Variational principles for topological pressures on subsets

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Abstract. The goal of this paper is to define and investigate those topological pressures, which is an extension of topological entropy presented by Feng and Huang [13], of continuous transformations. This study reveals the similarity between many known results of topological pressure. More precisely, the investigation of the variational principle is given and related propositions are also described. That is, this paper defines the measure theoretic pressure $P_{\mu}(T, f)$ for any $\mu \in \mathcal{M}(X)$, and shows $P_B(T, f, K) = \sup \{P_{\mu}(T, f) : \mu \in \mathcal{M}(X), \mu(K) = 1\}$, where $K \subseteq X$ is a non-empty compact subset and $P_B(T, f, K)$ is the Bowen topological pressure on $K$. Furthermore, if $Z \subseteq X$ is an analytic subset, then $P_B(T, f, Z) = \sup \{P_B(T, f, K) : K \subseteq Z$ is compact}. However, this analysis relies on more techniques of ergodic theory and topological dynamics.

Key words and phrases. Measure-theoretic pressure, Variational principle, Borel Probability measure, Topological pressure.

1 Introduction.

Throughout this paper, $(X, T)$ denotes a topological dynamical system (TDS), that is, $X$ is a compact metric space with a metric $d$, and $T : X \to X$ is a continuous transformation. Let $\mathcal{M}(X)$, $\mathcal{M}_T$ and $\mathcal{E}_T$ denote the sets of all Borel probability measures, $T$-invariant Borel probability measures on and $T$-invariant ergodic measures on $X$, respectively. For any $\mu \in \mathcal{M}_T$, let $h_{\mu}(T)$ denote the measure theoretic entropy of $\mu$ with respect to $T$ and let $h_{top}(T)$ denote the topological entropy of the system $(X, T)$, see [22] for precise definitions. It is well-known that entropies constitute essential invariants in the characterization of the complexity of a dynamical system. The classical measure-theoretic entropy for an invariant measure [17] and the topological entropy [1] are introduced. The basic relation between topological entropy and measure theoretic entropy is the variational principle, e.g., see [32].

Topological pressure is a non-trivial and natural generalization of topological entropy. Starting from ideas in the statistical mechanics of lattice systems, Ruelle [26] introduced topological pressure of a continuous function for $\mathbb{Z}^n$ actions on compact spaces and established the variational principle for topological pressure in this context when the action is expansive and satisfies the specification property. Later, Walters [31] proved the variational principle for a $\mathbb{Z}^+-$action without these assumptions. Misiurewicz [20] gave a elegant proof of the variational principle for $\mathbb{Z}^+_+\ $action. See [22] [23] [28] [29] [30]

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for the variational principle for amenable group actions and [11] for actions of sofic groups. Moreover, Barreira [2,3,4], Cao-Feng-Huang [8], Mummert [21], Zhao-Cheng [35,36] dealt with variational principle for topological pressure with nonadditive potentials, and Huang-Yi [15] and Zhang [33], also considered the variational principle for the local topological pressure. This paper conducts research for $\mathbb{Z}$ or $\mathbb{Z}^+$ actions.

From a viewpoint of dimension theory, Pesin and Pitskel’ [25] defined the topological pressure for noncompact sets which is a generalization of Bowen’s definition of topological entropy for noncompact sets [5], and they proved the variational principle under some supplementary conditions. The notions of the topological pressure, variational principle and equilibrium states play a fundamental role in statistical mechanics, ergodic theory and dynamical systems (see the books [6,32]).

Motivated by Feng and Huang’s recent work [13], where the authors studied the variational principle between Bowen topological entropy and measure theoretic entropy for an arbitrary subset. As a natural generalization of topological entropy, topological pressure is a quantity which belongs to one of the concepts in the thermodynamic formalism. This study defines measure theoretic pressure for a Borel probability measure and investigates its variational relation with the Bowen topological pressure. The outline of the paper is as follows. The main results, as well as those definitions of the measure theoretic pressure and topological pressures, are given in Section 2. The proof of the main results and related propositions are given in section 3.

2 Main results

One of the most fundamental dynamical invariants that associate to a continuous map is the topological pressure with a potential function. It roughly measures the orbit complexity of the iterated map on statistical mechanics, ergodic theory and dynamical systems (see the books [6,32]).

2.1 Measure theoretic pressure

Let $\mu \in \mathcal{M}(X)$ and $f \in C(X)$, the measure theoretic pressure of $\mu$ for $T$ (w.r.t. $f$) is defined by

$$P_\mu(T, f) := \int P_\mu(T, f, x) \, d\mu(x)$$

where $P_\mu(T, f, x) := \lim_{\epsilon \to 0} \liminf_{n \to \infty} \left( \frac{1}{n} \log \left| \frac{1}{n} \sum_{i=0}^{n-1} f(T^i x) \right| \right)$. Here, $C(X)$ denotes the Banach space of all continuous functions on $X$ equipped with the supremum norm $\| \cdot \|$.

$$h_\mu(T, x) = \lim_{\epsilon \to 0} \liminf_{n \to \infty} \frac{1}{n} \log \mu(B_n(x, \epsilon)).$$
Also \( f^* \circ T = f^* \), \( \int f^* \, d\mu = \int f \, d\mu \) and \( \int h_\mu(T, x) \, d\mu = h_\mu(T) \). Particularly, if \( \mu \in \mathcal{E}_T \) we have that \( P_\mu(T, f, x) = h_\mu(T, x) + f^*(x) = h_\mu(T) + \int f \, d\mu \) for \( \mu \)-almost every \( x \in X \). See [9, 10, 14, 34, 37] for more details on the measure theoretic pressure of invariant measures for a large class of potentials.

In the following subsections, we turn to give definitions of upper capacity topological pressure, Bowen topological pressure and weighted topological pressure. The main idea of those pressures is the extension from that of Feng and Huang’s approximations in [13].

### 2.2 Upper capacity topological pressure

Recall that the upper capacity topological pressure of \( T \) on a subset \( Z \subseteq X \) with respect to a continuous function \( f \) is given by

\[
P(T, f, Z) = \lim_{\epsilon \to 0} P(T, f, Z, \epsilon)
\]

where

\[
P(T, f, Z, \epsilon) = \limsup_{n \to \infty} \frac{1}{n} \log P_n(T, f, Z, \epsilon),
\]

\[
P_n(T, f, Z, \epsilon) = \sup \{ \sum_{x \in E} e^{f_n(x)} : E \text{ is an } (n, \epsilon)\text{-separated subset of } Z \}.
\]

This definition is equivalent to the Pesin and Pitskel’s definition which is the standard dynamically defined dimension characteristic, see [24] for details.

### 2.3 Bowen topological pressure

Let \( Z \subseteq X \) be a subset of \( X \), which neither has to be compact nor \( T \)-invariant. Fix \( \epsilon > 0 \), we call \( \Gamma = \{B_n(x_i, \epsilon)\} \) a cover of \( Z \) if \( Z \subseteq \bigcup_{i} B_n(x_i, \epsilon) \). For \( \Gamma = \{B_n_i(x_i, \epsilon)\}_i \), set \( n(\Gamma) = \min_i \{n_i\} \).

The theory of Carathéodory dimension characteristic ensures the following definitions.

