Tunneling in Quantum Wires: a Boundary Conformal Field Theory Approach

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Abstract

Tunneling through a localized barrier in a one-dimensional interacting electron gas has been studied recently using Luttinger liquid techniques. Stable phases with zero or unit transmission occur, as well as critical points with universal fractional transmission whose properties have only been calculated approximately, using a type of “\(\epsilon\)-expansion”. It may be possible to calculate the universal properties of these critical points exactly using the recent boundary conformal field theory technique, although difficulties arise from the \(\infty\) number of conformal towers in this \(c = 4\) theory and the absence of any apparent “fusion” principle. Here, we formulate the problem efficiently in this new language, and recover the critical properties of the stable phases.
I. INTRODUCTION

The low energy transmission of a 1-d interacting electrons through barriers, including resonant spin tunneling, was analyzed by mapping the problem to a Luttinger liquid where the gapless spin and charge degrees of freedom decouple, interacting with a barrier or a quantum impurity [1] [2]. A simple lattice version of the underlying microscopic Hamiltonian is given by the Hubbard model with a spin-spin interaction which breaks the \( SU(2) \) spin symmetry:

\[
H = \sum_i \left[ t_i \psi_i^\dagger \psi_{i+1} + \text{h.c.} + U (\psi_i^\dagger \psi_i)^2 + J S_i^z S_{i+1}^z + \mu_i \psi_i^\dagger \psi_i + h_i S_i^z \right]
\]

where \( S_i^z = \psi_i^\dagger (\sigma^3)_\alpha \psi_{\beta,i} \).

Here the hopping amplitude \( t_i \), chemical potential \( \mu_i \) and magnetic field \( h_i \) are constant, except in the vicinity of the origin. The Fermi energy is arbitrary except that we stay away from half-filling, so that both charge and spin excitations are gapless. We will often be interested in cases where there is parity symmetry (which may be reflection about a site \( \psi_i \to \psi_{-i} \) or about a link \( \psi_i \to \psi_{i+1} \)) that requires \( t_i \) to be real or where there is symmetry of spin rotation about the \( x \)-axis by \( \pi \): \( S_i^z \to -S_i^z \) which requires \( h_i = 0 \). In the simplest case of a local barrier, only \( t_0 \) differs from the other \( t \)'s, and we set \( \mu_i \) constant and \( h_i = 0 \). In the resonant tunneling case, we take \( t_{-1} = t_0 \) different from the other \( t \)'s. Now the site 0 is distinguished, so we choose \( \mu_0 \) different than \( \mu_i = \mu \) on all other sites. By fine-tuning these two parameters describing the double barrier, we can achieve resonances. We emphasize that we are concerned with universal, low energy properties so that the detailed form of the microscopic Hamiltonian is unimportant.

In the case of spinless fermions scattering off a potential barrier, Kane and Fisher [1] showed that at zero temperature, the charge conductance is zero if the bulk interactions are repulsive and perfect if attractive. More generally, for fermions with spin, the charge and spin conductances depend on two parameters which are related to the bulk interactions \( U \) and \( J \) of (1) in the charge and spin sectors. It was found that there are four possible stable phases whose stability depend on the strength of the bulk interactions: charge and spin with zero or perfect transmission. In addition, there exist unstable phases which have partial conductances separating pairs of the above phases in the region of overlap of the domains where the two phases are stable. These unstable phases were probed perturbatively, using a type of “\( \epsilon \)-expansion” based on the observation that when the bulk interaction constants approach certain values these fixed points become trivial. Our hope is that by using the recently developed boundary conformal field theory technique [3–9] we can determine their properties non-perturbatively. But before diving into the nontrivial unstable boundary fixed points, we have to verify that this method is applicable and that in the new formalism, it does reproduce all the basic features of the problem. This is the purpose of this paper.

The present problem fits into the more general setting where we have one-dimensional gapless degrees of freedom in the bulk coupled to a local potential or impurity degree of freedom. Such systems have been tackled by the boundary critical phenomenon approach in the Kondo problem and in the isotropic spin-\( \frac{1}{2} \) antiferromagnetic Heisenberg chain with an impurity [3–5,7,9]. However, in the Heisenberg chain after a Jordan Wigner transformation, the bulk is composed of interacting spinless fermions and in the Kondo case, free spinful
fermions. Here we are interested in effects of local interactions in an interacting spin-$\frac{1}{2}$ gas of fermions.

In general, the system of gapless degrees of freedom coupled to a local degree of freedom is a difficult problem to solve exactly even in 1-d. There exists exact solution from the Bethe Ansatz but the Hamiltonian must be fine tuned to become integrable. For a generic situation, we simplify the problem by asking what the low energy behavior of the system is. At long wavelengths and low energies, we can describe the bulk by a relativistic (1 + 1)-dimensional field theory with conformal invariance. In the boundary critical phenomenon, we do not integrate out the bulk degree of freedom as in Cardy's treatment, but propose that at low energies, the effects of local interactions with the barrier or the impurity can be summarized by an effective boundary condition on the bulk. The boundary condition must renormalize to a fixed point, so that it will be compatible with the bulk conformal symmetry. At such a boundary fixed point, conformal symmetry in (1 + 1)-dimension is powerful enough to give, for example, the finite size spectrum. By turning the present problem into a boundary critical phenomenon, the four stable phases and the unstable ones mentioned above will correspond to the various conformally invariant boundary conditions on the bulk. As in the Kondo and Heisenberg chain, we will follow Cardy's approach to boundary critical phenomenon to treat the present problem.

There are however two aspects that are new to this problem that were not present in Cardy's treatment nor in the Kondo or Heisenberg problem. The first concerns the symmetry of the problem. This eventually leads to irrational conformal field theories rather than rational ones like the other cases. Cardy concentrated only on the $c < 1$ conformal field theories. Viewing the problems as (1 + 1)-dimensional field theories, the states in the Hilbert space can be classified into a finite number of primary states and an infinite number of descendents. For instance, in the $c = \frac{1}{2}$ Ising case, we have three primary states. For the Kondo and Heisenberg problems, the bulk and the spin-spin interactions with the impurities are both spin $SU(2)$ invariant. Although we have $c \geq 1$ conformal field theories, the extra $SU(2)$ symmetry enlarge the symmetry group to $SU(2)$ Kac-Moody symmetry. Once again, the states in the Hilbert space can be classified into finite number of primaries and the rest descendents. A finite number of primary states has the nice feature that the modular transformation needed in the boundary critical phenomenon approach is linear and is given by a finite-dimensional matrix, known as the modular $S$-matrix. We will make this point clear in the next section. For the case at hand, we have a $U_C(1) \times U_S(1)$ symmetry for the conservation of the fermion’s charge and the $z$-component of the spin. The interactions with the local potential or the impurity are through both spin and charge. Any interactions must preserve this $U_C(1) \times U_S(1)$ symmetry. (In the special case when the bulk and boundary spin interaction preserve rotational symmetry, then we recover $U_C(1) \times SU_S(2)$ symmetry. For the spinless fermions, we only have the $U_C(1)$ symmetry.) However, the $U(1)$ Kac-Moody symmetries are not restrictive enough to group the spectrum of this $c \geq 1$ conformal field theory into a finite number of conformal towers. With an infinite number of primaries, the modular transformation is given by an integral equation and not a finite-dimensional $S$-matrix. However, we are able to generalize Cardy’s approach since it is the partition function that is important, not the fact that the number of primaries is finite. A similar problem was dealt with recently in the boundary conformal field theory of monopole-catalyzed baryon decay.
The other somewhat new aspect has to do with the fact that Cardy’s formalism assumes that the boundary conditions do not allow momentum to flow across the boundary. For instance, in Cardy’s treatment of the Ising model, the three conformally invariant boundary conditions are spin up, down and free. Here, in the extreme case when the electrons are perfectly transmitting across the barrier, we anticipate a fixed point at which all the charge coming in is being transmitted across the boundary to the other side. To transform such a fixed point into a boundary fixed point of Cardy’s type, we fold the system at the boundary so that we essentially turn the system into one defined on the half-line by doubling the number of bulk degrees of freedom. The same trick was used in the boundary conformal field theory treatment of the two-impurity Kondo effect [5].

The paper is organized as follows. In section II, we generalize Cardy’s boundary critical phenomenon approach to include the present problem. We first demonstrate how this is applied to the simpler case of spinless fermions in section III. In particular, we will give the finite size spectra corresponding to the various conformally invariant boundary conditions. We will then give the conductance formula and discuss the stability of these boundary fixed points by examining the operator contents. We will see that our results agree with [1]. We will also calculate the ground state degeneracy, $g$ for periodic and open boundary conditions, showing that the ‘$g$-theorem’ [4] is obeyed. In section IV, we extend our analysis for the spin-$\frac{1}{2}$ fermions. All our results are in full agreement with Ref. [1], after correcting a minor error in that work. We conclude in section IV.

II. GENERALIZATIONS OF CARDY’S APPROACH

We will recall briefly and generalize the ingredients of Cardy’s approach to boundary critical phenomenon [11]. Consider some conformally invariant boundary conditions imposed along the real axis of the complex plane $z = \tau + ix$, giving the geometry of the upper-half plane. The conformal field theory is invariant under infinitesimal coordinate transformation $z \rightarrow \sum_n a_n z^{n+1}$. In order to preserve the boundary $x = 0$, $a_n$ must be real. Truncating half of the infinitely many symmetry transformation leads to half as many conserved charges,

$$L_n = \frac{1}{2\pi i} \int_{C_+} z^{n+1} T(z) dz - \frac{1}{2\pi i} \int_{C_+} \bar{z}^{n+1} \bar{T}(\bar{z}) d\bar{z}$$

where $C_+$ is a semicircle contour in the upper-half plane and there are no $\bar{L}_n$’s. Here $T$ and $\bar{T}$ are the left and right-moving components of the Hamiltonian density. For the Ward identity to continue to be valid, we impose $T - \bar{T} = 0$ along the real axis so that there is no contribution from the integral along the real axis part of the contour $C_+$. In other words, $T_{xx} = 0$ is imposed at the boundary, meaning no momentum flux across it. This condition allows us to think of $\bar{T}(\bar{z})$ in the upper-half plane as $T(z)$ in the lower-half plane, yielding

$$L_n = \frac{1}{2\pi i} \int_{C} z^{n+1} T(z) dz$$

where $C$ is the circle at infinity. We therefore have a purely holomorphic (left moving) system when we extend from the half-plane to the entire complex plane. In addition to the
conformal symmetry, we also have $U(1)$ symmetries, corresponding to the charge and the $z$-component of spin. For a $U(1)$ current, $J(z)$, the analogous equations hold:

\begin{align}
J_n &= \frac{1}{2\pi i} \int_{C_+} z^n J(z) dz - \frac{1}{2\pi i} \int_{C_+} \bar{z}^n \bar{J}(\bar{z}) d\bar{z} \\
&= \frac{1}{2\pi i} \int_C z^n J(z) dz
\end{align}

(4)

where $J(z) - \bar{J}(\bar{z}) = 0$ along the real axis. By mapping the upper-half plane to an infinite strip of width $l$ by $w = \frac{l}{2} \ln z = u + iv$ where $u$ and $v$ are the coordinates for time and space respectively, one can show that [12]

\begin{equation}
H = \frac{1}{2\pi} \int_0^l \left( T(w) + \bar{T}(\bar{w}) \right) dv = \frac{\pi}{l} (L_0 - \frac{c}{24})
\end{equation}

(6) has been used to derive the above equation and we are going to determine $n_{iab}$, the number of times that the $i$th conformal tower appears in the system spectrum with boundary conditions $a$ and $b$ imposed at the two boundaries. The trace in $\chi_i$ is over the descendent states of the $i$th primary. In the $c = 1$ $U(1)$ Kac-Moody theory, we will see that $\chi_{i=Q}$ are of the form

\begin{equation}
\chi_Q(a, \delta; q) = \frac{1}{\eta(q)} q^{\frac{\delta}{24}(Q - \frac{\delta}{2\pi})^2}
\end{equation}

where $\eta(q) = q^{\frac{1}{24}} \prod_{k=1}^{\infty} (1 - q^k)$ is the Dedekind function, $Q$ is an integer, $a$ and $\delta$ are real parameters.

