Segal’s multisimplicial spaces

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Abstract

Some sufficient conditions on a simplicial space $X : \Delta^{op} \to \text{Top}$ guaranteeing that $X_1 \simeq \Omega |X|$ were given by Segal. We give a generalization of this result for multisimplicial spaces. This generalization is appropriate for the reduced bar construction, providing an $n$-fold delooping of the classifying space of a category.

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1 Introduction

This note makes no great claim to originality. It provides a complete inductive argument for a generalization of [17, Proposition 1.5], which was spelled out, not in a precise manner, in [2, paragraph preceding Theorem 2.1]. The authors of [2] considered this generalization trivial and did not even provide a sketch of a proof. Some related, but quite different, results are given in [4] and [3].

The main result of [5] reaches its full potential role in constructing a model for an $n$-fold delooping of the classifying space of a category only with the help of such a generalization of [17, Proposition 1.5]. Although we referred to [2], the referees of [5] were not convinced that our bar construction actually provides an appropriate model for delooping. The aim of this note is to fill in a gap in the literature concerning these matters.

Segal, [17, Proposition 1.5], gave conditions on a simplicial space $X : \Delta^{op} \to \text{Top}$ guaranteeing that $X_1 \simeq \Omega |X|$. His intention was to cover a more general class of simplicial spaces than we need for our purposes, therefore he worked with nonstandard geometric realizations of simplicial spaces. We generalize his result, in one direction, by passing from simplicial spaces to multisimplicial spaces, but staying in a class appropriate for the standard geometric realization. Our result is formulated to be directly applicable to the reduced bar construction of [5], providing an $n$-fold delooping of the classifying space of a category.
We work in the category (here denoted by $\text{Top}$) of compactly generated Hausdorff spaces. (This category is denoted by $\text{KCG}$ in [7] and by $\text{CGHaus}$ in [3].) The objects of $\text{Top}$ are called spaces and the arrows are called maps. Products are given the compactly generated topology. We adopt the following notation throughout: $\simeq$ for homotopy of maps or same homotopy type of spaces, $\approx$ for homeomorphism of spaces.

The category $\Delta$ (denoted by $\Delta^+$ in [3]) is the standard topologist’s simplicial category defined as in [3 VII.5]. We identify this category with the subcategory of $\text{Top}$ whose objects are the standard ordered simplices (one for each dimension), i.e., with the standard cosimplicial space $\Delta \to \text{Top}$.

The objects of $\Delta$ are the nonempty ordinals $1_0$, $1_1$, $1_2$, etc. The coface arrows from $[n-1]$ to $[n]$ are denoted by $\delta^n_i$, for $0 \leq i \leq n$, and the codegeneracy arrows from $[n]$ to $[n-1]$ are denoted by $\sigma^n_i$, for $0 \leq i \leq n-1$.

For the opposite category $\Delta^\text{op}$, we denote the arrow $(\delta^n_i)^\text{op}: [n] \to [n-1]$ by $d_i^n$ and $(\sigma^n_i)^\text{op}: [n-1] \to [n]$ by $s_i^n$. For $f$ an arrow of $\Delta^\text{op}$ (or $(\Delta^\text{op})^n$), we abbreviate $X(f)$ by $f$ whenever the (multi)simplicial object $X$ is determined by the context.

We consider all the monoidal structures to be strict, which is supported by the strictification given by [3 XI.3, Theorem 1]. Some proofs prepared for non-specialists are given in the appendix.

## 2 Multisimplicial spaces and their realization

A multisimplicial space is an object of the category $\text{Top}^{(\Delta^\text{op})^n}$, i.e., a functor from $(\Delta^\text{op})^n$ to $\text{Top}$. When $n = 0$, this is just a space and when $n = 1$, this is a simplicial space. As usual, for a multisimplicial space $X : (\Delta^\text{op})^n \to \text{Top}$, we abbreviate $X([k_1], \ldots, [k_n])$ by $X_{k_1 \ldots k_n}$.

We say that $X : (\Delta^\text{op})^n \to \text{Top}$ is a multisimplicial set when every $X_{k_1 \ldots k_n}$ is discrete. A multisimplicial map is an arrow of $\text{Top}^{(\Delta^\text{op})^n}$, i.e., a natural transformation between multisimplicial spaces. When $n = 1$, this is a simplicial map. Throughout this paper we use the standard geometric realization of (multi)simplicial spaces.

