S-Duality for surfaces with $A_n$-type singularities

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Abstract We show that the generating series of Euler characteristics of Hilbert schemes of points on any algebraic surface with at worst $A_n$-type singularities is described by the theta series determined by integer valued positive definite quadratic forms and the Dedekind eta function. In particular it is a Fourier development of a meromorphic modular form with possibly half integer weight. The key ingredient is to apply the flop transformation formula of Donaldson–Thomas type invariants counting two dimensional torsion sheaves on threefolds proved in the author’s previous paper.

1 Introduction

1.1 Background

For an algebraic variety $X$, the Hilbert scheme of $m$-points $\text{Hilb}^m(X)$ is defined to be the moduli space of zero dimensional subschemes $Z \subset X$ such that the length of $\mathcal{O}_Z$ equals to $m$. Its topological Euler characteristic $\chi(\text{Hilb}^m(X))$ has drawn much attention in connection with string theory. If $X$ is a (possibly singular) curve, then it is related to BPS state counting [23] and HOMFLY polynomials for links [17,21]. If $X$ is a non-singular surface, then $\text{Hilb}^m(X)$ is also non-singular and we have the remarkable formula by Göttsche [8]

$$
\sum_{m \geq 0} \chi(\text{Hilb}^m(X)) q^m - \frac{z(X)}{24} = \eta(q)^{-\chi(X)}.
$$

1 In this paper, all the varieties are defined over $\mathbb{C}$.
Here $\eta(q)$ is the Dedekind eta function

$$\eta(q) = q^{1/24} \prod_{m \geq 1} (1 - q^m).$$

In particular, the generating series (1) is a Fourier development of a meromorphic modular form of weight $-\chi(X)/2$. If $X$ is a smooth threefold, then $\chi(\text{Hilb}^m(X))$ is related to the Donaldson–Thomas (DT) invariants [18,26] and described in terms of MacMahon function [2,15,16].

The S-duality conjecture by Vafa and Witten [30] predicts the (at least almost) modularity of the generating series of Euler characteristics of moduli spaces of stable torsion free sheaves on algebraic surfaces. Since $\text{Hilb}^m(S)$ is regarded as a moduli space of rank one stable sheaves by the correspondence $Z \mapsto I_Z$, where $I_Z$ is the ideal sheaf of $Z$, the formula (2) gives an evidence of the S-duality conjecture. The S-duality conjecture in a higher rank case is still an open problem, but there exist several evidence, and we refer to [9] for the mathematical developments so far. Instead of stable torsion free sheaves on algebraic surfaces, let us consider pure two dimensional semistable torsion sheaves on Calabi–Yau threefolds and the generating series of Donaldson–Thomas invariants counting them. Similarly to the Vafa–Witten’s S-duality conjecture, such generating series are expected to have certain modular invariance property. This ‘3d S-duality conjecture’ also plays an important role in physics. For example, it is used in [7] to derive the Ooguri–Strominger–Vafa conjecture [22] in string theory. We also refer to [11,12] for the physics articles of this subject.

From a mathematical point of view, one of the differences of the 3d S-duality conjecture from Vafa–Witten’s S-duality conjecture is that sheaves we count may have singular supports inside a threefold. This issue motivates us to study $\chi(\text{Hilb}^m(S))$ for a singular surface $S$. If $S$ is singular, the number $\chi(\text{Hilb}^m(S))$ has not been studied in the literature. Because of the singularities of $S$, the scheme $\text{Hilb}^m(S)$ is no longer non-singular and $\chi(\text{Hilb}^m(S))$ reflects the complexity of the singularities of $S$. The behavior of the invariants $\chi(\text{Hilb}^m(S))$ is more complicated than the smooth case, and it seems to be difficult to see the modularity of their generating series. The purpose of this paper is to prove such a modularity for any singular surface $S$ with at worst $A_n$-type singularities, a simplest class of surface singularities. It gives a first definitive result for the modularity of the generating series of $\chi(\text{Hilb}^m(S))$ for a singular surface $S$.

1.2 Main result

Recall that an algebraic surface $S$ has an $A_n$-type singularity at $p \in S$ if the germ $(S, p)$ is analytically isomorphic to the affine singularity

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2 The Donaldson–Thomas invariants are the weighted Euler characteristics of the moduli spaces of semi-stable sheaves w.r.t. the Behrend function [1]. Although they are in general different from the naive Euler numbers of the moduli spaces, they are observed to share common properties (cf. [28]). From this point, we may expect that the 3d S-duality conjecture also holds for the Euler characteristics of moduli spaces.

3 In [10], the weighted Euler characteristics of $\text{Hilb}^m(S)$ for a K3 surface $S$ with $A_1$-type singularities is studied. The formula in [10, Example 3.26] involves Noether–Lefschetz numbers, and is different from ours in Theorem 1.1.
Table 1 Descriptions of $\Theta_n(q)$ for $1 \leq n \leq 4$

$\Theta_1(q) = -\frac{1}{2} \sum_{k \in \mathbb{Z}} q^{k^2} + \frac{3}{2} \sum_{k \in \mathbb{Z}} q^{9k^2}$

$\Theta_2(q) = -\sum_{(k_1, k_2) \in \mathbb{Z}^2} q^{3k_1^2+k_2^2} + 2 \sum_{(k_1, k_2) \in \mathbb{Z}^2} q^{4k_1^2+4k_2^2}$

$\Theta_3(q) = -\frac{1}{4} \sum_{(k_1, k_2, k_3) \in \mathbb{Z}^3} q^{k_1^2+k_2^2+k_3^2+k_1k_2+k_1k_3+k_2k_3}$

$+ \frac{5}{4} \sum_{(k_1, k_2, k_3) \in \mathbb{Z}^3} q^{25k_1^2+3k_2^2+7k_3^2-15k_1k_2-25k_1k_3+8k_2k_3}$

$\Theta_4(q) = \frac{1}{2} \sum_{(k_1, k_2, k_3, k_4) \in \mathbb{Z}^4} q^{k_1^2+k_2^2+k_3^2+k_4^2+k_1k_2+k_1k_3+k_2k_3+k_3k_4+k_1k_3+k_1k_4+k_2k_4}$

$- \sum_{(k_1, k_2, k_3, k_4) \in \mathbb{Z}^4} q^{4k_1^2+3k_2^2+7k_3^2+13k_4^2-6k_1k_2+8k_2k_3+18k_3k_4-10k_1k_3-14k_1k_4+11k_2k_4}$

$- \frac{3}{2} \sum_{(k_1, k_2, k_3, k_4) \in \mathbb{Z}^4} q^{9k_1^2+3k_2^2+7k_3^2+13k_4^2-9k_1k_2+8k_2k_3+18k_3k_4-15k_1k_3-21k_1k_4+11k_2k_4}$

$+ 3 \sum_{(k_1, k_2, k_3, k_4) \in \mathbb{Z}^4} q^{36k_1^2+3k_2^2+7k_3^2+13k_4^2-18k_1k_2+8k_2k_3+18k_3k_4-30k_1k_3-42k_1k_4+11k_2k_4}$

$A_n := \{xy - z^{n+1} = 0 : (x, y, z) \in \mathbb{C}^3\}$

at the origin. The following is the main result in this paper:

**Theorem 1.1** Let $S$ be a quasi-projective surface which is smooth except $A_{n_i}$-type singularities $p_i \in S$ for $1 \leq i \leq l$. Then we have the following formula:

$$
\sum_{m \geq 0} \chi(\text{Hilb}^m(S))q^m - \frac{\chi(\widetilde{S})}{2\pi} = \eta(q)^{-\frac{1}{2}} \cdot \prod_{i=1}^{l} \Theta_{n_i}(q).
$$

(4)

Here $\widetilde{S} \rightarrow S$ is the minimal resolution, and $\Theta_n(q)$ is defined by

$$
\Theta_n(q) := \sum_{(k_1, \ldots, k_n) \in \mathbb{Z}^n} q^{\sum_{1 \leq i \leq j \leq n} k_ik_j} e^{2\pi i (k_1+2k_2+\ldots+nk_n)}.
$$

(5)

By an elementary argument, we show that $\Theta_n(q)$ is a $\mathbb{Q}$-linear combination of the theta series determined by some integer valued positive definite quadratic forms on $\mathbb{Z}^n$ (cf. Proposition 3.2 and Table 1 for small $n$). In particular, $\Theta_n(q)$ is a modular form of weight $n/2$, and we obtain the following corollary:

**Corollary 1.2** The generating series (4) is a Fourier development of a meromorphic modular form of weight $-\chi(S)/2$ for some congruence subgroup in $\text{SL}_2(\mathbb{Z})$. 
1.3 Outline of the proof

Here is an outline of the arguments: in Sect. 2, we give a closed formula of the generating series of Euler characteristics of rank one Quot schemes of points on $A_{n-1}$ in terms of an infinite product. Let $D \subset A_{n-1}$ be the Weil divisor defined by

$$D := (x = z = 0) \subset A_{n-1}$$

in the notation of (3). For $j \in \mathbb{Z}$, we denote by $\mathcal{O}_{A_{n-1}}(jD)$ the rank one reflexive sheaf associated to the Weil divisor $jD$. Note that any rank one reflexive sheaf on $A_{n-1}$ is isomorphic to $\mathcal{O}_{A_{n-1}}(jD)$ for some $0 \leq j \leq n-1$. Let

$$\text{Quot}^m(\mathcal{O}_{A_{n-1}}(jD))$$

be the Quot scheme which parametrizes the quotients $\mathcal{O}_{A_{n-1}}(jD) \twoheadrightarrow Q$ where $Q$ is a zero dimensional coherent sheaf on $A_{n-1}$ with length $m$. Note that if $j = 0$, the scheme (6) coincides with $\text{Hilb}^m(A_{n-1})$. We will show the following formula in Sect. 2.3:

$$\sum_{0 \leq j \leq n-1} \chi(\text{Quot}^m(\mathcal{O}_{A_{n-1}}(jD))) q^{\frac{k^2n}{2} + (\frac{n}{2} - j)k + mt^kn - j} = \prod_{m \in \mathbb{Z}_{>0}} f_n(q^mt) \prod_{m \in \mathbb{Z}_{\geq0}} f_n(q^{m}t^{-1}).$$

Here $f_n(x)$ is given by

$$f_n(x) := 1 + x + \cdots + x^n.$$
obtain a contribution of the invariants from the singular point of $S^\dagger$, which gives the formula (7).

We note that the flop formula in [27] relies on Bridgeland’s equivalence of derived categories of coherent sheaves under threefold flops [3], and the Hall algebra method which is developed in recent years [4,6,13,14,28,29]. In turn, this indicates that the algebraic geometry involving flops, derived categories and Hall algebras provides an interesting application to a study of enumerative combinatorics. Also it may be worth pointing out that the formula (7) for $n = 1$ together with (1) show that

$$
\sum_{k \in \mathbb{Z}} q^{\frac{k^2}{2} + \frac{k}{2}} t^k = \prod_{m \geq 1} (1 - q^m) \prod_{m > 0} (1 + q^m t) \prod_{m \geq 0} (1 + q^m t^{-1}).
$$

The above formula is nothing but Jacobi triple product formula. It is surprising that the above classical result is also proved using threefold flops, derived categories, etc. We also refer to physic articles [19,20] on the derivation of the above formula from a similar wall-crossing argument of D4D2D0 bound states on the resolved conifold, using Kontsevich–Soibelman’s wall-crossing formula [14].

In Sect. 3, we prove Theorem 1.1. By a standard argument, the result is reduced to the case of $S = A_n$. In this case, the result follows by working with the formula (7) using Jacobi triple product formula. After that, we show the modularity of the series $\Theta_n(q)$ by describing $\Theta_n(q)$ as a $\mathbb{Q}$-linear combination of the theta series determined by integer valued positive definite quadratic forms. In Sect. 1, as an appendix, we provide a combinatorial description of $\chi(\text{Quot}^m(\mathcal{O}_{A_{n-1}}(jD)))$ in terms of $n$-tuples of Young diagrams.

The idea in this paper using the flop formula has possibilities to be applied for other surface singularities, but we leave them for a future work.

2 Euler characteristics of Quot schemes of points on $A_{n-1}$

2.1 Threefold flops

This subsection is devoted to a preliminary of the proof of the formula (7). We first fix a threefold flop whose exceptional locus has width $n$ in the sense of [24], satisfying some properties (cf. Fig. 1).

**Lemma 2.1** For each $n \geq 1$, there exist smooth projective threefolds $X$, $X^\dagger$ and a flop diagram

$$
(C \subset X) \xrightarrow{\phi} \xrightarrow{f} (X^\dagger \supset C^\dagger)
$$

satisfying the following conditions:
• There is a Zariski open neighborhood \( p \in U \subset Y \) which is isomorphic to the affine variety
\[
\{(xy + z^2 - w^{2n} = 0 : (x, y, z, w) \in \mathbb{C}^4)\}. \tag{10}
\]

In particular, the exceptional locus of \( f, f^\dagger \) are irreducible rational curves \( C, C^\dagger \) which are contracted to \( p \in Y \).

• There is an irreducible smooth divisor \( S \subset X \) such that \( S \cap C \) is scheme theoretically one point, and \( S^2 = S^3 = 0 \).

• The strict transform \( S^\dagger \subset X^\dagger \) of \( S \) contains \( C^\dagger \), has an \( A_{n-1} \)-type singularity at a point \( o \in S^\dagger \), and \( S^\dagger \setminus \{o\} \) is smooth.

**Proof** We take \( Y \) to be a projective compactification of the affine variety (10) which is smooth outside \( 0 \in \mathbb{C}^4 \). We take a flop diagram (9) by blowing up at the Weil divisors on \( Y \), given by the closures of the subschemes
\[
(x = z + w^n = 0) \subset U, \quad (x = z - w^n = 0) \subset U
\]
respectively. By the construction, there exists a divisor \( T \subset X \) with \( T \cap C \) scheme theoretically one point. Let \( H_Y \) be a sufficiently ample divisor on \( Y \). Then \( T + f^* H_Y \) is ample and globally generated by the base point free theorem. Let
\[
T_1, T_2 \in |T + f^* H_Y|
\]

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\( ^4 \) This last condition is required to make the computations of the Mukai vectors in Sect. 2.2 simpler, and not essential.
be general members. We replace $X$, $X^\dagger$, $Y$ by blow-ups at $T_1 \cap T_2$ which is smooth and lies outside $C$, $C^\dagger$, $p$ respectively. Then by setting $S$ to be the connected component of the strict transform of $T_1$ which intersects with $C$, we obtain a diagram (9) satisfying the first and the second conditions.