Cardy’s crucial idea is that this partition function can be calculated using the Hamiltonian with periodic boundary conditions $H^P$ which generates translation in the $v$ direction. By a conformal map
\[ \xi = \exp(-\frac{2\pi i}{\beta} w), \] (10)

we map the cylinder with \( H^P \) into the complex \( \xi \) plane where the Virasoro generators \( (L_n^P, \bar{L}_n^P) \) are defined. We find that the Hamiltonian with periodic boundary conditions is given by

\[ H^P = \frac{2\pi}{\beta} (L_0^P + \bar{L}_0^P - \frac{c}{12}). \] (11)

It has both left and right movers. Using (11), the partition function is then

\[ Z_{P}^{(a,b)}(\tilde{q}) = \langle a | \exp(-lH^P) | b \rangle \]
\[ = \tilde{q}^{-c/24} \langle a | \frac{1}{2} (L_0^P + \bar{L}_0^P) | b \rangle \]
\[ = \frac{4\pi l}{\beta}. \] (14)

\(|a\) and \(|b\) are boundary states that obey the following conditions. On the cylinder, the boundary conditions at \( v = 0, l \) are \( J(w) = \bar{J}(\bar{w}) \) and \( T(w) = \bar{T}(\bar{w}) \). Using the conformal map (10), we get

\[ (J_n^P + \bar{J}_{-n}^P) |b\rangle = 0 \] (15)
\[ (\bar{L}_n^P - L_{-n}^P) |b\rangle = 0. \] (16)

We will see later that with the \( U(1) \) Kac-Moody symmetry, (16) follows from (13). Here, it is the currents \((J, \bar{J})\) that classify the states into primaries, not the Virasoro generators. Ishibashi [16] and Cardy [11] showed that these boundary states \(|a\) can be built out of linear combinations of the Ishibashi states that by construction obey (13). An Ishibashi state is given symbolically by

\[ |j\rangle = \sum_N |j; N\rangle \otimes |j; \bar{N}\rangle \] (17)

where \( j \) denotes a primary state and \( N \) denotes its \( N^{th} \) descendent with normalization \( \langle j; N|j'; N'\rangle = \delta_{j,j'}\delta_{N,N'} \). We will give the Ishibashi states explicitly in our problem. Then we can rewrite the partition function using (8) and (16) as

\[ Z_{P}^{(a,b)}(\tilde{q}) = \sum_j \langle a | j; 0 \rangle \langle j; 0 | b \rangle \chi_j(\tilde{q}). \] (18)

We now equate the two partition functions (7) and (18) to constrain the operator content \( n_{ab}^i \). But note that one is a function of \( q \) and the other \( \tilde{q} \). We have to convert one into the other. The difference in having an irrational conformal field theory versus a rational one is that the sum in (7) and (18) run over infinite number of primary states in the irrational case. Furthermore, for a rational conformal field theory, we can turn the \( q \) dependence in (7) into \( \tilde{q} \) by a linear transformation given by the known modular \( S \) matrix. That is,

\[ \chi_i(q) = \sum_j S_{ij}^\dagger \chi_j(\tilde{q}). \] (19)
By equating the two partition functions, one obtains
\[ \sum_i n_{ab}^i S_i^j = \sum_j \langle a|j; 0\rangle \langle j; 0|b \rangle . \] (20)

Here, with infinite number of primaries, the transformation from \( q \) to \( \tilde{q} \) is not given by (19) but by
\[ \chi_Q(a, \delta; q) = \frac{1}{\sqrt{a}} \int dQ' e^{2\pi i (Q - \frac{\delta}{2\pi}) Q'} \chi_Q' \left( \frac{1}{a}, 0; \tilde{q} \right) \] (21)

where \( \chi_Q \) is defined in (9). This modular transformation can be derived using the Gaussian integral and by expressing \( q = \exp(-\pi \beta \lambda) \) and \( \tilde{q} = \exp(-\frac{4\pi l}{\beta}) \). Note that modular transformation requires the summation of a continuous set of conformal towers \( \chi_Q \). However, the modular transformation gives a discrete sum over a set of conformal towers when we sum up the contribution from each tower in the following way:
\[ \sum_{Q=-\infty}^{\infty} e^{i\delta_1 Q} \chi_Q(a, \delta_2; q) = e^{i\delta_1 \delta_2} \frac{1}{\sqrt{a}} \sum_{P=-\infty}^{\infty} e^{i\delta_2 P} \chi_Q' \left( \frac{1}{a}, -\delta_1; \tilde{q} \right) . \] (22)

We derive this equation in the appendix. Hence, we will express the partition functions in terms of
\[ \Omega(a, \delta_1, \delta_2; q) = \sum_{Q=-\infty}^{\infty} e^{i\delta_1 Q} \chi_Q(a, \delta_2; q) \] (23)

and the modular transformation (22) becomes
\[ \Omega(a, \delta_1, \delta_2; q) = e^{i\delta_1 \delta_2} \frac{1}{\sqrt{a}} \Omega \left( \frac{1}{a}, \delta_2, -\delta_1; \tilde{q} \right) . \] (24)

Without the \( S \) matrix, we do not have (24). But we can still equate the partition functions as Cardy did by using (24) and solve for \( n_{ab}^i \) of (7) in the irrational case. The solutions now rest on satisfying the following consistency conditions on \( n_{ab}^i \) of (7) for each of the infinite number of primaries \( i \):
\[ n_{ab}^i \text{ must be an integer for any pair } (a, b) \text{ and} \]
\[ n_{aa}^i = 1 \text{ for each } a . \] (25)

The second consistency condition comes from demanding a unique vacuum through the one to one correspondence between scaling dimensions of operators and the finite size spectrum with identical boundary conditions at both ends [13].

Recall that in the rational case, Cardy [1,2] found the boundary states corresponding to spin up, down and free boundary conditions for the Ising spins and gave the finite spectrum for any pair of boundary conditions. He showed that we can start from a spectrum determined by a set of boundary conditions and obtain the other spectra with other boundary conditions by the process of fusion. The case is similar in Kondo [3,5] and Heisenberg chain [6] where fusion with the impurity spin give the finite size spectra with the new boundary conditions.
conditions. In the present case where the $S$ modular transformation is given by an integral, fusion does not seem to work.

We see that $J(w) = \bar{J}(\bar{w})$ and $T(w) = \bar{T}(\bar{w})$ are imposed at the boundary in Cardy’s formalism and therefore exclude the periodic boundary condition since no momentum or charge can pass the boundary. To incorporate the possibility of the periodic boundary condition being one of the conformally invariant boundary fixed point, we fold the system in half and double the bulk degrees of freedom. In a finite size system with the two channels we have a set of boundary conditions at both ends of the system. The periodic boundary condition in the unfolded system would correspond to having all the momenta coming in through one channel go out the other. This give the perfect conductance case. The open boundary condition will have all the momenta coming in one channel reflected away in the same channel. This give the zero conductance scenario.

More precisely, consider a finite size system extending from $-l$ to $l$ where the scattering potential or impurity is placed at the origin. In order to fold the system in half, we impose the same boundary conditions at $-l$ and $l$ and identify the two points. We expect that the interaction at the origin with the potential or impurity will renormalize into a boundary condition at the origin. Let us now see what the periodic and open boundary conditions on the $U(1)$ currents at the origin of the unfolded system become for the two channel system. Before folding as in figure 1a, we have for open boundary $J(x = 0_+, t) = \bar{J}(x = 0_+, t)$, $J(x = 0_-, t) = \bar{J}(x = 0_-, t)$ and for periodic boundary $J(0_+, t) = J(0_-, t)$, $\bar{J}(0_+, t) = \bar{J}(0_-, t)$ where $J$ and $\bar{J}$ are the left and right moving currents. We folded the system about the origin in figure 1b so that the currents in the two channels are related to the unfolded system by

$$
J(x > 0) = J^1(x), \quad J(x > 0) = J^1(x) \\
J(x < 0) = J^2(-x), \quad \bar{J}(x < 0) = J^2(-x).
$$

Therefore, periodic and open boundary conditions at $x = 0$ in the two channel system are

$$
J^1(0) = \bar{J}^1(0), \quad J^2(0) = \bar{J}^2(0) \quad \text{open} \\
J^1(0) = J^2(0), \quad J^2(0) = J^1(0) \quad \text{periodic}.
$$

The finite size spectrum with appropriate boundary conditions at $-l$, $l$ and 0 is the same as the folded two channels system. Therefore, the folding process does not affect the calculation of the partition function (7). With two channels, $H^P$ in (11) now becomes

$$
H^P = \frac{2\pi}{\beta}(L_0^1 + \bar{L}_0^1 + L_0^2 + \bar{L}_0^2 - \frac{c}{6}).
$$

We have dropped the superscript $P$ in $L_0^P$ but understand that it is distinguished from $L_0$ in (4). (13) becomes

$$
(J_n^1 + \bar{J}_{-n}^1 + J_n^2 + \bar{J}_{-n}^2)|a\rangle = 0.
$$

In the following sections, we will solve (29) and work out the boundary states corresponding to zero and perfect conductances in both the spinless and the spin-$\frac{1}{2}$ fermion cases.
III. SPINLESS FERMIONS

In this section, we will illustrate our procedure in the spinless fermion case before generalizing to the spinful case. The plan is as follows. We will obtain the partition function (18) by constructing two boundary states that satisfy (29). We label the two boundary states by $|1\rangle$ and $|2\rangle$ but refer to them as “open” and “periodic” in anticipation that they correspond to zero and perfect conductance. We first need to obtain (28). To do that, we bosonize the two channels of interacting fermions and obtain the finite size spectrum for the bosons with periodic boundary conditions. We then change basis from the two channels to a more convenient even and odd basis. In the even and odd basis, we give explicitly the open and periodic boundary states. We will obtain $Z_{(a,b)}^{(p)}(\tilde{q})$ for the three combinations of pairs of the two boundary states. By modular transforming the three $Z_{(a,b)}^{(p)}(\tilde{q})$, we obtain three partition functions denoted by $Z_{pp}^{(q)}$, $Z_{oo}^{(q)}$ and $Z_{op}^{(q)}$ as defined in (7) where $p$ and $o$ denotes periodic and open boundary conditions. In other cases where we do not have simple boundary conditions on the fermions, we will have to rely on using the boundary states and the consistency conditions (25) to compute the various partition functions. Thus, we are able to verify with the appropriate normalizations that $Z_{oo}^{(q)} = Z_{11}^{(q)}$, $Z_{pp}^{(q)} = Z_{22}^{(q)}$ and $Z_{op}^{(q)} = Z_{12}^{(q)}$. We will then give the equation for computing conductance in terms of the boundary states. By examining the operator content and the ground state degeneracy, we find the conditions for the stability of the two boundary fixed points. This will be compared to [11]. We will end the section by discussing the resonant tunneling problem.

A. Calculation of the partition functions

The Hamiltonian can be considered to be as in Eq. (11) with only a nearest neighbor interaction term for the spinless fermions. We are interested in the low energy behavior and therefore we take the long wavelength limit by expanding the fermion field about the Fermi momentum $k_F$

$$\psi = e^{-ik_Fx} \psi_L + e^{ik_Fx} \psi_R.$$  \hspace{1cm} (30)

$\psi_L$ and $\psi_R$ are the low energy degrees of freedom. In the continuum limit, the Hamiltonian is a relativistic theory for $\psi_L$ and $\psi_R$ with a four fermi interaction [17]. One can bosonize and parametrize the strength of the interaction by a positive real number $R$. More precisely, we bosonize the left and right moving fermions by

$$\psi_L \sim \left[ \exp -i(\frac{\phi}{2R} + 2\pi R \tilde{\phi}) \right] \begin{array}{c} \psi_R \sim \left[ \exp i(\frac{\phi}{2R} - 2\pi R \tilde{\phi}) \right] \end{array}$$  \hspace{1cm} (31)

where $\phi = \phi_L + \phi_R$, $\tilde{\phi} = \phi_L - \phi_R$ and $R = \frac{1}{4\pi}$ at the free fermion point. Here, we have already rescaled $\phi \rightarrow \phi/\sqrt{4\pi R}$ and $\tilde{\phi} \rightarrow \sqrt{4\pi R} \tilde{\phi}$ so that the continuum Hamiltonian is simply the one for a free boson,
\[ H = \frac{1}{2} \int_0^L \left( (\partial_x \phi)^2 + (\partial_t \phi)^2 \right) dx. \] (32)

We will often compare our results with Kane and Fisher’s and \( R \) is related to their interaction strength \( g \) \( [1] \) by \( 4\pi R^2 = 1/g \) in the spinless case.

