1. Introduction

Let $V := \mathbb{C}^\mathbb{Z}$ be a countable dimensional vector space with fixed basis $\{ u_i | i \in \mathbb{Z} \}$. Consider the Lie algebra $\mathfrak{sl}(\infty)$ of all traceless linear operators in $\mathbb{C}^\mathbb{Z}$ annihilating almost all $u_i$. Clearly, $\mathfrak{sl}(\infty)$ can be identified with the Lie algebra of traceless infinite matrices with finitely many non-zero entries. We consider $\mathfrak{sl}(\infty)$ as a Kac-Moody Lie algebra associated with Dynkin diagram $A_\infty$. The Chevalley–Serre generators $e_a, f_a, a \in \mathbb{Z}$ of $\mathfrak{sl}(\infty)$ act on $V$ by

$$f_a u_b = \delta_{a,b} u_{b+1}, \quad e_a u_b = \delta_{a+1,b} u_{b-1}. \quad \text{for all } a \in \mathbb{Z}.\,$$

The fermionic Fock space $\mathfrak{F}$ is a simple $\mathfrak{sl}(\infty)$-module with fundamental highest weight $\omega_{-1}$. It has a realization as the “semi-infinite exterior power” $\Lambda^\infty/2 \mathbb{C}^\mathbb{Z}$ which is the span of all formal expressions $u_{i_1} \wedge u_{i_2} \wedge \ldots$ satisfying the conditions $i_j > i_{j+1}$ for all $j \geq 1$ and $i_k = -k$ for sufficiently large $k$. In this way the highest weight vector is $u_{-1} := u_{-1} \wedge u_{-2} \wedge \ldots$. The famous boson-fermion correspondence identifies $\mathfrak{F}$ with the space of symmetric functions. That in particular implies that $\mathfrak{F}$ has a natural basis $\{ u_\lambda \}$ enumerated by partitions $\lambda$ (this basis corresponds to Schur functions) where

$$u_\lambda := u_{\lambda_1+1} \wedge u_{\lambda_2+2} \wedge u_{\lambda_3+3} \wedge \ldots.\,$$

Let $t \in \mathbb{Z}$. We denote by $\mathfrak{F}^t$ the simple $\mathfrak{sl}(\infty)$-module with lowest weight $-\omega_{t-1}$. We will use the following realization of $\mathfrak{F}^t$. Set $V^t = \mathbb{C}^\mathbb{Z}$ with basis $\{ w_i | i \in \mathbb{Z} \}$ and define the action of $e_a, f_a$ on $V^t$ by

$$e_a w_b = \delta_{a,b} w_{b+1}, \quad f_a w_b = \delta_{a+1,b} w_{b-1}. \quad \text{for all } a \in \mathbb{Z}.\,$$

Then $\mathfrak{F}^t$ is the span of all formal expressions $w_{i_1} \wedge w_{i_2} \wedge \ldots$ satisfying the conditions $i_j > i_{j+1}$ for all $j \geq 1$ and $i_k = t - k$ for sufficiently large $k$. We can enumerate the elements of the basis of $\mathfrak{F}^t$ by partitions

$$w_\mu := w_{\mu_1+t-1} \wedge w_{\mu_2+t-2} \wedge w_{\mu_3+t-3} \wedge \ldots.\,$$

The goal of this paper is to describe the structure of $\mathfrak{F}^t \otimes \mathfrak{F}$. Let us consider $(m, n) \in \mathbb{Z}^2$ such that $m - n = t$. As follows from [PS] $\Lambda^m V^t \otimes \Lambda^n V^t$ is an indecomposable
\(\mathfrak{sl}(\infty)\)-module with simple socle \(S_{m,n}\). To describe this socle consider the contraction map \(c : V \otimes V^\vee \to \mathbb{C}\) given by \(c(w_i \otimes u_j) = (-1)^j \delta_{i,j}\) and extend it to \(c_{m,n} : \Lambda^m V^\vee \otimes \Lambda^n V \to \Lambda^{m-1} V^\vee \otimes \Lambda^{n-1} V\). Then \(S_{m,n}\) is the kernel of \(c_{m,n}\).

Theorem 1.1. (1) The \(\mathfrak{sl}(\infty)\)-module \(R := \mathfrak{F}^\vee_t \otimes \mathfrak{F}\) has an infinite decreasing filtration
\[
R := R^0 \supset R^1 \supset \ldots \supset R^k \supset \ldots
\]
such that \(\cap_k R^k = 0\) and
\[
R^k / R^{k+1} \simeq \begin{cases} 
S_{k+t,k} & \text{if } t \geq 0, \\
S_{k,k-t} & \text{if } t < 0.
\end{cases}
\]

(2) Every non-zero submodule of \(R\) coincides with \(R^r\) for some \(r \geq 0\).

The proof of this theorem is based on categorification of \(\mathfrak{F}^\vee_t \otimes \mathfrak{F}\) by the complexified Grothendieck group \(K[V_t]_\mathbb{C}\) of the abelian envelope \(V_t\) of the Deligne category \(\text{Rep}\ GL_t\) explained in [E] and Brundan categorification of \(\Lambda^m V^\vee \otimes \Lambda^n V\) via representation theory of the supergroup \(GL(m|n)\), [B]. We use the symmetric monoidal functor
\[
DS_{m,n} : V_t \to \text{Rep}\ GL(m|n)
\]
for \(m - n = t\). Existence of such functor follows from construction of \(V_t\), see [EHS]. While \(DS_{m,n}\) is not exact, it has a certain property, see Lemma 2.3 below, which allows to define the linear map
\[
ds_{m,n} : K[V_t]_\mathbb{C} \to K_{\text{red}}[\text{Rep}\ GL(m|n)]_\mathbb{C}
\]
where by \(K_{\text{red}}\) we denote the quotient of the Grothendieck group \(K\) by the relation \([\mathbb{C}^0] = -[\mathbb{C}]\) in the category \(\text{Rep}\ GL(m|n)\). Furthermore, \(ds_{m,n}\) is a homomorphism of rings and also a homomorphism of \(\mathfrak{sl}(\infty)\)-modules. We prove that the quotients \(K_{\text{red}}[\text{Rep}\ GL(m|n)]_\mathbb{C}\) form the layers of the radical filtration of \(\mathfrak{F}^\vee_t \otimes \mathfrak{F} \simeq K[V_t]_\mathbb{C}\). Let us warn the reader that the image of \(ds_{m,n}\) is not \(\Lambda^m V^\vee \otimes \Lambda^n V\) but another submodule in \(K_{\text{red}}[\text{Rep}\ GL(m|n)]_\mathbb{C}\). While this submodule has the same Jordan-Hoelder series as \(\Lambda^m V^\vee \otimes \Lambda^n V\), it is not isomorphic to \(\Lambda^m V^\vee \otimes \Lambda^n V\) as an \(\mathfrak{sl}(\infty)\)-module.

