Generalized Universal Covers of Uniform Spaces

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Abstract
We develop a generalized covering space theory for a class of uniform spaces called coverable spaces. Coverable spaces include all geodesic metric spaces, connected and locally pathwise connected compact topological spaces, in particular Peano continua, as well as more pathological spaces like the topologist’s sine curve. Each coverable space has a generalized universal cover with universal and lifting properties. Associated with this generalized universal cover is a functorial uniform space invariant called the deck group, which is related to the classical fundamental group by a natural homomorphism. We obtain some specific results for one-dimensional spaces.

Keywords: universal cover, uniform space, geodesic space, fundamental group
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1 Introduction
In this paper we construct a generalized universal cover for a very large class of uniform spaces called coverable spaces, which includes all geodesic metric spaces (Corollary 99), connected and locally pathwise connected compact topological spaces (Corollary 74), and in particular Peano continua (Corollary 97). Coverable spaces also include some more pathological spaces like the topologist’s
sine curve (see below) and totally disconnected spaces (1). Associated with this generalized universal cover is a functorial uniform space invariant called the deck group, which is related to the classical fundamental group by a natural homomorphism.

Three types of uniform spaces are of greatest importance: topological groups, compact topological spaces and metric spaces. Topological groups were considered in 1 and 2; the relationship to the present paper may be found in Section 5 along with a correction to 1. For compact topological spaces, which have a unique uniform structure compatible with the topology, the deck group is a topological invariant that coincides with the fundamental group in the setting of what we will call Poincaré spaces (i.e., connected, locally pathwise connected, semilocally simply connected spaces). The special case of metric spaces is of particular interest. In addition to more classical examples such as the Hawaiian earring and related spaces (11, 17, 8, 6, 3), generalized universal covers and fundamental groups have recently been studied in connection with Gromov-Hausdorff limits of Riemannian manifolds (23 and 24). Such limits are always geodesic spaces, and hence coverable. A recent example of Menguy shows that limits of Riemannian manifolds with positive Ricci curvature can have bad local topology (16) – precisely the sort of metric spaces at which the present work is aimed. We will consider metric spaces in more detail in an upcoming paper.

One of the main impediments to generalizing the classical construction of the universal cover is the traditional definition of covering map, the most important property of which is the ability to lift curves and homotopies. However, this lifting property is traditionally gained at the expense of requiring that a space and its cover be locally homeomorphic in a fairly strong way, and consequently traditional universal (or simply connected) covers exist only for Poincaré spaces. Earlier work concerning systems coverings of uniform spaces (cf. 15, 22) was also limited by considering only traditional covers, and therefore a universal object is impossible to obtain even for basic examples such as the countable product of circles or the Hawaiian earring.

One may take a hint from topological groups for how to proceed. In this category, a traditional covering map is a quotient homomorphism with discrete kernel. The action by the kernel is not only properly discontinuous, it is uniformly so. Moreover, as we showed in 1, one can exploit this more uniform kind of action to define generalized covers as quotients with central kernels that are (complete and) prodiscrete, i.e. are inverse limits of discrete groups. These generalized covers have the lifting properties of traditional covers, but are not in general local homeomorphisms. This allows one to abandon assumptions concerning (semi-)local simple connectivity. The kernel of the generalized universal cover is a kind of generalized fundamental group.

The key to generalizing the results of 1 is to build a group action right into the definition of generalized cover. Actually doing so is, unfortunately, somewhat technical, and this was carried out by the second author in 20. The basic idea is to consider a kind of “uniformly” properly discontinuous action called a discrete action on a uniform space. Inverse systems of discrete actions
define *prodiscrete actions* that generalize the notion of the action on a topological group via a prodiscrete, central subgroup. The action by a subgroup also preserves the uniform structure (as long as one matches the left or right action to the left or right uniformity); this property is generalized by something called an “isomorphic” action on a uniform structure, which also broadens the notion of an isometric action on a metric space. A generalized *cover* of uniform spaces is defined in \[20\] to be a quotient via a prodiscrete, isomorphic action. At the beginning of \[20\] is a review of basic definitions and properties of uniform spaces that are used for the construction; we will use the same notation in this paper.

What follows is a sketch of our construction and main results. All of these constructions involve a choice of basepoint(s), but basepoint choice has an impact in this setting similar to that in the traditional setting for pathwise connected spaces, and for simplicity we will save detailed discussion of basepoints for the body of the paper. For now, all functions are simply assumed to preserve some chosen basepoints (this is particularly important for the uniqueness statements, which are only true up to choice of basepoint).

For each uniform space \(X\) there is an inverse system \((X_E, \phi_{EF})\) indexed on the collection of entourages of \(X\) partially ordered by reverse inclusion, called the *fundamental inverse system* of \(X\). \(X_E\) consists of equivalence classes of finite \(E\)-chains starting at a fixed basepoint. An \(E\)-*chain* between points \(p, q \in X\) is an ordered set of points \(x_0 := p, \ldots, x_n := q\) such that for all \(i, (x_i, x_{i+1}) \in E\). An \(E\)-*loop* is an \(E\)-chain that starts and ends at the same point. Two \(E\)-chains from \(p\) to \(q\) are equivalent if one can be obtained from the other through finitely many steps, each of which involves removing or adding a point, always leaving the endpoints fixed and keeping an \(E\)-chain at each stage. The collection of all equivalence classes of \(E\)-loops at the basepoint forms a group \(\delta_E\) with respect to concatenation. This group is finitely generated when \(X\) is compact (Theorem \[35\]). When \(X_E\) is provided with a natural “lifted” uniformity, \(\delta_E\) acts discretely on \(X_E\) by concatenating an \(E\)-loop at the beginning of an \(E\)-chain. In a sense \(\delta_E\) detects “holes” in \(X\) that are, roughly speaking, “larger than \(E\)”. The restriction \(\theta_{EF}\) of the bonding map \(\phi_{EF}\) to \(\delta_E\) preserves concatenation and produces another inverse system \((\delta_E, \theta_{EF})\) of groups and homomorphisms; in fact these two systems form an inverse system of actions as defined in \[20\]. The inverse limit of the group system consists of a group \(\delta_1(X) := \varprojlim \delta_E\), called the *deck group* of \(X\), which acts prodiscretely and isomorphically on \(\tilde{X} := \varprojlim X_E\). Observe that \(\delta_1(X)\) is actually a (prodiscrete) topological group and the homomorphisms induced on the deck group are continuous homomorphisms. However, we do not know of examples of spaces having deck groups that are abstractly, but not continuously, isomorphic.

In general the projection \(\phi : \tilde{X} \to X\) may not be surjective, and hence not a cover. In fact \(\tilde{X}\) may be only a single point even when \(X\) is not. The next definition, which is central to the paper, deals with this issue.

**Definition 1** Let \(X\) be a uniform space. An entourage \(E\) such that the projection \(\phi_E : \tilde{X} \to X_E\) is surjective is called a covering entourage. A uniform
space \( X \) is called coverable if there is a uniformity base of covering entourages including \( X \times X \). The collection of all such entourages is called the covering base \( C(X) \).

If \( X \) is coverable then each \( \phi_E \) is a covering map (in the sense of \([20]\)) and we refer to the projection \( \phi : \tilde{X} \to X \) as the universal cover of \( X \). The next theorem combines results of Theorems \([61, 62]\).

**Theorem 2 (Induced Mapping)** Let \( X, Y \) be coverable spaces with universal covers \( \phi : \tilde{X} \to X \) and \( \psi : \tilde{Y} \to Y \). If \( f : X \to Y \) is uniformly continuous then there is a unique uniformly continuous function \( \tilde{f} : \tilde{X} \to \tilde{Y} \) such that \( f \circ \phi = \psi \circ \tilde{f} \). Moreover,

1. If \( f \) is a cover then \( \tilde{f} \) is a uniform homeomorphism.
2. If \( Z \) is a coverable space and \( g : Y \to Z \) is uniformly continuous then \( \tilde{g} \circ f = \tilde{g} \circ \tilde{f} \).

The restriction of \( \tilde{f} \) to \( \delta_1(X) \) in the above theorem is a homomorphism \( f_* : \delta_1(X) \to \delta_1(Y) \) (Theorem \([61]\)). Therefore the deck group is a functorial invariant of uniform structures. In the case of compact spaces the deck group is a topological invariant; if \( X \) is a compact Poincaré space then the deck group is naturally isomorphic to the fundamental group (Corollary \([20]\)). For non-compact spaces the deck group need not be a topological invariant; for example the surface of revolution \( S \) obtained by rotating the graph of \( e^x \) about the \( x \)-axis has trivial deck group even though it is homeomorphic to a standard cylinder—the deck group of which is \( \mathbb{Z} \). The problem is that a generator of \( \pi_1(S) \) may be represented by a path that extends down the cusp arbitrarily far, wraps around the small cusp, and then travels back to the basepoint. This generator will therefore not be detected by any of the groups \( \delta_E \), where \( E \) is the metric entourage of size \( \varepsilon \). The deck group does indicate that the standard cylinder and \( S \) are not uniformly homeomorphic and that \( S \) is not uniformly semilocally simply connected. The next theorem follows from Corollary \([59]\) and Theorem \([62]\).

**Theorem 3 (Universal Property)** If \( X \) and \( Y \) are coverable and \( f : X \to Y \) is a cover then there is a unique cover \( f_B : \tilde{Y} \to X \) such that \( f \circ f_B = \phi \), where \( \phi \) is the universal cover of \( Y \).

With regard to the universal property it should be pointed out that, in contrast to the situation for coverable topological groups, we do not know whether the composition of covers between coverable spaces (or uniform spaces in general) is a cover. (The situation for topological groups is significantly simpler because the deck group is actually a central subgroup of the universal cover and covers are simply quotient homomorphisms, making their composition easier to understand.) Recall that the composition of traditional covering maps between connected topological spaces need not be a traditional cover \([18]\). We resolve
define a category whose objects are covers \( p : Y \to X \) between coverable uniform spaces and whose morphisms are commutative diagrams

\[
\begin{array}{c}
X_1 \\ \searrow^{p_1} \\
\downarrow \phantom{f} \downarrow \phantom{p_2} \\
\downarrow f \phantom{\searrow X_2} \\
X_2 \\
\end{array}
\]

where \( p_1 \) and \( p_2 \) are covers and \( f \) is uniformly continuous. It is an immediate consequence of the preceding theorem that the universal cover \( \phi : \tilde{X} \to X \) is a universal object in this category.

**Definition 4** A uniform space \( X \) is called universal if there is a base \( U \) for the uniformity such that for any \( E \in U \), \( \phi_{XE} : X_E \to X \) is a uniform homeomorphism. The collection of all such uniformities is called the universal base of \( X \) (which always contains \( X \times X \)).

If \( X \) is coverable then \( \tilde{X} \) is universal (Theorem 49) and every universal space is coverable (Corollary 50). Moreover, a coverable space \( Y \) is universal if and only if \( \delta_1(X) \) is trivial, or equivalently every \( E \)-loop is \( E \)-homotopic to the trivial loop for entourages \( E \) in a particular basis (Corollary 50), a condition that is reminiscent of simply connected. The next theorem follows from Theorem 57.

**Theorem 5** (Lifting) Let \( X \) be universal, \( Y \) be coverable and \( f : X \to Y \) be uniformly continuous. Then there exists a unique uniformly continuous function \( f_L : X \to \tilde{Y} \) such that \( f = \phi \circ f_L \), where \( \phi \) is the universal cover of \( Y \).

Any connected, uniformly locally pathwise connected (see Definition 64), simply connected uniform space is universal—in particular any compact, connected, locally pathwise connected, simply connected topological space is universal (Theorem 60 and Corollary 61). Therefore we may apply the Lifting Theorem to paths and homotopies of paths. This allows one to define a natural mapping \( \lambda : \pi_1(X) \to \delta_1(X) \), for any coverable space \( X \), by lifting a loop that represents an element of the fundamental group and taking the deck transformation that takes the basepoint to the endpoint of the loop. Even though the action is not discrete this mapping is well defined, and a homomorphism.

There is a satisfying relationship between the two most basic algebraic properties of the map \( \lambda \) and topological properties of \( \tilde{X} \) when \( X \) is pathwise connected: (1) \( \lambda \) is injective if and only if \( \tilde{X} \) is simply connected (Proposition 78), and (2) \( \lambda \) is surjective if and only if \( \tilde{X} \) is pathwise connected. More precisely, for arbitrary coverable \( X \), the image of \( \lambda \) is the stabilizer in the deck group \( \delta_1(X) \) of the pathwise connected component of \( \tilde{X} \) that contains the basepoint (Theorem 77). Hence if \( \tilde{X} \) is both pathwise connected and simply connected then the deck group and the fundamental group are isomorphic. This is true as mentioned earlier when \( X \) is a compact Poincaré space—and also when \( X \) is a locally compact, pathwise connected topological group (cf. 2). If \( X \) is connected and uniformly locally pathwise connected then the pathwise connected
component of $\tilde{X}$ is dense in $\tilde{X}$ (Proposition 79) and therefore $\tilde{X}$ is connected. In this case it is also true that $\lambda(\pi_1(X))$ is dense in $\delta_1(X)$. Note that, in the case of the surface $S$ described above, $S$ is itself universal. $S$ is pathwise connected but not simply connected and correspondingly $\lambda : Z = \pi_1(X) \to \delta_1(X) = 0$ is surjective but not injective.

As another example, consider the (closed) topologist’s sine curve $T$, illustrated in Figure 1 with its universal cover. The deck group of $T$ is $Z$; the action shifts the universal cover in a way similar to the action of $Z$ on $\mathbb{R}$. Note that arbitrarily fine chains may wrap around $T$, while no path does. Hence the deck group indicates topology where the fundamental group, which is trivial in this case, does not. Finally, $\tilde{T}$ is simply connected but not pathwise connected, and correspondingly $\lambda : 0 = \pi_1(T) \to \delta_1(T) = Z$ is injective but not surjective (and the image of $\lambda$ need not be dense in the deck group because $T$ is not locally pathwise connected).

We show that if $X$ is coverable with uniform dimension $\leq n$ in a sense due to Isbell ([12]) then $X$ has the same property (Theorem 88), and we conjecture that the the uniform dimensions of $X$ and $\tilde{X}$ are the same. When $X$ has dimension 1, we show that $\tilde{X}$ is simply connected and contains no topological circles. As a consequence the function $\lambda$ is injective.

We prove some results concerning dimension and coverable spaces, although it seems much more may be said. Further progress may require additional work extending theorems concerning covering dimension to theorems about uniform dimension. However, this partial result is particularly useful when $n = 1$ because then $\tilde{X}$ is forced to be at most one dimensional—and if $X$ contains any non-trivial curve then the uniform dimension of $X$ must be exactly 1.

One-dimensional metric spaces are of interest since they include, for ex-
ample, planar fractals and metrized Cayley graphs, as well as many familiar pathological examples. It is known that the fundamental group of a compact, connected, 1-dimensional metrizable space embeds in an inverse limit of free groups. Given this fact, it seems very likely that for appropriate choice of $E$, the groups $\delta_E(X)$ are finitely generated free groups when $X$ is compact, connected, and 1-dimensional. We have verified this in some special cases (see Section 7). It would be very interesting to know whether for suitable choice of $E$, $\delta_E(X)$ is free when $X$ is coverable (or just connected?) of uniform dimension 1.

In this paper we show that, if $X$ is coverable of uniform dimension $\leq 1$ then $\tilde{X}$ has no topological circles (Proposition 89). It follows that $\tilde{X}$ is simply connected and hence (assuming $X$ is pathwise connected) $\lambda : \pi_1(X) \to \delta_1(X)$ is an embedding. In general, $\tilde{X}$ may not be pathwise connected and hence $\lambda$ may not be surjective, but at least in the case when $X$ is uniformly locally pathwise connected we know that the (isomorphic) image of the fundamental group is dense in the deck group. In other words, $X$ is a quotient via a free action of its fundamental group on a connected, simply connected, uniformly one dimensional space.

We conclude with a general result about metric spaces. Given a metric space $X$, $\tilde{X}$ is metrizable (Proposition 27), and if $X$ is coverable then by definition of cover the action of $\delta_1(X)$ on $\tilde{X}$ is isomorphic in the sense that there is a uniformity base for $\tilde{X}$ that is invariant with respect to $\delta_1(X)$. It is natural to ask whether there is a metric on $\tilde{X}$ with respect to which the action of $\delta_1(X)$ is isometric and such that the metric on $\tilde{X}$ is the quotient metric. (Since $\delta_1(X)$ is complete and acts prodiscretely, the orbits of $\delta_1(X)$ are closed and hence if $\tilde{X}$ has an invariant metric there is a well defined quotient metric on $X$, namely the distance between the corresponding orbits in $\tilde{X}$—see [21], 4.4 for details on quotient metrics.) We do not know the answer to this question in general. However, if one examines the proof in [4] that a uniform space with a countable base possesses a compatible pseudometric, it is clear that the explicitly constructed pseudometric is invariant with respect to the action of a group $G$ if the entourages used in the construction are invariant. In other words, if $X$ is a coverable metric space then we may put a metric on $\tilde{X}$ that is invariant with respect to the action of $\delta_1(X)$. Although the quotient metric on $X$ may not be the original, it is still uniformly equivalent to the original. This proves:

**Theorem 6** If $X$ is a coverable metric space then $X$ is uniformly homeomorphic to the metric quotient $Y/G$ of a universal metric space $Y$ with respect to the isometric prodiscrete action of a group $G$ isomorphic to $\delta_1(X)$.
2 The Fundamental Inverse System

Note that for any entourage \( E \),

\[
E^n = \{(x_0, x_n) \in X \times X : \text{for some } x_1, ..., x_{n-1}, (x_i, x_{i+1}) \in E \text{ for } 0 \leq i < n\}.
\]

(1)

That is, \( E^n \) is the set of all pairs of points joined by an \( E \)-chain of length \( n \).

