Discontinuity problem in the linear stability analysis of thin-shell wormholes

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Received: 5 March 2019 / Revised: 19 May 2019
Published online: 19 July 2019
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Abstract. We investigate the infinite discontinuity points of the stability diagram in thin-shell wormholes. The square of the speed of sound $\beta^2$, which is expressed in terms of pressure and energy density at equilibrium on the throat, arises with a divergent amplitude. As this is physically non-acceptable, we revise the equation of state, such that by fine-tuning of the pressure at static equilibrium, which is at our disposal, we eliminate such singularity. The efficacy of the method is shown in Schwarzschild, extremal Reissner-Nordström and dilaton thin-shell wormholes.

1 Introduction

The concept of thin-shell wormhole (TSW) was introduced by Visser in 1989 [1,2] in the hope of keeping the idea of wormholes alive by confining the exotic matter to a thin shell, called the throat of the TSW. Exotic matter, which inevitably emerges in the theories of wormholes, is an unwanted type of matter that violates the known energy conditions, such as the weak energy condition (WEC) [3]. Prior to Visser’s articles, theories had the exotic matter distributed on certain parts of the spacetime, if not all over it. However, Visser’s so-called cut-and-paste procedure allows us to confine such a notorious matter on a very limited part of the space, the TSW itself. Moreover, the cut-and-paste procedure has the advantage that can be applied to a vast variety of spacetimes [4–16] while, before Visser, only some certain spacetimes had the structure of a wormhole [17]. It is also worth mentioning that, while TSWs are categorized as traversable wormholes, not all the wormholes are considered to be traversable [18].

Recently we have systematically analyzed the asymmetric thin-shell wormholes (ATSWs) with different spacetimes on the two sides of the throat [19–21]. The role of asymmetry in the stability of TSWs, if any, has been studied. In the this article we address to a particular issue of stability which incorporates infinite discontinuity in the stability diagrams. From the Schwarzschild TSW, it was observed that at a certain equilibrium radius of the shell, \textit{i.e.} $a_0 = 3m$ (for Schwarzschild mass $m$), there arises a divergence in the asymptotes of the stability curves [22]. The divergence radius occurs at a finite radius which lies outside the event horizon and inside the cosmological horizon (if any), excluding the divergences at the center and at infinity. What gives rise to this behavior? Is it possible to eliminate these types of divergences?

In the barotropic equation of state (EoS), the pressure $p$ and the energy density $\sigma$ on the shell are related to the square of the speed of sound $\beta^2$ as $\beta^2 = \frac{dp}{d\sigma}$. This relation is expressed by $p'_0 = \beta^2_0 \sigma'_0$, where a prime denotes deriva-
tive with respect to the radius, all evaluated at the equilibrium radius $a_0$. It turns out that the mathematical structure of the speed of sound $\beta_0$ is given by a fractional expression, such that it diverges at a finite radius when its denominator approaches zero, for a nonzero numerator. This is precisely what happens at $a_0 = 3m$ in the Schwarzschild TSW. The same behavior is observed for other TSWs as well, at a certain radius of the shell. Once we identify this fact, we present a recipe to eliminate such types of divergences. This is achieved by considering a more general EoS (introduced under the name “variable EoS” [4,5]), in which the pressure, besides the energy density, depends explicitly on the radius of the shell, as well. With the new EoS, the speed of sound relation takes the form $\beta_0^2 = (p_0 + \gamma_0)/\sigma_0^2$, for $\gamma_0 =$ const., so that the choice $p_0 + \gamma_0 = 0$ will eliminate the singularity for the stability diagram, in the limit.

Additionally, Varela [5] has pointed out the solution for a symmetric Schwarzschild TSW without addressing the cause for such an anomaly, a general removal method for other TSWs, or considering asymmetric cases. Beside the Schwarzschild ATSW, we consider extremal Reissner-Nordström (ERN), and the dilaton TSWs as well, as examples to show that our method works perfectly for eliminating the infinite discontinuity. Let us add that since our shell occurs at finite radius, the remaining infinite divergences at the speed of sound at $a_0 = r_e$ (event horizon) and $a_0 = \infty$ are not of physical significance.

