Cusped Light-Like Wilson Loops in Gauge Theories
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Abstract—We propose and discuss a new approach to the analysis of the correlation functions which contain light-like Wilson null-polygons, the soft factors of the parton distribution and fragmentation functions, high-energy scattering amplitudes in the eikonal approximation, gravitational Wilson lines, etc. Our method is based on a generalization of the universal quantum dynamical principle by J. Schwinger and allows one to take care of extra singularities emerging due to lightlike or semi-light-like cusps. We show that such Wilson loops obey a differential equation which connects the area variations and renormalization group behavior of those objects and discuss the possible relation between geometrical structure of the loop space and area evolution of the light-like cusped Wilson loops.

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1. INTRODUCTION

Wilson lines (also known as gauge links or eikonal lines) can be naturally introduced in any gauge field theory. These objects are generically defined via traces of path-ordered exponentials of a gauge field evaluated along a given trajectory \( W(\Gamma) = \mathcal{P}\exp[-ig \int_{\Gamma} dz^\mu A_\mu(z)] \). The path \( \Gamma \) is a curve along which the gauge field \( A_\mu \) gets transported from the initial point to the final one. Wilson lines defined on closed contours are called Wilson loops. They are path-dependent non-local functionals of the gauge field, invariant under gauge group transformations. Putting the matter of question more mathematical, one can construct a space with its elements being gauge fields. Let the functionals \( \mathcal{O}(A) \) be the Wilson exponentials \( \Phi(\Gamma) \) (1). Then Eq. (2) turns into the Makeenko–Migdal (MM) equations [3]:

\[
\frac{\delta}{\delta \sigma_{\mu\nu}(x)} W_1(\Gamma) = N_c g^2 \int_{\Gamma} dz^\mu \delta(4) \times (x - z) W_2(\Gamma_{xz}, \Gamma_{xz}),
\]

where the basic operations are the area—\( \delta/\delta \sigma_{\mu\nu} \) and the path—\( \partial_\mu \) derivatives [3]:

\[
\frac{\delta}{\delta \sigma_{\mu\nu}(x)} \Phi(\Gamma) = \lim_{|\delta \sigma_{\mu\nu}(x)| \to 0} \frac{\Phi(\Gamma + \delta \Gamma) - \Phi(\Gamma)}{|\delta \sigma_{\mu\nu}(x)|}.
\]

The area derivative can be written as well in the so-called Polyakov form—see, e.g., [5] for a discussion of an alternative approach.
Note that the derivation of the MM equations from the Schwinger–Dyson equations is grounded on the Mandelstam formula
\[
\frac{\delta}{\delta \sigma_{\mu \nu}(x)} \Phi(\Gamma) = ig Tr [F_{\mu \nu} \Phi(\Gamma_x)]
\] (6)
and/or on the Stokes theorem, so that the Wilson functionals which do not satisfy the corresponding restrictions (such as, e.g., cusped light-like loops) apparently cannot be straightforwardly treated within the same scheme. There are several other issues limiting the predictive power of the MM equations. Namely, there exists an interesting class of Wilson loops which possess very specific singularities originating, in particular, from the cusps and/or self-intersections of the contours and, in addition, from the light-like segments of the integration paths. The simplest example is given by a Wilson exponential evaluated along a cusped contour with two semi-infinite light-like sides, Fig. 1. Already the leading order contribution to this Wilson exponential possesses all the peculiar singularities: the pure ultraviolet, the infrared (due to the infinite lengths of the sides), and the light-like cusp divergences. This simple contour will arise in what follows as a building unit of many important Wilson loops and correlation functions. Physically it corresponds to the soft part of the factorized quark form factor, which has been studied in detail in [6, 7].

