Global solution for the coagulation equation of water droplets in atmosphere between two horizontal planes.

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Abstract. In this paper we give a global existence and uniqueness theorem for an initial and boundary value problem (IBVP) relative to the coagulation equation of water droplets and we show the convergence of the global solution to the stationary solution. The coagulation equation is an integro-differential equation that describes the variation of the density \(\sigma\) of water droplets in the atmosphere. Furthermore, IBVP is considered on a strip limited by two horizontal planes and its boundary condition is such that rain fall from the strip. To obtain this result of global existence of the solution \(\sigma\) in the space of bounded continuous functions, through the method of characteristics, we assume bounded continuous and small data, whereas the vector field, besides being bounded continuous, has \(W^{1,\infty}\) regularity in space.

Key words: initial and boundary value problem, integro-PDE, method of characteristics, phase transitions, stationary solution, coagulation equation.

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1 Introduction

In \cite{17} has been introduced a model of motion of the air and the phase transition of water in the three states in the atmosphere. Since then, many papers have been produced in relation to this model (see for example \cite{15}, \cite{3} \cite{1}, \cite{18}, \cite{19}, etc.). In particular, in the paper \cite{15}, under some suitable conditions, has been proved the existence and the uniqueness of the stationary solution for the equation of water droplets, without taking into account the condensation or evaporation process, assuming a domain with two spatial dimensions and a horizontal constant wind. Moreover in \cite{3} has been shown, under weak assumptions, the global existence and uniqueness of the solution for the same equation,
without taking into account the wind, on a domain with one spatial dimension. On the other hand, in the paper [19] has been proved the existence and the uniqueness, in $W^{1,\infty}$, of the local solution of an IBVP for the hyperbolic part of the model introduced in [17], on a strip bounded by two horizontal planes with the boundary condition such that rain fall from the strip.

Now, a significant question which arises, is that to establish the global existence of the solution of IVBP studied in [19]. At the moment, this question is very difficult to treat, therefore, in this paper, we have decided to study only one equation of the hyperbolic part of the model seen in [17]. More precisely, we consider the coagulation equation of water droplets, including condensation or evaporation process and effects of wind, in atmosphere between two horizontal planes (or vertical strip), supposing known the initial density and the density on the upper horizontal plane. For this problem, assuming small data and that the vertical velocities of droplets are negative, we prove the global existence and uniqueness of the solution and we establish the existence of the stationary solution and the convergence of the global solution to the stationary solution.

Of course, we know that there exists a well-developed mathematical theory of coagulation (see for example [9]), but the results obtained in our paper, although they appear elementary, are not present in the literature about the coagulation equation.

Afterwards, let us say something about the sections of this paper. In section 2, we introduce the initial and boundary value problem for the coagulation equation of water droplets and its stationary equation associated. In section 3, after having introduced hypotheses about the velocities of water droplets, using the method of characteristics, we transform our IBVP in two sets of infinite Cauchy’s problems for ordinary differential equations (ODEs) and we give the definitions of generalized solutions for them. Hence, in section 4, we give the two main theorems of this paper. In section 5, we study the linearized version of the ODEs about the coagulation equation obtained in section 3 and we establish useful estimates to treat our IBVP. In section 6, using some fixed-point arguments, we are able to prove the first main theorem about the global existence and uniqueness of the solution for the coagulation equation. In section 7, we prove the second main theorem about stationary solution.

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2 Position of the problems.

We consider the integro-differential equation which describes the variation of the density $\sigma(t, x, m)$ of water droplets with mass $m$ in the atmosphere (see [17], [2])

\begin{equation}
\begin{aligned}
\partial_t \sigma(t, x, m) + \nabla_x \cdot (\sigma(t, x, m) u(t, x, m)) + \partial_m (m h_{gl}(t, x, m) \sigma(t, x, m)) = \\
= h_{gl}(t, x, m)\sigma(t, x, m) + \frac{m}{2} \int_0^m \beta(m - m', m')\sigma(t, x, m')\sigma(t, x, m - m')dm' + \\
- m \int_0^\infty \beta(m, m')\sigma(t, x, m)\sigma(t, x, m')dm' + g_0(m)\left[N_1 - \bar{N}(\sigma)\right]^+(t, x)\left[Q\right]^+(t, x) +
- g_1(m)\left[Q\right]^-(t, x)\sigma(t, x, m),
\end{aligned}
\end{equation}
on the domain

\[
\mathbb{R}^+ \times \Omega \times \mathbb{R}^+,
\]

where \( \Omega = \{ x \in \mathbb{R}^3 \mid (x_1, x_2) \in \mathbb{R}^2, \ 0 < x_3 < 1 \} \), \( t \in \mathbb{R}^+ \) is time, \( x = (x_1, x_2, x_3) \in \Omega \) is a point in three-dimensional space and \( m \in \mathbb{R}^+ \) is the mass of a droplet, whereas \([r]^+\) and \([r]^-\) are the positive and the negative part of \( r \in \mathbb{R} \). In (2.1), \( u \) and \( h_{gl} \) are given functions defined on \( \mathbb{R}^+ \times \Omega \times \mathbb{R}^+ \), \( Q \) is a given function on \( \mathbb{R}^+ \times \Omega \), \( g_0 \) and \( g_1 \) are given functions on \( \mathbb{R}^+ \) and \( \beta \) is a given function on \( \mathbb{R}^+ \times \mathbb{R}^+ \), whereas \( N_1 \) and \( \bar{N}(\sigma) \) are a positive constant and a linear functional of \( \sigma \) (that we define below) respectively. Now, we study the equation (2.1) with the following conditions

\[
\sigma(0, x, m) = \tilde{\sigma}_0(x, m) \quad \text{for} \quad x \in \Omega, \ m \in \mathbb{R}^+,
\]

\[
\sigma(t, x_1, x_2, 1, m) = \tilde{\sigma}_1(t, x_1, x_2, m) \quad \text{for} \quad t \in \mathbb{R}^+, \ (x_1, x_2) \in \mathbb{R}^2, \ m \in \mathbb{R}^+,
\]

where \( \tilde{\sigma}_0 \) and \( \tilde{\sigma}_1 \) are given functions defined on \( \Omega \times \mathbb{R}^+ \) and \( \mathbb{R}^2 \times \mathbb{R}^+ \) respectively. Moreover, assuming \( \Gamma_- = \{ 0 \} \times \Omega \times \mathbb{R}^+ \cup (\mathbb{R}^+ \times \mathbb{R}^2 \times \{ 0 \} \times \mathbb{R}^+) \), we can rewrite (2.3) and (2.4) as

\[
\sigma|_{\Gamma_-} = \tilde{\sigma} = \begin{cases} 
\tilde{\sigma}_0 & \text{on} \ \{ 0 \} \times \Omega \times \mathbb{R}^+, \\
\tilde{\sigma}_1 & \text{on} \ \mathbb{R}^+ \times \mathbb{R}^2 \times \{ 1 \} \times \mathbb{R}^+.
\end{cases}
\]