**Definition 2.1.** The realization functor $| | : \text{Top}^{\Delta^\text{op}} \to \text{Top}$ of simplicial spaces is defined so that for a simplicial space $X$, we have

$$|X| = \left( \coprod_n X_n \times \Delta^n \right) / \sim,$$

where the equivalence relation $\sim$ is generated by

$$(d_i^n x, t) \sim (x, \delta_i^n t) \quad \text{and} \quad (s_i^n x, t) \sim (x, \sigma_i^n t).$$

**Definition 2.2.** For $p \geq 0$, the functor $\mu^{(p)} : \text{Top}^{(\Delta^\text{op})^{n+p}} \to \text{Top}^{(\Delta^\text{op})^{n}}$ of partial realization is defined inductively as follows

$\mu^{(0)}$ is the identity functor, and $\mu^{(p+1)}$ is the composition

$$\text{Top}^{(\Delta^\text{op})^{n+p+1}} \xrightarrow{\simeq} \left( \text{Top}^{\Delta^\text{op}} \right)^{(\Delta^\text{op})^{n+p}} \quad | |^{(\Delta^\text{op})^{n+p}} \quad \text{Top}^{(\Delta^\text{op})^{n+p}} \xrightarrow{\mu^{(p)}} \text{Top}^{(\Delta^\text{op})^{n}},$$

where $| |^{(\Delta^\text{op})^{n+p}}$ is the geometric realization of the $n+p$-dimensional product of $\Delta^\text{op}$, and $\mu^{(p)}$ is the partial realization functor.
where the first isomorphism maps $X$ to $Y$ such that $(Y_{k_1\ldots k_{n+p}}) = X_{k_1\ldots k_{n+p}}$.

For a multisimplicial space $X : (\Delta^{op})^p \to \text{Top}$, we denote $X^{(p)}$ by $|X|$. Hence, for $X : (\Delta^{op})^n+p \to \text{Top}$, we have that $X_{k_1\ldots k_n} = \bigl|X_{k_1\ldots k_n} \bigr|$.

**Definition 2.3.** If $X = Y^{(p)}$, for $Y$ a multisimplicial set, then we say that $X$ is a partially realized multisimplicial set (PRmss).

**Definition 2.4.** For $n \geq 0$ and $X : (\Delta^{op})^n \to \text{Top}$, we denote $X^{(p)}$ by $|X|$.

Hence, for $X : (\Delta^{op})^{n+p} \to \text{Top}$, we have that $(X^{(p)})_{k_1\ldots k_n} = |X_{k_1\ldots k_n}|$.

**Definition 2.5.** For $X : (\Delta^{op})^n \to \text{Top}$, let the simplicial space $\text{diag}X : \Delta^{op} \to \text{Top}$ be such that $(\text{diag}X)_k = X_{k\ldots k}$.

In particular, when $n = 0$ and $X$ is just a topological space, we have that $(\text{diag}X)_k = X$ and all the face and degeneracy maps of $\text{diag}X$ are $1_X$.

The following lemma is a corollary of [15, Lemma, p. 94].

**Lemma 2.5.** For $X : (\Delta^{op})^n \to \text{Top}$, we have that $|X| \approx |\text{diag}X|$.

As a consequence of Lemma 2.5 and [10, Theorem 14.1] we have the following lemma.

**Lemma 2.6.** If $X$ is a PRmss, then $|X|$ is a CW-complex.

The following remark easily follows.

**Remark 2.7.** (a) If $X : (\Delta^{op})^{n+p} \to \text{Top}$ is a PRmss, then $X^{(p)}$ is a PRmss.

(b) If $X : (\Delta^{op})^n \to \text{Top}$ is a PRmss, then for every $k_1,\ldots,k_n$, the space $X_{k_1\ldots k_n}$ is a CW-complex.

(c) If $X : (\Delta^{op})^{n+p} \to \text{Top}$ is a PRmss, then for every $k \geq 0$, $X_{k\ldots k}$ is a PRmss.

(d) If $X : (\Delta^{op})^n \to \text{Top}$, for $n > 1$, is a PRmss, then $Y : \Delta^{op} \times \Delta^{op} \to \text{Top}$, defined so that $Y_{mk} = X_{mk\ldots k}$, is a PRmss.

**Definition 2.8.** A simplicial space $X : \Delta^{op} \to \text{Top}$ is good when for every $0 \leq i \leq n-1$, the map $s^i_n : X_{n-1} \to X_n$ is a closed cofibration. It is proper (Reedy cofibrant) when for every $n \geq 1$, the inclusion $sX_n \hookrightarrow X_n$, where $sX_n = \bigcup_i s^i_n (X_{n-1})$, is a closed cofibration.

**Proposition 2.9.** Every PRmss $X : \Delta^{op} \to \text{Top}$ is good.

**Proof.** Since $d^n_i \circ s^i_n = 1_{X_{n-1}}$, we may consider $X_{n-1}$ to be a retract of $X_n$. By Remark 2.7 (b), $X_n$ is a CW-complex and by [4, Corollary III.2] (see also [3, Corollary 2.4 (a)]) a locally equiconnected space. By [3, Lemma 3.1] and since every $X_k$ is Hausdorff, $s^i_n$ is a closed cofibration.

As a corollary of [18, Proposition 22] (see also references listed in [13, Section 6, p. 19]) we have the following result.

**Lemma 2.10.** Every good simplicial space is proper.

The following result is from [12, Appendix, Theorem A4 (ii)].
Lemma 2.11. Let $f : X \to Y$ be a simplicial map of proper simplicial spaces. If each $f_k : X_k \to Y_k$ is a homotopy equivalence, then $|f| : |X| \to |Y|$ is a homotopy equivalence.

