Composite Wavelet Transforms: Applications and Perspectives

Ilham A. Aliev, Boris Rubin, Sinem Sezer, and Simten B. Uyhan

Abstract. We introduce a new concept of the so-called composite wavelet transforms. These transforms are generated by two components, namely, a kernel function and a wavelet function (or a measure). The composite wavelet transforms and the relevant Calderón-type reproducing formulas constitute a unified approach to explicit inversion of the Riesz, Bessel, Flett, parabolic and some other operators of the potential type generated by ordinary (Euclidean) and generalized (Bessel) translations. This approach is exhibited in the paper. Another concern is application of the composite wavelet transforms to explicit inversion of the k-plane Radon transform on \(\mathbb{R}^n\). We also discuss in detail a series of open problems arising in wavelet analysis of \(L_p\)-functions of matrix argument.

Contents

1. Introduction.
2. Composite wavelet transforms for dilated kernels.
3. Wavelet transforms associated to one-parametric semigroups and inversion of potentials.
4. Wavelet transforms with the generalized translation operator.
5. Beta-semigroups.
6. Parabolic wavelet transforms.
7. Some applications to inversion of the k-plane Radon transform.
8. Higher-rank composite wavelet transforms and open problems.

References.

1. Introduction

Continuous wavelet transforms

\[ W_{f}(x, t) = t^{-n} \int_{\mathbb{R}^n} f(y) w \left( \frac{x - y}{t} \right) dy, \quad x \in \mathbb{R}^n, \quad t > 0, \]

2000 Mathematics Subject Classification. 42C40, 44A12, 47G10.

Key words and phrases. Wavelet transforms, potentials, semigroups, generalized translation, Radon transforms, inversion formulas, matrix spaces.

The research was supported by the Scientific Research Project Administration Unit of the Akdeniz University (Turkey) and TUBITAK (Turkey). The second author was also supported by the NSF grants EPS-0346411 (Louisiana Board of Regents) and DMS-0556157.
where \( w \) is an integrable radial function satisfying \( \int_{\mathbb{R}^n} w(x) dx = 0 \), have proved to be a powerful tool in analysis and applications. There is a vast literature on this subject (see, e.g., [Da], [HO], [M], just for few). Owing to the formula

\[
\int_0^\infty Wf(x,t) \frac{dt}{t^{1+\alpha}} = c_{\alpha,w}(-\Delta)^{\alpha/2} f(x), \quad \alpha \in \mathbb{C}, \quad \Delta = \sum_{k=1}^n \frac{\partial^2}{\partial x_k^2},
\]

that can be given precise meaning, continuous wavelet transforms enable us to resolve a variety of problems dealing with powers of differential operators. Such problems arise, e.g., in potential theory, fractional calculus, and integral geometry; see, [HO], [R1–R7], [Tr]. Dealing with functions of several variables, it is always tempting to reduce the dimension of the domain of the wavelet function \( w \) and find new tools to gain extra flexibility. This is actually a motivation for our article.

We introduce a new concept of the so-called \textit{composite wavelet transforms}. Loosely speaking, this is a class of wavelet-like transforms generated by two components, namely, a kernel function and a wavelet. Both are in our disposal. The first one depends on as many variables as we need for our problem. The second component, which is a wavelet function (or a measure), depends only on one variable. Such transforms are usually associated with one-parametric semigroups, like Poisson, Gauss-Weierstrass, or metaharmonic ones, and can be implemented to obtain explicit inversion formulas for diverse operators of the potential type and fractional integrals. These arise in integral geometry in a canonical way; see, e.g., [H1, R2, R6, R9].

In the present article we study different types of composite wavelet transforms in the framework of the \( L_p \)-theory and the relevant Fourier and Fourier-Bessel harmonic analysis. The main focus is reproducing formulas of Calderón’s type and explicit inversion of Riesz, Bessel, Flett, parabolic, and some other potentials. Apart of a brief review of recent developments in the area, the paper contains a series of new results. These include wavelet transforms for dilated kernels and wavelet transforms generated by Beta-semigroups associated to multiplication by \( \exp(-t|\xi|^\beta) \), \( \beta > 0 \), in terms of the Fourier transform. Such semigroups arise in the context of stable random processes in probability and enjoy a number of remarkable properties [Ko, La]. Special emphasis is made on detailed discussion of open problems arising in wavelet analysis of functions of matrix argument. Important results for \( L_2 \)-functions in this “higher-rank” set-up were obtained in [OOR] using the Fourier transform technique. The \( L_p \)-case for \( p \neq 2 \) is still mysterious. The main difficulties are related to correct definition and handling of admissible wavelet functions on the cone of positive definite symmetric matrices.

The paper is organized according to the Contents presented above.

2. Composite Wavelet Transforms for Dilated Kernels

2.1. Preliminaries. Let \( L_p \equiv L_p(\mathbb{R}^n) \), \( 1 \leq p < \infty \), be the standard space of functions with the norm

\[
\|f\|_p = \left( \int_{\mathbb{R}^n} |f(x)|^p dx \right)^{1/p} < \infty.
\]

For technical reasons, the notation \( L_{\infty} \) will be used for the space \( C_0 \equiv C_0(\mathbb{R}^n) \) of all continuous functions on \( \mathbb{R}^n \) vanishing at infinity. The Fourier transform of a
function $f$ on $\mathbb{R}^n$ is defined by
$$Ff(\xi) = \int_{\mathbb{R}^n} f(x) e^{ix \cdot \xi} dx, \quad x \cdot \xi = x_1 \xi_1 + \cdots + x_n \xi_n.$$ For $0 \leq a < b \leq \infty$, we write $\int_a^b f(\eta) d\mu(\eta)$ to denote the integral of the form $\int_{[a,b)} f(\eta) d\mu(\eta)$.

**Definition 2.1.** Let $q$ be a measurable function on $\mathbb{R}^n$ satisfying the following conditions:

(a) $q \in L_1 \cap L_r$ for some $r > 1$;
(b) the least radial decreasing majorant of $q$ is integrable, i.e.
$$\tilde{q}(x) = \sup_{|y| > |x|} |q(y)| \in L_1;$$
(c) $\int_{\mathbb{R}^n} q(x) dx = 1$.

We denote
(2.1) $q_t(x) = t^{-n} q(x/t)$, \quad $Q_t f(x) = (f * q_t)(x)$, \quad $t > 0$,
and set
(2.2) $Wf(x,t) = \int_0^\infty Q_{t\eta} f(x) d\mu(\eta),$
where $\mu$ is a finite Borel measure on $[0, \infty)$. If $\mu$ is a wavelet measure (i.e., $\mu$ has a certain number of vanishing moments and obeys suitable decay conditions) then (2.2) will be called the composite wavelet transform of $f$. The function $q$ will be called a kernel function and $Q_t$ a kernel operator of the composite transform $W$.

The integral (2.2) is well-defined for any function $f \in L_p$, and
$$||Wf(\cdot, t)||_p \leq ||\mu|| \cdot ||q||_1 \cdot ||f||_p,$$ where $||\mu|| = \int_{[0,\infty)} d|\mu|(\eta)$. We will also consider a more general weighted transform
(2.3) $W_a f(x, t) = \int_0^\infty Q_{t\eta} f(x) e^{-a t \eta} d\mu(\eta),$
where $a \geq 0$ is a fixed parameter.

The kernel function $q$, the wavelet measure $\mu$, and the parameter $a \geq 0$ are in our disposal. This feature makes the new transform convenient in applications.

**2.2. Calderón’s identity.** An analog of Calderón’s reproducing formula for $W_a f$ is given by the following theorem.

**Theorem 2.2.** Let $\mu$ be a finite Borel measure on $[0, \infty)$ satisfying
(2.4) $\mu([0, \infty)) = 0$ and $\int_0^\infty |\log \eta| d|\mu|(\eta) < \infty$.

If $f \in L_p$, $1 \leq p \leq \infty$ and
$$c_\mu = \int_0^\infty \log \frac{1}{\eta} d\mu(\eta),$$
then
(2.5) $\int_0^\infty W_a f(x, t) \frac{dt}{t} \equiv \lim_{\varepsilon \to 0} \int_\varepsilon^\infty W_a f(x, t) \frac{dt}{t} = c_\mu f(x)$

\[1\] We remind that $L_\infty$ is interpreted as the space $C_0$ with the uniform convergence.
where the limit exists in the $L_p$-norm and pointwise for almost all $x$. If $f \in C_0$, this limit is uniform on $\mathbb{R}^n$.

**Proof.** Consider the truncated integral

$$I_\varepsilon f(x) = \int_{\varepsilon}^{\infty} W_\alpha f(x, t) \frac{dt}{t}, \quad \varepsilon > 0.$$  

Our aim is to represent it in the form

$$I_\varepsilon f(x) = \int_{0}^{\infty} Q_{\varepsilon s} f(x) e^{-\alpha x} k(s) \, ds$$

where

$$k \in L_1(0, \infty) \quad \text{and} \quad \int_{0}^{\infty} k(s) ds = c_\mu.$$  

Once (2.7) is established, all the rest follows from properties (a)-(c) in Definition 2.1 according to the standard machinery of approximation to the identity; see [St].

Equality (2.7) can be formally obtained by changing the order of integration, namely,

$$I_\varepsilon f(x) = \int_{0}^{\infty} d\mu(\eta) \int_{\varepsilon}^{\infty} Q_{t\eta} f(x) e^{-at\eta} \frac{dt}{t}$$

$$= \int_{0}^{\infty} d\mu(\eta) \int_{0}^{\varepsilon} Q_{s\eta} f(x) e^{-as\eta} \frac{ds}{s}$$

$$= \int_{0}^{\infty} Q_{\varepsilon s} f(x) e^{-\alpha x} k(s) \, ds, \quad k(s) = s^{-1} \int_{0}^{s} d\mu(\eta).$$

Furthermore, since $\mu([0, \infty)) = 0$, then

$$\int_{0}^{\infty} |k(s)| ds = \int_{0}^{1} \left[ \int_{0}^{s} d\mu(\eta) \frac{ds}{s} \right] + \int_{1}^{\infty} \left[ \int_{1}^{s} d\mu(\eta) \frac{ds}{s} \right]$$

$$\leq \int_{0}^{1} d|\mu|(\eta) \int_{0}^{1} \frac{ds}{s} + \int_{1}^{\infty} d|\mu|(\eta) \int_{1}^{\infty} \frac{ds}{s}$$

$$= \int_{0}^{\infty} |\log \eta| d|\mu|(\eta) < \infty.$$  

Similarly we have

$$\int_{0}^{\infty} k(s) ds = \int_{0}^{\infty} \frac{1}{\eta} d\mu(\eta) = c_\mu,$$

which gives (2.8). Thus, to complete the proof, it remains to justify application of Fubini’s theorem leading to (2.7). To this end, it suffices to show that the repeated integral

$$\int_{\varepsilon}^{\infty} dt \int_{0}^{\infty} |Q_{t\eta} f(x)| \, d|\mu|(\eta)$$

is finite for almost all $x$ in $\mathbb{R}^n$. We write it as $A(x) + B(x)$, where

$$A(x) = \int_{\varepsilon}^{\infty} dt \int_{0}^{t} |Q_{t\eta} f(x)| \, d|\mu|(\eta), \quad B(x) = \int_{\varepsilon}^{\infty} dt \int_{t}^{\infty} |Q_{t\eta} f(x)| \, d|\mu|(\eta).$$

