THE GENUS OF A RANDOM BIPARTITE GRAPH

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Abstract. Archdeacon and Grable (1995) proved that the genus of the random graph \( G \in \mathcal{G}_{n,p} \) is almost surely close to \( pn^2/12 \) if \( p = p(n) \geq 3(\ln n)^2 n^{-1/2} \). In this paper we prove an analogous result for random bipartite graphs in \( \mathcal{G}_{n_1,n_2,p} \). If \( n_1 \geq n_2 \gg 1 \), phase transitions occur for every positive integer \( i \) when \( p = \Theta((n_1 n_2)^{-i/2}) \). A different behaviour is exhibited when one of the bipartite parts has constant size, \( n_1 \gg 1 \) and \( n_2 \) is a constant. In that case, phase transitions occur when \( p = \Theta(n_1^{-1/3}) \) and when \( p = \Theta(n_1^{-1/2}) \).

1. Introduction

For a simple graph \( G \), let \( g(G) \) be the genus of \( G \), that is, the minimum \( h \) such that \( G \) embeds into the orientable surface \( S_h \) of genus \( h \), and let \( \tilde{g}(G) \) be the non-orientable genus of \( G \) which is the minimum \( c \) such that \( G \) embeds into the non-orientable surface \( N_c \) with crosscap number \( c \). The surface here is a compact two-dimensional manifold without boundary. We say \( G \) is 2-cell embedded in a surface \( S \) if each face of \( G \) is homeomorphic to an open disk, and a \( k \)-gon embedding of \( G \) is when every face is bounded by a cycle of length \( k \).

Given a graph \( G \), determining the genus of \( G \) is one of the fundamental problems in topological graph theory. Youngs [16] showed that the problem of determining the genus of a connected graph \( G \) is the same as determining a 2-cell embedding of \( G \) with minimum genus. The same holds for the non-orientable genus [9]. It was proved by Thomassen [15] that the genus problem is NP-complete. For further background on topological graph theory, we refer to [8].

The random graph \( \mathcal{G}_{n,p} \) is a probability space whose objects are all (labelled) graphs defined on a vertex set \( V \) of cardinality \( n \), and each possible edge occurs with probability \( p \) independently, i.e., a graph \( G = (V,E) \in \mathcal{G}_{n,p} \) has probability \( p^{|E|}(1-p)^{(n^2)/2-|E|} \). Similarly, one can define random bipartite...
graphs $\mathcal{G}_{n_1, n_2, p}$ as the probability space of all bipartite graphs with (labelled) bipartition $X \sqcup Y$, $|X| = n_1$, $|Y| = n_2$, where each edge $xy$ ($x \in X$, $y \in Y$) appears with probability $p$. In this paper we will always assume $n_1 \geq n_2$ for the convenience. There are thousands of papers studying properties of random graphs; for more background about this fascinating area, see [1, 3].

Stahl [14] was the first to consider the genus (in fact, the average genus) of random graphs. Almost concurrently, Archdeacon and Grable [2] studied the genus of random graphs in $G_{n, p}$. They obtained the following result when $p = p(n)$ is not too small.

**Theorem 1.1** (Archdeacon and Grable [2]). Let $\varepsilon > 0$ and let $0 < p < 1$ with $p^2(1 - p^2) \geq 8(\ln n)^4/n$. Then almost every graph $G$ in $G_{n, p}$ satisfies

$$
(1 - \varepsilon)\frac{pn^2}{12} \leq g(G) \leq (1 + \varepsilon)\frac{pn^2}{12}
$$

and

$$
(1 - \varepsilon)\frac{pn^2}{6} \leq \tilde{g}(G) \leq (1 + \varepsilon)\frac{pn^2}{6}.
$$

They also conjectured that almost every graph in $G_{n, p}$ has an $\varepsilon$-near $k$-gon embedding (in which all but an $\varepsilon$-fraction of edges lie on the boundary of two $k$-gonal faces) on some orientable surface and on some non-orientable surface. Rödl and Thomas [13] resolved their conjecture and extended Theorem 1.1 to an even broader range of edge-probabilities.

**Theorem 1.2** (Rödl and Thomas [13]). Let $\varepsilon > 0$, let $i \geq 1$ be an integer and assume that $n^{-\frac{1}{i+1}} \ll p \ll n^{-\frac{1}{i-1}}$. Then $G \in G_{n, p}$ almost surely satisfies

$$
(1 - \varepsilon)\frac{i}{4(i + 2)}pn^2 \leq g(G) \leq (1 + \varepsilon)\frac{i}{4(i + 2)}pn^2
$$

and

$$
(1 - \varepsilon)\frac{i}{2(i + 2)}pn^2 \leq \tilde{g}(G) \leq (1 + \varepsilon)\frac{i}{2(i + 2)}pn^2.
$$

In this paper, we will study the genus of random bipartite graphs, which plays an important role in approximating the genus of dense graphs [7]. The main results of this paper show that a result similar to Theorems 1.1 and 1.2 is also true for random bipartite graphs.

**Theorem 1.3.** Let $\varepsilon > 0$ and $G \in G_{n_1, n_2, p}$ be a random bipartite graph and suppose that $i \geq 2$ is an integer. If $p$ satisfies $(n_1n_2)^{-\frac{1}{i+1}} \ll p \ll (n_1n_2)^{-\frac{1}{i-1}}$, $n_1/n_2 < c$ and $n_2/n_1 < c$ where $c$ is a positive real number, then we have a.a.s.

$$
(1 - \varepsilon)\frac{i}{2i + 2}pn_1n_2 \leq g(G) \leq (1 + \varepsilon)\frac{i}{2i + 2}pn_1n_2
$$
and

$$(1 - \varepsilon) \frac{i}{i+1} pm_1n_2 \leq \bar{g}(G) \leq (1 + \varepsilon) \frac{i}{i+1} pm_1n_2.$$ 

In particular, when $p$ is relatively large, $G \in \mathcal{G}_{n_1, n_2, p}$ will almost surely have an $\varepsilon$-near 4-gon embedding.

**Theorem 1.4.** Let $\varepsilon > 0$ and $G \in \mathcal{G}_{n_1, n_2, p}$ be a random bipartite graph. If $n_1 \geq n_2 \gg 1$ and $p \gg n_2^{-\frac{3}{2}}$, then we have a.a.s.

$$(1 - \varepsilon) \frac{pm_1n_2}{4} \leq g(G) \leq (1 + \varepsilon) \frac{pm_1n_2}{4}$$

and

$$(1 - \varepsilon) \frac{pm_1n_2}{2} \leq \bar{g}(G) \leq (1 + \varepsilon) \frac{pm_1n_2}{2}.$$ 

The above results exhibit phase transitions for every positive integer $i$, when $p = \Theta((n_1n_2)^{-\frac{i+1}{2i+4}})$. The genus in these critical ranges can be estimated within a constant factor as follows. Let $n = \sqrt{n_1n_2}$ and $\varepsilon > 0$. It is easy to see that the genus of a graph $G$ satisfies the edge-Lipschitz condition, i.e., if $G$ and $G'$ differ in only one edge, then $|g(G) - g(G')| \leq 1$. By [1 Chapter 7], when $n_2 = \Theta(n_1)$ and $p = cn^{-\frac{i+1}{2i+4}}$ for $i \geq 2$, there exists a number $f(c, n, p)$ with

$$(1 - \varepsilon)f(c, n, p)pn^2 \leq g(G) \leq (1 + \varepsilon)f(c, n, p)pn^2, \text{ a.a.s.}$$

When a random bipartite graph $G \in \mathcal{G}_{n_1, n_2, p}$ satisfies $n_1 \gg 1$ and $n_2$ is a constant, the genus of $G$ has different behaviour.