The second part of the paper contains calculation of dimensions of certain objects in \(V_t\).

The author was supported by NSF grant DMS-1701532. The author would like to thank Inna Entova-Aizenbud for reading the first version of the paper and pointing out typos and unclear arguments.

2. The category \(\text{Rep}\ GL(m|n)\) and \(DS\) functors

2.1. Translation functors. Let \(\text{Rep}\ GL(m|n)\) denote the category of finite-dimensional \(GL(m|n)\)-modules. Let \(\mu = (a_1, \ldots, a_m|b_1, \ldots, b_n) \in \mathbb{Z}^{m+n}\) satisfy the condition
\[a_1 \geq a_2 \geq \cdots \geq a_m, b_1 \geq b_2 \geq \cdots \geq b_n.\] For every such \(\mu\) there are three canonical objects in \(\text{Rep}\, GL(m|n)\):

1. The simple module \(S(\mu)\) with highest weight \(\mu\);
2. The Kac module \(K(\mu) := U(\mathfrak{gl}(m|n)) \otimes_{U(\mathfrak{p})} S_0(\mu)\), where \(\mathfrak{p}\) is the parabolic subalgebra with Levi subalgebra \(\mathfrak{gl}(m|n)_\mathfrak{p}\), \(S_0(\mu)\) is the simple \(\mathfrak{gl}(m|n)_\mathfrak{p}\)-module with highest weight \(\mu\);
3. The indecomposable projective cover \(P(\mu)\) of \(S(\mu)\).

The category \(\text{Rep}\, GL(m|n)\) is the highest weight category, \([Z]\). We denote by \(\mathcal{J}_\text{red}[\text{Rep}\, GL(m|n)]\) the reduced Grothendieck group of \(\text{Rep}\, GL(m|n)\) and set
\[
\mathcal{J}_{m|n} := \mathcal{J}_\text{red}[\text{Rep}\, GL(m|n)] \otimes_{\mathbb{Z}} \mathbb{C}.
\]

It was a remarkable discovery of J. Brundan that \(\mathcal{J}_{m|n}\) has a natural structure of \(\mathfrak{sl}(\infty)\)-module, \([B]\). To define it let us consider translation functors \(E_a, F_a : \text{Rep}\, GL(m|n) \to \text{Rep}\, GL(m|n)\) defined in the following way. There is a canonical \(\mathfrak{gl}(m|n)\)-invariant map \(\omega : \mathbb{C} \to \mathfrak{gl}(m|n) \otimes \mathfrak{gl}(m|n)\) usually called the Casimir element. Let \(V_{m|n}\) be the standard \(GL(m|n)\)-module and \(M\) be an arbitrary object of \(\text{Rep}\, GL(m|n)\). Let \(\Omega\) be the composition map
\[
\mathbb{C} \otimes M \otimes V_{m|n} \xrightarrow{\omega \otimes \text{id}} \mathfrak{gl}(m|n) \otimes \mathfrak{gl}(m|n) \otimes M \otimes V_{m|n} \xrightarrow{\text{id} \otimes s \otimes \text{id}} \mathfrak{gl}(m|n) \otimes M \otimes \mathfrak{gl}(m|n) \otimes V_{m|n} \xrightarrow{a_M \otimes a_{V_{m|n}}} M \otimes V_{m|n},
\]
where \(s\) is the braiding in \(\text{Rep}\, GL(m|n)\) defined by the sign rule and \(a_M, a_{V_{m|n}}\) are the action maps. Let \(E_a(M)\) be the generalized eigenspace of \(\Omega\) in \(M \otimes V_{m|n}\) with eigenvalue \(a\). Similarly, we define \(F_a(M)\) as the generalized eigenspace of \(\Omega'\) in \(M \otimes V_{m|n}^*\) with eigenvalue \(a\), where \(\Omega'\) is defined as above with substitution of \(V_{m|n}^*\) in place of \(V_{m|n}\).

The following theorem is a direct consequence of results in \([B]\).

**Theorem 2.1.** (1) \(E_a, F_a\) are non-zero only for \(a \in \mathbb{Z}\);
(2) \(E_a, F_a\) are biadjoint exact endofunctors of \(\text{Rep}\, GL(m|n)\);
(3) Let \(e_a, f_a : J_{m|n} \to J_{m|n}\) be the induced \(\mathbb{C}\)-linear maps. Then \(e_a, f_a\) satisfy the Chevalley-Serre relations for \(A_\infty\). Hence \(J_{m|n}\) is an \(\mathfrak{sl}(\infty)\)-module.
(4) The subspace of \(\Lambda_{m|n} \subset J_{m|n}\) generated by classes of all Kac modules \([K(\mu)]\) is an \(\mathfrak{sl}(\infty)\)-submodule isomorphic to \(\Lambda^m \mathcal{V} \otimes \Lambda^n \mathcal{V}\).

We need the exact description of the socle filtration of \(J_{m|n}\) obtained in \([B, HPS]\), Corollary 29.

