ON STEIN NEIGHBORHOOD BASIS OF REAL SURFACES

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Abstract. In this paper, we show that a compact real surface embedded in a complex surface has a regular Stein neighborhood basis, provided that there are only finitely many complex points on the surface, and that they are all flat and hyperbolic. An application to unions of totally real planes in $\mathbb{C}^2$ is then given.

1. Introduction

Let $S \hookrightarrow X$ be a real compact surface, smoothly embedded in a complex surface $X$. In a generic position, there are only finitely many complex points on $S$, which can be, following Bishop [1], classified as either elliptic or hyperbolic.

By a result of Bishop [1], the local holomorphic hull of $S$ at elliptic points contains a real one parameter family of pairwise disjoint holomorphic discs with boundaries in $S$. The presence of elliptic complex points on $S$ thus prevents the surface $S$ from having small Stein neighborhoods in $X$. On the other hand, by a result of Forstneriˇ c and Stout [5], $S$ is locally polynomially convex at hyperbolic complex points. Using this, one can easily construct a Stein neighborhood basis of a real surface with only hyperbolic complex points by just patching together local pseudoconvex defining functions. The problem is that one does not necessarily understand the topology of such a basis. For example, none of the members are a priori even of the same homotopy type as $S$. For this reason, we prefer to put some further restrictions on our basis.

Definition 1.1. Let $\pi : S \hookrightarrow M$ be an embedding of a manifold $S$ into a manifold $M$. A system $\{\Omega_\epsilon, \epsilon \in (0, 1)\}$ of open neighborhoods of $\pi(S)$ in $M$ is called a regular basis, if for every $\epsilon \in (0, 1)$, we have

1. $\Omega_\epsilon = \bigcup_{s<\epsilon} \Omega_s$,
2. $\bar{\Omega}_\epsilon = \bigcap_{s>\epsilon} \Omega_s$,
3. $\pi(S) = \bigcap_{s>0} \Omega_s$ is a strong deformation retract of $\Omega_\epsilon$.

It has been asked by Forstnerič in [1], whether the nonexistence of elliptic complex points on $S$ is sufficient for construction a regular Stein neighborhood basis of $S$ in $X$. We partially answer the question in the following theorem.

Theorem 1.2. Let $S$ be a compact real surface, embedded in a complex surface $X$, and having only finitely many complex points. Let all complex points on $S$ be flat hyperbolic complex points. Then $S$ has a regular strictly pseudoconvex Stein neighborhood basis in $X$.

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The condition of flatness, with the rest of the terms, will be defined in the next section. We would like to indicate that the flatness condition is in our view redundant, and is here only because of our inability to yet prove the theorem without it.

2. Local structure at complex points

Let \( S \hookrightarrow X \) be a real compact surface, embedded in a complex surface \( X \). We call \( p \in S \) a complex point if \( T_pS \subset T_pX \) is a complex line. If this is not the case, the point \( p \) is totally real. In a generic situation, there are only finitely many complex points on \( S \). Let \( p \in S \) be an isolated complex point on \( S \hookrightarrow X \). In local holomorphic coordinates around \( p = (0,0) \), we can write \( S \) as a graph \( w = f(z) \).

In a generic case, we can further change the coordinates, see [1], to locally have \( S \) given by

\[
2w = \alpha z\bar{z} + \frac{1}{2}(z^2 + \bar{z}^2) + o(|z|^3),
\]

where \( 0 \leq \alpha \leq \infty \). The case \( \alpha = \infty \) should be understood as the surface \( w = z\bar{z} + o(|z|^3) \). We call \( p \) a hyperbolic complex point if \( 0 \leq \alpha < 1 \) and elliptic if \( \alpha > 1 \). The parabolic case \( \alpha = 1 \) is not generic. A complex point \( p \) on a real surface \( S \) is called quadratic, if we can write the surface in the above form with \( o(|z|^3) \equiv 0 \). We call a complex point \( p \) a flat complex point, if local holomorphic coordinates can be chosen so that \( \text{Im} o(|z|^3) \equiv 0 \) in the form (2.1). Quadratic complex points are of course flat.

**Remark 2.1.** If \( S \) is embedded as a real analytic submanifold of a complex surface \( X \), then by a result of Moser and Webster in [3], every real analytic elliptic complex point is flat. On the other hand, they show the existence of real analytic hyperbolic complex points that are not flat.

Let us now take the case of a flat hyperbolic complex point. Locally, we assume the surface lies in \( \mathbb{C}^2 \) with coordinates \((z,w)\), the complex point corresponds to \((z,w) = (0,0)\) and that the surface \( S \) is given by

\[
S = \begin{cases} 
\text{Re} w = \frac{1}{2} \alpha z\bar{z} + \frac{1}{4}(z^2 + \bar{z}^2) + \tau \\
\text{Im} w = 0 
\end{cases}
\]

where \( \tau = o(|z|^3) \) is a real function, and \( 0 \leq \alpha < 1 \). We introduce new real coordinates in the neighborhood of the origin of \( \mathbb{C}^2 \), given by

\[
\begin{align*}
x &= \text{Re} z \\
y &= \text{Im} z \\
u &= \text{Re} w - \frac{1}{2} \alpha z\bar{z} - \frac{1}{4}(z^2 + \bar{z}^2) - \tau \\
v &= \text{Im} w.
\end{align*}
\]

These new coordinates are nonholomorphic and they depend on \( \alpha \), with the surface \( S \) corresponding to \( \{u = v = 0\} \). The partial derivatives in the Levi form
computation are expressed in these coordinates as
\[
4 \frac{\partial^2 f}{\partial z \partial w} = \triangle_{x,y} - 2((\alpha + 1)x \frac{\partial^2 f}{\partial x^2} + (\alpha - 1)y \frac{\partial^2 f}{\partial y^2} + \alpha) \frac{\partial f}{\partial y} \\
+ (\alpha + 1)^2 x^2 + (\alpha - 1)^2 y^2 \frac{\partial^2 f}{\partial y^2} \\
+ o(|z|^2) \frac{\partial f}{\partial y} + o(|z|) \frac{\partial^2 f}{\partial y^2} + o(|z|^3) \frac{\partial^3 f}{\partial y^3}
\]
(2.4)
\[
4 \frac{\partial^2 f}{\partial w \partial u} = \triangle_{u,w} + \frac{\partial^2 f}{\partial w^2} - (\alpha + 1)x \frac{\partial^2 f}{\partial u^2} + o(|z|^2) \frac{\partial^2 f}{\partial u^2} \\
+ i \left( - \frac{\partial^2 f}{\partial w \partial u} + (\alpha - 1)y \frac{\partial^2 f}{\partial y^2} + o(|z|^2) \frac{\partial^3 f}{\partial y^3} \right).
\]
The terms consist of derivatives of \(\tau\), and so vanish for quadratic complex points. The term \(\triangle_{x,y}\) denotes the Laplace operator with respect to \(x, y\) coordinates.

3. Local construction at hyperbolic complex points

Throughout this section, let \((z, w)\) be the standard holomorphic coordinates coordinates on \(\mathbb{C}^2\) and let \((x, y, u, v)\) be the non-holomorphic real coordinates on \(\mathbb{C}^2\), given by \((x^2, y^2, u^2, v^2)\), and are dependent on \(\alpha\).

