Finding Bipartite Partitions on Co-Chordal Graphs

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Abstract

In this paper, we show that the biclique partition number (bp) of a co-chordal (complementary graph of chordal) graph $G = (V, E)$ is less than the number of maximal cliques (mc) of its complementary graph: a chordal graph $G^c = (V, E^c)$. We first provide a general framework of the “divided and conquer” heuristic of finding minimum biclique partition on co-chordal graphs based on clique trees. Then, an $O(|V|(|V| + |E^c|))$-time heuristic is proposed by applying lexicographic breadth-first search. Either heuristic gives us a biclique partition of $G$ with a size of $\text{mc}(G^c) − 1$. Eventually, we prove that our heuristic can solve the minimum biclique partition problem on $G$ exactly if its complement $G^c$ is chordal and clique vertex irreducible. We also show that $\text{mc}(G^c) − 2 \leq \text{bp}(G) \leq \text{mc}(G^c) − 1$ if $G$ is a split graph.

Keywords: biclique partition, co-chordal graphs, clique vertex irreducible, split graphs

1. Introduction

The biclique partition number (bp) of a graph $G$ is referred to as the least number of complete bipartite (biclique) subgraphs that are required to cover the edges of the graph exactly once. Graham and Pollak first introduced this concept in network addressing \cite{10} and graph storage \cite{11}. Their famous Graham-Pollak Theorem proves the biclique partition numbers on complete graphs and

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it draws much attention from algebraic graph theory \[24, 18, 6, 25, 16\]. However, no purely combinatorial proof is known to the result \[17\]. Rawshdeh and Al-Ezeh \[20\] extend Graham-Pollak Theorem to find biclique partition numbers on line graphs and their complements of complete graphs and bicliques. The biclique partition also has a strong connection with biclique cover number (bc), where the edges of a graph are covered by bicliques but not necessarily disjointed. Pinto \[19\] shows that \(bp(G) \leq \frac{1}{2}(3^{bc(G)} - 1)\).

Moreover, finding a biclique partition with the minimum size is NP-complete even on the graphs without 4-cycles \[14\]. In our work, we focus on studying the biclique partition number on co-chordal (complement of chordal) and its important subclass, split graphs.

There are also many related research studies around biclique partitions. Motivated by a technique for clustering data on binary matrices, Beine et al. \[3\] consider a biclique vertex partition problem on a bipartite graph where each vertex is covered exactly once in a collection of biclique subgraphs. De Sousa Filho et al. \[8\] also study the biclique vertex partition problem and its variant bicluster editing problem, where they develop a polyhedral study on biclique vertex partitions on a complete bipartite graph. Groshaus et al. \[12\] give a polynomial-time algorithm to determine whether a graph is a biclique graph, an intersection graph of the bicliques, of a subclass of split graphs. Shigeta and Amano \[23\] provide an explicit construction of an ordered biclique partition, a variant of biclique partition, of \(K_n\) of size \(n^{1/2+o(1)}\), which improves the \(O(n^{2/3})\) bound shown by Amano \[2\].

Another related graph characteristics is \(bp_k(G)\) where the edges can be covered by at least one and at most \(k\) biclique subgraphs \[1\] and Alon shows that the minimum possible number for \(bp_k(K_n)\) is \(\Theta(kn^{1/k})\), where \(K_n\) is a complete graph with \(n\) vertices. A recent work by Rohatgi et al. \[21\] shows that if the

\[1\] All most all chordal graphs are split graphs. It means that as \(n\) goes to infinity, the fraction of \(n\)-vertex split graphs in \(n\)-vertex chordal graphs go to 1 \[4\]. Since the complement of a split graph is also split, a split graph is also co-chordal.
each edge is exactly covered by \( k \) bicliques, the number of bicliques required to cover \( K_n \) is \((1 + o(1))n\).

In this paper, we study a biclique (edge) partition problem on co-chordal graphs and split graphs using clique trees and lexicographic breadth-first search defined in Section 2. In Section 4, we provide a heuristic to find a biclique partition on a co-chordal graph given a clique tree of the complement of the co-chordal graph. We also give proof of the correctness and show the size of the biclique partition is exactly equal to the number of maximal cliques in the complementary graph of the co-chordal graph minus one. It then gives a Corollary that the biclique partition number of a co-chordal graph is less than the number of maximal cliques of its complement. In Section 5 we provide an efficient heuristic to obtain biclique partitions of co-chordal graphs based on lexicographic breadth-first search. We also show that two heuristics provide biclique partitions of the same size. In Section 6 we prove that our heuristic can find a minimum biclique partition of \( G \) if its complement \( G^c \) is chordal and clique vertex irreducible. We also derive a lower bound of the biclique partition number of split graphs and show that our heuristics can obtain a biclique partition on any split graph with a size no more than the biclique partition number plus one.

