Spherical Gravitational Collapse in 4D Einstein- Gauss- Bonnet theory

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Abstract

In this paper, we study spherical gravitational collapse of inhomogeneous pressureless matter in a well- defined $n \rightarrow 4d$ limit of the Einstein- Gauss- Bonnet gravity. The collapse leads to either a black hole or a massive naked singularity depending on time of formation of trapped surfaces. More precisely, horizon formation and its time development is controlled by relative strengths of the Gauss- Bonnet coupling ($\lambda$) and the Misner- Sharp mass function $F(r,t)$ of collapsing sphere. We find that, if there is no trapped surfaces on the initial Cauchy hypersurface and $F(r,t) < 2\sqrt{\lambda}$, the central singularity is massive and naked. When this inequality is equalised or reversed, the central singularity is always censored by spacelike/timelike spherical marginally trapped surface of topology $S^2 \times \mathbb{R}$, which eventually becomes null and coincides with the event horizon at equilibrium. These conclusions are verified for a wide class of mass profiles admitting different initial velocity conditions. Hence, our result implies that the 4d Einstein- Gauss- Bonnet generically violates the cosmic censorship conjuncture. Further implications of this violation from the perspective of visibility of causal signals from the spacetime singularity are also discussed.

1 Introduction

The $n$- dimensional Einstein- Gauss- Bonnet (EGB) theory has the following Lagrangian functional (we shall assume $G = 1, \ c = 1$):

$$16\pi L[\bar{g}, \psi] = \bar{R}[\bar{g}] + \lambda_n \bar{G}[ar{g}] + L_m[\psi] - 2\Lambda, \quad (1)$$

where, $\bar{R}$ is the Ricci scalar, $\bar{G}[ar{g}]$ is the Gauss- Bonnet (GB) functional in $n$- dimensions with the tensor field $\bar{G} = \bar{R}^2 - 4\bar{R}_{\mu\nu}\bar{R}^{\mu\nu} + \bar{R}_{\mu\nu\lambda\sigma}\bar{R}^{\mu\nu\lambda\sigma}$, $\lambda_n$ is the $n$ dimensional GB coupling constant, $L_m$ is the Lagrangian of matter field $\psi$, and $\Lambda$ is the cosmological constant. The

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GB Lagrangian in equation (1) is known to be a total derivative in 4- dimensions and hence, does not contribute to the equations of motion [1, 2, 3, 4, 5]. However, it was claimed in [6] that by choosing a metric ansatz of the form:

\[ ds^2 = -f(r) \, dt^2 + f(r)^{-1} \, dr^2 + r^2 \, d\Omega_{n-2}, \]  

(2)
a rescaling of the coupling \( \lambda \rightarrow \lambda/(n - 4) \) before taking the limit \( n \rightarrow 4 \), leads to a well defined theory in 4- dimensions. This theory is Ostragradskii stable with 2 dynamical gravitational degrees of freedom. Thus, at the level of the equations of motion, a consistent relabelling of the GB coupling constant seems to lead to well defined set of field equations which is identical to 4d limit of field equations of higher dimensional \((n \geq 5)\) EGB theory. This observation led to a surge of work to understand the possible physical implications of this 4d EGB theory of gravity [7, 8, 9, 10, 11, 12, 13, 14, 15]. The interest is justified since these modified higher curvature field equations may help in explaining deviations from GR arising in future experiments or, may present signatures of hereto unknown new physics. So, from a purely scientific point of view, it is instructive to study various aspects of these equations from multiple points of view.

As pointed out before, the 4d EGB theory, as proposed in [6], violates the Lovelock theorems [1, 2]. At the level of field equations, one does not have regular limit to 4d [16, 17, 18]. Nevertheless, since the higher curvature terms have interesting consequences in cosmology, astrophysics and quantum gravity, various alternate methods have been developed in which, starting from a different theory, with dimensional reductions and/or field redefinitions, one is led to some alternate version of a 4d model. These attempts have led to interesting proposals [19, 20, 21, 22, 23, 24]. In one of these suggestions, by Lu and Pang [19], one begins with a \( n \) dimensional EGB theory and carries out a Kaluza-Klein reduction to 4d, by compactifying over a \((n - 4)\) dimensional maximally symmetric space, and a simultaneous redefinition of \( \lambda \rightarrow \lambda/(n - 4) \). This results in a scalar-tensor type theory with 3 degrees of freedom (leaving aside the matter fields)- two gravitational, and one scalar. Thus, this theory is a well defined 4d limit of \( n \)- dimensional EGB theory consistent with the Lovelock theorems.

We are interested in the formation of spherically symmetric black holes from gravitational collapse in the Lu and Pang version of 4d EGB theory since it may illuminate questions related to the cosmic censorship hypothesis. To see this connection, note that the spherical black hole solution in the Lu- Pang theory is identical to \( n \rightarrow 4d \) limit of the Boulware- Deser solution of \( n \)- dimensional EGB theory [19]. This metric in 4d resembles the Reissner- Nordstrom spacetime (in general relativity) [25, 26], where the role of electric charge is played by square-root of the GB coupling \( (\lambda) \). So, if \( M \) is the ADM mass of the black hole, the condition \( M^2 > \lambda \) leads to a black hole solution with an inner and an outer event horizon; for \( M^2 = \lambda \), the two horizons merge to form one, whereas for \( M^2 < \lambda \), the central singularity is globally naked. Thus due to this correspondence, one can propose that the nature of central singularity and the horizon may have dependence on relative strengths of the EGB coupling constant and the (Misner-Sharp) mass inside a collapsing shell. In particular, the following behaviour should drive the gravitational collapse of spherical matter configurations leading to black hole formation: if the Misner-Sharp (MS) mass contained inside the collapsing spherically symmetric matter is less than \( \sqrt{\lambda} \), the collapse must necessarily proceed to a naked central singularity, that is, a spacetime where light is not trapped inside a horizon. On the other hand, if the MS mass of collapsing shell is greater than \( \sqrt{\lambda} \), regular black holes with two horizons should form. This qualitative response and interplay of coupling constant and mass of the collapsing
matter in black hole formation needs to be checked through a consistent and rigorous formulation of classical gravitational collapse through direct use of the 4d Lu-Pang EGB field equations along with the notion of trapped surfaces. Closely related are the following questions: (a) Does singularity/ trapped region begin to form from collapse of the first matter shells? (b) Does the formation of central singularity depend on the parameters controlling the initial velocity profile and density distribution profile of the matter fields? (c) How does one distinguish naked singularity from a clothed singularity using the formulation of trapped surfaces? (d) Is the relation between MS mass, GB coupling and formation of trapped regions, as envisaged above, are in accordance with the censorship hypothesis? (e) Does the presence of a naked singularity violate the censorship conjecture? (f) How does the entire mechanism change if there is already a black hole present on the initial Cauchy hypersurface? In this paper, we shall develop the theory for inhomogeneous gravitational collapse to answer these questions.

To understand the properties of horizons developing during the collapse process (in particular the Reissner- Nordstrom like nature), we shall use the formulation of trapped surfaces [31]. The basic idea is as follows: during the collapse of the matter cloud leading to a spacetime singularity, trapped surfaces should arise inevitably. These surfaces are defined as 2-dimensional closed surfaces (which we shall take as round spheres) such that the expansion of the outgoing null normal to the sphere ($\ell^\mu$), and that of the outgoing null normal ($n^\mu$) are both negative. Such surfaces form the trapped region, and since no light ray can extend out of this region, this forms the black hole region. The boundary of the trapped region is a three dimensional surface foliated by two dimensional spheres which have vanishing outgoing expansion, $\theta(\ell) = 0$, and negative ingoing expansion $\theta(n) < 0$. These spheres are called the marginally trapped spheres (MTS) and the cylinder foliated by MTS is called a marginally trapped tube (MTT). Following [32, 33, 34, 35], the black hole horizon, formed from the matter collapse, shall be defined as a MTT. A crucial feature of MTT is that they do not carry any specific signature, and hence, can be used to infer if the horizon is growing (for which the tangent vector to MTT shall be spacelike), or if it is in equilibrium (when the MTT carries null signature) [30, 36, 37]. Our task will be to locate the spherical MTT during the physical process of gravitational collapse of matter, and to track the variations in the parameter space which affect the dynamics of MTT. We must mention here that the marginally bound gravitational collapse model of homogeneous dust, using the original 4d EGB of [6], is discussed in [27]. However, there are several limitations in this work. First, the 4d gravitational theory used in [27] is incorrect; second, homogeneous, isotropic models cannot capture the intricacies of collapse scenarios, and third, the formulation leads to a simplistic model where the Reissner-Nordstrom like features, and the relation of the GB coupling constant $\lambda$ towards formation of naked singularity is not explored. Here, we consider the process of gravitational collapse of a distribution of inhomogeneous dust bounded by a spherical surface which are either marginally bound or are bounded. This shall allow us to explore the dynamics of gravitational collapse in finer detail.

