Anisotropic Friedmann-Robertson-Walker universe from nonlinear massive gravity

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Motivated by the recent no-go result of homogeneous and isotropic solutions in the nonlinear massive gravity, we study fixed points of evolution equations for a Bianchi type–I universe. We find a new attractor solution with non-vanishing anisotropy, on which the physical metric is isotropic but the Stückelberg configuration is anisotropic. As a result, at the background level, the solution describes a homogeneous and isotropic universe, while a statistical anisotropy is expected from perturbations, suppressed by smallness of the graviton mass.

**Introduction.** General relativity, which describes long range gravitational interactions, is in agreement with current experimental and observational data. On the other hand, from a theorist’s point of view, it is an interesting question whether the range of gravity can be consistently made to be finite, or equivalently, whether graviton can have a non-zero mass. This question has also phenomenological relevance, since the graviton mass may introduce new terms that mimic dark energy and thus may source the late-time acceleration of the universe.

Assigning a mass to graviton has been one of the most challenging problems in classical field theory for the last 70 years. The linear theory of Fierz and Pauli [1] gives rise to a discontinuity in the observables [2], which can be alleviated by nonlinear terms [3]. However, such terms generically introduce the so-called Boulware-Deser (BD) ghost [4], spoiling the stability of the theory.

Only recently, a nonlinear extension of the massive gravity theory has been introduced [5, 6], where the BD ghost is systematically removed by construction. The theoretical and phenomenological possibilities brought this theory a significant attention.

The nonlinear massive gravity allows self-accelerating open Friedmann-Robertson-Walker (FRW) solutions with the Minkowski fiducial metric [8] as well as flat/closed/open FRW solutions with general FRW fiducial metric [8]. Unlike the other branch of solutions [10, 11], these backgrounds evade the Higuchi bound [12] and thus are free from ghost at the linearized level even when the expansion rate is significantly higher than the graviton mass. This is because there are only two propagating modes on these backgrounds. However, these constructions exhibit a ghost instability at nonlinear order in perturbations [13]. This is a consequence of the FRW symmetries; in order to obtain a stable solution, some of these symmetries need to be broken.

An inhomogeneous background solution was obtained in [7], where the observable universe is approximately FRW for a horizon size smaller than the Compton length of graviton. Similar solutions with inhomogeneities in the Stückelberg sector, meaning that the physical metric and the fiducial metric do not have common isometries acting transitively, were found in [14]. Note that those inhomogeneous solutions cannot be isotropic everywhere since isotropy at every point implies homogeneity [13]. Note also that cosmological perturbations can in principle probe inhomogeneities in the Stückelberg sector. For example, generic spherically-symmetric solutions are isotropic only when they are observed from the center of the universe.

The goal of the present paper is to introduce an alternative option, where the assumption of isotropy is dropped but homogeneity, i.e. the cosmological principle, is kept. In a region with relatively large anisotropy, we find an attractor solution. On the attractor, the physical metric is still isotropic, and the background geometry is of FRW type. Hence, the thermal history of the standard cosmology can be accommodated in this class of solutions. However, the Stückelberg field configuration is anisotropic, which may lead to effects at the level of the perturbations, suppressed by smallness of the graviton mass.

**The action and background.** We consider a simple description of the universe at present time. We assume that the late-time acceleration is sourced by a cosmological constant $\Lambda$, as well as the contribution from the graviton mass. (Setting $\Lambda = 0$ corresponds to self-accelerating solutions as in the example shown in Fig.[1]). For this purpose, the vacuum configuration is sufficient. The action with the graviton mass term, constructed by imposing the absence of BD ghost in the decoupling limit [6], is

$$I = \frac{M^2}{2}\int d^4x \sqrt{-g}(R - 2\Lambda + m^2 g^2 (\mathcal{L}_2 + \alpha_3 \mathcal{L}_3 + \alpha_4 \mathcal{L}_4)),$$

