Electromagnetic Source Localization with Finite Set of Frequency Measurements

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Abstract

A phase conjugation algorithm for localizing an extended radiating electromagnetic source from boundary measurements of tangential components of the electric field is presented. Measurements are taken over a finite number of frequencies. The artifacts related to the finite frequency data are tackled with $l_1$-regularization blended with the fast iterative shrinkage-thresholding algorithm with backtracking of Beck & Teboulle.

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1 Introduction

Inverse source problems have been the subject of numerous studies over the recent past due to a plethora of applications in science and engineering, specially in biomedical imaging, non-destructive testing and geophysics; see, for instance, [1]-[10] and references therein. Several frameworks to recover spatial support of the stationary acoustic, elastic and electromagnetic sources in time and frequency domain have been developed [3, 4, 8, 11], including time reversal and phase conjugation algorithms [1, 2, 5, 12, 13, 14, 15].

This work aims to evince a radiating source for the Maxwell’s equations using boundary measurements over a finite set of frequencies using a phase conjugation sensitivity framework blended with fast iterative shrinkage-thresholding algorithm with backtracking of Beck and Teboulle for $l_1$-regularization; refer to [16, 4]. Phase conjugation and time reversal techniques have observed a significant success in the resolution of inverse problems, [17, 18], wherein the recorded wave is phase conjugated or time reversed and re-emitted into the medium [15, 12]. The re-emitted wave retraces its path backwards in chronology, due to the self-adjointness and reciprocity of the wave operator in lossless media, converging to the source location.

The inverse source problems are ill-posed having non-uniqueness issues generally due to the presence of non-radiating sources [19, 10, 8]. The stability and localization of electromagnetic radiating sources with single, multiple and entire frequency or time data have

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been extensively studied in, for instance, [10, 13, 20, 21, 22]. The well-posedness of problem undertaken in this study, that is, the source problem for Maxwell’s equation with a finite set of frequency surface measurements is established in particular in [8].

The investigation is sorted in the following order. The inverse problem is presented and a few key identities are collected in Section 2. In Section 3 an electromagnetic source is retrieved using entire frequency measurements with a phase conjugation functional. The functional is further adopted to locate a source using finite set of frequencies in Section 4. An initial guess is retrieved and then optimized using $l_1$–regularization. The principle contributions of the investigation are summarized in Section 5.

2 Problem Formulation

Let $\Omega \subset \mathbb{R}^d, d = 2, 3$, be an open bounded domain with a Lipschitz boundary $\Gamma$. Consider

\[
\begin{cases}
\nabla \times \mathbf{E} - i\omega \mu_0 \mathbf{H} = 0, & x \in \mathbb{R}^d, \\
\nabla \times \mathbf{H} + i\omega \epsilon_0 \mathbf{E} = \mathbf{J}(x), & x \in \mathbb{R}^d,
\end{cases}
\]

subject to the Silver-Müller radiation conditions

\[
\lim_{|x| \to \infty} |x|^\frac{d-1}{2} \left( \sqrt{\mu_0} \mathbf{H} \times \hat{x} - \sqrt{\epsilon_0} \mathbf{E} \right) = 0, \quad \text{where } \hat{x} := \frac{x}{|x|},
\]

with frequency pulsation $\omega$, electric permittivity $\epsilon_0 > 0$ and magnetic permeability $\mu_0 > 0$, where $\mathbf{E}$ and $\mathbf{H}$ are the time-harmonic electric and magnetic fields respectively. Here $\mathbf{J}(x) \in \mathbb{R}^d$ is the current source density, assumed to be sufficiently smooth and compactly supported in $\Omega$, that is, $\text{supp}\{\mathbf{J}\} \subset \subset \Omega$.

Let $\nu$ be the outward unit normal to $\Gamma$. Define the admissible set of frequencies and the boundary data consisting of tangential component of the electric field respectively by

\[
\mathcal{W} := (\omega_n)_{n=1}^N \quad \text{and} \quad \mathbf{d}(x, \omega) = \nu \times \mathbf{E}(x, \omega), \quad \forall (x, \omega) \in \Gamma \times \mathbb{R}.
\]

Then, the ultimate goal of this work is to tackle the following problem:

**Statement of Problem.** Given $\mathbf{d}_\mathcal{W} := \mathbf{d}|_{\Gamma \times \mathcal{W}}$ for $N$ sufficiently large, identify the support, $\text{supp}\{\mathbf{J}\}$, of current source density $\mathbf{J}$.

