DIRICHLET SERIES ASSOCIATED TO SUM-OF-DIGITS FUNCTIONS

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ABSTRACT. We study the Dirichlet series $F_b(s) = \sum_{n=1}^\infty d_b(n)n^{-s}$, where $d_b(n)$ is the sum of the base-$b$ digits of the integer $n$, and $G_b(s) = \sum_{n=1}^\infty S_b(n)n^{-s}$, where $S_b(n) = \sum_{m=1}^{n-1} d_b(m)$ is the summatory function of $d_b(n)$. We show that $F_b(s)$ and $G_b(s)$ have continuations to the plane $C$ as meromorphic functions of order at least 2, determine the locations of all poles, and give explicit formulas for the residues at the poles.

1. INTRODUCTION

There has been a great deal of study of properties of the radix expansions to an integer base $b \geq 2$ of integers $n$. For each integer base $b \geq 2$, every positive integer $n$ has a unique base-$b$ expansion

$$ n = \sum_{i \geq 0} \delta_{b,i}(n)b^i $$

with digits $\delta_{b,i} \in \{0, 1, \ldots, b-1\}$ given by

$$ \delta_{b,i} = \left\lfloor \frac{n}{b^i} \right\rfloor - b \left\lfloor \frac{n}{b^{i+1}} \right\rfloor. $$

This paper considers two summatory functions of base $b$ digits of $n$:

1. The base-$b$ sum-of-digits function $d_b(n)$ is

$$ d_b(n) = \sum_{i \geq 0} \delta_{b,i}(n). $$

2. The (base $b$) cumulative sum-of-digits function $S_b(n)$ is

$$ S_b(n) = \sum_{m=1}^{n-1} d_b(m). $$

We follow here the convention of previous authors (including [7] and [9]), with the sum defining $S_b(n)$ running to $n-1$ instead of $n$. We associate to the functions $d_b(n)$ and $S_b(n)$ the Dirichlet series generating functions

$$ F_b(s) = \sum_{n=1}^\infty \frac{d_b(n)}{n^s} $$

and

$$ G_b(s) = \sum_{n=1}^\infty \frac{S_b(n)}{n^s}. $$
These Dirichlet series have abscissa of convergence $\text{Re}(s) = 1$ and $\text{Re}(s) = 2$, respectively.

This paper studies the problem of the meromorphic continuation to $\mathbb{C}$ of Dirichlet series associated to the base-$b$ digit sums $d_b(n)$ and $S_b(n)$. Here we obtain the meromorphic continuation and determine its exact pole and residue structure. The pole structure contains half of a two-dimensional lattice and the residues involve Bernoulli numbers and values of the Riemann zeta function on the line $\text{Re}(s) = 0$. A meromorphic continuation of these functions was previously obtained in the thesis of Dumas [8] by a different method, which specified a half-lattice containing all the poles but did not determine the residues; in fact infinitely many of the residues on his possible pole set vanish.

The asymptotics of $S_b(n)$ have been extensively studied, see Section 1.2. We mention particularly work of Delange [7], given below as Theorem 1.6, which gives an exact formula of the Riemann zeta function on the line $\text{Re}(s) = 0$. The asymptotics of these functions was previously obtained in the thesis of Dumas [8] by a different method, which specified a half-lattice containing all the poles but did not determine the residues; in fact infinitely many of the residues on his possible pole set vanish.

1.1. Results. Our first results concern the meromorphic continuation of the functions $F_b(s)$ and $G_b(s)$ to the entire complex plane $\mathbb{C}$.

**Theorem 1.1.** For each integer base $b \geq 2$, the function $F_b(s) = \sum_{n=1}^{\infty} d_b(n) n^{-s}$ has a meromorphic continuation to $\mathbb{C}$. The poles of $F_b(s)$ consist of a double pole at $s = 1$ with Laurent expansion beginning

$$F_b(s) = \frac{b - 1}{2 \log b} (s - 1)^{-2} + \left( \frac{b - 1}{2 \log b} \log(2\pi) - \frac{b + 1}{4} \right) (s - 1)^{-1} + O(1),$$

simple poles at each other point $s = 1 + 2\pi im/\log b$ with $m \in \mathbb{Z} (m \neq 0)$ with residue

$$\text{Res} \left( F_b(s), s = 1 + \frac{2\pi im}{\log b} \right) = -\frac{b - 1}{2\pi im} \left( \frac{2\pi im}{\log b} \right),$$

and simple poles at each point $s = 1 - k + 2\pi im/\log b$ with $k = 1$ or $k \geq 2$ an even integer and with $m \in \mathbb{Z}$, with residue

$$\text{Res} \left( F_b(s), s = 1 - k + \frac{2\pi im}{\log b} \right) = (-1)^{k+1} \frac{b - 1}{\log b} \zeta \left( \frac{2\pi im}{\log b} \right) \frac{B_k}{k!} \prod_{j=1}^{k-1} \left( \frac{2\pi im}{\log b} - j \right),$$

where $B_k$ is the $k$th Bernoulli number.

Theorem 1.1 is proved by first considering the Dirichlet series $\sum (d_b(n) - d_b(n-1)) n^{-s}$ and then exploiting a relation between power series and Dirichlet series to recover $F_b(s)$. The proof is presented in Section 1.2.
Theorem 1.2. For each integer \( b \geq 2 \), the function \( G_b(s) = \sum_{n=1}^{\infty} S_b(n)n^{-s} \) has a meromorphic continuation to \( \mathbb{C} \). The poles of \( G_b(s) \) consist of a double pole at \( s = 2 \) with Laurent expansion

\[
G_b(s) = \frac{b - 1}{2 \log b} (s - 2)^{-2} + \left( \frac{b - 1}{2 \log b} (\log(2\pi) - 1) - \frac{b + 1}{4} \right) (s - 2)^{-1} + O(1), \tag{1.10}
\]

a simple pole at \( s = 1 \) with residue

\[
\text{Res}(G_b(s), s = 1) = \frac{b + 1}{12}. \tag{1.11}
\]

