CYCLE CLASS MAPS FOR CHOW GROUPS OF ZERO-CYCLES WITH MODULUS

KAY RÜLLING AND SHUJI SAITO

Abstract. For a smooth scheme $X$ of pure dimension $d$ over a field $k$ and an effective Cartier divisor $D \subset X$ whose support is a simple normal crossing divisor, we construct a cycle class map

$$cyc_{X|D} : CH_0(X|D) \to H^d_{\text{Nis}}(X, K^M_d(O_X, I_D))$$

from the Chow group of zero-cycles with modulus to the cohomology of the relative Milnor $K$-sheaf.

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1. INTRODUCTION

Let $U$ be a smooth variety of dimension $d$ over a finite field $k$ and $U \hookrightarrow X$ be an open immersion with $X$ normal and proper over $k$ such that $X\setminus U$ is the support of an effective Cartier divisor on $X$. Let $\pi_1^{ab}(U)$ be the abelianized fundamental group of $U$. Kato and Saito [KS86] established class field theory for $U$ by using the reciprocity map

$$\rho_U : \lim_D H^d(X_{\text{Nis}}, K^M_d(O_X, I_D)) \to \pi_1^{ab}(U),$$

where the limits are over all effective Cartier divisors $D$ supported on $X\setminus U$, and $K^M_d(O_X, I_D)$ with $I_D \subset O_X$ the ideal sheaf of $D$ is the relative Milnor $K$-sheaf on the Nisnevich site over $X$ (see (2.17.2)). A refinement of this statement was given by Kerz-Saito [KS16] using another reciprocity map

$$\phi_U : \lim_D CH_0(X|D) \to \pi_1^{ab}(U),$$

where $CH_0(X|D)$ is the Chow group of zero-cycles with modulus. By definition $CH_0(X|D)$ is the quotient of the group $Z_0(U)$ of zero-cycles on $U$ by the subgroup generated by divisors of rational functions on curves in $U$ which satisfy a certain modulus conditions with respect to $D$ (see 3.1). On the other hand, the Gersten resolution of the Milnor $K$-sheaf yields a natural map

$$\theta : Z_0(U) \longrightarrow H^d(X_{\text{Nis}}, K^M_d(O_X, I_D)).$$

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1They needed to assume $\text{ch}(k) \neq 2$. This assumption was removed by Binda-Krishna-Saito [BKS21] giving a simpler proof of the main result in [KS16] .
which is surjective by [KS86], see (3.1.3).

With the help of ramification theory, the main result of [KS16] implies that \( \phi_U \) induces a map for a fixed divisor \( D \):

\[
\phi_{X|D} : \text{CH}_0(X|D) \to \pi^0_{1}(X|D)
\]

whose restriction to the degree-zero part is an isomorphism of finite abelian groups. Here, \( \pi^0_{1}(X|D) \) is the quotient of \( \pi^0_{1}(U) \) which classifies abelian étale coverings of \( U \) with ramification over \( X \setminus U \) bounded by \( D \), see [KS14]. A natural question is whether the latter fact holds with \( \text{CH}_0(X|D) \) replaced by \( H^d(X_{\text{Nis}}, K^M_d(\mathcal{O}_X, I_D)) \).

A positive answer was given by Gupta-Krishna [GK22] assuming \( X \) is smooth and projective over the finite field \( k \). It implies that there is a natural isomorphism

\[
cyc_{X|D} : \text{CH}_0(X|D) \cong H^d_{\text{Nis}}(X, K^M_d(\mathcal{O}_X, I_D)),
\]

which gives a factorization of \( \theta \) from (1.0.1), namely fits into the following commutative diagram

\[
\begin{array}{ccc}
\text{CH}_0(X|D) & \xrightarrow{cyc_{X|D}} & H^d_{\text{Nis}}(X, K^M_d(\mathcal{O}_X, I_D)) \\
\pi & \downarrow{\theta} & \\
Z_0(U) & \text{ } & \\
\end{array}
\]

where \( \pi \) is the quotient map. The question whether such a factorization of \( \theta \) via \( \text{CH}_0(X|D) \) exists makes sense in case \( X \) is smooth (not necessarily projective) over an arbitrary field \( k \) and \( D \) is an effective Cartier divisor on \( X \). It was positively answered in the following cases in which \( X \) is always smooth and \( D \) is an effective Cartier divisor:

1. \( X \) is a quasi-projective surface over a field, see [Kri15, Theorem 1.2];
2. \( X \) is affine over an algebraically closed field, see [GK20, Theorem 1.3(1)];
3. \( X \) is projective over an algebraically closed field and \( D \) is integral, see [GK20, Theorem 1.3(2)].

In all the above cases the cycle map \( cyc_{X|D} \) is actually an isomorphism, by [BKS21] in the case (1) and by [GK20] in the cases (1), (2). The authors were furthermore informed by Krishna that the existence of the cycle map also follows from [Kri18] for smooth projective varieties over algebraically closed fields if \( D \) is reduced.

In this note we prove:

**Theorem 1.1 (Theorem 3.2).** Let \( X \) be a smooth scheme of pure dimension \( d \) over a field \( k \) and \( D \subset X \) be an effective Cartier divisor whose support is a simple normal crossing divisor. There exists a map \( cyc_{X|D} \) which makes (1.0.2) commutative.

We will use the cycle map from the above theorem in [RS21a], where it is important to allow an arbitrary base field. This is our main motivation.

The proof of Theorem 1.1 is different in spirit from the proofs in (1)-(3) above and follows instead the same strategy as in [RS18], in which a factorization

\[
(1.1.1) \quad \text{CH}_0(X|D) \to H^d_{\text{Nis}}(X, U_{d,X|D})
\]

is constructed, where \( U_{d,X|D} \) is a variant of \( K^M_d(\mathcal{O}_X, I_D) \) (see 2.1). There is another variant \( V_{d,X|D} \) and it is easy to prove that \( K^M_d(\mathcal{O}_X, I_D) \) and \( V_{d,X|D} \) have the same cohomology group in the top degree \( d \) (see (2.18.2)). Thus we are reduced to constructing a factorization

\[
(1.1.2) \quad \text{CH}_0(X|D) \to H^d_{\text{Nis}}(X, V_{d,X|D}).
\]
A key input in the construction of (1.1.1) is a projective bundle formula for the cohomology of $U_{d,X,D}$. In §2, we prove its variant for $V_{d,X,D}$. This is deduced from the original statement by comparing the cohomologies of $U_{d,X,D}$ and $V_{d,X,D}$. To this end we define a decreasing filtration on $V_{d,X,D}$, which ends in $U_{d,X,D}+D_{red}$ (see 2.6) and compute the associated graded algebra. Having this refined projective bundle formula at hand, we construct (1.1.2) in §3 using the same argument as for the construction of (1.1.1) in [RS18].

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## 2. The Projective Bundle Formula for Relative Milnor $K$-theory

In this section $k$ is a field of characteristic $p > 0$ and $\text{Sm}$ denotes the category of smooth $k$-schemes.

### 2.1. Let $X$ be a $k$-scheme. We denote by $K^M_{r,X}$ $(r \geq 0)$ the Nisnevich sheafification of the improved Milnor $K$-theory from [Ker10]. Let $D \subset X$ be a closed subscheme and denote by $j : U := X \setminus D \hookrightarrow X$ (resp. $i : D \hookrightarrow X$) the corresponding open (resp. closed) immersion. Set

$$\mathcal{O}_X^\times := \text{Ker}(\mathcal{O}_X^\times \to i_* \mathcal{O}_D^\times).$$

We consider the following Nisnevich sheaves for $r \geq 1$:

$$U_{r,X,D} := \text{Im}(\mathcal{O}_{X,D}^\times \otimes_{\mathcal{O}} j_*K^M_{r-1,U} \to j_*K^M_{r,U}), \quad V_{r,X,D} := \text{Im}(\mathcal{O}_{X,D}^\times \otimes_{\mathcal{O}} K^M_{r-1,X} \to K^M_{r,X}).$$

Note that

1. $U_{1,X,D} = V_{1,X,D} = \mathcal{O}_{X,D}^\times$ and the restrictions $K^M_{s,X} \to j_*K^M_{s,U}$, $s = r - 1, r$, induce a natural map $V_{r,X,D} \to U_{r,X,D}$, which is injective if $X$ is regular (by the Gersten resolution, see [Ker10, Proposition 10(8)]);

2. if $k$ is an infinite field, then $V_{r,X,D}$ is equal to the Nisnevich sheaf $\text{Ker}(K^M_{r,X} \to K^M_{r,D})$ considered in [KS86, (1.3)] (this follows from [KS86, Lemma 1.3.1] and [Ker10, Proposition 10(5)]);

3. the sheaf $U_{r,X,D}$ (for $D$ an effective Cartier divisor) is equal to the Nisnevich sheaf $K^M_{r,X,D,Nis}$ considered in [RS18, 2.3].

