When does a discrete-time random walk in $\mathbb{R}^n$ absorb the origin into its convex hull?

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Abstract

We connect this question to a problem of estimating the probability that the image of certain random matrices does not intersect with a subset of the unit sphere $\mathbb{S}^{n-1}$. In this way, the case of a discretized Brownian motion is related to Gordon’s escape theorem dealing with standard Gaussian matrices. The approach allows us to prove that with high probability, the $\pi/2$-covering time of certain random walks on $\mathbb{S}^{n-1}$ is of order $n$. For certain spherical simplices on $\mathbb{S}^{n-1}$, we extend the “escape” phenomenon to a broad class of random matrices; as an application, we show that $c^N$ steps are sufficient for the standard walk on $\mathbb{Z}^n$ to absorb the origin into its convex hull with a high probability.

1 Introduction

The goal of this paper is to study certain convexity aspects of high-dimensional random walks. Given a discrete-time random walk $W(i)$ with values in $\mathbb{R}^n$, we are interested in estimating the number of steps $N$ when the origin enters the convex hull of $\{W(i)\}_{i \leq N}$. This question was raised by I. Benjamini and considered in [4] by R. Eldan. Three models of random walks are treated in this paper: a random walk given by a discretization of the standard Brownian motion in $\mathbb{R}^n$, the standard random walk on $\mathbb{Z}^n$ and a random walk on the unit sphere $\mathbb{S}^{n-1}$. We follow a novel approach of reducing the problem to studying certain geometric properties of random matrices. The latter subject is of great interest in the area of Asymptotic Geometric Analysis (see for example [1] and [20]) and is related to Gordon’s escape theorem [7] and estimates of diameters of random sections of convex sets [14], [17]. The interconnection between random walks, Random Matrix Theory and High-dimensional Convex Geometry is at the heart of our paper.

The standard Brownian motion with values in $\mathbb{R}$ is a centered Gaussian process $BM_1(t)$, $t \in [0, \infty)$, such that $\text{cov}(BM_1(t), BM_1(s)) = \min(t, s)$ for all $t, s \in [0, \infty)$. The Brownian motion in $\mathbb{R}^n$, denoted by $BM_n$, is a vector of $n$ independent one-dimensional Brownian

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motions. We refer the reader to [16] for extensive information on the process BM\(_n\). Various properties of the convex hull of the Brownian motion in high dimensions were studied recently in [4], [5] and [9]. Our paper is motivated by the following question considered (in a slightly different form) by Eldan in [4]:

Let \(t_1, t_2, \ldots, t_N\) be points in \([0, 1]\). How is the probability that the origin belongs to the interior of \(\text{conv}\{\text{BM}_n(t_i) : i \leq N\}\) related to the structure of the set \(\{t_i\}_{i \leq N}\)?

Here, “\(\text{conv}\)” denotes the convex hull of a set. In [4], the numbers \(N\) and \(t_1, t_2, \ldots, t_N\) were generated by a homogeneous Poisson point process in \([0, 1]\). It was shown that when the expected number of generated points \(N\) is greater than \(e^{Cn\log(n)}\), the origin belongs to the interior of \(\text{conv}\{\text{BM}_n(t_i) : i \leq N\}\) with high probability [4, Theorem 3.1]. A related result of [4] dealing with the standard walk on \(\mathbb{Z}^n\) states that, with probability close to one, \(e^{Cn\log(n)}\) steps are sufficient for the convex hull of the walk to “absorb” the origin. It was not clear, however, whether the bound \(e^{Cn\log(n)}\) was sharp. This question is addressed in the first main theorem of our paper:

**Theorem A.** There exists a sufficiently large universal constant \(C > 0\) such that the following holds. For \(n \in \mathbb{N}\) and \(N \geq \exp(Cn)\), we have:

- Setting \(t_i := i/N, i = 1, 2, \ldots, N\), the set \(\text{conv}\{\text{BM}_n(t_i), i \leq N\}\) contains the origin in its interior with probability at least \(1 - \exp(-n)\).

- The convex hull of the first \(N\) steps of the standard random walk on \(\mathbb{Z}^n\) starting at 0, contains the origin in its interior with probability at least \(1 - \exp(-n)\).

The first part of the statement above can be reformulated for points \(t_i\) generated by a homogeneous Poisson point process in \([0, 1]\). Hence, our result strengthens [4, Theorem 3.1] and the upper bound in [4, Theorem 1.2].

The assertion of Theorem A is close to optimal: In fact, for the random walk on \(\mathbb{Z}^n\) and the “discretized” Brownian motion \(\text{BM}_n(t_i)\) with \(t_i = i/N\), it is proved in [4] that if the number of steps \(N\) is less than \(\exp(cn/\log n)\) then with high probability the origin is not “absorbed” by the convex hull.

The second main result of this paper deals with discrete-time random walks on the sphere. For any \(\theta \in (0, \pi/2)\), we consider a Markov chain \(W_\theta\) with values in \(\mathbb{S}^{n-1}\) such that the angle between two consecutive steps is \(\theta\) (i.e. \(\langle W_\theta(j), W_\theta(j+1) \rangle = \cos \theta, j \in \mathbb{N}\)) and the direction from \(W(j)\) to \(W(j+1)\) is chosen uniformly at random in the sense that for any \(u \in \mathbb{S}^{n-1}\), the distribution of \(W_\theta(j+1)\) conditioned on \(W_\theta(j) = u\) is uniform on the \((n-2)\)-sphere \(\mathbb{S}^{n-1} \cap \{x \in \mathbb{R}^n : \langle x, u \rangle = \cos \theta\}\).

**Theorem B.** For any \(\theta \in (0, \pi/2)\), there exist \(L = L(\theta)\) and \(n_0 = n_0(\theta)\) depending only on \(\theta\) such that the following holds: Let \(n \geq n_0\) and \(W_\theta\) be the process with values in \(\mathbb{S}^{n-1}\) described above. Then for all \(N \geq Ln\) we have

\[
P\{0 \text{ belongs to } \text{conv}\{W_\theta(i) : i \leq N\}\} \geq 1 - \exp(-n).
\]
Clearly, this estimate of the number of steps is optimal up to a factor depending only on $\theta$. We note here that a related problem for the standard spherical Brownian motion was studied in [4].

Let us discuss the main ideas of the proof of the above statements. The following simple observation relates the question about random walks to a problem dealing with random matrices: Let $X(t)$ ($t \in [0, \infty)$ or $t \in \mathbb{N} \cup \{0\}$) be a random process with values in $\mathbb{R}^n$, with $X(0) = 0$ and with independent increments, and let $0 = t_0 < t_1 < \cdots < t_N$ be a collection of non-random points. Define $A$ as the $N \times n$ random matrix with independent rows obtained by appropriately rescaling the increments $X(t_i) - X(t_{i-1})$, $i = 1, 2, \ldots, N$. Then there exists a non-random $N \times N$ lower-triangular matrix $F$ such that the rows of $FA$ are precisely $X(t_i)$, $i = 1, 2, \ldots, N$. Thus, we can restate our problem about the convex hull of $X(t_i)$’s in terms of certain properties of the matrix $FA$. Namely, the convex hull of $X(t_i)$’s contains the origin in its interior if and only if for any unit vector $y$ in $\mathbb{R}^n$, the vector $FAy$ has at least one negative coordinate. Geometrically, this problem is reduced to estimating the probability that the image of $A$ “escapes” (i.e. does not intersect) the set $F^{-1}(\mathbb{R}_+^N) \cap S^{N-1}$, where $\mathbb{R}_+^N$ denotes the cone of positive vectors. For the standard Brownian motion, $A$ is the $N \times n$ standard Gaussian matrix. In this case, we apply Gordon’s escape theorem [7] which estimates the probability that a random subspace uniformly distributed on the Grassmannian does not intersect with a given subset of $S^{N-1}$. In a more general case, when the image of $A$ is not uniformly distributed, Gordon’s theorem is not applicable. To work with the random walk on $\mathbb{Z}^n$, we prove a statement dealing with a broad class of random matrices, however, with considerable restrictions on the subsets of $S^{N-1}$. Our treatment of the random walks $W_\theta$ on the sphere follows the same scheme as for processes in $\mathbb{R}^n$ with independent increments, modulo some modifications.

