Verifying the upper bound on the speed of scrambling with the analogue Hawking radiation of trapped ions

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A general bound on the Lyapunov exponent of a quantum system is given by \( \lambda_L \leq \frac{2\pi T}{\hbar} \), where \( T \) is the system temperature, as established by Maldacena, Shenker, and Stanford (MSS). This upper bound is saturated when the system under consideration is the exact holographic dual of a black hole. It has also been shown that an inverted harmonic oscillator (IHO) may exhibit the behavior of thermal energy emission, in close analogy to the Hawking radiation emitted by black holes. We demonstrate that the Lyapunov exponent of the IHO indeed saturates the MSS bound, with an effective temperature equal to the analogue black hole radiation temperature, and propose using a trapped ion as a physical implementation of the IHO. We derive the corresponding out-of-time-ordered correlation function (OTOC) diagnosing quantum chaos, and theoretically show, for an experimentally realizable setup, that the effective temperature of the trapped-ion-IHO matches the upper MSS bound for the speed of scrambling.

The Hawking effect [1, 2], describing particle creation by quantum vacuum fluctuations in the presence of a black hole event horizon, can be mimicked in the laboratory, as argued by Unruh [3]. Since his seminal work, and with increasing experimental capabilities, a veritable explosion of interest occurred in what was dubbed analogue gravity [4], involving, for example, Bose-Einstein condensates [5–18] light in nonlinear media [19–21], magnets [22, 23], or superconducting circuits [24–28].

Gauge-gravity duality [29, 30] represents a powerful tool enabling the description of quantum gravity by certain classes of large-N gauge theories. Recently, it was conjectured that the Lyapunov exponent \( \lambda_L \) characterizing the rate of growth of chaos in thermal quantum many-body systems has a rigorous bound, \( \lambda_L \leq \frac{2\pi T}{\hbar} \) [31]. This bound is saturated when the quantum system is exactly holographic dual to a black hole [32–34], and thus may capture an essential feature of the gauge-gravity duality. Conversely, the bound might suggest that a quantum system with given \( \lambda_L \) has a minimal temperature \( T \geq \hbar \lambda_L / 2\pi \). Thus a system with \( \lambda_L \neq 0 \), although possessing zero temperature classically, may exhibit thermality in the semiclassical regime [35], that is \( \lambda_L \neq 0, T \neq 0 \), when \( \hbar \neq 0 \). This recalls a salient property of black holes: While being completely black classically, they semiclassically radiate at the Hawking temperature.

The quantum dynamics of the IHO exhibits thermal behavior with a temperature depending on the Lyapunov exponent \( \lambda_L \) [35–37]. The IHO model has, therefore, been used to explore the MSS bound on system temperature from a different angle, revealing its relation to analogue Hawking radiation [35], and to study the scattering outside a black hole horizon quantum mechanically [36, 37]. However, to experimentally verify whether the claimed duality between black holes and the fastest quantum scramblers indeed exists in nature, and how to extract the Lyapunov exponent within an experimentally accessible system are still open issues. To fill this gap, we propose below an implementation of the IHO with a trapped ion. Specifically, we propose to determine the Lyapunov exponent by measuring the IHO’s OTOC [31, 38]. To establish gauge-gravity duality, we show that IHO analogue Hawking temperature and Lyapunov exponent indeed fulfill \( \lambda_L = \frac{2\pi T}{\hbar} \).

Trapped ions, featuring a unique level of fidelity in preparation, control, and readout of quantum states, have been extensively used to simulate relativistic quantum physics, such as Zitterbewegung [39–41], the Klein paradox [42, 43], and (analogue) cosmological particle creation [44–47]. We demonstrate that our proposal furnishes an implementation of exact gauge-gravity duality within current experimental reach for trapped ions.

