Abstract

We compute the simplest non-trivial Operator Product Expansion of Wilson-’t Hooft loop operators in $\mathcal{N} = 4$ and $\mathcal{N} = 2$ Super-Yang-Mills theory with gauge group $G = PSU(3)$. This amounts to finding the Euler characters of certain vector bundles, describing electric degrees of freedom of loop operators entering the OPE, over moduli spaces of BPS states in the presence of loop operators.
Contents

1 Introduction 3

2 Review of moduli spaces $\mathcal{M}$ and $X$ 5

3 Computing OPE $WT_{w_1aw_1+bw_2}WT_{w_1,0}$ 7

3.1 $I_{N=2}(\mathcal{M}, \mathcal{V}_{a,b})$ and $I_{N=4}(\mathcal{M}, \mathcal{V}_{a,b})$ 7

3.2 $I_{N=2}(X, \mathcal{V}_{a,b}^{\text{bulk}})$ and $I_{N=4}(X, \mathcal{V}_{a,b}^{\text{bulk}})$ 9

3.3 Finding the OPE 12

4 Conclusion 13

A Computing $I_{N=4}(\mathcal{M}, \mathcal{V})$ 14

A.1 $R^p\pi_*\Omega_{\mathcal{M}}$ 14

A.2 $R^p\pi_*\Omega_{\mathcal{M}}^2$ 16

A.3 $H^j(\mathcal{M}, \Omega_{\mathcal{M}}^p \otimes \mathcal{V})$ 17

B $I_{N=4}(\mathbb{T}^2, \mathcal{O}_{\mathbb{T}^2}(3m))$ for $m \geq 1$ 18

B.1 $L^2$ Dolbeault cohomology of $\mathcal{O}_X(m)$ with $m \geq 1$ 18

B.2 $L^2$ Dolbeault cohomology of $\Omega_X^k(3m)$ with $m \geq 1$ 18

B.3 $L^2$ Dolbeault cohomology of $\Omega_X^2(3m)$ with $m \geq 1$ 20

B.4 $L^2$ Dolbeault cohomology of $\Omega_X^3(3m)$ and $\Omega_X^4(3m)$ with $m \geq 1$ 21

C Some evidence in support of the ‘vanishing assumption’ 23

C.1 $H^1_{D,L^2}(X, \mathcal{O}_X(n))$ 23

C.2 $H^1_{D,L^2}(X, \mathcal{O}_X(1))$ 25

C.3 $H^1_{D,L^2}(X, \mathcal{O}_X(-1))$ 26

C.4 $H^2_{D,L^2}(X, \mathcal{O}_X(-1))$ 27

C.5 $H^2_{D,L^2}(X, \mathcal{O}_X(-2))$ 28

C.6 $H^2_{D,L^2}(X, \mathcal{O}_X(3))$ for $j = 2, 3, 4$ 28

C.7 $H^2_{D,L^2}(X, \mathcal{O}_X(-3))$ 29

D Useful formulae for computing the norm 30
1 Introduction

Wilson loop operators [1] and ’t Hooft loop operators [2], [3] are famous examples of non-local observables in gauge theory. The Operator Product Expansion (OPE) of these operators contains important information about the theory. The product of parallel Wilson loops is determined by the representation ring of the gauge group $G$, while S-duality conjecture [4] predicts that product of parallel ’t Hooft loops is controlled by the representation ring of the Langlands dual group $L^G$. This prediction has been verified in [5] based on the earlier mathematical result [6].

Yang-Mills theory also admits mixed Wilson-’t Hooft (WH) loop operators. As explained in [3], at zero $\theta$–angle they are labeled by elements of the set

$$\hat{\Lambda}(G)/\mathcal{W} = \left(\Lambda_w(G) \oplus \Lambda_w(L^G)\right)/\mathcal{W},$$

where $\Lambda_w(G)$ is the weight lattice of $G$ and $\mathcal{W}$ is the Weyl group (which is the same for $G$ and $L^G$). In $\mathcal{N} = 4$ or $\mathcal{N} = 2$ Super-Yang-Mills (SYM) theory these mixed operators can be made supersymmetric preserving one quarter of the original supersymmetry.

In [7] an approach how to compute the product of WH loop operators in $\mathcal{N} = 2$ gauge theory was outlined and the OPEs were actually computed in $\mathcal{N} = 4$ SYM with gauge group $G = SU(2)$ and $G = PSU(2)$. More recently we determined the basic geometric ingredients [8] required for the computation of the OPE of Wilson-t Hooft loop operators in $\mathcal{N} = 4$ SYM and $\mathcal{N} = 2$ SYM with gauge group $G = PSU(3)$. In this paper we use these ingredients to obtain the simplest non-trivial OPE of WH operators in $\mathcal{N} = 4$ and $\mathcal{N} = 2$ SYM theory for $G = PSU(3)$:

$$WT_{w_1, \nu} \times WT_{w_2, 0} = WT_{2w_1, \nu} + \sum_j (-)^{s_j} WT_{w_2, \nu_j}$$

where magnetic charge $w_1(w_2)$ is the highest weight of a fundamental (anti-fundamental) representation of $L^G = SU(3)$ and electric charge $\nu = aw_1 + bw_2$ is the highest weight of $G$, i.e. $a + 2b = 0 \mod 3$. The electric weights $\nu_j$ and signs $(-)^{s_j}$ on the right side of (1) are explicitly determined in Section 3 for some values of $a, b$ and the prescription how to compute them for general $a, b$ is provided.

Our approach uses the holomorphic-topological twist [9] of the $\mathcal{N} = 2$ gauge theory and the connection between BPS configurations in the presence of ’t Hooft operators and solutions of 3d Bogomolny equations with magnetic sources [5], [10]. To determine the right side of (1) in $\mathcal{N} = 4$ SYM, we have to compute the Euler characters $I_{\mathcal{N}=4}(\mathcal{M}, V_{a,b})$ and $I_{\mathcal{N}=4}(X, V_{a,b}^{\text{bulk}})$ of certain vector bundles $V_{a,b}$ and $V_{a,b}^{\text{bulk}}$ on moduli spaces $\mathcal{M}$ and $X$ whose geometry was found in [8] and is reviewed in Section 2. Similarly, to compute (1) in $\mathcal{N} = 2$ SYM we compute the holomorphic Euler characters $I_{\mathcal{N}=2}(\mathcal{M}, V_{a,b})$ and $I_{\mathcal{N}=2}(X, V_{a,b}^{\text{bulk}})$. The (holomorphic) Euler characters compute with sign the ground states of appropriate supersymmetric quantum mechanics (SQM) with the BRST operator acting as the covariant Dolbeault operator. This SQM arises as the result of quantizing the gauge theory on $\mathbb{R} \times I \times \mathcal{C}$ in the presence of Wilson-’t Hooft operators along $\mathbb{R}$. Here $I$ is an interval and $\mathcal{C}$ is a Riemann surface, and boundary
conditions\footnote{For an explicit choice of boundary conditions see \cite{7}.} at the two ends of $I$ are such that without any magnetic sources there is unique vacuum. The WH operators $WT_{w_1,\nu}$ and $WT_{w_1,0}$ are taken to sit at the same point at $C$ and at different\footnote{The twist \cite{9} ensures that there is no dependence on the distance between points along $I$.} points along $I$.

The compact moduli space $\mathcal{M}$ is obtained by blowing-up certain singular complex 4-fold which is the compactification of the moduli space of solutions of 3d Bogomolny equations in $I \times C$ with a single source characterized by magnetic charge $2w_1$. The blow-up procedure produces exceptional divisor $D$ in $\mathcal{M}$. The non-compact moduli space $X$ - referred to as the ‘bulk part’ of $\mathcal{M}$ - is obtained by removing from $\mathcal{M}$ the vicinity of $D$, i.e. the total space of the normal bundle of $D$ in $\mathcal{M}$. As we review in Section 2, $\mathcal{M}$ is a $\mathbb{P}^2$ fibration over $\mathbb{P}^2$ and $X = T\mathbb{P}^2$. In computing the Euler characters of bundles over $\mathcal{M}$ (in Appendix A) we use the Leray spectral sequence. Meanwhile, to compute these characters for bundles over $X$ we use $L^2$ Dolbeault cohomology for the metric written down in \cite{8} and reviewed in equation (4) below.

The unknown electric weights $\nu_i$ and signs $s_i$ in the right side of (1) can be read off from the so called bubbled contribution:

\[
I^{\text{bubble}_{N=2}}(\nu_{a,b}) = I_{N=2}(\mathcal{M}, \nu_{a,b}) - I_{N=2}(X, \nu_{a,b}^{\text{bulk}})
\]

and

\[
I^{\text{bubble}_{N=4}}(\nu_{a,b}) = I_{N=4}(\mathcal{M}, \nu_{a,b}) - I_{N=4}(X, \nu_{a,b}^{\text{bulk}})
\]

decomposed into representations of the $SU(2)_{\alpha_1} \times U(1)$ which is a subgroup of the gauge group unbroken in the presence of ’t Hooft operator with magnetic charge $w_2$. The existence of bubbled contribution is due to monopole bubbling \cite{5} which occurs when the magnetic charge of the ’t Hooft operator decreases by absorbing a BPS monopole. This process is possible because the moduli space of solutions of 3d Bogomolny equations in the presence of magnetic source with charge $2w_1$ is non-compact.

Note that the computations in this paper can be viewed as a UV method. There are alternative IR methods of studying loop operators \cite{11},\cite{12},\cite{13},\cite{14}. Furthermore, loop operators in a certain class \cite{15} of $\mathcal{N} = 2$ SYM theories were studied \cite{16},\cite{17},\cite{18},\cite{19},\cite{20} using connection with 2d Conformal Field Theory \cite{21}. In the future we hope to compare our answers for the OPE of Wilson-’t Hooft operators in $\mathcal{N} = 2$ and $\mathcal{N} = 4$ SYM with gauge group $G = PSU(3)$ with the OPE which can be obtained from these alternative methods.

This note is organized as follows. In Section 2 we review the geometry of moduli spaces $\mathcal{M}$ and $X$ together with vector bundles over them. In Section 3 we explain how to compute the OPE (1) and provide explicit examples. We compute the Euler characters for vector bundles on $\mathcal{M}$ in Appendix A and, under certain vanishing assumption, for vector bundles on $X$ in Appendix B. We provide evidence supporting the vanishing assumption in Appendix C and collect useful formulae for computing the $L^2$ norms of differential forms taking values in line bundles on $X$ in Appendix D.
2 Review of moduli spaces $\mathcal{M}$ and $X$

Here we review basic geometric ingredients [8] required for the computation of the OPE of Wilson-t Hooft loop operators in $\mathcal{N} = 4$ SYM and $\mathcal{N} = 2$ SYM with gauge group $G = PSU(3)$. These include the geometry of $\mathcal{M}$ - the moduli space of BPS configurations in the presence of two ‘t Hooft operators each with fundamental magnetic weight, as well as the geometry of its ‘bulk part’ - open subspace $X$ obtained by excision of the vicinity of the certain blown-up region in $\mathcal{M}$. We also present vector bundles over $\mathcal{M}$ and $X$ which encode electric degrees of freedom of the loop operators in the OPE.

It was shown in [8] that $\mathcal{M}$ is defined by a hypersurface $y_a U^a = 0$ in a toric 5-fold $Y_5$. The weights under the two $\mathbb{C}^*$ actions are

$$\begin{array}{c|c|c|c|c|c|c}
\text{first} & U^1 & U^2 & U^3 & \Lambda & y_1 & y_2 & y_3 \\
\text{second} & 0 & 0 & 0 & 1 & 1 & 1 & 1
\end{array}$$

The 5-fold $Y_5$ is a $\mathbb{P}^3$ fibration over $\mathbb{P}^2$. Here $U^1, U^2, U^3$ are homogenous coordinates on the base $\mathbb{P}^2_U$ and $\Lambda, y_1, y_2, y_3$ are homogenous coordinates on the fiber $\mathbb{P}^3$. Meanwhile, the 4-fold $\mathcal{M}$ is $\mathbb{P}^2$ fibration over $\mathbb{P}^2_U$. General $PSU(3)$ invariant metric on $\mathcal{M}$ was written out and confirmed in [8] by reproducing correctly the OPE of ‘t Hooft operators.

Let $\pi: \mathcal{M} \mapsto \mathbb{P}^2_U$. The vector bundle $\mathcal{V}_{a,b}$ on $\mathcal{M}$, which arises in computing $WT_{w_1, a w_1 + b w_2} WT_{w_1, 0}$, is a pull-back from the base:

$$\mathcal{V}_{a,b} = \pi^* \mathcal{V}_{a,b}$$

where $\mathcal{V}_{a,b}$ was determined in [8] to be

$$\mathcal{V}_{a,b} = \mathcal{O}_{\mathbb{P}^2}(-(2a + b)) \otimes S^{[b]} \tilde{V}_1$$

with $\tilde{V}_1$ a vector bundle on satisfying the properties

$$r(\tilde{V}_1) = 2, \quad c_1(\tilde{V}_1) = 0, \quad c_2(\tilde{V}_1) = 1, \quad H^0(\mathbb{P}^2, \tilde{V}_1) = 1, \quad H^p(\mathbb{P}^2, \tilde{V}_1) = 0 \quad p = 1, 2.$$  \hspace{1cm} (3)

These properties are a consequence of the fact that $\tilde{V}_1$ admits the following explicit connection in the patch $U^1 \neq 0$ (see [8], section 6):

$$A_{(1,0)} = \left( A_{(0,1)} \right)^\dagger, \quad A_{(0,1)} = i (\mathcal{G}) G^{-1} \quad \mathcal{G} = \begin{pmatrix} \alpha & \alpha^{-1} \beta \\ \alpha^{-1} \beta & \alpha^{-1}(1 + \beta) \end{pmatrix}$$

with

$$\alpha = \frac{y^{1/4}}{(y - 1)^{1/2}}, \quad \beta = -\frac{\bar{z}_1}{(y - 1)z_2} \quad \text{or} \quad \beta = \frac{\bar{z}_2}{(y - 1)z_1},$$

where

$$y = 1 + |z_1|^2 + |z_2|^2, \quad z_1 = \frac{U^2}{U^1}, \quad z_2 = \frac{U^3}{U^1}.$$  \hspace{1cm} (3)

\footnote{Both choices of $\beta$ give the same connection.}
Note that the bulk part $X$ of $\mathcal{M}$ is $T\mathbb{P}^2_U$. Indeed, $X$ was defined in [8], section 4, by $t_iU^i = 0$ in the region $\Lambda \neq 0$ of $Y_5$, where $t_i = \frac{\alpha}{\bar{\chi}}$ is a (global) coordinate on the fibre of $\mathcal{O}_{\mathbb{P}^2}(2)$ for each $i = 1, 2, 3$.

