UNBOUNDED HANKEL OPERATORS AND MOMENT PROBLEMS

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Abstract. We find necessary and sufficient conditions for a non-negative Hankel quadratic form to admit the closure. We also describe the domain of the corresponding closed form. This allows us to define unbounded non-negative Hankel operators under optimal assumptions on their matrix elements. The results obtained supplement the classical Widom condition for a Hankel operator to be bounded.

1. Main results. Discussion

1.1. Hankel operators $Q$ can formally be defined in the space $\ell^2(\mathbb{Z}_+)$ of sequences $g = (g_0, g_1, \ldots)$ by the formula

$$ (Qg)_n = \sum_{m=0}^{\infty} q_{n+m}g_m, \quad n = 0, 1, \ldots $$

(1.1)

Thus the matrix elements of a Hankel operator depend on the sum of indices only.

The precise definition of the operator $Q$ requires some accuracy. Let $D \subset \ell^2(\mathbb{Z}_+)$ be the dense set of sequences $g = (g_0, g_1, \ldots)$ with only a finite number of non-zero components. If the sequence $q = (q_0, q_1, \ldots) \in \ell^2(\mathbb{Z}_+)$, then for $g \in D$ sequence (1.1) also belongs to $\ell^2(\mathbb{Z}_+)$. In this case the operator $Q$ is defined on $D$, and it is symmetric if $q_n = \overline{q_n}$. Without any a priori assumptions on $q_n$, only the quadratic form

$$ q[g, g] = \sum_{n,m \geq 0} q_{n+m}g_m\overline{g_n} $$

(1.2)

is well defined for $g \in D$.

The fundamental theorem of Nehari [6] guarantees that a Hankel operator $Q$ (defined, possibly, via its quadratic form (1.2)) is bounded if and only if $q_n$ are the Fourier coefficients of some bounded function on the unit circle $\mathbb{T}$. The theory of Hankel operators is a very well developed subject. We refer to the books [7, 8] for basic information on this theory. However to the best of our knowledge, it was always assumed that Hankel operators were bounded. The only exception is paper [13] where Hankel operators were realized as integral operators in the space $L^2(\mathbb{R}_+)$. The goal of this paper is to make first steps in the study of unbounded Hankel operators. We consider non-negative quadratic forms (1.2) (in particular, we always

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assume that \( q_n = \overline{q_n} \) so that we are tempted to define \( Q \) as a self-adjoint operator corresponding to the quadratic form \( q[g,g] \). Such an operator exists if the form \( q[g,g] \) admits the closure in the space \( \ell^2(\mathbb{Z}_+) \), but as is well known this is not always true. We refer to the book [2] for basic information concerning these notions.

1.2. Below we give necessary and sufficient conditions guaranteeing the existence of the closure of \( q[g,g] \), but previously we discuss the link of Hankel quadratic forms with the Hamburger moment problem. The following result was obtained in [5].

**Theorem 1.1.** The condition

\[
\sum_{n,m \geq 0} q_{n+m} g_n g_m \geq 0, \quad \forall g \in \mathcal{D},
\]  

is satisfied if and only if there exists a non-negative measure \( dM(\mu) \) on \( \mathbb{R} \) such that the coefficients \( q_n \) admit the representations

\[
q_n = \int_{-\infty}^{\infty} \mu^n dM(\mu), \quad \forall n = 0, 1, \ldots.
\]  

Note that the measure satisfying equations (1.4) is in general not unique (see the paper [11], for a comprehensive discussion of this phenomenon). Roughly speaking, the non-uniqueness of solutions of the Hamburger moment problem is due to a very rapid growth of the coefficients \( q_n \). Indeed, the famous Stieltjes example shows that the measures

\[
dM_\theta(\mu) = 1_{\mathbb{R}_+}(\mu)\mu^{-\ln R}(1 + \theta \sin(2\pi \ln \mu))d\mu, \quad \forall \theta \in [-1, 1],
\]  

solve equations (1.4) with \( q_n = \sqrt{\pi e^{(k+1)^2/4}} \). On the other hand, if \( |q_n| \leq R^n n! \) for some \( R > 0 \), then the solution of equations (1.4) for the measure \( dM(\mu) \) is unique.