The results about random matrices are given in Section 3, while the corollaries for the Brownian motion and the standard random walk on $\mathbb{Z}^n$ are stated in Section 4. Section 5 is devoted to random walks on the sphere.

2 Preliminaries

In this section we introduce notation and state some classical or elementary facts that will be useful for us further in the text.

By $\{e_i\}_{i=1}^N$ we denote the standard unit basis in $\mathbb{R}^N$, by $\|\cdot\|$ — the canonical Euclidean norm and by $\langle \cdot, \cdot \rangle$ — the corresponding inner product. Let $B_2^N$ and $S^{N-1}$ be the Euclidean ball of radius 1 in $\mathbb{R}^N$ and the unit sphere, respectively.

For $N \geq n$ and an $N \times n$ matrix $A$, let $s_{\max}(A)$ and $s_{\min}(A)$ be its largest and smallest singular values, respectively, i.e. $s_{\max}(A) = \|A\|$ (the operator norm of $A$) and $s_{\min}(A) = \inf_{y \in S^{n-1}} \|Ay\|$. When $A$ is an $N \times N$ invertible matrix, the condition number of $A$ is given by $\|A\| \cdot \|A^{-1}\|$. Note that the condition number is equal to the ratio of the largest and the smallest singular values of $A$.

Throughout the text, $g$ denotes the standard Gaussian variable. The following esti-
mate is well known (see, for example, [6, Lemma VII.1.2]):

$$\mathbb{P}\{g \geq t\} = \frac{1}{\sqrt{2\pi}} \int_t^\infty \exp(-r^2/2) \, dr < \frac{1}{\sqrt{2\pi t}} \exp(-t^2/2), \quad t > 0. \quad (1)$$

A random vector $X$ in $\mathbb{R}^n$ is isotropic if $\mathbb{E}X = 0$ and the covariance matrix of $X$ is the identity i.e. $\mathbb{E}XX^t = I$. The standard Gaussian vector $Y$ in $\mathbb{R}^n$ is a random vector with i.i.d. coordinates having the same law as $g$. As a corollary of a concentration inequality for Gaussian variables (see [18, Theorem 4.7] or [15, Theorem V.1]), we have for any $\varepsilon > 0$

$$\mathbb{P}\{ (1 - \varepsilon)\sqrt{n} \leq \|Y\| \leq (1 + \varepsilon)\sqrt{n} \} \geq 1 - 2 \exp(-\tilde{c}\varepsilon^2n) \quad (2)$$

for a universal constant $\tilde{c} > 0$. Let $N \geq n$ and let $G$ be the standard $N \times n$ Gaussian matrix. Then for any $t \geq 0$ we have

$$\mathbb{P}\{ \sqrt{N} - \sqrt{n} - t \leq s_{\min}(G) \leq s_{\max}(G) \leq \sqrt{N} + \sqrt{n} + t \} \geq 1 - 2 \exp(-t^2/2) \quad (3)$$

(see, for example, [21, Corollary 5.35]).

Given a vector $x \in \mathbb{R}^N$, we denote by $x_+$ and $x_-$ its positive and negative part, respectively, i.e.

$$x_+ = \sum_{i=1}^N \max(0, \langle x, e_i \rangle) e_i \quad \text{and} \quad x_- = \sum_{i=1}^N \max(0, -(x, e_i)) e_i.$$

The following simple observation will be useful in the proof of the main Theorems.

**Lemma 1.** Let $x, y \in \mathbb{R}^N$. Then $\|x_-\| \geq \|y_-\| - \|x - y\|$.

**Proof.** Writing $x = x_+ - x_-$ and $y = y_+ - y_-$, we have

$$\|x - y\|^2 = \|(x_+ - y_+) - (x_- - y_-)\|^2$$

$$= \|x_- - y_-\|^2 + \|x_+ - y_+\|^2 - 2\langle x_+ - y_+, x_- - y_- \rangle$$

$$\geq \|x_- - y_-\|^2$$

$$\geq (\|y_-\| - \|x_-\|)^2,$$

where the first inequality in the above formula holds since $\langle x_+ - y_+, x_- - y_- \rangle$ is non-positive. \qed

Given a compact set $S \subset \mathbb{R}^N$, the Gaussian width of $S$ is

$$w(S) := \mathbb{E} \sup_{x \in S} \langle Y, x \rangle,$$

where $Y$ is the standard Gaussian vector in $\mathbb{R}^N$ (see [2] or [3]). The following is a consequence of Urysohn’s inequality (see, for example, Corollary 1.4 in [18]) and the relation between the Gaussian and mean width:

$$\sqrt{N - 1} \left( \frac{\text{Vol}_N(S)}{\text{Vol}_N(B_2^N)} \right)^{1/N} \leq w(S). \quad (4)$$
Given a convex cone $C$ in $\mathbb{R}^N$, the polar cone of $C$ is defined as

$$C^* := \{ x \in \mathbb{R}^N, \langle x, y \rangle \leq 0 \text{ for any } y \in C \}.$$ 

The next Lemma provides a useful relation between the Gaussian width of the parts of a convex cone and its polar enclosed in the unit Euclidean ball. The Lemma is proved in [3] for intersections of cones with the unit sphere (see [3, Lemma 3.7]); the slightly modified version given below will be more convenient for us:

**Lemma 2.** Let $C \subset \mathbb{R}^N$ be a nonempty closed convex cone. Then we have that

$$w\left( C \cap B_2^N \right)^2 + w\left( C^* \cap B_2^N \right)^2 \leq N.$$ 

**Proof.** For any $x \in \mathbb{R}^N$, let $P_C x := \arg \inf_{y \in C} \| x - y \|$ be the projection of $x$ onto $C$. It can be checked that each vector $x \in \mathbb{R}^N$ can be decomposed in the form

$$x = P_C x + P_{C^*} x, \quad (5)$$

with $\langle P_C x, P_{C^*} x \rangle = 0$. Having this decomposition in mind, we can write

$$w(C \cap B_2^N) = \mathbb{E} \sup_{x \in C \cap B_2^N} \langle Y, x \rangle \leq \mathbb{E} \sup_{x \in C \cap B_2^N} \langle P_C Y, x \rangle,$$

where the last inequality holds since $\langle P_C Y, x \rangle \leq 0$ for all $x \in C$. We deduce that

$$w(C \cap B_2^N) \leq \mathbb{E} \| P_C Y \|. \quad (6)$$

Now using the decomposition (5) and the above inequality, we obtain

$$w(C \cap B_2^N)^2 \leq \mathbb{E} \| P_C Y \|^2 \leq \mathbb{E} \| Y \|^2 - \mathbb{E} \| P_{C^*} Y \|^2 = N - \mathbb{E} \| P_{C^*} Y \|^2. \quad (7)$$

Note that (6) applied to the cone $C^*$ yields $w(C^* \cap B_2^N)^2 \leq \mathbb{E} \| P_{C^*} Y \|^2$. Plugging it into (7), we complete the proof.

## 3 Escape theorems for random matrices

In this section, we estimate the probability that the image of a random $N \times n$ matrix $A$ escapes the intersection of a given cone with the unit sphere $S^{N-1}$ (we shall restrict ourselves to considering a special family of convex cones in $\mathbb{R}^N$). Similar questions have attracted considerable attention recently in connection with the theory of Compressed sensing [2].

Given a closed subset $S \subset S^{N-1}$, the problem of estimating the probability $\mathbb{P}\{ \text{Im}(A) \cap S = \emptyset \}$ can be treated in different ways. One may look at it as the question of bounding the diameter of the random section $\text{conv}(S, -S) \cap \text{Im}(A)$ of the convex set $\text{conv}(S, -S)$: clearly, $\text{Im}(A) \cap S = \emptyset$ if and only if $\text{diam}(\text{conv}(S, -S) \cap \text{Im}(A)) < 2$. The study of random sections of convex sets is a large topic within Asymptotic Geometric Analysis, influenced by Milman’s proof of Dvoretzky’s theorem [14], [15]. The question of estimating diameters of random sections of proportional dimension was originally considered in [14]
and [17] in the case when the corresponding random subspace is uniformly distributed on the Grassmannian (i.e. the randomness is given by a standard Gaussian matrix). More recently, results for much more general distributions of sections given by kernels and images of random matrices were obtained in [11] and [13]. Let us note, however, that these papers provide estimates for diameters up to a constant multiple, which is not sufficiently precise to be used in our case, as the sets $S$ which we consider are relatively “large”. For example, if $S = S^{N-1} \cap \mathbb{R}_+^N$ then it is not difficult to show that the diameter of any section of conv$(S, -S)$ is at least $\sqrt{2}$.

