Cellular chain complexes of universal covers of some 3-manifolds

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Abstract

For a closed 3-manifold $M$ in a certain class, we give a presentation of the cellular chain complex of the universal cover of $M$. The class includes all surface bundles, some surgeries of knots in $S^3$, some cyclic branched cover of $S^3$, and some Seifert manifolds. In application, we establish a formula for calculating the linking form of a cyclic branched cover of $S^3$, and develop procedures of computing some Dijkgraaf-Witten invariants.

Keywords

Universal covering, 3-manifold group, group homology, knot, branched coverings, linking form

1 Introduction

In order to investigate a connected CW-complex $X$ with a non-trivial fundamental group $\pi_1(X)$, it is important to give a concrete presentation of the cellular chain complex, $C_*(\tilde{X};Z)$, and the cup-products of the universal cover $\tilde{X}$. In fact, the homology of $X$ with local coefficients and the (twisted) Reidemeister torsion of $X$ are defined from $C_*(\tilde{X};Z)$. If $X$ is a $K(\pi, 1)$-space, the chain complex means a projective resolution of the group ring $\mathbb{Z}[\pi_1(X)]$. Thus, it is also of use for computing many invariants to concretely present $C_*(\tilde{X};Z)$.

This paper focuses on a class of closed 3-manifolds satisfying the following condition:

Assumption (†) A closed oriented 3-manifold $M$ satisfies that any closed 3-manifold $M'$ with a group isomorphism $\pi_1(M) \cong \pi_1(M')$ admits a homotopy equivalence $M \simeq M'$.

For example, $M$ satisfies this assumption if $M$ is an Eilenberg-MacLane space of type $(\pi_1(M), 1)$, which is equivalent to that $M$ is irreducible and has an infinite fundamental group. In Section 3 we examine many 3-manifolds, including all surface bundles, some surgeries of knots in $S^3$, spliced sums, cyclic branched covers of $S^3$ with Assumption (†), and some Seifert manifolds. For when $M$ is one of these, we describe presentations of the complex $C_*(\tilde{M};Z)$ and of the cup-product $H^1(M; N) \otimes H^2(M; N') \to H^3(M; N \otimes N')$ for any local coefficient modules $N, N'$. The procedure for obtaining such descriptions essentially follows from the work of [Sie, Tro] in terms of “identity”, which we review in Section 2. This procedure can also be used to describe the fundamental homology 3-class, $[M]$ of $M$; see Remark 2.4.

In application, we give a formula for the linking forms of cyclic branched covers of $S^3$ with Assumption (†) (see Propositions 4.1). Furthermore, we develop procedures of computing some Dijkgraaf-Witten invariants from the above descriptions; see §5. In addition, such descriptions of identities are used for computing knot concordance groups, Reidemeister torsions, and Casson invariants; see [MP, No1, Waki]. There might be other applications from the above presentations of the complexes $C_*(M; Z)$.

Conventional notation. In this paper, every manifold is understood to be smooth, connected, and orientable. By $M$, we mean a closed 3-manifold with orientation $[M]$.

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2 Taut identities and cup-products

2.1 Review: identities and cup-products

Let us recall the procedure of obtaining cellular chain complexes of some universal covers, as described in the papers [Sie] and [Tro]. There is nothing new in this section.

We will start by reviewing identities. Take a finitely presented group \( \langle x_1, \ldots, x_m \mid r_1, \ldots, r_m \rangle \) of deficiency zero. Setting up the free groups \( F := \langle x_1, \ldots, x_m \rangle \) and \( P := \langle \rho_1, \ldots, \rho_m \rangle \), let us consider the homomorphism,

\[ \psi : P \ast F \longrightarrow F \] defined by \( \psi(\rho_j) = r_j, \quad \psi(x_i) = x_i. \)

An element \( s \in P \ast F \) is an identity if \( s \in \text{Ker}(\psi) \) and \( s \) can be written as \( \prod_{k=1}^n \omega_k \rho_{jk} \omega_k^{-1} \) for some \( \omega_k \in F \), \( \epsilon_k \in \{ \pm 1 \} \) and indices \( j_k \)’s.

Given a closed 3-manifold \( M \) with a genus-\( m \) Heegaard splitting, let us review the cellular complex of the universal cover, \( \widetilde{M} \), of \( M \). A CW-complex structure of \( M \) induced by the splitting consists of a single zero-cell, \( m \) one-handles, \( m \) two-handles, and a single three-handle. Therefore, \( \pi_1(M) \) has a group presentation \( \langle x_1, \ldots, x_m \mid r_1, \ldots, r_m \rangle \), and the cellular complex of \( \widetilde{M} \) is described as

\[ C_* (\widetilde{M}; \mathbb{Z}) : 0 \rightarrow \mathbb{Z}[\pi_1(M)] \xrightarrow{\partial_n} \mathbb{Z}[\pi_1(M)]^m \xrightarrow{\partial_{n-1}} \mathbb{Z}[\pi_1(M)]^m \xrightarrow{\partial_{n-2}} \cdots \xrightarrow{\partial_1} \mathbb{Z}[\pi_1(M)]^m \xrightarrow{\partial_0} \mathbb{Z}[\pi_1(M)] \rightarrow 0. \] (1)

Here, \( \mathbb{Z}[\pi_1(M)] \) is the group ring of \( \pi_1(M) \). We will explain the boundary maps \( \partial_* \) in detail. Let \( \{ a_1, \ldots, a_m \}, \{ b_1, \ldots, b_m \}, \) and \( \{ c \} \) denote the canonical bases of \( C_1(\widetilde{M}; \mathbb{Z}) \), \( C_2(\widetilde{M}; \mathbb{Z}) \), and \( C_3(\widetilde{M}; \mathbb{Z}) \) as left \( \mathbb{Z}[\pi_1(M)] \)-modules, respectively. Then, as is shown in [Lyn], \( \partial_1(a_i) = 1 - x_i \), and \( \partial_2(b_i) = \sum_{k=1}^m [\frac{\partial \omega_k}{\partial x_k}]a_k \), where \( \frac{\partial \omega_k}{\partial x_k} \) is the Fox derivative. Moreover, the main result in [Sie] is that there exists an identity \( s \) such that \( \partial_3(c) = \sum_k [\psi(\frac{\partial \rho_k}{\partial \rho_k})]b_k. \)

Next, we will briefly give a formula for the cup-product in terms of the identity, which is a result of [Tro] §2.4. Let \( N \) and \( N' \) be left \( \mathbb{Z}[\pi_1(M)] \)-modules. We can define the cochain complex on \( C^* (M; N) := \text{Hom}_{\mathbb{Z}[\pi_1(M)]} (C_*(\widetilde{M}; \mathbb{Z}), N) \) with local coefficients. Recalling the definition of the identity \( s = \prod_{k=1}^m \omega_k \rho_{jk} \omega_k^{-1} \), define

\[ D^2 (c) = \sum_{k=1}^n \epsilon_k \sum_{i=1}^m \left[ \frac{\partial \omega_k}{\partial x_i} \right] a_i \otimes \omega_k b_{jk} \in C_1(\widetilde{M}; \mathbb{Z}) \otimes C_2(\widetilde{M}; \mathbb{Z}). \] (2)

Then, for cochains \( p \in C^1(M; N) \) and \( q \in C^2(M; N') \), we define a 3-cochain \( p \ast q \) by

\[ p \ast q (uc) := (p \otimes q)(u D^2 (c)) \in N \otimes_Z N'. \]

Here, \( u \in \mathbb{Z}[\pi_1(M)] \). Then, the map

\[ \sim : C^1(M; N) \otimes C^2(M; N') \rightarrow C^3(M; N \otimes_Z N'); \quad (p, q) \mapsto p \ast q, \]

induces the bilinear map on cohomology, which is known to be equal to the usual cup-product. Here, notice that, since the third \( \partial_3 \otimes_{\mathbb{Z}[\pi_1(M)]} \text{id}_\mathbb{Z} \) is zero, the 3-class \( s \otimes 1 \in C_3(\widetilde{M}) \otimes_{\mathbb{Z}[\pi_1(M)]} \mathbb{Z} \) is a generator of \( H_3(C_*(\widetilde{M}) \otimes \mathbb{Z}) \cong H_3(M; \mathbb{Z}) \cong \mathbb{Z} \), which represents the fundamental 3-class \([M]\); thus, given a \( \pi_1(M) \)-invariant bilinear map \( \psi : N \otimes N' \rightarrow A \) for some abelian group \( A \), we have the following equality on the pairing of \([M]\):

\[ \psi \circ \sim (p, q) = \psi (p \ast q, [M]) \in A, \] (3)

for any cochains \( p \in C^1(M; N) \) and \( q \in C^2(M; N') \).

In summary, for a description of the complex \( C_* (\widetilde{M}) \) and the cup-product, it is important to describe an identity from \( M \).
2.2 Taut identities

In order to find such identities giving the complex \([I]\), we review tautness from [Sie]; see also Waki Appendix for a brief explanation. Fix a finite presentation \(\langle x_1, \ldots, x_m \mid r_1, \ldots, r_m \rangle\).

