EXCEPTIONAL AND COSMETIC SURGERIES ON KNOTS

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Abstract. We show that the distance of a link $K$ with respect to a bridge surface of any genus determines a lower bound on the genus of essential surfaces and Heegaard surfaces in the manifolds that result from non-trivial Dehn surgeries on the knot. In particular, knots with high bridge distance do not admit non-trivial non-hyperbolic surgeries or non-trivial cosmetic surgeries. We further show that if a knot has bridge distance at least 3 then its bridge number is bounded above by a function of Seifert genus, or indeed by the genus of (almost) any essential surface or Heegaard surface in the surgered manifold.

1. Introduction

We begin with a very brief summary of some of the highlights in the paper for easy reference. In the next section we explain the origin and significance of these results in greater detail.

The main goal of this paper is to show that the genus of essential surfaces and Heegaard surfaces in a manifold obtained by Dehn surgery on a knot $L$ gives rise to an upper bound on the bridge distance of $L$. The bridge distance of a link $L$ in a 3-manifold $M$, like bridge number $b(L)$, is a non-negative integer invariant of links in 3-manifolds. In fact, there are two slightly different measures of bridge distance, $d_C$ and $d_{AC}$; we consider both of them.

We show that a knot with high bridge distance does not admit exceptional or cosmetic surgeries. In particular, we make significant progress towards solving the Cabling Conjecture by showing that if $K \subset S^3$ has a reducing surgery then $d_C(K) \leq 2$ (assuming $b(K) \geq 6$). In [7], we give a characterization of knots $K \subset S^3$ with $d_C(K) = 2$. Combined with the fact that if a knot $K \subset S^3$ has $d_C(K) = 1$ then it has an essential tangle decomposition, this gives a detailed description of any potential counterexample to the conjecture. On a related note, we show that the Berge Conjecture (concerning knots with lens space surgeries) is true for all $K \subset S^3$ with $d_C(K) \geq 6$. The following theorem provides a summary of these and related results.
**Theorem.** Let $M$ be a closed, orientable manifold and suppose that $L \subset M$ is a knot in minimal bridge position with respect to a minimal genus Heegaard surface for $M$. Assume that the exterior of $L$ is irreducible and $\partial$-irreducible. Then:

1. If $M = S^3$, $b(L) \geq 6$, and a non-trivial surgery on $L$ produces a reducible 3-manifold, then $d_C(L) \leq 2$.

2. If $M = S^3$, $b(L) \geq 7$, and a non-trivial surgery on $L$ produces a 3-manifold with an essential torus, then $d_C(L) \leq 2$.

3. If $M = S^3$, $b(L) \geq 3$, and a non-trivial surgery on $L$ produces a lens space, then $d_C(L) \leq 5$.

4. If $M = S^3$, $b(L) \geq 3$, and a non-trivial surgery on $L$ produces a small Seifert fibered space, then $d_C(L) \leq 6$.

5. If $M \neq S^3$ does not contain an essential sphere or torus and $L$ is a hyperbolic knot with a non-trivial non-hyperbolic surgery then $d_C(L) \leq 12$.

6. If $M \neq S^3$ has Heegaard genus $g$, and a non-trivial surgery on $L$ produces a 3-manifold with Heegaard genus at most $g$, then $d_C(L) \leq \max(6, 4g + 4)$.

**Remark 1.1.** Conclusions (1) and (2) can be found in Theorem 10.2. Conclusions (3) - (6) are contained in Theorem 9.4 below.

In fact, it turns out that there is a surprising relationship between bridge number, bridge distance, and the genus of essential surfaces in the knot exterior. Using this relationship, we also exhibit a startling connection between the Seifert genus and bridge number of a knot of high bridge distance:

**Theorem 10.3.** Suppose that $L$ is a knot in a homology sphere (with $b(L) \geq 3$ if $M = S^3$). If $d_C(L) \geq 3$, then

\[ b(L) \leq 4g(L) \]

where $g(L)$ is the Seifert genus of $L$.

2. **Background and results**

Every link $L$ in a 3-manifold $M$ can be put into bridge position with respect to a minimum genus Heegaard surface $H$ for $M$ [12]. The minimum $b(L)$ of $b(H) = |L \cap H|/2$ over all possible bridge positions
is a natural, and well-studied, natural number invariant of the link. As with Heegaard splittings [26], every bridge surface can be uniquely assigned a non-negative integer called “distance” (see Section 4.2). In an effort to provide the strongest possible results, we study two related notions of distance. The distance in the curve complex $d_C(L)$ is defined to be the minimum distance of a minimal bridge surface as measured in the curve complex. It is an integer valued link invariant as long as $b(L) \geq 3$ if $M = S^3$. If the exterior of $L$ is irreducible it is at least 1. Similarly, the distance in the arc and curve complex $d_{AC}(L)$, is defined to be the minimum distance of a minimal bridge surface as measured in the arc and curve complex. It is an integer valued link invariant as long as $b(L) \geq 2$ if $M = S^3$. If the exterior of $L$ is irreducible and $\partial$-irreducible, it also is at least 1. There exist knots and links of arbitrarily high distance [6, 29]. Indeed, if the well-known analogy between bridge surfaces and Heegaard surfaces continues to hold, high distance knots are likely “generic” [35,36].

Although bridge number and distance are independent of each other, we find a surprising relationship between their product and the genus of surfaces in 3-manifolds obtained by Dehn surgery on $L$. More precisely, we show:

**Theorem 2.1.** Suppose that $L \subset M$ is a knot with irreducible and $\partial$-irreducible exterior. Let $M'$ be the result of non-trivial Dehn surgery on $L$. Let $S \subset M'$ be a closed connected orientable surface of genus $g$. Then the following hold:

1. If $S$ is an essential surface and if $\Delta$ is the surgery distance, then either there is a closed essential surface of genus at most $g$ in the exterior of $L$ or
   \[ \Delta b(L)(d_{AC}(L) - 2) \leq \max(2, 4g - 2) \]

2. If $S$ is a Heegaard surface, then one of the following holds:
   - $b(L)(d_{AC}(L) - 2) \leq \max(1, 2g)$
   - There is a Heegaard surface of genus $g$ for the exterior of $L$
   - $M$ contains an essential surface of genus at most $g$ which intersects $L$ exactly twice.

The first conclusion is proved as part of Theorem 6.3 and the second as part of Theorem 8.1. Refinements of the bounds in Theorem 2.1 give very strong bounds on the distance and bridge numbers of hyperbolic knots $L$ with exceptional or cosmetic surgeries. We explore these consequences, and others, in the subsections which follow.
2.1. Distance and exceptional surgeries. The cabling conjecture [18] asserts that if a knot in $S^3$ has a reducing surgery then the knot is a cable knot and the slope of the surgery is the slope of the cabling annulus. The conjecture has been shown to hold for many classes of knots including satellite knots [45], symmetric knots [13, 22, 34], persistently laminar knots [9, 10], alternating knots [37], many knots with essential tangle decompositions [23], knots where a surgery produces a connected sum of lens spaces [17], knots with bridge number at most 4 [28] (with respect to a sphere), and knots that are band sums [50].

In this paper, we show:

**Theorem 10.2 (1).** No reducible 3-manifold can be obtained by surgery on a knot $L \subset S^3$ with $b(L) \geq 6$ and $d_C(L) \geq 3$.

This puts strong restrictions on any potential counterexample to the cabling conjecture [18]. For, suppose that a counterexample $K \subset S^3$ is in minimal bridge position with respect to a Heegaard sphere $T$. Hoffman [27] showed that $b(T) \geq 5$ and, in [28], claims he has also proved (in unpublished notes) that $b(T) \geq 6$. If that result is correct, then our result reduces the cabling conjecture to studying knots having bridge spheres $T$ satisfying the simple combinatorial condition $d_C(T) \leq 2$. Cable knots have distance at most 2, so we have substantial new evidence for the cabling conjecture. In fact, it suggests a program for proving the cabling conjecture: First, prove it for knots with minimal bridge spheres of distance 1 (which is partially done by Hayashi, since such knots have an essential meridional planar surface in the exterior). Second, prove it for knots with minimal bridge spheres of distance 2. Such bridge spheres have been extensively studied in [7].

A characterization of hyperbolic knots in $S^3$ having toroidal surgeries is more elusive and examples are easily constructed by considering knots lying on knotted genus two surfaces in $S^3$. Additionally, many other examples of knots with toroidal surgeries are known (e.g. [14, 52]). It is known that the surgery slope of a toroidal surgery must be integral or half-integral [20] and the punctured torus in the knot exterior can be assumed to have no more than 2 boundary components [21]. We show:

**Specialization of Theorem 10.2 (2).** Suppose that $L \subset S^3$ is a hyperbolic knot with $b(L) \geq 7$ and $d_C(L) \geq 3$, then $L$ does not admit a toroidal surgery.

The Berge conjecture (cf. [32, Problem 1.78]) states that every knot in $S^3$ with a lens space surgery is doubly primitive with respect to a
genus two Heegaard surface for \( S^3 \). Recently, a number of research programs to prove the Berge conjecture have been proposed and partially completed. In [4], a two step program using knot Floer homology is outlined and the first step of the program is completed in [25]. In [54], the first step in a three step program is completed that would result in the proof of the Berge conjecture for tunnel number one knots. We show:

**Theorem 9.4(1).** If \( L \subset S^3 \) is a knot with \( b(L) \geq 3 \) and \( d_C(L) \geq 6 \), then no surgery on \( L \) produces a lens space.

Using the fact that small Seifert fibered spaces have Heegaard genus at most 2 [8], we show:

**Theorem 9.4(2).** If \( L \subset S^3 \) is a knot with \( b(L) \geq 3 \) and \( d_C(L) \geq 7 \), then no non-trivial surgery on \( L \) produces a small Seifert-fibered space.

Theorem 9.4 also gives a version of these results for hyperbolic knots in other 3-manifolds. It uses the Geometrization Theorem [39–41] to show that a hyperbolic knot \( L \) with \( d_C(L) \geq 13 \) does not have any non-trivial non-hyperbolic surgeries.

Knots of distance at most 12 are certainly plentiful. But, as long as the distance of a knot \( L \) admitting an exceptional surgery is at least 3, we can also bound \( b(L) \). A modification (done in Section 10) of the techniques which produce Theorem 2.1 produces the following:

**Specialization of Corollary 10.4.** Suppose that \( L \subset S^3 \) has \( b(L) \geq 3 \) and \( d_C(L) \geq 3 \) and has tunnel number at least 2. If \( L \) admits an exceptional surgery then \( b(L) \leq 8 \).

(The tunnel number of \( L \) is one less than the minimum genus of a Heegaard surface for the exterior of \( L \).)

2.2. **Heegaard genus.** Hyperbolic knots not only have very few exceptional surgeries, they also have very few cosmetic surgeries [5, Theorem 1], i.e., surgeries along distinct slopes that produce the same 3-manifold. Indeed, it is conjectured [32, Problem 1.81] that hyperbolic knots have no so-called “exotic” cosmetic surgeries. For example, Gabai [16] showed that no non-trivial knot in \( S^2 \times S^1 \) admits a non-trivial cosmetic surgery. Gordon and Luecke’s solution to the knot complement conjecture shows that non-trivial knots in \( S^3 \) admit no cosmetic surgeries [19]. More generally, Lackenby has determined bounds on the denominator of the surgery coefficient of a cosmetic surgery [33]. The article [5] gives a thorough survey of known results on cosmetic surgery.
If \( L \) is a knot in a 3-manifold \( M \) of Heegaard genus \( g \), and if \( L \) has a non-trivial cosmetic surgery, then, obviously, \( L \) has a non-trivial surgery producing a 3-manifold of genus \( g \). We show the following (which is a combination of Theorem 9.4(4) and Theorem 10.4):

**Theorem.** Assume that \( M \neq S^3 \) is closed, has Heegaard genus \( g \), and that \( L \subset M \) is a knot. If \( L \) admits a non-trivial cosmetic surgery, then

\[
d_C(L) \leq \max(5, 4g + 4).
\]

Furthermore, if \( M \) is non-Haken and if the tunnel number of \( L \) is at least \( g \), then also

\[
b(L) \leq 2g + 4.
\]

More generally we show

**Corollary 10.4 (2).** Suppose that \( L \subset M \) is a knot in a closed non-Haken 3-manifold of Heegaard genus \( g \geq 1 \). Assume that \( d_C(L) \geq 3 \). Suppose that the tunnel number of \( L \) is at least \( g \) and that a non-trivial Dehn surgery on \( L \) produces a 3-manifold of Heegaard genus at most \( g \). Then \( b(L) \leq 2g + 4 \).

Finally, we note that a number of authors have studied the effect of Dehn surgery on Heegaard surfaces. Most of them have fixed a knot and studied the Heegaard genera which can result from surgery on the knot, often producing an inequality relating surgery distance to Heegaard genus. For example, Moriah and Rubinstein \[38\] have shown that most surgeries on most knots preserve the isotopy classes of Heegaard splittings of their exteriors. (See also \[15, 43\].) Likewise, Rieck \[42\] showed that most (but possibly not all) surgeries on an annular knot strictly increase the Heegaard genus.

On the other hand, we show that “most” knots do not admit Heegaard genus decreasing Dehn surgeries. A philosophy similar to the one presented in the current paper is evident in the recent paper by Baker, Gordon, and Luecke \[2\] that relates surgery distance, knot width, and the Heegaard genus of the surgered 3-manifold for certain knots in \( S^3 \).

### 2.3. Genus and Bridge number.

As a final application, we consider the relationship of Seifert genus to bridge number.

In general, the Seifert genus and the bridge number of a knot \( L \subset S^3 \) are unrelated. For example, let \( J \) be a non-trivial knot and let \( K_n \) be the \( n \)-th Whitehead double of \( J \). The knots \( K_n \) each have Seifert genus 1, but have bridge number going to infinity with \( n \) \[48, 49\]. Hyperbolic examples of this phenomena can likely be constructed using recently developed machinery of Baker, Gordon, and Luecke \[3\].
stark contrast, this cannot occur when the knots have bridge distance at least 3:

**Theorem 10.3.** Suppose that $L$ is a knot in a homology sphere (with $b(L) \geq 3$ if $M = S^3$). If $d_c(L) \geq 3$, then

$$b(L) \leq 4g(L)$$

where $g(L)$ is the Seifert genus of $L$.

