H-TWISTED COURANT ALGEBROIDS

MELCHIOR GRÜTZMANN
Department of Mathematics, Northwestern Polytechnical University,
Chang’an Campus, Xi’an 710129, Peoples Republic of China,
melchiorG@gmail.com

Received June 2012
Revised July 1901

We generalize Hansen–Strobl’s definition of Courant algebroids twisted by a 4-form on the base manifold such that the twist $H$ of the Jacobi identity is a four-form in the kernel of the anchor map and is closed under a naturally occurring exterior covariant derivative. We give examples and define a cohomology.

Keywords: twist of the Courant bracket; Courant algebroid; cohomology of algebroids.

1. Introduction

Courant algebroids were introduced by Liu, Weinstein, and Xu in [1] in order to describe the double of a Lie bialgebroid. They were further investigated by Roytenberg during his Ph.D. studies and a formulation in terms of a Dorfman bracket was discovered [2] as well as the fitting into a two-term $L_\infty$-algebra [3]. In [4] Hansen and Strobl discovered four-form twisted Courant algebroids arising naturally in the Courant sigma model with a Wess–Zumino boundary term. These $H$-twisted Courant algebroids were further investigated by Liu and Sheng in [5] where the observation was made that exact $H$-twisted Courant algebroids, they fit into a short exact sequence with the tangent and cotangent bundle, always have an exact four-form $H$. In this paper we want to generalize the notion of $H$-twist and exhibit examples that do not come from an exact or even closed four-form. The idea is analogous to $H$-twisted Lie algebroids (introduced in [6]) that guided from an exterior covariant derivative (Proposition [7]) that occurs naturally for strongly anchored almost Courant algebroids with anchor $\rho$ on the exterior algebra of sections of $\ker \rho$, one permits the Jacobiator to be a $\ker \rho$-four-form closed under the exterior covariant derivative. We will give examples of generalized exact four-forms, i.e. starting from a Courant algebroid with anchor $\rho$ and a $\ker \rho$-three-form with a certain integrability condition we define a Dorfman bracket together with a (non-trivial) $\ker \rho$-four-form $H$ that fit under the above idea.

Since already the definition of the closed generalized four-form requires sections of a possibly singular vector bundle, we also give a definition generalizing Roytenberg’s idea of Courant–Dorfman algebras in [8].
Furthermore, we carry over the idea of Stiénon and Xu [9] to define cochains as a subset of the exterior algebra of the $H$-twisted Courant algebroid such that the naive expression of a differential by the formula that holds for Lie algebroids actually gives a cochain again and squares to 0 in Theorem 15. We end the treatment with the obvious generalization of Dirac structures to $H$-twisted Courant algebroids and Strobl’s as well as Sheng–Liu’s idea [5] that such Dirac structures give $H$-twisted Lie algebroids.

In the meantime parallel developments have shown that it is possible to simplify the definition of $H$-twisted Courant algebroids, see [7].

The paper is organized as follows. In Section 2 we give a short summary of the definition of Courant algebroid, two-term $L_\infty$-algebra introduced by Baez and Crans [10] and Roytenberg–Weinstein’s observation that together with the skew-symmetric bracket the Courant algebroid gives such a two-term $L_\infty$-algebra. In Subsection 3.1 we begin with a definition of strongly anchored almost Courant algebroids and their natural covariant derivative on the kernel of the anchor map. We continue with the definition of $H$-twisted Courant algebroids and some examples. This part ends with the definition of an $H$-twisted Courant–Dorfman algebra. In Section 4 we define the naive cohomology of $H$-twisted Courant algebroids. In the last section we generalize the notion of Dirac structures and give examples of $H$-twisted Lie algebroids.

2. Preliminaries

Remember the definition of Courant algebroid. This goes back to Liu–Weinstein–Xu in [1]. We take the version of Roytenberg in [2, 2.6].

