Maxwell’s theory on a post-Riemannian spacetime and the equivalence principle

Roland A. Puntigam, Claus Lämmerzahl* and Friedrich W. Hehl

Institute for Theoretical Physics, University of Cologne, D-50923 Köln, Germany

The form of Maxwell’s theory is well known in the framework of general relativity, a fact that is related to the applicability of the principle of equivalence to electromagnetic phenomena. We pose the question whether this form changes if torsion and/or nonmetricity fields are allowed for in spacetime. Starting from the conservation laws of electric charge and magnetic flux, we recognize that the Maxwell equations themselves remain the same, but the constitutive law must depend on the metric and, additionally, may depend on quantities related to torsion and/or nonmetricity. We illustrate our results by putting an electric charge on top of a spherically symmetric exact solution of the metric-affine gauge theory of gravity (comprising torsion and nonmetricity). All this is compared to the recent results of Vandyck.

1. INTRODUCTION

It was Minkowski, in 1908, who formulated Maxwell’s theory in a four-dimensional flat pseudo-Euclidean spacetime, Minkowski’s special-relativistic ‘world’. The next step, generalizing Maxwell’s theory if gravity can no longer be neglected, was performed by Einstein and Grossmann in 1913. They ‘lifted’ Maxwell’s theory to a four-dimensional pseudo-Riemannian spacetime. This amounted to a successful application of the equivalence principle to Maxwell’s theory. Not too much later, after the creation of general relativity theory, Einstein [1] reformulated Maxwell’s theory such that it became apparent that the basic structure of Maxwell’s theory, namely the field equations, is left intact even when a metric is not used. Later Kottler [2], E. Cartan [3], van Dantzig [4], and others put forward the so-called metric-free formulation of electrodynamics, see Post [5,6] and Schouten [7]. It was recognized in this general framework that Maxwell’s equations can be understood as arising from the conservation laws of electric charge and magnetic flux, see Truesdell and Toupin [8].

These conservation laws can be reduced to counting statements, since electric charge comes in quantized portions of elementary charges (or rather as one thirds of them) and magnetic flux can also exist, in superconducting media, in a quantized form, the flux quantum or fluxoid $h/(2e)$, as was predicted (up to a factor 2) already by F. London [9]. Thus it is obvious that the formulation of these laws only requires a four-dimensional differentiable manifold and the possibility of a foliation of it into three-dimensional hypersurfaces. The constitutive laws do need a metric, in contrast to the Maxwellian field equations themselves, a point of view which has been repeatedly stressed by Post [6,10], see also Bamberg & Sternberg [11]. Any post-Riemannian geometry, that is, any spacetime geometry which has more geometrical field variables (‘gravitational fields’) than the metric, is irrelevant to the Maxwell equations. In particular, neither torsion nor nonmetricity couple to it, a point which has already been made by Benn, Dereli, and Tucker [4]. Only the constitutive law may depend in a very restricted way on torsion and nonmetricity structures.

In the Einstein-Cartan theory of gravity, spacetime carries an additional torsion, and, still more general, in the framework of metric-affine gravity [13], a nonmetricity enters the geometrical arena of spacetime. What can we predict about Maxwell’s theory under those more exotic circumstances? Can we again apply the equivalence principle? Should we write down Maxwell’s equations in the Minkowski world in Cartesian coordinates and replace the partial by covariant derivatives, or should we do that in the Lagrangian? What type of coupling to gravity should we assume?

Especially the application of the ‘comma goes to semicolon rule’ (see Misner, Thorne, and Wheeler [14]) creates difficulties for charge conservation if applied to the inhomogeneous Maxwell equation in post-Riemannian spacetimes.

*Permanent address: Fakultät für Physik, Universität Konstanz, D-78434 Konstanz, Germany
Some test theory for the coupling of the Maxwell equations to non-metric structures of spacetime has been investigated by Coley [15]. Vandyck has addressed these questions in a recent article [16]. We find his answers not totally convincing. Therefore we will try to argue that the axiomatic formulation of Maxwell’s theory, alluded to above, is sufficient for formulating Maxwell’s theory in such post-Riemannian spacetimes, including possibly torsion and nonmetricity.

As formalism we use exterior calculus, for our conventions see [13].

