Toward the Quantum Design of Multichannel Systems

The Inverse Problem Approach

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Abstract

The multichannel generalization of the theory of spectral, scattering and decay control is presented. New universal algorithms of construction of complex quantum systems with given properties are suggested. Particularly, transformations of interaction matrices leading to the concentration of waves in a chosen partial channel and spatial localization are shown. The limiting instructive cases illustrating different phenomena which occur with the combination of 'incompatible' properties are considered. For example, the scattering solutions with different resonance widths at the same energy for the same interaction are revealed. Analogously, a 'paradoxical' coexistence of both strong reflection and absolute transparency is explained. The case of the violation of 'natural' asymptotic behavior of partial wave function is demonstrated: it has a greater damping decrement for the channel with a lower threshold. Peculiarities of the multichannel periodic structures, bound states embedded into continuum, resonance tunneling and degeneracy of states are described.

1 INTRODUCTION

Many years ago this journal published Feshbach’s paper "Unified theory of nuclear reactions" [1] which attracted attention of a wide physical community. This approach is a mighty tool for describing quantum systems (e.g. few-body) with internal degrees of freedom. It can simplify the Schrödinger equation in partial derivatives reducing it to a system of coupled ordinary differential equations for the vector-valued wave functions, see (2).

The theory has gradually been developing in the direction of its greater unification. Generalization to reactions with the rearrangement of particles has been suggested by us in this journal [2-4]. A new contribution to the theory consists in employing the inverse problem (IP) and quantum supersymmetry (SUSYQ) approach. We get a deeper insight into a relationship between interactions (V) and
observables \( S \equiv \{E_n, C_{\alpha n}\} \): spectral weights \( C_{\alpha n} \) or \( M_{\alpha n} \), energy levels and their analogs, e.g., resonance positions.

In the IP and SUSYQ, the input data are \( S \). This is a remarkable advantage. It provides us with a complete set of elementary quantum transformations corresponding to variations of individual spectral parameters. They are given by explicit expressions: the exactly solvable models, see, e.g., [5-7] and references therein for the one-channel case. For comparison, in the direct problem, variation of only one chosen \( S \)-parameter is an extremely difficult problem.

There is in principle a possibility of converting any system into another one of a different nature. Specifically, a scattering state can be 'raked up' into a bound state embedded into continuum (\( S \)-transformation at a single energy point without any other spectral variation), see below.

A lot of qualitative physical information is revealed via a computer visualization of these transformations. So, in the one-dimensional and one-channel case we have developed a qualitative theory of quantum design [8-12]. It becomes clear what "bricks and blocks" are needed to build systems with given physical properties. A comparison with a children’s toy constructor-set suggests itself. In this way, the understanding of wave mechanics is raised to the intuitive level. We acquire the possibility of a qualitative foresight of the results of IP-SUSYQ transformations even without formulae and computer calculations. Namely, we look for universal rules of the structural and dynamic response of quantum complexes when varying \( S \). Particularly, an arbitrary energy level \( E_n \) can be created, removed or shifted. Any object is uniquely determined by a complete set of \( S \) data. So, it allows the choice of interactions providing the prescribed properties of the system.

The same status for the multichannel theory is still to be achieved. The one-channel IP scalar transformations represent a zero-measure subset of their multichannel matrix analogs. Thus, one can expect substantial advancement and greater diversity of new effects in comparison with the one-channel case.

In this paper, the latest results on the multichannel quantum design are discussed. First, we give short preliminary information about the systems of coupled Schrödinger and IP equations.

2 Multichannel equations of direct and inverse problems

The description of a wide class of quantum objects with the multidimensional wave functions \( \Psi \) can be reduced to the system of close coupled Schrödinger (multichannel) equations if we choose one variable \( x \) and expand \( \Psi \) over the set of known basis functions \( \Phi_\alpha(\xi) \) of all other variables \( \xi \)

\[
\Psi(x,\xi) = \sum_\alpha \Psi_\alpha(x)\Phi_\alpha(\xi).
\]  

The coefficients of these expansions, called channel wave functions, are determined by the system of equations

\[
-\Psi''_\alpha(x) + \sum_\beta V_{\alpha\beta}(x)\Psi_\beta(x) = E_\alpha \Psi_\alpha(x), \quad E_\alpha = E - \epsilon_\alpha,
\]
\[ V_{\alpha\beta}(x) = \int \Phi_\alpha^*(\xi)V(x, \xi)\Phi_\beta(\xi)d\xi \]  

where \( V(x, \xi) \) is the potential for the corresponding multi-dimensional problem and \( V_{\alpha\beta}(x) \) is the interaction matrix with non-diagonal elements coupling the equations. Greek indices \( \alpha, \beta \) mean the channel numbers, i.e., partial equation numbers. Symbol \( E \) stands for a total energy value and \( \epsilon_\alpha \) are the threshold energies. The potential matrices \( V_{\alpha\beta}(x) \) depend on \( x \) locally.

The IP gives the recipe of transformation from the initial \( \hat{\Psi}_\alpha(x), \hat{V}_{\alpha\beta}(x) \) to the new \( \Psi_\alpha(x), V_{\alpha\beta}(x) \) with the given spectral data – see, e.g., [13] for Eqs. (4)-(7)

\[ \Psi_\alpha(x) = \hat{\Psi}_\alpha(x) + \int_x^\infty \sum_\beta K_{\alpha\beta}(x, x') \hat{\Psi}_\beta(x')dx'; \]

\[ V_{\alpha\beta}(x) = \hat{V}_{\alpha\beta}(x) - 2\frac{d}{dx}K_{\alpha\beta}(x, x). \]

Here, the transformation matrix kernel \( K \) is determined by the IP equation

\[ K_{\alpha\beta}(x, x') + Q_{\alpha\beta}(x, x') + \int_x^\infty \sum_\gamma K_{\alpha\gamma}(x, y)Q_{\gamma\beta}(y, x')dy = 0. \]

The input spectral information is contained in its matrix valued kernel

\[ Q_{\alpha\beta}(x, x') = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk_1 dk_2 \sum_{\alpha'\beta'} \hat{F}_{\alpha'\beta'}(x, k)\{k^{-1/2}[(\hat{S}(k) - \hat{S}(\hat{k}))\hat{k}^{-1/2}]_{\beta'\alpha'} \hat{F}_{\beta'\alpha'}(x', \hat{k}) + \sum_n \sum_{\alpha'\beta'} \hat{F}_{\alpha'\beta'}(x, E_n)M_{\alpha'n}M_{\beta'n} \hat{F}_{\beta'\alpha'}(x', E_n) - \sum_n \sum_{\alpha'\beta'} \hat{F}_{\alpha'\beta'}(x, \hat{E}_n)\hat{M}_{\alpha'n}\hat{M}_{\beta'n} \hat{F}_{\beta'\alpha'}(x', \hat{E}_n). \]

In this expression the partial spectral parameter \( M_{\alpha n} \) stands for the factor in the asymptotic behavior of the partial wave in the \( n \)th channel of the normalized bound state at the energy \( E_n: \Psi_\alpha(x, E_n) \rightarrow M_{\alpha n}\exp(-\sqrt{\epsilon_\alpha - E_n} x), \) as \( x \rightarrow \infty \). We use \( F_{\alpha\beta}(x, E_n) \) to designate the matrix Jost solutions of (2) defined at the energies of bound states: \( F_{\alpha\beta}(x, E_n) \rightarrow \delta_{\alpha\beta}\exp(-\sqrt{\epsilon_\alpha - E_n} x), \) as \( x \rightarrow \infty \). Here, the first index means the channel number (i.e. partial equation number in (2)), the second one designates the type of a boundary condition. In (7) the scattering matrix is \( \hat{S}(\hat{k}) \) and \( F_{\alpha\beta}(x, \hat{k}) \) is the matrix Jost solution for the continuous spectrum. The sums in \( Q \) correspond to energy levels of bound states and the integral refers to the continuous spectrum. Symbol \( \hat{k} \) is a diagonal momentum matrix \( k_\alpha\delta_{\alpha\beta}, k_\alpha = \sqrt{\epsilon_\alpha} \).

