ON A RESONANT MEAN FIELD TYPE EQUATION: 
A “CRITICAL POINT AT INFINITY” APPROACH

MOHAMEDEN AHMEDOU*
Mathematisches Institut der Justus-Liebig-Universität Giessen
Arndtstrasse 2, D-35392 Giessen, Germany

MOHAMED BEN AYED
Université de Sfax, Faculté des Sciences
Département de Mathématiques
Route de Soukra, Sfax, Tunisia

MARCELLO LUCIA
The City University of New York, CSI
Mathematics Department
2800 Victory Boulevard
Staten Island New York 10314, USA

(Communicated by Yanyan Li)

In fond memory of Abbas Bahri

Abstract. We consider the following mean field type equations on domains
of \( \mathbb{R}^2 \) under Dirichlet boundary conditions:

\[
\begin{aligned}
-\Delta u &= \varrho \frac{K e^u}{\int_{\Omega} K e^u} \quad \text{in } \Omega, \\
 u &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]

where \( K \) is a smooth positive function and \( \varrho \) is a positive real parameter.

A “critical point theory at Infinity” approach of A. Bahri to the above
problem is developed for the resonant case, i.e. when the parameter \( \varrho \)
is a multiple of \( 8\pi \). Namely, we identify the so-called “critical points at infinity” of
the associated variational problem and compute their Morse indices. We then
prove some Bahri-Coron type results which can be seen as a generalization of
a degree formula in the non-resonant case due to C.C.Chen and C.S. Lin [13].

1. Introduction. Given a positive function \( K \) of class \( C^3(\bar{\Omega}) \) on a bounded smooth
domain \( \Omega \subset \mathbb{R}^2 \), and a positive parameter \( \varrho \), we consider the following non-local
equation:

\[
(P_\varrho) \quad \begin{cases}
-\Delta u &= \varrho \frac{K e^u}{\int_{\Omega} K e^u} \quad \text{in } \Omega, \\
 u &= 0 \quad \text{on } \partial \Omega.
\end{cases}
\]
Problem \((P_\varrho)\) arises as limiting equation in some mean field approximation in the study of limit point vortices of Euler flows, in Onsager vortex theory, and also in the description of self dual condensates in some Chern-Simons-Higgs models. See for example the papers \([12, 13, 17, 18, 22, 28, 39, 36]\), and the monographs of Tarantello \([37]\) and of Yang \([38]\), and the references therein.

Problem \((P_\varrho)\) has a variational structure. Indeed, its solutions are in a one to one correspondence with the critical points of its associated Euler-Lagrange functional \(J_\varrho\) defined on \(H^1_0(\Omega)\) through

\[
J_\varrho(u) := \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx - \varrho \ln \left( \int_{\Omega} Ke^u dx \right).
\]

It turns out that the analytic features of \(J_\varrho\) and hence the way of finding critical points of \((2)\) depend strongly on the range of values taken by the real parameter \(\varrho\). Namely depending on the range of values of the real parameter \(\varrho\), the noncompactness of the associated variational problem and therefore the way of finding critical points of the functional \(J_\varrho\) on \(H^1_0(\Omega)\) changes drastically. Indeed if \(\varrho < 8\pi\), then the functional \(J_\varrho\) is, thanks to the well known Moser-Trudinger Inequality, lower bounded, coercive and weakly lower semi continuous and hence achieves its minimum. While if \(\varrho = 8\pi\) it is still lower bounded but no more coercive and if \(\varrho > 8\pi\), it is neither lower or upper bounded. Moreover it turns out that, if \(\varrho\) is a multiple of \(8\pi\), the corresponding variational problem is not compact, in the sense that “critical point at Infinity” may occur, a situation which causes the break down of standard variational techniques.

As far as existence and compactness results are concerned, the non resonant situation (i.e. the cases \(\varrho \neq 8\pi m, m \in \mathbb{N}\)) is relatively well understood. First existence results for equation \((P_\varrho)\) have been obtained by Struwe-Tarantello \([36]\) and Ding-Jost-Li-Wang \([22]\). For example in \([22]\) it is proved that if \(8\pi < \varrho < 16\pi\) and \(\Omega\) is an annulus, then Equation \((P_\varrho)\) has at least one solution. Moreover a blow up analysis has been performed in \([11]\). A more detailed blow up analysis undertaken by Li and Shafrir \([29]\) allowed to show that for \(\varrho \neq 8\pi m, m \in \mathbb{N}\) the set of solutions of the equation \((P_\varrho)\) is bounded in \(C^{2,\alpha}(\Omega)\). Such a compactness result opened the way of proving existence of solutions through topological degree methods. Such a program of computing the degree of all solutions of \((P_\varrho)\) initiated by Y.Y. Li in \([30]\) has been completed by C.C. Chen and C.S. Lin in \([17, 18]\).

Namely they proved the following result:

**Theorem A** \([Chen-Lin\,03]\) Let \(\varrho \in (8\pi m, 8\pi(m + 1))\); \(m \in \mathbb{N}\). Then, the total Leray-Schauder degree \(deg_\varrho\) of solutions of \((P_\varrho)\) is given by

\[
deg_\varrho = \left( \frac{m - 1 + g}{m} \right)
\]  

where \(g\) denotes the number of holes of \(\Omega\). In particular, if the domain \(\Omega\) is not simply connected then the problem \((P_\varrho)\) for \(\varrho \neq 8\pi m; m \in \mathbb{N}\) has at least one solution.

The aim of this paper is to prove some existence and compactness results in the resonant case (i.e. when \(\varrho = 8\pi m\)). We will also provide under generic conditions, a general formula for the degree in this case. While in the non resonant case the degree depends only on the topology of the domain, our results show that it depends, for \(\varrho = 8\pi\) on the function \(K\), and in the supercritical one \(\varrho = 8\pi m, m \geq 2\) on the geometry and the topology of the domain as well as on the function \(K\).
1.1. The critical case. When $\varrho = 8\pi$, our first result shows that, under some conditions on $K$, the functional $J_\varrho$ satisfies the Palais-Smale condition below a certain level. Furthermore, when this possibility is excluded, using some tools of the theory of critical points at infinity [2] we provide an Euler-Hopf type criterium. To state our results more precisely, we introduce the following function

$$F^K_1(y) := \ln(K(y)) - 4\pi H(y, y), \quad y \in \Omega,$$

where $H$ is the regular part of the Green’s function of the Laplacian under Dirichlet boundary condition. Note that $F^K_1$ tends to $-\infty$ as $y$ goes to the boundary and therefore $F^K_1$ achieves its maximum.

**Theorem 1.1.** Let $\varrho = 8\pi$ and assume that there exists a maximum point $y$ of $F^K_1$ such that

$$\Delta(\ln K)(y) > 0.$$

Then the functional $J_{8\pi}$ achieves its minimum (hence Problem [1] has at least one solution).

**Remark 1.** Regarding the above theorem, some remarks are in order:

- In the case of non simply connected domain and under the assumption that $\Delta(\ln K)$ has a fixed sign, theorem 1.1 is already proved in [13]. Indeed Theorem 1.6 in [18] asserts that if the domain is not simply connected and $\Delta(\ln K)$ has a fixed sign, then (1) has a solution for any $\varrho$ (including $8\pi N$).
- As pointed out in [13] and deeply investigated in [16], [7], [8], the existence of solution in the critical case is very sensitive to the geometry of the domain.

We now introduce the following non degeneracy condition: We say that the function $K$ satisfies the condition $(ND_1)$ if $F^K_1$ has only non degenerate critical points and at each critical point of $F^K_1$ there holds that:

$$\Delta(\ln K)(y) \neq 0.$$

By considering the set

$$K^- := \{y \in \Omega : \nabla F^K_1(y) = 0 \text{ and } \Delta(\ln K)(y) < 0\},$$

and denoting by $\text{morse}(F^K_1, q)$ the Morse index of $F^K_1$ at a critical point $q$, our second result completes Theorem 1.1 as follows:

**Theorem 1.2.** Let $\varrho = 8\pi$. Assume that $K$ satisfies the condition $(ND_1)$, and that each maximum point $y$ of $F^K_1$ satisfies $y \in K^-$. If

$$\sum_{q \in K^-} (-1)^{\text{morse}(F^K_1, q)} \neq 1,$$

then Problem [1] has at least one solution.

Actually the above existence criterium provides, for generic $K$, a lower bound on the number of solutions to equation $(P_{8\pi})$. Namely we prove that:

**Theorem 1.3.** Let $\varrho = 8\pi$. For generic $K$’s the number of solutions of [1] is lower bounded by

$$\left| 1 - \sum_{q \in K^-} (-1)^{\text{morse}(F^K_1, q)} \right|.$$
Remark 2. It turns out that, when the function satisfies the non degeneracy condition \((ND_1)\), the set of solutions to the problem \((P_{8\pi})\) is bounded in \(C^{2,\alpha}(\Omega)\) and the Leray-Schauder degree is given by

\[
\text{deg}_{8\pi} = 1 - \sum_{q \in K_{-}^{-}} (-1)^{morse(F^K_m, q)}.
\]

This degree formula was first derived in [14], [15] and [26] in the case of the 2-sphere.

1.2. The supercritical case. Regarding the supercritical case \(\varrho = 8\pi m, m \in \mathbb{N}, m \geq 2\), we fix some notations. Let \(F_m(\Omega) := \{(x_1, \cdots, x_m) : \text{there exist } i, j \text{ such that } x_i = x_j\}\) denote the thick diagonal. We define the function \(F^K_m : \Omega \setminus F_m(\Omega) \to \mathbb{R}\), through

\[
F^K_m(y_1, \cdots, y_m) := \sum_{i=1}^{m} \left( \ln(K(y_i)) - 4\pi H(y_i, y_i) + 4\pi \sum_{j \neq i} G(y_i, y_j) \right).
\]

Moreover for \(q = (q_1, \cdots, q_m)\), we set for \(i \in \{1, \cdots, m\}\)

\[
F^q_i(x) := K(x) \exp \left( -8\pi H(q_i, x) + 8\pi \sum_{j \neq i} G(q_j, x) \right),
\]

where \(G(q_i, \cdot)\) is the Green’s function of \(-\Delta\) under Dirichlet boundary conditions with pole \(q_i\). We notice that the above functions are related by the following relation:

\[
\nabla F^K_m(q) = \left( \frac{\nabla F^q_1(q_1)}{F^K_m(q_1)}, \cdots, \frac{\nabla F^K_m(q_m)}{F^K_m(q_m)} \right).
\]

We say that \(K\) satisfies the condition \((ND_m)\) if the critical points of \(F^K_m\) are non degenerate and at each critical point \(q = (q_1, \cdots, q_m)\) there holds:

\[
l(q) := \sum_{i=1}^{m} (\Delta F^q_i)(q_i) \neq 0.
\]

We set

\[
K_m^{-} := \{q = (q_1, \cdots, q_m) \text{ a critical point of } F^K_m \text{ such that } l(q) < 0\}.
\]

To each point \(q \in K_m^{-}\), we associate an index defined by

\[
\iota : K_m^{-} \to \mathbb{N}, \quad q \mapsto 3m - 1 - \text{morse}(F^K_m, q),
\]

where \(\text{morse}(F^K_m, q)\) stands for the Morse index of \(F^K_m\) at its critical point \(q\).

We can now state our main results in the case \(\varrho = 8\pi m, m \geq 2\).

**Theorem 1.4.** Let \(\rho = 8\pi m\) with \(m \geq 2\). Assume that \(K\) satisfies the non degeneracy condition \((ND_m)\) and

\[
\sum_{q = (q_1, \cdots, q_m) \in K_m^{-}} (-1)^{\iota(q)} \neq \frac{1}{(m-1)!} \prod_{\ell=0}^{m-2} (\ell + g),
\]

where \(g\) denotes the number of holes inside \(\Omega\). Then the equation \((P_{8\pi m})\) has at least one solution.
Remark 3. We point out that, under the assumption that
\[ l(q) < 0 \text{ (resp. } > 0 \text{)} \text{ for all critical points } q \text{ of } F_K, \]
it follows from the work of Chen-Lin [18] that equation \((P_{8\pi m})\) has at least one solution. Hence the main interest in Theorem 1.4 lies in the fact that we do not assume that \(l(q)\) has a definite sign.

As a byproduct of our blow up analysis, it follows that when \(K\) satisfies the non degeneracy condition \((ND_m)\), then the set of solutions of \((P_{8\pi m})\) is bounded in \(C^{2,\alpha}(\Omega)\) and the Leray-Schauder degree is well defined and is given by:

\[
\deg_{8\pi m} = \sum_{q=(q_1,\ldots,q_m)\in K_m^-} (-1)^{\iota(q)} - \frac{1}{(m-1)!} \prod_{\ell=0}^{m-2} (\ell + g),
\]

where \(g\) denotes the number of holes inside \(\Omega\). Therefore the assumption stated in the theorem is equivalent to the fact that the degree is not zero. As already pointed out the formula of the degree formula in the non resonant case \(\varrho \not\in 8\pi N\) has been proved by Chen and Lin [18] based on their refined estimates in [17]. It has been also derived by A. Malchiodi [32] and [33] based on the precise characterization of the topology of very negative level of the Euler Lagrange functional \(J_\varrho\). All these approaches use in crucial way the a priori estimate for solutions derived by Brezis-Merle [11] and Li-Shafrir [29]. We emphasize that our approach does not rely on any a priori estimates and hence seems to be appropriate to handle the case where non compactness actually occurs.

As special case of the above theorem, we derive the following existence result, in the particular case where the domain is simply connected.

**Theorem 1.5.** Let \(\rho = 8\pi m\) with \(m \geq 2\), we assume that \(\Omega\) is a simply connected bounded domain in \(\mathbb{R}^2\) and \(K\) satisfies the non degeneracy condition \((ND_m)\). If

\[
\sum_{q=(q_1,\ldots,q_m)\in K_m^-} (-1)^{\iota(q)} \neq 0,
\]

then the equation \((P_{8\pi m})\) has at least one solution.

