Generic Hyperbolicity for the equilibria
of the one-dimensional parabolic equation
\[ u_t = (a(x)u_x)_x + f(u). \]

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Abstract

We show, for some classes of diffusion coefficients that, generically in
\( f \), all equilibria of the reaction-diffusion equation
\[ u_t = (a(x)u_x)_x + f(u) \quad 0 < x < 1 \]
with homogeneous Neumann boundary conditions are hyperbolic.

1 Introduction

We consider here the scalar parabolic equation with Neumann boundary conditions:

\[
\begin{cases}
    u_t = (a(x)u_x)_x + f(u), & 0 < x < 1 \\
    u_x(0,t) = u_x(1,t) = 0
\end{cases}
\tag{1}
\]

where \( f: \mathbb{R} \to \mathbb{R} \) is a \( C^2 \) function and \( a: [0, 1] \to \mathbb{R}^+ = (0, \infty) \) is sufficiently smooth.

An equilibrium point \( u \) of (1) is called hyperbolic if and only if \( 0 \) is not one of the eigenvalues of the linearized problem:

\[ (aw_x)_x + f'(u)w = \lambda w, \quad w_x(0) = w_x(1) = 0. \]

For constant diffusion coefficient \( a \), it has been proved by Brunovsky and Chow (2) that generically for \( f \) in the Whitney topology, all equilibria of (1) are hyperbolic. The same result was obtained by Henry (4) under slightly more general hypotheses.

For nonconstant diffusion coefficient, it has been shown by Rocha (6), that generically in the pair \((a, f)\), all equilibria are either hyperbolic or non-degenerate (with an appropriate definition of the latter concept). He also proved that the equilibria satisfying an additional hypothesis can be made hyperbolic or

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non-degenerate, by perturbing only $f$. We have shown in [5] that generic hyperbolicity can be obtained with respect to $a$ and also with respect to $f$ if additional properties are imposed on the equilibria. Our purpose here is to eliminate those additional requirements. To achieve this goal, we will need to impose some extra hypotheses in the diffusion coefficient.

We observe that, if explicit dependence of $f$ in $x$ is allowed such a result is much easier to obtain (see for example [3]). When the dependence on $x$ occurs only through $u$, the result is certainly expected to hold but to the extent of our knowledge a proof is not yet available.

We finally observe that, once hyperbolicity of equilibria has been obtained, the Morse-Smale property follows since transversality of the invariant manifolds is automatic as proved in [1] and [4].

2 Some technical lemmas

We state here some auxiliary results we will need in the proof.

**Lemma 1** Suppose $u : [a, b] \rightarrow \mathbb{R}$ is a $C^2$ function with nondegenerate critical points and $\phi : [a, b] \rightarrow \mathbb{R}$ is continuous. Let $R_q = \{ p \in [a, b] | u(p) = q, u'(p) \neq 0 \}$, $C_q = \{ p \in [a, b] | u(p) = q, u'(p) = 0 \}$. If $\int_a^b f(u)\phi\,dx = 0$, for any continuous real function $f$. then we have

$$\sum_{p \in R_q \cap (a,b)} \frac{\phi(p)}{|u'(p)|} = 0. \quad (2)$$

and

$$\sum_{p \in C_q \cap (a,b)} \frac{\phi(p)}{\sqrt{|u''(p)|}} + \frac{1}{2} \sum_{p \in C_q \cap \{ a,b \}} \frac{\phi(p)}{\sqrt{|u''(p)|}} = 0. \quad (3)$$

Reciprocally, if (2) holds for any $q \in R_q$ then $\int_a^b f(u)\phi\,dx = 0$, for any continuous function real function $f$.

**Proof.** Let $p$ be a point where $u'(p) \neq 0$ and $q = u(p)$. We can find a local inverse $u^{-1} : (q - \varepsilon, q + \varepsilon) \rightarrow (p - \delta_1, p + \delta_2)$, where $\delta_1, \delta_2, \varepsilon$ are positive numbers. We suppose first that $u$ is increasing in $(p - \delta_1, p + \delta_2)$ and, therefore, $u(p - \delta_1) = q - \varepsilon, u(p + \delta_2) = q + \varepsilon$.

Let $f_\varepsilon : \mathbb{R} \rightarrow \mathbb{R}$ be a $C^\infty$ function satisfying

$$f(\varepsilon) = \begin{cases} \frac{1}{\varepsilon} & \text{if } |y - q| \leq \varepsilon \\ \frac{1}{\varepsilon} & \text{if } \varepsilon \geq |y - q| \leq \varepsilon + \varepsilon^2 \\ 0 & \text{otherwise.} \end{cases}$$

Then we have

$$\int_{p-\delta_1}^{p+\delta_2} f_\varepsilon(u(x))\phi(x)\,dx = \int_{q-\varepsilon}^{q+\varepsilon} f_\varepsilon(y)\phi(u^{-1}(y)) \cdot \frac{1}{u'(u^{-1}(y))} \,dy$$
\[
\begin{align*}
&= \int_{q - \frac{\varepsilon}{2}}^{q + \frac{\varepsilon}{2}} \frac{1}{\varepsilon} \phi(u^{-1}(y)) \frac{1}{u'(u^{-1}(y))} \, dy \\
&+ \int_{q - \frac{\varepsilon}{2} - \varepsilon^2}^{q - \frac{\varepsilon}{2}} f_\varepsilon(y) \phi(u^{-1}(y)) \frac{1}{u'(u^{-1}(y))} \, dy \\
&+ \int_{q + \frac{\varepsilon}{2} + \varepsilon^2}^{q + \frac{\varepsilon}{2}} f_\varepsilon(y) \phi(u^{-1}(y)) \frac{1}{u'(u^{-1}(y))} \, dy
\end{align*}
\]

It is easy to see that the last two integrals go to zero as \( \varepsilon \to 0 \). For the first, we have

\[
\int_{q - \frac{\varepsilon}{2}}^{q + \frac{\varepsilon}{2}} \frac{1}{\varepsilon} \phi(u^{-1}(y)) \frac{1}{u'(u^{-1}(y))} \, dy = \frac{1}{\varepsilon} \cdot \phi(u^{-1}(\xi))u'(u^{-1}(\xi))
\]

where \( \xi \in [q - \frac{\varepsilon}{2}, q + \frac{\varepsilon}{2}] \).