2. ELECTRIC CHARGE CONSERVATION

Let be given the odd (or twisted, see [17]) electric current three-form $J$. We assume that a $(1 + 3)$-foliation of spacetime holds locally. The different three-dimensional hypersurfaces are labeled by a parameter $\tau$. We introduce a normal vector $n$ such that $n \cdot d\tau = 1$. Then we can decompose the current three-form according to

$$J = \rho - j \wedge d\tau,$$

(2.1)

where $\rho$ is the charge density three-form and $j$ the electric current two-form. As axiom 1 we assume electric charge conservation ($d =$ four-dimensional exterior derivative):

$$\oint_{\partial V_4} J = \int_{V_4} dJ = 0 \quad \text{(Axiom 1)}.$$  

(2.2)

Here $V_4$ is an arbitrary four-dimensional volume and $\partial V_4$ its three-dimensional boundary. If this is assumed to be valid for all three-cycles $c_3 = \partial V_4$, then $J$ is exact [10,18,19]:

$$dG = J.$$  

(2.3)

The electromagnetic excitation $G$ is an odd two-form which decomposes as

$$G = D - H \wedge d\tau.$$  

(2.4)

Therefore (2.3) is equivalent to the inhomogeneous Maxwell equations

$$dD = \rho \quad \text{(Gauss law)},$$  

(2.5)

$$dH - \dot{D} = j \quad \text{(Oersted-Ampère law)};$$  

(2.6)

here $d := d - n \cdot (n \cdot d)$ is the three-dimensional exterior derivative, and $\dot{D} := L_n D$ is defined via the Lie derivative, for details compare [20] and the literature given. Note that, up to now, only the differential structure of the spacetime was needed. The electric excitation $D$ can be measured by means of Maxwellian double plates (see Pohl [21]) as charge per unit area, the magnetic excitation by means of a small test coil, which compensates the $H$-field to be measured, as current per unit length. In other words, the extensive quantities $D$ and $H$ have an operational significance provided we know the characterizing properties of an ideal conductor.

3. LORENTZ FORCE

From mechanics we take the notion of an even covector-valued force density four-form $f_\alpha$. In the conventional manner, we define the electromagnetic field strength $F$ via axiom 2:

$$f_\alpha = (e_\alpha \cdot \mathcal{J} F) \wedge J \quad \text{(Axiom 2)}.$$  

(3.1)

From mechanics originates the notion of $f_\alpha$, from axiom 1 the current $J$, the $e_\alpha$’s denote the frame. The even two-form $F$ can be decomposed as

$$F = B + E \wedge d\tau,$$  

(3.2)

that is, for $a,b = 1,2,3$,

$$f_\alpha = -\rho (e_\alpha \cdot \mathcal{J} E) - j \wedge (e_\alpha \cdot \mathcal{J} B) \quad \text{with} \quad f_\alpha = f_\alpha \wedge d\tau.$$  

(3.3)
Therefore the Lorentz force (3.3), via (3.2), yields an operational definition of the electromagnetic field strength $F$ as a force field – and hence as an intensive quantity. Again no metric nor a connection is necessary for formulating axiom 2.

An alternative way of introducing $F$ – again metric etc. independent – is provided by quantum interference measurements of the Aharonov-Bohm type yielding an observable phase shift $\delta \varphi = \frac{e}{\hbar} \int_{\cal{V}_3} F$.

Now we have to impose some conditions on the newly defined field strength $F$.

4. MAGNETIC FLUX CONSERVATION

The field strength $F$ is a two-form. Thus we can postulate the conservation of magnetic flux as axiom 3:

$$\oint_{\partial V_3} F = \int_{V_3} dF = 0 \quad \text{(Axiom 3)}.$$  \hfill (4.1)

By Stokes’ theorem and the arbitrariness of the two-cycles $c_2 = \partial V_3$, we have

$$dF = 0.$$  \hfill (4.2)

In (1 + 3)-decomposition this reads

$$\begin{align*}
\frac{dB}{dt} &= 0 \quad \text{(magnetic field closed)}, \\
\frac{dE}{dt} + \dot{B} &= 0 \quad \text{(Faraday law)}.
\end{align*}$$  \hfill (4.3, 4.4)

Maxwell’s equations are represented by (2.3, 4.2) or, equivalently, by (2.5, 2.6, 4.3, 4.4). In this form, they are generally covariant, i.e. valid in arbitrary frames and arbitrary coordinates. Moreover, neither metric nor torsion or nonmetricity take part in this set-up. Therefore, if one starts from a four-dimensional differential manifold, which admits a (1 + 3)-foliation, and introduces a metric and a connection, then the structure of the Maxwell equations (2.3) and (4.2) is insensitive to it and does not change.

