GEOMETRY OF SUB-ALGEBRAS OF $\mathcal{H}ol(\Gamma \cup \text{Int}(\Gamma))$ AND ZEROS OF HOLOMORPHIC FUNCTIONS

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ABSTRACT. We study $\mathcal{H}ol(\Gamma \cup \text{Int}(\Gamma))$, the normed algebra of all holomorphic functions defined on some simply connected neighborhood of a simple closed curve $\Gamma$ in $\mathbb{C}$, equipped with the supremum norm on $\Gamma$. We explore the geometry of nowhere vanishing, point separating sub-algebras of $\mathcal{H}ol(\Gamma \cup \text{Int}(\Gamma))$. We characterize the extreme points and the exposed points of the unit balls of the said sub-algebras for $\Gamma$ analytic. We also characterize the smoothness of an element in these sub-algebras by using Birkhoff-James orthogonality techniques without any restriction on $\Gamma$. As a culmination of our study, we assimilate the geometry of the aforesaid sub-algebras with some classical concepts of complex analysis and establish a connection between Birkhoff-James orthogonality and zeros of holomorphic functions.

1. Introduction

The aim of the present article is to study the geometry of a family of normed algebras and to establish a connection between Birkhoff-James orthogonality and zeros of holomorphic functions. The mentioned normed algebras consist of holomorphic functions defined on any simply connected neighborhood of a simple closed curve in $\mathbb{C}$. We shall let $\mathcal{H}ol(K)$ denote the normed algebra of all continuous functions on a compact set $K \subset \mathbb{C}$ that can be extended to a holomorphic function on some neighborhood of $K$, equipped with the supremum norm on $K$. In the same spirit, we consider the normed algebra $\mathcal{H}ol(\Gamma \cup \text{Int}(\Gamma))$ for a simple closed curve $\Gamma \subset \mathbb{C}$, where Int($\Gamma$) denotes the simply connected...
region enclosed by $\Gamma$, i.e.,
\[ \mathcal{H}ol(\Gamma \cup \text{Int}(\Gamma)) := \{ f : \Gamma \cup \text{Int}(\Gamma) \to \mathbb{C} : \exists U_f \supset \Gamma \cup \text{Int}(\Gamma) \text{ open, } f \text{ holomorphic on } U_f \}, \]
with the norm defined as:
\[ \|f\| := \sup_{z \in \Gamma} |f(z)|, \quad f \in \mathcal{H}ol(\Gamma \cup \text{Int}(\Gamma)). \]
Clearly, if $f \in \mathcal{H}ol(\Gamma \cup \text{Int}(\Gamma))$, then $f^{(n)} \in \mathcal{H}ol(\Gamma \cup \text{Int}(\Gamma))$ for every $n \in \mathbb{N}$, where $f^{(n)}$ denotes the $n$-th order derivative of $f$.

A sub-algebra $\mathfrak{A}$ of $\mathcal{H}ol(\Gamma \cup \text{Int}(\Gamma))$ is said to be nowhere vanishing and point separating at $\Gamma$ if for every $z \in \Gamma$, there exists $f \in \mathfrak{A}$ such that $f(z) \neq 0$, and for every $z_1, z_2 \in \Gamma$, there exists $g \in \mathfrak{A}$ such that $g(z_1) \neq g(z_2)$. Throughout this article, we call these sub-algebras as nowhere vanishing point separating sub-algebras of $\mathcal{H}ol(\Gamma \cup \text{Int}(\Gamma))$, without any ambiguity.

For a given choice of $\Gamma \subset \mathbb{C}$, define:
\[ J(\Gamma) := \{ f \in \mathcal{H}ol(\Gamma \cup \text{Int}(\Gamma)) : |f(z)| = \|f\| \forall z \in \Gamma \}, \tag{1.1} \]
and for a given $f \in \mathcal{H}ol(\Gamma \cup \text{Int}(\Gamma))$, set:
\[ M_f := \{ z \in \Gamma : |f(z)| = \|f\| \}, \tag{1.2} \]
\[ Z_f := \{ z \in \Gamma \cup \text{Int}(\Gamma) : f(z) = 0 \}. \tag{1.3} \]
For example, if $\Gamma = \{ z \in \mathbb{C} : |z| = r \}$ for some $r > 0$, then $f \in J(\Gamma)$ if and only if $f$ is constant or
\[ f(z) = c \prod_{k=1}^{n} \left( \frac{z - ra_k}{r - \alpha_k} \right), \quad z \in \Gamma \cup \text{Int}(\Gamma), \]
for some $c \in \mathbb{C}\{0\}$, $n \in \mathbb{N}$ and $a_k \in \mathbb{D}$ for $1 \leq k \leq n$, where $\mathbb{D}$ denotes the open unit disc in $\mathbb{C}$.

For a given $z \in \mathbb{C}$, the real and the imaginary parts of $z$ are denoted by $\Re(z)$ and $\Im(z)$, respectively. If additionally, $z \neq 0$, let $\arg(z)$ denote the real number $\theta \in [0, 2\pi)$ such that $z = |z|e^{i\theta}$. Also, the sign function $\text{sgn} : \mathbb{C} \to \mathbb{C}$ is defined by
\[ \text{sgn}(z) := \begin{cases} \frac{z}{|z|}, & z \neq 0, \\ 0, & z = 0. \end{cases} \]

Given a normed linear space $X$, let $B_X$ and $S_X$ denote the closed unit ball and the unit sphere of $X$ respectively, i.e.,
\[ B_X = \{ x \in X : \|x\| \leq 1 \}, \quad S_X = \{ x \in X : \|x\| = 1 \}. \]
Let $X^*$ denote the continuous dual of $X$. For a non-zero element $x \in X$, we denote the collection of all support functionals at $x$ by $J(x)$, i.e.,
\[ J(x) = \{ \Psi \in S_{X^*} : \Psi(x) = \|x\| \}, \]
Let $Ext(X)$ denote the collection of all extreme points of $B_X$. An element $x \in B_X$ is said to be an exposed point of $B_X$ if $\|x\| = 1$ and there exists $\Psi \in J(x)$ such that
\[ |\Psi(y)| = \|y\| \text{ if and only if } y = cx \text{ for some } c \in \mathbb{C}. \]

We denote the collection of all exposed points of $B_X$ by $Exp(X)$. It is well-known that $Exp(X) \subseteq Ext(X)$. For $x, y \in X$, $x$ is said to be Birkhoff-James orthogonal to $y$, denoted by $x \perp_B y$, if
\[ \|x + \lambda y\| \geq \|x\| \text{ for all } \lambda \in \mathbb{C}. \]

In [8], James proved that for $x, y \in X$, $x \perp_B y$ if and only if $x = 0$ or there exists some $\Psi \in J(x)$ such that $\Psi(y) = 0$. A non-zero element $x \in X$ is said to be smooth if $J(x)$ is singleton. James [8] proved that a non-zero $x \in X$ is smooth if and only if $x \perp_B y, x \perp_B z \implies x \perp_B (y + z)$ for all $y, z \in X$.