**Definition 1.** A Courant algebroid is a vector bundle $E \to M$ together with an $\mathbb{R}$-bilinear (non-skewsymmetric) bracket $[\cdot,\cdot] : \Gamma(E) \otimes \Gamma(E) \to \Gamma(E)$, a morphism of vector bundles $\rho : E \to TM$, and a symmetric non-degenerate bilinear form $\langle \cdot,\cdot \rangle : E \otimes E \to \mathbb{R} \times M$ subject to the following axioms

\[
[\phi, [\psi_1, \psi_2]] = [[\phi, \psi_1], \psi_2] + [\psi_1, [\phi, \psi_2]], \\
[\phi, f \cdot \psi] = \rho(\phi)[f] \cdot \psi + f \cdot [\phi, \psi], \\
[\psi, \psi] = \frac{1}{2} \rho^* \text{d}\langle \psi, \psi \rangle, \\
\rho(\phi) \langle \psi, \psi \rangle = 2\langle [\phi, \psi], \psi \rangle.
\]

where $\phi, \psi \in \Gamma(E)$, $f \in C^\infty(M)$, and $\text{d}$ is the de Rham differential of the smooth manifold $M$.

In what follows we will identify $E^*$ with $E$ via the symmetric non-degenerate bilinear form $\langle \cdot, \cdot \rangle$.

From [10] we take the following definition of a two-term $L_\infty$-algebra.

**Definition 2.** A two-term $L_\infty$-algebra is a two-term complex $0 \to V_1 \xrightarrow{\partial} V_0 \to 0$.
together with three more maps

\[\ldots\colon V_0 \wedge V_0 \to V_0,\]
\[\triangleright \colon V_0 \otimes V_1 \to V_1,\]
\[l_3 \colon V_0 \wedge V_0 \wedge V_0 \to V_1\]

Subject to the rules

\[\begin{align*}
[\phi, \partial f] &= \partial(\phi \triangleright f) \\
(\partial f) \triangleright g + (\partial g) \triangleright f &= 0 \\
[\phi_1, [\phi_2, \phi_3]] + \text{cycl.} &= \partial l_3(\phi_1, \phi_2, \phi_3) \\
\phi_1 \triangleright (\phi_2 \triangleright f) - \phi_2 \triangleright (\phi_1 \triangleright f) - [\phi_1, \phi_2] \triangleright f &= l_3(\phi_1, \phi_2, \partial f) \\
l_3([\phi_1, \phi_2] \wedge \phi_3 \wedge \phi_4) + \phi_1 \triangleright l_3(\phi_2 \wedge \phi_3 \wedge \phi_4) + \text{unshuffles} &= 0
\end{align*}\]

where \(\phi_i \in V_0\) and \(f \in V_1\).

As Roytenberg–Weinstein observed, the Courant algebroid gives rise to a two-term \(L_\infty\)-algebra with the identifications \(V_0 = \Gamma(E), V_1 = C^\infty(M), \partial = l_1 = \rho^* \circ d, l_2(\psi_1, \psi_2) = [\psi_1, \psi_2] - \frac{1}{2} \rho^* d(\psi_1, \psi_2), \psi \triangleright f = \frac{1}{2} \langle \psi, \partial f \rangle,\) and \(l_3(\psi_1, \psi_2, \psi_3) = \frac{1}{6} \langle [\psi_1, \psi_2], \psi_3 \rangle + \text{cycl.}\).

Since in the treatment of \(H\)-twisted Courant algebroids we will encounter sections of possibly singular vector bundles, we will also introduce the notion of Lie–Rinehart \([\text{11}]\) as well as Courant–Dorfman algebras \([\text{8}]\). For this purpose let \(\mathbb{k}\) be a commutative ring (with unit 1) and \(R\) a commutative \(\mathbb{k}\)-algebra.

