Envelope Algebras of Partial Actions as Groupoid C*-Algebras

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Abstract

We describe the envelope C*-algebra associated to a partial action of a countable discrete group on a locally compact space as a groupoid C*-algebra (more precisely as a C*-algebra from an equivalence relation) and we use our approach to show that, for a large class of partial actions of $\mathbb{Z}$ on the Cantor set, the envelope C*-algebra is an AF-algebra. We also completely characterize partial actions of a countable discrete group on a compact space such that the envelope action acts in a Hausdorff space.

1 Introduction

The concept of partial actions was introduced in [4] and [9] and it has been a very important tool in C*-algebras and dynamical systems ever since. As the name suggests, partial actions generalize the notion of an action in a C*-algebra or in a topological space. The problem of deciding whether or not a given partial action is the restriction of some global action (called envelope action) was studied by F. Abadie in [1], where, among other things, he shows that the cross product of the envelope C*-algebra by the envelope action is Morita-Rieffel equivalent (previously known as strongly Morita equivalent) to the partial cross product.

In this paper we are interested in partial actions of a countable discrete group $G$ (in particular of $\mathbb{Z}$) on a locally compact space $X$ (particularly the Cantor set, that is, a compact, totally disconnected, with no isolated points, metric space). It is well known that there is a correspondence between partial actions on a Hausdorff locally compact space $X$ and the partial actions on the C*-algebra $C_0(X)$ (see proposition 3.3 of [4] or [8] for example). In [1], it is shown that a partial action on a topological space always has an envelope action, which may not act on a Hausdorff space (the odometer partial action for example). When the envelope space is Hausdorff the notion of the envelope action in the category of C*-algebras is a rather natural one, but when the envelope space in non Hausdorff the notion of the envelope action has to be reformulated with the use of C*-ternary rings and the introduction of the notion of strong Morita equivalence between partial actions. Although it seems that this approach can not be avoided in general, in the case of a partial action of a countable discrete group on a locally compact space we give a description of the envelope C*-algebra as a C*-algebra from an equivalence relation (viewed as a groupoid). Our approach has the advantage of working for either

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Hausdorff or non Hausdorff envelope spaces. We also use our description of the envelope algebra to show that it is an AF-algebra, provided we have a partial action of \( \mathbb{Z} \) on the Cantor set with some mild assumptions, namely that it arises as a "restriction" of a global action (we should warn the reader that we use the word restriction here with a slight different meaning then what usually appears in the literature).

The paper is structured as follows. In section 2 we make a quick review of the necessary notions on partial actions and envelope actions. We completely characterize the partial actions of a countable discrete group on a compact space such that the envelope space is Hausdorff in section 3 and finally in section 4 we describe the envelope algebra as a groupoid C*-algebra and show it is an AF-algebra under the assumptions mentioned above.

## 2 Partial Actions and Envelope Actions

**Definition 2.1.** A partial action of a group \( G \) on a set \( \Omega \) is a pair \( \theta = ([\Delta_t]_{t \in G}, [h_t]_{t \in G}) \), where for each \( t \in G \), \( \Delta_t \) is a subset of \( \Omega \) and \( h_t : \Delta_{t^{-1}} \to \Delta_t \) is a bijection such that:

1. \( \Delta_e = \Omega \) and \( h_e \) is the identity in \( \Omega \);
2. \( h_t(\Delta_{t^{-1}} \cap \Delta_s) = \Delta_t \cap \Delta_{ts} \);
3. \( h_t(h_s(x)) = h_{ts}(x), x \in \Delta_{s^{-1}} \cap \Delta_{s^{-1}t^{-1}}. \)

If \( \Omega \) is a topological space, we also require that each \( \Delta_t \) is an open subset of \( \Omega \) and that each \( h_t \) is a homeomorphism of \( \Delta_{t^{-1}} \) onto \( \Delta_t \).

Analogously, a pair \( \theta = ([D_t]_{t \in G}, [\alpha_t]_{t \in G}) \) is a partial action of \( G \) on a C*-algebra \( \Lambda \) if each \( D_t \) is a closed two sided ideal and each \( \alpha_t \) is a *-isomorphism of \( D_{t^{-1}} \) onto \( D_t \).

Since we are very interested in partial actions of \( \mathbb{Z} \), below we give the most important example of such partial actions.

**Example 2.2.** Let \( X \) be a locally compact space, \( U \) and \( V \) open subsets of \( X \) and \( h \) a homeomorphism from \( U \) to \( V \). Let \( X_{-n} = \text{dom}(h^n) \) and \( h_n : X_{-n} \to X_n \) be defined by \( h^n \), for \( n \in \mathbb{Z} \). Then \( \theta = ([X_n]_{n \in \mathbb{Z}}, [h_n]_{n \in \mathbb{Z}}) \) is a partial action of \( \mathbb{Z} \).

