Boundary value problem for discrete analytic functions

M. Skopenkov\textsuperscript{a,b}

\textsuperscript{a}Institute for information transmission problems of the Russian Academy of Sciences
\textsuperscript{b}King Abdullah university of science and technology

Abstract

This paper is on further development of discrete complex analysis introduced by R. Isaacs, R. Duffin, and C. Mercat. We consider a graph lying in the complex plane and having quadrilateral faces. A function on the vertices is called discrete analytic, if for each face the difference quotients along the two diagonals are equal.

We prove that the Dirichlet boundary value problem for the real part of a discrete analytic function has a unique solution. In the case when each face has orthogonal diagonals we prove that this solution converges to a harmonic function in the scaling limit (under certain regularity assumptions). This solves a problem of S. Smirnov from 2010. This was proved earlier by R. Courant–K. Friedrichs–H. Lewy for square lattices, by D. Chelkak–S. Smirnov and implicitly by P.G. Ciarlet–P.-A. Raviart for rhombic lattices.

In particular, our result implies uniform convergence of finite element method on Delauney triangulations. This solves a problem of A. Bobenko from 2011. The methodology is based on energy estimates inspired by alternating-current networks theory.

Keywords: discrete analytic function, boundary value problem, energy, alternating current

2010 MSC: 39A12, 65M60

1. Introduction

Various constructions of complex analysis on planar graphs were introduced by Isaacs, Duffin, Mercat \cite{18,15,13,22,23}, Dynnikov–Novikov \cite{14}, Bobenko–Mercat–Suris \cite{2}, Bobenko–Pinkall–Springborn \cite{4}. Recently this subject is developed extensively due to applications to statistical physics \cite{28}, numerical analysis \cite{17,3}, computer graphics \cite{1,29}, combinatorial geometry \cite{24}; see \cite{21,28} for recent surveys.

This paper concerns \textit{linear} complex analysis on quadrilateral lattices \cite{2}. A \textit{quadrilateral lattice} is a graph $Q \subset \mathbb{C}$ with rectilinear edges such that each bounded face is a quadrilateral (not necessarily convex). Depending of the shape of faces, one speaks about \textit{square}, \textit{rhombic}, \textit{kite}, or \textit{orthogonal} lattices (the latter are quadrilateral lattices such that the diagonals of each face are orthogonal); see Figure 1. Different types of lattices are required for different applications; see Section 5.

![Examples of lattices Q: (a) square; (b) rhombic; (c) orthogonal; (d) quadrilateral.](image)

Figure 1: Examples of lattices $Q$: (a) square; (b) rhombic; (c) orthogonal; (d) quadrilateral.

Email address: skopenkov@rambler.ru (M. Skopenkov)
A complex-valued function \( f \) on the vertices of \( Q \) is called \textit{discrete analytic} [23], if the difference quotients along the two diagonals of each face are equal, i.e.,

\[
\frac{f(z_1) - f(z_3)}{z_1 - z_3} = \frac{f(z_2) - f(z_4)}{z_2 - z_4}
\]

(1)

for each quadrilateral face \( z_1z_2z_3z_4 \); see Figure 1d. The motivation for this definition is that both sides of equation (1) approximate the derivative of an analytic function \( f \) inside this face. The real part of a discrete analytic function is called a \textit{discrete harmonic function}.

Discrete complex analysis is analogous to the classical complex analysis in many aspects [21]. One of the most natural and at the same time challenging problems is to prove convergence of discrete theory to the continuous one when the lattice becomes finer and finer [28]. A natural formalization of such \textit{convergence} is uniform convergence of the solution of the Dirichlet boundary value problem for a discrete harmonic function to a harmonic function in the scaling limit.

1.1. Previous work

Convergence in this sense was proved by R. Courant–K. Friedrichs–H. Lewy [11, §4] for square lattices, by D. Chelkak–S. Smirnov [8, Proposition 3.3] and implicitly by P.G. Ciarlet–P.-A. Raviart [10, Theorem 2] for rhombic lattices. In fact convergence for rhombic lattices is equivalent to convergence of the classical finite element method [13]. The latter subject is well-developed; see a survey [7] and a textbook [6]. Nonrhombic lattices cannot be accessed by known methods. Weaker convergence results not involving boundary value problems were obtained in [22, Theorem 3], [17, Theorem 2].

1.2. Contributions

We prove that the Dirichlet boundary value problem for a discrete harmonic function on a quadrilateral lattice has a unique solution. Our main result is that in the case of orthogonal lattices this solution converges to a harmonic function in the scaling limit; see Convergence Theorem 1.2 below. This solves a problem of S. Smirnov [28, Question 1]. In concert we get a simpler proof for the particular cases known before.

In particular, our main result implies uniform convergence of finite element method on Delauney triangulations; see Corollary 5.1 below. This solves a problem of A. Bobenko (private communication; see also [29, Table in Section 2]).

1.3. Statements

Let us give precise statements of main results.

The boundary \( \partial Q \) of the graph \( Q \) is the boundary of its outer face. Hereafter assume for simplicity that \( \partial Q \) is a closed curve without self-intersections. Denote by \( Q^0 \) the set of vertices of the graph \( Q \).

Let \( g: \mathbb{C} \to \mathbb{R} \) be a smooth function. The \textit{Dirichlet (boundary value) problem} on \( Q \) is to find a discrete harmonic function \( u_{Q,g}: Q^0 \to \mathbb{R} \) such that \( u_{Q,g}(z) = g(z) \) for each vertex \( z \in \partial Q \). The function \( u_{Q,g}: Q^0 \to \mathbb{R} \) is called a \textit{solution} of the Dirichlet problem.

**Uniqueness Theorem 1.1.** \textit{The Dirichlet boundary value problem on any finite quadrilateral lattice has a unique solution.}

This theorem is nontrivial because discrete harmonic functions do not satisfy the maximum principle in general; see Example 3.6 below.

Let \( \Omega \subset \mathbb{C} \) be a domain. The \textit{Dirichlet (boundary value) problem} on \( \Omega \) is to find a continuous function \( u_{\Omega,g}: \mathbb{C} \cap \Omega \to \mathbb{R} \) harmonic in \( \Omega \) and such that \( u_{\Omega,g}(z) = g(z) \) for each point \( z \in \partial \Omega \). The harmonic function \( u_{\Omega,g}: \Omega \to \mathbb{R} \) is called a \textit{solution} of the Dirichlet problem.

A sequence of lattices \( \{Q_n\} \) \textit{approximates the domain} \( \Omega \), if for \( n \to \infty \):

- the maximal distance from a point of \( \partial Q_n \) to the curve \( \partial \Omega \) tends to zero;
- the maximal edge length of \( Q_n \) tends to zero.
A sequence of lattices \( \{Q_n\} \) is nondegenerate uniform, if there is a constant \( \text{Const} \) (not depending on \( n \)) such that for each member of the sequence:

(D) the ratio of the diagonals of each face is less than \( \text{Const} \) and the angle between them is greater than \( 1/\text{Const} \);

(U) the number of vertices in an arbitrary disk of radius equal to the maximal edge length is less than \( \text{Const} \).

A sequence of functions \( u_n: Q_n^0 \rightarrow \mathbb{C} \) converges to a function \( u: \Omega \rightarrow \mathbb{C} \) uniformly on each compact subset, if for each compact set \( K \subset \Omega \) we have \( \max_{z \in \partial K \cap Q_n^0} |u_n(z) - u(z)| \rightarrow 0 \) as \( n \rightarrow \infty \).

**Convergence Theorem 1.2.** Let \( \Omega \subset \mathbb{C} \) be a domain bounded by a smooth closed curve \( \partial \Omega \) without self-intersections. Let \( g: \mathbb{C} \rightarrow \mathbb{R} \) be a smooth function. Let \( \{Q_n\} \) be a nondegenerate uniform sequence of finite orthogonal lattices approximating the domain \( \Omega \). Then the solution \( u_{Q_n,g}: Q_n^0 \rightarrow \mathbb{R} \) of the Dirichlet problem on \( Q_n \) converges to the solution \( u_{\Omega,g}: \Omega \rightarrow \mathbb{R} \) of the Dirichlet problem in \( \Omega \) uniformly on each compact subset of \( \Omega \).

1.4. Organization of the paper

In Section 2 we introduce main ideas of the proofs and state key lemmas. In Sections 3 and 4 we prove Uniqueness Theorem 1.1 and Convergence Theorem 1.2, respectively. In Section 5 we give applications of our results to numerical analysis, network theory, probability theory, and state some open problems.

2. Main ideas

2.1. Energy minimization

Our approach is based on energy estimates inspired by alternating-current networks theory. Recall that the (Dirichlet) energy of a continuous piecewise-smooth function \( u: \Omega \rightarrow \mathbb{R} \) is

\[
E_{\Omega}(u) := \int_{\Omega} |\nabla u|^2 \, dA.
\]

This is a convex functional on the space of continuous piecewise-smooth functions with fixed boundary values, and harmonic functions are characterized as minimizers of this functional.

Let us define a discrete counterpart of the energy, which is the main concept of the paper. The gradient of a function \( u: Q^0 \rightarrow \mathbb{R} \) at a face \( z_1z_2z_3z_4 \) of the quadrilateral lattice \( Q \) is the unique vector \( \nabla_Q u(z_1z_2z_3z_4) \in \mathbb{R}^2 \) such that

\[
\nabla_Q u(z_1z_2z_3z_4) \cdot \overrightarrow{z_1z_3} = u(z_1) - u(z_3),
\]

\[
\nabla_Q u(z_1z_2z_3z_4) \cdot \overrightarrow{z_2z_4} = u(z_2) - u(z_4).
\]

The energy of the function \( u: Q^0 \rightarrow \mathbb{R} \) is the number

\[
E(u) := \sum_{z_1z_2z_3z_4 \in Q} |\nabla_Q u(z_1z_2z_3z_4)|^2 \cdot \text{Area}(z_1z_2z_3z_4),
\]

where the sum is over all the faces \( z_1z_2z_3z_4 \) of the lattice \( Q \).

