Embedding construction based on amalgamations of group relators.

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Abstract

An embedding construction \( G \hookrightarrow H \) for groups \( G \) with length function was introduced by the author earlier. Here we obtain new properties of this embedding, answering some questions raised by M.V. Sapir. In particular, an analog of Tits' alternative holds for the subgroups of \( H \).

Key words: group relation, group embedding, free group, van Kampen diagram, small cancellation

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1 The embedding \( \gamma : G \hookrightarrow H \).

Let \( \mathcal{A}^{\pm 1} = \{a_1^{\pm 1}, \ldots, a_m^{\pm 1}\} \) be a group alphabet. The length of arbitrary word \( W \) in this alphabet is denoted by \( |W|_\mathcal{A} \) or just by \( |W| \). If the set \( \mathcal{A} \) generate a group \( G \) and \( W \) is a shortest word in the alphabet \( \mathcal{A}^{\pm 1} \) representing an element \( g \in G \), then \( |g| = |g|_\mathcal{A} = |W|_\mathcal{A} \).

Consider now a group \( G \) with an arbitrary length function, i.e., with a function \( \ell : G \to \mathbb{N} = \{0, 1, \ldots\} \) satisfying the conditions

- \( \ell(g) = \ell(g^{-1}) \) for all \( g \in G \), and \( \ell(g) = 0 \) iff \( g = 1 \);
- \( \ell(gh) \leq \ell(g) + \ell(h) \) for \( g, h \in G \);
- there exists a positive number \( c \) such that \( \text{card}\{g \in G \mid \ell(g) \leq r\} \leq c^r \) for any \( r \in \mathbb{N} \).

Obviously, \( G \) is a left-invariant metric space with respect to \( \text{dist}(g, h) = \ell(g^{-1}h) \), and it is clear that the length with respect to a finite set of generators satisfies these three conditions. Moreover, if \( G \) is a subgroup of a finitely generated group \( H = \langle \mathcal{A} \rangle \), then the restriction of the length \( | |_\mathcal{A} \) to \( G \) is a length function on \( G \). Up to equivalence such length functions do not depend on the choice of a finite generating set of \( H \). Two functions \( f_1, f_2 : G \to \mathbb{N} = \{0, 1, \ldots\} \) on a group \( G \) are called equivalent here if there is a positive constant \( c \) such that \( f_1(g) \leq cf_2(g) \) and \( f_2(g) \leq cf_1(g) \) for every \( g \in G \).

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One can show that there are uncountably many pairwise nonequivalent lengths functions on every infinite countable group (see Corollary 1 in [BO]). Nevertheless, it was proved that every length function on a group $G$ can be obtained by means of an embedding into a finitely generated group:

Proposition 1.1. ([O99]) Assume that $\ell$ is a length function on a group $G$. Then there is an embedding of $G$ into a group $H$ with two generators $a$ and $b$, such the restriction of the function $g \to |g|_{\{a,b\}}$ to the subgroup $G$ is equivalent to the function $\ell$.

This embedding theorem helped to answer a number of questions raised earlier (see [O99], [O97], [OS]). Here we establish some additional properties of the construction. These characteristics are useful in connection with “rapid decay” (RD) property of groups, and the author have been asked about their validity by Mark Sapir. The results of the present paper are needed in [S] to obtain 2-generated groups without RD, where all amenable subgroups are cyclic and undistorted.

At first we want to describe the embedding from [O95] and [O99]. The notation $U \equiv V$ means that the words $U$ and $V$ in a group alphabet $A \equiv^1$ are letter-by-letter equal; these two words are freely equal (freely conjugate) if they represent the same element (the conjugate elements) of the free group $F(A)$.

We shall say that a subset of words $X$ is an exponential set if there exist constants $C$ and $c > 1$ such that $\text{card}\{X \in X | |X| \leq i\} \geq c^i$ for every $i \geq C$.

A word $W$ in the ‘positive’ alphabet $A = \{a, b\}$ is called positive.

A reduced word $X$ is called $s$-aperiodic if it has no non-empty powers $Y^s$ as subwords.

Lemma 1.2. For any $\lambda > 0$ one can choose an exponential set $\mathcal{Y}$ of positive words over the alphabet $\{a,b\}$ with the following properties.

(\*) Let $V$ be a subword of some $W \in \mathcal{Y}$ and $|V| \geq \lambda|W|$. Then $V$ occurs in $W$, as a subword, only once. If this $V$ is a subword of some $U \in \mathcal{Y}$, then $U \equiv W$.

(\**) Every word in $\mathcal{Y}$ is 7-aperiodic.

Proof. We recall a construction from [O99], Lemma 4. Given a 2-letter alphabet $\{a, b\}$, there is an exponential set of 6-aperiodic positive words $X_1$ such that every word $X \in X_1$ starts and ends with the letter $b$. We enumerate the word in $X_1$ so that $i < j$ if $|X_i| < |X_j|$. It was shown in Lemma 4 of [O99] that there is an integer $N_0$ such that the set $\mathcal{Y}$ consisting of all words

\[ Y_j \equiv a_6^5X_{(j-1)N_0+1}a_6^5X_{(j-1)N_0+2} \ldots a_6^5X_{jN_0}, \quad j = 1, 2, \ldots , \]

is exponential and satisfies (\*). It remains to verify (\**).

Assume that $Y_j$ has a nonempty subword $Z$ of the form $A^7$. Since every $X_k$ is 6-aperiodic and starts/ends with $b$, the subwords $a^6X_ka^6$ are 7-aperiodic. It follows that $A^7$, and so some cyclic shift $A'$ of $A$, must contain the subword $a^6$. We have a 6-th power $(A')^6 \equiv (a^6U)^6$ as a subword of $Z$ and of $Y_j$. But this is not possible since all the words $X_k$ are pairwise distinct, a contradiction.

Let us fix $\lambda = 0.01$ and let $\mathcal{Y}$ be an exponential set of words in the alphabet $\{a,b\}$ satisfying the properties (\*) and (\**) from Lemma 1.2. Given a group $G$ with a length
function \( \ell \), there are a constant \( d = d(\mathcal{Y}, \ell) \) and a subset \( \mathcal{X} = \{ X_g \}_{g \in G} \subset \mathcal{Y} \) such that the inequalities

\[
\ell(g) \leq |X_g| < d \ell(g), \quad g \in G \setminus \{1\}
\]
hold (see [O99], Lemma 5; and note that the condition (*) for some \( \lambda \) implies the same condition for any \( \mu > \lambda \)).

The group \( G \) can be presented as a homomorphic image of the free group \( F_G \) with the basis \( \{ x_g \}_{g \in G \setminus \{1\}} \) under the epimorphism \( \varepsilon : x_g \mapsto g \). Denote by \( \delta \) the natural embedding of the kernel \( \ker \varepsilon = N \) into the group \( F_G \).

Below we explain the commutative diagram

\[
\begin{array}{cccccc}
1 & \rightarrow & N & \xrightarrow{\delta} & F_G & \xrightarrow{\varepsilon} & G & \rightarrow & 1 \\
\downarrow{\alpha} & & \downarrow{\beta} & & \downarrow{\gamma} & & \downarrow & & \\
1 & \rightarrow & L & \xrightarrow{\bar{\delta}} & F(a, b) & \xrightarrow{\bar{\varepsilon}} & H & \rightarrow & 1
\end{array}
\]

Namely, the group \( F(a, b) \) is free with the basis \( \{a, b\} \), and the homomorphism \( \beta \) is given by the formula \( \beta(x_g) = X_g \) for \( g \in G \setminus \{1\} \). By definition, the homomorphism \( \bar{\delta} \) is the canonical embedding of the normal closure \( L \) of the subgroup \( \beta \delta(N) \) in \( F(a, b) \). The homomorphism \( \alpha \) is well-defined by the equality \( \bar{\delta} \alpha = \beta \delta \). Define \( \bar{\varepsilon} \) as the natural epimorphism of the group \( F(a, b) \) onto its quotient \( H = F(a, b)/L \), and so one can regard the set \( \{a, b\} \) as a generating set for the group \( H \) too. Finally, in view of the condition \( \bar{\delta} \alpha = \beta \delta \), the homomorphism \( \gamma \) can be well-defined by the equality \( \gamma \varepsilon = \bar{\varepsilon} \beta \).

By Lemma 6 [O99], the homomorphism \( \gamma \) is injective, and so one can identify the group \( G \) with the subgroup \( \gamma(G) \leq H \), and by Theorem 1 [O99], the function \( g \mapsto |g|_{\{a,b\}} \) is equivalent to the function \( \ell \). (Only condition (*) with \( \lambda = 0.02 \) was used for \( \mathcal{X} \) in [O99] to obtain the embedding \( \gamma \).)

Here we show that the subgroups of \( H \) ‘avoiding’ the conjugates of \( G \) are big, with few trivial exceptions.

**Theorem 1.3.** Every subgroup \( K \) of \( H \) is either

1. conjugate to a subgroup of \( G \), or
2. infinite cyclic and has trivial intersection with any \( hGh^{-1} \) (\( h \in H \)) , or
3. infinite dihedral (the involutions from \( H \) are conjugate to the elements of \( G \)), or
4. contains a noncyclic free subgroup.

Every element, which is not conjugate with an element of \( G \), generates a quasi-geodesic cyclic subgroup:

**Theorem 1.4.** If an element \( g \) of \( H \) is not conjugate with an element of the subgroup \( G \), then \( |g^n|_{\{a,b\}} > cn|g|_{\{a,b\}} \) for some \( c = c(g) > 0 \) and every \( n > 0 \).