We can argue similarly in the complementary situation: In a Minkowski space, we formulate the Maxwell equations in the way we did. Then they keep their form with respect to an accelerated frame. Consequently, switching on gravity and requiring the equivalence principle to be valid, the Maxwell equations must not change either. In spite of the ‘deformation’ of spacetime by means of gravity, the Maxwell equations remain ‘stable’. Therefore, in this framework of deriving Maxwell’s theory from electric charge and magnetic flux conservation, the Maxwell equations stay the same ones in a Minkowskian, a Riemannian, or a post-Riemannian spacetime. No additional effort is needed in order to adapt the Maxwell equations if a spacetime is considered with additional geometrical attributes. This is the most straightforward application of the equivalence principle one can think of.

5. CONSTITUTIVE LAW

So far, the Maxwell equations (2.3) and (4.2) represent an underdetermined system of evolution equations for $G$ and $F$. In order to reduce the number of independent variables, we have to set up a relation between $G$ and $F$:

$$G = G(F) \, .$$  \hfill (5.1)

Special cases of this constitutive law are:

(i) Vacuum: The standard constitutive law for vacuum is

$$G = *F \, .$$  \hfill (5.2)

On the right-hand side of (5.2), the factor $(\varepsilon_0/\mu_0)^{1/2}$ has been absorbed for simplicity. Here, by means of the Hodge star, the metric enters the Maxwell theory for the first time. The appearance of the metric is necessary from a physical point of view in order to get the light cone as characteristic surface of the evolution equations for the Maxwellian field strength $F$. The law (5.2) is valid in Minkowski, Riemannian, and post-Riemannian spacetimes likewise.
(ii) **Axion:** The constitutive law of the vacuum (5.2) relates the ordinary two-form $F$, via the Hodge star (which is twisted), to the twisted two-form $G$. If we had a twisted zero-form $\theta$ (‘pseudo-scalar’) at our disposal, then we could supplement the right hand side of the vacuum law by the twisted term $\theta F$:

$$G = \star F + \theta F = (\star + \theta) F. \quad (5.3)$$

The exterior derivative of this equation, because of $dF = 0$, turns out to be

$$dG = d\star F + d\theta \wedge F. \quad (5.4)$$

Thus the inhomogeneous Maxwell equation, in terms of $F$, reads

$$\left( d\star + (d\theta) \wedge \right) F = J, \quad \text{with} \quad dJ = 0. \quad (5.5)$$

The ‘pseudo-scalar’ field $\theta$ is known in the literature as hypothetical axion field, see [22] and [23]. Its possible implications for cosmology are discussed in [24]. The axion-Maxwell interaction Lagrangian turns out to be

$$\sim \theta F \wedge F = \theta d(F \wedge A).$$

We can relate the axion field to the torsion of spacetime. The torsion $T^\alpha$ is an ordinary two-form. Its axial piece is proportional to the ordinary three-form $T^\alpha \wedge \vartheta^\alpha$. The dual of it is a twisted one-form $\star (T^\alpha \wedge \vartheta^\alpha)$. Therefore, with some constant $c$, we can make the identification

$$d\theta = c \star (T^\alpha \wedge \vartheta^\alpha), \quad (5.6)$$

which yields (Gasperini and de Sabbata [25], see also [26])

$$\left( d\star + c \star (T^\alpha \wedge \vartheta^\alpha) \wedge \right) F = J. \quad (5.7)$$

This equation describes the coupling of the inhomogeneous Maxwell equation to an axial piece of the torsion. However, this interpretation is not compulsory. Incidentally, Eq.(5.7) seems to represent the most general post-Riemannian coupling linear in $F$ which is compatible with charge conservation [27]; a piece with the Weyl covector, e.g., is excluded since it is an ordinary, but not a twisted form.

