LINEARLY REDUCTIVE QUOTIENT SINGULARITIES

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Abstract. We study isolated quotient singularities by finite and linearly reductive group schemes (lrq singularities for short) and show that they satisfy many, but not all, of the known properties of finite quotient singularities in characteristic zero:

1. From the lrq singularity we can recover the group scheme and the quotient presentation.
2. We establish canonical lifts to characteristic zero, which leads to a bijection between lrq singularities and certain characteristic zero counterparts.
3. We classify subgroup schemes of $GL_d$ and $SL_d$ that correspond to lrq singularities. For $d = 2$, this generalises results of Klein, Brieskorn, and Hashimoto [Kl84, Br67, Ha15]. Also, our classification is closely related to the spherical space form problem.
4. F-regular (resp. F-regular and Gorenstein) surface singularities are precisely the lrq singularities by finite and linearly reductive subgroup schemes of $GL_2$ (resp. $SL_2$). This generalises results of Klein and Du Val [Kl84, DV34].
5. Lrq singularities in dimension $\geq 4$ are infinitesimally rigid. We classify lrq singularities in dimension 3 that are not infinitesimally rigid and compute their deformation spaces. This generalises Schlessinger’s rigidity theorem [Sc71] to positive and mixed characteristic.

Finally, we study Riemenschneider’s conjecture [Ri74] in this context, that is, whether lrq singularities deform to lrq singularities.

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1. INTRODUCTION

1.1. **Linearly reductive group schemes.** Let $k$ be an algebraically closed field of characteristic $p > 0$, let $G$ be a finite group scheme over $k$, and let

$$1 \to G^0 \to G \to G^{\text{et}} \to 1$$

be the connected-étale sequence.

A particularly nice class of finite group schemes are the **linearly reductive** ones, which have been classified by Nagata [Na61]: these are precisely those $G$, where $G^{\text{et}}$ is of order prime to $p$ and where $G^0$, if non-trivial, is a finite subgroup scheme of $\mathbb{G}_m^N$ for some $N$. For example, the representation theory of a finite and linearly reductive group scheme is equivalent to that of a finite group over the complex numbers, see Corollary 2.11.

Moreover, we will see in Proposition 6.3 that if $G$ is a finite group scheme that acts on $\text{Spec } k[[u_1, \ldots, u_d]]$ freely outside the closed point and with schematic fixed locus equal to the closed point, then this action can be **linearized** if and only if $G$ is linearly reductive. This generalises a well-known lemma of Cartan [Ca53] in the complex case.

In this article, we study linearly reductive group schemes $G$, actions as just described, and the associated quotient singularities. Most of our results are known in the case where $G$ is reduced, that is, if $G^0$ is trivial, where they become statements about groups of order prime to $p$.

1.2. **Very small representations.** A linear representation

$$\rho : G \to \text{GL}_d$$

of a finite and linearly reductive group scheme is **very small** if for every $\mu_n \subset G$ with $n \geq 2$, the $\rho(\mu_n)$-invariant subspace is zero-dimensional. This is equivalent to the induced $G$-action on $\text{Spec } k[[u_1, \ldots, u_d]]$ being free outside the closed point (Proposition 6.5).

The first objective of this article is the classification of finite linearly reductive group schemes that admit very small representations. Rather than repeating the somewhat involved and technical results from Section 3 let us sketch the main ideas and reduction steps in this introduction:

(1) Every finite linearly reductive group scheme $G$ admits a distinguished (“canonical”) geometric lift to characteristic zero (Proposition 2.4), whose geometric points are a finite group that we denote $G_{\text{abs}}$, the **abstract group associated to $G$.**
(2) Linear representations of finite linearly reductive group schemes admit distinguished lifts (Proposition 2.9). Specialization yields a bijection

\[ \text{sp} : \text{Rep}_\mathbb{C}(G_{\text{abs}}) \rightarrow \text{Rep}_k(G) \]

between (very small) representations of a linearly reductive group scheme \( G \) and (very small) complex representations of the group \( G_{\text{abs}} \), see Corollary 2.11 and Remark 2.12.

(3) Admitting a very small representation puts strong restrictions on a finite linearly reductive group scheme \( G \): namely, every abelian subgroup scheme of \( G_{\text{abs}} \) must be cyclic (Theorem 3.2).

Finite groups all of whose abelian subgroups are cyclic have been studied in different contexts by Milnor, Suzuki, Thomas, Wall, Wolf, Zassenhaus, and others, see, for example, \([\text{Mi}57, \text{Wo}11, \text{Za}35]\).

Putting all these results and observations together, we obtain a (rough) classification of linearly reductive group schemes that admit very small representations. Let us highlight some of the results:

1. We recover the classification of finite, small, and linearly reductive subgroup schemes of \( \text{SL}_2 \), due to Klein \([\text{Kl}84]\) (characteristic zero) and Hashimoto \([\text{Ha}15]\) (positive characteristic), see Theorem 3.3.
2. We extend Brieskorn’s classification \([\text{Br}67]\) of finite and small subgroups of \( \text{GL}_2(\mathbb{C}) \) to finite and linearly reductive subgroup schemes of \( \text{GL}_2 \) (any characteristic), see Theorem 3.4.
3. We show that all very small subgroup schemes of \( \text{GL}_d \) for \( d \) odd are cyclic or split metacyclic, see Theorem 3.8.
4. We obtain a classification of very small subgroup schemes of \( \text{SL}_3 \) (any characteristic), see Corollary 3.9.
5. We show that if \( d \) is a power of \( \text{char}(k) \), then every finite and very small subgroup scheme of \( \text{GL}_{d,k} \) is cyclic.

We refer to Section 3 for precise definitions and statements and more applications, corollaries, and examples.

1.3. **Linearly reductive quotient singularities.** Associated to a finite and very small subgroup scheme \( G \subseteq \text{GL}_{d,k} \) over some algebraically closed field \( k \) of characteristic \( p \geq 0 \), we have a \( G \)-action on \( \mathbb{A}^d_k \) that is free outside the closed point. By definition, the quotient \( \mathbb{A}^d_k/G \) (and every \( k \)-scheme that is formally isomorphic to it) is called a linearly reductive quotient singularity or lrq singularity for short.

It is known that invariant rings by linearly reductive group schemes satisfy many nice properties. We will summarize some of them in the following proposition, where most of the statements are more or less well-known - if \( G^\circ \) is trivial, then they are definitely well-known.

**Proposition 1.1** (Proposition 7.1). Assume that \( G \) is a finite, linearly reductive, and very small subgroup scheme of \( \text{GL}_d \) that acts linearly on \( S := k[[u_1, ..., u_d]] \) and let \( R := S^G \) be the ring of invariants. Then
1. The ring $R$ is $F$-regular.
2. The class group of $R$ is isomorphic to $\text{Hom}(G, \mathbb{G}_m)$.
3. The $F$-signature of $R$ is equal to $\frac{1}{|G|}$.
4. The Hilbert-Kunz multiplicity of $(R, \mathfrak{m}_R)$ is equal to $\frac{1}{|G|} \text{length}(S/\mathfrak{m}_RS)$.
5. The local fundamental group of $\text{Spec } R$ is isomorphic to $G^\text{et}$.

In particular, the $F$-signature detects the length of $G$, whereas the Hilbert-Kunz multiplicity depends on the embedding of $G$ into $\text{GL}_d$, see Remark 7.4. Let us also note that isolated toric singularities are precisely the lrq singularities by linearly reductive group schemes of the form $\mu_n$, which might be more or less well-known:

**Proposition 1.2** (Proposition 7.3). If $x \in X$ be an isolated singularity of dimension $d$ over an algebraically closed field $k$. Then, the following are equivalent:

1. $x \in X$ is toric,
2. $x \in X$ is an lrq singularity by an abelian group scheme,
3. $x \in X$ is an lrq singularity by $\mu_n$ for some $n > 0$,
4. $x \in X$ is an lrq singularity by $\mu_n = \text{Cl}(X)^D$, where $\mu_n$ is embedded into $\text{GL}_d$ as $\zeta \mapsto \text{diag}(\zeta^{q_1}, \ldots, \zeta^{q_d})$ with $1 = q_1 \leq \ldots \leq q_d < n$.

1.4. Uniqueness. Over the complex numbers, the universal topological cover of a finite quotient and isolated singularity is smooth and the local fundamental group acts on it by deck transformations. This shows uniqueness of the group of a finite quotient singularity, as well as uniqueness of the action (up to conjugation). For linearly reductive quotient singularities, we have a similar uniqueness statement, but the proof is much more involved (the reason is that if $G$ is not étale, then topological methods do not give any information about $G^\text{et}$).

**Theorem 1.3** (Theorem 8.1). Let $R = S^G$ be as in Proposition 7.7. If $H$ is another finite group scheme acting on $S$, such that the $H$-action is free outside the closed point, $H$ fixes the closed point, and such that the invariant ring $S^H$ is isomorphic to $R$, then

1. $H$ is isomorphic to $G$,
2. the $H$-action is linearizable, and then,
3. the two actions are conjugate in $\text{GL}_d$.

Note that we do not assume $H$ to be linearly reductive - this is a consequence of the theorem.

1.5. Rigidity of lrq singularities. Over the complex numbers, a classical theorem of Schlessinger [Sc71] states that finite and isolated quotient singularities in dimension $\geq 3$ are infinitesimally rigid. Moreover, by loc.cit. this is still true in characteristic $p > 0$ for quotient singularities by finite groups of order prime to $p$.
For lrq singularities, we will prove that Schlessinger’s rigidity theorem is still true in dimension $\geq 4$, whereas (a little bit to our surprise) the situation is more complicated and subtle in dimension 3.

**Theorem 1.4** (Corollary 10.9). Let $x \in X = 0 \in (k^d/G)^\wedge$ be an lrq singularity.

1. If $d \geq 4$, then $X$ is infinitesimally rigid.
2. If $d = 3$, then $X$ is rigid, but not necessarily infinitesimally rigid.

Here, (infinitesimal) rigidity is defined in terms of deformations over equicharacteristic Artin rings (Definition 9.1). An lrq singularity over an algebraically closed field $k$ of characteristic $p > 0$ admits a canonical lift, see Corollary 10.10. In particular, lrq singularities in mixed characteristic are not rigid. Nevertheless, we refer to Section 10.3 for deformations in mixed characteristic and a version of arithmetic (infinitesimal) rigidity.

**1.6. Deformation spaces in dimension 3.** The previous theorem begs for the computation of the miniversal deformation spaces of 3-dimensional lrq singularities. To state the results, we consider the following two $G$-representations, which are at most one dimensional.

1. the adjoint representation $\chi_{\text{ad}} : G \to \text{Aut} \text{(Lie}(G^o)) \in \{\text{id}, \mathbb{G}_m\}$, which is at most one-dimensional and depends on $G$ only, and
2. the determinant $\chi_{\text{det}} : G \to \text{GL}_3 \to \mathbb{G}_m$, which depends on the $G$-action $\rho$.

Then, we can determine which 3-dimensional lrq singularities are infinitesimally rigid and compute the deformation spaces in the other cases:

**Theorem 1.5** (Theorem 10.2 and Theorem 10.8). Let $x \in X = (k^3/G)^\wedge$ be a 3-dimensional lrq singularity over an algebraically closed field $k$ of characteristic $p > 0$. Then,

1. $x \in X$ is infinitesimally rigid if and only if $G$ is étale or if $G$ is not étale and $\chi_{\text{ad}} \neq \chi_{\text{det}}$.
2. If $x \in X$ is not infinitesimally rigid, then $G^o = \mu_{p^n}$ for some $n \geq 1$ and the miniversal deformation space is isomorphic to

$$\text{Def}_X \cong \text{Spec} W(k)[\varepsilon]/(\varepsilon^2, p^n\varepsilon),$$

where $W(k)$ denotes the Witt ring.

In particular, miniversal deformation spaces of 3-dimensional lrq singularities can be arbitrarily nonreduced. We refer to Section 10 for precise statements, applications, corollaries, and explicit examples.

**1.7. Characterization of F-regular singularities in dimension 2.** Over the complex numbers and in dimension two, it is well-known that the canonical singularities (resp. klt singularities) are precisely the quotient singularities by finite subgroups of $\text{SL}_2$ (resp. $\text{GL}_2$).
Generalising results of Hashimoto [Ha15], Liedtke-Satria [LS14], and using the classification of finite linearly reductive group schemes of $\text{SL}_2$ and $\text{GL}_2$, we show the following.

**Theorem 1.6** (Theorem 11.2). For a surface singularity over an algebraically closed field of characteristic $p > 0$, the following are equivalent

1. it is an lrq quotient singularity with respect to a finite and very small subgroup scheme of $\text{GL}_2$ (resp. $\text{SL}_2$)
2. it is F-regular (resp. F-regular and Gorenstein).

Moreover, if $p \geq 7$, these are equivalent to

3. it is a normal klt-singularity (resp. a rational double point).

We note that, if $p \geq 7$, then the F-regular (resp. F-regular and Gorenstein) surface singularities are precisely the klt-singularities (resp. canonical singularities), as follows from Hara’s classification [Ha98]. Hence, if $p \geq 7$, then Theorem 1.6 establishes a similar picture as over the complex numbers. We refer to Section 3 for the subgroup schemes showing up and to Section 11 for further details, especially in small characteristics.

### 1.8. Deformations of lrq singularities in dimension 2.

Riemenschneider [Ri74] conjectured that finite quotient singularities over the complex numbers deform to finite quotient singularities. In dimension $d \geq 3$ this is true by Schlessinger’s rigidity theorem [Sc71]. In dimension 2, this conjecture was established by Esnault and Viehweg [EV85]. Moreover, Kollár and Shepherd-Barron [KSB88] showed that cyclic quotient singularities deform to cyclic quotient singularities. In the final section, we study the following analog for lrq singularities:

**Conjecture 1.7** (Conjecture 12.1). Let $B$ be the spectrum of a DVR with closed, generic, and geometric generic points $0$, $\eta$, and $\overline{\eta}$, respectively. Let $\mathcal{X} \to B$ be a flat family of $d$-dimensional singularities with special and geometric generic fiber $\mathcal{X}_0$ and $\mathcal{X}_\overline{\eta}$, respectively. Assume that the non-smooth locus of $\mathcal{X} \to B$ is proper over $B$.

1. If $\mathcal{X}_0$ is an lrq singularity, then $\mathcal{X}_\overline{\eta}$ contains at worst lrq singularities.
2. Let $G_0$ be the group scheme associated to $\mathcal{X}_0$ and let $G_\eta$ be the group scheme associated to an lrq singularity on $\mathcal{X}_\overline{\eta}$. Then, we have $|G_0| \geq |G_\eta|$, where $| \cdot |$ denotes the length of a group scheme.
3. If $\mathcal{X}_0$ is a cyclic lrq singularity, then $\mathcal{X}_\overline{\eta}$ contains at worst cyclic lrq singularities.

If $B$ is of equal characteristic zero, then Parts (1) and (3) are true by the already mentioned results of [EV85] and [KSB88]. As further evidence for this conjecture, we show the following.

1. We establish it in dimension $d \geq 3$ (Proposition 12.4).
2. We establish it if $\mathcal{X}_0$ is Gorenstein (Corollary 12.14).
3. We establish Part (1) if $X$ is $\mathbb{Q}$-Gorenstein (Proposition 12.8). In recent work of Sato and Takagi [ST21], they removed the $\mathbb{Q}$-Gorenstein condition, that is, Part (1) is now a theorem, see Remark 12.9.
4. We establish Part (2) if $B$ has equal characteristic zero and $X_0$ is cyclic, as well as in some cases where $B$ has equal characteristic $p$ (Proposition 12.11).
5. We give examples and counter-examples that illustrate that some more naive versions of this conjecture are not true.

1.9. Beyond lrq singularities. The assumption on linear reductivity in the previous results is crucial - rigidity in dimension $\geq 3$ fails, deformations of quotient singularities need no longer be quotient singularities, etc. We will see this in examples in this article, as well as in the companion article [LMM21], where we study canonical surface singularities in positive characteristic.

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2. Linearly reductive group schemes

In this section, we recall a couple results about finite group schemes over algebraically closed fields $k$ of positive characteristic $p$.

2.1. Generalities on (linearly reductive) group schemes. Let $G$ be a finite group scheme over an algebraically closed field $k$ of characteristic $p \geq 0$. Since $k$ is perfect, the canonical short exact sequence of finite group schemes over $k$

\[(1) \quad 1 \to G^o \to G \to G^{\text{ét}} \to 1\]

splits, where $G^o$ is the connected component of the identity, and $G^{\text{ét}}$ is an étale group scheme over $k$. Since $k$ is algebraically closed, $G^{\text{ét}}$ is the constant group scheme associated to a group and we will not distinguish between these two objects. Thus, we have a canonical isomorphism $G \cong G^o \times G^{\text{ét}}$. Moreover, $G^o$ is an infinitesimal group scheme of length equal to some power of $p$. In particular, if $p = 0$ or if the length of $G$ is prime to $p$, then $G^o$ is trivial and $G$ is étale.

If $M$ is a finitely generated abelian group, then the group algebra $k[M]$ carries a Hopf algebra structure, and the associated commutative group scheme is denoted $D(M) := \text{Spec } k[M]$. By definition, such group schemes are called diagonalizable. For example, we have $D(\mathbb{Z}) \cong \mathbb{G}_m$ and $D(\mathbb{C}_n) \cong \mu_n$, where $\mathbb{C}_n$ denotes the cyclic group of order $n$. Quite generally, every
diagonalizable group scheme can be embedded into $G^N_m$ for some $N \geq 1$. Moreover, $\mu_n$ is étale over $k$ if and only if $p \nmid n$.

A finite group scheme $G$ over $k$ is said to be linearly reductive if every (finite-dimensional) representation of $G$ is semi-simple. If $p = 0$, then all finite group schemes over $k$ are étale and linearly reductive. If $p > 0$, then, by a theorem of Nagata [Na61, Theorem 2] (but see also [AOV08, Proposition 2.10] and [Ha15, Section 2]), a finite group scheme over $k$ is linearly reductive if and only if it is an extension of a finite and étale group scheme, whose length is prime to $p$, by a diagonalizable group scheme.

In particular, diagonalizable group schemes are examples of linearly reductive group schemes. To obtain more examples of linearly reductive group schemes, we note the following. For a field $k$ and an abstract group $G$, we denote by $G_k$ the constant group scheme associated to $G$ over $k$.

**Lemma 2.1.** For every algebraically closed field $k$ of characteristic $p \geq 0$, the functor

$$G^o \times G^{\text{ét}} \mapsto \left(\left((G^o)^D(k)\right)_C\right)^D \otimes (\mathbb{C} \times G^{\text{ét}}(k))$$

is an equivalence of categories between the category of finite and linearly reductive group schemes over $k$ and the category of finite abstract groups with a unique abelian $p$-Sylow subgroup.

For $p = 0$, this reduces to the equivalence of categories of finite groups and finite constant group schemes over $k$.

**Proof.** First, we note that if $G$ is a group with a unique $p$-Sylow subgroup $G_p$, then $G_p$ is a normal subgroup. Moreover, the quotient $G' := G/G_p$ is of order prime to $p$ and hence, $G$ is isomorphic to the semidirect product $G_p \rtimes G'$ by the Schur-Zassenhaus theorem.

An essential inverse to the functor of the lemma is given by the functor that maps a group $G = G_p \rtimes G'$, where $G_p$ is the unique $p$-Sylow subgroup and $p \nmid |G'|$, to the linearly reductive group scheme $\left((G_p^o)^D(C)\right)_k^D \rtimes G'_k$. \qed

In this lemma and the proof, the two functors can also be written as follows

$$G^o \rtimes G^{\text{ét}} \mapsto \text{Hom}(\left((G^o)^D(k), \mathbb{C}^\times\right) \rtimes G^{\text{ét}}(k))$$

and

$$G_p \rtimes G' \mapsto \left((\text{Hom}(G_p, \mathbb{C}^\times))\right)_k^D \rtimes G'_k$$

but this is a matter of taste.

**Definition 2.2.** Let $k$ be an algebraically closed field.

1. If $G$ is a linearly reductive group scheme over $k$, we let $G_{\text{abs}}$ be the image of $G$ under the equivalence of Lemma 2.1 and call it the abstract group associated to $G$.

2. If $G$ is a finite and abstract group with a unique abelian $p$-Sylow subgroup, we let $G_{\text{lr}}$ be the preimage of $G$ under the equivalence of
Lemma 2.1 and call it the linearly reductive group scheme associated to \( G \) (over \( k \)).