In the present work we propose and discuss a new approach to these issues, having in mind, as an instructive example, a very special type of Wilson loops—planar rectangles with light-like sides. Considerable interest to cusped light-like Wilson polygons has arisen thanks to the recently conjectured duality between the \( n \)-gluon planar scattering amplitudes in the \( N = 4 \) super-Yang–Mills theory and the vacuum average of planar Wilson loops formed, correspondingly, by \( n \) light-like segments connecting space-time points \( x_i \), so that their “lengths” \( x_i - x_{i+1} = p_i \) are chosen equal to the external momenta of the \( n \)-gluon amplitude (see, e.g., [8] and references therein). It has been demonstrated that the infrared singularities of the former corresponds to the ultraviolet singularities of the latter, and the cusp anomalous dimension is the crucial constituent of the evolution equations [9].

Wilson exponentials possessing light-like segments (or that are fully light-like) have been studied also in a different context [6]. The main observation is that the renormalization properties of these Wilson loops are more intricate than those of cusped Wilson loops defined on off-light-cone integration contours. Namely, the light-cone cusped Wilson loops are not multiplicatively renormalizable because of the additional light-cone singularities (besides the standard ultraviolet and infrared ones). It is possible, however, to construct a combined renormalization-group equation taking into account ultraviolet as well as light-cone divergences. The cusp anomalous dimension, which is the principal ingredient of this equation, is remarkably universal: it controls, e.g., the infrared asymptotic behavior of such important quantities as the QCD and QED Sudakov form factors, the gluon Regge trajectory, the integrated (collinear) parton distribution functions at large-\( x \), the anomalous dimension of the heavy quark effective theory, etc. [6, 7, 9, 10].

Another interesting field of application of cusped light-cone Wilson lines could be transverse-momentum dependent parton densities (TMDs) [11, 12]. The latter are introduced to describe the intrinsic transverse momentum of partons inside the nucleon, which is needed in the study of semi-inclusive processes within the (generalization of) the QCD factorization formalism [11, 13].

2. EXAMPLE:
SINGULARITY STRUCTURE OF TMDs

Let us discuss the emergent singularities arising in TMDs beyond tree-approximation. At one-loop level, the following three classes of divergences appear: (i) standard ultraviolet poles, which are removable by a normal renormalization procedure; (ii) pure rapidity divergences, which depend on an additional rapidity cutoff, but do not violate renormalizability of TMDs; they can be resummed by means of the Collins-Soper evolution equation; (iii) very specific overlapping divergences: they contain the ultraviolet and rapidity poles simultaneously and thus break down the standard renormalizability of TMDs. This situation resembles the problems with renormalizability of the light-like Wilson loops discussed above. However, the structure of Wilson lines is quite involved already in the tree-approximation. The most straightforward definition of “a quark in a quark” TMD, which meets
the requirement of the parton number interpretation, reads

\[ \mathcal{F}_{\text{unsub}}(x, k_\perp) = \frac{1}{2} \int \frac{d^2\xi}{(2\pi)^2} e^{-ik\cdot\xi} \times \langle p|\bar{\psi}(\xi^-, 0^-) \gamma^\mu W^\mu_n(\xi^-, \xi_\perp; 0^-) |\psi(0^-, 0^-)\rangle \]

(7)

with \( \xi^+ = 0 \). Here we define the semi-infinite Wilson lines evaluated along a four-vector \( w \) as

\[ \mathcal{W}_w(\infty; \xi) \equiv \mathcal{P} \exp \left[ -ig \int d\tau w^\mu A_\mu^\tau(\xi + wt) \right], \]

where the vector \( w \) can be light-like \( w_L = n^+ \), \( (n^+)^2 = 0 \), or transverse \( w_T = l \). Formally, the integration of (7) over \( k_\perp \) is expected to give the collinear (also called integrated) PDF

\[ \left[ d^k_{\perp} \mathcal{F}_{\text{unsub}}(x, k_\perp) \right] = 1 \int \frac{d^2\xi}{(2\pi)^2} e^{-ik\cdot\xi} \times \langle p|\bar{\psi}(\xi^-, 0^-) \gamma^\mu W^\mu_n(\xi^-, 0^-; 0^-) |\psi(0^-, 0^-)\rangle \]