As we shall see, the models of black hole in this theory, constructed from collapse of inhomogeneous dust matter, presents a number of constraints on the parameters space: First, if there is no black hole on the initial hypersurface, the central singularity is always naked if mass function of the gravitating matter $F(r; t)$ is less than GB coupling constant, $\lambda$. This result holds true for all values of coupling, but for calculations in this paper, the $n \geq$ marginally bound collapse model of inhomogeneous dust is discussed in [28, 29], whereas, [30] discuss generic models.
we shall assume $\lambda < 1$. The formation of trapped surfaces is delayed until sufficient matter is concentrated. Secondly, when $F(r, t) = 2\sqrt{\lambda}$, a non-central marginally trapped surface appears covering the central singularity. Thereafter, for all $F(r, t) > 2\sqrt{\lambda}$, the MTT continues to cover the central singularity. Thirdly, this MTT may be spacelike or timelike depending on matter profile of the collapsing matter. However, it becomes null only when no matter falls through it, and eventually forms the event horizon of the black hole. Fourth, the initial velocity and density profile of the collapsing matter controls the time of formation of central singularity as well as the equilibration time of the horizon. To exhibit the robustness of these results, we illustrate several scenarios by assuming a wide variety of density profiles of collapsing matter. To ensure the correctness of our models, we have verified that (i) our matter profiles do not admit trapped surfaces on the initial hypersurfaces (except for the cases where we deliberately admit a black hole initially), and (ii) there are no shell crossing singularities during the process.

The paper is organised as follows: In the following section, we shall present a brief discussion on dimensional reduction to 4d Lu-Pang version of the EGB action. We shall also discuss the static black hole solution (to be considered as the external spacetime) to which the collapsing metric is to be matched. This is followed by a discussion of the 4d field equations for gravitational collapse. More specifically, using inhomogeneous collapse models, also called the Lemaitre-Tolman-Bondi (LTB) models of pressureless dust, we derive formal equations for the collapsing matter shells, the time of horizon formation, and the singularity formation time. This allows us to provide a complete picture of gravitational collapse leading to black holes. The section 3 contains application of these equations for various matter profiles, for marginally bound and bounded collapse models. Section 4 discusses the status of the cosmic censorship in this model. The section 5 ends with a discussion.

2 Solutions of the Field equations

In the approach advocated in [19], one begins with a $n \geq 5$ dimensional theory, eqn. (1). This theory is reduced over a warped metric given by:

$$ds_n^2 = ds_4^2 + \exp(2\phi) ds_{(n-4)}^2,$$

where the warping scalar $\phi$ depends on the 4- dimensional spacetime coordinates. If the compactified internal space of $(n-4)$ dimensions is flat, the reduced theory is given by the following 4d Lagrangian [19]:

$$L_4 = \sqrt{-g} \left[ R + \lambda \left\{ \phi G + 4G^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - 4(\partial_\mu \phi)^2 \Delta \phi + 2(\partial_\mu \phi)^4 \right\} \right],$$

where $G_{\mu\nu} = R_{\mu\nu} - (1/2) R g_{\mu\nu}$, is the usual Einstein tensor in 4- dimensions, $R_{\mu\nu}$, $R$, $G$ are the 4- dimensional Ricci tensor, Ricci scalar and the Gauss- Bonnet term. This Lagrangian also arises in the study of spontaneously broken CFTs: The 4- dimensional effective action of massless Goldstone scalar has terms in its effective action which are invariant under Diff × Weyl transformation, as well as terms arising from conformal trace anomalies. The $a$-trace anomaly Lagrangian is identical to eqn. (4) (see [38] for details).

This theory has three degrees of freedom, two gravitational and one scalar. The scalar has no free kinetic term in this theory; it is coupled to the Einstein tensor. Further, the $\lambda$ dependent terms all have quartic derivatives in each term. Apart form the Gauss- Bonnet term, this Lagrangian belongs to the class of most general gravity Lagrangians, linear in
the curvature scalar $R$, quadratic in $\phi$, and containing terms with quartic derivatives [39, 40, 41, 42, 43, 44, 45]. Such theories with non-minimal derivative coupling are used to model dark matter and dark energy in cosmological spacetimes [42, 43, 44, 45]. However, Lagrangians with additional Gauss-Bonnet coupling, have not been studied for gravitational collapse models, and our task in this paper is to model this phenomenon.

The equations of motion for the metric and the scalar field can be derived from the action in eqn. (4). They are given in detail in the Appendix 1 of this paper. The interesting equation is the field equation of the scalar field (given in eqn. (63) of Appendix 1). Since the kinetic term of the scalar is coupled to the Einstein tensor, this leads to a complete modification of the scalar field equations, which is now given by:

$$\Box \phi \left[ \left( \frac{R}{2} \right) + 8 (\nabla_\mu \phi)^2 \right] = R_{\mu\nu} \left[ \nabla^\mu \phi \nabla^\nu \phi + \nabla^\mu \nabla^\nu \phi \right] + 2 (\nabla_\mu \nabla_\nu \phi)(\nabla^\mu \phi)(\nabla^\nu \phi) + (\nabla_\mu \nabla_\nu \phi)^2 + G. \quad (5)$$

As we shall see below, the dynamics of the scalar field is dependent on isometries of the spacetimes under consideration, for static spacetime its behaviour is drastically different in comparison with dynamical spacetimes. The scalar equation is however very important since, the trace of the Einstein equations, and eqn.(5) implies that the constraint $2R = \lambda G$ holds on the space of solutions. This constraint implies that on the space of solutions, the field equations of the action eqn. (4), should have similarity with those of the GR or the EGB solutions. This expectation is borne out in both the static and dynamical solutions discussed below.

2.1 Static solution

The spherically symmetric black hole solution in this theory is also known [19]. The metric is given by:

$$ds^2 = -f(r) dt^2 + f(r)^{-1} dr^2 + r^2 d\Omega_2, \quad (6)$$

where $d\Omega_2 = d\theta^2 + \sin^2 \theta d\phi^2$, is the 2-dimensional round metric, and the metric function $f(r)$ is given by the following Boulware-Deser form [47, 48, 49]:

$$f(r) = 1 + \frac{r^2}{2\lambda} \left[ 1 - \left( 1 + \frac{8\lambda M}{r^3} \right)^{1/2} \right]. \quad (7)$$

The metric is a spherically symmetric, static, asymptotically flat black hole solution with ADM mass $M$. The black hole has a timelike singularity at $r = 0$, where the curvature scalars blow up. Interestingly however, the metric has a nonsingular limit at that surface. Note that as $\lambda \to 0$, the metric reduces to the Schwarzschild spacetime. We also note that, since $\lambda > 0$, there is an inner $(r_-)$ and an outer event horizon $(r_+)$, located at

$$r_\pm = M \left[ 1 \mp \left( 1 - \frac{\lambda}{M^2} \right)^{1/2} \right]. \quad (8)$$

As $\lambda = M^2$, the horizons merge, and we get an extremal black hole with a single horizon, at $r_\mp = M$. For $\lambda > M^2$, there is no event horizon, and the singularity is globally naked. In this sense, the metric has similarity with the Reissner-Nordstrom solution, with $\lambda$ playing the role of square of electric charge.

