A Tight Degree 4 Sum-of-Squares Lower Bound for the Sherrington-Kirkpatrick Hamiltonian

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Abstract

We show that, if $W \in \mathbb{R}^{N \times N}_{\text{sym}}$ is drawn from the gaussian orthogonal ensemble, then with high probability the degree 4 sum-of-squares relaxation cannot certify an upper bound on the objective $N^{-1} \cdot \mathbf{x}^\top W \mathbf{x}$ under the constraints $x_i^2 - 1 = 0$ (i.e. $\mathbf{x} \in \{\pm 1\}^N$) that is asymptotically smaller than $\lambda_{\text{max}}(W) \approx 2$. We also conjecture a proof technique for lower bounds against sum-of-squares relaxations of any degree held constant as $N \to \infty$, by proposing an approximate pseudomoment construction.

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## Contents

1 Introduction .......................................................... 3  
   1.1 The Sherrington-Kirkpatrick Hamiltonian ....................... 3  
   1.2 Conjectural Hardness of Certification ........................... 4  
   1.3 Sum-of-Squares Relaxations .................................. 5  
   1.4 The Montanari-Sen Degree 2 Lower Bound ...................... 6  

2 Results .................................................................. 7  

3 Preliminaries .......................................................... 8  
   3.1 Glossary: Variables, Parameters, and Constants ............... 8  
   3.2 Other Notation .................................................. 9  
   3.3 Basic Properties of the Montanari-Sen Witness ............... 9  

4 Degree 4 Pseudomoment Construction ............................ 10  
   4.1 Heuristic 1: Evidence from Equiangular Tight Frames ....... 11  
   4.2 Heuristic 2: Conditional Covariance of Gaussian Matrices .... 12  
   4.3 Precise Construction Details .................................. 14  

5 Conjectural Higher-Degree Extension ............................. 15  

6 Proof of Positive Semidefiniteness: First Steps ............... 17  

7 Proof of Positive Semidefiniteness: Main Term .................. 18  
   7.1 Bounding the Cross-Term \( T^{(1)} \) ............................ 20  
   7.2 Bounding the Projection Term \( T^{(2)} \): Unnormalized Case ... 21  
   7.3 Bounding the Projection Term \( T^{(2)} \): Normalization ........ 26  
   7.4 Final Main Term Bound: Proof of Lemma 6.1 ................. 27  

8 Proof of Positive Semidefiniteness: Correction Term .......... 28  

Acknowledgements ......................................................... 30  

References ................................................................ 31  

A Moments of \( \mathcal{O}(N) \) ............................................. 34  

B Entries of Random Projectors: Proof of Proposition 3.1 .... 34
1 Introduction

1.1 The Sherrington-Kirkpatrick Hamiltonian

This paper concerns convex relaxations of the following optimization problem:

\[
M(W) := \frac{1}{N} \max_{x \in \{\pm 1\}^N} x^\top W x. \tag{1}
\]

In particular, we are interested in the case where \( W \in \mathbb{R}_{\text{sym}}^{N \times N} \) is a random matrix drawn from the gaussian orthogonal ensemble: \( W_{ii} \sim \mathcal{N}(0, 2/N) \) and \( W_{ij} = W_{ji} \sim \mathcal{N}(0, 1/N) \), with the \( N(N + 1)/2 \) entries on and above the diagonal distributed independently. This distribution is the gaussian orthogonal ensemble (GOE), which we denote \( W \sim \text{GOE}(N) \). Under this model, the spectral radius of \( W \) is of constant order, and the normalization in (1) is chosen so that \( \mathbb{E}M(W) \) also remains of constant order as \( N \to \infty \), as we will describe shortly.

The problem \( M(W) \) for general \( W \) includes the problem of finding maximum cuts in graphs (MaxCut), when \( W \) is taken to be a graph Laplacian. The classical result of [Kar72] therefore implies that computing \( M(W) \) is \( \text{NP} \)-hard in the worst case. The case \( W \sim \text{GOE}(N) \) is a simple example with which we hope to probe the average-case complexity of the same problem, seeking to understand whether the worst-case complexity abates for specific random models of \( W \).

We are assisted in this task by the rich history of the random optimization problem \( M(W) \) in statistical physics: up to a change in sign, its value is the ground-state energy of the Sherrington-Kirkpatrick (SK) model, a mean-field model of spin glasses [SK75]. In particular, the asymptotics of its expected value have been well-understood at a non-rigorous level since the seminal work of Parisi [Par79], who developed a system of deep conjectures on the optimization landscape of \( M(W) \), which, among other results, allowed him to analytically predict the limit

\[
\lim_{N \to \infty} \mathbb{E}M(W) =: 2P_* \approx 1.5264. \tag{2}
\]

(Standard results from general gaussian process theory also imply strong concentration around the expectation.) Recent mathematical work has made the computation of this limit rigorous in a certain technical sense [Pan11, Pan13, Tal06].

From the perspective of computer science, perhaps the more natural random model of \( M(W) \) is the case where \( W \) is the adjacency matrix or graph Laplacian of a random graph, which gives randomized instances of MaxCut. An elegant pair of recent works [MS16, DMS+17] showed that, in fact, for sparse random graphs this problem is intimately related to the gaussian setting of the SK model: an interpolation argument may be used to control both the true value and the value of a certain simple semidefinite programming relaxation of \( M(W) \) for sparse random graphs in terms of the SK model.

Thus, whether motivated by the mathematical interest of the GOE and SK model or the application to MaxCut, we are led to consider the following algorithmic question:

\textit{Can }\( M(W) \)\textit{ be approximated accurately and efficiently when }\( W \sim \text{GOE}(N) \)\textit{?}

Of course, knowing the limiting expectation (2) and concentration around this value, it is simple to produce a vacuous algorithm that outputs the value \( 2P_* \). To capture the difficulty of solving instances of \( M(W) \) for specific random draws of \( W \), we must therefore refine our question.

One way to do this is to ask instead:

\textit{Can }\( x = x(W) \in \{\pm 1\}^N \)\textit{ be efficiently computed such that }\( \frac{1}{N}x^\top W x \approx 2P_* \)\textit{?}
This question was recently answered in the affirmative by [Mon18], using a method devised in a sequence of similar works on simpler models [ABM18, Sub18].

**Theorem 1.1 (Theorem 2 of [Mon18]).** Conditional on the widely-believed conjecture of full replica symmetry breaking in the SK model [MPV87], for any $\epsilon > 0$, there exists a polynomial-time algorithm computing $x = x(W) \in \{\pm 1\}^N$ such that

$$\lim_{N \to \infty} \mathbb{P} \left[ \frac{1}{N} x^\top W x \geq 2P_\star - \epsilon \right] = 1. \quad (3)$$

Another way to refine our question is to ask for certificates of upper bounds on $M(W)$:

Can $c(W) \in \mathbb{R}$ be efficiently computed with $c(W) \geq M(W)$ and $c(W) \approx 2P_\star$?

(\text{Note that we require } c(W) \geq M(W) \text{ to hold for every } W; \text{ the algorithm is not allowed to “cheat” the random setting by merely outputting a number slightly larger than } 2P_\star.) One simple but sub-optimal approach is to form the spectral certificate, which amounts to disregarding the constraint $x \in \{\pm 1\}^N$ by taking $c(W) := \lambda_{\text{max}}(W) \approx 2$. Recently, Montanari asked\footnote{The authors learned of this problem through private communications soon after [MS16] was published. More recently, it was also included in the problem list “AimPL: Phase transitions in randomized computational problems,” available online at http://aimpl.org/phaserandom.} whether any certification algorithm could improve on this performance, a problem which to the best of our knowledge, besides modest progress that we will review in the following sections, has since remained open.

Our contribution in this paper is to provide evidence that the spectral certificate is asymptotically optimal by showing that the degree 4 sum-of-squares relaxation, a much more sophisticated convex relaxation, achieves the same performance.

### 1.2 Conjectural Hardness of Certification

One step towards making a convincing prediction of whether better-than-spectral certification is possible in the SK model was taken in [BKW19], in which the authors participated. In this work, we first showed that, if efficient certification below 2 were possible for the SK model, then it would be possible to efficiently perform a certain hypothesis testing task in a variant of a spiked matrix model. Then, we provided evidence that this hypothesis testing task should be hard using a method based on the low-degree likelihood ratio. Roughly speaking, this technique takes low-degree polynomials as a proxy for all polynomial-time testing statistics and measures their performance in a convenient smoothed sense, which allows the optimal low-degree polynomial statistic to be identified and analyzed using orthogonal polynomials.

This line of reasoning suggests the following conjecture, which would hold conditional on another, quite broad conjecture of [HS17, Hop18] on the correctness of the low-degree likelihood ratio analysis for a large class of hypothesis testing problems.

**Conjecture 1.2.** For any $\epsilon > 0$, there does not exist a polynomial-time certification algorithm for $M(W)$ such that $c(W) \leq 2 - \epsilon$ with high probability.

Unfortunately, though the low-degree likelihood ratio method predicts many known computational thresholds in random problems correctly, at the moment it is only known to imply rather weak lower bounds against specific certification algorithms—either only lower bounds in expectation or a smoothed $L^2$ sense, or high-probability lower bounds under quite restrictive assumptions (see, e.g., the recent survey [KWB19] by the authors). In search of further evidence of hardness of certification, we therefore consider concrete algorithms and analyze their performance directly.
1.3 Sum-of-Squares Relaxations

The main algorithmic approach for certifying bounds on a problem like $M(W)$ is to form convex relaxations that may be solved efficiently by standard convex optimization techniques. First, note that, defining the cut polytope

$$
\mathcal{C}^N := \text{conv} \left( \left\{ xx^T : x \in \{ \pm 1 \}^N \right\} \right),
$$

we may rewrite $M(W)$ as a linear optimization problem over this set of matrices,

$$
M(W) = \frac{1}{N} \max_{M \in \mathcal{C}^N} \langle W, M \rangle.
$$

Though $\mathcal{C}^N$ is a convex set, it is complex to describe [DL09], and in particular does not admit a polynomial-time separation oracle unless $P = NP$ (by the same result of [Kar72] mentioned before). We thus pursue the idea of expanding $\mathcal{C}^N$ to a larger convex set that may be described more simply, and over which convex optimization is tractable.

Specifically, we will study the performance of semidefinite programming (SDP) relaxations of $M(W)$. Perhaps the simplest of these is based on the inclusion of sets

$$
\mathcal{C}_d^N \subseteq \{ M \in \mathbb{R}^{N \times N}_{\text{sym}} : M \succeq 0, \text{Tr}(M) = N \} =: \mathcal{C}_{\text{spec}}^N.
$$

Replacing $\mathcal{C}_d^N$ with $\mathcal{C}_{\text{spec}}^N$ in the definition of $M(W)$ just computes $\lambda_{\text{max}}(W)$, so this expresses the spectral certificate as an SDP relaxation. Thus one consequence of Conjecture 1.2 is that this naive relaxation is optimal among the wide variety of SDP relaxations that may be applied to $M(W)$.

One broad and successful framework for SDP relaxation of optimization problems through which one might hope to find an improvement is the sum-of-squares (SoS) hierarchy of relaxations [Las01, Lau09, BPT12, BS14]. This generates a sequence of convex sets we will denote by $\mathcal{C}_d^N$, indexed by a parameter $d$, an even natural number called the degree, which satisfy [Lau03, FSP16] the strict inclusions

$$
\mathcal{C}_2^N \supset \mathcal{C}_4^N \supset \cdots \supset \mathcal{C}_{N+1}(N\text{ odd}) = \mathcal{C}^N.
$$

Moreover, $\mathcal{C}_d^N$ is a projection of an affine slice of the positive semidefinite cone of $N^d \times N^d$ real symmetric matrices, and thus may be optimized over in time $N^{O(d)}$ with standard semidefinite programming methods. Optimization over these sets thus gives a hierarchy of certification algorithms; as one ascends the hierarchy, one pays a cost in runtime in hopes of obtaining better upper bounds by better approximating $\mathcal{C}^N$.

We now describe $\mathcal{C}_d^N$ in terms of the pseudomoment interpretation of SOS optimization (as derived from the general formulation of SOS in, e.g., [Lau03]).

**Definition 1.3.** For a finite set $A$, we write $\mathcal{M}(A)$ for the collection of finite multisets in the elements of $A$, and $\mathcal{M}^k(A) \subseteq \mathcal{M}(A)$ for those multisets of size exactly $k$. The set $\mathcal{M}(A)$ is in one-to-one correspondence with the functions $s : A \to \mathbb{N}$, where $s(a)$ gives the number of times $a$ occurs in the corresponding multiset. We will identify these objects, viewing a function $s$ as identical to its multiset, writing $s + t$ for multiset union, and writing $|s| := \sum_{a \in A} s(a)$ for the size. Lastly, we write $\text{odd}(s) \subseteq A$ for the (non-multi) set of symbols $a$ for which $s(a)$ is odd.

**Definition 1.4.** $\mathcal{C}_d^N \subseteq \mathbb{R}^{N \times N}_{\text{sym}}$ is the set of $M$ such that there exists $Z \in \mathbb{R}^{N^{d/2}([N]) \times N^{d/2}([N])}$ having

$$
Z_{\{1 \ldots i\}{1 \ldots j}} = M_{ij} \text{ for all } i, j \in [N]
$$

and satisfying the following properties:
1. \( Z \succeq 0 \).

2. \( Z_{st} \) only depends on \( \text{odd}(s + t) \).

3. \( Z_{st} = 1 \) whenever \( \text{odd}(s + t) = \emptyset \).

In this case, we say \( Z \) is a degree \( d \) pseudomoment matrix for the constraint polynomials \( \{x_i^2 - 1 : i \in [N]\} \subset \mathbb{R}[x_1, \ldots, x_N] \), which extends \( M \).