In the first section of this article, we characterize the extreme points and the exposed points of any nowhere vanishing, point separating sub-algebra of $\text{Hol}(\Gamma \cup \text{Int}(\Gamma))$ for $\Gamma$ analytic. In the next section, we characterize Birkhoff-James orthogonality in the said sub-algebras and also identify its smooth points but for any simple closed curve $\Gamma$, not necessarily analytic. Characterizing the extreme points, the exposed points and the smooth points of the closed unit ball of a given normed linear space is of fundamental importance in determining the geometry of the space. We refer the readers to [1], [3], [4], [5], [7], [9], [10], [11], [12], [13], [14], [15] for some of the illustrative works in this regard.

In the final section, we find an interrelation between Birkhoff-James orthogonality in nowhere vanishing, point separating sub-algebras of $\text{Hol}(\Gamma \cup \text{Int}(\Gamma))$ with the zeros of holomorphic functions using some classical concepts of complex analysis.

2. Geometry of $\text{Hol}(\Gamma \cup \text{Int}(\Gamma))$ for $\Gamma$ simple, closed and analytic

We begin with a couple of simple propositions. In the first one, we characterize the set $J(\Gamma)$ for $\Gamma \subset \mathbb{C}$ analytic.

**Proposition 2.1.** Let $\Gamma \subset \mathbb{C}$ be a simple closed analytic curve and let $f : \text{Int}(\Gamma) \to \mathbb{D}$ be an onto biholomorphic map. Then
(i) $f$ extends to a biholomorphic function $F : U \to \mathbb{C}$ for some neighborhood $U$ of $\Gamma \cup \text{Int}(\Gamma)$ such that $F(\Gamma \cup \text{Int}(\Gamma)) = \mathbb{D}$.
(ii) $h \in J(\Gamma)$ if and only if $h$ is constant or
\[ h(z) = c \prod_{k=1}^{n} \left( \frac{F(z) - a_k}{1 - \overline{a_k} F(z)} \right), \quad z \in \Gamma \cup \text{Int}(\Gamma), \]
for some $c \in \mathbb{C} \setminus \{0\}$, $n \in \mathbb{N}$ and $a_k \in \mathbb{D}$ for $1 \leq k \leq n$. 

Proof. The first part of the proposition follows from the observation that for every \( z \in \Gamma \), there exists \( U_z \subset \mathbb{C} \) open such that \( \Gamma \cap U_z \) is a one-sided free arc of \( \Gamma \) containing \( z \) and hence \( f \) can be extended biholomorphically to some neighborhood \( V_z \) of \( \Gamma \cap U_z \), using the Schwarz reflection principle (see [2, Theorem 4, p. 235] for details). The desired \( F \) is now obtained by considering the extensions on \( V_z \) for every \( z \in \Gamma \) and restricting the function on a sufficiently small neighborhood of \( \Gamma \cup \text{Int}(\Gamma) \) so that it remains injective.

The second part follows from the observation that \( T : \mathcal{H}ol(\overline{D}) \to \mathcal{H}ol(\Gamma \cup \text{Int}(\Gamma)) \) given by
\[
T(g)(z) := g \circ F(z), \quad z \in F^{-1}(U_g),
\]
where \( \overline{D} \subset U_g \) open and \( g : U_g \to \mathbb{C} \) holomorphic is an isometric isomorphism and \( T(g) \in \mathcal{J}(\Gamma) \) if and only if \( |g(z)| = \|g\| \) for every \( |z| = 1 \). \( \square \)

The second proposition pertains to the completion of \( \mathcal{H}ol(\Gamma \cup \text{Int}(\Gamma)) \) for arbitrary simple closed curves \( \Gamma \subset \mathbb{C} \) (not necessarily analytic). The result follows easily from the observation that \( \mathcal{H}ol(\Gamma \cup \text{Int}(\Gamma)) \) contains all the polynomials.

**Proposition 2.2.** If \( \Gamma \subset \mathbb{C} \) is a simple closed curve, then \( \mathcal{H}ol(\Gamma \cup \text{Int}(\Gamma)) \) is incomplete and its completion is isometrically isomorphic to
\[
\mathcal{A}(\text{Int}(\Gamma)) := \{ f \in \mathcal{H}ol(\text{Int}(\Gamma)) : f \text{ extends continuously on } \Gamma \cup \text{Int}(\Gamma) \},
\]
equipped with the supremum norm.

Now, we characterize the extreme points and the exposed points of the closed unit ball of any nowhere vanishing, point separating sub-algebra of \( \mathcal{H}ol(\Gamma \cup \text{Int}(\Gamma)) \) for \( \Gamma \) simple, closed and analytic. We begin with a preliminary lemma.

**Lemma 2.3.** Let \( \Gamma \subset \mathbb{C} \) be a simple closed analytic curve. Let \( X = \mathcal{H}ol(\Gamma \cup \text{Int}(\Gamma)) \) and let \( f \in S_X \). If \( z_0 \in \Gamma \) is an isolated point of \( M_f \) (see (1.2)) and \( f(z_0) = e^{i\theta_0} \) for some \( \theta_0 \in [0,2\pi) \), then
\[
\lim_{z \to z_0, z \in \Gamma} \frac{|z - z_0|}{|1 - f(z)|} = 0,
\]
for some natural number \( r \).

**Proof.** Replacing \( f \) by \( e^{-i\theta_0}f \), we may and do assume that \( f(z_0) = 1 \). Also, let \( f \) be holomorphic on the domain \( \Omega \supset \Gamma \cup \text{Int}(\Gamma) \). Choose a neighborhood \( N \) of \( z_0 \), contained in \( \Omega \) such that \( |f(z)| < 1 \) for every \( z \in (\Gamma \cap N) \setminus \{z_0\} \). Set \( g(z) := 1 - f(z) \) on \( N \). Then \( g \) is holomorphic on \( N \) and has a zero at \( z = z_0 \). Let \( m_0 \) be the multiplicity of the zero of \( g \) at \( z = z_0 \). Then there exists a
neighbourhood $N'$ of $z_0$ and a holomorphic function $h : N' \to \mathbb{C}$ with $h(z_0) \neq 0$ such that:

$$g(z) = (z - z_0)^{m_0}h(z), \, z \in N'.$$

Now, observe that

$$|z - z_0|^r \frac{|1 - f(z)|}{1 - |f(z)|} = |z - z_0|^r + m_0 \frac{|h(z)|}{1 - |g(z)|},$$

and

$$\frac{1}{1 - |1 - g(z)|} = \frac{1 + |1 - g(z)|}{1 - |1 - g(z)|^2} = \frac{1 + |1 - g(z)|}{2 \Re(g(z)) - |g(z)|^2}$$