2. Preliminaries

A simple graph is a pair \( G = (V, E) \) where \( V \) is a finite set of vertices and \( E \subseteq \{ uv : u, v \in V, u \neq v \} \). We use \( V(G) \) and \( E(G) \) to represent the vertex set and edge set of the graph \( G \). Two vertices are adjacent in \( G \) if there is an edge between them. A subgraph \( G' = (V', E') \) of \( G \) is a graph where \( V' \subseteq V \) and \( E' \subseteq \{ uv \in E : u, v \in V' \} \). An induced subgraph of \( G \) by only keeping vertices \( A \) is denoted as \( G(A) = (A, E_A) \), where \( E_A = \{ uv \in E : u, v \in A \} \). A graph is a complete graph or a clique if there is an edge between every two distinct vertices of the graph. We denote cliques on \( n \) vertices as \( K_n \). The neighborhood of a vertex \( v \in V \) of the graph \( G \) is denoted as \( N_G(v) \) (not including \( v \)). A maximal clique subgraph of \( G \) is an induced subgraph of \( G \) such that it isn’t a
proper subgraph of any clique subgraph of $G$. Note that a graph with only one vertex $K_1$ is also a clique. We denote the number of maximal clique subgraphs of $G$ as $mc(G)$ and we use $\mathcal{K}_G$ or $\mathcal{K}$ to denote the set of all maximal cliques of $G$. A maximum clique subgraph of $G$ is a clique subgraph of $G$ with the maximum number of vertices and we denote that number as the clique number of $G$, $\omega(G)$. An independent set of a graph $G$ is a set of vertices that none of them are adjacent with each other in $G$. Note that an independent set can be empty or only have one vertex.

A graph $C_n = (V, E)$ is a cycle if the vertices and edges:

\[ V = \{v_1, v_2, \ldots, v_n\} \]
\[ E = \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n, v_nv_1\}. \]

A graph is tree if it is connected and does not have any subgraph that is a cycle.

Given two vertex sets $U$ and $V$, we denote $U \times V$ to be the edge set $\{uv : u \in U, v \in V\}$. A bipartite graph $G = (L \cup R, E)$ is a graph where $L$ and $R$ are disjointed vertex sets and the edge set $E \subseteq L \times R$. A clique graph is a complete bipartite graph $G = (L \cup R, E)$ where $E = L \times R$ and we denote it as $\{L, R\}$ for short. A biclique partition of a graph $G$ is a collection of biclique subgraph of $G$ such that every edge of $G$ is in exactly one biqlue of the collection. Minimum biclique partition problem on $G$ is to find a biclique partition with the minimum size and we denote that size to be $bp(G)$.

We denote that $[n] = \{1, 2, \ldots, n\}$ where $n$ is a positive integer. A vertex is simplicial if its neighborhood is a clique. An ordering $v_1, v_2, \ldots, v_n$ of $V$ is a perfect elimination ordering if for all $i \in [n]$, $v_i$ is simplicial on the induced subgraph $G(\{v_j : j \in \{i, i+1, \ldots, n\}\})$. An ordering function $\sigma : [n] \rightarrow V$ is defined to describe the ordering of vertices $V$. A clique tree $T_K$ for a chordal graph $G$ is a tree where each vertex represents a maximal clique of $G$ and satisfies the clique-intersection property: given any two distinct maximal cliques $K_1$ and $K_2$ in the tree and every clique on the path between $K_1$ and $K_2$ in $T_K$ contains $V(K_1) \cap V(K_2)$. We also define the middle set of edge $e \in T_K$ to be the intersection of the vertex of cliques on its two ends.

A graph $G$ is clique vertex irreducible if every maximal clique in $G$ has a vertex which does not lie in any other maximal clique of $G$. [15]
A graph is chordal if there is no induced cycle subgraph of length greater than 3. A graph is chordal if and only if it has a clique tree \([5]\). In our work, we study biclique partitions on co-chordal graphs, so most of \(G^c\)’s in the later sections are chordal graphs. We use \(K^c\) and \(T_{K^c}\) to denote the set of all maximal cliques and a clique tree of \(G^c\).

Also, a graph is chordal if and only if it has a perfect elimination ordering \([9]\). The perfect elimination ordering of a chordal graph can be obtained by the reverse order of lexicographic breadth-first search (LexBFS) described in Algorithm \([22,7]\), where lexicographical order is defined in Definition \([13]\). Note that the LexBFS algorithm in Algorithm \([1]\) can be implemented in linear-time: \(O(|V| + |E|)\) with partition refinement \([13]\).

**Definition 1.** [Lexicographical Order] Let \(X\) be a set of all vectors of real numbers with a finite length. Then, we can define a lexicographical order on \(X\) where

1. \(\emptyset \in X\) and for any \(x \in X\), \(\emptyset \preceq x\).
2. \((x_1, x_2) \preceq (y_1, y_2)\) if and only if \(x_1 \leq y_1\) and \(x_2 \preceq y_2\) where \(x_1, y_1 \in \mathbb{R}\) and \(x_2, y_2 \in X\).

**Algorithm 1** Generic Lexicographic Breadth-First Search (LexBFS) \([7]\).

1: **Input:** Graph \(G = (V, E)\) and an arbitrary selected vertex \(v\) in \(V\).
2: **Output:** An ordering function \(\sigma : [n] \rightarrow V\) of the vertices \(V\).
3: label\((v) \leftarrow \{n\}\) where \(n = |V|\), and label\((u) \leftarrow \{\}\) for all \(u \in V \setminus \{v\}\).
4: for \(i \in \{n, n-1, \ldots, 1\}\) do
5: Select an unnumbered vertex \(u\) with lexicographically the largest label.
6: \(\sigma(n+1-i) \leftarrow u\).
7: for each unnumbered vertex \(w\) in \(N_G(u)\) do
8: label\((w) \leftarrow \text{label}(w) \cup \{i\}\).
9: end for
10: end for
11: return \(\sigma\).
3. Partitioned Biclique

We will introduce a new definition: partitioned biclique, which can naturally partition the edges of a graph into two edge disjointed induced subgraphs. We first start with a definition of partitioned biclique.

