A universal lower bound for certain quadratic integrals of automorphic $L$–functions

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With an Appendix by Laurent Clozel and Peter Sarnak

Introduction

1. Let $\pi$ be a unitary cuspidal representation of $GL(m, \mathbb{A})$ where $\mathbb{A}$ is the ring of adèles of $\mathbb{Q}$. Let $L(s, \pi)$ denote its $L$–function. More generally, we may consider a representation $\pi$ of the form $\pi_1 \times \pi_2 \times \cdots \times \pi_r$ where $m = m_1 + m_2 + \cdots + m_r$, $\pi_i$ is a cuspidal unitary representation of $GL(m_i, \mathbb{A})$ and $\times$ denotes induction by blocks. Thus

$$L(s, \pi) = L(s, \pi_1)L(s, \pi_2) \cdots L(s, \pi_r).$$

We will assume that $L(s, \pi)$ has at most a simple pole, at $s = 1$. Thus there is at most one occurrence of $m_i = 1$, $\pi_i(x) = |x|^c$ ($x \in \mathbb{A}^\times$) and we then assume that $c = 0$, so $L(s, \pi_i) = \zeta(s)$. (We can in fact consider more general $L$–functions, coming from a finite extension $F/\mathbb{Q}$. See § 2.1)

Assume $s$ is a zero of $L(s, \pi)$ in the critical strip: $s = \sigma + it$, $0 < \sigma < 1$. We first assume that $\pi$ contains no “zêta” factor. In this Introduction we assume that $s$ is on the critical line: $\sigma = 1/2$.

**Theorem A.** ($\pi_i = \mathbb{C}$ does not occur in $\pi$) ($\sigma = 1/2$)

$$\int_{-\infty}^{+\infty} \left| \frac{L(\frac{1}{2} + it, \pi)}{\frac{1}{2} + it - s} \right|^2 dt > 2\pi \log 2.$$

In the case where $L(s, \pi)$ has a zêta factor, the result is slightly different:

**Theorem B.** ($\sigma = 1/2$)

$$(i) \int_{-\infty}^{+\infty} \left| \frac{L(\frac{1}{2} + it, \pi)}{\frac{1}{2} + it - s} \right|^2 dt > 2\pi \left( \log 2 - \frac{2|\kappa|}{|1 - s|} \right)$$
where $\kappa$ is the residue of $L(s, \pi)$ at $s = 1$.

$$(ii) \int_{-\infty}^{+\infty} \left| \frac{L(\frac{1}{2} + it, \pi)}{\frac{1}{2} + it - s} \right|^2 dt > \pi \log 2.$$ 

Obviously Theorem B contains Theorem A. Note that $\kappa \ll D^\varepsilon$ for small $\varepsilon > 0$, $D$ being the conductor of $\pi$ and the implicit constant being uniform (for $m$ fixed) if $\pi$ satisfies the Ramanujan hypothesis. See Iwaniec–Kowalski [11, p. 160]. Thus the first lower bound is effective.

The two next results were suggested by Peter Sarnak. They correspond to $(s = 0)$ in the previous statements; note that $L(s, \pi)$ does not necessarily vanish at $(s = 0)$.

**Theorem C.** $(\pi_i = \mathbb{C}$ does not occur in $\pi)$

$$\int_{-\infty}^{+\infty} \left| \frac{L(\frac{1}{2} + it, \pi)}{\frac{1}{2} + it} \right|^2 dt > \pi.$$ 

This theorem implies a uniform 'pseudo-\(\Omega\)-result':

**Corollary.** Under the same assumptions, for any $\varepsilon < 1/2$,

$$\operatorname{Sup}_{t \in \mathbb{R}} \frac{|L(\frac{1}{2} + it, \pi)|}{|\frac{1}{2} + it|^{\varepsilon}} > \sqrt{2}/2(1 + O(\varepsilon))$$

where the remainder in $O(\varepsilon)$ is independent of $m$ and $\pi$.

Of course this is not an \(\Omega\)-result, as we do not obtain the inequality for arbitrarily large values of $t$. Note that the Sup is finite according to the Lindelöf conjecture. The proof follows directly by bounding the integral by the product of the supremum (squared) and of the integral of $|\frac{1}{2} + it|^{-2+2\varepsilon}$. By considering the powers $L(s, \pi)^r$, $r \geq 1$ (cf. the argument in the next paragraph), we can even deduce:

**Corollary.** Under the same assumptions,

$$\lim_{\varepsilon \to 0} \operatorname{Sup}_{t \in \mathbb{R}} \frac{|L(\frac{1}{2} + it, \pi)|}{|\frac{1}{2} + it|^{\varepsilon}} \geq 1$$

uniformly with respect to $m$ and $\pi$. 

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In the case where a residue is present, we obtain a slightly weaker result:

**Theorem D.**

\[
\int_{-\infty}^{+\infty} \left| \frac{L\left(\frac{1}{2} + it, \pi \right)}{\frac{1}{2} + it} \right|^2 dt > \frac{\pi}{2}.
\]

2. In the present state of our knowledge, it is not known that the integrals in Theorems A-D are finite. One may consider the Theorems true if they are infinite: so construed they will be proved in the Appendix by Sarnak and this author. In view of classical results for \( m = 1 \), and of recent subconvexity results for \( m = 2 \), they are finite in these cases. The main body of the paper is concerned with the cases where all the integrals considered are finite. We prove the Theorems for \( m = 1 \) (Riemann zeta function or Dirichlet \( L \)-series), see Theorem 1.2, Proposition 3.1. For \( m = 2 \), this includes the case where \( L(s, \pi) = \zeta_F(s) \) for \( F \) a quadratic extension of \( \mathbb{Q} \); and of \( L(s, \pi) \) for a cuspidal (unitary) representation of \( \text{GL}(2, \mathbb{A}_\mathbb{Q}) \). We assume that the Archimedean factor \( \pi_\infty \) is self-dual and that \( \pi \) verifies the Ramanujan Conjecture. In this case Theorems A and B are proved in Proposition 3.2; Theorems C and D are proved in §3.3.

Sarnak also pointed out that lower bounds, uniform in \( m \), such as in Theorem C cannot be obtained for small intervals. Let \( L(s, E) \) be the \( L \)-function of an elliptic curve \( E \) over \( \mathbb{Q} \), normalised as here: functional equation relating \( L(E, s) \) and \( L(E, 1-s) \). One can find \( E \) such that \( |L\left(\frac{1}{2} + it, E \right)| < 1 \) for \( t \in [-1, 1] \). See [18], and in particular the \( L \)-function of \( X_0(11) \) there\[^3\]. Consider \( L(s) = L(s, E)^r \). Then for \( r \) sufficiently large the integral in Theorem C, restricted to \([-1, 1] \), is arbitrarily small. Note that the arguments in this paper do not require \( \pi \) to be cuspidal.

3. The proof relies on an application, apparently new, of the Mellin transform. Consider

\[
L(s, \pi) = \sum_{n=1}^{\infty} a_n n^{-s}
\]

\[^{1}\text{The constants in Proposition 3.1 are different because we do not assume } \sigma = 1/2. \text{ For } \sigma = 1/2 \text{ one gets the indicated constants.} \]

\[^{2}\text{Likewise, the constants can be replaced by the constant } 2\pi \log 2 \text{ of Conjecture A if } \text{Re}(s) = 1/2. \]

\[^{3}\text{I thank Andrew Booker for this reference.}\]
and, for $X > 0$,

\begin{align}
0.1 \quad A_s(X) &= \sum_{n \leq X} a_n n^{-s} \\
0.2 \quad H_s(X) &= X^{s-1} \sum_{n \leq X} a_n n^{-s} - \frac{\kappa}{1-s}.
\end{align}

We were first led (by Tate’s thesis) to consider $H_s(X)$ when $\pi$ corresponds to the zeta function of a number field $F$ of degree $m$ over $\mathbb{Q}$; in this case,

$$H_s(X) = X^{s-1} \sum_{N a \leq X} N a^{-s} - \frac{\kappa}{1-s}$$

where $a$ ranges over integral ideals of $F$, different from $\{0\}$, and $\kappa$ is the usual residue. Note that if we consider $x \in F_\infty = \prod_{v|\infty} F_v$ and $X = |x| = \prod_{v|\infty} |x_v|$, $H_s(X)$ is a function of slow growth on $F_\infty$. In particular its Fourier transform $\mathcal{F}H_s$ is defined.

In this case, $H_s$ enjoys some remarkable properties. Let

$$K_s(X) = D^{-1/2} X^{s-1} \sum_{\sigma \leq D^{-1}} N a^{-s} - \frac{\kappa}{1-s}$$

Here $D^{-1}$ is the inverse different of the ring of integers of $F$ and $a$ ranges over fractional ideals, different from 0. We now have

\begin{equation}
0.3 \quad \text{For } 0 < \sigma < 1, \; \zeta_F(s) = 0 \text{ if, and only if, } \mathcal{F}(H_s) = -K_{1-s}.
\end{equation}

See Theorem 1.1.

Furthermore, using Perron’s formula, one can compute the Mellin transform of $H_s$:

$$\mathcal{M}H_s(w) = \int_0^\infty H_s(x)x^{w-1}dx.$$ 

One needs an estimate for $H_s(x)$ ($x \to \infty$); as a first step we use an estimate of Landau (1915) for $A_0(x)$. Assuming again $\sigma = \frac{1}{2}$, one finds that $\mathcal{M}H_s$ is absolutely convergent for $0 < Re(w) < \frac{1}{2}$ (Lemma 1.1). On the other hand, if $0 < c < 1, c \leq \frac{2}{m}$, and $\zeta_F(s) = 0$:

$$H_s^*(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\zeta_F(1-w)}{1-s-w} x^{-w} dw.$$
Here, as usual, \( H^*_s(x) \) is given by  
\[
x^s-1 \sum_{n \leq x} a_n n^{-s},
\]
the last term of the sum being weighted by \( \frac{1}{2} \) if \( x = n \) is an integer.

This suggests, of course, that more generally:

**Conjecture E.**

\[
MH_s(w) = \frac{L(1 - w, \pi)}{1 - s - w}
\]

when \( H_s \) is defined by (0.2), \( L(s, \pi) = 0 \), and \( MH_s \) is defined.

Returning to the case of \( \zeta_F \), one finds that for \( F \) quadratic \( \zeta_F(1 - w) \) is an \( L^2 \) function of \( t \ (w = 1/2 + it) \) and one can deduce (although \( Re(w) = 1/2 \) is the limit of the domain of convergence of \( MH_s \)) that

\[
\int_0^\infty |H_s(x)|^2 dx = \frac{1}{2\pi} \int_{1/2 - i\infty}^{1/2 + i\infty} |\zeta_F(w)|^2 \frac{|w - s|^2}{|w - s|^2} dw.
\]

This implies Theorem B by an easy computation of \( \int_1^2 |H_s(x)|^2 dx \) (Theorem 1.2). The proof is easier for \( \zeta = \zeta_Q \); in this case \( MH_s \) converges for \( 0 < Re w < 1 \) (Proposition 1.1, Theorem 1.2).

4. We now want to make sense, for the \( L \)-functions of § 0.1, of Conjecture E for \( w \) in some domain which ideally should include the line \( Re(w) = \frac{1}{2} \). For this we need a growth estimate for \( H_s(x) \) when \( x \to +\infty \) and \( L(s, \pi) = 0 \); equivalently, for \( A_s(x) - \kappa x \).

As we pointed out, the remainder \( A_0(x) - \kappa x \) has been majorised by Landau, at least if \( \pi \) verifies some conditions which are satisfied when \( \pi \) is associated to \( \zeta_F \) (see § 2.3). For general \( s \), we use the proof of a similar result by Friedlander and Iwaniec [8]. (They prove a stronger result, giving an expression of the remainder with an explicit constant depending on the conductor of \( \pi \).)

Friedlander and Iwaniec implicitly impose a condition on \( \pi \), namely, that the Archimedean factor \( \pi_\infty \) be self–dual; furthermore, they assume that \( \pi \) satisfies the Ramanujan Conjecture. (See § 2.3.) We now make these assumptions. It is then possible to adapt their proof (for \( A_0 \)) to the case of \( A_s \) for \( s \) in the critical strip. The final result is Theorem 2.2 and its Corollary: if \( s \) is a zero,

\[
H_s(x) = O\left(D^{-\frac{2}{m+1}}x^{-\frac{2}{m+1}+\varepsilon}\right).
\]
In particular, $H_s(x)$ is $L^2$ for $m \leq 2$.\footnote{We have given rather abundant details in this part of the paper, as the article of Friedlander–Iwaniec is elliptic.}

Having done this, we can return to the conjecture for $m \leq 2$. This is done in §3: in this case Conjecture E is well–defined, the functions on both sides are $L^2$, and we prove Theorem B. Furthermore the vertical integrals of $\left| \frac{L(w,\pi)}{w-s} \right|^2$ can be majorized for some values of $Re(w)$ different for $\frac{1}{2}$: see §3.2. §3.3 is devoted to the case of $s = 0$ (Theorems C,D.)

In §3.4, we pause to draw some consequences of the estimate of §2 for the abscissa of convergence of $L(s,\pi)$ (Theorem 3.1). The general result is that it converges for $Re(s) > 1 - \frac{2}{m+1}$. We then discuss, in light of known conjectures, the expected value of this abscissa.