(1)

where

$$\mathcal{L}_2 = [\mathcal{K}]^2 - [\mathcal{K}^2]$$

$$\mathcal{L}_3 = \frac{3}{2}([\mathcal{K}]^3 - 3[\mathcal{K}] [\mathcal{K}^2] + 2[\mathcal{K}^3]),$$

$$\mathcal{L}_4 = \frac{3}{4}([\mathcal{K}]^4 - 6[\mathcal{K}]^2 [\mathcal{K}^2] + 3[\mathcal{K}^2]^2 + 8[\mathcal{K}] [\mathcal{K}^3] - 6[\mathcal{K}^4]).$$

(2)

In the above, the square brackets denote trace operation
and
\[ K^\mu_\nu = \delta^\mu_\nu - \left( \sqrt{g^{-1}f} \right)_\nu^\mu. \] (3)

Here, \( g_{\mu\nu} \) is the physical metric, while the space-time tensor \( f_{\mu\nu} \) is the fiducial metric, whose vacuum expectation value gives rise to the breaking of general coordinate invariance.

For the physical metric, we adopt the axisymmetric Bianchi type–I metric, which is the simplest anisotropic extension of FRW ansatz
\[ g^{(0)}_{\mu\nu} dx^\mu dx'^\nu = -N^2 dt^2 + a^2 [e^{4\sigma} dx^2 + e^{-2\sigma} \delta_{pq} dy^p dy^q], \] (4)
where \( N, a \) and \( \sigma \) are functions of \( t \), Greek indices denote space-time coordinates, while \( i, j \) are functions of the temporal St"uckelberg field \( \phi^0 \). The form \( (5) \) includes a de Sitter fiducial as a special case, with \( H_f = \dot{\alpha}/\alpha n = \) constant.

Varying the St"uckelberg fields around the background value \( \phi^0 = x^a + \alpha^a \), the variation of the mass term up to first order is
\[ I = I^{(0)} + M P^2 m^2 \int d^4 x N a^3 n \pi^0 \mathcal{E}_\phi + O[(\pi^a)^2], \] (6)
giving the equation of motion
\[ \mathcal{E}_\phi = \mathcal{J}_\phi \left( H + 2 \Sigma - H_f e^{-2\sigma} X \right) + 2 \mathcal{J}_\phi \left( H - \Sigma - H_f e^{-2\sigma} X \right) = 0, \] (7)
where
\[ \mathcal{J}_\phi^{(x)} = \gamma_1 - 2 \gamma_2 e^{-\sigma} X + \gamma_3 e^{2\sigma} X^2, \]
\[ \mathcal{J}_\phi^{(y)} = \gamma_1 - \gamma_2 (e^{-2\sigma} + e^\sigma) X + \gamma_3 e^{-\sigma} X^2, \] (8)
with \( \gamma_1 = 3 + 3 \alpha_3 + \alpha_4, \gamma_2 = 1 + 2 \alpha_3 + \alpha_4, \gamma_3 = \alpha_3 + \alpha_4, H \equiv \frac{\dot{\alpha}}{\alpha n}, \Sigma \equiv \frac{\dot{\alpha}}{\gamma_3} \) and \( X \equiv \frac{\dot{\alpha}}{\gamma_3} \). The expansion rate for the fiducial metric \( H_f \) is related to the invariants of the field space metric, and is independent of the choice of the background values of \( \phi^a \). Thus, Eq.\( (7) \) can be interpreted as an algebraic equation for \( \alpha \) (or equivalently for \( X \)), instead of a differential equation.

Varying the action with respect to \( g_{\mu\nu} \), the field equations for the physical metric are obtained as
\[ 3 \left( H^2 - \Sigma^2 \right) - \Lambda = m^2 \left[ -3 (\gamma_1 - 3 \gamma_2 + \gamma_3) + \gamma_1 (2 e^\sigma + e^{-2\sigma}) X - 2 \gamma_2 (e^{2\sigma} + 2 e^{-\sigma}) \right]^2 + 3 \gamma_3 X^3, \]
\[ \frac{\dot{\Sigma}}{N} + 3 H \Sigma = \frac{m^2}{3} (e^{-2\sigma} - e^\sigma) X \left[ \gamma_1 - \gamma_2 (e^\sigma + r) X + \gamma_3 r e^\sigma X^2 \right], \] (9)
where
\[ r \equiv \frac{n a}{N \alpha} = \frac{1}{X H_f} \left( \frac{\dot{X}}{N X} + H \right). \] (10)

Additionally, there is also an equation for \( \dot{H} \), which can be recovered by combining Eq.\( (7) \) with Eq.\( (9) \).