Given a finite and infinitesimal group scheme \( G \) over \( k \), the \( k \)-linear Frobenius morphism \( F \) yields a canonical decomposition of \( G \)

\[
1 \leq G[F] \leq G[F^2] \leq \ldots \leq G[F^n] = G
\]

for some sufficiently large \( n \). By definition, the minimal \( n \) for which we have \( G = G[F^n] \) is called the height of \( G \). Each subquotient in this decomposition series is of height one.

**Lemma 2.3.** Let \( G \) be a finite \( k \)-group scheme. Then, \( G \) is linearly reductive if and only if it does not contain \( \alpha_p \) or \( C_p \).

**Proof.** By Nagata’s classification, a linearly reductive group scheme does not contain \( \alpha_p \) or \( C_p \), so we only have to prove the converse.

First, we recall from the exact sequence (1) that we have a semi-direct decomposition \( G = G^\circ \rtimes G^{\text{ét}} \). The group \( G^{\text{ét}} \) is linearly reductive if and only if it is of order prime to \( p \) if and only if \( G \) does not contain a subgroup isomorphic to \( C_p \). Thus, the claim is clear for \( G^{\text{ét}} \) and we may assume \( G = G^\circ \). We will prove the claim by induction on the height \( n \) of \( G \).

Consider the short exact sequence

\[
(2) \quad 1 \to G[F^{n-1}] \to G[F^n] \to G[F^n]/G[F^{n-1}] \to 1,
\]

where \( G[F^{n-1}] \) is linearly reductive by the induction hypothesis. Since extensions of linearly reductive groups are linearly reductive, it suffices to show that \( G[F^n]/G[F^{n-1}] \) is linearly reductive.

Assume that this is not the case. Then, either

- \( G[F^n]/G[F^{n-1}] \) is not abelian. In this case its Lie algebra contains a non-zero vector \( v \) such that \( v[p] = 0 \) (by Chwe’s theorem [Ch65]) and we can integrate \( v \) to a subgroup scheme \( \alpha_p \subseteq G[F^n]/G[F^{n-1}] \).
- \( G[F^n]/G[F^{n-1}] \) is abelian, connected and not linearly reductive. In particular, the \( p \)-operation on its Lie algebra is a semi-linear endomorphism which is not semi-simple, hence not injective (see [SGA7II, Exp. XXII, Section 1]). In particular, as in the previous case, we find a subgroup scheme \( \alpha_p \subseteq G[F^n]/G[F^{n-1}] \).

In any case, we find a subgroup scheme \( \alpha_p \subseteq G[F^n]/G[F^{n-1}] \). Then, we can restrict the sequence (2) to this \( \alpha_p \) and obtain an extension of \( \alpha_p \) by the linearly reductive group scheme \( G[F^{n-1}] \). Since \( k \) is perfect, all such extensions split by [SGA3II, Exp. XVII, Théorème 6.1.1. B)], so we find \( \alpha_p \subseteq G \), contradicting our hypothesis. Hence, \( G[F^n]/G[F^{n-1}] \) is linearly reductive and thus so is \( G[F^n] \).

2.2. **Deformations of linearly reductive group schemes.** The following result, which should be more or less well-known to the experts, states that finite linearly reductive group schemes lift to characteristic 0 and that any two lifts are geometrically isomorphic.
Proposition 2.4. Let \( k \) be an algebraically closed field of characteristic \( p > 0 \) and let \( R \) be a complete DVR with residue field \( k \). Let \( G = \prod \mu_{p^i} \times G^\text{\text{et}} \) be a finite and linearly reductive group scheme over \( k \). Then,

1. \( \tilde{G} := \prod \mu_{p^i, R} \times (G^\text{\text{et}}(k))_R \) is a deformation of \( G \) to \( R \),
2. its automorphism scheme satisfies \( \text{Aut}^\text{\text{et}} \tilde{G} \cong (\tilde{G}^\circ)/(\tilde{G}^\circ)^\text{\text{et}}, \)
3. every other deformation \( \tilde{G}^\prime \) of \( G \) to \( R \) is a twisted form of \( \tilde{G} \) that splits over a totally ramified, finite, and flat extension \( S \) of \( R \) of degree \( p^i \leq |G^\circ/(G^\circ)^\text{\text{et}}| \) for some \( i \geq 0 \). Moreover, if \( \text{char}(R) = p \), then the extension \( R \subseteq S \) can be chosen to be purely inseparable.

Proof. Claim (1) is clear, since \( \tilde{G} \) is flat over \( R \) with special fiber isomorphic to \( G \).

Next, consider Claim (2). Since \( \tilde{G}^\circ = \prod \mu_{p^i, R} \) is a characteristic subgroup scheme of \( \tilde{G} \), we have an exact sequence

\[
0 \to E \to \text{Aut}^\text{\text{et}} \tilde{G} \xrightarrow{\varphi} \text{Aut}^\text{\text{et}} \tilde{G}^\circ \times \text{Aut}^\text{\text{et}} \tilde{G}^\text{\text{et}},
\]

where \( E := \ker(\varphi) \) and \( \text{Im}(\varphi) \) is a subgroup scheme of the étale group scheme \( \text{Aut}^\text{\text{et}} \tilde{G}^\circ \times \text{Aut}^\text{\text{et}} \tilde{G}^\text{\text{et}} \). On the other hand, the conjugation action of \( \tilde{G}^\circ \) on \( \tilde{G} \) induces a homomorphism \( \psi : \tilde{G}^\circ \to E \), which induces an isomorphism of the closed fiber of \( \tilde{G}^\circ/(\tilde{G}^\circ)^\text{\text{et}} \) with the closed fiber of \( E \) by [AOV08, Lemma 2.24]. By the fiberwise criterion for flatness, we deduce that \( E \) is flat over \( R \). Since \( G^\circ/(G^\circ)^\text{\text{et}} \) is diagonalizable, hence rigid, both \( E \) and \( \tilde{G}^\circ/(\tilde{G}^\circ)^\text{\text{et}} \) coincide with the unique deformation of \( G^\circ/(G^\circ)^\text{\text{et}} \) to \( R \). Note that \( E \cong \text{Aut}^\text{\text{et}} \tilde{G}^\circ \), since \( \text{Im}(\varphi) \) is étale and \( E_k = \text{Aut}^\text{\text{et}} \tilde{G}^\circ \).

Finally, let us prove Claim (3). By [AOV08, Lemma 2.14] there is a \((\tilde{G}, \tilde{G}^\prime)\)-bitorsor \( I \to \text{Spec} R \). In particular, \( \tilde{G}^\prime \) is a twisted form of \( \tilde{G} \). Thus, \( S^\prime := \text{Isom}(\tilde{G}, \tilde{G}^\prime) \) is an \( \text{Aut}^\text{\text{et}} \tilde{G} \)-torsor over \( R \) and, since \( R \) is complete, \( S^\prime \) is a disjoint union of \( |\text{Aut}^\text{\text{et}} \tilde{G}| \) connected components, each of which is an \( \text{Aut}^\text{\text{et}} \tilde{G} \)-torsor over \( R \). Taking the normalization of one of these components, we obtain the desired \( S \). Indeed, \( R \) is complete, so \( S \) is totally ramified over \( R \), the fibers of \( \text{Aut}^\text{\text{et}} \tilde{G} \to \text{Spec} R \) are local group schemes if \( \text{char}(R) = p \), so \( S \) is purely inseparable over \( R \) if \( \text{char}(R) = p \), and the degree of \( S \) over \( R \) divides the length of \( \text{Aut}^\text{\text{et}} \tilde{G} \) over \( R \), which coincides with \( |G^\circ/(G^\circ)^\text{\text{et}}| \) by Claim (2). In particular, the degree of \( S \) over \( R \) is a power of \( p \).

In particular, Proposition 2.4 provides us with a “canonical” deformation of \( G \) to \( R \).

Definition 2.5. Let \( G \cong \prod \mu_{p^i} \times G^\text{\text{et}} \) be a finite and linearly reductive group scheme over an algebraically closed field \( k \) of characteristic \( p > 0 \). Let \( R \) be a complete DVR with residue field \( k \). The canonical deformation of \( G \) to \( R \) (or, canonical lift of \( G \) to characteristic 0, if \( R = W(k) \)) is the \( R \)-group scheme \( \prod \mu_{p^i, R} \times (G^\text{\text{et}}(k))_R \).
The following example shows that there exist linearly reductive group schemes that admit “non-canonical” deformations.

**Example 2.6.** Assume $p \geq 3$, let $R$ be as in Proposition 2.4, and consider the non-trivial semi-direct product $G := \mu_p \rtimes C_2$, where $C_2$ acts as inversion on $\mu_p$. Let $\tilde{G} := \mu_{p,R} \rtimes C_2$ be the canonical deformation of $G$ to $R$. By Proposition 2.4, we have $\text{Aut}_{\tilde{G}}^\circ \cong \mu_p$. Thus, every non-trivial element of $H_1^{fl}(R, \mu_p) = R/\langle R \rangle_p$ yields a twisted form of $\tilde{G}$ over $R$ and hence, a non-trivial deformation of $G$ to $R$. We note that $R/\langle R \rangle_p$ is non-trivial, since it contains the class of $1 + \pi$, where $\pi$ is a uniformizer of $R$, and this element is not a $p$-th power.

2.3. **Deformations of representations.** If $\tilde{G}$ is a group scheme over a ring $R$, we denote by $\text{Rep}^d_R(\tilde{G})$ the set of isomorphism classes of free $R$-modules of rank $d$ together with an action of $G$, that is, $\text{Rep}^d_R(\tilde{G})$ is the set of homomorphisms in $\text{Hom}(\tilde{G}, \text{GL}_d,R)$ up to conjugation in $\text{GL}_d,R$.

**Definition 2.7.** Let $V$ be a vector space over an algebraically closed field $k$ and let $\rho : G \to \text{GL}(V)$ be a representation of a linearly reductive group scheme $G$. We define the $\lambda$-invariant of $\rho$ as

$$\lambda(\rho) := \max_{\langle \text{id} \rangle \neq \mu_n \subseteq G} \dim V^{\mu_n},$$

where $V^{\mu_n} \subseteq V$ denotes the subspace of $\mu_n$-invariant vectors.

**Remark 2.8.** Although the $\lambda$-invariant may not have been studied before, let us note that the representation

1. $\rho$ is faithful if and only if $\lambda(\rho) \neq \dim V$ and that
2. $\rho$ contains no pseudo-reflections if and only if $\lambda(\rho) \leq \dim(V) - 2$,

see also Section 7.4. Such representations are called small.

3. If $G$ is étale, then the $\lambda$-invariant can be defined as

$$\max_{\langle \text{id} \rangle \neq g \in G} \dim(V_{g,1}),$$

where $V_{g,1}$ is the eigenspace of the $\rho(g)$-action for the eigenvalue 1.

In particular, we have $\lambda(\rho) = 0$ (resp. $\lambda(\rho) \leq \dim V - 2$) if and only if for every $g \in G \setminus \{e\}$, the linear map $\rho(g)$ does not fix a line (resp. a hyperplane).

We will give a geometric interpretation of $\lambda(\rho)$ in Proposition 6.5.

**Proposition 2.9.** Let $k$ be an algebraically closed field of characteristic $p > 0$ and let $R$ be a complete DVR with residue field $k$ and field of fractions $K$. Let $G$ be a finite and linearly reductive group scheme over $k$ and let $\tilde{G}$ be a deformation of $G$ to $R$, with generic fiber $\tilde{G}_\eta$. Then, for every $d \geq 0$, the following hold:

1. Restriction of $\tilde{G}$-representations to $K$ induces a bijection

$$\text{Rep}^d_R(\tilde{G}) \sim \text{Rep}^d_R(\tilde{G}_\eta)$$
(2) Restriction of \( \tilde{G} \)-representations to \( k \) induces a bijection

\[
\text{Rep}^d_R(\tilde{G}) \cong \text{Rep}^d_K(G)
\]

(3) There is a bijective specialization map

\[
\text{sp}_d : \text{Rep}^d_K(\tilde{G}_\eta) \rightarrow \text{Rep}^d_k(G),
\]

such that \( \text{sp} := \coprod_{d=0}^\infty \text{sp}_d \) is compatible with direct sums, tensor products, duals, \( \lambda \)-invariants, and characters. In particular, \( \text{sp} \) maps simple representations to simple representations.

Proof. We consider the functor \( \text{Hom}(\tilde{G}, \text{GL}_{d,R}) \) of homomorphisms of group schemes over \( R \) up to conjugation in \( \text{GL}_{d,R} \). This means that \( \text{Rep}^d_S(\tilde{G}_S) = \text{Hom}(\tilde{G}, \text{GL}_{d,R})(S) \) for every \( R \)-algebra \( S \).

Let us first prove Claim (1). By [Se68, Lemme 2], every \( G_\eta \)-representation admits a \( \tilde{G} \)-invariant \( R \)-sublattice of full rank, that is, the restriction map \( \text{Hom}(\tilde{G}, \text{GL}_{d,R})(R) \rightarrow \text{Hom}(\tilde{G}, \text{GL}_{d,R})(K) \) is surjective. Thus, to show (1), we have to show that two \( R \)-representations \( u, v \) of \( \tilde{G} \) are conjugate in \( \text{GL}_{d,R} \) if and only if their generic fibers \( u_K \) and \( v_K \) are conjugate in \( \text{GL}_{d,K} \). It is clear that \( u_K \) and \( v_K \) are conjugate if \( u \) and \( v \) are. For the converse, we use that the transporter functor \( \text{Transp}(u,v) \) of sections of \( \text{GL}_{d,R} \) conjugating \( u \) to \( v \) is representable by a smooth closed subscheme of \( \text{GL}_{d,R} \) by [Ma09, Theorem 4.5, Lemma 4.7]. In particular, if the generic fiber of \( \text{Transp}(u,v) \) is non-empty (that is, if \( u_K \) and \( v_K \) are conjugate), then the special fiber of \( \text{Transp}(u,v) \) is non-empty and Hensel’s Lemma provides us with an element of \( \text{GL}_{d,R}(R) \) conjugating \( u \) to \( v \).

Next, consider Claim (2). By [Ma09, Lemma 4.4, Theorem 4.5], the functor \( \text{Hom}(\tilde{G}, \text{GL}_{d,R}) \) is formally étale and locally of finite presentation. Since \( R \) is complete, we immediately obtain the claimed bijection.

For Claim (3), we note that Claims (1) and (2) apply to every finite field extension \( L \) of \( K \) with \( R \) replaced by the integral closure \( R_L \) of \( R \) in \( L \). We can thus define \( \text{sp}_d \) as the following chain of bijections:

\[
\text{Rep}^d_K(G_\eta) = \text{Hom}(\tilde{G}, \text{GL}_{d,R})(K) = \lim_{\substack{\text{finite} \ \ K \subseteq L}} \text{Hom}(\tilde{G}, \text{GL}_{d,R})(L) = \lim_{\substack{\text{finite} \ \ K \subseteq L}} \text{Hom}(\tilde{G}, \text{GL}_{d,R})(R_L) \rightarrow \text{Hom}(\tilde{G}, \text{GL}_{d,R})(k) = \text{Rep}^d_k(G).
\]

Since \( \text{sp} := \coprod_{d=0}^\infty \text{sp}_d \) is a coproduct of compositions of restriction maps and their inverses, both of which satisfy the stated compatibilities, \( \text{sp} \) satisfies the compatibilities as well. \( \square \)
2.4. Representation theory of linearly reductive group schemes and their associated abstract groups. The following proposition shows that the abstract group $G_{\text{abs}}$ associated to a linearly reductive group scheme $G$ over an algebraically closed field $k$ of characteristic $p > 0$ appears naturally as the geometric generic fiber of any lift of $G$ to characteristic 0.

**Proposition 2.10.** Let $G$ be a finite abstract group. Let $k$ be an algebraically closed field of characteristic $p > 0$, let $W(k)$ be the ring of Witt vectors, let $K := \text{Frac}(W(k))$, and let $\bar{K}$ be an algebraic closure. Then, the constant group scheme $\bar{G}_{K}$ is the geometric generic fiber of a lift of a linearly reductive group scheme $H$ over $k$ to characteristic 0 if and only if $G$ has a unique abelian $p$-Sylow subgroup and $H \cong G_{\text{lr}}$.

**Proof.** Using Nagata’s classification, we can write $H = H^{\circ} \times H^{\text{et}}$, where $H^\circ \cong \prod_{i=1}^{l} \mu_{p^{n_i}}$ for some $n_i \geq 0$ and where $H^{\text{et}}$ is finite étale of length prime to $p$. Let $\bar{H}$ be any lift of $H$ to $W(k)$. By Proposition 2.4, we have $G = \bar{H}(\bar{K}) \cong \prod_{i=1}^{l} C_{p^{n_i}} \times H^{\text{et}}(k)$. In particular, $G$ has a unique abelian $p$-Sylow subgroup and we have $G_{\text{lr}} \cong H$, which is what we had to prove.

For the converse, it suffices to note that $G_{K}$ is the geometric generic fiber of the canonical lift of $G_{\text{lr}}$ to characteristic 0 as in Definition 2.5. □

Since we already know from Proposition 2.9 that the representation theory of a linearly reductive group scheme $G_{\text{lr}}$ and of the geometric generic fiber of a lift of $G_{\text{lr}}$ to characteristic 0 coincide, we obtain the following corollary.

**Corollary 2.11.** Let $k$ be an algebraically closed field of characteristic $p \geq 0$ and let $G$ be a finite and linearly reductive group scheme over $k$. Then, for every $d \geq 0$, there is a bijection

$$
\text{sp} : \text{Rep}_{d}^{G}(G_{\text{abs}}) \cong \text{Rep}_{d}^{G}(G),
$$

such that $\text{sp} := \bigoplus_{d=0}^{\infty} \text{sp}_{d}$ is compatible with direct sums, tensor products, duals, $\lambda$-invariants, and characters. In particular, $\text{sp}$ maps simple representations to simple representations.

**Remark 2.12.** This can also be phrased in the language of cde-triangles as in [Se77, Part III]. We denote by $R_{F}(H)$ (resp. $P_{F}(H)$) the Grothendieck ring of finite (resp. finite projective) $H$-modules over some field $F$. Keeping the notations and assumptions of the previous Corollary, we have the following:

1. The Cartan morphism $c : P_{k}(G) \rightarrow R_{k}(G)$ is an isomorphism since $G$ is linearly reductive.
2. We define the decomposition $d : R_{C}(G_{\text{abs}}) \rightarrow R_{k}(G)$ via $\text{sp}$.
3. We define the extension $e : P_{k}(G) \rightarrow R_{C}(G_{\text{abs}})$ via the inverse of $\text{sp}$ or using the lifting of representations.

We leave the details including commutativity of the cde-triangle to the reader.
3. Very small linearly reductive subgroup schemes of $\text{GL}_d$

In this section, we explain how to classify representations of finite and linearly reductive group schemes admitting a representation with $\lambda = 0$ (see Definition 2.7) in arbitrary dimension $d$. We will make this classification explicit if $d \leq 3$, if $d$ is odd and if $d$ is a power of $\text{char}(k)$. We will see in Section 6 that these are precisely the representations leading to lrq singularities. In particular, the results of this section yield a complete classification of lrq singularities.

3.1. Very small representations.

**Definition 3.1.** Let $G$ be a linearly reductive group scheme over an algebraically closed field. A $d$-dimensional representation $\rho$ of $G$ is called very small if $\lambda(\rho) = 0$. In this case, we say that $\rho(G)$ is a very small subgroup scheme of $\text{GL}_d$.

We note that a very small representation is faithful and that the name very small is motivated by the terminology of $\rho$ being small if $\lambda(\rho) \leq d - 2$, see Remark 2.8. We refer to Proposition 6.5 for a geometric interpretation in terms of fixed loci.

Next, we have the following corollary of our analysis of representations of linearly reductive group schemes from Section 2, which puts strong restrictions on them and which is the key to our classification results.

**Theorem 3.2.** Let $G$ be a linearly reductive group scheme over an algebraically closed field of characteristic $p \geq 0$. Then, the following hold:

1. The bijection $\text{sp}_d : \text{Rep}_d^k(G) \rightarrow \text{Rep}_d^C(G_{\text{abs}})$ identifies the subsets of very small representations.
2. If $G$ admits a very small representation, then every abelian subgroup of $G_{\text{abs}}$ is cyclic.

