TITLE:
Special birational transformations of type (2,1)

AUTHOR(S):  
Fu, Baohua

CITATION:
Fu, Baohua. Special birational transformations of type (2,1). 代数幾何学シンポジウム記録 2014, 2014: 12-17

ISSUE DATE:
2014

URL:
http://hdl.handle.net/2433/215026

RIGHT:
SPECIAL BIRATIONAL TRANSFORMATIONS OF TYPE (2, 1)

BAOHUA FU
(JOINT WORK WITH JUN-MUK HWANG)

1. Introduction

This is a report on my joint work ([FH]) with Jun-Muk Hwang from KIAS. We are interested in the classification of special birational transformations: namely a birational transformation \( \Phi : \mathbb{P}^n \rightarrow Z \subset \mathbb{P}^N \) where \( Z \subset \mathbb{P}^N \) is a nonsingular projective variety of Picard number 1 such that the base locus \( S \subset \mathbb{P}^n \) of \( \Phi \) is irreducible and nonsingular. The map \( \Phi \) is called of type \((a, b)\) if the rational map \( \Phi \) is given by a linear system belonging to \( \mathcal{O}_{\mathbb{P}^n}(a) \) and the inverse rational map \( \Phi^{-1} \) is given by a linear system belonging to \( \mathcal{O}_Z(b) \).

When \( Z = \mathbb{P}^n \), such a transformation is called a special Cremona transformation. In [ES], Ein and Shepherd-Barron have classified special Cremona transformations of type \((2, 2)\), by proving that the base locus \( S \) is one of the four Severi varieties classified by Zak ([Z]). Recently, special Cremona transformations on \( \mathbb{P}^n \) of type \((2, 3)\) or of type \((2, 5)\) or when \( n \) is odd have been classified by Russo ([R]).

Let \( \Phi : \mathbb{P}^n \rightarrow Z \) be a special Cremona transformation of type \((2, 2)\). If we restrict \( \Phi \) to a general hyperplane section, we obtain a special birational transformation of type \((2, 2)\) \( \Phi' : \mathbb{P}^{n-1} \rightarrow \mathbb{Q}^{n-1} \) with base locus being a hyperplane section of a Severi variety. Conversely, every special birational transformation of type \((2, 2)\) from \( \mathbb{P}^n \) to \( \mathbb{Q}^n \) is of this form, as shown by Staglianò ([St]).

Motivated by the paper of [AS2] (and by the lack of examples), we have the following

**Speculation 1.1.** Let \( \Phi : \mathbb{P}^n \rightarrow Z \) be a special birational transformation of type \((2, b)\). If \( Z \) is different to \( \mathbb{P}^n \) and \( \mathbb{Q}^n \), then \( b \leq 2 \). Furthermore if \( b = 2 \), then \( n \) is even and \( Z \simeq \text{Gr}(2, n/2 + 2) \).

Here are some supporting results for this speculation: 1) If \( Z \) is a linear section of some rational homogeneous manifolds, then Speculation 1.1 holds by [AS2] (Corollary 3.21). 2) Speculation 1.1 with \( b = 2 \) holds if \( Z \) is a LQEL manifold by [AS2] (Theorem 3.6). 3) Speculation 1.1 with \( b = 2 \) holds if \( Z \) is a hypersurface by [Li].
In our paper ([FH]), we have obtained a complete classification for special birational transformations of type \((2, 1)\), hence confirming Speculation 1.1 for the case when \(b = 1\). This classification can be described in terms of the classification of the base locus \(S \subset \mathbb{P}^n\), which is contained in a hyperplane \(\mathbb{P}^{n-1} \subset \mathbb{P}^n\) for the type \((2, 1)\). Our main result is the following.