simple poles at \( s = 2 + 2\pi m/\log b \) with \( m \in \mathbb{Z} \) \((m \neq 0)\) with residue

\[
\text{Res}\left(G_b(s), s = 2 + \frac{2\pi im}{\log b}\right) = -\frac{b - 1}{2\pi im} \left( 1 + \frac{2\pi im}{\log b} \right)^{-1} \zeta\left( \frac{2\pi im}{\log b} \right), \tag{1.12}
\]

and simple poles at point \( s = 2 - k + 2\pi im/\log b \) with \( k \geq 2 \) an even integer and \( m \in \mathbb{Z} \) with residue

\[
\text{Res}\left(G_b(s), s = 2 - k + \frac{2\pi im}{\log b}\right) = \frac{b - 1}{\log b} \zeta\left( \frac{2\pi im}{\log b} \right) \left( \frac{B_k}{k(k - 2)!} \right) \prod_{j=1}^{k-2} \left( \frac{2\pi im}{\log b} - j \right). \tag{1.13}
\]

An interesting feature of the above theorems is the abundance of poles. Since each function \( F_b(s) \) and \( G_b(s) \) has \( r^2 \) poles in the disk \(|s| < r\), we have the following corollary, which we discuss further in Section 3.5.

Corollary 1.3. The functions \( F_b(s) \) and \( G_b(s) \) are meromorphic functions of order at least \( 2 \) on \( \mathbb{C} \).

The Riemann zeta function, the Dirichlet L-functions, and the Dirichlet series generating functions of many important arithmetic functions (such as the M"obius function \( \mu(n) \), the von Mangoldt function \( \Lambda(n) \), the Euler totient function \( \phi(n) \), and the sum-of-divisors functions \( \sigma_a(n) \)) analytically continue as meromorphic functions of order 1 on the complex plane. The Dirichlet series \( F_b(s) \) and \( G_b(s) \) thus have a different analytic character than many other Dirichlet series considered in number theory.

In Section 4 we use a formula of Delange [7] for \( S_b(n) \) to define continuous real-valued interpolations of the functions \( d_b(n) \) and \( S_b(n) \) from integer bases \( b \geq 2 \) to a real parameter \( \beta > 1 \). As before, we associate to these interpolated sum-of-digits functions the Dirichlet series

\[
F_{\beta}(s) = \sum_{n=1}^{\infty} \frac{d_{\beta}(n)}{n^s}, \tag{1.14}
\]

and

\[
G_{\beta}(s) = \sum_{n=1}^{\infty} \frac{S_{\beta}(n)}{n^s}. \tag{1.15}
\]

We prove that these Dirichlet series each have a meromorphic continuation one unit to the left of their halfplane of absolute convergence. For the function \( F_{\beta}(s) \) we have the following theorem.

Theorem 1.4. For each real \( \beta > 1 \), the function \( F_{\beta}(s) \) has a meromorphic continuation to the halfplane \( \text{Re}(s) > 0 \), with a double pole at \( s = 1 \) with Laurent expansion

\[
F_{\beta}(s) = \frac{\beta - 1}{2 \log \beta} (s - 1)^{-2} + \left( \frac{\beta - 1}{2 \log \beta} (\log(2\pi)) - \frac{\beta + 1}{4} \right) (s - 1)^{-1} + O(1) \tag{1.16}
\]
Theorem 1.5. For each real \( \beta > 1 \), the function \( G_\beta(s) \) is meromorphic in the region \( \Re(s) > 1 \) with a double pole at \( s = 2 \) with Laurent expansion

\[
G_\beta(s) = \frac{\beta - 1}{2 \log \beta} (s - 2)^{-2} + \left( \frac{\beta - 1}{2 \log \beta} (\log(2\pi) - 1) - \frac{\beta + 1}{4} \right)(s - 2)^{-1} + O(1)
\]

and simple poles at \( s = 2 + 2\pi m/\log \beta \) for \( m \in \mathbb{Z} \) with \( m \neq 0 \) with residue

\[
\text{Res} \left( G_\beta(s), s = 2 + \frac{2\pi im}{\log \beta} \right) = -\frac{\beta - 1}{2 \pi im} \zeta \left( \frac{2\pi im}{\log \beta} \right).
\]

To prove these theorems, we start by obtaining the continuation of the series \( G_\beta(s) \) by working directly with its Dirichlet series and then obtain the continuation of \( F_\beta(s) \) by studying the relation between these two Dirichlet series.

1.2. Previous work. There has been much previous work studying the functions \( d_b(n) \) and \( S_b(n) \). The function \( d_b(n) \) exhibits significant fluctuations as \( n \) changes to \( n + 1 \). It can only increase slowly, having \( d_b(n + 1) \leq d_b(n) + 1 \) but it can decrease by an arbitrarily large amount. The sequence \( d_b(n) \) is a \( b \)-regular sequence in the sense of Allouche and Shallit [4].

In 1968 Trollope [28] expressed the error term for the base-2 cumulative digit sum \( S_2(n) \) in terms of a continuous everywhere nondifferentiable function, the Takagi function—see [15] for a survey of the properties of this function. In 1975 Delange [7] proved the following formula for \( S_b(n) \), expressing the error term as a Fourier series with coefficients involving values of the Riemann zeta function on the imaginary axis.

