Impact of Compliant Wall Properties on Peristaltic Transport of a Compressible Non-Newtonian Maxwellian Fluid Through Axisymmetric Cylindrical Tube

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Abstract

In this article, Maxwell's viscous fluid movement is studied through a cylindrical symmetric tube (pore) with a compliant wall and this movement is peristaltic. The fluid’s compressibility and slip effect have been taken into consideration. The problem was solved with the perturbation approach in terms of small amount that triggers amplitude ratio. Second-order approximation, the net flux caused by the moving wave is determined for various physical parameters like the wall tension, damping force, wall rigidity, slip parameter, wave number, relaxation time and compressibility parameter. The impacts of interested parameters are discussed numerically and illustrated graphically. In various fields of study, such as biological sciences, there are many applications of this study represented in studying blood flow in living things and in industry as well as simulating fluid flow through flexible tubes.

Keywords: Compressible fluid, Wall features, Slip conditions, Peristaltic flow, Maxwellian fluid.

1. Introduction

The Peristalsis results from the change of pressure on the wall of the tubular organs, which leads to constriction and the expansion of the muscles continuously, working on the movement of fluids during these members. This phenomenon has many applications through the human body are represented in food is swallowed through the esophagus into the stomach, transmission of lymph into lymphatic vessels, transport of chyme during small intestine, and the flow through the urethra of urine from the kidneys to the urinary bladder. The phenomenon of peristalsis also is played a role in biomedical instruments is represented in dialysis machines, artificial devices for hearts and orthos, heart-lung machines, transport inside the sanitary ducts with toxic substance waste.

The mechanism of peristalsis action during the last four decades has attracted the attention of many investigators. Literature review is introduced to demonstrate the behaviour of peristaltic phenomena, Latham [1]’s initial attempt introduced some experimental studies by simulating the model of the peristalsis movement and compared the results obtained with theoretical analysis and verified its validity. Burns and Parkes [2] handeled a viscous fluid that moves peristatically in axisymmetric tube and wavy shaped channel, bearing in mind that the Reynolds number small. Shapiro [3] introduced the earlier mathematical
investigations. Shapiro et al. [4] studied two physiological theory" reflux, trapping" under the long wave length approximation with low Re. Barton and Raynor [5] discussed the peristaltic transport and obtained the solution for it in the tube. And the results of [3] checked by Fung and Yih [6]. The peristaltic fluid transport in tube and its application in the aerospace field have been studied by Eldesoky and Mousa [7].

Recently, scholars in this field have been researching various aspects of studying peristaltic flows such as Srivastava and Srivastava [8-11] interested in studying the wavy motion of a non-compressed fluid and also, in [12] introduced a two-phase fluid’s peristaltic motion. Hayat et al. [13] explained the peristaltic movement via a tube. The impact of slip and endoscopy effects on the peristaltic movement in an irregular annulus of blood containing suspended particles have been analyzed by Bhatti et al. [14]. Mekheimer et al. [15] expounded the peristaltic motion of two phase flow of fluid circulation by 2D channel. Eldesoky et al. [16] investigated the peristaltic motion fluid with suspended particles under the effect of heat transfer with MHD in porous medium. Abdelsalam et al. [17-21] also discussed the peristaltic movement under different topics. Eldesoky et al. [22] introduced the combined influence of heat transfer, porous media, slip state, and magnetic field on a fluid provided with suspended particles peristatically moving in catheterized tube.

For compressible fluid there are many article as Srinivas et al. Anderson [23] introduced the concept of compressible fluid. Aarts and Ooms [24] investigated the wavy transport of compressible fluids that flow under the impact of ultrasonic radiation via a porous conduit. Eldesoky et al. [25] displayed the effectiveness of suspension concentration on the wavy motion of compressible flow in channel. Elshehawey et al. [26] showed that compressibility parameter, space porosity and relaxation time have a visible role on net flow induced by a peristaltic locomotion in a tapered pore. Also, the action of porosity of the medium and magnetic field on compressible Maxwellian fluid moved though microchannel has been introduced by Mekheimer et al. [27].

Many researchers have studied the compliant nature of the channel or tube walls effect in net flow rate such as Pandey and Chaube [28], Javed et al. [29]. In addition, Shankar and Kumaran [30] Introduced the effect wall properties on the Couette flow by taking high value of Re number in account. The effects of wall characteristics on a couple stress fluid flowing peristaltically have been studied by Ellahi et al. [31]. Elnaby and Haroun [32] investigate the study of wavy movement of a non-compressible liquid through a duct having a flexible walls by taking into account the porosity of the medium. Abd-Alla et al. [33] took the effectiveness of rotation, heat transfer, space porosity and wall characteristics on an incompressible fluid's wavy flow in a duct. Mekheimer and Abdel-Wahab [34] discussed an analytical investigation for the movement of compressed fluid in a small channel. Abbas et al. [35] introduced the peristaltic locomotion of Ellis fluid under the influence of a channel's compliant walls. Eldesoky et al.[36] debated the combined impact of slip state, heat transfer and wall characteristics of peristaltic locomotion of compressible fluid in tube.

On the other hand, the Relaxation time also is taking some interest in recent studies such as Hayat et al. [37] improved the analysis of [36] for Jeffrey fluid. An investigation of peristaltic transportation for a Maxwell viscoelastic fluid and taken slip conditions in account have been debated by El-shehawy et al. [38]. Hina et al. [39] discussed the elastic features of a flexible duct on a Maxwell fluid moving peristaltically. Eldesoky et al. [40] illustrated the impact of slipping and relaxation time on compressible Maxwellian fluid
moved peristaltically in flexible duct. Eldesoky and Mousa [41] analyzed the influences of relaxation time and space porosity on the peristaltic transport of a compressible flowing in tube. Eldesoky. [42] presented enhancement to article [41] by adding the effect of slip parameter to the others. Haroun [43] analyzed the wavy flow of Oldroydian viscoelastic fluid under the effectiveness of retardation and relaxation time.

From the above, the previous studies did not take into account the combined impact of the various factors such as compliant wall features, slip effect and the relaxation time with viscoelastic parameters of compressible Maxwell fluid flow peristaltically in tube. This article has not been handeled in open literature, to the best of the understanding of authers. Actually we extend the analysis of El-shehawy et al. [38] by adding the wall properties's impact into account and using perturbation method with small amplitude to handling mathematical relations expressed the average axial velocity, pressure gradient and net flux in terms of different parameters.