2.4. **Structure of the Paper.** In Section 3 we introduce the notions of bridge surface, Heegaard surface and thin position for knots in 3-manifolds as well as present relevant background results. In Section 4, we define sloped Heegaard surface and introduce the two notions of distance that we investigate in the paper. In Section 5, we bound the product of bridge number and distance of bridge surfaces in terms of the genus of surfaces in the knot exterior that admit desirable embeddings. In Section 6, we use the results from Section 5 to bound the product of bridge number and distance of bridge surfaces in terms of the genus of essential surfaces. In Section 7, we analyze the graphic associated to the simultaneous sweepouts of two sloped Heegaard surfaces. In Section 8 and Section 9, we use the results of Section 7 and Section 5 to bound the product of bridge number and distance of bridge surfaces in terms of the genus of alternative sloped Heegaard surfaces. In Section 10, we employ additional techniques to improve the bounds found elsewhere in the paper and present applications to toroidal and reducible surgeries.

3. **Definitions**

If $X$ is an embedded submanifold of $M$, we let $\eta(X)$ denote an open regular neighborhood of $X$. The notation $|X|$ indicates the number of components of $X$. If $L$ is a link in $M$, then $M(L)$ denotes the exterior $M - \eta(L)$ of $L$.

A surface $F$ properly embedded in a 3-manifold $M$ is *essential* if it is incompressible, not boundary parallel, and not a 2–sphere bounding a 3-ball. We say $F$ is *boundary-compressible* if there exists an embedded disk in $M$ with interior disjoint from $F \cup \partial M$ and with boundary the endpoint union of an essential arc in $F$ and an arc in $\partial M$. We will denote the genus of $F$ by $g(F)$.

If $V$ is a union of tori then a *multislope* $\sigma$ on $V$ is an isotopy class in $V$ of a union of essential curves, one in each component of $V$. If $V \subset \partial M$ and if $F \subset M$ is a properly embedded surface, we say that $F$ defines a $\sigma$-slope on $V$ if $\sigma$ is represented by the union of components of $\partial F \cap V$. If $\sigma$ and $\tau$ are multislopes on $V$, their **intersection number**, denoted $\Delta(\sigma, \tau)$, is calculated by taking the minimum intersection number (over
all components of $V$) between minimally intersecting representatives of $\sigma$ and $\tau$ in a single component of $V$. In particular, if $\Delta(\sigma, \tau) > 0$ then the slopes are distinct in every component. The result of Dehn filling $M(L)$ along the multislope $\sigma$ is denoted $M(L)(\sigma)$.

3.1. Heegaard Splittings. A handlebody is a compact 3-manifold homeomorphic to a closed regular neighborhood of a graph $G$ properly embedded in $\mathbb{R}^3$. A compression body $H$ is a connected 3-manifold homeomorphic to any component of a regular neighborhood in a 3-manifold $N$ of $G \cup \partial N$. Here $N$ may have empty boundary and $G$ is an embedded graph, possibly with vertices in $\partial N$. The inclusion in $H$ of $\partial N$ is called the negative boundary of $H$ and is denoted $\partial_- H$. We call $\partial H \setminus \partial_- H$ the positive boundary of $H$ and denote it $\partial_+ H$. The inclusion into $H$ of $G \cup \partial N$ is called a spine of $H$.

A Heegaard splitting for a 3-manifold $M$ is a triple $(H, H_\downarrow, H_\uparrow)$ where $H$ is a connected, closed, embedded, separating surface and $H_\downarrow$, $H_\uparrow$ are compression bodies with disjoint interiors such that $M = H_\downarrow \cup H_\uparrow$, $H = \partial_+ H_\downarrow = \partial_+ H_\uparrow$ and $\partial M = \partial_- H_\downarrow \cup \partial_- H_\uparrow$. The surface $H$ is called a Heegaard surface.

3.2. Bridge Surfaces. The terminology presented in this section is an adaptation of that used by various authors including Hayashi-Shimokawa [24], Scharlemann-Tomova [47], and Taylor-Tomova [51]. Given a link $L$ embedded in a manifold $M$ we will denote by $M(L)$ the complement in $M$ of an open regular neighborhood of $L$ in $M$, i.e., $M(L) = M \setminus \eta(L)$.

We say that a Heegaard surface $H$ for a compact manifold $M$ transverse to a properly embedded 1-manifold $L$ is a bridge surface for $(M, L)$ if for each arc $\alpha$ of $L \cap H_\downarrow$ or $L \cap H_\uparrow$ with both endpoints on $H$ there is an embedded disk $D \subset M$ with interior disjoint from $H \cup L$ and with boundary the endpoint union of an essential arc in $H$ and an arc in $L$ (any such disk $D$ is called a bridge disk for $L$) and every other component of $L \cap H_\downarrow$ or $L \cap H_\uparrow$ is an arc isotopic to an I-fiber of $\partial_- H_\downarrow \times I$ or $\partial_- H_\uparrow \times I$.

A compressing disk for the bridge surface $H$ is an embedded disk in $M(L)$ whose interior is disjoint from $H$ and whose boundary is an essential loop in $H \setminus \eta(L)$. A bridge surface is stabilized if there is a pair of compressing disks $D_\downarrow$, $D_\uparrow$ on opposite sides of $H$ such that $D_\downarrow \cap D_\uparrow$ is a single point. A bridge surface $H$ is weakly reducible if there is a disjoint pair of compressing disks $D_\downarrow$, $D_\uparrow$ on opposite sides of $H$. A bridge surface that is not weakly reducible is called strongly irreducible.

Suppose that $D_\downarrow$ and $D_\uparrow$ are bridge disks on opposite sides of $H$ such that $D_\downarrow \cap D_\uparrow$ is contained in $L$. If $D_\downarrow \cap D_\uparrow$ is a single point, then $H$
is perturbed. If $H$ is perturbed, isotoping $H$ across either $D_\uparrow$ or $D_\downarrow$ produces a new bridge surface for $L$ intersecting $L$ two fewer times.

If $D_\downarrow \cap D_\uparrow$ is two points contained in $L$, then $H$ is cancellable and the component $K \subset \partial D_\downarrow \cup \partial D_\uparrow$ is called the cancellable component of $L$. Note that a cancellable or perturbed bridge surface may not be weakly reducible. If $H$ is cancellable, we can isotope $H$ across $D_\downarrow \cup D_\uparrow$ to a surface $H'$ containing $K$. The surface $R = H' \cap M(L)$ is called a cancelled bridge surface for $(M, L)$.

We say that a component of $L$ is removable if $H$ is cancellable with canceling disks $D_\downarrow, D_\uparrow$ and there is a compressing disk disjoint from one of $D_\downarrow, D_\uparrow$ and intersecting the other in a single point in its boundary. If a component $K \subset L$ is removable and non-perturbed, the surface $H$ can be isotoped to be a bridge surface for $(M - \hat{\eta}(K), L - K)$. See [47] for more details.

3.3. Thin position. Ever since Gabai’s introduction [16] of thin position, it has been an extremely useful tool in studying knots and 3-manifolds. We use a version of thin position due, in its original form, to Hayashi and Shimokawa [22]. Here are the relevant definitions and results.

A multiple bridge surface [22] for $(M, L)$ is a surface $F \cup S \subset M$ transverse to $L$ such that in each component $C$ of the closure of $M \setminus F$, there is a unique component of $S$ which is a bridge surface for $(C, C \cap L)$.

Taylor and Tomova [51] (generalizing work of Hayashi-Shimokawa) prove a more general version of the following theorem.

**Theorem 3.1.** Let $L \subset M$ be a link in a compact, orientable 3-manifold such that $M - L$ is irreducible and no sphere in $M$ intersects $L$ transversally exactly once. Suppose that $\overline{H}$ is a genus $g$ bridge surface for $(M, L)$ which is not removable, perturbed, stabilized, or $\partial$-stabilized. Then there is a multiple bridge surface $S \cup F$ for $(M, L)$ such that all of the following hold:

1. Each component of $F$ is essential in $M(L)$.
2. Each component of $S$ is strongly irreducible in $M \setminus F$
3. Each component of $S \cup F$ has genus no greater than $g$ and intersects $L$ no more than $|(S \cup F) \cap L|$ times.
4. No component of $S$ is removable, perturbed, stabilized, or $\partial$-stabilized.

**Proof.** Conclusions (1) - (2) follow directly from [51, Corollary 9.4] applied with $K = \overline{H}$, $T = L$, and $\Gamma = \emptyset$. Conclusion (3) follows from the definition of “thinning” given in [51]. Conclusion (4) follows from [51, Lemma 6.3].
Remark 3.2. As stated above, Theorem 3.1 also follows from the work of [22], although it is not immediately clear that all of Hayashi and Shimokawa’s thin surfaces are essential. It is also similar to Camplisi’s theorem in [11] which develops thin position for sloped Heegaard splittings (see below). Her theorem, however, also does not guarantee that all thin surfaces are essential.

Since we state all of our applications for knots in closed 3-manifolds, we have not defined the notion of “∂-stabilization”. For its definition see [51].

The next lemma records, for later reference, some properties of a cancelled bridge surface.

Lemma 3.3. Let $S \cup F$ be a multiple bridge surface for $(M, L)$. Suppose that a component $H \subset S$ is a cancellable bridge surface for a link $L \cap N$ where $N$ is the closure of the component of $M \setminus F$ that contains $H$. Let $K$ be a cancellable component of $L \cap N$ that is not the unknot and let $R$ be the cancelled bridge surface. Then all of the following hold:

- The slopes $\rho$ and $\tau$ in $\partial \eta(K)$ defined by $R$ and $H$, respectively, intersect in a single point.
- $R$ is an essential surface in $M(L)$.
- The sum of the genera of the components of $R$ is equal to $g(H)$ if $R$ disconnected and equal to $g(H) - 1$ if $R$ is connected.
- If $K = L$ and if $M$ is closed, then after Dehn filling $\partial \eta(K) \subset \partial M(L)$ with slope $\rho$ and capping $\partial R \cap \partial \eta(K)$ with disks, we obtain an incompressible surface in the filled manifold.

Proof. Suppose that $D_\downarrow$ and $D_\uparrow$ are bridge disks for the cancellable component $K \subset \partial D_\downarrow \cup \partial D_\uparrow$. We can obtain $R$ by simultaneously boundary compressing $H \cap M(L)$ using the disks $D_\downarrow$ and $D_\uparrow$. Since these disks each intersect a component of $\partial (H \cap M(L))$ exactly once, each component of $\partial R$ in $\partial \eta(K)$ intersects a meridian of $K$ exactly once. This is the first claim.

Suppose that $E$ is a compressing disk for $R \cap M(L)$. Recall that $H'$ is the surfaces obtained from $H$ by isotopying it along the cancelling pair of disks so that the cancelling component is contained in $H'$. Since $E$ is disjoint from $K$, we can assume $E$ is fixed by the isotopy taking $H'$ to $H$ and that $E$ is disjoint from $D_\downarrow \cup D_\uparrow$.

Suppose $\partial E$ is inessential in $H \cap M(L)$, then $\partial E$ bounds a disk in $H'$ that is disjoint from $L$. Since $E$ is disjoint from $D_\downarrow \cup D_\uparrow$ and $K \subset D_\downarrow \cup D_\uparrow$, then $D_\downarrow \cup D_\uparrow$ is disjoint from the disk $\partial E$ bounds in $H$. Thus, $\partial E$ is inessential in $R$, a contradiction. Hence, $\partial E$ is essential in $H \cap M(L)$ and therefore $E$ is a compressing disk for $H \cap M(L)$. Since
\[ \partial N \] consists of essential surfaces in \( F \), then \( E \) is a compressing disk for \( H \cap M(L) \) in \( N \cap M(L) \). Since one of \( D \downarrow \) \( D \uparrow \) is on the opposite side of \( H \) from \( E \) and \( E \) is disjoint from \( D \downarrow \cup D \uparrow \), then \( H \) is weakly reducible, a contradiction to Theorem 3.1. Therefore \( R \) must be incompressible.

Suppose that \( E \) is a boundary compressing disk for the incompressible surface \( R \). Since \( \partial R \) lies in torus components of \( \partial (M(L)) \), this implies that \( R \) is a boundary parallel annulus. Hence, \( L = K \) is a knot. Moreover, \( H \) is obtained from \( R \) by attaching the annulus \( \eta(K) \cap H \) and so \( H \) is a torus and \( K \) can be isotoped to lie on this torus. In fact, \( E \) is a compressing disk for \( H \) intersecting one of \( D \downarrow \) \( D \uparrow \) exactly once. Hence, \( H \) is removable, a contradiction to Theorem 3.1. We conclude that \( R \) is an essential surface in \( M(L) \). This is the second claim.

The loop \( K \) in \( H' \) is either separating or non-separating. If it is non-separating, then \( R \) is connected and has genus one less than the genus of \( H \). If \( K \) is separating, then \( R \) has two components. Since \( K \) is not the unknot, neither of these components is a disk and, therefore, they both must have genus strictly less than the genus of \( H \). This is the third claim.

Suppose that \( K = L \) and that \( M \) is closed. Since \( K \) is not the unknot in 1-bridge position, \( K \) is an essential loop in \( H' \). Since \( M \) is closed, \( H' \) is compressible to both sides in \( M \). Hence, the Jaco handle addition theorem [30], applied to the 3-manifolds on either side of \( H' \), implies that \( R(\rho) \) is incompressible in \( M(L)(\rho) \). This is the fourth claim. \( \square \)

4. Sloped Heegaard Surfaces

Rather than working with a \((3\text{-manifold, link})\) pair \((M, L)\), it is often advantageous to work entirely in the exterior of the link. To that end, we follow Campisi [11] and define the notion of a “sloped Heegaard splitting”.

