Contextuality and nonlocality of indistinguishable particles

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Abstract
Unlike in the case of distinguishable particles, the concept of entanglement—not to mention nonlocality—remains debated in the case of indistinguishable particles. Here, we show that certain existing all-versus-nothing type proofs of contextuality or nonlocality for distinguishable particles, based on a logical contradiction, may be carried over to indistinguishable particles.

Keywords: indistinguishable particles, nonlocality, contextuality, entanglement

1. Introduction

Quantum entanglement, an intriguing feature of quantum mechanics (QM), and indeed of a broad family of non-classical generalized probability theories [1], refers to multipartite states that cannot be expressed as a convex combination of product states. Nonlocality [2–4] refers to statistical correlations among particles of an entangled system, that prevent the assignment of properties to the particles even if they are well separated. Entanglement and nonlocality are now recognized as important resources and are applied in various areas of quantum information processing, among them quantum computation, cryptography, communication and metrology [5].

All the same, whereas quantum entanglement and nonlocality have a clear formulation in the case of distinguishable particle systems [6], the issue remains open to some debate in the case of identical particles. The state space of multiple identical particles lacks the usual tensor product structure, preventing the methods used for studying the usual notions of correlations and entanglement pertaining to distinguishable particles to be carried over. Identical particles do not possess an operational individuality, and enforcing the relevant exchange statistics creates a formal entanglement, whose operational significance is not obvious. The essential problem is that of eliminating the exchange contributions to the entanglement.

Conventional quantum information theory has normally ignored this issue, since typical practical situations involve distant particles with their wave functions having negligible spatial overlap, making them effectively distinguishable. However, with the advent of quantum computation technologies, which would require ever larger scale integration, the effect of exchange statistics on quantum correlations will no longer be insignificant.

Various authors have responded to this issue over the past two decades. It has been pointed out [7] that relinquishing such 'surplus ontology' as particle labels would clear up certain aspects of the quantum statistics in a way that is more faithful to the ontology of Fock space. In [8], entanglement in a pair of fermionic systems in a pure or mixed state is analyzed in terms of Slater rank and Slater number, which are based on the Slater determinant and are in analogy with Schmidt rank and number in the context of two distinguishable particles. In [9], this approach is generalized to the multipartite case. Reference [10] notes the basic contradictions that arise in quantifying the entanglement of identical particles, between the case where they are treated as individuals or as mode excitations, and provides an operational approach to resolve the problem. Articles [11–13] point out that only those bipartite states of identical particles that are not obtainable via (anti)symmetrization of product states can be considered as entangled. In [14], the problem of N fermionic pure states is considered, where N < D and D is the single-particle dimension. Since correlations that exist only to guarantee the fermionic antisymmetrization should be discounted as entanglement, multifermionic states of Slater rank 1 are considered separable, and entangled otherwise, in the context of pure states. On this basis, they provide separability...
criteria for $N$-fermionic pure states in terms of purity or entropy of the single-particle reduced density operator. The authors of [15] propose generalizing the concepts of entanglement and separability beyond the usual tensor decomposition of states to the corresponding algebra of observables. For a recent review of state-based or mode-based approaches to the entanglement of identical particles, see [16].

Cabello and Cunha [17] obtained Bell–Kochen–Specker (BKS) proofs of contextuality [18, 19] for identical particles, by formulating state-independent contextuality in the symmetrized or antisymmetrized space of bosons or fermions.

The difference between the non-classicality of properties and that of identities in QM is that the former is related to non-commutativity of observables in QM, whereas the latter is connected to counting statistics. This indeterminacy of identity, which has recently received a lot of attention [20–24], heightens the quantum ‘mystery’ already evident in the uncertainty in, and entanglement, of the properties.

In this work, we adopt a new approach to correlations among indistinguishable particles, in particular quantum non-locality and contextuality. In contrast to entanglement among indistinguishable particles, the issue of nonlocality among indistinguishable particles has received very little attention. Our approach revisits the foundations of Bell nonlocality, and makes an attempt to consider particle indistinguishability at that basic level. By establishing nonlocality for indistinguishable particles, we will have of course established entanglement for them. Our main result is that certain types of inequalities for contextuality or nonlocality among distinguishable particles can be modified in a simple way to yield corresponding inequalities for indistinguishable particles.

