The inner boundary of random walk range

Izumi Okada

Abstract

In this paper, we deal with the inner boundary of random walk range, that is, the set of those points in a random walk range which have at least one neighbor site outside the range. If $L_n$ be the number of the inner boundary points of random walk range in the $n$ steps, we prove $\lim_{n \to \infty} \frac{L_n}{n}$ exists with probability one. Also, we obtain some large deviation result for transient walk. We find that the expectation of the number of the inner boundary points of simple random walk on two dimensionnal square lattice is of the same order as $\frac{n}{(\log n)^2}$.

1 Introduction and Known results

Let $d$ be a positive integer and $X_1, X_2, \ldots$ be i.i.d. $\mathbb{Z}^d$-valued random variables, and put $S_k = S_0 + \sum_{i=1}^k X_i$ with some constant $S_0$, a random walk taking values in $\mathbb{Z}^d$ started from $S_0$. Let $P^a$ denote the probability law of the walk such that $S_0 = a$ a.s., and we simply write $P$ for $P^0$. Let $R_n$ be the cardinality of the range of the walk of length $n$. Namely, $R_n$ is the number of distinct points visited by the walk in the first $n$ steps. Many results of the asymptotic behavior of $R_n$ as $n \to \infty$ have been obtained by various authors. It was shown by Spitzer [12], pp 38 – 40 that for all random walks of any dimension,

$$\lim_{n \to \infty} \frac{R_n}{n} = p \text{ a.s.}$$

where $p = P(0 \notin \{S_k\}_{k=1}^\infty)$.

For simple random walk in three dimensions the following results are shown: by Dvoretzky and Erdős [3]

$$ER_n = pn + O(\sqrt{n}),$$

and, by Jain and Pruitt [9]

$$E[(R_n - ER_n)^4] = O(n^2(\log n)^2),$$

$$\frac{R_n - ER_n}{n^{1/2} \log n} \to cN,$$

where $c$ is a constant, $N$ is the standard normal distribution, and the convergence is in the sense of distribution. Also, for simple random walk in two dimensions it was shown by Jain and Pruitt [8, 10] that

$$ER_n = \pi \frac{n}{\log n} + O\left(\frac{n}{(\log n)^2}\right),$$

$$\text{Var}(R_n) = O\left(\frac{n^2}{(\log n)^4}\right).$$
The large deviations of $R_n$ are studied by Donsker and Varadhan \cite{2} and Hamana and Kesten \cite{6}. In \cite{6} it is shown that for any random walk on $d \geq 2$
\[
\psi_0(x) := \lim_{n \to \infty} - \frac{1}{n} \log P(R_n \geq nx)
\]
exists for all $x$, and $\psi_0(\cdot)$ has the following properties:
- $\psi_0(x) = 0$ for $x \leq p$,
- $0 < \psi_0(x) < \infty$ for $p < x \leq 1$,
- $\psi_0(x) = \infty$ for $1 < x$,
- $\psi_0$ is continuous on $x \in [0, 1]$,
- $\psi_0$ is convex on $x \in [0, 1]$, and
- $\psi_0$ is strictly increasing on $x \in [p, 1]$.

The assumption imposed in \cite{6} on the walk, is only the irreducibility condition:
\begin{equation}
\text{the group generated by the support of } X \text{ is all of } \mathbb{Z}^d.
\end{equation}

Next we describe the known result about the multiple points of random walk range. Let $Q_n^{(p)}$ the number of the strictly $p$-multiple points of random walk range in the $n$ steps. That is,
\[
Q_n^{(p)} = |\{S_i : 0 \leq i \leq n, \exists \{m : 0 \leq m \leq n, S_m = S_i\} = p\}|
\]

Then, it is shown by Flatto \cite{5} that for simple random walk on $d = 2$
\[
\frac{(\log n)^2 Q_n^{(p)}}{n} \to \pi^2 \quad a.s..
\]

In this paper, we deal with the inner boundary of random walk range. Let $L_n$ be the number of the inner boundary points of random walk range in the $n$ steps (see the next section for the definition). The lower bound of the expectation of the number of the inner boundary points is known by \cite{1} Lemma 5. More precisely, it was shown that for $d \geq 2$ there exists a constant $C_d > 0$ such that for all $n \geq 1$,
\[
EL_n \geq \frac{C_d n}{(\log n)^2} \quad d = 2,
\]
\[
EL_n \geq C_d n \quad d \geq 3.
\]

In \cite{1}, it is noticed that the entropy of random walk is essentially governed by the size of the boundary of the trace. In this paper, we consider the asymptotic behavior of number of the inner boundary points and obtain analogues for $L_n$ of those results that are mentioned above.

\section{Framework and Main Results}

\subsection{Framework}

We consider the $\mathbb{Z}^d$-valued random walk ($d \geq 1$) $S_n$ described in the introduction. Let $z$, $a$, $a_i$, $i \geq 0$ denote points in $\mathbb{Z}^d$. A neighbor of $a$ is a point $z$ that satisfies $\text{dist}(a, z) = 1$. Let $\mathcal{N}(a)$ denote the set of all neighbors of $a$:
\[
\mathcal{N}(a) = \{z : \text{dist}(a, z) = 1\}.
\]
So we may write \( \{a_0, a_1, ..., a_l\} \supset \mathcal{N}(a) \) if every neighbor of \( a \) is in \( \{a_0, a_1, ..., a_l\} \), and \( \{a_0, a_1, ..., a_l\} \not\supset \mathcal{N}(a) \) if not. The inner boundary of random walk range \( \{S_m\}_{m=0}^n \), denoted by \( H_n \), is defined by

\[
H_n = \{ S_i : 0 \leq i \leq n, \{S_m\}_{m=0}^n \not\supset \mathcal{N}(S_i) \}.
\]

Let \( L_n \) be the cardinality of the inner boundary by \( n \) steps, that is,

\[
L_n = \#H_n,
\]

where \( \#A \) denote the cardinality of \( A \). Let \( J'_n \) be the number of the \( p \)-multiple points in the inner boundary of random walk range in \( n \) steps, and \( J^{(p)}_n \) be the number of the strictly \( p \)-multiple points in the inner boundary of random walk range in the \( n \) steps. That is

\[
J'_n = \#\{ S_i \in H_n : \#\{ m : 0 \leq m \leq n, S_m = S_i \} \geq p \},
\]

\[
J^{(p)}_n = \#\{ S_i \in H_n : \#\{ m : 0 \leq m \leq n, S_m = S_i \} = p \}.
\]

### 2.2 Main Results

For \( i \geq 1 \) let \( \{S'_m\}_{m=0}^\infty \) be independent copy of \( \{S_m\}_{m=0}^\infty \), and define \( T_{n,i} = \inf\{ m \geq 1 : S'^i_m = a \} \) and \( T_n = \inf\{ m \geq 1 : S_m = a \} \), the corresponding passage times. Let \( \{S''_m\}_{m=0}^\infty \) denote an independent dual walk of \( \{S_m\}_{m=0}^\infty \), namely an independent copy of \( \{-S_m\}_{m=0}^\infty \).

