On some open problems concerning perfect powers

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Abstract: The starting point of our paper is Kashihara’s open problem number 30, concerning the sequence A001292 of the OEIS, asking how many terms are powers of integers. We confirm his last conjecture up to the 100128-th term and provide a general theorem that rules out $4/9$ of the candidates. Moreover, we formulate a new, provocative, conjecture involving the OEIS sequence A352991 (which includes all the terms of A001292). Our risky conjecture states that all the perfect powers belonging to the sequence A352991 are perfect squares and they cannot be written as higher order perfect powers if the given term of A352991 is not equal to one. This challenging conjecture has been checked for any integer smaller than $10111121314151617181920212223456789$ and no counterexample has been found so far.

Keywords: Open problem, Perfect power, Perfect square, Conjecture, Integer sequence.

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1 Introduction

In late 2010, the author of this paper found a recreative open problem by Kenichiro Kashihara (see [1], open problem number 30, p. 25) concerning the sequence A001292 of the On-Line Encyclopedia of Integer Sequences (OEIS) [2]. Kashihara’s problem number 30 consists of two independent parts and the author solved the first one quite easily at the time (the complete solution can be found in [4], Section 3.3, pp. 12–15) since it asks to find the probability $0 < p(c) < 1$ that the trailing digit of the generic term of the sequence A001292 is $c \in \{0, 1, 2, \ldots, 9\}$ and the formula provided in [4] shows that $p(c) = \frac{11-c}{55}$ for any $c \neq 0$, whereas $p(0) = 0.018$ (e.g., if $c = 7$, then $p(7) = \frac{1}{55} = 0.0182$).

In the present paper, we will focus on the second part of the above-mentioned Kashihara’s problem number 30, asking how many elements of the sequence A001292 are perfect powers since Kashihara conjectured that there are none.

Now, bearing in mind that a perfect power of an integer $d > 0$ is a natural number $k \geq 2$ such that $a^k = d$, where also a is a positive integer, we could point out that A001292(1) = 1 can be considered as a solution and argue how this disproves the conjecture, but (from here on) we will disregard this special case and assume that we are looking for a nontrivial counterexample to Kashihara’s conjecture.
Lastly, Section 3 is devoted to introducing a (quite improbable) conjecture concerning perfect powers belonging to the OEIS sequence A352991 \footnote{[5,6]}.

2 The exclusion criterion

To be clear on the invoked OEIS sequences, let us introduce a few useful definitions.

**Definition 2.1.** We define the \( m \)-th term of the OEIS sequence A007908 as
\[
A_{007908}(m) = 1 \cdot 2 \cdot 3 \cdots (m - 1) \cdot m, \quad m \in \mathbb{Z}^+.
\]

**Definition 2.2.** We define the sequence A001292 of the OEIS as the concatenations (sorted in ascending order) of every cyclic permutation of the elements of the sequence A007908 (e.g., given \( m = 3 \), \( A_{001292}(A_{007908}(3)) = 123, 231, 312 \)).

**Definition 2.3.** We define the OEIS sequence A352991 as the concatenation of all the distinct permutations of the first strictly positive \( m \) integers, sorted in ascending order (e.g., given \( m = 3 \), \( A_{352991}(m) = 12345671089 \), while \( 12345670189 \) does not belong to A352991, even if all the digits of the string 1, 2, 3, \ldots, 9, 10 appear once and only once, since “10” is missed).

After having checked the first 100128 terms of the sequence A001292 (see Appendix), exploring any exponent at or above two, we have not found any perfect power so that Kashihara’s conjecture has been verified up to \( 10^{1235} \) (i.e., the 100129-th term of A007908 is the smallest cyclic permutation of \( A_{007908}(448) \) and is greater than \( 10^{1235} \) by construction). Moreover, we can prove the following Theorem 2.1, concerning the sequence A352991 which includes every term of A001292.

