GEOMETRIC INTERPRETATION OF GENERALIZED
DISTANCE-SQUARED MAPPINGS OF $\mathbb{R}^2$ INTO $\mathbb{R}^\ell$ ($\ell \geq 3$)

SHUNSUKE ICHIKI

Abstract. Generalized distance-squared mappings are quadratic mappings of $\mathbb{R}^m$ into $\mathbb{R}^\ell$ of special type. In the case that matrices $A$ constructed by coefficients of generalized distance-squared mappings of $\mathbb{R}^2$ into $\mathbb{R}^\ell$ ($\ell \geq 3$) are full rank, the generalized distance-squared mappings having a generic central point have the following properties. In the case of $\ell = 3$, they have only one singular point. On the other hand, in the case of $\ell > 3$, they have no singular points. Hence, in this paper, the reason why in the case of $\ell = 3$ (resp., in the case of $\ell > 3$), they have only one singular point (resp., no singular points) is explained by giving a geometric interpretation to these phenomena.

1. Introduction

Throughout this paper, positive integers are expressed by $i$, $j$, $\ell$ and $m$. In this paper, unless otherwise stated, all mappings belong to class $C^\infty$. Two mappings $f : \mathbb{R}^m \to \mathbb{R}^\ell$ and $g : \mathbb{R}^m \to \mathbb{R}^\ell$ are said to be $A$-equivalent if there exist two diffeomorphisms $h : \mathbb{R}^m \to \mathbb{R}^m$ and $H : \mathbb{R}^\ell \to \mathbb{R}^\ell$ satisfying $f = H \circ g \circ h^{-1}$. Let $p_i = (p_{i1}, p_{i2}, \ldots, p_{im})$ ($1 \leq i \leq \ell$) (resp., $A = (a_{ij})_{1 \leq i \leq \ell, 1 \leq j \leq m}$) be a point of $\mathbb{R}^m$ (resp., an $\ell \times m$ matrix with non-zero entries). Set $p = (p_1, p_2, \ldots, p_\ell) \in (\mathbb{R}^m)^\ell$. Let $G_{(p,A)} : \mathbb{R}^m \to \mathbb{R}^\ell$ be the mapping defined by

$$G_{(p,A)}(x) = \left( \sum_{j=1}^m a_{1j} (x_j - p_{1j})^2, \sum_{j=1}^m a_{2j} (x_j - p_{2j})^2, \ldots, \sum_{j=1}^m a_{\ell j} (x_j - p_{\ell j})^2 \right),$$

where $x = (x_1, x_2, \ldots, x_m) \in \mathbb{R}^m$. The mapping $G_{(p,A)}$ is called a generalized distance-squared mapping, and the $\ell$-tuple of points $p = (p_1, p_\ell) \in (\mathbb{R}^m)^\ell$ is called the central point of the generalized distance-squared mapping $G_{(p,A)}$. For a given matrix $A$, a property of generalized distance-squared mappings will be said to be true for a generalized distance-squared mapping having a generic central point if there exists a subset $\Sigma$ with Lebesgue measure zero of $(\mathbb{R}^m)^\ell$ such that for any $p \in (\mathbb{R}^m)^\ell \setminus \Sigma$, the mapping $G_{(p,A)}$ has the property. A distance-squared mapping $D_p$ (resp., Lorentzian distance-squared mapping $L_p$) is the mapping $G_{(p,A)}$ satisfying that each entry of $A$ is 1 (resp., $a_{11} = -1$ and $a_{ij} = 1$ ($j \neq 1$)). In [3] (resp., [4]), a classification result on distance-squared mappings $D_p$ (resp., Lorentzian distance-squared mappings $L_p$) is given. Moreover, in [6] (resp., [5]), a classification result on generalized distance-squared mappings $G_{(p,A)}$ of $\mathbb{R}^2$ into $\mathbb{R}^2$ (resp., $\mathbb{R}^{m+1}$ into $\mathbb{R}^\ell$ ($\ell \geq 2m + 1$)) is given.