If initial and final systems differ in only a finite number of discrete spectral parameters, the kernel \( Q \) becomes degenerate and equation (2) reduces to exactly solvable algebraic equations (see, e.g. [13]). Indeed, in this case the integral term in (6) vanishes because \( \hat{S} = S \) and the terms in the sums with \( \hat{E}_n = E_n, \hat{M}_{\alpha n} \)
= $M_{\alpha n}$ are cancelled. So, there remain a finite number of terms with the factorized dependence on $x, x'$. Thus, IP gives rise to an infinite number of exactly solvable models which form a complete set, i.e. there is a possibility (at least theoretically) to fit any quantum system by them.

The above formulae correspond to the case when the spectral parameters $(S, M)$ characterize the asymptotic ($x \to \infty$) behavior of the wave functions. This is Marchenko’s approach (M). It is often convenient also to control the spectral parameters which determine the wave behavior at the origin according to Gelfand-Levitan’s approach (GL). Then, instead of $M_{\alpha n}$ the derivatives of normalized bound states at $x = 0$ are used

$$C_{\alpha n} = \frac{d}{dx} \Psi_{\alpha}(x, E_n)|_{x=0}. $$

The matrix-valued regular solutions $\Phi_{\alpha\beta}(x, E_n)$ enter in GL formalism instead of the Jost ones ($\Phi_{\alpha\beta}(x, E_n)|_{x=0} = 0, \quad \frac{d}{dx} \Phi_{\alpha\beta}(x, E_n)|_{x=0} = \delta_{\alpha\beta}$), and

$$\Psi_{\alpha}(x, E_n) = \sum_{\beta} \Phi_{\alpha\beta}(x, E_n)C_{\beta n}. \tag{8}$$

Furthermore, the minus sign in (5) is replaced by the opposite one and the integration is performed over the interval $[0, x]$. In what follows $C_{\beta n}$ will be referred to as a spectral weight vector (SWV) related to the $n$th eigenstate.

For the reader interested in the IP approach it would be useful to conceive the IP as the Gram-Schmidt orthogonalization of the set of eigenvectors generalized to the case of infinite dimensions and continuous basis. The bound state wave functions are considered as vectors in a special Hilbert space in which the coordinate $x$ numbers the vectors while the energy indices $n$ indicate their components. The inner product in that space is determined by a measure dependent on the SWV vectors $C$ or $M$ etc, [4]. In fact, Parseval’s completeness relation

$$\sum_n \Psi_{\alpha}(x, E_n)\Psi_{\beta}(x', E_n) = \delta_{\alpha\beta}\delta(x - x') \tag{9}$$

can be rewritten using (8) as follows :

$$\sum_n \sum_{\gamma\gamma'} \Phi_{\alpha\gamma}(x, E_n)C_{\gamma n} C_{\gamma' n} \Phi_{\beta\gamma'}(x', E_n) = \delta_{\alpha\beta}\delta(x - x'). \tag{10}$$

The last equality represents the orthogonality relation with the left part as an 'inner product' for vectors $\Phi_{\alpha\gamma}(x, E_n)$. It is determined via summing over the multi-index $(\alpha, n)$ with a weight $C_{\gamma n} C_{\gamma' n}$ which gives the measure of a new Hilbert space. The IP transformation $C_{\alpha n} \to C_{\alpha n}$ means the change of the measure. Then, the Hilbert space is changed so that the initial orthogonality relation is violated. For this relation to be restored, the Gram-Schmidt orthogonalization is performed. So the IP equation (6) itself is just the equation for the coefficients of such a procedure.

It turns out that the exactly solvable models of the IP can be alternatively derived within the framework of the supersymmetric quantum mechanics (SUSYQ) formalism [14] by E.Witten. The SUSYQ has been generalized to the multichannel case [15-19]. In many respects the SUSYQ scheme is the same in both one- and multichannel cases. The initial Hamiltonian, the second order differential operator, is
factorized into the simpler first order matrix operators $A^\pm = \pm \partial + W(x)$ ($\partial$ is a symbol of the derivative, $\text{Im}W(x) = 0$, $\{W\}^T(x) = W(x)$). The transformation consists in permutation of $A^\pm$:

$$H_0 = A^+A^- + \mathcal{E} \rightarrow H_1 = A^-A^+ + \mathcal{E},$$  \hspace{1cm} (11)

where $\mathcal{E}$ is a constant factorization energy.

The matrix $W(x)$ has the form as in the one-channel case:

$$W(x) = \Psi_0'(x, \mathcal{E})\Psi_0(x, \mathcal{E})^{-1}$$

where $\Psi_0(x, \mathcal{E})$ is an arbitrary matrix solution for the initial system at the factorization energy. It should be noted here that the choice of $\mathcal{E}$ and the selection of $\Psi_0(x, \mathcal{E})$ are crucial points predetermining the resulting properties of the transformed Hamiltonian and corresponding solutions. It is remarkable that there is a simple relation between the solutions $\Psi_0(x, E)$ and $\Psi_1(x, E)$ of the Schrödinger equation with the initial and new Hamiltonians $H_0, H_1$:

$$\Psi_1(x, E) = A^-\Psi_0(x, E) = (-\partial + W(x))\Psi_0(x, E).$$  \hspace{1cm} (12)

Let us perform the SUSYQ transformation twice at the same energy $\mathcal{E}$ which can be arbitrary. The resultant formulae can be equivalent to the IP ones for variation of spectral weight or for creation (removal) of any bound state while all other spectral parameters from the complete set $\{E_n, C_n\}$ or $\{E_n, M_n\}$ remain unchanged (see also references for the one-channel case). But there exist SUSYQ transformations which are non-equivalent to the IP ones. For example, one can accomplish the double-SUSYQ procedure at different energies so that the first step is made at a bound state energy while the second one is performed at another energy, see, e.g., for the one-channel case the work. Then a chosen bound state energy is shifted and the spectral weights for all the energy levels are changed. As infinite number of spectral parameters are involved, to fulfill such a transformation is infeasible in the framework of the IP. Thus, in some cases, the SUSYQ formalism and one-parametric IP lead to the same results, whereas in some cases, SUSYQ transformations are not equivalent to finite-parametric models of the IP.

The interaction matrices $V_{\alpha\beta}(x)$ can also correspond to non-local potentials $V(x, \xi, \xi')$ of the related multi-dimensional problem. So, wide classes of exactly solvable models mentioned above open new possibilities to investigate systems with non-local forces which are still very poorly understood.

### 3 Uncoupled channels

Let us start the consideration of the multichannel peculiarities with the limiting case of disconnected equations when $V_{\alpha\neq\beta} = 0$. The partial channel transformed potentials and the corresponding wave functions can be unified in the matrix, e.g., $2\times2$ form

$$\hat{V}(x) = \begin{pmatrix} V_{11}(x) & 0 \\ 0 & V_{22}(x) \end{pmatrix},$$  \hspace{1cm} (13)
\[\hat{\Psi}(x) = \begin{pmatrix} \Psi_1(x) & 0 \\ 0 & \Psi_2(x) \end{pmatrix}.\] (14)

For uncoupled channels it is especially clearly seen that the spectrum consists of several branches (each for its channel). By altering a chosen threshold value \( \epsilon_\alpha \), one can shift the \( \alpha \)th partial branch with respect to the others. Some bound state energy level of one spectral branch can occur in the continuous spectrum of another channel. It is the simplest case of a bound state embedded into the continuum (BSEC). In the general case of coupled channels, the spectrum becomes unified and its branches are not so clearly separated.

