Radio Geometric graceful graphs

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Abstract. A one-one mapping \( f : V(G) \to \mathbb{Z}^+ \) satisfying the condition \( d(u, v) + \left[ \sqrt{f(u)f(v)} \right] \geq diam(G) + 1 \), for every pair of distinct vertices in \( G \) is defined as radio geometric mean labeling of \( G \). The maximum number assigned to any vertex of \( G \) under the labeling \( f \) is called its radio geometric mean number of \( f \), denoted by \( r_{gm}(f) \). The least value of \( r_{gm}(f) \), taken over all such labelings \( f \) of \( G \) is defined as its radio geometric mean number and is denoted by \( r_{gm}(G) \). Clearly, \( r_{gm}(G) \geq |V(G)| \). Graphs for which \( r_{gm}(G) = |V(G)| \) are defined as radio geometric graceful. In this paper, we find the radio geometric mean number of certain classes of graphs like sunflowers, Helms, gear graphs and show that they are radio geometric graceful.

Introduction

Graphs considered in this paper are finite, simple, connected and undirected. Standard graph terminology has been used and the terms not defined here can be referred from [1,2]. The well-known channel assignment problem was introduced by Hale [3] in 1980. For a set of transmitting stations, a channel has to be assigned which is a nonnegative integer. The assignment should be done so as to reduce interference between stations. At the same time, the highest frequency assigned, called as the span of the labeling has to be minimized due to scarce availability of frequency spectrum. This was modelled as a graph theory problem called as a radio labeling problem [4] as follows. A one-one mapping \( f : V(G) \to \mathbb{Z}^+ \) such that \( |f(u) - f(v)| \geq diam(G) + 1 - d(u, v) \) holds or every pair of distinct vertices in \( G \) is called a radio labeling of \( G \). The radio number of \( f \), denoted by \( rn(f) \) is the maximum number assigned to any vertex of \( G \) under \( f \). The minimum value of \( rn(f) \) taken over all radio labelings \( f \) of \( G \) is called the radio number of \( G \), and is denoted by \( rn(G) \). It is obvious that \( rn(G) \geq |V(G)| \). Finding the radio number of a graph is highly non-trivial. Graphs for which \( rn(G) = |V(G)| \) are called radio graceful.

Radio labeling was originally introduced by Chartrand et al [10]. For a connected graph \( G \) of order \( n \) and diameter 2, they proved that \( n \leq rn(G) \leq 2n - 2 \). They also showed that for every pair \( k, n \) of integers with \( n \leq k \leq 2n - 2 \), there exists a connected graph \( G \) of order \( n \) and diameter 2 with \( rn(G) = k \). A characterization of connected graphs of order \( n \) and diameter 2 with a prescribed radio number was given by them. Zhang in 2002 [11] discussed the upper and lower bounds for the radio number of cycles and the bounds are shown to be tight for certain cycles. In 2004, Liu and Xie worked on the radio number of square of cycles [12]. In 2006, Liu obtained lower
bounds for the radio number of trees, and characterized the trees achieving this bound [15]. A lower bound for the radio number of trees with at most one vertex of degree more than two (called spiders) in terms of the lengths of their legs was obtained and a characterization of the spiders achieving this bound was found. The results of Liu generalize the radio number for paths obtained by Liu and Zhu [14]. Further Liu and Xie obtained radio labeling of square of paths [16]. The results obtained by Zhang on radio number of cycles [11] were shown to be inaccurate by Sooryanarayana and Raghunath. An independent proof for the results of Liu and Xie with better bounds was given by them in [13]. The radio labeling of cube and fourth power of cycles have been discussed by Sooryanarayana and Raghunath [17, 18] and the radio number of cube of a path [19] is obtained by them in [13]. The radio labeling of cube and fourth power of cycles have been discussed by Sooryanarayana and Vishukumar. Different variations of the problem has been studied by the authors in [7,8,9].