As \( \varepsilon \to 0 \) we then obtain

\[
\int_{p - \delta_1}^{p + \delta_2} f_\varepsilon(u(x))\phi(x) \, dx \to \phi(u^{-1}(q)) \frac{1}{u'(u^{-1}(q))} = \frac{\phi(p)}{u'(p)} = \frac{\phi(p)}{|u'(p)|}
\]

If \( u \) is decreasing in \((p - \delta_1, p + \delta_2)\) we obtain similarly that

\[
\int_{p - \delta_1}^{p + \delta_2} f_\varepsilon(u(x))\phi(x) \, dx \to -\phi(u^{-1}(q)) \frac{1}{u'(u^{-1}(q))} = \frac{-\phi(p)}{u'(p)} = \frac{-\phi(p)}{|u'(p)|}
\]

Now, let \( q \) be a regular value of \( u \) and denote by \( I_q \) the pre-image of \( q \) by \( u \), that is \( I_q = \{ p \in [a, b] \mid u(p) = q \} \).

Then, with \( f_\varepsilon \) as above, we obtain

\[
0 = \int_a^1 f_\varepsilon(u(x))\phi(x) \, dx \\
= \sum_{p \in I_q} \int_{p - \delta_1}^{p + \delta_2} f_\varepsilon(u(x))\phi(x) \, dx \\
\to \sum_{p \in I_q} \frac{\phi(p)}{|u'(p)|}
\]
(Of course the values of $\varepsilon$ above depend on the $p$. We chose not to write this dependence explicitly to simplify the notation).

Suppose now that $p \in (a, b)$ is a critical point of $u$. We can write $u$ in a neighborhood of $p$ as

$$u(x) = u(p) + \frac{u''(p)}{2}(x - p)^2 + O(x - p)^3.$$ 

Observe that $u''(p) \neq 0$ by hypothesis. Suppose $u''(p) < 0$. The image of a (small) interval $I$ around $p$ is then an interval of the form $[q - \varepsilon, q]$, where $q = u(p)$. We have, for $x \in I, y \in [q - \varepsilon, q]$

Write $\alpha = \frac{u''(p)}{2}, w = (x - p)^2, z = y - q$. Then

$$u(x) = y \Leftrightarrow \alpha w + O(|w|^{3/2}) = z$$

$$\Leftrightarrow w = \frac{z}{\alpha + O(|w|^{1/2})}$$

$$= \frac{z}{\alpha + O(|z|^{1/2})}$$

$$= \frac{z}{\alpha (1 + O(|z|^{1/2}))}$$

$$= \frac{z}{\alpha + O(|z|^{3/2})}$$

Therefore $u$ is one to one from each one of the intervals $[p - \delta_1, p]$ and $[p, p + \delta_2]$ into $[q - \varepsilon, q]$, with

$$(x - p) = \pm \sqrt{\frac{y - q}{\alpha}} + O(|y - q|^{3/2})$$

$$= \pm \sqrt{\frac{y - q}{\alpha}} \sqrt{1 + O(|y - q|^{1/2})}$$

$$= \pm \sqrt{\frac{y - q}{\alpha}} \left(1 + O(|y - q|^{1/2})\right)$$

$$= \pm \sqrt{\frac{y - q}{\alpha}} \left(1 + O(|y - q|^{1/2})\right)$$

$$= \pm \sqrt{\frac{y - q}{\alpha}} + O(|y - q|).$$

Let $g_\varepsilon : \mathbb{R} \to \mathbb{R}$ be a $C^\infty$ function satisfying

$$g_\varepsilon(x) = \begin{cases} \frac{1}{\sqrt{\varepsilon}} & \text{if } |y - q| \leq \varepsilon \\ \frac{1}{\sqrt{\varepsilon}} & \text{if } \frac{\varepsilon}{2} \geq |y - q| \leq \frac{\varepsilon + \varepsilon^2}{2} \\ 0 & \text{otherwise.} \end{cases}$$
Denoting by \( x(y) \) the inverse of \( u : [p - \delta_1, p] \to [q - \varepsilon, q] \), we obtain
\[
\int_{p-\delta_1}^{p} g_\varepsilon(u(x))\phi(x) \, dx = \int_{x(q-\varepsilon/2-\varepsilon^2/2)}^{x(q-\varepsilon/2)} g_\varepsilon(u(x))\phi(x) \, dx + \int_{x(q-\varepsilon/2)}^{p} g_\varepsilon(u(x))\phi(x) \, dx
\]

For the first integral, we have
\[
\left\| \int_{x(q-\varepsilon/2)}^{x(q-\varepsilon/2-\varepsilon^2/2)} f_\varepsilon(u(x))\phi(x) \, dx \right\| \leq \frac{1}{\sqrt{\varepsilon}} \|\phi\|_\infty \left( \sqrt{\frac{-\varepsilon}{2\alpha}} - \sqrt{\frac{-\varepsilon - \varepsilon^2}{2\alpha}} + O(\varepsilon) \right)
\]
\[
= \frac{1}{\sqrt{\varepsilon}} O(\varepsilon)
\]
\[
= O(\sqrt{\varepsilon})
\]

For the second integral, we have
\[
\int_{x(q-\varepsilon/2)}^{p} g_\varepsilon(u(x))\phi(x) \, dx = \frac{1}{\sqrt{\varepsilon}}(p - x(q-\varepsilon/2))\phi(\bar{x})
\]
\[
= \frac{1}{\sqrt{\varepsilon}} \left( \sqrt{\frac{-\varepsilon}{2\alpha}} \right) \phi(\bar{x})
\]
\[
= \frac{\phi(\bar{x})}{\sqrt{-2\alpha}}.
\]

where \( \bar{x} \in [q - \varepsilon/2, q] \).