For the sake of brevity, we will simply refer to such \( Z \) as a degree \( d \) pseudomoment matrix, since we only study optimization over \( \{-1, 1\}^N \).

**Definition 1.5.** The degree \( d \) sum-of-squares relaxation of the problem \( M(W) \) is

\[
\text{SOS}_d(W) := \frac{1}{N} \max_{M \in \mathcal{E}^N_d} \langle W, M \rangle.
\]

In addition to Montanari’s general question on certifying bounds on \( M(W) \) mentioned before, Jain, Risteski, and Koehler have independently posed the more specific question of determining the asymptotic values of \( \text{SOS}_d(W) \) when \( W \sim \text{GOE}(N) \) in [JKR19].

We will only study degree 2 and degree 4 pseudomoment matrices in detail, so we give more concrete versions of the above conditions for those cases.

**Proposition 1.6.** \( \mathcal{E}^N_2 = \{ M \in \mathbb{R}^{N \times N}_{\text{sym}} : M \succeq 0, M_{ii} = 1 \text{ for all } i \in [N] \} \).

**Proposition 1.7.** Let \( Z \in \mathbb{R}^{N(N+1)/2 \times N(N+1)/2} \), with the row and column indices of \( Z \) identified with unordered pairs \( \{ij\} \) with \( i, j \in [N] \) and possibly \( i = j \). Concretely, we order these pairs with \( \{11\}, \ldots, \{NN\} \) first, followed by \( \{ij\} \) for \( i < j \) ordered lexicographically. Then, \( Z \) is a degree 4 pseudomoment matrix if and only if the following conditions hold:

1. \( Z \succeq 0 \).

2. \( Z_{\{ij\}\{kk\}} \) does not depend on the index \( k \).

3. \( Z_{\{ii\}\{ii\}} = 1 \) for every \( i \in [N] \).

4. \( Z_{\{ij\}\{k\ell\}} \) is invariant under permutations of the indices \( i, j, k, \ell \).

### 1.4 The Montanari-Sen Degree 2 Lower Bound

The only previous result on SOS relaxations of \( M(W) \) under \( W \sim \text{GOE}(N) \) that we are aware of is the following result of Montanari and Sen [MS16], which establishes hardness of certification for degree 2 SOS.

**Theorem 1.8** (Theorem 5(a) of [MS16]). Let \( \epsilon > 0 \). Then,

\[
\lim_{N \to \infty} \mathbb{P} \left[ \text{SOS}_2(W) \geq 2 - \epsilon \right] = 1. \tag{10}
\]

The mechanics of this result will be crucial for proving ours, so let us also review how to construct feasible points achieving this bound. First, fix a parameter \( \delta \in (0, 1) \), and set \( r = r(N) = \delta N. \)

Then, given \( W \sim \text{GOE}(N) \), let \( V \subset \mathbb{R}^N \) be the subspace spanned by the \( r(N) \) eigenvectors of \( W \)

\[\text{following [MS16], we assume for the sake of simplicity that } \delta N \text{ is an integer; recovering the same results for } r = [\delta N] \text{ is tedious but straightforward.}\]
having the largest eigenvalues, and let $P$ be the orthogonal projector to $V$. Let $D \in \mathbb{R}^{N \times N}$ be the diagonal matrix with entries $D_{ii} = P_{ii}$, and define

$$M^{(\delta)} = M^{(\delta)}(W) := D^{-1/2}PD^{-1/2}. \quad (11)$$

(Note that $D_{ii} = 0$ if and only if $e_i \in V^\perp$, which almost surely does not occur.) Then, $M^{(\delta)}(W) \in \mathcal{E}_2^N$ for all $W \in \mathbb{R}^{N \times N}$ and $\delta \in (0, 1)$, and for each $\epsilon > 0$ there exists $\delta \in (0, 1)$ such that

$$\lim_{N \to \infty} P\left[\frac{1}{N}\langle M^{(\delta)}(W), W \rangle \geq 2 - \epsilon \right] = 1. \quad (12)$$

We call $M^{(\delta)}$ the Montanari-Sen witness. This construction will be the basis of ours; indeed, we will show that only a small correction is necessary to make this witness feasible for the degree 4 SOS relaxation as well.

2 Results

Our main result establishes hardness of certification for degree 4 SOS, by showing that a minor variant of the Montanari-Sen witness admits a degree 4 extension.

**Theorem 2.1.** Let $\alpha, \delta \in (0, 1)$, and let $M^{(\delta)}$ be as in Theorem 1.8. Define

$$M^{(\alpha, \delta)}(W) := (1 - \alpha)M^{(\delta)}(W) + \alpha I_N. \quad (13)$$

Then,

$$\lim_{N \to \infty} P\left[M^{(\alpha, \delta)}(W) \in \mathcal{E}_4^N \right] = 1. \quad (14)$$

The theorem says that an arbitrarily small adjustment in the direction the identity matrix (the barycenter of the vertices of the cut polytope) suffices to make the degree 2 Montanari-Sen primal witness admit a degree 4 extension with high probability. In Conjecture 5.2, we will also propose that the same membership holds with high probability for any SOS relaxation $\mathcal{E}_d^N$ with constant degree $d$ as $N \to \infty$.

Since the degree 2 part of our construction is a simple modification of the Montanari-Sen witness, it is straightforward to apply Theorem 1.8 and obtain a lower bound for the degree 4 SOS objective.

**Corollary 2.2.** Let $\epsilon > 0$. Then,

$$\lim_{N \to \infty} P\left[\text{SOS}_4(W) \geq 2 - \epsilon \right] = 1. \quad (15)$$

**Proof.** By Theorem 1.8, take $\delta \in (0, 1)$ such that, with high probability, $\frac{1}{N}\langle M^{(\delta)}(W), W \rangle \geq 2 - \epsilon/3$. Let $\alpha \in (0, 1)$ be sufficiently small that $(1 - \alpha)(2 - \epsilon/3) \geq 2 - 2\epsilon/3$. By Theorem 2.1, with high probability,

$$\text{SOS}_4(W) \geq \frac{1}{N}\langle M^{(\alpha, \delta)}(W), W \rangle \geq 2 - \frac{2\epsilon}{3} + \frac{1}{N}\alpha \text{Tr}(W). \quad (16)$$

The random variable $\text{Tr}(W)$ has law $\mathcal{N}(0, 2)$, so with high probability the last term is smaller than $\epsilon/3$, and the result follows. \hfill $\Box$

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3The specific choice of the identity matrix is probably not essential, but is convenient because, as we will see, the degree 4 extension of the identity matrix has an especially simple spectral structure.
Organization. The remainder of the paper gives the proof of Theorem 2.1, and conjectures an extension of the same proof technique to higher degree SOS relaxations. In Section 3 we review some basic preliminary facts and notations. In Section 4 we give the motivation and precise statement of our construction of a matrix that is with high probability a degree 4 pseudomoment matrix extending $M^{(\alpha, \delta)}(W)$. In Section 5, we present a natural conjecture for how this construction can be extended to higher degrees. Finally, in Sections 6, 7, and 8, we prove that our construction from Section 4 indeed furnishes a valid degree 4 pseudomoment matrix with high probability.

3 Preliminaries

3.1 Glossary: Variables, Parameters, and Constants

For reference, we summarize various symbols that will appear throughout. The following are the general scalar parameters involved.

- $N$ is a parameter indicating the size of $W \sim \text{GOE}(N)$ in the problem statement.
- $\delta \in (0, 1)$ is a fixed parameter not depending on $N$.
- $r = r(N) := \delta N$, which we assume for the sake of simplicity is an integer. This gives the rank of the Montanari-Sen witness that we extend (before modifications that make it have full rank)
- $\alpha \in (0, 1)$ is another fixed parameter not depending on $N$ or $\delta$.
- $K$ is a constant appearing in concentration inequalities, giving polynomial rates of decay of probabilities of the form $N^{-K}$. All of the concentration inequalities where $K$ appears hold with any choice of $K > 0$, but the other constants appearing in those results depend on both $K$ and $\delta$. Thus a typical inequality will take the form $\mathbb{P} \left[ \text{quantity} \leq O_{\delta, K}(N) \right] \geq 1 - O_{\delta, K}(N^{-K})$.

One may think of any concrete choice, e.g. $K = 100$, throughout.

The following are vectors and matrices associated to the Montanari-Sen construction applied to a specific instance $W$.

- $V \in \mathbb{R}^{r \times N}$ is the matrix having the top $r$ (unit norm) eigenvectors of $W$ as its rows.
- $v_1, \ldots, v_N \in \mathbb{R}^r$ are the columns of $V$ (not eigenvectors of $W$).
- $P = V^\top V$ is the orthogonal projector to the top $r$-dimensional eigenspace of $W$, and also the Gram matrix of the $v_i$.
- $\hat{v}_i = v_i / \|v_i\|_2$.
- $D \in \mathbb{R}^{N \times N}$ is a diagonal matrix with $D_{ii} = \|v_i\|_2^2 = P_{ii}$.
- $\tilde{V} = VD^{-1/2} \in \mathbb{R}^{r \times N}$ is the matrix having the $\hat{v}_i$ as its columns.
- $M = M^{(\delta)} = \tilde{V}^\top \tilde{V} = D^{-1/2}PD^{-1/2}$ is the Montanari-Sen witness, and also the Gram matrix of the $\hat{v}_i$. For the sake of brevity, we will often drop the $(\delta)$ superscript, as $\delta$ will be a constant carried throughout.
3.2 Other Notation

Linear algebra. The identity matrix in dimension \( n \) is denoted \( I_n \), and the all-ones vector is denoted \( 1_n \). The Frobenius or entrywise inner product of matrices having the same shape is denoted \( \langle A, B \rangle = \text{Tr}(A^\top B) = \sum_{i,j} A_{ij} B_{ij} \). The Hadamard or entrywise product of matrices having the same shape is denoted \( A \odot B \) and has entries \( (A \odot B)_{ij} = A_{ij} B_{ij} \). Repeated Hadamard products of a matrix with itself are denoted \( A \odot^k \). The matrix operator norm (the largest singular value) is denoted \( \|A\|_{\text{op}} = \max_{\|x\|_2 = 1} \|y\|_2 = 1 x^\top A y \). The matrix Frobenius norm is denoted \( \|A\|_F = \text{Tr}(AA^\top)^{1/2} \). The vectorized supremum norm is denoted \( \|A\|_{\ell\infty} = \max_{i,j} |A_{ij}| \).

Vectorization. Several notations that describe vectorizing symmetric matrices will be useful. For \( A \in \mathbb{R}^{n \times n}_{\text{sym}} \), we let \( \text{diag}(A) \in \mathbb{R}^n \) and \( \text{offdiag}(A) \in \mathbb{R}^{n(n-1)/2} \) be the vectorized diagonal and strict upper triangle of \( A \), with index pairs ordered lexicographically. These two vectors determine a symmetric matrix completely. The following is the most direct way to vectorize a symmetric matrix:

\[
\text{entryvec}(A) := \begin{bmatrix} \text{diag}(A) \\ \text{offdiag}(A) \end{bmatrix}.
\] (18)

It will also be useful to define an isometry between \( \mathbb{R}^{n \times n}_{\text{sym}} \) endowed with the Frobenius inner product \( \langle A, B \rangle = \text{Tr}(AB) = \sum_{i,j=1}^n A_{ij} B_{ij} \) and \( \mathbb{R}^{n(n+1)/2} \) endowed with the ordinary Euclidean inner product. This is given by

\[
\text{isovec}(A) := \begin{bmatrix} \text{diag}(A) \\ \sqrt{2} \cdot \text{offdiag}(A) \end{bmatrix}.
\] (19)

It is straightforward to check that isovec indeed satisfies

\[
\langle \text{isovec}(A), \text{isovec}(B) \rangle = \langle A, B \rangle = \sum_{i,j=1}^n A_{ij} B_{ij},
\] (20)

\[
\|\text{isovec}(A)\|_2^2 = \|A\|_F^2 = \sum_{i,j=1}^n A_{ij}^2.
\] (21)

We denote \( 1_{\text{diag}} := \text{isovec}(I_n) \) when the ambient dimension \( n \) is clear from context. We use this notation because this vector is the indicator vector of the diagonal index pairs \( \{ii\} \).

3.3 Basic Properties of the Montanari-Sen Witness

It will be useful to establish some preliminary bounds on and distributional properties of the Montanari-Sen construction described in Theorem 1.8.

First, we use an elegant geometric argument mentioned in [MS16] (but which seems to have been well-known folklore previously) to obtain bounds on the entries of \( P \).

Proposition 3.1. For all \( K > 0 \),

\[
P \left[ \max_{i,j \in [N]} \left\{ \frac{|P_{ii} - \delta|}{|P_{ij}|} \text{ if } i = j \right\} \leq \frac{O_{\delta,K} \left( \sqrt{\frac{\log N}{N}} \right)}{1 - O_{\delta,K}(N^{-K})} \right]
\] (22)

The proof is provided in Appendix B. Next, since \( M_{ij} = P_{ij}/\sqrt{P_{ii} P_{jj}} \), the following analogous inequality for \( M \) follows immediately.
Corollary 3.2. For all $K > 0$,

\[
P \left[ \max_{i,j \in [N]} \left\{ \begin{array}{ll} |M_{ii} - 1| & \text{if } i = j \\ |M_{ij}| & \text{if } i \neq j \end{array} \right\} \leq O_{\delta,K} \left( \sqrt{\frac{\log N}{N}} \right) \right] \geq 1 - O_{\delta,K}(N^{-K}).
\]

(These results are all one needs to reproduce the proof of Theorem 1.8 from [MS16].)