Since the least radial decreasing majorant of $q$ is integrable (see property (b) in Definition 2.1), then $\sup_{t > 0} |Q_{t\eta} f(x)| \leq c M_f(x)$ where $M_f(x)$ is the Hardy-Littlewood
maximal function, which is finite for almost $x$; see e.g., [St], Theorem 2, Section 2, Chapter III. Hence, for almost $x$,

$$A(x) \leq c_M f(x) \int_t^\infty \frac{dt}{t} \int_0^{1/t} d\mu(\eta) = c_M f(x) \int_0^{1/\varepsilon} \left( \log \frac{1}{\eta} - \log \varepsilon \right) d\mu(\eta) < \infty.$$ 

To estimate $B(x)$, we observe that since $q \in L_r$, $r > 1$, then, by Young’s inequality

$$||Q_t f||_s \leq ||f||_p ||q_t||_r = t^{-\delta} ||f||_p ||q||_r, \quad \delta = n(1 - 1/r) > 0, \quad \frac{1}{s} = \frac{1}{r} + \frac{1}{p} - 1.$$ 

This gives

$$\left\| \int_1^\infty |Q_t f(x)| d\mu(\eta) \right\|_s \leq t^{-\delta} ||f||_p \int_1^\infty \eta^{-\delta} d\mu(\eta),$$

and therefore,

$$||B||_s \leq ||f||_p \int_t^\infty \frac{dt}{t^{1+s}} \int_1^\infty \eta^{-\delta} d\mu(\eta) = ||f||_p \left( \int_0^{1/\varepsilon} d\mu(\eta) + \frac{1}{\varepsilon^\delta} \int_\varepsilon^\infty \eta^{-\delta} d\mu(\eta) \right) \leq \frac{||f||_p ||\mu||}{\delta} < \infty.$$ 

This completes the proof. \qed

3. Wavelet Transforms Associated to One-parametric Semigroups and Inversion of Potentials

In this section we consider an important subclass of wavelet transforms, generated by certain one-parametric semigroups of operators. Some composite wavelet transforms from the previous section belong to this subclass.

3.1. Basic examples.

**Example 3.1.** Consider the Poisson semigroup $P_t$ generated by the Poisson integral

$$P_t f(x) = \int_{\mathbb{R}^n} p(y, t) f(x - y) \, dy, \quad t > 0$$

with the Poisson kernel

$$p(y, t) = \frac{\Gamma((n + 1)/2)}{\pi^{(n+1)/2}} \frac{t}{(t^2 + |y|^2)^{(n+1)/2}} = t^{-n} p(y/t, 1);$$

see [SW2], [St]. In this specific case, the kernel function of the relevant composite wavelet transform is $q(x) \equiv p(x, 1)$ and on the Fourier transform side we have

$$F[P_t f](\xi) = e^{-t|\xi|} Ff(\xi).$$

**Example 3.2.** Another important example is the Gauss-Weierstrass semigroup $W_t$ defined by

$$W_t f(x) = \int_{\mathbb{R}^n} w(y, t) f(x - y) \, dy, \quad F[w(\cdot, t)](\xi) = e^{-t|\xi|^2}, \quad t > 0;$$

see [SW2]. The Gauss-Weierstrass kernel $w(y, t)$ is explicitly computed as

$$w(y, t) = (4\pi t)^{-n/2} \exp(-|y|^2/4t).$$
In comparison with [2.1], here the scaling parameter $t$ is replaced by $\sqrt{t}$, so that
\begin{equation}
(3.6) \quad w(y, t) = (\sqrt{t})^{-n}q(y/\sqrt{t}), \quad q(y) = w(y, 1) = (4\pi)^{-n/2}\exp(-|y|^2/4),
\end{equation}
and the corresponding wavelet transform has the form
\begin{equation}
(3.7) \quad Wf(x, t) = \int_{\mathbb{R}^n} \mathcal{W}_t f(x) e^{-atn} d\mu(\eta), \quad x \in \mathbb{R}^n, \ t > 0, \ a \geq 0.
\end{equation}
This agrees with [2.3] up to an obvious change of scaling parameters.

**Example 3.3.** The following interesting example does not fall into the scope of wavelet transforms in Section 2, however, it has a very close nature. Consider the metaharmonic semigroup $\mathcal{M}_t$ defined by
\begin{equation}
(3.8) \quad (\mathcal{M}_t f)(x) = \int_{\mathbb{R}^n} m(y, t) f(x-y) dy, \quad F[m(\cdot, t)](\xi) = e^{-t\sqrt{1+|\xi|^2}};
\end{equation}
see [R1] p. 257-258. The corresponding kernel has the form
\begin{equation}
(3.9) \quad m(y, t) = \frac{2t}{(2\pi)(n+1)/2} K_{(n+1)/2}(\sqrt{|y|^2 + t^2}),
\end{equation}
where $K_{(n+1)/2}(\cdot)$ is the McDonald function. The relevant wavelet transform is
\begin{equation}
(3.10) \quad Wf(x, t) = \int_{\mathbb{R}^n} \mathcal{M}_t f(x) d\mu(\eta), \quad x \in \mathbb{R}^n, \ t > 0.
\end{equation}
This list of examples can be continued [AR4].

**3.2. Operators of the potential type.** One of the most remarkable applications of wavelet transforms associated to the Poisson, Gauss-Weierstrass, and metaharmonic semigroups is that they pave the way to a series of explicit inversion formulas for operators of the potential type arising in analysis and mathematical physics. Typical examples of such operators are the following:
\begin{equation}
(3.11) \quad I^\alpha f = F^{-1}[|\xi|^{-\alpha} Ff \equiv (-\Delta)^{-\alpha/2} f \quad \text{(Riesz potentials)},
\end{equation}
\begin{equation}
(3.12) \quad J^\alpha f = F^{-1}(1 + |\xi|^2)^{-\alpha/2} Ff \equiv (E - \Delta)^{-\alpha/2} f \quad \text{(Bessel potentials)},
\end{equation}
\begin{equation}
(3.13) \quad F^\alpha f = F^{-1}(1 + |\xi|)^{-\alpha} Ff \equiv (E + \sqrt{\Delta})^{-\alpha} f \quad \text{(Flett potentials)}.
\end{equation}
Here $Re \alpha > 0$, $|\xi| = (\xi_1^2 + \cdots + \xi_n^2)^{1/2}$, $\Delta = \sum_{k=1}^n \frac{\partial^2}{\partial x_k}$ is the Laplacean, and $E$ is the identity operator. For $f \in L_p(\mathbb{R}^n)$, $1 \leq p < \infty$, these potentials have remarkable integral representations via the Poisson and Gauss-Weierstrass semigroups, namely,
\begin{equation}
(3.14) \quad I^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} \mathcal{P}_t f(x) dt, \quad 0 < Re \alpha < n/p;
\end{equation}
\begin{equation}
(3.15) \quad J^\alpha f(x) = \frac{1}{\Gamma(\alpha/2)} \int_0^\infty t^{\alpha/2-1} e^{-t} \mathcal{W}_t f(x) dt, \quad 0 < Re \alpha < \infty;
\end{equation}
\begin{equation}
(3.16) \quad F^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-t} \mathcal{P}_t f(x) dt, \quad 0 < Re \alpha < \infty;
\end{equation}
see [SW1], [R1], [F]. Regarding Flett potentials, see, in particular, [F] p. 446-447, [SKM] p. 541-542, [ASE]. We also mention another interesting representation of the Bessel potential, which is due to Lizorkin [L1] and employs the
metaharmonic semigroup, namely,

\begin{equation}
J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} \mathcal{M}_t f(x) \, dt, \quad 0 < \text{Re} \, \alpha < \infty.
\end{equation}

Equalities (3.14)-(3.17) have the same nature as classical Balakrishnan’s formulas for fractional powers of operators (see [SKM, p. 121]).

Let us show how these equalities generate wavelet inversion formulas for the corresponding potentials. The core of the method is the following statement which is a particular case of Lemma 1.3 from [R3].

**Lemma 3.4.** Given a finite Borel measure \( \mu \) on \([0, \infty)\) and a complex number \( \alpha, \alpha' = \text{Re} \, \alpha \geq 0 \), let

\begin{equation}
\lambda_\alpha(s) = s^{-1} I_{+}^{\alpha+1} \mu(s),
\end{equation}

where

\begin{equation}
I_{+}^{\alpha+1} \mu(s) = \frac{1}{\Gamma(\alpha+1)} \int_0^s (s-\eta)^\alpha d\mu(\eta)
\end{equation}

is the Riemann-Liouville fractional integral of order \( \alpha+1 \) of the measure \( \mu \). Suppose that \( \mu \) satisfies the following conditions:

\begin{equation}
\int_1^\infty \eta^\gamma d|\mu|(\eta) < \infty \quad \text{for some} \quad \gamma > \alpha';
\end{equation}

\begin{equation}
\int_0^\infty \eta^j d\mu(\eta) = 0 \quad \forall j = 0, 1, \ldots, \lfloor \text{Re} \, \alpha \rfloor \quad \text{(the integer part of} \ \alpha').
\end{equation}

Then

\begin{equation}
\lambda_\alpha(s) = \begin{cases} 
O(s^{\alpha'-1}), & \text{if} \quad 0 < s < 1, \\
O(s^{-1-\delta}) & \text{for some} \quad \delta > 0, \quad \text{if} \quad s > 1,
\end{cases}
\end{equation}

and

\begin{equation}
c_{\alpha,\mu} = \int_0^\infty \lambda_\alpha(s) \, ds = \int_0^\infty \frac{\hat{\mu}(t)}{t^{\alpha+1}} \, dt
\end{equation}

\begin{equation}
= \begin{cases} 
\Gamma(-\alpha) \int_0^\infty \eta^\alpha d\mu(\eta) & \text{if} \quad \alpha \notin \mathbb{N}_0 = \{0, 1, 2, \ldots\}, \\
(-1)^{\alpha+1} \alpha! \int_0^\infty \eta^\alpha \log \eta d\mu(\eta) & \text{if} \quad \alpha \in \mathbb{N}_0,
\end{cases}
\end{equation}

where \( \hat{\mu}(t) = \int_0^\infty e^{-t\eta} d\mu(\eta) \) is the Laplace transform of \( \mu \).

The estimate (3.22) is important in proving almost everywhere convergence in forthcoming inversion formulas.

Consider, for example, Flett potential (3.13), (3.16), and make use of the composite wavelet transform

\begin{equation}
W \varphi(x,t) = \int_0^\infty \mathcal{P}_{\eta} \varphi(x) e^{-t\eta} d\mu(\eta),
\end{equation}

cf. Example 3.1 and (2.3) with \( a = 1 \).
Let $f \in L^p$, $1 \leq p \leq \infty$, and let $\varphi = F^\alpha f$, $\alpha > 0$, be the Flett potentials of $f$. Suppose that $\mu$ is a finite Borel measure on $[0, \infty)$ satisfying (3.20) and (3.21). Then

$$\int_0^\infty W_{\mu} \varphi(x, t) \frac{dt}{t^{1+\alpha}} = \lim_{\varepsilon \to 0} \int_{\varepsilon}^\infty W_{\mu} \varphi(x, t) \frac{dt}{t^{1+\alpha}} = c_{\alpha, \mu} f(x),$$

where $c_{\alpha, \mu}$ is defined by (3.23) and the limit is interpreted in the $L^p$-norm and pointwise a.e. on $\mathbb{R}^n$. If $f \in C_0$, the statement remains true with the limit in (3.25) interpreted in the sup-norm.