From a physical point of view, the function \( u(t, x, m) \) is the velocity of the droplet located at \( x \) with mass \( m \) and has approximately the following expression

\[
u(t, x, m) = v(t, x) - \frac{g}{\alpha(m)}e_3, \quad e_3 = (0, 0, 1)^T, \quad (t, x, m) \in \mathbb{R}^+ \times \Omega \times \mathbb{R}^+,
\]

where \( v(t, x) \) is the velocity of air, \( g \) is the gravitational acceleration and \( \alpha(m) \) is determined by air friction on droplet with mass \( m \). However, in this paper, we do not assume the expression (2.6) for \( u \). On the other hand, the function \( h_{gl}(t, x, m) \) is the amount of \( H_2O \) that turns from gas to liquid condensing on a droplet with mass \( m \) and \( Q = Q(t, x) \) is the difference between the vapor density \( \pi(t, x) \) and the density \( \pi_{vs}(T) \) of a saturated vapor relative to the liquid state at temperature \( T \); more precisely we have

\[
Q(t, x) = \pi(t, x) - \pi_{vs}(T(t, x)), \quad (t, x) \in \mathbb{R}^+ \times \Omega;
\]

therefore a well approximation for \( h_{gl} \) is given by the relation

\[
h_{gl}(t, x, m) = \eta(m)Q(t, x), \quad (t, x, m) \in \mathbb{R}^+ \times \Omega \times \mathbb{R}^+,
\]

where the coefficient \( \eta(m) \geq 0 \) is a lipschitz function and has a compact support in \( \mathbb{R}^+ \). Moreover \( \beta(m_1, m_2) \) is the encounter probability between a droplet with mass \( m_1 \) and another with mass \( m_2 \), whereas \( g_0(m)[N_1 - \bar{N}(\sigma)]^+ \) is the coefficient of appearance for droplets with mass \( m \) and \( g_1(m) \) is that of disappearance for droplets with mass \( m \); \( N_1 \) and \( \bar{N}(\sigma) \) are the number of aerosol compared to the unit of air volume and the number of aerosol already present in droplets respectively. (For details of physical meaning of these functions, see [17, 2]).
To determine the distribution for $\sigma$, we introduce two numbers $\overline{m}_a$ and $\overline{m}_A$ (with $0 < \overline{m}_a < \overline{m}_A < \infty$) and we consider that droplets are absent apart from an interval $[\overline{m}_a, \overline{m}_A]$, then we have

$$\sigma(m) = 0 \quad \text{ for } m \in [0, \overline{m}_a] \cup \overline{m}_A, \infty,$$

where $\overline{m}_a$, $\overline{m}_A$ correspond respectively to the lower mass and the upper mass of droplets (see \([\text{14}]\)).

Now we suppose that $\beta$ is a continuous function defined on $\mathbb{R}_+ \times \mathbb{R}_+$ such that

$$\beta(m_1, m_2) \geq 0, \quad \beta(m_1, m_2) = \beta(m_2, m_1) \quad \forall (m_1, m_2) \in \mathbb{R}_+ \times \mathbb{R}_+,$$

$$\max \left[ \sup_{0 < m' < m < \infty} \frac{m}{2} \beta(m - m', m'), \sup_{m, m' \in \mathbb{R}_+} m \beta(m, m') \right] < \infty,$$

$$\beta(m_1, m_2) = 0 \quad \text{ if } m_1 + m_2 \geq \overline{m}_A.$$

For the functions $g_0$ and $g_1$ we suppose that

$$g_0 \in \mathcal{C}(\mathbb{R}_+, \mathbb{R}_+) \quad \text{or} \quad g_1 \in \mathcal{C}(\mathbb{R}_+, \mathbb{R}_+) \quad \text{or} \quad g_1 \in \mathcal{C}(\mathbb{R}_+, \mathbb{R}_+), \quad \int_{m_1}^{m_2} g_1(m)dm = \infty, \quad m_1, m_2 \in [0, \overline{m}_a],$$

$$g_1 \geq 0.$$

For $\tilde{N}(\sigma)$ we assume that

$$\tilde{N}(\sigma)(t, x) = \int_0^\infty n(m)\sigma(t, x, m)dm, \quad \forall (t, x) \in \mathbb{R}_+ \times \Omega,$$

$$n(\cdot) \in L^1(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+), \quad n(m) \geq 0 \quad \forall m \in \mathbb{R}_+,$$

whereas $N_1$ is a strictly positive constant.

Now, we consider the stationary equation associated to (2.1)

$$\nabla_x \cdot (\sigma^\infty (x, m) u^*(x, m)) + \partial_m (m h_{gl}^*(x, m) \sigma^\infty (x, m)) =$$

$$= h_{gl}^*(x, m) \sigma^\infty (x, m) + \frac{m}{2} \int_0^m \beta(m - m', m')\sigma^\infty(x, m')\sigma^\infty(x, m - m')dm' +$$

$$-m \int_0^\infty \beta(m, m')\sigma^\infty(x, m)\sigma^\infty(x, m')dm' + g_0(m) [N_1 - \tilde{N}(\sigma^\infty)](x)[Q^*](x) +$$

$$-g_1(m)[Q^*](x)\sigma^\infty(x, m), \quad (x, m) \in \Omega \times \mathbb{R}_+,$$

with the following conditions

$$\sigma^\infty(x_1, x_2, 1, m) = \bar{\sigma}_1^*(x_1, x_2, m) \quad \text{ for } (x_1, x_2) \in \mathbb{R}^2, \ m \in \mathbb{R}_+,$$

where $\bar{\sigma}_1^*$ is a given function defined on $\mathbb{R}^2 \times \mathbb{R}_+$ and $u^*, h_{gl}^*$ are the functions defined as in (2.6), (2.8) respectively and independent on $t$. 
Afterwards, we suppose that
\begin{equation}
Q^* \in W^{1,\infty}(\Omega), \quad h^*_{gl}(x, m) = \eta(m)Q^*(x) \quad \text{for} \ (x, m) \in \Omega \times \mathbb{R}_+.
\end{equation}