Theorem 1.6 (Delange [7]). The cumulative sum-of-digits function \( S_b(n) \) satisfies

\[
S_b(n) = \frac{b - 1}{2 \log b} n \log n + O(n).
\]

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Theorem 1.6 (Delange [7]). The cumulative sum-of-digits function \( S_b(n) \) satisfies

\[
S_b(n) = \frac{b - 1}{2 \log b} n \log n + h_b \left( \frac{\log n}{\log b} \right) n
\]

where \( h_b \) is a nowhere-differentiable function of period 1. The function \( h_b \) has a Fourier series

\[
h_b(x) = \sum_{k=-\infty}^{\infty} c_b(k) e^{2\pi ikx}
\]

with coefficients

\[
c_b(k) = \frac{b - 1}{2\pi ik} \left( 1 + \frac{2\pi ik}{\log b} \right)^{-1} \zeta \left( \frac{2\pi ik}{\log b} \right)
\]

for \( k \neq 0 \) and

\[
c_b(0) = \frac{b - 1}{2 \log b} (\log(2\pi) - 1) - \frac{b + 1}{4}.
\]
A complex-analytic proof of a summation formula for general \( q \)-additive functions, of which the base-\( q \) sum-of-digits function is an example, was given by Mauclaire and Murata in 1983 [16, 17, 21]. A shorter complex-analytic proof of (1.21) in the specific case of \( S_2(n) \) was given by Flajolet, Grabner, Kirschenhofer, Prodinger, and Tichy in 1994 [9]. The method of Flajolet et al. is based on applying a variant of Perron’s formula to the Dirichlet series

\[
\sum_{n=1}^{\infty} \left( d_2(n) - d_2(n-1) \right) n^{-s}.
\]

Grabner and Hwang [10] study higher moments of the sum-of-digits function by similar complex-analytic methods.

Our formulas for the residues of \( F_b(s) \) and \( G_b(s) \) involve the Bernoulli numbers. Kellner [13] and Kellner and Sondow [14] investigate another relation between sums of digits and Bernoulli numbers, proving that the least common multiple of the denominators of the coefficients of the polynomial \( \sum_{i=0}^{n} n^k \), which can be written in terms of a Bernoulli polynomial, can be expressed as a certain product of primes satisfying \( d_p(n+1) \geq p \).

2. Sum-of-digits Dirichlet series

First we consider basic properties of the Dirichlet series

\[
F_b(s) = \sum_{n=1}^{\infty} \frac{d_b(n)}{n^s} \tag{2.1}
\]

attached to the base-\( b \) digit sum of \( n \) and the Dirichlet series

\[
G_b(s) = \sum_{n=1}^{\infty} \frac{S_b(n)}{n^s} \tag{2.2}
\]

attached to the cumulative base-\( b \) digit sum. For standard references on the basic analytic properties of Dirichlet series, see the books of Hardy and Riesz [12] or Titchmarsh [26, Ch. IX].

Recall that each ordinary Dirichlet series \( \sum a_n n^{-s} \) has an abscissa of conditional convergence \( \sigma_c \) such that the Dirichlet series converges and defines a holomorphic function if \( \text{Re}(s) > \sigma_c \) and diverges if \( \text{Re}(s) < \sigma_c \). Each Dirichlet series also has an abscissa of absolute convergence \( \sigma_a \) such that the Dirichlet series converges absolutely if \( \text{Re}(s) > \sigma_a \) and does not converge absolutely if \( \text{Re}(s) < \sigma_a \). For ordinary Dirichlet series, one always has \( \sigma_a - 1 \leq \sigma_c \leq \sigma_a \), and \( \sigma_c = \sigma_a \) if the coefficients \( a_n \) are nonnegative reals.

**Proposition 2.1.** For each integer \( b \geq 2 \), the Dirichlet series

\[
F_b(s) = \sum_{n=1}^{\infty} \frac{d_b(n)}{n^s} \tag{2.3}
\]

converges and defines a holomorphic function for \( \text{Re}(s) > 1 \).

*Proof.* A positive integer \( n \) has \( \lfloor \log n / \log b \rfloor + 1 \) digits when written in base \( b \), each of which is at most \( b - 1 \), so

\[
d_b(n) \leq (b - 1) \left( \left\lfloor \frac{\log n}{\log b} \right\rfloor + 1 \right). \tag{2.4}
\]

We then obtain the estimate

\[
S_b(n) \ll n \log n \tag{2.5}
\]
with an implied constant depending on \( b \). This implies that the Dirichlet series (2.3) has abscissa of absolute convergence at most 1 and therefore defines a holomorphic function for \( \Re(s) > 1 \).

**Proposition 2.2.** For each integer \( b \geq 2 \), the Dirichlet series

\[
G_b(s) = \sum_{n=1}^{\infty} \frac{S_b(n)}{n^s}
\]  

(2.6)

converges and defines a holomorphic function for \( \Re(s) > 2 \).

**Proof.** The estimate (2.5) gives

\[
\sum_{m=1}^{n} |S_b(m)| \ll n^2 \log n,
\]  

(2.7)

which shows that the Dirichlet series (2.6) converges for \( \Re(s) > 2 \). \( \Box \)

It follows from Delange’s formula (1.21) that \( F_b(s) \) and \( G_b(s) \) have abscissa of absolute convergence \( \Re(s) = 1 \) and \( \Re(s) = 2 \), respectively, and this can be proven directly using a more careful estimate of the function \( d_b(n) \). We can also obtain the exact values of the abscissas of convergence as a corollary of our theorems on the meromorphic continuation of \( F_b(s) \) and \( G_b(s) \), since \( F_b(s) \) has a pole at \( s = 1 \) and \( G_b(s) \) has a pole at \( s = 2 \).

As in previous work on Dirichlet series associated to \( q \)-additive sequences, it is advantageous to consider the Dirichlet series

\[
Z_b(s) = \sum_{n=1}^{\infty} \left( d_b(n) - d_b(n-1) \right) n^{-s}
\]  

(2.8)

obtained by differencing the coefficients of the series \( F_b(s) \), setting \( d_b(0) = 0 \). Identity (2.9) in the following proposition appears in a more general form (for \( q \)-additive functions) in the work of Mauclaire and Murata [16, 17, 21] and is stated and proved explicitly for the sum-of-digits series by Allouche and Shallit [3]. We give a more direct proof of this result.

