Range results for some social choice correspondences

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December 16, 2017

Abstract

Determination of the range of several standard social choice correspondences; Borda rule, plurality rule, top-cycle, the Copeland rule, approval voting.

Social choice correspondences might be partially evaluated in terms of what they do, in terms of what sets get chosen. Presumably we would not think well of a rule that only selected either one specific singleton \( \{a\} \), or the set \( X \) of all alternatives. Range conditions could also be part of characterization results. In this paper we determine the range of several standard social choice correspondences.

The is a wide variety of such determinations. Assuming strong preference orderings, any resolute, neutral correspondence has a range consisting of \( m \) singleton sets. Many rules, like Pareto or Borda have a range consisting of all non-empty subsets of alternatives. The Copeland Rule falls just barely short of that full range property. In between, the range of plurality voting depends on detailed relations between the number of individuals and the number of alternatives.

Let \( X \) with cardinality \( |X| = m \geq 3 \) be the set of alternatives and let \( N = \{1, 2, ..., n\} \) with \( n \geq 2 \) be the set of individuals. A (strong) ordering on \( X \) is a complete, asymmetric, transitive relation on \( X \) (non-trivial individual indiffERENCE is disallowed). The highest ranked element of an ordering \( r \) is denoted \( r[1] \), the second highest is denoted \( r[2] \), etc. Also \( r[1 : k] \) is the set of alternatives in the top \( k \) ranks of \( r \). The set of all orderings on \( X \) is \( L(X) \). A profile \( u \) is an element \((u(1), u(2), ..., u(n))\) of the Cartesian product \( L(X)^N \). If \( x \) ranks above \( y \) in \( u(i) \), we write \( x >_{u(i)} y \).

A social choice correspondence \( G \) is a map from the domain \( L(X)^N \) to non-empty subsets of \( X \). For a discussion of most of the social choice correspondences in this paper, see, for example Schwartz (1986) and Sen (2017).
The **range** of social choice correspondence $G$ is the collection of all sets $S$ such that there exists a profile $u$ with $G(u) = S$.

Some range determinations are trivial.

1. A constant social choice correspondence, say $G(u) = \{x, y\}$ for all $u$ obviously has a singleton range.

2. With strong orderings, a dictatorial social choice correspondence has a range consisting of the $m$ different singleton sets.

3. If $G(u) = T(u) = \cup u(i)[1]$, the set $T(u)$ of the tops of all individual orderings at $u$, the range is all non-empty subsets of $X$ of size less than or equal to $n$.

For an example that is just above trivial, consider the Pareto optimal correspondence. Alternative $y$ is **Pareto dominated** by another alternative $x$ at profile $u$ if, for all $i$, individual $i$ ranks $x$ above $y$ in $u(i)$. Alternative $y$ is **Pareto optimal** at $u$ if it is Pareto dominated by no other alternative at $u$. Consider $G_{Pa}(u) = \{x : x \text{ is Pareto optimal at } u\}$. Then $G_{Pa}(u) \neq \emptyset$ since $T(u) \subseteq G(u)$. The range of the Pareto correspondence consists of all non-empty subsets $S$ of $X$. Suppose $|S| = k$. To construct a profile $u$ with $G_{Pa}(u) = S$, start by putting all of $X \setminus S$ in the bottom $m-k$ ranks in any order for everyone. Then, for #1, put $S$ in the top $k$ ranks in any ordering $O$. Follow that by putting $S$ in the top $k$ ranks for #2 but ordered as in $O^{-1}$. (Here $O^{-1}$ is the inverse of $O$: $xO^{-1}y$ if and only if $yOx$.) Finally fill in the rest of the profile in any manner.

For another example at this level, we treat the maximin correspondence which selects the alternatives whose worst ranking (over individuals) is highest. Given profile $u$, let $n(u, x)$ be the largest integer $p$ for which there is an individual $i$ with $u(i)[p] = x$. Then the maximin correspondence sets

$$G_M(u) = \{x : n(u, x) \leq n(u, y) \text{ for all } y \in X\}.$$  

By neutrality, if we can find a set of $k$ elements in the range of $G_M$, all sets of $k$ elements are in the range.

