SMALL HANKEL OPERATORS ON GENERALIZED FOCK SPACES

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ABSTRACT. We consider Fock spaces $F_{p,\ell}^{\alpha}$ of entire functions on $\mathbb{C}$ associated to the weights $e^{-\alpha|z|^2\ell}$, where $\alpha > 0$ and $\ell$ is a positive integer. We compute explicitly the corresponding Bergman kernel associated to $F_{2,\ell}^{\alpha}$ and, using an adequate factorization of this kernel, we characterize the boundedness and the compactness of the small Hankel operator $h_{b,\alpha}^\ell$ on $F_{2,\ell}^{\alpha}$. Moreover, we also determine when $h_{b,\alpha}^\ell$ is a Hilbert-Schmidt operator on $F_{2,\ell}^{\alpha}$.

1. INTRODUCTION

Let $\alpha > 0$. For $1 \leq p < \infty$, we denote by $L_{p,\ell}^{\alpha}$ the space of all measurable functions $f$ on $\mathbb{C}$ such that

$$
\|f\|_{L_{p,\ell}^{\alpha}}^p := \int_{\mathbb{C}} |f(z)e^{-\alpha|z|^2/2}|^p \, d\nu(z) < \infty,
$$

where $d\nu$ denotes the Lebesgue measure on $\mathbb{C}$.

For $p = \infty$, $L_{\infty,\ell}^{\alpha}$ denotes the space of all measurable functions $f$ on $\mathbb{C}$ such that

$$
\|f\|_{L_{\infty,\ell}^{\alpha}} := \text{ess sup}_{z \in \mathbb{C}} |f(z)e^{-\alpha|z|^2/2}| < \infty.
$$

Note that $L_{p,\ell}^{\alpha} = L^p(\mathbb{C}, e^{-\alpha|z|^2/2} \, d\nu), 1 \leq p < \infty$. So, for $1 \leq p \leq \infty$, $(L_{p,\ell}^{\alpha}, \| \cdot \|_{L_{p,\ell}^{\alpha}})$ is a Banach space and $L_{2,\ell}^{\alpha}$ is a Hilbert space with the inner product

$$
\langle f,g \rangle_{L_{2,\ell}^{\alpha}} := \int_{\mathbb{C}} f(z)\overline{g(z)}e^{-\alpha|z|^2/2} \, d\nu(z).
$$
The generalized Fock spaces are defined to be
\[ F^p,\ell_{\alpha} := H(\mathbb{C}) \cap L^p,\ell_{\alpha} \quad (1 \leq p \leq \infty), \]
where \( H(\mathbb{C}) \) denotes the space of entire functions. It is well known that the space of holomorphic polynomials is dense in \( F^p,\ell_{\alpha} \) for \( p < \infty \).

If \( p = 2 \), \( F^2,\ell_{\alpha} \) is a Hilbert space. We will denote by \( P_{\alpha}^\ell \) the orthogonal projection from \( L^2,\ell_{\alpha} \) to \( F^2,\ell_{\alpha} \), which is an integral operator whose kernel is \( K_{\alpha}^\ell \), the Bergman reproducing kernel for \( F^2,\ell_{\alpha} \).

It is also convenient to consider the little Fock space
\[ f^\infty,\ell_{\alpha} := \{ f \in H(\mathbb{C}) : \lim_{|z| \to \infty} |f(z)|e^{-\alpha|z|^{2\ell}/2} = 0 \}, \]
which is the closure of the space of all holomorphic polynomials in \( F^\infty,\ell_{\alpha} \).

Recall that for \( \ell = 1 \) one obtains the classical Fock spaces \( F^p_{\alpha} \) and \( f^\infty_{\alpha} \).

The main goal of this paper is to characterize the boundedness and the compactness of the small Hankel operators
\[ h_{b,\alpha}^\ell(f) := P_{\alpha}^\ell(bf) \]
on \( F^p,\ell_{\alpha} \) for the whole range \( 1 \leq p < \infty \).

For the classical case \( \ell = 1 \) and \( p = 2 \), it is well known that, if \( b \in F^2_{\alpha} \), the small Hankel operator \( h_{b,\alpha}^1(f) := P_{\alpha}^1(bf) \) is bounded (compact) from \( F^2_{\alpha} \) to \( F^2_{\alpha} \) if and only if \( b \in F^\infty_{\alpha/2} \) (\( b \in f^\infty_{\alpha/2} \)). Moreover, \( h_{b,\alpha}^1 \) is a Hilbert-Schmidt operator if and only if \( b \in F^2_{\alpha/2} \) (see \( [8] \) and \( [15] \)).

Up to our knowledge, there are not known results on small Hankel operators for \( \ell > 1 \). This is not the case for the big Hankel operator \( H_{\alpha}^\ell(f) := \overline{bf - P_{\alpha}^\ell(bf)} \). In \( [3] \) (see also \( [11] \)) the authors prove that \( H_{\alpha}^\ell \) is a bounded operator on \( F^2,\ell_{\alpha} \) if and only if \( b(z)(1 + |z|)^{1-\ell} \in L^\infty_{\alpha} \), that is, \( b \) is a polynomial of degree at most \( \ell \). It is also worth mentioning \( [13] \), where are described the bounded, compact and Schatten class big Hankel operators on Hilbert Fock spaces induced by radial rapidly decreasing weights.

Observe that \( (1 + |z|)^{1-\ell} \simeq (\Delta|z|^{2\ell})^{-1/2}, |z| \geq 1 \). It is well known that in the general theory of Fock spaces \( F^p_{\phi} \), the Laplacian of the subharmonic weight \( \phi \) plays an important role (see, for instance, the recent papers \( [9] \) and \( [11] \) and the references therein). A natural question from those observations, which will be solved by the main results of this paper, is whether or not the boundedness of \( h_{b,\alpha}^\ell \) on \( F^2,\ell_{\alpha} \) is described by conditions on \( b \) involving \( \Delta|z|^{2\ell} \).
In order to introduce a natural space of symbols to study the small Hankel operator acting on $F_{p,\ell}^p$, notice that if $h_{b,\alpha}^\ell$ is bounded on $F_{p,\ell}^p$, for some $b \in H(\mathbb{C})$, then $b = h_{b,\alpha}^\ell(1) \in F_{p,\ell}^p$. For the classical case, we have $F_{p}^p \subset F_{\infty}^\infty$, if $1 \leq p < \infty$. Those considerations suggest that the appropriate space of symbols in this classical setting is $F_{\infty}^\infty$. When $\ell > 1$ the inclusion $F_{p,\ell}^p \subset F_{\infty,\ell}^\infty$ is no longer true (see for instance [3, Corollary 2]). Instead, for any function $b$ in $F_{p,\ell}^p$ the pointwise estimate
\[ |b(z)| \lesssim \|b\|_{F_{p,\ell}^p} (1 + |z|)^{(2\ell-2)/p} e^{\alpha|z|^{2\ell/2}} \]
holds (see [2, Lemma 19(a)]). Hence, in the general setting we consider the space of holomorphic symbols given by
\[ H_{\infty,\ell}^\infty := \left\{ b \in H(\mathbb{C}) : |b(z)| = O((1 + |z|)^{(2\ell-2)/p} e^{\alpha|z|^{2\ell/2}}) \right\} . \]

Assuming $b \in H_{\infty,\ell}^\infty$, the operator $h_{b,\alpha}^\ell$ is well defined on the space $E$ of entire functions of order $\ell$ and finite type, that is,
\[ E := \{ f \in H(\mathbb{C}) : |f(z)| = O(e^{\beta|z|^{\ell}}) \quad \text{for some} \quad \beta > 0 \}. \]

Since $E$ contains the space of the holomorphic polynomials, $E$ is dense in $F_{\infty,\ell}^\infty$, for any $p < \infty$.

Our main results are the following.

**Theorem 1.1.** Let $\alpha > 0$, $\ell \in \mathbb{N}$, $b \in H_{\infty,\ell}^\infty$ and $1 \leq p < \infty$. Then $h_{b,\alpha}^\ell$ is a bounded operator from $F_{p,\ell}^p$ to $F_{p,\ell}^p$ if and only if $b \in F_{\infty,\ell}^{\alpha/2}$. In such case, $\|h_{b,\alpha}^\ell\|_{F_{p,\ell}^p} \simeq \|b\|_{F_{\infty,\ell}^{\alpha/2}}$.

Analogously, $h_{b,\alpha}^\ell$ is a bounded operator from $F_{1,\ell}^\infty$ to $F_{1,\ell}^\infty$ if and only if $b \in F_{\infty,\ell}^{\alpha/2}$ and $\|h_{b,\alpha}^\ell\|_{F_{1,\ell}^\infty} \simeq \|b\|_{F_{\infty,\ell}^{\alpha/2}}$.

Here and throughout the paper $\|h_{b,\alpha}^\ell\|_{F_{p,\ell}^p}$ denotes the norm of $h_{b,\alpha}^\ell$ as an operator from $F_{p,\ell}^p$ to $F_{p,\ell}^p$.

Since the boundedness of small Hankel operators is equivalent to the boundedness of the corresponding Hankel forms, as an application of Theorem 1.1 we obtain:

**Theorem 1.2.** Let $1 < p < \infty$, $\ell \in \mathbb{N}$, $\alpha > 0$ and $b \in H_{\infty,\ell}^\infty$.

(i) Let $\Lambda_{b,\alpha}^\ell$ be the Hankel bilinear form defined by
\[ \Lambda_{b,\alpha}^\ell(f, g) := \langle fg, b \rangle_{\alpha}^\ell \quad (f, g \in E). \]

Then, $\Lambda_{b,\alpha}^\ell$ extends to a bounded bilinear form either on $F_{p,\ell}^p \times F_{p,\ell}^p$ or on $F_{1,\ell}^1 \times F_{1,\ell}^\infty$ if and only if $b \in F_{\infty,\ell}^{\alpha/2}$.
(ii) The space \( F^\infty_{\alpha/2} \) coincides with \( F^\ell_\alpha(L^\infty) \) and also with the dual of \( F^{1,\ell}_{2\alpha} \) with respect to the pairing \( \langle \cdot, \cdot \rangle_\alpha \).

(iii) \( F^p_{\alpha,\ell} \cap F^{p',\ell}_{\alpha} = F^{1,\ell}_{\alpha} \cap F^{\infty,\ell}_{\alpha} = F^{1,\ell}_{2\alpha} \).

Here and throughout the paper, \( p' \) denotes the conjugate exponent of \( p \).

We recall that the weak product \( F^p_{\alpha,\ell} \cap F^{p',\ell}_{\alpha} \) consists of all entire functions \( h = \sum_{j=1}^{\infty} f_j g_j \), \( f_j \in F^p_{\alpha,\ell} \) and \( g_j \in F^{p',\ell}_{\alpha} \), such that

\[
\|h\|_{F^p_{\alpha,\ell} \cap F^{p',\ell}_{\alpha}} := \inf \left\{ \sum_{j=1}^{\infty} \|f_j\|_{F^p_{\alpha,\ell}} \|g_j\|_{F^{p',\ell}_{\alpha}} : h = \sum_{j=1}^{\infty} f_j g_j \right\} < \infty.
\]

**Theorem 1.3.** Let \( \alpha > 0 \), \( \ell \in \mathbb{N} \), \( b \in H^\infty_{\alpha,\ell} \) and \( 1 \leq p < \infty \). Then \( h^\ell_{b,\alpha} \) is compact from \( F^p_{\alpha,\ell} \) to \( F^{p,\ell}_{\alpha} \) if and only if \( b \in \overline{F}^\infty_{\alpha/2} \).

