AN L-A PAIR FOR THE APEL’ROT SYSTEM
AND A NEW INTEGRABLE CASE FOR THE
EULER-POISSON EQUATIONS ON $so(4) \times so(4)$

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We present an L-A pair for the Apel’rot case of a heavy rigid 3-dimensional body. Using it
we give an algebro-geometric integration procedure. Generalizing this L-A pair, we obtain a new
completely integrable case of the Euler-Poisson equations in dimension four. Explicit formulae for
integrals which are in involution are given. This system is a counterexample to one well known
Ratiu’s theorem. Corrected version of this classification theorem is proved.

1. Introduction

The rotations of a heavy rigid body fixed at a point are described by the Euler-
Poisson equations. It is well known that these equations are integrable in Euler,
Lagrange and Kovalevskaya cases. For a fixed value of one of the integrals, there
are additional integrable cases, for example Goryachev-Chaplygin and Apel’rot (see
[1], [5]). L-A pairs were known for all these problems except the last one (see [4],
[7], [10], [11]). In this paper, in section 2, we present an L-A pair for the Apel’rot
case. (Almost the same L-A pair serves in the Apel’rot gyrostat case.) Using
it, we give an algebro-geometric integration procedure in section 3. The spectral
curve is reducible and it consists of a sphere and a torus. The evolution of the
pole divisor of the eigen-function of the $L$ matrix is linearized on the Jacobian of
the torus. It leads to the solution of the initial problem up to rotation. For a
complete solution one needs one step more: to integrate the Riccati equation with
double-periodical coefficients. Generalizing this L-A pair, we obtain, in section 4, a new completely integrable system of a four-dimensional rigid body motion. This system is integrable even without reduction to any invariant submanifold. Explicit formulae for integrals which are in involution are given.

It turns out that the L-A pair, which we use, is of the form analysed by Ratiu and van Moerbeke in the study of the generalized Lagrange case and of the completely symmetrical case in [9], [10]. Our system is a counterexample to Ratiu’s well known theorem (see [9], [13]). This theorem claims that the Lagrange and the completely symmetric case have exhausted the list of the Euler-Poisson equations which are equivalent to an L-A pair of that form. We conclude the paper by giving a corrected version of this classification theorem.

2. Apel’rot case

The equations of rotation of a heavy rigid body fixed at a point in a moving frame are:

\[ \frac{d}{dt} \vec{M} = \vec{M} \times \vec{\omega} + \vec{\gamma} \times \vec{r}_C, \quad \frac{d}{dt} \vec{\gamma} = \vec{\gamma} \times \vec{\omega}, \]  

(1)

where \( \vec{\omega} \) is the vector of the angular velocity, \( \vec{M} = I \vec{\omega} \) is the kinetic momentum, \( I \) is the inertia operator, \( \vec{\gamma} \) is the unit vertical vector, and \( \vec{r}_C = (x_0, y_0, z_0) \) is the radius vector of the mass center according to the fixed point. One can choose the moving frame such that the inertia operator is diagonal, \( I = diag(I_1, I_2, I_3) \). The equations (1) have the following three integrals of motion (see [3]):

\[ F_1 = \frac{1}{2} \langle I \vec{\omega}, \vec{\omega} \rangle + \langle \vec{\gamma}, \vec{r}_C \rangle, \]
\[ F_2 = \langle \vec{\gamma}, \vec{\gamma} \rangle (= 1), \]
\[ F_3 = \langle I \vec{\omega}, \vec{\gamma} \rangle. \]

For complete integrability, we need one integral more. In 1894 Apel’rot noticed (see [1], [5]) that under the additional conditions:

\[ i) \quad y_0 = 0 \]
\[ ii) \quad x_0 \sqrt{I_1(I_2 - I_3)} + z_0 \sqrt{I_3(I_1 - I_2)} = 0 \]

(2)
the equations (1) became integrable on the hypersurface given with:

\[ I_1 x_0 \omega_1 + I_3 z_0 \omega_3 = 0. \]  

(3)

A geometric interpretation of these conditions was given by Zhukovski [6],[14].