**Proposition 2.3.** For each integer \( b \geq 2 \), the Dirichlet series \( Z_b(s) \) has abscissa of absolute convergence \( \sigma_a = 1 \), abscissa of conditional convergence \( \sigma_c = 0 \), and has a meromorphic continuation to \( \mathbb{C} \), satisfying

\[
Z_b(s) = \frac{b^s - b}{b^s - 1} \zeta(s).
\]  

(2.9)

**Proof.** For bases \( b \geq 3 \), we have \( |d_b(n) - d_b(n-1)| \geq 1 \) for all \( n \); if \( b = 2 \), we have \( |d_b(n) - d_b(n-1)| \geq 1 \) for at least all odd \( n \). Hence \( \sigma_a \geq 1 \). We also have \( d_b(n) - d_b(n-1) \ll \log n \), so \( \sigma_a \leq 1 \). The abscissa of conditional convergence \( \sigma_c = 0 \) follows from the bound

\[
\sum_{m \leq n} (d_b(m) - d_b(m-1)) = d_b(n) \ll \log n.
\]  

(2.10)

The effect of adding 1 on the digit sum in base-\( b \) arithmetic depends on the divisibility of \( n \) by \( b \); in particular, we have

\[
d_b(n) - d_b(n-1) = 1 - k(b-1)
\]  

(2.11)

where \( k \) is the largest integer such that \( b^k \mid n \). We may also express this as

\[
d_b(n) - d_b(n-1) = \sum_{m \mid n} \alpha(m)\beta(m/n)
\]  

(2.12)
Writing the right side as a product of two Dirichlet series, we have

\[
\beta(n) = \begin{cases} 
1 & \text{if } b \mid n \\
0 & \text{if } b \nmid n .
\end{cases}
\] (2.13)

Then we have, for Re(s) > 1,

\[
Z_b(s) = \sum_{n=1}^{\infty} \left( d_b(n) - d_b(n-1) \right) n^{-s} = \sum_{n=1}^{\infty} \left( \sum_{m \mid n} \alpha(m) \beta(m/n) \right) n^{-s} ;
\] (2.14)

Writing the right side as a product of two Dirichlet series, we have

\[
Z_b(s) = \sum_{n=1}^{\infty} \alpha(n)n^{-s} \sum_{n=1}^{\infty} \beta(n)n^{-s} = \sum_{n=1}^{\infty} b^{-ns} \sum_{n=1}^{\infty} (1 - b)(bn)^{-s} .
\] (2.15)

Summing the geometric series, we obtain

\[
Z_b(s) = \frac{1}{1 - b^{-s}} \left( \zeta(s) - b\theta^{-s} \zeta(s) \right) = \frac{b^s - b}{b^s - 1} \zeta(s)
\] (2.16)
as claimed. Equation (2.9) then provides a meromorphic continuation of \( Z_b(s) \) since the right side is meromorphic on \( \mathbb{C} \). \( \square \)

We will obtain information about the meromorphic continuation of \( F_b(s) \) and \( G_b(s) \) by considering the relation between these series and the series \( Z_b(s) \). For future use, we list the poles of the function \( Z_b(s) \).

**Lemma 2.4.** The function \( Z_b(s) \) is meromorphic on \( \mathbb{C} \), with simple poles at \( s = 2\pi im / \log b \) for \( m \in \mathbb{Z} \). The residue at each pole is

\[
\text{Res} \left( Z_b(s), s = \frac{2\pi im}{\log b} \right) = -b - 1 \frac{2\pi im}{\log b} \zeta \left( \frac{2\pi im}{\log b} \right) .
\] (2.17)

In particular, at \( s = 0 \), the function \( Z_b(s) \) has a Laurent expansion beginning

\[
Z_b(s) = \left( \frac{b - 1}{2 \log b} \right) s^{-1} + \left( -\frac{b + 1}{4} + \frac{b - 1}{2 \log b} \log(2n) \right) + O(s).
\] (2.18)

**Proof.** The function \( (b^s - b)/(b^s - 1) \) has simple poles at \( s = 2\pi im / \log b \) for each \( m \in \mathbb{Z} \), with residue

\[
\text{Res} \left( \frac{b^s - b}{b^s - 1}, s = \frac{2\pi im}{\log b} \right) = -b - 1 \frac{2\pi im}{\log b} .
\] (2.19)

The Laurent expansion at \( s = 0 \) follows from multiplying the expansions

\[
\frac{b^s - b}{b^s - 1} = -\frac{b - 1}{\log b} s^{-1} + \frac{b + 1}{2} + O(s)
\] (2.20)

and

\[
\zeta(s) = -\frac{1}{2} - \frac{1}{2} \log(2\pi)s + O(s^2).
\] (2.21)

The function \( \zeta(s) \) has only a simple pole at \( s = 1 \), cancelled by a zero of \( (b^s - b) \). \( \square \)

### 3. Meromorphic continuation of \( F_b(s) \) and \( G_b(s) \)

In this section, we show that for integers \( b \geq 2 \), the Dirichlet series \( F_b(s) \) and \( G_b(s) \) have a meromorphic continuation to \( \mathbb{C} \) and determine the structure of the poles, proving Theorems 1.1 and 1.2.
3.1. Bernoulli numbers. Our formulas for the meromorphic continuation of \( F_b(s) \) and \( G_b(s) \) involve Bernoulli numbers. For standard facts about the Bernoulli numbers and their basic properties, see [1, Ch. 23]. For a thorough reference on Bernoulli numbers, their history, and their relation to zeta functions, see [5].

The Bernoulli numbers \( B_k \) are the sequence of rational numbers defined by the generating function
\[
\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} \frac{B_k}{k!} x^k. \tag{3.1}
\]
If \( x \) is a complex variable, this series converges for \(|x| < 2\pi\).

Note that there are several competing notations for the Bernoulli numbers; with our definition, we have \( B_0 = 1 \), \( B_1 = -1/2 \), and \( B_2 = 1/6 \). Because the function
\[
\frac{x}{e^x - 1} + \frac{1}{2} x \tag{3.2}
\]
is an even function, we find that \( B_{2k+1} = 0 \) for all \( k \geq 1 \).

3.2. Power series and Dirichlet series. To prove the meromorphic continuation of \( F_b(s) \) and \( G_b(s) \), we make use of the following classical relation between Dirichlet series and power series.

**Proposition 3.1.** Let \( \sigma_c \) be the abscissa of conditional convergence of the Dirichlet series \( \sum_{n=1}^{\infty} a_n n^{-s} \). Then
\[
\Gamma(s) \sum_{n=1}^{\infty} a_n n^{-s} = \int_0^{\infty} \left( \sum_{n=1}^{\infty} a_n e^{-nx} \right) x^{s-1} \, dx \tag{3.3}
\]
for \( \text{Re}(s) > \max(\sigma_c, 0) \).