In the construction of Stein neighborhood basis near a hyperbolic complex point, we use the following lemma.

**Lemma 3.1.** Let \(S\) be a submanifold in \(M\) and let \(\Omega\) be an open neighborhood of \(S\) in \(M\). Let \(f: \Omega \to [0, 1)\) be a \(C^2\) function, having the property \(S = \{ f = 0 \} = \{ \nabla f = 0 \}\). Then \(\{ \Omega_{\epsilon} \}_{\epsilon<1} = \{ f < \epsilon \}_{\epsilon<1}\), defines a regular neighborhood basis for \(S\) in \(M\).

**Proof.** Let \(\Omega_{s} = \{ f < s \}\), and let \(\psi_t\) be a the flow of the vector field \(-\nabla f\) in some Riemannian metric on \(M\). This flow gives us a strong deformation retract of \(\Omega_{s}\) to \(S\). \(\square\)

Our goal is to find functions \(\phi(x, y, u) \geq 0\) and \(\psi(x, y, v) \geq 0\), defined in a small neighborhood of \((0, 0, 0)\) in \(\mathbb{R}^3\) and having the following properties

- \(\Phi(z, w) = \phi(x, y, u) + \psi(x, y, v)\) is plurisubharmonic in a small neighborhood \(U\) of \((0, 0)\) in \(\mathbb{C}^2\), and strictly plurisubharmonic in \(U\setminus\{(0,0)\}\),
- \(\{ \Phi = 0 \} = \{ \nabla \Phi = 0 \} = \{ u = v = 0 \} \cap U\).

### 3.1. The quadratic case

We first study quadratic complex points. Let \(S\) be written as in \((2.2)\) with vanishing \(\tau\). We are frequently going to use the following simple observation

**Lemma 3.2.** Let \(p(x, y, u) = b_2(x, y)u^2 + b_1(x, y)u + b_0\), where \(b_0, b_1, b_2\) are continuous functions, defined in a neighborhood of the origin. Assume \(b_1, b_0\) both vanish at \((x, y) = (0, 0)\) and \(b_2 > 0\). Then there exists a small neighborhood \(U\) of \((0, 0, 0)\) in \(\mathbb{R}^3\) with \(p\) strictly positive on \(U\setminus\{u = 0\}\), as long as \(b_1^2 < 4b_2b_0\) for small \((x, y) \neq (0, 0)\).

**Proof.** For fixed \(x, y\),
\[
-b_1 + \sqrt{b_1^2 - 4b_2b_0} \over 2b_2
\]
gives the zeros of the quadratic polynomial \(b_2(x, y)u^2 + b_1(x, y)u + b_0\). We notice that when \((x, y)\) approaches \((0, 0)\), both zeros converge to 0. The only way to achieve strict positivity of \(p\) in some \(U\setminus\{u = 0\}\), is for both zeros to be in \((\mathbb{C}\setminus\mathbb{R}) \cup \{0\}\) for small \((x, y)\). That is true if \(b_1^2 < 4b_2b_0\) for small nonzero \((x, y)\). \(\square\)
Lemma 3.3. For $\alpha < 0.52$, and any $M > 0$, there exists a homogeneous polynomial $P \in \mathbb{R}[x^2, y^2, u]$ of degree 6 and an open neighborhood $U \subset \mathbb{C}^2$ of $(0, 0)$, so that the function

$$\Phi(z, w) = P(x^2, y^2, u) + M(x^2 + y^2)u^6 + (1 + x^2 + y^2)v^2$$

has properties

(a) $\Phi$ is strictly plurisubharmonic for $(z, w) \in U \setminus \{(0, 0)\}$,
(b) $(u, v) \cdot \left(\frac{\partial \Phi}{\partial u}, \frac{\partial \Phi}{\partial v}\right) > 0$ for $(z, w) \in U \setminus \{u = v = 0\}$,
(c) $\Phi > 0$ for $(z, w) \in U \setminus \{u = v = 0\}$,
(d) $\{\Phi = 0\} = \{u = v = 0\} \cap U$.

Proof. We look at polynomials $\tilde{P}$ of the form

$$\tilde{P}(x^2, y^2, u) = u^6 + (6(c + a)x^2 - cy^2)u^5 + (Ax^4 + Bx^2y^2 + A'y^4)u^4 + (Cxy^2)u^3 + (Dx^6y^2 + Ex^4y^4)u^2,$$

where $c, A, A', B, C, D, E$ are strictly positive constants, to be determined later.

The computation of $\frac{\partial^2 \tilde{P}}{\partial u^2}$, using (2.4), gives us

(3.1)

$$12\frac{\partial^2 \tilde{P}}{\partial u^2} = [(30(\alpha + 1)^2 + 12\mathbf{A} + 2\mathbf{B} - 60\mathbf{A}(3\mathbf{A} + 2)\mathbf{a} - 10\mathbf{c}(3\mathbf{A} + 2))x^2$$

$$+ (120(\alpha - 1)^2 + 2[(12\mathbf{A}(\alpha + 1)^2 + 2\mathbf{D})x^6 + (12\mathbf{A}(\alpha - 1)^2 + 2\mathbf{D})x^6]y^2$$

$$= (6\mathbf{A}(\alpha - 1)^2 + 4\mathbf{D}(9\mathbf{A} + 4))x^6y^2$$

$$+ (6\mathbf{A}(\alpha - 1)^2 - 36\mathbf{E}x^4y^4)u + 2(Dx^6y^2 + Ex^4y^4)((\alpha + 1)^2x^2 + (\alpha - 1)^2y^2).$$

We want this expression to be nonnegative. To insure this, we set the coefficients so that polynomials in front of $u^4$, $u^3$ vanish, and polynomials in front of powers $u^4$, $u^2$, $u^0$ are positive for $(x, y)$ small and nonzero.

We first assume that $c = 0$, and so also $\mathbf{A}' = 0$. Setting polynomials in front of powers $u^4$, $u^3$ to zero, gives us the following conditions

1. $\mathbf{A} = 15\frac{\alpha(\alpha + 1)^2}{(\alpha + 1)^2} + \frac{c}{3(\alpha + 1)}$,
2. $\mathbf{B} = 3(\alpha - 1)^2 + \frac{2}{\alpha}$,
3. $\mathbf{D} = \frac{2\alpha(\alpha + 1)^2}{(\alpha + 1)^2}$,
4. $\mathbf{E} = \frac{1}{\alpha}(\alpha - 1)^2$.

Next we write inequalities, insuring the polynomials in front of $u^4, u^2, u^0$ are nonnegative. Satisfying the obvious requirement $\mathbf{C} > 0$, and using equalities $1 - 4$, we only need to satisfy the following inequalities

5. $15(\alpha + 1)^2 + 6\mathbf{A} + \mathbf{B} > 30\mathbf{A}(3\mathbf{A} + 2)$,
6. $2\mathbf{A}(\alpha - 1)^2 + 2\mathbf{B}(\alpha + 1)^2 + 5\mathbf{D} + 2\mathbf{E} > \mathbf{C}(7\mathbf{A} + 2)$.

