ON THE ECCENTRICITY DISTRIBUTION OF COALESCING BLACK HOLE BINARIES DRIVEN BY THE KOZAI MECHANISM IN GLOBULAR CLUSTERS

LINQING WEN
Division of Physics, Mathematics, and Astronomy, California Institute of Technology, MS 103-33, Pasadena, CA 91125; lwen@ligo.caltech.edu

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ABSTRACT

In a globular cluster, hierarchical triple black hole systems can be produced through binary-binary interaction. It has been proposed recently that the Kozai mechanism could drive the inner binary of the triple system to merge before it is interrupted by interactions with other field stars. We investigate qualitatively and numerically the evolution of the eccentricities in these binaries under gravitational radiation (GR) reaction. We predict that ∼30% of the systems will possess eccentricities greater than 0.1 when their emitted gravitational waves pass through 10 Hz frequency. The implications for gravitational wave detection, especially the relevance to data analyses for broadband laser interferometer gravitational wave detectors, are discussed.

Subject headings: binaries: close — gravitational waves — relativity — stellar dynamics

1. INTRODUCTION

Globular clusters (GCs) are excellent birthplaces for black hole (BH) binaries. In the bulk of our Galaxy, BH binaries can result from the evolution of two stars that are born in a close binary, that experience several phases of mass transfer, and that subsequently survive two supernovae. The event rates for BH merges from these binaries are generally predicted to be too low to be of observational interest to ground-based gravitational wave (GW) detectors (e.g., LIGO). In a dense cluster, however, BHs can become members of close binaries not through binary evolution but rather via dynamical interactions with other stars. As a result, the fraction of BHs ending up in compact binaries in these clusters could be overwhelmingly higher than those in the field of the Galaxy. It has been proposed that the BH merger rate from these binaries, specifically those hardened and ejected out of the clusters through three-body interactions, could be substantially greater than those of neutron star mergers and will be of great interest to ongoing GW detections (Portegies Zwart & McMillan 2000).

In our Galaxy, a GC contains typically \( N \approx 10^6 \) stars and resides normally in the Galactic halo. Typical ages of these clusters are estimated to be 12 billion years, and the clusters contain the oldest stars known in our Galaxy. Stellar evolution theory predicts that massive stars with mass above 10 \( M_\odot \) would evolve into BHs or neutron stars in \( \sim 10^7 \) yr. For a typical GC that follows the initial mass function (IMF) given by Scalo (1986), a significant number of BHs \((\sim 6 \times 10^{-4}N)\) for initial masses greater than 20–25 \( M_\odot \) should have been born. These BHs would be the most massive stars in the cluster, and hence dynamical two-body relaxation causes them to sink rapidly toward the core. In the core, BH binaries are expected to form. The single BHs form binaries preferentially with another BH, and BHs born in binaries with a lower mass stellar component rapidly exchange the companion for another BH through three-body relaxation processes. As a result of these dynamical processes, the fraction of BHs that end up in compact binaries could be overwhelmingly higher than in the bulk of the Galaxy. If these BH binaries merge within the Hubble time, they would become excellent sources for ongoing GW detectors (e.g., Thorne 1987).

It has been argued that GCs are unlikely places to harbor BH mergers (or subsequently to harbor a massive central BH) if we consider binary-single interactions alone (Portegies Zwart & McMillan 2000). Numerical simulations show that binary-single interactions tend to throw out these BH binaries and only 8% of BHs will be retained in the GC within its lifetime. Miller & Hamilton (2002) argued that this picture will be different if we consider binary-binary interactions. A substantial fraction of stable hierarchical triple BH systems can result from binary-binary interactions. The well-known Kozai mechanism (Kozai 1962) can then drive the inner binaries of a significant fraction of these triples to extreme eccentricities and thus reduce their time to coalescence to be shorter than the Hubble time. This is because the coalescence timescale \( \tau_{\text{GR}} \) due to gravitational radiation for a binary with fixed semimajor axis \( a \) has a strong dependence on the eccentricity \( e \) through the quantity \( \epsilon = 1 - e^2 \) (Peters 1964),

\[
\tau_{\text{GR}} \approx 6 \times 10^{10} \frac{(a/AU)^4 (\epsilon/0.01)^{3.5}}{(m_0 + m_1)m_0m_1/M_\odot} \text{ yr},
\]

where \( m_0 \) and \( m_1 \) are the masses of each binary component.

In their paper, Miller & Hamilton (2002) have studied the parameter space for successful mergers induced by the Kozai mechanism by comparing the coalescence timescale with the interaction timescale of a triple system with a field star. For an individual triple system of random orientation, the inner binary can merge successfully only in a restricted parameter space. However, the chance of mergers can be much higher for the following reasons. First, interaction with a field star may simply reorient the triple system. The inner binary of the reoriented triple system can still have a finite probability to merge if the new orientation is assumed to be random. Second, even if the inner binary does not merge successfully and is disintegrated from a triple system, it can interact with another binary and form a new hierarchical triple system. The inner binary of the new triple system can again merge under the Kozai mechanism. Such iteration
can occur many times before the orbit of the inner binary is hardened enough and is kicked out of the cluster by three-body interaction. Miller & Hamilton (2002) argued that, under this scenario, tens of percent of binaries may merge under the Kozai mechanism. The BH mergers may thus be retained in the GCs and a central intermediate-mass BH may form through subsequent mergers.

The event rate for this BH-BH coalescence induced by the Kozai mechanism depends on several effects that require further investigations. For instance, it is important to know if these merger events occur only in the early history of the cluster forming. As the cluster ages, it is possible, even though not clear, that most BH binaries get kicked out of the clusters, e.g., by three-body interactions; therefore, the BH binary population in the clusters reduces with time. However, young clusters are known to be in the process of forming in our observable universe. These and some low-density clusters that have not undergone core collapse can be candidate sites for these merger events. The importance of binary-binary interactions in the core of clusters, for instance, is another open issue. However, existing facts have supported the notion that these BH coalescence events could be very important in GCs and therefore are potentially important GW sources (Miller & Hamilton 2002).

We are motivated by the fact that the proposed mechanism promises stellar mass BH-BH mergers with extremely high initial eccentricities and that they are potential sources for upcoming GW detectors such as LIGO. We investigate in this paper whether or not these systems would still possess finite eccentricities when the emitted GWs enter LIGO’s frequency band. The eccentricities of these sources are particularly interesting as no astrophysical merger sources were expected before to possess significant eccentricities when they reach the LIGO band. Current efforts in detecting GWs from coalescing BH binary sources thus focus on waves from circular orbits (see Buonanno, Chen, & Vallisneri 2003 for recent development). On the other hand, it is well known that the gravitational waveform of an eccentric binary could be significantly different from that of a circular one and different detection strategies are required for optimal detections (Peters & Mathews 1963; Peters 1964; Lincoln & Will 1990; Gopakumar & Iyer 2002; Glampedakis, Hughes, & Kennefick 2002; Moreno-Garrido, Mediavilla, & Buitrago 1994, 1995; Glampedakis et al. 2002). Our work can thus help shed light on the importance of the eccentricities for detecting GWs from these sources.

In this paper we investigate in detail the eccentricity evolution of these merger systems under the impact of gravitational radiation reaction (simplified as the GR effect). This is in addition to the Kozai mechanism and the post-Newtonian periastron precession (simplified as the PN effect) considered by previous authors. It has been long realized (Peters & Mathews 1963; Peters 1964) that gravitational radiation reaction is very efficient in circularizing coalescing binaries. For the systems we are considering, this circularization effect would have to compete with the “eccentricity-enhancing” Kozai mechanism. Under the Kozai mechanism and the PN effect, the system tries to maintain secular cyclic eccentricity oscillations at fixed amplitude (with exceptions at critical inclination angles; see text). The GR effect operates most efficiently near the minimum \( \epsilon \) of each cycle. In some cases, it could drive a system near its minimum \( \epsilon \) directly toward merger. In other cases, it would affect the minimum \( \epsilon \) and alter the otherwise fairly strict periodic evolution trajectory of the system. In particular, it will enhance rapidly the importance of the PN effect, which in turn would introduce fast oscillations to destroy the Kozai resonance. The Kozai resonance will eventually be quenched either directly by the GR effect or by the PN effect before the GR effect takes over. We will discuss the details of these effects and their implications for the evolution of the eccentricities and GW frequencies.

