Correlation decay and deterministic FPTAS for counting list-colorings of a graph *

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Abstract

We propose a deterministic algorithm for approximately counting the number of list colorings of a graph. Under the assumption that the graph is triangle free, the size of every list is at least \( \alpha \Delta \), where \( \alpha \) is an arbitrary constant bigger than \( \alpha^* = 2.8432 \ldots \), and \( \Delta \) is the maximum degree of the graph, we obtain the following results. For the case when the size of each list is a large constant, we show the existence of a deterministic FPTAS for computing the total number of list colorings. The same deterministic algorithm has complexity \( 2^{O(\log^2 n)} \), without any assumptions on the sizes of the lists, where \( n \) is the instance size. We further extend our method to a discrete Markov random field (MRF) model. Under certain assumptions relating the size of the alphabet, the degree of the graph and the interacting potential we again construct a deterministic FPTAS for computing the partition function of a MRF.

Our results are not based on the most powerful existing counting technique – rapidly mixing Markov chain method. Rather we build upon concepts from statistical physics, in particular, the decay of correlation phenomena and its implication for the uniqueness of Gibbs measures in infinite graphs. This approach was proposed in two recent papers [?] and [?]. The principle insight of this approach is that the correlation decay property can be established with respect to certain computation tree, as opposed to the conventional correlation decay property with respect to graph theoretic neighborhoods of a given node. This allows truncation of computation at a logarithmic depth in order to obtain polynomial accuracy in polynomial time.

1 Introduction

This paper is devoted to the problem of computing the total number of list colorings of a graph. It is further concerned with the problem of computing a partition function corresponding to a Markov random field (also known as graphical) model. The setting for the list coloring problem is as follows. Each node of a given graph is associated with a list of colors. An assignment of nodes to colors is called list coloring if every node is assigned to some color from its list and no two nodes sharing an edge are assigned to the same color. When all the lists are identical, the problem reduces to the problem of coloring of a graph. The problem of determining whether a list coloring exists is NP-hard, but provided that the size of each list is strictly larger than the degree for each node, a simple greedy algorithm produces a coloring. We are concerned with the corresponding counting problem – compute the total

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number of list colorings of a given graph/list pair. This problem is known to be \#P hard even for the restricted problem of counting the colorings, and the focus is on the approximation algorithms. The existing approximation schemes are mostly based on the rapidly mixing Markov chain technique, also known as Glauber dynamics approach. It was established by Jerrum [?] that the Glauber dynamics corresponding to graphs where the ratio of the number of colors to degree satisfies \( q/\Delta \geq 2 \), mixes rapidly. This leads to a randomized approximation algorithm for enumerating the number of colorings. The 2-barrier was first broken by Vigoda [?], who lowered the ratio requirement to 11/6. Many further significant improvements were obtained subsequently. The state of the art is summarized in [?]. For a while the improvement over 11/6 ratio came at a cost of lower bound \( \Omega(\log n) \) on the maximum degree, where \( n \) is the number of nodes. This requirement was lifted by Dyer et al. [?].

In this paper we focus on a different approach to the counting list colorings problem. Our setting is a list coloring problem. We require that the size of every list is at least \( \alpha \epsilon - \) the unique solution to \( g \) restriction is \( \geq 6 \) ratio came at a cost of lower bound \( \Omega(\log n) \) on the maximum degree, where \( n \) is the number of nodes. This requirement was lifted by Dyer et al. [?].

In this paper we propose a general deterministic approximate counting algorithm which can be used for arbitrary multi-valued counting problem. We also by-pass the large girth assumption by considering a certain computation tree corresponding to the Gibbs (uniform for the case of colorings) measure. Our principal insight is establishing correlation decay for the computation tree as opposed
to the conventional correlation decay associated with the graph-theoretic structure of the graph. We provide a discussion explaining why it is crucial to establish the correlation decay in this way in order to obtain FPTAS. Contrast this with [?] where correlation decay is established for the coloring problem but in the conventional graph-theoretic distance sense. Our method is similar to the self-avoiding walk method of Weitz’s but somewhat more direct as the step of relating the marginal probability on a graph to the marginal probability on the tree is by-passed in our computation tree approach. The advantage of establishing correlation decay on a computation tree as opposed to the original graph has been highlighted also in [?] in the context of BP algorithms and the Dobrushin’s Uniqueness condition. More importantly our approach works for general, not necessarily two-valued model. We extend our approach to Markov random field model and also show that under some conditions the computation tree satisfies the correlation decay property and, as a result, one obtains a deterministic algorithm for computing approximately the associated partition function.

The remainder of the paper has the following structure. The model description and the main result are stated in Section 2. Some preliminary technical results are established in Section 3. The description of the algorithm and its complexity are subject of Section 4. The principal technical result is established in Section 5. The key result is Theorem 2 which establishes the correlation decay result on a computation tree arising in computing the marginals of the uniform distribution on the set of all list colorings. Section 6 is devoted to the extension of our approach to a Markov random field. Section 7 provides a brief comparison between the correlation decay on a computation tree and the correlation decay in a conventional sense. Some conclusions and open problems are in Section 8.

2 Definitions and the main result

We consider a simple graph $G$ with the node set $V = \{v_1, v_2, \ldots, v_{|V|}\}$. Our graph is assumed to be triangle-free. Namely the girth (the size of the smallest cycle) is at least $g \geq 4$. Let $E, \Delta$ denote respectively the set of edges and the maximum degree of the graph. $\Delta(v)$ denotes the degree of the node $v$. Each node $v$ is associated with a list of colors $L(v) \subset \{1, 2, \ldots, q\} = \cup_{v \in V} L(v)$, where $\{1, 2, \ldots, q\}$ is the total universe of colors. We let $L = (L(v), 1 \leq v \leq n)$ denote the vector of lists. We also let $\|L\| = \max_v |L(v)|$ the size of the largest list. The list-coloring problem on $G$ is formulated as follows: associate each node $v$ with a color $c(v) \in L(v)$ such that no two nodes sharing an edge are associated with the same color. When all the lists are identical and contain $q$ elements, the corresponding problem is the problem of coloring $G$ using $q$ colors. We let $|L(v)|$ denote the cardinality of $L(v)$. It is easy to see that if

$$|L(v)| \geq \Delta(v) + 1$$

for every node $v$, then a simple greedy procedure produces a list-coloring. We adopt here a stronger assumption

$$|L(v)| \geq \alpha \Delta(v) + \beta,$$

where $\alpha$ is an arbitrary constant strictly larger than $\alpha^{**}$, the unique solution of $\alpha^{**} \exp(-\frac{1}{\alpha^{**}}) = 2$. That is $\alpha^{**} \approx 2.8432 \ldots$. We also assume that $\beta$ is a large constant which depends on $\alpha$. To be more specific we assume that $\beta = \beta(\alpha)$ is large enough to satisfy

$$\frac{1}{\beta} \alpha e^{-\frac{1}{\alpha^{(1+\frac{1}{\beta})}}} > 2,$$

which is always possible when $\alpha > \alpha^{**}$.
Let \( Z(G, L) \) denote the total number of possible list-colorings of a graph/list pair \((G, L)\). The corresponding counting problem is to compute (approximately) \( Z(G, L) \). In statistical physics terminology, \( Z(G, L) \) is the partition function. We let \( Z(G, L, \chi) \) denote the number of list colorings of \((G, L)\) which satisfy some condition \( \chi \). For example \( Z(G, L, c(v) = i, c(u) = j) \) is the number of list colorings such that the color of \( v \) is \( i \) and the color of \( u \) is \( j \).

On the space of all list colorings of \( G \) we consider a uniform probability distribution, where each list coloring assumes weight \( 1/Z(G, L) \). For every node/color pair \( v \in V, i \in L(v) \), \( P_{G, L}(c(v) = i) \) denotes the probability that node \( v \) is colored \( i \) with respect to this probability measure. The size of the instance corresponding to a graph/list pair \((G, L)\) is defined to be \( n = \max\{|V|, |E|, q\} \).

**Definition 1.** An approximation algorithm \( \mathcal{A} \) is defined to be a Fully Polynomial Time Approximation Scheme for a computing \( Z(G, L) \) if given arbitrary \( \delta > 0 \) it produces a value \( \hat{Z} \) satisfying

\[ 1 - \delta \leq \frac{\hat{Z}}{Z(G, L)} \leq 1 + \delta, \]

in time which is polynomial in \( n, \frac{1}{\delta} \).

We now state our main result.

**Theorem 1.** There exist a deterministic algorithm which provides a FPTAS for computing \( Z(G, L) \) for arbitrary graph list pair \( G, L \) satisfying \( |L| \), when the size of the largest list \(|L|\) is constant. The same algorithm has complexity \( 2^O(\log^2 n) \), without any restriction on \(|L|\), where \( n \) is the size of the instance.

## 3 Preliminary technical results

### 3.1 Basic recursion

We begin by establishing a standard relationship between the partition function \( Z(G, L) \) and the marginals \( \mathbb{P}_{G, L}(c(v) = i) \). The relation, also known as cavity method, is also the basis of the Glauber dynamics approach for computing partition functions.