As \( \varepsilon \to 0 \) we obtain
\[
\int_{p-\delta_1}^{p} f_\varepsilon(u(x))\phi(x) \, dx \to \frac{\phi(p)}{\sqrt{-2\alpha}}.
\]

Similarly, we obtain
\[
\int_{p}^{p+\delta_2} g_\varepsilon(u(x))\phi(x) \, dx \to \frac{\phi(p)}{\sqrt{-2\alpha}}.
\]

Therefore,
\[
\int_{p-\delta_1}^{p+\delta_2} g_\varepsilon(u(x))\phi(x) \, dx \to \sqrt{2} \frac{\phi(p)}{\sqrt{|\alpha|}}.
\]

as \( \varepsilon \to 0 \), and the same is true if \( \alpha = \frac{u''(p)}{2} > 0 \).

If \( p = a \) or \( p = b \), the same computations can be done in the intervals to the right or left respectively, giving the same result divided by 2.
Now, let \( q \) be a critical value of \( u \) and denote by \( I_q \) the pre-image of \( q \) by \( u \), that is \( I_q = \{ p \in [a, b] \mid u(p) = q \} \).

\( C_q \) the critical points in \( I_q \) and \( R_q \) the regular points in \( I_q \), that is
\[
C_q = \{ p \in [a, b] \mid u(p) = q, u'(p) = 0 \}
\]
and \( R_q = I_q - C_q \) the regular points in \( I_q \).

Then, with \( g \) as above we obtain
\[
0 = \int_a^b g_x(u(x))\phi(x) \, dx
= \sum_{p \in I_q} \int_{p-\delta}^{p+\delta} g_x(u(x))\phi(x) \, dx
= \sum_{p \in C_q} \int_{p-\delta}^{p+\delta} g_x(u(x))\phi(x) \, dx + \sum_{p \in R_q} \int_{p-\delta}^{p+\delta} g_x(u(x))\phi(x) \, dx
= \sum_{p \in C_q} \int_{p-\delta}^{p+\delta} g_x(u(x))\phi(x) \, dx.
\]
(Of course the values of \( \delta_1, \delta_2 \) above depend on \( p \). If \( p = a \) (resp. \( p = b \)), then \( \delta_1 = 0 \) resp. \( \delta_2 = 0 \). We chose not to write this dependence explicitly to simplify the notation).

As \( \varepsilon \to 0 \) we obtain
\[
\sum_{p \in C_q \cap (a, b)} \sqrt{2} \frac{\phi(p)}{\sqrt{|u''(p)|}} + \sum_{p \in C_q \cap [a, b]} \sqrt{2} \frac{\phi(p)}{2 \sqrt{|u''(p)|}} = 0.
\]

To prove the converse, we first denote by \( 0 = x_0 < x_1 < \cdots < x_n = 1 \) the critical points of \( u \) in \([0, 1] \), and by \( y_0 < y_1 < \cdots < y_m \) the critical values.

Let \( I_j = [y_{j-1}, y_j] \). For a fixed \( j \) and any \( y \) in the interior of \( I_j \), let \( n_{i_j} \) be the number of points in \( u^{-1}(y) \) (observe that \( u^{-1}(I_j) \) is then the disjoint union of \( n_{i_j} \) intervals). By hypothesis, we have
\[
\sum_{i=1}^{n_{i_j}} \frac{\phi(\xi_i(y))}{|u'(\xi_i(y))|} = \sum_{i=1}^{n_{i_j}} \phi(\xi_i(y)) \cdot |\xi_i(y)| = 0.
\]

To simplify the notation, we write this sum as
\[
\sum_{i=1}^{n} \phi(\xi_i(y)) \cdot |\xi_i(y)| = 0.
\]
with the understanding that the summand is zero, if \( y \) is not in the range of \( \xi \).

If \( g \) is a continuous function in \( I_j \), we then have
0 = \sum_{i=1}^{n} \int_{I_i} g(y)\phi(\xi_i(y)) \cdot |\xi_i(y)| \, dy \\
= \sum_{i=1}^{n} \int_{\xi(I_i)} g(u(x))\phi(x) \, dx \tag{4}

Now, if \( I_i = [x_{i-1}, x_i], \ i = 1, 2, \cdots, n, \) we have \( I_i = \bigcup \xi(I_j). \) Therefore,

0 = \sum_{j=1}^{m} \sum_{i=1}^{n} \int_{\xi(I_j)} g(u(x))\phi(x) \, dx \\
= \sum_{i=1}^{n} \int_{(I_i)} g(u(x))\phi(x) \, dx \\
= \int_{0}^{u} g(u(x))\phi(x) \, dx. \tag{5}

If \( u \) is a (fixed) function in \([0, 1]\) and \([a, b] \subset [0, 1]\), we will say that \( \Phi : [0, 1] \mapsto \mathbb{R} \) is in the orbit of \( u \) in \([a, b]\), and write \( \Phi \in \mathcal{O}(u)\{a, b\} \) if \( \Phi(x) = g(u(x)) \) for some continuous function \( g : \mathbb{R} \mapsto \mathbb{R} \) for any \( x \in [a, b] \). We will also say that \( \Phi \) belongs to the \( q \)-orbit of \( u \) in \([a, b]\) if there exists a function \( g : \mathbb{R} \mapsto \mathbb{R} \) continuous, except maybe at the critical points of \( u \) with \( \Phi(x) = g(u(x)) \) for \( x \) regular point of \( u \). In this case, we write \( \Phi \in q - \mathcal{O}(u)\{a, b\} \).

**Lemma 2** Suppose \( u \) is analytic in the interval \([a, b]\), with nondegenerate critical points, \( u(a) = u(b) \), \( u'(a) = 0 \) (or \( u'(b) = 0 \)) and, if \( p_1 \), \( p_2 \) are critical points of \( u \) in \([a, b]\), then \( u(p_1) \neq u(a) \) (or \( u(p_1) \neq u(b) \)) and \( u(p_1) \neq u(p_2) \). Suppose also that \( \Phi \) is analytic in the interval \([a, b]\) except maybe at the critical points of \( u \). Then, with the notion of lemma [1], if \( \sum_{p \in \mathcal{R}_q^c(a, b)} \Phi(p) \frac{u'(p)}{|u'(p)|} = 0 \), \( \Phi \) belongs to the \( q \)-orbit of \( u \) in \([a, b]\). In particular, if \( \Phi \) is analytic in the interval \([a, b]\), then \( \Phi \in \mathcal{O}(u)\{a, b\} \).

