Non-local Correlations are Generic in Infinite-Dimensional Bipartite Systems

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It was recently shown that the nonseparable density operators on the Hilbert space \( \mathcal{H}_1 \otimes \mathcal{H}_2 \) are trace norm dense if either factor space has infinite dimension. We show here that non-local states—i.e., states whose correlations cannot be reproduced by any local hidden variable model—are also dense. Our constructions distinguish between the cases \( \dim \mathcal{H}_1 = \dim \mathcal{H}_2 = \infty \), where we show that states violating the CHSH inequality are dense, and \( \dim \mathcal{H}_1 < \dim \mathcal{H}_2 = \infty \), where we identify open neighborhoods of nonseparable states that do not violate the CHSH inequality but show that states with a subtler form of non-locality (often called ‘hidden’ non-locality) remain dense.

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I. INTRODUCTION

The observables of a bipartite quantum system are represented by the set of all self-adjoint operators on the tensor product of two Hilbert spaces \( \mathcal{H}_1 \otimes \mathcal{H}_2 \), whose dimensions we shall denote by \( d_1 \) and \( d_2 \), taking \( d_1 \leq d_2 \) without loss of generality. It is well-known that when \( d_1 \geq 2 \) the states of the system can be nonseparable, and it is this possibility that much of the new technology associated with quantum information and computation theory relies upon. Prompted by concerns about whether the very noisy mixed states exploited by certain models of NMR quantum computing are truly nonseparable [1,2], detailed investigations have shown that, whenever \( d_2 < \infty \), there is always an open neighborhood of separable states surrounding the maximally mixed state \( (d_1 d_2)^{-1} I \otimes I \) [3–5].

Complementing these results, two of us [6] have recently shown that if \( d_2 = \infty \), the set of nonseparable states is dense, and, therefore, there can be no open neighborhood of separable states in that case. It was then conjectured [6] that the same density result ought to hold for states which violate some Bell inequality, at least in the case \( d_1 = d_2 = \infty \). This does not follow immediately from the main theorem in [6], since the nonseparability of a mixed state (in contrast to the pure case [7,8]) is not known to imply that it violates a Bell inequality or that its correlations cannot be reproduced by a local hidden variables model. No counterexample is known either; however, Werner [9] has shown that a local hidden variables model can reproduce the correlations of a non-separable mixed state for single projective measurements on each component system.

We show here that the conjecture made in [6] is true. More precisely, we show that a bipartite system possesses a dense set of states violating the CHSH inequality for projective measurements if and only if \( d_1 = d_2 = \infty \), and that the system possesses a dense set of states with non-local correlations if \( d_1 < d_2 = \infty \). In the second case, we demonstrate that the states have non-local correlations for sequences of projective measurements: we do not exclude the possibility that they also violate a ‘higher order’ Bell inequality involving more than two measurement choices for each component system, nor
do we exclude violations which involve positive operator valued measurements. Our results also yield an elementary proof of the main result of [6].

II. PRELIMINARIES

We first establish some basic facts about nonseparability and non-locality necessary for the sequel.

Let $\mathfrak{B}(H_1 \otimes H_2)$ denote the set of all (bounded) operators on $H_1 \otimes H_2$, and let $\mathfrak{T} \equiv \mathfrak{T}(H_1 \otimes H_2)$ be the subset of (positive, trace-1) density operators. Throughout, we shall consider $\mathfrak{T}$ as endowed with the metric (and corresponding topology) induced by the trace norm, $\|A\|_T \equiv \text{Tr}((A^* A)^{1/2})$. We reserve the notation $\|A\|$ for the standard operator norm. An operator $A$ is called a contraction if $\|A\| \leq 1$. We denote the self-adjoint contractions acting on a Hilbert space $H$ by $\mathfrak{B}(H)_s$. The metric induced by the trace norm is appropriate physically for measuring the distance between quantum states, because $\|\mathfrak{B}(H_1 \otimes H_2)_s\|_T = \sup\{\|\text{Tr}(DA) - \text{Tr}(D'A)\| : A \in \mathfrak{B}(H_1 \otimes H_2)_s\}$

which implies that trace norm close states dictate close probabilities for the outcomes of measuring any observable.

For $D \in \mathfrak{T}$, $D$ is said to be a product state just in case there is a $D_1 \in \mathfrak{T}(H_1)$ and a $D_2 \in \mathfrak{T}(H_2)$ such that $D = D_1 \otimes D_2$. The separable density operators are then defined to be all members of $\mathfrak{T}$ that may be approximated (in trace norm) by convex combinations of product states [4]. In other words, the separable density operators are those in the closed convex hull of the set of all product states in $\mathfrak{T}$. By definition, then, the set of nonseparable density operators is open.