**Definition 2** (partitioned biclique). Given a graph $G = (V, E)$, a biclique subgraph of $G$, $\{L, R\}$, is a partitioned biclique subgraph of $G$ if the edge sets of $\{L, R\}$, $G(V \setminus L)$, and $G(V \setminus R)$ partition the edges of $G$, i.e. each edge of $G$ is exactly in one of $\{L, R\}$, $G(V \setminus L)$, and $G(V \setminus R)$.

![Figure 1](image.png)

Figure 1: Given a graph $G = (V, E)$, $\{L, R\}$ is a partitioned biclique of $G$, where each edge in $E$ is exactly in one of $\{L, R\}$, $G(V \setminus L)$, and $G(V \setminus R)$.

We then show that an arbitrary biclique subgraph can divide the edges of the original graphs into three parts: the biclique and two induced subgraphs.

**Lemma 1.** Given a graph $G = (V, E)$, let $\{L, R\}$ be an arbitrary biclique subgraph of $G$. Then, $E = E(\{L, R\}) \cup E(G(V \setminus L)) \cup E(G(V \setminus R))$.

**Proof.** We first show that $E(\{L, R\}) \cup E(G(V \setminus L)) \cup E(G(V \setminus R)) = E$. Since $\{L, R\}$, $G(V \setminus L)$, $G(V \setminus R)$ are all subgraphs of $G$, then $E(\{L, R\}) \cup E(G(V \setminus L)) \cup E(G(V \setminus R)) = E$. Therefore, $\{L, R\}$, $G(V \setminus L)$, $G(V \setminus R)$ partition the edges of $G$. Hence, $E = E(\{L, R\}) \cup E(G(V \setminus L)) \cup E(G(V \setminus R))$. 

\[ \boxed{\text{\vspace{1em}}} \]
L)) \cup E(G(V \setminus R)) \subseteq E.

Given an arbitrary \( uv \in E \). If \( u, v \in (V \setminus L) \), \( uv \in E(G(V \setminus L)) \). Similarly, if \( u, v \in (V \setminus R) \), \( uv \in E(G(V \setminus R)) \). If \( u, v \in (V \setminus L) \) and \( u, v \in (V \setminus R) \), then \( uv \in E\). Thus, \( E \subseteq E(\{L, R\}) \cup E(G(V \setminus L)) \cup E(G(V \setminus R)) \) and \( E(\{L, R\}) \cup E(G(V \setminus L)) \cup E(G(V \setminus R)) = E. \)

Next, we show that a biclique subgraph is a partitioned biclique of a graph \( G \) if and only if the vertices in \( G \) that is not in the biclique form an independent set in \( G \).

**Proposition 1.** Given a graph \( G = (V, E) \) and a biclique subgraph of \( G \): \( \{L, R\} \), then \( \{L, R\} \) is a partitioned biclique subgraph of \( G \) if and only if \( V \setminus (L \cup R) \) is an independent set in \( G \).

**Proof.** Denote \( C = V \setminus (L \cup R) \). In the backward direction, suppose that \( C \) is an independent set in \( G \). By Lemma 1, we know that \( E = E(\{L, R\}) \cup E(G(V \setminus L)) \cup E(G(V \setminus R)) \). Then, we need to show that \( E(\{L, R\}) \), \( E(G(V \setminus L)) \), \( E(G(V \setminus R)) \) are disjointed. Since every edge in \( \{L, R\} \) is between vertices in \( L \) and \( R \), then \( E(\{L, R\}) \cap E(G(V \setminus L)) = \emptyset = E(\{L, R\}) \cap E(G(V \setminus R)) \). Since \( C \) is an independent set of \( G \) and
\[
[(V \setminus L) \times (V \setminus L)] \cap [(V \setminus R) \times (V \setminus R)] = C \times C,
\]
we know that \( E(G(V \setminus L)) \cap E(G(V \setminus R)) = \emptyset. \)

In the forward direction, the proof is by contradiction. Suppose that \( C \) isn’t an independent set in \( G \). Then, there exists an edge \( uv \) such that \( u, v \in C \). Thus, \( uv \in E(G(V \setminus L)) \) and \( uv \in E(G(V \setminus R)) \) which is a contradiction. \( \Box \)

Note that an independent set can be empty. Since the partitioned biclique can partition the original graphs into a biclique subgraph and two induced subgraphs, it can be used to design a heuristic to find a biclique partition of a graph. In the next two sections, we will focus on a class of graph, co-chordal, where partitioned bicliques are easy to find since the complementary graph is chordal.
4. Heuristic Based on Clique Trees

In this section, we want to design a heuristic to find a biclique partition of a co-chordal graph $G$. Since $G^c$ is chordal, one of the good ways to represent $G^c$ is its clique tree where each vertex represents a maximal clique of $G$ and satisfies the clique-intersection property. We demonstrate a heuristic with an input of a clique tree $T_{K^c}$ of $G^c$ and an output of a biclique partition of $G$ in Algorithm 2. We also show that the size of that biclique partition is equal to $mc(G^c) - 1$, which provides us an upper bound of the biclique partition number of co-chordal graphs.