We also want $\tilde{P} > 0$ and $\frac{\partial \tilde{P}}{\partial u} \neq 0$, when $u \neq 0$. Let us, still assuming $c = 0$, split $\tilde{P} = \tilde{P}_1 + \tilde{P}_2$ with

$$\tilde{P}_1 = u^6 + 6\mathbf{A}x^2u^5 + Ax^4u^4$$

$$\tilde{P}_2 = Bx^2y^2u^4 + Cx^4y^2u^3 + (Dx^6 + Ex^4y^4)u^2.$$
We require \( \hat{P}_1 \) and \( u \cdot (\frac{\partial \hat{P}_1}{\partial u}) \) to be strictly positive when \( u \neq 0 \), and \( \hat{P}_2 \) and \( u \cdot (\frac{\partial \hat{P}_2}{\partial u}) \) nonnegative when \( u \neq 0 \). Applying Lemma 3.2 we get two new inequalities

7. \( 75\alpha^2 < 8A \),
8. \( 9C^2 < 32\beta \).

We want to find positive constants \( A, B, C, D, E \), satisfying equalities and inequalities \( 1 - 8 \). Applying equalities \( 1 - 4 \), we get constants \( A, B, D, E \) expressed in terms of \( \alpha \) and \( C \). Inequalities \( 5 - 8 \) then become

5. \( 30\alpha(3\alpha + 2)(5\alpha^2 + 2\alpha - 2) < (5\alpha + 2)\mu \)
6. \( 180\alpha(9\alpha + 4)(5\alpha + 2)(\alpha - 1)^2(\alpha + 1) > (7\alpha + 2)(495\alpha^3 + 518\alpha^2 - 64\alpha - 112)\mu \)
7. \( 15\alpha(17\alpha^2 + 4\alpha - 8) < 2\mu \)
8. \( 80\alpha(\alpha - 1)^2(\alpha + 1)^2 > (37\alpha^2 + 4\alpha - 8)\mu \).

We can quickly see that, for \( \alpha < 1 \), inequality 7 follows from 5. Let us look at the polynomial

\[ q_6 = 495\alpha^3 + 518\alpha^2 - 64\alpha - 112 \]

from the right hand side of inequality 6. We have \( q_6(-1) < 0 \), \( q_6(-0.7) > 0 \) and \( q_6(0) < 0 \), and so the polynomial \( q_6 \) has three real zeros, out of which only one, \( \alpha_0 \), is in the interval \([0, 1] \). Since \( q_6(0.43) < 0 \) and \( q_6(0.44) > 0 \), this zero is in the interval \((0.43, 0.44) \). Applying the quadratic formula, we see that the polynomial

\[ q_8 = 37\alpha^2 + 4\alpha - 8 \]

from the right hand side of 8 has two real zeros, of which only one, \( 0.41 < \alpha_1 < 0.42 \), is positive. For \( \alpha < \alpha_1 \), both equations 6 and 8 are trivially satisfied with any positive \( C \), since the coefficients in front of \( C \) are non-positive. For \( \alpha \leq \alpha_0 \), we only need to take \( C \) small enough to satisfy 8, since 5 is trivially satisfied there for any positive \( C \). This is true because the left hand side of 5 is negative for \( \alpha < \alpha_0 \), since the positive zero \( \frac{1}{5\alpha + 2} \) of \( 5\alpha^2 + 2\alpha - 2 \) is greater than \( \alpha_0 \).

For \( 0.52 > \alpha > \alpha_0 \), 6 follows from 8. To see this, we need to check that

\[
\frac{180\alpha(9\alpha + 4)(5\alpha + 2)(\alpha - 1)^2(\alpha + 1)^2}{(7\alpha + 2)(495\alpha^3 + 518\alpha^2 - 64\alpha - 112)} > \frac{80\alpha(\alpha - 1)^2(\alpha + 1)^2}{(37\alpha^2 + 4\alpha - 8)}
\]

holds for \( \alpha_0 < \alpha < 0.52 \). This is equivalent to

\[
9(9\alpha + 4)(5\alpha + 2)(37\alpha^2 + 4\alpha + 8) - 4(7\alpha + 2)(495\alpha^2 + 4\alpha + 8) = 5(\alpha - 4)(225\alpha^3 + 62\alpha^2 - 64\alpha - 16) > 0.
\]

Let \( q = 225\alpha^3 + 62\alpha^2 - 64\alpha - 16 \). Then \( q' = 675\alpha^2 + 124\alpha - 64 \), and \( q' \) has only one positive zero, as we can see from by using the quadratic formula. Since \( q(0) < 0 \) and the leading coefficient of \( q \) is positive, this implies that \( q \) has only one positive zero. Since \( q(0.52) < 0 \), this zero is greater than 0.52. So \( 5(\alpha - 4)q > 0 \) for \( 0 < \alpha < 0.52 \), as long as \( c \) is small enough.

We now only need to worry about 5 and 8, for \( \alpha > \alpha_0 \). We can simultaneously solve these inequalities if and only if

\[
30\alpha(3\alpha + 2)(5\alpha^2 + 2\alpha - 2)(37\alpha^2 + 4\alpha - 8) - 80\alpha(\alpha - 1)^2(\alpha + 1)^2(5\alpha + 2) < 0.
\]

By estimating the zeros of this polynomial, we see that the inequality holds for \( \alpha_0 < \alpha < 0.52 \). This shows that equalities and inequalities \( 1 - 8 \) can all simultaneously be satisfied, as long as \( 0 \leq \alpha \leq 0.52 \).

By a continuity argument, we can see that condition \( c = 0 \) can be dropped, as long as we take \( c \) small enough and positive. This forces \( A' > 0 \). The reason is that
equalities and inequalities, insuring positivity of $\frac{\partial^2 P}{\partial z \partial \bar{z}}$, depend continuously on $c$. Finally, we want to make $P$ more generic in the coefficient in front of $u^2$, so that $\frac{\partial^2 P}{\partial z \partial \bar{z}}|_{u=0}$ is positive for every $(x, y) \neq (0, 0)$. We can simply achieve this, by taking $P$ to be a small perturbation of $\tilde{P}$, of the form $P = \tilde{P} + \epsilon(x^3 + y^3)u^2$. We should again understand $P$ as a polynomial in variables $x^2$, $y^2$ and $u$. As long as $\epsilon$ is small enough, we do not spoil the positivity of $\frac{\partial^2 P}{\partial z \partial \bar{z}}$.

Next we compute

$$\frac{\partial^2 (M(x^2 + y^2)u^6)}{\partial z \partial \bar{z}} = 4Mu^6 - p_1u^5 + p_2u^4.$$ 

Here $p_1, p_2 \in \mathbb{R}[x^2, y^2]$ are homogeneous polynomials of respectively degrees 1 and 2. We have $\frac{\partial^2 P}{\partial z \partial \bar{z}} = q_1u^4 + q_3u^2 + q_5$, where $q_1 \in \mathbb{R}[x^2, y^2]$ is a polynomial with strictly positive coefficients of degree 1. For any small $\epsilon$, the polynomial

$$4Mu^6 + p_1u^5 + (p_2 + \epsilon q_1)u^4$$

is strictly positive for $u \neq 0$ and $(x, y)$ small enough, and since

$$4 \frac{\partial((1 + x^2 + y^2)v^2)}{\partial z \partial \bar{z}} = 4v^2 \geq 0,$$

we have that $4 \frac{\partial^2 \tilde{P}}{\partial z \partial \bar{z}} > 0$, whenever $(z, w) \neq 0$.