In sect. 2 we briefly explain how a TSW can be constructed by gluing two (generally not identical) spacetimes at a common hypersurface. Section 3 is devoted to the infinite discontinuity emerging in the stability diagram of TSWs. Therein, we clarify the subject by discussing examples from the Schwarzschild ATSW, ERN ATSW, and dilaton TSW. In sect. 4 we explain how replacing the barotropic EoS with the variable EoS contributes to the elimination of the infinite discontinuity. Finally, we conclude the paper in sect. 5. Throughout the article, we follow the unit convention $c = 8\pi G = 4\pi\varepsilon_0 = 1$, where $c$ is the speed of light, $G$ is the gravitational constant, and $\varepsilon_0$ is the permittivity of free space in 3 + 1 dimensions.

### 2 Constructing a TSW

To construct a TSW by Visser’s method in spherical coordinates, consider two distinct Lorentzian spacetimes denoted by $(\Sigma, g)^\pm$. Out of each spacetime, a subset is cut such that no singularities or event horizons of any sort are included, i.e. $(\Sigma, g)^\pm \subset (\Sigma, g)^\pm$ and $(\Sigma, g)^\pm = (x^\mu_\pm|r_\pm \geq a(r) > r_e)$, where $r_e$ is any existing event horizon, and $a$ is the proper time on the shell $r_\pm = a$. Then, by pasting these two cuts at their common timelike hypersurface $\partial T$, such that $\partial T \subset (\Sigma, g)^\pm$, one creates a complete Riemannian spacetime which provides a passage from one spacetime to the other. The hypersurface $\partial T$ is indeed the throat of the TSW and contains the exotic matter. Note that the coordinates of the two sides of the throat $x^\pm_t$ and, more generally, the very nature of the two spacetimes does not necessarily need to be the same. Although most authors consider thin shell wormholes admitting a mirror symmetry across the throat, such a mirror symmetry was broken in some studies recently [19–21] to introduce ATSWs.

Suppose that the line element of the bulks are given by the static general spherically symmetric metrics

$$ds^2_\pm = g^\pm_{\mu\nu}dx^\mu_\pm dx^\nu_\pm = -f_\pm(r_\pm)dt^2_\pm + f_\pm^{-1}(r_\pm)dr^2_\pm + h_\pm(r_\pm)d\Omega^2_\pm,$$

where $f_\pm(r)$ and $h_\pm(r)$ are positive functions of the radial coordinates $r_\pm$, and $d\Omega^2_\pm$ are the line elements of unit 2-spheres. The line element on the hypersurface $\partial T$ (the throat) is given by

$$ds^2_{\partial T} = q^\pm_{ij}d\xi^i d\xi^j,$$

where $\xi^i$ are the local coordinates on the shell and $q^\pm_{ij} = \frac{\partial x^\pm_\sigma}{\partial \xi^i}\frac{\partial x^\pm_\lambda}{\partial \xi^j}g^\pm_{\sigma\lambda}$ are the localized metric of $\partial T$. The unit spacelike normals to the surface are also given by $n^\pm_\mu = \frac{\partial x^\mu_\pm}{\partial \xi^\pm}$, provided $n^\pm_\mu n^{\pm\mu}_\pm = 1$. To account for the uniqueness of the TSW, $q^\pm_{ij} = q^\pm_{ij}$ must hold on the throat. In general relativity, this is called the first Israel-Darmois junction condition [23]. More particularly, this condition requires that $h_\pm(a) = h_+(a)$ at the throat. There also exists a second junction condition, which imposes a discontinuity on the extrinsic curvature tensor components, given by