**Proof.** We sketch the proof and address the reader to [ASE] for details. Changing the order of integration, owing to (3.24), (3.16), and the semigroup property of the Poisson integral, we get

$$W_{\varphi}(x, t) = \frac{1}{\Gamma(\alpha)} \int_0^\infty d\mu(\eta) \int_{t\eta}^{\infty} (\rho - t\eta)^{\alpha-1} e^{-\rho} P_{\rho} f(x) d\rho.$$  

Then further calculations give

$$\int_{\varepsilon}^\infty W_{\varphi}(x, t) \frac{dt}{t^{1+\alpha}} = \int_0^\infty e^{-\varepsilon s} P_{\varepsilon s} f(x) \lambda_{\alpha}(s) ds, \quad \lambda_{\alpha}(s) = s^{-1}s^{\alpha+1}\mu(s),$$

cf. (3.18). It remains to apply Lemma 3.4 combined with the standard machinery of approximation to the identity. \hfill \Box

**Potentials (3.11)-(3.13) and many others can be similarly inverted by making use of the wavelet transforms associated with suitable semigroups; see [AR4], [ASE].**

### 3.3. Examples of wavelet measures.

Examples of wavelet measures, that obey the conditions of Lemma 3.4 with $c_{\alpha, \mu} \neq 0$, are the following.

1. Fix an integer $m > Re\alpha$ and choose an even Schwartz function $h(\eta)$ on $\mathbb{R}^1$ so that

$$h^{(k)}(0) = 0 \quad \forall \ k = 0, 1, 2, \ldots, \quad \text{and} \quad \int_0^\infty \eta^{\alpha-m} h(\eta) d\eta \neq 0.$$  

One can take, for instance, $h(\eta) = \exp(-\eta^2 - 1/\eta^2)$, $h(0) = 0$. Set $d\mu(\eta) = h^{(m)}(\eta) d\eta$. It is not difficult to show that $\int_0^\infty \eta^k d\mu(\eta) = 0$, $\forall \ k = 0, 1, \ldots, |Re\alpha|$, and $c_{\alpha, \mu} \neq 0$.

2. Let $\mu = \sum_{j=0}^m \binom{m}{j} (-1)^j \delta_j$, where $m > Re\alpha$ is a fixed integer and $\delta_j = \delta_j(\eta)$ denotes the unit mass at the point $\eta = j$, i.e., $\langle \delta_j, f \rangle = f(j)$. It is known [SKM] p. 117], that

$$\int_0^\infty \eta^k d\mu(\eta) = \sum_{j=0}^m \binom{m}{j} \langle -1 \rangle^j k = 0, \quad \forall \ k = 0, 1, \ldots, m - 1 \quad (\text{we set } 0^0 = 1).$$

Moreover, $c_{\alpha, \mu} = \int_0^\infty t^{-\alpha-1} (1 - e^{-t})^m dt \neq 0$. 


4. Wavelet transforms with the generalized translation operator

Continuous wavelet transforms, studied in the previous sections, rely on the classical Fourier analysis on $\mathbb{R}^n$. Interesting modifications of these transforms and the corresponding potential operators arise in the framework of the Fourier-Bessel harmonic analysis associated to the Laplace-Bessel differential operator

(4.1) \[ \Delta_\nu = \sum_{k=1}^{n} \frac{\partial^2}{\partial x_k^2} + \frac{2\nu}{x_n} \frac{\partial}{\partial x_n}, \quad \nu > 0. \]

This analysis amounts to pioneering works by Delsarte [De] and Levitan [Le], and was extensively developed in subsequent publications; see [Kî, Tr, AR3], and references therein.

Let $\mathbb{R}^+_n = \{ x : x = (x_1, \ldots, x_n) \in \mathbb{R}^n, \ x_n > 0 \}$ and $x' = (x_1, \ldots, x_{n-1})$. Denote

\[ L_{p,\nu}(\mathbb{R}^+_n) = \left\{ f : ||f||_{p,\nu} = \left( \int_{\mathbb{R}^+_n} |f(x)|^p x_n^{2\nu} \, dx \right)^{1/p} < \infty \right\}. \]

The Fourier-Bessel harmonic analysis is adopted to the generalized convolutions

(4.2) \[ (f \ast g)(x) = \int_{\mathbb{R}^+_n} f(y)(T^y g)(x) y_n^{2\nu} \, dy, \quad x \in \mathbb{R}_+^n, \]

with the generalized translation operator

(4.3) \[ (T^y f)(x) = \frac{\Gamma(\nu + 1/2)}{\Gamma(\nu)\Gamma(1/2)} \int_0^\pi f(x' - y', \sqrt{x_n^2 - 2x_ny_n\cos \alpha + y_n^2}) \sin^{2\nu-1} \alpha \, d\alpha, \]

[Kî, Le, Tr]. The Fourier-Bessel transform $F_\nu$, for which $F_\nu (f \ast g) = F_\nu (f) F_\nu (g)$, is defined by

(4.4) \[ (F_\nu f)(\xi) = \int_{\mathbb{R}^+_n} f(x) e^{i\xi \cdot x'} j_{\nu-1/2}(\xi x_n) x_n^{2\nu} \, dx, \quad \xi \in \mathbb{R}^n_+. \]

Here \( j_{\lambda}(\tau) = 2^\lambda \Gamma(\lambda + 1) \tau^{-\lambda} J_\lambda(\tau) \), where $J_\lambda(\tau)$ is the Bessel function of the first kind. The generalized Gauss-Weierstrass, Poisson, and metaharmonic semigroups \{\( \mathcal{G}_t^{(\nu)} \), \{\( \mathcal{P}_t^{(\nu)} \), \{\( \mathcal{M}_t^{(\nu)} \)\} are defined as follows:

(4.5) \[ (\mathcal{G}_t^{(\nu)} f)(x) = \int_{\mathbb{R}^+_n} w^{(\nu)}(y, t)(T^y f)(x) y_n^{2\nu} \, dy, \]

\[ F_\nu[w^{(\nu)}(\cdot, t)][\xi] = e^{-t|\xi|^2}; \]

(4.6) \[ (\mathcal{P}_t^{(\nu)} f)(x) = \int_{\mathbb{R}^+_n} p^{(\nu)}(y, t)(T^y f)(x) y_n^{2\nu} \, dy, \]

\[ F_\nu[p^{(\nu)}(\cdot, t)][\xi] = e^{-t|\xi|}; \]

(4.7) \[ (\mathcal{M}_t^{(\nu)} f)(x) = \int_{\mathbb{R}^+_n} m^{(\nu)}(y, t)(T^y f)(x) y_n^{2\nu} \, dy, \]

\[ F_\nu[m^{(\nu)}(\cdot, t)][\xi] = e^{-t\sqrt{1+|\xi|^2}}. \]
The corresponding kernels \( w^{(\nu)}(y, t) \), \( p^{(\nu)}(y, t) \), and \( m^{(\nu)}(y, t) \) have the form

\[
(4.8) \quad w^{(\nu)}(y, t) = \frac{2\pi^{\nu+1/2}}{\Gamma(\nu + 1/2)} (4\pi t)^{-(n+2\nu)/2} e^{-|y|^2/4t},
\]

\[
(4.9) \quad p^{(\nu)}(y, t) = \frac{2\Gamma((n + 2\nu + 1)/2)}{\pi^{n/2}(\nu + 1/2)} \frac{t}{(|y|^2 + t^2)^{(n+2\nu+1)/2}},
\]

\[
(4.10) \quad m^{(\nu)}(y, t) = \frac{2^{-\nu+3/2} t}{\Gamma(\nu + 1/2)(2\pi)^{n/2}} K_{(n+2\nu+1)/2}(\sqrt{|y|^2 + t^2})
\]

More information about these semigroups and their modifications

\[
\{e^{-t}W^{(\nu)}_t\}, \quad \{e^{-t}P^{(\nu)}_t\}, \quad \{e^{-t}M^{(\nu)}_t\},
\]

can be found in [AB1, AB2, CA].

Modified Riesz, Bessel, and Flett potentials with the generalized translation operator \((4.3)\) are formally defined in terms of the Fourier-Bessel transform by

\[
(4.11) \quad I^\alpha f = F^{-1}_\nu |\xi|^{-\alpha} F_\nu f = (-\Delta_\nu)^{-\alpha/2} f,
\]

\[
(4.12) \quad J^{\alpha}_\nu f = F^{-1}_\nu (1 + |\xi|^2)^{-\alpha/2} F_\nu f = (E -\Delta_\nu)^{-\alpha/2} f,
\]

\[
(4.13) \quad \mathcal{F}^{\alpha}_\nu f = F^{-1}_\nu (1 + |\xi|)^{-\alpha} F_\nu f = \left(E + \sqrt{-\Delta_\nu}\right)^{-\alpha} f,
\]

respectively. Here \( Re \alpha > 0 \) and \( \Delta_\nu \) is the Laplace-Bessel differential operator \((4.1)\). These generalized potentials have analogous to \((3.14)-(3.16)\) representations in terms of the semigroups \((4.3)-(4.7)\), namely, if \( f \in L_{p, \nu}(\mathbb{R}^n_+) \) then

\[
(4.14) \quad I^\alpha_\nu f(x) = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} P^{(\nu)}_t f(x) \, dt, \quad 0 < Re \alpha < (n + 2\nu)/p,
\]

\[
(4.15) \quad J^{\alpha}_\nu f(x) = \frac{1}{\Gamma(\alpha/2)} \int_0^\infty t^{\alpha/2-1} e^{-t} W^{(\nu)}_t f(x) \, dt, \quad 0 < Re \alpha < \infty,
\]

\[
(4.16) \quad \mathcal{F}^{\alpha}_\nu f(x) = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-t} P^{(\nu)}_t f(x) \, dt, \quad 0 < Re \alpha < \infty.
\]

Moreover,

\[
(4.17) \quad J^{\alpha}_\nu f(x) = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} M^{(\nu)}_t f(x) \, dt, \quad 0 < Re \alpha < \infty.
\]

We denote by \( S^{(\nu)}_t \) any of the semigroups

\[
(4.18) \quad W^{(\nu)}_t, \quad e^{-t} W^{(\nu)}_t, \quad P^{(\nu)}_t, \quad e^{-t} P^{(\nu)}_t, \quad M^{(\nu)}_t, \quad e^{-t} M^{(\nu)}_t,
\]

and define the relevant wavelet transform (cf. \((2.2)\))

\[
(4.19) \quad S^{(\nu)} f(x, t) = \int_0^\infty S^{(\nu)}_t f(x) \, d\mu(\eta), \quad t > 0,
\]

generated by a finite Borel measure \( \mu \) on \([0, \infty)\).