When the matrix $A$ is Gaussian, a more suitable way of estimating the probability $\mathbb{P}\{\text{Im}(A) \cap S = \emptyset\}$ in our setting is by applying the following result of Gordon (see Corollary 3.4 in [7]):

**Theorem 3** (Gordon’s escape Theorem). Let $S$ be a subset of the unit Euclidean sphere $S^{N-1}$ in $\mathbb{R}^N$. Let $E$ be a random $n$-dimensional subspace of $\mathbb{R}^N$, distributed uniformly on the Grassmannian with respect to the associated Haar measure. Assume that $w(S) < \sqrt{N-n}$. Then $E \cap S = \emptyset$ with probability at least

$$1 - 3.5 \exp\left(-\frac{1}{18}\left(\frac{N-n}{\sqrt{N-n+1}} - w(S)\right)^2\right).$$

For the standard Gaussian matrix $G$, its image is uniformly distributed on the Grassmannian, and Gordon’s result allows to efficiently estimate the probability $\mathbb{P}\{\text{Im}(G) \cap S = \emptyset\}$, provided that we have a control of the Gaussian width of the set $S$. In our setting, the choice of $S$ is determined by the applications to random walks; in fact, $S$ shall always be a spherical simplex satisfying certain additional assumptions. A standard approach would be to bound $w(S)$ in terms of the covering numbers of $S$ using the classical Dudley’s inequality (see, for example, [10, Theorem 11.17]). However, in our case the set $S$ is relatively large making Dudley’s bound not applicable. Instead, we will estimate the Gaussian width of $S$ by considering a cone polar to the one generated by $S$, and applying Lemma 2.

**Theorem 4.** For any $\gamma \in (0, 1]$ there exist positive $L$, $\kappa$ and $\eta$ depending on $\gamma$ such that the following is true: For $N \geq Ln$, let $F$ be an $N \times N$ random matrix and $\tilde{F}$ be a deterministic invertible $N \times N$ matrix whose condition number satisfies $\|\tilde{F}\|\cdot\|\tilde{F}^{-1}\| \leq \gamma^{-1}$. If $G$ is the $N \times n$ standard Gaussian matrix, then

$$\mathbb{P}\{\exists y \in S^{n-1}, \ FGy \in \mathbb{R}_+^n\} \leq 5.5 \exp(-\kappa N) + \mathbb{P}\{\|F - \tilde{F}\| > \eta \|\tilde{F}\|\}.$$  

The statement holds with $L = 64/\gamma^2$, $\kappa = 2L^{-2}/9$ and $\eta = \gamma/4L$.

**Proof.** Let $\gamma \in (0, 1)$ and take $L, \kappa,$ and $\eta$ as stated above. In view of Lemma 4 we have

$$\mathbb{P}\{\exists y \in S^{n-1}, \ (FGy)_- = 0\}$$

$$\leq \mathbb{P}\{\exists y \in S^{n-1}, \ |(FGy)_-| \leq \|(F - \tilde{F})Gy\|\}$$

$$\leq \mathbb{P}\{\exists y \in S^{n-1}, \ |(FGy)_-| \leq \eta \|\tilde{F}\| \cdot \|G\|\} + \mathbb{P}\{\|F - \tilde{F}\| > \eta \|\tilde{F}\|\}.$$
Further,
\[
\mathbb{P}\left\{ \exists y \in S^{n-1}, \| (\tilde{F} G y)_- \| \leq \eta \| \tilde{F} \| \cdot \| G \| \right\} \\
\leq \mathbb{P}\left\{ \exists y \in S^{n-1}, \tilde{F} G y \in \mathbb{R}_+^N + \eta \| \tilde{F} \| \cdot \| G \| B_2^N \right\} \\
\leq \mathbb{P}\left\{ \exists y \in S^{n-1}, \frac{G y}{\| G y \|} \in \tilde{F}^{-1}(\mathbb{R}_+^N) + \eta \| \tilde{F} \| \frac{\| G \|}{s_{\min}(G)} \tilde{F}^{-1}(B_2^N) \right\} \\
\leq \mathbb{P}\left\{ \exists y \in S^{n-1}, \frac{G y}{\| G y \|} \in \tilde{F}^{-1}(\mathbb{R}_+^N) + 2\eta \cdot \gamma^{-1} B_2^N \right\} \\
\quad + \mathbb{P}\left\{ \| G \| > 2s_{\min}(G) \right\} \\
\leq \mathbb{P}\left\{ \text{Im}(G) \cap (\tilde{F}^{-1}(\mathbb{R}_+^N) + 2\eta \cdot \gamma^{-1} B_2^N) \cap S^{n-1} \neq \emptyset \right\} + 2e^{-N^{1/28}}, \quad (8)
\]
where the last estimate follows from (3).

To control the probability of “escaping” in (8) with help of Theorem 3, we have to estimate the Gaussian width of the set
\[
\Gamma := (\tilde{F}^{-1}(\mathbb{R}_+^N) + 2\eta \cdot \gamma^{-1} B_2^N) \cap S^{n-1}.
\]
Note that \( \Gamma \subset (1 + 2\eta \cdot \gamma^{-1})\tilde{F}^{-1}(\mathbb{R}_+^N) \cap B_2^N + 2\eta \cdot \gamma^{-1} B_2^N \). Therefore
\[
w(\Gamma) \leq (1 + 2\eta \cdot \gamma^{-1}) \cdot w\left( \tilde{F}^{-1}(\mathbb{R}_+^N) \cap B_2^N \right) + 2\eta \cdot \gamma^{-1}\sqrt{N}. \quad (9)
\]
It remains to bound the Gaussian width of \( \tilde{F}^{-1}(\mathbb{R}_+^N) \cap B_2^N \). Denote \( C := \tilde{F}^{-1}(\mathbb{R}_+^N) \) and note that \( C^* = \tilde{F}'(\mathbb{R}_-^N) \). Then we have
\[
\text{Vol}_N(\tilde{F}'(\mathbb{R}_-^N) \cap B_2^N) = |\det(\tilde{F})| \cdot \text{Vol}_N(\mathbb{R}_N^N \cap (\tilde{F}')^{-1}(B_2^N)) \\
\geq |\det(\tilde{F})| \cdot \| \tilde{F} \|^{-N} \cdot \text{Vol}_N(\mathbb{R}_N^N \cap B_2^N).
\]
Since \( |\det(\tilde{F})| \geq \| \tilde{F}^{-1} \|^{-N} \), then \( \text{Vol}_N(C^* \cap B_2^N) \geq (\gamma/2)^N \cdot \text{Vol}_N(B_2^N) \). Combined with (11), this implies that
\[
w(C^* \cap B_2^N) \geq \frac{\gamma}{2}\sqrt{N - 1}.
\]
Now applying Lemma 2, we deduce that
\[
w(C \cap B_2^N) \leq \sqrt{(1 - \gamma^2/8)N} \quad (10)
\]
Putting (9) and (10) together, we get that
\[
w(\Gamma) \leq (1 + 4\eta \cdot \gamma^{-1} - \gamma^2/16) \sqrt{N}.
\]
The proof is finished by a direct application of Theorem 3.

As we will see in the next sections, Theorem 4 provides a way to deal with the standard Brownian motion in \( \mathbb{R}^n \) and random walks \( W_\theta \) on the sphere. To treat the standard walk on \( \mathbb{Z}^n \), we shall derive a statement covering a rather broad class of random matrices. Let us introduce the following
**Definition 5.** A random variable $\xi$ is said to have property $P(\tau, \delta)$ (or satisfy condition $P(\tau, \delta)$) for some $\tau, \delta \in (0, 1]$ if $$\mathbb{P}\{\xi < -\tau\} \geq \delta.$$ A random vector $X$ in $\mathbb{R}^n$ is said to have property $P(\tau, \delta)$ for $\tau, \delta \in (0, 1]$ if for any $y \in S^{n-1}$, the random variable $\langle X, y \rangle$ satisfies $P(\tau, \delta)$.