(2.4)

Now, let $I \subset \mathbb{R}$ be some interval and $\gamma : I \to \mathbb{C}$ be an analytic parametrization of $\Gamma$ in some neighborhood of $z_0$. Since $g \circ \gamma$ is analytic on $I$, $\Re(g \circ \gamma)$ is a real-analytic function. Without loss of generality, we may and will assume that $0 \in I$ and $\gamma(0) = z_0$. Then $\Re(g \circ \gamma)$ has a zero at 0 and hence, there exist a natural number $k$, a neighbourhood $I' \subset I$ of 0 and a real-analytic function $\nu : I' \to \mathbb{R}$ with $\nu(0) \neq 0$ such that

$$g(\gamma(t)) = t^k \nu(t), \, t \in I'.$$

(2.5)

Also, by the analyticity of the function $t \mapsto \gamma(t) - z_0$, there exist a natural number $n$, a neighbourhood $I'' \subset I$ of 0 and an analytic map $\psi : I'' \to \mathbb{C}$ with $\psi(0) \neq 0$ such that

$$\gamma(t) - z_0 = t^n \psi(t), \, t \in I''.$$

(2.6)

Now, to show that (2.2) holds for some $r \in \mathbb{N}$, applying (2.3), (2.4),(2.5) and (2.6) we need to show:

$$\lim_{t \to 0} |\gamma(t) - z_0|^{r + m_0} \frac{h \circ \gamma(t) (1 + |1 - g \circ \gamma(t)|)}{2t^n \psi(t) - |\psi(t)|^{2m_0} |h \circ \gamma(t)|^2} = 0$$

$$\iff \lim_{t \to 0} |t^{n(r + m_0)} |\psi(t)|^{r + m_0} \frac{h \circ \gamma(t) (1 + |1 - g \circ \gamma(t)|)}{2t^n \psi(t) - |t^n \psi(t)|^{2m_0} |h \circ \gamma(t)|^2} = 0$$

Clearly, choosing $r \in \mathbb{N}$ sufficiently large yields the result. \hfill \Box

We now come to the first of our main results.

**Theorem 2.4.** Let $\Gamma \subset \mathbb{C}$ be a simple closed analytic curve and let $\mathfrak{A}$ be a nowhere vanishing, point separating sub-algebra of $\text{Hol}(\Gamma \cup \text{Int}(\Gamma))$. Then the following are equivalent:
(i) \( f \in \text{Exp}(\mathfrak{A}) \).
(ii) \( f \in \text{Ext}(\mathfrak{A}) \).
(iii) \( \|f\| = 1 \) and \( f \in J(\Gamma) \), i.e., \( |f(z)| = 1 \) for every \( z \in \Gamma \).

**Proof.** (i) \( \Rightarrow \) (ii)
Elementary as discussed before.
(ii) \( \Rightarrow \) (iii)
Suppose that \( \Gamma \cup \text{Int}(\Gamma) \subset \Omega \) is open and \( f : \Omega \to \mathbb{C} \) is holomorphic. If \( f \in \text{Ext}(\mathfrak{A}) \), clearly, \( f \in S_\mathfrak{A} \).

We first prove that \( f \in J(\Gamma) \) if and only if \( M_f \) is infinite. The only if part is clearly trivial and for the if part, let \( z_n \in M_f \) be an infinite sequence such that \( z_n \to z_0 \). Consider an interval \( I \subset \mathbb{R} \) such that \( 0 \in I \) and \( \gamma : I \to \mathbb{C} \) is an analytic parametrization of \( \Gamma \) in some neighbourhood of the point \( z_0 \) with \( \gamma(0) = z_0 \). Clearly, \( |f \circ \gamma(t)|^2 = (\Re(f \circ \gamma(t)))^2 + (\Im(f \circ \gamma(t)))^2 \) is real analytic on \( I \). Since \( |f \circ \gamma(t)|^2 = 1 \) for infinitely many \( t \in I \), by the identity theorem for real-analytic functions, \( |f \circ \gamma(t)| = 1 \) for every \( t \in I \). The result now follows using an elementary connectedness argument.

Now, assume for the sake of contradiction, \( f \in S_\mathfrak{A} \) and \( M_f \equiv \{ z_1, z_2, \ldots, z_n \} \) for some \( n \in \mathbb{N} \). Then by Lemma 2.3, there exist \( r_k \in \mathbb{N} \) such that

\[
\lim_{z \to z_k} |z - z_k|^{-r_k} \left| \frac{e^{i\theta_k} - f(z)}{1 - f(z)} \right| = 0,
\]

where \( f(z_k) = e^{i\theta_k} \) for every \( 1 \leq k \leq n \). Also, let \( m_k \) denote the multiplicity of the zero \( z_k \) of the holomorphic map \( e^{i\theta_k} - f(z) \), \( z \in \Omega \) for every \( 1 \leq k \leq n \).

Now, let \( w_1, w_2 \in \Gamma \) and \( c_1, c_2 \in \mathbb{C} \). Since \( \mathfrak{A} \) is nowhere vanishing and point separating, there exist \( h_1, h_2, h_3 \in \mathfrak{A} \) such that

\[
 h_1(w_1), h_2(w_2) \neq 0, \quad h_3(w_1) \neq h_3(w_2).
\]

Then

\[
 h := \frac{c_1(h_1 h_3 - h_3(w_2) h_1)}{h_1(w_1)(h_3(w_1) - h_3(w_2))} + \frac{c_2(h_2 h_3 - h_3(w_1) h_2)}{h_2(w_2)(h_3(w_2) - h_3(w_1))},
\]

is well-defined and belongs to \( \mathfrak{A} \) with \( h(w_1) = c_1 \) and \( h(w_2) = c_2 \).