For $k = 1$, we start with the profile $u_1$ 

|   | 1 | 2 | 3 | \cdots | n |
|---|---|---|---|-------|---|
| $x$ | $x$ | $x$ | $x$ | $x$ |
| $y$ | $y$ | $y$ | $\cdots$ | $y$ |
| $z$ | $z$ | $z$ | $z$ |
| $w$ | $w$ | $w$ | $\cdots$ | $w$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |

with $G_M(u_1) = \{x\}$. 

For $k = 2$, we construct $u_2$ from $u_1$ by switching the alternatives in first and second ranks for individual #1:

| 1 | 2 | 3 |   | n |
|---|---|---|---|---|
| y | x | x |   | x |
| x | y | y |   | y |
| z | z | z |   | z |
| w | w | w |   | w |
|   |   |   |   |   |

with $G_M(u_2) = \{x, y\}$.

For $k = 3$, we construct $u_3$ from $u_2$ by switching the alternatives in second and third ranks for individuals #1 and #2:

| 1 | 2 | 3 |   | n |
|---|---|---|---|---|
| y | x | x |   | x |
| z | z | y |   | y |
| x | y | z |   | z |
| w | w | w |   | w |
|   |   |   |   |   |

with $G_M(u_3) = \{x, y, z\}$. Continue in this fashion until $k$ is the smaller of $m$ and $n$.

Now we take up more complicated cases.

1 The Borda correspondence

At profile $u$, let $s(x, i, u)$ be the rank of alternative $x$ in $i$’s ordering (so if, at $u$, individual $i$ has $x$ top-ranked, $s(x, i, u) = 1$). The Borda score for $x$ at $u$ is the sum

$$B(x, u) = \sum_i s(x, i, u)$$

of those ranks over all individuals. Then the **Borda correspondence** selects the alternatives with minimal Borda score:

$$G_B(u) = \{x \in X : B(x, u) \leq B(y, u) \text{ for all } y \in X\}$$

If $n$ is even, the range of Borda consists of all possible subsets $S$ of $X$. To get $S$, construct the profile $u$ as follows: Let $P$ be an ordering of $S$ and $Q$ be an ordering of $X$. $u(1)$ will be $P$ above and $X \setminus S$ below, ordered in any way. $u(2)$ will be $P^{-1}$ above and $X \setminus S$ below, ordered in any way. Then half of the
remaining individuals have ordering $Q$ while the other half have ordering $Q^{-1}.$ If $n$ is odd, there is a profile $u$ with $G_B(u) = S$ unless $S = X$ where $|X|$ is even. For proof of this, see Kelly and Qi (2016), where also partial results are given for the range of the Borda social welfare function, which yields social rankings.

2 Plurality rule

Given a profile $u$ and an element $x$ of $X,$ let $n_x(u)$ be the number of individuals with $x$ top-ranked at $u.$ Alternative $x$ is a plurality winner at $u$ if $n_x(u) \geq n_z(u)$ for all $z$ in $X.$ Plurality rule is the social choice correspondence $G_{Pl}$ that, at each profile $u,$ has $G_{Pl}(u)$ equal to the set of all plurality winners at $u.$ The plurality number at $u,$ $N(u),$ is the value of $n_x$ for a plurality winner $x$ at $u.$ When profiles $u$ and $u^*$ are fixed for a discussion, we simplify $n_x(u)$ and $n_x(u^*)$ respectively to $n_x$ and $n_x^*.$

For the range of plurality rule, let $S$ be a subset of $X$ so $|S| = k \leq m.$ When is $S = \{x_1, x_2, ..., x_k\}$ in the range of plurality rule? We certainly need $k \leq n$ since there are at most $n$ tops. If $k|n,$ let $u$ be a profile where the first $n/k$ individuals have $x_1$ on top, the next $n/k$ have $x_2$ on top, and so on. $S = G_{Pl}(u).$ If $k \nmid n,$ let $n = kq + r$ with $0 < r < k.$ Let $u$ be a profile where the first $q$ individuals have $x_1$ on top, the next $q$ have $x_2$ on top, and so on. If $k = m$ and $r > 0,$ sets $S$ with $|S| = k$ are not in the range.