The role of the scalar field $\phi(r)$, which depends on the radial coordinate in this spacetime, is interesting. Note that the metric functions are independent of the $\phi(r)$ and
the fall-off conditions are such that the ADM mass of the spacetime is not changed. This can be understood as follows [19]: The field equation for $\phi(r)$, eqn. (5) reduces to:

$$\left(r \phi' - 1\right)^2 f(r) - 1 = 0. \tag{9}$$

Using $f(r)$ of eqn.(7), the regular solution of equation, (9) is given by:

$$\phi(r) = \log \left[ \frac{r}{L} \left( \cosh x - \sinh x \right) \right], \quad x(r) = \int_{r_+}^{r} \frac{dy}{y \sqrt{f(y)}}, \tag{10}$$

where $r_+$ is coordinate of the outer event horizon in (8), and $L$ is a constant of integration of eqn. (9), and plays the role of length scale. For the spacetimes under consideration, where compactification is carried over a flat metric, and the black hole is spherically symmetric round metric (6), the scalar field $\phi(r)$ vanishes asymptotically. Therefore, the conditions are such that the ADM mass of the spacetime is not affected by the presence of the scalar. For this specific case of spherically symmetric black hole solution, this scalar $\phi(r)$ is like a background field which satisfies fixed conditions on the horizon and at asymptotic infinity. Of course, the scalar field has a completely different behaviour for other solutions of this theory, but we are not interested in them here (see [19] for these solutions). In our analysis of gravitational collapse formalism in this theory, the black hole metric of eqn. (6) shall be taken as the external spacetime. The collapse of inhomogeneous matter shall be assumed to lead to this external spacetime metric for an external observer.

2.2 Time dependent inhomogeneous metric solutions

Let us now look into the gravitational collapse formalism for the theory in eqn. (4). Since the scalar field has an important role to play, the first task at hand is to look into its dynamics. The theories with scalar fields kinetically coupled to the Einstein tensor has been studied in great detail for cosmological spacetimes [43, 44, 45]. It arises, quite surprisingly that, in absence of any other matter sources or in presence of other pressureless matter, the scalar itself behaves as a dust [44]. Keeping this idea in mind, we shall consider the scalar to be a test scalar or a background spectator field, having a fixed value $\phi_0$, and add a dust matter as the energy momentum of spacetime. Then, the equations of motion, eqn. (59) of Appendix 1, reduce to the following

$$G_{\mu\nu} = 8\pi T_{\mu\nu} - \lambda H_{\mu\nu}, \tag{11}$$

where $T_{\mu\nu}$ is the energy momentum tensor of dust, (the coupling $\lambda$ is in fact the quantity $\lambda \phi_0$, but we shall continue to denote it as $\lambda$), and the $H_{\mu\nu}$ is due to the Gauss- Bonnet term, which shall be taken as the effective energy momentum tensor having form

$$H_{\mu\nu} = 2 \left[ RR_{\mu\nu} - 2 R_{\mu\rho} R^\rho_\nu - 2 R^\rho_\delta R_{\mu\rho\nu\delta} + R_{\mu}^{\sigma\delta\lambda} R_{\nu\sigma\delta\lambda} \right] - \frac{1}{2} \sigma_{\mu\nu} G. \tag{12}$$

The line element of spherical symmetric space-time geometry can be written as

$$ds^2 = -e^{2\Phi(r,t)} dt^2 + e^{2\Psi(r,t)} dr^2 + R(r,t)^2 d\theta^2 + R(r,t)^2 \sin^2 \theta d\phi^2, \tag{13}$$

where $\Phi(r,t)$, $\Psi(r,t)$ are the metric functions, $R(r,t)$ is the physical radius of the matter sphere, and $\theta$, $\phi$ are the angular coordinates on the sphere. The energy momentum tensor for pressureless dust is given by

$$T_{\mu\nu}(r,t) = \rho(r,t) u_\mu u_\nu, \tag{14}$$
where, \( u^\mu \) is the unit time-like vector satisfying \( u_\mu u^\mu = -1 \), and \( \rho (r,t) \) is the energy density respectively. In the comoving co-ordinates the four velocity is \( u^\mu = e^{-\Phi(t,r)} \delta_0^\mu \). The EGB field equations which govern the dynamical development of collapsing cloud takes the following form

\[
\begin{align*}
F'(r,t) &= 8\pi \rho R^2(r,t) R'(r,t), \\
F(r,t) &= 0 \\
\Phi(r,t)' &= 0 \\
\dot{R}(r,t)' &= \dot{R}(r,t) \Phi' + R'(r,t) \Psi \\
F(r,t) &= R(r,t) (1 - G + H) + \frac{\lambda}{R(r,t)} (1 - G + H)^2
\end{align*}
\]  

(15) (16) (17) (18) (19)

where \( F(r,t) \) is the Misner-Sharp mass function, and the dots, and the primes imply differentiation with respect to \( t \), and \( r \) respectively. The functions \( H(r,t) \) and \( G(r,t) \) are defined as \( H(r,t) = \exp[-2\Phi(r,t)] \dot{R}(r,t)^2 \), and \( G(r,t) = \exp[-2\Psi(r,t)] \dot{R}(r,t)^2 \). Gravitational collapse is obtained by requiring that \( \dot{R} < 0 \). These equations shall lead to solutions for the metric functions \( \Phi(r,t) \), \( \Psi(r,t) \), and determine the radius \( R(r,t) \) of the matter cloud. First, note from eqn. (16) that, \( F = F(r) \), and from eqn. (17) that \( \Phi(t,r) \) is a function of \( t \) only. Naturally, one may redefine the variable \( t \) to absorb the function \( \Phi(t) \) in eqn. (15). The function \( \Psi(t,r) \) similarly is obtained from eqn. (18), to give \( \Psi(t,r) = R'^2(r,t)/(1 - k(r)) \). All this exercise leads to the following interior metric:

\[
ds^2 = -dt^2 + \frac{R'^2(r,t)}{1 - k(r)} dr^2 + R(r,t)^2 d\Omega_2.
\]

(20)

The metric is completely determined by \( R(r,t) \), and \( k(r) \). \( R(r,t) \) is controlled by the density \( \rho(r) \) or \( F(r) \), and \( k(r) \) fixes the initial energy of the shells, as to whether they are bounded or otherwise. These parameters dictate if this metric leads to either a black hole or a naked singularity. Note that this metric needs to be matched to the eqn. (6), across a \( r = \text{constant hypersurface} \). The matching of the metric and extrinsic curvatures lead to the condition that \( F(r) = 2M \), see Appendix- 3 of this paper for details.

## 3 Horizon formation from gravitational collapse

As we have discussed earlier, we shall use the formulation of MTT to locate horizons. This is natural since MTT also forms the boundary of black hole region. The MTT is located from the conditions \( \theta_{(t)} = 0 \), and \( \theta_{(n)} < 0 \). For the metric (15), and the choice of following null normals:

\[
\begin{align*}
\ell^\mu &= (\partial_t)^\mu + e^{-\Psi(t,r)} (\partial_r)^\mu \\
n^\mu &= (1/2)(\partial_t)^\mu - (1/2) e^{-\Psi(t,r)} (\partial_r)^\mu
\end{align*}
\]  

(21) (22)

the eqn. (19), and the form of \( \Psi(t,r) \) lead to the following expression for radius of MTT:

\[
R(r,t)_{\text{MTT}} = \frac{F(r)}{2} \left[ 1 + \left\{ 1 - \frac{4\lambda}{F(r)^2} \right\}^{1/2} \right].
\]

(23)

This expression in eqn. (23) is of fundamental importance, since this value controls the time of formation of MTT. For each collapsing shell labelled by the coordinate \( r \), the
value of $R_{\text{MTT}}$ is obtained for $F(r) = 2M$, where $M$ is the total mass of the spacetime obtained from matching the external metric in eqn. (6). So, a matching of spacetimes across $r = \text{constant}$ implies that only for $F(r) \geq 2\sqrt{\lambda}$ the is singularity covered. Until that time for which $F(r) < 2\sqrt{\lambda}$, no trapped surface forms and the singularity remains naked.

One further advantage of the MTT formalism is that it allows a direct check of nature of growth of the horizon: For example, the as matter falls through, the MTT increases in radius. To study these changes, let $t^\mu = \ell^\mu - C n^\mu$ is tangent to the MTT. Then, $t^\mu$ becomes spacelike or null depending on the quantity $C$, which is given by [30, 46]:

$$C = \frac{8\pi T_{\mu\nu}l^\mu l^\nu - \lambda H_{\mu\nu}l^\mu l^\nu}{4\pi/A} - 8\pi T_{\mu\nu}n^\mu n^\nu + \lambda H_{\mu\nu}n^\mu n^\nu,$$

(24)

where $A$ is area of the MTT. We shall see that when no matter falls in, $C = 0$, and the MTT is null, and when matter falls in, $C > 0$, indicating that the MTT is spacelike.