### 2.2. Let $X \in \text{Sm}$. Denote by $\Omega_X^i$, $i \geq 0$, the Nisnevich sheaf of absolute differential $i$-forms on $X$ and set

$$B_0\Omega_X^i = 0, \quad B_1\Omega_X^i = d\Omega_X^{i-1}, \quad Z_1\Omega_X^i = \text{Ker}(d : \Omega_X^i \to \Omega_X^{i+1}), \quad Z_0\Omega_X^i = \Omega_X^i.$$

Recall that for $p > 0$ we have the inverse Cartier operator $C^{-1} : \Omega_X^i \to Z_1\Omega_X^i/B_1\Omega_X^i$ at disposal. It is an isomorphism of sheaves of abelian groups given by

$$C^{-1}(a \log b_1 \cdots \log b_i) = a^p \log b_1 \cdots \log b_i \mod B_1\Omega_X^i,$$

for local sections $a \in \mathcal{O}_X$, $b_1, \ldots, b_i \in \mathcal{O}_X$. For $s \geq 1$, the subsheaves $B_s\Omega_X^i$ and $Z_s\Omega_X^i$ of $\Omega_X^i$ are recursively defined by

$$C^{-1} : B_{s-1}\Omega_X^i \xrightarrow{\sim} B_s\Omega_X^i/B_1\Omega_X^i, \quad C^{-1} : Z_{s-1}\Omega_X^i \xrightarrow{\sim} Z_s\Omega_X^i/B_1\Omega_X^i.$$

In case no confusion can arise we simply write $B_s$ and $Z_s$ instead of $B_s\Omega_X^i$ and $Z_s\Omega_X^i$, respectively.
2.3. Let $X \in \text{Sm}$ and let $D = \text{div}(t)$ be a smooth integral and principal divisor on $X$. For $m \geq 1$ and $r \geq 1$ we set

$$U_{r,m} = U_{r,X|MD}, \quad V_{r,m} = V_{r,X|MD}.$$ 

If $p > 0$, write $m = p^am'$ with $(p,m') = 1$ and $s \geq 0$; if $p = 0$ we set $s = 0$ and $m' := m$. Set

$$m^{G^r} := \text{Coker} \left( \frac{\Omega^r_D}{B_s} \to \frac{\Omega^{r-1}_D}{B_s} \oplus \frac{\Omega^{r-2}_D}{B_s}, \quad \alpha \mapsto (C^{-s}(d\alpha), (-1)^mC^{-s}\alpha) \right),$$

where $C^{-s}$ is the $s$-fold iterated inverse Cartier operator, for $s \geq 1$, and $C^{-0} = \text{id}$. By [BK86, Remark (4.8)] we have an isomorphism

$$m^{G^r} \cong \frac{U_{r,m}}{U_{r,m+1}} \quad \text{(2.3.1)}
$$
given by

$$(a \cdot \text{dlog} b, e \cdot \text{dlog} f) \mapsto \{1 + t^m\tilde{a}, \tilde{b}\} + \{1 + t^m\tilde{e}, \tilde{f}, t\},$$

where $a,e \in \mathcal{O}_D$, $b = (b_1, \ldots, b_{r-1})$, and $f = (f_1, \ldots, f_{r-2})$, $b_i,f_j \in \mathcal{O}_X$ are local sections and $\tilde{a}, \tilde{e}, \tilde{b}, \tilde{f}$ are lifts to $\mathcal{O}_X$. See also [RS18, Proposition 2.15, Theorem 2.19], in particular for the sheaf theoretic statement considered here and the case $s = 0$.

**Lemma 2.4.** Let the assumption and notation be as in 2.3. There is an isomorphism of exact sequences of sheaves on $D$

$$\begin{array}{ccccccccc}
0 & \to & V_{r,m} & \to & U_{r,m} & \to & U_{r,m} & \to & 0 \\
& & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & 0 \\
0 & \to & \frac{\Omega^r_D}{B_s\Omega^r_D} & \to & m^{G^r} & \to & \frac{\Omega^{r-2}_D}{Z_s\Omega^{r-2}_D} & \to & 0.
\end{array} \quad \text{(2.4.1)}
$$

In particular in characteristic 0, we have $U_{r,m} = V_{r,m}$.

**Proof.** Note that by the definition of (2.3.1) the composition

$$\frac{\Omega^r_D}{B_s} \to m^{G^r} \to U_{r,m}/U_{r,m+1}$$

factors via a surjection

$$\frac{\Omega^r_D}{B_s} \to V_{r,m}/U_{r,m+1}, \quad a \cdot \text{dlog} b \mapsto \{1 + t^m\tilde{a}, \tilde{b}\}.$$ 

Therefore it suffices to show that the lower sequence is exact. This is immediate for $s = 0$. We assume $s \geq 1$ (hence $p > 0$). By [Ill79, I, Proposition 3.11] we have

$$B_s\Omega^r_D = F^{s-1}dW_s\Omega^r_D, \quad Z_s\Omega^r_D = F^sW_{s+1}\Omega^r_D,$$

where $W_n\Omega^r_D$ denotes the de Rham-Witt complex from [Ill79] and $F$ is the Frobenius on it. Using the relation $FdV = d$ and $FV = VF = p$, where $V$ is the Verschiebung, we find in $\Omega^{r-1}_D \oplus \Omega^{r-2}_D$

$$\left( F^s d\alpha_0 + F^{s-1}d\gamma, (-1)^mF^s(\alpha_0) + F^{s-1}d\beta \right)$$

$$= \left( F^s d\alpha, (-1)^mF^s(\alpha) + F^{s-1}d\beta \right), \quad \text{with } \alpha := \alpha_0 + V(\gamma),$$

where $\gamma$ is the Verschiebung.
where \( \alpha_0 \in W_{n+1}\Omega_D^{r-2}, \gamma \in W_n\Omega_D^{r-2} \), and \( \beta \in W_n\Omega_D^{r-3} \). Using that \( F^s : W_{n+1}\Omega_D \to \Omega_D' \) lifts the iterated inverse Cartier operator \( C^{-s} : \Omega_D' \to \Omega_D/B \), we see that the lower row in (2.4.1) becomes

\[
\begin{array}{ccc}
\Omega_D^{r-1} & \varphi & \Omega_D^{r-1} \oplus \Omega_D^{r-2} \\
\{F^s-1 d\gamma\} & \{(F^s d\alpha, (-1)^{i} m' F^s(\alpha) + F^s-1 d\beta)\} & \{(F^s(\delta))\}
\end{array}
\]

where,

\[
\varphi(x) = (x, 0), \quad \psi(x, y) = y.
\]

Note that \( \varphi \) clearly surjects onto the kernel of \( \psi \). We show that \( \varphi \) is injective. Indeed, \( \varphi(x) = 0 \) implies that there exist \( \alpha, \beta \) with \( x = F^s d\alpha \) and \( F^s \alpha = F^s-1 d\beta \). We obtain \( \alpha - dV\beta \in \text{Ker}(F^s) \). By [Ill79, I, (3.21.1.2)], we find a \( \gamma \) such that \( \alpha - dV\beta = V\gamma \). Thus

\[ x = F^s d\alpha = F^s dV(\gamma) + F^s ddV(\beta) = F^s-1 d\gamma. \]

Therefore the lower sequence is exact. \( \square \)

The following lemma is direct to check (see [RS18, Lemma 2.7]).

**Lemma 2.5.** Let \( R \) be a local integral domain with maximal ideal \( m \) and fraction field \( K \). Let \( a, b, c \in R \) and \( s, t \in m \). The following equalities hold in \( K_2^M(K) \)

\[
\begin{align*}
(1) \quad & \{1 + as, 1 + bt\} = \{-1 + \frac{ab}{1+st}, -as(1+bt)\}; \\
(2) \quad & \{1 + \frac{s-t}{1+st} e, 1 - \frac{st}{1+st} s\} = \{1 + c e, s\}.
\end{align*}
\]

**2.6.** Let \( X \in Sm \) and let \( D = \sum_{j=1}^{n} m_j D_j \) be an effective divisor, such that \( |D| = \sum_{j=1}^{n} D_j \) is a simple normal crossing divisor with smooth divisors \( D_j \) and \( m_j \geq 1 \). For \( i \geq 0 \) we set

\[ E_i := D_{i+1} + \ldots + D_n, \]

in particular \( E_0 = 0 \), and denote by \( j_i : U_i := X \setminus |E_i| \hookrightarrow X \) the open immersion (with the notation from 2.1 we have \( j = j_0 : U = U_0 \hookrightarrow X \)). For \( r \geq 1 \) we define recursively

\[ V_r^{-1} := 0 \quad \text{and} \quad V_r^i := V_r^{i-1} + \text{Im} \left( \mathcal{O}_{X,D+E_i}^* \otimes_{\mathbb{Z}} j_{i*} K_{r-1,U} \to j_{i*} K_{r,U} \right), \quad i \geq 0. \]

