VERTEX OPERATORS FOR THE BF SYSTEM
AND ITS SPIN-STATISTICS THEOREMS

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Abstract

Let $B$ and $F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu$ be two forms, $F_{\mu\nu}$ being the field strength of an abelian connection $A$. The topological $BF$ system is given by the integral of $B \wedge F$. With "kinetic energy" terms added for $B$ and $A$, it generates a mass for $A$ thereby suggesting an alternative to the Higgs mechanism, and also gives the London equations. The $BF$ action, being the large length and time scale limit of this augmented action, is thus of physical interest. In earlier work, it has been studied on spatial manifold $\Sigma$ with boundaries $\partial \Sigma$, and the existence of edge states localised at $\partial \Sigma$ has been established. They are analogous to the conformal family of edge states to be found in a Chern-Simons theory in a disc. Here we introduce charges and vortices (thin flux tubes) as sources in the $BF$ system and show that they acquire an infinite number of spin excitations due to renormalization, just as a charge coupled to a Chern-Simons potential acquires a conformal family of spin excitations. For a vortex, these spins are transverse and attached to each of its points, so that it resembles a ribbon. Vertex operators for the creation of these sources are constructed and interpreted in terms of a Wilson integral involving $A$ and a similar integral involving $B$. The standard spin-statistics theorem is proved for this sources. A new spin-statistics theorem, showing the equality of the “interchange” of two identical vortex loops and $2\pi$ rotation of the transverse spins of a constituent vortex, is established. Aharonov-Bohm interactions of charges and vortices are studied. The existence of topologically nontrivial vortex spins is pointed out and their vertex operators are also discussed.
1. INTRODUCTION

When a system coupled to an abelian gauge field $A = A_\mu dx^\mu$ has its U(1) symmetry spontaneously destroyed, its phenomenology at large length and time scales is well described by London’s constitutive equation

$$dJ = \lambda dA,$$

$$\lambda = \text{constant}, \quad J := J_\mu dx^\mu$$ (1.1)

involving the current $J_\mu$. The conventional Lagrangian description of (1.1) is based on the Nambu-Goldstone field $e^{i\phi}$ of unit modulus and charge $q$. In 3+1 dimensions, it reads

$$L_{NG} = \int d^3x L_{NG},$$

$$L_{NG} = -<H>^2 \left(D_\mu e^{i\phi}\right)^* \left(D^\mu e^{i\phi}\right) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu},$$

$$<H>^2 = \frac{\lambda}{8q^2}, \quad D_\mu e^{i\phi} = (\partial_\mu - 2iqA_\mu)e^{i\phi},$$ (1.2)

the metric being $(-1, 1, 1, 1)$ diagonal.

It has been known for some time that there is an alternative Lagrangian approach to London’s equation employing the two-form field $B$ instead of $e^{i\phi}$. It is best explained by first remarking that in the Nambu-Goldstone approach, which uses (1.1), the current

$$J_\mu = 4iq <H>^2 e^{i\phi} D_\mu e^{i\phi}$$ (1.3)

solves the constitutive equation (1.1) as an identity, whereas the continuity equation

$$\partial_\mu J_\mu = 0$$ (1.4)

is obtained as a field equation. In contrast, in the alternative approach, we solve the continuity equation (1.4) as an identity by setting

$$J_\mu = -\frac{1}{2} \epsilon^{\mu\nu\lambda\rho} \partial_\nu B_{\lambda\rho},$$ (1.5)
the Levi-Civita symbol being fixed by the convention $\epsilon^{0123} = +1$. The constitutive equation is then obtained as a field equation from the Lagrangian

$$L_0 = \int d^3x \mathcal{L}_0,$$

$$\mathcal{L}_0 = \frac{1}{4} \epsilon^{\mu\nu\lambda\rho} B_{\mu\nu} F_{\lambda\rho} - \frac{1}{12\chi} H^{\mu\nu\rho} H_{\mu\nu\rho} - \frac{1}{4} F^{\mu\nu} F_{\mu\nu},$$

$$H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}. \quad (1.6)$$

There has been a certain past interest in (1.6) when the spatial manifold has no boundary [1, 2, 3, 4]. In previous work [5, 6], we also initiated its study for manifolds $\Sigma$ with boundary $\partial\Sigma$ and established the existence of edge states localized at $\partial\Sigma$. They are mathematically analogous to the quantum Hall edge states [7, 8, 9]. We also argued [5, 6] that they are the modes of either of the Lagrangians

$$L_{\phi,\psi} = \int_{\partial\Sigma} \dot{\phi} d\psi, \quad (1.7)$$

$$L_{\phi} = \frac{1}{2} \int_{\partial\Sigma} \mu (\dot{\phi}^2 - \omega_0 \phi^2) \quad (1.8)$$

on $\partial\Sigma$, $\psi$ being a one form and $\mu$ a volume form on $\partial\Sigma$, and clarified the relation of (1.7) to a coadjoint orbit of a certain group and its symplectic form [8]. $L_{\phi,\psi}$ is invariant under all the diffeomorphisms (diffeos) of $\Sigma$ whereas $L_{\phi}$ is invariant under its subgroup $SDiff(\partial\Sigma)$ of diffeos preserving the form $\mu$. The Lie algebra of this latter group is related to the algebra $u_{1+\infty}$ [10] if $\Sigma$ is the solid cylinder and $\mu$ is $d\theta dx$ for the choice of coordinates $(e^{i\theta}e^1, xe^R)$ on $\partial\Sigma$. [The wedge symbols between differential forms will be omitted.] A generalized Sugawara construction [11] was described. It was also pointed out that the edge states were insensitive to excitations in its interior $\Sigma^0$ of $\Sigma$ and that their effective 3 + 1 dimensional Lagrangian associated with (1.6) is the $BF$ Lagrangian

$$L_{0}^* = \int d^3x \mathcal{L}_0^*,$$

$$\mathcal{L}_0^* = \frac{1}{4} \epsilon^{\mu\nu\lambda\rho} B_{\mu\nu} F_{\lambda\rho}. \quad (1.9)$$
when the energy density in $\Sigma^0$ is zero.

In this paper, we explore the $BF$ system with sources calling upon the experience from our earlier work \[8, 9\] on the Chern-Simons Lagrangian. Two natural sources for the $BF$ system are magnetic vortices and point charges, the interaction Lagrangian being

\[ L_I = \int d^3 x \mathcal{L}_I \]

\[ \mathcal{L}_I = -\frac{\lambda}{2} \int d\sigma^1 \delta^3 (x - y) \epsilon^{ab} B_{\mu\nu}(y) \partial_a y^\mu \partial_b y^\nu - e \delta^3 (x - y) A_i(z) \partial_0 z^i, \quad \partial_a \equiv \frac{\partial}{\partial \sigma^a}. \quad (1.10) \]

Here, $\lambda$ and $q$ are constants, $\epsilon^{ab} = -\epsilon^{ba}$ with $\epsilon^{01} = +1$, $i$ is a spatial index taking values 1,2,3,

\[ y : \mathbb{R}^1 \times S^1 \rightarrow \mathbb{R}^1 \times \Sigma, \]

\[ (\sigma^0, \sigma^1) \rightarrow y(\sigma^0, \sigma^1) \equiv y(\sigma) \quad (1.11) \]

is the vortex and

\[ z : \mathbb{R}^1 \rightarrow \mathbb{R}^1 \times \Sigma, \]

\[ \sigma^0 \rightarrow z(\sigma^0) \quad (1.12) \]

is the charge in spacetime. \([The \mathbb{R}^1 \text{ factors account for time, and } \sigma^1 \text{ and } \sigma^1 + 2\pi \text{ are to be identified, in these expressions. We also assume that } y^0(\sigma), z^0(\sigma^0) \text{ and } \sigma^0 \text{ are all equal to coordinate time } x^0.\] We can also contemplate charged magnetic vortices \[3\] and these too will be encountered in the course of our discussion. We will furthermore assume that the vortices are unlinked unknots.

In Section 2, we argue that the Lagrangian

\[ L^* = L_0^* + L_I \]

requires regularization already at the classical level (that being also the case with Chern-Simons sources \[8, 9\]). The regularization consists of first enclosing the location of the
vortex or the charge in the interior of a solid torus or a ball \( H \) and letting \( H \) shrink to these locations when all calculations are done. [The symbol \( H \) here is to be thought of as the first letter of the word “hole”.] The presence of this \( H \) means that \( L^* \) is not defined on \( \Sigma \), but rather on \( \Sigma \setminus H \), which is \( \Sigma \) with a hole. In this way, it gets associated with a new spatial slice \( \Sigma \setminus H \) with a new boundary \( \partial H \) which is a torus \( T^2 \) or a sphere \( S^2 \). This boundary too now acquires edge excitations exactly as \( \partial \Sigma \) does so that the available independent internal states for a source with a given geographical location are infinite in number. They are similar to the conformal family of internal states of a Chern-Simons source. They are spin excitations of the sources and of an uncommon transverse sort for vortices as will be seen in this Section and further elucidated in Section 3.

In Section 3, we discuss the observables of \( L^* \) in detail. There are first the class of observables localized at \( \partial H \) whose study was already initiated in the last Section. They are similar to the observables localized at \( \partial \Sigma \) investigated before \[5\]. In addition, there also certain new observables. They describe the charges of the sources and at \( \partial \Sigma \) (the total charge adding up to zero), the magnetic fluxes on the vortices, and their conjugate variables. The latter incidentally had turned up at \( \partial \Sigma \) before \[5\] when \( \Sigma \) was the solid torus \( T_3 \). The mode decomposition of these observables is also carried out in this Section.

Section 4 turns to the quantization of the preceding system and introduces vertex operators. The conjugate variables alluded to above lead to vertex operators for \( L^* \) just as the variable conjugate to charge enables us to construct vertex operators in Chern-Simons dynamics. They are the creation operators of charges and vortices. The work also shows the interpretation of the Wilson integral as a vertex operator for the creation of a charge, and an analogous expression involving \( B \) as the creation operator of a vortex, after suitable regularization. This interpretation is the generalization of a similar interpretation \[9\] of the Wilson integral in Chern-Simons theory.

In Section 5, we prove the spin-statistics theorem. The conventional spin-statistics
theorem is quickly shown. We define an operator for an “interchange” \[13\] of two vortices which is a combination of an exchange and a “slide” \[3, 14, 15\]. A new spin-statistics theorem is then established, showing the homotopy equivalence of interchange and the \(2\pi\) rotation of transverse spin. Its proof is similar to the proof of the spin-statistics theorem for Chern-Simons sources established in \[9\]. The corresponding quantum operators are then equal for quantizations using covering spaces \[16\]. The Section concludes with a discussion of phase changes of states when charges or charged vortices are transported in loops enclosing fluxes of charged vortices, or equivalently, strands of vortices. These phase changes may be thought of as describing “Aharonov-Bohm” interactions of charges and vortices or of charged vortices. Quantization conditions involving charges and fluxes which make this interaction vanish are derived. The spin-statistics theorems are also associated with transports of states, and for this reason, the insertion of these remarks on the Aharonov-Bohm interaction of charges and vortices in this Section seems appropriate.

In this paper, until this point, we largely limit our work to the Lagrangian \(L^*\) which omits the “kinetic energy” terms \(L_0 - L_0^*\) for the fields [which are proportional to integrals of \(\partial_\mu B_{\lambda\rho}\partial^{[\mu} B^{\lambda\rho]}\) and \(\partial_\mu A_{\nu}\partial^{[\mu} A^{\nu]}\). In ref.\[5\], in contrast, we included these terms from the start. This difference between the papers is not accidental. The inclusion of \(L_0 - L_0^*\) does not generate striking differences in the nature of gauge transformations or the structure of constraints, and these will in fact be our central pursuits until Section 5. Edge states of sources and their vertex operators can in consequence be formally treated just as previously. The trouble lies elsewhere: Sources have divergent self energies like in electrodynamics, and they require renormalisation, as we will indicate in Section 6.

Incidentally, by working along parallel lines, it is straightforward also to generalize the Chern-Simons work of \[8\] and \[4\] to the Lagrangian.

\[
L_{CS} = \int d^2 x L_{CS},
\]
and its nonabelian version. (Here the metric has the signature \((-, +, +)\) and \(\epsilon^{\mu\nu\lambda}\) is the Levi-Civita symbol with \(\epsilon^{012} = 1\).) This extension has been carried out in unpublished work \([17]\) elsewhere and is affected by self-energy divergences just as in the work here.

We have not included kinetic energy terms for sources in this paper as they do not affect our considerations.

Section 7 is the final one. It discusses certain topologically nontrivial configurations of transverse spins (“twisted” transverse spins) on vortices and their associated vertex operators.

As the final remark of this Section, we note that the conventions regarding certain factors adopted in this paper differ from \([5]\) and that some minor algebraic errors of \([5]\) have also been corrected here.

### 2. How the Source Acquires Spin

The source acquires spin (“transverse” for the vortex) because of renormalization just like to Chern-Simons source \([9]\). We will now show how this happens in detail. Before we do so, let us record the nonzero Poisson brackets involving \(A\) and \(B\) following from \((1.9)\):

\[
\{A_i(x), B_{jk}(y)\} = \epsilon_{ijk} \delta^{3}(x - y).
\]  

(2.1)

Here, and in what follows, \(\epsilon_{ijk}\) is the Levi-Civita symbol with \(\epsilon_{123} = 1\) while PB’s, and in fact all considerations, are at equal times.

#### 2.1 The Isolated Charge

The Lagrangian \(\mathcal{L}_0^\ast\) has the following two equations analogous to Gauss’s law in electrodynamics:

\[
\epsilon^{ijk} \partial_i B_{jk} = 0,
\]

(2.2)
\[ \epsilon^{ijk} F_{jk} = 0 . \] (2.3)

They are obtained by varying \( A_0 \) and \( B_{i0} \) respectively.

In the presence of an isolated point charge at \( z = (\vec{z}, z^0) \) [\( z^0 \) being time \( x^0 \)], the law (2.2) gets changed to

\[ \frac{1}{2} \epsilon^{ijk} \partial_i B_{jk}(x) = e \delta^3(x - z) \] (2.4)

as shown by (1.13). [All equations are at the equal time \( x^0 \) in accordance with a previous remark. We suppress the argument \( \sigma^0 \) in \( \vec{z}(\sigma^0) \) hereafter.] The law (2.3) is unaffected, if as we for the moment suppose, there are no vortices present.

Let \( \mathcal{B}_3 \) be a ball or a hole in \( \Sigma \) at time \( t \) enclosing \( \vec{z} \), with boundary \( \partial \mathcal{B}_3 \). [Earlier, we called \( \mathcal{B}_3 \) and \( \partial \mathcal{B}_3 \) as \( H \) and \( \partial H \) when dealing generically with a charge or a vortex.] According to (2.4), it is then the case that

\[ \int_{\partial \mathcal{B}_3} B = e , \quad B = \frac{B_{jk} dx^j dx^k}{2} \] (2.5)

where \( \partial \mathcal{B}_3 \) is positively oriented. [The orientation of \( \partial \mathcal{B}_3 \) is inherited from the orientation of the ambient three manifold which is a priori chosen. The wedge symbols between differential forms are being omitted in this paper.] On shrinking \( \mathcal{B}_3 \) to the point \( \vec{z} \), (2.5) shows that \( B(x) \) has no definite limit as \( \vec{x} \to \vec{z} \). We must thus regularize.

A good way to regularize is to keep \( \mathcal{B}_3 \) of finite but small size till all calculations are done, \( \vec{z} \) being in its interior. \( \mathcal{B}_3 \) is shrunk to \( \vec{z} \) only at the end of the calculations. [Such a limiting procedure on holes is hereafter to be understood whenever required.] When \( \mathcal{B}_3 \) is of nonzero size, the spatial manifold is not \( \Sigma \), but \( \Sigma \setminus \mathcal{B}_3 \) which is \( \Sigma \) with a hole. There is then a new boundary \( \partial \mathcal{B}_3 \). According to [3] (see also [3, 4]), the Gauss laws are then

\[ G_0(\lambda^{(0)}) = \int_{\Sigma \setminus \mathcal{B}_3} \lambda^{(0)} dB \approx 0 , \] (2.6)

\[ G_1(\lambda^{(1)}) = 2 \int_{\Sigma \setminus \mathcal{B}_3} \lambda^{(1)} dA \approx 0 , \quad \lambda^{(1)} = \lambda^{(1)}_j dx^j , \quad A = A_j dx^j \] (2.7)
where the test functions $\lambda^{(0)}$ and the test 1-forms $\lambda^{(1)}$ vanish on boundaries $\partial \Sigma$ and $\partial \mathcal{B}_3$. We have also introduced the symbol $\approx$ to denote Dirac’s weak equality \cite{18}. If additional boundaries appear in the problem, as they will later, $\lambda^{(j)}$ must vanish there too.