**Proposition 1.** Consider an arbitrary list coloring \( i_1, \ldots, i_{|V|} \) of the graph \( G \) (which can be constructed using a simple greedy procedure). For every \( k = 0, 1, \ldots, |V| - 1 \) consider a graph list pair \( G_k, L_k \), where \( (G_0, L_0) = (G, L) \), \( G_k = G \setminus \{v_1, \ldots, v_k\} \), \( k \geq 1 \) and the list \( L_k \) is obtained by deleting from each list \( L(v_l) \), \( l > k \) a color \( i_r, r \leq k \) if \((v_l, v_r) \in E\). Then

\[ Z(G, L) = \prod_{0 \leq k \leq |V| - 1} \mathbb{P}^{-1}_{G_k, L_k}(c(v_k) = i_k). \]

**Proof.** We have

\[ \mathbb{P}_{G, L}(c(v_1) = i_1) = \frac{Z(G, L, c(v_1) = i_1)}{Z(G, L)} = \frac{Z(G_1, L_1)}{Z(G, L)}, \]

from which we obtain

\[ Z(G, L) = \mathbb{P}_{G, L}(c(v_1) = i_1)^{-1} Z(G_1, L_1). \]

Iterating further for \( k \geq 2 \) we obtain the result.
Our algorithm is based on a recursive procedure which relates the number of list colorings of a given graph/list pair in terms of the number of list colorings of some reduced graph/list pairs.

Given a pair \((G, L)\) and a node \(v \in G\), let \(v_1, \ldots, v_m\) be the set of neighbors of \(v\). For every pair \((k, i) \in \{1, \ldots, m\} \times L(v)\) we define a new pair \((G_v, L_{k,j})\) as follows. The set of nodes of \(G\) is \(V_k = V \setminus \{v\}\) and \(L_{k,i}(v_r) = L(v_r) \setminus \{i\}\) for \(1 \leq r < k\), \(L_{k,j}(u) = L(u)\) for all other \(u\). Namely, we first delete node \(v\) from the graph. Then we delete color \(i\) from the lists corresponding to the nodes \(v_r, r < k\), and leave all the other lists intact.

**Lemma 1.** The graph/list pair \((G_v, L_{k,j})\) satisfies (3) for every \(1 \leq k \leq m, j \in L(v)\), provided that \((G, L)\) does.

**Proof.** When we create graph \(G_v\) from \(G\) the list size of every remaining node either stays the same or is reduced by one. The second event can only happen for neighbors \(v_1, \ldots, v_m\) of the deleted node \(v\). When the list is reduced by one the degree is reduced by one as well. Since \(\alpha > 1\), the assertion follows by observing that \(|L(v_k)| \geq \alpha \Delta(v_k) + \beta\) implies \(|L(v_k)| - 1 \geq \alpha(\Delta(v_k) - 1) + \beta\).

The basis of our algorithm is the following simple result.

**Proposition 2.** Given a graph/list pair \((G, L)\) and a node \(v\), suppose \(\Delta(v) = m > 0\). For every \(i \in L(v)\)

\[
\mathbb{P}_{G,L}(c(v) = i) = \frac{\prod_{1 \leq k \leq m} (1 - \mathbb{P}_{G_v,L_{k,i}}(c(v) = i))}{\sum_{j \in L(v)} \prod_{1 \leq k \leq m} (1 - \mathbb{P}_{G_v,L_{k,j}}(c(v) = j))}.
\]

(4)

The recursion as well as the proof is similar to the one used by Weitz in [?], except we bypass the construction of a self-avoiding tree, considered in [?].

**Proof.** Consider a graph/list \((G_v, L)\) obtained simply by removing node \(v\) from \(G\), and leaving \(L\) intact for the remaining nodes. We have

\[
\mathbb{P}_{G_v,L}(c(v) = i) = \frac{\mathbb{P}_{G_v,L}(c(v) = i)}{\sum_{j \in L(v)} \mathbb{P}_{G_v,L}(c(v) = j)}
\]

\[
= \frac{Z(G_v, L, c(v) = i)Z^{-1}(G_v, L)}{\sum_{j \in L(v)} Z(G_v, L, c(v) = j)Z^{-1}(G_v, L)}
\]

\[
= \frac{Z(G_v, L, c(v_k) \neq i, 1 \leq k \leq m)}{\sum_{j \in L(v)} \mathbb{P}_{G_v,L}(c(v_k) \neq j, 1 \leq k \leq m)}
\]

\[
= \frac{\mathbb{P}_{G_v,L}(c(v_k) \neq i, 1 \leq k \leq m)}{\sum_{j \in L(v)} \mathbb{P}_{G_v,L}(c(v_k) \neq j, 1 \leq k \leq m)}
\]

Now, for every \(j \in L(v)\)

\[
\mathbb{P}_{G_v,L}(c(v_k) \neq j, 1 \leq k \leq m) = \mathbb{P}_{G_v,L}(c(v_1) \neq j) \prod_{2 \leq k \leq m} \mathbb{P}_{G_v,L}(c(v_k) \neq j|c(v_r) \neq j, 1 \leq r < k)
\]

We observe that \(L_{1,j} = L\) for every \(j\) (no colors are removed due to the vacuous condition \(r < 1\)), and \(\mathbb{P}_{G_v,L}(c(v_k) \neq j|c(v_r) \neq j, 1 \leq r < k) = \mathbb{P}_{G_v,L_{k,j}}(c(v_k) \neq j)\). Namely

\[
\mathbb{P}_{G_v,L}(c(v_k) \neq j, 1 \leq k \leq m) = \prod_{1 \leq k \leq m} \mathbb{P}_{G_v,L_{k,j}}(c(v_k) \neq j) = \prod_{1 \leq k \leq m} (1 - \mathbb{P}_{G_v,L_{k,j}}(c(v_k) = j)).
\]

Substituting this expression we complete the proof. 

\[\square\]
3.2 Upper and lower bounds

The condition (2) allows us to obtain the following simple bounds.

**Lemma 2.** For every $G, L$, node $v$ and a color $i \in L(v)$

$$\mathbb{P}_{G,L}(c(v) = i) \leq \frac{1}{\beta}.$$  

**Proof.** Observe that given an arbitrary coloring of the neighbors $v_1, \ldots, v_m$ of $v$, there are at least $|L(v)| - \Delta(v) \geq \beta$ colors remaining. Then the upper bound holds.

From this simple bound we now establish a different upper bound and also a lower bound using the triangle free assumption.

**Lemma 3.** There exist $\epsilon_0 = \epsilon_0(\alpha) \in (0, 1)$ and $\beta > 0$ such that for every $G, L$, node $v$ and a color $i \in L(v)$

$$q^{-1}(1 - \beta^{-1})^\Delta \leq \mathbb{P}_{G,L}(c(v) = i) \leq \frac{1}{2\Delta(v)(1 + \epsilon_0)}.$$  

We note that the upper bounds of this lemma and Lemma 2 are not comparable, since values of $\Delta(v)$ could be smaller and larger than $\beta$.

**Proof.** We let $v_1, \ldots, v_m$ denote the neighbors of $v$, $m = \Delta(v)$ and let $v_{kr}$ denote the set of neighbors of $v_k$, other than $v$ for $k = 1, \ldots, m$. We will establish that for any coloring of nodes $(v_{kr})$, which we generically denote by $c$, we have

$$q^{-1}(1 - \beta^{-1})^\Delta \leq \mathbb{P}_{G,L}(c(v) = i|c) \leq \frac{1}{2m(1 + \epsilon_0)}.$$  

The corresponding inequality for the unconditional probability then follows immediately. Now observe that, since the girth is at least 4, then there are no edges between $v_k$. Then $\mathbb{P}_{G,L}(c(v) = i|c)$ is the probability $\mathbb{P}_T(c(v) = i)$ that $v$ is colored $i$ in a depth-1 tree $T \triangleq \{v, v_1, \ldots, v_m\}$, where the lists $\hat{L}(v_k)$ of $v_k$ are obtained from $L(v_k)$ by deleting the colors used by the neighbors $v_{kr}$ by coloring $c$. From the assumption (2) we have that the remaining lists $\hat{L}(v_k)$ have size at least $|L(v_k)| - \Delta(v_k) \geq \beta$ each. Let $t_i = \mathbb{P}_T(c(v) = i)$. For each color $j \in L(v)$ let $t_{j,k} = 1/|\hat{L}(v_k)|$ if $j \in \hat{L}(v_k)$ and $= 0$ otherwise. Proposition 2 then simplifies to

$$t_i = \frac{\prod_{1 \leq k \leq m} (1 - t_{i,k})}{\sum_{j \in L(v)} \prod_{1 \leq k \leq m} (1 - t_{j,k})} \leq \frac{1}{\sum_{j \in L(v)} \prod_{1 \leq k \leq m} (1 - t_{j,k})},$$  

for every $i \in L(v)$, where $\prod_{1 \leq k \leq m}$ is defined to be equal to unity when $m = 0$. From the equality part, applying $t_{j,k} \leq 1/\beta$, we get

$$t_i \geq |L(v)|^{-1}(1 - \beta^{-1})^m \geq q^{-1}(1 - \beta^{-1})^\Delta,$$

and the lower bound is established.

