Global solvability and convergence to stationary solutions in singular quasilinear stochastic PDEs

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This article is dedicated to István Gyöngy on the occasion of his 70th birthday.

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Abstract
We consider singular quasilinear stochastic partial differential equations (SPDEs) studied in Funaki et al. (Ann Inst Henri Poincaré Probab Stat 57:1702-1735, 2021), which are defined in paracontrolled sense. The main aim of the present article is to establish the global-in-time solvability for a particular class of SPDEs with origin in particle systems and, under a certain additional condition on the noise, prove the convergence of the solutions to stationary solutions as \( t \to \infty \). We apply the method of energy inequality and Poincaré inequality. It is essential that the Poincaré constant can be taken uniformly in an approximating sequence of the noise. We also use the continuity of the solutions in the enhanced noise, initial values and coefficients of the equation, which we prove in this article for general SPDEs discussed in Funaki et al. (Ann Inst Henri Poincaré Probab Stat 57:1702-1735, 2021) except that in the enhanced noise. Moreover, we apply the initial layer property of improving regularity of the solutions in a short time.

Keywords Singular SPDE · Quasilinear SPDE · Paracontrolled calculus · Global solvability · Stationary solution · Energy inequality

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1 Introduction

We studied in [7] the following quasilinear stochastic partial differential equation (SPDE) defined in paracontrolled sense

\[ \partial_t u = a(\nabla u)\Delta u + g(\nabla u) \cdot \xi, \quad (1.1) \]

on one dimensional torus \( \mathbb{T} \simeq [0, 1) \) having the spatial noise \( \xi \in C^{a-2}, \alpha \in (\frac{4}{3}, \frac{3}{2}) \), where \( \nabla = \partial_x, \Delta = \partial_x^2 \) and \( C^\alpha(\mathbb{T}) = B^\alpha_{p,\infty}(\mathbb{T}) \) denotes the Hölder-Besov space on \( \mathbb{T} \) with regularity exponent \( \alpha \in \mathbb{R} \) equipped with the norm \( \| \cdot \|_{C^\alpha} \). We showed the local-in-time solvability and the continuity of the solution in the enhanced noise \( \hat{\xi} \). More precisely, assuming that the coefficients satisfy \( a, g \in C^3(\mathbb{R}) \) and

\[ c_- \leq a(v) \leq c_+, \quad (1.2) \]

for some \( c_-, c_+ > 0 \), it was shown that (1.1) with the initial value \( u_0 \in C^\alpha \) has a solution \( u \) up to some \( T_* > 0 \) (see (2.23)) and if the enhanced noise \( \hat{\xi} = (\xi, \Pi(\nabla X, \xi)) \) converges in \( C^{a-2} \times C^{2a-3} \), then the corresponding solution \( u = u^{\hat{\xi}} \) converges in \( L_T^a \), where \( L_T^a = C([0, T], C^\alpha) \cap C^{a/2}((0, T], L^\infty) \) taking \( T > 0 \) uniformly in a neighborhood of some \( \hat{\xi} \) and \( L^\infty = L^\infty(\mathbb{T}) \). Here \( X = (-\Delta)^{-1}(\xi - \xi(\mathbb{T})), \xi(\mathbb{T}) \equiv \langle \xi, 1 \rangle \), and \( \Pi(\nabla X, \xi) \) denotes the resonant term in the paraproduct of \( \nabla X \) and \( \xi \). In particular, we see \( u(t) \in \cap_{\delta>0} C^{\frac{3}{2}-\delta} \) for \( t < T_* \), see [7] for details. We remark that the same result still holds under a weaker assumption: \( a, g \in C^3(\mathbb{R}) \) satisfying (1.2), see Lemma 1.4 below.

1.1 The aim of the article

The present article is a continuation of [7]. Assuming that two coefficients \( a \) and \( g \) satisfy the relation \( a = g' \), we establish the global-in-time solvability and convergence of the solution to a stationary solution as \( t \to \infty \), see Theorems 1.1 and 1.2. For the convergence to the stationary solution, we assume \( |\mu_\xi| \) is small enough for the noise \( \xi \), where \( \mu_\xi \) is the constant defined from \( \xi \) by (1.7) below. A typical example of the noise \( \xi \) is the derivative of a periodic Brownian motion \( w = w(x), x \in \mathbb{T}; \xi = \dot{w} \) and, for this \( \xi, \mu_\xi = 0 \) holds. Then, in general without assuming such conditions for the coefficients \( a, g \) and noise \( \xi \), we show the continuity of the local-in-time solution in initial values, see Theorem 1.3. This is used for the proofs of Theorems 1.1 and 1.2.

Let \( \varphi \in C^4(\mathbb{R}) \) satisfying

\[ c_- \leq \varphi'(v) \leq c_+ \quad (1.3) \]

for some \( c_-, c_+ > 0 \) and \( \chi \in C^3(\mathbb{R}) \) be given, and consider the SPDE

\[ \partial_t v = \Delta\{\varphi(v)\} + \nabla\{\chi(v)\hat{\xi}\}, \quad x \in \mathbb{T}. \quad (1.4) \]
Note that the linear case $\varphi(v) = \chi(v) = v$ is included. Then, for every $m \in \mathbb{R}$, if $u \equiv u_m$ is a solution of (1.1) with $a(v) = \varphi'(v + m)$, $g(v) = \chi(v + m)$, $v := \nabla u_m + m$ solves the Eq. (1.4) under the assumption $a, g \in C^3_b(\mathbb{R})$, see Sect. 1.2 of [7] and an indirect Definition 1.1 of the solution of (1.4) below. Note that, if $\xi$ is smooth, this is true in classical sense. In particular, (1.4) has a local-in-time solution $v$ in $L^T_T$. See Sect. 1.4 for more explanation on the relation between $u$ and $v$ under the weak assumption $a, g \in C^3(\mathbb{R})$. We note that the Eq. (1.4) has a mass conservation law:

$$\int T v(t, x) dx = m$$

(1.5)

for all $t \geq 0$ with a constant $m \in \mathbb{R}$, which is determined from its initial value $v_0$.

As we mentioned above, the main aim of the present article is to show the global-in-time solvability and establish the convergence of the solution to the stationary solution as $t \to \infty$ for the SPDE (1.1) when the coefficients satisfy $a = g'$. For this purpose, it turns out to be more convenient to study the SPDE in the form of (1.4). Due to the discussions in Sect. 1.4, especially by Definition 1.1, the result for (1.4) implies that for $\nabla u$ for the solution $u$ of (1.1). Our assumption $a = g'$ for (1.1) corresponds to $\chi = \varphi$ for (1.4) so that we consider the equation of the special form

$$\partial_t v = \Delta \{ \varphi(v) \} + \nabla \{ \varphi(v) \xi \},$$

(1.6)

on $\mathbb{T}$. The Eq. (1.6) has a physical meaning in the sense that it can be derived from a microscopic particle system in random environment, see [14].

Our another aim is to establish the continuity of the solution of (1.1) in initial values in general without assuming $a = g'$. We use such property in the study of the SPDE (1.6), but we show it for the general case. The solution determines a continuous flow on the state space $C^\alpha \cup \{ \Delta \}$ with a death point $\Delta$ added to cover the explosion of the solutions, see Remark 2.4 in Sect. 2.2. The Eq. (1.1) can be considered as if a deterministic PDE once $\xi$ is fixed, so that it is different from the SPDEs driven by the space-time white noise, but the continuity in initial value $u_0$ in our case corresponds to the strong Feller property for such SPDEs, cf. [11].

### 1.2 Global solution in time and convergence to stationary solutions

We consider the SPDE (1.6) with $\xi \in C^{\alpha - 2}, \alpha \in (\frac{4}{3}, \frac{3}{2})$. To describe its stationary solutions, for a given $\xi$, we define its integral $\eta(x) := \langle \xi, 1 \rangle := \int_0^x \xi(y) dy, x \in \mathbb{T}$. Note that $\eta$ is not periodic, but $\tilde{\eta}(x) := \langle \xi - \sigma, 1 \rangle = \eta(x) - \sigma x$ is periodic, where

$$\sigma := \sigma_\xi := \xi(\mathbb{T}) = \langle \xi, 1 \rangle = \eta(1) \in \mathbb{R}.$$

It is known that $\eta \in C^{\alpha - 1}(\mathbb{R})$ and especially $\eta \in C([0, 1])$, see Lemma A.10 of [9]. Typically, we can take $\xi = \dot{w}(x) + \sigma$ with a periodic Brownian motion $w(x), x \in \mathbb{T}$ and $\sigma \in \mathbb{R}$. The most interesting noise is $\dot{\xi} = \dot{\xi}$ and, in this case, $\sigma_\xi = 0$ holds.
Then, from $\eta$ or equivalently from $\xi$, we define a function $\theta(x) = \theta_\xi(x)$ on $\mathbb{T}$ and a constant $\mu = \mu_\xi \in \mathbb{R}$, respectively, by

$$\theta(x) := e^{-\eta(x)} \left\{ \mu \int_0^x e^{\eta(y)} \, dy + 1 \right\}, \quad x \in \mathbb{T},$$

$$\mu := \frac{e^{\eta(1)} - 1}{\int_0^1 e^{\eta(y)} \, dy}.$$  

(1.7)

Note that $\theta(x) > 0$ so it is uniformly positive by its continuity. Indeed, this is obvious if $\mu \geq 0$, while $\theta(x) \geq e^{-\eta(x)} \left\{ \mu \int_0^1 e^{\eta(y)} \, dy + 1 \right\} = e^{-\eta(x) + \eta(1)} > 0$ if $\mu < 0$. Moreover, $\theta$ is periodic: $\theta(0) = \theta(1) = 1$. In case $\sigma = 0$, we have $\mu = 0$ (and vice versa) and $\theta(x) = e^{-\eta(x)}$. Note that $\theta = \theta(x)$ satisfies

$$\nabla \theta + \xi \theta = \mu$$

at least if $\xi \in C(\mathbb{T})$, that is, $\eta \in C^1([0, 1])$, and therefore $\Delta(z \theta) + \nabla(z \theta \cdot \xi) = 0$ holds for every $z \in \mathbb{R}$. In particular, $v = \varphi^{-1}(z \theta)$ are stationary solutions of (1.6), where $\varphi^{-1}$ is the inverse function of $\varphi$.

For each conserved mass $m \in \mathbb{R}$ given as in (1.5), determine $z = z_m \in \mathbb{R}$ uniquely by the relation

$$m = \int_{\mathbb{T}} \varphi^{-1}(z \theta(x)) \, dx.$$  

(1.9)

Note that, since $\varphi$ satisfies (1.3) and in particular, it is strictly increasing, this determines a one to one relation between $z$ and $m$. Then, $\bar{v}(x) \equiv \bar{v}_m(x) := \varphi^{-1}(z_m \theta(x))$ (1.10) is a stationary solution of (1.6) satisfying $\int_{\mathbb{T}} v \, dx = m$ in distributional sense, or at least if $\xi \in C(\mathbb{T})$. Indeed, the mass conservation law is clear and, as we noted above,

$$\Delta \{ \varphi(\bar{v}_m) \} + \nabla \{ \varphi(\bar{v}_m) \xi \} = z_m \nabla \{ \nabla \theta + \theta \xi \} = z_m \nabla \mu = 0,$$

so that the right hand side of (1.6) vanishes for $v = \bar{v}_m$.

As we will see, at least if $|\mu_\xi|$ is small enough, $\bar{v}_m$ is the unique stationary solution of the SPDE (1.6) for each fixed $m$, where $\mu_\xi$ is the constant determined in (1.7) from the noise $\xi$. In fact, we will show in Sect. 2 that, for initial value $v_0(x)$ satisfying (1.5) (with $v(t)$ replaced by $v_0$), the solution $v(t, x)$ of the SPDE (1.6) converges to the stationary solution $\bar{v}_m$ as $t \to \infty$ at least if $|\mu_\xi|$ is small. Moreover, without assuming the smallness of $|\mu_\xi|$, the SPDE (1.6) has a global solution in time.

**Theorem 1.1** Let $\varphi \in C^4(\mathbb{R})$ satisfy (1.3) and $\alpha \in \left(\frac{13}{12}, \frac{3}{2}\right)$. Then, for every initial value $v_0 \in C^{\alpha-1}$, the SPDE (1.6) has a global-in-time solution $v(t) \in C^{\alpha-1}$ for all
\[ t \geq 0. \text{ Moreover, if } |\mu_\xi| \text{ is sufficiently small, } v(t) \text{ converges exponentially fast to } \bar{v}_m \text{ in } C^{\alpha-1} \text{ as } t \to \infty: \]

\[ \| v(t) - \bar{v}_m \|_{C^{\alpha-1}} \leq C e^{-ct}, \quad (1.11) \]

for some \( c, C > 0 \), where \( m \) is determined from \( v_0 \) as \( m = \int_T v_0(x) \mathrm{d}x \).

We apply the energy inequality, Poincaré inequality, the continuity of the solutions in enhanced noise and initial values, and also the initial layer property to show Theorem 1.1. See Remark 2.2 below for the SPDE (1.4) with general \( \chi \) instead of (1.6).

This theorem for the slope \( v(t) = \nabla u(t) \) of \( u(t) \) implies the following result for \( u(t) \) itself.

**Theorem 1.2** Assume \( a = g' \in C^3(\mathbb{R}) \) (so that \( g \in C^4(\mathbb{R}) \)), the condition (1.2) and \( \alpha \in \left( \frac{13}{9}, \frac{3}{2} \right) \). Then, the SPDE (1.1) has a global-in-time solution \( u(t) \in C^\alpha \) for all \( t \geq 0 \). Moreover, if \( |\mu_\xi| \) is sufficiently small as in Theorem 1.1, \( u(t) \) has the following uniform bound in \( t \):

\[ \sup_{t \geq 0} \| u(t) - z_0 \mu_\xi t \|_{C^\alpha} < \infty, \quad (1.12) \]

where \( z_0 \) is defined by (1.9) with \( m = 0 \). In particular, we have

\[ \lim_{t \to \infty} \frac{1}{t} u(t, x) = z_0 \mu_\xi \]

uniformly in \( x \in \mathbb{T} \).

**Remark 1.1** (i) In Theorems 1.1 and 1.2, we assume \( \alpha \in \left( \frac{13}{9}, \frac{3}{2} \right) \), which is slightly more restrictive than the original assumption \( \alpha \in \left( \frac{4}{3}, \frac{3}{2} \right) \) in [7]. This is because of Theorem 1.3, which has been used for the proofs of Theorems 1.1 and 1.2, see Proposition 2.7 in Sect. 2. For the reason for changing the range of \( \alpha \), see Remark 3.2. Except this point, the other statements in Sect. 2 hold for \( \alpha \in \left( \frac{4}{3}, \frac{3}{2} \right) \). So, unless otherwise noted, we still assume \( \alpha \in \left( \frac{4}{3}, \frac{3}{2} \right) \) throughout this article.

(ii) For the space-time white noise case as in [8], the average (i.e. integral on \( \mathbb{T} \)) of \( u(t) \) behaves as a Brownian motion and it never converges as \( t \to \infty \). But, in our case, noise is only spatially dependent and the situation is different. Removing the constant drift, \( u(t) \) stays bounded in \( t \).

We remark that the existence of global solutions of singular semilinear SPDEs are known, for example, for the following models. The linear Eq. (1.1) with \( a = 1, g = v \) on \( \mathbb{R}^d \) (i.e., the Eq. (1.6) with \( \varphi(v) = v \)) is studied in [4] and respectively, the generalized parabolic Anderson model (PAM) (i.e., the Eq. (1.6) with \( \varphi(v) = v \) and without \( \nabla \)) in [9] (Remark 5.4) and [2] by different approaches. For the nonlinear case, the dynamic \( \phi_3^4 \)-model on \( \mathbb{T}^3 \) is studied in [16] by establishing a priori estimate, and the complex Ginzburg–Landau equation on \( \mathbb{T}^3 \) in [12]. The global existence for multi-component coupled Kardar–Parisi–Zhang (KPZ) equation is shown in [6] under the trilinear condition by studying its stationary measure. In addition, there are few
works on the exponential decay in time of the solutions of singular semilinear SPDEs. For instance, [17] showed the exponential decay for the dynamic \( P(\phi)_2 \)-model on \( \mathbb{T}^2 \) to its unique invariant measure with respect to the total variation norm as \( t \to \infty \), and recently [10] showed the exponential \( L^2 \)-ergodicity of conservative stochastic Burgers equation on \( \mathbb{T} \) based on the approach of the martingale problem. Among these, to the best of our knowledge, our result is the first one in quasilinear case.

1.3 Continuity of the local solution in initial values and parameter \( m \)

Let \( u(t, \hat{x}, u_0), t \leq T \) denote the local-in-time paracontrolled solution of (1.1) for each fixed enhanced noise \( \hat{x} := (\xi, \Pi(\nabla X, \xi)) \) and initial value \( u_0 \). In [7], it is shown that the paracontrolled solution \( u(t, \hat{x}, u_0) \) is continuous in the enhanced noise \( \hat{x} \) for each fixed initial value \( u_0 \), see Theorem 3.1-(ii) of [7]. In Sect. 3, we show the joint continuity of \( u(t, \hat{x}, u_0) \) in \( (\xi, u_0) \) and also in the mass parameter \( m \) introduced as in (1.13) below without the restriction \( a = g' \).

Indeed, recalling the relation between SPDE (1.1) and SPDE (1.4), see Subsect. 1.4 for more explanation, to show Theorem 1.1, precisely speaking, Proposition 2.7, (i) For the special case \( a = g' \), we have that \( u(t, \hat{x}, u_0) \) of (1.1) is continuous in \( (t, \hat{x}, u_0) \) in the region \( \{t, \hat{x}, u_0\} \in [0, \infty) \times \mathbb{R} \times (C^{\alpha-2} \times C^{2\alpha-3}) \times C^\alpha; \ t \leq T \). In particular, for each \( 0 \leq t \leq T \), the solution \( u(t) \) of (1.1) is continuous in its initial values.

This theorem can be easily proved by Theorem 3.1 together with Remark 3.3 in Sect. 3.

Remark 1.2 (i) For the special case \( a = g' \), we have that \( u(t, \hat{x}, u_0) \) of (1.1) is continuous in \( (t, \hat{x}, u_0) \) in the region \( \{t, \hat{x}, u_0\} \in [0, \infty) \times (C^{\alpha-2} \times C^{2\alpha-3}) \times C^\alpha \) by Theorems 1.2 and 1.3.

