DYNAMICS FOR THE ENERGY CRITICAL NONLINEAR
SCHRÖDINGER EQUATION IN HIGH DIMENSIONS

DONG LI AND XIAOYI ZHANG

Abstract. In [7], T. Duyckaerts and F. Merle studied the variational
structure near the ground state solution $W$ of the energy critical NLS
and classified the solutions with the threshold energy $E(W)$ in dimen-
sions $d = 3, 4, 5$ under the radial assumption. In this paper, we extend
the results to all dimensions $d \geq 6$. The main issue in high dimensions
is the non-Lipschitz continuity of the nonlinearity which we get around
by making full use of the decay property of $W$.

1. INTRODUCTION

We consider the Cauchy problem of the focusing energy critical nonlinear
Schrödinger equation:

\[
\begin{cases}
    iu_t + \Delta u + |u|^{4/d}u = 0, \\
    u(0, x) = u_0(x),
\end{cases}
\]

where $u(t, x)$ is a complex function on $\mathbb{R} \times \mathbb{R}^d$, $d \geq 3$ and $u_0 \in \dot{H}^1_x(\mathbb{R}^d)$. The name ”energy critical” refers to the fact that the scaling

\[
u(t, x) \rightarrow u_\lambda(t, x) = \lambda^{-d/2} u(\lambda^{-2} t, \lambda^{-1} x).
\]

leaves both the equation and the energy invariant. Here, the energy is
defined by

\[
E(u(t)) = \frac{1}{2} \|\nabla u(t)\|_2^2 - \frac{d-2}{2d} \|u(t)\|^{2d/2-d}_{\frac{2d}{d-2}},
\]

and is conserved in time. We refer to the first part as ”kinetic energy” and
the second part as ”potential energy”.

From the classical local theory [5], for any $u_0 \in \dot{H}^1_x(\mathbb{R}^d)$, there exists a unique maximal-lifespan solution of (1.1) on a time interval $(-T_-, T_+)$ such
that the local scattering size

\[
S_I(u) = \|u\|_{L^{2(d+2)}_{t,x} (I \times \mathbb{R}^d)}^{2(d+2)} < \infty,
\]

for any compact interval $I \subset (-T_-, T_+)$. If $S_{[0,T^+]}(u) = \infty$, we say $u$ blows
up forward in time. Likewise $u$ blows up backward in time if $S_{(-T_,0]}(u) = \infty$. We also recall the fact that the non-blowup of $u$ in one direction implies scattering in that direction.

For the defocusing energy critical NLS, the global wellposedness and scat-
tering was established in [2, 19, 25, 6, 21]. In the focusing case, depending on
the size of the kinetic energy of the initial data, both scattering and blowup
may occur. One can refer to [4] for scattering of small kinetic energy solu-
tions and [9] for the existence of finite time blowup solutions. The threshold
between blowup and scattering is believed to be determined by the ground state solution of the equation \( (1.1) \):

\[
W(x) = \left(1 + \frac{|x|^2}{d(d-2)}\right)^{-\frac{d-2}{2}},
\]

which solves the static NLS

\[
\Delta W + W^{\frac{d+2}{d-2}} = 0.
\]

This was verified by Kenig-Merle \([12]\) in dimensions \(d = 3, 4, 5\) in the spherically symmetric case and by Killip-Visan \([16]\) in all dimensions \(d \geq 5\) without the radial assumption. To summarize, we have the following

**Theorem 1.1** (Global wellposedness and scattering \([12, 16]\)). Let \(u = u(t,x)\) be the maximal-lifespan solution of \((1.1)\) on \(I \times \mathbb{R}^d\) in dimension \(d \geq 3\), in the case when \(d = 3, 4\), we also require that \(u\) is spherically symmetric. If

\[
E_* := \sup_{t \in I} \|\nabla u(t)\|_2 < \|\nabla W\|_2,
\]

then \(I = \mathbb{R}\) and the scattering size of \(u\) is finite,

\[
S_I(u) = \|u\|_{\frac{2(d+2)}{d-2}} \left(I \times \mathbb{R}^d\right) < C(E_*).
\]

As a consequence of this theorem and the coercive property of \(W\) \([12]\), they also proved

**Corollary 1.2** \([12, 16]\). Let \(d \geq 3\) and \(u_0 \in \dot{H}^1_0(\mathbb{R}^d)\). In dimension \(d = 3, 4\) we also require \(u_0\) is spherically symmetric. If

\[
E(u_0) < E(W),
\]

\[
\|\nabla u_0\|_2 \leq \|\nabla W\|_2,
\]

then the corresponding solution \(u = u(t,x)\) exists globally and scatters in both time directions.

Theorem \([12]\) and Corollary \([12]\) confirmed that the threshold between blowup and scattering is given by the ground state \(W\). Our purpose of this paper is not to investigate the global wellposedness and scattering theory blow the threshold. Instead, we aim to continue the study in \([7]\) on what will happen if the solution has the threshold energy \(E(W)\). In that paper, T. Duyckaerts and F. Merle carried out a very detailed study of the dynamical structure around the ground solution \(W\). They were able to give the characterization of solutions with the threshold energy in dimensions \(d = 3, 4, 5\) under the radial assumption. Note that the energy-critical problem here can be compared with the focusing mass critical problem

\[
iu_t + \Delta u = -|u|^{\frac{4}{d}}u.
\]

There the ground state solution \(Q\) satisfies the equation

\[
\Delta Q - Q + Q^{1+\frac{4}{d}} = 0.
\]

And the mass of \(Q\) turns out to be the threshold between blowup and scattering. The characterization of the minimal mass blowup solution was established in \([18], [27], [13], [14]\).
In this paper, we aim to extend the results in [7] to all dimensions $d \geq 6$. Although the whole framework designed for low dimensions can also be used for the high dimensional setting, there are a couple of places where the arguments break down in high dimensions. Roughly speaking, this is caused by the non-smoothness of the nonlinearity; more precisely, in high dimensions, the nonlinearity $|u|^{\frac{4}{d-2}} u$ is no longer Lipschitz continuous in the usual Strichartz space $\dot{S}^1$ (see Section 2 for the definition). This reminds us of the similar problem one encountered in establishing the stability theory for high dimensional energy critical problem where this was gotten around by using exotic Strichartz estimates (see, for example, [23]).

However, the exotic Strichartz trick will inevitably cause the loss of derivatives and one cannot go back to the natural energy space $H^1_x$. On the other hand, the $H^1_x$ regularity is heavily used in the spectral analysis around the ground state $W$ (See for example the proof of Proposition 5.9 in [7]). To solve this problem, we will use a different technique where the decay property of $W$ is fully considered. When constructing the threshold solutions $W^\pm$ (see Theorem 1.3 below), we transform the problem into solving a perturbation equation with respect to $W$ using the fixed point argument. Although the nonlinearity of the perturbed equation is not Lipschitz continuous for general functions, it is for perturbations which are much smaller than $W$. The reason is that if we restrict ourselves to the regime $|z| \ll 1$, we can expand the real analytic function $|1 + z|^{\frac{4}{d-2}} (1 + z)$ (which corresponds to the form of energy critical nonlinearity) and get the Lipschitz continuity. This consideration leads us to working in the space of functions which have much better decay than $W$. The weighted Sobolev space $H^{m,m}$ (see (3.4) for the definition) turns out to be a good candidate for this purpose. By doing this, besides proving the existence of the threshold solutions $W^\pm$, we can actually show the difference $W^\pm - W$ has very high regularity and good decay properties.

This property also helps us in the next step where we have to show after extracting the linear term, the perturbed nonlinearity is superlinear with respect to the perturbations. The superlinearity is needed to show the rigidity of the threshold solutions $W^\pm$. This time again we make use of the decay estimate of $W$. We split $\mathbb{R}^d$ into regimes where the solution dominates $W$ and the complement. In the first regime, we can transform some portion of $W$ to increase the power of the solution, and get the superlinearity (cf. Lemma 2.3). In the regime where the solution is dominated by $W$, we simply use the real analytic expansion. The fact that the difference $W^\pm - W$ has enough decay in space and time plays a crucial role in the whole analysis.

In all, the material in this paper allows us to extend the argument in [7] to all dimensions $d \geq 6$. With some suitable modifications, the same technique can be used to treat the high dimensional energy critical nonlinear wave

---

1The main ingredient of exotic Strichartz trick is as follows: instead of using spaces $\dot{S}^1$, we use the space which has the same scaling but lower regularity. The nonlinearity can be shown to be Lipschitz continuous in such spaces. (See lecture notes [15], Section 3 for more details).
equation and we will address this problem elsewhere [17]. For NLS we have the following

**Theorem 1.3.** Let \( d \geq 6 \). There exists a spherically symmetric global solution \( W^- \) of (1.1) with \( E(W^-) = E(W) \) such that

\[
\| \nabla W^-(t) \|_2 < \| \nabla W \|_2, \quad \forall \ t \in \mathbb{R}.
\]

Moreover, \( W^- \) scatters in the negative time direction and blows up in the positive time direction, in which \( W^- \) is asymptotically close to \( W \):

\[
\lim_{t \to +\infty} \| W^-(t) - W \|_{H^1_x} = 0.
\]

There also exists a spherically symmetric solution \( W^+ \) with \( E(W^+) = E(W) \) such that

\[
\| \nabla W^+(t) \|_2 > \| \nabla W \|_2, \quad \forall \ t \in \mathbb{R}.
\]

Moreover in the positive time direction, \( W^+ \) blows up at infinite time and is asymptotically close to \( W \)

\[
\lim_{t \to +\infty} \| W^+(t) - W \|_{H^1_x} = 0.
\]

In the negative time direction, \( W^+ \) blows up at finite time.

Next, we classify solutions with the threshold energy. Since the equation is invariant under several symmetries, we can determine the solution only modulo these symmetries. In the spherically symmetric setting, when we say \( u = v \) up to symmetries, we mean there exist \( \theta_0, t_0 \in \mathbb{R}, \lambda_0 > 0 \) such that

\[
u(t, x) = e^{i\theta_0} \lambda_0^{-\frac{d-2}{2}} v\left(\frac{t+t_0}{\lambda_0}, \frac{x}{\lambda_0}\right).
\]

With this convention we have

**Theorem 1.4.** Let \( d \geq 6, u_0 \in \dot{H}_x^1(\mathbb{R}^d) \) be spherically symmetric and such that \( E(u_0) = E(W) \). Let \( u \) be the corresponding maximal-lifespan solution of (1.1) on \( I \times \mathbb{R}^d \). We have

1) If \( \| \nabla u_0 \|_2 < \| \nabla W \|_2 \), then either \( u = W^- \) up to symmetries or \( u \) scatters in both time directions.

2) If \( \| \nabla u_0 \|_2 = \| \nabla W \|_2 \), then \( u = W \) up to symmetries.

3) If \( \| \nabla u_0 \|_2 > \| \nabla W \|_2 \) and \( u_0 \in L^2_x(\mathbb{R}^d) \), then either \( |I| \) is finite or \( u = W^+ \) up to symmetries.

The proof of Theorem 1.3 and 1.4 will follow roughly the same strategy as in [7]. Here we make a remark about the proof of Theorem 1.4. The second point is a direct application of variational characterization of \( W \) (see the last section for more details). To prove 1) and 3), in [7], a large portion of the work was devoted to showing the exponential convergence of the solution to \( W \), which after several minor changes, also works for higher dimensions. For this reason, we do not repeat that part of the argument and build our starting point on the following

**Proposition 1.5 (Exponential convergence to \( W \)).** Suppose \( u_0, u \) satisfy the same conditions as in Theorem 1.3 and \( u \) blows up on \( I \) forward in time. If \( \| \nabla u_0 \|_2 > \| \nabla W \|_2 \), we assume \([0, \sup I) = [0, \infty)\). If \( \| \nabla u_0 \|_2 \leq \| \nabla W \|_2 \),
then the solution exists globally and $I = \mathbb{R}$. In all cases there exist $\theta_0 \in \mathbb{R}$, $\gamma_0 > 0$, $\mu_0 > 0$ such that

$$
\|u(t) - W[\theta_0, \mu_0]\|_{\dot{H}^1} \leq C e^{-\gamma_0 t}, \quad \forall \ t \geq 0,
$$

(1.4)

where

$$
W[\theta_0, \mu_0](x) = e^{i \theta_0 \mu_0} - d - 2 W(\frac{x}{\mu_0}).
$$

This paper is organized as follows. In Section 2, we introduce some notations and collect some basic estimates. Section 3 is devoted to proving Theorem 1.3. In Section 4, we give the proof of Theorem 1.4 by assuming Proposition 1.5.