**Theorem 2.1.** For any random walk and \( d \geq 1 \),

\[
\lim_{n \to \infty} \frac{L_n}{n} = q \quad \text{a.s.,}
\]

where

\[
q = P(\{S_m\}_{m=0}^\infty \cup \{S'_m\}_{m=0}^\infty \not\supset \mathcal{N}(0) \text{ and } 0 \notin \{S_m\}_{m=1}^\infty ).
\]

**Theorem 2.2.** For any random walk and \( d \geq 1 \),

\[
\lim_{n \to \infty} \frac{J^{(p)}_n}{n} = P(\{S_m\}_{m=0}^\infty \cup \{S'_m\}_{m=0}^\infty \cup (\cup_{i=1}^{p-1} \{S''_m\}_{m=0}^{T_{n,i}}) \not\supset \mathcal{N}(0),
\]

\[
0 \notin \{S_m\}_{m=1}^\infty \cup \{S'_m\}_{m=1}^\infty \text{ and } 0 \in \{S''_m\}_{m=1}^\infty \text{ for } i = 1, \ldots, p - 1 \} \quad \text{a.s.,}
\]

\[
\lim_{n \to \infty} \frac{J'_n}{n} = P(\{S_m\}_{m=0}^\infty \cup \{S'_m\}_{m=0}^\infty \cup (\cup_{i=1}^{p-1} \{S''_m\}_{m=0}^{T_{n,i}}) \not\supset \mathcal{N}(0),
\]

\[
0 \notin \{S_m\}_{m=1}^\infty \text{ and } 0 \in \{S''_m\}_{m=1}^\infty \text{ for } i = 1, \ldots, p - 1 \} \quad \text{a.s.}
\]

**Theorem 2.3.** For any random walk on \( d \geq 2 \) which satisfies (7),

\[
\psi(x) := \lim_{n \to \infty} \frac{-1}{n} \log P(L_n \geq nx) \quad \text{exists}
\]

(2)

for all \( x \), and \( \psi(\cdot) \) has the following properties:

\[
\psi(x) = 0 \quad \text{for } x \leq q, \quad \text{(3)}
\]

\[
0 < \psi(x) < \infty \quad \text{for } q < x \leq 1, \quad \text{(4)}
\]

\[
\psi(x) = \infty \quad \text{for } 1 < x, \quad \text{(5)}
\]

\[
\psi \text{ is continuous on } x \in [0, 1], \quad \text{(6)}
\]

\[
\psi \text{ is convex on } x \in [0, 1], \text{ and} \quad \text{(7)}
\]

\[
\psi \text{ is strictly increasing on } x \in [q, 1]. \quad \text{(8)}
\]
We call the random walk \( \{S_n\} \) simple if \( P[S_1 = b_j] = 1/2d \) where \( b_j, j \in \{\pm 1, \ldots, \pm d\} \) are neighbors of the origin in the square lattice \( \mathbb{Z}^d \).

**Theorem 2.4.** Let \( d = 2 \) and \( p \geq 1 \) and suppose the random walk to be simple. Then

\[
\lim_{n \to \infty} E L_n \times \frac{(\log n)^2}{n}, \quad \lim_{n \to \infty} E J_n^{(p)} \times \frac{(\log n)^2}{n}, \quad \lim_{n \to \infty} E J_n^p \times \frac{(\log n)^2}{n}
\]

exist. Moreover, it holds that

\[
\frac{\pi^2}{2} \leq \lim_{n \to \infty} E L_n \times \frac{(\log n)^2}{n} \leq 2\pi^2, \quad (9)
\]

\[
\frac{\bar{c}^{p-1}\pi^2}{4} \leq \lim_{n \to \infty} E J_n^{(p)} \times \frac{(\log n)^2}{n} \leq \bar{c}^{p-1}\pi^2, \quad (10)
\]

\[
\frac{\bar{c}^{p-1}\pi^2}{2} \leq \lim_{n \to \infty} E J_n^p \times \frac{(\log n)^2}{n} \leq 2\bar{c}^{p-1}\pi^2, \quad (11)
\]

where \( \bar{c} = P(T_0 < T_b) \) with some \( b \in \mathcal{N}(0) \).

## 3 Proof

### 3.1 Proof of Theorem 2.1

Let \( \{Z_n\}_{n \in \mathbb{Z}} \) be a sequence of random variables defined by \( Z_0 = 0, \{Z_n\}_{n=1}^\infty = \{S_n\}_{n=1}^\infty \), and \( \{Z_{-n}\}_{n=1}^\infty = \{S_n\}_{n=1}^\infty \), where \( \{S_n\}_{n=1}^\infty \) is independent dual walk of \( \{S_n\}_{n=1}^\infty \). We suppose \( \{Z_n\} \) to be a canonical realization, so that \( P \) is the probability measure on the product space \( (\Pi_n^{\infty} \Omega_n, \mathcal{F}) \) such that \( Z_n \) is the coordinate map from \( \Omega = \Pi_n^{\infty} \Omega_n \) into \( \Omega_n \), where \( \Omega_n \)'s are copies of \( \mathbb{Z}^d \) and \( \mathcal{F} = \sigma(\{Z_n\}_{n \in \mathbb{Z}}) \). Let \( \phi \) be the usual shift operator: \( \phi : \Omega \to \Omega \) and \( Z_n \circ \phi = Z_{n+1} \). Let \( \phi^m \) be the \( m \) times iterate of \( \phi \): formally \( \phi^0(\omega) = \omega \) and \( \phi^m = \phi \circ \phi^{m-1} \) (\( m \geq 1 \)). Since \( \phi \) is \( P \)-measure preserving, by the ergodic theorem it holds that for any \( A \in \mathcal{F} \)

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{m=0}^{n-1} 1_A(\phi^m \omega) = P(A) \quad \text{a.s.} \quad (12)
\]

**Proof of Theorem 2.1.** Let \( A \) be the event that \( \{S_n\}_{m=0}^\infty \cup \{S'_m\}_{m=0}^\infty \not\subset \mathcal{N}(0) \) and \( 0 \not\subset \{S_m\}_{m=1}^\infty \). In terms of \( Z_m \), \( A \) is expressed as \( \{Z_m\}_{m \in \mathbb{Z}} \not\subset \mathcal{N}(Z_0) \) and \( Z_0 \not\subset \{Z_m\}_{m=1}^\infty \). Note that we can write

\[
L_n = \#\{m : 0 \leq m \leq n, \{S_t\}_{t=0}^m \not\subset \mathcal{N}(S_m), S_m \not\subset \{S_t\}_{t=m+1}^n\}. \quad (13)
\]

Then

\[
L_n \geq \sum_{m=0}^{n} 1_A(\phi^m \omega)
\]

since the right hand side equals

\[
\#\{m : 0 \leq m \leq n, \{S_t\}_{t=0}^m \cup \{S'_t\}_{t=0}^m \not\subset \mathcal{N}(S_m), S_m \not\subset \{S_{m+1}\}_{t=m+1}^n\}.
\]
Noting that $A \in \mathcal{F}$, we apply (12) to see

$$\liminf_{n \to \infty} \frac{L_n}{n} \geq P(A) \quad a.s.$$  \hspace{1cm} (14)

To prove the inequality in opposite direction, let $A_k$ be the event that $\{Z_m\}_{m=0}^k \cup \{Z_m\}_{m=-k}^0 \not\subseteq \mathcal{N}(Z_0)$ and $Z_0 \notin \{Z_m\}_{m=1}^k$. Then, in view of (12) we obtain that for any $k < \infty$

$$L_n \leq 2k + \sum_{m=k}^{n-k} 1_{A_k}(\phi^m \omega)$$

since the sum on the right hand side equals

$$\{m : k \leq m \leq n - k, \{S_m\}_{i=k}^n \not\subseteq \mathcal{N}(S_m), S_m \notin \{S_m\}_{i=1}^k\}.$$ 

As before an application of (12) shows

$$\limsup_{n \to \infty} \frac{L_n}{n} \leq P(A_k) \quad a.s..$$

Since $\cap A_k \to A$, we now conclude

$$\limsup_{n \to \infty} \frac{L_n}{n} \leq P(A) \quad a.s..$$  \hspace{1cm} (15)