In [5], Ponraj et al introduced the concept of radio mean labeling as an injective mapping \( f : V(G) \to Z^+ \) where \( ind(u,v) + \left[ \frac{f(u)+f(v)}{2} \right] \geq diam(G) + 1 \) holds for every pair of distinct vertices in \( G \). The maximum number assigned to any vertex of \( G \) under \( f \) is called the span or radio mean number of \( f \), denoted by \( Rmn(f) \). The radio mean number of \( G \), denoted by \( Rmn(G) \), was defined as the minimum value of \( Rmn(f) \) taken over all radio mean labelings \( f \) of \( G \). It is obvious that \( Rmn(G) \geq |V(G)| \). In [5], the authors evaluated the radio mean number of certain classes of graphs and showed that this number is equal to the order of the graph.

In [6], Hemalatha et al introduced the concept of radio geometric mean labeling as an injective mapping \( f : V(G) \to Z^+ \) satisfying the condition that \( d(u,v) + \left[ \frac{f(u)+f(v)}{2} \right] \geq diam(G) + 1 \) for every pair of vertices in \( G \). The maximum number assigned to any vertex of \( G \) under the labeling \( f \) is called its span or the radio geometric mean number of \( f \), denoted by \( rgmn(f) \). The radio geometric mean number of \( G \), defined as the minimum value of \( rgmn(f) \) taken over all radio geometric mean labeling \( f \) of \( G \) was denoted by \( rgmn(G) \). Since any radio geometric mean is one to one it follows that \( rgmn(G) \geq |V(G)| \). The natural question that arises is what classes of graphs have \( rgmn(G) = |V(G)| \). In [6], the authors evaluated the radio geometric mean number of some subdivision graphs and showed that this number is equal to \( |V(G)| \). In this paper the authors prove that classes of graphs like sunflower, helm, gear graphs and lotus inside a circle also have \( rgmn(G) = |V(G)| \) and calls them radio geometric graceful graphs.

**Main Results**

For any graph of diameter 2, the condition \( d(u,v) + \left[ \frac{f(u)+f(v)}{2} \right] \geq diam(G) + 1 \), holds for every two distinct vertices \( u,v \in V(G) \). Hence graphs of diameter at least 3 are considered in the following results.

The sunflower graph \( SF_n, n \geq 3 \) is defined as follows: Consider the wheel graph with central vertex \( v_0 \) and let \( v_1, v_2, v_3, ..., v_n, v_1 \) be vertices on the rim of the wheel in the clockwise direction. New vertices \( w_1, w_2, w_3, ..., w_n, w_1 \) are added to the graph and each \( w_i \) is joined by edges to \( v_i \) and \( v_{i+1} \) where \( i + 1 \) is taken under modulo \( n \).

**Theorem 2.1:** The Sunflower graph \( SF_n, n \geq 3 \) is radio geometric graceful.

**Proof:** For \( n = 3 \), \( diam(SF_3) = 2 \), the result is obvious.

For \( n \geq 4 \), define the function \( f \) with co-domain, \( \{1, 2, ..., (2n + 1)\} \) as follows: \( f(v_0) = n + 1, f(w_1) = 1, f(w_2) = n, f(w_3) = 2 \) and \( f(w_i) = i - 1, \, 4 \leq i \leq n \).

Also, for \( n \equiv 0 (mod \, 2) \)
\[ f(v_i) = \begin{cases} 
 i + \frac{3n}{2}, & 1 \leq i \leq \frac{n}{2} + 1 \\
 i + \frac{n}{2}, & \frac{n}{2} + 2 \leq i \leq n 
\end{cases} \]

and for \( n \equiv 1(\text{mod } 2) \)

\[ f(v_i) = \begin{cases} 
 i + \frac{3n + 1}{2}, & 1 \leq i \leq \frac{n}{2} \\
 i + \frac{n + 1}{2}, & \frac{n}{2} + 1 \leq i \leq n 
\end{cases} \]

**Case (i):** \( n = 4, 5 \)

For \( n = 4, 5 \) \( \text{diam}(\mathcal{F}_n) = 3 \).

