The simplified self-consistent probabilities method for percolation and its application to interdependent networks

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Abstract
Interdependent networks in areas ranging from infrastructure to economics are ubiquitous in our society, and the study of their cascading behaviors using percolation theory has attracted much attention in recent years. To analyze the percolation phenomena of these systems, different mathematical frameworks have been proposed, including generating functions and eigenvalues, and others. These different frameworks approach phase transition behaviors from different angles and have been very successful in shaping the different quantities of interest, including critical threshold, size of the giant component, order of phase transition, and the dynamics of cascading. These methods also vary in their mathematical complexity in dealing with interdependent networks that have additional complexity in terms of the correlation among different layers of networks or links. In this work, we review a particular approach of simple, self-consistent probability equations, and we illustrate that this approach can greatly simplify the mathematical analysis for systems ranging from single-layer network to various different interdependent networks. We give an overview of the detailed framework to study the nature of the critical phase transition, the value of the critical threshold, and the size of the giant component for these different systems.

1. Introduction
Systems consisting of multiple inter connected networks with different types of links have received enormous attention in recent years [1–9] due to their ubiquitous applications in complex systems. Such networks appear in the literature as interdependent networks or multiplex networks. Studies have shown that interdependent networks show different percolation/phase transition behaviors than single networks. In particular, an interdependent network is more vulnerable to random attacks [10]. As many real-world infrastructure networks can be classified into interdependent networks [11–14], the understanding of their robustness carries great practical significance.

In a network consisting of links and nodes, one of the most important quantities used to analyze its robustness is the size of the giant component, which is defined as the largest set of nodes that are connected to each other. When a network is under attack (i.e., a fraction 1 − p of nodes (or links) are removed), the size of the largest cluster shrinks. Usually its size is a finite fraction of the total number of nodes in the network, unless more than a certain fraction, 1 − p, of nodes are removed. If that happens, then the largest cluster, which is known as the giant component, disappears and all of the clusters become negligibly small. This phase is associated with the disintegration of the network. Therefore, the size, \( \mu^\infty \), of the giant component serves as an order parameter that is very useful in studying the phase transition behaviors and the robustness of the network structure.

Some original works [1, 15] provided a precise and powerful analytical solution to the phase transition behaviors. In their mathematical analysis, recursive mapping was used to track the percolation process in each stage of cascading failures. In some systems where correlations exist in dependency links [7–9, 16, 17], this method, while having the advantage of following and yielding insight into the cascading process [18], could lead
to very complicated formulations that are not always easy to solve. To study different network constructions, some of the other studies used different methods to achieve a relatively simpler analytical framework. In particular, the works in [7, 8, 19–22] used self-consistent equations of the converging probabilities to have an alternative approach to analyzing the critical behaviors of certain types of interdependent networks. Some of these methods can be extended to other scenarios.

In this paper, we illustrate the use of one particular technique based on self-consistent probabilities [8, 19–21], and we demonstrate that this technique could be applied to a wide variety of different interdependent networks with minimum simplicity by surveying the literature in this field. This method focuses on the recursive representation of two central quantities, defined as the probabilities of finding a link/node in the giant component. It is able to give a set of straightforward self-consistent equations describing the percolation behaviors without going through the cascading process [1], and it also deals with many correlated systems with simpler mathematical formulations.

First we will illustrate the framework through the example of single-layer network. Next, we will extend it to multilayer networks without degree-degree correlations. Following that, we extend the analysis to more complicated scenarios of partially correlated networks and degree-degree correlated networks. More complications are added to the case when multiple dependency links per node are introduced along with correlations, as well as a single network with different types of links, which is also known as a multiplex network.

2. Single-layer network

The classic site-percolation problem in a random network [23, 24, 28, 29] gives rich phase transition phenomena for various network structures. In the simplest case, we consider a random network without any correlations, and its degree distribution, \( P(k) \), fully captures its structural property. We start by introducing a key quantity, \( x \), in to the system. This \( x \) will be similarly defined throughout this work and will play a central role in the mathematical analysis. If we randomly choose a link from the network and travel along one direction of the link, there is a probability, \( x \), that it would reach the giant component of the network, and a probability, \( 1 - x \), that it will not. (See figure 1 for an illustration.)

Suppose we randomly choose a link and find an arbitrary node, \( u \), by following this link in an arbitrary direction. The probability that the node, \( u \), has degree \( k' \) is

\[
\frac{P(k')}{\sum_k P(k)k} = \frac{P(k')}{\langle k \rangle}.
\]

For this node \( u \) to be part of the giant component, at least one of its other \( k - 1 \) out going links (other than the link we first picked) must lead to the giant component. By calculating this probability, we can write out the self-consistent equation for \( x \):

\[
x = \sum_k \frac{P(k)k}{\langle k \rangle} \left[ 1 - (1 - x)^{k-1} \right].
\]

Figure 1. Definition of \( x \). A link (red color) is chosen and a node \( u \) (in green) is found. From the three outgoing links (dashed lines) of node \( A \), one of them leads to the giant component, represented by the symbol \( \infty \). Since at least one of the three outgoing links of \( u \) leads to the giant component, the red link leads to the giant component. We define the probability of finding such a red link as \( x \).
Therefore, for a randomly chosen node \( u \), the probability that it is in the giant component is equal to the probability that at least one of its \( k \) links leads to the giant component. Thus we have:

\[
\mu^{\infty} = \sum_{k} P(k) \cdot \left[ 1 - (1 - x)^k \right],
\]

where \( 1 - (1 - x)^k \) is the probability that none of the \( k \) links of node \( u \) leads to the giant component, and \( P(k) \) is the probability that node \( u \) has degree \( k \). It is worth noting that \( \mu^{\infty} \) is also the normalized size of the giant component (i.e., the fraction of nodes in the giant component). The above equations exactly equal the results obtained by Newman et al in [28].