Let \(s = \prod_{k=1}^{2m} w_k \rho_{jk}^{\epsilon_k} \omega_k^{-1} \in P \ast F\) be an identity, where \(\rho_{jk}\) and \(w_k\) can be written in

\[
\rho_{jk} = a_{k,1}^{\epsilon_{k,1}} \cdots a_{k,\ell_k}^{\epsilon_{k,\ell_k}}, \quad w_k = b_{k,1}^{n_{k,1}} \cdots b_{k,n_k}^{n_{k,n_k}}, \quad (\epsilon_{i,j}, \eta_{i,j} \in \{\pm 1\}).
\]

Here, \(a_{k,\ell}\) and \(b_{k,\ell}\) lie in \(\{x_1, \ldots, x_m\}\). For each \(w_k \rho_{jk}^{\epsilon_k} \omega_k^{-1}\), take the \(\ell_k\)-gon \(D_{jk}\) whose \(i\)-th edge is labeled by \(a_{k,i}^{\epsilon_{k,i}}\), and the segment \(I_k = [0, n_k]\) such that \([i - 1, i]\) is labeled by \(b_{k,i}^{n_{k,i}}\).

**Definition 2.1 ([Sie])**

1. A self-bijection

\[
\mathcal{I} : \sqcup_{k=1}^{2m} \{(k, 1), \ldots, (k, \ell_k)\} \rightarrow \sqcup_{k=1}^{2m} \{(k, 1), \ldots, (k, \ell_k)\}
\]

is called a *syllable* if \(a_{\mathcal{I}(i,j)} = a_{i,j} \in F\) and \(\epsilon_{i,j} = -\epsilon_{\mathcal{I}(i,j)} \in \{\pm 1\}\).

2. For a syllable \(\mathcal{I}\), consider the following equivalence on the disjoint union \(\sqcup_{i=1}^{2m} D_r\); the interval with labeling \(a_{i,j}\) is identified with those with labeling \(a_{\mathcal{I}(i,j)}\).

3. An identity \(s\) is said to be *taut* if there is a syllable \(\mathcal{I}\) such that the quotient space \(\sqcup_{i=1}^{2m} D_r / \sim\) of \(\sqcup_{i=1}^{2m} D_r\) subject to the above equivalence \(\sim\) is homeomorphic to \(S^2\), and if there are injective continuous maps

\[
\kappa_k : I_k = [0, n_k] \rightarrow \sqcup_{i=1}^{2m} \partial D_r / \sim, \quad \lambda_k : [0, \ell_k] \rightarrow \partial D_r / \sim
\]

satisfying the following condition (*)

(*) For each \(k\), the image \(\kappa_k([i - 1, i])\) coincides with an edge labeled by \(b_{k,i}\) compatible with the orientations, and \(\lambda_k([j - 1, j])\) coincides with the \(j\)-th edge of \(D_{rk}\) compatible with the orientations. Furthermore, \(\kappa_k(n_k) = \lambda_k(0) = \lambda_k(\ell_k)\).

This paper is mainly based on the following theorem of Sieradski:

**Theorem 2.2 ([Sie]).** Given a group presentation \(\langle x_1, \ldots, x_m \mid r_1, \ldots, r_m \rangle\) with a taut identity \(s\), there exists a closed 3-manifold \(M\) with a genus-\(m\) Heegaard splitting such that the complex \(C_s(\hat{M}; \mathbb{Z})\) is isomorphic to the complex \([1]\).

In a concrete situation where an identity \(s\) is explicitly described, it is not so hard to find such a \(\mathcal{I}\) and show the tautness of \(s\) (in fact, this check is to construct a 2-sphere from the disjoint union \(\sqcup_{i=1}^{2m} D_r\) as a naive pasting). In all the statements in §3, we will claim that some identities satisfy the taut condition; however, we will also omit the check by elementary complexity, as in other papers on taut identities [BHI Sie Tro].

**Example 2.3.** As an easy example of the pasting, we focus on the 3-dimensional torus \(M = (S^1)^3\) with presentation \(\pi_1(M) = \langle x, y, z \mid r, s, u \rangle\), where \(r = [x, y], s = [y, z], u = [z, x]\). As in [Sie], consider the following identity.

\[
W_{(S^1)^3} = r(y^{-1}u^{-1}y)s(z^{-1}rz)u(x^{-1}z^{-1}x).
\]

Then, Figure [1] gives a self-bijection and \(\lambda_m, \kappa_m\) satisfy the tautness. Moreover, if we attach a 3-ball in the right hand side in the figure along the boundary of the 3-cube, the resulting space is equal to \((S^1)^3\).
Remark 2.4. Suppose that we find a taut identity $s$ from $\langle x_1, \ldots, x_m \mid r_1, \ldots, r_m \rangle$, and the resulting 3-manifold $M$ satisfies Assumption $(\dagger)$. Then, by Assumption $(\dagger)$, the resulting 3-manifold up to homotopy does not depend on the choice of $s$. In particular, we emphasize that, if $M$ satisfies Assumption $(\dagger)$ and we find a taut identity from $\pi_1(M) = \langle x_1, \ldots, x_m \mid r_1, \ldots, r_m \rangle$, then the third $\partial_3$ and the cup-product are uniquely determined, up to homotopy, by the identity. In fact, if we have another identity $\omega'$ and consider the associated $C_*(\tilde{M})'$, Assumption $(\dagger)$ ensures a chain map $C_*(\tilde{M}) \to C_*(\tilde{M})'$, which induces a homotopy equivalence.

3 Descriptions of taut identities of various 3-manifolds

In this section, we give several examples of identities from some classes of 3-manifolds. We will describe the cellular complexes of some universal covers.

3.1 Fibered 3-manifolds with surface fibers over the circle

First, we will focus on surface bundles over $S^1$. Let $\Sigma_g$ be an oriented closed surface of genus $g$ and $f : \Sigma_g \to \Sigma_g$ an orientation-preserving diffeomorphism. The mapping torus, $T_f$, is the quotient space of $\Sigma_g \times [0,1]$ subject to the relation $(y,0) \sim (f(y),1)$ for any $y \in \Sigma_g$. The homeomorphism type of $T_f$ depends on the mapping class of $f$. Conversely, if a closed 3-manifold $M$ is a fibered space over $S^1$, then $M$ is homeomorphic to $T_f$ for some $f$. Since $T_f$ is a $\Sigma_g$-bundle over $S^1$, it is a $K(\pi,1)$-space and therefore satisfies Assumption $(\dagger)$.

We will construct an identity. Choose a generating set $\{x_1, \ldots, x_{2g}\}$ of $\pi_1(\Sigma_g)$, which gives the isomorphism $\pi_1(\Sigma_g) \cong \langle x_1, \ldots, x_{2g} \mid [x_1,x_2] \cdots [x_{2g-1},x_{2g}] \rangle$. Following a van Kampen argument, we can verify the presentation of $\pi_1(T_f)$ as

$$\langle x_1, \ldots, x_{2g}, \gamma \mid r_i := \gamma f_s(x_i) \gamma^{-1} x_i^{-1}, \quad (i \leq 2g), \quad r_{2g+1} := [x_1,x_2] \cdots [x_{2g-1},x_{2g}] \rangle.$$ (4)

Here, $\gamma$ represents a generator of $\pi_1(S^1)$. For $i \leq 2g$, define $w_i = \prod_{j=1}^i [x_{2j-1},x_{2j}] \in F$, and

$$W_i := w_{i-1} \rho_{2i-1}^{-1} w_{i-1} \cdot (w_{i-1} x_{2i-1}) \rho_{2i} (w_{i-1} x_{2i-1})^{-1} \cdot (w_i x_{2i}) \rho_{2i-1} (w_i x_{2i})^{-1} \cdot w_i \rho_{2i}^{-1} w_i^{-1}.$$ Since $f$ can be isotoped so as to preserve a point $z \in \Sigma_g$, we regard the induced map $f_*$ as a homomorphism : $\pi_1(\Sigma_g \setminus \{z\}) \to \pi_1(\Sigma_g \setminus \{z\})$. Since $f_*$ is a group isomorphism, there exists a unique element $q_f \in \langle x_1, \ldots, x_{2g} \rangle$ satisfying

$$f_*([x_1,x_2] \cdots [x_{2g-1},x_{2g}]) = q_f([x_1,x_2] \cdots [x_{2g-1},x_{2g}] q_f^{-1} \in \langle x_1, \ldots, x_{2g} \rangle.$$ Theorem 3.1. Let $W$ be $\left(\prod_{i=1}^g W_{i}\right) \rho_{2g+1} (\gamma q_f \rho_{2g+1}^{-1} \gamma^{-1}) \in F \ast P$. Then, $W$ is an identity.
Proof. Direct calculation gives $\psi(W_i) = w_{i-1}\gamma[f_*(x_{2i-1}), f_*(x_{2i})]\gamma^{-1}w_i^{-1}$, which implies $\psi(\Pi_{i=1}^g W_i) = \gamma(\Pi_{i=1}^g [f_*(x_{2i-1}), f_*(x_{2i})])\gamma^{-1}w_i^{-1} = \gamma\Pi_{i=1}^g [f_*(x_{2i-1}), f_*(x_{2i})]\gamma^{-1}(\Pi_{i=1}^g [x_{2i-1}, x_{2i}])^{-1}$. Hence, $\psi(W) = 1$ by definition; that is, W turns out to be an identity.