Also, since in matrix notation $M = D^{-1/2}PD^{-1/2}$, the following operator norm bound is a simple consequence as well.

Corollary 3.3. For all $K > 0$,

\[
P \left\| M \right\|_{\text{op}} \leq \delta^{-1} + O_{\delta,K} \left( \sqrt{\frac{\log N}{N}} \right) \geq 1 - O_{\delta,K}(N^{-K}).
\]

Proof. We have $\|M\|_{\text{op}} \leq \|D^{-1/2}\|_{\text{op}}^2 \|P\|_{\text{op}}^2 = (\min_{i \in [N]} D_{ii})^{-1}$, and the result then follows by Proposition 3.1. \qed

Lastly, the following simple argument shows that the marginal distribution of each $\hat{v}_i$ is uniform over the unit sphere. We will later return to the question of what special structure is created by the coupling of the $\hat{v}_i$.

Proposition 3.4. For each $i \in [N]$, the law of $\hat{v}_i$ is $\text{Unif}(S^{r-1})$.

Proof. Viewing $V$ as the first $r$ rows of an orthogonal matrix drawn from $\text{Haar}(O(N))$, each column $v_i$ of $V$ has the distribution of a uniform unit vector in $\mathbb{R}^N$ truncated to the first $r$ coordinates. That is, if $g^{(r)} \sim \mathcal{N}(0, I_r)$ and $g^{(N-r)} \sim \mathcal{N}(0, I_{N-r})$ are independent, then

\[
v_i^{(d)} = \frac{g^{(r)}}{\sqrt{\|g^{(r)}\|_2^2 + \|g^{(N-r)}\|_2^2}}.
\]

Therefore,

\[
\hat{v}_i = \frac{v_i}{\|v_i\|_2} \overset{(d)}{=} \frac{g^{(r)}}{\|g^{(r)}\|_2}.
\]

and the result follows. \qed

4 Degree 4 Pseudomoment Construction

In this section we outline the main idea in the proof of Theorem 2.1, detailing the construction of a suitable degree 4 pseudomoment matrix and reducing Theorem 2.1 to verifying the positive semidefiniteness of this matrix.

First, we give two heuristic lines of reasoning supporting the construction we propose. The first derives a formula for the degree 4 pseudomoment extension of a highly structured collection of degree 2 pseudomoments and assumes that this formula may be transferred verbatim to the random case. The second recovers the same formula after applying some simplifying heuristics to a probabilistic argument.

Remark 4.1. We do not know if the “pseudocalibration” sum-of-squares heuristic of [BHK+19] (for some choice of a “planted solution” distribution in this problem, such as the one described in [BKW19]) would give the same prediction as these more problem-specific and somewhat geometric techniques. Such a reframing of our construction would be interesting but seems improbable, since the pseudocalibration approach has not been sharp enough in previous applications to identify the effectiveness of SOS to the level of precise asymptotic constants.
4.1 Heuristic 1: Evidence from Equiangular Tight Frames

In this section, we review a result from the works [BK18, BK19] of the authors’, which derived an explicit description of degree 4 extensions of degree 2 pseudomoment matrices for some very structured special cases, that resemble the Montanari-Sen witness in that their degree 2 pseudomoment matrices are constant multiples of orthogonal projectors.

The extra structure that allows this description of the degree 4 pseudomoments to be derived in closed form is described by the following notions from finite frame theory.

**Definition 4.2.** A collection of vectors \( \hat{v}_1, \ldots, \hat{v}_N \in \mathbb{R}^r \) forms a unit norm tight frame (UNTF) if the following conditions hold.

1. (Unit Norm) \( \| \hat{v}_i \|_2 = 1 \) for all \( i \in [N] \).
2. (Tight Frame) \( \sum_{i=1}^{N} \hat{v}_i \hat{v}_i^\top = \frac{N}{r} I_r \).

They moreover form an equiangular tight frame (ETF) if the following additional condition holds.

3. (Equiangular) There exists \( \mu \in [0,1] \) such that \( |\langle \hat{v}_i, \hat{v}_j \rangle| = \mu \) whenever \( i \neq j \).

ETFs are rare and combinatorially structured objects [STDHJ07, CRT08, FM15], which do not seem a priori related to the SK problem and the Montanari-Sen witness. However, it turns out that studying the degree 4 extensions of ETFs gives useful insight into the correct construction of pseudomoments even in the random case.

In general, we showed in [BK18] that degree 4 extensions are related to the following notion from convex geometry.

**Definition 4.3.** Let \( K \subseteq \mathbb{R}^d \) be a closed convex set. For \( M \in K \), the perturbation of \( M \) in \( K \) is the subspace

\[
\text{pert}_K(M) := \left\{ A \in \mathbb{R}^d : M \pm tA \in K \text{ for all } t > 0 \text{ sufficiently small} \right\}.
\]

The relevant case for degree 4 extensions is \( K = \mathcal{E}_2^N \). In this case, the following theorem characterizes the perturbation subspace.

**Proposition 4.4** (Theorem 1(a) of [LT94]). Let \( M \in \mathcal{E}_2^N \) have \( \text{rank}(M) = r \) and \( M = \hat{V}^\top \hat{V} \) for some \( \hat{V} \in \mathbb{R}^{r \times N} \) with unit vector columns \( \hat{v}_1, \ldots, \hat{v}_N \). Then,

\[
\text{pert}_{\mathcal{E}_2^N}(X) = \left\{ \hat{V}^\top S \hat{V} : S \in \mathbb{R}^{r \times r}_{\text{sym}} \cap \left\{ A \in \mathbb{R}^{N \times N} : \text{diag}(A) = 0 \right\} \right\} = \left\{ \hat{V}^\top S \hat{V} : S \in \mathbb{R}^{r \times r}_{\text{sym}}, \hat{v}_i^\top S \hat{v}_i = 0 \text{ for } i \in [N] \right\} = \hat{V}^\top \left( \text{span}\left( \{ \hat{v}_1^\top, \ldots, \hat{v}_N^\top \} \right) \right)^\perp \hat{V}.
\]

For ETFs, a degree 4 extension, when any exists, is given explicitly in both spectral and entrywise terms as follows.

**Theorem 4.5** (Theorem 2.19 of [BK18]). Let \( M \in \mathcal{E}_2^N \) be the Gram matrix of an ETF of \( N \) vectors in \( \mathbb{R}^r \). Then, \( M \in \mathcal{E}_4^N \) if and only if \( N < \frac{r(r+1)}{2} \). In this case, a degree 4 extension \( Z \in \mathbb{R}^{N(N+1)/2 \times N(N+1)/2} \) is given by

\[
Z = \text{entryvec}(M) \text{entryvec}(M)^\top + \frac{N^2(1 - \frac{1}{r})}{\frac{r(r+1)}{2} - N} \text{P}_{\text{entryvec}(\text{pert}_{\mathcal{E}_2^N}(M))}.
\]
(Note that entryvec(pert_{E^N}(M)) = isovec(pert_{E^N}(M)), since matrices of pert_{E^N}(M) are identically zero on the diagonal.) The entries of this matrix are given by

\[ Z_{\{ij\}\{k\ell\}} = \frac{r(r-1)}{r(r+1)}(M_{ij}M_{k\ell} + M_{ik}M_{j\ell} + M_{i\ell}M_{jk}) - \frac{r^2(1 - \frac{1}{N})}{r(r+1)} - \frac{N}{N} \sum_{m=1}^N M_{im}M_{jm}M_{km}M_{\ell m}. \]  

(30)

(The spectral description will be of interest later, to draw a connection to the second heuristic presented in the following section.)

The Montanari-Sen witness \( M \) is close to a constant multiple of a projection matrix, since by Proposition 3.1 all entries of the normalizing diagonal matrix \( D \) are close to \( \delta \), so \( M \) is a “near-UNTF Gram matrix.” Also, the off-diagonal entries of \( M \) are inner products of random unit vectors \( \hat{v}_i \), which it is reasonable to think are weakly dependent and therefore \( M \) should moreover behave like “an ETF in expectation.”

Thus to guess a degree 4 extension for the Montanari-Sen witness \( M \), we may be justified in simply trying to apply the combinatorial ETF construction directly. Since we take \( r = \delta N \) with \( r, N \to \infty \), we may also simplify the leading coefficients to their asymptotic values, which gives the prediction

\[ "Z_{\{ij\}\{k\ell\}} = M_{ij}M_{k\ell} + M_{ik}M_{j\ell} + M_{i\ell}M_{jk} - 2\sum_{m=1}^N M_{im}M_{jm}M_{km}M_{\ell m}.\]  

(31)

4.2 Heuristic 2: Conditional Covariance of Gaussian Matrices

We next show another, perhaps more principled argument through which we arrive at the same prediction of degree 4 pseudomoments. Let us suppose that \( M \) is exactly the Gram matrix of a UNTF, i.e. \( M = \delta^{-1}P = \delta^{-1}V^\top V \) and \( \text{diag}(P) = \delta 1_N \). As we will discuss below, this is approximately the case for the Montanari-Sen witness, but our heuristic derivation is much simplified if the correction by the diagonal matrix \( D \) to form the actual Montanari-Sen witness may be reduced to a constant scaling.

Now, we take the perspective that \( Z \), a positive semidefinite \( \frac{N(N+1)}{2} \times \frac{N(N+1)}{2} \) matrix, is the degree 2 moment matrix of the entries of a gaussian random matrix: there exists some \( A \in \mathbb{R}^{N \times N}_{\text{sym}} \) with random jointly gaussian entries on and above the diagonal, such that

\[ Z_{\{ij\}\{k\ell\}} = \mathbb{E}[A_{ij}A_{k\ell}]. \]  

(32)

We then design \( A \) so that \( Z \) automatically satisfies some of the necessary constraints, and hope that the remaining constraints will then be approximately satisfied as well.

We begin with a matrix \( A^{(0)} \) having a canonical gaussian distribution for symmetric matrices, the GOE, suitably rescaled to allow us a normalizing degree of freedom later: \( A_{ij}^{(0)} \sim \mathcal{N}(0,2\sigma^2) \) and \( A_{ij}^{(0)} = A_{ji}^{(0)} \sim \mathcal{N}(0,\sigma^2) \). Next, we take \( A \) to have the distribution of \( A^{(0)} \), conditional on the following two properties:

1. \((I_N - P)A = 0.\)
2. \(A_{ii} = 1\) for all \( i \in [N].\)
Property 1 ensures that any symmetric matrix formed from \( Z \) by “freezing” two indices \( i, j \) and letting the others vary, \( Z^{(ij)} := (Z_{\{i\}\{j\}\{k\}})_{k,l=1}^N \), has row (or column) space contained in that of \( M \). Every degree 4 pseudomoment matrix \( Z \) must satisfy this property: if \( Z \) extends \( M \), then \( Z \) has a principal minor of the form

\[
\begin{bmatrix}
M & Z^{(ij)} \\
Z^{(ij)} & M
\end{bmatrix}
\]

(with repeated rows and columns deleted),

(33)

whose positive semidefiniteness gives this condition on \( Z \). Property 2 ensures that \( Z_{\{i\}\{i\}\{k\}} \) does not depend on the index \( i \), one of the basic conditions on a degree 4 pseudomoment matrix per Proposition 1.7.

What is the law of the resulting gaussian matrix \( A \)? Conditioning on Property 1 yields the law of \( PA(0)P = V(V^T A(0)V)V^T \). By rotational invariance of the GOE, the inner matrix \( V^T A(0)V =: A^{(1)} \in \mathbb{R}^{r \times r}_{\text{sym}} \) has the same law as the upper left \( r \times r \) block of \( A(0) \), i.e. a smaller GOE matrix with the same variance scaling of \( \sigma^2 \).

It remains to condition on Property 2, or equivalently to condition \( A^{(1)} \) on \( \langle v_1^T A^{(1)} v_i, A^{(1)} \rangle = 1 \). \( A^{(1)} \) has the law of \( \text{isovec}^{-1}(a) \) for \( a \sim N(0, 2\sigma^2 I_{(r+1)/2}) \). Since \( \text{isovec} \) is an isometry, we may equivalently condition \( a \) on \( \langle a, \text{isovec}(v_i v_i^T) \rangle = 1 \) for each \( i \in [N] \). By basic properties of gaussian conditioning, the resulting law is

\[
\mathcal{N} \left( \sum_{i=1}^N ((P^{o2})^{-1} 1_i) \text{isovec}(v_i v_i^T), 2\sigma^2 (I - \tilde{P}) \right),
\]

where \( P^{o2} \) is the Gram matrix of the \( \text{isovec}(v_i v_i^T) \) or equivalently the entrywise square of \( P \), and \( \tilde{P} \) is the orthogonal projector to the span of the \( \text{isovec}(v_i v_i^T) \). Let \( A^{(2)} \) be a gaussian matrix with the law of \( \text{isovec}^{-1} \) applied to the law given in (34).

