POWERS OF A MATRIX AND COMBINATORIAL IDENTITIES

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Abstract. In this article we obtain a general polynomial identity in \( k \) variables, where \( k \geq 2 \) is an arbitrary positive integer. We use this identity to give a closed-form expression for the entries of the powers of a \( k \times k \) matrix. Finally, we use these results to derive various combinatorial identities.

1. Introduction

In [4], the second author had observed that the following ‘curious’ polynomial identity holds:

\[
\sum (-1)^i \binom{n-i}{i} (x+y)^{n-2i}(xy)^i = x^n + x^{n-1}y + \cdots + xy^{n-1} + y^n.
\]

The proof was simply observing that both sides satisfied the same recursion. He had also observed (but not published the result) that this recursion defines in a closed form the entries of the powers of a \( 2 \times 2 \) matrix in terms of its trace and determinant and the entries of the original matrix. The first author had independently discovered this fact and derived several combinatorial identities as consequences [2].

In this article, for a general \( k \), we obtain a polynomial identity and show how it gives a closed-form expression for the entries of the powers of a \( k \times k \) matrix. From these, we derive some combinatorial identities as consequences.

2. Main Results

Throughout the paper, let \( K \) be any fixed field of characteristic zero. We also fix a positive integer \( k \). The main results are the following two theorems:

Theorem 1. Let \( x_1, \ldots, x_k \) be independent variables and let \( s_1, \ldots, s_k \) denote the various symmetric polynomials in the \( x_i \)'s of degrees 1, 2, \ldots, \( k \) respectively. Then, in the polynomial ring \( K[x_1, \ldots, x_k] \), for each positive...
integer \( n \), one has the identity
\[
\sum_{r_1 + \cdots + r_k = n} x_1^{r_1} x_2^{r_2} \cdots x_k^{r_k} = \sum_{2i_2 + 3i_3 + \cdots + ki_k \leq n} c(i_2, \cdots, i_k, n) s_1^{n-2i_2-3i_3-\cdots-ki_k} \times (-s_2)^{i_2} s_3^{i_3} \cdots ((-1)^{k-1}s_k)^i_k,
\]
where
\[
c(i_2, \cdots, i_k, n) = \frac{(n - i_2 - 2i_3 - \cdots - (k - 1)i_k)!}{i_2! \cdots i_k!(n - 2i_2 - 3i_3 - \cdots - (ki_k)!)!}.
\]

**Theorem 2.** Suppose \( A \in M_k(K) \) and let
\[
T^k - s_1 T^{k-1} + s_2 T^{k-2} + \cdots + (-1)^k s_k I
\]
denote its characteristic polynomial. Then, for all \( n \geq k \), one has
\[
A^n = b_{k-1} A^{k-1} + b_{k-2} A^{k-2} + \cdots + b_0 I,
\]
where
\[
b_{k-1} = a(n - k + 1),
b_{k-2} = a(n - k + 2) - s_1 a(n - k + 1),
\]
and
\[
b_1 = a(n - 1) - s_1 a(n - 2) + \cdots + (-1)^{k-2} s_{k-2} a(n - k + 1),
b_0 = a(n) - s_1 a(n - 1) + \cdots + (-1)^{k-1} s_{k-1} a(n - k + 1)
= (-1)^{k-1} s_k a(n - k).
\]
and
\[
a(n) = c(i_2, \cdots, i_k, n) s_1^{n-i_2-2i_3-\cdots-(k-1)i_k} (-s_2)^{i_2} s_3^{i_3} \cdots ((-1)^{k-1} s_k)^i_k,
\]
with
\[
c(i_2, \cdots, i_k, n) = \frac{(n - i_2 - 2i_3 - \cdots - (k - 1)i_k)!}{i_2! \cdots i_k!(n - 2i_2 - 3i_3 - \cdots - (ki_k)!)!}.
\]
as in Theorem 1.

