MIXED HEGSELMANN-KRAUSE DYNAMICS

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Abstract. For typical Hegselmann Krause model, each agent updates his opinion via taking average over his and his neighbors’ opinions. Mixed Hegselmann Krause model covers both synchronous and asynchronous Hegselmann Krause models. Agents can decide the degree to play stubborn or open minded at each time step. Agents with the same opinion may depart the next time step. δ-equilibrium may not exist or be achieved in finite time. The study is to discuss under what circumstances agents reach consensus or their opinions are asymptotically stable.

1. Introduction

When it comes to discussion, there’s no lack of opinions. Hegselmann Krause model, briefly say HK model, is well-known to analyze the interactions of agents with their opinions. For a typical HK model, there are \( n \) agents and each agent updates his opinion via taking average over his and his neighbors’ opinions. The model is as follows: 

\[
x_i(t) = \frac{\sum_{j \in N_i(t)} x_j(t)}{|N_i(t)|},
\]

where \( x_i(t) \in \mathbb{R}^d \) is the opinion of agent \( i \) at time \( t \in \mathbb{N} \) and \( N_i(t) = \{ j \in [n] \mid \|x_i(t) - x_j(t)\| \leq \epsilon \} \) for \( [n] = \{1, 2, ..., n\} \). \( \epsilon > 0 \) is the confidence bound. In [1], an one dimensional modified HK model was introduced as follows: 

\[
x_i(t) = \alpha_i x_i(t) + (1 - \alpha_i) \frac{\sum_{j \in N_i(t)} x_j(t)}{|N_i(t)|}
\]

where \( x_i(t), \alpha_i \in [0, 1] \). Namely, agent \( i \)'s opinion at time \( t + 1 \) is between his opinion and the average of his and his neighbors’ opinions at time \( t \). The more stubborn agent \( i \) is, the larger \( \alpha_i \) is. In this say, we extend the modified HK model to higher dimensions and \( \alpha_i \) is replaced with \( \alpha_i(t) \). That is, agent \( i \) changes his opinion weight by time. The model we are going to study in matrix form is

\[
x(t + 1) = \text{diag}(\alpha(t)) x(t) + (I - \text{diag}(\alpha(t))) A(t) x(t)
\]

where \( A(t) \in \mathbb{R}^{n \times n} \) is row stochastic with \( A_{ij} = \frac{1}{|N_i(t)|} \mathbb{1}_{\{ j \in N_i(t) \}} \) and \( \alpha(t) = (\alpha_1(t), ..., \alpha_n(t))' \), the transpose of \( (\alpha_1(t), ..., \alpha_n(t)) \). Observe that it’s a mixed model covering both synchronous and asynchronous HK models. \( \alpha(t) \) differs by time implies agents can change their strategies by time, deciding the degree of playing stubborn or open minded. For instance, agent \( i \) could be stubborn initially and later on become open minded, which leads to decrease of \( \alpha_i(t) \) by time \( t \). Our aim is to study what strategies the agents play so that eventually consensus reached or their opinions are asymptotically stable. There are some properties in (1) different from synchronous HK models. Before we illustrate, some definitions are introduced.

Definition 1. An opinion profile at time \( t \) is an undirected graph \( \mathcal{G}(t) = (\mathcal{V}(t), \mathcal{E}(t)) \) with vertex set \( \mathcal{V}(t) = [n] \) and edge set \( \mathcal{E}(t) = \{ij : i \neq j; \|x_i(t) - x_j(t)\| \leq \epsilon\} \).

Apart from [2], the opinion profile here is simple.

Definition 2. The termination time of \( n \) agents \( T_n \) is the maximum number of iterations in (1) by reaching steady state over all initial profiles.

\( i.e. T_n = \inf \{ t : x(t) = x(s) \text{ for all } s \geq t \} \).