The relevant Conjecture here is due to Friedlander–Iwaniec in the same paper [8, Conjecture 2]. Assuming, as seems likely, that their conjectural estimate can be extended, as in §2, from $(s = 0)$ to an arbitrary value in the critical strip, it would imply (for $s$ a zero) that $H_s(x) \ll x^{1/2-\frac{1}{m+1}+\epsilon}$. On the other hand, the integral in Theorem B is, of course, finite if we assume an approximation of the Lindelöf hypothesis. Under these assumptions, Conjecture E would be meaningful and true, and the finite case of Theorem B would follow. See §3.5.

One interesting aspect of the duality introduced by Conjecture E is that it gives a relation between the growth of $L(s,\pi)$ in vertical lines (controlled by the Lindelöf hypothesis) and the growth of $H_s$ (for which a useful control is given only by the Friedlander–Iwaniec Conjecture: the generalised Riemann hypothesis does not suffice.) See 3.5.

Finally, in §3.6, we hint at the following problem. We consider $\zeta = \zeta_Q$ for simplicity; in this case Theorem B is true, the integral being finite: there is a lower bound for

$$I(s) = \int_{-\infty}^{+\infty} \left| \frac{\zeta(1/2+it)}{\frac{1}{2}+it-s} \right|^2 dt$$

when $s$ is a zero. Does this integral tend to infinity with $s$? This seems likely; a counter–intuitive consequence of the existence of a uniform bound for $I(s)$ is given in Proposition 3.3.

To conclude, we note that it would be desirable to rid the proof in §2 of the assumptions on $\pi$, but this seems difficult. Moreover, let $\theta : 0 \leq \theta \leq 1/2$ be the “deviation from the Ramanujan hypothesis” for $\pi$. (see e.g. [6, §3.1]).
Thus $\theta \leq 1/2 - \frac{1}{m+1}$ \cite{[19]} ; $\theta \leq \frac{7}{32}$ for $m = 2$ and Maass forms \cite{[15]}. The (known) abscissa of convergence for $L(s, \pi)$ should be shifted (positively) by $\theta$ ; similarly one can expect that $\mathcal{M}H_s$ is defined for $Re(w) < \frac{1}{2} + \frac{1}{2m} - \theta$. For Maass forms, this yields $Re(w) < \frac{1}{2} + \frac{1}{32}$. If so, one could consider the integral on $Re w = \frac{1}{2}$ and Theorem A would be accessible with the same proof.

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1 The Tate kernel for number fields, and its Fourier and Mellin transforms

1.1

Let $F$ be a number field. We denote by $v$ a place of $F$. Let $F_\infty = \prod_{v | \infty} F_v$.

The Fourier transform on $F_\infty$ is defined place by place: on $F_v \cong \mathbb{R}$,

$$\mathcal{F}f(y) = \hat{f}(y) = \int f(x)e^{-2\pi xy}dy.$$ 

On $F_v \cong \mathbb{C}$, variables $z = x + iy$, $w = \xi + i\eta$,

$$\mathcal{F}f(w) = \hat{f}(w) = \int f(z)e^{-4\pi \text{Re}(zw)}dz$$

where $dz = 2dxdy$. (This is Tate’s self-dual normalisation, cf [23]). This also defines the Fourier transform for distributions on $F_\infty$.

For $x = (x_v) \in F_\infty$, we write $X = \prod_v |x_v|$ where the absolute value is normalised [23]. Let $O = O_F$ be the integers of $F$, $D = D_F$ its different and $D^{-1}$ the inverse different. Let $D = |D_F|$ the absolute value of the discriminant; thus $D = |N_{F/\mathbb{Q}}D|$.

We denote by $a$ a non-zero fractional ideal in $F$, and by $N a$ its norm. Let $\kappa = \frac{2^{r_1(2r_2+1)h_{R}}}{w \sqrt{|D|}}$ be the residue at 1 of the zeta function $\zeta_F$. We define the following functions on $F_\infty$ (depending only on $X$) for $s \in \mathbb{C}$, $s \neq 0,1$:

**Definition 1.1.** For $x \in F_\infty$, $X = |x|$,

$$H_s(x) = X^{s-1} \sum_{Na \leq X} Na^{-s} - \frac{\kappa}{1-s}$$

where $a$ runs over non-zero integral ideals $a \subset O$.

$$A_s(X) = \sum_{Na \leq X} Na^{-s}.$$
Definition 1.2. For \( x \in F_\infty \), \( X = |x| \),

\[
K_s(x) = D^{-1/2}X^{s-1} \sum_{a \subset D^{-1} \atop Na \leq X} Na^{-s} - \kappa \frac{D^{1/2}}{1-s}.
\]

These are functions of slow growth on \( F_\infty \), and therefore tempered distributions. We call \( H_s \) the Tate kernel.

Theorem 1.1. Assume \( \sigma = \text{Re}(s) \in ]0,1[ \). Then \( \zeta_F(s) = 0 \) if, and only if

\[
\mathcal{F}(H_s) = -K_{1-s}.
\]

1.2

The proof follows easily from Tate’s functional equation. Let \( \mathbb{A} \) denote the adèles of \( F \), \( \mathbb{A} = F_\infty \times \prod_v F_v \) (restricted product), \( v \) running over the finite places. We write \( \mathbb{A} = F_\infty \times \mathbb{A}_f \). Let \( I \) be the group of idèles, \( I = F_\infty \times I_f \), and \( I^+ \) the set of elements in \( I \) of idèle norm \( > 1 \). We endow \( \mathbb{A} \) with the self–dual measure of Tate [23], associated to Tate’s construction of local additive characters. Let \( h = h_\infty \otimes h_f \), where \( h_\infty \in \mathcal{S}(F_\infty) \), and \( h_f = \bigotimes_v h_v \), \( h_v = \text{ch}(\mathcal{O}_v) \), \( \mathcal{O}_v \subset F_v \) being the integers. Let \( D_v \) be the local different in \( F_v \), \( D_v^{-1} \) its inverse. Then

\[
(1.1) \quad \hat{h}_v = |D_v|^{-1/2} \text{ch}(D_v^{-1})
\]

where \( \text{ch}(\ ) \) denotes the characteristic function. The self–dual measures \( dx_v \) define as usual Haar measures \( d^\times x_v \), on \( F_v^\times \), hence a measure on \( I \) (at the Archimedean places, \( d^\times x_v = \frac{dx_v}{|x_v|} \)). With \( h = \hat{h}_\infty \otimes \bigotimes_v \hat{h}_v \), Tate’s formula reads

\[
Z(h,s) = \int_{I^+} h(x)|x|^s d^\times x + \int_{I^+} \hat{h}(x)|x|^{1-s} d^\times x
\]

\[
- \frac{\kappa}{1-s} \hat{h}(0) - \frac{\kappa}{s} h(0).
\]

Here \( Z(h,s) = Z(h_\infty,s)D^{-1/2}\zeta_F(s) \), \( Z(h_\infty,s) \) being the product of the local integrals \( Z(h_v,s) \) (\( v \mid \infty \)). Assume then that \( \sigma \in ]0,1[ \). The local,
Archimedean, integrals, are holomorphic for $\sigma > 0$; for a suitable choice of the $h_v$, they are non–zero. Therefore $s$ is a zero if, and only if, for all values of $h_\infty$:

$$
(1.2) \quad \int_{I^+} h(x)|x|^s d^\times x - \frac{\kappa}{1-s} \hat{h}(0) + \int_{I^+} \hat{h}(x)|x|^{1-s} d^\times x - \frac{\kappa}{s} h(0) = 0.
$$

In this formula, we may replace the finite part of the measure $d^\times x$ by the “trivial” measure on $I_f$, giving mass 1 to the idelic units, $\prod_v \mathcal{O}_v^\times := \mathcal{O}_f^\times$. We denote by $d^\times x$ this new measure on $I_f$ (and $I$, the Archimedean components being unchanged.) Then $d^\times x = D^{-1/2} d^\times x$, cf. Tate [23, p.310]. Therefore (1.2) yields:

$$
(1.3) \quad \int_{I^+} h(x)|x|^s d^\times x - \frac{\kappa D^{1/2}}{1-s} \hat{h}(0) + \int_{I^+} \hat{h}(x)|x|^{1-s} d^\times x - \frac{\kappa D^{1/2}}{s} h(0) = 0.
$$

We now write the first part in classical terms. By (1.1) $D^{1/2} \hat{h}_f(0) = 1$. We have

$$
\int_{I^+} h(x)|x|^s d^\times x = \int_{F_\infty} h_\infty(x_\infty)|x_\infty|^{s-1} d x_\infty \int_{I_f} h_f(x_f)|x_f|^s d^\times x_f
$$

where the second integral runs over

$$
\{x_f \in I_f \mid |x_f| \geq X^{-1}\}.
$$

The integrand is invariant by $\mathcal{O}_f^\times$. We have

$$
(1.4) \quad I_f = \prod_{\alpha=(\alpha_v)} \left( \prod_v \mathcal{O}_v^{\alpha_v} \mathcal{O}_v^\times \right)
$$

where the $\alpha_v$ are almost all 0. They must be positive since $h_v$ is supported on the integers. If $x_f$ is in the $\alpha$–component of (1.4), $|x_f| = \prod_v q_v^{-\alpha_v}$. Thus $\prod q_v^{\alpha_v} \leq X$, and the corresponding value of $|x_f|^s$ is $\prod q_v^{-\alpha_v s}$. If $a = \prod q_v^{\alpha_v}$, we see that the inner integral is

$$
\sum_{na \leq X} Na^{-s}.
$$
Taking into account the residue term in (1.3), we see that the first two terms in (1.3) yield

\[ \int_{F_{\infty}} h_{\infty}(x_{\infty}) H_s(x_{\infty}) dx_{\infty}. \]

We compute similarly the second part of (1.3). The integral is

\[ \int_{F_{\infty}} \hat{h}_{\infty}(x_{\infty}) |x_{\infty}|^{-s} dx_{\infty} \int_{I_f} X \hat{h}_f(x_f) |x_f|^{-s} dx_f \]

with identical notation. Now in the decomposition (1.4) only terms such that \( \varpi_v^{\alpha_v} \in D_v^{-1} \) occur: thus \( \alpha_v \geq -\delta_v \), where \( \delta_v = \text{val}(D_v) \). Again \( \prod q_v^{\alpha_v} \leq X \), and the value of \( |x_f|^{1-s} \) is \( \prod q_v^{\alpha_v(s-1)} \). The inner integral is therefore — since \( \hat{h}_f = D^{-1/2} ch(D_f^{-1}) \), \( D_f^{-1} = \prod D_v^{-1} \) by (1.1) —

\[ D^{-1/2} \sum_{\begin{array}{c} a \subseteq D^{-1} \\text{and} \\sum_1 \text{is} N_a \leq X \end{array}} N a^{s-1}. \]

Finally, the second part of (1.3) is equal to the integral of \( \hat{h}_{\infty} \) against

\[ D^{-1/2} X^{-s} \sum_{\begin{array}{c} a \subseteq D^{-1} \\text{and} \\sum_1 \text{is} N_a \leq X \end{array}} N a^{s-1} \frac{D^{1/2} \kappa}{s}. \]

The equality (1.3) then implies Theorem 1.1.

1.3

We will make no analytic use of Theorem 1.1. We note, however, that a direct “classical” proof that \( \zeta(s) = 0 \) implies the identity of Fourier transforms is easily obtained for \( F = \mathbb{Q} \), where \( H_s = K_s \). In this case \( H_s \) is the even function defined for \( x \geq 0 \) by

\[ H_s(x) = x^{s-1} \sum_{n \leq x} n^{-s} - \frac{1}{1-s} \]

\( (n \in \mathbb{N} - \{0\}) \). Proceeding as in Titchmarsh [25, p.14] we get another expression of \( H_s \). By the Euler–Maclaurin formula, we have for \( 0 < x < y \)
and \( s \in \mathbb{C} \):

\[
\sum_{x < n \leq y} n^{-s} = \int_x^y t^{-s} dt - \psi(y)y^{-s} + \psi(x)x^{-s} - s \int_x^y \psi(t)t^{-s-1} dt,
\]

with \( \psi(x) = \{x\} - 1/2 \) (\( x \geq 0 \)).

For \( \sigma > 1 \) we get

\[
\sum_{x < n} n^{-s} = -\frac{x^{1-s}}{1-s} + \psi(x)x^{-1} - s \int_x^\infty \psi(t)t^{-s-1} dt
\]

so

\[
\sum_{n \leq x} n^{-s} = \zeta(s) + \frac{x^{1-s}}{1-s} - \psi(x)x^{-s} + s \int_x^\infty \psi(t)t^{-s-1} dt.
\]

This is true for \( \sigma > 0 \) (\( s \neq 1 \)) by analytic continuation. If \( \zeta(s) = 0 \), \( 0 < \sigma < 1 \)
we get

\[
(1.7) \quad H_s(x) = -\psi(x)x^{-1} + sx^{s-1} \int_x^\infty \psi(t)t^{-s-1} dt.
\]

Since the average value of \( \psi \) vanishes, the last term of (1.7) is an \( O(x^{-2}) \). In particular \( H_s \in L^2(\mathbb{R}) \).

On the other hand, (1.6) yields the differential equation

\[
(1.8) \quad DH_s = (s - 1/2)H_s + R,
\]

\[
R = -1 + \sum_n \delta_n
\]

where \( D = x \frac{4}{dx} + \frac{1}{2} \), \( \delta_n \) is the Dirac measure at \( n \in \mathbb{Z} \), \( \Sigma' \) runs over \( (n \neq 0) \),
and \( H_s \) is seen as a tempered distribution.