$$K^\pm_{ij} = -n^\pm_\lambda \left( \frac{\partial^2 x^\pm_\lambda}{\partial \xi^i \partial \xi^j} + \Gamma^\lambda_{\alpha\beta} \frac{\partial x^\pm_\alpha}{\partial \xi^i} \frac{\partial x^\pm_\beta}{\partial \xi^j} \right),$$

where $\Gamma^\lambda_{\alpha\beta}$ are the Christoffel symbols of the bulk spacetimes, compatible with $g^\pm_{\alpha\beta}$. By introducing $S^\pm_{ij} = \text{diag}(-\sigma, p, p)$ as the stress-energy tensor of the perfect fluid on the throat, with $\sigma$ and $p$ being the surface energy density and lateral pressure, respectively. The second junction condition admits

$$[K^+_i]_+ - [\delta^+_j K^+]_+ = -S^+_i,$$
where we symbolically have $|Ψ|^+ = Ψ_+ - Ψ_-$, for a jump in quantity $Ψ$ passing across the throat. Employing all the cumbersome calculations, one obtains

$$σ = -\frac{h'}{h} \left( \sqrt{f_+ + \dot{a}^2} + \sqrt{f_- + \dot{a}^2} \right)$$

(5)

and

$$p = \frac{\sqrt{f_+ + \dot{a}^2}}{2} \left( \frac{2\dot{a} + f_+}{f_+ + \dot{a}^2} + \frac{h'}{h} \right) + \frac{\sqrt{f_- + \dot{a}^2}}{2} \left( \frac{2\dot{a} + f_-}{f_- + \dot{a}^2} + \frac{h'}{h} \right),$$

(6)

where $h = h_+ = h_-$, due to the first junction condition. Here, an overdot and a prime represent a total derivative with respect to the proper time on the throat $τ$ and the corresponding radial coordinates $r_±$, respectively. Note that all the functions are evaluated at the location of the throat $r_± = a$. The conservation of energy, is identified as [11]

$$\sigma' + \frac{h'}{h} (σ + p) + \frac{h'^2 - 2hh''}{2h'^2} σ = 0.$$  

(7)

The latter is accompanied by an EoS to yield the so-called linear stability analysis. In this popular method, developed by Poisson and Visser [22], eq. (5) is recast into the form

$$\dot{a}^2 + V(a) = 0,$$  

(8)

to resemble the equation of conservation of mechanical energy with a kinetic term $\dot{a}^2$ and an effective potential term

$$V(a) = -\left( \frac{hσ}{2h'} \right)^2 - \left[ \frac{h'}{2hσ} (f_+ - f_-) \right]^2 + \frac{f_+ + f_-}{2}.$$  

(9)

The potential is then expanded using Taylor series about a hypothetical static equilibrium radius $a_0 > r_c$ to a quadratic term

$$V(a) = V(a_0) + V''(a_0) (a - a_0) + \frac{1}{2} V'''(a_0) (a - a_0)^2 + O^4(a).$$  

(10)

The first two terms on the right-hand side are zero due to the static version of eq. (8) and the assumption of $a_0$ being the equilibrium radius, respectively. Therefore, in the very vicinity of $a_0$, the effective potential $V(a)$ is approximated by the first non-zero term on the right-hand side, i.e. the third term which is proportional to $V''(a_0)$. If $V''(a_0) > 0$ ($V''(a_0) < 0$), the state of the ATSWS is said to be mechanically stable (unstable). In order to proceed with the stability analysis, $V''(a_0)$ must be explicitly calculated, and an EoS is required to do so. An EoS is an equation that relates the energy density $σ$ and the pressure $p$ of the throat. Of the EoS which appeared in the literature, one may enumerate the barotropic EoS [22], the EoS of a (generalized) Chaplygin gas [24], polytropic gas EoS [25], phantomlike EoS and the variable EoS [5]. Nonetheless, the barotropic EoS, mathematically given by $p = p(σ)$, due to its simple yet realistic nature, provides an useful model for the fluid's behavior on the throat of the TSW. Therefore, following [22], here we consider the barotropic EoS with $β^2 ≡ dp/ dσ$, which implies that