**Definition 2.1.** Let \( f \) be a continuous function and \( s \in \mathbb{R} \), put

\[
M(Z, f, s, N, \epsilon) = \inf \sum_i \exp(-sn_i + \sup_{y \in B_n(x_i, \epsilon)} f_{n_i}(y)),
\]

where the infimum is taken over all covers \( \Gamma \) of \( Z \) with \( n(\Gamma) \geq N \). Then let

\[
m(Z, f, s, \epsilon) = \lim_{N \to \infty} M(Z, f, s, N, \epsilon),
\]

\[
P_B(T, f, Z, \epsilon) = \inf \{ s : m(Z, f, s, \epsilon) = 0 \} = \sup \{ s : m(Z, f, s, \epsilon) = +\infty \},
\]

\[
P_B(T, f, Z) = \lim_{\epsilon \to 0} P_B(T, f, K, \epsilon).
\]

The term \( P_B(T, f, Z) \) is called the Bowen topological pressure of \( T \) on the set \( Z \) (w.r.t. \( f \)).

The Bowen topological pressure can be defined in an alternative way, see [2] or [24] for more details.

Suppose \( \mathcal{U} \) is a finite open cover of \( X \). Denote the diameter of the open cover by \( |\mathcal{U}| := \max \{ \text{diam}(U) : U \in \mathcal{U} \} \). For \( n \geq 1 \) we denote by \( W_n(\mathcal{U}) \) the collection of strings \( \mathcal{U} = U_1...U_n \) with \( U_i \in \mathcal{U} \). For \( \mathcal{U} \in W_n(\mathcal{U}) \) we call the integer \( m(\mathcal{U}) = n \) the length of \( \mathcal{U} \) and define

\[
X(\mathcal{U}) = U_1 \cap T^{-1}U_2 \cap ... \cap T^{-(n-1)}U_n = \{ x \in X : T^{-j}x \in U_j \text{ for } j = 1,...n \}.
\]

Let \( Z \subseteq X \). We say that \( \Lambda \subseteq \bigcup_{n \geq 1} W_n(\mathcal{U}) \) covers \( Z \) if \( \bigcup_{\mathcal{U} \in \Lambda} X(\mathcal{U}) \supseteq Z \). For \( s \in \mathbb{R} \), define

\[
M_n^R(\mathcal{U}, f, Z) = \inf \sum_{\mathcal{U} \in \Lambda} \exp(-sm(\mathcal{U}) + \sup_{y \in X(\mathcal{U})} f_{m(\mathcal{U})}(y))
\]

and define
where the infimum is taken over all $\Lambda \subset \bigcup_{n \geq 1} W_n(U)$ that cover $Z$ and $\sup_{y \in X(U)} f_m(y) = -\infty$ if $X(U) = \emptyset$. Clearly, $M_N^*(U, f, \cdot)$ is a finite outer measure on $X$, and

$$M_N^*(U, f, Z) = \inf \left\{ M_N^*(U, f, G), G \supset Z, G \text{ is open} \right\}.$$  

Note that $M_N^*(U, f, Z)$ increases as $N$ increases, define

$$M^*(U, f, Z) := \lim_{N \to \infty} M_N^*(U, f, Z)$$

and

$$P_B(T, f, U, Z) := \inf \left\{ s : M^*(U, f, Z) = 0 \right\} = \sup \left\{ s : M^*(U, f, Z) = +\infty \right\},$$

set

$$P_B(T, f, Z) := \sup_{U} P_B(T, f, U, Z)$$

From these notations, it is not difficult to prove that $\sup_{U} P_B(T, f, U, Z) = \lim_{|U| \to 0} P_B(T, f, U, Z)$.

### 2.4 Weighted topological pressure

For any bounded function $g : X \to \mathbb{R}$, $f \in C(X)$, $\epsilon > 0$ and $N \in \mathbb{N}$, define

$$W(g, f, s, N, \epsilon) = \inf \sum_i c_i \exp(-s_n + \sup_{y \in B_i} f_n(y))$$

where the infimum is taken over all finite or countable families $\{B_i, c_i, \epsilon \}$ such that $0 < c_i < \infty, x_i \in X, n_i \geq N$ and

$$\sum_i c_i \chi_{B_i} \geq g,$$

where $B_i := B_i(x_i, \epsilon)$ and $\chi_A$ denotes the characteristic function on a subset $A \subseteq X$. For $K \subseteq X$ and $g = \chi_K$ we set

$$W(K, f, s, N, \epsilon) := W(\chi_K, f, s, N, \epsilon).$$

The quantity $W(K, f, s, N, \epsilon)$ does not decreases as $N$ increases, hence the following limit exists:

$$w(K, f, s, \epsilon) = \lim_{N \to \infty} W(K, f, s, N, \epsilon).$$

Clearly, there exists a critical value of the parameter $s$. Hence, define

$$P_W(T, f, K, \epsilon) = \inf \left\{ s : w(K, f, s, \epsilon) = 0 \right\} = \sup \left\{ s : w(K, f, s, \epsilon) = \infty \right\}$$

It is easy to see that the quantity $P_W(T, f, K, \epsilon)$ is monotone with respect to $\epsilon$, thus the following limit exists:

$$P_W(T, f, K) = \lim_{\epsilon \to 0} P_W(T, f, K, \epsilon).$$

The term $P_W(T, f, K)$ is called a weighted topological pressure of $T$ on the set $K$ (with respect to $f$).

Now we collect some properties of the pressures, see [2] or [24] for proofs.

**Proposition 2.1.** Let $(X, T)$ be a TDS and $f \in C(X)$, then the following properties hold:

(i) For $Z_1 \subseteq Z_2$, $P(T, f, Z_1) \leq P(T, f, Z_2)$, where $P$ is $P, P_B$ or $P_W$;

(ii) For $Z = \bigcup_{i=1}^{\infty} Z_i$, $P_B(T, f, Z) = \sup_{i \geq 1} P_B(T, f, Z_i)$ and $P(T, f, Z) \leq \sup_{i \geq 1} P(T, f, Z_i)$;
(iii) For any $Z \subseteq X$, $P_B(T, f, Z) \leq P(T, f, Z)$. Moreover, we have $P_B(T, f, Z) = P(T, f, Z)$ if $Z$ is $T$-invariant and compact.

The following variational relation between the Bowen topological pressure and the measure theoretic pressure is the main finding of this paper. We give the statement first and postpone the proof to the next section. To formulate our results, we need to introduce an additional notion. A set in a metric space is said to be analytic if it is a continuous image of the set $N$ of infinite sequences of natural numbers (with its product topology). It is known that in a Polish space, the analytic subsets are closed under countable unions and intersections, and any Borel set is analytic (c.f. [12, 2.2.10]).

Theorem A. Let $(X, T)$ be a TDS and $f$ a continuous function on $X$.

(1) If $K \subseteq X$ is non-empty and compact, then

$$P_B(T, f, K) = \sup \{P_\mu(T, f) : \mu \in M(X), \mu(K) = 1\};$$

(2) If the topological entropy of the system is finite, i.e., $h_{top}(T) < \infty$, and $Z \subseteq X$ is analytic, then

$$P_B(T, f, Z) = \sup \{P_B(T, f, K) : K \subseteq Z, K \text{ is compact}\}.$$

3 Proof of the main result

In the academic study of a dynamical system $(X, T)$, the well-known variational principle of topological pressure provides the relationship among pressure, entropy invariants and potential energy from the probabilistic and topological versions. This section provides a proof of the variational principle for these pressures in Theorem A. To study the relations of the Bowen topological pressure with the weighted topological pressure, the following Vitali covering lemma is necessary.