2.1 Preliminaries

In the sequel, we refer to $\kappa_0 := \omega \sqrt{\epsilon_0 \mu_0} = \omega/c_0$ as the wave number with $c_0 := 1/\sqrt{\epsilon_0 \mu_0}$ being the wave speed in dielectrics. By virtue of (2.1), the time-harmonic electric field satisfies the Helmholtz equation

\[
\nabla \times \nabla \times \mathbf{E} - \kappa_0^2 \mathbf{E} = i\omega \mu_0 \mathbf{J}(x), \quad x \in \mathbb{R}^d,
\]

subject to the outgoing radiation condition (2.2).

Let $\mathbf{G}_0^{\text{reg}}(x, \omega)$ be the outgoing electric-electric Green’s function for the Maxwell’s equations (2.1) in $\mathbb{R}^d$, that is,

\[
\nabla \times \nabla \times \mathbf{G}_0^{\text{reg}}(x, \omega) - \kappa_0^2 \mathbf{G}_0^{\text{reg}}(x, \omega) = i\omega \mu_0 \delta_0(x), \quad x \in \mathbb{R}^d,
\]
where $\delta_0(x)$ is the Dirac mass at $x = 0$. Following spatial reciprocity can be proved for isotropic dielectrics; see [24]:

$$G_0^{ee}(x - y, \omega) = G_0^{ee}(y - x, \omega), \quad x, y \in \mathbb{R}^d, \quad \omega \in \mathbb{R}. \quad (2.6)$$

### 2.2 Electromagnetic Identities

The following identities are the key ingredients to elucidate the localization property of the imaging functional proposed in the next section. The variants of the identity in Lemma 2.1 can be found in literature; see, for instance, [12, 23]. The details are, however, provided here since it is one of the building blocks of our reconstruction algorithm.

**Lemma 2.1 (EM Helmholtz-Kirchhoff Identity).** Let $B(0, R)$ be an open ball in $\mathbb{R}^d$ with large radius $R \to \infty$ and boundary $\partial B(0, R)$. Then, for all $x, y \in \mathbb{R}^d$, we have

$$\lim_{R \to +\infty} \int_{\partial B(0, R)} G_0^{ee}(x - \xi, \omega) G_0^{ee}(\xi - y, \omega) d\sigma(\xi) = \mu_0 c_0 \Re \left\{ G_0^{ee}(x - y, \omega) \right\},$$

where the superposed bar indicates a complex conjugate.

**Proof.** Recall that we have for all constant vectors $p, q \in \mathbb{R}^d$, and $x, y \in \mathbb{R}^d$,

$$\nabla_\xi \times \nabla_\xi \times G_0^{ee}(x - \xi, \omega) p - \kappa_0^2 G_0^{ee}(x - \xi, \omega) p = i\omega \mu_0 \delta_x(\xi) p, \quad (2.7)$$

$$\nabla_\xi \times \nabla_\xi \times G_0^{ee}(y - \xi, \omega) q - \kappa_0^2 G_0^{ee}(y - \xi, \omega) q = -i\omega \mu_0 \delta_y(\xi) q. \quad (2.8)$$

Taking scalar product of (2.7) by $G_0^{ee}(y - \xi, \omega) q$ and (2.8) by $G_0^{ee}(x - \xi, \omega) p$, subtracting the resultant equations and finally integrating over $B(0, R)$, we arrive at

$$\int_B (\nabla_\xi \times \nabla_\xi \times G_0^{ee}(y - \xi, \omega) q) \cdot G_0^{ee}(x - \xi, \omega) p d\xi - \int_B G_0^{ee}(y - \xi, \omega) q \cdot (\nabla_\xi \times \nabla_\xi \times G_0^{ee}(x - \xi, \omega) p) d\xi = -i\omega \mu_0 \int_B G_0^{ee}(y - \xi, \omega) q \cdot \delta_x(\xi) p d\xi - i\omega \mu_0 \int_B G_0^{ee}(x - \xi, \omega) p \cdot \delta_y(\xi) q d\xi,$$

$$= -i\omega \mu_0 \int_B G_0^{ee}(y - \xi, \omega) q \cdot \delta_x(\xi) p d\xi - i\omega \mu_0 q \cdot G_0^{ee}(y - \xi, \omega) q d\xi,$$

$$= -2i\omega \mu_0 q \cdot \Re \left\{ G_0^{ee}(x - y, \omega) \right\} p. \quad (2.9)$$