Let $A_1, A_2$ be self-adjoint contractions in $\mathfrak{B}(H_1)_s$, and, similarly, let $B_1, B_2 \in \mathfrak{B}(H_2)_s$. Then the corresponding operator

$$R \equiv \frac{1}{2}\left(A_1 \otimes (B_1 + B_2) + A_2 \otimes (B_1 - B_2)\right)$$

(2)

is called a Bell operator for the system $H_1 \otimes H_2$. Fix a density operator $D \in \mathfrak{T}$. We can then define the Bell coefficient $\beta(D)$ of $D$ by

$$\beta(D) \equiv \sup\{\|\text{Tr}(DR)\| : R \text{ is a Bell operator for } H_1 \otimes H_2\}. \quad (3)$$

Bell’s theorem, as elaborated by Clauser-Horne-Shimony-Holt [13,14,19], implies that for any state $D$ and Bell operator $R$, a local hidden variable model of $D$’s correlations is committed to predicting the CHSH inequality $|\text{Tr}(DR)| \leq 1$. On the other hand, there are always states $D$ for which $\beta(D) > 1$. We say such states are CHSH violating.

Convexity arguments entail that $\beta(D)$ is in fact equivalent to the supremum taken over all Bell operators where $A_i, B_i$ are self-adjoint unitary operators satisfying $A_i^2 = B_i^2 = I$, i.e., generalized spin components [14, Prop. 3.2]. For completeness, we set out a detailed proof of this fact in Appendix A. Unless otherwise noted, we henceforth assume that all our Bell operators are constructed out of self-adjoint unitaries. Moreover, for such Bell operators we always have

$$R^2 = I \otimes I - \frac{1}{4}[A_1, A_2] \otimes [B_1, B_2], \quad (4)$$

from which it follows by an elementary calculation that $\|R\| \leq \sqrt{2}$. Thus, for any state $D$, $\beta(D) \leq \sqrt{2}$ since $|\text{Tr}(DR)| \leq \|R\|$. Moreover, $\beta(D) \geq 1$, since we may always take $A_i = B_i = I$.

If any of the four operators $A_i, B_i$ is $\pm I$, then (1) entails that $\|R\|^2 = \|R^2\| = 1$ and $R$ cannot indicate any CHSH violation. Thus, we will find it convenient to define $\gamma(D)$ in analogy to the definition of $\beta(D)$, but with the added restriction that the supremum be taken over all Bell operators constructed from nontrivial (i.e., not $\pm I$) self-adjoint unitary operators. It immediately follows that for any $D \in \mathfrak{T}$, $\gamma(D) \in [0, \sqrt{2}]$ and

$$\beta(D) = \max\{1, \gamma(D)\}. \quad (5)$$

Thus, any nonclassical CHSH violation indicated by $\beta(D) > 1$ is indicated just as well by $\gamma(D) > 1$.

Let $D, D' \in \mathfrak{T}$ be such that $\|D - D'\|_T \leq \epsilon$. Then, for any Bell operator $R \in \mathfrak{B}(H_1 \otimes H_2)$, it follows from (1) that

$$|\text{Tr}(DR) - \text{Tr}(D'R)| \leq \epsilon \|R\|. \quad (6)$$

In particular, since for any Bell operator $R$, $\|R\| \leq \sqrt{2}$,

$$|\text{Tr}(DR)| \leq \epsilon \sqrt{2} + |\text{Tr}(D'R)|. \quad (7)$$

Taking the supremum over nontrivial Bell operators $R$, first on the right-hand side of (6), and then on the left, we see that $\gamma(D) \leq \epsilon \sqrt{2} + \gamma(D')$. By symmetry, we have $\gamma(D') \leq \epsilon \sqrt{2} + \gamma(D)$, so that

$$|\gamma(D) - \gamma(D')| \leq \epsilon \sqrt{2} \quad (8)$$

and $\gamma$ is a continuous function from $\mathfrak{T}$ (in trace norm) into $[0, \sqrt{2}]$. It then follows from (3) that $\beta$ is a continuous function from $\mathfrak{T}$ into $[1, \sqrt{2}]$. Since the set of CHSH violating density operators is the pre-image of $(1, \sqrt{2})$ under $\beta$, this set is open in the trace norm topology.