Finally, we conclude this section rewriting the equations (2.1) and (2.16) respectively as
\begin{equation}
\partial_t \sigma + \tilde{U} \cdot \nabla_{(x, m)} \sigma = -\tilde{g} \sigma + \Phi[\sigma] - \sigma f[\sigma] + h[\sigma]
\end{equation}
and
\begin{equation}
\tilde{U}^* \cdot \nabla_{(x, m)} \sigma^\infty = -\tilde{g}^* \sigma^\infty + \Phi[\sigma^\infty] - \sigma^\infty f[\sigma^\infty] + h^*[\sigma^\infty],
\end{equation}
where
\begin{equation}
\nabla_{(x, m)} = (\partial_{x_1}, \partial_{x_2}, \partial_{x_3}, \partial_m)^T,
\end{equation}
\begin{equation}
\tilde{U}(t, x, m) = (u_1(t, x, m), u_2(t, x, m), u_3(t, x, m), mh_{gl}(t, x, m)),
\end{equation}
\begin{equation}
\tilde{g}(t, x, m) = \nabla_x \cdot u(t, x, m) + \partial_m(mh_{gl}(t, x, m)) - h_{gl}(t, x, m) + g_1(m)[Q(t, x)]^-,
\end{equation}
\begin{equation}
\Phi[\sigma](t, x, m) = \frac{m}{2} \int_0^m \beta(m - m', m') \sigma(t, x, m') \sigma(t, x, m - m') dm',
\end{equation}
\begin{equation}
f[\sigma](t, x, m) = m \int_0^\infty \beta(m, m') \sigma(t, x, m') dm',
\end{equation}
\begin{equation}
h[\sigma](t, x, m) = g_0(m)[N_1 - \tilde{N}(\sigma)(t, x)]^+[Q(t, x)]^+,
\end{equation}
\begin{equation}
\tilde{U}^*(x, m) = (u_1^*(x, m), u_2^*(x, m), u_3^*(x, m), mh_{gl}^*(x, m)),
\end{equation}
\begin{equation}
\tilde{g}^*(x, m) = \nabla_x \cdot u^*(x, m) + \partial_m(mh_{gl}^*(x, m)) - h_{gl}^*(x, m) + g_1(m)[Q^*(x)]^-,
\end{equation}
\begin{equation}
h^*[\sigma^\infty](x, m) = g_0(m)[N_1 - \tilde{N}(\sigma^\infty)(x)]^+[Q^*(x)]^+,
\end{equation}
with \((t, x, m) \in \mathbb{R}_+ \times \Omega \times \mathbb{R}_+\).
3 Assumptions and generalized solutions.

We make the following assumptions on the velocity $u$ and $Q$

\[(3.1) \quad u \in C_b(\mathbb{R}_+ \times \Omega \times \mathbb{R}_+^3) \cap L^1_{loc}(\mathbb{R}_+; W^{1,\infty}(\Omega \times \mathbb{R}_+^3)),\]
\[(3.2) \quad L^1_{loc}(\mathbb{R}_+; W^{1,\infty}(\Omega \times \mathbb{R}_+^3)) = \bigcap_{T>0} L^1((0,T); W^{1,\infty}(\Omega \times \mathbb{R}_+^3)),\]
\[(3.3) \quad L^1_{x_3}(0,1; W^{1,\infty}_{(t_{loc},x_1,x_2,m)}(\mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{R}_+)) = \bigcap_{T>0} L^1((0,1); W^{1,\infty}((0,T) \times \mathbb{R}^2 \times \mathbb{R}_+)) ;\]

and moreover there exists a strictly positive constant $A_0$ such that

\[(3.4) \quad u_3(t,x,m) \leq -A_0, \quad \forall (t,x,m) \in \mathbb{R}_+ \times \Omega \times \mathbb{R}_+\]

and we suppose that

\[(3.5) \quad Q \in L^1_{loc}(\mathbb{R}_+; W^{1,\infty}(\Omega)).\]

Therefore the condition (3.4) determines the fall of rain from the strip.

Afterwards, for data $\sigma_0$ and $\sigma_1$, we suppose that

\[(3.6) \quad \sigma_0 \in C_b(\Omega \times \mathbb{R}_+), \quad \sigma_0 \geq 0,\]
\[(3.7) \quad \sigma_1 \in C_b(\mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{R}_+), \quad \sigma_1 \geq 0,\]
\[(3.8) \quad \sigma_0(x,m) = 0, \quad \sigma_1(t,x_1,x_2,m) = 0 \quad \text{if} \quad (t,x) \in \mathbb{R}_+ \times \Omega, \quad m \notin [\overline{m}_q, \overline{m}_A].\]

Furthermore, we assume that

\[(3.9) \quad IK(1 - e^{-J}) < J,\]
\[(3.10) \quad \|\tilde{\sigma}\|_{L^\infty(\Gamma_-)} < \frac{IJ e^{-J}}{2(J + KI(1 - e^{-J}))},\]

where

\[(3.11) \quad I = \|\tilde{\sigma}\|_{L^\infty(\Gamma_-)} + (1/A_0) \|g_0\|_{L^\infty(\mathbb{R}_+)} N_1 \|Q\|_{L^\infty(\mathbb{R}_+ \times \Omega)} ;\]
\[(3.12) \quad J = (1/A_0) \left( \|
abla \cdot u\|_{L^\infty(\mathbb{R}_+ \times \Omega \times \mathbb{R}_+)} + \|\partial_m (mh_I)\|_{L^\infty(\mathbb{R}_+ \times \Omega \times \mathbb{R}_+)} + \|g_1\|_{L^\infty(\mathbb{R}_+)} \|Q\|_{L^\infty(\mathbb{R}_+ \times \Omega)} + \|g_0\|_{L^\infty(\mathbb{R}_+)} \|n\|_{L^1(\mathbb{R}_+)} \|Q\|_{L^\infty(\mathbb{R}_+ \times \Omega)} \right).\]
\( K = \frac{1/A_0}{\sup_{m \in \mathbb{R}_+} \left| m \int_0^{+\infty} \beta(m, m') \, dm' \right| + \sup_{m \in \mathbb{R}_+} \left| \frac{m}{2} \int_0^{m} \beta(m - m', m') \, dm' \right|} \).

Now, for the stationary speed \( u^* \) we make the following assumptions

\( u^* \in W^{1,\infty}(\Omega \times \mathbb{R}_+; \mathbb{R}^3), \quad \nabla \cdot u^* \in C_b(\Omega \times \mathbb{R}_+) \),

\( u_3^*(x, m) \leq -A_0 \quad \forall (x, m) \in \Omega \times \mathbb{R}_+ \).

Hence, for \( \tilde{\sigma}_1^* \) we assume that

\( \tilde{\sigma}_1^* \in C_b(\mathbb{R}^2 \times \mathbb{R}_+), \quad \tilde{\sigma}_1^* \geq 0, \)

\( \tilde{\sigma}_1^*(x_1, x_2, m) = 0 \quad \text{if} \quad (x_1, x_2) \in \mathbb{R}^2, \quad m \not\in [\bar{m}_a, \bar{m}_A], \)

\( \| \tilde{\sigma}_1^* \|_{L^\infty(\mathbb{R}^2 \times \mathbb{R}_+)} < \frac{I^* J^* e^{-J^*}}{2(J^* + K I^*(1 - e^{-J^*}))}, \)

where

\( I^* = \| \tilde{\sigma}_1^* \|_{L^\infty(\mathbb{R}^2 \times \mathbb{R}_+)} + (1/A_0) \| g_0 \|_{L^\infty(\mathbb{R}_+)} N_1 \| Q^* \|_{L^\infty(\Omega)}, \)

\( J^* = (1/A_0) \left( \| \nabla \cdot u^* \|_{L^\infty(\Omega \times \mathbb{R}_+)} + \| \partial_m (m h_{gl}) \|_{L^\infty(\Omega \times \mathbb{R}_+)} + \| g_1 \|_{L^\infty(\mathbb{R}_+)} \| Q^* \|_{L^\infty(\Omega)} + \| g_0 \|_{L^\infty(\mathbb{R}_+)} \| n \|_{L^1(\mathbb{R}_+)} \| Q^* \|_{L^\infty(\Omega)} \right), \)

\( I^* K (1 - e^{-J^*}) < J^*. \)

Now, after introducing key assumptions about the velocities and data, we are ready to give the definitions of generalized solutions for the problems \((2.11) \) (or \((2.19)\)), \((2.5) \) and \((2.16) \) (or \((2.20)\), \((2.17)\)).

