On the robustness of triaxial Schwarzschild modelling

The effects of correcting the orbit mirroring

Sabine Thater¹, Prashin Jethwa¹, Behzad Tahmasebzadeh² ³, Ling Zhu², Mark den Brok⁴, Giulia Santucci⁵ ⁶, Yuchen Ding² ³, Adriano Poci⁷, Edward Lilley⁸, P. Tim de Zeeuw⁶ ⁹, Alice Zocchi¹, Thomas I. Maindl¹ ¹° ¹¹, Fabio Rigamonti¹¹ ¹² ¹³, Glenn van de Ven¹, Meng Yang², and Katja Fahrion¹⁴

¹ Department of Astrophysics, University of Vienna, Türkenschanzstraße 17, 1180 Vienna
² e-mail: sabine.thater@univie.ac.at
³ Shanghai Astronomical Observatory, Chinese Academy of Sciences, 80 Nandan Road, Shanghai 200030, China
⁴ Department of Astronomy and Space Sciences, University of Chinese Academy of Sciences, 19A Yuquan Road, Beijing 100049, China
⁵ Leibniz-Institute for Astrophysics Potsdam (AIP), An der Sternwarte 16, 14482 Potsdam, Germany
⁶ School of Physics, University of New South Wales, NSW 2052, Australia
⁷ ARC center of Excellence for All Sky Astrophysics in 3 Dimensions (ASTRO 3D)
⁸ Center for Extragalactic Astronomy, University of Durham, Stockton Road, Durham DH1 3LE, United Kingdom
⁹ Sterrewacht Leiden, Leiden University, Postbus 9513, 2300 RA Leiden, The Netherlands
¹⁰ Max Planck Institute for extraterrestrial Physics, Giessenbachstraße 1, 85748 Garching, Germany
¹¹ SDB Science-driven Business Ltd, 85 Faneromenis Avenue, Ria Court 46, Suite 301, 6025 Larnaca, Cyprus
¹² DiSAT, Università degli Studi dell’Insubria, via Valleggio 11, 22100 Como, Italy
¹³ INAF, Osservatorio Astronomico di Brera, Via E. Bianchi 46, 2-23807 Merate, Italy
¹⁴ European Space Agency, European Space Research and Technology Centre, Keplerlaan 1, 2200 AG Noordwijk, Netherlands

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ABSTRACT

In the past 15 years, the triaxial Schwarzschild orbit-superposition code by van den Bosch et al. (2008) has been widely applied to study the dynamics of galaxies. Recently, Quenneville et al. (2022) reported a bug in the orbit calculation of this code, specifically in the mirroring procedure that is used to speed up the computation. We have fixed the incorrect mirroring in DYNAMITE, which is the publicly-released successor of the triaxial Schwarzschild code by van den Bosch et al. (2008). In this study, we provide a thorough quantification of how this bug has affected the results of dynamical analyses performed with this code. We compare results obtained with the original and corrected versions of DYNAMITE, and discuss the differences in the phase-space distribution of a single orbit and in the global stellar orbit distribution, in the mass estimate of the central black hole in the highly triaxial galaxy PGC 46832, and in the measurement of intrinsic shape and enclosed mass for more than 50 galaxies. Focusing on the typical scientific applications of a Schwarzschild triaxial code, in all our tests we find that differences are negligible with respect to the statistical and systematic uncertainties. We conclude that previous results with the van den Bosch et al. (2008) triaxial Schwarzschild code are not significantly affected by the incorrect mirroring.

Key words. galaxies: kinematics and dynamics – galaxies: supermassive black holes – galaxies: structure

1. Introduction

Dynamical modelling is a powerful technique to study the evolution of galaxies. Traditionally this approach was restricted to spherical and axisymmetric models with analytic distribution functions depending on the integrals of motion E and L (e.g., Nagai & Miyamoto 1976, Satoh 1980, Qian et al. 1995, Magorrian et al. 1998). Observational evidence and numerical investigations showed that such models do not capture the full solution space (e.g., Binney 1982). A popular alternative is to use the Jeans (1922) equations which connect the velocity dispersions to the mass density and gravitational potential, however without the assurance that the resulting models have a non-negative distribution function (Cappellari 2008, 2020).

Schwarzschild (1979) sidestepped explicit reliance on the integrals of motion or on the Jeans equations by numerically solving the problem of populating the large variety of orbits in an assumed potential in such a way that it reproduces the mass density. As the orbits and the number of stars on it are known, the distribution of stars over position and velocity is then known everywhere in the model. The orbit occupation numbers are therefore the equivalent of the phase-space distribution function $f(x, v) \geq 0$ that is the solution to the collisionless Boltzmann equation. Elliptical galaxies can broadly be described by cored triaxial density distributions. Schwarzschild (1982) demonstrated that these triaxial density distributions can be in dynamical equilibrium, even when the figure of the system is allowed to rotate slowly. Merritt & Fridman (1996) constructed triaxial systems with central density cusps in this way.