$$p_0 = β_0^2 σ_0$$  

(11)

holds on the throat, when it is at the equilibrium radius $a_0$ (the sub-index zero indicates the value of the parameter at $a_0$, i.e. $T_0 = \mathcal{T}(a_0)$ for each physical variable $\mathcal{T}(a_0)$). This in turn amounts to

$$σ''_0 = \frac{1}{4h_0^2 h''_0} \left\{ \left[ (2β^4_0 + 5)(3σ_0 + 2p_0) \right] h''_0 - 2 \left[ (2β^4_0 + 9)σ_0 + 4p_0 \right] h_0 h''_0 + 4h_0^2 h''_0 σ_0 \right\},$$  

(12)

for the second derivative of the energy density with respect to the radial coordinate $a$, at the equilibrium radius $a_0$. This expression for $σ''_0$ appears naturally in $V''(a_0)$. It is observed easily that for spacetimes with $h(r) = r^2$, which covers a large class, this expression reduces to $σ''_0 = \frac{2}{h_0^2} (2β^4_0 + 3)(σ_0 + 2p_0)$.

Using the static versions of eqs. (5) and (6), along with eqs. (7) and (12), one can calculate the expression for $V''(a_0)$ by taking the second derivative of $V(a)$ in eq. (9). According to the linear stability analysis method [22], $V''(a_0)$ is then set to zero to write $β_0^2$ in terms of $a_0$ (and possibly other parameters such as mass or charge). Afterwards, $β_0^2$ is plotted against (a redefined) $a_0$ and the regions of stability (regions wherein $V''(a_0)$ becomes positive) are specified; e.g., see fig. 4(a) for $β_0^2$ against $a_0/m$ plotted for a usual Schwarzschild TSW, when the matter on the throat is barotropic. The stable regions are indicated in the figure. Generally speaking, the graph of $β_0^2$ against $a_0$ may exhibit some infinite discontinuities at some specific radii, which are the main focus of this study. In what follows we address these infinite discontinuities and the reason of their emergence, followed by some examples for clarification.
Fig. 1. The graph shows $m(\sigma_0 + p_0)$ versus $a_0/m$ for a symmetric Schwarzschild TSW. At $a_0 = 3m$, $m(\sigma_0 + p_0) = 0$ which validates the previous results. Note that while $\sigma_0 + p_0$ is positive valued pre-3$m$, it is negative post-3$m$. This explains $\lim_{a_0 \to 3m} \beta_0^2 = \mp\infty$ in the original stability diagram.

3 Infinite discontinuity in the stability diagram

Previously, we have observed that for the barotropic EoS we have $p_0' = \beta_0^2 \sigma_0'$ on the throat. Exercising this on the static version of eq. (7) leads to

$$
\beta_0^2 = \frac{p_0'}{h_0' (\sigma_0 + p_0) + \frac{2h_0'^2 - h_0'^2 - \sigma_0}{2a_0 h_0'^2 - \sigma_0}},
$$

which goes to infinity once the denominator goes to zero, unless $p_0' \to 0$ faster. Therefore, the infinite discontinuity is fundamental and cannot be removed by, say, changing the coordinates. In the symmetric case where $h(a) = a^2$ on each side of the throat, the above equation reduces to the simpler form

$$
\beta_0^2 = \frac{p_0'}{-\frac{\sigma_0}{a_0} (\sigma_0 + p_0)},
$$

In such case, $\sigma_0 + p_0 \to 0$ faster than $p_0' \to 0$ leads to an infinite discontinuity. Here we proceed with some examples.