**Proposition 2.2.** The \(\mathfrak{sl}(\infty)\)-module \(J_{m|n}\) has finite length. Furthermore, the socle filtration of \(J_{m|n}\) is given by the formula
\[
\text{soc}^i(J_{m|n})/\text{soc}^{i-1}(J_{m|n}) \simeq S_{m-i+1|n-i+1}^i.
\]
In particular, the socle of \(J_{m|n}\) is a simple \(\mathfrak{sl}(\infty)\)-module isomorphic to \(S_{m|n}\). It is identified with the subspace generated by classes of all projective modules \([P(\mu)]\).
2.2. **DS-functor.** Fix an odd \(x \in \mathfrak{gl}(m|n)\) such that \([x, x] = 0\) and \(\text{rk } x = 1\). Define a functor \(DS_x\) from \(\text{Rep } GL(m|n)\) to the category of vector superspaces by setting
\[
DS_x(M) = \text{Ker} x_M / \text{Im} x_M.
\]
It is shown in [DS] that \(M_x\) has a natural structure of \(GL(m-1|n-1)\)-module and \(DS_x\) is a symmetric monoidal functor
\[
\text{Rep } GL(m|n) \to \text{Rep } GL(m-1|n-1).
\]
Furthermore, although \(DS_x\) is not an exact functor it has the following property pointed out by V. Hinich. For the proof see [HPS] Lemma 30.

**Lemma 2.3.** Every exact sequence \(0 \to N \to M \to K \to 0\) of \(GL(m|n)\)-modules induces the exact sequence
\[
0 \to E \to DS_x N \to DS_x M \to DS_x K \to E' \to 0
\]
for certain \(E \in \text{Rep } GL(m-1|n-1)\) and \(E' \simeq E \otimes C^{0|1}\).

It follows immediately from Lemma 2.3 that \(DS_x\) induces a homomorphism of complexified reduced Grothendieck groups \(ds_x : J_{m|n} \to J_{m-1|n-1}\). While \(DS_x\) and \(DS_y\) are not isomorphic if \(x\) and \(y\) are not conjugate by the adjoint action of \(GL(m) \times GL(n)\), the homomorphism \(ds_x\) does not depend on a choice of \(x\). In [HR] the homomorphism \(ds_x\) was constructed explicitly in terms of supercharacters and the kernel of \(ds_x\) was computed.

**Lemma 2.4.**

1. \(DS_x\) commutes with translation functors \(E_a, F_a\) and hence \(DS_x\) induces a homomorphism \(ds_x : J_{m|n} \to J_{m-1|n-1}\) of \(\mathfrak{sl}(\infty)\)-modules.

2. The kernel of \(ds_x\) coincides with \(\Lambda_{m|n}\).

**Proof.** For (1) see Lemma 32 in [HPS]. For (2) see [HR]. \(\square\)

3. **The category \(V_t\), translation functors and categorification**

3.1. **The Deligne category \(D_t\).** In [DM] Deligne and Milne constructed a family \(\{D_t = \text{Rep } GL_t \mid t \in \mathbb{C}\}\) of symmetric monoidal rigid categories satisfying the following properties:

1. \(D_t\) is a universal additive symmetric monoidal Karoubian category generated by a dualizable object \(V_t\) of dimension \(t\);

2. The indecomposable objects of \(D_t\) are in bijection with bipartitions \(\lambda = (\lambda^+, \lambda^-)\), we denote the corresponding indecomposable objects by \(T(\lambda)\);

3. If \(t \notin \mathbb{Z}\), then \(\dim \text{Hom}(T(\lambda), T(\nu)) = \delta_{\lambda,\mu}\) and hence the category \(D_t\) is an abelian semisimple category;

4. If \(t \in \mathbb{Z}\), and \(m - n = t\), then there exists a (unique up to isomorphism) symmetric monoidal functor \(F_{m|n} : D_t \to \text{Rep } GL(m|n)\) which sends \(V_t\) to \(V_{m|n}\). This functor is full.
The functor $F_{m|n}$ was studied in [CW]. In particular, it was computed on the indecomposable objects of $\mathcal{D}_t$. We call a bipartition $\lambda = (\lambda^\bullet, \lambda^\circ)$ an $(m|n)$-cross if for there exists $0 \leq k \leq m$ such that $\lambda^\bullet_{k+1} + (\lambda^\circ)_{m-k+1} \leq n$. Here $\mu^T$ stands for the conjugate of $\mu$. Denote by $C(m|n)$ the set of all $(m|n)$-crosses.

**Theorem 3.1.**

1. $F_{m|n}T(\lambda) \neq 0$ if and only if $\lambda \in C(m|n)$.
2. The set $\{F_{m|n}T(\lambda) \mid \lambda \in C(m|n)\}$ is a complete set of pairwise non-isomorphic indecomposable direct summands in tensor powers $V^\otimes_p (V^*_m)^\otimes_q$ for $p, q \geq 0$.

**Proof.** The first statement is Theorem 8.7.6 in [CW] and the second is the particular case of Theorem 4.7.1 in [CW]. \hfill $\Box$

### 3.2. The abelian envelope of $\mathcal{D}_t$

Let $t \in \mathbb{Z}$. Then $\mathcal{D}_t$ is not abelian. In [EHS] we construct an abelian envelope $\mathcal{V}_t$ of $\mathcal{D}_t$. We need here some particular features of this construction. Let $m - n = t$ and let $\text{Rep}^k GL(m|n)$ be the abelian full subcategory of $\text{Rep} GL(m|n)$ containing mixed tensor powers $V^\otimes_p (V^*_m)^\otimes_q$ for $p, q \leq k$. The following statement is crucial for our construction.

**Lemma 3.2.** Let $m, n >> k$ and $x \in \mathfrak{gl}(m|n)_1$ be a self-commuting element of rank 1. Then the restriction of $DS_x$ to $\text{Rep}^k GL(m|n)$ defines an equivalence of the categories $\text{Rep}^k GL(m|n) \to \text{Rep}^k GL(m-1|n-1)$.

That allows us to define the abelian category $\mathcal{V}_t^k$ as the inverse limit $\lim_{\leftarrow} \text{Rep}^k GL(m|n)$. Then set

$$\mathcal{V}_t := \lim_{\leftarrow} \mathcal{V}_t^k.$$ 

We have an exact fully faithful functor $I : \mathcal{D}_t \to \mathcal{V}_t$. Slightly abusing notation we write $T(\lambda) = IT(\lambda)$.