We recognize also in this example that there doesn’t seem to exist a chance to introduce other post-Riemannian structures in the axion-Maxwell equation (5.7) in an ad hoc way. We would like to stress that (5.7) is valid in a spacetime with arbitrary metric and connection.

(iii) **Born-Infeld:** The non-linear Born-Infeld theory [28] represents a classical generalization of Maxwell’s theory for accommodating stable solutions for the description of ‘electrons’. Its constitutive law reads (with a dimensionful parameter $f$, the so-called maximal field strength, see also [29]):

$$G = \frac{\star F - \frac{1}{2f^2} \star (F \wedge F) F}{\sqrt{1 + \frac{1}{2f^2} \star (F \wedge F) - \frac{1}{4f^4} [\star (F \wedge F)]^2}}. \quad (5.8)$$

It leads to a non-linear equation for the dynamical evolution of the field strength $F$. As a consequence, the characteristic surface, the light cone, depends on the field strength, and the superposition principle for the electromagnetic field doesn’t hold any longer.

(iv) **Heisenberg-Euler:** Quantum electrodynamical vacuum corrections to Maxwell’s theory can be accounted for by an effective constitutive law constructed by Heisenberg and Euler [30]. To second order in the fine structure constant $\alpha$, it is given by (see also [31])

$$G = \left[ 1 + \frac{4 \alpha^2}{45 m^4} (F \wedge \star F) \right] \star F + \frac{7 \alpha^2}{45 m^4} \star (F \wedge F) F, \quad (5.9)$$

where $m$ is the mass of the electron. Again, post-Riemannian structures don’t interfere here.
6. ENERGY-MOMENTUM CURRENT OF THE ELECTROMAGNETIC FIELD

For quantifying the gravitational effect of the electromagnetic field, we need its energy-momentum current. The Lagrangian four-form of Maxwell’s field reads

\[ L_{\text{Max}} = -\frac{1}{2} F \wedge G . \] (6.1)

The canonical energy-momentum current is computed from the Lagrangian four-form \( L_{\text{Max}} \) and can be represented by the odd covector-valued three-form

\[ \Sigma_{\alpha}^{\text{Max}} = e_{\alpha} \mathcal{I} L_{\text{Max}} + (e_{\alpha} \mathcal{J} F) \wedge G = \frac{1}{2} (e_{\alpha} \mathcal{J} F) \wedge G - \frac{1}{2} (e_{\alpha} \mathcal{J} G) \wedge F . \] (6.2)

This energy-momentum current will enter the right hand side of the first field equation, as we will see below.

7. AN ELECTRIC CHARGE IN EINSTEIN-DILATION-SHEAR GRAVITY

As a nontrivial example, let us consider the electromagnetic field in the framework of the metric-affine gauge theory (MAG) of gravity \([13]\), in particular its effect on an exact solution of this theory \([32]\), see also \([33]\). Similar solutions have been found by Tucker and Wang, see \([34]\) and \([35]\). The geometrical ingredients of MAG are the curvature two-form \( R_{\alpha \beta} = \frac{1}{2} R_{ij \alpha \beta} dx^i \wedge dx^j \), and, as post-Riemannian structures, the nonmetricity one-form \( Q_{\alpha \beta} = Q_{\alpha \beta \gamma} dx^\gamma \) and the torsion two-form \( T^\alpha = \frac{1}{2} T_{ij}^\alpha dx^i \wedge dx^j \). The simple toy model that we want to consider is specified by a gravitational gauge Lagrangian, quadratic in curvature, torsion, and nonmetricity, see \([32]\).

\[ V_{\text{dil-sh}} = -\frac{1}{2\kappa} (R_{\alpha \beta} \wedge \eta_{\alpha \beta} - 2\lambda T + \gamma T \wedge \gamma T) - \frac{\alpha}{8} R_{\alpha \beta} \wedge R_{\beta \gamma} , \] (7.1)
coupled to the Maxwell Lagrangian \((6.1)\) according to \( L_{\text{tot}} = V_{\text{dil-sh}} + L_{\text{Max}} \). In \((7.1)\) we have introduced the Weyl covector \( Q := Q_{\gamma \beta} \) and the covector piece of the torsion \( T := e_\alpha \mathcal{J} T^\alpha \). Einstein’s gravitational constant is denoted by \( \kappa = \ell^2/(\hbar c) \) (with the Planck length \( \ell \)), and \( \lambda \) is the cosmological constant. The coupling constants \( \alpha, \beta, \) and \( \gamma \) are dimensionless.