A general $PSU(3)$ invariant Kähler form on $X$ was obtained in [8], section 4:

$$(-iJ) = \sum_{j=1}^{4} \tilde{f}_j(\tilde{s})e_J \wedge \bar{e}_J$$

where in a patch $U^1 \neq 0$

$$\tilde{s} = \frac{\bar{x}}{y^2}, \quad \bar{x} = t_2\bar{\alpha}^2 + t_3\bar{\alpha}^3, \quad y = 1 + |z_1|^2 + |z_2|^2, \quad z_1 = \frac{U^2}{U^1}, \quad z_2 = \frac{U^3}{U^1}$$

with

$$\bar{\alpha}^2 = \bar{t}^2 + z^1(\bar{t}^2\bar{z}_1 + \bar{t}^3\bar{z}_2), \quad \bar{\alpha}^3 = \bar{t}^3 + z^1(\bar{t}^2\bar{z}_1 + \bar{t}^3\bar{z}_2).$$

Explicitly, we may take

$$\tilde{f}_1(\tilde{s}) = \frac{1}{2\int(1 + \tilde{s})^{3/2}}, \quad \tilde{f}_2(\tilde{s}) = \frac{1}{\sqrt{1 + \tilde{s}} + \frac{1}{\tilde{s}}}, \quad \tilde{f}_3(\tilde{s}) = \frac{1}{\tilde{s}^2}(1 - \frac{1}{\sqrt{1 + \tilde{s}}}), \quad \tilde{f}_4(\tilde{s}) = \frac{2}{\sqrt{1 + \tilde{s}}}$$

This metric is used in Appendix C for explicit computation of $L^2$ Dolbeault cohomology of vector bundles over $X$.

Note that the bundle $\mathcal{V}_{b,\text{bulk}}$ which describes electric degrees of freedom in Wilson-'t Hooft operator $WT_{2w_1, aw_1 + bw_2}$ is again a pull back from the base $\mathbb{P}^2_U$. We use the following connection on $\mathcal{V}_{b,\text{bulk}}$:

$$\mathcal{A}_{(1, 0)} = \left(\mathcal{A}_{(0, 1)}\right)^{\dagger}, \quad \mathcal{A}_{(0, 1)} = i\left(\partial \mathcal{G}_X\right) \mathcal{G}_X^{-1} \quad \mathcal{G}_X = \frac{h(n)(\tilde{s})}{y^2} \mathcal{G} \quad n = 2a + b$$

where the factor $h(n)(\tilde{s})$ in $\mathcal{G}_X$ describes the lift of the connection on $\mathcal{O}_{\mathbb{P}^2}(n)$ to $X$:

$$h(n)(\tilde{s}) \sim \tilde{s}^{-n/4} \quad \tilde{s} \rightarrow \infty, \quad h(n)(\tilde{s}) \sim 1 \quad \tilde{s} \rightarrow 0$$

The asymptotic at $\tilde{s} \rightarrow \infty$ is chosen in such a way that the norm (evaluated at the point $z_1 = z_2 = 0$ on the base $\mathbb{P}^2_U$) of the holomorphic section $s_{\text{hol}} = t_i$ of $\mathcal{O}_X(2)$ approaches a constant, i.e. we go to the unitary trivialization

$$s_{\text{unit}} = \mathcal{G}_X s_{\text{hol}}$$

and require the pointwise norm $|s_{\text{unit}}|^2$ at $z_1 = z_2 = 0$ to approach a constant. The reason is that $\tilde{s} \rightarrow \infty$ limit corresponds to approaching the blown-up region in $\mathcal{M}$ and, as shown in [8], $X$ behaves as $\mathbb{C}^2/\mathbb{Z}_2$ singularity fibered over $\mathbb{P}^2_U$, i.e. $\mathbb{P}^2_U$ effectively collapses to a point in this limit and, therefore, $t_i$ should become a section of a trivial bundle.

The postulated behavior at $\tilde{s} \rightarrow 0$ (i.e. away from the blown-up region) ensures that the connection on $X$ is the same as the connection on the total space $\mathcal{M}$ - a pull-back connection from $\mathbb{P}^2_U$.

---

4We use $e_1 = -\tilde{s}^2E_1$, $e_{2, 4} = \sqrt{\tilde{s}}E_{2, 4}$, $e_3 = \tilde{s}E_3$ to relate $\tilde{f}_K(\tilde{s})$ for $\tilde{s} = s^{-1}$ with $g_K(s)$ given in equation (27) in [8].

5$\tilde{s} \rightarrow \infty$ near $\Lambda = 0$ region in $\mathcal{M}$.
3 Computing OPE $WT_{w_1, aw_1 + bw_2}WT_{w_1, 0}$

In the approach [7, 8], to find the OPE $WT_{w_1, aw_1 + bw_2}WT_{w_1, 0}$ in $\mathcal{N} = 4$ SYM we first have to compute the Euler characters of vector bundles on $\mathcal{M}$ and $X$

$$I_{\mathcal{N}=4}(\mathcal{M}, V_{a,b}) = \sum_{\alpha=0}^{4} (-)^{\alpha} I^{(\alpha)}(\mathcal{M}, V_{(a,b)}), \quad I_{\mathcal{N}=4}(X, V_{a,b}^{bulk}) = \sum_{\alpha=0}^{4} (-)^{\alpha} I^{(\alpha)}(X, V_{a,b}^{bulk})$$

where for a vector bundle $V$ on a complex 4-fold $Y$ we denote

$$I^{(\alpha)}(Y, V) = \sum_{p=0}^{4} (-)^p H^p_{D, L^2}(Y, \Omega^\alpha_Y \otimes V).$$

Note that Kodaira-Serre duality implies

$$I_{\mathcal{N}=4}(\mathcal{M}, V_{-a,-b}) = I_{\mathcal{N}=4}(\mathcal{M}, V_{a,b}), \quad I_{\mathcal{N}=4}(X, V_{-a,-b}^{bulk}) = I_{\mathcal{N}=4}(X, V_{a,b}^{bulk}).$$

To determine the same OPE in $\mathcal{N} = 2$ SYM, we have to compute the holomorphic Euler characters of vector bundles on $\mathcal{M}$ and $X$

$$I_{\mathcal{N}=2}(\mathcal{M}, V_{a,b}) = I^{(0)}(\mathcal{M}, V_{a,b}), \quad I_{\mathcal{N}=2}(X, V_{a,b}^{bulk}) = I^{(0)}(X, V_{a,b}^{bulk}).$$

The characters for vector bundles on $\mathcal{M}$ and $X$ are computed in Section 3.1 and Section 3.2. We use these characters in Section 3.3 to obtain the bubbled contribution and read off the unknown electric weights $\nu_j$ and signs $s_j$ in (1).

3.1 $I_{\mathcal{N}=2}(\mathcal{M}, V_{a,b})$ and $I_{\mathcal{N}=4}(\mathcal{M}, V_{a,b})$

In Appendix A we applied the Leray spectral sequence to compute, for any pull-back bundle $V = \pi^* V$ on $\mathcal{M}$,

$$I_{\mathcal{N}=4}(\mathcal{M}, V) = \sum_{p,q} (-)^{p+q} H^p(\mathcal{M}, \Omega^q_M \otimes V) = 3 \sum_{q=0}^{2} (-)^q \hat{\chi}(\Omega^q_{\mathbb{P}^2} \otimes V)$$

$$I_{\mathcal{N}=2}(\mathcal{M}, V) = \sum_p (-)^p H^p(\mathcal{M}, V) = \hat{\chi}(V),$$

where $\hat{\chi}$ denotes the weighted sum

$$\hat{\chi}(V) := \sum_p (-)^p H^p(\mathbb{P}^2, V)$$

of bundle cohomologies on $\mathbb{P}^2$.  

7
Let us first compute $I_{N=4}(\mathcal{M}, \mathcal{V}_{-1,-1}) = I_{N=4}(\mathcal{M}, \mathcal{V}_{1,1})$ and $I_{N=2}(\mathcal{M}, \mathcal{V}_{-1,-1})$ for

$$\mathcal{V}_{-1,-1} = \pi^*\mathcal{V}_{-1,-1}, \quad \mathcal{V}_{-1,-1} = \mathcal{O}_{\mathbb{P}^2}(3) \otimes \tilde{V}_1.$$  

The properties (3) allow us to identify $\tilde{V}_1$ with the vector bundle that fits into the following exact sequence

$$0 \mapsto \mathcal{O}_{\mathbb{P}^2} \mapsto \tilde{V}_1 \mapsto I_p \mapsto 0,$$

where $I_p$ is the ideal sheaf of a point on $\mathbb{P}^2$. It is useful to recall that $I_p$ fits into another exact sequence

$$0 \mapsto I_p \mapsto \mathcal{O}_{\mathbb{P}^2} \mapsto \mathcal{O}_p \mapsto 0,$$

where $\mathcal{O}_p$ is the skyscraper sheaf supported on $p$. We further tensor (8) and (9) by $\mathcal{O}_{\mathbb{P}^2}(3) \otimes \Omega_{\mathbb{P}^2}$

and use the corresponding long exact sequences for cohomology groups together with Kodaira-Serre duality to find:

$$H^0(\mathbb{P}^2, \mathcal{V}_{-1,-1}) = 2\mathcal{V}_{(3,0)} - \mathcal{V}_{(0,0)}, \quad H^0(\mathbb{P}^2, \Omega_{\mathbb{P}^2}^{\otimes} \mathcal{V}_{-1,-1}) = 2\mathcal{V}_{(1,1)} - 2\mathcal{V}_{(0,0)}$$

(10)

$$H^0(\mathbb{P}^2, \Omega_{\mathbb{P}^2}^{\otimes} \mathcal{V}_{-1,-1}) = \mathcal{V}_{(0,0)}, \quad H^p(\mathbb{P}^2, \Omega_{\mathbb{P}^2}^{\otimes} \mathcal{V}_{-1,-1}) = 0 \quad p = 1, 2 \quad \forall q.$$

(11)

Here $\mathcal{V}_{(m,n)}$ is an irreducible representation of $PSU(3)$ with the highest weight $(m, n)$. In total

$$I_{N=2}(\mathcal{M}, \mathcal{V}_{-1,-1}) = 3 \sum_{q=0}^{2} (-1)^q \hat{\chi}(\Omega_{\mathbb{P}^2}^{\otimes} \mathcal{V}_{1,1}) = 6(\mathcal{V}_{(3,0)} - \mathcal{V}_{(1,1)} + \mathcal{V}_{(0,0)}).$$

(12)

Similarly, for $\mathcal{V}_{-2,-2} = \mathcal{O}_{\mathbb{P}^2}(6) \otimes S^2\tilde{V}_1$ we use

$$\hat{\chi}(\Omega_{\mathbb{P}^2}^{\otimes} \mathcal{V}_{-2,-2}) = \hat{\chi}(\Omega_{\mathbb{P}^2}^{\otimes} \mathcal{V}_{-1,-1} \otimes \mathcal{V}_{-1,-1}) - \hat{\chi}(\Omega_{\mathbb{P}^2}^{\otimes} \mathcal{O}_{\mathbb{P}^2}(6))$$

$p = 0, 1, 2$

to compute

$$\hat{\chi}(\mathbb{P}^2, \mathcal{V}_{-2,-2}) = 3\mathcal{V}_{(6,0)} - 4\mathcal{V}_{(0,0)}, \quad \hat{\chi}(\mathbb{P}^2, \Omega_{\mathbb{P}^2}^{(1)} \mathcal{V}_{-2,-2}) = 3\mathcal{V}_{(4,1)} - 8\mathcal{V}_{(0,0)};$$

$$\hat{\chi}(\mathbb{P}^2, \Omega_{\mathbb{P}^2}^{(2)} \mathcal{V}_{-2,-2}) = 3\mathcal{V}_{(3,0)} - 4\mathcal{V}_{(0,0)}$$

so that

$$I_{N=2}(\mathcal{M}, \mathcal{V}_{-2,-2}) = 3\mathcal{V}_{(6,0)} - 4\mathcal{V}_{(0,0)}$$

(14)

$$I_{N=4}(\mathcal{M}, \mathcal{V}_{2,2}) = I_{N=4}(\mathcal{M}, \mathcal{V}_{-2,-2}) = 9(\mathcal{V}_{(6,0)} - \mathcal{V}_{(4,1)} + \mathcal{V}_{(3,0)}).$$

(15)

Analogously, both the Euler and the holomorphic Euler characters can be computed straightforwardly for any $\mathcal{V}_{-a,-b} = \pi^*\mathcal{V}_{-a,-b}$ with $a > 0$, $b > 0$, $a + 2b = 0 \mod 3$ by using that
To evaluate cohomology groups involving $O_X$.