**Theorem 1.5.** Let $\varepsilon > 0$ and $G \in \mathcal{G}_{n_1, n_2, p}$ where $n_1 \gg 1$ and $n_2$ is a constant.

(a) If $p \gg n_1^{-\frac{1}{3}}$ we have a.a.s. (as $n_1 \to \infty$)

$$(1 - \varepsilon) \frac{n_1n_2p}{4} \Psi(p, n_2) \leq g(G) \leq (1 + \varepsilon) \frac{n_1n_2p}{4} \Psi(p, n_2)$$

and

$$(1 - \varepsilon) \frac{n_1n_2p}{2} \Psi(p, n_2) \leq \bar{g}(G) \leq (1 + \varepsilon) \frac{n_1n_2p}{2} \Psi(p, n_2),$$

where $\Psi(p, n_2) = \sum_{i=2}^{n_2-1} \frac{i-1}{i+1}(n_2-1)(-p)^i$.

(b) If $n_1^{-\frac{1}{2}} \ll p \ll n_1^{-\frac{1}{4}}$, then a.a.s.

$g(G) = \lfloor \frac{(n_2 - 3)(n_2 - 4)}{12} \rfloor$ and $\bar{g}(G) = \lfloor \frac{(n_2 - 3)(n_2 - 4)}{6} \rfloor$

with a single exception that $\bar{g}(G) = 3$ when $n_2 = 7$.

(c) If $p \ll n_1^{-\frac{1}{2}}$, then a.a.s. $g(G) = 0$. 

This result shows two phase transitions. When $p \ll n_1^{-1/2}$, a random bipartite graph is almost surely planar; after this first threshold, we obtain a subdivision of the complete graph on $n_2$ vertices (with additional vertices of degrees 0 or 1), and when $p \gg n_1^{-1/3}$, $G$ has an $\varepsilon$-near 4-gon embedding a.a.s.

The paper is organized as follows. In the next section, we give basic definitions and properties in topological graph theory and discuss random graphs. Also, our main tools used in the proofs are presented. In Section 3, we prove Theorems 1.3 and 1.4. Section 4 resolves the cases when one of the bipartition parts has constant size and contains the proof of Theorem 1.5.

2. Preliminaries

We will use standard definitions and notation for graphs and probabilistic methods as given in [4] and [1, 3]. We use the following notation: $A(n) \sim B(n)$ means $\lim_{n \to \infty} A(n)/B(n) = 1$, and $A(n) \ll B(n)$ means $\lim_{n \to \infty} A(n)/B(n) = 0$. By $X \sqcup Y$ we denote the disjoint union of $X$ and $Y$, and we set $X \oplus Y = (X \times Y) \sqcup (Y \times X)$. We say an event $A(n)$ happens asymptotically almost surely (abbreviated a.a.s.) if $\mathbb{P}(A(n)) \to 1$ as $n \to \infty$.

We consistently use $G$ to denote a simple undirected graph, $D$ is always a digraph and $H$ is a hypergraph. A vertex partition $P = \{V_i\}_{i=1}^k$ is equitable if $V_i \cap V_j = \emptyset$ for every $1 \leq i < j \leq k$ and the parts have size as equal as possible, i.e. $|V_i| - |V_j| \leq 1$ for all $i, j$. A trail in a graph $G$ (or a digraph $D$) is a (directed) walk that has no repeated edges. A closed trail is a trail that starts and ends at the same vertex. If $D$ is a digraph, then $D^{-1}$ is the digraph obtained from $D$ by replacing each arc $\overrightarrow{xy}$ with the reverse arc $\overrightarrow{yx}$.

Let $G$ be a simple graph. The corresponding digraph $D$ of $G$ is a random simple digraph obtained from $G$ by randomly orienting each edge. Specifically, each digraph $D \in \mathcal{D}$ has $V(D) = V(G)$ and if $uv \in E(G)$ then either $\overrightarrow{uv}$ or $\overrightarrow{vu}$ is an edge of $D$, each has probability 1/2 and the two events are exclusive. The corresponding digraph $D$ of a random graph $G$ is a family of digraphs defined on the same vertex set of graphs in $G$, and when two vertices $u, v$ produce an edge with probability $p$ in $G$, then $\overrightarrow{uv}$ occurs with probability $\frac{p}{2}$ and $\overrightarrow{vu}$ occurs with probability $\frac{1-p}{2}$ in $D$, and those two events are exclusive.

Now we focus on the 2-cell embeddings of a graph $G$. We say $\Pi = \{\pi_v \mid v \in V(G)\}$ is a rotation system if for each vertex $v$, $\pi_v$ is a cyclic permutation of the edges incident with $v$. The Heffter-Edmonds-Ringel rotation principle [8, Theorem 3.2.4] shows that every 2-cell embedding of a graph $G$ in an orientable surface is uniquely determined (up to homeomorphisms of the surface) by its rotation system. Let $g(G)$ be the orientable genus of $G$ and let $\tilde{g}(G)$ be the non-orientable genus of $G$. For 2-cell embeddings we have the famous Euler’s Formula.
Theorem 2.1 (Euler’s Formula). Let $G$ be a graph which is 2-cell embedded in a surface $S$. If $G$ has $n$ vertices, $e$ edges and $f$ faces in $S$, then

\[(1) \quad \chi(S) = n - e + f.\]

Here $\chi(S)$ is the Euler characteristic of the surface $S$, where $\chi(S) = 2 - 2h$ when $S = S_h$ and $\chi(S) = 2 - c$ when $S = S_c$.

Given a digraph $D$, a blossom of length $l$ with center $v$ and tips $\{v_1, v_2, \ldots, v_l\}$ is a set $C$ of $l$ directed cycles $\{C_1, C_2, \ldots, C_l\}$, where $\overrightarrow{v_i}v, \overrightarrow{vv_{i+1}} \in C_i$, for $i = 1, 2, \ldots, l$, with $v_{l+1} = v_1$. A $k$-blossom is a blossom, all of whose elements are directed $k$-cycles. A blossom of length $l$ is simple if either $l \geq 3$ or $l = 2$ and $C_1 \neq C_2^{-1}$.

![Figure 1. A 4-blossom of length 4 with center $v$ and tips $v_1, v_2, v_3, v_4$.](image)

Let $C$ be a family of arc-disjoint closed trails in $D \cup D^{-1}$. We say that $C$ is blossom-free if no subset of $C$ forms a blossom centered at some vertex. The following lemma is a slight strengthening of [13, Lemma 2.1]; the proof is elementary and we omit details.

**Lemma 2.2.** Let $G$ be a graph and let $D$ be the corresponding digraph. Suppose that $C_1$ and $C_2$ is a set of arc-disjoint closed trails in $D$ and $D^{-1}$ (respectively) such that their union $C_1 \cup C_2$ is blossom-free in $D \cup D^{-1}$. Then there exist a rotation system $\Pi$ of $G$ such that every closed trail in $C_1 \cup C_2$ is a face of $\Pi$.

For every $\varepsilon > 0$, an $\varepsilon$-near $k$-gon embedding $\Pi$ is a rotation system of $G$ such that $kf_k(\Pi) \geq 2(1 - \varepsilon)|E(G)|$, where $f_k(\Pi)$ is the number of faces of length $k$ of $\Pi$.