1.3. The definition of the Hankel operator requires essentially more restrictive assumptions on the matrix elements \( q_n \). Let us state our main result.

**Theorem 1.2.** Let assumption (1.3) be satisfied. Then the following conditions are equivalent:

(i) The form \( q[g,g] \) defined on \( \mathcal{D} \) admits the closure in \( \ell^2(\mathbb{Z}_+) \).
(ii) The matrix elements \( q_n \to 0 \) as \( n \to \infty \).
(iii) The measure \( dM(\mu) \) defined by equations (1.3) satisfies the condition

\[
M(\mathbb{R} \setminus (-1,1)) = 0 \tag{1.5}
\]  

(to put it differently, \( \text{supp} \ M \subset [-1, 1] \) and \( M(\{-1\}) = M(\{1\}) = 0 \)).

Note that for \( q \in \ell^2(\mathbb{Z}_+) \) the assertion (i) is obvious because in this case \( q[g,g] = (Qg,g) \) where \( Q \) is the symmetric operator \( Q \) defined on \( \mathcal{D} \) by (1.1). As far as the proof of Theorem 1.2 is concerned, we note that only the implication

\[
(i) \implies (ii) \text{ or (iii)}
\]  

is sufficiently non-trivial.
In Section 3 we also give (see Theorem 3.1) an efficient description of the closure of
the form (1.2). In Section 4 we discuss some consequences of our results for moment
problems.

1.4. Theorem 1.2 is to a large extent motivated by the following classical results of
H. Widom.

Theorem 1.3. [12, Theorem 3.1] Let the matrix elements \( q_n \) of the Hankel operator
(1.1) be given by the relations

\[
q_n = \int_{-1}^{1} \mu^n dM(\mu), \quad \forall n = 0, 1, \ldots, \quad M(\{-1\}) = M(\{1\}) = 0, \tag{1.6}
\]

with some non-negative measure \( dM(\mu) \). Then the following conditions are equivalent:

(i) The operator \( Q \) is bounded.
(ii) \( q_n = O(n^{-1}) \) as \( n \to \infty \).
(iii) \( M((1 - \varepsilon, 1)) = O(\varepsilon) \) and \( M((-1, -1 + \varepsilon)) = O(\varepsilon) \) as \( \varepsilon \to 0 \).

Theorem 1.4. [12, Theorem 3.2] Under the same a priori assumptions as in Theo-
rem 1.3 the following conditions are equivalent:

(i) The operator \( Q \) is compact.
(ii) \( q_n = o(n^{-1}) \) as \( n \to \infty \).
(iii) \( M((1 - \varepsilon, 1)) = o(\varepsilon) \) and \( M((-1, -1 + \varepsilon)) = o(\varepsilon) \) as \( \varepsilon \to 0 \).

Theorems 1.3 and 1.4 give optimal conditions for the Hankel operator \( Q \) with matrix
elements (1.6) to be bounded and compact. Roughly speaking, condition (iii) of The-
orem 1.3 means that the measure \( dM(\mu) \) is “subordinated” to the Lebesgue measure
near the end points 1 and \(-1\) of its support. Similarly, condition (iii) of Theorem 1.4
means that the measure \( dM(\mu) \) is “diluted” compared to the Lebesgue measure near
these end points.

2. Proof of Theorem 1.2

2.1. It is almost obvious that conditions (ii) and (iii) are equivalent. Indeed, if (iii)
is satisfied, then

\[
q_n = \int_{-1+\varepsilon}^{1-\varepsilon} \mu^n dM(\mu) + \int_{1-\varepsilon}^{1} \mu^n dM(\mu) + \int_{-1}^{-1+\varepsilon} \mu^n dM(\mu).
\]