**Fixed Points.** We consider a de Sitter fiducial metric \((H_f = \text{const.)}\) and seek solutions with \( H = \Sigma = \lambda X = 0 \). The constancy of \( X \) allows us to express \( H \) as \( H = H_f X r \). In this setup, the independent equations become
\[ 3 \lambda - (3 \gamma_1 - 3 \gamma_2 + \gamma_3 + \gamma_1 (2 e^\sigma + e^{-2\sigma}) X - \gamma_2 \left( 2 e^{-\sigma} + e^{2\sigma} \right) X^2 + \gamma_3 X^3 = 0, \] (11)
\[ (e^\sigma - 1) \left[ \gamma_1 - 2 \gamma_2 (r + e^\sigma) X + \gamma_3 e^\sigma r X^2 \right] = 0, \] (12)
\[ \gamma_1 (3 r - 2 e^\sigma - e^{-2\sigma}) - 2 \gamma_2 \left( 2 e^\sigma + e^{-2\sigma} \right) r - (e^2 \sigma + 2 e^{-\sigma}) X + \gamma_3 \left( e^{2\sigma} + 2 e^{-\sigma} r - 3 \right) X^2 = 0, \] (13)
where \( \lambda \equiv \frac{\Lambda}{m^2} \) and \( \mu \equiv \frac{m a}{H_f} \) are dimensionless parameters.

For \( \sigma = 0 \), the set of equations is reduced to that for isotropic configurations, which was already investigated in \( [8, 9] \). Assuming \( \sigma \neq 0 \) and using \( [12] \), Eq.\( (13) \) can be rewritten as
\[ (\gamma_1 - \gamma_2 X e^\sigma) (e^\sigma - r) (r e^{2\sigma} - 1) = 0. \] (14)
Considering \( (14) \) as an algebraic equation for \( e^\sigma \), there...
are three solutions:
\[ e^\sigma = \begin{cases} 
\gamma_1 \left\{ \frac{1}{\gamma_2 X}, r, r^{-1/2} \right\}. 
\end{cases} \tag{15} \]

We now consider each solution separately.

**Case I.** \( e^\sigma = \frac{2\gamma_1}{\gamma_2 X} \). Using this solution in Eq. (12) gives \( X = \gamma_1/\gamma_2 \), implying \( \sigma = 0 \). In other words, this solution is isotropic and thus is not of our interest.

**Case II.** \( e^\sigma = r \). In this case, Eq. (12) gives
\[ (r - 1) \left[ \gamma_1 - 2 \gamma_2 r X + \gamma_3 (r X^2) \right] = 0. \tag{16} \]
This equations have two solutions; the first solution is \( r = 1 \), and leads to isotropy \( \sigma = 0 \) which is not our interest. The second solution gives \( rX = (\gamma_2 \pm \sqrt{\gamma_2^2 - \gamma_1 \gamma_3})/\gamma_3 \), which reduces Eq. (11) to a nontrivial constraint between the parameters of the theory. Since this case requires a fine-tuning of a parameter, it is not of our interest either.