**Algorithm 2** Find a biclique partition of a co-chordal graph $G$ given a clique tree of $G^c$.

1. **Input**: A clique tree $T_{K^c}$ of a chordal graph $G^c$.
2. **Output**: A biclique partition $bp$ of the complementary graph $G$ of $G^c$.
3. **function** FINDPARTITION($T_{K^c}$)
4.   if $|V(T_{K^c})| \leq 1$ then
5.     return $\emptyset$.
6.   end if
7.   Select an arbitrary edge $e$ to cut $T_{K^c}$ into two components $T_{K^c_1}$ and $T_{K^c_2}$.
8.   $L = \bigcup_{K \in V(T_{K^c_1})} V(K) \setminus \text{mid}(e)$; $R = \bigcup_{K \in V(T_{K^c_2})} V(K) \setminus \text{mid}(e)$.
9.   return $\{\{L, R\}\} \cup \text{FINDPARTITION}(T_{K^c_1}) \cup \text{FINDPARTITION}(T_{K^c_2})$.
10. **end function**

We start with proving the output of Algorithm 2 is a biclique partition of a co-chordal graph $G$ by showing that at each recursion a nonempty partitioned biclique $\{L, R\}$ is found and two subtrees $T_{K^c_1}$ and $T_{K^c_2}$ are also clique trees of two induced subgraphs $G^c(V \setminus L)$, and $G^c(V \setminus R)$ of $G^c$ respectively. Note that the edge $e$ can be selected arbitrarily in Algorithm 2.

**Proposition 2.** Given a chordal graph $G^c = (V, E^c)$ and one of its clique trees $T_{K^c} = (K^c, E)$ where $V(T_{K^c}) > 1$, any edge $e$ of $T_{K^c}$ can partition $K^c$ into $K_1$
and $\mathcal{K}_2$ (trees $T_{\mathcal{K}_1}$ and $T_{\mathcal{K}_2}$ respectively) such that

1. The edges of $\{L, R\} = \{\bigcup_{K \in \mathcal{K}_1} V(K) \setminus \text{mid}(e), \bigcup_{K \in \mathcal{K}_2} V(K) \setminus \text{mid}(e)\}$, $G(V \setminus L)$, and $G(V \setminus R)$ partition the edges in $G$ where $G$ is the complementary graph of $G^c$, i.e. $\{L, R\}$ is a partitioned biclique subgraph of $G$.
2. $T_{\mathcal{K}_2}$ and $T_{\mathcal{K}_1}$ are clique trees of chordal graphs $G^c(V \setminus L)$, and $G^c(V \setminus R)$ respectively.
3. Both $L$ and $R$ are not empty.

Proof. (1) is proved by Claim 1 and definition of partitioned biclique, (2) is proved by Claim 2, and (3) is proved by Claim 3.

Before we can prove that $\{L, R\}$ is a partitioned biclique of $G$, we need to remark that two vertices in a graph are adjacent if and only if they are both in a maximal clique of $G$.

Remark 1. Given a graph $G = (V, E)$ and two distinct vertices $u, v \in V$, $uv \in E$ if and only if both $u$ and $v$ are the vertices of some maximal clique of $G$.

We next prove that $\{L, R\}$ is a partitioned biclique of $G$.

Claim 1. $\{L, R\}$ is a partitioned biclique of the complementary graph $G = (V, E)$ of $G^c$.

Proof. Since every edge in a tree is a cut, $e$ can partition $T_{\mathcal{K}^c}$ into two sets of vertices, $\mathcal{K}_1$ and $\mathcal{K}_2$, in $T_{\mathcal{K}^c}$. Let the two ends of edge $e$ in $T_{\mathcal{K}^c}$ to be $K'$ and $K''$. Since $\text{mid}(e) = V(K') \cap V(K'')$ and both $K'$ and $K''$ are clique subgraphs of $G^c$, then $\text{mid}(e)$ is an independent set of $G$.

Given an arbitrary $u \in \bigcup_{K \in \mathcal{K}_1} V(K) \setminus \text{mid}(e)$ and $v \in \bigcup_{K \in \mathcal{K}_2} V(K) \setminus \text{mid}(e)$, we assume that $uv \in G^c$. By Remark 1, we know that $\{u, v\} \subseteq V(K)$ for some $K \in \mathcal{K}$. Without loss of generality, we assume that $u \in K_1$ and $\{u, v\} \in K_2$ where $K_1 \in \mathcal{K}_1$ and $K_2 \in \mathcal{K}_2$. Since $K'$ and $K''$ both on the path between $K_1$ and $K_2$, $V(K_1) \cap V(K_2) \subseteq \text{mid}(e)$. Therefore, $u \notin K_2$, which is a contradiction.

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In Claim 2 we prove part (2) of Proposition 2. We first show that any subgraph of a chordal graph is chordal and then use it to prove part (2) in Claim 2.

**Remark 2.** Given a chordal graph \( G = (V, E) \), any induced subgraph of \( G \) is chordal.

**Claim 2.** \( T_{K_2^c} \) and \( T_{K_1^c} \) are clique trees of chordal graphs \( G^c(V \setminus L) \) and \( G^c(V \setminus R) \) respectively.

**Proof.** By Remark 2 we know that both \( G^c(V \setminus L) \) and \( G^c(V \setminus R) \) are chordal. Thus, there exist clique trees for both \( G^c(V \setminus L) \) and \( G^c(V \setminus R) \). Without loss of generality, we only need to prove \( T_{K_1^c} \) is a clique tree of \( G^c(V \setminus R) \). Since \( T_{K_1^c} \) is a subtree of \( T_{K_2^c} \), we only need to show that \( K_1 \) is the set of all maximal cliques of \( G^c(V \setminus R) \).