**Theorem 1.2** ([FH]). The base locus \(S^d \subset \mathbb{P}^{n-1}\) of a special birational transformations of type \((2, 1)\) is projectively equivalent to one of the following:

(a) \(Q^d \subset \mathbb{P}^{d+1}\) for \(d \geq 1\);
(b) \(\mathbb{P}^1 \times \mathbb{P}^{d-1} \subset \mathbb{P}^{2d-1}\) for \(d \geq 3\);
(c) the 6-dimensional Grassmannian \(\text{Gr}(2, 5) \subset \mathbb{P}^9\);
(d) the 10-dimensional Spinor variety \(S_5 \subset \mathbb{P}^{15}\);
(e) a nonsingular codimension \(\leq 2\) linear section of \(\mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5\);
(f) a nonsingular codimension \(\leq 3\) linear section of \(\text{Gr}(2, 5) \subset \mathbb{P}^9\).

The corresponding \(Z\) is given by

(a') \(Q^{d+2}\) for \(d \geq 1\);
(b') \(\text{Gr}(2, d + 2)\) for \(d \geq 3\);
(c') the 10-dimensional Spinor variety \(S_5\);
(d') the 16-dimensional \(E_6\)-variety \(\mathbb{O}\mathbb{P}^2\);
(e') a nonsingular codimension \(\leq 2\) linear section of \(\text{Gr}(2, 5)\);
(f') a \(\mathbb{P}^4\)-general linear section of \(S_5 \subset \mathbb{P}^{15}\) of codimension \(\leq 3\).

The birational map \(\Phi\) is given by \(H^0(\mathbb{P}^n, \mathcal{I}_S(2))\) except in case (b) with \(d \geq 6\), where it can be given by some subspaces of \(H^0(\mathbb{P}^n, \mathcal{I}_S(2))\).

By [IP] (Remark 3.3.2), all non-singular linear sections of \(\text{Gr}(2, 5)\) of a fixed codimension \(\leq 3\) are projectively equivalent. All non-singular hyperplane sections of \(S_5\) are projectively equivalent. We can show that smooth codimension 2 linear sections of \(S_5\) are isomorphic but smooth codimension 3 linear sections of \(S_5\) have moduli. Thus in the case (f'), these linear sections of codimension 3 are not general, but only general among those containing a \(\mathbb{P}^4\).

Consider the case (b) with \(d \geq 6\). The corresponding birational transformation is \(\phi^o : \mathbb{P}(V) \simeq \mathbb{P}^{2d} \dashrightarrow Z := \text{Gr}(2, d + 2) \subset \mathbb{P}(V \oplus W)\) with \(Y = \text{Gr}(2, d) \subset \mathbb{P}W = \mathbb{P}(\wedge^2 \mathbb{C}^d)\). Note that \(Z \cap \mathbb{P}W = Y\) and \(\text{Sec}(Z) \cap \mathbb{P}W = \text{Sec}(Y)\). Take any linear subspace \(L \subset W\) such that \(\mathbb{P}L \cap \text{Sec}(Y) = \emptyset\), then \(\mathbb{P}L \cap \text{Sec}(Z) = \emptyset\). Let \(p_L : \mathbb{P}(V \oplus W) \dashrightarrow \mathbb{P}(V \oplus W/L)\) be the projection from \(\mathbb{P}L\). Then \(p_L\) sends isomorphically \(Z\) to \(Z_L \subset \mathbb{P}(V \oplus W/L)\) and \(Y\) to \(Y_L \subset \mathbb{P}(W/L)\). The map \(\phi^o_L := p_L \circ \phi^o : \mathbb{P}(V) \dashrightarrow Z_L\) is a special birational transformation of type \((2, 1)\) with \(Y(\sigma_L) = Y_L \simeq Y \subset \mathbb{P}(W/L)\).
2. Outline of the proof

2.1. Basic properties of special birational transformation of type (2, 1). For a projective subvariety $S \subset \mathbb{P}^n$, its secant variety $\text{Sec}(S)$ is the closure of the loci of lines which intersect $S$ at two or more points. The secant defective is given by $\delta = 2 \dim(S) + 1 - \dim \text{Sec}(S)$. For $u \in \text{Sec}(S) \setminus S$, the entry locus $\Sigma_u$ is the closure of the locus on $S$ of secant lines through $u$. We call $S \subset \mathbb{P}^n$ is a QEL-manifold if it is smooth and $\Sigma_u$ is a smooth hyperquadric in its linear span $C_u := \langle \Sigma_u \rangle$.