Naturally, the MTT approaches the event horizon asymptotically when no matter falls through. To determine the value of $C$, the geometric conditions like $H_{\mu\nu}l^\mu l^\nu$ and $H_{\mu\nu}l^\mu n^\nu$ need to be rewritten in terms of the matter variables:

$$H_{\mu\nu}l^\mu l^\nu = \frac{2\rho}{R(r,t)^2} \left[ -\frac{4F}{R} + \frac{1}{R(r,t)} \left( \ddot{R}(r,t) - \dot{R}(r,t) \right) \right]$$

$$- \frac{6}{R^2} \left[ \left( \frac{F - R}{R} \right) \right]^2 + \frac{6}{R^2} \left[ \left( \frac{F - R}{R} \right) \right]'^2,$$

(25)

$$H_{\mu\nu}l^\mu n^\nu = - \left[ \rho^2 + \frac{7}{4} \left( \rho + \frac{4F}{R^2} \right)^2 \right] + \frac{2\rho}{R} \left( \frac{1}{R} \left( \frac{F - R}{R} \right) + \frac{1}{R^2} \left( \frac{F - R}{R} \right)' \right).$$

(26)

In the following section, we shall determine the nature of the MTT using this value of $C$, for a wide variety of density distribution.

Two matter of vital importance which has been implemented in each of the following examples are: (i) no crossing of shells take place during collapse, and that (ii) on the initial Cauchy hypersurface, no black hole exists (unless we deliberately introduce them). For the first objective, we require that $R'(r,t) > 0$. This condition ensures physical separation of $r = \text{constant}$ shells and, asserts that shells starting earlier in coordinate time (or proper time in this case), shall reach the singularity earlier. The second requirement is ensured if for the shells labelled by $r$, the no- trapping condition condition holds initially:

$$r > \frac{F(r)}{2} \left[ 1 + \left( 1 - \frac{4\lambda}{F(r)^2} \right)^{1/2} \right].$$

(27)

For all the matter collapse models considered below, the initial density parameters are adjusted so that these two abovementioned conditions hold correct.

### 3.1 Marginally bound collapse

Marginally bound collapse implies the choice $k(r) = 0$, and from the equation [19] we obtain the equation of motion for the radius of the collapsing sphere $R(r,t)$. The time
dependence of $R(r, t)$ and the time coordinates on shell are given by:

$$
\dot{R}(r, t) = -\frac{R(r, t)}{\sqrt{2\lambda}} \left[ \sqrt{1 + 4\lambda F(r)/R(r, t)^3} - 1 \right]^{1/2},
$$

(28)

$$
t_{\text{shell}} = \frac{2}{3} \sqrt{\lambda} \left[ \left( \frac{1}{Z_0} - \arctan(Z_0) \right) - \left( \frac{1}{Z} - \arctan(Z) \right) \right],
$$

(29)

where $Z = (1/\sqrt{2}) \left[ \sqrt{1 + 4\lambda F(r)/R^3} - 1 \right]^{1/2}$, and $Z_0 = Z(R = r)$ [27]. The time for the formation of the singularity is obtained when the shell reaches $R(r, t) = 0$ and is given by $t_s$, and the time for formation of MTT, for each shell, is obtained from eqn. (23), and shall be denoted by $t_{\text{MTT}}$:

$$
t_s = \frac{2}{3} \sqrt{\lambda} \left( \frac{1}{Z_0} - \arctan(Z_0) \right) - \frac{\pi}{3} \sqrt{\lambda},
$$

(30)

$$
t_{\text{MTT}} = \frac{2}{3} \sqrt{\lambda} \left[ \left( \frac{1}{Z_0} - \arctan(Z_0) \right) - \left( \frac{1}{Z_1} - \arctan(Z_1) \right) \right],
$$

(31)

where, $Z_1 = Z(R = R_{\text{MTT}})$.

Let us now study how, given an initial matter density distribution, the collapse of the matter cloud proceeds, along with the formation of MTT. For a given initial density distribution, from the Einstein equation (16), we can calculate the initial mass distribution of the cloud

$$
F(r) = 8\pi \int_0^r r'^2 \rho(r') \, dr'.
$$

(32)

The addition of the GB term, as can be seen from eqn. (19), leads to modification of the dependence of mass function $F(r)$. If the mass function arising in GR is labelled $F_1(r)$, the contribution of the Gauss Bonnet term and can be written as $F(r) = F_1(r) + \lambda F_1^2/R(r, t)^3$. To have better understanding of this collapsing phenomena we consider explicit examples of different initial density profiles. The main result of the study is to show that the MTT begins to form only when $F(r) \geq 2\sqrt{\lambda}$, and until $F(r)$ reaches this value, the collapse leads to globally naked massive singularities. For all the examples below, we take the GB coupling constant $\lambda = 0.1$.

Examples:

1. In this example, we consider the density of the cloud to be of the following form [36, 37]:

$$
\rho(r) = \frac{m_0 E(\sigma)}{r_0^3} \left[ 1 - \text{Erf} \left\{ \sigma \left( \frac{r}{r_0} - 1 \right) \right\} \right],
$$

(33)

where $m_0 = m(r \to \infty)$ is the total mass of the cloud, $r_0$ is the location on the cloud where it matches the Boulware-Deser radius eqn. (8), and the quantity $\sigma$ controls the approach to the OSD model; larger the value of $\sigma$, closer is the density to uniformity. $m_0$ is taken to be 1. The function $E(\sigma)$ has the following form:

$$
E(\sigma) = 3\sigma^3 \left[ 2\pi\sigma(2\sigma^2 + 3)(1 + \text{Erf} \sigma) + 4\sqrt{\pi} \exp(-\sigma^2)(1 + \sigma^2) \right]^{-1},
$$

(34)

and Erf is the usual error function. We consider the cases where $\sigma = 5$ and 15. The graphs are given in fig. [1]. The graphs show that as the shells begin to collapse,
the central singularity is formed but is not shrouded by MTT. The formation of trapped spheres are delayed. The MTT begins only after sufficient number of shells have fallen in. For example in the fig. (1d, only after the shell labelled by \( R = 1.36 \) fallen in, that the MTT begins. In fact, the MTT is noncentral and begins only after the mass content of the singularity is \( \geq 0.3 \). Before that, the spacetime has no marginally trapped surfaces. Note that until the MTT forms, the singularity is massive and globally naked: the interior metric is matched with the external metric in eqn. (6) admitting naked central singularity for each shell radius. The graph in fig. (1b) shows that the MTT has \( C > 0 \), meaning that the MTT is spacelike and becomes null only when no more matter falls through, after the shell labelled by \( r = 3 \). The MTT at equilibrium is at \( r = 1.95 \), which is exactly the Boulware-Deser event horizon.

![Graphs](image_url)

Figure 1: This gives the gravitational collapse for the OSD profile discussed above where (a) gives the density fall-off, (b) gives the values for \( C \) and hence the signature of MTT, (c) gives the formation of MTT for \( \sigma = 5 \), and \( \lambda = 0.1 \). Here, the MTT begins to form only after the shell labelled by \( R = 1.39 \) falls in. Until then, the singularity is naked and is matched with the external metric in eqn. (6) admitting naked central singularity. Similarly, The graph (d) shows the MTT for \( \sigma = 15 \) along with the shell coordinates for \( \lambda = 0.1 \). Here, the MTT begins only after the shell labelled by \( R = 1.36 \) falls in. The straight lines of MTT in (c) and (d) represents the isolated horizon phase obtained at \( R = 1.95 \), and when the horizon is null. The MTT looks like a timelike surface in (c), and (d), but that is only due to the choice of foliation.

2. Let us consider the case where the initial mass density of the collapsing matter is
of the following form:

$$\rho_1(r) = (3M/2500)(10 - r) \Theta(10 - r), \quad (35)$$

where $M = 1$. For this density profile too, initially, the central singularity is naked and massive. The trapped surfaces form only after some massive shells fall in. As can be seen from the graph in fig. (2), the MTT begins after shell $R(r,t) = 5$ has collapsed, and the mass content of the singularity $\geq 0.3$. The MTT becomes null and approaches the isolated phase, after the shell at $R(r,t) = 10$. The graph in fig. (2)a shows that after this, no more mass falls in.

![Graphs](image)

Figure 2: The graphs show the (a) density distribution, (b) values of $C$ which dictates the nature of MTT, and (c) formation of MTT along with the shells for $\lambda = 0.1$. The MTT begins non-centrally, and only when the mass content of the singularity is $\geq 0.3$. The straight lines of MTT in (c), after approximately $r = 10$, represents the isolated horizon phase when MTT becomes null.