Taking the Fourier transform yields

\[
D\hat{H}_s = (1/2 - s)\hat{H}_s - R
\]

whence

\[
D(\hat{H}_s + H_{1-s}) = (1/2 - s)(\hat{H}_s + H_{1-s})
\]

This implies that \( F = \hat{H}_s + H_{1-s} = Cx^{-s} \), say on \( \mathbb{R}_+ \), which is impossible,
since \( F \) is \( L^2 \), unless \( C = 0 \). Thus

\[
\mathcal{F}H_s = -H_{1-s}.
\]
1.4

We now return to the general case and consider the Mellin transform of \( H_s \), viewed as a function of \( X \in \mathbb{R}_+ \). In order to define

\[
(1.9) \quad \mathcal{M}H_s(w) = \int_0^\infty H_s(x) x^{w-1} dx
\]

(where we write \( x \geq 0 \) for \( X \)) we must control the order of growth of \( H_s \) at infinity.

However, we first consider (1.9) for \( F = \mathbb{Q} \) where the calculation is explicit. From (1.7) we see that the integral is absolutely convergent for \( 0 < \tau < 1 \), where \( \tau = \text{Re}(w) \). Using (1.6) we consider

\[
(1.10) \quad \int_0^X \left( x^{s-1} \sum_{n \leq x} n^{-s} - \frac{1}{1-s} \right) x^{w-1} dx.
\]

There is a first term equal to

(a) \[ -\frac{1}{1-s} \int_0^X x^{w-1} dx = -\frac{1}{w(1-s)} X^w. \]

The other term is

\[
\sum_{n \leq X} n^{-s} \left[ \frac{x^{s+w-1}}{s+w-1} \right]_n = \frac{1}{s+w-1} \sum_{n \leq X} \{ n^{-s} X^{s+w-1} - n^{w-1} \}.
\]

By the formulas in §1.3,

\[
\sum_{n \leq X} n^{w-1} = \zeta(1-w) + \frac{X^w}{w} + O(X^{\tau-1}).
\]

On the other hand, since \( \zeta(s) = 0 \),

\[
\sum_{n \leq X} n^{-s} = \frac{X^{1-s}}{1-s} + O(X^{-\sigma}),
\]

so

\[
X^{s+w-1} \sum_{n \leq X} n^{-s} = \frac{1}{1-s} X^w + O(X^{\tau-1}).
\]
The other term of the integral (1.10) is therefore equal to
\[
(b) \quad \frac{\zeta(1-w)}{s+w-1} + \frac{1}{s+w-1} \left\{ -\frac{1}{w} + \frac{1}{1-s} \right\} X^w + o(1).
\]

This yields

**Proposition 1.1.** \((F = \mathbb{Q})\). — For \(0 < \text{Re}(w) < 1\),
\[
\mathcal{M}H_s(w) = \frac{\zeta(1-w)}{1-s-w}.
\]

We will now see that the same formula is true for any number field (but valid analytically in a smaller domain.) However we must proceed differently. To control the growth of \(H_s\) for \(x \to \infty\) is to control the growth of \(A_s\).

In this section we will simply derive an estimate from the estimate of \(A_s(x)\) coming from a classical theorem of Landau \([16]^5\). We have for \(x \geq 1\):
\[
A_s(x) = \sum_{n \leq x} a_n n^{-s}
\]
where \(a_n = \sum_{N=n} 1\). The zêta function
\[
L(s) = \zeta_F(s) = \sum a_n n^{-s} \quad (\text{Re } s > 1)
\]
can be seen as an \(L\)-function over \(\mathbb{Q}\). For further reference we note that it is then an Euler product with factors
\[
L_p(s) = \prod_{i \leq m_p} (1 - \zeta_i p^{-s})^{-1}
\]
with \(m_p \leq m = [F : \mathbb{Q}]\), and \(m_p = m\) at the unramified primes, the \(\zeta_i\) being roots of unity. Landau’s assumptions are verified, and we get for
\[
A(x) = A_0(x) = \sum_{n \leq x} a_n
\]

\[5\text{Cf Michel [20, Thm 1.2]. We return to Landau’s theorem in § 2.3 (cf. Theorem 2.1)}\]
the estimate

\[(1.12) \quad A(x) = \kappa x + O(x^{m-1+\epsilon}).\]

By Stieltjes integration, we obtain the following estimate for \(A_s(x) = \int_{1-}^x t^{-s} dA\).

We fix \(\varepsilon\) (small) and set \(\mu = \frac{m-1}{m+1} + \varepsilon\). We assume \(\sigma = Re(s) < 1\). Then

\[
A_s(x) = \frac{\kappa}{1-s} x^{1-s} + O(x^{\mu-\sigma}) \quad (\sigma < \mu)
\]

\[
= \frac{\kappa}{1-s} x^{1-s} + O(1) \quad (\sigma > \mu).
\]

Thus

\[
H_s(x) = O(x^{\mu-1}) \quad (\sigma < \mu)
\]

\[
O(x^{\sigma-1}) \quad (\sigma > \mu),
\]

and the integral (1.9) converges in the following domain, setting \(\tau = Re(w)\):

\[
\tau < 1 - \mu \quad (\sigma < \mu)
\]

\[
\tau < 1 - \sigma \quad (\sigma > \mu).
\]

In particular we record that

**Lemma 1.1.** (i) For \(\sigma < \frac{m-1}{m+1}\), \(\mathcal{M}H_s(w)\) is given by an absolutely convergent integral for \(0 < \tau < \frac{2}{m+1}\).

(ii) For \(\sigma > \frac{m-1}{m+1}\), this is true for \(0 < \tau < 1 - \sigma\).

So far we have not assumed that \(s\) was a zero of \(\zeta_F\). We now do so; also \(0 < \sigma < 1\). Perron’s formula yields for \(x > 0\)

\[
A^*_s(x) = \sum_{n \leq x}^* a_n n^{-s} = \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} \zeta_F(s + w) \frac{x^w}{w} dw
\]

for \(c > 1 - \sigma\). Here \(\sum^*\) means as usual that the last term is pondered by \(\frac{1}{2}\) if \(x\) is an integer. The vertical integrals are Cauchy principal values. Thus

\[
x^{s-1}A^*_s(x) = \frac{1}{2i\pi} \int_c \zeta_F(s + w) \frac{x^{s+w-1}}{w} dw
\]

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(where $\int_c$ will denote $\int_{c-i\infty}^{c+i\infty}$)

$$
\frac{1}{2i\pi} \int_c \zeta_F(1-w) \frac{x^{-w}}{1-s-w} \, dw
$$

by the change of variable $w \mapsto 1-s-w$; here

$$
c' = 1-\sigma - c < 0.
$$

Consider now $c'' > 0$. We shift the integration to the abscissa $c''$; we assume $c'' < 1$. Then, formally,

$$
\int_c = \int_{c''} -2i\pi \sum \text{Res}
$$

where $\sum \text{Res}$ is the sum of the residues in the band bounded by the two lines. There is a residue equal to $\frac{1}{1-s} \kappa$, at $w = 0$; the denominator $\frac{1}{1-s-w}$ contributes a pole at $w = 1-s$, but $\zeta_F(w) = 0$. Therefore our last integral is

$$
\frac{1}{2i\pi} \int_{c''} \zeta_F(1-w) \frac{x^{-w}}{1-s-w} + \frac{\kappa}{1-s}.
$$

To justify the translation of the integral, we must now consider (for $T \to \infty$) the horizontal integrals on $[c'+iT, c''+iT]$. We have

$$
1-c'' \leq \text{Re}(1-w) \leq 1-c' \quad (c' < 0).
$$

We have assumed $c'' \in ]0, 1[$. Then the convexity estimate, for $w = \tau + it$, $\tau \in [c', c'']$, is

$$
|\zeta_F(1-w)| \ll t^{\frac{c''}{2}+\varepsilon},
$$

cf. Iwaniec-Kowalski [11, p.100, p.126]. However it is known [21, Thm.1.1] that subconvexity holds for the zêta functions of number fields, so in fact

$$
|\zeta_F(1-w)| \ll t^{\frac{c''}{2}-\delta}
$$

for some small $\delta > 0$. Consequently the horizontal integrals tend to 0 if

$$
c'' \leq \frac{2}{m}.
$$

Therefore :

---

6The statement of Theorem 5.30 in [11] is incorrect.
Lemma 1.2. Assume \( c'' \leq \frac{2}{m} \). Then, for \( x > 0 \),

\[
H_s^*(x) = \frac{1}{2i\pi} \int_{c''} \frac{\zeta_F(1-w)x^{-w}}{1-s-w}dw.
\]

In this paper we will be mostly concerned with the integral on the critical line \( \text{Re}(w) = \frac{1}{2} \). We must therefore assume \( m \leq 4 \). Furthermore, we will consider the quadratic integral \( \int |\zeta_F(1/2-it)|^2 dt \); even with the (known) subconvex estimates, this converges only for \( m \leq 2 \). We now assume this.

As a direct consequence of Lemma 1.2, we now have:

Proposition 1.2. Assume that \( F \) is \( \mathbb{Q} \) or a quadratic extension of \( \mathbb{Q} \). Let \( s \) a zero of \( \zeta_F \) with \( 0 < \sigma < 1 \). Then

(i) \( H_s \in L^2(\mathbb{R}_+, dx) \)

(ii) \( \mathcal{M}H_s(w) = \frac{\zeta_F(1-w)}{1-s-w} (\text{Re } w = 1/2) \).

In part (ii), \( \mathcal{M}H_s(w) \) is a priori defined as an \( L^2 \) function of \( w \); for \( w = \frac{1}{2} + it \),

\[
\mathcal{M}H_s(w) = \int_0^\infty H_s(x)x^{1/2+it}dx.
\]

i.e.

\[
(1.13) \quad \mathcal{M}H_s(w) = \int_{-\infty}^\infty (H_s(e^X)e^{X/2})e^{itX}dX,
\]

and (i) is equivalent to the fact that \( H_s(e^X)e^{X/2} \in L^2(\mathbb{R}, dX) \). Cf. Titchmarsh [26, Thm. 7.1].

The proof is clear: write

\[
(1.14) \quad H_s^*(x) = \frac{1}{2i\pi} \int_{-\infty}^{+\infty} \frac{\zeta_F(1/2-it)x^{-1/2-it}}{\frac{1}{2} - it - s} d(it)
\]

according to Lemma 1.2. On the other hand, in view of the last remarks, we can see the right-hand side as an additive Fourier transform (evaluated at \( X = \log x \)). Thus the Fourier transform (in \( L^2 \)) of the right-hand side of (1.14) coincides with \( e^{X/2}H_s^*(e^X) \). This implies (i), and (ii) by the involutivity of the Fourier transform.

\footnote{We will return to these statements, using known conjectures, in §3.4.}
It is interesting to notice the relation with the constraints in Lemma 1.1. Assume $m = 2$. If $\sigma < 1/3$, $\mathcal{M}H_s(w)$ is holomorphic for $\tau < \frac{2}{3}$. If $\sigma > \frac{1}{3}$, it is holomorphic for $\tau < 1 - \sigma$. In particular, $\tau = 1/2$ is attained only if $\sigma < 1/2$. For $\sigma = 1/2$, we are on the boundary of the domain of convergence.

**Corollary.** With the assumptions of Proposition 1.2,

$$\int_0^{\infty} |H_s(x)|^2 dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \frac{\zeta_F(1/2 + it)}{1/2 + it - s} \right|^2 dt.$$  

This follows from $\int_0^{\infty} |H_s(x)|^2 dx = \int_{-\infty}^{\infty} |H_s(e^X)e^{X/2}|^2 dX$, cf [26].

We can now prove the main theorem of this chapter:

**Theorem 1.2.** Assume $s$ is a zero of $\zeta_F$ in the critical strip, and consider

$$I(s) = \int_{-\infty}^{\infty} \left| \frac{\zeta_F(1/2 + it)}{1/2 + it - s} \right|^2 dt.$$  

Then

$$I(s) \geq 2\pi \left( \log 2 - \frac{2\kappa}{1 - s} \right)$$

We recall that $\kappa$ can be majorized, for any $\varepsilon > 0$, by $C(\varepsilon)D^\varepsilon$ where the constant is explicit (cf. e.g. Lang [17, Ch.XVI]). In particular the lower bound is effective.

The proof is now obvious: the integral of $|H_s|^2$ is larger than

$$\int_1^2 |H_s(x)|^2 = \int_1^2 \left| x^{s-1} - \frac{\kappa}{1 - s} \right|^2 dx,$$

which is bounded below by

$$\int_1^2 \left( x^{2\sigma - 2} - 2\kappa \frac{x^{\sigma-1}}{1 - s} \right) dx.$$  

The first term yields

$$\log 2 \quad \text{for } \sigma = 1/2$$

$$f(\sigma) = \frac{2^{2\sigma-1} - 1}{2\sigma - 1} \quad \text{for } \sigma \neq 1/2.$$  

\[\text{But recall that we are using only the crude estimates given by Stieltjes integration. Compare Theorem 2.2 and its Corollary.}\]
For $\sigma > \frac{1}{2}$, the value of (1.16) is larger than $\log 2$. However $I(\sigma) = I(1 - \sigma)$, as follows from $\zeta_F(s) = \zeta_F(\bar{s})$.

(If $\sigma = \text{Re}(s)$ is sufficiently close to 1, one can obtain better bounds by considering the integrals on the intervals $[N, N + 1], \ N \leq 3$, using that $\frac{1}{2\sigma} + \frac{1}{3\sigma} < 1$.)