(8)

However, this is only justified in tree approximation. It is worth noting that the normalization of the above TMD

\[ \mathcal{F}_{\text{unsub}}^{(0)}(x, k_\perp) = \frac{1}{2} \int \frac{d^2\xi}{(2\pi)^2} e^{-ik\cdot\xi + i\xi_\perp \cdot k_\perp} \times \langle p|\bar{\psi}(\xi^-, 0^-) \gamma^\mu W^\mu_n(0^-; 0^-) |\psi(0^-, 0^-)\rangle \]

(9)

can be most easily obtained by making use of the canonical quantization procedure in the light-cone gauge, where longitudinal Wilson lines become equal to unity and where equal-time commutation relations for creation and annihilation operators \{\( a^\dagger(k, \lambda), a(k, \lambda) \)\} immediately yield the parton number interpretation

\[ \mathcal{F}_{\text{unsub}}^{(0)}(x, k_\perp) \sim \langle p|a^\dagger(k^+, k_\perp; \lambda)a(k^+, k_\perp; \lambda) |p\rangle. \]

(10)

The usage of “tilted” gauge links in the operator definition of TMDs does not meet this requirement. We visualize the geometrical layout of various Wilson lines in the operator definition of TMDs in Figs. 2, 3, 4 and discuss relevant issues in their captions.

Beyond tree-approximation, the virtual diagrams producing terms with overlapping singularities are shown in Fig. 5. The typical extra divergency stems from the one-loop vertex-type graph Fig. 5a in covariant gauges or from the self-energy graph Fig. 5b in the light-cone gauge (in the large-\( N_c \) limit) and reads

\[ \text{TMD}_{\text{UV}\otimes\text{LC}} = -\frac{\alpha_s N_c}{2\pi} \Gamma(\epsilon) \left[ \frac{4\pi\mu^2}{-p} \right]^\epsilon \]

\[ \times \delta(1-x)\delta^{(2)}(k_\perp) \int_0^1 dx \frac{x^{1-\epsilon}}{(1-x)^{1+\epsilon}}. \]

(11)

The standard ultraviolet pole in the Gamma-function \( \Gamma(\epsilon) \) is accompanied by an additional singularity in the integral. The latter is due to the integration over infinite gluon rapidity and cannot be treated by dimensional regularization, calling for an extra (rapidity) cutoff. The reason for renormalizability violation in
the leading order contribution to TMDs is that light-like Wilson lines (or the “standard” quark self-energy in light-cone gauge) produce more singular terms than usual Green functions do.

To solve the problems with extra singularities and renormalizability in TMDs, a variety of (possibly non-equivalent) methods has been proposed. Working in the covariant Feynman gauge, Ji, Ma and Yuan proposed a scheme which utilizes tilted (off-light-cone) longitudinal Wilson lines directed along the vector $n_B^2 \neq 0$ [15]. Transverse Wilson lines at the light-cone infinity cancel in covariant gauges, while the rapidity cutoff $\zeta = (2p \cdot n_B)^2 / n_B^2$ marks the deviation of longitudinal Wilson lines from a pure light-like direction. A subtracted soft factor then contains non-light-like Wilson lines as well. Obviously, such off-light-cone unsubtracted TMDs with the light-like vector $n^-$ replaced by the vector $n_B = (-e^{2\eta_B}, 1, 0)$ do not obey the Eq. (8), not even at tree level. However, it is possible to formulate a “secondary factorization” method which allows one to express off-light-cone TMDs (in impact parameter space $\mathcal{F}(x, b_{\perp})$) as a convolution of
in the form of the “too singular” non-renormalizable quantities as TMDs, light-like Wilson polygons, etc., which originate in their light-cone structure and arise those properties shared by such apparently different loops [16]. To this end, it appears instructive to study approach to the problems of light-cone cusped Wilson definition. In the present work we discuss another strategy, trying to avoid using operations which implicitly assume the smoothness of the TMDs, how-