So it is enough to prove that for \( \forall u, v \in V(\mathcal{G}) \), \( d(u, v) + \lceil \sqrt{f(u)f(v)} \rceil \geq 4 \).

If \( d(u, v) = 1 \), then Eq (1) holds only if \( f(u)f(v) \geq 5 \). This means adjacent vertices cannot be given the labels (1, 2) or (1, 3) or (1, 4). By the definition of \( f \), we have \( f(u)f(v) \geq 7 \) in the case of \( n = 4 \) and \( f(u)f(v) \geq 9 \) in the case of \( n = 5 \) for every edge \( uv \in V(\mathcal{G}) \).

If \( d(u, v) = 2 \), or \( d(u, v) = 3 \), then Eq (1) obviously holds.

**Case (ii):** \( n \geq 6 \)

For \( n \geq 6 \), \( \text{diam}(\mathcal{F}_n) = 4 \).

Now it is enough to prove that for \( \forall u, v \in V(\mathcal{G}) \), \( d(u, v) + \lceil \sqrt{f(u)f(v)} \rceil \geq 5 \).

If \( d(u, v) = 1 \), then Eq (2) holds only if \( f(u)f(v) \geq 10 \).

Now, \( d(v_0, v_i) = 1, 1 \leq i \leq n \). By the definition of \( f \), the minimum value of \( f(v_i) \), \( 1 \leq i \leq n \) is \( n + 2 \). Hence \( f(v_0)f(v_i) \geq (n + 1)(n + 2) \geq 56 \) as \( n \geq 6 \).

Also for any edge \( v_iv_{i+1} \) on the cycle where \( i + 1 \) is taken under modulo \( n \), the labels \( n + 2 \) and \( n + 3 \) are the least, hence \( f(v_i)f(v_{i+1}) \geq (n + 2)(n + 3) \geq 72 \) as \( n \geq 6 \).

Also for any edge of the form \( w_iv_i \) and \( w_iv_{i+1} \) where \( i + 1 \) is taken under modulo \( n \), the labels \( 1 \) and \( i + \frac{3n}{2} \) in the case of even \( n \), the labels \( 1 \) and \( i + \frac{3n + 1}{2} \) in the case of odd \( n \) are the least, hence \( f(w_i)f(v_i) \geq \left(i + \frac{3n}{2}\right) \geq 10 \) for even \( n \) and \( f(w_i)f(v_{i+1}) \geq \left(i + \frac{3n + 1}{2}\right) \geq 12 \) for odd \( n \).

If \( d(v_0, v) = 2 \), then Eq (2) holds only if \( f(u)f(v) \geq 5 \). For any vertex \( v_i \) on the cycle all vertices of the form \( v_j \), where \( j \neq i \pm 1 \), taken under modulo \( n \) are at distance 2. The least of all labels given to such vertices are \( n + 2 \) and \( n + 4 \). Hence \( f(u)f(v) \geq 80 \) for all such vertices.

Also, for any vertex of the form \( v_i \), \( 1 \leq i \leq n \), the vertices \( v_0, w_{i+1}, v_{i-1}, v_{i+2} \) are at distance 2. Since \( f(v_0) = n + 1 \) and the least value of \( f(w_i) \) is 1, \( f(w_i)f(v_0) \geq n + 1 \geq 7 \) as \( n \geq 6 \).

For vertices of the form \( w_i \), \( 1 \leq i \leq n \), where \( i + 1 \) is taken under modulo \( n \), the least values of \( f(w_i) \) are 1 and \( n - 1 \). Hence \( f(w_i)f(w_{i+1}) \geq n - 1 \geq 5 \).

