}\), (ii) follows from (i).

(i) We have an exact sequence

\[
(P', \alpha', Q') \xrightarrow{\text{cone } f, \gamma, \text{cone } g} (P[1], \alpha[1], Q[1]).
\]

According to Lemma 2.8, we have

\[
\chi(\text{cone } f, \gamma, \text{cone } g) = \chi(P', \alpha', Q') + \chi(P[1], \alpha[1], Q[1])
\]

\[
= \chi(P', \alpha', Q') - \chi(P, \alpha, Q).
\]

Since \(H_* \text{ cone } f = H_* \text{ cone } g = 0\), it follows from Lemma 2.9 that

\[
\chi(\text{cone } f, \gamma, \text{cone } g) = \sum (-1)^i [(H_i \text{ cone } f, H_i \gamma, H_i \text{ cone } g)] = 0.
\]

\[\square\]
Lemma 2.11. Let \( P, Q, R \) be bounded complexes in \( \mathcal{A} \) with homotopy equivalences \( \alpha : F(P) \simto F(Q) \) and \( \beta : F(Q) \simto F(R) \). Then

\[
\chi(P, \beta \alpha, R) = \chi(P, \alpha, Q) + \chi(Q, \beta, R).
\]

Proof. From the exact sequence

\[
(P, \beta \alpha, R) \longrightarrow (P \oplus Q, (0 \ 1 \ 0), Q \oplus R) \longrightarrow (Q, 1, Q),
\]

it follows that

\[
\chi(P, \beta \alpha, R) = \chi(P \oplus Q, (0 \ 1 \ 0), Q \oplus R).
\]

Let \( \beta^{-1} \) be a homotopy inverse of \( \beta \). Then we have homotopy equivalences

\[
\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \sim \begin{pmatrix} 0 & \beta^{-1} \\ \beta \alpha & 0 \end{pmatrix} \sim \gamma \begin{pmatrix} 0 & 1 \\ \beta \alpha & 0 \end{pmatrix},
\]

where \( \gamma \) is a product of elementary transformations. Hence, by Corollary 2.10 (ii) and Lemma 2.6 (i), we have

\[
\chi(P \oplus Q, (0 \ 1 \ 0), Q \oplus R) = \chi(P \oplus Q, \alpha \oplus \beta, Q \oplus R) = \chi(P, \alpha, Q) + \chi(Q, \beta, R).
\]

\( \square \)

Proposition 2.12. The Euler characteristic (Definition 2.5) gives a group homomorphism \( \chi : K_0(\text{D}(F)) \rightarrow K_0(\text{F}) \).

Proof. Let \( X = (P, \bar{\alpha}, Q) \) be an object of \( \text{Rel}(K^b(\mathcal{A})/K^b(\mathcal{A}) \xrightarrow{D(F)} K^b(\mathcal{B})) \); \( P, Q \) are bounded complexes in \( \mathcal{A} \) and \( \bar{\alpha} \) is the homotopy equivalent class of a homotopy equivalence \( \alpha : F(P) \simto F(Q) \). Hence, by Corollary 2.10 (ii), the Euler characteristic of \( X \)

\[
\chi(X) := \chi(P, \alpha, Q)
\]

is well-defined. It remains to show that \( \chi \) kills the relations (a) (b) of \( K_0(\text{D}(F)) \).

(a) Let \( (P, \alpha, Q), (Q, \beta, R) \in \text{Rel}(\text{D}(F)) \). Then \( \bar{\alpha} \beta \) is a homotopy equivalent class of \( \beta \alpha \). Hence, by Lemma 2.11, we have

\[
\chi(P, \bar{\alpha} \beta, R) = \chi(P, \bar{\alpha}, Q) + \chi(Q, \beta, R).
\]

(b) Let \( (P, \bar{\alpha}, Q) \in \text{Rel}(\text{D}(F)) \). Then \( \beta \bar{\alpha} \) is a homotopy equivalent class of \( \beta \alpha \). Hence, by Lemma 2.11, we have

\[
\chi(P, \beta \bar{\alpha}, R) = \chi(P, \bar{\alpha}, Q) + \chi(Q, \beta, R).
\]

(a) Let

\[
(P_1, \alpha_1, Q_1) \xrightarrow{(f, g)} (P_2, \alpha_2, Q_2) \xrightarrow{} (P_3, \alpha_3, Q_3) \xrightarrow{} (P_1, \alpha_1, Q_1)[1]
\]

be an exact triangle in \( \text{Rel}(\text{D}(F)) \). According to Corollary 2.10 (i), we may assume that \( f, g \) are maps of complexes.