Since the restriction map \( K_{r,X} \to j_{i*} K_{r,U} \) is injective, by [Ker10, Proposition 10(8)], we have \( V_r^i \subset V_{r,X,D} \) for all \( i \), by Lemma 2.5(2). We obtain a filtration

\[ V_{r,X,D+E_0} = V_r^0 \subset V_r^1 \subset \ldots \subset V_r^n = V_{r,X,D}. \]

**2.7.** Let \( r \geq 1 \). Let \( X \in Sm \) and let \( D = \sum_{j=1}^{n} m_j D_j \) be as in 2.6. If \( p > 0 \), write \( m_j = p^{s_j} m_j' \) with \( (p,m_j') = 1 \) and \( s_j \geq 0 \); if \( p = 0 \) we set \( m_j' := m_j \) and \( s_j = 0 \). (We will use the convention \( 0^0 = 1 \).) After renumbering we can assume that

\[
(2.7.1) \quad s_1 \geq s_2 \geq \ldots \geq s_n \geq 0.
\]

Fix \( i \in \{1, \ldots, n\} \). If \( p > 0 \), then there is an inverse Cartier operator on the absolute logarithmic differential forms (defined by the same formula as in 2.2)

\[
C^{-1} : \Omega_D^{-1}_{r_i}(\log E_i) \to \frac{F_i^r \Omega_D^{r-1}_{r_i}(\log E_i)}{d\Omega_D^{r-2}_{r_i}(\log E)}.
\]

\(^2\text{We do not assume the } D_j \text{ to be connected to include the case where } D = u^* D_0 \text{ for some } \text{étale map } u : X \to X_0 \text{ and effective divisor } D_0 \text{ on } X_0 \text{ with SNCD support.} \)
where $F$ is the absolute Frobenius and we write $\Omega_{D_i}^{-1}(\log E_i)$ as a shorthand for $\Omega_{D_i}^{-1}(\log E_{i|D_i})$ etc. The $\mathcal{O}_{D_i}$-submodule $B_s\Omega_{D_i}^{-1}(\log E_i) \subset F^*\Omega_{D_i}^{-1}$ is recursively defined by

$$B_0\Omega_{D_i}^{-1}(\log E_i) := 0, \quad C^{-1} : B_{s-1}\Omega_{D_i}^{-1}(\log E_i) \xrightarrow{\sim} \frac{B_s\Omega_{D_i}^{-1}(\log E_i)}{d\Omega_{D_i}^{-2}(\log E_i)}.$$  

Denote by $D(i)$ the minimal divisor satisfying $p^nD(i) \geq D + E$, i.e., by (2.7.1)

$$D(i) = \sum_{j=1}^{i} p^{s_j-s_i}m_j^iD_j + \sum_{j=i+1}^{n} \left\lfloor \frac{m_j + 1}{p^n} \right\rfloor D_j.$$

We write $\mathcal{O}_{D_i}(-D) := \mathcal{O}_X(-D)|_{D_i}$ etc. and have natural inclusions of $\mathcal{O}_{D_i}^{p^n_i}$-modules

$$\mathcal{O}_{D_i}(-D(i))^{p^n_i} \otimes_{\mathcal{O}_{D_i}^{p^n_i}} B_s\Omega_{D_i}^{-1}(\log E_i) \subset \mathcal{O}_{D_i}(-D(i))^{p^n_i} \otimes_{\mathcal{O}_{D_i}^{p^n_i}} B_s\Omega_{D_i}^{-1}(\log E_i) 
\subset \mathcal{O}_{D_i}(-D - E_i) \otimes_{\mathcal{O}_{D_i}} \Omega_{D_i}^{-1}(\log E_i).$$

We obtain a sheaf of $\mathcal{O}_{D_i}^{p^n_i}$-modules

$$Q_{D_i}^{i,r-1} := \frac{\mathcal{O}_{D_i}(-D - E_i) \otimes_{\mathcal{O}_{D_i}} \Omega_{D_i}^{-1}(\log E_i)}{\mathcal{O}_{D_i}(-D(i))^{p^n_i} \otimes_{\mathcal{O}_{D_i}^{p^n_i}} B_s\Omega_{D_i}^{-1}(\log E_i)}.$$  

If $p = 0$, then $s_i = 0$ and we have $Q_{D_i}^{i,r-1} = \mathcal{O}_{D_i}(-D - E_i) \otimes_{\mathcal{O}_{D_i}} \Omega_{D_i}^{-1}(\log E_i)$.

**Lemma 2.8.** Let the notations and assumptions be as in 2.7. Assume $k$ is perfect.

1. Then $Q_{D_i}^{i,r-1}$ is a locally free $\mathcal{O}_{D_i}$-module.
2. Let $j_i : U_i = D_i \setminus |E_i| \hookrightarrow D_i$ be the open immersion. Then

$$j_i^*Q_{D_i}^{i,r-1} = \mathcal{O}_{U_i} \left( -\sum_{j=1}^{i} p^{s_j-s_i}m_j^iD_j \right)^{p^n_i} \otimes_{\mathcal{O}_{D_i}^{p^n_i}} \frac{\Omega_{U_i}^{-1}}{B_s\Omega_{U_i}^{-1}}$$

and the restriction map $Q_{D_i}^{i,r-1} \rightarrow j_i^*Q_{D_i}^{i,r-1}$ is injective.

**Proof.** The first statement in (2) is clear, the second statement follows from (1). Furthermore, (1) is immediate if $s_i = 0$. Thus we assume $s_i \geq 1$ and hence also $p > 0$. The statement is local. Let $x \in X$ be a point, up to replacing $\{D_j\}_j$ by some ordered subset we can assume that all $D_j$ contain $x$. Hence we may assume $X = \text{Spec} R$, with $R$ a local ring, essentially smooth over $k$ and with a regular sequence of parameters $t_1, \ldots, t_N$, where $t_j$ is a local equation for $D_j$, $j = 1, \ldots, n$. Then $D_i = \text{Spec} R_i$, with $R_i = R/(t_i)$. Set

$$\tau := \prod_{j=1}^{i} t_j^{m_j} \prod_{j=i+1}^{n} t_j^{m_j+1},$$

which is an equation for $D + E$. Since $k$ is perfect of positive characteristic, an $R_i^{p^n}$-basis of $\tau \cdot \Omega_{R_i}^{-1}(\log(t_1+1 \cdots t_n))$ is given by

$$\tau \cdot t_1^{l_1} \cdots t_N^{l_N} \cdot dt_{k_1} \cdots dt_{k_{r_1}} \cdot (\log t_1 \cdots \log t_{r_2}),$$

where $j_{\nu} \in [0, p^n - 1]$, $1 \leq k_1 < \cdots < k_{r_1} \leq i$, $i + 1 \leq l_1 < \cdots < l_{r_2} \leq n$, and $r_1, r_2 \geq 0$ with $r_1 + r_2 = r - 1$. By (2.7.1) we have a local equation for $p^nD(i)$ of the
form

\[(2.8.2) \quad \sigma_{\sigma^i} = \tau \cdot \prod_{j=i+1}^{n} t_{h_j}^j, \quad \text{with } h_j \in \{0, p^s - 1\}.\]

It follows from the definition of the inverse Cartier operator, that an $R_{t_i}^{i,r}$-basis of \(\sigma_{\sigma^i} \cdot B_s^i \Omega_{R_i}^{-1}(\log(t_{i+1} \cdots t_n))\) as submodule of \(\tau \cdot \Omega_{R_i}^{-1}(\log(t_{i+1} \cdots t_n))\) is given by

\[(2.8.3) \quad \sigma_{\sigma^i} \cdot t_{k_1}^{\mu_1} \cdots t_{k_r}^{\mu_r} \cdot dt_{k_1} \cdots dt_{k_r} \cdot d\log t_{i_{r_1}} \cdots d\log t_{i_2},\]

where \(k_\mu, l_\nu\) are as above, \(\mu \in \{0, s_i - 1\}\), and \(p_1 \geq 1, p_2 \geq 0\) with \(p_1 + p_2 = r - 1\). By (2.8.2) and direct inspection of the bases above, we find that the inclusion

\(\sigma_{\sigma^i} \cdot B_s^i \Omega_{R_i}^{-1}(\log(t_{i+1} \cdots t_n)) \hookrightarrow \tau \cdot \Omega_{R_i}^{-1}(\log(t_{i+1} \cdots t_n))\)

is a split-injection of $R_{t_i}^{i,r}$-modules. Hence $Q_{D^i}^{i,r-1}$ is a locally free $O_{D^i}$-module. \(\square\)

**Lemma 2.9.** Let \((R, m)\) be a regular local ring and \(t_1, \ldots, t_n \in m\) be part of a regular sequence of local parameters \((n \geq 1)\). Set \(R_n := R/\langle t_i \rangle\) and denote by \(\tilde{t}_i\) the image of \(t_i\) in \(R_n\). Then the following sequence is exact

\[0 \to 1 + t_n R \to (R[t_1^{-1}, \ldots, t_n^{-1}, \tau])^\times \to (R_n[t_1^{-1}, \ldots, t_n^{-1}, \tau])^\times \to 0.\]

Furthermore, \(1 + t_n R\) is multiplicatively generated by elements \(1 + t_n u\), with \(u \in R^\times\).