For vertices of the form \( w_i \), \( v_{i-1} \) on the cycle, the least value of the product \( f(w_i)f(v_{i-1}) \) is \( \frac{3n}{2} \) for even \( n \) and \( \frac{3n + 1}{2} \) for odd \( n \) respectively. Hence \( f(w_i)f(v_{i-1}) \geq \frac{3n}{2} \geq 9 \) for even \( n \) and \( f(w_i)f(v_{i-1}) \geq \frac{3n + 1}{2} \geq 9 \) and for odd \( n \) respectively.

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For vertices of the form \(w_i, v_{i+2}\) on the cycle, the least value of the product \(f(w_i)f(v_{i+2})\) is \(\frac{3n+6}{2}\) for even \(n\) and \(\frac{3n+7}{2}\) for odd \(n\) respectively. Hence \(f(w_i)f(v_{i+2}) \geq \frac{3n+6}{2} \geq 12\) for even \(n\) and \(f(w_i)f(v_{i+2}) \geq \frac{3n+7}{2} \geq 14\) and for odd \(n\) respectively.

If \(d(u, v) = 3\), or \(d(u, v) = 4\) then Eq (2) holds trivially. Hence, sunflower graphs are radio geometric mean graceful.

An illustration of the labelling for the sunflower graph \(SF_n\) is shown in Figure 1 below.

**Figure 1**

The Helm graph \(H_n, n \geq 3\) is obtained from a wheel graph \(W_n, n \geq 3\) by attaching a pendant edge at each vertex of the cycle \(C_n, n \geq 3\). The central vertex of the wheel is denoted by \(v_0\), the vertices on the rim by \(v_i, 1 \leq i \leq n\). The pendant vertices adjacent to \(v_i, 1 \leq i \leq n\) are denoted by \(w_i, 1 \leq i \leq n\).

**Theorem 2.2**: The helm graph \(H_n, n \geq 3, n \neq 4\) is radio geometric graceful.

**Proof:**
On the vertex set of the helm graph \(H_n, n \geq 3, n \neq 4\), define the function \(f\) with co-domain, \{1, 2, ..., (2n + 1)\} as follows: \(f(v_0) = n + 1, f(v_i) = 2n + 1, f(v_i) = n + i, 2 \leq i \leq n\) and \(f(w_i) = i, 1 \leq i \leq n\).

**Case (i) :** \(n = 3\)
For \(n = 3\), \(diam(H_n) = 3\).
So it is enough to prove that
\[\forall u, v \in V(G), d(u, v) + \sqrt{f(u)f(v)} \geq 4\]......Eq (3).
If \(d(u, v) = 1\), then Eq (3) holds only if \(f(u)f(v) \geq 5\). This means adjacent vertices cannot be given the labels \((1, 2)\) or \((1, 3)\) or \((1, 4)\). By the definition of \(f\), it can be easily seen that \(f(u)f(v) \geq 7\) for every edge \(uv \in E(G)\).
If \(d(u, v) = 2\), or \(d(u, v) = 3\), then Eq(3) obviously holds.
Case (ii): $n \geq 5$

For $n \geq 5$, $diam(H_n) = 4$.

Now it is enough to prove that
\[ \forall u, v \in V(G), \quad d(u, v) + \left\lceil \sqrt{f(u)f(v)} \right\rceil \geq 5 \quad \text{Eq (4)} \]

If $d(u, v) = 1$, then Eq (4) holds only if $f(u)f(v) \geq 10$.

Now, $d(v_0, v_i) = 1, 1 \leq i \leq n$. By the definition of $f$, the minimum value of $f(v_i), 1 \leq i \leq n$ is $n + 2$. Hence $f(v_0)f(w_1) \geq (n + 1)(n + 2) \geq 42$ as $n \geq 5$.

Also for any edge $v_i, v_{i+1}, 1 \leq i \leq n$, on the cycle where the suffix $i + 1$ is taken under modulo $n$, the labels $n + 2$ and $n + 3$ are the least, hence $f(v_0)f(w_1) \geq (n + 2)(n + 3) \geq 10$ as $n \geq 5$.

