Certain Integral Operators of Analytic Functions

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Abstract: In this paper, two new integral operators are defined using the operator $DR_{m,n}^\lambda$, introduced and studied in previously published papers, defined by the convolution product of the generalized Sălăgean operator and Ruscheweyh operator. The newly defined operators are used for introducing several new classes of functions, and properties of the integral operators on these classes are investigated. Subordination results for the differential operator $DR_{m,n}^\lambda$ are also obtained.

Keywords: analytic functions; Hadamard product; differential operator; integral operator; starlike and convex functions; differential subordination

1. Introduction

The field of study involving differential and integral operators has been a constant research topic from the beginning of the field of study on analytic functions, with the first integral operator being introduced by Alexander in 1915 [1]. Differential and integral operators or combinations of those forms of operators are still emerging [2,3]. Sălăgean and Ruscheweyh operators play a significant role in research, as can be exemplified by very recent papers, such as [4–7]. Differential and integral operators were studied recently from many points of view, and quantum calculus has also been included in those studies, generating interesting outcomes which can be applied to other areas of mathematics and physics. In the recent survey-cum-expository review article [8], the author points out the interesting applications emerging from such an approach on operators. Examples of papers related to extensions of Sălăgean differential operators include [9,10]; $q$-extensions of other operators are investigated in [11–13], to mention only a few papers which can lead to further interesting results investigating their references.

The theory of integrals and derivatives of an arbitrary real or complex order has been applied in geometric function theory and has also emerged as a potentially useful direction in the mathematical modeling and analysis of real-life problems in applied sciences. Such examples include the research related to the analysis of the transmission dynamics of the dengue infection [14] and the new model of the human liver, which uses Caputo—Fabrizio fractional derivatives with the exponential kernel proposed in [15]. An interesting and innovative fractional derivative operator beyond the singular kernel has been investigated in [16] concerning its application to a heat transfer model. Fractional operators are recommended for solving other real-life problems, as can be seen in [17].

A family of integral operators associated with the Lommel functions of the first kind is introduced in [18], which, in particular, plays a very important role in the study of pure and applied mathematical sciences. Various interesting mapping and geometric properties for the integral operators introduced in the paper are also derived.

It is in this line of research that two new integral operators are introduced in this paper using a differential operator previously introduced and studied. Since operators contribute to the study of differential equations in terms of operator theory and functional analysis, some role of those operators in giving solutions of partial differential equations might be found after further research.
The most important notations and classes of functions used in the research are first recalled.

\( U = \{ z \in \mathbb{C} : |z| < 1 \} \) represents the unit disc of the complex plane, and the space of holomorphic functions in \( U \) is denoted by \( \mathcal{H}(U) \).

Let \( \mathcal{A}_n = \{ f \in \mathcal{H}(U) : f(z) = z + a_{n+1}z^{n+1} + \ldots, z \in U \} \), for \( n = 1, A_1 = \mathcal{A} \) and \( \mathcal{T} = \{ f \in \mathcal{A} : f(z) = z - \sum_{j=2}^{\infty} a_j z^j, z \in U \} \).

For \( a < 1 \), let \( \mathcal{S}^*(a) = \{ f \in \mathcal{A} : \Re \left( \frac{zf'(z)}{f(z)} \right) > a \} \) denote the class of starlike functions of order \( a \). For \( a = 0 \), the class of starlike functions is denoted by \( \mathcal{S}^* \).

The Hadamard product of the functions \( f, g \in \mathcal{A} \), \( f(z) = z + \sum_{j=2}^{\infty} a_j z^j \) and \( g(z) = z + \sum_{j=2}^{\infty} b_j z^j, z \in U \), is defined by \( (f \ast g)(z) = z + \sum_{j=2}^{\infty} a_j b_j z^j, z \in U \).

The results presented in this paper follow the research done on the operator \( DR_{\lambda}^{m,n} \), which was introduced and studied in [19] and further investigated in [20,21]. Definitions of generalized S̆aligaean and Ruscheweyh operators are needed for presenting the operator \( DR_{\lambda}^{m,n} \), and they are next reminded to be:

**Definition 1** ([Al Oboudi [22]]) For \( \lambda \geq 0, n \in \mathbb{N}, \text{ and } f \in \mathcal{A}, \text{ the operator } D_{\lambda}^{n} \text{ is defined by } D_{\lambda}^{n} : \mathcal{A} \to \mathcal{A} ,

\[
D_{\lambda}^{0} f(z) = f(z) \\
D_{\lambda}^{1} f(z) = (1 - \lambda)f(z) + \lambda z f'(z) = D_{\lambda} f(z) \\
\ldots
\]

\[
D_{\lambda}^{n} f(z) = (1 - \lambda)D_{\lambda}^{n-1} f(z) + \lambda z \left( D_{\lambda}^{n-1} f(z) \right)' = D_{\lambda} \left( D_{\lambda}^{n-1} f(z) \right), \quad z \in U.
\]

**Remark 1.** If \( f(z) = z + \sum_{j=2}^{\infty} a_j z^j \in \mathcal{A} \), then \( D_{\lambda}^{n} f(z) = z + \sum_{j=2}^{\infty} [\lambda(j - 1) + 1] a_j z^j, z \in U \).

If \( f(z) = z - \sum_{j=2}^{\infty} a_j z^j \in \mathcal{T} \), then \( D_{\lambda}^{n} f(z) = z - \sum_{j=2}^{\infty} [\lambda(j - 1) + 1] a_j z^j, z \in U \).

The S̆aligaean differential operator [23] is obtained for \( \lambda = 1 \).

**Definition 2** ([Ruscheweyh [24]]) For \( n \in \mathbb{N} \) and \( f \in \mathcal{A} \), the operator \( R_{n} \) is defined by \( R_{n} : \mathcal{A} \to \mathcal{A} ,

\[
R_{0} f(z) = f(z) \\
R_{1} f(z) = zf'(z) \\
\ldots
\]

\[
(n + 1) R_{n} f(z) = z (R_{n} f(z))' + n R_{n} f(z), \quad z \in U.
\]

**Remark 2.** If \( f(z) = z + \sum_{j=2}^{\infty} a_j z^j \in \mathcal{A} \), then \( R_{n} f(z) = z + \sum_{j=2}^{\infty} (n+j-1)! \frac{m}{n!(n-j+1)!} a_j z^j, z \in U \).

If \( f(z) = z - \sum_{j=2}^{\infty} a_j z^j \in \mathcal{T} \), then \( R_{n} f(z) = z - \sum_{j=2}^{\infty} (n+j-1)! \frac{m}{n!(n-j+1)!} a_j z^j, z \in U \).