In § 2 we give a summary of the relevant features of the Kozai mechanism and derive an explicit general expression for the minimum \( \epsilon \) (or maximum eccentricity) the system can reach for arbitrary values of BH masses and initial eccentricities. In § 3 we discuss the evolution of the system parameters under gravitational radiation reaction (the GR effect). The equations of motion are described in § 3.1. We describe technical details to estimate the minimum \( \epsilon \) the system can reach in § 3.2. The PN effect and its role in determining the evolution of the GW frequency are described in § 3.3. A procedure to obtain the relation of the GW frequencies and the eccentricity at various evolution stages is given in § 3.4. Two numerical examples are presented in § 3.5. In § 4 we estimate the eccentricity distribution when the GWs enters the LIGO band. Our results are then summarized in § 5.

2. KOZAI MECHANISM

2.1. Principle and Notation

The Kozai mechanism was initially proposed (Kozai 1962) to describe the perturbation of asteroids due to Jupiter. This mechanism describes the secular evolution of the orbit of a hierarchical triple system as a result of cyclic exchange of angular momentum between the inner binary and orbit of the distant tertiary. One of the most interesting results of the Kozai mechanism is that the eccentricity of the inner binary can reach very large values while the semimajor axis remains unchanged, provided that the initial mutual inclination angle of the two binaries reaches a certain critical value. Although the mechanism was proposed for a system where the mass of one component in the inner binary is much smaller than those of the others, the mechanism has then been found to operate in hierarchical triple systems with arbitrary masses (Lidov & Ziglin 1976).

A hierarchical triple star system consists of a close binary of masses \( m_0, m_1 \), and a tertiary \( m_2 \) in a much wider outer orbit (Fig. 1). The dynamics of the system can be treated as if it consists of an inner binary of point masses \( m_0 \) and \( m_1 \) and an outer binary of point masses \( m_2 \) and \( M_1 = m_0 + m_1 \). The separations of the masses of each binary are denoted \( r_1 \) and \( r_2 \). The mutual inclination angle of the two binaries is denoted \( I \). For the inner and outer binaries, respectively, we denote the eccentricities \( e_1 \) and \( e_2 \), reduced masses \( \mu_1 = m_0 m_1 / M_1 \) and \( \mu_2 = m_2 M_1 / (M_1 + m_2) \), and semimajor axes \( a_1 \) and \( a_2 \). We define \( g_1 \) as the argument of pericenter of the inner binary relative to the line of the descending node.

Throughout this paper we frequently use quantities \( \epsilon = 1 - e_1^2 \) and \( \eta_2 = a_2 (1 - e_2^2)^{1/2} \) to simplify our discussion. Angular momentum is normalized to have dimension (length\(^3\))/2 by dividing by the quantity \( L_1 = \mu_1 (G M_1)^{1/2} \). We use notation \( \alpha_0, \beta_0, \) and \( \gamma \) for the normalized magnitude of total angular momentum of the triple system, of the outer binary, and of the inner binary, respectively. Our \( \alpha_0 \) and \( \beta_0 \) are related to dimensionless quantities \( \alpha \) and \( \beta \) used in previous literature by \( \alpha = \alpha_0 / \sqrt{a_1} \) and \( \beta = \beta_0 / \sqrt{a_1} \). It
follows that \( \gamma = \sqrt{\alpha_1 \epsilon} \), \( \beta_0 = \mu_2 / \mu_1 [M_2 a_2 / M_1 (1 - e_2^2)]^{1/2} \), and \( \alpha_0 \) is related to \( \beta_0 \) and \( \alpha_1 \epsilon \) through the trigonometric relation:

\[
\cos I = \frac{\alpha_0^2 - \beta_0^2 - \alpha_1 \epsilon}{2 \beta_0 \sqrt{\alpha_1 \epsilon}}. \tag{2}
\]

As we shall see, during the binary’s Kozai mechanism–
induced evolution, \( \alpha_0 \), \( \beta_0 \), and \( \alpha_1 \) are conserved, while \( \epsilon \) changes. It follows that \( \alpha_0 = \beta_0 \) is the critical condition for the system to be able to evolve to \( \epsilon = 0 \) \( (e_1 = 1) \). This implies a critical initial \( I \) of

\[
I_0 = I_\epsilon = \cos^{-1} \left( -\frac{\sqrt{\epsilon_0}}{2 \beta_0} \right), \tag{3}
\]

where \( \epsilon_0 \) is the initial \( \epsilon \). For this solution to exist, \( \beta \geq \sqrt{\epsilon_0 / 2} \) is required.

The total Hamiltonian of the system can be written as \( H = H_1 + H_2 + H' \), where \( H_1 \) and \( H_2 \) are the Hamiltonians for the inner and outer binaries, respectively, as if they were isolated binaries of point masses. \( H' \) is the perturbative Hamiltonian that includes perturbation of the inner binary by the presence of the tertiary \( M_1 \) and perturbation of the outer binary by the finite size of the binary component \( M_1 \). For \( r_1 / r_2 < 1 \) and \( (m_2 / M_1) (r_1 / r_2)^3 < 1 \), the Hamiltonian \( H' \) can be expanded in powers of \( r_1 / r_2 \). The first non-zero contribution comes from the term of order \( (r_1 / r_3)^2 \), labeled as the quadrupole term because of the degree of the Legendre polynomial associated with it. This term is sufficient for our discussion as we are only concerned about cases with large initial mutual inclination angle \( I_0 \) (see Ford, Kozinsky, & Rasio 2000 for discussions concerning the octupole terms of order \( (r_1 / r_3)^3 \)).

It is this perturbative Hamiltonian \( H' \) that leads to a secular evolution of the triple system. In the first approximation, the secular evolution can be described by \( H' \) doubly averaged over the mean anomaly of both the inner and outer orbits. It turns out that the doubly averaged \( H' \), the total angular momentum of the system \( (\alpha_0) \), the energy of each binary (or equivalently \( a_1 \) and \( a_2 \)), and the magnitude of the angular momentum of the outer binary \( (\beta_0, \text{or equivalently } e_2) \) are conserved quantities. The secular evolution of the triple system can therefore be fully described in terms of time-evolving \( \epsilon \) and \( g_1 \). A useful constant of integration can be derived from equation (2),

\[
a_1 \epsilon \left( \cos I + \frac{\sqrt{\epsilon}}{2 \beta_0} \right)^2 = \frac{\alpha_0^2 - \beta_0^2}{4 \beta_0^2} \tag{4}
\]

Note that the quantity on the left-hand side is conserved as long as the total angular momentum \( (\alpha_0) \) and that of the outer binary \( (\beta_0) \) are conserved. In addition, whenever \( \sqrt{\epsilon} \ll 2 \beta_0, \alpha_1 \epsilon \cos^2 I \) is a constant.

The secular evolution is a result of coherent additions of perturbations from the tertiary to the inner binary. Therefore, any mechanism that perturbs the phase relation of the system could modify the evolution of \( \epsilon \) and \( g_1 \). For BH systems, an important effect comes from the general relativistic periastron precession of the inner binary. Inclusion of this effect can be found in Miller & Hamilton (2002) and Blues, Lee, & Socrates (2002). In summary, contribution of the periastron precession in the first-order post-Newtonian approximation can be added to the doubly averaged Hamiltonian as \( H_{\text{PN}} = -k \theta_{\text{PN}} / \sqrt{\epsilon} \), where

\[
\theta_{\text{PN}} = 8 \times 10^{-8} \left( M_1 / M_\odot \right)^2 \left( b_2/a_1 \right)^3 \frac{1}{a_1 / \text{AU}} \tag{5}
\]

and \( k = 3 G m_0 m_1 m_2^2 / (8 M_\odot b_1^2) \) is a quantity related to the evolution timescale. Apparently the influence of the periastron precession (abbreviated as PN effect) is the largest (in the sense that \( \theta_{\text{PN}} \) is the largest) for systems consisting of a tight and massive inner binary and a light, far-out third component. With the addition of this PN effect, the total Hamiltonian remains conserved. It can be written as \( H = k W \) (Miller & Hamilton 2002), where

\[
W(\epsilon, g_1) = -2 \epsilon + \epsilon \cos^2 I + 5 (1 - \epsilon) \sin^2 g_1 \left( \cos^2 I - 1 \right) + \theta_{\text{PN}} / \epsilon^{1/2} \tag{6}
\]

is a conserved quantity.