To avoid confusion when we compute \( Z_P^{(a,b)}(\bar{q}) \), we will take the space to be periodic in \( 0 \leq x \leq L \) and \( t \) to be the Minkowski time. At the end we will substitute \( L = \beta \), the inverse temperature. Let us concentrate on one channel for the moment. To obtain \( H_P \), we need to specify the boundary conditions on the low energy fields. In computing \( Z_{ab}(q) \) of (7) at a finite temperature, we impose antiperiodic boundary condition on the fermions \( \psi_{L,R} \) or equivalently periodic boundary condition on the bosons \( \phi \) and \( \tilde{\phi} \) in the imaginary time direction. Therefore, when we compute \( Z_{ab}(q) \) now where we have interchanged the role of space and imaginary time, we impose (anti)periodic boundary condition on the (fermions) bosons in the space direction. By imposing antiperiodic boundary condition, \( \psi_{L,R}(x) = -\psi_{L,R}(x+L) \), we use (31) to obtain the boundary conditions on \( \phi \) and \( \tilde{\phi} \)1

\[ Q = \phi(L) - \phi(0) = 2\pi n R \quad \Pi = \tilde{\phi}(L) - \tilde{\phi}(0) = \frac{m}{2R} \] (33)

where \( n = m \mod 2 \). (34)

We refer to the restriction on the quantum numbers in (34) as the “gluing conditions”. We see that \( \phi \) and \( \tilde{\phi} \) are bosons compactified on circles with radii \( R \) and \( 1/4\pi R \) respectively. With these boundary conditions, we can write down a mode expansion for the boson \( \phi \) following \([9]\)

\[
\phi(x,t) = \phi_0 + \frac{1}{2} \left( \hat{\Pi} + \hat{Q} \right) \frac{t+x}{L} + \frac{1}{2} \left( \hat{\Pi} - \hat{Q} \right) \frac{t-x}{L} \\
+ \sum_{n=1}^{\infty} \frac{1}{\sqrt{4\pi n}} \left[ e^{-2\pi i n \frac{t+x}{L}} a_n^L + e^{-2\pi i n \frac{t-x}{L}} a_n^R + h.c. \right] 
\] (35)

where the eigenvalues of \( \hat{Q} \) and \( \hat{\Pi} \) are \( Q \) and \( \Pi \) defined in (33). \( \tilde{\phi}(x,t) \) has a similar mode expansion

\[
\tilde{\phi}(x,t) = \tilde{\phi}_0 + \frac{1}{2} \left( \hat{\Pi} + \hat{Q} \right) \frac{t+x}{L} - \frac{1}{2} \left( \hat{\Pi} - \hat{Q} \right) \frac{t-x}{L} \\
+ \sum_{n=1}^{\infty} \frac{1}{\sqrt{4\pi n}} \left[ e^{-2\pi i n \frac{t+x}{L}} a_n^L - e^{-2\pi i n \frac{t-x}{L}} a_n^R + h.c. \right]. 
\] (36)

The nonzero canonical commutation relations are \([a_n^L, a_m^L] = [a_n^R, a_m^R] = \delta_{nm} \) and \([\hat{Q}, \phi_0] = [\hat{\Pi}, \phi_0] = -i \). Substituting the mode expansion (33) into (32), the Hamiltonian becomes

\[ H = \frac{2\pi}{L} \left[ \frac{1}{4\pi} (\hat{\Pi}^2 + \hat{Q}^2) + \sum_{n=1}^{\infty} n(\hat{m}_n^L + \hat{m}_n^R) - \frac{1}{12} \right] 
\] (37)

1One need to use the fact \([\phi^L, \phi^R] = i/4 \).
where $\hat{n}_L^n = a_L^n a_L^{n\dagger}$ and $\hat{n}_R^n = a_R^n a_R^{n\dagger}$ are the boson occupation number operators and the $\frac{1}{12}$ term is the ground state energy from (11) with $c = 1$ for the boson theory.

The conserved Virasoro and $U(1)$ Kac-Moody currents are given by

$$T = : (\partial_+ \phi)^2 : , \quad \bar{T} = : (\partial_- \phi)^2 :$$

$$J = \frac{1}{2\pi R} \partial_+ \phi, \quad J_n = -\frac{1}{2\pi R} \partial_- \phi$$

(38)

where $\partial_\pm = \frac{1}{2}(\partial_t \pm \partial_x)$ and $: :$ denotes normal ordering. To get the condition (29), we substitute (35) into (38) and expand the current in terms of the boson operators at $t = 0$ by

$$J_n = \int_0^L dx \exp \left( \frac{2\pi i n x}{L} \right) J(x, 0) \quad \text{and} \quad \bar{J}_n = \int_0^L dx \exp \left( -\frac{2\pi i n x}{L} \right) \bar{J}(x, 0).$$

We obtain for $n > 0$

$$J_0 = \frac{1}{4\pi R} (\hat{\Pi} + \hat{Q}), \quad J_n = -\sqrt{\frac{n}{4\pi R^2}} i a_L^n, \quad J_{-n} = J_n^{\dagger}$$

$$\bar{J}_0 = \frac{1}{4\pi R} (-\hat{\Pi} + \hat{Q}), \quad \bar{J}_n = \sqrt{\frac{n}{4\pi R^2}} i a_R^n, \quad \bar{J}_{-n} = \bar{J}_n^{\dagger}. \quad (39)$$

A primary state with respect to the $U(1)$ Kac-Moody algebra is

$$|Q, \Pi\rangle = e^{i(\hat{\Pi} + \hat{Q}) \phi_0} |0\rangle \quad (40)$$

since we can see from (39) that

$$J_{n > 0} |Q, \Pi\rangle = 0, \quad \bar{J}_{n > 0} |Q, \Pi\rangle = 0, \quad J_0 |Q, \Pi\rangle = \frac{1}{4\pi R} (\hat{\Pi} + \hat{Q}) |Q, \Pi\rangle, \quad \text{and} \quad \bar{J}_0 |Q, \Pi\rangle = \frac{1}{4\pi R} (-\hat{\Pi} + \hat{Q}) |Q, \Pi\rangle.$$}

For $Q$ or $\Pi$ nonzero, this state is also a primary state with respect to the Virasoro algebra $(L_n, \bar{L}_n)$ where

$$L_n = 2\pi R^2 \sum_{m=-\infty}^{\infty} : J_m J_{n-m} : . \quad (41)$$

Since $L_n$ are bilinear in $J_n$, in the presence of the $U(1)$ symmetry we use the more fundamental $J_n$ to classify the states into primaries and descendents. The descendents of the primary state (10) are given by

$$e^{i(\hat{\Pi} + \hat{Q}) \phi_0} \prod_{n=1}^{\infty} \frac{(a_L^n)^{m_L^L} (a_R^n)^{m_R^R}}{(m_L^n!)^{1/2} (m_R^n!)^{1/2}} |0\rangle \quad (42)$$

where $m_L^n, m_R^n$ are the occupation numbers. An Ishibashi state is one that by construction satisfies (13). By using (39), we can show that

$$|Q, \Pi\rangle_I = e^{i\Pi \phi_0} \prod_{n=1}^{\infty} \sum_{m_n=0}^{\infty} \frac{(-a_L^n a_R^n)^{m_n}}{m_n!} |0\rangle \quad (43)$$

$$= e^{i\Pi \phi_0} \prod_{n=1}^{\infty} e^{-a_L^n a_R^n} |0\rangle$$
is the desired Ishibashi state. We can put the above Ishibashi state into the form (17) by rewriting the product of sums as sum of products as follows,

\[ \prod_{n=1}^{\infty} \sum_{m_n=0}^{\infty} = \sum_{N=0}^{\infty} \sum_{\{m_n\}'} \prod_{n=1}^{\infty} \]

where the prime in the sum denotes the restriction \( \sum_n m_n = N \). A boundary state is a linear combinations of such Ishibashi states. The open and periodic boundary states exist only in the two channel system which we will turn to next.

For a system with two channels, we work with two copies of bosons \( \phi^1 \) and \( \phi^2 \). We will see that it is advantageous to project the two channels into an even and odd basis similar to the two impurity Kondo problem [5]. In the interacting fermion picture, we have a problem since this will generate nonlocal interactions. However, we can do so now after bosonization since we have a free theory. To define and see the advantage of the even and odd basis, we go back to the current conservation for the two channel system

\[ J^1 + J^2 - \bar{J}^1 - \bar{J}^2 = 0, \quad T^1 + T^2 - \bar{T}^1 - \bar{T}^2 = 0 \] (44)

and substitute (38). By defining

\[ \phi^{e,o} = \frac{1}{\sqrt{2}}(\phi^1 \pm \phi^2), \] (45)

we get

\[ J^e - \bar{J}^e = 0, \quad T^e + T^o - \bar{T}^e - \bar{T}^o = 0 \]

where \( J^e = J^1 + J^2 = \frac{1}{\sqrt{2\pi R}} \partial_+ \phi^e, \quad T^e =: (\partial_+ \phi^e)^2 : \) and \( T^o =: (\partial_+ \phi^o)^2 : \). (46)

\( J^e \) is the total current of the two channels. Notice that \( J^o = J^1 - J^2 \) is not present in the constraints. By combining the two current equations in (46), we deduce that \( T^o = \bar{T}^o \). That is, in the even channel, the boundary preserves the \( U(1) \) Kac-Moody symmetry but in the odd channel, the boundary only preserves the smaller conformal symmetry. It is important to note that in the bulk, however, both \( J_e \) and \( J_o \) are conserved currents.

To get the Ishibashi states in the even and odd basis, we need to expand \( \phi^{e,o} \) in modes. We start with the mode expansions of \( \phi^1 \) and \( \phi^2 \) as in (35), noting that there are gluing conditions (34) between \( Q^1 \) and \( \Pi^1 \) and similarly \( Q^2 \) and \( \Pi^2 \). By (45), we obtain mode expansion for \( \phi^{e,o} \) as in (35) with

\[ a_n \rightarrow a^{e,o}_n = \frac{1}{\sqrt{2}}(a^1_n \pm a^2_n) \quad \text{and} \]

\[ \Pi \rightarrow \Pi^{e,o} = \frac{1}{\sqrt{2}}(\Pi^1 \pm \Pi^2) = \frac{1}{\sqrt{2}} \frac{m^{e,o}}{2R} \]

\[ Q \rightarrow Q^{e,o} = \frac{1}{\sqrt{2}}(Q^1 \pm Q^2) = \sqrt{2\pi R} n^{e,o} \] (47)

where \( m^{e,o} = m^1 + m^2 \) and \( n^{e,o} = n^1 + n^2 \). The gluing conditions between \( m^{e,o} \) and \( n^{e,o} \) can be derived from \( m^{1,2} = n^{1,2} (\mod 2) \). We obtain
Note that the even and odd channels are not decoupled.

The Ishibashi states for the even channel must be (13)

\[ |n^e = 0, m^e\rangle_I^e \equiv |Q^e = 0, \Pi^e\rangle_I^e = e^{iH^e\phi_0} \prod_{n=1}^{\infty} e^{-a_{eL}^n a_{eR}^n} |0\rangle^e. \] (49)

However, in the odd channel, we do not necessarily have such Ishibashi states because the odd \( U(1) \) current is not constrained. But for the open and periodic boundary conditions, something special happens. Transforming the boundary conditions (27) into even and odd sectors using (46), we arrive at

\[ J^e(0) - \bar{J}^e(0) = 0, \quad J^o(0) - \bar{J}^o(0) = 0 \quad \text{open} \] (50)

\[ J^e(0) - \bar{J}^e(0) = 0, \quad J^o(0) + \bar{J}^o(0) = 0 \quad \text{periodic}. \] (51)

Therefore, we further have \( U(1) \) conservation at the boundary in the odd channel for open boundary and maximal violation of the odd \( U(1) \) charge at the boundary for periodic boundary. The periodic odd channel here resembles the problem of monopole-catalyzed baryon decay where the baryon number conservation is maximally violated at the boundary. [14]

Using the conformal map (10), we require that the open Ishibashi state in the odd channel also be annihilated by \( J^o_n + \bar{J}^o_{-n} \). We then use the same Ishibashi state as (19) with \( e \rightarrow o \). Gluing this odd Ishibashi state with the even one leads to the open Ishibashi state. For the periodic Ishibashi state in the odd channel, we impose that it be annihilated by \( J^o_n - \bar{J}^o_{-n} \), giving us

\[ |n^o, m^o = 0\rangle^o_{\tilde{I}} \equiv |Q^o, \Pi^o = 0\rangle_{\tilde{I}}^o = e^{iQ^o\phi_0} \prod_{n=1}^{\infty} e^{-a_{oL}^n a_{oR}^n} |0\rangle^o. \] (52)

When glued with the even channel, this leads to the periodic Ishibashi state.