In the network percolation problem, when a fraction of \( 1 - p \) nodes are randomly removed from the network [28, 29] (i.e., there is a fraction of \( p \) nodes remaining), we could apply the previous equations with slight modifications. Assuming that the links of the removed nodes are still present on the network, the probability that a randomly selected link leads to the giant component is the same as before and is given by

\[
\sum_{k} \frac{P(k)k}{\langle k \rangle} \left[ 1 - (1 - x)^{k-1} \right].
\]

But since only a fraction, \( p \), of the nodes remain in the network, by calculating the probability that the randomly chosen link does not lead to the giant component, the self-consistent equation of \( x \) in equation (2) becomes:

\[
x = p \cdot \sum_{k} \frac{P(k)k}{\langle k \rangle} \cdot \left[ 1 - (1 - x)^{k-1} \right],
\]

where \( 1 - (1 - x)^{k-1} \) is the probability that at least one of the \( k - 1 \) outgoing links of node \( u \) leads to the giant component, and \( \frac{P(k)k}{\langle k \rangle} \) is the probability that \( u \) has degree \( k \), which is the same as before. The additional variable, \( p \), in front is due to the fact that only a fraction of \( p \) nodes remain in the network after removing \( 1 - p \) nodes.

Similarly, the probability that a randomly selected node is in the giant component is:

\[
\mu^{\infty} = p \cdot \sum_{k} P(k) \cdot \left[ 1 - (1 - x)^k \right].
\]

It is known that in a single network, we usually only have second-order phase transitions, such that there is no giant component when \( p \) is smaller than a critical probability, \( p_c \). Above the threshold, the giant component appears and its size increases continuously from 0 with increasing \( p \). This means that when \( p \to p_c^II \), we have \( x \to 0 \) and \( \mu \to 0 \). When \( x \to 0 \), by taking the Taylor expansion of equation (4), we obtain:

\[
x = p_c^II \cdot \sum_{k} \frac{P(k)k}{\langle k \rangle} \cdot (k - 1)x + o(x),
\]

which leads to

\[
p_c^II = \frac{\langle k \rangle}{\langle k(k-1) \rangle},
\]

which is exactly the analytical result discovered in [24]. When \( p = 1 \) (i.e., the network without attack), this is also in agreement with the results in [27]. For the Erdos–Renyi network, equation (7) yields \( p_c^II = 1/\langle k \rangle \). For a scale-free network with \( \gamma < 3 \), \( \langle k^2 \rangle \) diverges, so we would obtain \( p_c = 0 \), which is also in agreement with the known result [25, 26].

The above system is based on node percolation, in which nodes are randomly removed until the giant component disintegrates. An alternative scenario is link (bond) percolation, in which links are randomly removed from the network. In this case, we still have the same definition for \( x \), and its equation remains the same as equation (4), because a randomly selected link has the probability, \( p \), to remain in the network after removing a fraction of \( 1 - p \) links. The only difference is in equation (5), in which we need to remove \( p \) on the right side:

\[
\mu^{\infty} = \sum_{k} P(k) \cdot \left[ 1 - (1 - x)^k \right].
\]

This is due to the fact that all of the nodes remain in the network in link percolation, unlike the cases of node percolation, in which only a fraction of \( p \) remains. Hence we would obtain the same \( p_c^II \) value for both node and link percolation, but we would obtain different \( \mu^{\infty} \) values.

Before we proceed to interdependent networks, it is worth mentioning that other than the second-order phase transition mentioned above, there could also be a first-order phase transition, and the critical threshold can be labelled as \( p_c^I \). For such phase transitions, when \( p < p_c^I \), the size of the largest cluster is 0, and it abruptly jumps to a nonzero value at \( p = p_c^I \). We will see more of these examples later.
3. Multilayer interdependent network

3.1. Two-layer interdependent network

In the original and seminal work of Buldyrev et al [1], generating functions were used to study the phase transitions in a two-layer interdependent network. The system consists of two networks, A and B, with degree distributions \( P_A(k) \) and \( P_B(k) \), respectively. Networks A and B both have \( N \) nodes, and each node in network A is linked with exactly one node in network B by a dependency link, and vice versa. The dependency link is different from the connectivity links within each network, in that once a node on one end of the dependency link is removed, the other node on the other network is also removed. This corresponds to the case where the failure of a power plant in the grid network will make the connected computer system shut down due to lack of electricity. Also, any node outside the giant cluster of its own network would fail since it is disconnected from the majority of the other nodes. In the defined mutually connected giant component (MCGC), every node in the giant component is connected via the connectivity links in its own network, and its dependent node is in the giant component of the other network as well. Thus the MCGC is a steady state of the remaining network, such that no further cascading of failures would happen.

Here we present a simple method for studying the phase transition behaviors using the formulation extended from the previous section. Following the definition in equation (2), we define \( x \) as the probability that a randomly chosen link in network A leads to the giant component. Analogously, we define \( y \) as the probability that a randomly chosen link in network B leads to the giant component.

If a randomly chosen link in A leads to a node with degree \( k \), the node is in the MCGC only if at least one of its other \( k - 1 \) links leads to the giant component, and its dependent node in network B is also in the MCGC. Otherwise, this corresponds to the case where the failure of a power plant in the grid network will make the connected computer system shut down due to lack of electricity. Also, any node outside the giant cluster of its own network would fail since it is disconnected from the majority of the other nodes. In the defined mutually connected giant component (MCGC), every node in the giant component is connected via the connectivity links in its own network, and its dependent node is in the giant component of the other network as well. Thus the MCGC is a steady state of the remaining network, such that no further cascading of failures would happen.

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If a randomly chosen link in A leads to a node with degree \( k \), the node is in the MCGC only if at least one of its other \( k - 1 \) links leads to the giant component. Since both nodes \( u \) and \( u' \) are connected to the giant component, we can be sure that the initial red link leads to the MCGC. In network A, we define the probability of finding a connectivity link leading to the MCGC as \( x \). Similarly, in network B, we define the probability of finding a connectivity link leading to the MCGC as \( y \).