It is interesting that one can join the initially independent branches at energy \( E_n \) through creating at this point a common bound state (or BSEC if \( E_n \) is above at least one of the thresholds). This is attained in the IP formalism when both components \( C_{\alpha n} \) of the SWV for \( E = E_n \) are chosen to be non-zero. It results in some channel coupling \( V_{12}(x) \) according to, e.g., Eqs. (36)-(38), section 4.7. At other energies, spectral characteristics remain unchanged and correspond to initial separated channels despite arbitrarily strong coupling. And only at the chosen energy \( E_n \) the spectra are mixed completely. For more details on the SWV control, see section 4.3.

The system of uncoupled channels is a convenient bridge linking one- and multi-channel theories. Here the one-channel 'bricks and building blocks' \[8, 9, 11, 12, 25\] appear in their pure form.

As the first example, we shall consider the two-channel model with infinite rectangular potential wells \( V_{11}(x), V_{22}(x) \) having purely discrete partial spectra. The decrease in the spectral weight parameter \( C_{11} \) for a bound state in the first channel is demonstrated in Fig.1. This makes the chosen normalized partial wave smaller at the origin and bigger on the right due to conservation of the norm. It means shifting the chosen partial state in the configuration space (along \( x \)) to the right. The corresponding potential transformation \( \Delta V_{11}(x) \) has a barrier on the left which pushes the wave to the right. But this barrier alone would shift upwards all the energy levels of the first channel and violate the restriction that all other spectral parameters except for \( C_{11} \) remain unchanged. To compensate this undesirable influence of the barrier, the additional well is introduced on the right.

A general rule-I for the spatial shift of an arbitrary partial state in any potential was found \[8, 9\]. By decreasing (increasing) \( C_{\alpha n} \) the \( n \)th partial state is shifted to the right (to the left) by \( n \) analogous \( \Delta V_{\alpha \alpha}(x) \)-blocks: 'barrier+well' ('well+barrier') one for each bump of the wave function. Returning to the initial system obviously requires the inversion of the perturbation block: the first transformation (barrier+well) combined with the second (the reverse 'well+barrier') cancel one another and restore the initial rectangular potentials.

The increase (decrease) in the spectral parameter \( C_{11} \) at one boundary of the allowed interval of motion is equivalent to decrease (increase) in the adjoint spectral parameter \( M_{11} \) defined at another boundary (or for \( x \to \infty \)). The same is not valid when an interchannel coupling is switched on (see below).

Shifting a chosen energy level over the \( \alpha \)th branch of the spectrum is another group of the complete set of elementary transformations. The case of potential symmetry with respect to the center of the interaction interval is especially simple
Here, keeping the spectral weights unaltered is superfluous because nothing but energy levels form a complete set of spectral parameters for an uncoupled channel.

The qualitative rule-II. Upward (downward) shifts of the energy level $E_n$ require barriers (wells) in the potential $\Delta V_{\alpha\alpha}(x)$ at the regions of the most sensitivity of the $n$th chosen state: near the extremum of each anti-knot. Compensating wells (barriers) at positions of knots are necessary to keep other energy levels at the same positions. It is illustrated in Fig.1 for the lowest bound state in the second channel.

Let us give a notion of analytic expressions for these transformations. The formulae for the transformed potential and the wave function of the lowest bound state related to the case of changing $C_{11}$ are:

$$V_{11}(x) = \tilde{V}_{11}(x) - 2 \frac{d}{dx} \left( \frac{C_{11}^2 / \tilde{C}_{11}^2 - 1}{1 + (C_{11}^2 / \tilde{C}_{11}^2 - 1) \int_0^x \tilde{\Psi}_1(y) dy} \tilde{\Psi}_1(x) \right),$$  \hspace{1cm} (15)

and

$$\Psi_1(x) = \frac{(C_{11}^2 / \tilde{C}_{11}^2 - 1) \tilde{\Psi}_1(x)}{1 + (C_{11}^2 / \tilde{C}_{11}^2 - 1) \int_0^x \tilde{\Psi}_1(y) dy}.$$

It is remarkable that the new system is constructed of the initial $\tilde{\Psi}_1(x)$ and $\tilde{V}_{11}(x)$ as building material. Analogous simple formulae for the one-channel energy shift are given in [8, 24].

The same rule for spatial shifts is valid in the case when there exists continuous spectrum. If the external potential wall is of finite height, the last well of the $n$th block can be pressed out of the initial potential (carrier potential wells of the soliton-like form – the fundamental quantum brick). It is shown for the second channel in Fig.2.

In the limit $C_{\alpha n} \to 0$ or $M_{\alpha n} \to \infty$, the bound state is carried away to infinity inside the reflectionless well. It is equivalent to the effective removal of the chosen energy level from the initial spectrum [8].

A scattering state wave can be considered as a bound state in the infinite broad potential well. If we imagine it to be distributed over the infinite interval with norm 1, it has an infinite small 'spectral weight' $\tilde{C}_{\alpha E}$. By making $C_{\alpha E}$ finite, this state can be raked up to the origin and becomes BSEC. But there is an infinite number of bumps in the initial state. So we need an infinite set of the potential well-barrier blocks, as shown in Fig.2. It is the limiting case of the above-mentioned rule-I. The BSEC in one channel can coexist with a scattering partial component in another channel at the same energy. It is a non-trivial fact that this can be realized for strong coupling of channels as well, see below.

It is remarkable that energy levels can also be shifted in the "imaginary direction" [12]: $E \to E + i\Gamma$. The above stated qualitative rule is valid for $Im\Delta V_{\alpha\alpha}(x)$. So quadratically integrable decaying states (with the exponential factor $\exp(-\Gamma t)$) are new convenient exact models. By analogy with the optical model it is an effective account of coupling with virtual channels not exhibited explicitly.
4 Coupled channels

As in the one-channel case, we can perform here a complete set of utmost elementary transformations. Any chosen bound state energy level can be shifted and any partial channel component $C_{\alpha n}$ of the SWV can be changed using the closed analytic expressions, see [13, 25] and references therein. We shall consider some qualitative properties of multichannel solutions after some preliminary remarks.

Switching on non-diagonal elements of the interaction matrix in (2) leads to a wave exchange between the channels which resembles a fluid motion in communicating vessels. The conservation of a partial channel flux is usually violated and only the total current is conserved.

Below are some simplest examples illustrating the specific influence of channel coupling. If we introduce a constant coupling $V_{12} = V_{21} = W \equiv const$ into two free equivalent equations of motion with initial solutions $\sim \sin(kx), \ k = \sqrt{E}$

\[
-\Psi_1''(x) + W\Psi_2(x) = E\Psi_1(x) \\
-\Psi_2''(x) + W\Psi_1(x) = E\Psi_2(x),
\]

we get new solutions with frequency that depends on the boundary conditions. The partial waves coincide $\Psi_1(x) = \Psi_2(x) k = \sqrt{E - W}$ for identical partial boundary conditions. But if the partial boundary conditions are chosen with different signs the solutions $\Psi_1(x) = -\Psi_2(x) \sim \sin(\tilde{k}x)$ have another wave number $\tilde{k} = \sqrt{E + W}$.

Let us consider a system on the whole line with equal thresholds and rectangular interaction matrix elements $V_{11}(x) = V_{22}(x) = -V_{12}(x) = V, \ 0 \leq x \leq a \ (V_{\alpha\beta}(x) = 0, \ x \leq 0; \ a \leq x)$

\[
-\Psi_1''(x) + V\Psi_1(x) - V\Psi_2(x) = E\Psi_1(x) \\
-\Psi_2''(x) + V\Psi_2(x) - V\Psi_1(x) = E\Psi_2(x).
\]

In this case, we have the absolute transparency for identical boundary conditions in both the channels $\Psi_1(x) = \Psi_2(x)$. Moreover, there is free (!) motion due to complete cancellation of interactions. But if we choose different signs of the solutions $\Psi_1(x) = -\Psi_2(x)$, there is no such cancellation, which results in strong reflection of waves.