**Definition 7** A uniform space \( X \) is called chain connected (sometimes called uniformly connected) if every uniformly continuous function from \( X \) into a discrete uniform space is constant.

**Proposition 8** The following are equivalent:

1. \( X \) is chain connected
2. For any entourage \( E \), \( X \times X = \bigcup_{n=1}^{\infty} E^n \).
3. For any entourage \( E \), every pair of points in \( X \) is joined by an \( E \)-chain.
4. For any entourage \( E \) and \( x \in X \), \( X = \bigcup_{n=1}^{\infty} B(x, E^n) \).

**Proof.** The equivalence of the first two conditions is proved in [13], 9.34–the statement of 9.34 is wrong but the proof is right! The last three are equivalent by Formula (1). \( \square \)

**Corollary 9** If \( X \) is chain connected and \( f : X \to Y \) is a uniformly continuous surjection then \( Y \) is chain connected.

For a topological group \( G \), chain connected is equivalent to \( G \) being locally generated (generated by every neighborhood of the identity), and \( E \)-chains are the same as what we called \( U \)-chains in [1]. If one fixes a single point \( p \) and entourage \( E \) in an arbitrary uniform space \( X \) then if \( J_p \subset X \) is the set of all points that can be joined to \( p \) by a \( E \)-chain, it is easy to check that \( J_p \) is both open and closed. Therefore every connected uniform space is chain connected. On the other hand, the rational numbers are chain connected and totally disconnected (see [1] for related topics).

**Lemma 10** Let \( \{X_\alpha, \phi_{\alpha\beta}\}_{\alpha \in \Lambda} \) be an inverse system of sets such that each of the projections \( \phi_\alpha : \lim \leftarrow X_\alpha \to X_\alpha \) is surjective. Then for any subsets \( E, F \) of \( X_\alpha \times X_\alpha \), \( \phi_\alpha^{-1}(EF) = \phi_\alpha^{-1}(E)\phi_\alpha^{-1}(F) \). In particular for any \( n \), \( \phi_\alpha^{-1}(E^n) = (\phi_\alpha^{-1}(E))^n \).

**Proof.** We have \((x_\beta), (y_\beta)) \in \phi_\alpha^{-1}(EF)\) if and only if there exists some \( z_\alpha \in X_\alpha \) such that \((x_\alpha, z_\alpha) \in E \) and \((z_\alpha, y_\alpha) \in F \). Since \( \phi_\alpha \) is surjective this is equivalent to: for some \((z_\beta) \in X = \lim \leftarrow X_\beta\), \((x_\beta), (z_\beta)) \in \phi_\alpha^{-1}(E) \) and \((z_\beta), (y_\beta) \in \phi_\alpha^{-1}(F) \). But this is equivalent to \((x_\beta), (y_\beta)) \in \phi_\alpha^{-1}(E)\phi_\alpha^{-1}(F) \). \( \square \)
Lemma 11 Let \( \{X_\alpha, \phi_{\alpha\beta}\}_{\alpha \in \Delta} \) be an inverse system of chain connected uniform spaces. If each of the projections \( \phi_\alpha \) is surjective then \( X = \lim_{\leftarrow} X_\alpha \) is chain connected.

Proof. For any entourage \( \phi_\alpha^{-1}(E) \) in \( X \) and \( ((x_\beta), (y_\beta)) \in X \) we have \( (x_\alpha, y_\alpha) \in E^n \) for some \( n \). Then \( ((x_\beta), (y_\beta)) \in \phi_\alpha^{-1}((x_\alpha, y_\alpha)) \subset \phi_\alpha^{-1}(E^n) = (\phi_\alpha^{-1}(E))^n \).

An \( E \)-extension of an \( E \)-chain \( \{x_0, \ldots, x_n\} \) is an \( E \)-chain \( \{x_0, \ldots, x_i, x', x_{i+1}, \ldots, x_n\} \), where \( 0 \leq i < n \). Two \( E \)-chains from \( x_0 \) to \( x_n \) are said to be \( E \)-related if one is a \( E \)-extension of the other. An \( E \)-homotopy between \( E \)-chains \( \gamma_0 \) and \( \gamma_m \) is a sequence \( \{\gamma_0, \ldots, \gamma_m\} \) of \( E \)-chains such that \( \gamma_i \) is \( E \)-related to \( \gamma_{i-1} \) for all \( 1 \leq i \leq m \). The number \( m \) is called the length of the homotopy. We denote the \( E \)-homotopy class of an \( E \)-chain \( \gamma \) by \( [\gamma]_E \). Now fix a basepoint \( * \in X \). Let \( X_E \) denote the set of all \( E \)-homotopy classes of \( E \)-chains in \( X \) starting at \( * \) and define a function (the “endpoint map”) \( \phi_{XE} : X_E \rightarrow X \) by \( \phi_{XE}([* = x_0, \ldots, x_n]) = x_n \). The proof of the next lemma is immediate.

Lemma 12 If \( X \) is a uniform space and \( E \) is an entourage then \( \phi_{XE} \) is surjective if and only if every pair in \( X \) is joined by an \( E \)-chain; in particular \( \phi_{XE} \) is surjective if \( X \) is chain connected.

The next lemma will often be used without reference. The proof is straightforward but tedious.

Lemma 13 Let \( \alpha := \{* = a, x_1, \ldots, x_{n-1}, b\} \) and \( \beta := \{* = a, y_1, \ldots, y_{m-1}, b\} \) be \( E \)-chains for some entourage \( E \) in a uniform space \( X \). Then \( \alpha \) and \( \beta \) are \( E \)-homotopic if and only if the \( E \)-loop \( \alpha \ast \beta^{-1} \) is \( E \)-homotopic to the trivial chain \( \{\ast\} \).

Definition 14 Let \( X \) be a uniform space with entourage \( E \). For any entourage \( D \subset E \), define \( D^\ast \) as follows: let \( ([\alpha]_E, [\beta]_E) \in D^\ast \) if and only if
\[
([\alpha]_E, [\beta]_E) = ([* = x_0, \ldots, x_n, y]_E, [* = x_0, \ldots, x_n, z]_E) \quad \text{with} \quad (y, z) \in D.
\]

Lemma 15 Let \( X \) be a uniform space with entourage \( E \). For any entourages \( D, F \subset E \), \( ([\alpha]_E, [\beta]_E) \in D^\ast F^\ast \) if and only if for some \( [\gamma]_E = [* = x_0, \ldots, x_n]_E \) we have
\[
[\alpha]_E = [* = x_0, \ldots, x_n, y]_E \quad \text{and} \quad [\beta]_E = [* = x_0, \ldots, x_n, z]_E
\]
for some \( y, z \) such that \( (y, x_n) \in D \) and \( (z, x_n) \in F \). In particular, \( [\alpha]_E, [\beta]_E \in B([\gamma]_E, (D^\ast)^2) \) if and only if these conditions hold with \( D = F \).

Proof. The reverse implication is obvious. Conversely, if \( ([\alpha]_E, [\beta]_E) \in D^\ast F^\ast \) then we may take \( \alpha = \{* = x_0, \ldots, x_{n-1}, y\}, \gamma = \{* = x_0, \ldots, x_n\}, \gamma' = \{* = y_0, \ldots, y_m, x_n\} \) and \( \beta = \{* = y_0, \ldots, y_m, z\} \) where \( \gamma' \) is \( E \)-homotopic to \( \gamma \) and \( (z, x_n) \in F \) and \( (y, x_n) \in D \). Now \( \beta \) is \( E \)-related to \( \{* = y_0, \ldots, y_m, x_n, z\} \) and the latter is \( E \)-homotopic to \( \{* = x_0, \ldots, x_n, z\} \) using the \( E \)-homotopy from \( \gamma \) to \( \gamma' \). Finally, \( \alpha \) is \( E \)-related to \( \{* = x_0, \ldots, x_n, y\} \). ■
Proposition 16 Let $X$ be a chain connected uniform space with entourage $E$, and $\phi_{XE} : X_E \to X$ be the endpoint map. Then

1. $\phi_{XE}$ is injective when restricted to any $E^*$-ball.

2. For any entourage $D \subset E$ and $[\alpha]_E \in X_E$,

   $\phi_{XE}(B([\alpha]_E, D^*)) = B(\phi_{XE}([\alpha]_E), D)$

   and $\phi_{XE}(D^*) = D$.

3. The collection of all $D^*$ such that $D \subset E$ is a base for a uniformity of $X_E$.

4. $\phi_{XE}$ is a bi-uniformly continuous surjection with respect to this uniformity. In particular the restriction of $\phi_{XE}$ to any $D^*$-ball is a uniform homeomorphism onto the corresponding $D$-ball.

Proof. If $[\alpha]_E$ and $[\beta]_E$ are in $B([\gamma]_E, E^*)$ and $\phi_{XE}([\alpha]_E) = \phi_{XE}([\beta]_E)$ then by definition of $E^*$, $\alpha$ and $\beta$ are $E$-homotopic. This proves the first part.

For Part (2) let $\alpha = \{* = x_0, ..., x_n, x\}$; it is obvious from the definition of $\phi_{XE}$ and $D^*$ that

$\phi_{XE}(B([\alpha]_E, D^*)) \subset B(\phi_{XE}([\alpha]_E), D) = B(x, D)$.

Now suppose $(x, y) \in D$ and let $\beta = \{* = x_0, ..., x_n, x, y\}$. Since $\alpha$ is $E$-homotopic to $\{* = x_0, ..., x_n, x, x\}$, we have that $(\alpha, \beta) \subset D^*$ and

$\phi_{XE}((\alpha, \beta)) = (x, y)$.

This implies that $\phi_{XE}(D^*) = D$.

We will now check the conditions for a uniformity base. Clearly, for entourages $E$ and $F$ we have $(E \cap F)^* \subset E^* \cap F^*$ (and in fact they are equal but it is not necessary to prove this). Since $D$ is symmetric, so is $D^*$. Next, let $F$ be an entourage such that $F^2 \subset D$. Suppose that $([\alpha]_E, [\beta]_E) \in (F^*)^2$. Then for some $[\gamma]_E, [\alpha]_E, [\beta]_E \subset B([\gamma]_E, F^*)$. Applying Lemma [15] we may write $[\alpha]_E = [* = x_0, ..., x_n, y]_E$ and $[\beta]_E = [* = x_0, ..., x_n, z]_E$ with $(x_n, z), (x_n, y) \in F$ and hence $(y, z) \in F^2 \subset D$. By definition $([\alpha]_E, [\beta]_E) \in D^*$. Recall from [20] that $\phi_{XE}$ is bi-uniformly continuous by definition if the image or inverse image of any entourage with respect to $\phi_{XE}$ is again an entourage. Part (4) is now an immediate consequence of Parts (2) and (3). □

We can now see precisely how the choice of basepoints affect things.

Definition 17 Let $\beta = \{x_0, ..., x_n\}$ and $\alpha = \{y_0 = x_n, ..., y_m\}$ be $E$-chains for some entourage $E$ in a uniform space $X$. Define the concatenation of $\alpha$ to $\beta$ by

$\beta \ast \alpha := \{x_0, ..., x_n = y_0, ..., y_m\}$. 

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Remark 18 Let $X$ be a chain connected uniform space, $p_1, p_2 \in X$, and $E$ be an entourage. Let $X_E$ denote the space of $E$-homotopy classes of $E$-chains based at $p_1$. Let $\gamma$ be any $E$-chain from $p_2$ to $p_1$. There is a natural map from $X_E^1$ to $X_E^2$ defined by taking $[\alpha]_E$ to $[\gamma \ast \alpha]_E$, where $\gamma \ast \alpha$ is the $E$-chain obtained by concatenating $\alpha$ to the end of $\gamma$. It is easy to check that this function is a bijection with inverse $\eta \mapsto \gamma^{-1} \ast \eta$. For any entourage $F$ in $X$, there are corresponding entourages $F_1^* \subset X_E^1$ and $F_2^* \subset X_E^2$. Now $(\alpha, \beta) \in F_1^*$ if and only if $(\gamma \ast \alpha, \gamma \ast \beta) \in F_2^*$ and in particular, $X_E^1$ and $X_E^2$ are uniformly homeomorphic. Therefore, as in the case of traditional covering space theory, the choice of basepoint $*$ plays only a minor and predictable role.

Notation 19 Given a basepoint $*$ in $X$, we will always take $[\ast]_E$ for the basepoint in $X_E$. In general, for any function $f : X \to Y$ we will always suppose that $f$ is “pointed” in the sense that $f(*) = *$. For each chain $\gamma = \{x_0, ..., x_n\}$ in $X$ we denote the chain $\{f(x_0), ..., f(x_n)\}$ by $f(\gamma)$, with similar notation for any finite sequence of chains. We will always take the uniform structure on $X_E$ to be the one given by Proposition 16.

If $E, F$ are entourages of $X, Y$, respectively, such that $f(E) \subset F$ and $\gamma$ is an $E$-chain then $f(\gamma)$ is an $F$-chain in $Y$. Note that if $\eta$ is an $E$-homotopy between $E$-chains $\alpha$ and $\gamma$ then $f(\eta)$ is an $F$-homotopy between $f(\alpha)$ and $f(\gamma)$. In particular the function in the following definition is well defined (and pointed).

Definition 20 Given $f : X \to Y$ is a function between uniform spaces such that for entourages $E, F$ of $X, Y$, respectively, $f(E) \subset F$, we define $f_{EF} : X_E \to Y_F$ by $f([\gamma]_E) = [f(\gamma)]_F$.

Definition 21 For any entourages $D \subset E$ in a uniform space $X$, define $\phi_{ED} : X_D \to X_E$ by $\phi_{ED}([\alpha]_D) = [\alpha]_E$.

Note that by definition $\phi_{ED} = I_{DE}$, where $I : X \to X$ is the identity, and

$$\phi_{XE} \circ \phi_{ED} = \phi_{XD}.$$

Lemma 22 For any entourages $D \subset E$ in a uniform space $X$, $\phi_{ED}$ is uniformly continuous.

Proof. Let $F^*$ be an entourage in $X_E$ and $G \subset F \cap D$ be an entourage in $X$. We will need to distinguish between $G^* \subset X_D \times X_D$, which we will refer to as $G_D^*$ and $G^* \subset X_E \times X_E$, which we will refer to as $G_E^*$. if $([\alpha]_D, [\beta]_D) \in G_D^*$ then by definition we can take $\alpha = \{\ast = x_0, ..., x_n, y\}$ and $\beta = \{\ast = x_0, ..., x_n, z\}$ with $(y, z) \in G$. Now

$$\phi_{ED}([\alpha]_D, [\beta]_D)) = ([\alpha]_E, [\beta]_E)$$

and by definition $([\alpha]_E, [\beta]_E) \in G_E^*$. That is, $\phi_{ED}(G_D^*) \subset G_E^* \subset F^*$. ■
Proposition 23  Let $X$ be a uniform space and $D \subset E$ be entourages. Then

$$\phi_{D^* D} := (\phi_{XE})_{D^* D} : (X_E)_{D^*} \to X_D$$

is a uniform homeomorphism such that the following diagram commutes:

\[
\begin{array}{ccc}
(X_E)_{D^*} & \xrightarrow{\phi_{XE}_{D^*}} & X_E \\
\downarrow_{\phi_{D^* D}} & & \downarrow_{\phi_{XE}} \\
X_D & \xrightarrow{\phi_{XE}} & X
\end{array}
\]

Moreover, for any entourage $F \subset D$ in $X$, we have $\phi_{D^* D}((F^*)^*) = F^*$.

Proof. For simplicity denote $(\phi_{XE})_{D^* D}$ by $\phi_{D^* D}$. Let $C = \{[*]_E = [\alpha_0]_E, \ldots, [\alpha_n]_E\}$ be a $D^*$-chain in $X_E$, where each $\alpha_i$ ends at a point $z_i$. Then

$$\phi_{D^* D}([C]_{D^*}) = [\phi_{XE}([*])_E, \ldots, \phi_{XE}([\alpha_n]_E)]_D = [z_0 = *, z_1, \ldots, z_n]_D. \quad (2)$$

Since $([*]_E, [\alpha_1]_E) \in D^*$,

$$[\alpha_1]_E = [z_0 = *, z_1]_E.$$ Proceeding inductively, for all $i$ we have that

$$[\alpha_i]_E = [z_0 = *, z_1, \ldots, z_i]_E \quad (3)$$

which implies that $\phi_{D^* D}$ is injective.

Given any $D$-chain $\{z_0 = *, z_1, \ldots, z_n\}$ we can let $\alpha_i = \{*, z_1, \ldots, z_i\}$ for all $0 \leq i \leq n$. Since $\{z_0 = *, z_1, \ldots, z_n\}$ is a $D$-chain, $[[\alpha_0]_D, \ldots, [\alpha_n]_D]$ is a $D^*$-chain and

$$\phi_{D^* D}([[*]_E = [\alpha_0]_E, \ldots, [\alpha_n]_E]_{D^*}) = [z_0 = *, z_1, \ldots, z_n]_D$$

which shows that $\phi_{D^* D}$ is surjective.