By (14) and (15) the proof is complete. \hfill \square

**Remark 3.1.** We can rewrite Theorem 2.1 more generally. For any two finite sets $H_j \subset H \subset \mathbb{Z}^d$ for $j = 1, 2, ..., N$, let

$$L'_n = \mathbb{1} \cup_{j=1}^N \{S_i : 0 \leq i \leq n, \{S_m\}_{m=0}^n \cap (S_i + H) = (S_i + H_j)\}.$$ 

By the same argument as in the proof of Theorem 2.1, we can deduce that

$$\lim_{n \to \infty} \frac{L'_n}{n} = P((\{S_m\}_{m=0}^\infty \cup \{S'_m\}_{m=0}^\infty) \cap H = H_j \text{ for some } j = 1, 2, ..., N, 0 \notin \{S_m\}_{m=1}^\infty) \quad a.s..$$

**Proof of Theorem 2.2:** First, we prove the upper bound of the first formula. Note that if $l = \{l_i\}_{i=1}^p$ and $G_l$ is the event that

$$\{S_m\}_{m=0}^\infty \not\subseteq \mathcal{N}(S_{l_1}), S_{l_1} = S_{l_2} = ... = S_{l_p} \quad \text{and} \quad S_{l_1} \not\subseteq (\{S_m\}_{m=0}^\infty - \{S_{l_i}\}_{i=1}^p),$$

then for any $n \geq (p - 1)k$

$$J_n^{(p)} \geq \sum_{l_1=0}^{n-(p-1)k} \sum_{l_{1}+k}^{l_{2}+k} \sum_{l_{2}+k}^{l_{3}+k} ... \sum_{l_{p-1}+k}^{l_{p}+k} 1_{G_l}(\omega).$$

So it holds that if $h = \{h_i\}_{i=2}^p$ and $G'_h$ is the event that $\{Z_m\}_{m \in \mathbb{Z}} \not\subseteq \mathcal{N}(Z_0), Z_0 = Z_{h_2} = ... = Z_{h_p}$ and $Z_0 \notin (\{Z_m\}_{m \in \mathbb{Z}} - (\{Z_{h_i}\}_{i=2}^p \cup Z_0))$, then

$$J'_n^{(p)} \geq \sum_{l_1=0}^{n-(p-1)k} \sum_{h_2=1}^{h_2+k} \sum_{h_3=h_2+1}^{h_3+k} ... \sum_{h_{p-1}+k}^{h_{p-1}+k} 1_{G'_h}(\phi^h \omega).$$
Noting that $G'_h \in \mathcal{F}$, by (16) we get for any $k < \infty$

$$ \liminf_{n \to \infty} \frac{J_n^{(p)}}{n} \geq P(T^p_{Z_0} - T^p_{Z_0} \leq k \text{ for } i = 1, \ldots, p - 1, T^p_{Z_0} = \infty, \quad (16) $$

\{Z_m\}_{m \in \mathbb{Z}} \not\subset \mathcal{N}(Z_0) \text{ and } Z_0 \not\subset \{Z_m\}_{m = -\infty},

where $T^p_a = \inf\{m > T^p_{Z_0} : Z_m = a\}$, and $T^0_a = 0$. Therefore, by the strong Markov property we get

$$ \liminf_{n \to \infty} \frac{J_n^{(p)}}{n} \geq P(T^p_{Z_0} < \infty, T^p_{Z_0} = \infty, \{Z_m\}_{m \in \mathbb{Z}} \not\subset \mathcal{N}(Z_0) \text{ and } Z_0 \not\subset \{Z_m\}_{m = -\infty}. $$

By (16) and (17) we get the one side inequality. To prove the inequality in opposite direction, note that

$$ J_n^{(p)} \leq 2k + \sum_{l_1 = k}^{n} \sum_{l_2 = l_1 + 1}^{\infty} \cdots \sum_{l_p = l_{p-1} + 1}^{\infty} 1_{G_{l,k}}(\omega), $$

where $G_{l,k}$ is the event that $\{S_m\}_{m = l_1 - k}^{l_p} \not\subset \mathcal{N}(S_{l_1}), S_{l_1} = S_{l_2} = \ldots = S_{l_p}$, and $S_{l_1} \not\subset \{S_m\}_{m = l_1 - k}^{l_1}$ and $S_{l_1} \not\subset \{S_m\}_{m = l_1 - k}^{l_1}$. Hence, if we set $h = \{h_i\}_{i = 2}^p$, then

$$ J_n^{(p)} \leq 2k + \sum_{l_1 = k}^{n} \sum_{h_2 = l_1 + 1}^{\infty} \cdots \sum_{h_p = h_{p-1} + 1}^{\infty} 1_{G'_{h,k}}(\phi^l(\omega),$$

where $G'_{h,k}$ is the event that $\{Z_m\}_{m = -\infty}^{h_p+k} \not\subset \mathcal{N}(Z_0), Z_0 = Z_{k_2} = \ldots = Z_{h_p}$, and $Z_0 \not\subset \{Z_m\}_{m = -\infty}^{h_p+k} - \{Z_{h_i}\}_{i = 2}^p \cup Z_0\}. $ If we note that $G'_{h,k} \in \mathcal{F}$, by (12) we get for any $k < \infty$

$$ \limsup_{n \to \infty} \frac{J_n^{(p)}}{n} \leq P(T^p_{Z_0} < \infty, T^p_{Z_0} = \infty, \{Z_m\}_{m \in \mathbb{Z}} \not\subset \mathcal{N}(Z_0) \text{ and } Z_0 \not\subset \{Z_m\}_{m = -\infty}. $$

By the monotonicity in $k$, we find that as $k \to \infty$ the right hand side of the last formula converges to

$$ P(T^p_{Z_0} < \infty, T^p_{Z_0} = \infty, \{Z_m\}_{m \in \mathbb{Z}} \not\subset \mathcal{N}(Z_0) \text{ and } Z_0 \not\subset \{Z_m\}_{m = -\infty}. $$

By (16), (17), (18) and (19), the proof of the first formula is complete. Next we prove the second formula. Note that it holds that for any $n \geq (p - 1)k$

$$ J_n^p \geq \sum_{l_1 = 0}^{n-(p-1)k} \sum_{l_1 = k_1 + 1}^{l_1+k} \sum_{l_2 = l_1 + 1}^{l_1+2k} \cdots \sum_{l_p = l_{p-1} + 1}^{l_p+k} 1_{\tilde{G}_t}(\omega), $$

$$ J_n^p \leq 2k + \sum_{l_1 = k}^{n} \sum_{l_2 = l_1 + 1}^{\infty} \cdots \sum_{l_p = l_{p-1} + 1}^{\infty} 1_{\tilde{G}_{l,k}}(\omega), $$

where $\tilde{G}_t$ is the event that $\{S_m\}_{m = 0}^{l_p} \not\subset \mathcal{N}(S_{l_1}), S_{l_1} = S_{l_2} = \ldots = S_{l_p}$ and $S_{l_1} \not\subset \{S_m\}_{m = l_1+1}^{l_1} - \{S_{l_1}\}_{i = 2}^p$, and $\tilde{G}_{l,k}$ is the event that $\{S_m\}_{m = l_1 - k}^{l_p+k} \not\subset \mathcal{N}(S_{l_1}), S_{l_1} = S_{l_2} = \ldots = S_{l_p}$ and $S_{l_1} \not\subset \{S_m\}_{m = l_1+1}^{l_1} - \{S_{l_1}\}_{i = 2}^p$. Since the rest of proof of the second formula is the same as the first one, we omit it. □
3.2 Proof of Theorem 2.3