**Proof.** See [19, eq. 5.23]. \( \square \)

Proposition 3.1 allows us to translate the additive relations between the arithmetic functions \( d_b(n) - d_b(n-1) \), \( d_b(n) \), and \( S_b(n) \), which are easily expressed in terms of power series generating functions, into relations between their associated Dirichlet series.

3.3. Meromorphic continuation of \( F_b(s) \). We now prove the meromorphic continuation of the Dirichlet series \( F_b(s) \) by combining the relation between the Dirichlet series and power series generating functions of \( d_b(n) \) with the relation between the power series generating functions of \( d_b(n) \) and \( d_b(n) - d_b(n-1) \).

**Proof of Theorem 1.1** Let
\[
p(x) = \sum_{n=1}^{\infty} \{d_b(n) - d_b(n-1)\} x^n. \tag{3.4}
\]
We note that
\[
\sum_{n=1}^{\infty} d_b(n) x^n = \frac{p(x)}{1 - x} \tag{3.5}
\]
Then by Proposition 3.1, we have
\[
\Gamma(s) F_b(s) = \int_0^{\infty} \frac{1}{1 - e^{-x}} p(e^{-x}) x^{s-1} \, dx \tag{3.6}
\]
for $\text{Re}(s) > 1$. The series expansion

$$\frac{x}{1 - e^{-x}} = \sum_{k=0}^{\infty} \frac{(-1)^k B_k}{k!} x^k,$$

(3.7)

which follows from (3.1), holds for $|x| < 2\pi$. Since

$$\Gamma(s)Z_b(s) = \int_0^{\infty} p(e^{-x})x^{s-1} \, dx,$$

(3.8)

for $\text{Re}(s) > 0$, we can write

$$F_b(s) = \sum_{k=0}^{K} (-1)^k \frac{B_k}{k!} \frac{\Gamma(s)}{\Gamma(s-1)} Z_b(s-k) + R_K(s)$$

(3.9)

with

$$R_K(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \left( \frac{x}{1 - e^{-x}} - \sum_{k=0}^{K} \frac{(-1)^k B_k}{k!} x^k \right) p(e^{-x})x^{s-2} \, dx.$$

(3.10)

Note that

$$\frac{\Gamma(s-1)}{\Gamma(s)} = \frac{1}{s-1}$$

(3.11)

and

$$\frac{\Gamma(s-1+k)}{\Gamma(s)} = (s+1) \cdots (s+k-2).$$

(3.12)

Since

$$\frac{x}{1 - e^{-x}} = \sum_{k=0}^{K} \frac{(-1)^k B_k}{k!} x^k \ll x^{K+1}$$

(3.13)

as $x \to 0^+$, the integral in (3.10) converges and defines a holomorphic function in the region $\text{Re}(s) > 1 - K$. From Lemma 2.4 we know that $Z_b(s)$ has simple poles at $s = 2\pi ik / \log b$ for $k \in \mathbb{Z}$. The $k = 0$ term

$$\frac{1}{s-1} Z_b(s-1)$$

(3.14)

has a double pole at $s = 1$ with Laurent expansion beginning

$$\frac{1}{s-1} Z_b(s-1) = \frac{b-1}{2 \log b} (s-1)^{-2} + \left( \frac{b-1}{2 \log b} \log(2\pi) - \frac{b+1}{4} \right) (s-1)^{-1} + O(1),$$

(3.15)

and simple poles at each other point $s = 1 + 2\pi im / \log b$. Each term

$$(-1)^k \frac{B_k}{k!} \prod_{j=0}^{K-2} (s+j) \cdot Z_b(s-k)$$

(3.16)

with $k = 1$ or with $k$ an even integer with $k \geq 2$ has a simple pole at $s = 1 - k + 2\pi im / \log b$ for $m \in \mathbb{Z}$ with residue

$$(-1)^k \frac{B_k}{k!} \prod_{j=1}^{K-1} \frac{2\pi im}{\log b} - j \cdot \left( \frac{b-1}{\log b} \right)^2 \left( \frac{2\pi im}{\log b} \right).$$

(3.17)

Since $K$ can be taken arbitrarily large, this proves the theorem. $\square$
3.4. **Meromorphic continuation of** $G_b(s)$. We continue the function $G_b(s)$ to the plane in a similar fashion, using the fact that $S_b(n)$ is a double sum of the differences $d_b(n) - d_b(n - 1)$ appearing in the series $Z_b(n)$.

**Proof of Theorem 1.2**. Define $p(x)$ by (3.4). We make use of the identity of power series
\[
\frac{x}{(1 - x)^2} p(x) = \sum_{n=1}^{\infty} S_b(n) x^n. \tag{3.18}
\]
Then by Proposition 3.1, we have
\[
\Gamma(s) G_b(s) = \int_0^\infty \frac{e^{-x}}{(1 - e^{-x})^2} \Gamma(x) x^{s-1} \, dx \tag{3.19}
\]
for Re$(s) > 2$. From (3.1) and noting that
\[
\frac{e^{-x}}{(1 - e^{-x})^2} = -\frac{d}{dx} \left( \frac{1}{e^x - 1} \right), \tag{3.20}
\]
we have the power series expansion
\[
\frac{x^2 e^x}{(e^x - 1)^2} = 1 - \sum_{k=2}^{\infty} \frac{B_k}{k(k-2)!} x^k. \tag{3.21}
\]
Then for a fixed integer $K \geq 2$ can write $G_b(s)$ as
\[
G_b(s) = \frac{\Gamma(s - 2)}{\Gamma(s)} Z_b(s - 2) - \sum_{k=2}^{K} \frac{B_k}{k(k-2)!} \frac{\Gamma(s - 2 + k)}{\Gamma(s)} Z_b(s - 2 + k) + R_K(s) \tag{3.22}
\]
with remainder $R_K$ given by
\[
R_K(s) = \frac{1}{\Gamma(s)} \int_0^\infty \left( \frac{x^2 e^{-x}}{(1 - e^{-x})^2} - 1 + \sum_{k=2}^{K} \frac{B_k}{k(k-2)!} x^k \right) p(e^{-x}) x^{s-3} \, dx. \tag{3.23}
\]
The function $R_K(s)$ is holomorphic for Re$(s) > 2 - K$.