Using standard isomorphism between Lie algebras \( R^3 \) and \( so(3) \) (notation: \( \vec{A} \in R^3 \mapsto A \in so(3) \)), (see [2], [10]), the equations (1) can be transformed to the following equations on \( so(3) \times so(3) \):

\[ \frac{dM}{dt} = [M, \omega] + [\gamma, r_c], \quad \frac{d\gamma}{dt} = [\gamma, \omega] \]  

(4)

**Theorem 1.** Under the Apel’rot conditions (2), the equations (4) are equivalent to:

\[ \frac{d}{dt}(\lambda^2 C + \lambda M + \gamma) = [\lambda^2 C + \lambda M + \gamma, \omega + \lambda r_C], \]  

(5)

where \( \lambda \) is a spectral parameter and \( C \in so(3) \) is a constant matrix given by

\[ C = I_2 r_C. \]  

(6)

**Proof.** The equations (4) are equivalent to (5) if and only if

\[ [C, r_C] = 0, \quad [C, \omega] + [M, r_C] = 0. \]

The first equation is satisfied by (6). The second one can be reduced to the equation:

\[ (I_1 - I_2) \omega_1 z_0 + (I_2 - I_3) \omega_3 x_0 = 0. \]  

(7)

But (7) is a consequence of the Apel’rot conditions (2) and the equation of the hypersurface (3).

**Note.** A gyrostat is a system which consists of two rigid bodies. The second one, the gyroscope, is axially-symmetric and it rotates with a constant angular velocity about its symmetry axis which is fixed in the first rigid body. The equations of motion of that system are of the form (4) with

\[ M = I \omega + P \]
where \( P \) is a constant skew-symmetric matrix, which represents the kinetic momentum of the gyroscope. Sretenskiy in [12] proved that, if Apel’rot conditions (2) are satisfied, and \( P_2 = 0 \), then the equations of motion are integrable on the hypersurface given by:

\[
(I_1 - I_2)\omega_1 z_0 + (I_2 - I_3)\omega_3 x_0 - (P_3 x_0 - P_1 z_0) = 0.
\]

It can easily be proved that:

\[
L(\lambda) = \lambda^2 C + \lambda M + \gamma; \quad A(\lambda) = \omega + \lambda r_C,
\]

form the L-A pair for the Apel’rot gyrostat.

An L-A pair of the form

\[
L(\lambda) = \lambda^2 C + \lambda M + \gamma, \quad A(\lambda) = \omega + \lambda r_C
\]

was used in [10] for the Lagrange case (of course, the matrices \( r_C \) and \( C \) from [10] are different from the matrices \( r_C \) and \( C \) in this paper. Also, the conditions on the inertia operator are not the same).

3. Integration of the Apel’rot case

The next steps in algebro-geometric integration of the Apel’rot case follow the procedure from [10].

As usual, in a direct problem one corresponds to \( L(\lambda) \) some algebro-geometric data with simple evolution in time. After the integration of this evolution, in the inverse part, one has to reconstruct the matrix elements of \( L(\lambda) \), starting from the algebro-geometric data as known functions of time. The final part is the integration of the equations of motion of the initial problem by use of the known matrix entries of \( L(\lambda) \). A specific characteristic of the problem presented here, is that the algebro-geometric procedure gives a solution of the equations of motion of the Apel’rot case only up to rotations. For the full solution of the problem, we need one integration more to determine the angle variable \( \phi_x = \arg x \) - see theorem 5 below.
Let:

\[ \alpha = \frac{x_0}{\sqrt{x_0^2 + z_0^2}} \quad \beta = \frac{z_0}{\sqrt{x_0^2 + z_0^2}} \]  

(9)

The matrix

\[ U = \begin{bmatrix} \alpha & i\beta \\ \frac{1}{\sqrt{2}} & \frac{\beta}{\sqrt{2}} \\ \frac{i\alpha}{\sqrt{2}} & \frac{-\beta}{\sqrt{2}} \end{bmatrix} \]

diagonalises matrix \( C \). In the new basis, the matrix \( L(\lambda) = C\lambda^2 + M\lambda + \gamma \) transforms into:

\[ \tilde{L} = U^{-1}LU = \begin{bmatrix} 0 & \Delta & i\Delta^* \\ -\Delta^* & -\Omega & 0 \\ i\Delta & 0 & \Omega \end{bmatrix} \]

where:

\[ \Delta = y + \lambda x; \quad \Delta^* = \bar{y} + \lambda \bar{x}; \]

\[ y = \frac{1}{\sqrt{2}}(\beta\gamma_1 - \alpha\gamma_3 - i\gamma_2); \quad x = \frac{1}{\sqrt{2}}(\beta M_1 - \alpha M_3 - iM_2) \]

\[ \Omega = -i \left[ \alpha(C_1\lambda^2 + M_1\lambda + \gamma_1) + \beta(C_3\lambda^2 + M_3\lambda + \gamma_3) \right] \]