By definition,

$$\phi_{ED} (\phi_{D^* D}([C]_{D^*})) = [z_0 = *, z_1, \ldots, z_n]_E.$$ That is, the upper triangle of the diagram commutes, and we already know that the bottom triangle does. Finally, suppose that

$$([[*]_E = [\alpha_0]_E, \ldots, [\alpha_n]_E]_{D^*}, [[*]_E = [\beta_0]_E, \ldots, [\beta_m]_E]_{D^*}) \in (F^*)^*$$

where, according to Formula 3, we can suppose that for some $z_1, \ldots, z_n, w_1, \ldots, w_m \in X$, $\alpha_i = \{*, z_1, \ldots, z_i\}$ and $\beta_i = \{*, w_1, \ldots, w_i\}$ for all $i$. By definition this means that we can suppose that $m = n$ and $([\alpha_n]_E, [\beta_n]_E) \in F^*$. This in turn is equivalent to $(z_n, w_n) \in F$, which is equivalent to

$$\phi_{D^* D}([[*]_E = [\alpha_0]_E, \ldots, [\alpha_n]_E]_{D^*}, [[*]_E = [\beta_0]_E, \ldots, [\beta_m]_E]_{D^*})) \in F^*.$$
Remark 24 The preceding somewhat technical-looking proposition in fact has a very nice interpretation. Essentially it identifies \((X_E)_D\) with \(X_D\) by taking a \(D^*\)-chain of \(E\)-chains to the \(D\)-chain of their endpoints. In other words, \(X_E\) and \(X\) are "locally the same"; \(D^*\) and \(E^*\) are really just copies of \(D\) and \(E\) inside \(X_E \times X_E\). Dealing with \(X_D\) rather than \((X_E)_D\) means we are dealing with chains rather than chains of chains. At the same time, Proposition 23 identifies the mapping \(\phi_{X_E\rightarrow D}\) : \((X_E)_D\rightarrow X_E\) with the more easily understood mapping \(\phi_{ED} : X_D \rightarrow X_E\). With such identifications we can express the proposition as \((X_E)_D = X_D\). This proposition is very useful because it will allow us to immediately apply results proved for \(\phi_{X_E}\) to the more general functions \(\phi_{EF}\).

The next lemma illustrates this.

Lemma 25 Suppose that \(X\) is a uniform space and \(E\) is an entourage in \(X\) such that \(X_E\) is chain connected. Then for any entourage \(F\) and \(E\)-chain \(\gamma\) in \(X\), \(\gamma\) is \(E\)-homotopic to an \(F\)-chain.

Proof. Without loss of generality we can suppose that \(F \subset E\). Since \(X_E\) is chain connected, \(\phi_{X_E\rightarrow F} = \phi_{EF}\) is surjective by Lemma 12. But this means that if \(\gamma\) is an \(E\)-chain in \(X\) there is some \(F\)-chain \(\alpha\) in \(X\) such that \([\alpha]_E = \phi_{EF}([\alpha]_F) = [\gamma]_E\).

Given entourages \(D \subset E \subset F\) in \(X\), we have functions \(\phi_{ED} : X_D \rightarrow X_E\) and \(\phi_{FE} : X_E \rightarrow X_F\) with
\[
(\phi_{FE}(\phi_{ED}([x_0 = \ldots, x_n]_D))) = [x_0 = \ldots, x_n]_F = \phi_{FD}([x_0 = \ldots, x_n]_D).
\]
In other words, \(\phi_{FE} \circ \phi_{ED} = \phi_{FD}\) and \(\{X_E, \phi_{ED}\}\) forms an inverse system of uniformly continuous functions having as its indexing set the set of all entourages of \(X\) partially ordered by reverse inclusion.

Definition 26 We will call the inverse system \(\{X_E, \phi_{ED}\}\) the fundamental inverse system of \(X\) and let \(\tilde{X}\) denote the inverse limit of this inverse system with the inverse limit uniformity. We will let \(\phi_{X} : \tilde{X} \rightarrow X_E\) be the projection; we simply denote \(\phi_X\) by \(\phi\). We will always choose for our basepoint \(\ast\) in \(\tilde{X}\) the element having as each of its coordinates the basepoint \(\ast = [\ast]_E\) in \(X_E\).

Since a uniform space is metrizable if and only if it has a countable base for its uniformity (4), if \(X\) is metrizable then each \(X_E\) is metrizable. We may index the fundamental system with this countable base and conclude:

Proposition 27 If \(X\) is metrizable then \(\tilde{X}\) is metrizable.

Theorem 28 Let \(X\) and \(Y\) be uniform spaces, \(f : X \rightarrow Y\) be uniformly continuous, and \(E, F\) be entourages in \(X, Y\), respectively, such that \(f(E) \subset F\). Then \(f_{EF}\) is uniformly continuous and satisfies
\[
f \circ \phi_{XE} = \phi_{YE} \circ f_{EF}. \tag{4}
\]
Moreover, if \(X_E\) is chain connected then \(f_{EF}\) is the unique uniformly continuous function satisfying (4) such that \(f_{EF}([\ast]_E) = [f(\ast)]_F\).
**Proof.** To prove $f_{EF}$ is uniformly continuous we need only consider entourages of the form $D^*$ in $Y_F$, where $D \subset F$ is an entourage in $Y$. Since $f$ is uniformly continuous there exists some entourage $G \subset E$ such that $f(G) \subset D$. Let $\alpha := \{ * = x_0, ..., x_n, y \}$ and $\beta := \{ * = x_0, ..., x_n, z \}$ be such that

$$([\alpha]_E, [\beta]_E) \in G^*,$$

which means by definition $(y, z) \in G$. Then

$$(f(y), f(z)) \in D$$

and therefore

$$f_{EF}([\alpha]_E, [\beta]_F) = ([* = f(x_0), ..., f(x_n), f(y)]_F, [* = f(x_0), ..., f(x_n), f(z)]_F) \in D^*.$$  

This shows that $f_{EF}$ is uniformly continuous. That $f \circ \phi_{XE} = \phi_{YF} \circ f_{EF}$ is an immediate consequence of the definition.

To prove the last statement suppose $X_E$ is chain connected and $f' : X_E \to Y_F$ is a uniformly continuous function such that $f \circ \phi_{XE} = \phi_{YF} \circ f'$ and $f'([*]_E) = [f(*)]_F$. Let $G$ be an entourage in $X$ such that $f'(G^*) \cup f_{EF}(G^*) \subset F^*$. By way of Lemma 26 it is sufficient to show that if $\beta := \{ * = x_0, ..., x_n \}$ is a $G^*$-chain then $f'([\beta]_E) = f_{EF}([\beta]_F)$. We will prove it by induction on $n$. The case $n = 0$ is given; suppose the statement is true for $n \geq 0$ and consider $\beta := \{ * = x_0, ..., x_{n+1} \}$ with $\alpha := \{ * = x_0, ..., x_n \}$. Suppose that $f'([\beta]_E) := [f(*) = z_0, ..., z_m]_F$; then

$$f(x_{n+1}) = f \circ \phi_{XE}([* = x_0, ..., x_{n+1}]_E) = \phi_{YF} \circ f'([* = x_0, ..., x_{n+1}]_E) = z_m.$$  

By the inductive hypothesis, $f'([\alpha]_E) = f_{EF}([\alpha]_E)$ and by definition of $G^*$ we have

$$(f_{EF}([\alpha]_E), f'([\beta]_E)) = (f'([\alpha]_E), f'([\beta]_E)) \in F^*$$

and

$$(f_{EF}([\alpha]_E), f_{EF}([\beta]_E)) \in F^*.$$  

In other words, both $f_{EF}([\beta]_E)$ and $f'([\beta]_E)$ lie in $B(f_{EF}([\alpha]_E, F^*)$ and by Proposition 10, $\phi_{YF}$ is injective on this ball. The fact that $f_{EF}([\beta]_E) = f'([\beta]_E)$ now follows from

$$\phi_{YF}(f_{EF}([\beta]_E) = f(\phi_{XE}([\beta]_E)) = \phi_{YF}(f'(\beta]_E)).$$

\[\square\]

**Corollary 29** If $X$ is a uniform space and $F \subset E$ are entourages in $X$ such that $X_F$ is chain connected then $\phi_{EF}$ is the unique uniformly continuous function such that $\phi_{EF}([*]_F) = [*]_E$ and $\phi_{XE} = \phi_{XE} \circ \phi_{EF}$.

The proof of the next lemma is virtually identical to the proof of Lemma 65 in [1]; one need only replace statements like $xy^{-1} \in U$ with $(x, y) \in E$. 

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Lemma 30 (Chain Lifting) Let $X, Y$ be uniform spaces, $f : X \to Y$ be a uniformly continuous surjection, $F$ be an entourage in $Y$ and $E := f^{-1}(F)$. Let $c$ be an $E$-chain in $X$ and $\eta$ be an $F$-homotopy from the $F$-chain $d := f(c)$ to another $F$-chain $d'$. Then $\eta$ lifts to an $E$-homotopy from $c$ to an $E$-chain $c'$. That is, there exist an $E$-chain $c'$ and an $E$-homotopy $\kappa$ between $c$ and $c'$ such that $f(\kappa) = \eta$.

Proposition 31 Let $X, Y$ be uniform spaces, $f : X \to Y$ be a uniformly continuous surjection, $F$ be an entourage in $Y$ and $E := f^{-1}(F)$. If $\phi_{XE} : X_E \to X$ is surjective and there exists a uniformly continuous function $\psi : X \to Y_F$ such that the following diagram commutes

$$
\begin{array}{ccc}
X_E & \xrightarrow{\phi_{XE}} & X \\
\downarrow f_{EF} & & \downarrow f \\
Y_F & \xrightarrow{\phi_{YE}} & Y \\
\end{array}
$$

then $\phi_{XE}$ is a uniform homeomorphism.

Proof. We need only show that $\phi_{XE}$ is injective. Equivalently we need only show that if $c := \{* = x_0, \ldots, x_n = *\}$ is an $E$-loop in $X$ then $c$ is $E$-homotopic to the trivial loop $\{*\}$. Let $d := f(c)$, which is an $E$-loop in $Y$ and $f_{EF}([c]_E) := [h]_F$; by the commutativity of the diagram $\phi_{YE}([h]_F) = f(\phi_{XE}([c]_E) = f(*).$ In particular, $h$ is a loop in $Y$. Moreover, $[h]_F = \psi(\phi_{XE}([c]_E)) = \psi(*).$ On the other hand,

$$
[f(*)]_F = f_{EF}([*]_E) = \psi(\phi_{XE}([*]_E)) = \psi(*).
$$

That is, $[h]_F = [f(*)]_F$ and therefore $h$ is $F$-homotopic to the trivial loop $f(*)$. Now by definition,

$$
[h]_F = f_{EF}([c]_E) = [f(x_0), \ldots, f(x_n)]_F = [f(c)]_F = [d]_F
$$

and therefore $d$ is also $F$-homotopic to $[f(*)]_F$. The Chain Lifting Lemma now finishes the proof. ■

Corollary 32 Let $X$ and $Y$ be uniform spaces with $X$ chain connected, $f : X \to Y$ be a surjective uniformly continuous map, $F$ be an entourage in $Y$ and $E := f^{-1}(F)$. If $\phi_{YE} : Y_F \to Y$ is bijective then $\phi_{XE} : X_E \to X$ is a uniform homeomorphism.

Proof. Note that since $X$ is chain connected, $\phi_{XE}$ is surjective. Let $\psi := \phi_{YE}^{-1} \circ f$, which is uniformly continuous since $\phi_{YE}$ is bi-uniformly continuous. Certainly $f = \phi_{YE} \circ \psi$ and

$$
\phi_{YE} \circ (\psi \circ \phi_{XE}) = f \circ \phi_{XE} = \phi_{YE} \circ f_{EF}.
$$

Since $\phi_{YE}$ is bijective we conclude that $\psi \circ \phi_{XE} = f_{EF}$ and the conditions of Proposition 31 are satisfied, completing the proof. ■
Proposition 33 If $X$ is a coverable uniform space and $E$ is a covering entourage in $X$ then $X_E$ is coverable and hence chain connected.

Proof. If $E$ is a covering entourage then we may index the fundamental inverse system of $X_E$ using the set of all entourages $F^*$ where $F \subset E$ is a covering entourage in $X$. Then $\phi_F : X \to X_E$ is surjective for all such $F$. But $(X_E)^{F^*}$ is naturally identified with $X_F$ by Proposition 28 and therefore $\phi_{F^*} : X_E \to (X_E)^{F^*}$ is also surjective. Therefore the collection of all such $F^*$ is a covering basis for $X_E$. ■

It is straightforward but tedious to check that if $[\alpha]_E = [\alpha']_E$ and $[\beta]_E = [\beta']_E$ then $[\alpha \ast \beta]_E = [\alpha' \ast \beta']_E$.

Definition 34 For an entourage $E$ in a uniform space $X$, we define $\delta_E(X)$ to be the group of all $E$-homotopy classes of $E$-loops at * with the group operation induced by concatenation. We will call this the $E$-deck group of $X$.

That is, given $E$-loops $\alpha$ and $\beta$ based at *, we let

$$[\alpha]_E \ast [\beta]_E := [\alpha \ast \beta]_E.$$ 

Note that the identity chain is $[*]_E$ and if $\alpha = \{* = x_0, ..., x_n = *\}$ then $[\alpha]^{-1}_E = [\alpha^{-1}]_E$ where $\alpha^{-1} := \{* = x_n, ..., x_0 = *\}$. It is easy to check that $\delta_E(X)$ is in fact a group.

Theorem 35 If $X$ is a compact uniform space and $E$ is an entourage in $X$ such that $X_E$ is chain connected then $\delta_E(X)$ is finitely generated.

Proof. Let $F$ be an entourage in $X$ such that $F^3 \subset E$. Since $X$ is compact there exists some finite $F$-dense set, i.e., a set $A := \{x_1, ..., x_n\}$ such that for every $x \in X$ there exists some $x_i \in A$ such that $(x, x_i) \in F$. We will first show that if $\alpha := \{y_1, ..., y_m\}$ is any $E$-chain with $y_1, y_m \in A$ then $\alpha$ is $E$-homotopic to an $E$-chain $\{y_1 = z_1, z_2, ..., z_m = y_m\}$ such that $z_i \in A$ for all $i$. By Lemma 26 we may assume that $\alpha$ is in fact an $F$-chain. Now for each $y_i$, $1 < i < m$, there is some $z_i$ such that $(y_i, z_i) \in F$. We may now proceed iteratively, removing each $y_i$ and then replacing it by $z_i$. For example, since $(y_1, y_3) \in F^2 \subset E$, we may remove $y_2$ and still have an $E$-chain. Since $(y_1, y_2) \in F$ and $(y_2, z_2) \in F$, $(y_1, z_2) \in F^2 \subset E$. Similarly $(z_2, y_3) \in F^2$, so we may add $z_2$. Likewise, $(z_2, y_4) \in F^3 \subset E$, so we may remove $y_3$, and then $(z_2, z_2) \in F^3$ and $(z_3, y_4) \in F^2$, so we may add $z_3$. After finitely many steps we have the desired $E$-chain.

We may suppose that $* \in A$. We now claim that $\delta_E(X)$ is generated by elements of the form

$$[* = y_1, ..., y_k, y_{k+1}, ..., y_m = y_k, y_{k-1}, ..., y_1 = *]_E$$

where $y_i \in A$ for all $i$, and for $1 \leq i < j \leq m - 1$ we have $y_i \neq y_j$. We will call elements of this form minimal elements. That is, a minimal element is represented by a loop that consists of a chain made of distinct points of $A$, followed by a loop (which may be empty) of additional distinct points of
A, followed by the initial chain in reverse order. If we prove this claim then the proof of the theorem is finished because $A$, and hence the set of minimal elements, is finite.

Let $\gamma = \{\ast = z_1, \ldots, z_r = \ast\}$ be an arbitrary $E$-loop. We will show by induction on $r$ that $[\gamma]_E$ is a product of minimal elements. For $r = 1$ the proof is trivial; suppose it is true for $r - 1 \geq 1$. If all of the points $z_i$ $1 \leq i < r$ are distinct then $[\gamma]_E$ is already a minimal element. Otherwise, let $j < r$ be the smallest index such that for some $i<j$, $z_i = z_j$. Let

$\beta := \{z_1, \ldots, z_i, z_{j+1}, \ldots, z_r\}$.

We may apply the inductive hypothesis to conclude that $[\beta]_E$ is the product of minimal elements. On the other hand, let $\alpha := \{z_1, \ldots, z_i, z_i+1, \ldots, z_j = z_i, z_i-1, \ldots, z_1\}$.

By construction, $\alpha$ is minimal and since $[\gamma]_E = [\alpha \ast \beta]_E$, the proof is finished.

The following mapping is certainly well-defined.

**Definition 36** Let $X$ be a uniform space and $E$ be an entourage. For each $[\lambda]_E \in \delta_E(X)$, define a mapping $\overline{\lambda} : X_E \to X_E$ by $\overline{\lambda}([\alpha]_E) = [\lambda \ast \alpha]_E$.