Lemma 3.1. Let $(X, d)$ be a compact metric space and $B = \{B(x_i, r_i)\}_{i \in I}$ be a family of closed (or open) balls in $X$. Then there exists a finite or countable subfamily $B' = \{B(x_i, r_i)\}_{i \in I'}$ of pairwise disjoint balls in $B$ such that

$$\bigcup_{B \in B} B \subseteq \bigcup_{i \in I'} B(x_i, 5r_i).$$

Proof. See [19, Theorem 2.1].

Proposition 3.2. Let $K \subseteq X$. Then for any $s \in \mathbb{R}$ and $\epsilon, \delta > 0$, we have

$$M(K, f, s + \delta, N, 6\epsilon) \leq W(K, f, s, N, \epsilon) \leq M(K, f, s, N, \epsilon)$$

for all sufficiently large $N$, where $M(K, f, s + \delta, N, 6\epsilon) := \inf \sum_i \exp(-sn_i + f_{n_i}(x_i))$ and the infimum is taken over all covers $\Gamma = \{B_{n_i}(x_i, 6\epsilon)\}$ of $K$ with $n(\Gamma) \geq N$. Consequently, we have $P_B(T, f, K, 6\epsilon) \leq P_W(T, f, K, \epsilon) \leq P_B(T, f, K, \epsilon)$ and $P_B(T, f, K) = P_W(T, f, K)$.

Proof. We follow Feng and Huang’s argument [13, Proposition 3.2] to prove this result. Let $K \subseteq X, s \in \mathbb{R}, \epsilon, \delta > 0$, taking $c_i = 1$ in the definition of weighted topological pressure, we see that $W(K, f, s, N, \epsilon) \leq M(K, f, s, N, \epsilon)$ for each $N \in \mathbb{N}$. In the following, we show that $M(K, f, s + \delta, N, 6\epsilon) \leq W(K, f, s, N, \epsilon)$ for all sufficiently large $N$.

Assume that $N \geq 2$ is such that $n^2e^{-n\delta} \leq 1$ for $n \geq N$. Let $\{B_{n_i}(x_i, \epsilon), c_i\}_{i \in I}$ be a family so that $I \subseteq \mathbb{N}, x_i \in X, 0 < c_i < \infty, n_i \geq N$ and

$$\sum_i c_i x_{n_i} \geq \chi_K$$

[5]
where \( B_i := B_{n_i}(x_i, \epsilon) \). We show below that

\[
\mathcal{M}(K, f, s + \delta, N, 6\epsilon) \leq \sum_{i \in I} c_i \exp(-sn_i + \sup_{y \in B_{n_i}(x_i, \epsilon)} f_n(y))
\]

(3.1)

which implies \( \mathcal{M}(K, f, s + \delta, N, 6\epsilon) \leq W(K, f, s, N, \epsilon) \).

Denote \( I_n := \{ i \in I : n_i = n \} \) and \( I_{n,k} = \{ i \in I_n : i \leq k \} \) for \( n \geq N \) and \( k \in \mathbb{N} \). Write for brevity \( B_i := B_{n_i}(x_i, \epsilon) \) and \( 5B_i := B_{n_i}(x_i, 5\epsilon) \) for \( i \in I \). Obviously we may assume \( B_i \neq B_j \) for \( i \neq j \). For \( t > 0 \), set

\[
K_{n,t} = \left\{ x \in K : \sum_{i \in I_n} c_i \chi_{B_i}(x) > t \right\}
\]

and \( K_{n,k,t} = \left\{ x \in K : \sum_{i \in I_{n,k}} c_i \chi_{B_i}(x) > t \right\} \).

We divide the proof of (3.1) into the following three steps.

**Step 1.** This part differs slightly from [13], the construction goes through largely verbatim. We write out the details for collecting some constants. For each \( n \geq N \), \( k \in \mathbb{N} \) and \( t > 0 \), there exists a finite set \( I_{n,k,t} \subseteq I_{n,k} \) such that the balls \( B_i(i \in I_{n,k,t}) \) are pairwise disjoint, \( K_{n,k,t} \subseteq \bigcup_{i \in I_{n,k,t}} 5B_i \) and

\[
\sum_{i \in I_{n,k,t}} \exp(-sn + \sup_{y \in B_i} f_n(y)) \leq \frac{1}{t} \sum_{i \in I_{n,k}} c_i \exp(-sn + \sup_{y \in B_{n_i}(x_i, \epsilon)} f_n(y)).
\]

Since \( I_{n,k} \) is finite, by approximating \( c_i^\prime \) from above, we may assume that each \( c_i \) is a positive integer. Let \( m \) be the least integer with \( m \geq t \). Denote \( B = \{ B_i : i \in I_{n,k} \} \) and define \( u : B \to \mathbb{Z} \) by \( u(B_i) = c_i \). We define by induction integer-valued functions \( v_0, v_1, ..., v_m \) on \( B \) and subfamilies \( B_1, ..., B_m \) of \( B \) starting with \( v_0 = u \). Using Lemma 3.1 (in which we take the metric \( d_n \) instead of \( d \) we find a pairwise disjoint subfamily \( B_1 \) of \( B \) such that \( \bigcup_{B \in B_1} B \subseteq \bigcup_{B \in B_1} 5B, \) and hence \( K_{n,k,t} \subseteq \bigcup_{B \in B_1} 5B \). Then by repeatedly using Lemma 3.1 we can define inductively for \( j = 1, ..., m \), disjoint subfamilies \( B_j \) of \( B \) such that

\[
B_j \subseteq \{ B \in B : v_{j-1}(B) \geq 1 \}, \quad K_{n,k,t} \subseteq \bigcup_{B \in B_j} 5B
\]

and the function \( v_j \) such that

\[
v_j(B) = \begin{cases} 
  v_{j-1}(B) - 1 & \text{for } B \in B_j, \\
  v_{j-1}(B) & \text{for } B \in B \setminus B_j.
\end{cases}
\]

This is possible since \( K_{n,k,t} \subseteq \left\{ x : \sum_{B \in B : B \ni x} v_j(B) \geq m - j \right\} \) for \( j < m \), whence every \( x \in K_{n,k,t} \) belongs to some ball \( B \in B \) with \( v_j(B) \geq 1 \). Hence,

\[
\sum_{j=1}^m \sum_{B \in B_j} \exp(-sn + \sup_{y \in B} f_n(y)) = \sum_{j=1}^m \sum_{B \in B_j} (v_{j-1}(B) - v_j(B)) \exp(-sn + \sup_{y \in B} f_n(y)) \\
\leq \sum_{B \in B_j} \sum_{j=1}^m (v_{j-1}(B) - v_j(B)) \exp(-sn + \sup_{y \in B} f_n(y)) \\
\leq \sum_{B \in B} u(B) \exp(-sn + \sup_{y \in B} f_n(y)) \\
= \sum_{i \in I_{n,k}} c_i \exp(-sn + \sup_{y \in B_i} f_n(y)).
\]
Choose \( j_0 \in \{1, \ldots, m\} \) so that \( \sum_{B \in B_{j_0}} \exp(-sn + \sup_{y \in B} f_n(y)) \) is the smallest. Then
\[
\sum_{B \in B_{j_0}} \exp \left( -sn + \sup_{y \in B} f_n(y) \right) \leq \frac{1}{m} \sum_{i \in I_{n,k}} c_i \exp \left( -sn + \sup_{y \in B_n(x, \epsilon)} f_n(y) \right)
\]
\[
\leq \frac{1}{t} \sum_{i \in I_{n,k}} c_i \exp \left( -sn + \sup_{y \in B_n(x, \epsilon)} f_n(y) \right).
\]

Hence \( J_{n,k,t} = \{ i \in I_{n,k} : B_i \in B_{j_0} \} \) is as desired.

