EXPANDING LATTICE ORDERED ABELIAN GROUPS TO RIESZ SPACES

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ABSTRACT. First we give a necessary and sufficient condition for an abelian lattice ordered group to admit an expansion to a Riesz space (or vector lattice). Then we construct a totally ordered abelian group with two non-isomorphic Riesz space structures, thus improving a previous paper where the example was a non-totally ordered lattice ordered abelian group. This answers a question raised by Conrad in 1975. We give also a partial solution to another problem considered in the same paper. Finally, we apply our results to MV-algebras and Riesz MV-algebras.

1. Introduction

Our objects of interest are two kinds of algebraic structures: Riesz spaces (or vector lattices) and Riesz MV-algebras. See the preliminary section for the definitions.

Riesz spaces find applications in several fields like functional analysis, economy, etc. whereas Riesz MV-algebras find applications in many-valued logic, fuzzy logic, quantum mechanics, etc. For Riesz spaces, an example of monography is [10], whereas Riesz MV-algebras are more recent and we know no extensive treatment of the subject. These structures are enrichments of simpler structures: abelian l-groups and MV-algebras. For abelian l-groups see [1] or [2], whereas for MV-algebras see [4].

The relations between all these kinds of structures are very interesting and are studied, for instance, in [5,7] and [11]. This paper is devoted to the study of these relations.

This paper is in a sense a continuation of [9], whereas [5] is our source of inspiration. Our main theme is the relation between abelian lattice ordered groups (abelian l-groups) and Riesz spaces. Each Riesz space is also, by definition, an abelian l-group. The problem now is: given an abelian l-group, can it be expanded to a Riesz space? and in how many ways?

First of all, one would like to have simple necessary and sufficient conditions for an l-group to admit a Riesz space structure. A necessary and sufficient condition is given in [5]. We propose another one. More generally we attempt to make a theory of l-groups, or MV-algebras, which admit at least one Riesz structure (we call these structures extendible).

For instance, a countable l-group cannot become a Riesz space, and Archimedean l-groups can have at most one structure. On the other hand, [5] gives examples of l-groups $G$ with at least two Riesz space structures: for instance, $G = R \text{ lex } R$, where $R$ is the ordered group of the real numbers and $\text{ lex}$ denotes lexicographic product. More generally [5] proves that every totally...

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ordered, non-Archimedean l-group has either no structure, or at least two. Another example of
double structure is given by [12: Example 11.54].

In [5], several open problems are left. One of them (question II thereof) is whether all Riesz
spaces over a given abelian l-group \( G \) are isomorphic. This problem is solved in the negative in
[9] with an explicit counterexample of \( G \). However, \( G \) is not totally ordered, whereas [5] asks for
a totally ordered example. In this paper, we solve in the negative this problem, by exhibiting a
totally ordered abelian group \( G \) with two non-isomorphic Riesz space structures. The construction
of the example is similar to that of [9], but somewhat simpler. Another question of [5] is whether
every l-group with exactly one Riesz space structure is Archimedean. We give a partial positive
solution concerning the l-groups embedded in a product of totally ordered abelian groups which
are closed under finite variant.

Finally, we turn to MV-algebras. As it often happens, we can use the Mundici functor of [11]
to transfer information from l-groups to MV-algebras. This paper is no exception: the previous
results on l-groups can be transferred to MV-algebras. In particular, via the Mundici functor, we
can prove that there is a totally ordered MV-algebra with two non-isomorphic Riesz MV-algebra
structures.

2. Preliminaries

In this preliminary section, we mostly follow [9]. We denote by \( N, Z, Q, R \) the sets of natural
numbers (starting from 0), the integers, the rationals and the reals respectively.

2.1. l-groups

A lattice ordered abelian group (l-group) is a structure \((G, +, \leq)\) such that:

- \((G, +)\) is an abelian group;
- \((G, \leq)\) is a lattice;
- \(x \leq y\) implies \(x + z \leq y + z\).

The infimum and supremum of two elements \(x, y \in G\) will be denoted by \(x \wedge y\) and \(x \vee y\). A
particular case is when the lattice order is total, in which case we say that \((G, +, \leq)\) is a
totally ordered abelian group.

A strong unit of an l-group \(G\) is an element \(u \in G\) such that for every \(x \in G\) there is \(n \in N\)
such that \(x \leq nu\).