Let \( M \) be a compact, orientable manifold, possibly with boundary, containing a properly embedded link \( L \). Let \( N = M(L) \) and let \( \partial_L N = \partial N \setminus \partial M \) be a union of torus boundary components of \( N \) defined by \( L \) with \( \sigma \) a multislope on \( \partial_L N \).

In each component \( V_i \) of \( \partial_L N \), choose a collection of parallel simple closed curves \( \alpha_i \) representing the slope \( \sigma_i = \sigma \cap V_i \). Let \( \Gamma \) be a connected graph that is the union of the curves \( \alpha_i \) with a properly embedded graph \( \Gamma^- \) in \( N \) having the property that any vertex of \( \Gamma^- \) contained in any \( V_i \) is also contained in some component of some \( \alpha_i \). Let \( b_i \) be a regular neighborhood of a component of some \( \alpha_i \). Let \( S_i \) be a regular neighborhood of \( \Gamma \) together with any component of \( \partial N \) that contains only vertices of \( \Gamma \), as in Figure 1. We call \( S_i \) a boundary compression.
body with slope $\sigma$. The union of $\Gamma$ and any component of $\partial N \setminus \partial L N$ intersecting $\Gamma$ only in vertices is called a spine for $S \ddownarrow$. If the boundary compression body is either a 3–ball or the product of a component of $\partial N$ with the closed interval, we call it trivial.

If the closure of the complement of $S \ddownarrow \subset N$ is a second boundary compression body $S \uparrow$ then we will say that $(S, S \ddownarrow, S \uparrow)$ is a $\sigma$-sloped Heegaard splitting for $N$. The surface

$$S = S \ddownarrow \cap S \uparrow = \partial_+ S \ddownarrow = \partial_+ S \uparrow$$

is a properly embedded surface called a $\sigma$-sloped Heegaard surface. If the multislope $\sigma$ is clear from the context, we drop it from the terminology and refer to sloped Heegaard splittings and sloped Heegaard surfaces. The following straightforward lemma explains our interest in sloped Heegaard splittings. Its proof is left to the reader.

**Lemma 4.1.** If $H$ is a bridge surface for $(M, L)$ then $H = \overline{H} \cap M(L)$ is a meridian sloped Heegaard surface for $N = M(L)$. Conversely, suppose that $H$ is a $\sigma$-sloped Heegaard surface for $N$, and that $N(\sigma)$ is obtained by Dehn filling $\partial N$ with slope $\sigma$. If $L$ is the union of the cores of the filling tori then there is a bridge surface $\overline{H}$ for $(N(\sigma), L)$ such that $H = \overline{H} \cap M(L)$.

**Definition 4.2.** Henceforth, whenever $\overline{H}$ is a surface in a 3-manifold $M$ and whenever we are considering the exterior $M(L)$ of a link $L \subset M$ transverse to $\overline{H}$, we will let $H = \overline{H} \cap M(L)$.

A separating, properly embedded, possibly disconnected surface $S$ in $N$ is weakly $\partial$-reducible if there are disjoint disks $D \ddownarrow, D \uparrow$, each either a compressing disk or a boundary compressing disk, on opposite sides of $S$. If $S$ has compressing or boundary compressing disks on both sides but is not weakly boundary reducible then it is strongly $\partial$-irreducible. Note that if $\overline{H}$ is a bridge surface for $(M, L)$ that is perturbed or
cancellable then the two bridge disks that intersect in one or two points in \( L \) define disjoint \( \partial \)-compressing disks for \( H \). Thus, being weakly reducible is a more restrictive condition than being weakly \( \partial \)-reducible. However, \( H \) will be weakly \( \partial \)-irreducible if and only if \( \overline{H} \) is either weakly reducible, perturbed or cancellable.

Campisi [11] defines a structure called a \textit{sloped generalized Heegaard splitting} that generalizes sloped Heegaard surfaces in the same way that generalized Heegaard surfaces [46] generalize Heegaard surfaces. A \textit{sloped generalized Heegaard splitting} [11] of \( N \) is a decomposition

\[
N = ((S_1)_{\downarrow} \cup S_1(S_1)_{\uparrow}) \cup F_1( (S_2)_{\downarrow} \cup S_2(S_2)_{\uparrow}) \cup F_2 \cdots \cup F_{m-1}( (S_m)_{\downarrow} \cup S_m(S_m)_{\uparrow})
\]

such that each of the \((S_i)_{\uparrow}\) and \((S_i)_{\downarrow}\) are boundary compression bodies. The collection of surfaces \( \cup_i F_i = \mathcal{F} \) is called the \textit{thin surfaces} and the collection of surfaces \( \cup_i S_i = S \) is called the \textit{thick surfaces} of the decomposition. We denote the sloped generalized Heegaard splitting by \((S, \mathcal{F})\). As in Lemma 4.1, sloped generalized Heegaard splittings and multiple bridge surfaces are equivalent concepts.

4.1. \textbf{Sweep-outs}. Let \((S, S_{\uparrow}, S_{\downarrow})\) be a sloped Heegaard splitting for a manifold \( M \). From the definition of a spine one can construct a map \( \phi_S : M \to [0, 1] \) such that \( \phi_S^{-1}(0) \) is a spine for \( S_{\downarrow} \), \( \phi_S^{-1}(1) \) is a spine for \( S_{\uparrow} \) and \( \phi_S^{-1}(s) = S_s \) is properly isotopic to \( S \) for all \( s \in (0, 1) \). This function is called a \textit{sweep-out} representing \((S, S_{\uparrow}, S_{\downarrow})\).

More generally, if \( S_i \) is a thick surface with boundary in a sloped generalized Heegaard splitting that contains thin levels, then \((S_i, (S_i)_{\uparrow}, (S_i)_{\downarrow})\) is a sloped Heegaard splitting of the submanifold \( M_i = (S_i)_{\downarrow} \cup S_i(S_i)_{\uparrow} \). As above, one can construct a map \( \phi_i : M_i \to [0, 1] \) such that \( \phi_i^{-1}(0) \) is a spine for \( (S_i)_{\downarrow} \), \( \phi_i^{-1}(1) \) is a spine for \( (S_i)_{\uparrow} \) and \( \phi_i^{-1}(s) = (S_i)_s \) is properly isotopic to the surface \( S_i \) for all \( s \in (0, 1) \). This function is called a \textit{partial sweep-out} representing \((S_i, (S_i)_{\uparrow}, (S_i)_{\downarrow})\).

4.2. \textbf{Distance}. The \textit{curve complex} \( \mathcal{C}(S) \) of a compact surface \( S \) is the simplicial complex whose vertices are isotopy classes of essential (and not boundary parallel) simple closed curves in \( S \) and whose edges span pairs of isotopy classes of curves that have disjoint representatives. Higher dimensional simplices are defined by sets of pairwise disjoint curves. We make the vertex set of \( \mathcal{C}(S) \) a metric space by defining the distance between two vertices as the number of edges in the shortest edge path between them. The curve complex of a closed, connected surface is connected as long as the surface is not a sphere with four or fewer punctures or a torus with one or fewer punctures. We will only consider the curve complex for such surfaces.
Given a bridge surface $\overline{H}$ for a link $L \subset M$, we let $\mathcal{C}(\overline{H}) = \mathcal{C}(H)$, that is $\mathcal{C}(\overline{H})$ is the curve complex of the punctured surface. The disk sets $\mathcal{H}_\uparrow, \mathcal{H}_\downarrow$ for $H$ are the sets of loops that bound compressing disks for $H$ in $H_\uparrow \cap M(L)$ and $H_\downarrow \cap M(L)$.

**Definition 4.3.** The distance $d_{\mathcal{C}}(\overline{H})$ is the distance in $\mathcal{C}(\overline{H})$ between $\mathcal{H}_\uparrow$ and $\mathcal{H}_\downarrow$.

As mentioned in the introduction, this is a non-trivial definition. In [6], high distance bridge surfaces were constructed for knots and tangles in certain compact 3-manifolds. In [29], it was shown that all closed manifolds contain knots of arbitrarily high distance.

The arc and curve complex $\mathcal{AC}(S)$ for a compact surface $S$ is defined similarly. It is the simplicial complex whose vertices are isotopy classes of essential simple closed curves and essential properly embedded arcs. Edges span pairs of disjoint arcs/curves and higher dimensional simplices span larger sets of pairwise disjoint arcs/curves. As with the curve complex, we make its vertex set a metric space by declaring each edge of the curve complex to have length one and ignoring the higher dimensional simplices. Given a sloped Heegaard surface $S$, the collection of boundary compressing disks and compressing disks for $S$ intersect $S$ in arcs and loops, respectively. These arcs and loops define vertices in the arc and curve complex of $S$. The disk sets of $S$ are the sets $\mathcal{D}_\uparrow, \mathcal{D}_\downarrow$ of vertices of $\mathcal{AC}(S)$ defined by the intersection of all boundary compressing and compressing disks on the two sides of $S$. As before, if $\overline{H}$ is a bridge surface, we let $\mathcal{AC}(\overline{H}) = \mathcal{AC}(H)$. The arc-and-curve complex is connected for any closed, connected surface which is not a sphere with three or fewer punctures or a torus with no punctures. We will only consider the complex for such surfaces.

**Definition 4.4.** The distance $d_{\mathcal{AC}}(H)$ of a sloped Heegaard surface $H$ is the edge path distance in $\mathcal{AC}(S)$ from $\mathcal{D}_\uparrow$ to $\mathcal{D}_\downarrow$, i.e., the number of edges in the shortest edge path from a vertex of $\mathcal{D}_\uparrow$ to a vertex in $\mathcal{D}_\downarrow$. In general, the distance (in $\mathcal{C}(H)$ or $\mathcal{AC}(H)$) between two subsets is just the minimal path distance between the subsets.

Suppose that $H$ is a surface with $\mathcal{C}(H)$ connected. Given a path $\alpha = \alpha_1, \alpha_2, \ldots, \alpha_n$ in $\mathcal{AC}(H)$, there is a way to construct a path $\beta$ in $\mathcal{C}(H)$ of length at most twice the length of $\alpha$. If $\alpha_i$ is represented by a loop, we let $\beta_{2i-1} = \alpha_i$. If $\alpha_i$ is represented by an arc, we let $\beta_{2i-1}$ be the loop which is a frontier in $\overline{H}$ of a regular neighborhood of an arc representing $\alpha_i$ (so that $\beta_{2i-1}$ bounds in $\overline{H}$ a twice-punctured disc). If $\beta_{2i-1}$ and $\beta_{2i+1}$ are disjoint, we let $\beta_{2i} = \beta_{2i-1}$. If $\beta_{2i-1}$ and $\beta_{2i+1}$ are not disjoint, then $\alpha_i$ and $\alpha_{i+1}$ are represented by arcs in $H$. 

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Figure 2. The top row shows a path in $\mathcal{AC}$ and the second row shows the induced path in $\mathcal{C}$.

having disjoint interiors and sharing one or both endpoints. There is an essential loop $\beta_{2i}$ which is disjoint from the arc representing both $\alpha_i$ and $\alpha_{i+1}$. The path $\beta = \beta_1, \beta_1, \ldots, \beta_{2n-1}$ is then a path in $\mathcal{C}(H)$ of length at most twice the length of $\alpha$.

Definition 4.5. We call $\beta$ a path in $\mathcal{C}(H)$ induced by the path $\alpha$ in $\mathcal{AC}(H)$. See Figure 2 for an example.

The existence of induced paths gives the following well-known relationship between $d_{\mathcal{AC}}$ and $d_{\mathcal{C}}$.

Lemma 4.6. If $H$ is a bridge surface which is not a sphere with four or fewer punctures, then $d_{\mathcal{AC}}(H) \leq d_{\mathcal{C}}(H) \leq 2d_{\mathcal{AC}}(H)$.

5. Bounding distance

As before, let $M$ be a compact, orientable manifold containing a properly embedded link $L$ and let $N = M(L)$ and $\partial_L N = \partial N \setminus \partial M$. Let $\sigma$ and $\tau$ be multi-slopes on $\partial_L N$ such that $\Delta(\sigma, \tau) > 0$. Let $S \subset N$ be a properly embedded orientable surface with boundary slope $\sigma$ and let $T$ be a thick surface of a generalized sloped Heegaard splitting for $N$ such that $T$ has boundary slope $\tau$. Assume that no component of $S$ is a closed sphere. We let $\partial_L S$ denote the components of $\partial S$ lying on $\partial_L N$ and we let $\partial_0 S = \partial S \setminus \partial_L S$.

In this section, we prove the result which is key to our distance bounds. In order to apply the result in a number of different situations we state it as generally as possible. We begin with two definitions.

The first definition is essentially that of “Morse position” for the surface $S$ with respect to the sweepout by $T$. 
Definition 5.1. A (partial) sweepout \( \phi \) of \( N \) with level surfaces \( T_t = \phi^{-1}(t) \) is adapted to \( S \) if all of the following hold:

1. \( \partial T_t \) intersects \( \partial S \) minimally for all \( t \in (0, 1) \)
2. The restriction \( \phi|_S \) is Morse (i.e. has non-degenerate critical points)
3. For all but at most one critical value \( v \in (0, 1) \), there is a unique critical point of \( \phi|_S \) with critical value \( v \)
4. For the exceptional critical value \( v \in (0, 1) \), there are at most two critical points of \( \phi|_S \) with critical value \( v \)
5. If \( c_1 \) and \( c_2 \) are distinct critical points of \( \phi|_S \) with the same height \( v \), then every component of \( T_{v-\epsilon} \cap S \) can be isotoped (in \( T \)) to be disjoint from every component of \( T_{v+\epsilon} \cap S \).

The next definition hones in on the important values of the sweepout which allow us to infer the topology of \( S \) from the structure of the critical points of the sweepout. It is related to the idea of “mutuality” from [1].