The next example demonstrates the influence of channel coupling on energy level spacing. Switching on the constant coupling $W$ between two channels with equivalent thresholds and infinite rectangular potential wells $V_{11} = V_{22}$ results in splitting initially degenerated levels $E_n \rightarrow E_n \pm W$.

Now let us consider the multichannel spectral control by using the IP approach.

4.1 Variation of SWV components

Let us start with an arbitrary initial potential matrix $\hat{V}_{\alpha\beta}(x)$ for the system with bound states. We shall vary SVW $\hat{C}_{\alpha n} \rightarrow C_{\alpha n}$ and energy value $\hat{E}_n \rightarrow E_n$ of the chosen $n$th bound state. The relevant expressions for the transformed potential matrix and solutions are derived, e.g., within the framework of the multichannel GL
formalism. This is the case that corresponds to the exactly solvable models of IP discussed in section 2. Similar formulae are in the Marchenko approach as well.

The resulting potential matrix has the form

$$V_{\alpha\beta}(x) = \tilde{V}_{\alpha\beta}(x) - 2 \frac{d}{dx} \left\{ \tilde{\Upsilon}(x) \hat{P}(x)^{-1} \tilde{\Upsilon}(x) \right\}_{\alpha\beta}, \quad (19)$$

where

$$\tilde{\Upsilon}(x) = \left( \begin{array}{cc} \sum_{\beta} C_{\beta n} \hat{\Phi}_{1\beta}(x, E_n) & \sum_{\beta} C_{\beta n} \hat{\Phi}_{1\beta}(x, \tilde{E}_n) \\ \sum_{\beta} C_{\beta n} \hat{\Phi}_{2\beta}(x, E_n) & \sum_{\beta} C_{\beta n} \hat{\Phi}_{2\beta}(x, \tilde{E}_n) \end{array} \right), \quad (20)$$

$$\hat{\Upsilon}(x) = \left( \begin{array}{cc} \sum_{\beta} C_{\beta n} \hat{\Phi}_{1\beta}(x, E_n) & \sum_{\beta} C_{\beta n} \hat{\Phi}_{2\beta}(x, E_n) \\ -\sum_{\beta} C_{\beta n} \hat{\Phi}_{1\beta}(x, \tilde{E}_n) & -\sum_{\beta} C_{\beta n} \hat{\Phi}_{2\beta}(x, \tilde{E}_n) \end{array} \right), \quad (21)$$

and

$$P_{11}(x) = 1 + \int_{0}^{x} \left[ \sum_{\beta} C_{\beta n} \hat{\Phi}_{1\beta}(y, E_n) \sum_{\beta} C_{\beta n} \hat{\Phi}_{1\beta}(y, E_n) \\ + \sum_{\beta} C_{\beta n} \hat{\Phi}_{2\beta}(y, E_n) \sum_{\beta} C_{\beta n} \hat{\Phi}_{2\beta}(y, E_n) \right] dy$$

$$P_{12}(x) = -P_{21}(x) = \int_{0}^{x} \left[ \sum_{\beta} C_{\beta n} \hat{\Phi}_{1\beta}(y, E_n) \sum_{\beta} C_{\beta n} \hat{\Phi}_{1\beta}(y, \tilde{E}_n) \\ + \sum_{\beta} C_{\beta n} \hat{\Phi}_{2\beta}(y, E_n) \sum_{\beta} C_{\beta n} \hat{\Phi}_{2\beta}(y, \tilde{E}_n) \right] dy$$

$$P_{22}(x) = 1 - \int_{0}^{x} \left[ \sum_{\beta} \hat{C}_{\beta n} \hat{\Phi}_{1\beta}(y, \tilde{E}_n) \sum_{\beta} \hat{C}_{\beta n} \hat{\Phi}_{1\beta}(y, \tilde{E}_n) \\ + \sum_{\beta} \hat{C}_{\beta n} \hat{\Phi}_{2\beta}(y, \tilde{E}_n) \sum_{\beta} \hat{C}_{\beta n} \hat{\Phi}_{2\beta}(y, \tilde{E}_n) \right] dy. \quad (22)$$

The transformed regular solution at arbitrary energy E is

$$\Phi_{\alpha\beta}(x, E) = \tilde{\phi}_{\alpha\beta}(x, E) - \int_{0}^{x} \sum_{\gamma} \left\{ \tilde{\Upsilon}(x) \hat{P}(x)^{-1} \tilde{\Upsilon}(y) \right\}_{\alpha\gamma} \tilde{\phi}_{\gamma\beta}(y, E) dy. \quad (23)$$

The solution $\Psi_{\alpha}(x, E_n) = \sum_{\beta} C_{\beta n} \Phi_{\alpha\beta}(x, E_n) = \left\{ \tilde{\Upsilon}(x) \hat{P}(x)^{-1} \right\}_{\alpha 1}$ represents a normalized wave function of the pushed nth bound state

$$\int_{0}^{\infty} \sum_{a} [\Psi_{a}(x, E_n)]^2 dx = 1.$$

Computer visualization of these formulae shows curious peculiarities of the corresponding transformations of the potential matrix and wave functions some of which are presented here.

A gradual increase in only one partial channel component of the spectral weight vector $C_{\alpha n}$ or $M_{\alpha n}$ for the nth bound state results in progressive concentration of waves in the $\alpha$th channel by their transition from all
other partial channel components $\beta \neq \alpha$ of the whole $n$th wave function. So the waves are gathered from both the configurational and channel spaces while the total norm is conserved. In the limiting case $C_{\alpha n} \to \infty$ or $M_{\alpha n} \to \infty$, the whole wave function is pressed into the origin in the $\alpha$ channel or is shifted to infinity. It means that the chosen energy level effectively disappears from the spectrum. A typical behavior of $V_{\alpha\beta}(x)$ and the wave function is shown in Fig. 3 when $M_{11} \to \infty$.

The wave function is gradually moved away to infinity by the reflectionless soliton-like 'carrier' potential well. This process is accompanied by the new effect: the concentration of all waves in the chosen (first) channel at the expense of the others (second one). This is true for both equal and different channel thresholds. What is also amazing is that in spite of strong coupling of channels, all other channels are almost completely emptied. The 'carrier' interaction well is surprisingly similar to the one-channel case: we meet again with the soliton-like potential as a simple "brick" of quantum reconstruction. The discovery of this remarkable effect deepens our understanding of complex multichannel systems. So in comparison with the one-channel case, there is a strong magnification of the response of the system to increasing partial components $C_{\alpha n}$ or $M_{\alpha n}$.

On the contrary, the gradual decreasing to zero of only one partial channel component of spectral weight vector $C_{\alpha n}$ or $M_{\alpha n}$ does not completely empty the $\alpha$th channel, see Fig. 15 in [12] as a typical example of such a situation. The $\alpha$th channel wave is only partly shifted. Some part of it is pushed out into other channels. In the one-channel case, the $n$th state disappears if $C_n \to 0$. In the multi-channel case, the zero value of the partial function and its derivative at one point does not mean that this wave must disappear. It returns to this channel at other points due to the wave exchange between channels. Only if all the components $C_{\alpha n}$ become zero, the bound state is completely removed. Thus, in comparison with the one-channel case, we have the weakening of the response of the system to the decrease in the partial spectral weight parameter. Furthermore, the equivalence of increasing (decreasing) in $C_{\alpha n}$ and decreasing (increasing) in $M_{\alpha n}$ being valid in the one-channel case is violated when the channels are coupled.