2. Formulation of the problem
Consider that Maxwell's compressible fluid flows in an identical two-dimensional cylindrical tube with uniform radius \( R \) and length \( L \) with flexible walls shown in figure 1. The wall is considered as a wall of compliant on which we are imposed an elastic wave induces small amplitude travelling sinusoidal wave and the following form shows its displacement [24]

\[
W(z,t) = R + \eta(z,t), \tag{2.1a}
\]

and

\[
\eta(z,t) = a \cos \left( \frac{2\pi}{\lambda} (z - ct) \right), \tag{2.1b}
\]

where the amplitude of travelling wave, wavelength and its velocity are \( a, \lambda, c \) respectively.

Figure 1. Diagram of tube with compliant wall.

Let \( (r,z) \) be the polar coordinate and \( z \)-axis goes around the tube axis in which the basic flow is taken parallel to it in the orientation of a wave's propagation. If \( r \) and \( z \)-directions velocities are \( v_r \) and \( v_z \).

The momentum and conservation of mass equations are the equations governing the flow in the form of vector notation, as following [45]

\[
\frac{\partial p}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0, \tag{2.2a}
\]
\[
\rho \frac{\partial \vec{v}}{\partial t} + \rho (\vec{v} \cdot \nabla) \vec{v} = -\nabla p - \nabla \cdot \vec{\tau},
\]

(2.2b)

where \(\rho\) liquid density, \(p\) pressure of liquid, \(\vec{\tau}\) viscous stress tensor, and \(\vec{v}\) velocity vector.

By using Maxwell model which represent the viscoelastic properties of the fluid in which

\[
t_m \frac{\partial \vec{\tau}}{\partial t} = -\mu \nabla \vec{v} - \frac{\mu}{3} (\nabla \cdot \vec{v}) - \vec{\tau},
\]

(2.3)

where \(\mu\) dynamic viscosity and \(t_m\) relaxation time.

The compression property of the liquid is described by Anderson [23] as follows

\[
\frac{1}{\rho} \frac{\partial \rho}{\partial p} = k^*,
\]

(2.4a)

where \(k^*\) the liquid compressibility. The solution of equation (2.5a) which the density is given as a function of pressure by

\[
\rho = \rho_o e^{[k^* (p-p_0)]},
\]

(2.4b)

such that \(\rho_o\) is a fixed density with a reference pressure \(p_0\).

We can rewrite equation (2.4) by the way

\[
\left(1 + t_m \frac{\partial}{\partial t}\right) \vec{\tau} = -\mu \nabla \vec{v} - \frac{\mu}{3} (\nabla \cdot \vec{v}).
\]

(2.5)

By applying the operator \(1 + t_m \frac{\partial}{\partial t}\) to equation (2.2b) and eliminate \(\vec{\tau}\) from it using equation (2.5), then the momentum equation reads

\[
\left(1 + t_m \frac{\partial}{\partial t}\right) \left( \rho \frac{\partial \vec{v}}{\partial t} + \rho (\vec{v} \cdot \nabla) \vec{v} \right) = -\left(1 + t_m \frac{\partial}{\partial t}\right) \nabla p + \mu \nabla \vec{v} + \frac{\mu}{3} (\nabla \cdot \vec{v}).
\]

(2.6)

The flexible wall motion can be obtained from [34] as following equation

\[
L (\eta) = p - p_c = -T \frac{\partial^2 \eta}{\partial z^2} + m \frac{\partial^2 \eta}{\partial t^2} + D \frac{\partial \eta}{\partial t} + B \frac{\partial^4 \eta}{\partial z^4} + K \eta,
\]

(2.7)

where \(L\) is an operator, which describe the motion of complaint wall of the stretched membrane under the effect of viscous damping forces, \(T\) is longitudinal tension for each unit wall width, the mass of plate per unit area is \(m\), \(D\) is wall damping coefficient, \(B\) is plate flexural rigidity, \(K\) is stiffness of the spring, \(p_c\) is the pressure which applied on the wall's outer surface due to muscle's tension, and \(p\) the interaction pressure because of fluid particles' motion on the wall. It will supposed that \(p_c = 0\) and tube walls are not extendable so, the horizontal movement is negligible.

In polar coordinates, the conservation of mass equation becomes

\[
\frac{\partial \rho}{\partial t} + v_r \frac{\partial \rho}{\partial r} + v_z \frac{\partial \rho}{\partial z} + \rho \left( \frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{\partial v_z}{\partial z} \right) = 0.
\]

(2.8a)

And the Navier-Stokes quations reads
\[ (1 + t_m \frac{\partial}{\partial t}) \left( \rho \frac{\partial v_r}{\partial t} + \rho \left( v_r \frac{\partial v_r}{\partial r} + v_z \frac{\partial v_r}{\partial z} \right) \right) \]
\[ = -\left( 1 + t_m \frac{\partial}{\partial t} \right) \frac{\partial p}{\partial t} + \mu \left( \frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r} \frac{\partial v_r}{\partial r} - \frac{v_r}{r^2} + \frac{\partial^2 v_r}{\partial z^2} \right) \]
\[ + \frac{\mu}{3} \frac{\partial}{\partial r} \left( \frac{\partial v_r}{\partial r} + \frac{\partial v_r}{\partial r} + \frac{\partial v_z}{\partial z} \right). \]

\[ (2.8b) \]

\[ (1 + t_m \frac{\partial}{\partial t}) \left( \rho \frac{\partial v_z}{\partial t} + \rho \left( v_r \frac{\partial v_z}{\partial r} + v_z \frac{\partial v_z}{\partial z} \right) \right) \]
\[ = -\left( 1 + t_m \frac{\partial}{\partial t} \right) \frac{\partial p}{\partial t} + \mu \left( \frac{\partial^2 v_z}{\partial r^2} + \frac{1}{r} \frac{\partial v_z}{\partial r} + \frac{\partial^2 v_z}{\partial z^2} \right) + \frac{\mu}{3} \frac{\partial}{\partial z} \left( \frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{\partial v_z}{\partial z} \right). \]

\[ (2.8c) \]

According to the boundary conditions, the fluid on the wall must achieve the slip conditions and impermeability conditions, then the wall's boundary conditions become

\[ v_r (W, z, t) = \frac{\partial W}{\partial t}, v_z (W, z, t) = A \frac{\partial v_z}{\partial r}. \]