Now, fix \( z_0 \in \Gamma \setminus M_f \). Then, there exist \( f_1, f_2, \ldots, f_n \in \mathfrak{A} \), such that \( f_k(z_0) = 0 \) and \( f_k(z_0) = 1 \) for every \( 1 \leq k \leq n \). Suppose \( \Gamma \cup \text{Int}(\Gamma) \subset U_k \) is open such that \( f_k : U_k \to \mathbb{C} \) is holomorphic and \( \mu_k \) is the multiplicity of the zero \( z_k \) of the holomorphic map \( f_k \) for every \( 1 \leq k \leq n \). Define \( P : \Omega \cap \left( \bigcap_{k=1}^n U_k \right) \to \mathbb{C} \) by

\[
P(z) := \prod_{k=1}^n (f_k(z))^{\left[ \frac{m_k + r_k}{\mu_k} \right]}, \quad z \in \Omega \cap \left( \bigcap_{k=1}^n U_k \right),
\]
Clearly, for some $G_k : \Omega \cap \left( \bigcap_{k=1}^n U_k \right) \to \mathbb{C}$ holomorphic. Note that $G_k(z_k)$ may be zero for some $1 \leq k \leq n$. Therefore, for any $1 \leq k \leq n$, by (2.7),

$$
\lim_{z \to z_k} \frac{|P(z)|}{1 - |f(z)|} = G_k(z_k) \lim_{z \to z_k} \frac{|z - z_k|^{m_k}}{|e^{i\theta_k} - f(z)|} \frac{|z - z_k|^r_k}{1 - |f(z)|} = 0.
$$

Fix $\epsilon > 0$. Then there exist $N_1, N_2, \ldots, N_n \in \mathbb{C}$ open such that $z_k \in N_k$ and

$$
\frac{|P(z_k)|}{1 - |f(z_k)|} < \epsilon, \quad z \in \Gamma \cap N_k \setminus \{z_k\}, \quad 1 \leq k \leq n.
$$

Since $\Gamma \setminus \left( \bigcup_{k=1}^n N_k \right)$ is compact and $z \mapsto \frac{|P(z)|}{1 - |f(z)|}$ is continuous on $\Gamma \setminus \left( \bigcup_{k=1}^n N_k \right)$,

$$
\frac{|P(z)|}{1 - |f(z)|} \leq M, \quad z \in \Gamma \setminus \left( \bigcup_{k=1}^n N_k \right),
$$

for some $M > 0$. Set $K = \max\{M, \epsilon\}$. Then

$$
|P(z)| \leq K(1 - |f(z)|), \quad z \in \Gamma.
$$

Hence,

$$
|f(z)| + \left| \frac{1}{K} P(z) \right| \leq 1, \quad z \in \Gamma.
$$

Clearly, $\frac{1}{K} P \in \mathfrak{A}$ and $P(z_0) = 1$. Hence $P \neq 0$ proving $f \notin \text{Ext}(\mathfrak{A})$.

$(iii) \Rightarrow (i)$

Let $f \in S_{\mathfrak{A}} \cap J(\Gamma)$. Fix $\{z_n : n \in \mathbb{N}\} \subset M_f = \Gamma$ such that $z_n \to z_0$ for some $z_0$. Consider $\Psi : \mathfrak{A} \to \mathbb{C}$ given by

$$
\Psi(h) := \sum_{n=1}^{\infty} \frac{\text{sgn}(f(z_n))}{2^n} h(z_n), \quad h \in \mathfrak{A}.
$$

Then clearly, $\Psi$ is unit norm functional on $\mathfrak{A}$ and $\Psi(f) = ||f||$. Also, $||\Psi(h)|| = ||h||$ for some $h \in \mathfrak{A}$ if and only if

$$
h(z_n) = e^{i\theta_0} ||h|| f(z_n),
$$

for every $n \in \mathbb{N}$ and some fixed $\theta_0 \in [0, 2\pi)$. Since $f$ and $h$ are holomorphic on some connected neighborhood of $\Gamma \cup \text{Int}(\Gamma)$, by identity theorem, $h = e^{i\theta_0} ||h|| f$ proving $f \in \text{Exp}(\mathfrak{A})$. \qed
Clearly, if $\Gamma$ is not analytic, there exist functions $f \in \text{Ext}(\text{Hol}(\Gamma \cup \text{Int}(\Gamma)))$ such that $M_f \neq \Gamma$. The following simple example illustrates this:

**Example 2.5.** Let $\Gamma := \{e^{it} : t \in [0, \pi] \cup [-1, 1]\}$. Then any monomial $h_n := z^n$, $z \in \mathbb{C}$ is not in $\mathcal{F}(\Gamma)$ but is an extreme point of the closed unit ball of $\text{Hol}(\Gamma \cup \text{Int}(\Gamma))$, since $M_{h_n} = \{e^{it} : t \in [0, \pi]\}$ is an infinite set and therefore if $h_n = \frac{1}{2}(f + g)$ for some $f, g \in S_{\text{Hol}(\Gamma \cup \text{Int}(\Gamma))}$, then $h_n = f = g$ on $M_{h_n}$ proving $f = h_n = g$ by the identity theorem.

Proposition 2.1 and Theorem 2.4 completely characterize the extreme points of the closed unit ball of $\text{Hol}(\Gamma \cup \text{Int}(\Gamma))$ for simple closed analytic curves $\Gamma \subset \mathbb{C}$ as $B \circ F$ where $B$ is a finite Blaschke product or a unimodular constant function on $\mathbb{D}$ and $F$ is a holomorphic extension of the Riemann map $f : \text{Int}(\Gamma) \rightarrow \mathbb{D}$ on some neighborhood of $\Gamma \cup \text{Int}(\Gamma)$. Let us recall the characterization of the extreme points of the closed unit ball of the disc algebra $\mathcal{A}(\mathbb{D})$ from [6, p.139] given by:

**Theorem 2.6.** A function $f \in \text{Ext}(\mathcal{A}(\mathbb{D}))$ if and only if $|f| \leq 1$ on $\overline{\mathbb{D}}$ and
\[
\int_{0}^{2\pi} \ln(1 - |f(e^{it})|)dt = -\infty.
\]

Hence clearly, if $f$ is an extreme point of the closed unit ball of $\text{Hol}(\overline{\mathbb{D}})$, $f \in \text{Ext}(\mathcal{A}(\mathbb{D}))$. Also observing that the map $T$ defined in (2.1) can be extended to an isometric isomorphism from $\mathcal{A}(\mathbb{D})$ onto $\mathcal{A}(\text{Int}(\Gamma))$, we can conclude the following corollary:

**Corollary 2.7.** If $\Gamma \subset \mathbb{C}$ is a simple closed analytic curve, any extreme point of the closed unit ball of any nowhere vanishing, point separating sub-algebra of $\text{Hol}(\Gamma \cup \text{Int}(\Gamma))$ is an extreme point of the closed unit ball of its completion $\mathcal{A}(\text{Int}(\Gamma))$.