Furthermore, we can verify that W is taut by the definition of W. Hence, from the discussion in [2] we can readily prove the following corollary.

**Corollary 3.3.** Under the above terminology, the cellular chain complex of $\tilde{T}_f$ is given by

$$C_*(\tilde{T}_f; \mathbb{Z}) : 0 \to \mathbb{Z}[\pi_1(T_f)] \xrightarrow{\partial_1} \mathbb{Z}[\pi_1(T_f)]^{2g+1} \xrightarrow{\partial_2} \mathbb{Z}[\pi_1(T_f)]^{2g+1} \xrightarrow{\partial_3} \mathbb{Z}[\pi_1(T_f)] \to 0.$$ 

Here, $\partial_1(a_i) = 1 - x_i$, $\partial_1(\gamma) = 1 - \gamma$, and $\partial_2$ and $\partial_3$ have the matrix presentations,

$$\left( \begin{array}{cc} \gamma \frac{\partial f_*(x_i)}{\partial x_j} - \delta_{ij} & \{1 - x_i\} \text{transpose} \\
\{\partial_{x,j+1} \over \partial x_j\} & 0 \end{array} \right)_{1 \leq i, j \leq 2g},$$

$$\left( \begin{array}{c} w_{j-1} - w_j x_{2j}, w_{j-1} x_{2j-1} - w_j \\
(1 - \gamma q_f) \end{array} \right)_{1 \leq j \leq g}.$$ 

Furthermore, the diagonal map $D^g(c)$ is represented by

$$\left( \sum_{i=1}^g \sum_{k=1}^{2g} \partial w_{i-1} \over \partial x_k \right) a_k \otimes w_{i-1} b_{2i-1} - \partial(w_i x_{2i-1}) \over \partial x_k a_k \otimes w_i x_{2i-1} b_{2i-1} - \partial(w_{i-1} x_{2i-1}) \over \partial x_k a_k \otimes w_{i-1} x_{2i-1} b_{2i-1}$$

$$+ \partial w_i \over \partial x_k a_k \otimes w_i b_{2i} - \sum_{k=1}^{2g} \partial(\gamma q_f) \over \partial x_k a_k \otimes \gamma q_f b_{2g+1} + a_{2g+1} \otimes (1 - \gamma q_f) b_{2g+1}.$$ 

**Remark 3.3.** Corollary 3.2 for every g is a generalization of the result of [Mar]; the paper gives the cellular complexes of $\tilde{T}_f$ only in the case $g = 1$. We can verify that Corollary 3.2 with $g = 1$ coincides with the results in [Mar].

Finally, we mention the virtually fibered conjecture, which was eventually proven by Wise; see, e.g., [Wise]. This conjecture states that every closed, irreducible, atoroidal 3-manifold $M$ with an infinite fundamental group has a finite cover, which is homeomorphic to $T_f$ for some $f$. Let $d \in \mathbb{N}$ be the degree of the covering. Then, if we can find such a cover $p : T_f \to M$, the pushforward of the above identity $W$ gives an algebraic presentation of $d[M]$.

### 3.2 Spliced sums and ($p/1$)- and ($1/q$)-surgeries of $S^3$ along knots

We will focus on spliced sums and some surgeries of $S^3$ along knots and construct taut identities. This section supposes that the reader has basic knowledge of knot theory, as in [Lic] Chapters 1–11.

Let us review spliced sums. Take two knots $K, K' \subset S^3$ and an orientation-reversing homeomorphism $h : \partial(S^3 \setminus \nu K) \to \partial(S^3 \setminus \nu K')$, where $\nu K$ means an open tubular neighborhood of $K$. Then, we can define a closed 3-manifold, $\Sigma_h(K, K')$, as the attaching space $(S^3 \setminus \nu K) \cup_h (S^3 \setminus \nu K')$ with $\partial(S^3 \setminus \nu K)$ glued to $\partial(S^3 \setminus \nu K')$ by $h$. This space is commonly referred to as the **spliced sum of** $(K, K')$ **via** $h$. Spliced sums sometimes appear in discussions on additivity of topological invariants; see, e.g., [BC]. Further, choose the preferred meridian-longitude pair $(m, l)$ (resp. $(m', l')$) as a generating set of $\pi_1 \partial(S^3 \setminus \nu K)$ (resp. of $\partial(S^3 \setminus \nu K')$). If
\(h_*: \pi_1(\partial(S^3 \setminus \nu K)) \to \pi_1(\partial(S^3 \setminus \nu K'))\) is represented by \(\begin{pmatrix} 0 & 1 \\ 1 & p \end{pmatrix}\) (resp. \(\begin{pmatrix} 1 & 0 \\ q & -1 \end{pmatrix}\)) for some \(p, q \in \mathbb{Z}\), we denote \(\Sigma_p(K, K')\) by \(\Sigma_{p/1}(K, K')\) (resp. \(\Sigma_{1/q}(K, K')\)). In particular, if \(K'\) is the unknot, then \(\Sigma_{p/1}(K, K')\) and \(\Sigma_{1/q}(K, K')\) are the closed 3-manifolds obtained by \((p/1)\)- and \((1/q)\)-Dehn surgery on \(K\) in \(S^3\), respectively.

Since the identities of \(\Sigma_{p/1}(K, K')\) and \(\Sigma_{1/q}(K, K')\) will be constructed in an analogous way to \([\text{Tro}]\) Page 481], let us review the terminology in \([\text{Tro}]\). Choose a Seifert surface \(\Sigma\) of genus \(p, q\), \(\pi\times\ad\) admits such a bouquet. Choose a bicollar \(\Sigma\) is a free group. 21. Page. For example, any Seifert surface obtained by a Seifert algorithm yields a presentation \(\pi\) equal by definition, we should notice

\[\langle \pi, \pi, \langle u_1^i, u_2^i \rangle \rangle \text{ is a representative of a meridian in } \pi(\Sigma)\text{, and } w_l := (\langle u_1^i, u_2^i \rangle) \text{ and } l = \langle \pi(\Sigma) \rangle \text{ represents a loop of } \pi(\Sigma); \text{ a van Kampen argument yields a presentation}\]

\[
\langle x_1, \ldots, x_{2g}, m \mid r_i := mu_i^j m^{-1}(u_i^j)^{-1} \quad (1 \leq i \leq 2g) \rangle
\]

of \(\pi_1(S^3 \setminus K)\). Here, \(m\) is a representative of a meridian in \(\pi_1(S^3 \setminus K)\), and the \(x_i\)'s lie in the commutator subgroup of \(\pi_1(S^3 \setminus K)\). Since the boundary loops of \(\pi_1\Sigma\) and \(\pi_1(S^3 \setminus \Sigma)\) are equal by definition, we should notice

\[
[u_1^i, u_2^i] \cdots [u_{2g-1}^i, u_{2g}^i] = [u_1^i, u_2^i] \cdots [u_{2g-1}^i, u_{2g}^i] \in \pi_1(S^3 \setminus \Sigma),
\]

which we denote by \(I\). In other words, \(I\) means a preferred longitude of \(K\).

In a parallel way, concerning the other \(K'\), we have a generating set \(\{x_1, \ldots, x_{2g'}\}\) of \(\pi_1(S^3 \setminus \Sigma')\) and can define appropriate words \(u_1^i\) and \(u_2^i\) such that

\[
\pi_1(S^3 \setminus K') \cong \langle x_1', \ldots, x_{2g'}, m' \mid r_i := m' u_i^j (m')^{-1}(u_i^j)^{-1} \quad (1 \leq i \leq 2g') \rangle.
\]

We also redefine \(l'\) by \(\langle u_1^i, u_2^i \rangle \cdots [u_{2g-1}^i, u_{2g}^i]\).

