Spherical gauge fields

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We introduce the spherical field formalism for free gauge fields. We discuss the structure of the spherical Hamiltonian for both general covariant gauge and radial gauge and point out several new features not present in the scalar field case. We then use the evolution equations to compute gauge-field and field-strength correlators. [PACS numbers: 11.10.Kk, 11.15.Tk, 11.15.-q]

1 Overview

Spherical field theory is a new non-perturbative method for studying quantum field theory. It was introduced in [1] to describe scalar boson interactions. The method utilizes the spherical partial wave expansion to reduce a general $d$-dimensional Euclidean functional integral into a set of coupled one-dimensional, radial systems. The coupled one-dimensional systems are then converted to differential equations which then can be solved using standard numerical methods. The extension of the spherical field method to fermionic systems was described in [2]. In that analysis it was shown that the formalism avoids several difficulties which appear in the lattice treatment of fermions. These include fermion doubling, missing axial anomalies, and computational difficulties arising from internal fermion loops. This finding suggests that the spherical formalism could provide a useful method for studying gauge theories, especially those involving fermions. As a small but important initial step in this direction, we contribute the present work in which we introduce and discuss the spherical field method for free gauge

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4For free field theory these equations can be solved exactly, as we will demonstrate here for gauge fields.
The basic formalism for spherical boson fields was described in [1]. In this paper we will build on those results with most of our attention devoted to new features resulting from the intrinsic spin of the gauge field. We discuss the operator structure of the spherical Hamiltonian in detail, using two-dimensional Euclidean gauge fields as an explicit example. Like standard field theory gauge-fixing is essential in spherical field theory, and we have chosen to consider general covariant gauge and radial gauge. In each case we derive the spherical Hamiltonian and use the corresponding evolution equations to calculate the two-point correlators for the gauge field and the gauge-invariant field strength. Free gauge fields in higher dimensions can be described by a straightforward generalization of the methods presented here. The application of spherical field theory to non-perturbative interacting gauge systems and related issues are the subject of current research.

2 Covariant gauge

In this section we derive the spherical field Hamiltonian for general covariant gauge. We will use both polar and cartesian coordinates with the following conventions:

\[ \vec{t} = (t \cos \theta, t \sin \theta) = (t^1, t^2) = (x, y). \]  

(1)

In general covariant gauge the Euclidean functional integral is given by

\[ \int (\prod_i \mathcal{D}A^i) \exp \left[ \int_0^\infty \! dt \, L \right] \]  

(2)

where

\[ L = \int \! d\theta \, t \left[ -\frac{1}{2} F^{12} F^{12} - \frac{1}{2a} (\partial_i A^i)^2 \right]. \]  

(3)

We can write the field strength \( F^{12} \) as

\[ F^{12} = \frac{1}{2t} \left[ \left( \frac{\partial}{\partial x} + \frac{i}{\partial y} \right) (A^x - iA^y) - \left( \frac{\partial}{\partial x} - \frac{i}{\partial y} \right) (A^x + iA^y) \right] \]  

(4)

\[ = \frac{1}{\sqrt{2t}} \left[ e^{i\theta} \left( \frac{\partial}{\partial x} + \frac{i}{t \partial \theta} \right) A^+ - e^{-i\theta} \left( \frac{\partial}{\partial x} - \frac{i}{t \partial \theta} \right) A^- \right] \]

See [3] and references therein for a discussion of radial gauge.
\[ A^x \mp iA^y = \sqrt{2}A^{\pm 1}. \] (5)

We now decompose \( A^{\pm 1} \) into partial waves\(^6\)
\[ A^{\pm 1} = \frac{1}{\sqrt{2\pi}} \sum_{n=0,\pm 1,\ldots} A^{\pm 1}_n e^{in\theta}. \] (6)

Returning to our expression for the field strength, we have
\[ F^{12} = \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2i}} \sum_{n=0,\pm 1,\ldots} e^{in\theta} \left( F^{+1}_n - F^{-1}_n \right), \] (7)

where
\[
F^{+1}_n = \frac{\partial A^{+1}_n}{\partial t} - \frac{n-1}{t} A^{+1}_{n-1},
\]
\[
F^{-1}_n = \frac{\partial A^{-1}_{n+1}}{\partial t} + \frac{n+1}{t} A^{-1}_{n+1}.
\] (8) (9)

We can also express the gauge-fixing term in terms of \( F^{\pm 1}_n \),
\[ \partial_i A^i = \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2i}} \sum_{n=0,\pm 1,\ldots} e^{in\theta} \left( F^{+1}_n + F^{-1}_n \right). \] (10)

With these changes the Lagrangian, \( L \), is
\[
\frac{\ell}{4} \sum_{n=0,\pm 1,\ldots} \left( F^{+1}_{-n} - F^{-1}_{-n} \right) \left( F^{+1}_n - F^{-1}_n \right) - \frac{\ell}{4\alpha} \sum_{n=0,\pm 1,\ldots} \left( F^{+1}_{-n} + F^{-1}_{-n} \right) \left( F^{+1}_n + F^{-1}_n \right).
\] (11)

