Hadamard Powers and Kernel Perceptrons

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Abstract

We study a relation between Hadamard powers and polynomial kernel perceptrons. The rank of Hadamard powers for the special case of a Boolean matrix and for the generic case of a real matrix is computed explicitly. These results are interpreted in terms of the classification capacities of perceptrons.

Keywords: Hadamard power, kernel perceptron, Boolean matrix, generic rank

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1. Introduction

The $d$-th Hadamard power ($d \in \mathbb{N}$) of a matrix is obtained by taking the $d$-th power of each entry. A kernel perceptron is a nonlinear classifier that separates data in a feature space. In this note we discuss a relation between the rank of Hadamard powers of a matrix and the shattering properties of kernel perceptrons with polynomial kernel. These notions will be explained in the text. More precisely, we consider Hadamard powers of the Gramian matrix associated to a set of data vectors. If the rank of the $d$-th Hadamard power is maximal, then any separation of the data set can be classified by a polynomial kernel perceptron of degree $d$. In this context we provide a new proof for the fact that any Boolean function in $n$ variables can be realized by a kernel perceptron with polynomial kernel of degree $n$. We compare our rank calculations to lower bounds obtained quite recently in [14]. Moreover, we show that generically the rank of the $d$-th Hadamard power of a matrix $A \in \mathbb{R}^{n \times m}$ is

$$\min \left\{ \binom{\text{rank } A + d - 1}{d}, n, m \right\}.$$  \hspace{1cm} (1)

Properties of Hadamard products and their applications have received attention over many years, e.g., in [25, 2, 13, 10, 12, 18, 8]. A prominent result is Schur’s product theorem [24], which states that the Hadamard product preserves non-negative definiteness. Definiteness, rank, and other properties of Hadamard

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powers have been discussed in [17, 6, 9, 11, 15, 4, 14, 16, 20]. Applications of Hadamard products in artificial intelligence can be found in [28]. But to our knowledge, the relation of Hadamard powers to polynomial kernel perceptrons has not been accentuated so far, although the field of machine learning is very active now. Concerning kernel perceptrons, we largely follow the presentation in the monograph [23]. The kernelized approach, however, goes back to [1] and has been taken up, e.g., in [3, 7].

Our paper is structured as follows. In Section 2, we introduce the Hadamard product and Hadamard powers. In Section 3 we compute the rank of Hadamard powers explicitly for special matrices that play a role in the later discussion. In contrast, Section 4 is devoted to the generic case. We show that for all \( r \) and all matrices from an open and dense subset of \( M_r = \{ A \in \mathbb{R}^{n \times m} \mid \text{rank} A = r \} \) these ranks equal \( (1) \). From this, we also draw conclusions for non-integer Hadamard powers. In Section 5 we collect facts on kernel perceptrons and the associated dual optimization problem before we apply the obtained results in the context of Boolean and general classifications problems, respectively.

2. Hadamard products and powers

For arbitrary matrices \( A = (a_{ij}) \), \( B = (b_{ij}) \in \mathbb{R}^{n \times m} \) the Hadamard product is defined as the componentwise product \( A \circ B = (a_{ij}b_{ij}) \). In the same way, we define the Hadamard power \( A^{\circ d} = (a_{ij}^d) \) for \( d \in \mathbb{N} \), e.g., [13, 12]. The following simple identity for rank-1 matrices can be found in [25] or [5, Fact 9.6.2].

**Lemma 2.1.** If \( u, w \in \mathbb{R}^n \), \( v, z \in \mathbb{R}^m \) then

\[
(uv^T) \circ (wz^T) = (u \circ w)(v \circ z)^T.
\]

By this lemma it follows easily that the Hadamard product preserves non-negative definiteness. Namely, if \( A = A^T \in \mathbb{R}^{n \times n} \) and \( B = B^T \in \mathbb{R}^{n \times n} \) are nonnegative definite, \( A \succeq 0 \) and \( B \succeq 0 \) in short, they can be written as sums of rank-1 matrices, i.e.,

\[
A \circ B = \left( \sum_{i=1}^n a_i a_i^T \right) \circ \left( \sum_{j=1}^n b_j b_j^T \right) = \sum_{i,j=1}^n (a_i \circ b_j)(a_i \circ b_j)^T \succeq 0. \tag{2}
\]

We will also make use of the following identities. For \( C = [C_1, \ldots, C_r] \in \mathbb{R}^{n \times r} \) and \( D = [D_1, \ldots, D_r] \in \mathbb{R}^{m \times r} \) we can apply Lemma 2.1 to get

\[
\left( \sum_{j=1}^n C_j D_j^T \right)^{\circ 2} = \left( \sum_{i=1}^n C_i D_i^T \right) \circ \left( \sum_{j=1}^n C_j D_j^T \right) = \sum_{i,j=1}^n (C_i D_i^T) \circ (C_j D_j^T) = \sum_{i,j=1}^n (C_i \circ C_j)(D_i \circ D_j)^T,
\]
which for $d = 2, 3, \ldots$ inductively implies
\[
\left( \sum_{j=1}^{n} C_j D_j^T \right)^{od} = \sum_{\ell_1, \ldots, \ell_d=1}^{n} (C_{\ell_1} \circ \cdots \circ C_{\ell_d})(D_{\ell_1} \circ \cdots \circ D_{\ell_d})^T .
\] (3)

In Section 5 the rank of Hadamard powers of nonnegative definite matrices is important. Relations of this rank to the Kruskal rank and the Hadamard power rank of the matrix have been reported in [14].