Since this early work, Schwarzschild’s orbit superposition method has been extended to include kinematic data (e.g., Richstone & Tremaine 1984, Rix et al. 1997, van der Marel et al. 2000).
In a stationary, non-rotating, triaxial potential, the orbital code exploits the symmetries of the assumed galaxy potential. To minimize the computational costs, the Schwarzschild performance computing to facilitate expensive calculations. In a careful handling of state-of-the-art data and the use of high-performance computing to study their interactions. This de-

Table 1. Orbit mirroring scheme. For each mirror, the velocity components are listed, and the change in sign is clearly indicated: components marked in bold are those that have been corrected with the bugfix described in this paper. This table is adapted from [Quenneville et al. 2022]. Note that intermediate-axis tube orbits are not included, as they are not present in realistic models.

| # | octant | short-axis tube | long-axis tube | box |
|---|---|---|---|---|
| 1 | (x, y, z) | (vx, vy, vz) | (vx, vy, vz) | (vx, vy, vz) |
| 2 | (−x, y, z) | (−vx, vy, −vz) | (−vx, vy, −vz) | (−vx, vy, −vz) |
| 3 | (x, −y, z) | (−vx, −vy, vz) | (−vx, −vy, vz) | (−vx, −vy, vz) |
| 4 | (x, y, −z) | (vx, vy, −vz) | (vx, vy, −vz) | (vx, vy, −vz) |
| 5 | (−x, y, −z) | (−vx, vy, −vz) | (−vx, vy, −vz) | (−vx, vy, −vz) |
| 6 | (−x, −y, −z) | (−vx, −vy, −vz) | (−vx, −vy, −vz) | (−vx, −vy, −vz) |
| 7 | (x, −y, −z) | (vx, −vy, −vz) | (vx, −vy, −vz) | (vx, −vy, −vz) |
| 8 | (−x, −y, z) | (−vx, −vy, vz) | (−vx, −vy, vz) | (−vx, −vy, vz) |

1998; Cretton et al. 2000), and has been applied to spherical, ax-symmetric and triaxial models with a dark halo and a central density cusp plus supermassive black hole. Rather than deprojecting the observed galaxy properties, the comparison with the numerical model is done in the observed plane, by calculating the line-of-sight integrated properties for each orbit, including, e.g., the effect of finite detector pixels, seeing convolution and internal extinction. When only the surface density is given, the solutions are non-unique (e.g., Rybicki 1987; Gerhard & Binney 1996). Adding kinematic constraints shrinks the range of solutions, which are computed for a range of assumed black hole masses or halo profiles. The resulting distribution of stars in phase space in many cases reveals significant structure, which reflects the different components in the observed galaxy. This decomposition is not done ad hoc on, e.g., the surface brightness profile, but is done in phase space, constrained by all the observables, and allows exploring the formation history of the galaxy. Alternatives to Schwarzschild’s method include the made-to-measure models of Syer & Tremaine (1996); de Lorenzi et al. (2008); Bovy et al. (2018).

In the past two decades, the Schwarzschild method has been applied to multiple galaxies, initially in axisymmetric geometry and more recently also in triaxial geometry, to study their internal structure (e.g., Thomas et al. 2007; Cappellari et al. 2007; van de Ven et al. 2008; Feldmeier-Krause et al. 2017; Poci et al. 2019; Jin et al. 2020; Santucci et al. 2022; Pilawa et al. 2022), to determine their dark matter (DM) content (e.g., Thomas et al. 2007; Cappellari et al. 2012; Poci et al. 2016; Santucci et al. 2021), to weight their central massive black holes (e.g., van der Marel et al. 1998; Verolme et al. 2002; Gebhardt et al. 2003; Valleri et al. 2004; Gebhardt & Thomas 2009; Kra-movič et al. 2009; Walsh et al. 2012; Rusli et al. 2013; Thater et al. 2017; Krajnović et al. 2018; Ahn et al. 2018; Thater et al. 2019; Liepold et al. 2020; den Brok et al. 2021; Roberts et al. 2021; Thater et al. 2021; Pilawa et al. 2022) and to identify accreted galactic components (e.g., Zhu et al. 2020; Poci et al. 2021; Zhu et al. 2022). Independent implementations of the triaxial Schwarzschild method include [van den Bosch et al. 2008], Vasiliev & Valluri (2020) and Neureiter et al. 2021. These typically use numerically computed orbit libraries on the order of 10^5 orbits. The linear combination of these orbits is found which best represents the observed kinematics and luminosity of the galaxy under study. Each of the modelling steps requires a careful handling of state-of-the-art data and the use of high-performance computing to facilitate expensive calculations. In order to minimize the computational costs, the Schwarzschild code exploits the symmetries of the assumed galaxy potential. In a stationary, non-rotating, triaxial potential, the orbital prop-