**Proof.** Let \( a = x_0 < x_1 < x_2 < \cdots < x_n = b \) be the critical points of \( u \) in \([a, b]\), and \( \xi_i, \ i = 1, 2, \cdots, n \) be the inverse of \( u \) in the interval \([x_{i-1}, x_i]\). If \( q \) is a regular value of \( u \) and \( \xi_{i_1}, \xi_{i_2}, \cdots, \xi_{i_n} \) are its inverse images under \( u \), we have, by hypotheses \( \sum_{j=1}^{k} \Phi(\xi_{ij}) \text{ sign}(\xi_{ij}) = 0 \). Since the critical points are nondegenerate they must be local minima and maxima in succession. Suppose, for definiteness, that \( x_0, x_2, \cdots, x_n \) are local maxima and \( x_1, \cdots, x_{n-1} \) are local minima. Let \( i_1, i_2, \cdots, i_k \) \( k = \frac{(n-1)}{2} \) be such that \( u(x_{i_1}) < u(x_{i_2}) < \cdots < u(x_{i_k}) \). Now, for \( u(x_{i_1}) < q < u(x_{i_2}) \), we must have, by hypotheses, \( -\Phi \circ \xi_{i_1}(q) + \Phi \circ \xi_{i_1+1}(q) = 0 \), that is \( \Phi \circ \xi_{i_1}(q) = \Phi \circ \xi_{i_1+1}(q) \). Since \( \Phi \circ \xi_{i_1} \) and \( \Phi \circ \xi_{i_1+1} \) are analytic functions in their common (open) interval of definition, we must have the equality in this whole interval. Proceeding upward, we next find that
\[-\Phi \circ \xi_i(q) + \Phi \circ \xi_{i+1}(q) - \Phi \circ \xi_i(q) + \Phi \circ \xi_{i+1}(q) = 0.\] Since we already know that \(\Phi \circ \xi_i(q) = \Phi \circ \xi_{i+1}(q)\) this implies \(\Phi \circ \xi_i(q) = \Phi \circ \xi_{i+1}(q)\). In this way we find successively that \(\Phi \circ \xi_i = \Phi \circ \xi_{i+1}, \Phi \circ \xi_{i+2} = \Phi \circ \xi_{i+1}, \ldots, \Phi \circ \xi_{k-1} = \Phi \circ \xi_{k-1}\) or, returning to the old ordering \(\Phi \circ \xi_1 = \Phi \circ \xi_2, \Phi \circ \xi_3 = \Phi \circ \xi_4, \ldots, \Phi \circ \xi_{n-1} = \Phi \circ \xi_n\).

Now, starting from the maximum value and going downward, we find similarly that \(\Phi \circ \xi_2 = \Phi \circ \xi_3, \Phi \circ \xi_4 = \Phi \circ \xi_5, \ldots, \Phi \circ \xi_{n-2} = \Phi \circ \xi_{n-1}\) and also that \(\Phi \circ \xi_1 = \Phi \circ \xi_n\). From this, it follows that \(\Phi \circ \xi_i(q) = \Phi \circ \xi_j(q)\) for any \(1 \leq i, j \leq n\) and any \(q\) that belongs to the common interval of \(\Phi \circ \xi_i\) and \(\Phi \circ \xi_j\). In other words, we must have \(\Phi(p_1) = \Phi(p_2)\) whenever \(u(p_1) = u(p_2)\) and \(p_1, p_2\) are not critical points, which proves the claim.

**Lemma 3** Suppose \(a\) and \(f\) are analytic functions, \(u\) is a non-constant solution of (25) and \(a, b\) are points in \([0, 1]\) such that \(u(a) = u(b), u'(a) = 0\) (or \(u'(b) = 0\)) and, if \(p_1, p_2\) are critical point of \(u\) in \((a, b)\), then \(u(p_1) \neq u(a)\) (or \(u(p_1) \neq u(b)\)) and \(u(p_1) \neq u(p_2)\) Supposes there exists a solution \(\eta\) of the second order O.D.E. in (25), with \(\int_a^b g(u)\eta dx = 0\), for any continuous real function \(g\). Then, the following functions belong to \(O(u)\{a, b\}\)

\[
\begin{align*}
\frac{a\eta_x}{u_x} & \quad (6) \\
au_x \eta & \quad (7) \\
a\eta^2 & \quad (8) \\
au_x^2 & \quad (9) \\
a_x \eta & \quad (10) \\
a_x u_x & \quad (11) \\
auxx & \quad (12) \\
a_{xx} & \quad (13) \\
a & \quad (14) \\
\eta_x & \quad (15) \\
\eta^2 & \quad (16) \\
u_x^2 & \quad (17) \\
u_{xx} & \quad (18)
\end{align*}
\]

Also the following functions belong to \(q - O(u)\{a, b\}\)

\[
\begin{align*}
\frac{\eta}{u_x} & \quad (19) \\
\frac{(a_{xx} \eta + a_x \eta_x)}{u_x} & \quad (20) \\
\frac{a_{xx} + \frac{a_x \eta_x}{\eta}}{u_x} & \quad (21) \\
\frac{a_x}{u_x} & \quad (22)
\end{align*}
\]
Proof. We suppose that $u'(a) = 0$. The case $u'(b) = 0$ is similar. First, observe that (19) follows immediately from lemmas 1 and 2.

Multiplying (25) by any $C^1$ function $g(u) \in \mathcal{O}(u)$ and integrating by parts, we obtain

$$0 = \int_a^b g(u)(a\eta_x)xu_x \, dx$$

Since $u(a) = u(b)$, for an arbitrary continuous function $h$, we can find a $(C^1)$ function $g$, with $g(u(a)) = g(u(b)) = 0$, and $g' = h$.