We give a physical motivation for this definition in Section 5.2. A similar but nonequivalent definition was given in [23, Formula (12)]. Our energy has the same properties as its continuous counterpart:

**Convexity Principle 2.1.** The energy \( E(u) \) is a strictly convex functional on the affine space \( \mathbb{R}^{Q^0-\partial Q} \) of functions \( u: Q^0 \rightarrow \mathbb{R} \) having fixed values at the boundary \( \partial Q \).

**Variational Principle 2.2.** A function \( u: Q^0 \rightarrow \mathbb{R} \) has minimal energy \( E(u) \) among all the functions with the same boundary values if and only if it is discrete harmonic.

These principles are proved in Section 3. Uniqueness Theorem 1.1 is their direct consequence. After the “right” discrete energy has been guessed, these results are proved by standard methods.
2.2. Energy estimates

Let introduce more delicate energy estimates required for the proof of Convergence Theorem 1.2.

Joining the opposite vertices in each quadrilateral face of the lattice \( Q \), we get two connected graphs \( B \) and \( W \) associated to the lattice; see Figure 2. (The vertices are joined by a straight line segment, if the segment lies inside the face, and by the 2-segment broken line through the midpoint of the opposite diagonal, otherwise.) The eccentricity of a lattice \( Q \) is the infimum of the numbers \( \text{Const} \) such that the lattice satisfies conditions (D) and (U) from Section 1.3. Throughout the paper we use the following notation:

- \( \Omega \) is an arbitrary domain bounded by a smooth curve without self-intersections;
- \( g : \mathbb{C} \to \mathbb{R} \) is an arbitrary smooth function;
- \( B \) and \( W \) are the two graphs associated to the lattice \( Q \);
- \( e \) is the eccentricity of the lattice \( Q \);
- \( h \) is twice the maximal edge length of the lattice \( Q \).

![Figure 2: The graphs \( B \) and \( W \) associated to a quadrilateral lattice \( Q \).](image)

**Energy Convergence Lemma 2.3.** Let \( \{Q_n\} \) be a nondegenerate uniform sequence of quadrilateral lattices approximating the domain \( \Omega \). Then \( E(g \mid Q_n) \to E_\Omega(g) \) as \( n \to \infty \).

**Equicontinuity Lemma 2.4.** Let \( Q \) be an orthogonal lattice, \( K \) be a compact set inside \( \partial Q \), and \( u : Q^0 \to \mathbb{R} \) be a discrete harmonic function. Denote by \( r := \min_{z \in K} \text{Dist}(z, \partial Q) \). Then there is a constant \( \text{Const}_{K,r,e} \) depending only on \( K, r, e \) (but not on \( Q, u, z, w \)) such that

\[
|u(z) - u(w)| \leq \text{Const}_{K,r,e} \cdot E(u)^{1/2} \cdot \ln^{-1/2} \left( 1 + \frac{r}{|z - w|} \right) \tag{3}
\]

for any \( z, w \in K \cap B^0 \).

The laplacian \( \Delta_Q u : Q^0 \to \mathbb{R} \) of a function \( u : Q^0 \to \mathbb{R} \) is defined by the formula

\[
[\Delta_Q u](z) := -\frac{\partial E(u)}{\partial u(z)} \text{ for each } z \in Q^0.
\]

**Laplacian Approximation Lemma 2.5.** Let \( Q \) be a quadrilateral lattice and \( R \) be a square of side length \( r > h \) inside \( \partial Q \). Then there is a constant \( \text{Const}_e \) depending only on \( e \) (but not on \( Q, g, R, r, h \)) such that

\[
\left| \sum_{z \in R \cap B^0} \left( [\Delta_Q \left( g \mid Q^0 \right)](z) - \int_R \Delta g \, dA \right) \right| \leq \text{Const}_e \left( hr \max_{z \in R} |D^2 g(z)| + r^3 \max_{z \in R} |D^3 g(z)| \right).
\]
For a subset \( K \subset \mathbb{C} \) and a function \( u: B^0 \to \mathbb{R} \) denote \( L^2_{K}(u) := \sum_{z \in K \cap B^0} u^2(z) \).

**Friedrichs Inequality Lemma 2.6.** Let \( Q \) be a quadrilateral lattice, \( K \) be a compact set inside \( \partial Q \), and \( u: B^0 \to \mathbb{R} \) be an arbitrary function. Denote by \( r := \max_{z \in K} \text{Dist}(z, \partial Q) \). Assume that \( r > h \) and \( r > \max_{z \in \partial Q} \text{Dist}(z, \partial \Omega) \). Then there is a constant \( \text{Const}_{\Omega, e} \) depending only on \( \Omega \) and \( e \) (but not on \( Q, K, r, h, u \)) such that

\[
h^2 L^2_{\Omega - K}(u) \leq \text{Const}_{\Omega, e} \left( hr E^2_{\partial \Omega}(u) + r^2 E(u) \right).
\]

These four results are proved in Section 4. Lemmas 2.3, 2.5, 2.6 are proved using suitable modifications of the approaches of [12], [8], [11], respectively. Equicontinuity Lemma 2.4 is essentially novel. Estimates analogous to (3) were known for square lattices [11, equation (12) in §4.2], for rhombic ones [8, Corollary 2.9, Proposition 2.7 and Appendix A], and for orthogonal ones [25, Corollary 3.4]. The methods known for square and rhombic lattices do not generalize to more general ones (because in general there are no discrete exponentials \([19]\) and no higher derivatives of discrete analytic functions). Surprisingly, our proof is simpler than the ones in [8, 11] even for the particular types of lattices studied there.

**Sketch of the proof of Convergence Theorem 1.2 modulo the above lemmas.** (See the details in Section 4.7.) Take a compact set \( K \subset \Omega \). Restrict each function \( u_{Q_n,g} \) to the set \( B^0_n \cap K \). By Variational Principle 2.2 and Energy Convergence Lemma 2.3 we have \( E(u_{Q_n,g}) \leq E(g |_{Q^0_n}) \to E_\Omega(g) < \infty \) for \( n \to \infty \). Thus the sequence \( E(u_{Q_n,g}) \) is bounded. Then by Equicontinuity Lemma 2.4 it follows that the sequence of functions \( u_{Q_n,g} \) is equicontinuous. By the Arzelà-Ascoli theorem a subsequence \( u_{Q_k,g} \) of the sequence \( u_{Q_n,g} \) converges uniformly to some function \( u: \Omega \to \mathbb{R} \). Using the Weyl lemma and Laplacian Approximation Lemma 2.5 we show that the function \( u \) is harmonic. Using Friedrichs Inequality Lemma 2.6 we estimate \( \int_{\Omega - K} (u - g)^2 dA \) and thus verify the boundary condition. We conclude that \( u \) equals the unique solution \( u_{\Omega,g} \) of the Dirichlet problem. Thus the initial sequence \( u_{Q_n,g} \) (not just a subsequence) converges to \( u_{\Omega,g} \) uniformly on each compact subset of \( \Omega \).

3. Uniqueness

In this section we prove Uniqueness Theorem 1.1 and the results stated in Section 2.1. We also prove two results (Maximum Principle 3.5 and the Green Identity 3.7) required for the next section.

3.1. Convexity Principle

**Proof of Convexity Principle 2.1.** Consider the linear space \( \mathbb{R}^{Q^0} \) of functions \( u: Q^0 \to \mathbb{R} \). Let the coordinates of a function \( u: Q^0 \to \mathbb{R} \) be the values of the function. Clearly, the gradient \( \nabla u \) linearly depends on \( u \) and thus the energy \( E(u) \) is a quadratic form in \( u \). So it suffices to prove the convexity of \( E(u) \) in the case when the affine space \( \mathbb{R}^{Q^0-\partial Q} \subset \mathbb{R}^{Q^0} \) passes through the origin, that is, all the fixed boundary values equal zero.

Clearly, \( E(u) \geq 0 \) for each \( u \in \mathbb{R}^{Q^0-\partial Q} \). It remains to prove that \( E(u) = 0 \) only if \( u = 0 \). Assume that \( E(u) = 0 \). Then \( \nabla_{Q} u(z_1z_2z_3z_4) = 0 \) for each face \( z_1z_2z_3z_4 \) of the lattice \( Q \). This means that for each face \( z_1z_2z_3z_4 \) we have \( u(z_1) = u(z_3) \) and \( u(z_2) = u(z_4) \). Any face can be joined with the boundary \( \partial Q \) by a sequence of faces such that two neighboring ones share a common edge. Thus for each vertex \( z \in Q^0 \) there is a boundary vertex \( w \in \partial Q \) such that \( u(z) = u(w) \). Since the boundary values equal zero it follows that \( u = 0 \).

3.2. Variational Principle

Denote by \( *: \mathbb{R}^2 \to \mathbb{R}^2 \) the counterclockwise rotation through \( \pi/2 \) around the origin. Two functions \( u, v: Q^0 \to \mathbb{R} \) are conjugate, if \( \nabla_{Q} v = *\nabla_{Q} u \).