The one-dimensional IHO, with Hamiltonian \( (\alpha > 0) \)

\[
H = \frac{\hat{p}^2}{2m} - \frac{\alpha}{2} \hat{x}^2,
\]

(1)
yields the classical trajectory \( x(t) = c_1 e^{\sqrt{\alpha/ma} t} + c_2 e^{-\sqrt{\alpha/ma} t} \) (see discussion below for the relation of the IHO “particle” to the actual ion). This deterministic evolution is exponentially sensitive to the initial condition specified by \( c_1 \) and \( c_2 \), and the Lyapunov exponent...
\[ \lambda_L = \sqrt{\alpha/m}. \] In the quantum version of the IHO, the OTOC for momentum and position operators is given by:

\[ C(t) = -\langle [\hat{x}(t), \hat{p}(0)]^2 \rangle \sim \hbar^2 e^{2\lambda_L t}. \tag{2} \]

Applying the Lyapunov exponent bound [31], \( \lambda_L \leq 2\pi T/\hbar \), to the quantum IHO predicts the existence of a lower bound on temperature \( T \geq \hbar \lambda_L/2\pi \), as conjectured in [35]. For the classical IHO (\( \hbar \to 0 \)), which is both nonthermal and deterministic, this inequality becomes an equality as \( T = 0 \). Conversely, in the semiclassical regime, following [35], the classical Lyapunov exponent \( \lambda_L \) acquires a quantum correction \( O(\hbar) \), so that the right-hand side of the inequality becomes \( T \geq \hbar^2 \lambda_L/O(\hbar) = \hbar^2 \lambda_L + O(\hbar^2) \). This suggests that, at least on the semiclassical level, an effective temperature of \( O(\hbar) \) from the right may be reflected by the potential. Defining the light-cone-type operators \( \hat{\epsilon} \pm \) and \( \hat{\epsilon} \) and \( \hat{\epsilon} \), respectively. The in- and outgoing energy eigenstates are of the form

\[ \frac{1}{\sqrt{2\pi \hbar \lambda_L}} (\pm u^\pm) e^{\pi \hbar \lambda_L \over \lambda} \Theta(\pm u^\pm), \]

where \( \Theta(\pm u^\pm) \) is Heaviside step function, and \( \varepsilon \) is energy. Incoming and outgoing states are connected by a Mellin transform, which gives the scattering matrix (S matrix) in the form [36, 37]

\[ S = \frac{1}{\sqrt{2\pi}} \exp \left[ -\frac{i \varepsilon}{\hbar \lambda_L} \log \hbar \lambda_L \right] \left( \frac{1}{2} - \frac{i \varepsilon}{\hbar \lambda_L} \right) \left( e^{-i \frac{\pi \varepsilon}{\hbar \lambda_L}} e^{i \frac{\pi \varepsilon}{\hbar \lambda_L}} \right), \]

where \( \Gamma \) is the Gamma function. From the S matrix, the transmission and reflection probability are respectively given by, using \( \left| \Gamma \left( \frac{1}{2} - i \frac{\varepsilon}{\hbar \lambda_L} \right) \right|^2 = \pi \text{sech} \left( \frac{\pi \varepsilon}{\hbar \lambda_L} \right) \),

\[ |T(\varepsilon)|^2 = \frac{1}{1 + e^{2\pi \varepsilon \hbar \lambda_L}}, \quad |R(\varepsilon)|^2 = \frac{1}{1 + e^{-2\pi \varepsilon \hbar \lambda_L}} \tag{3} \]

corresponding to a thermal-mechanical, with a temperature given by \( T = \hbar \lambda_L/2\pi \).

As illustrated in Fig. 1, we can understand the transmission coefficient above by the quantum mechanical tunneling process through a barrier of the particle with negative energy \( \varepsilon \) moving towards the inverted harmonic potential from the left (\( x \to 0 \)), with a tunneling probability \( |T(\varepsilon)|^2 \). Similarly, the incoming positive energy particle (\( \varepsilon > 0 \)) from the right may be reflected by the incoming quantum mechanical tunnelling, with probability \( |R(\varepsilon)|^2 \).