Obviously, the above property holds (for some $\tau$ and $\delta$) for any non-zero r.v. $\xi$ with $\mathbb{E} \xi = 0$. As the next elementary lemma shows, with some additional assumptions on moments of $\xi$, the numbers $\tau$ and $\delta$ can be chosen as certain functions of the moments:

**Lemma 6.** Any random variable $\xi$ such that $\mathbb{E} \xi = 0$, $\mathbb{E} \xi^2 = 1$ and $\mathbb{E} |\xi|^{2+\varepsilon} \leq B < \infty$ for some $\varepsilon > 0$, has the property $P(\tau, \delta)$, with $\tau$ and $\delta$ depending only on $\varepsilon$ and $B$.

**Proof.** Indeed, an easy calculation shows that such $\xi$ satisfies $$\int_{L_{\xi}^2}^\infty \mathbb{P}\{\xi^2 \geq t\} \, dt \leq \frac{1}{2}$$ for some $L_{\xi} > 0$ depending only on $B$ and $\varepsilon$. Then $$\mathbb{E}|\xi| \geq \int_0^{L_{\xi}} \mathbb{P}\{|\xi| \geq t\} \, dt \geq \frac{1}{2 \sqrt{2}} \int_0^{L_{\xi}^2} \mathbb{P}\{\xi^2 \geq t\} \, dt \geq \frac{1}{4 L_{\xi}},$$ implying, as $\mathbb{E} \max(0, -\xi) = \frac{1}{2} \mathbb{E}|\xi|$,

$$\frac{1}{8 L_{\xi}} \leq \int_0^{L_{\xi}} \mathbb{P}\{\xi \leq -t\} \, dt$$

$$\leq \int_0^{8L_{\xi}} \mathbb{P}\{\xi \leq -t\} \, dt + \int_{8L_{\xi}}^{\infty} \frac{1}{2 \sqrt{t}} \mathbb{P}\{\xi^2 \geq t\} \, dt$$

$$\leq \int_0^{8L_{\xi}} \mathbb{P}\{\xi \leq -t\} \, dt + \frac{1}{16 L_{\xi}}.$$ Hence, $\mathbb{P}\{\xi < -2^{-5} L_{\xi}^{-1}\} \geq 2^{-8} L_{\xi}^{-2}$. 

**Theorem 7.** For any $\tau, \delta \in (0, 1]$ and any $K > 1$, there exist $L$ and $\eta > 0$ depending only on $\tau$, $\delta$ and $K$ with the following property: Let $N \geq Ln$ and let $A$ be an $N \times n$ random matrix with independent rows satisfying $P(\tau, \delta)$. Then for any $N \times N$ random matrix $F$, matrix $FA$ satisfies

$$\mathbb{P}\{\exists y \in S^{n-1}, FAy \in R^N\} \leq \exp(-\delta^2 N/4)$$

$$+ \mathbb{P}\{\|A\| > K \sqrt{N}\} + \mathbb{P}\{\|F - I\| > \eta\}. $$
Proof. Define $L$ as the smallest positive number satisfying
\[
\left(\frac{3}{\eta}\right)^{1/L} \leq \exp(\delta^2/4),
\]
where $\eta := \frac{\sqrt{\tau}}{2\sqrt{2k}}$. Now, take any admissible $N \geq Ln$ and let $A$ and $F$ be as stated above.

Let $\mathcal{N}$ be an $\eta$-net on $S^{n-1}$ of cardinality at most $\left(\frac{3}{\eta}\right)^n$. In view of Lemma 1 we have
\[
\mathbb{P}\left\{ \exists y \in S^{n-1}, \ FAy \in \mathbb{R}^N_+ \right\}
\leq \mathbb{P}\left\{ \exists y \in S^{n-1}, \ \|Ay\| \leq \|(F - I)Ay\| \right\}
\leq \mathbb{P}\left\{ \exists y \in S^{n-1}, \ \|Ay\| \leq \eta \|A\| \right\} + \mathbb{P}\{\|F - I\| > \eta\}
\leq \mathbb{P}\{\exists y' \in \mathcal{N}, \ \|Ay'\| \leq 2\eta \|A\|\} + \mathbb{P}\{\|F - I\| > \eta\}.
\]

Further,
\[
\mathbb{P}\{\exists y' \in \mathcal{N}, \ \|Ay'\| \leq 2\eta \|A\|\}
\leq \mathbb{P}\{\exists y' \in \mathcal{N}, \ \|Ay'\| \leq 2K\eta\sqrt{N}\} + \mathbb{P}\{\|A\| > K\sqrt{N}\}.
\]

Fix any $y' \in \mathcal{N}$. For all $i = 1, 2, \ldots, N$, the random variable $\langle Ay', e_i \rangle$ satisfies the property $\mathcal{P}(\tau, \delta)$. For any $i \leq N$, denote by $\chi_i$ the indicator function of the event $\{\langle Ay', e_i \rangle < -\tau\}$. Then $(\chi_i)_{i \leq N}$ are independent and $\mathbb{E}\chi_i \geq \delta$. Applying Hoeffding’s inequality (see [8, Theorem 1]), we get
\[
\mathbb{P}\left\{ \{|i \leq N : \langle Ay', e_i \rangle < -\tau\| \leq \frac{\delta N}{2} \right\} \leq \mathbb{P}\left\{ \frac{1}{N} \sum_{i \leq N} (\chi_i - \mathbb{E}\chi_i) \leq -\frac{\delta}{2} \right\}
\leq \exp(-\delta^2 N/2).
\]

Therefore for any fixed $y' \in \mathcal{N}$, we have
\[
\mathbb{P}\{\|Ay'\| \leq 2K\eta\sqrt{N}\} \leq \mathbb{P}\{\{|i \leq N : \langle Ay', e_i \rangle < -\tau\| \leq 4K^2\eta^2N/\tau^2 \}
\leq \exp(-\delta^2 N/2).
\]

Combining the last estimate with (11) and the upper estimate for $|\mathcal{N}|$, we get
\[
\mathbb{P}\{\exists y \in S^{n-1}, \ FAy \in \mathbb{R}^N_+ \}
\leq \left(\frac{3}{\eta}\right)^n \exp(-\delta^2 N/2) + \mathbb{P}\{\|A\| > K\sqrt{N}\} + \mathbb{P}\{\|F - I\| > \eta\}.
\]

The proof follows by the choice of $L$. \qed

Remark 1. Theorem 7, applied to the Gaussian matrix $G$, gives a weaker form of Theorem 4 (with more restrictions on the choice of $F$). Let us emphasize that the theorems do not require $F$ to be independent from $G$. This will be important in Section 5.
4 Applications to random walks in $\mathbb{R}^n$

In this section, we will apply the statements about random matrices to the Brownian motion and the standard walk on $\mathbb{Z}^n$.

**Corollary 8.** For any $K > 1$, there are constants $L$ and $\kappa$ depending only on $K$ such that the following holds. Let $N \geq Ln$ and $t_1, \ldots, t_N$ be such that $t_i \geq K \cdot t_{i-1}$ for any $i = 2 \ldots N$ and $t_1 > 0$. Then

$$\mathbb{P}\{0 \text{ belongs to the interior of } \text{conv}\{\text{BM}_n(t_i) : i \leq N\}\} \geq 1 - 5.5 \exp(-\kappa N).$$

**Proof.** Let $c_K := 1 + (K - 1)^{-1/2} \sum_{j \geq 0} K^{-j/2}$ and $\gamma := c_K^{-1} \cdot (1 + (K - 1)^{-1/2})^{-1}$ be two constants depending only on $K$ and take $L = 64/\gamma^2$ and $\kappa := 2L^{-2}/9$.

Denote $\delta_1 := \sqrt{t_1}$ and $\delta_i := \sqrt{t_i - t_{i-1}}$ for any $i = 2 \ldots N$. Observe that for any $j < i$, we have $\delta_i \geq K^{-1/2} \sqrt{K - 1 \cdot \delta_j}$.