The article is arranged as follows. In section 2, we explain the new approach adopted to create BKS type inequalities for indistinguishable particles. The method involves adapting all-versus-nothing type inequalities, which are based on a logical contradiction when classical value assignments are made to potential measurement outcomes [25]. A simple instance of contextuality or nonlocality based on such an inequality is discussed in section 3. This is then adapted to an instance of contextuality among indistinguishable particles in section 4, and to nonlocality among distinguishable particles in section 5. Finally, we conclude in section 6.

2. A new approach

The contradiction obtained by the BKS theorems has its origin in the fact that observables corresponding to properties are not required to commute in QM, rendering it impossible to embed the algebra of these observables in a commutative algebra, taken to represent the classical structure of the putative hidden variables.

Bell’s theorem tests whether the correlation $P(ab|xy)$ obtained by measuring inputs $x$ and $y$ on two distinguishable particles, to obtain corresponding outcomes $a$ and $b$, can be expressed as a joint probability distribution $P(a, b|x, y) = \sum_{\lambda} p(\lambda) P(a|x, \lambda) P(b|y, \lambda)$, with $\sum_{\lambda} p(\lambda) = 1$, where $\lambda$ parametrizes any local preparation of the two particles (‘localism’) and values $a$ and $b$ are assumed to pre-exist according to a probability distribution determined by $\lambda$ (‘realism’). Correlations that violate Bell’s inequality lack a local-realist explanation. The distinguishability of the particles is implicitly assumed here. To make this explicit, correlations that satisfy this inequality may be described as embodying two assumptions: (1a) localism of distinguishable particles; and (1b) realism of properties possessed by these distinguishable particles.

In the case of indistinguishable particles, extending the above, the natural assumptions to establish nonlocality would be: (2a) localism associated with particles; (2b) realism associated with them; and (2c) distinguishability of the particles. A proof of nonlocality of indistinguishable particles would then contradict a conjunction of these three assumptions. That is, an inequality based on this model would test the assertion that ‘the correlations can be explained by a model that assumes localism and realism and distinguishability’. A violation of the inequality would indicate that one or more of these three assumptions should be given up.

To our knowledge, no such inequality exists in the literature. Here we shall consider a somewhat different model, where assumption (2c) is absorbed into the other two assumptions. We shall do this by symmetrizing over the inputs, so that $P(ab|xy)$ is replaced by $P(\bar{a}, b|x, y)$. The overline indicates symmetrization over measurements and would correspond to the idea that we are only concerned with the probability associated with a combination of measurement outcomes, and not about which particle produces a specific outcome.

This modified underlying model would correspond to the twin assumptions of (3a) localism among non-distinguishable particles; and (3b) realism among non-distinguishable particles. An inequality based on (3a) and (3b) would test the assertion that ‘given particles that are indistinguishable (or simply, not distinguished), the correlations can be explained by a model that assumes localism and realism of the non-distinguished particles’. Thus, a violation of this inequality can be interpreted as signifying the nonlocality of indistinguishable particles. In applying such an inequality to a QM system, we implement the symmetrization procedure by not assigning values to measurements on specific particles, but jointly to the symmetrized measurement operator.

Rather than consider the problem in its full generality, we shall restrict ourselves to the question of adapting existing Bell type inequalities for distinguishable particles. How an arbitrary bipartite or multipartite Bell type inequality would be modified by such a symmetrization requirement would depend on the structure of the inequality. However, this process can be implemented in a simple way when applied to inequalities that are obtained from logical contradictions—the so-called all-versus-nothing type inequalities—and, furthermore, the set of measurements involved possesses a symmetry compatible with the symmetrization argument.

After introducing an illustrative example of an all-versus-nothing type contextuality or nonlocality in the following
section, we shall construct the corresponding inequality for identical particles in the subsequent section.