**Theorem 2.1.** For any \( m > 1 \), \( A_{352991}(n) \) cannot be a perfect power of an integer if \( A_{352991}(n) \) is a permutation of \( A_{007908}(m) \) and \( m : m \equiv \{2, 3, 5, 6\} \pmod{9} \).

**Proof.** By definition, \( A_{007908}(m) \) \footnote{[3]} cannot be a perfect power if \( 1 \cdot 2 \cdot 3 \cdots (m - 1) \cdot m \) is divisible by 3 and it is not divisible by \( 3^2 \). Thus, from the well-known divisibility by 3 and 9 criteria, \( m : (3 \mid \sum_{j=1}^{m} j) \wedge (3^2 \nmid \sum_{j=1}^{m} j) \) is a sufficient, but not necessary, condition for letting us disregard any permutation of \( 1 \cdot 2 \cdot 3 \cdots (m - 1) \cdot m \) (i.e., given \( m \), if a generic permutation of \( A_{007908}(m) \) is divisible by 3 and is not congruent to 0 \pmod{9}, then all the permutations of \( A_{007908}(m) \) are divisible by 3 once and only once, since the commutativity property holds for addition).

It follows that, for any \( n \in \mathbb{Z}^+ \), \( A_{352991}(n) \) cannot be a perfect power if it is a permutation of the string 1, 2, 3, \ldots, (m - 1), m, where m is such that \( A_{134804}(m) \) is divisible by 3. Therefore, the residue modulo 9 of every perfect power belonging to A352991 cannot be 2 or 3 or 5 or 6, and this concludes the proof of Theorem 2.1.

**Corollary 2.1.** Kashihara’s conjecture is true for the concatenation of any cyclic permutation of \( A_{007908}(m) \), where \( m : m \equiv \{2, 3, 5, 6\} \pmod{9} \) \lor \( m < 448 \).
Proof. We observe that \( A_{001292} \) is a subsequence of \( A_{352991} \). By invoking Theorem \ref{t1} we can state that every perfect power candidate has to be the concatenation of a (cyclic) permutation of \( A_{007908}(m) \), where \( m \) is such that \( m : m \equiv \{2, 3, 5, 6\} \pmod{9} \). On the other hand, all the remaining terms up to \( 99_{100}_{101} \cdots 445_{446}_{447}_{1}_{2}_{3} \cdots 96_{97}_{98} \) have been directly checked (see Appendix for details) and no perfect power has been found.

Therefore, Corollary \ref{c1} confirms Kashihara’s conjecture for any term of \( A_{001292} \) such that \( m \) is congruent to \( \{2, 3, 5, 6\} \pmod{9} \) or \( m \leq 447 \).

\begin{corollary}
\label{c2}
\forall n : \ A_{353025}(n) \equiv \{2, 3, 4, 5, 6, 7, 8\} \pmod{9}, \text{ and any term of } A_{001292} \text{ cannot be a perfect power if its digital root is not equal to 0 or 1.}
\end{corollary}

\begin{proof}
\begin{align}
0 \pmod{9} & \quad \text{if } m : m \equiv 0 \pmod{9} \\
1 \pmod{9} & \quad \text{if } m : m \equiv 1 \pmod{9} \\
1 \pmod{9} & \quad \text{if } m : m \equiv 4 \pmod{9} \\
1 \pmod{9} & \quad \text{if } m : m \equiv 7 \pmod{9} \\
0 \pmod{9} & \quad \text{if } m : m \equiv 8 \pmod{9}
\end{align}
\end{proof}

\begin{remark}
A well-known property of integers is that every perfect power that is congruent modulo 5 to 0 is also necessarily congruent to \( \{0, 25, 75\} \pmod{100} \), while if a perfect power is congruent modulo 10 to 6, then its second last digit is odd.