3.1 The Schwarzschild ATSW

It is well known that, for a symmetric Schwarzschild TSW, there exists an infinite discontinuity at $a_0 = 3m$, where $m$ is the central mass of the Schwarzschild spacetime [22]. More generally, for a Schwarzschild ATSW with metric functions

$$
\begin{align*}
  f_{\pm} & = 1 - \frac{2m_{\pm}}{r_{\pm}}, \\
  h_{\pm} & = r_{\pm}^2,
\end{align*}
$$

we obtain

$$
\sigma_0 + p_0 = -\frac{[a_0 - 3(1 + \epsilon)m][a_0 - 2m] \sqrt{a_0 - 2(1 + \epsilon)m} + [a_0 - 2(1 + \epsilon)m][a_0 - 3m] \sqrt{a_0 - 2m}}{a_0^{3/2}[a_0 - 2(1 + \epsilon)m][a_0 - 2m]},
$$

where $m_- \equiv m$ and $m_+ \equiv (1 + \epsilon)m$, and $\epsilon \in [0, \infty)$ is the mass asymmetry factor. This has a double root at

$$
a_{\text{ID}} = \frac{3m}{8} \left[3(\epsilon + 2) \pm \sqrt{9\epsilon^2 + 4\epsilon + 4}\right],
$$

where the sub-index “ID” stands for infinite discontinuity. However, for the admissible domain of $\epsilon$, the root with the minus sign falls behind the event horizon, i.e. $a_{\text{ID}_-} < r_\Sigma$, whereas $a_{\text{ID}_+} > r_\Sigma$ always holds. Hence, an infinite discontinuity is expected at $a_{\text{ID}_+}$, which obviously leads to $a_{\text{ID}} = 3m$ for a symmetric Schwarzschild TSW with $\epsilon = 0$. Figure 1, which illustrates $m(\sigma_0 + p_0)$ versus $a_0/m$ for the symmetric case, explains why $\lim_{a_0 \to 3m} \beta_0^2 = \mp\infty$. 

Fig. 2. The graph of $m(\sigma_0 + p_0)$ against $a_0/m$ for a symmetric ERN TSW. The zero of the vertical axis at $a_0 = 2m$ is expected according to the previous studies.

3.2 The extremal Reissner-Nordström TSW

The case of an extremal Reissner-Nordström (ERN) TSW has also been considered in the literature [14,26,27]. Having

$$
\begin{align*}
  f_{\pm} &= \left(1 - \frac{m_{\pm}}{r_{\pm}}\right)^2, \\
  h_{\pm} &= r_{\pm}^2,
\end{align*}
$$

as the metric function of ERN, we obtain

$$
\sigma_0 + p_0 = -\frac{2[a_0 - (\epsilon + 2)m]}{a_0^2}.
$$

Accordingly, there must be an infinite discontinuity at

$$
a_{ID} = (\epsilon + 2)m.
$$

As shown in [14], this also admits an infinite discontinuity at $a_0 = 2m$ for a symmetric ERN TSW, when $\epsilon = 0$. For a symmetric TSW analogous to the previous case, $\lim_{a_0 \to 2m} \beta_0^2 = \mp\infty$, according to fig. 2 plotted for $m(\sigma_0 + p_0)$ versus $a_0/m$.

3.3 The dilaton TSW

In [11], Eiroa studied a TSW constructed by two symmetric spacetimes which are the solutions of the action

$$
S = \int d^4x \sqrt{-g} \left[ -R + (\nabla \phi)^2 + e^{-2b\phi} F^2 \right].
$$

