NORMAL REDUCTION NUMBER OF NORMAL SURFACE SINGULARITIES

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Abstract. Let \((X, o)\) be a complex analytic normal surface singularity and let \(O_{X, o}\) be its local ring. We investigate the normal reduction number of \(O_{X, o}\) and related numerical analytical invariants via resolutions \(\tilde{X} \to X\) of \((X, o)\) and cohomology groups of different line bundles \(L \in \text{Pic}(\tilde{X})\).

The normal reduction number is the universal optimal bound from which powers of certain ideals have stabilization properties. Here we combine this with stability properties of the iterated Abel maps. Some of the main results provide topological upper bounds for both stabilization properties.

The present note was partially motivated by the open problems formulated in [O19]. Here we answer several of them.

1. Introduction

Let \((X, o)\) be a complex analytic normal surface singularity and let \(O_{X, o}\) be its local ring. Our goal is to investigate certain properties of the normal reduction number of \(O_{X, o}\) via certain numerical invariants associated with resolutions \(\tilde{X} \to X\) of \((X, o)\) and with the cohomology groups of different line bundles \(L \in \text{Pic}(\tilde{X})\).

1.1. Some ring–theoretical invariants of \(O_{X, o}\). (See e.g. [HS06, O19].) Let \(I\) be an \(m\)–primary ideal of \(O_{X, o}\), where \(m\) is the maximal ideal. The integral closure \(\overline{I}\) of \(I\) is the ideal consisting of all solutions of equations of type \(z^n + c_1 z^{n-1} + \cdots + c_{n-1} z + c_n = 0\) with coefficients \(c_i \in I^i\). Then \(I \subseteq \overline{I} \subseteq \sqrt{I}\). We say that \(I\) is integrally closed if \(I = \overline{I}\). In the sequel we will assume that \(I = \overline{I}\).

Recall that an ideal \(J \subseteq I\) is called a ‘reduction’ of \(I\) if \(I\) is integral over \(J\), or, equivalently, \(I^{r+1} = J I^r\) for a certain \(r\). An ideal \(Q \subseteq I\) is called ‘minimal reduction’ of \(I\) if \(Q\) is minimal among the reductions of \(I\). In the case of \(m\)–primary (integrally closed) ideals \(I\) of \(O_{X, o}\), a minimal reduction is a parameter ideal.

For any minimal reduction \(Q\) of \(I\), we have that \(\overline{I}^{r+1} = Q \overline{I}^r\) for all large \(n\). We define the normal reduction number \(\overline{r}(I)\) by \(\overline{r}(I) := \min\{r \mid \overline{I}^{r+1} = Q \overline{I}^r\}\); this integer does not depend on the choice of \(Q\) (cf. [HS7, O19]). The normal reduction number of \((X, o)\) is defined by

\[\overline{r}(X, o) := \max\{\overline{r}(I) \mid I \subset O_{X, o}, \sqrt{I} = m, I = \overline{I}\}.\]

Since the (normal) filtration \(\{\overline{I}^n\}\) contains several key information about the blow-up of \((X, o)\) (along \(I\)), we expect that the normal reduction numbers should encode important information about the ring \(O_{X, o}\) and also about the resolution spaces of \((X, o)\). However, the structure of the set of reductions is clarified only for very special cases (see e.g. [OWY19b, O19]).

Some major questions are the following:

(i) For any fixed \((X, o)\) find the possible values of \(\overline{r}(I)\) and \(\overline{r}(X, o)\). (The expectation is that in general the answer depends essentially on the analytic type of \((X, o)\).)

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(ii) Fix a topological type of a singularity. Show that the integers $\bar{r}(I)$ and $\bar{r}(X, o)$, associated with all the analytic structures supported on the fixed topological type, can universally be bounded (from above) by a topological invariant.

(iii) Find an optimal topological upper bound of (ii) (which is realized by a certain analytic structure).

1.2. Some singularity–theoretical invariants of $(X, o)$. Fix a resolution $\phi : (\tilde{X}, E) \to (X, o)$ and an effective cycle $Z \geq E$ supported on the exceptional curve $E$, and $l' \in H^2(\tilde{X}, \mathbb{Z})$ a Chern class with $-l', E_o \leq 0$ for any irreducible component $E_o$ of $E$ (denoted as $-l' \in S'$, cf. \ref{2}).

If $\mathcal{L} \in \text{Pic}^0(Z)$ is a line bundle on $Z$ with Chern class $l'$ and without fixed components then $h^1(Z, \mathcal{L}^n) \geq h^1(Z, \mathcal{L}^{n+1})$ for $n \geq 0$ (cf. section \ref{1}). Hence the sequence $n \mapsto h^1(Z, \mathcal{L}^n)$ is non–increasing and it is constant for $n \gg 0$. For such an $\mathcal{L}$ we define $n_0(Z, \mathcal{L})$ as the smallest integer $n$ such that $h^1(Z, \mathcal{L}^n) = h^1(Z, \mathcal{L}^{n+1})$. In fact, $n_0(Z, \mathcal{L}) = \min\{n \geq 0 : h^1(Z, \mathcal{L}^n) = h^1(Z, \mathcal{L}^m)\}$ as well, see e.g. [O17, Lemma 3.6].

Let us consider next another related stability problem as well. For fixed $Z$ and $l'$ as above, one can consider $\text{EC}_a(Z)$, the space of effective Cartier divisors with Chern class $l' \in L'$ and the Abel map $\text{c}^l(Z) : \text{EC}_a(Z) \to \text{Pic}^0(Z)$. Note that $\mathcal{L} \in \text{im}(\text{c}^l(Z))$ if and only if $\mathcal{L}$ has no fixed components, cf. \ref{1.1.6}.

Then one shows that the sequence $n \mapsto \dim(\text{im}(\text{c}^n(Z)))$ is a non–decreasing and it is bounded from above by $h^1(O_Z)$, hence it must stabilise, cf. \ref{1.1.8}. Let $n'_0(Z, l')$ be the smallest integer $n$ such that $\dim(\text{im}(\text{c}^n(Z))) = \dim(\text{im}(\text{c}^{n+1}(Z)))$. Again, $n'_0(Z, l') = \min\{n \geq 0 : \dim(\text{im}(\text{c}^n(Z))) = \dim(\text{im}(\text{c}^{n+1}(Z)))\}$, $\forall m > n$, cf. Remark \ref{1.1.6}.

The integers $n_0(Z, \mathcal{L})$ and $n'_0(Z, l')$ are important invariants of the singularity. If $Z \gg 0$ then they are independent of $Z$, and they will be denoted by $n_0(\mathcal{L})$ and $n'_0(l')$ respectively.

In Lemma \ref{1.1.8} we show that for any $Z$ and any $-l' \in S'$ as above

$$n'_0(Z, l') \leq \min\{n_0(Z, \mathcal{L}) : \mathcal{L} \in \text{im}(\text{c}^n(Z))\}.$$

In parallel to the questions formulated in \ref{1.1.6} we can formulate the very same type of questions, but now for the integers $h^1(Z, \mathcal{L})$, $n_0(Z, \mathcal{L})$ and $n'_0(Z, l')$. (Note that one of the goals of the theory of Abel maps is also to understand the possible values of $h^1(Z, \mathcal{L})$ and the corresponding stratification in $\text{Pic}(Z)$ induced by $h^1(Z, \mathcal{L})$, cf. \ref{NNI}, \ref{NNII}, \ref{NNIII}.)

1.3. The connection between the two approaches. By a result of Lipman \cite{Lipman69}, if $I$ is an integrally closed $m$–ideal then there exist a resolution $\phi : \tilde{X} \to X$ and an integral cycle $l \in S'$ on $\tilde{X}$ such that $I = (\phi_* \mathcal{O}_{\tilde{X}}(-l))_o = H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(-l))$ and $I \mathcal{O}_{\tilde{X}} = \mathcal{O}_{\tilde{X}}(-l)$ (that is, $\mathcal{O}_{\tilde{X}}(-l)$ is generated by global sections). Then we can define for every integer $n \geq 0$ a non-increasing chain of integers $q(nI) := h^1(\mathcal{O}_{\tilde{X}}(-nl))$, where $q(0I) := p_g(X, o)$ is the geometric genus of $(X, o)$. It turns out that $q(nI)$ is independent of the representation of $I$ as $\mathcal{O}_{\tilde{X}}(-l)$ (cf. [OWY13, \S3]). Furthermore, we have $0 \leq q(I) \leq p_g(X, o)$ and $\bar{r}(I) = \min\{n \in \mathbb{Z}_{\geq 0} \mid q((n-1)I) = q(nI)\}$ (cf. [OWY13a]).

This makes the bridge between the normal reduction number of ideals and the study of the cohomology of line bundles on resolution spaces. In particular, $\bar{r}(I) - 1 = n_0(\mathcal{O}_{\tilde{X}}(-l))$.

As we mentioned above, the normal reduction number $\bar{r}(X, o)$ plays a key role in the algebraic study of the ring $\mathcal{O}_{X, o} = (\phi_* \mathcal{O}_{\tilde{X}})_o$. Here are some exemplifications of certain key bounds:

- The natural homomorphism $\phi_* \mathcal{O}_{\tilde{X}}(-nl) \otimes \phi_* \mathcal{O}_{\tilde{X}}(-l) \to \phi_* \mathcal{O}_{\tilde{X}}(-(n + 1)l)$ is surjective for $n \geq \bar{r}(X, o)$. That is, the graded algebra $\bigoplus_{n \geq 0} \phi_* \mathcal{O}_{\tilde{X}}(-nl)$ is generated by parts of degree $\leq \bar{r}(X, o)$. 

Theorem C. Under the assumptions of Theorem B we also have on a fixed topological type).

The function \( \varphi(n) := \dim_{\mathbb{C}} H^0(\mathcal{O}_X)/H^0(\mathcal{O}_X(-nl)) \) is a polynomial function of \( n \) for \( n \geq \bar{r}(X, o) \); in fact, \( \varphi(n) = \chi(\mathcal{O}_n) + h^1(\mathcal{O}_X) - h^1(\mathcal{O}_X(-nl)) \) by Kato’s Riemann-Roch Theorem.

In [OWY15] it is proved that \( \bar{r}(X, o) = 1 \) if and only if \( (X, o) \) is rational. Furthermore, the third author proved that \( \bar{r}(X, o) = 2 \) for an elliptic singularity [O17].

1.4. The new results. The present note was partially motivated by the open problems formulated in [O19]. Here we answer several of them. In parallel we answer also some of the questions formulated in the previous subsections. The main results are the following.

**Theorem A.** Fix a complex analytic normal surface singularity \((X, o)\) such that all the irreducible exceptional curves (in some resolution) are rational. Fix also an arbitrary integer \(0 \leq q \leq p_g(X, o)\).

Then there exists a resolution \( \tilde{X} \to X \) and an effective integral cycle \( l > 0 \) such that \( \mathcal{O}_X(-l) \) is base point free and \( h^1(\mathcal{O}_X(-l)) = q \).

This answers Conjecture 2.8 of [O19]. For a slightly more general statement see Remark 3.3.

**Theorem B.** Assume that \((X, o)\) is a normal surface singularity whose link is a rational homology sphere and let us fix a resolution \( \tilde{X} \to X \) and an effective cycle \( Z \subset \tilde{X} \) such that \( \chi_{\tilde{X}} = 1 \) and \( \bar{r}(X, o) \leq 2 \).

This answers Problem 3.13 of [O19].

Since \( \sum_{l>0} \chi(l) \) is a topological invariant independent of the choice of the resolution, the above inequality provides a topological upper bound for \( \bar{r}(X, o) \) (valid for any analytic structure supported on a fixed topological type).

The analogue of Theorem B for the images of the Abel maps is the following.

**Theorem C.** Under the assumptions of Theorem B we also have \( n_0(Z, l') \leq 1 \) and \( \bar{r}(X, o) = 2 \).

Finally, we provide a criterion which guarantees that \( n_0(Z, \mathcal{L}) = \sum_{l>0} \chi(l) \).