**Lemma 3.3.** For every $(m|n)$ such that $m - n = t$ there exists a symmetric monoidal functor $DS_{m|n} : \mathcal{V}_t \to \text{Rep} GL(m|n)$. This functor is not exact but satisfies the condition of Lemma 2.3. Moreover, $DS_{m|n} \circ I$ is isomorphic to $F_{m|n}$.

**Proof.** It suffices to construct $DS_{m|n} : \mathcal{V}_t^k \to \text{Rep} GL(m|n)$. We identify $V_t^k$ with $\text{Rep}^k GL(m'|n')$ for sufficiently large $m', n'$ and define $DS_{m|n} : \text{Rep}^k GL(m'|n') \to \text{Rep}^k GL(m|n)$ as a composition of the functors $DS_{x_r} \circ DS_{x_{r-1}} \circ \ldots DS_{x_1}$ for some self-commuting rank 1 odd elements $x_i \in \mathfrak{gl}(m+i|n+i)$ with $r = m'-m = n'-n$. Lemma 3.2 ensures that this composition does not depend on the choice of $(m'|n')$ and that passing to the direct limit is well-defined. By construction $DS_{m|n}$ satisfies Lemma 2.3. Finally, $DS_{m|n} \circ I$ is a symmetric monoidal functor from $\mathcal{D}_t$ to $\text{Rep}^k GL(m|n)$ which maps $V_t$ to $V_m$. Hence by (4) it must be isomorphic to $F_{m|n}$. \hfill $\Box$

**Remark 3.4.** Construction of $DS_{m|n}$ given in the above proof depends on a choice of $x_s \in \mathfrak{gl}(m+s,n+s)_1$. Apriori there may be several non-isomorphic functors satisfying the condition of Lemma 3.3. We suspect however that all these functors
Proof. The second assertion is a consequence of (3.1) and (3.2). The first assertion follows from the fact that \( |\lambda| + |\lambda^\circ| \), and \( K(\lambda, \mu) \) has 1-s on the main diagonal.  

\[ K(\lambda, \mu) = \sum_{\mu \subset \lambda} K(\lambda, \mu)[L(\mu)], \quad [T(\lambda)] = \sum_{\mu \subset \lambda} K(\lambda, \mu)[V(\mu)]. \]

\textbf{3.4. Translation functors and categorical action of} \( \mathfrak{sl}(\infty) \). One readily sees that \( \mathfrak{gl}(V_t) := V_t \otimes V_t^* \) is a Lie algebra object in \( V_t \). Furthermore, there exists a unique canonical morphism \( \omega : 1 \to \mathfrak{gl}(V_t) \). For every \( X \in V_t \) we do have the action morphism \( a_X : \mathfrak{gl}(V_t) \otimes X \to X \). Hence in the same way as for \( \text{RepGL}(m|n) \) we
can define the translation functors $E_aX$ and $F_aX$ as generalized eigenspaces with eigenvalue $a$ for

$$\Omega : X \otimes V_t \xrightarrow{\omega \otimes \text{id}} \mathfrak{gl}(V_t) \otimes \mathfrak{gl}(V_t) \otimes X \otimes V_t \xrightarrow{\text{id} \otimes s \otimes \text{id}} \mathfrak{gl}(V_t) \otimes \mathfrak{gl}(V_t) \otimes X \otimes V_t$$

and

$$\Omega' : X \otimes V^*_t \xrightarrow{\omega \otimes \text{id}} \mathfrak{gl}(V_t) \otimes \mathfrak{gl}(V_t) \otimes X \otimes V^*_t \xrightarrow{\text{id} \otimes s \otimes \text{id}} \mathfrak{gl}(V_t) \otimes \mathfrak{gl}(V_t) \otimes X \otimes V^*_t,$$

respectively.

The following theorem is proven in [E]

**Theorem 3.6.** Let $t \in \mathbb{Z}$.

1. $E_a, F_a$ are non-zero only for $a \in \mathbb{Z}$;
2. $E_a, F_a$ are biadjoint exact endofunctors of $V_t$;
3. Let $e_a, f_a : K[V_t]_C \to K[V_t]_C$ be the induced $\mathbb{C}$-linear maps. Then $e_a, f_a$ satisfy the Chevalley-Serre relations for $A_\infty$. Hence $K[V_t]_C$ is an $\mathfrak{sl}_2(\infty)$-module.
4. There is a unique isomorphism $f : K[V_t]_C \to \mathfrak{g}_t^\vee \otimes \mathfrak{g}^\vee$ of $\mathfrak{sl}_2(\infty)$-modules such that $f([V(\lambda)]) = v_\lambda := w_\lambda \otimes u_\lambda$.

**4. Proof of the main theorem**

Recall the functor $DS_{m|n}$ defined in Lemma 3.3.

**Lemma 4.1.** We have the following commutative diagrams of functors:

$$
\begin{array}{ccc}
\mathcal{V}_t & \xrightarrow{E_aF_a} & \mathcal{V}_t \\
DS_{m|n} \downarrow & & \downarrow DS_{m|n} \\
\text{RepGL}(m|n) & \xrightarrow{E_aF_a} & \text{RepGL}(m|n)
\end{array}
$$

**Proof.** By Lemma 2.4 one has the following commutative diagram

$$
\begin{array}{ccc}
\text{RepGL}(m|n) & \xrightarrow{E_aF_a} & \text{RepGL}(m|n) \\
\downarrow DS & & \downarrow DS \\
\text{RepGL}(m-1|n-1) & \xrightarrow{E_aF_a} & \text{RepGL}(m-1|n-1)
\end{array}
$$

Hence the statement follows from definition of $\mathcal{V}_t$ and the proof of Lemma 3.3.

**Corollary 4.2.** The induced map $d_{m|n} : K[V_t]_C \to \text{RepGL}(m|n)_{\text{red}}$ is a homomorphism of $\mathfrak{sl}_2(\infty)$-modules.