The last statement can be directly checked by describing the birational map $\phi$ on each affine charts of crepant resolutions of (10). An alternative geometric argument is as follows: by [24], the birational map $\phi$ is given by the Pagoda diagram,

$$
X \xleftarrow{f_1} X_1 \xleftarrow{f_2} \cdots \xleftarrow{f_{n-1}} X_{n-1} \xleftarrow{f_n} X_n \xrightarrow{f^\dagger_n} X^\dagger_n \xrightarrow{f^\dagger_{n-1}} \cdots \xrightarrow{f^\dagger_2} X^\dagger_2 \xrightarrow{f^\dagger_1} X^\dagger_1 \xrightarrow{f^\dagger_0} X^\dagger.
$$

(11)

Here $f_i$, $f^\dagger_i$ for $1 \leq i \leq n-1$ are blow-ups at $(0, -2)$-curves, and $f_n$, $f^\dagger_n$ are blow-ups at $(-1, -1)$-curves. Hence the birational map $S \rightarrow S^\dagger$ decomposes into

$$
S = S_0 \xleftarrow{g_1} S_1 \xleftarrow{g_2} \cdots \xleftarrow{g_{n-1}} S_{n-1} \xleftarrow{g_n} S_n \xrightarrow{g^\dagger_n} S^\dagger_{n-1} \xrightarrow{g^\dagger_{n-1}} \cdots \xrightarrow{g^\dagger_2} S^\dagger_2 \xrightarrow{g^\dagger_1} S^\dagger_1 \xrightarrow{g^\dagger_0} S^\dagger = S^\dagger.
$$

Here each $g_i : S_i \rightarrow S_{i-1}$ is a blow-up at a point in $\text{Ex}(g_{i-1}) \setminus g_{i-1*}^{-1} \text{Ex}(g_1 \circ \cdots \circ g_{i-2})$, where $g_{i-1*}^{-1}$ is the strict transform. The exceptional locus of $S_n \rightarrow S$ is an $A_{n-1}$-configuration of $(-2)$-curves together with a tail of a $(-1)$-curve, given by $\text{Ex}(g_n)$. The birational morphism $S_n \rightarrow S^\dagger$ contracts the above $A_{n-1}$-configuration of $(-2)$-curves on $S_n$ to an $A_{n-1}$-singularity $o \in S^\dagger$, and the image of the tail $\text{Ex}(g_n)$ coincides with $C^\dagger$. \hfill $\Box$

In what follows, we fix a flop diagram (9). We next describe rank one torsion free sheaves on $S$ and $S^\dagger$.

**Lemma 2.2** (i) An object $E \in \text{Coh}(S)$ is a rank one torsion free sheaf with trivial determinant on $S \setminus C$ if and only if $E$ is an ideal sheaf $I_Z$ for some zero dimensional subscheme $Z \subset S$.

(ii) An object $E \in \text{Coh}(S^\dagger)$ is a rank one torsion free sheaf if and only if it fits into the exact sequence

$$
0 \rightarrow E \rightarrow \mathcal{L} \rightarrow Q \rightarrow 0
$$

where $\mathcal{L}$ is a rank one reflexive sheaf on $S^\dagger$, $Q$ is a zero dimensional sheaf on $S^\dagger$. Moreover $E$ has a trivial determinant on $S^\dagger \setminus C^\dagger$ if and only if $\mathcal{L}$ is of the form $\mathcal{O}_{S^\dagger}(jC^\dagger)$ for some $j \in \mathbb{Z}$.

**Proof** The proofs of (i) and (ii) are similar, so we only show (ii). Let $E \in \text{Coh}(S^\dagger)$ be a rank one torsion free sheaf. We have the exact sequence in $\text{Coh}(S^\dagger)$

$$
0 \rightarrow E \rightarrow E^{\vee \vee} \rightarrow Q \rightarrow 0.
$$

Since $S^\dagger$ is normal, $Q$ is a zero dimensional sheaf. By setting $\mathcal{L} = E^{\vee \vee}$, we obtain the exact sequence (12). Conversely if $E$ fits into (12), then obviously $E$ is a rank one torsion free sheaf. The last assertion is also obvious. \hfill $\Box$
2.2 Application of the flop formula

Let us consider a flop diagram (9). We denote by \( i, i^\dagger \) the closed embeddings \( S \subset X, S^\dagger \subset X^\dagger \) respectively, and fix an ample divisor \( \omega \) on \( Y \). The flop transformation formula of DT type invariants in [27] compares invariants counting \( f^*\omega \)-semistable torsion sheaves on \( X \) supported on \( S \) with those counting \( f^\dagger*\omega \)-semistable torsion sheaves on \( X^\dagger \) supported on \( S^\dagger \). For \( \beta \in H_2(X) \) and \( \gamma \in \mathbb{Q} \), let \( M_{\beta,\gamma}(S) \) be the moduli space of rank one torsion free sheaves \( E \) on \( S \) such that the Mukai vector of \( i_*E \) satisfies

\[
\text{ch}(i_*E)\sqrt{\text{td}_X} = (0, S, -\beta, -\gamma) \in H^0(X) \oplus H^2(X) \oplus H^4(X) \oplus H^6(X).
\]

Here we have identified \( H^4(X), H^6(X) \) with \( H^2(X) \) by the Poincaré duality. We note that the \( f^*\omega \)-semistable sheaves on \( X \) supported \( S \) with Mukai vector \( (0, S, -\beta, -\gamma) \) coincide with the sheaves \( i_*E \) for \( [E] \in M_{\beta,\gamma}(S) \). The similar statement also holds for \( f^\dagger*\omega \)-semistable sheaves on \( X^\dagger \) supported on \( S^\dagger \). Therefore in this situation, the flop formula in [27, Theorem 3.23 (ii)] is described as

\[
\sum_{\beta^\dagger \in H_2(X^\dagger), \gamma \in \mathbb{Q}} \chi(M_{\beta^\dagger,\gamma}(S^\dagger)) q^\gamma t^{\beta^\dagger} = \sum_{\beta \in H_2(X), \gamma \in \mathbb{Q}} \chi(M_{\beta,\gamma}(S)) q^\gamma t^{\beta^\dagger} \\
\times q^{\frac{n}{12}} t^{\frac{C}{2}} \prod_{m \in \mathbb{Z}_{>0}} f_n(q^m t^C) \prod_{m \in \mathbb{Z}_{\geq 0}} f_n(q^m t^{-C}).
\]

(14)