The second and third integrals on the right are bounded by \( M((1 - \varepsilon, 1)) \) and
\( M((-1, -1 + \varepsilon)) \), and hence they tend to zero as \( \varepsilon \to 0 \) uniformly in \( n \). The first
integral is bounded by \((1 - \varepsilon)^n M((-1, 1))\), and therefore it tends to zero as \( n \to \infty \)
for every \( \varepsilon > 0 \). Conversely, if there exists a set \( X \subset \mathbb{R} \setminus (-1, 1) \) such that \( M(X) > 0 \),
then \( q_{2n} \geq M(X) \), and hence condition (ii) cannot be satisfied.
It is convenient to reformulate condition (i) in an equivalent form. Let \( \mu \) be an arbitrary measure satisfying the condition
\[
\int_{-\infty}^{\infty} |\mu|^n dM(\mu) < \infty, \quad \forall n = 0, 1, \ldots,
\]
and let \( L^2(M) = L^2(\mathbb{R}; dM) \) be the space of functions \( u(\mu) \) with the norm \( \|u\|_{L^2(M)} \). We put
\[
(\mathcal{A}g)(\mu) = \sum_{n=0}^{\infty} g_n \mu^n
\]
and observe that under assumption (2.1), \( \mathcal{A}g \in L^2(M) \) for all \( g \in \mathcal{D} \). Therefore we can define an auxiliary operator \( \mathcal{A}: \ell^2(\mathbb{Z}_+) \to L^2(M) \) on domain \( \mathcal{D}(\mathcal{A}) = \mathcal{D} \) by the formula
\[
\mathcal{A}g = \sum_{n=0}^{\infty} g_n \mu^n,
\]
In view of equations (1.4) the form \( q[g, g] \) defined by relation (1.2) can be written as
\[
q[g, g] = \|Ag\|^2_{L^2(M)} = \int_{-\infty}^{\infty} |(\mathcal{A}g)(\mu)|^2 dM(\mu), \quad g \in \mathcal{D}.
\]
This yields the following result.

**Lemma 2.1.** The form \( q[g, g] \) defined on \( \mathcal{D} \) admits the closure in the space \( \ell^2(\mathbb{Z}_+) \) if and only if the operator \( \mathcal{A}: \ell^2(\mathbb{Z}_+) \to L^2(M) \) defined on the same set \( \mathcal{D} \) admits the closure.

Recall that the operator \( \mathcal{A} \) admits the closure if and only if its adjoint operator \( \mathcal{A}^*: L^2(M) \to \ell^2(\mathbb{Z}_+) \) is densely defined. So our next goal is to construct \( \mathcal{A}^* \). Observe that under assumption (2.1) for an arbitrary \( u \in L^2(M) \), all the integrals
\[
\int_{-\infty}^{\infty} u(\mu) \mu^n dM(\mu) =: u_n, \quad n \in \mathbb{Z}_+,
\]
are absolutely convergent. We denote by \( \mathcal{D}_* \subset L^2(M) \) the set of \( u \in L^2(M) \) such that the sequence \( \{u_n\}_{n=0}^{\infty} \in \ell^2(\mathbb{Z}_+) \).

**Lemma 2.2.** Under assumption (2.1) the operator \( \mathcal{A}^* \) is given by the equality
\[
(A^* u)_n = \int_{-\infty}^{\infty} u(\mu) \mu^n dM(\mu), \quad n \in \mathbb{Z}_+,
\]
on the domain \( \mathcal{D}(A^*) = \mathcal{D}_* \).

**Proof.** Obviously, for all \( g \in \mathcal{D} \) and all \( u \in L^2(M) \), we have
\[
(\mathcal{A}g, u)_{L^2(M)} = \sum_{n=0}^{\infty} g_n u_n
\]
where the numbers \( u_n \) are defined by relations (2.4). The right-hand side here equals \( (g, A^* u) \) provided \( u \in \mathcal{D}_* \). It follows that \( \mathcal{D}_* \subset \mathcal{D}(A^*) \).