**Claim 3.1.** Two functions \( u, v: Q^0 \to \mathbb{R} \) are conjugate if and only if the function \( u + iv: Q^0 \to \mathbb{C} \) is discrete analytic.
Proof. Identify the gradient $\nabla Q u \in \mathbb{R}^2$ with a complex number $\nabla u \in \mathbb{C}$. The function $u + iv : Q^0 \to \mathbb{C}$ is discrete analytic if and only if
\[
\frac{u(z_1) + iv(z_1) - u(z_3) - iv(z_3)}{z_1 - z_3} = \frac{u(z_2) + iv(z_2) - u(z_4) - iv(z_4)}{z_2 - z_4}.
\]
for each face $z_1 z_2 z_3 z_4$ of the lattice $Q$. Substitute the expression
\[
u(z_1) - u(z_3) = \nabla Q u \cdot \frac{(z_1 - z_3)}{2} + \nabla Q u \cdot \frac{(z_1 - z_3)}{2}
\]
and analogous ones for $v(z_1) - v(z_3)$, $u(z_2) - u(z_4)$, $u(z_2) - u(z_4)$ into the above equation. We get
\[
(\nabla Q u + i \nabla Q v) \cdot \frac{(z_1 - z_3)}{z_1 - z_3} = 0.
\]
Since the second factor in the left-hand side is nonzero, the equation is equivalent to $\nabla Q v = i \nabla Q u$. □

Claim 3.2. A function $u : Q^0 \to \mathbb{R}$ has a conjugate if and only if for each $z \in Q^0 - \partial Q$ we have
\[
\sum_{z_1 z_2 z_3 z_4 : z_1 = z} * \nabla Q u(z_1 z_2 z_3 z_4) \cdot \frac{z_4 - z_2}{z_3} = 0,
\]
where the sum is over all the faces $z_1 z_2 z_3 z_4$ of the lattice $Q$ such that $z_1 = z$.

Proof. Let us prove the “only if” part. Assume that $v : Q^0 \to \mathbb{R}$ is conjugate to $u$. Then
\[
\sum_{z_1 z_2 z_3 z_4 : z_1 = z} * \nabla Q u \cdot \frac{z_4 - z_2}{z_3} = \sum_{z_1 z_2 z_3 z_4 : z_1 = z} \nabla Q v \cdot \frac{z_4 - z_2}{z_3} = \sum_{z_1 z_2 z_3 z_4 : z_1 = z} (v(z_4) - v(z_2)) = 0
\]
because the diagonals of type $z_2 z_4$ form a closed cycle around a nonboundary vertex $z$.

Let us prove the “if” part. Denote by $V(z_2 z_4) := * \nabla Q u \cdot \frac{z_4 - z_2}{z_3}$. Assume that for each bounded face $w_1 w_2 \ldots w_m$, $w_{m+1} := w_1$, of the graph $W$ we have $\sum_{1 \leq k \leq m} V(w_k w_{k+1}) = 0$. Then a function $v : W^0 \to \mathbb{R}$ is well-defined by the formula $v(w_m) := \sum_{1 \leq k \leq m} V(w_k w_{k+1})$, where $w_1 w_2 \ldots w_m$ is a path in the graph $W$ joining $w_m$ with a fixed vertex $w_1$. Define a function $v : B^0 \to \mathbb{R}$ analogously. Consider the combined function $v : Q^0 \to \mathbb{R}$. Then for each face $z_1 z_2 z_3 z_4$ of the lattice $Q$ we have $v(z_4) - v(z_2) = V(z_2 z_4) = * \nabla Q u \cdot \frac{z_4 - z_2}{z_3}$ and analogously $v(z_3) - v(z_1) = * \nabla Q u \cdot \frac{z_3 - z_1}{z_4}$. Thus $\nabla Q v = * \nabla Q u$, and $v$ is conjugate to $u$. □

Claim 3.3. For each vertex $z \in Q^0$ we have
\[
[\Delta_Q u](z) = \sum_{z_1 z_2 z_3 z_4 : z_1 = z} * \nabla Q u(z_1 z_2 z_3 z_4) \cdot \frac{z_4 - z_2}{z_3}.
\]

Proof. Let $v : Q^0 \to \mathbb{R}$ be equal to 1 at the vertex $z$ and 0 at all the other vertices. Differentiating the energy $E(u)$ and applying the identity $(c \cdot d)(a \cdot b) = (*c \cdot b)(a \cdot d) - (*c \cdot a)(b \cdot d)$ we get
\[
[\Delta_Q u](z) = \sum_{z_1 z_2 z_3 z_4 \subset Q} \nabla Q u \cdot \nabla Q v(-2 \text{Area}(z_1 z_2 z_3 z_4))
\]
\[
= \sum_{z_1 z_2 z_3 z_4 \subset Q} \nabla Q u \cdot \nabla Q v(*\frac{z_4 - z_2}{z_3})
\]
\[
= \sum_{z_1 z_2 z_3 z_4 \subset Q} ((*\nabla Q u \cdot \frac{z_4 - z_2}{z_3}),(\nabla Q v \cdot \frac{z_4 - z_2}{z_3}) - (*\nabla Q u \cdot \frac{z_4 - z_2}{z_3}),(\nabla Q v \cdot \frac{z_4 - z_2}{z_3}))
\]
\[
= \sum_{z_1 z_2 z_3 z_4 \subset Q} ((*\nabla Q u \cdot \frac{z_4 - z_2}{z_3}),(v(z_4) - v(z_2)) - (*\nabla Q u \cdot \frac{z_4 - z_2}{z_3}),(v(z_4) - v(z_2)))
\]
\[
= \sum_{z_1 z_2 z_3 z_4 : z_1 = z} * \nabla Q u \cdot \frac{z_4 - z_2}{z_3}.
\]

□

Proof of Variational Principle 2.2. By Claims 3.1–3.3 a function is discrete harmonic if and only if its laplacian vanishes at nonboundary vertices. By Convexity Principle 2.1 the latter is equivalent to having minimal energy among the functions with the same boundary values. □
3.3. Uniqueness Theorem

Proof of Uniqueness Theorem 1.1. By Convexity Principle 2.1 the energy \( E: \mathbb{R}^{Q^0-\partial Q} \to \mathbb{R} \) has a unique global minimum \( u \in \mathbb{R}^{Q^0-\partial Q} \). By Variational Principle 2.2 the function \( u \) is the solution of the Dirichlet problem and it is unique. \( \square \)

Remark 3.4. Define a discrete Riemann surface to be a cell decomposition \( Q \) of a surface with quadrilateral faces together with an identification of each face with a quadrilateral \( z_1z_2z_3z_4 \subset \mathbb{C} \) by an orientation preserving homeomorphism. (No agreement of such identifications for different faces is assumed.) The results of Sections 2.1 and 3 remain true for an arbitrary simply-connected discrete Riemann surface, not necessarily a quadrilateral lattice in the complex plane.

3.4. Maximum Principle

Let us discuss the case of orthogonal lattices in more detail. For a face \( z_1z_2z_3z_4 \) with the vertices listed clockwise denote by \( c(z_1z_3) := i\frac{z_2-z_4}{z_1-z_3} \). In the case of an orthogonal lattice we have \( c(z_1z_3) > 0 \), and the energy and the laplacian take the usual form

\[
E(u) = \sum_{z_1z_2z_3z_4} \left( c(z_1z_3) (u(z_3) - u(z_1))^2 + c(z_1z_3)^{-1} (u(z_4) - u(z_2))^2 \right) / 2;
\]

\[
[\Delta_Q u](z_1) = \sum_{z_3: z_1z_3 \subset B} c(z_1z_3) (u(z_3) - u(z_1)).
\]

Thus the value of a discrete harmonic function \( u \) at a nonboundary vertex of \( B \) equals to the weighted mean of the values at the neighbors. This immediately implies the following known result.

Maximum Principle 3.5. Let \( Q \) be an orthogonal lattice and let \( u: Q^0 \to \mathbb{R} \) be a discrete harmonic function. Then

\[
\max_{z \in Q^0} u(z) = \max_{w \in Q^0 \cap \partial Q} u(w) \quad \text{and} \quad \max_{z \in B^0} u(z) = \max_{w \in B^0 \cap \partial Q} u(w).
\]

For an orthogonal lattice Uniqueness Theorem 1.1 is an immediate corollary of Maximum Principle 3.5 and the finite-dimensional Fredholm alternative. However, the principle does not hold for nonorthogonal lattices in general.

Example 3.6. (S. Tikhomirov) Let \( M > 1 \) and let \( Q \) be the lattice formed by the quadrilateral with the vertices \( 0, \tan(\pi/8), M(\tan(\pi/8) + i), i, \) and the 3 other quadrilaterals obtained by symmetries with respect to the origin and the coordinate axes; see Figure 3. Define a discrete analytic function \( f: Q^0 \to \mathbb{C} \) by the following table:

| \( z \)         | \( 0 \)   | \( \pm i \) | \( \pm \tan \frac{\pi}{8} \) | \( \pm M(\tan \frac{\pi}{8} + i) \) | \( \pm M(\tan \frac{\pi}{8} - i) \) |
|------------------|-----------|-------------|-----------------------------|----------------------------------|----------------------------------|
| \( f(z) \)       | \( M\frac{1+i}{2} \) | 1           | 0                           | 0                                | \( Mi \)                          |

Set \( u := \text{Re } f \). Then \( \max_{Q^0} u / \max_{\partial Q} u = M/2 \) and thus can be arbitrarily large for large \( M \). If \( B \) is the graph formed by 4 diagonals from the origin then \( \max_{\partial B} u > 0 \) whereas \( \max_{B \cap \partial Q} u = 0 \).

Figure 3: A discrete analytic function on a nonorthogonal quadrilateral lattice not satisfying the maximum principle. The values of the function are shown near the vertices.
3.5. The Green Identity

Let us state one more result specific for orthogonal lattices, which is required for the sequel.

**Green Identity 3.7.** Let $Q$ be an orthogonal lattice and $u, v: B^0 \to \mathbb{C}$ be arbitrary functions. Then

$$
\sum_{z \in B^0} [u \Delta_Q v - v \Delta_Q u](z) = 0.
$$

**Proof.** For an orthogonal lattice the energy splits as $E(u) = E_B(u) + E_W(u)$, where $E_B(u)$ and $E_W(u)$ depend only on the values of the function $u$ at the vertices of $B$ and $W$, respectively. For an arbitrary homogeneous quadratic form $E_B(u)$ we have $\sum_{z \in B^0} \left( u(z) \frac{\partial E_B(v)}{\partial u(z)} - v(z) \frac{\partial E_B(u)}{\partial u(z)} \right) = 0$, which is equivalent to the required identity.

For nonorthogonal lattices the Green identity does not remain true unless one replaces summation over $B^0$ by summation over $Q^0$.

