Boundedness of the base varieties of certain fibrations

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Abstract
It is conjectured that the base varieties of the Iitaka fibrations are bounded when the Iitaka volumes are bounded above. We confirm this conjecture for Iitaka $\varepsilon$-lc Fano type fibrations.

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1 | INTRODUCTION

Throughout this paper, we work with varieties defined over complex numbers.
By analogy with the definition of volumes of divisors, the Iitaka volume of a $\mathbb{Q}$-Cartier divisor is defined as follows.
**Definition 1.1** (Iitaka volume). Let $X$ be a normal projective variety and $D$ be a $\mathbb{Q}$-Cartier divisor. When the Iitaka dimension $\chi(D)$ of $D$ is nonnegative, then the Iitaka volume of $D$ is defined to be

$$\text{Ivol}(D) = \text{Ivol}(X, D) := \limsup_{m \to \infty} \frac{\chi(D)!h^0(X, \mathcal{O}_X([mD]))}{m^{\chi(D)}}.$$  \hspace{1cm} (1.1.1)

When $\chi(D) = -\infty$, then we put $\text{Ivol}(D) = 0$.

We study the following problem on the boundedness of base varieties of Iitaka fibrations with fixed (or bounded) Iitaka volumes. For relevant definitions and properties of singularities of pairs, boundedness, Iitaka dimensions/fibrations, DCC/ACC property of sets, and so on (see Section 2).

**Conjecture 1.2.** Let $d \in \mathbb{N}, v \in \mathbb{R}_{>0}$ be fixed numbers, and $I \subset [0,1] \cap \mathbb{Q}$ be a DCC set. Let $S(d, v, I)$ be the set of varieties $Z$ satisfying the following properties.

1. $(X, B)$ is klt with $\dim X = d$, and coefficients of $B$ are in $I$.
2. $\text{Ivol}(K_X + B) = v$.
3. $f : X \dashrightarrow Z$ is the Iitaka fibration associated with $K_X + B$, where

$$Z = \text{Proj} \bigoplus_{m=0}^{\infty} H^0(X, \mathcal{O}_X([m(K_X + B)])).$$

Then $S(d, v, I)$ is a bounded family.

When $K_X + B$ is big, that is, $\text{Ivol}(K_X + B) = \text{vol}(K_X + B) > 0$, Conjecture 1.2 is proved by [20, Theorem 1.1]. For variants of this conjecture, see Section 4.

Our main result is about the boundedness of the base varieties with further assumptions on singularities of $(X, B)$ and the Iitaka fibrations.

**Theorem 1.3.** Let $d \in \mathbb{N}, \varepsilon, v \in \mathbb{Q}_{>0}$ be fixed numbers, and $I \subset [0,1] \cap \mathbb{Q}$ be a DCC set. Let $F(d, \varepsilon, v, I)$ be the set of varieties $Z$ satisfying the following properties.

1. $(X, B)$ is $\varepsilon$-lc with $\dim X = d$, and coefficients of $B$ are in $I$.
2. $0 < \text{Ivol}(K_X + B) < v$.
3. $f : X \dashrightarrow Z$ is the Iitaka fibration associated with $K_X + B$, where

$$Z = \text{Proj} \bigoplus_{m=0}^{\infty} H^0(X, \mathcal{O}_X([m(K_X + B)])).$$

4. $B$ is big over the generic point $\eta_Z$ of $Z$.

Then $F(d, \varepsilon, v, I)$ is a bounded family.

Let $g : X' \to X$ be a birational morphism that resolves $f$ and let $\text{Exc}(g)$ be the sum of reduced exceptional divisors of $g$. If $A$ is a $\mathbb{Q}$-Cartier ample divisor over $\eta_Z$ such that $0 \leq A \leq B$, then there exists $t > 0$ such that $g^{-1}_* B + \text{Exc}(g) \geq tg^* A$ over $\eta_Z$. Thus, the assumption in (4) means that $g^{-1}_* B + \text{Exc}(g)$ is big over $\eta_Z$. 


Notice that we impose the strong assumption that $B$ is big over $\eta_Z$. This implies that a general fiber of $f$ is birationally bounded [22, Theorem 1.3; 9, Theorem 1.1]. However, it is desirable to obtain the boundedness of the base varieties regardless of the boundedness of fibers.

Jiang [23], Birkar [7], and Di Cerbo and Svaldi [12] studied similar fibrations (called $(d, r, \epsilon)$-Fano type fibrations, see Definition 2.3) where the boundedness of $Z$ is built in the definition. They show the boundedness of the total space under certain assumptions. Filipazzi [16] established the boundedness of varieties with fixed Iitaka volumes whose Iitaka fibrations are elliptic fibrations admitting multisecions of fixed degrees.

**Theorem 1.4** [7, Theorem 1.3]. Let $d, r$ be natural numbers and $\epsilon, \delta$ be positive real numbers. Consider the set of all $(d, r, \epsilon)$-Fano type fibrations $(X, B) \to Z$ and $\mathbb{R}$-divisors $0 \leq \Delta \leq B$ whose nonzero coefficients are $\geq \delta$. Then the set of such $(X, \Delta)$ is log bounded.

To compare Theorem 1.3 with Theorem 1.4, we note that Theorem 1.3 is about the boundedness of the base variety $Z$, while Theorem 1.4 is about the boundedness of the total space $(X, \Delta)$ assuming the boundedness of the base variety $Z$. Combining the two results, we have the following corollary.

**Corollary 1.5.** Let $d \in \mathbb{N}, \epsilon, \nu \in \mathbb{Q}_{>0}$ be fixed numbers, and $I \subset [0, 1] \cap \mathbb{Q}$ be a DCC set. Let $S(d, \epsilon, \nu, I)$ be the set of log pairs $(X, B)$ satisfying the following properties.

1. $(X, B)$ is $\epsilon$-lc with $\dim X = d$, and coefficients of $B$ are in $I$.
2. $K_X + B$ is semi-ample with $\text{Ivol}(K_X + B) < \nu$.
3. $f : X \to Z$ is the Iitaka fibration associated with $K_X + B$, where

$$Z = \text{Proj} \bigoplus_{m=0}^\infty H^0(X, \mathcal{O}_X(|m(K_X + B)|)).$$

4. $B$ is big over the generic point $\eta_Z$ of $Z$.

Then $S(d, \epsilon, \nu, I)$ is a log bounded family.

The distribution of Iitaka volumes is closely related to the above boundedness property. By analogy with [1, 19, 28], we propose the following conjecture.

**Conjecture 1.6.** Let $d \in \mathbb{N}$ be a fixed number, and $I \subset [0, 1] \cap \mathbb{Q}$ be a DCC set. Then the set of Iitaka volumes

$$\{\text{Ivol}(K_X + B) \mid (X, B) \text{ is klt, } \dim X = d, \text{ and coefficients of } B \text{ are in } I\}$$

is a DCC set.

When $K_X + B$ is big, this conjecture follows from [19, Theorem 1.3(1)] that was originally conjectured by Alexeev and Kollár. We show Conjecture 1.6 under suitable assumptions on the boundedness of the general fibers of the Iitaka fibrations.

**Corollary 1.7.** Let $d \in \mathbb{N}, \epsilon \in \mathbb{Q}_{>0}$ be fixed numbers, and $I \subset [0, 1] \cap \mathbb{Q}$ be a DCC set. Let $D(d, \epsilon, I)$ be the set of log pairs $(X, B)$ satisfying the following properties.
(1) \((X, B)\) is \(\varepsilon\)-lc with \(\dim X = d\), and coefficients of \(B\) are in \(I\).

(2) \(f : X \to Z\) is the Iitaka fibration associated with \(K_X + B\) (if it exists), where

\[
Z = \text{Proj} \bigoplus_{m=0}^{\infty} H^0(X, \mathcal{O}_X(\lfloor m(K_X + B) \rfloor)).
\]

(3) \(B\) is big over the generic point \(\eta_Z\) of \(Z\).

Then \(\{ \text{Ivol}(K_X + B) \mid (X, B) \in D(d, \varepsilon, I) \}\) is a DCC set.

We briefly explain the idea for the proof of Theorem 1.3. First, we reduce to the case where \(X \to Z\) is a morphism. Then we show that \(Z\) is birationally bounded. This means that \(Z\) is birational to a fiber of a projective morphism \(Z' \to T'\) between Noetherian schemes of finite type over \(\mathbb{C}\). Using the boundedness of a general fiber of \(f\), we show that \((X, B)\) is log birational to a fiber of a family of log pairs that is built upon \(Z' \to T'\). This is the most technical part of the argument. Finally, the boundedness of \(Z\) follows from the finiteness of ample models [4, Corollary 1.1.5].

Another possible approach is to apply the canonical bundle formula to \(f : (X, B) \to Z\), then there is a generalized polarized pair \((Z, \Delta_Z + M_Z)\) on \(Z\) such that \(K_X + B \sim_{\mathbb{Q}} f^*(K_Z + \Delta_Z + M_Z)\). One may expect to bound \(Z\) under the boundedness of the volumes of \(K_Z + \Delta_Z + M_Z\). Such result is only known for surfaces with fixed volumes and some extra conditions (see [15, Theorem 1.8]). However, it seems that even for \(\dim Z = 2\), Theorem 1.3 does not directly follow from [15, Theorem 1.8] (the existence of the universal \(r \in \mathbb{N}\) such that \(rM_Z\) is Cartier does not necessarily hold. On the other hand, this is known for a sufficiently high model of \(Z\)).

The paper is organized as follows. In Section 2, we introduce preliminary notation and results. Theorem 1.3 and its corollaries are proven in Section 3. In Section 4, we discuss possible variants of Conjecture 1.2, the relation to the effective adjunction conjecture, and establish some lower dimensional cases.

2  |  PRELIMINARIES

2.1  |  Notation and conventions

Let \(I \subset \mathbb{R}\) be a subset, then \(I\) is said to be a DCC set (resp., ACC set) if there is no strictly decreasing subsequence (resp., strictly increasing subsequence) in \(I\). Let \(B\) be an \(\mathbb{R}\)-divisor, then \(B \in I\) denotes the fact that the coefficients of \(B\) belong to \(I\). For a birational morphism \(f : Y \to X\) and a divisor \(B\) on \(X\), \(f^{-1}_*(B)\) denotes the strict transform of \(B\) on \(Y\), and \(\text{Exc}(f)\) denotes the sum of reduced exceptional divisors of \(f\). A fibration means a projective and surjective morphism with connected fibers. For an \(\mathbb{R}\)-divisor \(D\), we denote by \(|D|\) the linear system \(||D||\). For \(k = \mathbb{Z}, \mathbb{Q}, \mathbb{R}\), and two divisors \(A, B \in k\) on a variety \(X\) over \(Z\), \(A \sim_k B/Z\) means that \(A\) and \(B\) are \(k\)-linearly equivalent over \(Z\). When \(k = \mathbb{Z}\) or \(Z = \text{Spec} \mathbb{C}\), we omit \(k\) or \(Z\).

2.2  |  Singularities of pairs and boundedness

Let \(X\) be a normal projective variety and \(B\) be an \(\mathbb{R}\)-divisor on \(X\), then \((X, B)\) is called a log pair. We assume that \(K_X + B\) is an \(\mathbb{R}\)-Cartier divisor for a log pair \((X, B)\). For a divisor \(D\) over \(X\), if
f : Y → X is a birational morphism from a normal projective variety Y such that D is a divisor on Y, then the log discrepancy of D with respect to (X, B) is defined as mult_D(K_Y - f^*(K_X + B)) + 1. This definition is independent of the choice of Y. A log pair (X, B) is called sub-klt (resp., sub-lc) if the log discrepancy of any divisor over X is > 0 (resp., ≥ 0). If B ≥ 0, then a sub-klt (resp., sub-lc) pair (X, B) is called klt (resp., lc). For ε ∈ ℝ>0, a log pair (X, B) is called ε-lc if the log discrepancy of any divisor over X is > ε. For a log pair (X, B) over Z, see [4, Definition 3.6.5] for the definition of the ample model of K_X + B over Z. This is also called the ample model of (X, B) over Z. Let Z → T be a morphism and B be a subscheme of Z. If t ∈ T is a closed point, then Z_t and B_t denote the fiber of Z and B over t, respectively.