**Proof.** Claim (1) is a special case of Corollary 2.11.

For Claim (2), let $H \subseteq G_{\text{abs}}$ be an abelian subgroup and let $\rho : G_{\text{abs}} \rightarrow \text{GL}_d^C(\mathbb{C})$ be a very small representation of $G_{\text{abs}}$, which exists by Claim (1). Since $H$ is abelian, we may conjugate $\rho$ to assume that $H$ acts diagonally. Consider the restriction $f : H \rightarrow \mathbb{C}^*$ of the $H$-representation to a coordinate axis. Then, $\text{Ker}(f)$ is trivial since $\lambda(\rho) = 0$. Thus, $H$ is isomorphic to a finite, hence cyclic, subgroup of $\mathbb{C}^*$, which is what we wanted to show. \(\square\)

Thus, for every dimension $d$ and every algebraically closed field $k$ of characteristic $p \geq 0$, the classification of finite, linearly reductive, and very small subgroup schemes of $\text{GL}_{d,k}$ is the same (by passing to the associated abstract group) as the classification of finite and very small subgroups of $\text{GL}_{d,C}(\mathbb{C})$ that admit a unique cyclic normal $p$-Sylow subgroup.

Next, the study of finite groups that admit very small representations was initiated by Zassenhaus [Za35] (motivated by questions in near-fields), followed by important contributions by Suzuki and others, and a complete
classification was achieved (motivated by questions in differential geometry, see Section 3.4) by Milnor, Thomas, Wall, Wolf, and others, see [Wo11]. Instead of repeating this classification here, we will focus on interesting special cases, such as \( d = 2 \) or \( d \text{ odd} \).

Let us also note that the class of finite groups, all of whose abelian subgroups are cyclic, has interesting reformulations and characterizations: they are precisely the groups with periodic cohomology (which makes them interesting for topologists) and they can be characterized by the structure of their Sylow subgroups. We refer to [CE56, Chapter XII, Theorem 11.6] for details.

3.2. Very small linearly reductive subgroup schemes of \( \text{GL}_2 \). Let us first recall the classification of finite subgroup schemes of \( \text{SL}_2 \) over arbitrary fields from [Ha15], which extends the classification of finite subgroups of \( \text{SL}_2, \mathbb{C}(\mathbb{C}) \) from [Kl84]. We adapt these results to our setting.

**Theorem 3.3** (Hashimoto, Klein). Let \( G \) be a finite and linearly reductive subgroup scheme of \( \text{SL}_2,k \) over an algebraically closed field \( k \) of characteristic \( p \geq 0 \). Then, \( G \) is conjugate to one of the following, where \( \zeta_r \) denotes a primitive \( r \)-th root of unity.

1. \((n \geq 1)\) The group scheme \( \mu_n \) of length \( n \) embedded in \( \text{SL}_2,k \) as
   \[
   \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \ a \in \mu_n \right\}.
   \]
2. \((n \geq 2, p \geq 3)\) The binary dihedral group scheme \( \text{BD}_n \) of length \( 4n \) generated by \( \mu_{2n} \subset \text{SL}_2,k \) as in (1) and
   \[
   \begin{pmatrix} 0 & \zeta_4 \\ \zeta_4 & 0 \end{pmatrix}.
   \]
3. \((p \geq 5)\) The binary tetrahedral group scheme \( \text{BT}_{24} \) of length \( 24 \) generated by \( \text{BD}_2 \subset \text{SL}_2,k \) as in (2) and
   \[
   \frac{1}{\sqrt{2}} \begin{pmatrix} \zeta_8 & \zeta_8 \\ \zeta_8 & \zeta_8 \end{pmatrix}.
   \]
4. \((p \geq 5)\) The binary octahedral group scheme \( \text{BO}_{48} \) of length \( 48 \) generated by \( \text{BT}_{24} \subset \text{SL}_2,k \) as in (3) and \( \mu_8 \subset \text{SL}_2,k \) as in (1).
5. \((p \geq 7)\) The binary icosahedral group scheme \( \text{BI}_{120} \) of length \( 120 \) generated by \( \mu_{10} \subset \text{SL}_2,k \) as in (1),
   \[
   \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ \text{and} \ \frac{1}{\zeta_5^2 - \zeta_5^3} \begin{pmatrix} \zeta_5 + \zeta_5^{-1} & 1 \\ 1 & -(\zeta_5 + \zeta_5^{-1}) \end{pmatrix}.
   \]

Conversely, any of the above is a linearly reductive group scheme of \( \text{SL}_2,k \) in the indicated characteristics.

**Proof.** See [Ha15, Theorem 3.8]. Note also that, using Theorem 3.2, this follows immediately from the classification of finite subgroups of \( \text{SL}_2, \mathbb{C}(\mathbb{C}) \), which goes back to Klein [Kl84], by passing to the linearly reductive group.
schemes associated to the finite subgroups of $\text{SL}_{2,\mathbb{C}}(\mathbb{C})$ with a unique abelian $p$-Sylow subgroup.

To state the classification of very small linearly reductive subgroup schemes of $\text{GL}_{2,k}$, we follow Brieskorn [Br67 Section 2.4]. We let

$$\psi : \mathbb{G}_m \times \text{SL}_{2,k} \to \text{GL}_{2,k}$$

be the multiplication map, where $\mathbb{G}_m \subseteq \text{GL}_{2,k}$ is the diagonal torus. Let $H_1 \subseteq \mathbb{G}_m$ and $H_2 \subseteq \text{SL}_{2,k}$ be finite and linearly reductive subgroup schemes, let $N_i \subseteq H_i$ be normal subgroup schemes with projection maps $\pi_i : H_i \to H_i/N_i$, and assume that there exists an isomorphism $\varphi : H_2/N_2 \to H_1/N_1$. Then, we define

$$(H_1, N_1; H_2, N_2)_{\varphi} := \psi(H_1 \times_{\pi_1, H_1/N_1, \varphi \circ \pi_2} H_2),$$

where we consider $H_1 \times_{\pi_1, H_1/N_1, \varphi \circ \pi_2} H_2$ as a subgroup scheme of $\mathbb{G}_m \times \text{SL}_{2,k}$ via its natural embedding into $H_1 \times H_2$. In particular, since linear reductivity is stable under taking quotients and subgroups, the group scheme $(H_1, N_1; H_2, N_2)_{\varphi}$ is linearly reductive. If the length of $H_i/N_i$ is at most 3, then the conjugacy class of $(H_1, N_1; H_2, N_2)_{\varphi}$ does not depend on $\varphi$, so we will drop $\varphi$ from our notation in these cases.

Moreover, for $1 \leq q < n$ with $(n, q) = 1$, we define $\mu_{n,q}$ as the group scheme $\mu_n$ of length $n$ embedded in $\text{GL}_{2,k}$ as

$$\left\{ \begin{pmatrix} a & 0 \\ 0 & a^q \end{pmatrix}, a \in \mu_n \right\}.$$

After this preparation, the classification of small linearly reductive subgroup schemes of $\text{GL}_{2,k}$ follows from [Br67 Satz 2.9] and is as follows.

**Theorem 3.4.** Let $G$ be a finite, very small and linearly reductive subgroup scheme of $\text{GL}_{2,k}$ over an algebraically closed field $k$ of characteristic $p \geq 0$. Then, the following hold:

(i) $G$ is conjugate to one of the following.

1. $(n \geq 1, (n, q) = 1)$
   - The cyclic group scheme $\mu_{n,q}$
2a. $(n \geq 2, (m, 2) = (m, n) = 1, p \geq 3)$
   - The group scheme $(\mu_{2m,1}, \mu_{2m,1}; \text{BD}_n, \text{BD}_n)$.
3. $(n \geq 2, (m, 2) = 2, (m, n) = 1, p \geq 3)$
   - The group scheme $(\mu_{4m,1}, \mu_{2m,1}; \text{BD}_n, \mu_{2m,2n-1})$.
4a. $(m, 6) = 1, p \geq 5$
   - The group scheme $(\mu_{2m,1}, \mu_{2m,1}; \text{BT}_{24}, \text{BT}_{24})$.
4b. $(m, 6) = 3, p \geq 5$
   - The group scheme $(\mu_{6m,1}, \mu_{2m,1}; \text{BT}_{24}, \text{BD}_2)$.
5. $(m, 30) = 1, p \geq 7$
   - The group scheme $(\mu_{2m,1}, \mu_{2m,1}; \text{BI}_{120}, \text{BI}_{120})$. 
Two of these group schemes $G_1$ and $G_2$ are conjugate if and only if they are equal or $G_1 = \mu_{n,q}$ and $G_2 = \mu_{n,q'}$ with $qq' \equiv 1 \mod n$.

Conversely, any of the above is a small linearly reductive subgroup scheme of $GL_{2,k}$ in the indicated characteristics.

Proof. As already mentioned before, any of these subgroup schemes is linearly reductive and it easy to check that they are very small.

Conversely, assume that $G$ is a finite, very small and linearly reductive subgroup scheme of $GL_{2,k}$. By Theorem 3.2, the associated abstract group $G_{\text{abs}}$ is a finite and very small subgroup of $GL_{2,C}(C)$. By [Br67, Satz 2.9], $G_{\text{abs}}$ is the abstract group associated to one of the linearly reductive group schemes listed in Theorem 3.4 where the condition on $p$ comes from the condition that $G_{\text{abs}}$ admits a unique cyclic $p$-Sylow subgroup. It is easy to check that the the embeddings of $G$ into $GL_{2,k}$ given in Cases (1),..., (5) correspond to the embeddings of $G_{\text{abs}}$ into $GL_{2,C}(C)$ given by Brieskorn in [Br67, Satz 2.9]. Thus, Claim (i) and (ii) follow from [Br67, Satz 2.9]. □

3.3. Very small linearly reductive subgroup schemes of $GL_d$ in higher dimensions.

The simplest examples of finite and linearly reductive group schemes that admit very small representations are the following metacyclic group schemes.

Definition 3.5. Let $k$ be an algebraically closed field $k$ of characteristic $p \geq 0$. Let $m, n \geq 1$ and $0 < r \leq m$ be integers with $(n(r - 1), m) = 1$ and $r^n \equiv 1 \mod m$. Let $e := \text{ord}(r)$ be the multiplicative order of $r$ mod $m$ and assume $p \nmid e$.

(1) The split metacyclic group scheme with parameters $m, n, r$ is defined as $\mu(m, n, r) := \mu_m \rtimes \mu_n$ such that $\mu_n$ acts on $\mu_m$ through an epimorphism $\mu_n \rightarrow \langle r \rangle \subseteq \text{Aut}(\mu_m)$.

(2) We say that $\mu(m, n, r)$ is very small if and only if every prime divisor of $e$ divides $n/e$.

Remark 3.6. Note that the “cyclic” linearly reductive group schemes $\mu_n$ are examples of split metacyclic group schemes, since $\mu_n \cong \mu(1, n, 1)$.

Remark 3.7. The very small split metacyclic group schemes defined as above are precisely those linearly reductive group schemes whose associated abstract group is split metacyclic in the sense of [Wo11, Type I in Theorem 6.1.11] and which admits very small representations over $C$.

Theorem 3.8. Let $G$ be a very small, finite, and linearly reductive subgroup scheme of $GL_{d,k}$ over an algebraically closed field $k$ of characteristic $p \geq 0$.

(1) If $d$ is odd or $p = 2$, then $G \cong \mu(m, n, r)$ such that $G$ is very small and $\text{ord}(r) \mid d$.

(2) If $d = 3$, then $G$ is conjugate to one of the following subgroups:
metacyclic groups with ord($r$)

Let us first prove Claim (1). Since

Proof.

subgroup scheme of

subgroup of $GL$ with ord($r$)

7.2.18], the only very small subgroups of $SL$

small subgroups of $GL$

Corollary 3.9.

$N_{\mu}$ is embedded as in (a)

with ord($r$) = 3.

3 $|$ $N$, $f \geq 2$, $\mu_{m}$ is embedded as in (a)

with $q_{1} = r$ and $q_{2} = r^{2}$,

$\mu_{N}$ is embedded as in (a) with $q_{1} = q_{2} = 1$, and a generator of

$C_{3f}$ is mapped to

\[
\begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
\zeta_{3f-1} & 0 & 0
\end{pmatrix},
\]

where $\zeta_{3f-1}$ is a primitive $3f^{-1}$th root of unity.

(3) If $d = p^{i}$ for some $i \geq 0$, then $G \cong \mu_{n}$ for some $n \geq 1$.

Proof. Let us first prove Claim (1). Since $G$ is a very small linearly reductive subgroup scheme of $GL_{d,k}$, the associated abstract group $G_{abs}$ is a very small subgroup of $GL_{d,C}(\mathbb{C})$. First, assume that $d$ is odd. By [Wo11 Theorem 7.2.18], the only very small subgroups of $GL_{d,C}(\mathbb{C})$ for $d$ odd are the split metacyclic groups with ord($r$) $|$ $d$. Hence, $G$ is metacyclic and very small with ord($r$) $|$ $d$. Next, assume that $p = 2$. Then, the 2-Sylow subgroup of $G_{abs}$ is cyclic. By [Wo11 Theorem 6.1.11, Theorem 7.2.18], the only very small subgroups of $GL_{d,C}(\mathbb{C})$ with a cyclic 2-Sylow subgroup are the split metacyclic groups with ord($r$) $|$ $d$.

In particular, in the situation of Claim (2), we know that $G \cong \mu(m, n, r)$ is very small with ord($r$) $|$ 3. If ord($r$) = 1, then $G$ is cyclic and the embedding $G \to GL_{3,k}$ can be diagonalized as in Case (a). If ord($r$) = 3, then 3 $|$ $n$ and (3, $m$) = 1, so that $\mu(m, n, r) = \mu(m, 3fN, r)$ for some $f \geq 2$ and 3 $|$ $N$. The embeddings of $G$ into $GL_{3,k}$ given in Case (b) correspond to the embedding of $G_{abs}$ into $GL_{3,C}(\mathbb{C})$ given in [Wo11 Section 7.5].

Finally, for Claim (3), note that $G \cong \mu(m, n, r)$ with ord($r$) $|$ $p^{i}$ by Claim (1). Recall that $p \nmid$ ord($r$) by the definition of $\mu(m, n, r)$. Hence, ord($r$) = 1, so that $(n(r - 1), m) = 1$ implies that $m = 1$ and $\mu(m, n, r) = \mu(1, n, 1) = \mu_{n}$.

Similarly, it is straightforward to determine which of the above representations factors through $SL_{d,k}$, thereby giving a classification of finite, very small, and linearly reductive subgroup schemes of $SL_{d,k}$. This is particularly interesting if $d = 3$, where it implies the following.

Corollary 3.9. Let $G$ be a very small, finite, and linearly reductive subgroup scheme of $SL_{3,k}$ over an algebraically closed field $k$ of characteristic $p \geq 0$. 

Then, $G$ is conjugate to $\mu_m$ embedded as
\[
\begin{cases}
  \begin{pmatrix}
    a & 0 & 0 \\
    0 & a^{q_1} & 0 \\
    0 & 0 & a^{q_2}
  \end{pmatrix}, & a \in \mu_m
\end{cases}
\]
for some $q_1, q_2 \geq 1$ with $(m, q_1) = (m, q_2) = 1$ and $q_1 + q_2 + 1 = 0 \mod m$.

Finally, we can classify finite, very small, and abstract groups that have “everywhere good reduction”.

**Corollary 3.10.** Let $G$ be a finite abstract group. Assume that, for every $p > 0$ there exists an algebraically closed field $k$ of characteristic $p$ and a finite and linearly reductive group scheme $G_p$ over $k$ such that $G_p$ admits a very small representation and such that $G$ is the abstract group associated to $G_p$. Then, $G$ is cyclic.

**Proof.** By Theorem 3.2, $G$ admits a unique cyclic $p$-Sylow subgroup for every $p > 0$. In particular, $G$ is the direct product over its cyclic $p$-Sylow subgroups. Hence, $G$ itself is cyclic. $\square$

3.4. **Connection to differential geometry.** Let $\rho : G \to \text{GL}_{d,\mathbb{C}}(\mathbb{C})$ be a very small representation of a finite group. After fixing a $G$-invariant inner product, we may assume that $\rho$ is unitary. Then, $\rho(G)$ acts on the $(2d - 1)$-dimensional (dimension over $\mathbb{R}$) unit sphere $S^{2d-1} \subset \mathbb{C}^d$ by unitary transformations. Since $\rho$ is very small, $G$ acts without fixed points on $S^{2d-1}$ and thus, the quotient $S^{2d-1}/\rho(G)$ is a $(2d - 1)$-dimensional differential manifold that admits a metric of constant positive curvature. In fact, the classification of complete $n$-dimensional Riemannian manifolds of constant positive curvature (the so-called spherical space form problem) is equivalent to the classification of finite and very small subgroups of $O(n)$, see [Wo11, Chapter 5].

4. **Generalities on $F$-singularities**

In this section, we recall some results about $F$-injectivity, $F$-regularity, and $F$-signature of singularities, as well as tautness in dimension two.

4.1. **$F$-regularity and related properties.** Let $k$ be an algebraically closed field of positive characteristic $p > 0$ and let $(R, m)$ be a local, complete, and Noetherian $k$-algebra of dimension $d \geq 2$.

1. $R$ is $F$-injective if the action of Frobenius on the local cohomology groups $H^i_m(R)$ is injective for all $i$.
2. $R$ is weakly $F$-regular if all ideals of $R$ are tightly closed.
3. $R$ is strongly $F$-regular if for all $c \in R$ that is not in a minimal prime there exists $e > 0$ such that the map $R \to R^{1/p^e}, 1 \mapsto c^{1/p^e}$ splits.
(4) For each $e \geq 1$, we define $a_{pe}$ to be the maximal rank of a free summand of $R$, considered as a module over itself via the $e$-fold Frobenius. Then, the $F$-signature of $R$ is defined to be

$$s(R) := \lim_{e \to \infty} \frac{a_{pe}}{p^{ne}}$$

and we refer to [Tu12] for the existence of this limit.

(5) The Hilbert-Kunz multiplicity of $(R, m)$ is defined to be

$$e_{HK}(R) := e_{HK}(m, R) := \lim_{e \to \infty} \frac{\text{length}(R/m[p^e])}{p^{ed}}$$

and we refer to [Hu13] for the existence of this limit.

**Remark 4.1.** Strong F-regularity implies weak F-regularity by [HH89, Theorem 3.1 (d)] and they are conjectured to coincide in general. We will see in Proposition 7.1 below that they are both satisfied for lrq singularities, which is why we will simply write F-regular instead of strongly F-regular.

We recall the following implications between the above notions

$$s(R) > 0 \iff R \text{ is F-regular} \implies R \text{ is F-injective}.$$  

### 4.2. F-regular surface singularities.

Next, we recall the following theorem of Hara [Ha98, Theorem (1.1)] that gives a classification of F-regular singularities in dimension 2. Here, the notion star-shaped of type $(a, b, c)$ for a graph means that it is star shaped, that there are three components once the central vertex is removed, and that these components span lattices of discriminant $a$, $b$, and $c$, respectively.

**Theorem 4.2.** Let $(R, m)$ be a local, complete, Noetherian $k$-algebra of dimension 2 with $R/m = k$. Then, the following are equivalent:

1. $R$ is F-regular,
2. $R$ has rational singularities and the graph $\Gamma$ of the minimal resolution is one of the following:
   a. $\Gamma$ is a chain,
   b. $\Gamma$ is star shaped of type $(2, 2, d)$, $d \geq 2$, and $p \neq 2$,
   c. $\Gamma$ is star shaped of type $(2, 3, 3)$ or $(2, 3, 4)$ and $p \neq 2, 3$,
   d. $\Gamma$ is star shaped of type $(2, 3, 5)$ and $p \neq 2, 3, 5$.

In particular, if $R$ is F-regular, then $R$ is normal and klt, and the converse holds if $p \geq 7$.

Moreover, F-regular surface singularities are taut by work of Tanaka [Ta15, Theorem 1.3], that is, their formal isomorphism class is uniquely determined by the exceptional locus of their minimal resolution.

**Theorem 4.3.** Let $(R, m)$ be a local, complete, and Noetherian F-regular $k$-algebra of dimension 2 with $R/m = k$. Then, $R$ is taut.
5. Local fundamental groups and class groups

In this section, we recall a couple of results concerning the local étale fundamental group and the class group of a singularity.

Let $k$ be an algebraically closed field of characteristic $p \geq 0$ and let $(R, \mathfrak{m})$ be a local, complete, and Noetherian $k$-algebra of dimension $d \geq 2$. We denote the closed point of $X := \text{Spec } R$ corresponding to $\mathfrak{m}$ by $x$ and set $U := X \setminus \{x\}$. We will simply write $x \in X$ in this situation.