If $m > k,$ there are $r$ individuals whose tops at $u$ are still undetermined and $m - k$ alternatives to use for those tops. For $x_1, x_2, ..., x_k$ to be the plurality winners, we would need $q$ strictly larger than the highest number needed for a non-winner:

$$q > \left\lfloor \frac{r}{m - k} \right\rfloor = \left\lfloor \frac{n - qk}{m - k} \right\rfloor$$

where $\lfloor z \rfloor$ is the smallest integer not less than $z.$ Here $q$ is $\left\lceil \frac{n}{k} \right\rceil$ where $\lceil z \rceil$ is the largest integer not greater than $z.$ Combining, when

$$\left\lfloor \frac{n}{k} \right\rfloor > \left\lfloor \frac{n - \frac{n}{k}k}{m - k} \right\rfloor$$

any set of $k$ elements is in the range of plurality rule.

3 The top cycle correspondence

Here $G_T(u)$ is the set of maximal alternatives for the transitive closure of the simple majority voting relation. Given $x, y \in X,$ we define the simple majority
voting (SMV) relation as \( x \gtrapprox_{u} y \) just when \( |\{i : x \succ_{u(i)} y\}| \geq |\{i : y \succ_{u(i)} x\}|. \) Let \( R(u) \) be the transitive closure of \( \gtrapprox_{u}. \) That is, \( xR(u)y \) if there is a sequence \( x_1, x_2, \ldots, x_t \) such that \( x \gtrapprox_{u} x_1 \gtrapprox_{u} x_2 \gtrapprox_{u} \cdots \gtrapprox_{u} x_t \gtrapprox_{u} y. \)

If \( xR(u)y \) and \( yR(u)x, \) we write \( x \approx y. \) Then \( G_{T}(u) = \{x : xR(u)y \text{ for all } y\}. \)

Note first, this rule is neutral, so if we can find a set of \( k \) elements in the range, all sets of \( k \) elements are in the range.

**Case 1.** \( n \) odd. For odd \( n, \) any one element set is in the range, but no two element set is. Now we construct, for any subset \( S \) of \( X \) consisting of three or more alternatives, a profile \( u \) that has \( S \) in the range. First we observe that with three individuals, there is a profile on just \( S \) such that the transitive closure of SMV is social indifference on \( S. \) The standard voting paradox shows a profile \( v \) on three individuals and exactly three alternatives that exhibits a voting cycle: \( x \succ y \succ z \succ x, \) and the transitive closure is \( x \approx y \approx z \approx x. \) For \( k = 4, \) pick one alternative from \( \{x, y, z\}, \) say \( y, \) and a new alternative, \( w. \) Now insert \( w \) in each ordering in profile \( v \) just below \( y. \) That creates a 4-element cycle and the transitive closure of SMV is social indifference among these 4 alternatives. Continuing in this manner, we can construct a profile \( s \) such that the transitive closure of SMV is social indifference on \( S. \) Now let \( P \) be an arbitrary ordering of \( X \setminus S \) and \( Q \) is an arbitrary ordering of \( X, \) and construct \( u \) as follows:

A. For individuals #1-#3, put the elements of \( S \) in the top \( k \) ranks, ordered as in the profile \( s \) on \( S \) and put the elements of \( X \setminus S \) in the bottom \( m - k \) ranks ordered as in \( P; \)

B. For odd \( j, \) with \( 3 < j \leq n, \) set \( u(j) = Q; \)

C. For even \( j, \) with \( 2 < j \leq n - 1, \) set \( u(j) = Q^{-1}. \)

Then \( S \) is the maximal set of the transitive closure of SMV at \( u. \)

|   | 2 | 3 |
|---|---|---|
| 1 | x | z |
| 2 | y | x |
| 3 | w | y |
|   | v | w |
|   | z | v |
|   |   | x |

where \( G_{T}(u) = \{x, y, z, w, v\}. \) This example also illustrates that \( G \) does not satisfy the Pareto condition.