**Proof of Theorems 1 and 2.** In Theorem 1, if \( a(n) \) denotes either side, it is straightforward to verify that
\[
a(n) = s_1 a(n - 1) - s_2 a(n - 2) + \cdots + (-1)^{k-1} s_k a(n - k).
\]

Theorem 2 is a consequence of Theorem 1 on using induction on \( n \).

\[ \square \]

The special cases \( k = 2 \) and \( k = 3 \) are worth noting for it is easier to derive various combinatorial identities from them.
Corollary 1. (i) Let $A \in M_3(K)$ and let $X^3 = tX^2 - sX + d$ denote the characteristic polynomial of $A$. Then, for all $n \geq 3$,

\begin{equation}
A^n = a_{n-1}A + a_{n-2} \text{Adj}(A) + (a_n - ta_{n-1})I,
\end{equation}

where

\[ a_n = \sum_{2i + 3j \leq n} (-1)^i \binom{i+j}{j} \binom{n-i-2j}{i+j} \theta^{n-2i-3j} s^i d^j \]

for $n > 0$ and $a_0 = 1$.

(ii) Let $B \in M_2(K)$ and let $X^2 = tX - d$ denote the characteristic polynomial of $B$. Then, for all $n \geq 2$,

\[ B^n = b_n I + b_{n-1} \text{Adj}(B) \]

for all $n \geq 2$, where

\[ b_n = \sum \binom{n-i}{i} (-1)^i t^{n-2i} d^i. \]

Corollary 2. Let $\theta \in K$, $B \in M_2(K)$ and $t$ denote the trace and $d$ the determinant of $B$. We have the following identity in $M_2(K)$:

\[(a_{n-1} - \theta a_{n-2})B + (a_n - (\theta + t)a_{n-1} + \theta a_{n-2}t)I = y_{n-1}B + (y_n - ty_{n-1})I,\]

where

\[ a_n = \sum_{2i + 3j \leq n} (-1)^i \binom{i+j}{j} \binom{n-i-2j}{i+j} (\theta + t)^{n-2i-3j} (\theta t + d)^i (\theta d)^j \]

and

\[ y_n = \sum \binom{n-i}{i} (-1)^i t^{n-2i} d^i. \]

In particular, for any $\theta \in K$, one has

\[ b_n - (\theta + 1)b_{n-1} + \theta b_{n-2} = 1, \]

where

\[ b_n = \sum_{2i + 3j \leq n} (-1)^i \binom{i+j}{j} \binom{n-i-2j}{i+j} (\theta + 2)^{n-2i-3j} (1 + 2\theta)^i (2\theta)^j. \]

Corollary 3. The numbers $c_n = \sum_{2i+3j=n}(-1)^{i+j}2^i 3^j$ satisfy

\[ c_n + c_{n-1} - 2c_{n-2} = 1. \]

Proof. This is the special case of Corollary 2 where we take $\theta = -2$. Note that the sum defining $c_n$ is over only those $i,j$ for which $2i + 3j = n$. \qed

Note than when $k = 3$, Theorem 1 can be rewritten as follows:
Theorem 3. Let $n$ be a positive integer and $x, y, z$ be indeterminates. Then

\[(2.2) \sum_{2i+3j\leq n} (-1)^i \binom{i+j}{i} \binom{n-i-2j}{i+j} (x+y+z)^{n-2i-3j} (xy+yz+zx)^i (xyz)^j \]

\[= \frac{x y (x^n+1 - y^n+1) - x z (x^{n+1} - z^{n+1}) + y z (y^{n+1} - z^{n+1})}{(x-y) (x-z) (y-z)}. \]

Proof. In Corollary 1, let

\[A = \begin{pmatrix} x+y+z & 1 & 0 \\ -xy-xz-yz & 0 & 1 \\ xy & 0 & 0 \end{pmatrix}. \]

Then $t = x + y + z$, $s = xy + xz + yz$ and $d = xyz$. It is easy to show (by first diagonalizing $A$) that the $(1, 2)$ entry of $A^n$ equals the right side of (2.2), with $n + 1$ replaced by $n$, and the $(1, 2)$ entry on the right side of (2.1) is $a_{n-1}$. □

Corollary 4. Let $x$ and $z$ be indeterminates and $n$ a positive integer. Then

\[\sum_{2i+3j\leq n} (-1)^i \binom{i+j}{i} \binom{n-i-2j}{i+j} (2x+z)^{n-2i-3j} (x^2+2xz)^i (x^2z)^j \]

\[= \frac{x^{2+n} + nx^{1+n} (x-z) - 2x^{1+n} z + z^{2+n}}{(x-z)^2}. \]

Proof. Let $y \rightarrow x$ in Theorem 3. □

Some interesting identities can be derived by specializing the variables in Theorem 1. For instance, in [5], it was noted that Binet’s formula for the Fibonacci numbers is a consequence of Theorem 1 for $k = 2$. Here is a generalization.