Just as in the above case, under the condition of Theorem 1.5, the set of solutions of \((P_{8\pi m})\) is bounded in \(C^{2,\alpha}(\Omega)\) and the Leray-Schauder degree is given by

\[
\deg_{8\pi m} = \sum_{q=(q_1,\ldots,q_m)\in K_m^-} (-1)^{\iota(q)}.
\]

**Remark 4.** The above results have their counterparts on closed surfaces, and will appear elsewhere.

The reminder of this paper is organized as follows. Some notations and known facts are given in Section 2. In Section 3 we give refined expansion of the gradient near potential neighborhoods at Infinity. In Section 4 we perform a finite dimensional reduction, while Section 5 is devoted to the characterization of “critical points at Infinity”. The proofs of our main results are given in Section 6. Finally, in the last section we collect the technical computations needed in our proofs.
2. Neighborhood at Infinity and lack of compactness. For \( a \in \Omega \) and \( \lambda > 0 \), we define on \( \mathbb{R}^2 \) the following function:
\[
\delta_{a, \lambda}(x) := \ln \left( \frac{8\lambda^2}{(1 + \lambda^2|x-a|^2)^2} \right).
\]
Observe that this function satisfies:
\[
-\Delta \delta_{a, \lambda} = e^{\delta_{a, \lambda}} \text{ in } \mathbb{R}^2 \quad \text{and} \quad \int_{\mathbb{R}^2} e^{\delta_{a, \lambda}} = 8\pi.
\]
We introduce the function \( P_{\delta_{a, \lambda}} \) which is the unique solution of the following equation
\[
\begin{cases}
-\Delta P_{\delta_{a, \lambda}} = e^{\delta_{a, \lambda}} & \text{in } \Omega, \\
P_{\delta_{a, \lambda}} = 0 & \text{on } \partial\Omega.
\end{cases}
\]
For \( x \in \Omega \), let \( G(x,.) \) be the Green’s function of \( -\Delta \) under Dirichlet boundary condition and \( H(x,.) \) its regular part, that is:
\[
G(x,y) = \frac{1}{2\pi} \ln \left( \frac{1}{|x-y|} \right) - H(x,y).
\]
In order to detect critical points for the functional \( J_\rho \) with \( \rho = 8\pi m \), we have to find the obstruction in deforming sub-level sets \( J_\rho := \{ u : J_\rho(u) \leq a \} \). Using a deformation Lemma given in [31], it is known that for the functional \( J_\rho \), aside critical points, the obstruction to decrease the functional comes only from flow lines entering a set that is a neighborhood of solutions \( u_k \) which are solutions of the problem \( \rho_k \) with \( \rho_k \to \rho \).

Assuming that \( J_\rho \) does not have any critical point, then we derive that \( (u_k) \) has to blow up and the results of [11] combined with [29] imply \( \rho \) has to be \( 8\pi m \) with \( m \in \mathbb{N}^* \). Furthermore, from the works of [17] and [18], the solutions \( (u_k) \) have to belong to some set \( V(m,\varepsilon) \), called in the sequel neighborhood of potential critical points at Infinity, which has the following properties:
\[
V(m,\varepsilon) := \left\{ u \in H^1_0(\Omega) : \| \nabla J_\rho(u) \| < \varepsilon ; \ \exists \lambda_1, \cdots, \lambda_m > \varepsilon^{-1} \text{ with } \lambda_i < C_1 \lambda_j \ \forall i \neq j ; \ \exists a_1, \cdots, a_m \text{ with } |a_i - a_j| \geq 2\eta \ \forall i \neq j \right\},
\]
where \( m \in \mathbb{N} \), \( \varepsilon > 0 \), and \( C_1, \eta \) are fixed positive constants.

Hence, we are led to study if there are any obstructions to decrease the functional \( J_\rho \) in the set \( V(m,\varepsilon) \). A first step consists in finding an appropriate parametrization of this set. With this aim, following the ideas of A. Bahri and J.M. Coron in their proof of Proposition 7 in [4] (see also Chen and Lin [18, Lemma 3.2]), we consider the following minimization problem
\[
\min_{\alpha_1 > 0, \alpha_i \in \Omega ; \lambda_i > 0} \| u - \sum_{i=1}^{m} \alpha_i P_{\delta_{a_i, \lambda_i}} \|.
\]
Observe that we have $P_\delta$, the "normalized" bubbles two of them. Hence every $u \in V(m, \varepsilon)$ can be written as

$$u = \sum_{i=1}^{m} \alpha_i P_{\delta_{a_i, \lambda_i}} + w,$$

where $\alpha_i$, $w$ satisfy

$$|\alpha_i - 1| \leq c \varepsilon \quad \forall i, \quad \|w\| \leq c \varepsilon \quad (9)$$

$$\langle w, P_{\delta_{a_i, \lambda_i}} \rangle = \langle w, \frac{\partial P_{\delta_{a_i, \lambda_i}}}{\partial \lambda_i} \rangle = 0, \quad \langle w, \frac{\partial P_{\delta_{a_i, \lambda_i}}}{\partial a_i} \rangle = 0 \quad \forall i.$$ 

In the following, for $a = (a_1, \ldots, a_m)$ and $\Lambda = (\lambda_1, \ldots, \lambda_m)$, we denote

$$E_{a, \Lambda}^m := \{ w \in H_0^1(\Omega) : w \text{satisfies } (9) \}.$$

To keep the notation short we will write $P_\delta$ instead of $P_{\delta_{a_i, \lambda_i}}$.

Our aim will be to construct a vector field in this set that decreases the functional, and to use the deformation lemmas on the level sets of $J_\rho$.

3. **Expansion of the gradient in the neighborhood at Infinity.** In order to study the monotonocity of the functional $J_\rho$ in the set $V(m, \varepsilon)$, we need to compute the derivatives of $J_\rho$ with respect to each variables $\lambda_i, \alpha_i, a_i$.

We start with the gradient with respect to the concentration speeds $\lambda_i$’s.

**Proposition 1.** Let $u := \sum_{i=1}^{m} \alpha_i P_\delta + w := \pi + w \in V(m, \varepsilon)$ with $w \in E_{a, \Lambda}^m$. Then we have

$$\left\langle \nabla J_\rho(u), \lambda_i \frac{\partial P_\delta}{\partial \lambda_i} \right\rangle = 16\pi \alpha_i \tau_i + O \left( |\alpha_i - 1|^2 + \frac{1}{\lambda^2} + \|w\|^2 \right), \quad (11)$$

with

$$\tau_i = 1 - \frac{m\pi}{2\alpha_i - 1} \frac{\lambda_i^{4\alpha_i - 2} F_p(a_i) g_i(a_i)}{\int_{\Omega} K e^u dx}, \quad (12)$$

where $F_p$ and $g_i$ are defined in (17). Furthermore, we have the estimate

$$|\tau_i| = O(\varepsilon), \quad \forall i \in \{1, \ldots, m\}. \quad (13)$$

**Proof.** Using the orthogonality $\langle w, \frac{\partial P_\delta}{\partial \lambda_i} \rangle = 0$ (see [58]), we have

$$\langle \nabla J_\rho(u), \lambda_i \frac{\partial P_\delta}{\partial \lambda_i} \rangle = \sum_{j=1}^{m} \alpha_j \langle P_{\delta_j}, \lambda_i \frac{\partial P_\delta}{\partial \lambda_i} \rangle - \frac{g}{\int_{\Omega} K e^u} \int_{\Omega} K e^u \lambda_i \frac{\partial P_\delta}{\partial \lambda_i}.$$ 

Observe that

$$\int_{\Omega} K e^u \lambda_i \frac{\partial P_\delta}{\partial \lambda_i} = \left[ \int_{\Omega} K e^u \lambda_i \frac{\partial P_\delta}{\partial \lambda_i} \right]_{I_1} = \left[ \int_{\Omega} K e^u \lambda_i \frac{\partial P_\delta}{\partial \lambda_i} \right]_{I_2} + \left[ \int_{\Omega} K e^u \lambda_i \frac{\partial P_\delta}{\partial \lambda_i} \right]_{I_3}.$$

**Estimate of $I_1$:** Since $|\lambda_i \frac{\partial P_\delta}{\partial \lambda_i}| \leq c$, using [58], we deduce

$$|I_1| \leq c \int_{\Omega} K e^u |e^u - 1 - w| \leq c \|w\|^2 \sum_{k=1}^{m} \lambda_k^{4\alpha_k - 2}. \quad (15)$$
Estimate of $I_2$: Using Lemma A.6 and Lemma A.1 we derive
\[ I_2 = \int_{B_i} \frac{4\lambda_i^{4\alpha_i} F_i g_i}{1 + \lambda_i^2 |x-a_i|^2} w + O \left( \frac{1}{\lambda_i^2} \int_{\Omega} e^{|w|} \right). \] (16)

Observe that
\[
\int_{\Omega} e^{|w|} \leq c \sum_{k=1}^m \lambda_k^{4\alpha_k-2} \left( \int_{B_k} \frac{c\lambda_k^{4\alpha_k} |w|}{1 + \lambda_k^2 |x-a_k|^2} \right)^{1/2} + O(\|w\|)
\]
\[
\leq c \|w\| \sum_{k=1}^m \lambda_k^{4\alpha_k-2}.
\] (17)

Using the function defined in (48), we estimate the first integral of (16) as follows:
\[
\int_{B_i} \frac{4\lambda_i^{4\alpha_i} F_i g_i w}{1 + \lambda_i^2 |x-a_i|^2} \frac{\delta_i}{2} = \frac{1}{2} \lambda_i^{4\alpha_i-2} F_i g_i(a_i) \int_{B_i} \frac{\delta_i}{1 + \lambda_i^2 |x-a_i|^2} w
\]
\[
+ O \left( \lambda_i^{4\alpha_i-2} \int_{B_i} e^{\delta_i} |w| (|x-\xi_i| + |x-a_i|) \right).
\]

From (49) and Lemma A.5 we get
\[
\int_{B_i} e^{\delta_i} |w| (|x-\xi_i| + |x-a_i|) \leq c \|w\| \left( |\alpha_i - 1| + \frac{1}{\lambda_i} \right).
\] (18)

Furthermore, since $w \in E_{m, \Lambda}^m$, we get
\[
\int_{\Omega} e^{\delta_i} = 0 = \int_{\Omega} e^{\delta_i} \frac{1 - \lambda_i^2 |x-a_i|^2}{1 + \lambda_i^2 |x-a_i|^2}
\]
which implies that
\[
\int_{\Omega} e^{\delta_i} = 0.
\]

Thus we obtain
\[
\int_{B_i} \frac{4\lambda_i^{4\alpha_i} F_i g_i w}{1 + \lambda_i^2 |x-a_i|^2} = O \left( \|w\| \lambda_i^{4\alpha_i-2} (|\alpha_i - 1| + \frac{1}{\lambda_i}) \right).
\] (19)

Therefore, using (17), (19) in (16), we derive that
\[
I_2 = O \left( \|w\| \left( |\alpha_i - 1| + \frac{1}{\lambda_i} \sum_{k=1}^m \lambda_k^{4\alpha_k-2} \right) \right).
\] (20)

Estimate of $I_3$: For the last integral in (14), we note
\[
I_3 = \int_{B_i} \frac{4\lambda_i^{4\alpha_i} F_i g_i}{1 + \lambda_i^2 |x-a_i|^2} + O \left( \frac{1}{\lambda_i^2} \int_{\Omega} Ke^{\delta_i} \right)
\]
\[
= \frac{2\pi}{\alpha_i} \lambda_i^{4\alpha_i-2} F_i g_i(a_i) + O \left( \int_{B_i} \frac{\lambda_i^{4\alpha_i} |x-a_i|^2}{1 + \lambda_i^2 |x-a_i|^2} \right) + O \left( \frac{1}{\lambda_i^2} \int_{\Omega} Ke^{\delta_i} \right)
\]
\[
= \frac{2\pi}{\alpha_i} \lambda_i^{4\alpha_i-2} F_i g_i(a_i) + O \left( \frac{1}{\lambda_i^2} \sum_{k=1}^m \lambda_k^{4\alpha_k-2} \right).
\]

Summing up the above estimates $I_1 + I_2 + I_3$, and using Lemma A.2 with the fact that $(2\alpha_i - 1)\alpha_i^{-2} = 1 + O((\alpha_i - 1)^2)$, the proof of (11) follows.
Finally, using the facts that \( \| \nabla J_\varrho (u) \| \leq \varepsilon \) and \( \| \lambda_i \frac{\partial P_{\delta_i}}{\partial \lambda_i} \| = O(1) \), we obtain the estimate \( 13 \).

For elements of \( V(m, \varepsilon) \) for which \( |\alpha_i - 1| \ln \lambda_i \) is small, the estimate given in Lemma A.9 allows to restate Proposition 1 more precisely as follows:

**Corollary 1.** Let \( u = \sum_{i=1}^{m} \alpha_i P_{\delta_i} + w \in V(m, \varepsilon) \) with \( w \in E_m^{a, \Lambda} \). If \( |\alpha_i - 1| \ln \lambda_i = o(\varepsilon) \) for each \( i \), then

\[
\left\langle \nabla J_\varrho (u), \sum_{i=1}^{m} \frac{\lambda_i}{\alpha_i} \frac{\partial P_{\delta_i}}{\partial \lambda_i} \right\rangle = 8\pi^2 m \int_{\Omega} Ke^u \sum_{i=1}^{m} \Delta F^a_i (a_i) \ln \lambda_i + O\left( \sum_{k=1}^{m} \left\{ |\alpha_k - 1|^2 + \frac{1}{\lambda_k^2} \right\} + \|w\|^2 \right).
\]

Now we prove an expansion of the gradient with respect to the bubble’s weights \( \alpha_i \)’s.