We now focus on the upper bound and use the inequality part of (5). Thus it suffices to show that

$$\sum_{j \in L(v)} \prod_{k} (1 - t_{j,k}) \geq 2(1 + \epsilon_0)m$$  

(6)
for some constant $\epsilon_0 > 0$. Using the first order Taylor expansion for $\log z$ around $z = 1$,

$$
\prod_{1 \leq k \leq m} (1 - t_{j,k}) = \prod_{1 \leq k \leq m} e^{\log(1-t_{j,k})} = \prod_{1 \leq k \leq m} e^{-t_{j,k} - \frac{t_{j,k}^2}{2(1-\theta_{j,k})^2}},
$$

for some $0 \leq \theta_{j,k} \leq t_{j,k}$, since $-1/z^2$ is the second derivative of $\log z$. Again using the bound $t_{j,k} \leq 1/\beta$, we have $(1 - \theta_{j,k})^2 \geq (1 - 1/\beta)^2$. We assume that $\beta$ is a sufficiently large constant ensuring $(1 - 1/\beta)^2 > 1/2$. Thus we obtain the following lower bound

$$
\prod_{1 \leq k \leq m} (1 - t_{j,k}) \geq \prod_{1 \leq k \leq m} e^{-t_{j,k} - \frac{t_{j,k}^2}{2(1-\theta_{j,k})^2}} \geq e^{-(1+\frac{1}{\beta}) \sum_k t_{j,k}} \triangleq e^{-(1+\frac{1}{\beta})T_j},
$$

where $T_j$ stands for $\sum_k t_{j,k}$. Then

$$
\sum_{j \in L(v)} \prod_{1 \leq k \leq m} (1 - t_{j,k}) \geq \sum_{j \in L(v)} e^{-(1+\frac{1}{\beta})T_j} \geq |L(v)|e^{-\frac{1}{\beta}\sum_{j \in L(v)} T_j},
$$

where we have used an inequality between the average arithmetic and average geometric. Finally we observe

$$
\sum_{j \in L(v)} T_j = \sum_{j,k} t_{j,k} = \sum_{1 \leq k \leq m} \sum_{j \in L(v_k)} \frac{1}{|L(v_k)|} = m.
$$

Thus

$$
\sum_{j \in L(v)} \prod_{1 \leq k \leq m} (1 - t_{j,k}) \geq |L(v)|e^{-\frac{m}{\beta}} \geq (am + \beta)e^{-\frac{1}{\alpha}(1+\frac{1}{\beta})} > \alpha e^{-\frac{1}{\alpha}(1+\frac{1}{\beta})}
$$

The condition $\alpha > \alpha^{**}$ implies that there exists a sufficiently large $\beta$ such that $\alpha e^{-\frac{1}{\alpha}(1+\frac{1}{\beta})} > 2$. We find $0 < \epsilon_0 < .1$ such that $\alpha e^{-\frac{1}{\alpha}(1+\frac{1}{\beta})} = 2(1 + \epsilon_0)$. We obtain a required lower bound (6).

\[ \square \]

4 Algorithm and complexity

4.1 Description of an algorithm

Our algorithm is based on the idea of trying to approximate the value of $P_G(L(c(v) = i))$, by performing a certain recursive computation using (4) a fixed number of times $d$ and then using a correlation decay principle to guarantee the accuracy of the approximation. Specifically, introduce a function $\Phi$ which takes as an input a vector $(G, L, v, i, d)$ and takes some values $\Phi(G, L, v, i, d) \in [0, 1]$. The input $(G, L, v, i, d)$ to $\Phi$ is any vector, such that such that $v$ is a node in $G$, $i$ is an arbitrary color, and $d$ is an arbitrary non-negative integer. Function $\Phi$ is defined recursively in $d$. The quantity $\Phi$ "attempts" to approximate $P_G(L(c(v) = i))$. The quality of the approximation is controlled by $d$. We define $\Phi$ as follows. For every input $(G, L, v, i, d)$ such that $i \notin L(v)$ we set $\Phi(G, L, v, i, d) = 0$. Otherwise we set the values as follows.

- When $d = 0$, we set $\Phi(G, L, v, i, d) = 1/|L(v)|$ for every input $(G, L, v, i)$. (It turns out that for our application the initialization values are not important, due to the decay of correlations).
Lemma 4. We now establish some properties of $\Phi$.

4.2 Some properties

Set

$$\Phi(G, L, v, i, d) = \min \left[ \frac{1}{2(1 + \epsilon_0)m} \beta \cdot \sum_{j \in L(v)} \prod_{1 \leq k \leq m} \left( 1 - \frac{(1 - \Phi(G, L_{k,j}, v, i, d - 1))}{1 - \Phi(G, L_{k,j}, v, i, d - 1)} \right) \right]. \quad (7)$$

The last part of the expression inside $\min[\cdot]$ corresponds directly to the expression (4) of Proposition 2. Specifically, if it was true that $\Phi(G, L_{k,j}, v, i, d - 1) = \mathbb{P}_{G, L_{k,j}}(c(v_k) = j)$, then, by Lemmas 2,3, the minimum in (7) would be achieved by the third expression, and then the value of $\Phi(G, L, v, i, d)$ would be exactly $\mathbb{P}_{G,L}(c(v) = i)$.

We will use the correlation decay property to establish that the difference between the two values, modulo rescaling, is diminishing as $d \to \infty$. Note that the computation of $\Phi$ can be done recursively in $d$ and it involves a dynamic programming type recursion. The underlying computation is done essentially on a tree of graph list pairs $G_s, L_s$ generated during the recursion. We refer to this tree as computation tree with depth $d$.

We now describe our algorithm for approximately computing $Z(G, L)$. The algorithm is parametrized by the "quality" parameter $d$.

**Algorithm CountCOLOR**

**INPUT:** A graph/list pair $(G, L)$ and a positive integer $d$.

**BEGIN**

Set $\hat{Z} = 1, \hat{G} = G, \hat{L} = L$.

While $\hat{G} \neq \emptyset$, find an arbitrary node $v \in \hat{G}$ and a color $i \in \hat{L}(v)$. Compute

$$\hat{p}(v, i) \triangleq \Phi(\hat{G}, \hat{L}, v, i, d). \quad (8)$$

Set $\hat{Z} = \hat{p}^{-1}(v, i) \hat{Z}, \hat{G} = \hat{G} \setminus \{v\}, \hat{L}(u) = \hat{L}(u) \setminus \{i\}$ for all neighbors $u$ of $v$ in $\hat{G}$, and $\hat{L}(u)$ remains the same for all other nodes.

**END**

**OUTPUT:** $\hat{Z}$.

4.2 Some properties

We now establish some properties of $\Phi$.

**Lemma 4.** The following holds for every $G, L, v, i \in L(v), d \geq 0$.

$$\Phi(G, L, v, i, d) \leq 1, \quad (9)$$

$$\Phi(G, L, v, i, d) \leq \frac{1}{2(1 + \epsilon_0)\Delta(v)}, \quad (10)$$

$$\Phi(G, L, v, i, d) \geq q^{-1}(1 - 1/\beta)^\Delta. \quad (11)$$

**Proof.** (10) follows directly from the definition of $\Phi$. To show (9) we consider cases. For $d \geq 1$ this follows directly from the recursion (7). For $d = 0$, this follows since $\Phi(G, L, v, i, 0) = 1/|L(v)| \leq 1/(\alpha \Delta(v) + \beta)$ and $2(1 + \epsilon_0) < 2.2 < \alpha$. We now establish (11). For the case $d = 0$ this follows since $1/|L(v)| \geq 1/q$. For
the case \( d \geq 1 \) this follows from the recursion \((7)\) since \( 1/\beta, 1/(2(1 + \epsilon_0)\Delta(v)) > 1/q \) and the third term inside the minimum operator is at least \( q^{-1}(1 - 1/\beta)^{\Delta} \), using upper bound \( \Phi(G, L, v, i, d - 1) \leq 1/\beta \) which we have from \((8)\).

\[\square\]

4.3 Complexity

We begin by analyzing the complexity of computing function \( \Phi \). Recall that \( n = \max(|V|, |E|, q) \) is the size of the instance.

**Proposition 3.** For any given node \( v \), the function \( \Phi \) can be computed in time \( 2^{O(d\log \|L\| + \log \Delta)} \). In particular when \( d = O(\log n) \), the overall computation is \( 2^{O(\log^2 n)} \). If in addition the size of the largest list \( \|L\| \) is constant then the computation time is polynomial in \( n \).

**Proof.** Let \( T(d) \) denote the complexity of computing function \( \Phi(\cdot, d) \). Clearly, \( T(0) = O(\|L\|) \). We now express \( T(d) \) in terms of \( T(d - 1) \). Given a node \( v \), in order to compute \( \Phi(G, L, v, i, d) \) we first identify the neighbors \( v_1, \ldots, v_m \) of \( v \). Then we create graph/list pairs \( G_{v_k, L_j} \), \( 1 \leq k \leq m, j \in L(v) \), compute \( \Phi(\cdot, d - 1) \) for each of this graphs, and use this to compute \( \Phi(G, L, v, i, d) \). The overall computation effort is then

\[ T(d) = O(\|L\|\Delta T(d - 1)). \]

Iterating over \( d \) we obtain \( T(d) = O(\|L\|^{d+1}\Delta^d) = O(2^{(d+1)(\log \|L\| + \log \Delta)}) = 2^{O(d\log \|L\| + \Delta)} \). When \( d = O(\log n) \), we obtain a bound \( 2^{O(\log^2 n)} \). If in addition \( \|L\| = O(1) \), then the assumption \((2)\) implies \( \Delta = O(1) \), and then \( T(d) = n^{O(1)} \).

The following is then immediate.

**Corollary 1.** Suppose \( d = O(\log n) \). Then the complexity of the algorithm CountCOLOR is \( 2^{O(\log^2 n)} \). If in addition the size of the largest list \( \|L\| \) is constant, then CountCOLOR is a polynomial time algorithm.

5 Correlation decay

The following is the key correlation decay result.

**Theorem 2.** Consider a triangle-free graph/list pair \((G, L)\) satisfying \((2), (3)\). There exist constants \( 0 < \epsilon < 1 \) which depend only on \( \alpha \), such that for all nodes \( v \), colors \( i \in L(v) \) and \( d \geq 0 \)

\[ \max_{i \in L(v)} \left| \log \mathbb{P}_{G, L}(c(v) = i) - \log \Phi(G, L, v, i, d) \right| \leq O(n^2(1 - \epsilon)^d). \]  \((12)\)

This theorem is our key tool for using the values of \( \Phi \) for computing the marginals \( \mathbb{P}_{G, L}(c(v) = i) \). We first establish that this correlation decay result implies our main result, Theorem 1.