**Step 2.** For each \( n \geq N \) and \( t > 0 \), we have
\[
\mathcal{M}(K_{n,t}, f, s + \delta, N, 6\epsilon) \leq \frac{1}{n^\delta t} \sum_{i \in I_{n}} c_i \exp \left( -sn + \sup_{y \in B_n(x, \epsilon)} f_n(y) \right). \tag{3.2}
\]

To see this, assume \( K_{n,t} \neq \emptyset \); otherwise there is nothing to prove. It’s clear that \( K_{n,k,t} \uparrow K_{n,t} \), \( K_{n,k,t} \neq \emptyset \) when \( k \) is large enough. Let \( J_{n,k,t} \) be the sets constructed in step 1, then \( J_{n,k,t} \neq \emptyset \) when \( k \) is large enough. Define \( E_{n,k,t} = \{ x_i : i \in J_{n,k,t} \} \). Note that the family of all non-empty compact subsets of \( X \) is compact with respect to the Hausdorff distance (cf. [12, 2.10.21]). It follows that there is a subsequence \( (k_j) \) of natural numbers and a non-empty compact set \( E_{n,t} \subseteq X \) such that \( E_{n,k_j,t} \) converges to \( E_{n,t} \) in the Hausdorff distance as \( j \to \infty \). Since any two points in \( E_{n,k,t} \) have a distance (with respect to \( d_n \)) not less then \( \epsilon \), so do the points in \( E_{n,t} \). Thus \( E_{n,t} \) is a finite set and \( \sharp(E_{n,k_j,t}) = \sharp(E_{n,t}) \) when \( j \) is large enough. Hence
\[
\bigcup_{x \in E_{n,t}} B_n(x, 5.5\epsilon) \supseteq \bigcup_{x \in E_{n,k_j,t}} B_n(x, 5\epsilon) = \bigcup_{i \in J_{n,k_j,t}} 5B_i \supseteq K_{n,k_j,t}
\]
when \( j \) is large enough, and thus \( \bigcup_{x \in E_{n,t}} B_n(x, 6\epsilon) \supseteq K_{n,t} \). Since \( \sharp(E_{n,k_j,t}) = \sharp(E_{n,t}) \) when \( j \) is large enough, using the result in step 1 we have
\[
\sum_{x \in E_{n,t}} \exp(-ns + f_n(x)) \leq \sum_{x \in E_{n,k_j,t}} \exp(-ns + \sup_{y \in B_n(x, \epsilon)} f_n(y)) \leq \frac{1}{t} \sum_{i \in I_{n}} c_i \exp(-sn + \sup_{y \in B_n(x, \epsilon)} f_n(y)).
\]

Hence,
\[
\mathcal{M}(K_{n,t}, f, s + \delta, N, 6\epsilon) \leq \sum_{x \in E_{n,t}} \exp \left( -n(s + \delta) + f_n(x) \right)
\]
\[
\leq \frac{1}{n^\delta t} \sum_{i \in I_{n}} c_i \exp \left( -sn + \sup_{y \in B_n(x, \epsilon)} f_n(y) \right)
\]
\[
\leq \frac{1}{n^\delta t} \sum_{i \in I_{n}} c_i \exp \left( -sn + \sup_{y \in B_n(x, \epsilon)} f_n(y) \right).
\]

**Step 3.** For any \( t \in (0, 1) \), we have
\[
\mathcal{M}(K, f, s + \delta, N, 6\epsilon) \leq \frac{1}{t} \sum_{i \in I} c_i \exp(-sn + \sup_{y \in B_n(x, \epsilon)} f_n(y)).
\]

As a result, \((3.3)\) holds.
To see this, fix \( t \in (0, 1) \). Note that \( \sum_{n=N}^{\infty} n^{-2} < 1 \) and \( K \subseteq \bigcup_{n=N}^{\infty} K_{n,n^{-2}} \). By (3.2) we have

\[
\mathcal{M}(K, f, s + \delta, N, 6\epsilon) \leq \sum_{n=N}^{\infty} \mathcal{M}(K_{n,t}, f, s + \delta, N, 6\epsilon) \\
\leq \sum_{n=N}^{\infty} \frac{1}{t} \sum_{i \in I_n} c_i \exp\left(-sn_i + \sup_{y \in B_n(x, \epsilon)} f_n(y)\right) \\
\leq \frac{1}{t} \sum_{i \in I} c_i \exp\left(-sn_i + \sup_{y \in B_n(x, \epsilon)} f_n(y)\right).
\]

To end the proof of this proposition, note that the Bowen topological pressure does not change if we replace \( \sup_{y \in B_n(x, \epsilon)} f_n(y) \) by any number in the interval \( [\inf_{y \in B_n(x, \epsilon)} f_n(y), \sup_{y \in B_n(x, \epsilon)} f_n(y)] \) in the definition of the Bowen topological pressure, see [2, Corollary 1.2] or [24] for a proof of this fact. \( \square \)

The following lemma is an analogue of Feng and Huang’s approximation and classic Frostman’s lemma, see [13, Lemma 3.4].

**Lemma 3.3.** Let \( K \) be a nonempty compact subset of \( X \) and \( f \in C(X) \). Let \( s \in \mathbb{R}, \ N \in \mathbb{N} \) and \( \epsilon > 0 \). Suppose that \( c := W(K, f, s, N, \epsilon) > 0 \). Then there is a Borel probability measure \( \mu \) on \( X \) such that \( \mu(K) = 1 \) and

\[
\mu(B_n(x, \epsilon)) \leq \frac{1}{c} \exp\left[-ns + \sup_{y \in B_n(x, \epsilon)} f_n(y)\right], \quad \forall x \in X, n \geq N.
\]

**Proof.** Clearly \( c < \infty \). We define a function \( p \) on the Banach space \( C(X) \) by

\[
p(g) = \frac{1}{c} W(\chi_K \cdot g, f, s, N, \epsilon).
\]

Let \( 1 \in C(X) \) denote the constant function \( 1(x) = 1 \), it is easy to verify that

1. \( p(g + h) \leq p(g) + p(h) \) for any \( g, h \in C(X) \);
2. \( p(tg) = tp(g) \) for any \( t \geq 0 \) and \( g \in C(X) \);
3. \( p(1) = 1, 0 \leq p(g) \leq \|g\| \) for any \( g \in C(X) \), and \( p(h) = 0 \) for \( h \in C(X) \) with \( h \leq 0 \).