The IP and SUSYQ formalism allow one not only to remove energy levels but also to create them at given positions. With the initial free motion system this creation leads to reflectionless interaction matrices.

4.2 Transparency

It was once found that the Coulomb barrier between nuclei with large $Z$ is much more penetrable than was expected. Before our investigations on IP, we found the phenomena of intensified barrier transparency and even supertransparency in the multi-channel approach to quantum many-body systems, see e.g. [4].

Additional understanding of the phenomena can be achieved via consideration of the limiting model of total transparency. The isospectral transformation of a free motion system into the reflectionless one with a bound state gives [24]

$$V_{\alpha\beta}(x) = 2 \frac{d}{dx} M_{\alpha} M_{\beta} \exp[-(\kappa_{\alpha} + \kappa_{\beta}) x]$$

$$1 + \sum_{\gamma} \frac{M_{\gamma}^{2}}{2\kappa_{\gamma}} \exp(-2\kappa_{\gamma} x)$$

$$\kappa_{\alpha} = \sqrt{\epsilon_{\alpha} - E_{b}},$$

(24)
and

\[ \Psi_\alpha(x, E_b) = \frac{M_\alpha \exp(-\kappa_\alpha x)}{1 + \sum \frac{M_\gamma^2}{2\kappa_\gamma} \exp(-2\kappa_\gamma x)}. \]  

(25)

Here \( M_\alpha \) corresponds to the created bound state. In [26], the transparent interaction matrices in the case of different thresholds were derived and shown. For equal thresholds, the matrix elements of \( V_{\alpha\alpha}(x) \) have a simple soliton-like form. For different thresholds, there occurs an repulsion in \( V_{\alpha\alpha}(x) \) which is unexpectedly necessary for the complete transparency. The waves reflected disappear via destructive interference with the backward waves decaying from other channels. It is a non-trivial multichannel analog of the one-channel soliton-like potential well.

In (25), the first channel component of the bound state wave functions decreases "unnaturally" differently in the directions \( x \to \pm \infty \). It may seem that the channels become "disconnected" at large \( |x| \) values, and partial waves must have a natural asymptotic decrease \( \exp(\mp \sqrt{E - \epsilon_\alpha} |x|) \), \( x \to \pm \infty \) for \( V_{\alpha\beta}(x) \) rapidly decreasing in both the directions. But even a weak coupling \( V_{\alpha\beta}(x) \to 0 \) as \( x \to \infty \) can suck out the remainder waves from some channels into other ones violating the standard asymptotic behavior. Instead of the "naturally expected" behavior

\[ \Psi_\alpha(x, E_b) \to \exp(-\kappa_\alpha x), \ x \to \infty \]  

(26)

we have

\[ \Psi_1(x, E_b) \to \exp[(-\kappa_1 + 2\kappa_2)x], \ x \to -\infty \]  

(27)

and

\[ \Psi_2(x, E_b) \to \exp(\kappa_2 x), \ x \to -\infty. \]  

(28)

So, there is a phenomenon of partial inversion of the "degree of closeness" of different channels.

This can also be illustrated by reducing the two-channel system (2) to one equation for the partial channel wave function \( \Psi_1(x, E_b) \). This is done by substituting the explicit expression for \( \Psi_2(x, E_b) \) in terms of \( \Psi_1(x, E_b) \) (see Eq. (25))

\[ \Psi_2(x, E_b) = \frac{M_2}{M_1} \exp\{(\kappa_2 - \kappa_1)x\} \Psi_1(x, E_b). \]  

(29)

into the first equation in (2) with the potential matrix (24). As a result, we get the one-channel Schrödinger equation for \( \Psi_1(x, E_b) \) with the effective potential

\[ V(x) = 2 \frac{d}{dx} \left\{ \frac{M_1^2 \exp(-2(\kappa_1) x)}{1 + \sum \frac{M_\gamma^2}{2\kappa_\gamma} \exp(-2\kappa_\gamma x)} \right\} + \]

\[ -2 \frac{d}{dx} \left\{ \frac{M_1 M_2 \exp(-\kappa_1 x)}{1 + \sum \frac{M_\gamma^2}{2\kappa_\gamma} \exp(-2\kappa_\gamma x)} \right\} \exp\{(\kappa_2 - \kappa_1)x\} \frac{M_2}{M_1}. \]  

(30)

As \( x \to -\infty \), it becomes a non-zero constant \( 4\kappa_2(\kappa_2 - \kappa_1) > 0 \) whereas \( V_{22}(x) \) decreases as \( \exp[(2\kappa_2 - 2\kappa_1)x] \), which leads to the unnatural inversion of closeness of channels (on the left). This illustrates the fact that the rate of the asymptotic
decrease in the partial channel wave function can be strengthened even by exponentially decreasing coupling $V_{\alpha\beta}$. The interaction (30) is energy dependent.

In [16], we have also found transparent interaction matrices without bound states by employing the multichannel SUSYQ formalism which gives a broader class of exactly solvable models. These quantum systems not previously known have no one-channel analogs. They play a significant role in creating a bound state energy level close to the existing one in the initial spectrum (see the next section and the left part of Fig.5).

One of the multichannel peculiarities found by us is the right-left symmetry violation [4, 25] for the reflection of a complex particle by a non-symmetrical potential barrier. It is more penetrable in one direction for an incident particle in the ground state. And in the opposite direction (in two-channel approximation) it is more transparent in the exited state.

Let us now consider the model demonstrating shifts of energy levels up to their degeneracy.

4.3 Degeneracy of energy levels

In the one channel case, two bound states are forbidden to have the same energy. We have discovered [27] the phenomenon of “effective annihilation” of degenerating states. These states must both be equal, as solutions of the same equation with the same boundary conditions, and simultaneously orthogonal as different states. This contradiction is resolved in the following way. There is a separation of parts of the waves by the non-penetrable barrier (infinitely high or broad) under which $\Psi_\alpha \to 0$. So the wave functions become identical up to the sign in the limit of zero distance between the levels $E_n - E_{n+1} \to 0$. But in the case of $M$ channels, $M$ degenerate states can coexist provided that their SWV are linearly independent.

Let us use the IP formulae for creation of two energy levels $E_b$ and $E'_b$ which are very close to one another (including the case of degeneracy). As an initial two channel system on the whole line we choose the free motion with $V_{\alpha\beta}(x) = 0$. These formulae are similar to those in section 4.1:

$$V_{\alpha\beta}(x) = 2\frac{d}{dx}\{\hat{\Upsilon}(x)\hat{P}(x)^{-1}\hat{\Upsilon}(x)\}_{\alpha\beta},$$
where

$$\hat{\Upsilon}(x) = \begin{pmatrix} M_1 \exp(-\kappa_1 x) & M_1' \exp(-\kappa_1' x) \\ M_2 \exp(-\kappa_2 x) & M_2' \exp(-\kappa_2' x) \end{pmatrix},$$

$$\hat{\Upsilon}(x) = \begin{pmatrix} M_1 \exp(-\kappa_1 x) & M_2 \exp(-\kappa_2 x) \\ M_1' \exp(-\kappa_1' x) & M_2' \exp(-\kappa_2' x) \end{pmatrix},$$

and

$$P_{11}(x) = 1 + \sum_\alpha \frac{M_\alpha^2}{2\kappa_\alpha} \exp(-2\kappa_\alpha x)$$
\[ P_{12}(x) = P_{21}(x) = \sum_{\alpha} \frac{M_{\alpha}M'_{\alpha}}{\kappa_{\alpha}\kappa'_{\alpha}} \exp[-(\kappa_{\alpha} + \kappa'_{\alpha})x] \]

\[ P_{22}(x) = 1 + \sum_{\alpha} \frac{(M'_{\alpha})^2}{2\kappa'_{\alpha}} \exp(-2\kappa'_{\alpha}x). \]  