Now, having finished the conditioning calculations, we obtain the statistics of \( A \). Recall that \( A_{ij} = v_i^T A^{(2)} v_j = \langle \frac{1}{2}(v_i v_j^T + v_j v_i^T), A^{(2)} \rangle \). Applying \( \text{isovec} \) to each matrix and using the expression derived above, we find the mean and covariance

\[
\mathbb{E}[A_{ij}] = \sum_{m=1}^N ((P^{o2})^{-1} 1_k) P_{im} P_{jm},
\]

(35)

\[
\text{Cov}[A_{ij}, A_{kl}] = \frac{1}{2} \sigma^2 \text{isovec} \left( v_i v_j^T + v_j v_i^T \right)^T (I - \tilde{P}) \text{isovec} \left( v_k v_l^T + v_l v_k^T \right) .
\]

(36)

Now, we make two simplifying approximations. For the means, we approximate

\[
P^{o2} \approx \delta I + \delta \frac{1}{r} N 1_N^T ,
\]

(37)

which gives

\[
\mathbb{E}[A_{ij}] \approx \delta^{-1} P_{ij} = M_{ij}.
\]

(38)

For the covariances, since under our tight frame assumption we have \( \| \text{isovec}(v_i v_i^T) \|_2 = \| v_i v_i^T \|_F = \| v_i \|_2 = \delta \), we approximate

\[
\tilde{P} \approx \delta^{-2} \sum_{i=1}^N \text{isovec}(v_i v_i^T) \text{isovec}(v_i v_i^T)^T ,
\]

(39)
which gives
\[
\text{Cov}[A_{ij}, A_{k\ell}] \approx \sigma^2 \left( P_{ik} P_{j\ell} + P_{i\ell} P_{jk} - 2\delta^{-2} \sum_{m=1}^{N} P_{im} P_{jm} P_{km} P_{\ell m} \right).
\]

Finally, to recover what this predicts for the entries of \( Z \), we compute
\[
Z_{\{ij\}\{k\ell\}} = \mathbb{E}[A_{ij}A_{k\ell}] = \mathbb{E}[A_{ij}] \mathbb{E}[A_{k\ell}] + \text{Cov}[A_{ij}, A_{k\ell}]
= M_{ij}M_{k\ell} + \sigma^2 \left( P_{ik} P_{j\ell} + P_{i\ell} P_{jk} - 2\delta^{-2} \sum_{m=1}^{N} P_{im} P_{jm} P_{km} P_{\ell m} \right).
\]
We then choose \( \sigma^2 \) such that \( Z_{\{ii\}\{ii\}} = 1 \), which requires \( \sigma^2 = \delta^{-2} \), and we recover the same formula as (31):
\[
"Z_{\{ij\}\{k\ell\}} = M_{ij}M_{k\ell} + M_{ik}M_{j\ell} + M_{i\ell}M_{jk} - 2\sum_{m=1}^{N} M_{im}M_{jm}M_{km}M_{\ell m}."
\]

**Remark 4.6.** It is worth noting the intriguing geometric interpretation of the random matrix \( A \) we have constructed: we have \( \mathbb{E}A = M \), \( \text{diag}(A) = 1 \) deterministically, and \( A \) fluctuates in the linear subspace \( \text{pert}_{\delta^2}^N(M) \) (as may be verified from the covariance formula (36) and is intuitive by analogy with the ETF case of Section 4.1). Thus, \( A \) behaves, roughly speaking, like a random element of \( \delta^2_N \) (except that there is no enforcement of positive semidefiniteness), which lies on the same face of \( \delta^2_N \) as \( M \) and fluctuates gaussianly about \( M \) along this face.

### 4.3 Precise Construction Details

Having intuitively motivated the \textit{a priori} unusual degree 4 pseudomoment formula given (identically) in (31) and (42) in the previous two sections, we now give the precise details of how this may be adjusted to produce an actually valid degree 4 pseudomoment extension of \((1 - \alpha)M + \alpha I_N\), the “nudged” Montanari-Sen witness. It is instructive to view our construction as first attempting to build an extension of \( M \) itself, then introducing the adjustment towards \( I_N \) as a necessity to ensure positive semidefiniteness.

**Step 1: Heuristic pseudomoments.** We first build \( X \in \mathbb{R}^{N(N+1)/2 \times N(N+1)/2} \) that is a reasonable prediction of a degree 4 extension of \( M \). View \( X \) as a 2×2 block matrix, with block dimensions \( N \) and \( N(N-1)/2 \) corresponding to “diagonal” (\( \{ii\} \)) and “off-diagonal” (\( \{ij\} \) for \( i < j \)) index pairs, respectively. Then, the upper left, upper right, and bottom left blocks are prescribed by the property of extending \( M \):
\[
X = \begin{bmatrix}
1_N 1_N^\top & 1_N \text{offdiag}(M)^\top \\
\text{offdiag}(M) 1_N^\top & X^{[2,2]} \end{bmatrix}.
\]
We complete the definition by defining \( X^{[2,2]} \) using the heuristics described earlier. In particular, we take
\[
X^{[2,2]}_{\{ij\}\{k\ell\}} := M_{ij}M_{k\ell} + M_{ik}M_{j\ell} + M_{i\ell}M_{jk} - 2\sum_{m=1}^{N} M_{im}M_{jm}M_{km}M_{\ell m}.
\]
Step 2: Correction to satisfy linear constraints. We next correct $X$ to satisfy exactly all linear constraints required for a degree 4 pseudomoment matrix. Define an additive correction $\Delta \in \mathbb{R}^{N(N-1)/2 \times N(N-1)/2}$ by

$$\Delta_{\{ij\}\{k\ell\}} = 0 \text{ if } |\{i, j, k, \ell\}| = 4,$$

$$\Delta_{\{ik\}\{i\ell\}} = \sum_{m=1}^{N} \sum_{m \neq i} M_{im} M_{km} M_{\ell m}. \quad (46)$$

(Note that the second part of the definition is consistent when $k = \ell$ regardless of whether we view $i$ or $k$ as the repeated index.)

Then, we set

$$Y := X + \begin{bmatrix} 0 & 0 \\ 0 & 2\Delta \end{bmatrix},$$

and $Y$ satisfies all linear constraints on a degree 4 pseudomoment matrix exactly. However, as we will see, $X \succeq 0$ (which, as we will see, is nearly true), $Y \not\succeq 0$ regardless due to the fluctuations in $\Delta$.

Step 3: Correction to satisfy positive semidefiniteness. Finally, we introduce a second correction to counteract the fluctuations in the spectra of $X^{[2,2]}$ and $\Delta$. We use the following degree 4 pseudomoment matrix extending the identity matrix:

$$Y^{(id)} := \frac{1}{2^N} \sum_{x \in \{\pm 1\}^N} \text{entryvec}(xx^T) \text{entryvec}(xx^T)^T = \begin{bmatrix} 1_N 1_N^T & 0 \\ 0 & I_{N(N-1)/2} \end{bmatrix}. \quad (48)$$

$Y^{(id)}$ is a degree 4 pseudomoment matrix, which extends $I_N \in \mathcal{C}^N$. As (48) shows, $I_N$ is a natural choice of a point of $\mathcal{C}^N$ towards which to “push” $Y$ to enforce positive semidefiniteness, because it lies at the barycenter of the vertices of $\mathcal{C}^N$.

Then, for a given setting of the parameter $\alpha \in (0, 1)$, we set

$$Z := (1 - \alpha)Y + \alpha Y^{(id)}$$

$$= \begin{bmatrix} 1_N 1_N^T & (1 - \alpha)1_N \text{offdiag}(M)^T \\ (1 - \alpha)\text{offdiag}(M)1_N^T & \alpha I_{N(N-1)/2} + (1 - \alpha)(X^{[2,2]} + 2\Delta) \end{bmatrix}. \quad (49)$$

Clearly, $Z$ extends $(1 - \alpha)M + \alpha I_N \in \mathcal{E}^N_2$ and satisfies all linear constraints on a degree 4 pseudomoment matrix (since both $Y$ and $Y^{(id)}$ do so). Thus to show $(1 - \alpha)M + \alpha I_N \in \mathcal{E}^4_4$, it suffices to show $Z \succeq 0$, in which case $Z$ will be a degree 4 pseudomoment extension. Theorem 2.1 will then be proved if we show $Z \succeq 0$ with high probability.

5 Conjectural Higher-Degree Extension

Before proceeding to the proofs, we mention that there is a natural extension of the heuristic for pseudomoment construction of Section 4.2 that appears promising, though difficult to analyze, for higher-degree SOS relaxations.

The idea is to view higher-order pseudomoments as the second moments of symmetric gaussian tensors, which, as was done in Section 4.2 for matrices, are formed by conditioning a certain
canonical symmetric tensor distribution on desirable properties. Suppose we want to build a degree \( d = 2k \) pseudomoment extension \( \mathbf{Z}^{(k)} \in \mathbb{R}^{\mathcal{M}^k([N]) \times \mathcal{M}^k([N])} \) of the Montanari-Sen witness \( \mathbf{M} \in \mathcal{E}_2^N \) (which, as before, we assume to be an exact unit norm tight frame, i.e. a constant multiple of \( \mathbf{P} \)). We do this by building a symmetric tensor \( \mathbf{A}^{(k)} \in \text{Sym}^k(\mathbb{R}^N) \) with jointly gaussian entries, and setting, for \( s, t \in \mathcal{M}^k([N]) \),

\[
Z^{(2k)}_{st} = \mathbb{E}[A_s^{(k)} A_t^{(k)}].
\] (50)

(Note that since \( \mathbf{A}^{(k)} \) is a symmetric tensor, its indices, like those of \( \mathbf{Z} \), may be interpreted as multisets rather than ordered tuples.)

First, we define the following tensorial generalization of the GOE (see e.g. [RM14] for favorable properties of this distribution analogous to those of the GOE).

**Definition 5.1.** Let \( \mathbf{G} \in \bigotimes^k \mathbb{R}^N \) have i.i.d. entries distributed as \( \mathcal{N}(0,1) \). Then, write \( \mathcal{G}^{N,k}(\sigma^2) \) for the law of \( \mathbf{A} \in \text{Sym}^k(\mathbb{R}^N) \subset \bigotimes^k \mathbb{R}^N \) defined by

\[
A_i = \frac{\sigma}{k!} \sum_{\pi \in S_k} G_{i_{\pi(1)}i_{\pi(2)}\cdots i_{\pi(k)}}.
\] (51)

Now, we define \( \mathbf{A}^{(k)} \) inductively over \( k \) as a family of coupled gaussian tensors, and ensure that the pseudomoment matrices thus formed are consistent with one another. Namely, we proceed as follows.

1. Let \( A^{(0)} = 1 \).

2. For \( k \geq 1 \), let \( \mathbf{A}^{(k)} \) have the law \( \mathcal{G}^{N,k}(\sigma_k^2) \), conditional on the following two properties:
   - (Subspace Property) For \( s \in [N]^{k-1} \), define \( \mathbf{A}^{(k)}[s] := (A_{s+t})_{i=1}^N \in \mathbb{R}^N \). Then, for all \( s \in [N]^{k-1} \), \( (\mathbf{I} - \mathbf{P})\mathbf{A}^{(k)}[s] = \mathbf{0} \).
   - (Consistency Property) For all \( i \in [N] \) and \( s \in [N]^{k-2} \), \( A^{(k)}_{s+\{ii\}} = A^{(k-2)}_s \).

The constants \( \sigma_k^2 \) remain as free parameters to be tuned to ensure normalization, as in the case \( k = 2 \) from Section 4.2.

Based on this reasonable generalization, we offer two conjectures. First, we believe that whatever adjustments are necessary to this construction are already captured in the simple adjustment of the Montanari-Sen witness towards the identity matrix given in Theorem 2.1.

**Conjecture 5.2.** For any \( \alpha, \delta \in (0,1) \) and \( d \in 2\mathbb{N} \),

\[
\lim_{N \to \infty} \mathbb{P}\left[ \mathbf{M}^{(\alpha, \delta)}(\mathbf{W}) \in \mathcal{E}_d^N \right] = 1.
\] (52)

More specifically but less formally, we believe the construction presented above is approximately the correct degree \( d \) pseudomoment extension.

**Conjecture 5.3** (Informal). For \( \delta \in (0,1) \), let \( \mathbf{M}^{(\delta)} \) be the Montanari-Sen witness. Then, for fixed \( d \in 2\mathbb{N} \) and constants \( \sigma_k^2 \) depending only on \( \delta \), with high probability as \( N \to \infty \), \( \mathbf{Z}^{(d)} \) as defined above is “nearly” a degree \( d \) pseudomoment extension of \( \mathbf{M}^{(\delta)} \).

We have tested Conjecture 5.3 numerically on Laurent’s construction in [Lau03] of higher degree pseudomoment matrices extending a simple deterministic \( \mathbf{M} \in \mathcal{E}_2^N \), which indeed forms the Gram matrix of an ETF. Laurent’s construction shows that certain parity inequalities holding over \( \mathcal{E}_N^N \) are not certified by SOS until \( d \sim N \). We find that the results, for suitable tuning of \( \sigma_k^2 \),
agree with Laurent’s construction (with no further adjustment needed). Thus one pleasant consequence of verifying Conjecture 5.3 would be a simplified proof of Laurent’s theorem, whose proof in [Lau03] involves first predicting the entries of \( Z^{(d)} \) and then appealing to a technical analysis of hypergeometric functions to verify positive semidefiniteness of \( Z^{(d)} \).

Let us remark on what seems to be the major difficulty in analyzing this construction. By analogy with the analysis in Section 4.2, we are eventually led, in conditioning on the Consistency Property, to attempt to approximate the orthogonal projector to the “repeated indices subspace”

\[
V^{(k)} = \text{span} \left( \left\{ v_i \odot v_i \odot v_{j_1} \odot \cdots \odot v_{j_{k-2}} : i \in [N], j \in [N]^{k-2} \right\} \right) \subset \text{Sym}^k(\mathbb{R}^n).
\]

(Here \( \odot \) denotes the symmetric product of tensors; see e.g. [SKM89] for definitions. We mean “orthogonal projection” with respect to the Frobenius or entrywise inner product of general non-symmetric tensors, into which symmetric tensors are embedded by repeating entries.) When \( k = 2 \), the spanning set consists of the \( N \) linearly independent and roughly orthogonal tensors \( v_i^{\odot 2} \), which allows the orthogonal projection to be estimated by the sum of rank one projections as in (39). However, when \( k \geq 3 \), the spanning set is no longer even linearly independent, since the \( v_i \) themselves are an overcomplete set in \( \mathbb{R}^n \) (and the symmetric tensor product is distributive, so any dependence among the \( v_i \) is inherited by the \( A \odot v_i \) for any symmetric tensor \( A \)). Thus it is unclear what approximately-orthogonal basis to choose to carry out the calculation, or how to correct the approximation of (39) for these dependencies.