The absolute value of an element \(x \in G\) is \(|x| = x \vee -x\).

When \(G\) is totally ordered, we simply have \(|x| = x\) if \(x \geq 0\) and \(|x| = -x\) if \(x < 0\).

Given \(x, y \in G\), we say that \(x\) dominates \(y\) if there is \(n \in N\) such that \(|y| \leq n|x|\). We say that
\(x, y\) are equidominant if they dominate each other. Note that equidominance is an equivalence
relation (in literature the equidominance relation is often called Archimedean equivalence). We
say that \(x\) strictly dominates \(y\) if \(x\) dominates \(y\) but \(y\) does not dominate \(x\).

An l-group \(G\) is called Archimedean if any two non-zero elements of \(G\) are equidominant (id est
Archimedean equivalent).

Note that l-groups are an equational class, so there is a natural notion of homomorphism of
l-groups and a natural category of l-groups.

2.2. Riesz spaces

A Riesz space is a structure \((G, +, \leq, \rho)\) which is an l-group with a structure of vector space
over \(R\), formally a map \(+ : G \times G \to G\) and a map \(\rho : R \times G \to G\), satisfying the usual vector space
axioms, that is (letting \(rv = \rho(r, v)\)):
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- \( r(v + w) = rv + rw \)
- \( (r + s)v = rv + sv \)
- \( r(sv) = (rs)v \)
- \( 1v = v \)

and such that if \( v \geq 0 \) and \( r \) is a positive real, then \( rv \geq 0 \).

Like l-groups, Riesz spaces form an equational class, so there is a natural notion of homomorphism of Riesz spaces and of the category of Riesz spaces.

2.3. MV-algebras

An MV-algebra is a structure \((A, \oplus, 0, 1, \neg)\) where:

- \((A, \oplus, 0)\) is a commutative monoid where 0 is the neutral element;
- \(1 = \neg 0\);
- \(x \oplus 1 = 1\);
- \(\neg \neg x = x\);
- \(\neg (\neg x \oplus y) \oplus y = \neg (\neg y \oplus x) \oplus x\).

Intuitively, \(\oplus\) is a kind of sum, and \(\neg\) is a kind of negation.

The most important, and motivating, example of an MV-algebra is based on the unit real interval, where \(A = [0, 1]\), \(x \oplus y = \min(x + y, 1)\) and \(\neg x = 1 - x\).

Other derived connectives in MV-algebras are \(x \odot y = \neg(\neg x \oplus \neg y)\) (a kind of a product, dual to the sum) and \(x \ominus y = x \odot \neg y\) (a kind of difference).

Once again we have an equational class, a natural notion of homomorphism and category.

2.4. Riesz MV-algebras

A Riesz MV-algebra is a structure \((A, \oplus, 0, \neg, \rho)\), where \((A, \oplus, 0, \neg)\) is an MV-algebra and \(\rho: [0, 1] \times A \to A\) verifies the following axioms (where \(rx = \rho(r, x)\)):

- \(r(x \ominus y) = (rx) \ominus (ry)\);
- \((r \odot q)x = (rx) \odot (qx)\);
- \(r(qx) = (rq)x\);
- \(1x = x\).

Once again we have an equational class, a natural notion of homomorphism and category.

3. A categorial equivalence

In [5], a necessary and sufficient condition is given for an abelian l-group to admit an expansion to a Riesz space structure. In this paper, we give another one, and we build a category of “expanded l-groups” equivalent to the category of Riesz spaces.

We will say that an expanding family of a divisible group \(G\) is a family \(G(b)_{b \in G^+}\) of subgroups of \(G\) such that:

- \(b \in G(b)\);
- \(G(b)\) is isomorphic to the reals as an ordered group;
- if \(c \in G(b)\) and \(c \in G^+\) then \(G(c) = G(b)\);
- if \(x \in G(b)\), \(y \in G(c)\) and for every rational \(q, x < qb\) if and only if \(y < qc\), then \(x + y \in G(b + c)\) and \(x \wedge y \in G(b \wedge c)\).
We will say that an abelian l-group \( G \) is Riesz expandable if \( G \) is divisible and admits an expanding family.