4. Convergence

In this section we prove Convergence Theorem 1.2 and the results stated in Section 2.2.

4.1. Geometric preliminaries

Let us start with some basic estimates involving the lattice eccentricity. For a subgraph $R \subset B$ denote by $E_R(u)$ the sum (2) over all the faces $z_1z_2z_3z_4$ of $Q$ containing an edge of $R$.

**Path Energy Claim 4.1.** For any path $w_0w_1 \ldots w_m \subset B$ we have

$$
E_{w_0 \ldots w_m}(u) \geq \frac{(u(w_m) - u(w_0))^2}{2me^2}.
$$

**Proof.** Estimating $|\nabla_Q u|$ through its projection, using condition (D) from Section 1.3, the inequality $\sin(1/e) \geq 1/2e$, and the Schwarz inequality we get

$$
E_{w_0 \ldots w_m}(u) = \sum |\nabla_Q u|^2 \text{Area}(z_1z_2z_3z_4)
\geq \sum_{k=1}^{m} \frac{(u(w_k) - u(w_{k-1}))^2}{|z_1z_3|^2} \cdot |z_1z_3| \cdot |z_2z_4| \cdot \sin \angle(z_1z_3, z_2z_4)
\geq \sum_{k=1}^{m} \frac{(u(w_k) - u(w_{k-1}))^2}{e} \cdot \sin \frac{1}{e}
\geq \frac{(u(w_m) - u(w_0))^2}{2me^2},
$$

where the first sum is over all faces $z_1z_2z_3z_4$ of $Q$ containing an edge of the path $w_0 \ldots w_m$. \qed

**Projection Claim 4.2.** For any face $z_1z_2z_3z_4$ of the lattice $Q$ and any vector $\vec{v}$ we have

$$
|\vec{v}| \leq 4e \left| \frac{\vec{v} \cdot \overline{z_1z_3}}{|z_1z_3|} + 4e \frac{\vec{v} \cdot \overline{z_2z_4}}{|z_2z_4|} \right|.
$$

**Proof.** Since $\angle(\overline{z_1z_3}, \overline{z_2z_4}) = \pm \angle(\overline{\vec{v}, z_1z_3}) \pm \angle(\overline{\vec{v}, z_2z_4})$ it follows that at least one of the angles in the right-hand side, say, the first one, does not belong to the interval $\left( \frac{\pi}{8} - \frac{1}{2e}, \frac{\pi}{8} + \frac{1}{2e} \right)$. Then

$$
|\vec{v}| \leq \sec \frac{1}{2e} \left| \frac{\vec{v} \cdot \overline{z_1z_3}}{|z_1z_3|} \right| \leq 4e \left| \frac{\vec{v} \cdot \overline{z_1z_3}}{|z_1z_3|} \right|.
$$

\qed

8
Rectangle Capacity Claim 4.3. A rectangle $r \times h$ with the side $r > h$ contains at most $4er/h$ vertices of the graph $B$.

Proof. The rectangle $r \times h$ can be covered by $4[r/h]$ discs of radius $h/2$. Then by condition (U) from Section 1.3 the number of vertices in the rectangle is less than $4e[r/h]$. □

Diameter Claim 4.4. The diameter of each bounded face of the graphs $B$ and $W$ is at most $h$.

Proof. A bounded face of the graph $B$ contains a vertex of the graph $W$. The vertex is joined by edges of the graph $Q$ with all the vertices of the face (and by “half-edges” of the graph $W$ with the break points of the 2-segment edges of $B$ in nonconvex faces of $Q$). Since the edges of $Q$ have length at most $h/2$ it follows that the diameter of the face of $B$ is at most $h$. □

4.2. Convergence of energy

For the proof of Energy Convergence Lemma 2.3 we need the following claim.

Gradient Approximation Claim 4.5. We have $|\nabla g - \nabla Q(g | Q_0)| \leq 8h \max_{z \in \text{Conv}(Q)} |D^2g(z)|$.

Proof. Consider a face $z_1z_2z_3z_4$ of the lattice $Q$. By the Rolle theorem there is a point $z \in z_1z_3$ (possibly outside $z_1z_2z_4$ but inside the convex hull $\text{Conv}(Q)$) such that $(\nabla g(z) - [\nabla Q(g)](z_1z_2z_4)) \cdot \frac{z_1z_3}{|z_1z_3|} = 0$. Thus $(\nabla g - \nabla Qg) \cdot \frac{z_1z_3}{|z_1z_3|} \leq h \max_{z \in z_1z_2z_4} |D^2g(z)|$ in the face $z_1z_2z_3z_4$. The same inequality holds with $z_1z_3$ replaced by $z_2z_4$. By Projection Claim 4.2 the claim follows. □

Proof of Energy Convergence Lemma 2.3. Denote by $\hat{Q}$ the domain enclosed by the curve $\partial Q$. Since $Q_n$ approximates $\Omega$ it follows that some neighborhood $\Omega'$ of $\Omega$ contains all the lattices $Q_n$, and $\text{Area}(\Omega' - \hat{Q}_n), \text{Area}(\hat{Q}_n - \Omega) \to 0$ as $n \to \infty$. Since the domain $\Omega$ is bounded and the function $g: \mathbb{C} \to \mathbb{R}$ is smooth it follows that $\nabla g$ is bounded in $\text{Conv}(\Omega')$. Thus the integrals $E_\Omega(g)$, $E_{\hat{Q}_n}(g)$ exist and $E_\Omega(g) - E_{\hat{Q}_n}(g) = E_{\Omega - \hat{Q}_n}(g) - E_{\hat{Q}_n - \Omega}(g) \to 0$ as $n \to \infty$. By Gradient Approximation Claim 4.5 we get $E_{\hat{Q}_n}(g) - E(g | Q'_n) \to 0$ as $n \to \infty$, and the lemma follows. □

Remark 4.6. For a harmonic function $v: \Omega \to \mathbb{R}$ and a sequence of functions $v_n: \Omega \to \mathbb{R}$ conditions $v_n = v$ on $\partial \Omega$ and $E_\Omega(v^n) \to E_\Omega(v)$ do not necessarily imply that $v_n \to v$ pointwise. For instance, take a continuous function $v_n: \mathbb{D}^2 \to \mathbb{R}$ such that $v_n(z) = 0$ for $|z| = 1$, $v_n(z) = 1$ for $|z| \leq 1/n$, and $v_n(z)$ is harmonic in the ring $1/n < |z| < 1$. Then $E_{\mathbb{D}^2}(v_n) = 1/\ln n \to 0$ as $n \to \infty$ but $v_n(0) = 1 \not\to 0 = v(0)$.

4.3. Equicontinuity

Proof of Equicontinuity Lemma 2.4 for square lattices. First assume that $B$ is a square lattice and the segment joining $z$ and $w$ is contained in the graph $B \cap K$; see Figure 4. Assume without loss of generality that $u(z) \geq u(w)$.

For now denote by $h$ the step of the square lattice $B$. Let $R_m$ be the boundary of the rectangle $2mh \times (2mh + |z - w|)$ centered at the point $(z + w)/2$ with the side $2mh$ orthogonal to the segment $zw$; see Figure 4.

![Figure 4: The rectangles $R_m$ and the points $z_m, w_m$ from the proof of Lemma 2.4.](image-url)
Take \( m < r/2h \). Then the distance from \( R_m \) to the segment \( zw \subset K \) is less than \( r/\sqrt{2} \). So \( R_m \) lies inside \( \partial Q \) and thus \( R_m \subset B \). Then by Maximum Principle 3.5 there are points \( z_m, w_m \in R_m \) such that \( u(z_m) \geq u(z) \) and \( u(w_m) \leq u(w) \). The points \( z_m \) and \( w_m \) are joined in the graph \( R_m \) by a path of length at most \( 4m + |z - w|/h \). Since the eccentricity of a square lattice is 1, by Path Energy Claim 4.1 we get the following estimate for the energy on the subgraph \( R_m \)

\[
E_{R_m}(u) \geq \frac{|u(z_m) - u(w_m)|^2}{8m + 2|z - w|/h} \geq \frac{|u(z) - u(w)|^2}{8mh + 2|z - w|} \cdot h.
\]

Summing these inequalities for \( m \) from 0 to \( [r/2h] \) and estimating the sum via an integral we get

\[
E(u) \geq \sum_{m=0}^{[r/2h]} E_{R_m}(u) \geq \frac{1}{8} |u(z) - u(w)|^2 \ln \left( 1 + \frac{r}{|z - w|} \right).
\]

This is equivalent to the required inequality.

For nonsquare lattices the proof is essentially the same but requires more technical details.

Proof of Equicontinuity Lemma 2.4 for convex compact sets. Let us prove the lemma in the case when the segment \( zw \subset K \).

Consider an auxiliary square lattice with edges parallel and orthogonal to \( zw \) and with step equal to the maximal edge length \( h \) of the lattice \( Q \). Define the rectangles \( R_m \) literally as above.

For each \( m = 1, 2, \ldots, [r/2h] \) the boundaries of the rectangles \( R_m \) and \( R_{m-1} \) are separated by a simple closed path \( R_m \subset B \). Indeed, otherwise \( R_m \) and \( R_{m-1} \) would be joined by a path in the complement to the graph \( B \) and thus the graph \( B \) would have a face of diameter \( > h \), which contradicts to Diameter Claim 4.4.

Number of edges in the path \( R_m \) is not greater than the number of vertices of the graph \( B \) lying in the strip between \( R_m \) and \( R_{m+1} \). Thus by Rectangle Capacity Claim 4.3 the path \( R_m \) contains less than \( 4\epsilon(m + |z - w|/h) \) edges. Now the same energy estimates as for the square lattice (with \( R_m \) replaced by the path \( R_m \)) prove the required inequality.