Therefore, lemma 1 applies to $a\eta_xu_x$ in the place of $\phi$ and the result follows from lemma 2.

Also, taking now $g(u(a)) \neq 0$, we obtain $a\eta_x(a) = a\eta_x(b)$.

Multiplying (23) by any $C^1$ function $g(u) \in \mathcal{O}(u)$ times $\eta$, with $g(u(a)) = 0$ and integrating by parts, we obtain

$$0 = \int_1^a (au_x\eta) x g(u) \eta + f(u) g(u) \eta \, dx$$

where, to obtain the last equality, we used that $a\eta_x \in \mathcal{O}(u)$. Therefore (7) also follows from lemmas 1 and 2 that $au_x \eta \in \mathcal{O}(u)$. To obtain (9), we multiply $au_x \eta$ by $\frac{\partial}{\partial u}$, and (7) and (19) to obtain the result at the points where $\eta \neq 0$, and then conclude $u_x^2 \in \mathcal{O}(u)$ by continuity. Note that, in particular, we must have $u_x(b) = 0$.

To obtain (10), we can proceed in various ways. We can, for instance, multiply (6) by $g(u)\eta$, integrate by parts and use (7) (8) to obtain (10).

Now (11) and (12), follow from (10), (13) and equation (22) can also be obtained in various ways. For example, from (10)

$a_x \eta \in \mathcal{O}(u) \Rightarrow g(u)f'(u)a_x \eta \in \mathcal{O}(u) \Rightarrow g(u)a_x(\eta \eta_x) \in \mathcal{O}(u)$

for any $g(u)$. Multiplying by $\eta$ we arrive after integrating by parts and using (10) at

$$\int_0^1 (g(u)\eta)(a_{xx} \eta + a_x \eta_x) \, dx = 0.$$
Dividing (5) by (8) and then multiplying by (10) we obtain that \( \frac{a_x u}{\eta} \in q - \mathcal{O}(u)\{a, b\} \) and so (13) follows from (24).

Now multiplying \( a_{xx} \) by \( g(u)u_x \), it follows that

\[
0 = \int_a^b g(u)a_{xx}u_x \, dx
\]

\[
= -\int_a^b g'(u)u_x^2 a_x \, dx - \int_a^b g(u)u_x a_x
\]

\[
= -\int_a^b g(u)u_{xx} a_x
\]

Therefore \( \frac{a_x u}{\eta} \in q - \mathcal{O}(u)\{a, b\} \). From this, and (12), we conclude that there is a function \( g(u) \), with \( a_x = g(u)u_x \), at least at the points where \( u_x \neq 0 \).

Now, we claim that \( a_x = 0 \) if \( u_x = 0 \). In fact, from lemma 1, we know that \( n = 0 \) at those points and, from (10), \( a_x \eta + a_x \eta_x = 0 \). If \( a_x \neq 0 \), it follows then that \( \eta(x) = \eta(x) = 0 \) and \( \eta = 0 \) a contradiction. Therefore the equality must hold everywhere. Integrating, we obtain \( \ln(a) = G(u) \) where \( G \) is a primitive of \( g \), from which (14) and (22) follow immediately. (15), (16), (17) and (18) follow then from (14) and (6), (8), (9) and (12) respectively.

3 Hyperbolicity of the equilibria

Let \( f : \mathbb{R} \to \mathbb{R} \) be a \( C^2 \) function and \( a : [0, 1] \to \mathbb{R}^+ = (0, \infty) \) continuous. We denote by \( E(a, f) \) the set of equilibria of (1). Clearly, \( u \in E(a, f) \) if and only if it is a solution of the boundary value problem

\[
\begin{cases}
(a(x)u_x)_x + f(u) = 0, & 0 \leq x \leq 1 \\
u_x(0) = u_x(1) = 0
\end{cases} \tag{23}
\]

The initial value problem associated to (23) is

\[
\begin{cases}
(a(x)u_x)_x + f(u) = 0, & 0 \leq x \leq 1 \\
u_x(0) = 0 \quad u(0) = u_0
\end{cases} \tag{24}
\]

and the linear variational equation corresponding to (23) around \( u \) is

\[
\begin{cases}
(a(x)\phi_x)_x + f'(u) \cdot \phi = 0, & 0 \leq x \leq 1 \\
\phi_x(0) = \phi_x(1) = 0
\end{cases} \tag{25}
\]

A solution of (25) is therefore hyperbolic if, and only if, (25) has only the trivial solution.

It is easy to show (see [6]) that, if the set of solutions of (23) is non-empty, then it contains a constant solution \( \xi \). Also, since the change of variables \( u' = u - \xi \) does not affect the boundary conditions on (1) we can, and will, assume that \( f(0) = 0 \).

In what follows we denote by \( C^2_{\xi}(\mathbb{R}) \) the space of \( C^2 \) functions \( f : \mathbb{R} \to \mathbb{R} \) satisfying \( f(0) = 0 \) with the Whitney topology.
Let $H^2_N[0,1] = \{ u \in H^2[0,1] || u_x(0) = u_x(1) = 0 \}$ and consider the map

$$
\Psi_f : H^2_N[0,1] \rightarrow L^2[0,1] \\
u \mapsto (au_x)_x + f(u)
$$

**Lemma 4** An equilibrium of (1) is hyperbolic if, and only if, it is a regular point of $\Psi_f$.

**Proof.** $u$ is a regular point of $\Psi_f$ if and only if, the derivative

$$D_u\Psi_f : H^2_N[0,1] \rightarrow L^2[0,1] : \phi \mapsto (a\phi)_x + f'(u)\phi$$

is surjective. Now $D_u\Psi_f$ is Fredholm of index 0 and, therefore, it is surjective if and only it is an isomorphism.

Let us consider the equilibria of (1) with norm less than or equal to $N$, for some natural $N$. Our plan is to prove that they are all hyperbolic for an open dense set of $f$ in the Whitney topology and then take intersection. Now, for these equilibria, we can use the $C^2$ topology for $f$ since its values outside a compact set are irrelevant.

We first look at the constant equilibria.