**Proof of Theorem 1.** We consider an arbitrary instance \((G, L)\) with size \( n \) and arbitrary \( \delta > 0 \). We may assume without the loss of generality that \( n \) is at least a large constant bigger than \( C/\delta \), for any universal constant \( C \), since we can simply extend the size of the instance by adding isolated nodes. The proof uses a standard idea of approximating marginals \( \mathbb{P}_{G, L}(c(v) = i) \) and then using Proposition 1.
for computing $Z(G, L)$. From Proposition 4 if the algorithm CountCOLOR produces in every stage $k = 1, 2, \ldots, |V| - 1$ a value $\hat{p}(v, i)$ which approximates $\mathbb{P}_{G,v} L_k (c(v_k) = i)$ with accuracy

$$1 - \frac{\delta}{n} \leq \frac{\hat{p}(v, i)}{\mathbb{P}_{G,v} L_k (c(v_k) = i)} \leq 1 + \frac{\delta}{n}$$

then the output $\hat{Z}$ of the algorithm satisfies

$$\left(1 - \frac{\delta}{n}\right)^n \leq \left(1 - \frac{\delta}{n}\right)^{|V|} \leq \frac{Z(G, L)}{\hat{Z}} \leq \left(1 + \frac{\delta}{n}\right)^{|V|} \leq \left(1 + \frac{\delta}{n}\right)^n$$

Since $|V| \leq n$ and $n$ is at least a large constant, we obtain an arbitrary accuracy of the approximation. Thus it suffices to arrange for (13). We run the algorithm CountCOLOR with recursion (4). As before, let $v, v, \ldots, v$ be the neighbors of $v$ in $G$, $m = \Delta(v)$. Observe that (12) holds trivially when $m = 0$, since both expression inside the absolute value become $1/|L(v)|$ and the left-hand side becomes equal to zero. Thus we assume that $m \geq 1$. Denote by $m_k$ the degree of $v_k$ in the graph $G$. In order to ease the notations, we introduce

$$x_i = \mathbb{P}_{G,v} L (c(v) = i), \quad i \in L(v),$$
$$x_{i,k} = \mathbb{P}_{G,v} L_k (c(v_k) = i), \quad i \in L(v), 1 \leq k \leq m,$$
$$x_i^* = \Phi(G, v, L, v, i, d), \quad i \in L(v),$$
$$x_{i,k}^* = \Phi(G, v, L_k, v_k, i, d - 1), \quad i \in L(v) \cap L_{i,k}(v_k), 1 \leq k \leq m$$

**Proposition 4.** There exists a constant $\epsilon > 0$ which depends on $\alpha$ only such that

$$\frac{1}{m} \max_{i \in L(v)} \left| \log(x_i) - \log(x_i^*) \right| \leq (1 - \epsilon) \max_{j \in L(v), k > 0 \cdot m_k > 0} \frac{1}{m_k} \left| \log(x_{j,k}) - \log(x_{j,k}^*) \right|$$

First we show how this result implies Theorem 2.
Proof of Theorem 3. Applying this proposition \( d \) times and using the fact that we are summing over \( k : m_k > 0 \), we obtain

\[
\frac{1}{m} \max_{i \in L(v)} \left[ \log(x_i) - \log(x_i^*) \right] \leq M(1 - \epsilon)^d,
\]

where

\[
M = \max_{l,s} \left| \log \mathbb{P}_{G_s,L_s}(c(v) = l) - \log \Phi(G_s,L_s,v,l,0) \right|
\]

and the maximum is over all graph/list pairs \( G_s, L_s \) appearing during the computation of \( \Phi \) and over all colors \( l \). Recall that if \( l \) does not belong to the list associated with node \( v \) and list vector \( L_s \), then \( \mathbb{P}_{G_s,L_s}(c(v) = l) = \Phi(G_s,L_s,v,l,0) = 0 \) (the first is equal to zero by definition, the second by the way we set the values of \( \Phi \)). Otherwise we have from Lemma 2 and part (11) of Lemma 4 that absolute value of the difference is at most

\[
\log q + \Delta \log(\beta/(\beta - 1)).
\]

Since \( m \leq \Delta \leq n \), \( \beta \) is a constant which only depends on \( \alpha \), and \( q \leq n \), then we obtain \( M = O(n) \) and \( m M = O(n^2) \).

Thus we focus on establishing Proposition 4.

Proof of Proposition 4. Observe that for every \( i \in L(v) \setminus L_{k,j}(v_k) \) we have \( x_{i,k} = x^*_{i,k} = 0 \). This is because the probability of node \( v_k \) obtaining color \( i \) is zero when this color is not in its list. Similarly, the corresponding value of \( \Phi \) is zero, since we set it to zero for all colors not in the list. For every \( i \in L(v) \) introduce

\[
A_i \triangleq \prod_{1 \leq k \leq m} (1 - x_{i,k}) \tag{15}
\]

and

\[
A \triangleq \sum_{j \in L(v)} A_j \tag{16}
\]

Introduce \( A^*_i, A^* \) similarly. Applying Proposition 2 we obtain

\[
x_i = \frac{A_i}{A}, \tag{17}
\]

\[
x^*_i = \min \left[ \frac{1}{2(1 + \epsilon_0)m}, \frac{1}{\beta} \frac{A^*_i}{A^*} \right]. \tag{18}
\]

Let

\[
\bar{x}^*_i = \frac{A^*_i}{A^*}.
\]

We claim that in order to establish \( 14 \) it suffices to establish the bound

\[
\frac{1}{m} \left| \log x_i - \log \bar{x}^*_i \right| \leq (1 - \epsilon) \max_{j \in L(v), k : m_k > 0} \frac{1}{m_k} \left| \log(x_{j,k}) - \log(x^*_{j,k}) \right|
\]

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Indeed, if \( \tilde{x}_i^* \neq x_i^* \), then \( x_i^* = \min\left[ \frac{1}{2(1+\epsilon_0)m}, \frac{1}{j_i} \right] \). On the other hand, by Lemmas 23 we have \( x_i \leq \min\left[ \frac{1}{2(1+\epsilon_0)m}, \frac{1}{j_i} \right] \), implying \( x_i \leq x_i^* \leq \tilde{x}_i^* \), and the bound for \( \tilde{x}_i^* \) implies the bound (19).

We have
\[
\max_{i \in L(v)} |\log(x_i) - \log(x_i^*)| = \max_{i \in L(v)} |\log A_i - \log A_i^* - \log A + \log A^*|.
\]

(19)

We introduce auxiliary variables \( y_i = \log(x_i), y_{i,k} = \log(x_{i,k}) \). Similarly, let \( y_i^* = \log(\tilde{x}_i^*), y_{i,k}^* = \log(x_{i,k}^*) \).

Define \( y = (y_{i,k}), y^* = (y_{i,k}^*) \). Observe that if \( m_k = 0 \) then for every color \( i \) \( x_{i,k} = x_{i,k}^* \). This follows since both values are \( 1/|L_{i,k}| \) when \( i \in L_{i,k} \) and zero otherwise. This implies \( y_{i,k} = y_{i,k}^* \). Then we rewrite (19) as
\[
\max_{i \in L(v)} |y_i - y_i^*| = \max_{i \in L(v)} \left| \sum_{k:m_k > 0} \log(1 - \exp(y_{i,k})) - \sum_{k:m_k > 0} \log(1 - \exp(y_{i,k}^*)) \right|
\]

\[
- \log \left( \prod_{j \in L(v), 1 \leq k \leq m} (1 - \exp(y_{j,k})) \right) + \log \left( \prod_{j \in L(v), 1 \leq k \leq m} (1 - \exp(y_{j,k}^*)) \right)
\]

(20)

where the sums \( \sum_{1 \leq k \leq m} \) were replaced by \( \sum_{k:m_k > 0} \) due to our observation \( y_{i,k} = y_{i,k}^* \) when \( m_k = 0 \).

For every \( i \) denote the expression inside the absolute value in the right-hand side of equation (20) by \( G_i(y) \). That is we treat \( y^* \) as constant and \( y \) as a variable. It suffices to prove that for each \( i \)
\[
G_i(y) \leq (1 - \epsilon) \max_{j \in L(v), k:m_k > 0} \frac{1}{m_k} |\log(x_{j,k}) - \log(x_{j,k}^*)|
\]

(21)

Observe that \( G_i(y^*) = 0 \). Let \( g_i(t) = G_i(y^* + t(y - y^*)), t \in [0,1] \). Then \( g_i \) is a differentiable function interpolating between 0 and \( G_i(y) \). In particular, \( g_i(1) = G_i(y) \). Applying the Mean Value Theorem we obtain
\[
|g_i(1) - g_i(0)| = |g_i(1)| \leq \sup_{0 \leq t \leq 1} |\dot{g}_i(t)|
\]

\[
= \sup_{0 \leq t \leq 1} \left| \nabla G_i(y^* + t(y - y^*))^T (y - y^*) \right|
\]

where the supremum is over values of \( t \). We use a short-hand notation
\[
\Pi_j = \prod_{1 \leq k \leq m} (1 - \exp(y_{j,k} + t(y_{j,k} - y_{j,k}^*)))
\]

For each \( t \) we have
\[
\nabla G_i(y^* + t(y - y^*)) = \sum_{k:m_k > 0} \frac{-\exp(y_{i,k} + t(y_{i,k} - y_{i,k}^*))}{1 - \exp(y_{i,k} + t(y_{i,k} - y_{i,k}^*))} (y_{i,k} - y_{i,k}^*)
\]

\[
+ \sum_{j \in L(v)} \sum_{1 \leq k \leq m} \frac{\exp(y_{j,k} + t(y_{j,k} - y_{j,k}^*))}{1 - \exp(y_{j,k} + t(y_{j,k} - y_{j,k}^*))} (y_{j,k} - y_{j,k}^*) \Pi_j
\]