**Case 2.** \( n \) even. For even \( n, \) every subset is in the range. Let \( S \) be any \( k \)-element subset of \( X \) for \( 0 < k \leq m. \) Construct profile \( u \) as follows, where
O is an arbitrary ordering of S, P is an arbitrary ordering of X\S and Q is an arbitrary ordering of X:

A. For #1, put the elements of S in the top $k_r$ ranks, ordered as in $O$ and put the elements of $X\setminus S$ in the bottom $m-k_r$ ranks ordered as in $P$;

B. For #2, put the elements of $S$ in the top $k_r$ ranks, ordered as in $O^{-1}$ and put the elements of $X\setminus S$ in the bottom $m-k_r$ ranks ordered as in $P$;

C. For odd $j$, with $1 < j \leq n-1$, set $u(j) = Q$;

D. For even $j$, with $2 < j \leq n$, set $u(j) = Q^{-1}$.

Then $S$ is the maximal set of the transitive closure of SMV at $u$.

4 The Copeland rule

Given a profile $u$ and an element $x$ of $X$, let $n_x(u)$ be the Copeland score\footnote{There is a variant of the Copeland score that is the number of alternatives defeated by $x$ minus the number of alternatives that defeat $x$. The distinction between these variants is of no consequence for our range results.}, the number of alternatives defeated by $x$ under simple majority vote at $u$. Alternative $x$ is a Copeland winner at $u$ if $n_x(u) \geq n_z(u)$ for all $z$ in $X$. The Copeland rule is the social choice correspondence $G_C$ that, at each profile $u$, has $G_C(u)$ equal to the set of all Copeland winners at $u$.

Our focus is on the range of $G_C$. The Copeland correspondence is neutral and by neutrality, a set $S$ from $X$ is in the range if and only if there is at least one set $T \subseteq X$ in the range of $G_C$ with $|S| = |T|$. Thus, by an abuse of language, we will say that $k$ is in the range of $G_C$ if there is a $u$ with $|G_C(u)| = k$. If $k$ is in the range for $m$ and $n$, it is also in the range for $m+1$ and $n$ (just add an alternative at the bottom of everyone’s ordering). Also, if $k$ is in the range for $m$ and $n$, it is also in the range for $m$ and $n+2$ (just add two individuals with inverse orderings).

Even $n$

With even $n$, SMV ties are possible. That makes the determination of the range much easier. We want to show for any $m$, $n$, and any set $S$ of $k$ alternatives where $1 \leq k \leq m$, there is a profile at which the Copeland choice set is $S$. By the first remarks in this section, it suffices to show this for $n = 2$. Let $P$ be any strong ordering on $S$ and $Q$ be any strong ordering on $X\setminus S$. Construct $u$ by setting #1’s ordering to be $P$ on top followed by $Q$ and #2’s ordering to be $P^{-1}$ on top followed by $Q$.

Odd $n$

Now assume all preferences are strong orderings on $X$ and there are an odd number of individuals. Given profile $u$, consider the $m$-term non-increasing
sequence \((s_1, s_2, \ldots s_m)\) obtained from \(n_x(u), n_y(u), \ldots\) by rearranging according to magnitude, largest to smallest.

The set \(GC(u)\) of maximal elements of \(R(u)\) will have \(k\) elements just when the associated sequence \(s_1, s_2, \ldots s_m\) has an initial subsequence of \(k\)-elements, all the same, and then, if \(k < m\), a \(k+1\)st element different from the first \(k\). Note that since there are \(m(m-1)/2\) pairs of distinct alternatives, \(s_1 + s_2 + \ldots + s_m = m(m-1)/2\).