3. SCHWINGER DYNAMICAL PRINCIPLE AND AREA EVOLUTION FOR SMOOTH WILSON LOOPS

We made use of the observation that in the large-$N_c$ limit, in the transverse null-plane, for the light-like planar dimensionally regularized (not renormalized) Wilson rectangles, the area derivatives introduced in the previous sections can be reduced to the normal ones. The area variational equations in the coordinate representation describe the evolution of lightlike Wilson polygons and represent, therefore, the “equations of motion” in loop space, valid for a specific class of its elements. As a result, the obtained differential equations give us a closed set of dynamical equations for the loop functionals, and can in principle be solved in several interesting cases.

Let us start with the quantum dynamical principle proposed by Schwinger [17]: the quantum action operator $\hat{S}$ defines variations of arbitrary states, so that

$$\delta \langle \alpha'|\alpha'' \rangle = \frac{i}{\hbar} \langle \alpha'|\delta \hat{S}|\alpha'' \rangle. \quad (12)$$

The area variations (4) of field exponentials $\Phi(\Gamma)$ yield

$$\frac{\delta}{\delta \sigma}(\alpha'|\Phi(\Gamma)|\alpha'' \rangle = \frac{i}{\hbar} \langle \alpha'|\delta \hat{S}|\Phi(\Gamma)|\alpha'' \rangle, \quad (13)$$

where $\hat{S}$ is yet to be defined. The loop space consists of scalar objects with different geometrical and topological features, hence the equations of motion in this space must be the laws which state how those objects change their shape. It means that “motion” in loop space is equivalent to the variation of the integration contours in Wilson loops [3]. Therefore, we have to find the correct operator $\hat{S}$, which governs the shape variations of the light-like cusped loops (Wilson null-polygons).

Within the standard approach, one utilizes (12) in the form (2) and obtains the set of the MM Eqs. (3). We will follow another strategy, trying to avoid using operations which implicitly assume the smoothness of the Wilson loops under consideration. For the sake of clarity, consider at first a generic Wilson loop $W(\Gamma)$ without specifying whether it is smooth or not. Its perturbative expansion reads

$$W(\Gamma) = W^{(0)} + W^{(1)} = 1 - \frac{g^2 C_F}{2}$$

$$\times \int d\Gamma d\Gamma' D^{\mu\nu}(z-z') + O(g^4),$$

where $D^{\mu\nu}$ is the free dimensionally regularized ($\omega = 4 - 2\epsilon$) gluon propagator

$$D^{\mu\nu} = -g^{\mu\nu}\Delta(z-z'), \quad \Delta(z-z') = \frac{\Gamma(1-\epsilon)}{4\pi^2}$$

$$\times \frac{(\pi\mu^2)^{\epsilon}}{[-(z-z')^2 + i0]^{1-\epsilon}}.$$

For convenience’s sake, we work in the Feynman covariant gauge and separate out the scalar part of the propagator $\Delta(z)$. The issues related to gauge- and regularization independence of the calculations will be
4. SINGULARITIES OF WILSON RECTANGLES

We are now in a position to extend the Schwinger approach to a more complicated case and to try to derive the corresponding area evolution equations. The calculation of cusped light-cone Wilson loops beyond tree approximation in different gauges and the justification of gauge independence calls for a careful treatment of a variety of divergences already in leading order. Special attention must be paid to the separation of the light-cone singularities and the standard ultraviolet poles [2, 6, 18]. In the large-$N_c$ limit one obtains in coordinate space [6]

$$W(\Gamma^0) = 1 - \frac{1}{\epsilon^2} \left[ \frac{\alpha_s N_c}{\pi} \ln \left( \frac{-2 N^+ N^- \mu^2 + i 0}{-N^+ N^- + i 0} \right) + \text{finite terms} \right] + O(\alpha_s^2),$$