Lemma 3.1. There exist constants $\zeta \in (0,1)$, $c < \infty$ (depending only on $d$) such that for all integer $n,m \geq 0$ and $y,z \in [0,\infty)$, it holds that

$$P(L_{n+m} \geq y + z - c(nm)^{\frac{1}{d+1}}) \geq \frac{1}{2} \zeta^{d(nm)^{\frac{1}{d+1}} + d} P(L_n \geq y) P(L_m \geq z).$$

Proof. Let $\hat{X}_1, \hat{X}_2, \ldots$ be an independent copy of $X_1, X_2, \ldots$, $\hat{S}_0 = 0$, $\hat{S}_k = \sum_{i=1}^k \hat{X}_i$, and $\hat{L}_n$ be the number of the inner boundary points of $\{\hat{S}_0, \hat{S}_1, \ldots, \hat{S}_n\}$, that is,

$$\hat{L}_n = \# \{ \hat{S}_i : 0 \leq i \leq n, \{ \hat{S}_m \}_{m=0}^n \not\in \mathcal{N}(\hat{S}_i) \}.$$

We define $L\{a_1, \ldots, a_l\}$ to be the cardinality of the inner boundary of $\{a_i\}_{i=1}^l$, i.e.,

$$L\{a_1, \ldots, a_l\} = \# \{ a_i : 1 \leq i \leq l, \{ a_k \}_{k=1}^l \not\in \mathcal{N}(a_i) \},$$

and $U\{a_1, \ldots, a_l\}$ to be the union of the outer boundary and the inner boundary of the range of $\{a_i\}_{i=1}^l$, i.e.,

$$U\{a_1, \ldots, a_l\} = \{ a_i : 1 \leq i \leq l, \{ a_k \}_{k=1}^l \not\in \mathcal{N}(a_i) \} \cup \{ x \in \mathbb{Z}^d : x \not\in \{ a_k \}_{k=1}^l \text{ and there exists } y \in \{ a_k \}_{k=1}^l \text{ such that } \text{dist}(x,y) = 1 \}.$$

Moreover, we define

$$U[a, b] = U\{S_a, S_{a+1}, \ldots, S_b\}, \quad \hat{U}[a, b] = U\{\hat{S}_a, \hat{S}_{a+1}, \ldots, \hat{S}_b\}.$$

Next we define for $\lambda \in \mathbb{Z}^d$

$$N_{n,m}(\lambda) = \# \{ u \in \mathbb{Z}^d : u \in U[0,n] \text{ and } u \in S_n + \lambda + \hat{U}[0,n] \}.$$

For any fixed integers $p \geq 0$ and $n \geq 0$, consider the random walk defined by

$$T_k = \begin{cases} S_k & (k \leq n + p) \\ S_{n+p} + \hat{S}_{k-n-p} & (k > n + p). \end{cases}$$

Of course, $\{T_k\}_{k=0}^l$ has the same distribution as $\{S_k\}_{k=0}^l$, and hence also $P(L_{n+p+m} \geq l) = P(L\{T_0, \ldots, T_{n+p+m-1}\} \geq l)$. We claim that on the event

$$\{ S_{n+p} - S_n = \lambda \},$$

it holds that

$$L\{T_0, \ldots, T_{n+p+m-1}\} \geq L_n + \hat{L}_m - N_{n,m}(\lambda),$$

and

$$N_{n,m}(\lambda) = N_{n,m}(T_{n+p} - T_n) = \# U\{T_0, \ldots, T_{n-1}\} \cap U\{T_{n+p}, \ldots, T_{n+p+m-1}\}.$$ 

Owing to the assumption (II), we can pick $d$ linearly independent vectors $v_1, \ldots, v_d \in \mathbb{Z}^d$ for which $P(X = v_i) > 0$. We can then choose $0 < \zeta < 1$ such that $P(X = v_i) \geq \zeta$. We set

$$\Xi_q = \{ \sum_{i=1}^d k_i v_i : 0 \leq k_i \leq q \} \subset \mathbb{Z}^d.$$
For any $\lambda = \sum_{i=1}^{d} k_i v_i \in \Xi_\varrho$, we then have that for $p = p(\lambda) = \sum_{i=1}^{d} k_i \le d\varrho$, 

$$P(S_{n+p} - S_n = \lambda) = P(S_p = \lambda) \ge \zeta^p \ge \zeta^{d\varrho}.$$ 

Moreover, 

$$\#\Xi_\varrho = (\text{number of vectors } \omega \in \Xi_\varrho) = (q+1)^d.$$ 

We take 

$$q = q(n,m) = \lceil (nm)^{\frac{1}{d+1}} \rceil,$$

where $[a]$ denotes the smallest integer $\ge a$. Note that the simple monotonicity in $n$ of $P(L_n \ge y)$ does not hold, that is, it does not hold for any $n, y, v > 0$ $P(L_{n+v} \ge y) \ge P(L_n \ge y)$. But it holds that for any $n, y, v > 0$ 

$$P(L_{n+v} \ge y - 2dv) \ge P(L_n \ge y). \quad (22)$$

As a result of (21) and (22) for any $c < \infty$ and each $\lambda \in \Xi_\varrho$

$$P(L_{n+m} \ge y + z - c(nm)^{\frac{1}{d+1}})$$

$$\ge P(L_{n+dq+m} \ge y + z - (c-d)(nm)^{\frac{1}{d+1}})$$

$$\ge P(L_{n+p+m} \ge y + z - (c-d-2d^2)(nm)^{\frac{1}{d+1}})$$

$$\ge P(L_n \ge y, \hat{L}_m \ge z, S_{n+p} - S_n = \lambda, N_{n,m}(\lambda) \le \frac{1}{4d}(c-d-2d^2)(nm)^{\frac{1}{d+1}}).$$

The event (20) depends only on $X_i$ with $n < i \le n + p$, and is independent of the events $\{L_n \ge y\}, \{\hat{L}_m \ge z\}$ and of random variable $N_{n,m}(\lambda)$. Consequently, 

$$P(L_{n+m} \ge y + z - c(nm)^{\frac{1}{d+1}})$$

$$\ge P(S_{n+p} - S_n = \lambda) P(L_n \ge y, \hat{L}_m \ge z, N_{n,m}(\lambda) \le \frac{1}{4d}(c-d-2d^2)(nm)^{\frac{1}{d+1}})$$

$$\ge \zeta^{d\varrho} P(L_n \ge y, \hat{L}_m \ge z, N_{n,m}(\lambda) \le \frac{1}{4d}(c-d-2d^2)(nm)^{\frac{1}{d+1}}).$$

Since this inequality holds for all $\lambda \in \Xi_\varrho$, we can take its average over $\Xi_\varrho$ to obtain 

$$P(L_{n+m} \ge y + z - c(nm)^{\frac{1}{d+1}})$$

$$\ge \frac{\zeta^{d\varrho}}{|\Xi_\varrho|} \sum_{\lambda \in \Xi_\varrho} P(L_n \ge y, \hat{L}_m \ge z, N_{n,m}(\lambda) \le \frac{1}{4d}(c-d-2d^2)(nm)^{\frac{1}{d+1}})$$

$$= \frac{\zeta^{d\varrho}}{|\Xi_\varrho|} \mathbb{E}[\#\{\lambda \in \Xi_\varrho : N_{n,m}(\lambda) \le \frac{1}{4d}(c-d-2d^2)(nm)^{\frac{1}{d+1}}\} I_{\{L_n \ge y\}} I_{\{\hat{L}_m \ge z\}}]. \quad (23)$$