Since we have an infinite number of primaries, we will have a sum over infinite number of Ishibashi states to obtain a boundary state. Consider the following two boundary states, each a linear combinations of the Ishibashi states,

\[ |1\rangle = \sum_{m^e, m^o} C_{m^e, m^o} |0, m^e\rangle_{I}^e \otimes |0, m^o\rangle_{I}^o \] \[ \quad \text{and} \] (53)

\[ |2\rangle = \sum_{m^e, n^o} C_{m^e, n^o} |0, m^e\rangle_{I}^e \otimes |0, n^o\rangle_{I}^o \] \[ \quad \text{where the primes denote summing over the quantum numbers allowed by the gluing conditions (18). The boundary states} \] (54)

\[ |1\rangle \text{ and } |2\rangle \text{ determine } Z_{P}^{(1,1)}(q), Z_{P}^{(2,2)}(\tilde{q}) \text{ and } Z_{P}^{(1,2)}(\tilde{q}), \text{ which in turn give } Z_{11}(q), Z_{22}(q) \text{ and } Z_{12}(q) \text{ by a modular transformation. By imposing the consistency condition (25) on the partition functions } Z's, \text{ we find that } C_{m^e, m^o} \text{ and } C_{m^e, n^o} \text{ can at most be phases } \exp(\imath m^e \alpha + \imath m^o \beta) \text{ and } \exp(\imath m^e \alpha' + \imath n^o \beta'). \text{ These phases amount to the chemical potentials in the two bulk channels and do not affect the boundary physics.} \]

We will proceed with \( C_{m^e, m^o} = C_{m^e, n^o} = 1 \) to illustrate the calculation.

For two channels, \( H^P \) is given by (11) and expanded in modes, each channel is given by (37). In terms of even and odd basis, we can rewrite \( H^P \) using (17) as
\[ H^P = \frac{2\pi}{L} \left[ \frac{1}{4\pi} (\hat{\Pi}^{e2} + \hat{\Pi}^{o2} + \hat{Q}^{e2} + \hat{Q}^{o2}) + \sum_{n=1}^{\infty} n(m_n^{eL} + m_n^{eR} + m_n^{oL} + m_n^{oR}) - \frac{1}{6} \right] \] (55)

Using (14) generalized to the two channel problem and (55) with \( L = \beta \), we obtain for the boundary state (53)

\[ Z^{(1,1)}_P (\tilde{q}) = \langle 1 | e^{-iH^P} | 1 \rangle = \tilde{q}^{-\frac{1}{12}} \sum_{m^e, m^o}^\prime \sum_{m_1^{e,o}, m_2^{e,o}, \ldots = 0}^{\infty} \tilde{q}^{\frac{1}{16\pi R^2} (m^{e2} + m^{o2}) + \sum_{n=1}^{\infty} n(m_n^{e} + m_n^{o})} \] (56)

where \( \tilde{q} = \exp(-\frac{4\pi l}{\beta}) \). Once again, the prime in the sum denotes summing over quantum numbers allowed by the gluing conditions (48). The Ishibashi states sets \( Q^e \) and \( Q^o \) to zero and \( m_n^{eL} = m_n^{eR} = m_n^e \) and \( m_n^{oL} = m_n^{oR} = m_n^o \) in (55). Solving the gluing conditions (48) with \( n^e = n^o = 0 \), we see that \( m^{e,o} = 2k^{e,o} \) are even where \( k^{e,o} \in \mathbb{I} \) and \( k^e + k^o = 0 \pmod{2} \). Substituting into (55), we get

\[ Z^{(1,1)}_P (\tilde{q}) = \frac{1}{\eta(\tilde{q})^2} \sum_{k^e + k^o = 0 \pmod{2}} \tilde{q}^{\frac{1}{16\pi R^2} (k^{e2} + k^{o2})} \] (57)

where the Dedekind function \( \eta \) is obtained by summing over \( m^e_n \) and \( m^o_n \). Solving the constraints by letting \( k^{e,o} = k \pm l \), \( k, l \in \mathbb{I} \), we finally obtain in terms of \( \Omega \) defined in (23)

\[ Z^{(1,1)}_P (\tilde{q}) = \Omega(\frac{1}{4\pi R^2}, 0, 0; \tilde{q}) = 4\pi R^2 \Omega(4\pi R^2, 0, 0; \tilde{q}) = Z^{(1,1)} (\tilde{q}) \] (58)

Similarly, we obtain

\[ Z^{(2,2)}_P (\tilde{q}) = Z^{(2,2)} (\tilde{q}) = \Omega(\frac{1}{2\pi R^2}, 0, 0; \tilde{q}) \Omega(8\pi R^2, 0, 0; \tilde{q}) + \Omega(\frac{1}{2\pi R^2}, 0, \pi; \tilde{q}) \Omega(8\pi R^2, 0, \pi; \tilde{q}) \] (59)

This partition function is modular invariant, that is, \( Z^{(2,2)}_P (\tilde{q}) = Z^{(2,2)} (\tilde{q}) \). The reason is that we have periodic boundary conditions for the bosons in both the time and space directions, or equivalently, antiperiodic boundary conditions on the fermions \( \psi^{L,R} \) in both directions. It needs some explanation to compute \( Z^{(1,2)} \). The mixed matrix element between the boundary states (53) and (54) sets \( m^o = n^o = 0 \) giving

\[ Z^{(1,2)}_P (\tilde{q}) = \tilde{q}^{-\frac{1}{12}} \sum_{m^e}^\prime \sum_{m_1^{e,o}, m_2^{e,o}, \ldots = 0}^{\infty} (-1)^{m^o_n} \tilde{q}^{\frac{1}{16\pi R^2} (m^{e2}) + \sum_{n=1}^{\infty} n(m_n^{e} + m_n^{o})} \] (60)

The gluing condition sets \( m^e = 4k, k \in \mathbb{I} \). We then get

\[ Z^{(1,2)}_P (\tilde{q}) = \Omega(\frac{1}{2\pi R^2}, 0, 0; \tilde{q}) \tilde{q}^{-\frac{1}{12}} \sum_{m_1^{o}, m_2^{o}, \ldots = 0 n=1}^{\infty} \prod_{n=1}^{\infty} (-\tilde{q})^{m^o_n} \]

\[ = \Omega(\frac{1}{2\pi R^2}, 0, 0; \tilde{q}) W(\tilde{q}) \]

where \( W(\tilde{q}) = \tilde{q}^{-\frac{1}{12}} \prod_{n=1}^{\infty} \frac{1}{1 + \tilde{q}^n} = \Omega(2, \pi, 0; \tilde{q}) \). (61)
The last equality in \( W(\tilde{q}) \) is given by the Jacobi triple product identity \([18]\). (See appendix for a similar derivation). By modular transforming, we get

\[
Z_{(1,2)}(q) = \sqrt{4\pi R^2} \Omega(2\pi R^2, 0, 0; q) W_+(q)
\]

where \( W_+(q) = \frac{1}{2} \Omega(\frac{1}{2}, 0, \pi; q) \).

(62)

As argued before, we must be allowed to impose any pairs of valid boundary conditions to the bulk. The criterion is that for any pairs of boundary states, the partition functions \( Z_{(a,b)} \) generated must satisfy (25). We see that if we normalize the state \( |1\rangle \) by \( \frac{1}{\sqrt{4\pi R^2}} \), then all the partition functions have unit integer coefficients. Comparing these partition functions with \( Z_{op}(q) \) worked out in the appendix from imposing the boundary conditions directly, we conclude that

\[
|\text{periodic}\rangle = |2\rangle \quad \text{and} \quad |\text{open}\rangle = \frac{1}{\sqrt{4\pi R^2}}|1\rangle
\]

are the appropriate boundary states.

B. Conductance

We define the charge conductance beginning with the Kubo formula as in [1],

\[
G = \lim_{\omega \to 0} \frac{4\pi^2 e^2}{h(2l)^2 \omega} \int_{-l}^{l} dx \int_{-l}^{l} dy \int_{-\infty}^{\infty} d\tau e^{i\omega \tau} \langle j(x, \tau) j(y, 0) \rangle
\]

(64)

where \( j(x, \tau) = J - \bar{J} \) is the spatial component of the current in the unfolded system. Folding into a two channel system by (26) and going into the even and odd basis by \( J^{e,o} = J^1 \pm J^2 \) \([16]\), we see that \( \int_{-l}^{l} dx \ j(x, \tau) = \int_{0}^{l} dx (J^o - \bar{J}^o) \).

To evaluate the conductance, we use the following correlation functions:

\[
\langle J^o(x, \tau) J^o(y, 0) \rangle = \frac{1}{2\pi R^2} \frac{1}{4\pi^2[\tau + i(x-y)]^2}
\]

\[
\langle \bar{J}^o(x, \tau) \bar{J}^o(y, 0) \rangle = \frac{1}{2\pi R^2} \frac{1}{4\pi^2[\tau - i(x-y)]^2}
\]

\[
\langle J^o(x, \tau) \bar{J}^o(y, 0) \rangle = \frac{A}{4\pi^2[\tau + i(x+y)]^2}
\]

\[
\langle \bar{J}^o(x, \tau) J^o(y, 0) \rangle = \frac{A}{4\pi^2[\tau - i(x+y)]^2}
\]

\( A \) is sensitive to the boundary condition and is given by \([13]\) \([8]\)

\[
A = -\frac{\langle 0| J^o_{n=1} \bar{J}^o_{n=1} |B \rangle}{\langle 0|B \rangle},
\]

where \( |B\rangle \) is the boundary state and \( J^o_{n=1} \bar{J}^o_{n=1} |0\rangle \) is the first descendent state of the vacuum with respect to the \( U(1) \) algebra. We use only the first descendent because the \( U(1) \) charge
operator is the first descendent of the identity operator in the $U(1)$ theory and it is a primary operator with respect to the Virasoro algebra. After doing a contour integral in the complex $\tau$ plane, taking the zero frequency limit and then performing the spatial integrals, we find

$$G = \frac{(1 - 2\pi R^2 A)}{2} \frac{1}{4\pi R^2} \frac{e^2}{h}$$

(65)

For $|B\rangle = |\text{open}\rangle$, we see from (53) that $A = 1/2\pi R^2$, therefore the conductance vanishes. For $|B\rangle = |\text{periodic}\rangle$, $A = -1/2\pi R^2$ and we obtain

$$G = \frac{1}{4\pi R^2} \frac{e^2}{h}$$

(66)

which reproduces what Kane and Fisher got when we use $4\pi R^2 = g^{-1}$.

C. Operator content and ground state degeneracies

We can find out the operator content (the dimensions of the boundary operators) from the finite size spectrum or the partition function when identical boundary conditions are applied at both ends [20,12]. The partition functions equivalent to the ones given in the appendix are

$$Z_{(o,o)}(q) = \left[ \frac{1}{\eta(q)} \sum_k q^{2\pi R^2 k^2} \right]^2$$

$$Z_{(p,p)}(q) = \frac{1}{\eta(q)^2} \sum_{m+n=0 \mod 2} q^{\frac{m^2}{\cos R^2} + n^2 \pi R^2}$$

(67)

(68)

Let’s consider the open-open case first. We have two decoupled channels indicated by the complete square in $Z_{(o,o)}(q)$ and therefore double the boundary operators of the one channel case. Due to the decoupling of the two channels, it only make sense that we think of the channels as 1 and 2 but not even and odd. In the one channel case, we can read off from (61) that the lowest surface dimensions of the primary operators are $0 \ (k = 0), 2\pi R^2 \ (k = \pm 1), 8\pi R^2 \ (k = \pm 2)$, etc.. We will think of this problem in the purely left moving formalism as indicated in the introduction. Recall that the dimension of an operator $e^{i\alpha \phi_L}$ in a purely left moving system is $\frac{\alpha^2}{8\pi}$ [13]. From this we can write down the corresponding operators with the above dimensions, namely the identity operator, $e^{\pm 4\pi i R\phi_L}, e^{\pm 8\pi i R\phi_L}$, etc.. This is the operator content for the open boundary fixed point.

To analyze the stability of the boundary fixed point, we bring the two channels together and make a local perturbation about the open boundary. We couple two of the above boundary operators, one from each channel. These coupled operators are allowed perturbations only if they have the symmetries of the Hamiltonian, in particular, they must be real and $U(1)$ invariant. The $U(1)$ transformation is $\psi \rightarrow e^{-i\alpha} \psi$. Using equations (30) and (31), we see that to effect this transformation by the bose fields, we need $\phi \rightarrow \phi$ and $\tilde{\phi} \rightarrow \phi + \frac{\alpha}{4\pi R}$. Using $\phi = \phi_L + \phi_R$ and $\tilde{\phi} = \phi_L - \phi_R$, we find

$$\phi_L \rightarrow \phi_L + \frac{\alpha}{4\pi R} \quad \text{and} \quad \phi_R \rightarrow \phi_R - \frac{\alpha}{4\pi R}.$$  

(69)
The lowest dimensional coupled operators allowed by the $U(1)$ symmetry are $\mathcal{O} = R e^{i 4 \pi R \phi} \{ e^{\pm i 2 \pi R \phi} \}$ or $i \{ Im [ e^{4 \pi R \phi} \{ e^{\pm i 2 \pi R \phi} \}] \}$ which have dimensions $4 \pi R^2$. The latter operator would be eliminated if we impose parity symmetry. The operator $\mathcal{O}$ enters the Hamiltonian as $H_{\text{int}} = \lambda \int \delta (x) \mathcal{O} d x d t$, where $\lambda$ is the coupling. Counting the dimension of $\delta (x)$ as one, we see that this is a relevant perturbation about the open boundary when $\dim (\mathcal{O}) = 4 \pi R^2 < 1$. This corresponds to the term $[ q^{2 \pi R^2} ]$ in Eq. (67). In other words, the open boundary condition is stable as long as the bulk interaction is repulsive, $4 \pi R^2 > 1$.