In \[ \text{[1]} \] the spherical Hamiltonian for the scalar field was found by direct application of the Feynman-Kac formula. This is also possible here, but in view of the number of mixed terms (a result of the intrinsic spin degrees of freedom) we find it easier to use the method of canonical quantization. Let us define the conjugate momenta to the gauge fields,
\[
\pi^{+1}_{n-1} = \frac{\delta L}{\delta \frac{\partial A^{+1}_{n-1}}{\partial t}} = \frac{\ell}{2} \left[ (1 - \frac{1}{\alpha}) F^{+1}_{-n} + (-1 - \frac{1}{\alpha}) F^{-1}_{-n} \right]
\] (12)
\[
\pi^{-1}_{n+1} = \frac{\delta L}{\delta \frac{\partial A^{-1}_{n+1}}{\partial t}} = \frac{\ell}{2} \left[ (-1 - \frac{1}{\alpha}) F^{+1}_{-n} + (1 - \frac{1}{\alpha}) F^{-1}_{-n} \right].
\] (13)

\(^6\)In our notation \( A^{\pm 1}_n \) carries total spin quantum number \( n \pm 1 \).
Following through with the canonical quantization procedure, we find a Hamiltonian of the form

\[ H = \sum_{n=0,\pm 1,\ldots} H_n, \quad \text{where} \]

\[ H_n = -\frac{1}{4t} \left( \pi_{n+1}^{-1} - \pi_{n-1}^{+1} \right) \left( \pi_{n-1}^{+1} - \pi_{n+1}^{-1} \right) \]

\[ -\frac{\alpha}{4t} \left( \pi_{n+1}^{-1} + \pi_{n-1}^{+1} \right) \left( \pi_{n-1}^{+1} + \pi_{n+1}^{-1} \right) + \frac{n-1}{t} A_{n-1}^{+1} \pi_{n-1}^{+1} - \frac{n+1}{t} A_{n+1}^{-1} \pi_{n+1}^{-1}. \]

We obtain the corresponding Schrödinger time evolution generator by making the replacements

\[ A_{n\pm 1} \rightarrow z_{n\pm 1}, \quad \pi_{n\pm 1} \rightarrow \frac{\partial}{\partial z_{n\pm 1}}. \]

We then find

\[ H_n = -\frac{1}{4t} \left( \frac{\partial}{\partial z_{n-1}} - \frac{\partial}{\partial z_{n+1}} \right) \left( \frac{\partial}{\partial z_{n-1}} - \frac{\partial}{\partial z_{n+1}} \right) \]

\[ -\frac{\alpha}{4t} \left( \frac{\partial}{\partial z_{n-1}} + \frac{\partial}{\partial z_{n+1}} \right) \left( \frac{\partial}{\partial z_{n-1}} + \frac{\partial}{\partial z_{n+1}} \right) + \frac{n-1}{t} z_{n-1}^{+1} \frac{\partial}{\partial z_{n-1}} - \frac{n+1}{t} z_{n+1}^{-1} \frac{\partial}{\partial z_{n+1}}. \]

For the sake of future numerical calculations, it is convenient to re-express \( H \) in terms of real variables. Let us define

\[ u_n = \frac{1}{2} \left( z_{n+1}^{-1} + z_{n-1}^{+1} + z_{n-1}^{+1} + z_{n+1}^{-1} \right) \]

\[ v_n = \frac{1}{2} \left( z_{n+1}^{-1} + z_{n-1}^{+1} - z_{n-1}^{+1} - z_{n+1}^{-1} \right) \]

\[ x_n = \frac{1}{2t} \left( z_{n+1}^{-1} - z_{n-1}^{+1} - z_{n-1}^{+1} + z_{n+1}^{-1} \right) \]

\[ y_n = \frac{1}{2t} \left( z_{n+1}^{-1} - z_{n-1}^{+1} + z_{n-1}^{+1} - z_{n+1}^{-1} \right) \]

for \( n > 0 \) and

\[ u_0 = \frac{1}{\sqrt{2}} \left( z_1^{-1} + z_1^{+1} \right) \]

\[ x_0 = \frac{1}{\sqrt{2t}} \left( z_1^{-1} - z_1^{+1} \right). \]

The variables \( u_n \) and \( y_n \) correspond with combinations of radially polarized gauge fields with total spin \( \pm n \), while \( v_n \) and \( x_n \) correspond with tangentially polarized gauge fields with total spin \( \pm n \). In terms of these new variables,

\[ H = H_0 + \sum_{n=1,2,\ldots} H'_n, \quad \text{where} \]

\[ H'_n = \frac{1}{4t} \left( \frac{\partial}{\partial z_{n-1}} - \frac{\partial}{\partial z_{n+1}} \right) \left( \frac{\partial}{\partial z_{n-1}} - \frac{\partial}{\partial z_{n+1}} \right) \]