First of all, we consider the following Cauchy’s problem related to the flow \( X \) associated to the vector field \( \bar{U} \)

\[
\begin{aligned}
\frac{d}{ds} t(s) &= 1, \\
\frac{d}{ds} X(s) &= \bar{U}(t(s), X(s)), \\
(t(0), X(0)) &= (\bar{t}, \bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{m}) \in \Gamma_-, 
\end{aligned}
\]

where \( X(s) = (X_1(s), X_2(s), X_3(s), M(s)). \) We observe that the first equation of \((3.22)\) gives

\[ t(s) = \bar{t} + s. \]

On the other hand, thanks to the assumptions on \( u \) and \( h_{gd} \) (see also the hypothesis \((3.5)\) about \( Q \)), we deduce there exists one and only one solution \( X(\bar{t}, \bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{m}; \cdot) \) on
that is equivalent, in Caratheodory’s theory, to the integral equation (3.28) in the following form

\[
\frac{d}{ds}(\sigma(t + s, X(t, x_1, x_2, x_3, m; s)) = -\sigma(t + s, X(t, x_1, x_2, x_3, m; s)) \times \\
\times [g(t + s, X(t, x_1, x_2, x_3, m; s)) + f[\sigma](t + s, X(t, x_1, x_2, x_3, m; s))] + \\
+ \Phi[\sigma](t + s, X(t, x_1, x_2, x_3, m; s)) + h[\sigma](t + s, X(t, x_1, x_2, x_3, m; s)),
\]

(3.25)

\[
\sigma(t, x_1, x_2, x_3, m) = \tilde{\sigma}(t, x_1, x_2, x_3, m) \quad \forall (t, x_1, x_2, x_3, m) \in \Gamma_-, 
\]

that is equivalent, in Caratheodory’s theory, to the integral equation (3.26)

\[
\sigma(t + s, X(t, x_1, x_2, x_3, m; s)) = \tilde{\sigma}(t, x_1, x_2, x_3, m) - \int_0^s \left\{ \sigma(t + r, X(t, x_1, x_2, x_3, m; r)) \times \\
\times [\tilde{g}(t + r, X(\tilde{t}, \tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{m}; r)) + f[\sigma](t + r, X(\tilde{t}, \tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{m}; r))] + \\
+ \Phi[\sigma](t + r, X(\tilde{t}, \tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{m}; r)) + h[\sigma](t + r, X(\tilde{t}, \tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{m}; r)) \right\} dr.
\]

Hence, we give the following definition

**Definition 3.1.** A continuous solution \( \sigma \) for the integral equation (3.26) is called a generalized solution for the problem (2.1) and (2.5).

Now, in a similar way, we can transform the stationary problem (2.16)-(2.17) in the following form

\[
\frac{d}{ds}\sigma^\infty(X^*(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{m}; s)) = -\sigma^\infty(X^*(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{m}; s)) \times \\
\times [\tilde{g}^*(X^*(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{m}; s)) + f[\sigma^\infty](X^*(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{m}; s))] + \\
+ \Phi[\sigma^\infty](X^*(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{m}; s)) + h^*[\sigma^\infty](X^*(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{m}; s)),
\]

(3.28)

\[
\sigma^\infty(\tilde{x}_1, \tilde{x}_2, 1, \tilde{m}) = \tilde{\sigma}^*_1(\tilde{x}_1, \tilde{x}_2, \tilde{m}) \quad \forall (\tilde{x}_1, \tilde{x}_2, \tilde{m}) \in \mathbb{R}^2 \times \mathbb{R}_+,
\]

where the flow \( X^*(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{m}; s) \) is the solution of the following problem

\[
\begin{align*}
\frac{d}{ds}X^*(s) &= \tilde{U}^*(X^*(s)), \\
X(0) &= (\tilde{x}_1, \tilde{x}_2, 1, \tilde{m}),
\end{align*}
\]

(3.29)

with \( X^*(s) = (X_1^*(s), X_2^*(s), X_3^*(s), M^*(s)) \).

Of course, the problem (3.27)-(3.28) is equivalent, in Caratheodory’s theory, to the following integral equation

\[
\sigma^\infty(X^*(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{m}; s)) = \tilde{\sigma}^*_1(\tilde{x}_1, \tilde{x}_2, \tilde{m}) - \int_0^s \left\{ \sigma^\infty(X^*(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{m}; r)) \times \\
\times [\tilde{g}^*(X^*(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{m}; r)) + f[\sigma^\infty](X^*(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{m}; r))] + \\
+ \Phi[\sigma^\infty](X^*(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{m}; r)) + h^*[\sigma^\infty](X^*(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{m}; r)) \right\} dr.
\]

(3.30)
\[\times [g^*(X^*(\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{m}; r)) + f[\sigma^\infty](X^*(\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{m}; r))] + \\
+ \Phi[\sigma^\infty](X^*(\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{m}; r)) + h^*[\sigma^\infty](X^*(\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{m}; r)) \right\} dr.

Afterwards, we give the following definition of a generalized solution for the stationary equation

**Definition 3.2.** A continuous solution \(\sigma^\infty\) for the integral equation (3.30) is called a generalized solution for the problem (2.16) and (2.17).

### 4 Main results.

Now, we are ready to give the first important result of this paper.

**Theorem 4.1.** We assume all hypotheses stated above on all functions involved in the problem (2.1) and (2.5). Then there exists one and only one generalized solution \(\sigma\) for the problem (2.1) and (2.5), such that

\[(4.1) \quad \sigma \in C_b(\mathbb{R}_+ \times \Omega \times \mathbb{R}_+), \quad \sigma \geq 0, \quad \text{supp} \ \sigma \subseteq \mathbb{R}_+ \times \Omega \times [0, \bar{m}_B],\]

where

\[(4.2) \quad \bar{m}_B = \bar{m}_A \exp \left( (1/A_0) \left\| h_{\text{gl}}^{\infty} \right\|_{\infty} \right).

Furthermore the solution \(\sigma\) verifies the inequality

\[(4.3) \quad \left\| \sigma \right\|_{L^\infty(\mathbb{R}_+ \times \Omega \times \mathbb{R}_+)} \leq \frac{IJ e^{-J}}{J + KI(1 - e^{-J})}.

(for \(I, J, K\), see (3.11)-(3.13)).