Define $F$ as the $N \times N$ lower triangular matrix whose entries are given by $f_{ii} = 1$ for any $i \leq N$ and $f_{ij} = \frac{\delta_i}{\delta_j}$ for any $i > j$. One can easily check that $\|F\| \leq c_K$. Moreover, the inverse of $F$ is a lower bidiagonal matrix with 1 on the main diagonal and $(\delta_i/\delta_{i+1})_{i<N}$ on the diagonal below. Hence $\|F^{-1}\| \leq 1 + (K - 1)^{-1/2}$, and the condition number of $F$ satisfies

$$\|F\| \cdot \|F^{-1}\| \leq \gamma^{-1}.$$  

Let $(R_i)_{i \leq N}$ be the rows of $FG$. One can check that $R_i = \text{BM}_n(t_i)/\delta_i$ and therefore

$$0 \in \text{conv}\{\text{BM}_n(t_i) : i \leq N\} \Leftrightarrow 0 \in \text{conv}\{R_i : i \leq N\}$$

Note that, by a standard separation argument, 0 does not belong to the interior of $\text{conv}\{R_i : i \leq N\}$ if and only if $\text{rank}(FG) < n$ or there is a vector $y \in S^{n-1}$ such that $\langle FGy, e_i \rangle = \langle y, R_i \rangle \geq 0$ for any $i \leq N$, where $(e_i)_{i \leq N}$ denotes the canonical basis of $\mathbb{R}^N$. Since with probability one we have $\text{rank}(FG) = n$, the result follows by applying Theorem \[\] with $\bar{F} := F$. \[\]

Suppose $(t_i)$ is a finite increasing sequence of points in $[0,1]$. The above statement tells us that if $(t_i)$ contains a “geometrically growing” subsequence of length $Ln$ for an appropriate $L > 0$ then with high probability the origin of $\mathbb{R}^n$ is contained in the interior of $\text{BM}_n(t_i)$’s. We shall apply this result to the case when the $t_i$’s are generated by the Poisson point process independent from $\text{BM}_n$.

Recall that the homogeneous Poisson point process in $[0,1]$ of intensity $s > 0$ is a random discrete measure $\mathbb{N}_s$ on $[0,1]$ such that 1) for each Borel subset $B \subset [0,1]$, the random variable $\mathbb{N}_s(B)$ has the Poisson distribution with parameter $s\mu(B)$, where $\mu$ is the usual Lebesgue measure on $\mathbb{R}$, and 2) for any $j \in \mathbb{N}$ and pairwise disjoint Borel sets $B_1, B_2, \ldots, B_j \subset [0,1]$, the random variables $\mathbb{N}_s(B_1), \mathbb{N}_s(B_2), \ldots, \mathbb{N}_s(B_j)$ are jointly independent. The measure $\mathbb{N}_s$ admits a representation of the form

$$\mathbb{N}_s = \sum_{i=1}^{\tau} \delta_{\xi_i},$$

where $\xi_i$ are independent random variables uniformly distributed on $[0,1]$. \[\]
where \( \xi_1, \xi_2, \ldots \) are i.i.d. random variables uniformly distributed on \([0, 1]\), \( \delta_{\xi_i} \) is the Dirac measure with the mass at \( \xi_i \) and \( \tau \) is the random non-negative integer with the Poisson distribution with parameter \( s \).

Theorem 3.1 of [4] states that if \( \tau \) and the points \( \xi_1, \xi_2, \ldots, \xi_\tau \) are generated by the homogeneous PPP in \([0, 1]\) of intensity \( s \geq n^{\tilde{c}n} \) then the convex hull of \( BM_n(\xi_i)'s \) contains the origin in its interior with probability at least \( 1 - n^{-n^2} \). In our next statement, we weaken the assumptions on \( s \) at expense of decreasing the probability to \( 1 - \exp(-n) \):

**Corollary 9.** There is a universal constant \( \tilde{C} > 0 \) with the following property: Let \( n \in \mathbb{N} \) and let \( BM_n(t), t \in [0, \infty), \) be the standard Brownian motion in \( \mathbb{R}^n \). Further, let \( \tau \) and the points \( \xi_1, \xi_2, \ldots, \xi_\tau \) be given by the homogeneous Poisson process on \([0, 1]\) of intensity \( s \geq \exp(\tilde{C}n) \), which is independent from \( BM_n(t) \). Then

\[
\mathbb{P}\{0 \text{ belongs to the interior of } \text{conv}\{BM_n(\xi_i) : i \leq \tau}\} \geq 1 - \exp(-n).
\]

**Proof.** Let \( K := 2 \) and \( \kappa, L \) be as in Corollary \[3\] Then we define the constant \( \tilde{C} := \max(\frac{24}{\kappa}, 8L) \). Let \( n \in \mathbb{N} \) and let \( N_s \) be as stated above. Take \( m := [\tilde{C}n] \) and

\[
I_1 := [0, K^{-m+1}] \quad I_j := (K^{j-m-1}, K^{j-m}], \quad j = 2, 3, \ldots, m.
\]

From the definition of \( N_s \), we have

\[
\mathbb{P}\{N_s(I_j) > 0 \text{ for all } j = 1, 2, \ldots, m\} \geq 1 - \sum_{j=1}^{m} \exp(-s\mu(I_j)) \geq 1 - m \exp(-sK^{-m}).
\]

In particular, with probability at least \( 1 - m \exp(-sK^{-m}) \) the set \( \{\xi_i\}_{i=1}^\tau \) contains a subset \( \{\xi_{i_1}, \xi_{i_2}, \ldots, \xi_{i_m}\} \) such that \( \xi_{i_j} \in I_j \) for every admissible \( j \), hence \( \xi_{i_{j+2}} \geq K\xi_{i_j} \) for any \( j \leq m - 2 \). Conditioning on the realization of \( N_s \), we obtain by Corollary \[3\]

\[
\mathbb{P}\{0 \text{ belongs to the interior of } \text{conv}\{BM_n(\xi_i) : i \leq \tau}\} \geq 1 - m \exp(-sK^{-m}) - 5.5 \exp(-\kappa[m/2]) \geq 1 - \exp(-n),
\]

and the proof is complete. \( \square \)

The last result of this section is connected to the standard random walk \( W(j) \) on \( \mathbb{Z}^n \), which is defined as a walk with independent increments such that each increment \( W(j+1) - W(j) \) is uniformly distributed on the set \( \{\pm e_j\}_{j \leq n} \). We note that the increments of the walk are not subgaussian; to be more precise, their subgaussian moment depends on the dimension \( n \). At the same time, the vectors still have very strong concentration properties as the next lemma shows:

**Lemma 10.** Let \( W(j) \) \( (j \geq 0) \) be the standard walk on \( \mathbb{Z}^n \) starting at the origin, and \( m \geq n^4 \) be any fixed integer. Then the vector \( X := \sqrt{n/m}W(m) \) is isotropic and satisfies for any \( y \in \mathbb{S}^{n-1} \):

\[
\mathbb{P}\{|\langle X, y \rangle| \geq t\} \leq \exp(-2(mn)^{1/4} + 2 \exp(-t^2/4)), \quad t > 0.
\]

In particular, \( \mathbb{E}|\langle X, y \rangle|^3 \leq 100 \) for all \( y \in \mathbb{S}^{n-1} \), and \( X \) has the property \( \mathcal{P}(\tau, \delta) \) for some universal constants \( \tau, \delta \).
Proof. The isotropicity of $X$ can be easily checked. Fix for a moment any vector $y \in \mathbb{S}^{n-1}$. The random variable $\langle X, y \rangle$ can be represented as

$$\langle X, y \rangle = \sqrt{\frac{n}{m}} \sum_{k=1}^{m} s_k,$$

where the variables $s_1, s_2, \ldots, s_m$ are i.i.d. and each

$$s_k := \langle W(k) - W(k - 1), y \rangle$$

is symmetrically distributed, has variance $\mathbb{E}s_k^2 = \frac{1}{n}$ and takes values in the interval $[-1, 1]$. Applying Hoeffding’s inequality to the sum $\sum_{k=1}^{m} s_k^2$, we get

$$\mathbb{P}\left\{ \sum_{k=1}^{m} s_k^2 \geq \frac{2m}{n} \right\} \leq \exp(-2m/n^2). \quad (12)$$