The canonical transformation generated by $\mathcal{G}_i(\lambda^{(i)})$ are given by the PB’s

\[
\{\mathcal{G}_0(\lambda^{(0)}), A_i(x)\} = \partial_i \lambda^{(0)} ; \{\mathcal{G}_0(\lambda^{(0)}), B_{jk}(x)\} = 0 ,
\]  
\[
\{\mathcal{G}_1(\lambda^{(1)}), A_i(x)\} = 0 , \{\mathcal{G}_1(\lambda^{(1)}), B_{jk}(x)\} = 2(\partial_j \lambda^{(1)}_k - \partial_k \lambda^{(1)}_j)(x) .
\]  

Again, according to \cite{5, 8, 9}, the observables localised at $\partial \mathcal{B}_3$ are

\[
q(d\xi^{(0)}) = \int_{\Sigma \setminus \mathcal{B}_3} d\xi^{(0)} B ,
\]
\[
p(d\xi^{(1)}) = -\int_{\Sigma \setminus \mathcal{B}_3} d\xi^{(1)} A ,
\]

where $\xi^{(j)}$ are $j$ forms vanishing on $\partial \Sigma :

\[
\xi^{(j)}|_{\partial \Sigma} = 0 , \quad \xi^{(j)}|_{\partial \Sigma} \equiv \text{Pull back of} \ \xi^{(j)} \ \text{to} \ \partial \Sigma .
\]  

[There are also nonlocal observables to be considered later. When there are more boundaries than $\partial \Sigma$ and $\partial \mathcal{B}_3$, $\xi^{(j)}$ vanish on all boundaries except $\partial \mathcal{B}_3$.] They are observables because their Poisson brackets (PB’s) with $\mathcal{G}_j$ are weakly zero. Also all observables with the same boundary values for $\xi^{(j)}$ are weakly equal and can be identified:

\[
q(d\xi^{(0)}) \approx q(d\tilde{\xi}^{(0)}) \quad \text{if} \quad (\xi^{(0)} - \tilde{\xi}^{(0)})|_{\partial \mathcal{B}_3} \text{ and } \partial \Sigma = 0 ,
\]
\[
p(d\xi^{(1)}) \approx p(d\tilde{\xi}^{(1)}) \quad \text{if} \quad (\xi^{(1)} - \tilde{\xi}^{(1)})|_{\partial \mathcal{B}_3} \text{ and } \partial \Sigma = 0 .
\]  

This is because their difference becomes a Gauss law on partial integration.

The new degrees of freedom localized at $\partial \mathcal{B}_3$ are $q$ and $p$. It is easy to see that they can be regarded as spin excitations. Let $R$ be a diffeomorphism (diffeo) of $\Sigma \setminus \mathcal{B}_3$ which acts as a rotation of $\partial \mathcal{B}_3$ [for the choice of a round (rotationally invariant) metric on $\partial \mathcal{B}_3 = S^2$]
and becomes the identity on $\partial \Sigma$. Let $B \to R^*B$ and $A \to R^*A$ be its standard actions on the forms $B$ and $A$ by pull back. The response of the $q, p$ observables to $R$ is, then,

\begin{align*}
R : q(d\xi^0) &\to (Rq)(d\xi^0) = \int_{\Sigma \setminus \mathcal{B}_3} d\xi^0 R^*B = q(R^{-1}*d\xi^0), \\
R : p(d\xi^1) &\to (Rp)(d\xi^1) = \int_{\Sigma \setminus \mathcal{B}_3} d\xi^1 R^*A = p(R^{-1}*d\xi^1).
\end{align*}

(2.14)

Hence $q$ and $p$ change under rotations and carry spin localized at $\partial \mathcal{B}_3$. They describe not just spin of course, as they can be transformed in a similar way by any diffeo.

There is another way to view these observables. We can argue that renormalization has associated a direction, or equally well a point on $S^2 = \partial \mathcal{B}_3$ to the charge, $\xi^{(j)}$ being fields on these directions or on this $S^2$. The particle has thus got “framed,” or more precisely acquired a direction, as a “spin” degree of freedom. The spin variables of our source are not quite this however, being fields $\xi^{(j)}$ on these directions. The quantum source associated with $L^*$ is thus described by a first quantized position and a second quantized direction.

### 2.2 The Isolated Vortex

Next we suppose that there is an isolated vortex or a closed string $y$. [Here too, of course, all considerations are carried out at some fixed time $y^0(\sigma) = x^0$. We will suppress the argument $\sigma$ in $\vec{y}(\sigma)$ hereafter.]

In the presence of the vortex, the law (2.2) is not affected, whereas (2.3) is changed to

\begin{equation}
F_{ij} = \lambda \epsilon_{ijk} \int d\sigma^1 \partial_1 y^k \delta^3(x - y).
\end{equation}

(2.15)

Let $C$ be a closed positively oriented contour around the vortex as in Fig.1. From (2.15), we see that

\begin{equation}
\int_C A = \lambda
\end{equation}

(2.16)

It is thus the case that the vortex is a magnetic line with flux $= \lambda$. 

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The orientation of $C$ in (2.16) is obtained from that of the surface $S$ cutting the vortex it encloses. [It cuts it just once as in Figure 1.] As for the orientation of the latter, we can fix it as follows. We first choose an orientation $\epsilon$ in the ambient three manifold. Now the orientation $\epsilon_S$ of $S$ along with the direction of $\partial_1 \vec{y}$ where the vortex intersects $S$ also defines an orientation of the three manifold. $\epsilon_S$ is then determined by requiring that the latter is the same as $\epsilon$.

In (2.16), by shrinking $C$ towards the vortex, we learn that $A(x)$ has no well defined limit as $x$ approaches the vortex $y$. We must thus again regularize.

Regularization can be accomplished much as before by enclosing the vortex $\vec{y}$ in a solid torus $T_3$ as in Figure 2. $T_3$ is of tiny cross section, and is collapsed to the vortex after all calculations are over. [Just as for the hole $B_3$ for a charge, this limiting procedure on the hole $T_3$ for a vortex is hereafter to be understood whenever required.]

The Lagrangian $L^*$ is now defined on $\Sigma \setminus T_3$ with a new boundary $\partial T_3 = \text{the two-torus } T^2$. Therefore, the Gauss laws become \[ G_0(\lambda^{(0)}) = \int_{\Sigma \setminus T_3} \lambda^{(0)} dB \approx 0, \quad (2.17) \]

\[ G_1(\lambda^{(1)}) = 2 \int_{\Sigma \setminus T_3} \lambda^{(1)} dA \approx 0, \quad \lambda^{(1)} = \lambda_j^{(1)} dx^j \quad (2.18) \]

where

\[ \lambda^{(j)}|_{\partial \Sigma} = \lambda^{(j)}|_{\partial T_3} = 0. \quad (2.19) \]

Here as before, $\lambda^{(j)}|_{\partial T_3}$ for example stands for the pull back of $\lambda^{(j)}$ to $\partial T_3$. If additional boundaries are inserted, $\lambda^{(j)}$ must vanish there too.

The construction of the observables also follows \[ \] and the treatment of the point charge above. Thus the observables localized at $\partial T_3$ are essentially all given by

\[ q(d\bar{\xi}^{(0)}) = \int_{\Sigma \setminus T_3} d\bar{\xi}^{(0)} B, \quad (2.20) \]

\[ p(d\bar{\xi}^{(1)}) = -\int_{\Sigma \setminus T_3} d\bar{\xi}^{(1)} A \quad (2.21) \]
where the \( j \) forms \( \xi^{(j)} \) vanish on \( \partial \Sigma \),

\[
\xi^{(j)}|_{\partial \Sigma} = 0 \quad (2.22)
\]

and any other boundary (if such exist) except \( \partial T_3 \). Furthermore, as in (2.13), if two \( \xi^{(j)} \) agree on \( \partial T_3 \), the corresponding observables are (weakly) the same.

Let us choose coordinates \( \theta^i \) mod \( 2\pi \) for the two-torus \( T^2 = \partial T_3 \) with \( \theta^2 \) becoming the angular coordinate \( \sigma^1 \) on the vortex when \( T_3 \) shrinks to the vortex [see Fig. 2]. Then these observables respond to rotation of \( \theta^i \) just as in (2.14). Of these, the rotation of \( \theta^1 \) taking a point around a circle such as \( C \) of Figure 1, is of greatest interest. Such a rotation has naturally occurred in an earlier work [19] and is associated with a spin-statistics theorem to be proved later. We can thus say that (2.20),(2.21) describe excitations of this transverse spin.

For specific topologies of \( \Sigma \), such as when \( \Sigma \) is a three sphere \( S^3 \), there can be additional observables localized at \( \partial T_3 \). It is constructed along the lines of the observable \( P \) in [5]. Generically, there will also be nonlocal observables. We will discuss these local and nonlocal observables in Section 3.

3. Observables and Their Mode Decomposition

3.1. Local and Nonlocal Observables

In general, in \( \Sigma \), we would have several charges and vortices enclosed by balls \( B_3 \) and solid tori \( T_3 \). There is also the ball \( \Sigma \) with \( S^2 \) as boundary.

Let \( H \) denote the union of the balls and solid tori enclosing charges and vortices. The boundary \( \partial (\Sigma \setminus H) \) of \( \Sigma \setminus H \) is the union of the boundaries \( \partial \Sigma \) and \( \partial H \) of \( \Sigma \) and \( H \).

The constraints \( G_j \) are now

\[
G_0 (\lambda^{(0)}) = \int_{\Sigma \setminus H} \lambda^{(0)} dB \approx 0 ,
\]

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\[ G_1(\lambda^{(1)}) = 2 \int_{\Sigma \setminus \mathcal{H}} \lambda^{(1)} dA \approx 0 , \]
\[ \lambda^{(j)} \text{ forms with } \lambda^{(j)}|_{\partial(\Sigma \setminus \mathcal{H})} = 0 . \] (3.1)

Let \( w^{(j)} \) be closed \( j \) forms on \( \Sigma \setminus \mathcal{H} \) and consider
\[ q(w^{(1)}) = \int_{\Sigma \setminus \mathcal{H}} w^{(1)} B , \] (3.2)
\[ p(w^{(2)}) = - \int_{\Sigma \setminus \mathcal{H}} w^{(2)} A . \] (3.3)

We can easily verify that their PB’s with \( G_j \) weakly vanish and therefore that they are observables. For example, in view of (2.1),
\[ \{ G_0(\lambda^0), p(w^{(2)}) \} = - \int_{\Sigma \setminus \mathcal{H}} w^{(2)} d\lambda^0 \]
\[ = - \int_{\partial(\Sigma \setminus \mathcal{H})} w^{(2)} \lambda^0 \]
\[ = 0 , \] (3.4)

as the test forms \( \lambda^{(j)} \) now vanish on all the boundaries of \( \Sigma \setminus \mathcal{H} \).

Suppose that we replace \( w^{(j)} \) by \( u + dv^{(j-1)} \) where
\[ v^{(j-1)}|_{\partial(\Sigma \setminus \mathcal{H})} = 0 . \] (3.5)

It is then the case that
\[ q(w^{(1)} + dv^{(0)}) = q(w^{(1)}) - G_0(v^{(0)}) \approx q(w^{(1)}) , \]
\[ p(w^{(2)} + dv^{(1)}) = p(w^{(2)}) - \frac{1}{2} G_1(v^{(1)}) \approx p(w^{(2)}) . \] (3.6)

Hence \( w^{(j)} \) and \( w^{(j)} + dv^{(j-1)} \) define equivalent observables.

The condition (3.5) can be relaxed somewhat. Thus since \( d\hat{v}^{(j-1)} = dv^{(j-1)} + u^{(j-1)} \) if \( du^{(j-1)} = 0 \), we can say that \( w^{(j)} \) and \( w^{(j)} + d\hat{v}^{(j-1)} \) define equivalent observables if \( \hat{v}^{(j-1)} \) differs from a \( v^{(j-1)} \) fulfilling (3.5) by a closed form. For this purpose, it is sufficient that \( \hat{v}^{(j-1)}|_{\partial(\Sigma \setminus \mathcal{H})} \) is exact. In that case, \( \hat{v}^{(j-1)}|_{\partial(\Sigma \setminus \mathcal{H})} = d\hat{w}^{(j-2)}|_{\partial(\Sigma \setminus \mathcal{H})} \). The form \( \hat{w}^{(j-2)}|_{\partial(\Sigma \setminus \mathcal{H})} \)
can be extended to a form \( \hat{w}^{(j-2)} \) to all of \( \Sigma \setminus \mathcal{H} \), in fact in many ways, such that its restriction to \( \partial(\Sigma \setminus \mathcal{H}) \) is \( \hat{w}^{(j-2)}|_{\partial(\Sigma \setminus \mathcal{H})} \). We can now set \( v^{(j-1)} = \hat{v}^{(j-1)} - d\hat{w}^{(j-2)} \) to see the result.

Hereafter, we write \( \mathcal{H} = \bigcup_{\alpha} \mathcal{H}_{\alpha} \) where \( \partial \mathcal{H}_{\alpha} \) is connected. This means that an \( \mathcal{H}_{\alpha} \) encloses a single charge or a single vortex.

In what follows, we will find it useful to define our meaning of an observable localized on a set \( U \subseteq \partial \mathcal{H}_{\alpha} \). We will say that an observable is localized on \( U \subseteq \partial \mathcal{H}_{\alpha} \) iff, up to weak equivalence, it is a function of observables of type \( q(d\xi^{(0)}) \) or \( p(d\xi^{(1)}) \) [with integral representations like (3.2) and (3.3)] where

\[
\xi^{(j)}|_{\partial \mathcal{H}_{\beta}(\beta \neq \alpha)} = \xi^{(j)}|_{\partial \Sigma} = \xi^{(j)}|_{\partial \mathcal{H}(\alpha) \setminus U} = 0 . \tag{3.7}
\]

This is a good definition because with this condition, \( q(d\xi^{(0)}) \) for example is determined by \( \xi^{(0)}|_{U} \).

In general, there exist observables not accounted for by those localizable on contractable open sets \( U \subseteq \partial \mathcal{H}_{\alpha} \). For example, let \( \Sigma \setminus \mathcal{H} \) be a ball with two holes \( \mathcal{H}_{1} \) and \( \mathcal{H}_{2} \). Then there are closed two forms \( w^{(2)}_{\alpha}(\alpha = 1 \text{ or } 2) \) with their integrals over \( \partial \mathcal{H}_{\alpha} \) and \( \partial \Sigma \) being nonzero whereas their integrals over \( \partial \mathcal{H}_{\beta}(\beta \neq \alpha) \) vanish. There are in fact \( w^{(2)}_{\alpha} \) with \( w^{(2)}_{\alpha}|_{\partial \mathcal{H}_{\beta}} = 0 \) if \( \beta \neq \alpha \). How should we think about the localization properties of \( p(w^{(2)}_{\alpha}) \)?

We suggest the following in this regard. Consider the observables \( q(d\beta^{(0)}) \) and \( p(d\beta^{(1)}) \) of the sort examined in the last paragraph with \( \beta^{(j)}|_{\mathcal{H}_{\alpha}} = \beta^{(j)}|_{\partial \Sigma} = 0 \). They are thus localized at \( \partial \mathcal{H}_{\beta} \). If an observable has nonzero PB with any of such observables, then we propose that they should not be regarded as having null support at \( \partial \mathcal{H}_{\beta} \). Instead, they should be thought of as partly living on \( \partial \mathcal{H}_{\beta} \) as well. If all these PB’s instead vanish, as it does for example for \( p(w^{(2)}_{\alpha}) \), then it is to be thought of as leading to a superselection rule for \( \partial \mathcal{H}_{\beta} \)-localized observables after quantization. It could of course vanish because of the nature of its test functions and the Gauss law constraints.

We have refrained from asserting that observables having zero PB’s with \( q(d\beta^{(0)}) \) and
$p(d\beta^{(1)})$ are localized away from $\partial\mathcal{H}_\beta$. Such an assertion seems misleading. For example, if $\xi^{(0)}|_\beta$ is a constant, and it is zero on all boundaries except $\beta$, $q(d\xi^{(0)})$ has the mentioned feature, and defines the charge operator for $\partial\mathcal{H}_\beta$, but can not be reasonably claimed to be localized away from $\partial\mathcal{H}_\beta$.

The preceding paragraphs show that we have to consider two kinds of observables, namely local observables and those which are not local. They will be examined in turn below.

### 3.2. Local Observables and Their Mode Decomposition

We recall that two observables are weakly equal if their difference is one of the constraints in (3.1). Let $<x>$ denote the equivalence class of observables weakly equal to $x$. Hereafter we will call $<x>$ also as the observable and say that $<x>$ is localized on $\partial\mathcal{H}_\alpha$ if $x$ is localized on $\partial\mathcal{H}_\alpha$. In this subsection, we perform the Fourier decomposition of certain basic $<x>$ which are localized on $\partial\mathcal{H}_\alpha$.

It is convenient, for all subsequent discussion, to choose volume forms $\mu$ on spheres $S^2$ and tori $T^2$ which make up the boundary components of $\Sigma \setminus \mathcal{H}$. For the former, we fix polar coordinates $\theta, \phi$ and set

$$\mu = \frac{\Delta}{4\pi} d\cos \theta d\phi,$$

$$\int_{S^2} \mu = \Delta. \quad (3.8)$$

For the latter, we choose as before angular coordinates $\theta^i$ and set

$$\mu = \frac{\Delta}{4\pi^2} d\theta^1 d\theta^2,$$

$$\int_{T^2} \mu = \Delta. \quad (3.9)$$

Now $\mu$ defines a Hilbert space $L^2(\mu, \partial\mathcal{H}_\alpha)$ of functions on $\partial\mathcal{H}_\alpha$ with scalar product

$$(\chi_1, \chi_2) = \int_{\partial\mathcal{H}_\alpha} \mu \chi_1^* \chi_2. \quad (3.10)$$
Let \( e^{(\alpha)}_n, n \in \mathbb{Z} \), define an orthonormal basis of smooth functions for this space [\( Z \) being the set of integers] where \( e^{(\alpha)}_0 = 1/\sqrt{\Delta} \):

\[
(e^{(\alpha)}_n, e^{(\alpha)}_m) = \delta_{nm}, \quad e^{(\alpha)}_0 = \frac{1}{\sqrt{\Delta}}. \tag{3.11}
\]

When \( x_\alpha \) is localized on \( \partial \mathcal{H}_\alpha \), it is determined, up to weak equivalence, by the pull back of its test forms to \( \partial \mathcal{H}_\alpha \). Thus \( \langle x_\alpha \rangle \) is entirely determined by the pull back of its test forms to \( \partial \mathcal{H}_\alpha \). The Fourier decomposition of \( \langle x_\alpha \rangle \) is thus accomplished by the Fourier decomposition of these pull backs.