From now on, let $\Phi : S^d \subset \mathbb{P}^n \dashrightarrow Z$ be a special birational transformation of type $(2, 1)$. As easily seen, for a line $\ell \subset \mathbb{P}^n$ not contained in $S$, the image $\Phi(\ell)$ is a point (resp. a line, a conic) if $\ell \cdot C$ is two points (resp. one point, empty).

**Lemma 2.1** ([ES]). $S \subset \text{Sec}(S) = \mathbb{P}^{n-1}$ is a QEL-manifold. As a consequence, the secant defective of $S$ is $\delta = 2d + 2 - n$.

From this, we deduce that for $u \in \mathbb{P}^{n-1}$ general, $C_u \simeq \mathbb{P}^{d+1}$ is contracted to one point by $\Phi$. The following implies that for any $u \in \mathbb{P}^{n-1} \setminus S$, we always have $C_u \simeq \mathbb{P}^{d+1}$. This property is called strong QEL in [AS2].

**Lemma 2.2.** The subvariety $Y := \Phi(\mathbb{P}^{n-1}) \subset Z$ is smooth.

The proof uses in a crucial way the smoothness of $Z$. Take any point $y \in \mathbb{P}^n \setminus \text{Sec}(S)$, we consider the $\mathbb{C}^*$-action on $\mathbb{P}^n$ with orbits being lines through $y$. It fixes the point $y$ and the hyperplane $\mathbb{P}^{n-1} = \text{Sec}(S)$. This action induces a $\mathbb{C}^*$-action on $Z$, with $Y$ being an irreducible component of the fixed locus, hence $Y$ is smooth since $Z$ is smooth.

By [ES] (Theorem 1.1), $\text{Bl}_S(\mathbb{P}^n)$ is also the blow-up of $Z$ along $Y$, hence the map $\psi : \text{Bl}_S(\mathbb{P}^{n-1}) \to Y$ is a $\mathbb{P}^{d+1}$-bundle. Let $c := n - 1 - d$ be the codimension of $S$ in $\mathbb{P}^{n-1}$. The notations and basic properties of our situation are summarised in the following diagram.

An easy argument using previous diagram shows that
Proposition 2.3. The Euler numbers of $S$ and $Y$ are related by

$$\chi(S) = \frac{(\delta + 2)\chi(Y) - n}{n - d - 2}.$$ 

2.2. Classification of $Y$. A key observation is that the fibers of $p : \mathbb{P}N_{S|\mathbb{P}^n} \to S$ are mapped to linear subspaces in $Y$ and such $\mathbb{P}^{c-1}$'s cover $Y$. On the other hand, we have $\dim Y = n - 1 - (\delta + 1) = 2(c-1)$ by Lemma 2.1. By $[S]$ and note that $Y$ has Picard number 1, we obtain

Proposition 2.4. The projective subvariety $Y \subset \mathbb{P}^{a-1}$ is isomorphic to one of the following:

1. $\mathbb{P}^{2(c-1)} \cong Y = \mathbb{P}^{a-1};$
2. a nonsingular quadric hypersurface $\mathbb{Q}^{2(c-1)} \cong Y \subset \mathbb{P}^{a-1} \cong \mathbb{P}^{2c-1};$ or
3. a biregular projection of the Plücker embedding $\text{Gr}(2, \mathbb{C}^{c+1}) \subset \mathbb{P}(\wedge^2 \mathbb{C}^{c+1}).$

We have the following topological consequence.