Conversely, if \( u \in \mathcal{D}(A^*) \), then
\[
|(\mathcal{A}g, u)_{L^2(M)}| = |(g, A^* u)_{\ell^2(\mathbb{Z}_+)}| \leq \|g\|_{\ell^2(\mathbb{Z}_+)} \|A^* u\|_{\ell^2(\mathbb{Z}_+)}
\]
for all \( g \in \mathcal{D} \). Therefore it follows from equality (2.5) that
\[
\left| \sum_{n=0}^{\infty} g_n \bar{u}_n \right| \leq C(u) \| g \|_{\ell^2(\mathbb{Z}_+)} , \quad \forall g \in \mathcal{D}.
\]
Since \( \mathcal{D} \) is dense in \( \ell^2(\mathbb{Z}_+) \), we see that \( \{ u_n \}_{n=0}^{\infty} \in \ell^2(\mathbb{Z}_+) \), and hence \( u \in \mathcal{D}_* \). Thus \( \mathcal{D}(A^*) \subset \mathcal{D}_* \). \( \square \)

2.2. Next, we use the following analytical result.

**Theorem 2.3.** The set \( \mathcal{D}_* \) is dense in \( L^2(M) \) if and only if condition (1.5) is satisfied.

**Proof.** Let the set \( \mathcal{E} \) consist of \( u \in L^2(M) \) such that supp\( u \subset [-a, a] \) for some \( a < 1 \). According to definition (2.4) for \( u \in \mathcal{E} \), we have \( |u_n| \leq Ca^n \) whence \( \mathcal{E} \subset \mathcal{D}_* \). Under assumption (1.5) the set \( \mathcal{E} \) is dense in \( L^2(M) \) and so \( \mathcal{D}_* \) is also dense in this space.

Let us prove the converse statement. Suppose that \( \mathcal{D}_* \) is dense in \( L^2(M) \). For an arbitrary \( u \in L^2(M) \), we put
\[
f(x) = \int_{-\infty}^{\infty} e^{ix\mu} u(\mu) dM(\mu). \tag{2.6}
\]
Obviously, we have
\[
|f(x)| \leq \| u \|_{L^2(M)} \sqrt{M(\mathbb{R})}. \tag{2.7}
\]
It follows from condition (2.1) that \( f \in C^\infty(\mathbb{R}) \) and
\[
f^{(n)}(0) = i^n \int_{-\infty}^{\infty} \mu^n u(\mu) dM(\mu).
\]
Assume now that \( u \in \mathcal{D}_* \). Then this sequence is bounded and hence the function
\[
f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n
\]
is entire and satisfies the estimate \( |f(z)| \leq C \sum_{n=0}^{\infty} \frac{1}{n!} |z|^n = Ce^{|z|}, \quad z \in \mathbb{C}. \tag{2.8} \)

Let us now show that
\[
|f(z)| \leq Ce^{\text{Im } z}, \quad z \in \mathbb{C}. \tag{2.9}
\]
Consider, for example the angle \( \arg z \in [0, \pi/2] \) and put \( F(z) = f(z) e^{iz} \). Since \( |e^{iz}| = e^{-\text{Im } z} \), it follows from estimates (2.7) and (2.8) that \( |F(z)| \leq Ce^{|z|} \) for all \( z \) and that the function \( F(z) \) is bounded on the rays \( z = r \) and \( z = ir \) where \( r \geq 0 \). Therefore, by the Phragmén-Lindelöf principle (see, e.g., the book [1]), \( F(z) \) is bounded in the whole angle \( \arg z \in [0, \pi/2] \). This yields estimate (2.9) for \( f(z) = F(z) e^{-iz} \).

According to the Paley-Wiener theorem (see, e.g., Theorem IX.12 of [9]) it follows from estimate (2.9) that the Fourier transform of \( f(x) \) (considered as a distribution in the Schwartz class \( \mathcal{S'}(\mathbb{R}) \)) is supported by the interval \([-1, 1] \). Therefore formula (2.6)
implies that for every $u \in \mathcal{D}$, the distribution $u(\mu) dM(\mu)$ is also supported by $[-1, 1]$, that is
\[
\int_{-\infty}^{\infty} \varphi(\mu) u(\mu) dM(\mu) = 0, \quad \forall \varphi \in C_0^\infty(\mathbb{R} \setminus [-1, 1]). \tag{2.10}
\]
Since $\mathcal{D}$ is dense in $L^2(M)$, we can approximate 1 by functions $u \in \mathcal{D}$ in this space. Hence equality (2.10) is true with $u(\mu) = 1$. It follows that
\[
supp M \subset [-1, 1] \tag{2.11}
\]
because $\varphi \in C_0^\infty(\mathbb{R} \setminus [-1, 1])$ is arbitrary.