Before we state Theorem 3.4, we should notice from the van Kampen theorem that the fundamental groups \(\pi_1(\Sigma_{p/1}(K, K'))\) and \(\pi_1(\Sigma_{1/q}(K, K'))\) are presented by

\[
\langle x_1, \ldots, x_{2g}, m x_1', \ldots, x_{2g'}, r_1, r_2, \ldots, r_{2g}, r_1', \ldots, r_{2g'}, r_{2g+1} := l m^p(l')^{-1} \rangle,
\]

\[
\langle x_1, \ldots, x_{2g}, m x_1', \ldots, x_{2g'}, m' \mid r_1, r_2, \ldots, r_{2g}, r_1', \ldots, r_{2g'}, r_1 := m^q(l')^{-1}, r_* := m^p l^{-1} \rangle.
\]

Here, in (7), we identify \(m\) with \(m'\). Define \(w_i\) to be \(\prod_{j=1}^i [u_{2j-1}^j, u_{2j}^j]\), and

\[
W_i := w_{i-1} \rho_{2i-1} w_{i-1}^{-1} \cdot (w_{i-1} w_{2i-1}^j) \rho_{2i} (w_{i-1} w_{2i-1}^j)^{-1} \cdot (w_i w_{2i}^j) \rho_{2i}^{-1} (w_i w_{2i}^j)^{-1} \cdot w_i \rho_{2i}^{-1} w_i^{-1}.
\]

Likewise, we also define words \(w_i^j\) and \(W_i^j\). We consider the two words,

\[
W_{p/1}^{K, K'} := (\prod_{j=1}^q W_i^j) \cdot \rho_{2g+1} (\prod_{j=1}^q W_i^j)^{-1} (m \rho_{2g+1}^{-1} m^{-1}),
\]

\[
W_{1/q}^{K, K'} := (\prod_{j=1}^q W_i^j) \cdot \rho_{*} (m' \rho_{*} (m')^{-1}) (\prod_{j=1}^q W_i^j) \cdot (l' \rho_{*} (l')^{-1}) \cdot \rho_{*}^{-1}.
\]

**Theorem 3.4.** Then, \(W_{p/1}^{K, K'}\) and \(W_{1/q}^{K, K'}\) are taut identities with respect to the presentations (7) of \(\pi_1(\Sigma_{p/1}(K, K'))\) and \(\pi_1(\Sigma_{1/q}(K, K'))\), respectively.
Proof. An immediate computation gives \( \psi(W_i) = w_{i-1}^*m[u_{2i-1}, u_{2i}]m^{-1}w_i^{-1} \), so that \( \psi(\Pi_{i=1}^g W_i) = m\Pi_{i=1}^g[u_{2i-1}, u_{2i}]m^{-1}w_1^{-1} \). Then, \( W_{p/1}^{K,K'} \) and \( W_{1/q}^{K,K'} \) turn out to be identities by (7). Furthermore, by the definition of \( W_{\circ}^{K,K'} \), we verify that \( W_{\circ}^{K,K'} \) are taut. \( \square \)

As a corollary, if \( K' \) is the unknot, we have the complex \( C_*(\tilde{M};\mathbb{Z}) \), where \( M \) is the 3-manifold, \( M_{p/1}(K) \), obtained by \( p/1 \)-surgery of \( S^3 \) along \( K \):

**Corollary 3.5.** If \( M := M_{p/1}(K) \) satisfies Assumption (†), then the boundary maps \( \partial_2 \) and \( \partial_3 \) in the associated complex \( C_*(\tilde{M};\mathbb{Z}) \) in (1) are given by the following matrix presentations:

\[
\begin{pmatrix}
\{ m\frac{\partial u_i^*}{\partial x_j} - u_i^* \frac{\partial m}{\partial x_j} \}_{1 \leq i, j \leq 2g} & \{ 1 - \frac{\partial l}{\partial x_j} \}_{1 \leq j \leq 2g} \\
\{ \frac{\partial l}{\partial x_j} m^g \}_{1 \leq j \leq 2g} & \frac{\partial m^g}{\partial m}
\end{pmatrix},
\]

where \( \partial_3(s) = (1 - m)b_{2g+1} + \sum_{i=1}^g (w_{i-1} - w_i u_{2i})b_{2i-1} + (w_{i-1} u_{2i-1} - w_i)b_{2i} \).

**Remark 3.6.** We give a comparison to Theorem 3.9 in [MP]. The authors give an expression of the chain complex \( C_*(\tilde{M};\mathbb{Z}) \), where \( M = M_0/1(K) \). However, the numbers of basis of \( C_3, C_2, C_1 \) are 2, \( c + 1, c \), respectively, where \( c \) is the crossing number of \( K \), while those in Corollary 3.5 are fewer.

Let us recall the cabling conjecture, which predicts that if \( K \) is not a cabling knot, then \( M_0(K) \) is irreducible; this conjecture has been proven for some classes of knots. Since \( \pi_1(M_0(K)) \) is of infinite order, it is fair to say that most \( M_0(K) \) satisfy Assumption (†). Incidentally, it is a problem for the future to clarify a taut identity for the \((p/q)\)-surgery for any \( p/q \in \mathbb{Q} \).

### 3.3 Branched covering spaces of \( S^3 \) branched over a knot

Take a knot \( K \) in \( S^3 \), and \( d \in \mathbb{N} \). In this subsection, we will give a taut identity of \( \pi_1(B_K^d) \), where we mean by \( B_K^d \) the \( d \)-fold cyclic covering space of \( S^3 \) branched over \( K \). We should remark the fact that, if \( K \) is a prime knot and \( \pi_1(B_K^d) \) is of infinite order, then \( B_K^d \) is aspherical and therefore admits Assumption (†). Let \( p : E_K^d \to S^3 \setminus K \) be the \( d \)-fold cyclic covering. For \( k \in \mathbb{Z}/d \), let \( x_i^{(k)} \) be a copy of \( x_i \) and \( u_{i,k}^* \) be the word obtained by replacing \( x_i \) with \( x_i^{(k)} \) in the word \( u_i^* \). We similarly define the word \( u_{i,k}^- \). Then, by using the Reidemeister-Schreier method (see, e.g., [Kab Proposition 3.1]), it follows from presentation (5) that \( \pi_1(E_K^d) \) is presented by

\[
\langle x_1^{(k)}, \ldots, x_{2g}^{(k)}, m \mid m u_{i,k}^- m^{-1} (u_{i,k+1})^{-1} (1 \leq i \leq 2g, k \in \mathbb{Z}/d) \rangle.
\]

(8)

Since \( B_K^d \) is obtained from \( E_K^d \) by attaching a solid torus which annihilates the meridian \( m \), \( \pi_1(B_K^d) \) is presented by the quotient of \( \pi_1(E_K^d) \) subject to \( m = 1 \); that is,

\[
\pi_1(B_K^d) \cong \langle x_1^{(k)}, \ldots, x_{2g}^{(k)} \mid m = 1 \rangle.
\]

(9)

Let \( F \) be the free group \( \langle x_1^{(k)}, \ldots, x_{2g}^{(k)} \mid k \in \mathbb{Z}/d \rangle \). From (6), we should notice that \( [u_{1,k}^*, u_{2,k}^-] \cdots [u_{2g-1,k}^*, u_{2g, k}^-] = [u_{1,k}^*, u_{2,k}^-] \cdots [u_{2g-1,k}^*, u_{2g, k}^-] \in F \) for any \( k \in \mathbb{Z}/d \).

Similarly to §3.2 we will give an identity with respect to the presentation (9). For \( 1 \leq i \leq g, 1 \leq k \leq d \), define \( w_{i,k} = \prod_{j=1}^i [u_{2j-1, k+1}, u_{2j, k+1}] \), and

\[
W_{i,k} = w_{i-1,k}^{\rho_{2i-1,k} w_{i-1,k}^{-1}} (w_{i-1,k} u_{2i-1,k+1}) \rho_{2i,k} (w_{i-1,k} u_{2i-1,k+1})^{-1}.
\]

(9)
\[(w,_{i,k} u^b_{2i,k+1}) \rho^{-1}_{2i-1,k} (w,_{i,k} u^b_{2i,k+1})^{-1} \cdot (w,_{i,k} \rho^{-1}_{2i,k} u^{-1}_{i,k}).\]

**Proposition 3.7.** Define \( W \) to be \( \coprod_{k=1}^d W_{1,k} W_{2,k} \cdots W_{g,k} \), by the above equality in \( F \). Then, \( W \) is a taut identity. In particular, if \( B^d_K \) satisfies Assumption (\dagger), the associated complex in [1] is isomorphic to the cellular chain complex of the universal cover of \( B^d_K \).

**Proof.** Direct calculation gives \( \psi(W,_{i,k}) = w_{i-1,k} [u^b_{2i-1,k}, u^b_{2i,k}] w^{-1}_{i,k} \); which deduces

\[
\psi(\coprod_{i=1}^g W,_{i,k}) = (\coprod_{i=1}^g [u^b_{2i-1,k}, u^b_{2i,k}]) w^{-1}_{g,k} = \coprod_{i=1}^g [u^b_{2i-1,k}, u^b_{2i,k}] (\coprod_{i=1}^g [u^b_{2i-1,k+1}, u^b_{2i,k+1}])^{-1}.
\]

Thus, \( W \) turns out to be an identity. Furthermore, since we can verify that \( W \) is taut by the definition of \( W \), Remark 2.2 readily leads to the latter part.

**Example 3.8.** Let \( K \) be the figure-eight knot. It can be verified that the presentation (5) can be written as

\[
\langle x_1, x_2, m \mid mx_1x_2m^{-1} = x_1, mx_2x_1x_2m^{-1} = x_2 \rangle.
\]

Thus, by (9), we have

\[
\pi_1(B^d_K) \cong \langle x_1^{(i)}, x_2^{(i)} \mid (1 \leq i \leq d) \mid x_1^{(i)} x_2^{(i)} = x_1^{(i+1)}, x_2^{(i)} x_1^{(i)} x_2^{(i)} = x_2^{(i+1)} \rangle.
\]

Annihilating \( x_2^{(i)} \) by using the relation \( x_1^{(i)} x_2^{(i)} = x_1^{(i+1)} \), we have

\[
\pi_1(B^d_K) \cong \langle x_1^{(1)}, \ldots, x_1^{(d)} \mid (x_1^{(i)})^{-1}(x_1^{(i+1)})^2(x_1^{(i+2)})^{-1}x_1^{(i+1)} \rangle.
\]

This isomorphism coincides exactly with the result in [KKV] Page 963.