Suppose now that $D$ is a convex combination $D = (1 - \lambda)W + \lambda W'$ where $W, W' \in \mathfrak{T}$. Then, for any Bell operator $R$,
\[ |\text{Tr}(DR)| = |(1 - \lambda)\text{Tr}(WR) + \lambda\text{Tr}(W'R)| \leq (1 - \lambda)|\text{Tr}(WR)| + \lambda|\text{Tr}(W'R)|. \] (9)

Taking the supremum over nontrivial Bell operators first on the right-hand side of (8), and then on left, we may conclude that
\[ \gamma(D) \leq (1 - \lambda)\gamma(W) + \lambda\gamma(W'). \] (10)

Thus, \( \gamma \) is a convex function. It is easy to check that \( \gamma(D) \leq 1 \) for all product states \( D \), and therefore the same holds for any separable state, by continuity and convexity of \( \gamma \).

It follows from the work of Werner that when \( d_1 = d_2 = n \geq 2 \), there are nonseparable states that satisfy all CHSH inequalities. In the case where \( d_1 = d_2 = 2 \), the Werner state, which we shall denote by \( W_{22} \), can be written as
\[ W_{22} = \frac{1}{8}(I \otimes I) + \frac{1}{4}[(I \otimes I) - U], \] (11)
where \( U \) is the (self-adjoint, unitary) permutation operator. Werner observed that for any separable density operator \( D \), we must have \( \text{Tr}(UD) \geq 0 \). However, using the fact that \( U^2 = I \) and \( \text{Tr}(U) = 2 \), we have
\[ \text{Tr}(UW_{22}) = \frac{1}{8}\text{Tr}(U) + \frac{1}{4}\text{Tr}(U - I) = -\frac{1}{4} < 0. \] (12)

Thus, \( W_{22} \) is nonseparable. Moreover, using the fact that \( U = I \otimes I - 2P_s \), where \( P_s \) is the projection onto the singlet state, we may conveniently rewrite \( W_{22} \) in the form:
\[ W_{22} = \frac{1}{8}(I \otimes I) + \frac{1}{2}P_s. \] (13)

Since \( \gamma[(1/4)(I \otimes I)] = 0 \), and \( \gamma \) is convex,
\[ \gamma(W_{22}) \leq \frac{1}{2}\gamma(P_s) = 2^{-1/2} < 1, \] (14)
and \( W_{22} \) is not CHSH violating.

More generally, we define a state \( D \) to be CHSH insensitive whenever \( D \) is nonseparable yet not CHSH violating, i.e., \( \gamma(D) \leq 1 \). Such states may still violate Bell inequalities involving projective measurements of observables with spectral values lying outside \([-1, 1]\), or more than two pairs of projective measurements, or positive operator valued measurements. They may also contain “hidden” CHSH violations in the sense that they may violate generalized CHSH inequalities which involve performing consecutive projective measurements on each of the two subsystems. To make this precise, let \( \mathcal{H} \) be an arbitrary Hilbert space, and let \( \mathcal{H}(\mathcal{H}) \) be the set of density operators on \( \mathcal{H} \). For any \( D \in \mathcal{H}(\mathcal{H}) \) and \( A \in \mathfrak{B}(\mathcal{H}) \) such that \( ADA^* \neq 0 \), we may define the new density operator \( D^A \) by
\[ D^A = \frac{ADA^*}{\text{Tr}(ADA^*)}. \] (15)

Then \( D \in \mathcal{H}(\mathcal{H}_1 \otimes \mathcal{H}_2)(\equiv \mathfrak{T}) \) will violate a generalized CHSH inequality just in case there are projections \( Q_1 \) and \( Q_2 \) such that \( DQ_1 \otimes Q_2 \) is CHSH violating. (In such a case, the violation is ‘seen’ after first performing a pair of selective measurements on the component systems.) For example, Popescu has shown that when \( n \geq 5 \), the states constructed by Werner violate generalized CHSH inequalities. On the other hand, it is clear from (12) that \( W_{22} \) itself cannot violate a generalized CHSH inequality, since for nontrivial \( Q_1 \) or \( Q_2 \), \( W_{22}Q_1 \otimes Q_2 \) is always a product state.

A state which violates any Bell inequality, including generalized inequalities, must be nonseparable. Moreover, since the correlations in such states—whether or not they are CHSH sensitive—cannot be reproduced by any local hidden variable theory, one is justified in terming them non-local states.

For example, while Werner has shown that the correlations dictated by \( W_{22} \) between the outcomes of projective measurements admit a local hidden variable model, this does not imply that \( W_{22} \) is non-local; for he left it as a conjecture that the same is true for positive operator valued measurements [4, p. 4280].

III. CHSH VIOLATION AND INFINITE-DIMENSIONAL SYSTEMS

In this section, we establish that a bipartite system has a dense set of non-local states when either component is infinite-dimensional.