Finally, we can give the last main theorem of this paper.

**Theorem 4.2.** We assume all hypotheses stated above on all functions involved in the problem (2.16) and (2.17). Then there exists one and only one generalized solution for the problem (2.16) and (2.17), such that

\[\sigma^\infty \in C_b(\Omega \times \mathbb{R}_+), \quad \sigma^\infty \geq 0, \quad \text{supp} \ \sigma^\infty \subseteq \Omega \times [0, \bar{m}_B],\]

where

\[(4.5) \quad \bar{m}_B = \bar{m}_A \exp \left( (1/A_0) \left\| h_{\text{gl}}^{\infty} \right\|_{\infty} \right).

Furthermore the solution \(\sigma^\infty\) verifies the inequality

\[\|\sigma^\infty\|_{L^\infty(\Omega \times \mathbb{R}_+)} \leq \frac{I^* J^* e^{-J^*}}{J^* + KI^*(1 - e^{-J^*)}}.

(for \(I^*, J^*\), see (3.19)-(3.20)).

Finally, if we assume

\[g_1(m) = 0, \quad \eta(m) = 0 \quad \text{if} \ m \notin [\bar{m}_a, \bar{m}_B],\]
(4.8) \( \tilde{\sigma}_1 \to L^\infty(\Omega \times \mathbb{R}_+) \tilde{\sigma}_1^*, \quad Q(t, \cdot) \to L^\infty(\Omega) Q^*(\cdot), \)

(4.9) \( u(t, \cdot) \to L^\infty(\Omega \times \mathbb{R}_+; \mathbb{R}^3) u^*(\cdot), \quad \nabla\cdot u(t, \cdot) \to L^\infty(\Omega \times \mathbb{R}_+) \nabla\cdot u^*(\cdot), \) for \( t \to \infty, \)

then

(4.10) \( \sigma(t, \cdot) \to L^\infty(\Omega \times \mathbb{R}_+) \sigma^\infty(\cdot), \) for \( t \to \infty, \)

where \( \sigma \) is defined in Theorem 4.1.

5 Linearization.

First of all, we observe that from the second equation of Cauchy’s problem (3.22) follows

(5.1) \( \frac{dm}{dx_3} = \frac{m h_{gl}}{u_3} \leq m \frac{\|h_{gl}\|_{L^\infty(\mathbb{R}_+ \times \Omega \times \mathbb{R}_+)}}{A_0}. \)

If, we define \( m_B \) such that

(5.2) \( \int \frac{m}{m_A} \frac{dm}{dx_3} = \int_0^1 \frac{\|h_{gl}\|_{L^\infty(\mathbb{R}_+ \times \Omega \times \mathbb{R}_+)}}{A_0} dx_3, \)

then we deduce (4.2). After defining \( m_B \), we consider the following cone \( K_+ \) of \( C_b(\mathbb{R}_+ \times \Omega \times \mathbb{R}_+) \)

(5.3) \( K_+ = \{ \lambda \mid \lambda \in C_b(\mathbb{R}_+ \times \Omega \times \mathbb{R}_+), \lambda \geq 0, \text{ supp } \lambda \subseteq \mathbb{R}_+ \times \Omega \times [0, m_B] \}. \)

In this section we study the linear differential equation

(5.4) \( \frac{d}{ds} \sigma(t + s, X(t, \tilde{\tau}_1, \tilde{\tau}_2, \tilde{\tau}_3, \tilde{m}; s)) = -\sigma(t + s, X(t, \tilde{\tau}_1, \tilde{\tau}_2, \tilde{\tau}_3, \tilde{m}; s)) \times \)

\( \times [\tilde{g}(t + s, X(t, \tilde{\tau}_1, \tilde{\tau}_2, \tilde{\tau}_3, \tilde{m}; s)) + f[\overline{\sigma}](t + s, X(t, \tilde{\tau}_1, \tilde{\tau}_2, \tilde{\tau}_3, \tilde{m}; s)) + \]

\( + \Phi[\overline{\sigma}](t + s, X(t, \tilde{\tau}_1, \tilde{\tau}_2, \tilde{\tau}_3, \tilde{m}; s)) + h[\overline{\sigma}](t + s, X(t, \tilde{\tau}_1, \tilde{\tau}_2, \tilde{\tau}_3, \tilde{m}; s)), \)

where \( \overline{\sigma} \in K_+. \) A first result about this ODE is the following

Lemma 5.1. The equation (5.4) with the condition (3.25) has one and only one solution \( \sigma(t + \cdot, X(t, \tilde{\tau}_1, \tilde{\tau}_2, \tilde{\tau}_3, \tilde{m}; \cdot)) \) on the interval \([0, \overline{\tau}_1] \), where

(5.5) \( X_3(t, \tilde{\tau}_1, \tilde{\tau}_2, \tilde{\tau}_3, \tilde{m}; \overline{\tau}_1) = 0 \)

and \( \overline{\tau}_1 \) satisfies the inequality

(5.6) \( 0 < \overline{\tau}_1 \leq \frac{1}{A_0}. \)
Moreover we have

\begin{equation}
\sigma(\bar{t} + s, X(\bar{t}, \bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{m}; s)) \geq 0 \quad \forall s \in [0, \bar{\sigma}_1],
\end{equation}

\begin{equation}
\sigma(\bar{t} + s, X(\bar{t}, \bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{m}; s)) = 0 \quad \text{if } M(\bar{t}, \bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{m}; \bar{\sigma}_1) > \overline{m}_B.
\end{equation}

**Proof.** As \( X(\bar{t}, \bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{m}; \cdot) \) is well defined and the equation \((5.4)\) is linear, the existence and the uniqueness of the solution for the problem \((5.4), (3.25)\) on \([0, \bar{\sigma}_1]\) follow from the classical theory. To determine \((5.6)\), it is sufficient to observe that

\begin{equation}
\frac{d}{ds} X_3(\bar{t}, \bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{m}; s) = u_3(\bar{t} + s, X(\bar{t}, \bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{m}; s)) \leq -A_0.
\end{equation}

Therefore, the solution \(\sigma(\bar{t} + \cdot, X(\bar{t}, \bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{m}; \cdot))\) of the problem \((5.4), (3.25)\) can be so represented on the interval \([0, \bar{\sigma}_1]\) by the expression

\begin{equation}
\sigma(\bar{t} + s, X(\bar{t}, \bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{m}; s)) = \tilde{\sigma}(\bar{t}, X(\bar{t}, \bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{m})) \times
\end{equation}

\begin{equation}
\times \exp \left\{ \int_0^s (\tilde{g} + f[\sigma])(\bar{t} + s', X(\bar{t}, \bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{m}, s')) ds' \right\} +
\end{equation}

\begin{equation}
+ \int_0^s \left[ (\Phi[\sigma] + h[\sigma])(\bar{t} + s', X(\bar{t}, \bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{m}, s')) \times
\end{equation}

\begin{equation}
\times \exp \left( \int_{s'}^s (\tilde{g} + s'', X(\bar{t}, \bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{m}; s'')) ds'' \right) ds' \right\}.
\end{equation}

Consequently \((5.7)\) can be directly obtained from \((5.10)\).