The following result from [5] (see also [10, 13] where its current formulation appears) will be our main tool for constructing near-optimal embeddings of random graphs.
Theorem 2.3. Let $\varepsilon > 0$ be a real number and $d \geq 2$ be an integer. Then there exist a positive real number $\delta$ and an integer $N_0$ such that for every $N \geq N_0$ the following holds. If $\Delta$ is a real number and if $H$ is a $d$-uniform hypergraph with $|V(H)| = N$ such that

1. $|\{x \in V(H) \mid (1 - \delta)\Delta \leq \deg(x) \leq (1 + \delta)\Delta\}| \geq (1 - \delta)N$,

2. for every $x, y \in V(H)$, $|\{e \in E(H) \mid x, y \in e\}| < \delta\Delta$,

3. at most $\delta N\Delta$ hyperedges of $H$ contain a vertex $v \in V(H)$ with $\deg(v) > (1 + \delta)\Delta$,

then $H$ has a matching of size at least $(1 - \varepsilon)N/d$. Moreover, for every matching $M$ in $H$, there exists a matching $M'$ in $H$ with $M \cap M' = \emptyset$, and with $|M'| \geq (1 - \varepsilon)N/d$.

Similarly as for undirected graphs (see [8, Lemma 5.4.2]), we have the following property on digraphs.

Lemma 2.4. Let $D(V,A)$ be a simple digraph and $a,b \in A,a \neq b$. Let $f,g \in \mathbb{Z}^+$. Then there exists a positive integer $K = K(f,g)$, such that if $D$ contains at least $K$ closed trails of length $f$ containing both $a$ and $b$, then there exist two vertices $u,v \in V(D)$ and $g$ internally disjoint directed paths from $u$ to $v$, all of the same length $l$, where $2 \leq l \leq f - 2$.

Proof. Let $x \in V$ be the head of $a$ and let $y \in V$ be the tail of $b$. We may assume $x \neq y$. The proof is by induction on $f + g$, with $K(f,g) = \prod_{i=1}^{f-2}((f-i)(f-i-1)g)^{2^{i-1}}$. In the base case when $g = 0$ there is nothing to prove, and when $f = 3$, the claim is easy, so we move to the induction step. Assume now we have $K(f + 1,g)$ closed trails of length $f + 1$ containing both $a$ and $b$. Let $\overrightarrow{P_{xy}}$ be the set of paths from $x$ to $y$ on these closed trails. Note that $K(f + 1,g) = f(f - 1)gK(f,g)^2$. If one of the edges say $\overrightarrow{az}$, is used on $K(f,g)$ of the paths, we can consider the $K(f,g)$ subpaths from $z$ to $y$ and apply induction. Otherwise, there is a subset $\overrightarrow{P'_{xy}}$ of $\overrightarrow{P_{xy}}$ containing $f(f - 1)gK(f,g)$ paths, all of which start with different edges. Choose one path in $\overrightarrow{P'_{xy}}$ arbitrarily, call it $P$.

If at least $fK(f,g)$ of our paths intersect $P$, there exists $v \in V(P)$ such that at least $K(f,g)$ paths pass though $v$. Contract all of those directed paths from $a$ to $v$, we have $K(f,g)$ closed trails of length at most $f$ containing both $a$ and $b$. For those closed trails of length $f' < f$, we will add closed trails of length $f - f'$ containing $x$. By induction, we obtain $g$ internally disjoint directed paths.

Finally we suppose that we do not have $fK(f,g)$ paths of $\overrightarrow{P'_{xy}}$ intersecting $P$. Since $P$ is arbitrary, we may assume the same holds for any $P$. Then at least $(f - 1)g$ of our paths of length at most $f - 1$ are internally disjoint.
Therefore at least \( g \) internally disjoint directed paths having the same length \( l \), where \( l \leq f - 1 \).

\[ \square \]

3. Genus of random bipartite graphs

In this section we treat random bipartite graphs in \( G_{n_1,n_2,p} \). Let us first consider the case when \( n_1 \) and \( n_2 \) have about the same magnitude.

**Lemma 3.1.** Let \( \varepsilon > 0 \) and \( G \in G_{n_1,n_2,p} \) be a random bipartite graph on vertex set \( X \sqcup Y \) with \( |X| = n_1 \geq n_2 = |Y| \). If there exist a positive real number \( c \) and a positive integer \( i \) such that \( n_1/n_2 < c \), and \( p \gg (n_1n_2)^{-\frac{1}{2i+1}} \), then a.a.s. \( G \) has an \( \varepsilon \)-near \((2i + 2)\)-gon embedding.

**Proof.** Choose \( 0 < \varepsilon_1 < \frac{1}{2} \), \( \varepsilon_0 = \frac{3i+1}{1-\varepsilon_1} \varepsilon_1 \), such that \( \varepsilon_0 < 1/2 \) and \( \varepsilon \geq \frac{4\varepsilon_0}{1+\varepsilon_0} \).

Let \( n = \sqrt{n_1n_2} \). Then \( p \gg n^{-\frac{2i}{2i+1}} \). Let us first assume that \( p \ll n^{-\frac{2i}{2i+1}} \).

Let \( D \in \mathcal{D} \) be the corresponding digraph of \( G_{n_1,n_2,p} \). Consider the following hypergraph \( \mathcal{H} \), where \( V(\mathcal{H}) \) is the edge set of \( D \) and \( E(\mathcal{H}) \) is the set of closed trails of \( D \) of length \( 2i + 2 \). Let \( d = 2i + 2, \delta = \frac{\varepsilon_1}{1-\varepsilon_1} \) and \( \Delta = n_1^i n_2^{i} (\frac{p}{2})^{2i+1} \).

We claim that our hypergraph \( \mathcal{H} \) satisfies all three conditions in Theorem 2.3.

To prove that condition (1) holds, let \( N = |V(\mathcal{H})| \). We have

\[
\begin{align*}
\mathbb{E}(N) &= n_1n_2p, \\
\mathbb{E}(N^2) &= n_1n_2p(n_1 - 1)(n_2 - 1)p + O(n_1^2n_2^2p^2 + n_1^2n_1p^2).
\end{align*}
\]

By Chebyshev’s inequality,

\[
\begin{align*}
\mathbb{P}(|N - \mathbb{E}(N)| \geq \varepsilon_1n_1n_2p) &\leq \frac{\mathbb{E}(N^2) - \mathbb{E}^2(N)}{\varepsilon_1^2 \mathbb{E}^2(N)} = O \left( \frac{n_1 + n_2}{n_1n_2} \right) = o(1).
\end{align*}
\]

Therefore, we have a.a.s.

\[
(1 - \varepsilon_1)n_1n_2p < N < (1 + \varepsilon_1)n_1n_2p.
\]

For each pair of vertices \((a, b) \in X \oplus Y\), let \( \rho(b,a) \) be the number of directed paths in \( D \) from \( b \) to \( a \) of length \( 2i + 1 \), and let \( U \) be the number of edges \( \overrightarrow{uv} \) of \( D \) such that the number of directed paths from \( u \) to \( v \) of length \( 2i + 1 \) is at most \((1 - \delta)\Delta \) or at least \((1 + \delta)\Delta \). Similarly as above we have

\[
\begin{align*}
\mathbb{E}(\rho(b,a)) &= \binom{n_1 - 1}{i} \binom{n_2 - 1}{i} (i!)^2 \left( \frac{p}{2} \right)^{2i+1}, \\
\mathbb{E}(\rho^2(b,a)) &= \binom{n_1 - 1}{i} \binom{n_2 - 1}{i} \binom{n_1 - 1 - i}{i} \binom{n_2 - 1 - i}{i} (i!)^4 \left( \frac{p}{2} \right)^{4i+2} + O(n_1^{2i} n_2^{2i-1} p^{4i+1} + n_1^{2i-1} n_2^{2i} p^{4i+1}).
\end{align*}
\]
Using Chebyshev’s inequality, since $|\Delta - \mathbb{E}(\rho(b, a))| = o(\mathbb{E}(\rho(b, a)))$, for sufficiently large $n$,

$$
\mathbb{P}(|\rho(b, a) - \Delta| \geq \delta \Delta) \leq \mathbb{P}(\frac{|\rho(b, a) - \mathbb{E}(\rho(b, a))|}{\mathbb{E}(\rho(b, a))} \geq \frac{\varepsilon_1}{2} \mathbb{E}(\rho(b, a))) \\
\leq \frac{\mathbb{E}(\rho^2(b, a))}{(\varepsilon_1^2/2) \mathbb{E}^2(\rho(b, a))} = O\left(\frac{n_1 + n_2}{n_1 n_2 p}\right) = o(1).
$$