1. https://github.com/dynamics-of-stellar-systems/dynamite
obtain orbits starting from all octants, while only computing directly orbits that originate in one, thereby saving computational time.

Regular orbits obey a point inversion symmetry in the shape of the orbit. Initial conditions are chosen in one octant and then integrated to obtain the orbit, and this orbit is then mirrored in the other 7 octants; if the orbit already had point symmetry then it is simply sampled more densely. However, although the spatial mirroring is always carried out identically for all orbit types, the velocity components need to be treated separately, due to the different conserved quantities associated with each orbit and the need to preserve them when the effective orientation of the orbit is flipped. Families of orbits are distinguished by which component of angular momentum, if any, they conserve. If there is no net (orbit-averaged) angular momentum conservation, or the net angular momentum is zero, then the orbit is a regular box orbit, so the signs of the mirrored velocity components just follow that of the coordinates. If the angular momentum component associated with the long axis is conserved, then it is a long-axis tube (LAT) orbit, and so in each octant the signs of $v_y$ and $v_z$ must be adjusted such that $L_x$ remains unchanged, where

$$L_x = yv_z - zv_y$$

Similar considerations must be made for short-axis tube (SAT) orbits, which conserve $L_y$ (sometimes a distinction is also made between inner- and outer-long-axis tube orbits, but this is not needed for the mirroring procedure). These sign changes are summarised in Table 1.

In the incorrect mirroring scheme, 4 out of eight mirror cases (out of eight) that are changed compared to the original code (see Table 1). In one panel, rows from top to bottom represent the 2nd, 3rd, 6th, and 7th mirrors, respectively. We show the orbit for only 25 (out of 200) revolutions to avoid crowding. The time of one orbital period is $\sim 0.06$ Gyr.
of 8 velocity components of short and long-axis tube orbits had a flipped sign, meaning that the resulting mirrored orbits did not all correspond to the same (regular) orbit, and consequently should not have shared the same weight.

We analysed hundreds of individual orbits under the correct and incorrect mirroring scheme, calculated for two different galaxy potentials: a triaxial potential and a potential in the axisymmetric limit. The triaxial potential uses the luminosity model of PGC 046832 by den Brok et al. (2021) and the potential in the axisymmetric limit was derived from the luminosity model of the simulated Auriga galaxy halo 6, which was published in Zhu et al. (2020). The second potential is close to axisymmetric as the galaxy disk dominates, but triaxiality is allowed in the model. The results of the triaxial potential are explained in the Appendix.

Figures 1 and 2 show the impact of the incorrect orbit mirroring for a single orbit. We choose a typical short-axis tube orbit as the base orbit. We show the same plots for a short-axis tube orbit in the close to axisymmetric potential in Figure A.3 and A.4 for a long-axis tube orbit in the same triaxial potential in Figures A.3 and A.4.

For a SAT orbit, which circulates around the z axis, we expect all mirrored versions to have the same Lz as the base orbit. The top right panel of Figure 1 shows that this is true both for the original and corrected codes. This is in accordance with Table 1, since only vz was incorrect for SAT orbits, the z component of the angular momentum stays unchanged by the bugfix. The other two components can change under mirroring, but the changes must be consistent with unchanged Lx and Ly.

In order to preserve Lz, the mirroring should only ever induce sign-changes in components of the angular momentum. This is because, while Lz is invariant with respect to sign-flips in the individual angular momentum components Lx, it is not invariant with respect to arbitrary sign-flips in the individual velocity components vz. This can be seen from the formula

$$L_{z_{\text{tot}}} = ||x||^2 ||v||^2 - (x \cdot v)^2,$$

which implies that Lz_{tot} is preserved only if the combinations (xLx, yLz, zLy) each have the same overall sign-flip. The short-period oscillations in the ‘nofix’ versions of Lx and Ly are consequences of the incorrect mirroring, the exact cause of which is unknown but which is probably due to Lx and Ly no longer being constrained by the overall conservation of Lz_{tot}.