Indeed, we found the following non-zero cohomology groups:

$$\chi\left(\Omega^p_{\mathbb{P}^2} \otimes V_{a-b}\right) = \chi\left(\Omega^p_{\mathbb{P}^2} \otimes V_{-(a-1)-(b-1)} \otimes V_{-1-1}\right) - \chi\left(\Omega^p_{\mathbb{P}^2} \otimes V_{-(a-2)-(b-2)} \otimes O_{\mathbb{P}^2}(6)\right).$$

(16)

For example, in this way one finds

$$I_{N=4}\left(\mathcal{O}, V_{a,b}\right) = I_{N=4}\left(\mathcal{M}, V_{a,b}\right) = 3(b+1)\left(V_{(n,0)} - V_{(n-2,1)} + V_{(n-3,0)}\right) \quad n = 2a + b.$$

(17)

### 3.2 $I_{N=2}(X, V_{bulk}^{a,b})$ and $I_{N=4}(X, V_{bulk}^{a,b})$

Let us denote $\pi_{bulk} : X \mapsto \mathbb{P}^2$. In this section we first compute $I_{N=4}(X, V_{bulk}^{a,b})$ with $V_{bulk}^{a,b} = (\pi_{bulk})^* V_{-1-1}$.

Using $\mathcal{G}_X$ defined in (5), we find the norm of a section $\psi_{unit} = \mathcal{G}_X \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ of the vector bundle $V_{bulk}^{a,b}$:

$$||\psi_{unit}||^2 = \int \left(\frac{\overline{h}^{(3)}(\tilde{s})}{y^3}\right)^2 vol_X \left(\alpha^2|\psi_1 + \psi_2|^2 + \alpha^{-2}|\beta(\psi_1 + \psi_2) + \psi_2|^2\right)$$

(18)

with volume form on $X$ given by

$$vol_X = \tilde{f}_1(\tilde{s})\tilde{f}_2(\tilde{s})\tilde{f}_3(\tilde{s})\tilde{f}_4(\tilde{s})e_1 \wedge e_2 \wedge e_3 \wedge e_4 \wedge \overline{\tau}_1 \wedge \overline{\tau}_2 \wedge \overline{\tau}_3 \wedge \overline{\tau}_4.$$

This allows to think about $V_{bulk}^{a,b}$ as follows

$$0 \mapsto \mathcal{O}_X(3) \mapsto V_{bulk}^{a,b} \mapsto I_Z(3) \mapsto 0$$

(19)

$$0 \mapsto I_Z(3) \mapsto \mathcal{O}_X(3) \mapsto \mathcal{O}_Z(3) \mapsto 0,$$

(20)

where $\mathcal{O}_Z$ is the structure sheaf of the fiber $Z$ of $X$ at the point $p$.

Now we note

$$I_{N=4}\left(X, V_{bulk}^{a,b}\right) = 2I_{N=4}\left(X, \mathcal{O}_X(3)\right) - I_{N=4}\left(X, \mathcal{O}_Z(3)\right)$$

(21)

$$I_{N=2}\left(X, V_{bulk}^{a,b}\right) = 2I_{N=2}\left(X, \mathcal{O}_X(3)\right) - I_{N=2}\left(X, \mathcal{O}_Z(3)\right)$$

(22)

To evaluate $I_{N=4}(X, \mathcal{O}_Z(3))$ and $I_{N=2}(X, \mathcal{O}_Z(3))$, we use the fact that the only non-zero cohomology groups involving $\mathcal{O}_Z(3)$, are

$$H^0(X, \mathcal{O}_Z(3)) = 3\mathbb{V}_{(0,0)}, \quad H^0(X, \Omega^1_X \otimes \mathcal{O}_Z(3)) = 8\mathbb{V}_{(0,0)}, \quad H^0(X, \Omega^2_X \otimes \mathcal{O}_Z(3)) = \mathbb{V}_{(0,0)}$$

(23)

Indeed, we found the following non-zero cohomology groups:

---

*Higher cohomology groups vanish since $Z$ is topologically $\mathbb{C}^2$*
\[ H^0(X, \mathcal{O}_X(3)) = \mathbb{V}_{(3,0)} + \mathbb{V}_{(1,1)} \]
\[ H^0(X, \Omega^1_X(3)) = 2\mathbb{V}_{(1,1)} + \mathbb{V}_{(0,0)} \]
\[ H^0(X, \Omega^2_X(3)) = \mathbb{V}_{(0,0)} \]

Explicitly, sections in \( H^0(X, \mathcal{O}_X(3)) \) (in the holomorphic gauge) are

1. \( M^i_j t_i U^j \begin{pmatrix} 1 \\ -1 \end{pmatrix} \) with \( M^i_i = 0 \) transform in \( \mathbb{V}_{(1,1)} \)
2. \( C_{ijk} U^i U^j U^k \) transform in \( \mathbb{V}_{(3,0)} \)

Meanwhile, sections in \( H^0(X, \Omega^1_X(3)) \) are (in the holomorphic gauge)

1. \( h^i_k(U^k dt_j - 2t_j dU^k) \) transform in \( \mathbb{V}_{(1,1)} + \mathbb{V}_{(0,0)} \)
2. \( C_{ijk} U^i U^j dU^k \) transform in \( \mathbb{V}_{(1,1)} \)

Finally, \( H^0(X, \Omega^2_X(3)) \) is one-dimensional and generated by (in the holomorphic gauge)

\[ \epsilon_{ijk} U^i dU^j \wedge dU^k. \]

In (23) we simply count cohomology elements which are non-vanishing at \( z^1 = z^2 = 0 \). So we compute

\[ I_{N=1}(X, \mathcal{O}_Z(3)) = -4 \mathbb{V}_{(0,0)}, \quad I_{N=2}(X, \mathcal{O}_Z(3)) = 3 \mathbb{V}_{(0,0)}. \]

Meanwhile, the characters for \( \mathcal{O}_X(3) \) are computed in Appendix B:

\[
    \begin{array}{c}
    I_{N=2}(X, \mathcal{O}_X(3)) = \mathbb{V}_{(3,0)} + \mathbb{V}_{(1,1)} \\
    I_{N=4}(X, \mathcal{O}_X(3)) = \mathbb{V}_{(3,0)} - \mathbb{V}_{(1,1)} + \mathbb{V}_{(0,0)}
    \end{array}
\]

Hence, we find the Euler and the holomorphic Euler characters of \( \mathcal{V}^{bulk}_{-1,-1} \)

\[
    I_{N=4}(X, \mathcal{V}^{bulk}_{1,1}) = I_{N=4}(X, \mathcal{V}^{bulk}_{-1,-1}) = 2 \left( \mathbb{V}_{(3,0)} - \mathbb{V}_{(1,1)} + \mathbb{V}_{(0,0)} \right) + 4 \mathbb{V}_{(0,0)}
\]

\[
    I_{N=2}(X, \mathcal{V}^{bulk}_{-1,-1}) = 2 \left( \mathbb{V}_{(3,0)} + \mathbb{V}_{(1,1)} \right) - 3 \mathbb{V}_{(0,0)}
\]

Let us consider \( \mathcal{V}^{bulk}_{-2,-2} \) as another example. We compute

\[ I^\alpha(X, \mathcal{V}^{bulk}_{-2,-2}) = I^\alpha(X, \mathcal{V}^{bulk}_{-1,-1} \otimes 2) - I^\alpha(X, \mathcal{O}_X(6)) \]
Next we use, in addition to (19) and (20), the following short exact sequence
\[ 0 \rightarrow \mathcal{O}_X(3) \otimes \mathcal{V}_{-1,-1} \rightarrow \mathcal{V}_{-1,-1} \otimes^2 I_Z(3) \otimes \mathcal{V}_{-1,-1} \rightarrow 0 \tag{24} \]

So that
\[ I^\alpha(X, \mathcal{V}_{-2,-2}) = 3I^\alpha(X, \mathcal{O}_X(6)) - 3I^\alpha(X, \mathcal{O}_Z(6)) \tag{25} \]
where we used
\[ I^\alpha(X, \mathcal{V}_{-1,-1}(3) \otimes \mathcal{O}_Z) = I^\alpha(X, \mathcal{O}_Z(6)). \tag{26} \]

We compute \( I^\alpha(X, \mathcal{O}_Z(6)) \) by first listing all generators of \( H^0_{\mathcal{D},L^2}(X, \mathcal{O}_X(6)) \) and then picking up those which do not vanish at \( z_1 = z_2 = 0 \). Only 23 of these are non-vanishing at \( Z \) so that
\[ I^{(1)}(X, \mathcal{O}_Z(6)) = 23V_{(0,0)}. \]

Similarly we computed
\[ I^{(0)}(X, \mathcal{O}_Z(6)) = 10V_{(0,0)}, \quad I^{(2)}(X, \mathcal{O}_Z(6)) = 14V_{(0,0)}, \quad I^{(3)}(X, \mathcal{O}_Z(6)) = 11V_{(0,0)}, \quad I^{(4)}(X, \mathcal{O}_Z(6)) = 0 \]
so that
\[ I_{N=4}(X, \mathcal{O}_Z(6)) = -10V_{(0,0)}, \quad I_{N=2}(X, \mathcal{O}_Z(6)) = 10V_{(0,0)} \]

Using \( I_{N=4}(X, \mathcal{O}_X(6)) \) computed in Appendix B, we finally get
\[ I_{N=4}(X, \mathcal{V}_{-2,-2}) = 3\left(V_{(0,0)} - V_{(1,1)} + V_{(2,2)} + V_{(3,0)} - V_{(4,1)}\right) + 30V_{(0,0)} \tag{27} \]
\[ I_{N=2}(X, \mathcal{V}_{-2,-2}) = 3\left(V_{(0,0)} + V_{(1,1)} + V_{(2,2)} + V_{(3,0)}\right) - 30V_{(0,0)} \tag{28} \]

One may use an iterative procedure to write down the characters for general \( \mathcal{V}_{-a,-b} \):
\[ I^\alpha(X, \mathcal{V}_{-a,-b}) = 2I^\alpha(X, \mathcal{V}_{-a-1,-b-1} \otimes \mathcal{O}_X(3)) - I^\alpha(X, \mathcal{V}_{-a-1,-b-1} \otimes \mathcal{O}_Z(3)) \tag{29} \]
\[ -I^\alpha(X, \mathcal{V}_{-a-2,-b-2} \otimes \mathcal{O}_X(6)) \quad \alpha = 0, \ldots, 4. \]
3.3 Finding the OPE

To determine the unknown electric weights \( \nu_j \) and signs \( s_j \) on the right side of the OPE (1), we need to decompose the bubbled contributions

\[
\Gamma^{\text{bubble}}_{\mathcal{N}=2} (\mathbf{\nu}_{a,b}) = \mathbf{I}_{\mathcal{N}=2} (\mathcal{M}, \mathbf{\nu}_{a,b}) - \mathbf{I}_{\mathcal{N}=2} (X, \mathbf{\nu}_{a,b}^{\text{bulk}})
\]

and

\[
\Gamma^{\text{bubble}}_{\mathcal{N}=4} (\mathbf{\nu}_{a,b}) = \mathbf{I}_{\mathcal{N}=4} (\mathcal{M}, \mathbf{\nu}_{a,b}) - \mathbf{I}_{\mathcal{N}=4} (X, \mathbf{\nu}_{a,b}^{\text{bulk}})
\]

into representations of the \( SU(2) \) which we can bring (30) into the form suitable for reading off the answer for the OPE:

\[
\text{From (31) we obtain the following OPE in } \mathcal{N} = 2 \text{ SYM with gauge group } G = PSU(3): \]

\[
WT_{w_1,-w_1-w_2} \times WT_{w_1,0} = WT_{2w_1,-w_1-w_2} - 2 \left( WT_{w_2,w_1+w_2} + WT_{w_2,w_1-2w_2} + WT_{w_2,2w_1-w_2} \right)
\]

(33)

Meanwhile, in \( \mathcal{N} = 4 \) SYM with gauge group \( G = PSU(3) \) we find from (32):

\[
WT_{w_1,-w_1-w_2} \times WT_{w_1,0} = WT_{2w_1,-w_1-w_2} + 4 \left( WT_{w_2,3w_1} - WT_{w_2,w_1+w_2} + WT_{w_2,-3w_2} \right) + 2WT_{w_2,0}
\]

(34)

\[
WT_{w_1,w_1+w_2} \times WT_{w_1,0} = WT_{2w_1,w_1+w_2} + 4 \left( WT_{w_2,3w_1} - WT_{w_2,w_1+w_2} + WT_{w_2,-3w_2} \right) + 2WT_{w_2,0}
\]

(35)

Similarly, for

\[
\Gamma^{\text{bubble}}_{\mathcal{N}=2} (\mathbf{\nu}_{-2,-2}) = 26\mathbf{\nu}_{0,0} - 3 \left( \mathbf{\nu}_{4,1} + \mathbf{\nu}_{2,2} + \mathbf{\nu}_{0,3} \right)
\]

(36)
In this paper we computed the OPE (1) of Wilson–’t Hooft loop operators in $\mathcal{N} = 4$ SYM with magnetic sources [5], [10].