**Definition 2.2.** Let $A \in \mathbb{R}^{n \times m}$. If $A$ possesses a zero column, then its *Kruskal rank* $k_A$ is zero. Otherwise $k_A$ is defined as the largest positive integer $k$, such that any selection of $k$ distinct columns of $A$ is linearly independent.

If rank $A \geq 2$, then the *Hadamard power rank* $h_A$ of $A$ is the largest positive integer $h$, such that there exists a selection of $h$ distinct columns of $A$, which are pairwise linearly independent. If rank $A < 2$, then $h_A = \text{rank } A$.

With this notation, we can summarize [14, Prop. 6 and Cor. 9] as follows.

**Proposition 2.3.** Let $A \in \mathbb{R}^{n \times n}$ with $n \geq 2$ be nonnegative definite.

(a) $k_A \geq 2 \iff \forall d \geq n - 1 : A^{od}$ is positive definite.

(b) $\forall d \geq \max\{h_A - 1, 1\} : \text{rank } A^{od} = h_A$.

**3. The rank of Hadamard powers of a Boolean matrix**

In this section we consider a special matrix that plays a central role in our application to kernel perceptrons and Boolean functions in Section 5. Here we compute the ranks of all its Hadamard powers explicitly. For $j = 0, \ldots, 2^n - 1$ let
\[
\begin{bmatrix}
  x_{1,j} \\
  \vdots \\
  x_{n,j}
\end{bmatrix} \in \{0, 1\}^n
\]

contain a binary representation of $j$, i.e.,
\[
\begin{aligned}
  j &= \sum_{\ell=1}^{n} x_{\ell,j} 2^{\ell-1} .
\end{aligned}
\] (4)

We say that in the binary representation of $j$ the $\ell$-th bit is *active*, if $x_{\ell,j} = 1$. Using $X = [x_1, \ldots, x_{2^n-1}] \in \mathbb{R}^{n \times 2^n-1}$ we define $K = X^T X$. We aim to calculate the rank of $K^{od}$. In Section 5 the matrix $K^{od}$ will be related to the kernel of a perceptron. Since $x_0 = 0$, the first column and row of $K^{od}$ vanishes for all $d > 0$, i.e., $K^{od}e_1 = 0$. Hence rank $K^{od} \leq 2^n - 1$. 

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Example 3.1. For $n = 3$ we have

$$X^T = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad K^\circ d = (X^T X)^\circ d = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 2^d & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 2^d & 1 \\ 0 & 0 & 1 & 1 & 1 & 2^d & 2^d \\ 0 & 1 & 1 & 2^d & 1 & 2^d & 2^d & 3^d \end{bmatrix}$$

with rank $K^\circ 1 = 3$, rank $K^\circ 2 = 6$ and rank $K^\circ 3 = 7$. Because of the zero column in all $K^\circ d$, the rank cannot increase further. Note also that the Hadamard power rank $h_K$ equals 7, since the last seven columns of $K$ are pairwise linearly independent. According to Proposition 2.3 we know only that rank $K^\circ d = 7$ for all $d \geq 6$. Our observation that already rank $K^\circ 3 = 7$ thus is stronger. We will show that rank $K^\circ n = 2^n - 1 = h_K$ for all $n$.

Theorem 3.2. Let $n \in \mathbb{N}$ and $X = (x_{\ell,j}) \in \{0,1\}^{n \times (2^n-1)}$ be defined according to (4). Then

$$\text{rank } K^\circ d = \begin{cases} \sum_{p=1}^{d} \binom{n}{p}, & \text{if } d \leq n, \\ 2^n - 1, & \text{if } d > n. \end{cases}$$

Proof. Let us write $X^T = (\beta_{j,\ell}) = [B_1, \ldots, B_n]$, with $B_\ell \in \{0,1\}^{2^n}$. The $j$-th entry $\beta_{j,\ell}$ of $B_\ell$ is $x_{\ell,j}$. This equals 1, if and only if the $\ell$-th bit is active in the binary representation of the number $j$ as in (4). In particular,

$$\beta_{j,\ell} = \begin{cases} 1, & \text{if } j = 2^{\ell-1} \\ 0, & \text{if } j < 2^{\ell-1}. \end{cases}$$

(5)

For $j = 1, \ldots, 2^n-1$, we define the sets

$$U(j) = \{ u \in \mathbb{R}^{2^n} \mid u_{j+1} = 1 \text{ and } u_i = 0 \text{ for } i \leq j \}$$

with the property that vectors from different $U(j)$ are linearly independent. From (5), it follows that $B_\ell \in U(2^{\ell-1})$, as is illustrated for $n = 3$ in Example 3.1. We now consider Hadamard products of columns $B_{\ell_1}, \ldots, B_{\ell_p}$. If $\ell_1 = \ell_2$, then $B_{\ell_1} \circ B_{\ell_2} = B_{\ell_1}$. Therefore let now $\ell_1 < \cdots < \ell_p$. Note that this product corresponds to a logical AND operation: The $j$-th entry of $B_{\ell_1} \circ \cdots \circ B_{\ell_p}$ is non-zero, if and only if the bits $\ell_1$ to $\ell_p$ in the binary representation (4) of $j$ are active. The smallest number $j$ with this property is $j = 2^{\ell_1-1} + \cdots + 2^{\ell_p-1}$. Hence