Again using the fact \( y_{j,k} = y_{j,k}^* \) when \( m_k = 0 \), we can replace the sum \( \sum_{1 \leq k \leq m} \) by \( \sum_{k:m_k > 0} \) in the expression above. For each \( j \) we have from convexity of \( \exp \)
\[
\exp(y_{j,k} + t(y_{j,k} - y_{j,k}^*)) \leq (1 - t) \exp(y_{j,k}^*) + t \exp(y_{j,k})
\]

\[
= (1 - t)x_{j,k} + tx_{j,k}
\]

\[
\leq \frac{1}{2(1 + \epsilon_0)m_k}
\]

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where the last inequality follows from Lemma 3 and part (9) of Lemma 4. This bound is useful for terms with \( m_k > 0 \) (for this reason we only kept these terms in the sum \( \sum_{k:m_k>0} \)). Similarly using Lemma 2 and again part (9) of Lemma 4 we obtain

\[
\frac{1}{1 - \exp(y_{j,k} + t(y_{j,k} - y_{j,k}^*))} \leq \frac{1}{1 - (1 - t)\exp(y_{j,k}^*) - t\exp(y_{j,k})} = \frac{1}{1 - (1 - t)x_{j,k} - tx_{j,k}^*} \leq \frac{1}{1 - \beta}.
\]

We obtain

\[
\left| \nabla G_i(y^* + t(y - y^*))(y - y^*) \right| \leq \sum_{k:m_k>0} \frac{1}{(1 - \frac{1}{\beta})2(1 + \epsilon_0)m_k} |y_{i,k} - y_{i,k}^*| \leq \sum_{j \in L(v)} \sum_{k:m_k>0} \frac{1}{(1 - \frac{1}{\beta})2(1 + \epsilon_0)m_k} |y_{j,k} - y_{j,k}^*| |\Pi_j| \leq \frac{m}{(1 - \frac{1}{\beta})2(1 + \epsilon_0)} \max_{j \in L(v), k:m_k>0} \frac{|y_{j,k} - y_{j,k}^*|}{m_k}.
\]

Combining with (20) we conclude

\[
\max_{i \in L(v)} \frac{|y_i - y_i^*|}{m} \leq \frac{1}{(1 - \frac{1}{\beta})(1 + \epsilon_0)} \max_{j \in L(v), k:m_k>0} \frac{|y_{j,k} - y_{j,k}^*|}{m_k}.
\]

We now select a sufficiently large constant \( \beta = \beta(\epsilon_0) \) such that

\[
1 - \epsilon \triangleq \frac{1}{(1 - \frac{1}{\beta})(1 + \epsilon_0)} < 1.
\]

This completes the proof of Proposition 4.

6 Extensions: Markov random field and partition function

6.1 Model and the preliminary results

The main conceptual point of this paper, namely construction a recursion of the form (4), construction of a corresponding computation tree, establishing correlation decay property and application to a counting problem, can be extended to an arbitrary model of random constraint satisfaction problems with multiple values. In this section we provide details using a very general framework of Markov random fields (MRF), also known as graphical model [?], [?]. We show that generalizing (4) is straightforward. It is establishing the decay of correlation which presents the main technical difficulty. We provide a simple
and general sufficient condition and then illustrate the approach on specific statistical physics problem, namely q-state Potts model. Here we restrict ourselves for simplicity to MRF defined on simple graphs. Extensions to multi-graphs are possible as well.

A Markov random field (MRF) is given as a graph $G$ with node set $V = \{v_1, \ldots, v_{|V|}\}$, edge set $E$, an alphabet $\mathcal{X}$ and set of functions $\phi_v : \mathcal{X} \to \mathbb{R}_+$, $v \in V$, $f_{v,u} : \mathcal{X} \to \mathbb{R}_+$, $(v, u) \in E$. Consider a probability measure on $\mathcal{X}^{|V|}$ defined by

$$\mathbb{P}(X = x) = \frac{\prod_{v \in V} \phi_v(x_v) \prod_{(v,u) \in E} f_{v,u}(x_v, x_u)}{Z},$$

for every $x = (x_v) \in \mathcal{X}^{|V|}$, where $Z = \sum_x \prod_{v \in V} \phi_v(x_v) \prod_{(v,u) \in E} f_{v,u}(x_v, x_u)$ is the normalizing constant called the partition function. Here $X = (X_v)$ is the random vector selected according to this probability measure. In the case $Z = 0$, the MRF is not defined. From now on assume that $\prod_{v \in V} \phi_v(x_v) \prod_{(v,u) \in E} f_{v,u}(x_v, x_u) > 0$ for at least one $x = (x_v) \in \mathcal{X}^{|V|}$.

Let us see that the problem of list-coloring can be cast as a Markov random field, where $\mathbb{P}(\cdot)$ corresponds to the uniform probability distribution on the set of valid colorings. Given an instance of a list-coloring problem $(G, L)$ with a universe of colors $\{1, \ldots, q\}$, we set $\mathcal{X} = \{1, \ldots, q\}$, $\phi_v(i) = 1\{i \in L(v)\}$ for all node/color pairs $v, i$, and $f_{v,u}(i,j) = 1\{i \neq j\}$, where $1\{\cdot\}$ is the indicator function. It is not hard to see that $\mathbb{P}(x) = 1/Z$ if $x$ corresponds to a valid coloring and $= 0$ otherwise, and $Z = Z(G, L)$ is the total number of valid list-colorings. Thus this MRF corresponds to the uniform distribution on the set of proper colorings.

An instance of a MRF is denoted by $M = (G, \mathcal{X}, \phi, f)$, with $\phi = (\phi_v), f = (f_{v,u})$. We will write $\mathbb{P}_M$ and $Z_M$ for the corresponding probability measure and the partition function, respectively, in order to emphasize the dependence on the particular instance of the MRF. Computation of $Z_M$ is the principle goal of this section. As in the case of list-coloring model, denote by $Z_M[\chi]$ the sum of the terms in the partition function which satisfy some condition $\chi$.

Observe that if $\phi_v, f_{v,u} > 0$ for all nodes and edges than $\mathbb{P}_M(X = x) > 0$ for every $x = (x_v), v \in V$. Moreover, if $f_{v,u} = c$ for all edges for some constant $c$, then we obtain a product form solution

$$\mathbb{P}_M(X = x) = \prod_v \frac{\phi_v(x_v)}{\sum_{y \in \mathcal{X}} \phi_v(y)}.$$

Thus we might expect the correlation decay to take place when the values of $f_{v,u}$ are close to each other. This is the regime within which we will establish our results. Let $\phi_{\min} = \min_{v,x} \phi_v(x), \phi_{\max} = \max_{v,x} \phi_v(x)$ and $c_{\phi} = \phi_{\max}/\phi_{\min}$. Also let $f_{\min} = \min_{v,u} f_{v,u}(x,u), f_{\max} = \max_{v,u} f_{v,u}(x,u)$ and let $c_f = f_{\max}/f_{\min}$. From now on we assume that the following conditions hold

$$\phi_v(x) > 0, \ \forall v \in V, x \in \mathcal{X} \quad (22)$$

$$f_{v,u}(x,y) > 0, \ \forall (v,u) \in E, x, y \in \mathcal{X} \quad (23)$$

These conditions in particular ensure that $c_f < \infty$. The following condition will be used in lieu of (22)

$$\gamma \triangleq (c_f^\Delta - c_f^{-\Delta}) \Delta |X|^\Delta < 1. \quad (24)$$

The size of an instance $M$ is

$$n = \max \left( |V|, |E|, |\mathcal{X}|, |\log \phi_{\max}|, |\log \phi_{\min}|, |\log f_{\max}|, |\log f_{\min}| \right).$$

We now state the main result of this section.
Theorem 3. There exist a deterministic algorithm which provides an FPTAS for computing $Z_M$ for an arbitrary MRF instance $M$ satisfying (22), (23), (24), whenever $|X|$ and $\Delta$ are constants.

Our next task is obtaining a generalization of the cavity recursion given by Proposition 4. Given a MRF $M = (G, X, \phi, f)$, an arbitrary node $v$ and an arbitrary element $x^* \in X$ we consider a new MRF instance $T_{v, x^*}[M] = (\tilde{G}, \tilde{X}, \tilde{\phi}, \tilde{f})$ defined as follows. The graph $\tilde{G}$ is the subgraph of $G$ induced by all nodes other than $v$. $\tilde{X} = X$ and $\tilde{f} = f$. $\tilde{\phi}$ is defined as follows. For every $u$ which is a neighbor of $v$, $\tilde{\phi}_u(x) = \phi_v(x) f_{v,u}(x^*, x)$, where $\Delta(v)$ is the degree of $v$ in $G$. For all the remaining nodes $u$ we set $\tilde{\phi}_u = \phi_u$.

Given a MRF $M = (G, X, \phi, f)$ let $v_1, \ldots, v_{|V|}$ be an arbitrary enumeration of nodes. Consider an arbitrary $x^* = (x^*_1, \ldots, x^*_r)$ such that $P(x^*) > 0$. Define $M_0 = M$ and $M_k = T_{v_k, x^*_k}[M_{k-1}]$, $k = 1, 2, \ldots, |V|$, where $M_0$ is an empty MRF and its partition function is set by default to unity.