**Proof.** By the Auslander-Buchsbaum theorem, \(R\) is factorial. Since \(R/\langle t_i \rangle\) and \(R/\langle t_n, t_i \rangle\) are again local regular rings (in particular they are integral), we see that \(t_1, \ldots, t_n\) are prime elements in \(R\) and \(t_1, \ldots, t_{n-1}\) are prime elements in \(R_n\). It follows that every unit in \(v \in R[t_1^{-1}, \ldots, t_{n-1}^{-1}]\) can be uniquely written in the form

\[v = u t_1^{i_1} \cdots t_{n-1}^{i_{n-1}}, \quad \text{with } u \in R^\times, i_1, \ldots, i_{n-1} \in \mathbb{Z},\]

and similar with \(R_n[t_1^{-1}, \ldots, t_{n-1}^{-1}]\). Hence \(v \equiv 1 \mod t_n\) implies that \(i_1 = \ldots = i_{n-1} = 0\) and \(u \in 1 + t_n R\). This implies the exactness of the sequence in the statement. For the last part observe that if \(a \in m\), then \(1 + t_n a = (1 + t_n u) \cdot (1 + t_n v)\) with \(u = a - 1, v = 1/(1 + t_n u) \in R^\times\). \(\square\)

**Proposition 2.10.** Let the notations and assumptions be as in 2.7. Then there is an isomorphism of sheaves of abelian groups

\[\theta : Q_{D^i}^{i,r-1} \xrightarrow{\sim} V_{F^i}^{r}/V_{F^i}^{r-1}\]

given on local sections by

\[\theta(a \cdot d\log b_1 \cdots d\log b_{r-1}) = \{1 + \tilde{a}, \tilde{b}_1, \ldots, \tilde{b}_{r-1}\},\]

where \(\tilde{a} \in O_X(-D - E_i), \tilde{b}_j \in O_{X \setminus E_i}^\times\) are local lifts of \(a\) and \(b_j\), respectively.

**Proof.** For an étale \(X\)-scheme \(W\) let \(T(W)\) denote the free abelian group on the generators \([e, f_1, \ldots, f_{r-1}]\), with \(e \in \Gamma(W, O_X(-D - E_i)|_W)\) and \(f_j \in O^\times(W \setminus E_i|_W)\), \(j = 1, \ldots, r - 1\). Then \(W \mapsto T(W)\) defines a Nisnevich sheaf on \(X\) and we have surjective morphisms of sheaves

\[(2.10.1) \quad T \to Q_{D^i}^{i,r-1}, \quad [e, f_1, \ldots, f_{r-1}] \mapsto \tilde{e} \cdot d\log \tilde{f}_1 \cdots d\log \tilde{f}_{r-1},\]

where \(\tilde{e}\) and \(\tilde{f}_j\) denote the image of \(e\) and \(f_j\) in \(O_{D_i}(-D - E_i)\) and \(O_{D_i \setminus E_i}^\times\), respectively, and

\[(2.10.2) \quad T \to V_{F^i}^{r}/V_{F^i}^{r-1}, \quad [e, f_1, \ldots, f_{r-1}] \mapsto \{1 + e, f_1, \ldots, f_{r-1}\}.\]
The map \( \theta \) from the statement is well-defined and surjective when we show that the kernel of the surjection \((2.10.1)\) is mapped to zero under \((2.10.2)\). This is a local question and it suffices to consider stalks. Note that both sheaves in the statement have support in \( D_i \). Therefore it suffices to show the statement in every point \( x \in D_i \). Up to replacing \( \{ D_j \}_j \) by an ordered subset we may assume that all components of \( D \) meet in \( x \). Thus we can assume \( X = \text{Spec} \, R \), where \( R \) is a local ring with maximal ideal \( m \), which is essentially smooth over \( k \). Let \( t_j \in m \) be a local equation of \( D_j \), \( j = 1, \ldots, n \). Set \( R_i := R/(t_i) \).

We first show that a well-defined map \( \theta \) as in the statement exits. Set \( M^{\text{sp}} := R_i[\frac{1}{t_{i+1} \cdots t_n}]^\times \) and denote by \( \tau \) the equation \((2.8.1)\) for \( D + E_i \). There is a well-defined map

\[
(2.10.3) \quad \tau R_i \oplus \bigoplus_{j=1}^{r-1} M^{\text{sp}} \to \frac{V^i_r}{V^{i-1}_r}
\]
given by

\[
(a, (b_1, \ldots, b_{r-1})) \mapsto \{1 + \tilde{a}, \tilde{b}_1, \ldots, \tilde{b}_{r-1}\},
\]
where \( \tilde{a} \in \tau R \), \( \tilde{b}_j \in (R_i[\frac{1}{t_{i+1} \cdots t_n}]^\times) \) are lifts of \( a \) and \( b_j \), respectively. We have to check the independence of the choice of \( \tilde{a} \) follows, since \( \{1 + t_i \tau c, \tilde{b}_j\} \in V^{i-1}_r \). Two different lifts of \( b_1 \) differ by Lemma 2.9 by a product of element of the form \((1 + t_i c)\) with \( c \in R^\times \). For \( c \in R \) and \( \tilde{b}_j \) as above, we compute

\[
\{1 + \tau c, 1 + t_i c, \tilde{b}_j\} = \frac{1 + ec t_i + c \tau, \tilde{b}}{1 + ec t_i + c \tau, \tilde{b}} = 1 + \tau c, 1 + t_i c, \tilde{b}
\]
by 2.5(1),

\[
\in V_r^{i-1}.
\]

This shows the independence of the choice of the lift of \( \tilde{b}_i \) and similarly for \( \tilde{b}_j \). Clearly \((2.10.3)\) is \( \mathbb{Z} \)-linear in the \( b_j \); the linearity in \( a \) follows from

\[
(1 + \tilde{a}_1)(1 + \tilde{a}_2) = (1 + \tilde{a}_1 + \tilde{a}_2)(1 + t_i c),
\]
for \( \tilde{a}_i, c \in \tau R \). Furthermore, if \( p \neq 2 \), then

\[
\{1 + \tau c, \tilde{b}\} \equiv 2 \cdot \{1 + \frac{1}{2} \tau c, \tilde{b}\} \mod V_r^{i-1}.
\]

Thus the formula \( \{b, b\} = \{b, -1\} \) in \( K_2^M(k(X)) \) implies that \((2.10.3)\) is alternating in the \( b \)'s (also for \( p = 2 \)). Altogether we see that \((2.10.3)\) induces a surjective map

\[
(2.10.4) \quad \tau R_i \otimes_{\mathbb{Z}} \bigwedge_{j=1}^{r-1} M^{\text{sp}} \to \frac{V^i_r}{V^{i-1}_r}.
\]

We claim that this map factors via \( \tau \cdot \Omega_{R_i}^{r-1}(\log(t_{i+1} \cdots t_n)) \). By [Kat89, (1.7)] we have to show that \((2.10.4)\) maps the following elements to zero

\[
(2.10.5) \quad \tau a \otimes a \land b_1 \land \cdots \land b_{r-2} - \sum_{\nu} \tau u_\nu \otimes u_\nu \land b_1 \land \cdots \land b_{r-2},
\]
where \( a \in M = M^{\text{sp}} \cap R_i \) and \( u_\nu \in R_i^\times \) such that \( a = \sum_\nu u_\nu \), and \( b_j \in M^{\text{sp}} \). Choose lifts \( \tilde{u}_\nu \in R_i^\times \) of \( u_\nu \) and set \( \tilde{a} = \sum_\nu \tilde{u}_\nu \). Let \( \tilde{b} = (\tilde{b}_1, \ldots, b_{r-2}) \) with \( \tilde{b}_j \in R_i[\frac{1}{t_{i+1} \cdots t_n}]^\times \).
lifts of \( b_j \). Write
\[
(1 + \tau \sum \hat{u}_\nu) = \prod \nu (1 + \tau \hat{u}_\nu) \cdot (1 + \tau t_i e),
\]
for some \( e \in R \). Assume first \( e \in R^x \) is a unit. Then in \( K^M_r(k(X)) \)
\[
\{1 + \tau \hat{a}, \hat{a}, \hat{b}\} = \{1 + \tau \hat{a}, -\tau, \hat{b}\} = -\sum \nu \{1 + \tau \hat{u}_\nu, -\tau, \hat{b}\} - \{1 + \tau t_i e, -\tau, \hat{b}\} = \sum \nu \{1 + \tau \hat{u}_\nu, \hat{u}_\nu, \hat{b}\} + \underbrace{\{1 + \tau t_i e, t_i e, \hat{b}\}}_{\in V_i^{t_i - 1}}.
\]