### 2.2. Eccentricity Evolution in Absence of the GR Effect

The trajectories of \( e_1 \) and \( g_1 \) in the phase space are known to be determined by the values of \( \alpha \) and \( \beta \) for given initial system parameters (Lidov & Ziglin 1976). In the general case of initial \( e_1 \neq 0, 1 \) and in the absence of the GR effect, the quantities \( e_1, g_1 \) undergo cyclic oscillations. A necessary condition for a large change in \( e_1 \) is that \( I_0 > \cos^{-1}(3/5)^{1/2} \approx 39^\circ \) for a restricted case where \( m_0 \ll m_2 \ll m_1 \). This condition still holds for arbitrary masses when the PN effect is weak (as can be proved using eq. [8]). The timescale for the system to swing from \( e_1 \approx 0 \) to \( e_1 \approx 1 \) is estimated to be (Innanen et al. 1997)

\[
\tau_{\text{evol}} \approx 0.16 f \left( \frac{a_1}{m_2} \right)^{1/2} \left( \frac{a_2 / \text{AU}}{m_2 / M_\odot} \right)^{3/2} \left( a_1 / m_2 \right)^{1/2} \left( 1 - e_2^2 \right)^{3/2} \text{ yr}, \tag{7}
\]

where \( f \approx 0.42 \log(1/\epsilon_{10}) / \sin^2(I_0) - 0.4 \) is a quantity of magnitude a few, \( \epsilon_{10} \) is the initial value of \( e_1 \), and \( I_0 \) is the initial value of \( I \). This timescale should be much longer than
the orbital periods of both binaries because it is an orbital-averaged effect.

The minimum value of $\epsilon$ (denoted $\epsilon_{\text{min}}$) for arbitrary masses $m_0, m_1, m_2$, and arbitrary $e_0$ can be solved for using the fact that the Hamiltonian is conserved, that is, $W(\epsilon_0, g_0) = W(\epsilon_{\text{min}}, g_{\text{min}})$ (eq. [6]). We are interested in the cases $\epsilon_{\text{min}} \approx 0$ and $\epsilon_{\text{min}} \ll \epsilon_0$. It follows that $g_m = \pi/2$ for $I_0 \neq I_2$ (Lidov & Ziglin 1976), $\epsilon_{\text{min}} \ll \epsilon_0^2$, and $\epsilon_{\text{min}} \ll 2\beta$. The assumption that $\epsilon_{\text{min}} \ll 2\beta$ is valid as long as there exists a solution for the critical initial value $I_c = -\sqrt{\epsilon_0}/(2\beta)$. Under these conditions, the resulting quadratic equation as $\epsilon_{\text{min}}$ is then obtained by solving the resulting quadratic equation as

$$\epsilon_{\text{min}} = \frac{1}{2\Omega} \left[ \theta_{\text{PN}} + \sqrt{\theta_{\text{PN}}^2 + 20\Omega \epsilon_0 \left( \cos I_0 + \frac{\sqrt{\epsilon_0}}{2\beta} \right)^2} \right].$$

(8)

By using equation (4), we have

$$\epsilon_{\text{min}} \approx \frac{1}{2\Omega} \left[ \theta_{\text{PN}} + \sqrt{\theta_{\text{PN}}^2 + 5 \Omega \left( \frac{\alpha_0^2 - \beta_0^2}{\beta_0^2} \right)^2} \right].$$

(9)

Here

$$\Omega = 5 - 2\epsilon_0 + \epsilon_0 \cos^2 I_0 + \frac{\theta_{\text{PN}}}{\sqrt{\epsilon_0}} + 4\epsilon_0 \left( \cos I_0 + \frac{\sqrt{\epsilon_0}}{2\beta} \right)^2$$

$$+ 5(1 - \epsilon_0)(\cos^2 I_0 - 1) \sin^2(\theta_{\text{PN}}).$$

(10)

For the restricted case of $m_0 \ll m_1 \ll m_2$, we have $\beta \to \infty$, $I_2 \to 90^\circ$. We recover equation (8) of Miller & Hamilton (2002) using their assumptions that $\epsilon_0 \approx 1$, $\theta_{\text{PN}} \ll 3$, and $5 \cos^2 I_0 \ll 3$. We then also recover the classical limit $\epsilon_{\text{max}} = (1 - \epsilon_{\text{min}})^{1/2} \approx 1 - (5/3) \cos^2 I_0$ (e.g., Innanen et al. 1997) by setting $\theta_{\text{PN}} = 0$.

It follows from equation (8) that a condition for achieving extremely small $\epsilon_{\text{min}}$ is $\theta_{\text{PN}} \ll 1$ and $I_0 \approx I_1 > 90^\circ$, which corresponds to $\alpha \approx \beta$ (eq. [4]). Note that $I_2 > 90^\circ$ means that the outer binary should initially be in a retrograde orbit with the inner one. This is consistent with what we obtained in the classical limit ($\theta_{\text{PN}} = 0$), where a system can evolve to $e = 1$ starting from any initial eccentricity if $\alpha = \beta$ (or $I_0 = I_2$; Lidov & Ziglin 1976). The critical values of $I_0$ above which the system can have a significant change in $\epsilon$ can also be derived. If we demand $\epsilon_{\text{min}} \leq 0.5\epsilon_0$, we would obtain from equation (8) $\cos^2 I_0 \leq 1$, or $I_0 \geq 39^\circ$ for $\epsilon_0 \approx 1$ and $\theta_{\text{PN}} \approx 0$. This is the same as the known classical limit for the restricted problem.

Two important constants can be derived from equation (8) for triple systems with arbitrary masses but negligible PN effect. As $\theta_{\text{PN}} \to 0$, $\Omega$ is a constant $\sim 3$ for $\epsilon_0 \approx 1$ and $I_0 \approx I_c$.

$$a_1 \epsilon_{\text{min}} \approx \frac{5}{12} \left( \frac{\alpha_0^2 - \beta_0^2}{\beta_0^2} \right)^2.$$

(11)

It thus follows from equation (4) that the mutual inclination angle at $\epsilon_{\text{min}}$ (denoted $I_m$) satisfies

$$\cos^2 I_m \approx \frac{1}{2},$$

for $\epsilon_{\text{min}} \ll 2\beta$.

## 3.1. Equations of Motion

The presence of gravitational radiation has important impacts on the eccentricity evolution of the inner binary. First, the radiation carries away energy and angular momentum and tends to shrink and circularize the orbit. The decay of the orbit subsequently changes the values of $\alpha$ and $\beta$ and thus modifies the secular evolution trajectory of the system. In particular, the effect of periapsis precession becomes stronger rapidly with the decay of the orbit (see eq. [5]). This results in rapid oscillations of $g_1$ that could destroy the secular oscillation of $e$. As for the outer binary, the orbit remains much wider than the inner one, so its energy (equivalently $a_2$) and momentum of angular momentum ($\beta_0$, or equivalently $e_2$) can therefore still be treated as conserved quantities.