The following theorem implies that the infinite cyclic subgroup \( \langle g \rangle \) from Theorem 1.4 has the Morse property. i.e., the obstacles of linear size intercepting a path along this subgroup cannot be circumvented in linear time. So any asymptotic cone of the group \( H \) has cut points [DMS].
Theorem 1.5. If an element \( g \) of \( H \) is not conjugate with an element of the subgroup \( G \) and \( \theta \in (0,1) \), then the lengths of the paths \( q \) connecting the vertices \( g^n \) and \( g^{-n} \) in the Cayley graph of \( H \) and avoiding the ball of radius \( \theta n \) centered at 1, are not bounded from above by a linear function of \( n \). Moreover, there is \( c = c(g) > 0 \) such that \( |q| > cn^2 \) for every such a path \( q \).

The next theorem implies both the malnormality of the embedding \( \gamma \) (take \( n = 2 \) in part (2)) and the congruence extension property of the embedded subgroup \( G \) : Every normal in \( G \) subgroup \( A \) is an intersection \( G \cap B \), where the subgroup \( B \) is normal in \( H \) (choose \( B \) as the normal closure of \( A \) in \( H \) and apply part (1)). To formulate the theorem, we say that a family of elements \( (g_1, \ldots, g_n) \) is reluctant in a group \( G \) if there is no subfamily \( (g_{i_1}, \ldots, g_{i_m}) \), where \( m \in [1, \ldots, n - 1] \), such that the product of conjugacy classes \( \prod_{j=1}^{m} g_{i_j}^G \) contains 1.

Theorem 1.6. (1) If \( g_1, \ldots, g_n \in G \) and the equation \( \prod_{i=1}^{n} x_i g_i x_i^{-1} = 1 \) has a solution in \( H \), then it has a solution in \( G \).

(2) If \( n \geq 1 \) and for a reluctant in \( G \) family \( (g_1, \ldots, g_n) \), we have \( \prod_{i=1}^{n} x_i g_i x_i^{-1} = 1 \) in \( H \), then all the elements \( x_1, \ldots, x_n \) belong to a single left coset of \( G \) in \( H \).

2 Amalgamations of relators, small cancellations and diagrams.

We consider the following set \( \mathcal{R} \) of defining words for the presentation \( H = F(a, b)/L \). For any nonempty cyclically reduced word \( R = R(x_g, \ldots, x_h) \) over the alphabet \( \{x_g\}_{g \in G \setminus \{1\}} \), vanishing in \( G \), we include the word \( R(X_g, \ldots, X_h) \) over the alphabet \( \{a^{\pm 1}, b^{\pm 1}\} \) into the set \( \mathcal{R} \). It is clear from the definition of the subgroup \( L \) that \( L \) is the normal closure of the set \( \mathcal{R} \) in the group \( F(a, b) \).

It follows from the condition (*) that the set \( \mathcal{X} \) is a free basis of the subgroup \( \beta(F_G) \) (see Lemma 2 in \([O95]\)). In general, any word in the alphabet \( \{a^{\pm 1}, b^{\pm 1}\} \) of the form

\[
A_1 \ldots A_k,
\]

where \( A_i \equiv X_{g_i}^{\pm 1} \) for some \( g_i \in G \), will be called a \( G \)-word. A \( G \)-word will be considered with its decomposition [I] into the factors \( A_1, \ldots, A_k \), which we call entire factorization.

An \( \mathcal{R} \)-word is, by definition, a cyclically reduced form (over the alphabet \( \{a^{\pm 1}, b^{\pm 1}\} \)) of a word from \( \mathcal{R} \). Note that possible cancellations between neighbor entire factors \( A_i \) and \( A_{i+1} \) are small by the condition (*), provided \( A_{i+1} \neq A_i^{-1} \).

Below, \( G \)-words will be viewed as certain paths in the Cayley graph of \( H \) or in a van Kampen diagram. Suppose \( p = e_1 \ldots e_n \) is such a path of length \( |p| = n \), the edges \( e_s \) of which are labelled by the symbols \( \text{Lab}(e_s) \in \{a^{\pm 1}, b^{\pm 1}\} \). If a \( G \)-word happens to be the label \( \text{Lab}(p) \), then \( p \) splits into a product \( p = p_1 \ldots p_k \) with labels \( \text{Lab}(p_i) = A_i \equiv X_{g_i}^{\pm 1} \). In this case, the factorization \( p_1 \ldots p_k \) will be called the entire factorization of \( p \), and its vertices, which divide \( p \) into the segments \( p_i \), will be called entire vertices of \( p \).

We use van Kampen’s interpretation (see \([LS], [O89]\) of the deduction of consequences from defining relations, according to which for any word \( w = w(a, b) \) in the normal closure
$L$ of $\mathcal{R}$ in $F(a,b)$, there exists a finite connected and simply-connected, planar 2-complex (\textit{= disc diagram}) $\Delta$, the label of each edge of which is a letter from $\{a^\pm 1, b^\pm 1\}$, the label of the boundary contour $\partial \Pi$ of each 2-cell (or just cell) $\Pi$ is a word from $\mathcal{R}$, and the word $w$ is written on the outer contour $\partial \Delta$.

However, successive edges $e, f$ that are incident with an entire vertex of the contour $\partial \Pi$ can have mutual inverse labels, which enables us to assume that they are mutual inverse in the contour $\partial \Pi$. By condition (*), under such an identification one can contract at the end of each segment in $\partial \Pi$ at most $0.01$ of it. The cyclically reduced path with label $B_1 \ldots B_k$ (where $B_i$ is a subword of $A_i$) will be called the reduced contour $\partial' \Pi$ of $\Pi$. So $\partial' \Pi$ bounds a region containing the cell $\Pi$ and all the edges from $\partial \Pi \setminus \partial' \Pi$ as well (Fig. 1).

![Figure 1](image1.png)

![Figure 2](image2.png)

By a $G$-fragment of the (cyclic) word $[\Pi]$, we mean its subword of the form $A'A_s \ldots A_{s+t}A''$, where $A'$ is a suffix of $A_{s-1}, A''$ is a prefix of $A_{s+t+1}$. This factorization is considered entire. Analogously, we define a $G$-fragment of an $G$-path with label $[\Pi]$. If the label of the boundary of a diagram $\Delta$ (or of a subpath $p$ of $\partial \Delta$) is an $G$-word, then we also can distinguish entire vertices on $\partial \Delta$ (or on $p$) and introduce the reduced boundary $\partial' \Delta$ (or $p'$). The parts of $\partial \Delta$ (of $p$) vanishing when passing to $\partial' \Delta$ (to $p'$, resp.) are situated on the plane outside of the region bounded by $\partial' \Delta$.

As in [O95], here we use a modified small cancellation property related to an amalgamation of cells in van Kampen diagrams rather than to their annihilation. Two different cells $\Pi_1$ and $\Pi_2$ of a diagram $\Delta$ are compatible, if there are entire vertices $o_1$ on $\partial \Pi_1$, $o_2$ on $\partial \Pi_2$, and a simple path $x = o_1 - o_2$ connecting them, whose label is equal in $F(a,b)$ to an $G$-word (Fig. 2). It is easy to see that this property does not depend on the choice of the pair $(o_1, o_2)$. Similarly we define the compatibility of a cell with (a fragment of ) $\partial \Delta$.

If the clockwise boundary contours $q_1$ and $q_2$ of the compatible cells $\Pi_1$ and $\Pi_2$ start with $o_1$ and $o_2$, resp., then the path $q_1 x q_2 x^{-1}$ bounds a subdiagram $\Gamma$ with boundary label freely conjugate to a relator from $\mathcal{R}$ (or to 1). So one can replace this subdiagram by a single cell (or remove both $\Pi_1$ and $\Pi_2$). Similarly, if $\Pi$ is compatible with a subpath $p$ of $\partial \Delta$ labeled by a $G$-word, one can remove $\Pi$ from $\Delta$, replacing $p$ by another path labeled by a $G$-word.
A diagram having no pairs of compatible cells is called reduced.

We need the following property of reduced diagrams over the group $H$.

**Lemma 2.1.** Let $\Delta$ be a reduced diagram over the presentation $\langle a, b \mid R \rangle$. Then

1. there are no two different cells $\Pi_1, \Pi_2$ in $\Delta$ such that $|q| \geq 0.05|\partial\Pi_1|$ for some common subpath $q$ of the contours $\partial\Pi_1, \partial\Pi_2$;
2. if $q$ is a common subpath for a boundary path $\partial\Pi = q_0$ of a cell $\Pi$ and for a subpath $p = p_1q_2$ of the contour $\partial\Delta$, such that the label $P$ of $p$ is an $G$-word and $|q| \geq 0.05|\partial\Pi|$, then the label of the path $\bar{p} = p_1(q_0)^{-1}p_2$ is freely equal to a $G$-word $\bar{P}$ with $\bar{\epsilon}(\bar{P}) = \bar{\epsilon}(P)$.

**Proof.** Under the assumption $|q| \geq 0.05|\partial\Pi_1|$, the cells $\Pi_1$ and $\Pi_2$ become compatible due the condition (*). See details in Lemma 3 of [095], but now plug 0.01 for $\lambda$. The second statement is also explained in that lemma.