**Definition 3** ([19]). Let \( n, m \in \mathbb{N} \) and \( \lambda \geq 0 \). Denote by \( DR_{\lambda}^{m,n} : \mathcal{A} \to \mathcal{A} \) the operator given by the Hadamard product of the generalized S̆aligaean operator \( D_{\lambda}^{m} \) and the Ruscheweyh operator \( R_{n} \),

\[
DR_{\lambda}^{m,n} f(z) = (DR_{\lambda}^{m} + R_{n}) f(z), \quad z \in U.
\]

**Remark 3** ([19]). If \( f(z) = z + \sum_{j=2}^{\infty} a_j z^j \in \mathcal{A} \), then

\[
DR_{\lambda}^{m,n} f(z) = z + \sum_{j=2}^{\infty} [\lambda(j - 1) + 1] \frac{m(n+j-1)!}{n!(n-j+1)!} a_j z^j, z \in U.
\]

If \( f(z) = z - \sum_{j=2}^{\infty} a_j z^j \in \mathcal{T} \), then

\[
DR_{\lambda}^{m,n} f(z) = z - \sum_{j=2}^{\infty} [\lambda(j - 1) + 1] \frac{m(n+j-1)!}{n!(n-j+1)!} a_j z^j, z \in U.
\]
For $\lambda \geq 0, m, n \in \mathbb{N}, f \in \mathcal{T}$, we obtain, after some computations

$$DR_\lambda^{m+1}f(z) = (1 - \lambda)DR_\lambda^{m}f(z) + \lambda z(DR_\lambda^{m}f(z))'$$

(2)

The following integral operators were introduced and investigated by Breaz and Breaz [25] and Breaz, Owa and Breaz [26]:

**Definition 4** ([25]). Consider the new general integral operator defined by the formula

$$F_{\gamma_1, \ldots, \gamma_l}(z) = \int_0^z \left( \frac{f_1(t)}{t} \right)^{\gamma_1} \cdots \left( \frac{f_l(t)}{t} \right)^{\gamma_l} dt,$$

where $f_i \in \mathcal{A}, \gamma_i \in \mathbb{R}, \gamma_i > 0, i \in \{1, 2, \ldots, l\}$.

**Definition 5** ([26]). Consider the new general integral operator defined by the formula

$$G_{\gamma_1, \ldots, \gamma_l}(z) = \int_0^z \left( f_1(t) \right)^{\gamma_1} \cdots \left( f_l(t) \right)^{\gamma_l} dt,$$

where $f_i \in \mathcal{A}, \gamma_i \in \mathbb{R}, \gamma_i > 0, i \in \{1, 2, \ldots, l\}$.

For the study presented in the paper, results from the theory of differential subordination must be used.

**Definition 6** ([27]). If $f$ and $g$ are analytic functions in $U$, we say that $f$ is subordinate to $g$, written $f \prec g$, if there is a function $w$ analytic in $U$, with $w(0) = 0, |w(z)| < 1$, for all $z \in U$ such that $f(z) = g(w(z))$ for all $z \in U$. If $g$ is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subseteq g(U)$.

**Definition 7** ([27]). Let $\psi: \mathbb{C}^3 \times U \rightarrow \mathbb{C}$ and $h$ be a univalent function in $U$. If $p$ is analytic in $U$ and satisfies the (second-order) differential subordination

$$\psi(p(z), zp'(z), z^2p''(z); z) < h(z), \quad \text{for } z \in U,$$

(3)

then $p$ is called a solution of the differential subordination. The univalent function $q$ is called a dominant of the solutions of the differential subordination, or more simply a dominant, if $p \prec q$ for all $p$ satisfying (3).

A dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants $q$ of (3) is said to be the best dominant of (3). The best dominant is unique up to a rotation of $U$.

**Lemma 1** ([27]). Let $q$ be univalent in $U$ and let $\phi$ be analytic in a domain containing $q(U)$. If $\frac{zq'(z)}{\phi(q(z))}$ is starlike, then

$$z\psi'(z)\phi(q(z)) < zq'(z)\phi(q(z)), \quad z \in U,$$

then $\psi(z) \prec q(z)$ and $q(z)$ is the best dominant.

**Lemma 2** ([28]). Assume that $p$ and $q$ are analytic in $U$, $q$ is convex univalent, $\alpha, \beta$ and $\gamma$ are complex and $\gamma \neq 0$. Further assume that

$$\text{Re} \left[ \frac{\alpha}{\gamma} + \frac{2\beta}{\gamma}q(z) + \left( 1 + \frac{zq''(z)}{q(z)} \right) \right] > 0.$$

If $p(z) = 1 + cz + \ldots$ is analytic in $U$ and satisfies

$$\alpha p(z) + \beta p^2(z) + \gamma z p'(z) \prec \alpha q(z) + \beta q^2(z) + \gamma z q'(z),$$

then $p(z) \prec q(z)$ and $q(z)$ is the best dominant.
Theorem 1 ([29]). Let $q$ be convex univalent and $0 < \beta \leq 1$,
\[
\Re\left[\frac{1-\beta}{\beta} + 2q(z) + \left(1 + \frac{zq''(z)}{q(z)}\right)\right] > 0.
\]

If $f \in A$ satisfies
\[
\frac{zf'(z)}{f(z)} + \beta z f''(z) < (1 - \beta)q(z) + \beta q^2(z) + \beta zq'(z),
\]
then $\frac{zf(z)}{f'(z)} < q(z)$ and $q(z)$ is the best dominant.

Theorem 2 ([29]). Let $q$ be analytic in $U$, $q(0) = 1$ and $h(z) = \frac{zq'(z)}{q(z)}$ be starlike univalent in $U$. If $f \in A$ satisfies
\[
\frac{(zf(z))''}{f'(z)} - 2zf'(z) < h(z),
\]
then $\frac{zf(z)}{f'(z)} < q(z)$, and $q(z)$ is the best dominant.

Using the operator $DR^{m,n}_\lambda$ given in Definition 3 and inspired by the operators introduced in [25,26], two new integral operators are introduced in Section 2 of the paper alongside some new classes of analytic functions defined by means of these operators. Two original lemmas are proved, which are related to the new integral operators which will be used for the proofs of the original results in the following sections. Investigations on the newly introduced classes provide theorems containing conditions for functions in class $T$ to belong to them in Section 3. The new operators are connected to the newly introduced classes in Section 4. In the last section, Theorem 1 and Theorem 2 cited in the Introduction are extended to the operator $DR^{m,n}_\lambda$, giving the best dominants of certain differential subordinations.