**Case III.** \( r = e^{-2\sigma} \). With this solution, Eq. (12) is reduced to
\[ \gamma_1 e^\sigma - \gamma_2 (e^{2\sigma} + e^{-\sigma}) X + \gamma_3 X^2 = 0. \tag{17} \]

while Eq. (11) becomes
\[ (3\lambda - 3\gamma_1 + 3\gamma_2 - \gamma_3) + \gamma_1 (e^{-2\sigma} + 2e^\sigma) X - [\gamma_2 (2e^{-\sigma} + e^{2\sigma}) + 3e^{-\sigma} \mu^2] X^2 + \gamma_3 X^3 = 0. \tag{18} \]

Combining these two equations, we obtain an expression linear in \( X \),
\[ X = \frac{3\gamma_1 + \gamma_2 (\gamma_2 - \gamma_3) + \gamma_1 (\gamma_2 - \gamma_1 + \lambda)}{(e^\sigma + e^{-2\sigma})[3\gamma_2 + (\gamma_2^2 - \gamma_1 \gamma_3)] \mu^2 e^{3\sigma}}. \tag{19} \]

and an equation which only depends on \( \sigma \)
\[ c_0 + c_1 e^{3\sigma} + c_2 e^{6\sigma} + c_3 e^{9\sigma} = 0, \tag{20} \]
where
\[ c_0 = 3\gamma_2 (\gamma_1^2 + 3\gamma_2^2) - 3\gamma_1 \gamma_2 - 2\gamma_2 \gamma_3 + 2\gamma_2 \lambda) \mu^2 - 9\gamma_1^2, \]
\[ c_1 = (\gamma_2^2 - \gamma_1 \gamma_3) \left[ -6(\gamma_1 - \gamma_2 + \gamma_3 - 3\lambda) + (\gamma_1^2 + 3\gamma_2^2 - 3\gamma_1 \gamma_2 - 2\gamma_2 \gamma_3 + 3\gamma_2 \lambda) \mu^2 \right], \]
\[ c_2 = (\gamma_2^2 - 2\gamma_1 \gamma_3) \left[ (\gamma_1^2 + 3\gamma_2^2 - 3\gamma_1 \gamma_2 - 2\gamma_2 \gamma_3 + 3\gamma_2 \lambda) \mu^2 - (\gamma_1 \gamma_2 - 3\gamma_1 \gamma_3 + 2\gamma_2 \gamma_3 - \gamma_3^2 + 3\gamma_3 \lambda) \mu^4 \right], \]
\[ c_3 = (\gamma_2 - \gamma_1 \gamma_3) \left( \gamma_1^2 + 3\gamma_2^2 - 3\gamma_1 \gamma_2 - 2\gamma_2 \gamma_3 + 2\gamma_2 \lambda \right) \mu^4, \tag{21} \]

For a given set of parameters \((\alpha_3, \alpha_4, \lambda, \mu)\), one can solve the cubic equation (20) for \( e^{3\sigma} \) and then use (19) to calculate the corresponding value of \( X \).

**Local Stability.** We now introduce homogeneous perturbations around the fixed point described in the third case above.

\[ H = H_f[\gamma_0 X_0 + \epsilon h_1(t) + O(\epsilon^2)], \]
\[ \sigma = \sigma_0 + \epsilon \sigma_1(t) + O(\epsilon^2), \]
\[ X = X_0 + \epsilon X_1(t) + O(\epsilon^2), \tag{22} \]

where \((X_0, \sigma_0, r_0 = e^{-2\sigma_0})\) is the background representing the fixed point. Adopting this expansion, we calculate the equations of motion up to \( O(\epsilon) \). At linear order, \( \sigma_1 \) can be decoupled from the remaining \( O(\epsilon) \) quantities, and a second-order evolution equation is obtained as
\[ \frac{d^2 \sigma_1}{d\tau^2} + 3X_0 e^{-2\sigma} \frac{d\sigma_1}{d\tau} + M^2 \sigma_1 = 0, \tag{23} \]

where
\[ M^2 = \left[ X_0 \mu^2 e^{-4\sigma_0} \frac{d_1 (3d_1 - d_2)(6 + d_1 \mu^2)}{2d_2 - d_1 \mu^2} \right], \]
\[ d_1 = (e^{3\sigma_0} - 1) \left[ \gamma_2 - 3\gamma_3 e^{\sigma_0} X_0 \right], \]
\[ d_2 = (e^{3\sigma_0} - 1) \left[ \gamma_2(3 + 2e^{3\sigma_0}) - 5\gamma_3 e^{\sigma_0} X_0 \right], \tag{24} \]

while the dimensionless time coordinate \( \tau \) is defined by \( d\tau = H_f N dt \). The fixed point is locally stable if
\[ M^2 > 0. \tag{25} \]