Corollary 5. (Generalization of Binet’s formula)

Let the numbers $F_k(n)$ be defined by the recursion

\[F_k(0) = 1, F_k(r) = 0, \forall r < 0, \]

\[F_k(n) = F_k(n-1) + F_k(n-2) + \cdots + F_k(n-k). \]

Then, we have

\[F_k(n) = \sum_{2i_2 + \cdots + ki_k \leq n} \frac{(n-i_2-2i_3-\cdots-(k-1)i_k)!}{i_1!i_2!\cdots i_k!(n-2i_2-3i_3-\cdots-ki_k)!}. \]

Further, this equals $\sum_{r_1+\cdots+r_k=n} \lambda_1^{r_1} \cdots \lambda_k^{r_k}$ where $\lambda_i, 1 \leq i \leq k$ are the roots of the equation $T^k - T^{k-1} - T^{k-2} - \cdots - 1 = 0$.

Proof. The recursion defining $F_k(n)$’s corresponds to the case $s_1 = -s_2 = \cdots = (-1)^{k-1}s_k = 1$ of the theorem. □
Corollary 6.

\[
\sum c(i_2, \cdots, i_k, n) k^n \prod_{j=2}^{k} \left( (-1)^{j-1} k \cdot \binom{k}{j} \right)^{i_j} = \binom{n + k - 1}{k}.
\]

where

\[
c(i_2, \cdots, i_k, n) = \frac{(n - i_2 - 2i_3 - \cdots - (k - 1)i_k)!}{i_2! \cdots i_k!(n - 2i_2 - 3i_3 - \cdots - ki_k)!}.
\]

Proof. Take \( x_i = 1 \) for all \( i \) in Theorem 1. The left side of Theorem 1 is simply the sum \( \sum_{r_1+\cdots+r_k=n} 1 \).

From Theorem 3 we have the following binomial identities as special cases.

Proposition 1. (i) Let \( \lambda \) be the unique positive real number satisfying \( \lambda^3 = \lambda + 1 \). Let \( x, y \) denote the complex conjugates such that \( xy = \lambda, x + y = \lambda^2 \), and let \( z = -\frac{1}{\lambda} \). Then,

\[
\sum_{2i+3j \leq n} (-1)^j \binom{n-2j}{j} = \sum_{r+s+t=n} x^r y^s z^t
\]

\[
= \frac{xy(x^{n+1} - y^{n+1}) - xz(x^{n+1} - z^{n+1}) + yz(y^{n+1} - z^{n+1})}{(x-y)(x-z)(y-z)}.
\]

(ii)

\[
\sum_{2i+3j \leq n} (-1)^j \binom{i+j}{j} \binom{n-i-2j}{i+j} = [n + 2)/2].
\]

(iii)

\[
\sum \binom{n-2j}{j} (-4)^j 3^{n-3j} = \frac{(3n + 4)2^{n+1} + (-1)^n}{9}.
\]

(iv)

\[
\sum \binom{n-2j}{j} 3^{n-3j}(-2)^j
\]

\[
= \frac{(1 + \sqrt{3})^{n+1} - (1 - \sqrt{3})^{n+1}}{2\sqrt{3}} + \frac{(1 + \sqrt{3})^{n+1} + (1 - \sqrt{3})^{n+1}}{6} - \frac{1}{3}.
\]

3. Commuting Matrices

In this section we derive various combinatorial identities by writing a general \( 3 \times 3 \) matrix \( A \) as a product of commuting matrices.

Proposition 2. Let \( A \) be an arbitrary \( 3 \times 3 \) matrix with characteristic equation \( x^3 - tx^2 + sx - d = 0, \) \( d \neq 0 \). Suppose \( p \) is arbitrary, with
Then integer. Then

$$A^n = \left( \frac{pd}{p^3 + p^2t + sp + d} \right)^n \sum_{r=0}^{3n} \sum_{j=0}^{n} \sum_{k=0}^{n} \binom{n}{r} \binom{n}{j} \binom{n}{k} \left( \frac{j}{r-j-k} \right) \times \left( \frac{-p(p+t)^2}{d} \right)^j \left( \frac{-(p + t)}{p} \right)^k \left( \frac{-A}{p+t} \right)^r.$$