2. Main Results

Definition 8. We define the integral operators $F^{m,n}_{\lambda_1,\gamma_1,...,\gamma_l}$ and $G^{m,n}_{\lambda_1,\gamma_1,...,\gamma_l}$ for functions $f_i \in T$, $\gamma_i \in \mathbb{R}, i \in \{1,2,...,l\}$, as follows
\[
F^{m,n}_{\lambda_1,\gamma_1,...,\gamma_l}(z) = \int_0^z \left(\frac{DR^{m,n}_\lambda f_1(t)}{t}\right)^{\gamma_1} \cdots \left(\frac{DR^{m,n}_\lambda f_l(t)}{t}\right)^{\gamma_l} dt, \tag{4}
\]
\[
G^{m,n}_{\lambda_1,\gamma_1,...,\gamma_l}(z) = \int_0^z \left(\frac{(DR^{m,n}_\lambda f_1(t))'}{t}\right)^{\gamma_1} \cdots \left(\frac{(DR^{m,n}_\lambda f_l(t))'}{t}\right)^{\gamma_l} dt, \tag{5}
\]
for $\lambda \geq 0$, $m,n \in \mathbb{N}, z \in U$.

By using the differential operator $DR^{m,n}_\lambda$ given in Definition 3 and the integral operators $F^{m,n}_{\lambda,\gamma_1,...,\gamma_l}$ and $G^{m,n}_{\lambda,\gamma_1,...,\gamma_l}$, first we introduce some subclasses of analytic functions $f \in T$.

Definition 9. The class $R(\delta)$, $\delta > 1$, consists of the functions $f \in T$ that satisfy the inequality
\[
\Re\left(\frac{z(DR^{m,n}_\lambda f(z))'}{DR^{m,n}_\lambda f(z)}\right) < \delta, z \in U.
\]

Definition 10. The class $D(\delta)$, $\delta > 1$, consists of the functions $f \in T$ that satisfy the inequality
\[
\Re\left(\frac{z(DR^{m,n}_\lambda f(z))''}{(DR^{m,n}_\lambda f(z))'}\right) < \delta, z \in U.
\]
Definition 11. The class $\mathcal{R}A(\beta, \mu), 0 \leq \beta < 1, 0 < \mu \leq 1$, consists of the functions $f \in \mathcal{T}$ that satisfy the inequality

$$\left|z\left(\frac{DR_{\lambda}^{m,n} f(z)}{DR_{\lambda}^{m,n} f(z)}\right)' - 1\right| < \mu \left|z\left(\frac{DR_{\lambda}^{m,n} f(z)}{DR_{\lambda}^{m,n} f(z)}\right)' - 1\right|, z \in U.$$  

Definition 12. The class $\mathcal{D}A(\beta, \mu), 0 \leq \beta < 1, 0 < \mu \leq 1$, consists of the functions $f \in \mathcal{T}$ that satisfy the inequality

$$\left|z\left(\frac{DR_{\lambda}^{m,n} f(z)}{DR_{\lambda}^{m,n} f(z)}\right)'' \right| < \mu \left|1 + \frac{z\left(\frac{DR_{\lambda}^{m,n} f(z)}{DR_{\lambda}^{m,n} f(z)}\right)''}{\left(\frac{DR_{\lambda}^{m,n} f(z)}{DR_{\lambda}^{m,n} f(z)}\right)'} + 1\right|, z \in U.$$  

Definition 13. The class $\mathcal{LAF}(\lambda, \beta, \mu, \gamma_1, \ldots, \gamma_l), \lambda \geq 0, \beta \geq 0, -1 \leq \mu \leq 1$, consists of the family of functions $f_i, i \in \{1, 2, \ldots, l\}$ that satisfy the inequality

$$\text{Re}\left(1 + \frac{z\left(F_{\lambda,\gamma_1,\ldots,\gamma_l}^{m,n}(z)\right)''}{\left(F_{\lambda,\gamma_1,\ldots,\gamma_l}^{m,n}(z)\right)'}\right) \geq \beta \left|\frac{z\left(F_{\lambda,\gamma_1,\ldots,\gamma_l}^{m,n}(z)\right)''}{\left(F_{\lambda,\gamma_1,\ldots,\gamma_l}^{m,n}(z)\right)'} + \mu\right|,$$

where $F_{\lambda,\gamma_1,\ldots,\gamma_l}^{m,n}$ is defined in (4).

Definition 14. The class $\mathcal{LAG}(\lambda, \beta, \mu, \gamma_1, \ldots, \gamma_l), \lambda \geq 0, \beta \geq 0, -1 \leq \mu \leq 1$, consists of the family of functions $f_i, i \in \{1, 2, \ldots, l\}$ that satisfy the inequality

$$\text{Re}\left(1 + \frac{z\left(G_{\lambda,\gamma_1,\ldots,\gamma_l}^{m,n}(z)\right)''}{\left(G_{\lambda,\gamma_1,\ldots,\gamma_l}^{m,n}(z)\right)'}\right) \geq \beta \left|\frac{z\left(G_{\lambda,\gamma_1,\ldots,\gamma_l}^{m,n}(z)\right)''}{\left(G_{\lambda,\gamma_1,\ldots,\gamma_l}^{m,n}(z)\right)'} + \mu\right|,$$

where $G_{\lambda,\gamma_1,\ldots,\gamma_l}^{m,n}$ is defined in (5).

Lemma 3. For $f_1(z) = z - \sum_{j=2}^{\infty} a_{ij} z^j \in \mathcal{T}, i \in \{1, \ldots, l\}$, we get

$$\frac{z\left(F_{\lambda,\gamma_1,\ldots,\gamma_l}^{m,n}(z)\right)''}{\left(F_{\lambda,\gamma_1,\ldots,\gamma_l}^{m,n}(z)\right)'} = \sum_{i=1}^{l} \gamma_i \left[- \sum_{j=2}^{\infty} \lambda (j-1) + 1 |\frac{m(n+j-1)!}{n!(j-1)!} (a_{ij})^2 z^{j-1}\right],$$

where $F_{\lambda,\gamma_1,\ldots,\gamma_l}^{m,n}$ is introduced in (4).

Proof. Let $f_1(z) = z - \sum_{j=2}^{\infty} a_{ij} z^j, i \in \{1, \ldots, l\}$; then

$$(DR_{\lambda}^{m,n} f_1(z))' = 1 - \sum_{j=2}^{\infty} |\lambda (j-1) + 1 |\frac{m(n+j-1)!}{n!(j-1)!} (a_{ij})^2 z^{j-1}, z \in U.$$  

We obtain

$$F_{\lambda,\gamma_1,\ldots,\gamma_l}^{m,n}(z) = \left(\frac{DR_{\lambda}^{m,n} f_1(z)}{z}\right)^{\gamma_1} \cdots \left(\frac{DR_{\lambda}^{m,n} f_1(z)}{z}\right)^{\gamma_l},$$

so

$$F_{\lambda,\gamma_1,\ldots,\gamma_l}^{m,n}(z) = E_1 \left(\frac{F_{\lambda,\gamma_1,\ldots,\gamma_l}^{m,n}(z)}{DR_{\lambda}^{m,n} f_1(z)}\right)^{\gamma_1} + \ldots + E_l \left(\frac{F_{\lambda,\gamma_1,\ldots,\gamma_l}^{m,n}(z)}{DR_{\lambda}^{m,n} f_1(z)}\right)^{\gamma_l} \frac{z}{DR_{\lambda}^{m,n} f_1(z)},$$
where
\[ E_i = \gamma_i \frac{z(DR_{\lambda}^{m,n} f_i(z))'}{z^2} - DR_{\lambda}^{m,n} f_i(z), \quad i \in \{1, \ldots, l\}. \]