Definition 5.2. Suppose that \( \phi \) is a (partial) sweepout of \( N \) adapted to \( S \) with level surfaces \( T_t \). An interval \( [a, b] \subset (0, 1) \) with \( a < b \) is essential if all of the following hold:

- \( a \) and \( b \) are regular values of \( \phi|_S \),
- every arc of \( T_t \cap S \) is essential in both \( T_t \) and \( S \) for all regular values \( t \in [a, b] \)
- every circle of \( T_t \cap S \) that is essential in \( T_t \) is also essential in \( S \) for all \( t \in [a, b] \).

The interval \( [a, b] \) is maximally essential if it is essential and if there is an interval \( [u, v] \supset [a, b] \) with \( u \) and \( v \) regular values of \( \phi|_S \) such that there is exactly one critical point of \( \phi|_S \) in each of the intervals \( [u, a] \) and \( [b, v] \) and, additionally, there is an arc or circle of \( T_u \cap S \) that is essential in \( T_u \) but bounds a disc in \( (T_u)_\downarrow \) and there is an arc or circle of \( T_v \cap S \) that is essential in \( T_v \) but bounds a disc in \( (T_v)_\uparrow \). See Figure 3.

Here is the theorem which is key to our results:

**Theorem 5.3.** Let \( \phi \) be a (partial) sweepout of \( N \) adapted to \( S \) with level surfaces \( T_t \) and with maximally essential interval \( [a, b] \). Assume that no component of \( S \) is a sphere or disc. If \( T \) is not a sphere with three or fewer punctures or a torus with no punctures, then:

\[
b(T)(d_{AC}(T) - 2)\Delta \leq \left( \frac{4g(S) - 4|S| + 2|\partial_0 S|}{|\partial L S|} \right) + 2
\]
Figure 3. A schematic depiction of maximally essential interval. The “x”s represent the critical values of $\phi|_S$.

If $T$ is not a sphere with four or fewer punctures, then also

$$b(T)(d_c(T) - 4)\Delta \leq \left( \frac{8g(S) - 8|S| + 4|\partial_0 S|}{|\partial_L S|} \right) + 4$$

Remark 5.4. It is easy to see that the second inequality is, whenever $b(T) \geq 2$, an improvement on what can be obtained from the first inequality by simply using the fact that $d_c(T) \leq 2d_{AC}(T)$.

The basic outline of the proof is as follows: Since the sweepout is adapted to $S$, as $t$ varies from $a$ to $b$, the intersection $T_t \cap S$ is a collection of arcs and circles whenever $t$ is a regular value of $\phi|_S$. For heuristic purposes, assume that all these arcs and circles are essential in both $S$ and $T$ (this is close to being the case since the interval is essential). Then each critical point of $\phi|_S$ is a saddle singularity on $S$. These saddles are “essential” and so contribute to $-\chi(S)$ (as observed by Bachman and Schleimer [1]). This observation is formalized in Lemma 5.5.

As $T_t$ passes through a saddle singularity of $S$, the isotopy classes in $T$ of these arcs may change. We show that the total number of times we have such a change is at least the product of the length of a certain path in $AC(T)$ with quantities involving the bridge number and $|\partial_L S|$. This is formalized in Lemma 5.6. The length of the path is related in a straightforward way to $d_{AC}(T)$ and $d_c(T)$. The fact that $|\partial_L S|$ appears in the inequality allows us to obtain an inequality involving just the genus of $S$ and not merely its euler characteristic.

Before tackling the two parts of the proof, we make a few definitions.

Since the components of $\partial_L S$ are essential loops on the tori $\partial_L N$, each component of $\partial_L T$ intersects $\partial S$ at least $\Delta |\partial_L S|$ times. Let $\mathcal{L} = \{ b_1, \ldots, b_{|\partial_T \cap \partial S|} \}$ be the set of labels for the points in the intersection
\( \partial T \cap \partial S \) (exactly how this labelling is performed is irrelevant for our purposes). Since, for each regular value \( t \), the number \( |\partial T_t \cap \partial S| = |\partial T \cap \partial S| \), the natural isotopy from \( T \) to \( T_t \) induces labels on each point of \( \partial T_t \cap \partial S \). Henceforth, we do not distinguish between the labels in \( \mathcal{L} \) and the labels on \( \partial L T_t \).

Let \( v_1 < v_2 < \ldots < v_k \) be the critical values of \( \phi|_S \) in \([a,b]\) and let \( a = t_0 < t_1 < \ldots < t_k = b \) be regular values of \( \phi|_S \) so that \( v_i \) is the unique critical value of \( \phi|_S \) in the interval \([t_{i-1}, t_i]\). Define a label \( \lambda \) in \( \mathcal{L}(b) \) to be active at \( v_i \) if as \( t \) varies from \( t_{i-1} \) to \( t_i \) the isotopy class of arc with endpoints having label \( \lambda \) changes. The arc of \( T_{t_{i-1}} \cap S \) with label \( \lambda \) is said to be pre-active at \( v_i \) and the arc of \( T_{t_i} \cap S \) with label \( \lambda \) is said to be post-active at \( v_i \). An arc is active if it is pre-active or post-active. Since each active arc is in \( T_t \cap S \) for some regular value \( t \), each active arc is also an arc in \( S \). Let \( Q \) be the total number of post-active arcs in \( S \) (where the count is taken over all critical values \( v_i \)). The number \( Q \) is also half the total number of times that labels in \( \partial L T \) are active (observing that a label may be active more than once). A critical value \( v_i \) is active if there is an active arc at \( v_i \). A critical point \( p \) is active if \( \phi(p) \) is active. We note that, by the definition of essential interval, the pre-image of an active critical value contains an index 1 critical point. See Figure 4. Let \( C \) be the set of active critical points.

**Figure 4.** Active arcs occur before and after saddles.

Passing through the critical points at height \( v_i \), causes some arcs and circles of \( T_{t_{i-1}} \cap S \) to be banded to other arcs and circles. The result of the banding are the arcs and circles of \( T_{t_i} \cap S \). Define a circle component of \( T_{t_i} \cap S \) to be pre-active if, at \( v_{i+1} \), it is banded to a pre-active arc. Define it to be post-active if, at \( v_i \), it is banded to a post-active arc. A circle is active if it is pre-active or post-active. By the definition
of “pre-active” arc and “post-active” arc, each active circle is essential in $T_t$. By the definition of essential interval, each active circle is also essential in $S$.

Suppose that $c_1$ and $c_2$ are distinct index 1 critical points both having height $v_i$. As $t$ varies from $t_{i-1}$ to $t_i$, two disjoint bands are attached to arcs and circles of $T_{t_{i-1}} \cap S$ to produce $T_{t_i} \cap S$. Say that $c_1$ and $c_2$ are connected if the union of the pre-active arcs and circles with the two bands is connected. See Figure 5.

![Figure 5](image_url)

**Figure 5.** We show part of the foliation of $S$ induced by the sweepout by $T$. On the left, the two critical points at the same height are connected. On the right, they are not.

We now tackle the first part of the proof of Theorem 5.3.

**Lemma 5.5.** We have:

$$Q \leq -2 \chi(S)$$

where $Q$ is the total number of post-active arcs.

**Proof.** If $c$ is the unique index 1 critical point with height $v_i \in [a, b]$, then there are at most 2 post-active arcs in $T_{t_i} \cap S$. If $c_1$ and $c_2$ are index 1 critical points both having height $v_i$, then there are at most 4 post-active arcs in $T_{t_i} \cap S$. (If, as $t$ passes through $v$, the two bands are attached to four distinct arc components of $T_{t_-} \cap S$ then there are four post-active arcs at $v$. If the any of the four ends of the two bands are adjacent to the same component of $T_{t_-} \cap S$, then there will be fewer than 4 post-active arcs.) In $S$, the post-active arcs at $T_{t_i} \cap S$ are (pairwise) disjoint from the post-active arcs of $T_{t_j} \cap S$, for $j \neq i$, since $T_{t_i} \cap T_{t_j} = \emptyset$.

Thus, we have:

**Observation 1:** $Q \leq 2|C|$ where $C$ is the set of active critical points.

Let $P$ be the closure of the complement of the post-active arcs and circles in $S$. Denote its components by $P_1, \ldots, P_m$. The boundary of each $P_k$ is the union of post-active arcs and circles and arcs or circles
lying in $\partial S \subset \partial N$. Each $P_k$ contains at most 2 active index 1 critical points of $\phi|_S$, and there is at most one $P_k$ that contains 2 active index 1 critical points. Let $b_k$ be the number of active arcs in $\partial P_k$ and define the index of $P_k$ to be:

$$J(P_k) = \frac{b_k}{2} - \chi(P_k).$$

Since each active arc shows up twice in $\partial P$ and since euler characteristic increases by one when cutting along an arc, we have:

**Observation 2:** $-\chi(S) = \sum_k J(P_k)$

Suppose that $\alpha$ is a pre-active arc at an active critical value $v_i$. The post-active arcs at $v_i$ are obtained by either banding $\alpha$ to another pre-active arc $\beta$ at $v_i$, in which case we say that $\alpha$ is *paired*, or by banding $\alpha$ to itself or to a circle component of $T_{t_i-1} \cap S$, in which case we say that $\alpha$ is *solitary*.

Fix $k$. Since no component of $S$ is a sphere or disc, $P_k$ is not a sphere or disc disjoint from $\partial N$. If $P_k$ is a disk, its boundary cannot be an active circle or contain only a single active arc as active circles and arcs are essential in $S$. Thus, if $P_k$ is a disk, $J(P_k) \geq 0$. If $P_k$ is not a disk, $-\chi(P_k) \geq 0$. Thus, if $P_k$ does not contain an active critical point then its index is non-negative.

Suppose $P_k$ contains a unique active critical point $c \in P_k$ and let $\alpha$ be a pre-active arc at $c$. If $\alpha$ is a paired arc at $c$, then $b_k \geq 4$ since there must be at least two pre-active arcs and two post active arcs. Then $J(P_k) \geq (4/2) - 1 = 1$. If $\alpha$ is a solitary pre-active arc at $c \in C \cap P_k$, then let $\gamma$ be the circle that is either banded to $\alpha$ at $c$ or that results from banding $\alpha$ to itself at $c$. In this case, $\gamma$ is essential so $P_k$ is not a disk and there are two active arcs in the boundary of $P_k$ so $J(P_k) \geq (2/2) - 0 = 1$. Hence, if $P_k$ contains a single active index 1 critical point, then $J(P_k) \geq 1$.

The analysis when $P_k$ contains two active index 1 critical points is similar, but somewhat more delicate. In this case, the index 1 critical points must have the same height $v$ and be connected. Let $C_-$ and $C_+$ denote the number of pre-active and post-active circles at $v$. Note that any pre-active or post-active circle, along with at least one post-active arc, at $v$ lies in the boundary of $P_k$. Furthermore, the bands must lie in $P_k$ (since the bands, themselves, contain the index 1 critical points). Thus, if $P_k$ is a disc, then $C_- = C_+ = 0$ and $b_k$ must be 6, in which case $J(P_k) = 2$. Suppose that $P_k$ is not a disc. Then either $C_- + C_+ \geq 2$ and $b_k \geq 2$, $C_- + C_+ \geq 1$ and $b_k \geq 4$ or $b_k = 2$ and $P_k$ is not planar. In any of these cases, $J(P_k) \geq 2$. Consequently:

**Observation 3:** $\sum J(P_k) \geq |C|$.

Combining Observations 1, 2, and 3 completes the proof. □
We now show how to relate $Q$ to paths in $\mathcal{AC}(T)$ and $\mathcal{C}(T)$.

**Lemma 5.6.** There is a path

$$\alpha = \alpha_0, \alpha_1, \ldots, \alpha_n$$

in $\mathcal{AC}(T)$ from the vertices represented by the arcs of $T_a \cap S$ to the vertices represented by the arcs of $T_b \cap S$ such that:

1. Each $\alpha_i$ is represented by an arc of $T_{t_i} \cap S$.
2. All of the arcs represented by the $\alpha_i$ have an endpoint on the same component of $\partial T$.
3. Letting the length of $\alpha$ be $l$, we have

$$lb(T)|\partial L_S|\Delta \leq Q$$

**Proof.** Since arcs have two endpoints, the number $Q$ is equal to the half total number of times the labels in $L$ are active. For a label $\lambda \in L$ let $n_\lambda$ denote the number of times it is active. Thus,

$$\frac{1}{2} \sum_{\lambda \in L} n_\lambda = Q$$

Let $\lambda_0$ be a label $L$ which minimizes $n_\lambda$. The total number of labels is greater than or equal to $2b(T)|\partial L_S|\Delta$. Thus,

$$n_{\lambda_0} b(T)|\partial L_S|\Delta \leq Q.$$ 

Let $\alpha_i$ be the arc in $T_{t_i} \cap S$ with label $\lambda_0$ at one of its endpoints. Then the sequence

$$\alpha_0, \alpha_1, \ldots, \alpha_n$$

is a sequence of vertices in $\mathcal{AC}(T)$ representing a path from an arc in $T_a \cap S$ to an arc in $T_b \cap S$. It has length $l = n_{\lambda_0}$. Thus we have conclusions (1) - (3). \qed

We can now easily prove Theorem 5.3.

**Proof of Theorem 5.3.** By Lemma 5.5, $Q \leq -2\chi(S)$. The interval $[a, b]$ is maximal, so there is an interval $[u, v] \supset [a, b]$ such that $u, v$ are regular values of $\phi|_S$ and so that there is a unique critical value in each of the intervals $(u, a)$ and $(b, v)$. Furthermore, some component $\gamma_u$ of $T_u \cap S$ is essential in $T_u$ and bounds a disc below $T_u$ and some component $\gamma_v$ of $T_v \cap S$ is essential in $T_v$ and bounds a disc above $T_v$. Each component of $T_a \cap S$ is distance (in $\mathcal{AC}(T)$) at most 1 from each component of $T_u \cap S$ and each component of $T_b \cap S$ is distance at most 1 from each component of $T_v \cap S$.