**Proof.** See [4] or [8].

**Example 2.3.** The Odometer: Let \( X = \{0, 1\}^\mathbb{N} = \prod_{n=1}^\infty \{0, 1\} \). Let \( \text{max} = 1^\infty \) (sequence of all 1’s), \( \text{min} = 0^\infty \) (sequence of all 0’s), \( X_{-1} = X - \{\text{max}\}, X_1 = X - \{\text{min}\} \) and \( h : X_{-1} \to X_1 \) be addition of 1 with carryover to the right. Then \( \theta = ([X_n]_{n \in \mathbb{Z}}, [h_n]_{n \in \mathbb{Z}}), \) where \( X_{-n} = \text{dom}(h^n) \), is a topological partial action.

**Remark 2.4.** With \( D_t = \{f \in C_0(X) : f|_{X_t} = 0\} \), where \( X_t^c \) means the complement of \( X_t \) in \( X \), and \( \alpha_t : D_{t^{-1}} \to D_t \) defined by \( \alpha_t(f) = f \circ h_t^{-1} \), we have a partial action on the C*-algebra \( C(X) \).

We now recall the definition of the envelope action in the topological sense.
Definition 2.5. Let $\theta = (\{X_t\}_{t \in G}, \{h_t\}_{t \in G})$ be a partial action. The envelope space, denoted by $X^e$, is the topological quotient space $(G \times X)/\sim$, where $\sim$ is the equivalence relation given by

$$(r, x) \sim (s, y) \iff x \in X_{r^{-1}s} \text{ and } h_{s^{-1}r}(x) = y.$$ 

The envelope action, denoted by $h^e$, is the action induced in $X^e$ by the action $h^e_t(t, x) \mapsto (st, x)$.

Given a locally compact space $X$, the definition of the envelope action and space in the C*-algebraic sense is motivated by the 1-1 relation between partial actions on $X$ and partial actions on $C_0(X)$, (see [8] for a proof of this relation). Basically, if a partial action $\theta = (\{X_t\}_{t \in G}, \{h_t\}_{t \in G})$ on $X$ has a Hausdorff envelope space $X^e$ and envelope action $h^e$, then $C_0(X^e)$ is the envelope C*-algebra and $\alpha^e(f) = f \circ (h^e)^{-1}$ is the induced global action associated to the partial action in $C_0(X)$. In [1], it is proved that when $X^e$ is Hausdorff, the partial cross product of $C_0(X)$ by the partial action is Morita-Rieffel equivalent to the global cross product of the envelope algebra $C_0(X^e)$ by the envelope action. The problem arises when $X^e$ is not Hausdorff (as for example in the odometer partial action). The result mentioned above is not valid anymore, as we may have very few continuous functions in $C_0(X^e)$. Abadie, in [1], goes around this problem by making use of C*-ternary rings and introducing the notion of Morita equivalence between partial actions. Below we completely characterize partial actions of a countable discrete group on a compact space for which the envelope space is Hausdorff. Throughout the rest of the paper $G$ will denote a countable discrete group.

3 Partial Actions of $G$ on a compact space such that the envelope space is Hausdorff.

In [1] it is shown that a partial action has a Hausdorff envelope space if and only if the graph of the action is closed. Below we give a concrete characterisation of partial actions of a countable discrete group on compact spaces for which the envelope space is Hausdorff.

Let $X$ be a compact set, $\{X_t, \alpha_t\}$ a partial action of $G$ on $X$ and $(X^e, \alpha^e)$ the envelope space and action respectively. Recall that $X^e$ is the quotient of $G \times X$ by the equivalence relation $(r, x) \sim (s, y) \iff x \in X_{r^{-1}s}$ and $\alpha_t^{-1}s(x) = y$, with the quotient topology. We denote the equivalence class of $(n, x)$ in $X^e$ by $[n, x]$.

Proposition 3.1. Let $G$ be a countable discrete group with unit $e$. Then $X^e$ is Hausdorff if and only if the partial action $\{X_t, \alpha_t\}$ acts in clopen subsets of $X$, that is, if and only if $X_t$ is clopen for each $t \in G$.

Proof. First assume that $X^e$ is Hausdorff. We will show that each $X_t$ is closed (it is already open by the definition of a partial action).

Suppose there exists $t \in G$ such that $X_t$ is not closed (we will show that this implies that $X^e$ is non-Hausdorff).

Since $X_t$ is not closed, there exists a sequence $(x_k)_{k \in \mathbb{N}}$ such that $x_k \in X_t$ for all $k$ and such that $x_k \to x$, where $x \notin X_t$. By the compactness of $X$, $(\alpha_{t-1}(x_k))_{k \in \mathbb{N}}$ has a converging subsequence and we may pass to this subsequence. We may therefore assume that there exists a sequence $(x_k)_{k \in \mathbb{N}}$ in $X_t$ such that $x_k \to x$, where $x \notin X_t$ and such that $(\alpha_{t-1}(x_k))_{k \in \mathbb{N}}$ converges to a point $y \in X$.

We now have that the points $[t^{-1}, x]$ and $[0, y]$ can not be separated. Let’s see why:

Suppose that $U$ and $V$ are open, $[t^{-1}, x] \in U$ and $[0, y] \in V$. Remember that $U$ is open if and only if $q^{-1}(U)$ is open, where $q$ is the quotient map.