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Figure 2. Definition of \( x \) and \( y \) in an interdependent network. Networks A and B, both with \( N \) nodes, are connected by dependency links with one-to-one matching. Here node \( u \) is connected with \( u' \) via an interdependent link. A link (red solid line) in network A is chosen and a node, \( u \) (in green), is found following the link. Of the two outgoing connectivity links (black dashed lines) of \( u \), one leads to the giant component. Node \( u' \) in the lower network is connected with \( u \) via the dependency link (solid brown line), and it is also connected with the giant component via a connectivity link in network B. Since both nodes \( u \) and \( u' \) are connected to the giant component, we can be sure that the initial red link leads to the MCGC. In network A, we define the probability of finding a connectivity link leading to the MCGC as \( x \). Similarly, in network B, we define the probability of finding a connectivity link leading to the MCGC as \( y \).
Consequently, the probability that a randomly chosen node (either in network A or B) is in the MCGC is:

$$\mu_\infty = \sum_k P_k(k) \left[ 1 - (1 - x)^k \right] \cdot \sum_{k'} P_{k'}(k') \left[ 1 - (1 - y)^k \right],$$  \hspace{1cm} (11)

which again is the normalized size of the MCGC. Note that we do not distinguish this value in different networks, because there is a one-to-one matching between nodes in A and B, so that $\mu_\infty$ is identical for both networks.

When we randomly remove $1 - p$ fraction of nodes from network A, there is only $p$ fraction of nodes left in A. Hence, out of the original probability, $x$, that a randomly selected link leads to the MCGC, only a fraction of $p$ nodes actually remain. It is easy to write down the new expression for $x$ as:

$$x = p \cdot \sum_k \frac{P_k(k)}{\langle k \rangle} \left[ 1 - (1 - x)^{k-1} \right] \cdot \sum_{k'} P_{k'}(k') \left[ 1 - (1 - y)^k \right].$$  \hspace{1cm} (12)

Analogously, the equation for $y$ is

$$y = p \cdot \sum_k \frac{P_k(k)}{\langle k \rangle} \left[ 1 - (1 - y)^{k-1} \right] \cdot \sum_{k'} P_{k'}(k') \left[ 1 - (1 - x)^k \right].$$  \hspace{1cm} (13)

At last, we arrive at the equation of $\mu_\infty$, which is the probability that a randomly selected node in network A or B is in the MCGC:

$$\mu_\infty = p \cdot \sum_k P_k(k) \left[ 1 - (1 - x)^k \right] \cdot \sum_{k'} P_{k'}(k') \left[ 1 - (1 - y)^k \right].$$  \hspace{1cm} (14)

If we define $f_x(p)$ as the probability that a randomly selected link in the original network A(B) is not in the MCGC, with substitutions $x = p (1 - f_x)$ and $y = p (1 - f_y)$, equations (12), (13), and (14) lead to the same equations from generating functions in the seminal work [1].

In principle, equations (12) and (13) can be transformed into

$$x = F_1(p, y),$$  \hspace{1cm} (15)

and

$$y = F_2(p, x).$$  \hspace{1cm} (16)

If we cannot get the explicit formula above, the numerical computation can always be used successfully.

Usually, the phase transition for the above system is of first order at the critical point, $p^c_1$ (example given below). Therefore, at $p = p^c_1$, the two functions $x = F_1(p^c_1, y)$ and $y = F_2(p^c_1, x)$ meet tangentially with each other:

$$\frac{\partial F_1 \left( p^c_1, y \right)}{\partial y} \cdot \frac{\partial F_2 \left( p^c_1, x \right)}{\partial x} = 1. \hspace{1cm} (17)$$

For the first-order phase transition, at the the critical point $p^c_1$, the giant component is not 0, implying that we cannot use the Taylor expansion to simplify equations (12) and (13), but rather we have to solve the polynomial equations directly. It could be very difficult to obtain the explicit formula for $p^c_1$, except in the most simple distributions, but numerical methods are possible.

3.1.1. Example with random regular network

For a simple example, we assume that both networks A and B are random regular networks with $P_x(3) = P_y(3) = 1$. This means that every node in both networks has degree 3, and the nodes are randomly connected. The above equations (12), (13), and (14) then become:

$$x = p \left[ 1 - (1 - x)^2 \right] \left[ 1 - (1 - y)^2 \right],$$  \hspace{1cm} (18)

$$y = p \left[ 1 - (1 - y)^2 \right] \left[ 1 - (1 - x)^2 \right],$$  \hspace{1cm} (19)

$$\mu_\infty = p \left[ 1 - (1 - x)^3 \right] \left[ 1 - (1 - y)^3 \right].$$  \hspace{1cm} (20)

If $x \neq 0$ and $y \neq 0$, further simplification gives

$$x = F_1(p, y) = 2 - \frac{1}{p \left[ 1 - (1 - y)^2 \right]},$$  \hspace{1cm} (21)

$$y = F_2(p, x) = 2 - \frac{1}{p \left[ 1 - (1 - x)^2 \right].}$$  \hspace{1cm} (22)
Hence, the requirement for \( p_c \) of equation (17) can be written explicitly as

\[
y = F_1(x, p) = 3(1 - x)^2\left[1 - (1 - y)^3\right]^2.
\]

Solving the above three equations gives us \( x \approx 0.5446, p_c \approx 0.7588 \), and consequently the mutual giant component size, \( \mu_\infty \approx 0.6329 \).

The tangential requirement in equation (17) is presented in figure 3. When \( p = p_c \approx 0.7588 \), the curves from equations (21) and (22) touch each other at \( x = y \approx 0.5446 \), where the slopes of the two curves are equal.

Figure 3. Solving for critical value of \( p_c \) for a random regular interdependent network with degree 3. The two curves represent equations (21) and (22). At \( p = p_c \approx 0.7588 \), the curves from equations (21) and (22) touch each other at \( x = y \approx 0.5446 \), where the slopes of the two curves are equal.