$$B_{\ell_1} \circ \cdots \circ B_{\ell_p} \in U (2^{\ell_1-1} + \cdots + 2^{\ell_p-1}) ,$$
and the set
\[ \{ B_{\ell_1} \circ \cdots \circ B_{\ell_p} \mid 1 \leq p \leq n, 1 \leq \ell_1 < \cdots < \ell_p \leq n \} \]
is linearly independent. We apply these observations to the Hadamard powers of \( X^T X \). By Lemma 2.1 and Equation (3) we have
\[
\begin{pmatrix}
\sum_{j=1}^{n} B_j B_j^T \end{pmatrix}^d = \sum_{\ell_1, \ldots, \ell_d=1}^{n} (B_{\ell_1} \circ \cdots \circ B_{\ell_d})(B_{\ell_1} \circ \cdots \circ B_{\ell_d})^T.
\]
Note that \( \beta_{j,l} \in \{0, 1\} \) implies (for possibly repeating indices \( \hat{\ell}_1 \leq \cdots \leq \hat{\ell}_p \)) that
\[
\{ \hat{\ell}_1, \ldots, \hat{\ell}_p \} = \{ \ell_1, \ldots, \ell_p \} \Rightarrow B_{\ell_1} \circ \cdots \circ B_{\ell_p} = B_{\hat{\ell}_1} \circ \cdots \circ B_{\hat{\ell}_p}.
\]
Hence the image of this matrix is
\[
\text{Im} \begin{pmatrix}
\sum_{j=1}^{n} B_j B_j^T \end{pmatrix}^d = \bigoplus_{p=1}^{d} \text{span} \{ B_{\ell_1} \circ \cdots \circ B_{\ell_p} \mid 1 \leq \ell_1 < \cdots < \ell_p \leq n \}.
\]
Since each of the direct summands has dimension \( \binom{n}{p} \), the proof is complete.

4. The rank of Hadamard powers in the generic case

For the matrix \( X \in \{0, 1\}^{n \times (2^n - 1)} \) containing the binary representations of all numbers 0, 1, \ldots, \( 2^n - 1 \) we have computed the ranks of the Hadamard powers of \( K = X^T X \) explicitly and found them to be larger than the lower bounds in Proposition 2.3. Generically, we can expect even higher ranks. To make this precise, we now analyze sets of vectors whose Hadamard products are in general position (compare, e.g., [27]).

**Definition 4.1.** A set \( S \subset \mathbb{R}^n \) is in general position, if any subset of \( S \) with at most \( n \) elements is linearly independent.

For a given matrix \( A \in \mathbb{R}^{n \times m} \) with columns \( A_1, \ldots, A_m \) we define the sets of Hadamard products of given order \( d \in \mathbb{N} \) and of arbitrary order as
\[
\mathcal{H}_A^d = \{ A_{\ell_1} \circ \cdots \circ A_{\ell_d} \mid 1 \leq \ell_1 \leq \ell_2 \leq \cdots \leq \ell_d \leq m \} \quad \text{and} \quad \mathcal{H}_A = \bigcup_{d \in \mathbb{N}} \mathcal{H}_A^d.
\]
Note that \( \bigcup_{d=1}^{N} \mathcal{H}_A^d \subset \mathcal{H}_A^N \) holds only for binary \( A \). In the following example, we construct \( P \in \mathbb{R}^{n \times m} \) such that \( \mathcal{H}_P \) is in general position.

**Example 4.2.** Let \( p_j \in \mathbb{N} \) denote the \( j \)-th prime number, i.e., \( (p_1, p_2, p_3, \ldots) = (2, 3, 5, \ldots) \) and consider
\[
P = [P_1, \ldots, P_m] = (e^{(i-1) \sqrt{p_j}})_{i=1, \ldots, n} = \begin{bmatrix} 1 & \cdots & 1 \\
e^{\sqrt{p_1}} & \cdots & e^{\sqrt{p_m}} \\
\vdots & \ddots & \vdots \\
e^{(n-1) \sqrt{p_1}} & \cdots & e^{(n-1) \sqrt{p_m}} \end{bmatrix}.
\]
For \( d \in \mathbb{N} \) and a given \( d \)-tuple \((\ell_1, \ldots, \ell_d) \in \mathbb{N}^d \) with \( 1 \leq \ell_1 \leq \ell_2 \leq \cdots \leq \ell_d \leq m \) we set \( \alpha_{\ell_1, \ldots, \ell_d} = \exp \left( \sum_{j=1}^{d} \sqrt{\ell_j} \right) \). Since the numbers \( \sqrt{\ell_j} \) are rationally independent, it follows that the mapping \((\ell_1, \ldots, \ell_d) \mapsto \alpha_{\ell_1, \ldots, \ell_d} \) from the set of ordered \( d \)-tuples in \( \mathbb{N}^d \) to \( \mathbb{R} \) is injective. Hence, for different \( d \) and different \((\ell_1, \ldots, \ell_d)\), with \( 1 \leq \ell_1 \leq \ell_2 \leq \cdots \leq \ell_d \leq m \) the vectors

\[
P_{\ell_1} \circ \cdots \circ P_{\ell_d} = \begin{bmatrix} 1 & \alpha_{\ell_1, \ldots, \ell_d} & \alpha_{\ell_1, \ldots, \ell_d}^2 & \cdots & \alpha_{\ell_1, \ldots, \ell_d}^{n-1} \end{bmatrix}^T
\]

constitute different columns of an \( n \times n \) Vandermonde matrix, which is nonsingular. Hence \( \mathcal{H}_{P} \) is in general position.