First, we want to prove that \( \bigcup_{K \in K_1} V(K) = V \setminus R \). We proved in Claim 1 \( \{L, R\} = \{\bigcup_{K \in K_1} V(K) \setminus \text{mid}(e), \bigcup_{K \in K_2} V(K) \setminus \text{mid}(e)\} \) is a biclique. Thus, \( L \) and \( R \) are disjoint vertex sets. Since \( V = \bigcup_{K \in K} K \) and \( L \cup R = \bigcup_{K \in K} K \setminus \text{mid}(e) \), then \( \text{mid}(e) = V \setminus (L \cup R) \). Thus,

\[
\bigcup_{K \in K_1} V(K) = L \cup \text{mid}(e) = V \setminus R.
\]

Hence, \( K_1 \) is a maximal clique of \( G^c(V \setminus R) \) for any \( K_1 \in K_1 \). By the definition of clique tree, given an arbitrary \( K_2 \in K_2 \)

\[
V(K_2) \cap (V \setminus R) = V(K_2) \cap \left( \bigcup_{K \in K_1} V(K) \right) = \bigcup_{K \in K_1} (V(K_2) \cap V(K)) \subseteq \text{mid}(e).
\]

Since \( \text{mid}(e) \subset K' \) for some \( K' \in K_1 \), then \( V(K_2) \cap (V \setminus R) \) can not be a maximal clique of \( G^c \). Therefore, \( K_1 \) is the set of all maximal cliques of \( G^c(V \setminus R) \) and \( T_{K_1^c} \) is its clique tree.

Then, we show that the edge set of biclique \( \{L, R\} \) is not empty.

**Claim 3.** Both \( L \) and \( R \) are not empty.
Proof. Let the two ends of edge $e$ in $T_{K^c}$ be $K'$ and $K''$. Since both $K'$ and $K''$ are maximal cliques of $G^c$ and $\text{mid}(e) = V(K') \cap V(K'')$, then both $V(K') \setminus \text{mid}(e)$ and $V(K'') \setminus \text{mid}(e)$ are not empty. We can complete the proof since $K' \in K_1$ and $K'' \in K_2$. □

Next, we show that the output of Algorithm 2 is a biclique partition of a co-chordal graph $G$.

**Theorem 1.** Given a co-chordal graph $G$ and clique tree $T_{K^c}$ of its complement $G^c$, the output of $\text{FindPartition}(T_{K^c})$ is a biclique partition of $G$.

Proof. We will use induction to prove Theorem 1. In the basis step, if $T_{K^c}$ only has one vertex, then $G^c$ is a complete graph and $G$ is an empty graph. Thus, $\text{bp}(G) = 0$.

In the induction step, supposed that $\text{FindPartition}(T_{K^c})$ is a biclique partition of $G$ if $V(T_{K^c}) < k$. Let $T_{K^c}$ to have $k$ vertices. Proved by Proposition 2, an arbitrary edge $e$ to cut $T_{K^c}$ into two components $T_1$ and $T_2$, where the edges of $G$ can be partitioned into a biclique $\{L, R\}$ as shown in Algorithm 2 $G(V \setminus L)$, and $G(V \setminus R)$. Moreover, $T_1$ and $T_2$ are clique trees of $G^c(V \setminus R)$, and $G^c(V \setminus L)$ respectively. Then, $\text{FindPartition}(T_1)$ returns a biclique partition of $G(V \setminus R)$ and $\text{FindPartition}(T_2)$ returns a biclique partition of $G(V \setminus L)$. Therefore, $\{\{L, R\}\} \cup \text{FindPartition}(T_1) \cup \text{FindPartition}(T_2)$ is a biclique partition of $G$. □

The size of the output of Algorithm 2 is equal to the one less than the number of maximal cliques of $G^c$.

**Theorem 2.** Given a co-chordal graph $G$ (with at least one vertex) and clique tree $T_{K^c}$ of its complement $G^c$, the output of $\text{FindPartition}(T_{K^c})$ is a biclique partition of $G$ with size $\text{mc}(G^c) - 1$.

Proof. By Theorem 1, we know that the output of $\text{FindPartition}(T_{K^c})$ is a biclique partition of $G$. We then want to prove that the size of the output of $\text{FindPartition}(T_{K^c})$ is $\text{mc}(G^c) - 1$ by induction.
In the basis step, \(|V(T_{K^c})| = 1\), then \(mc(G^c) = 1\) since \(G^c\) is a complete graph. The output is an empty set, which has size of 0.

In the induction step, assume that the size of the output of \(|V(T_{K^c})|\) is \(|V(T_{K^c})| - 1\) when \(|V(T_{K^c})| \in \{1, 2, \ldots, k-1\}\). Then, if \(|V(T_{K^c})| = k\), the output is \(\{L, R\} \cup \text{FindPartition}(T_1) \cup \text{FindPartition}(T_2)\) where \(L, R, T_1\) and \(T_2\) are defined in Algorithm 2. Since \(|V(T_1)| + |V(T_2)| = k\) and \(|V(T_1)|, |V(T_2)| > 0\), we know that \(|V(T_1)| < k\) and \(|V(T_2)| < k\). Also both \(L\) and \(R\) are not empty by Claim 3, the size of the output of \(\text{FindPartition}(T_{K^c})\) is \(1 + |V(T_1)| - 1 + |V(T_2)| - 1 = k - 1\).

