A NOTE ON THE PROJECTION OF THE DIAGONAL SUBVARIETY

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Abstract. The diagonal subvariety on a product of two CM elliptic curves, is presented as an example of a dimension one subvariety, that is pre-periodic only if the respective projection on the product of two projective lines is also pre-periodic.

1. Introduction

In [2] the authors presented a contradiction to the Dynamical Manin-Mumford conjecture using product of CM-elliptic curves. They considered maps $[\omega] \times [\omega'] : E \times E \rightarrow E \times E$ on the product $E \times E$, where $E$ is an elliptic with complex multiplication by $\omega, \omega'$, and studied the orbit of the diagonal subvariety $\Delta = \{(P, P) : P \in E \}$. In order to state the result obtained in [2] we define a subvariety $Y \subset X$ to be pre-periodic for a map $\varphi$ if it has a finite forward orbit $\{Y, \varphi Y, \ldots \}$ under $\varphi$. The characterization of when is $\Delta$ pre-periodic for the map $\varphi = [\omega] \times [\omega']$ is given by (c.f. lemma 2.7):

Lemma 1.1. $\Delta$ is pre-periodic for $[\omega] \times [\omega'] : E \times E \rightarrow E \times E$ if and only if $\omega/\omega'$ is a root of unity.

Similar properties can be found for the diagonal subvariety $\Delta'$ inside the product of projective lines under the action of Lattés maps. Any map $[\omega] : E \rightarrow E$ on an elliptic curve $E$ can be projected to a map $\varphi[\omega] : \mathbb{P}^1 \rightarrow \mathbb{P}^1$, when we mod out by the hyperelliptic involution $\pi : E \rightarrow \mathbb{P}^1$. The analogous result characterizing when is the orbit of the diagonal $\Delta' = \{(P, P) : P \in \mathbb{P}^1 \}$ finite under the action of the product of two Lattés maps reads (c.f. lemma 2.8):

Lemma 1.2. $\Delta'$ is preperiodic for $\varphi[\omega] \times \varphi[\omega'] : \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$ if and only if $\omega/\omega'$ is a root $\pm 1$.

Combining these two results together we have an example of subvarieties $\Delta, \Delta'$ satisfying:

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Theorem 1.3. \( \Delta \) is pre-periodic for \([\omega] \times [\omega']\) if and only if \( \pi_* (\Delta) = \Delta' \) is pre-periodic for \( \varphi_{[\omega]} \times \varphi_{[\omega']} \).

Note that the maps \([\omega] \times [\omega'] : E \times E \rightarrow E \times E, \varphi_{[\omega]} \times \varphi_{[\omega']} : \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1 \times \mathbb{P}^1\) fit into a commutative diagram:

\[
\begin{array}{ccc}
\Delta \subset E \times E & \xrightarrow{[\omega] \times [\omega']} & E \times E \\
\downarrow{\pi, \pi} & & \downarrow{\pi, \pi} \\
\pi(\Delta) = \Delta' \subset \mathbb{P}^1 \times \mathbb{P}^1 & \xrightarrow{\varphi_{[\omega]} \times \varphi_{[\omega']}\pi} & \mathbb{P}^1 \times \mathbb{P}^1.
\end{array}
\]

In the next section we will put the above example in the context of commutative diagram of varieties. We will show that the orbits of \( \Delta \) and \( \Delta' \) for some values of \( \omega \) and \( \omega' \) does not have the same length.

2. Projection of subvarieties in a commutative diagram

Let \( X \) be a projective algebraic variety. An algebraic dynamical system is a self-map \( \varphi : X \rightarrow X \), defined on the algebraic variety \( X \). A subvariety \( Y \subset X \) is said to be pre-periodic for \( \varphi \) if the forward orbit under \( \varphi \) is finite, more precisely

**Definition 2.1.** The subvariety \( Y \subsetneq X \) is pre-periodic for \( \varphi : X \rightarrow X \) if there are natural numbers \( n, k \), with \( k > 0 \), such that \( \varphi_{*}^{n+k}(Y) = \varphi_{*}^{n}(Y) \). In this situation we say that \( Y \) is pre-periodic with pair \( (n, k) \).

**Example 2.2.** If \( E \) is an elliptic curve, \( P \) is a torsion point with \( [k]P = 0 \) and we take the point \( Q \) such that \( [n]Q = P \), the point \( Q \) will be pre-periodic for the map \( [n] : E \rightarrow E \) with pair \( (n, k) \).