where the energy variables in momentum space, $s = (p_1 + p_2)^2$ and $t = (p_2 + p_3)^2$, map onto the area variables in the coordinate transverse null-plane, so that $s/2 = -t/2 \rightarrow N^+ N^-$. We will show separately that the result (20) is not only gauge invariant, but is independent of any regularization of light-cone and ultraviolet divergences and of the way they are separated. This issue is of particular importance to understand the operator structure of transverse-momentum dependent parton densities and soft-collinear effective theory (see, e.g., [12, 19] and references therein). The problem of regularization-independence in next-to-leading order deserves its own dedicated study.

The transverse null-plane is defined by the condition $\tau_L = 0$; therefore, the area variations are well-defined

$$\delta \sigma^{+-} = N^+ \delta N^- , \quad \delta \sigma^{-+} = -N^- \delta N^+ .$$

These operations make sense only at the corner points $x_i$, and we distinguish between the “left” and “right” variations, as shown in Fig. 6.

$W(\Gamma^0)$ is one of the best studied examples of (partially) light-like objects which are known to lack multiplicative renormalizability [6]. In order to decrease the power of singularity that violates the renormalizability, one can follow the scheme proposed in [7].
Having in mind Eq. (20), we define the area logarithmic derivative as
\[
\frac{\delta}{\delta \ln \sigma} = \sigma_{\pm} \frac{\delta}{\delta \sigma_{\pm}} + \sigma \frac{\delta}{\delta \sigma} \tag{21}
\]
and apply this operator to the r.h.s. of the Eq. (20):
\[
\frac{\delta}{\delta \ln \sigma} \ln W(\Gamma_{\Box}) = -\frac{\alpha_s N_c}{2\pi} \epsilon \tag{22}
\]
\begin{align*}
&\times \left[ -2N^+N^- \mu^2 \pm i0 \right] \left[ 2N^+N^- \mu^2 \pm i0 \right].
\end{align*}
Then the finite cusp anomalous dimension results from:
\[
\frac{d}{d\mu} \frac{\delta}{\delta \ln \sigma} W(\Gamma_{\Box}) = -4\Gamma_{\text{cusp}}, \tag{23}
\]
\[
\Gamma_{\text{cusp}} = \frac{\alpha_s N_c}{2\pi} + O(\alpha_s^2).
\]
We get the finite result (23) by making use of the logarithmic area derivative (21), given that the infinitesimal area variations are defined as in (4). Eq. (23) describes the dynamical properties of the light-like Wilson loops [16]. We relate, therefore, the geometry describing the dynamical properties of the light-like Wilson loops [16].