The GR effect is the strongest at $e \sim \epsilon_{\text{min}}$ within each Kozai cycle because of the strong dependence of the merger timescale $\tau_{\text{GR}}$ on $\epsilon$ (eq. [1]). When the PN effect is negligible and the GR effect does not affect the Kozai cycles dramatically, we have $a_1 \propto \epsilon_{\text{min}}$ according to equation (11). The evolution of the parameters $\theta_{\text{PN}}, \tau_{\text{evol}}$, and $\tau_{\text{GR}}$ with the decay of the orbit can then be summarized as follows:

$$\theta_{\text{PN}} \propto a_1^{-4} \propto \epsilon_{\text{min}}^{-4},$$

(13)

$$\tau_{\text{evol}} \propto a_1^{3/2} \propto \epsilon_{\text{min}}^{3/2},$$

(14)

$$\tau_{\text{GR}} \propto a_1^{4/7} \propto \epsilon_{\text{min}}^{-1/5}.$$

(15)

It is apparent that, as the orbit of the inner binary shrinks and circularizes, the value of $\tau_{\text{GR}}$ decreases proportionally to $\epsilon_{\text{min}}^{-1/5}$, while the time the system spends at $\epsilon_{\text{min}}$ ($\tau_{\text{evol}} = \tau_{\text{evol}}(\epsilon_{\text{min}})$) increases proportionally to $\epsilon_{\text{min}}^{1/2}$. When $\tau_{\text{GR}}$ becomes of order $\tau_{\text{evol}}(\epsilon_{\text{min}})$, the gravitational merger will take place within one Kozai cycle near $e \sim \epsilon_{\text{min}}$.

The evolution of the system can be calculated through a set of hybrid equations that combines the equations of motion derived from the conserved Hamiltonian $W(\epsilon, g_1)$ (Lidov & Ziglin 1976, eqs. [32] and [33]) including the PN effect and that from quadrupole gravitational radiation (Peters 1964, eqs. [5.6] and [5.7]). These are the same equations used in Blaes et al. (2002). The evolution equations govern the four parameters $\epsilon, g_1, a_1$, and $a_0$ and are given by

$$\frac{d\epsilon}{dt} = -10 \kappa G a_1^{-5} \sqrt{1 - \epsilon}(1 - \cos^2 I) \sin(2g_1)

+ \frac{\kappa G}{a_1^3} \left( \frac{304}{340} \epsilon - \frac{121}{425} \right),$$

(16)
\[
\frac{dg_1}{dt} = \kappa_E a_1^{1.5} \left\{ \frac{1}{\sqrt{e}} \left[ 4 \cos^2 I + (5 \cos 2g_1 - 1)(\epsilon - \cos^2 I) \right] + \frac{\cos I}{\beta} \left[ 2 + (1 - \epsilon)(3 - 5 \cos 2g_1) \right] + \frac{\theta_{PN}}{\epsilon} \right\},
\]
(17)

\[
\frac{da_1}{dt} = -\frac{6}{19} \frac{\kappa_G}{a_1^{3.5}} \left( \frac{425}{96} \frac{61}{16} + \frac{37}{96} \epsilon^2 \right),
\]
(18)

\[
\frac{d\alpha_0}{dt} = -\frac{3}{19} \frac{\kappa_G}{a_1^{3.5} \epsilon^2} \left( \frac{15}{8} - \frac{7}{8} \epsilon^2 \right) \sqrt{a_1^3 + \beta_0 \cos^2 I},
\]
(19)

where \(\theta_{PN}\) is defined in equation (5) and

\[
\kappa_E = 7.4554 \times 10^{-8} \left( \frac{m_2 / M_1}{(a_2 / AU)^2} \right)^{0.5} \left( \frac{(m_2 / M_1)^{0.5}}{1 - e_2^2} \right)^{1.5} \frac{1}{AU^{1.5}},
\]
(20)

\[
\kappa_G = 7.8218 \times 10^{-26} \frac{m_0 m_1 M_1}{M_0^3} \text{AU}^4 .
\]
(21)

The evolution of the angular momentum \(\gamma\) can be written as (see eqs. [16] and [18])

\[
\frac{d\gamma}{dt} = -5\kappa_E a_1^2 \sin^2 I \sin 2g_1 - \frac{3}{19} \frac{\kappa_G a_1^{3.5} \epsilon^2}{\alpha_0} \left( \frac{15}{8} - \frac{7}{8} \epsilon^2 \right).
\]
(22)

Equations (16)-(19) provide a valid description of the evolution as long as the energy \(a_1\) is approximately conserved within each cycle and as the total angular momentum vector of the triple system is conserved. These are necessary conditions for the validity of the equations derived from the conserved Hamiltonian, and the following considerations justify them for the systems that interest us.

We have assumed that the triple systems in GCs consist of stellar mass BHs with initial orbital separations on the order of AU and moderate initial \(\epsilon \sim 1\) (see the discussion of parameters in § 4). The systems are required to obtain extremely small \(\epsilon_{\text{min}} \sim 0\) during their secular evolution in order to merge well before the system is disrupted by interactions with field stars (Miller & Hamilton 2002). Equations (16) and (17) indicate that at each Kozai cycle, such systems would spend most of their time at moderate \(\epsilon \sim 1\) and a very small fraction of the time at \((\epsilon_{\text{min}})^2\) at extremely small values (see also Innanen et al. 1997). Note that gravitational radiation reaction is negligible at moderate \(\epsilon\) and \(a_1\) as \(\kappa_G \ll \kappa_E\) (eqs. [20] and [21]) for such systems. The energy (equivalently \(a_1\)) can thus be treated approximately as conserved for most of the cycles (see also eq. [18]).

The total angular momentum vector can also be treated as conserved within each cycle as long as \(\epsilon_{\text{min}} \sim 0\). It is known that, for an isolated system under gravitational radiation reaction, the angular momentum loss is negligible \((\times \kappa_G a_1^{1.5})\) at \(\epsilon_{\text{min}} \sim 0\) (eq. [22], second term). Moreover, gravitational radiation reaction will not change the direction of the angular momentum of the inner system. It is thus conceivable that as long as \(\epsilon_{\text{min}} \sim 0\), the evolution of the angular momentum of the inner binary \((\gamma)\) is negligibly affected by the GR effect, and the vector of the total angular momentum can be treated as conserved throughout the cycles (note that near \(\epsilon_{\text{min}} \sim 0\), both \(a_1\) and \(\epsilon_{\text{min}}\) values are affected by the GR effects, even though the angular momentum changes very little). For our systems, the GR effect starts to dominate near \(\epsilon_{\text{min}} \sim 0\) where the merger timescale is the shortest within each Kozai cycle. The extremely small \(\epsilon_{\text{min}}\) required here is expected to be much shorter than the interaction timescale with a field star. Above arguments justify the validity of equations (16)-(19).

### 3.2. Estimation of \(\epsilon_{\text{min}}\)

The minimum value of \(\epsilon\) that a system can reach within its first Kozai cycle (denoted \(\epsilon_{\text{min}}\)) is best estimated using equation (16). At \(\epsilon = \epsilon_{\text{min}}, \frac{d\epsilon}{dt} = 0\), we obtain

\[
\sin^2 g_m = \frac{1}{2} \left( 1 - \left( \frac{\gamma_0 \epsilon_{\text{min}}^3 - \epsilon_1^5 \sin^2 I_m \right)^2 \right),
\]
(23)

where

\[
\gamma_0 = \frac{425}{3040} \frac{\kappa_G}{\kappa_E} = 1.467 \times 10^{-19} \frac{M_3^3}{m_0 m_1 M_1} \left( \frac{M_1}{m_2} \right)^{1/2} \times \left( \frac{a_2 / AU}{(m_2 / M_0)^{1/2}} \right)^{3/2} \text{AU}^{5.5}.
\]
(24)

and where \(\sin^2 I_m\) is related to \(\epsilon_{\text{min}}\) through equation (2). We estimate \(\epsilon_{\text{min}}\) by assuming that \(a_1\) is a constant and solving implicitly the equation

\[
\Delta W = W(g_0, \epsilon_0) - W(g_m, \epsilon_{\text{min}}) = 0.
\]
(25)

In our numerical work, we actually solve for \(\epsilon_{\text{min}}\) by minimizing \((\Delta W)^2\) using a downhill simplex method developed by Nelder & Mead (1965) (available in Matlab). This method has proved to be robust in finding local minima. However, because of the existence of multiple solutions to equation (25) (with at least one other solution \(\epsilon = \epsilon_0\) and \(a_1 = a_0\), a good initial guess is essential for a well-converged solution.