**Lemma 4.3.**

1. $d_{m|n}([T(\lambda)]) \neq 0$ if and only if $\lambda \in C(m|n)$. 
Theorem 3.1 (1). By Lemma 3.3 we have 
Proof. 

It can be written as 

Corollary 4.4. The quotient $\text{Ker} ds_{m-1|n-1} / \text{Ker} ds_m$ is isomorphic to $S_m$ as an $\mathfrak{sl}(\infty)$-module.

Proof. Let us write $ds_{m-1|n-1} = ds_x ds_m$. Then $\text{Ker} ds_{m-1|n-1} / \text{Ker} ds_m$ is isomorphic to $\text{Im} ds_m \cap \text{Ker} ds_x$. Furthermore Lemma [4.3] implies that $\text{Im} ds_m$ is spanned by $ds_m([T(\lambda)])$ for all $\lambda \in C(m|n)$ and $\text{Im} ds_m \cap \text{Ker} ds_x$ is spanned by classes of all indecomposable projective modules in $\text{Rep} GL(m|n)$. Therefore the statement follows from Proposition [2.2] \hfill \Box

Lemma 4.5. 

$\bigcap_{m-n=t} \text{Ker} ds_m = 0$.

Proof. Suppose $ds_m([X]) = 0$ for all $m,n$ such that $m - n = t$. There exists $k$ such that $[X] \in K[Y_t^k]_C$. But $ds_m : K[Y_t^k]_C \to K[\text{Rep} GL(m|n)]_C$ is injective for sufficiently large $m,n$. Therefore $[X] = 0$. \hfill \Box
Corollary 4.4 and Lemma 4.5 prove Theorem 1.1(1). Indeed, it suffices to put

\[ \mathcal{R}_{s}^t := \begin{cases} \ker ds_{k, t-1, k-1} & \text{if } t \geq 0, \\ \ker ds_{k-1, t-1, k-1} & \text{if } t < 0. \end{cases} \]

Now let us prove Theorem 1.1(2). We consider the case \( t \geq 0 \), the case of negative \( t \) is similar. Note that \( \mathcal{R} \) satisfies the following property: for any \( u \in \mathcal{R} \), \( e_a u = f_a u = 0 \) for all but finitely many \( a \). Let \( \mathfrak{t}^-(\text{resp. } \mathfrak{t}^+) \) be the Lie subalgebra of \( \mathfrak{sl}(\infty) \) generated by \( e_a, f_a \) for \( a < s \) (resp., \( a > s \)). Let \( M^+_s := M^v \). Then \( M^+_s \) is a \( \mathfrak{t}^+ \)-module. If \( M \) is a submodule of \( \mathcal{R} \) then \( M = \bigcup_{s \leq 0} M^+_s \) by the above property. In particular, if \( M, N \) are two submodules of \( \mathcal{R} \) such that \( M^+_s = N^+_s \) for all \( s \leq s_0 \), then \( M = N \). A simple computation shows that for any \( s < 0 \)

\[ \mathcal{R}_{s}^+ \simeq \Lambda^{-s-1}((V^v)_{s}^+) \otimes \Lambda^{t-s-1}(V^+_s). \]

Note that \( \mathfrak{t}^+ \) is isomorphic to \( \mathfrak{sl}(\infty) \) and \( (V^v)^+_s \) and \( V^+_s \) are isomorphic to the standard and costandard \( \mathfrak{t}^+ \)-modules respectively. A description of the lattice of all submodules of \( \mathcal{R}_{s}^+ \) follows immediately from the socle filtration of \( \mathcal{R}_{s}^+ \), see [PS]. Since every layer of this socle filtration is simple, the only submodules of \( \mathcal{R}_{s}^+ \) are members of the socle filtration \( \text{soc}^{r+1}(\mathcal{R}_{s}^+) \) for some \( 0 \leq r \leq -1 - s \). Furthermore, \( \text{soc}^{r+1}(\Lambda^{-s-1}((V^v)_{s}^+) \otimes \Lambda^{t-s-1}(V^+_s)) \) is cyclic and is generated by a monomial vector \( x \) such that \( c^{r+1}(x) = 0, c'(x) \neq 0 \) for the contraction map

\[ c : \Lambda^k((V^v)^-_s) \otimes \Lambda^{t+k}(V^+_s) \to \Lambda^{k-1}((V^v)^-_s) \otimes \Lambda^{t+k-1}(V^+_s). \]

For any \( p \geq 0 \) set

\[ v(p) := (w_{t-1} \land w_{t-2} \land \ldots) \otimes (u_{t+p} \land u_{t+p-1} \land \ldots \land u_{t+1} \land u_{t-p} \land u_{t-p-2} \land \ldots). \]

By above \( \text{soc}^{r+1}(\mathcal{R}_{s}^+) \) is generated by \( v(-r - s - 1) \). Passing to the direct limit for \( s \to -\infty \) we obtain that every submodule of \( \mathcal{R}_{s}^+ \) is generated by \( v(p) \) for some \( p \geq 0 \). Thus, we obtain that every submodule of \( \mathcal{R} \) is generated \( v(p) \). On the other hand, it is not difficult to see that \( \mathcal{R}^v \) is generated by \( v(r) \). The statement follows.

Remark 4.6. The last argument uses presentation of \( \mathcal{R} \) as a direct limit. Indeed, for the directed system of algebras \( \cdots \subset \mathfrak{t}^+_{s} \subset \mathfrak{t}^-_{s-1} \subset \cdots \) (here \( s \to -\infty \)) we get

\[ \mathcal{R} = \lim_{\to} \Lambda^{-s+t-1}((V^v)^-_s) \otimes \Lambda^{-s-1}(V^+_s) \]

for \( t \geq 0 \) and similarly

\[ \mathcal{R} = \lim_{\to} \Lambda^{-s-1}((V^v)^-_s) \otimes \Lambda^{-s-t-1}(V^+_s) \]

for \( t \leq 0 \).
5. Blocks in \( \mathcal{V}_t \) and dimensions of tilting and standard objects.