In this way we are able to construct non-elliptic singularities \((X, o)\) with \( \bar{r}(X, o) = 2 \); this answers negatively Problem 3.12 of [O19] (which asked whether \( \bar{r}(X, o) = 2 \) characterizes the elliptic germs).

This also shows that if we fix a positive integer \( k \geq 2 \), the classification of those analytic structures which satisfy \( \bar{r}(X, o) = k \) can be a very hard task.

1.5. Most of the techniques and the guiding ideas of the proofs are based on the theory of Abel maps developed by the first two authors in [NN1] [NN2] [NN3].

2. Preliminaries

2.1. The resolution. Let \((X, o)\) be the germ of a complex analytic normal surface singularity, and let us fix a good resolution \( \phi : \tilde{X} \to X \) of \((X, o)\). We denote the exceptional curve \( \phi^{-1}(0) \) by \( E \), and let \( \cup_{v \in \mathcal{V}} E_v \) be its irreducible components. Set also \( E_I := \sum_{v \in I} E_v \) for any subset \( I \subset \mathcal{V} \).

For a cycle \( l = \sum n_v E_v \), we write \( |l| = \cup_{n_v \neq 0} E_v \) for its support. For more details see [N07] [N12] [N99].

2.2. Topological invariants. Let \( \Gamma \) be the dual resolution graph associated with \( \phi \); it is a connected graph. Then \( M := \partial \tilde{X} \) can be identified with the link of \( (X, o) \), it is also an oriented plumbed 3–manifold associated with \( \Gamma \). We use the same notation \( \mathcal{V} \) for the set of vertices as well. Recall that \( M \) is a rational homology sphere, if and only if \( \Gamma \) is a tree and all the curves \( E_v \) are rational.

\( L := H_2(\tilde{X}, \mathbb{Z}) \), endowed with a negative definite intersection form \( I = (\ , \ ) \), is a lattice. It is freely generated by the classes of 2–spheres \( \{E_v\}_{v \in \mathcal{V}} \). The dual lattice \( L' := H^2(\tilde{X}, \mathbb{Z}) \) is generated by the (anti) dual classes \( \{E^*_v\}_{v \in \mathcal{V}} \) defined by \( (E^*_v, E_u) = -\delta_{uv} \), the opposite of the Kronecker symbol. The
intersection form embeds \( L \) into \( L' \). Then \( \text{Tors}(H_1(M, \mathbb{Z})) \cong L'/L \), abridged by \( H \). Usually one also identifies \( L' \) with those rational cycles \( l' \in L \otimes \mathbb{Q} \) for which \( (l', L) \in \mathbb{Z} \), or, \( L' = \text{Hom}_\mathbb{Z}(L, \mathbb{Z}) \).

All the \( E_v \)-coordinates of any \( E'_v \) are strict positive. We define the Lipman cone as \( S' := \{ l' \in L' : (l', E_v) \leq 0 \) for all \( v \}. \) As a monoid, it is generated over \( \mathbb{Z}_{\geq 0} \) by \( \{ E'_v \}_v \).

We set \( \chi(l') = -(l', L - Z_K)/2 \), where \( Z_K \in L' \) is the (anti)canonical cycle identified by adjunction formulae \( -(Z_K + E_v, E_v) = 2g_v - 2 \) for all \( v \), where \( g_v \) is the genus of \( E_v \). By Riemann-Roch theorem \( \chi(l) = \chi(O_1) \) for any \( l \in L_{>0}. \) (Here \( l > 0 \) means that \( l = \sum_v n_v E_v \) with all \( n_v \geq 0 \) and \( l \neq 0 \).)

2.3. Analytic invariants. The group \( \text{Pic}(\tilde{X}) \) is the group \( H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}^*) \) of isomorphism classes of analytic line bundles on \( \tilde{X} \). It appears in the exact sequence

\[
0 \to \text{Pic}^0(\tilde{X}) \to \text{Pic}(\tilde{X}) \xrightarrow{c_1} L' \to 0,
\]

where \( c_1 \) denotes the first Chern class. Here \( \text{Pic}^0(\tilde{X}) \cong H^1(\tilde{X}, \mathcal{O}_{\tilde{X}})/H^1(E, \mathbb{Z}) \), where \( H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) \cong \mathbb{C}^{p_g} \), \( p_g = p_g(X, o) \) being the geometric genus of \( (X, o) \). \( (X, o) \) is called rational if \( p_g = 0 \). Artin in \( \text{[A62, A66]} \) characterized rationality topologically via the graphs; such graphs are called ‘rational’.

By this criterion, \( \Gamma \) is rational if and only if \( \chi(l) \geq 1 \) for any effective non–zero cycle \( l \in L_{>0}. \) (Recall also that the link of any rational singularity is a rational homology sphere.)

2.3.2. Similarly, if \( Z \in L_{>0} \) is an effective non–zero integral cycle such that \( Z \geq E \), \( \mathcal{O}_Z^* \) denotes the sheaf of units of \( \mathcal{O}_Z \), then \( \text{Pic}(Z) = H^1(Z, \mathcal{O}_Z^*) \) is the group of isomorphism classes of invertible sheaves on \( Z \). It appears in the exact sequence

\[
0 \to \text{Pic}^0(Z) \to \text{Pic}(Z) \xrightarrow{c_1} L' \to 0,
\]

where \( \text{Pic}^0(Z) \cong H^1(Z, \mathcal{O}_Z)/H^1(E, \mathbb{Z}). \) If \( Z \geq Z_1 \) then there are natural restriction maps \( \text{Pic}(\tilde{X}) \to \text{Pic}(Z) \to \text{Pic}(Z_1) \). Similar restrictions are defined at \( \text{Pic}^0 \) level too. These restrictions are homomorphisms of the exact sequences \( 2.3.1 \) and \( 2.3.3 \).

We also use the notations \( \text{Pic}^0(\tilde{X}) := c_1^{-1}(l') \subset \text{Pic}(\tilde{X}) \) and \( \text{Pic}^0(Z) := c_1^{-1}(l') \subset \text{Pic}(Z) \) respectively. If \( H^1(E, \mathbb{Z}) = 0 \) (i.e., the link is a rational homology sphere) then they are affine spaces associated with the vector spaces \( \text{Pic}^0(\tilde{X}) \) and \( \text{Pic}^0(Z) \) respectively.

As usual, we say that \( \mathcal{L} \in \text{Pic}(Z) \) has no fixed components if

\[
H^0(Z, \mathcal{L})_{\text{reg}} := H^0(Z, \mathcal{L}) \setminus \bigcup_v H^0(Z - E_v, \mathcal{L}(F))
\]

is non–empty.

2.4. Cohomological cycles. \( \text{[Re97]} \) Assume that \( (X, o) \) is a non–rational singularity and let \( \phi : \tilde{X} \to X \) be one of its resolutions. Then, by definition, the cohomological cycle \( Z_{\text{coh}} \) is the (unique) minimal cycle \( Z \in L_{>0} \) with \( h^1(\mathcal{O}_Z) = p_g(X, o) \). This is equivalent with \( h^1(\mathcal{O}_{Z_{\text{coh}}}) = p_g(X, o) \) and \( h^1(\mathcal{O}_{Z'}) < p_g(X, o) \) whenever \( Z' \not\geq Z_{\text{coh}} \). If \( p_g(X, o) = 0 \) then, by definition, we set \( Z_{\text{coh}} = 0 \).

2.5. \( q \)-cohomological cycles. \( \text{[OWY15a, OWY15b, O17]} \). Fix a resolution \( \tilde{X} \to X \).

A cycle \( C \in L_{>0} \) is called a \( q \)-cohomological cycle if

\[
h^1(\mathcal{O}_C) = q = \max_{D > 0, |D| \leq C} h^1(\mathcal{O}_D),
\]

and \( h^1(\mathcal{O}_{C'}) < q \) for every nonzero effective cycle \( C' < C \). This basically says that \( C \) is the cohomological cycle on its support \( |C| \) and the geometric genus on this support is \( q \). Note that in general, \( q \)-cohomological cycle on a resolution is not unique.
2.6. Laufer’s Duality. Let us fix a good resolution $\tilde{X} \to X$ as above. Then there exists a perfect pairing (cf. [La72, La77, NNI])

$$\langle , \rangle : H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) \otimes (H^0(\tilde{X} \setminus E, \Omega^2_{\tilde{X}})/H^0(\tilde{X}, \Omega^2_{\tilde{X}})) \to \mathbb{C}. \tag{2.6.1}$$

Here $H^0(\tilde{X} \setminus E, \Omega^2_{\tilde{X}})$ can be replaced by $H^0(\tilde{X}, \Omega^2_{\tilde{X}}(Z))$ for $Z \gg 0$ (e.g. for any $Z$ with $Z \geq |Z_K|$), cf. [NNI 7.1.3], and for such $Z \gg 0$ one also has $H^1(Z, \mathcal{O}_Z) \cong H^1(\tilde{X}, \mathcal{O}_{\tilde{X}})$.

More generally, for any $Z > 0$, from the exact sequence $0 \to \Omega^2_{\tilde{X}} \to \Omega^2_{\tilde{X}}(Z) \to \mathcal{O}_Z(K_{\tilde{X}} + Z) \to 0$, vanishing $H^1(\tilde{X}, \Omega^2_{\tilde{X}}) = 0$ and Serre duality $H^0(\mathcal{O}_Z(K_{\tilde{X}} + Z)) = H^1(\mathcal{O}_Z)^*$, we obtain $H^0(\tilde{X}, \Omega^2_{\tilde{X}}(Z))/H^0(\tilde{X}, \Omega^2_{\tilde{X}}) \cong H^1(\mathcal{O}_Z)^*$. In particular, (see also [NNI 7.4]) we have a perfect pairing

$$\langle , \rangle : H^1(Z, \mathcal{O}_Z) \otimes (H^0(\tilde{X}, \Omega^2_{\tilde{X}}(Z))/H^0(\tilde{X}, \Omega^2_{\tilde{X}})) \to \mathbb{C}. \tag{2.6.2}$$

Let $I \subset \mathcal{V}$ be a subset. Then (2.6.2) can also be applied for the cycle $Z|_{\mathcal{V}\setminus I}$, the restriction of $Z$ to the components $\{E_v\}_{v \in I}$. Let $\Omega_Z(I)$ be the subspace of $H^0(\tilde{X}, \Omega^2_{\tilde{X}}(Z))/H^0(\tilde{X}, \Omega^2_{\tilde{X}})$ generated by differential forms which have no poles along $\cup_{v \in I} E_v \setminus \cup_{v \not\in I} E_v$. Then (see also [NNI §8])

$$h^1(\mathcal{O}_{Z|_{\mathcal{V}\setminus I}}) = \dim \Omega_Z(I). \tag{2.6.3}$$

By (2.6.2) a basis $[\omega_1], \ldots, [\omega_n]$ of $H^0(\Omega^2_{\tilde{X}}(Z))/H^0(\Omega^2_{\tilde{X}})$ provides $h = h^1(Z, \mathcal{O}_Z)$ global coordinates in $H^1(Z, \mathcal{O}_Z)$. In 2.7.3 (under the assumption that the link is a rational homology sphere) we will describe the above dualities in terms of integrations, cf. [NNI §7].

Lemma 2.6.4. (Compare with [OWY15a 2.6].) Let $\phi$ be a resolution as above. Fix an irreducible exceptional divisor $E_v$ such that the multiplicity of $Z_{\text{coh}}(\tilde{X})$ along $E_v$ is positive. Let $\pi : \tilde{X}_\text{new} \to \tilde{X}$ be the blow up of $\tilde{X}$ at a generic point of $E_v$, and let $E_{\text{new}}$ be the newly created exceptional curve. Then $Z_{\text{coh}}(\tilde{X}_{\text{new}}) = \pi^* Z_{\text{coh}}(\tilde{X}) - E_{\text{new}}$.