We calculate the expression
\[ \frac{z\left(F_{\lambda,1,\ldots,\gamma_1}(z)^{''}\right)}{\left(F_{\lambda,1,\ldots,\gamma_1}(z)^{''}\right)}, \]

\[ \frac{z\left(F_{\lambda,1,\ldots,\gamma_1}(z)^{''}\right)}{\left(F_{\lambda,1,\ldots,\gamma_1}(z)^{''}\right)} = \sum_{i=1}^l \gamma_i \left[ \frac{z(DR_{\lambda}^{m,n} f_i(z))'}{DR_{\lambda}^{m,n} f_i(z)} - 1 \right]. \]

We find
\[ \frac{z\left(F_{\lambda,1,\ldots,\gamma_1}(z)^{''}\right)}{\left(F_{\lambda,1,\ldots,\gamma_1}(z)^{''}\right)} = \sum_{i=1}^l \gamma_i \left[ \frac{z(DR_{\lambda}^{m,n} f_i(z))'}{DR_{\lambda}^{m,n} f_i(z)} - 1 \right]. \]

\[ \frac{z\left(F_{\lambda,1,\ldots,\gamma_1}(z)^{''}\right)}{\left(F_{\lambda,1,\ldots,\gamma_1}(z)^{''}\right)} = \sum_{i=1}^l \gamma_i \left[ \frac{z(DR_{\lambda}^{m,n} f_i(z))'}{DR_{\lambda}^{m,n} f_i(z)} - 1 \right]. \]

□

**Lemma 4.** For \( f_i(z) = z - \sum_{j=2}^\infty a_{ij} z^j, \ i \in \{1, \ldots, l\}, \) we obtain
\[ \frac{z\left(G_{\lambda,1,\ldots,\gamma_1}(z)^{''}\right)}{\left(G_{\lambda,1,\ldots,\gamma_1}(z)^{''}\right)} = - \sum_{i=1}^l \gamma_i \left[ \frac{\sum_{j=2}^\infty \lambda(j - 1) + 1} {1 - \sum_{j=2}^\infty \lambda(j - 1) + 1} \right] \]
where \( G_{\lambda,1,\ldots,\gamma_1}(z) \) is introduced in (5).

**Proof.** Let \( f_i(z) = z - \sum_{j=2}^\infty a_{ij} z^j, \ i \in \{1, \ldots, l\}. \) We deduce
\[ \left(G_{\lambda,1,\ldots,\gamma_1}(z)^{'}\right)^\gamma_i \left( (DR_{\lambda}^{m,n} f_i(z))^{'} \right)^\gamma_i \]
so
\[ \left(G_{\lambda,1,\ldots,\gamma_1}(z)^{''}\right)^{'} = \sum_{i=1}^l \gamma_i \left( G_{\lambda,1,\ldots,\gamma_1}(z) \right)^{'} (DR_{\lambda}^{m,n} f_i(z))^{''}. \]

Next, we calculate the expression
\[ \frac{z\left(G_{\lambda,1,\ldots,\gamma_1}(z)^{''}\right)}{\left(G_{\lambda,1,\ldots,\gamma_1}(z)^{''}\right)} = \sum_{i=1}^l \gamma_i \left[ \frac{z(DR_{\lambda}^{m,n} f_i(z))^{''}}{(DR_{\lambda}^{m,n} f_i(z))^{''}} \right]. \]

\[ \frac{z\left(G_{\lambda,1,\ldots,\gamma_1}(z)^{''}\right)}{\left(G_{\lambda,1,\ldots,\gamma_1}(z)^{''}\right)} = \sum_{i=1}^l \gamma_i \left[ \frac{z(DR_{\lambda}^{m,n} f_i(z))^{''}}{(DR_{\lambda}^{m,n} f_i(z))^{''}} \right]. \]
Hence
\[
\frac{z(G_{\lambda, \gamma_1, \ldots, \gamma_l}(z))''}{(G_{\lambda, \gamma_1, \ldots, \gamma_l}(z))'} = -\sum_{i=1}^{l} \gamma_i \left[ \frac{\sum_{j=2}^{\infty}[\lambda(j-1) + 1]^m(n+j-1)!}{n!(j-1)!} (a_{i,j})^2 z^{j-1} \right].
\]

\[\Box\]

3. The Classes $\mathcal{LAF}(\lambda, \beta, \mu, \gamma_1, \ldots, \gamma_l)$ and $\mathcal{LAG}(\lambda, \beta, \mu, \gamma_1, \ldots, \gamma_l)$

Next we give sufficient conditions for the functions $f_i$ to belong to the classes $\mathcal{LAF}(\lambda, \beta, \mu, \gamma_1, \ldots, \gamma_l)$ and $\mathcal{LAG}(\lambda, \beta, \mu, \gamma_1, \ldots, \gamma_l)$.

**Theorem 3.** Let $f_i \in T$, $i \in \{1, \ldots, l\}$. Then $f_i \in \mathcal{LAF}(\lambda, \beta, \mu, \gamma_1, \ldots, \gamma_l)$ for $i \in \{1, \ldots, l\}$ if and only if
\[
\sum_{i=1}^{l} \gamma_i (\beta + 1) \left[ \frac{\sum_{j=2}^{\infty}[\lambda(j-1) + 1]^m(n+j-1)!}{n!(j-1)!} (a_{i,j})^2 \right] \leq 1 - \mu,
\]
where $-1 \leq \mu < 1$, $\beta \geq 0$.