2010 Mathematics Subject Classification. 53A04, 57R45, 57R50.
Key words and phrases. generalized distance-squared mapping, geometric interpretation, singularity, $A$-equivalence.

The author is Research Fellow DC1 of Japan Society for the Promotion of Science.
The important motivation for these investigations is as follows. Height functions and distance-squared functions have been investigated in detail so far, and they are well known as a useful tool in the applications of singularity theory to differential geometry (for example, see [1]). The mapping in which each component is a height function is nothing but a projection. Projections as well as height functions or distance-squared functions have been investigated so far.

On the other hand, the mapping in which each component is a distance-squared function is a distance-squared mapping. Besides, the notion of generalized distance-squared mappings is an extension of the distance-squared mappings. Therefore, it is natural to investigate generalized distance-squared mappings as well as projections.

In [6], a classification result on generalized distance-squared mappings of $\mathbb{R}^2$ into $\mathbb{R}^2$ is given. If the rank of $A$ is two, the generalized distance-squared mapping having a generic central point is a mapping of which any singular point is a fold point except one cusp point. The singular set is a rectangular hyperbola. Moreover, in [6], a geometric interpretation of a singular set of generalized distance-squared mappings of $\mathbb{R}^2$ into $\mathbb{R}^2$ having a generic central point is also given in the case of rank $A = 2$. By the geometric interpretation, the reason why the mappings have only one cusp point is explained.

On the other hand, in [5], a classification result on generalized distance-squared mappings of $\mathbb{R}^{m+1}$ into $\mathbb{R}^\ell$ ($\ell \geq 2m + 1$) is given. As the special case of $m = 1$, we have the following.

**Theorem 1 ([5]).** Let $\ell$ be an integer satisfying $\ell \geq 3$. Let $A = (a_{ij})_{1 \leq i \leq \ell, 1 \leq j \leq 2}$ be an $\ell \times 2$ matrix with non-zero entries satisfying $\text{rank } A = 2$. Then, the following hold:

1. In the case of $\ell = 3$, there exists a proper algebraic subset $\Sigma_A \subset (\mathbb{R}^2)^3$ such that for any $p \in (\mathbb{R}^2)^3 - \Sigma_A$, the mapping $G(p, A)$ is $A$-equivalent to the normal form of Whitney umbrella $(x_1, x_2) \mapsto (x_1^2, x_1x_2, x_2)$.
2. In the case of $\ell > 3$, there exists a proper algebraic subset $\Sigma_A \subset (\mathbb{R}^2)^\ell$ such that for any $p \in (\mathbb{R}^2)^\ell - \Sigma_A$, the mapping $G(p, A)$ is $A$-equivalent to the inclusion $(x_1, x_2) \mapsto (x_1, x_2, 0, \ldots, 0)$.

As described above, in [6], a geometric interpretation of generalized distance-squared mappings of $\mathbb{R}^2$ into $\mathbb{R}^2$ having a generic central point is given in the case that the matrix $A$ is full rank. On the other hand, in this paper, a geometric interpretation of generalized distance-squared mappings of $\mathbb{R}^2$ into $\mathbb{R}^\ell$ ($\ell \geq 3$) having a generic central point is given in the case that the matrix $A$ is full rank (for the reason why we concentrate on the case that the matrix $A$ is full rank, see Remark 1.1). Hence, by [6] and this paper, geometric interpretations of generalized distance-squared mappings of the plane having a generic central point in the case that the matrix $A$ is full rank are completed.

The main purpose of this paper is to give a geometric interpretation of Theorem 1. Namely, the main purpose of this paper is to answer the following question.

**Question 1.1.** Let $\ell$ be an integer satisfying $\ell \geq 3$. Let $A = (a_{ij})_{1 \leq i \leq \ell, 1 \leq j \leq 2}$ be an $\ell \times 2$ matrix with non-zero entries satisfying $\text{rank } A = 2$.