On the other hand, recall that, for all $u, v \in C^2(\mathbb{B})^d$

$$\int_B (\nabla \times \nabla \times u) \cdot \nu dx - \int_B u \cdot (\nabla \times \nabla \times v) dx = -\int_{\partial B} (u \times \nu) \cdot (\nabla \times \nu) \cdot d\sigma(x) - \int_{\partial B} (\nabla \times u \times \nu) \cdot v d\sigma(x),$$

where $d\sigma$ is the surface element. Substitute $u = G_0^{ee}(y - \xi, \omega) q$ and $v = G_0^{ee}(x - \xi, \omega) p$ to
get
\[
\int_B (\nabla \times \nabla \times \mathcal{G}_0(y - \xi, \omega) q) \cdot \mathcal{G}_0^c(x - \xi, \omega) p \, d\xi
- \int_B \mathcal{G}_0^c(y - \xi, \omega) q \cdot (\nabla \times \nabla \times \mathcal{G}_0^c(x - \xi, \omega) p) \, d\xi
= - \int_{\partial B} (\mathcal{G}_0^c(y - \xi, \omega) q \times \nu) \cdot (\nabla \times \mathcal{G}_0^c(x - \xi, \omega) p) \, d\xi
- \int_{\partial B} (\nabla \times \mathcal{G}_0^c(y - \xi, \omega) q \times \nu) \cdot (\mathcal{G}_0^c(x - \xi, \omega) p) \, d\sigma(\xi),
\]
\[
= - \int_{\partial B} (\nabla \times \mathcal{G}_0^c(y - \xi, \omega) q) \cdot (\nu \times \nabla \times \mathcal{G}_0^c(x - \xi, \omega) p) \, d\xi
+ \int_{\partial B} (\nu \times \nabla \times \mathcal{G}_0^c(y - \xi, \omega) q) \cdot (\mathcal{G}_0^c(x - \xi, \omega) p) \, d\sigma(\xi).
\]

Now from Sommerfeld radiation conditions,
\[
\nu \times \nabla \times \mathcal{G}_0^c(x - \xi, \omega) p = i\kappa_0 \mathcal{G}_0^c(x - \xi, \omega) p + O\left(R^{-\frac{d+1}{2}}\right),
\]
where the order term vanishes as \( R \to \infty \). Therefore,
\[
\int_B (\nabla \times \nabla \times \mathcal{G}_0(y - \xi, \omega) q) \cdot \mathcal{G}_0^c(x - \xi, \omega) p \, d\xi
- \int_B \mathcal{G}_0^c(y - \xi, \omega) q \cdot (\nabla \times \nabla \times \mathcal{G}_0^c(x - \xi, \omega) p) \, d\xi
\approx -2i\kappa_0 \int_{\partial B} i\kappa_0 \mathcal{G}_0^c(x - \xi, \omega) p \cdot \mathcal{G}_0^c(y - \xi, \omega) q \, d\sigma(\xi)
= -2i\kappa_0 \int_{\partial B} q \cdot \mathcal{G}_0^c(y - \xi, \omega) \mathcal{G}_0^c(x - \xi, \omega) p \, d\sigma(\xi),
\]
where we have used the reciprocity relation (2.10).

Finally, comparing Equations (2.10) and (2.11), and by varying and choosing \( p \) and \( q \) as the basis vectors in \( \mathbb{R}^d \) we get
\[
\int_{\partial B} \mathcal{G}_0^c(y - \xi, \omega) \mathcal{G}_0^c(x - \xi, \omega) \, d\sigma(\xi) \approx \mu_0 c_0 \text{Re} \left\{ \mathcal{G}_0^c(x - y, \omega) \right\},
\]
which leads to the conclusion by tending \( R \to \infty \).