For the proof of (1.5), it remains to show that $M(\{-1\}) = M(\{1\}) = 0$. In view of (2.11) for an arbitrary $u \in L^2(M)$, sequence (2.4) admits the representation
\[
u_{n} = M(\{1\}) u(1) + (-1)^n M(\{-1\}) u(-1) + \int_{-1}^{1} u(\mu) \mu^n dM_0(\mu) \tag{2.12}
\]
where $M_0(X) = M(X \cap (-1, 1))$ is the restriction of the measure $M$ on the open interval $(-1, 1)$. Obviously, for any $\varepsilon \in (0, 1)$, we have
\[
\int_{-1}^{1} u(\mu) \mu^n dM_0(\mu) = \int_{-1+\varepsilon}^{1-\varepsilon} u(\mu) \mu^n dM_0(\mu) + \int_{1-\varepsilon}^{1} u(\mu) \mu^n dM_0(\mu) + \int_{-1+\varepsilon}^{-1} u(\mu) \mu^n dM_0(\mu). \tag{2.13}
\]
Applying the Schwarz inequality to each integral on the right, we estimate this expression by
\[
((1-\varepsilon)^n \sqrt{M_0((-1, 1))} + \sqrt{M_0((1-\varepsilon, 1))} + \sqrt{M_0((-1, 1+\varepsilon))}) \| u \|_{L^2(M_0)}.
\]
Since $M_0((1-\varepsilon, 1)) \to 0$ and $M_0((-1, 1+\varepsilon)) \to 0$ as $\varepsilon \to 0$, we see that integral in the left-hand side of (2.13) tends to zero as $n \to \infty$. Thus (2.12) implies that
\[
u_{n} = M(\{1\}) u(1) + (-1)^n M(\{-1\}) u(-1) + o(1)
\]
as $n \to \infty$. Therefore if $u \in \mathcal{D}$, or equivalently $\{u_n\}_{n=0}^{\infty} \in l^2(\mathbb{Z}^+)$, then necessarily $M(\{1\}) u(1) = M(\{-1\}) u(-1) = 0$. So if $M(\{1\}) \neq 0$ (or $M(\{-1\}) \neq 0$), then $u(1) = 0$ (or $u(-1) = 0$). It follows that the function $u$ such that $u(\mu) = 0$ for $\mu \neq 1$ and $u(1) = 1$ (or $u(\mu) = 0$ for $\mu \neq -1$ and $u(-1) = 1$) cannot be approximated by functions in $\mathcal{D}$. Hence $\mathcal{D}$ is not dense in $L^2(M)$. \qed

**Remark 2.4.** We have used the Phragmén-Lindelöf principle for the proof of estimate (2.9) only. Actually, relation (2.10) can be directly deduced from estimates (2.7) and (2.8) using the arguments given in the proof of Theorem 19.3 of the book [10]. However the intermediary estimate (2.9) makes the proof of (2.10) essentially more transparent.

### 2.3. Let us come back to the proof of Theorem 1.2
Putting together Lemma 2.2 and Theorem 2.3, we see that the operator $A^*$ is densely defined and hence $A$ admits the closure if and only if condition (1.5) is satisfied. In view of Lemma 2.1 this proves that the conditions (i) and (iii) of Theorem 1.2 are equivalent and thus concludes its proof.
3. The closure of the Hankel quadratic form

3.1. Let the condition (1.5) be satisfied, and let \( \bar{A} \) be the closure of the operator \( A \) defined by equation (2.2) on domain \( D(A) = D \). Then the form (2.3) admits the closure in the space \( \ell^2(\mathbb{Z}_+) \) and the form

\[
q[g, g] = \| \bar{A}g \|_{L^2(M)}^2
\]

is closed on domain \( D[q] = D(\bar{A}) \). Since \( \bar{A} = A^{**} \), it remains to describe the set \( D(A^{**}) \).