6 Proof of Positive Semidefiniteness: First Steps

Recall that, in Section 4.3, we built from the Montanari-Sen witness \( M \) and an additional constant \( \alpha \in (0, 1) \) the matrix

\[
Z = \begin{bmatrix}
Z^{[1,1]} & Z^{[1,2]} \\
Z^{[2,1]} & Z^{[2,2]}
\end{bmatrix}
= \begin{bmatrix}
1_N 1_N^\top \\
(1 - \alpha) \text{offdiag}(M) 1_N^\top
\end{bmatrix}
\begin{bmatrix}
(1 - \alpha) 1_N \text{offdiag}(M)^\top \\
\alpha I_{(N-1)/2} + (1 - \alpha)(X^{[2,2]} + 2\Delta)
\end{bmatrix},
\]

and found that to prove Theorem 2.1 it suffices to show that \( Z \succeq 0 \) with high probability. We now give some technical preliminaries for the proof of this.

First, we will reduce the dimensionality of the task of showing \( Z \succeq 0 \) by taking the Schur complement criterion for positive semidefiniteness with respect to the upper left block. Since \( Z \) and in particular \( Z^{[1,1]} \) are rank-deficient, we must first check compatibility of the null spaces, \((I_N - Z^{[1,1]} Z^{[1,1]^\top}) Z^{[1,2]} = 0\). This is straightforward since \( Z^{[1,1]} \) has rank one. The remaining condition is the positive semidefiniteness of the Schur complement,

\[
Z/Z^{[1,1]} = Z^{[2,2]} - Z^{[1,2]} Z^{[1,1]^\top} Z^{[1,2]}
= \alpha I_{(N-1)/2} + (1 - \alpha)(X^{[2,2]} + 2\Delta)
- (1 - \alpha)^2 \text{offdiag}(M) 1_N^\top (1_N 1_N^\top) 1_N \text{offdiag}(M)^\top
= \alpha I_{(N-1)/2} + (1 - \alpha)(X^{[2,2]} + 2\Delta) - (1 - \alpha)^2 \text{offdiag}(M) \text{offdiag}(M)^\top
\succeq 0.
\]


We reorganize this expression as

\[
Z/Z^{[1,1]} = Z^{(1)} + Z^{(2)}, \quad \text{where}
\]

\[
Z^{(1)} = \frac{1}{2} \alpha I_{N(N-1)/2} + (1 - \alpha) \left( X^{[2,2]} - (1 - \alpha) \text{offdiag}(M) \text{offdiag}(M)^\top \right),
\]

\[
Z^{(2)} = \frac{1}{2} \alpha I_{N(N-1)/2} + 2(1 - \alpha) \Delta.
\]

To show \(Z \succeq 0\), it then suffices to show that both \(Z^{(1)} \succeq 0\) and \(Z^{(2)} \succeq 0\). We refer to these as the “main term” and the “correction term,” respectively.

In this decomposition, we split between \(Z^{(1)}\) and \(Z^{(2)}\) the extra term \(\alpha I\) that we introduced when nudging our pseudomoment matrix towards the identity. This term will act as a barrier against small fluctuations that might spoil positive semidefiniteness. We will show that without this adjustment \(Z^{(1)}\) and \(Z^{(2)}\) are nearly positive semidefinite already, having the magnitude of their smallest (most negative) eigenvalue tending to zero as \(N \to \infty\) for any fixed \(\delta\). Thus any choice of \(\alpha > 0\) will suffice to ensure that \(Z \succeq 0\) with high probability.

More specifically, we will show the following results.

**Lemma 6.1 (Control of Main Term).** For all \(\delta \in (0, 1)\),

\[
\lim_{N \to \infty} \mathbb{P} \left[ \left\| \text{min} \left\{ 0, \lambda_{\text{min}} \left( Z^{(1)} \right) \right\} \right\| \leq O_\delta \left( \frac{\log N}{N^{1/4}} \right) \right] = 1.
\]

**Lemma 6.2 (Control of Correction Term).** For all \(\delta \in (0, 1)\),

\[
\lim_{N \to \infty} \mathbb{P} \left[ \|\Delta\|_{\text{op}} \leq O_\delta \left( \frac{\log^2 N}{N^{1/4}} \right) \right] = 1.
\]

From Lemmas 6.1 and 6.2, it follows that with high probability \(Z \succeq 0\) for any \(\alpha \in (0, 1)\) fixed as \(N \to \infty\) (or for \(\alpha = \alpha(N)\) decreasing sufficiently slowly with \(N\), though for the sake of simplicity we will not pursue this minor strengthening of the results). Theorem 2.1 then follows. It remains only to prove the Lemmas; in Section 7 we will prove Lemma 6.1, and in Section 8 we will prove Lemma 6.2.

### 7 Proof of Positive Semidefiniteness: Main Term

In this section we prove Lemma 6.1. We have

\[
Z^{(1)}_{\{ij\}\{kl\}} = X^{[2,2]}_{\{ij\}\{kl\}} - (1 - \alpha) M_{ij} M_{kl} = \alpha M_{ij} M_{kl} + M_{ik} M_{jl} + M_{il} M_{jk} - 2 \sum_{m=1}^{N} M_{im} M_{jm} M_{km} M_{\ell m}.
\]

Consider the quadratic form \(a^\top Z^{(1)} a\), where we think of \(a_{ij} = \sqrt{2} A_{ij}\) for some symmetric matrix \(A\) with \(\text{diag}(A) = 0\) (i.e. \(a = \sqrt{2} \cdot \text{offdiag}(A)\)). Writing \(m_1, \ldots, m_N \in \mathbb{R}^N\) for the columns of \(M\),

\[
a^\top Z^{(1)} a = \alpha \text{Tr}(A M)^2 + 2 \text{Tr}(A M A M) - 2 \sum_{i=1}^{N} (m_i^\top A m_i)^2
\]

\[
= \alpha \text{Tr}(\hat{V} A \hat{V}^\top)^2 + 2 \|\hat{V} A \hat{V}^\top\|_F^2 - 2 \sum_{i=1}^{N} (\hat{v}_i^\top \hat{V} A \hat{V}^\top \hat{v}_i)^2.
\]
Writing the above as a quadratic form in isovec($\hat{V}A\hat{V}^\top$), we obtain the following.

**Proposition 7.1.** Define

$$\bar{Z}^{(1a)} := \alpha I_{\text{diag}}^\top + 2I_{r(r+1)/2} - 2\sum_{i=1}^N \text{isodec}(\bar{v}_i\bar{v}_i^\top)\text{isodec}(\bar{v}_i\bar{v}_i^\top)^\top. \tag{63}$$

(Recall that $1_{\text{diag}} := \text{isodec}(I_r).$) Then,

$$\lambda_{\min}(Z^{(1a)}) \geq \min \left\{ 0, \frac{\lambda_{\min}(\bar{Z}^{(1a)})}{\min_{i \in [N]} D_{ii}^2} \right\}. \tag{64}$$

**Proof.** Since the right-hand side of (64) is at most zero, it suffices to consider the case that $\lambda_{\min}(Z^{(1a)}) < 0$. By (62), we have

$$\text{isodec}(\hat{A})^\top Z^{(1a)} \text{isodec}(\hat{A}) = \text{isodec}(\hat{V}A\hat{V}^\top)^\top \bar{Z}^{(1a)} \text{isodec}(\hat{V}A\hat{V}^\top). \tag{65}$$

Note first that

$$\|\hat{V}A\hat{V}^\top\|_F^2 = \text{Tr}(AMAM) = \text{Tr}(AD^{-1/2}PD^{-1/2}AD^{-1/2}PD^{-1/2}) = \langle P, D^{-1/2}AD^{-1/2}PD^{-1/2}AD^{-1/2} \rangle \leq \text{Tr}((D^{-1/2}AD^{-1/2})(D^{-1/2}AD^{-1/2})) = \|D^{-1/2}AD^{-1/2}\|_F^2 \leq \frac{\|A\|_F^2}{\min_{i \in [N]} D_{ii}^2}. \tag{66}$$

Then, recalling that $\|a\|_2^2 = \|A\|_F^2$ and $\|\text{isodec}(\hat{V}A\hat{V}^\top)\|_2^2 = \|\hat{V}A\hat{V}^\top\|_F^2$, by the variational description of the minimum eigenvalue we have

$$\lambda_{\min}(Z^{(1a)}) = \min_{a \in \mathbb{R}^N[N^{-1/2}\{0\}]} \frac{a^\top Z^{(1a)}a}{\|a\|_2^2} \geq \frac{1}{\min_{i \in [N]} D_{ii}^2} \min_{a \in \mathbb{R}^N[N^{-1/2}\{0\}]} \frac{a^\top \bar{Z}^{(1a)}a}{\|\hat{V}A\hat{V}^\top\|_F^2} \left(\text{by (66)}\right) \tag{67}$$

completing the proof.

We will thus focus our attention on $\bar{Z}^{(1a)}$. Analyzing the sample covariance or Wishart-type matrix formed by the third term of (63), $\sum_{i=1}^N \text{isodec}(\bar{v}_i\bar{v}_i^\top)\text{isodec}(\bar{v}_i\bar{v}_i^\top)^\top$, will be our main difficulty.
Since \( \mathbb{E} \hat{v}_i \hat{v}_i^\top = \frac{1}{r} I_r \), we center the vectors involved, and decompose this term as

\[
\tilde{Z}^{(1a)} = \left( \alpha - \frac{2N}{r^2} \right) 1_{\text{diag}} 1_{\text{diag}}^\top + 2 I_{r(r+1)/2} \\
+ \frac{2}{r} \left( \text{isovec} \left( \sum_{i=1}^N \hat{v}_i \hat{v}_i^\top - \frac{N}{r} I_r \right) \right) 1_{\text{diag}} 1_{\text{diag}} \text{isovec} \left( \sum_{i=1}^N \hat{v}_i \hat{v}_i^\top - \frac{N}{r} I_r \right) \\
- 2 \sum_{i=1}^N \text{isovec} \left( \hat{v}_i \hat{v}_i^\top - \frac{1}{r} I_r \right) \text{isovec} \left( \hat{v}_i \hat{v}_i^\top - \frac{1}{r} I_r \right)^\top 
\]

(68)

\[ T^{(1)} \]

\[ T^{(2)} \]

The remaining analysis involves a delicate balance between requirements in controlling \( T^{(1)} \) and \( T^{(2)} \). In order to bound the cross-term \( T^{(1)} \), we will rely on the strong concentration of the eigenvalues of \( \tilde{V} \tilde{V}^\top \) that is created by the dependencies among the \( \hat{v}_i \) (this is the “near-UNTF Gram matrix” behavior of \( M \)). In particular, this concentration is much stronger than if \( \hat{v}_i \) were replaced with any reasonable distribution of i.i.d. unit vectors, and this portion of our argument would fail for i.i.d. vectors (see Remark 7.3).

On the other hand, in order to bound the term \( T^{(2)} \), we will need to take advantage of the weak dependence of the \( \hat{v}_i \), and formalize the intuition that because \( N \ll r(r+1)/2 \) and \( T^{(2)} \) is a sum of weakly dependent rank-one orthogonal projectors, \( T^{(2)} \) should itself behave approximately as an orthogonal projector to a subspace of dimension \( N \) (though we will discuss one important caveat to this intuition in Remark 7.3). Technically, we will appeal to Lipschitz concentration inequalities for the Haar measure on Stiefel manifolds, which capture the heuristic weak dependence of entries of blocks of random orthogonal matrices under the Haar measure.

7.1 Bounding the Cross-Term \( T^{(1)} \)

Lemma 7.2. For all \( K > 0 \),

\[
P \left[ |T^{(1)}| \preceq O_{\delta,K} \left( \frac{\log N}{N} \right) I_{r(r+1)/2} + \frac{2}{r} 1_{\text{diag}} 1_{\text{diag}}^\top \right] \geq 1 - O_{\delta,K} (N^{-K}).
\]

(69)

Proof. Applying the matrix arithmetic-geometric mean inequality,

\[
|T^{(1)}| \preceq \frac{2}{r} \text{isovec} \left( \sum_{i=1}^N \hat{v}_i \hat{v}_i^\top - \frac{N}{r} I_r \right) \text{isovec} \left( \sum_{i=1}^N \hat{v}_i \hat{v}_i^\top - \frac{N}{r} I_r \right)^\top + \frac{2}{r} 1_{\text{diag}} 1_{\text{diag}}^\top \\
\leq \frac{2}{r} \left\| \sum_{i=1}^N \hat{v}_i \hat{v}_i^\top - \frac{N}{r} I_r \right\|_F^2 I_{r(r+1)/2} + \frac{2}{r} 1_{\text{diag}} 1_{\text{diag}}^\top.
\]

(70)

Rewriting the norm appearing in the first term,

\[
\left\| \sum_{i=1}^N \hat{v}_i \hat{v}_i^\top - \frac{N}{r} I_r \right\|_F^2 = \left\| \tilde{V} \tilde{V}^\top - \frac{N}{r} I_r \right\|_F^2 = \left\| V D^{-1} V^\top - \delta^{-1} I_r \right\|_F^2 = \left\| D^{-1} - \delta^{-1} I_N \right\|_F^2,
\]

(71)
since \( \mathbf{V} \mathbf{V}^\top = \mathbf{I}_r \). By Proposition 3.1,
\[
\mathbb{P} \left[ \left( \delta^{-1} - O_{\delta,K} \left( \sqrt{\frac{\log N}{N}} \right) \right) \mathbf{I}_N \preceq \mathbf{D}^{-1} \preceq \left( \delta^{-1} + O_{\delta,K} \left( \sqrt{\frac{\log N}{N}} \right) \right) \mathbf{I}_N \right] \\
\geq 1 - O_{\delta,K}(N^{-K}).
\]
Thus with at least the same probability we have
\[
\| \mathbf{D}^{-1} - \delta^{-1} \mathbf{I}_N \|_F^2 \leq O_{\delta,K}(\log N),
\]
and the result follows.