Also for $U$ we have

$$
\mathbb{E}(U) = p n_1 n_2 \mathbb{P}(|\rho(b, a) - \Delta| \geq \delta \Delta) \leq O(n_1 + n_2).
$$

Hence by Markov’s inequality,

$$
\mathbb{P}\left(U \geq \varepsilon_1 \frac{p}{2} n_1 n_2\right) \leq \frac{\mathbb{E}(U)}{\varepsilon_1 \frac{p}{2} n_1 n_2} = O\left(\frac{n_1 + n_2}{n_1 n_2 p}\right) = o(1).
$$

This means, together with (4) and (6), a.a.s. at least $|Y| \geq (1 + \delta)|\Delta|$. Each such closed trail of length 2 contains at least one directed edge that together belong to at least $\delta \Delta$ hyperedges in $\mathcal{H}$. This means that they are together in many closed trails of length 2, which contain both $e$ and $f$, there exist two vertices $u$ and $v$, and at least $8i + 2$ directed paths from $u$ to $v$ of length $l$, where $2 \leq l \leq 2i$.

Let $B$ be the number of vertex pairs $(u, v) \in V(D)^2$ such that there exist $8i + 2$ internally disjoint directed paths from $u$ to $v$ of length $l$. Note that $p < n^{-\frac{2i+1}{2i+2}} < n^{-\frac{i-1}{4i+1}}$ since $\varepsilon_1 < 1/2$. We have

$$
\mathbb{E}(B) = O\left(n_1^{(8i+2)\frac{l+1}{2}} n_2^{(8i+2)\frac{l+1}{2}} n_2^{8i+2} \frac{(8i+2)l}{p^{(8i+2)l}}\right) \leq o(n^{4l-8i}) = o(1), \text{ when } l \equiv 1 \pmod{2};
$$

$$
\mathbb{E}(B) = O\left(n_1^{(8i+2)\frac{l+1}{2}} n_2^{(8i+2)\frac{l+2}{2}} n_2^{8i+2} \frac{(8i+2)l}{p^{(8i+2)l}}\right) \leq o(n_1^{2l} n_2^{2l-8i}) = o(1), \text{ when } l \equiv 0 \pmod{2}.
$$

By Markov’s inequality, $\mathbb{P}(B \geq 1) = o(1)$, that implies that no more than $K$ closed trails of $D$ contain both $e$ and $f$, for every $e, f \in A(D)$, a.a.s. Therefore in our hypergraph $\mathcal{H}$, condition (2) holds for $\mathcal{H}$ when $n$ is large enough.

Finally, let us consider condition (3) of Theorem 2.3. Let $F$ be the number of closed trails of length $2i + 2$ in $D$ which contain at least one directed edge $\bar{u} \bar{v} \in P^\delta$, where $P^\delta$ is the set of pairs of vertices $(u, v) \in X \oplus Y$ such that the number of directed trails from $v$ to $u$ of length $2i + 1$ is at least $(1 + \delta)|\Delta|$. Each trail $R = x_1 y_1 x_2 y_2 \cdots y_{i+1} x_1$ contributing to $F$ is determined by two sequences of vertices $x_1, x_2, \ldots, x_{i+1} \in X$ and $y_1, y_2, \ldots, y_{i+1} \in Y$. Each such closed trail
$R$ has the same probability that it forms a trail contributing to $F$. There are $2i + 2$ candidates for an edge of $R$ being in $P^\delta$. This implies that

\begin{equation}
\mathbb{E}(F) \leq n_1^{i+1}n_2^{i+1}(2i + 2)\mathbb{P}(R \subseteq A(D)) \mathbb{P}(\overrightarrow{x_1y_1} \in P^\delta \mid R \subseteq A(D)).
\end{equation}

For $j = 1, \ldots, 2i - 1$, let $\alpha_j$ be the number of trails of length $2i + 1$ from $y_1$ to $x_1$ that contain precisely $j$ edges in $R$. Then $\alpha = \sum_{j=1}^{2i-1} \alpha_j$ is the number of trails of length $2i + 1$ from $y_1$ to $x_1$ different from $R$ which contain at least one edge in $R$. Since $n_2 = \Theta(n_1)$, we have

\begin{equation}
\mathbb{E}(\alpha \mid R \subseteq A(D)) = \sum_{j=1}^{2i-1} \mathbb{E}(\alpha_j \mid R \subseteq A(D))
\end{equation}

\begin{equation}
\leq \sum_{j=1}^{2i-1} \binom{2i + 1}{j} j!n_1^{2i-j} \left(\frac{p}{2}\right)^{2i+1-j}
\leq O(n^{2i-1}p^{2i+1}).
\end{equation}

Now, by Markov’s inequality,

\begin{equation}
\mathbb{P}(\alpha \geq 1 \mid R \subseteq A(D)) \leq O(n^{2i-1}p^{2i+1}) \ll O(n^{-\frac{2i}{4i}}) = o(1).
\end{equation}

In the next argument we will use the following events: $Q^\delta$ is the event that the number of trails of length $2i + 1$ from $y_1$ to $x_1$ that are different from $R$ is at least $(1 + \delta)\Delta - 1$; $R_E$ is the event that all edges in $R$ appear in $D$, possibly with different orientations. There are $2^{2i+2}$ different orientations $\omega_1, \ldots, \omega_{2^{2i+2}}$ of these edges. We denote by $R_E^j$ the event that these edges are present and have orientation $\omega_j$. Clearly, different events $R_E^j$ are mutually exclusive and $R_E$ is the union of all these events. Note that the following holds:

\begin{equation}
\mathbb{P}(\overrightarrow{x_1y_1} \in P^\delta, \alpha = 0 \mid R \subseteq A(D)) = \mathbb{P}(\overrightarrow{x_1y_1} \in Q^\delta, \alpha = 0 \mid R \subseteq A(D))
\end{equation}

\begin{align*}
&\leq \sum_{j=1}^{2^{2i+2}} \mathbb{P}(\overrightarrow{x_1y_1} \in Q^\delta, \alpha = 0 \mid R_E^j) \\
&= 2^{2i+1} \mathbb{P}(\overrightarrow{x_1y_1} \in Q^\delta, \alpha = 0 \mid R_E) \\
&\leq 2^{2i+1} \mathbb{P}(\overrightarrow{x_1y_1} \in Q^\delta, \alpha = 0) \\
&\leq 2^{2i+1} \mathbb{P}(\overrightarrow{x_1y_1} \in Q^\delta).
\end{align*}

We used the fact that $\alpha = 0$ is less likely to happen under the condition that $R_E$ holds and that $\overrightarrow{x_1y_1} \in Q^\delta$ is independent of $R_E$ when $\alpha = 0$. 