This Fourier decomposition can now proceed as in [5]. The observables \( q_n(\alpha) \) are defined by

\[
q_n(\alpha) := \langle q(d\xi^{(0)}_n) \rangle, \quad \xi^{(0)}_n|_{\partial \mathcal{H}_\alpha} = e^{(\alpha)}_n. \tag{3.12}
\]

[It is understood here that \( \xi^{(j)}_n \) vanish on all boundaries except \( \partial \mathcal{H}_\alpha \).] The observables \( p_n(\beta) \) are given by

\[
p_n(\alpha) := \langle p(d\xi^{(1)}_n) \rangle, \quad d\xi^{(1)}_n|_{\partial \mathcal{H}_\alpha} = e^{(\alpha)*}_n \mu, n \neq 0. \tag{3.13}
\]

The value \( n = 0 \) can not be excluded from (3.12). For although \( e^{(\alpha)}_0 \) is a constant, we can not set \( \xi^{(0)}_0 = e^{(\alpha)}_0 \) since \( \xi^{(0)}_0|_{\partial \mathcal{H}_\beta} (\beta \neq \alpha) \) and \( e^{(0)}_0|_{\partial \Sigma} \) must vanish. In contrast, the value \( n = 0 \) is excluded from (3.13) because \( e^{(\alpha)}_0 \mu = \frac{1}{\sqrt{\Delta}} \mu \) is not exact unlike \( d\xi^{(1)}_0|_{\mathcal{H}_\alpha} = d(\xi^{(1)}_0|_{\mathcal{H}_\alpha}) \).

The nonzero PB's involving \( q_n^{(\alpha)} \) and \( p_n^{(\beta)} \) are, by a natural definition, given by

\[
\{ q_n(\alpha), p_m(\beta) \} := \delta_{\alpha\beta} < \{ q(d\xi^{(0)}_n), p(d\xi^{(1)}_m) \} >= \delta_{\alpha\beta} \delta_{nm}. \tag{3.14}
\]

We here record our choices of \( e^{(\alpha)}_n \) for \( S^2 \) and \( T^2 \) made in [4], choices which we will adopt in this paper as well. For \( S^2 \), they are defined by the correspondences

\[
n \rightarrow Jm,
\]

\[
e^{(\alpha)}_n \rightarrow e^{JM} = \left( \frac{4\pi}{\Delta} \right)^{\frac{1}{2}} Y_{JM}, \quad Y_{JM} = \text{Spherical harmonics}, \tag{3.15}
\]
\(e_0^{(\alpha)}\) becoming \(\frac{1}{\sqrt{\Delta}}\).

For \(T^2\), we have the correspondences

\[
n \to \vec{N} = (N_1, N_2),
\]

\[
e_n^{(\alpha)}(\vec{\theta}) \to e_N(\vec{\theta}) = \frac{1}{\sqrt{\Delta}}e^{i\vec{N} \cdot \vec{\theta}},
\]

\[
\vec{\theta} = \theta^1, \theta^2, \quad \vec{N} \cdot \vec{\theta} = N_1 \theta^1 + N_2 \theta^2, \quad N_i \in \mathbb{Z}.
\] (3.16)

For general topologies, there are more \(p\)-type observables localized at \(\partial \mathcal{H}_\alpha\). This would be the case if \(\partial \mathcal{H}_\alpha\) admits closed but inexact one forms \(\omega_N^{(1)}|_{\partial \mathcal{H}_\alpha}\), the one forms \(\omega_N^{(1)}\) on \(\Sigma \setminus \mathcal{H}\) vanishing on \(\partial(\Sigma \setminus \mathcal{H}_\beta)\). Here as usual \(\omega_N^{(1)}|_{\partial \mathcal{H}_\alpha}\) is the pull back of \(\omega_N^{(1)}\) to \(\partial \mathcal{H}_\alpha\). We can then choose \(\omega_N^{(1)}\) and cycles \(C_M \subset \partial \mathcal{H}_\alpha\) for the generators of the homology group of \(\partial \mathcal{H}_\alpha\) such that

\[
\int_{C_M} \omega_N^{(1)} = \delta_{MN} \times \text{nonvanishing constant}.
\] (3.17)

It is our experience that \(C_M\), regarded as a cycle in \(\Sigma \setminus \mathcal{H}\), is contractible to a point (if there are no links) or in other words homologous to the trivial cycle consisting of a point. That being so, \(\omega_N^{(1)}\) can not as a rule be closed in \(\Sigma \setminus \mathcal{H}\), it is only its pull back to \(\partial \mathcal{H}_\alpha\) which can enjoy this property.

Referring to (3.13), we see that the preceding Fourier decomposition misses the observables

\[
P_N(\alpha) = < p(d\omega_N^{(1)}) >, \quad (3.18)
\]

\(d\omega_N^{(1)}|_{\partial \mathcal{H}_\alpha}\) being zero. They must thus be added to our list of observables.

It is to be noted no new observable is associated with \(\xi_N^{(1)}\) having \(\xi_N^{(1)}|_{\partial \mathcal{H}_\alpha}\) exact. This is so for the following reason: If \(\xi_N^{(1)}|_{\partial \mathcal{H}_\alpha}\) is exact, we can write \(\xi_N^{(1)}|_{\partial \mathcal{H}_\alpha} = d\hat{\xi}_N^{(0)}|_{\partial \mathcal{H}_\alpha}\). Now \(\hat{\xi}_N^{(0)}\) can be readily extended to a function \(\hat{\xi}_N^{(0)}\) defined in all of \(\Sigma \setminus \mathcal{H}\), and

\[
p(d\xi_N^{(1)}) \approx p(dd\xi_N^{(0)}) = 0. \quad \text{Thus } < p(d\xi_N^{(1)}) > = 0.
\]

An example of an observable of type \(P_N(\alpha)\) was encountered in [5] where the space \(\Sigma = T_3\) was considered without holes, and the corresponding \(P_N(\alpha)\) was called \(P\). This
space $\Sigma$ is the *interior* of the torus in Fig. 2. If $\omega^{(1)}$ is the one form leading to $P$, then

$$\omega^{(1)}|_{\partial \Sigma} = -\frac{\sqrt{\Delta}}{4\pi^2} d\theta^1.$$  

(3.19)

$[\theta^1$ and $\theta^2$ in Fig. 2 are to be indentified with $\theta^2$ and $\theta^1$ of [3].]

We can make up an example of this kind whenever there is a cycle $C$ in $\partial (\Sigma \setminus H)$ which can not be contracted to a point while staying within $\partial (\Sigma \setminus H)$. Thus $C$ here, when regarded as defining an element of the homology group $H_1(\partial (\Sigma \setminus H))$, defines a nontrivial element of that group. When there is such a $C$, we can find a closed one form $\omega^{(1)}|_{\partial H_\alpha}$ with a nonzero integral over $C$, and a zero integral over all other cycles in $\partial H_\alpha$. This form can also be clearly extended as a one form $\omega^{(1)}$ over $\Sigma \setminus H$ with $\omega^{(1)}|_{\partial H_\beta} = 0$ for $\beta \neq \alpha$. [This form $\omega^{(1)}$ over $\Sigma \setminus H$ must not be closed in order to get a nonzero observable, only its pull back to $\partial H_\alpha$ must be closed.] Such an $\omega^{(1)}$ gives us an example of the sort we want.

We next note that

$$\{ p_n(\alpha), P_N(\beta) \} = \{ P_N(\alpha), P_M(\beta) \} = 0,$$

(3.20)

$$\{ q_n(\alpha), P_N(\beta) \} = \delta_{\alpha\beta} \int_{\partial H_\alpha} e_n^{(\alpha)} d\omega_N^{(1)} = 0.$$  

(3.21)

(3.20) here is evident. Thus $P_N(\alpha)$ define superselection rules for observables at $\partial H_\alpha$. Their physical meaning will be addressed in Section 3.3.

$P_N(\alpha)$ are not the only observables leading to superselection rules. $q_0(\alpha)$ also has this property: its PB with all observables localized at $\partial H_\alpha$ vanish in view of (3.14) and because $m \neq 0$.

The physical meaning of $q_0(\alpha)$ is that it is a measure of charge contained by the hole $H_\alpha$ as we shall see in Section 3.3.

In suggesting that $q_0(\alpha)$ and $P_N(\alpha)$ are associated with superselection sectors, we have omitted an examination of their PB’s with the nonlocal observables. We will see below
that there are observables of this kind related to charge and vortex creation. In what follows, arguments will be presented to justify this neglect.

### 3.3. Nonlocal Observables

We find by examples that these observables are associated with charge and vortex creation, the former being conjugate to $q_0(\alpha)$ and the latter to $P_N(\alpha)$. It is also the case that the former can be interpreted in terms of line integrals of $A$ (Wilson line integrals) and the latter in terms of surface integrals of $B$.

Nonlocal observables, or rather their suitably regularized exponentials [see Section 4], are the analogues of vertex operators in conformal field theories (CFT’s) [11]. They have this relationship only in a generalized sense, being classical, in 3+1 dimensions and in a field theory distinct from a CFT. But they do resemble CFT vertex operators in spite of these differences.

Now in CFT, a vertex operator is not treated as an observable. It is, rather, an intertwining operator between inequivalent representations of the affine Lie algebra with distinct charges or momenta.

In a similar spirit, the nonlocal “observables” we now describe in their classical versions are perhaps more properly regarded in quantum theory as operators intertwining inequivalent representations of the algebra of the remaining observables, and not as observables themselves.

We tacitly accepted this interpretation when suggesting previously that $q_0(\alpha)$ and $P_N(\alpha)$ lead to superselection rules. Perhaps a physical argument can be found which explains why nonlocal “observables” are not in fact observables.

Hereafter, we will call nonlocal “observables” as nonlocal variables or nonlocal operators to emphasize that we do not treat them as observables.

#### i) The Conjugate of Charge

It is enough to consider the ball $\Sigma$ to have a single hole $\mathcal{H}_1 = \mathcal{B}_3$ to illustrate the
ideas behind this variable. The hole could have been put there to regularize a charge. In quantum theory, it could also have been punched as a preparatory move to create a charged state there using a vertex operator as in Section 4. [Such a procedure has been described in [8] for Chern-Simons theories.] For what follows here, $H_1$ can also be $T_3$ and punched in $B_3$ to regularize a vortex.

Now $\Sigma \setminus H_1$ admits a closed two form $\omega^{(2)}$ which is not exact. A closed $\omega^{(2)}$ with its pull back to $H_1$ being $\mu$ is one such $\omega^{(2)}$. If $S^2$ is a two-sphere enclosing $H_1$, $\omega^{(2)}$ is in general specified by the properties

$$d\omega^{(2)} = 0,$$  \hspace{1cm} (3.22)

$$\int_{S^2} \omega^{(2)} \neq 0.$$  \hspace{1cm} (3.23)

Consider

$$W(\omega^{(2)}) = \int_{\Sigma \setminus H_1} \omega^{(2)} A.$$  \hspace{1cm} (3.24)

Under a gauge transformation $A \rightarrow A + d\lambda^{(0)}$ where $\lambda^{(0)}|_{\partial(\Sigma \setminus H_1)} = 0$, the function $W$ does not change:

$$W(\omega^{(2)}) \rightarrow W(\omega^{(2)}) + \int_{\Sigma \setminus H_1} \omega^{(2)} d\lambda^{(0)}$$

$$= W(\omega^{(2)}) + \int_{\partial(\Sigma \setminus H_1)} \omega^{(2)} \lambda^{(0)}$$

$$= W(\omega^{(2)}).$$  \hspace{1cm} (3.25)

It thus has zero PB with $G_0(\lambda^{(0)})$, and evidently with $G_1(\lambda^{(1)})$ as well, and can be thought of as an observable. Alternatively, the observable is the equivalence class $< W(\omega^{(2)} >$.

Also if $q_0(1)$ is the charge in $H_1$, as we argue under iii) below, we find using (3.12) that

$$\{ W(\omega^{(2)}), q_0(1) \} = \int_{\Sigma \setminus H_1} \omega^{(2)} d\xi_0^{(0)} = \frac{1}{\Delta} \int_{H_1} \omega^{(2)}.$$  \hspace{1cm} (3.26)

Thus, $W(\omega^{(2)})$ is conjugate to charge and the charge creation operator is a suitable exponential constructed using it. [See Section 4.]
There is an interpretation of $W(\omega^{(2)})$ in terms of the Wilson integral as we now indicate.

Let us choose $\omega^{(2)} = \mathcal{W}^{(2)}$ where $\mathcal{W}^{(2)}$ has support on a thin tube $T$ of cross section $\delta$ as pictured in Fig. 3. In that figure, $L$ is a line in the middle of $T$. In the limit $\delta \to 0$, we then find,

$$W(\mathcal{W}^{(2)}) \to \tilde{\lambda} \int_L A,$$

$$\tilde{\lambda} = \int_{S^2} \mathcal{W}^{(2)}.$$  \hspace{1cm} (3.27)

The Wilson integral is thus a limiting form of $W(\mathcal{W}^{(2)})$.

The Wilson integral for two lines $L$ and $L'$ describe the same observable provided only that $L'$ can be deformed to $L$ holding $P$ and $Q$ fixed. Fig. 4 shows such an $L$ and an $L'$. This is so because the two integrals are seen to differ by a constraint on using Stokes' theorem. It follows that the integral of $A$ over $L$ is associated with a localized blip at $P$ and, in the limit of $\mathcal{H}_1$ shrinking to $C$, a direction, shown by an arrow in Figs. 3 and 4, attached to the particle position. It is this directional degree of freedom which leads to the spin excitations of the particle.

[Section 4 contains further discussion of the dependence of $\langle W(\omega^{(2)}) \rangle$ on $L$.]

The variable $\langle W(\omega^{(2)}) \rangle$ is nonlocal, its associated charges being at the two boundaries $\partial \mathcal{H}_1$ and $\partial \Sigma$.

The variable

$$\langle \int_L A \rangle$$

\hspace{1cm} (3.28)

can be Fourier analyzed. We reserve this task to Section 4. In the Chern-Simons field theory, a corresponding analysis is known to lead to the Fubini-Veneziano field and the associated vertex operator \[ \mathcal{B} \mathcal{B} \mathcal{B} \].

The PB of $\langle W(\omega^{(2)}) \rangle$ with the rest of the observables is sensitive to the choice of
\( \omega^{(2)}|_{\partial \mathcal{H}_1} \). The choice leading to simple answers is

\[
\omega^{(2)}|_{\partial \mathcal{H}_1} = \mu. \quad (3.29)
\]

Then

\[
\{ < W(\omega^{(2)}) >, q_n(1) \} = \int_{(\partial \Sigma \setminus \mathcal{H}_1)} \omega^{(2)} d\xi_n^{(0)} = \int_{\Sigma \setminus \mathcal{H}_1} \mu e^{(1)}_n = \sqrt{\Delta} \delta_{n0}, \quad (3.30)
\]

\[
\{ < W(\omega^{(2)}) >, p_n(\alpha) \} = 0. \quad (3.31)
\]

We will have to consider yet another variable, the conjugate flux, in ii). The PB involving it will be recorded there. A summary of all PB’s can be found in iv) below.

Note that (3.29) and (3.30) are correct even if \( \mathcal{H}_1 \) is \( \mathbb{T}_3 \), provided the closed form \( \omega^{(2)} \) satisfies (3.29).

For the manifold \( \Sigma \setminus \mathcal{H}_1 \) we are considering, a similar analysis can be done on \( \partial \Sigma = S^2 \).

The analogues of the observables \( q_n(\alpha) \) and \( p_n(\alpha) \) on this \( S^2 \) and their PB’s, have been found in [5]. One can readily check that relations like (3.30) and (3.31) also hold.

For more complicated or for alternative topologies too, such as for \( \Sigma \setminus \mathcal{H} \) with \( \mathcal{H} \) having several disconnected components \( \mathcal{H}_\alpha \), or with \( \partial \Sigma \) being \( \mathbb{T}_2 \), similar conclusions can be drawn by choosing \( \omega^{(2)} \) to be \( \mu \) on one \( \partial \mathcal{H}_\alpha \) and \( \partial \Sigma \), and zero on \( \partial \mathcal{H}_\beta (\beta \neq \alpha) \).

ii) The Conjugate of Flux

Experience with \( W(\omega^{(2)}) \) suggests that this nonlocal variable involves closed but inexact one forms \( \omega^{(1)} \), the variable then being the equivalence class of

\[
V(\omega^{(1)}) = \int \omega^{(1)} B. \quad (3.32)
\]

The integration here is over \( \Sigma \), or if it has holes, over \( \Sigma \setminus \mathcal{H} \).
$V(\omega^{(1)})$ has vanishing PB with the constraints so that $< V(\omega^{(1)}) >$ is first class \[18\]. We only need to examine the response of $V(\omega^{(1)})$ to the transformation $B \rightarrow B + d\lambda^{(1)}$ to verify this fact. Here $\lambda^{(1)}$ vanishes on all the boundaries. Since

$$
\int \omega^{(1)} d\lambda^{(1)} = 0, \quad \text{(3.33)}
$$

$\omega^{(1)}$ being closed, the result follows.

In \[5\], an example of $\omega^{(1)}$ was presented when $\Sigma$ was the solid torus interpreted as the interior of $T_3$ in Fig. 2. The integration for $V(\omega^{(1)})$ is then over this $\Sigma$ while

$$
\omega^{(1)}|_{\partial \Sigma} = d\theta^2. \quad \text{(3.34)}
$$

It is easy to see that there is a closed $\omega^{(1)}$ with the property \(3.34\). For instance, we can introduce $\theta^1$, $\theta^2$ and a third variable $r$ as coordinates for all of $T_3$, $r$ being a radial distance from the central dotted thread in Fig.2, with $r = 1$ say for $\partial T_3$. The meaning of $\theta^i$ are as shown in that Figure. We can then set

$$
\omega^{(1)} = d\theta^2 \quad \text{(3.35)}
$$
in all of $T_3$.