Definition 2.12. The product $X \times Y$ of simplicial spaces $X$ and $Y$ is defined componentwise, i.e. $(X \times Y)_k = X_k \times Y_k$, and for every arrow $f : k \to l$ of $\Delta^{op}$ and every $x \in X_k$ and $y \in Y_k$, we have that $f(x, y) = (fx, fy)$.

Since the product of two CW-complexes in Top is a CW-complex, by reasoning as in Proposition 2.9, we have the following.

Remark 2.13. If $X, Y : \Delta^{op} \to \text{Top}$ are PRmss, then $X \times Y$ is good.

The following lemma is a corollary of [11, Lemma 11.11].

Lemma 2.14. If the space $X_0$ of a simplicial space $X : \Delta^{op} \to \text{Top}$ is path-connected, then $|X|$ is path-connected.

3 Segal’s multisimplicial spaces

For $m \geq 1$, consider the arrows $i_1, \ldots, i_m : [m] \to [1]$ of $\Delta^{op}$ given by the following diagrams.

\[
\begin{array}{c}
i_1 : \\
0 \quad 1 \quad \cdots \quad m \\
\end{array}
\quad \begin{array}{c}
i_2 : \\
0 \quad 1 \quad 2 \quad \cdots \quad m \\
\end{array}
\quad \begin{array}{c}
i_m : \\
0 \quad 1 \quad \cdots \quad m-1 \quad m \\
\end{array}
\]

The following images of these arrows under the functor $\mathcal{J} : \Delta^{op} \to \Delta$ of [14, Section 6] may help the reader to see that $i_1, \ldots, i_m$ correspond to $m$ projections. (Note that 0 and 2 in the bottom line of the images serve to project away all but one element of the top line.)

\[
\begin{array}{c}
\begin{array}{c}
i_1 : \\
0 \quad 1 \quad 2 \quad \cdots \quad m \quad m+1 \\
\end{array}
\quad \begin{array}{c}
i_2 : \\
0 \quad 1 \quad 2 \quad 3 \quad \cdots \quad m+1 \\
\end{array}
\quad \begin{array}{c}
i_m : \\
0 \quad 1 \quad \cdots \quad m-1 \quad m \quad m+1 \\
\end{array}
\end{array}
\]

For maps $f_i : A \to B_i$, $1 \leq i \leq m$, we denote by $\langle f_1, \ldots, f_m \rangle : A \to B_1 \times \cdots \times B_m$ the map obtained by the Cartesian structure of Top. In particular, for the above-mentioned $i_1, \ldots, i_m$ and for a simplicial space $X : \Delta^{op} \to \text{Top}$ we have the map

\[p_m = \langle i_1, \ldots, i_m \rangle : X_m \to (X_1)^m.\]

(According to our convention from the introduction, $X(i_k)$ is abbreviated by $i_k$.) If $m = 0$, then $(X_1)^0 = \{\ast\}$ (a terminal object of Top) and let $p_0$ denote the unique arrow from $X_0$ to $(X_1)^0$. The following lemma is claimed in [17].

Lemma 3.1. If $X : \Delta^{op} \to \text{Top}$ is a simplicial space such that for every $m \geq 0$, the map $p_m$ is a homotopy equivalence, then $X_1$ is a homotopy associative H-space whose multiplication $m$ is given by the composition

\[X_1^2 \xrightarrow{p_2^{-1}} X_2 \xrightarrow{d_2^1} X_1,\]

4
where $p_2^{-1}$ is an arbitrary homotopy inverse to $p_2$, and whose unit $*$ is $s_0(x_0)$, for an arbitrary $x_0 \in X_0$.

**Definition 3.2.** We say that a PRmss $X : \Delta^{op} \to \text{Top}$ is Segal’s simplicial space when for every $m \geq 0$, the map $p_m : X_m \to (X_1)^m$ is a homotopy equivalence.

**Lemma 3.3.** Let $Y : \Delta^{op} \times \Delta^{op} \to \text{Top}$ be a PRmss. If for every $k \geq 0$, the simplicial space $Y_{-k}$ is Segal’s, then $Y^{(1)}$ is Segal’s simplicial space.

**Definition 3.4.** We say that a PRmss $X : (\Delta^{op})^n \to \text{Top}$, where $n \geq 1$, is Segal’s multisimplicial space, when for every $l \in \{0, \ldots, n - 1\}$ and every $k \geq 0$, the simplicial space $X_{1 \ldots 1 k_k \ldots k_{n-1}}$ is Segal’s.

Note that we do not require $X_{k_1 \ldots k_n \ldots k_{n-1}}$ to be Segal’s for arbitrary $k_1, \ldots, k_{n-1}$ (see the parenthetical remark in Section 5.)

**Remark 3.5.** If $X : (\Delta^{op})^n \to \text{Top}$ is Segal’s multisimplicial space, then for every $l \in \{0, \ldots, n - 1\}$, $X_{l \ldots 1}$ is homotopy associative H-space with respect to the structure obtained from Lemma 3.1 applied to $X_{1 \ldots 1 l \ldots 1} : \Delta^{op} \to \text{Top}$.