**Acknowledgements.** Both authors were supported by the National Science Foundation under agreement No. DMS-0635607. Dong Li was also supported by a start-up funding from the Mathematics Department of University of Iowa. X. Zhang was also supported by NSF grant No. 10601060 and project 973 in China.

### 2. Preliminaries

We use $X \lesssim Y$ or $Y \gtrsim X$ whenever $X \leq CY$ for some constant $C > 0$. We use $O(Y)$ to denote any quantity $X$ such that $|X| \lesssim Y$. We use the notation $X \sim Y$ whenever $X \lesssim Y \lesssim X$. We will add subscripts to $C$ to indicate the dependence of $C$ on the parameters. For example, $C_{i,j}$ means that the constant $C$ depends on $i, j$. The dependence of $C$ upon dimension will be suppressed.

We use the ‘Japanese bracket’ convention $\langle x \rangle := (1 + |x|^2)^{1/2}$.

Throughout this paper, we will use $p_c$ to denote the total power of non-linearity:

$$
p_c = \frac{d + 2}{d - 2}.
$$

We write $L_t^q L_x^r$ to denote the Banach space with norm

$$
\|u\|_{L_t^q L_x^r(\mathbb{R} \times \mathbb{R}^d)} := \left( \int_{\mathbb{R}} \left( \int_{\mathbb{R}^d} |u(t, x)|^r \, dx \right)^{q/r} \, dt \right)^{1/q},
$$

with the usual modifications when $q$ or $r$ are equal to infinity, or when the domain $\mathbb{R} \times \mathbb{R}^d$ is replaced by a smaller region of spacetime such as $I \times \Omega$. When $q = r$ we abbreviate $L_t^q L_x^q$ as $L_{t,x}^q$.

For a positive integer $k$, we use $W^{k,p}$ to denote the space with the norm

$$
\|u\|_{W^{k,p}} = \sum_{0 \leq j \leq k} \|\nabla^j u\|_{L_x^p}.
$$

when $p = 2$, we write $W^{k,2}$ as $H^k$. 
2.1. Strichartz estimates. Let the dimension $d \geq 6$. We say a couple $(q, r)$ is admissible if $2 \leq q \leq \infty$ and

$$\frac{2}{q} + \frac{d}{r} = \frac{d}{2}.$$  

Let $I$ be a time slab. We denote $\dot{S}^0(I) = \bigcap_{(q,r)}$ admissible $L^q_t L^r_x(I \times \mathbb{R}^d)$ and $\dot{N}^0(I)$ as its dual space. We will use $\dot{S}^1(I)$ and $\dot{N}^1(I)$ to denote the space of functions $u$ such that $\nabla u \in \dot{S}^0(I)$ and $\nabla u \in \dot{N}^0(I)$ respectively. By Sobolev embedding, it is easy to verify that

$$\|u\|_{L^q_t L^r_x} \lesssim \|u\|_{\dot{S}^1}, \quad (2.1)$$

for all $\dot{H}^1$ admissible pairs $(q, r)$ in the sense that $2 \leq q \leq \infty$, and $\frac{2}{q} + \frac{d}{r} = \frac{d}{2} - 1$. Two typical $\dot{H}^1$ admissible pairs are $(\infty, \frac{2d}{d-2})$, $(2, \frac{2d}{d-2})$. Other pairs will also be used in this paper without mentioning this embedding.

With the notations above, we record the standard Strichartz estimates as follows.

**Lemma 2.1** (Strichartz estimates, [20, 11]). Let $k = 0, 1$. Let $I$ be an interval, $t_0 \in I$, $u_0 \in \dot{H}^k$ and $f \in \dot{N}^k(I)$. Then, the function $u$ defined by

$$u(t) := e^{i(t-t_0)\Delta}u_0 - i \int_{t_0}^{t} e^{i(t-t')\Delta} f(t') \, dt'$$

obeys the estimate

$$\|u\|_{\dot{S}^k(I)} \lesssim \|u_0\|_{\dot{H}^k} + \|f\|_{\dot{N}^k(I)}.$$  

2.2. Derivation of the perturbation equation near $W$. Let $u$ be the solution of (1.1) and $v = u - W$, then $v$ satisfies the equation

$$i \partial_t v + \Delta v + \Gamma(v) + i R(v) = 0,$$

where,

$$\Gamma(v) = \frac{p_e + 1}{2} W^{p_e - 1} v + \frac{p_c - 1}{2} W^{p_c - 1} \bar{v},$$

$$i R(v) = |v + W|^{p_e - 1} (v + W) - W^{p_e} - \frac{p_e + 1}{2} W^{p_e - 1} v - \frac{p_c - 1}{2} W^{p_c - 1} \bar{v}.$$  

Define the linear operator $\mathcal{L}$ by

$$\mathcal{L}(v) = -i \Delta v - i \frac{p_e + 1}{2} W^{p_e - 1} v - i \frac{p_c - 1}{2} W^{p_c - 1} \bar{v}.$$  

We write the equation for $v$ equivalently as

$$\partial_t v + \mathcal{L}(v) + R(v) = 0.$$  

For the spectral properties of $\mathcal{L}$, we need the following lemma from [7].

**Lemma 2.2** (7). The operator $\mathcal{L}$ admits two eigenfunctions $\mathcal{Y}_+, \mathcal{Y}_- \in \mathcal{S}(\mathbb{R}^d)$ with real eigenvalues

$$\mathcal{L}\mathcal{Y}_+ = e_0 \mathcal{Y}_+, \mathcal{L}\mathcal{Y}_- = -e_0 \mathcal{Y}_-,$$

and

$$\mathcal{Y}_+ = \mathcal{Y}_-, \quad e_0 > 0.$$  

**Proof.** See Lemma 5.1 of [7].
2.3. Basic estimates. We will use the following lemma many times throughout this paper.

**Lemma 2.3.** Let $I$ be a time slab. We have

$$
\| |u|^{p_c-1} \nabla v|_{\dot{N}^0(I)} \| \lesssim \| |u|^{p_c-1} \nabla v|_{L_t^\infty L_x^{2d}(I \times \mathbb{R}^d)} \|_{S^1(I)},
$$

$$
\| \nabla W |v|^{p_c-1}|_{\dot{N}^0(I;|v| > \frac{1}{4} W)} \| \lesssim \| v\|_{S^1(I)},
$$

$$
\| W^{p_c-2} \nabla W |v|^{p_c-1}|_{\dot{N}^0(I;|v| > \frac{1}{4} W)} \| \lesssim \| v\|_{S^1(I)},
$$

$$
\| W^{p_c-3} \nabla W v^2|_{\dot{N}^0(I;|v| \leq \frac{1}{4} W)} \| \lesssim \| v\|_{S^1(I)},
$$

$$
\| W^{-1} \nabla W |v|^{p_c}|_{\dot{N}^0(I;|v| \leq \frac{1}{4} W)} \| \lesssim \| v\|_{S^1(I)}.
$$

Here $\dot{N}^0(I;|v| > \frac{1}{4} W)$ denotes $\dot{N}^0(I \times \Omega)$, where $\Omega = \{ x : |v(x)| > \frac{1}{4} W(x) \}$. Similar conventions apply to $\dot{N}^0(I;|v| < \frac{1}{4} W)$.

**Proof.** The first one follows directly from Hölder’s inequality, we have

$$
\| |u|^{p_c-1} \nabla v|_{\dot{N}^0(I)} \| \leq \| |u|^{p_c-1} \nabla v|_{L_t^\infty L_x^{2d}(I \times \mathbb{R}^d)} \|_{S^1(I)}.
$$

Now we verify the second one. Noting $|\nabla W(x)| \lesssim \langle x \rangle^{-(d-1)}$, we have

$$
\| \nabla W |v|^{p_c-1}|_{\dot{N}^0(I;|v| > \frac{1}{4} W)} \|
\lesssim \| |v|^{p_c-1} \langle x \rangle^{-(d-\frac{3}{2})} \langle x \rangle^{-\frac{3}{2}} |\dot{N}^0(I;|v| \geq \frac{1}{4} W)} \|
\lesssim \| |v|^{\frac{d+3}{2 d-2}} \langle x \rangle^{-\frac{3}{2}} |L_t^\infty L_x^{2d-2}(I \times \mathbb{R}^d)} \|
\leq \| \langle x \rangle^{-\frac{3}{2}} |L_t^\infty L_x^{2d-2}(I \times \mathbb{R}^d)} \|
\| |v|^{\frac{d+3}{d-2}} |L_t^\infty L_x^{4d-2}(I \times \mathbb{R}^d)} \|
\leq \| v|^{\frac{d+3}{d-2}} |L_t^\infty L_x^{4d-2}(I \times \mathbb{R}^d)} \|
\leq \| v\|_{S^1(I)}.
$$

To see the third one, we use the bound $W^{p_c-2} |\nabla W| \lesssim \langle x \rangle^{-5}$ to control

$$
W^{p_c-2} |\nabla W| \lesssim \langle x \rangle^{-\frac{3}{2}} |v|^{\frac{d+3}{2 d-2}}, \quad |v| > \frac{1}{4} W,
$$

the same argument in proving the second one yields the desired estimate.
We verify the fourth inequality:

\[ \| W^{p_c - 3} \nabla Wv^2 \|_{\dot{H}^0(I; |v| \leq \frac{1}{4}W)} \lesssim \| W^{\frac{d+4}{2(d-2)} - 2} \|_{L^2_t L^4_x(I \times \mathbb{R}^d)} \| v \|_{L^2_t L^{\frac{4(d+4)}{d+2}}(I \times \mathbb{R}^d)} \]

\[ \lesssim \| W^{\frac{3}{2(d-2)}} \|_{L^1_x} \| v \|_{L^2_t L^{\frac{8(d+4)}{5d-2}}[I \times \mathbb{R}^d]} \]

\[ \lesssim \| v \|_{L^2_t S^1(I \times \mathbb{R}^d)}. \]

The last one follows from the bound

\[ W_0 |\nabla W|^p \lesssim \langle x \rangle^{-\frac{d}{2}} |v|^{\frac{d+4}{d-2}} \]

for \( |v| \leq \frac{1}{4}W \) and Hölder inequality, as in the second one.

\[ \square \]

3. The existence of \( W^-, W^+ \).

As in \[7\], we will construct the threshold solutions \( W^-, W^+ \) as the limit of a sequence of near solutions \( W^a_k(t, x) \) in the positive time direction. It follows from this construction that both \( W^- \) and \( W^+ \) approach to the ground state \( W \) exponentially fast as \( t \to +\infty \). On the other hand, the asymptotic behaviors of \( W^- \) and \( W^+ \) are quite different in the negative time direction (see Remark 3.10). We begin with the following result:

**Lemma 3.1** \([7]\). Let \( a \in \mathbb{R} \). There exist functions \( \{ \Phi^a_j \}_{j \geq 1} \) in \( S(\mathbb{R}^d) \) such that \( \Phi^a_1 = a \mathcal{Y}_+ \) (see Lemma 2.2 for the definition of \( \mathcal{Y}_+ \)) and the function

\[ W^a_k(t, x) = W(x) + \sum_{j=1}^k e^{-je^{0t}} \Phi^a_j(x), \]

is a near solution of the equation (1.1) in the sense that

\[ (i\partial_t + \Delta) W^a_k + |W^a_k|^{\frac{4}{d-2}} W^a_k = \varepsilon^a_k, \]

where the error \( \varepsilon^a_k \) is exponentially small in \( S(\mathbb{R}^d) \). More precisely, \( \forall J, M \geq 0, J, M \) are integers, there exists a constant \( C_{J,M} \) such that

\[ \langle x \rangle^J |\nabla^M (\varepsilon^a_k)(x) \rangle \lesssim C_{J,M} e^{-(k+1)e^{0t}}. \]

**Remark 3.2.** Since all \( \Phi_j \) are Schwartz functions, we have the following properties for the difference

\[ v_k = W^a_k - W = \sum_{j=1}^k e^{-je^{0t}} \Phi^a_j(x). \]

For any \( j, m \geq 0 \), there exists \( C_{k,j,m} > 0 \) such that

\[ |\langle x \rangle^j \nabla^m v_k(t, x) | \leq C_{k,j,m} e^{-e^{0t}}. \]
Next we show that there exists a unique genuine solution \( W^a(t,x) \) of (1.1) which can be approximated by the above constructed near solutions \( W^a_k(t,x) \). The existence and uniqueness of the solution \( W^a \) can be transformed to that of \( h := W^a - W^a_k \) which satisfies the equation

\[
i \partial_t h + \Delta h = -\Gamma(h) - iR(v_k + h) + iR(v_k) + i\varepsilon^a_k. \tag{3.3}
\]

Remark that this is the first place where the proof in [7] breaks down in higher dimensions. In [7] for dimensions \( d = 3, 4, 5 \), they made use of the fact that the nonlinearity \( G(h) := -\Gamma(h) - iR(v_k + h) + iR(v_k) \) is Lipschitz\(^2\) in \( \dot{S}^1 \) to construct the solution to (3.3) by the fixed point argument. In higher dimensions \( d \geq 6 \), the Lipschitz continuity does not hold anymore. However, since \( v_k \) is small compared with \( W \), we can use real analytic expansion for the complex function \( |1 + z|^{p_c - 1}(1 + z) \) to show that \( R(v_k + h) - R(v_k) \) is actually Lipschitz in \( h \) once \( h \) is small. This observation motivates us to construct the solution in a certain space consisting of functions which decay much faster than \( W \). It turns out that the weighted Sobolev space \( H^{m,m} \) with the norm

\[
\|f\|_{H^{m,m}} = \sum_{0 \leq j \leq m} \|\langle x \rangle^{m-j} \nabla^j f\|_2 \tag{3.4}
\]

for large \( m \) serves this purpose.