Now if $L$ is as indicated in Fig. 2,

$$
\{ V(\omega^{(1)}), \int_L A \} = - \int_L \omega^{(1)} = -2\pi. \quad \text{(3.36)}
$$

As the integral of $A$ over $L$ measures the flux enclosed by $L$, $V(\omega^{(1)})$ is conjugate to magnetic flux. This flux is confined to $\partial T_3$ as $L$ can be anywhere in the interior of $T_3$. The creation operator of a vortex on $\partial T_3$ is thus a suitable exponential constructed from $V(\omega^{(1)})$. [See Section 4.]

Previously, in \[3.27\] $W(\omega^{(2)})$ was interpreted in terms of an integral over $A$. There is a similar interpretation of $V(\omega^{(1)})$ in terms of

$$
\int_{\Sigma} B
$$
where $\Sigma$ is the cross section of $\mathbf{T}_3$ for the constant value $\theta^2$ of $\theta^2$. We find this interpretation as follows. Let $\overline{\mathcal{S}}_\delta$ be the solid disc with $\theta^2$ in the range $\bar{\theta}^2 - \frac{\delta}{2} < \theta^2 < \bar{\theta}^2 + \frac{\delta}{2}$ as in Fig.5. Let us choose $\omega^{(1)} = \overline{\omega}^{(1)}$ where $\omega^{(1)}$ has support on $\overline{\mathcal{S}}_\delta$. In the limit $\delta \to 0$, we then find,

$$V(\overline{\omega}^{(1)}) = \Lambda \int_{\overline{\mathcal{S}}} B, \quad \overline{\mathcal{S}} = \overline{\mathcal{S}}_0,$$

$$\Lambda = \int_{L} \overline{\omega}^{(1)}. \quad (3.37)$$

In this example, we have called $V(\omega^{(1)})$ nonlocal even though it is associated with a vortex at a single boundary. This usage may not be inappropriate. Thus the form $\omega^{(1)}$ is closed but not exact and can not therefore be chosen to be zero outside a small neighbourhood of $\partial \mathbf{T}_3$. In contrast, in (3.12) and (3.13), the forms $\xi^{(j)}_{\alpha}$ characterizing an observable localized at $\mathcal{H}_\alpha$ can always be made zero outside a small neighbourhood of $\mathcal{H}_\alpha$ without affecting the observable.

Reference [5] can be consulted for further discussion of this example. It can be generalized. For illustrative purposes, we will now indicate a few such generalizations.

$\Sigma \setminus \mathcal{H}_1 = \mathcal{B}_3 \setminus \mathbf{T}_3$

Let $\Sigma = \mathcal{B}_3$ and let us dig a single hole $\mathcal{H}_1 = \mathbf{T}_3$ in $\Sigma$. This hole is pictured in Fig. 2. It could have been put there to regularize a vortex. In quantum theory, just as for the charge [9], it could also have been punched as a preparatory move to create a vortex state using a vertex operator. [See Section 4.]

Now $\Sigma \setminus \mathcal{H}_1$ admits a closed but inexact one form $\omega^{(1)}$. Its cohomology class is entirely described by saying that $\omega^{(1)}|_{\partial \mathcal{H}_1} = d\theta^1$.

Let $\mathcal{C}$ be a contour enclosing this $\mathbf{T}_3$ as shown in Fig. 6. Then

$$\{V(\omega^{(1)}), \int_{\mathcal{C}} A\} = -\int_{\mathcal{C}} \omega^{(1)} = -2\pi. \quad (3.38)$$

Since $\int_{\mathcal{C}} A$ is a measure of the flux through $\mathcal{C}$, and $\mathcal{C}$ can be as close to $\partial \mathbf{T}_3$ as we please without affecting (3.38), this shows that $V(\omega^{(1)})$ is conjugate to magnetic flux threading
$\mathcal{H}_1$ in the $\theta^2$ direction. The creation operator of a vortex along the line $L$ of Fig. 2 is hence an exponential involving $V(\omega^{(1)})$ in the limit that $\mathcal{H}_1$ shrinks to $L$.

Let $\sigma$ be a surface spanning from $\partial T_3$ to $\partial \Sigma$ as in Fig. 7. Just as in (3.36), we can then show by a suitable choice of $\omega^{(1)}$ and a limiting procedure that

$$V(\omega^{(1)}) = 2\pi \int_\sigma B$$

where we have assumed that $\omega^{(1)}|_{\partial T_3} = d\theta^1$ and hence that

$$\int_C \omega^{(1)} = 2\pi.$$ (3.40)

Just as we argued previously in the case of $< W(\omega^{(2)}) >$, we can argue here as well that the distortion of $\sigma$ to another surface $\tilde{\sigma}$ as in Fig. 7 without altering its boundaries does not affect $< V(\omega^{(1)}) >$. What this means is that $V(\omega^{(1)})$ describes a vortex at $\partial \sigma \cap \partial T_3$ and another one at $\partial \sigma \cap \partial \Sigma$, but does not describe any degree of freedom in the interior of $\Sigma \setminus \mathcal{H}_1$. Note that in the limit where $\mathcal{H}_1$ shrinks to $L$ (represented in Fig. 7 by a dotted line inside $T_3$), the integral of $B$ over $\sigma$ gets associated not merely with $L$, but also with a set of directions on $L$ pointing from $L$ to $\sigma \cap \partial T_3$ as shown in Fig. 7. This field of directions endows the vortex with a transverse spin degree of freedom along the lines discussed previously in [19]. Its direction can be defined by choosing a metric and using it to define the tangent to $\sigma$ at a point of the vortex which is normal to its slope there. Then the transverse spin can be said to point in the direction of this tangent.

It is possible to “Fourier analyse” $< V(\omega^{(1)}) >$ for the $V(\omega^{(1)})$ of (3.39) just as (3.28) can be subjected to such an analysis. The resultant field however is no ordinary field, but a string field dependent on $\partial \sigma \cap \partial T_3$ and $\partial \sigma \cap \partial \Sigma$. We discuss this field further in Section 4.

The PB of $< V(\omega^{(1)}) >$ with the rest of the observables at a boundary $\partial \mathcal{H}_\alpha$ is sensitive to the choice of $\omega^{(1)}|_{\partial \mathcal{H}_\alpha}$. For $\Sigma \setminus \mathcal{H}_1$, let us choose

$$\omega^{(1)}|_{\partial \mathcal{H}_1} = d\theta^1.$$ (3.41)
We then get
\[
\{< V(\omega^{(1)}) >, \ p_N(\alpha) \} = 0, \quad (3.42)
\]
\[
\{< V(\omega^{(1)}) >, \ p_N(\alpha) \} = \int d\theta^1 d\xi^1_N = -\int d(\theta^1 \xi^1_N) = -\int_{\partial H_1} d\theta^1 \xi^1_N = 0. \quad (3.43)
\]
Here \(N_1\) and \(N_2\) are not both zero and \(\alpha\) of course is 1. Also the conclusion in (3.43) follows from (3.13) which permits us to assume that \(\xi^1_N|_{\partial H_0} = \frac{e^{-i\delta \phi}}{(-iN_1)}d\theta^2\) or \(\frac{e^{-i\delta \phi}}{(-iN_2)}d\theta^1\) according as \(N_1\) or \(N_2\) is nonzero.

We have yet to look at \(\{< V(\omega^{(1)}) >, \ P_N(\alpha) \}\) and \(\{< W(\omega^{(2)}) >, < V(\omega^{(1)}) >\}\). We have
\[
\{< V(\omega^{(1)}) >, \ P_N(\alpha) \} = -\int_{\partial H_0} \omega^{(1)} \omega^{(1)}_N, \quad (3.44)
\]
\[
\{< W(\omega^{(2)}) >, < V(\omega^{(1)}) >\} = \int_{\Sigma \setminus H_1} \omega^{(2)} \omega^{(1)} \quad (3.45)
\]
where \(\omega^{(2)} \omega^{(1)}\) as usual denotes \(\omega^{(2)} \land \omega^{(1)}\).

We will consider the evaluation of (3.44) for two typical cases, namely \(\Sigma = B_3\) and \(T_3, \ H_1\) in both these instances being a solid torus. The choice \(\Sigma = T_3\) actually gives us two examples depending on the placement of \(H_1\). They are illustrated in Figs. 8 and 9. It is to be noted that Figs. 8 and 9 admit two independent \(\omega^{(1)}\) corresponding to the two cycles \(C_i\). As for \(\omega^{(2)}\), it can in all three cases be taken to be the closed two form which reduces to our canonical volume form when pulled back to \(\partial \Sigma\) and \(\partial H_1\).

Let \(\omega_0^{(j)}\) be closed \(j\) forms with same boundary values as \(\omega^{(j)}\),
\[
\omega_0^{(j)}|_{\partial (\Sigma \setminus H_1)} = \omega^{(j)}|_{\partial (\Sigma \setminus H_1)}, \quad (3.46)
\]
such that \(\int_{\Sigma \setminus H_1} \omega_0^{(2)} \omega_0^{(1)}\) can be calculated. We can write
\[
\int_{\Sigma \setminus H_1} \omega^{(2)} \omega^{(1)} = \int_{\Sigma \setminus H_1} \omega_0^{(2)} \omega_0^{(1)} + \int_{\Sigma \setminus H_1} \Delta \omega^{(2)} \omega_0^{(1)} + \int_{\Sigma \setminus H_1} \omega_0^{(2)} \Delta \omega^{(1)} + \int_{\Sigma \setminus H_1} \Delta \omega^{(2)} \Delta \omega^{(1)},
\]
\[
\Delta \omega^{(j)} = \omega^{(j)} - \omega_0^{(j)}. \quad (3.47)
\]
We now show that the last three terms here and hence (3.45), can be calculated.

We argue as follows to compute these terms. First note that \( \Delta \omega^{(j)} \) is closed and
vanishes on the boundaries:

\[
d\Delta \omega^{(j)} = 0, \quad (3.48)
\]
\[
\Delta \omega^{(j)}|_{\partial(\Sigma \setminus H_1)} = 0. \quad (3.49)
\]

For the topologies we consider, every cycle in \( \Sigma \setminus H_1 \) is homologous to a cycle on the
boundary. The integral of \( \Delta \omega^{(j)} \) over every such cycle is zero by (3.49), and in view of
(3.48), its integral over all cycles in \( \Sigma \setminus H_1 \) is zero. So \( \Delta \omega^{(j)} \) is exact:

\[
\Delta \omega^{(j)} = d\epsilon^{(j-1)}. \quad (3.50)
\]
Here by (3.49), \( \epsilon^{(j-1)}|_{\partial(\Sigma \setminus H_1)} \) is closed:

\[
d\epsilon^{(j-1)}|_{\partial(\Sigma \setminus H_1)} = 0. \quad (3.51)
\]

Using these results, we find,

\[
\int_{\Sigma \setminus H_1} \Delta \omega^{(2)} \omega^{(1)} = \int_{\partial(\Sigma \setminus H_1)} \epsilon^{(1)} \omega^{(1)} \equiv (3.52)
\]
\[
\int_{\Sigma \setminus H_1} \omega^{(2)} \Delta \omega^{(1)} = \int_{\partial(\Sigma \setminus H_1)} \omega^{(2)} \epsilon^{(0)} \equiv (3.53)
\]
\[
\int_{\Sigma \setminus H_1} \Delta \omega^{(2)} \Delta \omega^{(1)} = \int_{\Sigma \setminus H_1} d\epsilon^{(1)} d\epsilon^{(0)} = \int_{\partial(\Sigma \setminus H_1)} \epsilon^{(1)} d\epsilon^{(0)} = 0. \quad (3.54)
\]

There are two sorts of connected boundaries, namely \( S^2 \) and \( T^2 \). \( S^2 \) occurs as \( \partial \Sigma \) for
\( \Sigma = B_3 \) whereas \( T^2 \) is \( \partial H_1 \) and also \( \partial \Sigma \) if \( \Sigma = T_3 \). We consider the integral of \( \epsilon^{(1)} \omega^{(1)}_0 \)
over \( S^2 \) and \( T^2 \) separately. The results must finally be assembled together with the correct
signs to find (3.52). (3.53) will be considered after that.

Now \( \epsilon^{(1)} \) and \( \omega^{(1)}_0 \) are closed on \( \partial(\Sigma \setminus H_1) \). The integral of \( \epsilon^{(1)} \omega^{(1)}_0 \) over \( S^2 \) or \( T^2 \) is zero
if either \( \epsilon^{(1)} \) or \( \omega^{(1)} \) is also exact. A closed one form on \( S^2 \) being exact, we thus have

\[
\int_{S^2} \epsilon^{(1)} \omega^{(1)}_0 = 0. \quad (3.55)
\]
As for $T^2$, let $\theta^1$, $\theta^2$ be our canonical choice of coordinates thereon. Then
\[ \epsilon^{(1)}|_{T^2} = \alpha d\theta^1 + \beta d\theta^2 + d\eta^0, \quad \omega_0^{(1)}|_{T^2} = \overline{\alpha}d\theta^1 + \overline{\beta}d\theta^2 + \overline{d}\eta^0, \quad \alpha, \beta, \overline{\alpha}, \overline{\beta} = \text{constants}. \] (3.56)

Here $d\eta^0$ and $\overline{d}\eta^0$ are exact. Also one of the coefficients $\overline{\alpha}, \overline{\beta}$ for $\omega_0^{(1)}$ is zero for the case in Fig. 9. This is because one of the cycles on $T^2$ [the cycle not shown on $\partial H_1$ and $\partial \Sigma$ in that figure] is homologous to a point, and a closed one form on $\Sigma \setminus H_1$ with a nonzero integral over this cycle does not exist. In any case,
\[ \int_{T^2} \epsilon^{(1)} \omega_0^{(1)} = (\alpha \overline{\beta} - \overline{\alpha}\beta) \int_{T^2} d\theta^1 d\theta^2 = 4\pi^2 (\alpha \overline{\beta} - \overline{\alpha}\beta). \] (3.57)

We next look at (3.53), again for $S^2$ and $T^2$ separately. $\epsilon^{(0)}$ is a constant on $S^2$ or $T^2$. Hence
\[ \int_{S^2 \text{ or } T^2} \omega_0^{(2)} \epsilon^{(0)} = \epsilon^{(0)}|_{S^2 \text{ or } T^2} \times \int_{S^2 \text{ or } T^2} \omega_0^{(2)}. \] (3.58)

We will now argue that the integral of $\omega_0^{(2)} \omega_0^{(1)}$ can be chosen to be zero in our examples. Let us first consider Fig. 8. In that case, we can choose $\omega_0^{(2)}$ to be $d\theta^1 d\theta^2$ and $\omega_0^{(1)}$ to be $d\theta^1$ or $d\theta^2$. Then $\omega_0^{(2)} \omega_0^{(1)}$ is zero and so is its integral.

The result seems correct in the remaining case with $\Sigma = \mathcal{B}_3$ as well. To see this, we first fill up $\Sigma \setminus H_1$ with a family of tori and introduce coordinates $\theta^1$, $\theta^2$ for the tori and a coordinate $r$ labelling the tori. Fig. 10 shows how this is done for $\Sigma = \mathcal{B}_3$. $r, \theta^1, \theta^2$ give a coordinate system for $\Sigma \setminus H_1$. $\omega_0^{(2)}$ can then be identified with $d\theta^1 d\theta^2$ and $\omega_0^{(1)}$ with $d\theta^1$ or $d\theta^2$ giving $\omega_0^{(2)} \omega_0^{(1)} = 0$. It may be remarked that a torus touches in the middle of Fig. 10 so that a cycle on this “torus” is homologically trivial. The corresponding form, say $d\theta^2$, is thus ill defined on this torus. $\omega_0^{(1)}$ is not this form, but $d\theta^1$, so that perhaps this singularity is not significant for our purposes.
iii) Interpretation of $q_0(\alpha)$ and $P_N(\alpha)$

We had previously used the integral of $B$ over a surface such as $S^2$ enclosing a hole as a measure of the charge in the hole. We had also used the integral of $A$ on a loop $C$ as a measure of the flux through $C$.

But neither of these integrals is well defined either in the classical canonical formalism or in quantum field theory. This is because integrals of fields over submanifolds are not meaningful in these contexts without regularization or interpretation.

We now argue that $q_0(\alpha)$ is the correct expression for the charge operator for charge in $\mathcal{H}_\alpha$. Similarly, $P_N(\alpha)$ is the corrected version of

$$\langle \int_{\mathcal{C}_N} A \rangle .$$  \hspace{1cm} (3.59)

Let us first examine $q_0(\alpha)$. According to (3.2) and (3.12),

$$q_0(\alpha) = \frac{1}{\sqrt{\Delta}} \langle \int_{\partial \mathcal{H}_\alpha} B - \int_{\Sigma \setminus \mathcal{H}} \xi_0 \xi_0 dB \rangle .$$  \hspace{1cm} (3.60)

The second term here is at least numerically zero in classical physics, $dB$ being numerically zero in $\Sigma \setminus \mathcal{H}$ by (2.4), whereas the first term is the measure of charge in $\mathcal{H}_\alpha$ we considered in Section 2.1. In this way, we can relate $q_0(\alpha)$ to the integral of $B$. But this relation is necessarily imprecise since $q_0(\alpha)$ is well defined whereas neither of the two terms in (3.60) has a good meaning in the canonical approach or quantum field theory.

In any case, when need arises, the first term in (3.60) will hereafter be identified with $q_0(\alpha)$.