**Lemma 5** The constant equilibria $u$ of (1), with $||u||_{H^2} \leq N$ are all hyperbolic for $f$ in a open dense set of $C^2$.

**Proof.** As usual, openness is not a problem here, so we just have to prove density. Now, a constant $u_0$ is a non hyperbolic equilibrium of (1) if and only $f(u_0) = 0$ and $f'(u_0)$ is one of the eigenvalues of the operator

$$\phi \mapsto (a\phi)_x$$

in $L^2[0,1]$. We can first choose $f$ with only a finite number of zeroes in the ball of radius $N$ and center at the origin, so that the number of constant equilibria with $||u||_{H^2} \leq N$ is finite. Now, since the eigenvalues of the above operator form a discrete set it is easily seen that $f'$ can be modified, without changing the zeroes of $f$, in such a way that $f'(u_0)$ will not be an eigenvalue whenever $u_0$ is a zero.

Let now $Y_N$ be the open dense dense set in $C^2$, given by lemma (6), $B_N = \{ u \in H^2_N[0,1] | : ||u||_{H^2_N[0,1]} \leq N \}$, and consider the map

$$
\Psi : B_N \times Y_N \rightarrow L^2[0,1] \\
(u,f) \mapsto (au_x)_x + f(u)
$$

**Lemma 6** Suppose 0 is a not a regular value of the $\Psi$. Then there is an equilibrium $u$ of (1) and a nontrivial solution $\phi$ of the corresponding linearized equation (25) such that $\int_0^1 f(u)\phi = 0 \forall \hat{f} \in C^2(\mathbb{R}, \mathbb{R})$. 

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Proof. Suppose there exists \((u, f) \in F^{-1}(0)\) such that
\[
D \Psi(u, f) : H^2_N[0, 1] \times C^2(\mathbb{R}, \mathbb{R}) \to L^2[0, 1]
\]
\[
(u, f) \mapsto (a u_x)_x + f(u) \dot{u} + \dot{f}(u)
\]
is not surjective.

By hypothesis, there exists a nontrivial \(\phi \in L^2[0, 1]\) orthogonal to the range of \(DF(u, f)\), that is
\[
\int_0^1 \phi \{ (a u_x)_x + f'(u) \dot{u} + \dot{f}(u) \} = 0 \quad \forall \dot{u} \in H^2_N[0, 1] \text{ and } \forall \dot{f} \in C^2(\mathbb{R}, \mathbb{R}).
\]
Taking \(\dot{f} = 0\),
\[
\int_0^1 \phi \{ (a u_x)_x + f(u) \dot{u} \} = 0 \quad \forall \dot{u} \in H^2_N[0, 1] \text{ and } \phi \in H^2_N(0, 1) \cap C^2(0, 1) \text{ is a weak, therefore, strong solution of (25).}
\]
Taking now, \(\dot{u} = 0\),
\[
\int_0^1 \dot{f}(u) \phi = 0 \quad \forall \dot{f} \in C^2(\mathbb{R}, \mathbb{R}), \text{ as claimed.}
\]

We are now in a position to prove our main result if additional properties on the equilibria are assumed. We first show how the results for constant coefficients can be obtained with our approach.

**Theorem 1** Suppose the diffusion coefficient in (1) is constant. Then there exists a residual set \(F\) in \(C^2_S(\mathbb{R})\) such that, if \(f \in F\), all equilibria of (1) are hyperbolic.

**Proof.** We prove that, for an open dense set of \(f\) the equilibria satisfying \(||u||_{H^2[0, 1]} \leq N\) are all hyperbolic and then take intersection. Again, openness is no problem.

Let \(Y_N\) be the open dense set given by (5).

We apply the Transversality Theorem for the map
\[
\Psi : B_N \times Y_N \to L^2[0, 1]
\]
\[
(u, f) \mapsto (au_x)_x + f(u)
\]
If 0 is not a regular value of \(\Psi\) it follows from lemma 6 that there is a solution \(u\) of (23) and a nontrivial solution \(\phi\) of the corresponding linearized equation (25) such that
\[
\int_0^1 \dot{f}(u) \phi = 0 \quad \forall \dot{f} \in C^2(\mathbb{R}, \mathbb{R}).
\]
But then, it follows from lemma 1 that
\[
\sum_{p \in C^q} \phi(p) \sqrt{a(p)} + \frac{1}{2} \sum_{p \in C^q} \phi(p) \sqrt{a(p)} = 0.
\]
Since \(u_x\) and \(\phi\) are two linearly independent solutions of the second order linear differential equation in (25) their Wronskian must be a nonzero constant, that is,
\[
u_x \phi - u_{xx} \phi = k \neq 0.
\]
At critical points \(p\) of \(u\), we then have \(u_{xx}(p)\phi(p) = k\) and also \(u_{xx}(p) = -\frac{f(u(p))}{a(p)}\) and thus, \(\phi(p) = -\frac{k a}{f(u(p))}\).
But then, for any critical value $q$ of $u$, the above sum gives $\frac{\partial}{\partial u} \int_{0}^{q} f(q) = 0$, a contradiction.

Therefore 0 must be a regular point for $\Psi$. By the transversality Theorem, it must be also a regular point for $\Psi(\cdot, f)$ for $f$ in a residual, therefore dense set of $Y_N$ and thus, also of $C^2_N$.

Remark 1 The argument above can be pushed a little further to cover the case of monotonic ($C^1$) coefficients. In fact, suppose $a'(x) \neq 0$ and let $q = f(0)$. If $p$ is another critical point of $u$, with $u(p) = q$ then, as shown during the proof of lemma 4, $\frac{1}{2} \int_{0}^{p} a_x u_x^2 = 0$. If $u$ is not constant, this implies $a_x = 0$ in $[0, p]$. Therefore we can conclude, as above, that $\phi(p) = \phi(0)$ and obtain the same contradiction.

For general diffusion coefficients, our arguments encounter difficulties for solutions, which have many critical points for each critical value. We first define them precisely.

Definition 1 We say a nonconstant solution of (23) is an exceptional equilibrium if

1. $u$ is not hyperbolic.