Our goal is to generalize the following proposition, which stems from [17, Proposition 1.5 (b)]. (In the proof of that result, contractibility of $|PA|$ comes from the fact that $|PA| \simeq A_0$.)

**Proposition 3.6.** Let $X : \Delta^{op} \to \text{Top}$ be Segal’s simplicial space. If $X_1$ with respect to the H-space structure obtained by Lemma 3.1 is grouplike, then $X_1 \simeq \Omega |X|$.

Our generalization is the following.

**Proposition 3.7.** Let $X : (\Delta^{op})^n \to \text{Top}$ be Segal’s multisimplicial space. If $X_{1 \ldots 1 l}$, with respect to the H-space structure obtained by Remark 3.5 when $l = n - 1$, is grouplike, then $X_{1 \ldots 1 n-1} \simeq \Omega^n |X|$.

**Proof.** We proceed by induction on $n \geq 1$. If $n = 1$, the result follows from Proposition 3.6.

If $n > 1$, then we may apply the induction hypothesis to $X_{1 \ldots 1 n-1}$. Hence,

$$X_{1 \ldots 1} \simeq \Omega^{n-1} |X_{1 \ldots 1}|.$$

By Lemma 2.5, we have that $|X_{1 \ldots 1}| \approx |\text{diag}X_{1 \ldots 1}|$. By the assumption and Remark 2.7 (d), the multisimplicial space $Y : \Delta^{op} \times \Delta^{op} \to \text{Top}$, defined so that $Y_{mk} = X_{mk \ldots k}$, satisfies the conditions of Lemma 3.3. Let $Z$ be the simplicial space $Y^{(1)} : \Delta^{op} \to \text{Top}$, i.e.,

$$Z_m = |Y_{m}| = |\text{diag}X_{m \ldots m}|.$$

By Lemma 3.3, $Z$ is Segal’s simplicial space. By Remark 2.7 (b), $Z_1$ is a CW-complex. Since the space $Y_{10}$ (i.e., $X_{10 \ldots 0}$) is by the assumption homotopic to $(X_{10 \ldots 0})^0$, it is contractible, and hence, path-connected. By Lemma 2.14, we have that $Z_1$, which is equal to $|Y_{1}|$, is path-connected. Note also that $|Z| = |Y| \approx |\text{diag}X| \approx |X|$.
By Lemma 3.1, $Z_1$ is a homotopy associative H-space, and since it is a path-connected CW-complex, by [11 Proposition 8.4.4], it is grouplike. Applying Proposition 3.6 to $Z$, we obtain
\[ |X_{1\ldots 1}| \approx |\text{diag}X_{1\ldots 1}| = Z_1 \approx \Omega |Z| \approx \Omega |X| \]
Finally, we have
\[ X_{1\ldots 1} \simeq \Omega^{n-1} |X_{1\ldots 1}| \simeq \Omega^n |X| \]

4 Segal’s lax functors

Thomason, [20], was the first who noticed that the reduced bar construction based on a symmetric monoidal category produces a lax, instead of an ordinary, functor. The idea to use Street’s rectification in that case, also belongs to him.

We use the notions of lax functor, left and right lax transformation as defined in [19]. The following theorem is taken over from [19, Theorem 2].

**Theorem 4.1.** For every lax functor $W : \mathcal{A} \to \text{Cat}$ there exists a genuine functor $V : \mathcal{A} \to \text{Cat}$, a left lax transformation $E : V \to W$ and a right lax transformation $J : W \to V$ such that $J$ is the left adjoint to $E$ and $W = EVJ$.

We call $V$ a rectification of $W$. It is easy to see that if $W : \mathcal{A} \times \mathcal{B} \to \text{Cat}$ is a lax functor and $V$ is its rectification, then for every object $A$ of $\mathcal{A}$, $W_A$ is a lax functor and $V_A$ is its rectification. As for simplicial spaces, for a (lax) functor $W : \Delta^{op} \to \text{Cat}$, we denote the unique arrow $W_0 \to (W_1)^0$ by $p_0$, and when $m \geq 1$, we have $p_m = \langle i_1, \ldots, i_m \rangle : W_m \to (W_1)^m$.

**Definition 4.2.** We say that a lax functor $W : \Delta^{op} \to \text{Cat}$ is Segal’s, when for every $m \geq 0$, $p_m : W_m \to (W_1)^m$ is the identity. We say that a lax functor $W : (\Delta^{op})^n \to \text{Cat}$ is Segal’s, when for every $l \in \{0, \ldots, n-1\}$ and every $k \geq 0$, the lax functor $W_{\ldots 1 l 1 \ldots k \ldots k} : \Delta^{op} \to \text{Cat}$ is Segal’s.

We denote by $B : \text{Cat} \to \text{Top}$ the classifying space functor, i.e., the composition $| | \circ N$, where $N : \text{Cat} \to \text{Top}^{\Delta^{op}}$ is the nerve functor.