**Definition 3.** A Lie–Rinehart algebra \((R, \mathcal{E}, [\ldots], \rho)\) is an \(R\)-module \(\mathcal{E}\) together with a \(\mathbb{k}\)-Lie algebra structure \([\ldots]\) on \(\mathcal{E}\) and an \(R\)-linear representation \(\rho\colon \mathcal{E} \to \text{Der}(R)\) subject to the rules

\[\begin{align*}
0 &= [\psi_1, [\psi_2, \psi_3]] + \text{cycl.}, \\
[\psi, f \cdot \phi] &= \rho(\psi)[f] \cdot \phi + f \cdot [\psi, \phi], \\
\rho[\phi, \psi] &= [\rho(\phi), \rho(\psi)]_{\text{Der}(R)}.
\end{align*}\]

Examples are \(\mathcal{E}\) the sections of a Lie algebroid \(E \to M\) with \(R = C^\infty(M)\).

**Definition 4.** Let \(\mathbb{k}\) contain \(\frac{1}{2}\). A Courant–Dorfman algebra \((R, \mathcal{E}, [\ldots], \rho, [\ldots])\) consists of an \(R\)-module \(\mathcal{E}\), a symmetric \(R\)-bilinearform \(\langle \ldots, \rangle\colon \mathcal{E} \otimes_R \mathcal{E} \to R\), a derivation \(\partial\colon R \to \mathcal{E}\), and a \(\mathbb{k}\)-bilinear (non-skewsymmetric) bracket \([\ldots]\colon \mathcal{E} \otimes \mathcal{E} \to \mathcal{E}\).
subject to the rules

\[ [\psi, f \cdot \phi] = \rho(\psi)[f] \cdot \phi + f \cdot [\psi, \phi], \]
\[ \langle \psi, \partial(\phi, \phi) \rangle = 2([\psi, \phi], \phi), \]
\[ [\psi, \psi] = \frac{1}{2} \partial\langle \psi, \psi \rangle, \]
\[ [\phi, [\psi_1, \psi_2]] = [[\phi, \psi_1], \psi_2] + [\psi_1, [\phi, \psi_2]], \]
\[ [\partial f, \phi] = 0, \]
\[ \langle \partial f, \partial g \rangle = 0 \]

for all \( \phi, \psi \in E, f, g \in R \). We call it almost Courant–Dorfman algebra iff only the first three rules hold.

Examples are \( E \) the sections of a Courant algebroid \( E \to M, R = C^\infty(M) \), \( \partial = \rho^* \circ d \); but also Lie–Rinehart algebras with trivial pairing \( \langle ., . \rangle \equiv 0 \).

3. \( H \)-twisted Courant algebroids

3.1. Covariant derivative for strongly anchored almost Courant algebroids

Definition 5. A strongly anchored almost Courant algebroid is a vector bundle \( E \to M \) together with a bilinear (non-skewsymmetric) bracket \( [.,.] : \Gamma(E) \otimes \Gamma(E) \to \Gamma(E) \), a symmetric nondegenerate bilinearform \( \langle ., . \rangle : E \otimes E \to \mathbb{R} \times M \), and a vector bundle morphism \( \rho : E \to TM \), called the anchor subject to the axioms

\[ \rho(\phi, \psi) = [\rho(\phi), \rho(\psi)]_{TM}, \]
\[ [\phi, f \cdot \psi] = \rho(\phi)[f] \cdot \psi + f \cdot [\phi, \psi], \]
\[ [\psi, \psi] = \frac{1}{2} \rho^* \partial\langle \psi, \psi \rangle, \]
\[ [\partial f, \phi] = 0, \]
\[ \langle \partial f, \partial g \rangle = 0 \]

Given a smooth anchor map \( \rho : E \to TM \) we define the \( \Omega^*_\rho(M, \ker \rho) \) to be the smooth sections \( \Gamma(\wedge^*E) \) that lie in the kernel of \( \tilde{\rho} : \wedge^*E \to TM \otimes \wedge^{*+1}E : \psi_1 \wedge \psi_2 \mapsto \rho(\psi_1) \otimes \psi_2 - \rho(\psi_2) \otimes \psi_1 \) and extended correspondingly for more terms.