Proof. This follows from the identity

$$A = \frac{-1}{p^3 + p^2t + sp + d} (pA^2 - Ap(p + t) - dI) (A + pI),$$

after raising both sides to the \(n\)-th power and collecting powers of \(A\). Note that the two matrices \(pA^2 - Ap(p + t) - dI\) and \(A + pI\) commute. □

**Corollary 7.** Let \(p, x, y\) and \(z\) be indeterminates and let \(n\) be a positive integer. Then

$$\sum_{r=0}^{3n} \sum_{j=0}^{n} \sum_{k=0}^{n} \binom{n}{r} \binom{n}{j} \binom{n}{k} \left( \frac{j}{r-j-k} \right) (-1)^{j+k+r} \left( \frac{p(p+x+y+z)^2}{xyz} \right)^j \times \left( \frac{p+x+y+z}{p} \right)^k \frac{x y (x^r - y^r) - x z (x^r - z^r) + y z (y^r - z^r)}{(p + x + y + z)^r}$$

$$= (x y (x^n - y^n) - x z (x^n - z^n) + y z (y^n - z^n)) \times \left( \frac{p^3 + p^2 (x + y + z) + p (x y + x z + y z) + x y z}{p x y z} \right)^n.$$

Proof. Let \(A\) be the matrix from Theorem 3 and compare (1,1) entries on both sides of (3.1). □

**Corollary 8.** Let \(p, x\) and \(z\) be indeterminates and let \(n\) be a positive integer. Then

$$\sum_{r=0}^{3n} \sum_{j=0}^{n} \sum_{k=0}^{n} \binom{n}{r} \binom{n}{j} \binom{n}{k} \left( \frac{j}{r-j-k} \right) (-1)^{j+k+r} \left( \frac{p(p+2x+z)^2}{x^2z} \right)^j \times \left( \frac{p+2x+z}{p} \right)^k \frac{r x^{1+r} - x^r z - r x^r z + z^{1+r}}{(p+2x+z)^r}$$

$$= (n x^{1+n} - x^n z - n x^n z + z^{1+n}) \times \left( \frac{p^3 + p^2 (2x + z) + p (x^2 + 2xz) + x^2 z}{p x^2 z} \right)^n.$$

Proof. Divide both sides in the corollary above by \(x - y\) and let \(y \to x\). □

**Corollary 9.** Let \(p\) and \(x\) be indeterminates and let \(n\) be a positive integer. Then
\[
\sum_{r=0}^{3n} \sum_{j=0}^{n} \sum_{k=0}^{n} \binom{n}{j} \binom{n}{k} \binom{j}{r-j-k} (-1)^{j+k+r} \left( \frac{p(p + 3x)^2}{x^3} \right)^r x^{r+1}
\]
\[
\times \left( \frac{p + 3x}{p} \right)^k r (1 + r) x^{-1+r} = \frac{n (1 + n) x^{-1+n}}{2} \left( \frac{(p + x)^3}{p x^3} \right)^n.
\]

**Proof.** Divide both sides in the corollary above by \((x-z)^2\) and let \(z \to x\). □

**Corollary 10.** Let \(p\) be an indeterminate and let \(n\) be a positive integer. Then
\[
\sum_{r=0}^{3n} \sum_{j=0}^{n} \sum_{k=0}^{n} \binom{n}{j} \binom{n}{k} \binom{j}{r-j-k} (-1)^{j+k+r} p^{j-k}(p+3)^{2j+k-r} \frac{(1 + r)}{2} r
\]
\[
= \frac{n (1 + n) (p + 1)^{3n}}{p^n}.
\]

**Proof.** Replace \(p\) by \(px\) in the corollary above and simplify. □

Various combinatorial identities can be derived from Theorem 3 by considering matrices \(A\) such that particular entries in \(A^n\) have a simple closed form. We give four examples.

**Corollary 11.** Let \(n\) be a positive integer.

(i) If \(p \neq 0, -1\), then
\[
\sum_{r=0}^{3n} \sum_{j=0}^{n} \sum_{k=0}^{n} \binom{n}{j} \binom{n}{k} \binom{j}{r-j-k} (-1)^{j+k+r} p^{j-k}(p+3)^{2j+k-r} r = \frac{n (1 + p)^{3n}}{p^n}.
\]