Herein, $g = \det(g_{\mu\nu})$, $R$ is the Ricci scalar, $\phi$ is the scalar dilaton field, $F = F^{\mu\nu} F_{\mu\nu}$ with $F_{\mu\nu}$ being the electromagnetic field, and $b \in [0,1]$ is the coupling parameter between the dilaton and the electromagnetic field. In Schwarzschild coordinates, the spherically symmetric solution is given by

$$
ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + h(r)d\Omega^2,
$$

where the metric functions are [28,29]

$$
\begin{align*}
  f(r) &= \left(1 - \frac{A}{r}\right) \left(1 - \frac{B}{r}\right)^{(1-b^2)/(1+b^2)}, \\
  h(r) &= r^2 \left(1 - \frac{B}{r}\right)^{2b^2/(1+b^2)}.
\end{align*}
$$
The constants $A$ and $B$ are related with the mass $m$ and charge $q$ of the spacetime through

\[
\begin{align*}
A &= m \pm \sqrt{m^2 - (1 - b^2)q^2}, \\
B &= (1 + b^2)q^2/A.
\end{align*}
\] (24)

Here we consider only the plus sign, since it is the plus sign that corresponds to the Schwarzschild metric when $q = 0$. The solutions for $b = 0$ reduce to the normal RN solutions for the Einstein-Maxwell action with a scalar field. For $b = 1$ a family of static, spherically symmetric charged solutions in the context of low-energy string theory are recovered [28]. Moreover, for $0 \leq q^2/m^2 < (1 + b^2)$ the solution is a black hole with an event horizon at $r = A$ and an inner horizon at $r = B$. When $1 + b^2 \leq q^2/m^2 \leq 1/(1 - b^2)$, the inner horizon grows larger than the event horizon and the metric exhibits a naked singularity. Also, the spacetime is not well defined, if $q^2/m^2 > 1/(1 - b^2)$. In what follows $a_0$ is considered to be greater than $A$ and $B$, as it must be.

According to the solution in eqs. (22) and (23), it is expected that the root of the denominator in the expression for $\beta_0^2$ in eq. (13) denotes the infinite discontinuity in the stability diagram. Having considered the static energy density as

\[\sigma_0 = -\frac{h_0'}{h_0}\sqrt{f_0},\] (25)

and the static pressure as

\[p_0 = \frac{f_0 + h_0'}{(f_0 + h_0')^2}\sqrt{f_0},\] (26)

one calculates for the roots of the denominator of $\beta_0^2$ in eq. (13), to acquire $a_{ID}$. Due to the relatively complicated forms of $f(r)$ and $h(r)$ in eq. (23), the expression for $a_{ID}$ is complicated and lengthy, too. For this reason, we refrain from bringing its explicit form here. Instead, we summarize the results in fig. 3. Considering five different values for $b$, the subfigures display $a_{ID}$, the event horizon ($EH$) associated with $r = A$, and the inner horizon ($IH$) associated with $r = B$, in diagrams of $a_{ID}/m$ against $|q|/m$. The results are in complete agreement with the ones in [11]. For instance, for $b = 0$ in fig. 3(a), there always exists an infinite discontinuity beyond horizons. In the numerical examples, this discontinuity is $a_{ID}^0 = 3m$ for $|q|/m = 0$, and $a_{ID}^0 \approx 2.458m$ for $|q|/m = 0.8$, as expected. Note that $|q|/m > 1$ is not allowed due to the restricting conditions on the bulk spacetime mentioned above. On the other hand, when $b = 1$, there is no infinite discontinuity for $|q|/m \geq \sqrt{2}$. Again, for the sake of comparison to the results in [11], note that, for example, when $|q|/m = 0.8$ we obtain $a_{ID} \approx 2.594m$, and $a_{ID} \approx 2.860m$, when $b = 0.5$, and $b = 1$, respectively. Furthermore, remark that $a_{ID}$ is $3m$ in all cases, when $|q|/m = 0$, due to the simple fact that when $q = 0$, the metric functions in eq. (23) reduce to the Schwarzschild metric functions, regardless of the value of $b$.