**Proposition 4.3.** If $W : \Delta^{op} \to \text{Cat}$ is Segal’s lax functor and $V$ is its rectification, then $B \circ V$ is Segal’s simplicial space.

By Definitions 3.4 and 4.2, the following generalization of Proposition 4.3 is easily obtained.

**Corollary 4.4.** If $W : (\Delta^{op})^n \to \text{Cat}$ is Segal’s lax functor and $V$ is its rectification, then $B \circ V$ is Segal’s multisimplicial space.

For Corollary 4.4, we conclude that $B \circ V$ is a PRmss as in the proof of Proposition 4.3 given in the appendix.

5 An application

Let $\mathcal{M}$ be an $n$-fold strict monoidal category and let $\widetilde{\mathcal{M}} : (\Delta^{op})^n \to \text{Cat}$ be the $n$-fold reduced bar construction defined as in [5]. The main result of that
paper says that $\overline{W}M$ is a lax functor and it is easy to verify that it is Segal's. (Note that $\overline{W}M_{k_1...k_l}$ is not Segal's when $k_j > 1$, for some $1 \leq j \leq l$)

For $V$ being a rectification of $\overline{W}M$, we have the following.

**Theorem 5.1.** If $BV_{1...1}$, with respect to the H-space structure obtained by Remark 3.5 when $l = n - 1$, is grouplike, then $BM \simeq \Omega^n B \circ V$.

**Proof.** By Corollary 4.4, we have that $B \circ V$ is Segal's multisimplicial space. Hence, by Proposition 3.7, $BV_{1...1} \simeq \Omega^n [B \circ V]$. Since $V$ is a rectification of $\overline{W}M$, by relying on Remark A1 of the appendix, we conclude that $BV_{1...1} \simeq B\overline{W}M_{1...1}$. Together with the fact that $\overline{W}M_{1...1} = M$, we obtain that

$$BM \simeq \Omega^n [B \circ V].$$

This means that up to group completion (see [17] and [13]), for every $n$-fold strict monoidal category $M$, the classifying space $BM$ is an $n$-fold loop space. When $M$ contains a terminal or initial object, we have that $BM$, and hence $BV_{1...1}$, is path-connected. In that case, by [11 Proposition 8.4.4], $BV_{1...1}$ is grouplike, and $[B \circ V]$ is an $n$-fold delooping of $BM$.

6 Appendix

**Proof of Lemma 3.1.** First, we prove that $\langle X_1, m, * \rangle$ is an H-space. Let $j_1 : X_2 \to X_1 \times X_1$ be such that $j_1(x) = (x, *)$, and analogously, let $j_2 : X_1 \to X_1 \times X_1$ be such that $j_2(x) = (*, x)$. By the assumption, $X_0$ is contractible. Hence, $d_0^3$ is homotopic to the constant map to $x_0$ and therefore $s_0^3 \circ d_0^3$ is homotopic to the constant map to *. We conclude that

$$j_1 \simeq \langle 1_{X_1}, s_0^3 \circ d_0^3 \rangle = \langle d_2^3 \circ s_1^3, d_0^3 \circ s_1^3 \rangle = \langle d_2^3, d_0^3 \rangle \circ s_1^3 = p_2 \circ s_1^3,$$

i.e., $p_2^{-1} \circ j_1 \simeq s_2^3$. Hence,

$$m \circ j_1 = d_2^3 \circ p_2^{-1} \circ j_1 \simeq d_2^3 \circ s_1^3 = 1_{X_1}.$$

Analogously, $m \circ j_2 \simeq 1_{X_1}$ and we have that $\langle X_1, m, * \rangle$ is an H-space.

Next, we prove that $m$ is associative up to homotopy, i.e., that

$$m \circ (m \times 1) \simeq m \circ (1 \times m).$$

Consider $p_3 : X_3 \to (X_1)^3$ for which we have:

$$p_3 = \langle (i_1, i_2), i_3 \rangle = \langle (d_2^3 \circ d_3^3, d_0^3 \circ d_3^3), i_3 \rangle = \langle p_2 \circ d_3^3, i_3 \rangle = (p_2 \times 1) \circ (d_3^3, i_3),$$

and analogously

$$p_4 = \langle (1 \times p_2) \circ (i_1, d_0^3) \rangle.$$

Since $p_2$ and $p_4$ are homotopy equivalences, we have that $\langle d_3^3, i_3 \rangle$ and $\langle i_1, d_0^3 \rangle$ are homotopy equivalences, too. Moreover,

1. $\langle d_3^3, i_3 \rangle^{-1} \simeq p_3^{-1} \circ (p_2 \times 1)$, and

2. $\langle i_1, d_0^3 \rangle^{-1} \simeq p_3^{-1} \circ (1 \times p_2)$.