Before proceeding we will recall some notation and definitions from [20]. Let $X$ be a uniform space. We denote by $H_X$ the topological group of uniform homeomorphisms of $X$ with composition as the operation. Suppose $G$ is a subgroup of $H_X$. An entourage $E$ is called $G$-invariant if $f(E) = E$ for every $f \in G$. If $X$ has a uniformity base consisting of $G$-invariant entourages we say that $G$ acts isomorphically. The action of $G$ is said to be discrete provided there exists some entourage $E$ such that if $(g(x), x) \in E$ for some $x \in X$ and $g \in G$ then $g$ must be the identity. If $G$ acts discretely and isomorphically on $X$ then the natural mapping $\pi : X \to X/G$ is called a discrete cover.

**Theorem 37** Let $X$ be a uniform space and $E$ be an entourage.

1. For any $\lambda_1, \lambda_2$ we have $\overline{\lambda_1 \circ \lambda_2} = \overline{\lambda_1} \ast \overline{\lambda_2}$ and $\delta_E(X)$ is naturally isomorphic to a subgroup of $H_{X_E}$.

2. $\delta_E(X)$ acts discretely and isomorphically on $X_E$.

3. If $\phi_{X_E}$ is surjective (in particular if $X$ is chain connected) then $\phi_{X_E} : X_E \to X$ is a discrete cover with covering group $\delta_E(X)$.

**Proof.** We have

$\overline{\lambda_1 \circ \lambda_2}([\alpha]_E) = [\lambda_1 \ast \lambda_2 \ast \alpha]_E = \overline{\lambda_1} \ast \overline{\lambda_2}([\alpha]_E)$.

This implies that each $\overline{\lambda}$ is a bijection and therefore is an element of $H_{X_E}$, while the inclusion $\mu : \delta_E(X) \to H_{X_E}$ is a homomorphism. If $\overline{\lambda}([\alpha]_E) = [\alpha]_E$ for some $\alpha$, then

$[\ast]_E = [\lambda \ast \alpha]_E \ast [\alpha]_E = [\lambda \ast \alpha \ast \alpha^{-1}]_E = [\lambda]_E$. 

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This implies both that \( \mu \) is injective and the action is free. The action of \( \delta_E(X) \) is free. By the very definition, for any entourage \( F \subset E \) and all \( \lambda \in \delta_E(X), \overline{\lambda}(F^*) = F^* \) and therefore the action is isomorphic. Now suppose that \( (x, \overline{\lambda}(x)) \in E^* \) for some \( x \in X_E \). Then \( \phi_{XE}(x) = \phi_{XE}(\overline{\lambda}(x)) \) and since \( \phi_{XE} \) is injective on every \( E^* \)-ball by Proposition \ref{Proposition_injectivity}, \( \overline{\lambda}(x) = x \). Since we have shown the action is free, it follows that \( \overline{\lambda} \) is the identity and we have shown that the action is discrete.

For the third part note that by Theorem 11 (see also Remark 13), \( \delta \), we need only check first that \( \phi_{XE} \) is bi-uniformly continuous (which we already know) and second that if \( x \in X \) and \( y \in X_E \) are such that \( \phi_{XE}(y) = x \), then the preimage \( \phi_{XE}^{-1}(x) \) is precisely the orbit \( \delta_E(X)(y) \). For any \( x \in X \),

\[
\phi_{XE}^{-1}(x) = \{[* = x_0, ..., x_{n-1}, x]_E\}.
\]

Now there is some \( E \)-chain \( \alpha = \{* = x_0, ..., x_{n-1}, x\} \) since \( \phi_{XE} \) is surjective. Moreover, for any \( \lambda \), the endpoint of \( \overline{\lambda}([\alpha]_E) \) is still \( x \) and hence \( \overline{\lambda}([\alpha]_E) \in \phi_{XE}^{-1}(x) \). That is, the orbit \( \delta_E(X)[\alpha]_E \) is contained in \( \phi_{XE}^{-1}(x) \). On the other hand, if \( \beta \) is any other \( E \)-chain to \( x \) and we let \( \lambda := \beta \star \alpha^{-1} \) then \([\lambda]_E \in \delta_E(X)\) and \( \overline{\lambda}([\alpha]_E) = [\beta]_E \) and therefore \( \phi_{XE}^{-1}(x) \) is contained in the orbit \( \delta_E(X)[\alpha]_E \).

\( \square \)

**Corollary 38** If \( X \) is a uniform space then for any entourages \( E, F \) in \( X \) with \( F \subset E \) such that \( \phi_{EF} : X_F \to X_E \) is surjective (in particular when \( X_E \) is chain connected), \( \phi_{EF} \) is a discrete cover with covering group \( \delta_F(X_E) \). In particular \( \phi_{EF} \) is a uniform homeomorphism if and only if \( \delta_F(X_E) \) is trivial.

**Definition 39** Let \( X \) be a uniform space and \( \theta_{EF} : \delta_F(X) \to \delta_E(X) \) denote the restriction of \( \phi_{EF} \) to \( \delta_F(X) \). The collection \((\theta_{EF}, \delta_F(X))\) forms an inverse system. We denote \( \lim_{F \to E} \delta_E(X) \) by \( \delta_1(X) \) and call it the deck group of \( X \).

**Lemma 40** If \( X \) is a uniform space and \( E \) is an entourage such that \( X_E \) is chain connected then for any entourage \( F \subset E \), \( \theta_{EF} : \delta_F \to \delta_E \) is surjective.

**Proof.** According to Lemma \ref{Lemma_surjectivity} if \( \gamma \) is an \( E \)-loop in \( X \), \( \gamma \) is \( E \)-homotopic to an \( F \)-chain \( \alpha \). But since \( E \)-homotopy preserves endpoints, \( \alpha \) must also be an \( F \)-loop and by construction \( \theta_{EF}([\alpha]_E) = [\alpha]_E = [\gamma]_E \). \( \square \)

**Proposition 41** Let \( X \) be a uniform space. Then

1. For any entourage \( E \), the group \( \delta_E(X) \) is discrete with respect to both the topology of uniform convergence and the topology induced by the inclusion of \( \delta_E(X) \) in \( X_E \).

2. The group \( \delta_1(X) \) is prodiscrte with respect to the inverse limit topology, which is the same as the topology induced by the inclusion in \( \overline{X} \).
Proof. Since $\delta_E(X)$ acts discretely (Theorem 37) it follows from Corollary 32 of [20] that $\delta_E(X)$ is discrete with respect to the topology of uniform convergence. On the other hand, since $\phi_{XE}$ is injective on $B(\ast, E^*)$ and $\delta_E(X) = \phi_{XE}^{-1}(\ast)$, $\delta_E(X) \cap B(\ast, E^*) = \ast$, which shows that $\delta_E(X)$ is a discrete subset of $\phi_{XE}$.

The second part now follows from the definitions. ■

Before we consider coverable spaces in more detail we need to revisit the issue of basepoints. Note that the construction of $\tilde{X}$ is dependent on the initial choice of basepoint, and in fact $\tilde{X}$ itself may depend on the choice of the basepoint. For example, if one takes $X := \{0\} \cup [1, 2]$ with the subspace metric then one can check that $\tilde{X}$ based at $\{0\}$ will consist of a single point, while $\tilde{X}$ based at any point in $[1, 2]$ will be $[1, 2]$. The following lemma clears up this issue:

Lemma 42 Suppose that $X$ is a uniform space such that for some choice of basepoint $\ast$, the projection $\phi : \tilde{X} \to X$ is surjective. Then $X$ is chain connected. Moreover, if $X$ satisfies the definition of coverable for $\ast$ then

1. for any two basepoints there is a natural system of uniform homeomorphisms between the fundamental inverse systems of $X$ with respect to the two basepoints. In particular the spaces $\tilde{X}$ constructed with each basepoint are naturally uniformly homeomorphic.

2. $X$ satisfies the definition of coverable for any choice of basepoint.

Proof. If $\phi : \tilde{X} \to X$ is surjective then since $\phi = \phi_{XE} \circ \phi_X$, $\phi_{XE}$ is surjective. It now follows from Lemma 12 that $X$ is chain connected. The first statement now follows from the observations in Remark 18. The second statement follows from the first; essentially the two inverse systems are the same and so the projections are surjective in one if and only if they are surjective in the other. ■

Theorem 43 Let $X$ be coverable and $\Lambda := C(X)$. Then $\{X_E, \phi_{EF}\}_{E \in \Lambda}$ and $\{\delta_E(X), \theta_{EF}\}_{E \in \Lambda}$ comprise an inverse system of discrete covers and $\phi : \tilde{X} \to X$ is a cover with covering group $\delta_1(X)$.

Proof. Since $X$ is coverable we may take $\Lambda$ for our indexing set. If $\lambda$ is an $F$-loop and $\alpha$ is an $F$-chain then

$$\phi_{EF}(\lambda_F([\alpha]_F)) = \phi_{EF}(\lambda \ast \alpha)_F = [\lambda \ast \alpha]_E$$

$$= [\lambda]_E ([\alpha]_E) = \theta_{EF}([\lambda]_F)(\phi_{EF}([\alpha]_F)).$$

That is, the system is compatible in the sense of [20]. Since each $\phi_{EF}$ is surjective, so is each $\phi_{EF}$ and according to Corollary 38 each $\phi_{EF}$ is a discrete cover. The proof is now finished by [20], Theorem 44. ■

Definition 44 When $X$ is coverable we will refer to the projection $\phi : \tilde{X} \to X$ as the universal covering map of $X$.

Remark 45 We will always take $\delta_1(X)$ to have the (prodiscrete) inverse limit topology, with respect to which $\delta_1(X)$ is complete and Hausdorff. $\delta_1(X)$ is also prodiscrete with respect to the topology of uniform convergence since $\phi$ is a cover (see [20]). It may be that the topology of uniform convergence is the same as the induced topology, but we have no need for such a statement in this paper.
3 Universal and Lifting Properties

**Proposition 46** Let $X$ be a uniform space and $U$ be a uniformity base. The following are equivalent for a fixed basepoint $*$:

1. $U$ is universal.
2. $X \times X \in U$ and for any $F \subset E$ with $E, F \in U$, $\phi_{EF}$ is a uniform homeomorphism.
3. $X \times X \in U$ and for any $E \in U$, $\phi_E : \tilde{X} \to X_E$ is a uniform homeomorphism.

**Proof.** Suppose that $U$ is universal and $F \subset E$ are entourages in $U$. Then $\phi_{EF} : X_F \to X_E$ satisfies $\phi_{EF} = \phi_X^{-1} \circ \phi_X F$ and is therefore a uniform homeomorphism. Since $X \times X$ lies in any universal base, $1) \Rightarrow 2)$. If 2) holds then we may use $U$ as our indexing set for the fundamental inverse system of $X$ and since each of the bonding maps is a uniform homeomorphism the inverse system is in fact trivial. Therefore each $\phi_E : \tilde{X} \to X_E$ is a uniform homeomorphism and 3) follows. If 3) is given then note that for any $E \in U$ we have that $\phi_{XE} = \phi_X \circ \phi_E^{-1}$ is a uniform homeomorphism and we see that $U$ is universal. ■

Definition 4 implicitly involves the choice of a basepoint for the construction of the spaces $X_E$. However, as the next Corollary (the proof of which is immediate from the Proposition 46 and Lemma 42) shows that the definition is independent of basepoint.

**Corollary 47** Any universal space is coverable, hence chain connected. In particular, if $X$ is universal with respect to one basepoint then $X$ is universal with respect to any basepoint.

**Corollary 48** If $X$ is universal and $E$ is an entourage such that $X_E$ is chain connected then $\phi_{XE} : X_E \to X$ is a uniform homeomorphism.

**Proof.** Let $F \subset E$ be an entourage in the universal base. Since $X_E$ is chain connected, $\phi_{EF} : X_F \to X_E$ is surjective. Since $\phi_{XE} : X_F \to X$ is bijective and $\phi_{XE} = \phi_X E \circ \phi_{EF}$, $\phi_{EF}$ must also be injective. Since $X_E$ is chain connected, Lemma 12 implies that $\phi_{EF}$ is surjective, hence a uniform homeomorphism because $\phi_{EF}$ is bi-uniformly continuous. Therefore $\phi_{XE} = \phi_X E \circ \phi_{EF}^{-1}$ is a uniform homeomorphism. ■

**Theorem 49** If $X$ is coverable then $\tilde{X}$ is universal.

**Proof.** Applying Proposition 49 and Lemma 11 we have that $\tilde{X}$ is chain connected. A basis element for the uniformity of $X$ is of the form $\phi_{E}^{-1}(F^*)$, where $F \subset E$ are entourages in $X$. Let $D$ be a covering entourage contained in $E \cap F$. Then $\phi_D$ is surjective and

$$\phi_E(\phi_D^{-1}(D^*)) = \phi_{ED}(\phi_D^{-1}(D^*))$$
\[ \phi_{E,D}(D^*) = D^* \subset F^* \subset X_E \times X_E. \]

In other words, \( \phi_D^{-1}(D^*) \subset \phi_{E}^{-1}(F^*) \) and we may take for our basis elements of \( \tilde{X} \) entourages of the form \( G := \phi_D^{-1}(D^*) \). We will now apply Corollary 32, taking \( f \) to be the surjective map \( \phi_D : \tilde{X} \to X_D \). We have the following diagram:

\[
\begin{array}{ccc}
\tilde{X}_E & \xrightarrow{\phi_{E,D^*}} & (X_D)_{D^*} \\
\downarrow^{\phi_{X,D^*}} & & \downarrow^{\phi_{X,D^*}} \\
\tilde{X} & \xrightarrow{\phi_D} & X_D \\
\end{array}
\]

According to Proposition 28, the function \( \phi_{X,D^*} : (X_D)_{D^*} \to X_D \) is a uniform homeomorphism and we may conclude that \( \phi_{X,G} \) is also a uniform homeomorphism. The collection of all such \( G \) is therefore a universal base for \( \tilde{X} \).

**Corollary 50** The following are equivalent for a coverable space \( X \).

1. \( X \) is universal.
2. For every entourage \( E \) such that \( X_E \) is chain connected, \( \delta_E(X) \) is trivial (i.e., every \( E \)-loop based at * is \( E \)-homotopic to the trivial loop).
3. \( \delta_1(X) \) is trivial.

**Proof.** The implication (1) \( \Rightarrow \) (2) follows from Corollary 48. If (2) holds then for any coverable entourage \( E \), \( \phi_E : \tilde{X} \to X_E \) is a uniformly continuous surjection and hence \( X_E \) is chain connected. Therefore each \( \delta_E(X) \) is trivial and the inverse limit \( \delta_1(X) \) is trivial. Finally, if \( \delta_1(X) \) is trivial then the universal covering map is a uniform homeomorphism and since \( \tilde{X} \) is universal, so is \( X \).

**Example 51** We will see later (Theorem 70) that \( \mathbb{R} \) is universal. Consider the entourage \( E \) consisting of all \( (x, y) \) such that \( x - y \in W \), where \( W := (-1, 1) \cup (2, 4) \cup (-4, -2) \). It is not hard to see that \( \{0, 3, 0\} \) is an \( E \)-loop based at 0 that is not \( E \)-homotopic to the trivial loop. At the same time it is true that \( \mathbb{R}_E \) is uniformly homeomorphic to \( \mathbb{R} \times \mathbb{Z} \), where \( \mathbb{Z} \) has the discrete uniformity, and hence \( \mathbb{R}_E \) is not chain connected. This shows that one cannot expect \( \delta_E(X) \) to be trivial for every choice of \( E \) when \( X \) is universal.

**Proposition 52** If \( f : X \to Y \) is a uniformly continuous bijection between coverable spaces \( X \) and \( Y \) and \( Y \) is universal then \( X \) is universal.

**Proof.** Suppose \( Y \) is universal and let \( F \) be an entourage in the universal base for \( Y \). Then \( \phi_{Y,F} : Y_F \to Y \) is a uniform homeomorphism and by Corollary 32, \( \phi_{X,F} : X_E \to X \) is a uniform homeomorphism, where \( E := f^{-1}(F) \).