(ii) From the explanation in Sect. 1.4, we see that Theorem 1.3 implies that the solution \( v(t, \hat{x}, v_0) \) of (1.4) is continuous in the region \( \{t, \hat{x}, v_0\} \in [0, \infty) \times (C^{\alpha-2} \times C^{2\alpha-3}) \times C^{\alpha-1}; \ t \leq T \) and in particular, \( v(t, \hat{x}, v_0) \) is continuous in its initial values \( v_0 \) up to time \( T \). Moreover, by Theorem 1.1, for the special type of SPDE (1.6), we know that its solution \( v(t, v_0) \) is continuous in \( (t, v_0) \) in the region \( \{t, v_0\} \in [0, \infty) \times C^{\alpha-1} \).

Remark 1.3 Hairer and Mattingly [11] discussed as follows: Let \( U \) be the state space of solutions of certain SPDE and \( \hat{U} := U \cup \{\Delta\} \) by adding the death point \( \Delta \) to cover
blow-up of solutions. They consider the solution of SPDE as a random dynamical system \( \Phi_{s,t} : \bar{U} \times M \to \bar{U} \), where \( M \) denotes the space of admissible models for a given regularity structure. In our case, we may fix the noise (or more precisely, the enhanced noise), and consider the solution of SPDE as a deterministic dynamical system \( \Phi_{s,t} : \bar{U} \to \bar{U} \) would be enough. Their Assumption 1 is for the continuity of the map \( \Phi_{s,t} \). They first prove the strong Feller property under general setting, and then show the assumptions formulated in general setting for several singular SPDEs. In our case, once the noise is fixed, the solution is deterministic so that what we need is the continuity of the solution in the initial value.

1.4 Definition of the solution of (1.4) and relation to (1.1)

Recall that the solution \( u(t) \) of (1.1) was defined in paracontrolled sense by solving the fixed point problem for the map \( \Phi \) defined by (2.16) in the class \( \mathcal{B}_T(\lambda) \), see [7].

Let us first remark the following lemma which generalizes Theorem 1.1 of [7] and is shown by the cut-off argument.

**Lemma 1.4** Let \( \alpha \in \left( \frac{4}{3}, \frac{3}{2} \right) \), \( a, g \in C^3(\mathbb{R}) \) and the condition (1.2) be satisfied. Then, the SPDE (1.1) has a unique local-in-time solution \( u(t) \) defined in paracontrolled sense.

**Proof** For a given \( a \in C^3(\mathbb{R}) \), we can take a sequence of functions \( a^n \in C^3_b(\mathbb{R}) \) such that \( 0 < c_- \leq a^n \leq c_+ \), \( a^n(v) = a(v) \), \( |v| \leq n \) and \( a^n \) converges uniformly to \( a \) on each compact set. Similarly, for a given \( g \in C^3(\mathbb{R}) \), let us take a sequence of functions \( g^n \in C^3_0(\mathbb{R}) \) such that \( g^n(v) = g(v) \), \( |v| \leq n \) and \( g^n \) converges uniformly to \( g \) on each compact set. Let us consider the equation

\[
\partial_t u^n = a^n(\nabla u^n) \Delta u^n + g^n(\nabla u^n) \cdot \xi
\]

starting from \( u_0 \in C^\alpha \). Then, the assumptions of Theorem 1.1 of [7] are satisfied and we know that (1.14) has a unique paracontrolled solution \( u^n = u^n(t) \) up to a time \( T^n_* > 0 \) a.s., which is similarly defined by (2.23). Without loss of generality, we may assume \( T^n_* < \infty \) a.s. Let us now define \( \tau^n \) by \( \tau^n = \inf \{ t > 0 : \|u^n(t)\|_{C^\alpha} \geq n \} \). Then, is clear that \( \tau^n > 0 \) a.s., whenever \( \|u_0\|_{C^\alpha} < n \) and Lemma 2.5 below gives that \( \tau^n \leq T^n_* \). Moreover, we have \( u^m(t) = u^n(t) \), \( t \leq \tau^m \wedge \tau^n \). Therefore, one knows that (1.1) has a unique solution \( u(t) \) in the paracontrolled sense at least up to the time \( \lim_{n \to \infty} \tau^n > 0 \) a.s.

As we mentioned, (1.4) is obtained at least for a smooth noise from (1.1) with proper modification in \( m \) by differentiation. Motivated by this, we give the meaning to the Eq. (1.4) indirectly via the Eq. (1.1). Let an initial value \( v_0 \in C^{\alpha-1} \), \( \alpha \in \left( \frac{4}{3}, \frac{3}{2} \right) \) of (1.4) be given. Then, set \( m := \int_{\mathbb{T}} v_0(x)dx \) and define \( u_0 \in C^\alpha \) by integrating \( v_0 - m \) as

\[
u_0(x) = \int_0^x (v_0(y) - m)dy + C, \quad x \in \mathbb{R}, \quad (1.15)\]
for any constant $C \in \mathbb{R}$. We solve \eqref{eq:1.1} with $a(v) = \varphi'(v + m)$, $g(v) = \chi(v + m)$ and this initial value $u_0$ in paracontrolled sense. The solution is denoted by $u(t) = u(t; v_0, C)$. Recall $u(t) \in L^2_T$ for some $T > 0$.

**Definition 1.1** We call $v(t) := \nabla u(t; v_0, C) + m \in L^{a_T - 1}_T$ the solution of the SPDE \eqref{eq:1.4} with initial value $v_0$.

Note that, if the noise $\xi$ (and $v_0$) is smooth, $v(t)$ is a smooth classical solution of \eqref{eq:1.4}. Indeed, in such case, $u(t)$ is a smooth solution of \eqref{eq:1.1} so that this follows by differentiation. Note also that $v(t; v_0, C)$ does not depend on the choice of $C$. Indeed, again for smooth smeared noise $\xi^\varepsilon$, we easily see $u^\varepsilon(t; v_0, C) = u^\varepsilon(t; v_0, 0) + C$ for the corresponding solutions $u^\varepsilon$ of \eqref{eq:1.1}. Thus, by applying Theorem 1.1 of [7] and taking the limit $\varepsilon \downarrow 0$, we see that $u(t; v_0, C) = u(t; v_0, 0) + C$ holds for general noise $\xi$. This implies $\nabla u(t; v_0, C) = \nabla u(t; v_0, 0)$. In particular, $v(t)$ is well-defined.

Conversely, $u(t)$ can be recovered from $v(t) = \nabla u(t)$ (with $m = 0$) as follows. Assume $\xi \in C^\infty(\mathbb{T})$ and let the initial value $u_0 \in C^\alpha, \alpha \in \left(\frac{4}{7}, \frac{3}{2}\right)$ of \eqref{eq:1.1} be given. Then, we determine $v(t)$ by solving \eqref{eq:1.4} with initial value $v_0 := \nabla u_0$, and set

$$u(t, x) := \int_0^x v(t, y)dy + \int_T u_0(y)dy - \int_T (1 - y)v(t, y)dy + \int_0^t ds \int_T \chi(v(s, y)) \cdot \xi(y)dy.$$  

(In the right hand side, especially in the first and third terms, we regard $\mathbb{T} = [0, 1]$.) Then, one can show that $u(t)$ solves the Eq. \eqref{eq:1.1} with $a = \varphi', g = \chi$, see Lemma 2.10 below. At least if $\xi$ is smooth, the equivalence between \eqref{eq:1.1} and \eqref{eq:1.4} is established.

Moreover, concerning the renormalizations, the Eq. \eqref{eq:1.1} in integrated form does not require them, since the resonant term $\Pi(\nabla X, \xi)$ involves the derivative of $X$ as we discussed in [7] (though \eqref{eq:1.1} is an analog of KPZ equation). In particular, the Eq. \eqref{eq:1.4} in differentiated form does not require them too.

The remainder of this article is organized as follows. Section 2 is for the proofs of Theorems 1.1 and 1.2. We first formulate the energy estimate for \eqref{eq:1.6} driven by a smooth noise in Sect. 2.1, see Proposition 2.2. Then, the proof of Theorem 1.1 is given in Sect. 2.2. We note the continuity of the solution in the enhanced noise $\hat{\xi}$, the initial values and the parameter $m$ in the coefficients. We derive Poincaré inequality and show that the Poincaré constant can be taken uniformly in the approximating sequence of the noise. We also rely on the initial layer type property of the solution of the SPDE \eqref{eq:1.6}, that is, the regularity of the solution is improved in an arbitrary short time. Section 2.3 is devoted to the proof of Theorem 1.2 based on the relation between \eqref{eq:1.1} and \eqref{eq:1.4}. In Sect. 3, we show Theorem 1.3 by establishing Theorem 3.1. For the reader’s convenience, some fundamental properties in the paracontrolled calculus are summarized in Appendix.
2 Global solvability and convergence to stationary solution

We show the global solvability of (1.6) based on the energy inequality and Poincaré inequality. This gives the exponentially fast convergence in $C^{\alpha-1}$ of the solution $v(t)$ to the stationary one, first for the initial value $v(0) \in D$, at least if $|\mu_\xi|$ is sufficiently small. Here $D$ is the class of all functions $v \in C^{\alpha-1}$ satisfying $\varphi(v)\theta^{-1} \in H^1$, where $H^1 = H^1(\mathbb{T})$ is the Sobolev space on $\mathbb{T}$. Then, this result will be extended to general initial values $v(0) \in C^{\alpha-1}$ by the initial layer type property of the solution. Note that $v(0) \in C^\beta$, $\beta \in (\frac{1}{2}, \alpha - 1)$, is equivalent to writing $v(0) \in C^{\alpha-1}$, $\alpha \in (\frac{4}{3}, \frac{3}{2})$, by tuning in $\alpha$. In addition, we will use $C$ to denote a positive generic constant that may change from line to line in this section.

2.1 Method of energy inequality

In this subsection, we assume $\xi \in C^\infty(\mathbb{T})$ (or at least $\xi \in C^2(\mathbb{T})$) and consider a differentiable solution $v(t, x)$ of (1.6). More precisely, if $\xi \in C^2(\mathbb{T})$, the Eq. (1.4) is a classical PDE of divergence form:

$$\partial_t v = \nabla[a_1(v, \nabla v)] - a_2(x, v, \nabla v), \quad x \in \mathbb{T},$$

or, we can further rewrite it as

$$\partial_t v = a_3(v)\Delta v - A(x, v, \nabla v), \quad x \in \mathbb{T},$$

where $a_1(v, p) = \varphi'(v)p$, $a_2(x, v, p) = -\left(\xi(x)\chi(v) + \xi'(x)p\right)$, $a_3(v) = \varphi'(v)$ and $A(x, v, p) = -\varphi''(v)p^2 + a_2(x, v, p)$. Note that (2.1) and (2.2) are written in the forms of (6.1) and (6.4) in [13] (p. 449, p. 450), respectively. We actually consider (1.6) so that $\chi = \varphi$. Recall that $\varphi \in C^4(\mathbb{R})$ satisfies (1.3) and this, in particular, implies the linear growth property of $\varphi$: $|\varphi(v)| \leq C(|v| + 1)$. Thus, we see that the conditions a)–d) of Theorem 6.1 ([13], p. 452), especially,

$$\partial_pa_1(v, p) = \varphi'(v) \geq 0, \quad A(x, v, 0)v \geq -b_1v^2 - b_2,$$

$$0 < c_- \leq \partial_pa_1(v, p) \leq c_+, \quad (|a_1| + |\partial_\nu a_1|)(1 + |p|) + |a_2| \leq C_M(p^2 + 1)$$

hold for $x \in \mathbb{T}$, $|v| \leq M$, $p \in \mathbb{R}$. Indeed, the second bound follows from

$$|A(x, v, 0)v| = |a_2(x, v, 0)v| = |\xi(x)||\varphi(v)v| \leq C_2(v^2 + 1).$$

Note that $M > 0$ given in (6.8) of [13] can be taken in our situation by applying the maximum principle, see Remark 2.1-(i) below. The condition $\xi \in C^2(\mathbb{T})$ is required for the condition c). The condition d) is shown by the boundedness of $\partial_\nu a_1$, $\partial_p a_2$, $\partial_\nu a_2$ for $|v| \leq M$ and $|p| \leq M_1$ for each $M, M_1 > 0$. Therefore, by Theorem 6.1 of [13], (1.4) has a unique global-in-time classical solution $v(t, x) \in H^{1+\beta/2, 2+\beta}([0, T] \times \mathbb{T})$ if $\xi \in C^2(\mathbb{T})$ and the initial value $v(0) \in C^{1+\beta}(\mathbb{T})$ for some $\beta \in (0, 1)$. Furthermore, noting that $|a_1(v, p)| + |\partial_\nu a_1(v, p)| \leq C_M|p|, |v| \leq M, p \in \mathbb{R}$, by Theorem 6.4
(13), p. 460), the existence of global-in-time classical solution is known if \( \xi \in C^2(\mathbb{T}) \) and \( v(0) \in C^\beta(\mathbb{T}), \beta \in (0, 1) \). Note that, from examples in p. 99 of [1] or p. 62 -L. 8 of [9], we know that \( C^\alpha = H^\alpha \) (Hölder space used in [13]) for all \( \alpha \in \mathbb{R}^+ \setminus \mathbb{N} \).

One can easily check that the classical solution of (1.4) is a solution in paracontrolled sense. This shows that the life time of the solution \( v(t) \in C^{\alpha - 1} \) of (1.6) equals to infinity, i.e.,

\[
T_* \equiv T_*(\hat{\xi}, v(0)) := \sup\{t \geq 0; \text{ solution of (1.6) with initial value } v(0) \text{ exists}\} = \infty,
\]

if \( \xi \in C^2(\mathbb{T}) \) and \( v(0) \in C^{\alpha - 1}, \alpha > 1 \). We expect that \( T_* \) is lower semicontinuous in \( \hat{\xi} \) as in [6], but this combined with \( T_* = \infty \) for \( \xi \in C^2(\mathbb{T}) \) does not imply the same for general \( \xi \in C^{\alpha - 2} \).

As we mentioned, in this subsection, we assume \( \xi \in C^\infty(\mathbb{T}) \) and \( v(t, x) \) is a smooth global-in-time solution of (1.6), i.e., \( v \in C^{1, 2}((0, \infty) \times \mathbb{T}) \cap C([0, \infty) \times \mathbb{T}) \). We define \( f(t, x) \) as

\[
f(t, x) := \frac{\varphi(v(t, x))}{\theta(x)}, \quad (2.3)
\]

where \( \theta = \theta_\xi(x) \) is defined in (1.7). Note that \( f(t, x) \) is periodic in \( x \in \mathbb{T} \). Then, we have

\[
\nabla \varphi(v) = \nabla(f \theta) = \nabla f \cdot \theta + f \nabla \theta
\]

\[
= \nabla f \cdot \theta + f(-\xi \theta + \mu)
\]

\[
= \nabla f \cdot \theta + \mu f - \xi \varphi(v).
\]

Therefore, the Eq. (1.6) can be rewritten as

\[
\partial_t v = \nabla(\theta \nabla f + \mu f). \quad (2.4)
\]

Let \( L^2_\theta := L^2(\mathbb{T}, \theta dx) \) and \( H^1_\theta := H^1(\mathbb{T}, \theta dx) \) be the spaces equipped with the norms \( \| f \|_{L^2_\theta} := (\int_\mathbb{T} f^2 \theta dx)^{1/2} \) and \( \| f \|_{H^1_\theta} := (\int_\mathbb{T} (f^2 + (\nabla f)^2) \theta dx)^{1/2} \), respectively. We define the functional \( \Phi(f) \equiv \Phi_\theta(f) \) of \( f \in H^1_\theta \) as

\[
\Phi(f) \equiv \Phi_\theta(f) := \frac{1}{2} \int_\mathbb{T} (\nabla f)^2 \theta dx.
\]

**Lemma 2.1** The functional \( \Phi \) is Fréchet differentiable in \( H^1_\theta \) and its Fréchet derivative \( D\Phi(f) \in (H^1_\theta)^* \) is given by

\[
D\Phi(f)(\psi) \equiv (H^1_\theta)^*, (D\Phi(f), \psi)_{H^1_\theta} = \int_\mathbb{T} \nabla f \nabla \psi \theta dx, \quad (2.5)
\]

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for $\psi \in H^1_{0\theta}$. If $f \in C^2(\mathbb{T})$, this is further rewritten as
\[
\int_{\mathbb{T}} D\Phi(x, f) \psi(x) \theta dx,
\]
with
\[
D\Phi(x, f) = -\theta^{-1}\nabla(\theta \nabla f),
\tag{2.6}
\]
note that $\theta^{-1}$ means $\frac{1}{\theta}$.

**Proof** Take $\psi \in H^1_{0\theta}$ and define $D\Phi(f)(\psi)$ as (2.5). Then,
\[
\Phi(f + \psi) - \Phi(f) - D\Phi(f)(\psi) = \Phi(\psi) = o(\|\psi\|_{H^1_{0\theta}})
\]
as $\|\psi\|_{H^1_{0\theta}} \to 0$. This shows the Fréchet differentiability of $\Phi$ and the formula (2.5). The formula (2.6) for $D\Phi(x, f)$ is shown by a simple integration by parts:
\[
\int_{\mathbb{T}} \nabla f \nabla \psi \theta dx = -\int_{\mathbb{T}} \theta^{-1}\nabla(\theta \nabla f) \psi \theta dx.
\]
\[\square\]

Noting $v = \varphi^{-1}(f\theta)$ from (2.3) and using (2.6), (2.4) is rewritten as
\[
\partial_t(\varphi^{-1}(f\theta)) = -\theta D\Phi(x, f) + \mu \nabla f.
\tag{2.7}
\]
Set $G(x, f) = (\varphi^{-1})'(f\theta(x)) > 0$ and
\[
K(x, f) = \frac{1}{G(x, f)} = \varphi'(\varphi^{-1}(f\theta)) = \varphi'(v) \geq c_\theta > 0,
\]
recall the assumption (1.3). Then, since $\partial_t \varphi^{-1}(f\theta) = G(x, f)\partial_t f \cdot \theta$, (2.7) can be further rewritten as
\[
\partial_t f = K(x, f)(-D\Phi(x, f) + \mu \theta^{-1}\nabla f),
\tag{2.8}
\]
which is sometimes called Onsager equation at least when $\mu = 0$, see [15], p. 193. See also Remark 2.1 below for this equation.