Similarly, \( h^\ell_{b,\alpha} \) is compact from \( \overline{F}^\infty_{\alpha/2} \) to \( \overline{F}^\infty_{\alpha,\ell} \) if and only if \( b \in \overline{F}^\infty_{\alpha/2} \).

As far as we know, the techniques that have been used to prove characterizations of the boundedness and the compactness of small Hankel operators on the classical Fock spaces \( F^p_{\alpha} = F^{p,1}_{\alpha} \) (see \([8, 15]\)) are strongly based on the fact that the Bergman reproducing kernel of \( F^2_{\alpha} \) is given by the neat expression

\[
K^1_{\alpha}(z, w) = \frac{\pi}{\alpha} e^{\alpha z \bar{w}},
\]

which permits to factorize the kernel as

\[
K^1_{\alpha}(z, w) = \frac{\pi}{\alpha} K^1_{\alpha}(z/2, w) K^1_{\alpha}(z/2, w).
\]

Thus, the proof is quite easy since the integral operator with kernel \( K^1_{\alpha}(z/2, \cdot) \) maps the function \( f \) in the Fock space to the function \( f(\cdot/2) \). However, the general situation on \( F^2_{\alpha,\ell} \), \( \ell > 1 \), is much more involved because of the lack of such a simple expression for \( K^1_{\alpha} \). In this general case we use the factorization

\[
K^\ell_{\alpha}(w, z) = G_{\alpha,0}(w, z) G_{\alpha,1}(w, z),
\]

where

\[
G_{\alpha,0}(w, z) := e^{\frac{\pi}{\alpha} (w \bar{z})^\ell} \quad \text{and} \quad G_{\alpha,1}(w, z) := e^{-\frac{\pi}{\alpha} (w \bar{z})^\ell} K^\ell_{\alpha}(w, z),
\]

which for \( \ell = 1 \) is just \((1.2)\).

Note that \((1.3)\), which is given in terms of analytic functions, is possible because \( \ell \) is a positive integer. For other values of \( \ell \) it is not clear how to choose a suitable decomposition.

Finally, we characterize the membership of \( h^\ell_{b,\alpha} \) in the class \( S_2(F^2_{\alpha,\ell}) \) of Hilbert-Schmidt operators from \( F^2_{\alpha,\ell} \) to \( F^2_{\alpha,\ell} \).

For \( \ell = 1 \), \( h^1_{b,\alpha} \in S_2(F^2_{\alpha}) \) if and only if \( b \in F^2_{\alpha/2} \) (see \([8] \) or \([15]\)). For \( \ell > 1 \) the characterization is given in terms of the space \( F^2_{\alpha,\Delta} \) of all functions \( f \in H(\mathbb{C}) \).
such that
\[ \|f\|^2_{F^{2,\ell}_{\alpha,\Delta}} := \int_C |f(z)e^{-\frac{\alpha}{2}|z|^2}|^2 (1 + |z|)^{2(\ell-1)} d\nu(z) < \infty. \]

**Theorem 1.4.** Let \(\alpha > 0, \ell \in \mathbb{N}\) and \(b \in H^\infty_{\alpha,\Delta}\). Then, \(h^\ell_{b,\alpha} \in S_2(F^{2,\ell}_{\alpha,\Delta})\) if and only if \(b \in F^{2,\ell}_{\alpha/2,\Delta}\). Moreover,
\[ \|h^\ell_{b,\alpha}\|_{S_2(F^{2,\ell}_{\alpha,\Delta})} \simeq \|b\|_{F^{2,\ell}_{\alpha/2,\Delta}}. \]

Observe that, while the descriptions of the boundedness and compactness of the small Hankel operators on \(F^{p,\ell}_\alpha\) obtained in Theorems 1.1 and 1.3 do not depend on the Laplacian of \(|z|^{2\ell}\), this is not the case for Hilbert-Schmidt operators. Taking into account our results, it seems natural to conjecture analogous ones for weighted Fock spaces induced by weights \(e^{-\phi}\), where \(\phi\) is a subharmonic function such that \(\Delta \phi\) is a doubling measure.

The paper is organized as follows. In Section 2 we state some useful properties of the Bergman projection, as well as the main properties of the spaces \(F^{p,\ell}_\alpha\) and of the small Hankel operator. In Section 3 we prove Theorem 1.1. In Sections 4 and 5 we give the proof of Theorems 1.2 and 1.3 respectively. Finally, in Section 6 we provide a proof of Theorem 1.4 which follows from the definition of the Hilbert-Schmidt norm.

1.1. **Notations.** Throughout the paper, \(\mathbb{N}\) denotes the set of all positive integers. We denote by \(p'\) the conjugate exponent of \(p\). The letter \(C\) will denote a positive constant, which may vary from place to place. The notation \(A \lesssim B\) means that there exists a constant \(C > 0\), which does not depend on the involved variables, such that \(A \leq CB\). We write \(A \simeq B\) when \(A \lesssim B\) and \(B \lesssim A\). We will also say that \(h^\ell_{b,\alpha}\) is bounded (compact) on \(F^{p,\ell}_\alpha\) if it is bounded (compact) from \(F^{p,\ell}_\alpha\) to \(\mathcal{F}^{p,\ell}_\alpha\). We denote the norm of this operator by \(\|h^\ell_{b,\alpha}\|_{F^{p,\ell}_\alpha}\).

The same notations will be used replacing \(F^{p,\ell}_\alpha\) by \(\mathcal{F}^{\infty,\ell}_\alpha\).

2. **Preliminaries**

2.1. **Properties of the Fock spaces \(F^{p,\ell}_\alpha\).**

We begin the subsection recalling some useful embeddings of the generalized Fock spaces.

**Lemma 2.1.** Let \(1 \leq p, q \leq \infty\). If \(0 < \alpha < \beta < \gamma < \delta\) then we have the embeddings
\[ F^{\infty,\ell}_\alpha \hookrightarrow F^{p,\ell}_\beta \hookrightarrow H^{\infty,\ell}_\beta \hookrightarrow \mathcal{F}^{\infty,\ell}_\gamma \hookrightarrow F^{q,\ell}_\delta. \]
Proposition 2.2. Let 1 ≤ p ≤ ∞, α, λ > 0 and ℓ ∈ N. For any function f on C we define
\[ \Phi^p_\lambda f(z) := f(\lambda^{1/(2\ell)}z) \quad (z \in \mathbb{C}). \]
Then \( \Phi^p_\lambda := \lambda^{1/(2\ell)} \Phi_\lambda \) is a linear isometry from \( L^p_\alpha \) onto \( L^p_{\lambda \alpha} \) such that \( \Phi^p_\lambda(F^p_\alpha) = F^p_{\lambda \alpha} \) and \( \Phi^p_\lambda(f^{\infty}_\alpha) = f^{\infty}_{\lambda \alpha} \). In particular,
\[ \langle \Phi^p_\lambda f, \Phi^p_\lambda g \rangle_{\lambda \alpha} = \langle f, g \rangle_\alpha \quad (f, g \in L^2_{\lambda \alpha}). \]

Proof. The first assertion follows by making the change of variable \( w = \lambda^{1/(2\ell)}z \).
The second assertion is a direct consequence of the first one for \( p = 2 \). □

2.2. The Bergman kernel.

It is well-known that \( F^2_{2,\ell} \) with the inner product \( \langle \cdot, \cdot \rangle_\ell \) is a Hilbert space such that the pointwise evaluation \( f \mapsto f(z) \) is a bounded linear functional on \( F^2_{2,\ell} \), for any \( z \in \mathbb{C} \). Thus \( F^2_{2,\ell} \) is a reproducing kernel Hilbert space, that is, for any \( z \in \mathbb{C} \) there exists a unique function \( K^\ell_{\alpha,z} \) in \( F^2_{2,\ell} \) such that \( f(z) = \langle f, K^\ell_{\alpha,z} \rangle_\ell \), for every \( f \in F^2_{2,\ell} \). The Bergman kernel for \( F^2_{2,\ell} \) is the function
\[ K^\ell_{\alpha}(z, w) := K^\ell_{\alpha,z}(w) = K^\ell_{\alpha,w}(z) \quad (z, w \in \mathbb{C}). \]

The following result is well known (see for instance [2]).

Proposition 2.3. Let \( \alpha > 0 \) and \( \ell \in \mathbb{N} \). Then the sequence of monomials \( \{z^m\}_{m \geq 0} \) is an orthogonal basis of \( F^2_{2,\ell} \) and
\[ \|z^m\|^2_{F^2_{2,\ell}} = \frac{\pi}{\ell \alpha^{(m+1)/\ell}} \Gamma \left( \frac{m+1}{\ell} \right). \]
Therefore, the sequence
\[ \{e_m\}_{m \geq 0} := \left\{ \frac{z^m}{\|z^m\|_{F^2_{2,\ell}}} \right\} = \left\{ \frac{\ell \alpha^{(m+1)/\ell}}{\pi} \Gamma \left( \frac{m+1}{\ell} \right) z^m \right\}_{m \geq 0} \]
is an orthonormal basis of \( F^2_{2,\ell} \) and the Bergman kernel for \( F^2_{2,\ell} \) admits the representation
\[ K^\ell_{\alpha}(z, w) = \sum_{m=0}^\infty \frac{z^m \overline{w^m}}{\|w^m\|_{F^2_{2,\ell}} \|z^m\|_{F^2_{2,\ell}}} = \frac{\ell \alpha^{1/\ell}}{\pi} \sum_{m=0}^\infty \frac{\alpha^{m/\ell} z^m \overline{w^m}}{\Gamma \left( \frac{m+1}{\ell} \right)}. \]
In particular,

\begin{equation}
K_\alpha^\ell(z, w) = \alpha^{1/\ell} K_1^{\ell/\alpha}(\alpha^{1/(2\ell)} z, \alpha^{1/(2\ell)} w).
\end{equation}

Formula (2.5) shows that the Bergman kernel can be written in terms of the Mittag-Leffler functions. Namely,

\begin{equation}
K_\alpha^\ell(z, w) = H_{1,1}^\ell(\alpha^{1/\ell} z, \alpha^{1/\ell} w) = \ell\alpha \sum_{k=0}^{\infty} \frac{\lambda^k}{\Gamma(k+1)} (\lambda \in \mathbb{C}),
\end{equation}

where

\begin{equation}
E_{1,1}^\ell(\lambda) = \sum_{k=0}^{\infty} \frac{\lambda^k}{\Gamma(k+1)} (\lambda \in \mathbb{C}).
\end{equation}

It is known that the Mittag-Leffler function $E_{1,1}^\ell(\lambda)$ satisfies the following asymptotic expansion as $|\lambda| \to \infty$ (see [1, Chapter XVIII]):

\begin{equation}
E_{1,1}^\ell(\lambda) = \begin{cases} \ell\lambda e^{\lambda} + O(\lambda^{-1}), & \text{if } |\text{arg}(\lambda)| \leq \frac{\pi}{2\ell}, \\ O(\lambda^{-1}), & \text{if } |\text{arg}(\lambda)| > \frac{\pi}{2\ell}. \end{cases}
\end{equation}

Here $\text{arg}(\lambda)$ denotes the principal branch of the argument of $\lambda$, that is, $-\pi < \text{arg}(\lambda) \leq \pi$.