There exist analogs of Calderón’s reproducing formula for wavelet transforms \((4.19)\) of functions belonging to the weighted space \( L_{p, \nu}(\mathbb{R}^n_+) \) and inversion formulas for potentials \( I^\alpha f, \ J^{\alpha}_\nu f, \ \mathcal{F}^{\alpha}_\nu f \), when \( f \in L_{p, \nu}(\mathbb{R}^n_+) \). For example, the following statement holds.
5. Beta-semigroups

We remind basic formulas from Section 3.1 for the kernels of the Poisson and Gauss-Weierstrass semigroups:

\begin{align}
F[p(\cdot, t)](\xi) &= e^{-t|\xi|}, \quad p(y, t) = \frac{\Gamma \left( \frac{n+1}{2} \right)}{\pi^{(n+1)/2}} \frac{t}{(t^2 + |y|^2)^{(n+1)/2}}, \\
F[w(\cdot, t)](\xi) &= e^{-t|\xi|^2}, \quad w(y, t) = (4\pi t)^{-n/2} \exp(-|y|^2/4t).
\end{align}

It would be natural to consider a more general semigroup generated by the kernel $w^{(\beta)}(y, t)$ defined by

\begin{equation}
F[w^{(\beta)}(\cdot, t)](\xi) = e^{-t|\xi|^\beta}, \quad \beta > 0.
\end{equation}

This semigroup arises in diverse contexts of analysis, integral geometry, and probability; see, e.g., [Fe], [Ko], [La], [R8]. Unlike (5.1) and (5.2), the kernel function $w^{(\beta)}(y, t)$ cannot be computed explicitly, however, by taking into account that

\begin{equation}
w^{(\beta)}(y, t) = t^{-n/\beta}w^{(\beta)}(t^{-1/\beta}y), \quad w^{(\beta)}(y) \equiv w^{(\beta)}(y, 1),
\end{equation}

properties of $w^{(\beta)}(y, t)$ are well determined by the following lemma.

**Lemma 5.1.** The function

\begin{equation}
w^{(\beta)}(y) = F^{-1}[e^{-|\cdot|^\beta}](y) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{-|\xi|^\beta} e^{i\xi \cdot y} d\xi, \quad \beta > 0,
\end{equation}

is uniformly continuous on $\mathbb{R}^n$. If $\beta$ is an even integer, then $w^{(\beta)}(y)$ is infinitely smooth and rapidly decreasing. More generally, if $\beta \neq 2, 4, \ldots$, then $w^{(\beta)}(y)$ has the following behavior when $|y| \to \infty$:

\begin{equation}
w^{(\beta)}(y) = c_\beta |y|^{-n-\beta}(1 + o(|y|)), \quad c_\beta = -\frac{2^\beta \pi^{-n/2} \Gamma((n + \beta)/2)}{\Gamma(-\beta/2)}.
\end{equation}

If $0 < \beta \leq 2$, then $w^{(\beta)}(y) > 0$ for all $y \in \mathbb{R}^n$.

**Proof.** (Cf. [Ko] p. 44, for $n = 1$). The uniform continuity of $w^{(\beta)}(y)$ follows immediately from (5.5). Note that if $\beta$ is an even integer, then $e^{-|\cdot|^\beta}$ is a Schwartz function and therefore, $w^{(\beta)}(y)$ is infinitely smooth and rapidly decreasing. Let us prove positivity of $w^{(\beta)}(y)$ when $0 < \beta \leq 2$. For $y = 0$ and for the cases $\beta = 1$ and $\beta = 2$, this is obvious. Let $0 < \beta < 2$. By Bernstein’s theorem [Fe]...
Then the equality
\[ e^{-|\xi|^\alpha} = \int_0^\infty e^{-t|\xi|^2} \, d\mu_\beta(t). \]  
(5.7)

Then the equality
\[ [e^{-|\xi|^2}]^\alpha(y) = \pi^{n/2} t^{-n/2} e^{-|y|^2/4t}, \quad t > 0, \]
yields
\[ (2\pi)^n u^{(\beta)}(y) = \int_{\mathbb{R}^n} e^{i\xi \cdot y} \, d\xi \int_0^\infty e^{-t|\xi|^2} \, d\mu_\beta(t) = \int_{\mathbb{R}^n} \, d\mu_\beta(t) \int_0^\infty e^{i\xi \cdot y} e^{-t|\xi|^2} \, d\xi \]
\[ = \pi^{n/2} \int_0^\infty t^{-n/2} e^{-|y|^2/4t} \, d\mu_\beta(t) > 0. \]

The Fubini theorem is applicable here, because, by (5.7),
\[ \int_{\mathbb{R}^n} |e^{i\xi \cdot y}| \, d\xi \int_0^\infty e^{-t|\xi|^2} \, d\mu_\beta(t) = \int_{\mathbb{R}^n} e^{-|\xi|^\alpha} \, d\xi < \infty. \]

Let us prove (5.6). It suffices to show that
\[ \lim_{|y| \to \infty} |y|^{n+\beta} u^{(\beta)}(y) = 2^\beta \pi^{-n/2-1} \Gamma(1 + \beta/2) \Gamma((n + \beta)/2) \sin(\pi \beta/2) \]
(5.9)
(we leave to the reader to check that the right-hand side coincides with \( c_\beta \)). For \( n = 1 \), this statement can be found in [PS Chapter 3, Problem 154] and in [Ko p. 45]. In the general case, the proof is more sophisticated and relies on the properties of Bessel functions. By the well-known formula for the Fourier transform of a radial function (see, e.g., [SW2]), we write \((2\pi)^n u^{(\beta)}(y) = I(|y|)\), where
\[ I(s) = (2\pi)^{n/2} s^{1-n/2} \int_0^\infty e^{-r^2 s} J_{n/2-1}(rs) \, dr \]
\[ = (2\pi)^{n/2} s^{-n} \int_0^\infty e^{-r^2 s} \frac{d}{dr} [(rs)^{n/2} J_{n/2}(rs)] \, dr. \]
Integration by parts yields
\[ I(s) = \beta (2\pi)^{n/2} s^{-n/2} \int_0^\infty e^{-r^2 s} r^{n/2+\beta-1} J_{n/2}(rs) \, dr. \]
Changing variable \( z = s^{\beta} r, \) we obtain
\[ s^{n+\beta} I(s) = (2\pi)^{n/2} A(s^{-\beta}), \quad A(\delta) = \int_0^\infty e^{-z^2 s^{n/2}} J_{n/2}(z^{1/\beta}) \, dz. \]

We actually have to compute the limit \( A_0 = \lim_{\delta \to 0} A(\delta). \) To this end, we invoke Hankel functions \( H^{(1)}_\nu(z) \), so that \( J_\nu(z) = \Re H^{(1)}_\nu(z) \) if \( z \) is real \([F] \). Let \( h_\nu(z) = z^\nu H^{(1)}_\nu(z) \). This is a single-valued analytic function in the \( z \)-plane with cut \((-\infty, 0]\). Using the properties of the Hankel functions \([F] \), we get
\[ \lim_{z \to 0} h_\nu(z) = 2\nu \Gamma(\nu)/\pi i, \]
(5.10)
\[ h_\nu(z) \to \sqrt{2\pi} z^{-\nu-1/2} e^{i\pi \nu/2} (\nu + 1/2), \quad z \to \infty. \]
(5.11)

Then we write \( A(\delta) \) as \( A(\delta) = \Re \int_0^\infty e^{-z^2} h_{n/2}(z^{1/\beta}) \, dz \) and change the line of integration from \([0, \infty)\) to \( n_\theta = \{ z : z = re^{i\theta}, \ r > 0 \} \) for small \( \theta < \pi \beta/2. \) By Cauchy’s
theorem, owing to \[ \text{(7.10)} \] and \[ \text{(5.11)} \], we obtain \( A(\delta) = \Re \int_{n_{\alpha}} e^{-z^\delta} h_{n/2}(z^{1/\beta}) \, dz \). Since for \( z = re^{i\theta} \), \( h_{n/2}(z^{1/\beta}) = O(1) \) when \( r = |z| \to 0 \) and \( h_{n/2}(z^{1/\beta}) = O(r^{(n-1)/2\beta} e^{-r^{1/\beta} \sin(\theta/\beta)}) \) as \( r \to \infty \), by the Lebesgue theorem on dominated convergence, we get \( A_0 = \Re \int_{n_{\alpha}} h_{n/2}(z^{1/\beta}) \, dz \). To evaluate the last integral, we again use analyticity and replace \( n_{\beta} \) by \( n_{\pi/2} = \{z : z = re^{i\pi/2}, r > 0\} \) to get

\[
A_0 = \Re \left[ e^{i\pi/2} \int_0^\infty h_{n/2}(r^{1/\beta} e^{i\pi/2}) \, dr \right].
\]

To finalize calculations, we invoke McDonald’s function \( K_\nu(z) \) so that

\[
h_\nu(z) = z^\nu H_\nu^{(1)}(z) = -\frac{2i}{\pi} (ze^{-i\pi/2})^\nu K_\nu(z e^{-i\pi/2}).
\]

This gives

\[
A_0 = \frac{2\beta}{\pi} \sin(\pi/2) \int_0^\infty \frac{e^{s^{1/2+\beta} - s}}{\sin(\pi\beta/2)} ds.
\]

The last integral can be explicitly evaluated by the formula 2.16.2 (2) from [PBM], and we obtain the result.

The Beta-semigroup \( B_t \) generated by the kernel \( w(\beta)(y, t) \) (see \[ \text{(5.1)} \]) is defined by

\[
B_t f(x) = \int_{\mathbb{R}^n} w(\beta)(y, t) f(x - y) \, dy,
\]

and the corresponding weighted wavelet transform has the form

\[
W_{\alpha} f(x, t) = \int_0^\infty B_{t^{\alpha}} f(x) e^{-a t^n} d\mu(\eta),
\]

where \( a \geq 0 \) is a fixed number which is in our disposal; cf \[ \text{(5.4)} \]. Following [AI], we introduce Beta-potentials

\[
J_\beta^\nu f = (E + (-\Delta^{1/2}))^{-\alpha/\beta} f, \quad \alpha > 0, \quad \beta > 0,
\]

that can be realized through the Beta-semigroup as

\[
J_\beta^\nu f(x) = \frac{1}{\Gamma(\alpha/\beta)} \int_0^\infty t^{\alpha/\beta - 1} e^{-t} B_t f(x) \, dt.
\]

For \( \beta = 2 \), \[ \text{(5.14)} \] coincides with the classical Bessel potential \[ \text{(3.12)} \], and \[ \text{(5.15)} \] mimics \[ \text{(3.16)} \]. Similarly, for \( \beta = 1 \), the Beta-potentials coincide with the Flett potential \[ \text{(5.10)} \].

Explicit inversion formulas for Beta-potentials can be obtained with the aid of the wavelet transform \[ \text{(5.13)} \] as follows.

**Theorem 5.2.** Let \( f \in L_p(\mathbb{R}^n) \), \( 1 \leq p < \infty \), \( \alpha > 0 \), \( \beta > 0 \). Suppose that \( \mu \) is a finite Borel measure on \([0, \infty)\) satisfying

(a) \( \int_0^\infty \eta^\gamma \, d|\mu|(\eta) < \infty \) for some \( \gamma > \alpha/\beta \);

(b) \( \int_0^\infty \eta^\gamma \, d\mu(\eta) = 0 \), \( \forall j = 0, 1, ..., [\alpha/\beta] \).