**Proposition 2.2** Assume $\xi \in C^\infty(\mathbb{T})$ and $v(t, x)$ is a smooth global-in-time solution of (1.6). Then, for $f(t)$ defined by (2.3), if $f(0) \in H^1_{0\theta}$, we have the bound
\[
\Phi(f(t)) \leq \Phi(f(0)) e^{C(\theta)\gamma t},
\tag{2.9}
\]
\[\square\]
where \( \theta = \theta_\xi \),

\[
C(\theta) = -\frac{c_-}{2c_2(\theta)} + \frac{1}{2c_-}\mu^2 c_1(\theta)^2,
\]

\( c_1(\theta) = c_1(\min \theta) \) defined by (2.12) and \( c_2(\theta) > 0 \) is the constant given in (2.15) in Poincaré inequality. (Note that \( c_2(\theta) \) stays finite for every \( \eta \in C([0, 1]) \).) In particular, if \( |\mu| = |\mu_\xi| \) is small enough, \( C(\theta) < 0 \) and this shows the exponential decay of \( \Phi(f(t)) \) as \( t \to \infty \):

\[
\Phi(f(t)) \leq \Phi(f(0)) e^{-c_* t},
\]

(2.10)

for some \( c_* > 0 \). When \( \mu_\xi = 0 \), in particular, when \( \sigma = \langle \xi, 1 \rangle = 0 \), one can take \( c_* = \frac{c_-}{c_2(\theta)} \) (better than \( C(\theta) \) with \( \mu_\xi = 0 \)).

Proof Recalling \( \theta(x) > 0 \) and \( K(x, f) = \varphi'(v) \geq c_- > 0 \), we obtain from (2.8)

\[
\partial_t \Phi(f) = \langle \partial_t f, D\Phi(\cdot, f) \rangle_{L^2_\theta}
\]

\[
= -\int_T K(x, f) D\Phi(x, f)^2 \theta dx + \mu \int_T K(x, f) D\Phi(x, f) \nabla f dx
\]

\[
\leq -c_- \|D\Phi(\cdot, f)\|^2_{L^2_\theta} + \mu \int_T K(x, f) D\Phi(x, f) \nabla f dx.
\]

(2.11)

For the second term, since \( \varphi \) satisfies (1.3), we have \( c_- \leq K(x, f) \leq c_+ \) and this shows

\[
\leq |\mu| c_+ \|D\Phi(\cdot, f)\|_{L^2_\theta} \|\nabla f\|_{L^2_{\theta^{-1}}}.
\]

However, since \( \theta \) is uniformly positive, \( \theta^{-1}(x) \leq c(\theta)\theta(x) \) for \( c(\theta) := (\min_{x \in \mathbb{T}} \theta^2(x))^{-1} > 0 \) and therefore \( \|\nabla f\|_{L^2_{\theta^{-1}}} \leq \sqrt{c(\theta)} \|\nabla f\|_{L^2_\theta} = \sqrt{2c(\theta)} \Phi(f)^{\frac{1}{2}} \).

Thus, the second term is bounded by

\[
\leq |\mu| c_1(\theta) \|D\Phi(\cdot, f)\|_{L^2_\theta} \Phi(f)^{\frac{1}{2}}
\]

\[
\leq \frac{c_+}{2} \|D\Phi(\cdot, f)\|^2_{L^2_\theta} + \frac{1}{2c_-}\mu^2 c_1(\theta)^2 \Phi(f),
\]

where

\[
c_1(\theta) = c_1(\min \theta) := c_+ \sqrt{2c(\theta)} > 0.
\]

(2.12)

Therefore, we obtain

\[
\partial_t \Phi(f) \leq -\frac{c_-}{2} \|D\Phi(\cdot, f)\|^2_{L^2_\theta} + \frac{1}{2c_-}\mu^2 c_1(\theta)^2 \Phi(f).
\]

(2.13)
We now apply Poincaré inequality \( \Phi(f) \leq c_2(\theta)\|D\Phi(\cdot, f)\|_{L^2_\theta}^2 \) given in Lemma 2.3 below, and then (2.13) shows that
\[
\partial_t \Phi(f) \leq -\frac{c_-}{c_2(\theta)} \Phi(f) + \frac{1}{c_-^2} \mu^2 c_1(\theta)^2 \Phi(f) = C(\theta) \Phi(f).
\]
This implies \( \partial_t \left(e^{-C(\theta)t} \Phi(f)\right) \leq 0 \) and leads to the bound (2.9). (2.10) is immediate from (2.9). When \( \mu_\xi = 0 \), \( \partial_t \Phi(f) \leq -\frac{c_-}{\|D\Phi(\cdot, f)\|_{L^2_\theta}^2} \) holds by (2.11), which is simpler than (2.13). Therefore, one can take \( c_* = \frac{c_-}{c_2(\theta)} \) in this case by Lemma 2.3.

The following is the Poincaré inequality used in the proof of Proposition 2.2.

**Lemma 2.3** For every \( f \in C^2(\mathbb{T}) \), we have
\[
\Phi(f) \leq c_2(\theta)\|D\Phi(\cdot, f)\|_{L^2_\theta}^2,
\]
where
\[
c_2(\theta) := \frac{1}{2} \int_\mathbb{T} \theta^{-1}(x)dx \int_\mathbb{T} \theta(y)dy.
\]

**Proof** Set \( g := \theta \nabla f \) and note that
\[
\int_\mathbb{T} g(0)^{-1}dx = \int_\mathbb{T} \nabla f dx = 0
\]
holds by the periodicity of \( f \). Then, noting that
\[
\Phi(f) = \frac{1}{2} \int_\mathbb{T} (g(0)^{-1})^2 \theta dx = \frac{1}{2} \int_\mathbb{T} g^2 \theta^{-1}dx,
\]
\[
\|D\Phi(\cdot, f)\|_{L^2_\theta}^2 = \int_\mathbb{T} (\nabla g)^2 \theta^{-1}dx,
\]
and setting
\[
Z := \int_\mathbb{T} \theta^{-1}dx,
\]
we have
\[
\Phi(f) = \frac{1}{2} \int_\mathbb{T} \left( g(x) - \frac{1}{Z} \int_\mathbb{T} g(y)\theta^{-1}(y)dy \right)^2 \theta^{-1}(x)dx
\]
\[
= \frac{1}{2} \int_\mathbb{T} \left( \int_\mathbb{T} (g(x) - g(y)) \frac{1}{Z} \theta^{-1}(y)dy \right)^2 \theta^{-1}(x)dx
\]
\[
\leq \frac{1}{2} \int_\mathbb{T} \theta^{-1}(x)dx \int_\mathbb{T} (g(x) - g(y))^2 \frac{1}{Z} \theta^{-1}(y)dy
\]
\[\square\]
\[
\frac{1}{2Z} \int_T \theta^{-1}(x)dx \int_T \theta^{-1}(y)dy \left( \int_x^y \nabla g(z)dz \right)^2 \\
\leq \frac{1}{2Z} \int_T \theta^{-1}(x)dx \int_T \theta^{-1}(y)dy \int_T (\nabla g(z))^2 \theta^{-1}(z)dz \int_T \theta(z)dz \\
= c_2(\theta) \| D\Phi(\cdot, f) \|_{L_0^2}^2,
\]
where we have used Schwarz’s inequality twice. This shows the conclusion. \Box

**Remark 2.1**
(i) The Eq. (2.8) is rewritten as
\[
\partial_t f = K(x, f)\theta^{-1}\nabla(\theta \nabla f) + \mu K(x, f)\theta^{-1}\nabla f. \tag{2.16}
\]
As we saw above, under the assumption \( \xi \in C^\infty(\mathbb{T}) \), \( f \) exists globally in time and (2.16) can be considered as a linear PDE for \( f \) regarding the coefficient \( K(x, f) \) is given. Then, since the right hand side of (2.16) has no zeroth-order term in \( f \), it satisfies the maximum principle and we have
\[
\min_{x \in \mathbb{T}} f(0, x) \leq f(t, x) \leq \max_{x \in \mathbb{T}} f(0, x),
\]
see, for example, [5] p. 368. Based on this observation and taking the limit in \( \xi \), one can cover the case \( \theta \in C(\mathbb{T}) \) and show the global-in-time solvability of (1.6). This provides another proof of the first part of Theorem 1.1, though the present article relies on the method of energy inequality.

(ii) (Linear case) When \( \varphi(v) = v \), we have \( G(x, f) = K(x, f) = 1 \). In addition, if \( \mu_\xi = 0 \), or equivalently, if \( \sigma_\xi = 0 \) holds, (2.8) defines a simple gradient flow:
\[
\partial_t f = - D\Phi(x, f), \tag{2.17}
\]
and this implies \( \partial_t \Phi(f) = - \| D\Phi(\cdot, f) \|_{L_0^2}^2 \).

**Remark 2.2** For the SPDE (1.4) with general \( \chi \) and smooth \( \xi \), the stationary solution is a periodic solution \( v = v(x) \) of the ordinary differential equation
\[
\Delta[\varphi(v)] + \nabla[\chi(v)\xi] = 0.
\]
As before, setting \( \theta = \varphi(v) \), this equation is rewritten as
\[
\nabla \theta + \psi(\theta)\xi = \mu, \tag{2.18}
\]
where \( \psi(\theta) := \chi(\varphi^{-1}(\theta)) \) and \( \mu \) is any constant. If \( \mu = 0 \), the Eq. (2.18) is of separable type and solved as
\[
\Psi(\theta) \equiv \int_0^\theta \frac{d\theta'}{\psi(\theta')} = -\eta(x) + C, \quad x \in \mathbb{T}.
\]
For simplicity, if \( \chi > 0 \), then \( \Psi \) is increasing and (2.18) is solved as

\[
\theta_C(x) = \Psi^{-1}(-\eta(x) + C).
\]

For \( \theta_C \) to be periodic, \( \eta \) should satisfy \( \eta(0) = \eta(1) \), that is, \( \sigma = 0 \). In other words, the condition \( \mu = 0 \) implies \( \sigma = 0 \). On the other hand, the constant \( C = C_m \) is determined from the conservation law (1.5).

Once stationary solutions are found, to link them to the SPDE (1.4), we need to find a proper transformation like (2.3) from \( v \) to \( f \), which extracts the factor \( z \), that is \( C \) in the present setting for general \( \chi \), and also a proper functional \( \Phi(\cdot) \) of \( f \). However, this looks nontrivial.

Note that, in case \( \chi = \varphi \), \( \psi(\theta) = \theta \), \( \Psi_1(\theta) = \log \vert \theta \vert \) and \( \theta_C(x) = \pm e^{-\eta(x)+C} = ze^{-\eta(x)} \), assuming \( \mu = 0 \).

### 2.2 Proof of Theorem 1.1

Now we consider general \( \xi \in C^{\alpha-2} \), \( \alpha \in \left( \frac{4}{3}, \frac{3}{2} \right) \). We are discussing \( v(t) \), but here start with \( u(t) \), i.e., the unique local-in-time paracontrolled solution of (1.1) with the initial value \( u_0 \in C^\alpha \).

Let us recall Theorem 3.1 (i) in [7]. In that theorem, it is declared that the map \( \Phi \) defined by (1.6) in [7] (or see (3.1) below, which is defined in a little different setting from the original one) is contractive from \( B_T(\lambda) \) (a variant of (3.4) below) into itself for some large enough \( \lambda \) and small enough \( T > 0 \). But, the explicit choices of \( \lambda \) and \( T \) were not given. To show Lemma 2.5 below, let us explicitly choose \( \lambda \) and \( T \). They can be constructed easily by the estimates (3.48) and (3.50) obtained in the proof of Theorem 3.1 (i) in [7]. In fact, by these estimates, we know that there exists a large enough constant \( M > 0 \) such that for all \( u := (u, u') \in B_T(\lambda) \),

\[
\| \Phi(u) \|_{\alpha,\beta,\gamma} \leq M \left( T^{-\frac{\alpha+\beta-\gamma}{2}} K(\|u\|_{\alpha,\beta,\gamma}, K_1(X, \xi) + K_0(\|u_0\|_{C^\alpha})(1 + \|\xi\|_{C^{\alpha-2}}) \right),
\]

(2.19)

and

\[
\| \Phi(u_1) - \Phi(u_2) \|_{\alpha,\beta,\gamma} \leq MT^{-\frac{\alpha+\beta-\gamma}{2}} K(\|u_1\|_{\alpha,\beta,\gamma}, \|u_2\|_{\alpha,\beta,\gamma}) \|u_1 - u_2\|_{\alpha,\beta,\gamma}, \tilde{K}_2(X, \xi),
\]

(2.20)

where \( \beta \in \left( \frac{1}{3}, \alpha - 1 \right) \), \( \gamma \in (2\beta + 1, \alpha + \beta) \), \( K(\lambda) \), \( K(\lambda, \lambda) \) denote the increasing and positive functions in \( \lambda > 0 \) introduced at the end of Sect. 2 in [7], \( \tilde{K}_1(X, \xi) \) and \( \tilde{K}_2(X, \xi) \) are the positive polynomial functions used in (3.48) and (3.50) in [7]. Let us determine \( \lambda \) and \( T > 0 \) as follows.

\[
\lambda = 2M \left( \tilde{K}_1(X, \xi) + K_0(\|u_0\|_{C^\alpha})(1 + \|\xi\|_{C^{\alpha-2}}) \right),
\]

(2.21)

\[
T = \min \left\{ \left( K(\lambda) + MK(\lambda, \lambda)\tilde{K}_2(X, \xi) \right)^{-\frac{2}{\alpha+\beta-\gamma}}, 1 \right\},
\]

(2.22)
where \( M \) is same as in (2.19). Then, Theorem 3.1 (i) of [7] can be restated as follows.

**Theorem 2.4** (Theorem 3.1 (i) of [7]) Let \( \lambda \) and \( T \) be defined by (2.21) and (2.22), respectively. Then, for any \( u_0 \in C^\alpha \) (or equivalently \( u_0' \in C^\beta \) and \( u_0'' \in C^\alpha \)), \( \Phi \) is contractive from \( B_T(\lambda) \) into itself. In particular, \( \Phi \) has a unique fixed point on \([0, T]\), which is the unique solution of the system (2.17) and (2.18) in [7] and it solves the SPDE (1.1) on \([0, T]\) in the paracontrolled sense.

**Proof** It is enough to show \( \Phi \) is contractive from \( B_T(\lambda) \) into itself. By (2.19) and (2.21), we easily have

\[
\| \Phi(u) \|_{\alpha, \beta, \gamma} \leq \frac{\lambda}{2} \left( T^{\frac{\alpha + \beta - \gamma}{2}} K(\| u \|_{\alpha, \beta, \gamma}) + 1 \right).
\]

Then, noting that \( T \) is given as (2.22), we have \( T^{\frac{\alpha + \beta - \gamma}{2}} K(\| u \|_{\alpha, \beta, \gamma}) < 1 \) whenever \( \| u \|_{\alpha, \beta, \gamma} \leq \lambda \) and therefore, \( \Phi \) maps \( B_T(\lambda) \) into itself. The contractivity of the map \( \Phi \) on \( B_T(\lambda) \) is obvious by (2.20) and \( T^{\frac{\alpha + \beta - \gamma}{2}} MK(\lambda, \lambda) \tilde{K}_2(X, \xi) < 1 \).

**Remark 2.3** From Theorem 2.4, we see that the time \( T \) chosen as in (2.22) depends continuously on the norm \( \| u_0 \|_{C^\alpha} \) (or equivalently on \( \| u_0' \|_{C^\beta} \) and \( \| u_0'' \|_{C^\alpha} \)), which is vital for the proof of the next Lemma 2.5.

Let us define the explosion time \( T_* \) by

\[
T_* \equiv T_*(\xi, u_0) := \sup\{t \geq 0; \text{ solution } u(t) \in C^\alpha \text{ of (1.1) starting from } u_0 \text{ exists}\}
\]

We know that \( T_* > 0 \). Furthermore, we have the following result.

**Lemma 2.5** If \( T_* < \infty \), we have

\[
\lim_{t \uparrow T_*} \| u(t) \|_{C^\alpha} = \infty.
\]

**Proof** By the definition of \( T_* \), the solution exists and satisfies \( u(\cdot) \in C([0, T_*), C^\alpha) \). If the conclusion does not hold, one can find \( M > 0 \) and a sequence \( t_n \uparrow T_* \) such that \( \| u(t_n) \|_{C^\alpha} \leq M \).

However, by Theorem 2.4 and Remark 2.3, there exists \( \varepsilon = \varepsilon_M > 0 \), which is uniform in \( n \), such that one can solve (1.1) starting from \( u(t_n) \) on the time interval \([t_n, t_n + \varepsilon]\). Since \( t_n \uparrow T_* \), this shows that one can solve (1.1) beyond \( T_* \) and contradicts the definition of \( T_* \).

**Remark 2.4** In particular, let \( C^\alpha \cup \{\Delta\} \) be the one point compactification of \( C^\alpha \) and define \( u(t) := \Delta \) for \( t \geq T_* \). Then, \( u(t) \) is defined for all \( t \geq 0 \) and \( u(\cdot) \in C([0, \infty), C^\alpha \cup \{\Delta\}) \) by Lemma 2.5. Denoting \( u(t) \) with initial value \( u_0 \in C^\alpha \cup \{\Delta\} \) by \( u(t, u_0) \), it has the flow property: \( u(t, u(s, u_0)) = u(t + s, u_0) \) for all \( t, s \geq 0 \).

To extend the result of Proposition 2.2 to general noise \( \xi \in C^{\alpha-2}, \alpha \in \left(\frac{13}{9}, \frac{3}{2}\right) \), we first give the following lemma, which will be used in the proof of Proposition 2.7 and Corollary 2.8 below.
Lemma 2.6 Assume the following condition for \( f = f(x) \in H^1_\theta \), which comes from the conservation law:

\[
\int_\mathbb{T} (\varphi^{-1}(f \theta) - \varphi^{-1}(z_m \theta)) dx = 0. \tag{2.24}
\]

Then, we have

\[
\| f - z_m \|_{L^2_\theta} \leq C \| \nabla f \|_{L^2_\theta}, \tag{2.25}
\]

for some \( C > 0 \). In particular, under the condition (2.24), the Sobolev norm \( \| f - z_m \|_{H^1_\theta} \) is equivalent to \( \| \nabla f \|_{L^2_\theta} \).

**Proof** First, note that, by mean value theorem applied for (2.24), we see \( z_m = f(y_*) \) for some \( y_* \in \mathbb{T} \). Indeed, by noting the monotone increasing property of \( \varphi^{-1} \), we see

\[
\int_\mathbb{T} \varphi^{-1}(\min f \cdot \theta) dx \leq \int_\mathbb{T} \varphi^{-1}(f \theta) dx = \int_\mathbb{T} \varphi^{-1}(z_m \theta) dx \leq \int_\mathbb{T} \varphi^{-1}(\max f \cdot \theta) dx
\]

and this implies \( \min f \leq z_m \leq \max f \). Therefore, we have

\[
\| f - z_m \|_{L^2} = \left( \int_\mathbb{T} (f(x) - f(y_*))^2 dx \right)^{1/2}
= \left( \int_\mathbb{T} dx \left\{ \int_{y_*}^x \nabla f(y') dy' \right\}^2 \right)^{1/2}
\leq \| \nabla f \|_{L^2}.
\]

Since \( 0 < \theta \in C(\mathbb{T}) \), this implies (2.25).