We have several properties for \( H^{m,m} \).

**Lemma 3.3** (Linear estimate in \( H^{m,m} \)). For any \( m \geq 1 \), there exists a constant \( C \) depending on \( m \) such that

\[
\|e^{it\Delta} u_0\|_{H^{m,m}} \leq e^{C|t|} \|u_0\|_{H^{m,m}}, \forall t \in \mathbb{R}. \tag{3.5}
\]

Let \( t_0 > 0 \), \( \alpha > 2C \) and \( \Sigma_{t_0} \) be the space with the norm

\[
\|u\|_{\Sigma_{t_0}} = \sup_{t \geq t_0} e^{\alpha t} \|u(t)\|_{H^{m,m}}, \tag{3.6}
\]

then the following holds:

\[
\| \int_t^\infty e^{i(t-\tau)\Delta} F(\tau) d\tau \|_{\Sigma_{t_0}} \leq \frac{1}{\alpha - C} \| F \|_{\Sigma_{t_0}}. \tag{3.7}
\]

\(^2\)More precisely \( \|G(h_1) - G(h_2)\|_{\dot{S}^1} \lesssim \|h_1 - h_2\|_{\dot{S}^1} \).

\(^3\)Certainly the estimate (3.5) is not optimal. For example, one can improve it to: \( \|e^{it\Delta} u_0\|_{H^{m,m}} \lesssim (1 + |t|)^m \|u_0\|_{H^{m,m}} \). However, the rough estimate (3.5) is enough for our use.
Proof. (3.3) follows directly from the standard energy method. To obtain (3.7), we use (3.5) to estimate
\[
\left\| \int_t^\infty e^{i(t-\tau)\Delta} F(\tau) d\tau \right\|_{H^{m,m}} \\
\leq \int_t^\infty \left\| e^{i(t-\tau)\Delta} F(\tau) \right\|_{H^{m,m}} d\tau \\
\leq \int_t^\infty e^{C(\tau-t)} \left\| F(\tau) \right\|_{H^{m,m}} d\tau \\
\leq \int_t^\infty e^{C(\tau-t)} e^{-\alpha \tau} d\tau \left\| F \right\|_{\Sigma_{t_0}} \\
\leq \frac{1}{\alpha - C} e^{-\alpha t} \left\| F \right\|_{\Sigma_{t_0}}.
\]
(3.7) now follows immediately.

Lemma 3.4 (Embedding in $H^{m,m}$). Let $k_1$, $k_2$ be non-negative integers, then for any $m \geq k_1 + k_2 + \frac{d}{2} + 1$, we have
\[
\left\| \langle x \rangle^{k_1} \nabla^{k_2} f \right\|_{\infty} \lesssim \left\| f \right\|_{H^{m,m}},
\]
where the implicit constant depends only on $k_1$, $k_2$.

Proof. Denote $\left\lfloor \frac{d}{2} + \right\rfloor$ as the smallest integer strictly bigger than $\frac{d}{2}$. By Sobolev embedding
\[
\left\| f \right\|_{\infty} \lesssim \left\| f \right\|_{H^{\left\lfloor \frac{d}{2} + \right\rfloor}},
\]
we have
\[
\left\| \langle x \rangle^{k_1} \nabla^{k_2} f \right\|_{\infty} \\
\lesssim \left\| \langle x \rangle^{k_1} \nabla^{k_2} f \right\|_2 + \left\| \nabla^{\left\lfloor \frac{d}{2} + \right\rfloor} \langle x \rangle^{k_1} \nabla^{k_2} f \right\|_2 \\
\lesssim \left\| f \right\|_{H^{m,m}} + \sum_{0 \leq j \leq \left\lfloor \frac{d}{2} + \right\rfloor} \left\| \nabla^j \langle x \rangle^{k_1} \nabla^{\left\lfloor \frac{d}{2} + \right\rfloor + j} f \right\|_2 \\
\lesssim \left\| f \right\|_{H^{m,m}} + \sum_{0 \leq j \leq \left\lfloor \frac{d}{2} + \right\rfloor} \left\| \langle x \rangle^{k_1 - j} \nabla^{\left\lfloor \frac{d}{2} + \right\rfloor - j} f \right\|_2 \\
\lesssim \left\| f \right\|_{H^{m,m}}.
\]
The lemma is proved.

Lemma 3.5 (Bilinear estimate in $H^{m,m}$). We have
\[
\left\| fg \right\|_{H^{m,m}} \lesssim \left\| f \right\|_{W^{m,\infty}} \left\| g \right\|_{H^{m,m}},
\]
with the implicit constant depending only on $m$. 

Proof.
\[ \| fg \|_{H^{m,m}} \leq \sum_{0 \leq j \leq m} \| \langle x \rangle^{m-j} \nabla^j (fg) \|_2 \]
\[ \lesssim \sum_{0 \leq j \leq m} \sum_{0 \leq k \leq j} \| \langle x \rangle^{m-j} \nabla^{j-k} f \nabla^k g \|_2 \]
\[ \lesssim \sum_{0 \leq j \leq m} \sum_{0 \leq k \leq j} \| (\langle x \rangle^{m-k} \nabla^k g) (\langle x \rangle^{k-j-k} f) \|_2 \]
\[ \lesssim \sum_{0 \leq j \leq m} \sum_{0 \leq k \leq j} \| \langle x \rangle^{m-k} \nabla^k g \|_2 \| \nabla^{j-k} f \|_{\infty} \]
\[ \lesssim \| g \|_{H^{m,m}} \| f \|_{W^{m,\infty}}. \]

Lemma 3.6. Let \( C > 0 \), \( j \geq 2 \) and \( m \geq \frac{d}{2} + 1 + \frac{C_j}{j-1} \), then
\[ \| \langle x \rangle^{C_j} h^j \|_{H^{m,m}} \lesssim j^m \| h \|_{H^{m,m}}, \]
where the implicit constant depends only on \( m \).

Proof. From the definition and the chain rule, we estimate
\[ \| \langle x \rangle^{C_j} h^j \|_{H^{m,m}} \]
\[ \leq \sum_{0 \leq l \leq m} \| \langle x \rangle^{m-l} \nabla^l (\langle x \rangle^{C_j} h^j) \|_2 \]
\[ \lesssim \sum_{0 \leq l \leq m} j^l \| \langle x \rangle^{m-l+C_j-l_0} \nabla^l h \nabla^l \cdots \nabla^{l_0} h h^{j-l} \|_2 \]
\[ \lesssim j^m \sum_{0 \leq l \leq m} \| \langle x \rangle^{m-l+C_j-l_0} \langle x \rangle^{-(m-l_1+m-l_2-d-1)} \cdots \|_{\infty} \]
\[ \cdot \| \langle x \rangle^{m-l_1} \nabla^l h \|_2 \cdot \| \langle x \rangle^{m-l_2} \|_2 \|
\[ \cdot \| \langle x \rangle^{m-l_2} \|_2 \| \nabla^{l_1} h \|_2 \| \nabla^{l_2} h \|_2 \| \nabla^{j-l} h \|_2 \| \]
\[ \cdot \| \langle x \rangle^{m-l_2} \|_2 \| \nabla^{l_1} h \|_2 \| \nabla^{l_2} h \|_2 \| \nabla^{j-l} h \|_2 \| \]
\[ \cdot \| \langle x \rangle^{m-l_2} \|_2 \| \nabla^{l_2} h \|_2 \| \nabla^{j-l} h \|_2 \|. \]

Since \( m > \frac{d}{2} + 1 + \frac{C_j}{j-1} \), it is not difficult to verify that the exponent of \( \langle x \rangle \) is non-positive in the first factor of the last expression. This combined with Lemma 3.4 shows that
\[ \| \langle x \rangle^{C_j} h^j \|_{H^{m,m}} \lesssim j^m \| h \|_{H^{m,m}}, \]
(3.9)

We will prove the following

Proposition 3.7. Let \( a \in \mathbb{R} \). Let \( \mathcal{Y}_+ \) and \( W_0^a = W_0^a(t, x) \) be the same as in Lemma 2.1. Assume \( m \geq 3d \) is fixed. Then there exists \( k_0 > 0 \) and a unique solution \( W^a(t, x) \) for the equation in (1.1) which satisfies the following: for any \( k \geq k_0 \), there exists \( t_k \geq 0 \) such that \( \forall t \geq t_k \),
\[ \| W^a(t) - W_0^a(t) \|_{H^{m,m}} \leq e^{-\alpha t}, \quad \alpha = (k + \frac{1}{2}) \epsilon_0. \]
(3.10)
Moreover, we have

\[ \| W^a(t) - W - ae^{-et} Y_+ \|_{H^{m,m}} \leq e^{-\frac{3}{2}e_0 t}. \] (3.11)

**Proof.** Let \( h = W^a - W^a_k \), then \( W^a \) is the solution of (1.1) as long as \( h \) is a solution of the equation (3.3) which tends to 0 as \( t \to \infty \). From Duhamel's formula, it is equivalent to solve the following integral equation

\[
h(t) = i \int_t^\infty e^{i(t-s)\Delta} [-\Gamma(h) - iR(h + v_k) + iR(v_k) + i\varepsilon^a_k](s)ds
=: \Phi(h(t)). \tag{3.12}
\]

Define the space \( \Sigma_{t_k} \) by \( \| f \|_{\Sigma_{t_k}} = \sup_{t \geq t_k} e^{\alpha t} \| f(t) \|_{H^{m,m}} \) and introduce the unit ball

\[ B_k = \{ f = f(t, x) : \| f \|_{\Sigma_{t_k}} \leq 1 \}. \]

We shall show that \( \Phi \) is a contraction on \( B_k \). Taking \( h \in B_k \), we compute the \( H^{m,m} \) norm of \( \Phi(h(t)) \):

\[
\| \Phi(h(t)) \|_{H^{m,m}} \leq \int_t^\infty \| e^{i(t-s)\Delta} \Gamma(h(s)) \|_{H^{m,m}} ds \tag{3.13}
+ \int_t^\infty \| e^{i(t-s)\Delta} (R(h + v_k) - R(v_k))(s) \|_{H^{m,m}} ds \tag{3.14}
+ \int_t^\infty \| e^{i(t-s)\Delta} \varepsilon^a_k(s) \|_{H^{m,m}} ds \tag{3.15}
\]

To estimate (3.13), we use Lemma 3.5, Lemma 3.3 to get

\[
(3.13) \leq \int_t^\infty e^{C|t-s|} \| \Gamma(h(s)) \|_{H^{m,m}} ds \leq \frac{1}{\alpha - C} e^{-\alpha t} \| h \|_{\Sigma_{t_k}} \tag{3.16}
\]

Since \( \alpha = (k + \frac{1}{2})e_0 \), by taking \( k_0 \) sufficient large, we have

\[
(3.13) \leq \frac{1}{100} e^{-\alpha t} \| h \|_{\Sigma_{t_k}} \leq \frac{1}{100} e^{-\alpha t} \tag{3.17}
\]

for all \( k \geq k_0 \).