It is clear that (2.8) implies the following pointwise estimate of the Bergman kernel.

**Proposition 2.4.**

\[ |K_\alpha^\ell(z, w)| \lesssim (1 + |z w|)^{-\frac{1}{\ell}} \left( e^{\alpha \Re((z w)^{\ell})} + 1 \right) \quad (z, w \in \mathbb{C}). \]

Observe that (2.9) also gives pointwise estimates of $K_\alpha^\ell$ for $\ell$ not necessarily integer. However, in this non-integer case to obtain a factorization of the Bergman kernel as in (1.3) seems more difficult. Other estimates for more general radial weights are given in [13].

2.3. **The Bergman projection.** The orthogonal projection $P_\beta^\ell$ from $L_\beta^{2,\ell}$ onto $F_\beta^{2,\ell}$ admits the integral representation

\[ P_\beta^\ell f(z) := \int_{\mathbb{C}} K_\alpha^\ell(z, w) f(w) e^{-\alpha|w|^{2\ell}} d\nu(w) \quad (f \in L_\beta^{2,\ell}, z \in \mathbb{C}). \]

Note that if $f \in L_\beta^p, 1 \leq p < \infty, 0 < \beta < 2\alpha$, then $P_\beta^\ell f$ is well defined, that is, for any $z \in \mathbb{C}$, the function

\[ F_z(w) = K_\alpha^\ell(z, w) f(w) e^{-\alpha|w|^{2\ell}} = K_\alpha^\ell(w, z) f(w) e^{-\alpha|w|^{2\ell}} \]
Lemma 2.1, we have the embedding $F$ shows that $F$ such that $u$ remains to prove that theorem 13 and corollary 14, so we only have for any $b \in F^p_{\beta, \ell}$. In particular, since by lemma 2.1 $H^\infty_{\alpha, \ell} \subset F^p_{\beta, \ell}$, for $\beta > \alpha$, (2.9) also holds for any $b \in H^\infty_{\alpha, \ell}$.

Proposition 2.5. For $\ell \geq 1$ and $\alpha > 0$ we have:

(i) If $1 \leq p \leq \infty$, then $P^\ell_\alpha$ is a bounded projection from $L^p_{\alpha, \ell}$ onto $F^p_{\alpha, \ell}$.
(ii) If $1 \leq p < \infty$, then $(F^p_\alpha)^\ell \equiv F^p_{\alpha, \ell}$, with respect to the pairing $\langle \cdot, \cdot \rangle_\ell$.
(iii) $(f^\infty_{\alpha, \ell})^\ell \equiv F^1_{\alpha, \ell}$, with respect to the pairing $\langle \cdot, \cdot \rangle_\alpha$.

Proof. The proof of the first two assertions can be found, for instance, in [5] Theorem 13 and Corollary 14] and [11] Theorems 3.1 and 3.6], so we only have to prove the last one.

First note that if $b \in F^1_{\alpha, \ell}$ then $\langle \cdot, b \rangle_\ell \in (f^\infty_{\alpha, \ell})^\ell$ and $\|\langle \cdot, b \rangle_\ell\|_{(f^\infty_{\alpha, \ell})^\ell} \lesssim \|b\|_{F^1_{\alpha, \ell}}$.

Conversely, given $u \in (f^\infty_{\alpha, \ell})^\ell$, we are going to prove that there is $b \in F^1_{\alpha, \ell}$ such that $u = \langle \cdot, b \rangle_\ell$ and $\|b\|_{F^1_{\alpha, \ell}} \lesssim \|u\|_{(f^\infty_{\alpha, \ell})^\ell}$. Pick $\alpha/2 < \beta < \alpha$. Then, by lemma 2.1 we have the embedding $F^2_{\beta, \ell} \hookrightarrow f^\infty_{\alpha, \ell}$ and so the restriction of $u$ to $F^2_{\beta, \ell}$ is a bounded linear form on this space. It follows that there is $g \in F^2_{\beta, \ell}$ such that $u(f) = \langle f, g \rangle_{\beta, \ell}$ for every $f \in E$. Now Proposition 2.2 for $\lambda = \alpha/\beta$ shows that $b := \Phi^2_{\alpha, \ell} g \in F^2_{\alpha, \ell}$ satisfies

$$u(f) = \langle f, g \rangle_{\beta, \ell} = \langle \Phi^2_{\alpha} f, \Phi^2_{\alpha} g \rangle_{\alpha, \ell} = \langle f, b \rangle_\ell, \quad \text{for every } f \in E.$$

(Note that $(\ast)$ holds because both functions $f$ and $g$ are entire.) Thus it only remains to prove that $\|b\|_{L^1_{\alpha, \ell}} \lesssim \|u\|_{(f^\infty_{\alpha, \ell})^\ell}$. Recall that, by duality,

$$\|b\|_{L^1_{\alpha, \ell}} = \sup_{f \in C_c(\mathbb{C})} \left| \int_{\mathbb{C}} f(z) e^{-\frac{\alpha}{2}|z|^2} \overline{b(z)} \, d\nu(z) \right| = \sup_{f \in C_c(\mathbb{C})} \|T_{\alpha} f, b\|_{\alpha, \ell}.$$
where \( T_\alpha f(z) := f(z)e^{\frac{2\alpha}{\beta}z^2} \). Note that \( b = P^\ell_\alpha(b) \), because \( b \in F^{2,\ell}_{\alpha/\beta} \) and \( \alpha^2/\beta < 2\alpha \). Therefore, for any \( f \in C_c(\mathbb{C}) \), we have that
\[
\langle T_\alpha f, b \rangle_\alpha = \langle T_\alpha f, P^\ell_\alpha b \rangle_\alpha \overset{(1)}{=} \langle P^\ell_\alpha(T_\alpha f), b \rangle_\alpha \overset{(2)}{=} u(P^\ell_\alpha(T_\alpha f)),
\]
where (1) follows from Fubini’s theorem and (2) holds since \( P^\ell_\alpha(T_\alpha f) \in E \). And hence
\[
\|T_\alpha f\|_\alpha \leq \|u\|_{(L^\infty)^*} \|P^\ell_\alpha\|_{L^\infty} \|T_\alpha f\|_{L^\infty} = \|u\|_{(L^\infty)^*} \|P^\ell_\alpha\|_{L^\infty} \|f\|_{L^\infty},
\]
which gives that \( \|b\|_{L^\infty} \lesssim \|u\|_{(L^\infty)^*} \). □

The last result of this subsection states that the dilation operators \( \Phi^\ell_\lambda \), defined by (2.3), “commute” with the Bergman projections.

**Proposition 2.6.** Let \( 1 \leq p \leq \infty \), \( \ell \in \mathbb{N} \) and \( \alpha, \beta, \lambda > 0 \) such that \( \beta < 2\alpha \). Then
\[
\Phi^\ell_\lambda(P^\ell_\alpha f) = P^\ell_\alpha(\Phi^\ell_\lambda f) \quad (f \in L^p_\beta^\ell).
\]

**Proof.** Let \( f \in L^p_\beta^\ell \). Then
\[
\Phi^\ell_\lambda(P^\ell_\alpha f)(z) = \int_\mathbb{C} K^\ell_\alpha(\lambda^{1/(2\ell)}z, w)f(w)e^{-\alpha|w|^{2\ell}}\,d\nu(w).
\]
By making the change of variable \( w = \lambda^{1/(2\ell)}v \) and taking into account that
\[
\lambda^{1/(2\ell)}K^\ell_\alpha(\lambda^{1/(2\ell)}z, \lambda^{1/(2\ell)}v) = K^\ell_\lambda(z, v),
\]
which follows from (2.3), we conclude that \( \Phi^\ell_\lambda(P^\ell_\alpha f)(z) = P^\ell_\alpha(\Phi^\ell_\lambda f)(z) \). □

### 2.4. The small Hankel operator on \( F^{p,\ell}_\alpha \), \( 1 \leq p < \infty \).

The next lemma gives some properties of the subspace of entire functions \( E \) defined in (1.1).

**Lemma 2.7.** The space \( E \) satisfies the following properties:

(i) \( E \cdot E \subset E \).

(ii) \( E \subset F^{1,\ell}_\alpha \), for any \( \alpha > 0 \).

(iii) \( E \) contains the space of all the holomorphic polynomials.

(iv) \( E \) contains the space \( \text{Span}\{K^\ell_{\alpha, z} : z \in \mathbb{C}\} \), i.e. the set of finite linear combinations of functions \( K^\ell_{\alpha, z} \).

(v) \( E \) is dense in \( F^{\infty,\ell}_\alpha \) and in \( F^{p,\ell}_\alpha \), for any \( 1 \leq p < \infty \).

**Proof.** The first three assertions are a consequence of the definition of \( E \) and the fact that \( e^{\beta|w|^{2\ell} - \gamma|w|^{2\ell}} \in L^1 \), for any \( \beta, \gamma > 0 \).

The fourth assertion is a consequence of Proposition 2.4.
The density of $E$ in $F_{\alpha}^{p,\ell}$ is a consequence of the fact that the holomorphic polynomials are dense in $F_{\alpha}^{p,\ell}$ (see [11, Theorem 28]).

In order to define the small Hankel operator for a large class of symbols we consider the space $X_{\alpha}^{\infty,\ell}$ of all measurable functions $\varphi$ on $\mathbb{C}$ such that

$$
\|\varphi\|_{X_{\alpha}^{\infty,\ell}} := \text{ess sup}_{z \in \mathbb{C}} |\varphi(z)|(1 + |z|)^{2-2\ell} e^{-\frac{\beta}{2}|z|^{2\ell}} < \infty.
$$

Observe that $H_{\alpha}^{\infty,\ell} = H(\mathbb{C}) \cap X_{\alpha}^{\infty,\ell}$.

Let $\varphi$ be a function in $X_{\alpha}^{\infty,\ell}$. Since $X_{\alpha}^{\infty,\ell} \subset L_{\beta}^{1,\ell}$, for any $\beta > \alpha$, the small Hankel operator $h_{\varphi,\alpha}^\ell$ with symbol $\varphi$ is well defined on $E$ by

$$
h_{\varphi,\alpha}^\ell(f)(z) := P_{\alpha}^\ell(\overline{f \varphi})(z) = \int_{\mathbb{C}} K_{\alpha}^\ell(w, z) f(w) \overline{\varphi(w)} e^{-\alpha|w|^{2\ell}} d\nu(w).
$$

The next proposition states the relationship between the the small Hankel operator $h_{\varphi,\alpha}^\ell$ and the corresponding Hankel bilinear form defined by

$$
\Lambda_{\varphi,\alpha}^\ell(f, g) := \langle fg, \varphi \rangle_{\alpha}^\ell \quad (f, g \in E).
$$

**Proposition 2.8.** If $f, g \in E$ and $\varphi \in X_{\alpha}^{\infty,\ell}$, then we have

$$
\Lambda_{\varphi,\alpha}^\ell(f, g) = \langle g, h_{\varphi,\alpha}^\ell(f) \rangle_{\alpha} = \langle f, h_{\varphi,\alpha}^\ell(g) \rangle_{\alpha}.
$$

Moreover, if $b = P_{\alpha}^\ell(\varphi) \in H_{\alpha}^{\infty,\ell}$, then $h_{\varphi,\alpha}^\ell(f) = h_{b,\alpha}^\ell(f)$, $\Lambda_{b,\alpha}^\ell(f, g) = \Lambda_{\varphi,\alpha}^\ell(f, g)$ and $h_{b,\alpha}^\ell(f) = h_{\varphi,\alpha}^\ell(f)$.