If \( \varphi = J_\beta^\nu f \), then

\[
\int_0^\infty W\varphi(x, t) \frac{dt}{t^{1+\alpha/\beta}} = \lim_{\varepsilon \to 0} \int_\varepsilon^\infty W\varphi(x, t) \frac{dt}{t^{1+\alpha/\beta}} = c_{\alpha/\beta, \mu} f(x),
\]

\( \varepsilon \to 0 \).
where $c_{\alpha/\beta,\mu}$ is defined by (6.25) (with $\alpha$ replaced by $\alpha/\beta$). The limit in (5.17) exists in the $L_p$-norm and pointwise for almost all $x$. If $f \in C_0$, the convergence is uniform.

The proof of this theorem mimics that of Theorem 5.3; see [AI] for details.

**Remark 5.3.** The classical Riesz potential $I^\alpha f$ has an integral representation via the Beta-semigroup, namely,

$$I^\alpha f(x) = \frac{1}{\Gamma(\alpha/\beta)} \int_0^\infty t^{\alpha/\beta - 1} B_t f(x) \, dt.$$  

(Here $f \in L_p(\mathbb{R}^n)$, $1 \leq p < \infty$, and $0 < \Re \alpha < n/p$. For the cases $\beta = 1$ and $\beta = 2$ we have the representations in terms of the Poisson and Gauss-Weierstrass semigroups, respectively.

The potential $I^\alpha f$ can be inverted in the framework of the $L_p$-theory by making use of (5.17) and the composite wavelet transform (5.13) with $a = 0$.

6. Parabolic Wavelet Transforms

The following anisotropic wavelet transforms of the composite type, associated with the heat operators

$$\partial/\partial t - \Delta, \quad E + \partial/\partial t - \Delta,$$

were introduced by Aliev and Rubin [AR2]. These transforms are constructed using the Gauss-Weierstrass kernel $w(y, t) = (4\pi t)^{-n/2} e^{-\|y\|^2/(4t)}$ as follows. Let $\mathbb{R}^{n+1}$ be the $(n+1)$-dimensional Euclidean space of points $(x, t)$, $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $t \in \mathbb{R}$. We pick up a wavelet measure $\mu$ on $[0, \infty)$, a scaling parameter $a > 0$, and set

$$P_\mu f(x, t; a) = \int_{\mathbb{R}^n \times (0, \infty)} f(x - \sqrt{a}y, t - a\tau) w(y, \tau) \, dy d\mu(\tau),$$

and

$$P_\mu f(x, t; a) = \int_{\mathbb{R}^n \times (0, \infty)} f(x - a\tau, t - a\tau) w(y, \tau) e^{-a\tau} \, dy d\mu(\tau),$$

(to simplify the notation, without loss of generality we can assume $\mu(\{0\}) = 0$).

We call (6.2) and (6.3) the parabolic wavelet transform and the weighted parabolic wavelet transform, respectively.

Parabolic potentials $H^\alpha f$ and $H^\alpha f$, associated to differential operators in (6.1), are defined in the Fourier terms by

$$F[H^\alpha f](\xi, \tau) = (|\xi|^2 + i\tau)^{-\alpha/2} F[f](\xi, \tau),$$

and

$$F[H^\alpha f](\xi, \tau) = (1 + |\xi|^2 + i\tau)^{-\alpha/2} F[f](\xi, \tau),$$

where $F$ stands for the Fourier transform in $\mathbb{R}^{n+1}$. These potentials were introduced by Jones [Jo] and Sampson [Sa] and used as a tool for characterization of anisotropic function spaces of fractional smoothness; see [AR2] and references therein. For $\alpha > 0$, potentials $H^\alpha f$ and $H^\alpha f$ are representable by the integrals

$$H^\alpha f(x, t) = \frac{1}{\Gamma(\alpha/2)} \int_{\mathbb{R}^n \times (0, \infty)} \tau^{\alpha/2 - 1} w(y, \tau) f(x - y, t - \tau) \, dy d\tau,$$

and

$$H^\alpha f(x, t) = \frac{1}{\Gamma(\alpha/2)} \int_{\mathbb{R}^n \times (0, \infty)} \tau^{\alpha/2 - 1} e^{-\tau} w(y, \tau) f(x - y, t - \tau) \, dy d\tau.$$
Their behavior on functions \( f \in L_p \equiv L_p(\mathbb{R}^{n+1}) \) is characterized by the following theorem.

**Theorem 6.1.** \([Ba], [Ra]\)

I. Let \( f \in L_p, 1 \leq p < \infty, 0 < \alpha < (n + 2)/p, \quad q = (n + 2 - \alpha)p^{-1}(n + 2)p. \)
   (a) The integral \((H^\alpha f)(x, t)\) converges absolutely for almost all \((x, t) \in \mathbb{R}^{n+1}.\)
   (b) For \( p > 1, \) the operator \( H^\alpha \) is bounded from \( L_p \) into \( L_q. \)
   (c) For \( p = 1, \) \( H^\alpha \) is an operator of the weak \((1, q)\) type:
   \[
   |\{(x, t) : |(H^\alpha f)(x, t)| > \gamma\}| \leq \left(\frac{\|f\|}{\gamma}\right)^q.
   \]

II. The operator \( H^\alpha \) is bounded on \( L_p \) for all \( \alpha \geq 0, \quad 1 \leq p \leq \infty. \)

Explicit inversion formulas for parabolic potentials in terms of wavelet transforms \([02]\) and \([03]\) are given by the following theorem.

**Theorem 6.2.** \([AR2]\) Let \( \mu \) be a finite Borel measure on \([0, \infty)\) satisfying the following conditions:

\[
\int_0^\infty \tau^\gamma d\mu(\tau) < \infty \quad \text{for some} \quad \gamma > \alpha/2;
\]

\[
\int_0^\infty \tau^j d\mu(\tau) = 0, \quad \forall j = 0, \ldots, \lfloor \alpha/2 \rfloor.
\]

Suppose that \( \varphi = H^\alpha f, \quad f \in L_p, \quad 1 \leq p < \infty, \quad 0 < \alpha < (n + 2)/p. \) Then

\[
\int_0^\infty P_{\alpha}(x, t; a) \frac{da}{a^{1+\alpha/2}} \equiv \lim_{\varepsilon \to 0} \int_0^\infty (\ldots) = c_{\alpha/2, \mu} f(x, t),
\]

where \( c_{\alpha/2, \mu} \) is defined by \([3.23]\) (with \( \alpha \) replaced by \( \alpha/2).\)

The limit in \((6.10)\) is interpreted in the \( L_p\)-norm for \( 1 \leq p < \infty \) and a.e. on \( \mathbb{R}^{n+1}\) for \( 1 < p < \infty.\)

The same statement holds for all \( \alpha > 0 \) and \( 1 \leq p \leq \infty \) \((L_\infty \) is identified with \( C_0)\) provided that \( H^\alpha \) and \( P_{\alpha} \) are replaced by \( H^\alpha \) and \( P_{\mu}, \) respectively.

More general results for parabolic wavelet transforms with the generalized translation associated to singular heat operators

\[
\partial/\partial t - \Delta_{\nu}, \quad E + \partial/\partial t - \Delta_{\nu}, \quad \left(\Delta_{\nu} = \sum_{k=1}^{n} \frac{\partial^2}{\partial x_k^2} + \frac{2\nu}{x_n} \frac{\partial}{\partial x_n}\right),
\]

were obtained in \([AR1].\) These include the Calderón-type reproducing formula and explicit \( L_\nu\)-inversion formulas for parabolic potentials with the generalized translation defined by

\[
H_{\nu}^\alpha f(x, t) = F_{\nu}^{-1}[(|x|^2 + it)^{-\alpha/2} F_{\nu} f(x, t)],
\]

\[
\mathcal{H}_{\nu}^\alpha f(x, t) = F_{\nu}^{-1}[(1 + |x|^2 + it)^{-\alpha/2} F_{\nu} f(x, t)].
\]

In the last two expressions, \( x \in \mathbb{R}_+^n \equiv \{x \in \mathbb{R}^n : x_n > 0\}, \quad t \in \mathbb{R}, \) and \( F_{\nu} \) is the Fourier-Bessel transform, i.e., the Fourier transform with respect to the variables \( t \) and \( x' = (x_1, \ldots, x_{n-1}); \) and the Bessel transform with respect to \( x_n > 0.\)

These results were applied in \([AR1], [AR2]\) to wavelet-type characterization of the parabolic Lebesgue spaces.
7. Some Applications to Inversion of the $k$-plane Radon Transform

We recall some basic definitions. More information can be found in [GCG, H, E, R4, R5]. Let $G_{n,k}$ and $G_{n,k}$ be the affine Grassmann manifold of all non-oriented $k$-dimensional planes ($k$-planes) $\tau$ in $\mathbb{R}^n$ and the ordinary Grassmann manifold of $k$-dimensional linear subspaces $\zeta$ of $\mathbb{R}^n$, respectively. Each $k$-plane $\tau \in G_{n,k}$ is parameterized as $\tau = (\zeta, u)$, where $\zeta \in G_{n,k}$ and $u \in \zeta^\perp$ (the orthogonal complement of $\zeta$ in $\mathbb{R}^n$). We endow $G_{n,k}$ with the product measure $d\tau = d\zeta du$, where $d\zeta$ is the $O(n)$-invariant measure on $G_{n,k}$ of total mass 1, and $du$ denotes the Euclidean volume element on $\zeta^\perp$. The $k$-plane Radon transform of a function $f$ on $\mathbb{R}^n$ is defined by

$$\hat{f}(\tau) = \hat{f}(\zeta, u) = \int_{\zeta} f(y + u) \, dy,$$

(7.1)

where $dy$ is the induced Lebesgue measure on the subspace $\zeta \in G_{n,k}$. This transform assigns to a function $f$ a collection of integrals of $f$ over all $k$-planes in $\mathbb{R}^n$. The corresponding dual $k$-plane transform of a function $\varphi$ on $G_{n,k}$ is defined as the mean value of $\varphi(\tau)$ over all $k$-planes $\tau$ through $x \in \mathbb{R}^n$:

$$\check{\varphi}(x) = \int_{O(n)} \varphi(\sigma_0 + x) \, d\sigma, \quad x \in \mathbb{R}^n.$$

(7.2)

Here $\zeta_0 \in G_{n,k}$ is an arbitrary fixed $k$-plane through the origin. If $f \in L_p(\mathbb{R}^n)$, then $\check{f}$ is finite a.e. on $G_{n,k}$ if and only if $1 \leq p < n/k$.

Several inversion procedures are known for $\check{f}$. One of the most popular, which amounts to Blaschke and Radon, relies on the Fuglede formula [H, p. 29],

$$\langle \check{f} \rangle^\vee = d_{k,n}^\vee f, \quad d_{k,n} = (2\pi)^k \sigma_{n-k-1}/\sigma_{n-1},$$

(7.3)

and reduces reconstruction of $f$ to inversion of the Riesz potentials $I^k f$. The latter can also be inverted in many number of ways [S, SKM, R1]. In view of considerations in Section 3.2 and 5, one can employ a composite wavelet transform generated by the Poisson, Gauss-Weierstrass, or Beta semigroup and thus obtain new inversion formulas for the $k$-plane transform on $\mathbb{R}^n$ in terms of a wavelet measure on the one-dimensional set $[0, \infty)$. For instance, this way leads to the following

**Theorem 7.1.** Let $\varphi = \hat{f}$ be the $k$-plane Radon transform of a function $f \in L_p$, $1 \leq p < n/k$. Let $\mu$ be a finite Borel measure on $[0, \infty)$ satisfying

(a) $\int_1^\infty \eta^{\gamma} d|\mu|(\eta) < \infty$ for some $\gamma > k$;

(b) $\int_0^\infty \eta^j d\mu(\eta) = 0 \quad \forall \ j = 0, 1, \ldots, k$.