If $d(u, v) = 2$, then Eq (4) holds only if $f(u)f(v) \geq 5$.

Now, $d(v_0, w_i) = 2, 1 \leq i \leq n$. By the definition of $f$, the lowest value of the product $f(v_0)f(w_i)$ occurs when $i = 1$. Hence $f(v_0)f(w_1) \geq 2n + 1 \geq 11$ as $n \geq 5$.

Also for any edge $v_i, v_{i+1}, 1 \leq i \leq n$, by the definition of $f$, the lowest value of the product $f(v_0)f(w_i)$ occurs when $i = 1$. Hence $f(v_0)f(w_1) \geq n + 1 \geq 6$ as $n \geq 5$.

If $d(u, v) = 2$, then $f(u)f(v) \geq 5$ is true if the following holds:

If $f(u) = 1$, then $f(v) \geq 5$. If $f(u) = 2$, then $f(v) \geq 3$. If $f(u) = 3, 4$, then $f(v) \geq 2$.

Now $f(w_1) = 1$, the vertices at distance 2 from it namely $v_0, v_2, v_n$ are assigned labels $n + 1$, $n + 2$, $2n$ respectively. Hence $f(v_0)f(w_1) \geq n + 1 \geq 6$, $f(v_2)f(w_1) \geq n + 2 \geq 7$, $f(v_n)f(w_1) \geq 2n \geq 10$.

Now $f(w_2) = 2$, the vertices at distance 2 from it namely $v_0, v_1, v_3$ are assigned labels $n + 1$, $2n + 1$, $n + 3$ respectively. Hence $f(v_0)f(w_2) \geq 2(n + 1) \geq 12$, $f(v_1)f(w_2) \geq 2(2n + 1) \geq 22$, $f(v_3)f(w_2) \geq 2(n + 3) \geq 16$.

Now $f(w_3) = 3$, the vertices at distance 2 from it namely $v_0, v_2, v_4$ are assigned labels $n + 1$, $n + 2$, $n + 4$ respectively. Hence $f(v_0)f(w_3) \geq 3(n + 1) \geq 18$, $f(v_2)f(w_3) \geq 3(n + 2) \geq 21$, $f(v_4)f(w_3) \geq 3(n + 4) \geq 27$.

Now $f(w_4) = 4$, the vertices at distance 2 from it namely $v_0, v_3, v_5$ are assigned labels $n + 1$, $n + 3$, $n + 5$ respectively. Hence $f(v_0)f(w_4) \geq 4(n + 1) \geq 20$, $f(v_3)f(w_4) \geq 4(n + 3) \geq 28$, $f(v_5)f(w_4) \geq 4(n + 5) \geq 40$.

Case (iii): $n = 4$

For $n = 4$, $diam(H_n) = 4$.

If $d(u, v) = 1$, then Eq (4) holds only if $f(u)f(v) \geq 10$.

If the label 1 is assigned to any vertex then the vertex adjacent to it must be labelled at least $10 > 2n + 1 = 9$.

Hence $H_n$, is radio geometric graceful for all $n \geq 3$, except for $n = 4$.

An illustration of the labelling for the Helm graph $H_8$ is shown in Figure 2 below.
The graph Lotus inside circle $L_{n,n}$, $n \geq 3$, is defined as follows: Consider a star with central vertex $u$ and the end vertices $u_1, u_2, \ldots, u_n$ taken in the clockwise direction. Let $C_n : w_1 w_2 \ldots w_n w_1$ be a cycle where the vertices are considered in the clockwise direction. Now join each $u_i$ to $w_i$ and $w_{i+1 (mod n)}$.