We made our initial guess \(\epsilon_{\text{min}}\) by considering the following two limits for \(g_m\) based on equation (23). In the first limit, where gravitational radiation reaction is weak, or \(a_1^{11/2} \epsilon^3 \gg \gamma_0\), we have \(g_m \sim 90°\). Equation (8) can then serve as a very good approximation to \(\epsilon_{\text{min}}\). In the second limit, where gravitational radiation is very important, \(g_m\) could deviate significantly from 90°. The extreme limit occurs around \(I_0 = I_t\) or \(\alpha_0 = \beta_0\) (eq. [4]). As with the critical case of \(\alpha = \beta\) in the classical limit, a system starting with \(\epsilon_0 \sim 1\) will pass through an unstable stationary point \(\epsilon \sim 1\) and \(g_1 \sim \sin^{-1}(2/5 \sqrt{5})\) and move on toward a stationary (and stable in the classical limit) point \(\epsilon \sim 0\) and \(g_c = \sin^{-1}(1/(2\beta))\{[8\beta^2 - 1]/5\}^{1/2}\) (Lidov & Ziglin 1976, eq. [46]).

If it were not for the GR effect, the system would reach \(\epsilon_1 = 1\) at this stable point. With the presence of the GR effect, the angle \(g_1\) will be steered away from this point while \(\epsilon\) is steered away from a true zero value (see also Fig. 5). We therefore expect the value of \(g_m\) to be very close to \(g_c\) (at \(\epsilon_{\text{min}} \sim 0\), \(dg/\epsilon \sim \epsilon_{\text{min}} = 0\), eq. [17], ignoring the PN effect). In this limit, the minimum \(\epsilon\) can thus be estimated using \(g_m \approx g_c\) and assuming constant \(a_1\). Specifically, we again use the minimization program introduced previously to solve equation

\[
a_1^{5.5} \epsilon_{\text{min}} = \frac{\gamma_0}{\sin^2 I_m} \frac{1}{\sin 2g_c}
\]
(26)

for \(\epsilon_{\text{min}}\), where \(\sin^2 I_m\) is related to \(\epsilon_{\text{min}}\) by equation (2).

Without knowing the solution ahead of time, we simply feed values of \(\epsilon_{\text{min}}\) estimated for the two limits as initial
guesses to the minimization program and then select the solutions of $|\Delta W|^2 = 0$ with minimum residues. In all cases, the residues of the minimization are monitored to ensure correct convergence.

3.3. PN Effect

The importance of the PN effect increases rapidly with the decay of the orbit as $\theta_{\text{PN}} \propto a_1^{-4}$. This effect has a direct impact on the oscillatory behavior of $g_1$ (eq. [17]), through which it can affect the evolution of $\epsilon$ and thus $a_1\epsilon_{\text{min}}$ (eq. [16]). The importance of the PN effect to the evolution of $\epsilon$ and $a_1\epsilon_{\text{min}}$, however, depends on the role of the GR effect.

The PN effect can be neglected if the GR effect dominates in the evolution of $\epsilon$ before the PN effect becomes important in the evolution of $g_1$. Once the GR effect dominates, the evolution of $\epsilon$ decouples from that of $g_1$ and the Kozai cycle is terminated. In the cases we are interested in, this transition occurs near $\epsilon_{\text{min}} \sim 0$. When both the GR and the PN effects are not important, the quantity $a_1\epsilon_{\text{min}}$ is roughly a constant (eq. [11]). After the GR effect dominates, it evolves according to equation (5.11) of Peters (1964) and remains a constant as long as $\epsilon_{\text{min}} \sim 0$. The overall evolution of $a_1\epsilon_{\text{min}}$ can therefore be conveniently described using Peters’ equation with $a_1$ and $\epsilon_{\text{min}}$ of the first Kozai cycle (see § 3.2) as the initial data.

The PN effect has a significant impact on the evolution of $\epsilon$ and $a_1\epsilon_{\text{min}}$ if it dominates in the evolution of $g_1$ before the GR effect becomes important. In this case, a dominating PN effect will introduce a fast oscillation in $g_1$, and thus in $\epsilon$ (eq. [17]) and will terminate the Kozai cycle. The GR effect will eventually dominate in the evolution of $\epsilon$, and Peters’ formula can again describe the evolution of $a_1\epsilon_{\text{min}}$. However, when the PN effect is important, the quantity $a_1\epsilon_{\text{min}}$ can no longer be approximated as a constant before the GR effect dominates. It will evolve to a larger value than that in the first Kozai cycle (see § 3.2). The results based on Peters’ formula with initial values taken at the first Kozai cycle can therefore be used to set the lower bound on $a_1\epsilon_{\text{min}}$.

An upper limit for the quantity $a_1\epsilon_{\text{min}}$ when the PN effect is important can be set based on the following considerations. First, the fast oscillation induced by the PN effect will be damped away under the GR effect. In light of equation (9), the quantity $\Omega$ will evolve from a rough constant of $3-5$ toward $\Omega \sim \theta_{\text{PN}}/\sqrt{\epsilon_0} \to \infty$ as $\theta_{\text{PN}} \to \infty$ with the decay of the orbit. As a result, $\epsilon_{\text{min}} \to \epsilon_0$; that is, the oscillation will be quenched. The timescale for the system under the PN effect to swing from $g_1 \sim 0$ to $\pi/2$ with a small change in $\epsilon$ and $a_1$ is

$$\tau_{\text{PN}} \sim \frac{\epsilon}{\kappa \theta_{\text{PN}} 0.5} \sim 5 \times 10^6 \left(\frac{a_1}{\text{AU}}\right)^{2.5} \frac{\epsilon}{\left(M_1/M_0\right)^{1.5}} \text{ yr.}$$ (27)

Secondly, neglecting the GR effect, the evolutions of $I$ and $g_1$ under a dominating PN effect can be written as

$$\frac{dI}{dt} = -5 \kappa a_1^2 \sin I \cos I (a_1 \epsilon)^{-1/2} \sin 2g_1 ,$$ (28)

$$\frac{dg_1}{dt} = \kappa a_1^2 \frac{\theta_{\text{PN}}}{\epsilon} ,$$ (29)

where we have applied equations (2) and (22) to obtain equation (28). It is apparent that the oscillation of $I$ with $g_1$ is also damped rapidly as $\theta_{\text{PN}}/\sqrt{\epsilon} \to \infty$ with the decay of the orbit. Integrating over $g_1 = [0, \pi/2]$ for $dI/dg_1$, we obtained an estimation for the oscillation amplitude

$$\delta I \approx 2.5 \frac{\sqrt{\epsilon}}{\theta_{\text{PN}}} \sin 2I .$$ (30)

An upper limit of $\delta I$ can be estimated by the fact that $\theta_{\text{PN}}/\sqrt{\epsilon} > 10\cos^2 I_0 \sim 6$, as required for a dominating PN effect (eqs. [17] and [12]), and that $\sin 2I \leq 1$. Here we have used the fact that the PN effect becomes dominant near $\epsilon_{\text{min}}$ and thus $a_1 \epsilon_{\text{min}}$ is roughly a constant. This condition is obtained within the first Kozai cycle, when the PN effect starts to dominate the evolution of $g_1$ (eq. [17]), that is,

$$a_1^{1/2} \epsilon^{1/2} \epsilon^{1/2} \geq \gamma_0 ,$$ (33)

$$\theta_{\text{PN}} \geq 6 \sqrt{\epsilon} .$$ (34)

Equation (32) and the comparison of $\tau_{\text{PN}}$ (eq. [27]) with $\tau_{\text{GR}}$ (eq. [1]) indicate that the PN effect is important only for systems with sufficiently large $a_1$ and $\epsilon_{\text{min}}$. As a result, the PN effect is relevant only in low-eccentricity cases in the LIGO band.