Also, we show that
By Lemma 2.11, we have that
\[
(d^1_2 \times 1) \circ (d^2_2, d^1_2) = (d^2_2, d^0_2) \circ d^1_2
\]
which together with (1) delivers (3). Also,
\[
(1 \times d^2_2) \circ (i_1, d^0_0) = (i_1, d^2_2) \circ (d^1_2, d^0_2) = (d^2_2, d^0_0) \circ (1 \times d^2_2)
\]
which together with (2) delivers (4). Finally, we have
\[
m \circ (m \times 1) = d^1_1 \circ p_2^{-1} \circ (d^2_1 \times 1) \circ (p_2^{-1}) \simeq d^1_2 \circ d^1_2 \circ p_3^{-1}, \text{ by (3)}
\]
\[
= d^2_2 \circ d^1_2 \circ p_3^{-1} \simeq d_2 \circ p_2^{-1} \circ (1 \times d^2_1) \circ (1 \times p_2^{-1}), \text{ by (4)}
\]
\[
m \circ (1 \times m) = m \circ (1 \times m).
\]

**Proof of Lemma 3.3.** Let \( Z : \Delta^{op} \to \text{Top} \) be \( Y^{(1)} \). By Remark 2.7 (a), it is a PRmss. We have to show that for every \( m \geq 0 \), the map \( p_m : Z_m \to (Z_1)^m \) is a homotopy equivalence.

Let \( m = 0 \) and let \( T \) be the trivial simplicial space with \( T_k = \{ \ast \} \). Consider the simplicial space \( Y_0 : \Delta^{op} \to \text{Top} \), which is a PRmss by Remark 2.7 (c). By Proposition 2.9 and Lemma 2.10, both \( T \) and \( Y_0 \) are proper. The following simplicial map is obtained by the assumptions (the diagrams are commutative since \( \{ \ast \} \) is terminal).

\[
Y_0 : \quad \cdots \quad Y_{02} \quad \xrightarrow{\sim} \quad Y_{01} \quad \xrightarrow{\sim} \quad Y_{00}
\]
\[
\downarrow \simeq \quad \downarrow \simeq \quad \downarrow \simeq
\]
\[
T : \quad \cdots \quad \{ \ast \} \quad \xrightarrow{\sim} \quad \{ \ast \} \quad \xrightarrow{\sim} \quad \{ \ast \}
\]

By Lemma 2.11, we have that \(|Y_0| \simeq |T| = \{ \ast \} \) via the unique map. Since \( Z_0 = |Y_0| \) and \((Z_1)^0 = \{ \ast \}\), we are done.

Let \( m > 0 \). Consider the simplicial spaces \( Y_m \) and \((Y_1)^m\), which are proper by Remark 2.7 (c), Proposition 2.9, Lemma 2.10 and Remark 2.13. The following simplicial map is obtained by the assumptions (it is straightforward to verify that the diagrams are commutative).

\[
Y_m : \quad \cdots \quad Y_{m2} \quad \xrightarrow{\sim} \quad Y_{m1} \quad \xrightarrow{\sim} \quad Y_{m0}
\]
\[
\downarrow \simeq \quad \downarrow \simeq \quad \downarrow \simeq
\]
\[
(Y_1)^m : \quad \cdots \quad (Y_{12})^m \quad \xrightarrow{\sim} \quad (Y_{11})^m \quad \xrightarrow{\sim} \quad (Y_{10})^m
\]

By Lemma 2.11, we have that
\[
|\{Y(i_1, \_), \ldots, Y(i_m, \_)\}| : |Y_m| \to |(Y_1)^m|
\]
is a homotopy equivalence. Also, for Top, the realization functor | | preserves products (see [11] Theorem 14.3, [7] III.3, Theorem] and [11] Corollary 11.6]).
Namely, for \( p_k : (Y_i)_m \to Y_{\underline{i}} \), 1 \( \leq k \leq m \) being the \( k \)-th projection,
\[
\langle |\pi_1|, \ldots, |\pi_m| \rangle : |(Y_{\underline{i}})_m| \to |Y_{\underline{i}}|^m
\]
is a homeomorphism (\(| | \) is strong monoidal; see [16] Example 6.2.2). Hence,
\[
\langle |\pi_1|, \ldots, |\pi_m| \rangle \circ |(Y(i_1, \underline{\_}), \ldots, Y(i_m, \underline{\_}))| : |Y_m| \to |Y_{\underline{i}}|^m
\]
is a homotopy equivalence.

The following easy computation, in which \( \langle Y(i_1, \underline{\_}), \ldots, Y(i_m, \underline{\_}) \rangle \) abbreviates by \( \alpha \),
\[
\langle |\pi_1|, \ldots, |\pi_m| \rangle \circ |\alpha| = \langle |\pi_1| \circ |\alpha|, \ldots, |\pi_m| \circ |\alpha| \rangle = \langle |\pi_1 \circ \alpha|, \ldots, |\pi_m \circ \alpha| \rangle
\]

shows that the map \( p_m = \langle Z(\langle i_1 \rangle), \ldots, Z(\langle i_m \rangle) \rangle \) is a homotopy equivalence between
\( Z_m = |Y_m|, \) and \( (Z_\uparrow)^m = |Y_{\underline{i}}|^m \).

\[\square\]