Corollary 2.5. All odd Betti numbers of $S$ vanish. In particular, the Euler number of $S$ satisfies $\chi(S) \geq d + 1.$

2.3. Classification when $Y = \mathbb{P}^{2(c-1)}.$

Proposition 2.6. Assume that $Y = \mathbb{P}^{2(c-1)}$, then $S \subset \mathbb{P}^{n-1}$ is projectively equivalent to one of the following:

1. $\mathbb{Q}^d \subset \mathbb{P}^{d+1};$
2. $\mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5;$
3. $\text{Gr}(2, 5) \subset \mathbb{P}^9;$
4. a general hyperplane section of (ii);
5. a general codimension $\leq 2$ linear section of (iii);
6. a general codimension-3 linear section of (iii);
7. a general codimension-2 linear section of (ii).

Here is the idea for a proof different to that presented in [FH]. If $c = 1$, then $S \subset \mathbb{P}^{n-1}$ is a hyperquadric and we are done. Assume now $c \geq 2$. Let $g(S)$ be the sectional genus of $S$. In this case, we have $a - 1 = 2(c - 1)$, hence by Theorem 1 [AS1], we have either $\deg(S) = 2, c = 2, g(S) = 0$ or $\deg(S) = 5, c = 3, g(S) = 1$. Now the Proposition follows from the classification of low degree varieties or from the results on QEL-manifolds obtained by Russo in [R].

2.4. Classification when $Y = \mathbb{Q}^{2(c-1)}.$ The following result is an easy consequence of Proposition 2.3.
Proposition 2.7. For a special quadratic manifold $S^d \subset \mathbb{P}^{n-1}$ with codimension $c$ and $n \geq 3$, assume that $Y = \mathbb{Q}^{2(c-1)}$. Then either $\delta \geq \frac{d}{2}$ or $\delta = \frac{d}{2} - 1$.

Proposition 2.8. For a special quadratic manifold $S \subset \mathbb{P}^{n-1}$ of codimension $c$, assume that $Y = \mathbb{Q}^{2(c-1)}$. Then $S \subset \mathbb{P}^{n-1}$ is projectively equivalent to one of the following

(i) the Segre 4-fold $\mathbb{P}^1 \times \mathbb{P}^3 \subset \mathbb{P}^7$;
(ii) the 10-dimensional Spinor variety $S_5 \subset \mathbb{P}^{15}$.

By [R], $d$-dimensional QEL-manifolds with $\delta \geq d/2$ have been completely classified. Hence by Proposition 2.7, the remaining case is $\delta = \frac{d}{2} - 1$. If $\delta \leq 2$, then we get $d \leq 6$ and we do a case-by-case check. Now assume $\delta \geq 3$. By the divisibility theorem of [R], $d - \delta = \delta + 2$ is divisible by $2^{\lfloor \frac{d}{\delta} - 1 \rfloor}$ which implies that we have only two possibilities: $(\delta, d, n) = (4, 10, 18)$ or $(\delta, d, n) = (6, 14, 24)$. For the first case, we can show the VMRT of $S$ is $\mathbb{P}^1 \times \mathbb{P}^4 \subset \mathbb{P}^9$, hence $S$ is isomorphic to $\text{Gr}(2, 7)$ by [Mo], which cannot be embedded linearly normally into $\mathbb{P}^{17}$.

The case $(\delta, d, n) = (6, 14, 24)$ is difficult to be ruled out. The contradiction comes from a delicate structure on intersections of entry loci of $S$. At one hand, we show that every two general entry loci $\Sigma_i, i = 1, 2$ of $S$ through a general point $x \in S$ will intersect along $\mathbb{P}^1$, hence the lines on $\Sigma_i$ through $x$ give two entry loci on the VMRT $C_x$ of $S$ at $x$, intersecting at a single point (corresponding to the line $\Sigma_1 \cap \Sigma_2$). On the other hand, we can show that $C_x$ is a codimension 2 linear section of the 10-dimensional spinor variety $S_5 \subset \mathbb{P}^{15}$. Any two entry loci of such a linear section intersect along a $\mathbb{P}^k$ with $k \geq 1$ if non-empty, which is a contradiction.