Well, since $x_k \to x$ and $\alpha_{t-1}(x_k) \to y$, there exists $N \in \mathbb{N}$ such that $(t^{-1}, x_k) \in q^{-1}(U)$ and $(e, \alpha_{t-1}(x_k)) \in q^{-1}(V)$ for all $k > N$. Now notice that $(t^{-1}, x_k) \sim (e, \alpha_{t-1}(x_k))$ and hence $[t^{-1}, x_k] = [e, \alpha_{t-1}(x_k)]$ and $U \cap V \neq \emptyset$.

For the converse, we may now assume that each $X_t$ is a clopen subset of $X$.

Let $[r, x] \neq [s, y]$ in $X^e$. So $(r, x)$ is not equivalent to $(s, y)$. We have two possibilities:

- If $x \notin X_{r-1}$, then there exists $V_x$ such that $x \in V_x$ and $V_x \cap X_{r-1} = \emptyset$ (since $X_{r-1}$ is clopen).

We may assume $r \neq s$ (if $r = s$ then $x \neq y$ and we find the desired neighborhoods using the fact that $X$ is Hausdorff).

Take $V = (r, V_x)$ and $U = (s, X)$. Then $i_r(V)$ and $i_s(U)$ are the desired open sets. (Where $i_r(x) = q(r, x)$, $i_s(x) = q(s, x)$ and $q$ is the quotient map) (Also notice that $i_r$ is an open map and the proof is done analogous to what is done in [1] for the map $i$ in theorem 2.5).

- If $x \in X_{r-1}$, then $\alpha_{s-1}(x) \neq y$.

Let $z = \alpha_{s-1}(x)$. We have that $z \neq y$. Since $X$ is Hausdorff, there exists open sets $U_z$ and $U_y$ such that $U_z \cap U_y = \emptyset$, $z \in U_z$ and $y \in U_y$. Take $V_x = \alpha_{s-1}(U_z \cap X_{r-1})$, which is an open set. Then $i_r(V_x)$ and $i_s(U_y)$ have the desired properties.

\[\blacksquare\]

**Remark 3.2.** Dokuchaev and Exel have a result in the more general context of partial actions on associative algebras, see theorem 4.5 of [8], that implies the proposition above. Still we believe our proof above helps to give the reader a feeling for the space $X^e$.

**Proposition 3.3.** Let $X$ be the Cantor set, $G$ a countable discrete group and $\{X_t, \alpha_t\}$ a partial action of $G$ on $X$ such that $X_t$ is clopen for all $t \in G$. Then the envelope space $X^e$ is a locally compact Cantor set.

**Proof.** We need to show that $X^e$ is Hausdorff, locally compact, has a countable basis of clopen sets and has no isolated points (this is equivalent to the characterization of the Cantor set we gave before).

Before we proceed, we note that for each $t \in G$ the function $i_t(x) = q(t, x)$ is a continuous, open and closed map (we already know that $i_t$ is continuous and open by an argument similar to what is done in [1] in theorem 2.5). To see that it is a closed map, let $F$ be closed in $X$. Then
$F$ is compact and hence $i_\varepsilon(F)$ is compact. Since $X^e$ is Hausdorff we have that $i_\varepsilon(F)$ is closed.

Notice that $X^e$ is Hausdorff by proposition $\text{3.1}$. To prove that $X^e$ is locally compact, let $[(r,x)]$ in $X^e$. Since $X$ is compact, there exists a compact neighborhood, $U_x$, of $x$ in $X$. But then $i_\varepsilon(U_x)$ is a compact neighborhood of $[(r,x)]$ in $X^e$.

Now, let $\{U_n\}_{n \in G}$ be a countable basis of clopen sets of $X$. Then $\{i_\varepsilon(U_n)\}_{n,t \in G}$ is a countable basis of clopen subsets of $X^e$.

Finally, we have that $X^e$ has no isolated points, since if $[(t,x)] \in X^e$ and $V$ is an open set that contains $[(t,x)]$ then $(t,x) \in q^{-1}(V)$ (we may assume that $q^{-1}(V)$ is of the form $(t,U)$, where $U$ is open in $X$). So there exists $(y,t) \neq (x,t)$ such that $(y,t) \in q^{-1}(V)$ and hence $[(y,t)] \in V$ and $[(y,t)] \neq [(x,t)]$.

With the above propositions we completely characterized the envelope actions of partial actions of $G$ acting on clopen subsets of the Cantor set.

The cross product of their envelope C*-algebra by the envelope action is Morita-Rieffel equivalent to the partial cross product, see [1]. The problem is that most of the interesting examples, including the famous odometer (or adding machine), do not satisfy the conditions of the propositions above. Namely they fail to be partial actions on clopen sets. We note here that the majority of examples from partial actions arises as in example $\text{2.2}$.

In the next section we show how to deal with these examples in a different (and we believe easier) way from what was done in [1]. As a consequence of our approach we show that the envelope C*-algebra associated to a partial action of $\mathbb{Z}$ on the Cantor set, as in example $\text{2.2}$, is an AF-algebra.