Now we deal with (3.15). Note that by Lemma 3.1, \( \varepsilon^a_k(t) = O(e^{-(k+1)e_0 t}) \) in \( \mathcal{S}(\mathbb{R}^d) \). This implies

\[ \| \varepsilon^a_k(t) \|_{H^{m,m}} \leq C_k e^{-(k+1)e_0 t}. \]
Thus,
\[
\int_t^\infty e^{C|t-s|} \|\varepsilon^a_k(s)\|_{H^{m,m}} ds \leq C_k \int_t^\infty e^{C|t-s|} e^{-\frac{1}{2} (k+1) e_0 s} ds \\
\leq C_k \int_t^\infty e^{-\frac{1}{4} e_0 t} e^{-\frac{1}{2} (k+1) e_0 s} ds \\
\leq C_k e^{-\frac{1}{2} (k+1) e_0 t} \leq \frac{1}{100} e^{-\alpha t}
\]
if \( t \geq t_k \) and \( t_k \) is sufficiently large.

It remains to estimate (3.14). The reason that we can take \( m \) derivatives is that both \( v_k \) and \( h \) are small compared to \( W \). Indeed by Remark 3.2, we have
\[
|v_k(t, x)| < \frac{1}{2} W(x), \quad \forall \ t \geq t_k, \ x \in \mathbb{R}^d.
\]
Moreover, since \( h \in \Sigma_{t_k} \) and \( m \geq 3d \), by Lemma 3.4 we have
\[
\|\langle x \rangle^{d-2} h(t)\|_{\infty} \lesssim \|h(t)\|_{H^{m,m}} \leq e^{-\alpha t} \|h\|_{\Sigma_{t_k}}.
\]
As a consequence, we have
\[
|h(t, x)| \lesssim e^{-\alpha t} \langle x \rangle^{-(d-2)} \|h\|_{\Sigma_{t_k}} \leq \frac{1}{4} W(x).
\]
Using (3.19) and (3.20) together with the expansion for the real analytic function \( P(z) = |1 + z|^{p_c-1}(1 + z) \) for \( |z| \leq \frac{3}{4} \) which takes the form
\[
P(z) = 1 + \frac{p_c + 1}{2} z + \frac{p_c - 1}{2} \bar{z} + \sum_{j_1 + j_2 \geq 2} a_{j_1, j_2} z^{j_1} \bar{z}^{j_2},
\]
we write
\[
i(R(v_k + h) - R(v_k)) = W^{p_c}\left[ 1 + \frac{v_k + h}{W} \right]^{p_c-1}(1 + v_k + h) - 1 + \frac{v_k}{W} \right]^{p_c-1}(1 + v_k + h) \\
- \frac{p_c + 1}{2} \frac{h}{W} - \frac{p_c - 1}{2} \frac{\bar{h}}{W} \\
= W^{p_c}\left[ \sum_{j_1 + j_2 \geq 2} a_{j_1, j_2} \left( \frac{v_k + h}{W}\right)^{j_1} \left( \frac{v_k + \bar{h}}{W}\right)^{j_2} - \left( \frac{v_k}{W}\right)^{j_1} \left( \frac{\bar{v}_k + \bar{h}}{W}\right)^{j_2} \right] \\
= O\left( \sum_{j \geq 2, 1 \leq i \leq j} a_j C_{i,j} W^{p_c-j} O(v_k^{j-i} h^i) \right),
\]
where the last equality following from using binomial expansion and regrouping coefficients, and we have the bound
\[
a_j = O\left( \frac{p_c(p_c - 1) \cdots (p_c - j + 1)}{j!} \right) \text{ and } |a_j| \leq 1,
\]
\[
C_{i,j} = O\left( \frac{j!}{i!(j-i)!} \right) \text{ and } C_{i,j} \leq 2^j.
\]
The notation $O(v_k^{j-i}h^i)$ denotes terms of the form

$$v_k^\alpha e_k^\beta h^\delta_1 h^\delta_2$$

with $\alpha_1 + \alpha_2 = j - i$, $\beta_1 + \beta_2 = i$.

Now we use this expression to estimate $\|R(v_k + h) - R(v_k)\|_{H^{m,m}}$. Using Lemma 3.5 and Lemma 3.6, we have

$$\|R(v_k + h) - R(v_k)\|_{H^{m,m}} \leq \sum_{j \geq 2} a_j C_{i,j} \|W^{p_e-j} v_k^{j-i} h^i\|_{H^{m,m}}$$

$$\leq \sum_{j \geq 2} 2^j \|W^{-j} v_k^{j-1} h\|_{H^{m,m}}$$

$$+ \sum_{j \geq 2} 2^j \|(W^{-1} v_k)^{j-i} W^{-i} h^i\|_{H^{m,m}}$$

$$\leq \sum_{j \geq 2} 2^j \|W^{-j} v_k^{j-1}\|_{W^{m,\infty}} \|h\|_{H^{m,m}}$$

$$+ \sum_{j \geq 2} 2^j m \|(W^{-1} v_k)^{j-i}\|_{W^{m,\infty}} \|h\|_{H^{m,m}}.$$

Applying Remark 3.2 and in view of $h \in \Sigma_{t_k}$, we have

$$\|W^{-j} v_k^{j-1}\|_{W^{m,\infty}} \leq j^m C_{k,m} e^{-(j-1)eot},$$

$$\|(W^{-1} v_k)^{j-i}\|_{W^{m,\infty}} \leq j^m C_{k,m} e^{-(j-i)eot}.$$ 

Noting moreover that

$$\|h\|_{H^{m,m}} \leq e^{-\alpha t} \|h\|_{\Sigma_{t_k}},$$

we estimate

$$\|R(v_k + h)(t) - R(v_k)(t)\|_{H^{m,m}} \leq \sum_{j \geq 2} 2^j j^{2m} C_{k,m} (e^{-\alpha t} \|h\|_{\Sigma_{t_k}}) (e^{-eot^j} j^{-i})$$

$$\leq e^{-\alpha t} \|h\|_{\Sigma_{t_k}} \sum_{j \geq 2} 2^j j^{2m} C_{k,m} e^{-(\alpha(j-1) + (j-i)eot)t}$$

$$\leq e^{-\alpha t} \|h\|_{\Sigma_{t_k}} \sum_{j \geq 2} 2^j j^{2m} C_{m,k} e^{-(j-1)eot_k}$$

$$\leq \frac{1}{100} e^{-\alpha t} \|h\|_{\Sigma_{t_k}}.$$ 

The last inequality comes from the fact we can choose $t_k$ large enough such that the series converges. Now we are ready to estimate (3.14). Using lemma 3.3 we have

$$\int_t^\infty e^{C|t-s|} \|R(v_k + h)(s) - R(v_k)(s)\|_{H^{m,m}} ds \leq \frac{1}{100} \|h\|_{\Sigma_{t_k}} \int_t^\infty e^{C(s-t)} e^{-\alpha s} ds \leq \frac{1}{100} e^{-\alpha t} \|h\|_{\Sigma_{t_k}}$$

$$\leq \frac{1}{100} e^{-\alpha t}.$$
Collecting the estimates (3.17), (3.18) and (3.23), we obtain
\[ \|\Phi(h(t))\|_{H^m,m} \leq \frac{1}{10} e^{-\alpha t} \] (3.24)
for all \( k \geq k_0 \) and \( t \geq t_k \). Therefore
\[ \|\Phi(h)\|_{\Sigma_{t_k}} \leq \frac{1}{10}, \]
which shows that \( \Phi \) is a contraction in \( B \) and \( W \) where the constants only remain to show that
uniqueness of the solution to the equation in (1.1) such that (3.10) holds. It
Taking \( h_1 \) and \( h_2 \) in \( \Sigma_{t_k} \), we compute
\[ \|\Phi(h_1(t)) - \Phi(h_2(t))\|_{H^m,m} \]
\[ \leq \int_t^\infty \|e^{i(t-s)}\Delta (h_1(s) - h_2(s))\|_{H^m,m} ds \] (3.25)
\[ + \int_t^\infty \|e^{i(t-s)}\Delta (R(v_k + h_1) - R(v_k + h_2))(s)\|_{H^m,m} ds \] (3.26)
The estimate of (3.25) is the same as (3.16), we omit the details. To estimate (3.26), we write
\[ -i(R(v_k + h_1) - R(v_k + h_2)) \]
\[ = \sum_{j \geq 2} a_{j_1,j_2} W^{p-\epsilon-j} \left[ \left( \frac{v_k + h_1}{W} \right)^{j_1} \left( \frac{\bar{v}_k + \bar{h}_1}{W} \right)^{j_2} - \left( \frac{v_k + h_2}{W} \right)^{j_1} \left( \frac{\bar{v}_k + \bar{h}_2}{W} \right)^{j_2} \right] \]
\[ = O \left( \sum_{j \geq 1, 1 \leq i \leq j-1} a_j C_{i,j} W^{p-\epsilon-j} O \left( (h_1 - h_2) v_k^{j-i} h^i \right) \right), \]
where the constants \( a_j \), \( C_{i,j} \) are the same as in (3.22). We are in the same situation as before. Therefore, we obtain
\[ \|\Phi(h_1(t)) - \Phi(h_2(t))\|_{H^m,m} \]
\[ \leq \frac{1}{10} e^{-\alpha t} \|h_1 - h_2\|_{\Sigma_{t_k}}, \forall \ k \geq k_0, \ t \geq t_k, \] (3.27)
which shows that \( \Phi \) is a contraction in \( B_k \). This proves the existence and uniqueness of the solution to the equation in (1.1) such that (3.10) holds. It only remains to show that \( W^a(t,x) \) is independent of \( k \). Indeed, let \( k_1 < k_2 \) and \( W^a, \bar{W}^a \) be the corresponding solutions such that
\[ \|W^a(t) - W^a_{k_2}(t)\|_{H^m,m} \leq e^{-(k_2+\frac{1}{2})\alpha t}, \forall \ t \geq t_{k_2}, \]
Without lose of generality we also assume \( t_{k_1} \leq t_{k_2} \), then the triangle inequality gives that
\[ \|\bar{W}^a(t) - W^a_{k_1}(t)\|_{H^m,m} \leq \|\bar{W}^a(t) - W^a_{k_2}(t)\|_{H^m,m} + \sum_{k_1 \leq j < k_2} e^{-j\alpha t} \Phi_j \|_{H^m,m} \]
\[ \leq e^{-(k_2+\frac{1}{2})\alpha t}, \forall \ t \geq t_{k_2}. \]
Therefore \( W^a(t) = \bar{W}^a(t) \) on \( [t_{k_2}, \infty) \) and we conclude \( W^a \equiv \bar{W}^a \) by uniqueness of solutions to (1.1). This shows that \( W^a \) does not depend on \( k \). The existence and uniqueness of the solution to (3.12) is proved.
We finally verify (3.11). Let $k_0$ be the constant specified above, then by the triangle inequality and Lemma 3.1 we have
\[
\|W^a(t) - W - ae^{-e_0t}Y_+\|_{H^{m,m}} \\
\leq \|W^a(t) - W^a(t_0)\|_{H^{m,m}} + \|v(t) - ae^{-e_0t}Y_+\|_{H^{m,m}} \\
\leq e^{-(k_0 + 1)e_0t} + \sum_{2 \leq j \leq k_0} e^{-je_0t} \|b_j\|_{H^{m,m}} \\
\leq e^{-\frac{3}{2}e_0t}
\]
for all sufficiently large $t$. Proposition 3.7 is proved. \hfill \Box

Corollary 3.8. Let $w^a = W^a - W$, then there exists $t_0 > 0$ such that for all $t \geq t_0$ and all $2 \leq p \leq \infty$, we have
\[
\|\langle x \rangle^{l_1} \nabla^{l_2} w^a(t)\|_{L^p_x} \leq e^{-\frac{1}{2}e_0t},
\]
(3.28)
as long as $l_1 + l_2 + \frac{d}{2} + 1 \leq m$. In particular,
\[
\|w^a\|_{\dot{H}^{1}(t,\infty)} \leq e^{-\frac{3}{2}e_0t}.
\]

Proof. Let $k_0$ be the same as in Proposition 3.7, then by Remark 3.2 we have
\[
\|w^a(t)\|_{H^{m,m}} \leq \|w^a(t) - v(t_0)\|_{H^{m,m}} + \|v(t_0)\|_{H^{m,m}} \\
\leq e^{-(k_0 + 1)e_0t} + \sum_{1 \leq j \leq k_0} e^{-je_0t} \|b_j\|_{H^{m,m}}.
\]
Thus for $t_0$ sufficiently large and $t \geq t_0$, we obtain
\[
\|w^a(t)\|_{H^{m,m}} \leq e^{-\frac{3}{2}e_0t}.
\]
An application of Sobolev embedding gives that for any $p$ with $2 \leq p \leq \infty$,
\[
\|\langle x \rangle^{l_1} \nabla^{l_2} w^a(t)\|_{L^p_x} \preceq \|w^a\|_{H^{m,m}} \leq e^{-\frac{1}{2}e_0t},
\]
provided $l_1 + l_2 + \frac{d}{2} + 1 < m$. The Corollary is proved. \hfill \Box

Before finishing this section, we make the following two remarks.