3.4. Evolution of Gravitational Wave Frequency

The frequency of GWs ($f_{\text{GW}}$) emitted from an in-spiraling binary depends strongly on its orbital frequency $f_{\text{orb}}$ and eccentricity $e_1$, where

$$f_{\text{orb}} = \frac{\sqrt{G M_1}}{2\pi} a_1^{-1.5} .$$ (35)

In the limit of the quadrupole approximation, the power at the $n$th harmonic of $f_{\text{orb}}$ has been derived in equations (19) and (20) of Peters & Mathews (1963) in terms of spherical Bessel functions. For a circular orbit, the power is concentrated in the second harmonic. For an eccentric orbit, the power spreads over a broad frequency band and the maximum occurs at a harmonic $n_m \gg 1$ for $\epsilon \sim 0$. For instance, this peak frequency could occur at the $n_m = 10^6$ harmonic of the orbital frequency for $\epsilon = 2 \times 10^{-4}$. In this section we discuss the evolution of the peak frequency with the eccentricity $e_1$ (or $\epsilon$).

We have derived an approximate expression for the “peak harmonic” of order $n_m$ in terms of $\epsilon$ for convenience of our calculation as follows. We first used the formulae in
Peters & Mathews (1963) to calculate \( n_m \) for various values of \( \epsilon = 10^{-6} \) to 1, and we then applied a least-squares fit to the data. An excellent fit to \( n_m(\epsilon) \) was obtained,

\[
n_m = \frac{2(1 + \epsilon)}{(1 - \epsilon^2)^{1.5}}. \tag{36}
\]

This formula preserves the correct limit of \( n_m = 2 \) for the circular case of \( \epsilon_1 = 0 \). The relation of the peak frequency \( f_{GW}^m \) to \( \epsilon \) is obtained by combining equations (35) and (36),

\[
f_{GW}^m(\epsilon_1) = \frac{\sqrt{GM_1}}{\pi} \frac{1}{(a_1 \epsilon)^{1.5}}. \tag{37}
\]

It indicates that the peak frequency \( f_{GW}^m \) is inversely proportional to the period of a circular Keplerian orbit with radius equal to the actual orbit’s semi-latus rectum \( a_1 \epsilon \).

We actually are interested in \( \epsilon_1 \) as a function of \( f_{GW}^m \), rather than \( f_{GW}^m(\epsilon_1) \), since the focus of this article is the orbital eccentricity as the GW frequency evolves through the LIGO band. We compute \( \epsilon_1 \) from equation (37), by applying our knowledge of the evolution of \( a_1 \epsilon \). Let the initial system parameters be \( a_1 = a_{10}, \epsilon = \epsilon_0 (\epsilon_1 = \epsilon_{10}), \) and the minimum \( \epsilon \) expected for the first Kozai cycle be \( \epsilon_{\text{min}} \) (eq. [25]) and \( \epsilon_m = (1 - \epsilon_{\text{min}})^{1/2} \). We consider the following three possibilities separately:

1. The frequency enters the LIGO frequency band before \( \epsilon \) reaches \( \epsilon_{\text{min}} \). In this case, \( a_1 \sim a_{10} \), and \( \epsilon \) lies in the range \( 0 < \epsilon < \epsilon_{\text{min}} \).

2. The frequency enters the LIGO band after \( \epsilon_{\text{min}} \) was reached and the GR effect dominates in the evolution of \( \epsilon \) before the PN effect becomes relevant. The evolution of \( a_1 \epsilon \) is then the same as for an isolated system evolving under gravitational radiation reaction with initial values of \( a_1 = a_{10} \) and \( \epsilon = \epsilon_{\text{min}} \); that is (eq. [5.11] in Peters 1964),

\[
a_1 \epsilon = \left( a_{10} \epsilon_{\text{min}} \right) \left( \frac{\epsilon_1}{\epsilon_{\text{min}}} \right)^{12/19} \left( 1 + \frac{121/304 \epsilon_1^2}{1 + 121/304 \epsilon_{\text{min}}^2} \right)^{870/2299}. \tag{38}
\]

3. The frequency enters the LIGO band after \( \epsilon_{\text{min}} \) was reached, but the PN effect becomes important before the GR effect dominates in the evolution of \( \epsilon \). In this case, the lower limit of \( a_1 \epsilon \) is obtained from equation (38) and the upper limit is set to be 3 times this lower limit. The bounds for \( \epsilon_1 \) at a fixed \( f_{GW}^m \) can be obtained by applying these two limits.

3.5. Numerical Examples

The numerical evolutions of \( \epsilon \) and \( I \) are shown for two typical examples in Figure 2 and in Figure 4 below for illustration. These evolutions were obtained by integrating the ordinary differentiation equations (16)–(19) for given initial data and then using equation (2) to derive the mutual inclination angle \( I \). We used a medium-order Runge-Kutta method (available in Matlab as ode45) for the integration.

The first example (Fig. 2) represents a case in which the system merges after many Kozai cycles. This is also a case in which the PN effect becomes important before the GR effect dominates in the evolution of \( \epsilon \). The system parameters are chosen to be \( m_0 = m_1 = m_2 = 10^6 M_\odot, \epsilon_1 = \epsilon_2 = 0.1, a_1 = 10^6 \) AU, \( a_2/a_1 = 10, g_1 = 0, \) and \( I_0 = 95.3^\circ \). The gradual increase in the timescale of the Kozai cycle is apparent. For most of the cycles, \( \epsilon_{\text{max}} \sim 1 (\epsilon_{\text{min}} \sim 0) \). Near the \( \epsilon_{\text{min}} \) of the last Kozai cycle, fast damped oscillations of \( \epsilon (I) \) due to the PN periastron precession are visible before a monotonic increase of \( \epsilon \) due to the dominating GR effect (see also Fig. 3). The value of \( I \) experiences damped oscillation after the PN effect becomes important and converges to a constant after the GR effect dominates in the evolution. Note also that before the last cycle, the minimum \( I \) is around \( 39^\circ \) as predicted by equation (12).
We show in Figure 3, for the first example, the evolution of \( \epsilon \) with \( a_1 \) (solid line) and compare with the results based on the lower limit of \( a_1 \epsilon_{\text{min}} \) obtained with equation (38) and an upper limit that is 3 times this lower limit as discussed in § 3.3 (dashed lines). Also shown in dot-dashed lines are the evolutions of the timescales of the GR and the PN effects normalized by the Kozai timescale \( \tau_{\text{Kozai}} \). The phase diagram for the evolution path of \( \epsilon \) vs. \( g_1 \) is then quickly dominated by the gravitational radiation effect of \( a_1 \). The decay of \( a_1 \) occurs mainly near \( \epsilon_{\text{min}} \) of each cycle. Damped oscillations are apparent near \( \tau_{\text{PN}}/\tau_{\text{Kozai}} \). There is an excellent agreement between our numerical results and the prediction in § 3.2 and 3.3.

The second example (Fig. 4) represents a case in which the system merges within the first Kozai cycle. The system parameters are chosen to be \( m_0 = m_1 = m_2 = 5 \) \( M_\odot \), \( e_1 = 0.01 \), \( e_2 = 0.51 \), \( a_1 = 2.1920 \) AU, \( a_2/a_1 = 5 \), \( g_1 = 0 \), and \( I_0 = 99.2^\circ \). This is a case similar to the classical \( \alpha = \beta \) case; that is, the system evolves along \( \epsilon \approx 1 \) until it passes through an unstable stationary point at \( g_1 \sim 90^\circ \) and then evolves toward extremely small values of \( \epsilon \) (Fig. 5, solid line with open circles). The evolution is then quickly dominated by the gravitational radiation reaction near \( \epsilon_{\text{min}} \).

The evolution of \( \epsilon \) with \( a_1 \) (solid circles) for this second example is shown in Figure 6. The predicted \( \epsilon_{\text{min}} \) of the first Kozai cycle calculated with the method in § 3.2 at initial \( a_1 = a_{10} \) is shown as a cross. The expected evolution of \( \epsilon \) with \( a_1 \) by equation (38) starting from this predicted \( \epsilon_{\text{min}} \) and \( a_{10} \) is shown by the dashed line. It is clear that there is an excellent agreement between the predicted evolution of \( a_1 \epsilon_{\text{min}} \) and the numerical values. The same excellent agreement has been found for all such cases we have investigated.