In Fig. 2 we show the trajectories of the same SAT orbit shown in Fig. 1. The trajectories were integrated for 200 times the orbital period, and we sample 50000 particles from it with equal time steps. We project the orbit with inclination angles of 15°, 45° and 75°, respectively. The surface density of particles sampled from the orbit are overplotted as black contours. The changes in the mean velocity and velocity dispersion are up to ~40% in some regions of the orbit space, but they are usually ≤10% in regions where the star spends most of its life.
the orbit does not depend on the mirroring of the velocity components, thus we only show it once.

Fig. 2 shows that (for this orbit) the relative error in mean velocity and velocity dispersion can reach ~ 40% in some regions, consistent with the findings of Quenneville et al. (2022). Note however that the regions with large difference are those with very low surface density of the particle. The changes are usually small (< 10%) in regions of the orbit space where the star spends most of its life, and this is true for all projections. Comparing different projections, we see that the largest errors are incurred for an intermediate inclination of 45°. This is somewhat surprising for this SAT orbit where only $v_z$ has changed: the naive expectation would be that the face-on view shows the largest error, decreasing monotonically to the edge-on view. This naive logic fails to account for the fact that $v_z$ flips sign frequently throughout this orbit. For the face-on case, the incorrect sign introduced by the bug is on average cancelled out by the repeated sign-flips of the orbit. For the exactly face-on and edge-on cases, we have confirmed that this orbit shows no change in its kinematic-maps after the bug-fix.

Having inspected one single orbit here, it is important to remember that Schwarzschild models are typically built from thousands of orbits. We therefore investigate in the following section, whether this change of mean velocity and velocity dispersion within a single orbit also introduces significant changes in the inferred best-fit parameters of the Schwarzschild models that use several thousands of orbits.

3. Enclosed mass and shape parameters

Although triaxial Schwarzschild modelling is computational expensive, it is possible to study the dynamics of large galaxy samples (Zhu et al. 2018a; Jin et al. 2020; Santucci et al. 2022). In order to investigate potential systematic biases due to the incorrect mirroring, we have collected data from several galaxy surveys and run the triaxial Schwarzschild code with the correct and incorrect mirroring scheme. While the previous studies had heterogeneous model setups (i.e. different numbers of orbits, different MGE assumptions), we ensured that each individual galaxy was run in the same setup for the correct and incorrect mirroring. Our analysis shows no significant discrepancies in the inferred enclosed mass and in the shape parameters of the studied galaxies. Furthermore, differences in the stellar orbit distribution caused by the incorrect mirroring are negligible compared to other systematics.

Unless otherwise stated, all models in this section are run with six free parameters: the stellar mass-to-light ratio $M_*/L$, intrinsic stellar axis-length ratios $p$ (long-to-short) and $q$ (intermediate-to-short), the stellar projected-to-intrinsic scale-length ratio $u$, dark matter virial mass $M_{200}$, and the dark matter concentration $c$.

3.1. The sample

Our studied galaxy sample is a combination of 25 passive galaxies from the SAMI Data release 3 (Croom et al. 2021), 15 early-type galaxies (ETGs) from the ATLAS3D galaxy survey (Cappellari et al. 2011) and 12 early and late-type galaxies (LTGs) from the Fornax3D survey (Sarzi et al. 2018). We added 6 additional galaxies from miscellaneous other works: FCC 47 (Fahrión et al. 2019; Thater et al. in prep.), ESO286-G022 (Poci & Smith 2022), PGC 046832 (den Brok et al. 2021) and Auriga galaxy halo 6 at inclinations 40°, 60° and 80° (Zhu et al. 2020).

Our SAMI galaxies are a subset of the sample by Santucci et al. (2022) who used the van den Bosch et al. (2008) triaxial Schwarzschild code to study the inner orbital structure and mass distribution of 161 passive galaxies. We randomly chose a subset of 25 galaxies from these galaxies with kinematic data S/N>15 and best-fit model reduced $\chi^2<3.0$. These criteria are used to avoid getting strong parameter biases due to galaxies with weak kinematic constraints or poor fit quality. We then used the same setup for the dynamical models as in Santucci et al. (2022) but with correct and incorrect mirroring.