2. For any critical point $p$, there exists another critical point $\bar{p}$ with the same value.

3. There are critical points $p, q$ with $u(p) = u(0), u(q) = u(1)$, such that $\phi(p) \neq \phi(0), \phi(q) \neq \phi(1)$, for some (and therefore for any) nontrivial solution of (25).

We will also call nonexceptional any equilibrium that is not exceptional.

For a given $f$ we denote by $E^* = E^*(a, f)$ the set of exceptional solutions of (23).

We can prove that $E^*$ is really ‘exceptional’ in the following sense:

Lemma 7 For any $f \in C^2_N(\mathbb{R})$ there exist at most a finite number of exceptional equilibrium points $u$ with $||u|| \leq n$, where $||u||$ is the norm of $u$ in $H^1(\{0, 1\})$.

Proof. We introduce the linear variational equation around a solution $u$ of (23)

\[\begin{cases} (a(x)v_x)_x + f'(u)v = 0, & 0 \leq x \leq 1 \\ v_x(0) = 0 & v(0) = 1 \end{cases} \tag{30} \]

Then $\frac{\partial u}{\partial u_0}$ the derivative of solutions of (23) with respect to the initial condition is the unique solution of (30).

Suppose $u$ is a solution of (23), and $\phi$ a non-trivial solution of (23). Then $\frac{\partial u}{\partial u_0}$ of (30) around $u$ is a multiple of $\phi$. Reciprocally, if $\frac{\partial u}{\partial u_0}$ satisfies $\left(\frac{\partial u}{\partial u_0}\right)_x (1) = 0$ then $\frac{\partial u}{\partial u_0}$ is a non-trivial solution of (23).
Let $u$ be an exceptional solution of (23) and let $\phi \neq 0$ and $\frac{\partial u}{\partial u_0}$ the corresponding solutions of (26) and (30) respectively. As observed above, we have $\frac{\partial u}{\partial u_0} = k\phi$, where $k \neq 0$ is a constant. If $p \in [0,1]$ is such that $u(p) = u(0)$, $u'(p) = 0$, then we write $\bar{u}_0 = u(0)$ and denote by $u(u_0, \cdot)$ the unique solution of (24) with initial value $u_0$. Then, by the implicit function theorem ($u$ is not constant!) there exists a unique point $\bar{p}(u_0)$ for $u_0$ in a neighborhood of $\bar{u}_0$, such that $u_x(u_0, p(u_0)) = 0$. We then have

\[
u(u_0, p(u_0)) = u(\bar{u}_0, p) + \left( \frac{\partial u}{\partial u_0}(\bar{u}_0, p) + u_x(\bar{u}_0, p)p'(u_0) \right) (u_0 - \bar{u}_0)
+ o(u_0 - \bar{u}_0)
= \bar{u}_0 + k\phi(p)(u_0 - \bar{u}_0) + o(u_0 - \bar{u}_0)
\]

On the other hand

\[
u(u_0, 0) = u(\bar{u}_0, 0) + \left( \frac{\partial u}{\partial u_0}(\bar{u}_0, 0) \right) (u_0 - \bar{u}_0) + o(u_0 - \bar{u}_0)
= \bar{u}_0 + k\phi(0)(u_0 - \bar{u}_0) + o(u_0 - \bar{u}_0)
\]

Since, by hypothesis $\phi(p) \neq \phi(0)$ we have, for any solution of (24) with initial condition $u_0$ close to $\bar{u}_0$ $u(u_0, p(u_0)) \neq u(u_0, 0)$.

Of course, $u(u_0, \cdot)$ may be a solution of (23) but it will not be an exceptional solution. Thus, such solutions must be in a discrete set. Since the set of solutions of (23) with $\|u\| \leq n$ is also compact in $H^1([0,1])$, and the set of exceptional solutions is closed in $H^2([0,1])$ it must be finite as claimed.

Assuming analyticity we can show generic hyperbolicity for the nonexceptional equilibria.

**Lemma 8** Suppose the diffusion coefficient $a$ and $f$ in (11) is analytic and it is not an even function about the point $x = \frac{1}{2}$. Then, there is a residual set $F$ in $C^2_z(I\mathbb{R})$ such that, if $f \in F$, all nonexceptional equilibria of (11) are hyperbolic.

**Proof.** We proceed as in theorem 11 but now restricting $u$ to the (open) subset of functions $u$ in $B_N$ which do not satisfy at least one of the conditions in definition 11 (that it is, excluding the potential nonexceptional equilibria). Suppose $u$ is a critical point of (26), and $\phi$ is a solution of (26), with $\int_0^1 f(u)\phi = 0$ for any continuous $f$. If $u$ fails to satisfy (8) of definition 11 then, it follows from lemma 11 that $\phi(0) = 0$, so $\phi \equiv 0$, a contradiction. Suppose then that $u$ does not satisfy 112 and let $p$ be the required critical point of $u$, with no other critical point at the same level. As in the proof of lemma 12 we denote by $0 = x_0 < x_1 < x_2 < \cdots < x_n = 1$ the critical points of $u$ and by $\xi_i$, $i = 1, 2, \cdots, n$ be the inverse of $u$ in the interval $[x_{i-1}, x_i]$. Suppose that $p = x_i$ is a local minimum (for local maximum, the argument is similar). If $q$ is a regular value of $u$ and $\xi_1, \xi_2, \cdots, \xi_n$ are its inverse images under $u$, and $\Phi = \frac{1}{u_x}$, we have, by lemma 114 $S(q) = \sum_{j=1}^n \Phi(\xi_j)(q)\text{sign}u'(\xi_j)(q) = 0$. 

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When \( q \) passes through \( u(p) \), coming from below, the term \( \Phi(\xi_1)(q)\text{sign}u'(\xi_1)(q) + \Phi(\xi_{l+1})(q)\text{sign}u'(\xi_{l+1})(q) \) is added to the above sum. Therefore, we must have \( \Phi(\xi_i)(q)\text{sign}u'(\xi_i)(q) + \Phi(\xi_{l+1})(q)\text{sign}u'(\xi_{l+1})(q) = 0 \), for \( q \) slightly above \( u(p) \).