\textbf{Remark 7.3.} Let us contrast the result of this section with the same analysis for i.i.d. vectors. Recall from Proposition 3.4 that the law of each \( \tilde{\mathbf{v}}_i \) is uniform over \( \mathbb{S}^{r-1} \). Take \( \tilde{\mathbf{v}}_i \sim \text{Unif}(\mathbb{S}^{r-1}) \) independent. Then, we compute
\[
\mathbb{E} \left\| \sum_{i=1}^N \tilde{\mathbf{v}}_i \tilde{\mathbf{v}}_i^\top - \frac{N}{r} \mathbf{I}_r \right\|_F^2 = N\mathbb{E}\|\tilde{\mathbf{v}}_1\|_2^4 + N(N-1)\mathbb{E} \langle \tilde{\mathbf{v}}_1, \tilde{\mathbf{v}}_2 \rangle^2 - \frac{N^2}{r} + N = \Omega(N).
\]
Thus the corresponding cross-term would have largest eigenvalue of order \( \delta^{-1} \), which in particular would not decay with \( N \).

Consequently, our previous intuition that \( \mathbf{T}^{(2)} = \sum_{i=1}^N \text{isovec}(\tilde{\mathbf{v}}_i \tilde{\mathbf{v}}_i^\top - \frac{1}{r} \mathbf{I}_r) \) isovec\((\tilde{\mathbf{v}}_i \tilde{\mathbf{v}}_i^\top - \frac{1}{r} \mathbf{I}_r)^\top \) should be an approximate orthogonal projector of rank \( N \) cannot be correct, since the putative basis vectors \( \text{isovec}(\tilde{\mathbf{v}}_i \tilde{\mathbf{v}}_i^\top - \frac{1}{r} \mathbf{I}_r) \) almost sum to zero. In the following section, we will show that this is in fact the only linear near-dependence of these vectors, and \( \mathbf{T}^{(2)} \) is still an approximate orthogonal projector, only of rank \( N - 1 \).

### 7.2 Bounding the Projection Term \( \mathbf{T}^{(2)} \): Unnormalized Case

Our strategy for bounding \( \mathbf{T}^{(2)} \) will proceed in two steps: first, we will bound the same matrix but constructed from the approximately normalized vectors \( \delta^{-1/2} \mathbf{v}_1, \ldots, \delta^{-1/2} \mathbf{v}_N \) in place of the strictly normalized vectors \( \tilde{\mathbf{v}}_1, \ldots, \tilde{\mathbf{v}}_N \), and then we will show that this replacement does not significantly affect the spectrum. In this section we perform the first, more difficult of these tasks. We will show the following result.

\textbf{Lemma 7.4.} Let \( \mathbf{A}^{\text{orth}} \) have isovec\((\delta^{-1} \mathbf{v}_i \mathbf{v}_i^\top - \frac{1}{r} \mathbf{I}_r) \) as its columns. Let \( \mathbf{P}_{1_N} := \mathbf{I}_N - \frac{1}{N} \mathbf{1}_N \mathbf{1}_N^\top \), the orthogonal projector to the subspace orthogonal to \( \mathbf{1}_N \). Then,
\[
\mathbb{P} \left[ \| \mathbf{A}^{\text{orth}} \mathbf{A}^{\text{orth}} - \mathbf{P}_{1_N} \|_{\text{op}} \leq O_{\delta} \left( \frac{\log N}{N^{1/4}} \right) \right] \geq 1 - \exp \left( -\Omega_{\delta}(N^{1/2}) \right).
\]

The argument will use the technique of union bounding over a net. Our main technical tool will be the following Lipschitz concentration inequality for the Haar measure of the \textit{Stiefel manifolds}.

The Stiefel manifolds are defined as follows:
\[
\text{Stief}(N,r) := \{ \mathbf{V} \in \mathbb{R}^{r \times N} : \mathbf{V} \mathbf{V}^\top = \mathbf{I}_r \}.
\]
In words, \( \text{Stief}(N,r) \) consists of the \( r \times N \) matrices with orthonormal rows. The Haar measure \( \text{Haar}(\text{Stief}(N,r)) \) is the unique measure on \( \text{Stief}(N,r) \) that is invariant under the action of \( \mathcal{O}(N) \) on
Stief$(N,r)$ by multiplication on the right. Equivalently, Haar$(\text{Stief}(N,r))$ is the measure obtained by restricting Haar$(\mathbb{O}(N))$ to the upper $r \times N$ matrix block.

These measures enjoy the following concentration inequality when $r < N$, obtained by standard arguments from logarithmic Sobolev or isoperimetric inequalities for the special orthogonal group $SO(N)$, of which Stief$(N,r)$ is a quotient when $r < N$ (see, e.g., the discussion following Theorem 2.4 of [Led01]).

**Proposition 7.5.** Suppose $1 \leq r < N$, and $F : \text{Stief}(N,r) \to \mathbb{R}$ has Lipschitz constant at most $L$ when $\text{Stief}(N,r)$ is endowed with the metric of the Frobenius matrix norm. Then, for an absolute constant $C > 0$,

$$
\mathbb{P}_{V \sim \text{Haar}(\text{Stief}(N,r))} \left[ |F(V) - \mathbb{E}F(V)| \geq t \right] \leq 2 \exp \left( - \frac{CNt^2}{L^2} \right). \tag{77}
$$

Note that since we have $r = \delta N$ with $\delta < 1$, we will always satisfy the hypothesis $r < N$; we will use this implicitly without further comment for the remainder of the proof.

**Proof of Lemma 7.4.** Note that $A^{(\text{orth})}1_N = \text{isovec}(\frac{N}{r}VV^\top - \frac{N}{r}I_r) = 0$; thus as suggested already in Remark 7.3, it is impossible for $A^{(\text{orth})}$ to act on $\mathbb{R}^N$ as an approximate isometric embedding, as we might naively expect from its weakly dependent columns. Our argument is more natural to carry out if we remove this caveat; therefore, let us define $A_0^{(\text{orth})}$ to have columns isovec$(\frac{N}{r}v_i v_i^\top - \frac{1 - \sqrt{\delta}}{r}I_r)$.

One may check that $\|A_0^{(\text{orth})}1_N\|_2 = \|1_N\|_2 = \sqrt{N}$, and that

$$
A_0^{(\text{orth})} = A^{(\text{orth})} + \frac{1}{N}1_N 1_N^\top. \tag{78}
$$

In particular, $A_0^{(\text{orth})} = A_{1_N}^{(\text{orth})} - I_N = A^{(\text{orth})} - P_{1_N}$, so it suffices to show the operator norm bound of (77) for $A_0^{(\text{orth})} - I_N$.

For $x \in \mathbb{R}^N$, let us denote $D_x := \text{diag}(x)$ for the course of this proof. Then,

$$
A_0^{(\text{orth})}x = \text{isovec} \left( \delta^{-1} \sum_{i=1}^{N} x_i v_i v_i^\top - \frac{1 - \sqrt{\delta}}{r} (1_N, x) I_r \right)
= \delta^{-1} \text{isovec} \left( V D_x V^\top - \frac{1 - \sqrt{\delta}}{N} (1_N, x) I_r \right). \tag{79}
$$

For $x, y \in \mathbb{R}^N$, define

$$
F_{x,y}(V) := \langle A_0^{(\text{orth})} x, A_0^{(\text{orth})} y \rangle. \tag{80}
$$

Then, recalling that $P := V^\top V$ is the orthogonal projector to the row space of $V$,

$$
F_{x,y}(V) = \delta^{-2} \left( V D_x V^\top - \frac{1 - \sqrt{\delta}}{N} (1_N, x) I_r, V D_y V^\top - \frac{1 - \sqrt{\delta}}{N} (1_N, y) I_r \right)
= \delta^{-2} \left[ \text{Tr}(D_x P D_y P) - \frac{1 - \sqrt{\delta}}{N} (1_N, x) \text{Tr}(P D_y) + (1_N, y) \text{Tr}(P D_x)) \right]
+ \frac{\delta(1 - \sqrt{\delta})^2}{N} (1_N, x) (1_N, y). \tag{81}
$$
Let us denote balls in Euclidean space by
\[ B(x, r) := \{ y \in \mathbb{R}^N : \|x - y\|_2 \leq r \}. \] (82)

Our first goal will be to obtain concentration bounds on \( F_{x,y}(V) \) when \( V \sim \text{Haar}(\text{Stief}(N, r)) \) for each fixed pair \((x, y) \in B(0, 1)^2\), by applying the Lipschitz concentration inequality.

**Lipschitz Constant of \( F_{x,y} \).**

**Claim 1:** Let \( x, y \in B(0, 1) \). Then,
\[
\text{Lip}(F_{x,y}) \leq 4\delta^{-2} \left[ \min \{ \|x\|_\infty, \|y\|_\infty \} + \frac{1}{\sqrt{N}} \right]. \] (83)

**Proof.** For \( V_1, V_2 \in \text{Stief}(N, r) \), letting \( P_1 = V_i^\top V_i \), we have using (81) and the triangle inequality
\[
\begin{align*}
\delta^2 |F_{x,y}(V_1) - F_{x,y}(V_2)| &= |\text{Tr}(D_x P_1 D_y P_1) - \text{Tr}(D_x P_2 D_y P_2)| \\
&= \left| \frac{(1 - \sqrt{\delta})}{N} \langle (1_N, x) \rangle |\text{Tr}(P_1 D_y) - \text{Tr}(P_2 D_y)| \\
&\quad + \frac{(1 - \sqrt{\delta})}{N} \langle (1_N, y) \rangle |\text{Tr}(P_1 D_x) - \text{Tr}(P_2 D_x)| \right|
\end{align*}
\]
then using that \( |\langle (1_N, x) \rangle| \leq \|x\|_1 \leq \sqrt{N} \) and likewise for \( y \),
\[
\begin{align*}
\leq |\text{Tr}(D_x P_1 D_y (P_1 - P_2))| + |\text{Tr}(D_x (P_1 - P_2) D_y P_2)| \\
&\quad + \frac{1}{\sqrt{N}} (|\text{Tr}((P_1 - P_2) D_x)| + |\text{Tr}((P_1 - P_2) D_y)|)
\leq (\|D_x P_1\|_F + \|D_x P_2\|_F)\|D_y (P_1 - P_2)\|_F + \frac{2}{\sqrt{N}} \|P_1 - P_2\|_F.
\end{align*}
\] (84)

Since \( P_i \) is an orthogonal projector for \( i \in \{1, 2\} \),
\[
\|D_x P_i\|_F = \langle D_x^2, P_i \rangle^{1/2} \leq (\text{Tr}[D_x^2])^{1/2} \leq 1.
\] (85)

We bound the other term by
\[
\|D_y (P_1 - P_2)\|_F = \langle D_y^2, (P_1 - P_2)^2 \rangle^{1/2} \leq \|y\|_\infty \|P_1 - P_2\|_F.
\] (86)

Combining these observations and a symmetric argument with \( x \) and \( y \) in opposite roles gives
\[
|F_{x,y}(V_1) - F_{x,y}(V_2)| \leq 2\delta^{-2} \left[ \min \{ \|x\|_\infty, \|y\|_\infty \} + \frac{1}{\sqrt{N}} \right] \|P_1 - P_2\|_F.
\] (87)

Lastly, we bound
\[
\|P_1 - P_2\|_F = \|V_1^\top V_1 - V_2^\top V_2\|_F \\
= \|(V_1 - V_2)^\top V_1 + V_2^\top (V_1 - V_2)\|_F \\
\leq \|(V_1 - V_2)^\top V_1\|_F + \|V_2^\top (V_1 - V_2)\|_F \\
= \left( \text{Tr} \left[ (V_1 - V_2)^\top V_1 V_1^\top (V_1 - V_2) \right] \right)^{1/2} + \left( \text{Tr} \left[ (V_1 - V_2)^\top V_2 V_2^\top (V_1 - V_2) \right] \right)^{1/2} \\
= 2\|V_1 - V_2\|_F,
\] (88)
where we have used that \( V_1 V_1^\top = V_2 V_2^\top = I_r \), and the result follows.
Therefore, and what is crucial to our argument, while for the worst-case \( x \in B(0, 1) \), namely \( x = e_i \) a standard basis vector, \( F_{x,x} \) will have Lipschitz constant \( O(1) \), for typical \( x \in B(0, 1) \), \( F_{x,x} \) will rather have Lipschitz constant \( \tilde{O}(N^{-1/2}) \). Moreover, the Lipschitz constant of \( F_{x,y} \) is comparable to the smaller of the Lipschitz constants of \( F_{x,x} \) and \( F_{y,y} \).

**Expectation of** \( F_{x,y}(V) \).

**Claim 2:** For \( x, y \in B(0, 1) \), \( \mathbb{E}_{V \sim \text{Haar}(\text{Stief}(N,r))} F_{x,y}(V) = \langle x, y \rangle + O_\delta(N^{-1}) \).