Theorem 2 leads us to a directed result that the biclique partition number of a co-chordal graph \(G\) is less than the number of maximal cliques of \(G^c\).

**Corollary 1.** Suppose \(G\) is a co-chordal graph, then \(bp(G) \leq mc(G^c) - 1\).

5. Heuristic Based on Perfect Elimination Ordering

Although Algorithm 2 can give a biclique partition of a co-chordal graph \(G\), it requires a clique tree of its complementary chordal graph \(G^c\). In this section, we will present another heuristic to find biclique partitions on co-chordal graphs. Given a co-chordal graph \(G = (V, E)\), Algorithm 3 can return a biclique partition of \(G\) with a same size as Algorithm 2 in \(O(|V|(|V| + |E^c|))\) time.
Algorithm 3 A LexBFS-based heuristic to find small biclique partition on 
co-chordal graph $G$.

1: **Input**: Co-chordal graph $G = (V, E)$.
2: **Output**: A biclique partition $bp$ of $G$.
3: Initialize $bp ← \emptyset$.
4: $G′ ← G^c$ so it is a chordal graph.
5: Find a perfect elimination ordering $σ$ of $G'$.
6: $S ← \{σ(1), \ldots, σ(n)\}$.
7: while $S$ is not $\emptyset$ do
8: Find the first element $v$ of $S$.
9: $L ← S \setminus \{v\} \cup N_{G'}(v)$. \> $L$ is the set of vertices not in the clique.
10: $R ← \{v\} \cup \{u \in N_{G'}(v) : N_{G'}(u) \cap L = \emptyset\}$.
11: If $L$ is not empty, $bp ← bp \cup \{L, R\}$.
12: $S ← S \setminus R$ and remove vertices in $R$ from $G'$.
8: end while
14: return $bp$

We first show that in a perfect elimination order of a chordal graph, any two 
distinct maximal cliques cannot share a vertex which is a first vertex of those 
two maximal cliques in the ordering.

**Lemma 2.** Given a perfect elimination ordering $σ : [n] → V$ of a chordal graph $G^c = (V, E^c)$, suppose that there exist two distinct maximal cliques $K_1$ and $K_2$ of $G^c$. Then, first vertex of $K_1$ in the ordering $σ$ cannot be the same vertex as that of $K_2$.

**Proof.** Assume that the first vertex of both $K_1$ and $K_2$ is $v$ in the ordering $σ$. By the definition of perfect elimination ordering, we know that $v$ and all of its neighbors after $v$ in the ordering $σ$ form a clique. Since $K_1$ and $K_2$ are maximal cliques, then $K_1 = K_2$, which is a contradiction. \[\]

Before we can prove that Algorithm 3 returns a biclique partition of a co-
chordal graph, we need to show that $\{L, R\}$ is a partitioned biclique of $G^c$ where
one of the two induced subgraphs, $G_c(V' \setminus R)$, has an empty edge set and the
order $\sigma$ preserve a perfect elimination ordering of a chordal graph $G'(V' \setminus L)$ in
each iteration of the while loop (Line 7).

**Proposition 3.** Given a perfect elimination ordering $\sigma$ of a chordal graph $G' = (V', E')$ with at least one edge, let $L = V' \setminus (\{\sigma(1)\} \cup N_{G'}(\sigma(1)))$ and $R = \{\sigma(1)\} \cup \{u \in N_{G'}(\sigma(1)) : N_{G'}(u) \cap L = \emptyset\}$. Then, the edges of $G_c$, the complement of $G'$, can be partitioned into $E(\{L, R\})$ and $E[G_c(V' \setminus L)]$. Furthermore, let $\sigma'$ be the order of $V' \setminus L$ in $\sigma$. Then, $\sigma'$ is a perfect elimination ordering of a chordal graph $G'(V' \setminus L)$.

**Proof.** Since $\sigma$ is a perfect elimination ordering, then $\{\sigma(1)\} \cup N_{G'}(\sigma(1))$ forms a clique subgraph of $G'$. Thus, $V' \setminus (L \cup R)$ is an independent set of $G_c$. By Proposition 4 $\{L, R\}$ is a partitioned biclique of $G_c$. Thus, the edges of $G_c$ is partitioned into $E(\{L, R\})$, $E[G_c(V' \setminus L)]$, and $E[G_c(V' \setminus R)]$. Since $\{\sigma(1)\} \cup N_{G'}(\sigma(1))$ is an independent set of $G_c$, $E[G_c(V' \setminus R)] = \emptyset$. By Remark 2 $G'(V' \setminus L)$ is chordal. From the definition of perfect elimination ordering, $\sigma'$ is a perfect elimination ordering of a chordal graph $G'(V' \setminus L)$.

Next, we show that the output of Algorithm 3 is a biclique partition of a co-chordal graph $G$ with a size of $mc(G^c) - 1$.

**Theorem 3.** Algorithm 3 returns a biclique partition of $G$ with a size of $mc(G^c) - 1$ given $G$ is a co-chordal graph.

**Proof.** By Proposition 3 the output of Algorithm 3 is a biclique partition of $G$. Now, we just need to prove the size of the output is $mc(G^c) - 1$.

In order to be clear, we use $v_i, L_i, R_i, G'_i, S_i$ to denote those in the iteration $i$ of while loop (Line 7) where $G'_i$ and $S_i$ are $G'$ and $S$ before the operation in Line 12.