Following an idea of Stiénon and Xu \[9\] we define an exterior covariant derivative on these cochains by the formula that holds for Lie algebroids.

Proposition 6. The following is an exterior covariant derivative, i.e. \( C^\infty(M) \)-linear in the occurring \( \psi_i \in \Gamma(M) \). For \( \alpha \in \Omega^*_\rho(M, \ker \rho) \) define

\[ \langle \mathcal{D}\alpha, \psi_0 \wedge \ldots \psi_p \rangle = \sum_{i=0}^{p} (-1)^i \rho(\psi_i) \langle \alpha, \psi_0 \wedge \ldots \hat{\psi}_i \ldots \psi_p \rangle + \sum_{i<j} (-1)^{i+j} \langle \alpha, [\psi_i, \psi_j] \wedge \psi_0 \ldots \hat{\psi}_i \ldots \hat{\psi}_j \ldots \psi_p \rangle \]
\( \mathcal{D} \) maps \( \Omega^p(\ker \rho) \to \Omega^{p+1}(\ker \rho) \) and fulfills the Leibniz rule
\[ \mathcal{D}(\alpha \wedge \beta) = (\mathcal{D}\alpha) \wedge \beta + (-1)^{|\alpha|} \alpha \wedge \mathcal{D}\beta . \] (15)

**Proof.** The main difference to Lie algebroids is that the bracket is not skewsymmetric. However the non-skewsymmetric part of the bracket vanishes when inserted into \( \alpha \). The rest is now a straightforward calculation. For the last statement note that \( \mathcal{D} \) is a first order odd differential operator.

Note that it is also possible to split a \( \ker \rho \)-p-k-form \( \alpha \) as a \( \ker \rho \)-p-form with values in the \( k \)-fold exterior power of \( \ker \rho \). We will denote any possible splitting as \( \tilde{\alpha} \).

### 3.2. Definition and examples

**Definition 7.** An \( H \)-twisted Courant algebroid is a vector bundle \( E \to M \) together with an \( \mathbb{R} \)-bilinear (non-skewsymmetric) bracket \( \{.,.\} : \Gamma(E) \otimes \Gamma(E) \to \Gamma(E) \), a morphism of vector bundles \( \rho : E \to TM \), a symmetric non-degenerate bilinear-form \( \langle .,. \rangle : E \otimes E \to \mathbb{R} \times M \), and a \( \ker \rho \)-four-form \( H \in \Omega^4_M(\ker \rho) \) subject to the following axioms

\[
\tilde{H}(\phi,\psi_1,\psi_2) = [\phi, [\psi_1, \psi_2]] - [[\phi, \psi_1], \psi_2] - [\psi_1, [\phi, \psi_2]],
\]

(16)

\[
\mathcal{D}H = 0,
\]

(17)

\[
[\phi, f \cdot \psi] = \rho(\phi)[f] \cdot \psi + f \cdot [\phi, \psi],
\]

(18)

\[
[\psi, \psi] = \frac{1}{2} \mathcal{D}(\psi, \psi),
\]

(19)

\[
\rho(\phi)(\psi, \psi) = 2\langle [\phi, \psi], \psi \rangle.
\]

(20)

where \( \phi, \psi \in \Gamma(E) \), \( f \in \mathcal{C}^\infty(M) \), and \( \mathcal{D} \) is the covariant derivative defined in the previous subsection.

**Lemma 8.** \( \rho \) is a morphism of brackets, i.e.
\[ \rho[\phi, \psi] = [\rho(\phi), \rho(\psi)] . \] (21)

**Proof.** Start from \([\rho(\phi), \rho(\psi)][f] \cdot \chi\) for \( \phi, \psi, \chi \in \Gamma(E) \), \( f \in \mathcal{C}^\infty(M) \) and expand using the Leibniz rule to iterated brackets. Then use the Jacobi identity (16), and note that the H-contributions cancel, because \( H \) is \( \mathcal{C}^\infty(M) \)-linear.