**Proof.** When (6) holds, we have to show that
\[
\beta \left| \frac{z(F_{\lambda, \gamma_1, \ldots, \gamma_l}(z))''}{(F_{\lambda, \gamma_1, \ldots, \gamma_l}(z))'} \right| - \text{Re} \left( \frac{z(F_{\lambda, \gamma_1, \ldots, \gamma_l}(z))''}{(F_{\lambda, \gamma_1, \ldots, \gamma_l}(z))'} \right) \leq (\beta + 1) \left| \frac{z(F_{\lambda, \gamma_1, \ldots, \gamma_l}(z))''}{(F_{\lambda, \gamma_1, \ldots, \gamma_l}(z))'} \right|.
\]

We have
\[
\beta \left| \frac{z(F_{\lambda, \gamma_1, \ldots, \gamma_l}(z))''}{(F_{\lambda, \gamma_1, \ldots, \gamma_l}(z))'} \right| - \text{Re} \left( \frac{z(F_{\lambda, \gamma_1, \ldots, \gamma_l}(z))''}{(F_{\lambda, \gamma_1, \ldots, \gamma_l}(z))'} \right) \leq (\beta + 1) \left| \frac{z(F_{\lambda, \gamma_1, \ldots, \gamma_l}(z))''}{(F_{\lambda, \gamma_1, \ldots, \gamma_l}(z))'} \right|.
\]

Applying Lemma 3, we obtain
\[
(\beta + 1) \left| \frac{z(F_{\lambda, \gamma_1, \ldots, \gamma_l}(z))''}{(F_{\lambda, \gamma_1, \ldots, \gamma_l}(z))'} \right| \leq (\beta + 1) \left| \frac{z(F_{\lambda, \gamma_1, \ldots, \gamma_l}(z))''}{(F_{\lambda, \gamma_1, \ldots, \gamma_l}(z))'} \right| - \text{Re} \left( \frac{z(F_{\lambda, \gamma_1, \ldots, \gamma_l}(z))''}{(F_{\lambda, \gamma_1, \ldots, \gamma_l}(z))'} \right).
\]

So, we deduce
\[
\beta \left| \frac{z(F_{\lambda, \gamma_1, \ldots, \gamma_l}(z))''}{(F_{\lambda, \gamma_1, \ldots, \gamma_l}(z))'} \right| - \text{Re} \left( \frac{z(F_{\lambda, \gamma_1, \ldots, \gamma_l}(z))''}{(F_{\lambda, \gamma_1, \ldots, \gamma_l}(z))'} \right) \leq 1 - \mu.
\]
or equivalently

\[
\text{Re}\left(1 + \frac{z\left(F_{\lambda,\gamma_1,...,\gamma_l}^{m,n}(z)\right)''}{F_{\lambda,\gamma_1,...,\gamma_l}^{m,n}(z)}\right) \geq \beta \left|\frac{z\left(F_{\lambda,\gamma_1,...,\gamma_l}^{m,n}(z)\right)''}{F_{\lambda,\gamma_1,...,\gamma_l}^{m,n}(z)}\right| + \mu.
\]

Hence \(f_i \in \mathcal{LAF}(\lambda, \beta, \mu, \gamma_1, \ldots, \gamma_l)\).
Conversely, suppose \(f_i \in \mathcal{LAF}(\lambda, \beta, \mu, \gamma_1, \ldots, \gamma_l)\), \(i \in \{1, \ldots, l\}\). From Lemma 3 and (6), we get

\[
1 - \sum_{i=1}^{l} \gamma_i \left[ \frac{\sum_{j=2}^{\infty} [\lambda(j - 1) + 1][n(j-1)!^j](a_{ij})^2 |z|^j-1}{1 - \sum_{j=2}^{\infty} [\lambda(j - 1) + 1][n(j-1)!^j](a_{ij})^2 |z|^j-1} \right] \geq \beta \left[ \frac{\sum_{j=2}^{\infty} [\lambda(j - 1) + 1][n(j-1)!^j](a_{ij})^2 |z|^j-1}{1 - \sum_{j=2}^{\infty} [\lambda(j - 1) + 1][n(j-1)!^j](a_{ij})^2 |z|^j-1} \right] + \mu,
\]

which is equivalent to

\[
\sum_{i=1}^{l} \gamma_i \beta \left[ \frac{\sum_{j=2}^{\infty} [\lambda(j - 1) + 1][n(j-1)!^j](a_{ij})^2 |z|^j-1}{1 - \sum_{j=2}^{\infty} [\lambda(j - 1) + 1][n(j-1)!^j](a_{ij})^2 |z|^j-1} \right] \leq 1 - \mu,
\]

which reduces to

\[
\sum_{i=1}^{l} \gamma_i (\beta + 1) \left[ \frac{\sum_{j=2}^{\infty} [\lambda(j - 1) + 1][n(j-1)!^j](a_{ij})^2 |z|^j-1}{1 - \sum_{j=2}^{\infty} [\lambda(j - 1) + 1][n(j-1)!^j](a_{ij})^2 |z|^j-1} \right] \leq 1 - \mu,
\]

when \(z \to 1^+\) along the real axis, we deduce the inequality (6). \(\square\)

Applying Lemma 4 and the same technique as in the proof of Theorem 3, we obtain

**Theorem 4.** Let \(f_i \in \mathcal{T}, i \in \{1, \ldots, l\}\). Then \(f_i \in \mathcal{LAG}(\lambda, \beta, \mu, \gamma_1, \ldots, \gamma_l)\), \(i \in \{1, \ldots, l\}\) if and only if

\[
\sum_{i=1}^{l} \gamma_i (\beta + 1) \left[ \frac{\sum_{j=2}^{\infty} [\lambda(j - 1) + 1][n(j-1)!^j](a_{ij})^2 |z|^j-1}{1 - \sum_{j=2}^{\infty} [\lambda(j - 1) + 1][n(j-1)!^j](a_{ij})^2 |z|^j-1} \right] \leq 1 - \mu,
\]

where \(-1 \leq \mu < 1, \beta \geq 0\).

4. The Integral Operators \(F_{\lambda,\gamma_1,...,\gamma_l}^{m,n}\) and \(G_{\lambda,\gamma_1,...,\gamma_l}^{m,n}\) on the Classes \(\mathcal{R}(\delta), \mathcal{D}(\delta), \mathcal{DA}(\beta, \mu)\) and \(\mathcal{RA}(\beta, \mu)\)

We give some of the properties of the integral operators \(F_{\lambda,\gamma_1,...,\gamma_l}^{m,n}\) and \(G_{\lambda,\gamma_1,...,\gamma_l}^{m,n}\) on the classes \(\mathcal{R}(\delta), \mathcal{D}(\delta), \mathcal{DA}(\beta, \mu)\) and \(\mathcal{RA}(\beta, \mu)\).