We say that a (non-oriented) edge $e$ is inner if it separates two cells in a diagram. An edge of the boundary $\partial\Pi$ of a cell $\Pi$ is outer if it lies on $\partial\Delta$.

We also say that $q'$ is an outer arc of $\partial\Pi$ in a disc diagram $\Delta$ if $q'$ also belongs to the reduced contour $\partial\Delta$. By Lemma 2.1, a reduced diagram satisfies the condition $C''(\lambda)$ with $\lambda = 0.05$ (see [LS], Chapter 5). Therefore by Grindlinger Lemma [G] with $1 - 3\lambda = 0.85$, we have

**Lemma 2.2.** If a reduced disc diagram $\Delta$ has at least one cell, then

1. it has a cell $\Pi$ with an outer arc of length $> 0.85|\partial\Pi|$;
2. the number of inner edges in $\Delta$ is less than $0.15\Sigma$, where $\Sigma$ is the sum of the reduced perimeters $|\partial\Pi|$ of all cells in $\Delta$.

We say that the cell and the arc from Lemma 2.2 (1) are Grindlinger cell and arc.

**Lemma 2.3.** Let $\Delta$ be a reduced diagram with contour $p_1q_1p_2q_2$, where the paths $q_1$ and $q_2$ are reduced and $|p_1| + |p_2| \leq 0.01(|q_1| + |q_2|)$. Assume that no edge belongs to both $q_i$ and $q_2^{-1}$ and no cell $\Pi$ of $\Delta$ has at least $0.55|\partial\Pi|$ edges belonging to one the paths $q_i$, $i = 1, 2$. Then there is a cell $\Pi$ whose boundary $\partial\Pi$ has edges of both $q_1$ and $q_2$, and the number of edges of $\partial\Pi$ belonging either to $q_1$ or to $q_2$ is greater than $0.55|\partial\Pi|$.

**Proof.** There are no edges $e$ such that both $e$ and $e^{-1}$ belong to $q_i$, $i = 1, 2$. Indeed, otherwise we would have a subdiagram $\Gamma$ bounded by a part of $q_i$, contrary to Lemma 2.2 (1) applied to $\Gamma$, because $0.85 > 0.55$. The inequality $|p_1| + |p_2| \leq 0.01(|q_1| + |q_2|)$ implies that at most $0.01(|q_1| + |q_2|)$ non-oriented edges of $q_1$ and $q_2$ are shared with $p_1$ or $p_2$. Therefore the number of the non-oriented outer edges $Q$ from the reduced boundaries of the cells in $\Delta$, lying on $q_1$ or on $q_2$, is at least $0.99(|q_1| + |q_2|)$.

It follows from Lemma 2.2 (2) that $I < \frac{1}{2}O$, where $I$ and $O$ are the numbers of inner and outer edges in the reduced contours of the cells from $\Delta$, resp. Also $O \leq Q + |p_1| + |p_2| \leq 0.99Q$, and so $I + |p_1| + |p_2| \leq \frac{3}{100}Q + \frac{1}{10}Q < 0.45Q$. Therefore there exists a cell $\Pi$ having $> 0.55|\partial\Pi|$ edges on $q_1$ and $q_2$. By the assumption, these edges cannot belong only to $q_1$ or only to $q_2$, which proves the lemma. \[\Box\]
3 Periodic words with minimal periods.

We say that an element of the group $H$ (or a word $A$ representing this element) is free if it is not conjugate in $H$ to the element of the subgroup $G$. A free word $A$ is called minimal if it is not conjugate in $H$ to a shorter word. A word $W$ is called $A$-periodic (or periodic with period $A$) if it is a subword of some power $A^t$ with $t > 0$.

Lemma 3.1. Assume that $A$ is a minimal word and $W$ is an $A$-periodic word. Let also $W$ be a subword of a cyclically reduced form $V$ of a $G$-word.

1. If $|W| \geq 0.55|V|$, then $|W| < 1.1|A|$.

2. We have $|W| < 16|A|$.

Proof. (1) Let $W \equiv B'B_s\ldots B_{s+t}B''$, where $B_i$'s are obtained after (small) cancellations from the factors $A_i$'s of a $G$-fragment. Denote also $B_{s-1} \equiv B'$ and $B_{s+t+1} \equiv B''$. If $|W| \geq 1.1|A|$, then $W$ can be presented as $UW_1 \equiv W_1U'$, where $U$ and $U'$ are cyclic permutations of $A$ and $|W_1| \geq \frac{1}{11}|W| \geq 0.05|V|$. Now it follows from (*) that $W_1 \equiv W_2W_3W_4$, where for $i \neq j$, $W_3$ is a subword of two factors $B_i$ and $B_j$ with $|W_3| > |W_1|/4 > 0.011|B_i|$. Therefore $B_i \equiv B_j$ and $W_3$ occurs in $B_i$ only once. Now the equality $UW_2W_3W_4 \equiv W_2W_3W_4U'$ implies that $U$, and so $A$, is freely conjugate to the $G$-word $A_1\ldots A_{j-1}$. This contradiction with the assumption that $A$ is free, proves the first statement.

(2) Again, the word $W$ has the form $C'C_1\ldots C_mC''$, where every factor is a subword of a word from $A'$. Therefore $|C'|, |C''| < 7|A|$ by the property (**). But the middle part $C_1\ldots C_m$ is a reduced form of a $G$-word $V$ with $|C_1\ldots C_m| > 0.98|V|$ by the property (*). Hence $|C_1\ldots C_m| < 1.1|A|$ by the part (1). Therefore $|W| < (7 + 7 + 1.1)|A|$, and the statement (2) is proved.

Lemma 3.2. Let $A$ be a minimal word, $p$ be an outer arc of a cell $\Pi$ in a diagram $\Delta$, and $\text{Lab}(p)$ be an $A$-periodic word. Then $|p| < 0.55|\partial \Pi|$.

Proof. If $|p| \geq 0.55|\partial \Pi|$, then $|p| < 1.1|A|$ by Lemma 3.1. Then $p$ can be presented as $p_1p_2$, where $|p_1| \leq |A|$, and so $p_1$ is geodesic, and $|p_2| < 0.1|p|$. If $pq$ is the contour $\partial \Pi$, then $|p_1| \leq |p_2| + |q|$, whence $|q| \geq 0.9|p|$. Hence $|q| > 0.45(|q| + |p|) = 0.45|\partial \Pi|$, and so $|p| < (1 - 0.45)|\partial \Pi|$, a contradiction.

Lemma 3.3. Let $\Delta$ be a reduced diagram.

1. If $\Gamma$ is a subdiagram with contour $pq$, where $p$ and $q$ are subpaths of the reduced boundaries of two cells $\Pi_1$ and $\Pi_2$ of $\Delta$, and $\Gamma$ includes neither $\Pi_1$ nor $\Pi_2$, then $\Gamma$ has no cells, i.e., $p = q^{-1}$.

2. Let $p$ be a subpath with an $A$-periodic label in $\partial \Delta$, where $A$ is a minimal word. If the reduced boundary $\partial \Pi$ of a cell $\Pi$ from $\Delta$ has two different vertices $o_1$ and $o_2$ from $p$, then the whole subpath $o_1 - o_2$ of $p$ belongs to $\partial \Pi$.

Proof. (1) Assume that $\Gamma$ has a cell, and $\Gamma$ is a counter-example with minimal number of cells. Then $\Gamma$ has a Grindlinger cell $\pi$ by Lemma 2.2 (1). A unique maximal common subpath of $\partial \pi$ with $p$ (with $q$) can be of length at most $0.05|\partial \pi|$ by Lemma 2.2 (1). We obtain a contradiction since $0.05 + 0.05 < 0.85$. 

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(2) We consider the subdiagram $\Gamma$ situated between $\Pi$ and $p$. It suffices to prove that $\Gamma$ has no cells. Assume that $\Gamma$ has a cell, and $\Gamma$ is a counter-example with minimal number of cells. Then $\Gamma$ has a Grindlinger cell $\pi$ by Lemma 2.3 (1). A unique maximal common subpath of $\partial \pi$ with $p$ has length at most $0.55|\partial \pi|$ by Lemma 3.2. By statement (1), $x$ has at most one maximal common arc with $\partial \Pi$, and its length is $< 0.05|\partial \pi|$ by Lemma 2.3 (1). We come to a contradiction since $0.55 + 0.05 < 0.85$. \hfill $\Box$

**Lemma 3.4.** Assume that $W$ is a periodic word with minimal period $A$ and $W = V$ in $H$ for some word $V$.

(1) Then $|V| \geq 0.25|W|$. In particular, $A$ has infinite order in $H$.

(2) A power $A^n$ cannot be conjugate to a word of length $\leq 0.2n|A|$.

**Proof.** (1) We consider a reduced diagram $\Delta$ with contour $pq$, where $Lab(p) \equiv W$ and $Lab(q^{-1}) \equiv V$. By induction on $|W|$ one may assume that the path $p$ has no common edges with $q^{-1}$ and with $p^{-1}$. It follows that every (non-oriented) edge of $p$ is shared with a cell of $\Delta$, and by lemmas 3.3 (2) and 3.2 the sum $\Sigma$ of the reduced perimeters of all cells in $\Delta$ is at least $20|p|$. On the other hand, by Lemma 2.2 (2), the number of (non-oriented) inner edges in $\Delta$ is at most $0.15 \Sigma$. Hence the path $q$ has to have at least $(1 - \frac{11}{20} - 2 \times 0.15) \Sigma = 0.15 \Sigma > 0.25|p|$ edges, as required.