**Definition 2.1** [19, Bounded pairs, section 3.5]. We say that a set ℳ of varieties is birationally bounded if there is a projective morphism Z → T, where T is of finite type, such that for every X ∈ ℳ, there is a closed point t ∈ T and a birational map f : Z_t → X.

We say that a set ℳ of log pairs is log birationally bounded (resp., bounded) if there is a log pair (Z, B), where the coefficients of B are all one, and a projective morphism Z → T, where T is of finite type, such that for every (X, Δ) ∈ ℳ, there is a closed point t ∈ T and a birational map f : Z_t → X (resp., isomorphism of varieties) such that the support of B_t is not the whole of Z_t and yet B_t contains the support of the strict transform of Δ and any f-exceptional divisor (resp., f(B_t) = Supp Δ).

### 2.3 Iitaka fibrations and Iitaka volumes

For this part, we follow the exposition in [32, section 2.1]. Let X be a normal projective variety and D be a ℚ-Cartier divisor. Let

\[ \mathcal{N}(X, D) := \{ m \in \mathbb{N} \mid H^0(X, O_X(mD)) \neq 0 \text{ and } mD \text{ is Cartier} \}. \]

(In particular, \( \mathcal{N}(X, D) = \{0\} \) if for any \( m \in \mathbb{N}_{>0} \) such that \( mD \) Cartier, \( H^0(X, O_X(mD)) = 0. \) If \( \mathcal{N}(X, D) = \{0\} \), then one puts the Iitaka dimension of D to be \( -\infty \), and thus \( \text{Ivol}(D) = 0. \) Otherwise, for any \( m \in \mathcal{N}(X, D) \), the linear system \( |mD| \) defines a rational map \( \phi_{|mD|} : X \to \mathbb{P}H^0(X, O_X(mD)) \). Then the Iitaka dimension of D is defined to be

\[ \kappa(D) := \max_{m \in \mathcal{N}(X, D)} \{ \dim \phi_{|mD|}(X) \}. \]

In this case, there are constants \( a, A > 0 \) such that

\[ am^{\kappa(D)} \leq h^0(X, O_X(mD)) \leq Am^{\kappa(D)} \]

for all sufficiently large \( m \in \mathcal{N}(X, D) \) [32, Corollary 2.1.38]. Hence,

\[ \text{Ivol}(D) = \limsup_{m \to \infty} \frac{\kappa(D)h^0(X, O_X([mD]))}{m^{\kappa(D)}} \]

is a positive real number (cf. [32, Remark 2.1.39]).
Recall that the volume of $D$ is defined to be

$$\text{vol}(D) = \text{vol}(X, D) := \limsup_{m \to \infty} \frac{(\dim X)! h^0(X, \mathcal{O}_X([mD]))}{m^{\dim X}},$$

hence $D$ is big if and only if $\text{Ivol}(D) = \text{vol}(D) > 0$. A particular case is when $D = K_X + B$ is an adjoint divisor. If $K_X + B$ is semi-ample, then for sufficiently large $m \in \mathbb{N}(X, D)$, $\phi = \phi_{[m(K_X+B)]} : X \to Z$ is a morphism to the ample model of $(X, B)$. Suppose that $K_X + B = \phi^* D_Z$ for some $\mathbb{Q}$-divisor $D_Z$ on $Z$, then $\text{Ivol}(K_X + B) = \text{vol}(D_Z) = D_Z^{\dim Z}$ by definition.

When $(X, B)$ is a projective klt pair, by [4, Corollary 1.1.2],

$$\bigoplus_{m=0}^{\infty} H^0(X, \mathcal{O}_X([m(K_X+B)]))$$

is finitely generated. When $\kappa(K_X + B) \geq 0$, then the rational map

$$X \to \text{Proj} \bigoplus_{m=0}^{\infty} H^0(X, \mathcal{O}_X([m(K_X+B)]))$$

is called the Iitaka fibration associated with $K_X + B$. This is slightly different from [32, Definition 2.1.34], where an Iitaka fibration associated with a Cartier divisor is defined to be a morphism satisfying certain properties. However, the Iitaka fibration in [32, Definition 2.1.34] is unique only up to birational equivalence. Hence, in order to make sense of Conjecture 1.2, we call (2.3.1) the Iitaka fibration.

### 2.4 Fan type (log Calabi–Yau) fibrations

**Definition 2.2** ($\epsilon$-lc Fan type fibration). Let $\epsilon$ be a positive real number. An $\epsilon$-lc Fan type (log Calabi–Yau) fibration consists of a pair $(X, B)$ and a fibration $f : X \to Z$ between normal varieties such that we have the following.

1. $(X, B)$ is a projective $\epsilon$-lc pair.
2. $K_X + B \sim_{\mathbb{R}} 0/Z$.
3. $-K_X$ is big over $Z$, that is, $X$ is of Fan type over $Z$.

**Definition 2.3** ($(d, r, \epsilon)$-Fan type fibration). Let $d, r$ be natural numbers and $\epsilon$ be a positive real number. A $(d, r, \epsilon)$-Fan type (log Calabi–Yau) fibration consists of a pair $(X, B)$ and a contraction $f : X \to Z$ such that we have the following.

1. $(X, B)$ is a projective $\epsilon$-lc pair of dimension $d$.
2. $K_X + B \sim_{\mathbb{R}} f^* L$ for some $\mathbb{R}$-divisor $L$.
3. $-K_X$ is big over $Z$, that is, $X$ is of Fan type over $Z$.
4. $A$ is a very ample divisor on $Z$ with $A^{\dim Z} \leq r$.
5. $A - L$ is ample.

We emphasize that a very ample or base point free divisor is naturally a Cartier divisor.

**Remark 2.4.** A $(d, r, \epsilon)$-Fan type fibration $(X, B) \to Z$ can be viewed as a special case of the fibration map defined in Theorem 1.3 given $B \in I$. In fact, for a $(d, r, \epsilon)$-Fan type fibration $f : X \to Z$
as above, there exists a klt pair \((Z, \Delta)\) such that \(K_X + B \sim_Q f^*(K_Z + \Delta)\). Then \(A - (K_Z + \Delta)\) is ample, and \(A^{\dim Z} \leq r\). By length of extremal rays, \(K_Z + \Delta + 3(\dim Z)A\) is ample. By taking a general \(G \in |6(\dim Z)f^*A|\) and replacing \(\varepsilon\) by \(\min\{\varepsilon, \frac{1}{2}\}\), then \((X, B + \frac{1}{2}G)\) is \(\varepsilon\)-lc. Moreover, \(K_X + B + \frac{1}{2}G \sim_Q f^*(K_Z + \Delta + 3(\dim Z)A)\) with \(K_X + B + \frac{1}{2}G\) semi-ample. Hence, \(f : X \to Z\) is the Iitaka fibration associated with \(K_X + B + \frac{1}{2}G\). By definition, \(B\) is big over \(Z\). Moreover,

\[
\text{Ivol} \left( K_X + B + \frac{1}{2}G \right) = (K_Z + \Delta + 3(\dim Z)A)^{\dim Z}
\leq (3(\dim Z) + 1)^{\dim Z} A^{\dim Z} \leq (3(\dim Z) + 1)^{\dim Z} r.
\]

Thus, \((X, B + \frac{1}{2}G), Z\) and \(v = (3(\dim Z) + 1)^{\dim Z} r\) satisfy the assumptions of Theorem 1.3 (after possibly enlarging \(I\) to \(I \cup \{\frac{1}{2}\}\)).

### 2.5 Canonical bundle formula

We recall the construction of the canonical bundle formula in [26]. We follow the notions and notation in [14] that appear slightly different from [2]. For more about the canonical bundle formula, see [2, 3, 17, 26], and so on.

The notion of b-divisors is introduced by Shokurov. Let \(X\) be a normal variety. An integral b-divisor on \(X\) is an element:

\[
D \in \text{Div}X := \lim_{Y \to X} \text{Div}Y,
\]

where the projective limit is taken over all birational models \(f : Y \to X\) proper over \(X\), under the pushforward homomorphism \(f_* : \text{Div}Y \to \text{Div}X\). If \(D = \sum d_i \Gamma_i\) is a b-divisor on \(X\), and \(Y \to X\) is a birational model of \(X\), then the trace of \(D\) on \(Y\) is the divisor

\[
D_Y := \sum_{\Gamma_i \text{ is a divisor on } Y} d_i \Gamma_i.
\]

Let \(M\) be a Cartier divisor on \(X\), then \(\overline{M}\) is the b-divisor such that \((\overline{M})_Y = f^*M\) for any birational model \(f : Y \to X\). Divisors with coefficients in \(\mathbb{Q}\) or \(\mathbb{R}\) are defined similarly. For more details, see [1, section 2.3].

Suppose that \((X, B)\) is a log pair with \(B\) a \(\mathbb{Q}\)-divisor. The discrepancy b-divisor \(A = A(X, B)\) is the \(\mathbb{Q}\)-b-divisor of \(X\) with the trace \(A_Y\) defined by the formula \(K_Y = f_*(K_X + B) + A_Y\), where \(f : Y \to X\) is a proper birational morphism of normal varieties. Similarly, we define \(A^* = A^*(X, B)\) by

\[
A^*_Y = \sum_{a_i > -1} a_i A_i
\]

for \(K_Y = f^*(K_X + B) + \sum a_i A_i\), where \(f : Y \to X\) is a proper birational morphism of normal varieties.
Definition 2.5 (Klt-trivial and lc-trivial fibrations). A klt-trivial (resp., lc-trivial) fibration $f: (X, B) \to Z$ consists of a proper surjective morphism $f: X \to Z$ between normal varieties with connected fibers and a pair $(X, B)$ satisfying the following properties.

1. $(X, B)$ is sub-klt (resp., sub-lc) over the generic point of $Z$.
2. $\text{rank} f_* O_X([A(X, B)]) = 1$ (resp., $\text{rank} f_* O_X([A^*(X, B)]) = 1$).
3. There exists a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $D$ on $Z$ such that
   \[ K_X + B \sim_{\mathbb{Q}} f^* D. \]

Let $f: (X, B) \to Z$ be an lc-trivial fibration and $P$ be a prime divisor on $Z$. Because $Z$ is normal, after shrinking around $P$, we can assume that $P$ is Cartier. Define
\[ b_P := \max\{t \in \mathbb{R} \mid (X, B + tf^*P) \text{ is sub-lc over the generic point of } P\} \]
and set
\[ B_Z := \sum_P (1 - b_P)P, \quad M_Z := D - (K_Z + B_Z). \]

Then the following canonical bundle formula holds
\[ K_X + B \sim_{\mathbb{Q}} f^*(K_Z + B_Z + M_Z). \tag{2.5.1} \]

In this formula, $B_Z$ is called the divisorial part and $M_Z$ is called the moduli part. When $(X, B)$ is lc, there exist $\mathbb{Q}$-b-divisors $B$ and $M$ of $X$ such that $B_Z = B_Z$ and $M_Z = M_Z$. Moreover, $M$ is b-nef and b-abundant in the sense that there is a proper birational morphism $Z' \to Z$ and a proper surjective morphism $h: Z' \to W$ between normal varieties such that (1) $M_{Z'} \sim_{\mathbb{Q}} h^*H$ for some nef and big $\mathbb{Q}$-divisor $H$ on $W$, and (2) $M = \overline{M_{Z'}}$. For details, see [3, Theorem 3.3] and [14, Theorem 1.1].