Let $\alpha = \alpha_0, \ldots, \alpha_n$ be the path in $\mathcal{AC}(T)$ provided by Lemma 5.6 and let $l$ be its length. Since $d(\alpha_0, \gamma_u) = 1$ and $d(\alpha_n, \gamma_v) = 1$, we have $l \geq d_{\mathcal{AC}(T)} - 2$. 
Consequently, by Lemma 5.6, we have
\[(d_{AC}(T) - 2)b(T)|\partial LS|\Delta \leq -2\chi(S).\]
Using the fact that
\[-2\chi(S) = 4g(S) - 4|S| + 2|\partial_0 S| + 2|\partial_L S|,\]
we obtain Conclusion (1).

Suppose now that \(T\) is not a sphere with 4 or fewer punctures. Let \(\hat{\alpha}\) be the path in \(\partial(T)\) induced by the path \(\gamma_u, \alpha_0, \ldots, \alpha_n, \gamma_v\). The length of \(\hat{\alpha}\) is at most \(2l + 4\). Thus, \(\frac{d_{C}(T) - 4}{2} \leq l\). Consequently,
\[(d_{C}(T) - 4)b(T)|\partial LS|\Delta \leq -4\chi(S)\]
This produces conclusion (2). \(\square\)

For convenience we observe that the inequalities also hold when \(T\) is weakly-\(\partial\)-reducible, even in the absence of a maximally essential interval.

**Lemma 5.7.** Suppose that \(T\) is weakly \(\partial\)-reducible and that \(T\) is not an annulus. Assume that each component of \(S\) is incident to \(\partial_L N\) and that no component of \(S\) is a disc. Then:
\[\Delta b(T)(d_{AC}(T) - 2) \leq \frac{4g(S) - 4|S| + 2|\partial_0 S|}{|\partial_L S'|} + 2\]
If, additionally \(T\) is not a four-punctured sphere, then we also have:
\[\Delta b(T)(d_{C}(T) - 2) \leq \frac{8g(S) - 8|S| + 4|\partial_0 S|}{|\partial_L S'|} + 4\]

**Proof.** Since \(T\) is weakly \(\partial\)-reducible, \(d_{AC}(T) \leq 1\) and, by Lemma 4.6, \(d_{C}(T) \leq 2\).

Suppose that the first conclusion does not hold. Then
\[-2 > \frac{4g(S) - 4|S| + 2|\partial_0 S|}{|\partial_L S'|}\]
Thus,
\[0 > 4g(S) - 4|S| + 2|\partial_0 S| + 2|\partial_L S'|\]
There must be some component \(S'\) of \(S\) for which
\[4 > 4g(S') + 2|\partial_0 S'| + 2|\partial_L S'|.\]
Hence, \(g(S') = 0\), and
\[3 \geq 2|\partial_0 S'| + 2|\partial_L S'|.\]
Since \(|\partial_L S'| \geq 1\), we have \(|\partial_0 S'| = 0\) and \(1 = |\partial_L S'|\). Thus, \(S'\) is a disc, contrary to hypothesis.
Suppose that the second conclusion does not hold. Then there must be a component $S'$ of $S$ so that

$$8 > 8g(S') + 4|\partial_0 S'| + 4|\partial L S'|.$$ 

Thus, $g(S') = 0$. Since $|\partial L S'| \geq 1$, we must have $|\partial_0 S'| = 0$ and $|\partial L S'| = 1$. This again contradicts the fact that no component of $S$ is a disc. □

6. Non-meridional essential surfaces

In this section, assume that $S \subset N$ is essential – that is, each component is incompressible, not $\partial$-parallel, and not a sphere bounding a ball. We show that the genus of $S$ produces an upper bound on the product of the bridge number and distance of $T$.

We begin by showing that there is a maximal interval for the sweepout of $N$ by $T$, with respect to $S$. The proof is standard and can, in essence, be found in Gabai’s original use of thin position [16].

Lemma 6.1. If $T$ is strongly $\partial$-irreducible and if $S$ is essential, then there is a sweepout $\phi$ by $T$ adapted to $S$ such that each critical value of $\phi|_S$ has a unique critical point in its pre-image and so that there is a maximal interval for $S$.

Proof. Let $\phi$ be a sweepout of $N$ by $T$. By standard Morse theory arguments, we may perturb $\phi$ so that $\phi$ is adapted to $S$ and so that every critical value of $\phi|_S$ has a unique critical point in its preimage. Let

$$v_0 = 0 < v_1 < \ldots < v_{n-1} < 1 = v_n$$

be the critical values of $\phi|_S$ and let $I_i = (v_{i-1}, v_i)$. Label the interval with $\uparrow$ (respectively, $\downarrow$) if, for $t \in I_i$, some component of $T_t \cap S$ is essential in $T_t$ and bounds a disc above (respectively, below) $T_t$.

Since $T$ is strongly $\partial$-irreducible, no interval is labelled both $\uparrow$ and $\downarrow$. Similarly, adjacent intervals cannot have distinct labels.

When $t$ is close to 1, every arc and circle of $T_t \cap S$ bounds a disc above $T_t$. Thus, $I_n$ is labelled $\uparrow$. When $t$ is close to 0, every arc and circle of $T_t \cap S$ bounds a disc below $T_t$. Thus, $I_1$ is labelled $\downarrow$. Thus there are $i < i'$ so that $I_i$ is labelled $\downarrow$, $I_{i'}$ is labelled $\uparrow$, and for each $j$ with $i < j < i'$, the interval $I_j$ is unlabelled. Choose $u \in I_i$, $v \in I_{i'}$ and $a \in I_{i+1}$, $b \in I_{i'-1}$ with $a < b$. Then the interval $[a, b]$ is essential since each $I_j$ with $i < j < i'$ is unlabelled and the interval $[u, v] \supset [a, b]$ certifies the fact that $[a, b]$ is a maximally essential interval. □
Since we have a maximal essential interval for $S$, we can use the genus of $S$ to bound bridge number and distance. For simplicity, we state the following only for distance measured in $\mathcal{AC}(T)$.

**Theorem 6.2.** Suppose that $S \subset N$ is a connected $\sigma$-sloped essential surface and that $N$ is irreducible and $\partial$-irreducible. Suppose that $T \subset N$ is a $\tau$-sloped Heegaard surface with $\Delta(\sigma, \tau) \geq 1$ and that $T$ is not a sphere with three or fewer punctures. Then:

$$\Delta b(T)(d_{\mathcal{AC}}(T) - 2) \leq \frac{4g(S) - 4 + 2|\partial_0 S|}{|\partial L S|} + 2.$$  

**Proof.** By Lemma 5.7, we may assume that $T$ is strongly $\partial$-reducible. By Lemma 6.1 there is a sweepout of $N$ by $T$ so that $S$ has a maximal essential interval. By Theorem 5.3 we have our conclusion. $\square$

Specializing to the situation where we are doing Dehn surgery on a knot produces the following:

**Theorem 6.3.** Suppose that $M$ is a closed, orientable 3-manifold and that $L \subset M$ is a knot with irreducible and $\partial$-irreducible exterior. Suppose that every essential surface in the exterior of $L$ has genus at least $g + 1$, for some fixed $g$. Also, assume that if $M = S^3$, then $b(L) \geq 3$. Let $M'$ be a 3-manifold obtained by non-trivial Dehn surgery on $L$ with surgery distance $\Delta$. If $M'$ contains an essential surface $\overline{S}$ of genus $g$, then

$$\Delta b(L)(d_{\mathcal{AC}}(L) - 2) \leq \max(2, 4g - 2).$$

Furthermore, if $\overline{S}$ is null-homologous, then

$$\Delta b(L)(d_{\mathcal{AC}}(L) - 2) \leq \max(2, 2g).$$

**Proof.** Suppose that the exterior of $L$ does not contain an essential surface of genus $g$ or smaller. Out of all surfaces in the same homology class in $H_2(M', \partial M')$ as $\overline{S}$, choose one which intersects $L'$, the surgery dual to $L$, minimally. We may reappropriate notation and call that surface $\overline{S}$. It is easy to see that $S = \overline{S} \cap (M' - \overline{\eta}(L'))$ is essential in the exterior of $L'$. Let $\overline{T}$ be a bridge surface for $(M, L)$ minimizing the triple $(g(T), b(T), d(T))$ lexicographically, so that $b(L) = b(T)$ and $d(L) = d(T)$.

If $T$ is weakly $\partial$-reducible, we use Theorem 5.7. If $T$ is strongly $\partial$-irreducible, we use Lemma 6.1 and Theorem 5.3. In either case, we conclude

$$\Delta b(L)(d_{\mathcal{AC}}(L) - 2) \leq \Delta b(T)(d_{\mathcal{AC}}(T) - 2) \leq \frac{4g - 4}{|\partial L S|} + 2.$$  

**Case 1:** $g = 0$
In this case,
\[ \Delta b(L)(d_{AC}(L) - 2) \leq -\frac{4}{|\partial_L S|} + 2. \]
Since \( \Delta b(L)(d_{AC}(L) - 2) \) is an integer, we have
\[ \Delta b(L)(d_{AC}(L) - 2) \leq 1, \]
as desired. Since \( b(L) \geq 1 \), we have \( d_{AC}(L) \leq 3 \). If \( d_{AC}(L) = 3 \), we obtain \( b(L) \leq 1 \), a contradiction to our assumption that \( b(L) \geq 3 \). Hence, \( d_{AC}(L) \leq 2 \).

**Case 2:** \( g \geq 1 \).
Since \( |\partial_L S| \) and \( g \) are both at least 1, we have:
\[ \Delta b(L)(d_{AC}(L) - 2) \leq 4g - 2. \]
If \( \overline{S} \) is null-homologous in \( M' \), then \( |\partial_L S| \geq 2 \), implying the better bound of:
\[ \Delta b(L)(d_{AC}(L) - 2) \leq 2g. \]

7. **Non-meridional strongly \( \partial \)-irreducible bridge surfaces**

As before, let \( N \) be a compact, orientable 3-manifold with incompressible boundary and let \( \partial_L N \) be a union of torus components of \( \partial N \). Let \( T \) be a sloped Heegaard surface for \( N \) inducing a multislope \( \tau \) on a subset of \( \partial_L N \). Let \( S \) be either another sloped Heegaard surface for \( N \) or a thick surface of a sloped generalized Heegaard splitting of \( N \). Assume that \( S \) induces a multi-slope \( \sigma \) on a subset of \( \partial_L N \) and assume that \( \Delta(\sigma, \tau) > 0 \).

Let \( \phi_T \) be a sweep-out of \( N \) associated to \( T \) and let \( \phi_S \) be either a sweep-out or a partial sweep-out of \( N \) associated to \( S \). Assume that the boundary of each level surface (possibly disconnected) of \( \phi_S \) intersects the boundary of each level surface of \( \phi_T \) minimally in each \( V_i \). (This is possible, for example, by choosing a flat metric on each \( V_i \) and requiring that the boundaries of the level sets be geodesics.)

Consider the product map \( \phi_T \times \phi_S : N \to [0, 1] \times [0, 1] \). Each point \((t, s)\) in the square represents a pair of surfaces \( T_t = \phi_T^{-1}(t) \) isotopic to the surface \( T \) and \( S_s = \phi_S^{-1}(s) \) isotopic to the surface \( S \). The **graphic** is the subset of the square consisting of all points \((t, s)\) where \( T_t \) and \( S_s \) are tangent. We say that \( \phi_T \times \phi_S \) is **generic** if it is stable [31] on the complement of the spines and each arc \( \{t\} \times [0, 1] \) and \([0, 1] \times \{s\} \) contains at most one vertex of the graphic. The vertices in the interior of the graphic are valence four (crossings) and valence two (cusps). By general position of the spines, the graphic is incident to the boundary
of the square in only a finite number of points. The vertices in the corners of the boundary correspond to points of intersection in ∂₀N between a spine of φₛ and a spine of φₜ. These vertices will have valence |∂ₜ∩∂ₛ|/4. All other vertices in the boundary are valence one or two. We model our analysis of the graphic on that of Johnson [31].

For (t, s) ∈ (0, 1) × (0, 1) a regular value of φₜ × φₛ we say that Tₜ is essentially above Sₛ if there exists a component l ⊂ Sₛ ∩ Tₜ that is an essential arc or circle in Sₛ but bounds a compressing or boundary compressing disk for Sₛ contained in (Sₛ)↓. Similarly, we say that Tₜ is essentially below Sₛ if there exists a component l ⊂ Sₛ ∩ Tₜ that is an essential arc or circle in Sₛ and bounds a compressing or boundary compressing disk for Sₛ contained in (Sₛ)↑.

Let Qₐ (Qₜ) denote the points in (0, 1) × (0, 1) such that Tₜ is essentially above (below) Sₛ.

Remark 7.1. If φₛ is a sweep-out (rather than a partial sweep-out), then a neighborhood of the bottom of the graphic is contained in Qₐ and a neighborhood of the top of the graphic is contained in Qₜ, for if ϵ is small, the components of Tₜ ∩ Sₜ are essential in Sₜ and inessential in Tₜ for any t ∈ (0, 1).

If φₛ is a partial sweep-out, then any level surface for φₛ is isotopic to a thick surface for a sloped generalized Heegaard splitting for N. Hence, φₛ⁻¹([0, s₀]) = (Sₛ₀)↓ and φₛ⁻¹([s₀, 1]) = (Sₛ₀)↑ are boundary compression bodies. Let e be an edge of a spine of φₛ⁻¹([0, s₀]). Since we have assumed φₜ is a full sweep-out of M and φₜ × φₛ is generic, there exists a level surface, φ⁻¹ₜ(t₀), that intersects e transversely. Hence, for ϵ sufficiently small, (t₀, ϵ) is contained in Qₐ. Similarly, there exists t₁ such that for ϵ sufficiently small, (t₁, 1 − ϵ) is contained in Qₜ.

Figure 6. Spanning and splitting graphics.

7.1. Spanning.

Definition 7.2. The sweep-out or partial sweep-out φₛ essentially spans the sweep-out φₜ if there exist s₀, t₁, t₂ in (0, 1) such that (t₁, s₀) ∈
Proof. For convenience, let $Q_a$ and $(t_2, s_0) \in Q_b$, as on the left in Figure 6. Observe that if $(Q_a \cap Q_b)$ is not empty, then $\phi_S$ spans $\phi_T$ since $Q_a$ and $Q_b$ are open subsets of $(0, 1) \times (0, 1)$. Additionally, since $Q_a$ and $Q_b$ are open, we can assume $t_1 < t_2$.