## 4 The envelope C*-algebra as a groupoid C*-algebra.

In this section, we start by showing that the envelope C*-algebra associated to a partial action of a countable discrete group $G$ on a locally compact space can be seen as a C*-algebra of an equivalence relation (seen as a groupoid in the usual way). Before we proceed we need to introduce the notion of core subalgebras, which will be used in our proof that the envelope algebra can be realized as a groupoid C*-algebra.

**Definition 4.1.** Let $A$ be a C*-algebra and let $B \subseteq A$ be a (not necessarily closed) *-subalgebra. We shall say that $B$ is a core subalgebra of $A$ when every representation $\pi$ of $B$ is continuous relative to the norm induced from $A$.

Assuming that $B$ is a core subalgebra of $A$, and given a representation $\pi$ of $B$, we may therefore extend $\pi$ to a representation $\bar{\pi}$ of $\bar{B}$ (the closure of $B$ in $A$). Since $\bar{B}$ is a C*-algebra we have by [2] that $\bar{\pi}$ is necessarily contractive. Therefore we have:

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3By a representation of a *-algebra $B$ we mean a multiplicative, *-preserving, linear map $\pi : B \to B(H)$, where $H$ is a Hilbert space.
Proposition 4.2. $B$ is a core subalgebra of $A$ if and only if every representation of $B$ is contractive.

Examples 4.3.

(i) Every closed *-subalgebra of a C*-algebra is a core subalgebra by [2].

(ii) Let $B$ be a *-subalgebra of a C*-algebra $A$, such that $B = \bigcup_{i \in I} B_i$, where each $B_i$ is a core subalgebra of $A$. Then $B$ is a core subalgebra of $A$. This is because every representation of $B$ must be contractive on each $B_i$.

(iii) If $X$ is a locally compact space then $C_c(X)$ is a core subalgebra of $C_0(X)$. This follows from the fact that $C_c(X)$ is the union of the closed *-subalgebras $C_0(U)$, where $U$ ranges in the collection of all relatively compact open subsets of $X$.

(iv) Let $\mathcal{G}$ be a groupoid satisfying the hypotheses of corollary 1.22 of [11]. Then $C_c(\mathcal{G})$ is a core subalgebra of $C^*(\mathcal{G})$ by the same corollary of [11].

(v) Let $B$ be a *-algebra such that $||b|| := \sup \|\pi(b)\| < \infty$, $\forall \ b \in B$, (4.3.1) where the supremum is taken over the collection of all representations $\pi$ of $B$. Then one may define the enveloping C*-algebra, $C^*(B)$, by moding out the elements $b$ such that $||b|| = 0$, and completing under $||\cdot||$. The image of $B$ within $C^*(B)$ is therefore a dense core subalgebra.

(vi) If $G$ is a discrete group then the complex group algebra $C^G$ is a core subalgebra of the full group C*-algebra of $G$.

(vii) If $G$ is non-amenable then $C^G$ will not be a core subalgebra of the reduced group C*-algebra $C^*_r(G)$, since some representations of $C^G$ will extend to a representation of $C^*_r(G)$ which does not vanish on the kernel of the left regular representation.

Let $B$ be a core subalgebra of a C*-algebra $A$. It is then evident that $B$ satisfies (4.3.1) and moreover that $||b|| = ||b||$, where the right hand side refers to the norm of $b$ computed as an element of $A$. Supposing that $B$ is dense in $A$, it follows that $A$ is isomorphic to the enveloping C*-algebra $C^*(B)$. From this one immediately has:

Proposition 4.4. Suppose that $A_1$ and $A_2$ are C*-algebras, and that $B_i$ is a dense core subalgebra of $A_i$, for $i = 1, 2$. If $B_1$ and $B_2$ are isomorphic as *-algebras, then $A_1$ and $A_2$ are isometrically *-isomorphic.

Theorem 4.5. Let $A$ be a C*-algebra and let $\{p_i\}$ be a family of mutually orthogonal projections in the multiplier algebra of $A$, here denoted as $M(A)$. Also let $B$ be a *-subalgebra of $A$ such that

$$B \subseteq \bigoplus_{i,j \in I} B \cap (p_i A p_j),$$
and such that $B \cap (p_iAp_i)$ is a core subalgebra for every $i \in I$. Then $B$ is a core subalgebra.

**Proof.** Given $b \in B$, by hypothesis we have that

$$b = \sum_{k,l \in I} a_{kl},$$

where the nonzero summands are finite and each $a_{kl} \in B \cap (p_kAp_l)$. We therefore have for all $i, j \in I$, that

$$p_i bp_j = \sum_{k,l \in I} p_i a_{kl} p_j = a_{ij},$$

from where we see that $a_{ij} = p_i bp_j$. From now on we will adopt the notation $b_{ij} := p_i bp_j \ \forall \ b \in A, \ \forall \ i, j \in I$.

and hence we have for every $b \in B$ that $b_{ij} \in B \cap (p_iAp_j)$, while $b = \sum_{i,j \in I} b_{ij}$, a sum with finitely many nonzero terms.