Let $W\check{\varphi}$ be the wavelet transform of $\check{\varphi}$, associated with the Poisson semigroup [3,4], namely,

$$W\check{\varphi}(x, t) = \int_0^\infty \mathcal{P}_t \check{\varphi}(x) d\mu(\eta), \quad x \in \mathbb{R}^n, \ t > 0.$$

(7.4)

Then

$$\int_0^\infty W\check{\varphi}(x, t) \frac{dt}{t^{1+k}} \equiv \lim_{\varepsilon \to \infty} \int_0^\infty W\check{\varphi}(x, t) \frac{dt}{t^{1+k}} = c_{k,n} f(x),$$

(7.5)
where (cf. \(\text{(3.23)}\)),

\[ c_{k,\mu} = \frac{(-1)^{k+1}}{k!} \int_0^\infty t^k \log t \, d\mu(t). \]

The limit in \(\text{(7.5)}\) exists in the \(L_p\)-norm and pointwise almost everywhere. If \(f \in C_0 \cap L_p\), the convergence is uniform on \(\mathbb{R}^n\).

**Remark 7.2.** The following observation might be interesting. Let

\[ I_{\alpha}^- u(t) = \frac{1}{\Gamma(\alpha)} \int_t^\infty (t - s)^{\alpha - 1} u(s) \, ds, \quad t > 0, \]

be the Riemann-Liouville integral of \(u\). It is known \([\text{R1}]\) formula (16.9)] that the Poisson integral takes the Riesz potential \(I_{\alpha}^- f\) to the Riemann-Liouville integral of the function \(t \to P_t f\), namely,

\[ P_t I_{\alpha}^- f = I_{\alpha}^- P(\cdot) f. \]

Denoting by \(R\) and \(R^*\) the Radon \(k\)-plane transform and its dual, owing to Fuglede’s formula \(\text{(7.3)}\), we have

\[ R^* R f = d_{k,n} I_k f. \]

Combining \(\text{(7.8)}\) and \(\text{(7.7)}\), we get

\[ R^* R f = d_{k,n} I_k P(\cdot) f, \quad R^* \varphi(x) = (P_t R^* \varphi)(x). \]

This formula has the same nature as the following one in terms of the spherical means, that lies in the scope of the classical Funk-Radon-Helgason theory:

\[ (\hat{f}^s)^\flat(x) = \sigma_{k-1} \int_r^\infty (M_t f)(x)(t^2 - r^2)^{k/2 - 1} \, dt; \]

see Lemma 5.1 in \([\text{R4}]\). Here \(\sigma_{k-1}\) is the volume of the \((k - 1)\)-dimensional unit sphere,

\[ (M_t f)(x) = \frac{1}{\sigma_{n-1}} \int_{S^{n-1}} f(x + t\theta) \, d\theta, \quad t > 0, \]

and \((\hat{f}^s)^\flat(x)\) is the so-called shifted dual \(k\)-plane transform, which is the mean value of \(f(\tau)\) over all \(k\)-planes \(\tau\) at distance \(r\) from \(x\).

### 8. Higher-rank Composite Wavelet Transforms and Open Problems

Challenging perspectives and open problems for composite wavelet transforms are connected with functions of matrix argument and their application to integral geometry. This relatively new area encompasses the so-called higher-rank problems, when traditional scalar notions, like distance or scaling, become matrix-valued.

**8.1. Matrix spaces, preliminaries.** We remind basic notions, following \([\text{R9}]\).

Let \(\mathfrak{M}_{n,m} \sim \mathbb{R}^{nm}\) be the space of real matrices \(x = (x_{i,j})\) having \(n\) rows and \(m\) columns, \(n \geq m\); \(dx = \prod_{i=1}^n \prod_{j=1}^m dx_{i,j}\) is the volume element on \(\mathfrak{M}_{n,m}\), \(x^t\) denotes the transpose of \(x\), and \(I_m\) is the identity \(m \times m\) matrix. Given a square matrix \(a\), we denote by \(\det(a)\) the determinant of \(a\), and by \(|a|\) the absolute value of \(\det(a)\); \(\text{tr}(a)\) stands for the trace of \(a\). For \(x \in \mathfrak{M}_{n,m}\), we denote

\[ |x|_m = \det(x^t x)^{1/2}. \]

If \(m = 1\), this is the usual Euclidean norm on \(\mathbb{R}^n\). For \(m > 1\), \(|x|_m\) is the volume of the parallelepiped spanned by the column-vectors of \(x\). We use standard notations
\(O(n)\) and \(SO(n)\) for the orthogonal group and the special orthogonal group of \(\mathbb{R}^n\) with the normalized invariant measure of total mass 1. Let \(S_m \sim \mathbb{R}^{m(m+1)/2}\) be the space of \(m \times m\) real symmetric matrices \(s = (s_{i,j})\) with the volume element \(ds = \prod_{i\leq j} ds_{i,j}\). We denote by \(P_m\) the cone of positive definite matrices in \(S_m\); \(\overline{P}_m\) is the closure of \(P_m\), that is, the set of all positive semi-definite \(m \times m\) matrices. For \(r \in P_m\) (or \(\overline{P}_m\)), we write \(r > 0\) (or \(\geq 0\)). Given \(a\) and \(b\) in \(S_m\), the inequality \(a > b\) means \(a - b \in P_m\) and the symbol \(\int_a^b f(s)ds\) denotes the integral over the set \((a + P_m) \cap (b - P_m)\).

The group \(G = GL(m, \mathbb{R})\) of real non-singular \(m \times m\) matrices \(g\) acts transitively on \(P_m\) by the rule \(r \rightarrow grg'\). The corresponding \(G\)-invariant measure on \(P_m\) is

\[
dr = |r|^{-d} dr, \quad |r| = \det(r), \quad d = (m + 1)/2
\]

[Te], p. 18.

**Lemma 8.1.** [Mu] pp. 57-59]

(i) If \(x = ayb\) where \(y \in \mathcal{M}_{n,m}\), \(a \in GL(n, \mathbb{R})\), and \(b \in GL(m, \mathbb{R})\), then \(dx = |a|^n|b|^m dy\).

(ii) If \(r = q'sq\) where \(s \in S_m\), and \(q \in GL(m, \mathbb{R})\), then \(d\theta = |q|^{m+1} ds\).

(iii) If \(r = s^{-1}\) where \(s \in P_m\), then \(r \in P_m\), and \(dr = |s|^{-m-1} ds\).

For \(Re \alpha > d - 1\), the Siegel gamma function of \(P_m\) is defined by

\[
\Gamma_m(\alpha) = \int_{P_m} \exp(-\text{tr}(r))|r|^\alpha dr = \pi^{m(m-1)/2} \prod_{j=0}^{m-1} \Gamma(\alpha - j/2),
\]

[FK, Te]. The relevant beta function has the form

\[
B_m(\alpha, \beta) = \int_0^1 |r|^{\alpha-d} I_m - r|^{\beta-d} dr = \frac{\Gamma_m(\alpha)\Gamma_m(\beta)}{\Gamma_m(\alpha + \beta)}, \quad d = (m + 1)/2.
\]

This integral converges absolutely if and only if \(Re \alpha, Re \beta > d - 1\).

All function spaces on \(\mathcal{M}_{n,m}\) are identified with the corresponding spaces on \(\mathbb{R}^{nm}\). For instance, \(\mathcal{S}(\mathcal{M}_{n,m})\) denotes the Schwartz space of infinitely differentiable rapidly decreasing functions. The Fourier transform of a function \(f \in L_1(\mathcal{M}_{n,m})\) is defined by

\[
\mathcal{F}f(y) = \int_{\mathcal{M}_{n,m}} \exp(\text{tr}(iy'x))f(x)dx, \quad y \in \mathcal{M}_{n,m}.
\]

The Cayley-Laplace operator \(\Delta\) on \(\mathcal{M}_{n,m}\) is defined by

\[
\Delta = \det(\partial^2\partial_x), \quad \partial = (\partial/\partial x_{i,j}).
\]

In terms of the Fourier transform, the action of \(\Delta\) represents a multiplication by the homogeneous polynomial \((-1)^m|y|^2_m\) of degree \(2m\) of \(nm\) variables \(y_{i,j}\).

For the sake of simplicity, for some operators on functions of matrix argument we will use the same notation as in the previous sections.

The Gårding-Gindikin integrals of functions \(f\) on \(P_m\) are defined by

\[
(I^\alpha f)(s) = \frac{1}{\Gamma_m(\alpha)} \int_0^s f(r)s^{-\alpha-d} dr, \quad (I^{-\alpha} f)(s) = \frac{1}{\Gamma_m(\alpha)} \int_s^\infty f(r)r^{-s\alpha-d} dr,
\]

where \(s \in P_m\) in the first integral and \(s \in \overline{P}_m\) in the second one. We assume \(Re \alpha > d - 1, d = (m + 1)/2\) (this condition is necessary for absolute convergence.
of these integrals). The first integral exists a.e. for arbitrary locally integrable function \( f \). Existence of the second integral requires extra assumptions for \( f \) at infinity.

The **Riesz potential** of a function \( f \in \mathcal{S}(\mathbb{M}_{n,m}) \) is defined by

\[
(\mathcal{I}^\alpha f)(x) = \frac{1}{\gamma_{n,m}(\alpha)} \int_{\mathbb{M}_{n,m}} f(x - y)|y|^{\alpha - n}dy;
\]

\[
\gamma_{n,m}(\alpha) = \frac{2^{\alpha m} \pi^{nm/2} \Gamma_m(\alpha/2)}{\Gamma_m((n-\alpha)/2)}, \quad \text{Re} \alpha > m-1, \quad \alpha \neq n-m+1, n-m+2, \ldots
\]

This integral is finite a.e. for \( f \in L^p(\mathbb{M}_{n,m}) \) provided \( 1 \leq p < n(\text{Re} \alpha + m - 1)^{-1} \) [R9, Theorem 5.10].

An application of the Fourier transform gives

\[
\mathcal{F}[\mathcal{I}^\alpha f]\xi = |\xi|^{-\alpha} \mathcal{F}f\xi
\]

(as in the case of \( \mathbb{R}^n \)), so that \( \mathcal{I}^\alpha \) can be formally identified with the negative power of the Cayley-Laplace operator \( (8.6) \), namely, \( \mathcal{I}^\alpha = (-\Delta_m)^{-\alpha/2} \). Discussion of precise meaning of the equality \( (8.10) \) and related references can be found in [R9], [OR2].