Here \( f_n(x) \) is the polynomial (8). The formula (14) also holds after replacing \( M_{\beta,\gamma}(S), M_{\beta^\dagger,\gamma}(S^\dagger) \) by the subschemes

\[
M'_{\beta,\gamma}(S) \subset M_{\beta,\gamma}(S), \quad M'_{\beta^\dagger,\gamma}(S^\dagger) \subset M_{\beta^\dagger,\gamma}(S^\dagger)
\]

consisting of \( [E] \in M_{\beta,\gamma}(S) \), \( [E^\dagger] \in M_{\beta^\dagger,\gamma}(S^\dagger) \) which have trivial determinants on \( S \setminus C, S^\dagger \setminus C^\dagger \) respectively. (Indeed in the proof of [27, Theorem 3.23], it is enough to notice that \( E \in B_{f^*\omega}^{\mu,S} \) has a trivial determinant on \( S \setminus C \) if and only the same holds for \( E \in B_{f^*\omega}^{\mu,S} \) By Lemma 2.2 (i), the objects which contribute to \( \chi(M'_{\beta,\gamma}(S)) \) are of the form

\[
I_Z \subset O_S, \quad Z \subset S
\]

(15)

where \( Z \) is a zero dimensional subscheme and \( I_Z \) is the ideal sheaf of \( Z \). Also by Lemma 2.2 (ii), the objects which contribute to \( \chi(M'_{\beta^\dagger,\gamma}(S^\dagger)) \) are of the form

\[
\text{Ker}(O_{S^\dagger}(lC^\dagger) \to Q), \quad l \in \mathbb{Z}
\]

(16)
where \( Q \) is a zero dimensional sheaf on \( S^\dagger \). We need to compute the Mukai vectors of the push-forward of (15), (16) to \( X, X^\dagger \). As for (15), it is easily computed as

\[
\left( 0, S, \frac{c_1(X)}{4} S, \frac{c_1(X)^2}{96} S + \frac{c_2(X)}{24} S - |Z| \right)
\]

using the condition \( S^2 = S^3 = 0 \) in Lemma 2.1, the resolution

\[
0 \to \mathcal{O}_X(-S) \to \mathcal{O}_X \to \mathcal{O}_S \to 0
\]

and

\[
\sqrt{\text{td}_X} = \left( 1, \frac{c_1(X)}{4}, \frac{c_2(X)}{24} + \frac{c_1(X)^2}{96}, \frac{c_1(X)c_2(X)}{96} - \frac{c_1(X)^3}{384} \right).
\]

As for (16), it requires some more arguments:

**Lemma 2.3** For \( 0 \leq j \leq n - 1 \), we have

\[
\text{ch}(i^\dagger_* \mathcal{O}_{S^\dagger}(jC^\dagger)) = \left( 0, S^\dagger, \left( j - \frac{n}{2} \right) C^\dagger, -\frac{n}{6} \right).
\]

**Proof** We first recall Bridgeland’s perverse coherent sheaves [3] (also see [27, Subsection 2.2]) defined by

\[
-1\text{Per}_0(X^\dagger/Y) := \left\{ E \in D^b \text{Coh}(X^\dagger) : \begin{align*} \mathbb{R}f^\dagger_* E &\in \text{Coh}_0(Y) \\ \text{Hom}^{-1}(E, \mathcal{O}_{C^\dagger}(-1)) &= 0 \\ \text{Hom}^{-1}(\mathcal{O}_{C^\dagger}(-1), E) &= 0 \end{align*} \right\}.
\]

Here \( \text{Coh}_0(Y) \) is the category of zero dimensional sheaves on \( Y \). We set \(-1\mathcal{T}\) and \(-1\mathcal{F}\) to be

\[-1\mathcal{T} := -1\text{Per}_0(X^\dagger/Y) \cap \text{Coh}(X^\dagger), -1\mathcal{F} := -1\text{Per}_0(X^\dagger/Y)[-1] \cap \text{Coh}(X^\dagger)\]

Then by [31], \((-1\mathcal{T}, -1\mathcal{F})\) forms a torsion pair of the category

\[
\text{Coh}_0(X^\dagger/Y) := \{ F \in \text{Coh}(X^\dagger) : f^\dagger_* F \in \text{Coh}_0(Y) \}
\]

and \(-1\text{Per}_0(X^\dagger/Y)\) is the associated tilting, i.e. \((-1\mathcal{F}[1], -1\mathcal{T})\).

For \( k \geq 1 \), let \( kC^\dagger \subset S^\dagger \) be the subscheme defined by the ideal \( \mathcal{O}_{S^\dagger}(-kC^\dagger) \subset \mathcal{O}_{S^\dagger} \). We prove that \( \chi(\mathcal{O}_{kC^\dagger}) = k \) holds for \( 1 \leq k \leq n \). Since there is a surjection \( \mathcal{O}_{X^\dagger} \to \mathcal{O}_{kC^\dagger} \), it follows that

\[5\text{ Since } S^\dagger \text{ is singular, we cannot simply apply the Grothendieck Riemann-Roch theorem to compute ch}(i^\dagger_* \mathcal{O}_{S^\dagger}(jC^\dagger)). \text{ In fact, as } \mathcal{O}_{S^\dagger}(jC^\dagger) \text{ is not a perfect object, its Chern character on } S^\dagger \text{ is not defined in the usual way.} \]
Note that, since \( O_{f^{-1}(p)} = O_C \), any one dimensional stable sheaf on \( X^\dagger \) supported on \( C^\dagger \) must be of the form \( O_C(a) \) for some \( a \in \mathbb{Z} \). By [27, Lemma 2.4], the category \( -F \) is the extension closure of objects of the form \( O_C(a) \) with \( a \leq -1 \). Since \( (-T, -F) \) is a torsion pair of \( \text{Coh}_0(X^\dagger/Y) \), this implies that the stable factors of \( O_C \) consist of \( O_C(a_i) \) for \( 1 \leq i \leq k \) with \( a_i \geq 0 \). In particular, we have the inequality

\[
\chi(O_C) \geq k
\]
and the equality holds if and only if \( a_i = 0 \) for all \( i \).

On the other hand since \( S^2 = 0 \), a local computation easily shows that \( O_S((nC^\dagger)) \cong O_S \). Hence we have

\[
\text{ch}(O_{nC^\dagger}) = \text{ch}(O_S) - \text{ch}(O_S(-S^\dagger))
\]
which shows \( \chi(O_{nC^\dagger}) = -S^\dagger 3 \). Since \( S^{\dagger 3} = nC^\dagger \cdot S^\dagger = -n \), we obtain \( \chi(O_{nC^\dagger}) = n \). Hence the above argument shows that \( O_{nC^\dagger} \) is a \( n \)-step extensions of \( O_C \). Let \( O_{nC^\dagger} \to T_k \) be a surjection such that \( T_k \) is a \( k \)-step extensions of \( O_C \). Then \( T_k \) is a structure sheaf of a pure one dimensional subscheme \( kC^\dagger \subset S^\dagger \) with fundamental cycle \( k[C^\dagger] \). Since this is a characterizing property of \( kC^\dagger \), we have \( kC^\dagger = kC^\dagger \). Therefore we have \( O_{kC^\dagger} \cong T_k \), and \( \chi(O_{kC^\dagger}) = k \) holds.