**Example 9.**

0. Courant algebroids are exactly the \( H \)-twisted Courant algebroids where \( H = 0 \).

1. Analogously to the \( H \)-twisted Lie algebroids we start with an untwisted Courant algebroid \( (E, \langle .,. \rangle, \rho, [.,.]_0) \) and make the general ansatz
\[
[\phi, \psi]_B := [\phi, \psi]_0 + B(\phi, \psi)
\] (22)
where \( B \in \Omega^3_M(\ker \rho) \). The Jacobiator of this bracket is
\[
\tilde{H} := \tilde{D}_0 B + \tilde{B}^2
\] (23)
where \( \tilde{B}^2(\psi_1, \psi_2, \psi_3) := \tilde{B}(\tilde{B}(\psi_1, \psi_2), \psi_3) + \text{cycl.} \) and the condition \( \mathcal{D} H = 0 \) reads as
\[
0 = \mathcal{D}_B H = \mathcal{D}_0 \tilde{B}^2 + \tilde{B} \mathcal{D}_0 B + \tilde{B}^3.
\] (24)

In the computation we use the fact observed by Stiénon–Xu that the naive differential \( \mathcal{D}_0 \) squares to 0. If we start with a Courant algebroid with \( \ker \rho \) of rank at most 4, then every \( B \in \Omega^3_M(\ker \rho) \) gives a twisted Courant algebroid.

In general, if we can find nontrivial solutions of this nonlinear first order PDE, we can provide nontrivial examples of \( H \)-twisted Courant algebroids.

2. One particular case arises when we start with a Courant algebroid \((E, \rho, [,], h)\) twisted by a closed 4-form \( h \in \Omega^4(M) \) in the sense of Hansen–Strobl [4]. If we pull it back to \( \Omega^4_M(\ker \rho) \) via \( \rho^* \) we obtain an \( H \)-twisted Courant algebroid, because \( \text{im} \rho^* \subseteq \ker \rho \) as well as

\[
\mathcal{D} \circ \rho^* = \rho^* \circ d
\] (25)

which follows from the morphism property of the anchor map.

3. Given an \( H \)-twisted Lie algebra (an almost Lie algebra \( \mathfrak{g} \) whose Jacobi identity is twisted by a three-form with values in \( \mathfrak{g} \) and \( \mathcal{D} \mathfrak{g} = 0 \) for the corresponding \( \mathcal{D} \)), then this augments to an \( H \)-twisted Courant algebroid over a point if we can find an ad-invariant symmetric bilinearform \( \langle ., . \rangle \) for it and \( H \) is then skew-symmetric.

**Proposition 11.** The \( H \)-twisted Courant algebroid \((E, \rho, [,], H)\) is a two-term \( L_\infty \)-algebra with the identifications \( V_0 := \Gamma(E), V_1 := \Gamma(\ker \rho) \), and the operations
\[
\partial = l_1 : V_1 \subseteq V_0,
\] (26)
\[
l_2 : V_0 \wedge V_1 \to V_1 : (\psi_1, \psi_2) \mapsto [\psi_1, \psi_2] - \frac{1}{2} \mathcal{D}\langle \psi_1, \psi_2 \rangle,
\] (27)
\[
l_3 : \wedge^3 V_0 \to V_1 : (\psi_1, \psi_2, \psi_3) \mapsto H(\psi_1, \psi_2, \psi_3) + \frac{1}{6} \mathcal{D}\langle [\psi_1, \psi_2], \psi_3 \rangle + \text{cycl.}
\] (28)

The correction in the bracket \( l_2 \) and in the Jacobiator \( l_3 \) are analogous to Roytenberg [2] and therefore fit the Courant case.