From $u \frac{\partial P}{\partial u} > 0$ and $P > 0$ for $u \neq 0$ we also get (b), (c). This is true, since the term $M(x^2 + y^2)u^6$ is small with respect to $P$.

To conclude the proof, we check that $\Phi$ is indeed strictly plurisubharmonic. First, let us check that in a small neighborhood of the origin, $\rho = (\frac{1}{2} + x^2 + y^2)v^2$ is plurisubharmonic, and strictly plurisubharmonic if $v \neq 0$. Using (2.4), we have

$$\frac{\partial^2 \rho}{\partial z \partial \bar{z}} = 4v^2$$

$$\frac{\partial^2 \rho}{\partial z \partial \bar{w}} = 2(\frac{1}{2} + x^2 + y^2)$$

$$\frac{\partial^2 \rho}{\partial z \partial w} = 4yv + 4ixv.$$

Let us compute the determinant

$$\frac{\partial^2 \rho}{\partial z \partial \bar{z}} \frac{\partial^2 \rho}{\partial w \partial \bar{w}} - \left| \frac{\partial^2 \rho}{\partial z \partial \bar{w}} \right|^2 = \frac{1}{2}v^2(\frac{1}{2} + x^2 + y^2) - x^2v^2 - 16y^2v^2 = \frac{1}{2}(\frac{1}{2} - x^2 - y^2)v^2$$

For small $(x, y)$ this is positive, and strictly positive if $v \neq 0$.

We now check that $\tilde{\Phi}(z, w) = P(x^2, y^2, u) + M(x^2 + y^2)u^6 + \frac{1}{2}u^2$ is also plurisubharmonic. We have

$$4 \frac{\partial^2 \tilde{\Phi}}{\partial w \partial \bar{w}} = \triangle u, v \rho + 30M(x^2 + y^2)u^4 + 1 > \frac{1}{2},$$

as long as $(z, w)$ is small enough. So we have an estimate of the determinant

$$\frac{\partial^2 \tilde{\Phi}}{\partial z \partial \bar{z}} \frac{\partial^2 \tilde{\Phi}}{\partial w \partial \bar{w}} - \left| \frac{\partial^2 \tilde{\Phi}}{\partial z \partial \bar{w}} \right|^2 > \frac{1}{8} \frac{\partial^2 \tilde{\Phi}}{\partial z \partial \bar{w}}.$$

From computations above, we know that $\frac{\partial^2 \tilde{\Phi}}{\partial z \partial \bar{z}}$ is bounded from below by a strictly positive homogeneous polynomial

$$Q_5(x^2, y^2, u) = \sum a_{\kappa, \lambda, \mu}(x^2)^{\kappa}(y^2)^{\lambda}u^\mu$$
of degree 5. We have chosen $P$ nondegenerate, so that all $a_{\kappa, \lambda, \mu}$ with $\mu$ even are strictly positive. Let us also estimate the term $|4\frac{\partial^2 \Phi}{\partial z \partial w}|^2$, using (2.4).

\[
|4\frac{\partial^2 \Phi}{\partial z \partial w}|^2 = \left( \frac{\partial^2 \Phi}{\partial y^2} - (\alpha + 1)x\frac{\partial^2 \Phi}{\partial u^2} + 12Mxu^5 - 30M(\alpha + 1)x(x^2 + y^2)u^4 \right)^2 \\
+ \left( -\frac{\partial^2 \Phi}{\partial u^2} + (\alpha - 1)y\frac{\partial^2 \Phi}{\partial u^2} + 12Myu^5 - 30M(\alpha - 1)y(x^2 + y^2)u^4 \right)^2 < Q_9,
\]

where $Q_9(x^2, y^2, u)$ is some homogeneous polynomial of degree 9. Together, we have

\[
\left| \frac{\partial^2 \Phi}{\partial z \partial \bar{w}} - \frac{\partial^2 \Phi}{\partial z \partial w} \right|^2 \geq Q_5 - Q_9.
\]

Since $Q_5$ is nondegenerate enough, the polynomial $Q_5 - Q_9$ is positive, as long as $(x, y, u) \neq (0, 0, 0)$. Note that the sum of terms with odd powers of $u$ in $Q_9$ can be written as $uQ_8$, where $Q_8$ is homogeneous of degree 8, and are thus also dominated by $Q_5$. Since $\Phi = \tilde{\Phi} + \rho$, the function $\Phi$ is strictly plurisubharmonic away from the origin. This completes the proof. \qed

Using exactly the same method as above, we can also prove the following lemma.

**Lemma 3.4.** For $\alpha \leq 0.44$, and any $M > 0$, there exists a homogeneous polynomial $P \in \mathbb{R}[x^2, y^2, u]$ of degree 4 and an open neighborhood $U \subset \mathbb{C}^2$ of $(0, 0)$, so that the function

\[
\Phi(x, y, u) = M(x^2 + y^2)u^4 + P(x^2, y^2, u) + (1 + x^2 + y^2)v^2
\]

has properties

(a) $\frac{\partial^2 \Phi}{\partial z \partial \bar{z}} > 0$, $(x, y, u) \in U \setminus \{(0, 0)\}$

(b) $(u, v) \cdot \left( \frac{\partial \Phi}{\partial y}, \frac{\partial \Phi}{\partial u} \right) > 0$, $(z, w) \in U \setminus \{u = v = 0\}$,

(c) $\Phi > 0$, $(z, w) \in U \setminus \{u = v = 0\}$,

(d) $\{\Phi = 0\} = \{u = v = 0\} \cap U$.

This might be of some interest, since the lower the degree of the polynomial defining the neighborhoods, the lower the vanishing of Hessian at the origin, and thus one might expect the smaller the irregularity of the thickness of the neighborhoods.

**Remark 3.5.** One might try taking higher degrees of the polynomials in Lemma 3.3 to push the parameter space of $\alpha$ all the way to 1. This would quickly make the construction very messy. We will rather use a different construction.

We now modify the construction above, to prove an analog of Lemma 3.5 for $\alpha > \frac{1}{2}$. We want to use a construction, similar to the one in Lemma 3.3, but having fewer inequalities and equalities to satisfy. We will achieve that, by adding a function that has a sufficiently large Levi form, and vanishes on $u = 0$.

**Lemma 3.6.** Let $\frac{1}{2} < \alpha < 1$. Then there exists a positive integer $m$, such that for every $\delta > 0$ the function $\Phi_m(z, w) := y^{2m}u^2 + \delta v^2$ is plurisubharmonic in a small neighborhood $U$ of the origin, and strictly plurisubharmonic in $U \setminus \{y = 0\}$.