Observe that series (2.2) converges for all \( g \in \ell^2(\mathbb{Z}_+) \) and all \( \mu \in (-1, 1) \). The function \( (A^2 g)(\mu) \) depends continuously on \( \mu \), but only the estimate

\[
\sum_{n=0}^{\infty} |g_n| |\mu|^n \leq (1 - \mu^2)^{-1/2} \| g \|_{\ell^2(\mathbb{Z}_+)}
\]

holds. So it is of course possible that \( A^2 g \notin L^2(M) \). Now we define the operator \( A_{\text{max}} \) by the formula \( A_{\text{max}} g = A^2 g \) on the domain \( D(A_{\text{max}}) \) that consists of all \( g \in \ell^2(\mathbb{Z}_+) \) such that \( A^2 g \in L^2(M) \). Our goal is to show that

\[
A^{**} = A_{\text{max}}. \tag{3.2}
\]

A difficult part in the proof of (3.2) is the inclusion \( A_{\text{max}} \subset A^{**} \) that is equivalent to the relation

\[
(A_{\text{max}} g, u)_{L^2(M)} = (g, A^* u)_{\ell^2(\mathbb{Z}_+)} \tag{3.3}
\]

for all \( g \in D(A_{\text{max}}) \) and all \( u \in D(A^*) = D_+ \). In the detailed notation, relation (3.3) means that

\[
\int_{-1}^{1} \left( \sum_{n=0}^{\infty} g_n \mu^n \right) u(\mu) dM(\mu) = \sum_{n=0}^{\infty} g_n \left( \int_{-1}^{1} \mu^n u(\mu) dM(\mu) \right).
\]

The problem is that these integrals do not converge absolutely, and so the Fubini theorem cannot be applied.

3.2. The shortest way to prove (3.2) is to reduce the operator \( A \) by appropriate unitary transformations to the Laplace transform defined by the relation

\[
(\mathcal{B} f)(\lambda) = \int_{0}^{\infty} e^{-t^\lambda} f(t) dt. \tag{3.4}
\]

We consider it as a mapping \( \mathcal{B} : L^2(\mathbb{R}_+) \rightarrow L^2(\mathbb{R}_+; d\Sigma) \) where the non-negative measure \( d\Sigma(\lambda) \) on \( \mathbb{R}_+ \) satisfies the condition

\[
\int_{0}^{\infty} (\lambda + 1)^{-k} d\Sigma(\lambda) < \infty \tag{3.5}
\]

for some \( k > 0 \). Let the set \( D \subset L^2(\mathbb{R}_+) \) consist of functions compactly supported in \( \mathbb{R}_+ \). If \( f \in D \), then \( (\mathcal{B} f)(\lambda) \) is a continuous function for all \( \lambda \geq 0 \) and \( (\mathcal{B} f)(\lambda) = O(e^{-c\lambda}) \) with some \( c = c(f) > 0 \) as \( \lambda \to \infty \); in particular, \( \mathcal{B} f \in L^2(\mathbb{R}_+) \). We put \( Bf = \mathcal{B} f \) with \( D(B) = D \).
It is easy to show (see [13], for details) that the operator $B^*$ is given by the formula
\[(B^*v)(t) = \int_0^\infty e^{-t\lambda}v(\lambda)d\Sigma(\lambda),\] (3.6)
and its domain $\mathcal{D}(B^*)$ consists of all $v \in L^2(\mathbb{R}_+; d\Sigma)$ such that $B^*v \in L^2(\mathbb{R}_+)$. Obviously, this condition is satisfied if $v$ is compactly supported in $\mathbb{R}_+$. Since the set of such $v$ is dense in $L^2(\mathbb{R}_+; d\Sigma)$, the operator $B^*$ is densely defined. Thus $B$ admits the closure and $\overline{B} = B^{**}$.

The integral (3.4) converges for all $f \in L^2(\mathbb{R}_+)$ and $\lambda > 0$. The function $(Bf)(\lambda)$ is continuous, but of course the estimate
\[|(Bf)(\lambda)| \leq (2\lambda)^{-1/2} \|f\|_{L^2(\mathbb{R}_+)}\]
does not guarantee that $Bf \in L^2(\mathbb{R}_+; d\Sigma)$. Let us now define the operator $B_{\text{max}}$ by the formula $Bf = Bf$ on the domain $\mathcal{D}(B_{\text{max}})$ that consists of all $f \in L^2(\mathbb{R}_+)$ such that $Bf \in L^2(\mathbb{R}_+; d\Sigma)$. We use the following assertion.