For \(|X| = m\), there exists at least one \(k\), with \(1 \leq k \leq m\) such that \(k\) is not in the range of \(GC\). To be more precise

(1) If \(m\) is even, \(m\) is not in the range of \(GC\);
(2) If \(m\) is odd, \(m - 1\) is not in the range of \(GC\).

For (1), note that for \(m\) to be in the range of \(GC\), we would have to have a profile \(u\) with associated sequence \((s_1, s_2, \ldots s_m) = (a, a, \ldots, a)\) for some positive integer \(a\). But then we would have \(s_1 + s_2 + \ldots + s_m = ma = m(m-1)/2\) so \(a = (m-1)/2\) which is only possible if \(m\) is odd.

For (2), note that for \(m - 1\) to be in the range of \(GC\), we would have to have a profile \(u\) with associated sequence \((s_1, s_2, \ldots s_m) = (a, a, \ldots, a, b)\) for some positive integer \(a, b\) with \(b < a\). Then \(s_1 + s_2 + \ldots + s_m = (m-1)a + b = m(m-1)/2\) with \(0 \leq b < a\). Thus \((m-1)a \leq m(m-1)/2 < ma\) and so \((m-1)/2 < a \leq m/2\), which is not possible for an integer \(a\) if \(m\) is odd.

We now show that (1) and (2) are the only exceptions: For \(n = 3\) (and so also for all larger odd \(n\)), and even \(m > 2\), every \(k\), \(1 \leq k < m\) is in the range of \(GC\). For \(n = 3\) (and so also for all larger odd \(n\)), and odd \(m\), every \(k\), \(1 \leq k \leq m\) except \(m - 1\), is in the range of \(GC\).

By what we have already argued, it suffices to show that \(k\) appears in the range \"as soon as possible,\" i.e., for odd \(m\), we have \(k = m\) in the range for \(m\) alternatives and for even \(m\), we have \(k = m\) in the range for \(m + 2\) alternatives.

**Part 1: m odd.** To get \(k = m\) in the range for odd \(m\), we construct profiles where the Copeland score for all \(m\) alternatives is 1. First, note that \(k = 3\) is in the range for \(m = 3\) at the profile \(u_1:\)

|   | 1 | 2 | 3 |
|---|---|---|---|
| x | y | z |
| y | z | x |
| x | y |   |

(the classic voting paradox). For later comparison, we have emboldened the bottom two alternatives in \(u_1(2)\) and the bottom alternative in \(u_1(3)\). These emboldened alternatives make up the whole set of alternatives with no duplicates.

For \(m = 5\), we construct \(u_2\) from \(u_1\) by inserting two new alternatives, \(a\) and \(b\). Alternative \(a\) is inserted at the top of 1’s ordering and at the bottom -
emboldened - for #3.  b is inserted at the bottom of #1’s ordering and at the top for #2:

|   | 1  | 2  | 3  |
|---|---|---|---|
| a | b | z |   |
| x | y | x |   |
| y |   |   |   |
| z | z | y |   |
| b | x | a |   |

Finally, a is inserted at the middle position for #2 and b is inserted in the middle position for #3 (that b is emboldened):

|   | 1  | 2  | 3  |
|---|---|---|---|
| a | b | z |   |
| x | y | x |   |
| y | a | b |   |
| z | z | y |   |
| b | x | a |   |

The emboldened alternatives make up the whole set of alternatives with no duplicates. Each alternative has Copeland score of 2.