3.2. N-layer interdependent network

The case of N-layer interdependent networks [7, 8, 10, 30–33] is an extension of the two-layer scenario. Assuming that there are N networks of equal numbers of nodes, each node in a layer is mutually dependent on only one node in each of the two connected dependency layers with one-to-one matching. Extending from equation (9), we obtain the probability that a randomly chosen link in network i leads to the MCGC as:
\[ x_i = \sum_{k_i} \frac{P_i(k_i)}{\langle k_i \rangle} \left[ 1 - (1 - x_i)^{k_i-1} \right] \prod_{j \neq i} \left\{ \sum_{k_j} \frac{P_j(k_j)}{\langle k_j \rangle} \left[ 1 - (1 - x_j)^{k_j} \right] \right\}, \]  

(24)

where \( \frac{P_i(k_i)}{\langle k_i \rangle} \) is the probability that a randomly chosen link in network \( i \) leading to node \( u \) has degree \( k_i \), \( [1 - (1 - x_i)^{k_i-1}] \) is the probability that at least one of the other \( k_i - 1 \) outgoing connectivity links of node \( u \) in network \( i \) leads to the MCGC, and \( \sum_{k_j} \frac{P_j(k_j)}{\langle k_j \rangle} [1 - (1 - x_j)^{k_j}] \) is the probability that at least one of the \( k_j \) connectivity links of node \( w \) in network \( j \) (node \( w \) and node \( u \) are connected by a dependency link) leads to the MCGC.

Similarly, extending equation (11), we obtain the probability that a randomly selected node is in the MCGC as:

\[ \mu^\infty = \prod_{n=1}^{N} \left\{ \sum_{k_n} P_n(k_n) \left[ 1 - (1 - x_n)^{k_n} \right] \right\}. \]  

(25)

In the percolation problem, if \( 1 - p \) fraction of nodes are randomly removed from layer \( i \), we can simply multiply equations (24) and (25) by \( p \), which is the fraction of nodes remaining in the layer after the attack:

\[ x_i = p \cdot \sum_{k_i} \frac{P_i(k_i)}{\langle k_i \rangle} \left[ 1 - (1 - x_i)^{k_i-1} \right] \prod_{j \neq i} \left\{ \sum_{k_j} \frac{P_j(k_j)}{\langle k_j \rangle} \left[ 1 - (1 - x_j)^{k_j} \right] \right\} \]  

(26)

\[ = F_i (p, x_1, x_2, \ldots, x_N), \]

\[ \mu^\infty = p \cdot \prod_{n=1}^{N} \left\{ \sum_{k_n} P_n(k_n) \left[ 1 - (1 - x_n)^{k_n} \right] \right\}. \]  

(28)

Note that equations (26) and (28) are identical to the results of equations (29) and (30) in [10] with a substitution, \( x_i = 1 - z_i \).

3.2.1. Example with random regular networks

For a simple illustration, we let the networks have the same degree distribution:

\[ P_i(3) = P_j(3) = 1, \]  

(29)

(i.e., every network is a random regular network with degree 3).

Since the equations are symmetric, and there is a one-to-one matching between every node on each network, we have the relation

\[ x_1 = x_2 = \cdots = x_N = x. \]  

(30)

Now equation (26) simplifies into

\[ x = p \left[ 1 - (1 - x)^2 \right] \left[ 1 - (1 - x)^3 \right]^{N-1}. \]  

(31)

\[ \text{Note that when } N = 2, \text{ we get exactly the equation (18) with } x = y. \]

By bringing \( x \) to the right-hand side, we could transform the equation (31) into

\[ F_i = p \left[ 1 - (1 - x)^2 \right] \left[ 1 - (1 - x)^3 \right]^{N-1} - x = 0. \]  

(32)

In this case, the critical value of \( p_c \) can be understood as the smallest value of \( p \) such that equation (32) has a real solution of \( x \) in the meaningful range of \([0, 1]\). This means that when \( p < p_c \), equation (32) only has a trivial solution of \( x = 0 \), when \( p > p_c \), there is more than one solution of \( x > 0 \), and when \( p = p_c \), one unique solution of \( x \) exist. Thus, at the critical point \( x = x_c, p = p_c \), we would have the following relation fulfilled:

\[ \frac{dF_i}{dx} \bigg|_{x=x_c} = 0, \]  

(33)

which leads to

\[ p \left\{ 2(1 - x) \left[ 1 - (1 - x)^2 \right]^{N-1} + 3(N - 1)(1 - x)^2 \left[ 1 - (1 - x)^3 \right]^{N-2} \right\} = 1 = 0. \]  

(34)

Solving the simultaneous equations (32) and (34) numerically, we are able to find out the value of \( p_c^i \). Note that for any integer value of \( N > 1 \), we have a solution of \( p_c \) and \( x_c \) in the range of \([0, 1]\), so we always have a first-order phase transition, but no second-order phase transition.
4. Percolation on multilayer interdependent networks with degree-degree correlations

Usually, in real-world networks, the connection through dependency links may not be random [7, 8, 16, 17, 22, 34–36]. In a general form, we can assume a joint probability, $P(k_a, k_b)$, for the dependency links to connect a node, $u$, with degree $k_a$ in network A and a node, $u'$, with degree $k_b$ in network B. In this case, we still assume that each node is connected with one and only one node in the other network through a dependency link.

Instead of using the independent probabilities, $P_A(k)$ and $P_B(k)$, the joint probability $P(k_a, k_b)$ is used. Thus equations (12), (13), and (14) become:

$$x = p \cdot \sum_{k_a} \sum_{k_b} \frac{k_a}{\langle k_a \rangle} P(k_a, k_B) \left[ 1 - (1 - x)^{k_a-1} \right] \left[ 1 - (1 - y)^{k_b-1} \right]$$

(35)

$$y = p \cdot \sum_{k_a} \sum_{k_b} \frac{k_B}{\langle k_B \rangle} P(k_B, k_a) \left[ 1 - (1 - y)^{k_B-1} \right] \left[ 1 - (1 - x)^{k_a-1} \right]$$

(36)

$$\mu^\infty = p \cdot \sum_{k_a} \sum_{k_b} P(k_a, k_B) \left[ 1 - (1 - x)^{k_a} \right] \left[ 1 - (1 - y)^{k_b} \right].$$

(37)

In fact, equations (35), (36), and (37) are the more general representations of equations (12), (13) and (14). In the case of [35], there is perfect correlation between the degrees of the two networks (i.e., $P(k_a, k_B) = 1$ if $k_a = k_B$; else $P(k_a, k_B) = 0$). The above equations transform into:

$$x = p \cdot \sum_k \frac{k}{\langle k \rangle} P(k) \left[ 1 - (1 - x)^{k-1} \right] \left[ 1 - (1 - y)^{k} \right]$$

(38)

$$y = p \cdot \sum_k \frac{k}{\langle k \rangle} P(k) \left[ 1 - (1 - y)^{k-1} \right] \left[ 1 - (1 - x)^{k} \right]$$

(39)

$$\mu^\infty = p \cdot \sum_k P(k) \left[ 1 - (1 - x)^{k} \right] \left[ 1 - (1 - y)^{k} \right].$$

(40)

A special case is when we have random regular networks for both A and B. The results were discussed in the previous section, since $P(k_a, k_B) = P(k_A) = P(k_B) = 1$.