As before, we consider the poles of each term of (3.22). The first term
\[
\frac{1}{(s-1)(s-2)} Z_b(s-2) \tag{3.24}
\]
has a double pole at $s = 2$ with Laurent expansion
\[
\frac{b - 1}{2 \log b} (s-2)^{-2} + \left( \frac{b - 1}{2 \log b} (\log(2\pi) - 1) - \frac{b + 1}{4} \right) (s-2)^{-1} + \cdots. \tag{3.25}
\]
a simple pole at each point $s = 2 + 2\pi im/\log b$ with $m \neq 0$, and a simple pole at $s = 1$ with residue $(b+1)/12$. Each other term
\[
\frac{B_k}{k(k-2)!} \prod_{j=0}^{k-3} (s+j) \cdot Z_b(s-2+k). \tag{3.26}
\]
has simple poles at $s = 2 - k + 2\pi im/\log b$ for all $m$. \hfill \Box

**Remark.** Instead of relating $G_b(s)$ to the series $Z_b(s)$ as we did in this proof, we could have also used the relation
\[
\Gamma(s) G_b(s) = \int_0^\infty \frac{e^{-x}}{1 - e^{-x}} \left( \sum_{n=1}^{\infty} d_b(n) e^{-xn} \right) x^{s-1} \, dx,
\]
following the proof of Theorem 1.1 to write $G_b(s)$ in terms of $F_b(s)$.  

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**Corey Everlove**
3.5. **Order of** \( F_b(s) \) **and** \( G_b(s) \) **as meromorphic functions.** The functions \( F_b(s) \) and \( G_b(s) \) are meromorphic functions on the complex plane with infinitely many poles on a left half-lattice. We now raise a further question about the analytic properties of these functions. Recall that the **order** of an entire function \( f(z) \) on \( \mathbb{C} \) is

\[
\rho = \inf \{ \rho \geq 0 \mid f(z) = O(\exp(|z|^\rho)) \text{ as } |z| \to \infty \}.
\]

An entire function is of **finite order** if \( \rho < \infty \).

The order of a meromorphic function is defined as the order of growth of its associated Nevanlinna characteristic function. This definition is equivalent (by [24] Lemma 15.6 and Theorem on p. 91) to the following.

**Definition 3.2.** The **order** of a meromorphic function \( f(z) \) is the infimum of \( \rho \geq 0 \) such that \( f \) can be written as \( f(z) = g(z)/h(z) \) for entire functions \( g(z) \) and \( h(z) \) of order \( \rho \).

Many of the common Dirichlet series of analytic number theory are meromorphic (or entire) functions of order 1. Examples include the Riemann zeta function \( \zeta(s) \), the Dirichlet \( L \)-functions \( L(s, \chi) \), and their relatives; more generally, all Dirichlet series in the Selberg class (as introduced in Selberg [25]) are meromorphic functions of order 1.

By Proposition 3.3 \((\text{[26], p. 284})\), the function \( Z_b(s) \) is meromorphic of order 1. The functions \( F_b(s) \) and \( G_b(s) \), however, must have greater order by the following fact.

**Proposition 3.3** \((\text{[26], p. 284})\). Let \( f(z) \) be a meromorphic function of order \( \rho \) and let \( n(r, a) \) be the number of zeros of \( f(z) - a \) in the disc \( |z| < r \). Then \( n(r, a) = O(r^{\rho+\varepsilon}) \).

By Theorems 1.1 and 1.2, the functions \( F_b(s) \) and \( G_b(s) \) each have \( \gg r^2 \) poles in the disc \( |z| < r \). Hence we have the following corollary.

**Corollary 3.4.** The functions \( F_b(s) \) and \( G_b(s) \) are meromorphic functions of order at least 2.

A meromorphic function of order greater than 2 could still have only \( O(r^2) \) poles in \( |z| < r \), so without further information, we cannot deduce that \( F_b(s) \) and \( G_b(s) \) have order 2.

**Question 1.** Are the functions \( F_b(s) \) and \( G_b(s) \) meromorphic functions of order exactly 2?

Such a question has been answered positively in the related setting of strongly \( q \)-multiplicative functions: the Dirichlet series attached to such functions are entire of order exactly 2 (see Alkauskas [2]).

### 4. **Meromorphic Continuation of Dirichlet Series for Non-Integer Bases**

In this section, we consider the problem of extending the digit sums \( d_b(n) \) and \( S_b(n) \) from integer bases \( b \) to real parameters \( \beta > 1 \). There are a number of possible ways to do this. One natural approach concerns the notion of \( \beta \)-expansions introduced by Renyi [22] and studied at length by Parry [23]. However, for non-integer values of \( \beta \), the \( \beta \)-expansion of an integer generally has infinitely many digits, so the sum of the digits will generally be infinite. Digit sums related to a different digit expansion with respect to an irrational base were considered by Grabner and Tichy [11].

The approach which we consider in this section is to use the formula of Delange to define the cumulative digit sum \( S_{\beta}(n) \) for real parameters \( \beta > 1 \), from which we can define a digit sum \( d_{\beta}(n) \) by differencing. The resulting functions are continuous in the \( \beta \)-parameter.
4.1. **Extension to non-integer bases by Delange’s formula.** We begin by replacing the integer variable $b$ in Theorem 1.6 which gives a formula for $S_b(n)$ for integer bases $b \geq 2$, by a real parameter $\beta > 1$.