As we saw earlier, $q_0(1)$ is canonically conjugate to $W(\omega^{(2)})$:

$$\{ < W(\omega^{(2)}) > , \ q_0(1) \} = \frac{1}{\sqrt{\Delta}} \int_{\partial \mathcal{H}} \omega^{(2)} .$$  \hspace{1cm} (3.61)

As for the interpretation of $P_N(\alpha)$, let us assume that $\Sigma = \mathcal{B}_3$ and $\partial \mathcal{H}_\alpha = T^2$. The latter has coordinates $\theta^i$ of which the $\theta^2$ cycle for fixed $\theta^1$ can be taken to be $\mathcal{C}_1$. [See Fig.
11.] With $N = 1$ and $\alpha = 1$, we have

$$P_1(1) = \langle -\int_{\Sigma \setminus H} d\omega_1^{(1)} A \rangle = \langle -\int_{\partial H_1} \omega_1^{(1)} A - \int_{\Sigma \setminus H} \omega_1^{(1)} dA \rangle. \quad (3.62)$$

The last term is numerically zero by constraint. Since $\omega_1^{(1)} |_{\partial H_1} = d\theta_2$, we can write

$$P_1(1) = \langle -\int_{\partial H_1} d\theta^2 A_1 d\theta^1 \rangle, \quad A_1 |_{\partial H_1} := A_1 d\theta^1 + A_2 d\theta^2. \quad (3.63)$$

The integral

$$\int A_1 d\theta^1 \quad (3.64)$$

over $\theta^1$ is independent of $\theta^2$, at least classically. This is because an infinitesimal deformation of the $\theta^1$ loop changes $(3.64)$ by an integral of $dA$, and that is zero by constraint. Thus $P_1(1)$ is proportional to the flux through the $\theta^1$ loop, or the flux on the vortex enclosed within $H_1$.

### 3.4. Summary of Observables and their Poisson Brackets

We have encountered the following observables associated to $\partial H_{\alpha}$:

**Local Observables**: $q_n(\alpha), p_n(\alpha), P_N(\alpha)$. \quad (3.65)

**Nonlocal Observables**: $\langle W(\omega^{(2)}) \rangle, \langle V(\omega^{(1)}) \rangle$. \quad (3.66)

Of these, $q_0(\alpha)$ is a measure of charge contained in $H_\alpha$ and $P_N(\alpha)$ a measure of magnetic flux across the surface with boundary $C_N(\alpha)$. The nonlocal observables $\langle W(\omega^{(2)}) \rangle$ and $\langle V(\omega^{(1)}) \rangle$ are conjugate to charge and magnetic flux respectively. As we saw, they have an interpretation in terms of a Wilson line integral involving $A$ and integral of $B$ over a surface.

It is enough to list the PB’s involving these observables which are not evidently zero. They are as follows.

$$\{q_n(\alpha), p_m(\beta)\} = \delta_{\alpha\beta} \delta_{nm}, \quad m \neq 0,$$

$$\{q_n(\alpha), P_N(\beta)\} = 0,$$

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\[ \{< W(\omega^{(2)}), q_n(1) >, \sqrt{\Delta} \delta_{n0} \}, \]
\[ \{< V(\omega^{(1)}), p_N(\alpha) >, 0 \}, \]
\[ \{V(\omega^{(1)}), P_N(\alpha) \} = -i \int_{\partial H} \omega^{(1)} \omega^{(1)}_N, \]
\[ \{< W(\omega^{(2)}), < V(\omega^{(1)}) >, \} = i \int_{\Sigma \setminus H_1} \omega^{(2)} \omega^{(1)}. \quad (3.67) \]

An approach to the evaluation of the last integral here has also been outlined in Section 3.3., ii).

4. Quantization, Diffeomorphisms and Vertex Operators

4.1. Quantization and Diffeomorphisms

For economy of notation, we will use the same notation for a quantum operator and its classical counterpart.

The quantum version of (3.67) is given by Dirac’s prescription and reads

\[ [q_n(\alpha), p_m(\beta)] = i \delta_{\alpha \beta} \delta_{nm}, \quad m \neq 0, \]
\[ [q_n(\alpha), P_N(\beta)] = 0, \]
\[ [< W(\omega^{(2)}), q_n(1) > = i \sqrt{\Delta} \delta_{n0}, \]
\[ [< V(\omega^{(1)}), p_N(\alpha) > = 0, \]
\[ [< V(\omega^{(1)}), P_N(\alpha) ] = -i \int_{\partial H} \omega^{(1)} \omega^{(1)}_N, \]
\[ [< W(w^{(2)}), < V(w^{(1)}) > ] = i \int_{\Sigma \setminus H_1} \omega^{(2)} \omega^{(1)}. \quad (4.1) \]

As remarked previously, the operator algebra \( \mathcal{A} \) we realize on a Hilbert space is generated by \( q_n(\alpha), p_n(\alpha) \) and \( P_N(\alpha) \). \(< W(\omega^{(2)}) > \) and \(< V(\omega^{(1)} > \) will be treated in the way that the Fubini-Veneziano field \( [11] \) is treated in string theory. Thus suitably regularized exponentials thereof will be regarded as analogues of vertex operators intertwining
distinct representations of $\mathcal{A}$. Let us therefore temporarily set aside $< W(\omega^{(2)}) >$ and $< V(\omega^{(1)}) >$.

$q_0(\alpha)$ and $P_N(\alpha)$ are in the center of $\mathcal{A}$. Their meaning in terms of the charge and fluxes for $\mathcal{H}_\alpha$ has been examined before. These operators are diagonal in a representation of $\mathcal{A}$. Thus if $|\cdot >$ is a state in this representation space,

$$q_0(\alpha)|\cdot> = e_\alpha|\cdot> .$$

$$P_N(\alpha)|\cdot> = F_N(\alpha)|\cdot> .$$

These states are also of course annihilated by the Gauss law constraints:

$$\mathcal{G}_j(\lambda^{(j)})|\cdot> = 0 \text{ where } \lambda^{(j)}|_{\partial(\Sigma \setminus \mathcal{H})} = 0, \quad j = 1, 2 .$$

The algebra $\mathcal{A}$ is the direct sum of commuting subalgebras $\mathcal{A}_\alpha$, $\mathcal{A}_\alpha$ having generators $q_n(\alpha)$, $p_n(\alpha)$ and $P_N(\alpha)$ for fixed $\alpha$:

$$\mathcal{A} = \oplus \mathcal{A}_\alpha .$$

A representation of $\mathcal{A}$ can thus be obtained by taking tensor products of states carrying representations of $\mathcal{A}_\alpha$. A state in (4.2) and (4.3) is such a product. It is thus enough to consider the representation of one $\mathcal{A}_\alpha$. Let us therefore fix the value of $\alpha$ for the present.

For $\mathcal{H}_\alpha = \mathcal{B}_3$ and $\mathcal{T}_3$, our bases for $\partial\mathcal{H}_\alpha$ which define the modes $q_n(\alpha)$, $p_n(\alpha)$ are shown in (3.15) and (3.16). The basis elements enjoy the symmetries:

$$e^{*}_{Jm} = (-1)^m e_{J-m},$$

$$e^{*}_N = e_{-N} ,$$

Let

$$q_n(\alpha), p_n(\alpha) \rightarrow q_{Jm}(\alpha), p_{Jm}(\alpha)$$

when

$$n \rightarrow Jm$$
and

\[ q_n(\alpha), p_n(\alpha) \rightarrow q_{\vec{N}}(\alpha), p_{\vec{N}}(\alpha) \quad (4.8) \]

when

\[ n \rightarrow \vec{N}. \quad (4.9) \]

The reflection of (4.5) are then the symmetries

\[ q_{Jm}(\alpha)^* = (-1)^m q_{J, -m}(\alpha), \quad p_{Jm}(\alpha)^* = (-1)^m p_{J, -m}(\alpha), \]
\[ q_{\vec{N}}(\alpha)^* = q_{-\vec{N}}(\alpha), \quad p_{\vec{N}}(\alpha)^* = p_{-\vec{N}}(\alpha). \quad (4.10) \]

Let \( \omega(\alpha) : n \rightarrow \omega_n(\alpha)(>0) \) be a frequency function invariant under the substitution

\[ n = Jm \rightarrow n^* = J, -m \]

or

\[ n = \vec{N} \rightarrow n^* = -\vec{N}. \quad (4.11) \]

The dispersion relation is otherwise left arbitrary for the moment.

We can then construct the annihilation and creation operators

\[ a_n(\alpha) = \frac{1}{\sqrt{2}}[\omega_n(\alpha) q_n(\alpha) + ip_n(\alpha)^\dagger], \]
\[ a_n^\dagger(\alpha) = \frac{1}{\sqrt{2}}[\omega_n(\alpha) q_n(\alpha)^\dagger - ip_n(\alpha)]. \quad (4.12) \]

Their only nonzero commutator is

\[ [a_n(\alpha), a_m(\alpha)^\dagger] = \omega_n(\alpha) \delta_{nm}. \quad (4.13) \]

The algebra defined by (4.13) can be realized on a Fock space in the usual way.

In our previous work, which examined the BF system on \( \Sigma \) without holes, the diffeomorphism (diffeo) group acting on \( \partial \Sigma \) was shown to be a group of classical symmetries for the BF system. A generalized Sugawara construction \([11]\) of its generators in terms of
the analogues of $q_n(\alpha)$, $p_n(\alpha)$ and $P_N(\alpha)$ was derived. It was argued that the full group of diffeos can not be implemented on quantum states. It was also shown that the group $SDiff_0(\partial \Sigma)$ of “symplectic” diffeormophisms can be implemented on quantum states provided only that the frequency function $\omega(\partial \Sigma) : n \to \omega_n(\partial \Sigma)$ appropriate to $\partial \Sigma$ was independent of $n$ and equal to a constant $\omega_0$.

Similar conclusions can be drawn with $\partial H_\alpha$ substituting for $\partial \Sigma$. Thus let $\eta = \eta^i \partial_i$ be a vector field which on $\partial H_\alpha$ is tangent to $\partial H_\alpha$ and which vanishes on all other boundaries, and consider

$$\ell(\eta; \alpha) = \int_{\Sigma \cup H} (L_\eta A) B. \tag{4.14}$$

$\ell(\eta; \alpha)$ generates the infinitesimal diffeo on $\partial H_\alpha$ for the vector field $\eta|_{\partial H_\alpha}$, the latter being the restriction of $\eta$ to $\partial H_\alpha$. The PB’s of $\ell(\eta; \alpha)$ with $G_j(\lambda^{(j)})$ are weakly zero so that $\ell(\eta; \alpha)$ leads to an observable. It is also the case that $\ell(\eta; \alpha) \approx \ell(\eta + \Delta \eta; \alpha)$ if $\Delta \eta|_{\partial (\Sigma \cup H)} = 0$.

The equivalence class $< \ell(\eta; \alpha) >$ of all $\ell$’s weakly equal to $\ell(\eta; \alpha)$ can then be thought of as the observable generating the infinitesimal diffeo of the vector field $\eta|_{\partial H_\alpha}$ on $\partial \Sigma$.

The rest of the analysis of $< \ell(\eta; \alpha) >$ including its mode decomposition and quantization follows [5] with conclusions indicated above.

In [3], it was shown that the modes localized at $\partial \Sigma$ can be described by a scalar field theory. Following that work, it is easy to show that the modes localized at $\partial H_\alpha$ can also be described by a scalar field theory.

4.2. Vertex Operators

i) Vertex Operator for Charge Creation

Suppose that there is charge $e$ at $z_\alpha$ and we want to find an operator for creating a state for this charge from one with zero charge. Such an operator is the vertex operator for charge $e$.

This vertex operator creates not just charge $e$ in $H_\alpha$, but also charge $-e$ elsewhere. This is true for the $BF$ system just as it is so for Chern-Simons dynamics [3]. Let us
assume that the charge \(-e\) is created at a point \(P\) on \(\partial \Sigma\).

We can proceed as follows to construct the state with these charges \(e\) and \(-e\). Enclosing \(z_\alpha\), we have the associated hole \(\mathcal{H}_\alpha\), it being understood that \(\mathcal{H}_\alpha\) will be shrunk to \(z_\alpha\) at the end of the construction. Let us first assume that there is no charge in \(\mathcal{H}_\alpha\). We have thus just punched a hole at the point \(z_\alpha\) where we want to create the charge.

This hole \(\mathcal{H}_\alpha\), topologically a ball, has a boundary \(\partial \mathcal{H}_\alpha\) and an associated Fock space of states \(|\cdot ; z_\alpha\rangle\) localized there and carrying zero charge:

\[
q_0(\alpha)|\cdot ; z_\alpha\rangle = 0. \tag{4.15}
\]

Following earlier work on the Chern-Simons system [20], we can interpret these states as describing spin fluctuations localized at \(z_\alpha\) without a corresponding charge excitation.

The manifold \(\Sigma \setminus \mathcal{H}\) in general has several connected boundaries \(\partial \mathcal{H}_\beta, \partial \Sigma\) and states localized thereon. A physical state is a tensor product constructed using one state from each connected boundary. Let us denote it by \(|\cdot \rangle\). Let us also in particular denote any physical state which describes the Fock vacuum at \(z_\alpha\) by \(|0\rangle\). In this notation, the dependence of \(|0\rangle\) on \(z_\alpha\) has not been displayed. In what follows, for specificity, we concentrate on creating a charged state from \(|0\rangle\).

Next consider the Wilson line from the point \(P\) on \(\partial \Sigma\) to \(z_\alpha\), the integral being along a line \(L\):

\[
w(z_\alpha) = \exp ie \int_P^{z_\alpha} A. \tag{4.16}
\]

Its response

\[
w(z_\alpha) \to \exp[ie \xi_0(z_\alpha)]w(z) \exp[-ie \xi_0(P)] \tag{4.17}
\]

to the gauge transformation \(A \to A + d \xi_0\) shows that it creates a state of charge \(e\) at \(z_\alpha\):

\[
\sqrt{\Delta} \ q_0(\alpha)w(z_\alpha)|0\rangle = ew(z_\alpha)|0\rangle . \tag{4.18}
\]

[It also creates charge \(-e\) at \(P\).] If the tangent to \(L\) points in the direction of the unit vector \(\vec{n}\) from \(z_\alpha\), then the spin of this charge is localized for this state in direction \(\vec{n}\).
Now for reasons mentioned earlier in Section 3.3 iii), the operator \( w(z_\alpha) \) is not well defined, as it involves a quantum field on a line. Using the result (3.26), we are thus led to consider

\[
\exp i e W(\omega^{(2)})
\]

where \( \omega^{(2)} \) has support on \( L \). This \( \omega^{(2)} \) is the same as the \( \omega^{(2)} \) of Section 3.3 i) in the limit \( \delta \to 0 \) and with the normalization constant \( \bar{\lambda} \) in (3.27) equal to 1. As in Section 3.3, we will also assume for simplicity that there is only one hole \( \mathcal{H}_\alpha \) for now, although a little later, we will have occasion to briefly comment on the situation with several holes.

A few preliminary remarks are in order before studying (4.19) further. For a general topology, it is not correct to say \( \omega^{(2)}|_{\partial(\Sigma \setminus \mathcal{H})} \) determines \( W(\omega^{(2)}) \) upto weak equivalence. \([\omega^{(2)} \text{ need not be } \omega^{(2)}].\) For suppose that the closed two forms \( \omega^{(2)} \) and \( \omega_0^{(2)} \) have both the same boundary values:

\[
\omega^{(2)}|_{\partial(\Sigma \setminus \mathcal{H})} = \omega_0^{(2)}|_{\partial(\Sigma \setminus \mathcal{H})}.
\]

Then by the argument following (3.47),

\[
\omega^{(2)} - \omega_0^{(2)} = d\epsilon^{(1)},
\]

\[
d\epsilon^{(1)}|_{\partial(\Sigma \setminus \mathcal{H})} = 0,
\]

or

\[
\epsilon^{(1)}|_{\partial\Sigma}, \epsilon^{(1)}|_{\partial\mathcal{H}_\alpha} = \text{closed one forms}.
\]

Suppose next that there is another one form \( \pi^{(1)} \) with the same boundary value as \( \epsilon^{(1)} \). Then

\[
\int d(\epsilon^{(1)} - \pi^{(1)}) A = \int (\epsilon^{(1)} - \pi^{(1)}) dA \approx 0.
\]

Thus upto weak equivalence, \( W(\omega^{(2)} - \omega_0^{(2)}) \) is not always zero, but is instead entirely determined by \( \epsilon^{(1)}|_{\partial(\Sigma \setminus \mathcal{H})} \). Given this boundary value, we are at liberty to chose its extension to \( \Sigma \setminus \mathcal{H} \) in determining the equivalence class \(<W(\omega^{(2)} - \omega_0^{(2)})>\).
Hence if there exists a closed one form in all of $\Sigma \setminus \mathcal{H}$ such that its pull back to $\partial(\Sigma \setminus \mathcal{H})$ is $\epsilon^{(1)}|_{\partial(\Sigma \setminus \mathcal{H})}$, then $W(\omega^{(2)} - \omega_0^{(2)}) \approx 0$. This is for example the case if $\epsilon^{(1)}|_{\partial(\Sigma \setminus \mathcal{H})}$ is exact, so that it can be written as $d\eta^{(0)}|_{\partial(\Sigma \setminus \mathcal{H})}$. We can then extend $\eta^{(0)}|_{\partial(\Sigma \setminus \mathcal{H})}$ to a function $\eta^{(0)}$ in $\Sigma \setminus \mathcal{H}$ and define the closed form extension of $\epsilon^{(1)}|_{\partial(\Sigma \setminus \mathcal{H})}$ to $\Sigma \setminus \mathcal{H}$ to be $d\eta^{(0)}$. This is also the case if for $\epsilon^{(1)}$, we can use a closed but not exact form of Section 3.3 ii).

These considerations show that $W(\omega^{(2)} - \omega_0^{(2)})$ is a linear combination of observables of type $P_N(\beta)$ and similar observables $P_N(\partial\Sigma)$ for $\partial\Sigma$:

$$W(\omega^{(2)}) - W(\omega_0^{(2)}) = \sum_{N,\beta} C_N(\beta) P_N(\beta) + \sum_N C_N(\partial\Sigma) P_N(\partial\Sigma), \quad C_N(\beta), C_N(\partial\Sigma) = \text{Constants}.$$ (4.23)

We can see (4.23) in another way for those $\omega^{(2)}$ with support on a line $L$. In that case, the statement that $< W(\omega^{(2)}) >$ is determined by $\omega^{(2)}|_{\partial(\Sigma \setminus \mathcal{H})}$ is equivalent to the statement that it is determined by the end points of $L$, and in addition by its tangents there when they are at the inner boundaries $\partial\mathcal{H}_\alpha$. We can see that this statement may be incorrect in the following way. In Fig. 12 (a,b), we show examples of $L$ with these same characteristics, but which differ by noncontractible loops. The corresponding $W$’s hence differ (upto constants) by loop integrals of $A$, and the latter need not vanish. Classically they may admit an interpretation in terms of fluxes enclosed by the loops.