The more general case of correlated systems of a multiplex network with different types of links was studied in [8]. With a similar argument, in the case of correlated N-layer interdependent networks, we could write down the equations of $x_i$ and $\mu^\infty$ from equations (26) and (28):

$$x_i = p \cdot \sum_{k_i, k_{i+1}, \ldots} P(k_i, k_{i+1}, \ldots) \frac{k_i}{\langle k_i \rangle} \left[ 1 - (1 - x_i)^{k_i-1} \right] \prod_{j \neq i} \left[ 1 - \left(1 - x_j\right)^{k_j} \right]$$

(41)

$$\mu^\infty = p \cdot \sum_{k_i, k_{i+1}, \ldots} P(k_i, k_{i+1}, \ldots) \prod_{i=1}^{N} \left[ 1 - (1 - x_i)^{k_i} \right].$$

(42)

A special case is when we have random regular networks for both A and B. The results were discussed in the previous section, since $P(k_a, k_B) = P(k_A) = P(k_B) = 1$.

The more general case of correlated systems of a multiplex network with different types of links was studied in [8]. With a similar argument, in the case of correlated N-layer interdependent networks, we could write down the equations of $x_i$ and $\mu^\infty$ from equations (26) and (28):

$$x_i = p \cdot \sum_{k_i, k_{i+1}, \ldots} P(k_i, k_{i+1}, \ldots) \frac{k_i}{\langle k_i \rangle} \left[ 1 - (1 - x_i)^{k_i-1} \right] \prod_{j \neq i} \left[ 1 - \left(1 - x_j\right)^{k_j} \right]$$

(41)

$$\mu^\infty = p \cdot \sum_{k_i, k_{i+1}, \ldots} P(k_i, k_{i+1}, \ldots) \prod_{i=1}^{N} \left[ 1 - (1 - x_i)^{k_i} \right].$$

(42)

Reference [8] provided a more general mathematical tool to solve for the critical points by using the Jacobian of the equations:

$$\det[J - I] = 0,$$

(44)

where $J$ is the Jacobian matrix with $J_{ij} = \partial F_i / \partial x_j$. Solving equations (42) and (44) gives the critical point value of $p^c$ and $x_c$ for each layer of network in the system.

5. Percolation on two-layer partially interdependent networks

In certain interdependent networks, not every node has a dependency link. It is more realistic to assume that only a fraction of nodes from each network have dependency links [15, 17]. In such systems, both first- and second-order phase transitions may occur depending on the details of the networks’ structural properties.

Let us assume two networks, A and B with degree distributions $P_A(k)$ and $P_B(k)$, and let us also assume that only a fraction of $q$ nodes from each network is connected to nodes in the other network with dependency networks. For simplicity, we let each node be connected with at most one other node through a dependency link.

In order for a randomly selected link in A to lead to the MCGC, it must satisfy two conditions. First, the node, $u$, that it directly attaches to must have at least one of its outgoing connectivity links leading to the MCGC, and the probability is the same as in the case of full dependency links given by $\sum_k \frac{p(k)}{\langle k \rangle} \left[ 1 - (1 - x)^{k-1} \right]$. New J. Phys. 17 (2015) 063025 L. Feng et al
Second, for the case of network B, there are two scenarios: there is a probability, \(1 - q\), that node \(u\) is not connected to any node in B, and then \(u\) is in the MCGC. There is a probability, \(q\), that \(u\) is connected to a node \(u'\) in B, and then at least one of the connectivity links of \(u'\) must also lead to the MCGC; the probability for this is \(q \sum_{k} P_{h}(k)[1 - (1 - y)^{k}]\). Therefore, the original equation (12) becomes:

\[
x = p \cdot \sum_{k} \frac{P_{h}(k)}{\langle k \rangle} \left[ 1 - (1 - y)^{k} \right] \cdot \left( 1 - q + q \sum_{k'} P_{h}(k')[1 - (1 - y)^{k'}] \right).
\]

(45)

The parameter \(p\) on the right-hand side takes into account that after removal of \(1 - p\) nodes in network A in the beginning of the attack, only a fraction of \(p\) nodes remain.

It is worth noting that the calculation of \(y\) is not symmetric with \(x\), because there is one-to-one matching between a node in A and a node in B. For a node \(u'\) in B, the difference is that when it has a dependency node, \(u\), in A (probability \(q\)), at least one of the connectivity links of \(u\) must lead to the MCGC (probability \(\sum_{k} P_{h}(k)[1 - (1 - x)^{k}]\)), and \(u\) must not have been removed (probability \(p\)). Thus, equation (13) becomes:

\[
y = \sum_{k} \frac{P_{h}(k)}{\langle k \rangle} \left[ 1 - (1 - y)^{k} \right] \cdot \left( 1 - q + pq \sum_{k'} P_{h}(k')[1 - (1 - x)^{k'}] \right).
\]

(46)

There is no additional parameter \(p\) in front of the right-hand side since we are not removing nodes from network B in the beginning.