2.5. Classification when $Y = \text{Gr}(2, c + 1)$. To handle the case (Y3) of Proposition 2.4, we need the following result.

Proposition 2.9 (Theorem 2.17 [AS2]). Let $S' \subset \mathbb{P}^{2k}$ be a quadratic manifold. Assume that the rational map $\psi : S' \dashrightarrow Y(S')$ is birational and $Y(S')$ is biregular to $\text{Gr}(2, k + 2)$. Then $S' \subset \mathbb{P}^{2k}$ is one of the rational normal scrolls. In particular, $S'$ has Picard number 2 and is covered by lines.

Proposition 2.10. Let $S \subset \mathbb{P}^{n-1}$ be a special quadratic manifold of codimension $c \geq 4$ with $Y(S) = \text{Gr}(2, c + 1)$. Then $S \subset \mathbb{P}^{n-1}$ is projectively equivalent to $\mathbb{P}^1 \times \mathbb{P}^c \subset \mathbb{P}^{2c+1}$.

The idea of the proof is the following: we take a general codimension $(\delta + 1)$ linear section $S' \subset \mathbb{P}^{n-\delta-2}$ of $S \subset \mathbb{P}^{n-1}$. This gives a birational
map $\psi^c : S' \subset \mathbb{P}^{n-\delta-2} \rightarrow Y(S') = \text{Gr}(2, c+1)$, hence $S'$ is a rational normal scroll by 2.9. Except a few lower dimensional cases, this implies that $S$ has Picard number 2 with $\delta > 0$. But then $S$ becomes a conic-connected manifold with Picard number 2, which have been completely classified by [IR], and then we are done.

References

[AS1] A. Alzati, J. C. Sierra, A bound on the degree of schemes defined by quadratic equations, Forum Math. 24 (4) (2012) pp. 733-750

[AS2] A. Alzati and J. C. Sierra: Quadro-quadric special birational transformations of projective spaces, Int. Math. Res. Not. (2013), doi:10.1093/imrn/rnt173

[ES] L. Ein and N. Shepherd-Barron: Some special Cremona transformations, Amer. J. Math., 111 (1989), 783-800

[FH] B. Fu and J.-M. Hwang: Special birational transformations of type (2,1), preprint.

[IR] P. Ionescu and F. Russo: Conic-connected manifolds, J. Reine Angew. Math. 644(2010), 145-157

[IP] V. A. Iskovskikh and Yu. G. Prokhorov: Fano varieties. Algebraic geometry, V, Encyclopaedia Math. Sci., 47, Springer, Berlin, 1999.

[Li] Qifeng Li: Quadro-quadric special birational transformations from projective spaces to smooth complete intersections, preprint

[Mo] N. Mok: Recognizing certain rational homogeneous manifolds of Picard number 1 from their varieties of minimal rational tangents. Third International Congress of Chinese Mathematicians. Part 1, 2, 41–61, AMS/IP Stud. Adv. Math., 42, pt.1, 2, Amer. Math. Soc., Providence, RI, 2008

[M] S. Mukai: Biregular classification of Fano 3-folds and Fano manifolds of coindex 3, Proc. Natl. Acad. Sci. USA 86(1989), 3000-3002

[R] F. Russo: Varieties with quadratic entry locus, I, Math. Ann.344 (2009), 597-617

[S] E. Sato: Projective manifolds swept out by large-dimensional linear spaces. Tôhoku Math. J. 49 (1997), no. 3, 299–321

[St] G. Staglianò: On special quadratic birational transformations of a projective space into a hypersurface. Rendiconti del Circolo Matematico di Palermo (2) 61, no. 3 (2012): 40329.

[Z] F. L. Zak: Tangents and secants of algebraic varieties. Translations of Mathematical Monographs, 127. American Mathematical Society, Providence, RI, 1993

Baohua Fu
Institute of Mathematics, AMSS, Chinese Academy of Sciences,
55 ZhongGuanCun East Road, Beijing, 100190, China
bhfu@math.ac.cn