**Proof.** Formula (2.11) follows from Fubini’s theorem and the fact that

$$
\Psi_{f, g, \varphi}(z, w) := K_{\alpha}^\ell(w, z) f(w) \overline{\varphi(w)} e^{-\alpha|w|^{2\ell}} g(z) e^{-\alpha|z|^{2\ell}}
$$

is in $L^1(\mathbb{C} \times \mathbb{C})$.

This is a consequence of Proposition 2.4. Indeed, if $\lambda > 0$ we have that

$$
|\Psi_{f, g, \varphi}(z, w)| \lesssim \|\varphi\|_{X_{\alpha}^{\infty,\ell}} (1 + |w|)^{\beta - 3} |f(w)|(1 + |z|)^{\lambda - 1} |g(z)| e^{\alpha|z|^{2\ell}} e^{-\alpha|w|^{2\ell}} e^{\beta|z|^{2\ell}} e^{\beta|w|^{2\ell}} e^{\alpha|w|^{\lambda - 1} |z|^{2\ell}}
$$

for some $\beta > 0$. Therefore by choosing $1 < \lambda < \sqrt{2}$ we see that $\Psi_{f, g, \varphi} \in L^1(\mathbb{C} \times \mathbb{C})$.

By Lemma 2.7 if $f, g \in E$ then $fg \in E \subset F_{\alpha}^{1,\ell}$, and so $fg = P_{\alpha}^\ell(fg)$, by Proposition 2.5. Therefore

$$
\Lambda_{\varphi,\alpha}^\ell(f, g) = \int_{\mathbb{C}} P_{\alpha}^\ell(fg)(w) \overline{\varphi(w)} e^{-\alpha|w|^{2\ell}} d\nu(w)
$$

$$
= \int_{\mathbb{C}} (fg)(z) K_{\alpha}^\ell(w, z) e^{-\alpha|z|^{2\ell}} d\nu(z) \overline{\varphi(w)} e^{-\alpha|w|^{2\ell}} d\nu(w).
$$
Proposition 2.10. Since
\[(fg)(z)K^\ell_\alpha(w,z)e^{-|z|^2/2}\overline{\varphi(w)}e^{-|w|^2/2} = \Psi_{1,fg,\varphi}(w,z) \in L^1(\mathbb{C} \times \mathbb{C}),\]
Fubini's theorem gives \(\Lambda^\ell_\alpha(f,g) = \Lambda^\ell_{\varphi,\alpha}(f,g)\) and \(b^\ell_{\varphi,\alpha}(f) = b^\ell_{\varphi,\alpha}(f)\) for any \(f, g \in E\).

As a consequence of the above proposition and Proposition 2.5(ii)-(iii) we obtain:

**Corollary 2.9.**

(i) If \(1 < p < \infty\), the Hankel operator \(b^\ell_{\varphi,\alpha}\) defined on the space \(E\) extends to a bounded operator on \(F^p,\ell_\alpha\), also denoted by \(b^\ell_{\varphi,\alpha}\), if and only if the bilinear form \(\Lambda^\ell_{\varphi,\alpha}\) defined on \(E \times E\) extends to a bounded bilinear form on \(F^p,\ell_\alpha \times F^p,\ell_\alpha\). Moreover, \(\|b^\ell_{\varphi,\alpha}\|_{F^p,\ell} \simeq \|\Lambda^\ell_{\varphi,\alpha}\|_{F^p,\ell \times F^p,\ell}\).

(ii) The Hankel operator \(b^\ell_{\varphi,\alpha}\) defined on the space \(E\) extends to a bounded operator, also denoted by \(b^\ell_{\varphi,\alpha}\), either on \(F^{1,\ell}_\alpha\) or on \(F^{\infty,\ell}_\alpha\). Moreover, \(\|b^\ell_{\varphi,\alpha}\|_{F^{1,\ell}_\alpha} \simeq \|\Lambda^\ell_{\varphi,\alpha}\|_{F^{1,\ell}_\alpha \times F^{1,\ell}_\alpha}\).

(iii) The adjoint (in the sense of (2.11)) of \(b^\ell_{\varphi,\alpha} : F^p,\ell_\alpha \to F^{p',\ell}_\alpha\), \(1 < p < \infty\), is \(b^\ell_{\varphi,\alpha} : F^{p',\ell}_\alpha \to F^p,\ell_\alpha\) and the adjoint of \(b^\ell_{\varphi,\alpha} : F^{\infty,\ell}_\alpha \to F^{\infty,\ell}_\alpha\) is \(b^\ell_{\varphi,\alpha} : F^{1,\ell}_\alpha \to F^{1,\ell}_\alpha\).

The last result of this subsection shows that the dilation operators \(\Phi^\ell_\lambda\), defined by (2.31), “commute” with the small Hankel operators.

**Proposition 2.10.** Let \(1 \leq p < \infty\), \(\alpha, \lambda > 0\) and \(\ell \in \mathbb{N}\). Then:

(i) \(\Phi^\ell_\lambda(X^\infty_\alpha) = X^\infty_{\alpha,\lambda}\) and \(\Phi^\ell_\lambda(E) = E\).

(ii) If \(\varphi \in X^\infty_\alpha\), \(\psi = \Phi^\ell_\lambda \varphi\) then \(\Phi^\ell_\lambda(b^\ell_{\varphi,\alpha}f) = b^\ell_{\psi,\lambda}(\Phi^\ell_\lambda f)\), for every \(f \in E\), and so \(\|b^\ell_{\varphi,\alpha}\|_{F^{p,\ell}_\alpha} = \|b^\ell_{\psi,\lambda}\|_{F^{p,\ell}_\alpha}\) and \(\|b^\ell_{\varphi,\alpha}\|_{F^{p,\ell}_\alpha} = \|b^\ell_{\psi,\lambda}\|_{F^{p,\ell}_\alpha}\).

**Proof.** The proof of (i) is straightforward. Part (ii) follows from Proposition 2.2 and Proposition 2.6. Indeed,
\[
\Phi^\ell_\lambda(b^\ell_{\varphi,\alpha}f) = \Phi^\ell_\lambda(P^\ell_{\alpha}(f \varphi)) = \Phi^\ell_\lambda(P^\ell_{\alpha}(f \varphi)) \overset{(\ast)}{=} P^\ell_{\lambda}(\Phi^\ell_\lambda(f \varphi)) = P^\ell_{\lambda}(\Phi^\ell_\lambda(f \varphi)) = b^\ell_{\psi,\lambda}(\Phi^\ell_\lambda f),
\]
for every \(f \in E\), where \((\ast)\) holds by Proposition 2.6. Then the above identity and Proposition 2.2 directly imply that \(\|b^\ell_{\varphi,\alpha}\|_{F^{p,\ell}_\alpha} = \|b^\ell_{\psi,\lambda}\|_{F^{p,\ell}_\alpha}\).
3. Proof of Theorem

3.1. Proof of the sufficiency.

**Lemma 3.1.** If \( \varphi \in L^\infty \) and \( 1 \leq p \leq \infty \), then \( h_{\varphi,\alpha}^t \) is bounded on \( F_{p,\alpha}^t \) and \( \| h_{\varphi,\alpha}^t \| \lesssim \| \varphi \|_{L^\infty} \).

**Proof.** If \( \varphi \in L^\infty \) and \( f \in F_{p,\alpha}^t \), then \( \varphi f \in L_{p,\alpha}^t \) and \( \| \varphi f \|_{L_{p,\alpha}^t} \leq \| \varphi \|_{L^\infty} \| f \|_{F_{p,\alpha}^t} \).

By Proposition \([2.5][1]\) \( P_{\alpha}^t \) is bounded on \( L_{p,\alpha}^t \), so we conclude that
\[
\| h_{\varphi,\alpha}^t (f) \|_{L_{p,\alpha}^t} = \| P_{\alpha}^t (\varphi f) \|_{L_{p,\alpha}^t} \lesssim \| P_{\alpha}^t \|_{L_{p,\alpha}^t} \| \varphi \|_{L^\infty} \| f \|_{F_{p,\alpha}^t}.
\]

The following result is a corollary of Proposition \([2.5][1]\).

**Proposition 3.2.** The projection \( P_{\alpha}^t \) is bounded from \( L^\infty \) onto \( F_{p,\alpha}^{\infty,\ell} \). Moreover, \( \inf \{ \| \varphi \|_{L^\infty} : \varphi \in L^\infty, P_{\alpha}^t \varphi = f \} \leq 2^{1/t} \| f \|_{F_{p,\alpha}^{\infty,\ell}} \) for every \( f \in F_{p,\alpha}^{\infty,\ell} \).

**Proof.** It is clear that \( T_{\alpha} \varphi(z) := \varphi(z)e^{\alpha|z|^{2t}} \) defines a linear isometry from \( L^\infty \) onto \( L_{2\alpha}^{\infty,\ell} \). Then, for any \( \varphi \in L^\infty \), we have that
\[
P_{\alpha}^t \varphi(z) = \int_{\mathbb{C}} K_{\alpha}^t(z, w) T_{\alpha} \varphi(w) e^{-2\alpha|w|^{2t}} d\nu(w)
\]
\[
\leq 2^{-1/t} \int_{\mathbb{C}} K_{2\alpha}^t (2^{-1/t} z, w) T_{\alpha} \varphi(w) e^{-2\alpha|w|^{2t}} d\nu(w)
\]
\[
= 2^{-1/t} P_{2\alpha}^t (T_{\alpha} \varphi) (2^{-1/t} z) = 2^{-1/t} \Phi_1^t (P_{2\alpha}^t (T_{\alpha} \varphi))(z)
\]
where (1) and (2) follow from \([2.5][1]\) and \([2.3]\), respectively. In other words, the projection \( P_{\alpha}^t \) on \( L^\infty \) is the composition of the following three bounded linear exhaustive operators:

(i) \( T_{\alpha} : L^\infty \rightarrow L_{2\alpha}^{\infty,\ell} \);
(ii) \( P_{2\alpha}^t : L_{2\alpha}^{\infty,\ell} \rightarrow F_{2\alpha}^{\infty,\ell} \);
(iii) \( \Psi := 2^{-1/t} \Phi_1^t : F_{2\alpha}^{\infty,\ell} \rightarrow F_{\alpha/2}^{\infty,\ell} \).