3. All-versus-nothing type contextuality/nonlocality among distinguishable particles

A simple, and historically perhaps the first, instance of an all-versus-nothing type of Bell’s theorem was given for the a Greenberger–Horne–Zeilinger (GHZ) state of three distinguishable qubits in the state [26]

$$|\Psi\rangle = \frac{1}{\sqrt{2}} (|\uparrow \uparrow \uparrow \rangle - |\downarrow \downarrow \downarrow \rangle),$$ \hspace{1cm} (1)

on which the mutually commuting three-qubit measurements $XYY$, $YXY$, $YYX$ and $XXX$ may be performed, where $X$ and $Y$ refer to the Pauli operators in those directions, and a tensor product is assumed between any two operators. The state $|\Psi\rangle$ is a $+1$-eigenstate of the first three-particle-operators, and a $-1$-eigenstate of the last. A realist (value-definite) model to explain quantum indeterminacy of this system requires that there exist definite value assignments to the $X$ and $Y$ irrespective of their measurement context.

That no such assignment is possible is seen from the following table:

$$
\begin{array}{cccc}
X & Y & Y & +1 \\
Y & X & Y & +1 \\
Y & Y & X & +1 \\
X & X & X & -1.
\end{array}
$$

Any realist assignment of $\pm 1$ to the $X$ and $Y$ will yield a product of $+1$ along the first three columns because there are two copies of $X$ or $Y$ along each column. The product of these products is $+1$, which contradicts the $-1$ obtained above.

However a contextual-realistic explanation is possible: for example, let all the $X$ and $Y = 1$, except $X_1$, which will be $+1$ in the first row and $-1$ in the last. That in general a noncontextual-realistic explanation is impossible is the essential content of the BKS theorem. Any classical explanation of the QM experimental data must come with a ‘twist’ (context-dependence). Assuming that the distinguishable particles are localized (say ‘here’, ‘there’ or ‘yonder’, respectively), then the above contradiction also implies the impossibility of a local-realistic explanation of quantum phenomena [27].

By contrast, the following example of the ‘Mermin square’ [27], cannot be used to demonstrate nonlocality (but can be so used in the case of contextuality), because of the appearance of joint operations (in the third row and column below):

$$
\begin{array}{ccc}
IX & XI & XX \\
ZI & IZ & ZZ \\
ZX & XZ & YY.
\end{array}
$$

In equation (3), a tensor product is assumed between each pair of operators. The elements in a given row or column commute, and their products equal $III$ (thus giving eigenvalue $+1$ for any state), except the last column, where the product operator yields the eigenvalue $-1$. If the two-particle operators are considered to carry noncontextual (independent of the row or column in which they appear) value-definiteness, then this implies a contradiction between QM and noncontextual realism.

4. Contextuality among indistinguishable particles

Analogous to the case of nonlocality of indistinguishable particles, contextuality among indistinguishable particles in our approach follows the scheme outlined in section 2, except that the concept of noncontextuality (i.e. independence of measurement context) replaces locality. Accordingly, assumptions (3a) and (3b) are replaced by: (4a) noncontextuality among non-distinguishable particles; and (4b) realism among non-distinguishable particles.

A violation of an inequality based on these twin assumptions would demonstrate that any underlying model to explain the correlations between the three identical particles should be contextual, or non-real or both. We will now realize this inequality by extending the logical contradiction in the GHZ to indistinguishable particles. Suppose in place of $|\Psi\rangle$ in equation (1), we have the state of three indistinguishable fermionic particles in the state

$$|\Psi\rangle = \frac{1}{\sqrt{2}} (|\uparrow \uparrow \uparrow \rangle - |\downarrow \downarrow \downarrow \rangle) \otimes |h, h, h\rangle,$$ \hspace{1cm} (4)

where $|h\rangle$ is the single-particle spatial state vector of a particle that is ‘here’. Note that the spatial specification is not necessary in a proof of contextuality, but we have retained it in anticipation of the following discussion on nonlocality.

In equation (2), the symmetrized version of each of the first three operators is the same, which is the sum of all three (ignoring the projector in position space, which is common to all three). The last operator, $XXX$, remains the same.