Likewise, we can verify that some groups, called “cyclically presented groups” in [KKV] and references therein, are isomorphic to \( \pi_1(B^d_K) \) for some \( K \) and \( d \).

**Remark 3.9.** As the referee points out, it is reasonable to hope that Proposition 3.7 is true without Assumption (\dagger). In fact, as seen in [Sie], given a Heegaard diagram, we can construct a “squashing map” and a taut identity compatible with the complex [1]. Thus, it is a conjecture that we can find an appropriate Heegaard diagram of \( B^d_K \) such that the associated taut identity is equal to the above \( W \).

### 3.4 0-Surgery-like spaces from branched covering spaces of \( S^3 \)

Using the notation in the preceding subsection, we can examine the 3-manifold obtained by the 0-surgery on the knot \( p^{-1}(K) \subset B^d_K \). The 0-surgery appears in the topic of the concordance group including the Casson-Gordon invariant [CG]. More precisely, regarding the boundary of \( E^d_K \) as a knot in \( B^d_K \), we consider the 3-manifold obtained by 0-surgery on the knot in \( B^d_K \). Notice from (8) that the fundamental group canonically has a group presentation

\[
\langle x_1^{(k)}, \ldots, x_2^{(k)} \mid \mathfrak{m} r_{i}^{(k)} \mid (i \leq 2g, k \in \mathbb{Z}/d), \Pi_{i=1}^g [u^b_{2i-1,k}, u^b_{2i,k}] \rangle.
\]

Let \( l^{(k)} := \Pi_{i=1}^g [u^b_{2i-1,k}, u^b_{2i,k}] \), and consider an analogous presentation

\[
\langle x_1^{(k)}, \ldots, x_2^{(k)} \mid \mathfrak{m} r_{i}^{(k)}, (i \leq 2g, k \in \mathbb{Z}/d), r_{\ell} := l^{(1)} l^{(2)} \cdots l^{(d)} \rangle.
\]

Similarly to (3.3), we can construct an identity. For \( i \leq 2g, k \leq d \), define \( z_k = l^{(1)} l^{(2)} \cdots l^{(k)} \) and

\[
W,_{i,k} = z_k w_{i-1,k} \rho_{2i-1,k} w_{i-1,k}^{-1} \cdot (z_k w_{i-1,k} u^b_{2i-1,k+1}) \rho_{2i,k} (z_k w_{i-1,k} u^b_{2i-1,k+1})^{-1}.
\]

In the usual way, we can easily show the following:
Proposition 3.10. Define $W$ to be $(\Pi_{k=1}^{d} \Pi_{i=1}^{n} W_{i,k}) \cdot \rho_\ell \cdot (m \rho_\ell^{-1} m^{-1})$. Then, $W$ is a taut identity. In particular, Remark 2.3 ensures that if the fundamental group of a closed 3-manifold satisfying Assumption (†) is isomorphic to (1), then the cellular chain complex of the universal cover is isomorphic to the complex (1).

3.5 Some Seifert fibered spaces over $S^2$

In the last subsection, we will discuss some of the Seifert fibered spaces and Brieskorn manifolds. The theorem of Scott [Sc] shows that the homeomorphism types of such spaces with infinite $\pi_1$ can be detected by the fundamental groups; thus, the spaces satisfy Assumption (†).

Let us state Proposition 3.11. Take integers $a_1, \ldots, a_{n+1}$ with $a_i \geq 2$, and $\epsilon_1, \ldots, \epsilon_n \in \{\pm 1\}$. Let $M$ be a Seifert fibered space of the form

$$\Sigma(0; (0, 1), (a_1, \epsilon_1), (a_2, \epsilon_2), \ldots, (a_n, \epsilon_n), (a_{n+1}, 1)).$$

Then, as is classically known, the fundamental group has the presentation

$$\langle x_1, \ldots, x_{n+1}, h \mid h x_i h^{-1} x_i^{-1}, x_i^{a_i} h^{\epsilon_i} (i \leq n), x_{n+1}^{a_{n+1}} h, x_1 \cdots x_{n+1} \rangle.$$  (12)

Furthermore, let us consider a group $G$ with the presentation

$$\langle x_1, \ldots, x_n \mid r_i := (x_i x_{i+1} \cdots x_n x_1 \cdots x_{i-1})^{-a_i+1} x_i^{a_i} (i \leq n) \rangle.$$  (12)

We can easily check that the correspondence $x_i \mapsto x_i$, $x_{n+1} \mapsto (x_1 \cdots x_n)^{-1}$, $h \mapsto x_1^{a_1}$ gives rise to a group isomorphism $\pi_1(M) \cong G$. Therefore, we shall define a taut identity on the presentation (12):

Proposition 3.11. Suppose that $\pi_1(M)$ is of infinite order. Define $W$ to be

$$\rho_1(x_1^{-1} \rho_1^{-1} x_1) \rho_2(x_2^{-1} \rho_2^{-1} x_2) \cdots \rho_n(x_n^{-1} \rho_n^{-1} x_n).$$

Then, $W$ is a taut identity of the presentation (12).

The proof is similar to the ones above, so we will omit the details.

Remark 3.12. The taut identity when $n = 2$ is presented in [Sc], p. 127]. The paper does not mention the homeomorphism type of the associated 3-manifold; however, Proposition 3.11 implies that the homeomorphism type can be detected by a Seifert structure.

Finally, let us turn to the topic of Brieskorn 3-manifolds. Choose integers $a, b, p, q, m \in \mathbb{Z}$ and $\varepsilon \in \{\pm 1\}$ satisfying $ap + bq = 1$ and $p, q, m > 1$. We will focus on the Brieskorn 3-manifold of the form,

$$M := \Sigma(p, q, mpq + \varepsilon) := \{(x, y, z) \in \mathbb{C}^3 \mid x^p + y^q + z^{mpq+\varepsilon} = 0, |x|^2 + |y|^2 + |z|^2 = 1\},$$

which is an Eilenberg-MacLane space if $1/p + 1/q + 1/(mpq + \varepsilon) < 1$. The manifold is known to be homeomorphic to a 3-manifold obtained from $(\varepsilon/m)$-surgery on the $(p, q)$-torus knot $T_{p,q}$. Recall the presentation of $\pi_1(S^3 \setminus T_{p,q})$ as $\pi_1(S^3 \setminus T_{p,q}) \cong \langle x, y \mid x^a = y^b \rangle$, and that the meridian $m$ and the preferred longitude $l$ are identified with $x^a y^b$ and $(x^a y^b)^{-p} x^q$, respectively. Therefore, $\pi_1(M)$ admits a genus-two Heegaard decomposition and has the group presentation,

$$\pi_1(M) \cong \langle x, y \mid r_1 := x^{am}(x^a y^b)^{-mpq-\varepsilon}, r_2 := (x^a y^b)^{mpq+\varepsilon} y^{-p} x^{-qm-q} \rangle.$$  (13)

Likewise, we can show the following result:
Proposition 3.13. Suppose $1/p + 1/q + 1/(mpq + \varepsilon) < 1$ as above. Then the following word is a taut identity of the presentation [13].

$$\rho_1 \rho_2^{-1} \rho_1^{-1}(x^{qm} y^{-p} x^{-qm-q} \rho_2 x^{qm+q} y^p x^{-qm}).$$

4 First application to the linking forms of branched covers

4.1 Review of the linking form and a theorem

Here, we will review the linking form of $M$ for a closed 3-manifold $M$ with $H_*(M; \mathbb{Q}) \cong H_*(S^3; \mathbb{Q})$. Considering the short exact sequence

$$0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Q} \rightarrow \mathbb{Q}/\mathbb{Z} \rightarrow 0,$$

we can easily check that the Bockstein maps

$$\beta : H_2(M; \mathbb{Q}/\mathbb{Z}) \cong H_1(M; \mathbb{Z}), \quad \beta : H^1(M; \mathbb{Q}/\mathbb{Z}) \cong H^2(M; \mathbb{Z}),$$

are isomorphisms from the long exact homology sequences. Let $PD_M^\mathbb{Z}$ be the Poincaré duality on the integral (co)-homology. We denote by $\Omega$ the composite map defined by setting

$$H_1(M; \mathbb{Z}) \xrightarrow{PD_M^\mathbb{Z}} H^2(M; \mathbb{Z}) \xrightarrow{\beta^{-1}} H^1(M; \mathbb{Q}/\mathbb{Z}) \xrightarrow{ev} \hom(H_1(M; \mathbb{Z}); \mathbb{Q}/\mathbb{Z}),$$

where the last map is the Kronecker evaluation map. Then, the linking form of $M$ is

$$\lambda_M : H_1(M; \mathbb{Z}) \times H_1(M; \mathbb{Z}) \rightarrow \mathbb{Q}/\mathbb{Z}$$

defined by $\lambda_M(a, b) = \Omega(a)(b)$. This bilinear map is known to be symmetric and non-singular. This definition goes back to Seifert [Sei], and the form has sometimes appeared in the study of algebraic surgery theory (see, e.g., [Wall]) and the concordance groups of knots [CG]. Recently, the linking form of $M$ can be computed in terms of Heegaard splittings [CFH].