**Proof.** Straightforward but lengthy calculation.

---

**3.3. \( H \)-twisted Courant–Dorfman algebras**

Let \( \mathfrak{k} \) be a commutative ring (with unit 1) that contains \( \frac{1}{2} \). Analogously to Roytenberg [8] we define a strongly anchored almost Courant–Dorfman algebra as:
Definition 12. A strongly anchored almost Courant–Dorfman algebra \((R, E, \langle ., . \rangle, \mathcal{D}_0, [., .])\) is an \(R\)-module \(E\) together with a symmetric \(R\)-bilinear form \(\langle ., . \rangle : E \otimes_R E \to R\) such that \(\kappa : E \to E^\ast : \psi \mapsto \langle \phi, . \rangle\) is an isomorphism of \(R\)-modules, a derivation \(\mathcal{D}_0 : R \to E\), and a \(k\)-bilinear (non-skewsymmetric) bracket \([., .] : E \otimes E \to E\) subject to the rules

\[
[\psi, f \cdot \phi] = \langle \psi, \mathcal{D}_0 f \rangle \cdot \phi + f \cdot [\psi, \phi],
\]

\[
\langle \psi, \mathcal{D}_0 (\phi, \phi) \rangle = 2 \langle [\psi, \phi], \phi \rangle,
\]

\[
[\phi, \phi] = \frac{1}{2} \mathcal{D}_0 (\phi, \phi),
\]

\[
\langle [\psi, \phi], \mathcal{D}_0 f \rangle = \langle \phi, \mathcal{D}_0 (\psi, \mathcal{D}_0 f) \rangle - \langle \psi, \mathcal{D}_0 (\phi, \mathcal{D}_0 f) \rangle
\]

Examples are \(E\) the sections of a strongly anchored almost Courant algebroid \((E, \langle ., . \rangle, \rho, [., .])\).

These strongly anchored almost Courant–Dorfman algebras inherit a derivative of degree 1 on the exterior algebra \(C^p(E, \mathcal{D}_0) := E^p \cap \ker \mathcal{D}_0 R\) as before:

\[
\langle \mathcal{D}_0 \alpha, \psi_0 \wedge \ldots \psi_p \rangle := \sum_{i=0}^p (-1)^i \langle \psi_i, \mathcal{D}_0 (\alpha, \psi_0 \wedge \ldots \hat{\psi}_i \ldots \psi_n) \rangle + \sum_{i<j} (-1)^{i+j} \langle \alpha, [\psi_i, \psi_j] \wedge \psi_0 \ldots \hat{\psi}_i \ldots \hat{\psi}_j \ldots \psi_p \rangle
\]

Note that in particular \((\mathcal{D} | R) = \mathcal{D}_0\).

Therefore we can define \(H\)-twisted Courant–Dorfman algebras analogously to Roytenberg’s definition.

Definition 13. An \(H\)-twisted Courant–Dorfman algebra \((R, E, \langle ., . \rangle, \mathcal{D}_0, [., .], H)\) is an \(R\)-module \(E\) together with a symmetric \(R\)-bilinear form \(\langle ., . \rangle : E \otimes_R E \to R\) such that \(\kappa : E \otimes_R E \to R : \psi \mapsto \langle \phi, . \rangle\) is an isomorphism of \(R\)-modules, a derivation \(\mathcal{D}_0 : R \to E\), a \(k\)-bilinear (non-skewsymmetric) bracket \([., .] : E \otimes E \to E\) and a \(C^4(E, \mathcal{D}_0)\)-form \(H\) subject to the rules

\[
[\psi, f \cdot \phi] = \langle \psi, \mathcal{D}_0 f \rangle \cdot \phi + f \cdot [\psi, \phi],
\]

\[
\langle \psi, \mathcal{D}_0 (\phi, \phi) \rangle = 2 \langle [\psi, \phi], \phi \rangle,
\]

\[
[\phi, \phi] = \frac{1}{2} \mathcal{D}_0 (\phi, \phi),
\]

\[
\mathcal{H} (\phi, \psi_1, \psi_2) = [\phi, [\psi_1, \psi_2]] - [[\phi, \psi_1], \psi_2] - [\psi_1, [\phi, \psi_2]],
\]

\[
\mathcal{D} H = 0,
\]

\[
[\mathcal{D}_0 f, \phi] = 0,
\]

\[
\langle \mathcal{D}_0 f, \mathcal{D}_0 g \rangle = 0
\]

where \(\phi, \psi_i \in E\), \(f, g \in R\) and \(\mathcal{D}\) the extension of \(\mathcal{D}_0\) as defined above.