**Lemma 3.1.** [13, Theorem 3.9] Let $d\Sigma(\lambda)$ be a measure on $\mathbb{R}_+$ such that the condition (3.5) is satisfied for some $k > 0$. Then
\[B^{**} = B_{\text{max}}.\] (3.7)

3.3. Let us find a relation between the operators $A$ and $B$. Suppose that the measures $d\Sigma(\lambda)$ and $dM(\mu)$ are linked by the equality
\[dM(\mu) = (\lambda + 1/2)^{-2}d\Sigma(\lambda), \quad \mu = \frac{2\lambda - 1}{2\lambda + 1}.\] (3.8)
Thus if $M((-1, 1)) < \infty$, then the condition (3.3) holds with $k = 2$. Let us also set
\[(Vu)(\lambda) = \frac{1}{\lambda + 1/2}u\left(\frac{(2\lambda - 1)}{2\lambda + 1}\right).\] (3.9)

Obviously, $V : L^2((-1, 1); dM) \to L^2(\mathbb{R}_+; d\Sigma)$ is a unitary operator. If the measures $dM$ and $d\Sigma$ are absolutely continuous, that is
\[d\Sigma(\lambda) = \sigma(\lambda)d\lambda, \quad \lambda > 0, \quad dM(\mu) = \eta(\mu)d\mu, \quad \mu \in (-1, 1),\]
with some $\sigma \in L^1_{\text{loc}}(\mathbb{R}_+)$ and $\eta \in L^1(-1, 1)$, then relation (3.8) means that
\[\eta(\mu) = \sigma\left(\frac{1 + \mu}{2(1 - \mu)}\right).\]

Recall that the Laguerre polynomial (see the book [3], Chapter 10.12) of degree $n$ is defined by the formula
\[L_n(t) = n!^{-1}e^td^n(e^{-t}t^n)/dt^n = \sum_{m=0}^n \frac{n!}{(n-m)!(m)!}(-t)^m.\]
We need the identity (see formula (10.12.32) in [3])
\[ \int_0^\infty L_n(t)e^{-(1/2+\lambda)t}dt = \frac{1}{\lambda + 1/2} \left( \frac{2\lambda - 1}{2\lambda + 1} \right)^n, \quad \lambda > -1/2. \] (3.10)
It can be deduced from this fact that the functions \( L_n(t)e^{-t/2}, n = 0, 1, \ldots \), form an orthonormal basis in the space \( L^2(\mathbb{R}_+) \), and hence the operator \( U: l^2(\mathbb{Z}_+) \rightarrow L^2(\mathbb{R}_+) \) defined by the formula
\[ (Ug)(t) = \sum_{n=0}^\infty g_n L_n(t)e^{-t/2}, \quad g = (g_0, g_1, \ldots), \] (3.11)
is unitary.

A link of the operators \( A \) and \( B \) is stated in the following assertion.

**Lemma 3.2.** For all \( g \in \mathcal{D} \), the identity holds
\[ VAg = BUg. \] (3.12)

**Proof.** It follows from equalities (3.4), (3.10) and (3.11) that
\[ (BUg)(\lambda) = \sum_{n=0}^\infty g_n \int_0^\infty L_n(t)e^{-(1/2+\lambda)t}dt = \sum_{n=0}^\infty g_n \frac{1}{\lambda + 1/2} \left( \frac{2\lambda - 1}{2\lambda + 1} \right)^n. \]
In view of definitions (2.2), (3.9) this expression equals \((VAg)(\lambda)\). \( \square \)

Combining Lemmas 3.1 and 3.2 it is now easy to obtain the following result.