The equivalence of \( \| \nabla f \|_{L^2_\theta} \) to the Sobolev norm is now clear as

\[
\| \nabla f \|_{L^2_\theta} \leq \| f - z_m \|_{H^1_\theta} = \left( \| \nabla(f - z_m) \|_{L^2_\theta}^2 + \| f - z_m \|_{L^2_\theta}^2 \right)^{1/2} \leq C \| \nabla f \|_{L^2_\theta}.
\]

\( \square \)

Recall that \( D \) is the class of all functions \( v \in C^{\alpha-1} \) satisfying \( \varphi(v) \theta^{-1} \in H^1 \).

**Proposition 2.7** Assume that the initial value of the SPDE (1.6) satisfies \( v(0) \in D \). Then, the solution \( v(t) \) exists globally in time for all \( t \geq 0 \) and \( f(t) \) defined from \( v(t) \) by (2.3) satisfies

\[
\Phi(f(t)) \leq e^{Ct} \Phi(f(0)), \tag{2.26}
\]

for some \( C \in \mathbb{R} \). In particular, if \( |\mu| \) is small enough, one can take \( C < 0 \).
Proof First assume that $\xi \in C^\infty(\mathbb{T})$ and let $f(t, x)$ be as in (2.3) and Proposition 2.2 defined from $v(t, x) := \nabla u(t, x) + m$, where $m = \int_T v(0, x)dx$ and $u(t, x)$ is a (classical) global-in-time solution of (1.1) with initial value $u_0$ defined by (1.15), $a = \varphi'(\cdot + m)$ and $g = \varphi(\cdot + m)$. Then, we have the estimate (2.9) for $\Phi(f(t))$, if $f(0) \in H^1_0$.

Now, let $\xi \in C^{\alpha-2}$, $\alpha \in \left(\frac{13}{9}, \frac{3}{2}\right)$ be given and take a sequence of $\xi_n \in C^\infty(\mathbb{T})$ such that $\hat{\xi}_n = (\xi_n, \Pi(\nabla X_n, \xi_n))$ converges to $\hat{\xi} = (\xi, \Pi(\nabla X, \xi))$ in $C^{\alpha-2} \times C^{2\alpha-3}$ as $n \to \infty$, see Lemma 5.2 in [7] for details. Let $u_n(0) := \varphi^{-1}(f(0)\theta_n)$, $m_n = \int_T v_n(0, x)dx$, $a_n = \varphi'(\cdot + m_n)$ and $g_n = \varphi(\cdot + m_n)$, where $\theta_n = \theta_{\xi_n}$ is defined as in (1.7) from the integral $\eta_n$ of $\xi_n$. Note that $v_n(0)$ is chosen in such a way that $f_n(0) := \varphi(v_n(0))/\theta_n = f(0)$ holds for every $n$.

We consider the SPDE (1.6) with the initial value $v_n(0)$ and the SPDE (1.1) with the initial value $u_n(0)$ by replacing $a$, $g$ by $a_n$, $g_n$, associated with $\xi_n$ respectively, where $u_n(0)$ is determined similarly to $u(0)$ above, see (1.15). Then, we have smooth classical global-in-time solutions $v_n, u_n$ for such equations. Let $u$ and $v$ denote the solutions of (1.1) and (1.6) in paracontrolled sense associated with $\hat{\xi}$ for $t < T_*$.

We first show that $\theta_n \to \theta$ in $C^{\alpha-1}$ as $n \to \infty$. By the definition of $\theta$, see (1.7) and, noting that $\|\int_0^\alpha e^{\eta_n(y)}dy - \int_0^\alpha e^{\eta(y)}dy\|_C^\alpha \leq C\|e^{\eta_n} - e^{\eta}\|_{C^{\alpha-1}}$ by Lemma A.10 of [9], it is enough to show that $\lim_{n \to \infty} \|\psi(\eta_n) - \psi(\eta)\|_{C^{\alpha-1}} = 0$ for any $\psi \in C^2(\mathbb{R})$, where $\eta_n$ is the integral of $\xi_n$. Actually, we may take $\psi(\theta) = \epsilon^{\pm\theta}, \theta \in \mathbb{R}$ and then apply the last estimate in Lemma A.1. Under the above assumption, we have $\lim_{n \to \infty} \|\eta_n - \eta\|_{C^{\alpha-1}} = 0$, which implies that $\sup\{\|\eta_n\|_{C^{\alpha-1}}, \|\eta\|_{C^{\alpha-1}}; n \in \mathbb{N}\} \leq M$ holds for a constant $M > 0$. Taking now $\hat{\psi} \in C^2_b(\mathbb{R})$ such that $\hat{\psi}(\cdot) = \psi(\cdot)$ on $[-M, M]$ and then using Lemma 9 of [3], we have

$$\|\psi(\eta_n) - \psi(\eta)\|_{C^{\alpha-1}} \leq \|\hat{\psi}(\eta_n) - \hat{\psi}(\eta)\|_{C^{\alpha-1}} \leq C\|\hat{\psi}\|_{C^2(1 + \|\eta\|_{C^{\alpha-1}})}\|\eta_n - \eta\|_{C^{\alpha-1}},$$

which gives the desired result. Since $\theta_n \to \theta$ in $C^{\alpha-1}$, by assumptions on $\varphi'$, we also have, as $n \to \infty$, $v_n(0)$ converges to $v(0)$ in $C^{\alpha-1}$, $m_n \to m$ and in particular, $a_n, g_n$ converge to $a, g$ on each compact set of $\mathbb{R}$. Then, noting that $u_n(0) \to u(0)$ in $C^\alpha$ and using Theorem 1.3, we know that $u_n$ converges to $u$ in $L^{\alpha-1}_T$ and therefore $v_n$ to $v$ in $L^{\alpha-1}_T$, $\alpha - 1 < 1/T$ for $T < T_*$. Since the initial values $v_n(0)$ move, we use the continuity of solutions in initial values. Note that the coefficients $a_n$ and $g_n$ also move. So, we require the condition $\alpha \in \left(\frac{13}{9}, \frac{3}{2}\right)$, see Theorem 1.3 or Remark 1.2 for explanation.

In particular, recalling $\eta(x) = \langle \xi, 1_{\{0 \leq x \leq 1\}} \rangle \in C^{\alpha-1}(\{0, 1\})$ so that $\theta_\xi \in C^{\alpha-1}(\mathbb{T})$, $\theta_\xi > 0$ and also $\varphi \in C^4(\mathbb{R})$ satisfying (1.3), $f_n(t) := \varphi(v_n(t))\theta^{-1}_n$ converges to $f(t) := \varphi(v(t))\theta^{-1}$ in $L^{\alpha-1}_T$ and $f_n(0) = f(0)$ for all $n$. We note the lower semi-continuity of $\Phi(f)$ in $C^{\alpha-1}$, which follows from the variational formula for the Dirichlet form:

$$\Phi(f) \equiv \Phi_\theta(f) = \frac{1}{2} \int_T (\nabla f)^2 \theta dx = \frac{1}{2} \sup \left\{ - \int_T \frac{\nabla (\theta \nabla w)}{w} f^2 dx; w \in C^2(\mathbb{T}) \right\}.$$
when \( \theta \in C^1(\mathbb{T}) \), where \( w(x) = 0 \) can happen only at \( x \) such that \( f(x) = 0 \). Indeed, we may use the integration by parts and note \((\nabla f)^2 - \nabla w \nabla \left( \frac{f^2}{w} \right) = (\nabla f - f \frac{\nabla w}{w})^2 \geq 0\). Then, \( f_n \to f \) in \( C^{\alpha-1} \) implies \( \Phi_\theta(f) \leq \lim_{n \to \infty} \Phi_\theta(f_n) \) if \( \theta \in C^1(\mathbb{T}) \), but in the definition of \( \Phi_\theta(f) \), we only have \( \theta \) without its derivative so that this property holds also for non-smooth \( \theta \) by taking the limit \( \theta_m \to \theta \) (in \( L^\infty \)) or \( \eta_m \to \eta \) (in \( L^\infty \)) introduced as above. More precisely, noting that the constant \( C(\theta) \) in (2.9) can be taken uniformly in \( n \); \( C(\theta_n) \leq C \), since the constant \( C(\theta) = C(\theta_\xi) \) can be estimated only by \( \min \theta \) and \( \max \theta \), for \( t < T_n \), we see

\[
\Phi_\theta(f(t)) = \lim_{m \to \infty} \Phi_{\theta_m}(f(t)) \leq \lim_{m \to \infty} \lim_{n \to \infty} \Phi_{\theta_m}(f_n(t)) \leq e^{Ct} \Phi_\theta(f(0)),
\]

at least if \( f(0) \in H_\theta^1 \); recall that \( f_n(0) = f(0) \) for all \( n \) and \( \theta_n \to \theta \) in \( C^{\alpha-1} \). For the last inequality, we use \( \Phi_{\theta_n}(f_n(t)) \leq e^{Ct} \Phi_{\theta_n}(f_n(0)) \) by noting that, for arbitrary small \( \varepsilon > 0 \), \( \theta_m/\theta_n, \theta_n/\theta \leq 1 + \varepsilon \) for large enough \( n, m \). In particular, if \( |\mu_\xi| \) is small enough, one can take \( C < 0 \).

Finally, we prove \( T_n = \infty \), which shows the existence of \( v(t) \) for all \( t \geq 0 \). The above estimate on \( \Phi(f(t)) \) implies \( \| f(t) \|_{H_\theta^1} \leq M_1(e^{Ct/2} + 1) \) for \( t < T_n \), if \( f(0) \in H_\theta^1 \), for some \( M_1 > 0 \). Instead, this follows from \( \Phi(f) = \frac{1}{2} \| \nabla f \|_{L_\theta^2}^2 \) and Lemma 2.6 (equivalence of norms). However, by Sobolev’s imbedding theorem and noting that \( H_\theta^1 \equiv H^1(\mathbb{T}) \simeq H_\theta^1 \) from \( \theta \in C(\mathbb{T}) \), we have \( H_\theta^1 \subset C^{\alpha-1}, \alpha - 1 < \frac{1}{2} \) and this shows \( \| v(t) \|_{C^{\alpha-1}} \leq M_2(e^{Ct/2} + 1), t < T_n \). Therefore, noting that \( \| u(t) \|_{C^\alpha} \leq 2 \| v(t) \|_{C^{\alpha-1}} \), we see \( |u(t)|_{C^\alpha} \leq 2M_2(e^{Ct/2} + 1), t < T_n \). This proves \( T_n = \infty \) by Lemma 2.5 at least if \( f(0) = \varphi(v(0)) \theta^{-1} \in H^1_\theta \simeq H^1 \), that is, if \( v(0) \in D \).

If \( |\mu_\xi| \) is small enough, based on the estimate (2.26) obtained in Proposition 2.7 with \( c_* := -C > 0 \), we can show the exponential decay of \( v(t) \) to the unique stationary solution \( \bar{v}_m \) for each conserved quantity \( m \).

**Corollary 2.8** Assume \( v(0) \in D \) as in Proposition 2.7. Then, if \( |\mu_\xi| \) is small enough, \( f(t) := \varphi(v(t)) \theta^{-1} \) converges to the constant \( z_m \) in \( H^1(\mathbb{T}) \) exponentially fast as \( t \to \infty \):

\[
\| f(t) - z_m \|_{H^1} \leq Ce^{-c_* t/2} \| f(0) - z_m \|_{H^1}, \tag{2.27}
\]

where \( m = \int_T v(0, x)dx \). We also have

\[
\| v(t) - \bar{v}_m \|_{C^{\alpha-1}} \leq Ce^{-c_* t/2} \| f(0) - z_m \|_{H^1}. \tag{2.28}
\]

**Proof** By Lemma 2.6, under the conservation law, we obtain

\[
\| f(t) - z_m \|^2_{H^1_\theta} \leq C \| \nabla f(t) \|^2_{L_\theta^2} = 2C \Phi(f(t)) \leq 2Ce^{-c_* t} \Phi(f(0)) \leq Ce^{-c_* t} \| f(0) - z_m \|^2_{H^1_\theta}.
\]
This shows the desired estimate on $\|f(t) - z_m\|_{H^1}$, since the norm of $H^1_\Theta$ is equivalent to that of $H^1$ due to the boundedness of $\eta(x)$.

In order to give the estimate (2.28) on $v(t)$, we first show the uniform boundedness of $\|v(t)\|_{C^{\alpha-1}}$ in $t \geq 0$, i.e.,

$$\sup_{t \geq 0} \|v(t)\|_{C^{\alpha-1}} < \infty. \tag{2.29}$$

By the assumption (1.3) on $\varphi$, we have that $\|\varphi^{-1}(v)\|_{C^{\alpha-1}} \leq C(1 + \|v\|_{C^{\alpha-1}})$, $v \in C^{\alpha-1}$. Therefore, recalling that $v(t) = \varphi^{-1}(f(t)\theta)$ and using Sobolev’s imbedding theorem, we have that

$$\|v(t)\|_{C^{\alpha-1}} \leq C(1 + \|f(t)\|_{\Theta} \|v\|_{C^{\alpha-1}}) \leq C(1 + \|\theta\|_{C^{\alpha-1}} \|f(t)\|_{H^1}).$$

Recall that the constant $C$ changes from line to line. So, by (2.27), we obtain (2.29).

Thus, by noting (2.29), we take a function $\overline{\varphi^{-1}} \in C^2(\mathbb{R})$ with compact support such that $\varphi^{-1}(f(t)\theta) = v(t)$ for all $t > 0$ and $\varphi^{-1}(z_m\theta) = \overline{v}_m (= \varphi^{-1}(z_m\theta))$. Then, Lemma 9 of [3] gives that

$$\|v(t) - \overline{v}_m\|_{C^{\alpha-1}} = \|\overline{\varphi^{-1}}(f(t)\theta) - \overline{\varphi^{-1}}(z_m\theta)\|_{C^{\alpha-1}} \leq \|\varphi^{-1}\|_{C^2}(1 + \|z_m\|_{C^{\alpha-1}})\|f(t) - z_m\|_{C^{\alpha-1}} \|\theta\|_{C^{\alpha-1}} \|f(t) - z_m\|_{H^1}.$$

Consequently, noting that $\|\overline{\varphi^{-1}}\|_{C^2}$ is independent of $t \geq 0$, we obtain the desired result (2.28) by (2.27).

The next lemma gives an initial layer type result. This is used to remove the condition $v(0) \in D$ in Proposition 2.7 and Corollary 2.8. Recall that we only assume $\xi \in C^{\alpha-2}(\mathbb{T})$, $\alpha \in (\frac{13}{5}, \frac{3}{2})$.

**Lemma 2.9** For every initial value $v(0) \in C^{\alpha-1}$ and all $t \in (0, T_*)$, the solution $v(t) \in C^{\alpha-1}$ of the SPDE (1.6) in paracontrolled sense satisfies $f(t) := \varphi(v(t))\theta^{-1} \in H^1$, that is, $v(t) \in D$. In other words, even if $f(0) \notin H^1$, immediately after, we have $f(t) \in H^1$, $t > 0$ and this proves $T_* = \infty$ by Proposition 2.7.

**Proof** If $\xi$ is smooth, taking any $u_0$ such that $\nabla u_0 + m = v_0$ for given $v_0$, we have from (1.1) with $a = \varphi'(\cdot + m)$ and $g = \varphi(\cdot + m)$

$$\partial_t u = \varphi'(v)\nabla v + \varphi(v)\xi = \theta \nabla f + \mu f.$$

This is an integrated form of (2.4). Therefore, we have

$$u(t) - u(0) = \theta \int_0^t \nabla f(s)ds + \mu \int_0^t f(s)ds.$$
or we can rewrite this as
\[ \int_0^t \nabla f(s, x)ds = \theta(x)^{-1} \left\{ u(t, x) - u(0, x) - \mu \int_0^t f(s, x)ds \right\}. \quad (2.30) \]

The right hand side belongs to \( C^{\alpha-1} \), since \( \theta^{-1} \in C^{\alpha-1} \), \( u(t) \in C^\alpha \) and \( f(s) \in C([0, T], C^{\alpha-1}) \). On the other hand, in the left hand side, \( \nabla f(s) \in C([0, T], C^{\alpha-2}) \), \( T < T^* \), since \( \varphi(v(s)) \in L^{\alpha-1}_T \), \( \theta^{-1} \in C^{\alpha-1} \). Therefore, taking the limit in \( \xi \), we have \( (2.30) \) for \( t < T^* \), if we interpret the left hand side as a Bochner integral in \( C^{\alpha-2} \).

First note that
\[ \int_0^t \nabla f(s, x)ds = \nabla \int_0^t f(s, x)ds, \]

where the integrals are Bochner integrals in \( C^{\alpha-2} \) for the left hand side and \( C^{\alpha-1} \) for the right hand side. (This can be shown by regarding both sides as generalized functions and by multiplying test function \( \psi \).) Thus, \( (2.30) \) implies \( \nabla \int_0^t f(s, x)ds \in C([0, T], C^{\alpha-1}) \), since the right hand side of \( (2.30) \) is in this class. This shows, by also noting an obvious relation \( \int_0^t f(s, x)ds \in C([0, T], C^{\alpha-1}) \), that

\[ \int_0^t f(s, x)ds \in C([0, T], C^\alpha). \]

Recall that the left hand side is defined as a Bochner integral in \( C^{\alpha-1} \) and \( \xi \) is already general.

Since \( f \in C([0, T], C^{\alpha-1}) \), for every \( t \in (0, T^*) \), by mean value theorem applied in the space \( C^{\alpha-1} \), we see that there exists \( \tau = \tau_t \in (0, t) \) such that
\[ \frac{1}{t} \int_0^t f(s, x)ds = f(\tau_t) \quad (2.31) \]
holds. However, since the left hand side belongs to \( C^\alpha \) and \( \alpha \in \left( \frac{4}{3}, \frac{3}{2} \right) \), we see \( f(\tau_t) \in C^\alpha \) and, in particular, \( f(\tau_t) \in H^1 \). Once we have \( f(\tau_t) \in H^1 \), taking the initial value of \( v \) as \( v(\tau_t) \), by Proposition 2.7, we see \( f(s) \in H^1 \) for every \( s \geq \tau_t \) and the solution exists globally in time. Since we can take \( t \in (0, T^*) \) arbitrary small, this shows the conclusion, recalling the flow property, see Remark 2.4. \( \square \)

**Remark 2.5** We can decompose \( v \) through \( \varphi \) as \( \varphi(v) = f \cdot \theta_\xi \) and the bad regularity \( v \in C^{\alpha-1} \) comes only from \( \eta \) in \( \theta_\xi \) and, by Lemma 2.9, \( f \) is a good part. Recall that Tsatsoulis and Weber [17] decomposed the solution of \( P(\phi) \)-dynamics on \( \mathbb{T}^2 \) into the sum of OU-part and good part. Since we deal with quasilinear equation, our decomposition is nonlinear, but in a sense, similar to this.