Suppose now that we have algebraic dynamical systems \( \varphi, \tilde{\varphi} \) that fit into a commutative diagram as follows:

\[
\begin{array}{ccc}
X & \xrightarrow{\varphi} & X \\
\downarrow{\pi} & & \downarrow{\pi} \\
\tilde{X} & \xrightarrow{\tilde{\varphi}} & \tilde{X}
\end{array}
\]

The first thing to note is the following:

**Remark 2.3.** If a subvariety \( Y \subset X \) is pre-periodic for \( \varphi \) then the projection \( \pi_* Y \) is pre-periodic for \( \tilde{\varphi} \). Indeed, \( Y \) pre-periodic implies the existence of natural numbers \( n, k(k > 0) \) such that,

\[
\varphi_{*}^{n+k}(Y) = \varphi_{*}^{n}(Y) \Rightarrow \pi_*(\varphi_{*}^{n+k}(Y)) = \pi_*(\varphi_{*}^{n}(Y)) \Rightarrow \varphi_{*}^{n+k}(\pi_* Y) = \varphi_{*}^{n}(\pi_* Y),
\]

and therefore \( \pi_* Y \) will also be pre-periodic with the same pair \( (n, k) \).
We are interested then in studying the following question:

**Question 2.4.** Is it true that $Y \subseteq X$ is pre-periodic for $\varphi$ if and only if the projection $\pi_*Y \subseteq \tilde{X}$ is pre-periodic for $\tilde{\varphi}$? In case of a positive answer, does it happen with the same pair $(n, k)$?

**Remark 2.5.** For subvarieties of dimension zero $Y = P$ with reduced structure. The theory of canonical height functions developed by Silverman and other authors in [1] or [4] for polarized dynamical systems, allows to completely characterize the pre-periodic points as points with height $\hat{h}_{\varphi}(P) = 0$. The first part of the question 2.4 is settled this way. For higher dimensional subvarieties, the results of Ghioca, Tucker and Zhang [2] show how the vanishing of height is not a sufficient condition for a subvariety to be pre-periodic.

### 2.1. The diagonal subvariety inside the product of elliptic curves.

We consider the following situation of a subvariety of dimension one. Let $E$ be an elliptic curve with complex multiplication by a ring $R$ and $\omega, \omega' \in R$. Let $\pi : E \to \mathbb{P}^1$ be the map arising when we mod out by the hyperelliptic involution. We want to study Question 2.4 for the diagram:

$$
\begin{array}{ccc}
\Delta \subset E \times E & \xrightarrow{[\omega] \times [\omega']} & E \times E \\
\downarrow \scriptstyle{(\pi, \pi)} & & \downarrow \scriptstyle{(\pi, \pi)} \\
\pi(\Delta) = \Delta' \subset \mathbb{P}^1 \times \mathbb{P}^1 & \xrightarrow{\varphi_{[\omega]} \times \varphi_{[\omega']}} & \mathbb{P}^1 \times \mathbb{P}^1,
\end{array}
$$

where $\Delta$ and $\Delta'$ are respectively the diagonal subvarieties:

$$
\Delta = \{(P, P) : P \in E\}, \quad \Delta' = \{(P, P) : P \in \mathbb{P}^1\}
$$

The result is the following:

**Theorem 2.6.** Let $\Delta \subset E \times E$ and $\Delta' \subset \mathbb{P}^1 \times \mathbb{P}^1$ denote respectively the diagonal subvarieties. Suppose that $\omega, \omega' \in R$, then $\Delta$ is pre-periodic for $[\omega] \times [\omega'] : E \times E \to E \times E$ if and only if $\Delta'$ is preperiodic for $\varphi_{[\omega]} \times \varphi_{[\omega']} : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$.

It is a consequence of the two lemmas:

**Lemma 2.7.** $\Delta$ is preperiodic for $[\omega] \times [\omega'] : E \times E \to E \times E$ if and only if $\omega/\omega'$ is a root of unity.

**Proof.** We reproduce the proof of Ghioca and Tucker. Suppose that $([\omega]^{n+k}, [\omega']^{n+k})(\Delta) = ([\omega]^n, [\omega']^n)(\Delta)$ for some $n, k > 0$. Consider a non-torsion point $P \in E$, then there exist $Q \in E$ also non-torsion such that $([\omega]^{n+k}, [\omega']^{n+k})(P, P) = ([\omega]^n, [\omega']^n)(Q, Q)$. But then $[\omega]^{n+k}(P) =$...
[\omega]^n(Q) and [\omega']^{n+k}(P) = [\omega]^n(Q) or equivalently [\omega]^n([\omega]^k(P) - Q) = 0 and [\omega']^n([\omega']^k(P) - Q) = 0. These last two equations are saying that there are torsion points \(P_1, P_2\) such that [\omega]^k(P) - Q = P_1 and [\omega']^k(P) - Q = P_2 and therefore [\omega]^k(P) - [\omega']^k(P) will also be a torsion point, and that cannot be for \(P\) non-torsion unless [\omega]^k - [\omega']^k = 0 and therefore \(\omega/\omega'\) is a root of unity. Conversely, suppose that [\omega]^k = [\omega']^k, then ([\omega]^{n+k}, [\omega']^{n+k})(P, P) = ([\omega]^n, [\omega]^n)((\omega]^k(P), [\omega]^k(P)) and because [\omega]^k is surjective ([\omega]^{n+k}, [\omega']^{n+k})(\Delta) = ([\omega]^n, [\omega]^n)(\Delta). \qed

