Quantum Hall plateau transition in graphene with spatially correlated random hopping

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We investigate how the criticality of the quantum Hall plateau transition in disordered graphene differs from those in the ordinary quantum Hall systems, based on the honeycomb lattice with ripples modeled as random hoppings. The criticality of the graphene-specific $n = 0$ Landau level is found to change dramatically to an anomalous, almost exact fixed point as soon as we make the random hopping spatially correlated over a few bond lengths. We attribute this to the preserved chiral symmetry and suppressed scattering between K and K' points in the Brillouin zone. The results suggest that a fixed point for random Dirac fermions with chiral symmetry can be realized in free-standing, clean graphene with ripples.

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After the seminal observation of the anomalous quantum Hall effect (QHE) in graphene,[1-3] fascination expands with the graphene QHE. One crucial question that is not fully explored is: what exactly is the role of the chiral symmetry in the problem? This has to do with a most significant feature of double Dirac cones (at K and K' in the Brillouin zone) in graphene. Although a single Dirac cone would already imply a characteristic Landau level structure with the zero-energy level, if we really want to look at the effect of disorder on the graphene Landau levels, we have to go back to the honeycomb lattice for which the chiral (A-B sub-lattice) symmetry and the associated valley (K and K') degrees of freedom enter as an essential ingredient. The effect of disorder should then be sensitive to the nature of disorder, i.e., bond disorder or potential disorder, which determines the presence or otherwise of the chiral symmetry, and whether the disorder is short-ranged or long-ranged, which controls the scattering between K and K' points.

For Dirac fermions, effects of random gauge fields induced by ripples in the two-dimensional plane have been discussed,[4-7] and the stability of zero modes has been argued in terms of the index theorem and the chiral symmetry.[4-7,12] More recently, the plateau-to-plateau transition for random Dirac fermions has been discussed, where the particle-hole symmetry is shown to make the zero-energy Landau level robust.[12] As for the criticality, however, the result[12] shows nothing special about the $n = 0$ Landau level, but this is obtained for a model of the Dirac fermions for which the randomness is introduced as a scalar random potential, so the chiral symmetry is degraded.

On the other hand, the actual randomness in graphene, even when atomically clean, is known to have ripples, i.e., long-ranged corrugation of the graphene plane.[4] In fact, while a monolayer graphene naively contradicts with the well-known theorem that two-dimensional crystals should be thermodynamically unstable, one explanation attributes the stability to the ripples.[14] In this sense, we can take the disorder coming from ripples in graphene as an intrinsic disorder. Since the ripples consist of random bending of the honeycomb lattice, its main effect should be, in the tight-binding model, a modification of the hopping integral between neighboring sites.[11] Thus the question amounts to: how does the QHE criticality behave for a model with random hopping on the honeycomb lattice. The random hopping is of fundamental theoretical interest as well, since a bond randomness preserves the chiral symmetry, so its effect, particularly on the criticality, is of crucial interest. The chiral symmetry indeed plays a fundamental role in graphene[4,3], which protects the gapless double Dirac cones as well as the existence of characteristic zero-modes with/without magnetic fields. A special importance of the chiral symmetry in localization physics has also been discussed with a viewpoint of the universality[15-17].

Now, the length scale over which the lattice is warped should be reflected as the spatial correlation in the random hopping. In the case of the ordinary QHE systems, the importance of the spatial correlation of randomness has been discussed in various contexts, among which are the pioneering work by Ando and Uemura[18], the levitation of the critical states in the lattice model[19,20], a multifractal analysis of critical wave functions[21], and plateau transitions in narrow wires.[22] In graphene, on the other hand, the range of disorder plays an unusually important role, since the range dominates the inter-valley (K-K') scattering. Hence we conceive here that it is imperative to examine the honeycomb lattice (rather than an effective Dirac model) with bond randomness of various correlation lengths in understanding the Hall plateau transition in graphene.

This is exactly our motivation here to explore how the criticality in disordered graphene QHE transition, espe-
cially for \( n = 0 \) Landau level, depends on (i) the symmetry and (ii) the range of randomness in the honeycomb lattice. As for the Hall conductivity which has a topological origin and mathematically a Chern number in units of \( e^2/h \), an unusually accurate and efficient method is required for examining QHE around the Dirac point (band center) for random systems. Here we have adopted a non-Abelian extension of the Chern-number formalism as combined with a lattice-gauge technique. We shall show that, while the plateau transition for the \( n = 0 \) Landau level has an ordinary critical behavior for the uncorrelated random bonds, the criticality changes dramatically to an anomalous, almost exact fixed point with a step-function-like plateau transition and a concomitant delta-function-like Landau level, as soon as the spatial correlation in the random bonds exceeds only a few bond lengths. This can indeed be attributed to the preserved chiral symmetry, which is confirmed by adding site randomness to modify the symmetry.