Now to prove \((5.8)\) it is sufficient to show that

a) \(\sigma(\bar{t} + s, X(\bar{t}, \bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{m}; s)) = 0\) if \(\bar{m} > \overline{m}_A\);

b) \(M(\bar{t}, \bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{m}; s) \leq \overline{m}_B\) if \(\bar{m} \leq \overline{m}_A\).

The relation a) can be deduced from the consideration that \(M(\bar{t}, \bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{m}; \cdot)\) is non decreasing function and that the conditions \((3.3), (2.12), (2.13)\) are satisfied. To obtain b), we observe that

\begin{equation}
\frac{d}{ds} M(\bar{t}, \bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{m}; s) \leq \| h_{gl} \|_{L^\infty(\mathbb{R}_+ \times \Omega \times \mathbb{R}_+)} M(\bar{t}, \bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{m}; s),
\end{equation}

therefore we have

\begin{equation}
M(\bar{t}, \bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{m}; s) \leq \overline{m} \exp \left( s \| h_{gl} \|_{L^\infty(\mathbb{R}_+ \times \Omega \times \mathbb{R}_+)} \right) \leq \overline{m}_B.
\end{equation}

\[\square\]

Now we are ready to give the following important result

**Lemma 5.2.** Let us assume \(\sigma \in K_+\) and \(\sigma\) the solution of the problem \((5.4), (3.25)\). Then \(\sigma\) is a function on \(\mathbb{R}_+ \times \Omega \times \mathbb{R}_+\) that belongs to \(K_+\).

**Proof.** As the relations \((5.7)\) and \((5.8)\) are already established, we just have to show that \(\sigma\) is a continuous function on \(\mathbb{R}_+ \times \Omega \times \mathbb{R}_+\). If we denote by \(\tau_- (t, x_1, x_2, x_3, m)\) the
minimum time of existence for \(X(t, x_1, x_2, x_3, m; \cdot)\) (see \[19]\), then we have the following representation formula for \(\sigma\)

\[
\sigma(t, x_1, x_2, x_3, m) = \tilde{\sigma}(\tau_-(t, x_1, x_2, x_3, m), X(t, x_1, x_2, x_3, m; \tau_-(t, x_1, x_2, x_3, m))) + \\
+ \int_{\tau_-(t, x_1, x_2, x_3, m)}^t \left[ \sigma(-\tilde{g} - f[\varphi]) + \Phi[\varphi] + h[\varphi] \right](s, X(t, x_1, x_2, x_3, m; s)) \, ds.
\]

Now, from the hypotheses (3.1)-(3.5) follow that, thanks to Lemma 4.2 in \[19]\, \(\tau_- \in W^{1,\infty}(t_{loc}, x_1, x_2, x_3, m)(\mathbb{R}^+ \times \Omega \times \mathbb{R}^+) = \bigcap_{T>0} W^{1,\infty}((0, T) \times \Omega \times \mathbb{R}^+)\) and \(X \circ \tau_- \in W^{1,\infty}(t_{loc}, x_1, x_2, x_3, m)(\mathbb{R}^+ \times \Omega \times \mathbb{R}^+)\), therefore, remembering that the given functions appearing in (5.13) are continuous, we deduce that \(\sigma\) is continuous. \(\Box\)

**Lemma 5.3.** Let \(\delta\) be a number such that \(0 < \delta \leq 1\). We suppose that there exists a solution \(\sigma\) on \(\mathbb{R}^+ \times \Omega_\delta \times \mathbb{R}^+\), where \(\Omega_\delta = \mathbb{R}^2 \times (1 - \delta, 1)\), for the equation (3.24) with the following condition

\[
\sigma(\tilde{t}, \tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{m}) = \tilde{\sigma}(\tilde{t}, \tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{m}) \quad \forall(\tilde{t}, \tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{m}) \in \Gamma_{\delta_-},
\]

where \(\Gamma_{\delta_-} = [\{0\} \times \mathbb{R}^2 \times (1 - \delta, 1) \times \mathbb{R}^+] \cup [\mathbb{R}^+ \times \mathbb{R}^2 \times \{1\} \times \mathbb{R}^+]\). Under these hypotheses, we have

\[
\|\sigma\|_{L^\infty(\mathbb{R}^+ \times \Omega_\delta \times \mathbb{R}^+)} \leq \frac{IJ \, e^{-J}}{J + KI(1 - e^{-J})}.
\]

**Proof.** From the equations relative to the flow associated to the vector field \(\tilde{U}\) (see (3.22)), we have

\[
\frac{d}{dx_3} \sigma(s, X(\tilde{t}, \tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{m}; s)) = \frac{1}{u_3} ds \sigma(s, X(\tilde{t}, \tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{m}; s)).
\]

Therefore, performing integration on (3.24) suitably transformed and after some calculations, we obtain

\[
\|\sigma(\cdot, x_3, \cdot)\|_{L^\infty(\mathbb{R}^+ \times \mathbb{R}^2 \times \mathbb{R}^+)} \leq \|\tilde{\sigma}\|_{L^\infty(\Gamma_{\delta_-})} + \\
+ \int_{x_3}^1 \left( \frac{1}{A_0} \|\sigma(\cdot, z, \cdot)\|_{L^\infty(\mathbb{R}^+ \times \mathbb{R}^2 \times \mathbb{R}^+)} \left( \|\tilde{g}\|_{L^\infty(\mathbb{R}^+ \times \Omega_\delta \times \mathbb{R}^+)} + \|f[\sigma](\cdot, z, \cdot, \cdot)\|_{L^\infty(\mathbb{R}^+ \times \mathbb{R}^2 \times \mathbb{R}^+)} \right) dz + \\
+ \int_{x_3}^1 \left( \frac{1}{A_0} \|\Phi[\sigma](\cdot, z, \cdot)\|_{L^\infty(\mathbb{R}^+ \times \mathbb{R}^2 \times \mathbb{R}^+)} + \|h[\sigma](\cdot, z, \cdot)\|_{L^\infty(\mathbb{R}^+ \times \mathbb{R}^2 \times \mathbb{R}^+)} \right) dz \right.
\]

\[
\leq I + \int_{x_3}^1 \left( J \|\sigma(\cdot, z, \cdot)\|_{L^\infty(\mathbb{R}^+ \times \mathbb{R}^2 \times \mathbb{R}^+)} + K \|\sigma(\cdot, z, \cdot)\|_{L^\infty(\mathbb{R}^+ \times \mathbb{R}^2 \times \mathbb{R}^+)}^2 \right) dz.
\]

Finally, taking into account the condition (3.9) and using the techniques of comparison lemma we deduce (5.15). \(\Box\)

**Remark 5.4.** We observe that the estimate (5.15) does not depending on \(\delta\).
6 Proof of the Theorem 4.1.

We start to prove that there exists $0 < \delta < 1$ such that the equation (3.24) on $\Omega_\delta$ with the condition (5.14) has one and only one solution. For this purpose, we consider the following operator $\Lambda_\delta : K_\delta \to C_b(\mathbb{R}_+ \times \Omega_\delta \times \mathbb{R}_+)$, where the domain of $K_\delta$ is so defined

\begin{equation}
\tag{6.1}
K_\delta = \left\{ \bar{\sigma} \in C_b(\mathbb{R}_+ \times \Omega_\delta \times \mathbb{R}_+) \mid \bar{\sigma} \geq 0, \ \text{supp} \ \bar{\sigma} \subseteq \mathbb{R}_+ \times \Omega_\delta \times [0, \bar{m}_B] \right\},
\end{equation}

and $\Lambda_\delta$ is the operator that to each $\bar{\sigma} \in K_\delta$ associates the solution $\sigma$ to the linear equation (5.4) with the condition (5.14).