Applying the Hahn-Banach Theorem, we can extend the linear functional \( t \mapsto tp(1), t \in \mathbb{R} \), from the subspace of the constant functions to a linear functional \( L : C(X) \to \mathbb{R} \) satisfying

\[
L(1) = p(1) = 1 \quad \text{and} \quad -p(-g) \leq L(g) \leq p(g) \quad \text{for any} \ g \in C(X).
\]

If \( g \in C(X) \) with \( g \geq 0 \), then \( p(-g) = 0 \) and \( L(g) \geq 0 \). Hence, combining the fact that \( L(1) = 1 \), we can use the Riesz representation theorem to find a Borel probability measure \( \mu \) on \( X \) such that \( L(g) = \int g \, d\mu \) for \( g \in C(X) \).

Now we show that \( \mu(K) = 1 \). To see this, for any compact set \( E \subseteq X \setminus K \), by the Uryson lemma there is \( g \in C(X) \) such that \( 0 \leq g(x) \leq 1, g(x) = 1 \) for \( x \in E \) and \( g(x) = 0 \) for \( x \in K \). Then \( g \cdot \chi_K = 0 \) and thus \( p(g) = 0 \). Hence \( \mu(E) \leq L(g) \leq p(g) = 0 \). This shows \( \mu(X \setminus K) = 0 \), i.e., \( \mu(K) = 1 \).

In the end, we show that

\[
\mu(B_n(x, \epsilon)) \leq \frac{1}{c} \exp\left[-ns + \sup_{y \in B_n(x, \epsilon)} f_n(y)\right], \quad \forall x \in X, n \geq N.
\]
To see this, for any compact set \( E \subset B_n(x, \epsilon) \), by the Urysohn lemma, there exists \( g \in C(X) \) such that \( 0 \leq g \leq 1 \), \( g(y) = 1 \) for \( y \in E \) and \( g(y) = 0 \) for \( y \in X \setminus B_n(x, \epsilon) \). This implies that \( \mu(E) \leq L(g) \leq p(g) \). Since \( g \cdot \chi_K \leq \chi_{B_n(x, \epsilon)} \) and \( n \geq N \), we have

\[
W(\chi_K \cdot g, f, s, N, \epsilon) \leq \exp \left[ -ns + \sup_{y \in B_n(x, \epsilon)} f_n(y) \right]
\]

and thus \( p(g) \leq \frac{1}{c} \exp \left[ -ns + \sup_{y \in B_n(x, \epsilon)} f_n(y) \right] \). Therefore \( \mu(E) \leq \frac{1}{c} \exp \left[ -ns + \sup_{y \in B_n(x, \epsilon)} f_n(y) \right] \), it follows that

\[
\mu(B_n(x, \epsilon)) = \sup \{ \mu(E) : E \text{ is a compact subset of } B_n(x, \epsilon) \} \\
\leq \frac{1}{c} \exp \left[ -ns + \sup_{y \in B_n(x, \epsilon)} f_n(y) \right].
\]

This completes the proof of the lemma. \( \Box \)

Now it’s ready to prove the first result in Theorem A.

**Proof of Theorem A(i).** Let \( \mu \in \mathcal{M}(X) \) satisfying \( \mu(K) = 1 \). Write

\[
P_\mu(T, f, x, \epsilon) = \liminf_{n \to \infty} \frac{1}{n} \log \frac{\mu(B_n(x, \epsilon))}{\mu(B_n(x, \epsilon) - 1)}
\]

for \( x \in X \), \( n \in \mathbb{N} \) and \( \epsilon > 0 \). Since \( \frac{1}{n} \log \frac{\mu(B_n(x, \epsilon))}{\mu(B_n(x, \epsilon) - 1)} \geq -\|f\| \) for each \( n \), applying Fatou’s lemma we have

\[
\lim_{\epsilon \to 0} \int P_\mu(T, f, x, \epsilon) \, d\mu \geq \int P_\mu(T, f, x) \, d\mu = P_\mu(T, f).
\]

Thus, to show \( P_B(T, f, K) \geq P_\mu(T, f) \), it suffices to prove that \( P_B(T, f, x, \epsilon) \geq \int P_\mu(T, f, x, \epsilon) \, d\mu \) for each \( \epsilon > 0 \).

Fix \( \epsilon > 0 \) and \( l \in \mathbb{N} \). Denote \( u_l = \min \{ l, \int P_\mu(T, f, x, \epsilon) \, d\mu - \frac{1}{l} \} \). Then there exists a Borel set \( A_l \subseteq X \) with \( \mu(A_l) > 0 \) and \( N \in \mathbb{N} \) such that

\[
\mu(B_n(x, \epsilon)) \leq \exp \left( -nu_l + f_n(x) \right), \forall x \in A_l, n \geq N.
\]

Now let \( \{ B_n(x_i, \epsilon/2) \} \) be a countable or finite family such that \( x_i \in X \), \( n_i \geq N \) and \( \bigcup_i B_n(x_i, \epsilon/2) \supseteq K \cap A_l \). We may assume that \( B_n(x_i, \epsilon/2) \cap (K \cap A_l) \neq \emptyset \) for each \( i \), and choose \( y_i \in B_n(x_i, \epsilon/2) \cap (K \cap A_l) \), then

\[
\sum_i \exp \left[ -nu_i + \sup_{y \in B_n(x_i, \epsilon/2)} f_n(y) \right] \geq \sum_i \exp \left( -nu_i + f_n(y_i) \right) \geq \sum_i \mu(B_n(y_i, \epsilon)) \geq \sum_i \mu(B_n(x_i, \epsilon/2)) \geq \mu(K \cap A_l) = \mu(A_l) > 0.
\]

It follows that

\[
M(K \cap A_l, f, u_l, N, \epsilon/2) \geq \mu(A_l) > 0.
\]

Therefore \( P_B(T, f, K) \geq P_B(T, f, K \cap A_l) \geq u_l \). Letting \( l \to \infty \), we have \( P_B(T, f, K) \geq \int P_\mu(T, f, x, \epsilon) \, d\mu \).

Hence

\[
P_B(T, f, K) \geq P_\mu(T, f).
\]
We next show that

\[ P_B(T, f, K) \leq \sup \left\{ P_\mu(T, f) : \mu \in M(X), \mu(K) = 1 \right\}. \]  

(3.3)

We can assume that \( P_B(T, f, K) \neq -\infty \), otherwise we have nothing to prove. By Proposition 3.2 we have \( P_B(T, f, K) = P_W(T, f, K) \). Fix a small number \( \beta > 0 \). Let \( s = P_B(T, f, K) - \beta \). Since

\[
\lim_{\epsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \left[ f_n(x) - \sup_{y \in B_n(x, \epsilon)} f_n(y) \right] = 0
\]

for all \( x \in X \), we have that

\[
\lim_{n \to \infty} \frac{1}{n} \left[ f_n(x) - \sup_{y \in B_n(x, \epsilon)} f_n(y) \right] > -\beta, \ \forall x \in X
\]

for all sufficiently small \( \epsilon > 0 \). Take such an \( \epsilon > 0 \) and a \( N \in \mathbb{N} \) such that \( c := W(K, f, s, N, \epsilon) > 0 \). By Lemma 3.3 there exists \( \mu \in M(X) \) with \( \mu(K) = 1 \) such that \( \mu(B_n(x, \epsilon)) \leq \frac{1}{c} \exp \left[ -ns + \sup_{y \in B_n(x, \epsilon)} f_n(y) \right] \) for any \( x \in X \) and \( n \geq N \). Therefore

\[
P_\mu(T, f, x) \geq P_\mu(T, f, x, \epsilon) \geq s + \lim_{n \to \infty} \frac{1}{n} \left[ f_n(x) - \sup_{y \in B_n(x, \epsilon)} f_n(y) \right] \geq P_B(T, f, K) - 2\beta
\]

for all \( x \in X \). Hence,

\[
P_\mu(T, f) = \int P_\mu(T, f, x) d\mu \geq P_B(T, f, K) - 2\beta.
\]

Consequently, (3.3) is obtained immediately.

