Rare Decay $K^+ \to \pi^+ \nu \bar{\nu}$ at the Next-to-Next-to-Leading Order in QCD

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We calculate the charm quark contribution to the rare decay $K^+ \to \pi^+ \nu \bar{\nu}$ in the next-to-next-to-leading order of QCD. This new contribution reduces the theoretical uncertainty in the relevant parameter $P_\tau$ from $\pm 1.0\%$ down to $\pm 0.04\%$, corresponding to scale uncertainties of $\pm 1.3\%$, $\pm 1.0\%$, $\pm 0.006$ and $\pm 1.2\%$ in $\mathcal{B}(K^+ \to \pi^+ \nu \bar{\nu})$ and in $|V_{td}|$, $\sin 2\beta$ and $\gamma$ extracted from the $K \to \nu \bar{\nu}$ system. The error in $P_\tau = 0.37 \pm 0.04$ is now fully dominated by the current uncertainty of $\pm 3.8\%$ in the charm quark mass $m_c$. We find $\mathcal{B}(K^+ \to \pi^+ \nu \bar{\nu}) = (8.0 \pm 1.1) \times 10^{-11}$, where the quoted error stems almost entirely from the present uncertainties in $m_c$ and the Cabibbo-Kobayashi-Maskawa elements.

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The rare process $K^+ \to \pi^+ \nu \bar{\nu}$ belongs to the theoretically cleanest decays in the field of $K$ and $B$ mesons. As it offers in conjunction with $K_L \to \pi^0 \nu \bar{\nu}$ a very clean determination of the standard unitarity triangle [1], a comparison of the information obtained from the $K \to \nu \bar{\nu}$ system with the one from $B$ decays provides a critical and truly unique test of the Cabibbo-Kobayashi-Maskawa (CKM) mechanism in the standard model (SM) [2, 3]. Even if these $K$- and $B$-physics predictions agree, $K^+ \to \pi^+ \nu \bar{\nu}$ will allow to discriminate between different extensions of the SM [2, 3], by probing effective scales of new physics operators of up to a several TeV or even higher [4].

In the SM the decay $K^+ \to \pi^+ \nu \bar{\nu}$ proceeds through $Z$-penguin and electroweak box diagrams, which are sensitive to short-distance dynamics. As the required hadronic matrix elements can be extracted, including isospin breaking corrections [5], from the accurately measured leading semileptonic decay $K^+ \to \pi^0 e^+ \nu$, and the remaining long-distance contributions turn out to be small [6], and in principle calculable by means of lattice QCD [7], theoretical computations of the relevant decay rate can reach an exceptionally high degree of precision. After summation over the three neutrino flavors the resulting branching ratio for $K^+ \to \pi^+ \nu \bar{\nu}$ can be written as [3, 8, 9]

\[
\mathcal{B}(K^+ \to \pi^+ \nu \bar{\nu}) = \kappa^+ \left[ \left( \frac{\text{Im} \lambda}{\lambda^8} X(x_t) \right)^2 + \left( \frac{\text{Re} \lambda}{\lambda^8} X(x_t) + \frac{\text{Re} \lambda_t}{\lambda} (P_c + \delta P_{c,u}) \right)^2 \right],
\]

\[
\kappa^+ \equiv r_{K^+} \frac{3\alpha^2 B(K^+ \to \pi^0 e^+ \nu)}{2\pi^2 \sin^4 \theta_w} \lambda^8 = (5.04 \pm 0.17) \times 10^{-11} \left( \frac{\lambda}{0.2248} \right)^8.
\]

Here $\lambda \equiv V^\ast_{ts} V_{td}$ denote the relevant CKM factors, while $\delta P_{c,u} = 0.04 \pm 0.02$ encodes the long-distance contributions calculated recently in [4], and the parameter $r_{K^+} = 0.901 \pm 0.027$ summarizes isospin breaking corrections in relating $K^+ \to \pi^+ \nu \bar{\nu}$ to $K^+ \to \pi^0 e^+ \nu$ [5]. The apparent strong dependence of $\mathcal{B}(K^+ \to \pi^+ \nu \bar{\nu})$ on $\lambda \equiv |V_{ts}|$ is spurious as $P_c$ and $\delta P_{c,u}$ are proportional to $1/\lambda^4$. In quoting the value for $P_c$ and $\mathcal{B}(K^+ \to \pi^+ \nu \bar{\nu})$ we will set $\lambda = 0.2248$ [10]. The electromagnetic coupling $\alpha$ and the weak mixing angle $\sin^2 \theta_w$ entering $\mathcal{B}(K^+ \to \pi^+ \nu \bar{\nu})$ are naturally evaluated at the electroweak scale [11]. Then the leading term in the heavy top expansion of the electroweak two-loop corrections to $X(x_t)$ amounts to typically $-1\%$ for the modified minimal subtraction scheme (MS) definition of $\alpha$ and $\sin^2 \theta_w$ [12]. In obtaining the numerical value of Eq. (2) we have employed $\alpha \equiv \alpha_{\text{MS}}(M_Z) = 1/127.9$, $\sin^2 \theta_w \equiv \sin^2 \theta_{w,\text{MS}} = 0.231$, and $\mathcal{B}(K^+ \to \pi^0 e^+ \nu) = (4.93 \pm 0.07) \times 10^{-2}$ [13].