The first three rows of equation (2) are ‘collapsed’ into a single expression, so that the required assignment for the state (4) is

$$
XXY \equiv XYY + YXY + YYX \rightarrow +3, \\
XXX \rightarrow -1.
$$

where a projector to the support (assumed to be compact) of $|h\rangle \langle h|$ in $(\mathcal{L}(R^3))^\otimes 3$ is assumed to be appended to the two operators in the lhs of equation (5).

The product rule imposed on the hidden variable state of particles becomes another algebraic rule, combining a sum and product rule. In this case, it is readily seen that, as before, a BKS-like contradiction arises. The summands in the rhs of the first equation in equation (5) can only take values $\pm 1$. Since a measurement of $XXY$ must yield $+3$, the only value that can be assigned to each of the summands in the expression for $XXY$ is $+1$. Then, the same reasoning that leads to the BKS contradiction in equation (1) follows here.

Our above method of extending a correlation inequality among distinguishable particles to that between indistinguishable ones by symmetrizing the measurement operators is suitable (as discussed in section 2) to all-versus-
nothing type inequalities having certain symmetry in the input settings. It may not work in general. A negative example here would be the case of the Mermin square. If the operators in equation (3) are adapted to indistinguishable particles, whether bosons or fermions, then we must symmetrize them. This alters the structure of the Mermin square to a rectangle, by collapsing the first two columns into one:

\[
\begin{align*}
IX + XI &= XX \\
IZ + IZ &= ZZ \\
ZX + XZ &= YY,
\end{align*}
\]

However, the table in equation (6) lacks the product structure that was exploited by equation (3) to set up a contradiction. In what follows, we will denote symmetrization of operators by an overline, thus \(\overline{IX} = IX + XI\) and so on.

For each row, we find \((\overline{IX})(XX) = \overline{IX}(XX) = \overline{IX}\) and \((\overline{IZ})(ZZ) = \overline{IZ} = ZZ\). Thus a value assignment consistent with the product rule would require, for example, that either \(\overline{IX} \rightarrow 0\) or \(XX \rightarrow 1\), and so on for the other two rows. In each case, the state-independence of the BKS contradiction for the Mermin square is lost. Further, the symmetrized operators in the first column no longer commute, for \([\overline{IX}, \overline{IZ}] = \overline{2IZ} \neq 0\). \([\overline{IX}, \overline{IZ}] = \overline{2XY}\) and \([\overline{IZ}, \overline{IX}] = \overline{2YZ}\). No operator in equation (6) appears in more than one context, and the conditions for testing a contradiction between noncontextual realism and QM vanish.

5. Nonlocality among indistinguishable particles

We shall now adapt the above instance of GHZ-state-based proof of contextuality among indistinguishable particles to obtain a proof of nonlocality among three indistinguishable particles. As noted in section 2, our proof of nonlocality would correspond to assumptions (3a) and (3b), which assert the localism and realism among indistinguishable particles.

We consider three identical bosonic particles, localized in three non-overlapping regions, and existing in the state

\[
|\Psi\rangle \equiv \frac{1}{\sqrt{2}} (|1\uparrow\uparrow\uparrow\rangle_{123} - |1\downarrow\downarrow\downarrow\rangle_{123})
\]

\[
\otimes \frac{1}{\sqrt{6}} \left( \sum_p \prod_p (-1)^p |\psi_h(x_1), \psi_i(x_2), \psi_j(x_3)\rangle \right),
\]

where \(j = h, t, y\) indicates the position (being ‘here’, ‘there’ or ‘yonder’). The spatial part of the state vector has been antisymmetrized over all permutations \(P\) of the spatial indices, which can be explicitly given as the Slater determinant of the three tensor-product terms, namely

\[
|\tilde{\Psi}\rangle \equiv \frac{1}{\sqrt{6}} (|\psi_h, \psi_t, \psi_y\rangle - |\psi_t, \psi_y, \psi_h\rangle - |\psi_y, \psi_h, \psi_t\rangle + |\psi_t, \psi_h, \psi_y\rangle - |\psi_y, \psi_t, \psi_h\rangle + |\psi_h, \psi_t, \psi_y\rangle)
\]

The antisymmetrization of the spatial component, together with the antisymmetrized spin component in equation (7), ensures that the overall bosonic state vector is symmetric under particle label exchange.