(2) Using the same argument, we have $|V| > 0.25|A^m|$ if $V = A^n$ in $H$ and $m \geq 1$. Now if $U = XA^nX^{-1}$ in $H$ and $|U| \leq 0.2n|A|$, then $|U^s| \leq 0.2sn|A|$ for every $s > 0$, whence $A^m$ is equal to the word $X^{-1}U^sX$ having length at most $0.2sn|A| + 2|X|$. But this sum is less then $0.25sn|A|$ if $s$ is large enough, a contradiction. \hfill $\Box$

We will denote by $q_-$ (by $q_+$) the initial (resp., terminal) vertex of a path $q$.

**Lemma 3.5.** Let $A$ be a minimal word and $\Delta$ be a reduced diagram with contour $p_1q_1p_2q_2$, where $|p_1| + |p_2| \leq 0.01(|q_1| + |q_2|)$ and $Lab(q_1)$, $Lab(q_2)$ are $A^{\pm 1}$-periodic words. Then

(1) there is a path $x$ of length $< 27|A|$ connecting in $\Delta$ a vertex $o$ of $q_1$, $o \neq (q_1)_-$, with a vertex of $q_2$;

(2) assume that $q_1 = q'q''$, where $|q'| \geq 100(|p_1| + 27|A|)$ and $|q''| \geq 100(|p_2| + 27|A|)$. Then the vertex $q'_+\_1$ can be connected with a vertex of $q_2$ by a path of length $< 2800|A|$.

**Proof.** (1) It is nothing to prove if $q_1$ and $q_2^{-1}$ share an edge. So we may assume that they have no edges in common.

By Lemmas 3.3 (2) and 3.2 there is no cell $\Pi$ in $\Delta$ such that $\geq 0.55|\partial \Pi|$ of its boundary edges belong to some $q_i$, $i = 1, 2$. Therefore by Lemma 2.3 there exists a cell $\Pi$ having $> 0.55|\partial \Pi|$ edges on $q_1$ and $q_2$ but not on the one of these paths. Hence the reduced contour of $\Pi$ factorizes as $x_1y_1x_2y_2$ where $y_1$ and $y_2$ start and end on $q_1$ and $q_2$, resp., and $|x_1|, |x_2| < 0.45|\partial \Pi|$. One of the two paths $y_1, y_2$, call it $y$, has length at least $0.55|\partial \Pi|/2 > 0.27|\partial \Pi|$. Also $|y| < 16|A|$ by Lemma 3.1 (2). Therefore $|x_1|, |x_2| < \frac{27}{2800}|y| < 27|A|$. Finally, $x$ is either $x_1$ or $x_2$.

(2) Let $o$ be the first vertex on $q_1$ which can be connected with a vertex of the path $q_2$ by a path $x$ of length $< 27|A|$. Such a vertex exists by the statement (1). Moreover, if $q_1 = yz$, where $y_+ = o$, then $|y| < 100(|p_1| + 27|A|)$ since otherwise one can apply
statement (1) to the subdiagram with the boundary \( p_1yx'y' \), where \( y' \) is the subpath of \( q_2 \), and replace \( o \) by another vertex on \( y \).

Similarly, one can find the last vertex \( o' \) on \( q_1 \) connected with \( q_2 \) by a path \( x' \) of length \( < 27|\Delta| \), and so the vertex \( q'_+ \) lies between \( o \) and \( o' \). One can draw \( x \) and \( x' \) so that they do not cross each other. Consider now the subdiagram with contour \( x^{-1}q_ix'q_2 \), where \( q_i \) is a subpath of \( q_i \) \( (i = 1, 2) \), and repeat the same trick cutting up this diagram and obtaining subdiagrams with contours \( x^{-1}s_ix_{i+1}t_i \), where \( s_1 \ldots s_k \) is the subpath of \( q_1 \) starting at \( o \) and ending at \( o' \), \( |x_i| < 27|\Delta| \) and \( |s_i| \leq 5400|\Delta| \). Since the vertex \( q'_+ \) belongs to some \( s_i \), it can be connected with a vertex of \( t_i \) (and of \( q_2 \)) by a path of length \( < 2700|\Delta| + 27|\Delta| < 2800|\Delta| \).

**Lemma 3.6.** Let \( A \) be a minimal word and \( \Delta \) be a reduced diagram with contour \( p_1q_1 \ldots p_nq_n \), where the factors are reduced paths, the words \( \text{Lab}(q_i) \) are \( A \)- or \( A^{-1} \)-periodic \( (i = 1, \ldots, n) \) of length \( \geq 100|\Delta| \) and \( |p_i| \leq 0.01|q_i| \) for all \( i, j \). Then for some \( i \), \( \Delta \) has a subdiagram \( \Gamma \) with contour \( q'pq''x \), where \( q' \) and \( q'' \) are the subpaths of \( q_{i-1} \) and \( q_i \), resp. (indices are taken modulo \( n \)), \( |x| < 15|\Delta| \) and either \( |q'| > \frac{|q_{i-1}|}{3} \) or \( |q''| > \frac{|q_i|}{3} \).

**Proof.** We will construct a nested series of subdiagrams \( \Delta \supset \Delta_1 \supset \ldots \) using transformations of two types. The transformations (a) and (b) of the first type will affect only one of the 'long' subpaths \( q_i \)-s. The transformations (a) and (b) of the second type will affect two (cyclically) neighbors \( q_{i-1} \) and \( q_i \) for some \( i \).

**Type 1.** (a) Assume that we have \( p_i = p_i^1e \) and \( q_i = e^{-1}q_i^1 \) (or \( p_i = ep_i^1 \) and \( q_{i-1} = q_{i-1}^1e^{-1} \)), where \( e \) is an edge. Then we can replace the subpath \( p_iq_i \) in \( \partial\Delta \) by \( p_i^1q_i^1 \). The diagram \( \Delta_1 \) has contour \( p_1q_1^1 \ldots p_nq_n^1 \), where \( p_j^1 = p_j \) and \( q_j^1 = q_j \) for \( j \neq i \).

(b) Assume that the path \( p_iq_i \) (or \( q_{i-1}p_i \)) is reduced and it contains a Grindlinger arc with a cell \( \Pi \), i.e., \( px \) is the reduced contour of \( \Pi \) with \( |x| < 0.15|\partial\Pi| \).

If \( p \) is a subpath of \( p_i \), we just replace it by \( x^{-1}p_i \) and obtain the shorter path \( p_i^1 \) in the diagram \( \Delta_1 = \Delta \setminus \Pi \). Similar transformation works in any case if \( p \) and \( p_i \) have a common subpath of length \( > 0.5|\partial\Pi| \).

Then we assume that \( p = yz \), where \( p_i = uy \), \( q_i = zq_i^1 \) (Fig. 3) and \( |z| < 0.55|\partial\Pi| \) by Lemma 3.2. It follows that \( |y| > (0.85 - 0.55)|\partial\Pi| = 0.3|\partial\Pi| \) and \( |y| - |x| > 0.15|\partial\Pi| \).

Now we define \( p_i^1 = ux^{-1} \) and removing \( \Pi \), obtain a diagram \( \Delta_1 \), where

\[
|q_{i-1} - q_{i-1}^1| < \frac{0.55}{0.15}|p_i| - |p_i^1|) \leq 4(|p_i| - |p_i^1|)
\] (2)

**Type 2.** (a) Assume that \( |p_i| = 0 \) for some \( i \) and \( q_{i-1} = q_{i-1}^1e \), \( q_i = e^{-1}q_i^1 \) for an edge \( e \). Then we replace the subpath \( q_{i-1}p_iq_i \) by \( q_{i-1}^1p_iq_i^1 \), where \( |p_i^1| = 0 \).

(b) Let for some \( i \), we have a Grindlinger cell \( \Pi \) with the Grindlinger arc \( p = ypz \), where \( |p_i| \leq |\partial\Pi|/2 \) and \( q_{i-1} = q_{i-1}^1y \) and \( q_i = zq_i^1 \) (Fig. 4). Then we have \( |y| + |z| > (0.85 - 0.5)|\partial\Pi| = 0.35|\partial\Pi| \). But \( |y|, |z| \leq 16|\Delta| \) by Lemma 3.1 (2), and so \( 32|\Delta| > 0.35|\partial\Pi| \), whence \( |\partial\Pi| < 100|\Delta| \) and for the compliment \( x \) of \( p \) in \( \partial\Pi \), we have \( |x| < 0.15 \times 100|\Delta| = 15|\Delta| \).

In this case, we define the contour of \( \Delta_1 = (\Delta \setminus \Pi) \) as follows. We replace \( p_i \) by \( p_i^1 = x^{-1} \) and replace \( q_{i-1} \) and \( q_i \) by \( q_{i-1}^1 \) and \( q_i^1 \), resp.
Remark 3.7. One may assume that the paths $q_{i-1}^{-1}x^{-1}$ and $x^{-1}q_i^1$ are reduced after the transformation of type 2(b) since the Grindlinger arc $p$ can be made the longest possible. Therefore if $\Delta_1$ results from $\Delta$ after a transformation of the second type at the 'corner' $i$, then no transformation of the first type is applicable to the same $i$-th corner of $\Delta_1$. Indeed, the path $x$ was a part of the boundary of $\Pi$. Hence the boundary of a cell $\pi$ from $\Delta_1$ cannot share with $x$ a path of length $\geq 0.05|\partial \pi|$ by Lemma 2.1 (1). Since $\partial \pi$ cannot share a path of length $\geq 0.55|\partial'\pi|$ with $q_i^{-1}$ or with $q_i^1$ by Lemma 3.2 it cannot be a Grindlinger cell of type 1 in $\Delta_1$ because $0.05 + 0.55 < 0.85$.