(34)

Here \( \kappa_{\alpha} = \sqrt{\epsilon_{\alpha} - E_b} \), \( \kappa'_{\alpha} = \sqrt{\epsilon_{\alpha} - E'_{b}} \) and \( M_{\alpha}, M'_{\alpha} \) are the SWV of the states under creation with the energies \( E_b \) and \( E'_{b} \), respectively. Let us combine normalized vector-column wave functions at the energies \( E_b \) and \( E'_{b} \) to form the matrix \( \hat{\Psi}(x) \) for compactification of the notation

\[ \hat{\Psi}(x) = \hat{\tilde{\Upsilon}}(x) \hat{\tilde{P}}(x)^{-1}. \]

(35)

When two levels \( E_b \) and \( E'_{b} \) with linearly dependent SWV come closer and closer to each other, the features of the effective annihilation become apparent. The wave functions are split and removed (either their parts or the whole states), as in the one-channel annihilation case. The same happens if SWV of the degenerated states become more linearly dependent (see Fig.4). Rudiments of spatial separation (preparation to the effective annihilation) can be observed even without changing the energy distance between the levels. As in the one-channel case ([12], Fig. 13), the states with linearly dependent SWV even corresponding to different energy levels will resist the concentration in the narrow spatial region. There was found an exchange of knots between partial channel wave functions with flips of some of their bumps ([8], Fig.14). The annihilation phenomenon and the recoil of all states with quantum number \( n \neq m \) from the origin when the chosen mth state is concentrated to \( x = 0 \) ([8] [25] have much in common. They are connected with the resistance of system of different states to their concentration in phase space.

There also occurs 'splitting' of BSEC when \( E_{1BSEC} \rightarrow E_{2BSEC} \) (as has been shown in [12], Fig. 22).

Unexpectedly, when two levels even with \textit{linearly independent} SWV come close to each other, this is often followed by special evolution of the interaction matrix \( V_{\alpha\beta}(x) \). In the case of unequal thresholds, there occurs a spatial separation of some block of \( V_{\alpha\beta}(x) \), see Fig.5. This block is transparent and goes to infinity in the limit of the level degeneracy resembling one-channel annihilation [24]. This separated block, in contrast with the annihilation case, does not carry away any part of the bound states. All bound states remain completely inside the main part of \( V_{\alpha\beta}(x) \).

An explanation of this new effect is still an open problem.

It is interesting that, in the multichannel case, there is the possibility of moving an energy level through the other level positions without the appearance of singularities in the interaction matrix. Such an operation is feasible if the SWV’s of the shifted state and the crossed states are linearly independent. We have already mentioned this peculiar interaction matrix in the previous section.

4.4 Bound states embedded into the continuum

BSEC being a wonderful phenomenon in the one-channel case, has a diversity of new multichannel aspects. A possibility appears to control the asymptotic decrease
of BSEC: making BSEC short- or long-range. The examples of exponential BSEC’s falloff $V_{\alpha\beta}(x)$ and $\Psi_\alpha(x, E_{BSEC})$ for systems with BSEC below the threshold of the most closed channel were shown in [1]. Later, we have discovered that in special cases BSEC can decrease like $\sim 1/x$ [28] when its energy is between the threshold ones. It can be explained as follows. The regular solutions increase exponentially in closed channels as $x \to \infty$. The physical scattering solutions $\Phi_\alpha(x, E)$ are constructed as a linear combination of the columns of the matrix of regular solutions with the coefficients providing exponential decrease in the closed channels. Let us choose the SWV components for BSEC proportional to these coefficients for solutions in the initial system (with non-vanishing channel coupling) where BSEC is to be created at $E_{BSEC}$. Then the resultant $V_{\alpha\beta}(x)$ and $\Psi_\alpha(x, E_{BSEC})$ have $\sim 1/x$ behavior. It is caused by the linear increase of the integral

$$\sum_\alpha \int_0^x \left[ \sum_\beta C_\beta \Phi_{\alpha\beta}(y, E_{BSEC}) \right]^2 dy$$

in the denominator of the formulae for $V_{\alpha\beta}(x)$ and $\Psi_\alpha(x, E_{BSEC})$ which are direct GL analogs of Eqs. (36) - (38), section 4.7. In this expression exponentially growing terms are cancelled by the aforementioned choice of SWV components $C_\alpha$. Any violation of the ratio between $C_\alpha$ leads to the exponential growth in the denominator. This results in exponential falloff of $V_{\alpha\beta}(x)$ and $\Psi_\alpha(x, E_{BSEC})$.

As in the one-channel case [29], a BSEC solutions above the thresholds, despite the different frequencies of oscillations of partial wave components, have strict conformity between knots of functions and diagonal interaction matrix elements. For each partial function bump there is a relevant potential well-barrier block, see Fig.1 and [29].

We have considered in section 4.1 pumping waves into a chosen channel by increasing the corresponding component of SWV. Analogous effect can also be observed for BSEC. The BSEC can be obtained even by taking a single partial $\alpha$th component of SWV to be non-zero. Then the interchannel coupling provides the transformation of other partial scattering waves into BSEC.

### 4.5 Phenomena of coexistence of states with ’incompatible’ properties

The existence of several physically allowed linearly independent solutions of Eq. (2) make it possible, unlike the one-channel case, to combine at the same energy different peculiarities of wave dynamics. For example, in [10] it was revealed that a bound state can coexist with a scattering state even without a strict space separation of propagating and bound waves. The qualitative arguments to resolve this ”paradox” are the following. In the M-channel case, there exist M linearly independent regular solutions of (2) for the same energy point. Above all thresholds these solutions are scattering states. In the IP and SUSY approach one of the scattering states can be transformed into a BSEC at $E = E_b > \epsilon_i; i = 1, 2, \ldots, M$. Then there will remain $M - 1$ scattering state solutions. As a result, it is impossible to construct scattering solutions at BSEC energy with arbitrary asymptotic conditions, e.g., with the incident wave in one channel only.
By analogy with the effect considered above, there can exist scattering solutions with different resonance decay widths at the same energy. We can construct such an exactly solvable model. Let us start from two uncoupled channels with different thresholds and resonances at the same energy with unequal widths $\Gamma_1, \Gamma_2$. We can create a bound state with non-zero SWV components in each channel by using the GL analogs of Eqs.(36)-(38). There arises a strong coupling $V_{12}(x)$ between channels. For incident waves in the $\alpha$th channel only, we shall get resonance scattering with the width $\Gamma\alpha$ despite the intensive exchange of waves between the channels inside the interaction region. Outgoing waves survive in the entrance channel only. After the mixing of channel waves in region where $V_{12}(x) \neq 0$, they return to the entrance channel. It is a common opinion that compound states (resonances) in complex systems "forget" how they were generated. Here an extremely simple example has been demonstrated of direct dependence of the life-time of a quasi-bound state on boundary conditions. So in the general case the analogous phenomenon is also possible.

In section 4.7, the coexistence of transparency and strong reflection is shown.

### 4.6 M+1 spectra

The systems with the pure discrete spectrum in the one-channel case are uniquely determined by the set of $\{E_n, C_n\}$ or by two spectra $\{E_n, E_m\}$. Here the index $m$ is related to eigenvalues with different boundary conditions at one end point of an interval (e.g., at the origin $\Psi'_\alpha(0) = 0$ instead of $\Psi_\alpha(0) = 0$) [3]. Daskalov [32] has suggested exactly solvable models corresponding to variations of any eigenvalues from two spectra $\{E_n, E_m\}$. This enlarges the possibilities to control quantum systems. In the M-channel case, the set of eigenvalues and the sets of $M$ SWV components $\{E_n, C_{m\alpha}\}$ form the complete $(M+1)$-fold set of parameters. They are equivalent to M+1 spectra with different boundary conditions [3]. Exact models for M+1 spectra variations are now under construction.