The module \( \mathcal{R} \) is a weight \( \mathfrak{sl}(\infty) \)-module. To simplify bookkeeping we embed \( \mathfrak{sl}(\infty) \hookrightarrow \mathfrak{gl}(\infty) \) and define a \( \mathfrak{gl}(\infty) \)-action on \( \mathcal{R} \) in the natural way. We fix the Cartan subalgebra \( \mathfrak{h} \) of the diagonal matrices in \( \mathfrak{gl}(\infty) \), choose the basis \( \{ E_{i,i} \mid i \in \mathbb{Z} \} \) and denote by \( \{ \theta_i \mid i \in \mathbb{Z} \} \) the dual system in \( \mathfrak{h}^* \). It is easy to compute the weight \( \text{wt}(v_\lambda) \) of the monomial vector \( v_\lambda \). Precisely for a bipartition \( \lambda \) define the sets

\[
A(\lambda) := \{ \lambda_i^+ \mid \lambda_i^+ + t - i \neq \lambda_j^- - j \ \forall j \},
\]

\[
B(\lambda) := \{ \lambda_j^- \mid \lambda_j^- + t - j \neq \lambda_i^+ - i \ \forall i \}.
\]

It follows immediately from definition that \( A(\lambda) \) and \( B(\lambda) \) are finite subsets of \( \mathbb{Z} \) and \( |B(\lambda)| - |A(\lambda)| = t \).

**Example 5.1.** If \( \lambda = (0,0) \) then \( A(\lambda) = \emptyset \), \( B(\lambda) = \{0,1,\ldots,t-1\} \) for \( t > 0 \) and \( A(\lambda) = \{-1,\ldots,t\} \), \( B(\lambda) = \emptyset \) for \( t < 0 \). For \( t = 0 \) \( A(\lambda) = B(\lambda) = \emptyset \).

Then we have

\[
\text{wt}(v_\lambda) = - \sum_{a \in A(\lambda)} \theta_a + \sum_{b \in B(\lambda)} \theta_b.
\]

**Theorem 5.2.** For a weight \( \theta \) of \( \mathcal{R} \) let \( \mathcal{V}_t^\theta \) denote the full subcategory of \( \mathcal{V}_t \) consisting of objects with simple constituents isomorphic to \( L(\lambda) \) with \( \text{wt}(v_\lambda) = \theta \). Then \( \mathcal{V}_t \) is the direct sum of \( \mathcal{V}_t^\theta \). Moreover, \( \mathcal{V}_t^\theta \) is a block in \( \mathcal{V}_t \) for every \( \theta \).

**Proof.** Since \( \mathcal{V}_t^k \) is the highest weight category for every \( k \) we have

\[
\text{Ext}^1(L(\lambda), L(\mu)) \neq 0 \Rightarrow [V(\lambda) : L(\mu)] \neq 0 \text{ or } [V(\mu) : L(\lambda)] \neq 0.
\]

On the other hand, since \( V(\lambda) \) is indecomposable all its simple constituents lie in the same block of \( \mathcal{V}_t \). Combinatorial description of the multiplicities \( [V(\lambda) : L(\mu)] \neq 0 \) is given in \( \square \). It is clear from this description that \( [V(\lambda) : L(\mu)] \neq 0 \) implies \( \text{wt}(v_\lambda) = \text{wt}(v_\mu) \). Let \( \sim \) be the equivalence closure of \( [V(\lambda) : L(\mu)] \neq 0 \). Then a simple combinatorial argument implies that \( \lambda \sim \mu \) if and only if \( \text{wt}(v_\lambda) = \text{wt}(v_\mu) \). \( \square \)

Let us denote by \( \dim M \) the categorical dimension of an object \( M \) in \( \mathcal{V}_t \). Since \( DS_{m|n} \) is a symmetric monoidal functor it preserves categorical dimension. Therefore for every \( m, n \) such that \( m - n = t \) we have

\[
\dim M = \text{sdim} DS_{m|n} M.
\]

We call weight \( \theta \) positive (resp., negative) if \( \theta = \sum_{c \in C} \theta_c \) (resp., \( \theta = - \sum_{c \in C} \theta_c \)). In this definition \( \theta = 0 \) is both positive and negative.

**Lemma 5.3.** (1) If \( \theta \) is neither positive nor negative, then \( \dim M = 0 \) for every object \( M \) in \( \mathcal{V}_t^\theta \).
(2) If $t < 0$ and $\theta = \sum_{c \in C} \theta_c$ is positive (resp., $t \geq 0$ and $\theta = -\sum_{c \in C} \theta_c$ is negative), then for every object $M$ in $\mathcal{V}_t^\theta$ we have $\dim M = \kappa(M) q(\theta)$ for some integer $\kappa(M)$ and
\[ q(\theta) = \frac{\prod_{a < b, a, b \in C} (b - a)}{\prod_{j=1}^{[\kappa - 1]} j!}. \]

**Remark 5.4.** If $t = 0$ the only positive (and negative) weight $\theta$ is zero and $q(\theta) = 1$.

**Proof.** Say $t \geq 0$. All weights of $\Lambda_{t|0}$ are negative. Since $d_{s|0} : \mathcal{R} \to \Lambda_{t|0}$ is a homomorphism of $\mathfrak{sl}(\infty)$-modules $d_{s|0}[M] = 0$ for every $M \in \mathcal{V}_t^\theta$. Hence the statement is a consequence of (5.2). Similarly for $t < 0$ we have $d_{s|0^{-t}} : \mathcal{R} \to \Lambda_{0^{-t}}$ is zero since all weights of $\Lambda_{0^{-t}}$ are positive. The proof of (1) is complete.