**Theorem 3.1.** An abelian l-group \( G \) is the reduct of a Riesz space if and only if \( G \) is Riesz extendable.

**Proof.** All conditions are clearly necessary since extendable.\n
Conversely, suppose all conditions are met. If \( b \in G^+ \), and \( r \in R \), we let \( rb \) be the image of \( r \) in the unique ordered group isomorphism between \( R \) and \( G(b) \) sending 1 to \( b \). Note that if \( b, c \in G^+ \) then \( r(b + c) = rb + rc \).

If \( g \) is any element of \( G \), then \( g = b_1 - b_2 \) for some \( b_1, b_2 \in G^+ \), so we let \( rg = rb_1 - rb_2 \). Note that the decomposition of \( rg \) as a difference of two positive elements is not unique, but the definition of \( rg \) is independent of the decomposition: if \( g = a - b = c - d \) then \( a + d = b + c \), so \( ra + rd = r(a + d) = r(b + c) = rb + rc \) and \( ra - rb = rc - rd \).

This gives a Riesz space structure on \( G \). In fact, from the last item, it follows for every \( r \in R \) and for every \( b, c > 0 \) that \( rb + rc = r(b + c) \) and \( rb \land rc = r(b \land c) \). The properties extend from \( G^+ \) to \( G \). \( \square \)

We note that expanding families are not unique.

**Proposition 3.1.** There is an l-group \( G \) with at least two different expanding families.

**Proof.** By the examples given first (to our knowledge) in [5], there is an l-group \( G \) with at least two Riesz space structures \( \rho \neq \rho' \). But if \( G(b), G'(b) \) are the corresponding expanding families, then \( G(b) \neq G'(b) \) for some \( b \). In fact, suppose \( G(b) = G'(b) \) for every \( b \in G^+ \). Then \( \rho(q, b) = \rho'(q, b) \) for every \( q \in Q \), and for every real \( r \), \( \rho(r, b) \) is the unique element of \( G(b) \) such that \( \rho(q, b) < \rho(r, b) < \rho(q', b) \) for every pair of rationals \( q < r < q' \), and \( \rho'(r, b) \) is the unique element of \( G'(b) \) such that \( \rho(q, b) < \rho(r, b) < \rho(q', b) \) for every pair of rationals \( q < r < q' \). So \( \rho(r, b) = \rho'(r, b) \) for every real \( r \) and \( b \in G^+ \), so \( \rho = \rho' \). \( \square \)

Let us call expanded l-group a structure \((G, G(b))_{b \in G^+}\) where \( G \) is a divisible l-group and \( G(b) \) is an expanding family of subgroups of \( G \).

We can make a category of expanded l-groups by taking as morphisms between \((G, G(b))\) and \((H, H(c))\) the homomorphism of groups \( f \colon G \to H \) such that \( f(G(b)) = H(f(b)) \).

**Proposition 3.2.** The categories of Riesz spaces and expanded l-groups are equivalent.

**Proof.** Given a Riesz space \( G \) we associate the expanded l-group \((G, G(b))\) where \( G(b) \) is the group of the real multiples of \( G \). This association is functorial and is an equivalence by Theorem 3.1. \( \square \)

### 4. On convex expanded families

**Proposition 4.1.** Let \( G \) be an l-group such that \( G(b) \) is convex for every \( b \). Then \( G \) is Archimedean.

**Proof.** Suppose for a contradiction that \( G \) is not Archimedean. Then there are \( b, \epsilon \in G^+ \) such that \( n \epsilon \leq b \) for every \( n \in N \). Now \( b \leq b + \epsilon \leq 2b \), so \( b + \epsilon \in G(b) \) and \( \epsilon \in G(b) \). But every nonzero element of \( G(b) \) dominates \( b \), whereas \( \epsilon \) dominates \( b \). This contradiction concludes the proof. \( \square \)
Note that if \( G \) is Archimedean then \( G \) has at most one Riesz space structure, hence at most one expanding family.

Conversely, suppose \( G \) is an Archimedean l-group with an expanding family \( G(b) \). Then \( G(b) \) is not necessarily convex for every \( b \), for example: \( G = R \times R, b = (1, 1) \), we have \((2, 2) < (2, 3) < (3, 3) \) but \((2, 3) \notin G(b) \).

5. On l-groups closed under finite variant

An open question of [5] is whether a non-Archimedean Riesz space can have only one Riesz space structure (compatible with its l-group structure). The following is a partial answer (recall that every l-group is embeddable in a product of totally ordered l-groups).