We begin with an elementary observation about the action of \( A \) on \( D \) defined by (13). This action is a natural generalization of the action of an operator on unit vectors. Indeed, we may always add an ancillary Hilbert space \( K \) onto \( \mathcal{H} \) (with \( \text{dim } K \geq \text{dim } \mathcal{H} \)) such that \( D \) is the reduced density operator for a pure vector state \( x \in \mathcal{H} \otimes K \). In such a case, a straightforward verification shows that (when \( (A \otimes I)x \neq 0 \)) the reduced density operator for \( (A \otimes I)x/\|A \otimes I)x\| \) is just \( D^A \).

Let \( \Phi \) be the map that assigns a unit vector \( x \in \mathcal{H} \otimes K \) its reduced density operator \( \Phi(x) \) on \( \mathcal{H} \). It is easy to see that \( \Phi \) is trace-norm continuous [3]. Let \( \{P_n\} \) be any increasing sequence of projections in \( \mathfrak{B}(\mathcal{H}) \) with least upper bound \( I \). Then, \( (P_n \otimes I)x \rightarrow x \) and
\[ D^{P_n} = \Phi[(P_n \otimes I)x/\|P_n \otimes I)x\|], \] (16)
\[ \rightarrow \Phi[x] = D, \] (17)
where the convergence is in trace norm. We make use of this convergence in our arguments below.

**Proposition 1.** If \( d_1 = d_2 = \infty \), then the set of CHSH violating states is trace norm dense in the set of all density operators on \( \mathcal{H}_1 \otimes \mathcal{H}_2 \).
Proof: Fix an arbitrary density operator $D$ on $\mathcal{H}_1 \otimes \mathcal{H}_2$, and fix orthonormal bases for the factor spaces $\{e_i\}$ and $\{f_j\}$. Let $P_n$ be the projection onto the span of $\{e_i \otimes f_j\}_{i,j \leq n}$, and set

$$\psi_n = \frac{1}{\sqrt{2}}(e_{n+1}|f_{n+1}\rangle + e_{n+2}|f_{n+2}\rangle).$$

(18)

Consider the sequence of density operators $\{D_n\}$ defined by

$$D_n = (1 - \frac{1}{n})D^{P_n} + \frac{1}{n}P\psi_n,$$

(19)

where $P\psi$ projects onto the ray $\psi$ generates. Since $\lim_{n \to \infty} D_n = D$ in trace norm, all that remains to show is that each $D_n$ is CHSH violating. As $\psi_n$ is the pure singlet state, there are “spin components” (i.e. self-adjoint unitaries) $A_i^0, B_i^0$ ($i = 1, 2$) such that each $A_i^0$ leaves the subspace generated by $|e_{n+1}\rangle, |e_{n+2}\rangle$ invariant and acts like the identity on the complement; similarly for each $B_i^0$ and the subspace generated by $|f_{n+1}\rangle, |f_{n+2}\rangle$; and, moreover, the Bell operator

$$R_n = \frac{1}{2}(A_1^0 \otimes B_1^0 + A_1^0 \otimes B_2^0 + A_2^0 \otimes B_1^0 - A_2^0 \otimes B_2^0)$$

(20)

is such that $\text{Tr}(P\psi_n, R_n) > 1$. Therefore, in view of (19), to show that $\text{Tr}(D_n R_n) > 1$, and hence that $D_n$ is CHSH violating, it suffices to observe that $\text{Tr}(D^{P_n} R_n) = 1$. But this is immediate from the fact that $R_n$ acts as the identity on $P_n$’s range. QED

A similar argument shows that non-local states are dense in the case $d_1 < d_2 = \infty$.

**Proposition 2.** If $d_1 < d_2 = \infty$, then the set of non-local states is trace norm dense in the set of all density operators on $\mathcal{H}_1 \otimes \mathcal{H}_2$.

Proof: Fix an arbitrary density operator $D$ on $\mathcal{H}_1 \otimes \mathcal{H}_2$, and fix orthonormal bases for the factor spaces $\{e_i\}_{i=1}^{d_1}$ and $\{f_j\}_{j=1}^{d_2}$. Let $P_n^D$ be the projection onto the span of $\{e_i \otimes f_j\}_{i \leq d_1, j \leq d_2}$, and set

$$\psi_n' = \frac{1}{\sqrt{2}}(e_1|f_{n+1}\rangle + e_2|f_{n+2}\rangle).$$

(21)

Consider the sequence of density operators $\{D_n\}$ defined by

$$D_n = (1 - \frac{1}{n})D^{P_n^D} + \frac{1}{n}P\psi_n'.$$

(22)

As before, $\lim_{n \to \infty} D_n = D$ in trace norm, so it suffices to show that each $D_n$ is non-local. Define the projections $Q_1, Q_2$ onto the spans of $\{e_i\}_{i \leq d_1}$ and $\{f_j\}_{j \leq d_2}$, respectively. Then since $D_n^{Q_1 \otimes Q_2} = P\psi_n'$, $D_n$ violates a generalized CHSH inequality. QED

Note that Prop. 2 entails that when $d_2 = \infty$, the set of nonseparable states is dense. This reproduces, by quite different methods, the main result of [3].