(2.9)

The dynamic boundary condition is

\[ \frac{\partial L(\eta)}{\partial z} = \frac{\partial p}{\partial z}, \]

(2.10a)

then, using z-momentum equation we can rewrite dynamic boundary condition in the following form

\[ \left( 1 + t_m \frac{\partial}{\partial t} \right) \frac{\partial}{\partial z} \left\{ -T \frac{\partial^2 \eta}{\partial z^2} + m \frac{\partial^2 \eta}{\partial t^2} + D \frac{\partial \eta}{\partial t} + B \frac{\partial^4 \eta}{\partial z^4} + K \eta \right\} \]

\[ = \mu \left( \frac{\partial^2 v_z}{\partial r^2} + \frac{1}{r} \frac{\partial v_z}{\partial r} + \frac{\partial^2 v_z}{\partial z^2} \right) + \frac{\mu}{3} \frac{\partial}{\partial z} \left( \frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{\partial v_z}{\partial z} \right) \]
\[ - \left( 1 + t_m \frac{\partial}{\partial t} \right) \left( \rho \frac{\partial v_z}{\partial t} + \rho \left( v_r \frac{\partial v_z}{\partial r} + v_z \frac{\partial v_z}{\partial z} \right) \right). \]

(2.10b)

By adding a non-dimensional analysis, the dimensionless variables are introduced:

\[ \bar{W} = \frac{W}{R}, \quad \bar{v_r} = \frac{v_r}{c}, \quad \bar{v_z} = \frac{v_z}{c}, \quad \bar{\rho} = \frac{\rho}{\rho_o}, \quad \bar{p} = \frac{p}{\rho_o c^2}, \quad \bar{t} = \frac{t c}{R}, \quad \bar{m} = \frac{m}{\rho_o R^2}. \]

(2.11a)

The dimensionless parameters are the wave number \( \alpha \), the amplitude ratio \( \varepsilon \), the Reynolds number \( Re \), the compressibility number \( \chi \) and the Knudsen number \( kn \) are defined by

\[ \alpha = \frac{2\pi R}{\lambda}, \quad \varepsilon = \frac{a}{R}, \quad Re = \frac{\rho_o c R }{\mu}, \quad \chi = k^* \rho_o c^2, \quad kn = \frac{A}{R}. \]

(2.11b)

In the non-dimensional form, we can rewrite equations (2.4b) and (2.8)

\[ \frac{\partial \bar{p}}{\partial \bar{t}} + \bar{v}_r \frac{\partial \bar{p}}{\partial \bar{r}} + \bar{v}_z \frac{\partial \bar{p}}{\partial \bar{z}} + \rho \left( \frac{\partial \bar{v}_r}{\partial \bar{r}} + \frac{\partial \bar{v}_r}{\partial \bar{r}} + \frac{\partial \bar{v}_z}{\partial \bar{z}} \right) = 0, \]

(2.11.1a)
\[
\left(1 + t_m \frac{\partial}{\partial t}\right) \left( \rho \frac{\partial v_r}{\partial t} + \rho \left( v_r \frac{\partial v_r}{\partial r} + v_z \frac{\partial v_r}{\partial z} \right) \right)
= -\left(1 + t_m \frac{\partial}{\partial t}\right) \left( \frac{\partial p}{\partial t} + \frac{1}{Re} \left( \frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r} \frac{\partial v_r}{\partial r} - \frac{v_z}{r^2} + \frac{\partial^2 v_r}{\partial z^2} \right) \right)
+ \frac{1}{3Re} \frac{\partial}{\partial r} \left( \frac{\partial v_r}{\partial r} + \frac{\partial v_r}{\partial z} \right), \tag{2.11.1b}
\]
\[
\left(1 + t_m \frac{\partial}{\partial t}\right) \left( \rho \frac{\partial v_z}{\partial t} + \rho \left( v_r \frac{\partial v_z}{\partial r} + v_z \frac{\partial v_z}{\partial z} \right) \right)
= -\left(1 + t_m \frac{\partial}{\partial t}\right) \left( \frac{\partial p}{\partial t} + \frac{1}{Re} \left( \frac{\partial^2 v_z}{\partial r^2} + \frac{1}{r} \frac{\partial v_z}{\partial r} + \frac{\partial^2 v_z}{\partial z^2} \right) \right)
+ \frac{1}{3Re} \frac{\partial}{\partial z} \left( \frac{\partial v_r}{\partial r} + \frac{\partial v_z}{\partial z} \right), \quad \rho = e^\epsilon (\varphi - \varphi_o). \tag{2.11.1d}
\]
Also the boundary conditions (2.9), (2.10) becomes
\[
v_r \left( (1 + \eta), z, t \right) = \frac{\partial \eta(z, t)}{\partial t} \cdot v_z \left( (1 + \eta), z, t \right) = k \eta \frac{\partial v_z(r, z, t)}{\partial r}, \tag{2.11.2a}
\]
\[
\left(1 + t_m \frac{\partial}{\partial t}\right) \frac{\partial}{\partial z} \left( -\frac{T}{Re^2 \partial^2} + m \frac{\partial^2 \eta}{\partial t^2} + B \frac{\partial \eta}{\partial t} + \frac{K \eta}{Re} \right)
= -\left(1 + t_m \frac{\partial}{\partial t}\right) \left( \rho \frac{\partial v_r}{\partial t} + \rho \left( v_r \frac{\partial v_z}{\partial r} + v_z \frac{\partial v_z}{\partial z} \right) \right)
+ \frac{1}{Re} \left( \frac{\partial^2 v_z}{\partial r^2} + \frac{1}{r} \frac{\partial v_z}{\partial r} + \frac{\partial^2 v_z}{\partial z^2} \right)
+ \frac{1}{3Re} \frac{\partial}{\partial z} \left( \frac{\partial v_r}{\partial r} + \frac{\partial v_z}{\partial z} \right). \tag{2.11.2b}
\]