We shall shortly show that there exists $c < \infty$ such that for all integer $n, m \ge 0$

$$\#\{\lambda \in \Xi_\varrho : N_{n,m}(\lambda) \le \frac{1}{4d}(c-d-2d^2)(nm)^{\frac{1}{d+1}}\} \ge \frac{1}{2}(q+1)^d. \quad (24)$$
Taking this for granted and recalling that \( \{L_n \geq y\} \) and \( \{\hat{L}_m \geq z\} \) are independent, if (24) is true, then we infer from (23) that

\[
P(L_{n+m} \geq y + z - c(nm)^{\frac{1}{d+1}})
\]

\[
\geq P(L_{n+d+q+m} \geq y + z - (c - d)(nm)^{\frac{1}{d+1}})
\]

\[
\geq \frac{c^{dq}}{(q + 1)^d} \frac{1}{2} (q + 1)^d P(L_n \geq y) P(L_{m} \geq z)
\]

\[
\geq \frac{1}{2} c^{dq} P(L_n \geq y) P(L_{m} \geq z).
\]

(25)

which implies the inequality of the lemma. It remains to prove (24). We have

\[
\sum_{\lambda \in \mathbb{Z}^d} N_{n,m}(\lambda) \leq \sum_{\lambda \in \mathbb{Z}^d} N_{n,m}(\lambda)
\]

\[
= \sum_{u \in \Xi_q} \sum_{\lambda \in \mathbb{Z}^d} I[u \in U \{S_0, S_1, ..., S_{n-1}\}] \times I[u \in U \{S_n + \lambda + \{\hat{S}_0, \hat{S}_1, ..., \hat{S}_{n-1}\}\}]
\]

\[
= \sum_{u \in \Xi_q} I[u \in U \{S_0, S_1, ..., S_{n-1}\}] \times \sum_{\lambda \in \mathbb{Z}^d} I[\lambda \in U \{S_n + u + \{\hat{S}_0, \hat{S}_1, ..., \hat{S}_{n-1}\}\}]
\]

\[
= \sum_{u \in \Xi_q} I[u \in U \{S_0, S_1, ..., S_{n-1}\}] \times \sharp U \{u - S_n - \hat{S}_0, u - S_n - \hat{S}_1, ..., u - S_n - \hat{S}_{m-1}\}
\]

\[
= \sum_{u \in \Xi_q} I[u \in U \{S_0, S_1, ..., S_{n-1}\}] \times \sharp \hat{U}_m
\]

\[
\leq \sharp U_n \sharp \hat{U}_m \leq Mnm, \text{ where } M = (2d + 1)^2.
\]

So if we pick \( c < \infty \) such that \( \frac{1}{4d} (c - d - 2d^2) \geq 2M \), it holds that

\[
\sharp \{\lambda \in \mathbb{Z}^q : N_{n,m}(\lambda) \geq \frac{1}{4d} (c - d - 2d^2)(nm)^{\frac{1}{d+1}}\}
\]

\[
\leq \sum_{\lambda \in \Xi_q} \frac{N_{n,m}(\lambda)}{2M(nm)^{\frac{1}{d+1}}} \leq \frac{Mnm}{2M(nm)^{\frac{1}{d+1}}} = \frac{1}{2} (nm)^{\frac{1}{d+1}} \leq \frac{1}{2} \sharp \Xi_q,
\]

and hence

\[
\sharp \{\lambda \in \mathbb{Z}^q : N_{n,m}(\lambda) \leq \frac{1}{4d} (c - d - 2d^2)(nm)^{\frac{1}{d+1}}\} \geq \frac{1}{2} (q + 1)^d.
\]

The proof of Lemma 3.1 is completed.

For \( x \in \mathbb{R} \) we define

\[
\psi(x) = \liminf_{n \to \infty} \frac{-1}{n} \log P(L_n \geq nx).
\]

(26)

Observe that \( \psi(x) \) is nondecreasing in \( x \). Moreover, it is bounded on \([0, 1]\) because by (1) there exists \( a \in \mathbb{Z} \setminus \{0\} \) such that

\[
P(L_n \geq n) \geq P(X_1(i) = X_2(i) = ... = X_n(i) \neq 0) = [P(X_1(i) = a)]^n,
\]

and \( P(X_1(i) = a) > 0 \) for some \( 1 \leq i \leq d \),
where $X_j(i)$ denotes the $i$-th component of $X_j$. Hence, we find
\[ \psi(1) < \infty. \] (27)

We have to prove that $\liminf$ in (26) can be replaced by $\lim$. We first show that this is permissible for any $x \in [0, 1]$ at which $\psi$ is continuous from the right.

**Proposition 3.1.** If (1) holds and $d \geq 2$ and if $\psi$ is right continuous at a given $x \in [0, 1)$, then
\[ \psi(x) = \lim_{n \to \infty} \frac{-1}{n} \log P(L_n \geq nx). \]

**Proof.** Since the idea of this proof is the same as in [6], Proposition 2, we only give an outline of the proof. Owing to (25) we can choose a constant $1 < s < \infty$ so that
\[ P(L_{n+m+d} \geq y + z - s(nm)^{\frac{1}{d+1}}) \geq \frac{1}{2} \zeta^{d(nm)^{\frac{1}{d+1}} + d} P(L_n \geq y) P(L_m \geq z), \] (28)
where $q = \lceil (nm)^{\frac{1}{d+1}} \rceil$. We set $\eta = \frac{d-1}{d+1}$ and $\xi = \frac{2}{d+1}$. If we define for any integer $N \geq 1$,
\[ \sigma(0) = N, \]
\[ \sigma(k + 1) = 2\sigma(k) + d \lceil \sigma(k) \xi \rceil \quad k \geq 0, \]
the following holds:
\[ \frac{\sigma(i - 1)}{\sigma(i)} \leq \frac{1}{2}, \quad \sigma(i) \geq 2^i N, \] (29)
and for some constants $c_1$, $c_2$, $N_0 < \infty$ and $N \geq N_0$
\[ 1 \leq \frac{\sigma(k)}{2^k N} \leq 1 + \frac{c_1}{N^\eta} \leq 2 \] (30)
\[ \sum_{i=0}^{k-1} 2^k \lceil \sigma(i) \xi \rceil \leq c_2 N^{-\eta} \sigma(k). \] (31)

Now let $x \in [0, 1)$ be such that $\psi$ is right continuous at $x$ and let $\epsilon > 0$. Take $\delta \in (0, 1)$ such that
\[ \psi(x + 4 \delta) \leq \psi(x) + \epsilon. \]

Take $c_3 = \frac{\xi d}{2} < 1$ and fix $l \geq 2$ such that
\[ (1 - 2^{-l+2})(x + 2 \delta) \geq x + \delta, \] (32)
\[ \frac{\delta}{2} > 2d2^{-l+2}. \] (33)

Finally, fix $N \geq N_0$ so that
\[ P(R_N \geq N(x + 4 \delta)) \geq \exp[-N(\psi(x + 4 \delta) + \epsilon)] \geq \exp[-N(\psi(x) + 2 \epsilon)], \]
\[ 1 + \frac{c_1}{N^\eta} \leq \frac{x + 4 \delta}{x + 3 \delta} \] (34)
\[ N^{-\eta} < \min \left\{ \frac{\delta}{sc_2}, \frac{-2 \epsilon}{c_2 d \log \zeta}, \frac{1}{2d} \right\}, \] (35)
\[ 5sl(3d + 2)(N^\xi + 1) < \delta N^\frac{1}{2}, \] (36)
and
\[ \frac{2}{N} |\log c_3| < \epsilon. \] (37)
We shall first consider $P(L_n \geq nx)$ for $n \in \{\sigma(k)\}_{k=0}$. If we set $m = n = \sigma(k - 1)$ and

$$y = z = 2^{k-1}N(x + 4\delta) - s \sum_{i=0}^{k-2} 2^{k-1-i}[\sigma(i)]^\xi,$$

then (28) gives for $k \geq 1$

$$P(L_{\sigma(k)} \geq 2^kN(x + 4\delta) - s \sum_{i=0}^{k-1} 2^{k-1-i}[\sigma(i)]^\xi)$$

$$\geq c_3 s^{[\sigma(k-1)]]^\xi} [P(L_{\sigma(k-1)} \geq 2^{k-1}N(x + 4\delta) - s \sum_{i=0}^{k-2} 2^{k-1-i}[\sigma(i)]^\xi)]^2. \quad (38)$$