For each finite set of indices $F \subseteq I$, let

$$B_F = \bigoplus_{i,j \in F} B \cap (p_iAp_j).$$

It is easy to see that $B_F$ is a *-subalgebra of $A$ and we claim that it is a core subalgebra. In fact, given a representation $\pi$ of $B_F$, we have for all $i, j \in F$, and all $b_{ij} \in B \cap (p_iAp_j)$ that

$$\|\pi(b_{ij})\|^2 = \|\pi(b_{ij} b_{ij}^*)\| \leq \|b_{ij} b_{ij}^*\| = \|b_{ij}\|^2,$$

where the crucial second step follows from the fact that $b_{ij} b_{ij}^* \in B \cap (p_iAp_j)$, the latter being a core subalgebra by hypothesis. Given any $b \in B_F$, we then have

$$\|\pi(b)\| \leq \sum_{i,j \in F} \|\pi(b_{ij})\| \leq \sum_{i,j \in F} \|b_{ij}\| \leq \|F\|^2 \|b\|.$$

This proves that $\pi$ is bounded and hence that $B_F$ is a core subalgebra, as claimed.

Now observe that $B = \bigcup_F B_F$, where $F$ ranges in the collection of all finite subsets of $F$, so the conclusion follows from (4.3.ii).

\[\square\]

We can now focus again in realizing the envelope algebra as a groupoid C*-algebra.

For this we fix, as before, a partial action of the discrete group $G$ on a locally compact space $X$.

Recall that the envelope space is the quotient of $G \times X$ by the equivalence relation $(r, x) \sim (s, y) \iff x \in X_{s^{-1}r}$ and $\alpha_{s^{-1}r}(x) = y$. So instead of considering this quotient, we will consider the equivalence relation $R \subseteq G \times X \times G \times X$ given by

$$(r, x) \sim (s, y) \iff x \in X_{s^{-1}r} \text{ and } \alpha_{s^{-1}r}(x) = y,$$
with the product topology.

Notice that a neighborhood base for \( z = (t, x, s, y) \), in \( R \), is formed by neighborhoods of the form

\[
U_{txs} = \{ (t, x, s, \alpha_{s-1}(x)) : x \in U_x \subseteq X_{t-1} \}, \text{where } U_x \text{ is open.}
\]

Before we can consider the groupoid \( C^* \)-algebra of this equivalence relation we will show that \( R \) with this topology is étale, that is, we have to show that \( R \) can be equipped with two maps, called range and source, defined by \( r(t, x, s, y) = (t, x) \) and \( s(t, x, s, y) = (s, y) \) and such that \( R \) is \( \sigma \)-compact, \( \Delta = \{(t, x, t, x) \in R : (t, x) \in Z \times X \} \) is an open subset of \( R \) and for all \((t, x, s, y) \in R, \) there exists a neighborhood \( U \) of \((t, x, s, y) \) in \( R, \) such that \( r \) restricted to \( U \) and \( s \) restricted to \( U \) are homeomorphisms from \( U \) onto open subsets of \( X \times Z, \) see \([10]\) or \([11]\).

**Proposition 4.6.** \( R \) is étale.

**Proof.** To see that \( R \) is sigma compact, we notice that for each fixed \( s \) and \( t \in G, \) \( X_{t-1} \) is a countable union of compact sets and hence each \( U_{txs} \), with \( U_x = X_{t-1} \) is a countable union of compact sets.

For \((t, x, t, x) \in \Delta, \) we take \( U_{txs} \) with \( U_x = X \) to see that \( \Delta \) is open.

Finally, given \((t, x, s, y) \in R, \) it is not hard to see that the range and source map are homeomorphisms, once restricted to \( U_{txs}, \) with \( U_x = X_{t-1} \).

\[ \blacksquare \]

We are now able to consider the full groupoid \( C^* \)-algebra of \( R, \) which we denote by \( C^*(R) \) (see \([11]\) or \([2]\) for details on the groupoid \( C^* \)-algebra of étale equivalence relations). Next we show that \( C^*(R) \) is isomorphic to the Morita envelope algebra defined in \([3]\). In order to do so, we quickly remind the reader of the definitions in \([3]\) (adapted to the case at hand).

Given a Fell Bundle \( B = (B_t)_{t \in G} \) of a partial action \( \{X_t, h_t\}_{t \in G}, \) (in our case \( B_t = C_0(X_t) \delta_t \), we consider the linear space, \( k_c(B) \), of all continuous functions \( k : G \times G \to B, \) with finite support and such that \( k(r, s) \in B_{r,s-1}. \) We now equip \( k_c(B) \) with the involution \( k^*(r, s) = k(s, r)^*, \forall k \in k_c(B), \) the multiplication \( k_1 * k_2 (r, s) = \sum_{t \in G} k_1(r, t)k_2(t, s), \forall k_1, k_2 \in k_c(B) \)

and the norm \( \|k\| = \left( \sum_{r,s} \|k(r, s)\|^2 \right)^{1/2} \). The universal \( C^* \)-algebra of the completion of \( k_c(B) \) with respect to the norm above is the envelope algebra, \( k(B). \) Finally we notice that there exists a natural action of \( G \) on \( k_c(B), \) which can be extended to \( k(B), \) given by \( \hat{\beta}(k)(r, s) = k(rt, st). \) The pair \((k(B), \hat{\beta})\) is the envelope action as in \([1]\). We can now prove our main result.