**Proposition 53** Let \( f : X \to X/G = Y \) be a discrete cover, where \( X \) is chain connected. For any sufficiently small \( G \)-invariant entourage \( E \) and \( F := f(E) \), the function \( f_{EF} : X_E \to Y_F \) is a uniform homeomorphism.
That is, \((y,F)\) is invariant, we have that if \(\{F_e\}_{e \in E}\) to the trivial loop \(\{f(x_0)\}\) then \(g = e\) (so \(\gamma\) is an \(E\)-loop) and \(\gamma\) is \(E\)-homotopic to the trivial loop \(\{x_0\}\). Let \(f(\gamma) = \{y_0, \ldots, y_n\}\). We will prove the statement by induction on the minimal length \(m\) of an \(F\)-homotopy between \(f(\gamma)\) and \(\{y_0\}\). If \(m = 0\) then \(f(\gamma)\) is already trivial and so is \(\gamma\), and the proof is finished. Suppose we have proved it for some \(m - 1 \geq 0\), and there is some \(F\)-homotopy of \(f(\gamma)\) to \(\{y_0\}\) of length \(m\). Suppose that the first step in the \(F\)-homotopy is to add a point: \(f(\gamma)\) is \(F\)-related to \(\{y_0, \ldots, y_k, y_{k+1}, \ldots, y_n\}\). That is, \((y_k,y), (y,y_{k+1}) \in F = f(E). \) Now there exists \((a,b) \in E\) such that \(f(a) = y_k\) and \(f(b) = y\), and therefore some \(g \in G\) such that \(g(a) = x_k\). Since \(E\) is invariant, we have that if \(w := g(b)\) then \((w,x_k) \in E\) and \(f(w) = y\). Likewise there is some \(w' \in X\) such that \(f(w') = y\) and \((x_{k+1},w') \in E\). Since \((x_k,x_{k+1}) \in E\), \((w,x_k) \in E\), and \((x_{k+1},w') \in E\), it follows that \((w,w') \in E^3 \subset D\). But since \(w,w' \in f^{-1}(y)\), \(w = k(w')\) for some \(k \in G\) and by choice of \(D\), it must be that \(k = e\) and \(w = w'\). The inductive hypothesis now finishes the proof. Now suppose that the first step in the homotopy is to remove a point: \(f(\gamma)\) is \(F\)-related to \(\{y_0, \ldots, y_k, y_{k+1}, \ldots, y_n\}\); that is, \((y_{k-1},y_{k+1}) \in E\). As in the preceding argument there is some \(g \in G\) such that \((x_{k-1},g(x_{k+1})) \in E\). But \((x_{k-1},x_k), (x_k, x_{k+1}) \in E\), so \((g(x_{k+1}),x_{k+1}) \in E^3 \subset D\), and we conclude that \(g(x_{k+1}) = x_{k+1}\). Therefore \((x_{k-1},x_{k+1}) \in E\) and we may again apply the inductive hypothesis.

To see why \(f_{EF}\) is surjective, let \(\{* = y_0, \ldots, y_n\} \) be an \(E\)-chain in \(Y\). We will prove the statement by induction in \(n\). If \(n = 0\) then the proof is obvious since \(* = f(*)\) by assumption. Suppose we have proved it for \(n - 1 \geq 0\). Then we can find some \(E\)-chain \(\{* = x_0, \ldots, x_{n-1}\}\) such that \(y_i = f(x_i)\) for all \(i\). Since \((y_{n-1},y_n) \in F = f(E), \) and \(f(x_{n-1}) = y_{n-1}\), we may again use the invariance of \(E\) to see that there exists some \(x_n \in X\) such that \(f(x_n) = y_n\) and \((x_{n-1},x_n) \in E\). Then \(\{x_0, \ldots, x_n\}\) is an \(E\)-chain such that \(f(\{x_0, \ldots, x_n\}) = \{y_0, \ldots, y_n\}\).

To finish the proof of the proposition, let \(D^*\) be an entourage in \(X_E\), where \(D \subset E\) is an invariant entourage in \(X\) (\(D\) exists since \(G\) acts isomorphically). Since \(X\) is chain connected, \(\phi_{X_E}(D^*) = D\) by Proposition \(16\) and \(f(D) = K \subset F\) is an entourage in \(X/G\). The proof will be complete if we show that \(K^* \subset f_{EF}(D^*)\), which makes \(f_{EF}(D^*)\) an entourage. Let \(([\alpha]_F, [\beta]_F) \in K^*\), which means

\[ ([\alpha]_F, [\beta]_F) = ([* = y_0, \ldots, y_n, y]_F, [* = y_0, \ldots, y_n, z]_F) \]

with \((y,z) \in K = f(D)\). Using the invariance of \(E\) and \(D\) and proceeding inductively as we have done above, we can find \(E\)-chains \(\alpha' = \{* = x_0, \ldots, x_n, y'\}\) and \(\beta' = \{* = x_0, \ldots, x_n, z'\}\) such that \(f(\alpha') = \alpha\) and \(f(\beta') = \beta\), and \((x_n, y'), (x_n, z') \in D\). By definition, \(([\alpha']_E, [\beta']_E) \in D^*\) and \((f_{EF}([\alpha']_E), f_{EF}([\beta']_E)) = ([\alpha]_F, [\beta]_F)\).
Remark 54 We give one final comment about basepoints and lifts of functions following Lemma 42. The lifting theorems below are true for any choice of basepoints such that the functions involved are basepoint-preserving. For example, in the proposition below we may start with a basepoint \( * \) in \( X \), choose \( \ast = f(*)\) in \( Y \), use any basepoint \( \ast' \) to construct \( \widetilde{Y} \) and then choose another basepoint \( * \) in \( \widetilde{Y} \) so that \( \phi(\ast) = \ast \).

Proposition 55 Let \( X \) be universal, \( Y \) be uniform and \( f : X \to Y \) be uniformly continuous. Then

1. For any entourage \( E \) in \( Y \) there is a unique uniformly continuous function \( f_E : X \to Y_E \) such that \( \phi_{Y_E} \circ f_E = f \) and \( f_E(\ast) = \ast \).

2. There is a unique uniformly continuous function \( f_L : X \to \widetilde{Y} \) such that \( f_L(\ast) = \ast \) and \( \phi \circ f_L = f \), where \( \phi : \widetilde{Y} \to Y \) is the projection.

Proof. Define \( f_E : X \to Y_E \) as follows. Let \( F \) be an entourage in the universal base of \( X \) such that \( f(F) \subset E \) and \( k_E : X_F \to Y_E \) be the unique uniformly continuous function given by Theorem 28 (\( \phi_{X_F} : X_F \to X \) is a uniform homeomorphism and therefore \( X_F \) is chain connected). Define \( f_E := k_E \circ \phi_{X_F}^{-1} \).

If \( g \) were any such function then it follows from the uniqueness of \( k_E \) that \( k_E = g \circ \phi_{X_F} \) and hence that \( g = f_E \).

Note that by uniqueness, if \( E \subset F \) are entourages in \( Y \) we have that \( \phi_{Y_E} \circ f_E = f_F \) and by the universal property of the inverse limit there is a unique function \( f_L : X \to \widetilde{Y} \) such that \( \phi_{E} \circ f_L = f_E \) for every entourage \( E \) and \( f_L(\ast) = \ast \). Suppose that \( f' : X \to \widetilde{Y} \) is uniformly continuous such that \( f'(\ast) = \ast \) and \( \phi \circ f' = f \). Note that for any entourage \( E \) in \( Y \) we have

\[
\phi_{Y_E} \circ (\phi_E \circ f') = \phi \circ f' = f
\]

and therefore by uniqueness in Part (1), \( \phi_E \circ f' = f_E \). Since \( f' \) is also induced by the functions \( \phi_E \circ f' \), \( f = f' \). ■

Notation 56 The functions \( f_E \) and \( f_L \) will both be referred to as “lifts” of \( f \).

Theorem 57 Let \( X \) and \( Y \) be uniform, \( f : X \to Y \) be a cover, \( Z \) be universal, and \( g : Z \to Y \) be uniformly continuous. Then there exists a unique uniformly continuous function \( h : Z \to X \) such that \( f \circ h = g \) and \( h(\ast) = \ast \).

Proof. Suppose first that \( f \) is a discrete cover. According to Proposition 53 there is an entourage \( E \) in \( X \) such that if \( F := f(E) \) then \( f_{EF} : X_E \to Y_F \) is a uniform homeomorphism and \( f \circ \phi_{X_E} = \phi_{Y_F} \circ f_{EF} \). Define \( h : Z \to X \) by \( \phi_{X_E} \circ f_{EF}^{-1} \circ g_F \) (where \( g_F \) is the lift of \( g \) given by Proposition 53). Then \( h(\ast) = \ast \) and

\[
\begin{align*}
\phi_{Y_F} \circ f_{EF} \circ f_{EF}^{-1} \circ g_F &= \phi_{Y_F} \circ g_F = g.
\end{align*}
\]
To prove uniqueness, suppose that \( h' \) is any such function. Consider the lift \( h'_E : Z \to X_E \). We have

\[
\phi_{YF} \circ (f_{EF} \circ h'_E) = f \circ \phi_{XE} \circ h'_E = f \circ h' = g.
\]

By uniqueness of lifts (Proposition 53), \( f_{EF} \circ h'_E = g_F \). But then \( h'_E = f_{EF}^{-1} \circ g_F \) and

\[
h' = \phi_{XE} \circ h'_E = \phi_{XE} \circ f_{EF}^{-1} \circ g_F = h.
\]

Now suppose that \( f \) is an arbitrary cover. By Theorem 48 in [29] there exists an inverse system \( \{X_\alpha, f_{\alpha \beta}\} \) such that \( f_{\alpha \beta} : X_\beta \to X_\alpha \) is a discrete cover and \( X = \varprojlim X_\alpha \) and \( Y = X_i \) for some minimal element 1. According to what we proved above, for each \( \alpha \) there is a unique uniformly continuous function \( h_\alpha : Z \to X_\alpha \) such that \( h_\alpha(*) = * \) and \( f_1 \circ h_\alpha = g \). If \( \alpha \leq \beta \) we have that

\[
f_1 \circ (f_{\alpha \beta} \circ h_\beta) = f_1 \circ h_\beta = g
\]

and by uniqueness \( f_{\alpha \beta} \circ h_\beta = h_\alpha \). By the universal property of inverse limits there is a unique uniformly continuous function \( h : Z \to X = \varprojlim X_\alpha \) such that for all \( \alpha \), \( f_\alpha \circ h = h_\alpha \) and \( h(*) = * \). Now suppose \( h' : Z \to X \) is any uniformly continuous function such that \( h'(*) = * \) and \( f \circ h' = g \). Define \( h'_\alpha := f_\alpha \circ h' \).

We have that \( h'_\alpha(*) = * \) and

\[
f_1 \circ h'_\alpha = f_1 \circ f_\alpha \circ h' = f \circ h' = g.
\]

By the uniqueness of \( h_\alpha \), \( h_\alpha = h'_\alpha \) and therefore \( h = h' \).

If \( Y \) is coverable then by Theorem 49 \( \tilde{Y} \) is universal and we obtain:

**Corollary 58** Let \( f : X \to Y \) be a cover where \( X \) is uniform and \( Y \) is coverable, and \( \phi : \tilde{Y} \to Y \) be the projection. Then there exists a unique uniformly continuous function \( f_B : \tilde{Y} \to X \) such that \( f_B(*) = * \) and \( f \circ f_B = \phi \).

**Corollary 59** If \( f : X \to Y \) is a bi-uniformly continuous surjection where \( X \) is uniform and \( Y \) is uniform then \( Y \) is coverable.

**Proof.** Let \( U \) be a universal base for \( X \). Since \( f \) is a bi-uniformly continuous surjection, the set \( B \) of all \( f(F) \) such that \( F \in U \) is a base for the uniformity of \( Y \).

We may index the fundamental system for \( Y \) using \( B \), and for each \( E = f(F) \) in \( B \) we may use \( F \) in the construction of the lift \( f_E \). Since \( f_E = \phi_E \circ f_L \) we will be finished by the definition of coverable if we can show that each \( f_E \) is surjective.

Let \( \beta = \{ * = y_0, ..., y_n \} \) be an \( E \)-chain in \( Y \). We will show by induction on \( n \) that there is an \( F \)-chain \( \alpha = \{ * = x_0, ..., x_n \} \) such that \( f(\alpha) = \beta \) and, by definition of \( f_E \), this completes the proof. For \( n = 0 \) the proof is obvious. Now suppose we have an \( F \)-chain \( \gamma = \{ * = x_0, ..., x_{n-1} \} \) such that \( f(\gamma) = \{ * = y_0, ..., y_{n-1}\} \). Since \( f(F) = E \) there is some ordered pair \( (x_{n-1}, x_n) \in F \) such that \( f((x_{n-1}, x_n)) = (y_{n-1}, y_n) \). The \( F \)-chain \( \alpha = \{ * = x_0, ..., x_{n-1} \} \) now satisfies \( f(\alpha) = \beta \).
Corollary 60 If $X$ is coverable and $f : X \to Y$ is a bi-uniformly continuous surjection then $Y$ is coverable. In particular, any quotient by an equiuniform action, hence by an isomorphic action, on a coverable uniform space is coverable.

Proof. Since the universal cover $\phi : \tilde{X} \to X$ is a bi-uniformly continuous surjection, so is $f \circ \phi : \tilde{X} \to Y$. The proof is finished by Corollary 59 and Theorem 49.

Theorem 61 Let $X,Y$ be coverable spaces, $f : X \to Y$ be uniformly continuous, and $\phi : \tilde{X} \to X$ and $\psi : \tilde{Y} \to Y$ be the projections. Then there is a unique uniformly continuous function $\tilde{f} : \tilde{X} \to \tilde{Y}$, such that $\tilde{f}(*) = \ast$ and $f \circ \phi = \psi \circ \tilde{f}$. Moreover,

1. For any $x \in \tilde{X}$ and $g \in \delta_1(X)$, $\tilde{f}(g(x)) = \tilde{f}(g)(\tilde{f}(x))$.
2. The restriction $f_*$ of $\tilde{f}$ to $\delta_1(X)$ is a homomorphism into $\delta_1(Y)$.
3. If $f$ is a discrete cover then $\tilde{f}$ is a uniform homeomorphism.
4. If $Z$ is uniform and $g : Y \to Z$ is uniformly continuous then $\widetilde{g \circ f} = g \circ \tilde{f}$ (and in particular $(g \circ f)_* = g_\ast \circ f_\ast$).

Proof. For the main statement, define $\tilde{f} := (f \circ \phi)_L$. If $g : \tilde{X} \to \tilde{Y}$ is a uniformly continuous function with $\psi \circ g = f \circ \phi$ then by definition $g$ is a lift of $f \circ \phi$ and so $g = \tilde{f}$.

For Part (1), note that if $x \in \delta_1(X)$ then $\phi(x) = \ast$ and

$$\psi \circ \tilde{f}(x) = f \circ \phi(x) = f(\ast) = \ast.$$ 

Therefore $\tilde{f}(\delta_1(X)) \subset \delta_1(Y)$. For any entourage $F$ in $Y$ let $E$ be an entourage in $X$ such that $f(E) \subset F$. Now $\psi_F \circ \tilde{f}$ and $f_{EF} \circ \phi_E$ are both lifts of $f \circ \phi$ to $Y_F$ and therefore $\psi_F \circ \tilde{f} = f_{EF} \circ \phi_E$ (see Proposition 55). Now let $g \in \delta_1(X)$ and $x \in \tilde{X}$ with $\phi_E(g) = \gamma_E$ and $\phi_E(x) = \alpha_E$.

By definition of the action of the inverse limit group $G$ on the inverse limit space $\tilde{X}$,

$$\psi_F(\tilde{f}(g(x))) = f_{EF} \circ \phi_E(g(x)) = f_{EF}([\gamma_E \ast \alpha_E]).$$

The latter quantity, by definition of $f_{EF}$, is equal to

$$[f(\gamma_E \ast \alpha_E)]_F = [f(\gamma_E) \ast f(\alpha_E)]_F = [f(\gamma_E)]_F \ast [f(\alpha_E)]_F.$$

As a special case when the $g$ is the identity,

$$\psi_F(\tilde{f}(x)) = ([f(\alpha_E)]_F).$$

If $x = \ast$ then

$$\psi_F(\tilde{f}(g(*))) = [f(\gamma_E)]_F.$$
Combining these we obtain
\[ \tilde{f}(g(x)) = \tilde{f}(g)(\tilde{f}(x)). \]

When \( x \in \delta_1(X) \), we have
\[ \tilde{f}(gx) = \tilde{f}(g)\tilde{f}(x), \]
which gives the second statement.

If \( f : X \to X/G = Y \) is a discrete cover then \( f \) is bi-uniformly continuous and we may index the fundamental system of \( Y \) using entourages of the form \( f(E) \) where \( E \) is an invariant entourage in \( X \). Then by uniqueness, \( \tilde{f} \) is induced by the functions \( f_E \). Proposition 53 implies that the functions \( f_E \) are all uniform homeomorphisms and hence \( f \) must be a uniform homeomorphism.

The last part follows from uniqueness of \( g \circ f \). ■

**Theorem 62** If \( X \) and \( Y \) are coverable spaces and \( f : X \to Y \) is a cover then \( \tilde{f} : \tilde{X} \to \tilde{Y} \) is a uniform homeomorphism. If \( X \) is coverable then \( f_B : \tilde{Y} \to X \) is a cover with covering group \( f_\ast(\delta_1(X)) \subset \delta_1(Y) \).

**Proof.** Let \( \phi : \tilde{X} \to X \) and \( \psi : \tilde{Y} \to Y \) be the projections. We will show that the lift \( (f_B)_L : \tilde{Y} \to \tilde{X} \) given by Corollary 58 and Proposition 53 is an inverse to \( f \). First note that
\[ \psi \circ (\tilde{f} \circ (f_B)_L) = f \circ \phi \circ (f_B)_L = f \circ f_B \circ \psi. \]
That is, \( \tilde{f} \circ (f_B)_L \) is the unique lift of the identity on \( Y \) and hence must be the identity on \( \tilde{Y} \).