3. Methodology of solution

We will consider the absence of flow in the absence of a peristaltic wave (free pumping motion) to indicate the solution and using the perturbation approach for solving the equations system (2.11.1), (2.11.2) following [24], we seek the solution for the governing equations as a power series in the form in terms of \( \epsilon \)
\[
p = p_o + \epsilon p_1(r, z, t) + \epsilon^2 p_2(r, z, t) + \cdots, \tag{3.1a}
\]
\[
v_r = \epsilon v_1(r, z, t) + \epsilon^2 v_2(r, z, t) + \cdots, \tag{3.1b}
\]
\[
v_z = \epsilon v_1(r, z, t) + \epsilon^2 v_2(r, z, t) + \cdots, \tag{3.1c}
\]
\[
\rho = 1 + \epsilon \rho_1(r, z, t) + \epsilon^2 \rho_2(r, z, t) + \cdots. \tag{3.1d}
\]
Inserting equations (3.1) into equations (2.11.1), and two closed set of equations for the first (\( \epsilon \)) and second (\( \epsilon^2 \)) order are obtained as the following:
For \( \epsilon \):
\[
\frac{\partial \rho_1}{\partial t} + \frac{\partial u_1}{\partial r} + \frac{u_1}{r} + \frac{\partial v_1}{\partial z} = 0, \tag{3.1.1a}
\]
\[ (1 + t_m \frac{\partial}{\partial t}) \frac{\partial u_1}{\partial t} \]
\[ \quad = - \left( 1 + t_m \frac{\partial}{\partial t} \right) \frac{\partial p_1}{\partial r} + \frac{1}{\Re} \left( \frac{\partial^2 u_1}{\partial r^2} + \frac{1}{r} \frac{\partial u_1}{\partial r} - \frac{u_1}{r^2} + \frac{\partial^2 u_1}{\partial z^2} \right) 
\quad + \frac{1}{3 \Re} \frac{\partial}{\partial r} \left( \frac{\partial u_1}{\partial r} + \frac{u_1}{r} + \frac{\partial v_1}{\partial z} \right), \]
\[ (3.1.1b) \]
\[ (1 + t_m \frac{\partial}{\partial t}) \frac{\partial v_1}{\partial t} \]
\[ \quad = - \left( 1 + t_m \frac{\partial}{\partial t} \right) \frac{\partial p_1}{\partial z} + \frac{1}{\Re} \left( \frac{\partial^2 v_1}{\partial r^2} + \frac{1}{r} \frac{\partial v_1}{\partial r} + \frac{\partial^2 v_1}{\partial z^2} \right) 
\quad + \frac{1}{3 \Re} \frac{\partial}{\partial z} \left( \frac{\partial u_1}{\partial r} + \frac{u_1}{r} + \frac{\partial v_1}{\partial z} \right), \]
\[ (3.1.1c) \]
For \( \varepsilon^2 \):
\[ \frac{\partial p_2}{\partial t} + u_1 \frac{\partial p_1}{\partial r} + v_1 \frac{\partial p_1}{\partial z} + \frac{\partial u_2}{\partial r} + \frac{u_2}{r} + \frac{\partial v_2}{\partial z} + p_1 \left( \frac{\partial u_1}{\partial r} + \frac{u_1}{r} + \frac{\partial v_1}{\partial z} \right) = 0, \]
\[ (3.1.2a) \]
\[ (1 + t_m \frac{\partial}{\partial t}) \left( \frac{\partial u_2}{\partial t} + p_1 \frac{\partial u_1}{\partial r} + u_1 \frac{\partial u_1}{\partial r} + v_1 \frac{\partial u_1}{\partial z} \right) \]
\[ \quad = - \left( 1 + t_m \frac{\partial}{\partial t} \right) \frac{\partial p_2}{\partial r} + \frac{1}{\Re} \left( \frac{\partial^2 u_2}{\partial r^2} + \frac{1}{r} \frac{\partial u_2}{\partial r} - \frac{u_2}{r^2} + \frac{\partial^2 u_2}{\partial z^2} \right) 
\quad + \frac{1}{3 \Re} \frac{\partial}{\partial r} \left( \frac{\partial u_2}{\partial r} + \frac{u_2}{r} + \frac{\partial v_2}{\partial z} \right), \]
\[ (3.1.2b) \]
\[ (1 + t_m \frac{\partial}{\partial t}) \left( \frac{\partial v_2}{\partial t} + p_1 \frac{\partial v_1}{\partial r} + \left( u_1 \frac{\partial v_1}{\partial r} + v_1 \frac{\partial v_1}{\partial z} \right) \right) \]
\[ \quad = - \left( 1 + t_m \frac{\partial}{\partial t} \right) \frac{\partial p_2}{\partial z} + \frac{1}{\Re} \left( \frac{\partial^2 v_2}{\partial r^2} + \frac{1}{r} \frac{\partial v_2}{\partial r} + \frac{\partial^2 v_2}{\partial z^2} \right) 
\quad + \frac{1}{3 \Re} \frac{\partial}{\partial z} \left( \frac{\partial u_2}{\partial r} + \frac{u_2}{r} + \frac{\partial v_2}{\partial z} \right), \]
\[ (3.1.2c) \]
\[ \rho_2 = \chi p_2 + \frac{\chi^2}{\varepsilon^2} p_1^2. \]
\[ (3.1.2d) \]
Expanding equations (2.11.2) by taylor expansion around the mean position \((r = 1)\) and substituting the expansions (3.1), then the boundary conditions read as:

For \( \varepsilon \):
\[ u_1 (1, z, t) = \alpha \sin \alpha (z - t), \]
\[ (3.1.3a) \]
\[ v_1 (1, z, t) = k n \frac{\partial v_1 (1, z, t)}{\partial r}, \]
\[ (3.1.3b) \]
\( \left( 1 + t_m \frac{\partial}{\partial t} \right) \frac{\partial}{\partial z} \left( \frac{T \alpha^3}{Re^2} \sin(\alpha(z-t)) + m \alpha^3 \sin(\alpha(z-t)) + \frac{D \alpha^2}{Re} \cos(\alpha(z-t)) - \frac{B \alpha^5}{Re^2} \sin(\alpha(z-t)) \right) \)

\[- \frac{K \alpha}{Re^2} \sin(\alpha(z-t)) \]

\[= - \left( 1 + t_m \frac{\partial}{\partial t} \right) \frac{\partial v_1(1)}{\partial t} + \frac{1}{Re} \left( \frac{\partial^2 v_1(1)}{\partial r^2} + \frac{\partial v_1(1)}{\partial r} + \frac{\partial^2 v_1(1)}{\partial z^2} \right) \]

\[+ \frac{1}{3Re} \frac{\partial}{\partial z} \left( \frac{\partial u_1(1)}{\partial r} + u_1(1) + \frac{\partial v_1(1)}{\partial z} \right). \tag{3.1.3c} \]