Meanwhile, for $\mathcal{N} = 4$ SYM with gauge group $G = PSU(3)$ we obtain:

$$WT_{w_1; -2w_1 - 2w_2} \times WT_{w_1; 0} = WT_{2w_1; -2w_1 - 2w_2} + 6 WT_{w_2; 6w_1} - 6 WT_{w_2; 4w_1 + w_2} + 6 WT_{w_2; -6w_2}$$

$$-3 WT_{2w_1; 3w_1 - 3w_2} + 3 WT_{2w_1; w_1 - 2w_2} - 3 WT_{2w_1; 3w_2} - 27 WT_{2w_1; 0}$$

$$WT_{w_1; 2w_1 + 2w_2} \times WT_{w_1; 0} = WT_{2w_1; 2w_1 + 2w_2} + 6 WT_{w_2; 6w_1} - 6 WT_{w_2; 4w_1 + w_2} + 6 WT_{w_2; -6w_2}$$

$$-3 WT_{2w_1; 3w_1 - 3w_2} + 3 WT_{2w_1; w_1 - 2w_2} - 3 WT_{2w_1; 3w_2} - 27 WT_{2w_1; 0}$$

## 4 Conclusion

In this paper we computed the OPE (1) of Wilson–’t Hooft loop operators in $\mathcal{N} = 4$ and $\mathcal{N} = 2$ SYM theory with gauge group $G = PSU(3)$. This work is an extension of our approach [7], [8] which uses the holomorphic-topological twist [9] of the $\mathcal{N} = 2$ gauge theory and the connection between BPS configurations in the presence of ’t Hooft operators and solutions of 3d Bogomolny equations with magnetic sources [5], [10].

The crucial ingredients in our computation of the OPE in $\mathcal{N} = 4$ SYM are Euler characters $\mathbf{I}_{\mathcal{N}=4}(\mathcal{M}, \mathcal{V}_{a,b})$ and $\mathbf{I}_{\mathcal{N}=4}(X, \mathcal{V}_{a,b}^\text{bulk})$ of vector bundles $\mathcal{V}_{a,b}$ and $\mathcal{V}_{a,b}^\text{bulk}$ on moduli spaces $\mathcal{M}$ and $X = TP^2$. These Euler characters compute (with sign) the ground states of the supersymmetric quantum mechanics which arises as the result of quantizing $\mathcal{N} = 4$ SYM on $\mathbb{R} \times I \times \mathcal{C}$ in the
presence of Wilson-t’ Hooft operators along $\mathbb{R}$. To find the OPE in $\mathcal{N} = 2$ SYM, we compute the holomorphic Euler characters instead of Euler characters relevant for $\mathcal{N} = 4$ SYM.

In Section 3, we explained how to compute the right side of (11) for any $\nu = aw_1 + bw_2$. Namely, we used that $\mathcal{M}$ is $\mathbb{P}^2$ fibration over $\mathbb{P}^2$ and applied the Leray spectral sequence in Appendix A. Together with the iterative procedure (16) for vector bundles on $\mathbb{P}^2$, this allows to reduce the problem of finding $I_{\mathcal{N}=4}(\mathcal{M}, \mathcal{V}_{a,b})$ to computing the Euler characters of line bundles on $\mathbb{P}^2$. Further, due to the iterative procedure (29) for vector bundles on $X = T\mathbb{P}^2$, the problem of finding $I_{\mathcal{N}=4}(X, \mathcal{V}_{\text{bulk}})$ is reduced to computing the Euler character of $L^2$ Dolbeault cohomology of line bundles on $X$. This step is done in Appendix B by using the exact sequences for $\Omega^\alpha_X$ for $\alpha = 1, \ldots, 4$. Finally, to get the explicit answers for the OPE, given in equations (33-35) and (38-40), we used vanishing assumption (55). We provided evidence in support of this assumption in Appendix C.

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A Computing $I_{\mathcal{N}=4}(\mathcal{M}, \mathcal{V})$

Here we determine

$$I_{\mathcal{N}=4}(\mathcal{M}, \mathcal{V}) = \sum_{p,q} (-)^{p+q}H^p(\mathcal{M}, \Omega^q_M \otimes \mathcal{V}).$$

Let $\pi : \mathcal{M} \mapsto \mathbb{P}^2$ and $\mathcal{V} = \pi^* \mathcal{V}$ where $\mathcal{V}$ is a vector bundle on $\mathbb{P}^2$. To compute cohomology $H^j(\mathcal{M}, \Omega^a_M \otimes \mathcal{V})$ we use the Leray spectral sequence. This is easy enough since we are interested in pull-back bundles. The necessary ingredients are the right direct images of $\Omega^a_M$ and $\Omega^2_M$ which we compute in sections A.1 and A.2 respectively.

A.1 $R^p\pi_*\Omega_M$

The 4-fold $\mathcal{M}$ is defined by $y_a U^a = 0$ in the toric 5-fold $Y_5$. The weights under the two $\mathbb{C}^*$ actions are

| $U^1$ | $U^2$ | $U^3$ | $\Lambda$ | $y_1$ | $y_2$ | $y_3$ |
|-------|-------|-------|----------|-------|-------|-------|
| new   | 1     | 1     | 1        | -2    | 0     | 0     |
| old   | 0     | 0     | 0        | 1     | 1     | 1     |

(41)

Here $U^1, U^2, U^3$ are homogenous coordinates on the base $\mathbb{P}^2$, and $\Lambda, y_1, y_2, y_3$ are homogenous coordinates on the fiber $\mathbb{P}^3$.
Let us use that $Y_5$ is the projectivisation of the vector bundle
\[ Y_5 = \mathbb{P}(E), \quad E = \mathcal{O}_{\mathbb{P}^2}(-2) \oplus \mathcal{O}_{\mathbb{P}^2}^{\oplus 3} \]
to write the following two exact sequences:
\[ 0 \to \pi^* \Omega^1_{\mathbb{P}^2} \to \Omega^1_{Y_5} \to \Omega^1_{\text{vert}} \to 0 \tag{42} \]
\[ 0 \to \Omega^1_{\text{vert}} \to \pi^* E^* \otimes \mathcal{O}_{Y_5}(0, -1) \to \mathcal{O}_{Y_5} \to 0. \tag{43} \]
Here and below $\mathcal{O}_{Y_5}(b, f)$ stands for a line bundle on $Y_5$ with degree $b$ on the base $\mathbb{P}^2$ and degree $f$ on the fiber $\mathbb{P}^3$. More explicitly, the exact sequence (43) is
\[ 0 \to \Omega^1_{\text{vert}} \to \mathcal{O}_{Y_5}(2, -1) \oplus \mathcal{O}_{Y_5}(0, -1)^{\oplus 3} \to \mathcal{O}_{Y_5} \to 0. \tag{44} \]
Let us denote
\[ \pi : M \mapsto \mathbb{P}^2, \quad \hat{\pi} : Y_5 \mapsto \mathbb{P}^2. \]
We apply push-forward map to (44) and use
\[ H^i\left( \mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(-1) \right) = 0 \quad \text{for} \quad i = 0, \ldots, 3, \quad H^0\left( \mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3} \right) = \mathbb{C}, \quad H^i\left( \mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3} \right) = 0 \quad \text{for} \quad i > 0 \]
to compute the direct images of $\Omega^1_{\text{vert}}$:
\[ R^1\pi_* \left( \Omega^1_{\text{vert}} \right) = \mathcal{O}_{\mathbb{P}^2}, \quad R^k\pi_* \left( \Omega^1_{\text{vert}} \right) = 0 \quad \text{for} \quad k \neq 1. \tag{45} \]
Now we apply the push-forward map to (42) to find
\[ \hat{\pi}_* \left( \Omega^1_{Y_5} \right) = \Omega^1_{\mathbb{P}^2}, \quad R^1\hat{\pi}_* \left( \Omega^1_{Y_5} \right) = \mathcal{O}_{\mathbb{P}^2}, \quad R^k\hat{\pi}_* \left( \Omega^1_{Y_5} \right) = 0 \quad \text{for} \quad k > 1. \tag{46} \]
Next we use the adjunction formula
\[ 0 \to N^*_{M|Y_5} \to \Omega^1_{Y_5}|_M \to \Omega^1_M \to 0 \]
where $N^*_{M|Y_5} = \mathcal{O}_M(-1, -1)$ is the co-normal bundle of $M$ in $Y_5$.
Now we use $R^i\pi_* i^* = R^i\hat{\pi}_*$ and $H^p(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-1)) = 0$ for all $p$ to find
\[ R^i\pi_* \Omega^1_M = R^i\hat{\pi}_* \Omega^1_{Y_5}. \]
Therefore, we conclude
\[ \pi_* \Omega^1_M = \Omega^1_{\mathbb{P}^2}, \quad R^1\pi_* \Omega^1_M = \mathcal{O}_{\mathbb{P}^2}, \quad R^p\pi_* \Omega^1_M = 0 \quad p > 1. \tag{47} \]
We use these push-forwards to compute \( E_{pq}^{pq} = E_{pq}^{pq} \) with $p + q = n$ giving
\[ H^1(M, \Omega^1_M) = \mathbb{V}_1 + \mathbb{V}_1, \quad H^p(M, \Omega^1_M) = 0 \quad p \neq 1. \]
This agrees with the cohomology of $M$ computed before in a different way.
A.2 \( R^p \pi_* \Omega^2_{\mathcal{M}} \)

We first use three short exact sequences:

1. \( 0 \to N^*_{\mathcal{M}|Y_5} \otimes \Omega^1_{\mathcal{M}} \to \Omega^2_{\mathcal{M}|Y_5} \to \Omega^2_{\mathcal{M}} \to 0 \)
2. \( 0 \to \Omega^1_{\mathcal{P}^2} \to \mathcal{O}(-1)^{\oplus 3} \to \mathcal{O} \to 0 \)
3. \( 0 \to \mathcal{O}^{\oplus 2} \to \Omega^1_{\mathcal{M}|Y_5} \to \Omega^1_{\mathcal{P}^2} \to 0 \)

to compute

\[
R^p \pi_* (N^*_{\mathcal{M}|Y_5} \otimes \Omega^1_{\mathcal{M}}) = 0 \quad p = 0, 1, 2
\]

and

\[
R^p \hat{\pi} (\Omega^2_{Y_5}) = R^p \pi_* (\Omega^2_{\mathcal{M}}) \quad p = 0, 1, 2.
\] (48)

Now we use the following filtration on \( Y_5 \):

\[ 0 \subset F_1 \subset F_2 \subset F_3 = \Omega^2_{Y_5} \]

Here

\[ F_1 = \Lambda^2 (\hat{\pi}^* \Omega^1_{\mathcal{P}^2}), \quad F_2/F_1 = (\hat{\pi}^* \Omega^1_{\mathcal{P}^2}) \otimes \Omega^1_{\text{vert}}, \quad F_3/F_2 = \Lambda^2 \Omega^1_{\text{vert}}. \]

From the push-forward of the following exact sequence

\( 0 \to F_1 \to F_2 \to F_2/F_1 \to 0 \)

we find

\[
\hat{\pi}_*(F_2) = \Omega^2_{\mathcal{P}^2}, \quad R^1 \hat{\pi}_*(F_2) = \Omega^1_{\mathcal{P}^2}, \quad R^j \hat{\pi}_*(F_2) = 0 \quad j > 1.
\] (49)

Then, from the push-forward of the other exact sequence

\( 0 \to F_2 \to F_3 \to F_3/F_2 \to 0 \)

we compute

\[
\hat{\pi}_*(F_3) = \Omega^2_{\mathcal{P}^2}, \quad R^1 \hat{\pi}_*(F_3) = \Omega^1_{\mathcal{P}^2}, \quad R^2 \hat{\pi}_*(F_3) = \mathcal{O}_{\mathcal{P}^2}, \quad R^j \hat{\pi}_*(F_3) = 0 \quad j > 2.
\] (50)

We conclude using (48) and (50)

\[
\pi_*(\Omega^2_{\mathcal{M}}) = \Omega^2_{\mathcal{P}^2}, \quad R^1 \pi_*(\Omega^2_{\mathcal{M}}) = \Omega^1_{\mathcal{P}^2}, \quad R^2 \pi_*(\Omega^2_{\mathcal{M}}) = \mathcal{O}_{\mathcal{P}^2}, \quad R^j \pi_*(\Omega^2_{\mathcal{M}}) = 0 \quad j > 2.
\]

As a check, we find

\[
E^0_2 = H^0(\mathbb{P}^2, R^2 \pi_*(\Omega^2_{\mathcal{M}})) = H^0(\mathbb{P}^2, \mathcal{O}) = V_1
\]
\[
E^{20}_2 = H^2(\mathbb{P}^2, \pi_*(\Omega^2_{\mathcal{M}})) = H^0(\mathbb{P}^2, \mathcal{O}) = V_1
\]
\[
E^{11}_2 = H^1(\mathbb{P}^2, R^1 \pi_*(\Omega^2_{\mathcal{M}})) = H^0(\mathbb{P}^2, \mathcal{O}) = V_1
\]

giving

\[
H^2(\mathcal{M}, \Omega^2_{\mathcal{M}}) = V_1 \oplus V_1 \oplus V_1
\]
in complete agreement with the analysis in [8].
A.3 \( H^j \left( \mathcal{M}, \Omega^p_{\mathcal{M}} \otimes \mathcal{V} \right) \)