2 Estimates for $H_s$: general case

2.1

In this chapter we return to the problem, raised in § 1.4, of getting estimates for $H_s$ better than that given by Landau’s theorem and partial integration. However we find that the question can be posed — and partly solved, giving non trivial estimates — in a very general framework.

We will consider the function $H_s$ associated to very general Dirichlet series. For definiteness assume first that $\pi$ is a cuspidal, unitary representation of $GL(m, A_\mathbb{Q})$. Thus

$$\pi = \pi_\infty \otimes \bigotimes_p \pi_p.$$ 

Its $L$–function $L(s, \pi)$ is an Euler product

$$L(s, \pi) = \prod_p L(s, \pi_p) \quad (\text{Res} > 1)$$

convergent for $\text{Res} > 1$ [3]. We assume first that $\pi$ verifies the Ramanujan conjecture at all primes, i.e., that $\pi_p$ and $\pi_\infty$ are tempered. After the early work of Deligne and Serre, this is now known in many non–trivial cases, cf [3], [6]. The $L$–function can then be written

$$L(s, \pi) = \sum_{n=1}^{\infty} a_n n^{-s}$$

with

$$(2.1) \quad |a_n| \leq \tau_m(n) \ll n^\varepsilon$$
as in §1.4. (In particular the estimate is uniform for \( m \) and \( \varepsilon \) fixed.) There
is a well–defined \( L \)–function \( L(s, \pi_\infty) \), which can be written

\[
L(s, \pi_\infty) = c(\pi_\infty)\pi^{-\frac{ms}{2}} \prod_{i=1}^{m} \Gamma\left(\frac{s + c_i}{2}\right).
\]

This will be reviewed presently. There is also a conductor \( D \), a positive
integer. See Jacquet [12], as well as [13].

We write

\[
\Lambda(s, \pi) = D^{s/2}L(s, \pi_\infty)L(s, \pi).
\]

Then \( \Lambda(s, \pi) \) satisfies a functional equation :

\[
(2.3) \quad \Lambda(s, \pi) = \varepsilon(\pi)\Lambda(1 - s, \tilde{\pi}).
\]

where \( |\varepsilon(\pi)| = 1 \) and \( \tilde{\pi} \) is the dual representation. It is holomorphic in the
whole plane, unless \( m = 1 \) and \( \pi(x) = |x|^a \) (\( x \in \mathbb{A}_Q^\times \)) with \( a \in i \mathbb{R} \).

More generally, we can consider \( \pi = \pi_1 \times \cdots \times \pi_r \), \( \pi_i \) being cuspidal
tempered representations of \( GL(m_i, \mathbb{A}_Q) \). We have similar properties for
\( L(s, \pi) \); its poles are easily described.

There are other \( L \)–functions with the same properties. First let \( F/\mathbb{Q} \) be
a finite extension (not necessarily Galois) and let \( \pi_F \) be a cuspidal repre-
sentation of \( GL(\mu, \mathbb{A}_F) \). By the Langlands classification (now known at all
primes, cf.[10]) \( \pi_F \) defines a representation

\[
\pi_Q = \text{Ind}^Q_F \pi_F
\]

of \( GL(n, \mathbb{A}_Q) \), of degree \( m = \mu d \), \( d = [F : \mathbb{Q}] \). Unless \( F \) is a soluble Galois
extension of \( \mathbb{Q} \), \( \pi_Q \) is not known to be automorphic. We assume that \( \pi_F \)
verifies the Ramanujan conjecture (at all primes). Then so does \( \pi_Q \); when
base change is known [2] it is of the form \( \pi_1 \times \cdots \times \pi_r \) as before. (Note that
the local correspondence has to be normalised, as can be done, so induction
on the (local) Weil group side preserves tempered representations). We have,
for \( Re(s) > 1 \),

\[
L(s, \pi_F) = L(s, \pi_Q)
\]

by the inductivity of \( L \)–functions ; both sides can be seen as Euler prod-
ucts over \( \mathbb{Q} \). Defining the Archimedean factor \( L(s, \pi_{Q, \infty}) \) as before, we see

\[
\text{(2.2)} \quad L(s, \pi_\infty) = c(\pi_\infty)\pi^{-\frac{ms}{2}} \prod_{i=1}^{m} \Gamma\left(\frac{s + c_i}{2}\right).
\]
that \( L(s, \pi_Q) \) extends meromorphically to the complex plane; we can form \( \Lambda(s, \pi_Q) \) with the same properties as before.

If \( \mu = 1 \) and \( \pi_F \) is trivial \( L(s, \pi_F) = \zeta_F(s) \) so this accounts for the situation in Chapter 1. (For \( m \geq 2 \), in the cases where the Ramanujan conjecture is known, \( F \) will be totally real, or a \( CM \) field.)

Similarly, assume \( \rho_F \) is an irreducible Artin representation of \( \text{Gal}(\overline{F}/F) \) such that the Artin Conjecture is known for the Artin \( L \)-function \( L(s, \rho_F) \). Again we can consider \( \rho_Q = \text{ind}_{G_F}^{G_Q} \rho_F \). Its \( L \)-function is just \( L(s, \rho_F) \) (seen as an Euler product over \( \mathbb{Q} \)). It has the specified properties, even if the representation \( \pi_Q \) associated to \( \rho_Q \) is not known to exist.

### 2.2

In all these cases we consider

\[
L(s) = L(s, \pi_Q) = \sum_{n} a_n n^{-s},
\]

defined for \( \text{Re}(s) > 1 \), and the Archimedean factor \( L(s, \pi_\infty) \). The representation \( \pi_\infty \) is well-defined and tempered.

We now describe the factor \( L(s, \pi_\infty) \). By the Langlands classification\(^{11}\), \( \pi_\infty \) is associated to a representation \( r(\pi) : W_{\mathbb{R}} \to GL(m, \mathbb{C}) \) where \( W_{\mathbb{R}} \) is the real Weil group. We have \( m = m_1 + 2m_2 \), \( r = r(1) + r(2) \), \( r(1) \) is a sum of \( m_1 \) 1-dimensional representations associated to characters \( \nu_i \) of \( \mathbb{R} \times \mathbb{C} \), and \( r(2) \) is a sum of \( m_2 \) 2-dimensional representations

\[
r_j = \text{ind}_{W_{\mathbb{C}}}^{W_{\mathbb{R}}} \mu_j := r(\mu_j)
\]

where \( \mu_j \) is a unitary character of \( \mathbb{C}^\times \) such that \( \mu_j(\bar{z}) \neq \mu_j(z) \).

A unitary character \( \nu \) of \( \mathbb{R}^\times \) can be written

\[
\nu(x) = (\text{sgn } x)^{\epsilon} |x|^c \quad (\epsilon = 0, 1; \ c \in i\mathbb{R}).
\]

Its \( L \)-function is

\[
L(s, \nu) = \pi^{-\frac{s+c}{2}} \Gamma\left(\frac{s+c}{2}\right) \quad (\epsilon = 0)
\]

\[
L(s, \nu) = \pi^{-1/2-\frac{s+c+1}{2}} \Gamma\left(\frac{s+c+1}{2}\right) \quad (\epsilon = 1).
\]

\(^{10}\)I do not know if the Artin Conjecture is known for some \( \rho_F \) such that the associated representation \( \pi_F \) has not been shown to exist.

\(^{11}\)Again, this is Langlands’s normalisation, compatible with unitarity and induction, cf. [4].

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A unitary character $\mu$ of $\mathbb{C}^\times$ such that $\mu_j(\bar{z}) \neq \mu_j(z)$ can be written

$$\mu(z) = z^{p+q}, \ p \neq q, \ p - q \in \mathbb{Z}, \ p + q \in i\mathbb{R}. \ $$

Its $L$–function, equal to that of $r(\mu)$, is

(2.7) \hspace{2cm} L(s, \mu) = 2(2\pi)^{-q-s} \Gamma(q+s) \ (p-q < 0) \\
(2.8) \hspace{2cm} L(s, \mu) = 2(2\pi)^{-p-s} \Gamma(p+s) \ (p-q > 0).

(see Tate [24 § 3].)

Using the duplication formula, we see that the factor (2.7) is equal to

(2.9) \hspace{2cm} L(s, \mu) = \pi^{-1/2} \pi^{-s-q} \Gamma\left(\frac{s+q}{2}\right)\Gamma\left(\frac{s+q+1}{2}\right).

It is then easy to see that $L(s, \pi_{\infty})$ is given by an equality (2.2). We now rewrite the functional equation (2.3) as

$$L(1-s, \pi) = \varepsilon(\pi) \gamma(s) L(s, \tilde{\pi})$$

with

(2.10) \hspace{2cm} \gamma(s) = (\pi^{-m} D)^{s-1/2} \frac{c(\pi_{\infty}) \Gamma(s, \pi_{\infty})}{c(\pi_{\infty}) \Gamma(1-s, \pi_{\infty})},

$\Gamma(s, \pi_{\infty})$ being the product of the gamma factors.

2.3

Given an $L$–function $L(s) = L(s, \pi)$, as in § 2.1, we now consider for $x \geq 0$

(2.11) \hspace{2cm} A_s(x) = \sum_{n \leq x} a_n n^{-s} \\
(2.12) \hspace{2cm} H_s(x) = x^{s-1} \sum_{n \leq x} a_n n^{-s} - \frac{\kappa}{1-s}

with $\kappa = \text{Res}_{s=1} L(s)$. We will assume that $L(s)$ has at most a simple pole at $s = 1$. In many cases, Landau (in 1915 !) had already obtained an excellent estimate for $A_0(x) = \sum a_n$. It is straightforward to check that his conditions (I–VII), p. 2–4 of [16], are satisfied. However, he assumes that
the \( \Gamma \)-factors, \( \Gamma(\frac{\Delta_{\pi}}{2}) \), are given by real parameters \( c_i \). This means here that the real characters \( \nu_i \) are equal to 1 or to \( \text{sgn}(x) \); in the complex case, that \( \mu(z) = z^p \overline{z}^q \) with \( p, q \) real, i.e. that

\[
\mu(z) = (z/\overline{z})^p, \quad p \in \frac{1}{2} \mathbb{Z}.
\]

Landau does not assume that the \( a_n \) are real, but that \( |a_n| \leq d_n \) where

\[
L^*(s) = \sum_{n=1}^{\infty} d_n n^{-s}
\]

has a functional equation of the same type as that of \( L(s) \). Since our representation \( \pi \) (or the \( \pi_i \)) is tempered at all primes, \( L(s) \) is given by an Euler product

\[
L(s) = \prod_{p} L_p(s) \quad (Res > 1)
\]

with factors of type (1.11), the \( \zeta_i \) being now complex numbers with absolute value 1, at unramified primes. One can write \( \pi_p = \pi_1 \times \pi_2 \times \ldots \times \pi_r \), the local analogue of the global decomposition of \( \pi \) in § 2.1. Each \( \pi_j \) is a unitary discrete series representation of \( GL(m_j, \mathbb{Q}_p) \). Then \( L(\pi, s) = \prod L(\pi_j, s) \), the product running over the \( j \) such that \( \pi_j \) is a unitary twist of the Steinberg representation. The corresponding factor is

\[
(1 - p^{1-m_j} + \eta_j p^{-s})^{-1}
\]

with \( t_j \) real. Cf. [12], [9] 12. It follows that we can take \( L^*(s) = \zeta_Q(s)^m \). The following result was therefore known in 1915 (for \( m > 1 \), which is equivalent to condition (9) in [13]) :

**Theorem 2.1. (Landau)** Assume that \( m > 1 \) and that each real character of \( \mathbb{R}^\times \) occurring in \( \tau(\pi) \) is of finite order, and each complex character of the form \( (z/\overline{z})^p \), \( p \in \frac{1}{2} \mathbb{Z} \). Then

\[
A_0(x) = \kappa x + O(x^{\frac{m-1}{m+1} + \epsilon}).
\]

In particular, we see that Lemma 1.1 remains true under these assumptions. It is remarkable that this applies to the cuspidal algebraic representations of [4] (twisted so as to render them unitary) when they satisfy the Ramanujan hypothesis. However, we want to do better.

12However Jacquet’s expression in [12] seems incorrect. Compare [9], Theorem 7.11.
2.4

In order to obtain an estimate of $A_4(x)$ better than that obtained by Stieljes integration, we now use the method of Friedlander–Iwaniec [8].

Return to the equation (2.10). Friedlander–Iwaniec assume that $\gamma(s)$ is a product

$$ (\pi^{-m}D)^{s-1/2} \prod_{i=1}^{m} \frac{\Gamma\left(\frac{s+c_i}{2}\right)}{\Gamma\left(\frac{1-s+c_i}{2}\right)}. $$

For the real factors (2.5), (2.6), the parameter $c$ is sent to $-c$ when $\pi$ is sent to $\tilde{\pi}$; similarly for (2.7), (2.8), $(p,q)$ is sent to $(-p,-q)$. We therefore now make the following assumption

**Hypothesis 2.1.** $\pi_\infty$ is self–dual.

In this case $\gamma(s)$ is given by a product (2.13) with $c_i \in \mathbb{C}$ and furthermore

$$ Re(c_i) \geq 0. $$

(For the parameters $c_i$ coming from (2.7), the condition on $Re(c_i)$ follows from $Re(p+q) = 0$ and $p-q < 0$; *idem* for (2.8.).