By (30), (31), (34) and (35)

$$2^kN(x + 4\delta) - s \sum_{i=0}^{k-1} 2^{k-i}[\sigma(i)]^\xi \text{ we also have } \geq \sigma(k)(x + 2\delta). \quad (39)$$

Hence, by (35), (33), (39) we get

$$P(L_{\sigma(k)} \geq \sigma(k)(x + 2\delta)) \geq [c_3]^{2k+1} \exp[-2^kN(\psi(x) + 3\epsilon)]. \quad (40)$$

Next we expand $n$ into a linear combination of the $\sigma(k)$ in the same as in [6], Proposition 2. Recall that we have fixed $l$ in (32) and (33). Now let $n \geq \sigma(2l)$, and take

$$\hat{n} = n - 2dl[n^\xi].$$

Owing to (30) and (35) we can pick $k_r, \alpha_r \in \{1, 2\}, p \leq l$ such that

$$0 \leq \hat{n} - \sum_{i=1}^{p} \alpha_i \sigma(k_i) < 2^{-l+2}n. \quad (41)$$

We set $\beta := \sum_{i=1}^{p} \alpha_i$ and let $n_1 < n_2 < ... < n_\beta$ be number of the form $\sum_{i=1}^{j} \alpha_i \sigma(k_i)$ or $\sum_{i=1}^{j} \alpha_i \sigma(k_i) - \sigma(k_j)$; the latter form is included only if $\alpha_j = 2$. We now apply (28) with $y = n_\gamma(x + 2\delta) - 5s\gamma n^\xi$, $z = (n_{\gamma+1} - n_\gamma)(x + 2\delta)$, $n = n_\gamma + d\gamma[n^\xi]$ and $m = n_{\gamma+1} - n_\gamma$. Using (22) and (28) we then find for $s > 1$

$$P(L_{n_{\gamma+1+d[\gamma+1]}[n^\xi]} \geq n_{\gamma+1}(x + 2\delta) - 5s(\gamma + 1)n^\xi)$$

$$\geq \frac{1}{2} \zeta^{dn^\xi+d} P(L_{n_{\gamma+d\gamma}[n^\xi]} \geq n_\gamma(x + 2\delta) - 5s\gamma n^\xi) \times P(L_{n_{\gamma+1} - n_\gamma} \geq (n_{\gamma+1} - n_\gamma)(x + 2\delta)). \quad (42)$$

Consequently, by (38) and (42) we get

$$P(L_{n_\beta+d[\beta][n^\xi]} \geq n_\beta(x + 2\delta) - 5s\beta n^\xi) \geq [c_3]^{2l+\sum_{i=1}^{p} \alpha_i 2^{k_j+1}} \zeta^{2dn^\xi} \exp[-\sum_{j=1}^{p} \alpha_j 2^{k_j}N(\psi(x) + 3\epsilon)]. \quad (43)$$
Now we apply (32), (36) and (41) to see that
\[ n_\beta(x + 2\delta) - 5s\beta n^\xi \geq n(x + \frac{\delta}{2}). \] (44)

On the other hand, by (33) and (41) we get for sufficiently large \( n \),
\[ \frac{\delta}{2} n \geq 2d(2^{l+2}n + d(2l - \beta)[n^\xi]) \geq 2d(n - n_\beta - d\beta[n^\xi]). \] (45)

Hence, by (22) and (45) we get for sufficiently large \( n \),
\[ P(L_n \geq nx) \geq P(L_{n_\beta + d\beta[n^\xi]} \geq n(x + \frac{\delta}{2})). \] (46)

Since \( \sum_{j=1}^p \alpha_j 2^j < n \), by (37), (43), (44) and (46) we get the assertion of the proposition.

Lemma 3.2. For \( d \geq 2 \), \( \psi \) is convex and continuous on \((0, 1)\).

Proof. To prove the convexity on continuous points of \( \psi \), we can apply Lemma 3.1. The proof that \( \psi \) is continuous on \((0, 1)\) is the same as in [6], Lemma 3. The details are omitted.

Proof of Theorem 2.3. It is obvious that (5) holds. To prove that \( \psi \) is continuous at 0, note that \( \psi(x) = 0 \) for \( x \leq 0 \), while by (1) there exists \( a \in \mathbb{Z} \setminus \{0\} \) such that for sufficiently small \( \delta \in (0, 1) \),
\[ P(L_n \geq \delta n) \geq P(X_1 = X_2, \ldots, X_{[\delta n]} \neq 0) \geq P(X_1(i) = a)[\delta n]. \]

It follows that \( \psi(\delta) \leq -\delta \log P(X_1(i) = a) \) as in (27), hence, also \( \lim_{\delta \to 0} \psi(\delta) = 0 \). Then, the proof that \( \psi \) is continuous at 1 is the same as in [6], Proposition 4, and combined with Lemma 3.1 and (6) this continuity shows (2).

Now that we have continuity of \( \psi \) on \([0, 1]\), we obtain the convexity of \( \psi \) on \([0, 1]\) from lemma 3.2. We also have continuity of \( \psi \) at \( q \), so that also (3) holds.

We can show that the right derivative at \( \eta = 0 \) of \( \lim_{n \to \infty} \frac{-1}{n} \log E\eta L_n \) is \( q \) by the argument given in [7], hence we get (4). Also, the proof of (8) is the same as in [6], Proposition 4.

3.3 Proof of Theorem 2.4

In this subsection we consider simple random walk in two dimensions. We denote the neighbors of 0 by \( b_1, \ldots, b_4 \). In the following lemma, \( a_n \sim c_n \) means \( \frac{a_n}{c_n} \to 1 \ (n \to \infty) \) for sequences \( a_n \) and \( c_n \).

Lemma 3.3. For any \( i \),
\[ P^{b_i} \left( \{S_m\}_{m=1}^n \cap \{0, b_i\} = \emptyset \right) + P^0 \left( \{S_m\}_{m=1}^n \cap \{0, b_i\} = \emptyset \right) \sim \frac{\pi}{\log n}. \]

In particular, by symmetry of the roles played by 0 and \( b_i \), we have
\[ P^{b_i} \left( \{S_m\}_{m=1}^n \cap \{0, b_i\} = \emptyset \right) = P^0 \left( \{S_m\}_{m=1}^n \cap \{0, b_i\} = \emptyset \right) \sim \frac{\pi}{2 \log n}. \]
**Remark 3.2.** While this lemma has been already proven by using Corollary 2 and Eq(1.2) in [11], we give a direct (and hence simpler) proof.