\[ -\frac{\alpha}{4t} \left( \frac{\partial}{\partial z_{n-1}} + \frac{\partial}{\partial z_{n+1}} \right) \left( \frac{\partial}{\partial z_{n-1}} + \frac{\partial}{\partial z_{n+1}} \right) + \frac{n-1}{t} z_{n-1}^{+1} \frac{\partial}{\partial z_{n-1}} - \frac{n+1}{t} z_{n+1}^{-1} \frac{\partial}{\partial z_{n+1}}. \]
where
\[ H_0 = -\frac{1}{2t} \frac{\partial^2}{\partial x_0^2} - \frac{\alpha}{2t} \frac{\partial^2}{\partial u_0^2} - \frac{1}{t} \left( u_0 \frac{\partial}{\partial u_0} + x_0 \frac{\partial}{\partial x_0} \right), \] (25)
and
\[ H'_n = -\frac{1}{2t} \left( \frac{\partial^2}{\partial v_n^2} + \frac{\partial^2}{\partial x_n^2} \right) - \frac{\alpha}{2t} \left( \frac{\partial^2}{\partial u_n^2} + \frac{\partial^2}{\partial y_n^2} \right) - \frac{n}{t} \left( u_n \frac{\partial}{\partial u_n} + v_n \frac{\partial}{\partial v_n} + x_n \frac{\partial}{\partial x_n} + y_n \frac{\partial}{\partial y_n} \right) \] (26)
\[- \frac{1}{t} \left( u_n \frac{\partial}{\partial u_n} + v_n \frac{\partial}{\partial v_n} + x_n \frac{\partial}{\partial x_n} + y_n \frac{\partial}{\partial y_n} \right).\]

Despite the somewhat complicated form, we know that the spectrum of \( H \) should resemble that of harmonic oscillators. To see this explicitly let us define the following ladder operators:

\[ U^+_n = -\frac{1}{\sqrt{2}} \left( \frac{\partial}{\partial u_n} + \frac{\partial}{\partial v_n} \right), \quad U^-_n = \frac{1}{\sqrt{2}} \left( \frac{2\alpha - n + \alpha n}{4(n+1)} \frac{\partial}{\partial u_n} + \frac{2 + n - \alpha n}{4(n+1)} \frac{\partial}{\partial v_n} + u_n + v_n \right), \]
(27)
\[ V^+_n = \frac{1}{\sqrt{2}} \left( -\frac{2\alpha - n + \alpha n}{4(n-1)} \frac{\partial}{\partial u_n} + \frac{2 - n + \alpha n}{4(n-1)} \frac{\partial}{\partial v_n} + u_n - v_n \right), \quad V^-_n = \frac{1}{\sqrt{2}} \left( \frac{\partial}{\partial u_n} - \frac{\partial}{\partial v_n} \right), \]
(28)
\[ X^+_n = -\frac{1}{\sqrt{2}} \left( \frac{\partial}{\partial y_n} + \frac{\partial}{\partial x_n} \right), \quad X^-_n = \frac{1}{\sqrt{2}} \left( \frac{2\alpha - n + \alpha n}{4(n+1)} \frac{\partial}{\partial y_n} + \frac{2 + n - \alpha n}{4(n+1)} \frac{\partial}{\partial x_n} + y_n + x_n \right), \]
(29)
\[ Y^+_n = \frac{1}{\sqrt{2}} \left( -\frac{2\alpha - n + \alpha n}{4(n-1)} \frac{\partial}{\partial y_n} + \frac{2 - n + \alpha n}{4(n-1)} \frac{\partial}{\partial x_n} + y_n - x_n \right), \quad Y^-_n = \frac{1}{\sqrt{2}} \left( \frac{\partial}{\partial y_n} - \frac{\partial}{\partial x_n} \right), \]
(30)
for \( n > 1 \) and
\[ U^+_0 = -\frac{1}{\sqrt{2}} \frac{\partial}{\partial u_0}, \quad U^-_0 = \frac{\alpha}{\sqrt{2}} \frac{\partial}{\partial u_0} + \sqrt{2} u_0, \] (31)
\[ X^+_0 = -\frac{1}{\sqrt{2}} \frac{\partial}{\partial x_0}, \quad X^-_0 = \frac{1}{\sqrt{2}} \frac{\partial}{\partial x_0} + \sqrt{2} x_0. \] (32)

We have constructed these operators so that
\[ [U^-_n, U^+_n] = [V^-_n, V^+_n] = [X^-_n, X^+_n] = [Y^-_n, Y^+_n] = 1 \] (33)
for \( n > 1 \) and

\[
[U_0^-, U_0^+ ] = [X_0^-, X_0^+] = 1. \tag{34}
\]

All other commutators involving these operators vanish. We can now rewrite

\[
H_0 \quad \text{and} \quad H_n' \quad (\text{for} \quad n > 1)
\]

as

\[
H_0 = \frac{1}{t} U_0^+ U_0^- + \frac{1}{t} X_0^+ X_0^- + \frac{2}{t}, \tag{35}
\]

\[
H_n' = \frac{n+1}{t} U_n^+ U_n^- + \frac{n-1}{t} V_n^+ V_n^- + \frac{n+1}{t} X_n^+ X_n^- + \frac{n-1}{t} Y_n^+ Y_n^- + \frac{2n+2}{t}. \tag{36}
\]

