Singularities in Axisymmetric Free Boundaries for ElectroHydroDynamic Equations

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Abstract

We consider singularities in the ElectroHydroDynamic equations. In a regime where we are allowed to neglect surface tension, and assuming that the free surface is given by an injective curve and that either the fluid velocity or the electric field satisfies a certain non-degeneracy condition, we prove that either the fluid region or the gas region is asymptotically a cusp. Our proofs depend on a combination of monotonicity formulas and a non-vanishing result by Caffarelli and Friedman. As a by-product of our analysis we also obtain a special solution with convex conical air-phase which we believe to be new.

1. Introduction

In his pioneering paper of 1964, [16], Sir Geoffrey Taylor describes an experiment for the formation of a liquid cone by exposing a fluid jet to an electric field, and he formally derives a formula for the electric potential (referred to as the Taylor-cone solution in the sequel) of that field. At a critical voltage value, the surface of the fluid ruptures, and a fluid jet forms, a phenomenon which has found applications as various as electrospraying, electrospinning and, to give more concrete examples, ink jet printers, mass spectrometers and the production of lab-on-the-chips. Despite these various applications and extensive research from a physical point of view, the phenomenon of the Taylor cone as well as other singularities in electro-hydrodynamics remain in effect untouched by mathematical analysis, possibly due to the difficulty of the free boundary problem arising from the particular ElectroHydroDynamic equations (EHD equations) used as a model.

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We use as a basis for our analysis the simplest model available. We consider a stationary, irrotational flow of an incompressible, inviscid, perfectly conducting liquid; outside the fluid there is a dielectric gas, and the stationary electric field is driven by the potential difference between the perfectly conducting liquid and some fixed outer domain boundary or infinity. Motivated by particular singularities on which gravity is supposed to have no influence (this point will be underlined by the heuristic argument below) we will neglect the influence of gravity. Since the fluid is a perfect conductor, the stationary electric potential $\phi$ is constant in the fluid region, and we may assume it to have the value 0 there. In this setting, the ElectroHydroDynamic equations simplify (see [13, (10)] as well as [22, Section 2]) to

$$\Delta \phi = 0 \text{ in the gas region,} \quad (1.1)$$
$$\Delta V = 0 \text{ in the fluid region,} \quad (1.2)$$
$$|\nabla \phi|^2 - |\nabla V|^2 = \kappa + B \text{ on the free surface of the fluid,} \quad (1.3)$$
$$\phi = 0 \text{ on the free surface of the fluid,} \quad (1.4)$$
$$\langle \nabla V, \nu \rangle = 0 \text{ on the free surface of the fluid,} \quad (1.5)$$

where $B$ is a constant, $V$ is the velocity potential of the stationary fluid, $\nu$ is the outward pointing unit normal and $\kappa$ the mean curvature on the boundary of the fluid phase. Note that we choose the sign of the mean curvature of the boundary of the set $A$ so that $\kappa$ is positive on convex portions of $\partial A$.

Viewed as a free boundary problem, problem (1.1)–(1.5) is new, so there are no results from that perspective. Possible reasons for the lack of results may be the “bad” sign of the mean curvature (explained in more detail below) as well as the Neumann boundary condition (1.5). While there are many results concerning free boundaries that are level sets, there are relatively few results on problems without this property.

There are other related free boundary/free discontinuity problems we should mention. For example, in [4], the authors study the free boundary problem

$$\Delta u = 0 \text{ in } \Omega \cap (\{u > 0\} \cup \{u < 0\}), \quad (1.6)$$
$$|\nabla u^+|^2 - |\nabla u^-|^2 = -\kappa \text{ on the free surface } \partial \{u \leq 0\} \cap \Omega, \quad (1.7)$$

where $\Omega$ is a smooth domain of $\mathbb{R}^n$. However, even in the case $u^- \equiv 0$, problem (1.6)–(1.7) differs from (1.1)–(1.5) by the sign of the mean curvature. This becomes clearer when comparing the energy functionals associated to the two problems: in the case of one-phase solutions ($u^- \equiv 0$) of (1.6)–(1.7), the energy takes the form

$$P_\Omega(\{u > 0\}) + \int_{\Omega \cap \{u > 0\}} |\nabla u|^2,$$

where $P_\Omega(\{u > 0\})$ is the perimeter of the set $\{u > 0\}$ relative to the domain $\Omega$, while in the case of one-phase solutions ($V \equiv 0$) of problem (1.1)–(1.5), where we extend $\phi$ by the value 0 to the fluid phase and we consider $B = 0$, the energy takes the form

$$-P_\Omega(\{\phi > 0\}) + \int_{\Omega \cap \{\phi > 0\}} |\nabla \phi|^2. \quad (1.8)$$