Now according to Theorem 57, \( \phi \) is the unique lift of the function \( f \circ \phi \); that is, the unique uniformly continuous function \( \eta : \tilde{X} \to X \) such that such that \( \eta(*) = * \) and \( f \circ \eta = f \circ \phi \). But we also have that
\[ f \circ (f_B \circ \tilde{f}) = \psi \circ \tilde{f} = f \circ \phi. \]
In other words, \( f_B \circ \tilde{f} = \phi \). We now have
\[ \phi \circ ((f_B)_L \circ \tilde{f}) = f_B \circ \tilde{f} = \phi \]
and \((f_B)_L \circ \tilde{f}\) must be the unique lift of the identity on \( X \), hence the identity on \( \tilde{X} \). We have shown that both \((f_B)_L\) and \( \tilde{f} \) are uniform homeomorphisms and inverses of one another.

For the second part note that since \( X \) is coverable, \( \phi \) is bi-uniformly continuous and since \( f_B = \phi \circ (f_B)_L = \phi \circ \tilde{f}^{-1} \), \( f_B \) is bi-uniformly continuous. Now \( f_B(x) = f_B(y) \) for \( x, y \in \tilde{Y} \) if and only \( \phi \circ \tilde{f}^{-1}(x) = \phi \circ \tilde{f}^{-1}(y) \), or equivalently, letting \( w := \tilde{f}^{-1}(x) \) and \( z := \tilde{f}^{-1}(y) \) there is some \( g \in \delta_1(X) \) such that \( g(w) = z \). But this is equivalent to
\[ f_\ast(g)(x) = f_\ast(g)(\tilde{f}(w)) = \tilde{f}(g(w)) = \tilde{f}(z) = y \]
(the second equality comes from Theorem 61 (1)). That is, the orbits of $f_*(\delta_1(X))$ are precisely the preimages of points with respect to $f_B$. According to [20] this means that $f_B$ is the quotient with respect to the action of $f_*(\delta_1(X))$. Finally, since $f_*(\delta_1(X)) \subset \delta_1(Y)$, which acts prodiscretely and isomorphically, so does $f_*(\delta_1(X))$ and $f_B$ is a cover with covering group $f_*(\delta_1(X))$ (see Remark 13 in [20]).

Since a universal space is uniformly homeomorphic to its own universal cover by Proposition 46, we obtain:

**Corollary 63** The universal cover of a coverable space is unique up to uniform homeomorphism. More precisely, if $X$ is coverable, $Y$ is universal, $f : Y \to X$ is a cover and $\phi : X \to X$ is the universal cover then $f_L : Y \to \tilde{X}$ is a uniform homeomorphism.

4 Traditional Topological Properties

We will say that a topological space $X$ is simply connected if every loop in $X$ is null-homotopic, regardless of whether $X$ is pathwise connected. Recall that $X$ is called locally connected (resp. locally pathwise connected) if for every $x \in X$ and open set $U$ containing $x$ there is a connected (resp. pathwise connected) open set $V$ with $x \in V \subset U$. $X$ is semilocally simply connected if each $x \in X$ is contained in an open set $U$ such that every loop in $U$ based at $x$ is null-homotopic in $X$ (see [18]). In a uniform space it is natural to consider the situation when these local conditions are true uniformly.

**Definition 64** A uniform space $X$ is called uniformly locally connected (resp. uniformly locally pathwise connected) if for each entourage $E$ there is an entourage $F \subset E$ such that all $F$-balls are open and connected (resp. open and pathwise connected). $X$ is called uniformly semilocally simply connected if there exists an entourage $E$ such that any loop in $B(x, E)$ based at $x \in X$ is null-homotopic in $X$.

Note that given any $E$ as in the above definition, any entourage $F \subset E$ has the same property.

**Proposition 65** A uniform space $X$ is uniformly locally connected (resp. uniformly locally pathwise connected) if and only if for every entourage $E$ there exists an entourage $F$ with open balls such that for every $x \in X$ there exists some open set $U_x$ such that $B(x, F) \subset U_x \subset B(x, E)$ and $U_x$ is connected (resp. $U$ is pathwise connected).

**Proof.** Necessity is trivial; take an entourage $F$ with open balls and let $U_x := B(x, F)$. To prove the converse, let $E$ be any entourage in $X$ and $K$ be an entourage such that $K^2 \subset E$. Let $D$ be an entourage such that for each $x \in X$ there exists an open set $U_x$ that is connected (resp. pathwise connected), such that $B(x, D) \subset U_x \subset B(x, K)$. Define $F \subset X \times X$ by

$$F := \{(x, y) : x, y \in U_w \text{ for some } w\}.$$

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$F$ is symmetric by definition and certainly contains the diagonal. Now

$$B(x, F) = \{ y : y \in U_w \text{ for some } U_w \text{ containing } x \}$$

which, being the union of connected (resp. pathwise connected) open sets $U_w$ containing $x$, is open and connected (resp. pathwise connected). $F$ is an entourage since by definition $D \subset F$. Finally, if $(x, y) \in F$ then $y \in B(x, F)$ and $y \in U_w$ for some $U_w$ containing $x$. Since $U_w \subset B(w, K)$, $(x, y) \in K^2 \subset E$.

**Proposition 66** Let $X$ be a locally connected (resp. locally pathwise connected) compact topological space. Then $X$, with the unique uniformity compatible with its topology, is uniformly locally connected (resp. uniformly locally pathwise connected). If $X$ is both locally pathwise connected and semilocally simply connected then $X$ is uniformly semilocally simply connected.

**Proof.** Recall that the unique compatible uniformity on $X$ has as a basis all symmetric open subsets of $X \times X$ that contain the diagonal. Let $E$ be any entourage in $X$ and for any $x \in X$ let $U_x$ be an open and connected (resp. pathwise connected) neighborhood of $x$ such that $U_x \times U_x \subset E$. Let $F \subset E$ be the union of all sets $U_x \times U_x$. Then $F$ is open and symmetric, hence an entourage. If $y \in B(x, F)$ then $(x, y) \in U_x \times U_y$ for some $z$, and therefore $x$ and $y$ both lie in $U_z$. That is, $B(x, F)$ is the union of all sets $U_z$ such that $x \in U_z$. Since each $U_z$ is connected (resp. pathwise connected), so is $B(x, F)$.

If $X$ is both locally pathwise connected and semilocally simply connected then by what we proved above, for every $x \in X$ there is an arbitrarily small entourage $E_x$ such that the ball $B(x, E_x)$ is open and pathwise connected. By choosing $E_x$ small enough we may assume that every loop in $B(x, E_x)$ based at $x$ is null-homotopic in $X$. For each $x$ let $F_x$ be an entourage with open balls such that $F_x^2 \subset E_x$. Let $\{ B(x_i, F_{x_i}) \}_{i=1}^k$ be a finite open cover of $X$, and let $F := \bigcap_{i=1}^k F_{x_i}$. For any $x \in X$ there is some $x_i$ such that $(x, x_i) \in F_{x_i}$. If $y \in B(x, F)$ then $(x, y) \in F \subset F_{x_i}$ and $(y, x_i) \in F_{x_i}^2 \subset E_{x_i}$. That is, $B(x, F) \subset B(x_i, E_{x_i})$. Given any loop $\gamma$ in $B(x, F)$ based at $x$, join it to $x_i$ by a path in $B(x_i, E_{x_i})$ from $x$ to $x_i$. The resulting loop based at $x_i$ is null-homotopic in $X$, and hence so is $\gamma$.

**Lemma 67** Let $X$ be a chain connected uniform space. If there is an entourage $E$ of $X$ such that the $E$-balls of $X$ are connected (resp. pathwise connected) then $X$ is connected (resp. pathwise connected).

**Proof.** It will follow from Proposition 66 and a standard theorem from topology if we show by induction that all $E^n$-balls are connected. The $n = 1$ case is given. Suppose that $E^n$-balls and $E$-balls are all connected for some $n$. Let $y \in B(x, E^{n+1})$. By definition of $B(x, E^{n+1})$ there is some $z$ such that $z \in B(x, E^n)$ and $z \in B(y, E)$. Each of these balls is connected and they intersect in $z$, hence their union is connected set. We have shown that every element of $B(x, E^{n+1})$ is contained in a connected subset of $B(x, E^{n+1})$ containing $x$, and
so $B(x, E^{n+1})$ is connected. If the $E$-balls are all pathwise connected then by what we have just proved $X$ is connected and locally pathwise connected, hence pathwise connected.

**Corollary 68** If $X$ is chain connected and uniformly locally connected (resp. uniformly locally pathwise connected) then $X$ is connected (resp. pathwise connected).

**Proposition 69** Let $X$ be a uniform space and $E$ be an entourage in $X$ such that the $E$-balls have one of the following properties: chain connected, connected, pathwise connected. Then $X_E$ has the same property.

**Proof.** Suppose that the $E$-balls of $X$ are chain connected. Let $F \subseteq E$ be an entourage and $\alpha = \{ * = x_0, \ldots, x_n \}$ be an $E$-chain. Since the $E$-balls are chain connected we may suppose up to $E$-homotopy that $\alpha$ is an $F$-chain. For example, we may join $x_0$ and $x_1$ by an $F$-chain $\{ x_0, y_1, \ldots, y_m, x_1 \}$ that lies entirely in $B(x_0, E)$. We may remove the points $y_1, \ldots, y_m$ one at a time in order to obtain an $E$-homotopy from $\{ x_0, y_1, \ldots, y_m, x_1, \ldots, x_n \}$ to $\alpha$. Then letting $\alpha_i := \{ * = x_0, \ldots, x_i \}$ we have that $([\alpha_i]_E, [\alpha_{i+1}]_E) \in F^*$ for all $i$. In particular, $\{ [\alpha_0]_E, \ldots, [\alpha_n]_E = [\alpha]_E \}$ is an $F^*$-chain to $[*]_E$. This shows that $X_E$ is chain connected. Now if $E$ has connected (resp. pathwise connected) balls then these balls are chain connected and therefore $X_E$ is chain connected by what we just proved. According to Proposition 16, the $E^*$-balls in $X_E$ are connected (resp. pathwise connected), and Lemma 67 now shows that $X_E$ is connected (resp. pathwise connected).

**Theorem 70** If $X$ is a uniformly locally pathwise connected, connected and simply connected uniform space then $X$ is universal.

**Proof.** Let $E$ be an entourage with pathwise connected open balls. Then $X_E$ is pathwise connected by Proposition 69. Moreover, by Corollary 34 of 20, the action of $\delta_\delta(X)$ on $X_E$ is properly discontinuous. Since $X$ is a Poincaré space, $\phi_{XE} : X_E \to X$ is a traditional cover (cf. 13) and since $X$ is simply connected, $\phi_{XE}$ must be a trivial cover, hence bijective. This means that the bi-uniformly continuous mapping $\phi_{XE}$ is a uniform homeomorphism.

**Corollary 71** Every compact, connected, locally pathwise connected, simply connected topological space is universal.

Proposition 69 now implies:

**Corollary 72** If $c : [0, 1] \to X$ is a (continuous) path, where $X$ is uniform, with $c(0) = *$, then there is a unique lift $c_L : [0, 1] \to \tilde{X}$ such that $c_L(0) = *$ and $\phi \circ c_L = c$, where $\phi : \tilde{X} \to X$ is the projection. A similar statement holds for homotopies.

If $c : [0, 1] \to X$ is a path from $*$ to $x \in X$ then $x = \phi \circ c_L(1) \in \phi(\tilde{X})$. We have:

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Corollary 73 If $X$ is a uniform space with projection $\phi: \tilde{X} \to X$ then the pathwise connected component of $X$ at $*$ is contained in $\phi(P)$, where $P$ is the pathwise connected component of $*$ in $\tilde{X}$.

Corollary 74 If $X$ is a connected, uniformly locally pathwise connected uniform space then $X$ is coverable. In particular, every compact, connected, locally pathwise connected topological space is coverable.

Proof. We may index the fundamental system of $X$ using entourages having pathwise connected balls. But for each such $E$, $X_E$ is pathwise connected by Proposition 69 and therefore each $\phi_E: \tilde{X} \to X_E$ is surjective by the previous corollary. $\blacksquare$

Theorem 75 Suppose $X$ is a uniformly locally pathwise connected uniform space having an entourage $E$ such that for all $x \in X$ every loop in $B(x, E)$ is null-homotopic in $X$ and an entourage $F$ having pathwise connected balls such that $F^2 \subset E$. Then $X_F$ is pathwise connected, uniformly locally pathwise connected and simply connected, hence universal.

Proof. According to Proposition 69 $X_F$ is pathwise connected, and the $F^*$-balls, which are uniformly homeomorphic to the $F$-balls, are pathwise connected. By Proposition 26 the function $\phi_{F^*,X_F}: (X_F)_F \to X_F$ is a uniform homeomorphism. This implies that every $F^*$-loop in $X_F$ is $F^*$-homotopic to the trivial loop. Now let $c: [0, 1] \to X_F$ be a loop based at $*$. Since $c$ is uniformly continuous there is some $F^*$-loop $\gamma = \{ * = x_0, ..., x_n = * \}$ such that each $x_i = c(t_i)$ for some $i$ and $c([t_i, t_{i+1}]) \subset B(x_i, F^*)$ for all $i$. We may use any $F^*$-homotopy from $\gamma$ to $*$ to construct a null-homotopy from $c$ to the trivial loop as follows. Suppose that $(x_i, y), (x_{i+1}, y) \in F^*$ for some $y \in X$. We may join $x_i$ to $y$ and $y$ to $x_{i+1}$ by paths $\alpha_1$ and $\alpha_2$ in $B(y, F^*)$. Let $\alpha_3$ be the restriction of $c$ to $[t_i, t_{i+1}]$. Then we have a loop $\beta = \alpha_1 * \alpha_2 * \alpha_3^{-1}$, each segment of which lies in the $F^*$-ball centered at its endpoint. Now let $\psi := \phi_{X_F} \circ \beta$, which is a loop consisting of three segments, each of which lies in the $F$-ball centered at its endpoint. Since $F^2 \subset E$, $\psi$ lies entirely in the $E$-ball centered at a point, and hence is null-homotopic in $X$. According to Theorem 67 any null-homotopy of $\psi$ lifts to a null-homotopy of $\beta$. In other words, the path $c$ is homotopic to a path $c_1$ obtained by replacing $\alpha_3$ by $\alpha_3$ concatenated by $\alpha_2$. We may carry out a similar process when removing any point in $\gamma$ to form a new $F^*$-chain as part of an $F^*$-homotopy of $\gamma$. Finitely many such steps show that $c$ is null-homotopic. Since the $F$-balls of $X$, and hence the $F^*$-balls of $X_F$, are pathwise connected, $X_F$ is uniformly locally pathwise connected and pathwise connected, hence universal by Theorem 70. $\blacksquare$

5 The homomorphism $\lambda: \pi_1(X) \to \delta_1(X)$

Let $c: [0, 1] \to X$ be a path in a coverable space $X$ such that $c(0) = c(1) = *$. Since $c(1) = *$, $\phi(c_1(1)) = *$, where $\phi$ is the universal covering map of $X$ and
$c_L$ is the lift given by Proposition 55. That is, $c_L(1) \in \delta_1(X)$. Define a function
\[ \lambda : \pi_1(X) \to \delta_1(X) \] 
by $\lambda([c]) = c_L(1)$, where $[c]$ is the homotopy equivalence class of a loop $c$ based at $*$. This function is well defined. In fact, suppose $c$ and $d$ are loops based at $*$ that are homotopic via a homotopy $\eta : [0,1] \times [0,1] \to X$, with $\eta(0,t) = c(t)$ and $\eta(1,t) = d(t)$. Then $\eta_L(t,1)$ is a path joining $c_L(1)$ to $d_L(1)$, both of which lie in $\tilde{\phi}^{-1}([\ast])$. But $\tilde{\phi}^{-1}([\ast]) = \delta_1(X)$ is a prodiskrete group (hence totally disconnected) with respect to the subspace topology. Therefore $c_L(1) = d_L(1)$.

**Proposition 76** Let $X$ be coverable, $[c] \in \pi_1(X)$, where $c : [0,1] \to X$ is based at $*$, and $E$ be an entourage in $X$. If $0 = t_0 < \cdots < t_n = 1$ are such that for every $i$, $c(t) \in B(c(t_i), E))$ for all $t \in [t_i, t_{i+1}]$ then $\theta_E(\lambda([c])) = [\ast = c(t_0), \ldots, c(t_n)]_E$. (Such values $t_i$ always exist since $c$ is uniformly continuous.)