The function $X(x_t)$ in Eq. (11) depends on the top quark MS mass through $x_t \equiv m_t^2(\mu_t)/M^2_u$. It originates from $Z$-penguin and electroweak box diagrams with an internal top quark. As the relevant operator has a vanishing anomalous dimension and the energy scales involved are of the order of the electroweak scale or higher, the function $X(x_t)$ can be calculated within ordinary perturbation theory. It is known through next-to-leading order (NLO) [3, 14], with a scale uncertainty due to the top quark matching scale $\mu_t = O(m_t)$.
of only ±1%. Converting the top quark pole mass of \( m_t = (172.7 ± 2.9) \text{ GeV} \) at three loops to \( m_t(M_t) \) \(^{10}\) and relating \( m_t(M_t) \) to \( m_t(m_t) = (163.0 ± 2.8) \text{ GeV} \) using the one-loop renormalization group (RG), we find \( X(x_t) = 1.464 ± 0.041 \). The given uncertainty combines linearly an error of ±0.028 due to the error of \( m_t(m_t) \) and an error of ±0.013 obtained by varying \( \mu_t \) in the range 60 GeV ≤ \( \mu_t \) ≤ 240 GeV.

The calculable parameter \( P_c \) entering Eq. (1) results from Z-penguin and electroweak box diagrams involving internal charm quark exchange. As now both high- and low-energy scales, namely, \( \mu_w = O(M_w) \) and \( \mu_c = O(m_c) \), are involved, a complete RG analysis of this term is required. In this manner, large logarithms \( \ln(\mu_w^2/\mu_c^2) \) are resummed to all orders in \( \alpha_s \). At the leading order such an analysis has been performed in \(^{17}\). The large scale uncertainty due to \( \mu_c \) of ±26% in this result was a strong motivation for the NLO analysis of this contribution \(^{8,10}\).

Performing the RG running from \( \mu_w \) down to \( \mu_c = O(m_c) \) in an effective five-flavor theory and the subsequent evolution from \( \mu_b \) down to \( \mu_c \) in an effective four-flavor theory, we obtain at the NLO

\[
P_c = 0.367 ± 0.037_{\text{theor}} ± 0.033_{m_c} ± 0.009_{\alpha_s} = (0.37 ± 0.06) \left( \frac{0.2248}{\lambda} \right)^4, \tag{3}
\]

where the parametric errors correspond to the ranges of the charm quark \( m_c \) mass \( m_c \) and the strong coupling constant \( \alpha_s(M_Z) \) given in Table I. We note that the final error has only an illustrative character, since the partial uncertainties are not statistically distributed. Numerically, it is the mean of the values obtained by adding the individual errors once linearly and once in quadrature. The same way of combining errors will be applied in Eqs. (4), (8), and (9).

The dependence of \( P_c \) on \( \mu_c \) can be seen in Fig. 1. The solid line in the upper plot shows the NLO result obtained by evaluating \( \alpha_s(\mu_c) \) from \( \alpha_s(M_Z) \) solving the RG equation of \( \alpha_s \) numerically, while the dashed and dotted lines are obtained by first determining the scale parameter \( \Lambda_{\overline{MS}} \) from \( \alpha_s(M_Z) \), either using the explicit solution of the RG equation of \( \alpha_s \) or by solving the RG equation of \( \alpha_s \) iteratively for \( \Lambda_{\overline{MS}} \), and subsequently calculating \( \alpha_s(\mu_c) \) from \( \Lambda_{\overline{MS}} \). The corresponding two-loop values for \( \alpha_s(\mu_c) \) have been obtained with the program RUNDEC \(^{10}\). Obviously, the difference between the three curves is due to higher order terms and has to be regarded as part of the theoretical error. With its size of ±0.012 it is comparable to the variation of the NLO result due to \( \mu_c \), amounting to ±0.020. In \(^{3,8,10}\) larger values for the latter uncertainty have been quoted. The observed difference is related to the definition of the charm quark mass. Replacing \( m_c(m_c) \) in the logarithms \( \ln(\mu_w^2/\mu_c^2) \) of the one-loop matrix elements by the more appropriate \( m_t(\mu_c) \) leads to a significant reduction of the dependence of \( P_c \) on \( \mu_c \). A detailed discussion of this issue will be presented in \(^{20}\). Finally, while in \(^{3,8,10}\) only \( \mu_c \) was varied, the theoretical error given in Eq. (3) includes also the dependence on \( \mu_b \) and \( \mu_w \) of ±0.004 and ±0.001, respectively. The specified scale uncertainties correspond to the ranges 1 GeV ≤ \( \mu_c \) ≤ 3 GeV, 2.5 GeV ≤ \( \mu_b \) ≤ 10 GeV, and 40 GeV ≤ \( \mu_w \) ≤ 160 GeV.