**Theorem 5.1.** Let \( G \) be a non-Archimedean l-group embedded in a product of totally ordered l-groups and closed under finite variant. Then \( G \) admits either zero or more than one Riesz space structure.

**Proof.** Let \( G \subseteq \Pi_{i \in I} G_i \), where each \( G_i \) is a totally ordered abelian group. Since \( G \) is closed under finite variant, for every \( i \in I \), \( G \) contains the vector \( u_i \) consisting of 1 in position \( i \) and 0 elsewhere. Let \( \rho \) be a Riesz space structure on \( G \). Let \( r \in R^+ \) and \( q,q' \in Q \) such that \( 0 < q < r < q' \). Then \( \rho(r,u_i) \) is between \( \rho(q,u_i) \) and \( \rho(q',u_i) \), hence \( \rho(r,u_i) \) must have zero in all components different from \( i \). Moreover, suppose any \( v \in G \) has \( v_i = 0 \) for some \( i \in I \). Then \( v \) is orthogonal to \( u_i \) (i.e. \( v \wedge u_i = 0 \)), and by definition of Riesz structure, \( \rho(v) = 0 \). That is, \( \rho(v) \) has \( i \)-th coordinate equal to zero. By additivity, if \( v \) and \( w \) have the same \( i \)-th component, then \( \rho(r,v) \) and \( \rho(r,w) \) have the same \( i \)-th component. So, for every \( i \in I \), there is a map \( \rho_i : G_i \times R \to G_i \) such that \( \rho_i(r,v_i) = \rho(r,v)_i \), and \( \rho_i \) is a Riesz space structure on \( G_i \). In other words, all the Riesz space structures on \( G \) are products of Riesz space structures on \( G_i \).

Now suppose \( G \) has a Riesz space structure \( \rho \). By the argument above, every \( G_i \) has a Riesz space structure \( \rho_i \). Since \( G \) is closed under finite variant, every \( G_i \) must be non-Archimedean. Any such \( G_i \) must have another Riesz space structure \( \rho'_i \). Now, let \( \rho' : R \times G \to G \) be the map such that \( \rho'(r,v) = w \) if and only if \( w_i = \rho'_i(r,v_i) \) and \( w_j = \rho_i(r,v_j) \) for every \( j \neq i \). \( \rho' \) is well defined because \( \rho'(r,v) \) is a finite variant of \( \rho(r,v) \), and is a Riesz space structure on \( G \). Since \( \rho_i \neq \rho'_i \), we conclude \( \rho \neq \rho' \). \( \square \)

Like in [9], we call atom of an l-group \( G \) an element \( a \in G^+ \) such that for every \( b,c \in G^+ \) with \( b,c \leq a \) we have \( b \wedge c \neq 0 \).

Note that every l-group closed under finite variant is atomic (i.e. below every positive element there is a positive atom). In fact, every positive real multiple of \( u_i \) is an atom, and every positive element is above some positive real multiple of \( u_i \) for some \( i \in I \). We conjecture that the previous theorem generalizes to atomic l-groups.

6. A totally ordered example

We have said that [9] gives a construction of a unital l-group with two non-isomorphic Riesz space structures. In this section we adapt the construction of [9] to the case of totally ordered abelian groups, and we obtain:

**Theorem 6.1.** There is a totally ordered abelian group \( G \) with a strong unit \( u \), such that \( G \) has two non-isomorphic Riesz space structures \( \rho_1 \) and \( \rho_2 \).
Proof. Like in [9], the idea is to build a group with two “asymmetric” Riesz structures.

Let $R^n$ be the field of the real algebraic numbers. $R$ and $R^n$ are real closed fields, so they are elementarily equivalent. By Frayne’s Theorem there is an embedding $j_1: R \to +R^n$, where $+R^n = (R^n)^1/U$ is an ultrapower of $R^n$ (so $I$ is a set and $U$ is an ultrafilter over $I$).

Let $j_2: R \to R^I/U$ be the diagonal embedding. So the field $K = R^I/U$ has two natural Riesz space structures $\rho_1, \rho_2$, where $\rho_1(r, x) = j_1(r)x$ and $\rho_2(r, x) = j_2(r)x$.