Now, we determine an upper bound for $\delta$ such that $\Lambda_\delta(K_\delta) \subseteq K_\delta$. Using for the equation (5.4) an analogous method to that seen for (3.24) in lemma 5.3, we deduce

\begin{equation}
\|\sigma(\cdot, x_3, \cdot)\|_{L^\infty(\mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{R}_+)} \leq \|\bar{\sigma}\|_{L^\infty(\Gamma_\delta^-)} + \int_{x_3}^1 \mathcal{F}(\|\sigma(\cdot, z, \cdot)\|_{L^\infty(\mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{R}_+)})dz,
\end{equation}

where $x_3 \in \lbrack 1 - \delta, 1 \rbrack$ and $\mathcal{F}(\cdot)$ is

\begin{equation}
\mathcal{F}(\|\sigma(\cdot, z, \cdot)\|_{L^\infty(\mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{R}_+)}) = (1/A_0) \left( \|\sigma(\cdot, z, \cdot)\|_{L^\infty(\mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{R}_+)} \right.
\end{equation}

\begin{equation}
\times \left( \|\bar{\sigma}\|_{L^\infty(\mathbb{R}_+ \times \Omega_\delta \times \mathbb{R}_+)} + \|f[\bar{\sigma}](\cdot, z, \cdot)\|_{L^\infty(\mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{R}_+)} + \right.
\end{equation}

\begin{equation}
\left. \left( \|\Phi[\bar{\sigma}](\cdot, z, \cdot)\|_{L^\infty(\mathbb{R}_+ \times \Omega_\delta \times \mathbb{R}_+)} + \right) \right) + \|h[\bar{\sigma}](\cdot, z, \cdot)\|_{L^\infty(\mathbb{R}_+ \times \Omega_\delta \times \mathbb{R}_+)} \right)
\end{equation}

\begin{equation}
\leq C(1 + \|\sigma(\cdot, z, \cdot)\|_{L^\infty(\mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{R}_+)})
\end{equation}

where $C$ is a constant not depending on $\delta$ and $\bar{\sigma}$. Therefore, using Gronwall’s lemma we have

\begin{equation}
\|\sigma\|_{L^\infty(\mathbb{R}_+ \times \Omega_\delta \times \mathbb{R}_+)} \leq \exp(C\delta)(\|\bar{\sigma}\|_{L^\infty(\Gamma_\delta^-)} + C\delta).
\end{equation}

Hence, using (3.10), we prove that there exists $0 < \delta_1 < 1$ such that

\begin{equation}
\tag{6.5}
\|\sigma\|_{L^\infty(\mathbb{R}_+ \times \Omega_\delta \times \mathbb{R}_+)} \leq \frac{IJ e^{-J}}{J + KI(1 - e^{-J})},
\end{equation}

for every $0 < \delta < \delta_1$. Therefore, thanks to Lemma 5.2 and (6.5) we deduce that $\Lambda_\delta(K_\delta) \subseteq K_\delta$ if $0 < \delta < \delta_1$.

We proceed to study $\Lambda_\delta$ and in particular we want to establish an upper bound for $\delta$ such that this map is a contraction. For this purpose, we consider $\bar{\sigma}_1, \bar{\sigma}_2 \in K_\delta$ and we define $\sigma_j = \Lambda_\delta(\bar{\sigma}_j)$ with $j = 1, 2$. Representing the solutions $\sigma_1, \sigma_2$ in integral form, we deduce, after some transformations, the inequality

\begin{equation}
\|\sigma_2(\cdot, x_3, \cdot) - \sigma_1(\cdot, x_3, \cdot)\|_{L^\infty(\mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{R}_+)} \leq ...
\end{equation}
\[ \leq \frac{1}{A_0} \left[ \| \tilde{f} \|_{L^\infty(\mathbb{R}_+ \times \Omega_3 \times \mathbb{R}_+)} + \| f[\sigma_2] \|_{L^\infty(\mathbb{R}_+ \times \Omega_3 \times \mathbb{R}_+)} \right] \times \\
\times \int_{x_3}^{1} \| \sigma_2(\cdot, z, \cdot) - \sigma_1(\cdot, z, \cdot) \|_{L^\infty(\mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{R}_+)} dz + \\
+ \delta \left( \| \sigma_1 \|_{L^\infty(\mathbb{R}_+ \times \Omega_3 \times \mathbb{R}_+)} \| \tilde{f} \| - \| f[\sigma_1] \|_{L^\infty(\mathbb{R}_+ \times \Omega_3 \times \mathbb{R}_+)} + \\
+ \| \Phi(\sigma_2) - \Phi(\sigma_1) \|_{L^\infty(\mathbb{R}_+ \times \Omega_3 \times \mathbb{R}_+)} + \| h(\sigma_2) - h(\sigma_1) \|_{L^\infty(\mathbb{R}_+ \times \Omega_3 \times \mathbb{R}_+)} \right) \leq \\
\leq C\delta \| \sigma_2 - \sigma_1 \|_{L^\infty(\mathbb{R}_+ \times \Omega_3 \times \mathbb{R}_+)} + C \int_{x_3}^{1} \| \sigma_2(\cdot, x_3, \cdot) - \sigma_1(\cdot, z, \cdot) \|_{L^\infty(\mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{R}_+)} dz, \]

where \( C \) is a constant not depending on \( \delta, \sigma_1 \) and \( \sigma_2 \). Hence, a direct application of Gronwall’s lemma gives

\[ (6.7) \quad \| \sigma_2 - \sigma_1 \|_{L^\infty(\mathbb{R}_+ \times \Omega_3 \times \mathbb{R}_+)} \leq C\delta \exp(C\delta) \| \sigma_2 - \sigma_1 \|_{L^\infty(\mathbb{R}_+ \times \Omega_3 \times \mathbb{R}_+)} \cdot \]

Therefore there exists \( 0 < \delta_2 < \delta_1 \) such that \( \Lambda_\delta \) is a contraction for every \( 0 < \delta < \delta_2 \). Consequently, the map \( \Lambda_\delta \) admits one and only one fixed point if \( 0 < \delta < \delta_2 \).