**Proof.** Note that by definition,
\[ \theta_E(\lambda([c])) = \theta_E(c_L(1)) = c_E(1) \]
where $c_E$ is the lift given by Proposition 55. Let $\gamma = \{ \ast = c(t_0), \ldots, c(t_n) \}$, where the points $t_i$ satisfy the conditions of the proposition. We will show by induction on $i$ that $c_E(t_i) = [c(t_0), c(t_1), \ldots, c(t_i)]_E$ for all $0 \leq i \leq n$. The case $i = 0$ is trivial. Suppose that we have proved the statement for some $i$, $0 \leq i < n$. By definition,
\[ ([c(t_0), c(t_1), \ldots, c(t_i)]_E, [c(t_0), c(t_1), \ldots, c(t_i), c(t_{i+1})]_E) \in E^* \]
and $\phi_{X_E}([c(t_0), c(t_1), \ldots, c(t_i), c(t_{i+1})]_E) = c(t_{i+1})$. Recall from Proposition 16 that $\phi_{X_E}$ is a uniform homeomorphism when restricted to any $E^*$-ball of $X_E$, and since we also have $\phi_{X_E}(c_E(t_{i+1})) = c(t_{i+1})$ we will be finished if we can show
\[ ([c(t_0), c(t_1), \ldots, c(t_i)]_E, c_E(t_{i+1})) \in E^*. \]
But the unique lift $c_i$ of $c|[t_i, t_{i+1}]$ to $X_E$ starting at $c_E(t_i)$ must be $\eta \circ c|[t_i, t_{i+1}]$, where $\eta$ is the inverse of $\phi_{X_E}$ restricted to $B([c(t_0), c(t_1), \ldots, c(t_i)]_E, E^*)$. Since $c_E|[t_i, t_{i+1}]$ also satisfies the conditions for this lift, we must have that
\[ c_E(t_{i+1}) = c_i(t_{i+1}) = \eta \circ c(t_{i+1}) \in B([c(t_0), c(t_1), \ldots, c(t_i)]_E, E^*). \]

**Theorem 77** If $X$ is coverable then the natural mapping $\lambda : \pi_1(X) \to \delta_1(X)$ is a homomorphism, the image of which is the normal subgroup $\sigma(X)$ of $\delta_1(X)$ that leaves invariant the pathwise connected component of $\bar{X}$ containing $\ast$. In particular, $\lambda$ is surjective if and only if $\bar{X}$ is pathwise connected.

**Proof.** To see why $\lambda$ is a homomorphism note that $\phi \circ (c_L * (\lambda([c]) \circ d_L)) = c \circ d$ (here $\lambda([c]) \circ d_L$ is the “translate” via the uniform homeomorphism $\lambda([c])$ of $d_L$ to the endpoint of $c_L$ and $\phi$ is the universal covering map of $X$) and therefore

\[ c_E(t_{i+1}) = c_i(t_{i+1}) = \eta \circ c(t_{i+1}) \in B([c(t_0), c(t_1), \ldots, c(t_i)]_E, E^*). \]
by uniqueness, \( c_L \ast (\lambda([c]) \circ d_L) = (c \ast d)_L \). But the endpoint of \( c_L \ast \lambda([c]) \circ d_L \) is \( c_L(1)d_L(1) = \lambda([c])\lambda([d]) \). If \( g \in \delta_1(X) \) stabilizes the pathwise connected component of \( \tilde{X} \) containing \( * \) then \( * \) and \( g(\ast) \) are joined by an path \( \alpha \), and \( \phi(\alpha) \) is a loop based at \( * \) with \( \lambda([\alpha]) = g \). On the other hand, if \( g \) lies in the image of \( \lambda \) then \( * \) and \( g(\ast) \) are joined by an path (namely the lift of a loop, the image of whose equivalence class is \( g \)).

Now suppose \( x \) lies in the pathwise connected component of \( \tilde{X} \) containing \( * \). We may join \( x \) to \( * \) by an path \( c \). But then \( g \circ c \) joins \( g(x) \) and \( g(\ast) \), and we have already observed that the latter is joined to \( * \) by an path. \( \blacksquare \)

The kernel of \( \lambda \), by definition, consists of those \([c]\) such that \( c \) lifts to a loop in \( \tilde{X} \), or equivalently the lift of \( c \) to each \( X_E \) is a loop. It is of obvious interest when \( \ker \lambda \) is trivial. This can be checked in some special cases. For example, in the case of the Hawaiian earring, a problem that essentially amounts to the injectivity of \( \lambda \) occupied several papers (\([11, 17, 22]\)) and was simply stated in \([3]\) (along with an incorrect statement that amounts to surjectivity of \( \lambda \)).

**Proposition 78** Let \( X \) be coverable. Then

1. If \( \tilde{X} \) is simply connected then \( \lambda : \pi_1(X) \to \delta_1(X) \) is injective.

2. If \( X \) is pathwise connected and \( \lambda : \pi_1(X) \to \delta_1(X) \) is injective then \( \tilde{X} \) is simply connected.

**Proof.** If \( \tilde{X} \) is simply connected and \([c] \in \ker \lambda \) then \( c \) lifts to a loop \( c_L \) in \( \tilde{X} \) based at \( * \), which is then null-homotopic. The image of any null-homotopy of \( c_L \) via the universal covering map \( \phi \) is a null-homotopy of \( c \).

Now suppose that \( X \) is pathwise connected and \( \lambda : \pi_1(X) \to \delta_1(X) \) is injective. Let \( c : [0, 1] \to \tilde{X} \) be a loop based at some point \( x \). Then \( \phi \circ c \) is a loop in \( X \), where \( \phi \) is the universal covering map of \( X \). Let \( d : [-1, 2] \to X \) be the concatenation of a path \( \alpha \) from \( * \) to \( \phi(c(0)) \), followed by \( \alpha^{-1} \), parameterized so that the restriction to \([0, 1] \) is \( \phi \circ c \). So \( d \) represents an element of \( \pi_1(X) \). Let \( g \in \delta_1(X) \) be such that \( g(dL(0)) = c(0) \) (such a \( g \) exists since \( \phi(c(0)) = d(0) \)). Now the composition of \( g \) with \( d_L \mid_{[0,1]} \) is simply \( c \) and therefore \( d_L \mid_{[0,1]} \) is a loop. But then \( \lambda([d]) = 0 \) and since \( \lambda \) is injective, \( d \) must be null-homotopic. But then any null-homotopy of \( d \) lifts to one of \( d_L \), and the composition of the lifted homotopy with \( g \) gives rise to a null-homotopy of \( c \). \( \blacksquare \)

**Proposition 79** If \( X \) is a uniformly locally pathwise connected, connected uniform space then the pathwise connected component of \( \tilde{X} \) is dense in \( \tilde{X} \) and \( \lambda(\pi_1(X)) \) is dense in \( \delta_1(X) \).

**Proof.** These two statements follow from the following general result concerning induced functions on inverse limits: Let \((X_\alpha, \phi_{\alpha\beta})\) be an inverse system of topological spaces with continuous bonding maps and \( X := \varprojlim X_\alpha \). Let \( f_\alpha : Y \to X_\alpha \) be a collection of continuous surjections from a topological space \( Y \) such that \( \phi_{\alpha\beta} \circ f_\beta = f_\alpha \) for all \( \alpha \leq \beta \). Then the induced mapping \( f : Y \to X \) has dense image in \( X \). We do not have a reference for this exact statement but
the proof is straightforward and similar to the proof of III.7.3 Proposition 2 in [4]. Now if \( E \) has pathwise connected balls then \( X_E \) is pathwise connected and by Corollary 73 the restriction \( \psi \) of \( \phi_E \) to the pathwise connected component \( P \) of \( \tilde{X} \) is surjective onto \( X_E \) and the proof of the first part is finished by the above general statement.

On the other hand, if \( x \in \delta_E(X) \) then there is some path \( \alpha \) from \( * \) to \( x \), and \( \phi_{XE} \circ \alpha \) is a loop \( \gamma \) in \( X \) based at \( * \) such that the unique lift \( \gamma_E \) has \( x \) as its endpoint. But then \( \phi_E \circ \lambda(\gamma) = x \). In other words, \( \theta_E \circ \lambda \) is surjective, and the proof is finished by the above general statement.

Note that the above proof really only requires that there be a basis for the uniformity of \( X \) such that for each \( E \) in the basis, \( X_E \) is pathwise connected.

Since the closure of a connected set is connected we have:

**Corollary 80** If \( X \) is a uniformly locally pathwise connected, connected uniform space then \( \tilde{X} \) is connected.

**Proposition 81** If \( X \) is a locally pathwise connected and connected space and \( P \) is the pathwise connected component of \( \tilde{X} \) containing \( * \) then \( X \) is the quotient of \( P \) by the free isomorphic action of \( \lambda(\pi_1(X)) \).

**Proof.** We already know that \( \lambda(\pi_1(X)) \) acts freely and isomorphically on \( P \) since \( \delta_1(X) \) does. We need to check that the restriction \( \psi \) of \( \phi \) to \( P \) is bi-uniformly continuous and that for any \( x \in P \), \( \psi^{-1}(\psi(x)) \) is precisely the orbit \( \lambda(\pi_1(X))(x) \) of \( x \) (see Remark 13 of [20]). We know that \( \psi \) is uniformly continuous. Let \( D := \phi_E^{-1}(F^*) \cap P \) be an entourage in \( P \); we may assume that the \( E \)-balls are pathwise connected, hence \( X_E \) is pathwise connected, hence the restriction \( \psi_E \) of \( \phi_E \) to \( P \) is surjective by Corollary 73. Since \( \psi_E \) is surjective,

\[
F^* = \psi_E(\psi^{-1}_E(F^*)) = \psi_E(\phi^{-1}_E(F^*) \cap P) = \psi_E(D).
\]

Now

\[
\psi(D) = \phi(D \cap P) = \phi_{XE}(\phi_E(D \cap P)) = \phi_{XE}(\psi_E(D)) = \phi_{XE}(F^*)
\]

which is an entourage since \( \phi_{XE} \) is bi-uniformly continuous. This shows that \( \psi \) is bi-uniformly continuous. The statement about the orbits simply follows from the fact that the orbits of \( \psi \) are precisely the orbits of \( \phi \) intersected with \( P \) and that \( \psi^{-1}(y) = \phi^{-1}(y) \cap P \) for any \( y \in X \).

**Remark 82** Note that when \( \lambda \) is injective, the action in the above proposition is in fact an action by \( \pi_1(X) \). On the other hand, when \( \lambda \) is not surjective one has the disadvantage that \( P \) is not complete with the uniformity induced by \( \tilde{X} \).

**Definition 83** A uniform space \( X \) is called strongly coverable if \( X \) is chain connected and for some entourage \( E \), \( X_E \) is universal, hence the universal cover of \( X \).

It is clear from the discussion in the introduction that the Topologist’s Sine Curve is strongly coverable.
Lemma 84 If $X$ is strongly coverable and uniformly locally pathwise connected then $\lambda : \pi_1(X) \to \delta_1(X)$ is surjective.

Proof. Let $E$ be an entourage such that $X_E$ is universal and $F \subset E$ be an entourage having pathwise connected balls. Then $X_E$ is pathwise connected and $F^*$ is in the universal base of $X_E$ by Corollary 48. Hence $X_E$ and $X_F$ are uniformly homeomorphic and therefore $X_E = \tilde{X}$ is also pathwise connected. The proof is finished by Theorem 77.

From Theorem 75, Lemma 84, and Proposition 78 we obtain:

Theorem 85 Every connected, uniformly locally pathwise connected and uniformly semi-locally simply connected uniform space $X$ is strongly coverable with pathwise connected, simply connected universal cover and $\delta_1(X) = \pi_1(X)$.

Corollary 86 If $X$ is a compact Poincaré space then $\tilde{X}$ is the traditional universal cover of $X$ and $\lambda : \pi_1(X) \to \delta_1(X)$ is an isomorphism.

6 Dimension and the universal cover

We refer the reader to [12] for more background on dimension and uniform spaces. Suppose that $X$ has uniform dimension $\leq n$, which we denote by $u \dim X \leq n$. This means that any uniform open cover $\mathcal{V}$ of $X$ has a refinement by a uniform open cover of order $n + 1$ (a uniform open cover is an open cover that is refined by the cover of $X$ by $F$-balls for some fixed entourage $F$). Note that this particular notion of dimension is called “large dimension” in [12] and is denoted by $\Delta dX$. We will use theorems from [12] concerning another dimension, called “uniform dimension” in [12] and denoted by $\delta dX$. This particular notion of dimension uses finite covers and is somewhat more difficult to work with in the present situation. However, it is always true that $\delta dX = \Delta dX = u \dim X$ (Theorem V.5 in [12]—in fact the dimensions are equal if $\Delta dX$ is finite) and therefore if we know that $u \dim X \leq n$ then we may use theorems from [12] that require $\delta dX \leq n$. If $X$ is compact then both of these dimensions are equal to covering dimension, which we denote by $\dim X$.

Proposition 87 If $X$ is a uniform space with $u \dim X \leq n$ then for any entourage $E$, $u \dim X_E \leq n$.

Proof. Every uniform open cover of $X_E$ is refined by the cover of $X_E$ by $F^*$-balls for some entourage $F$ such that $F^2 \subset E$ and therefore we need only consider the open cover of $X_E$ by $F^*$-balls for such $F$. By definition of uniform dimension, the cover of $X$ by $F$-balls has a refinement by a uniform open cover $\mathcal{V}$ such that every $x \in X$ is contained in at most $n + 1$ sets in $\mathcal{V}$. Let $A \in \mathcal{V}$. Then $A \subset B(x, F)$ for some $x \in X$. Let $\mathcal{W}_A$ be the collection of all sets of the form $\phi_{X_E}^{-1}(A) \cap B(y, F^*)$ where $y \in \phi_{X_E}^{-1}(x)$ and $\mathcal{W} := \bigcup_{A \in \mathcal{V}} \mathcal{W}_A$. First, $\mathcal{W}$ is a cover. In fact, if $z \in X_E$, $\phi_{X_E}(z) := w \in A$ for some $A \in \mathcal{V}$, with $A \subset B(x, F)$. But $\phi_{X_E}$ restricted to $B(z, E^*)$ is a uniform homeomorphism.
onto \( B(w, E) \), which contains \( A \subset B(x, F) \) since \( F^2 \subset E \). Therefore there is some \( y \in B(z, F^*) \cap \phi_{XE}^{-1}(x) \). But \( \phi_{XE}(B(z, F^*)) = B(w, F) \) and \( (w, x) \in F \); therefore \( (z, y) \in F^* \). That is, \( z \in \phi_{XE}(A) \cap B(y, F^*) \). Since \( \mathcal{V} \) is a uniform cover, so is \( \mathcal{W} \). In fact, if the \( D \)-ball cover of \( X \) refines \( \mathcal{V} \) for some \( D \subset F \) then the \( D^* \)-ball cover refines \( \mathcal{W} \). By definition, \( \mathcal{W} \) refines the \( F^* \)-ball cover of \( X_E \).

Finally, to check that the order of \( \mathcal{W} \) is at most \( n + 1 \), we need only check that if \( y, z \in \phi_{XE}^{-1}(x) \) are distinct and \( A \in \mathcal{V} \) then

\[
\left[ \phi_{XE}^{-1}(A) \cap B(y, F^*) \right] \cap \left[ \phi_{XE}^{-1}(A) \cap B(z, F^*) \right] = \emptyset.
\]

But \( B(y, F^*) \) and \( B(z, F^*) \) are already disjoint, because otherwise for any \( w \in B(y, F^*) \cap B(z, F^*) \) we would have \( z \in B(y, (F^*)^2) \subset B(y, E^*) \), which contradicts the fact that \( \phi_{XE} \) is injective on \( E^* \)-balls. \( \blacksquare \)

In \cite{12}, Theorem IV.32 it is shown that the inverse limit of spaces with uniform dimension at most \( n \) must have uniform dimension at most \( n \). Therefore by Proposition 87.

**Theorem 88** If \( X \) is a coverable uniform space with \( u \dim X \leq n \) then \( u \dim \bar{X} \leq n \).

Note that essentially the same argument shows that if \( X \) and \( Y \) are uniform spaces, \( f : X \to Y \) is a cover, and \( u \dim Y \leq n \) then \( u \dim X \leq n \). We conjecture that in this situation \( X \) and \( Y \) have exactly the same dimension.

**Proposition 89** If \( X \) is coverable uniform space with \( u \dim X \leq 1 \), then \( \bar{X} \) contains no simple closed curve (i.e. topological circle).

**Proof.** Suppose that there is a topological embedding \( f : S^1 \to S \subset \bar{X} \) with the inverse homeomorphism \( g : S \to S^1 \). By Theorem 88, we have \( u \dim \bar{X} \leq 1 \) and therefore by Theorem V.13 of \cite{12} there is an extension of \( g \) to a uniformly continuous function \( G : \bar{X} \to S^1 \). Choose any point \( * \) as the basepoint in both \( S \) and \( \bar{X} \), and choose \( g(*) \) as the basepoint in \( S^1 \) (see Remark 54). Let \( \psi : \mathbb{R} = \hat{S^1} \to S^1 \) be the traditional universal cover of \( S^1 \) also with some choice of basepoint \( * \) in \( \psi^{-1}(*) \), which is also the universal cover in the sense of the present paper by Corollary 50. Since \( \bar{X} \) is universal by Theorem 13, Proposition 54 (2) implies that there is a unique lift \( G_L : \bar{X} \to \mathbb{R} \) such that \( G_L(*) = * \) and \( \psi \circ G_L = G \). Thus we get that

\[
\psi \circ G_L \circ f = G \circ f = g \circ f = \text{id}_{S^1}.
\]

This implies that \( G_L \circ f \) is a topological embedding of \( S^1 \) into \( \mathbb{R} \), which is impossible. \( \blacksquare \)

**Theorem 90** If \( X \) is coverable uniform space with \( u \dim X \leq 1 \) (in particular if \( X \) is compact with covering dimension \( \dim X \leq 1 \)), then \( \bar{X} \) is simply connected and if \( X \) is pathwise connected the homomorphism \( \lambda : \pi_1(X,*) \to \delta_1(X) \) is injective.
Proof. By Proposition 78 it is enough to prove that $\tilde{X}$ is simply connected. Suppose that $c : [0,1] \to \tilde{X}$ is a loop in $\tilde{X}$. Then its image $C$ is a Peano continuum that contains no simple closed curves by Proposition 89. By the Hahn-Mazurkiewicz Theorem, $C$ is locally connected, hence a dendrite (see section 51, VI in [14]). Then $C$ is contractible by Corollary 7 in Section 54, VII of [14]. Therefore $C$ has trivial fundamental group. This means that any loop at any basepoint in $C$ (including the loop $c$) is null-homotopic in $C$ hence in $\tilde{X}$.