Proof. Fix a generic differential form $\omega \in H^0(\tilde{X} \setminus E, \Omega^2_{\tilde{X}})$. By Laufer’s duality its pole is exactly $Z_{\text{coh}}(\tilde{X})$. (I.e., $\Omega^2_{\tilde{X}}(Z_{\text{coh}})$ has no fixed components, cf. [OWY15a 2.6] .) By a local computation, and using the fact that we blow up a generic point of $E_v$, the pole of $\pi^* \omega$ is $\pi^* Z_{\text{coh}}(\tilde{X}) - E_{\text{new}}$. But $\omega$ being generic, $\pi^* \omega$ is a generic form at $\tilde{X}_{\text{new}}$ level, hence its pole is the cohomological cycle on $\tilde{X}_{\text{new}}$. \qed

2.7. The Abel map. In the proofs we will use several results from the theory of Abel maps associated with resolutions of normal surface singularities, see [NNI, NNI, NNI, NNI]. Next we recall some material from this theory needed later in the proofs.

In this subsection 2.7 we assume that the link is a rational homology sphere.

Let us fix an effective integral cycle $Z \in I$, $Z \geq E$. Let $\text{ECA}(Z)$ be the space of effective Cartier divisors supported on $Z$. Note that they have zero-dimensional supports in $E$. Taking the class of a Cartier divisor $c : \text{ECA}(Z) \to \text{Pic}(Z)$, called the Abel map. Let $\text{ECA}'(Z)$ be the set of effective Cartier divisors with Chern class $c' \in L'$, that is, $\text{ECA}'(Z) := c^{-1}(\text{Pic}(Z))$. We consider the restriction of $c, c'(Z) : \text{ECA}'(Z) \to \text{Pic}(Z)$ too, sometimes still denoted by $c$.

The bundle $L \in \text{Pic}(Z)$ is in the image $\text{im}(c)$ of the Abel map if and only if it has no fixed components, that is, if and only if $H^0(Z, L)_{\text{reg}} \neq \emptyset$.

One verifies that $\text{ECA}'(Z) \neq \emptyset$ if and only if $-l' \in S' \setminus \{0\}$. Therefore, it is convenient to modify the definition of $\text{ECA}$ in the case $l' = 0$: we (re)define $\text{ECA}'(Z) = \{\emptyset\}$, as the one–element set consisting of the ‘empty divisor’. We also take $c(0)(\emptyset) := \mathcal{O}_Z$. Then we have

$$\text{ECA}'(Z) \neq \emptyset \iff l' \in S' \tag{2.7.1}$$
If \( l' \in -S' \) then \( \text{ECA}^{l'}(Z) \) is a smooth complex irreducible quasi–projective variety of dimension

\[
(2.7.2) \quad \dim \text{ECA}^{l'}(Z) = (l', Z)
\]

(see [NNI] Th. 3.1.10). Moreover, cf. [NNI] Lemma 3.1.7, if \( \mathcal{L} \in \text{im}(e^{l'}(Z)) \) then the fiber \( e^{-1}(\mathcal{L}) \) is a smooth, irreducible quasi–projective variety of dimension

\[
(2.7.3) \quad \dim(e^{-1}(\mathcal{L})) = h^0(Z, \mathcal{L}) - h^0(O_Z) = (l', Z) + h^1(Z, \mathcal{L}) - h^1(O_Z).
\]

Using this, one proves (see [NNI, 5.6]) that for any \( \mathcal{L} \in \text{im}(e^{l'}) \subset \text{Pic}^{l'}(Z) \) one has

\[
(2.7.4) \quad h^1(Z, \mathcal{L}) \geq h^1(O_Z) - \dim(\text{im}(e^{l'}(Z))),
\]

and equality holds whenever \( \mathcal{L} \) is generic in \( \text{im}(e^{l'}(Z)) \).

Recall that if all \( E_v \)-coefficients \((-E_v^*, Z)\) of \( Z \) are very large (a fact denoted by \( Z \gg 0 \)), then \( Z \) constitute a ‘finite model’ for \( \tilde{X} \). (Note that ‘ECA(\tilde{X})’ is ‘undefined infinite dimensional’). Additionally, for \( Z \gg 0 \) one also has \( h^1(Z, \mathcal{L}) = h^1(\tilde{X}, \mathcal{L}) \) for \( \mathcal{L} \in \text{Pic}(\tilde{X}) \) by Formal Function Theorem.

2.7.5. Consider again a Chern class \( l' = \sum_{v \in V} a_v E_v^* \in -S' \) as above. The \( E^* \)–support \( I(l') \subset V \) of \( l' \) is defined as \( \{ v : a_v \neq 0 \} \). Its role is the following.

Besides the Abel map \( e^{l'}(Z) \) one can consider its ‘multiples’ \( \{ e^{nl'}(Z) \}_{n \geq 1} \) as well. It turns out (cf. [NNI] §6), that \( n \mapsto \text{dim}(\text{im}(e^{nl'}(Z))) \) is a non-decreasing sequence, hence it becomes constant, and \( \text{im}(e^{nl'}(Z)) \) for \( n \gg 1 \) are affine subspaces parallel to each other and parallel with the affine closure of \( \text{im}(e^{l'}(Z)) \), all of the same dimension. This common dimension will be denoted by \( e_Z(l') = \lim_{n \to \infty} \text{dim}(\text{im}(e^{nl'}(Z))) \). It depends only on \( I(l') \). Moreover, by [NNI] Theorem 6.1.9,

\[
(2.7.6) \quad e_Z(l') = h^1(O_Z) - h^1(O_{Z|I(l')'}),
\]

where \( Z|_{V \setminus I(l')} \) is the restriction of the cycle \( Z \) to its \( \{ E_v \}_{v \in V \setminus I(l')} \) coordinates.

Furthermore, for \( \mathcal{L} \in \text{Pic}(\tilde{X}) \) (cf. [NNI] Th. 6.1.9(c)–(e))

\[
(2.7.7) \quad \text{if } \mathcal{L}|_Z \in \text{im}(e^{nl'}(Z)) \text{ (} n \gg 0, Z \gg 0 \text{) then}
\begin{align*}
\mathcal{L} \text{ is generated by global sections, and} \\
h^1(\tilde{X}, \mathcal{L}) = h^1(O_{\tilde{X}|_{V \setminus I(l')} }).
\end{align*}
\]

2.7.8. The Lauder integration. Consider the following situation. We fix a smooth point \( p \) on \( E \), a local bidisc \( B \ni p \) with local coordinates \((x, y)\) such that \( B \cap E = \{ x = 0 \} \). We assume that a certain form \( \omega \in H^0(\tilde{X}, \Omega^2_{\tilde{X}}(Z)) \) has local equation \( \omega = \sum_{i,j \geq 0} a_{i,j} x^i y^j dx \& dy \) in \( B \).

In the same time, we fix a divisor \( \tilde{D} \) on \( \tilde{X} \), whose unique component \( \tilde{D}_1 \) in \( B \) has local equation \( y \). Let \( \tilde{D}_t \) be another divisor, which is the same as \( \tilde{D} \) in the complement of \( B \) and its component \( \tilde{D}_{1,t} \) in \( B \) has local equation \( y + td(t, x, y) \).

Next, we identify \( H^1(\tilde{X}, \Omega^2_{\tilde{X}}) \) with \( \text{Pic}^0(\tilde{X}) \) by the exponential map and we consider the composition \( t \mapsto \tilde{D}_t - \tilde{D} \mapsto \mathcal{O}_X(\tilde{D}_t - \tilde{D}) \mapsto \exp^{-1} \mathcal{O}_X(\tilde{D}_t - \tilde{D}) \mapsto (\exp^{-1} \mathcal{O}_X(\tilde{D}_t - \tilde{D}), \omega) \). The next formula makes this expression explicit. (Here \( B = \{ |x|, |y| < \epsilon \} \) for a small \( \epsilon \), and \(|t| < \epsilon \).

\[
(2.7.9) \quad \langle (\tilde{D}_t, \omega) \rangle := \langle \exp^{-1} \mathcal{O}_X(\tilde{D}_t - \tilde{D}), \omega \rangle = \int_{|x| \leq \epsilon} \log \left( 1 + t \frac{d(t, x, y)}{y} \right) \cdot \sum_{i,j \geq 0} a_{i,j} x^i y^j dx \& dy.
\]

This restricted to any cycle \( Z \gg 0 \) can be reinterpreted as ‘\( \omega \)–coordinate’ of the Abel map restricted to the path \( t \mapsto D_t := \tilde{D}_t|_Z \) (and shifted by the image of \( D := \tilde{D}|_Z \)). If \( \omega \) has no pole along the divisor \( \{ x = 0 \} \) then \( \langle (\tilde{D}_t, \omega) \rangle = 0 \) for any path \( \tilde{D}_t \).

If more components of \( \tilde{D} \) are perturbed then \( \langle (\tilde{D}_t, \omega) \rangle \) is the sum of such contributions.
Definition 2.7.10. Consider the above situation and assume that \( \tilde{D}_1 \) has local equation \( y \). Then, by definition, the Leray residue of \( \omega \) along \( \tilde{D}_1 \) is the 1–form on \( \tilde{D}_1 \) (with possible poles at \( \tilde{D}_1 \cap E \)) defined by \( (\omega/dy)_{y=0} = \sum_{t \in \mathbb{A}} a_t x^t dx \). We denote it by \( \text{Res}_{\tilde{D}_1}(\omega) \).

Note that if in \( (2.7.10) \) \( \tilde{D}_{1.t} = y + t \in \mathbb{C}^* \) for some \( o \geq 1 \), then \( \frac{\partial}{\partial x}(\tilde{D}_t, \omega) = \lambda \cdot a_{-o}, o \) for a certain \( \lambda \in \mathbb{C}^* \). Therefore, the right hand side of \( (2.7.10) \) tests exactly the non–regular part of \( \text{Res}_{\tilde{D}_1}(\omega) \).

2.7.11. The sheaf \( \Omega^2_X(\mathcal{Z})^{\text{reg Res}^B} \). Consider again \( l' \in S' \) and a divisor \( D \in \text{ECa}^{-l'}(\mathcal{Z}) \), which is a union of \( -(l', E) \) disjoint divisors \( \{D_i\}_i \), each of them \( \mathcal{O}_Z \)–reduction of divisors \( \{D_i\}_i \), from \( \text{EC}^{-l'}(\tilde{X}) \) intersecting \( E \) transversally. Set \( \tilde{D} = \bigcup_i \tilde{D}_i \) and write \( Z = \sum_i m_i E_v \).

We introduce a subsheaf \( \Omega^2_{\tilde{X}}(\mathcal{Z})^{\text{reg Res}^B} \) of \( \Omega^2_{\tilde{X}}(\mathcal{Z}) \) consisting of those forms \( \omega \), which have the property that for every \( i \) the residue \( \text{Res}_{\tilde{D}_i}(\omega) \) has no pole at \( \tilde{D}_i \cap E \). For more see [NNI] 10.1.

Theorem 2.7.12. [NNI] Th. 10.1.1 In the above situation one has the following facts.

(a) The sheaves \( \Omega^2_{\tilde{X}}(\mathcal{Z})^{\text{reg Res}^B} / \Omega^2_{\tilde{X}}(\mathcal{Z}) \) and \( \mathcal{O}_{\tilde{X}}(K_{\tilde{X}} + Z - D) \) are isomorphic.

(b) \( H^0(\tilde{X}, \Omega^2_{\tilde{X}}(\mathcal{Z})^{\text{reg Res}^B})/H^0(\tilde{X}, \Omega^2_{\tilde{X}}(\mathcal{Z})) \simeq H^0(Z, \mathcal{O}_Z(K_{\tilde{X}} + Z - D)) \simeq H^1(Z, \mathcal{O}_Z(D))^* \).

3. The possible \( p_g \) and \( h^1(\mathcal{L}) \) values

3.1. ‘Subsingularities’. In the next discussions it is convenient to use the following terminology. Let \( \phi: \tilde{X} \to X \) be a resolution with dual graph \( \Gamma \). Assume that \( \Gamma_r \) is a full connected subgraph of \( \Gamma \) with vertices \( \mathcal{Y}_r \). Let \( \tilde{X}_r \) be a convenient small tubular neighbourhood of \( E_r := \bigcup_{v \in \mathcal{Y}_r} E_v \). By Grauert theorem \( \mathcal{E}_r \) can be contracted in \( \tilde{X}_r \), which give rise to a normal singularity \( (X_r, o_r) \).