**Proof.** We have

\[
\delta^2 \mathbb{E}_{F_{x,y}(V)} = \mathbb{E} \left( V D_x V^\top V D_y V^\top \right) - \frac{1 - \sqrt{\delta}}{N} \mathbb{E} \langle V^\top V D_y \rangle - \frac{1 - \sqrt{\delta}}{N} \mathbb{E} \langle V^\top V D_x \rangle + \frac{\delta(1 - \sqrt{\delta})^2}{N} \langle (1_N, x) \rangle \langle (1_N, y) \rangle,
\]

and by either the moment formulae of Proposition A.1 or an argument from orthogonal invariance of Haar measure, we have \( \mathbb{E} V^\top V = \delta I_N \), whereby

\[
\mathbb{E} \left( V D_x V^\top V D_y V^\top \right) - \frac{1 - \sqrt{\delta}}{N} \langle (1_N, x) \rangle \langle (1_N, y) \rangle.
\]

Let us view \( V \sim \text{Haar}(\text{Stief}(N,r)) \) as the top \( r \times N \) block of \( Q \sim \text{Haar}(O(N)) \). Then, expanding the first term with the moment formulae of Proposition A.1,

\[
\mathbb{E} \left( V D_x V^\top V D_y V^\top \right) = \sum_{i,j=1}^{N} x_i y_j \left( \sum_{a,b=1}^{r} \mathbb{E} [ Q_{ai} Q_{bi} Q_{aj} Q_{bj} ] \right) = \sum_{i=1}^{N} x_i y_i \left( 3r \frac{1}{N(N+2)} + \frac{r(r-1)}{N(N+2)} + \sum_{1 \leq i < j \leq N} x_i y_j \left( \frac{r}{N(N+2)} - \frac{r(r-1)}{(N-1)(N+2)} \right) \right)
\]

\[
= \frac{\delta}{N + 2} \left( r + 1 + \frac{r-1}{N-1} \right) \sum_{i=1}^{N} x_i y_i + \frac{\delta}{N+2} \left( 1 - \frac{r-1}{N-1} \right) \left( \sum_{i=1}^{N} x_i \right) \left( \sum_{i=1}^{N} y_i \right)
\]

\[
= \left( \delta^2 + O_\delta(N^{-1}) \right) \langle x, y \rangle + (1 + O_\delta(N^{-1}) \delta(1 - \delta) N \langle (1_N, x) \rangle \langle (1_N, y) \rangle,
\]

and since \( \langle (x, y) \rangle \leq \|x\|_2\|y\|_2 \leq 1 \) and \( \langle (1_N, x) \rangle \cdot \langle (1_N, y) \rangle \leq N \), the result follows.

Combining Claim 1, Claim 2, and the concentration result Proposition 7.5, we find the following corollary on pointwise concentration of \( F_{x,y}(V) \).
Concentration of $F_{x,y}(V)$.

Claim 3: There exist constants $C_1, C_2 > 0$ depending only on $\delta$ such that, for any $x, y \in B(0,1)$,

$$
\Pr_{V \sim \text{Haar}(\text{Stief}(N,r))} \left[ \left| \langle A_0^{(\text{orth})} x, A_0^{(\text{orth})} y \rangle - \langle x, y \rangle \right| \geq \frac{C_1}{N} + t \right] \
\leq 2 \exp \left( - \frac{C_2 N t^2}{\min\{\|x\|_\infty, \|y\|_\infty\} + N^{-1/2} t^2} \right).
$$

(92)

This concludes the first part of the argument.

The remaining part of the argument is to apply a union bound of the probabilities controlled in Claim 3 over suitable nets of $B(0,1)$. We divide our task into a bound over sparse vectors and vectors with bounded largest entry, very similar to the technique in [Rud08, RV08] and especially [Ver11]. Introduce a parameter $\rho \in (0,1)$ to be chosen later. Define

$$
B_s := \{ y \in B(0,1) : \|y\|_0 \leq \rho N \},
$$

(93)

$$
B_b := \left\{ z \in B(0,1) : \|z\|_\infty \leq \frac{1}{\sqrt{\rho N}} \right\}.
$$

(94)

For any $x \in B(0,1)$, we define $y = y(x)$ and $z = z(x)$ by thresholding the entries of $x$, setting $y_i := x_i \mathbb{I}\{|x_i| > \frac{1}{\sqrt{\rho N}}\}$ and $z_i := x_i \mathbb{I}\{|x_i| \leq \frac{1}{\sqrt{\rho N}}\}$. Then, $x = y + z$, $y, z \in B_s$, and $z \in B_b$.

Introduce another parameter $\gamma \in (0,1)$ to be chosen later. Let $N_s \subset B_s$ and $N_b \subset B_b$ be $\gamma$-nets. By a standard bound (see, e.g., Lemma 9.5 of [LT13]), we may choose $|N_b| \leq \exp(2N/\gamma)$, and by the same bound applied to each choice of $\rho N$ support coordinates for an element of $B_s$, we may choose

$$
|N_s| \leq \left( \frac{N}{\rho N} \right)^{\frac{2\rho N}{\gamma}} \leq \exp \left( \frac{2\rho N}{\gamma} + \rho N + \log \left( \frac{1}{\rho} \right) \rho N \right).
$$

(95)

To lighten the notation, let us set $S := A_0^{(\text{orth})\top} A_0^{(\text{orth})} - I_N$. The following is an adaptation to our setting of a standard technique for estimating a matrix norm over a net: we first bound

$$
\|S\|_{\text{op}} = \max_{x \in B(0,1)} |x^\top S x| \
\leq \max_{y \in B_s} \max_{z \in B_b} (y + z)^\top S(y + z) \
\leq \max_{y \in B_s} |y^\top S y| + \max_{z \in B_b} |z^\top S z| + 2 \max_{y \in B_s} \max_{z \in B_b} |y^\top S z| \
\leq \max_{y \in N_s} |y^\top S y| + \max_{z \in N_b} |z^\top S z| + 2 \max_{y \in N_s} \max_{z \in N_b} |y^\top S z| + 12\gamma \|S\|_{\text{op}}.
$$

(96)

Rearranging this, we obtain

$$
\|S\|_{\text{op}} \leq \frac{1}{1 - 12\gamma} \left[ \max_{y \in N_s} |y^\top S y| + \max_{z \in N_b} |z^\top S z| + 2 \max_{y \in N_s} \max_{z \in N_b} |y^\top S z| \right].
$$

(97)
Using Claim 3 and a union bound, we have that
\[
P \left[ \|S\|_{\text{op}} \geq \frac{4}{1 - 12\gamma} \left( \frac{C_1}{N} + t \right) \right]
\leq 2(|N_b| + |N_s| \cdot |N_b|) \exp \left( -C_2 \frac{\rho}{(1 + \sqrt{\rho})^2} N^2 t^2 \right) + 2|N_s| \exp \left( -\frac{C_2}{2} nt^2 \right)
\leq 3 \exp \left( N \left[ \frac{2}{\gamma} (1 + \rho) + \rho + \log \left( \frac{1}{\rho} \right) \rho - C_2 \frac{\rho}{(1 + \sqrt{\rho})^2} N^2 t^2 \right] \right)
+ 2 \exp \left( N \left[ \frac{2\rho}{\gamma} + \rho + \log \left( \frac{1}{\rho} \right) \rho - \frac{C_2}{2} t^2 \right] \right).
\] (98)

Taking \( \rho = N^{-1/2}, \ t = C_3 N^{-1/4} \log N \) for a large constant \( C_3 \), and \( \gamma < \frac{1}{12} \) a small constant, we obtain the result. \( \square \)

### 7.3 Bounding the Projection Term \( T^{(2)} \): Normalization

In this section, we show that the passing from the approximately normalized vectors \( \delta^{-1/2}v_i \) discussed in the previous section to the exactly normalized vectors \( \hat{v}_i = v_i/\|v_i\|_2 \) does not affect the construction of \( T^{(2)} \) very much, as measured by operator norm.

**Lemma 7.6.** Let \( A^{(\text{orth})} \in \mathbb{R}^{r(r+1)/2 \times N} \) have isovec\((\delta^{-1/2}v_i v_i^\top - \frac{1}{r} I_r)\) as its columns, and let \( A^{(\text{norm})} \in \mathbb{R}^{r(r+1)/2 \times N} \) have isovec\((\hat{v}_i \hat{v}_i^\top - \frac{1}{r} I_r)\) as its columns. Then, for all \( K > 0 \),
\[
P \left[ \|A^{(\text{orth})} - A^{(\text{norm})}\|_{\text{op}} \leq O_{\delta,K} \left( \frac{\log N}{N} \right) \right] \geq 1 - O_{\delta,K}(N^{-K}).
\] (99)

**Proof.** Recall that \( D \in \mathbb{R}^{N \times N} \) is diagonal with \( D_{ii} = \|v_i\|^2_2 \). Then, we may write
\[
A^{(\text{norm})} = \delta A^{(\text{orth})} D^{-1} + \frac{\delta}{r} 1_{\text{diag}}^\top 1_{N} D^{-1} - \frac{1}{r} 1_{\text{diag}} 1_{N}^\top
= A^{(\text{orth})}(\delta D^{-1} - I_N) + \frac{1}{r} 1_{\text{diag}} 1_{N}^\top (\delta D^{-1} - I_N).
\] (100)
Therefore,
\[
\|A^{(\text{norm})} - A^{(\text{orth})}\|_{\text{op}} \leq \|\delta D^{-1} - I_N\|_{\text{op}} \left( \|A^{(\text{orth})}\| + \delta^{-1/2} \right)
= \left( \max_{i \in [N]} \left| \frac{\delta}{D_{ii}} - 1 \right| \right) \left( \|A^{(\text{orth})}\| + \delta^{-1/2} \right).
\] (101)

By Lemma 7.4 from the previous section, the second term is \( O_{\delta}(1) \) with super-polynomially high probability. The result then follows by Proposition 3.1. \( \square \)

It is straightforward to translate this result to the Gram matrix \( A^{(\text{norm})\top} A^{(\text{norm})} \).

**Corollary 7.7.** In the same setting as Lemma 7.6, for all \( K > 0 \),
\[
P \left[ \|A^{(\text{orth})\top} A^{(\text{orth})} - A^{(\text{norm})\top} A^{(\text{norm})}\|_{\text{op}} \leq O_{\delta,K} \left( \frac{\log N}{N} \right) \right] \geq 1 - O_{\delta,K}(N^{-K}).
\] (102)
Proof. We may bound
\[ \| A^{(\text{orth})\top} - A^{(\text{orth})\top} A^{(\text{norm})\top} \|_{\text{op}} \]
\[ = \| A^{(\text{orth})\top} (A^{(\text{orth})} - A^{(\text{norm})}) + (A^{(\text{orth})} - A^{(\text{norm})})\top A^{(\text{norm})} \|_{\text{op}} \]
\[ \leq \left( \| A^{(\text{orth})} \|_{\text{op}} + \| A^{(\text{norm})} \|_{\text{op}} \right) \left\| A^{(\text{orth})} - A^{(\text{norm})} \right\|_{\text{op}} \]
\[ \leq \left( 2 \| A^{(\text{orth})} \|_{\text{op}} + \| A^{(\text{orth})} - A^{(\text{norm})} \|_{\text{op}} \right) \| A^{(\text{orth})} - A^{(\text{norm})} \|_{\text{op}} . \] (103)

Then, combining Lemma 7.4 and Lemma 7.6 gives the result. \(\square\)

7.4 Final Main Term Bound: Proof of Lemma 6.1

We are now ready to complete the proof of Lemma 6.1. First, we combine Lemma 7.4 and Corollary 7.7. Recall that these showed the following bounds, with high probability:
\[ \| A^{(\text{orth})\top} A^{(\text{orth})} \|_{\text{op}} \leq 1 + O_{\delta} \left( \frac{\log N}{N^{1/4}} \right) , \] (Lemma 7.4)
\[ \| A^{(\text{orth})\top} A^{(\text{orth})} - A^{(\text{norm})\top} A^{(\text{norm})} \|_{\text{op}} \leq O_{\delta} \left( \frac{\log N}{N} \right) . \] (Corollary 7.7)

Combining these, we obtain the following bound on \( T^{(2)} = A^{(\text{norm})} A^{(\text{norm})\top} \).