The definitions of $B_Z$ and $M_Z$ still make sense when $B$ (in Definition 2.5) is an $\mathbb{R}$-divisor. In this case, we still have b-divisors $B$ and $M$. When $(X, B)$ is lc over the generic point of $Z$, $M_Z$ is pseudo-effective [8, Theorem 3.6]. For more results in this setting, see [8, section 3.4] and [7, section 6.1].

Next, we have the following effective adjunction conjecture (see [36, Conjecture 7.13.3]).

Conjecture 2.6 (Effective adjunction). Let $f: (X, B) \to Z$ be an lc-trivial fibration. There exists a positive integer $m$ depending only on the dimension of $X$ and the horizontal multiplicities of $B$ (a finite set of rational numbers) such that $m M$ is a base point free b-divisor (i.e., there is a birational morphism between projective varieties $h: Y' \to Y$ such that $m M_{Y'}$ is base point free and $M = \overline{M_{Y'}}$).

This conjecture is known when general fibers of $f$ are curves (see [36, Theorem 8.1], and references therein).
3 PROOF OF THEOREM 1.3 AND ITS COROLLARIES

For convenience, we introduce the following additional notation. Let $D, L$ be two $\mathbb{R}$-divisors, then we write $D \leq_{\mathbb{R}} L$ to indicate that there exists an effective $\mathbb{R}$-divisor $E$ such that $D + E \sim_{\mathbb{R}} L$. If $D, L$ are $\mathbb{Q}$-Cartier, then $D \leq_{\mathbb{R}} L$ implies that $\text{vol}(D) \leq \text{vol}(L)$. We use the notation $m = m(a_1, \ldots, a_k)$ to emphasize that the natural number $m$ depends only on the choice of factors $a_1, \ldots, a_k$. For example, $m = m(\dim X, \varepsilon, I)$ means that $m$ depends only on the dimension of $X$, $\varepsilon$ and the coefficient set $I$.

The proof of Theorem 1.3 relies on Proposition 3.5 whose proof is a bit technical and thus will be given afterward. We list the maps that shall appear in the proof of Theorem 1.3 in the following diagram (starting from Step 2).

![Diagram of the proof](image)

**Proof of Theorem 1.3.** Let $k = \dim Z$. We can assume that $k > 0$. Replacing $\varepsilon$ by $\min\{\varepsilon, \frac{1}{2}\}$, we can assume that $\varepsilon < 1$.

Step 1. Replace $(X, B)$ by its good minimal model.

Take a log resolution $g : X' \to X$ of $(X, B)$ such that $f \circ g : X' \to Z$ is a morphism. Let $K_{X'} + B'' = g^*(K_X + B) + E'$, where $B'' = g_*^{-1}B + (1 - \varepsilon/2) \text{Exc}(g)$. Then $(X', B'')$ is $\varepsilon/2$-lc and $B''$ is big over $Z$. If $F'$ is a general fiber of $f \circ g$, then $(F', B''|_{F'})$ is klt and $B''|_{F'}$ is big. By [4, Theorem 1.2] and the base point free theorem, $(F', B''|_{F'})$ has a good minimal model. Applying [31, Theorem 4.4], $(X', B'')$ has a good minimal model. In [31, Theorem 4.4], this result is obtained for $X$ with terminal singularities, however, the same argument works for a klt pair. In fact, the terminal singularity assumption is only used to show that when $(X, B)$ is terminal, then a good minimal model of $(X', g_*^{-1}B)$ is also a good minimal model of $(X, B)$ (see [31, Lemma 2.2]). However, a good minimal model of $(X', B'')$ is still a good minimal model of $(X, B)$ (e.g., see [5, Remark 2.6 (i)]). Thus, one can work with $(X', B'')$ in the proof of [31, Theorem 4.4] and the result follows.

Replacing $(X, B)$ by a good minimal model of $(X', B'')$, $\varepsilon$ by $\varepsilon/2$, and possibly enlarge $I$ to $I \cup \{1 - \varepsilon/2\}$, we can assume that $f : X \to Z$ is a morphism and it is an $\varepsilon$-lc Fano type fibration for $(X, B)$.

Step 2. The construction of $Z'$.

By $K_X + B \sim_{\mathbb{Q}} 0/Z$ and the canonical bundle formula (see (2.5.1)), $K_X + B \sim_{\mathbb{Q}} f^*(K_Y + \Delta + M)$, where $\Delta$ is the divisorial part and $M$ is the moduli part. As $B \in I$ with $I$ a DCC set, by ACC for log canonical thresholds (see [19, Theorem 1.1]), there exists a DCC set $J = J(\dim X, I)$ such that $\Delta \in J$. For a general fiber $F_f$ of $f$, $(F_f, B|_{F_f})$ is $\varepsilon$-lc Fano and coefficients of $B|_{F_f}$ are bounded below (in fact, finite by [19, Theorem D]). Thus, $(F_f, B|_{F_f})$ belongs to a bounded family ([9, Theorem 1.1], or use [22, Theorem 1.3]).
We argue as [22, Theorem 1.4] to show that there is an \( m = m(d, I, \epsilon) \) such that \( |m(K_Z + \Delta + M)| \) defines a birational map \( \varphi : Z \to Z' \). Let \( \tau : Z \to Z \) be a log smooth model of \((Z, \Delta)\) such that \( \tau^{-1}_*\Delta \cup \text{Exc}(\tau) \) is an snc divisor and the trace \( M_Z \) of the moduli b-divisor \( M \) is nef on \( Z \). Then by the boundedness of \( F_f \), there exists an \( r = r(\dim F_f, \epsilon) \in \mathbb{N} \) such that \( |m(K_Z + \Delta + M)| \) defines a birational map \( \mu : Z \to Z' \). Let \( \tau : \tilde{Z} \to Z \) be a log smooth model of \((Z, \Delta)\) such that \( \tau^{-1}_*\Delta + \text{Exc}(\tau) \) is an snc divisor and the trace \( M_{\tilde{Z}} \) of the moduli b-divisor \( M \) is nef on \( \tilde{Z} \). Then by the boundedness of \( \tilde{F}_f \), there exists an \( r = r(\dim \tilde{F}_f, \epsilon) \in \mathbb{N} \) such that \( |rM_{\tilde{Z}}| \) is an integral divisor (see [22, Claim 3.2] or [37] that both rely on [17]). Let \( K_{\tilde{Z}} + \Delta + M_{\tilde{Z}} = \tau^* (K_Z + \Delta + M) + E' \) with \( \Delta = \tau^{-1}_*\Delta + \text{Exc}(\tau) \). Then \( E' \geq 0 \) is \( \tau \)-exceptional, and \((\tilde{Z}, \Delta + M_{\tilde{Z}})\) is generalized lc (see [10, Definition 4.1]) with \( \Delta \in J \cup \{1\} \). Moreover, \( K_{\tilde{Z}} + \Delta + M_{\tilde{Z}} \) is big. By [10, Theorem 1.3], there exists an \( m = m(d, I, \epsilon) \in \mathbb{N} \) such that \( |m(K_{\tilde{Z}} + \Delta + M_{\tilde{Z}})| \) defines a birational map. For an \( \mathbb{R} \)-Cartier divisor \( D \) on \( Z \), there exists a \( \tau \)-exceptional divisor \( F \) such that \( \lfloor \tau^* D \rfloor \leq \tau^{-1}_* \lfloor D \rfloor + F \). Hence, if \( F' \) is a \( \tau \)-exceptional divisor such that \( \lfloor \tau^* D + F' \rfloor \) defines a birational map, then \( |D| \) also defines a birational map. Apply this to \( D = m(K_Z + \Delta + M), F' = mE' \), then \( |D| \) also defines a birational map.

Take a resolution \( p : \tilde{Z} \to Z \) such that the movable part of \( p^* |m(K_Z + \Delta + M)| \) is base point free, and write

\[
p^* |m(K_Z + \Delta + M)| = |\tilde{R}| + F, \tag{3.1.1}
\]

where \( |\tilde{R}| \) is the movable part and \( F \) is the fixed part. This can be done by first passing to a small \( \mathbb{Q} \)-factorialization of \( Z \) (it exists because there exists \( \Delta_Z \) such that \((Z, \Delta_Z)\) is klt by [3, Theorem 0.2]), and then resolving the \( \mathbb{Q} \)-Cartier divisor that is the strict transform of \( \lfloor m(K_Z + \Delta + M) \rfloor \) on this small \( \mathbb{Q} \)-factorial model. Now, \( |\tilde{R}| \) defines a morphism \( q : \tilde{Z} \to Z' \) that resolves \( \mu \circ p \). Let \( A \) be the very ample divisor on \( Z' \) such that \( \tilde{R} = q^* A \). Then

\[
A^k \leq \text{vol}(\tilde{Z}, \tilde{R} + \tilde{F}) \leq \text{vol}(Z, m(K_Z + \Delta + M)) \leq m^k v. \tag{3.1.2}
\]

Thus, \( Z' \) belongs to a bounded family.

**Step 3. The construction of \((X', D')\).**

Let \( g : X' \to X \) be a log resolution of \((X, B)\) that also resolves \( X \to \tilde{Z} \). Let \( \pi : X' \to \tilde{Z} \) be the corresponding morphism and \( f' = q \circ \pi \). Set

\[
D' = \left(1 + \frac{\epsilon}{2}\right) g_*^{-1} B + \left(1 - \frac{\epsilon}{2}\right) \text{Exc}(g). \tag{3.1.3}
\]

As \((X, B)\) is \( \epsilon \)-lc, \((X', D')\) is \( \frac{\epsilon}{2} \)-lc. For \( 0 < \delta \ll 1 \),

\[
K_{X'} + D' \geq g^* (K_X + (1 + \delta) B), \tag{3.1.4}
\]

thus \( K_{X'} + D' \) is big on \( Z \). \( K_{X'} + D' \) is also big on \( Z' \) as \( Z \to Z' \) is birational. By [4, Theorem 1.2], there exists a minimal model \( \vartheta : X' \to Y / Z' \) of \((X', D') / Z' \). Let \( D_Y = \vartheta_* D' \), then \( K_Y + D_Y \) is semi-ample on \( Z' \) by the base point free theorem.

**Step 4. Log birational boundedness of \((X, B)\).**

We claim that \( K_Y + D_Y + 3dh^* A \) is a nef and big divisor. Recall that \( \dim Y = d \) and \( A \) is very ample on \( Z' \) such that \( \tilde{R} = q^* A \). If there is an extremal curve \( C \) such that \( C \cdot (K_Y + D_Y + 3dh^* A) < 0 \), then \( h^* C \neq 0 \) because \( K_Y + D_Y \) is nef on \( Z' \). Moreover, \( C \cdot (K_Y + D_Y) < 0 \). By the length of extremal rays, there exists a curve \( C' \) on the same ray class of \( C \) such that \( -(2d + 1) \leq C' \cdot (K_Y + D_Y) < 0 \). But \( C' \cdot (3dh^* A) \geq 3d \), we get \( C' \cdot (K_Y + D_Y + 3dh^* A) > 0 \), a contradiction. Next, \( K_X + B \) is
pseudo-effective and thus \( K_{X'} + D' \) is pseudo-effective. As the image of \( K_{X'} + D' \), \( K_Y + D_Y \) is also pseudo-effective. Let \((Y', D_{Y'})\) be the ample model of \((Y, D_Y)\) over \(Z'\) with \( w : Y \to Y' \) and \( u : Y' \to Z' \) the corresponding morphisms. Then \( K_{Y'} + D_{Y'} + N u^* A \) is ample for \( N \gg 1 \). Thus,

\[
K_Y + D_Y + 3dh^* A = \left(1 - \frac{1}{N}\right)(K_Y + D_Y) + \left(\frac{1}{N}(K_Y + D_Y) + 3dh^* A\right)
\]

is big.