By analyticity, this should also be true for \( u(p) < q < u(x_{l-1}) \), if \( u(x_{l-1}) \leq u(x_{l+1}) \) or \( u(p) < q < u(x_{l+1}) \) otherwise. Suppose wolog, we are in the first case. Let \( a = (x_{l-1}) \), and \( b \) the unique point in \( (x_{l-1}), (x_{l+1}) \) with \( u(a) = u(b) \). From the second part of lemma \( \text{9} \) we have then \( \int_a^b f(u)\phi = 0 \) for any continuous \( f \).

From lemma \( \text{8} \) it follows that \( a = g(u) \) for some analytic function \( u \) in the interval \([a, b]\) and, in particular \( a_x = 0 \) at the critical point \( p \). (This can also be obtained from \( a_xu_x \in \mathcal{O}(u) \) or \( a_x\phi \in \mathcal{O}(u) \)).

Write \( \xi_1 : [u(p), u(a)] \to [a, p] \) \( \xi_2 : [u(p), u(b)] \to [p, b] \) for the inverse of \( u \) in each interval. From \( \text{17} \) of lemma \( \text{2} \) it follows also that \( u_x(\xi_1(y)) = -u_x(\xi_2(y)) \) for any \( y \in [u(p), u(a)] \), hence \( \xi_1(y) = -\xi_2(y) \) for any \( y \in [u(p), u(a)] \). If \( \xi_1(y) = x_1 \), \( x_2(y) = x_2 \), then \( x_1 - p = \xi_1(y) - \xi_1(u(p)) = \int_{u(p)}^y \xi_1(y)\,dy = -\int_{u(p)}^y \xi_2(y)\,dy = -(x_2 - p) \). Thus \( p - \xi_1(y) = \xi_2(y) - p \) for any \( y \in [u(p), u(0)] \).

Therefore, \( a \) is an even function about the point \( p \) in the interval \([a, b]\). But, being analytic, \( a \) must be an even function about \( p \) in the whole interval \([0, 1]\). In particular \( p = \frac{1}{2} \), contradicting the hypothesis.

This result is not completely satisfactory, since it imposes conditions on the (unknown) solutions of the problem. The natural question is then if one could rule out beforehand the existence of nonexceptional solutions. It turns out that this is possible, if \( a \) has few oscillations.

We say that a \( \mathcal{C}^\infty \) function \( a : [0, 1] \to \mathbb{R} \) has \( n \) intervals of monotonicity if there exists a partition \( 0 = x_0 < x_1 < \cdots < x_n = 1 \), such that \( a'(x) \) is strictly positive or negative in each subinterval, \([x_{i-1}, x_i]\).

**Lemma 9** Suppose \( a \) has at most two intervals of monotonicity in \([0, 1]\). Then there are no exceptional solutions of \( \text{25} \).

**Proof.** If \( p, \tilde{p} \in [0, 1] \) are two critical points with the same value, then multiplying \( \text{24} \) by \( u_x \) and integrating, we get

\[
0 = \int_p^{\tilde{p}} \{ (au_x)_x \} u_x + f(u)u_x \, dx \\
= (au_x)^2(\tilde{p}) - (au_x)^2(p) - \frac{1}{2} \int_p^{\tilde{p}} a_x \frac{d}{dx}(u_x)^2 \, dx \\
= -\frac{1}{2}(au_x)^2(\tilde{p}) + \frac{1}{2}(au_x)^2(p) + \frac{1}{2} \int_p^{\tilde{p}} a_x(u_x)^2 \, dx \\
= \frac{1}{2} \int_p^{\tilde{p}} a_x(u_x)^2 \, dx \tag{31}
\]

By hypothesis we cannot have a critical point \( p \) of \( u \) in \([0, 1]\) with \( C_u(p) = \{p\} \) (that is, such that no other critical point with the same value exists).
Suppose there are three critical points $p_1 < p_2 < p_3$, with the same value $q$. Then, if $p_2 \leq c a_x$ has constant sign in $[p_1, p_2]$ so $\int_{p_1}^{p_2} a_x(u_x)^2 \, dx \neq 0$, contradicting (31). By a similar argument, we also cannot have $p_2 \geq c$, so there are at most and therefore, exactly two points in $u^{-1}(q)$. In particular, the number of critical points must be even.

If $u(0) \neq u(1)$, we must have two critical points $p_0$ and $p_1$ in $(0, 1)$ with $u(p_0) = u(0), u(p_1) = u(1)$. If $p_0 < c$ or $p_1 > c$ then, since $a_x$ has constant sign in at least one of the intervals $[0, p_0], [p_1, 1]$ then $\int_0^{p_0} a_x(u_x)^2 \, dx \neq 0$ or $\int_{p_1}^{1} a_x(u_x)^2 \, dx \neq 0$, in contradiction with (31).

If $p_0 > c > p_1$ then, again by (31) $0 = \int_0^{p_0} a_x(u_x)^2 \, dx = \int_0^{p_1} a_x(u_x)^2 \, dx + \int_{p_1}^{p_0} a_x(u_x)^2 \, dx$, and $0 = \int_{p_1}^{1} a_x(u_x)^2 \, dx = \int_0^{p_0} a_x(u_x)^2 \, dx + \int_{p_0}^{1} a_x(u_x)^2 \, dx$. Subtracting, we obtain $0 = \int_0^{1} a_x(u_x)^2 \, dx - \int_0^{p_0} a_x(u_x)^2 \, dx = 0$. But this is not possible, since $a_x$ has different signs in the intervals $[0, p_0], [p_1, 1]$.

If $u(0) = u(1)$, since the zeroes of $u$ are nondegenerate, we must have an odd number of critical points, contradicting the conclusion above. This proves the result.

As an immediate consequence, we have

**Corollary 1** Suppose the diffusion coefficient $a$ and $f$ in (1) is analytic, and has only two intervals of monotonicity in $[0, 1]$, and is not even about the point $x = \frac{1}{2}$. Then there exists a residual set $F$ in $C^2(I \mathbb{R})$ such that, if $f \in F$, all equilibria of (1) are hyperbolic.

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