**Proposition 2.** Let \( u = \sum_{i=1}^{m} \alpha_i P_{\delta_i} + w = \bar{u} + w \in V(m, \varepsilon) \) with \( w \in E_m^{a, \Lambda} \). There holds

\[
\left\langle \nabla J_\varrho (u), P_{\delta_i} \ln \lambda_i \right\rangle = 32\pi (|\alpha_i - 1| + \tau_i) + \frac{1}{\ln \lambda_i} O\left( \sum_{k=1}^{m} \left\{ |\alpha_k - 1|^2 + \frac{1}{\lambda_k^2} \right\} + \|w\|^2 \right)
\]

\[
+ O\left( \sum_{k=1}^{m} \left\{ |\alpha_k - 1|^2 + \frac{1}{\lambda_k^2} \right\} + \|w\|^2 \right).
\]

**Proof.** Using the orthogonality \( \langle w, P_{\delta_i} \rangle = 0 \), we get

\[
\left\langle \nabla J_\varrho (u), P_{\delta_i} \right\rangle = \sum_{j=1}^{m} \alpha_j \langle P_{\delta_j}, P_{\delta_i} \rangle - \frac{\theta}{\int_{\Omega} Ke^u} \int_{\Omega} Ke^u P_{\delta_i}.
\]

We split the last integral as follows:

\[
\int_{\Omega} Ke^u P_{\delta_i} = \int_{I_1} Ke^u P_{\delta_i} + \int_{I_2} Ke^u w P_{\delta_i} + \int_{I_3} Ke^u (e^w - 1 - w) P_{\delta_i}. \quad (21)
\]

**Estimate of \( I_1 \):** Since \( |P_{\delta_i}| \leq c \ln \lambda_i \) and using (53), we get

\[
|I_1| \leq c \ln \lambda_i \int_{\Omega} e^u |e^w - 1 - w| \leq c \ln \lambda_i \|w\|^2 \sum_{k=1}^{m} \lambda_k^{4\alpha_k - 2}.
\]

**Estimate of \( I_2 \):** We have

\[
I_2 = \int_{B_i} \cdots + O\left( \int_{\Omega \setminus B_i} e^u |w| \right) = \int_{B_i} \cdots + O\left( \|w\|^2 \sum_{k=1}^{m} \lambda_k^{4\alpha_k - 2} \right)
\]
where we have used (17). Furthermore using Lemma A.8, we estimate
\[
\int_{B_i} \ldots \\
= \int_{B_i} \frac{\lambda_i^{4\alpha_i} F_i \varrho_{\lambda_i} g_i}{(1 + \lambda_i^2 |x - a_i|^2)^{2\alpha_i}} \left(1 + O\left(\frac{1}{\lambda_i^2}\right)\right) w \left(4 \ln \lambda_i - 2 \ln \left(1 + \lambda_i^2 |x - a_i|^2\right) + O(1)\right) \\
= \frac{1}{2} \lambda_i^{4\alpha_i - 2} \ln \lambda_i \int_{B_i} e^{\delta_i \xi_i} F_i^\alpha g_i w \\
+ O\left(\int_{B_i} \frac{\lambda_i^{4\alpha_i} |w|}{(1 + \lambda_i^2 |x - a_i|^2)^{2\alpha_i}} + \int_{B_i} \frac{\lambda_i^{4\alpha_i} |w|}{(1 + \lambda_i^2 |x - a_i|^2)^{2\alpha_i}}\right).
\]

Arguing as in the derivation of (17) we obtain
\[
\int_{B_i} \frac{\lambda_i^{4\alpha_i} \ln(1 + \lambda_i^2 |x - a_i|^2)|w|}{(1 + \lambda_i^2 |x - a_i|^2)^{2\alpha_i}} + \int_{B_i} \frac{\lambda_i^{4\alpha_i} |w|}{(1 + \lambda_i^2 |x - a_i|^2)^{2\alpha_i}} = O\left(\lambda_i^{4\alpha_i - 2} \|w\|\right).
\]

For the other integral, we have
\[
\int_{B_i} e^{\delta_i \xi_i} F_i^\alpha g_i w = \int_{B_i} e^{\delta_i \xi_i} F_i r_{\lambda_i} g_i w + O\left(\int_{B_i} e^{\delta_i |w| \left|\xi_i - 1\right|}\right) \\
= F_i^\alpha(a_i) g_i(a_i) \int_{B_i} e^{\delta_i w} + O\left(\int_{B_i} e^{\delta_i |w| \left|\{x - a_i\} \pm |\xi_i - 1|\right|}\right).
\]

Since \(w\) is in \(E_{a,\Lambda}^{m}\), using (49) and Lemma A.5 we get
\[
\int_{B_i} e^{\delta_i \xi_i} F_i^\alpha g_i w = O\left(\|w\| \left(|\alpha_i - 1\right| + \frac{1}{\lambda_i}\right)\right).
\]

Hence, the second integral in (21) can be estimated as
\[
I_2 = O\left(\|w\| \left(1 + |\alpha_i - 1| \ln \lambda_i\right) \sum_{k=1}^{m} \lambda_i^{4\alpha_k - 2}\right).
\]

**Estimate of \(I_3\):** This can be done by arguing as in the proof of Lemma A.8.

We conclude by collecting the estimates above, and applying Lemma A.3.

Similarly, by using lemmata A.1, A.2 and the definition of the \(\tau_i\) (see (12)), we derive the following estimate of the gradient with respect to the concentration points \(a_i\):

**Proposition 3.** Let \(\varrho = 8\pi m\) and \(u = \sum_{i=1}^{m} \alpha_i P \delta_{a_i} + w \in V(m, \varepsilon)\) with \(w \in E_{a,\Lambda}^{m}\). Then,
\[
\left\langle \nabla J_\varrho(u), \frac{1}{\lambda_i} \frac{\partial P \delta_i}{\partial a_i} \right\rangle = -\frac{8\pi}{F_i^\alpha(a_i)} \nabla F_i^\alpha(a_i) \lambda_i + O\left(\sum_{k=1}^{m} \left|\alpha_k - 1\right|^2 + \tau^2 + \frac{1}{\lambda_k^2}\right) + \|w\|^2.
\]

4. **A finite dimensional reduction.** In this section we show that the functional \(J_\varrho\) is coercive in the direction \(w\). We take advantage of such a coercivity to perform a finite dimensional reduction.

We start with an expansion of the functional \(J_\varrho\) (\(\varrho = 8\pi m\)) in \(V(m, \varepsilon)\).

**Proposition 4.** For \(u = \sum_{i=1}^{m} \alpha_i P \delta_i + w = \tilde{u} + w \in V(m, \varepsilon)\) with \(w \in E_{a,\Lambda}^{m}\), we have:
\[
J_\varrho(u) = J_\varrho(\tilde{u}) + f(w) + (1/2)Q(w) + O(\varepsilon \|w\|^2),
\]
where
\begin{equation}
\begin{aligned}
f(w) = -\varrho \frac{\int_{\Omega} K e^u w}{\int_{\Omega} K e^u}, \\
Q(w) = \|w\|^2 - \varrho \frac{\int_{\Omega} K e^u w^2}{\int_{\Omega} K e^u} + \varrho \left( \frac{\int_{\Omega} K e^u}{\int_{\Omega} K e^u} \right)^2.
\end{aligned}
\end{equation}

Proof. The proof follows from a Taylor expansion combined with the uniform estimates in Lemma A.4. In particular
\[
\int_{\Omega} K e^u \left| w^2 - \frac{w^2}{2} \right| \leq c\|w\|^3 \sum_{k=1}^m \lambda_k^{4\alpha_k - 2}.
\]
\[\square\]

Now we prove that the second variation $\partial^2 J_\varrho(\bar{u})$ is coercive on the space $E_{u,\Lambda}^m$.

**Proposition 5.** Let $u = \sum_{i=1}^m \alpha_i P \delta_i \in V(m, \varepsilon)$. Then there exists a constant $C_0$ such that
\[
Q(w) \geq Q_0(w) + O(\varepsilon^2 \|w\|^2) \geq C_0 \|w\|^2, \quad \forall w \in E_{u,\Lambda}^m,
\]
where
\[
Q_0(w) := \|w\|^2 - \sum_{i=1}^m \int_{\Omega} e^{\delta_{\alpha_i,\lambda_i}} w^2.
\]

Proof. We first notice that, it follows from Lemma A.7 that the last term in the definition of $Q(w)$ is $O(\varepsilon \|w\|^2)$.

Now, we will focus on the second one. It follows from Lemma A.1 that $e^u$ is bounded outside of the union of the $B_i$’s. Hence we have,
\[
\int_{\Omega \setminus \bigcup B_i} K e^u w^2 = O(\|w\|^2).
\]

Secondly, using Lemma A.6 we have
\[
\int_{B_i} K e^u w^2 = \int_{B_i} \frac{\lambda_i^{4\alpha_i} F_i(x) g_i(x)}{(1 + \lambda_i^2 |x - a_i|^2)^{2\alpha_i}} \left( 1 + O \left( \frac{1}{\lambda_i^2} \right) \right) w^2
\]
\[
= \lambda_i^{4\alpha_i - 2} F_i(a_i) g_i(a_i) \int_{B_i} \frac{e^{\delta_i}}{8} \xi_i(x) w^2
\]
\[
+ \lambda_i^{4\alpha_i - 2} O \left( \int_{B_i} e^{\delta_i} |\xi_i(x)||x - a_i|^2 w^2 + \frac{\|w\|^2}{\lambda_i^2} \right)
\]
\[
= \lambda_i^{4\alpha_i - 2} F_i(a_i) g_i(a_i) \int_{B_i} \frac{e^{\delta_i}}{8} \xi_i(x) w^2 + O(\varepsilon \lambda_i^{4\alpha_i - 2} \|w\|^2).
\]

Using the fact that
\[
\lambda_i^{4\alpha_i - 2} F_i(a_i) g_i(a_i) = \frac{(2\alpha_i - 1)}{m \pi} (1 - \tau_i) \int_{\Omega} K e^u,
\]
we deduce using Lemma A.10 and the estimate (13) on $\tau_i$ that
\[
\frac{\varrho}{\int_{\Omega} K e^u} \int_{\Omega} K e^u w^2 = \sum_{i=1}^m \int_{\Omega} e^{\delta_i} w^2 + O \left( (|\alpha_i - 1| + |\tau_i| \|w\|^2) \right) + O(\varepsilon \|w\|^2).
\]
Therefore we have that
\[ Q(w) = \|w\|^2 - \sum_{i=1}^{m} \int_{\Omega} e^{\delta_i} w^2 + O(\varepsilon \|w\|^2). \]

Now the result follows from the fact that \( w \in E_{a,\Lambda}^m \) using standard blow up argument (see also [18, Lemma 6.4]).

As a consequence of Proposition 5 and Lemma A.7 we have the following proposition:

**Proposition 6.** Let \( u := \sum_{i=1}^{m} \alpha_i P_\delta \in V(m, \varepsilon) \). Then there exists a unique \( w := w(u) \) such that:

\[ J_\varrho(u + w) = \min \{ J_\varrho(u + w) : u + w \in V(m, \varepsilon), w \in E_{a,\Lambda}^m \}. \]

Furthermore, there exists a constant \( C \) such that
\[ \|w\| \leq C \sum_{i=1}^{m} \left( |\alpha_i - 1| + \frac{|\nabla F^a_i(a_i)|}{\lambda_i} + \frac{|\ln \lambda_i|^{3/2}}{\lambda_i^2} \right). \]  

(24)

Note that, since \( J_\varrho \) achieves its minimum in the space \( E_{a,\Lambda}^m \) for each fixed variables \((\alpha, a, \Lambda)\), by using the Morse lemma, we derive the existence of a change of variables \( w \rightarrow \omega \) such that
\[ J_\varrho(\sum_{i=1}^{m} \alpha_i P_\delta + w) = J_\varrho(\sum_{i=1}^{m} \alpha_i P_\delta + \omega) + \|V\|^2. \]

Hence, to decrease the functional \( J_\varrho \) we need to define
\[ \dot{V} = -V. \]

This vector field will bring the variable \( V \) to zero, i.e. it will bring \( w \) to \( \bar{w} \).

The aim of the next section will be to construct a vector field on the remaining variables \( \alpha_i, a_i \) and \( \lambda_i \).

5. Characterization of “the critical points at Infinity”. Critical points at Infinity are the \( \omega \)-limit set of the gradient (or any related pseudogradient) which are not critical points of the functional \( J_\varrho \). In our case they are the limit of orbits \( u(t_k) \) of the gradient which remains “trapped” in the neighborhood at Infinity \( V(m, \varepsilon) \). That is \( J_\varrho(u_k) \rightarrow C_\infty \) and \( \partial J_\varrho(u_k) \rightarrow 0 \) but \( (u_k) \) does not have a converging subsequence. Hence they impede the deformation of the upper level \( J_\varrho^{C_\infty + \varepsilon} \) onto the lower level \( J_\varrho^{C_\infty - \varepsilon} \) for some small \( \varepsilon \) just like “true” critical points do. Our aim in this section is to give a precise characterization of such critical points, to perform a Morse type reduction around them and to compute their Morse indices. Inspired by the ideas introduced by A. Bahri in his study of Yamabe and scalar curvature type equations see [2], [6], we look for a normal form of the gradient around such points. The way to find such a normal form is to construct a pseudogradient for the functional in the neighborhood at Infinity whose dynamic is easier to be understood.
As a starting point in our search for such an appropriate pseudogradient we provide an accurate expansion of the functional $J_\varepsilon$ in $V(m, \varepsilon)$. Using the functions $F_K^a$, $F_i^a$ and $g_i$ defined in (4) and (17) we prove:

**Proposition 7.** Let $u = \sum_{i=1}^m \alpha_i P \delta_i \in V(m, \varepsilon)$. If $\sum_{i=1}^m |\alpha_i - 1| \ln \lambda_i = o_\varepsilon(1)$, then

$$J_\varepsilon(u + \varpi) = -8\pi m (1 + \ln(m\pi)) - 8\pi F_K^a(a_1, \ldots, a_m) - 4\pi \sum_{i=1}^m \tilde{\tau}_i^2$$

$$+ 16\pi \sum_{i=1}^m (\alpha_i - 1) \ln \lambda_i - 4m\pi \sum_{i=1}^m \Delta F_i^a(a_i) \ln \lambda_i$$

$$+ O\left( \sum_{k=1}^m \left\{ (\alpha_k - 1)^2 + |\tilde{\tau}_k|^2 + \frac{1}{\lambda_k^2} \right\} + \|f\|^2 + \|\varpi\|^2 \right),$$

where

$$\tilde{\tau}_i = 1 - \frac{m}{2\alpha_i-1} \lambda^{4\alpha_i-2} F_i^a(a_i) g_i(a_i) - \frac{m}{2\alpha_i} \lambda^{4\alpha_i-2} F_i^a(a_k) g_k(a_k).$$

**Proof.** Using Lemma A.3 we have

$$\|u\|^2 = 16\pi \sum_{i=1}^m \alpha_i^2 (2 \ln \lambda_i - 1 - 4\pi H(a_i, a_i)) + 64\pi^2 \sum_{i \neq j} \alpha_i \alpha_j G(a_i, a_j) + O\left( \frac{1}{\lambda^2} \right). \quad (25)$$

Secondly, setting $\Gamma := \sum_{i=1}^m \frac{\lambda^{4\alpha_i-2} F_i^a(a_i) g_i(a_i)}{2\alpha_i-1}$, and using Lemma A.8, we have

$$\ln \left( \int_\Omega K^u \right) = \ln(\pi \Gamma) + \ln \left( 1 + \frac{1}{2\pi} \sum_{i=1}^m \Delta F_i^a g_i(a_i) \ln \lambda_i + O\left( \sum_{k=1}^m \left\{ |\alpha_k - 1|^2 + \frac{1}{\lambda_k^2} \right\} \right) \right)$$

$$= \ln \pi - \frac{1}{m} \sum_{i=1}^m \ln(1 - \tilde{\tau}_i) + \frac{1}{m} \sum_{i=1}^m \ln \left( \frac{m\lambda^{4\alpha_i-2} F_i^a g_i(a_i)}{2\alpha_i-1} \right)$$

$$+ \frac{1}{2\pi} \sum_{i=1}^m \Delta F_i^a g_i(a_i) \ln \lambda_i + O\left( \sum_{k=1}^m \left\{ |\alpha_k - 1|^2 + \frac{1}{\lambda_k^2} \right\} \right).$$