Of particular interest to us is an application to the Casson-Gordon invariant [CG] and a procedure for computing $\lambda_M$ in another way. In what follows, let $B^d_K$ be the $d$-fold cyclic covering space of $S^3$ branched over a knot $K$. In the context of the invariant, the linking form of $B^d_K$ plays an important role: more precisely, it is important to calculate metabolizers of the form; see, e.g., [CG].

Now let us give a matrix presentation of the homology $H_1(B^d_K; \mathbb{Z})$ and state the main theorem. Choose a Seifert surface $\Sigma$ of $K$ whose genus is $g$, as in §3.2. Then, we have the Seifert form $\alpha : H_1(\Sigma; \mathbb{Z}) \otimes H_1(\Sigma; \mathbb{Z}) \rightarrow \mathbb{Z}$; see [Lic, Chapter 6] for the definition. Let $J$ be the inverse matrix $(V - t^IV)^{-1}$, where $\det(V - t^IV) = 1$ is known (see [Lic, Theorem 9.7]). The matrix presentation is often written as $V \in \text{Mat}(2g \times 2g; \mathbb{Z})$ and is called the Seifert matrix. Consider the following matrices of size $(2gd \times 2gd)$:

$$A := \begin{pmatrix} -V & 0 & \cdots & 0 & t^IV \\ tV & -V & \cdots & 0 & 0 \\ \vdots & tV & \ddots & \vdots & \vdots \\ 0 & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & tV & -V \end{pmatrix}, \quad B := \begin{pmatrix} 0 & 0 & \cdots & 0 & J^tV \\ J^tV & 0 & \cdots & 0 & 0 \\ \vdots & J^tV & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & J^tV & 0 \end{pmatrix},$$

which appear in [Tro, Page 494]. As is known (see [Sei] [Tro] or [Lic, Theorem 9.7]), the first homology $H_1(B^d_K; \mathbb{Z})$ is isomorphic to the cokernel of $A$, i.e., $H_1(B^d_K; \mathbb{Z}) \cong \mathbb{Z}^{2gd} / \langle A \mathbb{Z}^{2gd} \rangle$. In
particular, \( \det(A) \neq 0 \) if and only if \( H_1(B^d_K; \mathbb{Q}) \cong 0 \). The linking formula of \( B^d_K \) can be algebraically formulated in the above notation as follows:

**Theorem 4.1.** Suppose that \( B^d_K \) satisfies Assumption (†) and \( H_1(B^d_K; \mathbb{Q}) \cong 0 \). Then, the matrix multiplication \( B : \mathbb{Z}^{2d} \to \mathbb{Z}^{2d} \) induces an isomorphism \( B : \mathbb{Z}^{2d} / t^1 \mathbb{A} \mathbb{Z}^{2d} \to \mathbb{Z}^{2d} / t^1 \mathbb{A} \mathbb{Z}^{2d} \) and the linking form \( \lambda_{B^d_K} \) of \( B^d_K \) is equal to the form,

\[
\mathbb{Z}^{2d} / t^1 \mathbb{A} \mathbb{Z}^{2d} \times \mathbb{Z}^{2d} / t^1 \mathbb{A} \mathbb{Z}^{2d} \to \mathbb{Q}/\mathbb{Z}; \quad (v, w) \mapsto t^1 \text{adj}(A)^tB^{-1}w/\Delta. \tag{15}
\]

Here, \( \text{adj}(A) \) is the adjugate matrix of \( A \), and \( \Delta \) is the order \( |H_1(B^d_K; \mathbb{Z})| \in \mathbb{N} \).

This statement is implicitly connoted in [Sei, Satz I] and [Tro, p. 496] \(^3\); however, there is no complete proof for this statement in the literature.

Here, let us make a few remarks. Whereas the matrix \( \text{adj}(A)^tB^{-1} \) is not always symmetric, the quotient on \( \mathbb{Z}^{2d} / t^1 \mathbb{A} \mathbb{Z}^{2d} \) is symmetric. Next, the second condition of \( H_1(B^d_K; \mathbb{Q}) \cong 0 \) is not so strong: indeed, according to [Lic, Corollary 9.8], if any \( d \)-th root of unity is not a zero point of the Alexander polynomial of \( K \) (e.g., the case \( d \) is a prime power), then \( H_1(B^d_K; \mathbb{Q}) \cong 0 \). Furthermore, as the proof and Remark 3.9 imply, one may hope that the theorem is true even if we drop the condition (†).

**Proof of Theorem 4.1.** It is known [CFH, Lemma 2.5] that the linking form can be formulated in the terminology of cohomology as

\[
\lambda_M(a, b) = \langle (\beta^{-1} \circ \text{PD}_M^\mathbb{Z})(a) \sim \text{PD}_M^\mathbb{Z}(b), [M] \rangle. \tag{16}
\]

Here, \( \sim \) is the cup-product \( H^1(M; \mathbb{Q}/\mathbb{Z}) \otimes H^2(M; \mathbb{Z}) \to H^3(M; \mathbb{Q}/\mathbb{Z}) \).

Let \( M \) be \( B^d_K \), and let \( R \) be one of \( \mathbb{Z}, \mathbb{Q} \) or \( \mathbb{Q}/\mathbb{Z} \) as trivial coefficients. Let \( \varepsilon : \mathbb{Z}[\pi_1(M)] \to \mathbb{Z} \) be the augmentation map. Then, as is known (see [Tro, Proposition 4.1]), by choosing a Seifert surface, the integral matrices \( \{\varepsilon(\partial \gamma_i^j)\}_{1 \leq i,j \leq 2g} \) and \( \{\varepsilon(\partial \gamma_i^j)\}_{1 \leq i,j \leq 2g} \) are equal to \( V \) and \( t^1V \), respectively. Let us identify the complex \( C^*(M; R) \) in the coefficients \( R \) with \( C^*(\tilde{M}; \mathbb{Z}) \otimes_{\mathbb{Z}[\pi_1(M)]} R \) via \( \varepsilon \). Then, by presentation \( [9] \), the complex \( C^*(M; R) \) reduces to

\[
0 \to C^0(M; \mathbb{Z}) \xrightarrow{0} C^1(M; R) \xrightarrow{A} C^2(M; R) \xrightarrow{0} C^3(M; R) \to 0. \tag{17}
\]

If \( R = \mathbb{Q} \), the matrix \( A \) is an isomorphic because of \( H^*(M; \mathbb{Q}) \cong H^*(S^3; \mathbb{Q}) \). Therefore, from the definition of the Bockstein inverse map \( \beta^{-1} : C^2(M; \mathbb{Z}) \to C^1(M; \mathbb{Q}/\mathbb{Z}) \) is identified with \( \mathbb{Z}^{2d} \to (\mathbb{Q}/\mathbb{Z})^{2d}; v \mapsto \text{adj}(A)v/\Delta \).

Meanwhile, from the formula for the identity \( W \) in Proposition 3.7 and the formula (3), the cup-product \( \sim : C^1(M; \mathbb{Q}/\mathbb{Z}) \times C^2(M; \mathbb{Z}) \to C^3(M; \mathbb{Q}/\mathbb{Z}) \cong \mathbb{Q}/\mathbb{Z} \) is considered to be \( (\mathbb{Q}/\mathbb{Z})^{2d} \times \mathbb{Z}^{2d} \to \mathbb{Q}/\mathbb{Z}; (v, w) \mapsto t^1vBw \). The Poincaré duality ensures the non-degeneracy of the cup product on cohomology. In particular, the desired induced map \( B \) is an isomorphism, and is identified with the duality \( H_1(M; \mathbb{Z}) \cong H^2(M; \mathbb{Z}) \), where \( H^2(M; \mathbb{Z}) \) is canonically regarded as \( \text{Coker}(A) = \mathbb{Z}^{2d} / t^1 \mathbb{A} \mathbb{Z}^{2d} \) by (17). Hence, upon the identification \( H^2(M; \mathbb{Z}) \cong H_1(M; \mathbb{Z}) \cong \mathbb{Z}^{2d} / t^1 \mathbb{A} \mathbb{Z}^{2d} \), the formula (16) immediately implies that the linking form is equal to the required (15).