Let us prove (2). Note in $\Lambda_{t|0}$ and $\Lambda_{0^{-t}}$ all weight spaces are one-dimensional and the corresponding categories of $GL(|t|)$-supermodules are semisimple. Therefore $D_{s|0}M$ (resp., $D_{s|0^{-t}}M$) is a direct sum of several copies of a certain irreducible representation $W(\theta)$ of $GL(|t|)$. The highest weight $\nu(\theta)$ of $W(\theta)$ can be easily expressed in terms of $C = \{c_1 > c_2 > \cdots > c_{|t|}\}$. For $t \geq 0$ $\nu(\theta) = (c_1 + 1 - t, c_2 + 2 - t, \ldots, c_t)$ and for $t < 0$ $\nu(\theta) = (c_1 + 1, \ldots, c_t - t)$. Then by the Weyl dimension formula we have $\text{sdim} W(\theta) = \pm q(\theta)$. This implies (b). \hfill \Box

**Remark 5.5.** It is proven in [DS] that $D_{s|0} \colon \text{Rep} GL(m|n) \to \text{Rep} GL(m - k|n - k)$ maps a block to a block corresponding to the same weight of $\mathfrak{sl}(\infty)$. Hence $D_{s|0^{-t}}$ induces a functor from a block $\mathcal{V}_t^\theta$ to the corresponding block $\text{Rep}^\theta GL(m|n)$. In particular, $D_{s|0}$ (resp., $D_{s|0^{-t}}$) annihilates any object in $\mathcal{V}_t^\theta$ if $\theta$ is not negative (resp., not positive).

**Lemma 5.6.** Let $t \geq 0$ (resp., $t < 0$). Then
\[ \text{Hom}_{\mathfrak{sl}(\infty)}(\mathcal{R}, \Lambda^t(\mathcal{V})) = \mathbb{C}, \] respectively,
\[ \text{Hom}_{\mathfrak{sl}(\infty)}(\mathcal{R}, \Lambda^{-t}(\mathcal{V})) = \mathbb{C}. \]

**Proof.** Immediate consequence of Theorem 1.1. \hfill \Box

Next we are going to construct a homomorphism $\varphi : \mathcal{R} \to \Lambda^t(\mathcal{V})$, (resp., $\varphi : \mathcal{R} \to \Lambda^{-t}(\mathcal{V})$) by defining it on the monomial basis $v_\lambda = w_\lambda \hat{\otimes} u_\lambda$. Let $t > 0$ and
\[ w_\lambda = u_i \wedge u_2 \wedge \ldots, \quad w_\lambda = w_{i_1} \wedge w_{j_2} \wedge \ldots. \]
If $\text{wt}(v_\lambda) = -\theta_{a_1} - \cdots - \theta_{a_t}$ is negative we can write
\[ w_\lambda = (-1)^{s(\lambda)} w_{a_1} \wedge \cdots \wedge w_{a_t} \wedge w_{i_1} \wedge \cdots w_{i_2} \wedge, \]
and then set
\[ \varphi(v_\lambda) := (-1)^{s(\lambda)} \prod_{i_k \neq k} (-1)^{i_k} w_{a_1} \wedge \cdots \wedge w_{a_t}. \]
If $\text{wt}(v_\lambda)$ is not negative we set $\varphi(v_\lambda) := 0$. The easiest way to see that $\varphi$ commutes with action of $\mathfrak{sl}(\infty)$ is to realize it as the direct limit as in Remark 4.6. Then $\varphi$ is the direct limit of contraction maps $\Lambda^{-s-t}(\mathcal{V}) \otimes \Lambda^{-s}(\mathcal{V}) \to \Lambda^t(\mathcal{V})$. 

**TENSOR PRODUCT OF THE FOCK REPRESENTATION WITH ITS DUAL**

11
Similarly, for negative $t$ with $\text{wt}(v_\lambda) = \theta_{a_1} + \cdots + \theta_{a_{-t}}$ we write

$$u_\lambda^o = (-1)^{s(\lambda)}u_{a_1} \wedge \cdots \wedge u_{a_{-t}} \wedge u_{j_1} \wedge \cdots \wedge w_{j_2} \wedge,$$

and we set $\varphi(v_\lambda) = (-1)^{r(\lambda)} \prod_{j_k \neq -k} (-1)^{j_k} u_{a_1} \wedge \cdots \wedge u_{a_{-t}}$. In both cases if $\theta = \text{wt}(\lambda)$ is positive or negative we can write

$$\varphi(v_\lambda) = (-1)^{r(\lambda)}[W(\theta)],$$

for certain $r(\lambda) \in \mathbb{Z}$.

**Proposition 5.7.** If $t \geq 0$ and $\theta$ is negative then dimension of $V(\lambda)$ in $\mathcal{V}_t^\theta$ equals $(-1)^{r(\lambda)}q(\theta)$.

If $t < 0$ and $\theta$ is positive then dimension of $V(\lambda)$ in $\mathcal{V}_t^\theta$ equals $(-1)^{r(\lambda)+\frac{t(t-1)}{2}+\sum_{i=1}^{t} a_i}q(\theta)$.

**Proof.** First let us see that $ds_{t|0}$ (resp., $ds_{0|-t}$) equals $\varphi$. Indeed, if $1$ denotes the unit object in $\mathcal{V}_t$ then $DS_{t|0}(1)$ (resp., $DS_{0|-t}(1)$) is the trivial module. Hence $ds_{t|0}$ (resp., $ds_{0|-t}$) coincides with $\varphi$ on the vacuum vector $v_{0|0}$. Then the statement follows from Lemma 5.6.

Let $t \geq 0$ then $ds_{t|0}(v_{\lambda}) = (-1)^{r(\lambda)}[W(\theta)]$ and $\text{sdim}W(\theta) = q(\theta)$ since $W(\theta)$ is even. This implies the lemma by (5.2).

Let $t < 0$ then $ds_{0|-t}(v_{\lambda}) = (-1)^{r(\lambda)}[W(\theta)]$ and the parity of $W(\theta)$ is equal to the parity of the highest weight $\nu(\theta)$. The latter is equal to the parity of $\sum_{i=1}^{t} a_i + \frac{t(t-1)}{2}$. Hence the lemma.