Lemma 7.3. Suppose that $\phi_S$ is a sweep-out or partial sweep-out that essentially spans the sweep-out $\phi_T$ and that $S$ is not an annulus. Then $S$ is weakly boundary reducible.

Proof. Choose $s_0, t_1, t_2$ as in Definition 7.2. Without loss of generality, $t_1 < t_2$. Let $T = T_{t_1} \cup T_{t_2}$. Let $l_k \subset S_{s_0} \cap T_{t_k}$ be a component such that $l_k$ is essential in $S_{s_0}$ and bounds a compressing or boundary compressing disk $D_{t_k}$ for $S_{s_0}$ lying in $(S_{s_0})_k$ if $k = 1$ and in $(S_{s_0})_l$ if $k = 2$. Since $T_{t_1}$ and $T_{t_2}$ are disjoint, $\partial D_1$ and $\partial D_2$ are disjoint.

7.2. Splitting.

Definition 7.4. A sweep-out or partial sweep-out $\phi_S$ weakly splits a sweep-out $\phi_T$ for $N$ at $s_0 \in (0, 1)$, if for all $t \in (0, 1)$, the point $(t, s_0) \not\in (Q_a \cup Q_b)$ and if there exists $t_0 \in (0, 1)$ such that $(t_0, s_0)$ is a valence 4 vertex of the graphic and is in the intersection of the closures of $Q_a$ and $Q_b$. We say that $\phi_S$ essentially splits $\phi_T$ at $s_0$ if for all $t$, the point $(t, s_0)$ is in neither $Q_a \cup Q_b$ nor in the set of vertices of the graphic. The right side of Figure 6 shows an example where $\phi_S$ essentially splits $\phi_T$ at some $s_0$.

Lemma 7.5. Suppose $\phi_S$ is a generic sweep-out or a partial sweep-out that essentially splits or weakly splits the sweep-out $\phi_T$ at $s_0$. If $\phi_S$ is a partial sweep-out, assume the surface portions $F_0$ and $F_1$ of the spines $\phi_S^{-1}(0)$ and $\phi_S^{-1}(1)$ respectively are incompressible and boundary incompressible. Assume also that both $T$ and $S$ are strongly $\partial$-irreducible. Then there exists a maximally essential interval $[a, b] \subset (0, 1)$ for the sweepout $\phi_S(s_0) \times \phi_T$ of $N$.

Proof. For convenience, let $h_T = \phi_S(s_0) \times \phi_T$. Note that, by construction for each regular value $t \in (0, 1)$, $T_t = h_T^{-1}(t)$ is a surface in $N$ properly isotopic to $T$.

Claim 1: $h_T$ is adapted to $S_{s_0}$.

By the properties of the graphic and the definition of “weakly splits” and “essentially splits”, $h_T$ satisfies conditions (1)-(4) of Definition 5.1. We show that it also satisfies condition (5).

Suppose, then, that $(t_0, s_0)$ is a valence four vertex of the graphic in the intersection of the closures of $Q_a$ and $Q_b$. Let $\epsilon > 0$ be small enough so that $(t_0, s_0)$ is the unique vertex of the graphic in the $\epsilon$ ball centered at $(t_0, s_0)$. Let $w = (t_0 - \epsilon, s_0)$, $e = (t_0 + \epsilon, s_0)$, $n = (t_0, s_0 + \epsilon)$
and $\sigma = (t_0, s_0 - \epsilon)$. (The letters stand for West, East, North, and South around the vertex $(t_0, s_0)$.) Either $n \in Q_b$ and $\sigma \in Q_a$ or vice versa. Since $S$ is strongly $\partial$-irreducible, this implies that, in $S$, there is some component of $N = T_{t_0} \cap S_{s_0+\epsilon}$ which cannot be isotoped to be disjoint from some component of $\Sigma = T_{t_0} \cap S_{s_0-\epsilon}$.

To show that (5) holds, we must show that each component of $W = T_{t_0-\epsilon} \cap S_{s_0}$ can be isotoped in $T$ to be disjoint from each component of $E = T_{t_0+\epsilon} \cap S_{s_0}$. (The following argument is essentially that of [44, Lemma 5.6].) To see this, note that to go from $N$ to $\Sigma$ we attach two bands $b_1$ and $b_2$. To go from $N$ to $E$ we attach $b_1$ and to go from $N$ to $W$ we attach $b_2$. To go from $E$ to $\Sigma$ we attach $b_2$ and to go from $W$ to $S$ we attach $b_1$. If the ends of $b_1$ and $b_2$ are adjacent to different components of $N$ or if the ends of $b_1$ and $b_2$ are adjacent to the same side of the same component, then by moving those components slightly in the direction opposite the side to which the bands are incident, we see that $N$ is disjoint from $\Sigma$ in $S$. But this contradicts the observation that some component of $N$ cannot be isotoped to be disjoint from some component of $\Sigma$. Then, an end of $b_1$ and an end of $b_2$ are adjacent to opposite sides of the same component $\nu$ of $N$. If the other ends of $b_1$ and $b_2$ are adjacent to a component $\nu'$ of $N$ then those ends must be adjacent to opposite sides of $\nu_2$, for else one of $b_1$ or $b_2$ would be a band with ends attached to opposite sides of a component of $E$ or $W$, a contradiction to the orientability of $S$ and $T$. Thus, the arcs and loops that result from attaching $b_1$ to $N$ can be isotoped to be disjoint from the arcs and loops that result from attaching $b_2$ to $N$ (push each component a little in the direction that the bands approach from). Hence, each component of $E$ is isotopically disjoint from each component of $W$.

**Claim 2:** Every arc or loop in the intersection $S_{s_0} \cap T_t$ that is trivial in $T_t$ must also be trivial in $S_{s_0}$.

If there were a trivial loop or arc of $S_{s_0} \cap T_t$ in $T_t$ that was essential in $S_{s_0}$, then an innermost such loop or outermost such arc would bound or cobound a disk $D$ in $T_t$. Isotope $D$ fixing $\partial D \cap S_{s_0}$ so that the number of components of $D \cap (F_0 \cup F_1 \cup S_{s_0})$ is minimized. Suppose $\alpha$ is an outermost arc or innermost loop of $D \cap (F_0 \cup F_1 \cup S_{s_0})$ in $D$. If $\alpha$ is contained in $F_0 \cup F_1$, then, by irreducibility of $N$ and incompressibility and boundary incompressibility of $F_0 \cup F_1$, we can isotope $D$ to decrease the number of components of $D \cap (F_0 \cup F_1 \cup S_{s_0})$. If $\alpha$ is contained in $S_{s_0}$, then it must be trivial in $S_{s_0}$. Since $\alpha$ is trivial in both $S_{s_0}$ and $D$ and $N$ is irreducible and boundary irreducible, there is an isotopy of $D$ that decreases the number of components of $D \cap (F_0 \cup F_1 \cup S_{s_0})$. In either case, we arrive at a contradiction, so we can assume the interior
of $D$ is disjoint from $F_0 \cup F_1 \cup S_{s_0}$. Hence, after an isotopy, the disk $D$ is completely contained in either $(S_{s_0})_\downarrow$ or $(S_{s_0})_\uparrow$, contradicting the assumption that $(t, s_0)$ is disjoint from $Q_a$ and $Q_b$ for all $t$. We conclude that every arc or loop of $S_{s_0} \cap T_t$ is either essential in $S_{s_0}$ or trivial in both surfaces. □ (Claim 2)

On the other hand, a loop or arc in the intersection may be trivial in $S_{s_0}$ but essential in $T_t$. In fact, for values of $t$ near 0, such loops/arc must exist and will bound compressing disks and boundary compressing disks, respectively, in $(T_t)_\downarrow$. For $t$ near 1, such loops and arcs will bound disks in $(T_t)_\uparrow$. Since $T$ is strongly $\partial$-irreducible, Gabai’s argument (as in the proof of Lemma 6.1) shows that there exists a maximally essential interval. □

The existence of a maximal essential interval means we can use the genus of $S$ to bound $b(T)$ and $d_{C}(T)$, as in the case when $S$ was an essential surface. For simplicity, we state the bound only for distance in the curve complex.

**Corollary 7.6.** Suppose that $T$ is a $\tau$-sloped Heegaard surfaces for $N$ and that $S$ is a thick surface for a generalized $\sigma$-sloped Heegaard splitting of $N$. Assume that $S$ is strongly $\partial$-irreducible and not a sphere with four or fewer punctures. Then

$$
\Delta b(T)(d_{AC}(T) - 2) \leq \frac{4g(S) - 4|S| + 2|\partial_0 S|}{|\partial_L S|} + 2
$$

**Proof.** By Lemma 5.7, we may assume that $T$ is also strongly $\partial$-irreducible. The surface $S$ cannot be an annulus, for if it were it would have to be weakly-$\partial$-reducible, contrary to hypothesis.

Let $\phi_S$ and $\phi_T$ be sweepouts corresponding to $S$ and $T$ so that $\phi_T \times \phi_S$ is generic. Since $S$ is strongly boundary irreducible and not an annulus, by Lemma 7.3, $\phi_S$ does not span $\phi_T$.

Suppose that $A$ and $B$ are two distinct regions of the graphic adjacent to a vertex $(t_0, s_0)$ of the graphic of valence 2. As $(t_0, s)$ passes through $(t_0, s_0)$, a center tangency and a saddle tangency between the surfaces $T_{t_0}$ and $S_s$ are cancelled. Hence, a point of $A$ belongs to $Q_a$ or $Q_b$ if and only if a point of $B$ belongs to $Q_a$ or $Q_b$ respectively.

Let $[0, s_0]$ be the largest interval such that $[0, 1] \times [0, s_0]$ is disjoint from $Q_b$. Since $\phi_S$ does not span $\phi_T$, $s_0 \neq 0$. Note that the line defined by $s = s_0$ must pass through a vertex of the graphic. By the previous paragraph, that vertex has valence 4. Let $\epsilon > 0$ be small enough so that there is no vertex of the graphic in the region $\{(t, s) : s_0 - \epsilon \leq s < s_0\}$. By the choice of $s_0$, the line $s = s_0 - \epsilon$ is disjoint from $Q_b$. If it is also disjoint from $Q_a$, then $\phi_S$ essentially splits $\phi_T$ at $s_0 - \epsilon$. If the
line $s = s_0 - \epsilon$ intersects $Q_a$, then $\phi_s$ weakly splits $\phi_T$ at $s_0$. In either case, by Lemma 7.5, there exists $s'_0 \in (0, 1)$ such that there exists a maximally essential interval $[a, b] \subset (0, 1)$ for the sweepout $\phi_S(s'_0) \times \phi_T$ of $N$. Our result then follows from Theorem 5.3.

\section{Non-meridional bridge surfaces}

Rather than examining elementary consequences of Corollary 7.6, we move on to consider the general case for when $S$ is a, possibly weakly-reducible, bridge surface. Once again, for simplicity, we examine only the bounds on distance in the arc and curve complex and only the case when $L$ is a knot in a closed 3-manifold.

\begin{theorem}
Assume that $L$ is a knot in a closed 3-manifold $M$, that $S$ is a minimal genus Heegaard surface for $M(L)(\sigma)$ and that $L(\sigma)$ is in minimal bridge position with respect to $S$. Assume also that $T$ is a $\tau$-sloped Heegaard surface. If $T$ is planar, assume that $b(T) \geq 2$. Then one of following holds:

1. We have $b(T)(d_{AC}(T) - 2) \leq \max(1, 2g(S))$
2. $M(L)$ has a Heegaard surface of genus $g(S)$.
3. $M$ contains an essential surface of genus strictly less than $g(S)$ intersecting $L$ transversally at most twice.

\end{theorem}

\begin{proof}
If $T$ is planar, then $|\partial T| \geq 6$. Thus, by Lemma 5.7, if $T$ is weakly $\partial$-reducible,

$$b(T)(d_{AC}(T) - 2) \leq \frac{4g(S) - 4}{|\partial_L S|} + 2.$$ 

Since $S$ is a bridge surface, $|\partial_L S| \geq 2$. Consequently, if $g(S) \geq 1$, we have conclusion (1). If $g(S) = 0$, then conclusion (1) follows from the fact that $b(T)(d_{AC}(T) - 2)$ is an integer. Thus, we may assume that $T$ is strongly $\partial$-irreducible.

By the minimality of the bridge surface $S$, it is not stabilized or perturbed. Since $\partial M = \emptyset$, it is not $\partial$-stabilized. If it is removable, $S$ is isotopic to a Heegaard surface $N$ and we have conclusion (2). Assume, therefore, that $S$ is not removable.

By Theorem 3.1 (applied to $S$) there is a generalized $\sigma$-sloped Heegaard splitting $S \cup F$ with the following properties:

- Each component of $F$ is incompressible and $\partial$-incompressible
- Each component of $S$ is strongly irreducible in $M \setminus F$. 

• Each component of $S \cup F$ has genus at most the genus of $S$
• If $\Sigma$ is a component of $S$ then $\Sigma$ is not removable, perturbed, stabilized, or $\partial$-stabilized.

We note also that $S$ is weakly reducible if and only if $F \neq \emptyset$. We first consider the case when there is a thin surface with boundary. In this case we use Theorem 6.2 to get the desired bound. Subsequently, we may then assume that $F$ is closed. We then single out a particular component $S^*$ of $S$. The remaining cases correspond to whether $S^*$ is strongly $\partial$-irreducible or cancellable.

**Case 1:** There is a component $F$ of $F$ with non-empty boundary.

Since $S$ is separating, and since each component of $F$ is obtained by compressing $S$, if $|\partial F| = 1$, there must exist another non-separating component $F' \neq F$ of $F$ such that $\partial F' \neq \emptyset$ and $g(F) + g(F') \leq g(S)$. If such is the case, change notation to let $F$ be the union of $F$ and $F'$. In particular this means that $|\partial F| \geq 2$.