We now see that \( H_0 \) is equivalent to two harmonic oscillators with level-spacing \( \frac{1}{t} \), and \( H_n' \) is equivalent to four harmonic oscillators, two with spacing \( \frac{n+1}{t} \) and two with spacing \( \frac{n-1}{t} \). It is important to note that \( H_1' \), however, is quite different. The \( s \)-wave configurations are independent of \( \theta \), and so \( H_1' \) does not have terms of the form \( \frac{\partial}{\partial z_0} \) and \( \frac{\partial}{\partial z_0} \). Furthermore, the gauge fields are massless and so \( H_1' \) depends only on the derivatives of \( z_0^+ \) and \( z_0^- \). One consequence of this is that the spectrum of \( H_1' \) is continuous, and calculations involving \( H_1' \) and relevant boundary conditions must be done carefully. We will discuss these issues later in our analysis of correlation functions.

In preparation for our calculation of gauge-field correlators, let us now couple an external source \( \vec{J} \) to the gauge field,

\[
L = \int d\theta \left[ -\frac{1}{2} F_{12} F_{12} \right] - \frac{1}{2\kappa} (\partial_i A^i)^2 + \vec{A} \cdot \vec{J}. \tag{37}
\]

The new spherical field Hamiltonian is then

\[
H(\vec{J}) = H - \sum_{n=0,\pm1,\ldots} (z_n^+ J_n^- + z_n^- J_n^+), \tag{38}
\]

As noted in [1], the vacuum persistence amplitude is given by

\[
Z(\vec{J}) \propto \lim_{t_{min} \rightarrow 0^+} \lim_{t_{max} \rightarrow \infty} \langle b | T \exp \left[ -\int_{t_{min}}^{t_{max}} dt H(\vec{J}) \right] | a \rangle, \tag{39}
\]

where \( |a\rangle \) and \( |b\rangle \) are any states satisfying certain criteria. These criteria are that \( |a\rangle \) is constant with respect to the \( s \)-wave variables \( z_0^- \) and \( z_0^+ \) and has
non-zero overlap with the ground state of $H$ as $t \to 0^+$, and $|b\rangle$ has non-zero overlap with the ground state of $H$ as $t \to \infty$. Because the spectrum of $H'_1$ is continuous, in numerical computations it is useful to include a small regulating mass, $\mu$, for the gauge fields and then take the limit $\mu \to 0$.

3 Radial gauge

We now derive the spherical field Hamiltonian for radial gauge. We take the gauge-fixing reference point, $\vec{t}_0$, to be the origin,

$$(\vec{t} - \vec{t}_0) \cdot \vec{A} = \vec{t} \cdot \vec{A} = 0. \quad (40)$$

We expect this gauge-fixing scheme to be convenient in spherical field theory calculations for several reasons. One is that non-abelian ghost fields in radial gauge decouple, as they do in axial gauge. In contrast with axial gauge, however, radial gauge also preserves rotational symmetry. As we will see, the spherical Hamiltonian and correlation functions in radial gauge are relatively simple. Since

$$\vec{t} \cdot \vec{A} = \frac{t}{\sqrt{2}} [e^{i\theta} A^{+1} + e^{-i\theta} A^{-1}], \quad (41)$$

we can impose the gauge-fixing condition by setting

$$A^{-1} = -e^{2i\theta} A^{+1}. \quad (42)$$

With this constraint we express the field strength as

$$F^{12} = \frac{1}{i\sqrt{2}} \sum_{n=0, \pm 1, \ldots} e^{in\theta} \left[ \frac{\partial A_{n-1}^{+1}}{\partial t} + \frac{1}{t} A_{n-1}^{+1} \right]. \quad (43)$$

The radial-gauge Lagrangian is then

$$L = \int d\theta t \left[ -\frac{1}{2} F^{12} F^{12} \right] \quad (44)$$

$$= t \sum_{n=0, \pm 1, \ldots} \left[ \frac{\partial A_{n-1}^{+1}}{\partial t} + \frac{1}{t} A_{n-1}^{+1} \right] \left[ \frac{\partial A_{n-1}^{+1}}{\partial t} + \frac{1}{t} A_{n-1}^{+1} \right].$$

One caveat here is that $|a\rangle$ must lie in a function space over which the spectrum of $H$ is bounded below.
We again follow the canonical quantization procedure. The conjugate momenta to the gauge fields are

\[ \pi_{n-1}^{+1} = \frac{\delta L}{\delta \partial A_{n-1}^+} = 2t \left[ \frac{\partial A_{n-1}^+}{\partial t} + \frac{1}{i} A_{n-1}^- \right], \quad (45) \]

and the radial-gauge Hamiltonian has the form

\[ H = \sum_{n=0, \pm 1, \ldots} \left[ \frac{1}{4t} \pi_{n-1}^{+1} \pi_{n-1}^{-1} - \frac{1}{i} A_{n-1}^+ \pi_{n-1}^{-1} \right]. \quad (46) \]