Proposition 5. The following identity holds.

$$Z_M = \prod_{1 \leq k \leq |V|} P_{M_{k-1}}(X_{v_k} = x^*_{v_k}).$$

Proof. We have

$$P_M(X_{v_1} = x^*_{v_1}) = \frac{\sum_{x \in X^{|V|}} x_{v_1} = x^*_1 \phi_{v_1}(x^*_1) \prod_{u \neq v_1, u \in E} f_{v_1,u}(x^*_1, x_u) \prod_{u \neq v_1} \phi(x_u) \prod_{(v,u) \in E, v,u \neq v_1} f_{v,u}(x, x_u)}{Z_M} \frac{Z_{M_1}}{Z_M} = \frac{Z_{M_1}}{Z_M},$$

where the second equality follows since $\phi(x^*_1) \prod_{u \neq v_1, u \in E} f(x^*_1, x_u) = \prod_u \phi^I(x_u)$ and the second product is over neighbors $u$ of $v_1$ in $G$. Iterating further for $k \geq 2$ we obtain the result. \hfill \square

The identity in Proposition 5 provides an important representation of the partition function in terms of marginal probabilities. Thus, if we compute (approximately) these marginal probabilities, we can use them to obtain the value of the underlying partition function.

6.2 Basic recursion and the algorithm

Our next task is constructing a generalization of $(G_v, L_{k,i})$ and extending Proposition 2 to MRF. Given a MRF $M = (G, X, \phi, f)$ and a node $v$ let $M_v$ denote the MRF instance obtained naturally by removing node $v$. Namely, we keep $\phi_u$ and $f_{u,w}$ intact for all the nodes $u \neq v$ and edges $(u, w), u, w \neq v$. Also, given a MRF $M = (G, X, \phi, f)$, a set of nodes $v_1, \ldots, v_r \subset V$ and a set of elements $x_1, \ldots, x_r \in X$ we construct a MRF denoted by $M[v_1, x_1, \ldots; v_r, x_r] = (\tilde{G}, \tilde{\phi}, \tilde{f})$ as follows. The corresponding graph $\tilde{G}$ is the subgraph induced by nodes $V \setminus \{v_1, \ldots, v_r\}$. For every node $u \in \tilde{G}$ which has at least one neighbor among $v_1, \ldots, v_r$ we set $\tilde{\phi}_u(x) = \prod_i f_{u, v_i}(x, x_v) \phi_u(x)$, where the product is over $i = 1, 2, \ldots, r$ such that $(v_i, u)$ is an edge in $G$. For all the remaining $u$ we set $\tilde{\phi}_u = \phi_u$. We also set $\tilde{f} = f$. The interpretation for $M[v_1, x_1, \ldots; v_r, x_r]$ comes from the following simple fact.

Lemma 5. For every event $E$ corresponding to the probability measure $P_M$, the following holds

$$P_M(E \cap k \leq r X_{v_k} = x_k) = P_{M[v_1, x_1; \ldots; v_r, x_r]}(E).$$
Proposition 6. of an algorithm. observing that the terms \( \phi_{v_k}(x_k) \) cancel in the ratio \( \mathbb{P}_M(\mathcal{E} \land \land_{k \leq r} X_{v_k} = x_k) / \mathbb{P}_M(\land_{k \leq r} X_{v_k} = x_k). \) □

Observe that the value of \( c_f \) corresponding to the MRF \( M[v_1, x_1; \ldots; v_r, x_r] \) is the same of \( M. \) Thus, should \( M \) satisfy conditions (22), (23), (24), so does the instance \( M[v_1, x_1; \ldots; v_r, x_r] \). Moreover, \( c_{\phi} \) defined for this MRF satisfies

\[
c_{\phi} \leq c_f c_{\Delta}^\lambda (25)
\]

Now we obtain a recursion which serves as a basis for our correlation decay analysis and construction of an algorithm.

**Proposition 6.** For every node \( v \) and its neighbors \( v_1, \ldots, v_m \), the following identity holds for every \( x_0 \in \mathcal{X} \):

\[
\mathbb{P}_M(X_v = x_0) = \frac{\phi_v(x_0) \sum_{x_1, \ldots, x_m \in \mathcal{X}} \prod_{k=1}^{m} f_{v,v_k}(x_0, x_k) \mathbb{P}_M[v_1, x_1; \ldots; v_{k-1}, x_{k-1}] (X_{v_k} = x_k)}{\sum_{x \in \mathcal{X}} \phi_v(x) \sum_{x_1, \ldots, x_m \in \mathcal{X}} \prod_{k=1}^{m} f_{v,v_k}(x, x_k) \mathbb{P}_M[v_1, x_1; \ldots; v_{k-1}, x_{k-1}] (X_{v_k} = x_k)},
\]

where the sum \( \sum_{x_1, \ldots, x_m \in \mathcal{X}} = 1 \) when \( m = 0 \).

**Proof.** The case \( m = 0 \) is immediate. Assume \( m > 0 \). For every \( x_0 \in \mathcal{X} \) we have the following identity

\[
\mathbb{P}_M(X_v = x_0) = \frac{\phi_v(x_0) \sum_{x_1, \ldots, x_m \in \mathcal{X}} Z_{M_v}[X_{v_1} = x_1, \ldots, X_{v_m} = x_m] \prod_{k=1}^{m} f_{v,v_k}(x_0, x_k)}{\sum_{x \in \mathcal{X}} \phi_v(x) \sum_{x_1, \ldots, x_m \in \mathcal{X}} Z_{M_v}[X_{v_1} = x_1, \ldots, X_{v_m} = x_m] \prod_{k=1}^{m} f_{v,v_k}(x, x_k)},
\]

We divide both parts by \( Z_{M_v} \) and write

\[
\frac{Z_{M_v}[X_{v_1} = x_1, \ldots, X_{v_m} = x_m]}{Z_{M_v}} = \prod_{k=1}^{m} \frac{Z_{M_v}[X_{v_1} = x_1, \ldots, X_{v_k} = x_k]}{Z_{M_v}[X_{v_1} = x_1, \ldots, X_{v_{k-1}} = x_{k-1}]},
\]

where the term corresponding to \( k = 0 \) is identified with \( Z_{M_v} \). Applying Lemma 5 we recognize the \( k \)-th term in this product as \( \mathbb{P}_{M[v_1, x_1; \ldots; v_{k-1}, x_{k-1}]}(X_{v_k} = x_k) \) (note that the terms \( \phi_{v_j}(x_j), j \leq k - 1 \) cancel out). □

Proposition 6 also allows us to obtain upper and lower bounds on the marginal probabilities:

**Lemma 6.** For every node \( v \) and \( x_0 \in \mathcal{X} \)

\[
c_f^{-\Delta} \frac{\phi_v(x_0)}{\sum_{x} \phi_v(x)} \leq \mathbb{P}_M(X_v = x_0) \leq c_f^\Delta \frac{\phi_v(x_0)}{\sum_{x} \phi_v(x)}.
\]

**Proof.** The proof follows from Proposition 6. We have for every \( x \in \mathcal{X} \), node \( v \) and its neighbors \( v_1, \ldots, v_m \) that \( f_{v,v_k}^m \leq \prod_{k=1}^{m} f_{v,v_k}(x, x_k) \leq c_f^m f_{v,v_k}^m. \) Applying this bound to the numerator of (26) for \( x = x_0 \) we obtain the required upper bound. Applying the same to the denominator, we obtain the required lower bound. □

We now provide sufficient conditions under which the construction of a computation tree for computing approximately marginal probabilities \( \mathbb{P}_M(X_v = x) \) as well as the partition function \( Z_M \) can be performed in polynomial time.

Similarly to the problem of coloring, we introduce \( \Phi(\cdot) \) – a surrogate for computing the marginal probabilities \( \mathbb{P}_M(\cdot) \). Consider a function \( \Phi_M(v, x, d) \) defined recursively for an arbitrary instance of a MRF \( M = (G, \mathcal{X}, \phi, f) \), arbitrary node \( v \), element \( x \in \mathcal{X} \) and a non-negative integer \( d \) as follows.
• We set $\Phi_M(v, x, 0) = 1$. As in the case of coloring, it turns out that the initialization values are not particularly important, due to the decay of correlations.

• For every node $v$ with neighbors $v_1, \ldots, v_m$, every $x_0 \in \mathcal{X}$ and $d \geq 1$

$$
\Phi_M(v, x_0, d) = \frac{\phi_v(x_0) \sum_{x_1, \ldots, x_m \in \mathcal{X}} \prod_{k=1}^{m} \Phi_M[v_1, x_1, \ldots, v_{k-1}, x_{k-1}] (v_k, x_k, d - 1) f_{v, v_k}(x_0, x_k)}{\sum_{x \in \mathcal{X}} \phi_v(x) \sum_{x_1, \ldots, x_m \in \mathcal{X}} \prod_{k=1}^{m} \Phi_M[v_1, x_1, \ldots, v_{k-1}, x_{k-1}] (v_k, x_k, d - 1) f_{v, v_k}(x, x_k)}, \tag{27}
$$

where the sum $\sum_{x_1, \ldots, x_m \in \mathcal{X}} = 1$ when $m = 0$.

Assumptions (22) and (23) guarantee that $\Phi > 0$.

We now describe our algorithm for approximately computing $Z_M$. The algorithm is parametrized by $d$. It is based on computing recursively the values of $\Phi_M$.

**Algorithm ComputeZ**

**INPUT:** A MRF instance $M = (G, \mathcal{X}, \phi, f)$ and a positive integer $d$.

**BEGIN**

Set $\hat{Z} = 1, \hat{M} = M$.

While $\hat{G} \neq \emptyset$, fix an arbitrary node $v \in \hat{G}$ and element $x \in \mathcal{X}$. Compute $\Phi_M^v(v, x, d)$.

Set $\hat{Z} = \Phi_M^{-1}(v, x, d) \hat{Z}$.

Set $\hat{M} = T_{v, x}[\hat{M}]$, where the operator $T$ was defined before Proposition 5.

**END**

**OUTPUT:** $\hat{Z}$.

6.3 Complexity

We begin by analyzing the complexity of computing function $\Phi$.