1. In the case of $\ell = 3$, why do generalized distance-squared mappings $G(p, A) : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ having a generic central point have only one singular point?
On the other hand, in the case of $\ell > 3$, why do generalized distance-squared mappings $G(p,A) : \mathbb{R}^2 \rightarrow \mathbb{R}^\ell$ having a generic central point have no singular points?

1.1. Remark. In the case that the matrix $A$ is not full rank (rank $A = 1$), for any $\ell \geq 3$, the generalized distance-squared mapping of $\mathbb{R}^2$ into $\mathbb{R}^\ell$ having a generic central point is $A$-equivalent to only the inclusion $(x_1, x_2) \mapsto (x_1, x_2, 0, \ldots, 0)$ (see Theorem 3 in [4]). On the other hand, in the case that the matrix $A$ is full rank (rank $A = 2$), the phenomenon in the case of $\ell = 3$ is completely different from the phenomenon in the case of $\ell > 3$ (see Theorem 1). Hence, we concentrate on the case that the matrix $A$ is full rank.

In Section 2, some assertions and definitions are prepared for answering Question 1.1, and the answer to Question 1.1 is stated. In Section 3, the proof of a lemma of Section 2 is given.

2. The answer to Question 1.1

Firstly, in order to answer Question 1.1, some assertions and definitions are prepared. By Theorem 1, it is clearly seen that the following assertion holds. The assertion is important for giving an geometric interpretation.

**Corollary 1.** Let $\ell$ be an integer satisfying $\ell \geq 3$. Let $A = (a_{ij})_{1 \leq i \leq \ell, 1 \leq j \leq 2}$ (resp., $B = (b_{ij})_{1 \leq i \leq \ell, 1 \leq j \leq 2}$) be an $\ell \times 2$ matrix with non-zero entries satisfying rank $A = 2$ (resp., rank $B = 2$). Then, there exist proper algebraic subsets $\Sigma_A$ and $\Sigma_B$ of $(\mathbb{R}^2)^\ell$ such that for any $p \in (\mathbb{R}^2)^\ell - \Sigma_A$ and for any $q \in (\mathbb{R}^2)^\ell - \Sigma_B$, the mapping $G(p,A)$ is $A$-equivalent to the mapping $G(q,B)$.

Let $F_p : \mathbb{R}^2 \rightarrow \mathbb{R}^\ell$ ($\ell \geq 3$) be the mapping defined by

$$F_p(x_1, x_2) = (a(x_1 - p_{11})^2 + b(x_2 - p_{12})^2, (x_1 - p_{21})^2 + (x_2 - p_{22})^2, \ldots, (x_1 - p_{\ell 1})^2 + (x_2 - p_{\ell 2})^2),$$

where $0 < a < b$ and $p = (p_{11}, p_{12}, \ldots, p_{\ell 1}, p_{\ell 2})$. Remark that the mapping $F_p$ is the generalized distance-squared mapping $G(p,B)$, where

$$B = \begin{pmatrix} a & b \\ 1 & 1 \\ \vdots & \vdots \\ 1 & 1 \end{pmatrix}.$$ 

Since the rank of the matrix $B$ is two, by Corollary 1, in order to answer Question 1.1, it is sufficient to answer the following question.

**Question 2.1.** Let $\ell$ be an integer satisfying $\ell \geq 3$.

1. In the case of $\ell = 3$, why do the mappings $F_p : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ having a generic central point have only one singular point?