The transformations of the first type $\Delta \to \Delta_1 \to \Delta_2 \ldots$ can shorten the path $q_i$ by at most $4(|p_i| + |p_{i+1}|)$, as it follows from (2). Therefore after a maximal chain of the transformations of the first type applied to all the corners $i = 1, \ldots, n$ in arbitrary order, we will have a subdiagram $\tilde{\Delta}$ with contour $\tilde{p}_1\tilde{q}_1 \ldots \tilde{p}_n\tilde{q}_n$, such that $|\tilde{q}_i| > |q_i| - 0.08|q_i| > 0.9|q_i|$, $|\tilde{p}_i| \leq |p_i|$ for every $i = 1, \ldots, n$.

By Remark 3.7 only transformations of the second type can appear in any decreasing chain of subdiagrams $\tilde{\Delta} \to \tilde{\Delta}_1 \to \ldots$. If $\tilde{\Delta}_1$ has contour $\tilde{p}_1\tilde{q}_1 \ldots \tilde{p}_n\tilde{q}_n$ and for each $i$ we have $|\tilde{q}_i| \geq |q_i|/3 > 33|A|$, then one more transformation of the second type decreases the perimeter of $\Delta_j$, because no Grindlinger arc of a cell can entirely cover some $\tilde{q}_i^1$ by Lemma 3.1 (2). By the same lemma, any such transformation affects only an initial or a terminal part of length $< 16|A|$ for some $\tilde{q}_i^1$ and $\tilde{q}_i^3$. Thus, after some transformation of the second type we can obtain $|\tilde{q}_i^1| < |q_i|/3$, and so $q_i = q''q_i^1q''$ where $|q''| > |q_i|/3$ or $|q''| > |q_i|/3$. Let us consider only the first event. Then the vertex $q_i''$ is connected with $q_{i-1}$ by a path $x$ of length $< 15|A|$ since this $x$ appears after a transformation of the second type. The desired subdiagram $\Gamma$ is obtained.

Below a reduced disc diagram $\Delta$ is called a box if its contour is a product $x_1y_1x_2y_2$, where the subpaths $x_1, y_1, y_2$ are reduced, $x_2$ is reduced or a $G$-fragment, $\text{lab}(x_1)$ is an $\Lambda$-periodic word with a minimal period $A$, and $y_1^{-1}, y_2$ are perpendicular to $x_1$, i.e., the distance in $\Delta$ between $(y_1)_+$ and $x_1$ (between $(y_2)_-$ and $x_1$) is equal to $|y_1|$ (to $y_2$, resp.).

A subdiagram $\Delta'$ of a box $\Delta$ with a contour $x'_1y'_1x'_2y'_2$ is a subbox if $x'_1 = x_1$, $y'_1$ (resp., $y'_2$) is the beginning of the path $y_1$ (the end of $y_2$, and $x'_2$ is a reduced path or a $G$-fragment.
So a subbox is a box itself.

**Lemma 3.8.** Let $\Delta$ be a box with $|x_1| \geq 16|A|$. Then $|x_2| > 1$, and $x_2$ cannot be a $G$-fragment.

**Proof.** Assume on the contrary that $|x_2| \leq 1$ or $x_2$ is a $G$-fragment. Then $\Delta$ has a subbox $\Delta'$, which is minimal with respect to the property that either $|x_2'| \leq 1$ or $x_2'$ is a $G$-fragment.

Note that the subpaths $x_1'y_1'$ and $y_2'x_1'$ are reduced since $(y_1')^{-1}$ and $y_2'$ are perpendicular to $x_1'$, while $y_1'x_2'$ and $x_2'y_2'$ (or $y_1'y_2'$ if $x_2' = 0$) are reduced due to the minimality of the subbox $\Delta'$. Also one cannot have $|y_1'| = |y_2'| = 0$ since $Lab(x_1) = Lab(x_1') \neq Lab(x_2')$ in the free group by Lemma 3.1 (2). Therefore the boundary $\partial \Delta'$ is not a completely cancellable path, whence $\Delta'$ must have $R$-cells, and so it has a Grindlinger cell $\pi$ by Lemma 2.2 (1).

The Grindlinger arc $p$ of $\partial'\pi$ cannot have common edges with both paths $y_1'$ and $y_2'$ since in this case one obtains a smaller subbox $\Delta'' = \Delta' \setminus \pi$, where $x_2''$ is a $G$-fragment (Fig. 5). Also $p$ does not include a part of $x_2'$ of length $\geq 0.05|\partial'\pi|$ since otherwise one can obtain a smaller counter-example using Lemma 2.1 (2). The path $p$ does not include a subpath of $y_1'$ (or of $y_2'$) with length $> 0.5|\partial'\pi|$ since the perpendicular $(y_1')^{-1}$ is geodesic in $\Delta$. Finally, the arc $p$ cannot cover the whole path $x_1'$ by Lemma 3.1 (2), and the subpath of $p$ shared with $(x_1')^{\pm 1}$ has length $< 0.55|\partial'\pi|$ by Lemma 3.2.

Thus, up to substitution $x_1' \leftrightarrow x_2'$, the arc $p$ has a subpath $uv$ of length $> (0.85 - 0.05)|\partial'\pi|$, where $u$ is a subpath of $x_1'$, $v$ is a subpath of $y_1'$ and $|v| > (0.8 - 0.55)|\partial'\pi| = 0.25|\partial'\pi|$ (Fig. 6). Hence we have that $\partial\pi = pq$, where $|q| < (1 - 0.8)|\partial'\pi| = 0.2|\partial'\pi|$, and $q$ connects the vertex $v_+$ with $x_1'$. This contradicts the assumption that $y_1^{-1}$ is perpendicular to $x_1$ because $|q| < |v|$. The lemma is proved. □

**Lemma 3.9.** Let $\Delta$ be a box with $|x_1| \geq 16|A|$. Then $|x_1| + |x_2| > 0.01(|y_1| + |y_2|)$.

**Proof.** Assume that $|x_1| + |x_2| \leq 0.01(|y_1| + |y_2|)$. Then we can apply Lemma 2.3 to $\Delta$. Indeed, (1) the paths $y_1$ and $y_2^{-1}$ have disjoint sets of edges because otherwise we should have a subbox with contour $x_1'y_1'x_2'y_2'$, where $|x_2'| = 0$, a contradiction with Lemma 3.2.
both these subpaths are the shortest ones between $y$ and the vertices $y_i$. Let $\delta \Pi$ and (2) no reduced boundary $\partial' \Pi$ has $> 0.5|\partial' \Pi|$ edges on $y_i$ ($i = 1, 2$) since the perpendiculars are geodesic paths.

So there is a cell $\Pi$ whose boundary has common edges with both $y'_1$ and $y'_2$. But an arc of $\partial' \Pi$ connecting some vertices of $y'_1$ and $y'_2$ is labeled by a $G$-fragment, which by Lemma 3.8, gives an impossible subbox, a contradiction.

**Lemma 3.10.** Let $A$ be a minimal word and $\theta \in (0, 1)$. Then there is a positive number $c = c(\theta)$ with the following property. Let $\Delta$ be a reduced diagram with a contour $pq$ such that $\text{Lab}(p) \equiv A^{2n}$ and the distance in $\Delta$ from the middle vertex $o$ of $p$ to the path $q$ is at least $\theta n|A|$. Then $|q| \geq cn^2$.

**Proof.** Let $v_0, v_1, \ldots$ be all the vertices of $q$ counted from $q_-$ to $q_+$. We consider the set of paths $y^0, y^1, \ldots$, where $y^i$ is a perpendicular to $p$ drawn in $\Delta$ from $v_i$. One may assume that the different $y^i$ and $y^j$ do not cross each other. Indeed, if they cross at a vertex $u$ and $y^i = (y')^i(y'')^i$, $y^j = (y')^j(y'')^j$ with $(y')^i_+ = (y')^j_+ = u$, then $|(y')^i| = |(y'')^j|$ since both these subpaths are the shortest ones between $u$ and $p$, and so one can replace $y^i$ by $(y')^j(y'')^i$, and so on. Hence the ends $y^0, y_1, \ldots$ are placed on $p$ from $p_+$ to $p_-$.

Let $\Delta_i$ be the box with contour $x^i_1(y^i)^{-1}x^i_2y^i+1$, where $x^i_1$ is a subpath of $p$ connecting $y^i+1$ and $y^i$, and $x^i_2$ is the path of length 1 connecting $v_i$ and $v_{i+1}$, $i = 0, 1, \ldots$ (Fig. 7). We have $p = \ldots x^i_1x^0_1$, where for every $i$, we obtain the inequality $|x^i_1| < 16|A|$ from Lemma 3.8 since $|x^i_2| = 1$. So it is possible to find a pair of indices $(i, j)$ such that the path $z = x^i_1 \ldots x^j_1$ contains the vertex $o$, $|z| > \theta n|A| - 32|A|$, and the distances between $o$ and the vertices $z_-$ and $z_+$ are less than $\theta n|A|/2$. (Here and below one may assume that $n$ is large enough in comparison with $\theta^{-1}|A|$.)