Let $K_0$ be the set of finite sums $\Sigma_{ij}j_1(r_i)j_2(s_i)$ where $r_i, s_i \in R$. $K_0$ is a Riesz subspace of $K$ in both Riesz structures $\rho_1$ and $\rho_2$, and has a strong unit $j_1(1)$ (note that $j_1(1) = j_2(1)$).

Note that $K_0$ has the cardinality of the continuum, so $K_0$ is included in at most $2^{\aleph_0}$ Archimedean classes (to our knowledge the exact number of Archimedean classes of $K_0$ is not known, note that $K_0$ is defined in an indirect way by an ultraproduct construction).

The idea is to consider certain sequences of elements of $K_0$ indexed by a regular cardinal $\Lambda$ sufficiently large. More precisely, we fix two regular cardinals $\eta, \Lambda$ such that $2^{\aleph_0} < \eta < \Lambda$.

Note that any two elements of $K_0$ have Archimedean distance less than $\eta$.

Let us equip the group $K_0^\Lambda$ with the lexicographic ordering. That is, we let $g < h$ if and only if the first nonzero component of $h - g$ is positive. In this way $K_0^\Lambda$ is a totally ordered abelian group.

Let $G \subseteq K_0^\Lambda$ be the set of all sequences $g \in K_0^\Lambda$ such that for every $\alpha < \Lambda$, $g(\alpha)$ can be written, in the vector space $(K_0, \rho_1)$, as a real linear combination of some finite set $F \subseteq K_0$ independent of $\alpha$. In other words, the range of $g$ has finite dimension in $(K_0, \rho_1)$. In symbols,

$$g(\alpha) = \Sigma_{i \in F} j_1(r_{i, \alpha})k_i$$

where $r_{i, \alpha} \in R$, $F$ is finite, and $k_i \in K_0$.

Note that $G$ inherits from $K_0^\Lambda$ (and from $K$) the two vector space structures above, which we will still call $\rho_1$ and $\rho_2$.

An example of strong unit of $G$ is simply $u = (j_1(1), 0, 0, \ldots)$. Note that in order to have a strong unit, we do not need the condition (present in [9]) that the components of the elements of $G$ are bounded.

Since $G$ is totally ordered, the absolute value of an element $g$ of $G$, written $|g|$, is simply $g$ if $g \geq 0$, and $-g$ if $g < 0$; and the Archimedean classes (id est equidominance classes) of $G$ are also totally ordered in the natural way.

Let us call Archimedean distance between $g, h \in G$ the number, possiby infinite, of Archimedean classes between $g$ and $h$.

More simply than [9], we let $\Gamma = j_1(R)^\Lambda$. Similarly to [9] we have:

**Lemma 6.1.** $\Gamma \subseteq G$ and $\Gamma$ generates the vector space $(G, \rho_2)$.

Proof. $\Gamma \subseteq G$ because we can take $F = \{1\}$ and $k_1 = 1$.

For the second point, consider $g \in G$. Then $g(\alpha) = \Sigma_{i \in F} j_1(r_{i, \alpha})k_i$, where $F$ is finite and $k_i \in K_0$.

Since $k_i \in K_0$ we have $k_i = \Sigma_{j \in J_i} j_1(r_{ij})j_2(s_{ij})$, where $J_i$ is finite. So

$$g(\alpha) = \Sigma_{i \in F} \Sigma_{j \in J_i} j_1(r_{i, \alpha})j_1(r_{ij})j_2(s_{ij}).$$

Let $\gamma_{ij}$ the $\Lambda$-sequence such that $\gamma_{ij}(\alpha) = j_1(r_{i, \alpha})j_1(r_{ij})$. Then $\gamma_{ij} \in \Gamma$. Moreover

$$g(\alpha) = \Sigma_{i \in F} \Sigma_{j \in J_i} \gamma_{ij}(\alpha)j_2(s_{ij})$$

and, letting $\alpha$ range over $\Lambda$, we have

$$g = \Sigma_{i \in F} \Sigma_{j \in J_i} \gamma_{ij}j_2(s_{ij})$$

that is, $g$ is a linear combination of $\Gamma$ in the vector space $(G, \rho_2)$. \(\square\)

The following corollary, instead, is new.
**Corollary 6.1.** Every element \( g \in G \) is generated in \( (G, \rho_2) \) by positive elements of \( \Gamma \) which have Archimedean distance less than \( \eta \) from \( g \).