**Lemma 3.3.** Let \( dM(\mu) \) be a finite measure on \((-1, 1)\). Then equality (3.2) holds.

**Proof.** Observe that the adjoint of the operator \( B \) defined by (3.4) on the set \( UD \) is still given by formula (3.6). Therefore it follows from (3.12) that \( A^*V = UB^* \) and hence
\[ VA^{**} = B^{**}U. \] (3.13)
Let \( g \in l^2(\mathbb{Z}_+) \) be arbitrary. Approximating it by functions \( g_n \in \mathcal{D} \) and using (3.12), we see that \((VAg)(\lambda) = (BUg)(\lambda)\) for all \( \lambda > 0 \). It follows that
\[ VA_{\text{max}} = B_{\text{max}}U. \] (3.14)
Comparing (3.13) and (3.14), we see that the identities (3.2) and (3.7) are equivalent. \( \square \)

In view of relation (3.2), formula (3.1) leads to the following result.

**Theorem 3.4.** Let assumption (1.3) and one of three equivalent conditions (i), (ii) or (iii) of Theorem 1.2 be satisfied. Let the form \( q[g, g] \) be defined on \( \mathcal{D} \) by (1.2), and let \( A \) be the operator (2.2). Then the closure of \( q[g, g] \) is given by the equality
\[ q[g, g] = \int_{-1}^1 |(Ag)(\mu)|^2dM(\mu) \] (3.15)
on the set \( \mathcal{D}[q] \) of all \( g \in l^2(\mathbb{Z}_+) \) such that the right-hand side of (3.15) is finite.
We note that the non-negative operator $Q$ corresponding to the form (3.15) satisfies the relations
\[ q[g, h] = (g, Qh), \quad \forall g \in D[q], \quad \forall h \in D(Q) \subset D[q], \]
\[ q[g, g] = \|\sqrt{Q}g\|^2, \quad \forall g \in D(\sqrt{Q}) = D[q], \]
but its domain $D(Q)$ does not admit an efficient description.

4. Moment problems

4.1. Comparing Theorems 1.1 and 1.2, we obtain the following result concerning moment problems.

Proposition 4.1. A non-negative measure $dM(\mu)$ satisfying conditions (1.4) and (1.5) exists if and only if inequality (1.3) holds and $q_n \to 0$ as $n \to \infty$ (or, equivalently, the form (1.2) admits the closure).

Instead of the interval $[-1, 1]$ we can consider an arbitrary finite interval $[-a, a]$. Our arguments proving the equivalence of conditions (ii) and (iii) in Theorem 1.2 lead to the following simple assertion.

Proposition 4.2. A non-negative measure $dM(\mu)$ satisfying the condition
\[ q_n = \int_{-a}^{a} \mu^n dM(\mu), \quad \forall n = 0, 1, \ldots, \]
exists if and only if inequality (1.3) holds and $q_n = O(a^n)$ as $n \to \infty$. Moreover, $M(\{1/\}) = M(\{a\}) = 0$ if and only if $q_n = o(a^n)$ as $n \to \infty$.

4.2. The results obtained above can be combined with the Stieltjes theorem which states that there exists a non-negative measure $dM(\mu)$ satisfying equations (1.4) and such that supp $M \subset [0, \infty)$ if and only if inequalities (1.3) and
\[ \sum_{n,m \geq 0} q_{n+m+1} g_{m+1} \bar{g}_n \geq 0, \quad \forall g \in D, \quad (4.1) \]
hold.

Let us state analogues of Propositions 4.1 and 4.2

Proposition 4.3. A non-negative measure $dM(\mu)$ satisfying conditions (1.4) and supp $M \subset [0, 1]$, $M(\{1\}) = 0$ exists if and only if inequalities (1.3) and (4.1) hold and $q_n \to 0$ as $n \to \infty$ (or, equivalently, the form $q[g, g]$ admits the closure).

Proposition 4.4. A non-negative measure $dM(\mu)$ satisfying the condition
\[ q_n = \int_{0}^{a} \mu^n dM(\mu), \quad \forall n = 0, 1, \ldots, \]
exists if and only if inequalities (1.3) and (1.1) hold and $q_n = O(a^n)$ as $n \to \infty$. Moreover, $M(\{a\}) = 0$ if and only if $q_n = o(a^n)$ as $n \to \infty$. 
Note that the moment problem (1.4) with the measure $dM(\mu)$ supported by a compact interval is called the Riesz problem. The necessary and sufficient conditions for the existence of its solution are well known (see, e.g., the book [1]), but there are stated in quite different terms compared to the results of this section.

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