Lemma 2.2. For all \( x, y \in \mathbb{R}^d, x \neq y \),
\[
\frac{c_0}{2\pi} \int_\mathbb{R} \text{Re} \left\{ \mathcal{G}_0^c(x - y, \omega) \right\} d\omega = \delta_x(y) I.
\]

Proof. Let \( \tilde{G} \) be the solution to
\[
\frac{1}{c_0^2} \frac{\partial^2 \tilde{G}}{\partial t^2} (x, t; y, \tau) + \nabla \times \nabla \times \tilde{G}(x, t; y, \tau) = -\delta_y(x) \delta_x(t) I, \quad x, y \in \mathbb{R}, t > \tau,
\]
(2.11)
and let $G(x, y, \omega)$ be the Fourier transform of $\hat{G}(x; t; y, 0)$. Further, the following causality conditions are imposed

$$\hat{G}(x; t; y, \tau) = 0 = \frac{\partial \hat{G}}{\partial t}(x; t; y, \tau), \quad x, y \in \mathbb{R}, t < \tau.$$  

Then, integrating (2.11) over an infinitesimal time interval from $\tau^{-}$ to $\tau^{+}$, using the causality conditions above and the continuity of $\hat{G}$ away from $t = \tau$, we find out that

$$\frac{\partial \hat{G}}{\partial t}(x; t; y, \tau) \bigg|_{t=\tau^{+}} = -c_0^2 \delta_y(x) I. \quad (2.12)$$

Therefore, integrating above equation over $t$ and using the Parseval’s identity, yields

$$\int_{\mathbb{R}} i\omega G(x, y, \omega) d\omega = 2c_0^2 \pi \delta_y(x) I \int_{\mathbb{R}} \delta_0(\omega) d\omega = 2c_0^2 \pi \delta_y(x) I, \quad (2.13)$$

where we have made use of the fact that the Fourier transform of 1 is $2\pi \delta_0(\omega)$. Finally, since $G_0^c(x - y, \omega) = -i\omega \mu_0 G(x, y, \omega)$, and $\delta_y(x) I$ is real, relation (2.13) leads to the conclusion.  

3 Reconstruction with Full Bandwidth Measurements

This section is in order to provide the building blocks to handle finite frequency data problem. As a first step toward the ultimate goal, we find the spatial support of the current source, $\text{supp}\{J\}$, from data $d(x, \omega)$ with $\omega \in \mathbb{R}$.

For a fixed frequency $\omega \in \mathbb{R}$, define the adjoint field $E^*$ to be the solution to

$$\nabla \times \nabla \times E^*(x, \omega) - \kappa_0^2 E^*(x, \omega) = i\omega \mu_0 \overline{\nabla} G(x, \omega) \chi_\Gamma(x), \quad (x, \omega) \in \mathbb{R}^3 \times \mathbb{R}, \quad (3.1)$$

where $\chi_\Gamma$ is the characteristic function of the boundary $\Gamma$. Then, the phase conjugation functional is defined by

$$I(x) := \frac{\epsilon_0}{2\pi c_0 \mu_0} \int_{\mathbb{R}} E^*(x, \omega) d\omega, \quad \forall x \in \mathbb{R}. \quad (3.2)$$

Then, $I(x)$ yields the approximate spatial support of the current density $J(x)$. In fact, we have the following theorem.

**Theorem 3.1.** For $x \in \Omega$ sufficiently far from $\Gamma$ compared to the wavelength of the wave impinging upon $\Omega$, $I(x) \simeq J(x)$.

**Proof.** Since $J$ is supported compactly inside $\Omega$, for all $x \in \Omega$ and $y \in \Gamma$, we have

$$\left\{ \begin{array}{l}
E^*(x, \omega) = \int_{\Gamma} G_0^e(y - x, \omega) \overline{\mathbf{d}}(y, \omega) d\sigma(y) \\
\mathbf{d}(y, \omega) = E(y, \omega) \big|_{y \in \Gamma} = \int_{\Omega} G_0^e(y - z, \omega) J(z) dz \bigg|_{y \in \Gamma}
\end{array} \right. \quad (3.3)$$

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Therefore, by using (3.3) in (3.2) we arrive at

\[ I(x) = \frac{\epsilon_0}{2\pi c_0 \mu_0} \int_{\mathbb{R}^d} \left( \int_{\Gamma} G_0^{ce}(x - y, \omega) \overline{G_0^{ce}(y - z, \omega)} d\sigma(y) \right) d\omega J(z) dz. \]