Let \( G \in |6dh^* A| \) be a general element, then \((Y, D_Y + \frac{1}{2}G)\) is still \( \frac{\varepsilon}{2} \)-lc with \( D_Y + \frac{1}{2}G \in I' \), where \( I' = (1 + \frac{\varepsilon}{2})I \cup \{1 - \frac{\varepsilon}{2}\} \cup \{\frac{1}{2}\} \) is a DCC set. By [19, Theorem 1.3(3)], there exists \( m' = m'(\dim Y, I') \in \mathbb{N} \) such that the linear system \(|m'(K_Y + D_Y + \frac{1}{2}G)|\) defines a birational map \( \varphi : Y \to Y'' \). By Proposition 3.5,

\[
\text{vol} \left( Y, \left( K_Y + D_Y + \frac{1}{2}G \right) \right) \leq N(I, \varepsilon, d, v),
\]

where \( N(I, \varepsilon, d, v) \) depends only on \( I, \varepsilon, d, v \). Moreover, by [19, Lemma 7.3], there is a rational number \( \beta = \beta(d, I') < 1 \) such that \( K_Y + \beta(D_Y + \frac{1}{2}G) \) is still big.

By [18, Lemma 2.4.2(3)], to show that the family \( \{(X, B)\} \) is log birationally bounded, it is enough to show that \( \{(Y, D_Y)\} \) is log birationally bounded. Then by [18, Lemma 2.4.2(4)], it is enough to show that

\[
\left( \text{Supp} D_{Y''} + \text{Exc}(\varphi^{-1}) \right) \cdot (H)^{d-1}
\]

is bounded above, where \( D_{Y''} \) is the strict transform of \( D_Y \) and \( H \) is the very ample divisor on \( Y'' \) determined by \( \varphi \). As \( I' \) is a DCC set, let \( 0 < \delta = \min I' \). Let \( \pi_1 : W \to Y, \pi_2 : W \to Y'' \) be a common resolution such that \( \varphi \circ \pi_1 = \pi_2 \) and the movable part of \( \pi_1^* |m'(K_Y + D_Y + \frac{1}{2}G)| \) is base point free. Let \( D_W = \pi_1^{-1} D_Y \). By [18, Lemma 3.2],

\[
\left( \text{Supp} D_{Y''} + \text{Exc}(\varphi^{-1}) \right) \cdot (H)^{d-1} \\
\leq \left( \text{Supp} D_W + \text{Exc}(\pi_1) \right) \cdot (\pi_1^* H)^{d-1}
\leq 2^d \text{vol}(W, K_W + \text{Supp} D_W + \text{Exc}(\pi_1) + 2(2d + 1)\pi_1^* H)
\leq 2^d \text{vol} \left( W, K_W + \frac{1}{\delta} D_W + \text{Exc}(\pi_1) + 2(2d + 1)\pi_1^* H \right)
\leq 2^d \text{vol} \left( W, \pi_1^* \left( K_Y + \frac{1}{2}D_Y \right) + E \\
+ 2(2d + 1)\pi_1^* \left( m' \left( K_Y + D_Y + \frac{1}{2}G \right) \right) \right),
\]

where \( E \) is a \( \pi_1 \)-exceptional divisor with sufficiently large coefficients. As \( K_Y + \beta(D_Y + \frac{1}{2}G) \) is big, there is \( \Theta \geq 0 \) such that \( \Theta + (1 - \beta)D_Y \sim_{\mathbb{R}} K_Y + D_Y + \frac{1}{2}G \). Hence, \( D_Y \leq_{\mathbb{R}} \frac{1}{1-\beta}(K_Y + D_Y + \frac{1}{2}G) \).
In particular, 

\[ K_Y + \frac{1}{\delta} D_Y \leq_{\mathbb{R}} \left( 1 + \frac{1}{\delta(1 - \beta)} \right) \left( K_Y + D_Y + \frac{1}{2} G \right). \]

Now, continue the estimates in (3.1.7), we have

\[
\begin{align*}
&\langle \text{Supp} D_{Y''} + \text{Exc} (\varphi^{-1}) \rangle \cdot (H)^{d-1} \\
\leq & 2^d \text{vol} \left( Y, \left( K_Y + \frac{1}{\delta} D_Y + 2(2d + 1) \left( m' \left( K_Y + D_Y + \frac{1}{2} G \right) \right) \right) \right) \\
\leq & 2^d \left( 1 + \frac{1}{\delta(1 - \beta)} + 2(2d + 1)m' \right) \left( K_Y + D_Y + \frac{1}{2} G \right) \\
\leq & 2^d \left( 1 + \frac{1}{\delta(1 - \beta)} + 2(2d + 1)m' \right)^d N(I, \epsilon, d, v). \quad (3.1.8)
\end{align*}
\]

This final estimate depends only on \( I, \epsilon, d, v \). Hence, the set of log pairs \( \{(X, B)\} \) in Theorem 1.3 is log birationally bounded.

Step 5. Boundedness of \( Z \).

The argument below follows the same lines as [19, Theorem 1.6]. However, [19, Theorem 1.6] deals with the \( K_X + B \) ample case (see [19, Lemma 9.1]) while \( K_X + B \) is only semi-ample here. Hence, we provide details on this part.

First, by Step 4, there exists a projective morphism \( \mathcal{X} \to T \) between schemes of finite type, and a reduced subshebe \( B \subset \mathcal{X} \) such that for any \( (X, B) \) in Theorem 1.3, there exists \( t \in T \) and a birational map \( i : X \to \mathcal{X}_t \) satisfying \( \text{Supp} i_!^{-1}(B) \cup \text{Exc}(i^{-1}) \subset \text{Supp} B \). Taking log resolutions and stratifying \( T \) into a disjoint union of locally closed subvarieties, we can assume that \( (\mathcal{X}, B) \) is log smooth over \( T \) (i.e., \( \mathcal{X} \) as well as any stratum of \( B \) are smooth over \( T \)). Passing to a finite cover of \( T \) (see [30, Claim 4.38.1]), we can assume that every stratum of \( B \) has irreducible fibers over \( T \). For \( (\mathcal{X}, (1 - \epsilon)B) \), by a sequence of blowups of strata, we extract all the divisors whose log discrepancies are \( \leq 1 \). Then one can show that \( i^{-1} : \mathcal{X}_t \to X \) is a birational contraction, that is, \( i \) does not contract any divisor on \( X \). This step is done in [19, Proof of (1.6), pp. 556–557].

Next, let \( 0 < \delta := \min I \). We define a \( \mathbb{Q} \)-divisor \( \tilde{B} \) on \( \mathcal{X} \) corresponding to \( (X, B) \) as follows. Let \( i^{-1} : \mathcal{X}_t \to X \) be the above birational contraction. Then \( \tilde{B} \) is the \( \mathbb{Q} \)-divisor such that \( \text{Supp} \tilde{B} \subset \text{Supp} B \). This \( \tilde{B} \) is uniquely determined because \( (\mathcal{X}, B) \) is log smooth over \( T \) and each stratum of \( B \) has irreducible fibers over \( T \). We use [4, Corollary 1.1.5(2)] to show the following claim:

Suppose that \( (\mathcal{X}_t, \tilde{B}_t) \) corresponds to some \( (X, B) \). Then there are finitely many ample models over \( T \) for

\[ \{(\mathcal{X}', B') \mid \text{Supp} B' = \text{Supp} \tilde{B}, \delta \text{ Supp} \tilde{B} \leq B' \leq (1 - \epsilon) \text{ Supp} \tilde{B} \}. \]

We can choose a rational number \( 0 < \tau \ll 1 \) such that \( K_X + (1 + \tau)B \) is big by the same argument as (3.1.5). Then \( K_X + (1 + \tau)\tilde{B} \) is big over \( T \) by [20, Theorem 1.2]. Let \( A + \mathcal{E} \sim_{\mathbb{Q}} K_X + (1 + \tau)\tilde{B} / T \) with \( \mathcal{E} \geq 0 \) and \( A > 0 \) an ample \( \mathbb{Q} \)-Cartier divisor that has no components in common with \( \mathcal{E} \) and \( \text{Supp} \tilde{B} \). Choose \( \alpha \in \mathbb{Q}_{>0} \) such that \( (\mathcal{X}, \alpha(A + \mathcal{E}) + (1 - \epsilon/2) \text{ Supp} \tilde{B}) \) is klt. By choosing
\(\gamma \in \mathbb{Q}_{>0}\) sufficiently small, we can assume that \(\Delta'\) in

\[(1 + \gamma \alpha)(K_X + B') = K_X + \gamma \alpha(K_X + (1 + \tau)\mathcal{B}) + \Delta'\]

is effective with coefficients \(< (1 - \epsilon)/2\). Indeed, \(\Delta' = (1 + \gamma \alpha)B' - \gamma \alpha(1 + \tau)\mathcal{B}\), where \(B'\) has coefficients in \([\delta, (1 - \epsilon)]\) and \(\text{Supp } B' = \text{Supp } \mathcal{B}\). Moreover, \(\text{Supp } \Delta' \subset \text{Supp } \mathcal{B}\). Now \((\mathcal{X}, \gamma \alpha A + \Delta' + \gamma \alpha \mathcal{E})\) is klt with

\[(1 + \gamma \alpha)(K_X + B') \sim_{\mathbb{Q}} K_X + \gamma \alpha A + \Delta' + \gamma \alpha \mathcal{E}/\mathcal{T}.

Then the ample model of \((\mathcal{X}, B')/\mathcal{T}\) is also the ample model of \((\mathcal{X}, \gamma \alpha A + \Delta' + \gamma \alpha \mathcal{E})/\mathcal{T}\). By [4, Corollary 1.1.5(2)], we have finitely many rational maps \(\psi_j : \mathcal{X} \rightarrow \mathcal{Z}_j\) over \(\mathcal{T}\) such that for a klt pair \((\mathcal{X}, \gamma \alpha A + \Delta' + \gamma \alpha \mathcal{E})\) with \(\text{Supp } \Delta' \subset \text{Supp } \mathcal{B}\) and \(K_X + \gamma \alpha A + \Delta' + \gamma \alpha \mathcal{E}\) pseudo-effective, there exists a \(\psi_j\) that is the ample model of \((\mathcal{X}, \gamma \alpha A + \Delta' + \gamma \alpha \mathcal{E})\) over \(\mathcal{T}\). Hence, \(\psi_j\) is also the ample model of \((\mathcal{X}, B')\) over \(\mathcal{T}\).