Now we can easily complete the proof of Theorem 1.1.
Proof of Theorem 1.1 The conclusion follows immediately by combining Proposition 2.7, Corollary 2.8 and Lemma 2.9. Here we only give a short explanation for (1.11). We first note that, since $v \in C([0, \infty), C^{\alpha-1})$, we have
\[
\sup_{t \in [0,1]} \|v(t) - \bar{v}_m\|_{C^{\alpha-1}} < \infty. \tag{2.32}
\]
On the other hand, by Lemma 2.9, we have $f(1) = \varphi(v(1))\theta^{-1} \in H^1$. In particular, $v(1) \equiv v(1,v_0) \in D$ so that the condition of Corollary 2.8 is satisfied. Therefore, by the flow property and (2.28), for $t > 1$, we obtain
\[
\|v(t) - \bar{v}_m\|_{C^{\alpha-1}} = \|v(t - 1, v(1, v_0)) - \bar{v}_m\|_{C^{\alpha-1}} \leq Ce^{-c_* (t-1)/2} \|f(1) - \bar{v}_m\|_{H^1}, \quad t > 1,
\]
which implies (1.11) by noting (2.32).

2.3 Global existence and asymptotic behavior of $u(t)$ as $t \to \infty$

We can apply the result for $v(t)$ to study the global-in-time existence and the asymptotic behavior of $u(t)$. First we note that $u(t)$ can be recovered from $v(t) := \nabla u(t)$ at least when $\xi \in C^\infty(\mathbb{T})$, cf. Sect. 1.4. Note that $m = 0$ under this choice.

Lemma 2.10 Assume $\xi \in C^\infty(\mathbb{T})$ and let the initial value $u_0 \in C^\alpha$, $\alpha \in \left(\frac{4}{7}, \frac{3}{2}\right)$ of (1.1) be given. We determine $v(t)$ by solving (1.4) with initial value $v_0 := \nabla u_0$, and set
\[
\begin{align*}
  u(t,x) := & \int_0^x v(t,y)dy + \int_T u_0(y)dy - \int_T (1-y) v(t,y)dy \\
  & + \int_0^t ds \int_T \chi(v(s,y))\xi(y)dy,
\end{align*} \tag{2.33}
\]
where we regard $T = [0, 1)$ especially in the first and third terms of the right hand side. Then, $u(t)$ solves the SPDE (1.1) with $a = \varphi'$ and $g = \chi$.

Proof First, note that $u(t,x)$ defined by (2.33) satisfies the initial condition:
\[
u(0,x) = \int_0^x \nabla u_0(y)dy + \int_T u_0(y)dy - \int_T (1-y) \nabla u_0(y)dy = u_0(x).
\]
To see that it satisfies the SPDE (1.1), writing the sum of the second to fourth terms in the right hand of (2.33) as $A(t)$, we have
\[
\partial_t u(t,x) = \int_0^x \partial_t v(t,y)dy + \partial_t A(t)
\]
\[
= \int_0^x \left[ \Delta \varphi(v) \right] dy + \partial_t A(t).
\]
\[ = \nabla \{ \varphi(v) \}(t, x) - \nabla \{ \varphi(v) \}(t, 0) + \chi(v(t, x))\xi(x) - \chi(v(t, 0))\xi(0) + \partial_t A(t). \]

Since \( \nabla u = v \) holds for \( u \) defined by (2.33), we have \( \nabla \{ \varphi(v) \} = a(\nabla u) \Delta u \) and \( \chi(v)\xi = g(\nabla u)\xi \). Therefore, to complete the proof of the lemma, it is enough to show
\[
\partial_t A(t) = \nabla \{ \varphi(v) \}(t, 0) + \chi(v(t, 0))\xi(0). \tag{2.34}
\]

However,
\[
\partial_t A(t) = -\int_T (1 - y)\partial_t v(t, y)dy + \int_T \chi(v(t, y))\xi(y)dy \\
= -\int_T (1 - y)[\nabla \{ \varphi(v) \} + \chi(v)\xi]dy + \int_T \chi(v(t, y))\xi(y)dy \\
= -\int_T [\nabla \{ \varphi(v) \} + \chi(v)\xi]dy - (1 - y)[\nabla \{ \varphi(v) \} + \chi(v)\xi] \bigg|_0^1 \\
+ \int_T \chi(v(t, y))\xi(y)dy \\
= [\nabla \{ \varphi(v) \} + \chi(v)\xi](t, 0),
\]
which shows (2.34).

**Remark 2.6** When \( \xi \in C^{\alpha-2} \), the last term in the right hand side of (2.33) is well-defined even for general \( \chi \) in the sense that the other four terms are all well-defined. But, to look at it by itself especially without time integral, the product \( \chi(v(s))\xi \) is ill-posed in a classical sense, since \( \chi(v(s)) \in C^{\alpha-1} \) and \( \xi \in C^{\alpha-2} \) with \( \alpha \in \left( \frac{4}{3}, \frac{3}{2} \right) \).

However, when \( \chi = \varphi \), this product turns out to be well-defined. Indeed, in this case, \( \varphi(v(t)) = f(t)\theta \) with \( f(t) \in H^1 \) for every \( t > 0 \) by Lemma 2.9. Accordingly, the last term of (2.33) without time integral can be interpreted as \( H^1(\langle f(t), \theta\xi \rangle_{H^{-1}}, t > 0 \) if \( \theta\xi \) can be regarded as an element of \( H^{-1} \). Noting that \( \theta \in C^{\alpha-1}, \xi \in C^{\alpha-2} \) and \( (\alpha - 1) + (\alpha - 2) < 0 \), in general, the product \( \theta\xi \) is ill-posed. However, in our special case, we will see that \( \theta\xi \) is well-defined as an element of \( C^{\alpha-2} \) and then, in particular, \( \theta\xi \in H^{-1} \) because of \( C^{\alpha-2} \subset H^{-1} \). This gives the motivation for the following proof of Theorem 1.2.

We now give the proof of Theorem 1.2.

**Proof of Theorem 1.2** The global-in-time existence of \( u(t) \) is clear from \( T_\alpha = \infty \) noted in Lemma 2.9. In the rest, we assume \( |\mu\xi| \) is sufficiently small as in Theorem 1.1.

As we mentioned in Remark 2.6, we have to show that the term \( H^1(\langle f(t), \theta\xi \rangle_{H^{-1}}, t > 0 \) (or equivalently the term \( \theta\xi \)) is well-defined. For this, we apply Lemma 2.10 by introducing an approximation of \( u(t) \) as in the proof of Proposition 2.7. Since Lemma 2.10 holds for \( v_0 \in C^{\alpha-1} \), in the following, we consider the initial values \( u_n(0) = u_0 \in C^{\alpha} \) for all \( n \). Note that, we just have \( v(0) = \nabla u_0 \in C^{\alpha-1} \), which does not imply \( v(0) \in D \).
Take a sequence of enhanced noises \( \hat{\xi}_n \) such that \( \xi_n \in C^\infty(\mathbb{T}) \) and \( \hat{\xi}_n \) converges to \( \hat{\xi} \) in \( C^{\alpha-2} \times C^{2\alpha-3} \) as \( n \to \infty \). Then, the associated solution \( u_n \) converges to \( u \) in \( L^a_T \). Since \( T_n = \infty \), one can take \( T > 0 \) arbitrarily large and, in particular, we have for every \( t \geq 0 \),

\[
\|u_n(t) - u(t)\|_{C^a} \to 0 \quad (n \to \infty).
\] (2.35)

Since \( \xi_n \in C^\infty(\mathbb{T}) \), by Lemma 2.10, we have the formula (2.33) for \( u_n(t, x) \) by replacing \( v, \xi \) by \( v_n := \nabla u_n, \xi_n \), respectively, in the right hand side, which we denote as \( A^1_n(t, x) + A_2 - A^3_n(t) + A^4_n(t) \). Recall that we take \( a = \varphi' \) and \( g = \chi = \varphi \) here. We also denote \( f_n(t, x) := \varphi(v_n(t, x))\theta_n^{-1}(x) \) and \( \theta_n := \theta_{\xi_n} \). Then, we see \( f_n(t, x) \to f(t, x) := \varphi(v(t, x))\theta_\xi^{-1}(x) \) in \( L^a_T \) as before, where \( \theta = \theta_\xi \).

Noting that \( \xi_n \in C^\infty(\mathbb{T}) \) and (1.8), we have \( \theta_n \xi_n = \mu_n - \nabla \theta_n \), which are well-defined as the elements of \( C^{\alpha-2} \) and then

\[
\|\theta_n \xi_n - \theta_m \xi_m\|_{C^{\alpha-2}} = \|\mu_n - \nabla \theta_n - (\mu_m - \nabla \theta_m)\|_{C^{\alpha-2}} \\
\leq |\mu_n - \mu_m| + \|\nabla \theta_n - \nabla \theta_m\|_{C^{\alpha-2}} \\
\leq |\mu_n - \mu_m| + C\|\theta_n - \theta_m\|_{C^{\alpha-2}},
\]

where \( \mu_n := \mu_{\xi_n} \). Therefore, we see that \( \theta_n \xi_n \) is a Cauchy sequence of \( C^{\alpha-2} \) by noting that \( \theta_n \to \theta \) in \( C^{\alpha-1} \) and \( \mu_n \to \mu = \mu_\xi \) as \( n \to \infty \). In the sequel, we denote by \( \theta_\xi \) the limit of the sequence \( \theta_n \xi_n \) in \( C^{\alpha-2} \). So, we have \( \theta_\xi \in C^{\alpha-2} \) and in particular, \( \theta_\xi \in H^{-1} \).

Using the above notation, we claim that

\[
\begin{align*}
\lim_{n \to \infty} \{A^1_n(t, x) & - A^3_n(t) + A^4_n(t)\} \\
= \int_0^x v(t, y)dy - \int_\mathbb{T} (1-y)v(t, y)dy + \int_0^t H^1(f(s), \theta_\xi)_H^{-1}ds \\
=: A_1(t, x) - A_3(t) + A_4(t). \quad (2.36)
\end{align*}
\]

By Lemma 2.9, we know \( f(t) \in H^1 \) and \( \int_0^t f(s, x)ds \in H^1 \) for \( t > 0 \). So, \( H^1(f(t), \theta_\xi)_H^{-1}, t > 0 \) is well-defined. In particular, we see \( A_4(t) \) is well-defined under the assumption \( a = g' \). Let us first show

\[
\lim_{n \to \infty} A^4_n(t) = A_4(t). \quad (2.37)
\]

In order to do it, we show \( \int_0^t f_n(s, x)ds \) converges to \( \int_0^t f(s, x)ds \) in \( H^1 \) for \( t > 0 \). By (2.30) and \( \theta_n \to \theta \) in \( C^{\alpha-1} \) (in particular in \( L^\infty \)), \( \inf_{x, n} \theta_n(x) > 0, \inf_x \theta(x) > 0 \), we have

\[
\begin{align*}
\left\|\nabla \int_0^t f_n(s)ds - \nabla \int_0^t f(s)ds\right\|_{L^2} \\
\leq \|\theta_n^{-1}(u_n(t) - u(t))\|_{L^2} + \left\|\theta_n^{-1} - \theta^{-1}\right\|(u(t) - u_0 + \mu \int_0^t f(s)ds)\right\|_{L^2}
\]
\[ + \left\| \theta_n^{-1} \mu_n \int_0^t f_n(s) ds - \theta_n^{-1} \mu \int_0^t f(s) ds \right\|_{L^2} \leq C \left\{ \|u_n(t) - u(t)\|_{L^2} + \left\| (\theta_n - \theta)(u(t) - u_0 + \mu \int_0^t f(s) ds) \right\|_{L^2} + \left\| \mu_n \int_0^t f_n(s) ds - \mu \int_0^t f(s) ds \right\|_{L^2} \right\}. \]

Since \( f_n(t) \to f(t) \in \mathcal{L}_T^{\alpha-1} \), \( T > 0 \), we have \( \lim_{n \to \infty} \left\| \int_0^t f_n(s) ds - \int_0^t f(s) ds \right\|_{L^2} = 0 \). Taking \( n \to \infty \) in both sides of the above inequality, we have \( \lim_{n \to \infty} \left\| \nabla \int_0^t f_n(s) ds - \nabla \int_0^t f(s) ds \right\|_{L^2}^2 = 0 \). Therefore, we obtain the desired result, from which (2.37) follows by noting that \( A_n^3(t) = \int_0^t ds \int_T f_n(s, y) \theta_n(y) \xi_n(y) dy = \int_0^t \underbrace{H^1(f_n(s, \theta_n s_n))}_{H^{-1}} ds \).

For \( A_n^3(t, x) \), we have \( A_n^3(t, x) \to A_1(t, x) \in C([0, T], C^\alpha) \), \( T > 0 \) as \( n \to \infty \). Indeed, noting that \( C^\alpha \) coincides with the usual \( \alpha \)-Hölder space for \( \alpha \in \left( \frac{13}{9}, \frac{3}{2} \right) \), we have

\[ \|A_1^n(t) - A_1(t)\|_{C^\alpha} = \|A_1^n(t) - A_1(t)\|_{L^\infty} + \|\nabla A_1^n(t) - \nabla A_1(t)\|_{L^\infty} \leq \|v_n(t) - v(t)\|_{C^\alpha-1} + \|A_1^n(t) - A_1(t)\|_{L^\infty} \leq 2\|v_n(t) - v(t)\|_{C^\alpha-1}, \]

where \( \nabla A_1^n(t) = v_n(t) \) and \( \nabla A_1(t) = v(t) \) have been used for the second equality. Therefore, noting that \( v_n(t) \to v(t) \) in \( \mathcal{L}_T^{\alpha-1} \), we obtain the desired result. Finally for \( A_2^3(t) \), by similar arguments, it is easy to see \( \|A_2^3(t) - A_3(t)\| \leq \frac{1}{2} \|v_n(t) - v(t)\|_{L^\infty} \leq \frac{1}{2} \|v_n(t) - v(t)\|_{C^\alpha-1} \), which implies that \( A_2^3(t) \) converges uniformly to \( A_3(t) \) on any compact interval \([0, T]\). As a consequence, the proof of (2.36) is completed.

Combining (2.35) with (2.36), we have

\[ u(t, x) = A_1(t, x) + A_2 - A_3(t) + A_4(t). \]

We now show the uniform boundedness of \( \|A_1(t)\|_{C^\alpha} + |A_3(t)| \) in \( t \geq 0 \), that is,

\[ \sup_{t \geq 0} \{\|A_1(t)\|_{C^\alpha} + |A_3(t)|\} < \infty. \tag{2.38} \]

Indeed, this can be easily shown by Theorem 1.1. By similar arguments for convergences above, we easily have

\[ \|A_1(t)\|_{C^\alpha} + |A_3(t)| \leq C \|v(t)\|_{C^\alpha-1}, \quad t \geq 0. \]
Noting that \( \int T v_0(y) dy = \int T \nabla u_0(y) dy = 0 (= m) \) and \( |\mu| = |\mu_\xi| \) is sufficiently small, Theorem 1.1 gives that

\[
\|v(t)\|_{C^{\alpha-1}} \leq \|\tilde{v}_0\|_{C^{\alpha-1}} + C e^{-ct},
\]

where \( \tilde{v}_0 \) is defined by (1.10) with \( m = 0 \). Therefore, we have \( \sup_{t \geq 0} \|v(t)\|_{C^\alpha} < \infty \) and (2.38) is shown.

Noting that \( A_2 \) is a constant, in order to show (1.12), it is sufficient to show

\[
\sup_{t \geq 0} |A_4(t) - z_0 \mu t| < \infty.
\]

Let us rewrite \( A_4(t) \) as follows.

\[
A_4(t) = z_0 t H^1 \langle 1, \theta \xi \rangle_{H^{-1}} + \int_0^t r(s) ds,
\]

where \( z_0 \) is determined by (1.9) with \( m = 0 \) and \( r(s) := H^1(f(s) - z_0, \theta \xi)_{H^{-1}} \). By analogous arguments for (1.11), see the proof of Theorem 1.1, we have

\[
|r(s)| \leq C e^{-c_* s/2} \|\theta \xi\|_{H^{-1}} \|f(1) - z_0\|_{H^1}, \quad s \geq 1.
\]

holds for some \( c_* > 0 \), where Corollary 2.8 and \( f(1) \in H^1 \) have been used; recall that \( |\mu_\xi| \) is sufficiently small. Noting that

\[
\left| \int_0^1 r(s) ds \right| \leq \left\| \int_0^1 (f(s) - z_0) ds \right\|_{H^1} \|\theta \xi\|_{H^{-1}}
\]

and using (2.41), we see

\[
\sup_{t \geq 0} \left| \int_0^t r(s) ds \right| < \infty.
\]

Finally for the first term in the right hand side of (2.40), recalling that \( \theta \xi \) denotes the limit of \( \theta_n \xi_n \) in \( C^{\alpha-2} \) and \( \theta_n \xi_n = \mu_n - \nabla \theta_n \), we see

\[
H^1 \langle 1, \theta \xi \rangle_{H^{-1}} = \lim_{n \to \infty} \int_T \theta_n \xi_n dy = \lim_{n \to \infty} \int_T (\mu_n - \nabla \theta_n) dy = \lim_{n \to \infty} \mu_n = \mu,
\]

where the periodicity of \( \theta_n \) has been used for the third equation. Consequently, summarizing the above estimates, we have (2.39) and complete the proof of Theorem 1.2.

\( \square \)
3 Continuity of the solution $u(t)$ of (1.1) in initial values

In this section, we give the proof of Theorem 1.3. As in [7] and stated in Remark 3.2, Theorem 1.3 follows from Theorem 3.1 and Remark 3.3 below. For simplicity of notation, we first, instead of (1.13), consider the special case $m = 0$ of (1.13), i.e., (1.1), and prove Theorem 3.1, which is the main result of this section. Then, Theorem 1.3 is easily obtained by Theorem 3.1 together with Remark 3.3.