For \( m = 7 \), we construct \( u_3 \) from \( u_2 \) by inserting two new alternatives, c and d.  c is inserted at the top of 1’s ordering and at the bottom - emboldened - for #2.  Alternative d is inserted at the bottom of #1’s ordering and at the top for #3:

|   | 1  | 2  | 3  |
|---|---|---|---|
| c | b | d |   |
| a | y | z |   |
| x | a | x |   |
| y |   |   |   |
| z | z | b |   |
| b | x | y |   |
| d | c | a |   |

Finally, d and c are inserted at the middle position for #2 (where d is emboldened) and #3:

|   | 1  | 2  | 3  |
|---|---|---|---|
| c | b | d |   |
| a | y | z |   |
| x | a | x |   |
| y | d | c |   |
| z | z | b |   |
| b | x | y |   |
| d | c | a |   |
The emboldened alternatives make up the whole set of alternatives with no duplicates. Each alternative has Copeland score of 3.

We show one more iteration. For \( m = 9 \), we construct \( u_4 \) from \( u_3 \) by inserting two new alternatives, \( r \) and \( s \). \( r \) is inserted at the top of 1’s ordering and at the bottom - emboldened - for \( #3 \). Alternative \( s \) is inserted at the bottom of \( #1 \)’s ordering and at the top for \( #2 \):

\[
\begin{array}{ccc}
1 & 2 & 3 \\
r & s & d \\
c & b & z \\
a & y & x \\
x & a & c \\
y & \_ & \_ \\
z & d & b \\
b & z & y \\
d & x & a \\
s & c & r \\
\end{array}
\]

Finally, \( r \) and \( s \) are inserted at the middle position for \( #2 \) and \( #3 \) (where \( s \) is emboldened):

\[
\begin{array}{ccc}
1 & 2 & 3 \\
r & s & d \\
c & b & z \\
a & y & x \\
x & a & c \\
y & r & s \\
z & d & b \\
b & z & y \\
d & x & a \\
s & c & r \\
\end{array}
\]

The emboldened alternatives make up the whole set of alternatives with no duplicates. Each alternative has Copeland score of 4. This process, alternating which of \( #2 \) and \( #3 \) has a new alternative inserted at the bottom, can be continued indefinitely. For general odd \( m \), at the resulting profile every alternative has Copeland score \( (m - 1)/2 \).

**Part 2: m even.** For even \( m \), \( k = m - 2 \) is in the range for \( m \) alternatives. This requires only a very simple construction. Suppose we use Part 1 to start with a profile \( u \) for \( m - 1 \) alternatives (making up set \( X^* \)) such that every alternative in \( X^* \) has Copeland score \( (m - 2)/2 \). Then construct profile \( u^* \) by inserting a new alternative, say \( t \), as follows:

1. \( t \) is made the top-most alternative in \( u^*(1) \);
2. \( t \) is made the bottom-most alternative in \( u^*(2) \);
3. For individual \#3, \( t \) is inserted into the next-to-bottom space, just above the bottom element, call it \( z \), in \( u^*(3) \).
Then the Copeland scores of the $m - 2$ alternatives in $X^*$ other than $z$ are increased by 1 since they all now defeat $t$. The Copeland score of $z$ is unchanged. Since $t$ only defeats $z$, the Copeland score of $t$ is 1 less than the maximal Copeland score. The image of $G_C$ at $u^*$ is $X^* \setminus \{z\}$, of cardinality $m - 2$.

5 Approval voting

Approval voting (Brams and Fishburn, 1982) is not a collective choice correspondence since different information is used. For a fixed $B$, where $1 \leq B \leq m$, an index vector is an element of $B^N$. An extended social choice correspondence is a correspondence $G : L(X)^N \times B^N \rightarrow X$. Given preference profile $u = (u_1, ..., u_n)$ and index vector $b = (b_1, ..., b_n)$, the approval voting score for alternative $x$ is $A_S(x, u, b) = |\{i : x \in u(i)[1: b_i]\}|$, the number of individuals $i$ who have $x$ in their top $b_i$ ranks. Approval voting then is the extended social choice correspondence $G_A$ that sets $G_A(u, b) = \{x : A_S(x, u, b) \geq A_S(y, u, b)\}$ for all $y \in X$ where $B = M$, i.e., individuals can approve any number of alternatives.