3. Let us consider another density profile with the following form:

$$\rho(r) = \frac{m_0}{8\pi r_0^4} \exp(-r/r_0), \quad (36)$$

where $m_0 = 1$ is the total mass of the matter cloud, $r_0$ is a parameter which indicates the distance where the density of the cloud decreases to $[\rho(0)/e]$. As seen from fig. (3)c, the MTT is non-central, and begins at approximately with the shell $R(r,t) = 20$, when mass of the singularity $\geq 0.3$. After this shell has collapsed,
MTT becomes spacelike (as seen from $C > 0$) and remains so throughout the collapse until it reaches the isolated phase, when $C = 0$, see fig. 3b. The isolated phase at approximately $R(r,t) = 100$, after which all shells are massless.

4. Here, the situation is as follows. A black hole exists on the initial hypersurface, and on it, a large shell of constant density falls. The nature of MTT here is drastically different in nature from that described in the previous examples. Here, we shall observe formation of timelike MTTs. The initial density distribution of the collapsing matter is [36, 37]

$$\rho(r) = \frac{3 m_0 \left[ \text{Erf} \left( \frac{\bar{r} - r_1}{M} \right) - \text{Erf} \left( \frac{\bar{r} - r_2}{M} \right) \right]}{4\pi (r_2 - r_1) (2r_1^2 + 2r_2^2 + 2r_1 r_2 + 3M^2)},$$

(37)

where $m_0 = 600M$, $r_1 = 100M$, $r_2 = 2000M$. The form of the density is given in fig. 4a. If we assume that the initial black hole has the Schwarzschild radius given by $\bar{r} = 2M$, then the mass for each shell of radius $r(r > \bar{r})$ is then $m(r) = M + 4\pi \int_{\bar{r}}^{r} \rho(\hat{r})\hat{r}^2 d\hat{r}$. The MTT forms at two places, (i) at the center of the collapsing sphere, at $t = 500$, when the shell labelled by $r = 100$ collapses, and (ii) at approximately $t = 1000$, with the collapse of the last shell at $r = 2000$. The
outer MTT then bifurcates, and proceeds as a timelike membrane (as can be seen from the values of $C$) to meet the inner MTT. The other part of the outer MTT becomes null and reaches equilibrium. The timelike/null nature of the MTT is also confirmed from the values of $C$ in fig. 4b. Note that since a black hole exists initially, the MTT begins centrally with the fall of the first shell, and the singularity is censored.

5. Let us consider another density profile with the following form:

$$\rho(r) = \frac{\alpha \mu}{2 \pi^2 r_0^3} \sin^2(\alpha r / r_0),$$

where $r_0 = 1$ is a length scale, $\mu = (8 \pi r_0^3 / 5)$, is the measure of mass between successive shells, and $\alpha = 1/10$. As seen from fig. 5c, the MTT is non-central, and begins at approximately with the shell $R(r, t) = 20$, when mass of the singularity $\geq 0.3$. Now, just as the density of matter approaches zero, the MTT tends to become null. This behavior is clear from the graphs of MTT in fig. 5c, and the values of $C$ in fig. 4b. The isolated phase continues to appear in between the spacelike phase, throughout the process.
these expressions to study MTT formation. We shall use

\[
x = -R(r, t)^2 - 2 \lambda k + \sqrt{R(r, t)^4 + 4 \lambda F(r) R(r, t)}
\]

To solve this equation, we assume \( k(r) = F(r)/r \), and define a new variable

\[
x_0 = -r^2 - 2 \lambda k + \sqrt{r^4 + 4 \lambda F r}
\]

Naturally, one defines the associated quantities: \( x_{0\mu} = -R_{\mu\tau}^2 - 2 \lambda k + \sqrt{R_{\mu\tau}^4 + 4 \lambda F R_{\mu\tau}} \). Then, the time of formation of the singularity and the MTT is given by a long expression, and is given in the appendix. We shall use these expressions to study MTT formation.

\[
\rho
\]

Figure 5: The graphs show the (a) density distribution, (b) values of \( C \), and (c) formation of MTT along with the shells for \( \lambda = 0.1 \). The MTT is oscillates between null and spacelike nature, in accordance with the fall of matter.

### 3.2 Bounded collapse

For the bounded collapse, we shall consider the choice of the function \( k(r) < 0 \). Just like the case for marginally bounded collapse, the dynamics the shell radius and the time coordinate on the shell can be easily obtained from the equations of motion:

\[
\dot{R}(r, t) = -\frac{R}{\sqrt{2\lambda}} \left[ \sqrt{1 + \frac{4 \lambda F}{R^3}} - \left( 1 + \frac{2 \lambda k}{R^2} \right) \right]^{1/2}.
\]  

(39)

To solve this equation, we assume \( k(r) = F(r)/r \), and define a new variable

\[
x = -R(r, t)^2 - 2 \lambda k + \sqrt{R(r, t)^4 + 4 \lambda F(r) R(r, t)}
\]

(40)
Examples:

(i) Let us consider a Gaussian profile with density given by the following form [36, 37]:

\[ \rho(r) = \frac{m_0}{\pi^{3/2} r_0^3} \exp(-r^2/r_0^2), \]  

where \( m_0 = 1 \) is the total mass of the matter cloud, \( r_0 \) is a parameter which indicates the distance where the density of the cloud decreases to \([\rho(0)/e]\). In our example, we have chosen \( r_0 = 100 m_0 \). Initially, the shells collapse to the central singularity. There the singularity is naked and massive. However, only after the shell labelled by \( r = 140 \) (approximately) has collapsed, and the mass of the collapsed shells \( \geq 0.3 \), that the MTT begins from, remains spacelike until all the mass has fallen in. After the shell \( r = 300 \), the MTT reaches equilibrium and becomes isolated and null, see fig. (6)c. The signature of MTT may be verified from figure (6)b.

(ii) Two shells falling consecutively on a black hole: Let us assume that a black hole of mass \( M = 1 \) exists, upon which a density profile of the following form falls:

\[ \rho(r) = \frac{8 (m_0 / r_0^3)}{[2\sigma + (3 + 2\sigma^2)\sqrt{\pi}\sigma^2 \{1 + \text{Erf}(\sigma)\}]} \exp[(2r/r_0)\sigma - (r/r_0)^2], \]  

Figure 6: The graphs show the (a) density distribution, (b) values of \( C \), and (c) formation of MTT along with the shells for \( \lambda = 0.6 \). The MTT begins only after sufficient mass has fallen in. The central central singularity is naked, and formation of trapped surfaces are delayed. The straight lines of MTT in (c), after the shell \( r = 250 \), shows that the black hole has reached equilibrium state.
where $m_0 = M/2$ is the mass of the shell, $2r_0$ is the width of each shell. Given the mass of initial black hole, the initial EH radius given by 1.6. the mass for each shell of radius $r(r > \bar{r})$ is then $m(r) = M + 4\pi \int_0^r \rho(\tilde{r}) \tilde{r}^2 \, d\tilde{r}$. The quantity $\sigma$ is a parameter which denotes the position where the density vanishes. Here, we have used $M = 1$, $r_0 = 10$ and $\sigma = 10$. Not surprisingly, as we have explained, there is no naked singularity here. Because of the presence of a black hole on the initial Cauchy hypersurface, the singularity remains covered throughout. At around $r = 70$, the MTT starts to grow in a spacelike fashion and reaches approximately at $R = 1.5$ at approximately $t = 1014$ when the $r = 100$th shell falls. Note that at this time the $C$ vanishes making the MTT null. This is expected since the density of the shell goes to zero here (see figure [7]a). Again, just as the next shell starts to fall, the MTT again begins to evolve in a spacelike fashion to reach $R = 2.8$ at $t = 1664$ when the shell denoted by $r = 150$ has fallen in.

4 Censorship and visibility of singularities

The eqn. (23) is fundamental since it relates the radius of the collapsing matter sphere with the mass function, and specifically implies that a trapped region forms if:

$$F(r) \geq R(r, t) + \frac{\lambda}{R(r, t)}.$$  \hfill (43)
The natural question is the following: For a large part of the trapped region, until \( F(r) \geq 2\sqrt{\lambda} \) the singularity is naked, and hence, this region violates the weak cosmic censorship. Null rays emanating from this region reach the future null infinity, although they may be highly redshifted to be detected. The central singularity at \( r = 0 \) on the other hand is a singularity even after MTT has covered it. The question may raise is: does this singular point violate the strong version of cosmic censorship? To answer this question, we shall apply the widely used root equation method \([51]\).