**Proof.** We can suppose \( g \neq 0 \). Let \( \alpha \) be the first nonzero component of \( g \). Given \( h_1, \ldots, h_n \in \Gamma \) positive elements which generate \( g \), define \( h'_1, \ldots, h'_n \) such that \( h'_i(\beta) = 0 \) if \( \beta < \alpha \), and \( h'_i(\beta) = |h_i(\beta)| \) otherwise.

Then \( h'_i \in \Gamma \), \( h'_i \) still generate \( g \) and either \( h'_i(\alpha) = 0 \), or \( h'_i \) has Archimedean distance less than \( \eta \) from \( g \). Since the \( h'_i \) generate \( g \), there must be some index \( i_1 \) such that \( h'_i(\alpha) \neq 0 \). So, for every \( i \) such that \( h'_i(\alpha) = 0 \), we replace \( h'_i \) with \( h'_i + h'_i \).

Let \( g_n \) be a sequence of elements of \( G \). A weak sum of \( g_n \), if it exists, is an element \( s \in G \) such that, for every \( n \), \( s - g_1 - g_2 - \cdots - g_n \) is dominated by \( g_n \) at a distance at least \( \eta \). Note that a weak sum is not necessarily unique, because the components \( \alpha < \Lambda \) of \( g \) beyond the first nonzero components of all \( g_n \) are not specified (and such components exist because \( \Lambda \) is an uncountable regular cardinal).

We have the following key lemma.

**Lemma 6.2.** Every positive decreasing sequence \( g_n \) of elements of \( \Gamma \) with distances at least \( \eta \) admits a weak sum \( s \).

**Proof.** Let \( \alpha_n \) be the first nonzero component of \( g_n \) for \( n \in N \) and let \( \alpha \) be the supremum of the \( \alpha_n \) (note \( \alpha < \Lambda \)). Then for \( \beta < \alpha \) we let \( s(\beta) = 0 \), for \( \alpha_n \leq \beta < \alpha_{n+1} \) we let \( s(\beta) = g_1(\beta) + \cdots + g_n(\beta) \), and for \( \beta \geq \alpha \) we let \( s(\beta) = 0 \).

Like in [9] we say that an enriched Riesz space is a triple \((G, \rho, B)\), where \((G, \rho)\) is a Riesz space and \(B\) is a subset of \(G\). An isomorphism of enriched Riesz spaces \((G, \rho, B)\) and \((G', \rho', B')\) is an isomorphism between the Riesz spaces \((G, \rho)\) and \((G', \rho')\) which sends \(B\) bijectively onto \(B'\).

Now suppose by contradiction that \((G, \rho_1)\) is isomorphic to \((G, \rho_2)\). Then for some subset \(\Delta\) of \(G\), the enriched Riesz space \((G, \rho_2, \Gamma)\) is isomorphic to \((G, \rho_1, \Delta)\). So \(\Delta\) must satisfy Corollary 6.1 and Lemma 6.2 (up to replacing \(\rho_2\) with \(\rho_1\)). Let us choose a sequence \((t_n)\) of real transcendental numbers linearly independent over the subfield \(R^a\) of \(R\). Then already in [9] it was observed:

**Lemma 6.3** ([9]). The sequence \((j_2(t_n))\) is linearly independent in the vector space \((K_0, \rho_1)\).

**Proof.** In fact, let us suppose that \(\sum_{n \in F} j_1(r_n)j_2(t_n) = 0\), where \(F \neq \emptyset\) and \(r_n \neq 0\) for every \(n \in F\).

Note that \(j_1(r_n) \in *R^a\) and that \(*R^a\) is the ultrapower \((R^a)^I/\mathcal{U}\).

Instead, \(j_2: R \rightarrow R^I/\mathcal{U}\) is the diagonal embedding, so \(j_2(t_n)\) is the \(\mathcal{U}\)-class of the constant sequence \(t_n\).