**Remark.** If $\pi$ is associated to a Galois representation of $Gal(F/F)$, $\pi_\infty$ is self–dual.

The expression (2.13) shows that $\gamma(s)$ is holomorphic for $\sigma > 0$. Write $Q = D^{1/m}/2\pi$, so $2\pi Q \geq 1$.

**Lemma 2.1.** There exist constants $\omega_1, \omega_2 \in \mathbb{C}$ such that for $\sigma \geq \frac{1}{2}, t \geq 1$

$$ \gamma(\sigma + it) = \omega_1(Qt)^{m(\sigma-1/2)}(Qt/e)^{imt}(1 + O(t^{-1})) $$
$$ \gamma(\sigma - it) = \omega_2(Qt)^{m(\sigma-1/2)}(Qt/e)^{-imt}(1 + O(t^{-1})) $$

the implicit constant in the $O$–terms being uniform (for $\pi$ given) for $1/2 \leq \sigma \leq A$, $A > \frac{1}{2}$.

(In fact the uniformity is obtained if the real parameters $(c_i)$ are fixed.)

---

13This assumption is also (implicitly) made by Iwaniec and Kowalski in [11]: see p. 94, after (5.3), and the equality $\gamma(\tilde{f},s) = \gamma(f,s)$, p. 94, l. -9.
This is essentially (1.6) in [8], except that we do not obtain the same constants \( \omega_1 \). We sketch the proof. For \( t \geq 1 \), we have by Stirling’s formula [7]

\[
\Gamma(\sigma + it) = \sqrt{2\pi} e^{(\sigma-1/2)\frac{t}{2}} t^{\sigma-1/2} e^{-\frac{\pi}{2} t} (t/e)^it (1 + O(1/t))
\]

\[
\Gamma(\sigma - it) = \sqrt{2\pi} e^{(1/2-\sigma)\frac{t}{2}} t^{\sigma-1/2} e^{-\frac{\pi}{2} t} (t/e)^-it (1 + O(1/t))
\]

(2.15) \( \Gamma(\sigma + it) \) and (2.16) \( \Gamma(\sigma - it) \)

It suffices to consider one of the quotients in the formula (2.13), associated to \( c \in \mathbb{C} \), with \( Re(c) \geq 0 \). Let \( q_1, q_2, q_3, q_4 \) be the quotient associated to each non-trivial factor in (2.15), (2.16).

Let \( c = \alpha + i\tau \). We find for example

\[
q_1 = e^{i\left(\frac{\alpha}{2} - \frac{1}{2}\right)\frac{\pi}{2}} = e^{i(\alpha-1/2)\frac{\pi}{2}}.
\]

One computes similarly \( q_2 = \frac{t}{2}\sigma^{-1/2}(1 + O(t^{-1})) \), \( q_3 = e^{-(\pi/2)\tau} \), \( q_4 = (t/2e)^it(1 + O(t^{-1})) \). Multiplying all the factors, one finds for the product of quotients of \( \Gamma \)-functions :

\[
q(t) = e^{-im\frac{\pi}{4}} e^{i\frac{\pi}{4}c(t/2e)^imt(t/2)^m(\sigma^{-1/2})(1 + O(t^{-1}))}
\]

where \( c \) is now equal to \( \Sigma c_i \).

Multiplying by the factor \((\pi^{-m}D)^{s-1/2}\), one obtains the expression for \( t > 0 \) (\( \geq 1 \)), with

\[
\omega_1 = e^{-m\frac{\pi}{4}} e^{i\frac{\pi}{4}c}.
\]

For \( t < 0 \), using that, with obvious notation :

\[
\gamma(s, c_i) = \gamma(s, c_i)
\]

one obtains the requested expression, with \( \omega_2 = e^{im\frac{\pi}{4}} e^{-i\frac{\pi}{4}c} \). If the \( c_i \) are real, we obtain the result of [8]. This will be the case for the algebraic representations of [9], which includes the only cases where the Ramanujan conjecture is known.

**Remark.** The function \( \gamma \) does not have the same expression if \( \pi_{\infty} \) is not self-dual. Assume for instance that a factor (2.5) occurs with \( c \neq 0 \), while the factor associated with \((-c)\) does not occur. Computing in the same fashion, we find for the factor \( \frac{\Gamma(\frac{s+c}{2})}{\Gamma(\frac{s-c}{2})} \) an expression of the form \( q_1 q_2 q_3 q_4 \), with \( q_1 = e^{-i\pi}, q_2 = (\frac{1}{2})^{\alpha+\sigma^{-1/2}}(1 + O(t^{-1})) \), \( q_3 = 1 \) and \( q_4 = (t/2e)^it e^{i\tau}(t/2e)^it(1 + O(t^{-1})) \). The complete quotient is therefore of the form

\[
q(t) = e^{-i\pi/4} (t/2e)^i(t+\tau)(t/2)^{\alpha+\sigma^{-1/2}} e^{i\tau}(1 + O(t^{-1})).
\]

We do not know if the argument of Friedlander–Iwaniec extends to this case.
2.5

We now imitate the proof of [8] in order to obtain an estimate for $A_s(x)$. We will assume that $0 < \sigma < 1$, the case of $s = 0$ being treated in [8]. We may further assume that $|x - N| \geq \frac{1}{4}$, $N$ being the integer closest to $x$.

Fix $\epsilon > 0$ (small) and let $c = 1 - \sigma + \epsilon$. For $1 \leq T \leq x$, we have according to [25, Lemma 3.12]:

$$A_s(x) = \frac{1}{2i\pi} \int_{c-iT}^{c+iT} L(s + w) \frac{x^w}{w} dw + O\left(\frac{x^{1-\sigma + \epsilon'}}{T}\right)$$

for any sufficiently small $\epsilon'$. The implicit constants depend only on $\epsilon$ and $\epsilon'$.

If $s = \sigma + i\tau$, we obtain by a change of variables

$$A_s(x) = \frac{1}{2i\pi} \int_{\alpha+i(-\tau - T)}^{\alpha+i(T+\tau)} L(w) \frac{x^{w-s}}{w-s} dw + O\left(\frac{x^{1-\sigma + \epsilon}}{T}\right).$$

We may assume $\tau \geq 0$: the computation for $\tau \leq 0$ is obviously similar. Here $\alpha = 1 + \epsilon$. We shift the integral to the line $Re(w) = -\epsilon$. We pass poles at $w = 1$, with residue $\frac{\kappa}{1-s} x^{1-s}$, and at $w = s$, with residue $L(s)$. We write $w = u + it$. We now have to evaluate the integrals in the segments $Im(w) = \tau \pm T$, $Re w \in [-\epsilon, 1 + \epsilon]$.

We first do so, using only the convexity estimate. For $Re(w) = 1 + \epsilon$,

$$|L(w)| \leq \zeta_0 (1 + \epsilon)^m \ll \epsilon^{-m} \ll 1.$$  

For $Re(w) = -\epsilon$, $t \geq 1$

$$L(w) \ll (Qt)^{m(1/2+\epsilon)}$$

by the functional equation. If $R = (Qt)^{m(1/2+\epsilon)} \geq 1$, we have for $u \in [-\epsilon, 1 + \epsilon]$,

$$L(w) \ll R^{\frac{1+\epsilon-u}{1+2\epsilon}}$$

whence

$$\int_{-\epsilon}^{1+\epsilon} |L(u + it)| x^u du \ll x^{1+\epsilon} + R x^{-\epsilon} \ll x^{1+\epsilon} + R.$$  

We will assume $T \geq 2\tau$, so the correction term is $O\left(\frac{x^{-\sigma}}{T}(x^{1+\epsilon} + (QT)^{m(1/2+\epsilon)})\right)$ and, $I_{-\epsilon}$ being the integral for $Re(w) = -\epsilon$:

$$(2.17) \quad A_s(x) = I_{-\epsilon} + \frac{\kappa}{1-s} x^{1-s} + L(s) + O\left(\frac{x^{1-\sigma + \epsilon}}{T}\right) + O\left(\frac{x^{-\sigma}(QT)^{m(1/2+\epsilon)}}{T}\right).$$
Applying the functional equation, we get

\[ I_{-\varepsilon} = \frac{\varepsilon(\pi)}{2i\pi} \int_{-\varepsilon - i\tau}^{\alpha + i(T - \tau)} \gamma(w)L(w) \frac{x^{1-w-s}}{1 - w - s} \, dw. \]

The series for \( \bar{L}(w) = L(w, \bar{\pi}) \) being absolutely convergent, we get, with \( \bar{L}(w) = \sum_{0}^{\infty} b_{n} n^{-w} \):

\[ I_{-\varepsilon} = \varepsilon(\pi)x^{1-s} \sum_{0}^{\infty} b_{n} c(nx), \]

(2.18)

\[ c(y) = \frac{1}{2i\pi} \int_{\alpha - iT}^{\alpha + iT'} \gamma(w) \frac{y^{-w}}{1 - w - s} \, dw. \]

(2.19)

(Cf. (2.5) in [8], for \( s = 0 \).)

Recall that \( T' = T - \tau, T'' = T + \tau \) and that we assume

\[ T \geq 2\tau. \]

(2.20)

As in [8] we first consider \( y > 2(QT)^{m} \). Up to a constant, the integral (2.19) is (Lemma 2.1)

\[ Q^{m(\alpha - 1/2)} \int_{-T''}^{T'} \left| t \right|^{m(\alpha - 1/2)} (Q\left| t \right| / e)^{it} y^{-\alpha - it} \frac{(1 + O(t^{-1}))}{-\varepsilon - it - s} \, dt. \]

The function \( F(t) = m \log(Q|t| - e) - t \log y \) has derivative

\[ F'(t) = m \log |Q|e| - \log y. \]

Since \( y > 2(QT)^{m} \), \( F'(t) < -\log 2 \) for \( t \in [-T, T'] \). Moreover, \( F' \) is monotone on \( (t > 0) \) and \( (t < 0) \). By [22] Thm 2, p. 104, we deduce that \( \int_{T_{1}}^{T_{2}} e^{iF(t)} \, dt \ll 1 \) for \( -T \leq T_{1} \leq T_{2} \leq T' \). Let \( G(t) = \int_{0}^{t} e^{iF(t)} \, dt \), which is thus absolutely bounded in the interval. The integral (2.19) has a first term equal to

\[ Q^{m(\alpha - 1/2)} y^{-\alpha} \int_{-T}^{T'} \left| t \right|^{m(\alpha - 1/2)} e^{iF(t)} \frac{dt}{-\varepsilon - \sigma - i(t + \tau)}. \]
The denominator, say $D(t)$, is $C^\infty$ and $\gg 1 + |t|$. Neglecting the constant factor, we have the sum of

$$\left[ G(t) \frac{|t|^{m(\alpha - 1/2)}}{D(t)} \right]_{-T'}^T \ll T^{m(\alpha - 1/2)}T^{-1},$$

and of

$$\int_{-T}^{T'} G(F) \frac{d}{dt} \left( \frac{|t|^{m(\alpha - 1/2)}}{D(t)} \right) dt,$$

computation justified if $\frac{|t|^{m(\alpha - 1/2)}}{D(t)}$ is $C^1$. Since $\frac{d}{dt} \left( \frac{|t|^{m(\alpha - 1/2)}}{D(t)} \right)$ is equal to $(Cst) |t|^{m/2 - 1 + m\varepsilon}$ for $t$ positive or negative, this condition is satisfied if $m \geq 2$, which we will have to assume presently. Differentiating the quotient $|t|^{m(\alpha - 1/2)}D(t)$, we find that the last integral is $\ll T^{m(\alpha - 1/2)}T^{-1}$, as the first. The integral on the missing segment $[-T - \tau, -T]$ admits the same bound. Finally, the integral (2.19) is $\ll (QT)^{m(\alpha - 1/2)}y^{-1}T^{-1}$. Summing these contributions for $n > 2(QT)^m / x$, we find that these terms contribute to (2.18) a term

(2.21) $O(x^{-\sigma} T^{-1}(QT)^{m/2}),$

already present in (2.17).

We now consider the remainder term

(2.22) $I_{-\varepsilon} = \varepsilon(\pi)x^{1-s} \sum_n b_n c(nx),$

where $nx \leq 2(QT)^m$.

Again we follow [8]. We move the integration giving $c(y)$ from $Re(w) = \alpha$ to $Re(w) = \beta = \frac{1}{2} + \frac{1}{m}$.

We assume $m \geq 2$. (This is implicit in [8].) However, we move only the segments with $1 \leq |t| \leq T', T''$. We have to estimate the contributions of the vertical segment $Re(w) = \alpha$, $|t| \leq 1$, and of the horizontal segments $Re(w) \in [\beta, \alpha]$, $t = T', -T''$ and $t = \pm 1$. The vertical integral, $\gamma$ being the product of $D^{w-1/2}y^{-w}$ and of a bounded function (uniformly if the $c_i$ are fixed) is an $O(D^{1/2+\varepsilon}y^{-\alpha})$. The horizontal integrals for $t = T', -T''$ are dominated by $\int_{\beta}^{\alpha} (QT)^{m(u-1/2)}y^{-u} T^{-1} du$, dominated by

(2.23) $((QT)^{m(\alpha - 1/2)}y^{-\alpha} + (QT)^{m(\beta - 1/2)}y^{-\beta})T^{-1}$

by a computation similar to the one preceding (2.12). Since $y \leq 2(QT)^m$, the first term is dominant. The horizontal integrals for $t = \pm 1$ are dominated by
$y^{-\beta}$. (It may occur that $1 - \sigma \in [{\beta, \alpha}]$ and $\tau = \pm 1$; in this case use a small indentation of the horizontal contour.) This is dominated by $(yT)^{-1}(QT)^{T/2}$; this amounts to $y^{1-\beta} \ll T^{-1}(QT)^{T/2}$, which follows from $y \ll (QT)^{m}$, $1 - \beta = 1/2 - 1/m$ and $Q \gg 1$.