Proof of Lemma 2.4 in the general case. For each point of \( K \) take a disc of radius \( r/2 \) around the point. The interiors of these discs form an open covering of \( K \). Since \( K \) is compact it follows that the covering contains a finite subcovering. Let \( D_1, \ldots, D_N \) be the closures of the discs of the finite subcovering. Denote \( K' := \bigcup_{k=1}^N D_k \). Then \( K \subset K' \) and \( K' \) is inside \( \partial Q \).

Assume without loss of generality that \( \partial K' \) does not have “cusps”, i.e., the discs \( D_1, \ldots, D_N \) do not touch each other. Assume also that \( K' \) is connected. Denote by \( \text{Dist}(z, w) \) the distance between points \( z \) and \( w \) in \( K' \), i.e., infimum of lengths of all broken lines joining \( z \) with \( w \) lying in \( K' \).

Let us prove the asymptotic form \( \text{Dist}(z, w) \leq \text{Const}_{K,r} |z - w| \) for some number \( \text{Const}_{K,r} \) depending on \( K \) and \( r \). Indeed, denote by \( \epsilon \) the minimal distance between two disjoint discs among \( D_1, \ldots, D_N \). If \( |z - w| < \epsilon \) then the points \( z \) and \( w \) are contained in a pair of intersecting discs \( D_i \) and \( D_j \) (or possibly in the same disc). Since \( D_i \) and \( D_j \) do not touch each other the required asymptotic form follows. For \( |z - w| \geq \epsilon \) denote \( \phi(z, w) := \text{Dist}(z, w)/|z - w| \). Since \( \phi \) is a continuous function on a compact set \( \{ (z, w) \in K' \times K' : |z - w| \geq \epsilon \} \), the required asymptotic form follows again.

Now take two points \( z, w \in K' \). Join them by a broken line of length less than \( \text{Dist}(z, w) + |z - w| \). Since \( K' \) is a union of \( N \) discs we may assume that the broken line has at most \( N \) edges, and each edge is contained in one of the discs. Since \( \text{Dist}(z, w) = \text{Const}_{K,r}(|z - w|) \) it follows that we can subdivide the broken line to get a broken line \( z = z_1, z_2, \ldots, z_{M+1} = w \) with \( M \leq \text{Const}_{K,r} + 1 \) edges.
of length $\leq |z - w|$ each. Then

$$|u(z) - u(w)| \leq \sum_{k=1}^{M} |u(z_k) - u(z_{k+1})|$$

$$\leq \sum_{k=1}^{M} \text{Const}_e \cdot E(u)^{1/2} \ln^{-1/2}(1 + r|z_k - z_{k+1}|^{-1})$$

$$\leq \text{Const}_{K,r,e} \cdot E(u)^{1/2} \ln^{-1/2}(1 + r|z - w|^{-1}).$$

Here the first inequality is obvious. The second asymptotic form follows from the particular case of the lemma proved before because each edge $z_kz_{k+1}$ is contained in one of the discs $D_1, \ldots, D_N$. The third asymptotic form holds because the function $\ln^{-1/2}(1+r|z|^{-1})$ is increasing and $M \leq \text{Const}_{K,r}+1$. \hfill $\Box$

Remark 4.7. Our proof of Equicontinuity Lemma 2.4 cannot be generalized to nonorthogonal lattices because it essentially uses Maximum Principle 3.5 not holding for them.

4.4. The Friedrichs inequality

For the proof of Friedrichs Inequality Lemma 2.6 we need some notation and auxiliary claims. Let $abcd$ be a rectangle with vertices $a, b, c, d \in \mathbb{C}$ listed clockwise such that $ab$ is outside $\partial Q$ and $|b - c| > h$; see Figure 5. Denote $r := |b - c|$. For a point $z \in abcd$ denote by $Rz$ the rectangle $(r + 2h) \times 2h$ centered at $(z' + w')/2$ with the side $2h$ parallel to $ab$, where $z' \in cd$ and $w' \in ab$ are the points such that $z \in z'w'$ and $z'w' \parallel bc$.

![Figure 5: The rectangle abcd and the neighborhood Rz.](image)

**Claim 4.8.** A vertex $z \in B^0 \cap abcd$ can be joined with $\partial Q$ by a path in the graph $Rz \cap B$.

**Proof.** Assume that there is no path as required. Then the point $z$ is separated from $\partial Q$ by a path $P$ in the complement $Rz - B$. Since $ab$ is outside $\partial Q$ it follows that $z$ is separated from the point $w'$ by the path $P$ as well. The path $P$ is not closed because the graph $B$ is connected. Thus the endpoints of the path $P$ belong to $\partial Rz$. The path $P$ intersects the segment $zw'$ because $P$ separates $z$ from $w'$. Thus the face of the graph $B$ containing $P$ has diameter greater than $h$, a contradiction to Diameter Claim 4.4. \hfill $\Box$

**Claim 4.9.** We have $w \in Rz$ if and only if $z \in Rw$.

**Proof.** Straightforward. \hfill $\Box$

**Claim 4.10.** $h^2L_{abcd}^2(u) \leq 8erhL_{\partial Q}^2(u) + 64e^4r^2E(u)$.
Proof. Take a vertex \( z \in abcd \cap B^0 \). By Claim 4.8 it follows that the vertex \( z \) is joined with a vertex \( w \in \partial Q \) by a path in the graph \( R_z \cap B \). Take such path with minimal number \( m \) of vertices. Then \( m \) is not greater than the total number of vertices of the graph \( B \) in the rectangle \( R_z \). Hence \( m \leq 4er/h \) by Rectangle Capacity Claim 4.3. Thus by Path Energy Claim 4.1 we get

\[
u(z)^2 \leq 2u(w)^2 + 2(u(z) - u(w))^2 \leq 2L^2_{\partial Q \cap R_z}(u) + 16e^3rE_{B \cap R_z}(u)/h.
\]

Sum these inequalities over all \( z \in abcd \cap B^0 \). A vertex \( w \in \partial Q \) can contribute to \( L^2_{\partial Q \cap R_z}(u) \) only if \( w \in R_z \). Thus by Claim 4.9 and Rectangle Capacity Claim 4.3 it contributes at most \( 4er/h \) times. Similarly, an edge \( z''w'' \) of the graph \( B \) can contribute to \( E_{B \cap R_z}(u) \) only if \( z'' \in R_z \). Thus by Claim 4.9 and Rectangle Capacity Claim 4.3 it contributes at most \( 4er/h \) times. Thus our summation leads to the required inequality.

Proof of Friedrichs Inequality Lemma 2.6. Since \( \partial \Omega \) is a smooth curve it follows that its \( r \)-neighborhood can be covered by finitely many rectangles \( abcd \) such that \( ab \) is outside the \( r \)-neighborhood of \( \Omega \) and \( |b - c| = 3r \). Moreover, the number of rectangles is bounded by a number depending only on the domain \( \Omega \) (but not on \( K \)). These rectangles cover the strip \( \Omega - K \) as well. Since \( r > \max_{z \in \partial Q} \text{Dist}(z, \partial \Omega) \) and \( r > h \) it follows that for each rectangle the side \( ab \) is outside \( \partial Q \) and \( |b - c| > h \). Summing the inequalities of Claim 4.10 for each of the rectangles, we get the required inequality.

Remark 4.11. Our proof of Lemma 2.6 does not generalize to domains with nonsmooth boundaries, e.g., to a domain bounded by a cardioid.

4.5. Approximation of laplacian

First we prove Laplacian Approximation Lemma 2.5 in several particular cases and then combine them together. Throughout this subsection \( z \) denotes the coordinate in the complex plane \( \mathbb{C} \), so that, e.g., \( \text{Re } z \) denotes the function \( z \mapsto \text{Re } z \).

Proof of Lemma 2.5 for \( g(z) = \text{Re } z \). Clearly, the function \( f(z) = z \) is discrete analytic. Thus \( g(z) = \text{Re } z \) is discrete harmonic. Then by Variational Principle 2.2 it follows that \( [\Delta_Q \text{Re } z](w) = 0 \) for each vertex \( w \in Q^0 - \partial Q \). On the other hand, \( \Delta \text{Re } z = 0 \) as well, and the lemma follows.

Proof of Lemma 2.5 for \( g(z) = \text{Im } z \). This is proved by the previous argument with \( iz \) instead of \( z \).

Proof of Lemma 2.5 for \( g(z) = |z|^2 \). For a face \( z_1z_2z_3z_4 \) of the lattice \( Q \) denote by \( z' \) the intersection point of the middle perpendiculars to the diagonals \( z_1z_3 \) and \( z_2z_4 \). Clearly, then \( \nabla_Q |z-z'|^2(z_1z_2z_3z_4) = 0 \). Using the two previous cases of the lemma and Claims 3.1, 3.3 we get

\[
[\Delta_Q |z|^2](w) = [\Delta_Q |z-w|^2](w) + [\Delta_Q (|w|^2 - 2 \text{Re } w\bar{z})](w)
\]

\[
= \sum_{z_1z_2z_3z_4 : z_1 = w} (\ast \nabla_Q |z-w|^2) \cdot \overline{z_2}
\]

\[
= \sum_{z_1z_2z_3z_4 : z_1 = w} \left( 2 \ast \nabla_Q \text{Re}((z-w)z) + \ast \nabla_Q |z-z'|^2 \right) \cdot \overline{z_2}
\]

\[
= \sum_{z_1z_2z_3z_4 : z_1 = w} 2\nabla_Q \text{Im}(\overline{(z-w)z}) \cdot \overline{z_2}
\]

\[
= \sum_{z_1z_2z_3z_4 : z_1 = w} 2 \text{Im} \left( \frac{(z-w)}{z-z_4} \right)
\]

\[
= \sum_{z_1z_2z_3z_4 : z_1 = w} 4 \text{Area}(wz_2z_3z_4).
\]

Here and in the next paragraph the area of a closed broken line is understood in oriented sense.
Denote by $R_Q$ the closed broken line formed by the edges $z_2z'$ and $z'z_4$ for all the faces $z_1z_2z_3z_4$ of $Q$ such that $z_1 \in R$ and $z_3 \not\in R$. By Projection Claim 4.2 it follows that $z'z_1 < eh$, thus $R_Q$ is contained in the $eh$-neighborhood of the curve $\partial R$. Summing up the above expressions over all the vertices $w \in B^0 \cap R$ we get

$$\left| \sum_{w \in B^0 \cap R} [\Delta_Q |z|^2](w) - \int_R \Delta |z|^2 dA \right| = 4\text{Area}(R_Q) - 4\text{Area}(R) \leq 16ehr.$$ 

For the next case of the lemma we need the following two claims.