Varying the coframe and the connection, we find the two relevant field equations of MAG \([13]\),

\[ DH_\alpha - E_\alpha = \Sigma_\alpha , \] (7.2)
\[ DH_{\alpha \beta} - E_{\alpha \beta} = \Delta_{\alpha \beta} , \] (7.3)
referred to as the first and the second field equation, respectively, with \( D \) as the covariant exterior derivative. In \((7.2)\) and \((7.3)\) there enter the canonical energy-momentum and hypermomentum currents of matter \( \Sigma_\alpha \) and \( \Delta_{\alpha \beta} \), the gravitational gauge field momenta

\[ H_\alpha := -\frac{\partial V_{\text{dil-sh}}}{\partial T^\alpha} \] and \( H_{\alpha \beta} := -\frac{\partial V_{\text{dil-sh}}}{\partial R^\alpha_{\beta \gamma}} , \] (7.4)
and the canonical energy-momentum and hypermomentum currents of the gauge fields

\[ E_\alpha = e_\alpha \mathcal{I} V_{\text{dil-sh}} + (e_\alpha \mathcal{J} T^\beta) \wedge H_\beta + (e_\alpha \mathcal{J} R_{\beta \gamma}) \wedge H^\beta_{\gamma \beta} + \frac{1}{2} (e_\alpha \mathcal{J} Q_{\beta \gamma} M_{\beta \gamma} \), \] (7.5)
\[ E_{\alpha \beta} = -\partial^\alpha \wedge H_{\beta} - M_{\alpha \beta} . \] (7.6)
The gravitational gauge field momentum \( M_{\alpha \beta} \) is coupled to the nonmetricity:

\[ M_{\alpha \beta} := -2 \frac{\partial V_{\text{dil-sh}}}{\partial Q_{\alpha \beta}} . \] (7.7)

We study only the behavior of the electromagnetic field in the metric-affine framework. Thus, for the matter currents in \((7.2)\) and \((7.3)\) we have \( \Sigma_\alpha = \Sigma^{\text{Max}}_\alpha \), cf. \((7.2)\), and \( \Delta_{\alpha \beta} = 0 \).

The formalism of MAG, as outlined in the present section, is not limited to the simple and very restricted Lagrangian \((7.1)\); more general choices for the Lagrangian are possible. It is our intention, however, not to look at MAG for its own interest, but to investigate the behavior of Maxwell’s theory within a non-Riemannian spacetime. This was our motivation for making the simplest possible choice of the metric-affine part of the Lagrangian that still allows for propagating torsion and nonmetricity, see Obukhov et al. \([24]\).
8. EXACT SOLUTION WITH SPHERICAL SYMMETRY

The field equations (7.2) and (7.3), together with Maxwell’s equations (2.3) and (4.2) – assuming the constitutive law (5.2) – are approached as follows. The spherically symmetric coframe
\[
\vartheta^0 = f \, dt, \quad \vartheta^1 = \frac{1}{f} \, dr, \quad \vartheta^2 = r \, d\theta, \quad \vartheta^3 = r \sin \theta \, d\phi,
\]
(8.1) contains the zero-form \( f = f(r) \) and is assumed to be orthonormal, i.e., the metric reads
\[
ds^2 = \eta_{\alpha\beta} \vartheta^\alpha \otimes \vartheta^\beta = -f^2 \, dt^2 + \frac{1}{f^2} \, dr^2 + r^2 \left( d\theta^2 + \sin^2 \theta \, d\phi^2 \right).
\]
(8.2)
The nonmetricity one-form is taken to contain only two irreducible pieces (see [13, Appendix B.1]),
\[
Q_{\alpha\beta} = (3) Q_{\alpha\beta} + (4) Q_{\alpha\beta},
\]
(8.3) namely the dilation (or Weyl) piece (4) \( Q_{\alpha\beta} = Q g_{\alpha\beta} \) and a proper shear piece
\[
(3) Q_{\alpha\beta} = \frac{4}{9} \left( \partial_\alpha e_\beta - \frac{1}{4} g_{\alpha\beta} \Lambda \right), \quad \text{with} \quad \Lambda := \partial^\alpha e^\beta \cdot \mathcal{J} Q_{\alpha\beta}.
\]
(8.4)
Furthermore, we allow only the covector piece (2) \( T^\alpha \) in the torsion two-form:
\[
T^\alpha = (2) T^\alpha = \frac{1}{3} \, \vartheta^\alpha \wedge T.
\]
(8.5)
Finally, we use a spherically symmetric electric (Coulomb) charge at the origin of the spatial coordinates with the corresponding field strength
\[
F = \frac{q}{r^2} \vartheta^1 \wedge \vartheta^0,
\]
(8.6) and we impose the constitutive law (5.2).