The above computation shows that

\[
\text{ch}(i^\dagger_* O_S((kn + j)C^\dagger)) = \left( 0, S^\dagger, -\frac{S^{\dagger 2}}{2} - kC^\dagger, \frac{S^{\dagger 3}}{6} - k \right)
\]
for \( 1 \leq k \leq n \). Setting \( j = n - k \) and noting \( S^{\dagger 2} = nC^\dagger \), \( S^{\dagger 3} = -n \), \( i^\dagger_* O_S(jC^\dagger) = i^\dagger_* O_S((-kC^\dagger) \otimes O_X(S^\dagger)) \), we obtain the result. \( \square \)

We write \( l = kn + j \) for \( k \in \mathbb{Z} \) and \( 0 \leq j \leq n - 1 \). Note that we have

\[
\text{ch}(i^\dagger_* O_S((kn + j)C^\dagger)) = e^{knS^\dagger} (\text{ch}(i^\dagger_* O_S((jC^\dagger)))).
\]

Together with Lemma 2.3 and (17), a little computation shows that the Mukai vector of \( i^\dagger_* \) of (16) is computed as

\[
\left( 0, S^\dagger, \left( kn + j - \frac{n}{2} \right) C^\dagger + \frac{c_1(X^\dagger)}{4} S^\dagger, \right.
\]

\[
-\frac{n}{6} - kj + \frac{kn}{2} - \frac{k^2 n}{2} + \left( \frac{c_1(X^\dagger)^2}{96} + \frac{c_2(X^\dagger)}{24} \right) S^\dagger - |Q| \big).
\]
2.3 Proof of the formula (7)

Proof For a variety $X$ and $\mathcal{L} \in \text{Coh}(X)$, we denote by $\text{Quot}^m(\mathcal{L})$ the Quot scheme which parametrizes the zero dimensional quotients

$$\mathcal{L} \rightarrow Q, \quad \text{length } Q = m.$$  \hspace{1cm} (18)

Also for $p \in X$, we denote by

$$\text{Quot}^m_p(\mathcal{L}) \subset \text{Quot}^m(\mathcal{L})$$

the subscheme consisting of quotients (18) such that $Q$ is supported on $p$. By (14) and the arguments in the previous subsection, we obtain

$$\sum_{m \geq 0, k \in \mathbb{Z}_{\geq 0}} \chi(\text{Quot}^m(\mathcal{O}_{S^\dagger}((kn + j)C^\dagger)))q^{m + \frac{kn}{2} + kj - \frac{k^2n}{2} - \frac{c_2(X^\dagger)}{24}S^\dagger (\frac{z}{2} - jkn)}$$

$$= \sum_{m \geq 0} \chi(\text{Hilb}^m(S))q^{m - \frac{c_2(X)}{24}} \cdot q^{\frac{n}{12}} t^{S^\dagger} \prod_{m \in \mathbb{Z}_{>0}} f_n(q^m t^{S^\dagger}) \prod_{m \in \mathbb{Z}_{\geq 0}} f_n(q^m t^{-S^\dagger}).$$

Here we have used that

$$\phi_* \left( \frac{c_1(X)}{4} S, \frac{c_1(X)}{96} S^\dagger \right) = \left( \frac{c_1(X^\dagger)}{4} S^\dagger, \frac{c_1(X^\dagger)}{96} S^\dagger \right)$$

since $c_1(X)$ and $c_1(X^\dagger)$ are pull-backs from divisor classes on $Y$.

We simplify both sides of the above equation. Since $S^\dagger^2 = nC^\dagger$, we have the isomorphism

$$\otimes \mathcal{O}_{X^\dagger}(kS^\dagger): \text{Quot}^m(\mathcal{O}_{S^\dagger}(jC^\dagger)) \xrightarrow{\cong} \text{Quot}^m(\mathcal{O}_{S^\dagger}((kn + j)C^\dagger)).$$

Hence the Euler characteristics of both sides coincide. Also note that the Weil divisor $D \subset A_{n-1}$ corresponds to $C^\dagger \subset S^\dagger$ under a local isomorphism between $0 \in A_{n-1}$ and $\circ \in S^\dagger$. Hence we have an isomorphism

$$\text{Quot}^m_0(\mathcal{O}_{S^\dagger}(jC^\dagger)) \cong \text{Quot}^m_0(\mathcal{O}_{A_{n-1}}(jD)).$$  \hspace{1cm} (19)

We also have the stratification

$$\text{Quot}^m(\mathcal{O}_{S^\dagger}(jC^\dagger)) = \coprod_{m_1 + m_2 = m} \text{Quot}^{m_1}_0(\mathcal{O}_{S^\dagger}(jC^\dagger)) \times \text{Hilb}^{m_2}(S^\dagger \setminus \{\circ\}).$$  \hspace{1cm} (20)
Combined these, we have the following equalities:

\[
\sum_{m \geq 0} \chi(\text{Quot}^m(\mathcal{O}_{S^\dagger}(jC^\dagger))) q^m = \sum_{m \geq 0} \chi(\text{Quot}_0^m(\mathcal{O}_{S^\dagger}(jC^\dagger))) q^m \\
\cdot \sum_{m \geq 0} \chi(\text{Hilb}^m(S^\dagger \setminus \{o\})) q^m \\
= \sum_{m \geq 0} \chi(\text{Quot}_0^m(\mathcal{O}_{A_{n-1}}(jD))) q^m \\
\cdot \sum_{m \geq 0} \chi(\text{Hilb}^m(S)) q^m \\
= \sum_{m \geq 0} \chi(\text{Quot}^m(\mathcal{O}_{A_{n-1}}(jD))) q^m \\
\cdot \sum_{m \geq 0} \chi(\text{Hilb}^m(S)) q^m.
\]

Here the first equality follows from (20), the second equality follows from Göttsche formula (1), \(\chi(S^\dagger \setminus \{o\}) = \chi(S)\) and (19), and the last equality follows from the torus localization on \(A_{n-1}\). Also an easy computation (cf. [27, Lemma 2.8, Proposition 2.9]) shows that

\[c_2(X) \cdot S = c_2(X^\dagger) \cdot S^\dagger - 2n.\]

Summing up, we arrive at the formula:

\[
\sum_{0 \leq j \leq n-1, m \geq 0, k \in \mathbb{Z}} \chi(\text{Quot}^m(\mathcal{O}_{A_{n-1}}(jD))) q^{\frac{k}{2}n + \left(j - \frac{n}{2}\right)k + m t^{-(kn+j)}C^\dagger} \\
= \prod_{m \in \mathbb{Z}_{>0}} f_n(q^m t^c) \prod_{m \in \mathbb{Z}_{\geq 0}} f_n(q^m t^{-c}).
\]

By replacing \(k\) by \(-k\), we obtain the desired formula (7). \(\square\)

The following is an obvious corollary of the formula (7):

**Corollary 2.4** We have the following formula:

\[
\sum_{m \geq 0} \chi(\text{Hilb}^m(A_n)) q^m = \text{Coeff}_{t^0}(\prod_{m > 0} f_{n+1}(q^m t^{m}) \prod_{m \geq 0} f_{n+1}(q^m t^{-1})). \quad (21)
\]

Here \(\text{Coeff}_{t^0}(\ast)\) means that taking the \(t^0\) coefficient of the formal series \(\ast\) with variables \(q, t\), and \(f_n(x)\) is given by (8).
3 Proof of the main result

3.1 Proof of Theorem 1.1

Proof In what follows, we set
\[ \xi_m := e^{\frac{2\pi \sqrt{-1}}{m}} \in \mathbb{C}. \]

We have the decomposition
\[ f_n(x) = \prod_{i=1}^{n} (1 - x \xi_n^i). \]