Examples are \(E\) the sections of an \(H\)-twisted Courant agebroid \((E, \langle ., . \rangle, \rho, [., .], H)\).
4. Naive Cohomology

Proposition 14. The covariant derivative $\mathcal{D}$ of Subsection 3.1 does not square to 0 in general, instead it fulfills for $H$-twisted Courant algebroids

$$\langle \mathcal{D}^2 f, \psi_0 \wedge \psi_1 \rangle = 0, \quad (41)$$

$$\langle \mathcal{D}^2 \phi, \psi_0 \wedge \psi_1 \rangle = H(\phi, \psi_0, \psi_1), \quad (42)$$

$$\mathcal{D}^2(\alpha \wedge \beta) = (\mathcal{D}^2 \alpha) \wedge \beta + \alpha \wedge \mathcal{D}^2 \beta \quad (43)$$

for $f \in \mathcal{C}^\infty(M)$, $\phi \in \Gamma(\ker \rho)$, $\alpha, \beta \in \Omega^\bullet_M(\ker \rho)$, and $\psi_i \in \Gamma(E)$.

Proof. The proof is analogous to the one for $H$-twisted Lie algebroids, namely the first statement follows from the morphism property of $\rho$, the second statement is a reformulation of the Leibniz rule, and the last statement follows from the graded Leibniz rule (15).

Theorem 15 (Naive cohomology). The cochains

$$C^p(E, \rho, H) := \Omega^p(\ker \rho) \cap \ker \tilde{H} \quad (44)$$

together with the derivative

$$d: C^p(E, \rho, H) \to C^{p+1}(E, \rho, H) : \alpha \mapsto \mathcal{D}\alpha \quad (45)$$

form a cochain complex.

Proof. It remains to check that $\mathcal{D}$ maps $\tilde{H}$-closed forms to $\tilde{H}$-closed forms. This follows from the property

$$[\mathcal{D}, \tilde{H}] = \tilde{\mathcal{D}}H = 0 \quad (46)$$

due to the axiom (17).

The corresponding notion of naive cochains for Courant–Dorfman algebras is

$$C^p(E, \mathcal{D}_0, H) := \ker \tilde{H} | \mathcal{X}^p \cap \ker i_{\mathcal{D}_0 R}. \quad (47)$$

5. Dirac Structures and $H$-twisted Lie Algebroids

Given an $H$-twisted Courant algebroid (with bilinearform) of split signature, we define a Dirac structure in the usual way.

Definition 16. Given an $H$-twisted Courant algebroid $(E, \langle \cdot, \cdot \rangle, \{\cdot,\cdot\}, \rho, H)$, we define

1. an isotropic subbundle $L \subseteq E$ as a vector subbundle over $M$ such that $(L, L) \equiv 0$. If the bilinearform is of split signature, we can consider maximal isotropic subbundles with respect to inclusion and call them Lagrangean subbundles.
2. an integrable subbundle \( L \subseteq E \) when the bracket closes on the sections of \( L \), i.e. \([\Gamma(L), \Gamma(L)] \subseteq \Gamma(L)\).

3. a Dirac structure as a maximal isotropic integrable subbundle in an \( H \)-twisted Courant algebroid of split signature.