Further, since $s_k$ is symmetric, the distribution of the sum $\sum_{k=1}^{m} s_k$ is the same as the distribution of $\sum_{k=1}^{m} r_k s_k$, where $r_1, r_2, \ldots, r_m$ are Rademacher variables jointly independent with $s_1, s_2, \ldots, s_m$. Conditioning on the values of $s_k$ and using (12) and the Khintchine inequality, we obtain for every $t > 0$:

$$\mathbb{P}\left\{ \left| \sum_{k=1}^{m} s_k \right| \geq mt \right\} = \mathbb{P}\left\{ \left| \sum_{k=1}^{m} r_k s_k \right| \geq mt \right\} \leq \mathbb{P}\left\{ \sum_{k=1}^{m} s_k^2 \geq \frac{2m}{n} \right\} + \mathbb{P}\left\{ \sum_{k=1}^{m} s_k^2 \leq \frac{2m}{n} \text{ and } \sum_{k=1}^{m} r_k s_k \geq mt \right\} \leq \exp(-2m/n^2) + 2 \exp(-mnt^2/4).$$

Whence, in view of the bound $m \geq n^2(mn)^{1/4}$, we get

$$\mathbb{P}\left\{ |\langle X, y \rangle| \geq t \right\} \leq \exp(-2(mn)^{1/4}) + 2 \exp(-t^2/4), \quad t > 0. \quad (13)$$

The condition (13), together with the bound $\|X\| \leq \sqrt{mn}$, gives $\mathbb{E}|\langle X, y \rangle|^3 \leq 100$. It remains to apply Lemma 6. \qed

The next lemma follows from well known concentration inequalities for subexponential random variables (see, for example, [21, Corollary 5.17]):

**Lemma 11.** There is a universal constant $\tilde{C} > 0$ such that for any $N \in \mathbb{N}$ and independent centered random variables $\xi_1, \xi_2, \ldots, \xi_N$, each satisfying

$$\mathbb{P}\left\{ \xi_i \geq t \right\} \leq 3 \exp(-t/4), \quad t > 0, \quad (14)$$

we have

$$\mathbb{P}\left\{ \sum_{i=1}^{N} \xi_i \geq \tilde{C}N \right\} \leq 40^{-N}. \quad (15)$$

12
In the next result, compared to Theorem 1.2 of [4], we decrease the lower bound on the number of steps \( N \) of the walk on \( \mathbb{Z}^n \) sufficient to absorb the origin with high probability.

**Corollary 12.** There is a universal constant \( C > 0 \) with the following property: Let \( n, R \in \mathbb{N}, R \geq \exp(Cn) \) and let \( W(j), j \geq 0, \) be the standard random walk on \( \mathbb{Z}^n \) starting at the origin. Then

\[
\mathbb{P}\{0 \text{ belongs to the interior of conv}\{W(j) : j = 1, \ldots, R\}\} \geq 1 - 2 \exp(-n).
\]

**Proof.** Definition of constants and the matrix \( A. \) Let \( \tau, \delta > 0 \) be taken from Lemma 10 and \( \tilde{C} \) — from Lemma 11. Now, we define \( K := 2\sqrt{\tilde{C}} \) and let \( L \) and \( \eta \) be taken from Theorem 7. Finally, we define \( C > 0 \) as the smallest positive number satisfying

\[
\exp(Cn) \geq (28N)^4 \left[ \frac{4}{\eta^2} + 1 \right]^N
\]

for any \( \eta \in \mathbb{N} \) and \( N = n[\max(L, 4/\delta^2)] \).

Fix any numbers \( n > 0 \) and \( R \geq \exp(Cn) \), and let \( N := n[\max(L, 4/\delta^2)] \). Further, let \( t_i \) \((i = 0, 1, \ldots, N)\) be numbers from \( \{0, 1, \ldots, R\} \), with \( t_0 = 0, t_1 = (28N)^4 \) and \( t_i = \left[ \frac{4}{\eta^2} + 1 \right]t_{i-1}, i = 2, 3, \ldots, N \). Denote

\[
X_i := \sqrt{n}(t_i - t_{i-1})^{-1/2}(W(t_i) - W(t_{i-1})), \quad i = 1, 2, \ldots, N.
\]

Then the vectors are isotropic, jointly independent and, in view of Lemma 10, satisfy

\[
\mathbb{P}\{|\langle X_i, y\rangle| \geq t\} \leq \exp(-2(nt_i - nt_{i-1})^{1/4}) + 2 \exp(-t^2/4), \quad t > 0
\]

for all \( y \in S^{n-1} \). We let \( A \) to be the \( N \times n \) random matrix with rows \( X_i \).

**Estimate the norm of \( A. \)** Let \( \mathcal{N} \) be a 1/2-net on \( S^{n-1} \) of cardinality at most \( 5^n \). Fix any \( y' \in \mathcal{N}. \) For each \( i = 1, 2, \ldots, N \), let \( \xi_i := \langle X_i, y'\rangle^2 \), and let \( \tilde{\xi}_i \) be its truncation at level \( (nt_i - nt_{i-1})^{1/4} \); i.e.

\[
\tilde{\xi}_i(\omega) = \begin{cases} 
\xi_i(\omega), & \text{if } \xi_i(\omega) \leq (nt_i - nt_{i-1})^{1/4}, \\
0, & \text{otherwise}.
\end{cases}
\]

Note that, in view of (16), the variables \( \tilde{\xi}_i \) satisfy (14), and

\[
\mathbb{P}\{\xi_i \neq \tilde{\xi}_i\} \leq 3 \exp(-(nt_i - nt_{i-1})^{1/4}/4).
\]

Hence, by (15) and the above estimate, we have

\[
\mathbb{P}\{\|Ay'\| \geq \sqrt{CN}\} = \mathbb{P}\left\{\sum_{i=1}^{N} \xi_i \geq \tilde{C}N\right\}
\]

\[
\leq 40^{-n} + \mathbb{P}\{\xi_i \neq \tilde{\xi}_i \text{ for some } i \in \{1, 2, \ldots, N\}\}
\]

\[
\leq 40^{-n} + 3 \sum_{i=1}^{N} \exp(-(nt_i - nt_{i-1})^{1/4}/4)
\]

\[
\leq 40^{-n} + 3N \exp(-7Nn^{1/4})
\]

\[
\leq 20^{-n}.
\]
Taking the union bound for all $y' \in \mathcal{N}$ and applying the standard approximation argument, we obtain $\|A\| \leq 2\sqrt{CN} = K\sqrt{N}$ with probability at least $1 - \exp(-n)$.

**Construction of the matrix $F$ and application of Theorem 7.** Let $F$ be the $N \times N$ non-random lower-triangular matrix, with the entries

$$ f_{ij} = \sqrt{\frac{t_j - t_{j-1}}{t_i - t_{i-1}}}, \quad i \geq j. $$

Obviously, $FA$ is the matrix whose $i$-th row ($i = 1, \ldots, N$) is precisely the vector

$$ \sqrt{\frac{n}{t_i - t_{i-1}}} W(t_i). $$

Then, in view of the definition of $t_i$'s, we have

$$ \|F - I\| \leq \frac{\eta/2}{1 - \eta/2} \leq \eta. $$

Finally, applying Theorem 7, we obtain

$$ \mathbb{P}\{0 \text{ belongs to the interior of } \text{conv}\{W(j) : j = 1, 2, \ldots, R\}\} $$

$$ \geq \mathbb{P}\{0 \text{ belongs to the interior of } \text{conv}\{W(t_i) : i = 1, 2, \ldots, N\}\} $$

$$ = \mathbb{P}\{\text{rank}A = n \text{ and } \text{Im}(FA) \cap \mathbb{R}_+^n = \{0\}\} $$

$$ \geq 1 - 2 \exp(-n). $$

\[\square\]

5 Random walks on the sphere

Let $n > 1$ and $\theta \in (0, \pi/2)$. Here, we consider the Markov chain $W_\theta$ taking values on $S^{n-1}$ such that the angle between two consecutive steps is $\theta$ i.e. for any $i \geq 1$ we have $\langle W_\theta(i), W_\theta(i+1) \rangle = \cos \theta$ a.s., and the direction from $W_\theta(i)$ to $W_\theta(i+1)$ is chosen uniformly at random. The latter condition means that for any $u \in S^{n-1}$, the distribution of $W_\theta(i+1)$ conditioned on $W_\theta(i) = u$, is uniform on the $(n-2)$-sphere $S^{n-1} \cap \{x \in \mathbb{R}^n : \langle x, u \rangle = \cos \theta\}$.