By Lemma 2 the set of all maximal cliques of $G^c$ is a subset of cliques formed by $\{u'\} \cup \{u \in N_{G^c}(u') : \sigma^{-1}(u) > \sigma^{-1}(u')\}$ (the neighbors of $u'$ after $u'$ in $\sigma$) for $u' \in V$. It is also not hard to see for any iteration $i$, $u' \in S_i$ if
\{u \in S_i \cap N_{G^c}(u') : \sigma^{-1}(u) < \sigma^{-1}(u')\} is not empty. Note that \(\sigma^{-1} : V \rightarrow [n]\) is an inverse function of \(\sigma\).

Then, we want to prove that the clique formed by \(\{w\} \cup \{u \in N_{G^c}(w) : \sigma^{-1}(w) > \sigma^{-1}(u)\}\) cannot be a maximal clique for any \(w \in \{u \in N_{G^c}(v_i) : N_{G^c}(u) \cap L_i = \emptyset\}\) in iteration \(i\). Since \(N_{G^c}(w) \cap L_i = \emptyset\), we know that \(N_{G^c}(w) \subseteq (\{v_i\} \cup N_{G^c}(v_i))\). Also, \(w \in N_{G^c}(v_i)\) and \(\{u \in N_{G^c}(w) : \sigma^{-1}(u) > \sigma^{-1}(w)\} \subseteq N_{G^c}(w)\). Thus, we proved that \(w\) and its neighbors in \(G^c\) after \(w\) in \(\sigma\) cannot be a maximal clique of \(G^c\). Hence, the number of iterations in the while loop (Line 7) is no less than \(mc(G^c)\).

Next, we need to show that \(\{v_i\} \cup N_{G^c}(v_i)\) forms a maximal clique of \(G^c\) in each iteration. Note that \(N_{G^c}(v_i) = \{u \in N_{G^c}(v_i) : \sigma^{-1}(u) > \sigma^{-1}(v_i)\}\). Assume \(\{v_i\} \cup N_{G^c}(v_i)\) is not a maximal clique of \(G^c\). Then, there must exist \(j < i\) such that \(\{v_i\} \cup N_{G^c}(v_i) \subset \{v_j\} \cup N_{G^c}(v_j)\). Let \(j\) be such one closest to \(i\). Then, it is not hard to see that any vertex in \(S_j\) between \(v_j\) and \(v_i\) is not adjacent to \(v_i\). Then, \(N_{G^c}(v_i) = N_{G^c}(v_j)\) and \(v_i \in \{u \in N_{G^c}(v_j) : N_{G^c}(u) \cap L_j = \emptyset\} \subseteq R_j\).

Thus, \(v_i\) is removed in Line 12 of iteration \(j\) which is a contradiction. The number of iteration is exactly \(mc(G^c)\).

Since the while loop (Line 7) terminates when \(S\) is empty. Thus, \(L = \emptyset\) in the last iteration and Algorithm 3 has a size of \(mc(G^c) - 1\).

We also show that Algorithm 3 is efficient: \(O([V]|[V] + |E^c|)\) time.

**Theorem 4.** Algorithm 3 runs in \(O([V]|[V] + |E^c|)\) time.

**Proof.** Take the complement of a graph requires \(O(|V|^2)\) time. Finding a perfect elimination ordering \(\sigma\) of a chordal graph \(G^c\) can be done in \(O(|V| + |E^c|)\) time by lexicographic breadth-first search with partition refinement. The outer while loop has at most \(O(|V|)\) iterations. The construction of \(L\) take \(O(|V|)\) time. The most costly operations in the while loop is in Line 10 where each edge in \(G'\) can be traversed in constant times in the worst case: \(O(E^c)\).
6. Lower Bounds of Biclique Partition Numbers on Some Subclasses of Co-Chordal Graphs

In this section, we will first prove that both of Algorithms 2 and 3 find a minimum biclique partition of \( G \) if its complement \( G^c \) is both chordal and clique vertex irreducible. We start with the well-known Graham–Pollak Theorem, which provides a lower bound of biclique partition number on complete graphs.

**Theorem 5** (Graham–Pollak Theorem [10, 11]). *The edge set of the complete graph, \( K_n \), cannot be partitioned into fewer than \( n - 1 \) biclique subgraphs.*

We first show that the biclique partition number of a graph is no less than the biclique partition number of its induced subgraph.

**Lemma 3.** Given a graph \( G = (V, E) \), \( V' \) is a subset of vertices in \( V \) and let induced subgraph \( G' = G(V') \). Then, \( \text{bp}(G') \leq \text{bp}(G) \).

**Proof.** Suppose that \( \{ \{ L^i, R^i \} \}_{i=1}^{\text{bp}(G)} \) is a minimum biclique partition of graph \( G \). Then, we claim that \( \{ \{ L^i \cap V', R^i \cap V' \} \}_{i=1}^{\text{bp}(G)} \) is a biclique partition of graph \( G' \). Note that \( L^i \) and \( R^i \) can be empty. If the claim holds, then we construct a biclique partition of \( G' \) with size \( \text{bp}(G) \) and therefore \( \text{bp}(G') \leq \text{bp}(G) \). Then, we just need to prove the claim instead.