**Definition 8.2.** For \( x \in \mathbb{M}_{n,m}, n \geq m, \) and \( t \in \mathcal{P}_m \), we define the (generalized) heat kernel \( h_t(x) \) by the formula

\[
h_t(x) = (4\pi)^{-nm/2}|t|^{-n/2} \exp(-\text{tr}(t^{-1}x'x)/4), \quad |t| = \text{det}(t),
\]

and set

\[
H_t f(x) = \int_{\mathbb{M}_{n,m}} h_t(x - y)f(y)dy = \int_{\mathbb{M}_{n,m}} h_{tn}(y)f(x - yt^{1/2})dy.
\]

Clearly, \( H_t f(x) \) is a generalization of the Gauss-Weierstrass integral \( (3.4) \).

**Lemma 8.3.** [R9]

(i) For each \( t \in \mathcal{P}_m \),

\[
\int_{\mathbb{M}_{n,m}} h_t(x)dx = 1.
\]

(ii) The Fourier transform of \( h_t(x) \) has the form

\[
\mathcal{F}h_t(y) = \exp(-\text{tr}(ty'y)),
\]

which implies the semi-group property

\[
h_t \ast h_r = h_{t+r}, \quad t, r \in \mathcal{P}_m.
\]

(iii) If \( f \in L^p(\mathbb{M}_{n,m}), 1 \leq p \leq \infty, \) then

\[
||H_t f||_p \leq ||f||_p, \quad H_t H_r f = H_{t+r} f,
\]

and

\[
\lim_{t \to 0} H_t f(x) = f(x)
\]

in the \( L^p \)-norm. If \( f \) is a continuous function vanishing at infinity, then \( (8.17) \) holds in the sup-norm.
Theorem 8.4. [R9] Let \( m - 1 < Re \alpha < n - m + 1 \), \( d = (m + 1)/2 \). Then
\[
(I^\alpha f)(x) = \frac{1}{\Gamma(m/2)} \int_{P_m} \left| t \right|^{\alpha/2} H_t f(x) \, ds, \quad ds = \left| t \right|^{-d} \, dt,
\]
\[
H_t[I^\alpha f](x) = I_t^{\alpha/2}[H_t f(x)](t),
\]
provided that integrals on either side of the corresponding equality exist in the Lebesgue sense.

8.2. Composite wavelet transforms: open problems. Formula (8.18) provokes a natural construction of the relevant composite wavelet transform on \( \mathcal{M}_{n,m} \) associated with the heat kernel and containing a \( \mathcal{P}_m \)-valued scaling parameter. To find this construction, we first obtain an auxiliary integral representation of a power function of the form \( \left| t \right|^{\lambda-d} \), \( d = (m + 1)/2 \).

Definition 8.5. A function \( w \) on \( \mathcal{P}_m \) is said to be symmetric if
\[
w(gg^{-1}) = w(\eta) \quad \text{for all} \quad g \in GL(m, \mathbb{R}), \quad \eta \in \mathcal{P}_m.
\]

Note that if \( w \) is symmetric, then for any \( s,t \in \mathcal{P}_m \),
\[
w(t^{1/2}s^{1/2}) = w(s^{1/2}t^{1/2}) \quad \text{and} \quad w(ts) = w(st).
\]

Indeed, the second equality follows from (8.20) if we set \( \eta = ts, \quad g = t^{-1} \). The first equality in (8.21) is a consequence of the second one:
\[
w(t^{1/2}s^{1/2}) = w(t^{-1/2}[t^{1/2}s^{1/2}]^{1/2}) = w(st) = w(ts) = w(s^{1/2}t^{1/2}).
\]

Lemma 8.6. Let \( w \) be a symmetric function on \( \mathcal{P}_m \) satisfying
\[
\int_{\mathcal{P}_m} \frac{|w(\eta)|}{|\eta|^{\lambda}} \, d\eta < \infty, \quad c = \int_{\mathcal{P}_m} \frac{w(\eta)}{|\eta|^{\lambda}} \, d\eta \neq 0, \quad |\eta| = \det(\eta).
\]

Then for \( t \in \mathcal{P}_m \),
\[
|t|^{\lambda-d} = c^{-1} \int_{\mathcal{P}_m} \frac{w(a^{-1}t)}{|a|^{m+1-\lambda}} \, da, \quad d = (m + 1)/2.
\]

Proof. By (8.21) we have (set \( a = \rho^{-1} \), \( da = \rho^{-2d} \, d\rho \))
\[
\int_{\mathcal{P}_m} \frac{w(a^{-1}t)}{|a|^{m+1-\lambda}} \, da = \int_{\mathcal{P}_m} \frac{w(t^{1/2}a^{-1}t^{1/2})}{|a|^{m+1-\lambda}} \, da = \int_{\mathcal{P}_m} \frac{w(t^{1/2}\rho t^{1/2})}{|\rho|^{\lambda-d}} \, d\rho = \left| t \right|^{\lambda-d} \int_{\mathcal{P}_m} \frac{w(\eta)}{|\eta|^{\lambda}} \, d\eta = c \left| t \right|^{\lambda-d}.
\]

Now we replace a power function in (8.18) according to (8.23) with \( \lambda = \alpha/2 \). For \( Re \alpha > (m-1)/2 \), we obtain
\[
(I^\alpha f)(x) = \frac{c^{-1}}{\Gamma(m/2)} \int_{\mathcal{P}_m} H_t f(x) \, dt \int_{\mathcal{P}_m} \frac{w(a^{-1}t)}{|a|^{m+1-\alpha/2}} \, da
\]
\[
= \frac{c^{-1}}{\Gamma(m/2)} \int_{\mathcal{P}_m} \frac{d_s a}{|a|^{d-\alpha/2}} \int_{\mathcal{P}_m} H_t f(x) w(a^{-1}t) \, dt.
\]

This gives
\[
(I^\alpha f)(x) = \frac{c^{-1}}{\Gamma(m/2)} \int_{\mathcal{P}_m} \mathcal{H} f(x, a)|a|^{\alpha/2} \, d_s a, \quad Re \alpha > (m - 1)/2,
\]
with

\[ \mathcal{H} f(x, a) = |a|^{-d} \int_{\mathcal{P}_m} H_t f(x) w(a^{-1} t) \, dt \]

or, by the symmetry of \( w \), after changing variable,

\[ \mathcal{H} f(x, a) = \int_{\mathcal{P}_m} H_{a^{1/2} \eta a^{1/2}} f(x) w(\eta) \, d\eta, \quad x \in \mathcal{M}_{n,m}, \quad a \in \mathcal{P}_m. \]

Taking into account an obvious similarity between (8.26) and the corresponding “rank-one” formula for \( m = 1 \), we call \( \mathcal{H} f(x, a) \) the composite wavelet transform of \( f \) associated to the heat semigroup \( H_t \). Here \( w \) is a symmetric integrable function (that will be endowed later with some cancelation properties) and \( a \) is a \( \mathcal{P}_m \)-valued scaling parameter. One can replace \( w \) by a more general wavelet measure, as we did in the previous sections, but here we want to minimize technicalities.

Owing to (8.10), it is natural to expect that the inverse of \( I^\alpha \) has the same form (8.24) with \( \alpha \) formally replaced by \( -\alpha \), and the case \( \alpha = 0 \) gives a variant of Calderón’s reproducing formula.

Thus, we encounter the following open problem:

**Problem A.** Give precise meaning to the inversion formula

\[ f(x) = c_{m, \alpha} \int_{\mathcal{P}_m} \mathcal{H}\varphi(x, a) \frac{|a|^{\alpha/2}}{d_s a}, \quad \varphi = I^\alpha f, \]

and the reproducing formula

\[ f(x) = c_m \int_{\mathcal{P}_m} \mathcal{H} f(x, a) d_s a, \]

say, for \( f \in L_p \) or any other “natural” function space. Give examples of wavelet functions \( w \) for which (8.27) and (8.25) hold. Find explicit formulas for the normalizing coefficients \( c_{m, \alpha} \) and \( c_m \), depending on \( w \).

Solution of this problem would give a series of pointwise inversion formulas for diverse Radon-like transforms on matrix spaces; see, e.g., [OR2, OR3, R9], where such formulas are available in terms of distributions. Justification of (8.27) and (8.28) would also bring new light to a variety of inversion formulas for Radon transforms on Grassmannians, cf. [GR2, GR3].

### 8.3. Some discussion.

Trying to solve Problem A, we come across new problems that are of independent interest. Let \( \Re \alpha > d - 1, \quad d = (m + 1)/2 \). Suppose, for instance, that \( f(x), \quad x \in \mathcal{M}_{n,m} \), is a Schwartz function and \( w(\eta), \quad \eta \in \mathcal{P}_m \), is “good enough”. We anticipate the following equality:

\[ I_s f(x) = \int_{\mathcal{P}_m} \mathcal{H}[I^\alpha f](x, a) \frac{|a|^{\alpha/2}}{d_s a} = \int_{\mathcal{P}_m} \Lambda_{\alpha/2}(s) H_{ss} f(x) \, ds, \]

where \( \Lambda_{\alpha/2}(s) \) expresses through the Gårding-Gindikin integral in (S.21) as

\[ \Lambda_{\alpha/2}(s) = \frac{\Gamma_m(d)}{|s|^{d}} I_{\alpha/2+d}^+ w(s), \quad s \in \mathcal{P}_m. \]

If \( m = 1 \) and \( \alpha/2 \) is replaced by \( \alpha \), then (8.30) coincides with the function \( \lambda_\alpha(s) = s^{-1} I_{\alpha/2+1}^+ \mu(s) \) in Lemma 3.4. Now, we give the following
DEFINITION 8.7. An integrable symmetric function \( w \) on \( P_m \) is called an admissible wavelet if
\[
(8.31) \quad \Lambda_{\alpha/2}(s) = \frac{\Gamma_m(d)}{|\lambda|^d} \lambda_+^{\alpha/2+d} w(s) \in L_1(P_m) \quad \text{and} \quad c_\alpha = \int_{P_m} \Lambda_{\alpha/2}(s) \, ds \neq 0.
\]

If \( w \) is admissible, then, by Lemma 5.3, the \( L_p \)-limit as \( \varepsilon \to 0 \) of the right-hand side of (8.29) is \( c_\alpha f \), and we are done. This discussion includes the case \( \alpha = 0 \) corresponding to the reproducing formula.

Thus, our attempt to solve Problem A rests upon the following

**Problem B.** Find examples of admissible wavelets (both for \( \alpha \neq 0 \) and \( \alpha = 0 \)) and compute \( c_\alpha \).

Now, let us try to prove (8.29). We say “try”, because along the way, we come across one more open problem related to application of the Fubini theorem; cf. justification of interchange of the order of integration in the proof of Theorem 2.2.