(10)

\[ = -i \left[ \alpha(C_1\lambda^2 + \gamma_1) + \beta(C_3\lambda^2 + \gamma_3) \right] \]

and \( C_1 = I_2x_0; \quad C_3 = I_2z_0 \), using (3).

The spectral curve is defined by:

\[ \Gamma : p(\mu, \lambda) := \det(L(\lambda) - \mu E) = 0, \]

where

\[ p(\mu, \lambda) = -\mu(\mu^2 - \Omega^2 + 2\Delta\Delta^*). \]  

(11)

The spectral curve \( \Gamma \) is reducible and it consists of two components: the sphere \( \Gamma_1 \) given by \( \mu = 0 \), and the torus \( \Gamma_2 \):

\[ \mu^2 = \Omega^2 - 2\Delta\Delta^* =: P_4(\lambda). \]

(12)

The coefficients of the spectral polynomial (12) are integrals of the motion. Substituting expressions (10) in (11), the following form of the equation of the spectral curve is obtained:

\[ p(\mu, \lambda) = -\mu(\mu^2 + A\lambda^4 + B\lambda^3 + D\lambda^2 + E\lambda + F), \]

(13)
where

\[ A = I_2^2 (x_0^2 + z_0^2) \]
\[ B = 2I_2(M_1x_0 + M_3z_0)(= 0) \]
\[ D = 2I_2 \left( \frac{M_1^2}{2I_2} + \frac{M_2^2}{2I_2} + \frac{M_3^2}{2I_2} + x_0\gamma_1 + z_0\gamma_3 \right) \]
\[ E = 2(M_1\gamma_1 + M_2\gamma_2 + M_3\gamma_3) \]
\[ F = \gamma_1^2 + \gamma_2^2 + \gamma_3^2 (= 1) \]

(14)

Using the Apel’rot conditions, the expression for \( D \) can be simplified:

\[ D = 2I_2 \left( \frac{M_1^2}{2I_1} + \frac{M_2^2}{2I_2} + \frac{M_3^2}{2I_3} + x_0\gamma_1 + z_0\gamma_3 \right). \]

This is the energy integral. So, in this case the L-A pair (5) gives four integrals of the motion, which is enough for integrability.

Let \((f_1, f_2, f_3)^T\) denote an eigenvector of the matrix \( L(\lambda) \), which corresponds to the eigenvalue \( \mu \). Fix a normalizing condition \( f_1 = 1 \). Then:

\[ f_2 = -\frac{\Delta^*}{\Omega + \mu}, \quad f_3 = -\frac{i\Delta}{\Omega - \mu}. \]

(15)

The restrictions to \( \Gamma_2 \) are:

\[ f_2 = -\frac{\Omega - \mu}{2\Delta}, \quad f_3 = \frac{\Omega + \mu}{2i\Delta^*}. \]

The relation between the divisors on \( \Gamma_2 \):

\[(f_2) + (f_3) = 0.\]

is a consequence of \( f_2 \cdot f_3 = \frac{i}{2} \).

From (12) the divisors of \( \lambda \) and \( \mu \) on \( \Gamma_2 \) are:

\[(\lambda) = -P_1 - P_2 + R_1 + R_2 \]
\[(\mu) = -2P_1 - 2P_2 + (\mu)_0, \quad \deg(\mu)_0 = 4 \]

where \( P_1 \) and \( P_2 \) are the points on \( \Gamma_2 \) over \( \lambda = \infty \). From (12), we obtain the developments in neighborhoods of \( P_1 \) and \( P_2 \):