Therefore the RHS of (21) coincides with the \( t^0 \)-coefficient of
\[ \prod_{i=1}^{n+1} \left( \prod_{m > 0} (1 - q^m t \xi_{n+2}^i) \prod_{m \geq 0} (1 - q^m t^{-1} \xi_{n+2}^i) \right). \] (22)

Using the Jacobi triple product formula
\[ \sum_{k \in \mathbb{Z}} q^{\frac{k^2}{2} + \frac{k}{2}} (-t)^k = \prod_{m \geq 1} (1 - q^m) \prod_{m > 0} (1 - q^m t) \prod_{m \geq 0} (1 - q^m t^{-1}) \]
the product (22) is written as
\[ \prod_{m \geq 1} (1 - q^m)^{-n-1} \prod_{i=1}^{n+1} \left( \sum_{k \in \mathbb{Z}} q^{\frac{k^2}{2} + \frac{k}{2}} (-t \xi_n^i)^k \right). \]

The \( t^0 \)-coefficient of the above product becomes
\[ \prod_{m \geq 1} (1 - q^m)^{-n-1} \cdot \left( \sum_{(k_1, \ldots, k_{n+1}) \in \mathbb{Z}^{n+1}} q^{\frac{k_1^2}{2} + \cdots + \frac{k_{n+1}^2}{2}} \xi_{n+2}^{k_1+2k_2+\cdots+(n+1)k_{n+1}} \right). \]

The right sum coincides with \( \Theta_n(q) \) defined by (5). Therefore we obtain
\[ \sum_{m \geq 0} \chi(\text{Hilb}^m(A_n)) q^n = \prod_{m \geq 1} (1 - q^m)^{-n-1} \cdot \Theta_n(q). \] (23)

For a variety \( X \) and \( p \in X \), we denote by \( \text{Hilb}_p^m(X) \) the subscheme of \( \text{Hilb}^m(X) \) corresponding to the zero dimensional subschemes \( Z \subset X \) with \( \text{Supp}(Z) = \{p\} \). Let
$S$ be an algebraic surface as in Theorem 1.1. We have the stratification

$$\text{Hilb}^m(S) = \prod_{m_0+m_1+\cdots+m_l=m} \text{Hilb}^{m_0}(S^0) \times \prod_{i=1}^l \text{Hilb}^{m_i}(S).$$

Here $S^0 \subset S$ is the smooth part of $S$. Noting that $p_i$ is an $A_{n_i}$-type singularity, the torus localization on $A_{n_i}$ shows that

$$\chi(\text{Hilb}_{p_i}^m(S)) = \chi(\text{Hilb}_0^m(A_{n_i})) = \chi(\text{Hilb}_i^m(A_{n_i})).$$

Combined with (1) and (23), we obtain the formula:

$$\sum_{m \geq 0} \chi(\text{Hilb}^m(S))q^m = \prod_{m \geq 1} (1 - q^m)^{-\chi(S^0) - \sum_{i=1}^l (n_i + 1)} \cdot \prod_{i=1}^l \Theta_{n_i}(q).$$

For the minimal resolution $\tilde{S} \to S$, we have

$$\chi(\tilde{S}) = \chi(S^0) + \sum_{i=1}^l (n_i + 1).$$

Combined with the definition of $\eta(q)$ in (2), we obtain the desired formula (4). \[\square\]

**Remark 3.1** The proof of Theorem 1.1 is largely motivic, and one would obtain a motivic version of this result if one can find a diagram (9) for projective Calabi–Yau threefold $X$, $X^\dagger$, and establish basic foundation of Kontsevich-Soibelman’s wall-crossing formula [14] for motivic DT invariants.

### 3.2 Modularity of $\Theta_n(q)$

In order to conclude Corollary 1.2, we need to check the modularity of $\Theta_n(q).$ Indeed, we show that $\Theta_n(q)$ is a $\mathbb{Q}$-linear combination of the theta series determined by integer valued positive definite quadratic forms on $\mathbb{Z}^n$. This fact may be known to number theorists, but we include the proof here as the author could not find an appropriate reference. We refer to [5,32] for the basic of the theta series.

For a positive definite quadratic form

$$Q: \mathbb{Z}^n \to \mathbb{Z}$$

let $\Theta_Q(q)$ be the associated theta series

$$\Theta_Q(q) := \sum_{(k_1,\ldots,k_n) \in \mathbb{Z}^n} q^{Q(k_1,\ldots,k_n)}.$$
It is well-known that $\Theta_Q(q)$ is a modular form of weight $n/2$ for some congruence subgroup in SL$_2(\mathbb{Z})$ (cf. [5, Section 3.2]).

**Proposition 3.2** The series $\Theta_n(q)$ is a $\mathbb{Q}$-linear combination of the theta series determined by integer valued positive definite quadratic forms on $\mathbb{Z}^n$, i.e. there exist $N \geq 1$, $a_i \in \mathbb{Q}$ and integer valued positive definite quadratic forms $Q_i$ on $\mathbb{Z}^n$ for $1 \leq i \leq N$ such that $\Theta_n(q)$ is written as

$$\Theta_n(q) = \sum_{i=1}^{N} a_i \Theta_{Q_i}(q).$$

The result of Corollary 1.2 follows from Theorem 1.1 together with the above proposition. In order to prove Proposition 3.2, we note the following:

- By the base change of $\mathbb{Z}^n$ given by $k_1 \mapsto k_1 + 2k_2 + \cdots + nk_n$, $k_i \mapsto k_i$ for $i \geq 2$, the series $\Theta_n(q)$ is written as

$$\Theta_n(q) = \sum_{(k_1, \ldots, k_n) \in \mathbb{Z}^n} q^{Q(1_1, \ldots, k_n)} \xi_{k_1}^{n+2}$$

for some integer valued positive definite quadratic form $Q$ on $\mathbb{Z}^n$.

- The series $\Theta_n(q)$ is invariant after replacing $\xi_{n+2}$ by $g(\xi_{n+2})$ for any element $g \in \text{Gal}(\mathbb{Q}(\xi_{n+2})/\mathbb{Q})$. This follows since the product expansion (22) also holds after replacing $\xi_{n+2}$ by $g(\xi_{n+2})$.

Therefore the result of Proposition 3.2 follows from the following proposition:

**Proposition 3.3** Let $n, m$ be the positive integers, and $Q$ an integer valued positive definite quadratic form on $\mathbb{Z}^n$. Suppose that the series

$$\Theta_{Q,m}(q) = \sum_{(k_1, \ldots, k_n) \in \mathbb{Z}^n} q^{Q(1_1, \ldots, k_n)} \xi_{k_1}^{m}$$

is invariant after replacing $\xi_{m}$ by $g(\xi_{m})$ for any $g \in \text{Gal}(\mathbb{Q}(\xi_{m})/\mathbb{Q})$. Then $\Theta_{Q,m}(q)$ is a $\mathbb{Q}$-linear combination of the theta series determined by integer valued positive definite quadratic forms on $\mathbb{Z}^n$.

The rest of this section is devoted to proving Proposition 3.3.