The converse of this result is not true, i.e., there exist extreme points of the closed unit ball of the disc algebra $\mathcal{A}(\mathbb{D})$ that are not holomorphic on any neighborhood of $\overline{\mathbb{D}}$ as illustrated by the following example. Note that any function $f \in \text{Ext}(\mathcal{A}(\mathbb{D}))$ which is holomorphic in some neighborhood of $\overline{\mathbb{D}}$ must be an extreme point of the closed unit ball of $\text{Hol}(\overline{\mathbb{D}})$, i.e., in particular, $f$ must be a finite Blaschke product or a unimodular constant.

**Example 2.8.** Let $g : [0, 2\pi] \rightarrow [0, 1)$ be a continuous function such that $g(t) = e^{-n}$ on $\left[\frac{1}{2n}, \frac{1}{2n-1}\right]$; $n \in \mathbb{N}$ with $g(0) = g(2\pi) = 0$, $g$ is increasing on $[0, 1]$ and strictly decreasing on $[1, 2\pi]$. Note that $\ln(1 - g(t))$ is continuous and of
bounded variation. Consider $f : \mathbb{D} \to \mathbb{C}$ given by
\[
f(z) := \exp \left[ \frac{1}{2\pi} \int_{0}^{2\pi} \left( \frac{e^{it} + z}{e^{it} - z} \right) \ln (1 - g(t)) dt \right], \quad z \in \mathbb{D}.
\]
Then $f$ is holomorphic on $\mathbb{D}$ and has absolutely summable Taylor coefficients by Hardy’s Theorem \cite[p.70]{6}. Hence $f$ can be extended continuously to the boundary of $\mathbb{D}$, i.e., $f \in \mathcal{A}(\mathbb{D})$. Also, $|f(e^{it})| = 1 - g(t)$ for $t \in [0, 2\pi)$ and so $|f| \leq 1$ on $\mathbb{D}$. Further,
\[
\int_{0}^{2\pi} \ln (1 - |f(e^{it})|) dt = \int_{0}^{2\pi} \ln (g(t)) dt \leq \sum_{n=1}^{\infty} \frac{-n}{2n(2n-1)} = -\infty,
\]
proving $f \in \text{Ext}(\mathcal{A}(\mathbb{D}))$. However, $f(e^{it}) = 1$ for $t \in [0, 2\pi)$ if and only if $t = 0$. Therefore, $f \not\in \text{Ext}(\mathcal{H}(\mathbb{D}))$, i.e., $f \not\in \mathcal{H}(\mathbb{D})$.

3. Birkhoff-James orthogonality and characterization of smooth points of \(\mathcal{H}(\Gamma \cup \text{Int}(\Gamma))\)

Now, we characterize Birkhoff-James orthogonality of two elements in any nowhere vanishing, point separating sub-algebra of $\mathcal{H}(\Gamma \cup \text{Int}(\Gamma))$ for a simple closed curve $\Gamma \subset \mathbb{C}$. We begin by providing two sufficient conditions.

**Proposition 3.1.** Let $\Gamma \subset \mathbb{C}$ be a simple closed curve, and $\mathfrak{A}$ be a nowhere vanishing, point separating sub-algebra of $\mathcal{H}(\Gamma \cup \text{Int}(\Gamma))$. Let $f_1, f_2 \in \mathfrak{A}$.

(i) If for every $\theta \in [0, 2\pi)$ there exists $z \in M_{f_1}$ such that $\arg \frac{f_2(z)}{f_1(z)} = \theta$, then $f_1 \perp_B f_2$.

(ii) If there exists $z \in M_{f_1} \cap Z_{f_2}$, then $f_1 \perp_B f_2$ (see (1.3) for definition of $Z_{f_2}$).

**Proof.** (i) For $\lambda = 0$, $||f_1 + \lambda f_2|| = ||f_1||$. If instead, $\lambda = r e^{i\phi}$ for some $r > 0$ and $\phi \in (0, 2\pi]$, then by the hypotheses, there exists $z \in M_{f_1}$ such that $\arg \frac{f_2(z)}{f_1(z)} = 2\pi - \phi$. Let $\frac{f_2(z)}{f_1(z)} = r_1 e^{i(2\pi - \phi)}$ for some $r_1 > 0$. Then clearly,
\[
||f_1 + \lambda f_2|| \geq |f_1(z) + \lambda f_2(z)| \geq ||f_1||.
\]
Since $\lambda \in \mathbb{C} \setminus \{0\}$ was chosen arbitrarily, $f_1 \perp_B f_2$.

(ii) This is immediate since for $\lambda \in \mathbb{C}$ arbitrary,
\[
||f_1 + \lambda f_2|| \geq |f_1(z) + \lambda f_2(z)| = ||f_1||.
\]

\[\square\]

The following corollary follows directly from the first part of the above proposition.
Corollary 3.2. Let $\Gamma := \{z \in \mathbb{C} : |z| = r\}$. Let $h_j(z) := z^j$, $z \in \mathbb{C}$ for $j \in \mathbb{N}$. Then $h_m \perp_B h_n$ for $m \neq n$ in any nowhere vanishing, point separating sub-algebra of $\operatorname{Hol}(\Gamma \cup \operatorname{Int}(\Gamma))$ containing all the monomials.

In order to completely characterize the Birkhoff-James orthogonality of two elements in any nowhere vanishing, point separating sub-algebra of $\operatorname{Hol}(\Gamma \cup \operatorname{Int}(\Gamma))$, we introduce the following definition.

Definition 3.3. A subset $A \subseteq \mathbb{C}^2$ is said to be an orthogonality covering set if

$$\mathcal{C} = \bigcup_{(u,v) \in A} \{\lambda \in \mathbb{C} : |u + \lambda v| \geq |u|\}.$$

We furnish the following two examples illustrating the idea.

Example 3.4. $A = \{(z_1, z_2)\} \subset \mathbb{C}^2$ is an orthogonality covering set if and only if $z_1 = 0$ or $z_2 = 0$.

Example 3.5. $A = \{(z_1, z_2), (w_1, w_2)\} \subset \mathbb{C}^2$ is an orthogonality covering set if and only if $\frac{z_1 z_2}{z_1 z_2 w_1 w_2} \in (-\infty, 0]$.