**IV. GENERIC CHSH VIOLATION CHARACTERIZES INFINITE-DIMENSIONAL SYSTEMS**

As mentioned in the introduction, when both $d_1, d_2 < \infty$, there is always an open neighborhood of separable states [1, 3]. Since separable states cannot display any non-local correlations, it follows that in this case the CHSH violating states cannot be dense. Note, however, that this same method of argument could not establish an open CHSH non-violating neighborhood in the case where $d_1 < d_2 = \infty$, for in that case we know that the separable states are nowhere dense. However, as we now show, such neighborhoods exist.

Let $D \in \mathcal{H}$ be a density operator with $\gamma(D) < 1$. It is not difficult to see that the distance from $D$ to the set of CHSH violating states is bounded below by $2^{-1/2}(1 - \gamma(D))$. Indeed, for any density operator $D'$, if

$$\|D - D'\|_T \leq 2^{-1/2}(1 - \gamma(D)),$$

then from (18),

$$\gamma(D') \leq 2^{1/2}[2^{-1/2}(1 - \gamma(D)) + \gamma(D)] = 1.$$

(24)

Thus any state $D$ with $\gamma(D) < 1$ is surrounded by a neighborhood of states that are again not CHSH violators.

**Proposition 3.** If $d_1 < \infty$ then, for any density operator $D_2 \in \mathcal{H}(\mathcal{H}_2)$, we have

$$\gamma[d_1^{-1}(I \otimes D_2)] \leq 1 - 2d_1^{-1} < 1.$$

(25)

Proof: Let $A$ be a self-adjoint unitary operator (not $I$) acting on $\mathcal{H}_1$. Then $A = P_1 - P_2$, where $P_i$ is a projection ($i = 1, 2$). Since $A \neq \pm I$, $P_1 \neq 0$ and $P_2 \neq 0$. Thus,

$$|\text{Tr}(d_1^{-1} A)| = d_1^{-1} |\text{Tr}(P_1) - \text{Tr}(P_2)| \leq d_1^{-1} (d_1 - 2) = 1 - 2d_1^{-1}.$$

(27)

Now let $R$ be any Bell-operator for $\mathcal{H}_1 \otimes \mathcal{H}_2$, constructed from (nontrivial) self-adjoint unitary operators. Then,

$$|\text{Tr}(d_1^{-1} (I \otimes D_2) R)|$$

$$= \frac{1}{2} |\text{Tr}(d_1^{-1} (A_1 + A_2)) \cdot \text{Tr}(D_2 B_1) + \text{Tr}(d_1^{-1} (A_1 - A_2)) \cdot \text{Tr}(D_2 B_2)|$$

(28)

$$\leq \frac{1}{2} |\text{Tr}(d_1^{-1} A_1) + \text{Tr}(d_1^{-1} A_2)| + \frac{1}{2} |\text{Tr}(d_1^{-1} A_1) - \text{Tr}(d_1^{-1} A_2)|$$

(29)

$$\leq 1 - 2d_1^{-1}.$$
The last inequality follows since
\[ |a_1 + a_2| + |a_1 - a_2| \leq 2 \max \{|a_i|\}, \quad \text{(32)} \]
for any two real numbers \(a_1, a_2\). \textit{QED}

Note that the considerations prior to this proposition entail that \(d_1^{-1}(I \otimes D_2)\) lies in a neighborhood of CHSH non-violating states of (trace norm) size at least \(d_1^{-1}\sqrt{2}\).
(Of course, this estimate could be improved if restrictions on \(D_2\) were also taken into account.)

**Proposition 4.** The set of CHSH violating density operators is trace norm dense in the set of all density operators on \(H_1 \otimes H_2\) (and its complement is nowhere dense) if and only if \(\dim H_1 = \dim H_2 = \infty\).