3.3 K-group of subsets of $\mathbb{Z}^n$

In what follows, we fix the notation in Proposition 3.3. We define the K-group of the subsets in $\mathbb{Z}^n$ to be

$$K^n := \bigoplus_{T \subset \mathbb{Z}^n} \mathbb{Z}[T]/\sim.$$
Here the relation ∼ is generated by

\[ [T_1] + [T_2] \sim [T_1 \cup T_2] - [T_1 \cap T_2]. \tag{24} \]

For any element

\[ \alpha = \sum_i a_i [T_i] \in K^n, \quad a_i \in \mathbb{Z} \]

the series

\[ \Theta_{Q, \alpha}(q) = \sum_i a_i \sum_{(k_1, \ldots, k_n) \in T_i} q^{Q(k_1, \ldots, k_n)} \tag{25} \]

is well-defined as it respects the relation (24). Let

\[ 1 = m_1 < m_2 < \cdots < m_l = m \]

be the set of divisors of m. We define the following subsets:

\[ S_i := \{(k_1, \ldots, k_n) \in \mathbb{Z}^n : \text{g.c.d.}(k_1, m) = m_i \} \]
\[ T_i := \{(k_1, \ldots, k_n) \in \mathbb{Z}^n : m_i | k_1 \}. \]

**Lemma 3.4** The element \([S_i] \in K^n\) is contained in the subgroup of \(K^n\) generated by \([T_1], \ldots, [T_l]\).

**Proof** We prove the claim by the induction on \(i\). For \(i = l\), we have \(S_l = T_l\), and the statement is obvious. Suppose that the claim holds for \([S_j]\) with \(j > i\). We have \(S_i \subset T_i\) and the complement is the disjoint union of \(S_j\) with \(j > i, m_i | m_j\). Therefore we obtain

\[ [S_i] = [T_i] - \sum_{j > i, m_i | m_j} [S_j]. \]

By the induction, the claim also holds for \([S_i]\). \( \square \)

**Lemma 3.5** Both of \(\Theta_{Q, T_i}(q)\), \(\Theta_{Q, S_i}(q)\) are \(\mathbb{Z}\)-linear combinations of the theta series determined by integer valued positive definite quadratic forms on \(\mathbb{Z}^n\).

**Proof** The claim for \(\Theta_{Q, T_i}(q)\) is obvious since

\[ \Theta_{Q, T_i}(q) = \Theta_{Q, i}(q), \quad Q_i(k_1, \ldots, k_n) = Q(m_i k_1, k_2, \ldots, k_n). \]

The claim for \(\Theta_{Q, S_i}(q)\) follows from the claim for \(\Theta_{Q, T_i}(q)\), Lemma 3.4 and the fact that (25) is well-defined. \( \square \)
3.4 Proof of Proposition 3.3

**Proof** Let $\varphi(m)$ be the Euler function given by the order of $\text{Gal}(\mathbb{Q}(\xi_m)/\mathbb{Q}) = (\mathbb{Z}/m\mathbb{Z})^*$. We write $m = m_i \cdot m_i'$ for $1 \leq i \leq l$. Since $\Theta_{Q,m}(q)$ is invariant under $\xi_m \mapsto g(\xi_m)$ for any element $g \in \text{Gal}(\mathbb{Q}(\xi_m)/\mathbb{Q})$, we have

$$
\varphi(m)\Theta_{Q,m}(q) = \sum_{(k_1, \ldots, k_n) \in \mathbb{Z}^n} q^{Q(k_1, \ldots, k_n)} \sum_{g \in \text{Gal}(\mathbb{Q}(\xi_m)/\mathbb{Q})} g(\xi_m^{k_1})
$$

$$
= \sum_{i=1}^l \sum_{(k_1, \ldots, k_n) \in S_i} q^{Q(k_1, \ldots, k_n)} \sum_{g \in \text{Gal}(\mathbb{Q}(\xi_m)/\mathbb{Q})} g(\xi_m^{k_1})
$$

$$
= \sum_{i=1}^l \sum_{(k_1, \ldots, k_n) \in \mathbb{Z}^n} q^{Q(m_i k_1, k_2, \ldots, k_n)} \sum_{g \in \text{Gal}(\mathbb{Q}(\xi_m)/\mathbb{Q})} g(\xi_m^{k_1})
$$

$$
= \sum_{i=1}^l \sum_{(k_1, \ldots, k_n) \in \mathbb{Z}^n} q^{Q(m_i k_1, k_2, \ldots, k_n)} [\mathbb{Q}(\xi_m) : \mathbb{Q}(\xi_m')] \sum_{g \in \text{Gal}(\mathbb{Q}(\xi_m')/\mathbb{Q})} g(\xi_m'_{k_1})
$$

Now the value

$$
A_i := [\mathbb{Q}(\xi_m) : \mathbb{Q}(\xi_m')] \sum_{g \in \text{Gal}(\mathbb{Q}(\xi_m')/\mathbb{Q})} g(\xi_m'_{k_1})
$$

is an integer and independent of $k_1 \in \mathbb{Z}$ with g.c.d.$(k_1, m_i') = 1$. By setting $Q_i(k_1, \ldots, k_n) = Q(m_i k_1, k_2, \ldots, k_n)$, we obtain

$$
\Theta_{Q,m}(q) = \frac{1}{\varphi(m)} \sum_{i=1}^l A_i \sum_{(k_1, \ldots, k_n) \in \mathbb{Z}^n} q^{Q_i(k_1, \ldots, k_n)}
$$

Therefore the result follows from Lemma 3.5.

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**Appendix: Combinatorics on Quot schemes of points on $A_{n-1}$**

In this appendix, we describe $\chi(\text{Quot}^m_{A_{n-1}}(O_{A_{n-1}}(j D)))$ in terms of certain combinatorial data on Young diagrams. In what follows, we regard a Young diagram as a subset in
in the usual way, say:

\[ Y = \begin{array}{ccc} \Box & \Box & \Box \\ \Box & \Box & \Box \end{array} \iff \{(0, 0), (1, 0), (2, 0), (1, 0), (1, 1)\}. \]

Recall that there is a one to one correspondence between the set of ideals \( I \subset \mathbb{C}[x, y] \) generated by monomials and that of Young diagrams, by assigning \( I \) with the Young diagram \( Y_I \):

\[ Y_I := \{(a, b) \in \mathbb{Z}^2_\geq 0 : x^ay^b \notin I\}. \quad (26) \]

For a Young diagram \( Y \), we introduce the following notation:

\[ Y^\rightarrow := \{Y + (1, 0)\} \cup \{(0) \times \mathbb{Z}_\geq 0\} \]
\[ Y^\leftarrow := \{Y + (1, 1)\} \cup \{(\mathbb{Z}_\geq 0 \times \{0\}) \cup \{(0) \times \mathbb{Z}_\geq 0\}. \]

Note that \( Y^\rightarrow \) and \( Y^\leftarrow \) are Young diagrams with infinite number of blocks. See the following picture:

\[ Y = \begin{array}{ccc} \Box & \Box & \Box \\ \Box & \Box & \Box \end{array} \Rightarrow Y^\rightarrow = \begin{array}{ccc} \Box & \Box & \Box \\ \Box & \Box & \Box \end{array} \]
\[ Y^\leftarrow = \begin{array}{ccc} \Box & \Box & \Box \\ \Box & \Box & \Box \end{array} \]