**Some preliminary remarks for Proposition 4.3.** Let 2 be the category with two objects (0 and 1) and one nonidentity arrow \( h : 0 \to 1 \). Let \( I_0, I_1 : C \to C \times 2 \) be the functors such that for every object \( C \) of \( C \), we have that
\( I_0(C) = (C, 0) \) and \( I_1(C) = (C, 1) \). Let \( F, G : C \to D \) be two functors. There is a bijection between the set of natural transformations \( \alpha : F \Rightarrow G \), and the set of functors \( A : C \times 2 \to D \) such that \( A \circ I_0 = F \) and \( A \circ I_1 = G \). This bijection maps \( \alpha : F \Rightarrow G \) to \( A : C \times 2 \to D \) such that
\[ A(C, 0) = FC, \quad A(C, 1) = GC, \quad A(f, 1_0) = Ff, \quad A(f, 1_1) = Gf, \]
and for \( f : C \to C' \),
\[ A(f, h) = Gf \circ \alpha_C = \alpha_{C'} \circ Ff. \]

Its inverse maps \( A : C \times 2 \to D \) to \( \alpha : F \Rightarrow G \) such that \( \alpha_C = A(1_C, h). \)

The nerve functor \( N \) preserves products on the nose, hence, the classifying space functor \( B = \Rightarrow \circ N \) preserves products too. Therefore, the spaces \( BC \times I \) (i.e., \( BC \times B2 \)) and \( B(C \times 2) \) are homeomorphic and we have the following.

**Remark A1.** Every natural transformation \( \alpha : F \Rightarrow G \) gives rise to the homotopy
\[ BC \times I \xrightarrow{\alpha} B(C \times 2) \xrightarrow{BA} BD \]
between the maps \( BF \) and \( BG \).

**Proof of Proposition 4.3.** By the isomorphism mentioned in Definition 2.2, we have that \( N \circ V \) corresponds to a multisimplicial set \( X : \Delta^{op} \times \Delta^{op} \to Top \) and \( B \circ V \) is \( X^{(1)} \). Hence, it is a PRmss.

We have to show that for every \( m \geq 0 \), \( p_m : BV_m \to (BV)\) is a homotopy equivalence, where we denote again by \( p_0 \) the unique map from \( BV_0 \) to \( (BV)_0 \) and by \( p_m \) the map \( (BV(\langle i_1 \rangle), \ldots, BV(\langle i_m \rangle)) \).

When \( m = 0 \), we show that \( BJ_0 : BW_0 \to BV_0 \) is a homotopy inverse to \( p_0 \). Since \( W_0 \) and \( (V_\uparrow)_0 \) are the same trivial category and \( BW_0 = (BV)_0 = \{*\} \), it
is easy to conclude that \( p_0 \circ B J_0 \simeq 1_{(BV)} \), and that \( p_0 = B E_0 \). The latter, by the adjunction \( J_0 \vdash E_0 \) and Remark A1, delivers \( B J_0 \circ p_0 \simeq 1_{BV} \).

When \( m \geq 1 \), we have for every \( 1 \leq j \leq m \), the following natural transformations.

\[
\begin{align*}
&\xymatrix{ E_m & W_m \\
& V_m \\
i_j & V_1 \\
i_j & E_1 \\
>_i & W_1 \\
}_{\langle E_i, \ldots, E_{i_m} \rangle} & \quad & \xymatrix{ J_m & V_m \\
& W_m \\
i_j & J_1 \\
i_j & V_1 \\
>_i & W_1 \\
}_{\langle J_i, \ldots, J_{i_m} \rangle}
\end{align*}
\]

By using the monoidal structure of \( \text{Cat} \) given by 2-products and the fact that \( \langle i_1, \ldots, i_m \rangle : W_m \rightarrow (W) \) is the identity, we obtain the following two natural transformations.

\[
\begin{align*}
&\xymatrix{ V_m & W_m \\
& (i_1, \ldots, i_m) \\
& (V_1)^m \\
& (V_1)^m \\
& (E_1)^m \\
& (E_1)^m \\
}_{\langle E_i, \ldots, E_{i_m} \rangle} & \quad & \xymatrix{ V_m & W_m \\
& (i_1, \ldots, i_m) \\
& (J_1)^m \\
& (J_1)^m \\
& (V_1)^m \\
& (V_1)^m \\
& (V_1)^m \\
& (V_1)^m \\
}_{\langle J_i, \ldots, J_{i_m} \rangle}
\end{align*}
\]

For \( \pi_k : C^m \rightarrow C \), \( 1 \leq k \leq m \) being the \( k \)th projection,

\[
\langle B \pi_1, \ldots, B \pi_m \rangle : BC^m \rightarrow (BC)^m
\]

is a homeomorphism whose inverse we denote by \( q_m(C) \). It is easy to verify that for \( F, F_1, \ldots, F_m : \text{C} \rightarrow \text{D} \) we have \( B \langle F_1, \ldots, F_m \rangle = q_m(\text{D}) \langle BF_1, \ldots, BF_m \rangle \) and \( BF^m \circ q_m(C) = q_m(\text{D}) \circ (BF)^m \).