See [19] for a study of these walks and some of their generalizations.

The question addressed in this section is how many steps it takes for $W_\theta$ to get the origin in its convex hull. Note that the answer does not depend on the distribution of the first vector $W_\theta(1)$, and we shall further assume that $W_\theta(1)$ is uniformly distributed on the sphere. The question can be equivalently reformulated as a problem of estimating $\pi/2$-covering time of $W_\theta$. For $\phi \in (0, \pi/2]$, a $\phi$-covering of $S^{n-1}$ is any subset $S$ of the sphere such that the geodesic distance from any point of the sphere to $S$ is at most $\phi$. Then the $\phi$-covering time for $W_\theta$ is the random variable

$$ T = \min\{N : \text{the set } \{W_\theta(i), i \leq N\} \text{ is a } \phi\text{-covering of } S^{n-1}\}. $$

A related problem of estimating $\phi$-covering time of the spherical Brownian motion was considered in [12] and [4], for $\phi \to 0$ and $\phi = \pi/2$, respectively. It is not clear whether the
argument developed in [4] can be adopted to the walks \( W_\theta \). Our approach to the above problem is based on the results of Section 3 and is completely different from the argument in [4].

The walk \( W_\theta \) can be constructively described as follows: Let \( Y_1, Y_2, \ldots \) be a sequence of independent standard Gaussian vectors in \( \mathbb{R}^n \). Let \( \beta_1 := \|Y_1\| \) and define

\[
W_\theta(1) := \frac{Y_1}{\|Y_1\|} = \frac{Y_1}{\beta_1}.
\]

Further, for any \( i \geq 1 \) let

\[
W_\theta(i + 1) := \frac{\alpha_{i+1}W_\theta(i) + Y_{i+1}}{\beta_{i+1}},
\]

where

\[
\beta_{i+1} := \|\alpha_{i+1}W_\theta(i) + Y_{i+1}\| \quad \text{and} \quad \alpha_{i+1} := \cot \theta \|P_iY_{i+1}\| - \langle Y_{i+1}, W_\theta(i) \rangle, \quad i \geq 1,
\]

with \( P_i \) denoting the (random) orthogonal projection onto the hyperplane orthogonal to \( W_\theta(i) \). It can be easily checked that

\[
\beta_i = \frac{\|P_{i-1}Y_i\|}{\sin \theta}, \quad i \geq 2,
\]

and that \( W_\theta \) is the Markov process described at the beginning of the section. For any \( i = 2, 3, \ldots \) the coefficients \( \alpha_i \) and \( \beta_i \) are random variables depending on \( Y_i \) and \( W_\theta(i-1) \).

Using (1) and (2), one can deduce the following concentration inequalities:

Lemma 13. There exist a universal constant \( c > 0 \) such that for \( \delta_\theta := c \min(1, \cot \theta) \) and for any \( i = 2, 3, \ldots \) and \( \varepsilon > 0 \) we have

\[
\mathbb{P}\{(1 - \varepsilon)\sqrt{n} \cot \theta \leq \alpha_i \leq (1 + \varepsilon)\sqrt{n} \cot \theta\} \geq 1 - 2 \exp(-\delta_\theta^2 \varepsilon^2 n)
\]

and

\[
\mathbb{P}\{(1 - \varepsilon)\sin \theta/\sqrt{n} \leq \beta_i^{-1} \leq (1 + \varepsilon)\sin \theta/\sqrt{n}\} \geq 1 - 2 \exp(-\delta_\theta^2 \varepsilon^2 n).
\]

Moreover, (2) immediately implies

\[
\mathbb{P}\{(1 - \varepsilon)/\sqrt{n} \leq \beta_i^{-1} \leq (1 + \varepsilon)/\sqrt{n}\} \geq 1 - 2 \exp(-c \varepsilon^2 n), \quad \varepsilon > 0,
\]

provided that the constant \( c \) is sufficiently small. Before we state the main result of the section, let us consider the following elementary lemma:

Lemma 14. For any \( q \in (0, 1) \) and \( 0 < \varepsilon \leq \frac{1-q}{8} \) we have

\[
\sum_{k=0}^{\infty}((1 + \varepsilon)^{2k+1} - 1)q^k \leq \frac{4\varepsilon}{(1 - q)^2}.
\]
Proof. First, note that the conditions on $\varepsilon$ and $q$ imply

$$q(1 + \varepsilon)^2 \leq \frac{81q}{64} - \frac{9q^2}{32} + \frac{q^3}{64} \leq q + \frac{17q^2}{64} \leq \frac{1 + q}{2},$$

whence

$$1 - q(1 + \varepsilon)^2 \geq \frac{1 - q}{2}.$$  

Using the last inequality, we obtain

$$\sum_{k=0}^{\infty} ((1 + \varepsilon)^{2k+1} - 1)q^k = (1 + \varepsilon) \sum_{k=0}^{\infty} (q(1 + \varepsilon)^2)^k - \sum_{k=0}^{\infty} q^k = \frac{(1 + \varepsilon)}{1 - q(1 + \varepsilon)^2} - \frac{1}{1 - q} = \frac{\varepsilon + \varepsilon q + \varepsilon^2 q}{(1 - q)(1 - q(1 + \varepsilon)^2)} \leq \frac{4\varepsilon}{(1 - q)^2}.$$

\[\square\]

**Theorem 15.** For any $\theta \in (0, \pi/2)$ there exist $n_0 = n_0(\theta)$ and $K = K(\theta)$ depending only on $\theta$ such that the following holds: Let $n \geq n_0$ and let $W_\theta$ be the random walk on $S^{n-1}$ defined above. Then for all $N \geq Kn$ we have

$$\mathbb{P}\{0 \text{ belongs to } \operatorname{conv}\{W_\theta(i) : i \leq N\}\} \geq 1 - \exp(-n).$$

**Proof.** Fix an angle $\theta \in (0, \pi/2)$. Let $\gamma := \frac{\sin \theta (1 - \cos \theta)}{1 + \cos \theta}$ and let $\eta, L$ and $\kappa$ be as in Theorem 4. Define $\varepsilon := \eta \sin \theta (1 - \cos \theta)^2/4$ and let $n_0$ be the smallest integer such that for all $n \geq n_0$ we have

$$5.5 \exp(-\kappa[L_n]) + 4[L_n] \exp(-\delta_\theta^2 \varepsilon^2 n) \leq \exp(-\mu n),$$

where $\mu = \frac{1}{2} \min(\kappa, \delta_\theta^2 \varepsilon^2)$ and $\delta_\theta$ is taken from Lemma 13.

Fix $n \geq n_0$. First, we show that $\tilde{N} := [Ln]$ steps is sufficient to get the origin in the convex hull of $W_\theta(i)$ ($i \leq \tilde{N}$) with probability $1 - \exp(-\mu n)$. This shall be done by using the representation (17) for the walk $W_\theta$ and by applying Theorem 4. Then we will augment the probability estimate to $1 - \exp(-n)$ by increasing the number of steps.

Let $G$ be the standard $\tilde{N} \times n$ Gaussian matrix with rows $Y_i$ ($i \leq \tilde{N}$). We shall construct a random lower-triangular $\tilde{N} \times \tilde{N}$ matrix $F$ such that the $i$-th row of $FG$ is $W_\theta(i)$. Define $F := (f_{ij})$ with

$$f_{ij} := \frac{\prod_{k=j+1}^{\tilde{N}} \alpha_k}{\prod_{k=j}^{\tilde{N}} \beta_k} \text{ for } j < i \leq \tilde{N} \quad \text{and} \quad f_{ii} := \frac{1}{\beta_i} \text{ for } i \leq \tilde{N},$$

16
where $\alpha_k$ and $\beta_k$ are given by (13). Since $FG = (W_\theta(1), W_\theta(2), \ldots, W_\theta(\bar{N}))^t$, the origin does not belong to $\text{conv}\{W_\theta(i) : i \leq \bar{N}\}$ only if there exists $y \in S^{n-1}$ such that $FGy \in \mathbb{R}_+^\bar{N}$.