In the Schrödinger language the Hamiltonian becomes

\[ H = \sum_{n=0, \pm 1, \ldots} \left[ \frac{1}{4t} \partial^2_{n-1} \partial^2_{n-1} + \frac{1}{2} x_{n-1} \partial_{n-1} - \frac{1}{2} v_{n-1} \partial_{n-1} \right]. \quad (47) \]

As before we now define real variables,

\[ x_n = \frac{i}{\sqrt{2}} \left( \pi_{n-1}^{+1} + \pi_{n-1}^{-1} \right) \quad (48) \]

\[ v_n = \frac{1}{\sqrt{2}} \left( \pi_{n-1}^{+1} - \pi_{n-1}^{-1} \right) \quad (49) \]

for \( n > 0 \) and

\[ x_0 = i \cdot z_{n-1}^{+1}. \quad (50) \]

Our Hamiltonian can be re-expressed as

\[ H = H_0 + \sum_{n=1, 2, \ldots} H'_n, \quad (51) \]

where

\[ H_0 = -\frac{1}{4t} \partial^2_{x_0} - \frac{1}{2} x_0 \frac{\partial}{\partial x_0}, \quad (52) \]

\[ H'_n = -\frac{1}{4t} \left[ \frac{\partial^2}{\partial x_n^2} + \frac{\partial^2}{\partial v_n^2} \right] - \frac{1}{t} \left[ x_n \frac{\partial}{\partial x_n} + v_n \frac{\partial}{\partial v_n} \right]. \quad (53) \]

Let us now define ladder operators

\[ X_n^+ = \frac{1}{2} \frac{\partial}{\partial x_n}, \quad X_n^- = -\frac{1}{2} \frac{\partial}{\partial x_n} - 2x_n, \quad (54) \]
for $n \geq 0$ and
\begin{equation}
V_n^+ = \frac{1}{2} \frac{\partial}{\partial v_n}, \quad V_n^- = -\frac{1}{2} \frac{\partial}{\partial v_n} - 2v_n,
\end{equation}
for $n > 0$. These ladder operators satisfy the relations
\begin{equation}
[X_n^-, X_n^+] = [V_n^-, V_n^+] = 1,
\end{equation}
while all other commutators vanish. In terms of these operators we have
\begin{equation}
H_0 = \frac{1}{t} X_0^+ X_0^- + \frac{1}{t},
\end{equation}
\begin{equation}
H_n' = \frac{1}{t} X_n^+ X_n^- + \frac{1}{t} V_n^+ V_n^- + \frac{2}{t}.
\end{equation}
We note the radial gauge constraint has removed the continuous spectrum from the $s$-wave sector, and the spectrum of $H$ is purely discrete. Furthermore the splitting between energy levels of $H_n'$ is independent of $n$.

We now couple an external source $\vec{J}$ to the gauge field,
\begin{equation}
L = \int d\theta \left[ -\frac{1}{2} F_{12}^2 + \bar{A} \cdot \vec{J} \right].
\end{equation}
The new Hamiltonian is then
\begin{equation}
H(\vec{J}) = H - \sum_{n=0,\pm 1,\ldots} z_n^{+1} (J_{n-1}^- + J_{n-2}^+).
\end{equation}
The vacuum persistence amplitude in radial gauge is given by
\begin{equation}
Z(\vec{J}) \propto \lim_{t_{\text{min}} \to 0^+} \lim_{t_{\text{max}} \to \infty} \langle b | T \exp \left[ -\int_{t_{\text{min}}}^{t_{\text{max}}} dt \ H(\vec{J}) \right] | a \rangle,
\end{equation}
where $|a\rangle$ has non-zero overlap with the ground state of $H$ as $t \to 0^+$, and $|b\rangle$ has non-zero overlap with the ground state of $H$ as $t \to \infty$. Since the level-spacing of $H_1'$ diverges as $t \to 0^+$ only the ground state projection of $H_1'$ at $t = 0$ contributes, and $|a\rangle$ no longer needs to be constant with respect to the $s$-wave variables $z_{0}^{\pm 1}$. This a rather important point since, as we recall from [1], the constant $s$-wave boundary condition at $t = 0$ follows from the fact that the value of field at the origin is not constrained. This is however not true in radial gauge as a result of the gauge-fixing constraint.

\[^8\]The reason is that in radial gauge the gauge fields are tangentially polarized, and so purely tangential excitations in two dimensions do not contribute to the field strength $F_{12}^2$. 