Theorem 3.6. Let $\Gamma \subset \mathbb{C}$ be a simple closed curve, and $\mathfrak{K}$ be a nowhere vanishing, point separating sub-algebra of $\operatorname{Hol}(\Gamma \cup \operatorname{Int}(\Gamma))$. Let $f_1, f_2 \in \mathfrak{K}$. Then

$$f_1 \perp_B f_2 \iff \{(f_1(z), f_2(z)) : z \in M_{f_1}\} \text{ is an orthogonality covering set.}$$

Proof. Let $A = \{(f_1(z), f_2(z)) : z \in M_{f_1}\}$. First we prove the necessity. Suppose by contradiction that $A$ is not an orthogonality covering set. Then there exists $\lambda \in \mathbb{C}$ such that

$$|f_1(z) + \lambda f_2(z)| < |f_1(z)| \text{ for all } z \in M_{f_1}.$$

We claim that $|f_1(z) + \mu \lambda f_2(z)| < |f_1(z)|$ for all $z \in M_{f_1}$ and for all $\mu \in (0, 1)$. Indeed for $z_0 \in M_{f_1}$ and $\mu_0 \in (0, 1)$,

$$f_1(z_0) + \mu_0 \lambda f_2(z_0) = (1 - \mu_0)f_1(z_0) + \mu_0(f_1(z_0) + \lambda f_2(z_0)).$$

The claim now follows:

$$|f_1(z_0) + \mu_0 \lambda f_2(z_0)| = |(1 - \mu_0)f_1(z_0) + \mu_0(f_1(z_0) + \lambda f_2(z_0))|,$$

$$< (1 - \mu_0)|f_1(z_0)| + \mu_0|f_1(z_0)|,$$

$$= |f_1(z_0)|.$$

Now, let $g : \Gamma \times [-1, 1] \rightarrow \mathbb{R}$ be defined by

$$g(z, \mu) := |f_1(z) + \mu \lambda f_2(z)|, \quad z \in \Gamma, \quad -1 \leq \mu \leq 1.$$

Clearly, $g$ is continuous. Let $z_0 \in M_{f_1}$ and $\mu_{z_0} \in (0, 1)$. Then from the claim,

$$g(z_0, \mu_{z_0}) < \|f_1\|.$$
From the continuity of $g$, there exist $\epsilon_{z_0} > 0$ and $\delta_{z_0} > 0$ such that
\[ g(z, \sigma) < \|f_1\| \text{ for all } (z, \sigma) \in B(z_0, \epsilon_{z_0}) \times (\mu_{z_0} - \delta_{z_0}, \mu_{z_0} + \delta_{z_0}). \]
In particular, $g(z, \mu_{z_0}) < \|f_1\|$ for all $z \in B(z_0, \epsilon_{z_0})$. Thus, clearly from our claim,
\[ g(z, \sigma) < \|f_1\| \text{ for all } (z, \sigma) \in B(z_0, \epsilon_{z_0}) \times (0, \mu_{z_0}). \]
Similarly, if $u_0 \in \Gamma \setminus M_{f_1}$, then $g(u_0, 0) < \|f_1\|$. Again, from the continuity of $g$, there exist $\epsilon_{u_0} > 0$ and $\delta_{u_0} > 0$ such that
\[ g(z, \alpha) < \|f_1\| \text{ for all } (z, \alpha) \in B(u_0, \epsilon_{u_0}) \times (-\delta_{u_0}, \delta_{u_0}). \]
Now,
\[ \Gamma \subseteq \bigcup_{z \in M_{f_1}} B(z, \epsilon_z) \cup \bigcup_{u \in \Gamma \setminus M_{f_1}} B(u, \epsilon_u). \]
Since $\Gamma$ is compact, there exist $z_1, z_2, \ldots, z_{k_1} \in M_{f_1}$ and $u_1, u_2, \ldots, u_{k_2} \in \Gamma \setminus M_{f_1}$ such that
\[ \Gamma \subseteq \bigcup_{i=1}^{k_1} B(z_i, \epsilon_{z_i}) \cup \bigcup_{j=1}^{k_2} B(u_j, \epsilon_{u_j}). \]
Choose $\sigma_0 > 0$ such that $\sigma_0 < \mu_{z_i}$ and $\sigma_0 < \delta_{u_j}$ for all $1 \leq i \leq k_1$ and $1 \leq j \leq k_2$. Clearly, $(f_1 + \sigma_0 \lambda f_2) \in \mathcal{H} \sigma \mathcal{L} (\Gamma \cup \text{Int}(\Gamma))$. Let $w_0 \in M(f_1 + \sigma_0 \lambda f_2)$.

Now, either $w_0 \in \bigcup_{i=1}^{k_1} B(z_i, \epsilon_{z_i})$, or $w_0 \in \bigcup_{j=1}^{k_2} B(u_j, \epsilon_{u_j})$. However, in either case, from the choice of $\sigma_0$,
\[ \|f_1 + \sigma_0 \lambda f_2\| = |f_1(w_0) + \sigma_0 \lambda f_2(w_0)| < \|f_1\|, \]
proving $f_1 \nmid_B f_2$.

We now prove the sufficiency. Let $\kappa \in \mathbb{C}$ be arbitrary. Since $A$ is an orthogonality covering set, there exists $v \in M_{f_1}$ such that $|f_1(v) + \kappa f_2(v)| \geq \|f_1(v)\|$. Thus,
\[ \|f_1 + \kappa f_2\| = \sup_{z \in \Gamma} |f_1(z) + \kappa f_2(z)| \geq |f_1(v) + \kappa f_2(v)| \geq \|f_1\|. \]
Since $\kappa$ was chosen arbitrarily, $f_1 \nmid_B f_2$.

Next, we characterize the smoothness of an element in any nowhere vanishing, point separating sub-algebra of $\mathcal{H} \sigma \mathcal{L} (\Gamma \cup \text{Int}(\Gamma))$.

**Theorem 3.7.** Let $\Gamma \subset \mathbb{C}$ be a simple closed curve, and $\mathfrak{A}$ be a nowhere vanishing, point separating sub-algebra of $\mathcal{H} \sigma \mathcal{L} (\Gamma \cup \text{Int}(\Gamma))$. Let $f \in \mathfrak{A}$ be non-zero. Then $f$ is a smooth point in $\mathcal{H} \sigma \mathcal{L} (\Gamma \cup \text{Int}(\Gamma))$ if and only if $M_f$ is a singleton set.
Proof. We first prove the necessity. Suppose by contradiction that \( z_1 \neq z_2 \in M_f \).

Define \( \Psi, \Phi : \mathfrak{A} \to \mathbb{C} \) by

\[
\Psi(h) := \text{sgn}(f(z_1))h(z_1), \quad \Phi(h) := \text{sgn}(f(z_2))h(z_2), \quad h \in \mathfrak{A}.
\]

Then \( \Psi \) and \( \Phi \) are two support functionals of \( f \). Also, \( \Psi \neq \Phi \) since \( \mathfrak{A} \) is nowhere vanishing and point separating proving \( f \) is not a smooth point of \( \mathfrak{A} \).

To prove the sufficiency, let \( M_f = \{ z_1 \} \). Let \( f \perp_B g \) and \( f \perp_B h \) for some \( g, h \in \mathcal{H}o\text{l}(\Gamma \cup \text{Int}(\Gamma)) \). If either of \( g, h \) is zero, then trivially \( f \perp_B (g + h) \).