**Claim 4.12.** For an arbitrary function $u : W^0 \to \mathbb{C}$ we have $\sum_{z_1z_2z_3z_4} u(z_2) - u(z_4) = 0$, where the sum is over all the faces $z_1z_2z_3z_4$ such that $z_1 \in B^0 \cap R$, $z_3 \not\in R$.

**Proof.** Take an arbitrary vertex $z_2 \in W^0 - \partial Q$. It is contained in a face $w_1 \ldots w_m$ of the graph $B$. Assume that the vertices of the face are listed in clockwise order. Let us move along the path $w_1 \ldots w_m$. Each time we cross the boundary $\partial R$ the value $u(z_2)$ appears in the considered sum with certain sign. The sign is positive, if we move from the exterior of $R$ to the interior at the moment, and negative — otherwise. Since both $\partial R$ and $w_1 \ldots w_m$ are closed curves it follows that the vertex $z_2$ appears in the considered sum equal number of times with positive sign and with negative sign, and the claim follows.

**Claim 4.13.** The number of faces $z_1z_2z_3z_4$ such that $z_1 \in B^0 \cap R$, $z_3 \not\in R$ is at most $96er/h$.

**Proof.** By Diameter Claim 4.4 it follows that the vertices of such faces are contained in the $h$-neighborhood of $\partial R$. By Rectangle Capacity Claim 4.3 the number of these vertices is at most $32er/h$. Then by the Euler formula for planar graphs it follows that the number of faces is at most $96er/h$.

**Proof of Lemma 2.5 for $g(z) = \text{Re } z^2$.** Analogously to the previous case of the lemma we get

$$[\Delta_Q \text{Re } z^2](w) = \sum_{z_1z_2z_3z_4 : z_1 = w} 2 \text{Im}((z' - w)(z_4 - z_2)).$$

Summing up this expressions over all $w \in B^0 \cap R$, canceling repeating terms and applying Claim 4.12 we get

$$\sum_{w \in R^0 \setminus B^0} [\Delta_Q \text{Re } z^2](w) = \sum_{z_1z_2z_3z_4} 2 \text{Im}(z'(z_4 - z_2)) = \sum_{z_1z_2z_3z_4} \text{Im}((2z' - z_2 - z_4)(z_4 - z_2)).$$

Here the first sum is over all the vertices $z_1 \in B^0 \cap R$ and the other sums are over the faces $z_1z_2z_3z_4$ of $Q$ such that $z_1 \in R$ and $z_3 \not\in R$. By Projection Claim 4.2 it follows that $|z' - z_2|, |z' - z_4| < eh$. Thus by Claim 4.13 we get

$$\left| \sum_{z_1 \in R^0 \setminus B^0} [\Delta_Q \text{Re } z^2](z_1) - \int_R \Delta \text{Re } z^2 dA \right| \leq \sum_{z_1z_2z_3z_4} |2z' - z_2 - z_4| \cdot |z_4 - z_2| \leq 96er/h \cdot 2eh \cdot h \leq 192e^2 hr.$$ 

**Proof of Lemma 2.5 for $g(z) = \text{Im } z^2$.** This is analogous to the previous case.
Proof of Lemma 2.5 in the case when \( D^k g = 0 \) at the center of \( R \) for \( k = 0, 1, 2 \). Using the estimate \( |\Delta g(z)| \leq 4r \max_{z \in R} |D^3 g(z)| \) we get

\[
\left| \int_R \Delta g \, dA \right| \leq 4r^3 \max_{z \in R} |D^3 g(z)|.
\]

Now applying Claim 3.3, canceling repeating terms, applying Claim 4.13, Gradient Approximation Claim 4.5, and the estimate \( |\nabla g(z)| \leq r^2 \max_{z \in R} |D^3 g(z)| \) we get

\[
\left| \sum_{z_1 \in R \cap B^0} |\Delta Qg|(z_1) \right| = \left| \sum_{z_1 \in R \cap B^0} \sum_{z_1 = z_2 \neq z_3 : z_1 = z} \star \nabla Qg \cdot z_4 z_2 \right|
\leq \left| \sum_{z_1 \in R \cap B^0} \star \nabla Qg \cdot z_4 z_2 \right|
\leq 96er/h \cdot (r^2 + 4ehr) \max_{z \in R} |D^3 g(z)| \cdot h
\leq 480e^3 r^3 \max_{z \in R} |D^3 g(z)|.
\]

\[\square\]

4.6. Uniform limit

The following lemma is the last result we need before the proof of Convergence Theorem 1.2. We write \( x_n \sim y_n \) and \( x_n \succeq y_n \), if \( \lim(x_n - y_n) = 0 \) and \( \lim(x_n - y_n) \geq 0 \), respectively.

Lemma 4.14. Suppose that under the assumptions of Convergence Theorem 1.2 the sequence \( u_{Q_n \ast} \big|_{B_n^0} \) converges to a continuous function \( u : \Omega \to \mathbb{R} \) uniformly on each compact subset of \( \Omega \). Then \( u = u_{\Omega, \ast} \).

To prove the lemma, first let us establish harmonicity of the limit function.

Claim 4.15. Let \( \{Q_n\} \) be a nondegenerate uniform sequence of orthogonal lattices approximating a domain \( \Omega \). Let \( u_n : Q_n^0 \to \mathbb{R} \) be a sequence of discrete harmonic functions. Suppose that the restrictions \( u_n \big|_{B_n^0} \) converge to a continuous function \( u : \Omega \to \mathbb{R} \) uniformly on each compact subset of \( \Omega \); then the function \( u \) is harmonic.

Proof of Claim 4.15. Take an arbitrary smooth function \( v : \Omega \to \mathbb{R} \) vanishing outside a compact subset \( K \subset \Omega \). By the Weyl lemma it suffices to prove that \( \int_\Omega u \Delta v \, dA = 0 \). Assume without loss of generality that \( \partial K \) is smooth and \( v = 0 \) also in a neighborhood of \( \partial K \).

Let us estimate the difference between \( \int_\Omega u \Delta v \, dA \) and its discrete counterpart. For each \( n \) take an auxiliary infinite square lattice with edge length \( r := \sqrt{h} \). For a face \( R \) of the \( n \)-th auxiliary lattice denote \( \tilde{u}_n(R) := \max_{z \in R \cap K} u(z) \). Then \( \tilde{u}_n \Rightarrow u \) on the compact set \( K \) because \( u \) is continuous on this set. Applying the convergence \( u_n, \tilde{u}_n \Rightarrow u \), the boundness of \( \Delta v, \Delta Q_n v \), and then Laplacian Approximation Lemma 2.5 we get

\[
\left| \int_\Omega u \Delta v \, dA - \sum_{z \in B_n^0} [u_n \Delta Q_n v](z) \right| \sim \sum_{R : R \cap K \neq \emptyset} |\tilde{u}_n(R)| \cdot \left| \int_R \Delta v \, dA - \sum_{z \in R \cap B_n^0} [\Delta Q_n v](z) \right| \leq Area(K) / r^2 \cdot \max_K |u| \cdot \text{Const}_e \cdot (rh + r^3) \max_K |D^2 v, D^3 v| \leq \text{Const}_{e, K, u, v} \cdot \sqrt{h} \to 0 \quad \text{as} \quad n \to \infty.
\]
It remains to estimate the discrete counterpart of \( \int_{\Omega} u \Delta v \, dA \). Take \( n \) large enough so that \( K \) is inside \( \partial Q_n \). Applying Green Identity 3.7 and the assumptions that \( u_n \) is discrete harmonic and \( v \) vanishes outside \( K \), we get

\[
\sum_{z \in B_n^0} [u_n \Delta Q_n v](z) = \sum_{z \in B_n^0} [v \Delta Q_n u_n](z) = 0.
\]

Thus \( \int_{\Omega} u \Delta v \, dA = 0 \), which proves the claim.

**Proof of Lemma 4.14.** By Claim 4.15 the function \( u \) is harmonic. It remains to verify the boundary condition.

Denote by \( S(r) \subset \Omega \) the set of points with distance to the boundary \( \partial Q \) less than \( r \). First let us prove the estimate \( h^2 L^2_{S(r)}(u_{Q_n,g} - g) \leq \text{Const}_{e,\Omega,g} r^2 \) for \( r > h \) and some number \( \text{Const}_{e,\Omega,g} \) not depending on \( r \). Apply Friedrichs Inequality Lemma 2.6 for the function \( u_{Q_n,g} - g \). The energy \( E(u_{Q_n,g} - g) \leq 2E(u_{Q_n,g}) + 2E(g|\partial Q_n) \) is bounded by Variational Principle 2.2 and Energy Convergence Lemma 2.3. We have \( L^2_{\partial Q_n}(u_{Q_n,g} - g) = 0 \) because \( u_{Q_n,g} = g \) at \( \partial Q_n \), and the required estimate follows.

Now by Diameter Claim 4.4 it follows that for each \( \rho < r \)

\[
\int_{S(r) - S(\rho)} (u - g)^2 \, dA \leq h^2 L^2_{S(r) - S(\rho)}(u_{Q_n,g} - g) \leq h^2 L^2_{S(r)}(u_{Q_n,g} - g) \leq \text{Const}_{e,\Omega,g} r^2.
\]

Approaching \( \rho \to 0 \) we get \( \frac{1}{r} \int_{S(r)} (u - g)^2 \, dA \to 0 \) as \( r \to 0 \). By [11, §4.1] this condition implies the boundary condition.