Finally, by [20, Corollary 1.4], if \(\psi_j : \mathcal{X} \rightarrow \mathcal{Z}_j/\mathcal{T}\) is the ample model of \((\mathcal{X}, B')\), then \(\psi_{j,t} : \mathcal{X}_{t} \rightarrow \mathcal{Z}_{j,t}\) is the ample model of \((\mathcal{X}_{t}, B_{t})\) for every closed point \(t \in \mathcal{T}\). Because \(\mathcal{X}_{t} \rightarrow X\) is a birational contraction, the ample model of \((X, B)\) is the ample model of \((\mathcal{X}_{t}, B_{t})\). In fact, let \(r_1 : W \rightarrow X\), \(r_2 : W \rightarrow \mathcal{X}_t\), be a common log resolution that also resolves \(i : X \rightarrow \mathcal{X}_{t}\). Then \(K_W + r_{1,*}B + (1 - \epsilon)\text{Exc}(r_1) = r_1^*(K_X + B) + E_1\) and \(K_W + r_{2,*}B_{t} + (1 - \epsilon)\text{Exc}(r_2) = r_2^*(K_{X_{t}} + B_{t}) + E_2\) with \(E_1, E_2 \geq 0\). But \(r_{1,*}B + (1 - \epsilon)\text{Exc}(r_1) = r_{2,*}B_{t} + (1 - \epsilon)\text{Exc}(r_2)\) by the construction of \(B\). Then the assertion follows because the ample model of \((W, r_{1,*}B + (1 - \epsilon)\text{Exc}(r_1))\) (resp., \((W, r_{2,*}B_{t} + (1 - \epsilon)\text{Exc}(r_2))\)) is the same as the ample model of \((X, B)\) (resp., \((\mathcal{X}_{t}, B_{t})\)). Because \(K_X + B\) is semiample, \(Z = \text{Proj } \bigoplus_{m=0}^{\infty} H^0(X, \mathcal{O}_X([m(K_X + B)])\) is exactly the ample model of \((X, B)\) [4, Lemma 3.6.6 (3)], this shows the boundedness for \(P(d, \epsilon, v, I)\). \(\square\)

Next, we show Proposition 3.5 that is used in Step 4 in the proof of Theorem 1.3. The strategy is similar to [23, Theorem 4.1] (also see [12, Proposition 4.1] and [7, Proposition 5.8]). However, it is much more subtle here because \(h : Y \rightarrow Z'\) in the proof of Theorem 1.3 may not be an \(\epsilon\)-lc Fano type fibration. Under the notation in the proof of Theorem 1.3, we have the following auxiliary constructions:

\[
\begin{array}{c}
\xymatrix{
X' & W \\
X & Z \\
\mathcal{X} & \mathcal{Z}
}
\end{array}
\]

Recall that \(g : X' \rightarrow X\) is a log resolution of \((X, B)\). Let \(B' = g^{-1}_{*}B + (1 - \epsilon/2)\text{Exc}(g)\), then

\(K_{X'} + B' = g^*(K_X + B) + E'\).
where $E' > 0$ is $g$-exceptional. There is a nonempty smooth open set $U_Z \subset \tilde{Z}$ such that $(X'_{|\pi^{-1}(U_Z)}, B'_{|\pi^{-1}(U_Z)})$ is log smooth over $U_Z$. Notice that $p : \tilde{Z} \rightarrow Z$ is birational. For a general closed point $t \in U_Z$, let $F_{\pi}$ be the fiber $\pi^{-1}(t)$ and $F_f$ be the fiber $f^{-1}(p(t))$. Then

$$K_{F_{\pi}} + B'_{|F_{\pi}} = (g_{|F_{\pi}})^{*}(K_{F_f} + B_{|F_f}) + E'_{|F_{\pi}},$$

where $E'_{|F_{\pi}} > 0$ is an exceptional divisor for $g_{|F_{\pi}} : F_{\pi} \rightarrow F_f$. Because $K_{F_f} + B_{|F_f} \sim_{\mathbb{Q}} 0$, $(F_{\pi}, B'_{|F_{\pi}})$ has a good minimal model. In fact, we can run a partial $(K_{F_{\pi}} + B'_{|F_{\pi}})$-MMP over $F_f$. After finitely many steps, $E'_{|F_{\pi}}$ must be contracted and thus the resulting log pair is crepant over $(F_f, B_{|F_f})$. This is a good minimal model for $(F_{\pi}, B'_{|F_{\pi}})$ (alternatively, this follows from [4, Theorem 1.2] and the base point free theorem as before). Then by [20, Theorem 1.2], $(X'_{|\pi^{-1}(U_Z)}, B'_{|\pi^{-1}(U_Z)})$ has a good minimal model over $U_Z$. By [21, Theorem 1.1] (also see [6, Theorem 1.4]), $(X', B')$ has a good minimal model $(W, B_W)$ over $Z$. Let $\psi : W \rightarrow V/Z$ be the morphism associated with $K_W + B_W$. Then $K_W + B_W \sim_{\mathbb{Q}} 0/V$. Let $\phi : V \rightarrow Z$ be the corresponding birational morphism.

Next, let

$$K_{X'} + B' = g^{*}(K_X + B),$$

then $(X', B')$ is a subklt pair such that $K_{X'} + B' \sim_{\mathbb{Q}} \pi^{*}p^{*}L$, where

$$L := K_Z + \Delta + M$$

(see Step 2 in the proof of Theorem 1.3). In particular, $K_{X'} + B' \sim_{\mathbb{Q}} 0/Z$. Let $K_W + B_W$ be the pushforward of $K_{X'} + B'$ on $W$. Then $(W, B_W)$ is subklt as $K_{X'} + B' \sim_{\mathbb{Q}} 0/Z$. By the canonical bundle formula,

$$K_W + B_W \sim_{\mathbb{Q}} \psi^{*}(K_V + \Delta_V + M_V),$$

where $\Delta_V$ is the divisorial part and $M_V = L_V - (K_V + \Delta_V)$ is the moduli part with $L_V := \phi^{*}p^{*}L$. Apply the canonical bundle formula to the klt-trivial fibration $\psi : (W, B_W) \rightarrow V$, we have

$$K_W + B_W \sim_{\mathbb{Q}} \psi^{*}(K_V + \Delta_V + M_V).$$

Notice that the moduli parts in (3.1.10) and (3.1.11) can be chosen as the same $\mathbb{Q}$-divisor by the following reason (also see [8, Lemma 3.5]).

By construction, $K_W + B_W = K_W + \bar{B}_W + E_W$ for some $E_W \geq 0$. Because $K_W + B_W \sim_{\mathbb{Q}} 0/V$ and $K_W + \bar{B}_W + E_W \sim_{\mathbb{Q}} 0/V$, we have $E_W \sim_{\mathbb{Q}} 0/V$. Then $E_W = \psi^{*}E_V$ for some $\mathbb{Q}$-divisor $E_V \geq 0$ on $V$. This can be derived from [27, Lemma 1.7]. Although $V$ may not be $\mathbb{Q}$-factorial here, it is enough to show our claim on the smooth locus of $V$ because we have $E_W \sim_{\mathbb{Q}} 0/V$ instead of merely $E_W \equiv 0/V$.

By the definition of the divisorial parts, $\Delta_V = \bar{\Delta}_V + E_V$. Notice that

$$K_W + B_W \sim_{\mathbb{Q}} \psi^{*}(L_V + E_V).$$
Hence, the moduli part for $(W, B_W) \to V$ can be chosen as

$$L_V + E_V - (K_V + \Delta_V),$$

which is exactly $L_V - (K_V + \Delta_V) = M_V$.

Recall that in Step 2 of the proof of Theorem 1.3, we have $K_X + B \sim_\mathbb{Q} f^*(K_Z + \Delta + M)$. By above discussion, for $\eta := p \circ \phi$,

$$K_V + \Delta_V + M_V = \eta^*(K_Z + \Delta + M) + E_V. \quad (3.1.12)$$

Next, recall that $p : Z \to Z$ is a log resolution such that $p^*|m(K_Z + \Delta + M)| = |\tilde{R}| + \tilde{F}$ (see (3.1.1)), where $|\tilde{R}|$ is base point free and defines a birational morphism $q : \tilde{Z} \to Z'$. Moreover, $Z'$ belongs to a bounded family that depends only on $I, \epsilon, d, v$ (see Step 2 in the proof of Theorem 1.3, notice that we do not use Proposition 3.5 until Step 4). Let $R_V = \phi^*\tilde{R}$ and $\dim V = \dim Z = k$. The key estimate is the following lemma.

**Lemma 3.1.** Under the assumptions in Theorem 1.3, using the notation introduced above, there exists $\mathcal{R} = \mathcal{R}(I, \epsilon, d, v) \in \mathbb{Q}_{>0}$ such that $(K_V + \Delta_V + M_V) \cdot R_V^{k-1} < \mathcal{R}$.

**Proof.** We estimate $K_V \cdot R_V^{k-1}, \Delta_V \cdot R_V^{k-1}$ and $M_V \cdot R_V^{k-1}$ separately.

First, we work with $K_V \cdot R_V^{k-1}$. Recall that $\tilde{R}$ defines $q : \tilde{Z} \to Z'$, and for the very ample divisor $A$ on $Z'$ such that $\tilde{R} = q^*A$, we have $A^k \leq m^kv$ (see (3.1.2)). Hence, $Z' \subset \mathbb{P}^l$ with $l \leq m^kv + k - 1$ (e.g., see [13, Proposition 0]), and $\mathcal{O}_{Z'}(A) \simeq \mathcal{O}_{Z'}(1)$. Thus, the degree of $Z'$ is bounded above by $m^kv$. Then by the construction of Chow varieties (e.g., see [29, chapter I.3]), there is a projective morphism between Noetherian schemes of finite type $U' \to C$, and a line bundle $A$ on $U'$ such that for any $Z'$, there is some $t_0$ whose fiber $U'_{t_0} \simeq Z'$ and $A |_{U'_{t_0}} \simeq \mathcal{O}_{Z'}(A)$. In fact, $A$ is the restriction of $pr_2^*\mathcal{O}_{\mathbb{P}^l}(1)$ to the universal family $U' \subset C \times \mathbb{P}^l$. In particular, $K_{Z'} \cdot A^{k-1}$ can only achieve finitely many values and thus be bounded above. By projection formula [29, chapter VI, Proposition 2.11], $K_V \cdot R_V^{k-1} = K_{Z'} \cdot A^{k-1}$ that is also bounded above depending only on $I, \epsilon, d, v$.

Next, we work with $\Delta_V \cdot R_V^{k-1}$. Notice that $B_W \in I \cup \{1 - \frac{\mathbb{H}}{2}\}$ that is a DCC set. By ACC for log canonical thresholds [19, Theorem 1.1], $\Delta_V$ belongs to a DCC set depending on $I, \epsilon, d$. In particular, each coefficient of $\Delta_V$ is greater than some $\delta = \delta(I, \epsilon, d) \in \mathbb{Q}_{>0}$. By [10, Theorem 8.1] (this is [19, Lemma 7.3] for the generalized polarized pairs), and use the same argument as Step 2 in the proof of Theorem 1.3 (i.e., take a smooth model $\mathcal{V} \to V$ such that the moduli part is a nef $\mathbb{Q}$-Cartier divisor, and use the boundedness of general fibers to bound the Betti numbers), there exists $e = e(\epsilon, I, d) \in (0, 1) \cap \mathbb{Q}$ such that $K_V + e\Delta_V + eM_V$ is still big. Because $M_V$ is pseudo-effective,

$$(1 - e)\Delta_V \leq \mathcal{R}(1 - e)\Delta_V + (K_V + e\Delta_V + M_V) = K_V + \Delta_V + M_V.$$ 

By [18, Lemma 3.2], if $H = 2(2k + 1)R_V$, then

$$\text{Supp} \Delta_V \cdot H^{k-1} \leq 2^k \text{vol}(V, K_V + \text{Supp} \Delta_V + H).$$
Hence, it is enough to show that \( \text{vol}(V, K_V + \text{Supp} \Delta_V + H) \) has an upper bound that depends only on \( I, \varepsilon, d, v \). Because coefficients of \( \Delta_V \) are greater than \( \delta \),

\[
\text{Supp} \Delta_V \leq \frac{1}{\delta} \Delta_V \leq \frac{1}{\delta(1 - e)} (K_V + \Delta_V + M_V).
\]

(3.1.13)

By construction, \( R_V \leq mL_V \), hence \( H = 2(2k + 1)R_V \leq 2(2k + 1)mL_V \). Then by (3.1.13),

\[
\begin{align*}
\text{vol}(V, K_V + \text{Supp} \Delta_V + H) & \leq \text{vol}\left(V, K_V + \frac{1}{\delta(1 - e)} (K_V + \Delta_V + M_V) + H\right) \\
& \leq \text{vol}\left(V, K_V + \Delta_V + M_V + \frac{1}{\delta(1 - e)} (K_V + \Delta_V + M_V) + 2(2k + 1)mL_V\right) \\
& \leq \text{vol}\left(V, \left(1 + \frac{1}{\delta(1 - e)}\right)(L_V + E_V) + 2(2k + 1)mL_V\right) \\
& \leq \text{vol}\left(V, \left(1 + \frac{1}{\delta(1 - e)} + 2(2k + 1)m\right)(L_V + E_V)\right),
\end{align*}
\]

where the third inequality uses (3.1.12). By construction,

\[
\text{vol}(V, L_V + E_V) = \text{vol}(V, K_V + \Delta_V + M_V) = \text{Ivol}(W, K_W + B_W) = \text{Ivol}(X', K_{X'} + B') = \text{Ivol}(X, K_X + B) \leq v,
\]

and thus \( \text{vol}(V, K_V + \text{Supp} \Delta_V + H) \) has an upper bound that depends only on \( I, \varepsilon, d, v \).