Hence

$$\ln \left( \int_\Omega K^u \right) = \ln \pi + \frac{1}{m} \sum_{i=1}^m \left\{ \tilde{\tau}_i + \frac{\tilde{\tau}_i^2}{2} \right\} + \ln m - \frac{1}{m} \sum_{i=1}^m \ln(2\alpha_i - 1)$$

$$+ \frac{1}{m} \sum_{i=1}^m (4\alpha_i - 2) \ln \lambda_i + \frac{1}{m} \sum_{i=1}^m \ln(F_i^a(a_i)) + \frac{1}{m} \sum_{i=1}^m \ln(g_i(a_i))$$

$$+ \frac{1}{2\pi} \sum_{i=1}^m g_i(a_i) \Delta F_i^a(a_i) \ln \lambda_i$$

$$+ O\left( \sum_{k=1}^m \left\{ |\alpha_k - 1|^2 + |\tilde{\tau}_k|^2 + \frac{1}{\lambda_k^2} \right\} \right). \quad (26)$$

We remark that $\sum_{i=1}^m \tilde{\tau}_i = 0$ and

$$\ln (2\alpha_i - 1) = \ln (1 + 2(\alpha_i - 1)) = 2(\alpha_i - 1) - 2(\alpha_i - 1)^2 + O(|\alpha_i - 1|^3).$$
Hence, using (25) and (26), we get
\[
J_\varrho(u) = 16\pi \sum_{i=1}^{m} (\alpha_i - 1)^2 \ln \lambda_i - 8\pi \sum_{i=1}^{m} \alpha_i^2 (1 + 4\pi H(\alpha_i, \alpha_i)) - 8\pi m \ln (m\pi)
+ 32\pi^2 \sum_{i \neq j} \alpha_i \alpha_j G(\alpha_i, \alpha_j) - 4\pi \sum_{i=1}^{m} \varphi_i^2 + 8\pi \sum_{i=1}^{m} 2(\alpha_i - 1)
- 8\pi \sum_{i=1}^{m} 2(\alpha_i - 1)^2 - 8\pi \sum_{i=1}^{m} \ln (F_\alpha(\alpha_i)) - 8\pi \sum_{i=1}^{m} \ln (g_i(\alpha_i))
- \frac{8\pi m}{2\Gamma} \sum_{i=1}^{m} g_i(\alpha_i) \Delta F_\alpha(\alpha_i) \ln \lambda_i + O\left(\sum_{k=1}^{m} (\alpha_k - 1)^2 + |\tau_k|^3 + \frac{1}{\lambda_k^2}\right).
\]

The result follows by noting
\[
g_i(\alpha_i) = 1 + O\left(\sum_{k=1}^{m} |\alpha_k - 1|\right), \quad \Gamma = \sum_{i=1}^{m} \lambda_i^2 F_\alpha(\alpha_i) + O\left(\sum_{k=1}^{m} |\alpha_k - 1|\lambda_k^2 \ln \lambda_k\right).
\]

From Proposition [7] it is easy to see that the functional \(J_\varrho\) will decrease by bringing the variables \(\alpha_i\) to 1 and increasing the parameters \(\tau_k\). In fact we have the following proposition which defines a vector field \(W\) along which the weights \(\alpha_i\) will be brought to 1, the \(\tau_k\) to zero, the points of the concentration to critical points of the functional \(F^K_m\), while the concentration speed will be kept under control of all but one specific region, where the concentrations \(\lambda_i\) will increase dramatically leading to a “critical point at Infinity.”

**Proposition 8.** Let \(\varrho = 8\pi m\) with \(m \geq 1\) and assume that the function \(K\) satisfies the condition \((ND_m)\). Then there exists a pseudogradient \(W\) defined in \(V(m, \varepsilon)\) and satisfying the following properties:

There exists a constant \(C\) independent of \(u = \sum_{i=1}^{m} \alpha_i \varrho \delta_{\alpha_i, \lambda_i} + \bar{w}\) such that
\[
(1) \quad \langle -\nabla J_\varrho(u), W \rangle \geq C \sum_{i=1}^{m} \left( |\alpha_i - 1| + |\tau_i| + \frac{|\nabla F_\alpha(\alpha_i)|}{\lambda_i} + \frac{\ln \lambda_i}{\lambda_i^2} \right),
\]
\[
(2) \quad \langle -\nabla J_\varrho(u), W + \frac{\partial F_K}{\partial (\alpha, \lambda, a)} \rangle \geq C \sum_{i=1}^{m} \left( |\alpha_i - 1| + |\tau_i| + \frac{|\nabla F_\alpha(\alpha_i)|}{\lambda_i} + \frac{\ln \lambda_i}{\lambda_i^2} \right),
\]
\[
(3) \quad |W| \text{ is bounded and the only region where the variables } \lambda_i \text{'s increase along the flow lines of } W \text{ is the region where } (\alpha_1, \cdots, \alpha_m) \text{ is very close to a critical point } q := (q_1, \cdots, q_m) \text{ of } F^K_m \text{ with } q \in K^-_m.
\]

**Proof.** We will focus on the case \(m \geq 2\). The case \((m = 1)\) can be proved in the same way and is even easier.

Let \(\zeta \in C^\infty(\mathbb{R})\) be a function with the property
\[
0 \leq \zeta \leq 1, \quad \zeta(s) = 0 \text{ if } |s| \leq \frac{1}{2}, \quad \zeta(s) = 1 \text{ if } |s| \geq 1,
\]
and consider for each \(i \in \{1, \cdots, m\}\) the following vector fields
\[
W_{\alpha_i} := \frac{(1 - \alpha_i)}{[\alpha_i - 1]} \zeta\left(\frac{\lambda_i^2}{\ln \lambda_i} |\alpha_i - 1| \right) \left(\frac{P \delta_i}{\ln \lambda_i} - \frac{2}{\alpha_i} \frac{\lambda_i \partial P \delta_i}{\partial \lambda_i}\right),
\]
\[ W_{\alpha_i} := -\frac{4}{\alpha_i} \lambda_i \frac{\partial P^{\delta_i}}{\partial \lambda_i}, \quad W_{\tau_i} := -2 \frac{\tau_i}{|\tau_i|} \zeta\left( \frac{\lambda^2_2}{\ln \lambda_i} |\tau_i| \right) \frac{1}{\alpha_i} \lambda_i \frac{\partial P^{\delta_i}}{\partial \lambda_i}, \]
\[ W_{a_i} := \frac{\nabla F_i(a_i)}{||\nabla F_i(a_i)||} \zeta\left( \frac{\lambda_i}{\ln \lambda_i} ||\nabla F_i(a_i)|| \right) \frac{1}{\lambda_i} \frac{\partial P^{\delta_i}}{\partial a_i}. \]

Using Propositions 2 and 1 (with \( \bar{w} \)) we derive
\[ \langle -\nabla J_\rho(u), W_{a_i} \rangle = \zeta\left( \frac{\lambda^2_2}{\ln \lambda_i} |\alpha_i - 1| \right) \left\{ 32\pi |\alpha_i - 1| + O\left( \sum_{k=1}^m \left( |\alpha_k - 1| + \frac{|\tau_k|}{\ln \lambda_i} + \frac{||\bar{w}||}{\ln \lambda_i} \right) \right) + O\left( |\alpha_i - 1|^2 + \frac{1}{\lambda^2_i} + ||\bar{w}||^2 \right) \right\}, \tag{27} \]
\[ \langle -\nabla J_\rho(u), W_{\alpha_i} \rangle = 64\pi \tau_i + O\left( \sum_{k=1}^m \left( |\alpha_k - 1|^2 + \frac{1}{\lambda^2_k} \right) + ||\bar{w}||^2 \right), \tag{28} \]
\[ \langle -\nabla J_\rho(u), W_{\tau_i} \rangle = \zeta\left( \frac{\lambda^2_2}{\ln \lambda_i} |\tau_i| \right) \left\{ 32\pi |\tau_i| + O\left( \sum_{k=1}^m \left( |\alpha_k - 1|^2 + \frac{1}{\lambda^2_k} \right) + ||\bar{w}||^2 \right) \right\}, \tag{29} \]
and Proposition 3 implies
\[ \langle -\nabla J_\rho(u), W_{a_i} \rangle = c \zeta\left( \frac{\lambda_i}{\ln \lambda_i} ||\nabla F_i(a_i)|| \right) \left\{ \frac{||\nabla F_i(a_i)||}{\lambda_i} \right\} + O\left( \sum_{k=1}^m \left( |\alpha_k - 1|^2 + \frac{1}{\lambda^2_k} \right) + ||\bar{w}||^2 \right). \tag{30} \]

We divide the set \( V(m, \varepsilon) \) into 2 subsets and on each one we will define a pseudogradient. The global vector field will be a convex combination of them. Let us define
\[ \mathcal{V}_1(m, \varepsilon) := \{ u \in V(m, \varepsilon) : \exists i \text{ s.t. } |\alpha_i - 1| + |\tau_i| + ||\nabla F_i(a_i)||/\lambda_i \geq 6M \ln \lambda_i/\lambda^2_i \}, \]
\[ \mathcal{V}_2(m, \varepsilon) := \{ u \in V(m, \varepsilon) : \forall i, |\alpha_i - 1| + |\tau_i| + ||\nabla F_i(a_i)||/\lambda_i \leq 9M \ln \lambda_i/\lambda^2_i \}, \]
where \( M \) is a large positive constant to be chosen later.

In \( \mathcal{V}_1(m, \varepsilon) \), we define
\[ W^1 := \sum_{i=1}^m \{ W_{\alpha_i} + W_{\lambda_i} + W_{\tau_i} + W_{a_i} \}. \]

Note that, along the flow lines generated by \( W^1 \), the variables \( \lambda_i \)'s are decreasing functions.

To prove the claim (1) in the subset \( \mathcal{V}_1(m, \varepsilon) \), we introduce
\[ I_\alpha := \{ i : |\alpha_i - 1| \geq \ln \lambda_i/\lambda^2_i \}, \quad I_\tau := \{ i : |\tau_i| \geq \ln \lambda_i/\lambda^2_i \}, \quad I_a := \{ i : \frac{||\nabla F_i(a_i)||}{\lambda_i} \geq \ln \lambda_i/\lambda^2_i \}. \]
We have
\[
\langle -\nabla J_\rho(u), W^1 \rangle 
\geq 32\pi \sum_{i \in I_\alpha} |\alpha_i - 1| + 32\pi \sum_{i \in I_*} |\tau_i| + c \sum_{i \in I_\alpha} \frac{|\nabla F_i(a_i)|}{\lambda_i} + 64\pi \sum_{i=1}^m \tau_i + O_1,
\]
where \(O_1\) is given by summing the remainders of \(27\) and \(30\):
\[
O_1 = O \left( \sum_{k=1}^m \left\{ |\alpha_k - 1|^2 + |\tau_k|^2 + \frac{|\alpha_k - 1| + |\tau_k|}{\ln \lambda_k} + \frac{1}{\lambda_k^2} \right\} + \|\varpi\|^2 + \frac{\|\varpi\|}{\ln \lambda_k} \right).
\]

By definition of the set \(V_2(m, \varepsilon)\), there exists \(z_{i_0} \in \{|\alpha_{i_0} - 1|, |\tau_{i_0}|, |\nabla F_{i_0}(a_{i_0})|/\lambda_{i_0}\}\) such that \(z_{i_0} \geq 2M \frac{\ln \lambda_{i_0}}{\lambda_{i_0}^2}\). Since the elements in \(V(m, \varepsilon)\) have comparable parameter of concentration \(\lambda_i\), we have
\[
z_{i_0} \geq 2M \frac{\ln \lambda_{i_0}}{\lambda_{i_0}^2} \geq 2MC \sum_{i=1}^m \frac{\ln \lambda_i}{\lambda_i^2},
\]
which also implies
\[
z_{i_0} \geq 2MC \left( \sum_{i \notin I_\alpha} |\alpha_i - 1| + \sum_{i \in I_*} |\tau_i| + \sum_{i \notin I_\alpha} \frac{|\nabla F_i(a_i)|}{\lambda_i} \right).
\]
Hence,
\[
\langle -\nabla J_\rho(u), W^1 \rangle \geq cM \sum_{i=1}^m \frac{\ln \lambda_i}{\lambda_i^2} + \sum_{k=1}^m \left\{ |\alpha_k - 1| + |\tau_k| + \frac{|\nabla F_k(a_k)|}{\lambda_k} \right\} + 64\pi \sum_{i=1}^m \tau_i.
\]
Note that from Lemma 1.8 we have
\[
\sum_{i=1}^m \tau_i = O \left( \frac{1}{\int_\Omega Ke^u} \right) + \sum_{k=1}^m \left\{ |\alpha_k - 1|^2 + \frac{\ln \lambda_k}{\lambda_k^2} \right\} + \|\varpi\|^2.
\]
The second and the last term are small with respect to the desired lower bound, whereas the third one can be controlled by choosing \(M\) large enough. To study the first one, we observe:

- if there exists \(j\) such that \(|\alpha_j - 1| \geq \frac{1}{\lambda_j}\), then
  \[
  \frac{1}{\int_\Omega Ke^u} \leq \frac{c}{\sum_{k=1}^m \lambda_k^{4\alpha_k - 2}} \leq \frac{c}{\lambda_j^{4\alpha_j - 2}} \leq \frac{c}{\lambda_j^{5/2}} = o(|\alpha_j - 1|).
  \]

- In the other case, we have \(|\alpha_i - 1| \ln \lambda_i\) is very small for each \(i\) and therefore
  \[
  \frac{1}{\int_\Omega Ke^u} \leq \frac{c}{\lambda^2} = o \left( \frac{\ln \lambda}{\lambda^2} \right).
  \]

The result follows.