4  Gauge-field correlators

In this section we calculate two-point gauge-field correlators using the spher-
ical field formalism. This calculation can be done in several ways, including
numerically. For future applications, however, it is useful to have exact
expressions for use in perturbative calculations (e.g., for evaluating counter-
terms). Here we will obtain results by decomposing the fields as a combi-
nation of the ladder operators. Let us start with radial gauge. We have

\[ z_{n-1}^+ = \frac{1}{2\sqrt{2}} (iX_n^+ + iX_n^- - V_n^+ - V_n^-) \]  \hspace{1cm} (62)

\[ z_{n-1}^- = \frac{1}{2\sqrt{2}} (iX_n^+ + iX_n^- + V_n^+ + V_n^-) \]  \hspace{1cm} (63)

for \( n > 0 \) and

\[ z_{-1}^+ = \frac{i}{2} (X_0^+ + X_0^-) \]  \hspace{1cm} (64)

We would like to calculate the correlation function,

\[ f_{n-1,n-1}^{rad}(t, t') = \langle 0 | A_{n-1}^+ (t) A_{n-1}^+ (t') | 0 \rangle_{rad}. \]  \hspace{1cm} (65)

By angular momentum conservation \( f_{n-1,n-1}^{rad} \) vanishes unless \( n' = -n \). Also

\[ f_{n-1,n-1}^{rad}(t, t') = f_{n'-1,n-1}^{rad}(t', t), \]  \hspace{1cm} (66)

and so without loss of generality it suffices to consider \( f_{n-1,-n-1}^{rad} \) for \( n \geq 0 \).

For typographical convenience let us define

\[ \langle \{F\}_t \{G\}_t' \rangle = \theta(t - t') \frac{\langle b[U(\infty,t)F(t,t')GU(t',0)]a \rangle}{\langle b[U(\infty,0)]a \rangle} + \theta(t' - t) \frac{\langle b[U(\infty,t')GU(t,0)]a \rangle}{\langle b[U(\infty,0)]a \rangle}, \]  \hspace{1cm} (67)

where

\[ U(t_2, t_1) = T \exp \left[ - \int_{t_1}^{t_2} dt \ H \right]. \]  \hspace{1cm} (68)

For \( n \neq 0 \), we have

\[ f_{n-1,-n-1}^{rad} = \frac{1}{8} \langle \{iX_n^+ + iX_n^- - V_n^+ - V_n^-\}_t \{iX_n^+ + iX_n^- + V_n^+ + V_n^-\}_t' \rangle. \]  \hspace{1cm} (69)
Using the commutation properties of these ladder operators with $H(t)$, we find

$$f_{n-1,-n-1}^{rad} = -\frac{1}{4} \left[ \theta(t' - t) \frac{t}{T} + \theta(t - t') \frac{t'}{T} \right]. \quad (70)$$

For the special case $n = 0$, we obtain

$$f_{-1,-1}^{rad} = \frac{1}{4} \left\{ iX_0^+ + iX_0^- \right\}_t \left\{ iX_0^+ + iX_0^- \right\}_t' = -\frac{1}{4} \left[ \theta(t' - t) \frac{t}{T} + \theta(t - t') \frac{t'}{T} \right]. \quad (71)$$

These correlation functions are in agreement with results we obtain by decomposing the following expression $^3$ into partial waves:

$$\langle 0 | A^1(x) A^j(y) | 0 \rangle_{rad} \quad (72)$$

$$= \frac{1}{4\pi} \lim_{\epsilon \to 0^+} \left[ \delta^{ij} \log \frac{(x-y)^2}{L^2} - \partial_i \int_0^1 ds x^j \log \frac{(s\bar{x}-y)^2 + \epsilon}{L^2} - \partial_j \int_0^1 ds y^i \log \frac{(s\bar{y} - x)^2 + \epsilon}{L^2} \right. \right.$$ \left.

$$+ \partial_i \partial_j \int_0^1 ds \int_0^1 dt \bar{x} \cdot \bar{y} \log \frac{(s\bar{x}-\bar{y})^2 + \epsilon}{L^2} \right]. \quad (73)$$

The length scale $L$ is used to render the argument of the logarithm dimensionless. Its purpose, however, is only cosmetic since the gauge-field correlator is not infrared divergent in radial gauge and the dependence on $L$ cancels.$^9$

These same methods can be applied to gauge-field correlators in covariant gauge. We find

$$\langle 0 | A_{n-1}^{-1}(t) A_{n+1}^{-1}(t') | 0 \rangle_{cov} = \langle 0 | A_{-n+1}(t) A_{n+1}^{-1}(t') | 0 \rangle_{cov} \quad (74)$$

$$= \frac{\alpha - 1}{4} \delta_{n,0} \left[ \theta(t - t') \frac{t'}{T} + \theta(t' - t) \frac{t}{T} \right] \right.$$ \left.

$$- \frac{\alpha - 1}{4} \left[ \theta(t - t') \delta_{n,1} + \theta(t' - t) \delta_{n,1} \right]. \quad (75)$$

and for $n \neq 1$,

$$\langle 0 | A_{n-1}^{-1}(t) A_{n+1}^{-1}(t') | 0 \rangle_{cov} = \frac{\alpha + 1}{4|n-1|} \left[ \theta(t - t') \left( \frac{t}{T} \right)^{|n-1|} + \theta(t' - t) \left( \frac{t'}{T} \right)^{|n-1|} \right]. \quad (76)$$

In covariant gauge the $s$-wave correlator is infrared divergent,

$$\langle 0 | A_0^{-1}(t) A_0^{-1}(t') | 0 \rangle_{cov} = -\frac{\alpha + 1}{2} \left[ \theta(t - t') \log \frac{t}{L} + \theta(t' - t) \log \frac{t'}{L} \right]. \quad (77)$$