**Theorem 4.7.** Given a partial action \( h \) of a countable discrete group \( G \) on a locally compact space \( X, \) the groupoid \( C^* \)-algebra \( C^*(R), \) as defined above, is isomorphic to the envelope \( C^* \)-algebra \( k(B). \)

**Proof.** Initially let us observe that, given any element \((r, x, s, y) \in R, \) one has that \( y = h_{s-1}(x). \) Therefore the fourth variable “\( y \)” is a function of
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the first three, and hence we may discard it. In more precise terms we have that
\[(r, x, s, y) \mapsto (x, r, s)\]
establishes a one-to-one correspondence from \(R\) to the set
\[R' = \{(x, r, s) \in X \times Z \times Z : x \in X_{r^{-1}}\} .\]

Moreover this correspondence is seen to be a homeomorphism if \(R'\) is viewed as a subspace of the topological product space \(X \times Z \times Z\).

Borrowing the groupoid structure from \(R\) we have that \(R'\) itself becomes an ’étale groupoid under the multiplication operation
\[(x, r, s) \cdot (y, t, u) = (x, r, u),\]
defined iff \(y = h_{s^{-1},r}(x)\), and \(s = t\), while the inversion operation is given by
\[(x, r, s)^{-1} = (h_{s^{-1},r}(x), s, r).\]
Since \(R\) and \(R'\) are isomorphic topological groupoids, it is enough to show that \(\mathcal{C}^*(R')\) and \(k(B)\) are isomorphic C*-algebras. We will derive this result from proposition \ref{proposition} by showing the existence of two isomorphic dense core subalgebras of \(\mathcal{C}^*(R')\) and \(k(B)\), respectively.

On the one hand recall that \(\mathcal{C}^c(R')\) is a dense core subalgebra of \(\mathcal{C}^*(R')\), as observed in \ref{proposition}.iv. To define the relevant dense core subalgebra of \(k(B)\), recall that \(B\) is the Fell bundle with fibers \(B_t = C_0(X_t)\delta_t\).

Put \(D_t = \mathcal{C}^c(X_t)\delta_t\), so that each \(D_t\) is a dense linear subspace of \(B_t\). Moreover it is easy to see that \(D_r D_s \subseteq D_{rs}\), and \(D_r^* = D_{r^{-1}}, (†)\) for all \(r, s \in G\).

Denote by \(k_c(D)\) the subset of \(k_c(B)\) formed by all \(k \in k_c(B)\) such that \(k(r, s) \in D_{r^{-1}}\), for all \(r\) and \(s\). As a consequence of (†) one easily proves that \(k_c(D)\) is a *-subalgebra of \(k(B)\), which is also easily seen to be dense.

We will next show that \(k_c(D)\) is a core subalgebra of \(k(B)\) by using theorem \ref{theorem}. With this in mind we must first define a family of projections \(\{p_t\}_{t \in G}\) in the multiplier algebra \(M(k(B))\). Given \(t \in G\), consider the maps
\[L_t, R_t : k_c(B) \to k_c(B),\]
given, for every \(k \in k_c(B)\), by
\[L_t(k)(r, s) = \begin{cases} k(r, s), & \text{if } r = t, \\ 0, & \text{otherwise} \end{cases}, \quad \forall r, s \in G.\]
and
\[R_t(k)(r, s) = \begin{cases} k(r, s), & \text{if } s = t, \\ 0, & \text{otherwise} \end{cases}, \quad \forall r, s \in G.\]
One may then prove that both \(L_t\) and \(R_t\) extend continuously to \(k(B)\) giving a multiplier
\[p_t = (L_t, R_t) \in M(k(B)),\]
which turns out to be self-adjoint and idempotent, thus producing a family 
\{p_t\}_{t \in G} of mutually orthogonal projections.

For every \( r, s \in G \) one has that \( p_r k_c(D)p_s \) consists of all the \( k \in k_c(D) \)
which are supported on the singleton \( \{(r, s)\} \). In particular \( p_t k_c(D)p_t = C_c(X_t) \),
which sits within \( p_t k(B)p_t = C_0(X_t) \) as a core subalgebra, as in
example 4.3 iii. One may then easily prove that theorem 4.5 applies
and hence we deduce that \( k_c(D) \) is a core subalgebra of \( k(B) \).