**Proof:** Suppose that \(d_1 = d_2 = \infty\). Then, from Prop. 1, the set of CHSH violating states is trace norm dense (and its closed complement must be nowhere dense). Conversely, if \(d_1 < \infty\), then Prop. 3 (and the discussion preceding it) ensures the existence of many open neighborhoods of states that satisfy the CHSH inequality. \textit{QED}

### V. CHSH INSENSITIVE STATES

Props. 1–4 establish that CHSH insensitive states exist in the case when \(d_1 < d_2 = \infty\). In particular, since there is at least one open set of states that do not violate the CHSH inequality, and since the nonseparable states are dense, there must be nonseparable states that are not CHSH violating. Indeed, Prop. 3 provides us with a class of states which we know have a surrounding neighborhood of states that are not CHSH violating, while Prop. 2 shows how, given any state, we may construct a sequence of nonseparable states which converges to that state. In Appendix B, we invoke the alternate method of constructing nonseparable states given in \(\text{[8]}\) to construct a sequence of CHSH insensitive states that converges continuously to a product state. (We do so only for the simplest case of a bipartite system with exactly one two-dimensional component—such as a spin-1/2 particle, distinguishing its internal and external degrees of freedom.)

We have not so far shown that there are CHSH insensitive states in the cases \(d_1 < d_2 < \infty\) and \(d_1 = d_2 = \infty\). We now proceed to show that in all relevant cases, i.e., whenever \(d_1, d_2 \geq 2\), CHSH insensitive states exist. Moreover, if \(d_1 < \infty\), there is always an open neighborhood of CHSH insensitive states.

CHSH insensitive states can be constructed simply by embedding the \(2 \times 2\) Werner state \(W_{22}\) into the higher-dimensional space. Let \(\{e_i \otimes f_j\}\) denote an orthonormal product basis for \(H_1 \otimes H_2\), and let \(K\) denote the \(2 \times 2\) subspace spanned by \(\{e_i \otimes f_j : i, j = 1, 2\}\). Note that the projection onto \(K\) is just the product \(P \otimes Q\) of the projections \(P\) onto \(\{e_i : i = 1, 2\}\) and \(Q\) onto \(\{f_j : j = 1, 2\}\). Corresponding to the permutation operator \(U\) of \(\mathbb{C}^2 \otimes \mathbb{C}^2\), we let \(U'\) denote the (partial isometry) operator on \(H_1 \otimes H_2\) which permutes the basis elements of \(K\) and maps \(K^\perp\) to 0. Then, by analogy with \(W_{22}\), we may define
\[ W'_{22} \equiv \frac{1}{8} (P \otimes Q) + \frac{1}{4} \left[(P \otimes Q) - U'\right]. \quad \text{(33)} \]
It is not difficult to see that \(W'_{22} \in \mathfrak{S}(H_1 \otimes H_2)\). We now verify that \(W'_{22}\), as a state of \(H_1 \otimes H_2\), is again CHSH insensitive.

For a density operator \(D \in \mathfrak{S}(H_1 \otimes H_2)\), let us say that \(D\) is \(K\)-separable just in case \(D\) is in the closed convex hull of product states all of whose ranges are contained in \(K\).

**Proposition 5.** Suppose that \(D \in \mathfrak{S}(H_1 \otimes H_2)\) and \(D\) has range contained in \(K\). If \(D\) is separable, then \(D\) is \(K\)-separable.

Before we give the proof of this proposition, we recall from \(\text{[8]}\) some basic facts concerning the operation \(D \rightarrow D_A\) on density operators defined in \(\text{(13)}\). Suppose that \(D \in \mathfrak{S}(H)\) is a convex combination of density operators
\[ D = \sum_{i=1}^{n} \lambda_i D_i. \quad \text{(34)} \]
Then, for any \(A \in \mathfrak{B}(H)\), if \(ADA^* \neq 0\), we may set
\[ \lambda_i^A \equiv \lambda_i \frac{\text{Tr}(DA_iA^*)}{\text{Tr}(DA^*)}, \quad \text{(35)} \]
and we have
\[ D^A \equiv \frac{ADA^*}{\text{Tr}(DA^*)} = \sum_{i=1}^{n} \lambda_i^A D_i^A, \quad \text{(36)} \]
where \(\sum_{i=1}^{n} \lambda_i^A = 1\). Thus, \(D^A\) is a convex combination of the \(D_i^A\).

Further specializing to the case where \(H \equiv H_1 \otimes H_2\), note that if \(D = D_1 \otimes D_2\) is a product state, and \(A \in \mathfrak{B}(H_1)\), \(B \in \mathfrak{B}(H_2)\) are arbitrary, then
\[ D^{A \otimes B} = (D_1 \otimes D_2)^{A \otimes B} = D_1^A \otimes D_2^B. \quad \text{(37)} \]