Now, there are isomorphism \( \beta : P_3 \xrightarrow{\cong} \text{cone} f \) and \( \gamma : Q_3 \xrightarrow{\cong} \text{cone} g \) in \( K^b(\mathcal{A})/K^b(\mathcal{A}) \) which make the diagrams

\[
\begin{array}{ccccccc}
P_2 & \longrightarrow & P_3 & \longrightarrow & P_1[1] & & \\
\downarrow \beta & & \downarrow & & \downarrow & & \\
\text{cone} f & \longrightarrow & P_1[1], & & & & \\
\end{array}
\quad
\begin{array}{ccccccc}
Q_2 & \longrightarrow & Q_3 & \longrightarrow & Q_1[1] & & \\
\downarrow \beta & & \downarrow & & \downarrow & & \\
\text{cone} g & \longrightarrow & Q_1[1], & & & & \\
\end{array}
\]

\( \square \)
commutative. We have a homotopy equivalence \( \alpha'_3 : F(\text{cone } f) \xrightarrow{\sim} F(\text{cone } g) \) fitting into the commutative diagram

\[
\begin{array}{cccc}
F(Q_1) & \xrightarrow{\alpha_1} & F(Q_2) & \xrightarrow{\alpha_2} & F(Q_3) & \xrightarrow{\alpha_3} & F(Q_4) \\
\| & \| & \| & (\gamma) & \| & \| & \| \\
F(P_1) & \xrightarrow{\alpha'_1} & F(P_2) & \xrightarrow{\alpha'_2} & F(P_3) & \xrightarrow{\alpha'_3} & F(P_4) \\
\| & \| & \| & \| & \| & \| & \| \\
F(P_1) & \xrightarrow{\alpha_1[1]} & F(P_2) & \xrightarrow{\alpha_2[1]} & F(P_3) & \xrightarrow{\alpha_3[1]} & F(P_4) \\
\end{array}
\]

By Corollary 2.10 (i), we have

\[ \chi(P_3, \alpha_3, Q_3) = \chi(\text{cone } f, \alpha'_3, \text{cone } g). \]

By Lemma 2.8, we conclude that

\[
\begin{align*}
\chi(\text{cone } f, \alpha'_3, \text{cone } g) &= \chi(P_2, \alpha_2, Q_2) + \chi(P_1[1], \alpha_1[1], Q_1[1]) \\
&= \chi(P_2, \alpha_2, Q_2) - \chi(P_1, \alpha_1, Q_1).
\end{align*}
\]

2.5. Proof of Theorem 0.8. We prove that the Euler characteristic \( \chi : K_0(\mathcal{D}(F)) \to K_0(F) \) is the inverse of the canonical map \( \iota : K_0(F) \to K_0(\mathcal{D}(F)) \). It is clear that, for \( X \in \text{Rel}(F) \), \( \chi(\iota[X]) = [X] \).

Hence, it remains to prove the following.

**Lemma 2.13.** For \( (P, \alpha, Q) \in \text{Rel}(\mathcal{D}(F)) \),

\[ [(P, \alpha, Q)] = \iota \chi(P, \alpha, Q) \]

in \( K_0(\mathcal{D}(F)) \).

**Proof.** We may assume that \( P_i = Q_i = 0 \) for \( i < 0 \). We prove the lemma by induction on \( N := \min\{n \mid P_i = Q_i = 0 \quad \forall i > n \geq 0 \} \). The case \( N = 0 \) is clear.

We use the following notation: Set

\[ \Omega_1 := \bigoplus_i (P_{2i} \oplus Q_{2i+1}) \quad \text{and} \quad \Omega_2 := \bigoplus_i (Q_{2i} \oplus P_{2i+1}), \]

so that \( \chi(P, \alpha, Q) = [(\Omega_1, \Phi, \Omega_2)] \). We sometimes talk about “elements” of objects in \( \mathcal{A} \) to construct maps, but we can easily avoid this. We shall denote by \( x_i \) (resp. \( y_i \)) elements of \( P_i \) (resp. \( Q_i \)).