Since this part is a continuation of [7], the same notations as in [7] will be mostly used. We assume $\beta \in (\frac{1}{2}, \alpha - 1)$, $\gamma \in (2\beta + 1, \alpha + \beta)$, and use notations $K_0$, $K(\|u\|_{\alpha, \beta, \gamma})$ and $\bar{K}(\|u_1\|_{\alpha, \beta, \gamma}, \|u_2\|_{\alpha, \beta, \gamma})$ introduced at the end of Sect. 2 of [7], which may change from line to line. We still write $a \lesssim b$ for two non-negative functions $a$ and $b$ if there exists a constant $C > 0$ independent of the variables under consideration such that $a \leq Cb$. In addition, to emphasize the initial value, we sometimes write $K_0(\|u_0\|_{C^\alpha})$, $K_0(\|u_0\|^2_{C^\alpha}, \|u_0^2\|_{C^\alpha})$ for $K_0$.

Note that Theorem 1.3 is about the continuity of the local-in-time solutions on initial values and $m \in \mathbb{R}$. Recalling Lemma 1.4, for simplicity but without loss of generality, we may assume the coefficients $a, g \in C^3_b(\mathbb{R})$ with (1.2) in the sequel, which are same as in [7].

Let $\bar{C}_{\alpha, \beta, \gamma}(X)$ be the family of functions controlled by $X$, that is,

$$\bar{C}_{\alpha, \beta, \gamma}(X) := \{(u', u^\sharp); u = \bar{\Pi}_u'X + u^\sharp, \|(u', u^\sharp)\|_{\alpha, \beta, \gamma} < \infty\},$$

where $\bar{\Pi}_u'X$ denotes the modified paraproduct of $u'$ and $X = (-\Delta)^{-\frac{1}{2}}Q\xi$, see Sect. 2 of [7] and

$$\|(u', u^\sharp)\|_{\alpha, \beta, \gamma} := \|u'\|_{L^\beta_T} + \|u^\sharp\|_{L^\gamma_T} + \sup_{0< t \leq T} t^{\frac{\gamma - \alpha}{2}}\|u^\sharp(t)\|_{C^\gamma}.$$ 

Noting that $u$ is a function of $u'$ and $u^\sharp$, we used the space $C_{\alpha, \beta, \gamma}(X)$ and the norm $\|(u, u')\|_{\alpha, \beta, \gamma}$ in [7], which are equivalent to the above. For the given initial value $u_0$, we impose additional conditions $u(0) = u_0, u'(0) = \frac{g(\nabla u_0)}{a(\nabla u_0)}$ and define the (skew product) space $\mathcal{X}$ by

$$\mathcal{X} := \{(u_0, u', u^\sharp) \in C^\alpha \times \bar{C}_{\alpha, \beta, \gamma}; (u', u^\sharp) \text{ satisfies } u(0) = u_0, u'(0) = \frac{g(\nabla u_0)}{a(\nabla u_0)}\},$$

which is viewed as a metric space embedded in the Banach space $C^\alpha \times \bar{C}_{\alpha, \beta, \gamma}$ equipped with the norm

$$\|(u_0, \mathbf{u})\|_{C^\alpha \times \bar{C}_{\alpha, \beta, \gamma}} := \|u_0\|_{C^\alpha} + \|\mathbf{u}\|_{\bar{C}_{\alpha, \beta, \gamma}}, \quad \mathbf{u} = (u', u^\sharp).$$

Let $L^0 := \partial_t - a(\nabla u_0^T)\Delta$, $u_0^T := e^{t\Delta}u_0$, where $e^{t\Delta}$ denotes the semigroup generated by $\Delta$ on $\mathbb{T}$. Let us now define the map $\Phi$ from $\mathcal{X}$ into itself by

$$\Phi : (u_0, u', u^\sharp) \in \mathcal{X} \mapsto (u_0, v', v^\sharp) \in \mathcal{X}, \quad (3.1)$$
where

\[ u' = \frac{g(\nabla u) - (a(\nabla u) - a(\nabla u_0^T))u'}{a(\nabla u_0^T)}, \quad (3.2) \]

\[ L^0v^\varepsilon = \Pi_{a(\nabla u_0^T)}v^\varepsilon - L^0(\hat{\Pi}v'X) + g'(\nabla u)\Pi(\nabla u^\varepsilon, \xi) - a'(\nabla u)\Pi(\nabla u^\varepsilon, \hat{\Pi}u^\varepsilon) \]

\[ + (a(\nabla u) - a(\nabla u_0^T))Du^\varepsilon + \varepsilon_1(u, u') + \varepsilon_2(u, u') + \varepsilon(u, u'), \quad (3.3) \]

where \( \varepsilon_1 \) denotes the difference of the third and the fourth terms, \( \varepsilon_2 \) denotes the fifth term in the right hand of (3.3), and \( \varepsilon \) denotes the remainder, see Lemma 3.4 and (2.33) of [7] for details.

**Remark 3.1** The map \( \Phi \) defined by (3.1) is a little different from the original one introduced in [7], where the map \( \Phi : (u, u') \rightarrow (v, v') \) defined on \( C_{\alpha, \beta, \gamma}(X) := \{(u, u') : u = \hat{\Pi}u^\varepsilon X + u^\varepsilon, \|\|u, u')\|_{\alpha, \beta, \gamma} < \infty \} \) (more precisely on the space \( B_T(\lambda) \), see Sect. 3.1 of [7] or equivalently (3.4) below) is introduced, see (2.16)-(2.18) in [7]. As we pointed out at the end of Sect. 3 of [7], the space \( C_{\alpha, \beta, \gamma}(X) \) is actually identical to that of pairs \( (u', u^\varepsilon) \) with the norm \( \|\|u, u')\|_{\alpha, \beta, \gamma} \) defined in terms of \( (u', u^\varepsilon) \). In this way, \( C_{\alpha, \beta, \gamma}(X) \) is identified with \( \bar{C}_{\alpha, \beta, \gamma} \). Therefore, Theorem 3.1 of [7] holds for \( \Phi \) defined by (3.1) whenever the initial value is fixed and we will not repeat this fact in the sequel.

For \( \lambda > 0 \) and \( T > 0 \), set

\[ B_T(\lambda) := \{(u_0, u) \in X; \ u_0 \in C^\alpha, \|u\|_{\alpha, \beta, \gamma} \leq \lambda \}. \quad (3.4) \]

We recall that for each fixed initial value \( u_0 \), in [7], it is proved that the map \( \Phi \) is contractive on \( B_T(\lambda) \) for a large enough \( \lambda \) and a small time \( T > 0 \), and the unique fixed point on \( B_T(\lambda) \) solves the paracontrolled SPDE (1.1) up to time \( T > 0 \), see Theorem 3.1 of [7] or Theorem 2.4 for details. In particular, one explicit choice of \( \lambda \) and \( T \) is given in Theorem 2.4. Let us now give the main result of this section.

**Theorem 3.1** Let \( \alpha \in \left( \frac{13}{15}, \frac{3}{2} \right) \). Then, we have that the map \( \Phi \) defined by (3.1) depends continuously on the initial value \( u_0 \) and its contractivity on \( B_T(\lambda) \), \( T > 0 \) is locally uniform in initial values. More precisely, there exists a unique continuous map \( u_0 \in C^\alpha \rightarrow (u'(u_0), u^\varepsilon(u_0)) \in \bar{C}_{\alpha, \beta, \gamma} \) up to time \( T \) such that

\[ \Phi(u_0, u'(u_0), u^\varepsilon(u_0)) = (u_0, u'(u_0), u^\varepsilon(u_0)) \]

and \( (u'(u_0), u^\varepsilon(u_0)) \) (or equivalently \( (u(u_0), u'(u_0)) \)) solves the SPDE (1.1) starting from \( u_0 \) up to time \( T \) in the paracontrolled sense. Moreover, \( T \) can be taken depending continuously on \( \|u_0\|_{\alpha, \gamma} \) and \( \|v\|_{\alpha, 2} \times C_{2\alpha - 3} \).

**Remark 3.2** (i) Combining Theorem 3.1-(ii) of [7] with Theorem 3.1 above, we see that the map \( \Phi \) depends continuously on both the initial value \( u_0 \in C^\alpha \) and the enhanced
noise $\hat{\xi}$. In particular, the unique fixed point of $\Phi$ in $B_T(\lambda)$ inherits the continuity in $(\hat{\xi}, u_0)$.

(ii) Theorem 1.3 immediately follows from Theorem 3.1 and the relation between $\Phi$ and the solution $u(t)$, see the proof of Theorem 1.1 in [7] for detailed explanation.

(iii) To obtain the estimate (3.13) in the proof of Lemma 3.8 below, which is important for the proof of Theorem 3.1, the original assumption $\alpha \in \left(\frac{4}{3}, \frac{3}{2}\right)$ in [7] is changed to $\alpha \in \left(\frac{13}{9}, \frac{3}{2}\right)$ due to our techniques. Although the assumption on $\alpha$ becomes slightly restrictive, the most important case of the spatial white noise on $\mathbb{T}$ is covered.

We can obtain immediately the next result by Theorem 3.1.

**Corollary 3.2** The SPDE (1.1) is solvable up to time $T > 0$ and one can take $T = T(\|u_0\|_{C^\alpha}, \|\hat{\xi}\|_{C^{\alpha-2} \times C^{2\alpha-3}})$, which depends continuously on $\|u_0\|_{C^\alpha}$ and $\|\hat{\xi}\|_{C^{\alpha-2} \times C^{2\alpha-3}}$.

To prove Theorem 3.1, thanks to Theorem 3.1 in [7] and the implicit function theorem, the main task is to show the continuity of the map $\Phi$. We will first give some lemmas as preparation and postpone the proof of Theorem 3.1 to the end of this section. In the sequel, we frequently use some fundamental properties of paracontrolled calculus. For the reader’s convenience, we summarize some of them in Appendix and recall some of them directly in proofs based on [7]. If necessary, see Appendix before reading the proofs of the results in this section.

Since the main purpose of this section is to prove the continuity of $u(t)$ in its initial values, we have to consider different initial values. So, we first generalize (2.15) in Lemma 2.9 of [7], i.e., (2.20), Lemma A.5 in Appendix, to the next lemma.

**Lemma 3.3** Let $F \in C^2_b(\mathbb{R})$, $\alpha \in (1, 2)$ and $\beta \in (0, \alpha - 1]$. Then, for any $u, v \in L^\alpha_T$ (without the restriction $u(0) = v(0)$), we have

$$
\|F(\nabla u) - F(\nabla v)\|_{L^\beta_T} \lesssim T^{\frac{\alpha - \beta - 1}{2}} \|F\|_{C^2(1 + \|\nabla u\|_{L^\alpha_T})} \|u - v\|_{L^\alpha_T} + \|F\|_{C^2(1 + \|u(0)\|_{C^\alpha})} \|u(0) - v(0)\|_{C^\alpha}.
$$

**Proof** This lemma can be easily shown by modifying the proof of Lemma 9 of [3]. Although we do not assume $u(0) = v(0)$, refining the proof of Lemma 9 of [3], we deduce that

$$
\|F(\nabla u) - F(\nabla v)\|_{L^\beta_T} \lesssim T^{\frac{\alpha - \beta - 1}{2}} \|F\|_{C^2(1 + \|\nabla u\|_{L^{\alpha-1}_T})} \|\nabla u - \nabla v\|_{L^{\alpha-1}_T} + \|F(\nabla u(0)) - F(\nabla v(0))\|_{C^\beta},
$$

which gives the desired result by the fact that $\|\nabla u\|_{L^{\alpha-1}_T} \lesssim \|u\|_{L^\alpha_T}$, see Lemma 2.3 of [7], and Lemma A.5.

As an application of Lemma 3.3, we have the following lemma, which will be used frequently.
Lemma 3.4 For any \((u_0^i, u_1^i, u_2^i) \in B_T(\lambda), i = 1, 2,\) we have

\[
\|a(\nabla u_1) - a(\nabla u_0^1, T) - (a(\nabla u_2) - a(\nabla u_0^2, T))\|_{L_T^\beta} \\
\lesssim T^{\frac{\alpha - \beta - 1}{2}} K(\|u_1\|_{\alpha, \beta, \gamma}) (1 + \|\xi\|_{C^{\alpha - 2}}) + K_0(\|u_0\|_{C^{\alpha}})\|u_1 - u_2\|_{C^{\alpha}}.
\]

(3.5)

Proof By Lemma A.5, the increasing property of \(\|\cdot\|_{C^\alpha}\) in \(\alpha \in \mathbb{R},\) and the Schauder estimate \(\|e^L u\|_{C^\alpha} \lesssim \|u\|_{C^\alpha},\) we have that for any \(\beta \in (0, \alpha - 1),\)

\[
\|a(\nabla u_0^1, T) - a(\nabla u_0^2, T)\|_{C^\beta} \lesssim (1 + \|u_0^1, T\|_{C^{\beta + 1}})\|u_0^1, T - u_0^2, T\|_{C^{\beta + 1}}
\]

\[
\lesssim (1 + \|u_0^1\|_{C^{\alpha}})\|u_1 - u_2\|_{C^{\alpha}}.
\]

(3.6)

On the other hand, Lemma 3.3 gives that

\[
\|a(\nabla u_1) - a(\nabla u_2)\|_{L_T^\beta} \lesssim T^{\frac{\alpha - \beta - 1}{2}} (1 + \|u_1\|_{L_T^\alpha})\|u_1 - u_2\|_{L_T^\alpha} + (1 + \|u_0^1\|_{C^{\alpha}})\|u_0^1 - u_0^2\|_{C^{\alpha}}.
\]

Therefore, by the estimate \(\|u\|_{L_T^\alpha} \lesssim (1 + \|X\|_{C^{\alpha}})\|u\|_{\alpha, \beta, \gamma},\) see (3.9) in [7], together with \(\|X\|_{C^{\alpha}} \lesssim \|\xi\|_{C^{\alpha - 2}},\) we immediately obtain (3.5) with \(K(\|u_1\|_{\alpha, \beta, \gamma}) = (1 + \|u_1\|_{\alpha, \beta, \gamma})\) and \(K_0(\|u_0\|_{C^{\alpha}}) = (1 + \|u_0\|_{C^{\alpha}}).\)

According to Theorem 3.1 of [7] and its proof, we have the following result.

Lemma 3.5 Let \(\Phi\) be the map on \(B_T(\lambda)\) defined by (3.1), that is, \(\Phi(u_0, u', u^\alpha) = (u_0, v', v^\alpha)\) for \((u_0, u', u^\alpha) \in B_T(\lambda).\) Then, we have

\[
\|(v', v^\alpha)\|_{\alpha, \beta, \gamma} \lesssim T^{\frac{\alpha + \beta - \gamma}{2}} K(\|u\|_{\alpha, \beta, \gamma})\tilde{K}_1(X, \xi) + K_0(\|u_0\|_{C^{\alpha}})(1 + \|\xi\|_{C^{\alpha - 2}}),
\]

where \(v'\) are \(v^\alpha\) are determined by (3.2) and (3.3) respectively, and \(\tilde{K}_1(X, \xi)\) denotes the same constant as that introduced in Proposition 3.8 of [7].

By Lemma 3.5, in particular, we know that \(v'\) is well-defined as an element of \(L_T^\beta\) for each \((u_0, u', u^\alpha) \in B_T(\lambda).\) In the following, we will show the local Lipschitz continuity of \(v'\) in \((u_0, u', u^\alpha).\) For \((u_0^i, u_1^i, u_2^i) \in B_T(\lambda), i = 1, 2,\) we set \(\Phi(u_0^i, u_1^i, u_2^i) = (u_0^i, v_1^i, v_2^i)\) in the following.

Lemma 3.6 We have that \(v'\) is locally Lipschitz in \((u_0, u', u^\alpha) \in B_T(\lambda).\) More precisely, we have

\[
\|v_1^i - v_2^i\|_{L_T^\beta} \lesssim T^{\frac{\alpha - \beta - 1}{2}} K(\|u_1\|_{\alpha, \beta, \gamma}, \|u_2\|_{\alpha, \beta, \gamma})(1 + \|\xi\|_{C^{\alpha - 2}})^2\|u_1 - u_2\|_{\alpha, \beta, \gamma} + K(\|u_1\|_{\alpha, \beta, \gamma})K_0(\|u_0\|_{C^{\alpha}}, \|u_0^2\|_{C^{\alpha}})\|u_0^1 - u_0^2\|_{C^{\alpha}},
\]

where \(u_i = (u_0^i, u_1^i, u_2^i)\) for \(i = 1, 2.\)
Proof For simplicity of notation, we set

\[ b(u_i) = a(\nabla u_i) - a(\nabla u_i^{i,T}), \quad i = 1, 2. \] (3.7)

Then, it is easy to know that \( \|v_1' - v_2'\|_{L_T^\beta} \) is bounded from above by the sum of the following three terms:

\[
I_1 = \left\| \frac{1}{a(\nabla u_0^{1,T})} \left( g(\nabla u_1) - b(u_1)u_1' \right) - \frac{1}{a(\nabla u_0^{2,T})} \left( g(\nabla u_2) - b(u_2)u_1' \right) \right\|_{L_T^\beta},
\]

\[
I_2 = \left\| \frac{1}{a(\nabla u_0^{2,T})} \left( g(\nabla u_1) - g(\nabla u_2) - (b(u_1) - b(u_2))u_1' \right) \right\|_{L_T^\beta},
\]

\[
I_3 = \left\| \frac{b(u_2)(u_1' - u_2')}{a(\nabla u_0^{2,T})} \right\|_{L_T^\beta}.
\]

Let us first deal with the first term \( I_1 \). From the proof of Lemma 3.5 of [7], it easily follows that

\[
\|g(\nabla u_1) - b(u_1)u_1'\|_{L_T^\beta} \lesssim T^{\frac{\alpha - \beta - 1}{2}} K(\|u_1\|_{C^{\alpha,\gamma}})(1 + \|X\|_{C^{\alpha}}) + K_0(\|u_0\|_{C^{\alpha}}).
\]

Recalling that \( a \in C^3_b(\mathbb{R}) \) satisfies (1.2) and using (3.6) together with Lemma A.5, we have that

\[
\left\| \frac{1}{a(\nabla u_0^{1,T})} - \frac{1}{a(\nabla u_0^{2,T})} \right\|_{C^\beta} \lesssim \left\| \frac{1}{a(\nabla u_0^{1,T})} - \frac{1}{a(\nabla u_0^{2,T})} \right\|_{C^\beta} + \|a(\nabla u_0^{1,T}) - a(\nabla u_0^{2,T})\|_{C^\beta}
\]

\[
\lesssim (1 + \|u_0\|_{C^{\alpha}})^2 (1 + \|u_0^2\|_{C^{\alpha}}) \|a(\nabla u_0^{1,T}) - a(\nabla u_0^{2,T})\|_{C^\beta}
\]

\[
\lesssim K_0(\|u_0\|_{C^{\alpha}}, \|u_0^2\|_{C^{\alpha}}) \|u_1' - u_2'\|_{C^{\alpha}},
\]

where we have used the estimate

\[
\left\| \frac{1}{a(\nabla u_0^{i,T})} \right\|_{C^\beta} \lesssim (1 + \|u_0^i\|_{C^{\alpha}}), \quad i = 1, 2, \quad (3.8)
\]

see (3.23) of [7] for its proof.