Remark 3.9. In next section we shall show for $a, b$ such that $ab > 0$, $W^a$ is just a time translation of $W^b$. This will allow us to define $W^\pm$ as $W^{\pm 1}$ and to classify the solutions with threshold energy.

The second remark concerns the behavior of $W^\pm$ in the negative time direction.

Remark 3.10. From the construction of $W^\pm(t)$, it is clear that they both approaches to the ground state $W$ exponentially fast as $t \to +\infty$. For the behavior of $W^\pm$ in negative time direction, we can apply the same argument in [7](see Corollary 3.2, Corollary 4.2 for instance) to conclude that $W^-$ scatters when $t \to -\infty$ and $W^+$ blows up at finite time. To get the blowup of $W^+$, we need the crucial property $W^+ \in L^2_x$ which is now available as we are in dimensions $d \geq 6$. 

4. Classification of the solution

Our purpose of this section is to prove Theorem 4.1. Following the argument in [7], the key step is to establish the following

**Theorem 4.1.** Let \( \gamma_0 > 0 \). Assume \( u \) is the solution of the equation in (1.1) satisfying \( E(u) = E(W) \) and
\[
\| u(t) - W \|_{\dot{H}^1} \leq C e^{-\gamma_0 t}, \quad \forall \ t \geq 0,
\]
then there exists \( a \in \mathbb{R} \) such that
\[
u = W^a.
\]

As a corollary of Theorem 4.1, we see that modulo time translation, all the \( \{W^a, a > 0\} \) and \( \{W^a, a < 0\} \) are same.

**Corollary 4.2.** For any \( a \neq 0 \), there exists \( T_a \in \mathbb{R} \) such that
\[
\begin{cases}
W^a(t) = W^+(t + T_a), & \text{if } a > 0, \\
W^a(t) = W^-(t + T_a), & \text{if } a < 0,
\end{cases}
\]

We now prove Theorem 4.1. The strategy is the following: we first prove that there exists \( a \in \mathbb{R} \) such that \( u(t) - W^a(t) \) has enough decay, then using the decay estimate to show that \( u(t) - W^a(t) \) is actually identically zero. To this end, we have to input the condition (4.1) and upgrade it to the desired decay estimate. At this point, we need the following crucial result from [7].

**Lemma 4.3.** Let \( h \) be the solution of the equation
\[
\partial_t h + L h = \varepsilon.
\]
And for \( t \geq 0 \),
\[
\| \varepsilon \|_{\dot{H}^1([t, \infty))} \leq C e^{-c_1 t}, \quad \| \frac{\partial}{\partial t} \varepsilon \|_{L^2_x} \leq C e^{-c_1 t},
\]
\[
\| h(t) \|_{\dot{H}^1} \leq C e^{-\varepsilon_0 t},
\]
where \( c_0 < c_1 \). Then the following statements hold true,
- If \( c_0 < c_1 \) or \( \varepsilon_0 < c_0 < c_1 \), then
  \[
  \| h \|_{\dot{S}^1([t, \infty))} \leq C \eta e^{-(c_1 - \eta) t},
  \]
- If \( c_0 \leq \varepsilon_0 < c_1 \), then there exists \( a \in \mathbb{R} \) such that
  \[
  \| h - ae^{-\varepsilon_0 t} \mathcal{Y}_+ \|_{\dot{S}^1([t, \infty))} \leq C \eta e^{-(c_1 - \eta) t}.
  \]

Let \( v = u - W \), then (4.1) gives that
\[
\| v(t) \|_{\dot{H}^1} \leq e^{-\varepsilon_0 t}, \quad t \geq 0.
\]
Without loss of generality we assume \( \gamma_0 < \varepsilon_0 \). We first show that this decay rate can be upgraded to \( e^{-\varepsilon_0 t} \). More precisely, we have

**Proposition 4.4.** Let \( v = u - W \), then there exists \( t_0 > 0 \) such that for all \( t \geq t_0 \),
\[
\| v \|_{\dot{S}^1([t, \infty))} \leq C e^{-\varepsilon_0 t},
\]
\[
\| R(v) \|_{\dot{N}^1([t, \infty))} \leq C e^{\frac{d+2}{2} \varepsilon_0 t}, \quad \| R(v)(t) \|_{L^2_x}^{\frac{d}{2}} \leq C e^{-\varepsilon_0 \gamma_0 t}.
\]
In particular, there exists \( a \in \mathbb{R} \) such that
\[
\|v - ae^{-\rho_0 t}Y\|_{\dot{S}^1(I,\infty)} \lesssim C_0 e^{-(d+\frac{d}{2})\gamma_0 t}. \tag{4.9}
\]

**Proof.** First we show that \((4.9)\) is a consequence of \((4.7), \ 4.8\). To see this, note that \( v \) satisfies the equation
\[
i\partial_t v + \Delta v + \Gamma(v) + iR(v) = 0, \tag{4.10}
\]
or equivalently,
\[
\partial_t v + \mathcal{L}v = -R(v). \tag{4.11}
\]

Applying Lemma 4.3 with \( h = v, \varepsilon = -R(v), \ c_0 = e_0, \ c_1 = \frac{d+\frac{d}{2}}{d+\frac{d}{2}}e_0 \) and using the estimates \((4.7)\) and \((4.8)\), we obtain \((4.9)\). So we only need to establish \((4.7)\) and \((4.8)\). This will be done in two steps. At the first step, we prove that the Strichartz norm \( v \) decays like \( e^{-\gamma_0 t} \) and the dual Strichartz norm of \( R(v) \) decays even faster. Secondly, we iterate this process and upgrade the decay estimate by using Lemma 4.3 finitely many times.

**Step 1.** We prove there exists \( t_0 > 0 \) such that for \( t \geq t_0 \),
\[
\|v\|_{\dot{S}^1((t,\infty))} \lesssim e^{-\gamma_0 t}, \quad \|R(v)\|_{\dot{N}^1((t,\infty))} \lesssim e^{-(d+\frac{d}{2})\gamma_0 t}. \tag{4.12}
\]

Let \( \tau \) be a small constant to be chosen later. Using Strichartz estimate on the time interval \([t, t + \tau]\), we have
\[
\|v\|_{\dot{S}^1((t, t+\tau])} \lesssim \|v(t)\|_{\dot{H}^1} + \|\Gamma(v)\|_{\dot{N}^1((t, t+\tau])} + \|R(v)\|_{\dot{N}^1((t, t+\tau])}. \tag{4.13}
\]

For the linear term, we have
\[
\|\Gamma(v)\|_{\dot{N}^1((t, t+\tau])} \lesssim \|W^{p_c - 1}\nabla v\|_{L^1_I L^{2}_x((t, t+\tau]) \times \mathbb{R}^d} + \|W^{p_c - 2}\nabla W\|_{L^1_I L^{2}_x((t, t+\tau]) \times \mathbb{R}^d} \tag{4.14}
\]
\[
\lesssim \tau \|W^{p_c - 1}\nabla^\tau v\|_{L^\infty_T L^2_x} + \tau \|W^{p_c - 2}\nabla W\|_{L^\infty_T L^{2}_x} \tag{4.15}
\]
\[
\lesssim \tau \|v\|_{\dot{S}^1((t, t+\tau])}. \tag{4.16}
\]

This is good for us. Now we deal with the term \( R(v) \). In lower dimensions, it is easy to see that \( R(v) \) is superlinear in \( v \). In higher dimensions \((d \geq 6)\), this is trickier. Here we will rely heavily on the fact that \( W \) has nice decay to show that \( R(v) \) is essentially superlinear in \( v \). We claim for any time interval \( I \), that
\[
\|R(v)\|_{\dot{N}^1(I)} \lesssim \|v\|_{\dot{S}^1(I)}^{\frac{d+\frac{d}{2}}{2}} + \|v\|_{\dot{S}^1(I)}^{p_c}. \tag{4.17}
\]

Assume the claim is true for the moment. By \((4.6), \ 4.13, \ 4.14\) and \((4.15)\), we have
\[
\|v\|_{\dot{S}^1((t, t+\tau])} \lesssim e^{-\gamma_0 t} + \tau \|v\|_{\dot{S}^1((t, t+\tau])} + \|v\|_{\dot{S}^1((t, t+\tau])}^{p_c} + \|v\|_{\dot{S}^1((t, t+\tau])}^{\frac{d+\frac{d}{2}}{2}} \tag{4.18}
\]
Taking \( \tau \) small enough, a continuity argument shows that there exists \( t_0 > 0 \), such that for all \( t \geq t_0 \),
\[
\|v\|_{\dot{S}^1((t, t+\tau])} \lesssim e^{-\gamma_0 t}. \tag{4.19}
\]
Therefore, we have
\[ \|v\|_{S^1([t,\infty))} \leq \sum_{j \geq 0} \|v\|_{S^1([t+\gamma j, t+\gamma (j+1)])} \]
\[ \lesssim \sum_{j \geq 0} e^{-\gamma_0 (t+\gamma j)} \leq e^{-\gamma_0 t} \frac{1}{1 - e^{-\gamma_0}} \]
\[ \lesssim e^{-\gamma_0 t}. \]

Plugging this estimate into (4.15), we have proved (4.12).

Now it remains to prove the claim (4.15). Recall that
\[ iR(v) = |v + W|^{p_c - 1} (v + W) - W^{p_c} - \frac{p_c + 1}{2} W^{p_c - 1} (v + W) - \frac{p_c - 1}{2} W^{p_c - 1} \Delta, \]
\[ = W^{p_c} J_v, \]
where
\[ J(z) = |1 + z|^{p_c - 1} (1 + z) - 1 - \frac{p_c + 1}{2} z - \frac{p_c - 1}{2} \bar{z}. \]

We write \( i\nabla R(v) \) as,
\[ i\nabla R(v) = p_c W^{p_c - 1} \nabla W J_v = W^{p_c} J_v(v) \nabla (\frac{v}{W}) + W^{p_c} J_v(\frac{v}{W}) \nabla (\bar{v}) \]
\[ = \frac{p_c + 1}{2} |v + W|^{p_c - 1} \nabla (v + W) - W^{p_c - 1} \nabla W - (p_c - 1) W^{p_c - 2} \nabla W v - W^{p_c - 1} \nabla v \]
\[ + \frac{p_c - 1}{2} |v + W|^{p_c - 3} (v + W)^2 \nabla (\bar{v}) + W^{p_c - 1} \nabla W \]
\[ - (p_c - 1) W^{p_c - 2} \nabla W \bar{v} - W^{p_c - 1} \nabla \bar{v}. \]

Note moreover for \( |z| < 1 \), \( J(z) \) is real analytic in \( z \). Thus for \( |z| \leq \frac{3}{4} \), we have
\[ |J(z)| \lesssim |z|^2, \]
\[ |J_z(z)|, |J_{zz}(z)| \lesssim |z|. \]

To estimate \( \|\nabla R(v)\|_{\dot{N}^0} \), we split \( \mathbb{R}^d \) into regimes \( \{ x : |v(x)| \leq \frac{1}{4} W(x) \} \) and \( \{ x : |v(x)| > \frac{1}{4} W(x) \} \). In the first regime, we use the expression (4.18), the estimate (4.19) and Lemma 2.3 to get
\[ \|\nabla R(v)\|_{\dot{N}^0(I,|v| \leq \frac{1}{4} W)} \lesssim \|W^{p_c - 1} \nabla W \|_{\dot{N}^0(I,|v| \leq \frac{1}{4} W)} + \|W^{p_c} \frac{v}{W} \nabla (\frac{v}{W})\|_{\dot{N}^0(I,|v| \leq \frac{1}{4} W)} \]
\[ \lesssim \|W^{p_c - 3} \nabla W v^2\|_{\dot{N}^0(I,|v| \leq \frac{1}{4} W)} + \|W^{p_c - 2} v \nabla v\|_{\dot{N}^0(I,|v| \leq \frac{1}{4} W)} \]
\[ \lesssim \|v\|_{S^1(I)} + \|v\|_{S^1(I)}. \]