The subset of the ATLAS3D galaxy sample considered here was modelled with the goal to study the inner orbital structure in massive galaxies (Thater et al. in prep.) and the nature of counter-rotating galaxies (Jethwa et al. in prep.). We obtained the photometric (Scott et al. 2013) and kinematic (up to $h_0$) (Cappellari et al. 2011) measurements directly from the ATLAS3D webpage. The Schwarzschild models were run in the same setup as described in Santucci et al. (2022) using an orbit library of $21 \times 10^7$ for both tube and box orbits with a dithering of $5^\circ$. We fixed the dark matter concentration $c$ with the $M_{200} = c$ relation by Dutton & Macciò (2014) to get a better handle on the degeneracy between $M_*/L$ and dark matter. The ATLAS3D kinematics do not have the spatial resolution to constrain the central black hole, therefore we fixed its mass using the empirical $M_{BH} = \sigma_e$ relation by van den Bosch (2016). All models were run with the same setup for the correct and incorrect mirroring.

Finally, 6 ETGs and 6 LTGs come from the Fornax3D survey and were modelled to understand the effect of the cluster environment on the growth of cold disks (Ding et al. in prep.). We obtained the kinematics from Hodicke et al. (2019). The Fornax3D galaxies have very high-quality kinematic data: high spatial resolution; a high quality of $V, \sigma, h_3$ and $h_4$ and large kinematic data coverage (at least 2$R_e$). The sample includes galaxies with morphology from highly disk-dominated to triaxial bulge dominated. Similar to the ATLAS3D galaxies, the black hole mass was fixed using the $M_{BH} = \sigma_e$ relation by van den Bosch (2016) and the dark matter concentration was fixed via the $M_{200} = c$ relation. We sampled the orbits with $55 \times 11 \times 11$ for both box and non-box orbits with dithering $5^\circ$.

Thus, we were able to directly compare the two different code versions over more than 50 galaxies of very diverse morphology, data quality and coverage, as well as Schwarzschild model complexity (orbit library size, fitted parameters).

3.2. Comparison of correct and incorrect mirroring

From a grid search of our dynamical models, we derive the best-fit parameters for each of our galaxies for the correct and incorrect mirroring. In Fig. 3 and 4, we show a comparison of the enclosed mass and in the shape parameters of the studied galaxies. The changes are usually small (< 10%) in regions of the orbit space where the star spends most of its life, and this is true for all projections. Comparing different projections, we see that the largest errors are incurred for an intermediate inclination of 45°. This is somewhat surprising for this SAT orbit where only $v_z$ has changed: the naive expectation would be that the face-on view shows the largest error, decreasing monotonically to the edge-on view. This naive logic fails to account for the fact that $v_z$ flips sign frequently throughout this orbit. For the face-on case, the incorrect sign introduced by the bug is on average cancelled out by the repeated sign-flips of the orbit. For the exactly face-on and edge-on cases, we have confirmed that this orbit shows no change in its kinematic-maps after the bug-fix.

Having inspected one single orbit here, it is important to remember that Schwarzschild models are typically built from thousands of orbits. We therefore investigate in the following section, whether this change of mean velocity and velocity dispersion within a single orbit also introduces significant changes in the inferred best-fit parameters of the Schwarzschild models that use several thousands of orbits.
Fig. 3. Comparison of enclosed mass properties of our galaxy samples separated in total mass at one effective radius (left) and $M_*/L$ that drives the derived stellar mass (right) for correct (‘bugfix’) and incorrect (‘nofix’) orbit mirroring. The galaxy sample is a subset of Fornax3D LTGs and ETGs (green squares), SAMI passive galaxies (purple stars) and ATLAS3D ETGs (orange circles). Also added as black diamonds are the massive lensed ETG ESO286-G022 by Poci & Smith (2022), FCC 47 by Fabrion et al. (2019) and Thater et al. (in prep.), PGC 046832 by den Brok et al. (2021), which is also discussed in Section 4, and the three simulated LTGs by Zhu et al. (2020), which are discussed in Section 5. The dashed line shows the 1-1 line between the different versions. Both the derived total and stellar mass are not significantly affected by the incorrect mirroring of the orbits. Differences between the versions are within the reported statistical uncertainties of the dynamical modelling. Uncertainties were calculated by including all models within $\sqrt{2N_{\text{mod}}}$ of the best-fit model.

Fig. 4. Comparison of derived galaxy shape parameters for our modelled galaxies. The galaxy sample is divided into Fornax3D LTGs and ETGs (squares), SAMI passive galaxies (stars) and ATLAS3D ETGs (circles). Also added as diamonds are the massive lensed ETG ESO286-G022 by Poci & Smith (2022), FCC 47 by Fabrion et al. (2019) and Thater et al. (in prep.), PGC 046832 by den Brok et al. (2021), which is also discussed in Section 4, and the three simulated LTGs by Zhu et al. (2020), which are discussed in Section 5. The intrinsic intermediate-to-major axis ratio $p = b/a$ and minor-to-major axis ratio $q = c/a$ show no significant change for the majority of the galaxies. There is no clear trend of increasing discrepancy with galaxy morphology or triaxiality $T = (1 - p^2)/(1 - q^2)$. Uncertainties were calculated by including all models within $\sqrt{2N_{\text{mod}}}$ of the best-fit model.