Again, due to the lack of symmetry in this case, the sizes of the MCGC in A and B are expressed differently:

\[
\mu_{A}^{\infty} = p \cdot \sum_{k} \frac{P_{h}(k)}{\langle k \rangle} \left[ 1 - (1 - x)^{k} \right] \cdot \left( 1 - q + q \sum_{k'} P_{h}(k')[1 - (1 - y)^{k'}] \right)
\]

(47)

\[
\mu_{B}^{\infty} = \sum_{k} \frac{P_{h}(k)}{\langle k \rangle} \left[ 1 - (1 - y)^{k} \right] \cdot \left( 1 - q + pq \sum_{k'} P_{h}(k')[1 - (1 - x)^{k'}] \right).
\]

(48)

5.1. Example with random regular networks

For a simple illustration, we use random regular networks for both A and B, and \(P_{h}(3) = P_{h}(3) = 1\). Equations (45) and (46) simplify into:

\[
x = p (2x - x^2) \left( 1 - q + q \left[ 1 - (1 - y)^3 \right] \right),
\]

(49)

\[
y = (2y - y^2) \left( 1 - q + pq \left[ 1 - (1 - x)^3 \right] \right).
\]

(50)

Note that the cascading dynamics is not symmetric for networks A and B for we only attack network A. For example, when \(q = 0\), there is no dependency link between networks A and B. If we remove all of the nodes in network A, none of the nodes in network B is affected, and the giant component stills exist in network B. Without a loss of generality, we study the case where \(1 - p\) nodes are removed from A, and we focus on the phase transition behavior in A.

For the second-order phase transition of network A, \(x = 0\) at the critical point, \(p = p_{c}^{\infty}\). This means that when \(p \rightarrow p_{c}^{\infty}\), we have \(x \rightarrow 0\). Note that this does not imply that \(y \rightarrow 0\); we use \(y_{0}\) denote this nonzero solution at the critical point. When \(x \rightarrow 0\), from equation (50) we have

\[
y_{0} = 2 - \frac{1}{1 - q}.
\]

(51)

Only the largest solution of \(y_{0}\) in the range \([0, 1]\) is the realistic solution, so \(q\) must be in the range \([0, 0.5]\) for second-order phase transition to occur. Usually, for more general cases, we could numerically get the solution of \(y_{0}\) by iterative calculations, starting from a value close to 1. Note however, that this solution only depends on \(q\). Submitting \(y_{0}\) to equation (49) and ignoring \(x^2\) when \(x \rightarrow 0\), we have

\[
1 = p_{c}^{\infty} 2 \left( 1 - q + q \left[ 1 - (1 - y_{0})^{3} \right] \right).
\]

(52)

Thus we can obtain the critical point value of \(p\) for the second-order phase transition:

\[
p_{c}^{\infty} = \frac{1}{2 - \left( 1 - q \left[ 1 - \left( \frac{q}{1 - q} \right) \right] \right)}.
\]

(53)

For the first-order phase transition of network A, again we cannot assume that \(x \rightarrow 0\), as in the case of the second-order phase transition. Instead, by transforming equations (45) and (46), we have
as argued earlier, the critical point value $p_c$ satisfies the tangential requirement:

$$\frac{\partial F_1(p', y)}{\partial y} = \frac{\partial F_2(p', x)}{\partial x} = 1,$$

which can be written explicitly as

$$\frac{3q(y-1)^2}{p_c q(1-y)^3+1} = \frac{3p_c q(x-1)^2}{(p_c q(1-x)^3+1)} = 1.$$

Solving equations (49), (50), and (57) for $x$, $y$, and $p_c'$, we are able to get the threshold value $p_c'$ numerically.

For any given value of $q$, which determines the fraction of dependency between networks $A$ and $B$, we would have either a first-order phase transition with critical threshold $p_c'$, or a second-order phase transition with critical threshold $p_c''$. Figure 5 shows the plot of $p_c'$ and $p_c''$ versus the change in $q$. The two curves of $p_c'$ and $p_c''$ intersect at $q = 0.4$. Hence the percolation behavior is separated into two regions: When $q < 0.4$ it is a second-order transition, but when $q > 0.4$, $q = 0.4$ is the boundary point for the system.

5.1.1. Example with ER networks

For a more general example, we suppose the degree distributions of networks $A$ and $B$ are both Poisson, with average degrees $\langle k_A \rangle$ and $\langle k_B \rangle$. In this case we can use the generating function formulation [28] to simplify the expressions.

For networks $A$ and $B$, the corresponding generating functions are $G^A(x) = \sum P_A(k) x^k = e^{\langle k_A \rangle x - 1}$ and $G^B(y) = \sum P_B(k) y^k = e^{\langle k_B \rangle y - 1}$. The generating functions relating to the branching process are $G_A(x) = \sum P_A(k) x^k = e^{\langle k_A \rangle x - 1}$ and $G_B(y) = \sum P_B(k) y^k = e^{\langle k_B \rangle y - 1}$. Note that the generating functions of degree distribution and the branching process are the same for Poisson degree distributions.

Thus, equations (45) and (46) can be written as

$$x = p \left( 1 - e^{-\langle k_A \rangle x} \right) \left[ 1 - q + (1 - e^{-\langle k_B \rangle y}) \right],$$

and

$$y = \left( 1 - e^{-\langle k_B \rangle y} \right) \left[ 1 - q + pq \left( 1 - e^{-\langle k_A \rangle x} \right) \right].$$

Figure 5. $p_c'$, $p_c''$, and the giant component size, $\mu^\infty (p_c)$, of network $A$ at the critical point. We have $p_c' = p_c''$ and $\mu^\infty (p_c) = 0$ when $q = 0.4$, $\mu^\infty (q) = 0$ and increases with $q$ when $q > 0.4$. This means that $q = 0.4$ is the boundary of the second- and first-order phase transitions. There is a second-order transition when $q < 0.4$ and a first-order transition when $q > 0.4$. $q = 0.4$ is the boundary point for the system.
Again, since to the nature of Poisson degree distributions, the generating functions of degree distribution and the branching process are the same, we have \( \mu_A^\infty = x \) and \( \mu_B^\infty = y \). However, for other degree distributions, this relation is generally invalid.

Equations (58) and (59) lead to the explicit formula

\[
x = F_1(p, y) = -\frac{1}{\langle k_A \rangle} \log \left( 1 + \frac{1 - q}{pq} \frac{y}{p'q'(1 - e^{-(k_A)y})} \right),
\]

and

\[
y = F_2(p, x) = -\frac{1}{\langle k_B \rangle} \log \left( 1 + \frac{x}{pq} \frac{1}{1 - e^{-(k_B)x}} \right).
\]

For the second-order phase transition of network A, \( x \to 0 \) at the critical point. Using equation (59), we have

\[
y_0 = (1 - q) \left( 1 - e^{-\langle k_B \rangle y_0} \right).
\]

Submitting \( y_0 \) to equation (61), we can explicitly obtain the formula of the second-order phase transition critical point

\[
p_I^{II} = \frac{1}{\langle k_A \rangle} \left[ 1 - q e^{-\langle k_A \rangle y_0} \right].
\]

Here we use the first-order term in the Taylor expansion of \( e^{-\langle k_A \rangle x} \).