Let us turn to the periodic case. Here, both left and right movers are present and the dimensions of the operators are the sum of the left and the right’s. We will think of this system as having one channel of left and right movers on length $2l$. Furthermore, only $U(1)$ invariant operators are allowed in perturbing the Hamiltonian. As described above, $\phi$ is $U(1)$ invariant. Any function of $\tilde{\phi}$ will not be $U(1)$ invariant and therefore not allowed to enter $H_{\text{int}}$. Noting that in (68), dimensions $\frac{n^2}{16 \pi R^2}$ correspond to operators $e^{\pm i m \phi/2R}$ and $n^2 \pi R^2$ correspond to $e^{\pm 2 i n \pi R \phi}$, we set $n = 0$. The lowest dimension allowed operators from $Z_{p,p}$ are the identity, $e^{\pm \phi/R}$ and $e^{\pm 2i \phi/R}$ with dimensions $0$, $\frac{1}{4 \pi R^2}$ and $\frac{1}{\pi R^2}$. If we perturb locally in the periodic system, the most important operators $e^{\pm \phi/R}$ will appear in $H_{\text{int}}$. Parity invariance does not have much of an effect here: it only restricts from the two possible linear combinations of $e^{\pm \phi/R}$ entering the Hamiltonian to one, $\cos (\phi/R)$. This operator is relevant when $\frac{1}{4 \pi R^2} < 1$ and therefore the periodic boundary condition is stable if $\frac{1}{4 \pi R^2} > 1$. We can patch this result nicely with the above open-open one and give precisely what is in [1].

We have one more tool to decide stability of the boundary fixed points. It is the ground state degeneracy theorem of [4]. The ground state degeneracy is the universal number appearing in front of the partition function in the limit $\beta \to \infty$. It is not necessarily an integer because we have an infinite length system. It is proposed that under renormalization of boundary interactions, the ground state degeneracy decreases. Equivalently, we can associate the ground state degeneracy as the normalization factor for the boundary states (13). In our problem, we have ground state degeneracy for the periodic case $g_p = 1$ and $g_o = \frac{1}{\sqrt{4 \pi R^2}}$ for the open boundary. The $g$-theorem nicely reproduce the above results: $g_p < g_o$ is the stability condition for the periodic boundary condition and vice versa.

D. Resonant tunneling

In the resonant tunneling case, there is just a minor adjustment to the above reasoning. There is now an extra impurity degree of freedom which has zero scaling dimension as far as renormalization goes. About open boundaries, the impurity couples to the lowest dimension operators $e^{\pm 4 \pi R \phi}$ via hopping: e.g. $(t_0 \psi_{\eta}^\dagger \eta + \text{h.c.})$, where $\eta$ is the impurity fermion field, corresponding to $\psi_0$ in Eq. (11). Hence the dimension of this perturbation interaction is $2\pi R^2$. About the periodic case with parity invariance, Kane and Fisher [1] showed that for resonant tunneling to take place, one has to adjust the chemical potential at the impurity site $\mu_0$ so that the probabilities of it being empty and occupied are equal. This fine tuning of the potential to achieve resonance is equivalent to fine tuning away the lowest dimension operator $\cos (\phi/R)$ in the periodic system. In the case without parity invariance, one has to adjust two parameters to eliminate $e^{\pm \phi/R}$ and achieve resonance. The next lowest dimension operator is $e^{\pm 2i \phi/R}$ of dimension $\frac{1}{\pi R^2}$. We see that the results about the two boundary fixed points do
not match nicely as before. The open fixed point is stable when $2\pi R^2 > 1$ but the periodic fixed point is stable when $\pi R^2 < 1$ as in [1]. Taking a hint from [1], we see that about the open fixed point, the descendent of the identity operator $J$ can also couple with the impurity and enter $H_{int}$ through an induced interaction $K \psi^\dagger \eta^\dagger \eta_l \sim K \partial_x \phi \eta^\dagger \eta_l$. The descendents usually have higher dimensions than the primaries and therefore not important. Here, the dimension of the interaction is one and therefore a marginal coupling of the descendent $J$ with the impurity. This together with the hopping terms generate a Kosterlitz-Thouless type renormalization flow on the $t - K$ plane which was used to explain the disparities between the different stability of the two boundary fixed points [1]. We agreed with Kane and Fisher’s analysis.

Let us see what the ground state degeneracy says about the resonant case. For the open boundary at resonance, $g_r = 2g_o$ where the factor of two is due to the two states (empty or filled) at the decoupled impurity site. Therefore, $g_r = \frac{1}{2\pi R^2}$. For the periodic boundary condition, the impurity is absorbed by the continuum. Thus $g_p = 1$. The stability of the periodic fixed point is given $g_p < g_r$ which agrees with the above results. There appear to be a discrepancy between the predictions for the stability of the open boundary from the g-theorem ($g_r < g_p$) and the operator content ($R^2 > \frac{1}{2\pi}$). But we see from the Kosterlitz-Thouless type renormalization flow on the $t - K$ plane [1] that for a given initial value of $K$ for $R^2 > \frac{1}{2\pi}$, there is a critical bare $t$ that flows into the open fixed point ($t = 0$) and then to the periodic fixed point ($t \to \infty$). This renormalization flow holds until $R^2 < \frac{1}{\pi}$ beyond which $t = 0$ is a stable fixed point. This flow is consistent with the g-theorem for $R^2 < \frac{1}{\pi}$, where it occurs.

IV. SPIN-$\frac{1}{2}$ FERMIONS

We will reproduce the same steps for the more complicated spinful case. One may begin with the lattice fermion model [1] and derive the low energy continuum free boson Luttinger liquid as in [21]. We will only give the bosonization rules.

A. calculation of partition functions

We concentrate on one channel for the moment. We first take the low energy long wavelength limit by expanding the fermion field about the Fermi momentum $k_F$

$$
\psi_\alpha = e^{-ik_F x} \psi_{La}(x) + e^{ik_F x} \psi_{Ra}(x) \tag{70}
$$

where $\alpha = \uparrow$ or $\downarrow$. From the interacting fermions in the bulk, we obtain a free theory of bosons with charge and spin interactions parametrized by the radii $R_c$ and $R_s$. That is, we bosonize the low energies left and right moving fermions by

$$
\psi_{L\uparrow,\downarrow} \sim \left[ \exp - i(\frac{\phi_c}{2R_c} + \pi R_c \bar{\phi}_c \pm \frac{\phi_s}{2R_s} \pm \pi R_s \bar{\phi}_s) \right] \tag{71}
$$

$$
\psi_{R\uparrow,\downarrow} \sim \left[ \exp i(\frac{\phi_c}{2R_c} - \pi R_c \bar{\phi}_c \pm \frac{\phi_s}{2R_s} \mp \pi R_s \bar{\phi}_s) \right].
$$
Here $\phi_{c,s}$ are linear combinations of the bosons, $\phi_{\uparrow,\downarrow}$ introduced to represent the two fermion fields $\psi_{\uparrow,\downarrow}$:

$$
\begin{align*}
\phi_c & \equiv (\phi_\uparrow + \phi_\downarrow)/\sqrt{2} \\
\phi_s & \equiv (\phi_\uparrow - \phi_\downarrow)/\sqrt{2}
\end{align*}
$$

They represent the charge and spin degrees of freedom respectively. At the free fermion point, $R_c = R_s = \frac{1}{\sqrt{2\pi}}$. We have already rescaled the fields $\phi_{c,s} \rightarrow \phi_{c,s}/\sqrt{2\pi R_{c,s}}$ and $\tilde{\phi}_{c,s} \rightarrow \sqrt{2\pi R_{c,s}} \tilde{\phi}_{c,s}$ so that the Hamiltonian is simply the one for two free bosons $\phi_{c,s}$, normalized as in Eq. (32). When compared with Kane and Fisher [1], our radii $R_{c,s}$ are related to their interaction strengths $g_{c,s}$ [1] by $\pi R_{c,s}^2 = 1/g_{c,s}$.

To obtain $H^P$ in $Z_P^{(a,b)}(\tilde{q})$, we need to specify the boundary conditions on the low-energy fields. Once again, we infer from the finite temperature calculation of $Z_{ab}(q)$ that we impose (anti)periodic boundary conditions on the (fermions) bosons in the imaginary time direction. By switching the roles of space and time, we now compute $Z_P^{(a,b)}(\tilde{q})$ which leads us to impose (anti)periodic boundary conditions on $(\psi_{L,Ra}) \phi_{c,s}$ and $\tilde{\phi}_{c,s}$. By imposing antiperiodic boundary condition, $\psi_{L,Ra}(x) = -\psi_{L,Ra}(x + L)$, we use (71) to obtain the periodic boundary conditions on $\phi_{c,s}$ and $\tilde{\phi}_{c,s}$:

$$
Q_f = \phi_f(L) - \phi_f(0) = \pi n_f R_f \quad \Pi_f = \tilde{\phi}_f(L) - \tilde{\phi}_f(0) = \frac{n_f}{2R_f}
$$

where $f = c$ or $s$

$$
n_c + \bar{n}_c + n_s + \bar{n}_s = 0 \pmod{4} \text{ and same parity for all } n's.
$$

With these boundary conditions, we can write down a mode expansion for the bosons $\phi_{c,s}$ and $\tilde{\phi}_{c,s}$ as in (72),

$$
H = \frac{2\pi}{L} \sum_{f=c,s} \left[ \frac{1}{4\pi}(\tilde{\Pi}_f^2 + \tilde{Q}_f^2) + \sum_{n=1}^{\infty} n(\hat{m}_f^L + \hat{m}_f^R) - \frac{1}{12} \right]
$$

where the eigenvalues of $\tilde{Q}_f$ and $\tilde{\Pi}_f$ are $Q_f$ and $\Pi_f$ defined in (73) and the $-\frac{\pi}{3L}$ term is the ground state energy from (74) for the $c = 2$ spin and charge bosons.

The conserved Virasoro and $U_c(1) \times U_s(1)$ Kac-Moody currents are given by

$$
T_f = : (\partial_+ \phi_f)^2 : \quad \bar{T}_f = : (\partial_- \phi_f)^2 : \quad J_f = \frac{1}{\pi R_f} \partial_+ \phi_f \quad \bar{J}_f = -\frac{1}{\pi R_f} \partial_- \phi_f
$$

where $f = c, s$. Expanding in modes, we obtain for $n > 0$, $f = c, s$

$$
J_{f0} = \frac{1}{2\pi R_f}(\tilde{\Pi}_f + \tilde{Q}_f), \quad J_{fn} = -\sqrt{\frac{n}{\pi R_f^2}} i a^L_{fn}, \quad J_{f-n} = J^\dagger_{fn} \quad \bar{J}_{f0} = \frac{1}{2\pi R_f}(-\tilde{\Pi}_f + \tilde{Q}_f), \quad \bar{J}_{fn} = \sqrt{\frac{n}{\pi R_f^2}} i a^R_{fn}, \quad \bar{J}_{f-n} = \bar{J}^\dagger_{fn}.
$$

We now turn to the two channel system and work with two copies of the spin and charge bosons $\phi_f^1$ and $\phi_f^2$. We will use the index $f$ to denote charge $c$ and spin $s$ without further
The current conservation for the two channel system at the boundary is given by three equations,

\[ J_1^f + J_2^f - \bar{J}_1^f - \bar{J}_2^f = 0 \quad \text{and} \quad \sum_{f=c,s} T_1^f + T_2^f - \bar{T}_1^f - \bar{T}_2^f = 0 \]  

(78)

The \( U(1) \) currents have to be conserved separately for charge and spin at the boundary. However, we only impose that the sum of spin and charge energy momentum be conserved.

Going to the even and odd basis, we substitute (76) into (78) and define

\[ \phi_{e,o}^f = \frac{1}{\sqrt{2}}(\phi_1^f \pm \phi_2^f) \]  

(79)

giving

\[ J_e^f - \bar{J}_e^f = 0, \quad \sum_{f=c,s} T_e^f + T_o^f - \bar{T}_e^f - \bar{T}_o^f = 0 \]

(80)

where \( J_e^f = J_1^f + J_2^f = \frac{\sqrt{2}}{\pi R_f} \partial_+ \phi_e^f \) and \( T_e^f =: (\partial_+ \phi_e^f)^2 \) and \( T_o^f =: (\partial_+ \phi_o^f)^2 \).