Combining the above inequalities with (6), we get

\[ P(\overrightarrow{xy} \in P | R \subseteq A(D)) = P(\overrightarrow{xy} \in P, \alpha \geq 1 | R \subseteq A(D)) + P(\overrightarrow{xy} \in P, \alpha = 0 | R \subseteq A(D)) \leq o(1) + 2^{2i+1} P(\overrightarrow{xy} \in Q) = o(1). \] (13)

Now, together with (10), \( E(F) \leq o(n^{i+1} n^{2i+2}), \) and by Markov’s inequality,

\[ P(F \geq \delta N \Delta) \leq \frac{2^{2i+1} o(n^{2i+2} p^{2i+2})}{\delta(1 - \varepsilon_1)n^{2i+2} p^{2i+2}} = o(1). \] (14)

This means condition (3) holds for \( H \) a.a.s.

We are now ready to apply Theorem 2.3. The theorem tells us that for sufficiently large \( n \), there exists a matching \( M \) of \( H \) of size at least \( (1 - \varepsilon_1) N^{2i+2} \). Therefore \( M^{-1} = \{ H^{-1} \mid H \in M \} \) is a matching on \( H^{-1} \) defined on \( D^{-1} \).

Again, by Theorem 2.3, we have another matching \( M' \) in \( H^{-1} \) of size at least \( (1 - \varepsilon_1) N^{2i+2} \) such that \( M' \cap M^{-1} = \emptyset \). This implies that \( M \cup M' \) does not have non-simple blossoms of length 2.

Next we will argue that there is only a small number of simple blossoms. Consider the digraph \( D \cup D^{-1} \). Let \( 2 \leq j \leq \frac{1}{\varepsilon_1} \) be an integer, and let \( T(j) \) be the number of simple \((2i + 2)\)-blossoms of length \( j \) in \( D \cup D^{-1} \). We have

\[ E(T(j)) \leq n_1(n_1^{ij} n_2^{ij} p^{ij}) + n_2(n_1^{ij} n_2^{ij} p^{ij}) \]
\[ \leq 2n_1(n_1^{ij} n_2^{ij} p^{ij}) \leq 2\sqrt{c n^{1+2ij} p^{ij}} \]
\[ < 2\sqrt{c n^{2} p n_1^{(2i-\varepsilon_1)} p_1^{(2i+1-\varepsilon_1)}} \]
\[ < 2\sqrt{c n^{2} p n_1^{(2i-\varepsilon_1)} n^{-\varepsilon_1(2i-\varepsilon_1)}} = O(n^{2} p). \] (15)

Hence by Markov’s inequality,

\[ P\left( \sum_{j=2}^{1/\varepsilon_1} T(j) \geq \varepsilon_1 p n^2 \right) \leq P(T(j) \geq \varepsilon_1 p n^2) \leq o(1). \] (16)

Therefore, a.a.s. the number of simple \((2i + 2)\)-blossoms of length at most \( 1/\varepsilon_1 \) in \( D \cup D^{-1} \) is at most \( \varepsilon_1 p n_1 n_2 = \varepsilon_1 p n^2 \). Since \( M \cup M' \) has size at least \( 2(1 - \varepsilon_1) N^{2i+2} \), it has a subset \( M_1 \) without simple \((2i + 2)\)-blossom of length at
most $1/\varepsilon_1$ after removing at most $\varepsilon_1pn^2$ closed trails. By using (3) we have:

$$|M_1| \geq 2(1 - \varepsilon_1)\frac{N}{2i + 2} - \varepsilon_1pn^2$$

(17) $$\geq (1 - \varepsilon_1)\frac{N}{i + 1} - \frac{\varepsilon_1N}{1 - \varepsilon_1}$$

$$\geq (1 - \frac{i + 2}{i + 1}\varepsilon_1)\frac{N}{i + 1}, \text{ a.a.s.}$$

Now we consider the $(2i + 2)$-blossoms of length at least $1/\varepsilon_1$ in $M_1$. If $C_1$ and $C_2$ are two blossoms of $M_1$ with center $v$, by the way we constructed $M_1$ we could see that the tips of $C_1$ and $C_2$ cannot intersect. Therefore, if $v$ has $m$ neighbours in $D_i$ at most $\varepsilon_1m$ different $(2i + 2)$-blossoms of length at least $1/\varepsilon_1$ have center $v$. Thus, the total number of such blossoms is at most

$$\sum_{v \in V(D_i)} \deg_G(v)/(1/\varepsilon_1) = 2\varepsilon_1N.\text{ By removing one of the trails from each such blossom we get a blossom-free subset } M_0 \subseteq M_1 \text{ which satisfies}$$

$$|M_0| \geq |M_1| - 2\varepsilon_1N$$

(18) $$\geq (1 - \frac{3i + 4}{1 - \varepsilon_1})\frac{N}{i + 1} = (1 - \varepsilon_0)\frac{N}{i + 1}.\text{ a.a.s.}$$

Finally, using $M_0$ we can obtain an $\varepsilon_0$-near $(2i + 2)$-gon embedding of $G$ by using Lemma 2.2. This completes the proof when $p \ll n^{-\frac{2i + 1}{2i + \varepsilon_1}}$.

For the case $p \geq \Theta(n^{-\frac{2i + 1}{2i + \varepsilon_1}})$, we use a similar argument as used in [13, Lemma 4.8]. Choose an integer $t = t(n)$, such that $n^{-\frac{2i}{2i + 1}} \ll p/t \ll n^{-\frac{2i - \varepsilon_1}{2i + 1}}$. Let $p_1 = p/t$. Now take a corresponding digraph $D$ of $G_{n_1, n_2, p}$ and partition its edges into $t$ parts, putting each edge in one of the parts uniformly at random. Then each of the resulting digraphs $D_1, D_2, \ldots, D_t$ is a corresponding digraph of $G_{n_1, n_2, p_1}$. By the above, for every $1 \leq j \leq t$, $D_j \cup D_j^{-1}$ has a collection of blossom-free directed $(2i + 2)$-trails of size at least $(1 - \varepsilon_0)\frac{|A(D_j)|}{i + 1}$ a.a.s. That means, if we let $q$ be the probability that $D_j \cup D_j^{-1}$ does not have such set of trails, then $q \to 0$ as $n \to \infty$.

Let $I \subseteq \{1, 2, \ldots, t\}$ be the index set, containing all $j$, $1 \leq j \leq t$, for which $D_j \cup D_j^{-1}$ does not have a collection of directed blossom-free $(2i + 2)$-trails of size at least $(1 - \varepsilon_0)\frac{|A(D_j)|}{i + 1}$. Then by Markov’s inequality, $\mathbb{P}(|I| \geq \sqrt{qt}) \leq \sqrt{q}$. Hence for sufficiently large $n$, $|I| \leq \varepsilon_0 t$ a.a.s.