Compare this with the definition of \( H \)-twisted Lie algebroids (taken from [6]):

**Definition 17.** An \( H \)-twisted Lie algebroid is a vector bundle \( E \to M \) together with a bundle map \( \rho: E \to TM \) (called the anchor), a section \( H \in \Omega^3_M(E, \ker \rho) \), and a skew-symmetric bracket \([\cdot, \cdot]: \Gamma(E) \wedge \Gamma(E) \to \Gamma(E)\) subject to the axioms

\[
\begin{align*}
[\phi, [\psi_1, \psi_2]] &= [[\phi, \psi_1], \psi_2] + [\psi_1, [\phi, \psi_2]] + H(\phi, \psi_1, \psi_2) \\
[\phi, f \cdot \psi] &= \rho(\phi)(f) \cdot \psi + f \cdot [\phi, \psi] \\
DH &= 0
\end{align*}
\]

where \( f \in C^\infty(M) \), \( \phi, \psi, \psi_i \in \Gamma(E) \) and \( D \) is the one defined for anchored almost Lie algebroids analogous to (14), but \( \rho \) replaced by

\[
\nabla_v \psi := [\psi, v]
\]

for every \( \psi \in \Gamma(E) \) and \( v \in \Gamma(\ker \rho) \) which is an \( E \)-connection on \( \ker \rho \).

We have the immediate consequence.

**Proposition 18.** Given an \( H \)-twisted Courant algebroid \((E, H)\) of split signature. Then every Dirac structure \( L \subseteq E \) is an \( H \)-twisted Lie algebroid. In particular the twist \( \tilde{H} \) induces a \( D \)-closed \( L \)-three-form with values in \( \ker \rho|L \).

**Acknowledgments**

Research on this paper was conducted during the stay at Sun Yat-sen University and partially supported by NSFC(10631050 and 10825105) and NKBRPC(2006CB805905). The paper was finished at Northwestern Polytechnical University. I am grateful to Z.-J. Liu for comments on an earlier version of this paper.

**References**

[1] Z.-J. Liu, A. Weinstein, and P. Xu: Manin triples for Lie bialgebroids, *J. Diff. Geom.*, vol. 45(3), (1997) 547–574. math.DG/9508013.

[2] D. Roytenberg: Courant algebroids, derived brackets, and even symplectic supermanifolds, Ph.D. thesis, University of California, Berkeley (1999). math.DG/9910078.

[3] D. Roytenberg and A. Weinstein: Courant algebroids and strongly homotopy Lie algebras, *Lett. Math. Phys.*, vol. 46/1, (1998) 81–93. math.QA/9802118.

[4] M. Hansen and T. Strobl: First Class Constrained Systems and Twisting of Courant Algebroids by a Closed 4-form, in *Memorial of W. Kummer* (2009). arXiv:0904.0711.

[5] Z. Liu and Y. Sheng: Leibniz 2-algebras and twisted Courant algebroids. arXiv:1012.5515.
[6] M. Grützmann: H-twisted Lie algebroids, *J. Geom. and Phys.*, pp. 476–484, DOI: 10.1016/j.geomphys.2010.10.016 [math.DG/1005.5680]

[7] Z.-J. Liu, J.-H. Sheng and X.-M. Xu, Pre-Courant algebroids and Associated Lie 2-Algebras, arXiv:1205.5898

[8] D. Roytenberg, Courant–Dorfman algebras and their cohomology, *Lett. Math. Phys.*, vol. 90(1-3), (2009) 311–351, ISSN 0377-9017, DOI: 10.1007/s11005-009-0342-3 [math.QA/0902.4862]

[9] M. Stiénon and P. Xu: Modular classes of Loday algebroids, *C. R. Acad. Sci. Paris, vol. Ser. I* 346. (2008) 193–198 [math.DG/0803.2047]

[10] J. C. Baez and A. S. Crans: Higher-dimensional algebra. VI. Lie 2-algebras, *Theory Appl. Categ.*, vol. 12, (2004) 492–538, ISSN 1201-561X [math.QA/0307263]

[11] G. S. Rinehart: Differential forms on general commutative algebras, *Trans. Amer. Math. Soc.*, vol. 108, (1963) 195–222, ISSN 0002-9947, DOI: 10.2307/1993603