![FIG. 1. (Color online) (a) Scattering off an effective potential approximately equal to an inverse harmonic potential, outside a black hole, where \( r_* \) is the “tortoise coordinate”, and the event horizon is located at \( r_* \to -\infty \). (b) Scattering via the inverse harmonic potential leads to classical trajectories of incoming particles (solid lines) and particle tunneling (broken lines) near the hyperbolic fixed point \((x, p) = (0, 0)\) in phase space. The dotted lines are the separatrices (\( \varepsilon = 0 \)), and \( \beta = 1/T \).](image)

A relativistic particle, when close to a black hole horizon (pulled by external scalar or electromagnetic forces so that it does not fall into the black hole), possesses an effective Lagrangian corresponding to the Hamiltonian (1) [48]. The maximal Lyapunov exponent of the particle’s motion is found to be the surface gravity of the black hole when the total potential presents unstable maximum [48]. The Lyapunov exponent is then given by \( \lambda_L = \kappa = \frac{1}{2} \sqrt{g(r_0)f'(r_0)} \), for the metric \( ds^2 = -f(r)dt^2 + dr^2/g(r) + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \). Here \( g(r_0) \) and \( f'(r_0) \) are derivatives with respect to \( r \) evaluated at the horizon \( r = r_0 \).

Numerous variants of (re-)deriving Hawking’s result exist [1, 2, 49–55], ranging, e.g., from Hawking calculating the Bogoliubov coefficients between the quantum scalar field modes of the “in” and “out” vacuum states [1, 2], to an open quantum system approach [55]. Here, we follow Purik and Wilczek [50], employing the tunnelling probability through a classically forbidden region.

Consider our particle tunneling through an effective black hole potential due to a general spherically symmetric metric \( ds^2 = -f(r)dt^2 + dr^2/g(r) + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \). On the semiclassical level under consideration here, we can employ the WKB approximation to calculate the tunnelling (transmission) probability through the horizon. To leading order in \( \hbar \), it is given by [56]

\[ \Gamma = \exp \left[ -\frac{4\pi}{\sqrt{g'(r_0)f'(r_0)}} \varepsilon \right] := \exp \left[ -\varepsilon/T \right]. \tag{4} \]

The above tunneling rate implies Hawking radiation with temperature \( T = \sqrt{g'(r_0)f'(r_0)/4\pi} \). We conclude that IHO and black hole thus share the property that while classically not behaving as if being in thermal equilibrium...
with a bath, thermality is acquired when quantum mechanics is taken into account. This formal analogy forms the basis of our proposal to observe analogue Hawking radiation in an experimentally feasible quantum IHO system, to which we now proceed.

We propose using a trapped ion to experimentally implement the IHO. In our scenario, we monitor one of the oscillation directions of the ion, with associated momentum and position operators in the lab frame denoted as $\hat{P}$ and $\hat{X}$, respectively. In particular, the ion’s oscillation frequency $\omega_0$ can be periodically modulated by an applied voltage on the trap electrodes [57]. The corresponding Hamiltonian is given by $H = \frac{P^2}{2M} + \frac{1}{2} M \omega^2(t) X^2$, where $\omega(t) = \omega_0 [1 - \xi \cos(\omega_m t + \phi)]$ denotes the time-dependent frequency, $\xi$ is the modulation depth, $\phi$ is the initial phase of the modulation, and $M$ is the mass of the ion. Imposing $0 < \xi \ll 1$, and $\omega_m = 2 \omega_0$, the Hamiltonian, in rotating-wave approximation, can be rewritten in the form of the IHO in Eq. (1), with $m = 2M/\xi$, and $\alpha = \frac{1}{4} m \omega^2 \xi^2 = m \omega_0^2$. The momentum and position operators have been redefined by $\hat{p} = -i \sqrt{m \hbar} (\hat{a} e^{i \phi/2} - \hat{a}^\dagger e^{-i \phi/2})$, and $\hat{x} = \sqrt{\frac{\hbar}{2m \omega}} (\hat{a} e^{i \phi/2} + \hat{a}^\dagger e^{-i \phi/2})$, respectively. Note that the phase $\phi$ is tunable by appropriately choosing the initial phase of driving.