For \( \varepsilon^2 \):

\[u_2(1, z, t) + \cos \alpha(z-t) \frac{\partial u_1(1, z, t)}{\partial t} = 0, \tag{3.1.4a} \]

\[v_2(1, z, t) + \cos \alpha(z-t) \frac{\partial v_1(1, z, t)}{\partial r} = \kappa n \frac{\partial}{\partial r} \left( v_2(1, z, t) + \cos \alpha(z-t) \frac{\partial v_1(1, z, t)}{\partial r} \right). \tag{3.1.4b} \]

To obtain the problem solution following Aarts and Ooms approach [24], it was chosen the solution in the form

\[u_1(r, z, t) = u_1(r) e^{ia(z-t)} + \overline{u_1}(r) e^{-ia(z-t)}, \tag{3.2a} \]

\[v_1(r, z, t) = v_1(r) e^{ia(z-t)} + \overline{v_1}(r) e^{-ia(z-t)}, \tag{3.2b} \]

\[p_1(r, z, t) = p_1(r) e^{ia(z-t)} + \overline{p_1}(r) e^{-ia(z-t)}, \tag{3.2c} \]

\[\rho_1(r, z, t) = \chi p_1(r) e^{ia(z-t)} + \chi \overline{p_1}(r) e^{-ia(z-t)}, \tag{3.2d} \]

and the second (\( \varepsilon^2 \)) order solution in the following form

\[u_2(r, z, t) = u_{20}(r) + u_2(r) e^{2ia(z-t)} + \overline{u_2}(r) e^{-2ia(z-t)}, \tag{3.3a} \]

\[v_2(r, z, t) = v_{20}(r) + v_2(r) e^{2ia(z-t)} + \overline{v_2}(r) e^{-2ia(z-t)}, \tag{3.3b} \]

\[p_2(r, z, t) = p_{20}(r) + p_2(r) e^{2ia(z-t)} + \overline{p_2}(r) e^{-2ia(z-t)}, \tag{3.3c} \]

\[\rho_2(r, z, t) = D_{20}(r) + D_2(r) e^{2ia(z-t)} + \overline{D_2}(r) e^{-2ia(z-t)}. \tag{3.3d} \]

In the previous equations, the bar indicates a complex conjugate.

The motivation for the final choice of solution is the fact that nonlinear (second-order) is the essentially effect for the peristaltic flow [24]. The implementation of a non-oscillatory term in the first order only offers a trivial solution. So, we should add non oscillatory terms, like \( u_{20}(r) \), \( v_{20}(r) \), \( p_{20}(r) \) and \( D_{20}(r) \) that does not cancel in solution after average time over the period, only in second and higher orders.

Substituting from equations (3.2) into equations (3.1.1), (3.1.3) we get system of differential equations and their boundary conditions

\[u_1'(r) + \frac{u_1(r)}{r} + i\alpha v_1(r) = i\alpha \chi p_1(r), \tag{3.2.1a} \]
\(-i\alpha(1 - i\alpha t_m)u_1(r)\)
\[=- (1 - i\alpha t_m)p_1(r) + \frac{1}{Re} \left( u_1^\prime(r) + \frac{u_1^\prime(r)}{r^2} - \left(\frac{1}{r^2} + \alpha^2\right) u_1(r)\right) + \frac{1}{3Re} \frac{d}{dr} \left( u_1(r) + \frac{u_1(r)}{r} + i\alpha v_1(r)\right),\]  
(3.2.1b)
\(-i\alpha(1 - i\alpha t_m)v_1(r)\)
\[=- i\alpha(1 - i\alpha t_m)p_1(r) + \frac{1}{Re} \left( v_1^\prime(r) + \frac{v_1^\prime(r)}{r} - \alpha^2 v_1(r)\right) + \frac{i\alpha}{3Re} \left( u_1^\prime(r) + \frac{u_1(r)}{r} + i\alpha v_1(r)\right).\]  
(3.2.1c)

B. Cs:

\[u_1(1) = \frac{-i\alpha}{2},\]  
(3.2.2a)
\[v_1(1) = k_n v_1^\prime(1),\]  
(3.2.2b)
\[v_1^\prime(1) + v_1^\prime(1) + \frac{i\alpha}{3} \left( u_1^\prime(1) + u_1(1)\right) + i\alpha v_1(1) \left(\frac{4}{3} i\alpha + (1 - i\alpha t_m)Re\right) = (1 - i\alpha t_m)Re\delta,\]  
(3.2.2c)

In which,

\[\delta = \frac{1}{2iRe} \left(-Ta^3 + ma^3 Re^2 + ia^2 DRe - B\alpha^5 - K\alpha\right).\]  
(3.2.2d)

The prime refers to a derivative with respect to \(r\).

Further, we rewrite the system of equations (3.2.1) in the form

\[u_1^\prime(r) + \frac{u_1(r)}{r^2} - \beta^2 u_1(r) - \gamma p_1^\prime(r) = 0,\]  
(3.2.3a)
\[v_1^\prime(r) - \beta^2 v_1(r) - i\alpha \gamma p_1(r) = 0,\]  
(3.2.3b)
where \(\beta, \gamma\) is given by

\[\beta^2 = \alpha^2 - i\alpha Re(1 - i\alpha t_m), \gamma = (1 - i\alpha t_m)Re - \frac{i\alpha \chi}{3}.\]  
(3.2.3c)

Using equation (3.2.1a) eliminate \(v_1(r)\) and rewrite equation (3.2.3a) as the form

\[-i\alpha \chi \left[p_1^\prime(r) + \frac{p_1(r)}{r} - \left(\beta^2 + \frac{i\alpha \gamma}{\chi}\right) p_1(r)\right] + \frac{1}{\alpha^2} \left(1 + \frac{d}{dr}\right) \left[u_1^\prime(r) + \frac{u_1(r)}{r} - \frac{u_1(r)}{r^2} - \beta^2 u_1(r)\right] = 0.\]  
(3.2.3d)

Differentiating equation (3.2.3d) with respect to \(r\) and multiply by \(\alpha^2\), hence we get the following equation

\[\left(1 - i\alpha \chi\right) \left[\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{1}{r^2} - \beta^2\right] \left\{\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{1}{r^2} - v^2\right\} u_1(r) = 0,\]  
(3.2.3e)

Where

\[v^2 = \alpha^2 \left[\frac{(1 - i\alpha t_m)Re(1 - \chi) - \frac{4}{3}(i\alpha \chi)}{(1 - i\alpha t_m)Re - \frac{4}{3}(i\alpha \chi)}\right].\]  
(3.2.3f)
Solving equation (3.2.3e) and get the master solution for \( u_1(r) \):

\[
  u_1(r) = c_1 l_1(\alpha r) + c_2 l_1(\beta r),
\]

where \( l_1 \) is modified function of Bessel of the first type of order 1.