\[ \frac{\mu}{\lambda^2} = \left\{ \begin{array}{ll}
  iI_2\sqrt{x_0^2 + z_0^2} + O(\lambda^{-2}), & \text{in a neighborhood of } P_1 \\
  -iI_2\sqrt{x_0^2 + z_0^2} + O(\lambda^{-2}), & \text{in a neighborhood of } P_2
\end{array} \right. \]
From (15), it follows that $f_2$ has a simple pole in $P_1$ and a simple zero in $P_2$; $f_3$ has a simple zero in $P_1$ and a simple pole in $P_2$. In the affine part, $f_2$ has a simple pole in $\nu$ given by

$$\nu : \Delta = 0, \quad \Omega + \mu = 0, \quad \text{i.e.} \quad \nu_\lambda = -\frac{y}{x}, \quad \nu_\mu = -\Omega \mid_{\lambda = -\frac{y}{x}}.$$  

It has a simple zero in $\tilde{\nu} = (\tilde{\nu}_\lambda, \tilde{\nu}_\mu)$, where $\tilde{\nu}_\lambda$ denotes the complex conjugate of $\nu_\lambda$.

The previous consideration gives us:

**Lemma 1.** The divisors of $f_2$ and $f_3$ on $\Gamma_2$ are:

$$(f_2) = -P_1 + P_2 - \nu + \tilde{\nu}$$

$$(f_3) = P_1 - P_2 + \nu - \tilde{\nu}$$

Now we are going to analyse the converse problem. Suppose the evolution in time of the point $\nu$ is known (see theorem 4 below). For reconstructing the matrix $L(\lambda)$, one needs $x = |x|e^{i \arg x}$, $y = |y|e^{i \arg y}$.

**Theorem 2.** The point $\nu \in \Gamma_2$ and the initial conditions for $M$ and $\gamma$ determine $|x|, \ |y|$ and $\arg y - \arg x$, where $x$ and $y$ are given by (10).

**Proof.** Let $\nu$ have coordinates $\nu_\mu$ and $\nu_\lambda$. From the expressions for $\nu$

$$\nu_\lambda = -\frac{y}{x}; \quad \nu_\mu = -\Omega \mid_{\lambda = -\frac{y}{x}}$$

and expression (10) for $\Omega$, it follows:

$$\alpha \gamma_1 + \beta \gamma_3 = -i \nu_\mu - I_2 \nu_\lambda^2 \sqrt{x_0^2 + z_0^2}. \quad (16)$$

From (10) and (16) we can determine $\Omega$ as a polynomial of $\lambda$:

$$\Omega = -i[I_2 \lambda^2 \sqrt{x_0^2 + z_0^2} - i \nu_\mu - I_2 \nu_\lambda^2 \sqrt{x_0^2 + z_0^2}]. \quad (17)$$

In the expression for the spectral curve $\mu^2 = \Omega^2 - 2\Delta \Delta^*$ understood as a polynomial in $\lambda$, the coefficients are integrals of the motion (they are determined and given by (14)).

So

$$\mu^2 - \Omega^2 = -2\Delta \Delta^* = -2(|y|^2 + (y \bar{x} + x \bar{y}) \lambda + |x|^2 \lambda^2)$$
is a known polynomial and we can determine $|x|^2$, $|y|^2$ and $\arg y - \arg x = \arg \left( \frac{y}{x} \right)$.

The previous theorem gives us a possibility for integration of the Apel’rot case if we know the motion of the point $\nu$. As we will show, the motion of the point $\nu$ is linearized on the Jacobian $\text{Jac}(\Gamma_2) \cong \Gamma_2$.

In all propositions until the end of this section, we will assume that the Apel’rot conditions are satisfied.

**Theorem 3.** If $\nu_\lambda$ and $\nu_\mu$ are coordinates of point $\nu \in \Gamma_2$, then:

$$\frac{d}{dt}\nu_\lambda = \frac{1}{I_2} \nu_\mu$$

**Lemma 2.** The functions $x$ and $y$ satisfy the system

$$\begin{align*}
\frac{dx}{dt} &= \left( \frac{1}{I_1} - \frac{1}{I_2} \right) \frac{M_1}{\alpha \sqrt{2}} x + i \sqrt{x_0^2 + z_0^2} y \\
\frac{dy}{dt} &= -i(\alpha \gamma_1 + \beta \gamma_3) \frac{M_1}{I_2} x + \frac{iM_1}{\alpha \sqrt{2}} \left( \frac{1}{I_1} - \frac{1}{I_2} \right) y.
\end{align*}$$