4 Variable EoS

In 2015, Varela demonstrated that the infinite discontinuity of a Schwarzschild TSW can be removed by using a rather different EoS, called the variable EoS [5]. Mathematically shown as $p = p(\sigma, a)$, the variable EoS grants the pressure an explicit radius dependency. As a consequence, eq. (11) will be replaced by

\[p_0' = \beta_0^2\sigma_0' - \gamma_0,\] (27)

where now $\gamma_0 \equiv -\partial p/\partial a|_{a_0}$. The mechanism of infinite discontinuity removal by the variable EoS is simply to null the numerator of

\[\beta_0^2 = \frac{p_0' + \gamma_0}{-\frac{h_0'}{h_0}(\sigma_0 + p_0) + \frac{2h_0'h_0'' - h_0'^2}{2\sigma_0'}\sigma_0},\] (28)

at $a_{ID}$, where the discontinuity had happened when $\gamma_0$ was zero. This means that, if we set $\gamma_0$ such that

\[\gamma_0 = -p_0',\] (29)

then $\lim_{a_0 \rightarrow a_{ID}} \beta_0^2 = \frac{a_0}{b}$ is indefinite, and it becomes well defined if the numerator approaches zero, at least, at the same rate as the denominator. As far as the unit convention that is applied here is concerned, $\beta_0^2$ is a dimensionless quantity ($\beta_0$ is of type speed, with the SI dimension $[LT^{-1}]$, which becomes dimensionless here, since length and time are looked at on an equal footing in general relativity). Hereupon, the numerator and the denominator of $\beta_0^2$ have the same dimension ($L^{-2}$), and in case the fine-tuning $\gamma_0 = -p_0'$ is exerted, $\beta_0^2$ can be well defined. Note that eq. (28) is somehow a generalization to eq. (13).
Fig. 3. The graphs show $a_{ID}/m$ against $|q|/m$ for a dilaton TSW for different values of $b$. In the legend, $a_{ID+}$ and $a_{ID-}$ correspond to the roots of the denominator of $\beta_0^2$, given by eq. (30). Also, $EH$ and $IH$ correspond to the event horizon and the inner horizon of the bulk universe.

For the case of a symmetric Schwarzschild TSW, one obtains

$$p_0'|_{a_0=3m} = \frac{2\sqrt{3}}{9m^2},$$

by taking the first derivative of eq. (6), applying eq. (15) and setting $\epsilon = 0$. This means that by fine-tuning $\gamma_0$ to $2\sqrt{3}/(9m^2)$ we might be free from infinite discontinuity in the stability diagram. This is particularly shown in fig. 4 for $\beta_0^2$ against $a_0/m$. As is evident, there is no sign of the infinite discontinuity anymore.

Applying the variable EoS to an ERN ATSW leads to the same result. In this case we obtain

$$p_0'|_{a_0=(\epsilon+2)m} = -\frac{2}{(\epsilon+2)^2m^2},$$

as the derivative of the angular pressure at the radius of infinite discontinuity occurrence. Correspondingly, the choice $\gamma_0 = 2/((\epsilon+2)^2m^2)$ is expected to remove the infinite discontinuity. Figure 5 shows how this happens for a symmetric ERN TSW, for which $\epsilon = 0$. 


Fig. 4. The stability diagram for a symmetric Schwarzschild TSW with (a) the barotropic EoS and (b) the variable EoS. It can be observed that the infinite discontinuity is simply removed by virtue of the variable EoS.

Fig. 5. The stability diagram for a symmetric ERN TSW with (a) the barotropic EoS and (b) the variable EoS. As expected, the infinite discontinuity is removed due to the fine-tuning of the variable EoS.