The uncertainties in $W(\omega^{(2)})$ as represented by (4.23) do not affect its commutation relations. The states created by (4.19) differ only by phases as a result of these uncertainties. Therefore, to simplify matters, let us assume for the present that $\Sigma$ and $\mathcal{H}$ are balls so that this phase is absent. Note that this $\mathcal{H}$ is now also our $\mathcal{H}_\alpha$.

Paranthetically, it may be remarked that these phases are important in determining the statistical and other features of quantum states which depend on the fundamental group of the configuration space. In Fig. 12(b) for example, the dotted loop can be interpreted as describing the transport of a charge in a noncontractible loop around a vortex. Section 5 will further address this sort of issues.
We now return to (4.19). The specification of its action on \(|0>\) requires the Fourier decomposition of \(W(\omega^{(2)})\). For now, we want to concentrate on modes at \(\partial H_\alpha\). It is then best to change \(\omega^{(2)}\) to another closed two form \(\Omega^{(2)}\), such that \(W(\Omega^{(2)})\) commutes with all the observables localized at \(\partial \Sigma\), except for the \(\partial \Sigma\) charge operator \(q_0(\partial \Sigma)\). [We will return to \(\omega^{(2)}\) later.] The method to achieve this objective is shown by (3.29), (3.30) and (3.31).

Following those equations, we set

\[
\Omega^{(2)}|_{\partial \Sigma} = -\frac{1}{\Delta} \mu
\]

(4.24)

[The minus sign arises from considerations involving orientation. For example, the integral of \(\omega^{(2)}\) over \(\partial H_\alpha \cup \partial \Sigma\) with positive orientations must vanish by Stokes’s theorem.] For this choice, \(W(\Omega^{(2)})\) describes a mode at \(\partial \Sigma\) which is conjugate to \(q_0(\partial \Sigma)\) and commuting with the remaining local observables at \(\partial \Sigma\).

We will continue to assume that the support of \(\Omega^{(2)}\) in a neighbourhood of \(z_\alpha\) is on \(L\).

With this choice of \(\Omega^{(2)}\), we have deviated from an \(\omega^{(2)}\) with \(L\) as support in all of \(\Sigma \setminus \mathcal{H}\). But earlier work \([9]\) shows that \(\Omega^{(2)}\) is the one most appropriate for generalizing the Fubini-Veneziano vertex operator \([11]\). We will return to the consideration of \(\omega^{(2)}\) later.

Since all \(W(\omega^{(2)})\) with the same boundary values for \(\omega^{(2)}\) are weakly equal for our chosen topology, the action of \(e^{i\omega W(\omega^{(2)})}\) on a state is fixed by \(\omega^{(2)}|_{\partial (\Sigma \setminus \mathcal{H})}\). What remains thus for the specification of the action of \(W(\Omega^{(2)})\) on a physical state is the display of its dependence on the modes localized at \(\partial \mathcal{H}_\alpha\). For this purpose, let us first define the closed two form \(\overline{\Omega}^{(2)}\) by

\[
\overline{\Omega}^{(2)}|_{\partial \Sigma} = -\frac{\mu}{\Delta},
\]

\[
\overline{\Omega}^{(2)}|_{\partial \mathcal{H}_\alpha} = \frac{\mu}{\Delta},
\]

(4.25)

the two \(\mu\)'s being the chosen volume forms on \(\partial \Sigma\) and \(\partial \mathcal{H}_\alpha\). [They are not of course equal.]
Also the details regarding \( \Omega^{(2)} \) away from the boundaries are not important here. Then

\[
(\Omega^{(2)} - \overline{\Omega}^{(2)})|_{\partial \Sigma} = 0, \tag{4.26}
\]

\[
(\Omega^{(2)} - \overline{\Omega}^{(2)})|_{\partial \mathcal{H}_\alpha} = d\xi^{(1)}(\alpha)|_{\partial \mathcal{H}_\alpha}, \tag{4.27}
\]

following from the observation that the integral of its left hand side over \( \partial \mathcal{H}_\alpha \) is zero.

After noting that there is no observable of type \( P_N(\alpha) \) for \( \mathcal{H}_\alpha = \mathcal{B}_3 \), we can expand \( d\xi^{(1)}(\alpha)|_{\partial \mathcal{H}_\alpha} \) in a series of \( e^*_N \mu \):

\[
d\xi^{(1)}(\alpha)|_{\partial \mathcal{H}_\alpha} = \sum_N a_N e^*_N \mu. \tag{4.28}
\]

The Fourier coefficients are given by

\[
a_M = \int_{\partial \mathcal{H}_\alpha} e_M (\Omega^{(2)} - \overline{\Omega}^{(2)}) . \tag{4.29}
\]

For \( M = 0 \), this gives

\[
a_0 = 0. \tag{4.30}
\]

For \( M \neq 0 \), we find instead,

\[
a_M = \int_{S^2} e_M \Omega^{(2)}. \tag{4.31}
\]

Let us introduce the standard polar coordinates on \( S^2 \) and let \( \Omega^{(2)} \) have support at \( \theta_0, \phi_0 \). Then

\[
\Omega^{(2)}(\theta, \phi)|_{S^2} = \delta(\cos \theta - \cos \theta_0) \delta(\phi - \phi_0) d \cos \theta d\phi. \tag{4.32}
\]

Using the correspondence \( M \rightarrow Jm, \ e_M \rightarrow \left(\frac{4\pi}{\Delta}\right)^{1/2} Y_{Jm} \), we have the following complete list of Fourier coefficients:

\[
a_{00} = 0, \ a_{Jm} = \left(\frac{4\pi}{\Delta}\right)^{1/2} Y_{Jm}(\theta_0, \phi_0), \text{ for } J \neq 0. \tag{4.33}
\]

In this way, we find the mode decomposition

\[
< W(\Omega^{(2)} - \overline{\Omega}^{(2)}) > = -\left(\frac{4\pi}{\Delta}\right)^{1/2} \sum_{J,m; J \neq 0} Y_{Jm}(\theta_0, \phi_0) p_{Jm}(\alpha) \tag{4.34}
\]
or
\[
< W(\Omega(2)) > = < W(\Omega'(2)) > - \left( \frac{4\pi}{\Delta} \right)^{1/2} \sum_{J,m;J \neq 0} Y_{Jm}(\theta_0, \phi_0) p_{Jm}(\alpha) .
\] (4.35)

In this expansion, all but the first term are localized at \( \partial \mathcal{H}_\alpha \) while the first term is conjugate to charge at both \( \partial \mathcal{H}_\alpha \) and \( \partial \Sigma \).

The vertex operator for creation of charge \( e \) at \( z_\alpha \) can now be defined. It is not quite (4.19) with \( \Omega(2) \) for \( \omega(2) \), but is its normal ordered form as in string theory:
\[
\mathcal{W}(\Omega(2)) = : e^{i e < W(\Omega(2)) >} :
\] (4.36)

The normal ordering is defined here by using the creation-annihilation operators of Section 4.1.

The Wilson line creates localized charge at both \( \partial \mathcal{H}_\alpha \) and \( \partial \Sigma \). It is thus associated to \( W(\overline{\omega}(2)), \overline{\omega}(2) \) being supported on \( L \). We must thus examine the mode expansion of \( W(\overline{\omega}(2) - \Omega(2)) \). Since \( \overline{\omega}(2) - \Omega(2) |_{\partial \Sigma} \neq 0 \), the expansion has an additional series of terms localized at \( \partial \Sigma \), similar to the last group of terms in (4.35). Let \( q_{Jm}(\partial \Sigma), p_{JM}(\partial \Sigma) \) be the modes localized at \( \partial \Sigma \) which are the counterparts of \( q_{Jm}(\alpha), p_{JM}(\alpha) \) and let \( \Delta(\partial \Sigma) \) be the area of \( \partial \Sigma \) defined as in (3.8). Then we find
\[
< W(\overline{\omega}(2)) > = < W(\Omega'(2)) > - \left( \frac{4\pi}{\Delta} \right)^{1/2} \sum_{J,m;J \neq 0} Y_{Jm}(\theta_0, \phi_0) p_{Jm}(\alpha)
+ \left( \frac{4\pi}{\Delta} \right)^{1/2} \sum_{J,m;J \neq 0} Y_{Jm}(\theta'_0, \phi'_0) p_{Jm}(\partial \Sigma) .
\] (4.37)

Here, we have introduced polar coordinates \( \theta', \phi' \) on \( \partial \Sigma \) and \( \theta'_0, \phi'_0 \) is the point where \( L \) joins \( \partial \Sigma \).

We thus have the result
\[
\text{Regularized Wilson line} = : e^{i e W(\overline{\omega}(2))} :
\] (4.38)

The construction leading to (4.36) and (4.38) has relied on particular choices of \( \Sigma \) and \( \mathcal{H} \). For a more general situation, we can proceed as follows. Given a direction at \( z_\alpha \) and
a point $P$ on $\partial \Sigma$, we first choose a particular line $L$ from $P$ to $z_\alpha$ with its tangent at $z_\alpha$ being in the given direction. We then define the mode analysis following what we did above, assuming that the $P_N(\alpha)$ terms are absent, say. [The basis functions $e^{(\alpha)}_m$ should of course be appropriate for the topologies of $\partial \Sigma$ and $\partial H_\alpha$. Also there is no way to tell whether or not there are such terms for a given $L$, their choice being a convention.] The regularized Wilson integral or the vertex operator can then be constructed. If there is another line $L'$ and $L'$ also originates at $P$ and ends at $z_\alpha$ with the same direction of tangent, then $L$ can be smoothly changed to $L'$ keeping its end at $P$ fixed, but changing its other end smoothly. The vertex operator for $L'$ can then be expressed as the one for $L$ plus factors involving $P_N(\beta)$. [Cf. (4.23).] This adiabatic transport also relates the corresponding states they create from $|0\rangle$. Note that although the $P_N(\alpha)$ factors for one $L$ is a matter of choice, the additional such factors created when $L$ is changed $L'$ can be determined and has an intrinsic meaning.

The classical configuration of charges and vortices is specified by their locations, charges, fluxes and spin directions. $L$ and $L'$ are thus both associated with the same point of this configuration space $Q$, and they give states which differ by a phase. The smooth deformation of $L$ to $L'$ corresponds to parallel transport of quantum states in a loop in $Q$. The phase above is thus the holonomy for this loop, and the quantum states are really to be thought of as sections of vector bundles over $Q$.

**ii) Vertex Operator for Vortex Creation**

Let us suppose that there is a magnetic vortex along a loop $C_\alpha$ which we assume for simplicity to be an unknot. Our task is to find an operator which creates this loop just as (4.36) creates charge.

We can follow Section 4.2. i) in order to define this operator. Thus we begin with a manifold where no magnetic vortex is present at $C_\alpha$ and then punch a hole $H_\alpha$ enclosing $C_\alpha$. $H_\alpha$ is a solid torus and it is eventually to be shrunk to $C_\alpha$. 

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The hole $\mathcal{H}_\alpha$ has boundary $\partial \mathcal{H}_\alpha$ and Fock states $|\cdot;\mathcal{C}_\alpha>$ localized there. They are states which carry zero magnetic flux. Just as for the charge problem, they describe spin fluctuations which are not associated with a flux excitation. Let $|0;\mathcal{C}_\alpha>$ be the vacuum state at $\partial \mathcal{H}_\alpha$.

The manifold $\Sigma \setminus \mathcal{H}$ in general has many boundaries and there is a family of Fock states localized on each of its connected components. A physical state is a tensor product formed of these states as described in Section 4.2.i). As before, let us call a physical state, with the vacuum at $\partial \mathcal{H}_\alpha$ as a factor, as $|0>$, suppressing its dependence on $\mathcal{C}_\alpha$.

Let $S_\alpha$ be a surface with $\mathcal{C}_\alpha$ as boundary as in Fig. 13 and consider

$$e^{-i\Phi_\alpha \int_{S_\alpha} B} |0> .$$

(4.39)

The operator which measures flux on $\mathcal{C}_\alpha$ is the integral

$$\int_{L_\alpha} A$$

(4.40)

along the loop $L_\alpha$ of Fig. 13. As $A$ and $B$ are conjugate operators, we find that the state (4.39) describes a vortex of flux $\Phi_\alpha$:

$$\left( \int_{L_\alpha} A \right) \left( e^{-i\Phi_\alpha \int_{S_\alpha} B} \right) |0> = \Phi_\alpha e^{-i\Phi_\alpha \int_{S_\alpha} B} |0> .$$

(4.41)

Just as (4.16), the operator in (4.39) is not well defined. We are thus led to consider the exponential constructed from

$$V(\bar{\omega}^{(1)}) = \int \bar{\omega}^{(1)} B ,$$

(4.42)

$\bar{\omega}^{(1)}$ being a closed one form with support on $S_\alpha$ and

$$\int_{L_\alpha} \bar{\omega}^{(1)} = 1 .$$

(4.43)

Our next task is to examine the dependence of (4.42) on the choice of $S_\alpha$ for a given $\mathcal{C}_\alpha$. For continuous deformations of $S_\alpha$ with tangent directions to $S_\alpha$ at $\mathcal{C}_\alpha$ held fixed,
$V(\mathfrak{w}^{(1)})$ changes only by constraints as an application of Stokes’ theorem shows. But there are in general surfaces $S_\alpha$ and $S'_\alpha$ which are not mutually homotopic and which have same boundaries and tangent directions there. An example is shown in Fig. 14. The two surfaces in this figure give $V(\mathfrak{w}^{(1)})$ differing by a term proportional to the charge operator $q_0(\beta)$ and this term need not be zero if the $\partial \mathcal{H}_\beta$ state has nonzero charge.

We can understand such ambiguities in general by considering $V(\omega^{(1)})$ and $V(\omega_0^{(1)})$ where the closed forms here need not be supported on surfaces. The only condition we will impose is that their pull backs on boundaries agree:

$$\omega^{(1)}|_{\partial (\Sigma \setminus \mathcal{H})} = \omega_0^{(1)}|_{\partial (\Sigma \setminus \mathcal{H})}. \quad (4.44)$$

Our task is to determine the nature of $\omega^{(1)} - \omega_0^{(1)}$ given (4.44).

Using what follows (3.47), we can conclude from (4.44) that

$$\omega^{(1)} - \omega_0^{(1)} = d\epsilon^{(0)},$$

$$d\epsilon^{(0)}|_{\partial (\Sigma \setminus \mathcal{H})} = 0,$$

or

$$\epsilon^{(0)}|_{\partial \Sigma}, \quad \epsilon^{(0)}|_{\partial \mathcal{H}_\alpha} = \text{constant functions}. \quad (4.45)$$

Now if $\epsilon^{(0)}$ and $\bar{\epsilon}^{(0)}$ have the same boundary value, then

$$\int d(\epsilon^{(0)} - \bar{\epsilon}^{(0)})B \approx 0. \quad (4.46)$$

Hence, upto weak equivalence, $V(\omega^{(1)} - \omega_0^{(1)})$ is determined by $\epsilon^{(0)}|_{\partial (\Sigma \setminus \mathcal{H})}$. Given this boundary value, we are at liberty to choose any one of its extensions in determining the equivalence class $< V(\omega^{(1)} - \omega_0^{(1)}) >$.

Since $\epsilon^{(0)}$ is a constant on each connected component of the boundary, it follows that

$$< V(\omega^{(1)} - \omega_0^{(1)}) > = \sum_{\beta} D(\beta)q_0(\beta) + D(\partial \Sigma)q_0(\partial \Sigma),$$

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\[ D(\beta) = \sqrt{\Delta \epsilon(0)}|_{\partial \mathcal{H}_\beta}, \]
\[ D(\partial \Sigma) = \sqrt{\Delta \epsilon(0)}|_{\partial \Sigma}. \]  
(4.47)

(4.47) is similar to (4.23). Just as in that case, such extra terms \( < V(\omega^{(1)} - \omega_0^{(1)}) > \) in \( V \) do not affect its commutation relations with local observables, \( q_0(\beta), q_0(\partial \Sigma) \) being super-selected operators. Nevertheless they are important in determining adiabatic transport properties of states, just as in Section 4.2 i).

Our next task is the mode decomposition of \( V(\vec{\omega}^{(1)}) \). For this purpose, given a \( \mathcal{C}_\alpha \) and a field of directions at \( \mathcal{C}_\alpha \), we first choose an \( S_\alpha \) with \( \partial S_\alpha = \mathcal{C}_\alpha \) and with its tangents at \( \mathcal{C}_\alpha \) pointing in the given directions. For this \( S_\alpha \), we then arbitrarily assume that we can ignore charge terms like those in (4.47), this assumption amounting to the choice of a phase convention just as in the analogous situation in Section 4.2 i). The Fourier analysis of \( < V(\vec{\omega}^{(1)}) > \) can then be accomplished by Fourier analysing \( \vec{\omega}^{(1)}|_{\partial \mathcal{H}_\alpha} \) using the differentials \( d\vec{e}_{\vec{N}} \) of our chosen basis.

Let us choose coordinates \( \theta^i \) on \( \partial \mathcal{H}_\alpha = T^2 \) as indicated in Fig. 11. Then \( \vec{\omega}^{(1)}|_{\partial \mathcal{H}_\alpha} \) is supported on a loop \( S_\alpha \cap \partial \mathcal{H}_\alpha \) with coordinates \( (\theta^1(\theta^2), \theta^2) \). [ Here we are assuming for clarity that \( \mathcal{H}_\alpha \) has a finite cross section, although finally we must let it become a point.]