Let \( g, h \) be non-zero. From Theorem 3.6, \( \{(f(z), g(z)) : z \in M_f \} \) is a singleton orthogonality covering set. Therefore, by Example 3.4, \( g(z_1) = 0 \). A similar argument shows \( h(z_1) = 0 \). Hence \( z_1 \in Z_{g+h} \) and consequently by Proposition 3.1, \( f \perp_B (g + h) \).

\[ \square \]

4. \( \mathcal{H}o\text{l}(\Gamma \cup \text{Int}(\Gamma)) \) and Zeros of Holomorphic Functions

We begin this section with a simple observation:

**Proposition 4.1.** Let \( \Gamma \subset \mathbb{C} \) be a simple closed curve. Let \( f \in \mathcal{J}(\Gamma) \) (see (1.1)). Then either \( f \) is constant on \( \Gamma \cup \text{Int}(\Gamma) \) or \( f \) has a zero enclosed by \( \Gamma \).

**Proof.** Suppose by contradiction that \( f \) is non-zero on \( \Omega' := \text{Int}(\Gamma) \). Hence, the minimum modulus principle yields

\[
\min_{z \in \Gamma \cup \Omega'} |f(z)| = \min_{z \in \Gamma} |f(z)| = \|f\|.
\]

Thus, \( f(\Omega') \subseteq \{ \|f\|e^{i\theta} : \theta \in [0, 2\pi) \} \). Clearly, \( f(\Omega') \) is not open in \( \mathbb{C} \) while \( \Omega' \) is and hence, \( f \) must be constant on \( \Omega \).

\[ \square \]

The first result of this section shows that Birkhoff-James orthogonality in nowhere vanishing, point separating sub-algebras of \( \mathcal{H}o\text{l}(\Gamma \cup \text{Int}(\Gamma)) \) has a deep connection with the zeros of holomorphic functions.

**Theorem 4.2.** Let \( \Gamma \subset \mathbb{C} \) be a simple closed curve, and \( \mathfrak{A} \) be a nowhere vanishing, point separating sub-algebra of \( \mathcal{H}o\text{l}(\Gamma \cup \text{Int}(\Gamma)) \). Let \( f \in \mathcal{J}(\Gamma) \cap \mathfrak{A} \) and let \( g \in \mathfrak{A} \). Then \( f \perp_B g \) implies \( f \) and \( g \) have the same number of zeros enclosed by \( \Gamma \).

**Proof.** Since \( f \perp_B g \), there exists \( \lambda \in \mathbb{C} \setminus \{0\} \) such that \( \|f + \lambda g\| < \|f\| \). Now, since \( f \in \mathcal{J}(\Gamma) \),

\[
|f(z) + \lambda g(z)| \leq \|f + \lambda g\| < \|f\| = |f(z)|, \quad \forall \ z \in \Gamma.
\]

Since, \( f \) and \(-\lambda g\) are holomorphic within and on \( \Gamma \), by Rouche’s Theorem, they have the same number of zeros enclosed by \( \Gamma \), establishing the result.

\[ \square \]

The converse of the above theorem is not true as illustrated in the following example:
Example 4.3. Let $\Gamma := \{z \in \mathbb{C} : |z| = 1\}$. Let $f_1(z) = z$ and let $f_2(z) = z(z-1)$. Clearly, $f_1 \in \mathcal{J}(\Gamma)$. Since $1 \in M_{f_1} \cap Z_{f_2}$, by Proposition 3.1, $f_1 \perp_B f_2$. However, $f_1$ and $f_2$ have the same number of zeros enclosed by $\Gamma$.

It follows from the above theorem that if $\mathfrak{A}$ is any nowhere vanishing, point separating sub-algebra of $\mathcal{H}(\Gamma \cup \text{Int}(\Gamma))$ with $f \in \mathcal{J}(\Gamma) \cap \mathfrak{A}$, then for any $g \in \mathfrak{A}$ with $f$ and $g$ having different number of zeros enclosed by $\Gamma$, $f \perp_B g$. We record this observation in form of the following theorem.

Theorem 4.4. Let $\Gamma \subset \mathbb{C}$ be a simple closed curve, and $\mathfrak{A}$ be a nowhere vanishing, point separating sub-algebra of $\mathcal{H}(\Gamma \cup \text{Int}(\Gamma))$. Let $f \in \mathcal{J}(\Gamma) \cap \mathfrak{A}$ and let $g \in \mathfrak{A}$. If $f$ and $g$ have different number of zeros enclosed by $\Gamma$, then $f \perp_B g$.

It is easy to see that Corollary 3.2 is also a direct consequence of Theorem 4.4. We derive from Theorem 4.4 the following important inequality regarding polynomials.

Corollary 4.5. Let $\Gamma := \{z \in \mathbb{C} : |z| = r\}$. Let $h_j(z) := z^j$, $z \in \mathbb{C}$ for $j \in \mathbb{N}$. Then

$$\inf_{\lambda \in \mathbb{C}} \max_{z \in \Gamma} |h_n(z) + \lambda Q_m(z)| \geq r^n$$

for every polynomial $Q_m$ of degree $m < n$.

Proof. Clearly, $h_j \in \mathcal{J}(\Gamma)$ for every $j \in \mathbb{N}$. Now, $Q_m$ and $h_n$ have different number of zeros enclosed by $\Gamma$. Therefore, by Theorem 4.4, $h_n \perp_B P_m$ in $\mathcal{H}(\Gamma \cup \text{Int}(\Gamma))$ for all $m < n$. Thus, we have for every $\lambda \in \mathbb{C}$,

$$\max_{z \in \Gamma} |h_n(z) + \lambda Q_m(z)| = \|h_n + \lambda Q_m\| \geq \|h_n\| = r^n,$$

and the result follows.

Our next result relates Birkhoff-James orthogonality in $\mathcal{H}(\Gamma \cup \text{Int}(\Gamma))$ for a simple closed curve $\Gamma \subset \mathbb{C}$ with the fundamental theorem of algebra.

Theorem 4.6. Let $\Gamma := \{z \in \mathbb{C} : |z| = r\}$ and $h_j(z) := z^j$, $z \in \mathbb{C}$ for $j \in \mathbb{N}$. If $Q(z) := a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0$, $z \in \mathbb{C}$ with $a_n \neq 0$ and $r > \max \{1, \frac{1}{|a_n|} \sum_{k=0}^{n-1} |a_k|\}$, then $h_n \not\perp_B P$ in $\mathcal{H}(\Gamma \cup \text{Int}(\Gamma))$.