Suppose the \(\mathcal{U}\)-class \(j_1(r_n)\) contains a tuple of real algebraic numbers \((r_{n,i})_{i \in I}\). By Los’s Theorem on ultraproducts, we obtain \(\Sigma_{n \in F} r_{n,i}t_n = 0\) and \(r_{n,i} \neq 0\) for every \(n \in F\) and for almost all \(i \in I\) with respect to \(\mathcal{U}\). So, for some \(i \in \mathcal{U}\), we have \(\Sigma_{n \in F} r_{n,i}t_n = 0\) and \(r_{n,i} \in R^a \setminus \{0\}\). But this is not possible since the sequence \((t_n)\) is linearly independent over \(R^a\).

The idea of the following lemma (which gives the main construction) is to use the sequence \(j_2(t_n)\) and define a sequence \(\delta_n\) of positive elements of \(\Delta\) such that Lemma 6.2 may be applied to \(\delta_n\).

We denote by \(n\eta\) the ordinal \(\eta + \eta + \cdots + \eta\) (a sum with \(n\) occurrences of \(\eta\)).
Lemma 6.4. There is a sequence of “quasiconstant” elements $f_n \in G$, a sequence of integers $k_n \in N$, a sequence of finite sets $\Delta_n \subseteq \Delta$ and positive elements $\delta_n \in \Delta_n$ such that:

- for $\beta < n\eta$, $f_n(\beta) = 0$;
- for $n\eta \leq \beta < \Lambda$, $f_n(\beta) = j_2(t_{k_n})$;
- the elements of $\Delta_n$ generate $f_n$ in $(G, \rho_1)$ and all of them have distance less than $\eta$ from $f_n$;
- $\delta_n(n\eta)$ is linearly independent in $(K_0, \rho_1)$ from the components of all elements of $\Delta_1 \cup \cdots \cup \Delta_{n-1}$.

Proof. The proof goes by complete induction.

As a base step, we let $k_1 = 1$. Let $f_1$ be the corresponding quasiconstant element of $G$. By Corollary 6.1, $f_1$ is generated in $(G, \rho_1)$ by a finite set $\Delta_1$ of positive elements of $\Delta$, which cannot be empty. Let $\delta_1$ be any element of $\Delta_1$.

The inductive step $n + 1$ is as follows. We have $\Delta_i, \delta_i, k_i$ for $1 \leq i \leq n$. By definition of $G$, the components of every element of $\Delta_1 \cup \cdots \Delta_n$ have finite dimension in $(K_0, \rho_1)$, whereas the sequence $j_2(t_{k_n})$ has infinite dimension. So we can find a number $k_{n+1} \in N$ so high that $j_2(t_{k_{n+1}})$ is not generated by $\Delta_1 \cup \Delta_2 \cup \cdots \cup \Delta_n$.

Let $f_{n+1}$ be the quasiconstant element of $G$ associated to $k_{n+1}$. By Corollary 6.1, $f_{n+1}$ is generated in $(G, \rho_1)$ by a finite set $\Delta_{n+1}$ of positive elements of $\Delta$ with distance less than $\eta$ from $f_{n+1}$. In particular, $f_{n+1}(n+1)\eta) = j_2(t_{k_{n+1}})$ is generated in $(K_0, \rho_1)$ by the elements $\delta((n+1)\eta)$ with $\delta \in \Delta_{n+1}$. So, by definition of $k_{n+1}$, there must be $\delta_{n+1} \in \Delta_{n+1}$ such that $\delta_{n+1}(n+1)\eta)$ is also linearly independent in $(K_0, \rho_1)$ from the components of $\Delta_1 \cup \Delta_2 \cup \cdots \Delta_n$.

The inductive construction is thus completed.

Now by Lemma 6.2 the positive sequence $\delta_n \in \Delta$ constructed in the previous lemma admits a weak sum $s \in G$. By definition of weak sum, for every $n \in N$, we have $s(n\eta) = \delta_1(n\eta) + \cdots + \delta_n(n\eta)$.

By construction, for every $n \in N$, $\delta_n(n\eta)$ is linearly independent in $(K_0, \rho_1)$ from all components of the sequences $\delta_1, \ldots, \delta_{n-1}$. So also $s(n\eta)$ is linearly independent in $(K_0, \rho_1)$ from $s(\eta), \ldots, s(n-1)\eta)$. Summing up, we conclude that the range of $s$ has infinite dimension in $(K_0, \rho_1)$, so $s \notin G$, a contradiction.