Next we turn to prove the second result in Theorem A. We will first prove this result in the case of that \( X \) is zero-dimensional, and then prove it in general. Now we prove a useful lemma first.

**Lemma 3.4.** Assume that \( \mathcal{U} \) is a closed-open partition of \( X \). Let \( N \in \mathbb{N} \) and \( f \in C(X) \).

(i) If \( E_{i+1} \supseteq E_i \) and \( \bigcup_i E_i = E \), then \( M_N^s(\mathcal{U}, f, E) = \lim_{i \to \infty} M_N^s(\mathcal{U}, f, E_i) \);

(ii) Assume \( Z \subset X \) is analytic. Then \( M_N^s(\mathcal{U}, f, Z) = \sup \{ M_N^s(\mathcal{U}, f, K) : K \subset Z, K \text{ is compact} \} \).

**Proof.** We first show that (i) implies (ii). Assume that (i) holds. Let \( Z \) be analytic, i.e., there exists a continuous surjective map \( \Phi : \mathcal{N} \to Z \). Let \( \Gamma_{n_1, n_2, \ldots, n_p} \) be the set of \( (m_1, m_2, \ldots) \in \mathcal{N} \) such that \( n_1 \leq m_1, m_2 \leq n_2, \ldots, m_p \leq n_p \) and let \( Z_{n_1, \ldots, n_p} \) be the image of \( \Gamma_{n_1, \ldots, n_p} \) under \( \Phi \). Let \( \{ \epsilon_p \}_{p \geq 1} \) be a sequence of positive numbers. Due to (i), we can pick a sequence \( \{ n_p \}_{p \geq 1} \) of positive integers recursively so that

\[
M_N^s(\mathcal{U}, f, Z_{n_1, \ldots, n_p}) \geq M_N^s(\mathcal{U}, f, Z_{n_1, \ldots, n_{p-1}}) - \epsilon_p, \ p = 2, 3, \ldots
\]

Hence,

\[
M_N^s(\mathcal{U}, f, Z_{n_1, \ldots, n_p}) \geq M_N^s(\mathcal{U}, f, Z) - \sum_{i=1}^{\infty} \epsilon_i, \ \forall p \in \mathbb{N}.
\]

Let

\[
K = \bigcap_{p=1}^{\infty} Z_{n_1, \ldots, n_p}.
\]
Since $\Phi$ is continuous, we can show that $\bigcap_{p=1}^{\infty} Z_{n_1,\ldots,n_p} = \bigcap_{p=1}^{\infty} Z_{n_1,\ldots,n_p}$ by applying Cantor’s diagonal argument. Hence $K$ is a compact subset of $Z$. If $\Lambda \subset \bigcup_{j \geq N} \mathcal{W}_j(U)$ is a cover of $K$ (of course it is an open cover), then it is a cover of $Z_{n_1,\ldots,n_p}$ when $p$ is large enough, which implies

$$\sum_{y \in X(U)} \exp \left( -sm(U) + \sup_{y \in X(U)} f_{m(U)}(y) \right) \geq \lim_{p \to \infty} M_N^p(U, f, Z_{n_1,\ldots,n_p}) \geq M_N^p(U, f, Z) - \sum_{i=1}^{\infty} \epsilon_i.$$ 

Hence $M_N^p(U, f, K) \geq M_N^p(U, f, Z) - \sum_{i=1}^{\infty} \epsilon_i$. Since $\sum_{i=1}^{\infty} \epsilon_i$ can be chosen arbitrarily small, we have proved (ii).

Now we turn to prove (i). Note that any two non-empty elements in $\mathcal{W}_n(U)$ are disjoint, and each element in $\mathcal{W}_{n+1}(U)$ is a subset of some element in $\mathcal{W}_n(U)$. We call this the net property of $\{\mathcal{W}_n(U)\}_{n \geq 1}$. Let $E_i \uparrow E$ be given. Let $\{\delta_i\}_{i \geq 1}$ be a sequence of positive numbers to be specified later and for each $i$, choose a cover $\Lambda_i \subset \bigcup_{j \geq N} \mathcal{W}_j(U)$ of $E_i$ such that

$$\sum_{U \in \Lambda_i} \exp \left( -sm(U) + \sup_{y \in X(U)} f_{m(U)}(y) \right) \leq M_N^\infty(U, f, E_i) + \delta_i. \tag{3.4}$$

By the net property of $\{\mathcal{W}_n(U)\}_{n \geq 1}$, we may assume these elements in $\Lambda_i$ are disjoint for each $i$.

For any $x \in E_i$, choose $U_x \subset \bigcup_{n=1}^{\infty} \Lambda_i$ such that $X(U_x)$ containing $x$ and $m(U_x)$ is the smallest. By the net property of $\{\mathcal{W}_n(U)\}_{n \geq 1}$, the collection $\{U_x : x \in E\}$ consists of countable many disjoint elements. Relabel these elements as $U_i’$s. Clearly $E \subset \bigcup_i X(U_i)$.

We now choose an integer $k$. Let $A_1$ denote the collection of those $U_i’$s that are taken from $\Lambda_1$. They cover a certain subset $Q_1$ of $E_k$. The same subset is covered by a certain sub-collection of $\Lambda_k$, denoted as $\Lambda_{k,1}$, since $\Lambda_{k,1}$ also covers the smaller set $Q_1 \cap E_1$, by (3.4)

$$\sum_{U \in A_1} \exp \left( -sm(U) + \sup_{y \in X(U)} f_{m(U)}(y) \right) \leq \sum_{U \in \Lambda_{k,1}} \exp \left( -sm(U) + \sup_{y \in X(U)} f_{m(U)}(y) \right) + \delta_1. \tag{3.5}$$

To see this, assume that (3.5) is false. Then by (3.4),

$$\sum_{U \in (\Lambda_1 \setminus A_1) \cup \Lambda_{k,1}} \exp \left( -sm(U) + \sup_{y \in X(U)} f_{m(U)}(y) \right) < M_N^\infty(U, f, E_1)$$

which contradicts the fact that $(A_1 \setminus A_1) \cup \Lambda_{k,1} \subset \bigcup_{j \geq N} \mathcal{W}_j(U)$ is a open cover of $E_1$. Next we use $A_2$ to denote the collection of those $U_i’$s that are taken from $\Lambda_2$ but not from $\Lambda_1$. Define $\Lambda_{k,2}$ similarly. As above, we find

$$\sum_{U \in A_2} \exp \left( -sm(U) + \sup_{y \in X(U)} f_{m(U)}(y) \right) \leq \sum_{U \in \Lambda_{k,2}} \exp \left( -sm(U) + \sup_{y \in X(U)} f_{m(U)}(y) \right) + \delta_2. \tag{3.6}$$