**Theorem 5.** Let \(\gamma_i \in \mathbb{R}, \gamma_i > 0, i \in \{1, \ldots, l\}\), \(f_i \in \mathcal{T}\) and \(\left| \frac{\partial F_{\lambda,\gamma_1,...,\gamma_l}^{m,n}(z)}{\partial \lambda} \right| < M_i\). If \(f_i \in \mathcal{RA}(\beta_i, \mu_i)\), then \(F_{\lambda,\gamma_1,...,\gamma_l}^{m,n}(z) \in \mathcal{D}(\delta')\), where \(\delta' = 1 + \sum_{i=1}^{l} \gamma_i \left[ \beta_i M_i + 1 \right], z \in U\).
Proof. It is clear from (4) that $F_{\lambda,\gamma_1,\ldots,\gamma_l}^{m,n} \in T$.
On differentiating $F_{\lambda,\gamma_1,\ldots,\gamma_l}^{m,n}$ given by (4), we get
\[
\left( F_{\lambda,\gamma_1,\ldots,\gamma_l}^{m,n}(z) \right)' = \prod_{i=1}^{l} \left( \frac{DR_{\lambda,i}^{m,n} f_i(z)}{z} \right)^{\gamma_i}.
\] (8)
Differentiating (8) logarithmically and multiplying by $z$, we obtain
\[
z \left( F_{\lambda,\gamma_1,\ldots,\gamma_l}^{m,n}(z) \right)'' = \sum_{i=1}^{l} \gamma_i \left[ z \left( DR_{\lambda,i}^{m,n} f_i(z) \right)' - 1 \right],
\]
equivalently
\[
1 + z \left( F_{\lambda,\gamma_1,\ldots,\gamma_l}^{m,n}(z) \right)'' = 1 + \sum_{i=1}^{l} \gamma_i \left[ z \left( DR_{\lambda,i}^{m,n} f_i(z) \right)' - 1 \right].
\] (9)
Taking a real part of both sides of (9), we have
\[
\text{Re} \left[ 1 + z \left( F_{\lambda,\gamma_1,\ldots,\gamma_l}^{m,n}(z) \right)'' \right] = 1 + \sum_{i=1}^{l} \gamma_i \left| z \left( DR_{\lambda,i}^{m,n} f_i(z) \right)' - 1 \right|.
\]
Since $f_i \in R.A(\beta_i, \mu_i)$, for $i \in \{1, \ldots, l\}$, we deduce
\[
\text{Re} \left[ 1 + z \left( F_{\lambda,\gamma_1,\ldots,\gamma_l}^{m,n}(z) \right)'' \right] < 1 + \sum_{i=1}^{l} \gamma_i \left| \beta_i \frac{DR_{\lambda,i}^{m,n} f_i(z)}{DR_{\lambda,i}^{m,n} f_i(z)} + 1 \right| < 1 + \sum_{i=1}^{l} \gamma_i \beta_i \left| \frac{DR_{\lambda,i}^{m,n} f_i(z)}{DR_{\lambda,i}^{m,n} f_i(z)} \right| + \sum_{i=1}^{l} \gamma_i \mu_i < 1 + \sum_{i=1}^{l} \gamma_i \mu_i (\beta_i M_i + 1).
\]
As $\sum_{i=1}^{l} \gamma_i \mu_i (\beta_i M_i + 1) > 0$, $F_{\lambda,\gamma_1,\ldots,\gamma_l}^{m,n}(z) \in D(\delta')$, where $\delta' = 1 + \sum_{i=1}^{l} \gamma_i \mu_i (\beta_i M_i + 1)$.
\]
In Theorem 5, substituting $l = 1$, $\gamma_1 = \gamma$, $M_1 = M$, $f_1 = f$, we get the following corollary:

**Corollary 1.** Let $\gamma \in \mathbb{R}$, $\gamma > 0$, $f \in T$ and $\left\{ \frac{f(z)}{f'(z)} \right\} < M$, where $M$ is fixed. If $f \in R.A(\beta, \mu)$, then $\int_{0}^{z} \left( \frac{f(t)}{f'(t)} \right)^{\gamma} dt \in D(\delta')$, where $\delta' = \mu \gamma (1 + \beta M) + 1$, $z \in U$.

**Theorem 6.** Let $\gamma_i > 0$, $f_i \in T$, $i \in \{1, \ldots, l\}$, with $\delta_i > 1$. Then $F_{\lambda,\gamma_1,\ldots,\gamma_l}^{m,n}(z) \in D(\delta')$, where $\delta' = \sum_{i=1}^{l} \gamma_i (\delta_i - 1) + 1$, $z \in U$.

**Proof.** From (9), we have
\[
\text{Re} \left[ 1 + z \left( F_{\lambda,\gamma_1,\ldots,\gamma_l}^{m,n}(z) \right)'' \right] = \sum_{i=1}^{l} \gamma_i \text{Re} \left( z \left( DR_{\lambda,i}^{m,n} f_i(z) \right)' \right) - \sum_{i=1}^{l} \gamma_i + 1
\]
Let \( i \leq 1 \gamma i(i - 1) + 1 = \sum_{i = 1}^{l} \gamma i(\delta i - 1) + 1. \)

Since \( \delta i > 1 \), evidently \( \sum_{i = 1}^{l} \gamma i(\delta i - 1) > 0 \) and hence \( F_{\lambda, \gamma_1, \ldots, \gamma_l}^{m,n}(z) \in \mathcal{D}(\delta') \), with \( \delta' = \sum_{i = 1}^{l} \gamma i(\delta i - 1) + 1. \) \end{proof}

In Theorem 6 letting \( l = 1, \gamma_1 = \gamma, \delta_1 = \delta \) and \( f_1 = f \), we get the following corollary

**Corollary 2.** Let \( \gamma > 0, f \in \mathcal{R}(\delta) \) with \( \delta > 1 \). Then \( \int_{0}^{1}(\frac{f'(t)}{t})^{\gamma} dt \in \mathcal{D}(\delta'), \) where \( \delta' = \gamma(\delta - 1) + 1, z \in \mathcal{U}. \)

**Theorem 7.** Let \( \gamma i > 0 \) and \( f_i \in \mathcal{D}(\delta_i) \), for \( i \in \{1, 2, \ldots, l\} \), with \( \delta i > 1 \). Then \( G_{\lambda, \gamma_1, \ldots, \gamma_l}^{m,n}(z) \in \mathcal{D}(\delta'), \) with \( \delta' = \sum_{i = 1}^{l} \gamma i(\delta i - 1) + 1, z \in \mathcal{U}. \)

**Proof.** From the definition of \( G_{\lambda, \gamma_1, \ldots, \gamma_l}^{m,n} \) given by (5), we have

\[
\text{Re} \left( 1 + \frac{z(\frac{G_{\lambda, \gamma_1, \ldots, \gamma_l}^{m,n}(z)}{G_{\lambda, \gamma_1, \ldots, \gamma_l}^{m,n}(z)}')'}{G_{\lambda, \gamma_1, \ldots, \gamma_l}^{m,n}(z)} \right) = \sum_{i = 1}^{l} \gamma i \left\{ \text{Re} \left[ 1 + \left( \frac{z(\frac{G_{\lambda, \gamma_1, \ldots, \gamma_l}^{m,n}(z)}{G_{\lambda, \gamma_1, \ldots, \gamma_l}^{m,n}(z)}')'}{G_{\lambda, \gamma_1, \ldots, \gamma_l}^{m,n}(z)} \right) \right] \right\} - \sum_{i = 1}^{l} \gamma i + 1
\]

\[
< \sum_{i = 1}^{l} \gamma i(\delta i - 1) + 1.
\]