**Proof.** Let

\[
\gamma(n) = P^{b_i}(\{S_m\}_{m=1}^{2n} \cap \{0, b_1\} = \emptyset) + P^0(\{S_m\}_{m=1}^{2n} \cap \{0, b_1\} = \emptyset).
\]

If we consider the last return time to the set \(\{0, b_1\}\) in the first \(2n\) steps, we get

\[
1 = \sum_{k=0}^{n} P(S_{2k} = 0)P^0(\{S_m\}_{m=1}^{2n-2k} \cap \{0, b_1\} = \emptyset) + \sum_{k=0}^{n-1} P(S_{2k+1} = b_i)P^{b_i}(\{S_m\}_{m=1}^{2n-2k-1} \cap \{0, b_1\} = \emptyset).
\]  \(\text{(47)}\)

We first show the upper bound. By local central limit theorem (cf., for example, Theorem 1.2.1 in [11]), it holds that for each \(i\),

\[
P(S_{2k} = 0) \sim \frac{1}{\pi k}, \quad P(S_{2k+1} = b_i) \sim \frac{1}{\pi k}, \quad \text{when } k \to \infty.
\]  \(\text{(48)}\)

So we can rewrite (47) as

\[
1 = \sum_{k=1}^{n} \frac{1}{\pi k}P^0(\{S_m\}_{m=1}^{2n-2k} \cap \{0, b_1\} = \emptyset)(1 + o(1)) + \sum_{k=1}^{n-1} \frac{1}{\pi k}P^{b_i}(\{S_m\}_{m=1}^{2n-2k-1} \cap \{0, b_1\} = \emptyset)(1 + o(1)).
\]

Since \(\gamma(n)\) is nonincreasing, it holds that

\[
1 \geq \sum_{k=1}^{n} \frac{1}{\pi k} \gamma(n)(1 + o(1)).
\]

So we get

\[
\gamma(n) \leq \frac{\pi}{\log n}(1 + o(1)).
\]  \(\text{(49)}\)

Next we show the lower bound. For any \(0 \leq l \leq n\), it holds that

\[
1 \leq \sum_{k=0}^{l} P(S_{2k} = 0)P^0(\{S_m\}_{m=1}^{2n-2k} \cap \{0, b_1\} = \emptyset) + \sum_{k=0}^{l} P(S_{2k+1} = b_i)P^{b_i}(\{S_m\}_{m=1}^{2n-2k-1} \cap \{0, b_1\} = \emptyset)
\]

\[
+ \sum_{k=l+1}^{n} P(S_{2k} = 0) + \sum_{k=l+1}^{n-1} P(S_{2k+1} = b_i)
\]

\[
\leq \sum_{k=0}^{l} P(S_{2k} = 0)P^0(\{S_m\}_{m=1}^{2n-2l} \cap \{0, b_1\} = \emptyset) + \sum_{k=0}^{l} P(S_{2k+1} = b_i)P^{b_i}(\{S_m\}_{m=1}^{2n-2l-1} \cap \{0, b_1\} = \emptyset)
\]

\[
+ \sum_{k=l+1}^{n} P(S_{2k} = 0) + \sum_{k=l+1}^{n-1} P(S_{2k+1} = b_i).
\]

Again by [48], it holds that

\[
1 \leq \gamma(n-l+1)\frac{\log n}{\pi}(1 + o(1)) + \frac{2}{\pi} \log \frac{n}{l}(1 + o(1)).
\]

\[
1 \leq \gamma(n-l+1)\frac{\log n}{\pi}(1 + o(1)) + \frac{2}{\pi} \log \frac{n}{l}(1 + o(1)).
\]
If we pick \( l = n - \left\lceil \frac{n}{\log n} \right\rceil \), it holds that

\[
1 \leq \gamma \left( \left\lceil \frac{n}{\log n} \right\rceil + 1 \right) \frac{\log n}{\pi} (1 + o(1)) + O(1/\log n).
\]

So we get

\[
\gamma \left( \left\lceil \frac{n}{\log n} \right\rceil + 1 \right) \geq \frac{\pi}{\log n} (1 + o(1)). \tag{50}
\]

By (49) and (50) we get the result. \( \square \)

**Proof of Theorem 2.4.** We write the ergodic formula (12) in the form

\[
L_n = \sum_{k=0}^{n} C_{k,n}(\omega),
\]

where \( C_{k,n} \) is the event that \( \{S_m\}_{m=0}^{n} \not\supset N(S_k) \) and \( S_k \not\supset \{S_m\}_{m=k+1}^{n} \). If we denote by \( C'_{k,n,j} \) the event that \( \{S_m\}_{m=1}^{k-1} \cap \{S_k, S_k + b_j\} = \emptyset \) and \( S_k + b_j \not\supset \{S_m\}_{m=k+1}^{n} \), we find that

\[
P(C_{k,n}) = P(\bigcup_{j=1}^{4} C'_{k,n,j}). \tag{51}
\]

So we get

\[
P(C'_{k,n,j}) \leq P(C_{k,n}) \leq \sum_{j=1}^{4} P(C'_{k,n,j}). \tag{52}
\]

It holds that

\[
P(C'_{k,n,j}) = P(S_k + b_j \not\supset \{S_m\}_{m=0}^{k-1}) \times P(\{S_m\}_{m=k+1}^{n} \cap \{S_k, S_k + b_j\} = \emptyset)
\]

\[
= P(b_j \not\supset \{S_m\}_{m=1}^{k}) \times P(\{S_m\}_{m=1}^{n-k} \cap \{0, b_j\} = \emptyset)
\]

Therefore, by summing over \( k \) in (52) with the help of Lemma 3.3 we get (9), provided that the limit in it exists. Next we show (10) (in the same sense as for (9)). If \( l = \{l_i\}_{i=1}^{p} \) and \( D_{l,n} \) is the event that \( \{S_m\}_{m=0}^{n} \not\supset N(S_{l_1}), S_{l_1} = S_{l_2} = \ldots = S_{l_p} \) and \( S_{l_1} \not\supset (\{S_m\}_{m=0}^{n} - \{S_{l_1}\}_{i=1}^{p}) \), then it holds that

\[
J_{n}^{(p)} = \sum_{l_1=0}^{n} \sum_{l_2=l_1+2}^{n} \ldots \sum_{l_p=l_{p-1}+2}^{n} 1_{D_{l,n}}(\omega).
\]

If \( D'_{l,n,j} \) denotes the event that

\[
S_{l_1} = S_{l_2} = \ldots = S_{l_p} \quad \text{and} \quad (\{S_m\}_{m=0}^{n} - \{S_{l_1}\}_{i=1}^{p}) \cap \{S_{l_1}, S_{l_1} + b_j\} = \emptyset,
\]

then

\[
P(D_{l,n}) = P(\bigcup_{j=1}^{4} D'_{l,n,j}). \tag{53}
\]

So we get

\[
P(D'_{l,n,j}) \leq P(D_{l,n}) \leq \sum_{j=1}^{4} P(D'_{l,n,j}). \tag{54}
\]
It holds that
\begin{align*}
P(D_{l,n,j}^1) &= P\left(\{S_m\}_{m=0}^{l_1-1} \cap \{S_{l_1}, S_{l_1} + b_j\} = \emptyset\right) \\
&\times P\left(\{S_m\}_{m=l_1+1}^{l_2-1} \text{ firstly hit } S_{l_1} \text{ at the time } l_2 \text{ and } S_{l_1} + b_j \notin \{S_m\}_{m=l_1+1}^{l_2-1}\right) \\
&\times \ldots \times P\left(\{S_m\}_{m=l_p-1}^{l_p-1} \text{ firstly hit } S_{l_p-1} \text{ at the time } l_p \text{ and } S_{l_p-1} + b_j \notin \{S_m\}_{m=l_p-1}^{l_p-1}\right) \\
&\times P\left(\{S_m\}_{m=l_p+1}^{l_1} \cap \{S_{l_p}, S_{l_p} + b_j\} = \emptyset\right) \\
&= P\left(\{S_m\}_{m=1}^{l_1-1} \cap \{0, b_j\} = \emptyset\right) \\
&\times P\left(\{S_m\}_{m=1}^{l_2-1} \text{ firstly hit } 0 \text{ at the time } l_2 - l_1 \text{ and } b_j \notin \{S_m\}_{m=1}^{l_2-1}\right) \\
&\times \ldots \times P\left(\{S_m\}_{m=1}^{l_p-1} \text{ firstly hit } 0 \text{ at the time } l_p - l_p - 1 \text{ and } b_j \notin \{S_m\}_{m=1}^{l_p-1}\right) \\
&\times P\left(\{S_m\}_{m=1}^{n-l_p} \cap \{0, b_j\} = \emptyset\right).
\end{align*}