### 4.7 Resonance tunneling

Almost non-penetrable barriers could be combined in a system transparent at discrete energy values, see e.g. [4, 31]. This phenomenon in the one-channel case has already found numerous applications, see [32] and references therein. But we have not seen any papers on multichannel resonance tunneling. The waves accumulated between the barriers in the $V_{\alpha\beta}(x)$ decay in both directions. The decaying waves going backward can have the opposite phases and the same amplitudes as the reflected waves in all open channels. In this case reflected and decaying waves cancel one another totally, which results in resonance tunneling. It appears that the multichannel analog of this phenomenon is restricted by additional conditions. We have found that it can be achieved only for special proportions of amplitudes of incident waves in different channels at discrete resonance energies $E_{res}$.

So, there can coexist at the same energy solutions without reflection at all and with strong reflection depending on boundary conditions. The relevant example (exactly solvable IP model) can be constructed of the two initially uncoupled channels ($\tilde{V}_{12}(x) = 0$). We can choose the potential $\tilde{V}_{11}(x)$ in the first channel having
"one-channel" resonance tunneling at $E = E_{\text{res}}$ and in the second channel the potential $V_{22}(x)$ being weakly penetrable at the same energy. Let us now create a bound state at $E = E_b < 0$ common for two channels by the IP transformation $\hat{V}_{\alpha \beta}(x) \to V_{\alpha \beta}(x)$ that mixes intensively partial channel waves (i.e., leads to $V_{12}(x) \neq 0$). Then the non-trivial two-channel system occurs for which the waves incident in the first channel (i.e., $\psi_\alpha(x) \to \delta_\alpha \exp(-ik_1x), x \to \infty$) propagate without reflection at $E = E_{\text{res}}$ whereas the waves incident in the second channel (i.e., $\psi_\alpha(x) \to \delta_2 \exp(-ik_2x), x \to \infty$) are strongly reflected. This is possible because the IP transformations keep the continuous spectrum characteristics unchanged. The corresponding analytic expressions have the form:

$$V_{\alpha \beta}(x) = \hat{V}_{\alpha \beta}(x) + 2 \frac{d}{dx} \frac{\sum_{\alpha' \beta'} \hat{F}_{\alpha \beta'}(x, E_b) M_{\alpha'} M_{\beta'} \hat{F}_{\beta \alpha'}(x, E_b)}{1 + \int_x^{\infty} \sum_{\gamma} \sum_{\alpha' \beta'} \hat{F}_{\gamma \beta'}(y, E_b) M_{\alpha'} M_{\beta'} \hat{F}_{\gamma \alpha'}(y, E_b) dy},$$

(36)

where $\kappa_\alpha = \sqrt{\epsilon_\alpha - E_b}$;

$$F_{\alpha \beta}(x, E) = \hat{F}_{\alpha \beta}(x, E) - \int_x^{\infty} \sum_{\gamma} \sum_{\alpha' \beta'} \hat{F}_{\gamma \beta'}(y, E_b) M_{\alpha'} M_{\beta'} \hat{F}_{\gamma \alpha'}(y, E_b) dy \frac{\sum_{\beta} M_{\beta} \hat{F}_{\alpha \beta}(x, E_b)}{1 + \int_x^{\infty} \sum_{\gamma} \sum_{\alpha' \beta'} \hat{F}_{\gamma \beta'}(y, E_b) M_{\alpha'} M_{\beta'} \hat{F}_{\gamma \alpha'}(y, E_b) dy},$$

(37)

where $F_{\alpha \beta}(x, E)$ is the Jost solution at arbitrary energy $E$ obeying the asymptotic condition $F_{\alpha \beta}(x, E) \to \delta_{\alpha \beta} \exp(-i\sqrt{E - \epsilon_\alpha} x)$ as $x \to \infty$. The created bound state has the form

$$\Psi_\alpha(x, E_b) = \frac{\sum_{\beta} M_{\beta} \hat{F}_{\alpha \beta}(x, E_b)}{1 + \int_x^{\infty} \sum_{\gamma} \sum_{\alpha' \beta'} \hat{F}_{\gamma \beta'}(y, E_b) M_{\alpha'} M_{\beta'} \hat{F}_{\gamma \alpha'}(y, E_b) dy}.$$
One could suspect that switching on a weak coupling would hinder wave propagation at the energies belonging to a forbidden zone in one channel and to an allowed zone in another one. Indeed, it may seem that the exponential increase in the wave function component in the forbidden zone would prevail and cause physically unacceptable asymptotical divergence of solutions. But really it appears that switching on the coupling changes slightly the allowed zones of the uncoupled channels. So just the regime of oscillations, characteristic to the allowed zones, dominates. It results in damping the exponential increase corresponding to forbidden zones. In fact, there are two types of two-channel generalized Bloch’s waves

\[
\Psi_{1\alpha}^{(1,2)}(x) = \exp(\pm i\mathcal{K}(^{(1,2)}a)x)\Psi_{1\alpha}^{(1,2)}(x-a)
\]

with the quasi-momenta \(\mathcal{K}^{(1,2)}(E)\) common for both the channels. We can find such quasi-momenta by using, e.g., the exact model with the following periodic \(\delta\)-interaction matrix:

\[
V_{\alpha\beta}(x) = \sum_{n=-\infty}^{\infty} V_{\alpha\beta}(x+na).
\]

With such a choice of \(V_{\alpha\beta}(x)\) we have

\[
\cos(\mathcal{K}^{(1,2)}a) = \frac{1}{2}(\cos(K_1a) + \cos(K_2a)) \\
\pm \sqrt{[\cos(K_1a) - \cos(K_2a)]^2 + \frac{W^2\sin(k_1a)\sin(k_2a)}{k_1k_2}}.
\]

where \(W \equiv V_{12} = V_{21}, V_{\alpha} \equiv V_{\alpha\alpha},\) and the expressions for \(\cos(K_{1,2}a)\) are given by (29). So the condition \(\left|\cos(\mathcal{K}^{(1,2)}a)\right| \leq 1\) serves for the criterion of searching the allowed zones. It is shown in Fig.6 that the quasi-momenta \(\mathcal{K}^{(1,2)}\) are close to ones for the uncoupled channels (for \(W = 1\)).

Another peculiarity is that we can create a gap in the given allowed spectral zone constructing a special multichannel periodic potential. In fact, let us consider at first wave motion on a finite interval (period) [0, \(a\)] with homogeneous boundary conditions. Further, we transform a constant interaction matrix \(V_{\alpha\beta}(x)\) defined on this interval to \(V_{\alpha\beta}(x)\) so that a chosen bound state at the energy \(E_n\) related to \(V_{\alpha\beta}^{\circ}(x)\) is raked up to the right boundary by scalar decrease in the SWV \(C_{\alpha n}\). Next, let us continue the resulting matrix block to the whole axis with the step \(a\)

\[
V_{\alpha\beta}^{\text{per}}(x+la) = V_{\alpha\beta}(x), \ l = 0, \pm 1, \pm 2,\ldots, \ 0 \leq x \leq a.
\]