Definition 3. A convex set generated by \( v_1, ..., v_n \) in \( \mathbb{R}^d \), denotes \( C(\{v_1, ..., v_n\}) \), is the smallest convex set containing \( v_1, ..., v_n \). i.e.

\[
C(\{v_1, ..., v_n\}) = \{v : v = \sum_{i=1}^{n} \lambda_i v_i; (\lambda_i)_{i=1}^{n} \text{ is stochastic}\}
\]

Definition 4. A graph \( \mathcal{G}(t) \) is \( \epsilon \)-trivial if any two vertices are of distance no more than \( \epsilon \). i.e. \( \mathcal{G}(t) \) is complete.
**Definition 5.** \( x(t) \) in (1) is \( \delta \)-equilibrium if there is a partition \( \{ G_1, ..., G_m \} \) of \( \{ x_1(t), ..., x_n(t) \} \) such that \( \text{dist}(C(G_i), C(G_j)) > \epsilon \) if \( i \neq j \) and \( \text{diam}(C(G_i)) \leq \delta \) for all \( i \).

**Definition 6.** Merging time is the time \( t \) that two agents with different opinions at time \( t-1 \) but have the same opinion at time \( t \). i.e. \( x_i(t) = x_j(t) \) and \( x_i(t-1) \neq x_j(t-1) \) for some \( i, j \) in \( [n] \).

The following are some properties distinct from synchronous HK models.

**Property 1.** Termination time is not finite.

**Example 1.** Consider \( n = d = 2, x_1(0) = (0, 0), x_2(0) = (\epsilon, 0), \alpha_1(t) = \alpha_2(t) = \frac{1}{2} \) for all \( t \geq 0 \). Then each time \( x_1 \) and \( x_2 \) approach each other but never will they reach steady state in finite time.

**Property 2.** The agents merge at time \( t \) may depart at time \( t+1 \).

**Example 2.** Consider \( n = 3, d = 2, x_1(0) = (0, 0), x_2(0) = (\epsilon, 0), x_3(0) = (\frac{1}{2}, \epsilon), \alpha_1(0) = \alpha_2(0) = 0, \alpha_1(1) = \frac{1}{3}, \alpha_2(1) = \frac{2}{3} \). Then \( x_1 \) and \( x_2 \) merge at time \( t = 1 \) but depart at time \( t = 2 \).

**Property 3.** The opinion profile \( \mathcal{G}(t) \) is \( \epsilon \)-trivial may not imply \( x(t+1) \) in (1) is steady state.

**Example 3.** Consider \( n = d = 2, x_1(0) = (0, 0), x_2(0) = (\epsilon, 0), \alpha_1(t) = \alpha_2(t) = \frac{1}{2} \) for all \( t \geq 0 \). Then \( \mathcal{G}(0) \) is \( \epsilon \)-trivial but \( x(1) \) is not steady state.

**Property 4.** \( \delta \)-equilibrium may not exist for all \( \delta < \epsilon \).

**Example 4.** Consider \( n = d = 2, x_1(0) = (0, 0), x_2(0) = (\epsilon, 0), \alpha_1(t) = \alpha_2(t) = 1 \) for all \( t \geq 0 \). Then \( x \) has no \( \delta \)-equilibrium for all \( \delta < \epsilon \).

**Property 5.** \( \delta \)-equilibrium may not exist in finite time for all \( \delta \leq \epsilon \).

**Example 5.** Consider \( n = 3, d = 2, x_1(0) = (0, 0), x_2(0) = (\epsilon, 0), x_3(0) = (\frac{1}{2}, \epsilon), \alpha_1(t) = \alpha_2(t) = \frac{1}{2}, \alpha_3(t) = 1 \) for all \( t \geq 0 \). Then \( x \) can not achieve \( \delta \)-equilibrium in finite time for all \( \delta \leq \epsilon \).

Lemma 1 plays an important role in the proof of the main theorems.

**Lemma 1.** Given \( \lambda_1, ..., \lambda_n \) in \( \mathbb{R} \) with \( \sum_{i=1}^{n} \lambda_i = 0 \) and \( x_1, ..., x_n \) in \( \mathbb{R}^d \). Then

\[
\sum_{i=1}^{n} \lambda_i x_i = \sum_{i,j,k} c_i (x_j - x_k) \text{ where } c_i > 0 \text{ for all } i \text{ and } \sum_{i,j} c_i = \sum_{j, \lambda_j > 0} \lambda_j
\]

**Proof.** Argue by induction on \( n \). Without loss of generality, assume \( \lambda_i \)'s are non-increasing. For \( n = 2 \),

\[
\lambda_1 x_1 + \lambda_2 x_2 = \lambda_1 (x_1 - x_2)
\]

So it's true for \( n = 2 \). For \( n > 2 \), let \( \lambda_n = -\lambda \) and let \( i \) be the smallest integer such that \( \sum_{k=1}^{i} \lambda_k \geq \lambda \).