The phase diagram for the evolution path of \( \epsilon \) versus \( g_1 \) for systems with the parameters of our second example but with different initial values \( I_0 \) is shown in Figure 5. It is apparent that as long as \( I_0 \) is away from the critical values \( I_c \), the \( \epsilon_{\text{min}} \) is reached around \( g_1 \sim 90^\circ \). Near the critical value, \( g_{\epsilon_m} \sim g_c \).

The evolutions of eccentricities \( e_1 \) as a function of GW frequencies \( f_{\text{GW}} \) for our first and second examples are shown in Figures 7 and 8. In both cases, the GW frequency spans a wide range of 8 orders of magnitude. This means that the

\[
\begin{align*}
\text{Fig. 4.—Secular evolution for } \epsilon \text{ of the inner binary and of } I, \text{ as computed by integrating the evolution eqs. (16)–(19), which include contributions from the Kozai mechanism, PN periastron precession, and gravitational radiation reaction. The initial system parameters are } m_0 = m_1 = m_2 = 5 \text{ } M_\odot, \text{ } e_1 = 0.01, \text{ } e_2 = 0.51, \text{ } a_1 = 2.1920 \text{ } \text{AU, } a_2/a_1 = 5, \text{ } g_1 = 0, \text{ } \text{and } I_0 = 99.2^\circ. \text{ This is a typical case in which the GR effect dominates in the evolution within one Kozai cycle.}
\end{align*}
\]
GWs from these sources can be good candidates for detections in the LISA band, as well as LIGO. For the case of many Kozai cycles in Figure 7, the eccentricity drops sharply from $e_0 \sim 1$ at $f_{GW} \sim 0.1$ Hz (the end of the LISA band), where the GR effect starts to dominate, and is nearly zero at 10 Hz (the beginning of the LIGO band). This is expected as equations (37) and (38) predict that the eccentricities drop with the frequency roughly proportional to $f_{GW}^{3/4}$. For extreme cases as shown in Figure 8, the eccentricity remains at the significant value of $\sim 0.9$ at $f_{GW} = 10$ Hz and could be extremely high at lower frequencies.

4. ECCENTRICITY DISTRIBUTION

The Kozai mechanism can drive the inner binaries of triple systems to extremely small $\epsilon$ and merge before disruption by interactions with field stars. The timescale for disruption (the same as the stellar encounter timescale) is given as (Miller & Hamilton 2002)

$$\tau_{enc} \approx 6 \times 10^5 n_0^{-1} \frac{AU}{a_2} \frac{10 M_\odot}{M_2} \text{ yr} ,$$

(39)

where the number density of stars in the GC is $n = 10^6 n_0$ parsec$^{-3}$. We assume $n_0 = 1$ in this paper. Successful mergers of this sort therefore require

$$\tau_{enc} < \tau_{evol} ,$$

(40)

$$\tau_{GR}(a_1, \epsilon_{min}) < \tau_{enc} .$$

(41)

Equation (40) ensures that the system has completed at least a half Kozai cycle to reach small $\epsilon$. Equation (41) ensures that the system can merge successfully before the next interruption. The term on the left-hand side of equation (41) represents the total lifetime of the system, which is roughly a factor of $1/e_{min}^{1/2}$, the merger timescale $\tau_{GR}$. This is because the system only spends a significant fraction of $e_{min}$'s time near $e_{min}$ where the GR effect is the strongest.

4.1. Distribution of Initial $\epsilon_{min}$ and $a_1$

We have explored the parameter space of $\epsilon_{min}$ and $a_1$ for successful mergers driven by the Kozai mechanism in GCs, restricting ourselves to a triple system consisting of three 10 $M_\odot$ BHs. We fix the initial parameters $g_1 = 0$ and $e_{10} = e_{20} = 0.01$, and we have chosen $a_2/a_1 = [20, 10, 5, 3]$ and assume a uniform distribution in the initial mutual inclination angles $I_0$ and in the semimajor axis $a_1$. The minimum initial value of $a_1$ was set to 0.2 AU, so that during its interaction with a third body, the recoil velocity associated with binary hardening is low enough for the system to remain in the cluster (Miller & Hamilton 2002). There was no presumed upper limit for $a_1$. For each given $a_1$, $I_0$, and $a_2/a_1$, we first calculated $\epsilon_{min}$ with equation (25) for each given $a_1$ and $I_0$. Equations (40) and (41) were then evaluated to determine the permitted parameter space.

We show in Figure 9 the permitted values for $a_1$ (upper set) and corresponding $\epsilon_{min}$ (lower set) versus $I_0$. There are four sets of cone-shaped distributions, each corresponding to a given $a_2/a_1$, with value decreasing from left to right. Each distribution centers on the critical value $I_0 \sim I_c$, which depends solely on $a_2/a_1$ (eq. [3]) as $e_0$ is fixed. There is a cutoff at a maximum $a_1$. This cutoff in $a_1$ is set by the constraint from equation (40). For a fixed $a_2/a_1$, larger $a_1$ implies larger $a_2$ and therefore shorter $\tau_{enc}$ (eq. [39]). Beyond the maximum $a_1$, the timescale $\tau_{enc}$ becomes so short that the system is disrupted before it can finish a half Kozai cycle to reach $\epsilon_{min}$ For the same reason, a larger $a_2/a_1$ leads to smaller cutoff in $a_1$.

The overall cone shape in $a_1$ and $\epsilon_{min}$ results from the fact that the merger timescale should be less than the encounter timescale (eq. [41]). Near the center of the cone where $I_0 \sim I_c$, the system can reach extremely small $\epsilon_{min}$ (eq. [8]), which allows maximum possible values of $a_1$ under the constraint of equation (41). The farther $I_0$ is away from the center ($I_c$), the larger $\epsilon_{min}$ is, and so the smaller are the $a_1$ values that satisfy equation (41).

The range of $\epsilon_{min}$ within each distribution is caused by the PN effect. In the classical limit (with no GR or PN effect), $\epsilon_{min}$ depends only on the values of $I_0$ and $a_2/a_1$. It is
independent of $a_1$ for each distribution, making $\epsilon_{\text{min}}$ have a unique value at each $I_0$. This is apparent for $\epsilon$ at $I_0$ far away from the center of each distribution for $a_2/a_1 = 3, 5$, and 10 in Figure 9. A significant contribution from the PN effect makes $\epsilon_{\text{min}}$ depend on $a_1$, giving $\epsilon_{\text{min}}$ a finite range (eqs. [5] and [8]). The PN effect is especially pronounced near the center of each cone ($I_0 \sim 50$), where the contribution from the classical Hamiltonian is relatively smaller (eq. [8]). The finite range of $\epsilon_{\text{min}}$ is more pronounced for cones with larger $a_2/a_1$ values as the PN effect is more pronounced at larger $a_2/a_1$. This also explains the overall large $\epsilon_{\text{min}}$ at larger $a_2/a_1$.

The $\epsilon_{\text{min}}$ values estimated including the GR effect can be several orders of magnitude larger than those without the GR effect in the region $I_0 \sim 50$, where the GR effect is the strongest. This makes little difference in finding the permitted parameter space for $a_1$, as $\epsilon_{\text{min}}$ will be extremely small in this region and equation (41) will be satisfied in either case. However, the evolution of GW frequency depends sensitively on our knowledge of $a_1\epsilon_{\text{min}}$. It is therefore necessary to include the GR effect.

4.2. Distribution of Eccentricities in the LIGO Band

We proceed to investigate the distribution of eccentricities when the emitted GW wave enters the LIGO band. We restrict attention to stellar mass BHs in triple systems and show only a representative case of triple systems consisting of three $10 M_\odot$ BHs. We consider a uniform distribution of initial mutual inclination angle $I_0$, initial eccentricities of the inner and outer binaries, ratios of the semimajor axis $a_2/a_1$, and initial $g_1$. A summary of the parameter space we have investigated and the representative values we have included can be found in Table 1.