Therefore, by Lemma A.1 and the above estimates, we obtain

\[
I_1 \lesssim \|g(\nabla u_1) - b(u_1)u_1'\|_{L_T^\beta} \left\| \frac{1}{a(\nabla u_0^{1,T})} - \frac{1}{a(\nabla u_0^{2,T})} \right\|_{C^\beta}
\]

\[
\lesssim \left( T^{\frac{\alpha - \beta - 1}{2}} K(\|u_1\|_{C^{\alpha,\gamma}})(1 + \|\xi\|_{C^{\alpha-2}}) + 1 \right) K_0(\|u_0\|_{C^{\alpha}}, \|u_0^2\|_{C^{\alpha}}) \|u_1' - u_2'\|_{C^{\alpha}}.
\]
Next, let us evaluate the second term $I_2$. By Lemma 3.4, we have

$$
\| (b(u_1) - b(u_2))u'_1 \|_{L^\infty_T}
\lesssim T^{\frac{a-\beta-1}{2}} K(\|u_1\|_{\alpha, \beta, \gamma}) (1 + \|\xi\|_{C^{\alpha-2}}^2) \|u_1 - u_2\|_{\alpha, \beta, \gamma}
+ K_0(\|u_0^1\|_{C^\alpha}) \|u_1\|_{\alpha, \beta, \gamma} \|u_0^1 - u_0^2\|_{C^\alpha}.
$$

In addition, we have the following estimate more easily.

$$
\| g(\nabla u_1) - g(\nabla u_2) \|_{L^\infty_T} \lesssim T^{\frac{a-\beta-1}{2}} K(\|u_1\|_{\alpha, \beta, \gamma}) (1 + \|\xi\|_{C^{\alpha-2}}^2) \|u_1 - u_2\|_{\alpha, \beta, \gamma}
+ K_0(\|u_0^1\|_{C^\alpha}) \|u_0^1 - u_0^2\|_{C^\alpha}.
$$

Therefore, noting (3.8), we have

$$
I_2 \lesssim T^{\frac{a-\beta-1}{2}} K(\|u_1\|_{\alpha, \beta, \gamma}, \|u_0^2\|_{C^\alpha}) (1 + \|\xi\|_{C^{\alpha-2}}^2) \|u_1 - u_2\|_{\alpha, \beta, \gamma}
+ K_0(\|u_0^1\|_{C^\alpha}, \|u_0^2\|_{C^\alpha}) (1 + \|u_1\|_{\alpha, \beta, \gamma}) \|u_0^1 - u_0^2\|_{C^\alpha}.
$$

Finally, using (3.17) of [7] together with (3.8), we easily have

$$
I_3 \lesssim T^{\frac{a-\beta-1}{2}} K(\|u_2\|_{\alpha, \beta, \gamma}) (1 + \|\xi\|_{C^{\alpha-2}}^2) \|u_1 - u_2\|_{\alpha, \beta, \gamma}.
$$

Consequently, the proof of this lemma is completed by the above estimates. \hfill \Box

In the next two lemmas, we give the bounds for the terms involving both $u_0^T$ or (and) $v^\sharp$, which should be evaluated in the weighted space in time.

**Lemma 3.7** We have

$$
\sup_{0 < t \leq T} t^{\frac{\gamma-\alpha}{2}} \left\| \left( a(\nabla u_0^1, T) - a(\nabla u_0^2, T) \right) \Delta v_1^\sharp (t) \right\|_{C^{\gamma-2}}
\lesssim K_0(\|u_0^1\|_{C^\alpha}) (v'_1, v_1^\sharp)_{\alpha, \beta, \gamma} \|u_0^1 - u_0^2\|_{C^\alpha}.
$$

**Proof** By Lemma 3.5, we know $(v'_1, v_1^\sharp) \in \tilde{C}_{\alpha, \beta, \gamma}$. In particular, we have that for $t > 0$, $v_1^\sharp \in C^{\gamma}$. Then, noting that $0 < \beta + \gamma - 2 < \beta$ and using Lemma A.2, we have

$$
\left\| \left( a(\nabla u_0^1, T) - a(\nabla u_0^2, T) \right) \Delta v_1^\sharp (t) \right\|_{C^{\gamma-2}}
\lesssim \left\| \left( a(\nabla u_0^1, T) - a(\nabla u_0^2, T) \right) \Delta v_1^\sharp (t) \right\|_{C^{\beta+\gamma-2}}
\lesssim \left\| a(\nabla u_0^1, T) - a(\nabla u_0^2, T) \right\|_{C^\beta} \|v_1^\sharp (t)\|_{C^{\gamma}}.
$$

Now, we conclude our proof by (3.6) together with the fact $\sup_{0 < t \leq T} t^{\frac{\gamma-\alpha}{2}} \|v_1^\sharp (t)\|_{C^{\gamma}} \leq \|(v'_1, v_1^\sharp)\|_{\alpha, \beta, \gamma}$. \hfill \Box
Lemma 3.8 Suppose further $\alpha \in \left( \frac{13}{12}, \frac{3}{2} \right)$. Then we have

$$\sup_{0 < t \leq T} t^{\gamma - \alpha} \left\| \left( a(\nabla u_0^{1, T}) - a(\nabla u_0^{2, T}) \right) \Delta \tilde{\Pi}_{v_1}' X + \Pi (a(\nabla u_0^{1, T}) - a(\nabla u_0^{2, T})) v_1' \xi \right\|_{C^{\alpha + \beta - 2}} \lesssim K_0 \left( \| u_0^1 \|_{C^\alpha} \right) \|(v_1', \tilde{v}_1^i)\|_{\alpha, \beta, \gamma} \| \xi \|_{C^{\alpha - 2}} \| u_0^1 - u_0^2 \|_{C^\alpha}. \quad (3.9)$$

**Proof** Set $\tilde{a}_0 := a(\nabla u_0^{1, T}) - a(\nabla u_0^{2, T})$ for the sake of brevity. Using the the commutator $R_2(v_1', X) = [\Delta, \tilde{\Pi}_{v_1}'] X$ and $\Delta X = -Q \xi = - (\xi - \xi(T))$, we have

$$\| R_2(v_1', X) \|_{C^{\alpha + \beta - 2}} \lesssim \| v_1' \|_{C X} \| \xi \|_{C^{\alpha - 2}} \quad (3.10)$$

by Lemma A.3 and

$$\tilde{a}_0 \Delta \tilde{\Pi}_{v_1}' X = \tilde{a}_0 \left( \tilde{\Pi}_{v_1}' (\Delta X) + R_2(v_1', X) \right) = - \tilde{a}_0 \tilde{\Pi}_{v_1}' \xi + \tilde{a}_0 R_2(v_1', X).$$

Then, an analogous argument for (2.27) of [7] shows that

$$\tilde{a}_0 \Delta \tilde{\Pi}_{v_1}' X = - \Pi (\tilde{a}_0, \tilde{\Pi}_{v_1}' \xi) - R(\tilde{a}_0, v_1'; \xi) - \Pi \tilde{\Pi}_{v_1}' \xi \tilde{a}_0 + \tilde{a}_0 R_2(v_1', X),$$

where $R(\tilde{a}_0, v_1'; \xi) = \Pi (\tilde{a}_0, \tilde{\Pi}_{v_1}' \xi) - \Pi \tilde{a}_0 \tilde{\Pi}_{v_1}' \xi$, see Lemma A.4. Therefore, the term inside of the norm \( \| \cdot \|_{C^{\alpha + \beta - 2}} \) of the left hand side of (3.9) equals to

$$- \Pi (\tilde{a}_0, \tilde{\Pi}_{v_1}' \xi) - R(\tilde{a}_0, v_1'; \xi) - \Pi \tilde{\Pi}_{v_1}' \xi \tilde{a}_0 + \tilde{a}_0 R_2(v_1', X). \quad (3.11)$$

Let us first deal with the first term of (3.11), i.e., the resonant term $\Pi (\tilde{a}_0, \tilde{\Pi}_{v_1}' \xi)$. By Lemma A.5 and the Schauder estimate $\| e^{t \Delta} u \|_{C^{\alpha + \kappa}} \lesssim t^{-\frac{\alpha}{2}} \| u \|_{C^\alpha}$ for any $\alpha \in \mathbb{R}, \kappa \geq 0$ and $t \in (0, T]$, we have for $\alpha < \gamma' \leq \gamma$

$$\| \tilde{a}_0 \|_{C^{\gamma' - 1}} \lesssim (1 + \| u_0^{1, T} \|_{C^{\gamma'}}) \| u_0^{1, T} - u_0^{2, T} \|_{C^{\gamma'}} \lesssim T^{-\gamma' - \alpha} (1 + T^{-\frac{\gamma' - \alpha}{2}} \| u_0^1 \|_{C^\alpha}) \| u_0^1 - u_0^2 \|_{C^\alpha} \lesssim T^{-(\gamma' - \alpha)} (T^{-\frac{\gamma' - \alpha}{2}} + \| u_0^1 \|_{C^\alpha}) \| u_0^1 - u_0^2 \|_{C^\alpha}. \quad (3.12)$$

Let $\gamma' = \frac{\gamma + \alpha}{2}$. Then, we have $\alpha + \gamma' - 3 > 0$ and $\gamma' > \alpha$ by noting that $\alpha \in \left( \frac{13}{12}, \frac{3}{2} \right)$. Using Lemmas A.1 and A.2, and (3.12) with $\gamma' = \frac{\gamma + \alpha}{2}$, we have

$$\sup_{0 < t \leq T} t^{\gamma - \alpha} \| \Pi (\tilde{a}_0, \tilde{\Pi}_{v_1}' \xi) \|_{C^{\alpha + \gamma' - 3}} \lesssim \sup_{0 < t \leq T} t^{\gamma - \alpha} \| \Pi (\tilde{a}_0, \tilde{\Pi}_{v_1}' \xi) \|_{C^{\alpha + \gamma' - 3}} \lesssim \sup_{0 < t \leq T} t^{\gamma - \alpha} \| \tilde{a}_0 \|_{C^{\gamma' - 1}} \| \tilde{\Pi}_{v_1}' \xi \|_{C^{\alpha - 2}} \| u_0^1 - u_0^2 \|_{C^\alpha} \lesssim (1 + \| u_0^1 \|_{C^\alpha}) \| v_1' \|_{C^{\gamma'}} \| \xi \|_{C^{\alpha - 2}} \| u_0^1 - u_0^2 \|_{C^\alpha}, \quad (3.13)$$
where we have used $\alpha + \gamma' - 3 > 0$ for the first inequality and the relation $\gamma' - \alpha = \frac{\gamma - \alpha}{2} > 0$ for the last inequality.

Next, we deal with the last three terms of (3.11). By (3.6) and the similar arguments to Lemma 4.3 of [7], we easily have

$$
\| - R(\tilde{a}_0, v', \xi) - \Pi_{\tilde{a}_0} \tilde{a}_0 + \tilde{a}_0 R_2(v', X)\|_{C_T^{\gamma + \beta - 2}} \lesssim (1 + \|u_0\|_{C^\alpha})\|v'\|_{L_T^{\beta}}\|\xi\|_{C^{\alpha - 2}}\|u_0' - u_0^2\|_{C^\alpha}.
$$

(3.14)

We omit the details and just give the estimate on the last term $\tilde{a}_0 R_2(v', X)$ as an example. Noting the relation $0 < 2\alpha + \beta - 3 < \alpha - 1$ and using (3.10), we have

$$
\|\tilde{a}_0 R_2(v', X)\|_{C_T^{\gamma + \beta - 2}} \lesssim \|\tilde{a}_0 R_2(v', X)\|_{C_T^{2\alpha + \beta - 3}} \lesssim \|\tilde{a}_0\|_{C^{\alpha - 1}}\|R_2(v', X)\|_{C_T^{\gamma + \beta - 2}} \lesssim (1 + \|u_0\|_{C^\alpha})\|v'\|_{L_T^{\beta}}\|\xi\|_{C^{\alpha - 2}}\|u_0' - u_0^2\|_{C^\alpha}.
$$

where (3.6) has been used for the last inequality. As a consequence, we conclude the proof of this lemma by (3.13) and (3.14) together with $\gamma > \alpha$. □

Next, we reevaluate the term $\varepsilon_2$ involving $u^x$ according to our purpose.

**Lemma 3.9**  We have the local Lipschitz estimate for $\varepsilon_2$:

$$
\sup_{0 < t \leq T} t^{\frac{\gamma - \alpha}{2}} \|\varepsilon_2(u_1) - \varepsilon_2(u_2)\|_{C_T^{\gamma - 2}} \lesssim T^{\frac{\alpha - \beta - 1}{2}} K(\|u_1\|_{\alpha, \beta, \gamma}, \|u_2\|_{\alpha, \beta, \gamma})(1 + \|\xi\|_{C^{\alpha - 2}})^2\|u_1 - u_2\|_{\alpha, \beta, \gamma}
$$

$$
+ K_0(\|u_0\|_{C^\alpha})\|u_1\|_{\alpha, \beta, \gamma}\|u_0' - u_0^2\|_{C^\alpha}.
$$

(3.15)

**Proof**  This lemma can be shown essentially by analogous arguments to Lemma 3.4 of [7]. However, in that proof, the initial value is fixed and the explicit relation between the constants and its initial value is not written down. Hence, for the reader’s convenience, we give proof of (3.15) briefly. In the following, we use the notation $b(u_i)$ introduced in the proof of Lemma 3.6, see (3.7). By the definition of $\varepsilon_2$, we have

$$
\|(\varepsilon_2(u_1) - \varepsilon_2(u_2))(t)\|_{C_T^{\gamma - 2}} \lesssim \|(\varepsilon_2(u_1) - \varepsilon_2(u_2))(t)\|_{C_T^{\beta + \gamma - 2}} \lesssim \|b(u_1) - b(u_2)\|_{L_T^{\beta}}\|u_1^x(t)\|_{C_T^{\gamma}} + \|b(u_2)\|_{L_T^{\beta}}\|(u_1^x - u_2^x)(t)\|_{C_T^{\gamma}}.
$$

Therefore, the estimate (3.15) is immediately obtained by Lemma 3.4 above and (3.17) of [7]. □

As the final preparation, we give the estimate on the remainder $\zeta$ in (3.3).
Lemma 3.10 We have that \( \zeta \) is locally Lipschitz continuous in \((u_0, u', u^\sharp) \in B_T(\lambda)\). In fact, we have the following estimate

\[
\sup_{0 < t \leq T} t^{\gamma - a} \left\| (\zeta(u_1) - \zeta(u_2))(t) \right\|_{C^{\alpha + \beta - 2}} 
\leq K \left( \|u_1\|_{C^{\alpha,\beta,\gamma}} + \|u_2\|_{C^{\alpha,\beta,\gamma}} \right) K_2(X, \xi)\|u_1 - u_2\|_{C^{\alpha,\beta,\gamma}} + K_0(\|u_0\|_{C^\alpha}) \|u_1\|_{C^{\alpha,\beta,\gamma}} \|\xi\|_{C^{\alpha - 2}} \|u_0 - u_0^\sharp\|_{C^\alpha},
\]

where \( \tilde{K}_2(X, \xi) \) is the constant introduced in Proposition 3.8 of [7], which depends on \( \|\xi\|_{C^{\alpha - 2}} \) and \( \|\Pi(\nabla X, \xi)\|_{C^{2\alpha - 3}} \).

**Proof** This can be shown by similar arguments to Proposition 3.8 of [7]. Although many terms should be dealt with, from the proof of Proposition 3.8 of [7], we only have to reevaluate the terms involving \( u_0^T \), that is, \( \Pi(a(\nabla u_0^T), \tilde{u}_\lambda \xi) \) and the term \( A_2 \in C_T C^{\alpha + \beta - 2} \) defined by (2.28) of [7], that is,

\[
A_2 = -R(a(\nabla u) - a(\nabla u_0^T), u'; \xi) - \Pi_{\tilde{u}_\lambda \xi}(a(\nabla u) - a(\nabla u_0^T))
+ (a(\nabla u) - a(\nabla u_0^T))R_2,
\]

with \( R_2 := [\Delta, \tilde{u}_\lambda \xi] \in C_T C^{\alpha + \beta - 2} \), see Lemma A.4 for the definition of \( R \). Since we use the norm of \( \|u_1 - u_2\|_{C^{\alpha,\beta,\gamma}} \), the initial value will only affect our estimate when the estimates of \( \|b(u_1)\|_{C_T^{\alpha}} \) and \( \|b(u_1) - b(u_2)\|_{C_T^{\alpha}} \) are used in the proof of Proposition 3.8 of [7], where \( b(u_i) \) is defined by (3.7). However, the terms \( \Pi(a(\nabla u_0^T), \tilde{u}_\lambda \xi) \) and \( A_2 \) can be handled by the analogous arguments to Lemma 3.8 above, and Lemma 4.3 of [7]. Noting \( u' \in L_T^\beta \), a similar argument to (3.13) yields that

\[
\sup_{0 < t \leq T} t^{\gamma - a} \left\| \left( \Pi(a(\nabla u_0^1)^T, \tilde{u}_1 \xi) - \Pi(a(\nabla u_0^2)^T, \tilde{u}_2 \xi) \right)(t) \right\|_{C^{\alpha + \beta - 2}} 
\leq (1 + \|u_0^1\|_{C^\alpha})\|u_1\|_{L_T^\beta} \|\xi\|_{C^{\alpha - 2}} \|u_0^1 - u_0^2\|_{C^\alpha}.
\]

Then, recalling the estimate deduced in the proof of Lemma 4.1 of [7] and the bilinearity of the resonant term, we easily have

\[
\sup_{0 < t \leq T} t^{\gamma - a} \left\| \left( \Pi(a(\nabla u_0^1)^T, \tilde{u}_1 \xi) - \Pi(a(\nabla u_0^2)^T, \tilde{u}_2 \xi) \right)(t) \right\|_{C^{\alpha + \beta - 2}} 
\leq K_0(\|u_0^1\|_{C^\alpha}) \|u_1\|_{C^{\alpha,\beta,\gamma}} \|\xi\|_{C^{\alpha - 2}} \|u_0^1 - u_0^2\|_{C^\alpha}.
\]

On the other hand, \( A_2 \) can be essentially evaluated by the arguments for Lemma 4.3 of [7] thanks to Lemma 3.4. Here, we give the estimate on the first term of \( A_2 \) as an example. By bilinearity of \( R \), we have

\[
\|R(b(u_1), u_1'; \xi) - R(b(u_2), u_2'; \xi)\|_{C_T^{\alpha + \beta - 2}}
\leq \|R(b(u_1) - b(u_2), u_1'; \xi)\|_{C_T^{\alpha + \beta - 2}} + \|R(b(u_2), u_1' - u_2'; \xi)\|_{C_T^{\alpha + \beta - 2}}
\]
\[
\lesssim K(\|u_1\|_{\alpha,\beta,\gamma}, \|u_2\|_{\alpha,\beta,\gamma})(1 + \|\xi\|_{C^\alpha-2})^2\|\xi\|_{C^\alpha-2}\|u_1 - u_2\|_{\alpha,\beta,\gamma}
+ K(\|u_1\|_{C^\alpha})\|u_1\|_{\alpha,\beta,\gamma}\|\xi\|_{C^\alpha-2}\|u_0^1 - u_0^2\|_{C^\alpha},
\]
where Lemmas 3.4, A.4 and \(\alpha - \beta - 1 > 0\) have been used for the last inequality. Consequently, we complete the proof. \(\square\)

In the end, let us give the proof of Theorem 3.1 based on the above lemmas.