Now, we invoke the Helmholtz-Kirchhoff identity from Lemma 2.1 to have

\[ \int_{\Gamma} G_0^{ce}(x - y, \omega) \overline{G_0^{ce}(y - z, \omega)} d\sigma(y) \simeq \mu_0 c_0 \Re \{ G_0^{ce}(x - z, \omega) \}, \tag{3.4} \]

leading us to

\[ I(x) \simeq \int_{\mathbb{R}^d} \frac{\epsilon_0}{2\pi} \int_{\mathbb{R}} \Re \{ G_0^{ce}(x - z, \omega) \} d\omega J(z) dz. \]

Finally, using Lemma 2.2, we conclude that

\[ I(x) \simeq \int_{\mathbb{R}^d} \delta_x(z) J(z) dz = J(x). \]

\[ \square \]

4 Source Localization with Finite Set of Frequencies

In this section, we address the electromagnetic inverse source problem using the boundary data \( d_W = d(x, \omega)|_{\Gamma \times W} \). First an initial guess is retrieved and then subsequently optimized using an \( l_1 \)-regularization technique.

4.1 Initial Guess Retrieval

Inspired by the functional \( I \) defined in (3.2), we define a single frequency functional \( I_n \) by (4.2). However, since we are dealing with a finite set of frequency measurements, the lack of information over entire spectrum induces noise and blurring in the reconstruction. In order to fix the problem, in this section, an initial guess to the current source density is identified, which will be optimized providing an improved approximation to \( \text{supp}\{J\} \).

Let \( 0 \leq \kappa_1^0 \leq \kappa_2^0 \leq \cdots \leq \kappa_N^0 \) be \( N \) wave numbers corresponding to \( \omega_n \in W \) for \( n = 1, 2, \cdots, N \). Let us define the adjoint field \( E_n^* \) corresponding to a fixed frequency \( \omega_n \in W \) to be the solution to the Helmholtz equation

\[ \nabla \times \nabla \times E_n^*(x, \omega_n) - (\kappa_n^0)^2 E_n^*(x, \omega_n) = i\omega_n \mu_0 d_W(x, \omega_n) \chi_\Gamma(x), \quad (x, \omega_n) \in \mathbb{R}^3 \times \mathbb{R}, \tag{4.1} \]

and the \textit{single frequency phase conjugation} functional by

\[ I_n(x) := \frac{\epsilon_0}{2\pi c_0 \mu_0} E_n^*(x, \omega_n). \tag{4.2} \]

Our first result of this section is the following Lemma.

\textbf{Lemma 4.1.} For all \( x \in \Omega \) sufficiently far from \( \Gamma \), compared to the wavelength of the wave impinging upon \( \Omega \),

\[ I_n(x) \simeq \frac{\epsilon_0}{2\pi} \int_{\Omega} \Re \{ \hat{G}_0^{ce}(x - y, \omega_n) \} J(y) dy. \]
Proof. The proof is very similar to that of Theorem 3.1. Recall that
\[
\begin{align*}
E_n^*(x, \omega_n) &= \int_{\Gamma} G_0^\infty(\xi - x, \omega_n) dV(\xi, \omega_n) d\sigma(\xi), \\
\mathbf{d}_W(\xi, \omega_n) &= \left. \mathbf{E}(\xi, \omega_n) \right|_{\Gamma} = \int_{\Omega} G_0^\infty(y - \xi, \omega_n) dJ(y) dy \bigg|_{\xi \in \partial \Omega}.
\end{align*}
\]
Therefore,
\[
\mathcal{I}_n(x) = \frac{\epsilon_0}{2\pi\epsilon_0\mu_0} E_n^*(x, \omega_n),
\]
\[
= \frac{\epsilon_0}{2\pi\epsilon_0\mu_0} \int_{\Omega} \int_{\Gamma} G_0^\infty(\xi - x, \omega_n) \overline{G_0^\infty(y - \xi, \omega_n)} d\sigma(\xi) J(y) dy.
\]
Finally, we conclude, again using the electromagnetic Helmholtz-Kirchhoff identity from Lemma 2.1.

4.2 Regularization

In this section, an optimized reconstruction to the current source density is provided using $d_W$. The aim is to explore an $l_1$-regularization to optimize the localization of the support of the source.