Let us begin by redefining \( R(r,t) = r \bar{v}(r,t) \), where the function \( v(r,t_i) = 1 \), and \( v(r,t_s) = 0 \), where \( t_i/s \) are the initial and the singularity times respectively. The condition for collapse \( \dot{R}(r,t) < 0 \), shall now be replaced by the requirement \( \dot{v}(r,t) < 0 \). From the equation \([39]\), the equation for the function \( v(r,t) \) is obtained to be:

\[
\dot{v}(t,r) = - \left[ -k(r) - \frac{1}{2\lambda} \left( \bar{v}^2 - \sqrt{\bar{v}^4 + 4\lambda F(r) \bar{v}} \right) \right]^{1/2}.
\]

The equation of the time curve of shells is obtained by integrating this equation with respect to \( v \), leading to:

\[
t(v,r) = \int_0^1 \frac{d\bar{v}}{\left[ -k(r) - \frac{1}{2\lambda} \left( \bar{v}^2 - \sqrt{\bar{v}^4 + 4\lambda F(r) \bar{v}} \right) \right]^{1/2}}.
\]

The singularity time \( (t_s) \) is determined by using the value of \( v = 0 \) in the above integral.

\[
t_s(r) = \int_0^1 \frac{d\bar{v}}{\left[ -k(r) - \frac{1}{2\lambda} \left( \bar{v}^2 - \sqrt{\bar{v}^4 + 4\lambda F(r) \bar{v}} \right) \right]^{1/2}},
\]

whereas the time for the shells to reach the MTT, denoted by \( t_{\text{MTT}} \) is obtained by using the appropriate time values of the shell labelled by \( r \) to reach that particular MTT radius \( R_{\text{MTT}} = rv_{\text{MTT}} \). This equation is given by:

\[
t_{\text{MTT}}(r) = \int_{v_{\text{MTT}}}^1 \frac{d\bar{v}}{\left[ -k(r) - \frac{1}{2\lambda} \left( \bar{v}^2 - \sqrt{\bar{v}^4 + 4\lambda F(r) \bar{v}} \right) \right]^{1/2}}.
\]

A slight rearrangement of this equation lead to the following expression in terms of \( t_s(r) \):

\[
t_{\text{MTT}}(r) = t_s(r) - \int_{v_{\text{MTT}}}^0 \frac{d\bar{v}}{\left[ -k(r) - \frac{1}{2\lambda} \left( \bar{v}^2 - \sqrt{\bar{v}^4 + 4\lambda F(r) \bar{v}} \right) \right]^{1/2}}.
\]

This shows in a quite straightforward way that \( t_s(r) \) is usually higher than \( t_{\text{MTT}} \) provided that the result of integration is positive. We have shown in quite detail that unless \( F \geq 2\sqrt{\lambda} \), no MTT forms, and in this bound, when MTT is formed, \( t_{\text{MTT}}(r) < t_s(r) \).

Now, using these equations let us inquire the visibility of the singularities, near the matter center \( r = 0 \). Near \( r = 0 \), all the functions considered above may be taken to be a smooth function of \( r \) (note for example that all the density and mass functions considered in the previous sections have a smooth behaviour near \( r = 0 \)), and hence are Taylor expandable:

\[
F(r) = F_0 + F_2 r^2 + F_4 r^4 + \cdots,
\]

\[
k(r) = k_0 + k_2 r^2 + k_4 r^4 + \cdots,
\]
where, $F_i$s and $k_i$s are constants, and only the even powers survive since the system is symmetric about $r = 0$. Similarly, the equation for the time curve $t(v, r)$, may be expanded:

$$t(v, r) = t(v, 0) + (1/2) r^2 \chi_2(v) + \cdots ,$$

where the function $\chi_2(v)$ is obtained from the eqn. (45), and gives:

$$\chi_2(v) = \int_v^1 \frac{k_2 - F_2 (\bar{v}^2 + 4 \lambda F_0 \bar{v}^{-1})^{-1/2}}{[-k_0 - \frac{1}{2\lambda} (\bar{v}^2 - \sqrt{\bar{v}^4 + 4 \lambda F_0 \bar{v}})]^{3/2}} d\bar{v}.$$  

(52)

For understanding the visibility of the singularities, we require future-directed outgoing radial null geodesics to emerge from the singularity at $r = r_s = 0$. The fixed point method requires that we define a quantity:

$$x = \frac{R(r, t)}{r^q} = \frac{v(r, t)}{r^{q-1}},$$

(53)

where $q \geq 0$ is a constant. We use the root equation method, and accordingly, near the central singularity,

$$x_0 = \lim_{r \to 0} \frac{R}{r^q} = \lim_{r \to 0} \frac{1}{q r^{q-1}} \times \frac{dR}{dr} = \lim_{r \to 0} \left( \frac{v + rv'}{q r^{q-1}} \right) \left( 1 + \frac{rv'}{\sqrt{1 - kr^2}} \right).$$

(54)

Using the equations (44), and the expansions in (49), we get the desired root equation, which is to be evaluated at $v = x_0 r^{(q-1)}$:

$$x_0 = \lim_{r \to 0} \frac{1}{q r^{q-1}} \left[ 1 - r \left( \frac{v F_0}{\lambda} \right)^{1/4} \right] \times \left[ v + \left( \frac{v F_0}{\lambda} \right)^{1/4} r^2 \chi_2(0) \right].$$

(55)

The finite positive root of this equation is obtained with $q = (11/3) > 1$, such that

$$x_0^{3/4} = (3/8) \left( \frac{F_0}{\lambda} \right)^4 \chi_2(0).$$

(56)

This implies that, near the central singularity, the future directed radial null geodesic behaves as $R(r, t) = x_0 r^{11/3}$. The density again diverges at $r = 0$, since from eqn. (15), the expression for density contains $R'(r, t) = (v + rv')$. The expression for $v'$ is given by:

$$v' = \left( \frac{x_0 F_0}{\lambda} \right)^{1/4} r^{5/3} \chi_2(0).$$

(57)

Hence $r v' \to 0$ as $r \to 0$, leading to the divergence of the density as well as the curvature along this null geodesic. This establishes that the singularity at $r = 0$ is at least locally naked, and hence the strong version of the cosmic censorship is violated.

## 5 Discussion

In this paper, we studied gravitational collapse of inhomogeneous dust in 4d EGB gravity. The matter is assumed to be, initially, either bounded or marginally bounded. The
collapse phenomenon presents some surprising results which are absent in 4d general relativity. Using a wide variety of matter profiles, we show that if there are no black holes on the Cauchy hypersurface, (i) the central singularity is initially naked: the formation of trapped surfaces are delayed until mass contained inside the collapsed sphere exceeds $\sqrt{\lambda}$, where $\lambda$ is the GB coupling constant, (ii) MTT begins non-centrally when the equality is satisfied, and (iii) the MTT covers the singularity if $F(r) > 2\sqrt{\lambda}$. For the situation when there exists a black hole on the initial data set, the central singularity is censored, and the MTT begins from the first collapsing shell itself. Thus this theory provides a fertile ground to study effects of naked singularities.

Given these results, several queries arise. The first concerns the nature of naked singularity formed initially. For a large class of smooth matter profiles, one can obtain null geodesics emanating from the central singularity. For our case, initially for $F(r) < 2\sqrt{\lambda}$, the metric to which the naked singularity needs to be matched, is globally naked. Hence, until the mass function satisfies this bound, the singularity is globally naked, and violates the weak censorship. On the other hand, if the singularity is locally naked, it may be considered to violate the strong version cosmic censorship. As we saw in the previous section, the central singularity is always locally naked, whether it is covered or not. Indeed, when $F(r) \geq 2\sqrt{\lambda}$, the singularity violates the strong censorship. However, we must also note that, the central singularity at the least, is not strong, which, according to the analysis carried out in \cite{52, 53}, is essential for proving non-extendibility of spacetimes. The sufficient condition for a strong curvature singularity is that for at least one causal geodesic $\tau^\mu$, with affine parameter $s$, in the limit to singularity must obey:

$$\lim_{s \to s_0}(s - s_0)^2 R_{\mu\nu}\tau^\mu\tau^\nu > 0,$$

For a normalised radial timelike geodesic corresponding to the metric in eqn. (6), near the center, and with $\tau^\mu = \delta^\mu_t$, the above expression is zero. Hence, in 4d, the EGB theory, the singularity does not satisfy the strong curvature condition.