Now we use \( \mathcal{V} = \pi^*(\mathcal{V}) \) and the right direct images computed in A.1 and A.2

\[
\pi_*\Omega^1_\mathcal{M} = \Omega^1_{\mathcal{P}^2}, \quad R^1\pi_* \left( \Omega^1_{\mathcal{M}} \right) = \mathcal{O}_{\mathcal{P}^2}, \quad R^j\pi_* \left( \Omega^1_{\mathcal{M}} \right) = 0 \quad j \geq 2
\]

\[
\pi_* \left( \Omega^2_\mathcal{M} \right) = \Omega^2_{\mathcal{P}^2}, \quad R^1\pi_* \left( \Omega^2_{\mathcal{M}} \right) = \Omega^2_{\mathcal{P}^2}, \quad R^2\pi_* \left( \Omega^2_{\mathcal{M}} \right) = \mathcal{O}_{\mathcal{P}^2}, \quad R^j\pi_* \left( \Omega^2_{\mathcal{M}} \right) = 0 \quad j \geq 3
\]

to compute bundle cohomology:

\[
H^0 \left( \mathcal{M}, \Omega^1_{\mathcal{M}} \otimes \mathcal{V} \right) = H^0 \left( \mathbb{P}^2, \Omega^1_{\mathcal{P}^2} \otimes \mathcal{V} \right), \quad H^1 \left( \mathcal{M}, \Omega^1_{\mathcal{M}} \otimes \mathcal{V} \right) = H^1 \left( \mathbb{P}^2, \Omega^1_{\mathcal{P}^2} \otimes \mathcal{V} \right) + H^0 \left( \mathbb{P}^2, \mathcal{V} \right)
\]

\[
H^2 \left( \mathcal{M}, \Omega^1_{\mathcal{M}} \otimes \mathcal{V} \right) = H^2 \left( \mathbb{P}^2, \Omega^1_{\mathcal{P}^2} \otimes \mathcal{V} \right) + H^1 \left( \mathbb{P}^2, \mathcal{V} \right) + H^0 \left( \mathbb{P}^2, \mathcal{V} \right)
\]

\[
H^3 \left( \mathcal{M}, \Omega^1_{\mathcal{M}} \otimes \mathcal{V} \right) = H^2 \left( \mathbb{P}^2, \mathcal{V} \right) + H^2 \left( \mathbb{P}^2, \Omega^1_{\mathcal{P}^2} \otimes \mathcal{V} \right), \quad H^4 \left( \mathcal{M}, \Omega^1_{\mathcal{M}} \otimes \mathcal{V} \right) = H^2 \left( \mathbb{P}^2, \mathcal{V} \right)
\]

Now we use Kodaira-Serre duality

\[
H^n \left( \mathcal{M}, \Omega^3_{\mathcal{M}} \otimes \mathcal{V} \right) = H^{4-n} \left( \mathcal{M}, \Omega^1_{\mathcal{M}} \otimes \mathcal{V}^* \right), \quad H^n \left( \mathcal{M}, \Omega^4_{\mathcal{M}} \otimes \mathcal{V} \right) = H^{4-n} \left( \mathcal{M}, \mathcal{V}^* \right)
\]

For any vector bundle \( \mathcal{V} \) on \( \mathbb{P}^2 \) let us denote the weighted sum of cohomologies as

\[
\hat{\chi}(\mathcal{V}) := \sum_p (-)^p H^p(\mathbb{P}^2, \mathcal{V}).
\]

We now compute the desired expression

\[
\mathbf{I}_{N=4} (\mathcal{M}, \mathcal{V}) = \sum_{p,q} (-)^{p+q} H^p \left( \mathcal{M}, \Omega^q_{\mathcal{M}} \otimes \mathcal{V} \right) = 3 \sum_{q=0}^2 (-)^q \hat{\chi} \left( \Omega^q_{\mathcal{P}^2} \otimes \mathcal{V} \right)
\]

(53)

Note that

\[
\mathbf{I}_{N=4} (\mathcal{M}, \mathcal{V}) = \mathbf{I}_{N=4} (\mathcal{M}, \mathcal{V}^*)
\]
B \hspace{1em} I_{N=4}(\mathbb{T}P^2, \mathcal{O}_{\mathbb{T}P^2}(3m)) \text{ for } m \geq 1

B.1 \hspace{1em} L^2 \text{ Dolbeault cohomology of } \mathcal{O}_X(m) \text{ with } m \geq 1

Recall that \( X = \mathbb{T}P^2 \) is defined by a hypersurface \( t_iU^i = 0 \) in \( Y := W\mathbb{P}_{111222}/\{U^i = 0\} \). By explicit counting of global holomorphic sections, we find

\[
H^0_{\mathcal{D},L^2}(X, \mathcal{O}_X(1)) = \mathbb{V}_{(1,0)}, \quad H^0_{\mathcal{D},L^2}(X, \mathcal{O}_X(2)) = \mathbb{V}_{(2,0)} + \mathbb{V}_{(0,1)}, \quad H^0_{\mathcal{D},L^2}(X, \mathcal{O}_X(3)) = \mathbb{V}_{(3,0)} + \mathbb{V}_{(1,1)}
\]

\[
H^0_{\mathcal{D},L^2}(X, \mathcal{O}_X(4)) = \mathbb{V}_{(4,0)} + \mathbb{V}_{(2,1)} + \mathbb{V}_{(0,2)}
\]

More generally,

\[
H^0_{\mathcal{D},L^2}(X, \mathcal{O}_X(n)) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \mathbb{V}_{(n-2j,j)} \quad n \geq 1 \tag{54}
\]

We checked that these sections have finite norm by using the Kähler form on \( X \) and connection (73-74) on line bundle \( \mathcal{O}_X(n) \).

In this paper we \textit{assume} the following vanishing of \( L^2 \) Dolbeault cohomology:

\[
H^2_{\mathcal{D},L^2}(X, \mathcal{O}_X(n)) = 0 \quad n \geq -3 \quad j > 0 \tag{55}
\]

For \( n = 0 \) this was proved in [8], we checked this statement for several values of \( n \geq -3 \) and \( j > 0 \) and sample computations are presented in Appendix C.

B.2 \hspace{1em} L^2 \text{ Dolbeault cohomology of } \Omega^1_X(3m) \text{ with } m \geq 1

Here we express \( I^{(1)}(X, \mathcal{O}_X(3m)) \) for \( m \geq 1 \) in terms of holomorphic Euler characters of line bundles \( I^{(0)}(X, \mathcal{O}_X(3m-n)) \) for \( n = 0, \ldots, 3 \). Further, using the assumption (55), we compute some explicit answers.

Let us use the adjunction formula

\[
0 \mapsto \mathcal{O}_X(-3) \mapsto \Omega^1_X \mapsto \Omega^1_X \mapsto 0 \tag{56}
\]

as well as the short exact sequence which defines restriction of \( \Omega^1_Y \) to \( X \)

\[
0 \mapsto \Omega^1_Y \mapsto \mathbb{V}_{(1,0)} \otimes \mathcal{O}_X(-1) \oplus \mathbb{V}_{(0,1)} \otimes \mathcal{O}_X(-2) \mapsto \mathcal{O}_X \mapsto 0 \tag{57}
\]

Note that (57) simply states

\[
\omega \in \Omega^1_Y \text{ \ if \ } \omega = a^i dt_i + b_j dU^j \quad \text{s. t. } \quad 2a^i t_i + b_j U^j = 0
\]
From long exact sequences for cohomologies following from (56,57) we find

\[ I^{(1)}(X, \mathcal{O}_X(3m)) = \mathbb{V}_{(1,0)} \otimes I^{(0)}(X, \mathcal{O}_X(3m-1)) + \mathbb{V}_{(0,1)} \otimes I^{(0)}(X, \mathcal{O}_X(3m-2)) \]

\[ -I^{(0)}(X, \mathcal{O}_X(3m)) - I^{(0)}(X, \mathcal{O}_X(3m-3)) \]  

(58)

For example, we find using the assumption (55)

\[ I^{(1)}(X, \mathcal{O}_X(3)) = 2 \times \mathbb{V}_{(1,1)} + \mathbb{V}_{(0,0)} \]

\[ I^{(1)}(X, \mathcal{O}_X(6)) = 2 \times \left( \mathbb{V}_{(4,1)} + \mathbb{V}_{(2,2)} + \mathbb{V}_{(3,0)} + \mathbb{V}_{(1,1)} \right) + \mathbb{V}_{(0,3)} \]

\[ I^{(1)}(X, \mathcal{O}_X(12)) = 2 \times \left( \mathbb{V}_{(10,1)} + \mathbb{V}_{(9,0)} + \mathbb{V}_{(8,2)} + \mathbb{V}_{(7,1)} + \mathbb{V}_{(6,3)} + \mathbb{V}_{(5,2)} + \mathbb{V}_{(4,4)} + \mathbb{V}_{(3,3)} + \mathbb{V}_{(2,5)} + \mathbb{V}_{(1,4)} \right) + \mathbb{V}_{(0,6)} \]

Independently of (58), we computed (by listing all the sections and checking the finiteness of their norm) the following cohomology groups

\[ H^0_{\mathbb{D},L^2}(X, \Omega^1_X(3)) = 2 \times \mathbb{V}_{(1,1)} + \mathbb{V}_{(0,0)} \]

(59)

\[ H^0_{\mathbb{D},L^2}(X, \Omega^1_X(6)) = 2 \times \left( \mathbb{V}_{(4,1)} + \mathbb{V}_{(2,2)} + \mathbb{V}_{(3,0)} + \mathbb{V}_{(1,1)} \right) + \mathbb{V}_{(0,3)} \]

(60)

\[ H^0_{\mathbb{D},L^2}(X, \Omega^1_X(12)) = 2 \times \left( \mathbb{V}_{(10,1)} + \mathbb{V}_{(9,0)} + \mathbb{V}_{(8,2)} + \mathbb{V}_{(7,1)} + \mathbb{V}_{(6,3)} + \mathbb{V}_{(5,2)} + \mathbb{V}_{(4,4)} + \mathbb{V}_{(3,3)} + \mathbb{V}_{(2,5)} + \mathbb{V}_{(1,4)} \right) + \mathbb{V}_{(0,6)} \]

(61)

The equations (59,61) can be put into the form

\[ H^0_{\mathbb{D},L^2}(X, \Omega^1_X \otimes \mathcal{O}_X(3m)) = \mathbb{V}_{(1,0)} \otimes H^0_{\mathbb{D},L^2}(X, \mathcal{O}_X(3m-1)) + \mathbb{V}_{(0,1)} \otimes H^0_{\mathbb{D},L^2}(X, \mathcal{O}_X(3m-2)) \]

\[ -H^0_{\mathbb{D},L^2}(X, \mathcal{O}_X(3m)) - H^0_{\mathbb{D},L^2}(X, \mathcal{O}_X(3m-3)) \]  

(62)

Explicitly, the basis of sections in \( H^0(X, \Omega^1_X(3)) \) is given by (in the holomorphic gauge)

1. \( h^i_k(U^k dt_j - 2 t_j dU^k) \) transform in \( \mathbb{V}_{(1,1)} + \mathbb{V}_{(0,0)} \).

2. \( C_{i[jk]} U^i U^j dU^k \) transform in \( \mathbb{V}_{(1,1)} \).
Meanwhile, general \( \omega \in H^{0,\Omega}_{L^2}(X, \Omega^1_X(6)) \) is

\[
\omega = a^i_{(j_1j_2j_3j_4)}(dt^iU^{j_1} - 2t_idU^{j_1}) U^{j_2} U^{j_3} U^{j_4} + b^{(ik)}_{(j_1j_2)} t_k (dt^iU^{j_1} - 2t_idU^{j_1}) U^{j_2} + C_{(kmn)} dt^i t_j \epsilon^{ijk} U^m U^n
\]

\[
+ \tilde{f}_{i_1i_2i_3i_4[i_5i_6]} U^{i_1} U^{i_2} U^{i_3} U^{i_4} U^{i_5} dU^{i_6} + \tilde{c}_{pkn} \epsilon_{ijp} t_k t_n U^{i} dU^{j} + \tilde{b}^{ik}_{mn} U^m U^n t_k \epsilon_{ijp} U^{i} dU^{p} + d^{ijk} t_i t_j dt_k
\]

where the following identification is due to \( U^{j_1} t_i = 0 \):

\[
a^i_{(j_1j_2j_3j_4)} \sim a^i_{(j_1j_2j_3j_4)}, \quad b^{(ik)}_{(j_1j_2)} \sim b^{(ik)}_{(j_1j_2)}, \quad \tilde{b}^{ik}_{mn} \sim \tilde{b}^{ik}_{mn} + \delta^i_{(m} \alpha^j_{n)}\]

Similarly, sections in \( H^{0,\Omega}_{L^2}(X, \Omega^1_X(12)) \) are given (in the holomorphic gauge) by

\[
\omega = \epsilon^{ijp} t_i t_j dU^{m_1} \ldots U^{m_n} C_{(pn_1\ldots n_k)} + \epsilon^{ijp} t_i t_j t_k U^{m_1} \ldots U^{m_n} C_{(pn_1\ldots n_k)} + \epsilon^{ijp} t_i t_j t_k t_l U^{m_1} \ldots U^{m_n} C_{(pn_1\ldots n_k)} +
\]

\[
+ \epsilon^{ijp} t_i t_j t_k t_l U^{m_1} U^{m_2} C_{(pn_1n_2)} + \epsilon^{ijp} t_i t_j t_k t_l t_m t_n C_{(pn_1n_2)} + \epsilon^{ijp} t_i t_j t_k t_l t_n C_{(pn_1n_2)} +
\]

\[
a^i_{(j_1\ldots j_k)} (dt^iU^{j_1} - 2t_idU^{j_1}) U^{j_2} \ldots U^{j_k} + a^{(ik)}_{(j_1\ldots j_k)} (dt^iU^{j_1} - 2t_idU^{j_1}) U^{j_2} \ldots U^{j_k}
\]

\[
+ a^{(k_1k_2k_3)}_{(j_1\ldots j_k)} (dt^kU^{j_1} - 2t_kdU^{j_1}) t_k U^{j_2} \ldots U^{j_k} + a^{(k_1\ldots k_5)}_{(j_1j_2)} (dt^kU^{j_1} - 2t_kdU^{j_1}) t_k t_l U^{j_3} \ldots U^{j_5} + a^{(k_1\ldots k_5)}_{(j_1j_2)} (dt^kU^{j_1} - 2t_kdU^{j_1}) t_k t_l t_m U^{j_3} \ldots U^{j_5}
\]

where coefficients are subject to identifications due to \( t_i U^i = 0 \) constraint.