Let $G$ be a finite group scheme over $k$. Recall that isomorphism classes of $G$-torsors over $X$ (resp. $U$) are in bijection with the flat cohomology $H^1_{\text{fl}}(X, G)$ (resp. $H^1_{\text{fl}}(U, G)$). In general, this is a pointed set and if $G$ is abelian, then $H^1_{\text{fl}}(-, G)$ is an abelian group. Next, the canonical inclusion $\iota : U \to X$ induces a pullback map

$$
\iota^* : H^1_{\text{fl}}(X, G) \to H^1_{\text{fl}}(U, G)
$$

of pointed sets and of abelian groups if $G$ is abelian. We will now describe the structure of the cokernel of $\iota^*$ in some cases.

5.1. The local étale fundamental group. Let $G$ be a finite and étale group scheme over $k$. Since $k$ is assumed to be algebraically closed, $G$ is the constant group scheme associated to a finite group. By Hensel’s lemma [Na50, Proposition 5] and since $k$ is algebraically closed, $X$ admits no non-trivial finite étale covers by [EGA4, Proposition 18.8.1]. In particular, $H^1_{\text{fl}}(X, G)$ is trivial.

By [SGA1], there exist Galois categories classifying torsors under all finite and étale group schemes over $k$ over $X$ and $U$, which leads to the étale fundamental groups $\pi_1^\text{ét}(X, x)$ and $\pi_1^\text{ét}(U, u)$. Here, $x \in X$ and $u \in U$ are base-points that have to be chosen to begin with. For example, if we choose the generic points $\eta$ of $U$ and $X$ as base-points, then the inclusion $\iota : U \to X$ induces a map

$$
\iota_* : \pi_1^\text{ét}(U, \eta) \to \pi_1^\text{ét}(X, \eta),
$$

which is a continuous homomorphism of profinite groups. Since there are no non-trivial torsors under finite and étale group schemes over $X$, we have $\pi_1^\text{ét}(X, \eta) = \{e\}$, that is, $X$ is algebraically simply connected. The group

$$
\pi_1^\text{ét}(X) := \pi_1^\text{ét}(U, \eta)
$$

is called the local (étale) fundamental group of $X$. Then, the following is well-known and follows immediately from the definition of $\pi_1^\text{ét}(X)$.

**Proposition 5.1.** If $G$ is finite and étale, then there is a canonical bijection

$$
H^1_{\text{fl}}(U, G)/H^1_{\text{fl}}(X, G) = H^1_{\text{fl}}(U, G) \leftrightarrow \text{Hom}(\pi_1^\text{ét}(X), G),
$$

where the right hand side denotes morphisms of groups.
5.2. The class group. Now, let $G$ be a finite and abelian group scheme over $k$, such that $G^D$ is étale. By Boutot [Bo78, III Lemme 4.2., III Corollaire 4.9.], the class group of $X$ parametrizes equivalence classes of $G$-torsors over $U$ in the following sense.

**Proposition 5.2.** If $G$ is finite and abelian and $G^D$ is étale, then there is a canonical bijection

$$H^1_{\text{fl}}(U,G)/H^1_{\text{fl}}(X,G) \leftrightarrow \text{Hom}(G^D, \text{Cl}(X)),$$

where the right hand side denotes morphisms of abelian groups.

**Example 5.3.** If in Proposition 5.2, we assume additionally that $G$ is étale, then $H^1_{\text{fl}}(U,G)/H^1_{\text{fl}}(X,G) = \text{Hom}(G^D, \text{Cl}(X))$ by the discussion in the previous paragraph and thus, $\text{Hom}(G^D, \text{Cl}(X))$ parametrizes $G$-torsors over $U$. However, this is not true in general: for example, if $G = \mu_p^n$, then $H^1_{\text{fl}}(X,G)$ is usually much bigger than $\text{Hom}(G^D, \text{Cl}(X))$ in positive characteristics.

6. Quotient singularities and linearization

In this section, we introduce quotient singularities by finite group schemes, that is, quotients by a finite group scheme $G$ with respect to actions on the spectrum of formal a power series ring $k[[u_1, \ldots, u_d]]$, such that the $G$-action is free outside the closed point. The main result is that such $G$-actions are linearizable if and only if $G$ is linearly reductive, which leads to the notion of a linearly reductive quotient singularity (lrq singularity for short).

6.1. Quotient singularities. Quite generally, if $G$ is a finite group scheme over an algebraically closed field $k$, if $Y = \text{Spec } S$ is an affine scheme over $k$, and if we have an action $G \times_k Y \to Y$ over $k$, then the geometric quotient $Y/G$ exists, and it is isomorphic to $\text{Spec } S^G$, where $S^G \subseteq S$ denotes the ring of invariants, see [Mu70, Theorem 12.1]. The $G$-action on $Y$ is said to be free, if the quotient morphism $\pi : Y \to Y/G$ is a $G$-torsor over $Y/G$. Recall that a ($k$-valued) fixed point of the $G$-action is a point in $Y$, whose stabilizer is all of $G$.

**Definition 6.1.** Let $G$ be a finite group scheme over $k$. A faithful action of $G$ on $k[[u_1, \ldots, u_d]]$ with $d \geq 2$ is called very small if it is free outside the closed point and the closed point is a fixed point of the action.

We will see in Proposition 6.5 that this notion is compatible with the notion of very small representations in the case where $G$ is linearly reductive.

**Definition 6.2.** A quotient singularity $x \in X$ by a finite group scheme $G$ over $k$ is a pair $(X, x)$, where $X = \text{Spec } R$ is the spectrum of a local $k$-algebra of dimension $d \geq 2$ with closed point $x$ such that

$$R \cong k[[u_1, \ldots, u_d]]^G,$$

where $G$ acts on $k[[u_1, \ldots, u_d]]$ via a very small action.
6.2. **Linearization.** If the action of a group scheme $G$ over $k$ on $k[[u_1,\ldots,u_d]]$ factors over the obvious $\text{GL}_d$-action via a homomorphism $\rho: G \to \text{GL}_d$ of group schemes over $k$, then the $G$-action on $k[[u_1,\ldots,u_d]]$ is called **linear**. If the $G$-action becomes linear after some change of coordinates, it is called **linearizable**.

Over the complex numbers, all finite group actions as in Definition 6.2 are linearizable by a lemma of Cartan [Ca53, Lemma 1]. Using the correspondence between $\mu_p$-actions and multiplicative vector fields in characteristic $p > 0$, linearization of $\mu_p$-actions as in Definition 6.2 follows from a theorem of Rudakov and Shafarevich [RS76, Theorem 2]. The common feature shared by all the above situations is that all group schemes that occur are linearly reductive. This is no coincidence, as the following proposition shows.

**Proposition 6.3.** In the situation of Definition 6.2, the $G$-action on $Y = \text{Spec } k[[u_1,\ldots,u_d]]$ is linearizable if and only if $G$ is linearly reductive.

**Proof.** A proof of the fact that linearly reductive group scheme actions can be linearized can be found, for example in [Sa09, proof of Corollary 1.8].

For the converse, assume that $G$ is not linearly reductive. Seeking a contradiction, we assume that the $G$-action on $Y$ can be linearized, and thus, $G \subseteq \text{GL}_d$ compatible with the action of the latter on $Y$. Then, we have $C_p \subseteq G \subseteq \text{GL}_d$ for some $\mu_p \subseteq G \subseteq \text{GL}_d$ by Lemma 2.3. Since both of these group schemes are unipotent, they are conjugate in $\text{GL}_d$ to subgroup schemes of the group scheme of upper triangular matrices with ones on the diagonal. In particular, their fixed loci are not isolated and therefore, the $G$-action on $Y$ is not free outside the closed point, a contradiction. \[\square\]

**Definition 6.4.** A **linearly reductive quotient singularity**, or, **lrq singularity** for short, is a quotient singularity by a finite and linearly reductive group scheme.

The following proposition bridges the gap between very small actions in the sense of Definition 6.1 and very small representations in the sense of Definition 3.1 for finite and linearly reductive group schemes.

**Proposition 6.5.** Let $k$ be an algebraically closed field and let $\rho: G \to \text{GL}_{d,k}$ be a representation of a finite and linearly reductive group scheme $G$ over $k$. Then, $\lambda(\rho)$ coincides with the dimension of the non-free locus of the induced $G$-action $\rho'$ on $\text{Spec } k[[u_1,\ldots,u_d]]$.

**Proof.** Let $m$ be the dimension of the non-free locus of $\rho'$ and let $V$ be a $k$-vector space of dimension $d$ so that $(\text{Sym}(V))^\wedge = k[[u_1,\ldots,u_d]]$. We have $\lambda(\rho) \leq m$, since $\text{Spec } (\text{Sym}(V^H))^\wedge \subseteq \text{Spec } (\text{Sym}(V))^\wedge$ is contained in the non-free locus of $\rho'$ for every $\{\text{id}\} \neq \mu_n \subseteq G$. Conversely, let $\eta$ be the generic point of an irreducible component of the non-free locus of $\rho'$ such that $\eta$ has height $d - m$. By our choice of $\eta$, the stabilizer $\text{Stab}_\eta$ is non-trivial. Since $G_\eta$ is linearly reductive, the inclusion $\text{Stab}_\eta \subseteq G_\eta$ descends to $k$ and we find some $\mu_n \subseteq G$ that fixes $\eta$. Thus, we have $\eta \subseteq \text{Spec } (\text{Sym}(V^H))^\wedge \subseteq \text{Spec } (\text{Sym}(V))^\wedge$, so $m \leq \lambda(\rho)$, which is what we wanted to show. \[\square\]
Corollary 6.6. Let $k$ be an algebraically closed field and let $G$ be a finite and linearly reductive group scheme over $k$. An action of $G$ on $\text{Spec } k[[u_1, \ldots, u_d]]$ is very small if and only if, after a change of coordinates, it coincides with an action induced by a very small linear representation of $G$.

6.3. Very small actions of connected group schemes. To understand torsors over the punctured spectrum of lrq singularities, we also need some information on very small actions of not necessarily linearly reductive group schemes. It turns out that the class of connected group schemes admitting very small actions is rather restricted. The key observation is the following.

Lemma 6.7. If $G$ is a nontrivial, connected, and finite group scheme of height 1 that admits a very small action, then $\text{length}(G) = p$.

Proof. Seeking a contradiction, assume that $\text{length}(G) > p$. We write $G = \text{Spec } T$ with $T = k[t_1, \ldots, t_l]/(t_1^p, \ldots, t_l^p)$, where $l \geq 2$ and $S := k[[u_1, \ldots, u_d]]$. Write the very small action as $a : S \to T \otimes_k S$, $b \mapsto a(b)$ with

$$a(b) \equiv 1 \otimes b + \sum_{h=1}^l t_h \otimes D_h(b) \pmod{(t_1, \ldots, t_l)^2}.$$

Then, the $D_h$ are (not necessarily commuting) derivations on $S$. Now let $C$ be the vanishing locus of the ideal $I_C \subseteq S$ generated by the $2 \times 2$ minors of $M$ (so that $\text{rank } M \leq 1$ on $C$), where $M$ is the $d \times 2$ matrix

$$M = \begin{pmatrix} D_1(u_1) & D_2(u_1) \\ \vdots & \vdots \\ D_1(u_d) & D_2(u_d) \end{pmatrix}.$$ 

Note that all the $D_i(u_j)$ must vanish at 0, since $0 \in \mathbb{A}^d_k$ is a fixed point of the $G$-action, so $C$ is non-empty. Therefore, by the formula for the expected dimension of degeneracy loci (see, for example, [Fu98, Section 14]), we have $\dim(C) \geq d - (d - 1)(2 - 1) = 1$. We claim that this implies that the action of $G$ on $U$ is not free, where $U$ is the complement of the origin.

The map $(a, i_2) : S \otimes_k S \to T \otimes_k S$ induces a map $(a, i_2)|_U : O_U \otimes_k O_U \to T \otimes_k O_U$. Then, for every local section $\beta \in O_U \otimes O_U$, we have $c_1(\beta)D_2(u_j) - c_2(\beta)D_1(u_j) \in I_C$ for all $j = 1, \ldots, d$, where $c_h(\beta)$ is the coefficient of $t_h$ in $(a, i_2)(\beta)$ and $I_C$ is the ideal defining $C$. Indeed, we can reduce to the case $\beta = b_1 \otimes b_2 = u_k \otimes 1$, and this case is clear. Since $C$ has positive dimension, the ideal $I_C|_U$ is non-trivial. If the ideal $I_1 \subset O_U$ generated by the image of $D_1$ is equal to $O_U$, then it follows that $(a, i_2)|_U$ is not surjective. If $I_1 \subset O_U$, then again it follows that $(a, i_2)|_U$ is not surjective. Hence, the $G$-action on $U$ is not free, contradicting our assumption.

Proposition 6.8. Let $G$ be a finite group scheme admitting a very small action. Then the scheme underlying $G^n$ is isomorphic to $\text{Spec } k[t]/(t^n)$ for some $n \geq 0$. In particular, if $n \geq 1$, then

(1) either $G^n$ is linearly reductive of height $n$, hence isomorphic to $\mu_{p^n}$,

...
or else $G^o$ is unipotent of height $n$, hence $G^o[F^{i+1}]/G^o[F^i] \cong \alpha_p$ for all $0 \leq i < n$.

**Proof.** By Lemma 6.7 and our definition of very small actions, the subgroup scheme $G^o[F] \subseteq G^o$ of height 1 is of length $\leq p$. From this, the first assertion follows. Note that the height of $G^o = \text{Spec} k[t]/(t^n)$ is $n$ and that, if $n \geq 1$, we have $G^o[F] \in \{\alpha_p, \mu_p\}$, since these are the only connected group schemes of length $p$ over $k$.

If $G^o[F] \neq \alpha_p$, then $G^o$ is linearly reductive by Lemma 2.3 and thus, $G^o \cong \mu_p$, by Nagata’s classification. If $G^o[F] \neq \mu_p$, then $G^o$ is unipotent by [SGA3II, Théorème 4.6.1. iv), Exp. XVII]. Since $G^o$ has height $n$, the subquotient $G^o[F^{i+1}]/G^o[F^i]$ is unipotent of length $p$ for every $i < n$, hence $G^o[F^{i+1}]/G^o[F^i] \cong \alpha_p$. □

**Remark 6.9.** Both cases occur: the two-dimensional rational double points of type $A_p^{n-1}$ are lrq singularities by $G = G^o = \mu_p^n$. For examples with $G = G^o = \alpha_p$, we refer to Section 7.2 and examples with $G = G^o = \mathbb{M}_2$ (a non-split extension of $\alpha_p$ by $\alpha_p$) can be found in [LMM21].

The local Picard scheme and local fundamental group scheme of arbitrary quotient singularities can be very complicated and we refer to [LMM21] for examples. However, in the case of linearizable actions, the situation becomes simpler, as we will see in the next section.

## 7. Properties of lrq singularities

In this section, we will compute some basic invariants of lrq singularities and contrast them with properties of more general quotient singularities in Section 7.3 and Section 7.4.

### 7.1. Basic properties of lrq singularities

Given a $d$-dimensional (very small) representation $\rho : G \to \text{GL}_{d,k}$ of a finite $k$-group scheme $G$, we obtain an induced $G$-action on the symmetric algebra $\text{Sym}^*(k^d)$, and passing to spectra on $d$-dimensional affine space $A^d$. By Proposition 6.3 every lrq singularity is formally isomorphic to some $(A^d/G)^\wedge$.

The following proposition is a collection of well-known results, or results that are easily deduced from known results.

**Proposition 7.1.** Let $k$ be an algebraically closed field of characteristic $p > 0$, let $G$ be a finite and linearly reductive group scheme over $k$, and let $\rho : G \to \text{GL}_{d,k}$ be a very small representation of $G$. Let

$$x \in X := 0 \in (A^d_k/G)^\wedge$$

be the associated lrq singularity.

1. The singularity is $F$-regular and $\mathbb{Q}$-Gorenstein. In particular, it is Cohen-Macaulay, normal and log terminal.
(2) The class group $\text{Cl}(X)$ is finite and there exists an isomorphism of abelian groups

$$\text{Cl}(X) \cong (G^{\text{ab}})^D,$$

where $G^{\text{ab}}$ is the abelianization of $G$. In particular, the $p$-primary part of the class group $\text{Cl}(X)$ is cyclic of order $p^m \leq |G^\circ|$.

(3) The $F$-signature $s(X)$ is finite and satisfies

$$s(X) = \frac{1}{|G|}.$$ 

(4) The Hilbert-Kunz multiplicity $e_{\text{HK}}(X)$ satisfies

$$e_{\text{HK}}(X) = \frac{1}{|G|} \text{length}(S/\mathfrak{m}_RS),$$

where $(R, \mathfrak{m}_R)$ denotes the local ring of $x \in X$ and $S = k[[u_1, \ldots, u_d]]$ is the local ring of $(\mathbb{A}^d)^\wedge$. In particular, it is a rational number, whose denominator divides the length of $G$.

(5) The étale local fundamental group satisfies

$$\pi^\text{ét}_{\text{loc}}(X) \cong G^\text{ét}.$$ 

Proof. Let $S = k[[u_1, \ldots, u_d]]$ and $R := S^G \subseteq S$. Since $G$ is linearly reductive and since the action on $S$ is linear, the inclusion $R \subseteq S$ is split, that is, $R$ is a direct summand, see also [BH93 Section 6.5]. Being a direct summand of an $F$-regular ring, $R$ is $F$-regular, see [HH89 Theorem 3.1(e)].

The assertion that $\text{Cl}(X) \cong (G^{\text{ab}})^D$ is shown in [Be93 Theorem 3.9.2]. In fact, it is stated there for finite groups rather than group schemes, but the proof also works in our case. By Proposition 6.8, we have $G^\circ \cong \mu_{p^n}$ for some $n$ and since $G^\circ$ is normal in $G$, the group scheme $C_{p^n} = \mu_{p^n}^D$ surjects onto the $p$-primary part of $\text{Cl}(X)$. In particular, the latter is cyclic of order $p^m \leq |G^\circ|$.

Since the class group is finite, the class of a canonical Weil divisor has finite order in $\text{Cl}(X)$, that is, $X$ is $\mathbb{Q}$-Gorenstein.

The assertion on the $F$-signature follows from [CST18 Theorem B] and [Ca18 Theorem C].

Assertion (4) in the case of dimension 2 and if $G$ is étale is [WY00 Theorem 5.4]. However, as explained in [Hm13 Example 18 in Section 3], this formula also holds if $G$ is finite and étale and using these arguments, it is easy to see that the formula also holds in our case.

Finally, consider Assertion (5). The étale fundamental group of $Y = \text{Spec } k[[u_1, \ldots, u_d]]$ is trivial. Since $G^\circ$ is connected, the quotient map $Y \to Y/G^\circ$ is purely inseparable, which implies that the local fundamental group of $Y/G^\circ$ is trivial, see also [SGA1 Exposé IX, Théorème 4.10]. Since $G^\text{ét}$ is étale and acts freely on the pointed space $Y/G^\circ - \{0\}$, which is simply connected (since the dimension is $\geq 2$), where $\{0\}$ denotes the closed point, the local fundamental group of $Y/G$ is isomorphic to $G^\text{ét}$. □
As an application of Proposition 6.8 and Proposition 7.1, we deduce that lrq singularities are precisely the F-regular quotient singularities in positive characteristic.

**Proposition 7.2.** Let \( x \in X = \text{Spec} R \) be a quotient singularity in characteristic \( p \geq 0 \). Then, the following are equivalent:

1. \( x \in X \) is an lrq singularity.
2. Either \( p = 0 \) or \( p > 0 \) and \( p \nmid |\pi_{\text{loc}}(X)| \), and the integral closure \( \bar{x} \in \bar{X} \) of the universal étale cover of \( X - x \) is F-injective.
3. Either \( p = 0 \) or \( p > 0 \) and \( X \) is F-regular.

**Proof.** Since every quotient singularity is lrq if \( p = 0 \), we may assume that \( p > 0 \).

The implication \((1) \Rightarrow (3)\) is Proposition 7.1 (1).

Next, for \((3) \Rightarrow (2)\), note that \( p \nmid |\pi_{\text{loc}}(X)| \) by [CST18, Theorem A]. The F-signature of \( \bar{X} \) is positive by [CST18, Theorem B], so \( \bar{X} \) is F-regular and, in particular, \( \bar{X} \) is F-injective.