For the first-order phase transition of network A, using the tangential attachment of equations (60) and (61), we have

\[
\frac{\partial F_1(p', y)}{\partial y} \cdot \frac{\partial F_2(p', x)}{\partial x} = 1.
\]

Again, equations (58), (59), and (64) allow us obtain \( p_I' \) numerically. Similar to the previous example, we can find out the value of \( q \) such that

\[
p_I' = p_I^{II}.
\]

This would allow us to find the boundary between the first- and second-order phase transitions (i.e., the triple-point value).

Usually, the above three equations (60), (60), and (64) have no explicit formula to allow us to solve them directly. To detect \( p_I' \), we could brut-search for \( p \in [0, 1] \) according to the following calculations. For a given \( p \), we run equations (60) and (61) iteratively and obtain the solutions (fixed point). Then we put this fixed point, \( x, y, \) into equation (64). If equation (64) is equal to 1, this \( p \) should be \( p_I' \). For many more general cases, such as when the degree distribution is scale free, we have no explicit formula for equation (64), so we must use the numerical way to the degree partial derivative at the fixed point, \( x, y \).

Numerical simulation could help us to find the critical points without solving the equations [37]. As demonstrated in figure 6, for second-order phase transitions, the second-largest cluster size, \( \mu_2 \), is maximum at the critical point; repeated simulations for different values of \( p \) can be carried out to find out the peak \( \mu_2 \), to identify \( p_c \). For first-order phase transitions, the number of iterations (NOI) is at maximum, so one can identify the \( p_c \) as the point where the maximum NOI is located. Here iteration refers to the cascading of failures from one network to the other.

### 6. Single network with different types of links

Dependency links could also exist in single-layer networks [9, 37, 38], and the form of dependency could vary. This means that while the nodes in a single network are generally connected via connectivity links, some of the nodes have mutual dependencies.

Suppose that for a given network, certain pairs of nodes are mutually dependent on each other [37]. In this case, if node \( w \) is dependent on node \( u \), \( w \) must depend and only depend on node \( u \). Since the dependent nodes are in the same network, we could get the equation of \( x \) simply by replacing \( y \) with \( x \) in equation (12) of the two interdependent networks:
Note that the second $p$ on the right side is there because both the node itself and its dependent node have the probability, $p$, of remaining after the initial attack because they are in the same single network. Correspondingly, we can write down the size of the giant component as:

$$x = \left\{ p \sum_k \frac{P(k)}{\langle k \rangle} \left[ 1 - (1 - x)^{k-1} \right] \right\} \cdot \left\{ p \sum k P(k) \left[ 1 - (1 - x)^k \right] \right\}.$$  

(66)

Note that the second $p$ on the right side is there because both the node itself and its dependent node have the probability, $p$, of remaining after the initial attack because they are in the same single network. Correspondingly, we can write down the size of the giant component as:

$$\mu^* = \left\{ p \sum P(k) \left[ 1 - (1 - x)^k \right] \right\}^2.$$  

(67)

This corresponds to the special case in [37], with every node in a dependency pair.

For a more general case, a network could have dependency groups [38], which are certain groups of nodes that have dependency relations (as shown in figure 7), such that the removal of any one node would result in the removal of all the other nodes in the group. Given the probability distribution of the group size, $g(s)$, where $s$ is the number of nodes in a group, we would obtain the following equation of $x$:

$$x = \left\{ p \sum_k \frac{k P(k)}{\langle k \rangle} \left[ 1 - (1 - x)^{k-1} \right] \right\} \cdot \sum_s \frac{s \cdot g(s)}{\langle s \rangle} \left\{ p \sum_k P(k) \left[ 1 - (1 - x)^k \right] \right\}^{s-1}.$$  

(68)

Note that in this case, every node in the dependency group of size $s$ needs to be in the giant component in order for the group to be in the giant cluster, or else the whole dependency group would be removed. Here, $\sum s g(s) / \langle s \rangle$ is the probability that a node is a group of size $s$, and $\left\{ p \sum_k P(k) \left[ 1 - (1 - x)^k \right] \right\}^{s-1}$ is the probability that every other $s - 1$ node is in the giant cluster. Using similar arguments, we could write down the equation for $\mu^*$.
\[ \mu^A = \sum_{s} \frac{Z \cdot g(s)}{\langle s \rangle} \left\{ P_A \left( k_{in}^A, k_{out}^A \right) \frac{1 - (1 - x_A)^{k_{in}^A - 1}}{\langle k_{in}^A \rangle} \cdot \left[ 1 - \left( 1 - y_{k_{in}^A} \right)^{k_{out}^A} \right] \right\}^s. \] (69)

From here, we could study the critical phase transition behaviors by solving the equations using the techniques in the previous sections.

7. Percolation on two layer with many correlated links

In certain networks, including brain networks, it is observed that the dependency links are not always one-to-one matching, but could be one-to-many, and extensive correlations and dependencies between nodes could exist, as proposed by [39]. Reference [7] discovered that brain networks are wired in such a way that stability is maximized.

To examine the critical phase transition behaviors of such networks, additional parameters need to be defined to specify the structure. Assuming two networks, A and B, we denote \(k_n^A\) (\(k_n^B\)) as the degree of connectivity links of a node in A (B), and \(k_{out}^A\) (\(k_{out}^B\)) \(\geq 1\) as its dependency degree, which leads to nodes of the other network. The joint probability that a node \(u\) in A has \(k_{in}^A\) connectivity links and \(k_{out}^A\) dependency links is denoted by \(P_A\left(k_{in}^A, k_{out}^A\right)\). The conditional probability, \(P_{AB}\left(k_{in}^B, k_{out}^B\right)\), is that given a node \(u\) in A with degree \(k_{in}^A\), the probability that any of its dependent nodes in B is of degree \(k_{in}B\). Similar definitions carry over to network B for \(P_B\left(k_{in}^B, k_{out}^B\right)\) and \(P_{AB}\left(k_{in}^A, k_{out}^B\right)\). During an attack, a fraction of \(1 - p_1\) nodes in A and \(1 - p_2\) nodes in B are removed. This is contrary to our previous examples, in which only one network is being attacked at the beginning.