(18)

**Proof.** From the Apelrot conditions (2) we can get

$$\frac{\beta^2}{\alpha^2} \left( \frac{1}{I_1} - \frac{1}{I_2} \right) = \frac{1}{I_2} - \frac{1}{I_3}.$$  \hspace{1cm} (19)

From (19), (3), and the identity $\alpha^2 + \beta^2 = 1$ one obtains

$$\begin{align*}
\frac{dx}{dt} &= \left( \frac{1}{I_1} - \frac{1}{I_2} \right) \frac{M_1}{\alpha \sqrt{2}} (\alpha M_2 - iM_3) + i \sqrt{x_0^2 + z_0^2} y \\
\frac{dy}{dt} &= -i(\alpha \gamma_1 + \beta \gamma_3) \frac{M_1}{I_2} x + \frac{iM_1}{\alpha \sqrt{2}} \left( \frac{1}{I_1} - \frac{1}{I_2} \right) y.
\end{align*}$$

The claim follows applying (3).

As a consequence we get

**Lemma 3.** The function $\nu_\lambda = -\frac{y}{x}$ satisfies the equation:

$$\frac{d}{dt} \left( -\frac{y}{x} \right) = \frac{1}{x^2} \left( \frac{dx}{dt} y - \frac{dy}{dt} x \right) = \frac{1}{x^2} \left[ i \sqrt{x_0^2 + z_0^2} y + \frac{i(\alpha \gamma_1 + \beta \gamma_3)}{I_2} x^2 \right].$$

**Proof of theorem 3.** The divisor $\nu$ is given by $\nu_\lambda = -\frac{y}{x}$, $\nu_\mu = -\Omega \mid_{\lambda = -\frac{y}{x}}$. From lemma 2 and lemma 3, we have:

$$\frac{d}{dt}\nu_\lambda = \frac{d}{dt} \left( -\frac{y}{x} \right) = \frac{1}{x^2} \left( \frac{dx}{dt} y - \frac{dy}{dt} x \right) = \frac{1}{x^2} \left[ i \sqrt{x_0^2 + z_0^2} y + \frac{i(\alpha \gamma_1 + \beta \gamma_3)}{I_2} x^2 \right] = \frac{1}{I_2} \left( -\Omega \mid_{\lambda = -\frac{y}{x}} \right) = \frac{1}{I_2} \nu_\mu.$$  \hspace{1cm} (20)
Since $\Gamma_2$ is a curve of genus 1, there is only one linearly independent holomorphic differential
doing it.

So any linear flow on $\text{Jac}(\Gamma_2)$ is of the form $\frac{d\lambda}{dt} = \text{const} \cdot \mu$.

A consequence of (20) is that the flow of the point $\nu$ is linear on $\text{Jac}(\Gamma_2)$. We have the next theorem:

**Theorem 4.** The integration of the motion of the point $\nu$ reduces to the inversion of the elliptical integral

$$\int_{\nu_0}^{\nu} \frac{d\lambda}{\sqrt{\Omega^2 - 2\Delta \Delta^*}} = \frac{1}{I_2} t.$$  

From theorem 2 and expression (10) for $x$ and $y$, it follows that, if we know the motion of the point $\nu$, in order to determine the matrix $L(\lambda)$ as a function of time, we need arg $x$ as a function of time. Let us denote by $\phi_x = \text{arg} x$, and $u = \tan \phi_x$.

**Theorem 5.** The function $u(t)$ satisfies the Riccati equation:

$$\frac{du}{dt} = [(f(t) - g(t))u^2 + [f(t) + g(t)],$$  

where

$$f(t) = \frac{K}{2|x|^2} \quad g(t) = \frac{Q|x|}{2}$$

$$K = \frac{\langle M, \gamma \rangle}{2\sqrt{x_0^2 + z_0^2}}$$  

$$Q = \frac{\beta}{\alpha} \sqrt{2} \left( \frac{1}{I_2} - \frac{1}{I_1} \right)$$

and $|x|$ is a known function of time.