In \(V_2(m, \varepsilon)\), we observe that the point \((a_1, ..., a_m)\) is very close to a critical point \(q\) of \(F_k^u\). Note that, since the \(\lambda_i\)'s are of the same order, we get that \(\ln \lambda_i = \ln \lambda_1 + O(1)\) for each \(i\) and therefore \(\sum_{i=1}^m \ln \lambda_i \Delta F_i(a_i) = \ln \lambda_1 (l(q) + o(1)) + O(1)\). Moreover, since all the \(|\alpha_i - 1| \ln \lambda_i\) are small, we get that \(\int_\Omega Ke^u = c\lambda_1^2 (1 + o(1))\). Hence, the principal term in Corollary 1 becomes equivalent to \(c l(q) \ln \lambda_1 / \lambda_1^2\). We define
\[
W_\alpha^2 := - \text{sign}(l(q)) \sum_{i=1}^m \frac{\lambda_i}{\alpha_i} \frac{\partial P \delta_i}{\partial \lambda_i}.
\]
Using Corollary 1, we derive that for some positive constant $c$ we have that:

$$\langle -\nabla J_\varrho(u), W^3_\lambda \rangle \geq c \frac{\ln \lambda_i}{\lambda_i^2} + O \left( \sum_{k=1}^{m} \left\{ |\alpha_k - 1|^2 + \frac{1}{\lambda_k^2} \right\} + \|w\|^2 \right). \quad (31)$$

Note that, since the $\lambda_i$'s are of the same order, we can make appear all the $\ln \lambda_i/\lambda_i^2$'s in the lower bound of (31) and using the definition of $\mathcal{V}_2(m, \varepsilon)$, we can make appear all the terms $|\alpha_i - 1|, |\tau_i|$ and $|\nabla \mathcal{F}_i(a_i)|/\lambda_i$ in the lower bound of (31).

Now, we define the pseudogradients $W$ by a convex combination of $W^1$ and $W^3_\lambda$. We notice that $|W|$ is bounded and the only region where the concentration speeds $\lambda_i$'s increase along the flow lines of $W$ is the subset $\mathcal{V}_2^-(m, \varepsilon) \subset \mathcal{V}_2(m, \varepsilon)$ where the concentration points $(a_1, \cdots, a_m)$ are very close to a critical point $q := (q_1, \cdots, q_m)$ of $F^K_\mu$ with $l(q) < 0$ that is $q \in K^m$. Hence Claim (1) follows.

Concerning Claim (2), we remark that $\partial \bar{w}/\partial s$ does not necessary belong to $E^m_{a,\lambda}$. Then we write

$$\frac{\partial \bar{w}}{\partial s} = \sum_{i=1}^{m} \left\{ t_i \frac{P\delta_i}{\sqrt{\ln \lambda_i}} + \mu_i \lambda_i \frac{\partial P\delta_i}{\partial \lambda_i} + \sum_{j=1,2} \nu_{i,j} \frac{1}{\lambda_i} \frac{\partial P\delta_i}{\partial (a_{i,j})} \right\} + \bar{w}. \quad (32)$$

where $\bar{w} \in E^m_{a,\lambda}$. Hence, for $u = \sum \alpha_i P\delta_i + \bar{w}$, we have

$$\langle \nabla J_\varrho(u), \frac{\partial \bar{w}}{\partial s} \rangle = \sum_{i=1}^{m} \left\{ t_i \langle \nabla J_\varrho(u), \frac{P\delta_i}{\sqrt{\ln \lambda_i}} \rangle + \mu_i \lambda_i \langle \nabla J_\varrho(u), \lambda_i \frac{\partial P\delta_i}{\partial \lambda_i} \rangle \right\}$$

$$+ \sum_{j=1,2} \nu_{i,j} \langle \nabla J_\varrho(u), \frac{1}{\lambda_i} \frac{\partial P\delta_i}{\partial (a_{i,j})} \rangle \right\}, \quad (33)$$

(since $\bar{w}$ minimizes $J_\varrho(\sum \alpha_i P\delta_i + \bar{w})$ in $E^m_{a,\lambda}$). It remains to estimate the variables $t_i, \mu_i$ and $\nu_{i,j}$. For this purpose, we multiply (32) by $P\delta_i/\sqrt{\ln \lambda_i}$, $\lambda_i (\partial P\delta_i/\partial \lambda_i)$ and $(1/\lambda_i) (\partial P\delta_i/\partial (a_{i,j}))$ respectively (the norm of each term is of the order $c + o(1)$ where $c$ is a positive constant and the scalar product of two of them is of order $o(1)$). We get the following system

$$\begin{cases}
  c_i t_i + o(\sum |t_j| + |\mu_i| + |\nu_{k,j}|) = \langle \partial \bar{w}/\partial s, P\delta_i/\sqrt{\ln \lambda_i} \rangle, & i = 1, \ldots, m \\
  c'_i \mu_i + o(\sum |t_j| + |\mu_j| + |\nu_{k,j}|) = \langle \partial \bar{w}/\partial s, \lambda_i \frac{\partial P\delta_i}{\partial \lambda_i} \rangle, & i = 1, \ldots, m \\
  c''_{i,j} \nu_{i,j} + o(\sum |t_j| + |\mu_j| + |\nu_{k,j}|) = \langle \partial \bar{w}/\partial s, \lambda_i^{-1} \partial P\delta_{a_{i,j}}/\partial a_{i,j} \rangle, & 1 \leq i, j = 1, 2,
\end{cases}$$

where $c_i, c'_i$ and $c''_{i,j}$ are fixed positive constants. Recalling that $\bar{w}$ is in $E^m_{a,\lambda}$, we deduce that

$$\frac{\partial}{\partial s} \langle \bar{w}, P\delta_i \rangle = 0$$

which implies $\langle \partial \bar{w}/\partial s, P\delta_i \rangle = 0$.

Furthermore, in the same way we get (recall that $\dot{\lambda}_i = O(\lambda_i)$, and $\dot{a}_{i,j} = O(\lambda_i^{-1})$)

$$\langle \partial \bar{w}/\partial s, \frac{\partial P\delta_i}{\partial \lambda_i} \rangle = -\langle \bar{w}, \frac{\partial^2 P\delta_i}{\partial \lambda_i^2} \rangle \dot{\lambda}_i = -\langle \bar{w}, \frac{\partial^2 P\delta_i}{\partial \lambda_i \partial a_{i,j}} \rangle \dot{a}_{i,j} = O(\lambda_i^{-1} \|w\|).$$

Here we get

$$t_i, \mu_i, \nu_{i,j} = O(\|w\|).$$
This implies that the term $\langle \nabla J_\varrho(u), (\partial \bar{w})/(\partial s) \rangle$ is small with respect the lower bound in claim (1). Hence the second claim follows.

This completes the proof of our proposition in the case where $m \geq 2$. \hfill $\square$

As a consequence of Propositions 7 and 8 we are able to identify the critical points at Infinity of $J_\varrho$ and to compute their Morse indices. Indeed we have

**Corollary 2.** Let $\varrho = 8\pi m$, $m \geq 1$. The critical points at Infinity of $J_\varrho$ are in a one to one correspondence with critical points $(q_1, \ldots, q_m)$ of $\mathcal{F}_m^K$ satisfying $(q_1, \ldots, q_m) \in \mathcal{K}^-_m$, that will be denoted by $(q_1, \ldots, q_m)_\infty$.

Furthermore the energy level of such a critical point at Infinity $(q_1, \ldots, q_m)_\infty$ denoted $C_\infty(q_1, \ldots, q_m)_\infty$ is given by:

$$C_\infty(q_1, \ldots, q_m)_\infty = -8\pi m(1 + \ln(m\pi)) - 8\pi \sum_{i=1}^{m} \ln(K(q_i)) + 32\pi^2 \sum_{i=1}^{m} \left( H(q_i, q_i) - \sum_{j \neq i} G(q_i, q_j) (\text{ if } m \geq 2) \right).$$

Moreover the Morse index of such a critical point at Infinity $(q_1, \ldots, q_m)_\infty$ is given by:

$$3m - 1 - \text{morse}(\mathcal{F}_m^K, (q_1, \ldots, q_m)).$$

**Proof.** The first part is a consequence of Proposition 8. In fact, in $\mathcal{V}_1 \cup \mathcal{V}_2^+$, where $\mathcal{V}_2^+$ is the subset of $\mathcal{V}_2(m, \varepsilon)$ where the concentration points $(a_1, \ldots, a_m)$ converge to a critical point $q$ of $\mathcal{F}_m^K$ with $l(q) > 0$, the concentration speeds $\lambda_i$‘s are non-increasing functions and therefore, they are bounded in this set. However, in $\mathcal{V}_2^-$, the point $A := (a_1, \ldots, a_m)$ converges to a critical point $q$ of $\mathcal{F}_m^K$ with $l(q) < 0$, each variable $a_i$ converges to 1 and the $\lambda_i$’s go to $+\infty$. Hence we obtain a critical point at infinity. Its energy level obtained from Proposition 7. Concerning the Morse index, again from Proposition 7, since the variables $\tilde{\tau}_i$‘s satisfy: $\sum_{i=1}^{m} \tilde{\tau}_i = 0$, we need to cancel one variable. For that aim we decompose:

$$\sum_{i=1}^{m} \tilde{\tau}_i^2 = 2 \sum_{i=2}^{m} \tilde{\tau}_i^2 + \sum_{2 \leq j \neq i \leq m} \tilde{\tau}_i \tilde{\tau}_j = \Lambda^t M \Lambda \quad \text{with } \Lambda^t = (\tilde{\tau}_2, \ldots, \tilde{\tau}_m)$$

and $M$ is a matrix defined by

$$M = (m_{ij})_{2 \leq i, j \leq m} \quad \text{with } \quad m_{ii} = 2 \quad \text{and } \quad m_{ij} = 1, \quad \text{for } i \neq j.$$ 

It is easy to see that the eigenvalues of this matrix are $m$ (multiplicity one) and 1 (with multiplicity $m - 2$). Now, we consider the new variables

$$\beta_i := (\alpha_i - 1) / \ln \lambda_i, \quad i \leq m, \quad \zeta_1 := \sqrt{\sum_{j=1}^{m} \lambda_j F_j(a_j)} \quad \text{and} \quad \zeta_k := \tilde{\tau}_k, \quad 2 \leq k \leq m.$$ 

These variables are independent. Moreover, since $\lambda_j / \lambda_1$ is bounded, we deduce that $\ln \lambda_i = \ln \lambda_1 + O(1)$, for each $i$. Hence, from Proposition 7, since $l(q) < 0$, we derive that the Morse index is equal to:

$$m - 1 + \text{coindex } \mathcal{F}_m^K(q) = 3m - 1 - \text{index } \mathcal{F}_m^K(q).$$

The result follows. \hfill $\square$
Corollary 3. Let \((u_k)_k\) be a sequence of solutions of \((P_\varrho)\) for \(\varrho = 8\pi m; m \in \mathbb{N}\). Under the nondegeneracy condition \((ND_m)\) we have that \(u_k\) is bounded in \(C^{0,\alpha}(\Omega)\) for some \(\alpha \in (0,1)\). In particular for \(L\) large, every solution \(u\) of \((P_{8\pi m})\) satisfies that: 

\(-L < J_\varrho(u) < L\).

Proof. First of all we notice that it follows from elliptic estimates and Moser Trudinger Inequality that it suffices to prove that \((u_k)_k\) is uniformly bounded in \(H_0^1\). Arguing by contradiction, it follows from blow up analysis (see \([11, 29]\)) that \(u_k\) will concentrate at \(m\)-points \((a_1, \ldots, a_m) \in \Omega\). More precisely \(u_k \in V(m,\varepsilon)\). But under the nondegeneracy assumption \((ND_m)\), Proposition 8 rules out the existence of solutions (that is \(\nabla J(u_k) = 0\)) in \(V(m,\varepsilon)\).

6. Proof of the main results.

6.1. Results for the critical case. We first observe that by applying the Proposition 7 the energy values of the functional \(J_{8\pi}\) in \(V(1,\eta)\) can be estimated as follows:

Proposition 9. Let \(a\) be a critical point of \(F^K_1\) and let \(\lambda\) be such that \(P\delta_{a,\lambda} \in V(1,\eta)\). We have

\[
J_{8\pi}(P\delta_{a,\lambda}) = -8\pi(1 + \ln \pi) - 8\pi F^K_1(a) - 4\pi \ln(\lambda) \Delta \ln(K)(a) + O\left(\frac{1}{\lambda^2}\right).
\]

Proof of Theorem 1.1. First, we recall that the functional \(J_{8\pi}\) is lower bounded. Let

\[
c_\infty = -8\pi(1 + \ln(\pi)) - 8\pi \max F^K_1
\]

and let \(y_0\) be a maximum point of \(F^K_1\) such that \(\Delta \ln(K)(y_0) > 0\). Applying Proposition 9 with \(a = y_0\), we derive that \(J_{8\pi}(P\delta_{y_0,\lambda}) < c_\infty\). Therefore we deduce that

\[
\inf J_{8\pi} < c_\infty.
\]

(34)

Now, we approach our functional by the family \(J_{8\pi-\varepsilon}\) with \(\varepsilon > 0\). Note that

\[
\inf J_{8\pi-\varepsilon} \to \inf J_{8\pi} \quad \text{when} \quad \varepsilon \to 0.
\]

(35)

Furthermore, it is known that \(J_{8\pi-\varepsilon}\) achieves its minimum. Let \(u_\varepsilon\) be such that \(J_{8\pi-\varepsilon}(u_\varepsilon) = \min J_{8\pi-\varepsilon}\). We claim that \(u_\varepsilon\) is bounded. Indeed, arguing by contradiction and assuming that it is not bounded, then it follows from the blow up analysis that \(u_\varepsilon\) has to enter \(V(1,\eta)\) (for a small \(\eta\)). Thus, we will have

\[
u_\varepsilon = P\delta_{a_\varepsilon,\lambda_\varepsilon} + v_\varepsilon \quad \text{with} \quad \|v_\varepsilon\| \to 0.
\]

Now, it is easy to see that

\[
J_{8\pi-\varepsilon}(u_\varepsilon) = J_{8\pi}(u_\varepsilon) + \varepsilon \ln \left(\int_{\Omega} K e^{u_\varepsilon}\right)
\]

\[
\geq J_{8\pi}(u_\varepsilon) + \varepsilon \ln \left(\int_{\Omega} K\right) = -8\pi(1 + \ln(\pi)) - 8\pi F^K_1(a_\varepsilon) + o_\varepsilon(1)
\]

\[
\geq c_\infty + o_\varepsilon(1).
\]

Thus, using (35), we derive that \(\inf J_{8\pi} \geq c_\infty\) which contradicts (34). Hence our claim is proved. Finally, since \(u_\varepsilon\) is bounded, it will converge to a solution of \((P_{8\pi})\). The proof is thereby completed.
Proof of Theorem 1.2. Recall that \( J_{8\pi} \) is a lower bounded functional. Furthermore, by the assumption of the theorem, \( \inf J_{8\pi} \) corresponds to a level of some critical points at infinity. Moreover, assuming that \( J_{8\pi} \) does not have any critical point, then, using [5] and Corollary 2 we know that \( H^1_0(\Omega) \) retracts by deformation onto \( \bigcup_{q \in K^-} W_u(q_{\infty}) \).