2. On the other hand, in the case of $\ell > 3$, why do the mappings $F_p : \mathbb{R}^2 \rightarrow \mathbb{R}^\ell$ having a generic central point have no singular points?
The mapping $F_p = (F_1, \ldots, F_\ell)$ determines $\ell$-foliations $C_{p_i}(c_1), \ldots, C_{p_\ell}(c_\ell)$ in the plane defined by
\[ C_{p_i}(c_i) = \{(x_1, x_2) \in \mathbb{R}^2 \mid F_i(x_1, x_2) = c_i\}, \]
where $c_i \geq 0$ ($1 \leq i \leq \ell$) and $p = (p_1, \ldots, p_\ell) \in (\mathbb{R}^2)^\ell$. For a given central point $p = (p_1, \ldots, p_\ell) \in (\mathbb{R}^2)^\ell$, a point $q \in \mathbb{R}^2$ is a singular point of the mapping $F_p$ if and only if the $\ell$-foliations $C_{p_i}(c_i)$ ($1 \leq i \leq \ell$) defined by the point $p$ are tangent at the point $q$, where $(c_1, \ldots, c_\ell) = F_p(q)$.

For a given central point $p = (p_1, \ldots, p_\ell) \in (\mathbb{R}^2)^\ell$, in the case that a point $q \in \mathbb{R}^2$ is a singular point of the mapping $F_p$, there may exist an integer $i$ such that the foliation $C_{p_i}(c_i)$ is merely a point, where $c_i = F_i(q)$. However, by the following lemma, we see that the trivial phenomenon seldom occurs (for the proof of Lemma 2.1, see Section 3).

**Lemma 2.1.** Let $\ell$ be an integer satisfying $\ell \geq 3$. Then, there exists a proper algebraic subset $\Sigma \subset (\mathbb{R}^2)^\ell$ such that for any central point $p = (p_1, \ldots, p_\ell) \in (\mathbb{R}^2)^\ell - \Sigma$, if a point $q \in \mathbb{R}^2$ is a singular point of the mapping $F_p$, then the $\ell$-foliations $C_{p_i}(c_1)$ and $C_{p_\ell}(c_i)$ ($2 \leq i \leq \ell$) are an ellipse and $(\ell - 1)$-circles, respectively, where $(c_1, \ldots, c_\ell) = F_p(q)$.

### 2.1. Answer to Question 1.1

As described above, in order to answer Question 1.1, it is sufficient to answer Question 2.1.

1. We will answer (1) of Question 2.1. The phenomenon that the mapping $F_p : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ having a generic central point has only one singular point can be explained by the following geometric interpretation. Namely, constants $c_i \geq 0$ ($i = 1, 2, 3$) such that three foliations $C_{p_1}(c_1), C_{p_2}(c_2)$ and $C_{p_3}(c_3)$ defined by the central point $p = (p_1, p_2, p_3) \in (\mathbb{R}^2)^3$ are tangent are uniquely determined, and the tangent point is also unique. Moreover, in the case, remark that by Lemma 2.1 the three foliations $C_{p_1}(c_1), C_{p_2}(c_2)$ and $C_{p_3}(c_3)$ defined by almost all (in the sense of Lebesgue measure) $(p_1, p_2, p_3) \in (\mathbb{R}^2)^3$ are an ellipse and two circles, respectively.

Furthermore, by the geometric interpretation, we can also see the location of the singular point of the mapping $F_p$ having a generic central point (for example, see Figure 1).

2. We will answer (2) of Question 2.1. The phenomenon that the mapping $F_p : \mathbb{R}^2 \rightarrow \mathbb{R}^\ell$ ($\ell > 3$) having a generic central point has no singular points can be explained by the following geometric interpretation. Namely, for any constants $c_i \geq 0$ ($1 \leq i \leq \ell$), $\ell$-foliations $C_{p_1}(c_1), \ldots, C_{p_\ell}(c_\ell)$ defined by the central point $p = (p_1, \ldots, p_\ell) \in (\mathbb{R}^2)^\ell$ are not tangent at any points (for example, see Figure 2).
2.2. Remark. The geometric interpretation (the answer to Question 1.1) has one more advantage. By the interpretation, we get the following assertion from the viewpoint of the contacts amongst one ellipse and some circles.