Since the starting term and the final term are in \( V_i^t \), the equality of these two terms holds in \( V_i^t \). If \( e \) is not a unit we can write
\[
(1 + \tau t_i e) = (1 + \tau t_i (e - 1)) \cdot (1 + \tau t_i \frac{1}{1 + \tau t_i (e - 1)})
\]
and argue similarly. Hence (2.10.4) sends the relation (2.10.5) to zero and induces a surjection
\[
(2.10.6) \quad \tau \cdot \Omega_{R_i}^{-1}(\log(t_{i+1} \cdots t_n)) \to V_i^t / V_i^{t_i - 1}.
\]
Assume \( s_i \geq 1 \). Let \( \sigma^{p_i} \) be the equation (2.8.2) for \( p_i D(i) \). We want to show that (2.10.6) maps \( \sigma^{p_i} \cdot B_i \cdot \Omega_{R_i}^{-1}(\log(t_{i+1} \cdots t_n)) \) to zero. The latter module is generated as abelian group by the elements (cf. [II79, 0, Proposition 2.2.8])
\[
(2.10.7) \quad (\sigma^{p_i + \nu} u) p_i \log u \log v_1 \cdots \log v_{t-2}, \quad u \in R_i^x, v_j \in M_i^{\text{sp}}, \quad 0 \leq \nu \leq s_i - 1.
\]
We compute in \( K^M_r(k(X)) \) (cf. [BK86, Lemma (4.5)])
\[
\{1 + (\sigma^{p_i + \nu} \hat{u}) p_i \log \hat{u} \log \hat{v}_{\nu}, \hat{u}, \hat{v}_{\nu}\} = p_i^{\nu} \{1 + \sigma^{p_i + \nu} \hat{u}, \hat{u}, \hat{v}_{\nu}\}
\]
\[
= -p_i^{\nu} \{1 + \sigma^{p_i + \nu} \hat{u}, -\sigma^{p_i + \nu} \hat{u}, \hat{v}_{\nu}\}
\]
\[
= -\{1 + \sigma^{p_i + \nu} \hat{u}, \hat{u}, \hat{v}_{\nu}\}
\]
\[
= \{1 + \sigma^{p_i + \nu - 1} w_1, w_2, \hat{v}_{\nu}\}, \quad \text{with } w_j \in R^x,
\]
where the last equality holds by Lemma 2.5(2). Since \( p_i^{2s_i - \nu} - 1 \geq 2 \), for all \( \nu \leq s_i - 1 \), the last term lies in \( V_i^{t_i - 1} \). Thus (2.10.6) maps the element (2.10.7) to zero. We obtain a surjective morphism \( \theta \) as in the statement. Set \( S_i := R_i[i_1, \ldots, i_{s_i}] \) (we omit \( t_i \) in the denominator). Consider the following diagram
\[
Q_i^{r, t_i - 1}(R_i) = \frac{\tau \cdot \Omega_i^{-1}(\log(t_{i+1} \cdots t_n))}{\sigma^{p_i} \cdot B_i \cdot \Omega_i^{-1}(\log(t_{i+1} \cdots t_n))} \xrightarrow{\theta \text{ restr.}} \frac{\Omega_i^{-1}}{B_i \Omega_i^{-1} S_i} \xrightarrow{t_i - 1} \Omega_i^{-1} S_i \xrightarrow{2.4.1} \Omega_i^{-1} S_i.
\]
The diagram is commutative and the vertical arrow on the right hand side is injective by Lemma 2.4. We can write \( R \) as a directed limit of rings \( A \) which are smooth over \( F_p \) and on which \( t_1 \cdots t_n \) defines a simple normal crossing divisor. For \( R \) replaced by such an \( A \) the top horizontal map is injective by Lemma 2.8; since directed limits are exact this holds for \( R \) as well. Hence \( \theta \) is injective, which yields the statement. \( \square \)
Remark 2.11. Let $X$, $D$, $D_j$, $E_i$, and $V^i_r = V^i_{r,X|D}$ be as in 2.6 and $Q^{i,r-1}_D$ be as in 2.7. Let $f : Y \to X$ be a smooth morphism and define $V^i_{r,Y|f^*D}$ and $Q^{i,r-1}_{f^*D}$ similarly, where we replace $D_j$ by $f^*D_j$ (which is smooth but possibly not connected) and $E_i$ by $f^*E_i$. We obtain a commutative diagram on $X_{Nis}$

$$
\begin{array}{c}
\larrow{f_* Q^{i,r-1}_{f^*D}} \downarrow{\theta} \larrow{f_* (V^i_{r,Y|f^*D}/V^i_{r,Y|f^*D})} \\
Q^{i,r-1}_D \downarrow{\theta} V^i_{r-1,X|D}/V^i_{r,X|D},
\end{array}
$$

where the vertical maps are the natural pullback maps and the horizontal maps are the isomorphisms constructed in Proposition 2.10. By the explicit formula for $\theta$ the diagram obviously commutes.

The following corollary is not used in this paper but we include it here for use in [RS21b].

Corollary 2.12. Let $X \in \text{Sm}$ with $\dim X = d$ and let $D$ be an effective Cartier divisor with simple normal crossing support. Set $\mathcal{G} := V^i_{r,X|D}/V^i_{r,X|D + D_{red}}$, $r \geq 1$, which we view as a sheaf on $Y = D_{red}$. Then the natural morphism $H^{d-1}(Y_{\text{Zar}}, \mathcal{G}) \to H^{d-1}(Y_{\text{Nis}}, \mathcal{G})$ is surjective.

Proof. With the notation from 2.1 we set

$$
\mathcal{F} := \frac{U^i_{r,X|D + D_{red}}}{U^i_{r,X|D + 2D_{red}}} \quad \text{and} \quad \mathcal{H} := \frac{V^i_{r,X|D}}{U^i_{r,X|D + D_{red}}}.
$$

We have an exact sequence of Nisnevich sheaves

$$
(2.12.1) \quad \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0.
$$

By Grothendieck-Nisnevich vanishing we obtain exact sequences for $\tau \in \{\text{Zar, Nis}\}$

$$
H^{d-1}(Y_\tau, \mathcal{F}) \to H^{d-1}(Y_\tau, \mathcal{G}) \to H^{d-1}(Y_\tau, \mathcal{H}) \to 0.
$$

By 2.6 and Proposition 2.10 the quotient $\mathcal{H}$ admits a finite decreasing filtration whose successive quotients are coherent $\mathcal{O}_Y$-modules. Hence the natural morphism $H^{d-1}(Y_{\text{Zar}}, \mathcal{H}) \to H^{d-1}(Y_{\text{Nis}}, \mathcal{H})$ is an isomorphism. By [RS18, Proposition 2.15, Theorem 2.19] (cf. also 2.15 below) the same holds for $\mathcal{F}$. This yields the statement. \hfill \Box

Theorem 2.13. Let $X$ be a smooth $k$-scheme of pure dimension $d$ and $D$ an effective divisor on $X$ whose support has simple normal crossings. Let $E$ be a vector bundle of dimension $N + 1$ on $X$ and denote by $\pi : P := \text{Proj}(\text{Sym} E^\vee) \to X$ the projection from the structure map. Let

$$
\eta^t := c_1(\mathcal{O}_P(1))^t \in H^t_{\text{Nis}}(P, K^M_{\mathcal{L}P}) = \text{Hom}_{D(P_{\text{Nis}})}(\mathbb{Z}, K^M_{\mathcal{L}P}[t]), \quad t = 0 \ldots , N.
$$

The map in $D(X_{\text{Nis}})$, the derived category of abelian Nisnevich sheaves on $X$,

$$
(2.13.1) \quad - \cup \eta^t : \bigoplus_{t=0}^{N} V_{r-t,X|D\lceil -t} \xrightarrow{\cong} R\pi_* V_{r,P|\pi^*D}, \quad r \geq 0,
$$

is an isomorphism. If $D_{\text{red}}$ is smooth the same is true for the Zariski site.
Proof. Write $D = \sum_{j=1}^{n} m_j D_j$ as in 2.6. It suffices to show that $\cup \eta^i$ induces an isomorphism $V_{r-t,X|D} \xrightarrow{\sim} R^t\pi_*V_{r,P|\pi^*D}$, for all $0 \leq t \leq N$, and $R^t\pi_*V_{r,P|\pi^*D} = 0$, for all $t > N$. This is a local question and we may therefore assume $P = P_X$. The corresponding statement for $U_{r,X|D+D_{red}}$ holds by [RS18, Theorem 2.28] (see also 2.16) below. By Proposition 2.10 (and Remark 2.11) we are reduced to show:

(1) $\cup d\log (\eta^i) : Q^r_D^{i,-1} \to R^i\pi_*Q^{i,-1}_D$ is an isomorphism, for all $0 \leq t \leq N$ and $1 \leq i \leq n$;

(2) $R^t\pi_*Q^{i,-1}_D = 0$, for all $t > N$ and all $1 \leq i \leq n$,

where $Q^{i,-1}_D$ is defined in 2.7. By the definition of $Q^{r,-1}_D$ and the projection formula (e.g. [EGAIII, Proposition (0.12.2.3)]), this follows from the corresponding statements for $\Omega^{r,-1}_D(\log E_i)$ and - if $p > 0$ - for $B_s\Omega^{r,-1}_D(\log E_i)$ (where $E_i = D_{i+1} + \ldots + D_n$). These are well-known, cf. (2.16.2) and the proof of (2.16.4) (in the case $\text{char}(k) > 0$) below.

\[\square\]

Definition 2.14. Let the notations and assumptions be as in Theorem 2.13. We define

$$\text{tr}_\pi : R\pi_*V_{N+r,P|\pi^*D}[N] \to V_{r,X|D}$$

to be the composition of the natural map $R\pi_*V_{N+r,P|\pi^*D}[N] \to R^N\pi_*V_{N+r,P|\pi^*D}$ followed by the inverse of the isomorphism $- \cup \eta^N : V_{r,X|D} \xrightarrow{\sim} R^N\pi_*V_{N+r,P|\pi^*D}$.