For example:

\[
C_{(pn_1n_2n_3n_4)} = C_{(pn_1n_2n_3n_4)} + \delta^{(k_1}_{(p} B_{n_1n_2n_3n_4)}^{k_2}\]

\[
C_{(pn_1n_2n_3n_4)} = C_{(pn_1n_2n_3n_4)} + \delta^{(k_1}_{(p} B^{k_2}_{n_1n_2n_3n_4)} + \delta^{(k_1}_{(p} B^{k_2}_{n_1n_2n_3n_4)}
\]

\[
C_{(pn_1n_2n_3n_4)} = C_{(pn_1n_2n_3n_4)} + \delta^{(k_1}_{(p} B_{n_1n_2n_3n_4)} + \delta^{(k_1}_{(p} B_{n_1n_2n_3n_4)}
\]

\[
C_{(pn_1n_2n_3n_4)} = C_{(pn_1n_2n_3n_4)} + \delta^{(k_1}_{(p} B_{n_1n_2n_3n_4)} + \delta^{(k_1}_{(p} B_{n_1n_2n_3n_4)}
\]

B.3 \( L^2 \) Dolbeault cohomology of \( \Omega^2_X(3m) \) with \( m \geq 1 \)

Here we compute \( I^{(2)}(X, \mathcal{O}_X(3m)) \) for \( m \geq 1 \). We use the adjunction formula

\[
0 \mapsto \mathcal{O}_X(-3) \otimes \Omega^1_X \mapsto \Omega^2_{X}|_X \mapsto \Omega^2_X \mapsto 0
\]

as well as the short exact sequence for the restriction of \( \Omega^2_X \) to \( X \):

\[
0 \mapsto \Omega^2_{X}|_X \mapsto \mathbb{V}_{(0,1)} \otimes \mathcal{O}_X(-2) \oplus \mathbb{V}_{(1,0)} \otimes \mathcal{O}_X(-4) \oplus \left( \mathbb{V}_{(1,1)} + \mathbb{V}_{(0,0)} \right) \otimes \mathcal{O}_X(-3) \mapsto \Omega^1_X \mapsto 0
\]

We find

\[
I^{(2)}(X, \mathcal{O}_X(3m)) = \mathbb{V}_{(1,0)} \otimes I^{(0)}(X, \mathcal{O}_X(3m - 4)) + \left( \mathbb{V}_{(1,1)} + \mathbb{V}_{(0,0)} \right) \otimes I^{(0)}(X, \mathcal{O}_X(3m - 3))
\]
\[-V_{(1,0)} \otimes I^{(0)}(X, \mathcal{O}_X(3m - 1)) + I^{(0)}(X, \mathcal{O}_X(3m)) - I^{(1)}(X, \mathcal{O}_X(3m - 3))\].

For special cases \(m = 1, 2\) we find, using the vanishing assumption (53):

\[I^{(2)}(X, \mathcal{O}_X(3)) = V_{(0,0)}, \quad I^{(2)}(X, \mathcal{O}_X(6)) = 3V_{(3,0)} + V_{(2,2)} + V_{(0,3)} + 3V_{(1,1)} + V_{(0,0)}.

Independently of (65), we compute

\[H^0_{\mathcal{D},L^2}(X, \Omega^2_X(3)) = V_{(0,0)}\]
\[H^0_{\mathcal{D},L^2}(X, \Omega^2_X(6)) = 3V_{(3,0)} + V_{(2,2)} + V_{(0,3)} + 3V_{(1,1)} + V_{(0,0)}.

Explicitly, \(H^0_{\mathcal{D},L^2}(X, \Omega^2_X(3))\) is one-dimensional and generated by

\[\epsilon_{ijk}U^i dU^j \wedge dU^k.

Meanwhile, sections in \(H^0_{\mathcal{D},L^2}(X, \Omega^2_X(6))\) are

\[\omega = \rho^j_i \wedge \rho^i_k \epsilon^{ijk} C_{qjn} + \rho^j_i \wedge \mu_k \tilde{C}^{ki}_j U^n + \zeta^k \wedge \mu_n \tilde{B}^m_k + \alpha \zeta^k \wedge dt_k + \mu_k \wedge dU^k \left( \beta_{mnp} U^m U^n U^p + \beta^k_n U^n t_k \right)\]

where we use holomorphic 1-forms well-defined on \(X = TP^2\)

\[\rho^j_i = U^j dt_i - 2t_i dU^j, \quad \zeta^k = \epsilon^{kij} t_i dt_j, \quad \mu_k = \epsilon_{kij} U^i dU^j\]

and coefficients are subject to constraints due to \(t_i U^i = 0\).

### B.4 \(L^2\) Dolbeault cohomology of \(\Omega^3_X(3m)\) and \(\Omega^4_X(3m)\) with \(m \geq 1\)

We use the adjunction formula

\[0 \leftrightarrow \mathcal{O}_X(-3) \otimes \Omega^2_X \leftrightarrow \Omega^3_Y |_X \leftrightarrow \Omega^3_X \rightarrow 0\]
\[0 \leftrightarrow \Omega^3_Y |_X \leftrightarrow \mathcal{O}_X(-3) \oplus \mathcal{O}_X(-6) + \left( V_{(0,2)} + V_{(1,0)} \right) \otimes \mathcal{O}_X(-4) \oplus \left( V_{(2,0)} + V_{(0,1)} \right) \otimes \mathcal{O}_X(-5) \rightarrow \Omega^2_Y |_X \rightarrow 0\]
\[0 \leftrightarrow \Omega^3_Y |_X \leftrightarrow \mathcal{O}_X(-3) \oplus \mathcal{O}_X(-6) + \left( V_{(0,2)} + V_{(1,0)} \right) \otimes \mathcal{O}_X(-4) \oplus \left( V_{(2,0)} + V_{(0,1)} \right) \otimes \mathcal{O}_X(-5) \rightarrow \Omega^2_Y |_X \rightarrow 0\]

Now using (68) and (69) we find

\[I^{(3)}(\mathcal{O}_X(3m + 3)) = -I^{(2)}(\mathcal{O}_X(3m)) + I^{(0)}(\mathcal{O}_X(3m)) + I^{(0)}(\mathcal{O}_X(3m - 3)) - I^{(2)}(\mathcal{O}_X(3m - 3))\]
\[+ \left( V_{(0,2)} + V_{(1,0)} \right) \otimes I^{(0)}(\mathcal{O}_X(3m - 1)) + \left( V_{(2,0)} + V_{(0,1)} \right) \otimes I^{(0)}(\mathcal{O}_X(3m - 2)) - I^{(1)}(\mathcal{O}_X(3m)).\]
For example, using the vanishing assumption (55), we find:

\[
I^{(3)}(\mathcal{O}_X(3)) = -V_{(0,0)} \quad \quad I^{(3)}(\mathcal{O}_X(6)) = 2V_{(1,1)} + V_{(0,0)}.
\]

Independently of (68) and (69), we compute

\[
H^0_{\overline{\partial},L^2}(X, \Omega^3_X(6)) = 2V_{(1,1)} + V_{(0,0)}
\]

where general section of \(H^0_{\overline{\partial},L^2}(X, \Omega^3_X(6))\) is given by

\[
\omega = \left(\mu_k \wedge d\zeta^p + 2d(\mu_k) \wedge \zeta^p\right)C^k_p + B^k_i \rho^i_k \wedge d(\mu_m)U^m
\]

with \(B^i_k \sim B^i_k + \delta^i_k \gamma\).

Similarly, for \(\Omega^4_Y\) we use the adjunction formula

\[
0 \mapsto \mathcal{O}_X(-3) \otimes \Omega^3_X(6) \mapsto \Omega^4_Y|_X \mapsto \Omega^4_X \mapsto 0
\]

as well as the short exact sequence for the restriction of \(\Omega^4_Y\) to \(X\):

\[
0 \mapsto \Omega^4_Y|_X \mapsto V_{(0,1)} \otimes \mathcal{O}_X(-5) \oplus \left(V_{(0,0)} + V_{(1,1)}\right) \otimes \mathcal{O}_X(-6) \oplus V_{(1,0)} \otimes \mathcal{O}_X(-7) \mapsto \Omega^3_Y|_X \mapsto 0
\]

This allows to identify

\[
\Omega^4_X = \mathcal{O}_X(-9)
\]

and compute

\[
I^{(4)}(\mathcal{O}_X(3m)) = I^{(0)}(X, \mathcal{O}_X(3m - 9)).
\]

The vanishing assumption (55) implies\(^7\)

\[
I^{(4)}(X, \mathcal{O}_X(3)) = I^{(0)}(X, \mathcal{O}_X(-3)) = 0 \quad \quad I^{(4)}(X, \mathcal{O}_X(6)) = I^{(0)}(X, \mathcal{O}_X(-3)) = 0
\]

Collecting the pieces together we find the Euler characters of \(\mathcal{O}_X(3)\) and \(\mathcal{O}_X(6)\) bundles

\[
\mathbf{I}_{N=4}(X, \mathcal{O}_X(3)) = V_{(3,0)} - V_{(1,1)} + V_{(0,0)}
\]

\[
\mathbf{I}_{N=4}(X, \mathcal{O}_X(6)) = V_{(6,0)} - V_{(4,1)} + V_{(0,3)} + V_{(3,0)} - V_{(1,1)}
\]

\(^7\)In the first equation we used Kodaira-Serre duality for \(L^2\) Dobeault cohomology.
C Some evidence in support of the ‘vanishing assumption’

Here we explain how to compute $H^i_{D,L^2}(X,\mathcal{O}_X(n))$ for $i > 0$ and give some explicit examples in support of the vanishing assumption (55).

C.1 $H^1_{D,L^2}(X,\mathcal{O}_X(n))$

Let us define

$$\tilde{s} = \frac{x}{y^2}, \quad \tilde{x} = t_a t^a, \quad Y = \bar{U}_a U^a$$

so that in $U^1 \neq 0$ patch with inhomogenous coordinates $z^1 = \frac{t_1}{\bar{U}}$ and $z^2 = \frac{t_3}{\bar{U}}$ on $\mathbb{P}^2$ we solve $t_1 = -(t_2 z^1 + t_3 z^2)$ and write

$$\tilde{s} = \frac{x}{y^2}, \quad \tilde{x} = t_2 \bar{\alpha}^2 + t_3 \bar{\alpha}^3, \quad y = 1 + |z_1|^2 + |z_2|^2$$

with

$$\bar{\alpha}^2 = \bar{t}^2 + z^1 (t^2 \bar{z}_1 + t^3 \bar{z}_2)$$

$$\bar{\alpha}^3 = \bar{t}^3 + z^2 (t^2 \bar{z}_1 + t^3 \bar{z}_2).$$

Let us write general $\omega \in \Omega_X^{(0,1)} \otimes \mathcal{O}_X(n)$ as

$$\omega = \sum_{I=1}^4 \alpha_I \tilde{e}_I$$

where

$$e_1 = \partial \tilde{s}, \quad e_2 = \frac{t_a dU^a}{Y^{3/2}}, \quad e_3 = \frac{e a c U^a t_e d U^c}{Y^{5/2}}, \quad e_4 = \frac{\epsilon_{abc} t^a U^b d U^c}{y^2}$$

In $U^1 \neq 0$ patch we find

$$e_1 = \frac{\partial \tilde{x}}{y^2} - \frac{2 \partial y}{y} \tilde{s}, \quad e_2 = \frac{t_2 d z^1 + t_3 d z^2}{y^{3/2}}$$

$$e_3 = \frac{t_2 d t_3 - t_3 d t_2}{y^{3/2}} + \frac{t_2 \bar{z}_2 - t_3 \bar{z}_1}{y} e_2, \quad e_4 = \frac{\bar{\alpha}^3 d z^1 - \bar{\alpha}^2 d z^2}{y^2}.$$
For this reason \( \alpha_{1,4} \in \Gamma\left(\mathcal{O}_X(n)\right), \) \( \alpha_{2,3} \in \Gamma\left(\mathcal{O}_X(n + 3)\right) \).

Let us consider differential equations

\[
\overline{D}^{(n)}\omega = 0, \quad D^{(n)}(\ast \omega) = 0
\]

with

\[
D^{(n)} = \nabla + \frac{h\,'^{(n)}}{h^{(n)}} e_1, \quad \overline{D}^{(n)} = \nabla - \frac{h\,'^{(n)}}{h^{(n)}} \bar{e}_1
\]

(73)

Here

\[
\nabla = \partial - \frac{n}{2} \frac{\partial y}{y}, \quad \overline{\nabla} = \overline{\partial} + \frac{n}{2} \frac{\partial y}{y}
\]

and

\[
h^{(n)}(\bar{s}) \sim \bar{s}^{-1} \bar{s} \mapsto \infty, \quad h^{(n)} \sim 1 \bar{s} \mapsto 0
\]

(74)

We define \( \beta_{I;K} \) and \( \gamma_{I;K} \) as

\[
\nabla \alpha_I = \sum_K \beta_{I;K} e_K
\]
\[
\nabla \alpha_I = \sum_K \gamma_{I;K} e_K
\]

and use

\[
\nabla e_1 = 0, \quad \nabla e_2 = 0, \quad \nabla e_3 = \frac{1}{s} e_1 \wedge \bar{e}_1 - e_2 \wedge \bar{e}_2 + \frac{1}{s} e_3 \wedge \bar{e}_3 - 2 e_4 \wedge \bar{e}_4
\]
\[
\nabla e_4 = 0, \quad \nabla \bar{e}_4 = \frac{1}{s} e_1 \wedge \bar{e}_4 + \frac{1}{s} e_3 \wedge \bar{e}_2, \quad \nabla e_2 = \frac{1}{s} e_1 \wedge e_2 - \frac{1}{s} e_3 \wedge e_4
\]
\[
\nabla e_3 = \frac{2}{s} e_1 \wedge e_3, \quad \nabla \bar{e}_3 = - e_4 \wedge \bar{e}_2
\]

to show that \( \overline{D}^{(n)}\omega = 0 \) is equivalent to

\[
\beta_{1;4} = \beta_{4;1} - \frac{h\,'^{(n)}}{h^{(n)}} \alpha_4, \quad \beta_{2;3} = \beta_{3;2}, \quad \beta_{2;4} = \beta_{4;2}
\]