Thus, we are free to combine these additional constraints with Corollary \ref{c2} in order to reduce the number of perfect power candidates among the terms of \( A_{352991} \).
\end{remark}

\section{Perfect cubes in \( A_{353025} \)}

In the first half of April 2022, playing with Kashihara’s conjecture, a more risky (very likely false but hard to disprove by brute force) conjecture arose, it is as follows.

\begin{conjecture}
Let \( n \in \mathbb{N} - \{0, 1\} \) be given. We (provocatively) conjecture that if \( n \) is such that \( A_{352991}(n) \) is a perfect power of an integer, then \( \not\exists k \in \mathbb{N} - \{0, 1, 2\} : A_{352991}(n) = c^k, \ c \in \mathbb{N} \).
\end{conjecture}

On April 16, 2022, a direct search was performed by the author on the first \( 10^7 \) terms of the sequence and no counterexample has been found (42 perfect squares only). A few days later, Aldo Roberto Pessolano, performed a deeper search running the Mathematica codes published in
the Appendix, without finding any counterexample and thus confirming Conjecture 3.1 (at least) up to the smallest permutation of $A_{007908}(22)$ (i.e., for any term of $A_{352991}$ which is strictly greater than 1 and smaller than $1011121314151617181920212223456789$).

**Remark 3.1.** If confirmed, Conjecture 3.1 would imply that all the perfect powers (strictly greater than 1) in $A_{352991}$ are perfect squares and nothing more (no cube, no square of square, and so forth). Nevertheless, under the (arbitrary, but perfectly reasonable) assumption of a standard probability distribution of the cubes in $A_{352991}$ (i.e., we are assuming that $|n \in \mathbb{N} : A_{352991}(n) \leq m \wedge A_{352991}(n) \neq m| \approx |n \in \mathbb{N} : A_{000578}(n) \leq m|$ holds for any sufficiently large $m \in \mathbb{N}$), we would guess the existence of infinitely many counterexamples to Conjecture 3.1, even if the smallest one is expected to occur in the interval $[10^{58}, 10^{65}]$. On the other hand, the same argument would corroborate Kashihara’s conjecture, since the number of perfect powers belonging to $A_{001292}$ cannot probabilistically exceed

$$2 \cdot \sum_{k=2}^{+\infty} \left( \sum_{j=309}^{+\infty} \frac{4 \cdot j \cdot (10^{\frac{j}{k}} - 10^{\frac{j}{k} - 1})}{9 \cdot 10^{(4j-1)}} \right) \approx \frac{8}{9} \sum_{j=309}^{+\infty} j \cdot (10^{\frac{j}{k}} - 10^{\frac{j}{k} - 1}) \approx 0. \quad (2)$$

**Additional open problems.** Does the sequence $A_{353025}$ have infinitely many perfect squares, infinitely many perfect cubes, infinitely many perfect squares of squares, and so forth? Which is the smallest nontrivial perfect cube (if any) belonging to $A_{353025}$ (we point out that all the terms greater than one and below $(1.01 \cdot 10)^{40}$ have been checked without finding any cube)?

### 4 Conclusion

Kashihara’s open problem number 30 has not been completely solved yet. Even if the first part, concerning the probability that the trailing digit of $A_{001292}(n)$ is $c = 1, 2, \ldots, 9$, was solved by the author a dozen years ago [4], the conjecture in the second part still need a proof or a nontrivial counterexample (the smallest candidate has 1236 digits). Moreover, in the present paper, we have introduced a wider speculation that allows us to ask ourselves how to find a term of the OEIS sequence $A_{353025}$ (disregarding $A_{353025}(1)$) which is not a perfect square; a challenging open problem, considering that there is not any perfect cube among the terms of $A_{352991}$ in the interval $(1, 10^{40})$.

### 5 Appendix

Aldo Roberto Pessolano helped the author of the present paper by verifying Kashihara’s conjecture and Conjecture 3.1 for a very large number of terms. All the provided Mathematica codes run on the M1 processor of his Apple MacBook Air (2020). Kashihara’s conjecture has been currently tested up to the 100128-th term of $A_{001292}$ and we confirm that it holds for every element of the set $A_{001292}(2), A_{001292}(3), \ldots, A_{001292}(100128)$.