2.1 Small erratum to [RS18]. We use the opportunity to slightly correct an assumption and a proof from [RS18], see 2.15 and 2.16 below. Please note that these corrections do not have any effect on the rest of loc. cit., in particular not on any of the main results.

2.15. In [RS18, 2.1.4] we consider the following situation: Let $X$ be a smooth and separated scheme over $k$. Let $\sum_{\lambda \in \Lambda} D_\lambda$ be a simple normal crossing divisor (with smooth components $D_\lambda$). For $m = (m_\lambda)_{\lambda \in \Lambda} \in \mathbb{N}^{[\Lambda]}$ we define $D_m = \sum_{\lambda} m_\lambda D_\lambda$ and set for $\nu \in \Lambda$

$$\text{gr}_m^\nu K^M_{r,X} := U_{r,X|D_m}/U_{r,X|D_m+\delta_\nu},$$

where $\delta_\nu = (0, \ldots, 0, 1, 0, \ldots, 0)$ with 1 in the $\nu$’s place. (In loc. cit. $U_{r,X|D}$ is denoted by $K^M_{r,X|D}$.) In the Propositions 2.14, 2.15, Theorem 2.19, and Corollary 2.20 of [RS18] we give formulas for $\text{gr}_m^\nu K^M_{r,X}$. In loc. cit. we allow also some of the $m_\lambda$ to be zero and in this case the formulas are not correct as written. However, if $m_\lambda \geq 1$, for all $\lambda \in \Lambda$, these formulas are correct and this is the only case used in the rest of [RS18]. (The formulas are used to understand the difference between $U_{r,X|D}$ and $U_{r,X|D_{red}}$; the latter group can then be studied using Corollary 2.10, which is correct as written, i.e., the $m_\lambda$ may be zero.)

2.16. Let the notation and assumptions be as in [RS18, Lemma 2.25]. We correct the proof of the isomorphisms

(2.16.1)

$$- \cup d\log(c_1(\mathcal{O}(1))^i) : \omega^{q,-i}_{X|D,m,\nu}/B^{q,-i}_{X,D,m,\nu} \xrightarrow{\sim} R^i\pi_*\omega^{q}_{D,m,\nu}/B^q_{P_X|D,m,\nu},$$

for $i \geq 0$. To this end replace on page 1013 everything after “Now we prove the statement for $\omega^{q}_{m,\nu}$” in line 8 until “...from Lemma 2.26 below.” in line 25 by the following:

We have a well-known isomorphism

(2.16.2)

$$- \cup d\log(c_1(\mathcal{O}(1))^i) : \Omega^{q,-i}_X(\log D) \xrightarrow{\sim} R^i\pi_*\Omega^{q}_{X,\nu}(\log P_D).$$
induces an isomorphism $w$ results from the recursive definition of and it is an isomorphism at the stalks $x$.

More precisely, let $w_j = w_j \Omega_X^j(\log D)$ be the $\mathcal{O}_X$-submodule of $\Omega_X^j(\log D)$ whose sections have poles along at least $j$ different components of $D$. The Poincaré residue induces an isomorphism $w_j / w_{j-1} \cong \Omega_{D_j}^{j-1}$, where $D_j = \cup_{i_1 < \cdots < i_j} D_{i_1} \cap \cdots \cap D_{i_j}$; cf. [De71, (3.1.5.2)]. This isomorphism is compatible with the pullback to $P_X$ and we can deduce the isomorphism (2.16.2).) Using projection formula and base-change (2.16.2) yields an isomorphism

\begin{equation}
\label{eq:isomorphism}
- \cup \mathrm{dlog}(c_1(\mathcal{O}(1))) : \omega_{X|D,m,\nu}^{q-i} \cong R^i \pi_{X *} \omega_{P_X|P_D,m,\nu}^{q}, \quad i \geq 0.
\end{equation}

It remains to show

\begin{equation}
\label{eq:isomorphism2}
- \cup \mathrm{dlog}(c_1(\mathcal{O}(1))) : B_{X|D,r,m,\nu}^{q-i} \cong R^i \pi_{X *} B_{P_X|P_D,r,m,\nu}^{q}, \quad i \geq 0.
\end{equation}

By a limit argument we can reduce to the case that $k$ is finitely generated over its prime field, so that $\Omega_X^j(\log D) = 0$, for $q$ large enough.

**First case:** $\operatorname{char}(k) = 0$. In this case the statement follows by descending induction on $q$ from the exact sequence

\begin{equation}
\label{eq:exact_sequence}
0 \to Z_{X|D,m,\nu}^q \to \omega_{X|D,m,\nu}^q \to B_{X|D,m,\nu}^{q+1} \to 0,
\end{equation}

the equality $Z_{X|D,m,\nu}^q = B_{X|D,m,\nu}^q$ (see [BS19, Lemma 6.2]), and (2.16.3). (Lemma 2.26 is not needed.)

**Second case:** $\operatorname{char}(k) = p$. For $r = 1$ the statement follows by descending induction on $q$ from the exact sequence (2.16.5) together with the exact sequence

\begin{equation}
0 \to B_{X|D,m,\nu}^q \to Z_{X|D,m,\nu}^q \to \omega_{m',\nu}^q \to 0,
\end{equation}

coming from the Cartier isomorphism [RS18, Theorem 2.16]. For higher $r$ it follows from the recursive definition of $B_{X|D,r,m,\nu}$ in [RS18, 2.4.4].

### 2.2. Comparison of different relative Milnor $K$-sheaves

We will have to use results from [KS86] in which a slightly different version of the relative Milnor $K$-theory is used. Here we observe that these different versions give rise to the same top degree Nisnevich cohomology, so that the difference won’t play a role for us.

#### 2.17. Let $X$ be a $k$-scheme and $D \subset X$ a nowhere dense closed subscheme. Denote by

\[ K_{r,X}^{M,\text{naive}} = (\mathcal{O}_X^\times)^{\otimes n} / J, \]

the naive Milnor $K$-sheaf on $X_{\text{Nis}}$, where $J \subset (\mathcal{O}_X^\times)^{\otimes r}$ denotes the abelian subsheaf locally generated by elements of the form $a_1 \otimes \ldots \otimes a_r$, all $a_i \in \mathcal{O}_X^\times$, and $a_i + a_j = 1$, for some $i \neq j$. By [Ker10, Proposition 10 and Theorem 13] there is a surjective map

\begin{equation}
\label{eq:surjective_map}
K_{r,X}^{M,\text{naive}} \twoheadrightarrow K_{r,X}^M
\end{equation}

and it is an isomorphism at the stalks $x$ with infinite residue field $k(x)$. In [KS86, (1.3)] the relative Milnor $K$-sheaf of $(X, D)$ is defined by the formula

\begin{equation}
\label{eq:definition}
K_{(X, D)}^M = \text{Ker}(K_{r,X}^{M,\text{naive}} \rightarrow i_* K_{r,D}^{M,\text{naive}}),
\end{equation}

where $I$ is the ideal sheaf of $D$ and $i : D \rightarrow X$ denotes the closed immersion.
Lemma 2.18. Let the notations and assumptions be as in 2.17. Assume $X$ is noetherian, reduced, and of pure dimension $d < \infty$. Then (2.17.1) induces a surjection
\[(2.18.1) \quad K^M_r (O_X, I) \twoheadrightarrow V_{r, X|D},\]
and the induced map on top-degree Nisnevich cohomology is an isomorphism
\[(2.18.2) \quad H^d(X_{\text{Nis}}, K^M_r (O_X, I)) \xrightarrow{\sim} H^d(X_{\text{Nis}}, V_{r, X|D}).\]

Proof. The surjection in (2.18.1) holds by [KS86, Lemma 1.3.1]. By [Ker10, Proposition 10(4)] the map (2.17.1) is an isomorphism on fields. Hence (2.18.1) has kernel supported in codimension $\geq 1$. The isomorphism in (2.18.2) follows therefore from Grothendieck-Nisnevich vanishing. \(\square\)

3. THE CYCLE CLASS MAP

In this section $k$ denotes a field. For a scheme $Z$ we denote by $Z^{(i)}$ (resp. $Z^{(0)}$) the set of points of $Z$ whose closure have (co)dimension $i$.