**Proof.** Let $\alpha < 1$ be fixed. We check the condition for the function $\phi_m = y^{2m}u^2$ to have the property $\frac{\partial^2 \phi_m}{\partial z \partial \bar{z}} > 0$ for small $(z, w)$ and $y \neq 0$.

\[
4\frac{\partial^2 \phi_m}{\partial z \partial \bar{z}} = 2m(2m - 1)y^{2m-2}u^2 - 4((2m + 1)\alpha - 2m)y^{2m}u + 2(\alpha - 1)^2y^{2m+2} + 2(\alpha + 1)^2y^{2m}x^2.
\]
We get the desired positivity if and only if
\[
2m(2m-1)y^{m-2}a^2 - 4(2m+1)\alpha - 2m)y^m u + 2(\alpha - 1)^2 y^{2m+2}
\]
is positive for \(y \neq 0\). Using Lemma 3.2, the condition is equivalent to
\[
((2m+1)\alpha - 2m)^2 < m(2m-1)(\alpha - 1)^2
\]
\[
\Longleftrightarrow \frac{2m - \sqrt{(2m-1)m}}{2m+1 - \sqrt{(2m-1)m}} < \alpha < \frac{2m + \sqrt{(2m-1)m}}{2m+1 + \sqrt{(2m-1)m}}.
\]
Let \(a_m := \frac{2m - \sqrt{(2m-1)m}}{2m+1 - \sqrt{(2m-1)m}}\) and \(b_m := \frac{2m + \sqrt{(2m-1)m}}{2m+1 + \sqrt{(2m-1)m}}\). Thus Lemma 3.2 holds for some \(m\) and \(a_0 < \alpha < \lim_{m \to \infty} b_m = 1\), if we show

1. \(a_m, b_m\) are both increasing sequences,
2. \(a_{m+1} < b_m\).

We first show that \(a_m\) is an increasing sequence.
\[
a_m < a_{m+1}
\]
\[
\Longleftrightarrow \left(2m + 2 - \sqrt{(2m+1)(m+1)}\right) \left(2m + 1 - \sqrt{(2m-1)m}\right) > \left(2m + 3 - \sqrt{(2m+1)(m+1)}\right) \left(2m - \sqrt{(2m-1)m}\right)
\]
\[
\Longleftrightarrow 2 > \sqrt{(2m+1)(m+1)} - \sqrt{(2m-1)m} \iff 16m^2 + 8m - 9 > 0.
\]

Showing \(b_m\) is increasing is exactly the same, but it also follows from
\[
a_{m+1} < b_m
\]
\[
\Longleftrightarrow \left(2m + \sqrt{(2m-1)m}\right) \left(2m + 3 - \sqrt{(2m+1)(m+1)}\right) > \left(2m + 1 + \sqrt{(2m-1)m}\right) \left(2m + 2 - \sqrt{(2m+1)(m+1)}\right)
\]
\[
\Longleftrightarrow 2 < \sqrt{(2m+1)(m+1)} + \sqrt{(2m-1)m}.
\]

This obviously holds for \(m \geq 1\). Since \(a_1 = \frac{1}{2}\), we have 3.3 for \(\frac{1}{2} < \alpha < 1\).

We now show that \(\Phi_m\) is plurisubharmonic in a small neighborhood of \((0,0)\), as long as \(m\) is chosen so that \(a_m < \alpha < b_m\). We already have shown that \(\frac{\partial^2 \Phi_m}{\partial z \partial \bar{z}}\) is nonnegative. For \(y\) small, we have
\[
\left|\frac{\partial^2 \Phi_m}{\partial z \partial \bar{w}}\right|^2 - \left|\frac{\partial^2 \Phi_m}{\partial z \partial \bar{w}}\right|^2.
\]

Similar than in Lemma 3.3, monomials in \(\left|\frac{\partial^2 \Phi_m}{\partial z \partial \bar{w}}\right|^2\) are dominated by monomials in \(\frac{\delta}{\partial z \partial \bar{w}}\). This completes the proof of this lemma. \(\square\)

We now combine our first approach with Lemma 3.6 above, to get desired functions for all quadratic hyperbolic points. Let us first prove this elementary lemma.

**Lemma 3.7.** The function
\[
q(x, y, u) = u^2 + a|x|^\gamma|y|^\delta u^l + b|x|^\gamma_1|y|^\delta_1
\]
is strictly positive for small \((x, y, u) \neq (0, 0, 0)\), as long as \(b > 0, \gamma_1 < \frac{2k}{2k-\gamma}, \delta_1 < \frac{2k}{2k-\gamma} \).

**Proof.** For fixed \((x, y)\), the minimum of our function is achieved at the critical point:
\[
\frac{\partial q}{\partial u} = u^{l-1}(2ku^{2k-1} + al|x|^\gamma|y|^\delta) = 0.
\]
If the minimum is obtained at \( u = 0 \), we are done. If not, then at the minimum

\[
u = o\left(|x|^\frac{2}{2m-1} |y|^\frac{2k}{2k-1}\right),
\]

and so

\[
\min_u q(x, y, u) = o\left(|x|^\frac{2k}{2k-1} |y|^\frac{2k}{2k-1}\right) + b|x|^\gamma |y|^\delta.
\]

The value is strictly positive for \( x, y \) nonzero and small. \( \square \)

**Lemma 3.8.** Let \( 1 > \alpha > \frac{1}{2} \). Then there exist positive integers \( n, m, k \), a homogeneous polynomial \( P \in \mathbb{R}[x^2, y^2, u] \) of degree \( 2n \), such that for every \( M > 0 \), the function

\[
\Phi(z, w) = P(x^2, y^2, u) + y^{2m}u^2 + x^{2k}u^{2} + M(x^2 + y^2)u^{2n} + (1 + x^2 + y^2)v^2
\]

has the following properties on a small neighborhood \( U \) of \( (0, 0) \in \mathbb{C}^2 \)

(a) \( \Phi \) is plurisubharmonic in \( U \) and strictly plurisubharmonic in \( U \setminus \{(0, 0)\} \),

(b) \( (u, v) \cdot \left(\frac{\partial P}{\partial z}, \frac{\partial P}{\partial w}\right) > 0 \), \( (z, w) \in U \setminus \{(u = v = 0)\} \),

(c) \( \Phi > 0 \), \( (z, w) \in U \setminus \{u = v = 0\} \),

(d) \( \{\Phi = 0\} = \{u = v = 0\} \cap U \).

**Proof.** The proof is very similar to the proof of Lemma 3.3. Let us choose \( \frac{1}{2} < \alpha < 1 \). We set \( m \) as in Lemma 3.3 so that \( \frac{\partial^2(P^{2m}u^2)}{\partial z^2} \) is strictly positive for \( y \neq 0 \). Let \( n \) be any positive integer with \( 2n > m + 2 \) and \( k \) any integer with \( k + 2 > 2n \). Let \( P \in \mathbb{R}[x^2, y^2, u] \) be a homogeneous polynomial of degree \( 2n \) of the form

\[
u^{2n} + (ax^2 + cy^2)u^{2n-1} + (A x^4 + B x^2 y^2 + Cy^4)u^{2n-2},
\]

where \( a, c, A, B, C \) are going to be appropriately chosen. Using Lemmas 3.2 and 3.7, we get (a) and (b) to hold for \( \Phi \) as long as

1. \( B > 0 \)
2. \( 8(2n - 2)A > (2n - 1)^2a^2 \).