Now define $\tilde{F}$ as the $\bar{N} \times \bar{N}$ lower triangular matrix whose entries are given by

$$
\tilde{f}_{i1} = \frac{(\cos \theta)^{-1}}{\sqrt{n}} \text{ for any } i \leq \bar{N} \text{ and } \tilde{f}_{ij} := \sin \theta \frac{(\cos \theta)^{i-j}}{\sqrt{n}} \text{ for } 2 \leq j \leq i.
$$

It is not difficult to see that

$$
\frac{\sin \theta}{\sqrt{n}} \leq \|\tilde{F}\| \leq \frac{1}{(1 - \cos \theta)\sqrt{n}}. \tag{20}
$$

Further, let $Q$ be the matrix obtained from $\tilde{F}$ by multiplying the first column of $\tilde{F}$ by $\sin \theta$ and leaving the other columns unchanged. Then, clearly, $s_{\min}(Q) \leq s_{\min}(\tilde{F})$ implying $\|\tilde{F}^{-1}\| \leq \|Q^{-1}\|$. On the other hand, the inverse of $Q$ is a lower bidiagonal matrix with $\frac{\sqrt{n}}{\sin \theta}$ on the main diagonal and $-\cos \theta \frac{\sqrt{n}}{\sin \theta}$ on the diagonal below. Hence, $\|\tilde{F}^{-1}\| \leq \|Q^{-1}\| \leq (1 + \cos \theta)\frac{\sqrt{n}}{\sin \theta}$, and the condition number of $\tilde{F}$ satisfies

$$
\|\tilde{F}\| \cdot \|\tilde{F}^{-1}\| \leq \frac{1 + \cos \theta}{\sin \theta (1 - \cos \theta)} = \gamma^{-1}.
$$

Applying Theorem 4 we get

$$
\mathbb{P}\{\exists y \in S^{n-1}, \ FGy \in \mathbb{R}_+^\bar{N}\} \leq 5.5 \exp(-\kappa \bar{N}) + \mathbb{P}\{\|F - \tilde{F}\| > \eta \|\tilde{F}\|\}.
$$

It remains to bound the probability $\mathbb{P}\{\|F - \tilde{F}\| > \eta \|\tilde{F}\|\}$. In view of Lemma 13 and (19), with probability at least $1 - 4\bar{N} \exp(-\delta_\theta^2 \varepsilon^2 n)$ we have

$$
|f_{ij} - \tilde{f}_{ij}| \leq ((1 + \varepsilon)^{2(i-j)+1} - 1) \tilde{f}_{ij} \text{ for any } j \leq i.
$$

This, together with Lemma 14 and (20), implies that

$$
\|F - \tilde{F}\| \leq \frac{1}{\sqrt{n}} \sum_{k=0}^{\infty} \left( (1 + \varepsilon)^{2k+1} - 1 \right) (\cos \theta)^k \leq \frac{4\varepsilon}{(1 - \cos \theta)^2 \sqrt{n}} \leq \eta \|\tilde{F}\|
$$

with probability at least $1 - 4\bar{N} \exp(-\delta_\theta^2 \varepsilon^2 n)$. Hence, by the restriction on $n_0$,

$$
\mathbb{P}\{\exists y \in S^{n-1}, \ FGy \in \mathbb{R}_+^\bar{N}\} \leq 5.5 \exp(-\kappa \bar{N}) + 4\bar{N} \exp(-\delta_\theta^2 \varepsilon^2 n) \leq \exp(-\mu n),
$$

where $\mu = \frac{1}{2} \min(\kappa, \delta_\theta^2 \varepsilon^2)$. Finally, if $N \geq \lceil \mu^{-1} \rceil \bar{N}$ then the above estimate implies

$$
\mathbb{P}\{0 \text{ does not belong to } \text{conv}\{W_\theta(i) : i \leq N\}\}
\leq \mathbb{P}\{0 \text{ does not belong to } \text{conv}\{W_\theta(i) : i \leq \bar{N}\}\}^{\lceil \mu^{-1} \rceil}
\leq \exp(-n).
$$

\[\Box\]

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References

[1] D. Chafaï, O. Guédon, G. Lecure, and A. Pajor. *Interactions between compressed sensing random matrices and high dimensional geometry*, volume 37 of *Panoramas et Synthèses [Panoramas and Syntheses]*. Société Mathématique de France, Paris, 2012.

[2] E. J. Candès. Mathematics of sparsity (and a few other things). *Proceedings of the International Congress of Mathematicians*, Seoul, South Korea, 2014.

[3] V. Chandrasekaran, B. Recht, P. A. Parrilo, and A. S. Willsky. The convex geometry of linear inverse problems. *Found. Comput. Math.*, 12(6):805–849, 2012.

[4] R. Eldan. Extremal points of high-dimensional random walks and mixing times of a Brownian motion on the sphere. *Ann. Inst. Henri Poincaré Probab. Stat.*, 50(1):95–110, 2014.

[5] R. Eldan. Volumetric properties of the convex hull of an $n$-dimensional Brownian motion. *Electron. J. Probab.*, 19:no. 45, 34, 2014.

[6] Feller, W., An Introduction to Probability Theory and Its Applications, Vol. 1, 3rd Edition, Wiley, 1968.

[7] Y. Gordon. On Milman’s inequality and random subspaces which escape through a mesh in $\mathbb{R}^n$. In *Geometric aspects of functional analysis (1986/87)*, volume 1317 of *Lecture Notes in Math.*, pages 84–106. Springer, Berlin, 1988.

[8] W. Hoeffding. Probability inequalities for sums of bounded random variables. *J. Amer. Statist. Assoc.*, 58:13–30, 1963.

[9] Z. Kabluchko, D. Zaporozhets Intrinsic volumes of Sobolev balls with applications to Brownian convex hulls. Available at arXiv:1404.6113.

[10] M. Ledoux and M. Talagrand. *Probability in Banach spaces*, volume 23 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1991. Isoperimetry and processes.

[11] A. E. Litvak, A. Pajor, and N. Tomczak-Jaegermann. Diameters of sections and coverings of convex bodies. *J. Funct. Anal.*, 231(2):438–457, 2006.

[12] P. Matthews. Covering problems for Brownian motion on spheres. *Ann. Probab.*, 16(1):189–199, 1988.

[13] S. Mendelson. A Remark on the Diameter of Random Sections of Convex Bodies, Geometric Aspects of Functional Analysis, Lecture Notes in Mathematics Volume 2116, 2014, 395–404.
[14] V. D. Milman. Random subspaces of proportional dimension of finite-dimensional normed spaces: approach through the isoperimetric inequality. Banach spaces (Columbia, Mo., 1984), 106–115, Lecture Notes in Math., 1166, Springer, Berlin, 1985.

[15] V. D. Milman, G. Schechtman, Asymptotic theory of finite-dimensional normed spaces, Lecture Notes in Math., vol. 1200, Springer-Verlag, Berlin, 1986.

[16] P. Mörters and Y. Peres. Brownian motion. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, Cambridge, 2010. With an appendix by Oded Schramm and Wendelin Werner.

[17] A. Pajor and N. Tomczak-Jaegermann. Subspaces of small codimension of finite-dimensional Banach spaces. Proc. Amer. Math. Soc., 97(4):637–642, 1986.

[18] G. Pisier. The volume of convex bodies and Banach space geometry, volume 94 of Cambridge Tracts in Mathematics. Cambridge University Press, Cambridge, 1989.

[19] P. H. Roberts and H. D. Ursell. Random walk on a sphere and on a Riemannian manifold. Philos. Trans. Roy. Soc. London. Ser. A, 252:317–356, 1960.

[20] R. Vershynin. Estimation in high dimensions: a geometric perspective. Available at arXiv:1405.5103.

[21] R. Vershynin. Introduction to the non-asymptotic analysis of random matrices. In: Compressed Sensing: Theory and Applications, Yonina Eldar and Gitta Kutyniok (eds), 210–268, Cambridge University Press, 2012.