**Theorem 2.3:** The graph lotus inside circle $L_{n,n}$, $n \geq 3$ is radio geometric graceful.

**Proof:**

**Case (i):** When $n = 3, 4$ we have $diam(L_{n,n}) = 2$ and result trivially holds.

**Case (ii):** For $n = 5, 6, 7$ $diam(L_{n,n}) = 3$. Define the function $f$ with co-domain, $\{1, 2, \ldots, (2n + 1)\}$ as follows: $f(u) = 2n + 1$, $f(u_i) = 1$, $f(w_i) = n - 1 + i$, $1 \leq i \leq 2$, $f(u_i) = 1$, $3 \leq i \leq n$, $f(u_i) = n + i$, $2 \leq i \leq n$.

Now it is enough to prove that $\forall u,v \in V(G), d(u,v) + \lceil \sqrt{f(u)f(v)} \rceil \geq 4 \ldots \ldots \text{Eq (5)}$

If $d(u,v) = 1$, Eq (5) holds only if $f(u)f(v) \geq 5$. In other words, edges cannot be assigned labels $(1,2), (1,3)$ and $(1,4)$. This can be easily seen by the definition of $f$.

**Case (iii):** For $n \geq 8$, $diam(L_{n,n}) = 4$.

Define the function $f: V(G) \rightarrow \{1, 2, \ldots, (2n + 1)\}$ as follows: $f(u) = 2n + 1$, $f(u_i) = 1$, $f(u_i) = 3 + i$, $2 \leq i \leq n$, $f(w_i) = n + 3 + i$, $1 \leq i \leq n - 5$, $f(w_{n-2}) = 4$, $f(w_{n-3}) = 2n - 1$, $f(w_{n-4}) = 2$.

Now it is enough to prove that $\forall u,v \in V(G), d(u,v) + \lceil \sqrt{f(u)f(v)} \rceil \geq 5 \ldots \ldots \text{Eq (6)}$

If $d(u,v) = 1$, Eq (6) holds only if $f(u)f(v) \geq 10$.

If $f(u) = 1$, then we must have $f(u)f(v) \geq 10$. If $f(u) = 2$, then we must have $f(u)f(v) \geq 5$. If $f(u) = 3$, then we must have $f(u)f(v) \geq 4$. If $f(u) = 4$, then we must have $f(u)f(v) \geq 3$. If $f(u) = 5, 6, 7, 8, 9$, then we must have $f(u)f(v) \geq 2$.

Now $f(u_1) = 1$ and $u_1$ is adjacent with $w_1, w_2$, whose labels are $n + 4 \geq 12$ and $n + 5 \geq 13$ respectively hence Eq (6) holds.
Now $f(w_{(n-4)}) = 2$ and $w_{(n-4)}$ is adjacent with $w_{n-3}, w_{n-5}, u_{n-4}, u_{n-5}$. The least of their labels are $n - 2 \geq 6$ as required and hence Eq (6) holds.

Now $f(w_{(n-1)}) = 3$ and $w_{(n-1)}$ is adjacent with $w_n, w_{n-2}, u_{n-1}, u_{n-2}$. The least of their labels are 4 as required and hence Eq (6) holds.

Now $f(w_{(n-2)}) = 4$ and $w_{(n-2)}$ is adjacent with $w_{n-3}, w_{n-1}, u_{n-3}, u_{n-2}$. The least of their labels are 3 as required and hence Eq (6) holds.

None of the vertices having labels 5, 6, 7, 8, 9 are adjacent with vertex labelled 1 namely $u_1$.

If $d(u,v) = 2$, Eq (6) holds only if $f(u)f(v) \geq 5$.

Now $f(u_1) = 1$ and $u_1$ is at distance 2 from $w_n, w_3, u_i, 2 \leq i \leq n$ and the least of the labels assigned to these vertices is 5, as required.

Now $f(w_{(n-4)}) = 2$. Vertices at distance two from this are $w_{n-2}, w_{n-6}, u, u_{n-3}, u_{n-6}$ and the least of the labels assigned to these vertices is 4, as required.