We will next prove that \( C_c(R') \) and \( k_c(D) \) are isomorphic as \(*\)-algebras
and hence the result will follow from proposition 4.4. Given \( r, s \in G \), let

\[ R'_{r,s} = R' \cap (X \times \{r\} \times \{s\}) \]
or, equivalently,

\[ R'_{r,s} = \{(x, r, s) : x \in X_{r-1} \} , \]

so \( R'_{r,s} \) naturally identifies with \( X_{r-1} \). Given \( f \in C_c(R') \), denote by \( f_{r,s} \)
the restriction of \( f \) to \( R'_{r,s} \), seen as an element of \( C_c(X_{r-1}) \). Since \( f \)
is compactly supported, only finitely many \( f_{r,s} \) will be nonzero. Define \( \psi : C_c(R') \rightarrow k_c(D) \) by

\[ \psi(f)(r, s) = f_{r-1,s-1} \delta_{r,s-1}, \quad \forall f \in C_c(R'), \quad \forall r, s \in G . \]

Observing that \( R' \) is the disjoint union of the \( R'_{r,s} \), it should be obvious
that \( \psi \) is a well defined vector space isomorphism. The proof will then be
concluded once we show that \( \psi \) is a \(*\)-homomorphism.

In order to prove that \( \psi(f * g) = \psi(f) \psi(g) \), we may suppose without
loss of generality that \( f \) is supported in \( R'_{r,s} \) and that \( g \) is supported in
\( R'_{t,u} \). When \( s \neq t \), the product in \( R' \) of \((x, r, s)\) and \((y, t, u)\) is never
defined, so \( f * g = 0 \).

Otherwise, if \( s = t \), we have that \( f * g \) is supported in \( R'_{r,u} \). Moreover,
given \((x, r, u) \in R'_{r,u} \), the only way of writting \((x, r, u)\) as a product of an
element of \( R'_{r,s} \) and an element of \( R'_{s,u} \) is

\[ (x, r, u) = (x, r, s) (h_{s-1,r}(x), s, u) , \]
as long as \( x \in X_{r-1} \). Thus

\[ (f * g)(x, r, u) = f(x, r, s) g(h_{s-1,r}(x), s, u) = f_{r,s}(x) g_{s,u}(h_{s-1,r}(x)) . \]

On the other hand, since \( \psi(f) \) is supported on the singleton \( \{(r^{-1}, s^{-1})\} \),
and \( \psi(g) \) is supported on \( \{(s^{-1}, u^{-1})\} \), we have that \( \psi(f) \psi(g) \) is supported
on \( \{(r^{-1}, u^{-1})\} \), and

\[ \psi(f)(r^{-1}, s^{-1}) \psi(g)(s^{-1}, u^{-1}) = (f_{r,s} \delta_{r-1,s}) (g_{s,u} \delta_{s-1,u}) = f_{r,s} (g_{s,u} \circ h_{s-1,r}) \delta_{r-1,u} , \]

from where it is easily seen that \( \psi(f * g) = \psi(f) \psi(g) \). We leave it for the
reader to prove that \( \psi \) preserves the adjoint operation.

\[ \blacksquare \]

**Corollary 4.8.** Let \( \alpha \) be the action on \( C^*(R) \) given by \( \alpha_t(f)(r, x, s, y) = f(rt, x, st, y) \). Then \( C^*(R) \rtimes_\alpha G \) is isomorphic to \( k(B) \rtimes_\beta G \), which is
strong morita equivalent to the partial cross product \( C(X) \rtimes G \).
Proof. It is clear that the actions $\alpha$ and $\beta$ commute and hence the isomorphism follows. The second part is done in [1].

We finish the paper showing that for partial actions of $\mathbb{Z}$ on the Cantor set, $X$, as in 2.2 with $X_{-1} \neq X$, $R$ is an approximately proper equivalence relation and $C^*_p(R)$ (and hence the envelope $C^*$-algebra) is an AF-algebra.

Recall that an equivalence relation is said to be proper when the quotient space is Hausdorff. In [12], Renault defines approximately proper and a approximately finite equivalence relations as below.

Definition 4.9. An equivalence relation $R$, on a locally compact, second countable, Hausdorff space $X$, is said to be approximately proper if there exists an increasing sequence of proper equivalence relations $\{R_n\}_{n \in \mathbb{N}}$ such that $R = \bigcup_{n \in \mathbb{N}} R_n$. An approximately proper equivalence relation on a totally disconnected space is called an AF equivalence relation.

Remark 4.10. In [10], Giordano, Putnam and Skau define an AF equivalence relation as an equivalence relation that can be written as an increasing union of compact open étale subequivalence relations. They also mention that their definition is equivalent to the definition above.

To prove that $R$ is approximately proper we will come up with a sequence of partial actions by clopen sets (so that their envelope space is Hausdorff) such that the union of the induced equivalence relations is $R$.

Recall that $R$ is associated with a partial action $\theta = \{X_{-n}, h^n\}$ on $X$, as in example 2.2. That is, $h$ is a homeomorphism from $U$ to $V$ (where $U$ is a proper open subset of $X$), $X_{-n} = \text{dom}(h^n)$ and $h_n = h^n$.

To create the partial actions, let $\{U_k\}_{k=0,1,...}$ be an increasing sequence of clopen sets such that their union is $X_{-1} = U \neq X$. For each $U_k$, denote the partial action by clopen sets obtained by restricting $h$ to $U_k$ and proceeding as in example 2.2 by $\theta_k = \{X_{-n}^k, h_n\}_{n \in \mathbb{Z}}$, where $X_{-1}^k = U_k$ and $h_1$ is $h$ restricted to $U_k$.