In fact, \( \mathcal{H}_A \) is in general position for most matrices \( A \in \mathbb{R}^{n \times m} \) in the following sense.

**Lemma 4.3.** Consider the normed matrix space \( \mathbb{R}^{n \times m} \) with \( m \geq n \geq 2 \).

(a) Let \( N \in \mathbb{N} \). The set of matrices \( A \in \mathbb{R}^{n \times m} \) for which \( \bigcup_{d=1}^{N} \mathcal{H}_{A}^{d} \) is in general position is open and dense in \( \mathbb{R}^{n \times m} \).

(b) The set of matrices \( A \in \mathbb{R}^{n \times m} \) for which \( \mathcal{H}_A \) is not in general position is of the first category (in the sense of Baire, e.g., [19, page 40]).

**Proof.** (a) We first fix \( d \in \mathbb{N} \). Let \( A \in \mathbb{R}^{n \times m} \). To test whether \( \mathcal{H}_A^{d} \) is in general position amounts to considering determinants of a finite number of \( n \times n \) matrices with pairwise different columns from \( \mathcal{H}_A^{d} \). Each of these determinants is a multivariate polynomial in the entries of \( A \) and thus continuous in \( A \). If therefore all the determinants are nonzero for the given \( A \), then the same holds on an open neighbourhood of \( A \). This proves the openness statement.

To prove denseness, choose a matrix \( P \) such that \( \mathcal{H}_{P}^{d} \) is in general position (e.g., as in Example 4.2). For an arbitrary \( A \in \mathbb{R}^{n \times m} \) we will construct a number \( \epsilon_A > 0 \), such that \( \mathcal{H}_{A+E}^{d} \) is in general position for all \( \epsilon < \epsilon_A \).

As before, we consider the determinants of all \( n \times n \) matrices with pairwise different columns from \( \mathcal{H}_{A+E}^{d} \). Let \( \hat{A} \) be an \( n \times n \) submatrix of \( A \) and \( \hat{P} \) the corresponding submatrix of \( P \). The determinant

\[
\det(\hat{A} + \epsilon \hat{P}) = \epsilon^n \det \left( \frac{1}{\epsilon} \hat{A} + \hat{P} \right) =: p(\epsilon)
\]

is a polynomial in \( \epsilon \). Note that \( \frac{1}{\epsilon} p(\epsilon) = \det(\frac{1}{\epsilon} \hat{A} + \hat{P}) \xrightarrow{\epsilon \to \infty} \det \hat{P} \), which is nonzero, because \( P \) is in general position. Hence \( p \) is not the zero polynomial and in particular its positive real roots do not accumulate at zero, i.e.,

\[
\epsilon_\hat{A} := \inf \{ \lambda \mid \lambda > 0, \ p(\lambda) = 0 \} > 0,
\]

where possibly \( \epsilon_\hat{A} = \infty \). Since the number of submatrices of \( A \) is finite, also

\[
\epsilon_A = \inf \{ \epsilon_\hat{A} \mid \hat{A} \text{ is an } n \times n \text{ submatrix of } A \} > 0.
\]
Thus all \( n \times n \) submatrices of \( A + \epsilon P \) are nonsingular, provided \( 0 < \epsilon < \epsilon_A \). This concludes the proof of (a).

(b) It follows from (a) that the set of matrices \( A \in \mathbb{R}^{n \times m} \) for which \( \bigcup_{d=1}^{N} \mathcal{H}_A^d \) is in general position is a finite intersection of open dense subsets and therefore open and dense itself. The complement of an open and dense set is nowhere dense. Therefore the set of matrices \( A \in \mathbb{R}^{n \times m} \), for which \( \mathcal{H}_A \) is not in general position, is a countable union of nowhere dense sets. By definition, the set is thus of the first category.

We will also need the following general statement on matrices of fixed rank \( r \).

Let us consider

\[
M_r = \{ A \in \mathbb{R}^{n \times m} \mid \text{rank } A = r \}
\]

with the topology inherited from \( \mathbb{R}^{n \times m} \).

**Lemma 4.4.** For arbitrary \( m, n \), let \( S_{n,m} \) be an open and dense subset of \( \mathbb{R}^{n \times m} \). Then

\[
S_{n,r} S_{r,m} = \{ CD \mid C \in S_{n,r}, D \in S_{r,m} \}
\]

is open and dense in \( M_r \).

**Proof.** If \( A \in M_r \), then it possesses a nonsingular \( r \times r \) submatrix. For suitable permutation matrices, we have \( \Pi_1^T A \Pi_2 = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \) with \( A_{11} \in \mathbb{R}^{r \times r} \) nonsingular. Since \( \text{rank } A = r \), we have

\[
\begin{bmatrix} A_{12} \\ A_{22} \end{bmatrix} = \begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix} A_{11}^{-1} A_{12} \quad \text{and}
\]

\[
A = CD \quad \text{with} \quad C = \Pi_1 \begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix} T, \quad D = T^{-1} \begin{bmatrix} I & A_{11}^{-1} A_{12} \end{bmatrix} \Pi_2^T,
\]

where \( T \in \mathbb{R}^{r \times r} \) is an arbitrary nonsingular matrix. Moreover, all possible factorizations with \( C \in \mathbb{R}^{n \times r}, D \in \mathbb{R}^{r \times m} \) are of this form. Hence, if \( A = CD \) with \( C \in S_{n,r}, D \in S_{r,m} \), then \( C \) and \( D \) are as in (8) with a given \( T = T_A \). Consider now \( \tilde{A} \) with \( \text{rank } \tilde{A} = r \) and \( \| \tilde{A} - A \| < \epsilon \). Then also \( \tilde{A} = \tilde{C} \tilde{D} \) with \( \tilde{C} = \Pi_1 \begin{bmatrix} \tilde{A}_{11} \\ \tilde{A}_{21} \end{bmatrix} T_A \) and \( \tilde{D} = T_A^{-1} \begin{bmatrix} I & \tilde{A}_{11}^{-1} \tilde{A}_{12} \end{bmatrix} \Pi_2^T \). For sufficiently small \( \epsilon > 0 \), it follows that \( \tilde{C} \in S_{n,r} \) and \( \tilde{D} \in S_{r,m} \). This proves openness of \( S_{n,r} S_{r,m} \).