First of all, we construct \( (P', \alpha', Q') \in \text{Rel}(\mathcal{D}(F)) \) with morphisms

\[ (\Omega_1, \Phi, \Omega_2) \xrightarrow{\theta} (P', \alpha', Q') \xrightarrow{\sim} (P, \alpha, Q). \]

Here, \( Q' := Q \) and

\[ P' := P \oplus [\cdots \xrightarrow{1} Q_1 \xrightarrow{0} Q_1 \xrightarrow{1} Q_1 \cdots]. \]

The quasi-isomorphism \( \alpha' : F(P') \to F(Q) \) is given by

\[
\begin{array}{cccc}
\cdots & F(P_2) & \xrightarrow{\alpha_2} & F(P_1) \oplus F(Q_1) \xrightarrow{\alpha_1} F(P_0) \oplus F(Q_1) \xrightarrow{\alpha_0} \cdots \\
\| & \| & \| & \| & \| \\
\cdots & F(Q_2) & \xrightarrow{\alpha_0} & F(Q_1) \xrightarrow{\alpha_1 \oplus 1} F(Q_0).
\end{array}
\]
The canonical inclusion $P \rightarrow P'$ is a quasi-isomorphism, and it yields an isomorphism $(P, \alpha, Q) \xrightarrow{\sim} (P', \alpha', Q)$ in $\text{Rel}(D(F))$. The map $\theta$ is given by

\[
\begin{align*}
\Omega_1 &= P_0 \oplus Q_1 \oplus P_2 + \cdots \rightarrow P'_0 = P_0 \oplus Q_1 \quad (x_0, y_1, x_2, \ldots) \mapsto (-x_0, y_1) \\
\Omega_2 &= Q_0 \oplus P_1 \oplus Q_2 + \cdots \rightarrow Q_0 \quad (y_0, x_1, y_2, \ldots) \mapsto y_0.
\end{align*}
\]

We show that $[\text{cone } \theta] = 0$ in $K_0(D(F))$, which proves the lemma. First few degrees of cone $\theta$ look like

\[
\begin{array}{cc}
\vdots & \vdots \\
P_2 & Q_2 \\
(P_0 \oplus Q_1 \oplus P_2 + \cdots) \oplus P_1 + Q_1 & (Q_0 \oplus P_1 \oplus Q_2 + \cdots) \oplus Q_1 \\
\downarrow & \downarrow \\
P_0 \oplus Q_1 & Q_0.
\end{array}
\]

It follows that the class of cone $\theta$ in $K_0(D(F))$ is equal to $-[(R, \beta, S)]$ with

\[
\begin{align*}
R &= [\cdots \rightarrow P_3 \xrightarrow{d_3} P_2 \xrightarrow{0: \Phi P} (Q_1 \oplus P_2 + Q_3 + \cdots) \oplus P_1] \\
S &= [\cdots \rightarrow Q_3 \xrightarrow{Q_2 \Phi} Q_2 \xrightarrow{0: \Phi S} (P_1 \oplus Q_2 + P_3 + \cdots) \oplus Q_1].
\end{align*}
\]

$\beta_i = \alpha_{i+1}$ for $i \geq 1$ and $\beta_1$ is given by

\[
\beta_1((y_1, x_2, y_3, \ldots), x_1) = (\text{pr}_2\Phi(-dx_1, y_1, x_2, y_3, \ldots), \alpha x_1 + y_1).
\]

The induction hypothesis implies that $[(R, \beta, S)] = i\chi(R, \beta, S)$. We show that $\chi(R, \beta, S) = 0$.

We write

\[
\begin{align*}
\Omega'_1 := \bigoplus_{i \geq 1} (Q_{2i-1} \oplus P_{2i}) \\
\Omega'_2 := \bigoplus_{i \geq 1} (P_{2i-1} \oplus Q_{2i}),
\end{align*}
\]

and denote the projections $\Omega_i \rightarrow \Omega'_i$ by $\text{pr}_i$. Then we have

\[
\chi(R, \beta, S) = [(\Omega_1' \oplus \Omega_2', \Phi', \Omega_2' \oplus \Omega_1')],
\]

where $\Phi'$ is given by

\[
(\{(y_1, x_2, y_3, \ldots), (x_1, y_2, x_3, \ldots)\}) \mapsto (-\text{pr}_2\Phi(-dx_1, y_1, x_2, y_3, \ldots), \omega - y_1 + \text{pr}_1\Phi^{-1}(y_0, x_1, y_2, x_3, \ldots)).
\]