**Corollary 2.** Let $a, b$ be real numbers satisfying $0 < a < b$.

1. There exists a proper algebraic subset $\Sigma$ of $(\mathbb{R}^2)^3$ such that for any $(p_1, p_2, p_3) \in (\mathbb{R}^2)^3 - \Sigma$, constants $c_i > 0$ ($i = 1, 2, 3$) such that one ellipse $a(x_1 - p_{11})^2 + b(x_2 - p_{12})^2 = c_1$ and two circles $(x_1 - p_{i1})^2 + (x_2 - p_{i2})^2 = c_i$ ($i = 2, 3$) are tangent are uniquely determined, where $p_i = (p_{i1}, p_{i2})$. Moreover, the tangent point is also unique.

2. On the other hand, in the case of $\ell > 3$, there exists a proper algebraic subset $\Sigma$ of $(\mathbb{R}^2)^\ell$ such that for any $(p_1, \ldots, p_\ell) \in (\mathbb{R}^2)^\ell - \Sigma$, for any $c_i > 0$
is a singular point of \( F \), the one ellipse \( a(x_1 - p_{11})^2 + b(x_2 - p_{12})^2 = c_1 \) and the \((\ell - 1)\)-circles \((x_1 - p_{i1})^2 + (x_2 - p_{i2})^2 = c_i \) \((i = 2, \ldots, \ell)\) are not tangent at any points, where \( p_i = (p_{i1}, p_{i2}) \).

3. Proof of Lemma 2.1

The Jacobian matrix of the mapping \( F_p \) at \((x_1, x_2)\) is the following.

\[
JF_p(x_1, x_2) = 2 \begin{pmatrix}
a(x_1 - p_{11}) & b(x_2 - p_{12}) \\
\vdots & \vdots \\
x_1 - p_{\ell1} & x_2 - p_{\ell2}
\end{pmatrix}.
\]

Let \( \Sigma_i \) be a subset of \((\mathbb{R}^2)^\ell\) consisting of \( p = (p_1, \ldots, p_\ell) \in (\mathbb{R}^2)^\ell \) satisfying \( p_i \in \mathbb{R}^2 \) is a singular point of \( F_p \) \((1 \leq i \leq \ell)\). Namely, for example, \( \Sigma_1 \) is the subset of \((\mathbb{R}^2)^\ell\) consisting of \( p = (p_1, \ldots, p_\ell) \in (\mathbb{R}^2)^\ell \) satisfying

\[
\text{rank} \begin{pmatrix}
0 & 0 & \vdots & \vdots & \vdots \\
p_{11} - p_{21} & p_{12} - p_{22} & \vdots & \vdots & p_{1\ell} - p_{2\ell}
\end{pmatrix} < 2.
\]

By \( \ell \geq 3 \), it is clearly seen that \( \Sigma_i \) is a proper algebraic subset of \((\mathbb{R}^2)^\ell\). Similarly, for any \( i \) \((2 \leq i \leq \ell)\), we see that \( \Sigma_i \) is also a proper algebraic subset of \((\mathbb{R}^2)^\ell\). Set \( \Sigma = \bigcup_{i=1}^{\ell} \Sigma_i \). Then, \( \Sigma \) is also a proper algebraic subset of \((\mathbb{R}^2)^\ell\).

Let \( p = (p_1, \ldots, p_\ell) \in (\mathbb{R}^2)^\ell - \Sigma \) be a central point, and let \( q \) be a singular point of the mapping \( F_p \) defined by the central point. Then, suppose that there exists an integer \( i \) such that the foliation \( C_{p_i}(c_i) \) is not an ellipse or a circle, where \( c_i = F_i(q) \) \((F_p = (F_1, \ldots, F_\ell))\). Then, we get \( c_i = 0 \). Hence, we have \( q = p_i \). This contradicts the assumption \( p \in (\mathbb{R}^2)^\ell - \Sigma \).

\[ \square \]

Acknowledgements

The author is grateful to Takashi Nishimura for his kind advices. The author is supported by JSPS KAKENHI Grant Number 16J06911.

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Graduate School of Environment and Information Sciences, Yokohama National University, Yokohama 240-8501, Japan

E-mail address: ichiki-shunsuke-jb@ynu.jp