By (8.26) and (8.19),
\[
H \mathcal{I}_x^{\alpha} f(x, a) = \int_{P_m} H_{x^{1/2} \eta^{1/2}} I^{\alpha} f(x) \, w(\eta) \, d\eta
\]
Assume that \( x \) is fixed and denote \( \psi(s) = H_s f(x) \). Then
\[
H I^{\alpha}_x f(x, a) = \int_{P_m} w(\eta) I_{x^{1/2}}^{\alpha/2} \psi(\alpha/2 \eta^{1/2}) \, d\eta
\]
\[
= \frac{1}{\Gamma_m(\alpha/2)} \int_{P_m} w(\eta) \, d\eta \int_0^{\infty} \psi(s) |s - \alpha/2 \eta^{1/2}|^{\alpha/2-d} \, ds
\]
\[
= \frac{1}{\Gamma_m(\alpha/2)} \int_{P_m} \psi(s) \, ds \int_0^{\alpha/2} w(\eta) |s - \alpha/2 \eta^{1/2}|^{\alpha/2-d} \, d\eta
\]
\[
= |a|^{\alpha/2-d} \int_{P_m} \psi(s) I_{x^{1/2}}^{\alpha/2} \psi(a^{1/2} \eta^{1/2}) \, ds.
\]

Hence, the left-hand side of (8.29) transforms as follows.
\[
I_x f(x) = \int_{\ell_m} \int_{P_m} \psi(s) I_{x^{1/2}}^{\alpha/2} w(a^{1/2} \eta^{1/2}) \, da \quad \text{(set \( a = \tau^{-1} \))}
\]
\[
= \int_{\ell_m} \int_{P_m} \psi(s) I_{x^{1/2}}^{\alpha/2} w(\tau^{1/2} \eta^{1/2}) \, d\tau
\]
\[
= \varepsilon^{md} \int_{\ell_m} \psi(s) \, ds \int_0^{\varepsilon^{-1} m} I_{x^{1/2}}^{\alpha/2} w(\tau^{1/2} \varepsilon^{1/2}) \, d\tau.
\]

Thus we have
\[
(8.32) \quad I_x f(x) = \int_{P_m} \psi(s) k(s) \, ds = \int_{P_m} H_{x s} f(x) \, k(s) \, ds,
\]
where
\[
k(s) = \int_0^{m} I_{x^{1/2}}^{\alpha/2} w(\lambda^{1/2} s^{1/2} \lambda^{1/2}) \, d\lambda.
\]
To get (8.29), it remains to show that \( k(s) \) coincides with the function (8.30). We have

\[
k(s) = \frac{1}{\Gamma_m(\alpha/2)} \int_0^{I_m} d\lambda \int_0^{\lambda^{1/2} s^{1/2}} w(s)\lambda^{1/2} s^{1/2} - s^{\alpha/2-d} ds
\]

(set \( s = \lambda^{1/2} z^{1/2} \) and note that \( w(\lambda^{1/2} z^{1/2}) = w(z^{1/2} \lambda z^{1/2}) \))

\[
= \frac{1}{\Gamma_m(\alpha/2)} \int_0^{I_m} |\lambda|^{\alpha/2} d\lambda \int_0^{s} |s - z|^{\alpha/2-d} w(z^{1/2} \lambda z^{1/2}) dz
\]

\[
= \frac{1}{\Gamma_m(\alpha/2)} \int_0^{s} |s - z|^{\alpha/2-d} dz \int_0^{I_m} |\lambda|^{\alpha/2} w(z^{1/2} \lambda z^{1/2}) d\lambda
\]

\[
= \frac{1}{\Gamma_m(\alpha/2)} \int_0^{s} |s - z|^{\alpha/2-d} \frac{dz}{|z^{1/2} + d|} \int_z^b w(b) |b|^{\alpha/2} db
\]

\[
= \frac{1}{\Gamma_m(\alpha/2)} \int_b^s w(b) |b|^{\alpha/2} u(b, s) db,
\]

where

\[
u(b, s) = \int_b^s |s - r|^{\alpha/2-d} \frac{dz}{|z^{1/2} + d|} \quad (\text{set } z = r^{-1})
\]

\[
= \int_{b^{-1}}^s |r - I_m|^{\alpha/2-d} dr = |s|^{\alpha/2-d} \int_{s^{-1}}^{b^{-1}} |r - s^{-1}|^{\alpha/2-d} dr.
\]

The last integral can be easily computed using the well-known formula for Siegel Beta functions

\[
(8.33) \quad \int_a^b |r - a|^{\alpha-d} |b - r|^{\beta-d} dr = B_m(\alpha, \beta)|b - a|^{\alpha+\beta-d}
\]

(many such formulas can be found, e.g., in [OR2]), and we have

\[
u(b, s) = B_m(\alpha/2, d) \frac{|s - b|^{\alpha/2}}{|s|^{d} |b|^{\alpha/2}}, \quad B_m(\alpha/2, d) = \frac{\Gamma_m(\alpha/2) \Gamma_m(d)}{\Gamma_m(\alpha/2 + d)}.
\]

Finally, we get

\[
k(s) = \frac{\Gamma_m(d)}{|s|^d \Gamma_m(\alpha/2 + d)} \int_0^{s} w(b) |s - b|^{\alpha/2} ds = \frac{\Gamma_m(d)}{2 \Gamma_m(s^d) |s|^{d}} \int_{s^{-1}}^{b^{-1}} \frac{\Gamma_m(d)}{2 \Gamma_m(s^d) |s|^{d}} I_{s^{-1}}^{s^{-2}} w(s) = \Lambda_{\alpha/2}(s).
\]

**Problem C.** Although all calculations above go through smoothly, interchange of the order of integration remains unjustified. We do not know how to justify it and what additional requirements on the wavelet \( w \) should be imposed (if any). One of the obstacles is that \( \int_0^\infty \neq \int_0^{s^*} + \int_s^\infty \), when we integrate over the higher-rank cone.

**References**

[Al] I.A. Aliev, *On the Bessel type potentials and associated function spaces*, Preprint, 2007.

[AB1] I.A. Aliev and S. Bayrakci, *On inversion of B-elliptic potentials by the method of Balakrishnan-Rubin*, Fract. Calc. Appl. Anal., 1 (1998), 365-384.

[AB2] __________, *On inversion of Bessel potentials associated with the Laplace-Bessel differential operator*, Acta Math. Hungar., 95 (2002), 125-145.

[AE1] I.A. Aliev and M. Eryigit, *Inversion of Bessel potentials with the aid of weighted wavelet transforms*, Math. Nachr. 242 (2002), 27-37.

[AE2] __________, *Wavelet-type transform and Bessel potentials associated with the generalized translation*, Integr. Equation and Operator Theory, 51 (2005), 303-317.
I.A. Aliev and B. Rubin, Parabolic potentials and wavelet transforms with the generalized translations, Studia Mathematica, 145(2001), 1-16.

Parabolic wavelet transforms and Lebesgue spaces of parabolic potentials, Rocky Mountain J. of Math., 32 (2002), 391-408.

Spherical harmonics associated to the Laplace-Bessel operator and generalized spherical convolutions, Anal. Appl. (Singap.), 1 (2003), 81–109.

Wavelet-like transforms for admissible semi-groups; Inversion formulas for potentials and Radon transforms, J. of Fourier Anal. and Appl., 11, (2005), 333-352.

I.A. Aliev , S. Sezer, and M. Eryigit, An integral transform associated to the Poisson integral and inversion of Flett potentials, J. of Math. Anal. and Appl., 321 (2006), 691-704.

R. Bagby, Lebesgue spaces of parabolic potentials, Illinois J. Math., 15 (1971), 610-634.

Daubechies, I., Ten lectures on wavelets, CBMS-NSF Series in Appl. Math., SIAM Publ., Philadelphia, 1992.

J. Delsarte, Sur une extension de la formule de Taylor, J. Math. Pures Appl., 17, Fasc. III (1938) 213-231.

E. Grinberg and B. Rubin, Radon inversion on Grassmannians via Gårding-Gindikin fractional integrals, Annals of Math. 159 (2004), 809–843.
[OR3] _____, Semyanistyi’s integrals and Radon transforms on matrix spaces, J. of Fourier Anal. and Appl., (in press).

[PS] G. Polya and G. Szego, Aufgaben und lehrsatzes aus der analysis, Springer-Verlag, Berlin-New York, 1964.

[PBM] A. P. Prudnikov, Y. A. Brychkov, and O. I. Marichev, Integrals and series: special functions, Gordon and Breach Sci. Publ., New York - London, 1986.

[Ra] V.R. Gopala Rao, A characterization of parabolic function spaces, Amer. J. Math., 99 (1977), 985-993.

[R1] B. Rubin, Fractional integrals and potentials, Pitman monographs and Surveys in Pure and Applied Mathematics, 82, Longman, Harlow, 1996.

[R2] _____, The Calderón reproducing formula, windowed X-ray transforms and Radon transforms in $L_p$-spaces, The Journal of Fourier Anal. and Appl. 4, 175-197, 1998.

[R3] _____, Fractional Integrals and wavelet transforms associated with Blaschke-Levy representations on the sphere, Israel J. Math. 114 (1999), 1-27.

[R4] _____, Reconstruction of functions from their integrals over $k$-planes, Israel J. Math. 141 (2004), 93-117.

[R5] _____, The convolution-backprojection method for $k$-plane transforms, and Calderón's identity for ridgelet transforms, Appl. Comput. Harmon. Anal. 16 (2004), 231-242.

[R6] _____, Fractional calculus and wavelet transforms in integral geometry, Frac. Calc. Appl. Anal. 1 (1998), no. 2, 193-219.

[R7] _____, Calderon-type reproducing formula, in: Encyclopaedia Math. Supplement II, Kluwer, 2000, 104-105; reprinted in Frac. Calc. Appl. Anal., 3 (2000), 103-106.

[R8] _____, Intersection bodies and generalized cosine transforms, Preprint, 2007, arXiv:0704.0061v2.

[R9] _____, Riesz potentials and integral geometry in the space of rectangular matrices, Advances in Math. 205 (2006), 549-598.

[S] S.G. Samko, Hypersingular integrals and their applications, Analytical methods and Special functions, 5, Taylor and Francis, Ltd. London, 2002.

[SKM] S.G. Samko, A.A. Kilbas, and O.I. Marichev, Fractional Integrals and Derivatives, Theory and Applications, Gordon and Breach Science Publishers, 1993.

[Sa] C.H. Sampson, A characterization of parabolic Lebesgue spaces, Dissertation, Rice Univ. 1968.

[St] E. Stein, Singular integrals and differentiability properties of functions, Princeton Univ. Press, Princeton, N.J., 1970.

[SW1] E. Stein and G. Weiss, On the theory of harmonic functions of several variables, I. The theory of $H^p$ spaces, Acta Math. 103 (1960), 25-62.

[SW2] _____, Introduction to Fourier Analysis on Euclidean Spaces, Princeton University Press, Princeton, NJ, 1971.

[Te] A. Terras, Harmonic analysis on symmetric spaces and applications, Vol. II, Springer, Berlin, 1988.

[Tr] K. Trimèche, Generalized Wavelets and Hypergroups, Gordon and Breach Sci. Publ., New York-London, 1997.

DEPARTMENT OF MATHEMATICS, AKDENIZ UNIVERSITY, 07058 ANTALYA TURKEY
E-mail address: ialiev@akdeniz.edu.tr

DEPARTMENT OF MATHEMATICS, LOUISIANA STATE UNIVERSITY, BATON ROUGE, LOUISIANA 70803
E-mail address: borisr@math.lsu.edu

FACULTY OF EDUCATION, AKDENIZ UNIVERSITY, 07058 ANTALYA TURKEY
E-mail address: sinemsener@akdeniz.edu.tr

DEPARTMENT OF MATHEMATICS, AKDENIZ UNIVERSITY, 07058 ANTALYA TURKEY
E-mail address: simten@akdeniz.edu.tr