Finally, we bound \( M_V \cdot R_V^{k-1} \). By the above discussion, \( M_V \) is the moduli part for both \( (W, \tilde{B}_W) \to V \) and \( (W, B_W) \to V \), hence \( \eta_* M_V = M \). Because \( (X, B) \to Z \) is a klt-trivial fibration, there is a klt pair \( (Z, \Delta_Z) \) such that \( K_X + B \sim_{\mathbb{Q}} f^*(K_Z + \Delta_Z) \) [3, Theorem 0.2]. Thus, there is a small \( \mathbb{Q} \)-factorialization \( Z^q \to Z \). Moreover, \( p \) can be assumed to factor through \( Z^q \), and let \( p' : \tilde{Z} \to Z^q \) be the corresponding morphism. Let \( \Delta^q, M^q \) be the strict transforms of \( \Delta, M \). We claim that \( (p' \circ \phi)^* M^q = M_V + F_V \), where \( F_V \) is effective. In fact, because the moduli b-divisor \( \mathbf{M} \) is b-nef, there is a birational model \( \tilde{\phi} : \tilde{V} \to V \) such that the trace \( \mathbf{M}_V \) is nef. By the negativity lemma, there exists an effective divisor \( F_V \) such that \( \mathbf{M}_V + F_V = (p' \circ \phi \circ \tilde{\phi})^* M^q \). Thus, \( \mathbf{M}_V + \tilde{\phi}_* F_V = \tilde{\phi}_*(\mathbf{M}_V + F_V) = (p' \circ \phi)^* M^q \), where \( F_V = \tilde{\phi}_* F_V \geq 0 \). By the same argument as before, there exists \( e' = e'(\varepsilon, I, d) \in (0, 1) \cap \mathbb{Q} \) such that \( K_{Z^q} + e' \Delta^q + e'M^q \) is big. Hence,

\[
\begin{align*}
(1 - e')M_V & \leq_{\mathbb{R}} (1 - e')(p' \circ \phi)^* M^q \\
& \leq_{\mathbb{R}} (p' \circ \phi)^* (K_{Z^q} + \Delta^q + e'M^q + (1 - e')M^q) = L_V.
\end{align*}
\]

(3.1.14)

Because \( L_V, R_V \) are nef, and \( R_V \leq mL_V \), we have

\[
L_V \cdot R_V^{k-1} \leq L_V \cdot (mL_V) \cdot R_V^{k-2} \leq \cdots \leq L_V \cdot (mL_V)^{k-1} \leq m^{k-1} v.
\]
Thus, by (3.1.14),

\[ M_V \cdot R_V^{k-1} \leq m^{k-1}v \left(1 - e'\right). \]

Put the above estimates together and notice that \( m = m(d, I, \varepsilon) \) (see Step 2 in the proof of Theorem 1.3), we have \( \mathcal{R} = \mathcal{R}(I, \varepsilon, d, v) \in \mathbb{Q}_{>0} \) such that \((K_V + \Delta_V + M_V) \cdot R_V^{k-1} < \mathcal{R} \). \( \square \)

**Remark 3.2.**

1. One cannot apply the method for estimating \( M_V \cdot R_V^{k-1} \) to \( B_V \cdot R_V^{k-1} \). The reason is that we do not have a bounded \( \zeta = \zeta(I, \varepsilon, d, v) \in \mathbb{R}_{>0} \) such that \( B_V \leq \zeta(p' \circ \phi)^* B^d \).
2. One can show that \( E_V \) in (3.1.12) is \( \eta \)-exceptional. Hence,

\[ (K_V + \Delta_V + M_V) \cdot L_V^{k-1} = (K_Z + \Delta + M) \cdot L^{k-1} \leq v. \]

However, such estimate breaks in the induction argument of Proposition 3.5 (see (3.1.17)). Hence, we have to use the above more complicated estimate.

In the proof of Proposition 3.5, we use the following lemma.

**Lemma 3.3** [23, Lemma 2.5]. Let \( X \) be a projective normal variety and let \( D \) be an \( \mathbb{R} \)-Cartier \( \mathbb{R} \)-divisor on \( X \). Let \( S \) be a base point free Cartier prime divisor on \( X \). Then for any rational number \( q > 0 \),

\[ \text{vol}(X, D + qS) \leq \text{vol}(X, D) + q \cdot \text{dim} X \cdot \text{vol}(S, D|_S + qS|_S). \]

**Lemma 3.4.** Let \( d \in \mathbb{N}, \varepsilon, \sigma, \nu, l \in \mathbb{Q}_{>0} \) be fixed numbers. Let \( I \subset [0, 1] \cap \mathbb{Q} \) be a DCC set. Suppose that \( \psi : (W, B_W) \to V \) is an \( \varepsilon \)-lc Fano type fibration between projective normal varieties, and that there is a big and base point free Cartier divisor \( R_V \) on \( V \) such that they satisfy the following properties.

1. \( \dim W = d \) and \( B_W \in I \).
2. \( R_{\dim V}^{\dim V} \leq \sigma \).
3. In the canonical bundle formula \( K_W + B_W \sim_{\mathbb{Q}} \psi^*(K_V + \Delta_V + M_V), \ (K_V + \Delta_V + M_V) \cdot R_V^{\dim V - 1} < \nu. \)

Then there exists \( \mathcal{B} = \mathcal{B}(d, \varepsilon, \sigma, \nu, l, I) \in \mathbb{Q}_{>0} \) such that

\[ \text{vol}(K_W + 2B_W + l^* R_V) \leq \mathcal{B}. \]

**Proof.** We prove the lemma by induction on \( \dim V \). First, we show the following claim.

**Claim.** There exists \( q = q(d, \varepsilon, \sigma, \nu, l, I) \in \mathbb{Q}_{>0} \) such that

\[ \text{vol}(W, K_W + 2B_W + l^* R_V - q^* R_V) = 0. \]

\( \square \)
Proof of the Claim. This can be shown as [7, Proposition 5.8]. In fact, suppose that there exists

\[ 0 \leq \Theta \sim_{\mathbb{Q}} K_W + 2B_W + l\psi^*R_V - q\psi^*R_V, \]

then for a general fiber \( F_\psi \) of \( \psi : W \to V \),

\[ \Theta|_{F_\psi} \sim_{\mathbb{Q}} K_{F_\psi} + 2B_W|_{F_\psi} \sim_{\mathbb{Q}} -K_{F_\psi}. \]

Hence, by [9, Theorem 1.6], there is \( 1 > \tau = \tau(\epsilon, d) \in \mathbb{Q} > 0 \) such that \( (F_\psi, B_W|_{F_\psi} + \tau\Theta|_{F_\psi}) \) is klt.

By inversion of adjunction, \( (W, B_W + \tau\Theta) \) is klt over the generic point of \( V \). Then

\[ K_W + (1 - \tau)B_W + \tau\Theta = (1 + \tau) \left( K_W + B_W + \frac{\tau(l - q)}{1 + \tau}\psi^*R_V \right) \sim_{\mathbb{Q}} 0/V. \]

By the canonical bundle formula,

\[ K_W + (1 - \tau)B_W + \tau\Theta \sim_{\mathbb{Q}} \psi^*(K_V + \Delta'_V + M'_V), \]

with \( \Delta'_V \geq 0 \) and \( M'_V \) pseudo-effective [8, Theorem 3.6]. It is well-known that \( K_V + (\dim V + 2)R_V \) is big (e.g., see [34, Proposition 3.3]). By

\[ (1 + \tau) \left( K_W + B_W + \frac{\tau(l - q)}{1 + \tau}\psi^*R_V \right) \]

\[ \sim_{\mathbb{Q}} (1 + \tau)\psi^* \left( K_V + \Delta_V + M_V + \frac{\tau(l - q)}{1 + \tau}R_V \right) \]

\[ \sim_{\mathbb{Q}} \psi^*(K_V + \Delta'_V + M'_V), \]

then \((1 + \tau)(K_V + \Delta_V + M_V + \frac{\tau(l - q)}{1 + \tau}R_V) + (d + 2)R_V \) is big. Thus,

\[
0 \leq \left( (1 + \tau) \left( K_V + \Delta_V + M_V + \frac{\tau(l - q)}{1 + \tau}R_V \right) + (d + 2)R_V \right) \cdot R_V^{\dim V - 1}
= (1 + \tau)(K_V + \Delta_V + M_V) \cdot R_V^{\dim V - 1} + (\tau(l - q) + (d + 2)) \cdot R_V^{\dim V}
\leq (1 + \tau)\nu + (\tau(l - q) + (d + 2)) \cdot R_V^{\dim V}.
\]

Because \( R_V \) is a nef Cartier divisor that is big, \( R_V^{\dim V} \geq 1 \). Then (3.1.15) implies that when \( q > l \),

\[ 0 \leq (1 + \tau)\nu + (d + 2)R_V^{\dim V} + \tau(l - q). \]

By \( R_V^{\dim V} \leq \sigma \), we have

\[ q \leq \frac{(1 + \tau)\nu + (d + 2)\sigma}{\tau} + l. \]
Hence, if we choose \( q = q(d, \varepsilon, \sigma, \nu, l, I) = \frac{(1+\nu)+d+2}{\sigma} + 1 \), then \( \text{vol}(W, K_W + 2B_W + l\psi^*R_V - q\psi^*R_V) = 0 \).

If \( \dim V = 0 \), then

\[
\text{vol}(W, K_W + 2B_W + l\psi^*R_V) = \text{vol}(W, B_W) = \text{vol}(W, -K_W).
\]

Because \((W, B_W)\) is \( \varepsilon \)-lc and \( B_W \) is big, \( W \) belongs to a bounded family that only depends on \( \varepsilon \) and \( d \) [9, Corollary 1.2]. Hence, \( \text{vol}(W, -K_W) \) is bounded above.

If \( \dim V = 1 \), then take a general \( S \in |\psi^*R_V| \). Because \( \deg V R_V \leq \sigma \), \( S = \bigcup_{i=1}^{\zeta} S_i \) with \( \zeta \leq \sigma \). Because \( S \) is a disjoint union of prime divisors, Lemma 3.3 still holds by the same argument. Thus, for any \( q \in \mathbb{Q}_{>0} \),

\[
\text{vol}(W, K_W + 2B_W + l\psi^*R_V) \leq \text{vol}(W, K_W + 2B_W + l\psi^*R_V - q\psi^*R_V) + q d \text{vol}(S, (K_W + 2B_W + l\psi^*R_V)|_S).
\]

(3.1.16)

Taking \( q = q(d, \varepsilon, \sigma, \nu, l, I) \) as in the above claim, by (3.1.16), we have

\[
\text{vol}(W, K_W + 2B_W + l\psi^*R_V) \leq q d \sum_{i=1}^{\zeta} \text{vol}(S_i, K_{S_i} + 2B_W|_{S_i})
\]

\[
= q d \sum_{i=1}^{\zeta} \text{vol}(S_i, -K_{S_i}).
\]

\( \text{vol}(S_i, -K_{S_i}) \) is bounded above because \( S_i \) belongs to a bounded family (see the \( \dim V = 0 \) case). Then by \( \zeta \leq \sigma \), we get the desired result.