We compute the upper bound of (10). Summing over \(l_2, \ldots, l_p\) in (54) we get
\begin{align*}
\sum_{l_2=l_1+2}^{l_1+n} \ldots \sum_{l_p=l_p-1+2}^{l_p+n} P(D_{l,n,j}^1) &\leq P\left(\{S_m\}_{m=1}^{l_1-1} \cap \{0, b_j\} = \emptyset\right) \times P(0 \in \{S_m\}_{m=1}^{\infty}, b_j \notin \{S_m\}_{m=1}^{T_0}) \\
&\times \ldots \times P(0 \in \{S_m\}_{m=1}^{\infty}, b_j \notin \{S_m\}_{m=1}^{T_0}) \\
&\times P\left(\{S_m\}_{m=1}^{n-(p-1)(\frac{n}{\log n}-l_1)} \cap \{0, b_j\} = \emptyset\right).
\end{align*}

It is shown in [3] (see (2.4) of it) that
\begin{equation}
P\left(\frac{n}{\log n} < T_0 \leq n\right) = P(T_0 > \frac{n}{\log n}) - P(T_0 > n) \\
\leq \left(\frac{\pi}{\log n - \log \log n} + \frac{C}{(\log n - \log \log n)^2}\right) - \left(\frac{\pi}{\log n} - \frac{C}{(\log n)^2}\right) \leq \frac{C' \log \log n}{(\log n)^2},
\end{equation}
for some constants \(C\) and \(C'\). Hence we can obtain the bound
\begin{align*}
&\sum_{l_2=l_1+2}^{l_1+n} \ldots \sum_{l_p=l_p-1+2}^{l_p+n} P(D_{l,n,j}^1) - \sum_{l_2=l_1+2}^{l_1+n} \ldots \sum_{l_p=l_p-1+2}^{l_p+n} P(D_{l,n,j}^1) \\
= &\sum_{v=2}^{p} \sum_{l_2=l_1+2}^{l_1+n} \ldots \sum_{l_v=l_v-1+2}^{l_v+n} \ldots \sum_{l_p=l_p-1+2}^{l_p+n} P(D_{l,n,j}^1) \\
&\leq (p-1) P(\{S_m\}_{m=1}^{l_1-1} \cap \{0, b_j\} = \emptyset) \times P\left(\frac{n}{\log n} < T_0 \leq n\right).
\end{align*}

Summing over \(l_1\) in (55) and (56) with the help of Lemma 3.3 we get the upper bound of (10).

To compute the lower bound of (10), for \(\varepsilon > 0\) pick \(s < \infty\) such that \(P(T_0 < T_b, T_0 < s) > P(T_0 < T_b) - \varepsilon\). It holds that for \(n \geq (p-1)s\)
\begin{align*}
\sum_{l_2=l_1+2}^{l_1+n} \ldots \sum_{l_p=l_p-1+2}^{l_p+n} P(D_{l,n,j}^1) &\geq P(\{S_m\}_{m=1}^{l_1} \cap \{0, b_j\} = \emptyset) \\
&\times P(0 \in \{S_m\}_{m=1}^{s}, b_j \notin \{S_m\}_{m=1}^{T_0}) \times \ldots \times P(0 \in \{S_m\}_{m=1}^{s}, b_j \notin \{S_m\}_{m=1}^{T_0}) \\
&\times P(\{S_m\}_{m=1}^{n-s-l_1} \cap \{0, b_j\} = \emptyset) \\
&\geq P(\{S_m\}_{m=1}^{l_1} \cap \{0, b_j\} = \emptyset) \times (c - \varepsilon)^{p-1} \times P(\{S_m\}_{m=1}^{n-l_1} \cap \{0, b_j\} = \emptyset).
\end{align*}
Therefore, by summing over $l_1$, by Lemma 3.3 we get the lower bound of (10).

Also, if we set $l = \{l_i\}_{i=1}^p$, then

$$J_n^p = \sum_{l_1=0}^n \sum_{l_2=l_1+2}^n \ldots \sum_{l_p=l_{p-1}+2}^n 1_{E_{l,n}}(\omega),$$

where $E_{l,n}$ is the event that $\{S_m\}_{m=0}^n \not\ni N(S_{l_1}), S_{l_1} = S_{l_2} = \ldots = S_{l_p}$ and $S_{l_1} \not\in (\{S_m\}_{m=l_1+1}^n - \{S_l\}_{l=2}^p)$. So we can verify (11) by the argument given for (10). We finally prove the existence of the limits. [13], Theorem 2 tells us that for any $a \in \mathbb{Z}^2$, $\lim_{n \to \infty} P(a \not\in \{S_l\}_{l=1}^p) \times (\log n)$ exists. Since by applying inclusion-exclusion formula (e.g., [4], Exercise 1.6.9) (51) can be divided, it holds that

$$\lim_{n \to \infty} EL_n \times \frac{(\log n)^2}{n}$$
exists.

Also, by the same argument it is easy to see that

$$\lim_{n \to \infty} EJ_n^{(p)} \times \frac{(\log n)^2}{n}, \quad \lim_{n \to \infty} EJ_n^p \times \frac{(\log n)^2}{n}$$
exist.

References

[1] BENJAMINI,I. and KOZMA,G. and YADIN,A. and YEHUDAYOFF,A.(2010). Entropy of random walk range. Ann. Inst. H. Poincaré Probab. Statist.Volume 46, Number 4, 895-1194.

[2] DONSKER,M.D.and VARADHAN,S.R.S.(1979). On the number of distinct sites visited by a random walk. Comm. Pure Appl. Math. 32, 721-747.

[3] DVORETZKY,A. and ERDŐS,P.(1951). Some problems on random walk in space. Proc. Second Berkeley Symp. Math. Statist. Probab. 353-367. Univ. California Press, Berkeley.

[4] DURRET,R.(2010). Probability: theory and example, Edition 4.Cambridge Series.

[5] FLATTO,L.(1976).The Multiple range of two-dimensional recurrent walk. Ann. Probab. Volume 4, Number 2, 155-338.

[6] HAMANA,Y. and KESTEN,H.(2001). A large-deviation result for the range of random walk and for the Wiener sausage. Probability Theory and Related Fields, June, Volume 120, Issue 2, 183-208.

[7] HAMANA,Y.(2001). Asymptotics of the moment generating function for the range of random walks. Journal of Theoretical Probability January, Volume 14, Issue 1, 189-197.

[8] JAIN,N.C. and PRUITT,W.E.(1970). The range of recurrent random walk in the plane. Z. Wahrsch. Verw. Gebiete 16 279-292.
[9] JAIN, N.C. and PRUITT, W.E. (1971). The range of transient random walk. J. Anal. Math. 24. 369-393.

[10] JAIN, N.C. and PRUITT, W.E. (1972). The range of random walk. Proc. Sixth Berkeley Symp. Math. Statist. Probab. 3. 31-50. Univ. California Press, Berkeley.

[11] LAWLER, G.F. (1991). Intersections of Random Walks. Birkhauser, Boston.

[12] SPITZER, F. (1976). Principles of Random Walk. Springer, Berlin.

[13] KESTEN, H. and SPITZER, F. (1963). Ratio theorems for random walks I. Journal d’ Analyse Mathématique December, Volume 11, Issue 1, 285-322.