We will call \( (X_r, o_r) \) a ‘subsingularity’ of \( (X, o) \) with respect to \( \phi \) and \( \Gamma_r \). Or, just we say simply that \( \tilde{X}_r \) is a subsingularity of \( \tilde{X} \) associated with \( \Gamma_r \).

3.2. The next lemma will be useful in the next proofs. For a slightly different version see [N19].

Lemma 3.2.1. Consider \( \phi: \tilde{X} \to X \) as above with cohomological cycle \( Z_{coh} = \sum_v m_v E_v \). Assume that \( m_v \geq 2 \) for some \( E_u \subset \{Z_{coh}\} \). We blow up \( E_u \) sequentially along generic points \( m_v - 1 \) times (this means that we blow up \( E_u \) at a generic point, say at \( p \), then we blow up the newly created exceptional curve at a generic point, etc.). Let the last newly created curve be \( E_u' \). Let us denote the new modification by \( \pi: \tilde{X}_{\text{new}} \to \tilde{X} \), the new resolution by \( \tilde{X}_{\text{new}} \to X \), whose vertex set will be denoted by \( \mathcal{Y}_{\text{new}} \). Let \( \tilde{X}_{\text{new}} \) denote a convenient small neighbourhood of the exceptional divisors indexed by \( \mathcal{Y}_{\text{new}} \setminus \{u'\} \). Then \( h^1(\mathcal{O}_{\tilde{X}_{\text{new}}}) = h^1(\mathcal{O}_{\tilde{X}_{\text{new}}}) - 1 \). This means that if \( (X', o) \) denotes the singularity obtained from \( \tilde{X}_{\text{new}} \), by contracting its exceptional divisors, then \( p_g(X', o) = p_g(X, o) - 1 \).

(Note that \( u' \) is an end vertex, hence the dual graph of \( \tilde{X}_{\text{new}} \) is connected.)

Proof. Let \( Z_{coh} = Z_{coh}(\tilde{X}) \) (resp. \( Z_{coh}(\tilde{X}_{\text{new}}) \)) denote the cohomological cycle of \( \tilde{X} \) (resp. \( \tilde{X}_{\text{new}} \)).

First we claim that \( p_g(X', o) < p_g(X, o) \).

Indeed, by Lemma 2.6.4 the cohomological cycle of \( \tilde{X}_{coh} \) is not supported in \( \tilde{X}_{coh}' \), hence, by the definition of the cohomological cycle \( p_g(X', o) = h^1(\mathcal{O}_{\tilde{X}_{\text{new}}}) < h^1(\mathcal{O}_{Z_{coh}(\tilde{X}_{\text{new}})}) = p_g(X, o) \).

For the opposite inequality, let us compare in the sequence of blow ups the last step \( \tilde{X}_{\text{new}} \) with the previous step whose resolution space will be denoted by \( \tilde{X}_{\text{new}}^{pr} \). Let \( b: \tilde{X}_{\text{new}} \to \tilde{X}_{\text{new}}^{pr} \) be the blow up. Let \( Z_{pr} \) and \( Z \) be the cohomological cycle of \( \tilde{X}_{\text{new}}^{pr} \) and \( \tilde{X}_{\text{new}} \) respectively. Then by Lemma 2.6.4 \( Z = b^* Z_{pr} - E_u' \) and the \( E_u' \)–multiplicity of \( Z \) is one. Then one sees that the cohomological cycle of \( \tilde{X}_{\text{new}}' \) is smaller than \( Z - E_u' \). Then using the surjection \( \mathcal{O}_Z \to \mathcal{O}_Z - E_u' \) we get that \( p_g(X, o) - p_g(X', o) \leq h^1(\mathcal{O}_{E_u'}(2E_u' - b^*(Z_{pr}))) = 1 \).
Remark 3.2.2. In Lemma \[3.2.1\] the assumption \(m_u \geq 2\) cannot be replaced by \(m_u \geq 1\), see e.g. the cone-like case presented in Example \[3.3.8\].

3.3. In the sequel we prove a conjecture of the third author formulated in [O19, Conjecture 2.8] as follows: if \((X, o)\) is a complex normal surface singularity and we fix an arbitrary integer \(0 \leq q \leq p_g(X)\) then there exists a resolution \(\tilde{X} \to X\) and an effective integral cycle \(Z > 0\) such that \(\mathcal{O}_{\tilde{X}}(-Z)\) is base point free and \(h^1(\mathcal{O}_{\tilde{X}}(-Z)) = q\).

Theorem 3.3.1. Fix a complex normal surface singularity \((X, o)\) such that all the irreducible exceptional divisors are rational (however \(\Gamma\) is not necessarily a tree). Fix also an arbitrary integer \(0 \leq q \leq p_g(X, o)\). Then the following facts hold.

1. There exists a resolution \(\tilde{X} \to X\), and a subsingularity \(X_r\) with resolution \(\tilde{X}_r \subset \tilde{X}\) associated with a connected full subgraph \(\Gamma_r \subset \Gamma\) such that \(p_g(X_r, o_r) = q\).

2. There exists a resolution \(\tilde{X} \to X\), which admits a \(q\)-cohomological cycle.

3. There exists a resolution \(\tilde{X} \to X\) and an effective integral cycle \(l > 0\) such that \(\mathcal{O}_{\tilde{X}}(-l)\) is base point free and \(h^1(\mathcal{O}_{\tilde{X}}(-l)) = q\).

Proof. We prove part (1) by a decreasing induction on \(q\). If \(q = p_g(X, o)\), then the statement is trivial, since we can take any resolution \(\tilde{X} \to X\) with \(\Gamma_r = \Gamma\).

Next, assume that for a certain \(0 < q \leq p_g(X, o)\) we already know the validity of the statement, that is, we have a resolution \(\tilde{X} \to X\), and a subresolution/subsingularity \(\tilde{X}_r\) associated with the subgraph \(\Gamma_r \subset \Gamma\), such that \(p_g(X_r, o) = q\).

Let us denote the cohomological cycle of \(\tilde{X}_r\) by \(Z_r = \sum_{v \in \mathcal{V}(\Gamma_r)} m_v E_v\).

First, assume that there exists a vertex \(u \in \mathcal{V}(\Gamma_r)\) such that \(m_u > 1\). Then we perform the construction of Lemma \[3.2.1\] we blow up \(E_u\) sequentially along generic points \(m_u - 1\) times and let the last created curve be \(E_w\). Let us denote the new vertex set by \(\mathcal{V}_{\text{new}}\), and the new resolution by \(\tilde{X}_{\text{new}}\). Furthermore, we have its subsingularity \(\tilde{X}_{r,\text{new}}\) with vertex set \(\mathcal{V}_{r,\text{new}}\), where \(\tilde{X}_{r,\text{new}}\) is the modification of \(\tilde{X}_r\) by the previous sequence of blow ups. Clearly, \(h^1(\mathcal{O}_{\tilde{X}_{r,\text{new}}}) = p_g(X_r) = q\).

Additionally, let us denote the subresolution of \(\tilde{X}_{r,\text{new}}\) corresponding to the vertex set \(\mathcal{V}_{r,\text{new}}\setminus\{u'\}\) by \(\tilde{X}'_{r,\text{new}}\). Its dual graph is connected and by Lemma \[3.2.1\] \(h^1(\mathcal{O}_{\tilde{X}'_{r,\text{new}}}) = h^1(\mathcal{O}_{\tilde{X}_{r,\text{new}}}) - 1\), which finishes the induction step and the proof of part (1) in this case.

Second, assume that \(m_v \leq 1\) for all \(v \in \mathcal{V}(\Gamma_r)\). Then \(p_g(\tilde{X}_r)\) equals the number of independent 1–cycles in \(\Gamma_r\). Since \(p_g(\tilde{X}_r) > 0\), we necessarily have at least one such cycle in \(\Gamma_r\). Let us fix an edge \(e\) of \(\Gamma_r\) such that the graph obtained from \(\Gamma_r\) by deleting of \(e\) has one less independent 1–cycle. Then, if we blow up \(e\) end we delete the newly created irreducible exceptional divisor with its adjacent edges we get a full connected subgraph with the required property.

This ends the proof of part (1).

For part (2), take a resolution \(\tilde{X}\) and \(\tilde{X}_r \subset \tilde{X}\) with \(h^1(\tilde{X}_r) = q\) as in (1). Then the cohomological cycle of \(\tilde{X}_r\) is a \(q\)-cohomological cycle of \(\tilde{X}\).

Finally, part (3) follows from (2) via the following lemma (with slightly stronger statement).

Lemma 3.3.2. If there exists a \(q\)-cohomological cycle on \(\tilde{X}\), then there exists a resolution \(\tilde{X}_{\text{new}} \to X\), which factors through \(\tilde{X}\), and a cycle \(l > 0\) on \(\tilde{X}_{\text{new}}\) such that \(\mathcal{O}_{\tilde{X}_{\text{new}}}(-l)\) is base point free and \(h^1(\mathcal{O}_{\tilde{X}_{\text{new}}}(-l)) = q\).

Proof. Let \(C\) be a \(q\)-cohomological cycle of \(\tilde{X}\), and write \(C = \sum_{E_u \subset \mathcal{C}(\tilde{X})} m_u E_u\).

Next, we fix a cycle \(l\) such that \((l, E_u) < 0\) for all \(E_u\) and \(\mathcal{O}_{\tilde{X}}(-l)\) is base point free. Let \(f \in H^0(\mathcal{O}_{\tilde{X}}(-l))\) be a general element and write \(\tilde{f} = f + D\), where \(D\) has no exceptional
components. We write $D$ as $D_1 + D_2$ with disjoint union supports, such that all the components \{D_{1,i}\}, of $D_1$ intersect $|C|$, while $D_2 \cap |C| = \emptyset$. Since $\mathcal{O}_X(-l)$ is base point free, we can chose $f$ such that the intersection points $D_1 \cap |C|$ are generic with respect to the sections of $H^0(X, \mathcal{O}_X(K_X + C))$. Assume that $p_i := D_1 \cap |C| \in E_a_i$. For each $i$, we blow up $\tilde{X}$ at $m_{a_i}$ infinitely near point at $p_i$, let the created exceptional curves be \{\tilde{E}_{i,j}\}_{1 \leq j \leq m_{a_i}} (\tilde{E}_{i,1}$ being the very first one). Denote the strict transforms of $E_v$ by the same symbol $E_v$ ($v \in V$), the strict transform of $D_1$ by $\tilde{D}_1$. Let $\tilde{X}_{new}$ be the new resolution, set $\tilde{\pi} : \tilde{X}_{new} \to \tilde{X}$.

Then by the fact that the points $p_i$ are not base points of $\mathcal{O}_X(K_X + C)$, we obtain that the support $S = \cup_i E_v \cup \cup_{i,j \leq m_{a_i}} \tilde{F}_{i,j}$ has the following properties (cf. Lemma 2.6.3 and its proof).

1. $S$ consists of those exceptional curves of $\tilde{X}_{new}$ which do not intersect $\tilde{D}_1 \cup \tilde{D}_2$.
2. The cohomological cycle $\mathcal{C}'$ on $S$ is $\pi^*(C) - \sum_{i,j \leq m_{a_i}} j \tilde{F}_{i,j}$, and its support is $S$.