**Remark 5.8.** Let us explain how to compute $r(\lambda)$ in terms of weight diagram $f_\lambda$ (see Section 4.1 in [E]). Recall that $f_\lambda : \mathbb{Z} \to \{<,>,\times,\circ\}$ is defined as follows:

- $f_\lambda(i) = \circ$ if $u_i$ and $w_i$ do not occur in $v_\lambda$;
- $f_\lambda(i) = <$ if $u_i$ occurs in $v_\lambda$ and $w_i$ does not;
- $f_\lambda(i) = >$ if $w_i$ occurs in $v_\lambda$ and $u_i$ does not;
- $f_\lambda(i) = \times$ if both $u_i$ and $w_i$ occur in $v_\lambda$.

We represent $f_\lambda$ graphically by putting symbol $f_\lambda(i)$ into position $i$ on the number line. By definition $f_\lambda(i) = \circ$ for $i >> 0$ and $f_\lambda(i) = \times$ for $i << 0$. If $\theta = \text{wt}(\lambda)$ is positive then there are no symbols $>$ and if it is negative there are no symbol $<$. Symbols $<, >$ are called the core symbols. The core diagram is obtained from $f_\lambda$ by replacing all $\times$-s by $\circ$-s. Furthermore, $L(\lambda)$ and $L(\mu)$ are in the same block if and only if the core diagrams of $\lambda$ and $\mu$ coincide. Then $s(\lambda)$ equals the sum over all core symbols of the number of $\times$ to the right of that symbol. Now let

$$u(\lambda) = \begin{cases} \sum_{i \geq 0, f_\lambda(i) = \times} i & \text{for } t \geq 0, \\ \sum_{i <- t, f_\lambda(i) = \times} i & \text{for } t < 0 \end{cases}.$$

Then $r(\lambda) = u(\lambda) + s(\lambda)$.

**Proposition 5.9.** Let $\theta$ be negative or positive. There is exactly one up to isomorphism tilting object $T(\lambda)$ in the block $\mathcal{V}_t^\theta$ such that $\dim T(\lambda) \neq 0$. This is a unique tilting object in $\mathcal{V}_t^\theta$ such that $T(\lambda) \simeq V(\lambda) \simeq L(\lambda)$. 
Proof. We start with proving that \( \dim T(\lambda) \neq 0 \) implies \( T(\lambda) \simeq V(\lambda) \) and deal with the case \( t \geq 0 \). The other case is similar. Every \( T(\lambda) \) is a direct summand in \( V_t^{\otimes p} \otimes (V^*_t)^{\otimes q} \), therefore it is an indecomposable summand in \( F_{a_1} \ldots F_{a_q} E_{b_1} \ldots E_{b_q} 1 \). Note that \( 1 = V(\emptyset, \emptyset) \). An easy computation shows that for every \( \kappa e_a(v_\kappa) \) and \( f_a(v_\kappa) \) is zero, \( v_\mu \) or a sum \( v_\mu + v_\nu \). Moreover, the latter case is only possible if \( \text{wt}(\kappa) \) is not positive. If \( T(\lambda) \) is not isomorphic to \( V(\lambda) \) then for some \( k \)

\[ F_{a_k} \ldots F_{a_q} E_{b_1} \ldots E_{b_q} 1 \in V^0_t \]

for non-positive \( \theta \). Then by Remark 5.5 for some \( k \geq 1 \)

\[ DS_{t|0} F_{a_k} \ldots F_{a_q} E_{b_1} \ldots E_{b_q} 1 = 0 \]

and hence

\[ DS_{t|0} F_{a_1} \ldots F_{a_q} E_{b_1} \ldots E_{b_q} 1 = 0 \]

But then \( DS_{t|0}(T_\lambda) = 0 \) which implies \( \dim T(\lambda) = 0 \).

From combinatorial description of \( K(\lambda, \mu) \) given in [E] we see that if in \( f_\lambda \) there is \( \circ \) to the left of some \( \times \) then \( K(\lambda, \mu) = 1 \) for at least one \( \mu \neq \lambda \). If the core diagram is fixed then the re is exactly one diagram such that all \( \times \)-s lie to the left of all \( \circ \)-s. That implies uniqueness of \( \lambda \) in every block. We can also characterize \( \lambda \) as the minimal weight in the block. \( \square \)

References

B. J. Brundan, Kazhdan-Lusztig polynomials and character formulae for the Lie superalgebra \( \mathfrak{gl}(m|n) \), J. Amer. Math. Soc. 16 (2003), no. 1, 185–231.

CW. J. Comes, B. Wilson, Deligne’s category \( \text{Rep}(GL_\delta) \) and representations of general linear supergroups, Represent. Theory 16 (2012), 568–609; arXiv:1108.0652.

DM. P. Deligne, J.S. Milne, Tannakian Categories, Hodge Cycles, Motives, and Shimura Varieties, Lecture Notes in Math. 900 (1982), 101–228.

DS. M. Duflo, V. Serganova, On associated variety for Lie superalgebras, arXiv:math/0507198.

E. I. Entova-Aizenbud, Categorical actions on Deligne’s categories, J. of Algebra 504 (2018), 391–431.

EHS. I. Entova-Aizenbud, V. Serganova, V. Hinich, Deligne categories and the limit of categories \( \text{Rep}(GL(m|n)) \), to appear in IMRN; arXiv:1511.07699.

HR. Crystal Hoyt, Shifra Reif, Grothendieck rings for Lie superalgebras and the Duflo–Serganova functor, Algebra Number Theory 12 (2018), no. 9, 2167–2184.

HPS. Crystal Hoyt, Ivan Penkov, Vera Serganova, Integrable \( \mathfrak{sl}(\infty) \)-modules and the category \( \mathcal{O} \) for \( \mathfrak{gl}(m|n) \). Journal LMS, DOI:10.112/jlms.12176.

PS. I. Penkov, K. Styrkas, Tensor representations of infinite-dimensional root-reductive Lie algebras, in Developments and Trends in Infinite-Dimensional Lie Theory, Progr. Math. 288, Birkhäuser (2011), 127–150.

Z. Y. M. Zou, Categories of finite-dimensional weight modules over type I classical Lie superalgebras, J. of Algebra, 180 (1996),459–482.

DEPT. OF MATHEMATICS, UNIVERSITY OF CALIFORNIA AT BERKELEY, BERKELEY, CA 94720

E-mail address: serganov@math.berkeley.edu