(ii) Let \(F_n\) denote the \(n\)-th Fibonacci number. If \(p \neq 0, -1, \phi\) or \(1/\phi\) (where \(\phi\) is the golden ratio), then
\[
\sum_{r=0}^{3n} \sum_{j=0}^{n} \sum_{k=0}^{n} \binom{n}{j} \binom{n}{k} \binom{j}{r-j-k} (-1)^{k+r} p^{j-k}(p+2)^{2j+k-r} F_r
\]
\[
= F_n \frac{(1 + p)^n (-1 + p + p^2)^n}{(-p)^n}.
\]

(iii) If \(p \neq 0, -1\) or \(-2\), then
\[
\sum_{r=0}^{3n} \sum_{j=0}^{n} \sum_{k=0}^{n} \binom{n}{j} \binom{n}{k} \binom{j}{r-j-k} (-1)^{j+k+r} p^{j-k}(p+4)^{2j+k-r} 2^{-j} (2^r - 1)
\]
\[
= (2^n - 1) \left( \frac{(1 + p)^2 (p + 2)}{2p} \right)^n.
\]
(iv) If \( p \neq 0, -1, -g \) or \(-h\) and \( gh \neq 0\), then

\[
\sum_{r=0}^{3n} \sum_{j=0}^{n} \sum_{k=0}^{n} \binom{n}{j} \binom{n}{k} \binom{r}{r-j-k} (-1)^{j+k+r} p^{j-k} \\
\times (p + 1 + g + h)^{2j+k-r} g^r + h^r \frac{1}{(gh)^j} = (g^n + h^n) \left( \frac{(1 + p)(g + p)(h + p)}{ghp} \right)^n.
\]

Proof. The results follow from considering the \((1, 2)\) entries on both sides in Theorem 3 for the matrices

\[
\begin{pmatrix}
1 & 1 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad
\begin{pmatrix}
1 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad
\begin{pmatrix}
3 & 1 & 0 \\
-2 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad
\begin{pmatrix}
g + h \\
2 \\
1 \\
\frac{g - h}{4} \\
2 \\
1
\end{pmatrix},
\]

respectively. \(\square\)

4. A Result of Bernstein

In [1] Bernstein showed that the only zeros of the integer function

\[
f(n) := \sum_{j \geq 0} (-1)^j \binom{n-2j}{j}
\]

are at \( n = 3 \) and \( n = 12 \). We use Corollary [1] to relate the zeros of this function to solutions of a certain cubic Thue equation and hence to derive Bernstein’s result.

Let

\[
A = \begin{pmatrix}
1 & 1 & 0 \\
0 & 0 & 1 \\
-1 & 0 & 0
\end{pmatrix}.
\]

With the notation of Corollary [1], \( t = 1, s = 0, d = -1 \), so that

\[
a_n = \sum_{3j \leq n} (-1)^j \binom{n-2j}{j} = f(n),
\]

and, for \( n \geq 4 \),

\[
A^n = f(n-2)A^2 + (f(n) - f(n-2))A + (f(n) - f(n-1))I
\]

\[
= \begin{pmatrix}
f(n) & f(n-1) & f(n-2) \\
-f(n-2) & -f(n-3) & -f(n-4) \\
-f(n-1) & -f(n-2) & -f(n-3)
\end{pmatrix}.
\]

The last equality follows from the fact that \( f(k+1) = f(k) - f(k-2) \), for \( k \geq 2 \).
Now suppose $f(n - 2) = 0$. Since the recurrence relation above gives that $f(n - 4) = -f(n - 1)$ and $f(n) = f(n - 1) - f(n - 3)$, it follows that

$$(-1)^n = \det(A^n) = \begin{vmatrix}
  f(n - 1) - f(n - 3) & f(n - 1) & 0 \\
  0 & -f(n - 3) & f(n - 1) \\
  -f(n - 1) & 0 & -f(n - 3)
\end{vmatrix}$$

$$= -f(n - 1)^3 - f(n - 3)^3 + f(n - 1)f(n - 3)^2.$$  

Thus $(x, y) = \pm (f(n - 1), f(n - 3))$ is a solution of the Thue equation

$$x^3 + y^3 - x y^2 = 1.$$  

One could solve this equation in the usual manner of finding bounds on powers of fundamental units in the cubic number field defined by the equation $x^3 - x + 1 = 0$. Alternatively, the Thue equation solver in PARI/GP [3] gives unconditionally (in less than a second) that the only solutions to this equation are

$$(x, y) \in \{ (4, -3), (-1, 1), (1, 0), (0, 1), (1, 1) \},$$

leading to Bernstein’s result once again.

References

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