For reasons we will see later, we can not think of a neat way of explicitly displaying the mode analysis for a general dependence of \( \theta^1 \) on \( \theta^2 \). Let us therefore assume that \( \theta^1 \) has the same value \( \theta^1_0 \) for all \( \theta^2 \):
\[ \theta^1(\theta^2) = \theta^1_0. \]  
(4.48)

Let \( \Omega^{(1)} \) be a closed one form which has the following properties:
\[ \Omega^{(1)}|_{\partial \mathcal{H}_\alpha} = \frac{d\theta^1}{2\pi}, \]
\[ \Omega^{(1)}|_{\Sigma \setminus \overline{\mathcal{H}_\alpha}} = \overline{\omega}^{(1)}|_{\Sigma \setminus \overline{\mathcal{H}_\alpha}}. \]  
(4.49)

Here \( \overline{\mathcal{H}_\alpha} \) is a solid torus enclosing \( \mathcal{H}_\alpha \) (and no other holes) as in Fig. 15. Such an \( \Omega^{(1)} \) is
readily seen to exist. Now since

$$\left(\omega^{(1)} - \Omega^{(1)}\right)|_{\partial H_\beta} \text{ or } \partial \Sigma = 0, \quad \beta \neq \alpha,$$

$$V\left(\omega^{(1)} - \Omega^{(1)}\right) \text{ depends up to weak equivalence [and charge terms which will be set equal to zero] only on } (\omega^{(1)} - \Omega^{(1)})|_{\partial H_\alpha}. $$

The mode expansion of $$(\omega^{(1)} - \Omega^{(1)})|_{\partial H_\alpha}$$ can be accomplished by first writing

$$\left(\omega^{(1)} - \Omega^{(1)}\right)|_{\partial H_\alpha} = d\xi^{(0)}(\alpha).$$

We then expand $\xi^{(0)}(\alpha)$ assuming that its constant mode is absent:

$$\xi^{(0)}(\alpha) = \sum_{\vec{N} \neq 0} b_{\vec{N}} e_{\vec{N}}.$$

Hence

$$d\xi^{(0)}(\alpha) = i \sum_{\vec{N} \neq 0} b_{\vec{N}} e_{\vec{N}} N_j d\theta^j.$$

This is also equal to

$$(\omega^{(1)} - \Omega^{(1)})|_{\partial H_\alpha} = \delta(\theta^1 - \theta^1_0) d\theta^1 - \frac{d\theta^1}{2\pi}.$$

Therefore

$$b_{\vec{N}} = 0 \text{ for } N_2 \neq 0,$$

$$b_{N_1,0} = -i \frac{\sqrt{\Delta}}{2\pi N_1} e^{-iN_1\theta^1_0} \text{ for } N_1 \neq 0.$$

We thus find,

$$\langle V(\omega^1) \rangle = \langle V(\Omega^{(1)}) \rangle = -i \frac{\sqrt{\Delta}}{2\pi} \sum_{N_1 \neq 0} \frac{e^{-iN_1\theta^1_0}}{N_1} q_{N_1,0}(\alpha).$$

In this expansion, all but the first term are localized at $\partial H_\alpha$, whereas the first term can change by charge terms if the surface $S_\alpha$ entering its definition is changed to another homotopically different surface.
The vertex operator for the creation of a vortex is

\[ V(\omega^{(1)}) := e^{-i\Phi_{\alpha}\langle V(\omega^{(1)}) \rangle} , \]  

the creation and annihilation operators of Section 4.1 being used to define the normal ordering.

If the vortex were more general than what (4.48) describes and has coordinates \((\theta^1, \theta^2)\), the clumsiness in analysis will occur when we try to compute \(b_{\vec{N}}, \theta_0\) in (4.54) having to be replaced by \(\theta^1(\theta^2)\). Although integral formulae for \(b_{\vec{N}}\) can be readily written down, they can not always be neatly evaluated.

Remarks similar to those in Section 4.2 i) leading upto the assertion that states are sections of vector bundles apply with equal force here as well.

5. Spin and Statistics, Aharonov-Bohm Interactions

5.1. Preliminares

If \(\sigma\) is the operator for the exchange of two identical constituents of a system, and \(R_{2\pi}\) is the operator for the \(2\pi\) rotation of one of the constituents, the spin-statistics theorem asserts that these two are identical operators on quantum states:

\[ \sigma = R_{2\pi} . \]  

For extended systems such as a vortex [2], or for dynamics supported on an underlying manifold with nontrivial connectivity [21], there are in general several distinct ways of performing the exchange. The exchange operator \(\sigma\) in (5.1) then corresponds to the adiabatic transport of the constituents when they are confined in a contractible open set (say the interior of a ball \(\mathcal{F}_3\)) of the underlying manifold. The open set is assumed to contain only these constituents, and certain nontrivial motions available for extended systems are also excluded.
In Section 5.2, we will discuss how (5.1) can be proved for charges and vortices in the $BF$ system, explaining at the same time the particular exchange transport which enters (5.1). In Section 5.3, we then establish a genuinely new spin-statistics theorem for vortices which shows the identity of two operators on quantum states. The first is the operator $\hat{\sigma}$ of interchange $[3, 13]$, the latter involving an exchange transport of vortices which is not the same as the one for $\sigma$. It is explained by figures in Section 5.3. As for the second, let the loop $C$ be the location of the vortex in $\Sigma$. For a state created by a vertex operator such as (4.57), we can then associate a (spin) direction to each point $p$ of $C$. Let $\hat{R}_{2\pi}$ denote the $2\pi$ rotation of all these directions around their $p$’s, the axis of rotation being the tangent to $C$ at $p$. [It is also illustrated in Section 5.3.] Let us call this operator as “internal $2\pi$ rotation.” The result we find is then the identity of $\hat{\sigma}$ and $\hat{R}_{2\pi}$:

$$\hat{\sigma} = \hat{R}_{2\pi}. \quad (5.2)$$

There is one remarkable feature of these theorems which merits emphasis here. The possibility of creation-annihilation processes was an important ingredient in certain earlier work on spin and statistics including our own $[22, 19]$. In contrast, the proofs constructed here do not seem to use creation-annihilation processes, at least in any manifest way. Further understanding of these apparently distinct proofs is thus indicated.

In Section 5.4 we briefly consider what may be called the Aharonov-Bohm interaction of a charge and a vortex. The interaction phase comes about when the charge is transported in a loop enclosing the vortex flux, much as in the usual Aharonov-Bohm effect. This short discussion is included here because the spin-statistics theorems too are associated with transports of charges and vortices.

The discussion in this Section will assume for simplicity that the charges and vortices have sharp spin states, that is that they are created at $\partial H_\alpha$ by vertex operators like (1.30) and (4.57).
5.2. The Standard Spin-Statistics Theorem

i) Identical Charges

Let us suppose that the charges are located at positions $z_1$ and $z_2$ in $\Sigma$ and enclosed by small balls $H_1$ and $H_2$, their radii eventually becoming zero. There may also be other charges and vortices in $\Sigma$ in addition to these charges.

Without loss of generality, we can assume that $z_1$ and $z_2$ are located in the interior of a ball $B_3$ which excludes all other charges and vortices, and does not touch $\partial \Sigma$, as shown in Fig. 16. The state of these two charges with identical internal states can then be written as

$$\left(:\exp i e \int_{L_1} A :\right)\left(:\exp i e \int_{L_2} A :\right)|0>$$

(5.3)

the tangents to $L_i$ at $z_i$ being parallel. [This parallelism of vectors and hence identity of internal states at distinct points can be defined as follows. We first fix a flat metric in the interior of $\Sigma$. We then use its connection to parallel transport vectors. If directions of two vectors are related by this parallel transport, we declare them to be parallel.]

The Wilson integrals of (5.3) are the regularized Wilson integrals of Section 4.2 i). [Cf. (4.38).] The state (5.3) also contains “image” charges at $z_i'$ (shown in Fig. 16) which for convenience we assume are located in the interior of $\Sigma$. The positions and internal states of these image charges will be held fixed throughout the considerations below.

The spin-statistics theorem can now be proved following [8]. If (5.3) is represented by Fig. 16, then the theorem here is the identity of Fig. 17. [In Fig. 17, $\Sigma$ and holes except those of $z_i$ and $z_i'$ are omitted.]

For completeness, we next show that the $2\pi$ rotation of the charge at $z_2$ is actually trivial. In other words, $R_{2\pi}$ and hence $\sigma$ are unit operators, and the charges are integral spin (or tensorial) bosons. This result is not surprising since according to the remarks in Section 4.1, these charges are described by scalar fields.

The proof that $R_{2\pi} = 1$ is accomplished by continuously changing $L_2'$ to the configura-
tion $L''_2$ of Fig. 18 without ever changing its end points or tangents there, or touching $\mathcal{H}_2$. This can be done by first lifting and then sliding the interior of $L_2$ Fig. 18, the dotted portion there lying well above $\mathcal{H}_2$ and then shifting the lifted portion so as to get $L''_2$. 
ii) Identical Vortices

The vortices are assumed to occupy loops $C_1$ and $C_2$ and to be enclosed in a ball $\overline{B}_3$ which contains no other charge or vortex and which does not touch $\partial \Sigma$. They are to have identical internal “spin states” created by regularized vertex operators of type (4.57). The identity of vortices is defined here as follows. We choose a flat metric and its connection within $\Sigma$ as in Section 5.2 i) above. Now there are several ways we can continuously being illustrated in Fig. 19.

Suppose we can find one such special motion with the following property: A point $p_2$ of $C_2$ for any of these motions traces a curve $L$ and ends up at a point $p_1$ of $C_1$. Consider the parallel transport of the internal vector at $p_2$ along $L$ to $p_1$. Then for this special motion, this vector at $p_1$ must be parallel to the internal vector of the $C_1$ vortex at $p_1$. If one such special motion can be found, we will say that these two vortices have identical internal states.

It is convenient to display the two disks (assumed circular for the chosen metric) by taking their sections along the equator. Fig. 19(b) represents Fig. 19(a) in this fashion.

In Fig. 19(a), the shaded regions are the surfaces over which $B$ is integrated while constructing the vertex operators.

In Fig. 20, we have displayed a path for the adiabatic exchange of two identical loops. The final state is exactly the same as the initial state. Therefore, defining $\sigma$ to be the operator producing this final state from the initial state, we find

$$\sigma = 1.$$  \hspace{1cm} (5.4)

These loops are thus bosons.

Figure 21 shows the path for $2\pi$ rotation of a vortex. $R_{2\pi}$ is the operator producing the final state in this figure from the initial state. But clearly the final and initial states are identical, and therefore

$$R_{2\pi} = 1$$  \hspace{1cm} (5.5)
Thus the vortex has integral spin or tensorial states. Also the spin-statistics theorem (5.1) is satisfied.
5.3. A New Spin-Statistics Theorem: Interchange=Internal $2\pi$ Rotation

Interchange \cite{3} is the operation of exchanging identical vortices where one vortex is first taken through the middle of the other vortex. Representing a vortex by a loop, this motion is illustrated in Fig. 22. $\hat{\sigma}$ is the operator producing the final state here from the initial state.

The internal $2\pi$ rotation describes $2\pi$ rotation of the spin directions at every point $p$ of the vortex around the tangent to vortex at $p$. The history of a spin direction at a particular point $p$ is illustrated in Fig. 23.

Figure 24 shows the interchange on a two-vortex state in detail including the way we can distort the surfaces as this “adiabatic” process is being performed. The interior of the surfaces should not touch vortex locations, and for this reason, the surfaces must necessarily be deformed during the process. In the passage from (d) to (e), we have distorted a surface in the direction of the double arrows till it touched itself, and then pinched off the resultant bubble. In going from (g) to (h), we have deformed a surface in the direction of the double arrow till its middle portion touched the middle portion of the other surface. The integral of $B$ over these middle portions cancel leading to (h).

Figure 24(h) shows that interchange is internal $2\pi$ rotation of spin frames establishing (5.2).

There is an operation called slide defined in \cite{3} and illustrated in Fig. 25 using a presentation similar to Fig. 22. In this operation, the left vortex is spatially stationary whereas the right vortex is taken in a loop which passes through the middle of the left one. The operator producing the final state of this sequence from the initial state is the operator $S$ of slide. It is easy to see that the path which is the composition of slide and exchange is interchange. Hence

\[
\hat{\sigma} = \sigma S.
\] (5.6)
But as $\sigma = 1$ by (5.4), the result (5.2) implies that

$$S = \hat{R}_{2\pi}.$$  \hfill (5.7)

Internal spin directions had occurred in an earlier work on spin-statistics theorem for vortices [19]. It is significant that Srivastava [23] has succeeded in proving (5.2) using the approach of that paper and without appealing to the $BF$ Lagrangian used in this paper.
5.4. Aharonov-Bohm Interactions

Let \( |0 > \) be the "vacuum" state as defined previously. Let us apply vertex operators to \( |0 > \) to create therefrom a vortex at \( C_\alpha \) and a charge at \( z_\beta \) as in Fig. 26(a). The figure also shows the surface \( S_\alpha \) and the line \( L_\beta \) involved in the definition of vertex operators, the state being

\[
|\Psi_I > = e^{i\epsilon_\beta \int_{L_\beta} A} \cdot e^{-i\Phi_\alpha \int_{S_\alpha} B} |0 >
\]  

(5.8)

(The figure does not show other charges and vortices, the holes or \( \partial \Sigma \).)

Now consider the transport of the charge in a loop \( \tilde{L} \) around \( C_\alpha \) without changing its spin state. This loop is shown in Fig. 26(b). The successive stages of (5.8) for this transport can also be realized as in Figs. 26(c-h). The passage from Fig. 26(d) to Fig. 26(h) should be clear. Now the integral of \( A \) over a line does not change if it is distorted in its interior without crossing \( C_\alpha \). By the equality of Fig. 26(d) and 26(e), this means that \( L' \) can pierce the surface in Fig. 26(e) anywhere without changing the state. Note that for Fig. 26(h), the \( B \) integral goes over \( S_\alpha \) as well as over the bubble with surface \( S_\alpha'' \) enclosing the charge. When the transport around \( \tilde{L} \) is completed, we do not recover \( |\Psi_I > \), because of the additional phase from the integral of \( A \) over \( L'' \). The integral of \( B \) over \( S_\alpha'' \) gives no additional phase as the \( B \) integral in (5.8) is next to \( |0 > \) and the \( dB \) integral over the ball enclosed by \( S_\alpha'' \) annihilates \( |0 > \). The additional phase in question follows from (4.41) so that

\[
|\Psi_I > \xrightarrow{\text{on transport around } \tilde{L}} e^{-i\Phi_\alpha \int_{S_\alpha} B} |\Psi_I >
\]  

(5.9)

The state \( |\Psi_I > \) can equally well be written with the factors involving \( A \) and \( B \) interchanged:

\[
|\Psi_I > = e^{-i\Phi_\alpha \int_{S_\alpha} B} \cdot e^{i\epsilon_\beta \int_{L_\beta} A} |0 >
\]  

(5.10)

For this form of \( |\Psi_I > \), it is the phase from the integral of \( A \) over \( \tilde{L} \) which becomes 1 whereas the integral of \( B \) over \( S_\alpha'' \) leads to (5.9).
The phase in (5.9) becomes 1 and the Aharonov-Bohm interaction vanishes if the quantization condition
\[ e_\beta \Phi_\alpha = 2\pi \times \text{Integer} \] (5.11)
is fulfilled.

Suppose now that we replace the vortex in (5.8) with a charged vortex with charge \( e_\alpha \) and flux \( \Phi_\alpha \). Suppose also that the charge in (5.9) is also replaced by a charged vortex with charge \( e_\beta \) and flux \( \Phi_\beta \). The state \( |\Psi_I> \) then becomes
\[ |\Psi'_I> = e^{-i\Phi_\beta \int_{S_\beta} B} \cdot e^{i\Phi_\alpha \int_{S\alpha} A} \cdot e^{ie_\beta \int_{L_\beta} A} \cdot e^{ie_\alpha \int_{L\alpha} A} \cdot |0> \] (5.12)
where \( S_\alpha, L_\alpha, S_\beta, L_\beta \) are shown in Fig.27. The transport around \( \tilde{L} \) now becomes a slide and the phase change is readily computed to be \( e^{ie_\beta \Phi_\alpha} \),
\[ |\Psi'_I> \xrightarrow{\text{After a slide}} e^{ie_\beta \Phi_\alpha} |\Psi'_I> . \] (5.13)

It thus becomes 1 if
\[ e_\beta \Phi_\alpha = 2\pi \times \text{Integer} . \] (5.14)

(5.14) is the analogue of Dirac quantization condition for dyons [24].

6. Self Energies of Sources and When They Diverge

In our earlier work [3], we have discussed the edge states of the Lagrangian \( L_0 \) of (1.6). This Lagrangian has the virtue of including the Maxwell Lagrangian of \( A \) and the corresponding Lagrangian of \( B \). In that paper, it was shown that the structure of edge states was not sensitive to these terms and that they occur equally well in \( L_0 \) and \( L_0^* \).

The edge states treated in this paper are those at \( \partial \Sigma \). There are also these states at source boundaries, and as mentioned in the Introduction, they too are present in the Lagrangian
\[ L = L_0 + L_I \] (6.1)
obtained by replacing $L_0$ in (1.13) by $L_0^\ast$.

But as stated in the Introduction, the Hamiltonian for (6.1) diverges in the presence of sources much as in electrodynamics. Here we briefly indicate how this happens for (6.1) for charges. Divergences are present for vortices too much as in the work of Lund and Regge [25].