**Proof of Theorem 3.1** According to the proof of Theorem 3.1 of [7], we also know that the choice of \(T\) depends locally uniformly on \(\|\tilde{u}_0 - u_0\|_{C^\alpha}\) according to the estimates (3.48) and (3.50) of [7]. In fact, noting that constants \(K(\|u\|_{\alpha,\beta,\gamma})\) in (3.48) of [7] and \(K(\|u_1\|_{\alpha,\beta,\gamma}, \|u_1\|_{\alpha,\beta,\gamma})\) in (3.50) of [7] can be controlled by some polynomial, we can choose the time \(T\) as the function \(T = T(r)\) for all \(\|\tilde{u}_0 - u_0\|_{C^\alpha} < r\) with a small enough \(r > 0\). In particular, we know that \(\Phi\) is contractive on \(B_T(\lambda)\) for a large enough \(\lambda\) and a small enough \(T > 0\), that is, there exists \(\kappa \in (0, 1)\) such that

\[
\|\Phi(u_0, u_1, u_2) - \Phi(u_0, u_1', u_2')\|_{\alpha,\beta,\gamma} \leq \kappa\|u_1 - u_2\|_{\alpha,\beta,\gamma},
\]

Note that \(u_0\) in \(\Phi\) is the same in this estimate. Therefore, thanks to the implicit function theorem, it is enough for us to show the continuity of the map \(\Phi\) on \(B_T(\lambda)\). In fact, we can show that the map \(\Phi\) is locally Lipschitz continuous on \(B_T(\lambda)\). Let us take two elements \((u_0^i, u_i) \in B_T(\lambda), \ i = 1, 2\) and denote their images by \((u_0^i, v_1^i, v_2^i)\) under the map \(\Phi\). Then by (3.1)-(3.3), we have that the different \(v_1^i - v_2^i\) satisfies the equation

\[
\mathcal{L}_1^0 v_1^i - \mathcal{L}_2^0 v_2^i = \Pi_{\alpha(\nabla u_0^1, T)} v_1^i \xi - \Pi_{\alpha(\nabla u_0^2, T)} v_2^i \xi - (\mathcal{L}_1^0(\tilde{\Pi} v_1^i X) - \mathcal{L}_2^0(\tilde{\Pi} v_2^i X))
+ \varepsilon_1(u_1) - \varepsilon_1(u_2) + \varepsilon_2(u_1) - \varepsilon_2(u_2) + \xi(u_1) - \xi(u_2)
\]

with the initial value \((v_1^i - v_2^i)(0) = u_0^1 - u_0^2 - \Pi_{\alpha(\nabla u_0^1, T)} u_0^1; \ i = 1, 2\); recall that \(v_0^i = u_0^i, \ i = 0\) of \((u_0^i, v_1^i, v_2^i)\), \(i = 1, 2\), where \(\mathcal{L}_i^0 := \partial_t - a(\nabla u_0^i, T)\Delta\). We easily see that it can be rewritten to the next equation.

\[
\mathcal{L}_1^0(v_1^i - v_2^i) = \Pi_{\alpha(\nabla u_0^1, T)}(v_1^i - v_2^i) \xi - \mathcal{L}_2^0(\tilde{\Pi} v_1^i - v_2^i X) + (a(\nabla u_0^1, T) - a(\nabla u_0^2, T))\Delta v_1^i
+ (a(\nabla u_0^1, T) - a(\nabla u_0^2, T))\Delta v_1^i X + \Pi_{\alpha(\nabla u_0^1, T) - a(\nabla u_0^2, T)} v_1^i \xi
+ \varepsilon_1(u_1) - \varepsilon_1(u_2) + \varepsilon_2(u_1) - \varepsilon_2(u_2) + \xi(u_1) - \xi(u_2).
\]

By Lemma A.6 and \(v_1^i - v_2^i \in \mathcal{L}_1^\beta\), we see the first two terms of (3.16) can be estimated as follows:

\[
\sup_{0 < t \leq T} \int_0^\gamma \|\Pi_{\alpha(\nabla u_0^1, T)}(v_1^i - v_2^i) \xi - \mathcal{L}_2^0(\tilde{\Pi} v_1^i - v_2^i X))\|_{C^\alpha+\beta-2}
\lesssim (1 + \|u_0^1\|_{C^\alpha})\|v_1^i - v_2^i\|_{\mathcal{L}_1^\beta} X\|_{C^\alpha}.
\]
As we explained in Lemma 3.10, the estimate (3.7) in Lemma 3.4 of [7] still holds for \( \varepsilon_1(u_1) - \varepsilon_1(u_2) \) in the framework of this theorem. More precisely, we have

\[
\begin{align*}
\sup_{0 < t \leq T} \left| \frac{1}{t} \int_0^t \left( \varepsilon_1(u_1) - \varepsilon_1(u_2) \right)(s) \, ds \right| \leq K \left( \|u_1\|_{\alpha, \beta, \gamma}, \|u_2\|_{\alpha, \beta, \gamma}, (1 + \|\xi\|_{C^{a-2}})^2 \|\xi\|_{C^{a-2}} \varepsilon_1(u_1) - \varepsilon_1(u_2) \right), \quad (3.18)
\end{align*}
\]

Now let us denote by \( \phi_1(t) \) the sum of \( (a(\nabla u_1)^{1,T} - a(\nabla u_2)^{2,T}) \Delta v_1^2(t) \) and \( (\varepsilon_2(u_1) - \varepsilon_2(u_2))(t) \), and by \( \phi_2(t) \) all of the other terms in the right hand side of (3.16). Then by (3.17), (3.18) and Lemmas 3.5-3.10, we easily known that \( \phi_1 \in C((0, T), C^{a-2}) \) and \( \phi_2 \in C((0, T), C^{a+\beta-2}) \) satisfy the assumptions formulated in Lemma 3.6 of [7]. So we obtain

\[
\begin{align*}
\sup_{0 < t \leq T} \left| \frac{1}{t} \int_0^t \left( \varepsilon_1(u_1) - \varepsilon_1(u_2) \right)(s) \, ds \right| \leq K \left( \|u_1\|_{\alpha, \beta, \gamma}, \|u_2\|_{\alpha, \beta, \gamma}, (1 + \|\xi\|_{C^{a-2}})^2 \|\xi\|_{C^{a-2}} \varepsilon_1(u_1) - \varepsilon_1(u_2) \right), \quad (3.19)
\end{align*}
\]

On the other hand, using Lemmas A.1 and A.5, we have

\[
\begin{align*}
\|\Pi_{u_1(0) - u_2(0)}X\|_{C^a} \lesssim \left\| \frac{1}{a(\nabla u_1) \cdot (\nabla u_2)} \right\|_{C^\beta} \|g(\nabla u_1)^2 a(\nabla u_2)^2 - g(\nabla u_1^2) a(\nabla u_2^2)\|_{C^\beta} \|X\|_{C^a}
\end{align*}
\]

\[
\begin{align*}
\lesssim K_0(\|u_1\|_{C^a}, \|u_2\|_{C^a}) \|u_1 - u_2\|_{C^a} \|\xi\|_{C^{a-2}}.
\end{align*}
\]

Therefore, thanks to Lemmas 3.5 and 3.6, by (3.19) together with \( 0 < \alpha + \beta - \gamma < \alpha - \beta - 1 \), we have that

\[
\begin{align*}
\|\Phi(u_1, u_2, u_1', u_2') - \Phi(u_0, u_0', u_0')\|_{\alpha, \beta, \gamma}
\end{align*}
\]

\[
\begin{align*}
\lesssim K(\|u_1\|_{\alpha, \beta, \gamma}, \|u_2\|_{\alpha, \beta, \gamma}, (1 + \|\xi\|_{C^a}) \|u_1 - u_2\|_{C^a}
\end{align*}
\]

\[
\begin{align*}
+ T^{\alpha+\beta-\gamma} K(\|u_1\|_{\alpha, \beta, \gamma}, \|u_2\|_{\alpha, \beta, \gamma}, (1 + \|\xi\|_{C^a}) \|u_1 - u_2\|_{C^a}
\end{align*}
\]

which implies the local Lipschitz continuity of \( \Phi \) on \( B_T(r, \lambda) \). Consequently, the proof is completed. \( \square \)

**Remark 3.3** In the proof of Proposition 2.7, we used the continuity of the solutions in \( (m, u_0, \hat{\xi}) \) of the SPDE (1.13).

Although we took \( m = 0 \) and fixed it in Theorem 3.1, the continuity of the solutions in \( (m, u_0, \hat{\xi}) \) of the SPDE (1.13) can be shown by similar arguments. Roughly speaking, instead of the map \( \Phi \) defined by (3.1), it is natural to study the map \( \Phi : (m, u_0, u', v') \mapsto (u', v') \), where \( u', v' \) are determined by (3.2), (3.3) by replacing \( a(\cdot), g(\cdot) \) by \( a(\cdot + m), g(\cdot + m) \) respectively. Then, all of the estimates in Lemmas 3.4-3.10 still hold if we replace \( \|u_i\|_{\alpha, \beta, \gamma}, \|u_1 - u_2\|_{\alpha, \beta, \gamma}, \|u'_0\|_{C^a}, \|u'_0 - u'^{2}\|_{C^a} \) by

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\[ \|u_i\|_{\alpha,\beta,\gamma} + |m^i|, \|u_1 - u_2\|_{\alpha,\beta,\gamma} + |m^1 - m^2|, \|u_0^i\|_{C^\alpha} + |m^i|, \|u_0^i - u_0^2\|_{C^\alpha} + |m^1 - m^2| \]

respectively. For example, if \((m^i, u_0^i, u_i', u_i^1)\), \(i = 1, 2\) are given, then instead of (3.5) in Lemma 3.4, we have
\[
\left\| a(\nabla u_1 + m^1) - a(\nabla u_0^{1,T} + m^1) - \left( a(\nabla u_2 + m^2) - a(\nabla u_0^{2,T} + m^2) \right) \right\|_{L^2_T} \leq T^{\alpha - \beta - \gamma} K (\|u_1\|_{\alpha,\beta,\gamma}, |m_1|) (1 + \|\xi\|_{C^{\alpha - 2}})^2 (\|u_1 - u_2\|_{\alpha,\beta,\gamma} + |m^1 - m^2|)
\]
\[
+ K_0(\|u_0^1\|_{C^\alpha}, |m_1|) (\|u_0^1 - u_0^2\|_{C^\alpha} + |m^1 - m^2|),
\]

because we just replaced the functions \(a(\nabla u_i), a(\nabla u_0^{i,T})\) by \(a(\nabla u_i + m^i), a(\nabla u_0^{i,T} + m^i)\) and \(a(\cdot + m)\) satisfies (1.2) for all \(m \in \mathbb{R}\). Consequently, we can testify that \(\Phi\) satisfies the conditions of the implicit function theorem by the analogous arguments to the proof of Theorem 3.1 and therefore show the desired result.

One can interpret the continuity in \(m\) as that in the boundary condition, modified as in (1.6) of [7], for the original SPDE (1.1).

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Appendix

As we pointed out that the present article is a continuation of [7]. Especially, in Sect. 3, many results on paracontrolled calculus machinery have been frequently used. For the reader’s convenience and making this work self-contained, we summarize some of them in this part. For their proofs, see [7] and references therein.

We denote by \(\Pi u v = (u < v \text{ in } [9])\) the paraproduct, by \(\Pi(u, v) = (u \circ v)\) the resonant term and, by \(\Pi(u, v) = (u \leftrightarrow v)\) the modified paraproduct, respectively. Then, for two distributions \(u\) and \(v\), the general product \(uv\) can be (at least formally) written as \(uv = \Pi u v + \Pi (u, v) + \Pi u v\), which is called the Littlewood-Paley decomposition.

Lemma A.1 (Lemma 2.1, [7]) (i) \(\|\cdot\|_{C^\alpha} \lesssim \|\cdot\|_{C^\beta}\) holds for any \(\alpha \leq \beta\). Moreover, \(\|\cdot\|_{L^\infty} \lesssim \|\cdot\|_{C^\alpha}\) for \(\alpha > 0\) and conversely \(\|\cdot\|_{C^\alpha} \lesssim \|\cdot\|_{L^\infty}\) for \(\alpha \leq 0\).

(ii) The following Bony’s estimates hold:

- For \(\alpha > 0\) and \(\beta \in \mathbb{R}\), \(\|\Pi u v\|_{C^\beta} \lesssim \|u\|_{L^\infty} \|v\|_{C^\beta}\).
- For \(\alpha \neq 0\) and \(\beta \in \mathbb{R}\), \(\|\Pi u v\|_{C^{(\alpha,0)\beta}} \lesssim \|u\|_{C^\alpha} \|v\|_{C^\beta}\).
- For \(\alpha + \beta > 0\), \(\|\Pi (u, v)\|_{C^{\alpha + \beta}} \lesssim \|u\|_{C^\alpha} \|v\|_{C^\beta}\).

In particular, the product \(uv\) is well-defined if and only if \(\alpha + \beta > 0\), in this case \(uv \in C^{\alpha \land \beta}\) and whenever \(\alpha \beta \neq 0\), we have
\[
\|uv\|_{C^{\alpha \land \beta}} \lesssim \|u\|_{C^\alpha} \|v\|_{C^\beta}.
\]

The estimate of the modified paraproduct \(\Pi\) and its relation with paraproduct \(\Pi\) are given in the next lemma.
Lemma A.2 (Lemma 2.4, [7]) Let $\beta \in \mathbb{R}$. Then the following hold.

(i) If $\alpha \neq 0$, then for any $u \in C_T C^\alpha$ and $v \in C_T C^\beta$, we have
\[
\|\tilde{\Pi}_u v\|_{C_T C^{(\alpha \wedge 0) + \beta}} \lesssim \|u\|_{C_T C^\alpha} \|v\|_{C_T C^\beta}.
\]

(ii) If $\alpha \in (0, 2)$, then for any $u \in L_T^\alpha$ and $v \in C_T C^\beta$, we have
\[
\|\tilde{\Pi}_u v - \Pi_u v\|_{C_T C^{\alpha + \beta}} \lesssim \|u\|_{L_T^\alpha} \|v\|_{C_T C^\beta}.
\]

The next lemma gives estimates on the commutators $[\nabla, \tilde{\Pi}]$ and $[\Delta, \tilde{\Pi}]$.

Lemma A.3 (Lemma 2.6, [7]) Let $T > 0$, $\alpha \in (0, 1)$, $\beta \in \mathbb{R}$. For any $u \in C_T C^\alpha$ and $v \in C_T C^\beta$, we have
\[
\|[\nabla, \tilde{\Pi}_u]v\|_{C_T C^{\alpha + \beta - 1}} \lesssim \|u\|_{C_T C^\alpha} \|v\|_{C_T C^\beta} \quad \text{and} \quad \|[\Delta, \tilde{\Pi}_u]v\|_{C_T C^{\alpha + \beta - 2}} \lesssim \|u\|_{C_T C^\alpha} \|v\|_{C_T C^\beta}.
\]

The next lemma is the associative property for the paraproduct.

Lemma A.4 (Lemma 2.8, [7]) Let $\alpha \in (0, 1)$ and $\beta \in \mathbb{R}$, and let us define
\[
R(u, v; w) = \Pi_u (\tilde{\Pi}_v w) - \Pi_{uv} w
\]
for $u \in C_T C^\alpha$, $v \in L_T^\alpha$ and $w \in C^\beta$. Then we have
\[
\|R(u, v; w)\|_{C_T C^{\alpha + \beta}} \lesssim \|u\|_{C_T C^\alpha} \|v\|_{L_T^\alpha} \|w\|_{C^\beta}.
\]

Lemma A.5 (Lemma 2.9, [7]) Let $F \in C_b^2(\mathbb{R})$ and $\alpha \in (1, 2)$. Then the following hold.

(i) For any $u \in C^\alpha$, we have
\[
\|F(\nabla u)\|_{C^{\alpha - 1}} \lesssim \|F\|_{C^1(1 + \|u\|_{C^\alpha})}.
\]

and for any $u, v \in C^\alpha$, we have the local Lipschitz estimate
\[
\|F(\nabla u) - F(\nabla v)\|_{C^{\alpha - 1}} \lesssim \|F\|_{C^2(1 + \|u\|_{C^\alpha})} \|u - v\|_{C^\alpha}.
\]

(ii) Let $0 < \beta \leq \alpha - 1$. Then, for any $u, v \in L_T^\alpha$ satisfying $u(0) = v(0)$, we have the local Lipschitz estimate
\[
\|F(\nabla u) - F(\nabla v)\|_{L_T^\beta} \lesssim T^{\frac{\alpha - \beta - 1}{2}} \|F\|_{C^2(1 + \|u\|_{L_T^\alpha})} \|u - v\|_{L_T^\alpha}.
\]

The next lemma is about intertwining continuity estimate.
Lemma A.6 (Lemma 2.10, [7]) For any $f \in \mathcal{L}^\beta_T$ and $u_0 \in C^\alpha$, we have

$$\|\mathcal{L}^0\left(\bar{\Pi}_f X\right) - \Pi_{a(\nabla u_0^T)}f(-\Delta X)\|_{C^\alpha_{T+C^\alpha+\beta-2}} \lesssim \left(1 + T^{-\frac{\gamma}{2}}\|u_0\|_{C^\alpha}\right)\|f\|_{\mathcal{L}^\beta_T}\|X\|_{C^\alpha},$$

where $X = (-\Delta)^{-1}(\xi - \xi(\mathbb{T}))$ with $\xi(\mathbb{T}) \equiv \langle \xi, 1 \rangle$.

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