There is strong evidence that previous dynamical mass measurements are not severely affected by the mirroring bug.

In Figure 4 we show a comparison of the galaxy intrinsic shape parameters: the intrinsic intermediate-to-major axis ratio $p = b/a$, and the intrinsic minor-to-major axis ratio $q = c/a$. These parameters are much more difficult to constrain in dynamical models than the enclosed mass. Nevertheless, our comparison shows that $p$ and $q$ are robust against the incorrect mirroring. Almost all measurements are consistent within their uncertainties. Discrepancies are again independent of galaxy morphology, inclination or triaxiality parameter $T = (1 - p^2)/(1 - q^2)$. As $p$ and $q$ enter $T$ via their squares, the discrepancy in the derived $T$ values shows more scatter than in either individual axis-ratio, but again there is no clear trend with galaxy properties. Some galaxies are more driven to a prolate shape, others more oblate, while others did not change. From this comparison, we conclude that galaxy intrinsic shapes in previous results likely do not suffer from systematic biases. We point out that intrinsic shape parameters depend on the quality of the data, and larger discrepancies are found for kinematic data with S/N < 15.
4. A black hole mass estimate in a triaxial galaxy

Den Brok et al. (2021) used the triaxial code van den Bosch et al. (2008) in its original version to model the VLT/MUSE observations of PGC 046832. PGC 046832 is the brightest cluster galaxy in one of the subclusters of the Shapley Supercluster, at a distance of \( \sim 200 \) Mpc. Because of its complex structure, this galaxy poses a challenge for modellers, requiring several inversions in the direction of its angular momentum and a radial change in triaxiality.

The dynamical modeling by Den Brok et al. (2021) showed that a) the black hole mass determined with the triaxial Schwarzschild models was lower than the one determined using axisymmetric models and that b) the intrinsic shape of the galaxy changes from almost prolate in the centre to almost oblate in the outer parts. Here we show that correcting the orbit mirroring in the triaxial Schwarzschild code by van den Bosch et al. (2008) does not lead to a detection of the central supermassive black hole and only marginally changes the best-fit viewing angles. We have added the stellar orbit distribution (see also Section 5) of PGC 046832 to the Appendix (Fig. B.1). Changes in these plots compared to Den Brok et al. (2021) are not significant.

4.1. Black hole mass

To determine the direct influence of the orbit mirroring correction on the black hole mass of PGC 046832, we assume the same viewing angles as used by Den Brok et al. (2021) and the same dark matter halo mass. We re-run the Schwarzschild models in the exact same setup as in Den Brok et al. (2021), but additionally supplement the grid with points at \( M/L \) between 2.9 and 3.0, as the \( \chi^2 \) contours imply a 4\% lower \( M/L \).

We show the \( \chi^2 \) contours in Fig. 5, where the contours from the correctly mirrored code are given in red, and those from the incorrectly mirrored code in blue. The blue contours correspond the correctly mirrored code are given in red, and those from the incorrect intrinsic shape. After correcting the orbit mirroring bug, the derived \( \chi^2 \) surfaces at different black hole masses \( M_{\text{BH}} \) and mass to light ratios \( M/L \). Blue contours show the \( \chi^2 \) surface using the original version of the Schwarzschild code, and were previously presented in Den Brok et al. (2021). Red contours show the \( \chi^2 \) surface after correcting the mirroring bug in the code. Grey dots show the locations at which models were calculated. The thick contours contain models within \( \chi^2_{\text{min}} \) of the best-fit model. The contours do not close for lower-mass black holes and we thus obtain an upper limit. The sphere of influence of a 10\% \( M_{\odot} \) black hole is about 0.05 arcsec which is well below the spatial resolution of the data (seeing fwhm \( \sim 0.67 \) arcsec).

The deprojection of the MGE using these new viewing angles does not lead to a significantly different intrinsic shape. After correcting the mirroring bug, a preference towards a massive black hole with mass log\( (M_{\text{BH}}/M_\odot) \approx 9 \) is still predicted by the fits. This does not mean that the new models show no changes at all; the mass of the dark matter halo, expressed as a dimensionless scaling of the MGE mass, is higher than before (207\% vs 1477\%\%), whereas the stellar \( M/L \) is somewhat lower (3.0\% vs 3.1\% vs 3.1\%\%). The total galaxy mass within the radii that can be constrained by the kinematics is however consistent as was also found for other galaxies in Fig. 3. We note that the changes on these quantities are likely caused by the intrinsic degeneracy between dark matter and stellar \( M/L \), and the limited kinematic field-of-view that cannot constrain the dark matter very well.