Finally, to see \((2) \Rightarrow (1)\), let \( G \) be a finite group scheme, let \( S := k[[u_1, \ldots, u_d]] \), and assume that \( X \) is a quotient singularity via a very small action of \( G \) on \( \text{Spec} S \). Note that Spec \( S \to \text{Spec} S^{G^0} \) is a homeomorphism in the étale topology, so Spec \( S^{G^0} \cong \bar{X} \) and \( G^{\text{ét}} \cong \pi_{\text{loc}}(X) \). In particular, we have \( p \nmid |G^{\text{ét}}| \), so \( G^{\text{ét}} \) is linearly reductive. By Proposition 6.8, we know that \( G^0 \) is either linearly reductive or unipotent. If \( G^0 \) is unipotent, then it admits a quotient isomorphic to \( \alpha_p \) and the associated \( \alpha_p \)-torsor over \( \bar{X} - \bar{x} \) induces a non-trivial class in \( H^1(\bar{X} - \bar{x}, \mathcal{O}_{\bar{X} - \bar{x}}) = H^2_{\text{ét}}(\bar{X}) \), which is killed by Frobenius. By the definition of F-injectivity, this is impossible, so \( G^0 \) is linearly reductive, hence so is \( G = G^0 \rtimes G^{\text{ét}} \). \( \square \)

7.2. **Relation to toric singularities.** Toric singularities are examples of lrq singularities and we have the following relation between these classes of singularities.

**Proposition 7.3.** Let \( x \in X \) be an isolated singularity of dimension \( d \) over an algebraically closed field \( k \). Then, the following are equivalent:

1. \( x \in X \) is toric,
2. \( x \in X \) is an lrq singularity by an abelian group scheme,
3. \( x \in X \) is an lrq singularity by \( \mu_n \) for some \( n > 0 \),
4. \( x \in X \) is an lrq singularity by \( \mu_n = \text{Cl}(X)^D \), where \( \mu_n \) is embedded into \( \text{GL}_d \) as \( \zeta \mapsto \text{diag}(\zeta^{q_1}, \ldots, \zeta^{q_d}) \) with \( 1 = q_1 \leq \ldots \leq q_d < n \).

Moreover, the \( q_i \) in (4) are uniquely determined by \( X \) up to automorphisms of \( \mu_n \) and every linear action of \( \mu_n = \text{Cl}(X)^D \) on \( Y = \text{Spec} k[[u_1, \ldots, u_d]] \) with quotient \( X \) is conjugate in \( \text{GL}_d \) to an action as in (4).

**Proof.** Clearly, we have \((4) \Rightarrow (3) \Rightarrow (2)\). Conversely, the implications \((2) \Rightarrow (3) \Rightarrow (4)\) follow from Theorem 3.2 and the representation theory of cyclic groups over the complex numbers.
Let us prove (4) $\Rightarrow$ (1). Since $\mu_n$ acts diagonally, the diagonal torus action descends to $X$, showing that $X$ is toric.

As for (1) $\Rightarrow$ (2), assume that $X = \text{Spec } k[M]^\wedge$ for some affine semigroup $M$. Since the claim is clear for smooth $X$, we may assume that $X$ is singular. Then, the toric variety $\text{Spec } k[M]$ has no torus factors, so the Cox construction (see, for example, [3.1]) realizes $\text{Spec } k[M]$ as a quotient of $\mathbb{A}^d$ by a diagonal action of the abelian group $G = \text{Cl}(k[M])^D = \text{Cl}(X)^D$ (see Corollary 5.9) for the latter equality) with fixed locus of codimension at least 2. In fact, by the Chevalley-Shephard-Todd Theorem for linearly reductive group schemes [Sa09], the $G$-action on $\mathbb{A}^d$ is very small, since $x \in X$ is an isolated singularity.

Finally, note that a realization of $X$ as a $\mu_n$-quotient as in (4) yields an identification $X \sim \left(\text{Spec } k[M]\right)^\wedge$, where the embedding of $\mu_n$ into the diagonal torus $G^d_m$ defines the affine semigroup $M := \text{Ker}(\mathbb{Z}^d = (G^d_m)^D(k) \to (\mu_n)^D(k) = C_n) \cap \mathbb{N}^d$.

If $M' \subseteq \mathbb{N}^d$ is another affine semigroup such that $X \cong \left(\text{Spec } k[M']\right)^\wedge$ via the above construction, then there is an isomorphism $\varphi : M \cong M'$ of submonoids of $\mathbb{N}^d$ by Demushkin’s Theorem (see, for example, Theorem 4.6.1). It is elementary to check that every such $\varphi$ is induced by a permutation of the generators of $\mathbb{N}^d$. Hence, the two embeddings $\mu_n \to G^d_m$ corresponding to $M$ and $M'$ coincide up to permutation of the coordinates in $Y$ and up to automorphisms of $\mu_n$. □

Remark 7.4. By Proposition 7.1, the F-signature of an lrq singularity by $\mu_n$ is equal to $\frac{1}{n}$, regardless of the embedding $\mu_n \to \text{GL}_d$. On the other hand, the Hilbert-Kunz multiplicity depends on the embedding of $\mu_n$ inside $\text{GL}_d$, that is, the integers $1 = q_1 \leq \cdots \leq q_d < n$, as the following examples in dimension $d = 2$ show:

(1) If $q_1 = q_2 = 1$, then $x \in X$ is the cone over a rational normal curve of degree $n$ and $e_{HK}(X) = \frac{n+1}{2}$ by [Example 2.8].

(2) If $q_1 = 1$ and $q_2 = n - 1$, then $x \in X$ is a rational double point of type $A_{n-1}$ and $e_{HK}(X) = 2 - \frac{1}{n}$ by [Theorem 5.4].

Combining Corollary 3.10 and Proposition 7.3, we conclude that the only lrq singularities with “lrq reduction” modulo every prime are the toric singularities. More precisely:

Definition 7.5. Let $X$ be an lrq singularity over an algebraically closed field $K$ of characteristic 0. Let $p \in \mathbb{Z}$ be a prime. We say that $X$ has good reduction modulo $p$ if there exists a discrete valuation ring $R$ of mixed characteristic $(p,0)$, a finite and flat $R$-group scheme $\mathcal{G}$ such that all geometric fibers of $\mathcal{G}$ are linearly reductive, and an action of $\mathcal{G}$ on $Y := \text{Spec } R[[u_1, \ldots, u_d]]$ that is very small on every geometric fiber and such that the generic fiber of the quotient $X := Y/\mathcal{G}$ coincides with $X$ over a common field extension of $\text{Frac}(R)$ and $K$. 


Proposition 7.6. Let $X$ be an lrq singularity by a finite group $G$ over an algebraically closed field $K$ of characteristic $0$. Then, $X$ has good reduction modulo every prime $p \in \mathbb{Z}$ if and only if $X$ is toric and then, $X$ can be defined over $\mathbb{Z}$ (as an lrq singularity).

Proof. Let $G$ be the finite group by which $X$ is a quotient singularity.

First, assume that $X$ has good reduction modulo every prime $p$ and choose a model as in Definition 7.5. Then, the geometric special fiber $G_p$ of $G$ is a linearly reductive group scheme over an algebraically closed field of characteristic $p$ that admits a very small action. Moreover, by Proposition 2.10 we have $(G_p)_{abs} \cong G$. By Corollary 3.10 this implies that $G$ is cyclic, hence $X$ is toric by Proposition 7.3.

Conversely, assume $X$ is toric. By Proposition 7.3, we can write $X$ as a quotient of $K[[u_1, \ldots, u_d]]$ by a diagonal action of $\mu_n$ with weights $(q_1, \ldots, q_d)$. Now, both $\mu_n$ and this very small diagonal action descend to $\mathbb{Z}$. This shows that $X$ can be defined over $\mathbb{Z}$ and has good reduction modulo every prime. □

Remark 7.7. If, in the setting of Definition 7.5, we assume $\dim(X) \geq 3$, then $X$ has good reduction modulo $p$ if and only if we find a DVR $R$ of mixed characteristic $(p, 0)$ and a flat model $X$ of $X$ over $R$ whose geometric special fiber is an lrq singularity. This will follow from Corollary 10.10.

7.3. Cohen-Macaulayness of quotient singularities. We have seen in Proposition 7.1, that lrq singularities, being direct summands of regular local rings, are Cohen-Macaulay (see also [HE71, Proposition 12]). In this section, we will show that this fails for many, but not all, quotient singularities which are not lrq singularities. The case of wild quotient singularities by abstract groups is due to Fogarty [Fo81].

Lemma 7.8. Let $x \in X$ be a quotient singularity by a finite group scheme $G$ in dimension $d \geq 2$. If $G$ admits a quotient isomorphic to $\alpha_p$ or $\mathbb{C}_p$ (for example, if $G$ is unipotent), then $\text{depth}(X) = 2$. In particular, if additionally $d \geq 3$, then $X$ is not Cohen–Macaulay.

Proof. Since $X$ is normal, we have $\text{depth}(X) \geq 2$ and thus, it remains to show $\text{depth}(X) \leq 2$. First, assume that $G$ admits a normal subgroup scheme $N$ such that $G/N \cong \mathbb{C}_p$. By [Po81, Proposition 4(a)] and using that $Y/N$ is normal, we have $\text{depth}(Y/N) = \text{depth}((Y/N)/\mathbb{C}_p) \leq 2$. Next, suppose that $G$ admits a normal subgroup scheme $N$ such that $G/N \cong \alpha_p$. Then $Y/N \to X$ is an $\alpha_p$-torsor over $X - x$ that does not extend to $X$. As in the proof of Proposition 7.2, this shows that $H^2_{(x)}(X, \mathcal{O}_X)$ is non-trivial, which implies $\text{depth}(X) \leq 2$.

To finish the proof, it suffices to recall that if $G$ is unipotent, then it admits a decomposition series with quotients isomorphic to $\mathbb{C}_p$ or $\alpha_p$, see [SGA3II, Exp. XVII, Théorème 3.5]. □
Example 7.9. There exists an example of a 3-dimensional quotient singularity by a finite and non linearly reductive group scheme that is Cohen-Macaulay.

Suppose \( p = 2 \) and let \( H = \langle h \rangle \) be a cyclic group of odd order \( N \geq 5 \). Consider the action on \( A = k[[u_1, u_2, u_3]] \) of \( \alpha_2 \) corresponding to the derivation \( D(u_i) = u_i^2 \) and the action of \( H \) on \( A \) given by \( h(u_i) = \zeta u_i \), where \( \zeta \) is a primitive \( N \)-th root of unity. This induces an action of the semi-direct product \( G = \alpha_2 \rtimes H \) on \( A \). We note that \( G \) is neither linearly reductive nor unipotent and that it does not admit a quotient isomorphic to \( \alpha_2 \) or \( C_2 \). We claim that \( A \) is Cohen-Macaulay.

Proof. Since \( A \) is 3-dimensional, it suffices to show \( \text{depth}(A^G) \geq 3 \). First, we observe that \( B := A^D \) is generated by \( 1, y_1, y_2, y_3, z \) as a module over \( A^{(p)} = k[[x_1^2, x_2^2, x_3^2]] \), where

\[
y_i := u_i^2 + u_{i+1}u_{i+2}^2 \quad \text{and} \quad z := u_1u_2u_3(u_1 + u_2 + u_3)
\]

(consider the indices modulo 3), subject to the single relation \( \sum_i u_i^2y_i = 0 \), that is,

\[
B \cong \text{Coker} \left( A^{(p)} \xrightarrow{(u_1^2u_2^2u_3^2,0,0)} (A^{(p)})^\oplus 5 \right).
\]

Then we have \( H^2_m(B) \cong \text{Ker}(H^3_m(A^{(p)}) \to H^3_m(A^{(p)})^\oplus 5) \cong H^3_m(A^{(p)})[A^{(p)}] \), which is 1-dimensional over \( k \). With respect to the regular sequence \( u_1^2, u_2^2, u_3^2 \) of \( A^{(p)} \), this cohomology group is generated by the class \([u_1^{-2}u_2^{-2}u_3^{-2}]\]. Since \( h \) acts on this space by \( \zeta^{-6} \neq 1 \), we have \( H^2_m(B^H) = H^2_m(B)^H = 0 \), hence the depth of \( A^G = B^H \) is \( \geq 3 \). \( \square \)

7.4. More general quotients by linearly reductive group schemes.

Let \( G \subset \text{GL}_d \) be a finite and linearly reductive group scheme over the algebraically closed field \( k \) of characteristic \( p \geq 0 \). In Definition 6.4, we considered quotient singularities \( x \in X := 0 \in (\mathbb{A}^d/G)^\wedge \) such that the \( G \)-action is very small. We have seen in Corollary 6.5 that these correspond to \( G \)-representations with \( \lambda \)-invariant 0.

Now, one could also consider \( G \)-actions with larger fixed locus, (or, equivalently, \( G \)-representations with non-trivial \( \lambda \)-invariant) and refer to the singularities considered in Definition 6.4 and in this article as isolated brq singularities. In this case, not all conclusions of Proposition 7.1 may hold for \( x \in X \). In fact, \( x \in X \) might even be a smooth point, in particular, \( s(X) = 1 \) and \( \pi^\text{et}(X) = \{ e \} \), even if \( G^\text{et} \) is non-trivial.

To put this into perspective, we recall that a subgroup scheme of \( G \) acts via pseudo-reflections if its fixed locus has codimension 1 in \( \mathbb{A}^d \) - here, we follow Satriano [Sa09, Definition 1.2] since this definition also works in the case where \( G \) is not necessarily étale. In particular, a pseudo-reflection fixes a hyperplane in \( \mathbb{A}^d \). Using the representation \( \rho \) associated to the linear \( G \)-action on \( \mathbb{A}^d \), Proposition 6.5 shows that \( G \) contains a subgroup scheme that acts via pseudo-reflections if and only if \( \lambda(\rho) = d - 1 \), see also Remark 2.8.
It is easy to see that pseudo-reflections generate a normal subgroup scheme $N \unlhd G$. By the theorem of Chevalley-Satriano-Shephard-Todd [Sa09], the quotient $\mathbb{A}^d/G$ is smooth if and only if $G$ is generated by pseudo-reflections, that is, if and only if $N = G$. In particular, after replacing $G$ by $G/N$ and $\mathbb{A}^d$ by $\mathbb{A}^d/N \cong \mathbb{A}^d$, we may assume that $G$ acts without pseudo-reflections and that the $G$-action is free outside a closed subset of codimension $\geq 2$.

In particular, if $d = 2$, then every singularity $x \in X = 0 \in (\mathbb{A}^2/G)^\wedge$ arising in this more liberal sense is an lrq singularity in the sense of Definition 6.4. For higher $d$, the singularities that arise in this more general sense form a strictly bigger class than the lrq singularities we study in this article. However, these singularities are all still strongly F-regular, so the interested reader will find many interesting properties of these singularities in [Ca18].

8. Uniqueness of the quotient presentation

If $x \in X$ is an lrq singularity with respect to the finite and linearly reductive group scheme $G$, then Proposition 7.1 shows that the length of $G$, the maximal étale quotient $G_{\text{ét}}$ of $G$, and the character group $\text{Hom}(G, \mathbb{G}_m)$ are invariants of $x \in X$. This suggests that an lrq singularity determines the linearly reductive group scheme $G$ together with a linear action on a smooth $k$-algebra. This is indeed the case by the following uniqueness result, which is the main result of this section.

**Theorem 8.1.** Let $x \in X = 0 \in (\mathbb{A}^d/G_1)^\wedge$ be a $d$-dimensional lrq singularity with notations and assumptions as in Proposition 7.1. Let $G_2$ be a finite $k$-group scheme such that $x \in X$ is a quotient singularity by $G_2$ in the sense of Definition 6.2. Then,

1. $G_2$ is isomorphic to $G_1$ as finite $k$-group scheme,
2. the $G_2$-action on $\mathbb{A}^d$ is linearizable, and
3. with respect to linearizations, $G_1$ and $G_2$ are conjugate as subgroup schemes of $\text{GL}_d$.

**Proof.** For the proof, we set $S := k[[u_1, \ldots, u_d]]$ with $\mathbb{A}^d := \text{Spec } S$ and assume that $G_2$ acts on $\mathbb{A}^d$, such that the action of $G_2$ is very small and with quotient $\mathbb{A}^d/G_2$ isomorphic to $X$.

Since $X$ is an lrq singularity, we know from Proposition 7.1 (1) that $X$ is F-regular. Then, the same argument as in the proof of Proposition 7.2 shows that $G_2$ is linearly reductive.

To prove the theorem, it suffices to prove Claim (3) and to do this, we may assume that $G_2$ acts linearly on $\mathbb{A}^d$. By Proposition 7.1 (5) we have $G_{1, \text{ét}}^\text{loc}(X) \cong G_{2, \text{ét}}^\text{loc}$. Thus, $\mathbb{A}^d/G_1^\text{loc}$ and $\mathbb{A}^d/G_2^\text{loc}$ both induce the universal étale cover of $X-x$, from which we obtain a $\pi_{1, \text{loc}}(X)$-equivariant isomorphism $\phi: \mathbb{A}^d/G_1^\text{loc} \cong \mathbb{A}^d/G_2^\text{loc}$. By Proposition 7.1 (2), we have

$$\text{Hom}(G_1^\text{loc}, \mathbb{G}_m) = \text{Cl}(\mathbb{A}^d/G_1) \cong \text{Cl}(\mathbb{A}^d/G_2) = \text{Hom}(G_2^\text{loc}, \mathbb{G}_m),$$
which implies $G_1 \cong G_2$. Considering the action of $\pi_{\text{loc}}^\text{ét}(X)$ on the class groups, it follows that the two actions $G_i^\text{ét} \to \text{Aut}(G_i^\text{ét})$ coincide. But these actions determine the semi-direct product structure $G_i \cong G_i^\text{ét} \rtimes G_i^\text{ét}$, which implies that $G_1 \cong G_2$ as $k$-group schemes. Now, we apply Lemma 8.2 and Claim (3) follows. 

It remains to establish the following technical result, which is slightly stronger than what we need, since we do not assume that $G^\text{ét}$ acts with isolated fixed locus.

**Lemma 8.2.** Let $G$ be a finite and linearly reductive group scheme over $k$, let $\hat{k}^d := \text{Spec} k[[u_1, \ldots, u_d]]$, and assume that we have two actions $\rho_h, \ h = 1, 2, \text{ of } G$ on $\hat{k}^d$ with quotient morphisms $f_h : \hat{k}^d \to \hat{k}^d/\rho_h(G)$ such that the $\hat{k}^d/\rho_h(G)$ are lrq singularities by $G^\text{ét}$.

If there is a $G^\text{ét}$-equivariant isomorphism $\hat{k}^d/\rho_1(G) \cong \hat{k}^d/\rho_2(G)$, then there is a commutative diagram of $G$-equivariant morphisms

$$
\begin{array}{ccc}
\hat{k}^d & \xrightarrow{\psi} & \hat{k}^d \\
\downarrow{f_1} & & \downarrow{f_2} \\
\hat{k}^d/\rho_1(G) & \xrightarrow{\varphi} & \hat{k}^d/\rho_2(G),
\end{array}
$$

where $\psi$ and $\varphi$ are isomorphisms. In particular, the actions $\rho_1$ and $\rho_2$ are conjugate via $\psi$.

**Proof.** First, since $\hat{k}^d/\rho_h(G)$ is an lrq singularity by $G^\text{ét}$, we have $G^0 = \mu_{p^n}$ for some $n \geq 0$ by Proposition 7.3. Next, we note that we can conjugate the $\rho_h$ by automorphisms of $\hat{k}^d$ without changing the assertion. Thus, we may assume that $\rho_1$ is a linear action, that $\rho_1(G^\text{ét})$ acts diagonally, that $X := \hat{k}^d/\rho_1(G) = \hat{k}^d/\rho_2(G) = \text{Spec} k[[M]]$ for an affine semigroup $M$, that the two $G$-actions on $X$ coincide, and that $f_1$ is induced by an inclusion of monoids $M \subseteq N = \mathbb{N}^d$ with cokernel $N/M \cong \text{Cl}(X) = (G^\text{ét})^0$. In particular, we can write $(f_1)_*\mathcal{O}_{\hat{k}^d} = k[[N]] = \bigoplus_{i \in N/M} I_i$, where the $I_i$ are the eigenspaces for the action of $G^\text{ét}$ on $k[[u_1, \ldots, u_d]]$. Note that we have $g(I_i) = I_{g(i)}$ for all $g \in G^\text{ét}$, since $G^\text{ét}$ normalizes $G^\text{ét}$.