Observe that we have $\chi(R, \beta, S) \Rightarrow (\Omega_1 \oplus \Omega_2, \Psi, \Omega_2 \oplus \Omega_1)$ (cf. Definition 1.2) with

\[
\Psi: ((x_0, y_1, x_2, \ldots), (y_0, x_1, y_2, \ldots)) \\
\mapsto ((y_0, -\text{pr}_2\Phi(-dx_1 + x_0, y_1, x_2, y_3, \ldots)), (-x_0, -y_1 + \text{pr}_1\Phi^{-1}(y_0, x_1, y_2, x_3, \ldots))).
\]

Since the class of $(\Omega_2 \oplus \Omega_1, -\Phi^{-1} \oplus \Phi, \Omega_2 \oplus \Omega_1)$ is zero, we have

\[
\chi(R, \beta, S) = [(\Omega_2 \oplus \Omega_1, \Psi(-\Phi^{-1} \oplus \Phi), \Omega_2 \oplus \Omega_1)].
\]

Now, $\Psi(-\Phi^{-1} \oplus \Phi)$ is given by

\[
(\{(y_0, x_1, y_2, x_3, \ldots), (x_0, y_1, x_2, y_3, \ldots)\}) \mapsto (-\omega x_0 + dy_1, x_1 + A, y_2 + B, x_3, y_4, \ldots), (s(y_0)p_0 - dx_1, y_1 - \omega x_1 + dy_2 + s(y_0)q_1, x_2, y_3, \ldots),
\]

where $A := s((d_p s(x_0, y_1) p_1, 0)) p_1$, $B := s((d_p s(x_0, y_1) p_1, 0)) q_2$, and $s$ is the split $Q_0 \rightarrow P_0 \oplus Q_1$ or $P_0 \oplus Q_1 \rightarrow P_1 \oplus Q_2$. Hence,

\[
(\{(Q_0 \oplus P_1 \oplus Q_2) \oplus (P_0 \oplus Q_1), \Psi', (Q_0 \oplus P_1 \oplus Q_2) \oplus (P_0 \oplus Q_1)) \\
\mapsto (\Omega_2 \oplus \Omega_1, \Psi(-\Phi^{-1} \oplus \Phi), \Omega_2 \oplus \Omega_1),
\]

where $\Psi'$ is the restriction of $\Psi(-\Phi^{-1} \oplus \Phi)$. We calculate the class of the left hand side in $K_0(F)$ and show that it is zero.
We set $M_0 := Q_0$, $M_2 := P_1 \oplus Q_2$, $M_1 := P_0 \oplus Q_1$, and denote by $\delta: M_l \to M_{l-1}$ the differential of the cone of $\alpha$. Since $-d_{\rho}s(x_0, y_1)\rho_1 + (s\delta(x_0, y_1))\rho_0 = x_0$, we have

\[(A, B) = -s(x_0) + s(s(\delta(x_0, y_1))\rho_0).\]

Let $p$ and $q$ be the projections $M_1 \to P_0$ and $M_1 \to Q_1$ respectively. Then $\Psi'$ is expressed by the matrix (an endomorphism of $F(M_0) \oplus F(M_2) \oplus F(M_1)$)

\[
\Psi' = \begin{pmatrix}
0 & 0 & \delta \\
0 & 1 & -sp + sps\delta \\
s & \delta & pQ_1
\end{pmatrix}
\]

and we have

\[
\begin{pmatrix}
1 & 0 & \delta \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
-\delta & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
-\delta & 0 & 0 \\
0 & -1 & 0
\end{pmatrix}
\begin{pmatrix}
-1 & 0 & \delta p \\
0 & 1 & -sp \\
0 & 0 & 1
\end{pmatrix}.
\]

Therefore, $\Psi'$ lifts to an automorphism of $M_0 \oplus M_2 \oplus M_1$ modulo elementary transformations, and thus $[(M_0 \oplus M_2 \oplus M_1, \Psi', M_0 \oplus M_2 \oplus M_1)] = 0$.

\[\square\]

### 3. Relative cycle class map

3.1. Relative $K$-theory of schemes. For a scheme $X$, we use the following notation:

1. $\text{Vec}(X)$ is the category of algebraic vector bundles on $X$.
2. $D^b(X)$ is the derived category of bounded complexes of $\mathcal{O}_X$-modules.
3. $D^{\text{perf}}(X) \subset D^b(X)$ is the full subcategory of perfect complexes.