By Theorem 6.2 applied to $F$ in place of $S$, one of the following holds:

1. $g(F) \geq 1$ and $b(T)(d_{AC}(T) - 2) \leq 2g(F) \leq 2g(S)$.
2. $g(F) = 0$ and $b(T)(d_{AC}(T) - 2) < 2$.

Since the left hand side is an integer, in fact $b(T)(d_{AC}(T) - 2) \leq 1$.

Thus, we have conclusion (1) of the theorem. \[\square\](Case 1)

Henceforth, we assume that $F$ is closed. The 3-manifold $N - F$ has a component $N'$ which intersects $L$. Let $S^* = S \cap N'$. Observe that $\partial S^*$ has slope $\sigma$ on $\partial_L N$ (since $L$ is connected). Of course, if $F = \emptyset$, then $S^* = S$.

If $S^*$ were an annulus, then $\partial N' - \partial_L N$ is the union of 2-spheres disjoint from $L$. Since the exterior of $L$ is irreducible, $N' = N$ and $L'$ is the unknot in $S^3$. This contradicts the fact that $\partial N$ is incompressible.

**Case 2:** $S^*$ is weakly boundary reducible.

Since $S^*$ is weakly boundary reducible and strongly irreducible, then $S^*$ must be cancellable or perturbed. By Theorem 3.1, $S^*$ cannot be perturbed, so it must be cancelable. Let $R$ be the “cancelled bridge surface” given by the conclusion to Lemma 3.3 and let $\rho$ be its boundary slope. The surface $R$ is essential in $N$ and, after capping off $\partial R$ with disks, remains essential in the manifold obtained by filling $\partial N'$ by slope $\rho$. If $R$ is connected, then it has genus one less than the genus of $S^*$. If $R$ is disconnected, then the sum of the genera of its components is
equal to the genus of $S^*$. Since $L$ is connected and since its exterior is ∂-irreducible, no component of $R$ is a disc. Thus, if $\Delta(\rho, \tau) = 0$, then $\rho = \tau$ and we have conclusion (3).

Otherwise, if $\Delta(\rho, \tau) \geq 1$, we can apply Theorem 6.2 to $R$. In this case, we conclude that
\[ b(T)(d_{AC}(T) - 2) \leq \max(1, 2g(R)) \leq \max(1, 2g(S)) \]
since $|\partial R| \geq 2$ and $g(R) = g(S^*) \leq g(S)$. □(Case 2)

Case 3: $S^*$ is strongly ∂-irreducible.

By Corollary 7.6, we have
\[ b(T)(d_{AC}(T) - 2) \leq \frac{4g(S^*) - 4}{|\partial_L S^*|} + 2 \]
Since $|\partial_L S^*| \geq 2$ and $g(S^*) \leq g(S)$, we get obtain Conclusions (1), as in Case 1. □(Case 3)

Since lens spaces and small Seifert Fibred spaces have Heegaard genus at most 2, the following corollary is of independent interest. It follows immediately from Theorem 8.1.

**Corollary 8.2.** Suppose that $L \subset M$ is a knot in a closed, orientable 3-manifold with irreducible and ∂-irreducible exterior. If non-trivial Dehn surgery on $M$ produces a 3-manifold of Heegaard genus $g$, then one of the following holds:

1. We have
   \[ b(L)(d_{AC}(L) - 2) \leq \max(1, 2g) \]
2. The exterior of $L$ has a Heegaard surface of genus $g$.
3. $M$ contains an essential surface of genus at most $g$ which intersects $L$ transversally at most two times.

In particular, if $M$ is non-Haken, if $g \leq 2$, and if the tunnel number of $L$ is at least 2, then
\[ b(L)(d_{AC}(L) - 2) \leq 4. \]

Since $S^3$ is non-Haken, by combining Corollary 8.2 with Theorem 6.3, we immediately obtain:

**Corollary 8.3.** Suppose that $L \subset S^3$ is a hyperbolic knot of tunnel number at least 2. If $L$ has an exceptional surgery then
\[ b(L)(d_{AC}(L) - 2) \leq 4 \]

**Remark 8.4.** We recall from the proof of Theorem 6.3 that if the $L$ in Corollary 8.3 has a toroidal surgery, then $b(L)(d_{AC}(L) - 2) \leq 2$. By Corollary 8.2 if $L$ has a lens space surgery then also $b(L)(d_{AC}(L) - 2) \leq 2$ and if $L$ has a small Seifert-fibred surgery then $b(L)(d_{AC}(L) - 2) \leq 4$. 

9. Bounding Distance

In Theorem 8.1, the possibilities that there is an essential meridional surface of genus \( g \) with two boundary components or that there is a Heegaard surface of genus \( g \) for the complement of \( L \) ruin the ability to bound the distance of \( L \) purely in terms of the genus \( g(S) \). In this section, we consider those possibilities. Here it is most natural to make estimates in terms of distance in the curve complex.

The following result is an adaptation of Theorem 5.1 of [1] and a generalization of Proposition 4.3 of [53]. This result requires additional definitions: A surface \( S \) in \( M(L) \) is \( c \)-essential if it is essential, boundary incompressible, and there is no essential curve \( \delta \) in \( S \) that bounds a once punctured disk in \( M(L) \). If there is such a curve, the disk it bounds is a cut-disk and the surface is said to be cut-compressible. We use the terminology \( c \)-disk to denote a compressing disk or a cut-disk.

**Theorem 9.1.** Let \( L \) be a link embedded in a compact oriented manifold \( M \) so that the exterior of \( L \) is irreducible, \( \partial \)-irreducible, and contains no essential meridional annulus. Suppose \( T \) is a bridge surface for \((M,L)\) and suppose \( F \) is a meridional \( c \)-essential surface that is not an annulus. Then \( d_C(T) \leq \max(3, 2g(F) + |F \cap L|) \). Furthermore, if \( F \) is a closed surface, \( d_C(T) \leq \max(1, 2g(F)) \).

**Proof.** Consider a sweet-out \( \phi \) of \( M \) induced by the bridge surface \( T \). By standard arguments, we may assume that \( \phi|_F \) is Morse with critical points at distinct heights and, in particular, each surface \( T_t = \phi^{-1}(t) \) for fixed \( t \) intersects \( F \) in at most one figure-eight curve or in at most one curve that also contains a point of \( F \cap L \).

Let \( T_\uparrow \) and \( T_\downarrow \) be the compression bodies \( T \) bounds in \( M \). As in the proof of Theorem 6.3, for small \( \epsilon \), every curve of \( F \cap T_\epsilon \) bounds a disk in \( F \) that is a compressing disk for \( T_\downarrow \) (after the identification of \( T_\epsilon \) with \( T \)) and every curve of \( F \cap T_{1-\epsilon} \) bounds a disk in \( F \) that is a compressing disk for \( T_\uparrow \) (after the identification of \( T_{1-\epsilon} \) with \( T \)). An interval \([u, v]\) is maximally \( c \)-essential if it satisfies:

- a curve in \( F \cap T_u \) bounds a \( c \)-disk for \( T \) contained in \( T_\downarrow \) (after identifying \( T_u \) with \( T \))
- a curve in \( F \cap T_v \) bounds a \( c \)-disk for \( T \) contained in \( T_\uparrow \) (after identifying \( T_v \) with \( T \))
- If \( v_0 < v_1 < \ldots < v_n \) are the critical values of \( \phi|_S \) in \([u, v]\), then for every regular value \( t \) of \( \phi|_S \) in \([v_0, v_n]\) no circle of \( F \cap T_t \) bounds a \( c \)-disk for \( T_t \).

As in the proof of Theorem 6.3, if there is no maximally \( c \)-essential interval, then there are \( c \)-disks on opposite sides of \( T \) with disjoint
boundaries. By Proposition 4.1 of [53], which shows that every cut-
disk is distance at most one from some compressing disk on the same
side, it follows that \( d(T) \leq 3 \). In the special case when \( F \) is closed, we
know that both c-disks are compressing disks and \( d_c(T) \leq 1 \).

Let \( t_i \in (v_{i-1}, v_i) \). Let \( \gamma \) be a component of \( F \cap T_{t_i} \). If \( \gamma \) is inessential
in \( T_{t_i} \), then it must also be inessential in \( F \), as \( F \) is c-incompressible.
Since there is no essential meridional annulus for \( L \), we may isotope \( F \)
so as to eliminate \( \gamma \). We can, therefore, eliminate it by an isotopy of
\( F \). By Lemma 2.9 of [53], \( F \cap T_{t_i} \) for \( 0 \leq i \leq n \) always contains curves
essential in \( F \).

Cut \( F \) along the collection of curves of \( F \cap (T_{t_1} \cup \ldots \cup T_{t_n}) \) which
are essential in \( F \). Consider first the set of all components that don’t
lie entirely above \( T_{t_n} \) or entirely below \( T_{t_1} \). By construction, none
of these components are disks or punctured disks. Each component
has boundary on at most two levels \( T_{t_i} \) and \( T_{t_{i+1}} \) and thus can be
associated with a unique critical point \( v_i \). Furthermore, at least one
of the components associated with \( v_i \) must have boundary in both \( T_{t_i} \)
and \( T_{t_{i+1}} \), as otherwise \( F \) could be isotoped to be disjoint from \( T \), a
contradiction. Let \( \{P_i\} \) be the set of all components associated with \( v_i \)
that have boundary in both \( T_{t_i} \) and \( T_{t_{i+1}} \).

Let \( \gamma_0, \ldots, \gamma_n \) be a path in the curve complex from a curve in \( F \cap T_{t_0} \)
that bounds a c-disk for \( T \) in \( T_{t_1} \) to a curve in \( F \cap T_{t_n} \) that bounds a
c-disk for \( T \) in \( T_{t_{i+1}} \). Choose the path so that \( \gamma_i \) is a curve in \( F \cap T_{t_i} \).
We can do this since, for each \( i \), \( F \cap T_{t_i} \) contains a curve that is essential
in \( F \) and \( d_c(\gamma_i, \gamma_{i+1}) \leq 1 \) for all \( i \).

Claim: If each of \( \{P_i\}, \{P_{i+1}\}, \ldots, \{P_j\} \) consists entirely of annuli,
then, after possibly rechoosing \( \gamma_j \), \( d(\gamma_i, \gamma_j) = 0 \).

Proof: As \( \{P_k\} \) consists entirely of annuli for \( i \leq k \leq j \), then each
component of \( \cup_{k=i}^{j} \{P_i\} \) is an annulus. Hence, for any essential curve
\( \gamma_i \) in \( F \cap \phi^{-1}(r_i) \), there is an essential curve \( \gamma'_j \) in \( F \cap \phi^{-1}(r_j) \) so that
\( d(\gamma_i, \gamma'_j) = 0 \), namely the other boundary component of the annulus in
\( \cup_{k=i}^{j} \{P_i\} \) bounded by \( \gamma_i \). As the distance between any two curves in
\( F \cap T_{t_j} \) and \( F \cap T_{t_{j+1}} \) is at most one, we can replace \( \gamma_j \) by \( \gamma'_j \) in the
path \( \gamma_0, \ldots, \gamma_n \). \( \square \)

Let \( l \) be the number of indices \( 1 \leq i \leq n \) for which \( \{P_i\} \) does not
consist entirely of annuli. By the Claim, \( d(\gamma_0, \gamma_n) \leq l \).

Furthermore, each \( \{P_i\} \) that does not consist entirely of annuli has
a total Euler characteristic at most -1. The Euler characteristic of \( F \)
is equal to the sum of the Euler characteristics of the \( \{P_i\} \) and the Euler
characteristic of the components of \( F \) above \( T_{t_n} \) and those below \( T_{t_0} \).
Note that each of the latter components can have at most one disk so
\[ \chi(F) \leq -l + \delta \] where \( \delta \) is the total number of disk components. We have \( \delta \in \{0, 1, 2\} \).

Let \( D_\uparrow \) and \( D_\downarrow \) be the collection of all curves in the curve complex of the punctured surface \( T \) that bound compressing disks for \( T \) in the exterior of \( L \) above and below \( T \) respectively. Then

\[
d_{C}(T) \leq d_{C}(\gamma_0, D_\downarrow) + d_{C}(\gamma_0, \gamma_n) + d_{C}(\gamma_n, D_\uparrow) \\
\leq d_{C}(\gamma_0, D_\downarrow) + l + d_{C}(\gamma_n, D_\uparrow) \\
\leq d_{C}(\gamma_0, D_\downarrow) - \chi(F) + \delta + d_{C}(\gamma_n, D_\uparrow).
\]

If the components of \( F \) above \( T \) contain a disk component, then that contributes to the value of \( \delta \); however, \( d(\gamma_n, D_\uparrow) = 0 \), otherwise \( d(\gamma_n, D_\uparrow) = 1 \). Similarly for the components of \( F \) below \( T \).

Thus

\[
d_{C}(T) \leq 2 - \chi(F) = 2g(F) + |L \cap F|.
\]

\[ \square \]

Tomova demonstrated a bound for the distance of \( T \) if \( S \) is a Heegaard surface for \( M(L) \).

**Theorem 9.2** (Theorem 10.3 of [53]). Suppose that \( L \) is a non-trivial knot in a closed, irreducible, orientable 3-manifold \( M \). Let \( T \) be a bridge surface for \((M, L)\). If \( T \) is a sphere, assume that \(|T \cap L| \geq 6\). If \( S \) is a Heegaard surface for \( M(L) \), then

\[
d_{C}(T) \leq 2g(S).
\]

We can put these results together with our previous work to obtain a bound on distance.