Since $f_2 : \hat{k}^d \to X$ is an integral $G^\text{ét}$-torsor over the smooth locus of $X$ and $\{I_i\}_{i \in N/M}$ is a full set of representatives for $\text{Cl}(X)$, by Proposition 7.3 we have an isomorphism $(f_2)_*\mathcal{O}_{\hat{k}^d} \cong \bigoplus_{i \in N/M} I_i$ of $\mathcal{O}_X$-modules with $G$-action (but not necessarily of $\mathcal{O}_X$-algebras). Denote by $\chi_{h,i_1,i_2} : I_{i_1} \otimes I_{i_2} \to I_{i_1 + i_2}$ the morphisms induced by multiplication on $(f_h)_*\mathcal{O}_{\hat{k}^d}$. To prove the lemma, we have to construct $k$-linear automorphisms $\psi_i : I_i \to I_i$ such that

1. $\chi_{2,i_1,i_2} \circ (\psi_{i_1} \otimes \psi_{i_2}) = \psi_{i_1 + i_2} \circ \chi_{1,i_1,i_2}$ for all $i_1, i_2 \in \text{Cl}(X)$, and
2. $g \circ \psi_i = \psi_{g(i)} \circ g$ for all $g \in G^\text{ét}$.

The first condition ensures that the morphism $\psi := \bigoplus_{i \in \text{Cl}(X)} \psi_i : (f_1)_*\mathcal{O}_{\hat{k}^d} \to (f_2)_*\mathcal{O}_{\hat{k}^d}$ is an isomorphism of $k$-algebras and the second condition ensures
that $\psi$ is $G$-equivariant (as $\psi$ is compatible with the $G^\circ$-action by construction). The sought automorphism of $X$ is then induced by $\varphi := \psi_0$.

Consider the maps $\phi_{h,i,i}: I_i^{\otimes p^m} \to O_X$ induced by $\chi_{h,i,i}$. Then, since the cokernels of the $\phi_{h,i,i}$'s are supported on $\{m\}$, there exist units $c_i \in O_X$ with $\phi_{2,i} = c_i \phi_{1,i}$. This implies that we have $\chi_{2,i_1,i_2} = (c_i c_i c_i c_i^{-1})^{1/p^n} \chi_{1,i_1,i_2}$. In particular, we have $c_{i+j} = c_i c_j \pmod{(O_X^\circ)^{p^n}}$. Since the $f_h$ are $G^\delta_\et$-equivariant, the family $(c_i)$ is $G^\delta_\et$-equivariant, that is, $c_{g(i)} = g(c_i)$ for $g \in G^\delta_\et$.

Let $e_1, \ldots, e_d$ be the canonical basis of $N = \mathbb{N}^d$. Define a homomorphism of monoids $\delta: N \to O_X^\times$ by $\delta(e_j) = c_{[e_j]}$, where $[e_j] \in N/M$ denotes the class of $e_j \in N$. By construction, we have $\delta(\nu) \equiv c_{[\nu]} \pmod{(O_X^\circ)^{p^n}}$ for all $\nu \in N$.

Now, observe that every element of $I_i$, when considered as an element of $k[[N]]$ via $f_1$, is a (possibly infinite) sum of monomials in $I_i$, because $G^\circ$ acts diagonally. Hence, we can define $k$-linear morphisms $\psi_i: I_i \to I_i$ by mapping a monomial $\nu_i \in I_i \subseteq k[[N]]$ to $(\delta(\nu_i) c_i^{-1})^{1/p^n} \nu_i$ and extending linearly. We claim that $\psi := \bigoplus_{i \in C_k(X)} \psi_i$ satisfies properties (1) and (2) above. Both properties can be checked on monomials $\nu_{ij} \in I_{ij}$.

For (1), we compute

\[
\begin{align*}
\chi_{2,i_1,i_2} \circ (\psi_{i_1} \otimes \psi_{i_2})(\nu_{i_1} \otimes \nu_{i_2}) &= (c_{i_1} c_{i_2} c_{i_1+i_2})^{1/p^n} \chi_{1,i_1,i_2}((\delta(\nu_{i_1}) c_{i_1}^{-1})^{1/p^n} \nu_{i_1} \otimes (\delta(\nu_{i_2}) c_{i_2}^{-1})^{1/p^n} \nu_{i_2}) \\
&= (\delta(\nu_{i_1}) \nu_{i_2})^{1/p^n} \chi_{1,i_1,i_2}(\nu_{i_1} \otimes \nu_{i_2}) \\
&= \psi_{i_1+i_2} \circ \chi_{1,i_1,i_2}(\nu_{i_1} \otimes \nu_{i_2}).
\end{align*}
\]

As for (2), we use that for any monomial $\lambda$ appearing in $g(\nu)$ we have $\delta(\lambda) = g(\delta(\nu))$ (it suffices to check the case where $\nu \in \{e_1, \ldots, e_d\}$, which is clear). Setting $\delta(g(\nu)) := \delta(\lambda)$ and using $g(c_i) = c_{g(i)}$, we obtain

\[
g \circ \psi_i(\nu_i) = g((\delta(\nu_i) c_{g(i)}^{-1})^{1/p^n} \nu_i) = (\delta(g(\nu_i)) c_{g(i)}^{-1})^{1/p^n} g(\nu_i) = \psi_i(g(\nu_i)).
\]

Therefore, (1) and (2) hold for $\psi$ and thus, $\psi$ and $\varphi := \psi_0$ induce the stated commutative diagram. \qed

**Remark 8.3.** If $G$ is connected, that is, $G = G^\circ$, and using the terminology of local torsors as in [LMM21], Lemma 8.2 can be rephrased as follows: Let $X$ be an lrq singularity by a finite, connected, and linearly reductive group scheme $G$. Then, the pullback action of $\text{Aut}(X)$ on the set of strictly local $G$-torsors over $X$ is transitive.

However, we warn the reader that this does not imply that any two strictly local $G$-torsors over $X$ are isomorphic over $X$, but merely that they are $G$-equivariantly isomorphic as $k$-schemes.

**Corollary 8.4.** For every algebraically closed field $k$, there is a bijection between

\[
\text{Aut}(X) \times O_X \to O_X \times \text{Aut}(X).
\]
(1) the set of isomorphism classes of (Gorenstein) lrq singularities of dimension \(d\), and
(2) the set of conjugacy classes of finite and very small subgroup schemes of \(\text{GL}_{d,k}\) (resp. \(\text{SL}_{d,k}\)).

Proof. The statement for \(\text{GL}_{d,k}\) follows from Theorem [8.1]. Thus, it suffices to note that \(x \in X = 0 \in (\mathbb{A}^d/G)^{\wedge}\) is Gorenstein if and only if \(G\) preserves the \(d\)-form \(\omega = du_1 \wedge \ldots \wedge du_d\). Since \(G\) acts on \(\langle \omega \rangle\) through the determinant \(G \to \text{GL}_{d,k} \to \mathbb{G}_m\), we see that \(X\) is Gorenstein if and only if \(G \subseteq \text{SL}_{d,k}\). \(\square\)

We will use Corollary [8.4] to give an explicit description of all lrq singularities in dimension \(d = 2\) in Section [11] and show that, in this case, the lrq singularities are precisely the F-regular singularities.

Remark 8.5. Let \(x \in X = 0 \in (\mathbb{A}^d/G)^{\wedge}\) be a quotient singularity over an algebraically closed field \(k\). If \(k = \mathbb{C}\), then the group \(G\) can be recovered as the local étale fundamental group \(G = \pi_1^{\text{ét}}(X)\) and the action of \(G\) on \((\mathbb{A}^d)^{\wedge}\) can be linearized and two actions of \(G\) yield the same quotient \(X\) if and only if the actions are conjugate in \(\text{GL}_{d,\mathbb{C}}\) (see [Pr67]).

This cannot be true in positive characteristic, since \(G\) is not necessarily étale, and one has to take into account infinitesimal torsors over \(X \setminus \{x\}\). However, we remark that in [EV10], Esnault and Viehweg used Nori’s fundamental group scheme to define an analogue \(\pi_1^{\text{N}}(X)\) of \(\pi_1^{\text{loc}}(X)\) that also takes into account torsors under finite group schemes over \(X \setminus \{x\}\). However, at least if \(G\) is not linearly reductive, the obvious guess that \(G = \pi_1^{\text{N}}(X)\) holds is not true in positive characteristic, even if \(X\) is a rational double point and \(G\) is étale. For a more in-depth discussion of \(\pi_1^{\text{N}}(X)\) for surface singularities, including counterexamples to \(G = \pi_1^{\text{loc}}(X)\) and geometric reasons for the failure of this equality, we refer the reader to [LMM21].

Thus, Theorem [8.1] begs for the following question:

Question 8.6. Let \(x \in X = 0 \in (\mathbb{A}^d/G)^{\wedge}\) be an lrq-singularity. Does \(\pi_1^{\text{loc}}(X) = G\) hold?

9. Rigidity of lrq singularities in dimension \(d \geq 4\)

In [Sc71], Schlessinger proved infinitesimal rigidity of isolated quotient singularities by finite groups of order prime to the characteristic and in dimension \(\geq 3\). We will show in this section that his proof can be adjusted to lrq singularities in dimension \(d \geq 4\) and we will deal with \(d = 3\) in Section [10].

Convention 9.1. We say that an lrq singularity \(X\) is rigid (resp. infinitesimally rigid) if all deformations of \(X\) over equicharacteristic complete DVRs \((R, m)\) (resp. equicharacteristic Artinian local rings) are trivial modulo \(m^n\) for every \(n > 0\) (resp. are trivial).
Here, a deformation of an lrq singularity $X$ over a DVR $R$ is a flat family $X \to \text{Spec } R$ together with an identification of the special fiber of $X$ with $X$ and such that the non-smooth locus of $X \to \text{Spec } R$ is proper over $\text{Spec } R$.

Indeed, we already know from Proposition 2.4 that lrq singularities are not arithmetically rigid (that is, rigid in mixed characteristic).

9.1. Infinitesimal rigidity in dimension $\geq 4$.

**Proposition 9.2.** Let $x \in X = 0 \in (\mathbb{A}^d/G)^\wedge$ be a $d$-dimensional lrq singularity. If $d \geq 4$, then $X$ is infinitesimally rigid.

**Proof.** We follow the proof given in Schlessinger [Sc71] for the case where $G$ is a finite group of order prime to the characteristic. Setting $S = k[[u_1, \ldots, u_d]]$ and $R = S^G$ with punctured spectra $V$ and $U$, respectively, Schlessinger’s article shows that it suffices to prove that $h^1(U, Tu) = 0$.

Since $T_{V/G}^{G}$ = $Tu$ by [Sc71, Section 2, Proposition], we may assume that $G$ is infinitesimal, that is, $G = G^o$. Moreover, since the $G$-action on $\mathbb{A}^d$ has an isolated fixed locus, we may assume that $G^o = \mu_{p^n}$ for some $n \geq 0$ by Proposition 6.8.

We let $f : V \to U$ be the quotient map and denote by $T_V$ and $Tu$ their tangent sheaves. By [Ek87, Corollary 3.4] (see also [Ma18, Lemma 2.11]), there is an exact sequence

$$0 \to T_{V/U} \to T_V \to f^*Tu \to T_{V/U}^{\mu_{p^n}} \to 0$$

of $G$-equivariant sheaves on $V$, where $T_{V/U}$ is locally free and where $T_{V/U}^{\mu_{p^n}}$ is the pullback of $T_{V/U}$ along the $n$-th iterate of the absolute Frobenius. More precisely, by loc.cit., $\log \deg(f)$ is equal to the height of the purely inseparable morphism $f$ times the rank of $T_{V/U}$, which implies that $T_{V/U}$ is of rank 1.

Since Spec $S$ is smooth over $k$, the reflexive hull of $T_{V/U}$ on Spec $S$ is invertible, hence $T_{V/U}$ is free. Since $G$ is linearly reductive, applying $f_*$ and taking $G$-invariants is exact. Thus, we obtain an exact sequence

$$0 \to (f_*T_{V/U})^G \to (f_*T_V)^G \to Tu \to (f_*T_{V/U}^{\mu_{p^n}})^G \to 0.$$

Using once more that $G$ is linearly reductive, we know that $(f_*T_{V/U})^G$, $(f_*T_{V/U}^{\mu_{p^n}})^G$, and $(f_*T_V)^G$ are direct summands of $f_*T_{V/U}$, $f_*T_{V/U}^{\mu_{p^n}}$, and $f_*T_V$, respectively. Since $S$ is regular and the latter three sheaves are pushforwards of free sheaves on $V$, we deduce that

$$h^i(U, (f_*T_{V/U})^G) = h^i(U, (f_*T_{V/U}^{\mu_{p^n}})^G) = h^i(U, (f_*T_V)^G) = 0$$

for $0 < i < \dim(V) - 1$. We assumed $\dim(V) = \dim(X) = d \geq 4$ and plugging this into the exact sequence (41), we obtain the desired result. $\square$
9.2. Deformations of non-lrq quotient singularities. To complete the picture, let us note that rigidity fails for quotient singularities by group schemes that are not linearly reductive in every positive characteristic $p$ and in every dimension $d \geq 3$.

Let $B := \text{Spec } \mathbb{F}_p[[t]]$ with closed point 0 and generic point $\eta$, and let $G \to B$ be a finite and flat group scheme of length $p$ given by Oort-Tate parameters $a = t^p - 1$ and $b = 0$ as described in [OT70, Theorem 2]. Thus, the fiber of $G$ over 0 (resp. $\eta$) is $\alpha_p$ (resp. $C_p$). In [IM20, Part II, Chapter 7], this group scheme is called the unified $p$-group scheme.

**Example 9.3.** Let $p$ be a prime and let $d \geq 3$ be an integer. Then, there exists a $G$-action on $S = \mathbb{F}_p[[u_1, \ldots, u_d]]$ over $B$ that is free outside the closed point. The quotient $(\text{Spec } S)_B/G \to B$ is a family of quotient singularities, whose generic (resp. special) fiber is a quotient singularity by $C_p$ (resp. $\alpha_p$) and with local étale fundamental group $C_p$ (resp. trivial).

**Proof.** We use that $G$ admits a natural embedding $\iota$ into $G_{a,B}$. Consider any faithful action $\rho$ of $G_a$ on $\mathbb{P}^1$ that fixes $\infty$. Now, the action of $G$ on $(\mathbb{P}^1_B)^d$ given via

$$G \xrightarrow{\iota} G_{a,B} \xrightarrow{\Delta} (G_{a,B})^{d/2} (\text{PGL}_{2,B})^d$$

fixes $\infty_B^d$ and is free on a punctured neighborhood. This yields the desired example. □

**Remark 9.4.** We note that for $p = d = 2$, the family in Example 9.3 is a family of $G$-quotients with generic (resp. special) fiber a rational double point singularity of type $D_4$ (resp. $D_3$).

10. Deformation theory of lrq-threefolds

In this section, we establish a necessary and sufficient criterion for a 3-dimensional lrq singularity to be infinitesimally rigid. Moreover, we also compute the miniversal deformation spaces of 3-dimensional lrq singularities.

Recall that we have classified very small subgroup schemes of $\text{GL}_3$ in Theorem 3.8 (2). To simplify the statements of this section, we will introduce the following notation for the associated lrq singularities.

**Definition 10.1.** Let $x \in X = 0 \in (\mathbb{A}^3/G)^\wedge$ be a 3-dimensional lrq singularity by a linearly reductive group scheme $G$.

1. If $G \cong \mu_n$ and $G$ acts as in Theorem 3.8 (2a), we say that $X$ is of type $\frac{1}{n}(1, q_1, q_2)$.
2. If $G \cong \mu(m, 3^f N, r)$ and $G$ acts as in Theorem 3.8 (2b), we say that $X$ is of type $\frac{1}{m}(1, r, r^2) \times_{c_{j-1}} \mu_{3^f N}$.

10.1. The infinitesimal rigidity criterion. To state this criterion, we introduce two linear representations associated to a $d$-dimensional lrq singularity $x \in X$: let $G$ be the linearly reductive group scheme associated to $x \in X$ and let $\rho : G \to \text{GL}_d$ be a very small linear action on $\mathbb{A}^d$ such that $x \in X = 0 \in (\mathbb{A}^d/G)^\wedge$. Then,
(1) $G$ acts via conjugation on $G$ and $G^\circ$, which gives rise to the adjoint representation $\chi_{\text{ad}}$ of $G$ on the Lie algebras $\text{Lie}(G) = \text{Lie}(G^\circ)$. We note that this representation depends on $G$ only.

(a) If $G$ is étale, then $\text{Lie}(G)$ is zero.

(b) If $G$ is not étale, then $G^\circ = \mu_{p^n}$ for some $n \geq 1$ by Proposition 6.8 and thus, $\chi_{\text{ad}}$ is a one-dimensional representation. In particular, $\chi_{\text{ad}}$ can be regarded as a homomorphism from $G$ to $\text{Aut}(\text{Lie}(G)) = \mathbb{G}_m$.

(2) By composing the action with the determinant, we obtain the one-dimensional representation $\chi_{\text{det}} := \det \circ \rho : G \to \text{GL}_d \to \mathbb{G}_m$, which depends on $G$ as well as on its action $\rho$ on $\mathbb{A}^d$.

We note that by Theorem 5.1, $G$ is unique up to isomorphism and $\rho$ is unique up to conjugation, that is, the two representations $\chi_{\text{ad}}$ and $\chi_{\text{det}}$ depend only on the lrq singularity $x \in X$ and not on any choices. We refer to Example 10.3 for explicit computations of these representations.

**Theorem 10.2.** Let $x \in X = 0 \in (\mathbb{A}^3/G)^\wedge$ be a 3-dimensional lrq singularity by a linearly reductive group scheme $G$. Then, $X$ is infinitesimally rigid if and only if

1. $G^\circ$ is trivial or
2. $G^\circ$ is non-trivial and $\chi_{\text{ad}} \neq \chi_{\text{det}}$.

We note that the first part is precisely Schlessinger’s result [Sc71] that a 3-dimensional isolated quotient singularity by a group of order prime to the characteristic is infinitesimally rigid.

**Proof.** By Nagata’s classification, we have $G = G^\circ \times G^\text{et}$, where $G^\text{et}$ is of order prime to $p$. In particular, if $G^\circ$ is trivial, then $X$ is infinitesimally rigid by Schlessinger’s theorem [Sc71]. We may now assume that $G^\circ$ is non-trivial.

We keep the notations of the proof of Proposition 9.2 and let $Y = \text{Spec} S$ and let $y \in Y$ be the closed point. Then, we have an identification

$$H^2(V,-) \cong H^0(Y,-) \otimes H^2(V,\mathcal{O}_V)$$

$$\cong H^0(Y,-) \otimes H^3_y(Y,\mathcal{O}_Y)$$

$$\cong H^0(Y,-) \otimes \frac{S[(u_1u_2u_3)^{-1}]}{S[(u_2u_3)^{-1}] + S[(u_1u_3)^{-1}] + S[(u_1u_2)^{-1}]}$$

of free sheaves on $Y$, where $u_1, u_2, u_3$ are coordinates of $Y$. Next, let $D$ be a $p$-closed generator of $T_{V/U}$. Then, we have a surjection

$$\text{Ker} \left( H^2(V, T_{V/U}) \to H^2(V, T_V) \right) \to H^1(V, f^*T_U),$$

where the left-hand side is one-dimensional with basis $D \otimes (u_1u_2u_3)^{-1}$ under the above identification. The surjection in (5) is in fact an isomorphism and its injectivity follows from the surjectivity of $H^0(V, f^*T_U) \to H^0(V, T_{V/U}^{p^n})$, which we will prove now. We may assume the action of $G^0 = \mu_{p^n}$ is given by diag$(q_1, q_2, q_3)$, $q_i \in \mathbb{Z}$, and let $D_0 \in H^0(U, T_U)$ be the element characterized
by the property that $D_0(u^e) = ((\sum q_i e_i)/p^n)u^e$. Then the image of $D_0$ is a generator.

Now, observe that $GL_3$ acts on the one-dimensional subspace

$$\langle (u_1 u_2 u_3)^{-1}\rangle \cong H^2(V, \mathcal{O}_V)[m] \subseteq H^2(V, \mathcal{O}_V)$$

via the inverse of the determinant: $GL_3 \overset{\text{det}}{\rightarrow} \mathbb{G}_m \overset{-1}{\rightarrow} \mathbb{G}_m$. Indeed, it suffices to observe that the $GL_3$-action preserves this subspace and that diagonal matrices act via $\text{det}^{-1}$.