We have thus proved that the problem (4.24), (5.14) has one and only one solution on \( \Omega_3 \) if we assume \( 0 < \delta < \delta_2 \). Now, taking into account the estimate (5.15) and using a simple absurd reasoning about the maximal width of the strip on which the solution is defined, we arrive to show that the solution is defined on the whole strip \( \mathbb{R}_+ \times \Omega \times \mathbb{R}_+ \). Therefore, the main result has been proved. \( \square \)

7 Proof of the Theorem 4.2.

The proof of the existence and the uniqueness of generalized solution \( \sigma^\infty \) together with the properties (4.4)-(4.6) can be showed in analogous way to that seen for the generalized solution \( \sigma \) in Theorem 4.1. Therefore, we only prove the last part of this theorem, i.e. we show the convergence \( \sigma(t, \cdot) \to \sigma^\infty \), in \( L^\infty(\Omega \times \mathbb{R}_+) \), for \( t \to \infty \).

Let \( \delta \) be a number such that \( \tilde{t} > \frac{1}{A_0} \). Hence, we have \( \sigma(\tilde{t}, \bar{x}_1, \bar{x}_2, 1, \bar{m}) = \bar{\sigma}_1(\tilde{t}, \bar{x}_1, \bar{x}_2, \bar{m}) \) in the integral equation (4.20). Therefore, we can deduce

\[ (7.1) \quad \sigma(\tilde{t} + s, X(\tilde{t}, \bar{x}, \bar{m}, s)) = \bar{\sigma}_1(\tilde{t}, \bar{x}_1, \bar{x}_2, \bar{m}) + \\
\quad \quad - \int_{0}^{s} \left( (f[\sigma](\tilde{t} + s', X(\tilde{t}, \bar{x}, \bar{m}, s'))) + \bar{g}(\tilde{t} + s', X(\tilde{t}, \bar{x}, \bar{m}, s')) \right) \sigma(\tilde{t} + s', X(\tilde{t}, \bar{x}, \bar{m}, s')) + \\
\quad \quad + (\Phi[\sigma] + h[\sigma])(\tilde{t} + s', X(\tilde{t}, \bar{x}, \bar{m}, s')) ds', \quad \text{if} \quad \tilde{t} > \frac{1}{A_0}. \]

On the other hand, assuming \( X(0) = X^*(0) \) and recalling the definitions of the characteristics \( X(\tilde{t}, \bar{x}, \bar{m}, s) \) and \( X^*(\bar{x}, \bar{m}, s) \) given in (3.22) and (3.29), we obtain the following estimate

\[ (7.2) \quad |X(\tilde{t}, \bar{x}, \bar{m}, s) - X^*(\bar{x}, \bar{m}, s)| \leq \]
\[ \leq \int_0^s \left( |\tilde{U}(\tilde{t} + s', X(\tilde{t}, \tilde{x}, \tilde{\mu}, s')) - \tilde{U}^*(X(\tilde{t}, \tilde{x}, \tilde{\mu}, s'))| + |\tilde{U}^*(X(\tilde{t}, \tilde{x}, \tilde{\mu}, s')) - \tilde{U}^*(X^*(\tilde{x}, \tilde{\mu}, s'))| \right) ds'. \]

Remembering (2.18), (3.14), we have that there exists a constant \( C > 0 \) such that

\[ |X(\tilde{t}, \tilde{x}, \tilde{\mu}, s) - X^*(\tilde{x}, \tilde{\mu}, s)| \leq \int_0^s \left( \|u(\tilde{t} + s', \cdot) - u^*(\cdot)\|_{L^\infty(\Omega \times \mathbb{R}^3)} + C\|Q(\tilde{t} + s', \cdot) - Q^*(\cdot)\|_{L^\infty(\Omega)} + C|X(\tilde{t}, \tilde{x}, \tilde{\mu}, s') - X^*(\tilde{x}, \tilde{\mu}, s')| \right) ds'. \]

Now, applying the comparison lemma to (7.3) and using (4.9), we obtain the following useful convergence

\[ |X(\tilde{t}, \tilde{x}, \tilde{\mu}, s) - X^*(\tilde{x}, \tilde{\mu}, s)| \to_{\tilde{t} \to \infty} 0. \]

Afterwards, a direct application of (7.4) into (7.1) gives

\[ \sigma(\tilde{t} + s, X^*(\tilde{x}, \tilde{\mu}, s)) = \sigma_1(\tilde{t}, \tilde{x}_1, \tilde{x}_2, \tilde{\mu}) + \]

\[ - \int_0^s \left( (f[\sigma](\tilde{t} + s', X^*(\tilde{x}, \tilde{\mu}, s'))) + \tilde{g}((\tilde{t} + s', X^*(\tilde{x}, \tilde{\mu}, s'))) \right) ds', \text{ if } \tilde{t} > \frac{1}{A_0}. \]

Hence, making the difference between the equation (7.5) and the equation (3.30), after simple algebraic manipulations, we obtain

\[ |\sigma(\tilde{t} + s, X^*(\tilde{x}, \tilde{\mu}, s)) - \sigma^\infty(X^*(\tilde{x}, \tilde{\mu}, s))| \leq \|\sigma_1(\tilde{t}, \cdot) - \tilde{\sigma}_1^*(\cdot)\|_{L^\infty(\Omega \times \mathbb{R}^3)} + \]

\[ + \int_0^s \left( \|\nabla \cdot (u(\tilde{t} + s', \cdot) - u^*(\cdot))\|_{L^\infty(\Omega \times \mathbb{R}^3)} + C'\|Q(\tilde{t} + s', \cdot) - Q^*(\cdot)\|_{L^\infty(\Omega)} + \right. \]

\[ + C'\|\sigma(\tilde{t} + s', \cdot) - \sigma^\infty(\cdot)\|_{L^\infty(\Omega \times \mathbb{R}^3)} \|\sigma(\tilde{t} + s', \cdot)\|_{L^\infty(\Omega \times \mathbb{R}^3)} + N \|Q(\tilde{t} + s', \cdot) - Q^*(\cdot)\|_{L^\infty(\Omega)} + \]

\[ + \left( \|\nabla \cdot u^*(\cdot)\|_{L^\infty(\Omega)} + C'\|Q^*(\cdot)\|_{L^\infty(\Omega \times \mathbb{R}^3)} + \|\sigma^\infty(\cdot)\|_{L^\infty(\Omega \times \mathbb{R}^3)} + \|\sigma(\tilde{t} + s', \cdot)\|_{L^\infty(\Omega \times \mathbb{R}^3)} \right) \times \]

\[ \times \|\sigma(\tilde{t} + s', \cdot) - \sigma^\infty(\cdot)\|_{L^\infty(\Omega \times \mathbb{R}^3)} \right) ds', \]

where \( C' \) is a positive constant. Now, remembering (4.8) and applying the comparison lemma, we immediately prove the convergence (4.10). \( \square \)

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