To diagnose the quantum chaotic motion of the IHO, we propose to measure its OTOC using the protocol proposed in Ref. [58] which does not require the reversal of time evolution. Here, the OTOC is defined by $C(t) = \text{Tr}[\hat{p}_0 \hat{W}^\dagger(t) \hat{W}(t) \hat{V} V]$, where $W(t) = U^\dagger(t, 0) W U(t, 0)$ and $V = V(0)$ are operators evaluated in the Heisenberg picture at times 0 and $t$, respectively [31, 38].

Assuming the operator $V$ to be a projection operator onto an initially pure state, $V = \rho_0$ [58], the OTOC can be reduced to $C(t) = |\langle W(t) \rangle|^2$. To measure the OTOC, we therefore need to prepare a pure initial state for the degree of freedom of the ion’s motion. For our experimental proposal, we use for concreteness a single $^{9}$Be$^+$ ion trapped in a strong radio-frequency Paul trap with a pseudopotential trap frequency of $\omega_0/2\pi = 10$ MHz. The motional ground state can be prepared by a two-stage laser cooling process. By Doppler cooling, all the motional modes of the ion could be cooled down to near the Doppler limit (with an average phonon number $\bar{n}$ of typically a few to ten quanta), through driving the $^2S_{1/2}$ to $^2P_{3/2}$ dipole transition. We can further cool the motional mode of interest to its ground state using sideband cooling with stimulated Raman transitions between the ion’s two hyperfine ground states, i.e., between $^2S_{1/2} (F = 2, m_F = 2)$ and $^2S_{1/2} (F = 1, m_F = 1)$, denoted by $|\downarrow\rangle$ and $|\uparrow\rangle$, respectively, and are separated by $\sim 1.25$ GHz [59, 60]. The spin initialization and ground state cooling allow us to prepare the ion in the $|S = 1/2, n = 0\rangle$ state eventually. By applying appropriate laser pulse sequences on blue and red sidebands or the carrier, we can create arbitrary quantum superpositions of Fock states [61–64].

For the proposed experiment, we assume that the perturbation operator $W$ is an ion displacement measurement, $\hat{x} = \sqrt{\frac{\hbar}{2 m \omega}} (\hat{a} + \hat{a}^\dagger)$, and the initial pure state is prepared as, $|\Psi_{\text{in}}\rangle = \frac{1}{\sqrt{2}} (|n\rangle + |n+1\rangle)$, [63, 64]. We then readily find that the OTOC is given by

$$C(t) = |\langle \Psi_{\text{in}} | U^\dagger(t, 0) \hat{x} U(t, 0) | \Psi_{\text{in}} \rangle|^2 = \frac{(n + 1) \hbar}{2 m \omega} \cosh^2(\lambda_L t),$$

where $U(t, 0) = \exp[-iHt/\hbar]$. Measuring OTOC then reduces to determining directly accessible ion-motional quadratures [40, 65, 66]. We find that the OTOC exhibits exponential growth, $\propto e^{2\lambda_L t}$ (terms growing more slowly than $e^{2\lambda_L t}$), with a Lyapunov exponent $\lambda_L = \sqrt{\alpha/m}$. Fig. 2 shows the OTOC for various pure initial states and Lyapunov exponents.