The following theorem is a consequence of results in §1 and §2.

**Theorem 3.1.** Let $X$ be a scheme with an ample family of line bundles, $Y$ an affine scheme and $f: Y \to X$ be a morphism of schemes. Then there exists natural isomorphism

\[
\pi_0 \text{hofib}(K(\text{Vec}(X))) \xrightarrow{\iota^*} K(\text{Vec}(Y)) \simeq K_0(D^{\text{perf}}(X)) \xrightarrow{Lf^*} D^{\text{perf}}(Y).
\]

See Definition 0.3 for the definition of the right group.

**Proof.** Since $X$ has an ample family of line bundles, every perfect complex is quasi-isomorphic to a bounded complex of algebraic vector bundles, and thus there is an equivalence of triangulated categories

\[
K^b(\text{Vec}(X))/K^b(\text{Vec}(X)) \xrightarrow{\sim} D^{\text{perf}}(X).
\]

Since $Y$ is affine, $\text{Vec}(Y)$ is split exact and $D^{\text{perf}}(Y) \simeq K^b(\text{Vec}(Y))$. Now, the triangulated functor $Lf^*: D^{\text{perf}}(X) \to D^{\text{perf}}(Y)$ is identified with the functor $D(f^*)$ induced from the exact functor $f^*: \text{Vec}(X) \to \text{Vec}(Y)$, cf. §2.2. Therefore, by Theorem 0.8, we have an isomorphism

\[
K_0(\text{Vec}(X)) \xrightarrow{\iota^*} \text{Vec}(Y)) \simeq K_0(D^{\text{perf}}(X)) \xrightarrow{Lf^*} D^{\text{perf}}(Y).
\]

By Theorem 0.6, the right hand side is isomorphic to $\pi_0 \text{hofib}(K(\text{Vec}(X))) \xrightarrow{\iota^*} K(\text{Vec}(Y))$, and we get the theorem. \[\square\]

In the situation of Theorem 3.1, we denote by $K_0(X, Y)$ the relative $K_0$-group with respect to the triangulated functor $Lf^*: D^{\text{perf}}(X) \to D^{\text{perf}}(Y)$. This notation is abusive since $K_0(X, Y)$ depends on $f$. We adapted it because our main interest is the case $Y$ is a closed subscheme of $X$; in this case, the map $f: Y \to X$ is the canonical inclusion.

It is clear from the definition that $K_0(X, Y)$ is contravariant functorial, i.e. a commutative diagram

\[
\begin{array}{ccc}
Y' & \xrightarrow{f'} & X' \\
\downarrow & & \downarrow \\
Y & \xrightarrow{f} & X
\end{array}
\]
induces a group homomorphism $K_0(X', Y') \to K_0(X, Y)$.

According to the base change theorem [SGA6, IV 3.1], we have a proper transfer of $K_0(X, Y)$ in the following case.

**Proposition 3.2.** Suppose we are given a cartesian diagram

$$
\begin{array}{ccc}
Y' & \xrightarrow{f'} & X' \\
\downarrow{g'} & & \downarrow{g} \\
Y & \xrightarrow{f} & X
\end{array}
$$

of schemes. Assume that:

(i) $X, X'$ have ample family of line bundles and $Y, Y'$ are affine.

(ii) $f$ and $g$ are Tor-independent over $X$.

(iii) $g$ is proper and perfect.

Then there is a map

$$K_0(X', Y') \to K_0(X, Y)$$

which sends $(P, \alpha, Q) \in \text{Rel}(Lf^*)$ to $(Rg_*P, Rg'_*\alpha, Rg_*Q) \in \text{Rel}(Lf^*)$.

3.2. **Coniveau filtration.** Let $X$ be a scheme of dimension $d$ which has an ample family of line bundles, $Y$ an affine closed subscheme of $X$, and we denote the inclusion $Y \hookrightarrow X$ by $i$. Suppose that $X \setminus Y$ is regular.

**Definition 3.3.**

(i) For $\mathfrak{A} = (P, \alpha, Q) \in \text{Rel}(L_i^*)$, let $S_{\mathfrak{A}}$ be the set of open neighborhoods $U$ of $Y$ in $X$ such that there exits an isomorphism $\tilde{\alpha}: P|_U \cong Q|_U$ in $D_{\text{perf}}^+(U)$ which lifts $\alpha$.