3.1. Let $X$ be an equidimensional $k$-scheme of finite type and $D$ an effective Cartier divisor on $X$ such that $U = X \setminus |D|$ is smooth over $k$. Set $d = \dim X$. For $C \subseteq X$ an integral curve not contained in the support of $D$ and with normalization $\nu : \widetilde{C} \to C$, we set
\[(3.1.1) \quad G(C, D) := \bigcap_{x \in \widetilde{C} \cap \nu^{-1}(D)} \ker (\mathcal{O}^x_{\tilde{C}, x} \to \mathcal{O}^x_{\tilde{C} \times C, D, x}) \subseteq k(C)^\times.\]

We define a map
\[\partial_C : G(C, D) \xrightarrow{\text{div}} Z_0(\widetilde{C}) \xrightarrow{\nu_*} Z_0(U),\]
and put
\[\text{CH}_0(X|D) = \text{Coker} \left( \partial = \sum_C \partial_C : \bigoplus_C G(C, D) \to Z_0(U) \right),\]
where the sum is over all $C \subseteq X$ as above and $Z_0(Y)$ denotes the group of zero-cycles on $Y$.

For a closed point $x \in U$ the Gersten resolution ([Ker10, Proposition 10(8)]) yields an isomorphism
\[(3.1.2) \quad \theta_x : \mathbb{Z} \xrightarrow{\sim} H^d_{x}(U_{\text{Nis}}, K^M_{d,U}) \cong H^d_{x}(X_{\text{Nis}}, V_{d,X|D}),\]
where we use the notation from 2.1. By [KS86, Theorem 2.5] and Lemma 2.18 we obtain a surjective map
\[(3.1.3) \quad \theta = \sum_x \theta_x : Z_0(U) = \bigoplus_{x \in U^{(0)}} \mathbb{Z} \twoheadrightarrow H^d(X_{\text{Nis}}, V_{d,X|D}).\]

Similarly, we can define a map $Z_0(U) \to H^d(X_{\text{Nis}}, U_{d,X|D})$. As a special case of [RS18, Proposition 3.3] this map factors via $\text{CH}_0(X|D)$, if $X$ is smooth and $D$ has simple normal crossing support. The following theorem is a refinement of this statement with $U_{d,X|D}$ replaced by $V_{d,X|D}$. It implies Theorem 1.1, by (2.18.2).

Theorem 3.2. Let the notation and assumptions be as in 3.1 above. We assume additionally that $X$ is smooth and that the support of $D$ is a simple normal crossing divisor. Then (3.1.3) factors to give a surjective map
\[\text{CH}_0(X|D) \twoheadrightarrow H^d(X_{\text{Nis}}, V_{d,X|D}).\]
See the introduction for a discussion on under which assumptions on $X,D,k$ the theorem was previously known (in view of (2.18.2)). Before we can prove the theorem, we need to recall some notations and results from [RS18].

3.3. Let $Y = \text{Spec } A$ be an affine scheme. Let $s_1, \ldots, s_c \in A$ and set $Z = \text{Spec } A/(s_1, \ldots, s_c)$. Then $\mathfrak{V} = \{V_1, \ldots, V_c\}$ with $V_i = \text{Spec } A[\frac{1}{s_i}]$ is an open covering of $Y \setminus Z$. Let $F$ be a sheaf of abelian groups on $Y_{\text{Nis}}$ and denote by $C^\bullet(\mathfrak{V}, F)$ the Čech complex of $F$ with respect to $\mathcal{V}$. We obtain natural maps

\begin{equation}
(3.3.1) \quad F(V_1 \cap \ldots \cap V_c) = C^{c-1}(\mathfrak{V}, F) \to H^{c-1}(C^\bullet(\mathfrak{V}, F)) \to H^{c-1}((Y \setminus Z)_{\text{Zar}}, F) \to H^{c-1}((Y \setminus Z)_{\text{Nis}}, F) = H^c_Z(Y_{\text{Nis}}, F),
\end{equation}

where the last map is the boundary map of the localization sequence. For $a \in F(V_1 \cap \ldots \cap V_c)$ we denote by

\[
\begin{bmatrix}
a \\
\end{bmatrix}
\begin{bmatrix}
s_1, \ldots, s_c \\
\end{bmatrix} \in H^c_Z(Y_{\text{Nis}}, F)
\]

the image of $a$ under (3.3.1). We note the following two obvious functoriality properties of this symbol:

(1) if $h : F \to G$ is a morphism of sheaves then

\[
h\left(\begin{bmatrix}
a \\
\end{bmatrix}
\begin{bmatrix}
s_1, \ldots, s_c \\
\end{bmatrix}\right) = \begin{bmatrix}
h(a) \\
\end{bmatrix}
\begin{bmatrix}
s_1, \ldots, s_c \\
\end{bmatrix}
\text{ in } H^c_Z(Y_{\text{Nis}}, G);
\]

(2) if $j : U \hookrightarrow Y$ is an affine open immersion, then

\[
j^*\left(\begin{bmatrix}
a \\
\end{bmatrix}
\begin{bmatrix}
s_1, \ldots, s_c \\
\end{bmatrix}\right) = \begin{bmatrix}
j^*a \\
\end{bmatrix}
\begin{bmatrix}
j^*s_1, \ldots, j^*s_c \\
\end{bmatrix}
\text{ in } H^c_{Z \cap U}(U_{\text{Nis}}, F|_U).
\]

Now assume $Y$ is a regular $k$-scheme of pure dimension $d$ and $y \in Y^{(d-1)}$. In this case the Gersten complex of $K^{d,Y}_{d,Y}$ is a flasque resolution and we obtain an isomorphism

\begin{equation}
(3.3.2) \quad k(y)^\times \overset{\cong}{\to} H^{d-1}_y(Y_{\text{Zar}}, K^{M}_{Y,d}) = H^{d-1}_y(Y_{\text{Nis}}, K^{M}_{Y,d}) = H^{d-1}_y(Y_{(y),\text{Nis}}, K^{M}_{Y,d}),
\end{equation}

where $Y_{(y)} = \text{Spec } \mathcal{O}_{Y,y}$. Let $s_1, \ldots, s_{d-1} \in \mathcal{O}_{Y,y}$ be a regular sequence of parameters. Under the isomorphism (3.3.2) a function $f \in k(y)^\times$ is mapped to the symbol

\[
\begin{pmatrix}
\pm \{\hat{f}, s_1, \ldots, s_{d-1}\} \\
s_1, \ldots, s_{d-1}
\end{pmatrix},
\]

where $\hat{f} \in \mathcal{O}_{Y,y}$ is any lift of $f$, see [RS18, Corollary 2.3].

(To see this one considers the Čech complex of the Gersten resolution of $K^{M}_{Y,d}$. This yields a double complex whose associated complex comes with natural augmentation maps from the Gersten complex and the Čech complex of $K^{M}_{Y,d}$, which both are quasi-isomorphisms. If one applies the image of $f$ under the induced isomorphisms on cohomology one obtains the above description, for details see loc. cit.)

Proof of Theorem 3.2. The proof is similar to [RS18, Proposition 3.3], though we will need the projective bundle formula for $V_{r,X/L}$ proved in the previous section and we take a shortcut around the Cousin resolution used in loc. cit.

3The sign depends on the choice of sign in the definition of the tame symbols which appear in the Gersten resolution.
We have the spectral sequence
\[ E_{1}^{i,j} = \bigoplus_{x \in X^{(0)}} H_{x}^{i+j}(X_{\text{Nis}}, V_{d,x|D}) \Rightarrow H^{i+j}(X_{\text{Nis}}, V_{d,x|D}). \]

Grothendieck vanishing ([Nis89, Corollary 1.3.3]) yields
\[ H^{d}(X_{\text{Nis}}, V_{d,x|D}) = E_{2}^{d,0} \]
\[ = \text{Coker} \left( \bigoplus_{x \in X^{(0)}} H_{x}^{d-1}(X_{\text{Nis}}, V_{d,x|D}) \rightarrow \bigoplus_{x \in X^{(0)}} H_{x}^{d}(X_{\text{Nis}}, V_{d,x|D}) \right). \]

**Claim 3.3.1.** Assume \( C \subset X \) is an integral regular curve not contained in \( D \) and \( f \in k(C)^{\times} \) with \( f \equiv 1 \mod D_{|C} \). Then
\[ \text{div}(f) \in \bigoplus_{D(0)} \mathbb{Z} \subset \bigoplus_{x \in X^{(0)}} H_{x}^{d}(X_{\text{Nis}}, V_{d,x|D}), \]
lies in the image of
\[ \partial = (\partial_{x})_{x} : k(C)^{\times} \cong H_{c}^{d-1}(X_{\text{Nis}}, V_{d,x|D}) \rightarrow \bigoplus_{x \in X^{(0)}} H_{x}^{d}(X_{\text{Nis}}, V_{d,x|D}), \]
where \( c \) is the generic point of \( C \).