**Proposition 7.** For every $v \in V, x \in \mathcal{X}$, the function $\Phi_M(v, x, d)$ can be computed in time $O(2^{d\Delta \log |\mathcal{X}|}n^2)$.

In particular when $d = O(\log n)$, and $|\mathcal{X}| = O(1)$, the computation is polynomial in $n$.

We note that the dependence on $\Delta$ is not as nice as in the case of the list-coloring problem, as it appears as $\Delta$ not $\log \Delta$ in the exponent. Thus we can no longer claim that the computation time is $2^{O(\log^2 n)}$ in this case.

**Proof.** Let $T(d)$ denote the complexity of computing function $\Phi(\cdot, d)$. Clearly, $T(0) = O(1)$. We now express $T(d)$ in terms of $T(d - 1)$. Given a node $v$, in order to compute $\Phi_M(v, x, d)$ we identify the neighbors $v_1, \ldots, v_m$ of $v$. For every sequence $x_1, \ldots, x_m \in \mathcal{X}$ and every $k = 1, 2, \ldots, m$ we compute $\Phi_M[v_1, x_1, \ldots, v_{k-1}, x_{k-1}] (v_k, x_k, d - 1)$. The computation of each such quantity is $T(d - 1)$. We use the obtained values to compute $\Phi_M(v, x, d)$ via (27). We also need $O(n^2)$ time to "take care" of multiplying by $f_{v, v_k}$ and by $\phi_v$. The overall computation effort then satisfies

$$
T(d) = O(|\mathcal{X}|^{d}T(d - 1) + n^2).
$$

Iterating over $d$ we obtain $T(d) = O(|\mathcal{X}|^{d\Delta}n^2) = O(2^{d\Delta \log |\mathcal{X}|}n^2)$, and the first part is established. When $d = O(\log n)$ and $\Delta, |\mathcal{X}|$ are constants, we obtain $T(d) = n^{O(1)}$. \qed

The following is then immediate.

**Corollary 2.** Suppose $d = O(\log n)$ and $|\mathcal{X}|, \Delta = O(1)$. Then ComputeZ is a polynomial time algorithm.
6.4 Correlation decay analysis

We now establish a correlation decay result which is a key to proving our main result, Theorem 3.

**Theorem 4.** Given an arbitrary MRF satisfying conditions (22), (23), (24), the following holds for every node \( v \) and \( d \geq 1 \)

\[
\max_{x \in \mathcal{X}} \left| \log P_M(X_v = x) - \log \Phi_M(v, x, d) \right| \\
\leq (1 - \gamma) \max_{1 \leq k \leq m, y \in \mathcal{X}} \left| P_M[v_1, x_1; \ldots; v_{k-1}, x_{k-1}](X_{v_k} = y) - \Phi_M[v_1, x_1; \ldots; v_{k-1}, x_{k-1}](v_k, y, d - 1) \right| 
\]

(28)

where \( v_1, \ldots, v_m \) are the neighbors of \( v \).

We first show how this theorem implies our main algorithmic result.

**Proof of Theorem 3.** We claim that ComputeZ provides FPTAS for computing partition function \( Z_M \) when \( d = O(\log n) \) under the setting of Theorem 3. We have already established in Corollary 2 that the algorithm is polynomial time.

Consider any MRF instance \( \tilde{M} \) obtained during the computation of \( \Phi_M(\cdot) \) as a part of performing algorithm ComputeZ. Applying (25) and Lemma 6, we obtain that for every node \( v \) in \( \tilde{M} \) and every \( x \in \mathcal{X} \)

\[
P_{\tilde{M}}(X_v = x) \geq c_f^\Delta \frac{1}{|\mathcal{X}|^c_\phi} \geq c_f^\Delta \frac{1}{|\mathcal{X}|} c_f^{-d} c_\phi^{-1} = c_f^d \frac{1}{|\mathcal{X}|} c_\phi^{-1}.
\]

Then applying the result of Theorem 4 \( d \) times and recalling \( \Phi_M(v, x, 0) = 1 \), we obtain for \( d = O(\log n) \)

\[
\left| \log P_M(X_v = x) - \log \Phi_M(v, x, d) \right| \leq (1 - \gamma)^d \left( \Delta(d + 1) + \log c_f + \log |\mathcal{X}| + \log c_\phi \right) \\
= (1 - \gamma)^d O(d \log n) \\
= \frac{1}{n^O(1)} O(n \log^2 n) \\
= \frac{1}{n^O(1)}
\]

where the last step is obtained by selecting \( d = C \log n \) for sufficiently large constant \( C \).

We conclude that \( \Phi_M(v, x, d) \) provides an approximation of marginal probability \( P_{\tilde{M}}(X_v = x) \) with an inverse polynomial error. The remainder of the proof is the same as for Theorem 2. \( \square \)

**Proof of Theorem 4.** Fix a node \( v \) and an element \( x_0 \in \mathcal{X} \). Let \( v_1, \ldots, v_m \) be neighbors of \( v \). When \( m = 0 \) we let the left-hand side of (25) is zero. Thus assume \( m > 0 \). In order to ease the exposition we introduce some notations. Set \( z = \log P_{\tilde{M}}(X_v = x_0), z_{x_1}, \ldots, z_{x_k} = \log P_{\tilde{M}[v_1, x_1; \ldots; v_{k-1}, x_{k-1}]}(X_{v_k} = x_k) \).

Similarly \( \tilde{z} = \log \Phi_M(v, x_0, d), \tilde{z}_{x_1}, \ldots, \tilde{z}_{x_k} = \log \Phi_M[v_1, x_1; \ldots; v_{k-1}, x_{k-1}](v_k, x_k, d - 1) \).

Also let \( z \) denote the vector \( (z_{x_1}, \ldots, z_{x_k}), 1 \leq k \leq m, x_1, \ldots, x_m \in \mathcal{X} \) and \( \tilde{z} \) denote the vector \( (\tilde{z}_{x_1}, \ldots, \tilde{z}_{x_k}), 1 \leq k \leq m, x_1, \ldots, x_m \in \mathcal{X} \). Both vectors have dimension \( \sum_{1 \leq k \leq m} |\mathcal{X}|^m \). Then we can rewrite (20) as

\[
z = \log \frac{\phi_v(x_0) \sum_{x_1, \ldots, x_m \in \mathcal{X}} f_{v, x_k}(x_0, x_k) \exp(z_{x_1}, \ldots, z_{x_k})}{\sum_{x \in \mathcal{X}} \phi_v(x) \sum_{x_1, \ldots, x_m \in \mathcal{X}} f_{v, x_k}(x, x_k) \exp(z_{x_1}, \ldots, z_{x_k})},
\]

(29)
and rewrite (27) as
\[
\tilde{z} = \log \sum_{x \in \mathcal{X}} \frac{\sum_{x_{1}, \ldots, x_{m} \in \mathcal{X}} \prod_{k=1}^{m} f_{v, v_k}(x_{0}, x_{k}) \exp (\tilde{z}_{x_{1}, \ldots, x_{k}}) }{\sum_{x \in \mathcal{X}} \sum_{x_{1}, \ldots, x_{m} \in \mathcal{X}} \prod_{k=1}^{m} f_{v, v_k}(x, x_{k}) \exp (\tilde{z}_{x_{1}, \ldots, x_{k}})},
\]
(30)

Introduce a function \( \mathcal{G} \) defined on a vector \( w = (w_{x_{1}, \ldots, x_{k}}), 1 \leq k \leq m, x_{1}, \ldots, x_{m} \in \mathcal{X} \) with the same dimension \( \sum_{1 \leq k \leq m} |\mathcal{X}|^{m} \) as follows:
\[
\mathcal{G}(w) = \log \sum_{x \in \mathcal{X}} \frac{\sum_{x_{1}, \ldots, x_{m} \in \mathcal{X}} \prod_{k=1}^{m} f_{v, v_k}(x_{0}, x_{k}) \exp (w_{x_{1}, \ldots, x_{k}}) }{\sum_{x \in \mathcal{X}} \sum_{x_{1}, \ldots, x_{m} \in \mathcal{X}} \prod_{k=1}^{m} f_{v, v_k}(x, x_{k}) \exp (w_{x_{1}, \ldots, x_{k}})},
\]
(31)

which we rewrite as
\[
\log \phi_{v}(x_{0}) + \log \mathcal{G}_{1}(w) - \log \mathcal{G}_{2}(w)
\]
where the definition of \( \mathcal{G}_{1} \) and \( \mathcal{G}_{2} \) is immediate.
\[
\mathcal{G}_{1}(w) = \sum_{x \in \mathcal{X}} \phi_{v}(x) \sum_{x_{1}, \ldots, x_{m} \in \mathcal{X}} \prod_{k=1}^{m} f_{v, v_k}(x_{0}, x_{k}) \exp (w_{x_{1}, \ldots, x_{k}})
\]
\[
\mathcal{G}_{2}(w) = \sum_{x \in \mathcal{X}} \sum_{x_{1}, \ldots, x_{m} \in \mathcal{X}} \prod_{k=1}^{m} f_{v, v_k}(x, x_{k}) \exp (w_{x_{1}, \ldots, x_{k}})
\]

We have \( z - \tilde{z} = \mathcal{G}(z) - \mathcal{G}(\tilde{z}) \). Thus establishing (28) reduces to showing
\[
|\mathcal{G}(z) - \mathcal{G}(\tilde{z})| \leq (1 - \gamma) \| z - \tilde{z} \|_{L_{\infty}}.
\]

Applying Mean Value Theorem, there exists \( t \in [0, 1] \) such that
\[
z - \tilde{z} = \nabla \mathcal{G}(tz + (1 - t)\tilde{z})^{T}(z - \tilde{z})
\]
further implying
\[
|z - \tilde{z}| \leq \| \nabla \mathcal{G}(tz + (1 - t)\tilde{z}) \|_{L_{1}} \| z - \tilde{z} \|_{L_{\infty}}.
\]

It then suffices to establish
\[
\| \nabla \mathcal{G}(tz + (1 - t)\tilde{z}) \|_{L_{1}} \leq 1 - \gamma.
\]

In the following lemma we show that this bound holds for an arbitrary input vector \( w \) and thus complete the proof of Theorem 4.