---

\(^3\)To be precise, the original statements implicitly claim that the linking form \( \lambda_{B^d_K} \) is equal to the matrix presentation \( B \text{adj}(A) \) up to isomorphisms. However, for applications to the Casson-Gordon invariants, we should describe the linking form from a basis of \( H_1(B^d_K; \mathbb{Z}) \).
4.2 Example computations

It is easier to quantitatively compute kernels rather than cokernels. Let us examine Corollary 4.2 below. Let $\text{Ker}(A)_{\mathbb{Z}/\Delta}$ be $\{v \in (\mathbb{Z}/\Delta\mathbb{Z})^{2d} \mid Av = 0 \in (\mathbb{Z}/\Delta\mathbb{Z})^{2d}\}$. Consider the linear map

$$\mathbb{Z}^{2gd}/A\mathbb{Z}^{2gd} \rightarrow \text{Ker}(A)_{\mathbb{Z}/\Delta}; \quad v \mapsto \text{adj}(A)v.$$ 

This map is an isomorphism if $|\Delta| \neq 0$: in fact, with a choice of the section $s : \mathbb{Z}^{2gd}/A\mathbb{Z}^{2gd} \rightarrow \mathbb{Z}^{2gd}$, the inverse map is defined by $w \mapsto (As(w))/\Delta$. In summary, from Theorem 4.1 we immediately have the following:

**Corollary 4.2.** Let $\Delta$, $A$, $B$ and $\text{adj}(A)$ be as in Theorem 4.1. Under the supposition in Theorem 4.1, the linking form $\lambda_{B^d_K}$ of $B^d_K$ is isomorphic to the bilinear form

$$\text{Ker}(A)_{\mathbb{Z}/\Delta} \times \text{Ker}(A)_{\mathbb{Z}/\Delta} \rightarrow \mathbb{Q}/\mathbb{Z}; \quad (v, w) \mapsto t(s(v))^tABw/\Delta^2.$$ 

**Example 4.3.** Let $p, q, r \in \mathbb{Z}$ be odd numbers. Let $K$ be the Pretzel knot $P(p, q, r)$. When $d = 2$, the branched cover $B^2_K$ is known to be a Seifert fibered space of type $\Sigma(p, q, r)$ over $S^2$. Furthermore, we can choose a Seifert matrix of the form $V = \frac{1}{2}\begin{pmatrix} p + q & q + 1 \\ q - 1 & q + r \end{pmatrix}$, and $\Delta = pq + qr + rp$; see [Lic] Example 6.9.

First, consider the case where $p, q, r$ are relatively prime. Then, $\text{Ker}(A)$ is generated by $(-r - q, q, -r - q, q)$, and we can easily verify that the linking form equal to $2(q + r)/\Delta$.

However, if $p, q, r$ are not relatively prime, $\text{Ker}(A)$ and the linking form are complicated. For example, if $(p, q, r) = (p, -p, p)$, then $\text{Ker}(A) \cong (\mathbb{Z}/p)^2$ is generated by $(0, p, 0, p)$ and $(p, 0, p, 0)$; the linking matrix is equal to $\frac{2}{p}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Meanwhile, if $(p, q, r) = (p, p, p)$ and $p$ is not divisible by 3, then $\text{Ker}(A) \cong \mathbb{Z}/p \oplus \mathbb{Z}/3p$ possesses a basis, $v = (0, 3p, 0, 3p)$, $w = (3p + p^2, p^2, 3p + p^2, p^2)$. Hence, $\begin{pmatrix} \text{lk}(v, v) & \text{lk}(v, w) \\ \text{lk}(w, v) & \text{lk}(w, w) \end{pmatrix}$ can be computed as $\frac{2}{p}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

In a similar way, we can compute many linking forms of $d$-fold branched covering spaces for small $d$ with the help of a computer program.

5 Second application to Dijkgraaf-Witten invariants

As another application, we develop procedures of computing some Dijkgraaf-Witten invariants in terms of identities.

We start by reviewing the Dijkgraaf-Witten invariant [DW]. Let $G$ be a finite group, $A$ a commutative ring, and $\psi$ a group 3-cocycle of $G$. Denoting by $BG$ an Eilenberg-MacLane space of type $(G, 1)$, we have a classifying map $\iota : M \hookrightarrow B\pi_1(M)$ uniquely up to homotopy. Then, as is known, $\psi$ can be regarded as a 3-cocycle of $H^3(BG; A)$, and any group homomorphism $f : M \rightarrow G$ canonically gives rise to the composite

$$\iota^* \circ f^* : H^*(BG; A) \rightarrow H^*(B\pi_1(M); A) = H^*(\pi_1(M); A) \rightarrow H^*(M; A).$$

Then, the *Dijkgraaf-Witten invariant* of $M$ is defined as a formal sum in the group ring $\mathbb{Z}[A]$ by setting

$$\text{DW}_\psi(M) := \sum_{f \in \text{Hom}(\pi_1(M), G)} \langle \iota^* \circ f^*(\psi), [M] \rangle \in \mathbb{Z}[A].$$
Although the definition seems rather simple or direct, it is not easy to compute $\text{DW}_\psi(M)$ except in the case where $G$ is abelian, because it is not trivial to explicitly express $[M]$ and $f^*$ (however, see [DW, Wakui] for the abelian case and [No2] for a partially non-abelian case). To the knowledge of the author, there are few examples of Dijkgraaf-Witten invariants when $G$ is non-abelian.

This section develops a method for computing the invariants, and gives non-abelian examples. First, for simplicity, we now restrict on the case $\psi = \gamma \setminus \delta$ for some $\gamma \in H^1(G;A)$ and $\delta \in H^2(G;A)$. Take a group homomorphism $f : \pi_1(M) \to G$ and a group presentation $G = \langle y_1, \ldots, y_n \mid s_1, \ldots, s_4 \rangle$. Then, as in [1], we have a commutative diagram:

$$
\begin{align*}
C_*(M;A) : & \xrightarrow{\partial^1} A \otimes \mathbb{Z}[\pi_1(M)]^m \xrightarrow{\partial^2} A \otimes \mathbb{Z}[\pi_1(M)]^m \xrightarrow{0} A \otimes \mathbb{Z}[\pi_1(M)] \\
C_*(G;A) : & \xrightarrow{\partial^1} A \otimes \mathbb{Z}[G]^\ell \xrightarrow{\partial^2} A \otimes \mathbb{Z}[G]^n \xrightarrow{\partial^3 = 0} A \otimes \mathbb{Z}[G].
\end{align*}
$$

Here, the tensors are over $\mathbb{Z}[G]$, and $\partial^2(b'_i) = \sum_{k=1}^n [\partial_{b'_k}]a'_k$.

**Example 5.1.** Suppose $p, q \in \mathbb{N}$ such that $(p, q) = 1$. Let $A = G = \mathbb{Z}/p$, and $M$ be the lens space $L(p, q)$. Then, as is known, $H^*(G;A) \cong \mathbb{Z}/p$, and we can choose appropriate generators $\alpha_i \in H^i(G;A) \cong \mathbb{Z}/p$ such that $\alpha_3 = \alpha_1 \setminus \alpha_2$. We fix a presentation $G = \pi_1(M) = \langle x \mid s := x^p \rangle$. Then, the taut identity of $L(p, q)$ is known to be $W_{p,q} = sx^{-q}s^{-1}x^{q}$; see [Sic]. Then, for $i \leq 3$, we can regard $\alpha_i$ as a map $\mathbb{Z}/p = C_*(M;\mathbb{Z}/p) \to \mathbb{Z}/p$ that sends a generator to 1. Then it follows from [2] that the cup product $\setminus : H^1(L(p,q);\mathbb{Z}/p) \times H^2(L(p,q);\mathbb{Z}/p) \to \mathbb{Z}/p$ is computed as $(a,b) \mapsto qab$. Moreover, for $a \in \mathbb{Z}/p$, if we define $f_a : \pi_1(M) \to G$ by setting $x \mapsto a$, then $\text{Hom}(\pi_1(M),G)$ is equal to $\{f_a\mid a \in \mathbb{Z}/p\}$, and we can compute

$$
\langle f_a^*(\alpha_3), \iota_*[M] \rangle = \langle f_a^*(\alpha_1 \setminus \alpha_2), \iota_*[M] \rangle = \langle a\alpha_1 \setminus a\alpha_2, \iota_*[M] \rangle = qa^2.
$$

In conclusion,

$$
\text{DW}_{a\alpha_3}(L(p,q)) = \sum_{a \in \mathbb{Z}/p} 1\{qa^2\} \in \mathbb{Z}/p.
$$

In a similar way, if $M$ is another manifold such that the cohomology ring is known, we can compute $\text{DW}_{a\alpha_3}(M)$ for $G = \mathbb{Z}/p$. Comparing with [DW, Wakui] as original computations, the above computation seems easier.