Now $f(w_{(n-1)}) = 3$ and $f(w_{(n-2)}) = 4$. Now vertices at distance two from it are assigned a label greater than 1 as $f(u_1) = 1$ is at a distance three from it.

Hence the result.

An illustration of the labelling for the lotus inside circle graph $L_{C_B^8}$ is shown in Figure 3 below.

![Figure 3](image-url)

The gear graph $G_n, n \geq 3$ is obtained from the wheel $W_n, n \geq 3$ by inserting a vertex of degree 2 between every pair of adjacent vertices on the rim of the wheel. We denote the central vertex by
Theorem 2.4: The gear graph $G_n, n \geq 3, n \neq 4$ is radio geometric graceful.

Proof:

Case (i): For $n = 3$, $diam(G_n) = 3$. For $n = 5$, $diam(G_n) = 4$.

The proof of these cases is shown in the Figures 4 and 5 respectively.

Case (ii): For $n \geq 6$, $diam(G_n) = 4$.

Define the function $f$ with co-domain, $\{1, 2, \ldots, (2n + 1)\}$ as follows:

$$f(u) = 2n + 1, \quad f(u_1) = 1, \quad f(u_2) = n, \quad f(u_i) = f(u_{i-1}) + 1, \quad 3 \leq i \leq n, \quad f(v_1) = 2n - 1, \quad f(v_2) = 2n, \quad f(v_3) = 2, \quad f(v_4) = f(v_{i-1}) + 1, \quad 4 \leq i \leq n.$$

It is now enough to show that for $\forall \ u, v \in V(G)$

$$d(u, v) + \left\lceil \sqrt{f(u)f(v)} \right\rceil \geq 5 \ldots \ldots \text{Eq (7) holds.}$$

If $d(u, v) = 1$, Eq (7) holds only if $f(u)f(v) \geq 10$.

Now $f(u_1) = 1$. The vertices adjacent to this are $v_1, v_2$ with $f(v_1) = 2n - 1, f(v_2) = 2n$. The least of this is $2n - 1 \geq 11$. Hence Eq(7) holds.

Now $f(v_5) = 4$, vertices adjacent to this are $u_4, u_5$ and $f(u_4) = n + 2$, $f(u_5) = n + 3$. $f(u_4)f((v_5)) \geq 4n + 8 \geq 32$ and $f(u_5)f((v_5)) \geq 4n + 12 \geq 36$.

Now vertices with labels 5, 6, 7, 8, 9 are not adjacent to $u_1$, since $f(u_1) = 1$ as required.

If $d(u, v) = 2$, Eq(7) holds only if $f(u)f(v) \geq 5$.

Now $f(u_1) = 1$. The vertices $u, u_n, u_2$ are at distance 2 from it. The least of the labels assigned to these vertices is $n$ to the vertex $u_2$. Hence, $f(u_1)f(u_2) \geq n \geq 6$.

Now $f(v_2) = 2$ and vertices at distance two from it are $v_i, 2 \leq i \leq n$ and the least of the labels given to these vertices is 3 as required.

Now vertices with labels 3 and 4 are not adjacent to vertex labelled 1 as required.

Case (iii): For $n = 4$, $diam(G_n) = 4$. Any labeling $f$, defined on $V(G_n)$ should satisfy the condition $d(u, v) + \left\lceil \sqrt{f(u)f(v)} \right\rceil \geq 5$ for every pair of distinct vertices $u, v \in V(G)$. If $d(u, v) = 1$, then this condition holds only if $f(u)f(v) \geq 10$. If $f(u) = 1$, then $f(v) \geq 10$, is not possible since highest label available is 9.

Hence $G_n, n \geq 3, n \neq 4$ are radio geometric graceful. An illustration of the labelling is shown in Figure 6 below for the case of $n = 8$. 
3. Conclusion

The paper graph classes which are radio geometric graceful are studied. This definition of labeling which has applications in mobile communication can be studied for other classes of graphs for efficient transmission.
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