Finally, write $\mathcal{C}'_{\tilde{X}_{new}} = (f \circ \tilde{\pi})_{new} + \tilde{D}_1 + \tilde{D}_2$. Then $\mathcal{O}_{\tilde{X}_{new}}(-n_{\tilde{X}_{new}})$ has no fixed components, hence (1) and (2) together with [17] 3.6 imply that for $n \gg 0$ the bundle $\mathcal{O}_{\tilde{X}_{new}}(-n \cdot l_{\tilde{X}_{new}})$ is base point free and $h^1(\mathcal{O}_{\tilde{X}_{new}}(-n \cdot l_{\tilde{X}_{new}})) = h^1(\mathcal{O}_{\tilde{C}'})$. But $h^1(\mathcal{O}_{\tilde{C}'}) = h^1(\mathcal{O}_C) = q$. \hfill \Box

### Example 3.3.3.

(a) Consider a cone–like singularity whose minimal resolution is a smooth curve $C$ of genus $g \geq 2$ with self–intersection very negative (compared with $g$). Then, using the exact sequence $0 \to \mathcal{O}_C(-mC) \to \mathcal{O}_{(m+1)C} \to \mathcal{O}_C \to 0$ and the vanishing $h^1(\mathcal{O}_C(-mC)) = 0$ (for all $m \geq 1$) we have $p_g(X,o) = h^1(C, \mathcal{O}_C) = g$. Moreover, for an arbitrary resolution $\tilde{X}$, if we denote the strict transform of $C$ by the same symbol $C$, then $h^1(\tilde{X}, \mathcal{O}_C) = p_g(X,o) = h^1(C, \mathcal{O}_C) = g$ too.

We claim that part (1) of Theorem 3.3.1 does not hold for any $0 < k < p_g(X,o)$. Indeed, take an arbitrary resolution $\tilde{X}$ and $\tilde{X}_r \subset \tilde{X}$ as in (1). Then, if the strict transform of $C$ is in $\Gamma_r$ then $h^1(\mathcal{O}_{\tilde{X}_r}) = g$, otherwise it is zero.

(b) Let us start with a cone–like singularity as in part (a). Let $\tilde{X}$ be any resolution and fix a line bundle $\mathcal{L} \in \text{Pic}(\tilde{X})$ without fixed components. We claim that $h^1(\tilde{X}, \mathcal{L}) = h^1(C, \mathcal{L}|_C)$. Indeed, by the exact sequence $0 \to \mathcal{O}_{\tilde{X}}(-C) \to \mathcal{O}_{\tilde{X}} \to \mathcal{O}_C \to 0$ (and part (a)) we get that $h^1(\mathcal{O}_{\tilde{X}}(-C)) = 0$. On the other hand, by multiplication with a generic global section of $\mathcal{L}$ we have an exact sequence $0 \to \mathcal{O}_{\tilde{X}}(-C) \to \mathcal{L}(-C) \to \mathcal{A} \to 0$, such that the support of $A$ is Stein. Hence $H^1(\mathcal{O}_{\tilde{X}}(-C)) \to H^1(\mathcal{L}(-C))$ is onto, hence $H^1(\tilde{X}, \mathcal{L}(-C)) = 0$. This proves the claim.

Let us analyse the validity of part (2) of Theorem 3.3.1 in this case. Fix any resolution $\tilde{X}$ and a base point free line bundle $\mathcal{L} \in \text{Pic}(\tilde{X})$. If the degree $d$ of the restriction $\mathcal{L}|_C \in \text{Pic}(C)$ is zero then $h^1(\tilde{X}, \mathcal{L}) = g$. Otherwise the degree is necessarily $d \geq 2$. Assume next this case. Next we analyse the line bundle $\mathcal{L}|_C \in \text{Pic}(C)$ of degree $d \geq 2$.

If $d > 2g - 2$ then $h^1(C, \mathcal{L}|_C) = 0$. If $d = 2g - 2$ then $h^1(C, \mathcal{L}|_C) = h^0(C, \mathcal{L}^{-1}|_C \otimes K_C)$ is 0 or 1. If $d < 2g - 2$ then by Clifford theorem (and Riemann–Roch) $h^1(C, \mathcal{L}|_C) \leq g - d/2$ and if the equality holds then $C$ should be hyperelliptic. In particular, if $C$ is not hyperelliptic, and $g \geq 3$ then $h^1(\tilde{X}, \mathcal{L}) = g - 1$ can be realized.

If the degree $d$ is larger (than 2) then we get even a larger gap for $h^1(\tilde{X}, \mathcal{L})$.

### Remark 3.3.4.

In Theorem 3.3.1 in fact we proved the following fact. Fix a complex normal surface singularity $(X,o)$ and an arbitrary integer $q$ such that $\sum_v g_v \leq q \leq p_g(X,o)$. Then there exists a resolution $\tilde{X} \to X$ and a subsingularity $X_r$ with resolution $\tilde{X}_r \subset \tilde{X}$ associated with a connected full subgraph $\Gamma_r \subset \Gamma$ such that $p_g(X_r,or) = q$.

Indeed, in this case the induction runs as follows. If $Z_{coh}$ has a coefficient with $m_q > 1$ then we proceed as in the above proof and we can find a subsingularity with geometric genus one less. If $Z_{coh}$ is reduced then $p_g = \sum_v g_v + c_r$, where $c_r$ is the number of independent cycles in $\Gamma$. But the
number of such cycles can also be decreased one by one (by deleting the exceptional curve of a blow up at a conveniently chosen singular point of $E$).

4. Stability bound for $n \mapsto h^1(\tilde{X}, L^n)$.

4.1. Before we state the next result we wish to make the following preparation.

4.1.1. Let $\tilde{X}$ be a resolution and $Z \in L_{>0}$ an effective cycle and $L \in \text{Pic}^l(Z)$ a line bundle without fixed components. Fix a generic section $s \in H^0(Z, L)$. Then the cohomology long exact sequence of $0 \to L^n \xrightarrow{\times s} L^{n+1} \to A \to 0$ (where $\times s$ is the multiplication by $s$ and the support of $A$ is zero-dimensional) shows that $h^1(Z, L^n) \geq h^1(Z, L^{n+1})$. Hence the sequence $n \mapsto h^1(Z, L^n)$ is non-increasing and it is constant for $n \gg 0$.

**Definition 4.1.2.** For any line bundle $L \in \text{Pic}^l(Z)$ without fixed components let $n_0(Z, L)$ be the smallest integer $n$ such that $h^1(Z, L^n) = h^1(Z, L^{n+1})$.

**Remark 4.1.3.** Using the relevant exact sequences one verifies that $n_0(Z, L) = \min\{n \geq 0 : h^1(Z, L^n) = h^1(Z, L^m), \forall m > n\}$, see e.g. [3] Lemma 3.6.

4.1.4. Assume that the link is a rational homology sphere, hence the theory of Abel maps can be applied. Similarly as above, if we fix a Chern class $l' \in -S'$, then $n \mapsto \dim(\text{im}(\text{c}^nl')(Z))$ is a non-decreasing sequence bounded from above by $h^1(O_Z)$, hence it must stabilise, cf. [27.4]. The limit $e_Z(l')$ depends only on the $E^*$-support $I = I(l')$ of $l'$, hence will also be denote by $e_Z(I)$.

**Definition 4.1.5.** For any Chern class $l' \in -S'$ let $n'_0(Z, l')$ be the smallest integer $n$ such that \[\dim(\text{im}(\text{c}^nl')(Z)) = \dim(\text{im}(\text{c}^{(n+1)}l')(Z)).\]

**Remark 4.1.6.** A similar type of stability is valid for $n'_0$ as in Remark 4.1.3. Namely, $n'_0(Z, l') = \min\{n \geq 0 : \dim(\text{im}(\text{c}^nl')(Z)) = \dim(\text{im}(\text{c}^{(n+1)}l')(Z)), \forall m > n\}$. This follows from [3] §6: \[\dim(\text{im}(\text{c}^nl')(Z)) = \dim(\text{im}(\text{c}^{(n+1)}l')(Z))\] holds if and only if $\dim(\text{im}(\text{c}^nl')(Z))$ equals the dimension of the affine closure of $\text{im}^{nl'}(Z)$.

4.1.7. One has the following relations between the ‘stabilized dimensions’. By [3] Remark 3.8 or [3] Th. 6.1.9 we know that for $L$ without fixed components $\lim_{n \to \infty} h^1(Z, L^n) = h^1(O_Z|_{V\setminus l'(\tilde{c})})$, and by (27.4) $e_Z(I) = h^1(O_Z) - h^1(O_Z|_{V\setminus l'(\tilde{c})})$. In other words, whenever $L \in \text{im}(\text{c}^l(Z))$, we have

$$\lim_{n \to \infty} h^1(Z, L^n) = \lim_{n \to \infty} \text{codim}(\text{im}(\text{c}^nl')(Z)).$$

Furthermore, we have the following geometric interpretations in terms of differential forms as well. By [3] (8.3.1), see also (26.3), $h^1(O_Z|_{V\setminus l'(\tilde{c})}) = h^1(O_Z) - e_Z(I)$ equals $\dim\Omega_Z(I)$, where $\Omega_Z(I)$ is that subspace $H^0(\tilde{X}, \Omega^2_{\tilde{X}}(Z|_{V\setminus l'(\tilde{c})})/H^0(\tilde{X}, \Omega^2_{\tilde{X}}(Z))$ of $H^0(\tilde{X}, \Omega^2_{\tilde{X}}(Z))$, which is generated by forms which have no poles along (the generic points of) $\{E_u\}_{u \in I}$. Furthermore, by Theorem 27.12, the dimension of a cohomology group $H^1(Z, L)$ can be determined via forms whose Leray residues have no pole along $D$, where $D$ is a transversal sections as in (27.11).

This is complemented with the following comparison.

**Lemma 4.1.8.** Assume that for certain integer $n_0$ there exists a line bundle $L \in \text{Pic}^l(Z)$ without fixed components such that $n \mapsto h^1(Z, L^n)$ is constant for any $n \geq n_0$. Then $n \mapsto \dim(\text{im}(\text{c}^nl')(Z))$ is constant for $n \geq n_0$ as well. In other words, for any $l' \in -S'$,

$$n'_0(Z, l') \leq \min\{n_0(Z, L) : L \in \text{im}(\text{c}^l(Z))\}.$$
Proof. By assumption and the above discussion, for any \( n \geq n_0 \) one has \( h^1(O_Z) - e_Z(I) = h^1(Z, \mathcal{L}^n) \). Moreover, by (2.7.12), \( h^1(Z, \mathcal{L}^n) \geq h^1(O_Z) - \dim (\text{im}(c^{nd'}(Z))) \), hence \( \dim (\text{im}(c^{nd'}(Z))) \geq e_Z(I) \). But \( n \mapsto \dim (\text{im}(c^{nd'}(Z))) \) is non-decreasing with limit \( e_Z(I) \), hence \( \dim (\text{im}(c^{nd'}(Z))) = e_Z(I) \). \( \square \)

4.2. In this subsection we prove positively a question of the third author formulated as Problem 3.13 in [O19] as follows: if \( \tilde{X} \rightarrow X \) is an arbitrary resolution and \( l_0 \in L_{r>0} \) is an effective integral cycle such that the line bundle \( O_{\tilde{X}}(-l_0) \) is base point free, then \( n \mapsto h^1(O_{\tilde{X}}(-nl_0)) \) is constant for \( n \geq 1 - \min_{Z \geq E} \chi(l) \).

Note that \( \min_{Z \geq E} \chi(l) = \min_{Z > 0} \chi(l) \). In the literature the integer \( \min_{Z > 0} \chi(l) \) was already considered in rather different situations. In [Wa70] Wagreich called \( 1 - \min_{Z > 0} \chi(l) \) arithmetical genus \( p_a \) of \((X, o)\), and for any non-regular germ he proved that \( p_a \leq p_g \) (see [Wa70] p. 425). Furthermore, the bound \( 1 - \min_{Z > 0} \chi(l) \) has the following remarkable appearance too under the assumption that the link is a rational homology sphere: it is the smallest possible geometric genus of any singularity (analytic type) with the same topological type fixed by the dual graph \( \Gamma \) of \( \tilde{X} \), see [NNI]. Similarly, for any \( Z \geq E \), the integer \( 1 - \min_{Z \geq E} \chi(l) \) is the smallest possible value of \( h^1(O_Z) \) associated with any analytic singularity type with the same topological type fixed by the dual graph \( \Gamma \) see [NNII]. Both are realized by the generic analytic structure.