Calculating \( \frac{\partial^2 P}{\partial z \partial u} \) we get

\[
\frac{\partial^2 P}{\partial z \partial u} = [2a + 2b - 4na] u^{2n-1} + \left[12A + 2B + 2n(2n - 1)(\alpha + 1) - 2(2n - 1)a(3\alpha + 2)\right] x^2 + \left[(2n - 1)(2n - 2)u(a + 1)^2 - 2(2n - 1)c(3\alpha - 2)\right] y^2 u^{2n-2} + \left[(2n - 1)(2n - 2)u(a + 1)^2 - 2(2n - 2)(5\alpha + 4)A\right] x^4 + \left[(2n - 1)(2n - 2)u(a + 1)^2 - 2(2n - 1)c(\alpha - 1)^2 - 2(2n - 2)B\right] x^2 y^2 + \left[(2n - 1)(2n - 2)c(\alpha - 1)^2 - 2(2n - 2)(5\alpha - 4)C\right] y^4 u^{2n-3} + [(Ax^4 + Bx^2 y^2 + Cy^4)((\alpha + 1)^2 + (\alpha - 1)^2)] u^{2n-4}
\]

We proceed by setting some of the terms equal to 0 and some of the terms to be positive, but we will see later, that we do not have to worry about what happens to terms expressed purely by \( y \) and \( u \). So we just want the following equalities and inequalities to hold

3. \( a + c = 2na \)
4. \( (2n - 1)a(\alpha + 1) = 2(5\alpha + 4)A \)
5. \( (2n - 1)a(\alpha - 1)^2 + (2n - 1)c(\alpha + 1)^2 = 10(2n - 2)B \)
6. \( 12A + 2B + 2n(2n - 1)(\alpha + 1)^2 > 2(2n - 1)a(3\alpha + 2) \)
As long as \( a \) is chosen positive and small enough, and we set \( c = 2n\alpha - a \), all equalities and inequalities 1. – 6. are satisfied. Choosing also \( C \) large enough, we have that
\[
\frac{\partial^2 P}{\partial z \partial \bar{z}} = (\bar{A}x^2 + C y^2)u^{2n-2} + \bar{D}y^4u^{2n-3} + (\bar{E}x^4 + \bar{F}x^2y^2 + \bar{G}y^4),
\]
where \( \bar{A}, \bar{B}, \bar{E}, \bar{F}, \bar{G} > 0 \). Putting it all together, we get
\[
\frac{\partial^2 \Phi}{\partial z \partial \bar{z}} - \frac{\partial^2 \Phi}{\partial w \partial \bar{w}} \geq 4Mu^{2n} - 4nM((3\alpha + 2)x^2 + (3\alpha - 2)x^2)u^{2n-1} + (\bar{A}x^2 + \bar{C}y^2)u^{2n-2}
\]
\[
+ [Mu^{2n} - 4nM((3\alpha + 2)x^2 + (3\alpha - 2)x^2)u^{2n-1} + (\bar{A}x^2 + \bar{C}y^2)u^{2n-2}]
\]
\[
+ [Mu^{2n} - 4((2k + 1)\alpha + 2k)x^{2k+2}u + 2(\alpha + 1)^2x^{2k+2}]
\]
\[
+ [Mu^{2n} + \bar{D}y^4u^{2n-3} + \bar{C}^2u^{2n-3}] + Mu^{2n}.
\]
Each of the summands in the last equality are positive, by using Lemma 3.2.

To prove that the determinant
\[
\frac{\partial^2 \Phi}{\partial z \partial \bar{z}} \frac{\partial^2 \Phi}{\partial w \partial \bar{w}} - \left| \frac{\partial^2 \Phi}{\partial z \partial \bar{w}} \right|^2
\]
is positive, we proceed exactly the same way as in Lemma 3.3 by comparing the orders of monomials. This proves the lemma. \( \square \)

Putting things together, we have proven the following proposition.

**Proposition 3.9.** Let \( S \) be a real surface in \( \mathbb{C}^2 \) given by the equation
\[
w = \frac{1}{2} \alpha z \bar{z} + \frac{1}{4} (\bar{z}^2 + \bar{z}^2), \quad 0 \leq \alpha < 1.
\]
Then there exists a \( C^\infty \) function \( \Phi \), defined in a small neighborhood \( U \) of \((0,0) \in \mathbb{C}^2 \) such that

(a) \( \Phi \) is plurisubharmonic in \( U \) and strictly plurisubharmonic in \( U \setminus\{(0,0)\} \),
(b) \( \Phi \geq 0 \),
(c) \( \{\Phi = 0\} = \{\nabla \Phi = 0\} = S \cap U \).

3.2. The flat case.

**Lemma 3.10.** Let \( \pi: S \to X \) be a real surface imbedded in a complex surface \( X \). Let \( p \) be a hyperbolic flat complex point on \( S \). Then there exists a neighborhood \( V \) of \( p \) in \( X \) and a smooth function \( \phi: V \to \mathbb{R} \) with properties

(a) \( S \cap V = \{\phi = 0\} = \{\nabla \phi = 0\} \),
(b) \( \phi \geq 0 \),
(c) \( \phi \) is strictly plurisubharmonic in \( V \setminus\{p\} \).

**Proof.** Let \( U \) be a neighborhood of \( p \) and \((z,w)\) be holomorphic coordinates on \( U \) in which \( S \) is written as in (2.2), where \( 0 \leq \alpha < 1 \) and \( \tau = o(|z|^3) \). We use the non-holomorphic coordinates (2.3) to compute the Levi form. Let first \( \alpha \leq \frac{1}{2} \), and let \( P \in \mathbb{R}[x^2,y^2,u] \) be the homogeneous polynomial, constructed in Lemma 3.3. We take
\[
\Phi(z,w) = P(x^2,y^2,u) + M(x^2+y^2)u^6 + v^2,
\]
where \( M \) is to be chosen later. Similar calculations as in Lemma 3.3 show that
\[
4\frac{\partial^2 \Phi}{\partial z \partial \bar{z}} = 4Mu^6 + q_1u^5 + q_2u^4 + q_3u^3 + q_6u^2 + q_9u + q_{10}
\]
where \( q_2, q_6, q_{10} \in \mathbb{R}[x, y] \) are positive polynomials respectively of degrees 2, 6, 10, vanishing only at the origin, and \( q_1, q_5, q_9 \in \mathbb{R}[x, y] \) are some polynomials respectively of degrees 1, 5, 9. These last terms are the error terms, coming from

\[
((o(2) \frac{\partial}{\partial x} + o(2) \frac{\partial}{\partial y} + o(1)) \frac{\partial}{\partial u} + o(3) \frac{\partial^2}{\partial u^2}) \Phi.
\]

Using Lemma 3.2, \( \frac{\partial^2 \Phi}{\partial z \partial \bar{z}} \) is positive away from the origin, as long as \( q_2^2 < 16Mq_2 \).

Choosing \( M \) large enough, this is satisfied. Since we have that \( 4 \frac{\partial^2 \Phi}{\partial w \partial \bar{w}} = 2 + o(1) \), the expression \( \frac{\partial^2 \Phi}{\partial z \partial \bar{z}} \frac{\partial^2 \Phi}{\partial w \partial \bar{w}} \) approximately equals \( \frac{1}{2} \frac{\partial^2 \Phi}{\partial z \partial \bar{z}} \) near the origin. Comparing degrees of monomials as we did in Lemma 3.3, we see that

\[
\left| \frac{\partial^2 \Phi}{\partial z \partial \bar{z}} \frac{\partial^2 \Phi}{\partial w \partial \bar{w}} - \left| \frac{\partial^2 \Phi}{\partial z \partial \bar{w}} \right|^2 \right| > 0
\]

away from the origin.