With these prescriptions and the ansatz
\[
Q = u(r) \, \vartheta^0, \quad \Lambda = v(r) \, \vartheta^0, \quad T = \tau(r) \, \vartheta^0
\]
(8.7) for the one-form triplet \((Q, \Lambda, T)\), the solution is expressed by
\[
f = \sqrt{1 - \frac{2\kappa M}{r} + \frac{\lambda r^2}{3} + \kappa q^2 \frac{2r^2}{2r^2} + \alpha \kappa N^2 \frac{2}{2r^2}}
\]
(8.8) and
\[
u = \frac{\tilde{N}}{fr}, \quad v = \frac{3\beta}{2} \frac{\tilde{N}}{fr}, \quad \tau = -\frac{\beta + 6}{4} \frac{\tilde{N}}{fr},
\]
(8.9) where \( \tilde{N} \) is an integration constant. The dimensionless coupling constants are subject to the constraint
\[
\gamma = \frac{8}{3} \frac{\beta}{\beta + 6},
\]
(8.10) i.e., only two of the post-Riemannian coupling constants \((\alpha, \beta, \gamma)\) in (7.1) remain independent, while the third one, \( \gamma \), is determined by (8.10). These results have been found with the help of the computer algebra system REDUCE [30] making also use of its Excalc package [37], see [38].

Let us summarize the properties of the MAG-Maxwell solution that is presented here. The zero-form \( f \), that fixes the orthonormal coframe (8.1), has four contributions, see (8.8). The terms containing the mass parameter \( M \), the cosmological constant \( \lambda \), and the electric charge \( q \) correspond exactly to the (general relativistic) Reissner-Nordström solution with cosmological constant. The additional term with the dilation charge \( \tilde{N} \) has a similar structure as the previous term with the electric charge \( q \). The nonmetricity has the explicit form...
\[ Q^{\alpha \beta} = \frac{\tilde{N}}{fr} \left[ \vartheta^{\alpha} e^{\beta} \mathbf{J} - \frac{1}{4} \vartheta^{\alpha \beta} \right] \vartheta^0 \]  

(8.11)

carrying, besides the dilation piece, a shear part – the second term in \([8.11]\) with the factor \(\beta\). The torsion two-form evaluates to

\[ T^\alpha = -\frac{\beta + 6}{12} \frac{\tilde{N}}{fr} \vartheta^\alpha \wedge \vartheta^0, \]

(8.12)

and the Faraday two-form

\[ F = \frac{q}{r^2} \vartheta^1 \wedge \vartheta^0 \]

(8.13)

has the same innocent appearance as that of a point charge in flat Minkowski space. It is clear, however, that all relevant geometric objects, coframe, connection, torsion, curvature, etc., ‘feel’ – via the zero-form \(f\) – the presence of the electric charge. However, as one can recognize from \((8.11, 8.12, 8.13)\), the Maxwell field is disconnected from nonmetricity and torsion otherwise. This exemplifies and is in full accordance with our general statement concerning the coupling of the Maxwell equations to post-Riemannian structures.

9. DISCUSSION

There is so much experimental evidence in favor of the conservation laws of electric charge and magnetic flux that one can hardly doubt the correctness of axiom 1 and axiom 3 from a physical point of view. If so, then the form of the Maxwell equations is fixed, and we have no trouble in predicting how they change in spacetimes with Riemannian and post-Riemannian geometrical structure: They don't change at all. They are stable against such ‘deformations’. Thereby the equivalence principle turns out to be rather trivial in this context. The only ‘freedom’ one has is to modify the constitutive law. Incidentally, if the limits of classical physics are reached, then, on the level of quantum mechanics, a fresh look at the equivalence principle is needed, see \([39]\).