(ii) For $-1 \leq i \leq d$, we define $F_i K_0(X, Y)$ to be the subgroup of $K_0(X, Y)$ generated by elements $\mathfrak{A}$ for which there exists $U \in S_{\mathfrak{A}}$ with $\dim(X \setminus U) \leq i$.

By the definition, $F_i K_0(X, Y) \subset F_{i+1} K_0(X, Y)$. $F_{-1} K_0(X, Y)$ is generated by $(P, \alpha, Q)$ for which there exists $\tilde{\alpha}: P \cong Q$ such that $L_i^* \tilde{\alpha} = \alpha$, and it follows from Lemma 2.1 that $F_{-1} K_0(X, Y) = 0$. In general, $F_i K_0(X, Y)$ may not be equal to $K_0(X, Y)$. However, we have:

**Lemma 3.4.** If $Y$ has an affine open neighborhood in $X$, then $F_d K_0(X, Y) = K_0(X, Y)$.

**Proof.** Let $\mathfrak{A} \in \text{Rel}(L_i^*)$. According to Theorem 0.8, the class of $\mathfrak{A}$ in $K_0(X, Y)$ is equal to the one of some $(P, \alpha, Q)$ with $P, Q \in \text{Vec}(X)$. It suffices to show that $S(P, \alpha, Q) \neq \emptyset$, i.e. there exists an open neighborhood $U$ of $Y$ in $X$ and an isomorphism $P|_U \cong Q|_U$ which lifts $\alpha$.

By our assumption, we may assume that $X$ is affine, say $X = \text{Spec} A$ and $Y = \text{Spec} A/I$. Since $Q$ is a projective $A$-module, we have an $A$-homomorphism $\gamma: P \to Q$ which fits into the commutative diagram

$$
\begin{array}{ccc}
P & \xrightarrow{\gamma} & Q \\
\downarrow & & \downarrow \\
P \otimes_A A/I & \xrightarrow{m_y} & Q \otimes_A A/I.
\end{array}
$$

Let $K$ and $L$ be the kernel and the cokernel of $\gamma$ respectively. We claim that for every $y \in Y$, $K_y = L_y = 0$. Since $- \otimes_A A_y/m_y$ (m_y is the maximal ideal) is right exact and $\gamma_y \otimes_A A_y/m_y$ is an isomorphism, we have $L_y \otimes_A A_y/m_y = 0$. By Nakayama’s lemma, $L_y = 0$. Since $Q_y$ is projective, the exact sequence

$$
0 \xrightarrow{m_y} K_y \xrightarrow{P_y} Q_y \xrightarrow{m_y} 0
$$

is split exact. Hence, $K_y \otimes_A A_y/m_y = 0$ and $K_y = 0$.

Since the supports of $K$ and $L$ are closed, it follows from the claim that there exists some open neighborhood of $Y$ on which $\gamma$ is an isomorphism. This completes the proof. □
Let $C_k(X|Y)$ be the set of integral closed subschemes of $X$ of dimension $k$ which do not meet $Y$, and $Z_k(X|Y)$ the free abelian group generated by $C_k(X|Y)$. For $V \in C_k(X|Y)$, the triple $(O_V, 0, 0)$ defines an element of $F_kK_0(X, Y)$, which we denote by $\text{cyc}(V)$.

**Lemma 3.5.** The map $\text{cyc}: Z_k(X|Y) \to F_kK_0(X, Y)$ is surjective for all $k \geq 0$.

**Proof.** Suppose we are given $\mathcal{A} \in \text{Rel}(L_t^*)$ whose class is in $F_kK_0(X, Y)$ but not in $F_{k-1}$, so that there exists $U \in S_\mathcal{A}$ such that $\dim(X \setminus U) = k$. Set $V := X \setminus U$ equipped with the reduced scheme structure. Since $V$ does not meet $Y$ and $X \setminus Y$ is regular, we have $G_0(V) \simeq K_0^V(X, Y)$. Hence, we have an exact sequence

$$
\begin{array}{c}
G_0(V) \\
\longrightarrow
\end{array}
K_0(X, Y) \\
\longrightarrow
K_0(U, Y).
$$