The tight-binding Hamiltonian for the honeycomb lattice is \( H = \sum_{i,j} c_{i} \theta_{ij} c_{j} \), in standard notations, where the Peierls phase \( \theta_{ij} \) is determined such that the sum of the phases around a hexagon is equal to the magnetic flux \( -2\pi \phi \) piercing the hexagon in units of the flux quantum \( \phi_0 = h/e \). The spin degrees of freedom are neglected for simplicity. We introduce randomness in the nearest-neighbor transfer energy as \( t_{ij} = t + \delta t_{ij} \), where the disordered component \( \delta t_{ij} \) is assumed to be Gaussian distributed, \( P(\delta t) = e^{-\delta^2/2\sigma^2}/\sqrt{2\pi\sigma^2} \), with a variance \( \sigma \). Next we specify the spatial correlation \( \eta \) in the random components by requiring \( \langle \delta t_{ij}\delta t_{kl} \rangle = (\delta^2_e) e^{-r_{ij}-r_{kl}/2\eta^2} \), where \( r_{ij} \) denotes the position of the bond \( t_{ij} \), and \( \langle \rangle \) the ensemble average. We take the \( x \) - and \( y \) -axes as shown in Fig. 4 for \( L_x \times L_y \) rectangular systems, where a typical spatial landscape of the random hopping is displayed. All lengths are measured hereafter in units of the bond length \( a \) in the honeycomb lattice.

Even with such random transfers, the Hamiltonian respects the chiral symmetry, that is, there exists a local unitary operator \( \gamma \) (with \( \gamma^2 = 1 \)), which anti-commutes with the Hamiltonian, \( \{H, \gamma \} = 0 \). In real space we can decompose the honeycomb lattice into two sub-lattices \( A \) and \( B \), for which the fermion operators are transformed as \( \gamma c_i \gamma^{-1} = sc_i \) with \( s = +1(-1) \) for \( i \in A(B) \). Obviously, this symmetry is destroyed by a potential disorder, while the random hopping preserves it, even in magnetic fields. Since the eigenstates appear in chiral pairs \( |\psi, \gamma\psi \rangle \) with eigenenergies \( \pm E \), it is clear that the zero-energy states are special. If a zero-energy state \( \psi \) is not an eigenstate of the chiral operator, we can use the zero-energy chiral pairs, \( \psi \) and \( \gamma\psi \), to make eigenstates of the chiral operator, \( \gamma \psi_{\pm} = \pm \psi_{\pm} \) with \( \psi_{\pm} = \psi \pm \gamma\psi \). Then all the zero modes are eigenstates of \( \gamma \) with amplitudes residing only on one of the \( A \) and \( B \) sub-lattices. Hence the topologically protected zero-energy Landau levels, particularly their criticality, can be very sensitive to whether the disorder respects the chiral symmetry or not.

Let us first look at the result for the density of states around \( E = 0 \). A key interest is how the \( n = 0 \) Landau level is broadened by randomness as compared with \( n \neq 0 \) levels. In the calculation of the density of states we adopt the Landau gauge for the corresponding bricklayer lattice with periodic boundaries in \( y \)-direction and armchair edges in \( x \) to remove the contribution from zigzag edges. The density of states \( \langle \rho_e \rangle = -\sum_i \text{Im}G_{ii}(E + i\epsilon)/N\pi \), is obtained in terms of the Green’s function, \( G_{ii}(E+i\epsilon) = \langle \psi | (E-H+i\epsilon)^{-1} | \psi \rangle \), where \( N \) is the total number of sites, and \( \epsilon \) a small imaginary part in energy to evaluate the Green function numerically. We have performed the calculation for \( 6.3 \times 10^{-4} \leq \epsilon/t \leq 1.0 \times 10^{-2} \) and confirmed that the anomaly at \( E = 0 \) described below is not affected by the value of \( \epsilon \).

The result for the density of states with the disorder strength \( \sigma/t = 0.12 \) and a magnetic field \( \phi/\phi_0 = 1/50 \) for various values of the correlation length \( \eta \) is shown in Fig. 2. It has been shown that the \( n = 0 \) and several adjacent Landau levels characteristic to the relativistic electrons are captured even with this value of \( \phi \), which, when directly translated, corresponds to a large magnetic field, so the model should be adequate for the analysis of the criticality at the \( n = 0 \) Landau level. We can immediately see that the \( n = 0 \) Landau level is anomalously sharp, but that the sharpness depends sensitively on the correlation length \( \eta \) of the random hopping. More precisely, as soon as we have \( \eta/a \geq 1 \), the \( n = 0 \) Landau level becomes remarkably sharp, while this does not occur for \( n \neq 0 \) Landau levels. Indeed, the shape of the \( n = 0 \) Landau level for \( \eta/a \geq 3 \) is delta-function-like within the numerical accuracy in that its shape coincides almost exactly with the Lorentzian density of states in the clean limit, \( \rho(E) = \frac{1}{\pi} \frac{\epsilon}{E^2 + \epsilon^2} \). In this sense the \( n = 0 \) Landau
level in the presence of the correlated bond randomness is delta-function-like for \( \eta/a \geq 3 \). If we examine the dependence of the density of states on the disorder strength, with a fixed \( \eta/a = 3 \) (Fig. 2, inset), we can confirm that this anomaly at \( n = 0(E = 0) \) remains insensitive to the disorder strength as far as \( \eta/a \gtrsim 1 \), whereas other Landau levels are broadened by disorder.