**Proof of the proposition:** Let \(P \otimes Q\) denote the projection onto \(K\). Since \(D\) has range contained in \(K\), we have \(D^{P \otimes Q} = D\). Suppose now that \(D\) is separable. That is, \(D = \lim_{n} D_n\), where each \(D_n \in \mathfrak{S}(H_1 \otimes H_2)\) is a convex combination of product states. Thus, by continuity we have
\[ D = D^{P \otimes Q} = \lim_{n} D_n^{P \otimes Q}. \quad \text{(38)} \]
By the preceding considerations, each \(D_n^{P \otimes Q}\) is a convex combination of product states, each of which has range contained in \(K\). Thus, \(D\) is \(K\)-separable. \textit{QED}
Moreover, from Prop. 3, there is a \( D \in \mathcal{K} \) whose range lies in \( H \) insensitive for \( d \leq n \), let \( D \in \mathcal{K} \) be such that \( \gamma(W_n) \leq 1 - n^{-1} \gamma(W') + n^{-1} \gamma(D) \) for all \( n \). However, since \( W' \) is nonseparable, and the nonseparable states are open, there is an \( m \in \mathbb{N} \) such that \( W_n \) is nonseparable for all \( n \geq m \). Thus, setting \( W \equiv W_m \) gives the desired nonseparable state with \( \gamma(W) < 1. \) \textit{QED}

VI. CONCLUSION

We have established the conjecture made in [1] that bipartite systems whose components are both infinite-dimensional (e.g., a pair of particles, neglecting their spins) have states that generically violate the CHSH inequality. We also established that even if one of the components is finite-dimensional (e.g., a spin-1/2 particle), non-locally correlated states remain dense. Finally, we have identified new classes of CHSH insensitive states for finite by infinite systems, and established that such states can only be neglected, for all practical purposes, in the infinite by infinite case.

Infinite-dimensional systems thus provide a resource of nonlocality which — practically speaking — cannot be completely destroyed by noise or by errors in preparation or measurement. In this they differ from finite-dimensional systems, where entangled mixed states can always be reduced to separable states by sufficient noise. One might naively conclude that, to the extent that it is practicable in quantum information and computation theory to exploit infinite-dimensional systems, it would be advantageous to do so. But in fact we can never exploit all the degrees of freedom in an infinite-dimensional system. So, though we hope the above results may be useful in developing the theory of entanglement in large finite-dimensional systems, we doubt that they themselves can lead to direct practical application.

Even in the case of large finite-dimensional systems, there is a potential pitfall. It may well be that nonlocality becomes harder and harder to destroy, by some sensible quantitative measure, as the size of the system becomes larger. However, the nonlocality results we have outlined give no indication of a general procedure for extracting or demonstrating nonlocality. Protecting some form of nonlocality is less useful if it is achieved at the cost of making it harder and harder to find. It would thus be very interesting to quantify the trade-offs which can usefully be made in this direction when large finite-dimensional systems are used to counter-off against noise on a highly noisy channel.
APPENDIX A:

We give here a self-contained version of Summers and Werner’s argument [3] that $\beta(D)$ is equal to the supremum of $|\text{Tr}(DR)|$, where $R$ only runs over the Bell operators for $\mathcal{H}_1 \otimes \mathcal{H}_2$ that are constructed from self-adjoint unitary operators.

Recall that the weak-operator topology on $\mathcal{B}(\mathcal{H})$ is the coarsest topology for which all functionals of the form

$$T \to |\langle Tx, y \rangle|, \quad x, y \in \mathcal{H}, \quad (A1)$$

are continuous at 0. It then follows that the unit ball of $\mathcal{B}(\mathcal{H})$ is compact in the weak-operator topology [24, Thm. 5.1.3]. (Of course, if $\dim \mathcal{H} < \infty$, the unit ball of $\mathcal{B}(\mathcal{H})$ is also compact in the operator-norm topology.) Moreover, since the adjoint operation is weak-operator continuous, the set of self-adjoint operators is weak-operator closed in $\mathcal{B}(\mathcal{H})$, and $\mathcal{B}(\mathcal{H})$ is weak-operator compact (as well as convex).

Fix $A_2 \in \mathcal{B}(\mathcal{H}_1)_s$ and $B_1, B_2 \in \mathcal{B}(\mathcal{H}_2)_s$. We show that the map $\Psi_D : \mathcal{B}(\mathcal{H}_1)_s \to \mathbb{R}$ defined by

$$\Psi_D(A_1) = \frac{1}{2} \text{Tr} \left( D(A_1 \otimes (B_1 + B_2) + A_2 \otimes (B_1 - B_2)) \right), \quad (A2)$$

is affine and weak-operator continuous. From this it will follow that $\Psi_D$ attains its extremal values on extreme points of $\mathcal{B}(\mathcal{H}_1)_s$ [24, Prop. 7.9]. These, however, consist precisely of the self-adjoint unitary operators [24, Prop. 7.4.6].