Using the input parameters listed in Table I, we find

![Figure 1: P_c as a function of \( \mu_c \) at NLO (upper plot) and NNLO (lower plot). The three different lines correspond to three different methods of computing \( \alpha_s(\mu_c) \) from \( \alpha_s(M_Z) \) (see text).](image-url)
from Eqs. (1)–(3) at the NLO  27

\[ \mathcal{B}(K^+ \rightarrow \pi^+\bar{\nu}\nu) = (7.93 \pm 0.77P_c \pm 0.84_{\text{other}}) \times 10^{-11} \]

\[ = (7.9 \pm 1.3) \times 10^{-11}, \quad (4) \]

where the second error in the first line collects the uncertainties due to \( \kappa_{+}, \delta P_{\text{GWS}}, X(x_t) \), and the CKM elements. The final error has only an illustrative character, as the individual uncertainties have no statistical interpretation. Numerically, the enhancement of \( \mathcal{B}(K^+ \rightarrow \pi^+\nu\bar{\nu}) \) coming from \( \delta P_{\text{GWS}} \) has been compensated by the suppression due to the decrease of \( M_{K} \).

Provided \( P_{c} \) is known with a sufficient precision, a measurement of \( K^+ \rightarrow \pi^+\nu\bar{\nu} \), either alone or together with one of \( K_{L} \rightarrow \pi^0\nu\bar{\nu} \), allows for precise determinations of the CKM parameters  1. The comparison of this unitarity triangle with the one from \( B \) physics offers a stringent and unique test of the SM. In particular, the measurement of \( K_{L} \rightarrow \pi^0\nu\bar{\nu} \) due to the decrease of \( \delta P_{\text{GWS}} \) makes it necessary to introduce a Chern-Simons operator describing the interaction of neutrinos with similar formulas given in 3. Here \( V_{td} \) is the element of the CKM matrix and \( \beta \) and \( \gamma \) are the angles in the standard unitarity triangle. As the uncertainties in Eqs. (3) and (4) coming from the charm quark mass and the CKM parameters should be decreased in the coming years it is also desirable to reduce the theoretical uncertainty in \( P_{c} \). To this end, we here extend the NLO analysis of \( P_{c} \) presented in 3, 6 to the next-to-next-to-leading order (NNLO). This requires the computation of three-loop anomalous dimensions of certain operators and of certain two-loop contributions.

The main components of the NNLO calculation  20, which aims at resuming all \( \mathcal{O}(\alpha^6_{\text{em}}\ln n^{-1}(\mu^2_{\nu}/\mu^2_{\text{w}})) \) logarithms in \( P_{c} \), are (i) the \( \mathcal{O}(\alpha^2_{\text{em}}) \) matching corrections to the relevant Wilson coefficients arising at \( \mu_{\text{w}} \), (ii) the \( \mathcal{O}(\alpha^3_{\text{em}}) \) anomalous dimensions describing the mixing of the dimension-six and -eight operators, (iii) the \( \mathcal{O}(\alpha^2_{\text{em}}) \) threshold corrections to the Wilson coefficients originating at \( \mu_{b} \), and (iv) the \( \mathcal{O}(\alpha^2_{\text{em}}) \) matrix elements of some of the operators emerging at \( \mu_{c} \).

Conceptual new features in our NNLO calculation are (a) the appearance of the vector component of the effective neutral-current coupling describing the interaction of neutrinos and quarks mediated by \( Z \)-boson exchange, (b) the presence of anomalous triangle contributions which make it necessary to introduce a Chern-Simons operator in order to obtain the correct anomalous Ward identity involving the axial-vector coupling of the \( Z \) boson, and (c) the existence of nontrivial two-loop matching corrections to the Wilson coefficients of the current-current operators at the bottom quark threshold.

To determine the contributions of type (i), (iii), and (iv) one must calculate two-loop Green functions in the full SM and in effective theories with five or four flavors. Sample diagrams for steps (i) and (iv) are shown in the left and right columns of Fig. 2. The contributions (ii) are found by calculating three-loop Green functions with operator insertions. Sample diagrams with a double insertion of dimension-six operators are shown in the center column of Fig. 2. The corresponding three-loop amplitudes are evaluated using the method that has been described in  21, 22. A comprehensive discussion of the technical details of the matching, the renormalization of the effective theory and the actual calculation will be given in  21.