In the second regime, we use the second equality in (4.18). Using the triangle inequality, Lemma 2.3 and noting that the conjugate counterpart will give
the same contribution we obtain
\[ \| \nabla R(v) \|_{\dot{S}^1(I;|v|>\frac{1}{2}W)} \lesssim \| (v + W|^{p_c-1} - W^{p_c-1}) \nabla v \|_{\dot{S}^0(I;|v|>\frac{1}{2}W)} \\
+ \| (v + W|^{p_c-1} - W^{p_c-1}) \nabla W \|_{\dot{S}^0(I;|v|>\frac{1}{2}W)} \\
+ \| W^{p_c-2} \nabla W v \|_{\dot{S}^0(I;|v|>\frac{1}{2}W)} \]
\[ \lesssim \| v|^{p_c-1} \nabla v \|_{\dot{S}^0(I)} + \| \nabla W|v|^{p_c-1} \|_{\dot{S}^0(I;|v|>\frac{1}{2}W)} \\
+ \| W^{p_c-2} \nabla W v \|_{\dot{S}^0(I;|v|>\frac{1}{2}W)} \]
\[ \lesssim \| v\|^{|p_c}\dot{S}^1(I) + \| v\|_{\dot{S}^{\frac{4}{2}}(I)}. \]

Combining the two estimates together, \([4.15]\) is verified. Finally, we quickly show that
\[ \| R(v) \|_{L^2_x} \lesssim e^{-p_c\gamma_0 t}. \]
Indeed, note \( R(v) = -i W^{p_c} J(\frac{v}{W}) \) and \( J(z) \lesssim |z|^{p_c} \), we estimate
\[ \| R(v)(t) \|_{L^2_x} \lesssim \| v|^{p_c} \|_{L^2_x} \]
\[ \lesssim \| v(t) \|_{L^2_x} \lesssim \| v(t) \|_{\dot{H}^1} \]
\[ \lesssim e^{-p_c\gamma_0 t}. \]

**Step 2.** Iteration. Since \( v \) satisfies the equation \([4.11]\), we can use Lemma \([4.3]\) with \( h = v \) and \( \varepsilon = -R(v) \) to get
\[ \| v(t) \|_{\dot{H}^1} \leq C(e^{-\gamma_0 t} + e^{\frac{d+4}{2} \gamma_0 t}). \]
If \( \frac{d+4}{2} \gamma_0 \geq \gamma_0 \), then by repeating the same arguments as above, we have
\[ \| v\|_{\dot{S}^1((t,\infty))} \lesssim e^{-\gamma_0 t}, \]
and the proposition is proved. Otherwise, we are at the same situation as the first step with \( \gamma_0 \) now replaced by \( \frac{d+4}{2} \gamma_0 \). Iterating this process finitely many times, we obtain the proposition.

Based on this result, we now show that \( u - W^a \) decays arbitrarily fast.

**Proposition 4.5.** For any \( m \) > 0, there exists \( t_m > 0 \) such that
\[ \| u - W^a \|_{\dot{S}^1((t,\infty))} \leq e^{-mt}, \quad \forall \ t \geq t_m. \]  \((4.20)\)

**Proof.** **Step 1.** We first remark that as a consequence of Proposition 4.4 we can prove
\[ \| u - W^a \|_{\dot{S}^1((t,\infty))} \leq e^{-\frac{d}{m} \gamma_0 t}, \quad \forall \ t \geq t_0. \]
Indeed by the triangle inequality and recalling that \( v = u - W \), we estimate
\[ \| u - W^a \|_{\dot{S}^1((t,\infty))} \leq \| v - ae^{-\gamma_0 t} \|_{\dot{S}^1((t,\infty))} + \| W^a - v_{k_0} \|_{\dot{S}^1((t,\infty))} + \| v_{k_0} - ae^{-\gamma_0 t} \|_{\dot{S}^1((t,\infty))}. \]
For the first term, we use \((4.9)\) to get
\[
\|v - ae^{-\frac{d}{d-2}e_0 t}\|_{\dot{S}^1([t,\infty))} \leq \frac{1}{2}e^{-\frac{d}{d-2}e_0 t}.
\]

For the last two terms, we use the definition of \(v_k\) (see \((3.1)\)) and Proposition \((3.7)\) to obtain for any \(2 \leq p \leq \infty\):
\[
\|
abla (v_0 - ae^{-\frac{d}{d-2}e_0 t}Y_\mp)\|_{L^p_x} \leq e^{-\frac{d}{d-2}e_0 t},
\]
\[
\|
abla (w^a - v_0)\|_{L^p_x} \leq \|w^a - v_0\|_{H^{m,m}},
\]
\[
\leq \|W^a - W^a_0\|_{H^{m,m}},
\]
\[
\leq e^{-\frac{d}{d-2}e_0 t}.
\]

Integrating in the time variable over \([t,\infty),\) we have
\[
\|w^a - v_0\|_{\dot{S}^1([t,\infty))} + \|v_0 - ae^{-\frac{d}{d-2}e_0 t}Y_\mp\|_{\dot{S}^1([t,\infty))} \leq \frac{1}{2}e^{-\frac{d}{d-2}e_0 t} \leq \frac{1}{2}e^{-\frac{d}{d-2}e_0 t}.
\]

Hence
\[
\|u - W^a\|_{\dot{S}^1([t,\infty))} \leq e^{-\frac{d}{d-2}e_0 t}.
\]

**Step 2.** We will prove \((4.20)\) by induction. More precisely, suppose there exists \(t_m > 0\) such that
\[
\|u - W^a\|_{\dot{S}^1([t,\infty))} \leq e^{-m_1 t}, \quad \forall t \geq t_m, \tag{4.21}
\]
we aim to prove for \(t\) large enough that
\[
\|u - W^a\|_{\dot{S}^1([t,\infty))} \leq e^{-\frac{d}{d-2}e_0 t}e^{-m_1 t}. \tag{4.22}
\]

From Step 1, we can assume that \((4.21)\) holds with \(m_1 \geq \frac{d}{d-2}e_0.\) Let \(h = u - W^a,\) then \(h\) solves the equation
\[
\partial_t h + \mathcal{L}h = -R(h + w^a) + R(w^a), \tag{4.23}
\]
with
\[
\|h\|_{\dot{S}^1([t,\infty))} \leq e^{-m_1 t}, \quad m_1 > e_0.
\]

An application of Lemma \((4.3)\) gives immediately \((4.22)\) if we establish the following
\[
\|R(h + w^a)(t) - R(w^a)(t)\|_{L^p_x} \leq e^{-\frac{d}{d-2}t - m_1 t}, \tag{4.24}
\]
\[
\|R(h + w^a) - R(w^a)\|_{N^1([t,\infty))} \leq e^{-\frac{d}{d-2}t e_0 t}h\|_{\dot{S}^1([t,\infty))} \leq e^{-\frac{d}{d-2}t - m_1 t}, \tag{4.25}
\]
for \(t\) large enough.

The remaining part of the proof is devoted to showing \((4.25),\) \((4.24)\). The idea is similar to the proof of \((4.15).\) We split the space into two regimes. In the regime where \(h\) is large, we use the decay estimate of \(W\) to show that \(R(h + w^a) - R(w^a)\) is superlinear in \(h.\) In the regime where \(h\) is small, we simply use the real analytic expansion of the complex function \(P(z) = |1 + z|^{p-1}(1 + z).\) However, the argument here is more involved than the proof of \((4.15).\)
We first show (4.24). To begin with, we recall the exact form of \( R(h + w^a) - R(w^a) \). We have

\[
i(R(h + w^a) - R(w^a)) = |w^a + h + W|^{p_c - 1}(w^a + h + W) - |w^a + W|^{p_c - 1}(w^a + W) - \frac{p_c + 1}{2}W^{p_c - 1}h - \frac{p_c - 1}{2}W^{p_c - 1}\tilde{h}.
\]

By triangle inequality, we estimate

\[
\|R(h + w^a)(t) - R(w^a)(t)\|_{L^2_x} \lesssim \|R(h + w^a)(t) - R(w^a)(t)\|_{L^2_x}^{2d/(2d+2)}(|h| > \frac{1}{2}W) + \|R(h + w^a)(t) - R(w^a)(t)\|_{L^2_x}^{2d/(2d+2)}(|h| \leq \frac{1}{2}W).
\]

For (4.27), we use the fact that \( |w^a(t,x)| \leq \frac{1}{2}W(x) \) which follows from Corollary 3.8 to estimate

\[
\|R(h + w^a)(t) - R(w^a)(t)\|_{L^2_x} \lesssim \|h|^{p_c - 1}(w^a + W)\|_{L^2_x}^{2d/(2d+2)}(|h| > \frac{1}{2}W) + \|w^a + W + h|^{p_c - 1}h\|_{L^2_x}^{2d/(2d+2)}(|h| > \frac{1}{2}W) \lesssim \|h(t)|^{p_c - 1}h(t)\|_{L^2_x}^{2d/(2d+2)} \lesssim e^{-m_1 p_c t}.
\]

For (4.28), we use \( P(z) = |1 + z|^{p_c - 1}(1 + |z|) \) to rewrite (4.26) into

\[
i(R(h + w^a) - R(w^a)) = W^{p_c}\left(P\left(\frac{h + w^a}{W}\right) - P\left(\frac{w^a}{W}\right) - \frac{p_c + 1}{2} \frac{h}{W} - \frac{p_c - 1}{2} \frac{\tilde{h}}{W}\right).
\]

Note that

\[
\frac{|w^a + h|}{W} \leq \frac{3}{4}, \quad \frac{|w^a|}{W} \leq \frac{1}{2},
\]

We use the expansion for \( P(z) \) (see (3.21)) to write

\[
i(R(h + w^a) - R(w^a)) = \sum_{j_1+j_2 \geq 2} a_{j_1,j_2}\left(\frac{h + w^a}{W}\right)^{j_1} \left(\frac{\tilde{h} + \overline{u}^a}{W}\right)^{j_2} - \left(\frac{w^a}{W}\right)^{j_1} \left(\frac{\overline{u}^a}{W}\right)^{j_2}\right)
\]

\[
= O\left(\sum_{j \geq 2, 1 \leq i \leq j} a_j C_{i,j} W^{p_c - 1 - j} \nabla W(u^a)^{j-i} \tilde{h}^i\right),
\]
where \(|a_j| \lesssim 1\) and \(C_{i,j} \lesssim 2^j\). Therefore by triangle inequality we have

\[
\| R(h + w^a)(t) - R(w^a)(t) \|_{L^2_x(\{ |h| \leq \tfrac{1}{4} W \})} \lesssim \sum_{j \geq 2, \ 1 \leq i \leq j} 2^j \| W^{p_c-j}(w^a(t))^{j-i} h(t)^i \|_{L^2_x(\{ |h(t)| \leq \tfrac{1}{4} W \})} \tag{4.33}
\]

\[
\lesssim \sum_{j \geq 2} 2^j \| h(t) \|_{L^2_x} \| W^{p_c-j}(w^a(t))^{j-1} \|_{L^2_x} + \sum_{j \geq 2, \ 2 \leq i \leq j} 2^j \| h(t) \|_{L^2_x} \| W^{p_c-j}(w^a(t))^{j-i} h(t)^{i-p_c} \|_{L^2_x(\{ |h| \leq \tfrac{1}{4} W \})} \tag{4.34}
\]

\[
\lesssim \sum_{j \geq 2} \| h(t) \|_{H^1_x} \| W^{-1} w^a(t) \|_{L^2_x(j-1)} + \sum_{j \geq 2, \ 2 \leq i \leq j} 2^j \| h(t) \|_{H^1_x} \| W^{-1} \|_{L^\infty_x} \| w^a W^{-1} \|_{L^\infty_x} \lesssim e^{-\epsilon_0(j-1)t} \tag{4.25}
\]

Collecting estimates (4.29) and (4.33) we obtain (4.24).

