Strong gravitational lensing by Schwarzschild black hole

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Schwarzschild black holes can produce strong gravitational lensing. The relativistic images are produced due to bending of light around the black hole. We propose a model equation to study the strong gravitational lensing. The model equation can well describe the bending angle from a range very close to the photon sphere to the first relativistic image.

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I. INTRODUCTION

It is a challenging job to compute the bending angle in the strong gravitational field. Since the work of Darwin [1, 2] there were various attempts to find the deflection angle in the strong field limit. Also recent works by V. Bozza, S. Capozziello, G. Iovane and G. Scarpetta [3], V. Bozza, S. Capozziello, G. Iovane and M. Cervantes [4], K. S. Virbhadra and G. F. R. Ellis [5], S. V. Iyer and A. O. Petters [6] made considerable progress in calculating the deflection angle in the strong field limit. Also recent works by V. Bozza, S. Capozziello, G. Iovane and G. Scarpetta [3], P. Amore and S. Arceo [6], P. Amore and M. Cervantes [7], K. S. Virbhadra and G. F. R. Ellis [8], S. V. Iyer and A. O. Petters [9] made considerable progress in calculating the deflection angle in strong gravitational field. Relativistic images produced by strong gravitational lensing (SGL) will provide a good observational test for the general theory of relativity in very near the photon sphere to the first relativistic image.

Very Long Baseline Interferometry (VLBI) [10, 11] may resolve relativistic images. An analytical expression for the deflection angle will be very helpful in describing the sizes of the relativistic images. In this paper we have proposed a model equation for a Schwarzschild object which fit exact deflection angle with minor error. The range of validity is very near the photon sphere to the first relativistic image. So numerical integration is the only method to find the exact deflection angle. But an analytical expression is very handy to get the deflection angle. We propose the following equation which gives deflection angle very accurately near the photon sphere.

\[ \hat{\alpha}(r_0) = 2 \int_{r_0}^{\infty} \frac{dr}{r \sqrt{\left( \frac{r}{r_0} \right)^2 \left( 1 - \frac{2M}{r} \right) - \left( 1 - \frac{2M}{r_0} \right)}} - \pi \] (2)

and the impact parameter \( J \) of the light ray is given by

\[ J(r_0) = \frac{r_0}{\sqrt{1 - \frac{2M}{r_0}}} \] (3)

Now we define the dimensionless parameter \( x \) and \( x_0 \) by

\[ x = \frac{r}{2M}, \quad x_0 = \frac{r_0}{2M}. \] (4)

The deflection angle (Eq. 2) and impact parameter (Eq. 3) become

\[ \hat{\alpha}(x_0) = 2 \int_{x_0}^{\infty} \frac{dx}{x \sqrt{\left( \frac{x}{x_0} \right)^2 \left( 1 - \frac{1}{x} \right) - \left( 1 - \frac{1}{x_0} \right)}} - \pi \] (5)

and

\[ J(x_0) = \frac{2Mx_0}{\sqrt{1 - \frac{1}{x_0}}} \] (6)

The deflection angle is an elliptical integral. So numerical integration is the only method to find the exact deflection angle. But an analytical expression is very handy to get the deflection angle. We propose the following equation which gives deflection angle very accurately near the photon sphere.

\[ \hat{\alpha}_D (x_0) = a \text{ ArcCsch}(b x_0 + c) + d \] (7)

Here we consider \( G = c = 1 \) and \( M \) is the enclosed mass. The Schwarzschild radius is \( R_{sch} = 2M \). The bending angle \( \hat{\alpha} \) with closest distance of approach \( r_0 \) is given by [12]

\[ \hat{\alpha}(r_0) = 2 \int_{r_0}^{\infty} \frac{dr}{r \sqrt{\left( \frac{r}{r_0} \right)^2 \left( 1 - \frac{2M}{r} \right) - \left( 1 - \frac{2M}{r_0} \right)}} - \pi \] (2)

where

\[ a = -1.9904142396804585 \quad b = -7.152403979116706 \quad c = 10.728609274134325 \quad d = 2.603634899810115 \]

The constants \( a, b, c \) and \( d \) are calculated using the software Mathematica. For Schwarzschild lens the photon sphere is at \( x_{ph} = 1.5 \). The range of validity of this equation is that the value of \( x_0 \) is from very near the photon sphere to 1.55. Eq. 2 is an elliptic integral, but Eq. 7 is

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We also draw graphs of the deflection angle as proposed by different authors [3, 4] with that of our model equation to show how our equation describe the deflection angle in the strong field limit. In the following we shall write down the expression of deflection angle as given by Bozza and Amore.