Coming back to the article of Vandyck \([16]\), we recognize that the different options for generalizing the Maxwell equations are artificial ones in the sense that they violate the well-established axioms 1 and 3, namely the conservation of electric charge and magnetic flux. These options can only emerge, if one forgets the underlying physical structure of Maxwell’s theory. Clearly, whether one uses the calculus of tensor analysis (see the Appendix) or that of exterior differential forms, doesn’t make any difference, if one starts off with our axioms.

In the framework of the Poincaré gauge theory of gravitation, the spacetime of which carries, besides the metric, a propagating torsion, we also found exact electrically charged solutions, see \([40]\). In this latter context, as well as in the case of the new charged solution of MAG that was presented in Sect. 8, we used Maxwell’s theory as described in Sects. 2 to 5. And everything is well-behaved and consistent with our analysis of how to couple the Maxwell equations to post-Riemannian structures. There is almost no freedom for an alternative coupling of Maxwell’s equations to gravity within Riemannian or post-Riemannian spacetimes. The equations \((2.3), (4.4), \) and \((5.2)\) solve the problem completely.

APPENDIX: THE TENSOR ANALYSIS VERSION OF METRIC-FREE ELECTRODYNAMICS

We decompose excitation, field strength, and current into (holonomic) coordinate components:

\[ G = \frac{1}{2!} G_{ij} \, dx^i \wedge dx^j, \quad F = \frac{1}{2!} F_{ij} \, dx^i \wedge dx^j, \quad J = \frac{1}{3!} J_{ijk} \, dx^i \wedge dx^j \wedge dx^k. \]

(A.1)

If we use the Levi-Civita antisymmetric unit tensor density \(\epsilon^{ijkl} = \pm 1, 0\), which is metric-free,

\[ G^{ij} := \frac{1}{2!} \epsilon^{ijkl} G_{kl}, \quad J^i := \frac{1}{3!} \epsilon^{ijkl} J_{jkl}, \]

(A.2)

then Maxwell’s equations read

\[ \partial_k G^{ij} = J^i, \quad \partial_{[i} F_{j]k} = 0. \]

(A.3)
The constitutive law for vacuum can be put in the linear form
\[ G^{ij} = \frac{1}{2} \chi^{ijkl} F_{kl}, \quad \chi^{(ij)kl} = \chi^{(ij)kl} = 0, \quad \chi^{ijkl} = \chi^{ijlk}, \quad (A.4) \]
with the specific metric dependent “modulus”
\[ \chi^{ijkl} := 2 \sqrt{\left| \det g_{mn} \right|} g^{iki} g^{jli}. \quad (A.5) \]
Eqs. (A.3) to (A.5) are unchangedly valid in post-Riemannian spacetimes. Note that the representation of the electromagnetic excitation \( G^{ij} \) as density, see Schrödinger [41], is vital for these considerations and distinguishes our approach from that of Vandyck.

ACKNOWLEDGMENTS

We are grateful to Werner Esser, Eckehard Mielke, Yuri Obukhov, and Norbert Straumann (Zürich) for helpful remarks. The first named author (RAP) is supported by the Graduiertenkolleg Scientific Computing, Cologne-St.Augustin, the second named author (CL) thanks the Deutsche Forschungsgemeinschaft, Bonn for financial support.