The second is about the mass of the singularity. Unlike GR, massive singularities are no unheard of in higher curvature theories. Our solutions give explicit constructions of these objects. How does one quantify them, and what role shall they play in quantum theory? One also should determine the stability and experimental signatures of such spacetime structures \cite{54, 55}. These studies, with a detail description of the dynamics of the scalar field in the 4d EGB action shall be carried out in the future.

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Appendix 1: The field equations

The Lagrangian in eqn. (4), belongs to the class of most general gravity Lagrangians, linear in the curvature scalar $R$, quadratic in $\phi$, and containing terms with quartic derivatives \cite{39, 40, 41, 42, 43, 44, 45}. One further interesting point is that the kinetic term of the scalar is coupled to the Einstein tensor. Such theories have been considered in great
detail as useful models for both the dark energy and the dark matter [43, 44, 45]. In the
following we write the field equations for the Lagrangian in eqn. (4) [45, 57]:

\[ L_4 = \sqrt{-g} \left[ R + \lambda \left\{ \phi \mathcal{G} + 4G^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - 4(\partial_\mu \phi)^2 \Box \phi + 2 (\partial_\mu \phi)^4 \right\} \right]. \]

The metric variation of the action gives the following terms. Note that no matter Lagrangian has been added here, however, if they are added, they would contribute an additional factor of \(8\pi T_{\mu\nu}^{\text{matter}}\):

\[ T_{\mu\nu}^{(1)} = \lambda T_{\mu\nu}^{(1)} + 2\lambda T_{\mu\nu}^{(2)} - \lambda T_{\mu\nu}^{(3)}, \quad (59) \]

where \(T_{\mu\nu}^{(1)}\) is due to the variation of \(\sqrt{-g} (G^{\mu\nu} \partial_\mu \phi \partial_\nu \phi)\), \(T_{\mu\nu}^{(2)}\) is due to the variation of the quantity \(\sqrt{-g} (G^{\mu\nu} \partial_\mu \phi \partial_\nu \phi)\), whereas \(T_{\mu\nu}^{(3)}\) is due to the variation of \(\sqrt{-g} [-4(\partial_\mu \phi)^2 \Box \phi + 2 (\partial_\mu \phi)^4]\). The quantities are listed below:

\[ T_{\mu\nu}^{(1)} = 2 \left[ R \nabla_\mu \nabla_\nu \phi + 2G_{\mu\nu} \Box \phi + 2(R_{\mu\nu\lambda\sigma} - g_{\mu\nu}R_{\lambda\sigma}) \nabla^{\lambda\sigma} \phi - 4R(\lambda^{\mu} \nabla^{\nu} \sigma) \phi \right] - \phi H_{\mu\nu}, \quad (60) \]

where, \(H_{\mu\nu}\) is given by eqn. (12), and

\[ T_{\mu\nu}^{(2)} = G_{\mu\nu}(\nabla_\rho \phi)^2 + R\nabla_\mu \phi \nabla_\nu \phi - 4R(\nabla_\rho \phi \nabla_\nu \phi)\nabla_\rho \phi + 2(\nabla_\mu \nabla_\nu \phi) \Box \phi - 2(\nabla_\mu \nabla_\nu \phi)(\nabla_\rho \nabla_\sigma \phi) \nabla_\rho \nabla_\sigma \phi - g_{\mu\nu} \Box \phi \nabla_\rho \phi \nabla_\sigma \phi + 2g_{\mu\nu} \nabla_\rho \phi \nabla_\sigma \phi + 2g_{\mu\nu} \nabla_\rho \phi \nabla_\sigma \phi, \quad (61) \]

\[ T_{\mu\nu}^{(3)} = 4\nabla_\mu \phi \nabla_\nu \phi (\nabla_\rho \phi)^2 - g_{\mu\nu} \{ (\nabla_\rho \phi)^2 \}^2 - 4(\nabla_\mu \nabla_\nu \phi) \Box \phi - 4g_{\mu\nu} (\nabla_\rho \nabla_\sigma \phi)(\nabla_\rho \phi \nabla_\sigma \phi) + 8 \nabla_\rho (\nabla_\mu \phi \nabla_\nu \phi) \nabla_\rho \phi, \quad (62) \]

The equation of motion of the scalar field is obtained by variation over \(\phi\). This gives the following equation:

\[ \Box \phi \left[ (R/2) + \Box \phi + 8(\nabla_\rho \phi)^2 \right] = R_{\mu\nu} \{ \nabla^\mu \phi \nabla^\nu \phi + \nabla^\mu \nabla^\nu \phi \} + 2(\nabla_\mu \nabla_\nu \phi)(\nabla^\mu \phi)(\nabla^\nu \phi) + (\nabla_\mu \nabla_\nu \phi)^2 + \mathcal{G}, \quad (63) \]

The trace of the Einstein equation eqn. (59) leads to the following constraint:

\[ R + (\text{EoM of scalar field}) + (\lambda/2)\mathcal{G} = 0, \quad (64) \]

and on-shell, this leads to the equation relating the Ricci scalar and the GB scalars the Lagrangian of the theory.

**Appendix 2: Equations for shell collapse**

Here, we give the detailed expressions for the time of formation of the MTT for each collapsing shell, and the singularity formation time. These equations have been used in
the section 3.2, for plots of bounded collapse of matter configurations.

\[
t_c = \sqrt{2 \lambda} \left[ I_1 + I_2 + \frac{r^2 \tan^{-1} \left( \frac{r \pi}{2F \lambda} \right)}{2 \sqrt{2 F \lambda}} + \frac{I_3 I_{13} (-1)^{\frac{1}{2}} Q_3 \text{EllipticF} \left[ \tan^{-1} [I_6], I_7 \right]}{I_{15} I_{14}} \right. \\
- \frac{I_3}{I_{15} I_{12}} \left\{ I_4 + I_5 \left( -1 \right)^{\frac{1}{2}} + \left( -1 \right)^{\frac{3}{2}} \right\} \frac{\text{EllipticE} \left[ \tan^{-1} [I_6], I_7 \right]}{\left\{2 F \lambda \right\}/r - Q_3} \\
+ \frac{I_5 I_8 \text{EllipticF} \left[ \tan^{-1} [I_6], I_7 \right]}{I_9} - \frac{I_5 I_{10} \text{EllipticPi} \left[ I_{11}, \tan^{-1} [I_6], I_7 \right]}{\left\{2 F \lambda \right\}/r - Q_3} \\
+ \frac{I_3 I_{13} \left\{ 1 + (-1)^{\frac{1}{2}} \right\} Q_3 \text{EllipticPi} \left[ I_{11}, \tan^{-1} [I_6], I_7 \right]}{I_{14} I_{15}} \Bigg] - \left[ \cdots \right]_{R=r}, \quad (65)
\]

where \([\cdots]_{R=r}\) is the same expression as in the previous braces, however, now determined at \(R = r\).

\[
t_{\text{urr}} = \sqrt{2 \lambda} \left[ I_1 + I_2 + \frac{r^2 \tan^{-1} \left( \frac{r \pi}{2F \lambda} \right)}{2 \sqrt{2 F \lambda}} + \frac{I_3 I_{13} (-1)^{\frac{1}{2}} Q_3 \text{EllipticF} \left[ \tan^{-1} [I_6], I_7 \right]}{I_{15} I_{14}} \right. \\
- \frac{I_3}{I_{15} I_{12}} \left\{ I_4 + I_5 \left( -1 \right)^{\frac{1}{2}} + \left( -1 \right)^{\frac{3}{2}} \right\} \frac{\text{EllipticE} \left[ \tan^{-1} [I_6], I_7 \right]}{\left\{2 F \lambda \right\}/r - Q_3} \\
+ \frac{I_5 I_8 \text{EllipticF} \left[ \tan^{-1} [I_6], I_7 \right]}{I_9} - \frac{I_5 I_{10} \text{EllipticPi} \left[ I_{11}, \tan^{-1} [I_6], I_7 \right]}{\left\{2 F \lambda \right\}/r - Q_3} \\
+ \frac{I_3 I_{13} \left\{ 1 + (-1)^{\frac{1}{2}} \right\} Q_3 \text{EllipticPi} \left[ I_{11}, \tan^{-1} [I_6], I_7 \right]}{I_{14} I_{15}} \Bigg] \quad R_{\text{urr}} = \left[ \cdots \right]_{R=r}. \quad (66)
\]