$^9$The radial gauge constraint $A^{-1} = -\epsilon^{2\alpha} A^{-1}$ pairs $s$-wave configurations with $d$-wave configurations. Since the $d$-wave is not infrared divergent, neither is the $s$-wave.
where \( \log L \) is infinite. This divergence is specific to two-dimensional gauge fields and does not occur in higher dimensions. If we include a regulating gauge-field mass, \( \mu \), we find that \( L \) scales as \( 1/\mu \) as \( \mu \to 0 \). These correlation functions are in agreement with the results we obtain by decomposing the following known expression into partial waves:

\[
\langle 0 | A^i(\vec{x}) A^j(\vec{y}) | 0 \rangle_{\text{cov}} = -\frac{1}{4\pi} \left[ \frac{\alpha+1}{2} \delta^{ij} \log \frac{(x-y)^2}{L^2} + \frac{(\alpha-1)(x-y)^i(x-y)^j}{(x-y)^2} \right].
\]

(76)

5 Gauge-invariant correlators

Let us now consider the two-point correlator of the gauge-invariant field strength \( F^{12} \). We can calculate the \( F^{12} \) correlator by differentiating the gauge-field correlators calculated in the previous section, but it is instructive to redo the calculation by coupling a source to \( F^{12} \). This time we describe the calculation in detail for covariant gauge. The same calculation can be done for radial gauge using similar methods. Let us start by quoting the result we expect. From free field theory we know

\[
\langle 0 | F^{12}(\vec{t}) F^{12}(\vec{t}') | 0 \rangle = \delta^2(\vec{t} - \vec{t}').
\]

(77)

The \( F^{12} \) correlator has a simple local structure, a consequence of the fact that in two dimensions gauge fields can be decomposed into scalar and longitudinal polarizations (borrowing Minkowski space terminology). There are no transverse polarizations to produce non-local contributions to the \( F^{12} \) correlator. We will see this happen explicitly in the calculations to follow.

The two-point correlator for partial waves of \( F^{12} \) is given by

\[
\langle 0 | F^{12}_n(t) F^{12}_{n'}(t') | 0 \rangle = \frac{1}{2\pi} \int d\theta d\theta' e^{-in\theta} e^{-in'\theta'} \langle 0 | F^{12}(\vec{t}) F^{12}(\vec{t}') | 0 \rangle,
\]

(78)

and we deduce

\[
\langle 0 | F^{12}_n(t) F^{12}_{n'}(t') | 0 \rangle = \frac{1}{2} \delta_{n,-n'} \delta(t - t').
\]

(79)

Let us now reproduce this result using the spherical field method. We return to the covariant gauge Lagrangian and couple a source \( K \) to \( F^{12} \),

\[
L = \int d\theta \text{d}t \left[ -\frac{1}{2} F^{12} F^{12} - \frac{1}{2\alpha} (\partial_i A^i)^2 + F^{12} K \right].
\]

(80)
The conjugate momenta are now
\[
\pi_{n+1}^{\pm} = \frac{i}{2} \left[ (1 - \frac{\pi}{\alpha}) F_{n+1}^{\pm} + (-1 - \frac{\pi}{\alpha}) F_{n}^{\pm} - i\sqrt{2} K_{n} \right],
\]
and the new Hamiltonian is
\[
H(K) = \sum_{n=0,\pm,\cdots} H_{n}(K),
\]
where
\[
H_{n}(K) = H_{n}(0) + \frac{i}{\sqrt{2}} K_{n} \pi_{n+1}^{\pm} - \frac{i}{\sqrt{2}} K_{n} \pi_{n}^{\pm} - \frac{t}{2} K_{n} K_{n}.
\]
In the Schrödinger language we have
\[
H_{n}(K) = H_{n}(0) + \frac{i}{\sqrt{2}} K_{n} \left( -\frac{n}{|n|} \frac{\partial}{\partial v_{|n|}} + i \frac{\partial}{\partial x_{|n|}} \right) - \frac{t}{2} K_{n} K_{n}
\]
for \( n \neq 0 \) and
\[
H_{0}(K) = H_{0}(0) - K_{0} \frac{\partial}{\partial x_{0}} - \frac{t}{2} K_{0} K_{0}.
\]
The vacuum persistence amplitude is
\[
Z(K) = \langle b | T \exp \left[ -\int_{0}^{\infty} dt H(K) \right] | a \rangle,
\]
and we evaluate the \( F_{n}^{12} \) correlator by functional differentiation with respect to \( K \),
\[
\langle 0 | F_{n}^{12}(t) F_{n}^{12}(t') | 0 \rangle = \frac{1}{Z(0)} \left. \frac{\partial}{\partial K_{n}(t)} \frac{\partial}{\partial K_{n}(t')} Z(K) \right|_{K=0}.
\]
It is clear from angular momentum conservation that this correlator vanishes unless \( n = -n' \), and so it suffices to compute
\[
\langle 0 | F_{n}^{12}(t) F_{-n}^{12}(t') | 0 \rangle.
\]
Since (89) is symmetric under the interchange \( n, t \leftrightarrow -n, t' \), we can also restrict \( n \geq 0 \). Differentiating with respect to the sources, we obtain, for \( n > 0 \),
\[
\langle 0 | F_{n}^{12}(t) F_{-n}^{12}(t') | 0 \rangle = -\frac{1}{2it} \left\{ \frac{\partial}{\partial v_{n}} + i \frac{\partial}{\partial x_{n}} \right\}_{t} \left\{ -\frac{\partial}{\partial v_{n}} + i \frac{\partial}{\partial x_{n}} \right\}_{t'} + \frac{1}{t} \delta(t - t').
\]
When \( n > 1 \) we can write \( \pm \frac{\partial}{\partial v_n} + i \frac{\partial}{\partial x_n} \) as a linear combination of \( U_n^+ \), \( V_n^- \), \( X_n^+ \), and \( Y_n^- \). These ladder operators are, however, acting in four different spaces. The matrix element of the operator