Let \(\Pi_h\) denote the projector to the compact support of \(|\psi_h\rangle\), and similarly with \(\Pi_t\) and \(\Pi_y\). Further, let \(X_\alpha \equiv X \otimes \Pi_\alpha\). Thus, the operator \(XYY\) on distinguishable particles becomes the symmetrized version of \(X_\alpha \otimes Y_\beta \otimes Y_\gamma\), given by

\[
X_\alpha \otimes Y_\beta \otimes Y_\gamma = X_\alpha Y_\beta Y_\gamma + X_\beta Y_\gamma Y_\alpha + Y_\alpha Y_\beta X_\gamma + Y_\gamma Y_\alpha X_\beta + Y_\beta Y_\gamma X_\alpha + Y_\alpha Y_\beta X_\gamma,
\]

and so on for \(Y_\alpha X_\beta Y_\gamma\), \(Y_\beta Y_\gamma X_\alpha\) and \(X_\alpha X_\beta X_\gamma\), where a tensor product between the operators is omitted where there is no confusion. The eigenvalues of the symmetrized operators acting on \(|\tilde{\Psi}\rangle\) are as follows:

\[
\langle\tilde{\Psi}|X_\alpha Y_\beta Y_\gamma \rangle = +1,
\]

\[
\langle\tilde{\Psi}|Y_\alpha X_\beta Y_\gamma \rangle = +1,
\]

\[
\langle\tilde{\Psi}|X_\alpha Y_\beta X_\gamma \rangle = +1,
\]

which mirror table (2) in the case of unsymmetrized operators acting on state (1). Thus a BKS contradiction follows precisely as in the original case of distinguishable particles.

Further, let \(\Pi_1^j \equiv I_2 \otimes \Pi_j\) \((j = h, t, y)\), where \(I_2\) is the identity operation on the spin part. These projectors have orthogonal support. It follows from equations (7) and (8) that

\[
\langle\tilde{\Psi}|\Pi_h^j \Pi_t^j \Pi_y^j |\tilde{\Psi}\rangle = 1,
\]

which can be interpreted as asserting that precisely one particle (though one cannot say which one) is ‘here’ precisely one particle is ‘there’, and precisely one particle is ‘yonder’, which is to say that particle positions are simultaneously objective [11–13]. One can thus talk of the proximal (‘here’) particle, the distal (‘there’) particle and the other (‘yonder’) particle. These identities are not ontological, i.e. they are not the haecceities. Instead, they are spatial identities that are phenomenological, and account for exactly three particles. The data of equation (10) can be viewed as a contradiction between local realism of spin properties of these three effectively geographically individuated particles.

6. Discussion and conclusions

The nonlocality of identical particles, as against the weaker resource of entanglement, has hardly been studied, whilst in fact conceptual issues even in the study of entanglement among identical particles remain debated. The present paper explores a specific method to obtain contextuality or nonlocality inequalities for identical particles. In the characterization of entanglement among identical particles, two distinct approaches for both bosons and fermions may be discerned: one based on the particle aspect and employing first quantization (for example [11–13]); and the other based on the mode aspect and employing second quantization (for example [8, 9, 14–16]).

Our work is closer in spirit to the first approach, but uses a novel, axiomatic model in the manner of the derivation of the Bell [3] or Clauser–Horne–Shimony–Holt [4] inequalities. Here, a symmetrization over single particles is used to capture the notion of indistinguishability in the underlying model.
As our main result, we showed that certain existing logical-contradiction-based proofs of contextuality or non-locality for distinguishable particles can be adapted to indistinguishable particles. The physical interpretation of this result is that any objective value assignment of properties to particles, which remain non-distinguished even at the ontological level, produces a contradiction.

A future direction would be to study the possibility of constructing more general inequalities for nonlocality or contextuality of indistinguishable particles to be derived from those for distinguishable particles, in particular, inequalities that are not of the all-versus-nothing type.

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