For this periodic potential, Bloch’s solution at the energy \(E_n\) can be constructed in each lth interval \([la, (l+1)a]\) from the solutions obtained via the \(la\)-transfer of \(\Psi_\alpha(x, E_n)\) over the coordinate line: \(\Psi_{\alpha}^{(l)}(x+la, E_n) = \Psi_{\alpha}(x, E_n), \ 0 \leq x \leq a\). We have the following equalities: \(\Psi_{\alpha}(0, E_n) = \Psi_{\alpha}(a, E_n) = 0\) and \(\Psi_{1}(0, E_n)/\Psi_{2}(0, E_n) = \Psi_{1}(a, E_n)/\Psi_{2}(a, E_n)\). This guarantees the possibility of smooth continuation of the wave function under the construction \(\Psi_{\alpha}^{\text{per}}(x, E_n)\) \((-\infty \leq x \leq \infty)\) at the points \(la\) provided that the module of the solution \(\Psi_{\alpha}^{(l+1)}(x, E_n)\) for the \((l+1)\)th subsequent sector is each time multiplied by a scalar (i.e., independent of \(\alpha\)) factor \(\Theta \equiv \Psi_{\alpha}(a, E_n)/\Psi_{\alpha}(0, E_n) > 1\) \((\alpha = 1, 2)\)

\[
\Psi_{\alpha}^{\text{per}}(x+la, E_n) = (-1)^l\Theta^{l}\Psi_{\alpha}(x, E_n), \ l = 0, \pm 1, \pm 2,\ldots, \ 0 \leq x \leq a.
\]
Eventually, this leads to an exponential swinging divergence of $\Psi_{\alpha}^{\text{per}}(x, E_n)$ as $x \to \infty$ at the energy $E_n$. It means that this energy belongs to a forbidden zone created with the above procedure.

We can transform the periodical multichannel system by creating bound states inside the allowed or forbidden zones as, e.g., was shown in [34]. It is also interesting that using the SUSYQ transformations of (41) we have an additional flip effect of all the delta-peaks, as in the one-channel case [34]. In fact, after the single SUSYQ transformation (see, e.g. [16]) the potential matrix has the form

$$\hat{V}(x) = \hat{V}_0(x) - 2 \frac{d}{dx} [\hat{\Psi}_0(x) \hat{\Psi}_0(x)^{-1}] = -\hat{V}_0(x) + 2\mathcal{E} + 2[\hat{\Psi}_0(x) \hat{\Psi}_0(x)^{-1}]^2,$$

(43)

where $\hat{V}_0(x)$ is given by (41); $\hat{\Psi}_0(x)$ represents the corresponding matrix-valued solution at the factorization energy $\mathcal{E}$. The term $-\hat{V}_0(x)$ with the minus sign occurs in the last equality, which means the flip of all the original $\delta$-peaks which cannot be compensated for all $x$ by other finite terms.

5 Conclusion

The quintessence of the quantum mechanics is the special relationship of observables $S$ and interactions $V$ for wave systems. The universal simple rules for $V$ elementary transformations caused by elementary $S$ variations in the one-channel case were established earlier [8, 25].

For a long time the multichannel theory has been a 'black box' connecting input and output data through cumbersome computer calculations. The examples presented in this paper help one to look into intimate physical details of different new effects inherent in coupled Schrödinger equations.

The considered elements of the multichannel quantum design give some notion about elementary transformations from their complete set for systems with the complicated inner structure. No doubt, the multichannel theory must be richer and more wonderful than in the one-channel case.

From a historical point of view, it can be worth mentioning that the IP for multichannel systems with the same thresholds was considered in the book by Agranovich and Marchenko on multichannel IP (see references in [3]). See also the interesting papers [13, 17, 18, 23, 35, 36] and papers by Cox (see references in [5]).

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**FIGURE CAPTIONS**

Fig.1. Two uncoupled channels with the initial infinite rectangular potential wells \( V_{aa}(x) \) transformed in the following way. The lowest bound state in the first channel is shifted over the coordinate axis \( x \). The lowest energy level in the second channel is shifted over the \( E \) axis. The ground state of the first channel spectral branch was shifted to the right. All the energy levels in this branch and all other spectral weights \( C_n \neq 1 \) remain unperturbed. To decrease only one spectral weight \( \psi_1'(x = 0) \) of the ground state of the first channel, a potential perturbation block \( \Delta V_1(x) \) (barrier + well) is needed. The lowest energy level of the second channel branch was lifted upward without shifting other levels of the second branch.

Fig.2. A scattering state is transformed into a bound state at the origin in the first uncoupled channel by increasing its spectral weight \( C(E_{BSEC}) \).

Soliton-like well in the second channel serves as a carrier for the chosen bound state moving out of the original well when the partial channel spectral weight vector component decreases.

Fig.3. Increasing first channel parameter \( (M_{11} = 10^7, M_{21} = 0) \) for the ground state. Both the initial wave functions \( \hat{\Psi}_1(x, E) \) and \( \hat{\Psi}_2(x, E) \) are concentrated after the transformation in the first channel wave component \( \hat{\Psi}_1(x, E) \) and pooled to the right inside the separate soliton-like potential well \( V_1(x) \). The initial potential matrix is defined on the half-line and has the form \( \hat{V}_{11}(x) = \hat{V}_{22}(x) = -5, \hat{V}_{12}(x) = \hat{V}_{21}(x) = 0.3, \) \( 0 < x < \pi \). \( V_{a\beta}(x) = 0, x > \pi \), the thresholds are \( \epsilon_1 = 0, \epsilon_2 = 1 \). The remaining part of \( \hat{\Psi}_2(x, E) \) in the second channel is so small that is not visible.

Fig.4. Two degenerated bound states created at \( E_b = -0.5 \) become 'more' linearly dependent. This results in splitting of a) the interaction matrix and b) wave functions into two parts. The initial system corresponds to free motion along the whole line. a) Two blocks of \( V_{a\beta}(x) \) for almost equal SWV: \( M_1 = M_2 = 1; M_1' = 1, M_2' = 1.01 \). The left part of \( V_{a\beta}(x) \), a soliton-like well 'moves' to \( -\infty \) in the limit \( M' \rightarrow M_2 \) as is schematically shown by dashed lines. b) This well carries also the left part of wave functions to \( -\infty \).

Fig.5. a) A potential block (left part of the interaction matrix) without a bound state is separated and shifted to the left when two closely spaced bound states are created : \( E_b = 0.5; M_1 = 0, M_2 = 1, E_b' = 0.501; M_1' = 1, M_2' = 0.1 \). This block 'moves' to \( -\infty \) in the limit \( E_b \rightarrow E_b' \) as is schematically shown by dashed lines. b) Wave functions remain almost unchanged. They don’t "live" inside this potential block, which is almost transparent if considered separately of other part of potential matrix, see [28] and Fig.8 there.

Fig.6. a) The functions \( \cos(K^{(1,2)} \pi) \) (solid line) specifying the location of spectral zones for the two-channel system are shown. Allowed and forbidden zones are determined by \( |\cos(K^{(1,2)} \pi)| \leq 1 \) and \( |\cos(K^{(1,2)} \pi)| \geq 1 \), respectively. The parameters are \( V_1 = 6; V_2 = 5; W = 1; a = \pi; \epsilon_1 = 0; \epsilon_2 = 1 \). In the case of uncoupled channels, the corresponding curves \( \cos(K_{1,2} \pi) \) for separate channels are shifted from one another due to the threshold difference and have crossings shown by the dashed lines.
Switching on the coupling transforms the crossings (dashed lines) into the quasi-crossings so that there are the upper and the lower solid lines for two branches of band spectra. b) Solid line intervals are forbidden spectral zones of the coupled equations (their allowed zones are outside these intervals). The solid and the dashed line intervals together represent the forbidden zones for any uncoupled equation. Only blank intervals between vertical dashes correspond to the common allowed zones for both uncoupled equations.
Fig. 1
Fig. 2
Fig. 3
Fig. 4
Fig. 6

\[ \cos(K\pi) \]

\[ \cos(K_1\pi) \]

\[ \cos(K_2\pi) \]