**Proof.** From

$$\tan \phi_x = \frac{\text{Im} x}{\text{Re} x} = -\frac{M_2}{\beta M_1 - \alpha M_3},$$

by differentiating and using the equations of motion (4), we get:

$$\frac{d\phi_x}{dt} \frac{1}{\cos^2 \phi_x} = -\frac{M_1 \langle M, M \rangle \frac{1}{\alpha} (\frac{1}{I_2} - \frac{1}{I_1}) + \langle M, \gamma \rangle}{\langle M, M \rangle - M_2^2 \sqrt{\frac{2}{x_0^2 + z_0^2}}}. \quad (23)$$
From the system
\[
\alpha M_1 + \beta M_3 = 0
\]
\[
\beta M_1 - \alpha M_3 = \sqrt{2}\text{Re}x = \sqrt{2}|x| \cos \phi_x,
\]
\(M_1\) can be expressed as a function of \(|x|\), and \(\phi_x\)
\[
M_1 = \sqrt{2}\beta|x| \cos \phi_x.
\]
Also, from (10) we have:
\[
M_2 = -\sqrt{2}\text{Im}x = -\sqrt{2}|x| \sin \phi_x.
\]
From (10), using (3) we have that \(2|x|^2 = \langle M, M \rangle\). Substituting the last three expressions into (23) gives
\[
\frac{d\phi_x}{dt} = \frac{K - Q|x|^3 \cos \phi_x}{|x|^2}.
\]
By the change of variables \(u = tg\frac{\phi_x}{2}\) the equation (24) takes the form (21).

The classical integration procedure in the so-called Hesse coordinates [5] also yields the Riccati equation (see [5]). By Nekrasov (see [5], [8]) it was reduced to a second order linear differential equation with double-periodical coefficients.

4. A new integrable case on \(so(4) \times so(4)\)

The equations of rotations of a heavy \(n\)-dimensional rigid body fixed at a point on \(so(n) \times so(n)\) are given in [9]. In the moving frame, these equations are:
\[
\frac{dM}{dt} = [M, \Omega] + [\Gamma, X],
\]
\[
\frac{d\Gamma}{dt} = [\Gamma, \Omega].
\]
\(M = I\Omega + \Omega I \in so(n)\) is the kinetic momentum, \(\Omega \in so(n)\) is the angular velocity, \(I\) is a symmetric \(n \times n\) matrix, \(\Gamma \in so(n)\), and \(X \in so(n)\) is a given constant matrix. By choosing the moving frame with \(I\) diagonal, \(I = diag(I_1, \ldots, I_n)\), we have \(M_{ij} = (I_i + I_j)\Omega_{ij}\), and \(I_i + I_j\) are the principal inertia momenta.

In [9], the Lagrange case was defined by \(I_1 = I_2 = a; I_3 = \cdots = I_n = b, X_{12} \neq 0, X_{ij} = 0, i, j \neq 1, 2, i < j\). The completely symmetric case was defined there by
\( I_1 = \cdots = I_n = a \), where \( X \in \mathfrak{so}(n) \) is an arbitrary constant matrix. It was shown in [9] that the equations (25) in these cases could be represented by the following L-A pair:

\[
\frac{d}{dt}(\lambda^2 C + \lambda M + \Gamma) = [\lambda^2 C + \lambda M + \Gamma, \lambda X + \Omega],
\]

where in the Lagrange case \( C = (a + b)X \), and in the symmetric case \( C = aX \).

The theorem 4.1 in [9], states that there are no other cases of \( n \)-dimensional rigid bodies, with the equations equivalent to (26).

However, let us consider a 4-dimensional rigid body with the conditions:

\[
I_1 = I_2 = a, I_3 = I_4 = b,
X_{12} \neq 0, X_{34} \neq 0, X_{ij} = 0, i, j \neq \{1, 2\}, \{3, 4\}, i < j.
\]

Note that this is neither the Lagrange nor the completely symmetrical case. The next theorem shows that this example satisfies all assumptions of the theorem 4.1 from [9]. So, the system (27) is a counterexample to that theorem.

**Theorem 6.** The equations (25) for a 4-dimensional rigid body which satisfies (27) are equivalent to (26) with the matrix \( C = (a + b)X \).