Combining the above theorem with Proposition 81 we have the following:

Corollary 91 Let $X$ be a pathwise connected, uniformly locally pathwise connected uniform space with $\dim X = 1$. Then $X$ is the quotient of a one-dimensional pathwise connected, simply connected uniform space via a free isomorphic action of $\pi_1(X)$.

7 Pseudometric spaces

Definition 92 Let $X$ be a uniform space. We define an entourage $E$ to be chain connected if every $E$-ball in $X$ is chain connected. We say $X$ is totally chain connected if $X$ has a uniformity base that includes $X \times X$ such that each entourage in the base is chain connected.

From Proposition 80 we immediately have:

Lemma 93 If $X$ is a totally chain connected uniform space and $E$ is a chain connected entourage then $X_E$ is totally chain connected. If moreover $X$ is uniformly locally connected (resp. uniformly locally pathwise connected) then $X_E$ has the same property.

Example 94 In the proof of Theorem 9, [2], it was shown that the character group of $\mathbb{Z}^N$ is not coverable. But this group is known to be connected and locally connected, hence totally chain connected.

Theorem 95 Every totally chain connected pseudometric space $X$ is coverable.

Proof. Since $X$ is a pseudometric space we can find a countable sequence $\{E_i\}$ of chain connected entourages forming a base for the uniformity of $X$ (hence cofinal in the set of all entourages of $X$). Therefore $\tilde{X} = \lim X_{E_i}$. Each of the spaces $X_{E_i}$ is chain connected by Lemma 94 and when $j \geq i$ we may identify $X_{E_j}$ with $(X_{E_i})_{E_j}$ by Proposition 24. Therefore, Lemma 12 implies that each bonding map $\phi_{E_i,E_j}$ is surjective. Since the inverse system is countable it follows that the projections $\phi_{E_i} : \tilde{X} \to X_{E_i}$ are all surjective. ■

Since connected sets are chain connected, we have the following two corollaries:

Corollary 96 Every connected, uniformly locally connected pseudometric space is coverable.
Recall that a Peano continuum is a Hausdorff topological space that is the continuous image of an interval. Equivalently (by the Hahn-Mazurkiewicz Theorem), a Peano continuum is a compact, connected, locally (pathwise) connected metrizable space.

**Corollary 97** Every Peano continuum $X$ is coverable. Moreover, there is a compact subset $S$ of the pathwise connected component of $\tilde{X}$ such that the restriction of the universal covering map $\phi$ to $S$ is surjective.

**Proof.** The first statement follows from Corollary 74. Let $c : [0, 1] \to X$ be a continuous surjection. Then $S := c_L([0, 1])$ has the desired properties. ■

**Notation 98** To simplify matters, when $X$ is a metric space we refer to an $E_\varepsilon$-loop, where $E_\varepsilon$ is the metric entourage having open $\varepsilon$-balls as $E_\varepsilon$-balls, as an $\varepsilon$-loop ($\varepsilon > 0$), $E_\varepsilon$-homotopies as $\varepsilon$-homotopies, etc. We will denote $X_{E_\varepsilon}$ by $X_\varepsilon$, $\phi_{E_\varepsilon}$ by $\phi_\varepsilon$, $\phi_{X_{E_\varepsilon}}$ by $\phi_{X_\varepsilon}$, and $\phi_{E_\varepsilon}$ by $\phi_\varepsilon$.

Note that for a metric space $X$ and $\varepsilon > 0$, the cover $\phi_{X_\varepsilon} : X_\varepsilon \to X$ is a broadening of the notion of “$\varepsilon$-cover”, something that goes back at least to Spanier’s book ([24]), and has been used in [23] and [24] to study universal covers of limits of Riemannian manifolds. However, the construction of $\varepsilon$-covers uses paths and standard homotopies rather than chains.

It is not hard to construct examples of uniformly locally pathwise connected metric spaces having metric balls that are not necessarily pathwise connected, and so that the metric entourages are not covering entourages. Recall that an inner metric space is a metric space such that the distance between any two points is the infimum of lengths of curves joining them. A geodesic space further has the property that the distance is realized as the length of some curve, called a minimal geodesic. The metric balls of any inner metric space are pathwise connected and therefore the metric entourages are covering entourages. We have:

**Corollary 99** Every inner metric space $X$ is coverable. Moreover, $\tilde{X} = \lim_{\leftarrow} X_\varepsilon$, where $\varepsilon_i$ is any sequence of positive values decreasing to 0.

From Lemma 93 and Corollary 50 we have:

**Corollary 100** An inner metric space $X$ is universal if and only if for any $\varepsilon > 0$, every $\varepsilon$-loop based at some point is $\varepsilon$-homotopic to the trivial loop.

Note that the topologist’s sine curve, discussed in the introduction, is coverable but not totally chain connected because at some points every small neighborhood contains pairs of points that cannot be joined by an arbitrarily fine chain that stays inside the neighborhood.

We next give a unified calculation of the deck groups $\delta_1(X)$ of the Hawaiian earring $X = H$, the Sierpinski gasket $X = \Delta_S$, and Sierpinski carpet $X = C_S$. 

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This unified calculation is possible because each of these spaces admits a description as a countable intersection

\[ X = \bigcap_{n=0}^{\infty} X_n. \]  

Let us begin with the case \( X = H \). Define \( X_0 := H_0 \) as the closed disc \( D \) of diameter 1, lying in the upper Euclidean half-plane \( y \geq 0 \) and tangent to the \( x \)-axis at the origin \( O = (0,0) \). Let \( C = \partial D \) be its boundary circle. Denote by \( \psi_{1,c} \) the \( \alpha \)-homothety of Euclidean plane \( E^2 \) with the center \( c \). We omit \( c \) in the notation if \( c = O \). Now define \( X_n, n \geq 0 \), iteratively by the formulas

\[ X_0 := D, \quad X_{n+1} := \psi_{\frac{1}{n+1}}(X_n) \cup C. \]  

Then \( X = H \) is defined by Formulas (6) and (5).

For \( X = \Delta_S \) let \( X_0 := \Delta_0 \) be the isosceles triangle of diameter 1 in the Euclidean quarter-plane \( x \geq 0, y \geq 0 \), with one side on the \( x \)-axis and one vertex \( v_0 = O \); denote the other vertices by \( v_1, v_2 \). Now define \( X_n \) iteratively by the formulas

\[ X_0 = \Delta_0, \quad X_{n+1} := \bigcup_{i=0}^{2} \psi_{\frac{i}{n+1}}(X_n). \]  

Then \( X = \Delta_S \) is defined by Formulas (7) and (6).

For \( X = C_S \) let \( X_0 := C_0 \) be the square of diameter 1 in the Euclidean quarter-plane \( x \geq 0, y \geq 0 \) with one side on each of the coordinate axes. Denote by \( v_0, v_1, v_2, v_3 \) its vertices and \( w_0, w_1, w_2, w_3 \) the midpoints of its sides. Now define \( X_n \) iteratively by the formulas

\[ X_0 = C_0, \quad X_{n+1} := \bigcup_{i=0}^{3} \psi_{\frac{i}{n}}(X_n). \]  

Then \( X = C_S \) is defined by Formulas (8) and (7).

Denote by \( \varepsilon_n \) the number \( \frac{1}{2^{n+1}} \) in the cases \( X = H \) and \( X = \Delta_S \) and the number \( \frac{1}{2^n} \left( \frac{1}{2^n} \right) \) in the case \( X = C_S \). Moreover, by our choice of \( \varepsilon_n \), \( \phi_{X_n, \varepsilon_n} : (X_n)_{\varepsilon_n} \rightarrow X_n \) (see Notation 19) is the universal cover of \( X_n \), which means that the projection \( \tau_n : \delta_1(X_n) \rightarrow \delta_n(X_n) := \delta_{\varepsilon_n}(X_n) \) is an isomorphism. One can easily see that in all cases \( X_n \) is a compact Poincaré space and \( \delta_{\varepsilon_n}(X) \) is isomorphic to \( \delta_{\varepsilon_n}(X_n) \) via the map \( j_n = (i_n)_{E\varepsilon_n} \) defined in Definition 20, where \( i_n : X \rightarrow X_n \) is the inclusion. According to Corollary 20 \( \lambda : \pi_1(X_n) \rightarrow \delta_1(X_n) \) is an isomorphism. Therefore for any fixed \( n \), the function \( \omega_n := \tau_n \circ \lambda : \pi_1(X_n) \rightarrow \delta_n(X_n) \) is an isomorphism.

According to Proposition 20 the image with respect to \( \omega_n \) of an equivalence class of a loop \( c \) based at \( * \) is the \( E \)-homotopy class of any sufficiently fine \( E \)-chain of the form \( \{ c(t_0), \ldots, c(t_k) \} \). Therefore, if \( n \geq m \) we have the following
commutative diagram:

\[
\begin{align*}
\pi_1(X_n) & \xrightarrow{\omega_n} \delta_n(X_n) & \xrightarrow{j_n^{-1}} & \delta_{\epsilon_n}(X) \\
\pi_1(X_m) & \xrightarrow{\omega_m} \delta_m(X_m) & \xrightarrow{j_m^{-1}} & \delta_{\epsilon_m}(X)
\end{align*}
\]

where \(i_{mn}: X_n \to X_m\) is inclusion and \(j = h \circ \theta_{\epsilon_m, \epsilon_n}\) with \(h: (X_n)_{\epsilon_n} \to (X_m)_{\epsilon_m}\) being the inclusion-induced mapping given by Definition 20.

But \(X_n\) is homotopic to a wedge product of \(q_n\) circles, where \(q_n = n\) for \(X = H\), \(q_n = \sum_{k=0}^{n-1} 3^k\) for \(X = \Delta S\), and \(q_n = \sum_{k=0}^{n-1} 8^k\) for \(X = C S\). So, in any case, \(\pi_1(X_n) \cong F_{q_n}\), where \(F_q\) is the free group with \(q\) generators. Note that \(\delta_1(X) = \lim_{\leftarrow} \delta_{\epsilon_n}(X)\) and for all three examples the inverse sequence \((\delta_{\epsilon_n}(X), \theta_{mn})\) is a cofinal sequence in the inverse system \((F_n, \pi_{mn})\) where \(\pi_{mn}: F_n \to F_m\) is the unique surjective homomorphism that kills one extra generator.

Although all three spaces have the same deck group, the spaces \(H, \Delta S,\) and \(C S\) are mutually non-homeomorphic one-dimensional Peano continua. In fact, the last two spaces are not semi-locally simply connected at any point, while \(H\) has unique point \((O)\), at which it is not semi-locally simply connected. The spaces \(\Delta S\) and \(C S\) are not homeomorphic, because the first space have a countable subset \(M\) such that \(\Delta S - M\) is null-homotopic in itself, while the second one has no such subset. We can take

\[
M = \bigcup_{n=0}^{\infty} \psi^n(\{m\}),
\]

where \(m\) is the midpoint of the segment \([v_1, v_2]\), and

\[
\psi(A) := \bigcup_{i=0}^{2^n} \psi_{1/2^n}(A)
\]

for every subset \(A \subset E^n\).

Since \(\Delta S\) and \(C S\) are non-homeomorphic one-dimensional Peano continua that are not semi-locally simply connected at any point, then by a result of \[9\], \(\pi_1(\Delta S)\) and \(\pi_1(C S)\) are not isomorphic as abstract groups. Thus it follows from this and the above result \(\delta_1(\Delta S) \cong \delta_1(C S)\) that the homomorphism \(\pi_1(X) \to \delta_1(X)\) cannot be a bijection for all three examples. Finally, note that the conditions of Proposition \[89\] are satisfied and hence the image of the fundamental group with respect to \(\lambda\) is dense in \(\delta_1(X)\) in each case.

**Remark 101** A similar discussion may be applied to the Sierpiński sponge \(S_S\) (also known as the universal Sierpiński curve) which is formed by successively removing “middle cores” from the unit cube, implying that \(\delta_1(S_S)\) is also the inverse limit of finitely generated free groups. It is well-known that every 1-dimensional locally connected metrizable continuum can be topologically embedded in \(S_S\). By a result of Curtis and Fort \[10\], these embeddings induce inclusions of the fundamental group, and it follows that the fundamental group of any such continuum embeds in an inverse limit of finitely generated free groups. This fact has been previously established by various authors, most recently \[7\].
8 Topological Groups

The construction, in [1], of the group \( \tilde{G} \) for a (Hausdorff) topological group \( G \) is the same as the construction that we use in the present paper. However, the construction in the prior paper includes a compatible group structure on each group in the fundamental inverse system, which induces a group structure on \( \tilde{G} \) so that the natural homomorphism \( \phi : \tilde{G} \to G \) is a quotient map with closed, central, profinite kernel. (There are also several results in [1] that have no analogs for uniform spaces in general, such as results concerning extensions of local homomorphisms.) For a topological group \( G \), the condition that we call “universal” in this paper is equivalent to what was called “locally defined” in [1], by Proposition 61 of [1]. A “coverable group” was defined to be the quotient of a locally defined group via a closed normal subgroup. According to Corollary 60 in the present paper, such a group is a coverable uniform space.

The converse of this statement involves Theorem 90 in [1], which requires a correction. In fact, Professor Helge Glöckner of T. U. Darmstadt has pointed out that the proof of Lemma 42 in [1] has a gap, and we do not know whether this lemma is true. The only direct reference to Lemma 42 is Theorem 90, and in light of this gap part (2) of Theorem 90 should be restated as the following stronger condition: \( (2') \ G \) has a basis for its topology at \( e \) consisting of locally generated symmetric neighborhoods, and \( \phi_U \) is surjective for all \( U \) in this basis. The proof may be modified to show (1) implies (2') as follows. In the first part of the proof it is shown that \( \phi \) is surjective and that \( G_U \) is coverable. The same argument may then be applied to show that \( \phi_U \) is surjective, proving (2'). In the proof of (2)\( \Rightarrow \) (3), Lemma 42 is only used to prove that all of the homomorphisms \( \phi_U \) are surjective, and so (2') eliminates the need for Lemma 42. Note that (2') is actually a stronger condition than the definition of coverable in the present paper, which now completes the proof that a topological group is coverable in the sense of [1] if and only if it is coverable in the present sense.

We know of no example of a uniform space \( X \) (let alone topological group) that is not coverable and \( \phi : \tilde{X} \to X \) is surjective, hence it is still possible that Theorem 90 (or its generalization to uniform spaces) is true as stated. However, the revised version is sufficient for all applications in [1] and [2] except for three. The first is Corollary 91 of [1], which is not used elsewhere. The second exception is that the alternate hypothesis in Theorem 15 “or \( \phi : \tilde{G} \to G \) is surjective” needs to be taken out (or replaced by a stronger assumption related to the new condition (2')). The third exception is Corollary 107, which isn’t used elsewhere.

In the interest of completeness we will now address every reference to Theorem 90 both in [1] and [2] to show that the new version is sufficient. Since (2') is stronger than (2), there is no problem with any statement that doesn’t involve (2)\( \Rightarrow \) (1) or (2)\( \Rightarrow \) (3). Theorem 90 is used this way in proofs of the following statements in [1]: Theorem 4, Theorem 5, Proposition 10, Example 99, Theorem 92 (necessity), Theorem 101, and Theorem 15 (second reference).

Now consider the remaining references: In Theorem 92 (sufficiency) of [1],
all of the projections are surjective and therefore (2′) holds. In [2], Theorem 24 is simply a restatement of Theorem 90 (with an additional statement about metric spaces added) and therefore (2) must be replaced by (2′). Theorem 24 is only used in the proof of Theorem 7, and the only problematic usage is that Theorem 24, (2)⇒(3), is used to prove Theorem 7, (6)⇒(1). However, it is well-known that an pathwise connected, locally compact group is locally pathwise connected. Therefore for any pathwise connected symmetric open set \( U \) containing the origin, \( G_U \) is pathwise connected and hence \( \phi_U : \tilde{G} \rightarrow G_U \) is surjective. That is, the conditions for (2′) are satisfied.

The relationship between covers in the present sense and covers in the sense of [1] is considered in [20]. Note that our paper partially answers Problem 152 in [1], which asks whether the generalized fundamental group in that paper, which is the same as the deck group in the present paper, is a topological invariant. From the present paper we know that the deck group is in fact an invariant of uniform structures and for compact groups a topological invariant. There are several other theorems from [1] that likely can be generalized to the more setting of uniform spaces, and these will be considered in a future paper. Some of the questions of that paper have analogs for uniform spaces, most notably:

**Problem 102** If \( X \) is a uniform space, is \( \tilde{X} \) always universal?

**Problem 103** If \( X \) is a uniform space and \( \phi : \tilde{X} \rightarrow X \) is a uniform homeomorphism, is \( X \) coverable?

**Problem 104** If \( X \) is a simply connected, pathwise connected coverable space, is \( X \) universal?

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