If $m \geq 2$, $$D^{1/2+\varepsilon} \asymp Q^{m(1/2+\varepsilon)} \ll (QT)^{m(1/2+\varepsilon)}T^{-1}.$$ We now assume (as will be fulfilled later)

**Hypothesis 2.2. (Auxiliary).**

$$(QT)^{m} \leq x^{2}.$$ Since $y \geq 1$,$$T^{-1}(QT)^{m(1/2+\varepsilon)}y^{-\alpha} \leq (yT)^{-1}(QT)^{m} (QT)^{m\varepsilon} \leq (yT)^{-1}(QT)^{m/2} x^{2\varepsilon}.$$ Writing $\int_{\beta-iT''}^{\beta+iT'} (T', T'' > 1)$ for the integral with the segment $[-1, 1]$ deleted, we have for $y \leq 2(QT)^{m}$:

$$(2.24) \quad c(y) = \frac{1}{2i\pi} \int_{\beta-iT''}^{\beta+iT'} \gamma(w) \frac{y^{-w}}{1 - w - s} dw + O((yT)^{-1}(QT)^{m/2} x^{2\varepsilon}),$$

compare [8, (2.8)].

However, the translation of integrals is justified only if we do not encounter zeroes of $1 - w - s$ for $w = u + it$ in the upper rectangle $\beta \leq u \leq \alpha$, $t \in [1; T']$ and the lower one. Since $w = 1 - s$, this means that $\sigma \in [0, 1/2 - 1/m]$; for the upper rectangle, $t = -\tau$ (where we have chosen $\tau \geq 0$) is impossible. For the lower rectangle, we get the value $t = \tau$.

In the latter case, we obtain a residue of order $|y^{-w}\gamma| \ll y^{-u}(Q\tau)^{m(u-1/2)}$ by Lemma 2.1. Since we will not pursue the dependence on $\tau$, this is dominated by the remainder term in (2.24). Indeed this amounts to

$$y^{1-u}Q^{m(u-1/2)} \ll (QT)^{m/2} T^{-1} x^{2\varepsilon};$$

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Since \( y \ll (QT)^m \), this is implied by
\[
(QT)^{m(1-u)}Q^{m(u-1/2)} \ll (QT)^{m/2}T^{-1}x^{2e},
\]
so by
\[
Q^{m/2}T^{m(1-u)} \ll T^{-1}(QT)^{m/2}
\]
i.e. \( T^{m(1-u)} \ll T^{m/2-1} \), which is true since \( 1 - u \leq 1/2 - 1/m \). (If \( \sigma = 1/2 - 1/m \), use a small indentation of the contour.)

Consider now the upper integral in (2.24). This is equal to
\[
c_1(y) = \frac{1}{2\pi} \omega_1 Q y^{-\beta} \int_1^{T'} \frac{t(Qt/e)^{imt}y^{-it}}{1-s-(\beta+it)}(1 + O(t^{-1})) dt
\]
\[
= \frac{i}{2\pi} \omega_1 Q y^{-\beta} \int_1^{T'} (Qt/e)^{imt}y^{-it} dt + O((yT)^{-1}(QT)^{m/2} \log T)
\]
\((T' \geq 2)\), using that \( y \ll (QT)^m \) and
\[
(2.25) \quad \frac{1}{1-s-(\beta+it)} = \frac{i}{t}(1 + O(t^{-1})).
\]
(We have used that \( T' \asymp T, T'' \asymp T \)).

Since \( T \leq x \), the remainder term is dominated by the last remainder term, \((yT)^{-1}(QT)^{m/2}x^{2e}\). The lower integral yields
\[
c_2(y) = \frac{1}{2\pi} \omega_2 Q y^{-\beta} \int_1^{T''} \frac{t(Qt/e)^{-imt}y^{-it}}{1-s-(\beta-it)}(1 + O(t^{-1})) dt
\]
\[
= \frac{-i}{2\pi} \omega_2 Q y^{-\beta} \int_1^{T''} (Qt/e)^{-imt}y^{-it} dt + O((yT)^{-1}(QT)^{m/2} \log T).
\]

We are now able to reduce the computation to that in [8] for \( c(y) \). To be precise, in the case where \( \omega_2 = \bar{\omega}_1 = \bar{\omega} \), \( c(y) \) is given by [8 (2.8)] :
\[
c(y) = \pi^{-1} y^{-\beta} Q \Re(i\omega \int_1^T (QT/e)^{imt}y^{-it} dt) + \text{remainder term}.
\]
(The parentheses are missing in the original.) Here this is replaced by
\[
c(y) = c_1'(y) + c_2'(y) + \text{remainder term},
\]
\[
c_1'(y) = \frac{1}{2\pi} y^{-\beta} Q i\omega_1 \int_1^{T'} (Qt/e)^{imt}y^{-it} dt,
\]
\[
c_2'(y) = \frac{1}{2\pi} y^{-\beta} Q (-i\omega_2) \int_1^{T''} (Qt/e)^{-imt}y^{-it} dt
\]
We majorize the contributions of the intervals \([T', T]\) and \([T, T'']\): write \(C_1'(y), C_2'(y)\) for the integrals with end value \(T\).

**Lemma 2.2.** (1) \(|c_1'(y) + c_2'(y) - C_1'(y) - C_2'(y)| \ll y^{-\beta} Q \tau\)

(2) \(\sum_{n x \leq 2(QT)^m} b_n \tau Q (nx)^{-\beta} \ll \tau(xT)^{-1}(QT)^{m/2} x^\varepsilon\).

The first inequality is clear. The second follows by an easy summation, using that \(b_n \ll n^{\varepsilon'}\) for any \(\varepsilon' > 0\).

Now the integrals \(C_1'(y), C_2'(y),\) and their sum over \(n \leq 2(QT)^m\), are estimated in [8]. For \(1 \leq T \leq x\) let

\[
B_0(x, T) = \sum_{n x \leq (QT)^m} b_n n^{-m+1} \left\{ \omega_1 e^{3i\pi/4 - \frac{i\pi}{2m} (nx)^{1/m}} + \omega_2 e^{-\frac{3\pi}{4} + \frac{i\pi}{2m} (nx)^{1/m}} \right\}.
\]

We have transformed the expression of \(B_0(x, T)\) in [8] according to \((\omega, \bar{\omega}) \rightarrow (\omega_1, \omega_2)\). Let

\[
(2.26) \quad J_{-\varepsilon}^r(x) = \varepsilon(\pi) x^{1-s} \sum_n b_n c_{FI}(n x),
\]

where, as before Lemma 2.2, we have replaced \(c_1', c_2'\) by \(C_1', C_2';\) thus \(c_{FI}\) is the function "c" of Friedlander-Iwaniec (with \(\omega, \bar{\omega}\) replaced by \(\omega_1, \omega_2\).) Note the factor \(x^{1-s}\) in (2.26), as opposed to \(x\) in [8].

We now have, using the estimate in [8] for \(C_1', C_2'\):

**Lemma 2.3.** [8] - For \(1 \leq T \leq x\), with \((QT)^m \leq x^2\),

\[
J_{-\varepsilon}^r(x) = \varepsilon(\pi) x^{-s} \left( \frac{2Q}{\pi m} \right)^{1/2} x^{m-1} B_0(x, T) + O(x^{1-\sigma} T^{-1/2} (QT)^{-m/2}) + O(x^{2s-\sigma} T^{-1} (QT)^{m/2}).
\]

We still have to replace \(J_{-\varepsilon}^r\) by \(I_{-\varepsilon}^r\). The difference, according to Lemma 2.2, is dominated by the second remainder term in Lemma 2.3 (multiplied by \(\tau\), but we assume here \(s\) fixed.). We therefore obtain the same estimate.

Collecting the remainder term (2.17), and noting that

\[
x T^{-1/2} (QT)^{-m/2} \ll x Q^{-m/2} T^{-\frac{1}{2} - \frac{m}{2}} \ll x T^{-1}
\]
since \( Q \geq \frac{1}{2\pi} \), we obtain:

\begin{equation}
A_s(x) = \frac{\kappa}{1-s} x^{1-s} + L(s) + \varepsilon(\pi) x^{-s} \left( \frac{2Q}{\pi m} \right)^{1/2} x^{\frac{m-1}{2m}} \cdot B_0(x, T) + R(x, T),
\end{equation}

\begin{equation}
R(x, T) \ll x^{-\sigma} \{ x^{1+\varepsilon} + (QT)^{\frac{m}{2}} x^{2\varepsilon} \} T^{-1}
\end{equation}

\begin{equation}
\ll x^{-\sigma+\varepsilon} \{ x + (QT)^{\frac{m}{2}} x^\varepsilon \} T^{-1}.
\end{equation}

This is essentially the expression in [8, p. 499], the remainder term being of course multiplied by \( x^{-\sigma} \).

Friedlander and Iwaniec then set \( N = x^{-1}(QT)^m \). Assuming \( N \leq x \), \( R(x, T) \ll x^{-\sigma+\varepsilon} T^{-1} \). We have \( B_0(x, T) = B(x, N) \) where [8 (1.12)]

\[ B(x, N) = \sum_{n \leq N} b_n n^{-\frac{m+1}{2m}} \{ \omega_1 e^{-i\pi/4 - 2i\pi m(nD)^{1/m}} + \omega_2 e^{i\pi/4 + 2i\pi m(nD)^{1/m}} \}. \]

Obviously the estimates for \( B(x, N) \) and for its variant given in [8] by (1.12) are the same. However we must check that \( 1 \leq N \leq x \), as in [8, Theorem 1.2] are allowed values. For \( 1 \leq T \leq x \),

\[ Q^m / x \leq N \leq Q^m x^{m-1} = D/(2\pi)^m \cdot x^{m-1}. \]

Since \( m \geq 2 \), \( N \leq x \) (essentially) satisfies the upper bound. However \( N = 1 \) is allowed only if \( x \geq D/(2\pi)^m \), not for \( x \geq D^{1/2} \) as stated. Note also that Auxiliary hypothesis 2.2 is satisfied. Under this assumption, we get the statement of Theorem 1.2 in [8], the remainder term being multiplied by \( x^{-\sigma} \).

Namely, for \( x \geq D \),

\begin{equation}
A_s(x) = \frac{\kappa}{1-s} x^{1-s} + L(s) + \varepsilon(\pi) x^{-s} \left( \frac{Q}{2\pi m} \right)^{1/2} x^{\frac{m-1}{2m}} \cdot B(x, N) + O(D^{1/m} N^{-1/m} x^{-\sigma + \frac{m-1}{m} + \varepsilon})
\end{equation}

for \( 1 \leq N \leq (2\pi)^{-m} x \).

We want to evaluate this, in order to get Proposition 1.1 in [8], at \( N = D^{1/(m+1)} x^{m-1} \). We must therefore have

\[ D^{1/(m+1)} x^{m-1} \geq x^{-1} Q^m = x^{-1} D^{1/(2\pi)^m}, \]
which yields by a simple computation

\[ x^2 \geq D(1/2\pi)^{m+1}. \]

Thus the stated condition, \( x^2 \geq D \), is adequate. Under this assumption, Friedlander-Iwaniec show that the full remainder term for \( s = 0 \):

\[ \varepsilon(\pi) \left( \frac{Q}{2\pi m} \right)^{1/2} x^{\frac{m-1}{2m}} B(x, N) + O(D^{1/m}N^{-1/m}x^{\frac{m-1}{m+1}+\varepsilon}) \]

is \( \ll D^{\frac{1}{m+1}x^{\frac{m-1}{m+1}+\varepsilon}} \).

We have introduced another condition in the proof, i.e. \( T \geq 2\tau \), so for \( N = x^{-1}(QT)^m \):

\[ N \geq x^{-1}(D/(2\pi)^m)(2\tau)^m \]

which yields

\[ x \geq D^{1/2}\left| \frac{\tau}{\pi} \right|^{\frac{m+1}{2}}. \]

**Theorem 2.2.** Assume \( s = \sigma + i\tau \), \( 0 < \sigma < 1 \). Then, if

\[ x^2 \geq \text{Max}(D(1/2\pi)^{m+1}, D|\tau/\pi|^m), \]

\[ A_s(x) = \frac{\kappa}{1-s} x^{1-s} + L(s) + O(D^{\frac{1}{m+1}} x^{\frac{m-1}{m+1} - \sigma + \varepsilon}) \]

the implicit constant depending only on \( \varepsilon \), \( \pi_\infty \) and \( \tau \).

**Corollary.** Under the same assumptions,

\[ H_s(x) = O(D^{1/m+1} x^{-\frac{2}{m+1} + \varepsilon}) \]

if \( L(s) = 0 \).

As expected, we have improved on the estimates on \( H_s \) (before Lemma 1.1) : the new estimates do not depend on \( \sigma \).