4.3. In fact, we prove the statement for any base point free line bundle and any \( Z \geq E \).

**Theorem 4.3.1.** Assume that \((X, o)\) is a normal surface singularity whose link is a rational homology sphere and let us fix a resolution \( \tilde{X} \rightarrow X \) and an effective cycle \( Z \in L, Z \geq E \).

Assume that \( \mathcal{L} \in \text{Pic}^Z(Z), l' \in -S', \) is a base point free line bundle on \( Z \), and write \( I := l' = \{ v \in \mathcal{O} : (c_1 \mathcal{L}, E_v) \neq 0 \} \) as above. Then \( h^1(Z, \mathcal{L}^n) = h^1(O_Z) - e_Z(I) \) whenever \( n \geq 1 - \min_{Z \geq E} \chi(l) \).

**Proof.** If \( \mathcal{L} \in \text{Pic}^Z(Z) \) is base point free, then the divisor of a generic section of \( \mathcal{L} \) is the restriction of a divisor \( \tilde{D} \) of \( \tilde{X} \), which intersects \( E \) transversally (for \( Z \gg 0 \) see e.g. [NNIV], Remark 9.1.3), the general case follows similarly). Note that \( \tilde{D} \) usually has many components.

Assume that for a certain \( n \geq 2 \) with

\[
(4.3.2) \quad n \geq 1 - \min_{Z \geq E} \chi(l)
\]

one has \( h^1(Z, \mathcal{L}^n) > h^1(O_Z) - e_Z(I) \). Recall that \( h^1(O_Z) - e_Z(I) = \dim \Omega_Z(I) \), is the dimension of that subspace of \( H^0(\tilde{X}, \Omega^2_X(Z))/H^0(\tilde{X}, \Omega^2_X) \), which is generated by forms which have no poles along \( \{ E_v \}_{v \in I} \), cf. (2.6.3) and (2.7.10). Also, \( h^1(Z, \mathcal{L}^n) \) is the dimension of the subspace of classes of forms whose Leray residues along a set of fixed transversal sections of \( \mathcal{L}^n \) have no poles, cf. Theorem 2.7.12.

Consider any set of generic sections \( s_1, s_2, \ldots, s_n \in H^0(Z, \mathcal{L}) \) such that for any \( i \) the divisor of \( s_i \) is the restriction of a divisor \( D_i \) of \( \tilde{X} \), and each \( D_i \) intersects \( E \) transversally, and \( D_i \cap D_j = \emptyset \) for \( i \neq j \). Then, by the above numerical identifications, the inequality \( h^1(Z, \mathcal{L}^n) > h^1(O_Z) - e_Z(I) \) can happen only if there exists a differential form \( \omega \in H^0(\tilde{X}, \Omega^2_X(Z))/H^0(\tilde{X}, \Omega^2_X) \) such that the Leray residues along each component of \( \cup_i D_i \) has no pole, but \( \omega \) has a non-trivial pole along a certain \( E_u \) with \( u \in I \) (cf. the previous paragraph).

Since the set of poles of relevant forms runs over a finite set (hence for a generic choice of the sections \( \{ s_i \} \), the pole of \( \omega \) is stable) we obtain that there exists a fixed non-zero effective integral cycle \( Z' \leq Z \) with \( Z' \geq E_u \) for a certain \( u \in I \), such that for any generic choice of \( s_1, \ldots, s_n \in H^0(Z, \mathcal{L}) \) (with transversal sections \( D_i \), as above) we can find a differential form \( \omega \) with pole \( Z' \) which has regular Leray residue along each component of \( \cup_i D_i \).
Let us fix such a cycle $Z'$ and $E_u$. Write $m_u = -(E_u^*, Z)$ for the $E_u$-multiplicity of $Z$.

Since $(E_u, c_i L) \neq 0$, for any $i$ one of the components of $D_i$ intersects $E_u$. Hence, we can choose a point $p_i \in E_u$ from the support $|D_i|$ of $D_i$. Since $L$ is base point free, this intersection point is even generic on $E_u$.

Fix some local coordinates $(x, y)$ of the germ $(\tilde{X}, p_i)$ with $\{ x = 0 \} = E_u$, $\{ y = 0 \} = D_{p_i}$, the corresponding component $D_{p_i}$ of $D_i$. Write $\omega$ in coordinates $(x, y)$, $\omega = \varphi(x, y) dx \wedge dy/x^o$ (modulo $x^{m_u}$), where $\varphi(x, y)$ is locally holomorphic, and $\alpha > 0$ is the pole order along $E_u$, $x \{ \varphi$. Since the Leray residue $\varphi|_{y\neq0}dx/x^o$ has no pole along this $x$-axis, necessarily $\varphi$ has the form $x^\alpha \alpha(x) + y\beta(x, y)$, hence $\varphi(0, 0) = 0$. In such a case we say that $\omega$ vanishes at $p_i$.

Since $L$ is base point free, these points $p_i$ can be chosen freely on $E_u$. (Note that the point $p_i$ determines and is determined by $D_{p_i}$. Hence, generic $s_i$ provides a generic point $p_i$. Furthermore, the choices of $s_i$ and $s_j$ for $i \neq j$ — i.e. the choices of $p_i$ and $p_j$ — are independent.) Therefore, we obtain that for generic points $p_1, \ldots, p_n$ of $E_u$ there exists a certain $\omega$ as above, which vanishes at $p_1, \ldots, p_n$. In other words, for generic points $p_1, \ldots, p_n \in E_u$, there is a section $s \in H^0(\tilde{X}, \Omega^2_{\tilde{X}}(Z')) \setminus H^0(\tilde{X}, \Omega^2_{\tilde{X}}(Z' - E_u))$, which vanishes at the points $p_1, \ldots, p_n$.

Consider next the exact sequence

\begin{equation}
0 \to H^0(\tilde{X}, \Omega^2_{\tilde{X}}(Z' - E_u)) \to H^0(\tilde{X}, \Omega^2_{\tilde{X}}(Z')) \xrightarrow{\alpha} H^0(E_u, \Omega^2_{\tilde{X}}(Z'))
\end{equation}

and let $V := \text{im}(\alpha)$. It is the vector space of a linear system on $E_u$ such that for any generic $p_1, \ldots, p_n \in E_u$, there exists $s \in V$, $s \neq 0$, which vanishes at all the points $p_1, \ldots, p_n$. In particular, $\dim(V) \geq n + 1$. This combined with the assumption \eqref{4.3.2} we get $\dim(V) \geq 2 - \min_{Z \geq l > 0} \chi(l)$.

Hence, we get that $\dim(H^0(\tilde{X}, \Omega^2_{\tilde{X}}(Z'))/H^0(\tilde{X}, \Omega^2_{\tilde{X}}(Z' - E_u))) \geq 2 - \min_{Z \geq l > 0} \chi(l)$.

Since $\dim H^0(\tilde{X}, \Omega^2_{\tilde{X}}(Z'))/H^0(\tilde{X}, \Omega^2_{\tilde{X}}) = h^1(\mathcal{O}_{Z'})$ (see e.g. \cite{22} or \cite{NN} (7.1.40)), and similarly $\dim H^0(\tilde{X}, \Omega^2_{\tilde{X}}(Z' - E_u))/H^0(\tilde{X}, \Omega^2_{\tilde{X}}) = h^1(\mathcal{O}_{Z' - E_u})$ we obtain that

\begin{equation}
\tag{4.3.4}
h^1(\mathcal{O}_{Z'}) - h^1(\mathcal{O}_{Z' - E_u}) \geq 2 - \min_{Z \geq l > 0} \chi(l).
\end{equation}

On the other hand, from the following exact sequence

\begin{equation}
0 \to H^0(\mathcal{O}_{Z' - E_u}(-E_u)) \to H^0(\mathcal{O}_{Z'}) \xrightarrow{r} H^0(\mathcal{O}_{E_u}) \to H^1(\mathcal{O}_{Z' - E_u}(-E_u)) \to H^1(\mathcal{O}_{Z'}) \to 0
\end{equation}

and from the fact that $r$ is onto, we get that

\begin{equation}
\tag{4.3.5}
h^1(\mathcal{O}_{Z'}) = h^1(\mathcal{O}_{Z' - E_u}(-E_u)).
\end{equation}

Furthermore, let us denote the fixed component cycle of the line bundle $\mathcal{O}_{Z' - E_u}(-E_u)$ by $0 \leq A \leq Z' - E_u$. Then $H^0(\mathcal{O}_{Z' - E_u}(-E_u)) = H^0(\mathcal{O}_{Z' - E_u - A}(-E_u - A))$ and $H^0(\mathcal{O}_{Z' - E_u - A}(-E_u - A))_{\text{reg}} \neq 0$. Moreover, from the cohomological long exact sequence of

\begin{equation}
0 \to \mathcal{O}_{Z' - E_u - A}(-E_u - A) \to \mathcal{O}_{Z' - E_u}(-E_u) \to \mathcal{O}_A(-E_u) \to 0
\end{equation}

we get

\begin{equation}
\tag{4.3.6}
h^1(\mathcal{O}_{Z' - E_u}(-E_u)) = h^1(\mathcal{O}_{Z' - E_u - A}(-E_u - A)) + 1 - \chi(E_u + A).
\end{equation}

Finally, if $\mathcal{G} \in \text{Pic}(\tilde{Z})$ is a line bundle without fixed components then $h^1(\tilde{Z}, \mathcal{G}) \leq h^1(\mathcal{O}_{\tilde{Z}})$ (see e.g. \cite{NN} Proposition 5.7.1(b)), hence $h^1(\mathcal{O}_{Z' - E_u - A}(-E_u - A)) \leq h^1(\mathcal{O}_{Z' - E_u - A})$. But we also have $h^1(\mathcal{O}_{Z' - E_u - A}) \leq h^1(\mathcal{O}_{Z' - E_u})$. This combined with \eqref{4.3.5} and \eqref{4.3.6} gives

\begin{equation}
\tag{4.3.7}
h^1(\mathcal{O}_{Z'}) \leq h^1(\mathcal{O}_{Z' - E_u}) + 1 - \chi(E_u + A),
\end{equation}

which contradicts \eqref{4.3.3}. \qed
5. Stability bound for $n \mapsto \dim(\text{im}(c^{nl}_n(Z)))$.

5.1. The next theorem establishes the analogue of Theorem 13.3.1 for $n \mapsto \dim(\text{im}(c^{nl}_n))$.

Theorem 5.1.1. We fix an arbitrary singularity $(X, o)$, a resolution $\tilde{X}$, a Chern class $l' \in -S'$ with $E^*$--support $I = I(l')$, and a cycle $Z \in L, Z \geq E$. Then the dimension of the image of the Abel map $n \mapsto \dim(\text{im}(c^{nl}_n(Z)))$ is the constant $c_Z(I)$ whenever $n \geq 1 - \min_{Z \geq 1} \chi(l)$. 

Note that if there exists a base point free line bundle $L \in \text{Pic}^0(Z)$ then Theorem 5.1.1 follows from Lemma 13.1.8 and Theorem 13.3.1. The proof of the general case is based on different arguments.

Regarding the restriction $Z \geq E$ note that the above theorem remains true for any $Z > 0$ as well. Indeed, in this general case we apply the above version restricted to the connected subgraphs of the support $|Z|$ of $Z$.

Proof. If we fix some $L_0 \in \text{Pic}^{nl}_n(Z)$, then each $\text{Pic}^{nl}_n(Z)$ can be identified with the linear space $\text{Pic}^0(Z) = H^1(O_Z)$ via $L \mapsto L \otimes L_0^{-n}$, and the Abel map $c^{nl}_n(Z) : E\text{Ca}^{nl}_n(Z) \to \text{Pic}^{nl}_n(Z)$ with $\overline{\text{im}(c^{nl}_n(Z))} : E\text{Ca}^{nl}_n(Z) \to H^1(O_Z)$. (In [NN] $L_0$ is the ‘natural line bundle’ associated with $l'$.)