7. Applications to MV-algebras

We note that a condition for the MV-algebra reducts of Riesz MV-algebras can be inferred from Theorem 3.1 by applying the results of [2], where the Mundici equivalence $(\Gamma, \Xi)$ of [11] between MV-algebras and abelian unital l-groups is specialized to an equivalence $(\Gamma', \Xi')$ between Riesz MV-algebras and unital Riesz spaces. In fact, we have:

Lemma 7.1. An MV-algebra $A$ is the reduct of a Riesz MV-algebra if and only if the abelian l-group image $\Xi(A)$ of $A$ in the Mundici equivalence $(\Gamma, \Xi)$ is the reduct of a Riesz space.

Proof. Suppose $A$ is the reduct of a Riesz MV-algebra $R$. Then by [2], $\Xi(A)$ is the l-group reduct of the Riesz space $\Xi'(R)$.

Conversely, if the l-group $\Xi(A)$ is the reduct of a Riesz space $S = \Xi'(R)$, then $A$ is the reduct of the Riesz MV-algebra $R$. 

8
EXPANDING LATTICE ORDERED ABELIAN GROUPS TO RIESZ SPACES

However we have also a direct characterization in terms of MV-algebras. For this aim we call difference structure a structure \((A, \ominus)\) where \(\ominus\) is a binary operation on \(A\). For instance, every MV-algebra is a difference structure with respect to its usual truncated difference operation.

What we call difference structures are related to the D-posets of [3]. We will say that an MV-algebra \(A\) is Riesz-extendable if \(A\) is divisible and there is a family of difference substructures of \(A\), \(R(a)\) \(a \in A\) such that:

- \(R(0) = 0\);
- if \(a > 0\) then \(R(a)\) is isomorphic to \([0, 1]\) as a difference structure, and the (unique) isomorphism sends \(a\) to 1;
- if \(a \in R(a')\) then \(R(a) \subseteq R(a')\);
- if \(x \in R(a), x' \in R(a')\) and for every rational \(q \in [0, 1], x < qa\) if and only if \(x' < qa'\), then \(x \ominus x' \in R(a \ominus a')\).

**Theorem 7.1.** An MV-algebra \(A\) is the reduct of a Riesz MV-algebra if and only if \(A\) is Riesz-extendable.

**Proof.** The conditions are necessary since \(R(a)\) is the set of \(ra\) for \(r \in [0, 1]\).

Conversely, suppose all conditions are met. Let \(r \in [0, 1]\) and \(a \in A\). If \(a = 0\) then we let \(ra = 0\). If \(a > 0\) then we let \(ra\) be the image of \(r\) in the unique difference isomorphism from \([0, 1]\) to \(R(a)\). This is a Riesz MV-algebra structure on \(A\). \(\square\)

We note that, by [3], the Riesz MV-algebra structure on an MV-algebra, when it exists, is not necessarily unique, not even up to isomorphism. This means that the family \(R(a)\) is not uniquely determined by \(A\).

As a corollary of Theorem 6.1, we obtain:

**Corollary 7.1.** There is a totally ordered MV-algebra with two non-isomorphic Riesz MV-algebra structures.

**Proof.** Let \((G, u)\) be a totally ordered abelian group with a strong unit \(u\) such that \(G\) has two non-isomorphic Riesz space structures (such a group exists by Theorem 6.1). Consider the MV-algebra \(A = \Gamma_M(G, u)\), where \(\Gamma_M\) is the Mundici functor of [11]. So the universe of \(A\) is the set \(\{x \in G | 0 \leq x \leq u\}\) and the MV-algebra operations are \(x \oplus y = \min(x + y, u)\) and \(\neg x = u - x\).

We note (see [7] Theorem 3) that every Riesz space structure on \(G\) gives a Riesz MV-algebra structure on \(A\). Actually, in [7] there is an equivalence \(\Gamma_{DL}\) between the category of Riesz spaces with strong unit and Riesz MV-algebras, which coincides with \(\Gamma_M\) when restricted to the MV-algebra reducts of Riesz MV-algebras and the abelian l-group reducts of Riesz spaces.

So, the structures \(\rho_1\) and \(\rho_2\) on \(A\) cannot be isomorphic, otherwise by the functor \(\Gamma_{DL}\) we should have an isomorphism between \((G, \rho_1)\) and \((G, \rho_2)\), contrary to Theorem 6.1. \(\square\)

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