The range of $G_A$ is all non-empty subsets of $X$. Given subset $S \subseteq X$, with $|S| = k \geq 1$, let $u$ be a profile where every individual has $S$ in their top $k$ ranks (in any order) and $b = (k, k, ..., k)$. Then $G_A(u, b) = S$.

But one can generally get $G_A(u, b) = S$ for index vectors much smaller than $(k, k, ..., k)$. Given index vector $b$, let the gauge of $b$ be the maximal value of the $b_i$ components.

**Proposition.** Let $|S| = k \geq 1$.

1. If $k \geq n$, there exists a profile $u$ and an index vector $b$ of gauge $g = \lceil k/n \rceil$ with $G_A(u, b) = S$; $g \leq 2$ with $G_A(u, b) = S$.

2. If $k < n$, there exists a profile $u$ and an index vector $b$ of gauge $g \leq 2$ with $G_A(u, b) = S$.

**Proof:** For (1), let $u$ be a profile where the $k$ elements of $S$ are strung out in the top $\lceil k/n \rceil$ ranks, with one occurrence of each (here illustrated with $k = 2n + 2$):

|   | 1  | 2  | 3  | 4  | ... | n  |
|---|----|----|----|----|-----|----|
| $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_n$ |
| $x_{n+1}$ | $x_{n+2}$ | $x_{n+3}$ | $x_{n+4}$ | ... | $x_{2n}$ |
| $x_{2n+1}$ | $x_{2n+2}$ | ... | ... | ... | ... |
| ... | ... | ... | ... | ... | ... |

and set $b = ([k/n], [k/n], [k/n] - 1, ..., [k/n] - 1)$.

For (2), where $n = kq + r$ for $0 \leq r < n$ some care must be taken with assumptions about $m$. 

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**Example.** Suppose \( n = 10 \) and \( k = 4 \). Here \( q = 2 \) and you might consider setting \( u \) to be

\[
\begin{array}{cccccccccc}
1 & 2 & 3 & 4 & \cdots & 7 & 8 & 9 & 10 \\
x_1 & x_1 & x_2 & x_2 & \cdots & x_4 & x_4 & x_5 & x_6 \\
\vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\
\end{array}
\]

with \( b = (1, 1, \ldots, 1) \). The problem with this is that while we know \( m \geq 4 \) (since \( k = 4 \)), we don’t know \( m \geq 6 \).

We deal with this by enlarging the number of occurrences of an element of \( S \) from \( q \) to \( q + 1 \) and spilling over to the second rank:

\[
\begin{array}{cccccccccc}
1 & 2 & 3 & 4 & \cdots & 7 & 8 & 9 & 10 \\
x_1 & x_1 & x_1 & x_2 & \cdots & x_3 & x_3 & x_3 & x_4 \\
x_4 & x_4 & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\
\end{array}
\]

with \( b = (2, 2, 1, \ldots, 1) \).

Generalizing, consider first the case of \( k < n/2 \). Thus \( 2k < n \) and in \( n = kq + r \) we have \( q \geq 2 \) and \( r < k \). We get bounds on \( k(q+1) \).

(i) \( n = kq + r < kq + k = k(q + 1) \);
(ii) \( 2n = 2kq + 2r > k(q + 1) + 2r > k(q + 1) \).

Thus we can string \( q + 1 \) occurrences of each of \( k \) alternatives along the top rank and then extend part way along the second rank and set \( b = (2, 2, \ldots, 2, 1, \ldots, 1) \) as in the way we dealt with the example just above.

Finally, we treat the case where \( n/2 < k < n \) and \( q = 1 \). We again establish bounds:

(i) \( 2k > n \) since \( n/2 < k \);
(ii) But \( 2k < 2n \) since \( k < n \).

Thus we can string \( 2 = q + 1 \) occurrences of each of \( k \) alternatives along the top rank and then extend part way along the second rank and set \( b = (2, 2, \ldots, 2, 1, \ldots, 1) \). \( \square \)

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