**3 Automorphic \( L \)-functions (mostly) of degree \( \leq 2 \) over \( \mathbb{Q} \)**

**3.1**

In this chapter we use the results of Chapter 2 to extend Theorem 1.2. We consider \( L \)-series associated to automorphic representations of \( GL(m, \mathbb{A}_\mathbb{Q}) \)
Consider first the case of $\zeta_Q$, or of a classical Dirichlet series $L(s, \chi)$. (We will always consider primitive characters.) In this case we saw that $\mathcal{M}H_s$ (if $\zeta(s)$ or $L(s, \chi) = 0$) is defined (and holomorphic) for $0 < \text{Re}(w) < 1$.\footnote{Strictly speaking we did not do this for $L(s, \chi)$ but the same estimate $H_s(x) \ll x^{-1}$ is obtained by extending the computation in § 1.3.} We can apply the formula of Mellin–Parseval in this domain.

Assume $f(x) (x \geq 0)$ is a function bounded in 0 and such that $f(x) \ll x^{-\lambda+\varepsilon}$, $\lambda > 0$ (for any $\varepsilon > 0$) if $x \to \infty$. Then $\mathcal{M}f(w)$ is defined and holomorphic for $0 < \text{Re}(w) < \lambda$. Moreover, for $\tau < \lambda$,

$$\int_0^\infty |x^\tau f(x)|^2 \frac{dx}{x} < \infty.$$  

We are then in the domain of Titchmarsh’s Theorem 71\footnote{Thus $\tau$ no longer denotes $\text{Im}(s)$ as in Chapter 2!} : $\mathcal{M}f$ is $L^2$ on the line $\text{Re}(w) = \tau < \lambda$, and

$$\int_0^\infty |x^\tau f(x)|^2 \frac{dx}{x} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |\mathcal{M}f(\tau + it)|^2 dt.$$  

For $-2 \leq a \leq 2$,

$$\int_1^2 x^{a-1} \, dx \geq \frac{3}{8}.$$  

Using the argument for the proof of Theorem 1.2 $(i)$, we deduce :

**Proposition 3.1.** Assume $s$ is a zero of $L(s) = \zeta_Q(s)$ or $L(s, \chi)$. Then, for any $\tau \in ]0, 1[$,

$$\int_{-\infty}^{+\infty} \left| \frac{L(\tau + it)}{\tau + it - s} \right|^2 \, dt > \frac{3\pi}{4} - \frac{4\pi}{|1-s|}.$$
Consider now an automorphic representation $\pi$ of $GL(2, \mathbb{A}_\mathbb{Q})$ verifying the assumptions of § 2.1. (In particular $\pi_\infty$ is self-dual). This includes the case of $L_F(s, \chi)$ where $F/\mathbb{Q}$ is quadratic and, if $F$ is real, $\chi$ is an Artin character and, if $F$ is imaginary, $\chi$ is an algebraic Hecke character (possibly multiplied by $| \frac{\mu}{F} |$), and in particular the case of $\zeta_F$. By the Corollary to Theorem 2.1, $H_s(x) \ll x^{-\frac{2}{3}+\varepsilon}$. In particular (see after Proposition 1.2) we see that $\mathcal{M}H_s(w)$ is convergent for $\tau = 1/2$, and that $H_s \in L^2$ (Proposition 1.2(i)) follows from the growth estimate.

In fact $\mathcal{M}H_s(w) = L(1-w)\frac{1}{1-w-s}$ is convergent in the domain $\tau < \frac{2}{3}$, and is $L^2$ on the vertical lines. Arguing as before, we have:

**Proposition 3.2.** Assume $\pi$ is a tempered, automorphic representation of $GL(2, \mathbb{A}_\mathbb{Q})$ with $\pi_\infty$ self-dual. Let $s$ be a zero of $L(s, \pi)$, with $0 < \sigma < 1$. Then, for any $\tau > \frac{1}{3}$, $L(\tau + it)\tau + it - s$ is $L^2$ and

$$\int_{-\infty}^{+\infty} \frac{L(\tau + it)}{|\tau + it - s|^2} dt > \frac{3\pi}{4} - 4\pi \left| \frac{\kappa}{1 - s} \right|.$$  

(For $\tau = 1/2$, the constant $\frac{3\pi}{4}$ can be replaced by $2\pi \log 2$.) Moreover, for $\sigma = 1/2$:

$$\int_{-\infty}^{+\infty} \frac{L(1/2 + it)}{1/2 + it - s}^2 dt > \pi \log 2.$$

The argument for the second part is as follows. We seek a lower bound on

$$\int_0^2 |H_s(x)|^2 dx = \left| \frac{\kappa}{1 - s} \right|^2 + \int_1^2 |x^{s-1} - \frac{\kappa}{1 - s}|^2 dx$$

Let $f(x) = x^{s-1}, g(x) = \frac{\kappa}{1 - s}$ on $[1,2]$. The full integral is then equal to $||f - g||^2 + ||g||^2$ ($L^2$-norm on $[1,2]$). However this is greater than $||f||^2 - 2||f|| ||g|| + 2||g||^2 \geq \frac{1}{2}||f||^2 = 1/2 \log 2$, whence the result.

It is interesting to compare this with the subconvex estimate. Let $\mu(\tau)$ be defined by

$$|L(\tau + it)| \ll t^{\mu(\tau) + \varepsilon}$$

and minimal. Then $\mu(\frac{1}{2}) \leq \frac{1}{2} - \delta$ ($\delta > 0$), $\mu(0) = 1$ and therefore $\mu(\tau) \leq 1 - (1 + 2\delta)\tau$ for $\tau \in [0, \frac{1}{2}]$. In particular $\frac{L(\tau + it)}{t}$ ($t \geq 1$) is $L^2$, as a consequence.
of the convexity estimate, if $\tau > \frac{1}{2(1+2\delta)}$. Since $\delta$ is small, this does not imply the convergence of the quadratic integral in the range $\tau > \frac{1}{3}$.

### 3.3 The case $s = 0$

In this section we prove Theorem C and Theorem D in the context of this paper, i.e. when the integrals are convergent. We make the assumptions in § 2. Thus

$$A_0(x) = \sum_{n \leq x} a_n = \kappa x + O\left(D^{\frac{1}{m+1}} x^{1-\frac{2}{m+1}+\varepsilon}\right),$$

a priori for $x \geq D^{1/2}$. (However $|a_n| \leq \tau_m(n)$ and so

$$\sum_{n \leq x} |a_n| \ll x(\log x)^{m-1},$$

cf. [8, p. 23], and a trivial computation shows that, if $\kappa = 0$, this estimate is true for all $x \geq 1$.)

We have $H_0(x) = x^{-1}A_0(x) - \kappa$; note that $(s = 0)$ is not necessarily a zero of $L(s)$. As before we see that

$$\mathcal{M}H_0(w) = \int_1^\infty H_0(x) x^{w-1} dx$$

is convergent if $\tau = \text{Re}(w) < \frac{2}{m+1}$. We will now assume that $m = 1, 2$ and consider $\tau = 1/2$. Assume first that $\kappa = 0$. We can proceed as in § 1.3 and compute directly $\mathcal{M}H_0$

$$\int_1^T x^{-1} \sum_{n \leq x} a_n x^{w-1} dx$$

$$= \sum_{n \leq T} a_n \int_n^T x^{w-2} dx$$

$$= \frac{1}{w-1} \left\{ T^{w-1} \sum_{n \leq T} a_n - \sum_{n \leq T} a_n n^{w-1} \right\}.$$

---

16This was the case considered by Titchmarsh [25, p. 321]. Titchmarsh, studying the divisor problem, considers $L(s) = \zeta Q(s)^2$, a case we have excluded here. He uses the Mellin transform in the same fashion.
The first bracketed term is \( \ll T^{\frac{2}{m+1}} \to 0 \) \((T \to \infty)\), and the second is \( \sum_{n \leq T} a_n n^{w-1} \to L(1-w) \) by Theorem 2.2 (cf. also Theorem 3.1). Thus

\[
\mathcal{MH}_0(w) = \frac{L(1-w)}{1-w}
\]

for \( \tau < \frac{2}{m+1} \) and in particular for \( \tau = \frac{1}{2} \).

In the case where \( L(s) \) has a pole (at \( s = 1 \)) we must proceed as in § 1.4. We have

\[
A_0^s(x) = \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} L(w) \frac{x^w}{w} dw
\]

for \( c > 1 \),

\[
x^{-1}A_0^s(x) = \frac{1}{2i\pi} \int_{c'} L(1-w) \frac{x^{-w}}{1-w} dw
\]

for \( c' = 1 - c < 0 \), by a change of variables. We move the integral to \( c'' > 0 \), \( c'' < 1 \) (e.g. \( c'' = \frac{1}{2} \)) and deduce

\[
x^{-1}A_0^s(x) = \frac{1}{2i\pi} \int_{1/2}^{1} L(1-w)x^{-w} \frac{1}{1-w} dw + \kappa
\]

using the same estimates as in § 1.4, taking account of the residue at \((w = 0)\). Thus

\[
H_s^*(x) = x^{-1}A_0^s(x) - \kappa
\]

is given by the vertical integral.

The first computation yields Theorem C by the Parseval formula, since

\[
\int_1^\infty |x^{1/2} H_0(x)|^2 \frac{dx}{x} = \int_1^\infty |H_0(x)|^2 dx
\]

\[
\geq \int_1^2 x^{-2} dx = \frac{1}{2}.
\]

The second case yields

\[
\frac{1}{2\pi} \int \left| \frac{L(1/2 + it)}{1/2 + it} \right|^2 dt \geq \int_0^1 |\kappa|^2 dx + \int_0^2 |x^{-1} - \kappa|^2 dx
\]

The argument given in the proof of Proposition 3.2 shows that this sum of integrals is larger than \(1/4\), proving Theorem D.

Note the special case of Theorem D:
Theorem 3.1. Assume $F$ is a quadratic field. Then

$$\int \left| \frac{\zeta_F\left(\frac{1}{2} + it\right)}{1/2 + it} \right|^2 dt > \pi / 2.$$ 

We now consider the case of $m = 1$, and of a non-trivial Dirichlet character $\chi$. We assume that $\chi$ is a character mod $q$, $q$ being prime. Here the Friedlander–Iwaniec estimate is simply replaced by the Polyà–Vinogradov inequality (Schur’s version) $A_0(x) \leq q^{1/2}$. The vertical integral is bounded below by $\pi$. However, under the Lindelöf hypothesis (Schur’s version) the integrand admits a majoration by $\frac{(q|s|)^{2\varepsilon}}{1 + t^2}$, so the integral $\ll q^{2\varepsilon}$. We now consider the quadratic character $\chi$ of $(\mathbb{Z}/q)^\times$.

Let $\nu = \nu(q)$ be the least quadratic non-residue. We have

$$\int_1^\infty |H_0(x)|^2 dx \geq \int_1^\nu (x^{-1}[x])^2 dx \geq \int_1^\nu \left( 1 - \frac{2}{x} \right) dx = \nu - 2 \log \nu$$

using that $[x] \geq x - 1$. Thus

$$\nu - 2 \log \nu \ll q^{2\varepsilon},$$

i.e. $\nu \ll q^\varepsilon$. Thus this conjecture of Vinogradov follows directly from the Lindelöf hypothesis (including its dependence on $q$.) The proof of Ankeny [1] depends on the generalised Riemann hypothesis.

3.4

We now consider the convergence abscissa of a cuspidal representation $\pi$ of $GL(m,A)$. We assume $\pi$ tempered and $\pi_{\infty}$ self-dual.

Theorem 3.2. (i) The abscissa of absolute convergence $\sigma_a(\Pi)$ of the Euler product for $L(s,\pi)$ satisfies $0 \leq \sigma_a(\Pi) \leq 1$.

(ii) The abscissa of absolute convergence of the Dirichlet series for $L(s,\pi)$ satisfies $0 \leq \sigma_a \leq 1$.

(iii) The abscissa of convergence $\sigma_c$ of the $L$-series $L(s,\pi)$ verifies $\sigma_c \leq 1 - \frac{2}{m+1}$.

That $\sigma_a(\Pi) \leq 1$ is due to Jacquet–Shalika [3]. Let $A_p$ be the Hecke matrix of $\pi$ at an unramified prime $p$, so $a_p = \text{trace}(A_p)$. If $\sum |a_p|p^{-s} < \infty$ for
σ < 0, |trace(A_p)|^2 p^{-2σ} ≤ C, which implies that \( \sum |trace(A_p)|^2 p^{-1} < \infty \). This contradicts the divergence of the logarithm of the Rankin L-function at \( s = 1 \). Thus \( σ_a(\Pi) ≥ 0 \). Again, by [3], \( σ_a ≤ 1 \). Then \( σ_a ≥ 0 \) follows from the usual argument: if \( L(σ, π) \) were absolutely convergent for \( σ < 0 \), we would have

\[
\sum p \sum_{α≥1} |a_{p^α}| p^{-ασ} < ∞
\]

and this would imply that the Euler product is convergent.

Finally, Theorem 2.2 implies that \( \sum a_n n^{-s} \) converges to \( L(s) \) if \( σ < \frac{m-1}{m+1} \).

**Remark.**- If \( π \) verifies the generalised Sato-Tate conjecture, it is easy to see that \( σ_a(\Pi) = 1 \). This applies, in particular, when \( m = 2 \) and \( π \) is associated to classical modular forms.