The deflection angle due to a Schwarzschild object in the strong deflection limit as proposed by Amore [7] is

\[ \alpha_A = \frac{12}{-3\sqrt{3} - 4} + \sqrt{4\zeta + 1} \log \left( \frac{\zeta}{\zeta - 1} \right) \] (10)

and that due to Bozza [3] is

\[ \alpha_B = -\log \left( \frac{J}{J - 1} \right) + \log[216(7 - 4\sqrt{3}) - \pi] \] (11)

where \( J \) is the smallest impact parameter and its value is \( 3\sqrt{3}M/2 \) and \( \zeta = 2x_0/3 \). Fig 3 shows the comparison of the curves drawn using Eq. (10) (dotted green curve), Eq. (11) (thick dashed black curve) and the model Eq. (7) (continuous red curve). It is seen that the curve of the deflection angle due to Bozza is the same as due to the model equation but a little difference with the curve due to Amore et al. This shows that Eq. (7) can equally describe the deflection angle near the photon sphere.

To calculate the relativistic images we use the following lens equation [3]

\[ \tan \beta = \tan \theta - \frac{D_{ds}}{D_d} \left( \tan \theta - \tan(\hat{\alpha}_D - \theta) \right) \] (12)

From the lens diagram (Fig 4) we have

\[ \sin \theta = \frac{J}{D_d} \] (13)

Here \( \beta \) and \( \theta \) be the source and the image position measured from the optic axis, \( D_{ds} \) and \( D_d \) be the source-lens and source-observer distance respectively. The total magnification of a circularly symmetric Gravitational lens is given by

\[ \mu = \left( \frac{\sin \beta d\beta}{\sin \theta d\theta} \right)^{-1} \] (14)
and the tangential and radial magnification are given by

\[ \mu_t = \left( \frac{\sin \beta}{\sin \theta} \right)^{-1}, \quad \mu_r = \left( \frac{d\beta}{d\theta} \right)^{-1}. \] (15)

Here we shall take an example as given in the reference [8] to calculate the relativistic images and Einstein rings etc. using the model equation (7). The mass of the lens \( M = 2.8 \times 10^6 M_\odot \) and the distance \( D_d = 8.5 \) kpc. The lens is halfway between the point source and the observer, i.e. \( D_{ds}/D_s = 1/2 \).

The values given in Table II shows that all the entries are approximately the same as given in Table III of Ref. [8]. We also plot the tangential magnification \( \mu_t \) Fig. 5 and total magnification \( \mu \) Fig. 6.

### III. CONCLUSION

The observation of relativistic images will provide a test for the success of general theory of relativity in the strong field. We should have accurate theoretical value of deflection angle in the strong field limit to compare with observational result. We propose a model equation for the deflection angle and found that it can correctly describe the deflection angle in the strong field limit of a Schwarzschild object. We have compared the values of deflection angle, relativistic Einstein ring, magnification etc with the values given by Bozza [4], Virbhadra [8] and others. Our results almost match with those values. So we are confident that the model equation can be used to calculate different observable.
[1] C. Darwin, Proc. R. Soc. Lond. A 249, 180 (1959).
[2] C. Darwin, Proc. R. Soc. Lond. A 263, 39 (1961).
[3] V. Bozza, S. Capozziello, G. Iovane, and G. Scarpetta, Gen Rel Grav 33, 1535 (2001).
[4] V. Bozza, Phy Rev D 66, 103001 (2002).
[5] V. Bozza, Gen Rel Grav 42, 2269 (2010).
[6] P. Amore and S. Arceo, Phy Rev D 73, 083004 (2006).
[7] P. Amore and M. Cervantes, Phy Rev D 75, 083005 (2007).
[8] K. S. Virbhadra and G. F. R. Ellis, Phy Rev D 62, 084003 (2000).
[9] S. V. Iyer and A. O. Petters, Gen Rel Grav 39, 1563 (2007).
[10] “Constellation-X web page: constellation.gsfc.nasa.gov; maxim web page: maxim.gsfc.nasa.gov.”.
[11] J. S. Ulvestad, New Astron Rev 43, 531 (1999).
[12] S. Weinberg, Gravitation and Cosmology (Wiley, 1972).
[13] G. S. Bisnovatya-Kogan and T. O. Yu, APJ 51, 99 (2008).
[14] C. W. Misner, K. S. Thorne, and J. A. Wheeler, Gravitation (W A Freeman and Company, 1973).