We are left with \( \alpha > \frac{1}{2} \). In this case, we use the function \( \Phi \) constructed in Lemma 3.8. The rest follows exactly the same way. \( \square \)

4. Stein neighborhoods of real surface

Using the results from the previous section, we are ready to prove Theorem 1.2. First we need this classical result.

**Proposition 4.1.** Let \( S^n \) be a \( C^\infty \) smooth totally real regularly embedded submanifold in a complex manifold \( X^n \). Let \( \langle \cdot , \cdot \rangle \) be any \( C^\infty \) smooth Riemannian metric on \( X \). The function \( \Phi : X \to \mathbb{R} \), defined as

\[
\Phi(z) = \text{dist}^2(z, S)
\]

is \( C^\infty \) smooth and strictly plurisubharmonic in an open neighborhood of \( S \) in \( M \).

**Proof.** Let \( p \in S \) be any point, and let \( Z \) be a smooth vector field, defined in a neighborhood of \( p \) in \( X \), with \( Z(p) \neq 0 \). We need to show \( dd^c \Phi \) is a positive form in a neighborhood of \( S \). Let \( J \) be the complex structure on \( X \).

\[
\begin{align*}
dd^c \Phi(p)(Z, JZ) &= (Zd^c \Phi(JZ))(p) - ((JZ)d^c \Phi(Z))(p) - (d^c \Phi([Z, JZ]))(p) \\
&= (Zd\Phi(Z))(p) + ((JZ)d\Phi(JZ))(p) + (d\Phi(J[Z, JZ]))(p) \\
&= < \text{Hess} \Phi(p) Z, Z >_p + < \text{Hess} \Phi(JZ), (JZ) >_p,
\end{align*}
\]

where \( \text{Hess} \Phi \) is the real Hessian form. Since \( S \) is maximally real, either \( Z \) or \( JZ \) must have a normal component. Since \( S \) is regularly embedded, \( \text{Hess} \Phi \) is a non-degenerate positive form on the normal bundle of \( S \) in \( X \). This completes the proof. \( \square \)

**Theorem 4.2.** Let \( S \hookrightarrow X \) be a compact real surface, \( C^\infty \) embedded into a complex surface \( X \) and having only flat hyperbolic complex points \( \{p_1, \ldots, p_k\} \). Then there exists a \( C^\infty \) function \( \psi \), defined in a neighborhood \( U \) of \( S \) in \( X \), such that

(a) \( S = \{\psi = 0\} = \{\nabla \psi = 0\} \),

(b) \( \psi \) is strictly plurisubharmonic on \( U \setminus \{p_1, \ldots, p_k\} \).

Sublevel sets \( \Omega_\epsilon = \{\psi < \epsilon\} \) define a regular, strictly pseudoconvex Stein neighborhood basis of \( S \) in \( M \).
Proof. Let $\mathcal{S} = S \setminus \{p_1, \ldots, p_k\}$. For every $1 \leq j \leq k$ let $\Phi_j : U_j \to \mathbb{R}$ be the plurisubharmonic function, constructed in the previous section and defined in a small neighborhood $U_j$ of $p_j$. All $U_j$ are assumed to be pairwise disjoint. If $U_j$ is taken small enough, $\Phi_j$ is strictly plurisubharmonic in $U_j \setminus \{p_j\}$. Let $< \cdot, \cdot >$ be any Riemannian metric on $X$ and let $\Phi_0(z) = \text{dist}^2(z, S)$. Then $\Phi$ is strictly plurisubharmonic in a neighborhood $U_0$ of $\mathcal{S}$ in $X$. Let $V = \bigcup_{0 \leq j \leq k} U_j$ and let $\pi : V \to S$ be the map defined as $\pi(z) = p$, $\text{dist}(z, p) = \text{dist}(z, S)$. Provided the neighborhoods are chosen small enough, the map $\pi$ is well defined and $C^\infty$. Let furthermore $\{\rho_j\}_{0 \leq j \leq k}$ be a $C^\infty$ partition of unity for $\{U_j \cap S\}_{0 \leq j \leq k}$. We define

$$\Phi(z) := \sum_{j=0}^k \rho_j(\pi(z))\Phi_j(z).$$

For every $p \in S$, we have

$$dd^c\Phi(p) = \sum_{0}^{k} \rho_j(p) dd^c\Phi_j(p).$$

This expression is strictly positive away from the points $\{p_1, \ldots, p_j\}$. We also have that $dd^c\Phi = dd^c\Phi_j$ near $p_j$. After possibly shrinking $V$, $\Phi$ is plurisubharmonic in a neighborhood $U$ of $S$ in $X$ and strictly plurisubharmonic in $U \setminus \{p_1, \ldots, p_k\}$. Since $\nabla \Phi = \nabla \Phi_j$ near $p_j$, we also have $\nabla \Phi$ nonvanishing near $S$.

What is left is to show that the neighborhoods are indeed Stein. By a result of Grauert [2], strictly pseudoconvex domains in a complex manifold are Stein, if and only if they contain no compact complex analytic sets of positive dimension. The restrictions of strictly plurisubharmonic functions to analytic sets are again strictly plurisubharmonic. Since compact analytic sets do not have any nonconstant plurisubharmonic functions and the defining function $\Phi$ is strictly pseudoconvex everywhere, but at finitely many points, there can not be any compact positive dimensional analytic sets in our neighborhood. This completes the proof. \hfill \Box

The above theorem, together with Lemma 3.1 proves Theorem 1.2.

It would be nice to know if such functions can be constructed without the assumption of flatness of hyperbolic points. For now, we satisfy ourself with the next result.

**Corollary 4.3.** Let $\pi : S \to X$ be any generically embedded real compact surface without elliptic points in a complex surface $X$. Then there exists an embedding $\pi' : S \to X, C^2$ close to $\pi$ and isotopic to $\pi$, such that $\pi'(S)$ has a regular basis of Stein neighborhoods.

**Proof.** In local coordinates $(z, w)$ near a hyperbolic complex point $p$, the surface $S$ can be written as

$$\text{Re } w = \frac{1}{2} \alpha z \bar{z} + \frac{1}{4} (z^2 + \bar{z}^2) + \tau_1(z), \quad \text{Im } w = \tau_2(z)$$

where $\tau_1, \tau_2 = o(|z|^3)$ and real. Let $\rho : S \to [0, 1]$ be a smooth function with $\rho \equiv 1$ near $p$ and $\rho \equiv 0$ outside a small neighborhood of $p$. Then $\pi_t := \pi(t) - it\rho \tau_2$ defines an isotopy of $\pi = \pi_0$ to $\pi' = \pi_1$ with a flat complex point at $p$. Repeating this process for every complex point, and using Theorem 4.2 we get the required result. \hfill \Box
5. Application to unions of totally real planes in \( \mathbb{C}^2 \)

Using results from the previous sections, we construct Stein neighborhood basis for certain unions of two totally real planes \( L_1, L_2 \subset \mathbb{C}^2 \), with \( L_1 \cap L_2 = \{0\} \). Every such union is linearly holomorphically equivalent to \( M(B) := \mathbb{R}^2 \cup A(B) \), where \( A(B) \) is the real span of the columns of the matrix \( B + iI \), with \( B \) real and \( (B - iI) \) invertible. Furthermore, \( B \) is determined only up to real conjugacy. For the proofs of these simple statements, see \([6]\).