Denseness is inherited from the factors, because in any neighbourhood of a matrix \( A = CD \) there exists a matrix \( \tilde{A} = \tilde{C} \tilde{D} \) with \( \tilde{C} \in S_{n,r} \) and \( \tilde{D} \in S_{r,m} \). □

**Theorem 4.5.** For all \( r \leq \min\{m, n\} \) there exists an open and dense subset \( A_r \) of \( M_r \subset \mathbb{R}^{m \times n} \) such that for all \( A \in A_r \) and all \( d \in \mathbb{N} \) the rank of the \( d \)-th Hadamard power equals

\[
\text{rank } A^d = \min \left\{ \binom{r + d - 1}{d}, n, m \right\}.
\]
Proof. If rank $A = 0$ or rank $A = 1$, then rank $A^{rd} = \text{rank } A$ for all $d = 1, 2, \ldots$. This is consistent with the convention that $(d-1)/d = 0$ and $(d)/d = 1$. Hence we can set $A_0 = M_0 = \{0\}$ and $A_1 = M_1$.

For $r \geq 2$ and arbitrary $m, n \geq r$, we set

$$S_{n,r} = \{C \in \mathbb{R}^{n \times r} | \bigcup_{d=1}^{n} \mathcal{H}_{d}^{d} \text{ is in general position}\}$$

$$S_{r,m} = \{D \in \mathbb{R}^{r \times m} | \bigcup_{d=1}^{m} \mathcal{H}_{d}^{d} \text{ is in general position}\}.$$

By Lemma 4.3 both sets are open and dense in their spaces, and by Lemma 4.4 the set $A_r = S_{n,r} \cup S_{r,m}$ is open and dense in $M_r$. Every $A \in M_r$ can be written as $A = CD$, where $C = [C_1, \ldots, C_r] \in S_{n,r}$ and $D = [D_1, \ldots, D_t]^T \in S_{r,m}$.

Equation (3) yields

$$A^{rd} = \sum_{\ell_1, \ldots, \ell_d = 1}^{r} (C_{\ell_1} \circ \ldots \circ C_{\ell_d})(D_{\ell_1} \circ \ldots \circ D_{\ell_d})^T.$$

For all $d < \min\{n, m\}$ and all $d$-tuples $1 \leq \ell_1 \leq \ell_2 \leq \cdots \leq \ell_d \leq r$ the vectors $C_{\ell_1} \circ \cdots \circ C_{\ell_d} \in \mathbb{R}^n$ and the vectors $D_{\ell_1} \circ \cdots \circ D_{\ell_d} \in \mathbb{R}^m$ are in general position. Therefore as long as $\text{rank } A^{rd} < \min\{n, m\}$ it equals the number of $d$-tuples $(\ell_1, \ldots, \ell_d)$ with $1 \leq \ell_1 \leq \ell_2 \leq \cdots \leq \ell_d \leq r$. Choosing such a $d$-tuple is an instance of unordered sampling with replacement. Consequently the number of choices equals $\binom{r+d-1}{d}$, see [21, Thm. 2.1].

Since $\binom{r+d-1}{d} \geq \binom{d+1}{d} = d+1$, for $d \geq \min\{n, m\}$ the rank of $A^{rd}$ is guaranteed to be maximal and we do not need the assumption on the general position for these $d$. Therefore (9) holds for all $A \in A_r$ and all $d \in \mathbb{N}$. \hfill \qed

Remark 4.6. (a) In the proof, we have used the bound $\binom{r+d-1}{d} \geq d+1$, which is sharp only for $r = 2$. For larger $r$ and a given $n$ it is useful to find a small $d$ satisfying $\binom{r+d-1}{d} \geq n$. Note that

$$\binom{r+d-1}{d} = \frac{\prod_{j=1}^{r-1}(d+j)}{(r-1)!} \geq \frac{(d+1)^{r-1}}{(r-1)!}.$$ 

Hence, it suffices to choose $d \geq \left\lceil \sqrt[r]{n/(r-1)!} \right\rceil - 1$.

(b) For $r = \min\{n, m\}$ the set $A_r$ is open and dense in $\mathbb{R}^{m \times n}$. Therefore the set $A = \bigcup_{r=0}^{\min\{n, m\}} A_r$ is also dense in $\mathbb{R}^{m \times n}$, but not open. For all $A \in A$ and all $d \in \mathbb{N}$, we have rank $A^{rd} = \min \left\{ \frac{\text{rank } A^{d+1}}{d}, n, m \right\}$.