[1] Einstein A 1916 Sitzungsber. Königl. Preuss. Akad. Wiss. (Berlin) pp 184-87
[2] Kottler F 1922 Sitzungsber. Akad. Wien Ia 131 119-46
[3] Cartan É 1986 On Manifolds with an Affine Connection and the Theory of General Relativity, English translation of the French original of 1923/24 (Napoli: Bibliopolis)
[4] van Dantzig D 1934 Proc. Cambridge Phil. Soc. 30 421-27
[5] Post E J 1962 Formal Structure of Electromagnetics (Amsterdam: North Holland), soon available from New York: Dover
[6] Post E J 1980 Phys. Lett. A79 288-90
[7] Schouten J A 1989 Tensor Analysis for Physicists 2nd edn printed (New York: Dover)
[8] Truesdell C and Toupin R A 1960 The classical field theories Handbuch der Physik vol III/1 ed S Flügge (Berlin: Springer) pp 226-793
[9] London F 1950 Superfluids vol 1. Macroscopic Theory of Superconductivity (New York: Wiley)
[10] Post E J 1995 Quantum Reprogramming – Ensembles and Single Systems: A Two-Tier Approach to Quantum Mechanics (Dordrecht: Kluver)
[11] Bamberg P and Sternberg S 1990 A Course in Mathematics for Students of Physics vol 2 (Cambridge: Cambridge University Press)
[12] Benn I M, Dereli T and Tucker R W 1980 Phys. Lett. B96 (1980) 100-04
[13] Hehl F W, McCrea J D, Mielke E W and Ne’emann Y 1995 Phys. Rep. 258 1-171
[14] Misner C W, Thorne K S and Wheeler J A 1973 Gravitation. (San Francisco: Freeman)
[15] Coley A A 1983 Phys. Rev. D27 728-39
[16] Vandyck M A 1996 J. Phys. A: Math. Gen. 29 2245-55
[17] Burke W L 1985 Applied Differential Geometry (Cambridge: Cambridge University Press)
[18] Post E J 1979 Found. Phys. 9 619-40
[19] Post E J 1982 Found. Phys. 12 169-95
[20] Hehl F W, Lember M and Mielke E W 1991 Two lectures on fermions and gravity Geometry and Theoretical Physics, Proc. of the Bad Honnef School 12-16 Feb 1990 ed J Debrus and A C Hirshfeld (Heidelberg: Springer) pp 55-140
[21] Pohl R W 1975 Elektrizitätslehre 21st edn (Berlin: Springer)
[22] Weinberg S 1978 Phys. Rev. Lett. 40 223-26
[23] Wilczek F 1978 Phys. Rev. Lett. 40 279-82
[24] Kolb E W and Turner M S 1990 The Early Universe (Redwood City, CA: Addison-Wesley)
[25] Gasperini M and De Sabbata V 1981 Phys. Rev. D23 2116-20
[26] de Sabbata V and Sivaram C 1994 Spin and Torsion in Gravitation (Singapore: World Scientific)
[27] Lämmerzahl C et al 1997 ‘Reasons for the electromagnetic field to obey the Maxwell equations’ (in preparation)
[28] Born M and Infeld L 1934 Proc. Roy. Soc. (London) A144 425-51
[29] Gibbons G W and Rasheed D A 1995 Nucl. Phys. B454 185-206
[30] Heisenberg W and Euler H 1936 Z. Phys. 98 714-32
[31] Itzykson C and Zuber J-B 1985 Quantum Field Theory (New York: McGraw Hill)
[32] Obukhov Yu N, Vlachynsky E J, Esser W, Tresguerres R and Hehl F W 1996 Phys. Lett. A220 1-9
[33] Vlachynsky E J, Tresguerres R, Obukhov Yu N and Hehl F W 1996 Class. Quantum Grav. 13 3253-59
[34] Tucker R W and Wang C 1995 Class. Quantum Grav. 12 2587-605
[35] Tucker R W and Wang C 1996 Non-Riemannian Gravitational Interactions electronic archive Los Alamos gr-qc/9608055
[36] Hearn A C 1993 REDUCE User’s Manual, Version 3.5 RAND Publication CP78 (Rev. 10/93) The RAND Corporation, Santa Monica, CA 90407-2138, USA
[37] Schrüfer E 1994 EXCALC: A System for Doing Calculations in the Calculus of Modern Differential Geometry GMD-SCAI, D-53757 St.Augustin, Germany
[38] Puntigam R A, Schrüfer E and Hehl F W 1995 The use of computer algebra in Maxwell’s theory Computer Algebra in Science and Engineering. Proceedings of the ZiF Workshop, Bielefeld, 28-31 Aug 1994 ed J Fleischer et al (Singapore: World Scientific) pp 195-211
[39] Lämmerzahl C 1996 Gen. Rel. Grav. 28 1043-70
[40] Baekler P, Gürses M, Hehl F W and McCrea J D 1988 Phys. Lett. A128 245-50
[41] Schrödinger E 1954 Space-Time Structure (Cambridge: Cambridge University Press)