The general solution for \( v_1(r) \) and \( p_1(r) \):

\[
  v_1(r) = \frac{i\alpha c_1}{v} \ln(\alpha r) + \frac{i\beta c_2}{a} l_0(\beta r),
\]

\[
  p_1(r) = \frac{c_1(v^2 - \beta^2)}{v \gamma} l_0(\alpha r),
\]

where, \( l_0 \) is the modified Bessel function of the first kind of order 0, \( c_1 \) and \( c_2 \) are complex constants calculated using the boundary conditions (3.2.2) and defined by

\[
  c_1 = \frac{R_3(1 - i \alpha t_m) Re \delta}{R_1 R_3 + R_2}, \quad c_2 = \frac{1 - i \alpha t_m) Re \delta}{R_1 R_3 + R_2},
\]

where

\[
  R_1 = \frac{\frac{4}{3} i \alpha v + i \alpha q}{v} l_0(v), \quad R_2 = \frac{\frac{1}{3} i \alpha \beta + \frac{i \beta^3}{\alpha}}{v} l_0(\beta),
\]

\[
  R_3 = \frac{b v}{a^2} \left( \frac{k n \beta}{l_0(v) - k n v l_1(v)} \right), \quad q = i(1 - t_m) Re - \frac{4}{3} \frac{\alpha^2}{a^2}.
\]

The second order system can be obtained also by substituting from equations (3.3) into equations (3.1.2), (3.1.4) we get system of differential equations and their B. Cs

\[
  u_20(r) + \frac{u_20(r)}{r} = -\chi \left[ \frac{1}{r} + \frac{d}{dr} \right] [p_1(r) \bar{u}_1(r) + \bar{p}_1(r) u_1(r)],
\]

\[
  - p_20(r) + \frac{4}{3 Re} \left( u_20(r) + \frac{u_20(r)}{r} - \frac{u_20(r)}{r^2} \right)
  = -i \alpha \chi \left[ p_1(r) \bar{u}_1(r) - \bar{p}_1(r) u_1(r) \right] + \left[ u_1(r) \bar{u}_1'(r) + \bar{u}_1(r) u_1'(r) \right]
  + i\alpha [u_1(r) \bar{v}_1'(r) - \bar{u}_1(r) v_1'(r)],
\]

\[
  \frac{1}{Re} \left( v_20'(r) + \frac{v_20'(r)}{r} \right)
  = -i \alpha \chi \left[ v_1(r) \bar{p}_1'(r) - \bar{v}_1(r) p_1(r) \right] + \left[ u_1(r) \bar{v}_1'(r) + \bar{u}_1(r) v_1'(r) \right]
  + \chi^2 p_1(r) \bar{v}_1(r).
\]

B. Cs:

\[
  u_20(1) + \frac{1}{2} \left( u_1'(1) + u_1(1) \right) = 0,
\]

\[
  v_20(1) + \frac{1}{2} \left( v_1'(1) + v_1(1) \right) = k n \left[ v_20'(1) + \frac{1}{2} \left( v_1''(1) + v_1'(1) \right) \right].
\]

In the net flow is considered, the functions \( u_2, v_2, p_2 \) and \( D_2 \) don’t contribute to the net flow, only the functions \( u_20, v_20, p_20 \) and \( D_20 \) contribute to it as long as terms up to \( o(\varepsilon^2) \) are retained. So, We continue only with the solutions for \( u_20, v_20, p_20 \) and \( D_20 \) [24]. To that end, the second-order \( u_20 \) can also be written as follows:
\[ u_{20}(r) = \frac{D_1}{r} + \chi \left[ p_1(r) \overline{u_1}(r) + \overline{p_1}(r) u_1(r) \right], \quad (3.3.3a) \]

we can obtain the complex number \( D_1 \) from the boundary condition (3.3.2a) as the following form
\[ D_1 = \frac{\text{i} \alpha \kappa n}{2} \left[ \iota c_1 l_1(\nu) + \beta^2 c_2 l_1(\beta) + \iota c_1 l_1(\nu) + \beta^2 c_2 l_1(\beta) \right], \quad (3.3.3b) \]

And the general solution for \( v_{20}(r) \) reads
\[ v_{20}(r) = D_2 - \text{Re} \int_r^1 u_1(\zeta) \overline{v_1}(\zeta) + \overline{u_1}(\zeta) v_1(\zeta) \, d\zeta, \quad (3.3.4a) \]

where \( D_2 \) is defined by
\[ D_2 = \kappa n \left[ v_2'(1) + \frac{1}{2} \left( v_1'(1) + \overline{v_1}'(1) \right) - \frac{1}{2} \left[ v_1'(1) + v_1'(1) \right] \right], \quad (3.3.4b) \]

in which, \( v_2(1), v_1'(1) \) are defined by
\[ v_2'(1) = \text{Re} \left[ -\frac{\iota c_1 \overline{c}_1}{\nu} l_0(\nu) l_1(\nu) - \frac{\iota c_1 \overline{c}_1}{\nu} l_0(\overline{\beta}) l_1(\nu) - \frac{\iota c_1 \overline{c}_1}{\nu} l_0(\nu) l_1(\overline{\beta}) - \frac{\iota c_1 \overline{c}_1}{\nu} l_0(\nu) l_1(\overline{\beta}) \right] \]
\[ + \frac{\iota c_1 \overline{c}_1}{\nu} l_0(\nu) l_1(\nu) + \frac{\iota c_1 \overline{c}_1}{\nu} l_0(\overline{\beta}) l_1(\nu) + \frac{\iota c_1 \overline{c}_1}{\nu} l_0(\nu) l_1(\overline{\beta}) \]
\[ + \frac{\iota c_1 \overline{c}_1}{\nu} l_0(\overline{\beta}) l_1(\overline{\beta}) \], \quad (3.3.4c)