4.1. Non-integer Hadamard powers

It is also an interesting task to analyze the rank of non-integer Hadamard powers. These are defined after taking absolute values of each entry see, e.g., [6].
For \( d \in \mathbb{R}, d \geq 0 \) and \( A = (a_{ij}) \in \mathbb{R}^{m \times n} \) we set \( |A|^d = (|a_{ij}|^d) \). Actually, our investigations were motivated by the application to kernel perceptrons in the next section. The case of non-integer \( d \), however, is not related to positive kernels, because \( A \succeq 0 \) does not imply \( |A| \succeq 0 \), see [13, Problem 7.5.P4], or [6]. Nevertheless, we are happy to take up the suggestion of an anonymous referee and extend Theorem 4.5 in this new direction.

Numerical experiments suggest that for almost all \( d > 0 \) the rank of \( |A|^d \) is maximal, if \( A \in M_r \subset \mathbb{R}^{n \times m} \) with \( r \geq 2 \).

**Example 4.7.** Consider the Boolean matrix from Example 3.1 without the first zero column, i.e., \( \hat{X} = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ \end{bmatrix} \). The condition number of \( \hat{K}^d = (\hat{X}^T \hat{X})^d \) for \( d > 0 \) is shown in Fig. 1 on the left. The matrix is singular only for \( d \in \{1, 2\} \).

As another (arbitrary) example, we use a Hankel matrix containing the first Fibonacci numbers, \( H = \begin{bmatrix} 1 & 2 & 3 & 5 & 8 \\ 2 & 3 & 5 & 8 & 13 \\ 3 & 5 & 8 & 13 & 21 \\ 5 & 8 & 13 & 21 & 34 \\ 8 & 13 & 21 & 34 & 55 \end{bmatrix} \). This matrix has rank 2 because the sum of any two neighboring columns equals the next column to the right. The Hadamard powers \( H^d \) apparently are singular only for \( d \in \{1, 2, 3\} \). Their condition numbers are shown in the right plot.

To substantiate these observations, we show that in a generic sense almost all non-integer Hadamard powers have maximal rank.

**Corollary 4.8.** Let \( \mathcal{A}_r \subset M_r \subset \mathbb{R}^{n \times m} \) be defined as in Theorem 4.5. If \( r \geq 2 \) and \( |A| \in \mathcal{A}_r \), then rank \( |A|^d = \min\{m, n\} \) except for at most a finite number of \( d > 0 \).

**Proof.** Without loss of generality let \( m \geq n \). From Theorem 4.5, we know that rank \(|A|^{0n} = n\). Hence there exists an \( n \times n \) submatrix \( \hat{A} = (\hat{a}_{ij}) \) of \( A \) such that \( f(d) = \det |\hat{A}|^d \) is nonzero for \( d = n \). The function \( f : [0, \infty[ \rightarrow \mathbb{R} \) is a multivariate polynomial in all \( |\hat{a}_{ij}|^d = e^{\log(|\hat{a}_{ij}|)d} \) for which \( \hat{a}_{ij} \neq 0 \). More precisely, we can write \( f(d) = \sum_{\ell=1}^{N} e^{\alpha_{\ell} d} \) with \( N \leq n! \) and coefficients \( \alpha_{\ell} \in \mathbb{R} \) given as sums of suitable \( \log(|\hat{a}_{ij}|) \). By [22, Problem 75 in Part V] such a function \( f \) has at most \( N - 1 \) real zeros. \( \square \)
5. Kernel perceptrons and Hadamard powers

We apply the results obtained in Sections 3 and 4 in the context of kernel perceptrons. For that purpose, we collect basic facts from [23]. A symmetric mapping \( k : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \) is called a positive kernel, if any \( m \in \mathbb{N} \) with \( m > 0 \) and any set of vectors \( x_1, \ldots, x_m \in \mathbb{R}^n \) yields a nonnegative definite matrix \( K_k = (k(x_i, x_j))_{i,j=1,\ldots,m} \), i.e., \( K_k \succeq 0 \). The most natural example of a positive kernel is the canonical scalar product \( \langle \cdot, \cdot \rangle \), where as previously we set

\[
K = K_{\langle \cdot, \cdot \rangle} = \langle (x_i, x_j) \rangle_{i,j=1,\ldots,m} = X^T X .
\]  

(10)

Given two positive kernels \( k_1 \) and \( k_2 \) with the matrices \( K_{k_1}, K_{k_2} \succeq 0 \) for a fixed set of vectors \( x_j \), the product \( k = k_1 k_2 \) is associated to the Hadamard product \( K_{k_1} \odot K_{k_2} \). Thus, by equation (2) the product of two positive kernels and any natural power of a positive kernel is again a positive kernel. Hence, also the polynomial mappings \( (v, w) \mapsto ((v, w) + c)^d \), with \( c \geq 0, d \in \mathbb{N} \) define positive kernels. In the following we deal with such polynomial kernels. To simplify the notation, we consider only the case \( c = 0 \), i.e., the kernels \( k(v, w) = (v, w)^d \) for which \( K_k = K^{\odot d} \) with \( K \) from (10). The results carry over to the case \( c > 0 \).

Let now a set of vectors \( \{x_1, \ldots, x_m\} \subset \mathbb{R}^n \) with corresponding labels \( y_1, \ldots, y_m \in \{\pm 1\} \) be fixed. In the language of pattern recognition, the \( x_j \) represent features in the feature space \( \mathbb{R}^n \).