**Definition 4.1.** For $\beta \in \mathbb{R}$ with $\beta > 1$, define a generalized cumulative sum-of-digits function $S_\beta(n)$ by

$$S_\beta(n) := \frac{\beta - 1}{2 \log \beta} n \log n + h_\beta \left( \frac{\log n}{\log \beta} \right) n, \quad (4.1)$$

where the function $h_\beta(x)$ is defined by the Fourier series

$$h_\beta(x) = \sum_{k=-\infty}^{\infty} c_\beta(k) e^{2\pi i k x} \quad (4.2)$$

with coefficients

$$c_\beta(k) = \frac{\beta - 1}{2\pi i k} \left( 1 + \frac{2\pi i k}{\log \beta} \right)^{-1} \zeta \left( \frac{2\pi i k}{\log \beta} \right) \quad (4.3)$$

for $k \neq 0$ and

$$c_\beta(0) = \frac{\beta - 1}{2 \log \beta} (\log 2\pi - 1) - \frac{\beta + 1}{4}. \quad (4.4)$$

**Definition 4.2.** Define the generalized sum-of-digits function $d_\beta(n)$ for real $\beta > 1$ by

$$d_\beta(n) := S_\beta(n + 1) - S_\beta(n). \quad (4.5)$$

A plot of $S_\beta(10)$ as a function of $\beta$ for $1 \leq \beta \leq 15$ is shown in Figure 4.1. Note that $S_\beta(n)$ is approximately constant for $\beta \geq 10$.

4.2. **The function** $h_\beta(x)$. In this section, we study properties of the function $h_\beta(x)$ appearing in Definition 4.1 as a function of the variable $\beta > 1$ and as a function of the variable $x$. When $\beta = b \in \mathbb{N}$, Delange showed that $h_b(x)$ is a continuous but everywhere non-differentiable real-valued function of $x$ with period 1.
Lemma 4.3. For each fixed $\beta > 1$, the function $h_\beta(x)$ is a real-valued continuous function of $x$ on $\mathbb{R}$.

Proof. The zeta function satisfies the bound

$$|\zeta(it)| \ll t^{1/2+\varepsilon}$$

for $t \in \mathbb{R}$ (see for example [27] eq. 5.1.3), so the Fourier coefficients of $h_\beta$ satisfy

$$c_\beta(k) = -\frac{\beta-1}{2\pi i k} \left(1 + \frac{2\pi i k}{\log \beta}\right)^{-1} \zeta \left(\frac{2\pi i k}{\log \beta}\right) \ll \beta^{-3/2+\varepsilon}.$$  

This estimate shows that the Fourier series (4.2) is absolutely and uniformly convergent for $x \in \mathbb{R}$, so gives a continuous function of $x$.

The function $h_\beta(x)$ is real-valued for $x \in \mathbb{R}$ since the Fourier coefficients $c_\beta(k)$ satisfy $c_\beta(k) = c_\beta(-k)$.  \hfill \Box

A plot of $h_\beta(2)$ as a function of the real parameter $\beta$ for $1 \leq \beta \leq 8$ is shown in Figure 4.2. From the plot, it also appears that $h_\beta$ might be non-differentiable as a function of the real parameter $\beta$.

Question 2. For fixed $x \in \mathbb{R}$, is the function $h_\beta(x)$ everywhere non-differentiable as a function of the real variable $\beta$?

4.3. Meromorphic continuation of $G_\beta(s)$. Our proofs of the meromorphic continuation of $F_\beta(s)$ and $G_\beta(s)$ for integer bases relied on the identity

$$Z_\beta(s) = \sum_{n=1}^{\infty} (d_\beta(n) - d_\beta(n-1)) n^{-s} = \frac{b^s - b}{b^s - 1} \zeta(s).$$  

(4.8)

If for non-integer $\beta > 1$ we define

$$Z_\beta(s) := \sum_{n=1}^{\infty} (d_\beta(n) - d_\beta(n-1)) n^{-s},$$  

(4.9)
then $Z_\beta(s)$ is not equal to

$$\frac{\beta^s - \beta}{\beta^s - 1} \zeta(s)$$

(4.10)
as (4.10) is not an ordinary Dirichlet series. We must therefore take a different approach. We first consider the Dirichlet series generating function

$$G_\beta(s) := \sum_{n=1}^{\infty} \frac{S_\beta(n)}{n^s}$$

(4.11)

for $\beta \in \mathbb{R}$ with $\beta > 1$. Since the coefficients satisfy

$$S_\beta(n) \sim n \log n,$$

(4.12)
the Dirichlet series $G_\beta(s)$ has abscissa of absolute convergence $\sigma_a = 2$. We show that the function $G_\beta(s)$ can be analytically continued to a larger halfplane.

**Theorem 4.4.** For each real $\beta > 1$, the function $G_\beta(s)$ is meromorphic in the region $\text{Re}(s) > 1$ with a double pole at $s = 2$ with Laurent expansion

$$G_\beta(s) = \frac{\beta - 1}{2 \log \beta} (s - 2)^{-2} + c_\beta(0) (s - 2)^{-1} + O(1)$$

(4.13)
and simple poles at $s = 2 + 2\pi ik/\log \beta$ for $k \in \mathbb{Z}$ with $k \neq 0$ with residue

$$\text{Res} \left( G_\beta(s), s = 2 + \frac{2\pi ik}{\log \beta} \right) = c_\beta(k),$$

(4.14)
where the numbers $c_\beta(k)$ are those in Definition 4.1

**Proof.** Using the definition (4.1) of $S_\beta$, we have

$$G_\beta(s) = \sum_{n=1}^{\infty} \left( \frac{\beta - 1}{2 \log \beta} n \log n + h_\beta \left( \frac{\log n}{\log \beta} \right) n^{-s} \right)$$

$$= -\frac{\beta - 1}{2 \log \beta} \zeta'(s - 1) + \sum_{n=1}^{\infty} h_\beta \left( \frac{\log n}{\log \beta} \right) n^{-(s-1)}.$$  