Similarly as in the proof of [4], we see that for each $0 \leq j \leq t$ a.a.s.

$$\frac{1}{2}n^2p_1 \leq |A(D_j)| \leq (1 + \varepsilon_0)\frac{1}{2}n^2p_1.$$
Now let \( \Gamma \) be the union of collections of directed blossom-free \((2i + 2)\)-trails of size at least \((1 - \varepsilon_0)\frac{|A(D_j)|}{i + 1}\) for \( j \notin I \). We have:

\[
|\Gamma| \geq (1 - \varepsilon_0) \sum_{j \notin I} \frac{|A(D_j)|}{i + 1} \geq (1 - \varepsilon_0)^2 t (1 - \varepsilon_0) \frac{p_i n^2}{2i + 2} \\
\geq (1 - \varepsilon_0)^3 \frac{pm^2}{2i + 2} \\
\geq (1 - \varepsilon)(1 + \varepsilon_0) \frac{pm^2}{2} \geq (1 - \varepsilon) \frac{|A(D)|}{i + 1}.
\]

(20)

Since the directed closed trails of \( \Gamma \) that belong to any \( D_j \) \((j \notin I)\) are blossom-free and any \( D_k \) and \( D_j \) are edge disjoint for \( k \neq j \), \( \Gamma \) is blossom-free. By Lemma 2.2 we get a rotation system \( \Pi \) in which every closed trail in \( \Gamma \) is a face of \( \Pi \). Let \( f_{2i+2} \) be the number of faces of length \( 2i + 2 \) of \( \Pi \). We have \((2i + 2)f_{2i+2} \geq 2(1 - \varepsilon)|E(G)|\), thus \( \Pi \) is an \( \varepsilon \)-near \((2i + 2)\)-gon embedding. \( \square \)

The result of Lemma 3.1 has been proved under the assumption that \( n_2 = \Theta(n_1) \). However, that assumption can be omitted as long as \( n_2 \gg 1 \).

**Lemma 3.2.** Let \( \varepsilon > 0 \) and \( G \in \mathcal{G}_{n_1, n_2, p} \) be a random bipartite graph on vertex set \( X \cup Y \) with \(|X| = n_1 \geq n_2 = |Y|\). If \( p \gg n_2^{-\frac{2i}{2i+1}} \) where \( i \) is a fixed positive integer and \( n_2 \gg 1 \), then a.a.s. \((as n_2 \to \infty)\) \( G \) has an \( \varepsilon \)-near \((2i + 2)\)-gon embedding.

**Proof.** It is sufficient to consider the case \( n_1/n_2 \gg 1 \). Let \( t = \lfloor \frac{n_1}{n_2} \rfloor \), and let \( \mathcal{P} = \{X_j\}_{j \in J} \) be the equitable partition of \( X \) into \( t \) parts, where \( J = [t] \). Note that \(|X_j| = N_j\) is between \( n_2 \) and \( 2n_2 \), for every \( j \in J \). Let \( G_j \) be the bipartite graph \( G[X_j \cup Y] \) and let \( D_j \) be its corresponding digraph. Choose \( \varepsilon_0 > 0 \) such that \( \varepsilon \gg \frac{4\varepsilon_0}{1 + \varepsilon_0} \). By Lemma 3.1 there exists a set \( M_j \) of closed trails of length \( 2i + 2 \) in \( D_j \cup D_j^{-1} \), such that \(|M_j| \geq (1 - \varepsilon_0)\frac{|A(D_j)|}{i + 1}\) and \( M_j \) is blossom-free, for each \( j \in J \) a.a.s. That means, if we let \( q_j \) be the probability that \( D_j \cup D_j^{-1} \) does not have such set of closed trails, we have \( q_j \to 0 \) when \( n_2 \to \infty \). The probabilities \( q_j \) are almost the same since \(|X_j|\) only take at most two different values. We let \( q = \max\{q_j \mid j \in J\} \). Define the index set \( I \subseteq J \) containing those \( j \in J \), for which \( D_j \cup D_j^{-1} \) does not have a set of closed trails satisfying the conditions stated above. By Markov’s inequality, we have \( \mathbb{P}(|I| \geq \sqrt{qt}) \leq \sqrt{q} \). Then, when \( n \) is large enough, \(|I| \leq \varepsilon_0 t \). Similarly as in the proof of (4) we have a.a.s.

\[
(1 - \varepsilon_0)N_jn_2p \leq |A(D_j)| \leq (1 + \varepsilon_0)N_jn_2p, \quad \forall j \in J, \\
(1 - \varepsilon_0)n_1n_2p \leq |E(G)| \leq (1 + \varepsilon_0)n_1n_2p.
\]

(21)
Let $M = \bigcup_{j \in J \setminus I} M_j$. Since each $M_j \ (j \in J \setminus I)$ is blossom-free and the edge-sets of different $D_j$ are disjoint, $M$ is also blossom-free. We also have:

$$|M| = \sum_{j \in J \setminus I} |M_j| \geq t(1 - \varepsilon_0)(1 - \varepsilon_0)\frac{|A(D_j)|}{i + 1}$$

(22)

$$\geq t(1 - \varepsilon_0)^3\frac{\mathcal{N}_j n_2p}{i + 1} \geq (1 - \varepsilon_0)^3\frac{n_1 n_2 p}{i + 1}$$

$$\geq (1 + \varepsilon_0)(1 - \varepsilon)\frac{n_1 n_2 p}{i + 1} \geq (1 - \varepsilon)\frac{|E(G)|}{i + 1}.$$ 

Therefore, by Lemma 2.2 we get the desired $\varepsilon$-near $(2i + 2)$-gon embedding $\Pi$ a.a.s. $\square$

We are ready to complete the proof of our first main result.

**Theorem 3.3.** Let $\varepsilon > 0$ and $G \in \mathcal{G}_{n_1, n_2, p}$ be a random bipartite graph and suppose that $i \geq 2$ is an integer. If $p$ satisfies $(n_1 n_2)^{-\frac{i - 1}{2i + 1}} \ll p \ll (n_1 n_2)^{-\frac{i - 1}{2i - 1}}$, $n_1/n_2 < c$ and $n_2/n_1 < c$ where $c$ is a positive real number, then we have a.a.s.

$$(1 - \varepsilon)\frac{i}{2i + 2}p n_1 n_2 \leq g(G) \leq (1 + \varepsilon)\frac{i}{2i + 2}p n_1 n_2$$

and

$$(1 - \varepsilon)\frac{i}{i + 1}p n_1 n_2 \leq \bar{g}(G) \leq (1 + \varepsilon)\frac{i}{i + 1}p n_1 n_2.$$ 

**Proof.** To prove the lower bound, we count the number of closed trails of $G$ of length at most $2i$. Let $C$ be the number of such closed trails. We have

$$\mathbb{E}(C) \leq \sum_{j=2}^{i} n_1^j n_2^{j+1} p^{2j} = o(n_1 n_2 p).$$

Then by Markov’s inequality, a.a.s. at most $\frac{1}{4(i-1)}\varepsilon p n_1 n_2$ closed trails of $G$ have length at most $2i$. Similarly as in the proof of (4) we get $|E(G)| \geq (1 - \frac{1}{2i}\varepsilon)p n_1 n_2$, a.a.s. Let $\Pi$ be a rotation system of $G$, and let $f(\Pi)$ be the number of faces, and $f'$ be the number of faces of $\Pi$ with length at most $2i$. Then $2|E(G)| \geq (2i + 2)(f(\Pi) - f') + 4f' \geq (2i + 2)f(\Pi) - (2i - 2)f'$. By the
above, $f' \leq 2C \leq \frac{1}{2(2i-2)}\varepsilon pn_1 n_2$. Now we have a.a.s.

$$g(G, \Pi) = \frac{1}{2}(|E(G)| - f(\Pi) - |V(G)|) + 1 \sim \frac{1}{2}(|E(G)| - f(\Pi))$$

$$\geq \frac{i}{2i + 2}|E(G)| - \frac{i - 1}{2i + 2}f'$$

(24)

$$\geq \left(1 - \frac{1}{2i}\varepsilon\right) \frac{i}{2i + 2}pn_1 n_2 - \frac{i - 1}{2i + 2} \frac{1}{2i - 1} \varepsilon pn_1 n_2$$

$$= (1 - \varepsilon) \frac{i}{2i + 2}pn_1 n_2.$$

For the upper bound, by Lemma 3.1 we have an $\varepsilon'$-near $(2i + 2)$-gon embedding $\Pi$, with $\varepsilon' = \frac{i \varepsilon}{2i + 2}$, and let $f(\Pi)$ be the number of faces. Also, we have $|E(G)| \leq (1 + \frac{1}{2}\varepsilon)pn_1 n_2$. Therefore,

$$g(G, \Pi) = \frac{1}{2}(|E(G)| - f(\Pi) - |V(G)|) + 1 \sim \frac{1}{2}(|E(G)| - f(\Pi))$$