**Lemma 4.1** For \( 0 \leq j \leq n-1 \), the number \( \chi(\text{Quot}^m(\mathcal{O}_{\mathcal{A}_{n-1}}(-jD))) \) coincides with the number of \( n \)-tuples of Young diagrams \((Y_0, Y_1, \ldots, Y_{n-1})\) satisfying

\[ Y_{n-1} \subset \cdots \subset Y_j \subset Y_{j-1}^\rightarrow \subset \cdots \subset Y_0^\rightarrow \subset Y_{n-1}^\leftarrow, \quad \sum_{i=0}^{n-1} |Y_i| = m. \quad (27) \]

**Proof** Giving a point in \( \text{Quot}^m(\mathcal{O}_{\mathcal{A}_{n-1}}(-jD)) \) is equivalent to giving an ideal \( I \subset \mathcal{O}_{\mathcal{A}_{n-1}} \) such that \( I \subset \mathcal{O}_{\mathcal{A}_{n-1}}(-jD) \) and \( \mathcal{O}_{\mathcal{A}_{n-1}}(-jD)/I \) is a \( m \)-dimensional \( \mathbb{C} \)-vector space. As a \( \mathbb{C} \)-vector space, we have the decomposition

\[ I = \bigoplus_{k=0}^{n-1} I_k \cdot z^k \quad (28) \]

for sub vector spaces \( I_k \subset \mathbb{C}[x, y] \). Since \( I \) is an ideal in \( \mathcal{O}_{\mathcal{A}_{n-1}} \), each \( I_k \) is an ideal in \( \mathbb{C}[x, y] \). Moreover since \( I \) must be closed under the multiplication by \( z \), we have

\[ xyI_{n-1} \subset I_0 \subset I_1 \subset \cdots \subset I_{n-1}. \quad (29) \]
Since $\mathcal{O}_{A_{n-1}}(-jD) = (x, z^j)$, the condition $I \subset \mathcal{O}_{A_{n-1}}(-jD)$ is equivalent to $I_k \subset (x)$ for $0 \leq k \leq j - 1$. Hence for $0 \leq k \leq j - 1$, we have $I_k = I_k' \cdot (x)$ for some ideal $I_k' \subset \mathbb{C}[x, y]$. We obtain the sequence of ideals in $\mathbb{C}[x, y]$:

$$I_0', \ldots, I_j', I_{j+1}, \ldots I_{n-1}.$$ \hfill (30)

The condition that $\mathcal{O}_{A_{n-1}}(-jD)/I$ is $m$-dimensional is equivalent to

$$\sum_{k=0}^{j-1} \dim \mathbb{C}[x, y]/I_k' + \sum_{k=j}^{n-1} \dim \mathbb{C}[x, y]/I_k = m.$$ \hfill (31)

Conversely suppose that we have a sequence of ideals (30) in $\mathbb{C}[x, y]$ satisfying (31) and (29) for $I_k = I_k' \cdot (x)$ with $0 \leq k \leq j - 1$. Then we obtain an ideal $I \subset \mathcal{O}_{A_{n-1}}$ by setting (28), which gives a point in Quot$^m(\mathcal{O}_{A_{n-1}}(-jD))$. Note that $T = (\mathbb{C}^*)^2$ acts on $A_{n-1}$ via $(t_1, t_2) \cdot (x, y, z) = (t_1^n x, t_2^n y, t_1 t_2 z)$, and the ideal (28) is $T$-fixed if and only if the corresponding ideals in (30) are generated by monomials. Therefore the $T$-fixed locus of Quot$^m(\mathcal{O}_{A_{n-1}}(-jD))$ is identified with the set of $n$-tuples of Young diagrams $(Y_0, \ldots, Y_n)$ satisfying (27), by assigning a sequence (30) with

$$(Y_0, \ldots, Y_n) = (Y_{I_0}', \ldots, Y_{I_{j-1}}', Y_{I_j}, \ldots, Y_{I_{n-1}})$$

as in (26). By the $T$-localization, we obtain the desired result. \hfill $\square$

**Remark 4.2** The number $\chi(\text{Quot}^m(\mathcal{O}_{A_{n-1}}(-jD)))$ in Lemma 4.1 and the coefficients in the LHS of (7) are related by

$$\chi(\text{Quot}^m(\mathcal{O}_{A_{n-1}}(-jD))) = \chi(\text{Quot}^m(\mathcal{O}_{A_{n-1}}((n-j)D)))$$ \hfill (32)

as $\mathcal{O}_{A_{n-1}}(nD) \cong \mathcal{O}_{A_{n-1}}$.

We compare the formula (7) with the numbers of $n$-tuples of Young diagrams in Lemma 4.1 in examples:

**Example 4.3** (i) If $n = 2$ and $j = 0$, then the formula (7) implies

$$\sum_{m \geq 0} \chi(\text{Hilb}^m(A_1))q^m = 1 + q + 3q^2 + 5q^3 + 9q^4 + 14q^5 + \cdots.$$

For instance, $\chi(\text{Hilb}^5(A_1))$ corresponds to the following 14 pairs of Young diagrams $(Y_0, Y_1)$:

$$\begin{pmatrix} \square \\ \square, \varnothing \end{pmatrix}, \begin{pmatrix} \square, \varnothing \\ \square \end{pmatrix}, \begin{pmatrix} \square, \varnothing \\ \square \end{pmatrix}, \begin{pmatrix} \square \square, \varnothing \\ \square \end{pmatrix}, \begin{pmatrix} \square \square, \varnothing \\ \square \end{pmatrix}.$$
(i) If $n = 2$ and $j = 1$, then (7) and (32) yield
\[
\sum_{m \geq 0} \chi(\text{Quot}^m(\mathcal{O}_{A_1}(-D)))q^m = 1 + 2q + 3q^2 + 6q^3 + 10q^4 + 16q^5 + \cdots.
\]

Similarly to (i), $\chi(\text{Quot}^5(\mathcal{O}_{A_1}(-D)))$ corresponds to the following 16 pairs of Young diagrams $(Y_0, Y_1)$:

\[
(\begin{array}{cccc}
\square & \square & \square & \square \n\end{array}, \emptyset),
(\begin{array}{c}
\square \n\end{array}, \begin{array}{c}
\square \n\end{array}),
(\begin{array}{cc}
\square & \square \n\end{array}, \begin{array}{c}
\square \n\end{array}),
(\begin{array}{c}
\square \n\end{array}, \begin{array}{cc}
\square & \square \n\end{array}),
(\begin{array}{cc}
\square & \square \n\end{array}, \emptyset),
(\begin{array}{c}
\square \n\end{array}, \begin{array}{c}
\square \n\end{array}),
(\begin{array}{c}
\square \n\end{array}, \begin{array}{cc}
\square & \square \n\end{array}),
(\begin{array}{c}
\square \n\end{array}, \begin{array}{c}
\square \n\end{array}),
(\begin{array}{c}
\square \n\end{array}, \begin{array}{cc}
\square & \square \n\end{array}),
(\begin{array}{c}
\square \n\end{array}, \begin{array}{cc}
\square & \square \n\end{array}),
(\begin{array}{c}
\square \n\end{array}, \begin{array}{c}
\square \n\end{array}),
(\begin{array}{c}
\square \n\end{array}, \begin{array}{cc}
\square & \square \n\end{array}),
(\begin{array}{c}
\square \n\end{array}, \emptyset),
(\begin{array}{c}
\square \n\end{array}, \emptyset),
(\begin{array}{c}
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