Next we prove (4.25). To this end, we take the gradient and regroup the term, we have

\[
i \nabla (R(h + w^a) - R(w^a)) \]

\[
= \frac{p_c + 1}{2} \left( (|w^a| + h + W)^{p_c-1} - W^{p_c-1} \right) \nabla h \]

\[
+ \left( |w^a| + h + W^{p_c-1} - |w^a| + W^{p_c-1} \right) \nabla (w^a + W) \]

\[
+ (p_c - 1) W^{p_c-2} \nabla W h \]

\[
+ \frac{p_c - 1}{2} \left( (|w^a| + h + W)^{p_c-3} (w^a + h + W) - W^{p_c-1} \right) \nabla \tilde{h} \]

\[
+ (p_c - 1) W^{p_c-3} \nabla (w^a + W) \]

\[
= \frac{d}{4} \| h \|_{S^{\theta}(\{ \tilde{t} < \infty \})} + \frac{d}{4} \| h \|_{S^{\theta}(\{ \tilde{t} = \infty \})} + \frac{d}{4} \| h \|_{S^{\theta}(\{ \tilde{t} = \infty \})} \tag{4.35}
\]

By Lemma 2.3, Corollary 3.8 and the triangle inequality we have

\[
\| \nabla (R(h + w^a) - R(w^a)) \|_{\dot{H}^{\theta}(\{ \tilde{t} < \infty \}; |h| > \tfrac{1}{4} W)} \]

\[
\lesssim \| h + w^a \|_{H^{\theta}(\{ \tilde{t} < \infty \}; |h| > \tfrac{1}{4} W)} \| \nabla (w^a + W) \|_{\dot{H}^{\theta}(\{ \tilde{t} = \infty \}; |h| > \tfrac{1}{4} W)} + \| W^{p_c-2} \nabla W h \|_{\dot{H}^{\theta}(\{ \tilde{t} = \infty \}; |h| > \tfrac{1}{4} W)} \]

\[
\lesssim \| w^a \|_{S^{\theta}(\{ \tilde{t} < \infty \})} \| h \|_{S^{\theta}(\{ \tilde{t} = \infty \})} + \| h \|_{S^{\theta}(\{ \tilde{t} = \infty \})} + \| h \|_{S^{\theta}(\{ \tilde{t} = \infty \})} \]

\[
\lesssim e^{-\epsilon_0(p_c-1)t} \| h \|_{S^{\theta}(\{ \tilde{t} < \infty \})}.
\]
To get the estimate in the regime where \(|h|\) is small, we adopt the form (4.30). By chain rule we have

\[
i \nabla (R(h + w^a) - R(w^a))
= p_c W^{p_c-1} \nabla W \left[ P \left( \frac{h + w^a}{W} \right) - P \left( \frac{w^a}{W} \right) - \frac{p_c + 1}{2} \frac{h}{W} - \frac{p_c - 1}{2} \frac{\tilde{h}}{W} \right]
\tag{4.36}
\]

\[
+ W^{p_c} \nabla \left[ P \left( \frac{h + w^a}{W} \right) - P \left( \frac{w^a}{W} \right) - \frac{p_c + 1}{2} \frac{h}{W} - \frac{p_c - 1}{2} \frac{\tilde{h}}{W} \right].
\tag{4.37}
\]

In view of (4.32), we can use the expansion for \(P(z)\) (see (3.21)) to write (4.36) as

\[
(4.36) = p_c W^{p_c-1} \nabla W \sum_{j_1 + j_2 \geq 2} a_{j_1 j_2} \left[ \left( \frac{h + w^a}{W} \right)^{j_1} \left( \frac{h + \tilde{w}^a}{W} \right)^{j_2} - \left( \frac{w^a}{W} \right)^{j_1} \left( \frac{\tilde{w}^a}{W} \right)^{j_2} \right]
\]

\[
= O \left( \sum_{j \geq 2, 1 \leq i \leq j} a_j C_{i,j} W^{p_c-1-j} \nabla W (w^a)^{j-i} h^i \right),
\tag{4.38}
\]

where the constants \(a_j, C_{i,j}\) are the same as those in (3.22). Now we deal with the second term (4.37). Applying the chain rule and regrouping the terms, we eventually get

\[
(4.37) = \frac{p_c + 1}{2} W^{p_c-1} \left[ \left( 1 + \frac{h + w^a}{W} \right)^{p_c-1} - 1 \right] \nabla h
\tag{4.39}
\]

\[
- \frac{p_c + 1}{2} W^{p_c-2} \nabla W \left[ \left( 1 + \frac{h + w^a}{W} \right)^{p_c-1} - 1 \right] h
\tag{4.40}
\]

\[
+ \frac{p_c + 1}{2} W^{p_c-1} \left[ \left( 1 + \frac{h + w^a}{W} \right)^{p_c-1} - 1 + \frac{w^a}{W} \right] \nabla w^a
\tag{4.41}
\]

\[
- \frac{p_c + 1}{2} W^{p_c-2} \nabla W \left[ \left( 1 + \frac{h + w^a}{W} \right)^{p_c-1} - 1 + \frac{w^a}{W} \right] w^a
\tag{4.42}
\]

\[
+ \frac{p_c - 1}{2} W^{p_c-1} \left[ \left( 1 + \frac{h + w^a}{W} \right)^{p_c-1} \left( 1 + \frac{h + w^a}{W} \right)^2 - 1 \right] \nabla \tilde{h}
\tag{4.43}
\]

\[
+ \left[ \left( 1 + \frac{h + w^a}{W} \right)^{p_c-3} \left( 1 + \frac{h + w^a}{W} \right)^2 - 1 \right] \nabla \tilde{w}^a
\tag{4.44}
\]

\[
- \frac{p_c - 1}{2} W^{p_c-2} \nabla W \left[ \left( 1 + \frac{h + w^a}{W} \right)^{p_c-3} \left( 1 + \frac{h + w^a}{W} \right)^2 - 1 \right] \tilde{h}
\tag{4.45}
\]

\[
+ \left[ \left( 1 + \frac{h + w^a}{W} \right)^{p_c-3} \left( 1 + \frac{h + w^a}{W} \right)^2 - 1 \right] \tilde{w}^a
\tag{4.46}
\]

For (4.39) and (4.40), we use the fact

\[|1 + z|^{p_c-1} - 1 \leq |z|^{p_c-1}\]
to bound them as:  
\[
|1 + z|^{p_c - 1} = 1 + \frac{p_c - 1}{2} z + \frac{p_c - 1}{2} \bar{z} + \sum_{j_1 + j_2 \geq 2} b_{j_1, j_2} z^{j_1} \bar{z}^{j_2}
\]

to write
\[
\frac{p_c^2 - 1}{4} W^{p_c - 2} (h \nabla w^a + \bar{h} \nabla w^a) + \frac{p_c + 1}{2} W^{p_c - 1} \nabla w^a \sum_{j_1 + j_2 \geq 2} b_{j_1, j_2} 
\times \left( \frac{h + w^a}{W} \right)^{j_1} \left( \frac{\bar{h} + w^a}{W} \right)^{j_2} - \left( \frac{w^a}{W} \right)^{j_1} \left( \frac{\bar{w}^a}{W} \right)^{j_2} \right]
\]
\[
= O \left( W^{p_c - 2} h \nabla w^a \right) + O \left( \sum_{j \geq 2, 1 \leq i \leq j} b_{j} c_{i, j} W^{p_c - 1 - j} \nabla w^a \mathcal{O} \left( (w^a)^{j - i} h^i \right) \right),
\]

where in the last equality we use the same conventions as in (3.22). In particular the constants $|b_j| \lesssim 1$ and $C_{i, j} \lesssim 2^j$. We therefore have the bound
\[
|4.41| \lesssim W^{p_c - 2} |\nabla w^a h| + \sum_{j \geq 2, 1 \leq i \leq j} 2^j \left| W^{p_c - 1 - j} \nabla \nabla (w^a)^{j - i} h^i \right|.
\]

Similarly for (4.42) we have
\[
|4.42| \lesssim W^{p_c - 3} \nabla W |\nabla w^a h| + \sum_{j \geq 2, 1 \leq i \leq j} 2^j \left| W^{p_c - 2 - j} \nabla W (w^a)^{j + 1 - i} h^i \right|.
\]

Collecting all the estimates and noticing that (4.39) through (4.40) are just complex conjugates of (4.39) through (4.42), we therefore can bound (4.37) as follows
\[
|4.37| \lesssim |h + w^a|^{p_c - 1} \nabla h| + W^{-1} |\nabla W| \cdot |h + w^a|^{p_c - 1} \cdot |h| \quad (4.47)
\]
\[
+ W^{p_c - 2} |\nabla w^a h| \quad (4.48)
\]
\[
+ W^{p_c - 3} |\nabla w^a h| \quad (4.49)
\]
\[
+ \sum_{j \geq 2, 1 \leq i \leq j} 2^j \left| W^{p_c - 1 - j} \nabla w^a (w^a)^{j - i} h^i \right| \quad (4.50)
\]
\[
+ \sum_{j \geq 2, 1 \leq i \leq j} 2^j \left| W^{p_c - 2 - j} \nabla W (w^a)^{j + 1 - i} h^i \right|. \quad (4.51)
\]

Now our task is reduced to bounding the $\hat{N}^0$ norm of (4.38) and (4.47) through (4.52). We start from (4.47), using Lemma 2.3 and Corollary 3.8.
we have
\[ \left\| N^0(t,\infty) \right\|_{L^p_t L^{\frac{d}{2}}_x} \lesssim \left\| h \right\|_{S^1_t L^\infty_x} + \left\| u^{\alpha} \right\|_{S^1_t L^\infty_x} \left\| h \right\|_{S^1_t L^\infty_x} \tag{4.53} \]
\[ \lesssim e^{-\frac{t}{2}(p_c-1)t}\|h\|_{S^1_t L^\infty_x}. \]

Similarly we have
\[ \left\| N^0(t,\infty) \right\|_{L^p_t L^{\frac{d}{2}}_x} \lesssim \left\| h \right\|_{S^1_t L^\infty_x} + \left\| u^{\alpha} \right\|_{S^1_t L^\infty_x} \left\| h \right\|_{S^1_t L^\infty_x} \tag{4.54} \]
\[ \lesssim e^{-\frac{t}{2}(p_c-1)t}\|h\|_{S^1_t L^\infty_x}. \]

For (4.59), (4.60), we use Hölder’s inequality and Corollary 3.8 to get
\[ \left\| N^0(t,\infty) \right\|_{L^p_t L^{\frac{d}{2}}_x} \lesssim \left\| W^{p_c-2} \nabla u^\alpha h \right\|_{L^p_t L^{\frac{d}{2}}_x} \tag{4.55} \]
\[ \lesssim \left\| W^{p_c-2} \nabla u^\alpha \right\|_{L^p_t L^{\frac{d}{2}}_x} \left\| h \right\|_{L^p_t L^{\frac{d}{2}}_x} \tag{4.56} \]
\[ \lesssim e^{-\frac{t}{2}p_c t}\|h\|_{S^1_t L^\infty_x}. \]

Now we are left with the estimates of the summation terms (4.38), (4.51) and (4.52). We first treat (4.38). We have
\[ \left\| N^0(t,\infty) \right\|_{L^p_t L^{\frac{d}{2}}_x} \lesssim \sum_{j \geq 2, 1 \leq i \leq j} 2^j \left\| W^{\frac{d+3}{4}} \nabla u^\alpha \right\|_{L^2_t L^{\frac{d}{2}}_x} \left\| h \right\|_{L^\infty_t L^{\frac{d}{2}}_x} \tag{4.57} \]
\[ \lesssim \sum_{j \geq 2} 2^j \left\| W^{\frac{d+3}{4}} \nabla u^\alpha \right\|_{L^2_t L^{\frac{d}{2}}_x} \left\| h \right\|_{L^\infty_t L^{\frac{d}{2}}_x} \tag{4.58} \]
\[ + \sum_{j \geq 2, 1 \leq i \leq j-1} 2^j \left\| W^{\frac{d+3}{4}} \nabla u^\alpha \right\|_{L^2_t L^{\frac{d}{2}}_x} \left\| h \right\|_{L^\infty_t L^{\frac{d}{2}}_x}. \tag{4.59} \]

For (4.58) we have by Lemma 2.3
\[ \left\| N^0(t,\infty) \right\|_{L^p_t L^{\frac{d}{2}}_x} \lesssim \sum_{j \geq 2} 2^j \left\| W^{\frac{d+3}{4}} \nabla u^\alpha \right\|_{L^2_t L^{\frac{d}{2}}_x} \left\| h \right\|_{L^\infty_t L^{\frac{d}{2}}_x} \tag{4.60} \]
\[ \lesssim \sum_{j \geq 2} 2^j \left\| h \right\|_{S^1_t L^\infty_x} \left( \frac{1}{4} \right)^{j-2} \}
\[ \lesssim \left\| h \right\|_{S^1_t L^\infty_x} \lesssim e^{-\frac{t}{2}(d-2)t}\|h\|_{S^1_t L^\infty_x}. \]
For (4.59) we estimate

$$
\|W^{-1}h\|_{L^\infty_t L_x^4([t,\infty))} \leq \sum_{j \geq 2, 1 \leq j \leq 1} 2^j \|h\|_{L^2_x (\mathbb{R}^d \setminus \{0\})} \|W^{-1}h\|_{L^\infty_t L_x^4([t,\infty))} \|W^{\frac{d-3}{2} - 1 + j} (w^a)_{j^{-1}}\|_{L^\infty_t L_x^2([t,\infty))}
$$

$$
\lesssim \sum_{j \geq 2, 1 \leq j \leq 1} 2^j \|h\|_{\dot{S}^1([t,\infty))} \left( \frac{1}{4} \right)^{j-1} e^{-\frac{1}{2}(j-1)t}
$$

This ends the estimate of (4.38). Using the fact that $|\nabla w^a| \leq |\nabla W|$ and $|w^a| \leq W$, (4.50) and (4.52) can be bounded by (4.38), thus has the same estimate

$$
(4.50) + (4.52) \lesssim e^{-\frac{\epsilon_0}{2} t} \|h\|_{\dot{S}^1([t,\infty))},
$$

Collecting the estimates (4.35), (4.53) through (4.60), we have

$$
\|R(h + w^a) - R(w^a)\|_{\dot{N}^1([t,\infty))} \lesssim e^{-\frac{7}{4(d-2)}t} e^{\epsilon_0 t} \|h\|_{\dot{S}^1([t,\infty))} \lesssim e^{(-m_1 - \frac{\epsilon_0}{4(d-2)})t}.
$$

(4.25) is proved and we conclude the proof of the Proposition.