(75)

\[
\alpha_2 = \bar{s}\left(\beta_{1;2} - \beta_{2;1} + \frac{h\,'^{(n)}}{h^{(n)}} \alpha_2\right), \quad 2 \alpha_3 = \bar{s}\left(\beta_{1;3} - \beta_{3;1} + \frac{h\,'^{(n)}}{h^{(n)}} \alpha_3\right), \quad \alpha_2 = \bar{s}\left(\beta_{4;3} - \beta_{3;4}\right)
\]

while \( D^{(n)}(\ast \omega) = 0 \) is equivalent to

\[
\alpha_1 \left(\frac{5}{\bar{s}} + \frac{h\,'^{(n)}}{h^{(n)}}\right) \tilde{f}_2 \tilde{f}_3 \tilde{f}_4 + \left(\tilde{f}_2 \tilde{f}_3 \tilde{f}_4\right)' + \sum_{K=1}^4 \gamma_{K;K} \prod_{J \neq K} \tilde{f}_J = 0
\]

(76)
Here functions $\tilde{f}_i(\tilde{s})$ are coefficients of expanding Kähler form (4) on $X$. Finally, we have to compute the norm squared
\[
||\omega||^2 = \int_X \omega \wedge ^* \omega = \sum_{K=1}^4 \int_X |\alpha_K|^2 vol_X \tilde{f}_K
\]
where the volume form can be written as
\[
vol_X = \tilde{s}^5 \prod_{j=1}^4 \tilde{f}_j(\tilde{s}) \, d\tilde{s} \wedge d\varphi \wedge \frac{dv \wedge d\bar{v}}{(1 + |v|^2 + |z^1v + z^2|^2)^2} \wedge \frac{dz^1 \wedge d\bar{z}_1 \wedge dz^2 \wedge d\bar{z}_2}{y^2}
\]
Note that $\alpha_K$ can be written as polynomials in $\nu^i_1 = \frac{U^i}{y^{1/2}}, \nu^i_2 = \frac{\bar{t}^i}{y}, \nu^i_3 = \frac{\epsilon^{ijk} \bar{U}^k}{y^{3/2}}$ and their conjugates. In computing $\beta_{I,K}$ and $\gamma_{I,K}$ we use
\[
\tilde{s} \nabla (\nu^i_1) = \nu^i_2 e_2 - \nu^i_3 e_4, \quad \bar{\nabla} (\nu^i_1) = 0, \quad \nabla (\nu^i_2) = 0, \quad \tilde{s} \bar{\nabla} (\nu^i_2) = \nu^i_2 \bar{e}_1 - \nu^i_1 \tilde{s} \bar{e}_2 + \nu^i_3 \bar{e}_3 \quad (79)
\]
\[
\tilde{s} \nabla (\nu^i_3) = \nu^i_4 e_1 - \nu^i_2 e_3, \quad \tilde{s} \bar{\nabla} (\nu^i_3) = \nu^i_1 \bar{e}_4 \quad (80)
\]
\section*{C.2 $H^1_{D,L^2}(X, \mathcal{O}_X(1))$}
Let us first look for $\omega \in H^1_{D,L^2}(X, \mathcal{O}_X(1))$ that transforms in irreducible representation $\mathbb{V}_{(1,0)}$. This implies $\alpha^i_2 = \alpha^i_3 = 0$ and
\[
\alpha^i_1 = c_1(\tilde{s}) \nu^i_1 + b_1(\tilde{s}) \nu^i_3, \quad \alpha^i_4 = c_4(\tilde{s}) \nu^i_1 + b_4(\tilde{s}) \nu^i_3
\]
Now we use (79) and (80) to compute $\beta^i_{1,2} = \beta^i_{1,3} = \beta^i_{4,2} = \beta^i_{4,3} = 0$ and
\[
\beta^i_{1,4} = b_1(\tilde{s}) \nu^i_1, \quad \beta^i_{4,1} = c_4 \nu^i_1 + b_4 \nu^i_3
\] as well as
\[
\gamma^i_{1,1} = c_1 \nu^i_1 + b_1 \nu^i_3, \quad \gamma^i_{4,4} = -\frac{c_4}{\tilde{s}} \nu^i_3
\]
Now from (75) and (76) we obtain
\[
b_1 = c_4 - \frac{h^{(1)'}}{h^{(1)}} c_4, \quad b_4 - \frac{h^{(1)'}}{h^{(1)}} b_4 = 0
\]
\[8t_2 = t_3 v, t_3 = r_3 e^{i\varphi}, \tilde{x} = r_3^2 (1 + |v|^2 + |z^1v + z^2|^2)\]
Let us look for $\gamma$ conclude that there is no representation
\[4 = \left(\frac{0}{K}\right)\]
In the limit $s \mapsto \infty$ (i.e., $s \mapsto 0$) both $b_4$ and $c_1$ behave as $s^{-1/4}$ while the two solutions for $c_4$ behave as $s^{-1/4}$ and $s^{-3/4}$ in this limit. We checked using the norm (77) that all of these solutions have divergent contribution to their norm squared from the region $s \mapsto \infty$. Hence, we conclude that there is no representation $\mathbb{V}_{(1,0)}$ in $H^1_{D,L^2}(X, \mathcal{O}_X(1))$.

Now let us look for $\omega \in H^1_{D,L^2}(X, \mathcal{O}_X(1))$ that transforms in irreducible representation $\mathbb{V}_{(0,2)}$:

$$\alpha_J = a_j(\bar{s})\bar{v}_{1}(j\bar{v}_{2}k) + b_J(\bar{s})\bar{v}_{2}(j\bar{v}_{3}k) \quad J = 1, 4; \quad \alpha_J = a_J(\bar{s})\bar{v}_{2} j\bar{v}_{2}k \quad J = 2, 3.$$

Then, we compute $\beta_{J;K}$ and $\gamma_{K;K}$ (we only write non-zero components)

$$\beta_{J,1} = a'_J \bar{v}_1(j\bar{v}_{2}k) + \left( b'_J + \frac{b_J}{s} \right) \bar{v}_{2}(j\bar{v}_{3}k) \quad J = 1, 4; \quad \beta_{J,1} = a'_J \bar{v}_{2} j\bar{v}_{2}k \quad J = 2, 3;$$

$$\beta_{J,2} = a_j(\bar{s})\bar{v}_{3}(j\bar{v}_{2}k), \quad \beta_{J,3} = -\frac{b_J}{s} \bar{v}_{2}(j\bar{v}_{3}k), \quad \beta_{J,4} = -a_j(\bar{s})\bar{v}_{2}(j\bar{v}_{3}k), \quad J = 1, 4;$$

$$\gamma_{1,1} = \left( a'_J + \frac{a_J}{s} \right) \bar{v}_1(j\bar{v}_{2}k) + \left( b'_J + \frac{b_J}{s} \right) \bar{v}_{2}(j\bar{v}_{3}k), \quad \gamma_{2,2} = -2a_2 \bar{v}_1(j\bar{v}_{2}k), \quad \gamma_{3,3} = \frac{2a_3}{s} \bar{v}_{2}(j\bar{v}_{3}k), \quad \gamma_{4,4} = \frac{b_4}{s} \bar{v}_1(j\bar{v}_{2}k)$$

Now from (76) and (76) we obtain

$$a_4 = 0, \quad a_2 = -b_4, \quad a_1 = a_2 + \bar{s}\left( a'_2 - \frac{h'(1)}{h(1)} a_2 \right), \quad b_1 = -2a_3 - \bar{s}\left( a'_3 - \frac{h'(1)}{h(1)} a_3 \right)$$

$$b_1 \left( \frac{6}{s} + \frac{h'(1)}{h(1)} \right) \bar{f}_2 \bar{f}_3 \bar{f}_4 + \left( \bar{f}_2 \bar{f}_3 \bar{f}_4 \right)' + b'_1 \bar{f}_2 \bar{f}_3 \bar{f}_4 + \frac{2a_3}{s} \bar{f}_2 \bar{f}_3 \bar{f}_4 = 0$$

$$a_1 \left( \frac{6}{s} + \frac{h'(1)}{h(1)} \right) \bar{f}_2 \bar{f}_3 \bar{f}_4 + \left( \bar{f}_2 \bar{f}_3 \bar{f}_4 \right)' + a'_1 \bar{f}_2 \bar{f}_3 \bar{f}_4 - 2a_2 \bar{f}_1 \bar{f}_3 \bar{f}_4 + \frac{b_4}{s} \bar{f}_1 \bar{f}_2 \bar{f}_3 = 0.$$

We checked that there are no well-behaved solutions with finite norm.

**C.3 $H^1_{D,L^2}(X, \mathcal{O}_X(-1))$**

Let us look for $\omega \in H^1_{D,L^2}(X, \mathcal{O}_X(-1))$ that transforms in irreducible representation $\mathbb{V}_{(2,0)}$:

$$\omega = \left( K(\bar{s})\nu_1 \nu_2 + L(\bar{s})\nu_2 \nu_3 \right) \tilde{e}_1 + \left( A(\bar{s})\nu_1 \nu_3 + B(\bar{s})\nu_2^2 + E(\bar{s})\nu_3^2 \right) \tilde{e}_2 + \left( \tilde{A}(\bar{s})\nu_1 \nu_3 + \tilde{B}(\bar{s})\nu_2^2 + \tilde{E}(\bar{s})\nu_3^2 \right) \tilde{e}_3$$

26
Further, we checked that the solution for \( \tilde{\omega} \) is \( \tilde{\omega} \) and (76) gives a second order equation for \( \tilde{C} \).

We compute
\[
\beta_{2;1} = A' \nu_1 \nu_3 + B' \nu_1^2 + E' \nu_3^2, \quad \beta_{3;1} = \tilde{A'} \nu_1 \nu_3 + \tilde{B'} \nu_1^2 + \tilde{E'} \nu_3^2
\]
\[
\beta_{4;1} = (\tilde{K}' + \tilde{s}^{-1} \tilde{K}) \nu_1 \nu_2 + (\tilde{L}' + \tilde{s}^{-1} \tilde{L}) \nu_2 \nu_3
\]
\[
\beta_{1;2} = -(K \nu_1^2 + L \nu_1 \nu_3), \quad \beta_{1;3} = \tilde{s}^{-1} \ast (K \nu_1 \nu_3 + L \nu_3^2), \quad \beta_{1;4} = \tilde{s}^{-1} L \nu_1 \nu_2
\]
\[
\beta_{2;2} = -(\tilde{K} \nu_1^2 + \tilde{L} \nu_1 \nu_3), \quad \beta_{2;4} = \tilde{s}^{-1} \ast (A \nu_1^2 + 2E \nu_1 \nu_3)
\]
\[
\beta_{4;3} = \tilde{s}^{-1} \ast (K \nu_1 \nu_3 + \tilde{L} \nu_3^2), \quad \beta_{3;4} = \tilde{s}^{-1} \ast (\tilde{A} \nu_1^2 + 2\tilde{E} \nu_1 \nu_3)
\]

Plugging this into (75) gives rise to
\[
A = E = L = \tilde{L} = \tilde{K} = \tilde{E} = 0, \quad B = -\tilde{A}
\]
\[
K = -\tilde{s} \nabla_\tilde{s} \tilde{A} + 2\tilde{A}, \quad \tilde{K} = -\nabla_\tilde{s} \tilde{A} + \frac{\tilde{A}}{\tilde{s}}
\]
\[
2\tilde{B} + \tilde{s} \nabla_\tilde{s} \tilde{B} = 0.
\]

Note that the two different expression for \( K \) imply the first order differential equation for \( \tilde{A} \). Further, we checked that the solution for \( B \) has divergent norm. Meanwhile,
\[
\gamma_{1;1} = K' \nu_1 \nu_2, \quad \gamma_{2;2} = -2\frac{\tilde{A}}{\tilde{s}} \nu_1 \nu_2, \quad \gamma_{3;3} = -\frac{\tilde{A}}{\tilde{s}} \nu_1 \nu_2
\]

and (76) gives a second order equation for \( \tilde{A} \). The only solution of the two differential equations is \( \tilde{A} = 0 \). We conclude that there is no \((2, 0)\) in \( H^2_{D, L^2}(X, O_X(-1)) \).