A classification problem consists in finding a mapping \( f_c : \mathbb{R}^n \to \{\pm 1\} \), called a classifier, such that \( f_c(x_i) = y_i \) for \( i = 1, \ldots, m \). Each classifier \( f_c \) partitions the feature space into the classes \( f_c^{-1}(1) \) and \( f_c^{-1}(-1) \). A polynomial kernel perceptron of degree \( d \in \mathbb{N} \), \( d \geq 1 \) is a particular classifier defined by

\[
f_p : \mathbb{R}^n \to \{\pm 1\} \quad \text{that is of the form}
\]

\[
f_p(x) = \sigma \left( \sum_{i=1}^{m} \alpha_i y_i (x, x)^d + b \right) \quad \text{with} \quad \sigma(z) = \begin{cases} 1, & z \geq 0 \\ -1, & z < 0 \end{cases} .
\]

(11)

For \( d = 1 \) this is a classical perceptron. The function \( \sigma \) is the activation function, \( b \in \mathbb{R} \) is the bias, and \( f_p \) is also called transfer function of the (kernel) perceptron. The task is to construct the real parameters \( b \) and \( \alpha_i \), such that

\[
f_p(x_j) = y_j \quad \text{for all} \quad j = 1, \ldots, m .
\]

(12)

5.1. Boolean functions

In the particular case, where \( m = 2^n \) and \( \{x_1, \ldots, x_m\} = \{0,1\}^n \), the selection of labels \( y_i \in \{\pm 1\} \) defines a Boolean function \( f : \{0,1\}^n \to \{\pm 1\} \), \( f(x_i) = y_i \) in \( n \) variables. One says that a kernel perceptron realizes the Boolean function \( f \), if its transfer function \( f_p \) coincides with \( f \) on \( \{0,1\}^n \). The construction of a kernel perceptron can be reformulated as an optimization problem.
Theorem 5.1. The given task (12) is feasible, if and only if the constrained maximization problem

$$\max_{a \in \mathbb{R}^m} \left( \|a\|_1 - \frac{1}{2} \sum_{i,j=1}^m a_i a_j y_i y_j \langle x_i, x_j \rangle^d \right)$$  \hspace{1cm} (13a)$$

subject to \( \sum_{i=1}^m y_i a_i = 0, a_1, \ldots, a_m \geq 0 \) \hspace{1cm} (13b)

has a solution. In this case \( \alpha_i = a_i \) for \( i = 1, \ldots, m \) and

$$b = \frac{1}{2} \left( \min_{\{j \mid y_j = 1\}} \sum_{i=1}^m \alpha_i y_i \langle x_i, x_j \rangle^d + \max_{\{j \mid y_j = -1\}} \sum_{i=1}^m \alpha_i y_i \langle x_i, x_j \rangle^d \right)$$

are suitable.

Proof. See [23, Section 7.4].

Setting \( y = [y_1, \ldots, y_m] \in \mathbb{R}^{1 \times m}, Y = \text{diag}(y_1, \ldots, y_m) \in \mathbb{R}^{m \times m} \) and using \( K \) from (10), we can rewrite the sum in (13a) as

$$\sum_{i,j=1}^m a_i a_j y_i y_j \langle x_i, x_j \rangle^d = a^T Y K^{od} Y a .$$

With this notation we derive the following criterion.

Corollary 5.2. The constrained maximization problem (13) is infeasible if and only if

$$\exists a \in \mathbb{R}_+^m \setminus \{0\} : ya = \sum_{i=1}^m y_i a_i = 0 \text{ and } K^{od} Y a = 0 .$$ \hspace{1cm} (14)

Proof. If \( a \) satisfies (14), then for all \( t > 0 \) the vector \( ta \) satisfies the constraint (13b) while the target function in (13a) simplifies to \( \|ta\|_1 = t\|a\|_1 \). Since \( \|a\|_1 \neq 0 \), there is no maximum.

Conversely let \( K^{od} Y a \neq 0 \) for all \( a \neq 0 \) satisfying (13b). Since \( K^{od} \) is nonnegative definite, also \( a^T Y K^{od} Y a > 0 \) and by compactness

$$\mu = \min \{a^T Y K^{od} Y a \mid a \in \mathbb{R}_+^m, ya = 0, \|a\|_1 = 1\} > 0 .$$ \hspace{1cm} (15)

Hence, for all \( a \) satisfying (13b) the target function in (13a) is bounded by

$$\|a\|_1 - \frac{1}{2} a^T Y K^{od} Y a \leq \|a\|_1 - \frac{1}{2} \mu \|a\|_1^2 .$$

Thus, if \( a_0 \) is a minimizer for (15), then \( a = \frac{a_0}{\mu} \) solves (13).
Example 5.3. Let us illustrate this for the famous XOR-classification problem, where the vectors $x_i$ are given as the columns of the matrix $X = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ and the labels $y_i$ as the entries of the vector $y = \begin{bmatrix} -1 & 1 & 1 & -1 \end{bmatrix}$.

We compare the kernels $k(x, y) = \langle x, y \rangle^d$ for $d = 1$ and $d = 2$.

- If $d = 1$ then $K^{\circ 2}Y = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$ with $v = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \in \ker K^{\circ 2}Y$. Since $yv = 0$ and $\text{span}\{x_1, \ldots, x_4\} = \mathbb{R}^2$, this implies that the XOR function cannot be realized by a classical perceptron. Note that here rank $K^{\circ 2}Y = 2$.