Having described the general steps of our calculation, we now present our results. Using the general RG formalism 22, 23, we find at the NNLO

\[ P_{c} = 0.371 \pm 0.009_{\text{theor}} \pm 0.031_{m_{e}} \pm 0.009_{\alpha_{e}} \]

\[ = (0.37 \pm 0.04) \left( \frac{0.2248}{\lambda} \right)^4, \quad (8) \]

where the final error is fully dominated by the uncertainty in \( m_{e}(m_{c}) \). Comparing these numbers with Eq. 3 we observe that our NNLO calculation reduces the theoretical uncertainty by a factor of 4.

As can be nicely seen in the lower plot of Fig. 1, \( P_{c} \) depends very weakly on \( \mu_{c} \) at the NNLO, varying by only \( \pm 0.0047 \). Furthermore, the three different treatments of \( \alpha_{e} \) affect the NNLO result by as little as \( \pm 0.0005 \). The three-loop values of \( \alpha_{e}(m_{e}) \) used in the numerical anal-
yxis have been obtained with the program RUNDEC \cite{22}. The theoretical error quoted in Eq. (8) includes also the dependence on $\mu_0$ and $\mu_w$ of $\pm 0.0028$ and $\pm 0.0007$, respectively. The presented scale uncertainties correspond to the ranges given earlier.

Using Eqs. (11), (2), and (5) the result in Eq. (4) is modified to the NNLO value [27]

$$B(K^+ \to \pi^+ \nu \bar{\nu}) = (7.96 \pm 0.49_{\text{stat}} \pm 0.84_{\text{other}}) \times 10^{-11}$$

$$= (8.0 \pm 1.1) \times 10^{-11}.$$

Employing Eqs. (5)–(7) the reduction of the theoretical error in $P_\tau$ from $\pm 10.1\%$ down to $\pm 2.4\%$ translates into the following uncertainties:

$$\sigma ([V_{td}]) = \begin{cases} 4.1\%, & \text{NLO}, \\ 1.0\%, & \text{NNLO}, \end{cases}$$

$$\sigma (\sin 2\beta) = \begin{cases} 0.025, & \text{NLO}, \\ 0.006, & \text{NNLO}, \end{cases}$$

$$\sigma (\gamma) = \begin{cases} 4.9^\circ, & \text{NLO}, \\ 1.2^\circ, & \text{NNLO}, \end{cases}$$

implying a very significant improvement of the NNLO over the NLO results. In obtaining these numbers we have used $\sin 2\beta = 0.724$ and $\gamma = 58.6^\circ$, and included only the theoretical errors quoted in Eqs. (5) and (6).

On the experimental side the Alternating Gradient Synchrotron E787 and E949 Collaborations at Brookhaven observed the decay $K^+ \to \pi^+ \nu \bar{\nu}$ finding three events so far \cite{24}. The resulting branching ratio is

$$B(K^+ \to \pi^+ \nu \bar{\nu}) = (14.7_{-8.9}^{+13.0}) \times 10^{-11}.$$

Within theoretical, parametric, and experimental uncertainties, Eq. (13) is fully consistent with the data. The prospects for the future measurements of $B(K^+ \to \pi^+ \nu \bar{\nu})$ can be found in \cite{25}.

To conclude, we have evaluated the complete NNLO correction of thecharm quark contribution to $B(K^+ \to \pi^+ \nu \bar{\nu})$. The inclusion of these contributions leads to a drastic reduction of the theoretical uncertainty in the relevant parameter $P_\tau$. This strengthens the power of the $K \to \pi \nu \bar{\nu}$ system in determining the CKM parameters and increases its reach to new physics, in particular, if future experimental values of $B(K^+ \to \pi^+ \nu \bar{\nu})$ will not differ much from the SM prediction.

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\textbf{Note added:} Contrary to the statement made in point (b) in the second paragraph after Eq. (7) the initial condition of the Chern-Simons operator vanishes. The numerical effect of this mistake is completely negligible, see the erratum of \cite{26} for details.

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[27] Using the input parameters of the UTfit Collaboration that are available at http://utfit.roma1.infn.it/ we find $B(K^+ \to \pi^+ \nu \bar{\nu}) = (7.86 \pm 0.76_{P_c} \pm 0.72_{other}) \times 10^{-11}$ and $B(K^+ \to \pi^+ \nu \bar{\nu}) = (7.89 \pm 0.48_{P_c} \pm 0.72_{other}) \times 10^{-11}$ at the NLO and the NNLO, respectively.