In this way we can use the vector space structure of $H^1(O_Z)$. In particular,

\[(5.1.2) \quad \text{im}(\overline{\text{im}(c^{nl}_n)}) + \text{im}(\overline{\text{im}(c^{nl}_n + m'l)}) \subset \text{im}(\overline{\text{im}(c^{nl}_n)}) + \text{im}(\overline{\text{im}(c^{nl}_n)}),\]

where $\overline{}$ denotes the topological closure. Clearly, $\text{im}(c^{nl}_n)$ and $\text{im}(\overline{\text{im}(c^{nl}_n)})$ can also be identified as subspaces. Let us denote by $A(\text{im}(c^{nl}_n))$ the affine closure of $\text{im}(c^{nl}_n)$. Note that up to an affine translation $A(\text{im}(c^{nl}_n))$ is the same as the affine closure $A(\text{im}(c^{nl}_n))$ for any $n$. Therefore, the ‘stabilized limit’ $\text{im}(c^{nl}_n)) \to_{\infty} A(\text{im}(c^{nl}_n))$ (up to an affine translation). Hence, the dimension $n \mapsto \dim(\text{im}(c^{nl}_n))$ stabilizes exactly when $\text{im}(\overline{\text{im}(c^{nl}_n(Z)))} = A(\text{im}(c^{nl}_n(Z)))$. (For more see [NN] 6.1.)

Furthermore, using again the analogue of (5.1.2) for arbitrary two Chern classes, if $l' = \sum_{v \in V} -a_v E_v$, then $\text{im}(c^{nl}_n(Z)) = \sum_{v \in V} \text{im}(c^{-nE_v}(Z))$ and $A(\text{im}(c^{nl}_n(Z))) = \sum_{v \in V} A(\text{im}(c^{-nE_v}(Z)))$.

This means that it is enough to prove, independently for each $v \in V$, that $A(\text{im}(c^{-nE_v}(Z))) = \text{im}(\overline{\text{im}(c^{nl}_n(Z))))$ for any $n \geq 1 - \min_{Z \geq 1} \chi(l)$.

Fix some $v \in V$ and let $\{\Gamma_i\}$ be the connected full subgraphs of $\Gamma \setminus v$. For any cycle $W \in L(\Gamma)$ of the form $W = Ie_v + \sum W_i$ with $W_i \in L(\Gamma_i)$, we set $W_\Gamma := \sum W_i$, the restriction of $W$ to $\cup_i \Gamma_i$. Write also $l' = -E_v$ and $I = I(l') = \{v\}$.

Consider the restriction $r_n : \text{Pic}^{nl}_n(Z) \to \text{Pic}^0(Z_\Gamma)$. $\tilde{L} \mapsto L|_{Z_\Gamma}$. This is the affine projection associated with the vector space projection $H^1(O_Z) \to H^1(O_{Z_\Gamma})$. Hence the fiber $r_n^{-1}(O_{Z_\Gamma})$ has dimension $h^1(O_{Z_\Gamma}) - h^1(O_{Z_\lambda}) = e_Z(I)$ (cf. 2.7.6). Note also that $\text{im}(c^{nl}_n(Z)) \subset r_n^{-1}(O_{Z_\Gamma})$. In particular, $\dim(\text{im}(c^{nl}_n(Z))) = e_Z(I)$ if and only if $\text{im}(c^{nl}_n(Z))$ contains a Zariski open subset in $r_n^{-1}(O_{Z_\Gamma})$. This in [NR] is formulated as ‘the pair $(nl', O_{Z_\Gamma})$ is relative dominant’. In [NR] Theorem 4.1.6] the following criteria is proved for relative dominance. This is what we will use in the proof.

For the convenience of the reader we reproduce that part of [NR] Theorem 4.1.6] what is needed.

Claim: If for certain fixed $n \geq 1$

\[(5.1.3) \quad \chi(nE_v) - h^1(O_{Z_\Gamma}) < \chi(nE_v + l) - h^1(O_{Z_\lambda}(-l)),\]

for every cycle $0 < l \leq Z$, then $\dim(\text{im}(c^{nl}_n(Z))) = e(Z) = h^1(O_{Z_\Gamma}) - h^1(O_{Z_\lambda})$.

Proof of the Claim.

Choose $L$ a generic element of $\text{Pic}^{nl}_n(Z)$ with $L|_{Z_\Gamma} = O_{Z_\Gamma}$ (that is, generic in $r_n^{-1}(O_{Z_\Gamma})$). By a Chern class computation (recall that $l' = -E_v$) we obtain that (5.1.3) is equivalent with

\[(5.1.4) \quad h^1(O_{Z_\lambda}(-l)) + \chi(Z)|_{L(-l)} < h^1(O_{Z_\Gamma}) + \chi(Z, L).\]
Finally, this can be compared with (2.7.3) applied for \( c \). But from the epimorphism of sheaves \( L \rightarrow L_{\mid Z} \) we have
\[
(5.1.5) \quad h^1(Z, L) \geq h^1(\mathcal{O}_{Z\text{-}l})
\]
hence \( h^0(Z, L) > 0 \).

Next we discuss \( H^0(Z, L)_{\text{reg}} \). If \( H^0(Z, L)_{\text{reg}} \neq \emptyset \), then the chosen generic element \( \mathcal{L} \) of \( r_n^{-1}(\mathcal{O}_{Z\text{-}l}) \) is in the image of \( c_{\text{nf}}(Z) \), hence \( \dim(\text{im}(c_{\text{nf}}(Z))) = \dim(r_n^{-1}(\mathcal{O}_{Z\text{-}l})) = c_{\text{nf}}(I) \).

Next, assume that \( H^0(Z, L)_{\text{reg}} = \emptyset \). Then there exists \( 0 < l \leq Z \) (the cycle of fixed components of \( H^0(Z, L) \)) such that
\[
(5.1.6) \quad H^0(Z - l, \mathcal{L}(-l)) = H^0(Z, L) \quad \text{and} \quad H^0(Z - l, \mathcal{L}(-l))_{\text{reg}} \neq \emptyset.
\]

Consider the diagram
\[
\begin{array}{ccc}
\text{EC}_{\text{nf}}^{-1}(Z - l) & \xrightarrow{c} & \text{Pic}_{\text{nf}}^{-1}(Z - l) \\
\uparrow r_1 & & \downarrow r_2 \\
\text{EC}_{\text{nf}}^{-1}(Z - l)_{\text{reg}} & \xrightarrow{c} & \text{Pic}_{\text{nf}}^{-1}(Z - l)_{\text{reg}}
\end{array}
\]
By a local computation on charts of the Cartier divisors we get that \( r_1 \) is dominant (in fact, it is a submersion over any point of \( \text{im}(r_1) \)), hence the dimension of the generic fiber over \( (c_{\text{nf}}^{-1})(\mathcal{O}_{Z\text{-}l}(-l)) \) is \( (5.1.6) \)
\[
(5.1.7) \quad \dim \text{EC}_{\text{nf}}^{-1}(Z - l) - \text{EC}_{\text{nf}}^{-1}(Z - l)_{\text{reg}} = (n'l' - l, Z - l) - (n'l' - l, Z - l)_{\text{reg}}.
\]
On the other hand, by \( (2.7.2) \) we also have
\[
(5.1.8) \quad \dim (c_{\text{nf}}^{-1})(\mathcal{O}_{Z\text{-}l}(-l)) = h^1((Z - l)_{\text{reg}}, \mathcal{O}_{(Z - l)_{\text{reg}}(-l)}) - h^1(\mathcal{O}_{(Z - l)_{\text{reg}}}) + (n'l' - l, Z - l)_{\text{reg}}.
\]

Next we use the above commutative diagram: \( c_{\text{nf}} \circ r_1 = r_2 \circ c \).

The affine map \( r_2 \) is associated with the linear projection \( H^1(\mathcal{O}_{Z\text{-}l}) \rightarrow H^1(\mathcal{O}_{Z\text{-}l})_{\text{reg}} \), hence
\[
(5.1.9) \quad \dim r_2^{-1}(\mathcal{O}_{(Z - l)_{\text{reg}}(-l)}) = h^1(\mathcal{O}_{Z\text{-}l}) - h^1(\mathcal{O}_{(Z - l)_{\text{reg}}}).
\]

Note also that \( \mathcal{L} \rightarrow \mathcal{L}(-l)_{\mid Z - l} \) corresponds to an affine projection, hence if \( \mathcal{L} \) is generic in \( \text{Pic}_{\text{nf}}(Z) \) with \( \mathcal{L}_{\mid Z - l} = \mathcal{O}_{Z\text{-}l} \), then \( \mathcal{L}(-l)_{\mid Z - l} \) is generic in \( r_2^{-1}(\mathcal{O}_{(Z - l)_{\text{reg}}(-l)}) \). But by \( (5.1.0) \) it is also an element of \( \text{im}(c_{\text{nf}}^{-1}(Z - l)) \). Since we know from \( (5.10) \) that \( (c_{\text{nf}} \circ r_1)^{-1}(\mathcal{O}_{(Z - l)_{\text{reg}}(-l)}) \) is irreducible, we can compute its dimension two ways following the commutative diagram above. It means that the dimension of the fiber \( c^{-1}(\mathcal{L}(-l))_{\mid Z - l} \) can be computed by combination of \( (5.1.7), (5.1.8), (5.1.9) \), and it is
\[
(5.1.10) \quad \dim c^{-1}(\mathcal{L}(-l)_{\mid Z - l}) = h^1((Z - l)_{\text{reg}}, \mathcal{O}_{(Z - l)_{\text{reg}}(-l)}) - h^1(\mathcal{O}_{Z\text{-}l}) + (n'l' - l, Z - l)
\]
Finally, this can be compared with \( (2.7.3) \) applied for \( c = c_{\text{nf}}^{-1}(Z - l) \), which gives
\[
(5.1.11) \quad \dim c^{-1}(\mathcal{L}(-l)_{\mid Z - l}) = h^1(Z - l, \mathcal{L}(-l)) - h^1(\mathcal{O}_{Z\text{-}l}) + (n'l' - l, Z - l).
\]
Therefore, \( (5.1.10) \) and \( (5.1.11) \) show that
\[
(5.1.12) \quad h^1(Z - l, \mathcal{L}(-l)) = h^1((Z - l)_{\text{reg}}, \mathcal{O}_{(Z - l)_{\text{reg}}(-l)}).
\]
Now, \( h^0(Z, \mathcal{L}) \leq h^0(Z - l, \mathcal{L}(-l)) \leq h^1(\mathcal{O}_{(Z - l)_{\text{reg}}(-l)}) + \chi(Z - l, \mathcal{L}(-l)) < h^1(\mathcal{O}_{Z\text{-}l}) + \chi(Z, \mathcal{L}) \). Hence \( h^1(Z, \mathcal{L}) < h^1(\mathcal{O}_{Z\text{-}l}) \), which is certainly false, cf. \( (5.1.3) \).

This ends the proof of the Claim.
Therefore, in order to finish the proof of theorem, we have to prove that (5.1.13) holds whenever $0 < l \leq Z$ and $n \geq 1 - \min_{Z \geq l > 0} \chi(l)$. With the notation $l = l_+ + tE_v$ this reads as
\begin{equation}
- h^1(O_{Z_-}) < \chi(l_+ + tE_v) + t \cdot n - h^1(O_{Z_+}(-l_+ - tE_v)).
\end{equation}
Now notice that $H^0(O_{Z_+}(-l_+ - tE_v)) \subseteq H^0(O_{Z_+}(-tE_v))$ and the inclusion is proper if $t = 0$ (hence $l_+ > 0$). Via a computation this yields
\begin{equation}
\chi(tE_v) - h^1(O_{Z_+}(-tE_v)) \leq \chi(l_+ + tE_v) - h^1(O_{Z_+}(-l_+ - tE_v)),
\end{equation}
such that the inequality is strict if $t = 0$. This proves (5.1.13) for $t = 0$. Moreover, in order to prove (5.1.13) for $t \neq 0$ it is enough to verify
\begin{equation}
- h^1(O_{Z_-}) < \chi(tE_v) + t \cdot n - h^1(O_{Z_+}(-tE_v))
\end{equation}
for any $0 < t \leq -(Z, E_v^\ast)$ (and $n$ as in the assumption) since (5.1.14) and (5.1.15) imply (5.1.13).