Then

\[
\lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_n x_n = \lambda_1 (x_1 - x_n) + \lambda_2 (x_2 - x_n) + \ldots + \lambda_{i-1} (x_{i-1} - x_n)
\]

\[
+ (\lambda - \sum_{k=1}^{i-1} \lambda_k) (x_i - x_n) + \sum_{k=i}^{n} \lambda_k - \lambda) x_i + \lambda_{i+1} x_{i+1} + \ldots + \lambda_{n-1} x_{n-1}
\]

\[
= \sum_{i,j,k} c_i (x_j - x_k)
\]

where \( c_i > 0 \) for all \( i \) and \( \sum_{i} c_i = \sum_{j, \lambda_j > 0} \lambda_j \) by induction hypothesis. Hence we derive the result. \( \square \)

The result enables us to observe interactions between agents and derive a better upper bound.

**Lemma 2.**

\[
\text{diam}(C(\{v_1, ..., v_n\})) \leq \max_{i,j} \|v_i - v_j\| \text{ for all } v_i \text{'s in } \mathbb{R}^d.
\]
Lemma 3. \( \epsilon \)-trivial is a preserving property for (1).

Proof. Assume \( \mathcal{G}(t) \) is \( \epsilon \)-trivial then \( \mathcal{G}(t+1) \) is \( \epsilon \)-trivial.

Proof. Given \( i, j \in [n], x_i(t+1) \) and \( x_j(t+1) \) are in \( C([x_i(t),...,x_n(t)]) \). By Lemma 2,
\[
\|x_i(t+1) - x_j(t+1)\| \leq \max_{i,j} \|x_i - x_j\| \leq \epsilon.
\]

Thus \( \epsilon \)-trivial is a preserving property for (1).

Lemma 4. Assume \( \mathcal{G}(t) \) is \( \epsilon \)-trivial then
\[
\max_{i,j} \|x_i(t+1) - x_j(t+1)\| \leq \max_{i,j} \|x_i(t) - x_j(t)\|.
\]

Proof. Let \( d_t = \max_{i,j} \|x_i(t) - x_j(t)\| \) then \( d_{t+1} \leq \beta_t d_t \) by Lemma 4. By assumption there exists \( (t_i)_{i=1}^{\infty} \) strictly increasing such that \( \beta_i \leq \delta < 1 \) for some \( \delta \) and for all \( i \). For every \( s \geq t_1 \), \( t_i \leq s < t_{i+1} \) for some \( i_s \in \mathbb{Z}^+ \).
\[
d_s \leq \beta_{s-1} \beta_{s-2} \ldots \beta_1 d_t \leq \delta^s d_t
\]
As \( s \to \infty \), \( i_s \to \infty \). Thus
\[
\limsup_{s \to \infty} d_s \leq 0
\]
Hence the limit exists and so we get the result.

In an \( \epsilon \)-trivial profile, agents need not always be open minded. As long as there are infinite \( \beta_i \)'s having an upper bound less than 1, eventually will they reach consensus. Next theorem shows that even though the profile is not \( \epsilon \)-trivial, still could each agent’s opinion converges.

Theorem 1. Assume \( \limsup_{t \to \infty} \beta_t < 1 \) and \( \mathcal{G}(t) \) is \( \epsilon \)-trivial then
\[
\lim_{t \to \infty} \max_{i,j} \|x_i(t) - x_j(t)\| = 0.
\]