Since \( H^1_0(\Omega) \) is a contractible set, we derive that its Euler-Poincaré characteristic is equal to 1. Furthermore, the contribution of a critical point at infinity \( q_{\infty} \) is equal to \( (−1)^{\text{morse}(J_{8\pi}, q_{\infty})} \).

Hence, we get
\[
1 = \sum_{q \in K^-} (−1)^{2−\text{morse}(F^K_1, q)} + \sum_{\{w : \nabla J_{8\pi}(w) = 0\}} (−1)\text{morse}(J_{8\pi}, w),
\]
which is a contradiction with the assumption of the theorem. Therefore, we derive the existence of at least one solution to \((P_{8\pi})\).

Proof of Theorem 1.3. In this proof, the functional \( J_{8\pi} \) may have some critical points (which we can assume non-degenerate by perturbing \( K \) if necessary). In this case, \( H^1_0(\Omega) \) retracts by deformation onto (see [5])
\[
\bigcup_{q \in K^-} W_u(q_{\infty}) \cup \bigcup_{\{w : \nabla J_{8\pi}(w) = 0\}} W_u(w)
\]
and therefore, by using the Euler-Poincaré characteristic (see [31]), we get
\[
1 = \sum_{q \in K^-} (−1)^{\text{morse}(J_{8\pi}, q_{\infty})} + \sum_{w : \nabla J_{8\pi}(w) = 0} (−1)^{\text{morse}(J_{8\pi}, w)},
\]
which implies that
\[
\#\{w : \nabla J_{8\pi}(w) = 0\} \geq \sum_{w : \nabla J_{8\pi}(w) = 0} (−1)^{\text{morse}(J_{8\pi}, w)}
\]
\[
= 1 - \sum_{q \in K^-} (−1)^{\text{morse}(J_{8\pi}, q_{\infty})}.
\]
Hence the result follows.

6.2. Results for the supercritical case. We now consider the case \( \rho = 8\pi m \) with \( m \geq 2 \). The proof of Theorem 1.4 makes use of the following claim

Lemma 6.1. For \( L \) very large, the Euler Characteristic of the level sets \( J^{-L}_{\varphi} \) is given by:
\[
\chi(J^{-L}_{\varphi}) = 1 - \frac{1}{(m-1)!} \prod_{\ell = 0}^{m-2} (\ell + g).
\]

Proof. Recall that for \( \varepsilon \) small, \( J_{8\pi m - \varepsilon} \) does not have any critical point neither critical point at Infinity under the level \( -L \) or above the level \( L \). This follows for \( \varepsilon \neq 0 \) from the compactness result of Li-Shafrir [29] and for \( \varepsilon = 0 \) from Corollary 2 and Corollary 3. It follows then from the main result in [31] that the level sets \( J^{-L}_{\varphi} \).
and $J_{8\pi m-\epsilon}^{-L}$ are homotopically equivalent and the level set $J_{8\pi m}^L$ is contractible. Therefore using the additivity of the Euler-Characteristic we derive that

$$\chi(J_{8\pi m-\epsilon}^L) = \chi(J_{8\pi m-\epsilon}^L, J_{8\pi m-\epsilon}^{-L}) + \chi(J_{8\pi m-\epsilon}^{-L}).$$

Now using again the compactness result of Li-Shafrir [29] and Theorem 1.2 in [18](see also [32]) we derive that

$$\chi(J_{8\pi m-\epsilon}^L, J_{8\pi m-\epsilon}^{-L}) = \text{deg}_{8\pi m-\epsilon} = 1 - \frac{1}{(m-1)!} \prod_{\ell=0}^{m-2} (\ell + g).$$

Hence the claim follows. \hfill \Box

We have now all the tools for proving our main theorem 1.4:

Proof of Theorem 1.4. Recall that we are considering the case $\varrho = 8\pi m; m \geq 2$. Arguing by contradiction, we assume that $(P_{8\pi m})$ does not have solution. Thus, $H^1_0(\Omega)$ retracts by deformation onto

$$\bigcup_{w_\infty} W_u(w_\infty) \cup J_{\varrho}^-$$

for a large positive constant $L$, where $W_u(w_\infty)$ denotes the unstable manifold of the critical point at Infinity $w_\infty$. Now it follows from the above claim that the Euler Characteristic of very negative level sets is given by

$$\chi(J_{\varrho}^-) = 1 - \frac{1}{(m-1)!} \prod_{\ell=0}^{m-2} (\ell + g).$$

On the other hand, it follows from Corollary 2 that the critical points at Infinity are in a one to one correspondence to the set of blow up points $K^-$. Moreover if $q = (q_1, \ldots, q_m) \in K^-$ corresponds to the critical point at Infinity $w_\infty$ then the index $\iota(q)$ coincides with the Morse index of the critical point at Infinity $w_\infty$.

Now computing the Euler Characteristic of some sublevel $J^L$, for $L$ very large we derive that

$$1 = \chi(J^L_{\varrho}) = \sum_{q \in K^-} (-1)^{\iota(q)} + 1 - \frac{1}{(m-1)!} \prod_{\ell=0}^{m-2} (\ell + g).$$

Hence we obtain

$$\sum_{q \in K^-} (-1)^{\iota(q)} = \frac{1}{(m-1)!} \prod_{\ell=0}^{m-2} (\ell + g),$$

which contradicts the assumption of our theorem. \hfill \Box

Appendix A. Some estimates. In this appendix, the concentration points are assumed to be in a compact set of $\Omega$ and the concentration speeds are of the same order and large enough. Furthermore, for sake of simplicity, $O(1/\lambda^\alpha)$ designs some quantities like $O(\sum 1/\lambda^\alpha)$.

Lemma A.1. (i) On $\Omega$, there holds

$$P\delta_{a,\lambda} = \ln \left( \frac{\lambda^4}{1 + \lambda^2|x-a|^2} \right) - 8\pi H(x, a) + O\left( \frac{1}{\lambda^2} \right),$$

$$\lambda \frac{\partial P\delta_{a,\lambda}}{\partial \lambda} = \frac{4}{1 + \lambda^2|x-a|^2} + O\left( \frac{1}{\lambda^2} \right).$$
\[
\frac{1}{\lambda} \frac{\partial P\delta_{a,\lambda}}{\partial a} = \frac{4\lambda(x-a)}{1 + \lambda^2|x-a|^2} - \frac{8\pi}{\lambda} \frac{\partial H(a, x)}{\partial a} + O\left(\frac{1}{\lambda^3}\right).
\]

(ii) Let \(0 < \eta < d(a, \partial \Omega)\) such that \(\lambda \eta\) is very large. On \(\Omega \setminus B(a, \eta)\) there holds

\[
P\delta_{a,\lambda} = 8\pi G(a, x) + O\left(\frac{1}{\lambda^2 \eta^2}\right),
\]

\[
\lambda \frac{\partial P\delta_{a,\lambda}}{\partial \lambda} = O\left(\frac{1}{\lambda^2 \eta^2}\right),
\]

\[
\frac{1}{\lambda} \frac{\partial P\delta_{a,\lambda}}{\partial a} = \frac{8\pi}{\lambda} \frac{\partial G(a, x)}{\partial a} + O\left(\frac{1}{\lambda^3 \eta}\right).
\]

Proof. Part (i) follows from the maximum principle, and (ii) follows from (i). \(\square\)

Lemma A.2.

\[
\left\langle P\delta_{a,\lambda}, \lambda \frac{\partial P\delta_{a,\lambda}}{\partial \lambda} \right\rangle = 16\pi + O\left(\frac{1}{\lambda^2}\right),
\]

\[
\left\langle P\delta_{a,\lambda}, \frac{1}{\lambda} \frac{\partial P\delta_{a,\lambda}}{\partial a} \right\rangle = -64\pi^2 \frac{1}{\lambda} \frac{\partial H(a, a)}{\partial a} + O\left(\frac{\ln \lambda}{\lambda^3}\right),
\]

where \(\partial H/\partial a\) denotes the derivative with respect to the first variable.

For \(i \neq j\) and \(|a_i - a_j| \geq \epsilon > 0\), there hold

\[
\left\langle P\delta_{a_i,\lambda_i}, \frac{1}{\lambda_i} \frac{\partial P\delta_{a_i,\lambda_i}}{\partial a_i} \right\rangle = O\left(\frac{1}{\lambda^2}\right),
\]

\[
\left\langle P\delta_{a_i,\lambda_i}, \lambda_i \frac{\partial P\delta_{a_i,\lambda_i}}{\partial \lambda_i} \right\rangle = 64\pi^2 \frac{1}{\lambda_i} \frac{\partial G(a_i, a_j)}{\partial a_i} + O\left(\frac{\ln \lambda}{\lambda^3}\right).
\]

Proof. Using Lemma A.1, the estimate (37) is obtained as follows

\[
\left\langle P\delta_{a,\lambda}, \lambda \frac{\partial P\delta_{a,\lambda}}{\partial \lambda} \right\rangle = \int_{\Omega} \frac{8\lambda^2}{(1 + \lambda^2|x-a|^2)^2} \left( \frac{4\lambda(x-a)}{1 + \lambda^2|x-a|^2} + O\left(\frac{1}{\lambda^2}\right) \right) = 16\pi + O\left(\frac{1}{\lambda^2}\right).
\]

For the estimate (38) we have (choose \(d > 0\) such that \(B(a, d) \subset \Omega\))

\[
\left\langle P\delta_{a,\lambda}, \frac{1}{\lambda} \frac{\partial P\delta_{a,\lambda}}{\partial a} \right\rangle = \int_{B(a, d)} \frac{8\lambda^2}{(1 + \lambda^2|x-a|^2)^2} \left( 4\lambda(x-a) \right) \left( \frac{1}{1 + \lambda^2|x-a|^2} \right) - O\left(\frac{1}{\lambda^2}\right) \approx \int_{B(a, d)} \frac{8\lambda^2}{(1 + \lambda^2|x-a|^2)^2} \left( 4\lambda(x-a) \right) \left( \frac{1}{1 + \lambda^2|x-a|^2} \right) + O\left(\frac{1}{\lambda^2}\right).
\]

In \(\Omega \setminus B(a, d)\), the function \(\partial H(a, x)/\partial a\) is bounded thus we get that the integral is of the order of \(O\left(\frac{1}{\lambda^2}\right)\). In \(B(a, d)\), the first term is zero and for the second, we expand the function \(\partial H(a, x)/\partial a\) around \(a\):

\[
\frac{\partial H(a, x)}{\partial a} = \frac{\partial H(a, a)}{\partial a} + \nabla \frac{\partial H(a, x)}{\partial a} (x-a) + O\left(|x-a|^2\right).
\]

Concerning (39) and (40), choosing \(d\) such that \(B(a_i, d) \cap B(a_j, d) = \emptyset\), we have

\[
\left\langle P\delta_{a_i,\lambda_i}, \lambda_i \frac{\partial P\delta_{a_i,\lambda_i}}{\partial \lambda_i} \right\rangle = \int_{B(a_i, d)} e^{\delta_{a_i,\lambda_i}} O\left(\frac{1}{\lambda_i^2}\right) + \int_{\Omega \setminus B(a_i, d)} e^{\delta_{a_i,\lambda_i}} O(1)
\]
Now, we expand $\frac{\partial G}{\partial \alpha_i} (a_i, x)$ as above for $\frac{\partial H}{\partial \alpha_i} (a_i, x)$. The result follows. \hfill \Box

Lemma A.3.

$$\int_{\Omega} |\nabla P_{\delta a_i, \lambda_i}|^2 dx = 32 \pi \ln \lambda - 16 \pi - 64 \pi^2 H(a_i, a) + O\left(\frac{1}{\lambda^2}\right).$$

For $i \neq j$ and $|\lambda_i - \lambda_j|$ large enough, there holds

$$\langle P_{\delta a_i, \lambda_i}, P_{\delta a_j, \lambda_j} \rangle = 64 \pi^2 G(a_i, a_j) + O\left(\frac{1}{\lambda^2}\right).$$

Proof. From Lemma \[A.1\] we have

$$\int_{\Omega} e^{\delta a_i, \lambda_i} P_{\delta a_i, \lambda_i} = \int_{\Omega} e^{\delta a_i, \lambda_i} \left(4 \ln \lambda - 2 \ln(1 + \lambda^2|x-a|^2) - 8 \pi H(a_i, x) + O\left(\frac{1}{\lambda^2}\right)\right)$$

$$= \int_{\mathbb{R}^2} e^{\delta a_i, \lambda_i} \left(4 \ln \lambda - 2 \ln(1 + \lambda^2|x-a|^2)\right) - \int_{\mathbb{R}^2} e^{\delta a_i, \lambda_i} \ln \left(\frac{\lambda^4}{(1 + \lambda^2|x-a|^2)^2}\right)$$

$$+ \int_{\Omega} e^{\delta a_i, \lambda_i} H(a_i, x) + O\left(\frac{1}{\lambda^2}\right)$$

$$= 32 \pi \ln \lambda - 16 \pi - 64 \pi^2 H(a, a) + O\left(\frac{1}{\lambda^2}\right).$$

Indeed, the first integral gives $32 \pi \ln \lambda - 16 \pi$. The second one is $O\left(\frac{1}{\lambda^2}\right)$ (since $d(a_i, \partial \Omega) \geq c > 0$). Concerning the last one, we have

$$\int_{\Omega \setminus B(a,d)} e^{\delta a_i, \lambda_i} H(a, x) = O\left(\frac{1}{\lambda^2}\right).$$

$$\int_{B(a,d)} e^{\delta a_i, \lambda_i} H(a, x)$$

$$= \int_{B(a,d)} e^{\delta a_i, \lambda_i} \left(H(a, x) + \frac{1}{2} D^2 H(a, a)(x-a, x-a) + O(|x-a|^3)\right)$$