We repeat the argument until all coverings $\Lambda_n, n \leq k$, have been considered. Note that $\bigcup_{U \in \Lambda_{k,n}} U \subset \bigcup_{k=1}^{n} A_k U$ for $i \leq k$. For different $i, i’ \leq k$, the elements in $\Lambda_{k,i}$ are disjoint from those in $\Lambda_{k,i’}$. The $k$ inequalities (3.5), (3.6), ..., are added which yield

$$\sum_{U \in \Lambda_{k,1}} \exp \left( -sm(U) + \sup_{y \in X(U)} f_{m(U)}(y) \right) \leq \sum_{U \in \Lambda_{k,n}} \exp \left( -sm(U) + \sup_{y \in X(U)} f_{m(U)}(y) \right) + \sum_{n=1}^{k} \delta_n$$

$$\leq M_N^\infty(U, f, E_k) + \sum_{n=1}^{k} \delta_n + \delta_k.$$
Let $k \to \infty$, we have
\[
\sum_i \exp \left( - sm(U_i) + \sup_{y \in X(U_i)} f_m(U_i)(y) \right) \leq \lim_{k \to \infty} M_N^k(U, f, E_k) + \sum_{n=1}^{\infty} \delta_n
\]
Since $\sum_{n=1}^{\infty} \delta_n$ can be chosen arbitrarily small, it follows that
\[
M_N^k(U, f, E) \leq \lim_{k \to \infty} M_N^k(U, f, E_k).
\]
Clearly, the opposite inequality is trivial, thus (i) is proven.

**Theorem 3.5.** Let $(X, T)$ be a TDS. Assume that $X$ is zero-dimensional, i.e., for any $\delta > 0$, $X$ has a closed-open partition with diameter less then $\delta$. Then, for any analytic set $Z \subseteq X$,
\[
P_B(T, f, Z) = \sup \{ P_B(T, f, K) : K \subseteq Z, K \text{ is compact} \}.
\]

**Proof.** Let $Z$ be an analytic subset of $X$ with $P_B(T, f, Z) \neq -\infty$, otherwise there is nothing to prove. Let $s < P_B(T, f, Z)$. Since $P_B(T, f, Z) = \sup_{\mathcal{U}} P_B(T, f, \mathcal{U}, Z) = \lim_{k \to \infty} P_B(T, f, \mathcal{U}, Z)$, there exists a closed-open partition $\mathcal{U}$ so that $P_B(T, \mathcal{U}, f, Z) > s$ and thus $M^*(\mathcal{U}, f, Z) = \infty$. Hence $M_N^*(\mathcal{U}, f, Z) > 0$ for some $N \in \mathbb{N}$. By Lemma [3.3] we can find a compact set $K \subseteq Z$ such that $M_N^*(\mathcal{U}, f, K) > 0$. This implies $P_B(T, f, K) \geq P_B(T, \mathcal{U}, f, K) \geq s$. This is the result that we need.

**Proposition 3.6.** Let $(X, T)$ be a TDS with $h_{top}(T) < \infty$ and $f \in C(X)$, then there exists a factor $\pi : (Y, S) \to (X, T)$ such that $(Y, S)$ is zero-dimensional and
\[
\sup_{x \in X} P(S, f \circ \pi, \pi^{-1}(x)) \leq \|f\|.
\]

**Proof.** Assume that $(X, T)$ is a TDS with $h_{top}(T) < \infty$. By Lemma 3.13 in [13], there exists a factor $\pi : (Y, S) \to (X, T)$ such that $(Y, S)$ is zero-dimensional and
\[
\sup_{x \in X} P(S, 0, \pi^{-1}(x)) = 0.
\]
This immediately implies that $\sup_{x \in X} P(S, f \circ \pi, \pi^{-1}(x)) \leq \|f\|$.

**Proposition 3.7.** If $\pi : (Y, S) \to (X, T)$ is a factor map and $f$ is a continuous function on $X$, then for $E \subseteq Y$ we have
\[
P_B(T, f, \pi(E)) \leq P_B(S, f \circ \pi, E) \leq P_B(T, f, \pi(E)) + \sup_{x \in X} P(S, f \circ \pi, \pi^{-1}(x)).
\]

**Proof.** See [18, Theorem 2.1] for the proof of the second inequality. It is left to prove the first inequality. Fix $\epsilon > 0$. By the uniform continuity of the map $\pi$, there exists $\delta > 0$ such that
\[
d(x, y) < \delta \implies d(\pi(x), \pi(y)) < \epsilon.
\]
Fix a positive integer $N$, consider a cover of $E$ with Bowen balls $\{B_{n_i}(x_i, \epsilon)\}$, where $n_i \geq N$ for each $i$. Then it is easy to see that $\{B_{n_i}(\pi(x_i), \epsilon)\}$ is a cover of $\pi(E)$, and $M(\pi(E), f, s, N, \epsilon) \leq M(E, f \circ \pi, s, N, \delta)$. This implies that
\[
M(\pi(E), f, s, \epsilon) \leq M(E, f \circ \pi, s, \delta).
\]
Hence, $P_B(T, f, \pi(E), \epsilon) \leq P_B(S, f \circ \pi, E, \delta)$. Since $\epsilon \to 0$ implies $\delta \to 0$, let $\epsilon \to 0$ we have
\[
P_B(T, f, \pi(E)) \leq P_B(S, f \circ \pi, E).
\]
This completes the proof of the theorem.
Proposition 3.8. Let \((X, T)\) be a TDS with \(h_{\text{top}}(T) < \infty\), then there exists a factor \(\pi : (Y, S) \to (X, T)\) such that \((Y, S)\) is zero-dimensional and
\[
P_B(T, f, \pi(E)) = P_B(S, f \circ \pi, E), \quad \forall E \subseteq Y.
\]

Proof. By Proposition 3.6, there exists a factor \(\pi : (Y, S) \to (X, T)\) such that \((Y, S)\) is zero-dimensional and \(\sup_{x \in X} P(S, f \circ \pi, \pi^{-1}(x)) \leq \|f\|\).

Since for any \(c \in \mathbb{R}\) and \(f \in C(X)\), we have \(P_B(T, f + c, Z) = P_B(T, f, Z) + c\) and \(P(S, f \circ \pi + c, \pi^{-1}(x)) = P(S, f \circ \pi, \pi^{-1}(x)) + c\). Applying Proposition 3.7 for the function \(f - \|f\|\), we have
\[
P_B(T, f, \pi(E)) \leq P_B(S, f \circ \pi, E) \leq P_B(T, f, \pi(E)) + \sup_{x \in X} P(S, f \circ \pi, \pi^{-1}(x)) - \|f\|.
\]
This implies that
\[
P_B(T, f, \pi(E)) = P_B(S, f \circ \pi, E).
\]

Now we turn to prove the second result in Theorem A.