It directly follows that \( P_{\alpha}^t \) is bounded from \( L^\infty \) onto \( F_{\alpha/2}^{\infty,\ell} \). Moreover, since \( P_{2\alpha}^t \) is a projection from \( L_{2\alpha}^{\infty,\ell} \) onto \( F_{2\alpha}^{\infty,\ell} \) (by Proposition \([2.5][1]\)) and the operator \( \Psi := 2^{-1/t} \Phi_1^t : F_{2\alpha}^{\infty,\ell} \rightarrow F_{\alpha/2}^{\infty,\ell} \) is an isomorphism such that \( \Psi^{-1} = 2^{1/t} \Phi_4^t \) satisfies \( \| \Psi^{-1}(f) \|_{F_{\alpha/2}^{\infty,\ell}} = 2^{1/t} \| f \|_{F_{\alpha/2}^{\infty,\ell}} \), for every \( f \in F_{\alpha/2}^{\infty,\ell} \), (by Proposition \([2.2]\)) we conclude that
\[
\inf \{ \| \varphi \|_{L^\infty} : \varphi \in L^\infty, P_{\alpha}^t \varphi = f \} \leq 2^{1/t} \| f \|_{F_{\alpha/2}^{\infty,\ell}} \text{ for every } f \in F_{\alpha/2}^{\infty,\ell}.
\]

**Proposition 3.3.** Let \( b \in F_{\alpha/2}^{\infty,\ell} \).

(i) If \( 1 \leq p < \infty \), then \( h_{b,\alpha}^t \) extends to a bounded operator on \( F_{p,\alpha}^t \).
Since (i) Stirling’s formula gives

Moreover, \( \|h_{b,\alpha}^f\|_{F_{b,\alpha}^\ell, \ell} \leq \|b\|_{F_{b,\alpha}^\ell, \ell} \), for any \( 1 \leq \ell < \infty \), and \( \|h_{b,\alpha}^f\|_{F_{b,\alpha}^\ell, \ell} \leq \|b\|_{F_{b,\alpha}^\ell, \ell} \).

Proof. In order to prove (i), we show that for \( 1 \leq \ell < \infty \),

\[
(3.12) \quad \|h_{b,\alpha}^f(f)\|_{L_{b,\alpha}^\ell, \ell} \leq \|b\|_{F_{b,\alpha}^\ell, \ell} \|f\|_{F_{b,\alpha}^\ell, \ell} \quad (f \in E).
\]

By Proposition 3.2, \( b = P_\alpha^f(\varphi) \) for some \( \varphi \in L^\infty \) such that \( \|\varphi\|_{L^\infty} \leq 3\|b\|_{F_{b,\alpha}^\ell, \ell} \).

If \( f \in E \), Proposition 2.8 gives \( h_{b,\alpha}^f(f) = h_{\varphi,\alpha}^f(f) \), so Lemma 3.1 implies (3.12).

Taking into account (3.12) for \( p = \infty \), the proof of (ii) will follow after checking \( h_{b,\alpha}^f(E) \subset F_{b,\alpha}^\ell, \ell \). Indeed, by Proposition 2.4, for \( f \in E \) and \( 0 < \lambda < 1 \), we have

\[
|h_{b,\alpha}^f(f)(z)| \leq (1 + |z|)^{\ell-1}(1 + |w|)^{\ell-1}e^{\alpha|z|^2|w|}\cdot e^{\beta|\varphi|}\cdot e^{\beta|w|^2/4}e^{-\alpha|w|^2}d\nu(w)
\]

\[
\leq (1 + |z|)^{\ell-1}\int_C e^{\alpha\lambda^2|z|^2/2}e^{\beta|\varphi|}\cdot e^{\beta|w|^2/2\lambda^2}e^{-3\alpha|w|^2/4}d\nu(w)
\]

\[
\leq (1 + |z|)^{\ell-1}e^{\alpha\lambda^2/2}\int_C e^{2\beta|w|^2/3\lambda^2}e^{-\alpha(3/2-1/\lambda^2)|w|^2/3}d\nu(w).
\]

Choosing \( \sqrt{2/3} < \lambda < 1 \), the last integral is finite and we get

\[
\lim_{|z| \to \infty} |h_{b,\alpha}^f(f)(z)|e^{-\alpha|z|^2/2} = 0. \quad \Box
\]

3.2. Proof of the necessity.

In order to prove the necessity we need some technical results.

The first one is a simple consequence of Stirling’s formula.

Lemma 3.4. Let \( \delta \) be a positive number. Then

(i) \( \Gamma(s + t) \simeq s^{\delta}\Gamma(s) \quad (s \geq 2\delta, \ |t| \leq \delta) \).

(ii) Let \( a \) be a real number. Then

\[
\sum_{k=0}^{\infty} \frac{s^k}{k!(k+1)^a} \simeq \frac{e^s}{(1+s)^a} \quad (s \geq 0).
\]

All the constants in the above equivalences only depend on \( \delta \) and \( a \).

Proof. \( \Box \) Stirling’s formula gives

\[
\Gamma(x) \simeq x^{x-1/2}e^{-x} \quad (x \geq \delta),
\]

so

\[
\Gamma(s + t) \simeq (s + t)^{s+t-1/2}e^{-s-t} \simeq (s + t)^t(s + t)^{s-1/2}e^{-s}.
\]

Since \( \frac{s}{2} \leq s + t \leq 2s \) and \( |t| \leq \eta \), we have \( (s + t)^t \simeq s^t \) and \( (s + t)^{s-1/2} \simeq s^{s-1/2} \).
Note that both terms of the estimate are positive continuous functions of \( s \geq 0 \). So it is clear that we only have to prove that

\[
(3.13) \quad f_a(x) := \sum_{k=0}^{\infty} \frac{s^k}{k!} \frac{s^a}{(k+1)^a} \simeq e^s \quad (s \geq 1).
\]

But

\[
f_{a+1}(s) = \sum_{k=0}^{\infty} \frac{s^{k+1}}{(k+1)!} \frac{s^a}{(k+1)^a} = \sum_{k=1}^{\infty} \frac{s^k s^a}{k! \cdot k} \simeq \sum_{k=1}^{\infty} \frac{s^k}{k!} \frac{s^a}{(k+1)^a} \simeq f_a(s),
\]

and so we may assume that \( 0 \leq a < 1 \). Let \( s \geq 1 \) and let \( j \in \mathbb{N} \) be its integer part. Then

\[
f_a(s) = \sum_{k=0}^{j-1} \frac{s^k}{k!} \left( \frac{s}{k+1} \right)^a + \sum_{k=j}^{\infty} \frac{s^k}{k!} \left( \frac{s}{k+1} \right)^a.
\]

Now

\[
1 \leq \left( \frac{s}{k+1} \right)^a \leq \frac{s}{k+1} \quad (0 \leq k < j)
\]

and

\[
\frac{s}{k+1} \leq \left( \frac{s}{k+1} \right)^a \leq 1 \quad (j \leq k).
\]

It follows that

\[
\sum_{k=0}^{j-1} \frac{s^k}{k!} + \sum_{k=j}^{\infty} \frac{s^{k+1}}{(k+1)!} \leq f_a(s) \leq \sum_{k=0}^{j-1} \frac{s^k}{k!} + \sum_{k=j}^{\infty} \frac{s^{k+1}}{(k+1)!},
\]

and therefore

\[
e^s \left( 1 - \frac{2}{e} \right) \leq e^s \left( 1 - \frac{(j+1)^j}{e^j j!} \right) \leq e^s - \frac{s^j}{j!} \leq f_a(s) \leq 2e^s,
\]

since the sequence \( c_j = \frac{(j+1)^j}{e^j j!} \) is decreasing. Hence (3.13) holds. \( \square \)

The following lemma is an essential tool to prove the necessity.

**Lemma 3.5.** For \( \ell \in \mathbb{N} \), \( a, b > 0 \) and \( c \geq 0 \), let

\[
\mathcal{I}_{a,b,c}^{\ell}(z) := \int_{\mathbb{C}} \left| e^{a(z\bar{w})^\ell} \right|^2 e^{-b|w|^{2\ell}} (1 + |w|)^c d\nu(w) \quad (z \in \mathbb{C}).
\]

Then

\[
(3.14) \quad \mathcal{I}_{a,b,c}^{\ell}(z) \simeq e^{\frac{a}{b} z^{2\ell}/(1 + |z|)^{c+2-2\ell}} \quad (z \in \mathbb{C}).
\]
Proof. It is enough to prove the estimate (3.14) for \( |z| \geq 1 \). Observe that

\[
\mathcal{J}^\ell_{a,b,c}(z) \simeq \mathcal{I}^\ell_{a,b,c}(z) + \mathcal{J}^\ell_{a,b,c}(z),
\]

where

\[
\mathcal{J}^\ell_{a,b,c}(z) := \int_C e^{azw} |w|^{2\ell} e^{-|w|^2} |w|^c d\nu(w).
\]

Thus we only have to show that

\[
\mathcal{J}^\ell_{a,b,c}(z) \simeq e^{a^2|z|^{2\ell}/b} |z|^{c+2-2\ell} \quad (|z| \geq 1).
\]

Indeed, by integrating in polar coordinates and orthogonality,

\[
\mathcal{J}^\ell_{a,b,c}(z) \simeq \sum_{k=0}^{\infty} \int_0^{\infty} \frac{a^{2k} |z|^{2k\ell}}{(k!)^2} e^{-br^2} r^{2k\ell+c+1} dr \int_0^{\infty} e^{-t(2k\ell+c+2)/(2\ell)-1} dt
\]

\[
\simeq \sum_{k=0}^{\infty} \frac{a^{2k} |z|^{2k\ell}}{b^{k+c+2}/(2\ell)} \frac{1}{(k!)^2} \frac{\Gamma(k + (c + 2)/(2\ell))}{(k!)^{(2\ell-2-2\ell)/(2\ell)}}.
\]

Therefore Lemma 3.4 completes the proof:

\[
\mathcal{J}^\ell_{a,b,c}(z) \simeq \sum_{k=0}^{\infty} \frac{a^{2k} |z|^{2k\ell}}{b^k k!(k+1)^{(2\ell-2-2\ell)/(2\ell)}} \simeq e^{a^2|z|^{2\ell}/b} |z|^{c+2-2\ell}.
\]

□

Proof of the necessity. Let \( 1 \leq p \leq \infty \) and \( b \in H^{\infty,\ell}_\alpha \). Suppose that \( h^\ell_{b,\alpha} : (E, \| \cdot \|_{F_p^\ell}) \to L^p_\alpha \) is bounded and we want to prove that \( b \in F^{\infty,\ell}_{\alpha/2} \) and

\[
\|b\|_{F_{\alpha/2}^{\infty,\ell}} \lesssim \|h^\ell_{b,\alpha}\|_{F_{p}^{\ell}}.
\]

First of all, by Proposition 2.10 we may assume that \( \alpha = 1 \). Now (2.9) gives that

(3.15) \[
\bar{b}(z) = \int_C K^\ell_1(w, z) \bar{b}(w) e^{-|w|^{2\ell}} d\nu(w) = \langle K^\ell_1(\cdot, z), b \rangle^\ell_1.
\]

We decompose the Bergman kernel as

\[
K^\ell_1(w, z) = G_0(w, z)G^\ell_1(w, z),
\]

where

(3.16) \[
G_0(w, z) := e^{i\pi \ell w} \quad \text{and} \quad G^\ell_1(w, z) := e^{-i\pi \ell w} K^\ell_1(w, z).
\]

By Proposition 2.4 \( G_0(\cdot, z), G^\ell_1(\cdot, z) \in E \), and so (3.15) and Proposition 2.8 show