Therefore, $D \otimes (u_1 u_2 u_3)^{-1}$ is $G$-invariant if and only if the action of $G$ on $H^0(k, G^\circ) \cong (D) \subseteq H^0(Y, T_Y)$ coincides with $\text{det}$. Identifying $H^0(k, G^\circ)$ with $\text{Lie}(G) = \text{Lie}(G^\circ)$, this implies that $D \otimes (u_1 u_2 u_3)^{-1}$ is $G$-invariant if and only if $\chi_{\text{ad}} = \chi_{\text{det}}$. In particular, $D \otimes (u_1 u_2 u_3)^{-1}$ descends to a cohomology class in $H^1(U, T_U)$ if and only if $\chi_{\text{ad}} = \chi_{\text{det}}$.

Putting all these observations together, we see that the cohomology group $H^1(U, T_U) = H^1(V, f^* T_Y)^G$ is zero (resp. one-dimensional) if and only if $\chi_{\text{ad}} \neq \chi_{\text{det}}$ (resp. $\chi_{\text{ad}} = \chi_{\text{det}}$). In particular, $X$ is infinitesimally rigid if and only if $\chi_{\text{ad}} \neq \chi_{\text{det}}$.

**Remark 10.3.** We note that the proof of Theorem 10.2 shows more generally that, for $d \geq 3$, the vector space $H^{d-2}(U, T_U)$ is non-trivial if and only if $\chi_{\text{ad}} = \chi_{\text{det}}$, in which case it is one-dimensional.

**Example 10.4.** Assume that $X$ is an lqr singularity in dimension 3 by a group scheme $G$ with non-trivial $G^\circ$.

1. If $X$ is of type $\frac{1}{n}(1, q_1, q_2)$, then $\chi_{\text{ad}}$ is trivial. The character $\chi_{\text{det}} : \mu_n \rightarrow \mathbb{G}_m$ is induced by the $(1 + q_1 + q_2)$-th power map on $\mu_n$, hence it is trivial if and only if $n \mid (1 + q_1 + q_2)$.

2. If $X$ is of type $\frac{1}{m}(1, r, r^2) \rtimes_{\zeta_3} \mu_3$, then

$$\chi_{\text{ad}}|_{\mu_m}(a) = \chi_{\text{det}}|_{\mu_m}(a) = 1, \quad \chi_{\text{ad}}|_{\mu_N}(a) = 1, \quad \chi_{\text{det}}|_{\mu_N}(a) = a^3, \quad \chi_{\text{ad}}|_{\mathcal{C}_\beta^+}(1) = \tilde{r}, \quad \chi_{\text{det}}|_{\mathcal{C}_\beta^+}(1) = \zeta_3^{\tilde{r}-1}.$$ 

Here, $\tilde{r}$ is the image of $r$ under $(\mathbb{Z}/m\mathbb{Z})^\times \rightarrow (\mathbb{Z}/p\mathbb{Z})^\times \cong \mu_{p-1} \subseteq \mathbb{G}_m$.

Applying Theorem 10.2 to the above example, we obtain the following classification of non-rigid 3-dimensional lqr singularities.

**Corollary 10.5.** Let $x \in X$ be a 3-dimensional lqr singularity. Then, $X$ is not infinitesimally rigid if and only if one of the following conditions hold.

1. $X$ is of type $\frac{1}{n}(1, q_1, q_2)$ with $p \mid n$ and $n \mid (1 + q_1 + q_2)$.

2. $X$ is of type $\frac{1}{m}(1, r, r^2) \rtimes_{\zeta_3} \mu_3$ with $p \mid m$ and such that $\tilde{r} = \zeta_3$. In particular, $p \equiv 1 \pmod{3}$ in this case.
Quite surprisingly, this corollary shows that 3-dimensional lrq singularities in characteristic 2 are infinitesimally rigid.

**Corollary 10.6.** Let $X$ be a 3-dimensional lrq singularity. If $p \in \{0, 2\}$, then $X$ is infinitesimally rigid.

**Proof.** The case $p = 0$ is a special case of Schlessinger’s result. If $p = 2$ and $X$ is not infinitesimally rigid, then Corollary 10.5 shows that $X$ must be of type $\frac{1}{n}(1, q_1, q_2)$ with $2 \mid n$ and $n \mid (1 + q_1 + q_2)$, which is impossible since $2 \nmid q_i$. □

**Remark 10.7.** Since $G$ acts on $\omega := du_1 \wedge \ldots \wedge du_d$ via $\chi_{\text{det}}$, the lrq singularity $X$ is Gorenstein if and only if $\chi_{\text{det}}$ is trivial. Hence, we obtain the following table for 3-dimensional lrq singularities $(\mathbb{A}^3/G)\wedge$, where IR stands for infinitesimally rigid, Gor for Gorenstein, and ✓ and - denote respectively the existence and non-existence of such a singularity.

| $p$     | $G^\circ$ | IR, Gor | IR, not Gor | not IR, Gor | not IR, not Gor |
|---------|-----------|---------|-------------|-------------|-----------------|
| any     | 1         | ✓       | ✓           | -           | -               |
| 2       | $\neq 1$ | -       | ✓           | ✓           | -               |
| 3, 5 (mod 6) | $\neq 1$ | -       | ✓           | ✓           | ✓               |
| 1 (mod 6) | $\neq 1$ | -       | ✓           | ✓           | ✓               |

10.2. **Deformation spaces.** By Corollary 10.5, we know that there exist lrq singularities in dimension 3 that are not infinitesimally rigid. Being isolated singularities, their deformation functors have a pro-representable hull, the miniversal deformation space, which is represented by a local complete algebra over the Witt ring, see [Sc68, Proposition 3.10]. This begs for the question for the deformation spaces in the cases where infinitesimal rigidity fails.

**Theorem 10.8.** Let $x \in X = 0 \in (\mathbb{A}^3/G)\wedge$ be a 3-dimensional lrq singularity by a linearly reductive group scheme $G$. Assume $\text{length } G^\circ = p^n$. If $X$ is not infinitesimally rigid, then the miniversal deformation space $\text{Def}_X$ is given by

$$\text{Def}_X \cong \text{Def}_{\mathbb{A}^3/G^\circ} \cong \text{Spec } W(k)[\varepsilon]/(\varepsilon^2, p^n \varepsilon),$$

where $W(k)$ denotes the Witt ring.

In particular, the deformation space of $X$ in equicharacteristic $p$ is isomorphic to the spectrum of $k[\varepsilon]/(\varepsilon^2)$, which is of length 2. Before giving the proof, we combine the results of this section to obtain the following analog of Schlessinger’s theorem for lrq singularities.

**Corollary 10.9.** Let $x \in X = 0 \in (\mathbb{A}^d/G)\wedge$ be a $d$-dimensional lrq singularity by a linearly reductive group scheme $G$. If $d \geq 3$, then $X$ is rigid (but not necessarily infinitesimally rigid if $d = 3$).

**Proof.** If $d \geq 4$, then this is Proposition 9.2. If $d = 3$, then this is Theorem 10.2 and Theorem 10.8. □
As another corollary to Theorem 10.8 we obtain that for an lrq singularity of dimension \( d \geq 3 \), there is a canonical lift of \( X \) to characteristic 0:

**Corollary 10.10.** Let \( x \in X = 0 \in (\mathbb{A}^d/G)^\wedge \) be a \( d \)-dimensional lrq singularity by a linearly reductive group scheme \( G \) over an algebraically closed field \( k \) of characteristic \( p > 0 \). If \( d \geq 3 \), then the following hold:

1. We have \((\text{Def}_X)_{\text{red}} \cong \text{Spec } W(k)\).
2. The restriction of a miniversal deformation of \( X \) to \((\text{Def}_X)_{\text{red}}\) is isomorphic to the quotient \( X' \) of \( \mathbb{A}^d_{W(k)} \) by the natural linear action of the canonical deformation \( \tilde{G} \) of \( G \) (Definition 2.5).

**Proof.** By Proposition 2.9, there is a unique linear \( \tilde{G} \)-action on \( \mathbb{A}^d_{W(k)} \) lifting the linear action of \( G \) on \( \mathbb{A}^d \). The quotient \( X' \) of \( \mathbb{A}^d_{W(k)} \) by this \( \tilde{G} \)-action is a deformation of \( X \) over \( W(k) \), so \( X \) lifts to \( W(k) \).

Now Claim (1) follows from Proposition 9.2 and Theorem 10.8. Indeed, these results show that the special fiber of \((\text{Def}_X)_{\text{red}}\) has length 1 over \( k \). Since \( X \) lifts to characteristic 0, every irreducible component of \((\text{Def}_X)_{\text{red}}\) dominates \( W(k) \), which shows \((\text{Def}_X)_{\text{red}} \cong \text{Spec } W(k)\).

As for Claim (2), the deformation \( X' \) over \( W(k) \) is induced by a map \( \text{Spec } W(k) \to \text{Def}_X \), which factors through \((\text{Def}_X)_{\text{red}} \cong \text{Spec } W(k)\) since \( W(k) \) is reduced. \(\square\)

**Remark 10.11.** The reader should compare this result with the fact that a lift of \( G \) over the Witt ring \( W(k) \) is, in general, not unique, see Example 2.6. More precisely, any two lifts \( \tilde{G}_i, i = 1, 2 \) of \( G \) over \( K := \text{Frac}(W(k)) \) become isomorphic over \( K \) (Proposition 2.4). An embedding \( G \to \text{GL}_{d,k} \) lifts uniquely to an embedding \( \tilde{G}_i \to \text{GL}_{d,K} \) (Proposition 2.9) and thus, we obtain two quotient singularities \( \mathbb{A}^d_K/\tilde{G}_i \) over \( K \), which become isomorphic over \( K \). It is therefore surprising that Corollary 10.10 ensures that these two quotient singularities are already isomorphic over \( K \). In particular, the uniqueness result of Theorem 8.1 does not hold over fields that are not algebraically closed. In fact, Example 2.6 shows that it does not hold over \( K \), which is of characteristic zero and perfect.

**Proof of Theorem 10.8.** By Corollary 10.6 we may assume that \( p \geq 3 \). Next, we note that \( H^i(U,T_U) = H^i(V/G^o,T_{V/G^o})^{G^o} \) for \( i = 1, 2 \) follows from [Sc71, Section 2, Proposition]. The natural maps \( \text{Def}_X \to \text{Def}_U \) and \( \text{Def}_{V/G^o} \to \text{Def}_{\mathbb{A}^3/G^o} \) are isomorphisms, because the total spaces \( X \) and \( \mathbb{A}^3/G^o \) are of depth \( \geq 3 \), see Proposition 7.1. In particular, the map in the middle of

\[
\text{Def}_X = \text{Def}_U = (\text{Def}_{V/G^o})^{G^o} \to \text{Def}_{V/G^o} = \text{Def}_{\mathbb{A}^3/G^o}
\]

is an isomorphism, as it is formally smooth and since it induces an isomorphism on tangent spaces. Hence, we may assume that \( X \) is of type \( p^n(q_1,q_2,q_3) \) with \( p^n \mid (q_1 + q_2 + q_3) \) and we have to show that \( \text{Def}_X \cong \)
Claim 1: Suppose \( U \) there exists a deformation \( S/I \) isomorphism, we have to show that \( I/I \) lemma, that \( I \) extends \( U \) the underlying spaces of \( I \) for some \( k \) assumption that \( 3.10 \) that there is a miniversal deformation space, which is of the form \( S = U \beta \) satisfies \( \text{the universal deformation of } U \) deformation \( U \) the induced 1-cocycle \( (f_{ij}) \). One easily checks that

\[
\begin{align*}
R & = k[[u^e | q \cdot e = 0 \pmod{p^n}]], \quad \text{and} \\
\tilde{R} & = W(k)[u^e | q \cdot e = 0 \pmod{p^n}],
\end{align*}
\]

where \( e \) and \( q \) are multi-indices. We will use the open affine cover of \( \tilde{U} := \text{Spec } \tilde{R} - V(u_1, u_2, u_3) \) given by the \( \tilde{U}_i := \text{Spec } \tilde{R}_i \), where \( \tilde{R}_i := \tilde{R}[(w^n_i)^{-1}] \). Since \( \text{Def}_X = \text{Def}_U \), it suffices to study deformations of \( U \).

First, we construct a deformation \( U_{\text{univ}} \) of \( U \) over \( W(k)[\varepsilon]/(\varepsilon^2, p^n\varepsilon) \), which will turn out to be a miniversal deformation of \( U \). For this, consider indices modulo 3 and consider the alternating 1-cochain in \( C^1(\tilde{U}, T) \) defined by

\[
f_{i+1,i+2} := \frac{g_{ij}u_i}{u_1u_2u_3} \frac{\partial}{\partial u_i} \in T_{\tilde{R}_{i+1,i+2}/W(k)}.
\]

One easily checks that

\[
f_{12} + f_{23} + f_{31} = p^n D_0 \frac{u_1 u_2 u_3}{u_1 u_2 u_3} \in T_{\tilde{R}_{123}/W(k)},
\]

where \( D_0 \) is defined by the property \( D_0(u^e) = ((q \cdot e)/p^n)u^e \). In particular, we can use the \( 1 + \varepsilon f_{ij} \) to glue the \( \tilde{U}_i \otimes W(k)[\varepsilon]/(\varepsilon^2, p^n\varepsilon) \) and obtain a deformation \( U_{\text{univ}} \) of \( U \) over \( W(k)[\varepsilon]/(\varepsilon^2, p^n\varepsilon) \). By the proof of Theorem 10.2, the induced 1-cocycle \( (f_{ij} \pmod{p}) \) generates \( H^1(U, T_U) \), so \( U_{\text{univ}} \) induces the universal deformation of \( U \) over \( k[\varepsilon]/(\varepsilon^2) \).

Now, we show that \( U_{\text{univ}} \) is indeed a miniversal deformation of \( U \). Write \( S = W(k)[\varepsilon] \) and \( I = (\varepsilon^2, p^n\varepsilon) \subset W(k)[\varepsilon] \). We know by [Sc08 Proposition 3.10] that there is a miniversal deformation space, which is of the form \( S/I' \) for some \( I' \). There is a \( W(k) \)-homomorphism \( S/I' \to S/I \) that induces a \( k \)-isomorphism \( k[\varepsilon]/(\varepsilon^2) \to k[\varepsilon]/(\varepsilon^2) \). By replacing \( \varepsilon \), we may assume that \( S/I' \to S/I \) is a \( S \)-homomorphism and hence, \( I' \subset I \). To show that it is an isomorphism, we have to show that \( I/I' = 0 \) or, equivalently by Nakayama’s lemma, that \( I' + mI = I \). Therefore, it suffices to show the following:

Claim 1: Suppose \( I' \subset S \) is an ideal satisfying \( mI \subset I' \subset I \) and such that there exists a deformation \( U' \to \text{Spec } S/I' \) of \( U_{\text{univ}} \). Then, \( I' = I \).

Proof of Claim 1: Let \( U'_i \subset U' \) be the open subschemes corresponding to the underlying spaces of \( U_i \subset U \). For simplicity, we write \( A' = S/I' \). By the assumption that \( U' \) extends \( U_{\text{univ}} \), we can choose isomorphisms \( \alpha_i: U_i \to \tilde{U}_i \otimes A' \), such that the cocycle defined by \( \beta_{ij} := \alpha_i \alpha_j^{-1} \in \text{Aut}(\tilde{U}_{ij} \otimes A') \) satisfies \( \beta_{ij} \equiv 1 + \varepsilon f_{ij} \pmod{I} \).
Thus, we can write
\[ \beta_{ij} = \exp(\varepsilon_{fi,j} + \theta_{ij}), \]
with \( \theta_{ij} \in \text{Der}(\tilde{R}_{ij} \otimes A', I(\tilde{R}_{ij} \otimes A')) \cong \text{Der}(R_{ij}, R_{ij}) \otimes I/I'. \) A straightforward calculation shows that
\[ \text{id} = \beta_{12} \beta_{23} \beta_{31} = \text{id} + \varepsilon p^n \frac{D_0}{u_1 u_2 u_3} + \varepsilon^2 \frac{D_1}{(u_1 u_2 u_3)^2} + \theta_{12} + \theta_{23} + \theta_{31}, \]
where
\[ D_1 = q_1 q_2 \left( \frac{\partial}{\partial u_2} - \frac{\partial}{\partial u_1} \right). \]
Hence, the obstruction to extending \( U_{\text{univ}} \) to \( A' \) is
\[ \varepsilon p^n \frac{D_0}{u_1 u_2 u_3} + \varepsilon^2 \frac{D_1}{(u_1 u_2 u_3)^2} \in H^2(U, T_U) \otimes I/I'. \]
Thus, to show that \( I' = I \), it suffices to show that \( \frac{D_0}{u_1 u_2 u_3} \) and \( \frac{D_1}{(u_1 u_2 u_3)^2} \) are \( k \)-linearly independent, because then Equation (6) shows that \( p^n \varepsilon = \varepsilon^2 = 0 \in I/I' \), hence \( I = I' \), which is precisely Claim 1. Therefore, let us finish the proof by showing the following:

**Claim 2:** \( \frac{D_0}{u_1 u_2 u_3} \) and \( \frac{D_1}{(u_1 u_2 u_3)^2} \) are \( k \)-linearly independent in \( H^2(U, T_U) \).

**Proof of Claim 2:** To see this, recall that the image of \( D_0 \) under the map \( H^0(U, T_U) \to H^0(V, T_{p^n} V \otimes U)^G \) is a generator, so that \( \frac{D_0}{u_1 u_2 u_3} \) is non-trivial in \( H^2(V, T_{p^n} V \otimes U)^G \) under the identification given in the proof of Theorem [10.2](#footnote-10.2). Next, \( D_1 \) is a \( G \)-invariant derivation on \( k[[u_1, u_2, u_3]] \) that is not contained in \( \text{Lie}(G) \subseteq H^0(Y, T_Y) \). Hence, \( \frac{D_1}{(u_1 u_2 u_3)^2} \) is a non-trivial element in \( \text{Coker}(H^2(V, T_{p^n} V \otimes U)^G) \to H^2(V, T_V)^G \). Therefore, the exact sequence
\[ H^2(V, T_{Y/U})^G \to H^2(V, T_V)^G \to H^2(U, T_U) \to H^2(V, T_{p^n} V \otimes U)^G \to 0 \]
from the proof of Proposition [9.2](#footnote-9.2) shows Claim 2 and thus, finishes the proof. \( \square \)

### 10.3. Concerning arithmetic rigidity.

For an lrq singularity in positive characteristic, there exists a canonical lift over the Witt ring. In particular, lrq singularities are not arithmetically rigid, that is, rigid in mixed characteristic. On the other hand, the canonical lift together with Theorem [10.8](#footnote-10.8) and Corollary [10.10](#footnote-10.10) could be viewed as a sort of (infinitesimal) rigidity result in mixed characteristic.

### 11. F-regular surfaces are lrq-surfaces

Over the complex numbers and in dimension two, it is classical that the klt (resp. canonical) singularities are precisely the quotient singularities by finite subgroups of \( \text{GL}_2 \) (resp. \( \text{SL}_2 \)). In this section, we extend this result to positive characteristic and show that F-regular (resp. F-regular and Gorenstein) surface singularities are precisely the quotient singularities by finite and linearly reductive subgroup schemes of \( \text{GL}_2 \) (resp. \( \text{SL}_2 \)).


11.1. **F-regular Gorenstein surface singularities.** We recall that the F-regular Gorenstein surface singularities are precisely the F-regular rational double points. All of them have been independently realized as lrq singularities by Hashimoto and Liedtke-Satriano.

**Theorem 11.1.** The following rational double points over an algebraically closed field \( k \) of characteristic \( p \geq 0 \)

| Group Scheme | Length | Étale | Characteristic |
|--------------|--------|-------|---------------|
| \( A_n \)    | \( \mu_{n+1} \) | \( n + 1 \) | yes | all \( p \) |
| \( D_n \)    | \( BD_{n-2} \) | \( 4(n - 2) \) | yes | \( p \neq 2 \) |
| \( E_6 \)    | \( BT_{24} \) | 24    | yes | \( p \neq 2, 3 \) |
| \( E_7 \)    | \( BO_{48} \) | 48    | yes | \( p \neq 2, 3 \) |
| \( E_8 \)    | \( BI_{120} \) | 120   | yes | \( p \neq 2, 3, 5 \) |

are quotient singularities by the indicated linearly reductive subgroup schemes of \( SL_2 \) in the indicated characteristics.

**Proof.** See [Ha15, Corollary 3.10] or [LS14, Proposition 4.2].

Combining Theorem 11.1 with the classification lists of Artin [Ar77] and Hara [Ha98] and the classification of finite, very small, and linearly reductive subgroup schemes of \( SL_2 \) (see Theorem 3.3), we have the following characteristic \( p \) analog of Klein’s theorem [Kl84].