Consider the boxes $\Delta_i, \ldots, \Delta_j$. One can unite some neighbors to obtain larger boxes $\Delta_k$ ($k = 1, \ldots, m$ for some $m$) with contours $x^k_1(y^k)^{-1}x^k_2y^{k+1}$, where $16|A| \leq |x^k_1| < 32|A|$ (Fig. 8). Therefore $m \geq \frac{\theta n|A| - 32|A|}{32|A|} > \frac{9n}{40}$. We also derive $|y^k| > \theta n|A|/2$ from the triangle inequality because the distance from $o$ to $(y^k)_-$ is at least $\theta n|A|$.

Now by Lemma 3.9, we obtain $|x^k_1| + |x^k_2| > 0.01(\theta n|A|/2 + \theta n|A|/2) = 0.01\theta n|A|$, whence $|x^k_2| > 0.01\theta n|A| - 32|A| > 0.005\theta n|A|$. It follows that $|q| \geq m(0.005\theta n|A|) > \frac{9n}{40}(0.005\theta n|A|) > 10^{-4}\theta^2|A|n^2 \geq cn^2$ for $c = 10^{-4}\theta^2$, and the lemma is proved.

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**Figure 7**

**Figure 8**
4 Some non-simply-connected diagrams.

We need a few types of non-simply-connected diagrams on surfaces. They appear when one identifies some parts of the boundaries of simply-connected (= disc) diagrams. However if a simply-connected diagram $\Delta$ is not homeomorphic to the standard disc, we can obtain undesired singularities after such identifications. To avoid them, one can consider only van Kampen diagrams homeomorphic to disc, where every cell is also homeomorphic to a disc. For this goal, we suggest using so called 0-cells corresponding to trivial relations. By definition, every 0-edge is labeled by the empty word 1, and we assign zero length to 0-edges. The boundary contour of every 0-cell (which is a disc too) is of the form $e_1 \ldots e_n$, where all the edges $e_i$-s are 0-edges or there are two edges $e_i$ and $e_j$ labeled by mutual inverse letters from the alphabet $\{a^{\pm1}, b^{\pm1}\}$, while the remaining edges are 0-edges.

Under such an agreement, one can use only surfaces without singularities. For instance, every disc (spherical, annual, toric) diagram is now homeomorphic to a disc (to a sphere, to an annulus, to a torus, resp.) The van Kampen lemma (and other related lemmas) can be obviously reformulated for the diagrams with 0-cells. (See more details in Section 11 of [O89].) However instead of common arcs between $R$-cells $\Pi_1$ and $\Pi_2$, we consider now contiguity subdiagrams consisting of 0-cells only and having a contour of the form $p_1q_1p_2q_2$, where $|p_1| = |p_2| = 0$, $q_1$ is a subpath of $\partial\Pi_1$, and $q_2$ is a subpath in $\partial\Pi_2$ (Fig. 9). All the lemmas proved in the previous sections make clear sense for diagrams with 0-cells.

If a diagram $\Delta$ has a simple path $x$ connecting two vertices $o_1$ and $o_2$, and a diagram $\Delta'$ results from $\Delta$ after some amalgamation or cancellation of cells, which does not affect $o_1$ and $o_2$, then using 0-cells one can construct a simple path $x'$ connecting the same vertices in $\Delta$ with the label $\text{Lab}(x')$ freely equal to $\text{Lab}(x)$. (See details in Section 13.5 of [O89].)

One more distinction in comparison with simply-connected case is that a cell can be compatible with itself in a non-simply-connected diagram $\Delta$. This means that in the definition of compatibility, we now allow the equality $\Pi_1 = \Pi_2$, i.e., a cell $\Pi_1$ is compatible with itself if the path $o_1-o_2$ together with a subpath of $\partial\Pi$ connecting the entire vertices $o_1$ and $o_2$, give a closed path, which is not 0-homotopic in $\Delta$. This path is labeled by a $G$-word.

We call a diagram $\Delta$ singular if it contains a simple closed path, which is not 0-homotopic in $\Delta$ but has trivial in the free group label.
Lemma 4.1. Let $\Delta$ be a reduced diagram on a sphere with $n \geq 2$ holes, i.e., $\Delta$ has $n$ boundary components $p_1, \ldots, p_n$ with clockwise labels $P_1, \ldots, P_n$.

(1) If $\Delta$ is singular, then there are a positive integer $m \in [1, n-1]$, indices $i_1 < \cdots < i_m$, and words $Q_1, \ldots, Q_m$ such that $\prod_{j=1}^m Q_jP_jQ_j^{-1} = 1$ in $H$.

(2) If $\Delta$ is non-singular and the words $P_1, \ldots, P_n$ are $G$-words, then the label of arbitrary path connecting in $\Delta$ any two entire vertices of the cells or of the boundary components, is freely equal to a $G$-word.

Proof. (1) We have a simple closed path $q$, such that $\text{Lab}(q) = 1$ in $F(a, b)$, and $q$ bounds a subdiagram $\Gamma$ with $m$ holes, where $1 \leq m < n$. The holes are bounded by some $p_{i_1}, \ldots, p_{i_m}$. Then, after cancellations in $q$, $\Gamma$ becomes a diagram on a sphere with $m$ holes. The standard application of the version of van Kampen – Schupp Lemma (see Chapter V in [LS]) provides us with an equality $\prod_{j=1}^m Q_jP_jQ_j^{-1} = 1$ in $H$.

(2) Let us consider the bigger set of relations $Q$ which consists of all $G$-words. Then we obtain a spherical $Q$-diagram $\Gamma$ from $\Delta$ if we patch up every hole by a new cell. It suffices to prove the property (2) for $\Gamma$. We will distinguish the old cells, i.e. the cells of $\Delta$ and the new ones. By Lemma 2.2 (1) (where the length of the boundary is 0), the diagram $\Gamma$ is not reduced. Hence it has two different compatible cells $\Pi_1$ and $\Pi_2$ which can be replaced by a single $Q$-cell $\Pi$ in the spherical diagram $\Gamma'$ with fewer cells.

Note that both $\Pi_1$ and $\Pi_2$ cannot be old since then the original diagram could not be reduced. If one of them is old and another one is new, then we say that $\Pi$ is new. This cell $\Pi$ cannot have trivial label in the free group since the original diagram $\Delta$ was non-singular.

If both $\Pi_1$ and $\Pi_2$ are new, we say that $\Pi$ is also new but with multiplicity 2. Again $\Pi$ is labeled by a freely non-trivial $G$-word unless $n = 2$, because $\Delta$ was non-singular. If $\Pi$ is non-trivial, i.e., $\Pi_1$ and $\Pi_2$ do not annihilate, then $\Pi$ will have at least one entire vertex.

If this cell $\Pi$ is non-trivial, then it suffices to prove the property (2) for the diagram $\Gamma'$ with fewer cells because the cells $\Pi_1$ and $\Pi_2$ of $\Delta$ were compatible and the compliment of a subpath between two entire vertices in $\partial \Pi$ is also labeled by a $G$-word.

Proceeding this way, we obtain a chain $\Gamma = \Gamma^{(0)} \to \Gamma' \to \cdots \to \Gamma^{(s)}$, where $\Gamma^{(s)}$ consists of new and old $Q$-cells. Every step $\Gamma^{(i-1)} \to \Gamma^{(i)}$ decreases the number of old cells or replaces two new cells of multiplicities $m_1$ and $m_2$ by a single cell of multiplicity $m_1 + m_2 < n$. Thus, soon or later one achieves a transition $\Gamma^{(i-1)} \to \Gamma^{(i)}$, where two new cells $\Pi_1$ and $\Pi_2$ of $\Gamma^{(i-1)}$ with multiplicities $m_1$ and $m_2$ are compatible and $m_1 + m_2 = n$.

If $\Pi_1$ and $\Pi_2$ annihilates in $\Gamma^{(i)}$, then $\Gamma^{(i)}$ has no old cells by Lemma 2.2 (1), being a spherical reduced diagram. The same is true for $\Gamma^{(i-1)}$, and so $\Gamma^{(i-1)}$ has no entire vertices except for the vertices of $\Pi_1$ and $\Pi_2$. Since these two cells are compatible, the hypothesis (b) holds for $\Gamma^{(i-1)}$. Hence it holds for $\Delta$.

If $\Pi_1$ and $\Pi_2$ nontrivially amalgamate in $\Gamma^{(i)}$, then one can remove their amalgamation $\Pi$ from $\Gamma^{(i)}$ and obtain a disc diagram $E$ over $H$, whose contour is an $G$-word. Again one can remove the cells from $E$ one-by-one changing the contour of $E$ and stop when a diagram $E^{(i)}$ has only one (Grindlinger) cell.

\[\square\]
Lemma 4.2. Let $\Delta$ be a reduced toric diagram.

(1) If $\Delta$ is singular, then the boundary labels of all closed paths starting at a fixed vertex of $\Delta$ belong to the same cyclic subgroup of $H$.

(2) If $\Delta$ is not singular, then the label of arbitrary path connecting the entire vertices of its cells is freely equal to a $G$-word.

Proof. (1) There is a simple closed path $x$ in $\Delta$ such that it represents non-trivial element of the fundamental group $G_T$ of the torus, but $\text{Lab}(x)$ is freely equal to 1. We can find a simple closed path $y$, such that $x$ and $y$ represent a canonical generating set of $G_T$. In this case the subgroup $\langle \text{Lab}(x), \text{Lab}(y) \rangle$ of $G$ is cyclic and so the image of $G_T$ is cyclic under the homomorphism $\text{Lab}$ from $G_T$ to $H$. The first statement is proved since changing the origin, one replaces this subgroup by a conjugate one.