(3.17) \[
\bar{b}(z) = \langle G_1(\cdot, z), h^\ell_{b,1}(G_0(\cdot, z)) \rangle^\ell_1.
\]
Therefore the boundedness of $\mathfrak{b}_{b,1}^\ell$ implies that
\begin{equation}
|b(z)| \lesssim \|\mathfrak{b}_{b,1}^\ell\|_{F_{p,\ell}^w} \|G_0(\cdot, z)\|_{F_{p,\ell}^w} \|G_1(\cdot, z)\|_{F_{p,\ell}^w}.
\end{equation}
We claim that:
\begin{align}
&\|G_0(\cdot, z)\|_{F_{p,\ell}^w} \simeq (1 + |z|)^{2(1-\ell)/p} e^{|z|^{2\ell}/8} \\
&\|G_1(\cdot, z)\|_{F_{p,\ell}^w} \lesssim (1 + |z|)^{2(\ell-1)/p} e^{|z|^{2\ell}/8}
\end{align}
These norm-estimates together with (3.18) give $|b(z)| \lesssim \|\mathfrak{b}_{b,1}^\ell\|_{F_{p,\ell}^w} e^{\epsilon |z|^{2\ell}/4}$. Now, for $1 \leq p < \infty$, (3.19) is a consequence of Lemma 3.5:
\[\|G_0(\cdot, z)\|_{F_{p,\ell}^w} = \int_\mathbb{C} |e^{p(z\overline{w})^\ell/4}|^2 e^{-p|w|^{2\ell}/2} d\nu(w) = \mathcal{I}_p^\ell\]
\[\simeq (1 + |z|)^{2(1-\ell)/p} e^{p|z|^{2\ell}/8}.
\]
If $p = \infty$, using the identity
\begin{equation}
\text{Re}((z\overline{w})^\ell) - |w|^{2\ell} = -|w|^{\ell - \ell/2} + |z|^{2\ell/4},
\end{equation}
we obtain
\[\|G_0(\cdot, z)\|_{F_{\infty,\ell}^w} = \sup_{w \in \mathbb{C}} |e^{z\overline{w})^\ell/2/2} e^{-|w|^{2\ell/2}} = e^{1/8}.
\]
On the other hand, by Proposition 2.4
\[|G_1(w, z)| \lesssim (1 + |zw|)^{\ell-1} \left(e^{\text{Re}(z\overline{w})^\ell/2} + e^{-\text{Re}(z\overline{w})^\ell/2}\right)
\lesssim (1 + |z|)^{\ell-1}(1 + |w|)^{\ell-1} \left(e^{\text{Re}(z\overline{w})^\ell/2} + e^{-\text{Re}(z\overline{w})^\ell/2}\right).
\]
Therefore, for $1 \leq p' < \infty$, we have
\[\|G_1(\cdot, z)\|_{F_{p',\ell}^w} \lesssim J_1(z)^{1/p'} + J_2(z)^{1/p'},
\]
where
\[J_1(z) := (1 + |z|)^{\ell-1} \int_\mathbb{C} |e^{p(z\overline{w})^\ell/4}|^2 e^{-p'|z|^{2\ell/2}(1 + |w|)^{\ell-1} d\nu(w)}
\]
and
\[J_2(z) := (1 + |z|)^{\ell-1} \int_\mathbb{C} |e^{-p'(z\overline{w})^\ell/4}|^2 e^{-p'|z|^{2\ell/2}(1 + |w|)^{\ell-1} d\nu(w)}.
\]
By Lemma 3.5
\[J_1(z) = (1 + |z|)^{\ell-1} \int_{p'/4,p'/2} \mathcal{I}_{p'/4,p'/2}^\ell(\cdot) (z) \simeq (1 + |z|)^{2(\ell-1)(\ell-1)} e^{p'|z|^{2\ell}/8}.
\]
Since $J_2(z) = J_1(e^{i\pi/\ell} z)$, we obtain the estimate (3.20).
If $p' = \infty$, by using (3.21), we have
\[\|G_1(\cdot, z)\|_{F_{\infty,\ell}^w} = \sup_{w \in \mathbb{C}} |G_1(w, z)| e^{-|w|^{2\ell/2}} \lesssim (1 + |z|)^{2(\ell-1)} e^{1/8}.
\]
4. Proof of Theorem 1.2

The next proposition will be used to prove Theorem 1.2.

Proposition 4.1. The dual of $F_{2a}^{1,\ell}$ with respect to the pairing $\langle \cdot, \cdot \rangle^\ell$ is $F_{a/2}^{\infty,\ell}$.

Proof. By Proposition 2.5, if $\Phi \in \left( F_{2a}^{1,\ell} \right)^*$, there exists a unique $h \in F_{2a}^{\infty,\ell}$ such that

$$\Phi(f) = \langle f, h \rangle_{2a}^\ell = \langle f(z), h(z)e^{-\alpha|z|^{2\ell}} \rangle_{\alpha}^\ell, \quad \text{for any } f \in E.$$ 

Since $\varphi(z) = h(z)e^{-\alpha|z|^{2\ell}} \in L_{\infty}$, Proposition 3.2 gives $g = P_{a}^\ell(\varphi) \in F_{a/2}^{\infty,\ell}$, so

$$\Phi(f) = \langle f, g \rangle_{a/2}^\ell, \quad \text{for any } f \in E.$$ 

Conversely, if $g \in F_{a/2}^{\infty,\ell}$, by Proposition 3.2 there exists $\varphi \in L_{\infty}$ such that $P_{a}^\ell(\varphi) = g$ and $\|\varphi\|_{L_{\infty}} \simeq \|g\|_{F_{a/2}^{\infty,\ell}}$. Thus, for $f \in E$, we have

$$|\langle f, g \rangle_{a/2}^\ell| = |\langle f, \varphi \rangle_{a}^\ell| \leq \|\varphi\|_{L_{\infty}} \|f\|_{F_{a/2}^{1,\ell}}.$$ 

This ends the proof. □

Proof of Theorem 1.2 The proof of this result follows from standard arguments used in the setting of classical spaces of holomorphic functions. We only include a sketch of the proof for the sake of completeness.

(i) It is a consequence of Corollary 2.9 and Theorem 1.1

(ii) It is a consequence of Propositions 3.2 and 4.1

(iii) First we consider the case $1 < p < \infty$. By (ii), in order to show that $F_{a}^{p,\ell} \cap F_{a}^{p',\ell} = F_{2a}^{1,\ell}$, it is enough to prove that the dual of $F_{a}^{p,\ell} \cap F_{a}^{p',\ell}$ with respect to the pairing $\langle \cdot, \cdot \rangle^\ell$ is $F_{a/2}^{\infty,\ell}$.

By (i), if $b \in F_{a/2}^{\infty,\ell}$ then $\Lambda_{a}^b(h, 1)$ defines a bounded bilinear form on $F_{a}^{p,\ell} \times F_{a}^{p',\ell}$, so $h \mapsto \Lambda_{a}^b(h, 1)$ is a bounded linear form on $F_{a}^{p,\ell} \cap F_{a}^{p',\ell}$.

Conversely, it is clear that any form $\Phi$ on $F_{a}^{p,\ell} \cap F_{a}^{p',\ell}$ defines a bounded linear form on $F_{a}^{p,\ell}$. Thus, by Proposition 2.5(ii), there exists $b \in F_{a}^{p',\ell}$ such that $\Phi(h) = \Lambda_{a}^b(h, 1)$, for any $h \in E$. Since the space $E$ is dense in $F_{a}^{p,\ell}$ and $F_{a}^{p',\ell}$, the bilinear form $\Lambda_{a}^b$ extends boundedly to $F_{a}^{p,\ell} \cap F_{a}^{p',\ell}$. Thus, by part (i), $b \in F_{a}^{\infty,\ell}$.

Similar arguments, using Proposition 2.5(iii), prove that $F_{a}^{1,\ell} \cap F_{a}^{\infty,\ell} = F_{2a}^{1,\ell}$. Since $F_{a}^{1,\ell} \cap F_{a}^{\infty,\ell} \subset F_{2a}^{1,\ell}$, we have

$$F_{2a}^{1,\ell} = F_{a}^{1,\ell} \cap F_{a}^{\infty,\ell} \subset F_{a}^{1,\ell} \cap F_{a}^{\infty,\ell} \subset F_{2a}^{1,\ell},$$

which ends the proof. □
5. Proof of Theorem 1.3

In order to prove Theorem 1.3 we will use a standard technique based on the following lemma.

**Lemma 5.1.** Let \( 1 < p \leq \infty, \ell \in \mathbb{N} \) and \( \alpha > 0 \). Let \( \{g_n\}_{n \in \mathbb{N}} \) be a sequence of functions in \( E \). Then, the following conditions are equivalent:

(i) \( g_n \to 0 \) weakly in \( F^p_\alpha,\ell \), if \( p < \infty \), and in \( F^\infty_\alpha,\ell \), if \( p = \infty \).

(ii) \( g_n \to 0 \) uniformly on compact subsets of \( \mathbb{C} \) and \( \sup_{n \in \mathbb{N}} \|g_n\|_{F^p_\alpha,\ell} < \infty \).

**Proof.** Assume that (i) holds. Then it is well known that \( \sup_{n \in \mathbb{N}} \|g_n\|_{F^p_\alpha,\ell} < \infty \), so \( \{g_n\} \) is uniformly bounded on compact subsets of \( \mathbb{C} \). Moreover, since \( g_n \to 0 \) weakly in \( F^p_\alpha,\ell \), then, for each \( z \in \mathbb{C} \),

\[
g_n(z) = \langle g_n, K^\ell_{\alpha,z} \rangle_\alpha \to 0, \quad \text{as } n \to \infty.
\]

Consequently, \( g_n \to 0 \) uniformly on compact subsets of \( \mathbb{C} \), by Montel’s theorem.

Reciprocally, assume that (ii) holds. By Proposition 2.5(ii)-(iii), we have to show that \( \langle f, g_n \rangle_\alpha \to 0 \), as \( n \to \infty \), for every \( f \in F^p_\alpha,\ell' \).

Let \( f \in F^p_\alpha,\ell' \). Then, for every \( R > 0 \), we have

\[
\langle f, g_n \rangle_\alpha = \left\{ \int_{|w| \leq R} + \int_{|w| > R} \right\} f(w)g_n(w)e^{-|w|^2/p} d\nu(w) = I_n(R) + J_n(R).
\]

Since \( p' < \infty \), we have that \( \int_{|w| > R} |f(w)e^{-|w|^2/2}|^{p'} d\nu(w) \to 0 \), as \( R \to \infty \), so

\[
\lim_{R \to \infty} \sup_{n \in \mathbb{N}} J_n(R) = 0,
\]

by Hölder’s inequality and the fact that \( \sup_{n \in \mathbb{N}} \|g_n\|_{F^p_\alpha,\ell} < \infty \). Moreover, since \( g_n \to 0 \) uniformly on compact subsets of \( \mathbb{C} \) then \( I_n(R) \to 0 \), as \( n \to \infty \), for every \( R > 0 \). It turns out that \( \langle f, g_n \rangle_\alpha \to 0 \), as \( n \to \infty \), and the proof is complete. \( \square \)

**Proof of Theorem 1.3.** By Proposition 2.10 we only have to prove Theorem 1.3 for \( \alpha = 1 \).

First we prove that, if either \( b_{b_{1,1}} : F^p_1 \to F^{p'}_1 \), \( 1 < p < \infty \), or \( b_{b_{1,1}} : f^\infty_1 \to f^\infty_1 \) is compact, then \( b \in f^\infty_1/2^\ell \).