Let $A$ denote the fixed component cycle of $O_{Z_{-}}(-tE_v)$, i.e. $0 < A \leq Z_{-}$ is the unique cycle, such that $H^0(O_{Z_+}(-tE_v)) = H^0(O_{Z_{-} - A}(-tE_v - A))$ and $H^0(O_{Z_{-} - A}(-tE_v - A))_{\text{reg}} \neq \emptyset$. Then
\[ h^1(O_{Z_+}(-tE_v)) = h^1(O_{Z_+}(-E_v + A)) + \chi(tE_v) - \chi(tE_v + A). \]
Moreover, since the line bundle $O_{Z_{-} - A}(-E_v - A)$ has no fixed components, by [NNII Th. 5.7.1] we get $h^1(O_{Z_{-} - A}(-E_v - A)) \leq h^1(O_{Z_+}(-E_v))$. But $h^1(O_{Z_{-} - A}) \leq h^1(O_{Z_+})$ too. This means that:
\begin{equation}
h^1(O_{Z_+}(-tE_v)) \leq h^1(O_{Z_-}) + \chi(tE_v) - \chi(tE_v + A).
\end{equation}
This compared with (5.1.15) shows that we have to prove that for $0 < t \leq -(Z, E_v^\ast)$
\[ 1 - \chi(tE_v + A) \leq t \cdot \left( 1 - \min_{Z \geq l > 0} \chi(l) \right), \]
which clearly holds. \[ \square \]

6. Singularities with ‘distinct pole property’

6.1. Let $(X, o)$ be an arbitrary singularity and let $\phi : \tilde{X} \to X$ be a fixed resolution. Fix also $I \subset \mathcal{V}$ and $Z \geq E$.

We say that $\phi$ satisfies the ‘distinct pole property’ with respect to $I$ if for any $v \in I$ there exists
(i) a basis $\{[\omega_1], \ldots, [\omega_n]\}$ of $H^0(\tilde{X}, \Omega^2_{\tilde{X}}(Z))/H^0(\tilde{X}, \Omega^2_{\tilde{X}})$ with representatives $\{\omega_n\}_n$, and
(ii) a partition $J \cup K$ of $\{1, 2, \ldots, h\}$ ($J \cap K = \emptyset$)
such that the forms $\{\omega_j\}_{j \in J}$ have no pole along $E_v$, while the pole orders along $E_v$ of the forms $\{\omega_k\}_{k \in K}$ are all non-trivial and different. (Note that this property is independent of the choice of the representatives $\{\omega_n\}_n$ in $H^0(\tilde{X}, \Omega^2_{\tilde{X}}(Z))$.)

Usually, in several examples, when this property holds, the forms $\{\omega_j\}_{j}$ are determined by some geometric property and can be chosen independently of the choice of $I$ and $v$. (See e.g. the case of elliptic singularities [NNIII 3.4].)

The main point in this definition is the following. If one can find a basis with ‘distinct pole property’ with respect to $\{v\}$, then for a generic divisor $D$ intersecting $E_v$ transversally, the Leray residue of $\omega = \sum_{n} \lambda_n \omega_n$ ($\lambda_n \in \mathbb{C}$) along $D$ is regular if and only if $\omega$ has no pole along $E_v$.

In particular, if we fix some $l' \in \mathcal{S}^\prime$ with $I = I(l')$, and we can find a basis with ‘distinct pole property’ with respect to $I$, then for any base point free line bundle $\mathcal{L} \in \text{im}(c^l(Z))$ by Theorem 2.7.12 and 2.6.3 and 2.7.6 one has $h^1(Z, \mathcal{L}^n) = h^1(Z, \mathcal{L}) = \dim \Omega_Z(I) = h^1(O_{Z}(-c_Z(I))).$ That is, $\lim_{n \to \infty} h^1(Z, \mathcal{L}^n) = h^1(Z, \mathcal{L}).$ In particular, by Lemma 4.1.8 $\dim(\text{im}(c^l(Z))) = c_Z(I) = \dim(\text{im}(c^l(Z)))$.

This applied for $I = \mathcal{V}$ reads as follows (see also Corollary 2.2.12 and Remark 5.1.2 of [NNIII]).
Proposition 6.1.1. Assume that \((X, o)\) is not rational and \(Z \geq 0\). Fix a resolution \(\phi\) and assume that \(\phi\) satisfies the distinct pole property with respect to \(V\). Then the following properties hold:

1. for any base point free \(L \in \text{Pic}^0(Z)\) we have \(n_0(Z, L) = 1\);
2. for any Chern class \(l' \in -S' \setminus \{0\}\) we have \(n_0(Z, l') = 1\);

In the sequel let us assume that \(Z \geq 0\).

In parallel to the definitions \(\bar{r}(I) = 1 + n_0(O_X(-l))\) (where \(I = H^0(\tilde{X}, O_X(-l))\)), \(I \subset \mathcal{O}_X(-l)\) for a certain resolution \(\tilde{X}\) and \(l \in S' \cap L\) and \(\bar{r}(X, o) = \max\{\bar{r}(I) : I\text{ as above}\}\) we can define the following objects as well. For any resolution \(\tilde{X}\) and base point free line bundle \(L \in \text{Pic}(\tilde{X})\) set \(\bar{r}(\tilde{X}, L) = 1 + n_0(L)\) and \(\bar{r}_{\text{free}}(X, o) = \max\{\bar{r}(\tilde{X}, L) : \tilde{X}\text{ some resolution and }L \in \text{Pic}(\tilde{X})\text{ is base point free}\}\).

From definitions \(\bar{r}_{\text{free}}(X, o) \geq \bar{r}(X, o)\).

Recall that \(\bar{r}(X, o) = 1\) if and only if \((X, o)\) is rational, cf. [OY15].

Note that under the assumption of Proposition 6.1.1 (formulated for a fixed resolution) we cannot deduce automatically that \(\bar{r}_{\text{free}}(X, o) = 2\). The point is that even if a certain resolution satisfies the distinct pole property, from this fact does not follow that it is satisfies for any resolution.

A typical situation which might appear is the following. Assume that \(E_v \cap E_w \neq \emptyset\), the pole orders of \(\omega_1\) (respectively of \(\omega_2\)) along \(E_v\) and \(E_w\) are 2 and 1 (respectively 1 and 2). Then \(\{\omega_1, \omega_2\}\) satisfies the distinct pole property with respect to \(\{v, w\}\). On the other hand, if we blow up an intersection point of \(E_v \cap E_w\), then along the new exceptional divisor the pole orders are the same (2 and 2).

However, in the presence of some additional properties of the pole cycles we have the following.

Lemma 6.1.2. Let \(\tilde{X}_{\text{min}}\) be the minimal resolution of \((X, o)\). Assume that in this resolution \(\{\omega_n\}_{n=1}^P\) satisfies the distinct pole property with respect to \(V\). Let \(P_n\) be the pole cycle of \(\omega_n\). If any of the following two properties hold

1. \(|P_n| \cap |P_m| = \emptyset\) for any \(n \neq m\), or
2. \(P_1 \leq P_2 \leq \cdots \leq P_P\),

then the distinct pole property holds for any resolution \(\tilde{X}\) (with respect to the set of vertices of that resolution).

Proof. Use induction with respect to the number of blow ups needed to obtain \(\tilde{X} \to \tilde{X}_{\text{min}}\). \(\square\)

In particular, Proposition 6.1.1 and Lemma 6.1.2 implies the following.

Corollary 6.1.3. Assume that \((X, o)\) is not rational. If any resolution \(\phi\) satisfies the distinct pole property with respect to \(V\) then \(\bar{r}_{\text{free}}(X, o) = 2\). In particular \(\bar{r}(X, o) = 2\) too.

This statement applies whenever the assumptions of Lemma 6.1.2 hold.

We need QHS.

Example 6.1.4. Assume that \((X, o)\) is an elliptic singularity. In this case \(p_g\) might depend on the analytic structure supported on the elliptic topological type. However, for the minimal resolution the distinct pole property with respect to \(V\) together with property (ii) from Lemma 6.1.2 are satisfied (see e.g. [NN11, 3.4]). In particular \(\bar{r}_{\text{free}}(X, o) = \bar{r}(X, o) = 2\). This provides a new proof of the identity \(\bar{r}(X, o) = 2\), valid for elliptic germs, proved originally in [O17].

Example 6.1.5. Consider the following minimal good resolution graph:

```
\[ E_0 \quad \cdots \quad -1 \quad \cdots \quad -1 \quad -1 \]
\]
where $E_0$ has $n \geq 2$ adjacent edges, all $g_i = 0$, and all the unmarked vertices have self-intersection number $-N$, where $N$ is very large with respect to $n$. Let us denote the $(-1)$–vertices by $v_1, \ldots , v_n$.

Set $F = \sum_{i=1}^n E_{v_i}$. A computation shows that $|Z_K| = E + F$. Note also that the Artin minimal cycle is $Z_{min} = E + 2F$, hence $|Z_K| \leq Z_{min}$. Let us fix an arbitrary analytic structure $(X, o)$ supported by the topological type given by this graph, and a resolution $\tilde{X}$ with the above dual graph. Then $h^1(\mathcal{O}_{Z_{min}})$ can be computed by Laufer algorithm [La72], and it turns out that $p_g = h^1(\mathcal{O}_{E + F}) = h^1(\mathcal{O}_{Z_{min}}) = n$ and the cohomological cycle $Z_{coh} = E + F - E_0$.

For any $v_i$ consider the minimal star–shaped subgraph whose node is this vertex. Then it determines a minimally elliptic graph $\Gamma_i$ and singularity with $p_g = 1$ and it admits a unique differential form with nontrivial pole. Using this we obtain that there exists a collection of forms $\omega_1, \ldots , \omega_n$ on $\tilde{X}$ such that the pole cycle of $\omega_i$ is non–trivial and it is supported on $\Gamma_i$. In particular, they satisfy the distinct pole property together with the additional property (i) of Lemma 6.1.2. Therefore $\bar{r}_{free}(X, o) = \bar{r}(X, o) = 2$ (for any analytic structure supported on the above $\Gamma$).

On the other hand, since $n \geq 2$, the graph is not elliptic: $\chi(Z_{min}) = 1 - n < 0$.

This answers negatively [O15] Problem 3.12 of the third author (which asked whether the elliptic singularities are characterized by the property $\bar{r}(X, o) = 2$). In fact, we proved that there exists a singularity with $\bar{r}(X, o) = 2$ but with arbitrary small min $\chi$ (or with arbitrary high $h^1(\mathcal{O}_{Z_{min}})$).

Example 6.1.6. One can find non–elliptic singularities with $\bar{r}_{free}(X, o) = 2$ even among the Gorenstein germs. Consider the following resolution graph (cf. [NNIII Example 5.1.3]).

![Resolution Graph](attachment:resolution_graph.png)

The graph is not elliptic, min $\chi = -1$.

It is realized e.g. by the hypersurface singularity with non–degenerate Newton boundary $\{x^3 + x^{13} + y^{13} + x^2y^2 = 0\}$. This analytic structure has $p_g = 5$ and it is clearly Gorenstein. Let $\omega$ be the Gorenstein form (with pole $Z_K$). Then the classes of the five forms $\omega, \omega x, \omega x^2, \omega y, \omega y^2$ constitute a basis of $H^0(\Omega^2_X(Z))/H^0(\Omega^2_X)$, and they satisfy the ‘distinct pole property’ for $Z \gg 0$ (the verification is left to the reader; the divisor of $x$ is $E_1^\ast$, while the divisor of $y$ is $E_2^\ast$). A verification shows that the distinct pole property survives even if we blow up (several times) this $\tilde{X}$.

This example shows that Lemma 6.1.2 can be generalized to a more general situation regarding the structure of the poles (a combination of properties (i) and (ii)). (These conditions can be compared with the GCD property from [O15].)

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