Let \( G^c \) be the complement of \( G' \). For an arbitrary edge \( e \) in \( G^c \), since it is not in \( G \), it is not in the bicliques \( \{ L^i, R^i \} \) for any \( i \in [\text{bp}(G)] \). It also implies that \( e \) is not in the bicliques \( \{ L^i \cap V', R^i \cap V' \} \) for any \( i \in [\text{bp}(G)] \). Hence, \( \{ L^i \cap V', R^i \cap V' \} \) is a subgraph of \( G' \) for any \( i \in [\text{bp}(G)] \).

Let \( e \) be an arbitrary edge in \( G' \). Since \( \{ \{ L^i, R^i \} \}_{i=1}^{\text{bp}(G)} \) is a biclique partition of \( G \), then there exists a unique \( j \in [\text{bp}(G)] \) such that \( e \) is in \( \{ L^j, R^j \} \). It implies that \( e \) is in the biclique \( \{ L^j \cap V', R^j \cap V' \} \).

Then, we can show that the biclique partition number of an arbitrary graph is no less than its clique number minus one.

**Proposition 4.** *Given a graph \( G = (V, E) \), \( \text{bp}(G) \geq \omega(G) - 1. \)
Proof. Let $K$ be a maximum clique subgraph of $G$ with size $\omega(G)$. By Graham–Pollak Theorem, we know that $bp(K) \geq \omega(G) - 1$. By Lemma 3, $bp(G) \geq bp(K) \geq \omega(G) - 1$.

Next, we prove that Algorithms 2 and 3 are exact when the input graph $G$ has a chordal and clique vertex irreducible complement.

**Theorem 6.** Given a graph $G$ where its complement $G^c$ is chordal and clique vertex irreducible, $bp(G) = mc(G^c) - 1$.

**Proof.** By Corollary 1, $bp(G) \leq mc(G^c) - 1$ since $G$ is a co-chordal graph. Since $G^c$ is clique vertex irreducible, then $mc(G^c) = \omega(G)$. By Proposition 4, $bp(G) \geq mc(G^c) - 1$.

Then, we derive a lower bound of the bp number on any split graph and prove that $mc(G^c) - 2 \leq bp(G) \leq mc(G^c) - 1$ if $G$ is a split graph.

**Lemma 4.** Given a split graph $G = (V, E)$, $V$ can be partitioned into two sets $V_1, V_2$ such that $G(V_1)$ is a clique and $V_2$ is an independent set. Then, $mc(G) \leq |V_2| + 1$.

**Proof.** Since $V_2$ is an independent set, any maximal clique subgraph of $G$ can include at most one vertex in $V_2$. There is at most one maximal clique subgraph of $G$ not including any vertex in $V_2$, which is $G(V_1)$.

We then want to prove that given an arbitrary vertex $v \in V_2$, there is at most one maximal clique subgraph of $G$ containing $v$. Assume that there exists two distinct maximal clique subgraphs of $G$, $K_1$ and $K_2$, containing $v$. Then, it is safe to conclude that both $V(K_1) \setminus \{v\} \in V_1$ and $V(K_2) \setminus \{v\} \in V_1$. Since $G(V_1)$ is a clique subgraph of $G$, then we can construct a larger clique subgraphs of $G$ by including all the vertices in $K_1$ and $K_2$, which is a contradiction. Therefore, $mc(G) \leq |V_2| + 1$.

**Theorem 7.** Given a split graph $G$, then $bp(G) \geq mc(G^c) - 2$. 

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Proof. We first claim that the clique number of $G$, $\omega(G)$, is no less than the number of maximal cliques, $mc(G^c) - 1$. Then, by Proposition 4, $mc(G^c) - 2 \leq \omega(G) - 1 \leq bp(G)$.

To prove the claim, we first start with the definition of split graphs. Since $G = (V, E)$ is a split graph, then $V$ can be partitioned into two sets $V_1, V_2$ such that the induced subgraph $G(V_1)$ is a complete graph and $V_2$ is an independent set of $G$. Then, we know that $|V_1| \leq \omega(G)$.

Furthermore, $G^c$ is also a split graph, $V_1$ is an independent set of $G^c$, and $G(V_2)$ is a clique subgraph of $G^c$. By Lemma 4, we know that $mc(G^c) \leq |V_1| + 1$. Therefore, $mc(G^c) - 1 \leq |V_1| \leq \omega(G)$.

7. Final remarks

If a graph $G = (V, E)$ is a co-chordal graph, the biclique partition number of $G$ is less than the number of maximal cliques of its complement $G^c$. Additionally, we provided two heuristics, one based on clique trees and one based on lexicographic breadth-first search, to find an explicit construction of a biclique partition with a size of $mc(G^c) - 1$. We also showed that the computational time of heuristic based on the lexicographic breadth-first search is $O(|V|(|V| + |E^c|))$.

If a graph $G$ where its complement $G^c$ is both chordal and clique vertex irreducible, then $bp(G) = mc(G^c) - 1$. If a graph $G$ is a split graph, another subclass of co-chordal, $mc(G^c) - 2 \leq bp(G) \leq mc(G^c) - 1$.

In Section 4, we showed that given a co-chordal graph $G$ and a clique tree of $G^c$, $T_{K^c}$, Algorithm 2 can return a biclique partition with a size of $mc(G^c) - 1$ no matter which edge of $T_{K^c}$ is selected in each recursion. An open question is whether $bp(G) = mc(G^c) - 1$ if $G$ is a co-chordal graph or split graph. If it is the case, it is an extension of Graham-Pollak theorem to a more general class of graphs.
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