Suppose that \( b_{b_{1,1}} : F^p_1 \to L^p_1 \) is compact and we want to prove that \( b \in f^\infty_1/2^\ell \).

Let \( G_0, G_1 \) be the functions defined by (3.16).
Since \( h_{b,1}^\ell : F_1^{p,\ell} \rightarrow L_1^{p,\ell} \) is bounded, the proof of the necessity in Theorem 1.1 (see §3.2) implies that (3.17) holds, and so

\[
|b(z)| \lesssim \|b_{b,1}^\ell(G_0(\cdot, z))\|_{L_1^{p,\ell}} \|G_1(\cdot, z)\|_{F_1^{p,\ell}}
\]

\[
= \|b_{b,1}^\ell(g_0(\cdot, z))\|_{L_1^{p,\ell}} \|G_0(\cdot, z)\|_{F_1^{p,\ell}} \|G_1(\cdot, z)\|_{F_1^{p,\ell}}
\]

where \( g_0(w, z) = G_0(w, z) / \|G_0(\cdot, z)\|_{F_1^{p,\ell}}. \) Then (3.19) and (3.20) show that

\[
|b(z)| e^{-|z|^2/4} \lesssim \|b_{b,1}^\ell(g_0(\cdot, z))\|_{L_1^{p,\ell}}.
\]

It is easy to check that \( g_0(\cdot, z) \rightarrow 0 \) uniformly on compact subsets of \( \mathbb{C} \), as \( |z| \rightarrow \infty \). By Lemma 5.1 it follows that \( g_0(\cdot, z) \rightarrow 0 \) weakly in \( F_1^{p,\ell} \), as \( |z| \rightarrow \infty \). Note that, if \( p = \infty \), the same arguments show that \( g_0(\cdot, z) \rightarrow 0 \) weakly in \( f_1^{\infty,\ell} \).

Then the compactness of \( b_{b,1}^\ell : F_1^{p,\ell} \rightarrow L_1^{p,\ell} \), \( 1 < p < \infty \), (\( b_{b,1}^\ell : f_1^{\infty,\ell} \rightarrow L_1^{\infty,\ell} \), respectively) shows that

\[
\lim_{|z| \rightarrow \infty} \|b_{b,1}^\ell(g_0(\cdot, z))\|_{L_1^{p,\ell}} = 0.
\]

and so (5.22) gives that \( |b(z)| e^{-|z|^2/4} \rightarrow 0 \), as \( |z| \rightarrow \infty \).

Now we consider the case \( p = 1 \). By Corollary 2.9 the operator \( b_{b,1}^\ell : F_1^{1,\ell} \rightarrow F_1^{1,\ell} \) is the adjoint of \( b_{b,1}^\ell : f_1^{\infty,\ell} \rightarrow f_1^{\infty,\ell}. \) Thus the compactness of the first operator implies the compactness of the second operator and, as we have just shown, this implies that \( b \in f_1^{\infty,\ell}. \)

Now assume that \( b \in f_1^{\infty,\ell} \) and \( 1 \leq p < \infty \). Then, by Theorem 1.1 \( b_{b,1}^\ell \) is a bounded operator from \( F_1^{p,\ell} \) to \( F_1^{p,\ell}. \) Moreover, since \( f_1^{\infty,\ell} \) is the closure of the polynomials in \( F_1^{\infty,\ell} \), there is a sequence of polynomials \( \{P_n\}_{n \in \mathbb{N}} \) such that \( \|P_n - b\|_{F_1^{\infty,\ell}} \rightarrow 0 \). Therefore \( \|b_{b,1}^\ell - b_{P_n,1}^\ell\|_{F_1^{p,\ell}} \rightarrow 0, \) because

\[
\|b_{P_n,1}^\ell - b_{b,1}^\ell\|_{F_1^{p,\ell}} = \|b_{P_n,1}^\ell - b_{P_n,1}^\ell\|_{F_1^{p,\ell}} \lesssim \|P_n - b\|_{F_1^{\infty,\ell}},
\]

by Theorem 1.1 again. Since \( \{b_{P_n,1}^\ell\}_{n \geq 0} \) is a sequence of finite rank operators, it follows that \( b_{b,1}^\ell : F_1^{p,\ell} \rightarrow F_1^{1,\ell} \) is compact.

Note that the above argument also works by replacing the space \( F_1^{p,\ell} \) by \( f_1^{\infty,\ell} \), and hence the proof of Theorem 1.3 is complete.

\[
\square
\]

6. Proof of Theorem 1.4

6.1. The small Hankel operator on \( F_\alpha^{2,\ell}. \)
By Proposition 2.10 it is enough to prove the result for $\alpha = 1$, that is, to prove

\begin{equation}
\|h_{b,1}\|_{S_2^2(F_1^{2,\ell})}^2 \simeq \|b\|_{F_1^{2,\ell}}^2.
\end{equation}

In order to do that, first we estimate $\|h_{b,1}\|_{S_2^2(F_1^{2,\ell})}$ and $\|b\|_{F_1^{2,\ell}}^2$ in terms of the Taylor coefficients of $b$.

**Lemma 6.1.** Let $\ell \in \mathbb{N}$ and let $b(z) = \sum_{m=0}^{\infty} c_m z^m$ be a function in $H_1^\infty,\ell$. Then

\begin{equation}
\|h_{b,1}\|_{S_2^2(F_1^{2,\ell})}^2 \simeq \sum_{m=0}^{\infty} |c_m|^2 \Gamma\left(\frac{m+1}{\ell}\right)^2 \sum_{k=0}^{m} \frac{1}{\Gamma\left(\frac{k+1}{\ell}\right) \Gamma\left(\frac{m-k+1}{\ell}\right)},
\end{equation}

and

\begin{equation}
\|b\|_{F_1^{2,\ell}}^2 \simeq \sum_{m=0}^{\infty} |c_m|^2 \Gamma\left(\frac{m}{\ell} + 1\right).
\end{equation}

**Proof.** We begin proving (6.24). Let $e_n(z) = z^n/\|z^n\|_{F_1^{2,\ell}}$, $n = 0, 1, \ldots$. It is easy to check that

\[ h_{b,1}(e_n)(z) = \sum_{m=0}^{\infty} c_{n+m} \frac{\|w^{m+n}\|_{F_1^{2,\ell}}^2}{\|w^m\|_{F_1^{2,\ell}} \|w^n\|_{F_1^{2,\ell}}} e_m(z). \]

Thus

\[ h_{b,1}(e_n)(z) = \sum_{m=0}^{\infty} c_{n+m} \frac{\|w^{m+n}\|_{F_1^{2,\ell}}^2}{\|w^m\|_{F_1^{2,\ell}} \|w^n\|_{F_1^{2,\ell}}} e_m(z). \]

Therefore,

\[ \|h_{b,1}\|_{S_2^2(F_1^{2,\ell})}^2 = \sum_{n=0}^{\infty} \|h_{b,1}(e_n)\|_{F_1^{2,\ell}}^2 = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} |c_{n+m}|^2 \frac{\|w^{m+n}\|_{F_1^{2,\ell}}^4}{\|w^m\|_{F_1^{2,\ell}}^2 \|w^n\|_{F_1^{2,\ell}}^2} \]

\[ = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} |c_m|^2 \frac{\|w^m\|_{F_1^{2,\ell}}^4}{\|w^m\|_{F_1^{2,\ell}}^2 \|w^n\|_{F_1^{2,\ell}}^2} \]

\[ = \sum_{m=0}^{\infty} |c_m|^2 \Gamma\left(\frac{m+1}{\ell}\right)^2 \sum_{k=0}^{m} \frac{1}{\Gamma\left(\frac{k+1}{\ell}\right) \Gamma\left(\frac{m-k+1}{\ell}\right)}. \]
Next we prove (6.25):

\[ \| b \|_{F,2}^2 = \sum_{m=0}^{\infty} |c_m|^2 \int_{\mathbb{C}} |z|^{2m} e^{-|z|^2/2} (1 + |z|^2 - 2) \, d\nu(z) \]

\[ \simeq \sum_{m=0}^{\infty} |c_m|^2 \int_{0}^{\infty} r^{2m+1} (1 + r^2) e^{-r^2/2} \, dr \]

\[ \simeq \sum_{m=0}^{\infty} |c_m|^2 2^{m/\ell} \left\{ \Gamma \left( \frac{m+1}{\ell} \right) + \Gamma \left( \frac{m}{\ell} + 1 \right) \right\} \]

\[ \simeq \sum_{m=0}^{\infty} |c_m|^2 2^{m/\ell} \Gamma \left( \frac{m}{\ell} + 1 \right). \]

\[ \square \]

From Lemma 6.1 it is clear that (6.23) is equivalent to

\[ \Gamma \left( \frac{m+1}{\ell} \right) \sum_{k=0}^{m} \frac{1}{\Gamma \left( \frac{k+1}{\ell} \right) \Gamma \left( \frac{m-k+1}{\ell} \right)} \simeq 2^{m/\ell} \Gamma \left( \frac{m}{\ell} + 1 \right) \quad (m \geq 0), \]

which can be written as

\[ (6.26) \quad \sum_{k=0}^{m} \frac{\Gamma \left( \frac{m+2-h}{\ell} \right)}{\Gamma \left( \frac{k+1}{\ell} \right) \Gamma \left( \frac{m-k+1}{\ell} \right)} \simeq 2^{m/\ell} \frac{\Gamma \left( \frac{m+2-h}{\ell} \right) \Gamma \left( \frac{m}{\ell} + 1 \right)}{\Gamma \left( \frac{m}{\ell} + 1 \right)^2} \quad (m \geq 8\ell). \]

Now, by Stirling’s formula,

\[ \frac{\Gamma \left( \frac{m+2-h}{\ell} \right) \Gamma \left( \frac{m}{\ell} + 1 \right)}{\Gamma \left( \frac{m}{\ell} + 1 \right)^2} \simeq 1 \quad (m \geq 8\ell). \]

Hence (6.26) follows from the following lemma.

**Lemma 6.2.**

\[ \sum_{k=0}^{m} \frac{\Gamma \left( \frac{m+2-h}{\ell} \right)}{\Gamma \left( \frac{k+1}{\ell} \right) \Gamma \left( \frac{m-k+1}{\ell} \right)} \simeq 2^{m/\ell} \quad (m \geq 8\ell). \]

The key ingredient to prove Lemma 6.2 is the following important inequality.