(the term $\nabla H$ does not appear since its contribution is zero). The first one is equal to $8 \pi H(a, a) + O\left(\frac{1}{\lambda^2}\right)$. The last one is $O\left(\frac{1}{\lambda^2}\right)$. For the second, we derive that

$$\int_{B(a,d)} e^{\delta a_i, \lambda_i} D^2 H(a, a)(x-a, x-a)$$

$$= \sum_{i,j} \frac{\partial^2 H(a, a)}{\partial x_i \partial x_j} \int_{B(a,d)} \frac{8 \lambda^2(x-a)_i(x-a)_j}{(1 + \lambda^2|x-a|^2)^2}$$

$$= \sum_{i} \frac{\partial^2 H(a, a)}{\partial x_i^2} \int_{B(a,d)} \frac{8 \lambda^2(x-a)_i^2}{(1 + \lambda^2|x-a|^2)^2}$$

we notice that the integrals for $i = 1, 2$ have the same value, it follows that the integral vanishes since $\Delta H(a,.,|x=a|) = 0$. Hence the first claim is proved. The second estimate can be derived using similar arguments. \hfill \Box

Lemma A.4. There exists a constant $C_0 := C(\Omega) > 0$ such that:

$$\left(\int_{\Omega} e^{\delta a_i, \lambda_i} |w|^q \right)^{\frac{1}{q}} \leq C_0 \sqrt{\lambda} \|w\|, \quad \forall w \in E_{a_i, \lambda_i}^m, \quad \forall q \in [1, \infty),$$

(41)
\[
\int_{\Omega} e^{\delta_{u,\lambda}} h |e^w - 1 - w| \leq C_0 \left( \int_{\Omega} e^{\delta_{u,\lambda}} h^2 \right)^{1/2} \|w\|^2, \quad \forall w \in E_{a,\Lambda}, \quad \forall h \in L^\infty(\Omega),
\]

(42)

\[
\int_{\Omega} e^{\delta_{u,\lambda}} |e^w - 1 - w - \frac{w^2}{2}| \leq C_0 \|w\|^3, \quad \forall w \in E_{a,\Lambda}.
\]

(43)

**Proof.** We first observe that from the Sobolev embedding \( H^1(\mathbb{S}^2) \to L^2(\mathbb{S}^2) \), there exists a constant \( C_q \) such that

\[
\|w\|_{L^q} \leq C_q \{ \|\nabla w\|_{L^2} + \|w\|_{L^2} \}. \tag{44}
\]

It is known that \( C_q \leq \sqrt{q} C_0 \) where \( C_0 \) depends only on the domain (see for instance [35, Lemma 1.5, p.192]). Using then a stereographic projection we see that inequality (44) is equivalent to

\[
\left( \int_{\mathbb{R}^2} e^{\delta_{0,1}} |w|^q \right)^{\frac{1}{q}} \leq C_q \left\{ \left( \int_{\mathbb{R}^2} |\nabla w|^2 \right)^{\frac{1}{2}} + \left( \int_{\mathbb{R}^2} e^{\delta_{0,1}} |w|^2 \right)^{\frac{1}{2}} \right\}. \tag{45}
\]

Now by scaling \( x \mapsto \lambda(x-a) \), we derive that

\[
\left( \int_{\mathbb{R}^2} e^{\delta_{u,\lambda}} |w|^q \right)^{\frac{1}{q}} \leq C_q \left\{ \left( \int_{\mathbb{R}^2} |\nabla w|^2 \right)^{\frac{1}{2}} + \left( \int_{\mathbb{R}^2} e^{\delta_{u,\lambda}} |w|^2 \right)^{\frac{1}{2}} \right\}.
\]

Hence for any \( w \in H^1_0(\Omega) \) with \( \int_{\Omega} e^{\delta_{u,\lambda}} w = 0 \) we derive from the Poincaré Inequality

\[
\left( \int_{\Omega} e^{\delta_{u,\lambda}} |w|^q \right)^{\frac{1}{q}} \leq C_q \left( \int_{\Omega} |\nabla w|^2 \right)^{\frac{1}{2}}.
\]

Hence the proof of (41) follows.

To prove (42), we argue as follows

\[
\int_{\Omega} e^{\delta_{u,\lambda}} |e^w - 1 - w|h \leq \int_{\Omega} e^{\delta_{u,\lambda}} h \sum_{k=2}^{+\infty} \frac{|w|^k}{k!} \leq \sum_{k=2}^{+\infty} \frac{1}{k!} \int_{\Omega} e^{\delta_{u,\lambda}} h |w|^k
\]

\[
\leq \sum_{k=2}^{+\infty} \frac{1}{k!} \left( \int_{\Omega} e^{\delta_{u,\lambda}} h^2 \right)^{1/2} \left( \int_{\Omega} e^{\delta_{u,\lambda}} |w|^{2k} \right)^{1/2}
\]

\[
\leq \left( \int_{\Omega} e^{\delta_{u,\lambda}} h^2 \right)^{1/2} \|w\|^2 \sum_{k=2}^{+\infty} \frac{1}{k!} (C_0 \sqrt{2k})^k \|w\|^k
\]

\[
\leq \left( \int_{\Omega} e^{\delta_{u,\lambda}} h^2 \right)^{1/2} \|w\|^2 \sum_{j=0}^{+\infty} \frac{(C_1 \sqrt{j+2})^{j+2}}{(j+2)!} \|w\|^j.
\]

Now observe that the series \( \sum_{j=0}^{+\infty} \frac{(C_1 \sqrt{j+2})^{j+2}}{(j+2)!} z^j \) is convergent with an infinite radius of convergence. Since \( \|w\| \leq \varepsilon \), the claimed estimate follows. The proof of (43) is similar. \( \square \)

**Lemma A.5.** For each \( \beta \in [0,2] \) there exists a constant \( C_\beta \) such that:

\[
\int_{\Omega} e^{\delta_{u,\lambda}} |x-a|^{\beta} |w| \leq \begin{cases} C_\beta \frac{\|w\|}{\lambda^\beta} & \text{if } \beta \in [0,2), \\ C_2 \frac{\|\ln \lambda\|^{\frac{3}{2}} \|w\|}{\lambda^\beta} & \text{if } \beta = 2, \end{cases} \quad \forall w \in E_{a,\Lambda}.
\]

(44)
Lemma A.6. Let $\tilde{\xi}$ be any function such that $\Omega$. In general, we have

$$\int_{\Omega} e^{\delta_u} \, |x-a|^2 \, \|\nu\|^2 \leq c \|\nu\|^2, \quad \forall \nu \in E^{m}_{a,\Lambda}. \quad (46)$$

If $\beta \in [0, 2]$, by choosing $\gamma := (2 + \beta)/2 \in (\beta, 2)$, then H"older Inequality and Lemma A.4 yield:

$$\int_{\Omega} e^{\delta_u} |x-a|^\beta \, |w| \leq \left( \int_{\Omega} e^{\delta_u} |w|^{2/\gamma} \right)^{2-\beta/\gamma} \left( \int_{\Omega} e^{\delta_u} |x-a|^\gamma \right)^{\beta/\gamma} \leq C \|w\| \lambda^\beta,$$

The proof of the first claim follows. If $\beta = 2$, then

$$\int_{\Omega} e^{\delta_u} |x-a|^2 \, |w| \leq \left( \int_{\Omega} e^{2\delta_u} |x-a|^6 \, \|\nu\|^2 \right)^{1/2} \left( \int_{\Omega} |x-a|^2 \, \|\nu\|^2 \right)^{1/2}.$$

Now observe that

$$\int_{\Omega} e^{2\delta_u} |x-a|^6 \, \|\nu\|^2 = O\left( \frac{|\ln \lambda|^3}{\lambda^4} \right),$$

and using the two dimensional Hardy Inequality, we derive

$$\int_{\Omega} |x-a|^2 \, \|\nu\|^2 \leq c \|\nu\|^2.$$

Hence the claim follows.

Using the expansion of $P_{a,\Lambda}$ given in Lemma A.1, we obtain

**Lemma A.6.** Let $\tilde{\xi} := \sum_{i=1}^m \alpha_i P_{a,\Lambda}$. Then, on $B(a, \eta)$, there holds

$$K e^{\tilde{\xi}} = \frac{\lambda^{4\alpha} F_{a}(x) g(x)}{(1 + \lambda^2 |x-a|^2)^{2\alpha}} \left( 1 + O\left( \frac{1}{\lambda^2} \right) \right) = \frac{\lambda^{4\alpha} F_{a}(x) g(x)}{(1 + \lambda^2 |x-a|^2)^{2\alpha}} + O\left( \frac{|\tilde{\xi}|^2}{\lambda^2} \right),$$

where the functions $F_{a}$ (with $a = (a_1, \cdots, a_m)$) and $g_i$ are defined as follows

$$F_{a}(x) := K(x) \exp \left( -8\pi H(a, x) + 8\pi \sum_{j \neq i} G(a, j) \right),$$

$$g_{i}(x) := \exp \left( -8\pi (\alpha_i - 1) H(a, x) + 8\pi \sum_{j \neq i} (\alpha_j - 1) G(a, j) \right). \quad (47)$$

We introduce the following function

$$\xi(x) = \frac{1}{(1 + \lambda^2 |x-a|^2)^{2(\alpha-1)}}. \quad (48)$$

We remark that if $(\alpha - 1) \ln \lambda$ is small, it is easy to obtain that $\xi = 1 + O(1)$ uniformly on $\Omega$. In general, we have

$$|\xi(x) - 1| = |\xi(x) - \xi(a)| = \left| \int_0^1 \frac{4(1-\alpha)\lambda^2 |x-a|^2}{(1 + \lambda^2 |x-a|^2)^{2\alpha-1}} dt \right| \leq c |\alpha - 1| \sqrt{\lambda |x-a|}. \quad (49)$$

To state our next results, we will write $\xi_i(x) := \frac{1}{(1 + \lambda^2 |x-a_i|^2)^{2(\alpha_i-1)}}$. 
Lemma A.7. Let \( u := \sum_{i=1}^{m} \alpha_i P \delta_i + w := \tilde{u} + w \in V(m, \varepsilon) \) with \( w \in E_{a, \Lambda}^m \). Then,
\[
\int_{B_i} e^{\delta_i, \lambda_i} w \xi_i = O\left(\|w\| (|\alpha_i - 1| + \frac{1}{\lambda_i^2})\right),
\]
\[
\int_{\Omega} K e^w = O\left(\|w\| \sum_{k=1}^{m} \lambda_k^{4\alpha_k - 2} \left(|\alpha_k - 1| + \frac{|\nabla F_k(a_k)|}{\lambda_k} + \frac{|\ln \lambda_k^2|}{\lambda_k^2}\right)\right) + O(\|w\|).
\]

Proof. Using (49), we derive
\[
\int_{B_i} e^{\delta_i} w \xi_i = \int_{B_i} e^{\delta_i} w + O\left(|\alpha_i - 1| \int_{B_i} e^{\delta_i} |w| \sqrt{\lambda_i |x - a_i|}\right).
\]
The first claim follows from Lemma A.5 and the fact that \( \langle w, P \delta \rangle = 0 \) which implies
\[
\int_{B_i} e^{\delta_i} w = -\int_{\Omega \setminus B_i} e^{\delta_i} w = O \left(\frac{1}{\lambda_i^2} \int_{B_i} |w|\right) = O \left(\frac{\|w\|}{\lambda_i^2}\right).
\]
Concerning the second one, using Lemma A.6 we derive
\[
\int_{\Omega} K e^w = \sum_{i=1}^{m} \lambda_i^{4\alpha_i - 2} \int_{B_i} e^{\delta_i} \xi_i F_k^a g_i \left(1 + O \left(\frac{1}{\lambda_i^2}\right)\right) w + O(\|w\|). \tag{50}
\]
Using (49) and expanding the function \( F_k^a g_i \) around \( a_i \), we get
\[
\int_{B_i} e^{\delta_i} \xi_i F_k^a g_i w = \int_{B_i} e^{\delta_i} w + O \left(|\nabla (F_k^a g_i)(a_i)| \int_{B_i} e^{\delta_i} |x - a_i| |w| \right)
+ O \left(\int_{B_i} e^{\delta_i} |x - a_i|^2 |w| + |\alpha_i - 1| \int_{B_i} e^{\delta_i} \sqrt{\lambda_i |x - a_i|} |w|\right).
\]
The claim follows from (50) and Lemma A.5. \(\square\)

Lemma A.8. Let \( u := \sum_{i=1}^{m} \alpha_i P \delta_i + w := \tilde{u} + w \in V(m, \varepsilon) \) with \( w \in E_{a, \Lambda}^m \). Then,
\[
\int_{\Omega} K e^w = \pi \sum_{i=1}^{m} \lambda_i^{4\alpha_i - 2} \left(F_k^a g_i(a_i) \right)
+ O \left(1 + \sum_{k=1}^{m} \lambda_k^{4\alpha_k - 2} \left(|\alpha_k - 1|^2 + \frac{|\ln \lambda_k|}{\lambda_k^2} + \|w\|^2\right)\right). \tag{51}
\]
If \( \sum_{i=1}^{m} |\alpha_i - 1| \ln \lambda_i = o_4(1) \), the expansion (51) can be improved as follows
\[
\int_{\Omega} K e^w = \pi \sum_{i=1}^{m} \left\{\lambda_i^{4\alpha_i - 2} \left(F_k^a g_i(a_i) + \frac{1}{2} \Delta(F_k^a g_i)(a_i) \ln \lambda_i\right)\right\}
+ O \left(1 + \sum_{k=1}^{m} \lambda_k^2 \left(|\alpha_k - 1|^2 + \|w\|^2\right)\right). \tag{52}
\]

Proof. We split the integral as follows
\[
\int_{\Omega} K e^w = \int_{\Omega} K e^w + \int_{\Omega} K e^w w + \int_{\Omega} K e^w (e^w - 1 - w).
\]
Estimate of \( I_1 \): Applying Lemma A.6 we get
\[
\int_{\Omega} K e^w (e^w - 1 - w) \leq \int_{\Omega} P e^{\delta_i} w \leq c \sum_{i=1}^{m} \lambda_i^{4\alpha_i - 2} \int_{B_i} e^{\delta_i} \xi_i |w - 1 - w| + O \left(\int_{\Omega} |w - 1 - w|\right).
\]
Arguing as in (42), we have
\[ \int_{\Omega} e^{\delta_i \xi_i} |e^w - 1 - w| \leq c \left( \int_{\Omega} e^{\delta_i \xi_i} \right)^{1/2} \|w\|^2 \leq c \|w\|^2. \]

Arguing as for the estimate (42) we obtain
\[ \int_{\Omega} |e^w - 1 - w| \leq c \|w\|^2. \]

Thus, we derive that
\[ |I_1| \leq c \sum_{i=1}^{m} \lambda_i^{4\alpha_i - 2} \|w\|^2. \] (53)

**Estimate of I_2:** This integral has been computed in Lemma A.7 and we get
\[ |I_2| = O\left( \sum_{k=1}^{m} \lambda_k^{4\alpha_k - 2} \left( |\alpha_k - 1|^2 + \frac{1}{\lambda_k^2} + \frac{\ln \lambda_k^3}{\lambda_k^2} + \|w\|^2 \right) \right) + O(\|w\|). \]