By Remark A1, the transformations mentioned above give rise to

\[
\begin{align*}
\langle \dagger \rangle \quad & BE_m \simeq B(E_1)^m \circ B[V(i_1), \ldots, V(i_m)] \\
& = q_m(W_1) \circ (BE_1)^m \circ (BV(i_1), \ldots, BV(i_m)) \\
& = q_m(W_1) \circ (BE_1)^m \circ p_m, \text{ and}
\end{align*}
\]

\[
\begin{align*}
\langle \ddagger \ddagger \rangle \quad & BJ_1^m \simeq B[V(i_1), \ldots, V(i_m)] \circ BJ_m \\
& = q_m(V_1) \circ (BV(i_1), \ldots, BV(i_m)) \circ BJ_m \\
& = q_m(V_1) \circ p_m \circ BJ_m.
\end{align*}
\]

The following calculation shows that

\[
BJ_m \circ q_m(W_1) \circ (BE_1)^m : (BV_1)^m \rightarrow BV_m
\]

is a homotopy inverse to \( p_m \).

\[
\begin{align*}
1_{BV_1} & \simeq BJ_m \circ BE_m, \quad \text{by } J_m \vdash E_m, \text{Remark A1} \\
& \simeq BJ_m \circ q_m(W_1) \circ (BE_1)^m \circ p_m, \quad \text{by } \langle \dagger \rangle \\
1_{(BV)_1} & \simeq q_m^{-1}(V_1) \circ BJ_1^m \circ B(E_1)^m \circ q_m(V_1), \quad \text{by } J_1 \vdash E_1, \text{Remark A1} \\
& \simeq p_m \circ BJ_m \circ q_m(W_1) \circ (BE_1)^m, \quad \text{by } \langle \ddagger \ddagger \rangle.
\end{align*}
\]

\[\text{def} \]
References

[1] M. Arkowitz, *Introduction to Homotopy Theory*, Springer, Berlin, 2011

[2] C. Balteanu, Z. Fiedorowicz, R. Schwänzl and R. Vogt, *Iterated monoidal categories*, Advances in Mathematics, vol. 176 (2003), pp. 277-349

[3] A.K. Bousfield, *The simplicial homotopy theory of iterated loop spaces*, unpublished (1992)

[4] P.V.Z. Cobb, *Pn-spaces and n-fold loop spaces*, Bulletin of the American Mathematical Society, vol. 80 (1974), pp. 910-914

[5] S.LJ. Ćukić and Z. Petrić, *The n-fold reduced bar construction* (the old title was “n-fold monoidal categories”), preprint (2013) (arXiv:1309.6209)

[6] E. Dyer and S. Eilenberg, *An adjunction theorem for locally equiconnected spaces*, Pacific Journal of Mathematics, vol. 41 (1972), pp. 669-685

[7] P. Gabriel and M. Zisman, *Calculus of Fractions and Homotopy Theory*, Ergebnisse der Mathematik und Ihrer Grenzgebiete, vol. 35, Springer, Berlin, 1967

[8] L.G. Lewis, *When is the Natural Map X → ΩΣX a Cofibration?*, Transactions of the American Mathematical Society, vol. 273 (1982), pp. 147-155

[9] S. Mac Lane, *Categories for the Working Mathematician*, Springer, Berlin, 1971 (expanded second edition, 1998)

[10] J.P. May, *Simplicial Objects in Algebraic Topology*, The University of Chicago Press, Chicago, 1967

[11] ———, *The Geometry of Iterated Loop Spaces*, Lecture Notes in Mathematics, vol. 271, Springer, Berlin, 1972

[12] ———, *E∞-spaces, group completions and permutative categories*, New Developments in Topology (G. Segal, editor), London Mathematical Society Lecture Notes Series, vol. 11, Cambridge University Press, 1974, pp. 153-231

[13] D. McDuff and G. Segal, *Homology fibrations and the “group-completion” theorem*, Inventiones mathematicae, vol. 31 (1976), pp. 279-284

[14] Z. Petrić and T. Trimble, *Symmetric bimonoidal intermuting categories and ω × ω reduced bar constructions*, Applied Categorical Structures, vol. 22 (2014), pp. 467-499 (arXiv:0906.2954)

[15] D. Quillen, *Higher algebraic K-theory: I*, Higher K-Theory (H. Bass, editor), Lecture Notes in Mathematics, vol. 341, Springer, Berlin, 1973, pp. 85-147
[16] E. Riehl, *Categorical Homotopy Theory*, Cambridge University Press, New York, 2014

[17] G. Segal, *Categories and cohomology theories*, *Topology*, vol. 13 (1974), pp. 293-312

[18] D. Stevenson and D. Roberts, *Simplicial principal bundles in parametrized spaces*, preprint (2012) (arXiv:1203.2460)

[19] R. Street, *Two constructions on lax functors*, *Cahiers de topologie et géométrie différentielle*, vol. 13 (1972), pp. 217-264

[20] R.W. Thomason, *Homotopy colimits in the category of small categories*, *Mathematical Proceedings of the Cambridge Philosophical Society*, vol. 85, 91 (1979), pp. 91-109