As \( \delta i > 1 \), it is clear that \( \sum_{i = 1}^{l} \gamma i(\delta i - 1) > 0 \) and hence we obtain that \( G_{\lambda, \gamma_1, \ldots, \gamma_l}^{m,n}(z) \in \mathcal{D}(\delta'), \) where \( \delta' = \gamma(\delta - 1) + 1, z \in \mathcal{U}. \) \end{proof}

In Theorem 7, substituting \( l = 1, \gamma_1 = \gamma, \delta_1 = \delta \) and \( f_1 = f \), we get the following corollary

**Corollary 3.** Let \( \gamma > 0 \) and \( f \in \mathcal{D}(\delta) \) with \( \delta > 1 \). Then \( \int_{0}^{1}(f'(t))^{\gamma} dt \in \mathcal{D}(\delta'), \) where \( \delta' = \gamma(\delta - 1) + 1, z \in \mathcal{U}. \)

**Theorem 8.** Let \( f_i \in \mathcal{D}(\mathcal{A}(\beta_i, \mu_i)), \gamma_i \in \mathcal{R} \) with \( \gamma_i > 0 \) and \( \left| \frac{(\frac{D\mathcal{R}^{m,n}_i f_i(z)}{D\mathcal{R}^{m,n}_i f_i(z)})'}{(\frac{D\mathcal{R}^{m,n}_i f_i(z)}{D\mathcal{R}^{m,n}_i f_i(z)})'} \right| < M_i, i \in \{1, \ldots, l\}. \)

Then \( G_{\lambda, \gamma_1, \ldots, \gamma_l}^{m,n}(z) \in \mathcal{D}(\delta'), \) where \( \delta' = \sum_{i = 1}^{l} \gamma_i |\beta_i(1 + M_i) + 1| + 1, z \in \mathcal{U}. \)

**Proof.** From the definition of \( G_{\lambda, \gamma_1, \ldots, \gamma_l}^{m,n} \) given by (5), we have

\[
\text{Re} \left( 1 + \frac{z(\frac{G_{\lambda, \gamma_1, \ldots, \gamma_l}^{m,n}(z)}{G_{\lambda, \gamma_1, \ldots, \gamma_l}^{m,n}(z)}')'}{G_{\lambda, \gamma_1, \ldots, \gamma_l}^{m,n}(z)} \right) \leq \sum_{i = 1}^{l} \gamma i \left| \frac{z(\frac{D\mathcal{R}^{m,n}_i f_i(z)}{D\mathcal{R}^{m,n}_i f_i(z)})'}{(\frac{D\mathcal{R}^{m,n}_i f_i(z)}{D\mathcal{R}^{m,n}_i f_i(z)})'} \right|
\]

\[
< \sum_{i = 1}^{l} \gamma i \beta_i \left( 1 + \frac{z(\frac{D\mathcal{R}^{m,n}_i f_i(z)}{D\mathcal{R}^{m,n}_i f_i(z)})'}{(\frac{D\mathcal{R}^{m,n}_i f_i(z)}{D\mathcal{R}^{m,n}_i f_i(z)})'} \right) + 1 + 1
\]

\[
< \sum_{i = 1}^{l} \gamma i \beta_i \left( 1 + \frac{z(\frac{D\mathcal{R}^{m,n}_i f_i(z)}{D\mathcal{R}^{m,n}_i f_i(z)})'}{(\frac{D\mathcal{R}^{m,n}_i f_i(z)}{D\mathcal{R}^{m,n}_i f_i(z)})'} \right) + \sum_{i = 1}^{l} \gamma i \beta_i + 1 < \sum_{i = 1}^{l} \gamma i |\beta_i(1 + M_i) + 1| + 1.
\]

As \( \sum_{i = 1}^{l} \gamma i |\beta_i(1 + M_i) + 1| > 0 \), we conclude that \( G_{\lambda, \gamma_1, \ldots, \gamma_l}^{m,n}(z) \in \mathcal{D}(\delta'), \) where \( \delta' = \sum_{i = 1}^{l} \gamma i |\beta_i(1 + M_i) + 1| + 1. \) \end{proof}

In Theorem 8 taking \( l = 1, \gamma_1 = \gamma, M_1 = 1, f_1 = f \), we get the following corollary.
Corollary 4. Let $\gamma \in \mathbb{R}$, $\gamma > 0$, $f \in \mathcal{D}(\beta, \mu)$ and $\left| \frac{f''(z)}{f'(z)} \right| < M$, where $M$ is fixed. Then $\int_{0}^{t} (f'(t))^2 \, dt \in \mathcal{D}(\delta')$, where $\delta' = \mu \gamma [(1 + M) \beta + 1] + 1$, $z \in U$.

5. Subordination Results

We extend Theorem 1 and Theorem 2 to the operator $DR_{\lambda}^{m,n} f$.

Theorem 9. Let $q$ be convex and univalent, $\gamma \neq 0$ and

$$Re \left\{ \frac{1 - \gamma}{\lambda \gamma} + \frac{2}{\lambda} q(z) + \left( \frac{z q''(z)}{q'(z)} + 1 \right) \right\} > 0.$$

If $f \in \mathcal{T}$ satisfies the differential subordination

$$\frac{DR_{\lambda}^{m+1,n} f(z)}{DR_{\lambda}^{m,n} f(z)} \left( 1 - \gamma + \frac{DR_{\lambda}^{m+2,n} f(z)}{DR_{\lambda}^{m+1,n} f(z)} \right) < (1 - \gamma) q(z) + \gamma q^2(z) + \lambda \gamma z q'(z), \quad z \in U, \quad (10)$$

then

$$\frac{DR_{\lambda}^{m+1,n} f(z)}{DR_{\lambda}^{m,n} f(z)} \prec q(z), \quad z \in U, \quad (11)$$

and the best dominant is the function $q$.

Proof. Consider

$$p(z) = \frac{DR_{\lambda}^{m+1,n} f(z)}{DR_{\lambda}^{m,n} f(z)}, \quad z \in U. \quad (12)$$

We obtain

$$p'(z) = \frac{DR_{\lambda}^{m,n} f(z)}{DR_{\lambda}^{m+1,n} f(z)} \left[ \frac{DR_{\lambda}^{m+1,n} f(z)}{DR_{\lambda}^{m,n} f(z)} \right]' - \frac{DR_{\lambda}^{m,n} f(z)}{DR_{\lambda}^{m+1,n} f(z)} \left[ \frac{DR_{\lambda}^{m+1,n} f(z)}{DR_{\lambda}^{m,n} f(z)} \right]' \left( \frac{DR_{\lambda}^{m,n} f(z)}{DR_{\lambda}^{m+1,n} f(z)} \right)^2.$$  

Therefore,

$$zp'(z) = z \left( \frac{DR_{\lambda}^{m+1,n} f(z)}{DR_{\lambda}^{m+1,n} f(z)} \right)' - z \left( \frac{DR_{\lambda}^{m,n} f(z)}{DR_{\lambda}^{m+1,n} f(z)} \right)' \quad (13)$$