Proof of Theorem A(ii). By Proposition 3.8, there exists a factor map \(\pi : (Y, S) \to (X, T)\) such that \((Y, S)\) is zero-dimensional and \(P_B(T, f, \pi(E)) = P_B(S, f \circ \pi, E)\) for any \(f \in C(X)\) and \(E \subseteq Y\).

Let \(Z\) be an analytic subset of \(X\). Then \(\pi^{-1}(Z)\) is also an analytic subset of \(Y\). Using Proposition 3.5, we have
\[
P_B(T, f, Z) = \sup \{ P_B(T, f, E) : E \subseteq \pi^{-1}(Z), E \text{ is compact} \}
= \sup \{ P_B(T, f, \pi(E)) : E \subseteq \pi^{-1}(Z), E \text{ is compact} \}
\leq \sup \{ P_B(T, f, K) : K \subseteq Z, K \text{ is compact} \}
\]
By Proposition 2.4, the reverse inequality is trivial. Hence,
\[
P_B(T, f, Z) = \sup \{ P_B(T, f, K) : K \subseteq Z, K \text{ is compact} \}.
\]

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References

[1] R. Adler, A. Konheim and M. McAndrew, Topological entropy, Trans Amer Math Soc, 114 (1965), 309-319.

[2] L. Barreira, A non-additive thermodynamic formalism and applications to dimension theory of hyperbolic dynamical systems, Ergodic Theory and Dynamical Systems, 16 (1996), 871-927.

[3] L. Barreira, Nonadditive thermodynamic formalism: equilibrium and Gibbs measures, Disc. Contin. Dyn. Syst., 16 (2006), 279-305.

[4] L. Barreira, Almost additive thermodynamic formalism: some recent developments, Rev. Math. Phys., 22(10) (2010), 1147-1179.
[5] R. Bowen, Topological entropy for noncompact sets, *Trans. Amer. Math. Soc.*, **184** (1973), 125-136.

[6] R. Bowen, Equilibrium states and the ergodic theory of Anosov diffeomorphisms, *Lecture notes in Math.*, 470, Springer-Verlag, 1975.

[7] M. Brin and A. Katok, On local entropy, *Lecture Notes in Mathematics*, 1007, Springer-Verlag, 1983.

[8] Y. Cao, D. Feng and W. Huang, The thermodynamic formalism for sub-multiplicative potentials, *Discrete Contin. Dyn. Syst. Ser. A*, **20** (2008), 639-657.

[9] Y. Cao, H, Hu and Y. Zhao, Nonadditive Measure-theoretic Pressure and Applications to Dimensions of an Ergodic Measure, *Ergod. Th. & Dynam. Sys.*, **33** (2013), 831-850.

[10] W. Cheng, Y. Zhao and Y. Cao, Pressures for asymptotically subadditive potentials under a mistake function, *Discrete Contin. Dyn. Syst. Ser. A*, **32**(2) (2012), 487-497.

[11] N. Chung, Topological pressure and the variational principle for actions of sofic groups, *Ergod. Th. & Dynam. Sys.* doi:10.1017/S0143385712000429

[12] H. Federer, Geometric measure theory, Mathematical Foundations and Applications, second edition, John Wiley & Sons, Inc., Hoboken, NJ, 2003.

[13] D. Feng, W. Huang, Variational principles for topological entropies of subsets, *Journal of Functional Analysis.*, **263** (2012), 2228-2254.

[14] L. He, J. Lv and L. Zhou, Definition of measure-theoretic pressure using spanning sets, *Acta Math. Sinica, Engl. Ser.*, **20** (2004), 709-718.

[15] W. Huang, Y. Yi, A local variational principle of pressure and its applications to equilibrium states, *Israel J. Math.*, **161** (2007), 29-94.

[16] A. Katok, Lyapunov exponents, entropy and periodic orbits for diffeomorphisms, *Inst. Hautes Études Sci. Publ. Math.*, **51** (1980), 137-173.

[17] A. Kolomogorov, A new metric invariant of transient dynamical systems and automorphisms of lebesgue spaces, *Dokl Akad Soc S SSR*,**119** (1958), 861-864 (Russian).

[18] Q. Li, E. Chen and X. Zhou, A note of topological pressure for non-compact sets of a factor map, *Chaos, Solitons and Fractals*, **49** (2013),72-77.

[19] P. Mattila,Geometry of sets and Measures in Euclideans,Cambridge University Press,1995.

[20] M. A. Misiurewicz, A short proof of the variational principle for a $\mathbb{Z}^n$ action on a compact space, *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astron. Phys.*, **24**(12) (1976), 1069-1075.

[21] A. Mummert, The thermodynamic formalism for almost-additive sequences, *Discrete Contin. Dyn. Syst.*, **16** (2006), 435-454.

[22] J. M. Ollagnier, D. Pinchon, The variational principle, *Studia Math.*, **72**(2) (1982), 151-159.

[23] J. M. Ollagnier, Ergodic Theory and Statistical Mechanics (Lecture Notes in Mathematics, 1115), Springer, Berlin, 1985.
[24] Ya. Pesin, Dimension theory in dynamical systems, Contemporary Views and Applications, University of Chicago Press, Chicago, 1997.

[25] Ya. Pesin and B. Pitskel', Topological pressure and the variational principle for noncompact sets, Functional Anal. Appl., 18 (1984), 307-318.

[26] D. Ruelle, Statistical mechanics on a compact set with $Z^\nu$ action satisfying expansiveness and specification, Trans. Amer. Math. Soc., 187 (1973), 237-251.

[27] D. Ruelle, Repellers for real analytic maps, Ergodic Theory Dynamical Systems, 2 (1982), 99-107.

[28] A. M. Stepin, A. T. Tagi-Zade, Variational characterization of topological pressure of the amenable groups of transformations, Dokl. Akad. Nauk SSSR 254(3) (1980), 545-549. In Russian, translated in Sov. Math. Dokl. 22(2) (1980), 405-409.

[29] A. A. Tempelman, Specific characteristics and variational principle for homogeneous random fields, Z. Wahrscheinlichkeitstheor. Verw. Geb. 65(3) (1984), 341-365.

[30] A. A. Tempelman, Ergodic Theorems for Group Actions, Informational and Thermodynamical Aspects (Mathematics and its Applications, 78), Kluwer Academic, Dordrecht, 1992. Translated and revised from the 1986 Russian original.

[31] P. Walters, A variational principle for the pressure of continuous transformations, Amer. J. Math., 97 (1975), 937-971.

[32] P. Walters, An introduction to ergodic theory, Springer-Verlag, New York, 1982.

[33] G. Zhang, Variational principles of pressure, Discrete Contin. Dyn. Syst., 24(4) (2009), 1409-1435.

[34] Y. Zhao, A note on the measure-theoretic pressure in subadditive case, Chinese Annals of Math., Series A, No. 3 (2008), 325-332.

[35] Y. Zhao, W. Cheng, Variational principle for conditional pressure with subadditive potential, Open Syst. Inf. Dyn., 18(4) (2011), 389-404.

[36] Y. Zhao, W. Cheng, Coset pressure with sub-additive potentials, Stoch. Dyn., (2013), DOI: 10.1142/S0219493713500123.

[37] Y. Zhao, Y. Cao, Measure-theoretic pressure for subadditive potentials, Nonlinear Analysis, 70 (2009), 2237-2247.