\[
U(\infty, t)(\frac{\partial}{\partial v_n} + i \frac{\partial}{\partial x_n})U(t, t')(\frac{\partial}{\partial v_n} + i \frac{\partial}{\partial x_n})U(t', 0)
\]  

(91)

from the ground state at \( t = 0 \) to the ground state at \( t = \infty \) vanishes. Consequently for \( n > 1 \) only the delta function contributes to the correlation function in (90). The same arguments apply for the case \( n = 0 \), and only the delta function contributes here as well.

We now turn to the special case \( n = 1 \). The relevant part of the Hamiltonian is

\[
H_1' = -\frac{1}{8t} \left( \frac{\partial^2}{\partial u_1^2} + \frac{\partial^2}{\partial v_1^2} \right) - \frac{\alpha}{8t} \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial y_1^2} \right) - \frac{1}{t} \left[ (u_1 + v_1) \left( \frac{\partial}{\partial u_1} + \frac{\partial}{\partial v_1} \right) + (x_1 + y_1) \left( \frac{\partial}{\partial x_1} + \frac{\partial}{\partial y_1} \right) \right].
\]  

(92)

The combinations \( u_1 - v_1 \) and \( x_1 - y_1 \) correspond with linear combinations of the s-wave variables \( z_0^{+1} \) and \( z_0^{-1} \). The initial configuration \( |a\rangle \) at \( t = 0 \) is constant with respect to \( z_0^{+1} \) and \( z_0^{-1} \), and therefore constant with respect to \( u_1 - v_1 \) and \( x_1 - y_1 \). We note that when \( H_1' \) and/or \( \pm \frac{\partial}{\partial u_1} + i \frac{\partial}{\partial x_1} \) acts upon \( |a\rangle \), the result is again a state constant in \( u_1 - v_1 \) and \( x_1 - y_1 \). It therefore suffices to compute the correlator restricted to the subspace which is constant in \( u_1 - v_1 \) and \( x_1 - y_1 \). In this space \( H_1' \) has the form

\[
H_1' \rightarrow -\frac{\alpha + 1}{8t} \left( \frac{\partial}{\partial u_1} + \frac{\partial}{\partial v_1} \right)^2 - \frac{\alpha + 1}{8t} \left( \frac{\partial}{\partial x_1} + \frac{\partial}{\partial y_1} \right)^2 - \frac{1}{t} \left[ (u_1 + v_1) \left( \frac{\partial}{\partial u_1} + \frac{\partial}{\partial v_1} \right) + (x_1 + y_1) \left( \frac{\partial}{\partial x_1} + \frac{\partial}{\partial y_1} \right) \right].
\]  

(93)

Comparing with (25), we see that this is analogous with the previous case for \( n = 0 \). We again find the result

\[
\langle 0 | F_{12}^{12}(t) F_{12}^{12}(t') | 0 \rangle = \frac{1}{t} \delta(t - t').
\]  

(94)

6 Summary

In this work we applied the methods of spherical field theory to free gauge fields. We analyzed two dimensional gauge fields in general covariant gauge.

\footnote{As noted before, this is due to the fact that in two dimensions there are no transverse polarizations.}
and radial gauge. In the process we have discussed several new aspects which resulted from the spin degrees of freedom as well as the masslessness of the gauge field. As we have seen, polarization mixing complicates the structure of the spherical field Hamiltonian. Nevertheless in radial gauge we were able to decompose the spherical field Hamiltonian as a sum of harmonic oscillators. We did the same for covariant gauge, but found that the $s$-wave part of the Hamiltonian has continuous spectrum. In relation to these differences, we also discussed issues regarding the $s$-wave boundary condition at $t = 0$.

We then used the spherical field evolution equations to calculate two-point correlators for the gauge fields and field-strength tensors $F^{12}$. Our presentation here is intended as a first introduction to the application of spherical field methods to gauge theories. Free gauge fields in higher dimensions can be treated by a straightforward generalization of these methods. Interacting gauge systems, however, include many interesting theoretical and computational issues not discussed here, and these are the subject of active research.

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