By using (2) in (13), we get

$$zp'(z) = \frac{1}{\lambda} \frac{DR_{\lambda}^{m+2,n} f(z)}{DR_{\lambda}^{m+1,n} f(z)} - \frac{1 - \lambda}{\lambda} - \frac{1}{\lambda} \frac{DR_{\lambda}^{m+1,n} f(z)}{DR_{\lambda}^{m,n} f(z)} + \frac{1 - \lambda}{\lambda}.$$  

We obtain

$$z \frac{p'(z)}{p(z)} = \frac{DR_{\lambda}^{m+2,n} f(z)}{DR_{\lambda}^{m+1,n} f(z)} - p(z)$$

so

$$\frac{DR_{\lambda}^{m+2,n} f(z)}{DR_{\lambda}^{m+1,n} f(z)} = \lambda \left[ z \frac{p'(z)}{p(z)} + \frac{1}{\lambda} p(z) \right]. \quad (14)$$

We deduce

$$\frac{DR_{\lambda}^{m+1,n} f(z)}{DR_{\lambda}^{m,n} f(z)} \left[ \frac{DR_{\lambda}^{m+2,n} f(z)}{DR_{\lambda}^{m+1,n} f(z)} + 1 - \gamma \right].$$
\[
(1 - \gamma)p(z) + \gamma p^2(z) + \gamma \lambda z p'(z).
\]

Hence, the differential subordination (10) becomes

\[
(1 - \gamma)p(z) + \gamma p^2(z) + \gamma \lambda z p'(z) \prec (1 - \gamma)q(z) + \gamma q^2(z) + \gamma \lambda z q'(z).
\]

Applying Lemma 2, we obtain that

\[
\frac{\text{DR}^{m+1,n}_{\lambda} f(z)}{\text{DR}^{m,n}_{\lambda} f(z)} \prec q(z), \quad z \in U,
\]

and \(q\) is the best dominant. \(\square\)

**Theorem 10.** Let \(q\) be univalent in \(U\), \(q(0) \neq 0\), \(\gamma \neq 0\) and \(\frac{zq'(z)}{q(z)}\) be univalent and starlike in \(U\). If \(f \in \mathcal{T}\) satisfies the differential subordination

\[
\frac{\text{DR}^{m+2,n}_{\lambda} f(z)}{\text{DR}^{m+1,n}_{\lambda} f(z)} - \gamma \frac{\text{DR}^{m+1,n}_{\lambda} f(z)}{\text{DR}^{m,n}_{\lambda} f(z)} \prec \frac{\lambda z q'(z)}{q(z)} + 1 - \gamma, \quad z \in U,
\]

then

\[
\frac{z^{\gamma - 1} \text{DR}^{m+1,n}_{\lambda} f(z)}{(\text{DR}^{m,n}_{\lambda} f(z))^\gamma} \prec q(z), \quad z \in U,
\]

and the best dominant is the function \(q\).

**Proof.** Define

\[
p(z) = \frac{z^{\gamma - 1} \text{DR}^{m+1,n}_{\lambda} f(z)}{(\text{DR}^{m,n}_{\lambda} f(z))^\gamma}, \quad z \in U,
\]

and by differentiating it, we get

\[
p'(z) = \frac{z^{\gamma - 2}(\gamma - 1) \text{DR}^{m+1,n}_{\lambda} f(z) + z^{\gamma - 1}\left(\text{DR}^{m+1,n}_{\lambda} f(z)\right)'}{(\text{DR}^{m,n}_{\lambda} f(z))^\gamma} - \gamma \frac{z^{\gamma - 1} \text{DR}^{m+1,n}_{\lambda} f(z)\left(\text{DR}^{m,n}_{\lambda} f(z)\right)'}{(\text{DR}^{m,n}_{\lambda} f(z))^\gamma}\left(\text{DR}^{m,n}_{\lambda} f(z)\right)'.
\]

Therefore,

\[
\frac{zp'(z)}{p(z)} = (\gamma - 1) + \frac{z \left(\text{DR}^{m+1,n}_{\lambda} f(z)\right)'}{\text{DR}^{m+1,n}_{\lambda} f(z)} - \gamma \frac{z \left(\text{DR}^{m,n}_{\lambda} f(z)\right)'}{\text{DR}^{m,n}_{\lambda} f(z)}.
\]

We deduce

\[
\frac{zp'(z)}{p(z)} = \frac{1}{\lambda} \frac{\text{DR}^{m+2,n}_{\lambda} f(z)}{\text{DR}^{m+1,n}_{\lambda} f(z)} - \gamma \frac{1}{\lambda} \frac{\text{DR}^{m+1,n}_{\lambda} f(z)}{\text{DR}^{m,n}_{\lambda} f(z)} + \frac{\gamma - 1}{\lambda},
\]

which is equivalent to

\[
\frac{\text{DR}^{m+2,n}_{\lambda} f(z)}{\text{DR}^{m+1,n}_{\lambda} f(z)} - \gamma \frac{\text{DR}^{m+1,n}_{\lambda} f(z)}{\text{DR}^{m,n}_{\lambda} f(z)} = \lambda \frac{zp'(z)}{p(z)} + 1 - \gamma.
\]
By hypothesis (16), we have \( z p'(z) \frac{z p(z)}{q(z)} \). From Lemma 1 we obtain

\[
\frac{z^{\gamma-1} DR_{m,n}^{\lambda} f(z)}{(DR_{m,n}^{\lambda} f(z))^\gamma} < q(z), \quad z \in U,
\]

and \( q \) is the best dominant. \( \square \)

6. Conclusions

Continuing the study involving the operator \( DR_{m,n}^{\lambda} \) defined in [19], two new integral operators denoted by \( F_{m,n}^{\lambda,\gamma} \) and \( G_{m,n}^{\lambda,\gamma} \) are defined. Using those operators, certain classes of functions are introduced and studied in order to obtain conditions for the functions from class \( T \) to be part of those classes. The integral operators are also investigated on classes \( R(\delta), R(\delta), DA(\beta,\mu) \) and \( RA(\beta,\mu) \), also introduced in this paper. Subordination results are given for functions from class \( T \) associated with the operator \( DR_{m,n}^{\lambda,\gamma} \).

As further research proposals, the implementation of the dual theory of differential superordination for obtaining results involving those newly defined operators can be named. Additionally, the newly introduced classes might have interesting properties related to starlikeness or convexity. Study on those classes may be conducted in many other aspects, conditioned only by the imagination of the researchers and how they are inspired by the results presented here.

The introduction of differential operators allows differential equations in terms of operator theory and functional analysis to be investigated. Properties of differential operators are used in the operator method of solution of differential equations. A study could be conducted in the future to find out whether these operators can be used for giving solutions of PDEs. Applications to the physical sciences or other applied sciences which can be found for many differential operators could be searched for these particular newly introduced operators.

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