\( J_e^f \) are the total spin and charge currents of the two channels. Notice that \( J_o^f \) are not present in the constraints. By combining the two current equations in (80), we deduce that \( T_e^o + T_s^o = \bar{T}_e^o + \bar{T}_s^o \). That is, in the even channel, the boundary preserves the \( U_c(1) \times U_s(1) \) Kac-Moody symmetries but in the odd channel, the boundary only preserves the smaller conformal symmetry.

To get the Ishibashi states in the even and odd basis, we need to expand \( \phi_{e,o}^f \) in modes. We start with the mode expansions of \( \phi_1^f \) and \( \phi_2^f \) as in (35). By (79), we obtain mode expansion for \( \phi_{e,o}^f \) as in (35) with

\[ a_{e,o}^f = \frac{1}{\sqrt{2}}(a_{1f,n} \pm a_{2f,n}) \quad \text{and} \]

\[ \Pi_{e,o}^f = \frac{1}{\sqrt{2}}(\Pi_1^f \pm \Pi_2^f) \equiv \frac{1}{\sqrt{2} R_f} \tilde{n}_{e,o}^f, \quad Q_{e,o}^f = \frac{1}{\sqrt{2}}(Q_1^f \pm Q_2^f) \equiv \frac{1}{\sqrt{2} R_f} n_{e,o}^f \]

(81)

where \( \tilde{n}_{e,o}^f = \tilde{n}_1^f \pm \tilde{n}_2^f \) and \( n_{e,o}^f = n_1^f \pm n_2^f \). The gluing conditions between \( \tilde{n}_{e,o}^f \) and \( n_{e,o}^f \) can be derived from (74) which holds for each channel. We obtain

\[ \sum_{f=c,s} n_{e,o}^f + \tilde{n}_{e,o}^f = 0 \quad \text{(mod 4)}, \quad \sum_{i=e,o} n_i^s + \tilde{n}_i^c = 0 \quad \text{(mod 4)}, \quad \sum_{i=e,o} n_i^c + \tilde{n}_i^s = 0 \quad \text{(mod 4)}, \quad \sum_{f=c,s} n_{e,o}^f + \tilde{n}_{e,o}^f + n_{e,o}^c = 0 \quad \text{(mod 8)} \]

(82) \hspace{1cm} (83)

and all \( n_i^s \) have the same parity. Note that the even and odd channels are not decoupled.

The Ishibashi states for the even channel must be (43)

\[ |n_{e}^f = 0, \tilde{n}_{e}^f \rangle_f = e^{i \Pi_{e,o}^f} \phi_{e,o}^f \prod_{n=1}^{\infty} e^{-a_{e,R,f,n}^f a_{e,L,f,n}^f} |0 \rangle \]  

(84)
because of the even $U(1)$ charge and spin symmetries at the boundary. In the odd channel, we have at least two choices analogous to the spinless case for each $f$: the Ishibashi state (81) with $e \to o$ which preserve the odd $U(1)$ charge and spin current conservations or

$$|n_f^e, \bar{n}_f^o = 0\rangle_f^o = e^{iQ_f^o \phi_f^o} \prod_{n=1}^{\infty} a_{f,n}^{o\dagger} a_{f,n}^o |0\rangle$$ (85)

which maximally violate the odd $U(1)$ spin and charge conservation.

With these two possibilities for the Ishibashi states in the odd channel for each $f = c, s$, we construct four boundary states corresponding to perfect or zero conductances for charge and spin. Since the even Ishibashi states are the same for the four cases, we let

$$|n^e_{c,s} = 0, \bar{n}^e_{c,s}\rangle_I^e = |n^c_e = 0, \bar{n}^e_{c,s}\rangle_I^e \otimes |n^o_s = 0, \bar{n}^e_{c,s}\rangle_I^e$$ (86)

and the four boundary states are

$$|1\rangle = \sum_{\bar{n}^e_{c,s}, n^o_s} \langle n^e_{c,s} = 0, \bar{n}^e_{c,s}\rangle_I^e \otimes |n^e_c = 0, \bar{n}^o_c\rangle_I^e \otimes |n^o_s = 0, \bar{n}^o_s\rangle_I^e \sim |co, so\rangle$$ (87)

$$|2\rangle = \sum_{\bar{n}^e_{c,s}, n^o_s} \langle n^e_{c,s} = 0, \bar{n}^e_{c,s}\rangle_I^e \otimes |n^e_c, \bar{n}^o_c\rangle_I^e \otimes |n^o_s, \bar{n}^o_s = 0\rangle_I^e \sim |cp, sp\rangle$$ (88)

$$|3\rangle = \sum_{\bar{n}^e_{c,s}, n^o_s} \langle n^e_{c,s} = 0, \bar{n}^e_{c,s}\rangle_I^e \otimes |n^e_c, \bar{n}^o_c\rangle_I^e \otimes |n^o_s, \bar{n}^o_s\rangle_I^e \sim |co, sp\rangle$$ (89)

$$|4\rangle = \sum_{\bar{n}^e_{c,s}, n^o_s} \langle n^e_{c,s} = 0, \bar{n}^e_{c,s}\rangle_I^e \otimes |n^o_s, \bar{n}^o_s = 0\rangle_I^e \otimes |n^o_s = 0, \bar{n}^o_s\rangle_I^e \sim |cp, so\rangle$$ (90)

where the primes denote summing over the quantum numbers allowed by the gluing conditions (83) and $|cp, so\rangle$ denotes charge-open and spin-periodic boundary state, etc..

We will give the finite size spectrums for all possible pairs of the above boundary states. In particular, we have worked out in the appendix for comparison the partition functions $Z_{(cp,sp,cp,sp)}(q)$, $Z_{(co,co,co,so)}(q)$ and $Z_{(co,so,cp,sp)}(q)$ by directly imposing the corresponding boundary conditions. We find that by normalizing the boundary states $|1\rangle$ and $|2\rangle$ to get integer coefficients, we can recover these partition functions. We also normalize boundary states $|3\rangle$ and $|4\rangle$ to give partition functions from the seven other pairs of boundary states with integer coefficients. These are predictions for the finite size spectrums with the corresponding pairs of boundary conditions.

Before computing the partition functions, we need the Hamiltonian for the $c = 4$ two channel spin and charge bosons in the even and odd basis. Following the procedure in the spinless case, we obtain

$$H = \frac{2\pi}{L} \sum_{i=e,o} \sum_{f = c,s} \left[ \frac{1}{4\pi} (\hat{\Pi}_f^2 + \hat{Q}_f^2) + \sum_{n=1}^{\infty} n(\hat{m}_{fn}^L + \hat{m}_{fn}^R) - \frac{1}{12} \right]$$ (91)

We find

$$Z^{(1,1)}_P(q) = \langle 1| e^{-iH_F} |1\rangle = \frac{1}{\eta(q)} \sum_{\bar{n}^o_{c,s}} q^{\frac{1}{4\pi}} \left[ \frac{1}{R_2^4} (\bar{n}^o_{e^2} + \bar{n}^o_{c}^2) + \frac{1}{R_2^4} (\bar{n}^o_{s^2} + \bar{n}^o_{s}^2) \right]$$ (92)
Solving the gluing constraints by setting $n^i_f = 0$ where $i = e, o, f = c, s$ and letting $\tilde{n}^{e,o}_f = \tilde{n}^1_f \pm \tilde{n}^1_f$, we find $\tilde{n}^1_c = \tilde{n}^1_s \pmod{4}$ and $\tilde{n}^1_{c,s}$ are even and the same conditions for $\tilde{n}^2_f$. It now can be written as a complete square,

$$Z_P^{(1,1)}(\tilde{q}) = \left\{ \frac{1}{\eta(\tilde{q})^2} \sum_{\tilde{n}^1_c, \tilde{n}^1_s} \tilde{q}^{\frac{1}{8\pi} \left( \frac{\tilde{n}^1_c^2}{R^2_c} + \frac{\tilde{n}^1_s^2}{R^2_s} \right)} \right\}^2$$

where we have let $\tilde{q} = \tilde{c}, \tilde{o}$. Normalizing the boundary states by $\tilde{f} = \tilde{e}, \tilde{o}$, we reproduce the same spectrum as the ones from the appendix.

For these to be consistent partition functions, we need to normalize away the noninteger coefficients. Normalizing the boundary states by

$$| co, so \rangle = \frac{1}{2\pi R_c R_s} | 1 \rangle \quad \text{and} \quad | cp, sp \rangle = | 2 \rangle,$$

we reproduce the same spectrum as the ones from the appendix.

We predict the following partition functions:

$$Z_P^{(2,2)}(\tilde{q}) = \frac{1}{\eta(\tilde{q})^4} \sum_{n_c + n_s = 0} \tilde{q}^{\frac{1}{4\pi R^2_c n^2_c + \frac{a^2}{16\pi R^2_c} + \frac{1}{4\pi R^2_s n^2_s + \frac{a^2}{16\pi R^2_s}}}} \equiv Z_{(2,2)}(\tilde{q})$$

where the gluing conditions in this sum are $n_c + n_s + \tilde{n}_c + \tilde{n}_s = 0 \pmod{4}$ and all the $n$’s have the same parity. Just like the spinless case, this partition function is modular invariant because the boundary conditions on the bosons are periodic (fermions $\psi_{L,R\alpha}$ are antiperiodic) in both the time and space directions. We also find from the boundary states that

$$Z_P^{(1,2)}(\tilde{q}) = 2\pi R_c R_s \frac{1}{\eta(\tilde{q})^2} \sum_{n_c + n_s = 0} \tilde{q}^{\frac{1}{4\pi R^2_c n^2_c + \frac{1}{4\pi R^2_s n^2_s}} W_+(q)^2} \equiv Z_{(1,2)}(q)$$

For these to be consistent partition functions, we need to normalize away the noninteger coefficients. Normalizing the boundary states by

$$| 1 \rangle = \left( | co, so \rangle + | cp, sp \rangle \right)$$

we reproduce the same spectrum as the ones from the appendix.
By exchanging charge and spin, we obtain the other three spectra $Z_P^{(4,4)}$, $Z_P^{(1,4)}$ and $Z_P^{(2,4)}$. Finally, we showed

$$Z_P^{(3,4)} = Z_P^{(1,2)}$$ (104)

We find that if we normalized the boundary states with (100) and

$$|co, sp⟩ = \frac{1}{\sqrt{\pi R_c^2}}|3⟩$$ and $$|cp, so⟩ = \frac{1}{\sqrt{\pi R_s^2}}|4⟩$$ (105)

then all the partition functions will have integer coefficients. With the normalizations (100) and (105), we see from (104) that the ground state is two-fold degenerate with charge-open spin-periodic boundary condition at one end and charge periodic spin open boundary condition at the other.

The spin and charge conductances are defined as in (64) where the currents are the spin or charge currents in (77). We find that the spin and charge conductances are zero or $G_f = e^2/2\pi R_c^2 h$ depending on the boundary state (87)-(90) we use. That is, for the open and periodic boundaries, we get zero and perfect conductances respectively.

**B. operator content and ground state degeneracies**

Once again, we can determine the operator content from the finite size spectrum with the same boundary conditions at both ends. For these operators to enter the interaction Hamiltonian, they must have all the symmetry properties of the system. In particular, the operator must be real, $U(1)$ spin and charge invariant, and other additional symmetries that we wish to impose like parity, etc..