We now turn to the quantized Hall plateau transition. The Hall conductivity \( \sigma_{xy} \) is related to the Chern number \( n_C \) as \( \sigma_{xy} = n_C (e^2/h) \) provided that an energy gap exists above the Fermi energy \([24, 25, 26] \). In random systems the Chern number differs from sample to sample, so we should look at the ensemble-averaged quantity for each energy bin, which gives the Hall conductance as a function of \( E \) \([18, 28] \).

Since the Hall current is dissipationless, not only the state near the Fermi energy but all the filled states contribute. A specialty of the graphene QHE is that the region of interest is around \( E = 0 \), which implies that we have to question many Landau levels below the Fermi energy whose contributions almost cancel with each other to a value of order unity. So we obviously confront a numerically difficult situation, especially if we want to look at a criticality around \( E = 0 \). We have previously shown that such a situation can be treated with a non-Abelian formulation of the Hall conductivity as a Chern number for multi-dimensional multiplets of fermions \([4, 29] \). The Berry connection is then defined as a matrix, which is spanned by the Landau sub-bands, split by the randomness. We can then adopt an extended unit cell for each realization of the randomness to apply the formula \([25] \).

The required stable energy gap at the Fermi energy is mostly guaranteed by the level repulsion in finite, random systems. Level crossings below the Fermi energy do not cause any problem either in this formulation. In the numerical evaluation of the topological numbers, a technique developed in the lattice gauge theory has turned out to be useful, which is a two-dimensional generalization of the King-Smith-Vanderbilt formula for polarization \([30, 31] \). For this we employ twisted boundary conditions, \( \psi(x + L_x, y) = e^{i \phi_x} \psi(x, y) \), where \( \phi_x = 2 \pi n_x(y)/N \) with \( n_x(y) = 0, 1, 2, \ldots, N - 1 \) being discretized phases. Here the string gauge is used to treat weak magnetic fields with the twisted boundary condition \([19] \).

The Chern number is then evaluated as \([13, 24, 30] \)

\[
n_C = \frac{1}{2 \pi} \sum_{\phi} \text{arg} (\det \Psi (\phi)) ,
\]

where

\[
\Psi (\phi) = \begin{bmatrix} \psi_1 (\phi) \\ \vdots \\ \psi_M (\phi) \end{bmatrix}.
\]

The Chern number averaged over 300 realizations of randomness for a bond disorder strength \( \sigma/t = 0.12 \) is shown in Figs. 3 and 4 plotted within the energy region \((|E/|t| < 1) \), where the plateaus have the Dirac behavior, \((2m + 1)(e^2/h) \) with \( m \) an integer \([3] \). If we first look at Fig. 3 for the correlated disorder with \( \eta/a = 1.5 \), we immediately notice that the plateau transition between \( n_C = -1 \) and \( 1 \) around \( E = 0 \) is anomalously abrupt, i.e., \( n_C \) behaves like a step function, in sharp contrast with other transitions for \( n \neq 0 \). For the uncorrelated bond randomness \( \eta/a = 0 \), on the other hand, the transition for \( n = 0 \) is as smeared as those for \( n \neq 0 \). The anomalously sharp step for \( n = 0 \) agrees with the anomalously sharp \( n = 0 \) Landau level seen in the density of states.
We can confirm the behavior for \( n = 0 \) is indeed unusual by looking at a system-size dependence for a correlated disorder in Fig. 4 in which the results for two sizes coincide with each other within the numerical accuracy for the transition at \( E = 0 \), while other transitions exhibit the usual behavior of narrower transition widths for larger systems. We can further test our picture that the anomalous behavior is connected to the preserved chiral symmetry. For this purpose, we have added a potential disorder to the bond disorder (Fig. 4, insets). The result clearly shows that the addition of a potential disorder that destroys the chiral symmetry does wash out the anomalous (step-function-like) transition at \( E = 0 \) into a normal behavior (Fig. 4(c)), while other transitions remain essentially the same (Fig. 4(d)).

In summary we have revealed that the quantum Hall transition at \( E = 0 \) is anomalously sensitive to the spatial correlation of the random bonds, where concomitantly with the Landau level width, it becomes exact fixed-point-like as soon as the correlation length exceeds a few times the bond length. This sharply contrasts with the case of the generic random Dirac fermions, where the broadening of the Landau level occurs also for \( E = 0 \). The singular behavior may correspond to the fixed point for the random Dirac fermions with chiral symmetry discussed by Ludwig et al. \(^{12}\), where a generic instability of the fixed point is discussed.

Experimentally, the length scale of ripples is estimated to be several nanometers\(^{6,14}\). Since this is much greater than the correlation length adopted here, the bond disorder by such ripples should not broaden the \( n = 0 \) graphene Landau level. Conversely, the broadening at the \( n = 0 \) level as observed in experiments should be caused by other types of disorder, such as potential disorders by charged impurities\(^{10}\). The message here amounts to that the fixed-point behavior should be experimentally observed in free-standing clean graphene samples where the ripple is the only disorder.

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