Corollary 7.8. For all \( \delta \in (0, 1) \),
\[ \lim_{N \to \infty} P \left[ \| T^{(2)} \|_{\text{op}} \leq 1 + O_{\delta} \left( \frac{\log N}{N^{1/4}} \right) \right] = 1 . \] (104)

Next, we complete the proof of Lemma 6.1. Recall that
\[ \tilde{Z}^{(1a)} = \left( \alpha - \frac{2N}{p^2} \right) I_{\text{diag}} 1_{\text{diag}}\top + 2I_{r(r+1)/2} + T^{(1)} - 2T^{(2)} , \] (105)
and we have gathered the following bounds, holding with high probability:
\[ |T^{(1)}| \leq O_{\delta} \left( \frac{\log N}{N} \right) I_{r(r+1)/2} + \frac{2}{r} 1_{\text{diag}} 1_{\text{diag}}\top \] (Lemma 7.2)
\[ 0 \leq T^{(2)} \leq \left( 1 + O_{\delta} \left( \frac{\log N}{N^{1/4}} \right) \right) I_{r(r+1)/2} \] (Corollary 7.8)

We can therefore control the minimum eigenvalue of \( \tilde{Z}^{(1a)} \) by, with high probability,
\[ \tilde{Z}^{(1a)} \succeq \left( \alpha - \frac{2N}{p^2} - \frac{2}{r} \right) 1_{\text{diag}} 1_{\text{diag}}\top - O_{\delta} \left( \frac{\log N}{N^{1/4}} \right) I_{r(r+1)/2} . \] (106)

The first term is positive semidefinite for sufficiently large \( N \) since \( N/r^2 \to 0 \), and thus we find
\[ \lim_{N \to \infty} P \left[ \lambda_{\min} (\tilde{Z}^{(1a)}) \geq -O_{\delta} \left( \frac{\log N}{N^{1/4}} \right) \right] = 1 . \] (107)
Finally, we must convert this to a bound on the smallest eigenvalue of $Z^{(1a)}$. By Proposition 7.1, letting $P = V^\top V$, we have

$$
\lambda_{\min}(Z^{(1a)}) \geq \min \left\{ 0, \frac{\lambda_{\min}(\tilde{Z}^{(1a)})}{\min_{i \in [N]} \hat{P}^2_i} \right\}.
$$

(108)

By Proposition 3.1, with high probability $P_{ii} \geq \delta^{-1} - O_\delta(\sqrt{\log N})$ for all $i \in [N]$. Substituting this above, we thus find the result of Lemma 6.1,

$$
\lim_{N \to \infty} \mathbb{P} \left[ \lambda_{\min}(Z^{(1a)}) \geq -O_\delta \left( \frac{\log N}{N^{1/4}} \right) \right] = 1.
$$

(109)

8 Proof of Positive Semidefiniteness: Correction Term

Proof of Lemma 6.2. Recall that $\Delta$ is non-zero only on index pairs $\{ij\}$ and $\{ik\}$ that share an index. The non-zero entries are given by

$$
\Delta_{\{ij\}\{ik\}} = \sum_{m=1}^{N} M^2_{im} M_{jm} M_{km} - M_{ij} M_{ik}.
$$

(110)

Let us compute the quadratic form of $\Delta$ with $a \in \mathbb{R}^{(N-1)/2}$, which we view as having entries $a_{\{ij\}} = A_{ij}$ for some $A \in \mathbb{R}^{r \times N}$ with $\text{diag}(A) = 0$ (i.e. $a = \text{offdiag}(A)$). We then have, expanding with a correction for double-counting the diagonal terms,

$$
a^\top \Delta a = \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{k=1}^{N} A_{ij} \left\{ \left( \sum_{m=1}^{N} M^2_{im} M_{jm} M_{km} \right) - M_{ij} M_{ik} \right\} A_{ik}
$$

$$
- \sum_{1 \leq i < j \leq N} A_{ij}^2 \left\{ \left( \sum_{m=1}^{N} M^2_{im} M_{jm} \right) - M_{ij}^2 \right\}
$$

$$
= \sum_{i=1}^{N} \left\{ \left( \sum_{m=1}^{N} M^2_{im} (AM)^2_{im} \right) - (AM)^2_{ii} \right\} - \sum_{1 \leq i < j \leq N} A_{ij}^2 ((M^{o2})^2 - M^{o2})_{ij}
$$

$$
= \langle M^{o2}, (AM)^{o2} \rangle - \text{Tr}((AM)^{o2}) - \langle A^{o2}, (M^{o2})^2 - M^{o2} \rangle
$$

$$
= \langle M^{o2} - I_N, (AM)^{o2} \rangle - \langle A^{o2}, (M^{o2})^2 - M^{o2} \rangle,
$$

(111)

and, following the notation of Sections 7.2 and 7.3, we let $R := A_{(\text{norm})}^\top A_{(\text{norm})}$ be the Gram matrix of $\text{isovec}(\tilde{v}_i, \tilde{n}_i^\top - \frac{1}{r}I_N)$, whereby $M^{o2} = R + \frac{1}{r}1_N \frac{1}{r}1_N^\top$, allowing us to continue by expanding the second term

$$
= \langle M^{o2} - I_N, (AM)^{o2} \rangle - \frac{1}{r^2} (1_N^\top R 1_N^\top) ||A||_F^2 + \frac{1}{r} ||A||_{F^2}
$$

$$
- \left( \langle A^{o2}, R^2 - R + \frac{1}{r}1_N 1_N^\top R + \frac{1}{r} R 1_N 1_N^\top \rangle \right).
$$

(112)

We will now use the following inequality that allows us to bound inner products with the Schur square of a matrix by its Frobenius norm:

$$
|\langle X^{o2}, Y \rangle| \leq \sum_{i,j} |Y_{ij}| X_{ij}^2 \leq \left( \max_{i,j} |Y_{ij}| \right) ||X||_F^2 = ||Y||_{\infty} ||X||_F^2 \leq ||Y||_{\text{op}} ||X||_F^2.
$$

(113)
We denote by $\|Y\|_{\ell_\infty}$ the vectorized supremum norm. This will result in one term involving $\|AM\|_F^2$, which we bound by

$$\|AM\|_F^2 = \text{Tr}(AM^2A^\top) = \text{Tr}(A^\top AM^2) \leq \|M\|^2_{\text{op}} \text{Tr}(A^\top A) = \|M\|^2_{\text{op}} \|A\|^2_F.$$  \hfill (114)

Combining these inequalities and noting that $\|a\|^2 = 1/2\|A\|^2_F$, we find (for $a \neq 0$) that

$$\left|a^\top \Delta a\right| \leq 2 \left(\|M\|^{o2} - I_N\|e\| \|M\|^{2}_{\text{op}} + \frac{1}{r} + \frac{1}{r^2} \|R1_N\| + \|R^2 - R\|_{\text{op}} + \frac{2\sqrt{N}}{r} \|R1_N\|_2\right).$$ \hfill (115)

Since $\text{diag}(M) = 1$, we have by Corollary 3.2 that, with high probability,

$$\|M\|^{o2} - I_N\|e\| = \max_{i,j \in [N]} |M_{ij}|^2 \leq O_{\delta}\left(\frac{\log N}{N}\right),$$ \hfill (116)

and by Corollary 3.3, $\|M\|_{\text{op}} = O_{\delta}(1)$ with high probability.

To control the terms involving $R$, recall that from Lemmas 7.4 and 7.6 it follows that with high probability

$$\|R - P_{1_N}^\perp\|_{\text{op}} \leq O_{\delta}\left(\frac{\log N}{N^{1/4}}\right).$$ \hfill (117)

Thus we have, with high probability,

$$\left|\frac{1}{r} \|R1_N\|\right| \leq \frac{N}{r^2} \frac{1}{\|R - P_{1_N}^\perp\|_{\text{op}}} = O_{\delta}\left(\frac{\log N}{N^{5/4}}\right),$$ \hfill (118)

$$\|R^2 - R\|_{\text{op}} \leq 3 \|R - P_{1_N}^\perp\|_{\text{op}} + \|R - P_{1_N}^\perp\|_{\text{op}}^2 = O_{\delta}\left(\frac{\log^2 N}{N^{1/4}}\right),$$ \hfill (119)

$$\frac{2\sqrt{N}}{r} \|R1_N\|_2 \leq \frac{2N}{r} \|R - P_{1_N}^\perp\|_{\text{op}} = O_{\delta}\left(\frac{\log N}{N^{1/4}}\right).$$ \hfill (120)

Combining these results, Lemma 6.2 follows.

\[\Box\]

**Remark 8.1.** The argument in the proof above of Lemma 6.2 is conceptually unsatisfying, and may produce a suboptimal bound. Here we outline a simpler argument which suggests a sharper estimate, though it seems more difficult to formalize due to the dependency structure of $M$.

Since $\Delta$ is sparse (with non-zero entries only when the row and column index pairs $\{ij\}$ and $\{k\ell\}$ share an element), our strategy will be to apply the Gershgorin circle theorem. Thus we must bound the diagonal and off-diagonal entries of $\Delta$.

The summation defining the diagonal entries contains only positive summands, so we have

$$\Delta_{(ij)(ij)} = \sum_{m=1}^{N} M_{im}^2 M_{jm}^2 = M_{ij}^2 + \sum_{m=1}^{N} M_{im}^2 M_{jm}^2 = \tilde{O}(N^{-1}),$$ \hfill (121)
using Proposition 3.1 to control each term of the sum. (We omit bounds expressing “with high probability” statements and indulge in the logarithm-concealing $\tilde{O}$ notation in this informal discussion.)

The off-diagonal entries of $\Delta$ are more difficult to control. They are given by

\[ \Delta_{(ij)(ik)} = \sum_{m=1}^{N} M_{im}^2 M_{jm} M_{km} = \sum_{m=1}^{N} M_{im}^2 M_{jm} M_{km}. \]  

(122)

Here, we must capture the cancellations due to random signs, which we expect to improve our bound on this term by a factor of $\sqrt{N}$: a naive application of the triangle inequality would give $|\Delta_{(ij)(ik)}| = \tilde{O}(N^{-1})$, which would be insufficient for the Gershgorin circle theorem argument since there are $\Omega(N)$ non-zero entries in each row of $\Delta$. On the other hand, a scaling like that in the central limit theorem with independent signs would give $|\Delta_{(ij)(ik)}| = \tilde{O}(N^{-3/2})$, which would suffice, and would give $\|\Delta\|_{op} = \tilde{O}(N^{-1/2})$, stronger by a factor of $N^{-1/4}$ than the result of Lemma 6.2 (up to logarithmic factors).

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The following gives the low-degree moments of Haar(\(\mathcal{O}(N)\)). These values are derived in the references given by simple geometric arguments, but there is also an elegant and general combinatorial approach. Let \(Q \sim \text{Haar}(\mathcal{O}(N))\). The moment \(\mathbb{E} \prod_{k=1}^{d} Q_{i_{k}j_{k}}\) is zero if any index occurs an odd number of times among either the \(i_{k}\) or \(j_{k}\). The non-zero degree 2 and 4 moments are given by

\[
\begin{align*}
\mathbb{E}Q_{11}^2 &= \frac{1}{N}, \\
\mathbb{E}Q_{11}^4 &= \frac{3}{N(N+2)}, \\
\mathbb{E}Q_{11}^2Q_{12} &= \frac{1}{N(N+2)}, \\
\mathbb{E}Q_{11}^2Q_{22} &= \frac{N+1}{(N-1)N(N+2)}, \\
\mathbb{E}Q_{11}Q_{12}Q_{21}Q_{22} &= -\frac{1}{(N-1)N(N+2)}.
\end{align*}
\]

\(B\) Entries of Random Projectors: Proof of Proposition 3.1

**Proof of Proposition 3.1.** For the diagonal entries, we write \(P_{ii} = \|P_{0}e_{i}\|_{2}^2 = \|P_{0}v\|_{2}^2\) for \(v \in S^{N-1}\) a uniform random unit vector and \(P_{0}\) the orthogonal projector onto \(\text{span}(e_{1}, \ldots, e_{r})\). Letting \(g^{(N)}(N) \sim \mathcal{N}(0, I_{N})\), we have \(v \overset{(d)}{=} g/\|g\|_{2}\). Writing \(g^{(r)}\) and \(g^{(N-r)}\) for the first \(r\) and last \(N-r\) coordinates of \(g\) respectively, we then have

\[
P_{ii} \overset{(d)}{=} \frac{\|g^{(r)}\|_{2}^2}{\|g\|_{2}^{2} + \|g^{(N-r)}\|_{2}^{2}},
\]

where \(\|g^{(r)}\|_{2}^{2}\) and \(\|g^{(N-r)}\|_{2}^{2}\) are distributed as independent \(\chi^{2}\) random variables with \(r\) and \(N-r\) degrees of freedom respectively. The result then follows from the concentration inequality of [LM00] for \(\chi^{2}\) random variables.

For the off-diagonal case we may likewise write \(P_{ij} = \langle Pe_{i}, Pe_{j}\rangle \overset{(d)}{=} \langle P_{0}v, P_{0}w\rangle\), where now \((v, w)\) are a Haar-distributed two-dimensional orthonormal frame. If we draw \(g, h \sim \mathcal{N}(0, I_{N})\), then by performing two steps of Gram-Schmidt orthonormalization,

\[
(v, w) \overset{(d)}{=} \left(\frac{g}{\|g\|_{2}}, \frac{h - \langle g, h \|g\|_{2}^{-1}g}{\|h - \langle g, h \|g\|_{2}^{-1}g\|_{2}}\right).
\]

Thus computing as before we find

\[
P_{ij} \overset{(d)}{=} \frac{\langle g^{(r)}, h^{(r)} \rangle - \langle g, h \rangle \|g^{(r)}\|_{2}^{2} \|h - \langle g, h \|g\|_{2}^{-1}g\|_{2}^{2}}{\|g\|_{2}^{2} \|h - \langle g, h \|g\|_{2}^{-1}g\|_{2}^{2}}.
\]
To control these quantities, we first bound

\[
\left| \langle g^{(r)}, h^{(r)} \rangle - \langle g, h \rangle \frac{\|g^{(r)}\|_2^2}{\|g^{(r)}\|_2^2 + \|g^{(N-r)}\|_2^2} \right| \leq \frac{\|g^{(r)}\|_2^2}{\|g\|_2^2} \left( |\langle g^{(r)}, h^{(r)} \rangle| + |\langle g, h \rangle| \right)
\]

\[
\leq \frac{\|g^{(r)}\|_2^2}{\|h\|_2^2} \left( |\langle g^{(r)}, h^{(r)} \rangle| + |\langle g, h \rangle| \right),
\]

(131)

where we have used that \(\|g^{(r)}\|_2 \leq \|g\|_2\). In this expression, every inner product term appearing is the inner product of a uniformly distributed unit vector with an independent standard gaussian vector, and thus simply has law \(\mathcal{N}(0,1)\). Controlling \(\|h\|_2\) using the same concentration inequalities of [LM00] as before and applying standard union bounds for maxima of gaussian random variables then gives the result. \(\square\)