Now, we consider the sub equivalence relation $R_k \subseteq \mathbb{Z} \times X \times \mathbb{Z} \times X$ given by $(r, x, s, y) \in R_k \iff x \in X_{-1}^{k-1}$ and $h_{s-1}(x) = y$. Since each $R_k$ is associated to a partial action on clopen sets, we have by proposition 3.1 that the quotient $\xrightarrow{R_k}$ is Hausdorff for every $k$. With this set up we can prove that $R$ is approximately proper.

Proposition 4.11. $R$ is approximately proper.

Proof. It remains to show that $R = \bigcup_{k \in \mathbb{N}} R_k$ (it is clear that $R_k \subseteq G_{k+1}$ for $k = 0, 1, \ldots$). It follows promptly that $R_k \subseteq R$ for all $k$. Next we show that $R \subseteq \bigcup_{k \in \mathbb{N}} R_k$.

Let $(r, x, s, y) \in R$ (which happens iff $x \in X_{r-1}s$ and $h_{s-1r}(x) = y$). All we need to do is find a $K$ such that $x \in X_{r-1}^K$, since this would imply that $(r, x, s, y) \in R_K$. Now recall that $X_{-n} = \text{dom}(h^n)$ and assuming that $r^{-1} s \geq 0$ (the case $r^{-1} s \leq 0$ is analogous) we have that

$$X_{r-1s} = \text{dom}(h^{r^{-1}s}) = U \cap h^{-1}(U) \cap \ldots \cap h^{s-1r+1}(U).$$
So \( x, h(x), h^2(x), \ldots, h^{s-1+r+1} \) belong to \( U \) and hence we can find a \( K \) such that \( x, h(x), h^2(x), \ldots, h^{s-1+r+1} \) all belong to the same \( U_K \), since \( U = \bigcup_{k \in \mathbb{N}} U_k \) with \( U_k \subseteq U_{k+1} \). We conclude that \( x \in U_K \cap h^{-1}(U_K) \cap \ldots \cap h^{-1+s}(U_K) = X_{t-1+s}^K \) as desired.

Since we have shown that \( R \) is approximately proper, it is natural to consider \( R = \bigcup R_k \) with the inductive limit topology. This approach will allow us to write \( C^*_r(R) \) as an inductive limit \( C^* \)-algebra. But first we need to show that the inductive limit and product topology agree on \( R \).

**Proposition 4.12.** Let \( R = \bigcup_{k \in \mathbb{Z}} R_k \) above. Then the inductive limit topology and the product topology on \( R \) are the same.

**Proof.** Suppose \( U \neq \emptyset \) is open in the inductive limit topology. Then \( U \cap R_k \) is open for all \( k \). Let \( (t, x, s, y) \in U \) and find \( K \) such that \( (t, x, s, y) \in R_K \). Then \( U \cap R_K \) contain an open neighborhood of \( (t, x, s, y) \) of the form

\[
\{(t, z, s, h_{s-1}(z)) : z \in U^K \subseteq X_{t-1+s}^K \subseteq X_{t-1}^K\},
\]

where \( U^K \) is open in \( X_{t-1+s}^K \) and hence open in \( X_{t-1}^K \). So \( U \) is open in the product topology.

Now, notice that \( U_{t+s} \cap R_k = \{(t, x, s, h_{s-1}(x)) : x \in U_x \subseteq X_{t-1+s}, \text{where } U_x \text{ is open} \} \cap R_k \) is open in \( R_k \) for all \( k \) and hence \( U_{t+s} \) is open in the inductive limit topology.

**Corollary 4.13.** \( C^*_r(R) = \lim_{\rightarrow} C^*_r(R_k) \).

**Proof.** The proof is analogous to what is done in [7] for \( C^* \)-algebras from substitution tilings.

**Proposition 4.14.** \( C^*_r(R) \) is an AF-algebra.

**Proof.** We already know, by proposition 4.11, that \( R \) is approximately proper. Also, \( R \) is an equivalence relation in \( \mathbb{Z} \times X \) and since \( X \) is the Cantor set it is clear that \( R \) is and AF equivalence relation. Then by theorem 3.9 of [6] we have that \( R \) is isomorphic to tail equivalence in some Bratteli diagram and by [5] we have that the associated \( C^* \)-algebra is an AF algebra.

Another way to prove this proposition would be to show that each \( R_k \) is an AF equivalence relation, (as defined in [5]), so that \( C^*_r(R_k) \) is an AF \( C^* \)-algebra. Since inductive limits of AF \( C^* \)-algebras are again AF (via the local characterization of AF algebras) this will yield that \( C^*_r(R) \) is also AF. To see that each \( R_k \) is AF, notice that \( \cup_n R_k^n = R_k \), where \( R_k^n = \{(r, x, s, y) \in \mathbb{Z} \times X \times \mathbb{Z} : |r|, |s| \leq n \} \cap R_k \).
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