5. COMBINED EVOLUTION FROM THE SCHWINGER PRINCIPLE

The very possibility to obtain a finite result by means of Eqs. (22, 23) is a direct consequence of the geometrical properties of loop space, whose constituents are non-renormalizable cusped light-like Wilson loops. To show this explicitly we restrict ourselves to area variations (20), and apply the area derivative to a Wilson rectangle
\[
\frac{\delta W(\Gamma_{\Box})}{\delta \sigma_{\mu \nu}} = \frac{g^2 C_F}{2} \Gamma(1 - \epsilon)(\pi\mu^2)^\epsilon \frac{\delta}{\delta \sigma_{\mu \nu}} \sum_{i,j} \left( \nu_i^j \nu_j^i \right) \tag{24}
\]
\begin{align*}
&\times \left[ \int_{0}^{1} \int_{0}^{1} d\tau d\tau' \left( x_i^j - x_j^i - \tau \nu_i^j + \tau' \nu_j^i \right)^2 + i0 \right]^{1/\epsilon},
\end{align*}
where the sides of the rectangle are parameterized as \( \nu_i^j = x_i^j - \nu_i^j \tau \) with vectors \( \nu_i^j \) having dimension [mass\(^{-1}\)] [6]. A remarkable feature of light-like loops is that the area dependence factorizes out from the integrals and can be evaluated explicitly (taking into account that \( 2(\nu_1 \nu_2) = 2N^+N^- \), see Eq. (20))
\[
W^{(1)}(\Gamma_{\Box}) = -\frac{\alpha_s N_c}{2\pi} \Gamma(1 - \epsilon)(\pi\mu^2)^\epsilon \tag{25}
\]
\begin{align*}
&\times \left( -2N^+N^- \right)^{\epsilon/2} \int_{0}^{1} \int_{0}^{1} d\tau d\tau' \left[ (1 - \tau + \tau')^{1/\epsilon} \right].
\end{align*}
On the other hand, light-like Wilson lines with \( \nu_i^j = 0 \) produce an extra singularity, which shows up in the form of the second-order pole \( \sim \epsilon^{-2} \), while the cusps violate conformal invariance of the Wilson loop because the “skewed” scalar products \( \nu_i \nu_j \neq 0 \) replace the conformal ones \( \nu_i^j \). Then, performing the area \( \delta / \delta \ln \sigma = \delta / \delta \ln (2N^+N^-) \) and the mass logarithmic differentiation of Eq. (25) and collecting all relevant terms, we come to the result
\[
\frac{d}{d\mu} \left[ \frac{\delta}{\delta \ln \sigma} \ln W(\Gamma) \right] = -\sum \Gamma_{\text{cusp}}, \tag{26}
\]
which was anticipated in Eq. (23) and which is derived now as a direct consequence of the Schwinger approach. It is not surprising that this result resembles, in some sense, the situation in 2D QCD considered above. The area derivative turns into the ordinary derivative for the same reason: the null-plane is effectively a two-dimensional space, where the set of MM Eqs. becomes closed and—at least in principle—solvable [3, 4].

Note that the r.h.s. of Eq. (26) is given by the cusp anomalous dimension, which is an universal quantity (independent of the form of the contour) and which is known perturbatively up to the \( O(\alpha_s^3) \) order. It is therefore worth analyzing if the above result is only a leading order approximation, or if it is expected to be valid in the higher orders as well. Let us take into account the property of linearity of the (angle-dependent) cusp anomalous dimension in the large-angle asymptotic regime with respect to the logarithm of the cusp angle \( \chi \rightarrow \frac{1}{2} \ln \left( \frac{2 \nu_i \nu_j}{\nu_i^j} \right) \) [9]:
\[
\lim_{\chi \rightarrow \infty} \Gamma_{\text{cusp}}(\chi, \alpha_s) = \sum \alpha_s^k C_n(W) a_n(W) \ln \left( \frac{2 \nu_i \nu_j}{\nu_i^j} \right), \tag{27}
\]
\[
\text{where the “maximally non-Abelian” numerical coefficients are}
\]
\[
C_k \sim C_F N_c^{k-1} \frac{N_c^2}{2}, \tag{28}
\]
and \( a_n \) are cusp-independent factors. This regime corresponds exactly to the light-cone case with the angle-dependent logarithms being transformed into addi-
tional poles in $\epsilon$: $\chi \rightarrow (v_i, v_j)^{\epsilon}_\epsilon$, see [6, 9]. More specifically, the area variable $\sim (v_i, v_j)$ enters the regularized area-dependent cusp anomalous dimension in the light-cone limit as

$$\Gamma_{\text{cusp}}(\text{area}, \epsilon, \alpha_s) = \sum \alpha_s^n C_n(W) a_n(W) \frac{\text{area}}{\epsilon},$$

(29)

and, after logarithmic area differentiation, one gets the finite perturbative expansion of the cusp anomalous dimension

$$\lim_{\epsilon \rightarrow 0} \frac{d \Gamma_{\text{cusp}}(\text{area}, \epsilon, \alpha_s)}{d \ln \text{area}} = \sum \alpha_s^n C_n(W) a_n(W),$$