We prove the claim. We have \( \partial(f)|_{U} = \pm \text{div}_{C\cap U}(f) \) (e.g. [RS18, (3.3.1)]). Thus it remains to show \( \partial_{x}(f) = 0 \), for all \( x \in D(0) \cap C \). To this end it suffices to show that \( f \) lies in the image
\[ H_{C(x)}^{d-1}(\text{Spec}\mathcal{O}_{X,x}, V_{d,x|D}) \rightarrow H_{C(x)}^{d-1}(X, V_{d,x|D}), \]
for \( x \in D(0) \cap C \), where \( C(x) = \text{Spec}\mathcal{O}_{C,x} \). Under the isomorphism
\[ k(C)^{\times} \cong H_{c}^{d-1}(X, V_{d,x|D}) = H_{c}^{d-1}(X, K_{d,X}^{M}) \]
f corresponds to (see 3.3)
\[ \pm \left[ \{ \hat{f}, s_{1}, \ldots, s_{d-1} \} \right], \]
where \( \hat{f} \in \mathcal{O}_{X,c} \) is a lift of \( f \) and \( s_{1}, \ldots, s_{d-1} \in \mathfrak{m}_{c} \) is a regular system of parameters. In fact since \( C \) and \( X \) are regular, the closed immersion \( C \hookrightarrow X \) is regular and hence we can choose the \( s_{i} \)'s to be a regular sequence in \( \mathcal{O}_{X,c} \) generating the ideal sheaf of \( C \) at \( x \). Furthermore, since \( f \in \mathcal{O}_{C(D\cap C,x)}^{\times} \) by assumption, we can choose the lift \( \hat{f} \) to lie in \( \mathcal{O}_{X,D,x}^{\times} \). Together with 3.3(1), (2) this shows that (3.3.6) lies in the image of (3.3.5) and proves Claim 3.3.1.

Assume \( C \subset X \) is an arbitrary integral curve not contained in \( D \). Let \( \nu : \widetilde{C} \rightarrow C \) be the normalization. We embed \( \widetilde{C} \) in \( P_{X} = P_{X}^{d} \) over \( X \). Denote by \( \pi : P_{X} \rightarrow X \) the projection and by \( \pi_{U} \) its base change over \( U \). Consider the following diagram
\[
\begin{array}{c}
Z_{0}(\widetilde{C} \cap P_{U}) & \xrightarrow{\nu_{*}} & H_{(P_{U})_{0}}^{d+n}(P_{U,Nis}, K_{d+n,P_{U}}^{M}) \xrightarrow{\text{tr}_{\pi_{U}}} & H_{d}(P_{X,Nis}, V_{d+n,P_{X}|P_{D}}) \\
Z_{0}(C \cap U) & \xrightarrow{\text{tr}_{\pi_{U}}} & H_{U(0)}^{d}(U_{Nis}, K_{d,U}^{M}) & \xrightarrow{\text{tr}_{x}} & H_{d}(X_{\text{Nis}}, V_{d,x|D}) \\
\end{array}
\]
where $\text{tr}_\pi$ and $\text{tr}_{\pi_U}$ are defined in 2.14 and we view $U_{(0)}$ and $(P_U)_{(0)}$ as families of closed supports on $U$ and $P_U$, respectively. The left square commutes by [RS18, Lemma 2.27], the right square by construction, and the composite horizontal maps are equal to (3.1.3). Since $\widetilde{C}$ is regular Claim 3.3.1 and (3.3.3) yield that $\text{div}_\widetilde{C}(f)$ is mapped to zero in $H^{d+n}_{\text{dR}}(X_{\text{Nis}}, V_{d+n,P_X|P_D})$, for $f \equiv 1 \mod D_{\widetilde{C}}$. Hence $\text{div}_C(f) = \nu_* \text{div}_\widetilde{C}(f)$ is mapped to zero in $H^{d}(X_{\text{Nis}}, V_{d,X|D})$. This completes the proof. □

Remark 3.4. The assumption that the support of $D$ is a simple normal crossing divisor, was only used to apply the projective bundle formula, Theorem 2.13.

References

[BK86] S. Bloch and K. Kato, $p$-adic étale cohomology, Institut des Hautes Études Scientifiques. Publications Mathématiques, Inst. Hautes Études Sci. Publ. Math., (1986) 63, 107–152.

[BKS21] F. Binda, A. Krishna, and S. Saito, Bloch’s formula for 0-cycles with modulus and higher dimensional class field theory, to appear in J. Algebraic Geom., https://arxiv.org/abs/2002.01856.

[BS19] F. Binda and S. Saito, Relative cycles with moduli and regulator maps, J. Inst. Math. Jussieu 18 (2019), no. 6, 1233–1293.

[De71] P. Deligne, Théorie de Hodge. II, Inst. Hautes Études Sci. Publ. Math., 40 (1971), 5–57.

[EGAIII] A. Grothendieck, Éléments de géométrie algébrique. III. Étude cohomologique des faisceaux cohérents. I, Inst. Hautes Études Sci. Publ. Math., 11 (1961), 1–167.

[GK22] R. Gupta and A. Krishna, Ïdele class groups with modulus, Adv. Math., 404 (2022) 75 pp.

[GK20] R. Gupta and A. Krishna, K-theory and 0-cycles on schemes, J. Algebraic Geom., 29 (2020), no. 3, 547–601.

[Kat89] K. Kato, Logarithmic structures of Fontaine-Illusie, Algebraic analysis, geometry, and number theory (Baltimore, MD, 1988) (1989), 191–224, Johns Hopkins Univ. Press, Baltimore, MD.

[Ill79] L. Illusie, Complexe de de Rham-Witt et cohomologie cristalline, Ann. Sci. École Norm. Sup. (4) 12 (1979), no. 4, 501–661.

[KS86] K. Kato and S. Saito, Global class field theory of arithmetic schemes, Applications of algebraic $K$-theory to algebraic geometry and number theory, Part I, II (Boulder, Colo., 1983), Contemp. Math., vol. 55, Amer. Math. Soc., Providence, RI, 1986, pp. 255–331. MR 862639

[Ker10] M. Kerz, Milnor $K$-theory of local rings with finite residue fields, J. Algebraic Geom. 19 (2010), no. 1, 173–191.

[KS16] M. Kerz and S. Saito, Chow group of 0-cycles with modulus and higher-dimensional class field theory, Duke Math. J. 165 (2016), no. 15, 2811–2897. MR 3557274

[KS14] M. Kerz and S. Saito, Lefschetz theorem for abelian fundamental group with modulus, Algebra Number Theory 8 (2014), no. 3, 689–701.

[Kri15] A. Krishna, On 0-cycles with modulus, Algebra Number Theory 9 (2015), no. 10, 2397–2415.

[Kri18] A. Krishna, Torsion in the 0-cycle group with modulus, Algebra Number Theory, 12 (2018), no. 6, 1431–1469.

[Nis89] Y. A. Nisnevich, The completely decomposed topology on schemes and associated descent spectral sequences in algebraic $K$-theory, Algebraic $K$-theory: connections with geometry and topology (Lake Louise, AB, 1987), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol. 279, Kluwer Acad. Publ., Dordrecht, 1989, pp. 241–342.

[RS21a] K. Rülling and S. Saito, Ramification theory of reciprocity sheaves, I, Zariski-Nagata Purity. Preprint 2021, https://arxiv.org/abs/2111.01459.

[RS21b] K. Rülling and S. Saito, Ramification theory of reciprocity sheaves, II, Higher local symbols. Preprint 2021, https://arxiv.org/abs/2111.13373.
[RS18] K. Rülling and S. Saito, Higher Chow groups with modulus and relative Milnor $K$-theory, Trans. Amer. Math. Soc. 370 (2018), no. 2, 987–1043.

[SGA7II] Groupes de monodromie en géométrie algébrique. II, Lecture Notes in Mathematics, Vol. 340, Séminaire de Géométrie Algébrique du Bois-Marie 1967–1969 (SGA 7 II), Dirigé par P. Deligne et N. Katz, Springer-Verlag, Berlin-New York, (1973).

Bergische Universität Wuppertal, Gaussstr. 20, D-42119 Wuppertal, Germany
Email address: ruelling@uni-wuppertal.de

Graduate School of Mathematical Sciences, University of Tokyo, 3-8-1 Komaba, Tokyo 153-8941, Japan
Email address: sshuji@msb.biglobe.ne.jp