**Example 5.2.** Let $m, n$ be natural numbers such that $m$ is relatively prime to $6n$. Let $G$ be the non-abelian group of order $m^3$ which has a group presentation

$$
\langle x, y, z \mid x^m, y^m, z^m, s := xzx^{-1}z^{-1}, t := yzy^{-1}z^{-1}, u := zyx^{-1}x^{-1} \rangle.
$$

(18)

The (co)-homology of $G$ is known (see, e.g., [Lea]). As a result, $H_1(G;\mathbb{Z}) \cong (\mathbb{Z}/m)^2$. Dually, the first cohomology $H^1(G;\mathbb{Z}/m) \cong (\mathbb{Z}/m)^2$ is generated by the maps $\alpha$ and $\beta$ defined by $\alpha(x) = \beta(y) = 1$ and $\alpha(y) = \beta(x) = 0$. Furthermore, the Massey product $\langle \alpha, \beta, \alpha \rangle$ and the product $\psi := \beta \setminus \langle \alpha, \beta, \alpha \rangle$ are known to be non-trivial. The equality $\psi = -\alpha \setminus \langle \beta, \alpha, \beta \rangle$ is also known. Since the cup product $C^1 \otimes C^1 \to C^2$ is well described in [Ir6, §2.4], the Massey product $\langle \alpha, \beta, \alpha \rangle$ can be, by definition, regarded as the map $C_2(G;\mathbb{Z}/m) \to \mathbb{Z}/m$ by setting

$$
\begin{align*}
x^m & \mapsto 0, \quad y^m \mapsto 0, \quad z^m \mapsto 0, \quad s \mapsto 0, \quad t \mapsto 0, \quad u \mapsto 2.
\end{align*}
$$

(19)
On the other hand, for simplicity, we specialize to the Seifert manifolds of type $M_{m,n} := \Sigma(0, (1, 0), (m, 1), (m, -1), (n, -1))$ over $S^2$, whose fundamental groups are presented by

$$\pi_1(M_m) = \langle x_1, x_2 \mid r_1 := x_1^m (x_1^{-1} x_2^{-1})^n, r_2 := x_2^m (x_2^{-1} x_1^{-1})^n \rangle.$$  

By Proposition 3.11 the identity is $W := r_2 x_2 r_2^{-1} x_2^{-1} r_1 x_1 r_1^{-1} x_1^{-1}$. We further analyze the set $\text{Hom}(\pi_1(M_{m,n}), G)$. For $a, b, c \in \mathbb{Z}/m$, consider the homomorphism $f_{a,b,c} : \pi_1(M_{m,n}) \to G$ defined by

$$f_{a,b,c}(x_1) := x^a y^b z^c, \quad f_{a,b,c}(x_2) := x^{-a} y^{-b} z^{-c + a b}.$$  

It is not so hard to check the bijectivity of $(\mathbb{Z}/m)^3 \leftrightarrow \text{Hom}(\pi_1(M_{m,n}), G)$ which sends $(a, b, c)$ to $f_{a,b,c}$. Then, the conclusion is as follows:

**Proposition 5.3.** Let $\psi$ be $\beta \sim \langle \alpha, \beta, \alpha \rangle \in H^3(G, \mathbb{Z}/m)$. Let $m \in \mathbb{Z}$ be relatively prime to $6n$. Then, upon the identification $(\mathbb{Z}/m)^3 \leftrightarrow \text{Hom}(\pi_1(M_{m,n}), G)$, the Dijkgraaf-Witten invariant is equal to

$$\text{DW}_\psi(M_{m,n}) = \sum_{(a, b, c) \in (\mathbb{Z}/m)^3} 1\{n(2abc - a(a - 1)b(b - 1))\} \in \mathbb{Z}[\mathbb{Z}/m].$$

**Proof.** Recall from [1] that the basis of $C_2(M_{m,n}) \cong \mathbb{Z}[\pi_1(M_m)]^2$ is denoted by $b_1, b_2$, where $b_i$ corresponds to the relator $r_i$. We now analyse $(f_{a,b,c})_*(b_1) \in C_2(G; \mathbb{Z}/m)$. We can easily check that $f_{a,b,c}(r_i)$ is transformed to $x^m y^{bm} z^{cm(m+1)/2}$ by the above relators $s, t, u$. Let us define $N_{b_i} \in \mathbb{Z}$ to be the numbers of applying $u$ when we transform $(f_{a,b,c})(r_i)$ by $x^m y^{bm} z^{cm(m+1)/2}$. Then, by [19], the pairing $\langle \langle \alpha, \beta, \alpha \rangle, (f_{a,b,c})_*(b_1) \rangle$ is equal to $2N_{b_1}$. From the definition of $N_{b_1}$, a little complicated computation can lead to

$$N_{b_1} = \frac{m(m + 1)ac}{2} + \left(\sum_{i=1}^{m-1} \frac{ia(ia - 1)}{2}\right) + nac - \frac{na(a-1)(b-1)}{2} \in \mathbb{Z}.$$  

Since $m$ is relatively prime to $6n$, we can easily check the first and second terms to be zero modulo $m$. Hence, using the above description of $W$ and the formula (2), we have

$$\langle \psi, (f_{a,b,c})_*[M_{m,n}] \rangle = 0 + b \cdot 2N_{b_1} - 0 \cdot N_{b_2} + 0 = b(2nac - na(a-1)(b-1)) \in \mathbb{Z}/m,$$

which immediately leads to the conclusion. \qed

The above computation is relatively simple, since so are the presentations of $\pi_1(M)$ and $G$; however, a similar computation seems to be harder if $\pi_1(M)$ is complicated.

In contrast, we conclude this paper by suggesting another procedure of computing $\text{DW}_\psi(M)$, which is implicitly discussed in [No1, §4]. Hereafter $\psi \in H^3(G; A)$ may be an arbitrary 3-cocycle.

Let $C_n^{\text{nh}}(G; \mathbb{Z})$ be the normalized homogenous complex of $G$, which is defined as the quotient $\mathbb{Z}$-free module of $\mathbb{Z}[G^{n+1}]$ subject to the relation $(g_0, \ldots, g_n) \sim 0$ if $g_i = g_{i+1}$ for some $i$; see [Bro, 19 page]. Assume that we know an explicit expression of $\psi : G^4 \to A$ as an element of $C_3^{\text{nh}}(G, A)$. When $* \leq 3$, we now define a chain map $c_* : C_n(\hat{M}; \mathbb{Z}) \to C_n^{\text{nh}}(\pi_1(M); \mathbb{Z})$ as follows.

Let $c_0$ be the identity map. Let $A \in \mathbb{Z}[\pi_1(M)]$ be any element. Define $c_1(Ax_i) := (A, Ax_i)$. If $r_i$ is expanded as $x_{i_1}^{\epsilon_1} x_{i_2}^{\epsilon_2} \cdots x_{i_m}^{\epsilon_m}$ for some $\epsilon_k \in \{\pm 1\}$, we define

$$c_2(Ar_i) = \sum_{m:1 \leq m \leq n} \epsilon_m(A, Ax_{i_1}^{\epsilon_1} x_{i_2}^{\epsilon_2} \cdots x_{i_m-1}^{\epsilon_{m-1}} x_{i_m}^{(\epsilon_m-1)/2}, Ax_{i_1}^{\epsilon_1} x_{i_2}^{\epsilon_2} \cdots x_{i_m-1}^{\epsilon_{m-1}} x_{i_m}^{(\epsilon_m+1)/2}) \in C_2^{\text{nh}}(\pi_1(M); \mathbb{Z}).$$
Then, we can easily verify $\partial_{L}^{A} \circ c_{1} = c_{0} \circ \partial_{1}$ and $\partial_{L}^{A} \circ c_{2} = c_{1} \circ \partial_{2}$. Let $O_{M} \subset C_{3}^{\ast}(\tilde{M}; Z)$ be the basis. Notice that $\partial_{L}^{A} \circ c_{2} \circ \partial_{3}(O_{M}) = c_{1} \circ \partial_{2} \circ \partial_{4}(O_{M}) = 0$, that is, $c_{2} \circ \partial_{3}(O_{M})$ is a 2-cycle. If we expand $c_{2} \circ \partial_{3}(O_{M})$ as $\sum n_{i}(g_{i}^{0}, g_{i}^{1}, g_{i}^{2})$ for some $n_{i} \in Z, g_{i}^{j} \in G$, then $O_{M}^{\prime} := - \sum n_{i}(1, g_{i}^{0}, g_{i}^{1}, g_{i}^{2})$ satisfies $\partial_{L}^{A}(O_{M}^{\prime}) = c_{2} \circ \partial_{3}(O_{M})$. Therefore, the correspondence $O_{M} \mapsto O_{M}^{\prime}$ gives rise to a chain map $c_{3} : C_{3}^{\ast}(\tilde{M}) \rightarrow C_{3}^{\ast}(\pi_{1}(M); Z)$, as desired. In conclusion, the above discussion can be summarized as follows:

**Proposition 5.4.** For any homomorphism $f : \pi_{1}(M) \rightarrow G$, the pushforward $f_{\ast} \circ \iota_{\ast}[M]$ is equal to $1 \otimes_{\pi_{1}(M)} f_{\ast} \circ c_{3}(O_{M})$ in $H_{3}^{\ast}(G; Z)$.

To conclude, if we know an explicit presentation of $\pi_{1}(M)$ and a representative of the 3-cocycle $\psi : G^{4} \rightarrow A$, in principle, we can compute $DW_{\psi}(M)$ in terms of the chain map $c_{\ast}$ (with the help of computer program).

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