In the end, we turn our attention to the dilaton TSW. Here, we take a closer look at three cases for which $b = 0$, $b = 0.5$, and $b = 1$. The first is selected because it defines the RN spacetime, and the last is selected for its importance in string theory. The choice $b = 0.5$ is rather random, as an intermediate value in the $b$-spectrum. Also, for all three cases, without loss of generality, we have randomly chosen $|q|/m = 0.5$. Our numerical analyses show that for the three case we have

$$p'_{\text{ID}}|_{a_0} = 0.41309864432/m^2$$ when $b = 0$,
$$p'_{\text{ID}}|_{a_0} = 0.4146553639/m^2$$ when $b = 0.5$,
$$p'_{\text{ID}}|_{a_0} = 0.4162095590/m^2$$ when $b = 1$, (32)

which denotes that if we fine-tune $\gamma_0$ such that $\gamma_0|_{b=0} = 0.41309864432/m^2$, $\gamma_0|_{b=0.5} = 0.4146553639/m^2$, and $\gamma_0|_{b=1} = 0.4162095590/m^2$, the existing discontinuities will be removed. This can be seen clearly in fig. 6, where the related mechanical stability diagrams are plotted for the three cases, once when $\gamma_0 = 0$ (barotropic EoS), and once it is fine-tuned to remove the discontinuity. In all the subfigures, the horizontal axis starts at $A/m$, and the stable regions are marked with an “S”. Note that a similar analysis can be applied to other admissible values of $b$ and/or $|q|/m$.

5 Conclusion

The emergence of infinitely branching discontinuity, resembling a phase transition, seemed peculiar enough to attract attention since the stability analyses for TSWs were incepted. The prototype example was the Schwarzschild TSW, which had a similar discontinuity at the stability radius $a_0 = 3m$, as pointed out by Poisson and Visser [22]. In analogy, other TSWs also exhibited similar behavior. We identified the cause of such type of discontinuities: they arise from the vanishing of $-(b_0/h_0)(\sigma_0 + p_0) + (2h_0h''_0 - h'_0^2)\sigma_0/2h_0h'_0$ at equilibrium radius (eq. (13)). In consequence, the speed of sound $\beta_0$, which is inversely proportional to this expression, and is expected to be finite, subsequently diverges.
Fig. 6. The stability diagram for a symmetric dilaton TSW for (a) $b = 0$ with a barotropic EoS, (b) $b = 0$ with a fine-tuned variable EoS, (c) $b = 0.5$ with a barotropic EoS, (d) $b = 0.5$ with a fine-tuned variable EoS, (e) $b = 1$ with a barotropic EoS, and (f) $b = 1$ with a fine-tuned variable EoS. The value of $|q|/m$ is set half for all the cases. The fine-tuned values of $\gamma_0$ are given at the top of each diagram. The regions with “S” are where the TSW is mechanically stable.

In an attempt to resolve such a discontinuity in the symmetric Schwarzschild TSW, Varela employs a more general, modified EoS, i.e. the variable EoS, to replace its barotropic counterpart. In this rather general EoS, beside $\sigma$, the pressure $p$ also depends on the radius of the shell which creates an extra degree of freedom to be used as an advantage. We have precisely shown that such a generalization can be systematically applied to all the TSWs, by pointing out to the reason of the emergence of the discontinuities. Our investigation is generalized to all spherically symmetric spacetimes, with the generic line element in eq. (1), including non-asymptotically flat ones such as the dilaton TSW in sect. 3.3. This shows that the method is applicable even to those TSWs which are strongly coupled with the non-linear, non-asymptotically flat, dilatonic bulk spacetimes. In sect. 4, the logic was illustrated by representing three examples (see fig. 4 for the Schwarzschild TSW, fig. 5 for ERN TSW and fig. 6 for the dilaton TSW), where consequently, the discontinuities in question have been eliminated. It is not difficult to anticipate that the same technique can be applied to other TSWs as well, including the ones in alternative theories. Finally, let us add that in this article, we have used asymmetric TSWs in the sense that the spacetimes on different sides of the throat differ only parametrically. Undoubtedly, the spacetimes that differ in $r$-dependence also can be considered within the range of application.
We would like to kindly thank Justin C. Feng for his constructive comments and propositions which led to great improvement in this article.

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