To observe the Hawking radiation associated to the S matrix (3), we need to prepare a suitable incident wave packet. Specifically, we prepare the initial state of the ion’s motion as a superposition of energy states, $|\Psi_{\text{in}}\rangle = \int dx \psi(x) |\epsilon\rangle$. When the distribution $\psi(x)$ is assumed to be a Gaussian centered at an energy $\epsilon_0$ with small width $\Delta$, i.e., $\psi(x) = \frac{1}{\sqrt{2 \pi \Delta}} \exp \left[ -\frac{(x-\epsilon_0)^2}{2 \Delta^2} \right]$, the incident wave packet can be written as [67]

$$|\Psi_{\text{in}}(x, t) = |\epsilon\rangle |x\rangle^{-1/2} e^{-ixx - i\phi_0} F(t + \log[\sqrt{2}|x|]),$$

where $F(z) = (\Delta / \pi^{1/2})^{1/2} \exp \left[ -z^2 / 2 \Delta^2 \right]$, $\Phi_0 = \epsilon_0 \log[\sqrt{2}|x|] + x^2 / 2 + \varphi_0 / 2 + \pi / 4$, with $\varphi_0 = \arg \Gamma(\frac{1}{2} - i \epsilon_0)$. Generating arbitrary harmonic-oscillator states has been experimentally demonstrated recently [63]. To prepare this wave packet in experiment, it is convenient to represent it in a phononic Fock basis, $|\Psi_{\text{in}}\rangle = \sum_{n=0}^{\infty} \langle n | \Psi_{\text{in}} \rangle |n\rangle$. We note that the overlap $\langle n | \Psi_{\text{in}} \rangle$ is nonzero only when $n$ is even. Using the experimental techniques of Refs. [63, 64], the incident wave packet (6) can in principle be prepared experimentally. In Fig. 2, we plot the fidelity $F_{\text{in}} = \sum_{n=0}^{L} \langle n | \Psi_{\text{in}} \rangle^2$ for preparing the incident wave packet, with $L$ being the truncation level in Fock space. For the initial state (6), we derive an approximate analytical expression for the transmitted and reflected probability densities as follows [67]:

$$|\Psi_R|^2 = \left| R(\epsilon_0) \right|^2 \frac{1}{2x} \left| F \left( t - \log[\sqrt{2}|x|] - \varphi' (\epsilon_0) \right) \right|^2,$$

$$|\Psi_T|^2 = \left| T(\epsilon_0) \right|^2 \frac{1}{2x} \left| F \left( t - \log[\sqrt{2}|x|] - \varphi' (\epsilon_0) \right) \right|^2.$$
 Thermal state parameters (driving strengths) \( \hbar \) at time \( a \sigma \), respectively, and thus with the total Hamiltonian \( H_b + H_r = \hbar \Omega (\hat{a} e^{i \phi_0} + \hat{a}^\dagger e^{-i \phi_0}) \). Here, we set \( \eta \Omega_b = \eta \Omega_r = \Omega \), \( 2 \phi_a = \phi_r \pm \phi_b \) by tuning the amplitude and phase of the applied driving field for the sidebands, and \( \sigma^\pm = (\sigma_x \pm i \sigma_y)/2 \). To measure the scattering state, we first need to prepare the internal levels of the ion as \( |\uparrow\rangle = |\uparrow\rangle, |\downarrow\rangle \rangle \), and then apply the Raman-induced interaction coupling \( H_I \). We then record the probability \( P_\downarrow(t) \) of occupation in \( |\downarrow\rangle \). The expected signal is \( P_\downarrow(t) = \int (1 - \sin(2 \Omega t x)) |\Psi(x)|^2 dx \) and the scattering probability \( |\Psi(x)|^2 \) in Eq. (7) is then obtained by applying an inverse Fourier transform to \( P_\downarrow(t) \).

Taking the initial energy, \( \varepsilon_0 < 0 \), we plot transmission and reflection probabilities in Fig. 2. Negative \( \varepsilon_0 \) implies that the particle classically does not penetrate the barrier when coming from the left; Fig. 2 however clearly shows that both nonzero transmission and reflection probability exist simultaneously. The transmission probability, after integrating over position, satisfies a thermal distribution with temperature \( T = \hbar \lambda_L/2 \pi \). This mathematical relation between the temperature \( T \) and the Lyapunov exponent \( \lambda_L \) suggests that the conjectured MSS bound \( [31] \) is indeed satisfied for the ion’s motion. We have thus shown that the ion trap IHO is the dual of a black hole.