Proof. Since $r > \max \{1, \frac{1}{|a_n|} \sum_{k=0}^{n-1} |a_k|\}$,

$$\|a_n h_n - Q\| = \|-(a_{n-1} h_{n-1} + a_{n-2} h_{n-2} + \cdots + a_0)\|$$

$$\leq \|a_{n-1} h_{n-1}\| + \|a_{n-2} h_{n-2}\| + \cdots + \|a_1 h_1\| + |a_0|$$

$$\leq r^{n-1} (|a_{n-1}| + |a_{n-2}| + \cdots + |a_0|)$$

$$< r^n |a_n|$$

$$= |a_n| \|h_n\|.$$
Thus, \( \|h_n - \frac{1}{\alpha_n} Q\| < \|h_n\| \), giving \( h_n \perp B Q \). \( \square \)

We now obtain the fundamental theorem of algebra:

**Corollary 4.7** (Fundamental Theorem of Algebra). A polynomial of degree \( n \) has \( n \) zeros in \( \mathbb{C} \).

**Proof.** Let \( Q(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0, a_n \neq 0 \) and \( r > \max\{1, \frac{1}{|a_n|} \sum_{k=0}^{n-1} |a_k|\} \).

Set \( \Gamma := \{z \in \mathbb{C} : |z| = r\} \) and \( h_j(z) := z^j, z \in \mathbb{C} \) for \( j \in \mathbb{N} \). Then by Theorem 4.6, \( h_n \perp B Q \). Since \( h_n \in \mathcal{F}(\Gamma) \), by Theorem 4.2, \( h_n \) and \( Q \) have the same number of zeros enclosed by \( \Gamma \). Since \( r > \max\{1, \frac{1}{|a_n|} \sum_{k=0}^{n-1} |a_k|\} \) was chosen arbitrarily, \( P \) has \( n \) number of zeros in \( \mathbb{C} \). \( \square \)

Finally, we obtain a result regarding Birkhoff-James orthogonality between the \( n \)-th order derivatives of two holomorphic functions in \( \mathcal{H}(\Gamma \cup \text{Int}(\Gamma)) \) for a simple closed curve \( \Gamma \subset \mathbb{C} \). The result also has a connection between Birkhoff-James orthogonality and zeros of the \( n \)-th order derivatives of holomorphic functions.

**Theorem 4.8.** Let \( \Gamma_1, \Gamma_2 \subset \mathbb{C} \) be two simple closed curves such that \( \Gamma_2 \subset \text{Int}(\Gamma_1) \) and fix \( 0 < r < \text{dist}(\Gamma_1, \Gamma_2) \). Let \( f, g \in \mathcal{H}(\Gamma_1 \cup \text{Int}(\Gamma_1)) \) be such that \( g^{(n)} \neq 0 \) for some fixed natural number \( n \) and

\[
\max_{z \in \Gamma_1} |f(z) + \lambda_0 g(z)| < \frac{r^n}{n!} \max_{z \in \Gamma_2} |f^{(n)}(z)| \quad \text{for some } \lambda_0 \in \mathbb{C}.
\]

Then \( f^{(n)} \perp B g^{(n)} \) in any nowhere vanishing, point separating sub-algebra of \( \mathcal{H}(\Gamma_2 \cup \text{Int}(\Gamma_2)) \) containing both the functions.

Using Theorem 4.2, we immediately derive the following:

**Corollary 4.9.** Let \( \Gamma_1, \Gamma_2, \Gamma \) be as in Theorem 4.8. Let \( f, g \in \mathcal{H}(\Gamma_1 \cup \text{Int}(\Gamma_1)) \) be such that \( f^{(n)} \in \mathcal{F}(\Gamma_2), g^{(n)} \neq 0 \) for some natural number \( n \) and

\[
\max_{z \in \Gamma_1} |f(z) + \lambda_0 g(z)| < \frac{r^n}{n!} \max_{z \in \Gamma_2} |f^{(n)}(z)| \quad \text{for some } \lambda_0 \in \mathbb{C}.
\]

Then \( f^{(n)} \) and \( g^{(n)} \) have the same number of zeros enclosed by \( \Gamma_2 \).

**Proof of Theorem 4.8.** We need to show that there exists some \( \lambda \in \mathbb{C} \) such that

\[
\max_{z \in \Gamma_2} |f^{(n)}(z) + \lambda g^{(n)}(z)| < \max_{z \in \Gamma_2} |f^{(n)}(z)|.
\]

Since \( \Gamma_2 \) is compact,

\[
\max_{z \in \Gamma_2} |f^{(n)}(z) + \lambda g^{(n)}(z)| = |f^{(n)}(z_0) + \lambda g^{(n)}(z_0)| \quad \text{for some } z_0 \in \Gamma_2.
\]
Denote $\text{Int}(\Gamma_1)$ by $\Omega'$. Let $\gamma_r$ be the circle of radius $r$ about $z_0$. From hypotheses, $\gamma_r \subset \Omega'$. Now, by Cauchy’s integral formula for the derivative, we have that

$$f^{(n)}(z_0) + \lambda_0 g^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{\gamma_r} \frac{f(z) + \lambda_0 g(z)}{(z-z_0)^{n+1}} dz.$$ 

Taking modulus on both sides, we get

$$|f^{(n)}(z_0) + \lambda_0 g^{(n)}(z_0)| \leq \frac{n!}{r^n} \max_{z \in \gamma_r} |f(z) + \lambda_0 g(z)|,$$

since $\int_{\gamma_r} \frac{dz}{(z-z_0)^{n+1}} = \frac{2\pi}{r^n}$. Invoking the maximum modulus principle and noting $\gamma_r \subset \Omega'$ yields:

$$\max_{z \in \gamma_r} |f(z) + \lambda_0 g(z)| \leq \max_{z \in \Gamma_1} |f(z) + \lambda_0 g(z)|.$$ 

Thus,

$$|f^{(n)}(z_0) + \lambda_0 g^{(n)}(z_0)| < \frac{n!}{r^n} \max_{z \in \Gamma_1} |f(z) + \lambda_0 g(z)|.$$ 

Now, from the hypotheses,

$$\max_{z \in \Gamma_2} |f^{(n)}(z) + \lambda_0 g^{(n)}(z)| = |f^{(n)}(z_0) + \lambda_0 g^{(n)}(z_0)| < \max_{z \in \Gamma_2} |f^{(n)}(z)|.$$ 

Thus, $f^{(n)} \not\perp B g^{(n)}$ in any nowhere vanishing, point separating sub-algebra of $\mathcal{H}ol(\Gamma_2 \cup \text{Int}(\Gamma_2))$ containing both the functions. \hfill \Box

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