The Hamiltonian for (6.1) was derived in [5] and reads

$$H = \int d^3x \left[ \frac{1}{2} [\pi_i - \epsilon_{ijk} B_{jk}]^2 + \frac{\lambda}{16} P_{ij}^2 + \frac{1}{4} F_{ij}^2 + \frac{1}{3\lambda} H_{ijk}^2 \right. \\
\left. - A_0 (\partial_i \pi_i - e \delta^3(x - z)) - B_{0i} (\partial_j P_{ji} + 2 \epsilon_{ijk} \partial_j A_k) - \lambda \int d\sigma^1 \frac{\partial y^i}{\partial \sigma^1} \delta^3(x - y) \right. \\
\left. + \psi^0 \pi_0 + \psi^i P_{0i} \right],$$

(6.2)

the coefficients of $A_0$ and $B_{0i}$ being constraints.

Now in the presence of charge, $\partial_i \pi_i$ has a $\delta$-function singularity and hence $H$ is classically divergent.

There is a similar divergence in quantum theory too. Thus suppose that $H$ has been properly normal ordered and vanishes on the state $|0>$ of Section 4.2 i), $|0>$ having no charge or vortex. It is thus annihilated in particular by the operator $\partial_i \pi_i$ associated with the Gauss law constraint for $e = 0$. Then a state with a single charge, such as

$$W(\Omega^{(2)})|0>$$

(6.3)

is annihilated only by the operator associated with the Gauss law constraint containing also the point charge contribution. $H$ is therefore divergent on this state.

7. Twisted Spins on Vortices

There is a spin direction attached at each point of the vortex and the latter topologically is a circle $S^1$. We can thus conceive of the spin direction rotating by $2\pi$ as one
goes around the vortex as illustrated in Fig. 28. More generally, we can conceive of this
direction rotating by \(2\pi N\) \((N \in \mathbb{Z})\) as one goes around the vortex, thereby suggesting the
configuration of a soliton for winding number \(N\). In this concluding Section, we briefly
discuss when vertex operators can be found for such twisted spins.

First we explain the precise definition of the winding number of the vortex spin. We
assume for simplicity that the vortices have sharp transverse spins. Now, if we enclose
a vortex inside a solid torus \(T_3\) in the usual way, then there are two cycles \(Y_1\) and \(Y_2\)
on its boundary \(T^2 = \partial T_3\), \(Y_i\) corresponding to \(\theta_i\) of Fig. 2 increasing by \(2\pi\). If lines
(geodesics) are drawn in the direction of the transverse spins, they will pierce \(T^2\) along
the curve \(Y\) which is homologous to the curve obtained by traversing \(Y_1\) \(N\) times and \(Y_2\)
one, \(Y \sim NY_1 + Y_2\). The integer \(N\) is then defined to be the winding number of the
vortex spin. [Homologous curves are here defined by regarding them as curves on \(T^2\) and
not in \(T_3\).]

Referring to Fig. 13 or Eq. (4.39), we see that the construction of a vertex operator
for a vortex with spatial location \(C_\alpha\) involves the existence of a surface \(S_\alpha\) with boundary
\(\partial S_\alpha = C_\alpha\) and with directions of its tangents at \(C_\alpha\) giving the spin directions. Such a
surfaces \(S_\alpha\) must be orientable as well so that the integration of \(B\) over \(S_\alpha\) can be defined.
We are able to find these surfaces only under particular conditions, suggesting that vertex
operators exist only under special circumstances of this sort.

Below, we will discuss some surfaces \(S_\alpha\) associated with twisted spins (which by defi-
nition have winding number \(N\)). When it exists, the vertex operator can be constructed
starting with an expression like (4.39) and regularising it following Section 4.2 ii). We
will also write \(C\) and \(S\) for \(C_\alpha\) and \(S_\alpha\) when it is convenient to do so.

The simplest construction of twisted spins is as follows. We start with a ribbon with
\(L\) and \(L'\) as borders and a flat surface interpolating them as in Fig. 29 (a). We then twist
one end by \(2\pi M\) and then identify the ends. \(M\) here is half integral or integral. The
resultant configurations for \( M = \frac{1}{2} \) and 1 are shown in Figs. 29 (b) and (c).

Let us first consider Fig. 29(c). It shows an orientable surface \( S \) with vortices \( C \) and \( C' \) as borders. If the spin of \( C \) twists by +2\( \pi \) say, so that it has winding number \( N = 1 \), then the spin of \( C' \) twists by \(-2\pi\) and has \( N = -1 \).

Fig. 29(b) is a Möbius band. It is not orientable. For this reason, it does not seem possible to create the vortices of this figure by a vertex operator.

The method outlined here is capable of generalisations. One such would be to first knot the ribbon, for example in the shape of a trefoil cut at a point, as in Fig. 30(a). The loose ends of the ribbon are then identified after \( N \) twists. As \( S \) becomes nonorientable if \( N \in \mathbb{Z} + \frac{1}{2} \), \( N \) here is restricted to be an integer. \( \pm N \) are then the winding numbers of the spins of the vortices located at the borders \( C \) and \( C' \) indicated in Fig. 30(b).

This example can be generalised by creating links and knots using several ribbons, with knots having twisted spins. Fig. 31 illustrates an example of this sort. The general idea here is the same as the one governing the passage from braids to linked and unlinked knots \[26\]. The surface \( S \) going into the definition of the vertex operator is the surface on ribbons. It would have disconnected components if the knot has several links.

There is one further generalisation of this idea which gives the previous constructions as special cases and also shows how to create new types of states. We recall that to create a state at \( C_\alpha \), we first dig a hole \( \mathcal{H}_\alpha \) enclosing \( C_\alpha \), which hole is eventually shrunk to \( C_\alpha \). There is a Fock space of states localised at \( \partial \mathcal{H}_\alpha \) with the Fock vacuum \( \left| 0 \right> \). The application of the vertex operator of a vortex involving a surface with a boundary \( \mathcal{C}_\alpha \) gives the required vortex state.

Suppose now that there are holes \( \mathcal{H}_\alpha \) and \( \mathcal{H}_\beta \) enclosing loops \( C_\alpha \) and \( C_\beta \) respectively. Let the cycles \( Y_i \) introduced previously be denoted by \( Y_i(\alpha) \) and \( Y_i(\beta) \) when they are associated with \( \mathcal{H}_\alpha \) and \( \mathcal{H}_\beta \). Let \( Y(\rho) \left[ \rho = \alpha, \beta \right] \) be a loop on \( \partial \mathcal{H}_\rho \) which is homologous to \( N_1(\rho)Y_1(\rho) + N_2(\rho)Y_2(\rho) \), \( N_1(\rho)Y_1(\rho) + N_2(\rho)Y_2(\rho) \) denoting the loop where \( Y_1(\rho) \) is
traversed $N_1(\rho)$ times and $Y_2(\rho)$ is traversed $N_2(\rho)$ times. Homologous curves for a given $\rho$ are again defined by regarding them as curves confined to $\partial \mathcal{H}_\rho$. Figure 32 exhibits a typical $Y(\rho)$ for a hole $\mathcal{H}_\rho$.

Assume that there exists a surface $S$ with one or more connected boundaries. For a moment assume for specificity that there are two connected boundaries $Y(\alpha)$ and $Y(\beta)$. If $|0>$ is the tensor product of the Fock vacua for $\partial \mathcal{H}_\alpha$, we can create a state $|Y(\alpha), Y(\beta)>$ by applying the vertex operator involving $S$ to $|0>$. 

Now if for example $N_1(\alpha) = N_2(\alpha) = -N_1(\beta) = -N_2(\beta) = 1$, we get the example of Fig. 29(c). Simple generalisations of the construction leading to this state will also yield all previous examples.

But we can also create new states now as the number of boundary components need not be two and $N_i(\rho)$ are not restricted to have the values suitable for Fig.29(c). The surface $S_\alpha$ of Fig. 13 for example is an instance where there is only one connected boundary. As another example, suppose that $\mathcal{H}_\alpha$ and $\mathcal{H}_\beta$ are linked while $S$ has the property that $N_1(\alpha) = 0, N_2(\alpha) = 1, N_1(\beta) = -1, N_2(\beta) = 0$. Fig.33 shows how to realise this situation. In this case, when $\mathcal{H}_\rho$ finally shrinks to $\mathcal{C}_\rho$, there is a vortex with flux associated with it at $\mathcal{C}_\alpha$, but not at $\mathcal{C}_\beta$. Instead, we have created winding number 1 spin excitations at $Y(\beta)$, the definition of this winding number being similar to its definition for a vortex with flux.

In this example, we can clearly deform $S$ so that $Y(\rho)$ become any curve homologous to the corresponding curve in Fig.33.

Note that $N_1(\rho)$ measures the spin twist at $\mathcal{C}_\rho$. As for $N_2(\rho)$, suppose that for a particular $S$, $|N_2(\rho)|$ is neither 0 or 1. Now, $Y(\rho)$ winds $N_2(\rho)$ times around $\mathcal{C}_\rho$ and therefore after $\mathcal{H}_\rho$ shrinks to $\mathcal{C}_\rho$, the state describes a vortex at $\mathcal{C}_\rho$ with flux $N_2(\rho)\Phi$ if $\Phi_\alpha$ is $\Phi$ in the vertex operator defined by (4.39) or (4.57).

The brief considerations presented in this Section show that we can create several
types of states by using suitable surfaces. But we will not pursue their study further here.

Finally we point out that there is an operator which measures spin twist. It can be constructed as follows. The state \( |Y(\alpha), Y(\beta)\rangle \) discussed above is given by

\[
|Y(\alpha), Y(\beta)\rangle = e^{-i\varphi \int_{S} B} |0\rangle
\]

(7.1)

where \( \Phi \) has been substituted for the \( \Phi_{\alpha} \) of (4.39). [More precisely, we should define the state using the regularised version of the vertex operator in (7.1)]. Let \( \hat{C}_{\rho} \) a curve just outside the hole \( H_{\rho} \) and not touching \( \partial H_{\rho} \) obtained by deforming \( C_{\rho} \) as shown in Fig.34. Then we claim that

\[
\left[ \int_{\hat{C}_{\rho}} A, \int_{S} B \right] = iN_{1}(\rho)
\]

(7.2)

and therefore that

\[
e^{i\varphi \int_{S} B} \left( \int_{\hat{C}_{\rho}} A \right) e^{-i\varphi \int_{S} B} = \Phi N_{1}(\rho).
\]

(7.3)

Since the expectation value of

\[
T(\rho) = \frac{1}{\Phi} \int_{\hat{C}_{\rho}} A
\]

(7.4)

in \( |0\rangle \) is zero, it follows that its expectation value in \( |Y(\alpha), Y(\beta)\rangle \) measures the spin twist. For this reason, it is natural to regard \( T(\rho) \) as the spin twist operator for \( C_{\rho} \).

Before showing (7.2), let us note that we can let \( \hat{C}_{\rho} \) tend to \( C_{\rho} \) after \( H_{\rho} \) is shrunk to \( C_{\rho} \). Also, the mode expansion of (7.4) should be discussed, but we will not do so here.

As the first step in showing (7.2), let us deform \( \hat{C}_{\rho} \) to a curve \( \tilde{C}_{\rho} \) lying on \( \partial H_{\rho} \). \( \tilde{C}_{\rho} \) is a cycle homologous to \( Y_{2}(\rho) \) and is shown in Figures 34 and 35. Now it is well known that \( \tilde{C}_{\rho} \) will intersect \( Y(\rho) \) exactly \( N_{1}(\rho) \) times if an intersection is counted as +1 or −1 according to its orientation. Now the surface \( S \) terminates at \( Y(\rho) \). Therefore, as illustrated in Fig.36, the loop \( \hat{C}_{\rho} \), obtained by lifting \( \tilde{C}_{\rho} \) a little bit off \( \partial H_{\rho} \), will also intersect \( S \) exactly \( N_{1}(\rho) \) times. As each such intersection contributes an \( i \) or \( -i \) to the left hand side of (7.2) depending on its orientation, the result (7.2) is immediate.
If the expectation value of $T(\rho)$ in $|Y(\alpha), Y(\beta)>$ is not zero, then the definition (7.4) of $T(\rho)$ suggests that $\hat{C}_\rho$ encloses flux. This implies in particular that a single unknotted vortex with zero flux passing through its middle can not have spin twist.
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Fig. 1. $T^2$ is the boundary of the solid torus $T_3$ which encloses the vortex. The vortex is the dotted line. The angular coordinates $\theta^i$ on $T^2$ are shown.
Fig. 3. When $v^{(2)}$ is chosen to have support on $T$, $W(v^{(2)})$ leads to the Wilson line integral.

Fig. 4. $L'$ differs from $L$ only by the distortion shown by the dotted line.
Fig. 5. This figure shows what $\bar{S}_\delta$ is.

Fig. 6.
Fig. 7. A solid torus $\tilde{T}_3$ enclosing a solid torus $\mathcal{H}_1$ indicated by the shaded region. The loops $C_1$ and $C_2$ are not homologous and cannot be shrunk to points. Hence there are distinct closed one forms with nonzero integrals over $C_i$.

Fig. 8. $\Sigma$ here is a solid torus enclosing a solid torus $\mathcal{H}_1$ indicated by the shaded region. The loops $C_1$ and $C_2$ are not homologous and cannot be shrunk to points. Hence there are distinct closed one forms with nonzero integrals over $C_i$. 

$\partial \Sigma = S^2$
Fig. 9. The placement of $\mathcal{H}_1$ in $\Sigma$ is different here from its placement in Fig. 8. Just as in Fig. 8, neither of the loops $C_1$ and $C_2$ can be shrunk to a point or deformed to each other, and there are distinct closed forms with nonzero integrals over $C_i$.

Fig. 10. The figure shows how to fill up $\Sigma \setminus \mathcal{H}_1$ with a family of two-tori which are homologically nontrivial.
In (a), $\Sigma = T_3$ and there is only one $\mathcal{H}_\alpha$. In (b), $\Sigma = B_3$ and there is an $\mathcal{H}_\beta = T_3$ besides the $\mathcal{H}_\alpha$. In (a) or (b), there are two L's. The first is the straight line from P to $\partial \mathcal{H}_\alpha$. In the second, we first go from P to Q in a straight line, then along the dotted loop to Q and finally from Q to $\partial \mathcal{H}_\alpha$. The line integral of $A$ over the dotted loop need not be zero.
Fig. 13. The figure shows two surfaces \( S_\alpha \) and \( S'_\alpha \) with the same boundary \( c_\alpha \) and tangent directions at \( c_\alpha \). Their union encloses \( \mathcal{H}_\beta \) so that they are not homotopic.
Fig. 15. Here $H_\alpha$ is a solid torus enclosing $H_\alpha$ (and no other holes). $\Omega^{(1)}$ is equal to $\bar{\Omega}^{(1)}$ in the complement of $H_\alpha$.

Fig. 16. The figure shows the two charges at $z_1$ and $z_2$ with their holes $H_1$ and $H_2$, all enclosed within a ball $B_3$. $B_3$ contains no other charges or vortices. The figure shows the line $L_i$ used in the definition of the Wilson line integrals which create these charges.
Fig. 17. The figure displays only the right side portion of the final figure in Fig. 17 for convenience. The dotted portion of the line in (a) is above $\mathcal{H}_2$ and does not touch it.
Fig. 19. Figure (a) shows the vortices at $C_i$ enclosed in a ball $B_3$ and a particular transport of $C_2$ to $C_1$. For this transport, $p_2$ is moved along $L$ to $p_1$. It can also be represented by (b) where only the cross section of the two disks in (a) are shown.

Fig. 20. (a) to (d) show the successive stages of the path defining the exchange operator $\sigma$ for identical loops. The final state (d) here is the same as the initial state (a) so that $\sigma = 1$. 
Fig. 21. (a) to (g) show the successive stages of the path defining the $2\pi$ rotation operator $R_{2\pi}$. The final state (g) is the same as the initial state so that $R_{2\pi} = 1$.

Fig. 22. The figure shows the path for the interchange of two identical vortices.
Fig. 23. The figure shows the history of the spin direction at a particular point $p$ of a vortex under the internal $2\pi$ rotation.

Fig. 24. These figures show the process of interchange in detail. The arrows $\leftarrow$ and $\rightarrow$ indicate the relative orientations of the integrals of $B$ over the surfaces.
Fig. 25. The slide $S$.

Fig. 26. These figures display the adiabatic transport of a charge in a loop around a vortex.
The "spin" directions wind once around the vortex.

Fig. 27.

Fig. 28. The "spin" directions wind once around the vortex.
How to generalise the ribbon construction

Fig. 29.

Fig. 30. How to generalise the ribbon construction
Fig. 31. How to generalise the ribbon construction

Fig. 32. The figure shows the curve with winding numbers $N_1(\rho) = 3$ and $N_2(\rho) = 2$ on $\mathcal{H}_\rho$. 
Fig. 33. The figure shows an arrangement of vortices and the surface $S$ which leads to a vortex with flux and no spin winding at $\mathcal{H}_\alpha$ and a 'vortex' with zero flux and spin winding number 1 at $\mathcal{H}_\beta$.

Fig. 34. $\hat{C}_\rho$ and $\tilde{C}_\rho$ are curves obtained by deforming $C_\rho$. $\hat{C}_\rho$ lies just outside $\mathcal{H}_\rho$ while $\tilde{C}_\rho$ lies on $\partial \mathcal{H}_\rho$. 
Fig. 35. The figure shows $\bar{C}_\rho$ lying on $\partial \mathcal{H}_\rho$ and also $Y(\rho)$. The intersections of $\bar{C}_\rho$ and $Y(\rho)$ are represented by dots.

Fig. 36. The figure shows a piece of the surface $S$ terminating at $Y(\rho)$, $\bar{C}_\rho$, $Y(\rho)$ and an intersection of $\bar{C}_\rho$ and $Y(\rho)$. The latter is represented by a dot. When $\bar{C}_\rho$ is lifted to $\hat{C}_\rho$, this intersection becomes the intersection of $\hat{C}_\rho$ with $S$. 