4.7. Convergence Theorem

**Proof of Theorem 1.2.** Take an arbitrary subsequence \( Q_{nk} \) of the given sequence of lattices \( Q_n \). For brevity denote \( Q_k := Q_{nk}, B_k := B_{nk} \). Take a sequence of compact sets \( K_1 \subset K_2 \subset \cdots \subset \Omega \) such that \( \Omega = \bigcup_{j=1}^{\infty} K_j \).

Let us estimate \( |u_{Q_k,g}| \). Since the sequence \( Q_k \) approximates the domain \( \Omega \) it follows that there is a disk \( \Omega' \) containing all lattices \( Q_k \). By Maximum Principle 3.5 we have

\[
\max_{z \in B_r^0} |u_{Q_k,g}(z)| = \max_{w \in B_r^0 \cap \partial Q_k} |u_{Q_k,g}(w)| \leq \max_{w \in \partial \Omega'} |g(w)| < \infty
\]

because \( u_{Q_k,g} = g \) at \( \partial Q_k \) and \( g: \mathbb{C} \to \mathbb{R} \) is continuous. So the sequence \( u_{Q_k,g} \) is uniformly bounded.

Let us estimate the right-hand side of inequality (3) from Equicontinuity Lemma 2.4 for \( K := K_1 \) and \( u := u_{Q_k,g} \). Since the sequence \( Q_k \) approximates the domain \( \Omega \) it follows that there is a number \( k_1 \) such that for each \( k > k_1 \) the set \( K_1 \) is contained inside \( \partial Q_k \) and \( \text{Dist}(K_1, \partial Q_k) \) is bounded from zero. By Variational Principle 2.2 and Energy Convergence Lemma 2.3 we have \( E(u_{Q_k,g}) \leq E(g|Q_k) \to E|\Omega(g) < \infty \) for \( k \to \infty \). Thus the sequence \( E(u_{Q_k,g}) \) is bounded.

Then by Equicontinuity Lemma 2.4 it follows that \( u_{Q_k,g} \big|_{B_k^0} \) is equicontinuous, i. e., there is a positive function \( \delta(\epsilon) \) not depending on \( k \) and such that for each \( z, w \in K \cap B_k^0 \) with \( |z - w| < \delta(\epsilon) \) and \( k > k_1 \) we have \( |u_{Q_k,g}(z) - u_{Q_k,g}(w)| < \epsilon \). By the Arzelà-Ascoli theorem it follows that there is a function \( u_1: K_1 \to \mathbb{C} \) and a subsequence \( \{l_k\} \) of the sequence \( 1, 2, \ldots \) such that \( k_1 \) is as before and \( u_{Q_l,g} \) converges to \( u_1 \) uniformly in \( K_1 \).

Now proceed to the next compact set \( K_2 \). Analogously, there is a function \( u_2: K_2 \to \mathbb{C} \) and a subsequence \( \{m_k\} \) of the sequence \( \{l_k\} \) such that \( m_1 = l_1, m_2 = l_2, \) and \( u_{Q_{m_k},g} \) converges to \( u_2 \) uniformly on \( K_2 \). Clearly, \( u_1 = u_2 \) on \( K_1 \). Thus the extension can be continued, and eventually we get a function \( u: \Omega \to \mathbb{C} \) and a subsequence \( \{p_k\} \) of the sequence \( 1, 2, \ldots \) such that \( u_{Q_{p_k},g} \) converges to \( u \) uniformly on each compact subset of \( \Omega \).

By Lemma 4.14 the function \( u \) is the solution \( u_{\Omega,g} \) of the Dirichlet problem in \( \Omega \). Since the solution \( u_{\Omega,g} \) is unique it follows that the initial sequence \( u_{Q_{n,g}}: B_0^n \to \mathbb{R} \) converges to \( u_{\Omega,g} \) uniformly on each compact subset of \( \Omega \). Analogously, \( u_{Q_{n,g}}: W^n \to \mathbb{R} \) converges to \( u_{\Omega,g} \) uniformly on compact sets. This completes the proof of main results.

\[ \square \]
5. Applications and open problems

5.1. Application to numerical analysis

Convergence Theorem 1.2 provides a new approximation algorithm for numerical solution of the Dirichlet boundary value problem. It also gives a new convergence result for the classical finite element method, which we are going to state now.

Let $\hat{B}$ be a triangulation of a polygon $\hat{B}$. The finite element method approximates the solution of the Dirichlet problem on $\Omega$ by a continuous function $u_{B,g}: \hat{B} \rightarrow \mathbb{R}$ which is linear on each face of $B$, equal to the given function $g$ on the boundary $B^0 \cap \partial \hat{B}$, and has minimal energy $E_B(u) = \int_{\hat{B}} |\nabla u|^2 dA$ (among such functions). Equivalently, the restriction $u_{B,g}: B^0 \rightarrow \mathbb{R}$ can be defined as the unique function equal to $g$ on $B^0 \cap \partial \hat{B}$ and such that for each $z_1 \in B^0 - \partial \hat{B}$ we have

$$\sum_{z_3} c(z_1z_3)(u_{B,g}(z_1) - u_{B,g}(z_3)) = 0,$$

where the sum is over all neighbors $z_3$ of $z_1$, and $c(z_1z_3)$ is given by the formula

$$c(z_1z_3) := (\cot \alpha + \cot \beta)/2,$$

where $\alpha$ and $\beta$ are the angles opposite to the edge $z_1z_3$ in the two triangles of $B$ sharing the edge; see Figure 6. The function $u_{B,g}: B^0 \rightarrow \mathbb{R}$ is called the solution of the Dirichlet problem on $B$.

![Figure 6: A kite lattice $Q$ associated to a Delauney triangulation $B$.](image)

Usually one proves convergence of the finite element method under certain assumptions on individual triangles [7]. For instance, it was proved in [9, Theorem 3.3.7] that $u_{B_n,g}$ converges uniformly to $u_{B,g}$, if there is a constant Const such that

(A) the minimal angle of each triangle is greater than $1/\text{Const}$;
(R) the ratio of any two edges of each triangulation is less than $\text{Const}$.

According to [7] no uniform convergence results without assumptions (A) and (R) were available.

Following [5] we suggest a new approach measuring “triangulation quality” via configuration of neighboring triangles rather than the shape of individual ones. A triangulation $B$ is called Delauney, if $\alpha + \beta < \pi$ for each pair of adjacent triangular faces (and thus $c(z_1z_3) > 0$ above). A Delauney triangulation exists for any prescribed set of vertices $B^0$ [5]. A sequence of triangulations is nondegenerate uniform, if there is a constant Const such that for each member of the sequence

(D) for each edge the sum of opposite angles in the (two or one) triangles containing the edge is less than $\pi - 1/\text{Const}$ (in particular, the triangulation is Delauney);
(U) the number of vertices in an arbitrary disk of radius equal to the maximal edge length is less than $\text{Const}$.

Assumption (U) is weaker than (R); neither (D) nor (A) is weaker than the other one. We prove convergence of the finite element method for triangulations satisfying (D) and (U):
Corollary 5.1. Let Ω ⊂ C be a domain bounded by a smooth closed curve ∂Ω without self-intersections and let u: C → R be a smooth function. Let {B_n} be a nondegenerate uniform sequence of triangulations approximating the domain Ω. Then the solution u_{B_n,g}: B_n^0 → R of the Dirichlet problem on B_n converges to the solution u_{Ω,g}: Ω → R of the Dirichlet problem on Ω uniformly on each compact subset of Ω.

Proof. In each triangle of B_n, draw 3 segments joining the circumcenter with the vertices. Erase hanging edges from the obtained graph. We get an orthogonal quadrilateral lattice Q_n. Formula (4) gives the same values c(z_1z_3) as in Section 3.4; see Figure 6. Thus u_{B_n,g}: B_n^0 → R is the restriction of a discrete harmonic function Q_n^0 → R. By Uniqueness Theorem 1.1 we have u_{B_n,g}(z) = u_{Q_n,g}(z) for each z ∈ B_n^0. Since {B_n} is a nondegenerate uniform sequence of triangulations approximating the domain Ω, the same is true for {Q_n}. By Convergence Theorem 1.2 the corollary follows.

Vice verse, the finite element method can be applied to establish convergence of discrete harmonic functions on quadrilateral lattices. Using standard finite element described above one can approach only rhombic lattices (and also kite ones at the cost of establishing convergence only at the vertices of the graph B but not W).

The following nonconforming finite element might be useful in the case of general quadrilateral lattices. Given a function u: Q^0 → R define its interpolation I_Q u: z_1z_2z_3z_4 → R to be the linear function on a face z_1z_2z_3z_4 of Q such that [I_Q u](z_1) = u(z_1), [I_Q u](z_3) = u(z_3), and [I_Q u](z_2) − [I_Q u](z_4) = u(z_2) − u(z_4). Combining such linear functions together we get a (discontinuous) function I_Q u: Q → R on the union Q of all the quadrilateral faces of Q. Clearly, then E(u) = E_Q(I_Q u).

Remark 5.2. In the case when all the bounded faces of Q are convex one can prove that E(u) = E_Q(I_B u, I_W u), where I_B u, I_W u: Q → R are piecewise linear extensions of the function u: Q^0 → R to the faces of the graphs B ∪ Q and W ∪ Q, respectively.

Problem 5.3. Give an effective approximation algorithm for finding the solution of the Dirichlet problem on a (nonorthogonal) quadrilateral lattice.

Problem 5.4. Estimate the rate of convergence in Theorem 1.2.

5.2. Physical interpretation

Classical physical interpretation of complex analysis on orthogonal lattices uses direct-current networks [13] (for elementary introduction to networks see [26, 27, 24]). Let us give a new physical interpretation for arbitrary quadrilateral lattices involving alternating-current networks. This gives some insight and also interesting in itself.

Define admittance of an edge z_1z_3 ⊂ B by the formula

\[ c(z_1z_3) := \frac{z_2 - z_4}{z_1 - z_3}, \]

where z_1z_2z_3z_4 is the face containing z_1z_3 with the vertices listed clockwise. Clearly, this number has positive real part (and in case of an orthogonal lattice it is simply a positive number).