Let us first discuss the $U_c(1) × U_s(1)$ symmetries. The appropriate transformations are

$$\psi_{\uparrow, \downarrow}^{U_c(1)} → e^{-iα_c} \psi_{\uparrow, \downarrow}^{U_c(1)}$$ and $$\psi_{\uparrow, \downarrow}^{U_s(1)} → e^{±iα_s} \psi_{\uparrow, \downarrow}^{U_s(1)}$$ (106)

The $U_c(1)$ transformation is spin blind and the generator of $U_s(1)$ rotation about the z-axis is $e^{(-iα_s σ_z)}$ acting on the two component spinor $ψ_{\uparrow, \downarrow}$. From (74), we see that these transformations can be achieved by

$$\begin{align*}
φ_c &→ φ_c \\
\tilde{φ}_c &→ \tilde{φ}_c + \frac{α_c}{π R_c} \\
φ_s &→ φ_s \\
\tilde{φ}_s &→ \tilde{φ}_s + \frac{α_s}{π R_s}
\end{align*}$$ (107)
Using $\phi_{c,s} = \phi_{c,s}^L + \phi_{c,s}^R$ and $\tilde{\phi}_{c,s} = \phi_{c,s}^L - \phi_{c,s}^R$, we find
\[
\phi_j^L \to \phi_j^L + \frac{\alpha f}{2\pi R_f} \quad \text{and} \quad \phi_j^R \to \phi_j^R - \frac{\alpha f}{2\pi R_f}.
\] (108)

From the charge and spin-open partition function [97], we see that the lowest dimensional operators have dimensions 0, $\frac{\pi R_c^2}{2}$, $\frac{\pi R_s^2}{2}$, $2\pi R_c^2$, $2\pi R_s^2$, etc.. In the two channels pure left moving interpretation, they correspond to the operators the identity, $e^{\pm 2\pi i R_c \phi_L^c \pm 2\pi i R_s \phi^s_L}$, $e^{\pm 4\pi i R_c \phi_L^L}$, $e^{\pm 4\pi i R_s \phi^s_L}$, etc.. The two decoupled channels can interact with each other via these boundary operators. Preserving the $U_{c,s}(1)$ symmetries, the lowest dimension candidates are $e^{\pm 2\pi i R_c (\phi_L^c - \phi^s_L)}$, $e^{\pm 4\pi i R_c (\phi_L^c - \phi^s_L)}$, $e^{\pm 4\pi i R_s (\phi^s_L - \phi_L^c)}$, and $e^{\pm 4\pi i R_s (\phi^s_L - \phi_L^c)}$. For simplicity, we will proceed with the symmetry $S^z_i \to -S^z_i$. In the fermion language, they correspond to hopping of one fermion between the two channels

\[
t_c (\psi_{L1}^{\dagger} \psi_{L2}^i + \psi_{L1}^{\dagger} \psi_{L2}^j) + \text{h.c.} \sim \cos 2\pi R_s (\phi^s_{s1} - \phi^s_{s2}) [t_c e^{i 2\pi R_c (\phi^c_{c1} - \phi^c_{c2})} + \text{h.c.}],
\] (109)

hopping of a charge two spin singlet,

\[
t_c (\psi_{L1}^{\dagger} \psi_{L1}^j + \psi_{L1}^{\dagger} \psi_{L1}^j) + \text{h.c.} \sim t_c e^{i 4\pi R_c (\phi^c_{c1} - \phi^c_{c2})} + \text{h.c.}
\] (110)

and hopping of a neutral spin one object

\[
t_s (\psi_{L1}^{\dagger} \psi_{L1}^j + \psi_{L1}^{\dagger} \psi_{L1}^j) + \text{h.c.} \sim t_s e^{i 4\pi R_s (\phi^s_{s1} - \phi^s_{s2})} + \text{h.c.}
\] (111)

The stability of the open fixed point is governed by the relevance of these operators. For it to be stable, all operators must have dimensions greater than one. That is $\pi R_c^2 + \pi R_s^2 > 1$ and $4\pi R_{c,s} > 1$.

From the charge and spin-periodic partition function [98], the lowest $U_{c,s}(1)$ invariant operators are the identity, $e^{\pm i \phi_c / R_c \pm i \phi_s / R_s}$, $e^{\pm 2i \phi_c / R_c}$ and $e^{\pm 2i \phi_s / R_s}$ with dimensions 0, $1/4\pi R_c^2$, $1/4\pi R_s^2$, $1/\pi R_c^2$ and $1/\pi R_s^2$. In the fermion language, they correspond to the backscattering of a fermion

\[
v_c (\psi_{L1}^{\dagger} \psi_{R1}^j + \psi_{L1}^{\dagger} \psi_{R1}^j) + \text{h.c.} \sim [v_c e^{i \phi_c / R_c} + \text{h.c.}] \cos (\phi_s / R_s),
\] (112)

backscattering of a charge two spin singlet object

\[
v_c (\psi_{L1}^{\dagger} \psi_{L1}^j \psi_{R1}^j + \psi_{L1}^{\dagger} \psi_{R1}^j) + \text{h.c.} \sim v_c e^{i 2\phi_c / R_c} + \text{h.c.}
\] (113)

and backscattering of a spin one charge neutral object

\[
v_s (\psi_{L1}^{\dagger} \psi_{L1}^j \psi_{R1}^j + \psi_{L1}^{\dagger} \psi_{R1}^j) + \text{h.c.} \sim v_s e^{i 2\phi_s / R_s} + \text{h.c.}
\] (114)

respectively. For the charge and spin-periodic fixed point to be stable, the above operators must be irrelevant with dimensions greater than one.

About the charge-open and spin-periodic fixed point, we see from [101] that the lowest dimensions of the operators are 0, $1/4\pi R_c^2$, $1/\pi R_c^2$ and $\pi R_s^2$. They correspond to the identity operator, $e^{\pm i \phi_c / R_c}$, $e^{\pm 2i \phi_s / R_s}$ and $e^{\pm 2i \phi_s / R_s}$ since we expect that the spin field $\phi_s$ to be periodic across the boundary but not the charge $\phi_c$. In the fermion language, they correspond
to backscattering of a fermion $v_e$ (112) which in this case reduces to backscattering of spin, backscattering of a spin one charge neutral object $v_s$ (114) and the hopping of a fermion across the impurity $t_e$ (109) which reduces to hopping of charge. Essentially the operators $e^{i\phi_c(0)/R_c}$ and $e^{i2\pi R_s[\phi_{s1}^{(0)} - \phi_{s2}^{(0)}]}$ develop non-zero expectation values in this phase because we expect $|v_c|$ and $|t_s|$ to be infinite at this fixed point. This reduces the dimension of the $t_e$ and $v_e$ operators. The stability of this fixed point is governed by the irrelevance of the lowest dimensional operators $t_e$ and $v_e$. We note that the $t_e$ operator was overlooked in Ref. 1. We further note, following Kane and Fisher, that, if we impose parity, all these $t$ and $v$ parameters become real. Choosing a particular sign for $v_c$ then leads to $\langle e^{i\phi_c(0)/R_c} \rangle = 0$ so that the $v_e$ term vanishes.\footnote{Choosing a sign for $v_c$ can be seen as choosing a microscopic model.} Similarly, with parity, an appropriate choice of the sign of $t_s$ causes the $t_e$ term to vanish.

The results of a similar analysis of the charge-periodic spin-open fixed point can be obtained by exchanging charge and spin in the above case. The stability of the four fixed points are shown in figure 2, in the generic case where the $v_e$ and $t_e$ operators are non-vanishing in the mixed phase.

The ground state degeneracies also give relative stability for these four fixed points. The one fixed point that is stable has its ground state degeneracy smaller than the rest. The ground state degeneracies for the four boundary states are precisely the respective normalization coefficients in (100) and (105). We obtain a phase diagram very similar to the results of the operator content analysis as shown in figure 3. If there were no nontrivial unstable fixed points with higher ground state degeneracies, then this would be the correct phase diagram.

C. Resonant tunneling

By using the operator contents of the various boundary fixed points, we can find out the stability of the fixed points as a function of $R_c$ and $R_s$.

About the charge and spin open boundary, the lowest dimensional operators that can couple to the impurity are $e^{\pm 2\pi i R_c \phi_c^e} \pm 2\pi i R_s \phi_s^e$, $e^{\pm 4\pi i R_c \phi_c^e}$ and $e^{\pm 4\pi i R_s \phi_s^e}$ with dimensions $\frac{\pi R_c^2}{2} + \frac{\pi R_s^2}{2}$, $2\pi R_c^2$ and $2\pi R_s^2$. For instance, the first operator arises from $[\psi_{L1}^\dagger \eta_1 + \text{h.c.}]$ and the second and third arise from $(\psi_{L1}^\dagger \psi_{L2}^\dagger) \eta_1^{\dagger} \eta_L$ and $(\psi_{L1}^\dagger \psi_{L2}^\dagger) \eta_1^{\dagger} \eta_L^{\dagger}$. The stability of this fixed point is governed by the condition that all these operators have dimensions greater than unity.

About the periodic case, we fine tune away the backscattering of a single electron corresponding to the operator $v_e$. For a symmetric potential, we only need to fine tune one parameter, for instance the chemical potential at the impurity site. For an asymmetric potential, we need to fine tune two parameters to eliminate this backscattering term to achieve resonance. After fine tuning away this backscattering term, The next two lowest dimensional operators are $v_c$ and $v_s$ with dimensions $1/\pi R_c^2$ and $1/\pi R_s^2$.\footnote{Choosing a sign for $v_c$ can be seen as choosing a microscopic model.}
About the charge-open spin-periodic fixed point, we continue to fine tune $v_e$ to zero. From the partition function \( \langle 101 \rangle \) we find those operators, either by themselves or when coupled to the impurity $\eta_\alpha$, that are not eliminated by the various symmetries. The lowest dimensional operators allowed by the symmetries are $v_e$ \( \langle 114 \rangle \) and the hopping operator that connects the chain to the resonant site $[\psi_1^{\dagger}\eta_\alpha + \text{h.c.}]$. We conclude that the stability of this fixed point is now determined by the irrelevance of these two operators, that is, $1/\pi R_2^2 > 1$ and $\pi R_2^2/2 + 1/16\pi R_2^2 + \pi R_2^2/4 > 1$. By exchanging charge and spin, we obtain similar conclusions for the spin-open charge-periodic fixed point.

The relative stability of these resonant fixed points determined from their ground state degeneracies give a phase diagram same as the one about the charge and spin periodic fixed point.

V. CONCLUSIONS

We have shown that at low energies, interacting fermions coupled to a local potential or an impurity can be turned into a boundary critical phenomenon problem. In the spinful case we are able to analyze the charge open and spin periodic boundary fixed point in a somewhat more systematic way than in \[ 2, 1 \]. We agree completely with these papers after correcting a minor error concerning the stability of this charge open and spin periodic fixed point.

A possible way of finding the nontrivial fixed points of \[ 1 \] is to guess a boundary state that by construction partially conducts and compute the partition functions by taking all matrix elements with the known boundary states. When modular transformed, we require that these partition functions must have integer coefficients. So far, we have failed to guess such a state. Another way to proceed may be to fix the radius of interaction to a rational value in the region where we expect nontrivial fixed points and then use fusion to go from a trivial fixed point to a nontrivial one.

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APPENDIX A:

We wish to derive \[ 24 \] in this appendix. Substituting the Poisson sum formula

\[
\sum_{Q=-\infty}^{\infty} f(Q) = \int dx \sum_{P=-\infty}^{\infty} e^{2\pi i P x} f(x) \tag{A1}
\]

and \( \langle 4 \rangle \) into the left hand side of \( \langle 24 \rangle \), we obtain

\[
\sum_{Q=-\infty}^{\infty} e^{i\delta_1 Q} \chi_Q(a, \delta_2; q) = \frac{1}{\eta(q)} \int dx \sum_{P=-\infty}^{\infty} e^{2\pi i P x} e^{i\delta_1 x} q^{2(x-\frac{\delta_2}{\eta(q)})^2} \tag{A2}
\]
Shifting $x$ to $x + \frac{x_1}{2}\pi$ and using $q = e^{-\frac{x_2}{2\pi}}$, the above becomes

\[ \frac{1}{\eta(q)} e^{i\frac{x_1}{2}\pi} \sum_{P=-\infty}^{\infty} e^{iP\delta_2} \int dx \ e^{-\pi a \frac{x_2}{2\pi} (x^2 - i4R^2 \beta a)} \tag{A3} \]

\[ = \frac{1}{\eta(q)} e^{i\frac{x_1}{2}\pi} \sum_{P=-\infty}^{\infty} e^{iP\delta_2} \sqrt{\frac{2\pi}{a\beta}} e^{-\frac{2\pi}{a\beta} (P + \frac{x_1}{2\pi})^2}. \tag{A4} \]

Using the fact that under a modular transform, the Dedekind function transform as

\[ \eta(q) = \sqrt{2l}\eta(\tilde{q}) \tag{13}, \]

we get

\[ e^{i\frac{x_1}{2}\pi} \frac{1}{\sqrt{a\eta(q)}} \sum_{P=-\infty}^{\infty} e^{iP\delta_2} \tilde{q}^{\frac{1}{4\pi}}(P + \frac{x_1}{2\pi})^2 \tag{A5} \]

\[ = e^{i\frac{x_1}{2}\pi} \frac{1}{\sqrt{a}} \sum_{P=-\infty}^{\infty} e^{iP\delta_2} \chi_P(\frac{1}{a}, -\delta_1; \tilde{q}) \tag{A6} \]

which is the right hand side of (22).

**APPENDIX B:**

In this appendix, we derive the finite size spectrums for the spinless fermions by directly imposing the periodic and open boundary conditions. We will choose appropriate boundary conditions at $x = 0, \pm 1$ for the one channel system of length $2l$ to give the finite size spectra of the two channel folded system of length $l$ with periodic or open boundary conditions placed at $x = 0, l$. We choose, $2k_F l = 2\pi (N + \frac{1}{2})$.