4.2. Viewing angles

Den Brok et al. (2021) assumed that the density of the galaxy could be modelled as a sum of aligned concentric Gaussians with different axis ratios \( p_i \) and \( q_i \) and scale lengths \( \sigma_i \). This assumption allows an analytic deprojection of the observed light distribution given a set of viewing angles. Viewing angles therefore affect the gravitational potential in which orbits are calculated, and thus can be dynamically constrained. Den Brok et al. (2021) used Schwarzschild models to constrain the viewing angles as \( (\theta, \phi, \Psi) = (61.0^{+3.9}_{-3.8}, -59.4^{+4.9}_{-4.5}, 74.9^{+3.2}_{-3.6}) \) in degrees.

We re-run Schwarzschild models with the same assumptions as in Den Brok et al. (2021), i.e. fixed black hole mass and 3 different masses for the DM halo, to recreate the grid shown in their Fig. 6. We also explored the parameter space with a free DM halo mass and black hole mass using the same Gaussian sampling approach used in that paper. For the best-fit viewing angles we find \( (\theta, \phi, \Psi) = (60.2^{+3.8}_{-3.5}, -61.9^{+4.7}_{-4.3}, 75.9^{+4.5}_{-4.3}) \), consistent with those obtained with the previous version of the code.

4.3. Other black hole mass measurements

Few black hole mass measurements have been derived with the triaxial Schwarzschild code by van den Bosch et al. (2008). It has been applied to the two mildly triaxial fast-rotating early-type galaxies M32 and NGC 3379 (van den Bosch & de Zeeuw 2010), the moderately triaxial early-type galaxy NGC 3998 (Walsh et al. 2012), the nuclear star cluster in the Milky Way (Feldmeier-Kräuse et al. 2017) and the ultracompact dwarf galaxy M59-UCD3 (Ahn et al. 2018). Some of these studies resulted in black hole mass measurements that were inconsistent with other methods. Our investigation suggests that this inconsistency is not driven by the incorrect mirroring bug, but by other systematics, e.g. radially varying versus constant mass-to-light ratio (Thater et al. 2017, 2019, 2022), the inclusion of dark matter into the models (Gebhardt & Thomas 2009; Rusli et al. 2013; Thater et al. 2022) or other assumptions of the modelling techniques.
Fig. 6. The stellar orbit distribution in intrinsic radius $r$ vs. circularity $\lambda_z \equiv \mathcal{L}_z / (r \times V_c)$ for mock spiral galaxies with inclination angles of 40°, 60° and 80° from top to bottom. The top panel shows the true stellar orbit distribution from the simulation. In the following three rows, the left panels are for models with incorrect mirroring, the right panels are for models with correct mirroring. The side panels compare the $\lambda_z$ distribution for all the orbits at $r < 15$ kpc. The orbit distributions from the models with correct and incorrect mirroring are nearly identical, and they both present some difference with respect to the true distribution. The differences caused by the incorrect mirroring are negligible compared to the other systematic errors.

Furthermore, the incorrect mirroring bug is not present in the Leiden version of the axisymmetric Schwarzschild code that was used to derive several black hole mass measurements (e.g., Krainovic et al. [2009, 2018], Thater et al. [2017, 2019, 2022] and the axisymmetric measurement in den Brok et al. [2021]). Cross-checks between the different code versions are extremely valuable to find systematic differences and mistakes in the codes.

5. Stellar orbit distribution

In addition to constraints on the gravitational potential, a useful result of Schwarzschild modelling is the stellar orbit distribution. This distribution is often shown in the space of circularity, $\lambda_z = \mathcal{L}_z / (r \times V_c)$, i.e. the orbit angular momentum $\mathcal{L}_z$ normalized by the angular momentum of a circular orbit with the same binding energy (Zhu et al. [2018b]). This means that $|\lambda_z| = 1$ represents highly-rotating short-axis tube orbits (circular orbits), while $\lambda_z =$
0 represents mostly dynamically hot box or radial orbits. The circularity distribution has been used in the past to disentangle dynamical cold, warm and hot components and learn about the accretion history of nearby galaxies (e.g., Zhu et al.

We construct triaxial Schwarzschild models for three mock late-type galaxies, created from the same simulation Auriga halo 6, but with inclination angles of 40°, 60° and 80°, respectively. The advantage of using mock galaxies is that we know the underlying true orbit distribution. The creation of the mock data and dynamical models were performed in the same setup described in Zhu et al. (2020), but using the triaxial Schwarzschild code version with correct and incorrect mirroring.