A graph B with edge admittances having positive real parts can be considered as an alternating-current network; see [24, §2.4] and [16].

Given a discrete analytic function f: Q^0 → C, define the voltage drop V(z_1z_3) and the current I(z_1z_3) on an oriented edge z_1z_3 ⊂ B by the formula

\[ V(z_1z_3) := f(z_1) - f(z_3), \quad I(z_1z_3) := if(z_2) - if(z_4), \]

where z_1z_2z_3z_4 is the face of the lattice Q with the vertices listed clockwise. Boundary voltage drops are the differences f(z_1) − f(z_3) for all pairs of consecutive boundary vertices z_1, z_3 ∈ B^0 ∩ ∂Q. Boundary currents (or incoming currents) are the values if(z_4) − if(z_2) for all the pairs of consecutive boundary vertices z_2, z_4 ∈ W^0 ∩ ∂Q. The voltage drop at a moment t is the number Re(V(z_1z_3) exp(it)); the current at the moment t is defined analogously.

A reformulation of Uniqueness Theorem 1.1 is the following result.
**Corollary 5.5.** Boundary voltage drops at the initial moment and boundary currents after one quarter of the period uniquely determine all the voltage drops and currents in an alternating-current network at all the moments of time.

**Problem 5.6.** What is the physical meaning of the boundary condition, which is the Dirichlet one at the initial moment and the Neuman one after one quarter of the period?

The physical meaning of Convergence Theorem 1.2 is that the voltage in a distributed direct-current network can be approximated by voltages in lumped direct-current networks.

This physical interpretation gives also one more motivation for the definition of energy from Section 2.1. The energy of the network (dissipated per period) is

\[ E(f) := \text{Re} \sum_{z_1z_3} V(z_1z_3)\bar{I}(z_1z_3), \quad (6) \]

where the sum is over all the edges \( z_1z_3 \subset B \). Expressing the energy through \( u = \text{Re} f \) and the admittances \( c(z_1z_3) \), we arrive at the definition from in Section 2.1.

5.3. **Probabilistic interpretation**

Let \( Q \) be an orthogonal lattice and let \( B \) be one of the graphs obtained by joining the opposite points in each quadrilateral face of \( Q \). A random walk on the vertices of \( B \) is defined as follows. At each moment of time the walker moves from his current position to one of the neighboring vertices with the probability proportional to the weights of the corresponding edges given by formula (5).

The results of the present paper allow to generalize many estimates from [8] to nonrhombic lattices. Let us sketch two particular problems; see [8, 20] for accurate definitions. The probability that a random walk starting at a vertex \( z \in B^0 \) first exits the domain \( \Omega \) through an arc \( A \subset \partial \Omega \) is called the discrete harmonic measure \( \omega_{A,\Omega}(z) \). A trajectory of a loop-erased random walk is obtained from a trajectory of a random walk by deleting loops in chronological order.

**Problem 5.7.** Prove that for a nondegenerate uniform sequence of orthogonal lattices \( \{Q_n\} \) approximating the domain \( \Omega \) (so that \( \Omega \) is inside each curve \( \partial Q_n \)) the discrete harmonic measure \( \omega_{A,\Omega,Q_n}(z) \) converges uniformly to its continuous counterpart.

**Problem 5.8.** Prove that the trajectories of loop-erased random walks on orthogonal lattice converge to SLE curves in the scaling limit.

5.4. **Generalizations**

**Problem 5.9.** Generalize Theorem 1.2 to:

1. nonorthogonal quadrilateral lattices;
2. nonuniform sequences, i.e., not satisfying condition (U) from Section 1.3 (for adaptive meshes);
3. singular boundary values (for convergence of discrete harmonic measure, the Green function, the Cauchy and the Poisson kernels, abelian integrals);
4. domains with rough boundaries (for probabilistic applications);
5. other types of boundary conditions;
6. other Riemannian surfaces;
7. other elliptic PDE.

**Problem 5.10.** Prove that under the assumptions of Convergence Theorem 1.2 the gradient \( \nabla Q u_{Q_n,g} \) converges to \( \nabla u_{\Omega,g} \) uniformly on each compact subset of \( \Omega \).

**Problem 5.11.** Construct a sequence of quadrilateral lattices approximating a planar domain such that the solutions of the Dirichlet problem on the lattices do not converge uniformly to the solution of the Dirichlet problem in the domain.

**Problem 5.12.** For which nonorthogonal lattices \( Q \) Maximum Principle 3.5 remains true?
Acknowledgements

The author is grateful to A. Bobenko, D. Chelkak, C. Mercat, S. Tikhomirov, and A. Ustinov for useful discussions. This work has been presented at the seminars of A. Bobenko, Ya. Sinai, S. Smirnov.

[1] M. Alexa, M. Wardetzky, Discrete Laplacians on general polygonal meshes, ACM Trans. Graph. 30:4 (2011), 102:1–102:10.

[2] A. I. Bobenko, C. Mercat, and Y. B. Suris, Linear and nonlinear theories of discrete analytic functions. Integrable structure and isomonodromic Green’s function, J. Reine Angew. Math. 583 (2005), 117–161.

[3] A. I. Bobenko, C. Mercat, and M. Schmies, Period matrices of polyhedral surfaces. In: Computational Approach to Riemann Surfaces, A. I. Bobenko, C. Klein (Eds), Lect. Notes Math. (2013), to appear; http://arxiv.org/abs/0909.1305

[4] A. I. Bobenko, U. Pinkall, and B. A. Springborn, Discrete conformal maps and ideal hyperbolic polyhedra, preprint (2010); http://arxiv.org/pdf/1005.2698v1.

[5] A. I. Bobenko and B. A. Springborn, A discrete Laplace–Beltrami operator for simplicial surfaces, Discrete Comput. Geom. 38 (2007), 740–756.

[6] D. Braess, Finite elements. Theory, fast solvers, and applications in elasticity theory, transl. by L. L. Schumaker, Cambridge Univ. Press, 2007.

[7] J. Brandts, A. Hannukainen, S. Korotov, M. Krížek, On angle conditions in the finite element method, preprint (2011).

[8] D. Chelkak and S. Smirnov, Discrete complex analysis on isoradial graphs, Adv. Math., to appear, http://arxiv.org/abs/0810.2188v2.

[9] P. G. Ciarlet, The finite element method for elliptic problems, North-Holland, Amsterdam, 1978, 530 p.

[10] P. G. Ciarlet and P.-A. Raviart, Maximum principle and uniform convergence for the finite element method, Computer Methods Appl. Mech. Engin. 2 (1973), 17–31.

[11] R. Courant, K. Friedrichs, H. Lewy, Über die partiellen Differenzengleichungen der mathematischen Physik, Math. Ann., 100, (1928), 32–74. English transl.: IBM Journal (1967), 215–234. Russian transl.: Russ. Math. Surveys 8 (1941), 125–160. http://www.stanford.edu/class/cme324/classics/courant-friedrichs-lewy.pdf

[12] R. J. Duffin, Distributed and lumped networks, J. Math. Mech. 8:5 (1959), 793–826.

[13] R. J. Duffin, Potential theory on a Rhombic Lattice, J. Combin. Theory 5 (1968), 258–272.

[14] I. A. Dynnikov and S. P. Novikov, Geometry of the triangle equation on two-manifolds, Moscow Math. J., 3 (2003), 419–438.

[15] J. Ferrand, Fonctions préharmoniques et fonctions préholomorphes, Bull. Sci. Math. 68 (1944), 152–180.

[16] R. M. Foster, Academic and Theoretical Aspects of Circuit Theory, Proc. IRE 50:5 (1962), 866–871.

[17] K. Hildebrandt, K. Polthier, M. Wardetzky, On the convergence of metric and geometric properties of polyhedral surfaces, Geom. Dedicata 123 (2006), 89–112.
[18] R. Ph. Isaacs, A finite difference function theory, Univ. Nac. Tucumán. Revista A. 2 (1941), 177–201.

[19] R. Kenyon, The Laplacian and Dirac operators on critical planar graphs, Invent. Math. 150:2 (2002), 409–439.

[20] G. F. Lawler, O. Schramm, W. Werner, Conformal invariance of planar loop-erased random walks and uniform spanning trees, Ann. Probab. 32:1B (2004), 939–995.

[21] L. Lovasz, Discrete analytic functions: an exposition, In: Surv. Differ. Geom., IX, 241–273, Int. Press, Somerville, MA, 2004.

[22] C. Mercat, Discrete Riemann surfaces and the Ising model, Comm. Math. Phys. 218:1 (2001), 177–216.

[23] C. Mercat, Discrete complex structure on surfel surfaces. In: Discrete Geometry for Computer Imagery, Coeurjolly et al. (Eds.), Lect. Notes Computer Sc. 4992 (2008), 153–164.

[24] M. Prasolov and M. Skopenkov, Tilings by rectangles and alternating current, J. Combin. Theory A 118:3 (2011), 920–937, http://arxiv.org/abs/1002.1356.

[25] L. Saloff-Coste, Some inequalities for superharmonic functions on graphs, Potential Analysis 6 (1997) 163–181.

[26] M. Skopenkov, M. Prasolov, S. Dorichenko, Dissections of a metal rectangle, Kvant 3 (2011), 10–16 (in Russian) http://arxiv.org/abs/1011.3180.

[27] M. Skopenkov, V. Smykalov, and A. Ustinov, Random walks and electric networks, in preparation (2011).

[28] S. Smirnov, Discrete Complex Analysis and Probability, Proc. Intern. Cong. Math. Hyderabad, India, 2010, http://arxiv.org/abs/1009.6077.

[29] M. Wardetzky, S. Mathur, F. Käberer, E. Grinspun, Discrete Laplace operators: no free lunch, Eurographics Symp. Geom. Processing, A. Belyaev, M. Garland (eds.), 2007.