Also, the general solution for \( p_{20}(r) \) can be written as the following form
\[ p_{20}(r) = -\frac{4\chi}{3 \text{Re}} F_1(r) - \int F_2(r) \, dr + D_3, \quad (3.3.5a) \]

where
\[ F_1(r) = \left( \frac{1}{r} + \frac{d}{dr} \right) \left[ p_1(r) \overline{u_1}(r) + \overline{p_1}(r) u_1(r) \right], \quad (3.3.5b) \]
\[ F_2(r) = \iota \alpha \chi \left[ p_1(r) \overline{u_1}(r) - \overline{p_1}(r) u_1(r) \right] + \left[ u_1(r) \overline{u_1}(r) + \overline{u_1}(r) u_1'(r) \right] \]
\[ + \iota \alpha \left[ u_1(r) \overline{v_1}(r) - \overline{u_1}(r) v_1(r) \right], \quad (3.3.5c) \]
\[ D_3 = \frac{p_{20}(r)}{3 \text{Re}} F_1(r) + \int F_2(r) \, dr \quad \text{at } r = 1. \quad (3.3.5d) \]

The average of any variable \( X \) over one period \( T \) of time \( t \) is
\[ < X > = \frac{1}{T} \int_0^T X(r, z, t) \, dt, \quad (3.3.6a) \]

at \( T = \frac{2\pi}{\alpha} \), then the mean net axial velocity \( < v_x > \) may now be written as
\[ < v_x > = \varepsilon^2 v_{20}(r). \quad (3.3.6b) \]

The fluid flow rate can be written in the dimensionless form [24] as the following
\[ Q(z, t) = 2\pi \left[ \int_0^1 e v_1(r, z, t) \, rdr + \int_0^1 e^2 v_2(r, z, t) \, rdr \right] + o(\varepsilon^3), \quad (3.3.7a) \]

by neglecting \( o(\varepsilon^3) \) the net flow rate is given by.
\[< Q > = \pi \, e^2 [D_2 - Re \int_0^1 r^2 (u_1(r) \, \overline{v}_1(r) + \overline{u}_1(r) \, v_1(r)) \, dr].\] (3.3.7b)

4. Numerical results and discussion

4.1 Validation section
A Comparison with El-shehawy et al. [38] at \(t_m = 0\) the eq.(46) and eq.(3.3.7b) in our work are the same but the main difference is the constants in eqs. (31), (34) in El-shehawy et al. [38] where he relied on finding constants through the slip and non-premeability boundary conditions and to validate the solution in our work, if used the same boundary conditions we got the same results and all curves as shown in figure 2(a), (b) is identical to Fig. 1, Fig. 7 in El-shehawy et al. [38] under the same parameter taken from table 1.

![Figure 2(a). The dimensionless flow rate verses \(\chi\).](image1)

![Figure 2(b). The dimensionless flow rate verses \(\alpha\).](image2)

Table 1. Corresponding to figure 2.

|       | (a)      | (b)      |
|-------|----------|----------|
| \(Re\) | 10000    | 10000    |
| \(tm\) | 0        | 0        |
| \(\varepsilon\) | 0.001    | 0.001    |
4.2 Results
In this section, we present the graphical results to show the different impact of the interested parameters which appeared in the analysis (wall parameters, compressibility parameter, slip parameter and relaxation time) on the net flow rate and the reversal flow. We choose on our solution $\varepsilon \ll 1$ and the solution is valid under the condition $\varepsilon \alpha 2Re \ll 1$. So, the fluid flow is suggested to be laminar, also the value of Re is small in range [10:50], slip parameter in range [0:0.15] where at $kn = 0$ this mean no slip condition and compressibility parameter in range [0:1] in which, at $\chi = 0$ the liquid called incompressible fluid.

This article is considered the first attempt in this context, since we discuss in this paper combined effects such as wall properties, relaxation time, slip boundary condition and compressibility of the fluid and neglecting the effect of elastic wall properties getting the same mathematical relations and results which appeared in article is done with El-shehawy et al. [38].

Figure 3 Show the relation between the net flow rate and wave number for several values of wall parameters "$T, D, K, B$" and flow parameters "$\chi, kn, tm$" coressponding to table 2.

Wall Tension influence on the net flow rate is seen at figure 3(a), in which at $T=5000$ the flow rate varies in proportional relation reaching to maximum value ($Q_{max} = 1.99 * 10^{-5}$) at $\alpha = 0.7$, then the relation become inversely proportional after this value and the reversal flow appears at $\alpha = 0.89$. We notice that the effect of $T$ is weak when $\alpha < 0.7$ and great at $\alpha > 0.7$ (flux decreases as $T$ rises) and for large value of $T$ ($T = 9000$) the reversal flow appears early ($\alpha = 0.77$). Figure 3(b) shows that the damping force effect in flow rate profile, since for specified value of $\alpha$ the net flux profile is inversely proportional with $D$. There for, $Q$ increases as increasing the value of wave number and for all curves $Q$ reaches to its maximum value at $\alpha = 1$. The backward flow appears for $\alpha \leq 0.4$ at $D = 3.3$ and decreases as $D$ decreases.

In figure 3(c), the wall stiffness $K$ affects clearly on $Q$ and the relation between them direct proportion. In addition, as $\alpha$ increases the net flow also rises. In figure 3(d) the trend of the flux profile under the effect of flexural rigidity $B$ as the same in the previous figure but at $\alpha < 0.5$ for different value of $B$ the flux profile doesn't change and at $\alpha > 0.5$ the influence of $B$ becomes visible.

The relaxation time has great influence in net flow rate which shown in figure 3(e). The net flux profile show that, increasing the value of $tm$ lead to deceasing $Q$ and increasing the reversal flow but the previous action appears only when $\alpha > 0.4$. From previous the net flow rate independent on $tm$ when $\alpha < 0.4$. The compressibility parameter also affects in flow rate as shown in figure 3(f) at in which, by taking the wall properties in our account the net flow rate grows as the result of increasing the compressibility $\chi$ of the fluid but this growth in net flow rate is invisible when $\alpha \leq 0.3$. At $\chi = 0.2, 0.3, 0.5$ the net flux is in proportional
relation with $\alpha$ but at $\chi=0.001$ the relation become inversely proportion and the reversal flow appeared at $\alpha = 0.75$ and at $\chi=0.1$ the trend of flux profile is slightly sensible.