- If $d = 2$, then $K^{\circ 2}Y = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$. It is evident that rank $K^{\circ 2}Y = 3$ and $\ker K^{\circ 2}Y = \text{span}\{e_1\}$ with the first canonical unit vector $e_1$. Thus $K^{\circ 2}Yv \neq 0$ for all $v \neq 0$ with $yv = 0$. In fact, with $k(v, w) = \langle v, w \rangle^2$ the XOR-function is realized by (11) with $\alpha_1 = 6$, $\alpha_2 = \alpha_3 = 4$, $\alpha_4 = 2$, and $b = -1$.

By Corollary 5.2 and the argument in Example 5.3, there exists a kernel perceptron

$$f_p(x) = \sigma \left( \sum_{i=1}^m \alpha_i y_i \langle x_i, x \rangle^d + b \right),$$

such that $f_p(x_j) = y_j$ for all $j = 0, \ldots, 2^n - 1$, if

$$\text{rank} \left( \langle x_i, x_j \rangle^d \right)_{i,j=1}^{2^n-1} = \text{rank}(X^TX)^{\circ d} = 2^n - 1. \quad (16)$$

More generally, it is known that any Boolean function in $n$ variables can be realized by a kernel perceptron with $k(v, w) = \langle v, w \rangle^n$ (e.g., [26]). This is now a direct consequence of Theorem 3.2.

Corollary 5.4. Any Boolean function in $n$ variables can be realized by a perceptron with polynomial kernel of degree $n$.

For $d < n$ there exists a Boolean function that cannot be realized by a perceptron with polynomial kernel of degree $d$.

Proof. Since rank $K^{\circ n} = 2^n - 1$ by Theorem 3.2, the null space of $K^{\circ n}Y$ contains only scalar multiples $a = te_1$ of the first canonical unit vector. But then $ya = ty_1 \neq 0$, if $a \neq 0$, i.e., (14) does not hold.

If $d < n$, then rank $K^{\circ d} < 2^n - 1$ by Theorem 3.2. Hence $\ker Y K^{\circ d}$ contains the unit vector $e_1$ and another vector $v \in \mathbb{R}^m$ with first component $v_1 = 0$. We define labels $y_1 = -\sigma \left( \sum_{i=2}^m v_i \right)$ and $y_j = \sigma(v_j)$ for $j > 1$. Then for all $t \geq 0$, the vector $a_t = Yv + te_1 \in \mathbb{R}^n_+ \setminus \{0\}$ is contained in the kernel of $K^{\circ d}Y$. Moreover $ya_0 = \sum_{i=2}^m a_i y_i = \sum_{i=2}^m v_i$. If this is zero, then (14) holds for $a = a_0$. Otherwise, by construction of $y_1$, for large $t$ the signs of $ya_0$ and $ya_t$ differ. Hence there exists $t_0 > 0$, such that $a = a_{t_0}$ satisfies (14). \qed
5.2. General classification in \( \mathbb{R}^n \)

If we interpret a matrix \( A = [x_1, \ldots, x_m] \in \mathbb{R}^{n \times m} \) considered in Section 4 as a set of \( m \) sample vectors \( x_j \in \mathbb{R}^n \), we can state the following consequence of Theorem 4.5.

**Corollary 5.5.** Let \( m \geq n \). There is an open and dense subset \( A \subset \mathbb{R}^{n \times m} \), such that for all \( A \in A \), all choices of labels \( y_j \in \{-1, 1\} \), \( j = 1, \ldots, m \), and all \( d \) satisfying \( \binom{n+d-1}{d} \geq m \), there exists a classifier (11) with \( k(v, w) = \langle v, w \rangle^d \) solving the task (12).

**Proof.** Let \( A = \{ A \in \mathbb{R}^{n \times m} \mid \binom{n+d-1}{d} \leq m \Rightarrow H_A^d \text{ is in general position} \} \). By Lemma 4.3, the set \( A \) is open and dense in \( \mathbb{R}^{n \times m} \). In particular rank \( A = n \) for \( A \in A \). Therefore, as in the proof of Theorem 4.5, it follows that

\[
K^{od} = (A^T A)^{od} = \sum_{\ell_1, \ldots, \ell_d=1}^r (A_{\ell_1} \circ \cdots \circ A_{\ell_d})(A_{\ell_1} \circ \cdots \circ A_{\ell_d})^T
\]

is nonsingular, if \( \binom{n+d-1}{d} \geq m \). By Corollary 5.2 the task is feasible.

**Remark 5.6.** Usually, with a positive kernel \( k \), one associates a reproducing Hilbert space \( \mathcal{R}_k \) as the new feature space. The previous result is consistent with \( \dim \mathcal{R}_k = \binom{n+d-1}{d} \) for \( k(\cdot, \cdot) = \langle \cdot, \cdot \rangle^d \), see [23]. One also says that every \( A \in A \) is shattered by the set of kernel perceptrons with this \( k \).

**Example 5.7.** From Section 3, recall the matrix \( X \in \mathbb{R}^{n \times m} \) of Boolean vectors, where \( m = 2^n \). We have shown that this set of vectors is shattered by polynomial kernel perceptrons of degree \( d = n \). For arbitrarily small perturbations of \( X \) it suffices to choose the minimal \( d = d_n \), so that \( \binom{n+d-1}{d} \geq 2^n \). For instance, we have \( d_{10} = 5, d_{20} = 8, d_{40} = 14, d_{80} = 26, d_{160} = 50 \).

6. Conclusion

The main goal of this note was to point out a relation between kernel perceptrons with polynomial kernel and the rank of Hadamard powers. This relation provided us with a new way of deriving some known results on the capacity of such perceptrons. Moreover, we computed the generic rank of Hadamard powers directly without resorting to abstract results on algebraic varieties.

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