Finally, when \( \dim V \geq 2 \), choose a general \( V_1 \in |R_V| \). \( V_1 \) is irreducible because \( R_V \) defines a morphism to a variety with dimension \( \geq 2 \). Let \( W_1 = \psi^*V_1 \) that is an irreducible divisor such that \((W, W_1 + B_W)\) is plt, in particular, \( W_1 \) is normal. By adjunction, \( K_{W_1} + B_{W_1} = (K_W + W_1 + B_W)|_{W_1} \), \((W_1, B_{W_1})\) is still \( \varepsilon \)-lc by inversion of adjunction (e.g., see [4, Corollary 1.4.5]). Because \( W_1 \) is a general Cartier divisor, \( B_{W_1} = B_W|_{W_1} \in I \) (see [25, Corollary 16.7]). Let \( R_{V_1} = R_V|_{V_1} \) that is still big and base point free. Let \( \psi_1 : W_1 \to V_1 \) be the induced morphism. Notice that a general fiber of \( \psi_1 \) is also a general fiber of \( \psi \), hence \( W_1 \) is Fano type over \( V_1 \). Thus, \( \psi_1 : W_1 \to V_1 \) is an \( \varepsilon \)-lc Fano type fibration. By canonical bundle formula,

\[
K_{W_1} + B_{W_1} \sim_Q \psi_1^*(K_{V_1} + \Delta_{V_1} + M_{V_1}) \sim_Q \psi_1^*((K_V + \Delta_V + M_V + R_V)|_{V_1}).
\]

Hence,

\[
(K_{V_1} + \Delta_{V_1} + M_{V_1}) \cdot R_{V_1}^{\dim V_1 - 1}
\]

\[
= (K_V + \Delta_V + M_V + R_V) \cdot V_1 \cdot R_V^{\dim V_1 - 1}
\]

\[
= (K_V + \Delta_V + M_V) \cdot R_V^{\dim V - 1} + R_V^{\dim V}
\]
\[
\leq \nu + \sigma. \quad (3.1.17)
\]

In summary, \(\psi_1 : (W_1, B_{W_1}) \to V_1\) satisfies the assumptions of Lemma 3.4 with \(d, \epsilon, \sigma, \nu, I\) replaced by \(d - 1, \epsilon, \sigma, \sigma + \nu, I\). Hence, by the induction hypothesis, there exists \(B = B_1(d, \epsilon, \sigma, \nu, l, I) \in Q_{>0}\) such that

\[
\text{vol}(W_1, K_{W_1} + 2B_{W_1} + l\psi_1^*R_{V_1}) \leq B_1.
\]

By Lemma 3.3, for any \(q \in Q_{>0}\), we have

\[
\text{vol}(W, K_W + 2B_W + l\psi^*R_V) \leq \text{vol}(W, K_W + 2B_W + l\psi^*R_V - q\psi^*R_V) + qd \text{vol}(W_1, K_{W_1} + 2B_{W_1} + l\psi_1^*R_{V_1}).
\]

Take \(q = q(d, \epsilon, \sigma, \nu, l, I) \in Q_{>0}\) as in the above claim. Then

\[
\text{vol}(W, K_W + 2B_W + l\psi^*R_V) \leq qdB_1,
\]

and thus the induction step is completed. \(\square\)

**Proposition 3.5.** Under the notation in the proof of Theorem 1.3, for any \(l \in \mathbb{N}\), there exists \(N = N(I, \epsilon, d, \nu, l) \in \mathbb{R}_{>0}\) such that

\[
\text{vol}(Y, K_Y + D_Y + lh^*A) < N.
\]

**Proof.** Because \((Y, D_Y)\) is a minimal model of \((X', D')/Z'\),

\[
\text{vol}(Y, K_Y + D_Y + lh^*A) = \text{vol}(X', K_{X'} + D' + l f'^*A).
\]

Because \(D' = (1 + \epsilon/2)g_*^{-1}B + (1 - \epsilon/2)\text{Exc}(g), B' = g_*^{-1}B + (1 - \epsilon/2)\text{Exc}(g)\) and \(\epsilon < 1\), we have \(D' \leq 2B'\). Hence, \(\text{vol}(Y, K_Y + D_Y + lh^*A) \leq \text{vol}(X', K_{X'} + 2B' + l f'^*A)\). Recall that \(\tilde{R} = q^*A\), and because \(K_W + 2B_W = \text{pr}_{2*}\text{pr}_1^*(K_{X'} + 2B')\) where \(X' \xleftarrow{\text{pr}_1} X'' \xrightarrow{\text{pr}_2} W\) is a common resolution, we have

\[
\text{vol}(X', K_{X'} + 2B' + l f'^*A) \leq \text{vol}(W, K_W + 2B_W + l\psi^*R_V).
\]

Hence, it is enough to bound \(\text{vol}(W, K_W + 2B_W + l\psi^*R_V)\). Notice that \(R^\text{dim V}_V \leq (mL_V)^{\text{dim V}} \leq m^\text{dim V}V\) where \(m = m(I, \epsilon, d, \nu)\), and \((K_V + B_V + M_V) \cdot R^\text{dim V}V - 1 < R(I, \epsilon, d, \nu)\) by Lemma 3.1. Applying Lemma 3.4 to \(\psi : (W, B_W) \to V\), and taking \(d, \epsilon, \sigma, \nu, l, I\) to be \(d, \epsilon, m^\text{dim V}V, \mathfrak{R}, l, I\), respectively, we have an upper bound for \(\text{vol}(W, K_W + 2B_W + l\psi^*R_V)\) that depends only on \(I, \epsilon, d, \nu, l\). \(\square\)

**Proof of Corollary 1.5.** We show that \((X, B) \in S(d, \epsilon, \nu, I)\) is a \((d, r, \epsilon)\)-Fano type fibration for some \(d, r, \epsilon\). This certainly holds when \(\dim Z = 0\). Hence, we can assume \(\dim Z > 0\).

By the definition of \(S(d, \epsilon, \nu, I), (X, B) \to Z\) satisfies (1), (2), (3) in Definition 2.3 with exactly the same \(d\) and \(\epsilon\). Now, we show (4) and (5).
By the argument for the boundedness of $Z$ (see proof of Theorem 1.3, Step 5), there are finitely many projective morphisms $\mathcal{X}^i \to \mathcal{T}$ and divisors $B'_i$ on $\mathcal{X}^i$ whose coefficients vary in $[\delta,(1-\epsilon)]$ (but $\text{Supp } B'_i$ is fixed) satisfying the following properties.

(1) There are finitely many birational maps $\varphi^i_j: \mathcal{X}^i \to Y^i_j/\mathcal{T}$ such that if $\varphi: (\mathcal{X}^i, B'_i) \to (Y^i_j, B''_j)/\mathcal{T}$ is a log terminal model of $(\mathcal{X}^i, B'_i)/\mathcal{T}$, then there is a $j$ satisfying $\varphi^i_j = \varphi$ (this can be obtained by the same argument as Theorem 1.3 Step 5 using [4, Corollary 1.1.5(1)] instead of [4, Corollary 1.1.5(2)]).

(2) There are finitely many $\phi^i_{jk}: Y^i_j \to Z^i_{jk}/\mathcal{T}$ that is a morphism to the ample model of some $(Y^i_j, B''_j)/\mathcal{T}$. Hence, it is also the ample model of some $(\mathcal{X}^i, B'_i)/\mathcal{T}$ . Notice that $\phi^i_{jk} \circ \varphi^i_j : \mathcal{X}^i \to Z^i_{jk}/\mathcal{T}$ in the proof of Theorem 1.3, Step 5.

For simplicity, we fix some $\mathcal{X}^i$, $Y^i_j$, $Z^i_{jk}$, and so on, and denote them by $\mathcal{X}$, $Y$, $Z$, and so on. By the construction of $B''_j$, it belongs to a rational polytope of divisors $\mathcal{P}$ such that for any $D \in \mathcal{P}$, $K_Y + D \sim_\mathbb{Q} 0/\mathcal{Z}$ (this is [4, Corollary 1.1.5(2)(3)] with $\mathcal{A}_j$ replaced by $\mathcal{P}$). Suppose that $D^{(s)}$ corresponds to a vertex of $\mathcal{P}$, and assume $K_Y + D^{(s)} \sim_\mathbb{Q} K \varphi^i \mathcal{L}^{(s)}$, where $\mathcal{L}^{(s)}$ is a divisor on $\mathcal{Z}$. Take a very ample divisor $A$ on $\mathcal{Z}$ such that $A - L^{(s)}$ is ample for each $s$. Then for each $D \in \mathcal{P}$, $D = \sum a(s)D^{(s)}$ with $a(s) \geq 0$ and $\sum a(s) = 1$. Hence, $K_Y + D \sim_\mathbb{Q} K \varphi^i (\sum a(s)\mathcal{L}^{(s)})$, and $A - (\sum a(s)\mathcal{L}^{(s)})$ is still ample. Now suppose that $f: (X, B) \to Z$ is the Iitaka fibration in the definition of $S(d, \epsilon, v, I)$ with $K_X + B \sim_\mathbb{Q} f^*L$. If $Z \simeq Z_t$ for some $t \in \mathcal{T}$, then $B \sim_\mathbb{Q} D_t$ for some $D \in \mathcal{P}$. Hence, $A_t - L$ is ample.

Because $\mathcal{A}_i^{\dim Z}$ can only achieve finitely many values, we can take $r$ to be the maximal value. Then for such $Z$, $A = A_t$ and $r$ satisfy the requirements of (4) and (5) in Definition 2.3. Because there are finitely many $\mathcal{X}$, $Y$, $Z$, we can choose a uniform $r = r(d, \epsilon, v, I) \in \mathbb{N}$ satisfying the requirements for each $Z$.

Finally, because $(X, B) \in S(d, \epsilon, v, I)$ satisfies the conditions in [7, Theorem 1.3] (see Theorem 1.4), $S(d, \epsilon, v, I)$ is log bounded. 

Corollary 1.7 follows from Corollary 1.5.

Proof of Corollary 1.7. Just as in Step 1 in the proof of Theorem 1.3, we can replace $(X, B)$ by its good minimal model and assume that $f: X \to Z$ is a morphism defined by the semi-ample divisor $K_X + B$. Suppose that $[\text{Ivol}(K_X + B_i)]_{i \in \mathbb{N}}$ is a strictly decreasing sequence satisfying properties of Corollary 1.7. Let $v \in \mathbb{Q}_{>0}$ such that $\text{Ivol}(K_X + B_i) < v$ for each $i$. Then by Corollary 1.5, $(X_i, B_i), i \in \mathbb{N}$ belong to a bounded family $(\mathcal{X}, B) \to \mathcal{T}$.