Now, the class of $\mathcal{A}$ dies in $K_0(U, Y)$, and thus it comes from $G_0(V)$. Since the usual cycle map

$$
\bigoplus_{i=0}^k Z_i(V) \to G_0(V), \quad W \mapsto \mathcal{O}_W
$$

is surjective, the class of $\mathcal{A}$ is in the image of

$$
\bigoplus_{i=0}^k Z_i(V) \to \bigoplus_{i=0}^k Z_i(X|Y) \to F_kK_0(X, Y).
$$

This proves the lemma. \qed

### 3.3. Chow group with modulus

Let $X$ be a scheme separated of finite type over a field $k$ and $D$ an effective Cartier divisor on $X$. We denote the inclusion $D \hookrightarrow X$ by $\iota$. We recall the definition of the Chow group with modulus by Binda-Saito [BS14].

Let $k \geq 0$. Let $R_k(X|D)$ be the set of integral closed subschemes $V$ of $X \times \mathbb{P}^1$ of dimension $k + 1$ which is dominant over $\mathbb{P}^1$ and satisfies the following condition (modulus condition): Let $V^N$ be the normalization of $V$ and $\phi$ the canonical map $V^N \to X \times \mathbb{P}^1$, then we have an inequality of Cartier divisors

$$
\phi^*(D \times \mathbb{P}^1) \leq \phi^*(X \times \{\infty\}).
$$

For each $V \in R_k(X|D)$, the inverse images $V_t$ of $t \in \mathbb{P}^1$ are purely of dimension $k$ and do not meet $D$, and thus define elements of $Z_k(X|D)$ in the standard way.

**Definition 3.6** (Binda-Saito [BS14]). The **Chow group with modulus** $\text{CH}_k(X|D)$ is defined to be the quotient of $Z_k(X|D)$ by the relations $[V_0] = [V_1]$ for all $V \in R_k(X|D)$.

**Theorem 3.7.** Suppose that $X$ is regular and that $D$ is affine. Then there is a surjective group homomorphism

$$
\text{cyc}: \text{CH}_k(X|D) \to F_kK_0(X, D)/F_{k-1}K_0(X, D)
$$

for every $k \geq 0$.

**Proof.** By Lemma 3.5, we have a surjective homomorphism

$$
\text{cyc}: Z_k(X|D) \to F_kK_0(X, D)/F_{k-1}K_0(X, D).
$$

We show this map factors through $\text{CH}_k(X|D)$. \qed
We have to show that, for all $V \in R_k(X|D)$, $\text{cyc}(V_0) = \text{cyc}(V_1)$ in $F_kK_0(X, D)/F_{k-1}K_0(X, D)$. Let $V \in R_k(X|D)$ and consider the diagram

$$
\begin{array}{c}
V^N \\
\pi \\
q \\
p_N \\
X \times \mathbb{P}^1 \\
p \\
\mathbb{P}^1
\end{array}
$$

where all maps are the obvious ones. We fix a parameter $t$ of $\mathbb{P}^1 \setminus \{\infty\}$. Let $j_0: \mathcal{O}(-1) \to \mathcal{O}_{\mathbb{P}^1}$ be the canonical inclusion (sending $t$ to $t$) and $j_1: \mathcal{O}(-1) \to \mathcal{O}_{\mathbb{P}^1}$ the map sending $t$ to $t-1$. Then $V_t$, $t = 0, 1$, are the subscheme of $V$ defined by $p^*j_t$ and map isomorphically to closed subschemes of $X$ by $q$.

Let $V_t'$ be the scheme theoretic inverse image of $V_t$ by $\pi: V^N \to V$. First, we show that

$$[(\mathcal{O}_{V_t'}, 0, 0)] = [(\mathcal{O}_{V_t'}, 0, 0)]$$

in $K_0(V^N, q^*_N D) = K_0(Lt^*)$, where $t': q^*_N D \to V^N$ is the canonical inclusion. Now, we have an exact sequence

$$0 \to p^*_N \mathcal{O}(-1) \to p^*_{Nj_t} \mathcal{O}_V \to V_t' \to 0$$

of $\mathcal{O}_V$-modules. Hence, the class of $(\mathcal{O}_{V_t'}, 0, 0)$ in $K_0(V^N, q^*_N D)$ is equal to the one of

$$(p^*_N \mathcal{O}(-1), p^*_{Nj_t}, \mathcal{O}_{V_t'}).$$

Let $\theta$ be the multiplication of $(1-t)/t$, which is defined on $\mathbb{P}^1 \setminus \{0\}$ and an automorphism on $\mathbb{P}^1 \setminus \{0, 1\}$; $\theta$ fits into the commutative diagram

$$
\begin{array}{c}
\mathcal{O}(-1) \\
\theta \\
\mathcal{O}_{\mathbb{P}^1}
\end{array}
\xrightarrow{j_0} \xrightarrow{j_1} \xrightarrow{} \xrightarrow{}
$$