We choose the upper limit of $a_2/a_1$ to be 30 based on equation (40), as the parameter range for $a_1$ diminishes with larger $a_2/a_1$ (see Fig. 9). A general upper limit for the ratio $a_2/a_1$ can also be set based on the requirement that $\epsilon_0 \geq \epsilon_{\text{min}}$ (e.g., Blaes et al. 2002, eq. [5]). The lower limit for $a_2/a_1$ is set to be 3 based on the fact that the triple system is stable only if (Mardling & Aarseth 2001)

$$ \frac{a_2}{a_1} > 2.8 \left[ \frac{(1 + m_2/\mathcal{M}_1) - 1 + e_2}{(1 - e_2)^{1/2}} \right]^{2/5}. \quad (42) $$

We follow the same procedure described in the previous subsection. We first calculate the permitted parameter space for $\epsilon_{\text{min}}$ and $a_1$. The results (not shown) are many versions of the same distributions of those shown in Figure 9 for various $a_2/a_1$. The span of permissible $I_0$ is roughly $90^\circ - 106^\circ$, as required to obtain extremely small $\epsilon_{\text{min}}$. We then calculate the expected eccentricity $e_I$ or its upper and lower limits at a given frequency following the numerical procedures described in §3.4, taking into account both the GR and PN effects.

The number distribution of eccentricities at $f_{\text{GW}} \sim 10$ Hz is shown as a histogram in the left-hand panel of Figure 10. The distribution is shown in percentage relative to the total number of systems that would merge before interruption by field stars. The gray histogram represents the distribution without considering the impact of the PN effect on the evolution of $a_2/a_1\epsilon_{\text{min}}$ (see discussions in §3.3). The black histogram includes the upper limit due to the impact of the PN effect. The $x$-axes of the histograms are shifted for a better view. It is apparent that the PN effect affects mostly the low-eccentricity systems and makes little difference in the histogram.

At $f_{\text{GW}} = 10$ Hz, around 70% of the merger systems have eccentricity less than 0.1, and a little more than 50% have eccentricities less than 0.05. About $\sim 2\%$ have eccentricities $e_I \sim 1$, most of which are those with initial $I_0$ very close to the critical angle $I_c$, as the GWs reach 10 Hz before the systems reach $\epsilon_{\text{min}}$ within the first Kozai cycle. These are the type of systems studied in Figures 4, 5, 6, and 8. We also show eccentricity distributions in the right-hand panel of Figure 10 for $f_{\text{GW}} = 40$ Hz (gray histogram) and $f_{\text{GW}} = 200$ Hz (black histogram). At 40 Hz, all eccentricities are well below 0.2. At 200 Hz, all are well below 0.02. This is not surprising as equations (37) and (38) predict that the eccentricities drop with the frequency roughly proportional to $f_{\text{GW}}^{-10/11}$. This is consistent with the fact that the GR effect circularizes the system in a very rapid fashion.

| Parameter | Value         |
|-----------|---------------|
| $M_{\text{BH}}$ | $10 M_\odot$  |
| $a_1$     | 0.2–30 AU     |
| $a_2/a_1$ | 3, 5, 10, 20, 30 |
| $e_0$     | 0.01–0.901    |
| $g_1$     | $0^\circ$–$90^\circ$ |
| $I_0$     | $85^\circ$–$110^\circ$ |
| $a_{\text{min}}$   | 1             |

Notes.—$M_{\text{BH}}$ refers to BH masses for all three components, $e_0$ refers to eccentricities for both inner and outer binaries. Numbers of steps for $a_1$, $e_0$, $g_1$, and $I_0$ are 60, 4, 4, and 100, respectively. See the text for definitions.
5. DISCUSSION

We have studied the evolution of eccentricities of BH mergers driven by the Kozai mechanism in triple systems residing within GCs. Eccentricity distributions at GW frequencies relevant to LIGO have been presented. The evolution of these systems was investigated including the Kozai mechanism, the post-Newtonian periastron precession effect, and gravitational radiation reaction.

We conclude that around 30\% of the systems possess eccentricities greater than 0.1 when the emitted GWs reach 10 Hz. Around 2\% of the systems are extremely eccentric at 10 Hz. However, almost all our merger systems possess eccentricities well below 0.2 at 40 Hz and below 0.02 at 200 Hz (see also a brief discussion in Miller 2002). These merger systems, on the other hand, promise extremely high eccentricities at the lower frequencies of the \textit{LISA} band (see Figs. 7 and 8).

About 30\% of our merger systems possess $e > 0.1$ at 10 Hz, the lower end of the advanced LIGO frequency band (Fritschel 2002\textsuperscript{1}). It is thus important to determine whether it is possible to use LIGO’s current circular-binary search templates to detect such systems. For highly eccentric systems, the higher harmonics probably need to be taken into account when optimizing the search templates. It is plausible that the eccentricities associated with these systems are not important at 200 Hz or perhaps even at 40 Hz. It is important, however, to determine the limit of $e_1$ below which circular-binary templates need to be replaced by new, eccentric-binary templates (Martel & Poisson 1999).

The eccentricity distribution was calculated for triple systems consisting of three 10 $M_\odot$ BHs. Similar eccentricity distribution, however, can be found for triple systems with individual masses in the range of 3–25 $M_\odot$ known for the observed Galactic stellar mass BHs (Bailyn et al. 1998). We have assumed that, in GCs, triple systems with one or more BHs of mass greater than 25 $M_\odot$ are rare. The mass parameters affect very little the values of $\epsilon_{\text{min}}$ a system can reach (eqs. [8], [5], and [24]). Its effect to the shape of eccentricity distribution is also very weak compared with other parameters such as $a_1$, $a_2/a_1$, and $e$ (eqs. [40] and [41]). Any effects due to different masses will be further averaged out if we include a uniform distribution of masses within this range.

Our calculation of the gravitational radiation reaction is based on the Newtonian quadrupole approximation. At 10 Hz, this approximation is still valid as $v/c \sim 0.1$ at the periastron. However, at higher frequencies, especially near 200 Hz in the LIGO frequency band, this approximation breaks down as the speed of the binary orbit is approaching that of light. However, all our binaries become so circular well before they reach 200 Hz that our results are probably still relevant.

\textsuperscript{1} Available at http://www.ligo.caltech.edu/docs/P/P020016-00.pdf.
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REFERENCES

Bailyn, C. D., Jain, R. K., Coppi, P., & Orosz, J. A. 1998, ApJ, 499, 367
Blaes, O., Lee, M. H., & Socrates, A. 2002, ApJ, 578, 775
Buonanno, A., Chen, Y., & Vallisneri, M. 2003, Phys. Rev. D, 67, 24016
Ford, E. B., Kozinsky, B., & Rasio, F. A. 2000, ApJ, 535, 385
Fritschel, P. 2002, Second Generation Instruments for the Laser Interferometer Gravitational Wave Observatory (LIGO)
Glampedakis, K., Hughes, S., & Kennefick, D. 2002, Phys. Rev. D, 66, 64005
Gopakumar, A., & Iyer, B. R. 2002, Phys. Rev. D, 65, 084011
Innanen, K. A., Zheng, J. Q., Mikkola, S., & Valtonen, M. J. 1997, AJ, 113, 1915
Kozai, Y. 1962, AJ, 67, 591
Lidov, M. L., & Ziglin, S. L. 1976, Celest. Mech., 13, 471
Lincoln, C. W., & Will, C. M. 1990, Phys. Rev. D, 42, 1123
Mardling, R. A., & Aarseth, S. J. 2001, MNRAS, 321, 398
Martel, K., & Poisson, E. 1999, Phys. Rev. D, 60, 124008
Miller, C. 2002, ApJ, 581, 438
Miller, C., & Hamilton, D. P. 2002, ApJ, 576, 894
Moreno-Garrido, C., Mediavilla, E., & Buitrago, J. 1994, MNRAS, 266, 16
———. 1995, MNRAS, 274, 115
Nelder, J. A., & Mead, R. 1965, Comput. J., 7, 308
Peters, P. C. 1964, Phys. Rev. B, 136, 1224
Peters, P. C., & Mathews, J. 1963, Phys. Rev., 131, 435
Portegies Zwart, S. F., & McMillan, S. L. W. 2000, ApJ, 528, L17
Scalo, J. M. 1986, Fundam. Cosmic Phys., 11, 1
Thorne, K. 1987, in 300 Years of Gravitation, ed. S. W. Hawking & W. Israel (Cambridge: Cambridge Univ. Press), 697