Figure 3(a). The dimensionless net flux verses wave number under the effect of wall tension. Figure 3(b). The dimensionless net flux verses wave number under the effect of wall damping.

Figure 3(c). The dimensionless net flux verses wave number under the effect of wall stiffness. Figure 3(d). The dimensionless net flux verses wave number under the effect of wall rigidity.
Figure 3(e). The dimensionless net flux versus wave number under the effect of relaxation time.

Figure 3(f). The dimensionless net flux versus wave number under the effect of liquid compressibility.

Table 2. Corresponding to figure 3.

|       | (a)    | (b)    | (c)    | (d)    | (e)    | (f)    |
|-------|--------|--------|--------|--------|--------|--------|
| $T$   | 5000, 7000, 7574.365, 9000 | 200    | 20     | 200    | 5581   | 9866.7 |
| $D$   | 0.5    | 0.5, 1.4, 2.5, 3.3 | 0.1    | 0.1    | 0.5    | 0.5    |
| $K$   | 1      | 0.1    | 5, 10, 15, 20 | 0.1    | 1      | 1      |
| $B$   | 2      | 20     | 20     | 0, 40, 80 | 2      | 2      |
| $m$   | 0.01   | 0.01   | 0.01   | 0.01   | 0.01   | 0.01   |
| $tm$  | 0.1    | 0.1    | 0.1    | 0.1    | 0.0, 0.1, 0.2, 0.3 | 0.1 |
| $Re$  | 10     | 10     | 10     | 10     | 10     | 10     |
| $\varepsilon$ | 0.001 | 0.001 | 0.001 | 0.001 | 0.001 | 0.001 |
Figure 4 show the net flux variation against the interested parameters such as damping force coefficient $D$, wall tension $T$ corresponding to table 3.

Figure 4(a) shows that a slip parameter reduces the net flow and rises the backward flow. At $kn = 0$ "no slip condition" the relation between net flux and the damping coefficient is inversely proportional up to damping factor $D = 620$ at which $Q = -1.75 * 10^{-5}$ after that the relation becomes proportional. The influence of $kn$ on $Q$ does not appear when $D < 30$ after this value the previous effect appear.

The influence of the liquid compressibility on the net flux is illustrated in figure 4(b) in which increasing the damping force coefficient resists the net flow rate. On the other hand, for constant value of $D$ by increasing the compressibility parameter causes increasing the net flux and reducing the backward flow.

Figure 4(c) illustrate the relation between the net flux and wall tension under the effect of compressibility parameter which shows that at fixed value of wall tension $T$ the profile of the net flux is increased by increasing the compressibility factor. Also the net flow rate rises by increasing the wall tension but the reversal flow decreases.

| $kn$  | 0.15 | 0.05 | 0.15 | 0.15 | 0.15 | 0.15 |
|-------|------|------|------|------|------|------|
| $\chi$ | 0.001 | 0.5   | 0.5   | 0.5   | 0.001 | 0.001, 0.1, 0.2, 0.3, 0.5 |

Figure 4(a). The net flux with the damping coefficient under the effect of slip condition.

Figure 4(b). The net flux with the damping coefficient under the effect of liquid compressibility.
Figure 4(c). The net flux with the wall tension under the effect of liquid compressibility.

Table 3. Corresponding to figure 4.

|       | (a)     | (b)     | (c)     |
|-------|---------|---------|---------|
| $T$   | 1000    | 1000    | axis    |
| $D$   | axis    | axis    | 0.1     |
| $m$   | 0.01    | 0.01    | 0.01    |
| $K$   | 0.1     | 0.1     | 0.1     |
| $B$   | 20      | 10      | 20      |
| $\alpha$ | 0.3  | 0.3     | 0.3     |
| $tm$  | 0.1     | 0.1     | 0.1     |
| $Re$  | 10      | 10      | 10      |
| $\varepsilon$ | 0.001 | 0.001   | 0.001   |
| $kn$  | 0.0, 0.05, 0.1 | 0.15   | 0.15   |
| $\chi$ | 0.5     | 0.1, 0.2, 0.3, 0.4 | 0.1, 0.2, 0.3, 0.4 |
Figure 5 show the distribution of the net flow rate against the liquid compressibility for several values of wall parameters "K, D" corresponding to table 4.

From figure 5(a) we note that the net flux rises with enhancing the value of the wall stiffness "K" and compressibility parameter. In figure 5(b) the reversal flow appears at damping factor D > 0.5 and it observed that the streamlines decreasing by boasts the value of D.

**Figure 5(a).** Distribution of the net flow rate under the influence of wall stiffness.

**Figure 5(b).** Distribution of the net flow rate under the influence of wall damping.

|       | (a) | (b)          |
|-------|-----|--------------|
| T     | 20  | 20           |
| D     | 0.5 | 0.5, 0.6, 0.7|
| m     | 0.01| 0.01         |
| K     | 0.5, 1, 1.5 | 0.5         |
| B     | 20  | 20           |
| α     | 0.6 | 0.6          |
| tm    | 0.3 | 0.3          |
5. Conclusion
This paper is presented the peristaltic locomotion of compressible non-Newtonian Maxwellian fluid in an axisymmetric cylindrical tube under the combined influences of elastic wall features \((T, B, D, K)\), relaxation time, slip factor and liquid's compressibility. We introduced the analytical solution and extracted the flow rate and the average axial velocity using perturbation technique and also the behaviour of flow rate is plotted under the effects of the previous parameters. We can focus on the main points which are obtained from present study as the following:
- Increasing wall elasticity factor \(K\) and flexural rigidity factor \(B\) lead to increasing the net flux but the influence of \(K\) more effective than \(B\).
- In contrast, the Damping force coefficient \(D\) resists the net flow rate because appearing the reversal flow.
- Increasing wall tension \(T\) causes increasing in flow rate and the reversal flow decreased.
- The Damping force coefficient and wall tension have strong influence on the compressibility of the fluid, since rising \(T, D\) cause enhancing the effect of \(\chi\) on the net flow rate similarly increasing \(D\) boats the effect of \(kn\) on flow rate.
- Increasing the compressibility factor causes increasing in flow rate.
- By deleting the impact of the elastic wall properties from this paper then you will get the same mathematical relations and curves introduced by El-shehawy et al. [38].

Conflict of interest
The authors declare that they have no conflict of interest.

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