Rewriting Eq. (1) with phononic annihilation and creation operators, we find \( H = -\frac{\hbar \lambda_L}{2} (\hat{a}^2 \sigma + \hat{a}^\dagger \sigma^*) \) which corresponds to a squeezing operation [69]. We can thus alternatively interpret the analogue Hawking radiation via a squeezed state of the ion’s motion. Phonon number increases as \( \langle \hat{n} \rangle = \sinh^2(\lambda_L t) \), and the squeezed vacuum state population distribution is restricted to even states, \( P_2n = (2n)! \tanh^{2n}(\lambda_L t)/(2^n n!)^2 \cosh(\lambda_L t) \) [70]. The squeezed state can be detected by the evolution of the ion’s internal levels under a Jaynes-Cummings interaction [68]. The population distribution is shown in Fig. 3, and compared with a slightly thermal state close to the vacuum. The increased population of higher Fock states are evidence of squeezing; only even states populated corresponds to the creation of pairs of phonons.

In conclusion, we propose using well established quantum mechanical ion motion can be detected by observing the evolution of its internal level populations according to the Hamiltonian \( H_I = h \Omega \sigma_y (\hat{a} e^{i \phi_0}/2 + \hat{a}^\dagger e^{-i \phi_0}/2) \) [68]. This interaction can be implemented, in the Lamb-Dicke limit, by driving the ion with both blue and red sidebands, \( H_b = \hbar \Omega_b (\hat{a}^\dagger \sigma^+ e^{i \phi_0} + \sigma^- e^{-i \phi_0}) \) and \( H_r = \hbar \Omega_r (\hat{a} \sigma^- e^{i \phi_0} + \hat{a}^\dagger \sigma^+ e^{-i \phi_0}) \), respectively, and thus with the total Hamiltonian \( H_b + H_r = h \Omega (\hat{a} e^{i \phi_0} + \hat{a}^\dagger e^{-i \phi_0})(\sigma^+ e^{i \phi_0} + \sigma^- e^{-i \phi_0}) \). Here, we set \( \eta \Omega_b = \eta \Omega_r = \Omega \), \( 2 \phi_a = \phi_r \pm \phi_b \) by tuning the amplitude and phase of the applied driving field for the sidebands, and \( \sigma^\pm = (\sigma_x \pm i \sigma_y)/2 \). To measure the scattering state, we first need to prepare the internal levels of

\( 2m, (\hbar/2m \omega)^{1/2}, \omega^{-1}, \) and \( \hbar \omega \), respectively. Integrating Eq. (7) over space, we obtain the energy-dependent transmission and reflection probabilities in Eq. (3).

In an experiment, the quantum mechanical ion motion is shown in Fig. 3. (Color online) Phonon number distribution \( P_n \) for a squeezed state (yellow) and thermal state (light blue) at very low temperatures (average phonon number \( \bar{n} = 0.02 \)). The squeezing parameter is assumed to be \( \lambda_L t \approx 0.88 \). The vacuum squeezed state is created by the action of the Hamiltonian (1) on the phonon vacuum. Although there are no phonons at the beginning, phonons are created during the evolution, as a result of analogue Hawking radiation.

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tum optical tools for trapped ions to implement the IHO as the dual of a black hole. Chaotic quantum motion of the ion can be accessed by measuring the OTOC, which exhibits to leading order exponential growth, \( \propto e^{2\lambda_L t} \), with Lyapunov exponent \( \lambda_L = \sqrt{\alpha/m} \), and analogue Hawking radiation temperature \( T = \hbar \lambda_L / 2\pi \).

Numerous theoretical approaches employed the IHO concept to establish links between quantum mechanics and general relativity see, e.g., [35–37, 71–73]. Our IHO implementation proposal paves the way to experimentally explore these theoretical concepts in a highly controllable quantum optical environment.

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