As the last step of the argument, we show that any solution $h$ of the equation (4.23) which has enough exponential decay must be identically 0. This would imply $u = W^a$ and we can conclude the proof of Theorem 4.1.

To this end, we have

**Proposition 4.6.** Let $h$ be the solution of the equation (4.23) satisfying the following: $\forall \ m > 0$, there exists $t_m > 0$ such that

$$
\|h\|_{\dot{S}^1([t,\infty))} \leq e^{-mt}, \ \forall \ t > t_m.
$$

Then $h \equiv 0$.

**Proof.** Note first that in an equivalent form, $h$ satisfies

$$
i \partial_t h + \Delta h = -\Gamma(h) + i(-R(v + w^a) + R(w^a)),
$$

hence the following Duhamel’s formula holds

$$
h(t) = i \int_t^\infty e^{i(t-s)\Delta} (\Gamma(h) - iR(h + w^a) + iR(w^a)) (s) ds,
$$

since $\|h(t)\|_{\dot{H}^1} \to 0$ as $t \to \infty$. Using Strichartz estimate we then have

$$
\|h\|_{\dot{S}^1([t,\infty))} \leq \|\Gamma(h)\|_{\dot{N}^1([t,\infty))} + \|R(h + w^a) - R(w^a)\|_{\dot{N}^1([t,\infty))}.
$$
Denote $\|h\|_{\Sigma_t} := \sup_{s \geq t} e^{\eta s} \|h\|_{\dot{S}^1((s, \infty))}$, and we have for $\eta > 0$ small enough

$$\|\Gamma(h)\|_{N^1((t, \infty))} \leq \sum_{j \geq 0} \|\Gamma(h)\|_{N^1([t + \eta j, t + \eta(j + 1)])}$$

$$\leq \sum_{j \geq 0} \eta \|h\|_{\dot{S}^1([t + \eta j, t + \eta(j + 1)])}$$

$$\leq \sum_{j \geq 0} \eta e^{-m(t + \eta j)} \|h\|_{\Sigma_{t_m}}$$

$$\leq e^{-mt} \|h\|_{\Sigma_{t_m}} \frac{\eta}{1 - e^{-\eta m}}$$

$$\leq \frac{2}{m} e^{-mt} \|h\|_{\Sigma_{t_m}}.$$

From the estimate (4.25), we get

$$\|R(w^a + h) - R(w^a)\|_{N^1((t, \infty))} \leq \frac{1}{10} e^{-mt} \|h\|_{\Sigma_{t_m}}.$$

Combining these two estimates, we get for $m$ large enough that

$$\|h\|_{\Sigma_{t_m}} \leq \frac{1}{2} \|h\|_{\Sigma_{t_m}},$$

which implies that $h = 0$ on $[t_m, \infty)$. Recall that $h = u - W^a$ we obtain $u = W^a$ on $[t_m, \infty)$. Therefore $u$ is $W^a$ by uniqueness of solutions to (1.1).

The Proposition is proved and we have Theorem 4.1.

Proof of Corollary 4.2. The proof is almost the same as Corollary 6.6 in [7]. Let $a \neq 0$ and $T_a$ be such that $|a| e^{-e_0 T_a} = 1$. By (3.11) we have

$$\|W^a(-T_a) - W \pm e^{-e_0 T_a}\|_{H^{m, \infty}} \leq e^{-\frac{3}{4} e_0 t}. \tag{4.63}$$

Moreover $W^a(\cdot + T_a)$ satisfies the assumption in Theorem 4.1, thus there exists $a'$ such that $W^a(\cdot + T_a) = W^{a'}$. By (4.63), $a' = 1$ if $a > 0$ and $a' = -1$ if $a < 0$. Corollary 4.2 is proved.

Finally, we give the proof of the main theorem 1.4.

Proof of Theorem 1.4. We first note that (2) is just the variational characterization of $W$. More precisely we have

Theorem 4.7. [1][24] Let $c(d)$ denote the sharp constant in Sobolev-embedding

$$\|f\|_{d^{\frac{d}{2}} \leq c(d) \|\nabla f\|_2}.$$

Then the equality holds iff $f$ is $W$ up to symmetries. More precisely, there exists $(\theta_0, \lambda_0, x_0) \in \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^d$ such that

$$f(x) = e^{i\theta_0} \lambda_0^{-\frac{d-2}{2}} W \left( \frac{x - x_0}{\lambda_0} \right).$$

In particular, if $u_0$ satisfies

$$E(u_0) = E(W), \quad \|\nabla u_0\|_2 = \|\nabla W\|_2,$$

then $u_0$ coincides with $W$ up to symmetries, hence the corresponding solution $u$ coincides with $W$ up to symmetries.
It remains for us to show (1), (3). We first prove (1). Let \( u \) be the maximal-lifespan solution of (1.1) on \( I \) satisfying \( E(u) = E(W) \), \( \| \nabla u_0 \|_2 < \| \nabla W \|_2 \). Then by the Proposition 1.5, we have \( I = \mathbb{R} \). Assume that \( u \) blows up forward in time. Applying Proposition 1.5 again, we conclude that there exist \( \theta_0, \mu_0, \gamma_0 \) such that
\[
\| u(t) - W_{[\theta_0, \mu_0]} \|_{H^1} \leq e^{-\gamma_0 t}.
\]
This implies
\[
\| u_{[-\theta_0, \mu_0^{-1}]}(t) - W \|_{H^1} \leq e^{-\gamma_0 \mu_0^2 t}
\]
where
\[
u_{[-\theta_0, \mu_0^{-1}]}(t, x) = e^{-i\theta_0 \mu_0^{-1} t} u_0(\mu_0^2 t, \mu_0 x)
\]
is also a solution of the equation (1.1). By Theorem 4.1 with \( \gamma_0 \) now replaced by \( \gamma_0 \mu_0^2 \), we conclude there exists \( a < 0 \) such that \( u_{[-\theta_0, \mu_0^{-1}]} = W^a \).

Using Corollary 4.2, we get
\[
u(t, x) = e^{i\theta_0 \mu_0^{-d/2}} W^a(\mu_0^{-2} t + T_0, \mu_0^{-1} x).
\]
This shows that \( u = W^a \) up to symmetries. The proof of (3) is similar so we omit it. This ends the proof of Theorem 1.4.

References

[1] T. Aubin, *Équations différentielles non linéaires et problème de Yamabe concernant la courbure scalaire*. J. Math. Pures Appl. (9), 55(3): 269-296, 1976.

[2] J. Bourgain, *Global well-posedness of defocusing 3D critical NLS in the radial case*, J. Amer. Math. Soc. 12 (1999), 145–171.

[3] J. Bourgain, *Global solutions of nonlinear Schrödinger equations*. American Mathematical Society Colloquium Publications, 46. American Mathematical Society, Providence, RI, 1999.

[4] T. Cazenave and F. B. Weissler, *Some remarks on the nonlinear Schrödinger equation in the subcritical case*, in “New methods and results in nonlinear field equations (Bielefeld, 1987),” 59–69, Lecture Notes in Phys. 347, Springer, Berlin, 1989.

[5] T. Cazenave, *Semilinear Schrödinger equations*, Courant Lecture Notes in Mathematics, 10, American Mathematical Society, 2003.

[6] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao, *Global well-posedness and scattering for the energy-critical nonlinear Schrödinger equation in \( \mathbb{R}^3 \), to appear in Invent. Math.*

[7] T. Duyckaerts, F. Merle, *Dynamic of threshold solutions for energy-critical NLS*, Preprint math.ap/0710.5915.

[8] T. Duyckaerts, F. Merle, *Dynamic of threshold solutions for energy-critical wave equation*, Preprint math.ap/0710.5934.

[9] R. T. Glassey, *On the blowing up of solution to the Cauchy problem for nonlinear Schrödinger operators*, J. Math. Phys. 8 (1977), 1794–1797.

[10] J. Ginibre and G. Velo, *Smoothing properties and retarded estimates for some dispersive evolution equations*, Comm. Math. Phys. 144 (1992), 163–188.

[11] M. Keel and T. Tao, *Endpoint Strichartz Estimates*, Amer. Math. J. 120 (1998), 955–980.

[12] C. Kenig and F. Merle, *Global well-posedness, scattering, and blowup for the energy-critical, focusing, non-linear Schrödinger equation in the radial case*, Invent. Math. 166 (2006), 645–675.

[13] S. Keraani, *On the blow-up phenomenon of the critical nonlinear Schrödinger equation*, J. Funct. Anal. 235 (2006), 171–192.
[14] Rowan Killip, Dong Li, Monica Visan, Xiaoyi Zhang, Characterization of minimal-mass blowup solutions to the focusing mass-critical NLS, Preprint, math.ap/0804.1124.
[15] Rowan Killip, M. Visan, Nonlinear Schrödinger equations at critical regularity. Preprint.
[16] R. Killip, M. Visan, The focusing energy-critical nonlinear Schrödinger equation in dimensions five and higher, Preprint, math.ap/0804.1018.
[17] D. Li, X. Zhang, Dynamics for the energy critical nonlinear wave equation in dimensions $d \geq 6$. In preparation.
[18] F. Merle, Determination of blow-up solutions with minimal mass for nonlinear Schrödinger equation with critical power, Duke Math. J. 69 (1993), 427–453.
[19] E. Ryckman and M. Visan, Global well-posedness and scattering for the defocusing energy-critical nonlinear Schrödinger equation in $\mathbb{R}^{1+4}$, Amer. J. Math. 129 (2007), 1–60.
[20] R. S. Strichartz, Restriction of Fourier Transform to Quadratic Surfaces and Decay of Solutions of Wave Equations, Duke Math. J. 44 (1977), 705–774.
[21] T. Tao, Global well-posedness and scattering for the higher-dimensional energy-critical non-linear Schrödinger equation for radial data, New York Journal of Math. 11 (2005), 57–80.
[22] T. Tao, A pseudoconformal compactification of the nonlinear Schrödinger equation and applications, preprint math.AP/0606254.
[23] T. Tao, M. Visan, Stability of energy-critical nonlinear Schrödinger equations in high dimensions, Electron. J. Diff. Eqns. 118 (2005), 1-28.
[24] G. Talenti, Best constant in Sobolev inequality, Ann. Mat. Pura. Appl. (4), 110: 353–372, 1976.
[25] M. Visan, The defocusing energy-critical nonlinear Schrödinger equation in higher dimensions, to appear Duke Math. J.
[26] M. Weinstein, Nonlinear Schrödinger equations and sharp interpolation estimates, Comm. Math. Phys. 87 (1983), 567–576.
[27] M. Weinstein, The nonlinear Schrödinger equation—singularity formation, stability and dispersion, in “The connection between infinite-dimensional and finite-dimensional dynamical systems (Boulder, 1987),” 213–232, Contemp. Math. 99, Amer. Math. Soc., Providence, RI, 1989.

Institute for Advanced Study, 1st Einstein Drive, Princeton NJ, 08540
Institute for Advanced Study, 1st Einstein Drive, Princeton NJ, 08540.
Academy of Mathematics and System Sciences, Beijing 100080