Proof. Define \( d_t^1 = \max_{j \in N_i(t)} \|x_i(t) - x_j(t)\| \). If
\[
\sum_{t=1}^{\infty} (1 - \alpha_i(t))(1 - \frac{1}{|N_i(t)|})d_t^1 < \infty \text{ then } x_i(t) \to x_i \text{ as } t \to \infty.
\]
Lemma 7. A symmetric matrix \( M \) is a generalized Laplacian of a graph \( G = (V, E) \) if for \( x, y \in V \), \( M_{xy} = 0 \) for \( x \neq y \) and \( xy \notin E \), and \( M_{xy} < 0 \) for \( x \neq y \) and \( xy \in E \). A Laplacian of \( G \) is defined as \( \mathcal{L} = D_G - A_G \) where \( D_G = \text{diag}(d_G(x))_{x \in V(G)} \). \( d_G(x) \) denotes the degree of \( x \) in \( G \), \( V(G) \) denotes vertex set of \( G \), and \( E(G) \) denotes edge set of \( G \).

Note that diagonal entries of \( M \) have no restrictions. Clearly Laplacian of \( G \) is a generalized Laplacian.

Lemma 5 (Perron-Frobenius for Laplacians [5]). \( M \) is a generalized Laplacian of a connected graph. Then the smallest eigenvalue of \( M \) is simple and the corresponding eigenvector can be taken with all entries positive.

Lemma 6 (Courant-Fischer Formula [7]). Assume \( Q \) is a symmetric matrix with eigenvalues \( \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \) and corresponding eigenvectors \( v_1, \ldots, v_n \). Define \( S_k = \text{span}(\{v_i\}_{i=1}^{k-1}) \) with \( S_0 = \{0\} \). Then

\[
\lambda_k = \min_{\|x\|_1 = 1} x'Qx.
\]

Lemma 7 (Cheeger’s Inequality [6]). Assume \( G = (V, E) \) is an undirected graph with Laplacian \( \mathcal{L} \). Define \( i(G) = \min\left\{ \frac{e(S, S^c)}{|S|} : S \subset V, 0 < |S| \leq |G| \right\} \) where \( e(S, S^c) = \{uv \in E : u \in S, v \in S^c \} \). Then

\[
2i(G) \geq \lambda_2(\mathcal{L}) \geq \frac{j^2(G)}{2\Delta(G)}
\]

where \( \Delta(G) \) is the maximum degree of \( G \).

Lemma 8 ([4]). Let \( U_i(x_1, x_2, \ldots, x_n) = (n-1)e^2 - \sum_{j=1}^{n} \min(\|x_i - x_j\|^2, \epsilon^2) \). Define \( U(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} U_i(x_1, x_2, \ldots, x_n) \). Then

\[
U(x_1, x_{-i}) - U(x_1^*, x_{-i}) \leq \sum_{j \in N_i} (\|x_{j} - x_{j}^*\|^2 - \|x_j - x_i\|^2)
\]

where \( x_j^* \) is a deviation of \( x_i \), \( x_{-i} \) denotes \( x_j \)'s other then \( x_i \), and \( N_i = \{j \in [n] : \|x_j - x_i\| \leq \epsilon \} \).
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**Proof.** Let \( N_i^* = \{ j \in [n] : \| x_j - x_i^* \| \leq \epsilon \} \).

\[
U(x_i, x_{-i}) - U(x_i^*, x_{-i}) = \sum_{k=1}^{n} [U_k(x_i, x_{-i}) - U_k(x_i^*, x_{-i})]
\]

\[
= \sum_{k=1}^{n} (\| x_k - x_i^* \|^2 + \epsilon - \| x_k - x_i \|^2)
\]

\[
= \sum_{k \in N_i \cap N_i^*} (\| x_k - x_i^* \|^2 - \| x_k - x_i \|^2) + \sum_{k \in N_i - N_i^*} (\epsilon^2 - \| x_k - x_i \|^2)
\]

\[
= \sum_{k \in N_i - N_i^*} (\| x_k - x_i^* \|^2 - \| x_k - x_i \|^2)
\]

This completes the proof. \( \square \)

**Lemma 9.** For \( \alpha_i < 1 \),

\[
U(x_i, x_{-i}) - U(x_i^*, x_{-i}) \leq -|N_i|\left( \frac{2}{1 - \alpha_i} - 1 \right) \| x_i - x_i^* \|^2.
\]