The functions defined above, as Elliptic \cdots are the elliptic functions of various kinds \[\]. The variables of these functions are expressions given by:

\[
Q_1 = \left( x + \frac{2 F \lambda}{r} \right), \quad Q_2 = (4 F \lambda)^{1/3}, \quad Q_3 = (2 F^2 \lambda^2)^{1/3} \\
Q_4 = \sqrt{Q_2 - r}; \quad Q_5 = \sqrt{Q_2 + (-1)^{\frac{3}{2}} r}.
\]

The expressions \(I_i\) in the equations \(65\), and \(66\), are given by:

\[
I_1 = \frac{r \sqrt{x}}{2 Q_1}; \quad I_2 = -\frac{r \sqrt{x} \sqrt{2 F^2 \lambda^2 - Q_4^3}}{2 \sqrt{2 F \lambda} Q_1}; \quad I_3 = -\frac{r x \sqrt{Q_1^3 - 2 F^2 \lambda^2}}{2 Q_1 \left(4 F^2 \lambda^2 + Q_4^3\right)^{-\frac{1}{2}}} \\
I_4 = \frac{x \left( Q_1 + (-1)^{\frac{3}{2}} Q_3 \right)}{\left( Q_1 - (-1)^{\frac{3}{2}} Q_3 \right)}; \quad I_5 = \frac{(-1)^{\frac{3}{2}} \left( 1 + (-1)^{\frac{3}{2}} \right)}{\sqrt{2} 3^\frac{3}{2} Q_5 Q_2^{-1} Q_4^{-1}} \frac{\sqrt{Q_1 - (-1)^{\frac{3}{2}} Q_3}}{\left[ \frac{2 F \lambda}{r} + (-1)^{\frac{3}{2}} Q_3 \right]^{\frac{3}{2}}} \\
I_6 = \frac{Q_4 \sqrt{Q_1 + (-1)^{\frac{3}{2}} Q_3}}{Q_5 \sqrt{Q_1 - Q_3}}; \quad I_7 = \frac{-(-1)^{\frac{3}{2}} \left( -1 + (-1)^{\frac{3}{2}} \right) Q_5^2}{Q_4^2}.
\]
\[ I_8 = Q_3^2 \left[ (-1 + (-1)^{\frac{1}{2}}) - \frac{Q_2}{r} (1 + (-1)^{\frac{1}{2}}) \right] ; I_9 = -(1 + (-1)^{\frac{1}{2}}) Q_3^2 \left( 1 - \frac{Q_2}{r} \right) \]

\[ I_{10} = -\frac{2F \lambda}{r} ; I_{11} = \frac{Q_2^2}{Q_4^2} ; I_{12} = \sqrt{x} \sqrt{Q_3^3 - 2F^2 \lambda^2} \]

\[ I_{13} = \frac{(1)^{\frac{3}{2}} \sqrt{2x} (1 + (-1)^{\frac{1}{2}})}{3^{\frac{1}{2}} Q_5 F^{-1} \lambda^{-1} Q_4^{-1}} \frac{\sqrt{Q_1 - Q_3} \sqrt{Q_1 - (-1)^{\frac{1}{2}} Q_3}}{\left[ Q_1 + (-1)^{\frac{1}{2}} Q_3 \right]^{\frac{1}{2}}} \left[ \frac{2F \lambda}{r} + (-1)^{\frac{1}{2}} Q_3 \right]^{\frac{1}{2}} \]

\[ I_{14} = \sqrt{x} Q_3^3 (1 + (-1)^{\frac{1}{2}}) \left( 1 - \frac{Q_2}{r} \right) \sqrt{Q_1^3 - 2F^2 \lambda^2} \]

\[ I_{15} = \sqrt{2} \frac{F \lambda \sqrt{x}}{Q_1} \sqrt{2F^2 \lambda^2 - Q_1^3 (4F^2 \lambda^2 + Q_1^3)} \]

These equations have been heavily used to plot the MTTs for various matter field models.

6 Appendix 3: Matching conditions

The interior LTB metric of the spacetime \( M^- \) is given by \( \text{eqn. } [20] \). The metric of the external spacetime \( M^+ \) is \( \text{eqn. } [6] \). The matching is to be carried out at the timelike hypersurface \( \Sigma \) given by \( r_b \). Let us denote the coordinates on this surface \( \Sigma \) to be \( (\tau, \theta, \phi, \psi) \). From \( M^- \), we can write down the surface \( \Sigma \) as \( f(r, t) = r - r_b = 0 \), and hence, the induced metric on \( \Sigma \) is

\[ ds^2 = -d\tau^2 + r_b^2 d\Omega_2 . \]  

(67)

From the point of view of the exterior spacetime, the hypersurface may be described by \( r = \bar{R}_\Sigma(\tau) \) and \( t = T_\Sigma(\tau) \), with no change in the angular variables. The line element of the hypersurface is then given by

\[ ds^2 = -\left[ f(\bar{R}_\Sigma) \dot{T}_\Sigma^2 - f(\bar{R}_\Sigma)^{-1} \dot{R}_\Sigma^2 \right] d\tau^2 + \bar{R}_\Sigma(\tau)^2 d\Omega_2 , \]  

(68)

where the dots imply derivative with respect to \( \tau \).

The induced metric in equations in \( \text{eqns. } [67] \) and \( \text{eqns. } [68] \) must have matched metric functions. This implies that:

\[ f(\bar{R}_\Sigma) \dot{T}_\Sigma^2 - f(\bar{R}_\Sigma)^{-1} \dot{R}_\Sigma^2 = 1 \]  

(69)

Now, let \( u^\mu \) and \( n^\mu \) denote the velocity of the matter variables and the normal to the \( \Sigma \) respectively. They must satisfy the conditions \( u^\mu u_\mu = -1, n^\mu n_\mu = 1 \), whereas, \( u^\mu n_\mu = 0 \). From the interior spacetime, the expressions of these vectors is easily obtained:

\[ u^\mu = \delta_0^\mu \equiv (\partial_\tau)^\mu , \quad n_\mu = \frac{R'}{\sqrt{1 - k(r)}} (dr)_\mu . \]  

(70)

From the exterior spacetime, these vectors are also obtained similarly to give:

\[ u^\mu = \dot{T}_\Sigma (\partial_\tau)^\mu + \dot{R}_\Sigma (\partial_\tau)^\mu , \quad n_\mu = -\dot{R}_\Sigma (d\tau)_\mu + \dot{T}_\Sigma (dr)_\mu . \]  

(71)
The extrinsic curvatures are easily determined from these normals for the exterior as well the interior spacetimes:

\[ K_{\tau\tau} = 0, \quad K_{\theta\theta} = \bar{R}_\Sigma \sqrt{1 - k(r_b)} \]  \hspace{1cm} (72)

\[ K_{\tau\rho} = \hat{R}_\Sigma^{-1} \left[ \dot{f}(\bar{R}_\Sigma) \bar{T}_\Sigma + f(\bar{R}_\Sigma) \bar{T}_\Sigma \right], \quad K_{\theta\rho} = \bar{R}_\Sigma f(\bar{R}_\Sigma) \bar{T}_\Sigma. \]  \hspace{1cm} (73)

The \( K_{\theta\theta} \) equations imply the following relation:

\[ \frac{dT_\Sigma}{d\tau} = \sqrt{1 - k(r_b)} \frac{f(\bar{R}_\Sigma)}{f(\bar{R}_\Sigma)}, \] \hspace{1cm} (74)

whereas the equation (69) gives the following equation for the function \( \bar{R}_\Sigma \):

\[ \frac{d\bar{R}_\Sigma}{d\tau} = \left[ 1 - k(r_b) - f(\bar{R}_\Sigma) \right]^{1/2}. \] \hspace{1cm} (75)

This implies that the following relation hold good:

\[ \left( \frac{d\bar{R}_\Sigma}{d\tau} \right)^2 = -k(r_b) + \frac{\bar{R}_\Sigma^2}{4\lambda} \left[ 1 \mp \sqrt{1 + \frac{8\lambda M}{\bar{R}_\Sigma^3}} \right]. \] \hspace{1cm} (76)

A simple comparison with equation (39) implies that the condition \( 2M = F(r_b) \) must be satisfied at the boundary.

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