By taking an irreducible component of $\mathcal{T}$ and the corresponding $(\mathcal{X}, B)$, we can assume that $\mathcal{X}$, $\mathcal{T}$ are irreducible. Moreover, we can assume that the points that parameterize $(X_i, B_i)$ are dense in $\mathcal{T}$. Take a log resolution $\pi: \mathcal{X}' \to \mathcal{X}$ of $(\mathcal{X}, B)$ and set $B' = \pi^{-1}_* B + \text{Exc}(\pi)$. For $(X_i, B_i)$ that is parameterized by a general point $t_i \in \mathcal{T}$ (i.e., $X_i \simeq \mathcal{X}'_{t_i}$), set $X'_{t_i} = \mathcal{X}'_{t_i}$ and $B'_{t_i} = (\pi|_{\mathcal{X}'_{t_i}})^{-1}(B_i) + (1 - \epsilon)(\text{Exc}(\pi)|_{t_i})$. Then

$$K_{X'_{t_i}} + B'_{t_i} = (\pi|_{X'_{t_i}})^* (K_X + B_i) + E'_{t_i}$$

with $E'_{t_i} \geq 0$ a $\pi|_{X'_{t_i}}$-exceptional divisor. Thus, $\text{Ivol}(K_{X'_{t_i}} + B'_{t_i}) = \text{Ivol}(K_X + B_i)$. Moreover, $(X'_{t_i}, B'_{t_i})$ is $\epsilon$-lc and belongs to a bounded family $(\mathcal{X}', B') \to \mathcal{T}$. Replacing $(\mathcal{X}, B), (X_i, B_i)$ by
(\(X', B'\)), \((X'_i, B'_i)\), respectively, and stratifying \(T\) if necessary, we can assume that \((X', B)\) is log smooth over \(T\), and \(T\) is smooth. Passing to a finite cover of \(T\) (see [30, Claim 4.38.1]), we can further assume that every stratum of \(B\) has irreducible fibers over \(T\).

For \((X_i, B_i)\), let \(B'_i\) be a divisor on \(X\) whose components choose the same coefficients as those of \(B_i\). By [20, Theorem 4.2], for fixed \(m \in \mathbb{N}\) and any \(t \in T\), \(h^0(\mathcal{O}_X, m(K_X + (B'_i)_t))\) is invariant. Because \(I\) is a DCC set, we can assume that the coefficient of each component of \(\{B'_i\}_{i \in \mathbb{N}}\) is non-decreasing by passing to a subsequence. Moreover, we can assume that \(x(K_{X_i} + B_i) \geq 0\) is the same for each \(i \in \mathbb{N}\). Then, for any \(t \in T\), \(\{\text{Ivol}(K_{X_i} + (B'_i)_t)\}_{i \in \mathbb{N}}\) is nondecreasing. However, when \(i < j\),

\[
\text{Ivol}(K_{X_i} + B_i) = \text{Ivol}(K_{X_j} + (B'_i)_t) \leq \text{Ivol}(K_{X_j} + (B'_j)_t) = \text{Ivol}(K_{X_j} + B_j),
\]

which is a contradiction to the strictly decreasing of \(\{\text{Ivol}(K_{X_i} + B_i)\}_{i \in \mathbb{N}}\). \(\square\)

Remark 3.6. By the same argument as above, if further assume that \(I\) is a finite set, then \(\{\text{Ivol}(K_X + B)\}\) is a discrete set.

4 Further Discussions

In this section, we explain the relation between Conjecture 1.2 and the effective adjunction conjecture (see Conjecture 2.6). Along the way, some lower dimensional cases are established. We also discuss possible variants of Conjecture 1.2.

Proposition 4.1. Assuming Conjecture 2.6 and the existence of good minimal models, then Conjectures 1.2 and 1.6 hold.

Proof. By the existence of good minimal models, we can assume that \(K_X + B\) is semi-ample with \(f : X \to Z\) the morphism induced by \(K_X + B\). By the canonical bundle formula, \(K_X + B \sim Q f^*(K_Z + \Delta_Z + M_Z)\) where \(\Delta_Z\) belongs to a DCC set that depends only on \(d\) and \(I\). For a general fiber \(F\), \(K_F + B|_F \equiv 0\) and \(B|_F \in I\). Hence, by global ACC [19, Theorem D], \(B|_F\) belongs to a finite set \(I_0\) depending on \(\dim F\) and \(I\). Applying Conjecture 2.6, there is an \(m = m(d, I_0) \in \mathbb{N}_{>2}\) such that for the moduli b-divisor \(M, mM\) is a base point free b-divisor. Hence, there exists \(0 \leq G_Z \sim Q M_Z\) with \(G_Z \in \{\frac{k}{m} \mid 1 \leq k \leq m - 1, k \in \mathbb{Z}\}\) such that \((Z, \Delta_Z + G_Z)\) is klt. Notice that \(K_Z + \Delta_Z + G_Z\) is ample with \(\text{vol}(K_Z + \Delta_Z + G_Z) = \text{Ivol}(K_X + B)\), and the coefficients of \(\Delta_Z + G_Z\) belong to a DCC set depending only on \(d\) and \(I\). Then Conjecture 1.2 follows from [20, Theorem 1.1] and Conjecture 1.6 follows from [19, Theorem 1.3(1)].
\(\square\)

Lemma 4.2 [38, Theorem 1.3]. Let \(f : (X, B) \to Z\) be a klt-trivial fibration. When the relative dimension of \(f\) is two, there is a natural number \(m\) depending only on the coefficients of the horizontal part of \(B\), such that for the moduli b-divisor \(M, mM\) is b-Cartier.

Remark 4.3. The dependence only on the coefficients of the horizontal part of \(B\) is not explicitly stated in [38, Theorem 1.3], but it is easily seen from the proof of [38, Theorem 1.3] where \(m\) depends only on \((F, B|_F)\) with \(F\) a general fiber of \(f\).
Corollary 4.4. Conjectures 1.2 and 1.6 hold when $d \leq 3$.

Proof. We only show the case $d = 3$ as the easier case $d < 3$ can be obtained by the same argument. When $d = 3$, we only need to consider the cases when $\dim Z = 1, 2$.

If $\dim Z = 2$, then the relative dimension is 1, and Conjecture 2.6 holds by [36, Theorem 8.1]. Hence, the claim follows from Proposition 4.1.

If $\dim Z = 1$, then adopting the same notation as in the proof of Proposition 4.1, we have $\deg_Z K_Z \leq \deg_Z (K_Z + \Delta_Z + M_Z) = v$. Hence, $Z$ belongs to a bounded family. For Conjecture 1.6, by global ACC [19, Theorem D], $B|_F$ belongs to a finite set depending on $\dim F$ and $I$, then by Lemma 4.2, there exists $m \in \mathbb{N}$ such that $mM_Z$ is nef and Cartier. Thus, $\deg_Z M_Z \in \frac{1}{m} \mathbb{Z}_{\geq 0}$. Because $\Delta_Z$ belongs to a DCC set, $\text{Ivol}(K_X + B) = \deg_Z (K_Z + \Delta_Z + M_Z)$ belongs to a DCC set. □

It is well-known that in Conjecture 1.2, condition (2) cannot be replaced by $\text{Ivol}(K_X + B) \leq v$. In fact, even when $K_X + \Delta$ is ample, we have the following unbounded family of varieties.

Example 4.5 cf. [1, Example 2.15]. Let $X$ be a smooth surface and $\Delta = A + B$ be an snc divisor with reduced irreducible components $A$ and $B$ such that $A \cap B \neq \emptyset$. Suppose that $K_X + \Delta$ is ample. First, choose $0 < a_0 < 1$ close to 1, and then blow up an intersection point of $A \cap B$, we have a crepant model $(X_1, a_0(A_1 + B_1) + (2a_0 - 1)E)$, where $A_1, B_1$ are strict transforms of $A, B$. Choose $0 < \epsilon_1 \ll 1$, then $K_{X_1} + \Delta_1 := K_{X_1} + a_0(A_1 + B_1) + (2a_0 - 1 - \epsilon_1)E$ is ample. The coefficients of $\Delta_1$ are in $\{a_0, 2a_0 - 1 - \epsilon_1\}$. Notice that $\text{vol}(K_{X_1} + a_0(A_1 + B_1) + (2a_0 - 1 - \epsilon_1)E) \leq \text{vol}(K_X + \Delta)$. Next, choose $a_1 < 1$ even more close to 1. First blow up a point of $A \cap B$, then blow up a point of $A_1 \cap E$, we get a crepant model

$$(X_2, a_1(A_2 + B_2) + (2a_1 - 1)E + (3a_1 - 2)F).$$

Take $0 < \epsilon_2, \epsilon_2' \ll 1$ such that

$$K_{X_2} + \Delta_2 := K_{X_2} + a_1(A_2 + B_2) + (2a_1 - 1 - \epsilon_2)E + (3a_1 - 2 - \epsilon_2')F$$

is ample. Its volume is still bounded above by $\text{vol}(K_X + \Delta)$. Besides, $a_1, \epsilon_2, \epsilon_2'$ can be chosen such that $\min\{a_1, 2a_1 - 1 - \epsilon_2, 3a_1 - 2 - \epsilon_2'\} \geq \max\{a_0, 2a_0 - 1 - \epsilon_1\}$. We can continue this process to construct klt log pairs $(X_1, \Delta_i), i \in \mathbb{N}$. The coefficients of $\Delta_i$ are in a DCC set and $\text{vol}(K_{X_1} + \Delta_i) \leq \text{vol}(K_X + \Delta)$, but $\{X_i\}$ is unbounded because $\rho(X_i) = \rho(X) + i$.

However, under the $\epsilon$-lc assumption on the total space $(X, B)$, we propose the following conjecture.

Conjecture 4.6. Let $d \in \mathbb{N}, \epsilon, v \in \mathbb{Q}_{>0}$ be fixed numbers, and $I \subset [0, 1] \cap \mathbb{Q}$ be a DCC set. Let $\mathcal{Z}(d, \epsilon, v, I)$ be the set of varieties $Z$ satisfying the following properties.

1. $(X, B)$ is $\epsilon$-lc with $\dim X = d$, and coefficients of $B$ are in $I$.
2. $0 < \text{Ivol}(K_X + B) < v$.
3. $f : X \to Z$ is the Iitaka fibration associated with $K_X + B$, where

$$Z = \text{Proj} \bigoplus_{m=0}^{\infty} H^0(X, \mathcal{O}_X(m(K_X + B))).$$
Then $\mathcal{F}(d, \epsilon, v, I)$ is a bounded family.

When $K_X + B$ is big, that is, $\text{Ivol}(K_X + B) = \text{vol}(K_X + B) > 0$, Conjecture 4.6 follows from [19, Theorems 1.3 and 1.6]. Unfortunately, I do not know whether the conjecture holds when $\dim X = 3, \dim Z = 2$ and $B^h = 0$. In this case, $(Z, B_Z)$ may not be $\delta$-lc for any $\delta > 0$. Hence, a priori, $Z$ may not be in a bounded family (see Example 4.5). Our main result (Theorem 1.3) is about Conjecture 4.6 under the assumption that $-K_X$ is big over $Z$. Under this extra condition, $(Z, B_Z)$ is $\delta$-lc for some $\delta = \delta(\epsilon, d, I) > 0$ (see [7, Theorem 1.9]).

Remark 4.7. As mentioned above, it is desirable to obtain the boundedness of the bases regardless of the boundedness of fibers. From this perspective, we can even ask whether $Z$ belongs to a bounded family under the assumption that $\dim Z$ is fixed in Conjectures 1.2 and 4.6 (i.e., $\dim X$ can be arbitrarily large).

Besides, in Conjectures 1.2 and 4.6, one can consider $\mathbb{R}$-divisors instead of $\mathbb{Q}$-divisors. In this scenario, $\text{Proj} \bigoplus_{m=0}^{\infty} H^0(X, \mathcal{O}_X(\lfloor m(K_X + B) \rfloor))$ may not make sense. Instead, $Z$ should be replaced by the ample model of $(X, B)$ whose existence is known under certain conditions (see [24, 35]). For Conjecture 1.6, one can also require that $(X, B)$ to have lc singularities with real coefficients. See [2024, Section 6] for the precise statement of the conjectures and their relation to the $\Gamma$-effective adjunction conjecture for lc-trivial fibrations.

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