### C.4 \( H^2_{D, L^2}(X, O_X(-1)) \)

Let us look for representation \( \mathbb{V}_{(0, 1)} \) in \( H^2_{D, L^2}(X, O_X(-1)) \):
\[
\omega = (K(\tilde{s}) \tilde{v}_1 + L(\tilde{s}) \tilde{v}_3) \tilde{e}_1 \wedge \tilde{e}_4 + N(\tilde{s}) \tilde{v}_2 \tilde{e}_1 \wedge \tilde{e}_2 + P(\tilde{s}) \tilde{v}_2 \tilde{e}_1 \wedge \tilde{e}_3 + G(\tilde{s}) \tilde{v}_2 \tilde{e}_3 \wedge \tilde{e}_4 + M(\tilde{s}) \tilde{v}_2 \tilde{e}_3 \wedge \tilde{e}_4
\]

So that
\[
\ast \omega = \left(K(\tilde{s}) \tilde{v}_1 + \tilde{L}(\tilde{s}) \tilde{v}_3\right) \tilde{e}_1 \wedge \tilde{e}_4 \wedge e_2 \wedge e_3 \wedge e_4 + \tilde{N}(\tilde{s}) \tilde{v}_2 \tilde{e}_1 \wedge \tilde{e}_2 \wedge e_3 \wedge e_4 \wedge \tilde{e}_4
\]
\[
+ \tilde{P}(\tilde{s}) \tilde{v}_2 \tilde{e}_1 \wedge \tilde{e}_3 \wedge e_2 \wedge e_4 \wedge \tilde{e}_4
\]
\[
+ \tilde{G}(\tilde{s}) \tilde{v}_2 \tilde{e}_2 \wedge \tilde{e}_4 \wedge e_1 \wedge \tilde{e}_1 \wedge e_3 \wedge \tilde{e}_3 + \tilde{M}(\tilde{s}) \tilde{v}_2 \tilde{e}_3 \wedge \tilde{e}_4 \wedge e_1 \wedge \tilde{e}_1 \wedge e_2 \wedge \tilde{e}_2
\]
where
\[
\hat{K} = K\hat{f}_2\hat{f}_3, \quad \hat{N} = N\hat{f}_3\hat{f}_4, \quad \hat{L} = L\hat{f}_2\hat{f}_3, \quad \hat{P} = P\hat{f}_2\hat{f}_4, \quad \hat{G} = G\hat{f}_1\hat{f}_3, \quad \hat{M} = M\hat{f}_1\hat{f}_2.
\]
Imposing \(\mathcal{D}^{-1}(\omega) = 0\), \(\mathcal{D}^{-1}(\omega) = 0\) gives
\[
\tilde{s}\nabla_{\tilde{s}}^+ G + G = K, \quad \tilde{s}\hat{G} = \tilde{s}\nabla_{\tilde{s}}^{+\frac{1}{2}}\hat{K} + 5\tilde{K}, \quad L = N = M = P = 0
\]
(81)
We checked that there no solutions for \(G, K\) with finite norm. We conclude that there is no \(\mathbb{V}_{(0,1)}\) in \(H^2_{D,L^2}(X, \mathcal{O}_X(-1))\).

C.5 \(H^2_{D,L^2}(X, \mathcal{O}_X(-2))\)

Let us look for representation \(\mathbb{V}_{(1,0)}\) in \(H^2_{D,L^2}(X, \mathcal{O}_X(-2))\):
\[
\omega = A(\tilde{s})\nu_2\bar{e}_1 \wedge \bar{e}_4 + (B(\tilde{s})\nu_1 + C(\tilde{s})\nu_3)\bar{e}_1 \wedge \bar{e}_2 + (D(\tilde{s})\nu_1 + E(\tilde{s})\nu_3)\bar{e}_1 \wedge \bar{e}_3 + (K(\tilde{s})\nu_1 + L(\tilde{s})\nu_3)\bar{e}_2 \wedge \bar{e}_4 + (M(\tilde{s})\nu_1 + N(\tilde{s})\nu_3)\bar{e}_3 \wedge \bar{e}_4
\]
So that
\[
*\omega = A(\tilde{s})\nu_2\bar{e}_1 \wedge \bar{e}_4 \wedge \bar{e}_2 \wedge \bar{e}_3 \wedge \bar{e}_3 + \hat{B}(\tilde{s})\nu_1 + \hat{C}(\tilde{s})\nu_3)\bar{e}_1 \wedge \bar{e}_2 \wedge \bar{e}_3 \wedge \bar{e}_4 + \hat{D}(\tilde{s})\nu_1 + \hat{E}(\tilde{s})\nu_3)\bar{e}_1 \wedge \bar{e}_2 \wedge \bar{e}_3 \wedge \bar{e}_4 + (\hat{K}(\tilde{s})\nu_1 + \hat{L}(\tilde{s})\nu_3)\bar{e}_2 \wedge \bar{e}_4 \wedge \bar{e}_1 \wedge \bar{e}_3 \wedge \bar{e}_3 + (\hat{M}(\tilde{s})\nu_1 + \hat{N}(\tilde{s})\nu_3)\bar{e}_3 \wedge \bar{e}_4 \wedge \bar{e}_1 \wedge \bar{e}_2 \wedge \bar{e}_2
\]
where
\[
\hat{A} = A\hat{f}_2\hat{f}_3, \quad \hat{B} = B\hat{f}_3\hat{f}_4, \quad \hat{C} = C\hat{f}_3\hat{f}_4, \quad \hat{D} = D\hat{f}_2\hat{f}_4, \quad \hat{E} = E\hat{f}_2\hat{f}_4,
\]
\[
\hat{K} = K\hat{f}_1\hat{f}_3, \quad \hat{L} = L\hat{f}_1\hat{f}_3, \quad \hat{M} = M\hat{f}_1\hat{f}_2, \quad \hat{N} = N\hat{f}_1\hat{f}_2.
\]
Imposing \(\mathcal{D}^{-2}(\omega) = 0\), \(\mathcal{D}^{(-2)} \ast \omega = 0\) gives
\[
B = M = E = 0
\]
\[
\tilde{s}\nabla_{\tilde{s}}^+ L + L = 0, \quad \tilde{s}\nabla_{\tilde{s}}^{+\frac{1}{2}}\hat{D} + 3\hat{D} = 0
\]
\[
(1 + \tilde{s})A = \tilde{s}\nabla_{\tilde{s}}^+ (N - K) + 2N - K, \quad (1 + \tilde{s})C = -\tilde{s}\nabla_{\tilde{s}}^+ K - K - \tilde{s}\left(\tilde{s}\nabla_{\tilde{s}}^+ N + 2N\right)
\]
(82)
\[
(1 + \tilde{s})A = \tilde{s}\nabla_{\tilde{s}}^+ (N - K) + 2N - K, \quad (1 + \tilde{s})C = -\tilde{s}\nabla_{\tilde{s}}^+ K - K - \tilde{s}\left(\tilde{s}\nabla_{\tilde{s}}^+ N + 2N\right)
\]
(83)
We checked that solutions for \(L\) and \(D\) have divergent norm. Moreover, the system of equations
\[
(83)
\]
has only one dimensional family of solutions\(^9\) with good properties at \(\tilde{s} \to 0\):
\[
N = A = \frac{C_0}{\tilde{s}}, \quad K = -\frac{4C_0}{\tilde{s}}, \quad C = -C_0
\]
However, we checked using Mathematica, that this solution interpolates into a solution with \(C \sim \tilde{s}^{-5/4}\) at \(\tilde{s} \to \infty\) which has divergent norm. We conclude that there is no \(\mathbb{V}_{(1,0)}\) in \(H^2_{D,L^2}(X, \mathcal{O}_X(-2))\).

\(^9\)The other solutions at \(\tilde{s} \to 0\) are either too divergent \(K \sim \tilde{s}^{-4}\) or non-analytic in \(\tilde{s}\).
Let us first look for $\omega \in H^2_{D,L^2}(X, O_X(3))$ that transforms in representation $V_{(3,0)}$. General ansatz is given by
\[
\omega = \left( C(\tilde{s}) \nu_1^3 + B(\tilde{s}) \nu_1^2 \nu_3 + E(\tilde{s}) \nu_1 \nu_3^2 + \nu_3^3 F(\tilde{s}) \right) \tilde{e}_1 \wedge \tilde{e}_4
\]
\[
\overline{D}^{(3)} \omega = 0 \text{ is satisfied automatically, but } D^{(3)}(\ast \omega) = 0 \text{ gives}
\]
\[
C = B = E = 0, \quad \tilde{s} \nabla_{\tilde{s}}^+ \hat{F} + 8 \hat{F} = 0
\]
where $\hat{F} = F \tilde{f}_2 \tilde{f}_3$. We checked that the solution for $F$ has divergent norm.

Now we look for $V_{(1,1)}$ in $H^2_{D,L^2}(X, O_X(3))$. General ansatz is given by
\[
\omega = a(\tilde{s}) \nu_1 \nu_2 \tilde{e}_1 \tilde{e}_4
\]
\[
\overline{D}^{(3)} \omega = 0 \text{ is satisfied automatically, but } D^{(3)}(\ast \omega) = 0 \text{ gives}
\]
\[
\tilde{s} \nabla_{\tilde{s}}^+ \hat{a} + 6 \hat{a} = 0 \quad \hat{a} = a \tilde{f}_2 \tilde{f}_3
\]
We checked that the solution for $a$ has divergent norm. We conclude that there is neither $V_{(3,0)}$ nor $V_{(1,1)}$ in $H^2_{D,L^2}(X, O_X(3))$. Similarly, there is no $V_{(0,0)}$ in $H^2_{D,L^2}(X, O_X(3))$.

Moreover, there are no $V_{(3,0)}, V_{(1,1)}, V_{(0,0)}$ in $H^2_{D,L^2}(X, O_X(3))$ for $j = 3, 4$ since one cannot even write down an ansatz for $\omega$ in these cases.

Let us look for $\omega \in H^2_{D,L^2}(X, O_X(-3))$ that transforms in irreducible representation $V_{(0,0)}$:
\[
\omega = a(\tilde{s}) \tilde{e}_1 \wedge \tilde{e}_2 + b(\tilde{s}) \tilde{e}_1 \wedge \tilde{e}_3 + c(\tilde{s}) \tilde{e}_2 \wedge \tilde{e}_4 + d(\tilde{s}) \tilde{e}_3 \wedge \tilde{e}_4
\]
Then, $\overline{D} \omega = 0$ gives
\[
\tilde{s} \left( c' - \frac{h^{(-3)'}}{h^{(-3)}} c \right) + c = 0, \quad \tilde{s} \left( d' - \frac{h^{(-3)'}}{h^{(-3)}} d \right) + 2d + a = 0
\]
We further use
\[
* \left( \tilde{e}_I \wedge \tilde{e}_J \right) = \tilde{e}_I \wedge \tilde{e}_J \wedge \prod_{K \neq I, K \neq J} \tilde{f}_K(\tilde{s}) e_K \wedge \tilde{e}_K
\]
to show that $D(*\omega) = 0$ is equivalent to

$$\left(a\tilde{f}_3\tilde{f}_4\right)' + \left(\frac{4}{\tilde{s}} + \frac{h^{(-3)'}(\omega)}{h^{(-3)}}\right)a\tilde{f}_3\tilde{f}_4 + \frac{d}{\tilde{s}}f_1\tilde{f}_2 = 0$$

$$\left(b\tilde{f}_2\tilde{f}_4\right)' + \left(\frac{3}{\tilde{s}} + \frac{h^{(-3)'}(\omega)}{h^{(-3)}}\right)b\tilde{f}_2\tilde{f}_4 = 0.$$ 

In the limit $\tilde{s} \to \infty$ (i.e. $s \to 0$)

$$c \sim \tilde{s}^{-1/4}, \quad b \sim \tilde{s}^{-5/4} \quad \tilde{s} \to \infty$$

We checked using the norm (77) that both $b$ and $c$ solutions have divergent contribution to their norm squared from the region $\tilde{s} \to \infty$. The two solutions for $a$ behave as

$$a \sim C_1\tilde{s}^{-7/4} + C_2\tilde{s}^{-3/4}$$

The solution with $C_2 \neq 0$ has divergent contribution to the norm squared from $\tilde{s} \to \infty$. While in the other limit

$$a \sim C_3\tilde{s}^{-3} + C_4\tilde{s}^{-1} \quad \tilde{s} \to 0$$

and solution with $C_3 \neq 0$ has divergent contribution to the norm squared from $\tilde{s} \to 0$. We further checked, using Mathematica, that a good solution at $\tilde{s} \to 0$ behaves badly at $\tilde{s} \to \infty$. Hence, we conclude that there is no representation $\mathbb{V}_{(0,0)}$ in $H^2_{D,L^2}(X, \mathcal{O}_X(-3))$.

### D Useful formulae for computing the norm

We work in a patch $U^1 \neq 0$ and introduce 'polar coordinates':

$$t_2 = v t_3, \quad t_3 = |t_3| e^{i\varphi}, \quad z_1 = r_1 e^{i\phi_1}, \quad z_2 = r_2 e^{i\phi_2}, \quad v = r_v e^{i\phi_v}$$

and denote

$$a = 1 + T_2 + T_v(1 + T_1), \quad b = 2v r_1 r_2 v, \quad \Phi = \phi_2 - \phi_1 - \phi_v, \quad y = 1 + T_1 + T_2$$

where

$$T_v = r_v^2, \quad T_1 = r_1^2, \quad T_2 = r_2^2.$$ 

To evaluate the integrals arising in the computation of the norm we use

$$\int_0^{2\pi} \frac{d\Phi}{a + b \cos \Phi} = \frac{2\pi}{(a^2 - b^2)^{1/2}}, \quad \int_0^{2\pi} \frac{d\Phi}{(a + b \cos \Phi)^2} = \frac{2\pi a}{(a^2 - b^2)^{3/2}}, \quad a > b$$

$$\int_0^{2\pi} \frac{d\Phi}{(a + b \cos \Phi)^3} = \frac{\pi(b^2 + 2a^2)}{(a^2 - b^2)^{5/2}}, \quad \int_0^{2\pi} \frac{d\Phi}{(a + b \cos \Phi)^4} = \frac{\pi(2a^3 + 3ab)}{(a^2 - b^2)^{7/2}}, \quad a > b$$
\[ \int_0^\infty \frac{T_v dT_v}{(\beta T_v^2 + 2\gamma T_v + \delta)^{3/2}} = \frac{1}{2y(1+T_1)}, \quad \int_0^\infty \frac{dT_v}{(\beta T_v^2 + 2\gamma T_v + \delta)^{3/2}} = \frac{1}{2y(1+T_2)} \]
\[ \int_0^\infty \frac{T_v^2 dT_v}{(\beta T_v^2 + 2\gamma T_v + \delta)^{5/2}} = \frac{1}{12y^2(1+T_1)}, \quad \int_0^\infty \frac{T_v dT_v}{(\beta T_v^2 + 2\gamma T_v + \delta)^{5/2}} = \frac{1}{12y^2(1+T_2)} \]

where
\[ \beta = (1 + T_1)^2, \quad \gamma = 1 + T_1 + T_2 - T_1 T_2, \quad \delta = (1 + T_2)^2. \]

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