**Theorem 9.3.** Assume that \( M \) is closed and that \( L \subset M \) is a knot with irreducible exterior \( N \) and with no essential meridional annulus. Suppose that \( S \) and \( T \) are \( \sigma \) and \( \tau \)-sloped Heegaard surfaces for \( N \), respectively. If \( T \) is planar, assume that \( b(T) \geq 3 \).

\[
d_{C}(T) \leq \max \left( \frac{2}{b(T)} + 4, \frac{4g(S)}{b(T)} + 4, 2g(S) + 2 \right).
\]

**Proof.** By Theorem 8.1, one of the following occurs:

1. We have

\[
b(T)(d_{C}(T) - 4) \leq 2b(T)(d_{AC}(T) - 2) \leq \max(2, 4g(S))
\]

2. The exterior of \( L \) in \( M \) has a Heegaard surface of genus no more than \( g(S) \).

3. \( M \) contains an essential surface of genus strictly less than \( g(S) \) intersecting \( L \) transversally at most twice.
If conclusion (1) occurs, then
\[ d_{c}(T) \leq \frac{\max(2, 4g(S))}{b(T)} + 4, \]
as desired.

If conclusion (2) occurs, then the result follows from Theorem 9.2.

Assume, therefore, that conclusion (3) occurs. Among all surfaces of genus at most \( g(S) - 1 \) which intersect \( L \) at most twice and which are essential in \( M \), choose \( F \) to minimize the pair \( (g(F), |F \cap L|) \) lexicographically.

Suppose that \( F = \overline{F} \cap N \) is compressible by a compressing disc \( D \). If \( \partial D \) bounds a once-punctured disc in \( F \), then \( L \subset M \) intersects a sphere in \( M \) exactly once transversally. This contradicts the assumption that \( N \) is \( \partial \)-irreducible. Since \( \overline{F} \) is incompressible in \( M \), \( \partial D \) bounds a twice-punctured disc in \( \overline{F} \). Compressing \( F \) using \( D \) creates a meridional annulus \( A \) for \( L \). By hypothesis, it is \( \partial \)-parallel. There is, therefore, an isotopy of \( \overline{F} \) to be disjoint from \( L \), contradicting our choice of \( \overline{F} \).

If \( F \) is \( \partial \)-compressible then, since \( \partial N \) is a torus, either \( F \) is compressible or \( F \) is a \( \partial \)-parallel annulus. The former case contradicts the previous paragraph and the latter case means that we can isotope \( \overline{F} \) to be disjoint from \( L \). That also contradicts our choice of \( \overline{F} \). Hence, \( F \) is \( \partial \)-incompressible.

Suppose that \( F \) is cut-compressible by a cut-disc \( D \). Since \( \overline{F} \) is essential in \( M \), \( \partial D \) bounds a disc \( E \subset \overline{F} \). By the definition of meridional compressing disc, \( |E \cap L| \geq 2 \). One component \( \overline{G} \) resulting from cut-compressing \( \overline{F} \) using \( D \) has the same genus as \( \overline{F} \) and intersects \( L \) just once. This component must be essential in \( M \) since it is isotopic to \( \overline{F} \) in \( M \). Hence, we have contradicted our choice of \( \overline{F} \). Thus, \( F \) is incompressible, \( \partial \)-incompressible, and cut-incompressible. It cannot be an annulus by hypothesis. By Theorem 9.1, \( d_{c}(T) \leq \max(3, 2g(\overline{F}) + 2) \leq \max(3, 2g(S) + 2). \)

We can now bound the distance of knots admitting exceptional and cosmetic surgeries.

**Theorem 9.4.** Suppose that \( L \subset M \) is a knot in a closed, orientable 3-manifold. Let \( M' \) be a 3-manifold obtained by non-trivial Dehn surgery on \( L \). Then the following hold:

1. If \( M = S^{3} \), if \( b(L) \geq 3 \), and if \( M' \) is a lens space, then \( d_{c}(L) \leq 5 \). If \( b(L) \geq 4 \), then \( d_{c}(L) \leq 4 \).
2. If \( M = S^{3} \), if \( b(L) \geq 3 \), and if \( M' \) is a small Seifert fibered space, then \( d_{c}(L) \leq 6 \).
3. If \( M \) is hyperbolic and if \( M' \) is not hyperbolic, then \( d_{c}(L) \leq 12 \).
(4) Assume that if \( M = S^3 \), then \( b(L) \geq 3 \). Then, if the Heegaard genus of \( M' \) is \( g \), we have \( d_C(L) \leq \max(6, 4g + 4) \).

**Proof.** Note that if \( L \) is a non-hyperbolic knot then \( d_C(L) \leq 2 \) by [1] and the proof is complete. Otherwise, we can assume that \( L \) is hyperbolic. By the Geometrization Theorem of Thurston and Perelman, if \( M' \) is non-hyperbolic, then it is reducible, toroidal, or a small Seifert-fibered space. Small Seifert fibered spaces have Heegaard genus at most 2. If \( M' \) is reducible or toroidal, conclusion (3) follows from Theorem 6.2 and Lemma 4.6. If \( M' \) is a lens space, then conclusions (1) and (3) follow from Theorem 9.3 (and the fact that \( d_C \) is an integer) by setting \( g(S) = 1 \). Similarly, if \( M' \) is a small Seifert fibered space, conclusions (2) and (3) follow from Theorem 9.3 by setting \( g(S) = 2 \). Finally, we note that if \( M' \) has Heegaard genus \( g \), then

\[
\max \left( \frac{2}{b(T)} + 4, \frac{4g(S)}{b(T)} + 4, 2g(S) + 2 \right) \leq \max(6, 4g + 4),
\]

and so conclusion (4) also follow from Theorems 6.2 and 9.3 and Lemma 4.6. □

10. LOWERING THE BOUND ON BRIDGE NUMBER

By a slight variation of the argument in Section 5, we can halve the upper bound on bridge number as follows. We use the notation of that section freely. This will give us our best results on reducing and toroidal surgeries.

**Theorem 10.1.** Let \( T \) be a connected bridge surface for a knot \( L \) in a manifold \( M \) and let \( S \) be a property embedded surface in \( M \). Assume that \( C(T) \) is connected, \( d_C(T) \geq 3 \), and that there is a maximally essential interval for \( S \) relative to \( T \). Then,

\[
(b(T) - 4)\Delta \leq \frac{4g(S) - 4|S| + 2|\partial_0 S| - 2}{|\partial_L S|} + 2.
\]

**Proof.** Recall that \( Q \) is the number of post-active arcs, and that by Lemma 5.5,

\[
Q \leq -2\chi(S).
\]

Let \( [a, b] \subset [u, v] \) be a maximally essential interval for \( S \) relative to \( T \). Recall that

\[
v_1 < v_2 < \ldots < v_{n-1}
\]

are the critical values of \( h|_S \) in \((a, b)\) and set \( t_0 = a \) and \( t_n = b \). Let \( t_i = (v_i + v_{i-1})/2 \). Let \( \alpha \) be the arc or loop in \( T_u \cap S \) that is essential in \( T_u \) but bounds a compressing or boundary compressing disc for \( T_u \).
that lies in $T_v$ and let $\beta$ be the arc or loop in $T_v \cap S$ that is essential in $T_v$ but bounds a compressing or boundary compressing disc for $T_v$ that lies in $T_\dagger$.

A constant path is a sequence of inactive arcs $\kappa_1, \ldots, \kappa_m$ with $\kappa_i \subset T_{t_i} \cap S$ and all $\kappa_i$ mutually isotopic in both $S$ and $T$. Although we will not need this fact, if a constant path exists, then $d_{AC}(T) \leq 2$.

Suppose that $(\kappa_i)$ is a constant path and identify each $T_{t_i}$ with $T$. Let $\gamma_1$ be the frontier of a regular neighborhood of $\kappa_1$ in $T$. If $\alpha$ is a circle, let $\gamma_0 = \alpha$; otherwise let $\gamma_0$ be the frontier of a regular neighborhood in $T$ of $\alpha$. If $\beta$ is a circle, let $\gamma_2 = \beta$, otherwise let $\gamma_2$ be the frontier of a regular neighborhood of $\beta$ in $T$. Note that $\gamma_0$, $\gamma_1$, and $\gamma_2$ are all essential circles in $T$. Recall also that the interior of $\kappa_1$ is disjoint from $\alpha$, the interior of $\kappa_m$ is disjoint from $\beta$, and $\kappa_1$ and $\kappa_m$ are isotopic in $T$. Thus, if neither endpoint of $\kappa_1$ is on the same boundary component of $T_K$ as an endpoint of either $\alpha$ or $\beta$, then $\gamma_0, \gamma_1, \gamma_2$ is a path of length 2 in $C(T)$. See Figure 7. However, $\gamma_0 \in D_\dagger$ and $\gamma_2 \in D_\dagger$, so $d_C(T) \leq 2$, contradicting the hypotheses of the theorem. Consequently, whenever $(\kappa_i)$ is a constant path, one endpoint of $\kappa_1$ lies on a component of $\partial T_K$ adjacent to $\alpha$ or $\beta$.

![Figure 7](image_url)

**Figure 7.** The loops enclosing $\alpha$, $\beta$, and $\kappa_1$ and the boundary components of $T_K$ adjacent to their endpoints form a path of length 2 in $C(T)$.

Recall that $\mathcal{L}$ is the set of labels on $\partial T \cap \partial S$ and that at a critical point a label is active if the adjacent arc of $T \cap S$ changes isotopy class
in \( T \). A label is inactive if it is not active at any critical point. As previously argued, each inactive label corresponds to an endpoint of an arc in a constant path. Each arc in a constant path is adjacent to one of the components of \( \partial T \) incident to either \( \alpha \) or \( \beta \), but is distinct from \( \alpha \) and \( \beta \). Hence, there are at most \( 8|\partial L_S|\Delta - 4 \) inactive labels in \( \partial T \cap \partial S \). Since \( L \) is a knot in a closed manifold, there are exactly \( 2b(T)|\partial L_S|\Delta \) labels in \( \partial T \cap \partial S \), so there are at least \( (2b(T) - 8)|\partial L_S|\Delta + 4 \) active labels. Each active arc is adjacent to two active labels. Thus, by Inequality (1),

\[
(b(T) - 4)|\partial L_S|\Delta + 2 \leq Q \leq -2\chi(S).
\]

We have \(-2\chi(S) = 4g(S) - 4|S| + 2|\partial_0 S| + 2|\partial L_S|\). Consequently,

\[
(b(T) - 4)\Delta \leq (4g(S) - 4|S| + 2|\partial_0 S| - 2) / |\partial L_S| + 2.
\]

We can now obtain an improved bound on knots admitting non-trivial reducible or toroidal surgeries.

**Theorem 10.2.** Suppose that \( L \subset S^3 \) is a knot with \( b(L) \geq 3 \) and \( d_C(L) \geq 3 \). Then

1. If non-trivial Dehn surgery on \( L \) produces a reducible 3-manifold, then \( b(L) \leq 5 \).
2. If non-trivial Dehn surgery on \( L \) produces a toroidal 3-manifold, then \( b(L) \leq 6 \). Furthermore, if the surgery slope is non-longitudinal, then \( b(L) \leq 5 \).

**Proof.** Note that if \( L \) is a non-hyperbolic knot then \( d_C(L) \leq 2 \) by [1] and the proof is complete. Otherwise, we can assume that \( L \) is hyperbolic. Isotope \( L \) to be into a bridge position with respect to a Heegaard sphere \( \overline{T} \) in \( S^3 \) minimizing \((b(L), d_C(L))\) lexicographically. If non-trivial Dehn surgery on \( L \) produces a reducible 3-manifold, in the surgered manifold choose a reducing sphere intersecting the core of the surgery torus minimally. Since the exteriors of knots in \( S^3 \) are irreducible, the intersection \( S \) of the reducing sphere with \( S^3(L) \) is an essential planar surface in \( S^3(L) \). Since \( \overline{T} \) has \( d_C(L) \geq 3 \), the surface \( T \) is strongly \( \partial \)-irreducible. By Lemma 6.1, there is a sweepout by \( T \) adapted to \( S \) so that there is a maximal interval for \( S \). Since \( b(L) \geq 3 \), \( C(T) \) is connected. By Theorem 10.1,

\[
b(L) - 4 \leq (b(L) - 4)\Delta \leq \frac{4g(S) - 6}{|\partial L_S|} + 2 < 2.
\]

Since \( b(L) \) is an integer, \( b(L) \leq 5 \).

The proof for the case when \( L \) has a toroidal surgery is similar. \( \Box \)
Likewise, we can also obtain a bound on the bridge number of high distance knots in terms of the genus of the knot.

**Theorem 10.3.** Suppose that $L$ is a knot in a homology sphere $M$, and that if $M = S^3$, then $b(L) \geq 3$. If $d_C(L) \geq 3$, then
\[ b(L) \leq 4g(L), \]
where $g(L)$ is the Seifert genus of $L$.

The proof is an easy combination of Theorem 10.1 and Lemma 6.1, as in the proof of Theorem 10.2.

We conclude with a few consequences for exceptional surgeries and cosmetic surgeries.

**Corollary 10.4.** Suppose that $L \subset M$ is a knot in a closed hyperbolic non-Haken 3-manifold such that $d_C(L) \geq 3$. Let $M'$ be a 3-manifold obtained by non-trivial Dehn surgery on $M$. Then the following hold:

1. If $M'$ is non-hyperbolic and if the tunnel number of $L$ is at least 2, then $b(L) \leq 8$
2. If $M'$ has Heegaard genus at most the Heegaard genus $g$ of $M$ and if the tunnel number of $L$ is at least $g$, then $b(L) \leq 2g + 4$.

**Proof.** Since $M$ is non-Haken, there is no essential surface in $M$ intersecting $L$ at least twice transversally. As in Theorem 9.4, if $M(L)(\sigma)$ is non-hyperbolic either it contains an essential sphere, and essential torus, or has Heegaard genus at most 2. Under the hypotheses of (1), the tunnel number of $L$ is at least 2, which implies that there is no Heegaard surface for the exterior of $L$ having genus at most 2. Similarly, if the tunnel number of $L$ is at least $g$, then there is no Heegaard surface for the exterior of $L$ having genus at most $g$. The theorem then follows from the proof of Theorem 8.1 except using the bound from Theorem 10.1 in place of the inequalities from Theorems 6.2 and 7.6.  

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