**Theorem 11.2.** For a Gorenstein surface singularity \( x \in X \) over an algebraically closed field \( k \) of characteristic \( p > 0 \), the following are equivalent

1. \( x \in X \) is an lrq singularity,
2. \( x \in X \) is an lrq singularity by a subgroup scheme of \( SL_2, k \),
3. \( x \in X \) is F-regular.

Moreover, if \( p \geq 7 \), then these are equivalent to

4. \( x \in X \) is a rational double point.

**Proof.** First, note that the equivalence (3) ⇔ (4) in characteristic \( p \geq 7 \) follows from Theorem 4.2, so that we can focus on the equivalence of the first three properties.

Clearly, we have (2) ⇒ (1). To prove (1) ⇒ (2), note first that the singularity is an lrq singularity by a subgroup scheme of \( GL_2 \) by Proposition 6.3. Now, it suffices to note that a linearly reductive subgroup scheme of \( GL_2 \) that is not contained in \( SL_2 \) acts non-trivially on a regular 2-form, so the quotient will not be Gorenstein.

The implication (2) ⇒ (3) is contained in Proposition 7.1. Conversely, for (3) ⇒ (2) we use that F-regular surface singularities are rational by Theorem 4.2. Since \( X \) was assumed to be Gorenstein, this implies that \( X \) is a rational double point. Thus, the claim follows by noting that the rational double points in Theorem 11.1 are precisely the F-regular rational double points. □
Remark 11.3. If $p = 0$, then the equivalence $(1) \iff (4)$ is classical, see, for example [Br67] or [Du79].

Remark 11.4. It would be interesting to have a proof of these equivalences without using classification lists.

11.2. F-regular surface singularities. Next, we study also non-Gorenstein surface singularities and consider quotients by finite and linearly reductive subgroup schemes of $\text{GL}_2$.

Theorem 11.5. For a surface singularity $x \in X$ over an algebraically closed field $k$ of characteristic $p > 0$, the following are equivalent

1. $x \in X$ is an lrq singularity,
2. $x \in X$ is an lrq singularity by a subgroup scheme of $\text{GL}_{2,k}$,
3. $x \in X$ is F-regular.

Moreover, if $p \geq 7$, then these are equivalent to
4. $x \in X$ is a normal klt singularity,

Proof. First, note that the equivalence $(3) \iff (4)$ in characteristic $p \geq 7$ follows from Theorem 4.2, so that we can focus on the equivalence of the first three properties.

The implication $(2) \implies (1)$ is clear and the converse follows from Proposition 6.3.

Note that $(2) \implies (3)$ follows from Proposition 7.1, so that we can focus on the implication $(3) \implies (2)$. We thus have to show that every F-regular surface singularity is an lrq singularity. It suffices to show that the lrq singularities by the group schemes that occur in Theorem 3.4 exhaust all F-regular surface singularities. Since F-regular surface singularities are taut by Theorem 4.3, it suffices to show that all dual graphs allowed by Theorem 4.2 occur as dual graphs of the resolution of $X = (\mathbb{A}^2/G) \wedge$ for some $G$ in the list of Theorem 3.4.

The resolution graph $\Gamma$ of $X$ is the same as the resolution of the corresponding quotient singularity by $G_{\text{abs}}$ over the complex numbers. This is well-known if $G = \mu_{n,q}$ (and thus, $X$ is toric). In the general case, one studies the induced action of $G$ on the blow-up $Y$ of the origin in $\mathbb{A}^2$. Let $E \cong \mathbb{P}^1$ be the exceptional divisor of this blow-up and let $K$ (resp. $H$) be the kernel (resp. image) of the induced homomorphism $G \to \text{Aut}_E$. By the Chevalley-Shephard-Todd theorem [Sa09], $Y' := Y/K$ is smooth and the image $E'$ of $E$ in $Y'$ is still isomorphic to $\mathbb{P}^1$. The action of $H$ on $Y'$ preserves $E'$ and acts faithfully on it. Since $H$ is linearly reductive and $E'$ is $H$-stable, the action of every stabilizer can be linearized on $E'$, hence every such stabilizer is isomorphic to some $\mu_n$. This reduces the calculation of $\Gamma$ to the toric case. We refer the reader to [Br67, Satz 2.11] for a list and the description of $\Gamma$, which carries over without change to positive characteristic. Comparing this list with the list of resolution graphs of F-regular surface singularities given in Theorem 4.2, one easily checks that all dual
graphs are realized. As explained in the previous paragraph, this finishes the proof. □

**Remark 11.6.** If $p = 0$, then the equivalence (1) $\iff$ (4) is classical, see [Ka84, Theorem 9.6].

**Remark 11.7.** In [Br67, Satz 2.10], Brieskorn showed that 2-dimensional quotient singularities over the complex numbers are taut. Since 2-dimensional lrq singularities in characteristic $p > 0$ are F-regular by Proposition 7.1, it follows from Tanaka’s result in Theorem 4.3 that they are taut, which generalises Brieskorn’s theorem to positive characteristic.

**Remark 11.8.** In characteristic 2, Theorem 11.5 together with the list in Theorem 3.4 and Proposition 7.3 shows that F-regular surface singularities are the same as toric surface singularities. This emphasizes the fact that there is a huge difference between the notions of klt- and F-regular singularities in small characteristics.

**Remark 11.9.** The proof of Theorem 11.5 shows that the dual resolution graph of an lrq surface singularity $X$ uniquely determines the very small subgroup scheme $G \subseteq \text{GL}_{2,k}$ by which $X$ is a quotient singularity. We note that this observation, together with the rigidity of lrq singularities in dimension $d \geq 3$ that we proved in Proposition 9.2 and Theorem 10.8 can be used to give another proof of Theorem 8.1 (3) by reducing the problem to characteristic 0, where the classical topological proof applies.

12. Riemenschneider’s conjecture for lrq singularities

Over the complex numbers, Riemenschneider [Ri74] conjectured that every deformation of an isolated quotient singularity is again an isolated quotient singularity. While this follows from Schlessinger’s rigidity result [Sc71], if the dimension is at least 3, the dimension 2 case is more subtle and was later settled by Esnault and Viehweg in [EV85]. Moreover, Kollár and Shepherd-Barron [KSB88] showed that cyclic quotient singularities deform to cyclic quotient singularities.

12.1. The conjecture. First, we propose a version of Riemenschneider’s conjecture for lrq singularities in positive and mixed characteristic. We also propose a conjectural version of the result of Kollár and Shepherd-Barron for cyclic lrq singularities in positive and mixed characteristic. Here, we call an lrq singularity cyclic if it satisfies any of the equivalent conditions of Proposition 7.3. Recall that by Theorem 8.1 an lrq singularity $x \in X$ uniquely determines the linearly reductive group scheme $G$ by which $X$ is a quotient singularity. Thus, it makes sense to say that $G$ is associated to $x \in X$.

**Conjecture 12.1.** Let $B$ be the spectrum of a DVR with closed, generic, and geometric generic points $0$, $\eta$, and $\overline{\eta}$, respectively. Let $\mathcal{X} \to B$ be a flat
morphism with special and geometric generic fiber $X_0$ and $X_\eta$, respectively. Assume that the non-smooth locus of $X \to B$ is proper over $B$.

1. If $X_0$ is an lrq singularity, then $X_\eta$ contains at worst lrq singularities.
2. Let $G_0$ be the group scheme associated to $X_0$ and let $G_\eta$ be the group scheme associated to an lrq singularity on $X_\eta$. Then, we have $|G_0| \geq |G_\eta|$, where $| - |$ denotes the length of a group scheme.
3. If $X_0$ is a cyclic lrq singularity, then $X_\eta$ contains at worst cyclic lrq singularities.

Here, containing at worst (cyclic) lrq singularities means that the non-smooth locus of $X_\eta$ if non-empty is a finite set of (cyclic) lrq singularities.

**Remark 12.2.** In view of Proposition 7.3 Part (3) of Conjecture 12.1 is equivalent to the following statement: if $X_0$ is an isolated toric singularity, then $X_\eta$ contains at worst isolated toric singularities.

**Remark 12.3.** We note that Conjecture 12.1 (1) and (3) are true if the residue characteristic of $B$ is zero as indicated above: Part (1) is [EV85, Theorem 2.5] and Part (3) is [KSB88, Corollary 7.15]. We do not know whether Conjecture 12.1 (2) is known in characteristic 0 in the stated generality, but we will see in Proposition 12.11 that it holds if $X_0$ is Gorenstein or if it is a cyclic quotient singularities in characteristic 0.

**Proposition 12.4.** Conjecture 12.1 is true in dimension $d \geq 3$.

**Proof.** Let $p_0$ (resp. $p_\eta$) be the residue characteristic of the special (resp. generic) point of $B$. First, we treat the equicharacteristic case: if $p_0 = p_\eta = 0$, then the assertion follows from Schlessinger’s rigidity theorem [Sc71]. Moreover, if $p_0 = p_\eta > 0$, then it follows from Corollary 10.9. Second, we treat the mixed characteristic case, that is, $p_0 > 0$ and $p_\eta = 0$. Here, the assertion follows from Corollary 10.10. □

Thus, we may now focus on dimension two in Conjecture 12.1. In this case, we can rephrase Parts (1) and (2) of Conjecture 12.1 in more well-known terms.

**Lemma 12.5.** In the setting of Conjecture 12.1, assume that $\dim(X_0) = 2$ and that the residue characteristic of $B$ is $p > 0$. Then,

- Part (1) of Conjecture 12.1 is equivalent to the following:
  1. Assume char($B$) = $p$. If $X_0$ is F-regular, then $X_\eta$ contains at worst F-regular singularities.
  2. Assume char($B$) = 0. If $X_0$ is F-regular, then $X_\eta$ contains at worst klt singularities.

- Part (2) of Conjecture 12.1 is equivalent to the following:
  1. Assume char($B$) = $p$. Then, $s_0 \leq s_\eta$. Then, $s_0 \leq |\pi_{\text{et},\eta}|^{-1}$.

Here, $s_0$ and $s_\eta$ denote the F-signatures of $X_0$ and an lrq singularity on $X_\eta$, respectively, and $\pi_{\text{et},\eta}$ denotes the local fundamental group of an lrq singularity on $X_\eta$.
Proof. Equivalence of (1) with (1)' and (1)'' follows immediately from Theorem 11.5.

For the second part, note that by Proposition 7.1 the inverse F-signature $s_0^{-1}$ (resp. $s_0^{-1}$) coincides with the length of the group scheme by which $X_0$ (resp. $X_\eta$) is a quotient singularity. This shows that (2) is equivalent to (2)'.

In characteristic 0, the group scheme by which $X_\eta$ is a quotient singularity is $\pi^{\text{et,loc}},\eta$, so we have (2) $\iff$ (2)'.. □

12.2. Examples. First, we note that Conjecture 12.1 fails in every positive characteristic if we do not allow quotient singularities by group schemes, as the following two-dimensional examples shows: the special fiber is a cyclic lrq singularity by an étale group scheme and the generic fiber is a cyclic lrq singularity by an infinitesimal group scheme.

Example 12.6. Consider the family of rational double points

$$z^{p+1} + t z^p + xy$$

over $k[[t]]$. Its special fiber $X_0$ is a singularity of type $A_p$, while its generic fiber $X_\eta$ is of type $A_{p-1}$. Thus, $X_0$ (resp. $X_\eta$) is a cyclic quotient singularity by $\mu_{p+1}$ (resp. $\mu_p$). The local fundamental group of $X_0$ (resp. $X_\eta$) is $C_{p+1}$ (resp. trivial). Thus, $X_0$ is a quotient singularity by a finite group, while $X_\eta$, being simply connected and non-smooth, is not.

It is also easy to give an example where $X_0$ is a quotient singularity by a connected group scheme and $X_\eta$ is a quotient singularity by a finite group.

The following example shows that Conjecture 12.1 also fails for deformations of quotient singularities by finite group schemes that are not linearly reductive.

Example 12.7. In characteristic $p = 2$, consider the family of rational double points

$$z^2 + x^3 + y^5 + t x y^3 z$$

over $k[[t]]$. Its special fiber $X_0$ is a singularity of type $E_8^0$, while its generic fiber $X_\eta$ is of type $E_8^1$. We will see in [LMM21] that $X_0$ is a quotient singularity by $\alpha_2$, while $X_\eta$ is not a quotient singularity. Similarly, there exists a deformation of $E_8^2$, which is a quotient singularity by $C_2$, to $E_8^3$, which is not a quotient singularity.

12.3. Evidence. In the final section of this article, we collect further evidence for Conjecture 12.1. In particular, we will prove that all parts of Conjecture 12.1 hold if $X_0$ is Gorenstein. We start with Part (1).

Proposition 12.8. Part (1) of Conjecture 12.1 is true if the total space $X$ is $\mathbb{Q}$-Gorenstein.

Proof. Since the conjecture is known in dimension $d \geq 3$ by Proposition 12.4, we may assume $d = 2$. Furthermore, by Remark 12.3, we may assume that the residue characteristic $p_0$ of the special point of $B$ is positive. In
particular, by Proposition 7.1, the special fiber $X_0$ is F-regular. Let $p_\eta$ be the residue characteristic of the generic point of $B$.

First, assume that $p_\eta = p$. Then, the claim follows from Theorem 11.5 and the equivalence of (1) with (1)' in Lemma 12.5 since F-regularity is open in $\mathbb{Q}$-Gorenstein rings [AKM98].

Next, assume that $p_\eta = 0$. We will use the notion of (pure) BCM-regularity of [MS18] and [MS+20]. Since $X_0$ is F-regular, it is BCM-regular by [MS+20, Corollary 6.8]. Since $X$ is $\mathbb{Q}$-Gorenstein, it follows from inversion of adjunction [MS+20, Corollary 3.3] that the pair $(X, X_0)$ is purely BCM-regular and so, $X$ is BCM-regular. By [MS18 Corollary 6.22], this implies that $X$ is klt (note that resolution of singularities for the threefold $X$ exists by [CP19]). Since the definition of being klt is local (see, for example [KO13, Section 2.16]), this implies that $X_\eta$ is klt. By Lemma 12.5 (1)”, this proves the assertion.

In particular, Part (1) of Conjecture 12.1 holds if $X_0$ is Gorenstein, because then $X$ will be Gorenstein as well and the proposition applies.

**Remark 12.9.** After finishing the first version of this article, we were informed by Sato and Takagi that they are able to prove Proposition 12.8 without the $\mathbb{Q}$-Gorenstein assumption, thereby proving Conjecture 12.1 (1) in full generality. Their proof can be found in [ST21, Corollary 4.8].

In the proof of Proposition 12.8, we used that the F-regular locus is open in $\mathbb{Q}$-Gorenstein families. This openness is false in general as shown by examples of Singh [Si99]. Let us recall Singh’s example using our terminology and check that it does not give a counter-example to Conjecture 12.1 (2).

**Example 12.10.** Consider the deformation $X \to \text{Spec } \mathbb{F}_p[[t]]$ from [Si99, Theorem 1.1].

1. The generic fiber $X_\eta$ has a rational double point of type $A_1$, which is an lrq singularity with respect to the group scheme $\mu_2$.
2. The special fiber $X_0$ has a singularity of $D_{4n+1,2n}$ (in Riemenschneider’s notation [Ri76]), which is of index $2n + 1$ and which is an lrq singularity with respect to $(\mu_{2(2n+1)}, 1, \mu_{2(2n+1)}, 1; \text{BD}_n, \text{BD}_n)$, that is, case (2a) of Theorem 3.4.

The total space $X$ is not F-regular and not $\mathbb{Q}$-Gorenstein. On the other hand, this example does satisfy Conjecture 12.1.

Next, we give several special cases for which Part (2) holds.

**Proposition 12.11.** Part (2) of Conjecture 12.1 is true if

1. $B$ has equal characteristic 0 and $X_0$ is cyclic, or
2. $B$ has equal characteristic $p$ and $X$ is $\mathbb{Q}$-Gorenstein, or
3. $X_0$ is Gorenstein.

**Proof.** By Proposition 12.4, we may assume that $d = 2$. 

Let us first prove (1). Since $X_0$ is cyclic, it is a quotient by $\mu_n$ acting as $\mu_{n,q}$ for some $n, q \geq 1$ with $(n, q) = 1$ (see Theorem 3.3). Let $[a_1; a_2, \ldots, a_k]$ be the Hirzebruch-Jung continued fraction for $n/q$ and let $A_i$ be the $i$-th continuant of $[a_1; a_2, \ldots, a_k]$, that is, the numerator of the continued fraction $[a_1; a_2, \ldots, a_i]$ and set $A_0 = 1$ and $A_{-1} = 0$. Then, the $A_i$ satisfy the recursion $A_i = a_i A_{i-1} - A_{i-2}$.

By [157] Proposition 3.6, the geometric generic fiber $X_\eta$ is a cyclic quotient singularity by some $\mu_n$ acting as $\mu_{n', q'}$ such that $n'/q'$ admits a representation as a continued fraction $[a_1'; a_2', \ldots, a_k']$ such that $a_i' \leq a_i$ for all $i$. Now, a straightforward proof via induction shows that this implies $A_i' \leq A_i$ and $A_i' - A_{i-1}' \leq A_i - A_{i-1}$ for all $i$, where the $A_i'$ are the continuants of $[a_1'; a_2', \ldots, a_k']$. In particular, we have $n' \leq n$, so Conjecture 12.1 (2) holds in this case.

Next, let us prove (2). Since $X$ is $\mathbb{Q}$-Gorenstein, it follows from [119, Corollary 1.2] that the $F$-signature of $X$ is at least as big as the $F$-signature of $X_0$. Since the $F$-signature is lower semi-continuous on $X$, the same is true for $X_\eta$. Hence, the claim follows from the equivalence of (2) with (2)' in Lemma 12.5.

Finally, let us prove (3). By assumption, $X_0$ is a rational double point. It is well-known that $X_\eta$ is also a rational double point in this case. After possibly replacing $B$ by a finite extension, we may assume that $X \to B$ admits a minimal simultaneous resolution of singularities $Y \to X \to B$. Let $\Gamma_\eta$ (resp. $\Gamma_0$) be the dual resolution graph associated to a singularity on $Y_\eta$ (resp. to $Y_0$). Let $\Lambda(\Gamma_\eta)$ and $\Lambda(\Gamma_0)$ be the lattices associated to $\Gamma_\eta$ and $\Gamma_0$, respectively. Moreover, let $G_\eta$ and $G_0$ be the finite and linearly reductive group schemes associated to lrq rational singularities of type $\Gamma_\eta$ and $\Gamma_0$ (see Theorem 11.1 for the group schemes and their lengths). Specialization induces an injection of lattices $\Lambda(\Gamma_\eta) \subseteq \text{Pic}(Y_\eta) \to \text{Pic}(Y_0) = \Lambda(\Gamma_0)$. Using [103, Exercise 4.6.2, Theorem 4.6.12], which tells us which $\Lambda(\Gamma_\eta)$’s can occur for a given $\Lambda(\Gamma_0)$, it is straightforward to check that $|G_\eta| \leq |G_0|$. □

We note that the assumption on the index of $X_0$ in the proposition is satisfied if it is an lrq singularity with respect to an étale group scheme.

Remark 12.12. In dimension $d = 2$, Conjecture 12.1 (2) is related to the following more general fact: let $X \to B$ is a family of rational double point singularities. Let $\Gamma_\eta$ and $\Gamma_0$ be the dual resolution graphs (disjoint union of Dynkin diagrams) associated to the minimal resolutions of singularities of $X_\eta$ and $X_0$, respectively. If the residue characteristic of $B$ is zero or positive and large enough depending on $\Gamma_0$ (more precisely, if it is “good” in the sense of Slodowy), then $\Gamma_\eta$ is a subgraph of $\Gamma_0$, see [180, Section 8.10]. From this, we get another proof of Proposition 12.11 for good residue characteristics.

Finally, we note that Part (3) holds in the Gorenstein case.

Proposition 12.13. Part (3) of Conjecture 12.1 is true if $X_0$ is Gorenstein.
Proof. In the proof of Proposition [12.11], we have seen that every singularity on $X_0$ is a rational double point, whose associated lattice $\Lambda(\Gamma'_\eta)$ embeds into the lattice $\Lambda(\Gamma_0)$ associated to $X_0$. If $X_0$ is cyclic, then $\Lambda(\Gamma_0)$ is of type $A_n$. Since every root sublattice of $A_n$ is of type $A_k$ for some $k \leq n$ (see, for example, [Ma03, Exercise 4.6.2]), the claim follows.

Combining the previous propositions, we conclude the following.

**Corollary 12.14.** Conjecture [12.1] is true if $X_0$ is Gorenstein.

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