**Estimate of I_3:** Firstly, note that Lemma A.1 shows that \( e^u \) is bounded on the complement of the union of the \( B_i \)'s. Hence the integral in this set is bounded. Now, using Lemma A.6 we have
\[ \int_{B_i} K e^\pi = \int_{B_i} \lambda_i^{4\alpha_i}(F_i^a)(x) g_i(x) + O\left( \frac{1}{\lambda_i^2} \int_{B_i} K e^\pi \right). \]

Finally, we have
\[ \int_{B_i} \lambda_i^{4\alpha_i}(F_i^a)(x) g_i(x) = \lambda_i^{4\alpha_i - 2}(F_i^a)(a_i) \int_{B_i} \lambda_i^2 \left( \frac{1 + \lambda_i^2 |x - a_i|^2}{\lambda_i^2 |x - a_i|^2} \right)^{2\alpha_i} \]
\[ + \lambda_i^{4\alpha_i - 2} O\left( \int_{B_i} \lambda_i^2 |x - a_i|^2 \left( \frac{1 + \lambda_i^2 |x - a_i|^2}{\lambda_i^2 |x - a_i|^2} \right)^{2\alpha_i} \right). \]

Easy computations imply
\[ \int_{B_i} \lambda_i^2 \left( \frac{1 + \lambda_i^2 |x - a_i|^2}{\lambda_i^2 |x - a_i|^2} \right)^{2\alpha_i} = \frac{\pi}{2\alpha_i - 1} + O\left( \frac{1}{\lambda_i^{4\alpha_i - 2}} \right), \]
and (by using (49))
\[ \int_{B_i} \lambda_i^2 |x - a_i|^2 \left( \frac{1 + \lambda_i^2 |x - a_i|^2}{\lambda_i^2 |x - a_i|^2} \right)^{2\alpha_i} = \int_{B_i} \lambda_i^2 |x - a_i|^2 \left( \frac{1 + \lambda_i^2 |x - a_i|^2}{\lambda_i^2 |x - a_i|^2} \right)^{2\alpha_i} \xi_i \]
\[ = \int_{B_i} \lambda_i^2 |x - a_i|^2 \left( 1 + \lambda_i^2 |x - a_i|^2 \right) + \int_{B_i} \lambda_i^2 |x - a_i|^2 \left( \xi_i - 1 \right) \]
\[ = O\left( \frac{\ln \lambda_i}{\lambda_i^2} \right) + O\left( \frac{|\alpha_i - 1|}{\lambda_i^{3/2}} \right). \]

The proof of (51) follows by adding the above estimates for \( I_1 + I_2 + I_3 \).

To prove (52), we need to be more precise in the last expansion. Hence, we consider
\[ (F_i^a g_i)(x) \]
\[ = (F_i^a g_i)(a_i) + \nabla (F_i^a g_i)(a_i)(x - a_i) + \frac{1}{2} D^2(F_i^a g_i)(a_i)(x - a_i, x - a_i) \]
\[ + O(|x - a_i|^3). \]
Note that the second term in this expansion is odd in $B_i$, and therefore it does not contribute in the integral we want to estimate. Furthermore, since $|\alpha_i - 1| \ln \lambda_i = o(1)$ for each $i$, we get

$$
\int_{B_i} \frac{\lambda_i^{4\alpha_i}(x - a_i)^3}{(1 + \lambda_i^2|x - a_i|^2)^{2\alpha_i}} = \int_{B_i} \frac{\lambda^{4\alpha_i}(x - a_i)^3}{(1 + \lambda^2|x - a_i|^2)^{2\alpha_i}} = \frac{1}{2} \int_{B_i} \frac{\lambda^{4\alpha_i}|x - a_i|^2}{(1 + \lambda^2|x - a_i|^2)^{2\alpha_i}}
$$

$$
= \pi \ln \lambda_i + O(1 + |\alpha_i - 1| \ln^2(\lambda_i)),
$$

$$
\int_{B_i} \frac{\lambda^{4\alpha_i}|x - a_i|^3}{(1 + \lambda^2|x - a_i|^2)^{2\alpha_i}} = O(1) \quad \text{and} \quad \int_{B_i} \frac{\lambda^{4\alpha_i}(x - a_i)(x - a_i)^2}{(1 + \lambda^2|x - a_i|^2)^{2\alpha_i}} = 0.
$$

Then we deduce

$$
\int_{B_i} \frac{\lambda_i^{4\alpha_i}F_i^n(x)g_i(x)}{(1 + \lambda_i^2|x - a_i|^2)^{2\alpha_i}}
$$

$$
= \frac{\pi \lambda_i^{4\alpha_i - 2}}{2\alpha_i - 1} F_i^n(a_i)g_i(a_i) + \frac{\pi}{2} \Delta(F_i^n g_i)(a_i) \ln \lambda_i + O(1 + |\alpha_i - 1| \ln^2 \lambda_i).
$$

This concludes the proof of (52). \( \square \)

**Lemma A.9.** Let $u = \sum_{i=1}^m \alpha_i P \delta_i + w \in V(m, \varepsilon)$ with $w \in E_{a, \Lambda}^m$. If $\sum_{i=1}^m |\alpha_i - 1| \ln \lambda_i = o(1)$, then

$$
\sum_{i=1}^m \tau_i = \frac{\pi}{2} \int_\Omega K^u \sum_{i=1}^m \Delta F_i^n(a_i) \ln \lambda_i + O\left(\sum_{k=1}^m \left( |\alpha_k - 1|^2 + \frac{1}{\lambda_k^2} \right) + \|w\|^2 \right).
$$

**Proof.** From the definition of $\tau_i$ (see (12)),

$$
\sum_{i=1}^m \tau_i = \frac{\pi}{2} \int_\Omega K^u \left( \int_\Omega K^u - \sum_{i=1}^m \frac{\pi}{2\alpha_i - 1} \lambda_i^{4\alpha_i - 2} F_i^n(a_i)g_i(a_i) \right).
$$

Using Lemma A.8 we derive the claimed estimate. \( \square \)

**Lemma A.10.** Let $u = \sum_{i=1}^m \alpha_i P \delta_i + w \in E_{a, \Lambda}^m$. Then,

$$
\sum_{i=1}^m \lambda_i^{4\alpha_i - 2} \int_\Omega e^{\delta_i w^2} |\xi_i - 1| = O\left(\sum_{k=1}^m |\alpha_k - 1| \|w\|^2 \right)
$$

**Proof.** Applying (49), the Cauchy-Schwarz inequality and (41) we get

$$
\int_\Omega e^{\delta_i w^2} |\xi_i - 1|
$$

$$
\leq C|\alpha_i - 1| \left( \int_\Omega e^{\delta_i w^4} \right)^{1/2} \left( \int_\Omega e^{\delta_i \lambda_i |x - a_i|^2} \right)^{1/2}
$$

\leq C|\alpha_i - 1| \|w\|^2.
$$

The conclusion follows from this inequality combined with Lemma A.8. \( \square \)

**Acknowledgments.** The authors thank Prof. A. Bahri for many discussions about critical point theory at Infinity. Ben Ayed worked on this project while he was visiting IHES in Paris. He is very grateful to this institution for support and good working conditions. The second and third authors worked mainly on this project during several visits at the Mathematical Institute at Giessen University. They take the opportunity to thank the department for its support and very nice hospitality.
REFERENCES

[1] T. Aubin, *Some Nonlinear Problems in Riemannian Geometry*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 1998.

[2] A. Bahri, *Critical Points at Infinity in Some Variational Problems*, Pitman Research Notes in Mathematics Series, 182, Longman Scientific & Technical, Harlow; copublished in the United States with John Wiley & Sons, Inc., New York, 1989.

[3] A. Bahri and J.-M. Coron, *On a nonlinear elliptic equation involving the critical Sobolev exponent: the effect of the topology of the domain* Comm. Pure Appl. Math., 41 (1988), 253–294.

[4] A. Bahri and J.-M. Coron, *The scalar-curvature problem on the standard three-dimensional sphere* J. Funct. Anal., 95 (1991), 106–172.

[5] A. Bahri and P. Rabinowitz, Periodic solutions of Hamiltonian systems of 3-body type, Ann. Inst. H. Poincaré Anal. Non Linéaire, 8 (1991), 561–649.

[6] A. Bahri, *An invariant for Yamabe-type flows with applications to scalar-curvature problems in high dimension*. A celebration of John F. Nash, Jr Duke Math. J., 81 (1996), 253–466.

[7] D. Bartolucci and C. S. Lin, *Existence and Uniqueness of mean field equation on multiply connected domains* Arch. Ration. Mech. Anal., 217 (2015), 525–570.

[8] D. Bartolucci and F. De Marchis, *Supercritical mean field equations on convex domains and the Onsager’s statistical description of two dimensional turbulence* Arch. Ration. Mech. Anal., 95 (1991), 106–172.

[9] M. Ben Ayed, Y. Chen, H. Chtioui and M. Hammami, *On the prescribed scalar curvature problem on 4-manifolds* Duke Math. J., 84 (1996), 633–677.

[10] M. Ben Ayed and M. Ould Ahmedou, *Existence and multiplicity results for a fourth order mean field equation* J. Funct. Anal., 258 (2010), 3165–3194.

[11] H. Brézis and F. Merle, *Uniform estimates and blow-up behavior for solutions of $-\Delta u = V(x)e^u$ in two dimensions* Comm. Partial Differential Equations, 16 (1991), 1223–1253.

[12] E. Caglioti, P.-L. Lions, C. Marchioro and M. Pulvirenti, *A special class of stationary flows for two dimensional Euler equations: a statistical mechanics description* Comm. Math. Phys., 143 (1992), 501–525.

[13] E. Caglioti, P.-L. Lions, C. Marchioro and M. Pulvirenti, *A special class of stationary flows for two dimensional Euler equations: a statistical mechanics description, Part II* Comm. Math. Phys., 174 (1995), 229–260.

[14] A. Chang and P. Yang, *Prescribing Gaussian curvature on $S^2$* Acta Math., 159 (1987), 215–259.

[15] A. Chang and P. Yang, *Conformal deformations of metrics on $S^2$*, J. Diff. Geom., 27 (1988), 259–296.

[16] A. Chang, C. C. Chen and C. S. Lin, *Extremal function of a mean field equation in two dimension*, in *Lectures on Partial Differential Equations*, New Stud. Adv. Math, 2 (2003), 61–93.

[17] C. C. Chen and C. S. Lin, *Sharp estimates for solutions of multi-bubbles in compact Riemann surfaces* Comm. Pure Appl. Math., 55 (2002), 728–771.

[18] C. C. Chen and C. S. Lin, *Topological degree for a mean field equation on Riemann surfaces* Comm. Pure Appl. Math., 56 (2003), 1667–1727.

[19] F. De Marchis, *Multiplicity result for a scalar field equation on compact surfaces* Comm. PDE, 33 (2008), 2208–2224.

[20] F. De Marchis, *Generic multiplicity for a scalar field equation on compact surfaces* J. Funct. Anal., 259 (2010), 2165–2192.

[21] F. De Marchis, *Multiplicity of solutions for a mean field equation on compact surfaces*, Boll. Unione Mat. Ital., 4 (2011), 245–257.

[22] W. Ding, J. Jost, J. Li and G. Wang, *Existence results for mean field equations* Ann. Inst. Henri Poincaré Anal. Non Linéaire, 16 (1999), 653–666.

[23] Z. Djadli, *Existence result for the mean field problem on Riemann surfaces of all genus* Commun. Contemp. Math., 10 (2008), 205–220.

[24] Z. Djadli and A. Malchiodi, *Existence of conformal metrics with constant $Q$-curvature* Ann. of Math., 168 (2008), 813–858.

[25] P. Esposito, M. Grossi and A. Pistoia, *On the existence of blowing up solutions for a mean field equation* Ann. Inst. H. Poincaré Anal. Non Linéaire, 22 (2005), 227–257.

[26] Z. C. Han, *Prescribing Gaussian curvature on $S^2$* Duke Math. J., 61 (1990), 679–703.
[27] Z. C. Han, Asymptotic approach to singular solutions for nonlinear elliptic equations involving critical Sobolev exponent, *Ann. Inst. Poincaré Anal. non linéaire*, 8 (1991), 159–174.

[28] M. K. H. Kiessling, Statistical mechanics of classical particles with logarithmic interactions, *Comm. Pure Appl. Math.*, 46 (1993), 27–56.

[29] Y. Y. Li and I. Shafrir, Blow-up analysis for solutions of $-\Delta u = Ve^u$ in dimension two, *Indiana Univ. Math. J.*, 43 (1994), 1255–1270.

[30] Y. Y. Li, Harnack type inequality: The method of moving planes, *Comm. Math. Phys.*, 200 (1999), 421–444.

[31] M. Lucia, A deformation Lemma with an application to a mean field equation, *Topol. Methods Nonlinear Anal.*, 30 (2007), 113–138.

[32] A. Malchiodi, Morse theory and a scalar field equation on compact surfaces, *Adv. Differential Equations*, 13 (2008), 1109–1129.

[33] A. Malchiodi, Topological Methods for an elliptic equation with exponential nonlinearities, *Discrete Contin. Dyn. Syst.*, 21 (2008), 277–294.

[34] J. Milnor, *Lectures on the H-Cobordism Theorem*, Princeton University Press, Princeton, N.J., 1965.

[35] R. Schoen and S. T. Yau, *Lectures on Differential Geometry*, International Press, 1994.

[36] M. Struwe and G. Tarantello, On multivortex solutions in Chern-Simons gauge theory, *Boll. Unione. Math. Ital. Sez. B Artic. Ric. Mat.*, (8) 1 (1998), 109–121.

[37] G. Tarantello, *Selfdual Gauge Field Vortices: An Analytic Approach*, Progress in Nonlinear differential equations, 72, Birkhäuser Boston, Inc. Boston, MA, 2008.

[38] Y. Yang, *Solitons in Field Theory and Nonlinear Analysis*, Springer monographs in Mathematics, Springer Verlag, New York, Inc, 2001.

[39] L. Zhang, Blow up solutions of some nonlinear elliptic equation involving exponential nonlinearities, *Com. Math. Phys.*, 268 (2006), 105–133.

Received February 2016; revised November 2016.

E-mail address: Mohameden.Ahmedou@math.uni-giessen.de
E-mail address: Mohamed.Benayed@fss.rnu.tn
E-mail address: marcello.lucia@math.csi.cuny.edu