**Lemma 7.** For every vector \( w \)
\[
\| \nabla \mathcal{G}(w) \|_{L_{1}} \leq 1 - \gamma.
\]

**Proof.** Fix an arbitrary sequence \( x_{1}^{0}, \ldots, x_{k_{0}}^{0} \in \mathcal{X} \) and the corresponding variable \( w_{x_{1}^{0}, \ldots, x_{k_{0}}^{0}} \). We have
\[
\frac{\partial \mathcal{G}}{\partial w_{x_{1}^{0}, \ldots, x_{k_{0}}^{0}}} = \mathcal{G}_{1}^{-1} \frac{\partial \mathcal{G}_{1}}{\partial w_{x_{1}^{0}, \ldots, x_{k_{0}}^{0}}} - \mathcal{G}_{2}^{-1} \frac{\partial \mathcal{G}_{2}}{\partial w_{x_{1}^{0}, \ldots, x_{k_{0}}^{0}}}
\]
We have
\[ G_1^{-1} \frac{\partial G_1}{\partial w_{x_1,\ldots,x_{k_0}}} = \frac{\left( \prod_{k=1}^{k_0} f_{v,v_k} (x_0, x_k^0) \exp(w_{x_1,\ldots,x_{k_0}}) \right) \sum_{x_{k+1},\ldots,x_m} f_{v,v_k} (x_0, x_k) \exp(w_{x_1,\ldots,x_{k_0+1},\ldots,x_{k}}) \sum_{x_1,\ldots,x_m} \prod_{k=1}^{m} f_{v,v_k} (x_0, x_k) \exp(w_{x_1,\ldots,x_{k}})}{\sum_{x_1,\ldots,x_m} \prod_{k=1}^{m} f_{v,v_k} (x_0, x_k) \exp(w_{x_1,\ldots,x_{k}})} \]

Using \( f_{\min} \leq f_{v,v_k} (x_0, x_k^0) \leq c_f f_{\min} \), we obtain
\[ G_1^{-1} \frac{\partial G_1}{\partial w_{x_1,\ldots,x_{k}}} \leq c_f^m \leq c_f. \]

Similarly, we obtain
\[ G_1^{-1} \frac{\partial G_1}{\partial w_{x_1,\ldots,x_{k}}} \geq c_f^{-\Delta}. \]

Using again \( f_{\min} \leq f_{v,u} (x, y) \leq c_f f_{\min} \) we also obtain
\[ G_2^{-1} \frac{\partial G_2}{\partial w_{x_1,\ldots,x_{k_0}}} = \frac{\sum_{x \in X} \phi_v (x) \left( \prod_{k=1}^{k_0} f_{v,v_k} (x, x_k^0) \exp(w_{x_1,\ldots,x_k}) \right) \sum_{x_{k+1},\ldots,x_m} f_{v,v_k} (x, x_k) \exp(w_{x_1,\ldots,x_{k_0+1},\ldots,x_{k}}) \sum_{x_1,\ldots,x_m} \prod_{k=1}^{m} f_{v,v_k} (x, x_k) \exp(w_{x_1,\ldots,x_{k}})}{\sum_{x_1,\ldots,x_m} \prod_{k=1}^{m} f_{v,v_k} (x, x_k) \exp(w_{x_1,\ldots,x_{k}})} \leq c_f^m \leq c_f. \]

Similarly,
\[ G_2^{-1} \frac{\partial G_2}{\partial w_{x_1,\ldots,x_{k_0}}} \geq c_f^{-\Delta}. \]

Since the dimension of the argument \( w \) is \( \sum_{1 \leq k \leq m} |X|^m < \Delta \|X\|^{\Delta} \), then we conclude
\[ \|\nabla G(w)\|_{L_1} \leq (c_f^\Delta - c_f^{-\Delta}) \Delta |X|^\Delta \leq 1 - \gamma. \]

This concludes the proof. \( \square \)
6.5 Example: Potts model

One of the most widely studied objects in the statistical physics is $q$-state Potts model. It is described in the terminology of MRF as follows. Given a graph $G$ we set $\phi_v = 1$ for all nodes $v$. $X = \{1, 2, \ldots, q\}$. A parameter $\beta$ called inverse temperature is fixed. The coupling functions $f$ are set as $f_{u,v}(x, y) = \exp(\beta \{x = y\})$ for all nodes $u,v$ and all elements $x,y \in X$. The case $\beta > 0$ corresponds to the ferromagnetic Potts model. In this case the distribution $P_M(\cdot)$ "favors" assignments which select the same element along the edges. The case $\beta < 0$ corresponds to the anti-ferromagnetic Potts model, and in this case the distribution favors assignments with different elements along the edges. The extreme case $\beta = -\infty$ corresponds to the usual coloring problem, where monochromatic coloring are simply forbidden. The special case $q = 2$ is called Ising model - one of the cornerstone models of the statistical physics.

It is immediate that conditions (22) and (23) are satisfied by this model provided $|\beta| < \infty$. Thus an immediate corollary of Theorem 3 is the following algorithmic result.

**Corollary 3.** There exists a deterministic FPTAS for computing a partition function for a family of Potts model $(G, q, \beta)$ with constant constant degree $\Delta$, constant number of colors $q$, and satisfying
\[
(e^{\beta \Delta} - e^{-\beta \Delta})\Delta q^\Delta < 1.
\]

Observe that for large $\Delta$, the largest inverse temperature $\beta$ satisfying this condition behaves like $O\left(\frac{1}{\Delta q^\Delta}\right)$. We believe that this is an overly conservative estimate. We conjecture that in fact the correlation decay property can be established in the regime
\[
\beta = O\left(\frac{1}{\Delta}\right),
\]
leading to a deterministic FPTAS.

7 Comparison of the correlation decay on a computation tree and the spatial correlation decay property

As we have mentioned above, the (spatial) correlation decay is known to hold for the coloring problem in a stronger regime $\alpha > \alpha^* \approx 1.763 \ldots$, then the regime $\alpha > \alpha^{**}$ considered in this paper [7]. This decay of correlation is established in a conventional sense: for every node $v$ the marginal probability $P(c(v) = i)$ is asymptotically independent from changing a color on a boundary of the depth-$d$ neighborhood $B(v,d)$ of $v$ in the underlying graph. In fact it is established that the decay of correlation is exponential in $d$. It is natural to try to use this result directly as a method for computing approximately the marginals $P(c(v) = i)$, for example by computing the marginal $P_{B(v,d)}(c(v) = i)$ corresponding to the neighborhood $B(v,d)$, say using brute force computation. Unfortunately, this conventional correlation decay result is not useful because of the computation growth. In order to obtain $\epsilon$-approximation of the partition function, we need order $O(\epsilon/n)$ approximation of the marginals, which means the depth $d$ of the neighborhood $B(v,d)$ needs to be at least $O(\log n)$. Here $n$ is the number of nodes. But the resulting cardinality of $B(v,d)$, even for the case of constant degree graphs is $O(\Delta \log n) = n^{O(1)}$ - polynomial in $n$ and the brute-force computation effort would be exponential in $n$. Notice that even if the underlying graph has a polynomial expansion $|B(v,d)| \leq d^r$, for some power $r \geq 1$, the brute-force computation would still be $O(\exp(\log^r n))$ which is super-polynomial. This is where having correlation decay on computation tree as opposed to the conventional graph theoretic sense helps.
8 Conclusions

We have established the existence of a deterministic approximation algorithm for counting the number of list colorings for certain classes of graphs. We have further extended our approach to constructing deterministic approximation algorithm for computing a partition function of a Markov random field satisfying certain conditions. Along with [?] and [?] this work is another step in the direction of developing a new powerful method for solving counting problems using insights from statistical physics. This method provides an important alternative to the existing MCMC sampling based method as it leads to a deterministic as opposed to a randomized algorithm. Since the conference version of the paper [?] appeared, several new developments happened in this direction. A deterministic approximation algorithm for counting the number of partial matchings in constant degree graphs was constructed in Bayati et al. [?]. The result was further used by Gamarnik and Katz [?] for constructing a deterministic subexponential algorithm for computing a permanent of an arbitrary 0,1 matrix. Recently Nair and Tetali [?] introduced a somewhat different way of constructing a computation tree, closer to the original self-avoiding walk based construction of Weitz [?]. Furthermore, they established that a strong form of correlation decay (called very strong spatial mixing in the paper) implies correlation decay on the computation tree and ultimately leads to a polynomial time algorithm for computing a partition function of a MRF. Their setting also allows for a hypergraph structure. It would be interesting to use their result to perhaps tighten the condition (2) used in the present paper.

The principle insight from this work, along with the work of Weitz [?] is the advantage of establishing the correlation decay property on the computation tree as opposed to the original graph theoretic structure. While we have established such correlation decay only in the regime \( \alpha > 2.8432... \), we conjecture that it holds for much lower values of \( \alpha \). In fact, just as it is conjectured that the Markov chain is rapidly mixing in the regime \( q \geq \Delta + 2 \), we conjecture that the correlation decay on the computation tree holds in this regime as well, at least for the case of constant number of colors \( q \). Finally, we conjecture that the polynomial time algorithms for computing partition function of a MRF can be constructed under weaker a assumption than (24). Specifically, we conjecture that for the case of Potts model, the critical inverse temperature \( \beta^* \) under which the correlation decay can be established on a computation tree behaves like (32).

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