We now derive the finite size spectrum for the two channel system of length $l$ with periodic-periodic boundary conditions at the two ends. Unfolding this into a one channel system, we have a periodic system of length $2l$. For the one channel periodic system on $2l$, we simply obtain its spectrum from (37) with the periodic quantization conditions (33).

Substituting $L = 2l$ and (33) into (37), we obtain

\[ E_P = \pi \left[ \frac{1}{4\pi} \left( \frac{m^2}{4R^2} + 4\pi^2 n^2 R^2 \right) + \sum_{n=1}^{\infty} n(m_n^L + m_n^R) - \frac{1}{12} \right]. \tag{B1} \]

By (7), we have

\[ Z_{pp}(q) = \sum_{m,n} \sum_{m_1^L,R,m_2^L,R,} e^{-\beta E_P} \tag{B2} \]

where the gluing constraint is (34). Splitting the sum into when both $m = 2k$ and $n = 2p$ are even and when both $m = 2k + 1$ and $n = 2p + 1$ are odd, we obtain for $k, p \in \mathbb{I}$

\[ Z_{pp}(q) = \frac{1}{\eta(q)^2} \sum_{m,n} q^{\frac{m^2}{16\pi R^2} + n^2 \pi R^2} \tag{B3} \]

\[ = \frac{1}{\eta^2} \sum_{k,p} q^{\frac{k^2}{4\pi R^2} + 4\pi^2 R^2} + \frac{1}{\eta^2} \sum_{k,p} q^{\frac{(k+\frac{1}{2})^2}{4\pi R^2} + 4\pi^2 (p+\frac{1}{2})^2 R^2} \tag{B4} \]

\[ = \Omega(\frac{1}{2\pi R^2}, 0, 0; q) \Omega(8\pi R^2, 0, 0; q) + \Omega(\frac{1}{2\pi R^2}, 0, \pi; q) \Omega(8\pi R^2, 0, \pi; q), \tag{B5} \]

\[ 27 \]
in agreement with (59).

We now consider the finite size spectrum for the two channel open boundaries system. This system is equivalent to two independent copies of the one channel open boundary system. We will first work out the finite size spectrum for the one channel case. For open boundary conditions, we impose at the boundaries \( \partial_x \psi(0) = \partial_x \psi(l) = 0 \) for the one channel case. Using (30) and the fact that the low energy modes have \( k < k_F \), we obtain \( \psi_L(0) = \psi_R(0) \) and \( \psi_L(l) = e^{2ik_Fl} \psi_R(l) \). Bosonizing and setting \( 2k_Fl = 2\pi(N + \frac{1}{2}) \) we obtain

\[
\phi(0) = 0 \quad \text{and} \quad \phi(l) = 2\pi n R
\]

where we took

\[
[\phi_L(x), \phi_R(y)] = \begin{cases} 0 & x, y = 0 \\ \frac{1}{2} & 0 < x, y < l \\ 0 & x, y = l \end{cases}
\] (B6)

The boundary conditions in \( \phi \) and the commutation relation are compatible with the mode expansion

\[
\phi(x, t) = \hat{Q} \frac{x}{l} + \sum_{n=1}^{\infty} \frac{1}{\sqrt{\pi n}} \sin \left( \frac{\pi n x}{l} \right) \left[ e^{-\frac{imn}{l}} a_n + \text{h.c.} \right]
\] (B7)

where eigenvalues of \( \hat{Q} \) is \( Q = 2\pi n R \). The boundary condition restricts the zero mode \( \phi_0 = \phi_0^L + \phi_0^R = 0 \) but \( \phi_0 = \phi_0^L - \phi_0^R \) is nonzero and is conjugate to \( \hat{Q} \). Using (32), we obtain

\[
E = \frac{\pi}{l} \left[ \frac{1}{2} Q^2 + \sum_{n=1}^{\infty} nm_n - \frac{1}{24} \right]
\] (B8)

where \( m_n \) is the eigenvalue of \( a_n^\dagger a_n \). For one channel, the partition function is

\[
Z_{oo}^1(q) = \sum_Q \sum_{m_1, m_2...} e^{-\frac{Q^2}{4} + \sum_{p=1}^{\infty} \frac{pm_p}{q}}
\] (B9)

\[
= \frac{1}{\eta(q)} \sum_n q^{2\pi n^2 R^2}
\] (B10)

\[
= \Omega(4\pi R^2, 0, 0; q)
\] (B11)

For two channels with open boundaries, we simply have two uncoupled copies of the one channel problem. Therefore the partition function for the two channel open boundaries system is

\[
Z_{oo}(q) = \Omega(4\pi R^2, 0, 0; q)^2
\]

in agreement with (38) after normalization (9).

For the two channel periodic-open case, we equivalently have periodic boundary condition at \( x = 0 \) and open boundary conditions at \( x = \pm l \) in the unfolded one channel system. In other words, we have open boundary conditions at \( x = \pm l \) for the one channel system of length \( 2l \). Therefore, the spectrum is (38) with \( l \to 2l \) and the partition function is

\[
Z_{op}(q) = q^{-\frac{1}{48}} \sum_n \sum_{m_1, m_2...} q^{\pi R^2 n^2 + \frac{1}{2} \sum_{p=1}^{\infty} pm_p}
\] (B12)
We split the descendents $\frac{1}{2} \sum_{p=1}^\infty p m_p = \sum_{p=1}^\infty p m_{2p} + \sum_{p=1}^\infty (p - \frac{1}{2}) m_{2p-1}$ and then sum over the $m_{2p}$ to get the Dedekind function. Then,

$$Z_{op}(q) = \Omega(2\pi R^2, 0, 0; q) \sum_{m_1, m_2, \ldots} \prod_{p=1}^\infty q^{(p-\frac{1}{2})m_{2p-1}}$$

$$= \Omega(2\pi R^2, 0, 0; q) \sum_{m_1, m_2} \frac{1}{1 - q^{p-\frac{1}{2}}}.$$

Using Euler’s identity \[\prod_{n=1}^\infty (1 - x^{n-1})(1 + x^n) = 1\] with $x = \sqrt{q}$ and extracting out a Dedekind function, we rewrite

$$q^{\frac{1}{12}} \prod_{p=1}^\infty \frac{1}{1 - q^{p-\frac{1}{2}}} = q^{\frac{1}{12}} \frac{1}{\eta(q)} \prod_{p=1}^\infty (1 - q^p)(1 + q^{\frac{p}{2}})$$

$$= q^{\frac{1}{12}} \frac{1}{2\eta(q)} \prod_{p=1}^\infty (1 - q^p)(1 + q^{\frac{p}{2}})(1 + q^{\frac{p}{2}-\frac{1}{2}}).$$

We now use the Jacobi triple product identity \[\prod_{n=1}^\infty (1 - x^{2n-1})(1 + y x^{2n-1})(1 + y^{-1} x^{2n-1}) = \sum_{k=-\infty}^\infty y^k x^{k^2}\]

with $x = y = q^{1/4}$ and turn the above product into a sum. We finally arrive at

$$Z_{op}(q) = \Omega(2\pi R^2, 0, 0; q) W_+(q)$$

where $W_+(q)$ is given by (B12).

**APPENDIX C:**

In this appendix, we derive the finite size spectrums for the spin $\frac{1}{2}$ fermions by imposing periodic and open boundary conditions. For a two channel periodic-periodic system of length $l$, it is equivalent to a one channel periodic system of length $2l$. We obtain the finite size spectrum by substituting (73) and $L = 2l$ into (75), giving

$$E_p = \frac{2\pi}{2l} \sum_{f=c,s} \left[ \frac{1}{4\pi} (\tilde{n}_f^2 + \pi^2 n_f^2 R_f^2) + \sum_{n=1}^\infty n(m_{fn}^L + m_{fn}^R) - \frac{1}{12} \right]$$

Therefore, the partition function for this periodic-periodic system is

$$Z_{pp}(q) = \sum_{\tilde{n}_f, n_f} \sum'_{m_{fn}^L, m_{fn}^R} e^{-\beta E_p}$$

$$= \frac{1}{\eta(q)^4} \sum_{\tilde{n}_f, n_f} \sum' q^{\frac{1}{4\pi} \tilde{n}_f^2 R_f^2 + \frac{n_f^2}{16\pi R_f^2}}$$

(C1)
where $n_{c,s}$ and $\tilde{n}_{c,s}$ obey the gluing conditions (74). This agrees with (98).

Consider now the two channel open-open case. This decouples into two one channel system each of length $l$. For each of the one channel system, we again impose boundary conditions $\partial_x \psi_\alpha(0) = \partial_x \psi_\alpha(l) = 0$ leading to $\psi_{La}(0) = \psi_{Ra}(0)$ and $\psi_{La}(l) = e^{2ik_F l} \psi_{Ra}(l)$ when we use (70). We then bosonize these boundary conditions with (71) to obtain boundary conditions on the bosons and just like the spinless case, we use the commutation relations (B6) for the charge and spin bosons. We get

$$\left( \frac{\phi_c}{R_c} \pm \frac{\phi_s}{R_s} \right)(0) = 0 \quad \text{and} \quad \left( \frac{\phi_c}{R_c} \pm \frac{\phi_s}{R_s} \right)(l) = 2\pi n_{\pm}$$

where $n_{\pm} \in \mathbb{I}$ and we have chosen $2k_F l = 2\pi(N + \frac{1}{2})$. It follows that

$$\phi_{c,s}(0) = 0 \quad \text{and} \quad \phi_{c,s}(l) = \pi R_{c,s} n_{c,s}$$

where $n_{c,s} = n_+ \pm n_- \text{ or } n_{c,s} = n_+ \pm n_-$ or equivalently we simply impose gluing conditions $n_c = n_s \pmod{2}$. Expanding $\phi_{c,s}$ in modes compatible with the above boundary conditions as in (B7), we obtain for $Q_{c,s} = \phi_{c,s}(l) - \phi_{c,s}(0) = \pi R_{c,s} n_{c,s}$,

$$E_o = \frac{\pi}{l} \left[ \frac{1}{2\pi} (Q_c^2 + Q_s^2) + \sum_{p=1}^{\infty} p(m_{cp} + m_{sp}) - \frac{1}{12} \right]. \quad \text{(C2)}$$

Hence for a one channel system with open boundaries, the partition function is

$$Z_{1oo}^1(q) = \sum_{n_c = n_s \pmod{2}} \sum_{\{m_{cp}, m_{sp}\}} e^{-\beta E_o}$$

$$= \eta(q)^2 \sum_{n_c = n_s \pmod{2}} q^{\frac{1}{4} n_c^2 R_c^2 + \frac{1}{4} n_s^2 R_s^2 + \frac{1}{2} \sum_{p=1}^{\infty} p(m_{cp} + m_{sp})} \quad \text{(C3)}$$

Hence the partition function for the two channel open boundaries system is

$$Z_{oo}(q) = (Z_{1oo}^1)^2,$$

which agrees with (97) after normalization (100).

To obtain the finite size spectrum of the two channel periodic-open system, we again use the fact that it is equivalent to a one channel system of length $2l$ with open boundaries. The spectrum is therefore (C2) with $l \to 2l$ and the partition function becomes

$$Z_{op}(q) = q^{-1} \sum_{n_c = n_s \pmod{2}} \sum_{\{m_{cp}, m_{sp}\}} q^{\frac{1}{4} n_c^2 R_c^2 + \frac{1}{4} n_s^2 R_s^2 + \frac{1}{2} \sum_{p=1}^{\infty} p(m_{cp} + m_{sp})} \quad \text{(C4)}$$

Following the steps in the spinless case from (B12) to (B14) for both the charge and spin descendents, we obtain

$$Z_{op}(q) = \frac{1}{\eta(q)^2} \sum_{n_c = n_s \pmod{2}} q^{\frac{1}{2} n_c^2 R_c^2 + \frac{1}{2} n_s^2 R_s^2} W_+(q)^2, \quad \text{(C5)}$$

in agreement with (99).
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FIGURES

Figure 1ab: We fold the system of length $2L$ into a two-channel system of length $L$ after we identify the boundary conditions at $-L$ and $L$.

Figure 2: The phase diagram in the space of the charge and spin interaction strength $R_c$ and $R_s$ without imposing parity invariance. (cp,sp), (co,so), (co,sp) and (cp,so) denote the four stable boundary fixed points: charge (c) and spin (s) periodic (p) or open (o). The unshaded region is where both (cp,sp) and (co,so) fixed points are stable. An unstable fixed point should separate these stable phases.

Figure 3: The phase diagram according to the ground state degeneracies. Since the unstable fixed point is expected to have a higher ground state degeneracy than the stable phases, it does not show up when we only compare the relative stability of the four phases.