The next two lemmas show the connection between certain unions of totally real planes and complex points on real surfaces. We only prove the first one, since it is the one we use later.

**Lemma 5.1.** Let \( B = \begin{pmatrix} \mu & 0 \\ 0 & -\mu \end{pmatrix} \), with \( \mu \geq 0 \). Let

\[
\Psi(z, w) = \left( \text{ie}^{-i\frac{\theta}{2}}z + \text{ie}^{i\frac{\theta}{2}}w, -\left(\sin^2 \theta\right)zw \right),
\]

where \( \theta = \arcsin \sqrt{\frac{1}{1+\mu^2}} \). Then

\[
\Psi(M(B)) = \{(z, w), w = \frac{1}{2}\alpha z \bar{z} + \frac{1}{4} z^2 + \frac{1}{4} \bar{z}^2, \}
\]

with \( \alpha = \cos \theta \).

**Proof.** The set \( \Psi(M(B)) \) can be parameterized by

\[
(\text{ie}^{-i\frac{\theta}{2}}(i + \mu)x + \text{ie}^{i\frac{\theta}{2}}(i - \mu)y, -\left(\sin^2 \theta\right)(i + \mu)(i - \mu)xy).
\]

Let us use this to compute \( \frac{1}{2}\alpha z \bar{z} + \frac{1}{4} z^2 + \frac{1}{4} \bar{z}^2 \) on \( \Psi(M(B)) \).

\[
\frac{1}{2}\alpha z \bar{z} + \frac{1}{4} z^2 + \frac{1}{4} \bar{z}^2 = \\
\frac{1}{2} \alpha [\text{ie}^{-i\frac{\theta}{2}}(i + \mu)x + \text{ie}^{i\frac{\theta}{2}}(i - \mu)y]^2 + \text{Re} \frac{1}{2} (\text{ie}^{-i\frac{\theta}{2}}(i + \mu)x + \text{ie}^{i\frac{\theta}{2}}(i - \mu)y)^2 = \\
\frac{1}{2} \alpha (\mu^2 + 1) - \frac{1}{2} (\mu + i)^2 e^{-i\theta} - \frac{1}{2} (\mu - i)^2 e^{i\theta} (x^2 + y^2) + \left(\frac{1}{2} \alpha (\mu^2 + 1) e^{-i\theta} - (\mu - i)^2 e^{i\theta} + (\mu^2 + 1) \right) xy = xy
\]

In the above, we used that \( \alpha = \frac{1}{\sqrt{\mu^2 + 1}} \). On the other hand, the second coordinate on \( \Psi(M(B)) \) equals

\[-(\sin^2 \theta)(i + \mu)(i - \mu)xy = -\frac{1}{1 + \mu^2}(-1 - \mu^2)xy = xy.
\]

So we have concluded that

\[
\frac{1}{2}\alpha z \bar{z} + \frac{1}{4} z^2 + \frac{1}{4} \bar{z}^2 = w
\]

on \( \Psi(M(B)) \). This concludes the proof.

**Lemma 5.2.** Let \( B = \begin{pmatrix} 0 & \mu \\ -\mu & 0 \end{pmatrix} \) with \( \mu > 1 \). Let

\[
\Psi(z, w) = \left( \text{ie}^{-i\frac{\theta}{2}}z + \text{ie}^{i\frac{\theta}{2}}w, \frac{1}{2} \left(\tan \theta \sin \theta\right)(z^2 + w^2) \right),
\]

where \( \theta = \arcsin \frac{1}{\mu} \). Then

\[
\Psi(M(B)) = \{(z, w), w = \frac{1}{2}\alpha z \bar{z} + \frac{1}{4} z^2 + \frac{1}{4} \bar{z}^2 \}
\]
with $\alpha = \frac{1}{\cos \theta}$.

Remark 5.3. The maps constructed above were, in a slightly different context, found by Burns (personal communication). Lemma 5.2 can be used to pull-back Bishop discs, [1], from a neighborhood of an elliptic complex point. This gives us analytic annuli with boundaries in $M(B)$ with trace $B = 0$ and det $B > 1$, shrinking towards the origin. Weinstock [6] showed that this is the only non-polynomially convex case among the unions $M(B)$. One would thus expect to be able to find a regular Stein neighborhood basis for all other unions $M(B)$. Unfortunately, we are at the point only able to show this for a smaller class of unions of totally real planes.

Proposition 5.4. Let $B$ be a real $2 \times 2$ matrix, diagonalizable over $\mathbb{R}$ and with the property that trace $B = 0$. Then the union $M(B)$ has a regular Stein neighborhood basis.

Proof. We can assume $B = \begin{pmatrix} \mu & 0 \\ 0 & -\mu \end{pmatrix}$. By Lemma 5.1, the map

$$
\Psi(z, w) = \left( \begin{array}{c} ie^{-i\frac{\theta}{2}}z + ie^{i\frac{\theta}{2}}w, -\sin^2 \theta zw \end{array} \right)
$$

with $\theta = \arcsin \sqrt{1 + \mu^2}$ maps the union $M(B)$ of totally real planes onto the surface

$$
S_\alpha = \{(z, w), w = \frac{1}{2} \alpha z \bar{z} + \frac{1}{4} z^2 + \frac{1}{4} \bar{z}^2 \}.
$$

Let $\Phi: U \to \mathbb{R}$ be the map, constructed in Lemma 5.3 or Lemma 3.8, depending on the size of $\alpha = \cos \theta$. Let $\tilde{\Phi} = \Phi \circ \Psi$. The small sublevel sets of $\tilde{\Phi}$ are pseudoconvex. We only need to check that $\nabla \tilde{\Phi}(z, w) \neq 0$, for $(z, w) \neq M(B)$. Since $\nabla \Phi$ is nonzero away from the surface $S_\alpha$, this happen if $\nabla \Phi \notin \ker(D\Psi)^T$. We have

$$
D\Psi = \begin{pmatrix} ie^{-i\frac{\theta}{2}} & ie^{i\frac{\theta}{2}} \\ -(\sin^2 \theta)w & -(\sin^2 \theta)z \end{pmatrix}.
$$

$D\Psi$ is nondegenerate outside of $z = e^{i\theta}w$ and ker $D\Psi(e^{i\theta}w, w) = \mathbb{C}\{(e^{i\frac{\theta}{2}}, -e^{-i\frac{\theta}{2}})\}$. Let us assume that, at some point $(z_0, w_0) \neq (0, 0)$ in the image of the critical set, $\nabla \Phi \in \ker(D\Psi)^T$. We know from constructions of $\Phi$, that

$$
\Phi + N|z|^2(\text{Rew} - \frac{1}{2} \alpha z \bar{z} - \frac{1}{4} z^2 - \frac{1}{4} \bar{z}^2)^{2n}
$$

is also plurisubharmonic. Here $2n$ is the degree of the homogeneous polynomial in the definition of $\Phi$. So by possibly substituting $\Phi$ by

$$
\Phi + N|z|^2(\text{Rew} - \frac{1}{2} \alpha z \bar{z} - \frac{1}{4} z^2 - \frac{1}{4} \bar{z}^2)^{2n}
$$

for an appropriate $N$, ensures us to have no critical points of $\tilde{\Phi}$ away from $M(B)$. This completes the proof. \qed

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