The search reached the term $99_{100} \ldots 446_{447}_{1}2_{\ldots 97\_98} \approx 9.91 \cdot 10^{1232}$ in 28823 seconds (about 8 hours of calculations) and the code is as follows:
c = True;
p = Table[Prime[q], {q, 1, 565}];
Do[rn = Range[k];
    n = ToExpression[StringJoin[ToString[#] & /@ rn]];
    If[And[Mod[n, 9] != 3, Mod[n, 9] != 6],
        Do[r = RotateLeft[rn, i - 1];
            nk = ToExpression[StringJoin[ToString[#] & /@ r]];
            If[IntegerQ[nk^((1/#))],
                Print[nk, " = ", nk^((1/#)), "^", #]; c = False; Break[]]
        ] &/@ p,
        {i, 1, k}]
    ];
    If[c, Print["1..", k, " checked."], Break[]],
    {k, 2, 447}]

About our investigation on the perfect powers in A352991, Pessolano has recently completed the direct check of every term of A352991 which falls in the interval (1,98765432120191817161514131211110] (see the code below).

As expected, the test has not returned any perfect power above two.

z = False;
h = 3;
p = Table[Prime[q], {q, 2, 10}];
q[x_, k_, d_, m_] := (y = x^k;
    If[DigitCount[y] == d, c = True;
        Do[If[Not[StringContainsQ[ToString[x], ToString[i]]], c = False;
            Break[], c = True], {i, 10, m}], c = False]
    Return[c])
Do[r = Range[k];
    n = ToExpression[StringJoin[ToString[#] & /@ r]];
    If[And[Mod[n, 9] != 3, Mod[n, 9] != 6],
        d = DigitCount[n];
        f = IntegerPart[(10^(IntegerLength[n]) - 1)^(1/#)];
        Do[If[q[x, #, d, k], Print[x, "^", #, " = ", y];
            z = True; Break[]], {x, s, f}] &/@ p;
        g = 2^h;
        While[g < n, If[q[#, h, d, k],
            Print[x, "^", h, " = ", y];
            z = True;
            Break[]], &/@ {2, 3, 5, 6, 7};
        h++; g = 2^h];
    If[z, Break[]];
    Print["1..", k, " checked."],
    {k, 2, 21}]
On the other hand, the following code returns the complete list of the smallest 42 perfect squares belonging to $A_{352991}$.

```plaintext
z = 1;
Do[r = Range[k];
  n = ToExpression[StringJoin[ToString[#] & /@ r]];
  If[And[Mod[n, 9] != 3, Mod[n, 9] != 6],
    d = DigitCount[n];
    s = IntegerPart[Sqrt[10^(IntegerLength[n] - 1)]];
    f = IntegerPart[Sqrt[10^(IntegerLength[n])]];
    Do[y = x^2;
      If[DigitCount[y] == d,
        c = True;
        Do[
          If[Not[StringContainsQ[ToString[y], ToString[i]]],
            c = False
          ],
          {i, 10, k}]
        If[c, Print[z, " ", y]; z++]
      ],
      {x, s, f}]
    ],
    {k, 2, 10}]
```

These 42 perfect squares correspond to all the perfect powers in $(1, 10^{16}]$ belonging to $A_{352991}$, while the next perfect square is $1013568174231129$ (we observe that $100676123^2$ is a permutation of $123\ldots16$, as suggested by the statement of Theorem 2.1).

1 13527684
2 34857216
3 65318724
4 73256481
5 81432576
6 139854276
7 152843769
8 157326849
9 215384976
10 245893761
11 254817369
12 326597184
13 361874529
In the end, our tests have finally confirmed that all the perfect powers that are smaller than $10^{34}$ and that belong to the OEIS sequence A352991 are perfect squares (only). At present, Conjecture 3.1 has been tested for every integer smaller than $10^{1235}$ and no counterexample has been found yet.

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