(25)

$$\leq \frac{1}{2}(|E(G)| - \frac{2(1 - \varepsilon')}{2i + 2}|E(G)|)$$

$$\leq \left(1 + \frac{1}{2i}\varepsilon\right) \frac{i + \varepsilon'}{2i + 2}pn_1 n_2 = (1 + \varepsilon) \frac{i}{2i + 2}pn_1 n_2.$$

This completes the proof for the orientable genus. The proof for $\tilde{g}(G)$ is essentially the same, where the lower bound uses Euler's Formula as in (24), while for the upper bound we just observe that $\tilde{g}(G) \leq 2g(G) + 1$, see [8]. □

**Theorem 3.4.** Let $\varepsilon > 0$ and $G \in G_{n_1, n_2, p}$ be a random bipartite graph. If $n_1 \geq n_2 \gg 1$ and $p \gg n_2^{-\frac{1}{3}}$, then we have a.a.s.

$$(1 - \varepsilon) \frac{pn_1 n_2}{4} \leq g(G) \leq (1 + \varepsilon) \frac{pn_1 n_2}{4}$$

and

$$(1 - \varepsilon) \frac{pn_1 n_2}{2} \leq \tilde{g}(G) \leq (1 + \varepsilon) \frac{pn_1 n_2}{2}.$$
the set of neighbours of \( x \). Suppose that \( c \) is some constant. Then we say that an embedding \( \Pi \) of \( G \) is a near \( k \)-gon embedding (with respect to \( c \)) if \( 2|E(G)| - kf_k(\Pi) \leq c \).

**Lemma 4.1.** Let \( S \) be the standard graph of \( \mathcal{G}_{n_1,n_2,p} \) where \( n_1 \gg 1 \) and \( n_2 \) is a constant. Suppose that \( p \gg n_1^{-\frac{1}{3}} \) and let \( S' \) be the bipartite graph obtained by removing all vertices of degree at most one in \( S \). Then \( S' \) has a near 4-gon embedding with respect to the constant \( c = (4n_2 + 14)2^{n_2} \).

**Proof.** Let \( V(S) = X(S) \sqcup Y(S) \). Note that \( n_1 - 2^{n_2} \leq |X(S)| \leq n_1 \) and \( |Y(S)| = n_2 \). For every \( Y' \subseteq Y(S) \), let \( F_S(Y') = \{ x \in X(S) \mid N(x) = Y' \} \). Now consider all of the \( 2^{n_2} \) subsets of \( Y(S) \), they give us a partition of \( X(S) = \bigcup_{Y' \subseteq Y(S)} F_S(Y') \). Note that \( S[Y' \sqcup F_S(Y')] \) is a complete bipartite graph for every \( Y' \subseteq Y(S) \). If \( |Y'| \geq 2 \), by Lemma 11, we have a near 4-gon embedding of \( S[Y' \sqcup F_S(Y')] \). Moreover, there is always a near 4-gon embedding with respect to the constant 14 since in the worst case, we may have one 6-gon and one 8-gon apart from the 4-gons. Let \( C(Y') \) be the set of all facial walks of length 4 in the optimal embedding of \( S[Y' \sqcup F_S(Y')] \). We can remove from \( C(Y') \) a collection of at most \( |Y'| \) closed trails to make \( C(Y') \) free of blossoms with center in \( Y' \). Therefore, we can remove at most \( 2^{n_2}n_2 \) closed trails of length 4 to make \( \bigcup_{Y' \in Y(S), |Y'| \geq 2} C(Y') \) free of blossoms centered in \( Y' \). An obvious extension of Lemma 2.2 shows that the union of these sets for all \( Y' \) with \( |Y'| \geq 2 \) gives rise to a near 4-gon embedding of \( S' \) with respect to the constant \( c = (4n_2 + 14)2^{n_2} \). \( \square \)

**Lemma 4.2.** Let \( S \) be the standard graph of \( \mathcal{G}_{n_1,n_2,p} \) where \( n_1 \gg 1 \) and \( n_2 \) is a constant. Suppose that \( p \gg n_1^{-\frac{1}{3}} \), then

\[
g(S) \sim \frac{n_1n_2p}{4} \sum_{i=2}^{n_2-1} \frac{i-1}{i+1} \binom{n_2-1}{i} (-p)^i.
\]

In particular, when \( n_1^{-\frac{1}{3}} \ll p \ll 1 \), \( g(S) = (1 + o(1)) \frac{n_1p^3}{4} \binom{n_2}{3} \).
Lemma 4.3. Let \( p > 0 \) and \( G \in \mathcal{G}_{n_1,n_2,p} \) where \( n_1 \gg 1 \) and \( n_2 \) is a constant. If \( p \gg n_1^{-\frac{1}{3}} \) and \( S \) is the standard graph of \( \mathcal{G}_{n_1,n_2,p} \), we have a.a.s. (as \( n_1 \to \infty \))

\[
(1 - \varepsilon)g(S) \leq g(G) \leq (1 + \varepsilon)g(S).
\]

Proof. Let \( \Pi \) be the rotation system of \( S' \) given by Lemma 4.1. Since this gives a near 4-gon embedding, we have

\[
g(S') \sim \frac{1}{2} \left( 2 + |E(S')| - f(\Pi) - |V(S')| \right)
\]

\[
\sim \frac{1}{2} \left( \frac{|E(S')|}{2} - n_1 + (1 - p)^{n_2}n_1 + (1 - p)^{n_2-1}n_1n_2p \right)
\]

\[
\sim \frac{1}{2} \left( \frac{|E(S')|}{2} - n_1 + (1 - p)^{n_2}n_1 + (1 - p)^{n_2-1}n_1n_2p \right)
\]

\[
= \frac{1}{2} \left( \frac{1}{2} n_1n_2p - (1 - p)^{n_2-1}n_1n_2p \right) - n_1 + (1 - p)^{n_2}n_1 + (1 - p)^{n_2-1}n_1n_2p
\]

\[
= \frac{n_1n_2p}{4} \sum_{i=2}^{n_2-1} \binom{n_2-1}{i} \frac{(-p)^i}{i} + n_1 \sum_{i=2}^{n_2} \binom{n_2}{i} (-p)^i
\]

Since \( g(S) = g(S') \), this completes the proof. \( \square \)

Lemma 4.3. Let \( \varepsilon > 0 \) and \( G \in \mathcal{G}_{n_1,n_2,p} \) where \( n_1 \gg 1 \) and \( n_2 \) is a constant. If \( p \gg n_1^{-\frac{1}{3}} \) and \( S \) is the standard graph of \( \mathcal{G}_{n_1,n_2,p} \), we have a.a.s. (as \( n_1 \to \infty \))

\[
(1 - \varepsilon)g(S) \leq g(G) \leq (1 + \varepsilon)g(S).
\]

Proof. Let \( V(G) = X(G) \cup Y(G) \) with \( |X(G)| = n_1 \) and \( |Y(G)| = n_2 \). For every \( Y' \subseteq Y(G) \), where \( |Y'| = m \geq 1 \), let \( F_G(Y') = \{ x \in X(G) \mid N(x) = Y' \} \).