It follows that

$$[(p^*_N \mathcal{O}(-1), p^*_{Nj_0}, \mathcal{O}_{V^N})] + [(\mathcal{O}_{V^N}, p^*_N \theta, \mathcal{O}_{V^N})] = [(p^*_N \mathcal{O}(-1), p^*_{Nj_1}, \mathcal{O}_{V^N})].$$

It is clear that the restriction of $p^*_N \theta$ on $q^*_N(\{\infty\}) = \phi^*(X \times \{\infty\})$ is the identity. Hence, by the modulus condition, the restriction of $p^*_N \theta$ on $\phi^*(D \times \mathbb{P}^1) = q^*_N D$ is the identity. This implies that the second term of the above equation is zero; in other words,

$$[(p^*_N \mathcal{O}(-1), p^*_{Nj_0}, \mathcal{O}_{V^N})] = [(p^*_N \mathcal{O}(-1), p^*_{Nj_1}, \mathcal{O}_{V^N})].$$

This proves $[(\mathcal{O}_{V_t'}, 0, 0)] = [(\mathcal{O}_{V_t'}, 0, 0)]$ in $K_0(V^N, q^*_N D)$.

Now, $\iota: D \hookrightarrow X$ and $q^N: V^N \to X$ are Tor-independent, and $q^N$ is proper and perfect since $X$ is regular. Hence, by Proposition 3.2, we have a transfer map

$$q_{N*}: K_0(V^N, q^*_N D) \to K_0(X, D).$$

Consequently, we have

$$[(q_{N*} \mathcal{O}_{V_t'}, 0, 0)] = [(q_{N*} \mathcal{O}_{V_t'}, 0, 0)]$$

in $K_0(X, D)$.

Finally, we claim that

$$\text{cyc}(V_t) \equiv [(\mathcal{O}_{V_t}, 0, 0)] \equiv [(q_{N*} \mathcal{O}_{V_t'}, 0, 0)]$$

modulo $F_{k-1}K_0(X, D)$, which completes the proof. The first term is $\sum_i m_i [(\mathcal{O}_{V_{t,i}}, 0, 0)]$ by definition, where $V_{t,i}$ are irreducible components of $V_t$ and $m_i$ are their multiplicity. By Lemma 3.8 below, it suffices
to compare the length of $\mathcal{O}_{V_t}$ and $q_{N,*}\mathcal{O}_{V'_t}$ at the generic point of $V_{t,i}$. This is clear because $V_t \hookrightarrow V$ and $V'_t \hookrightarrow V^N$ are defined by the same rational function. □

Lemma 3.8. Let $\mathcal{F}$ be a coherent sheaf on $X$ whose support is of dimension $k$ and disjoint from $D$. Then

$$[(\mathcal{F}, 0, 0)] = \sum_{\dim V = k} m_V(\mathcal{F})[(\mathcal{O}_V, 0, 0)]$$

in $F_k K_0(X, D)/F_{k-1} K_0(X, D)$. Here, $V$ runs over all integral closed subschemes of $X$ of dimension $k$ and $m_V(\mathcal{F})$ is the length of the stalk of $\mathcal{F}$ at the generic point of $V$.

Proof. Let $j : Z \hookrightarrow X$ be the scheme theoretic support of $\mathcal{F}$, i.e. $\mathcal{F} = j_* \mathcal{G}$ for some coherent module $\mathcal{G}$ of $Z$. The map $G_0(Z) \rightarrow K_0(X, D)$ which sends $[\mathcal{G}]$ to $[(j_* \mathcal{G}, 0, 0)]$ is compatible with the coniveau filtration. Hence, it suffices to show that

$$[\mathcal{G}] = \sum_{\dim V = k} m_V(\mathcal{G})[(\mathcal{O}_V)]$$

in $F_k G_0(Z)/F_{k-1} G_0(Z)$. This is easily verified by induction. □

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