The objective is to resolve the following optimization problem:
\[
J_\lambda(x) := \arg\min_{J \in \mathbb{R}^d} \mathcal{M}(\tilde{J}) + \mathcal{R}(\tilde{J}),
\]
\[
\mathcal{M}(\tilde{J}) := \sum_{n=1}^{N} \frac{1}{2N} \left\| \mathcal{I}_n(x) - \frac{\epsilon_0}{2\pi} \int_{\Omega} \Re\left\{ G_0^\infty(x - y, \omega_n) \right\} \tilde{J}(y) dy \right\|^2, \tag{4.3}
\]
\[
\mathcal{R}(\tilde{J}) := \lambda \left\| \tilde{J}(x) \right\|_{l_1}, \tag{4.4}
\]
where the first term $\mathcal{M}$ is the data fidelity term and the second term $\mathcal{R}$ accounts for the $l_1$-regularization. It is precisely that $\lambda$ is a regularization parameter controlling the relative weights of the two terms and provides a trade-off between fidelity to the measurements and noise sensitivity. Here $\| \cdot \|$ denotes the Euclidean norm in $\mathbb{R}^d$.

4.2.1 Fast Iterative-Shrinkage Thrashing Algorithm with Backtracking

The direct computations of the solution $J_\lambda$ to the minimization problem (4.3) is not trivial, indeed, the $l_1$-term is not smooth, at least not differentiable. Thus, in order to obtain $J_\lambda$ explicitly, approximation schemes are indispensable. In order to do so, we follow Beck & Teboulle [16] and use their fast iterative shrinkage thresholding algorithm with backtracking. This method belongs to the class of split gradient descent iterative schemes with a global convergence rate $O(k^{-2})$, where $k$ is the iteration counter.

For any $\gamma > 0$, define the quadratic approximation of the Lagrangian
\[
\mathcal{L}(\tilde{J}, \lambda) = \mathcal{M}(\tilde{J}) + \mathcal{R}(\tilde{J})
\]
by

\[ P_\gamma(x, y) := M(y) + \langle x - y, \nabla M(y) \rangle + \frac{\gamma}{2} \| x - y \|^2 + \mathcal{R}(x) \]  

(4.6)

We also define

\[ T_\gamma(y) := \arg \min_{x \in \mathbb{R}^d} \left\{ P_\gamma(x, y) \right\}, \]

(4.7)

where \( y \in \mathbb{R}^d \) and \( \langle \cdot, \cdot \rangle \) is the standard Euclidean inner product in \( \mathbb{R}^d \). Then the Algorithm converges to the global minimum; see \([16]\):

**Algorithm 1** Fast Iterative-Shrinkage Thresholding with Backtracking.

**Require:** Set \( \gamma_0 > 0, \quad \eta > 1, \quad x_0 = 0, \quad y_1 = x_0, \quad s_1 = 1. \)

1. for \( k \geq 1 \) do
   2. Set \( i_k = 1, \quad \beta = \eta \gamma_{k-1}. \)
   3. while \( \mathcal{L}(T_\beta(y_k), \lambda) > P_\beta(T_\beta(y_k), y_k) \) do
      4. Update \( i_k = i_k + 1, \quad \beta = \eta^{i_k} \gamma_{k-1}. \)
   5. end while
   6. Set \( \gamma_k = \beta, \quad x_k = T_{\gamma_k}(y_k). \)
   7. Update \( s_{k+1} = \frac{1}{2} \left( 1 + \sqrt{1 + 4s_k^2} \right), \quad y_{k+1} = x_k + \frac{s_{k-1}}{s_{k+1}} (x_k - x_{k-1}), \quad i_k = 0, \)
   8. \( k = k + 1. \)
8. end for

return \( \hat{J} = x_k. \)

5 Conclusion

In this investigation, electromagnetic inverse source problem is tackled using boundary measurements of the tangential component of electric field over a finite set of frequencies. A phase conjugation algorithm is proposed in order to deal with the problem associated with full frequency spectrum which subsequently inspired an imaging functional for that with finite set of frequencies. Since the information is lost due to incomplete frequency spectrum, an \( l_1 \)-regularization blended with the fast iterative shrinkage thresholding algorithm with backtracking of Beck and Teboulle is deployed.

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