(4.15)
The function $\zeta'(s - 1)$ is meromorphic on $\mathbb{C}$ with only singularity a double pole at $s = 2$ with Laurent expansion $\zeta'(s - 1) = -(s - 1)^{-2} + O(1)$. Using the Fourier series (4.2) for $h_{\beta}$, we have

$$
\sum_{n=1}^{\infty} h_{\beta} \left( \frac{\log n}{\log \beta} \right) n^{-(s-1)} = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} c_{\beta}(k) \exp \left( 2\pi i k \frac{\log n}{\log \beta} \right) n^{-(s-1)} \tag{4.17}
$$

$$
= \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} c_{\beta}(k) n^{-1} n^{-(s - 2\pi i k/\log \beta)} \tag{4.18}
$$

This double sum is absolutely convergent, so we may exchange the sums, giving

$$
\sum_{n=1}^{\infty} h_{\beta} \left( \frac{\log n}{\log \beta} \right) n^{-(s-1)} = \sum_{k=1}^{\infty} c_{\beta}(k) \zeta \left(s - 1 - \frac{2\pi i k}{\log \beta}\right) \tag{4.19}
$$

If $\Re(s) > 1$, then

$$
\zeta \left(s - 1 - \frac{2\pi i k}{\log \beta}\right) \ll k^{1/2 + \epsilon} \tag{4.20}
$$

for any $\epsilon > 0$. On any compact set $K$ in the halfplane $\Re(s) > 1$ not containing a point $s = 2 + 2\pi i k / \log \beta$ for any $k \in \mathbb{Z}$, the sum

$$
\sum_{k=0}^{\infty} c_{\beta}(k) \zeta \left(s - 1 - \frac{2\pi i k}{\log \beta}\right) \tag{4.21}
$$

is uniformly convergent on $K$; if the compact set $K$ contains a point of the form $s = 2 + 2\pi i k_0 / \log \beta$, then one term of the sum has a simple pole with residue $c_{\beta}(k_0)$ while the remaining sum is uniformly convergent. □

When $\beta \geq 2$ is an integer, we know that the function $G_{\beta}(s)$ has a meromorphic continuation to the entire complex plane.

**Question 3.** For noninteger $\beta > 1$, does the Dirichlet series $G_{\beta}(s)$ have a meromorphic continuation beyond $\Re(s) > 1$?

### 4.4. Meromorphic continuation of $F_{\beta}(s)$

We now consider the Dirichlet series

$$
F_{\beta}(s) = \sum_{n=1}^{\infty} \frac{d_{\beta}(n)}{n^s} \tag{4.22}
$$

for real $\beta > 1$. We already know that this series has a meromorphic continuation to $\mathbb{C}$ when $\beta \geq 2$ is an integer. We show that for each real $\beta > 1$, the Dirichlet series $F_{\beta}(s)$ has a meromorphic continuation to the halfplane $\Re(s) > 0$.

**Theorem 4.5.** The function $F_{\beta}(s)$ has a meromorphic continuation to the halfplane $\Re(s) > 0$, with a double pole at $s = 1$ with Laurent expansion

$$
F_{\beta}(s) = \frac{\beta - 1}{2 \log \beta} (s - 1)^{-2} + \left(c_{\beta}(0) + \frac{\beta - 1}{2 \log \beta}\right) (s - 1)^{-1} + O(1) \tag{4.23}
$$

and simple poles at $s = 1 + 2\pi i k / \log \beta$ for $k \in \mathbb{Z}$ with $k \neq 0$ with residue

$$
\text{Res} \left(F_{\beta}(s), s = 1 + \frac{2\pi i k}{\log \beta}\right) = \left(1 + \frac{2\pi i k}{\log \beta}\right) c_{\beta}(k). \tag{4.24}
$$
Proof. Let
\[ p(x) = \sum_{n=2}^{\infty} S_p(n)x^n, \]  
(4.25)
since
\[ \Gamma(s)(G_p(s) - S_p(1)) = \int_0^{\infty} p(e^{-x})x^{s-1} \, dx. \]  
(4.26)
By our definition of \( d_p(n) \), we have
\[ \sum_{n=1}^{\infty} d_p(n)x^n + S_p(1) = (x^{-1} - 1)p(x). \]  
(4.27)
Hence by Proposition 3.1 we have
\[ \Gamma(s)(F_p(s) + S_p(1)) = \int_0^{\infty} (e^{x} - 1)p(e^{-x})x^{s-1} \, dx \]  
(4.28)
for \( \text{Re}(s) > 1 \). Using the power series expansion Then we write
\[ \Gamma(s)(F_p(s) + S_p(1)) = \Gamma(s+1)(G_p(s+1) - S_p(1)) + \int_0^{\infty} (e^{x} - 1 - x)p(e^{-x})x^{s-1} \, dx. \]  
(4.29)
Dividing by \( \Gamma(s) \) and rearranging, we obtain
\[ F_p(s) = -S_p(1)(s+1) + sG_p(s+1) + R(s) \]  
(4.30)
where the remainder term
\[ R(s) = \frac{1}{\Gamma(s)}\int_0^{\infty} (e^{x} - 1 - x)p(e^{-x})x^{s-1} \, dx \]  
(4.31)
is holomorphic in \( \text{Re}(s) > 0 \) since \( e^{x} - 1 - x \ll x^{2} \) as \( x \to 0^{+} \). Since \( G_p(s+1) \) is meromorphic in \( \text{Re}(s) > 0 \), we find that \( F_p(s) \) is meromorphic in \( \text{Re}(s) > 0 \), with poles coming from the poles of \( G_p(s+1) \). Since
\[ sG_p(s+1) = (s-1)G_p(s+1) + G_p(s+1), \]  
(4.32)
we find that \( F_p(s) \) has a double pole at \( s = 1 \) with Laurent expansion as given in the theorem. At each other point \( s = 1 + 2\pi im/\log \beta \), \( F_p(s) \) has a simple pole. \( \square \)

Meromorphic continuation of \( F_p(s) \) to a larger halfplane would follow from continuation of \( G_p(s) \) to a larger halfplane; in particular, by using more terms of the power series for \( e^{x} \) in formula (4.29), we find that if \( G_p(s) \) is meromorphic in \( \text{Re}(s) > c \) for some \( c \), then \( F_p(s) \) is meromorphic in \( \text{Re}(s) > c - 1 \).

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