**Global Stability.** To study the global stability of the fixed point, we consider an example with
\[ \lambda = 0, \quad \mu = 20, \quad \alpha_3 = -1/20, \quad \alpha_4 = 1, \tag{26} \]
for which the local stability condition (25) is satisfied. For this parameter set, there is only one set of real solutions to the equations of motion (11) to (13)
\[ X \simeq 4, \quad e^\sigma \simeq 4^{1/2}, \quad r \simeq 4. \tag{27} \]

In order to determine the phase flow, we first reduce the system of equations. Using Eq. (11), the first of Eq. (12) and
their time derivatives, we can express $X$, $H$ and their derivatives in terms of $\sigma$ and $\Sigma$. Since these equations are nonlinear, there are always more than one solution. For the parameter set $[26]$, we find that there are three branches of solutions which give $X > 0$ and $H > 0$. For each branch, we use this solution with the second of Eq. (11). As a result we obtain, for a set of $(\sigma, \Sigma)$, the corresponding set of $(\dot{\sigma}, \Sigma)$ pair. Out of the three branches, only one contains an attractor. The flow corresponding to this branch is shown in Fig. 1. The flow proceeds towards the fixed point we found in Eq. (27).

Discussion. The recently introduced nonlinear massive gravity theory [5, 6] provides a new framework to address some of the intriguing issues in theoretical physics and cosmology, such as the possibility of finite range gravity and the mystery of dark energy. Although the theory admits homogeneous and isotropic solutions, these suffer from an unavoidable nonlinear ghost [13] or a linear ghost [10]. Since this is a consequence of the FRW symmetries, either homogeneity or isotropy needs to be broken in order to obtain a stable solution.

In the present paper, for the first time in the nonlinear massive gravity theory, we explored regions with relatively large anisotropy for homogeneous attractor solutions. The classification of fixed points revealed the existence of a single anisotropic attractor. The local and global stability analyses indicate that, a universe with a sufficiently large anisotropy at the onset of the late-time accelerated expansion should flow to this point.

A very interesting implication is that the scale factors corresponding to the two directions differ only by a constant normalization, thus the expansion rate is completely isotropic. In general relativity, such a solution will be identical to an isotropic universe, up to a coordinate redefinition. Conversely, in nonlinear massive gravity, such a redefinition cannot remove the anisotropy completely; it is merely shifted from the physical metric to the fiducial metric. Although the background metric is of FRW type, the signature from the anisotropy will be imprinted on the spectrum of cosmological perturbations. The statistical anisotropy signal is expected to be suppressed by smallness of the graviton mass $m_g$. The type of anisotropy, i.e. statistical anisotropy for perturbations without anisotropic background expansion, is totally new. For example, none of the anisotropic inflation scenarios [10] has this type of anisotropy. Detailed analysis of perturbations and comparison with observational data are worthwhile pursuing. As the first step, a preliminary analysis of perturbations indicates that the anisotropic attractor solutions found in this paper are free from ghost for a range of parameters [17].

As a sensible effective theory, the theory can be reliable only below a cutoff scale $\Lambda_n = (M_{Pl} m_g^{n-1})^{1/n}$, which is much larger than the graviton mass $m_g$ for $n > 0$. The exact form of this scale will be determined by a detailed analysis of perturbations in a future publication [17]. For the scenario discussed in the present work, we associated the late time acceleration with the graviton mass term, thus the present expansion rate is taken to be of the order of the mass term. For instance, for the case given in Eqs. (20, 27), the constant expansion rate on the fixed point is $H = 0.8 m_g$. This is clearly well below the order of the cutoff scale $\Lambda_n$ for $n > 0$.

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