Figure 6 illustrates the stellar orbit distributions in radius $r$ vs. circularity $\lambda_z$ of the three mock galaxies derived by our models with the correct and incorrect mirroring. For each mock galaxy, we only use the best-fitting model. Their orbit distributions are compared to the true orbit distribution from the simulation. The darker colour indicates a higher phase space density. Being LTGs, the mock galaxies are dominated by a dynamical lation. The advantage of using mock galaxies is that we know the underlying true orbit distribution. The creation of the mock data and dynamical models were performed in the same setup described in Zhu et al. (2020), but using the triaxial Schwarzschild code version with correct and incorrect mirroring.

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6. Conclusion and future work

Recently, Quennessville et al. (2022) reported a bug in the original triaxial Schwarzschild code published by van den Bosch et al. (2008) where some orbits had been incorrectly mirrored. After correcting the mirroring in our open-source triaxial Schwarzschild code (DYNAMITE, Jethwa et al. 2020), we carefully checked for systematic changes regarding the estimate of a black hole mass, enclosed mass, intrinsic shape and the stellar orbit distribution. We noticed small effects in the shape of the $\chi^2$ distribution, but the best-fit parameters with the correct and incorrect mirroring were in almost all cases consistent within their uncertainties. We did not see any noticeable trends with galaxy morphology, inclination angle or triaxiality.

The advantage of using mock galaxies is that we know the underlying true orbit distribution. The creation of the mock data and dynamical models were performed in the same setup described in Zhu et al. (2020), but using the triaxial Schwarzschild code version with correct and incorrect mirroring.

We can therefore conclude that the incorrect mirroring did not systematically bias previous results obtained with the triaxial Schwarzschild code by van den Bosch et al. (2008).

Several other developments are planned for DYNAMITE in the near future. We will implement the orbit-colouring technique to incorporate stellar population information (Poci et al. 2019; Zhu et al. 2020) and offer more sophisticated parameter search algorithms (Gration & Wilkinson 2019). Finally, we recognize the need for a more thorough treatment of uncertainties for orbit-based modelling. An important step in this direction has been made in Lipka & Thomas (2021), who presented a novel technique to optimize the amount of regularization used for orbit-weight solving, which can have a significant impact on constraints on physical parameters such as galaxy mass. How to account for uncertainties on the orbit-weights themselves is an open question which has been largely ignored due to the difficulty of assigning meaningful uncertainties in the high-dimensional and degenerate space of orbit-weights, however, getting a handle on these uncertainties is vital if we wish to associate clumpiness in an orbit-distribution to the presence of merged galactic components. Addressing these concerns is the focus of our future research.

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Appendix A: Impact of incorrect orbit mirroring for a model with close to axisymmetric potential

We consider here the effects of the correct and incorrect mirroring scheme for a galactic potential in the axisymmetric limit, derived from the luminosity model of the simulated Auriga galaxy halo of Zhu et al. (2020). Figures A.1 and A.2 show the impact of the incorrect orbit mirroring for a single orbit, in an analogous way to what is shown in Fig. 1 and 2 in Section 2.

To further explore the more complex triaxial case shown in Section 2, we show in Fig. A.3 and A.4 the impact of the mirroring on a long-axis tube orbit.

Appendix B: Orbit distribution of PGC 046832

In Fig. B.1 we show the stellar orbit distribution obtained for the best-fit model of PGC 046832, when using a Schwarzschild code with correct orbit mirroring.
Fig. A.1. Similar to Fig. 1 but for a short-axis tube orbit in a model with axisymmetric potential. The orbit is shown for 9 (out of 200) revolutions. The time of one orbital period is $\sim 0.44$ Gyr.

Fig. A.2. Similar to Fig. 2 but for a short-axis tube orbit in a model with axisymmetric potential. This orbit has a large weight in the model that is discussed in section 5.
Fig. A.3. Similar to Fig. 1 but for an outer long-axis tube orbit in the model with triaxial potential. The orbit is shown for 40 (out of 200) revolutions. The time of one orbital period is $\sim 0.018$ Gyr.

Fig. A.4. Similar to Fig. 2 but for an outer long-axis tube orbit in the model with triaxial potential.
Fig. B.1. Average orbital circularity of tube orbits as a function of radius for all models consistent with the best fit model of PGC 046832 (discussed in Section 4). Darker colours imply a higher density of orbits. The dashed lines separate hot orbits, warm orbits and cold orbits.