Then

\[
\mathbb{E}(|F_G(Y')|) = p^m(1 - p)^{n_2-m}n_1,
\]

\[
\mathbb{E}(|F_G(Y')|^2) = p^{2m}(1 - p)^{2n_2-2m}n_1(n_1-1) + p^m(1 - p)^{n_2-m}n_1.
\]

For every \( t > 0 \), by Chebyshev’s inequality, we have

\[
P\left( |F_G(Y')| - \mathbb{E}(|F_G(Y')|) \right) \geq t \mathbb{E}(|F_G(Y')|) \leq \frac{\mathbb{E}(|F_G(Y')|^2) - \mathbb{E}^2(|F_G(Y')|)}{t^2 \mathbb{E}(|F_G(Y')|) ^2}
\]

\[
\sim \frac{p^m(1 - p)^{n_2-m}n_1}{t^2 p^{2m}(1 - p)^{2n_2-2m}n_1^2} = \frac{1}{t^2 p^m(1 - p)^{n_2-m}n_1}.
\]
Suppose now that \( p \gg n_1^{-\frac{1}{2}} \) and \( m \geq 3 \). By taking \( t = \frac{\varepsilon}{10n_2^{2n_2}}p^{3-m}(1 - p)^{m-n_2/2} \) in (28) we obtain that
\[
\mathbb{P}\left( \left| |F_G(Y')| - \mathbb{E}(|F_G(Y')|) \right| \geq \frac{\varepsilon}{10n_2^{2n_2}}p^{3}(1 - p)^{n_2/2}n_1 \right) \leq \frac{100n_2^{4n_2}}{\varepsilon^2 p^{6-m}n_1} \leq o(1).
\]

Let \( S \) be the standard graph of \( \mathcal{G}_{n_1,n_2,p} \) with \( V(S) = X(S) \sqcup Y(S) \). We may assume that \( Y(S) = Y(G) = [n_2] \). Let \( G' \) be the subgraph obtained from \( G \) by deleting all vertices of degree at most 2 in \( p \). Remove less than 2
\[
\sum_{|Y'| \geq 3} \left( |F_G(Y')| - |F_S(Y')| \right) \leq \sum_{|Y'| \geq 3} \left( |F_G(Y')| - \mathbb{E}(|F_G(Y')|) \right) + 1
\]
\[
\leq 2^{n_2}(1 + \frac{2^{n_2}\varepsilon}{10n_2}p^{3}(1 - p)^{n_2/2}n_1) \leq 2^{n_2} + \frac{\varepsilon}{10n_2}p^{3}n_1.
\]

If \( p \ll 1 \), then (31) implies, in particular, that \( S \) can be obtained from \( G' \) by adding and deleting at most \( n_22^{n_2} + \frac{1}{10}p^{3}n_1 \) edges a.a.s. Since adding an edge changes the genus by at most 1, and by Lemma 4.2 \( g(S) \geq \frac{1}{2}p^{3}n_1 \gg 1 \) (if \( n_1 \) is large), we obtain that \( (1 - \frac{1}{2}\varepsilon)g(S) \leq g(G') \leq (1 + \frac{1}{2}\varepsilon)g(S) \) a.a.s. Together with (30) this implies the lemma.

Finally, suppose that \( p \gg n_1^{-1/n_2} \). In this case we take \( t = \frac{\varepsilon p\Psi(p,n_2)}{15n_2^{2n_2}} \) in (28), where \( \Psi(p,n_2) \) is defined in Theorem 1.5. Therefrom we conclude that with high probability
\[
\left| |F_G(Y')| - |F_S(Y')| \right| \leq 2 + \frac{\varepsilon p\Psi(p,n_2)}{15n_2^{2n_2}}|F_S(Y')|.
\]

Now we derive similarly as above that \( S \) can be obtained from \( G' \) by adding and removing less than \( 2n_22^{n_2} + \frac{\varepsilon}{10}p\Psi(p,n_2)n_1 \) edges a.a.s., which is less than \( \frac{\varepsilon}{10}p\Psi(p,n_2)n_1 \) when \( n_1 \) is sufficiently large. The same conclusion as above follows.

We have all tools to prove the last main statement.

**Theorem 4.4.** Let \( \varepsilon > 0 \) and \( G \in \mathcal{G}_{n_1,n_2,p} \) where \( n_1 \gg 1 \) and \( n_2 \) is a constant.
(a) If \( p \gg n_1^{-1/2} \) we have a.a.s. (as \( n_1 \to \infty \))
\[
(1 - \varepsilon) \frac{n_1 n_2 p}{4} \Psi(p, n_2) \leq g(G) \leq (1 + \varepsilon) \frac{n_1 n_2 p}{4} \Psi(p, n_2)
\]
and
\[
(1 - \varepsilon) \frac{n_1 n_2 p}{2} \Psi(p, n_2) \leq \tilde{g}(G) \leq (1 + \varepsilon) \frac{n_1 n_2 p}{2} \Psi(p, n_2),
\]
where \( \Psi(p, n_2) = \sum_{i=2}^{n_2-1} \frac{(-p)^i}{i!} \binom{n_2-1}{i} \).

(b) If \( n_1^{-1/2} \ll p \ll n_1^{-1/2} \), then a.a.s.
\[
g(G) = \left\lceil \frac{(n_2 - 3)(n_2 - 4)}{12} \right\rceil \quad \text{and} \quad \tilde{g}(G) = \left\lceil \frac{(n_2 - 3)(n_2 - 4)}{6} \right\rceil
\]
with a single exception that \( \tilde{g}(G) = 3 \) when \( n_2 = 7 \).

(c) If \( p \ll n_1^{-1/2} \), then a.a.s. \( g(G) = 0 \).

Proof. To prove part (a), we just combine Lemmas 4.1, 4.2 and 4.3.

For case (b), when \( Y' \subseteq Y(G) \) with \( |Y'| = m \geq 3 \), we have
\[
\mathbb{E}(|F_G(Y')|) = p^m (1 - p)^{n_2 - m} n_1 = o(1).
\]
Then by Markov’s inequality, \( \mathbb{P}(|F_G(Y')| \geq 1) = o(1) \). For the sets \( Y_2 \subseteq Y(G) \) with \( |Y_2| = 2 \), by (28) we can see that for every \( t > 0 \), \( (1 - t)p^2 n_1 \leq |F_G(Y_2)| \leq (1 + t)p^2 n_1 \) a.a.s. That means if we remove all vertices with degree 1 in \( G \), we will obtain the complete graph \( K_{n_2} \), in which each edge is replaced by roughly \( p^2 n_1 \) internally disjoint paths of length 2. By (12) we have \( g(G) = g(K_{n_2}) = \left\lceil \frac{(n_2 - 3)(n_2 - 4)}{12} \right\rceil \) a.a.s. (and similarly for \( \tilde{g}(G) \), where the exception occurs when \( n_2 = 7 \)). This proves part (b).

To prove (c), note that when \( p \ll n_1^{-1/2} \), none of the subdivided edges of \( K_{n_2} \) from case (b) will occur (a.a.s.), and with high probability, every vertex in \( X(G) \) will be of degree at most 1. Thus, \( g(G) = 0 \) a.a.s. \( \square \)

Note that in Theorem 4.4, when \( p = \Theta(n_1^{-1/2}) \), a.a.s. the graph \( G \) will be the Levi graph of \( mK_{n_1}^3 \), where \( mK_{n_1}^3 \) is the complete 3-uniform multi-hypergraph of order \( n_1 \), and each triple has \( m \) edges. This problem is hard and of independent interest as a generalization of Ringel-Youngs Theorem. We will discuss it in a separate paper [6].

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