**Lemma 2.8.** \(\Delta'\) is pre-periodic for \(\varphi[\omega] \times \varphi[\omega'] : \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1 \times \mathbb{P}^1\) if and only if \(\omega/\omega'\) is a root \(\pm 1\).

**Proof.** The proof is analogous to the case of \(\Delta\). Suppose that we have \((\varphi[\omega], \varphi[\omega'])^{n+k}(\Delta') = (\varphi[\omega], \varphi[\omega'])^{n}(\Delta')\) for some \(n \geq 0\) and \(k > 0\). Then \((\pi, \pi)((\omega]^{n+k}, [\omega']^{n+k})(\Delta) = (\pi, \pi)((\omega]^n, [\omega]^n)(\Delta)\) and for each \(P \in E\) there will be \(Q \in E\) with \((\omega]^{n+k}, [\omega']^{n+k})(P, P) = ([\omega]^n, [\omega]^n)(Q, \pm Q).\)

But then \([\omega]^{n+k}(P) = [\omega]^n(Q)\) and \([\omega']^{n+k}(P) = [\omega]^n(\pm Q)\) or equivalently \([\omega]^n([\omega]^k(P) - Q) = 0\) and \([\omega']^n([\omega']^k(P) \mp Q) = 0\) These last two equations are saying that there are torsion points \(P_1, P_2\) such that \([\omega]^k(P) - Q = P_1\) and \([\omega']^k(P) \mp Q = P_2\) and therefore \(P_1 \mp P_2 = [\omega]^k(P) \mp [\omega']^k(P) = ([\omega]^k \mp [\omega']^k)(P)\) will also be a torsion point, and that cannot be if we choose \(P\) non-torsion unless \([\omega]^k \mp [\omega']^k = 0\) and \((\omega/\omega')^k = \pm 1\). \qed

**Corollary 2.9.** If \((\omega/\omega')^k = -1\), \(\Delta'\) is pre-periodic with pair \((n, k)\) but \(\Delta\) is not pre-periodic with the same pair. If \((\omega/\omega')^k = 1\), both \(\Delta\) and \(\Delta'\) are pre-periodic with the same pair \((n, k)\).

**Example 2.10.** The elliptic curve \(E : y^2 = x^3 + x\) admits complex multiplication by \(R = \mathbb{Z}[i]\). When we take \(\omega = 2 + i\) and \(\omega' = 1 - 2i\) in \(R\) we have \(\omega/\omega' = -i\), therefore in the diagram:

\[
\begin{array}{ccc}
\Delta \subset E \times E & \xrightarrow{[2+i] \times [1-2i]} & E \times E \\
(\pi, \pi) \downarrow & & (\pi, \pi) \\
\pi(\Delta) = \Delta' \subset \mathbb{P}^1 \times \mathbb{P}^1 & \xrightarrow{\varphi[2+i] \times \varphi[1-2i]} & \mathbb{P}^1 \times \mathbb{P}^1
\end{array}
\]

we will have that \(\Delta\) is pre-periodic for \([2 + i] \times [1 - 2i]\) with pair \((0, 4)\) (but not \((0, 2)\)), while \(\Delta'\) is pre-periodic for \(\varphi[2+i] \times \varphi[1-2i]\) with pair \((0, 2)\). We refer to [3] for the explicit computation of the maps \(\varphi[\omega], \varphi[\omega']\).

**Example 2.11.** The curve \(E : y^2 = x^3 + 1\) admits complex multiplication by the ring \(R = \mathbb{Z}[\rho]\) where \(\rho = (\sqrt{-3} - 1)/2\). For \(\omega = \sqrt{-3}\rho\) and \(\omega' = \sqrt{-3}\) we will get \((\omega/\omega')^3 = 1\) and \(\Delta, \Delta'\) will be pre-periodic with
the same pair \((n, 3)\). Again we refer to \([5]\) for explicit computations of the maps \(\varphi_\omega, \varphi'_\omega\).

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