**Proof.** We have \([C, X] = 0\) from (27). Also, \([C, \Omega] + [M, X] = 0\), so (26) is equivalent to (25).

One can naturally consider (25) for \( n = 4 \) as an equation on the semidirect product \( \mathfrak{so}(4) \times \mathfrak{so}(4) \) with the following Poisson structure on the orbits of the coadjoint action (see [9]):

\[
\{\tilde{f}, \tilde{g}\}(\mu, \nu) = -\mu([d_1 f(\mu, \nu), d_1 g(\mu, \nu)])
- \nu([d_1 f(\mu, \nu), d_2 g(\mu, \nu)])
- \nu([d_2 f(\mu, \nu), d_1 g(\mu, \nu)]),
\]

where \( \tilde{f}, \tilde{g} \) are restrictions of the functions \( f \) and \( g \) to the orbit of the coadjoint action, and \( d_i f \) are the partial derivatives of \( df \). From the L-A pair representation (26), we have that integrals of the motion for the system given by (27), are the coefficients in the \( \lambda \) polynomials \( \text{Tr}(\lambda^2 C + \lambda M + \Gamma)^2 \), and \( (\text{Det}(\lambda^2 C + \lambda M + \Gamma))^{1/2} \).
Using the same arguments as in [9], it can be proved that the system given by (27) has the following four Casimir functions:

\[ J_1 = M_{34}\Gamma_{12} + M_{12}\Gamma_{34} + M_{14}\Gamma_{23} + M_{23}\Gamma_{14} - M_{24}\Gamma_{13} - M_{13}\Gamma_{24} \]

\[ J_2 = \Gamma_{34}\Gamma_{12} + \Gamma_{23}\Gamma_{14} - \Gamma_{13}\Gamma_{24} \]

\[ J_3 = \Gamma_{12}^2 + \Gamma_{13}^2 + \Gamma_{14}^2 + \Gamma_{23}^2 + \Gamma_{24}^2 + \Gamma_{34}^2 \]

\[ J_4 = M_{12}\Gamma_{12} + M_{13}\Gamma_{13} + M_{14}\Gamma_{14} + M_{23}\Gamma_{23} + M_{24}\Gamma_{24} + M_{34}\Gamma_{34} . \]

Also, it follows that this system has four integrals:

\[ F_1 = C_{12}M_{12} + C_{34}M_{34} \]

\[ F_2 = C_{34}M_{12} + C_{12}M_{34} \]

\[ F_3 = M_{12}M_{34} + M_{23}M_{14} - M_{13}M_{24} + C_{34}\Gamma_{12} + C_{12}\Gamma_{34} \]

\[ F_4 = M_{12}^2 + M_{13}^2 + M_{14}^2 + M_{23}^2 + M_{24}^2 + M_{34}^2 + 2C_{12}\Gamma_{12} + 2C_{34}\Gamma_{34} , \]

which are in involution. Thus, we have

**Proposition.** The system (27) is completely integrable.

Except the three cases mentioned above, there are no other systems of the type (25) which are equivalent to (26). This is proved in the following:

**Theorem 7.** Let us suppose that \( X_{12} \neq 0 \). The Euler-Poisson equations (25) can be written in the form (26) if and only if the equations (25) describe:

a) for \( n = 4 \), the motion of the Lagrange top, the completely symmetric top or a rigid body which satisfies (27).

b) for \( n \neq 4 \), the motion of the Lagrange or completely symmetric top.

**Proof.** The equations (26) are equivalent to (25), if and only if \([C,X] = 0 \) and \([C,\Omega] + [M,X] = 0 \). The second relation is equivalent to

\[ C_{ij} = (I_i + I_k)X_{ij}, \quad C_{ij} = (I_j + I_k)X_{ij}, \quad k \neq i,j. \]

(28)

From \( C_{12} \neq 0 \) and (28), for \( i = 1, j = 2 \), it follows that \( I_1 = I_2 \), \( I_k = I_l \) for all \( k,l \neq 1,2 \). Let us fix \( i,j \) different from 1 and 2. From (28), we have \( I_i = I_j, I_k = I_l \), for all \( k,l \neq i,j \). For \( n \neq 4 \), this means that \( I_1 = \cdots = I_n \) and \( X \) is an arbitrary
matrix, or $I_1 = I_2 = a$, $I_3 = \cdots = I_n$ and $X_{ij} = 0$ for $i, j \neq 1, 2$. For $n = 4$, we get one more case which satisfies the conditions (27). For such $C_{ij}$, the relation $[C, X] = 0$ is satisfied.

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