Now, to establish that $\Psi_D$ is affine and weak-operator continuous, let $\Lambda_D : \mathcal{B}(\mathcal{H}_1)_s \to \mathbb{R}$ denote the linear functional defined by

$$\Lambda_D(A_1) = \text{Tr} \left( D(A_1 \otimes (1/2)(B_1 + B_2)) \right). \quad (A3)$$

Then, $\Lambda_D$ is the composition of the map

$$A_1 \to A_1 \otimes (1/2)(B_1 + B_2), \quad (A4)$$

from $\mathcal{B}(\mathcal{H}_1)_s$ into $\mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)_s$, with the functional $\text{Tr}(D \cdot)$. However, the former is continuous (when both algebras are equipped with the weak-operator topology) since multiplication by a fixed operator is weakly continuous. Moreover, $\text{Tr}(D \cdot)$ is weak-operator continuous on the unit ball of $\mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$. Thus, $\Lambda_D$ is weak-operator continuous. Now, let

$$r_D \equiv \text{Tr} \left( D(A_2 \otimes (1/2)(B_1 - B_2)) \right).$$

Then, $\Psi_D = \Lambda_D + r_D$ is affine and weak-operator continuous.

From the above considerations it now follows that for every $A_1 \in \mathcal{B}(\mathcal{H}_1)_s$ and Bell operator $R$ constructed using $A_1$, there is a Bell operator $R'$ constructed from the same elements as $R$, except with $A_1$ replaced by a self-adjoint unitary operator, and such that $|\text{Tr}(DR)| \leq |\text{Tr}(DR')|$. By symmetry, the same conclusion applies to $A_2, B_1$ and $B_2$. Thus, for any given Bell operator $R$, there is a Bell operator $R'$ constructed entirely from self-adjoint unitaries, and such that $|\text{Tr}(DR)| \leq |\text{Tr}(DR')|$. 

APPENDIX B:

In this appendix, we use the results of the current paper and of [3] to construct a continuous “path” of CHSH insensitive states with endpoint a product state. Reversing the convention $d_1 \leq d_2$ of the current paper (to align with that chosen in [4]), we examine the case where $d_1 = \infty$ and $d_2 = 2$.

Let $\{e_i\} \subseteq \mathcal{H}_1$ and $\{f_1, f_2\} \subseteq \mathcal{H}_2$ be orthonormal bases. Attaching an ancillary Hilbert space $\mathcal{H}_3$, with infinite orthonormal basis $\{g_k\}$, we may define a unit vector $v_0 \in \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$ by

$$v_0 \equiv \frac{1}{2} \left( |e_1|f_1|g_1\rangle + |e_2|f_2|g_2\rangle + |e_2|f_1|g_3\rangle + |e_1|f_2|g_4\rangle \right). \quad (B1)$$

Note that the reduced density operator $\Phi(v_0) \in \mathcal{S}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ for $v_0$ is just $\frac{1}{2} P \otimes \frac{1}{2} I$, where $P$ is the projection onto the subspace of $\mathcal{H}_1$ spanned by $\{e_1, e_2\}$. Thus, from Prop. 3 (interchanging 1 with 2), there is a CHSH non-violating neighborhood surrounding $\Phi(v_0)$.

Now, for each $\lambda \in [0, 1]$, define the unit vector $v_\lambda \in \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$ by

$$v_\lambda \equiv (1-\lambda)v_0 + \lambda(2-\lambda)^{1/2}u \quad (B2)$$

where $u$ is the unit vector

$$u \equiv \sum_{n=1}^{\infty} 2^{-(n+1)/2} \left( |e_{2n+1}|f_1|g_n\rangle + |e_{2n+2}|f_2|g_n\rangle \right). \quad (B3)$$

Clearly, $v_\lambda \to v_0$ as $\lambda \to 0$. Furthermore, by the continuity of $\Phi$, $\Phi(v_\lambda) \to \Phi(v_0)$. It then follows that there is an $\epsilon > 0$ such that $\Phi(v_\lambda)$ is not CHSH violating for all $\lambda < \epsilon$. However, by construction $v_\lambda$ is separating for the subalgebra $I \otimes \mathcal{B}(\mathcal{H}_2 \otimes \mathcal{H}_3)$, for all $\lambda \in [0, 1]$. That is, for any $A \in \mathcal{B}(\mathcal{H}_2 \otimes \mathcal{H}_3)$, if $(I \otimes A)v_\lambda = 0$, then $A = 0$. (To see this, observe that any such $A$ would have to annihilate all the basis vectors $\{f_j \otimes g_k\}$ due to the orthogonality of the $\{e_i\}$.) Thus, invoking [3, Lemmas 1,2], each $\Phi(v_\lambda)$ is nonseparable, and, for all $0 < \lambda < \epsilon$, CHSH insensitive.

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