**Proof.** Set \( x_i^* = \alpha_i x_i + \frac{1}{|N_i|} \sum_{j \in N_i} x_j \) and \( x_i' = \frac{1}{|N_i|} \sum_{j \in N_i} x_j \). Observe that

\[
\sum_{j \in N_i} (\| x_j - x_i^* \|^2 - \| x_j - x_i \|^2)
\]

\[
= \sum_{j \in N_i} (\| x_i - x_i^* \|^2 + 2 < x_j - x_i, x_i - x_i^* >)
\]

\[
= |N_i| (\| x_i - x_i^* \|^2 - 2 < x_i - x_i^*, x_i - x_i^*>)
\]

\[
= -|N_i|\left( \frac{2}{1 - \alpha_i} - 1 \right) \| x_i - x_i^* \|^2
\]

This derives the result by Lemma 8. \( \square \)

**Lemma 10.** Assume \( Q \) is a real square matrix and \( V \) is invertible such that \( VQ = L \) where \( L \) is the Laplacian of some connected graph. Then 0 is a simple eigenvalue of \( Q'Q \) corresponding to eigenvector \( \| x \| = 1, x \perp \| \)\). In particular, \( \lambda_2(Q'Q) = \min \{ x'Q'Qx : \| x \| = 1, x \perp \| \} \).

**Proof.** Observe that

\[
Q'Qx = 0 \iff Qx = 0 \iff Lx = 0.
\]

Note that a real symmetric matrix is diagonalizable, its algebraic multiplicity equals geometric multiplicity. Since \( L \) is positive definite, has eigenvalue 0 corresponding to eigenvector \( \| \), by Perron-Frobenius Lemma, \( Q'Q \) has a simple eigenvalue 0 corresponding to eigenvector \( \| \). Since \( x'Q'Qx = \| Qx \|^2 \geq 0, Q'Q \) is positive definite. By Courant-Fischer Lemma, \( \lambda_2(Q'Q) = \min \{ x'Q'Qx : \| x \| = 1, x \perp \| \} \). This completes the proof. \( \square \)

Now we show that under some circumstances the time that every component of \( \mathcal{H} \) is \( \delta \)-trivial is finite over all initial profiles; hence x in (1) is asymptotically stable.

**Theorem 3.** Assume \( \sup \max_{\alpha_i(t)} < 1 \) then the maximum time in (1) over all initial profiles such that every component of \( \mathcal{H}(t) \) is \( \delta \)-trivial is finite. Hence x in (1) is asymptotically stable.
Proof. If every component of $\mathcal{G}(t)$ is $\delta$-trivial then we’re done. Suppose $\mathcal{G}(t)$ has a $\delta$-nontrivial component. May assume $\mathcal{G}(t)$ is connected; else we could restrict on a $\delta$-nontrivial component. Let $W = \text{span}(\{1\})$ then $\mathbb{R}^n = W \oplus W^\perp$. Write

$$x(t) = [c_1 \mathbb{1}[c_2 \mathbb{1}]...[c_d \mathbb{1}]] + [\hat{c}_1 u^{(1)}]...[\hat{c}_d u^{(d)}]$$

where $c_i, \hat{c}_i$ are constants and $u^{(i)} \in \mathbb{1}^+$ is a unit vector for all $i$.

Claim: $\sum_{k=1}^{d} \hat{c}_k^2 > \frac{\delta^2}{2}$

Suppose not, $\sum_{k=1}^{d} \hat{c}_k^2 \leq \frac{\delta^2}{2}$ then for any $i, j \in [n],$

$$\|x_i(t) - x_j(t)\|^2 = \sum_{k=1}^{d} \hat{c}_k^2 (u_i^{(k)} - u_j^{(k)})^2$$

$$\leq \sum_{k=1}^{d} \hat{c}_k^2 2((u_i^{(k)})^2 + (u_j^{(k)})^2)$$

$$\leq 2 \sum_{k=1}^{d} \hat{c}_k^2 \leq \frac{\delta^2}{2},$$

contradicting that $\mathcal{G}(t)$ is $\delta$-nontrivial. On the other hand,

$$\begin{align*}
x(t) - x(t + 1) &= (I - B(t))x(t) \\
&= [\hat{c}_1 (I - B(t))u^{(1)}]...[\hat{c}_d (I - B(t))u^{(d)}] \\
&\text{so}\ \\
\sum_{t=1}^{n} \|x_i(t) - x_i(t + 1)\|^2 &= \sum_{j=1}^{d} \hat{c}_j^2 \| (I - B(t))u^{(j)} \|^2.
\end{align*}$$