(2) Let us contract every 0-edge of $\Delta$ to a vertex and, thus, contract every 0-cell to a vertex or to an edge. We obtain a reduced diagram $\Gamma$ on a surface homeomorphic to a torus since no closed path, which is not 0-homotopic, will be contracted to a vertex. So it suffices to prove the statement for $\Gamma$.

If for every cells $\Pi_1$ and $\Pi_2$, the length of any common arc $q$ of their reduced boundaries $\leq 0.05|\partial'\Pi_1|$, then every cell of $\Gamma$ is an $n$-gon with $n > 20$, but the torus cannot be tessellated by such polygons, as this follows from Euler's formula, a contradiction. Otherwise one can find a pair of compatible cells $\Pi_1$ and $\Pi_2$. In fact $\Pi_1 = \Pi_2$ since the diagram $\Gamma$ is reduced. In other words, we have a self-compatible cell $\Pi$ in $\Gamma$.

A part of $\partial\Pi$ and the path $x$ defining the self-compatibility, form a closed path $p$, which is not contractible to a vertex along $\Gamma$. Therefore, if we cut the torus along $p$, we obtain an annular diagram $E$ whose boundary labels are freely equal to $G$-words. The diagram $E$ is not singular since $\Gamma$ is non-singular. The statement (2) holds for $\Gamma$ iff it holds for $E$ since $\Pi$ is a self-compatible cell. However the required property of $E$ has been already proved in Lemma 4.1 (2). This completes the proof of Lemma 4.2.

5 Algebraic properties of the embedding $\gamma$.

Lemma 5.1. If a word $W$ is free and $WV = VW$ in $H$ for some word $V$, then the subgroup $\langle W, V \rangle$ of $H$ is cyclic.

Proof. By van Kampen’s Lemma, we have a disc diagram $\Delta$ over $G$ with contour $p_1q_1p_2q_2$, where $\text{Lab}(p_1) \equiv \text{Lab}(p_2)^{-1} \equiv W$ and $\text{Lab}(q_1) \equiv \text{Lab}(q_2)^{-1} \equiv V$. Identifying $p_1$ with $p_2^{-1}$ and $q_1$ with $q_2^{-1}$, we have a toric diagram $\Gamma$. Let $\Gamma_0$ be a reduced toric diagram resulting from $\Gamma$ after possible amalgamations and annihilations of some cells. Using auxiliary 0-cells, one may assume that it also has closed paths with labels freely equal to $V$ and $W$. Consider now two cases.

If $\Gamma_0$ is singular, then the statement follows from Lemma 4.2 (1). If $\Gamma_0$ is nonsingular, then by Lemma 4.2 (2), the words $V$ and $W$ are freely conjugate to some $G$-words, a contradiction.

Lemma 5.2. If $A$ is a free word, then so is $A^t$ for $t \neq 0$.  

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Proof. One may assume that $A$ is minimal and $t \geq 2$. Arguing by contradiction, assume that the word $A^t$ is not free, and therefore there is a reduced annular diagram $\Delta$ with contours $p_1$ and $p_2$ labeled by $A^t$ and by a $G$-word, respectively. This diagram is non-singular since $A^t \neq 1$ in $H$ by Lemma 3.4 (1). So, as in the proof of Lemma 4.2 one may assume that $\Delta$ has no 0-edges. If $\Delta$ has no cells, then we can compare the labels of $p_1$ and $p_2$ using Lemma 3.2 (1), and obtain a contradiction since $t \geq 2 > 1.1$. Therefore $\Delta$ has an $R$-cell $\Pi$.

A cell $\Pi$ cannot have a common boundary arc of length $\geq 0.05|\partial \Pi|$ with $p_2$ since this would make possible to modify $p_2$ using Lemma 2.1 (2) and to reduce the number of cells in $\Delta$. By Lemma 2.1 (1), $\Pi$ has no common boundary arcs of length $\geq 0.05|\partial \Pi|$ with other cells. The unique maximal common boundary arc of $\Pi$ and $p_1$ has length $\leq 0.55|\partial \Pi|$ by Lemmas 3.3 and 3.2. Thus, either the polygon $\Pi$ has at least 11 sides, or the cell $\Pi$ is self-compatible. The former case for all the cells of $\Delta$ gives a contradiction with Euler’ formula. If we have the latter case for some $\Pi$, then we cut up $\Delta$ by a path labeled by a $G$-word, as we did that in the proof of Lemma 4.2 (2). This gives an annular subdiagram $\Delta_1$, where one contour is $p_1$, and another one is again labeled by an $G$-word. Since $\Delta_1$ has fewer $R$-cells than $\Delta$, the statement is proved by induction.

Lemma 5.3. Let $A$ be a free word. Then any equality $XA^sX^{-1} = A^t$ implies that $|s| = |t|$.  

Proof. One may assume that the word $A$ is minimal. Since $X^kA^sX^{-k} = A^{sk}$ for any natural $k$, we have $|s^k| > 0.2|t^k|$ for every $k$ by Lemma 3.4 (2), whence $|s| \geq |t|$. By the symmetry, we also have $|t| \geq |s|$. The lemma is proved.

If a path $q$ has an $A$-periodic label, then one can select vertices along $q$ such that they divide $q$ into subpaths of length $\leq |A|$ and any two vertices of this system are connected by a subpath of $q$ labeled by a power of $A$. The vertices of this system will be called the phase vertices of $q$.

Lemma 5.4. Let $A$ be a minimal word. Then for every constant $C$, there is a natural number $m_0 = m_0(C, |A|)$ with the following property. If $|P_1Q_1P_2Q_2 = 1$ in $H$, where $|P_1|, |P_2| \leq C$, $Q_1$ and $Q_2^{-1}$ are $A$- or $A^{-1}$-periodic words starting with $A^{\pm 1}$ and $|Q_1| \geq m_0|A|$, then $P_1A^lP_2^{-1} = A^{\pm t}$ in $H$ for some $t \neq 0$.

Proof. Consider a reduced diagram with contour $p_1q_1p_2q_2$, where $Lab(p_i) \equiv P_i$, $Lab(q_i) \equiv Q_i$ ($i = 1, 2$). If $m_0$ is large enough, then by Lemma 3.4 (1), there is a phase vertex $o$ on $q_1$ connected with a phase vertex $o'$ on $q_2$ by a path of length $< 2800|A| + 2|A| < 3000|A|$. Moreover by Lemma 3.5 (2), we may assume that the number of such phase vertices is sufficiently large to guarantee that there are different phase vertices $o(1)$ and $o(2)$ of $q_1$ connected with some phase vertices of $q_2$ by paths having the same label $Z$. Therefore there is a closed path in $\Delta$ with label $Z^{-1}A^lZA^s$, where $t \neq 0$. So this product is trivial in $H$ and $s = \pm t$ by Lemma 5.3. Since the vertices $(p_1)_\pm$ are phase vertices two, we also have $Z^{-1} = A^kP_1A^t$ in $H$ for some integers $k$ and $l$. Therefore $(A^kP_1A^t)(A^kP_1A^t)^{-1}A^{\pm t} = 1$ in $H$, which proves the lemma.
For a free element $A$, we introduce its \textit{elementary closure} $E(A)$:

$$E(A) = \{ X \in H \mid XA^tX^{-1} = A^\pm t \text{ for some } t = t(X) \neq 0 \}$$

Clearly, $E(A)$ is a subgroup of $H$ containing the cyclic subgroup $\langle A \rangle$.

\textbf{Lemma 5.5.} The index $[E(A) : \langle A \rangle]$ is finite.

\textit{Proof.} Since $E(BAB^{-1}) = BE(A)B^{-1}$ for any $B$, we may assume that $A$ is a minimal word. Let $X \in E(A)$. Since $XA^tX^{-1} = A^\pm t$ for some $t > 0$, we have $XA^mX^{-1} = A^\pm m$ for any large $m$, which is a multiple of $t$. We consider a reduced disc diagram $\Delta$ for the equality $XA^{2m}X^{-1} = A^{\pm 2m}$ with contour $p_1q_1p_2q_2$, where $\text{Lab}(p_1) \equiv \text{Lab}(p_2)^{-1} \equiv X$, $\text{Lab}(q_1) \equiv A^{2m}$ and $\text{Lab}(q_2) \equiv A^{\pm 2m}$.

Let $o$ be the middle phase vertex of $q_1$ corresponding to the factorization $A^mA^m$ of $\text{Lab}(q_1)$. There is a path connecting $o$ with the middle phase vertex $o' \in q_2$ having label $A^{-1}X^{-1}A^\pm t = X^{-1}$ in $H$. On the other hand, if $m = m(|X|, |A|)$ is very large, then by Lemma 3.3 (2), the vertex $o$ can be connected with a phase vertex $o''$ on $q_2$ by a path having label $Y$ of length $< 3000|A|$. So the closed path $o - o' - o'' - o$ gives the equality $X^{-1}A^kY^{-1} = 1$ in $H$, where the section $o' - o''$ of $q_2$ is labeled by $A^k$.

Hence $X = A^kY^{-1}$ in $H$, and so every element $X \in E(A)$ belongs to a right coset $\langle A \rangle Y^{-1}$ with $|Y| < 3000|A|$. Therefore the set of right cosets of $\langle A \rangle$ in $E(A)$ is finite, as required. \hfill $\Box$

We define $E^+(A) = \{ X \in G \mid XA^tX^{-1} = A^t \text{ for some } t \neq 0 \}$. It is easy to see that $E^+(A)$ is a subgroup of $E(A)$ of index $\leq 2$.