(30)

which supports the validity of the previous result (26) in the higher orders by virtue that

$$\Gamma_{\text{cusp}} = -d \ln W / d \ln \mu.$$

This means that the result (26) should be understood as an all-order one, akin the MM Eq. (3): they both are exact and non-perturbative, while the r.h.s.s of each one can be evaluated order by order in perturbation theory. It is worth noting that Eq. (23) is consistent with the non-Abelian exponentiation of the dimensionally regularized Wilson loops with cusps

$$W(\Gamma_{\text{cusp}}, \epsilon) = \exp \left[ \sum_{\lambda = 1}^{\delta \sigma = N \delta N^-} \alpha_s^\lambda C_\lambda(W) F_\lambda(W) \right],$$

(31)

where the summation goes over all two-particle irreducible diagrams, whose contribution is given by the “web” functions $F_\lambda$ [9, 20]. Therefore, Eq. (23) can be applied, in principle, for computing the higher-order perturbative corrections to the cusp anomalous dimension, given that we have a closed recursion of the perturbative equations.

Besides for rectangular light-like Wilson loops on the null-plane, Eq. (26) is valid for transverse-

momentum densities with longitudinal gauge links on the light-cone $\Phi(x, k_\perp)$, such that

$$\mu \frac{d}{d \mu} \left[ \frac{d}{d \ln \theta} \ln \Phi(x, k_\perp) \right] = 2 \Gamma_{\text{cusp}},$$

(32)

where the corresponding area is encoded in the rapidity cutoff parameter $\theta \sim (N^+ N^-)^{-1}$ [12]. Another interesting example is given by the $\Pi$-shape loop with one (finite) segment lying on the light-cone [21]. In the one-loop order one has in the large-$N_c$ limit

$$W(\Gamma_{\Pi}) = 1 + \frac{\alpha_s N_c}{2\pi} \left[ -L^2(N N^-) + L(N N^-) - \frac{5\pi^2}{24} \right],$$

(33)

$$L(N N^-) = \frac{1}{2} \left( \ln (\mu NN^- + i\theta) + \ln (-\mu NN^- + i\theta) \right)^2,$$

where the area is defined by the product of the light-like $N^-$ and non-light-like $N$ vectors in the coordinate space, see Fig. 7. The $\Pi$-shaped Wilson loop (34) also satisfies Eq. (26):

$$\mu \frac{d}{d \mu} \left[ \frac{d}{d \ln \sigma} \ln W(\Gamma_{\Pi}) \right] = -2 \Gamma_{\text{cusp}},$$

(34)

the latter being responsible for the renormalization-group behavior of the collinear parton densities in the large-$x$ regime and for the anomalous dimensions of conformal operators with large Lorentz spin [21]. The $\Pi$-shape contour can be split and moved apart to separate two planes by the transverse distance $\xi_\perp$. The Wilson loop obtained in such a way is expected to be “dual” to the TMD, see Fig. 8. The detailed analysis of this configuration will be presented elsewhere.

6. CONCLUSIONS AND OUTLOOK

The universal quantum dynamical approach formulated by Schwinger provides a relevant description
of the geometrical and dynamical properties of loop space. Wilson loops of arbitrary shape are considered then as fundamental degrees of freedom, and the Makeenko–Migdal Eqs. (3) can be derived from the Schwinger–Dyson equations for renormalizable loop equations (which can be treated as the non-perturbative nature of the dynamical loop equations enables us, in principle, to construct a chain of equations for, e.g., the cusp dynamical exponentials on the light-cone. We implemented the program only in one of the simplest cases, a rectangular contour on the transverse null-plane. In Fig. 9, a more involving configuration is visualized, an arbitrary quadrilateral integration contour, of which the area evolution is far from being trivial and deserves a separate study.

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