**Chernoff’s inequality** ([17, (1.3.10) p.16]).

\[ \sum_{0 \leq i \leq n/4} \binom{n}{i} \leq 2^n e^{-n/8}, \quad \text{for every } n \geq 0. \]
Proof of Lemma 6.2. Let $m = n\ell + r$, where $n \geq 8$ and $0 \leq r < \ell$. Then we may decompose the sum $S(m)$ of the statement as

$$S(m) = \sum_{j=0}^{n-1} \sum_{s=0}^{\ell-1} \frac{\Gamma\left(\frac{m+2-j}{\ell}\right)}{\Gamma\left(\frac{j+s+1}{\ell}\right)\Gamma\left(\frac{(j+s)+1}{\ell}\right)} + \sum_{s=0}^{r} \frac{\Gamma\left(\frac{m+2-r}{\ell}\right)}{\Gamma\left(\frac{n\ell+s+1}{\ell}\right)\Gamma\left(\frac{m-(n\ell+s)+1}{\ell}\right)}$$

$$= \sum_{s=0}^{\ell-1} \sum_{j=0}^{n-1} \frac{\Gamma\left(n-1 + \frac{r+2}{\ell}\right)}{\Gamma\left(j + \frac{s+1}{\ell}\right)\Gamma\left(n-j + \frac{r-s+1}{\ell}\right)} + \sum_{s=0}^{r} \frac{\Gamma\left(n-1 + \frac{r+2}{\ell}\right)}{\Gamma\left(n + \frac{s+1}{\ell}\right)\Gamma\left(\frac{r-s+1}{\ell}\right)}$$

$$+ \sum_{s=0}^{\ell-1} \left( \sum_{1 \leq j \leq \frac{n}{4}} + \sum_{\frac{n}{4} < j < \frac{n}{2}} + \sum_{\frac{n}{2} \leq j \leq n-1} \right) \frac{\Gamma\left(n-1 + \frac{r+2}{\ell}\right)}{\Gamma\left(j + \frac{s+1}{\ell}\right)\Gamma\left(n-j + \frac{r-s+1}{\ell}\right)}$$

$$= S_1(m) + S_2(m) + S_3(m) + S_4(m) + S_5(m).$$

In order to estimate the above five sums we recall that $\Gamma$ is an increasing function on $[2, \infty)$. Then, since $\frac{r+2}{\ell} \leq 2$, we have that

$$\Gamma\left(n-1 + \frac{r+2}{\ell}\right) \leq \Gamma(n+1). \quad (6.27)$$

On the other hand, since

$$\frac{2}{\ell} - 1 \leq \frac{r-s+1}{\ell} \leq 1 \quad \text{and} \quad \frac{1}{\ell} \leq \frac{s+1}{\ell} \leq 1 \quad (0 \leq s < \ell),$$

we also have that

$$\Gamma(n-j-1) \leq \Gamma(n-j + \frac{r-s+1}{\ell}) \quad (0 \leq s < \ell, 0 \leq j \leq n-3), \quad (6.28)$$

and

$$\Gamma(j) \leq \Gamma\left(j + \frac{s+1}{\ell}\right) \quad (2 \leq j, 0 \leq s < \ell). \quad (6.29)$$

Now (6.27) and (6.28) imply that

$$S_1(m) \lesssim \frac{\Gamma(n+1)}{\Gamma(n-1)} = n(n-1) \lesssim 2^n \sim 2^{m/\ell}, \quad (6.30)$$

and, in particular,

$$S_2(m) = \sum_{s=0}^{r} \frac{\Gamma\left(n-1 + \frac{r+2}{\ell}\right)}{\Gamma\left(n + \frac{r-s+1}{\ell}\right)\Gamma\left(\frac{s+1}{\ell}\right)} \leq S_1(m) \lesssim 2^{m/\ell}. \quad (6.31)$$
Moreover, by (6.27), (6.28) and (6.29) we have that
\[ S_3(m) + S_5(m) \lesssim \sum_{1 \leq j \leq n} \frac{\Gamma(n+1)}{\Gamma(j)\Gamma(n-j+1)} + \sum_{n \leq j < n-1} \frac{\Gamma(n+1)}{\Gamma(j)\Gamma(n-j+1)} \]
\[ = \sum_{1 \leq j \leq \frac{n}{2}} \frac{\Gamma(n+1)}{\Gamma(j)\Gamma(n-j+1)} \left( 1 + \frac{n-j}{j} \right) \]
\[ = n \sum_{1 \leq j \leq \frac{n}{2}} \frac{\Gamma(n+1)}{\Gamma(j+1)\Gamma(n-j+1)} \leq n \sum_{0 \leq j \leq \frac{n}{2}} \binom{n}{j}. \]

So Chernoff’s inequality gives
\[(6.32) \quad S_3(m) + S_5(m) \lesssim n e^{-n/8} 2^n \lesssim 2^n \simeq 2^{m/\ell}.\]

To estimate \( S_4(m) \) we apply Lemma 3.4 and we obtain that
\[ S_4(m) \simeq \sum_{s=0}^{\ell-1} \sum_{\frac{n-s}{2} < j < \frac{n-s}{2}} \frac{(n-1)^{s+2}}{j^{s+1}(n-j)^{s+1}} \frac{\Gamma(n-1)}{\Gamma(j)\Gamma(n-j)} \]
\[ \simeq \sum_{\frac{n-s}{2} < j < \frac{n-s}{2}} \frac{\Gamma(n-1)}{\Gamma(j)\Gamma(n-j)} \simeq \sum_{\frac{n-s}{2} < j < \frac{n-s}{2}} \binom{n}{j} = 2^n - 2 \sum_{0 \leq j \leq \frac{n}{2}} \binom{n}{j}. \]

Therefore Chernoff’s inequality shows that
\[(6.33) \quad S_4(m) \simeq 2^n \simeq 2^{m/\ell}.\]

By (6.30), (6.31), (6.32) and (6.33) we conclude that \( S(m) \simeq 2^{m/\ell} \), and the proof is complete. \( \square \)

6.2. The small Hankel operator on \( L^{2,\ell}_{\alpha} \).

In this section we characterize the membership of \( h_{\varphi,\alpha}^{\ell} \) to the Hilbert-Schmidt class \( S_2(L^{2,\ell}_{\alpha}) \) of \( L^{2,\ell}_{\alpha} \).

Let \( L^{2}_{\Delta} := L^2(\mathbb{C}, (1+|z|)^{2(\ell-1)} \, dv) \). Then we have:

**Theorem 6.3.** \( h_{\varphi,\alpha}^{\ell} \in S_2(L^{2,\ell}_{\alpha}) \) if and only if \( \varphi \in L^{2}_{\Delta} \). Moreover,
\[ \| h_{\varphi,\alpha}^{\ell} \|_{S_2(L^{2,\ell}_{\alpha})} \simeq \| \varphi \|_{L^{2}_{\Delta}}. \]

In particular, if \( \varphi \in L^{2}_{\Delta} \), then \( h_{\varphi,\alpha}^{\ell} \in S_2(F^{2,\ell}_{\alpha}) \) and \( \| h_{\varphi,\alpha}^{\ell} \|_{S_2(F^{2,\ell}_{\alpha})} \lesssim \| \varphi \|_{L^{2}_{\Delta}}. \)

**Proof.** Note that
\[ h_{\varphi,\alpha}^{\ell}(f)(z) := P_{\alpha}^{\ell}(f \varphi)(z) = \int_{\mathbb{C}} K_{\alpha}^{\ell}(w,z) f(w) \overline{\varphi}(w) e^{-\alpha|w|^2} \, dv(w) \]
is an integral operator with respect to the positive measure $e^{-\alpha|w|^{2\ell}}\,d\nu(w)$ and whose integral kernel is $K^\ell_\alpha(w, z)\varphi(w)$. So it is well known (see [14] Theorem 3.5, for example) that

$$\|\mathbf{b}^\ell_{\varphi, \alpha}\|^2_{\mathcal{S}_2(L^{2, \ell}_0)} = \int_{\mathbb{C}} \left( \int_{\mathbb{C}} |K^\ell_\alpha(w, z)|^2|\varphi(w)|^2 \,e^{-\alpha|w|^{2\ell}}\,d\nu(w) \right) \,e^{-\alpha|z|^{2\ell}}\,d\nu(z)$$

$$= \int_{\mathbb{C}} |\varphi(w)|^2 \left( \int_{\mathbb{C}} |K^\ell_\alpha(w, z)|^2 \,e^{-\alpha|z|^{2\ell}}\,d\nu(z) \right) \,e^{-\alpha|w|^{2\ell}}\,d\nu(w)$$

$$= \int_{\mathbb{C}} |\varphi(w)|^2 K^\ell_\alpha(w, w) \,e^{-\alpha|w|^{2\ell}}\,d\nu(w)$$

$$\simeq \int_{\mathbb{C}} |\varphi(w)|^2 (1 + |w|)^{2(\ell-1)}\,d\nu(w),$$

where the last equivalence follows from $K^\ell_\alpha(w, w) = H^\ell_\alpha(|w|^2) \simeq (1 + |w|)^{\ell-1}e^{\alpha|w|^{2\ell}}$ (see (2.7) and (2.8)). And that’s all. \hfill \Box

Finally we show that the space of Hilbert-Schmidt symbols for $F^{2, \ell}_\alpha$ is just the projection of the space of Hilbert-Schmidt symbols for $L^{2, \ell}_0$.

**Proposition 6.4.** The projection $P^\ell_\alpha$ is bounded from $L^{2, \ell}_0$ onto $F^{2, \ell}_{\alpha/2, \Delta}$.

**Proof.** Let $\{e_m\}_{m \in \mathbb{N}}$ be an orthonormal basis of $F^{2, \ell}_\alpha$ and let $\{u_m\}_{m \in \mathbb{N}}$ be an orthonormal basis of the orthogonal of $F^{2, \ell}_\alpha$ in $L^{2, \ell}_0$.

By Theorems 3.3 and 1.4, we have, for any $\varphi \in L^{2, \ell}_0$,

$$\|\varphi\|^2_{L^{2, \ell}_0} \simeq \|\varphi\|^2_{\mathcal{S}_2(L^{2, \ell}_0)} = \sum_{m=1}^{\infty} \|\mathbf{b}^\ell_{\varphi, \alpha}(e_m)\|^2_{L^{2, \ell}_0} + \sum_{m=1}^{\infty} \|\mathbf{b}^\ell_{\varphi, \alpha}(u_m)\|^2_{L^{2, \ell}_0}$$

$$\geq \sum_{m=1}^{\infty} \|P^\ell_\alpha(e_m)\|^2_{F^{2, \ell}_{\alpha/2, \Delta}} \simeq \|P^\ell_\alpha(\varphi)\|^2_{F^{2, \ell}_{\alpha/2, \Delta}}.$$}

So we have just proved that $P^\ell_\alpha : L^{2, \ell}_0 \to F^{2, \ell}_{\alpha/2, \Delta}$ is bounded.

Let $b \in F^{2, \ell}_{\alpha/2, \Delta}$. Since $F^{2, \ell}_{\alpha/2, \Delta} \subset F^{2, \ell}_\alpha$, we have $b = P^\ell_\alpha(b)$, by (2.17). By (2.5), $K^\ell_{\alpha/2}(z, w) = 2^{-1/\ell}K^\ell_\alpha(z, 2^{-1/\ell}w)$, so

$$b(z) = 2^{-1/\ell} \int_{\mathbb{C}} K^\ell_\alpha(z, 2^{-1/\ell}w)b(w)\,e^{-\frac{2\alpha}{\ell}|w|^{2\ell}}\,d\nu(w)$$

$$= 2^{1/\ell} \int_{\mathbb{C}} K^\ell_{\alpha/2}(z, w)b(2^{1/\ell}w)\,e^{-2\alpha|w|^{2\ell}}\,d\nu(w) = P^\ell_\alpha(\varphi)(z),$$

where $\varphi(u) = 2^{1/\ell}b(2^{1/\ell}u)e^{-\frac{2\alpha}{\ell}|2^{1/\ell}u|^{2\ell}}$, which clearly belongs to $L^{2, \ell}_0$. \hfill \Box
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