\textbf{Lemma 5.6.} The subgroup $E^+(A)$ is infinite cyclic for a free word $A$.

\textit{Proof.} Let $X \in E^+(A) \setminus \{1\}$. Then for some $t > 0$, the subgroup $\langle A^t, X \rangle$ is infinite cyclic by Lemmas 5.2 and 5.1. Therefore $X$ has infinite order. Thus, the group $E^+(A)$ is torsion free and contains the cyclic group $\langle A \rangle$ of finite index by Lemma 5.3. It follows that $E^+(A)$ is cyclic itself by Schur’s theorem (See [R], 10.1.4.) \hfill $\Box$

\textbf{Lemma 5.7.} Let $A$ be a minimal word and $B \in H \setminus E(A)$. Then for sufficiently large $m$ the powers $A^m$ and $BA^mB^{-1}$ generate a free subgroup of rank 2 in $H$.

\textit{Proof.} Proving by contradiction, we have a nontrivial relation $r(A^m, BA^mB^{-1}) = 1$, which gives a reduced diagram $\Delta$ with contour $p_1q_1\ldots p_nq_n$, where $\text{Lab}(p_i) \equiv B^\pm 1$ and $\text{Lab}(q_i) \equiv A^{k_im}$ for some integer $k_i \neq 0$ ($i = 1, \ldots, n$).

If $m$ is large enough, we can apply Lemma 3.6 to $\Delta$ and obtain a subdiagram $\Gamma$ with boundary label of the form $P_1Q_1P_2Q_2$, where $\text{Lab}(P_1) \equiv B^\pm 1$, $|P_2| < 15|A|$, $Q_1$ and $Q_2$ are periodic words with periods $A^\pm 1$, $|Q_1| > m|A|/3$ and the words $Q_1$ and $Q_2^{-1}$ have prefixes $A$ or $A^{-1}$.

Then we may apply Lemma 5.3 to the equality $P_1Q_1P_2Q_2 = 1$ in $H$ if $m/4$ is greater than the constant $m_0$ depending on the lengths of the words $A$ and $B$. It says that $B \in E(A)$, a contradiction. \hfill $\Box$
6 Proofs of the theorems.

Proof of Theorem 1.3. Assume that $K$ is not conjugate to a subgroup of $G$. At first we also assume that $K$ has a free element $A$. We may replace $K$ by a conjugate subgroup and assume that $A$ is minimal. If $K$ has an element $B \notin E(A)$, then $K$ has a free subgroup of rank 2 by Lemma 5.7.

Suppose $K \leq E(A)$. The subgroup $E(A)$ has an infinite cyclic subgroup $E^+(A)$ of index $\leq 2$ by Lemma 5.6. But $E(A)$ has no elements of order $2$ in its center by Lemma 5.1 applied to $A$. Hence $E(A)$ is either an infinite cyclic or an infinite dihedral group. So is its subgroup $K$.

If $K$ is an infinite dihedral group, then it is generated by two involutions conjugated to some elements from $H$, since free elements have infinite order by Lemma 3.4 (1).

If $K$ is infinite cyclic, then $K \cap hGh^{-1} = \{1\}$ by Lemma 5.2.

Assume now that the subgroup $K$ has no free elements. Again, passing to a conjugate subgroup, we may assume that $K$ contains a nontrivial element $A$ from $G$. Arbitrary nontrivial element from $K \setminus \{A\}$ is equal in $H$ to $CBC^{-1}$, where $B \in G \setminus \{1\}$. Since $ACBC^{-1} \in K$, this product is conjugate to a non-trivial element $D$ from $G$.

Thus we have an annular diagram $\Delta$ for the conjugation of $ACBC^{-1}$ and $D$. Identifying the boundary subpaths labeled by $C$ and $C^{-1}$, we obtain a diagram $\Delta_1$ on a sphere with three holes bounded by closed paths $p_1$, $p_2$ and $p_3$ with labels $A, B$ and $D$, resp. The vertices $(p_1)_-$ and $(p_2)_-$ are connected by a simple path $x$ labeled by $C$.

Let $\Delta_2$ is a reduced singular since the words $A, B$ and $D$ are non-trivial in $H$. The vertices $(p_1)_-$ and $(p_2)_-$ can be connected by a simple path $y$ such that $\text{Lab}(y) = \text{Lab}(x) = C$ in $H$. It follows now from Lemma 1.1 (2), that $C$ is an $G$-word. Hence $CBC^{-1} \in G$, and so $K \leq G$. □

Proof of Theorem 1.4. The statement follows from Lemma 3.4 (1) because any free element of $H$ is conjugate to a minimal element. □

Proof of Theorem 1.5. Since a conjugation by a fixed element is a bi-Lipschitz mapping $H \to H$, we may assume that the element $g$ is presented by a minimal word $A$. Now consider a path $p$ from $g^n$ to $g^{-n}$ labeled by $A^{-2n}$. One can construct a reduced diagram $\Delta$ corresponding to the closed path $pq$. Since the distance in the Cayley graph between any two vertices of $pq$ do not exceed the distance between the corresponding vertices of $\partial \Delta$, we draw the required inequality from Lemma 3.10 □

Proof of Theorem 1.6. We will prove both parts of the theorem using simultaneous induction on $n$ with obvious base $n = 1$. Let us start with the second claim.

(2) Assume that we have a reluctant family $g_1, \ldots, g_n \in G$ and the elements $x_1, \ldots, x_n \in H$ such that $\prod_{i=1}^{n} x_i g_i x_i^{-1} = 1$. We can construct a diagram $\Delta$ with boundary label $\prod_{i=1}^{n} X_i P_i X_i^{-1}$, where the reduced words $P_1, \ldots, X_1, \ldots$ represent the elements $g_1, \ldots, x_1, \ldots$. Then for every $i$, we identify the subpaths of $\partial \Delta$ labeled by $X_i$ and $X_i^{-1}$ and obtain a diagram $\Delta_1$ on a sphere with $n$ holes. The $n$ contours $p_1, \ldots p_n$ of $\Delta_1$ are labeled by $P_1, \ldots P_n$. Now the path $x_i$ connects some vertex $o$ with the vertex $(p_i)_-$. If one moves $o$ to another point, say to $(p_i)_-$, then all the labels $X_i$ are multiplied from the left by the same word. Hence it suffices to prove that the label of every path $z_i$ connecting $(p_i)_-$ with $(p_i)_-$ is freely equal to a $G$-word.
Denote by $\Delta_2$ the reduced form of $\Delta_1$. The vertex $(p_i)_-$ can be connected in $\Delta_2$ with each $(p_i)_-$ by a path $u_i$ such that $\text{Lab}(u_i) = \text{Lab}(z_i)$ in $H$.

Note that the diagram $\Delta_2$ is non-singular. Indeed, otherwise we should have a simple closed path $q$ which bounds a subdiagram with $m$ holes, where $1 \leq m \leq n - 1$, and $\text{Lab}(q) = 1$ in the free group. But this means that for some $1 \leq i_1 \leq \cdots \leq i_m \leq n$, the product of conjugacy classes $\prod_{j=1}^{m} g_{i_j}^H$ contains the identity element. Then applying the statement (1) for $m < n$, we conclude that the product $\prod_{j=1}^{m} g_{i_j}^G$ also contains 1, which is impossible since the family $(g_1, \ldots, g_n)$ is reluctant.

Since the paths $u_i$ connect some pairs of entire vertices in $\Delta_2$, $\text{Lab}(u_i) \in G$ by Lemma 4.1 (2), as desired.

(1) Assuming that $\prod_{i=1}^{n} x_i g_i x_i^{-1} = 1$, we construct the diagrams $\Delta$, $\Delta_1$ and $\Delta_2$ as in the proof of the part (2).

If $\Delta_2$ is non-singular, then as in the proof of the claim (2), we obtain that the elements $x_1, \ldots, x_n$ belong to a coset $hG$ for some $h \in H$. Therefore $h^{-1} x_1, \ldots, h^{-1} x_n \in G$ and

$$\prod_{i=1}^{n} (h^{-1} x_i) g_i (h^{-1} x_i)^{-1} = h^{-1} (\prod_{i=1}^{n} x_i g_i x_i^{-1}) h = 1.$$  

If $\Delta_2$ is singular, then we have a simple closed path $q$ with trivial in the free group label, and $q$ cut up $\Delta_2$. One of the two obtained parts gives us an equality of the form $\prod_{j=1}^{m} y_j g_j y_j^{-1} = 1$ in $H$ for some $m \in [1, n - 1]$ and $y_j \in H$. Another one gives an equality $\prod_{j=1}^{n-m} z_j g_j z_j^{-1} = 1$, where $\{k_1, \ldots, k_{n-m}\} = \{1, \ldots, n\} \setminus \{i_1, \ldots, i_m\}$. In other words, both products $\prod_{j=1}^{m} g_{i_j}^H$ and $\prod_{j=1}^{n-m} g_{k_j}^H$ contain 1. By the statement (1), the same is true for the products $\prod_{j=1}^{m} g_{i_j}^G$ and $\prod_{j=1}^{n-m} g_{k_j}^G$. Hence the product of $n$ conjugacy classes $g_i^G$ of the group $G$ contains the identity too, as required.

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