Here we let \(x_A(x_B)\) be the probability that on following an arbitrary connectivity link in network A (B), we reach a node leading to the giant component. For a dependency link between node u with connectivity degree \(k_{in}^A\) and node v in B, \(y_{k_{in}^B}\) is defined as the probability that this link from \(u\) leads to the giant component, and \(y_{k_{in}^B}\) is defined similarly for a dependency link from B to A. Thus we have the following self-consistent equations:

\[ x_A = p_1 \sum_{k_{in}^A, k_{out}^A} \left\{ P_A\left(k_{in}^A, k_{out}^A\right) \frac{1 - (1 - x_A)^{k_{in}^A - 1}}{\langle k_{in}^A \rangle} \cdot \left[ 1 - \left( 1 - y_{k_{in}^A} \right)^{k_{out}^A} \right] \right\}, \] (70)

\[ x_B = p_2 \sum_{k_{in}^B, k_{out}^B} \left\{ P_B\left(k_{in}^B, k_{out}^B\right) \frac{1 - (1 - x_B)^{k_{in}^B - 1}}{\langle k_{in}^B \rangle} \cdot \left[ 1 - \left( 1 - y_{k_{in}^B} \right)^{k_{out}^B} \right] \right\}, \] (71)

where \(P_A\left(k_{in}^A, k_{out}^A\right)\) is the probability that a randomly chosen link in network A leading to node \(u\) has connectivity degree \(k_{in}^A\) and a dependency degree \(k_{out}^A\), \(1 - (1 - x_A)^{k_{in}^A - 1}\) is the probability that at least one of the other \(k_{in}^A - 1\) outgoing connectivity links of node \(u\) leads to the giant component, and \(1 - (1 - y_{k_{in}^A})^{k_{out}^A}\) is the probability that at least one of the \(k_{out}^A\) dependency links of node \(u\) leads to the giant component.

For the probability \(y_{k_{in}^A}\) and \(y_{k_{in}^B}\), we have

\[ y_{k_{in}^A} = p_2 \sum_{k_{in}^A} P_{AB}\left(k_{in}^B | k_{in}^A\right) \frac{1 - (1 - x_B)^{k_{in}^B}}{\langle k_{in}^B \rangle}, \] (72)

\[ y_{k_{in}^B} = p_1 \sum_{k_{in}^B} P_{AB}\left(k_{in}^A | k_{in}^B\right) \frac{1 - (1 - x_A)^{k_{in}^A}}{\langle k_{in}^A \rangle}, \] (73)

where \(1 - (1 - x_A)^{k_{in}^A}\) is the probability that at least one of the \(k_{in}^A\) dependency neighbors (in network B) of node \(u\) (which has degree \(k_{in}^A\)) is in the giant component. Finally, we can write down the probabilities that a randomly chosen node is in the giant component:

\[ \mu_A^* = p_1 \sum_{k_{in}^A, k_{out}^A} P_A\left(k_{in}^A, k_{out}^A\right) \frac{1 - (1 - x_A)^{k_{in}^A} \cdot \left[ 1 - \left( 1 - y_{k_{in}^A} \right)^{k_{out}^A} \right]}{\langle k_{in}^A \rangle}, \] (74)

\[ \mu_B^* = p_2 \sum_{k_{in}^B, k_{out}^B} P_B\left(k_{in}^B, k_{out}^B\right) \frac{1 - (1 - x_B)^{k_{in}^B} \cdot \left[ 1 - \left( 1 - y_{k_{in}^B} \right)^{k_{out}^B} \right]}{\langle k_{in}^B \rangle}, \] (75)

which are straightforward.

In general, since the above system is too complicated to have analytical solutions, numerical methods are usually preferred. Here we illustrate a simple yet efficient numerical method called, binary search, to detect the
critical point for network A. The same method can be easily applied to network B. First we set up the initial starting points with \( p_{-} = 0 \) and \( p_{+} = 1 \), and we let \( p_{i} = \frac{p_{-} + p_{+}}{2} \). If \( \mu_{A}(p) = 0 \), we change the value of \( p_{+} \) by letting \( p_{-} = p_{i} \); otherwise, we change \( p_{-} \) by letting \( p_{+} = p_{i} \). A \( p_{c} \) value with high precision could usually be reached with 20 such iterations, as this algorithm converges exponentially.

8. Conclusion

In this work, we have provided a specific mathematical framework to review the critical phase transition behavior of interdependent networks, otherwise known as the network of networks. Starting from single random networks, we have shown that by defining two key mathematical quantities—the probabilities of finding a link/node in the final giant component—one is able to directly write down the sets of self-consistent equations of these quantities without going through the iterative process of cascading failures in stages. This methodology greatly simplifies the mathematical analysis in complicated network structures, especially in very complex systems involving correlations and multiple dependency links per node. In this self-consistent probabilities approach, we focus on the final equilibrium state of the networks; the original generating function framework in [1] managed to lend insight on the process of cascading failures, which is not covered in this paper.

There have been many other works that we have not included here. For example, [9] analyzed multiplex-directed networks in the context of social networks. [40–42] studies evolutionary games. Interdependent networks with spatial constraint [43, 44] have been shown to exhibit unique phase transition behaviors, though we have not discussed them due to their analytical difficulties. In this work, our focus is to provide a mathematical overview using the specific technique of simplified self-consistent probabilities. The recursive mapping method [1] yields the same results, but we demonstrated that this particular method could greatly simplify the mathematical derivations for a wide range of complicated systems.

Although this method proves to be applicable to a wide range of network systems in studying their percolation behaviors, caution must be taken when implementing it. It is crucial that the self-consistent equations be carefully constructed, such that every component of the equations strictly follows the branching process underlying the percolation behaviors.

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