Let $B(t) = \text{diag}(\alpha(t)) + (I - \alpha(t))A(t).$ Observe that $I - B(t) = (I - \text{diag}(\alpha(t)))(I + D(t))^{-1} \mathcal{L}$ where $\mathcal{L}$ is Laplacian of $\mathcal{G}(t)$ and $D(t)$ is diagonal with $D_{ii}(t) = d_i(t)$, degree of vertex $i$. Via Lemma 10 and Cheeger’s inequality,

$$\begin{align*}
\| (I - B(t))u^{(j)} \|^2 &= u^{(j)}(I - B(t))'(I - B(t))u^{(j)} \\
&\geq \lambda_2((I - B(t))'(I - B(t))) \\
&\geq \left(1 - \max_i \alpha_i(t)\right)2\lambda_2(\mathcal{L}^2) = \left(1 - \max_i \alpha_i(t)\right)^2 \frac{\lambda_2(\mathcal{L})}{n} \\
&> \frac{4(1 - \max_i \alpha_i(t))^2}{n^8}.
\end{align*}$$

Thus

$$\sum_{t=1}^{n} \|x_i(t) - x_i(t + 1)\|^2 > \frac{2\delta^2(1 - \max_i \alpha_i(t))^2}{n^8}.$$ 

Let $\tau = \inf \{t \geq 0: \text{every component of } \mathcal{G}(t) \text{ is } \delta\text{-trivial}\}$ and $Z(t) = U(x_1(t), ..., x_n(t))$. Then by Lemma 9,

$$n^2 \epsilon^2 \geq Z(\tau) - Z(0) = \sum_{t=0}^{\tau-1} (Z(t + 1) - Z(t))$$

$$\geq \sum_{t=0}^{\tau-1} \sum_{i=1}^{n} |N_i(t)| \left(\frac{2}{1 - \alpha_i(t)} - 1\right) \|x_i(t) - x_i(t + 1)\|^2$$

$$\geq \sum_{t=0}^{\tau-1} \sum_{i=1}^{n} \|x_i(t) - x_i(t + 1)\|^2$$

$$> \sum_{t=0}^{\tau-1} \frac{2\delta^2(1 - \max_i \alpha_i(t))^2}{n^8}$$

$$\geq \frac{2\tau \delta^2(1 - \sup_i \max_i \alpha_i(t))^2}{n^8}.$$
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Hence

\[ \tau < \frac{n^{10}}{2(1 - \sup_i \max_j \alpha_i(t))^{2/\delta}}. \]

As \( \tau \to \infty, \delta \to 0 \). This shows \( x \) in (1) is asymptotically stable.

Note that the assumption \( \sup \) couldn’t be replaced with \( \lim \sup \). Consider an initial profile \( G(0) \) with a \( \delta \)-nontrivial component. For \( k \in \mathbb{Z}^+ \), set \( \alpha^{(k)}(t) = 1 \) for \( t \leq k \). Via different choices of \( \alpha \), the time to all components of \( G \) are \( \delta \)-trivial is infinite. From theorem 3, given the assumption, we know that for any \( \delta > 0 \), there exists \( \tau_\delta \in \mathbb{N} \) such that every component of \( G(\tau_\delta) \) is \( \delta \)-trivial. Thus if \( G(\tau_\delta) \) is connected for some \( \delta \leq \epsilon \) for some initial profile \( G(0) \) then by theorem 1, consensus reached eventually.

2. Conclusion

Mixed HK model covers synchronous and asynchronous HK models, which is more complicated. Each time agents can choose the degree to play stubborn or open minded. Agents with the same opinion may depart later, depicting the changeability of agents, which is closer to real world circumstances. Given the givens, make it more difficult to reach steady state or \( \delta \)–equilibrium. However, if meet some criteria, agents could reach consensus eventually or their opinions are asymptotically stable.

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