Anticipating on the next paragraph, we note that a stronger result should hold under a Conjecture of Friedlander–Iwaniec [8, Conjecture 2], which would imply (we forget the dependence on \( D \)):

\[
A_0(x) ≪ x^{\frac{m-1}{2m} + \varepsilon}.
\]

One should expect that the proof in Chapter 2 would then extend to yield

\[
A_s(x) - L(s) ≪ x^{\frac{m-1}{2m} - σ + \varepsilon},
\]

whence

\[
σ_c ≤ \frac{1}{2} - \frac{1}{2m}.
\]

Assume \( σ_c ≤ \frac{1}{2} \). By a classical result \( μ(σ_c) ≤ 1 \). On the other hand [27, 9.41], \( μ(σ) = 0 \) for \( σ > 1 \) and the functional equation implies \( μ(σ) = m(1/2 - σ) \) for \( σ < 0 \) and therefore \( μ(σ) ≥ m(1/2 - σ) \) by convexity for \( σ ≤ 1/2 \). Thus we see (unconditionally) that

\[
σ_c ≥ \frac{1}{2} - \frac{1}{m}.
\]

We do not know, even hypothetically, what should be the correct value of \( σ_c \).
3.5

We now describe, in any degree $m$, what should be expected of the square-integrability of $H_s$ and $\frac{L(1-w)}{1-s-w}$, in which case the lower bound would be effective. It is interesting to consider, as in §3.2, values of $Re(w) \neq \frac{1}{2}$. As we will see, we need the full force of the Friedlander–Iwaniec conjecture, i.e. (3.1). [Here $L(s)$ is an arbitrary $L$–function $L(s, \pi)$ of degree $m$ over $\mathbb{Q}$, where $\pi$ satisfies the conditions in §2.1.]

Consider first the convergence of $M H_s(w)$. One expects (3.2), which implies

$$H_s(x) \ll x^{-1/2 - \frac{1}{2m} + \varepsilon}.$$ 

Thus $M H_s(w)$ is well-defined for $\tau = Re(w) < \frac{1}{2} + \frac{1}{2m}$. Moreover, $M H_s(w)$ will be $L^2$ on the line $Re(w) = \tau$ if (see §3.2)

$$\int_0^\infty x^{2\tau} |H_s(x)|^2 \frac{dx}{x} < \infty,$$

which gives the same condition. Thus we would have by the substitution $w \to 1 - w$:

Assume $\tau > \frac{1}{2} - \frac{1}{2m}$. Then, for any zero of $L(s)$, $M H_s(1 - w)$ is defined, and equal to $\frac{L(w)}{\pi^{1-w}}$ for $Re(w) = \tau$. Moreover,

$$\frac{1}{2\pi} \int \frac{L(\tau + it)}{\tau + it - s} dt = \int_0^\infty x^{2-2\tau} |H_s(x)|^2 \frac{dx}{x}.$$

Both integrals are finite. In particular, $\frac{L(\tau + it)}{\pi^{1+|t|}}$ is $L^2$.

Assume $\frac{1}{2} - \frac{1}{2m} < \tau \leq \frac{1}{2}$. Then, under the generalized Lindelöf conjecture,

$$\mu(\tau) = m\left(\frac{1}{2} - \tau\right) < \frac{1}{2},$$

This implies that $\frac{L(\tau+i\gamma)}{\tau+i\gamma-s} \in L^2$, in conformity with the previous (conjectural) description.

It is easy to see that the estimate coming from the generalised Riemann hypothesis, namely, cf. [11, p. 500] ; [12, p. 116]:

$$(3.5) \quad A_0(x) = \kappa x + O(x^{1/2+\varepsilon})$$

does not allow, in general, for reaching the line $\tau = \frac{1}{2}$.

40
Remark.- According to Iwaniec-Kowalski [11, p.86], the effective estimate (of Landau or Friedlander-Iwaniec) for $A_0$ can be improved. Any small improvement would allow one to prove the results in this paper (with finite integrals) for $m = 3$.

3.6

Since we expect an absolute lower bound, when $L(s) = 0$, on the quadratic integral of $\frac{L(w)}{w}$ on the line $Re(w) = 1/2$, it is now a natural question whether this integral will tend to $\infty$ when $s \to \infty$ in the critical strip. Even in the case of the Riemann zeta function, this seems a difficult problem. We will only make a few remarks. As in chapter 1, let

$$I(s) = \int_{-\infty}^{+\infty} \left| \frac{\zeta(\frac{1}{2} + it)}{1/2 + it - s} \right|^2 \, dt.$$ 

Suppose that $I(s)$ is bounded. It is trivial that $\int_a^b \frac{\zeta(\frac{1}{2} + it)}{1/2 + it - s} \, dt \to 0$ for $s \to \infty$ for any finite interval $[a, b]$. Since the linear combinations of characteristic functions of such intervals are dense, we see that under this assumption

$$\frac{\zeta(\frac{1}{2} + it)}{1/2 + it - s} \text{ and } H_s \text{ tend to 0 weakly in } L^2.$$

The boundedness of $I(s)$ would also have a puzzling consequence. We consider only values of $s$ on the critical line. Writing $z(t)$ for $\zeta(1/2 + it)$ and $s = 1/2 + i\tau$, we would then have

$$\int_{-\infty}^{+\infty} \left| \frac{z(t)}{t - \tau} \right|^2 \, dt \leq C.$$

We consider the integral on a small interval around $\tau$. We have $z(\tau + h) = h z'(\tau) + \frac{h^2}{2} (z''_r(\tau + \theta_1 h) + i z''_i(\tau + \theta_2 h))$ where we have decomposed $z''$ in its real and imaginary parts. If $h \ll \tau$ and $\mu$ is an order of $\zeta$ on $(\sigma = 1/2)$, for example $\mu = 1/6$, $z''$, which is essentially $\zeta''$, is dominated by $\tau^{\mu + \varepsilon}$ by the residue formula. Replacing $\varepsilon$ by $2\varepsilon$ and taking $\tau$ sufficiently large, we deduce that for $u \leq h$:

$$\left| \frac{z(\tau + u)}{u} \right| \geq |z'(\tau)| - \frac{u}{2} \tau^{\mu + \varepsilon}.$$
whence
\[ \left| \frac{z(\tau + u)}{u} \right|^2 \geq \left| z'(\tau) \right|^2 - u \tau^{\mu+\varepsilon} |z'(\tau)| \]
and
\[ \int_{0}^{h} \left| \frac{z(\tau + u)}{u} \right|^2 du \geq \left| z'(\tau) \right| (h|z'(\tau)| - \frac{h^2}{2} \tau^{\mu+\varepsilon}). \]

Choosing \( h = \frac{|z'(\tau)|}{\tau^{\mu+\varepsilon}} \), so clearly \( h \ll \tau \), we see that the integral is larger than \( \frac{1}{2} |z'(\tau)|^3 \tau^{-\mu-\varepsilon} \). Since this is uniformly bounded, we deduce

**Proposition 3.3.** Assume \( I(s) \) is uniformly bounded for \( s \) a zero on the critical line. Then, for any such zero \( s = 1/2 + i\tau \),
\[ \zeta'(s) \ll \tau^{\mu/3 + \varepsilon}. \]

Of course the zeroes are too sparse to use this argument to improve the estimate on \( \zeta(s) \) on the critical line. Although the estimate on \( \zeta' \) is undoubtedly true, it seems unlikely that it can be so inferred.
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Appendix. Variances for $\lambda_\pi(n)$ and $L(s, \pi)$

L. Clozel and P. Sarnak

Introduction

The universal lower bounds for variances of $L$–functions on their critical lines that are established in this paper are based on a Parseval identity. The $L^2$–duality allows one to express a variance on the one side in terms of a quadratic form on the other side, facilitating an estimation. In this appendix we examine variances for the variations of $\lambda_\pi(n)$ in $n$ and $L(\frac{1}{2} + it, \pi)$ in $t$, and place the results of this paper in this more general context.

We restrict to $\pi$’s which are cuspidal on $GL_m/\mathbb{Q}$, $m \geq 1$, including the case that $\pi$ is the trivial representation and $L(s, \pi) = \zeta(s)$. We also specialize the points to be $1/2$ in the notation of the paper, which we adopt. The analytic conductor $c(\pi)$ (see [2]) is defined to be

$$D \cdot \prod_{j=1}^{m} (1 + |c_j|)$$

where $D = D(\pi) \in \mathbb{N}$ is the usual conductor of $\pi$ (see §2.1) and $c_j = c_j(\pi)$ are as in (2.2). So $D(\pi)$ measures the finite ramification of $\pi$ and the $c_j$’s correspond to the Archimedian component $\pi_\infty$ of $\pi$. $c(\pi)$ is a measure of the complexity of $\pi$ and it enters when estimating quantities associated with $\pi$. Note that for $m$ fixed, the number of $\pi$’s with $c(\pi)$ at most a given limit, is finite. In fact this count satisfies a “Weyl–Schanuel” asymptotic law (see [1]).

In the main paper, only the arithmetic conductor occurs. We note however that the Friedlander-Iwaniec estimate (Prop. 1.1) can likely be formulated in terms of the analytic conductor, and doing so would be worthwhile.

Let $L(s, \pi) = \sum_{n=1}^{\infty} \lambda_\pi(n)n^{-s}$. We are interested in the fluctuations of the function $\lambda_\pi(n)$, or rather of the summatory function $\sum_{n \leq x} \lambda_\pi(n)$ ($x \geq 1$).

On the dual side are the fluctuations of the function $L(\frac{1}{2} + it, \pi)$ as functions of $t \in \mathbb{R}$. We fixate on their fluctuations about their value at $t = 0$. In Theorems A and B, this corresponds to $s = 1/2$; however we do not want to

\footnotesize{\begin{itemize}
  \item[17] The unspecified references are to the main paper.
\end{itemize}}
assume that $L(\frac{1}{2}, \pi) = 0$. The variance $V$ introduced in this paper is defined by :

\begin{equation}
V(\pi) = V(\frac{1}{2}, \pi) := \int_{-\infty}^{\infty} \left| \frac{|L(\frac{1}{2} + it, \pi) - L(\frac{1}{2}, \pi)|^2}{t^2} \right| dt.
\end{equation}

As discussed in detail in the paper, $V(\pi)$ is expected to be finite. We will allow it to be infinite; as a consequence, we make no use of any unproven hypotheses.

The key Parseval relation proved in the paper may be extended to prove (assuming $L(s, \pi)$ has no pole) that :

\begin{equation}
V(\pi) = 2\pi \int_{0}^{\infty} \left| \sum_{n \leq x} \frac{\lambda_\pi(n)}{\sqrt{n}} - L(\frac{1}{2}, \pi) \right| \frac{2dx}{x}.
\end{equation}

The meaning in (3) is that both sides are finite or infinite together, and if finite they are equal. Note that (3) expresses variance $V(\pi)$ in terms of mean–square of $\lambda_\pi(n)/\sqrt{n}$ with $n$ varying over $\mathbb{N}$.

The means and variance of our fluctuating variable give us a good picture of its distribution. How do these vary with $\pi$? The size of the central value $L(\frac{1}{2}, \pi)$, is a much studied problem (see [4] for a survey). Any improvement in the exponent of the bound ($m$ fixed)

\begin{equation}
L(\frac{1}{2}, \pi) \ll c(\pi)^{1/4}
\end{equation}

is known as a subconvex bound. For $m = 1$ and 2 such bounds have been established, but for $m \geq 3$ the problem remains a central and widely open one.

For the variances $V(\pi)$, upper bounds and even their finiteness is problematic, as is discussed in detail in the paper. For $m$ fixed we expect that $V(\pi)$ is of order $\log c(\pi)$. For various families of $\pi$’s that are studied in [5] one can show that the typical member has $V(\pi)$ of order $\log c(\pi)$, but individual growing lower bounds seem difficult to prove, even conditionally assuming the Riemann hypothesis. The main result of the paper is the universal lower bound for $V(\pi)$ in (2) \footnote{Assume $L(s, \pi)$ has no pole.} :

\begin{equation}
V(\pi) > 2\pi \log 2 \left| 1 - L(\frac{1}{2}, \pi) \right|^2.
\end{equation}
If $L(\frac{1}{2}, \pi) = 0$ this recovers the form that is stated and proved in the paper (the proof is the same and uses (3) after restricting the $x$ integral to $[1, 2]$).

Similarly one can prove a universal lower bound for the mean—square.

For any $\pi$:

$$I(\pi) = \int_{-\infty}^{\infty} \frac{|L(\frac{1}{2} + it, \pi)|^2}{\frac{1}{4} + t^2} dt > \pi$$

(the right-hand side is $\frac{\pi}{2}$ if the L-function has a pole.) This follows from the following Parseval identity which generalizes (12.5.4) in Titchmarsh [6] (with the usual meaning if either is infinite)

$$LHS \ of \ (6) = 2\pi \int_0^{\infty} \left| \sum_{n \leq x} \lambda_\pi(n) - \delta_\pi x \right|^2 dx$$

where $\delta_\pi = 1$ if $m = 1$ and $\pi$ is the trivial representation, and 0 otherwise.

Perhaps the most basic question that presents itself is whether

$$\lim_{\pi \to \infty} I(\pi) = \infty$$

when $\pi$ tends to infinity in a suitable sense.

**Proof of (5) and its variant**

*We first assume that $L(s, \pi)$ has no pole. Let*

$$J(x) = \begin{cases} 1 & \text{if } 0 < x \leq 1 \\ 0 & \text{if } x > 1. \end{cases}$$

We smooth $J$ as a function on the multiplicative group $\mathbb{R}_{>0}$ as follows: fix $\varphi \in C^\infty_c(\mathbb{R})$, $\varphi \geq 0$, $\int_{-\infty}^{\infty} \varphi(X) dX = 1$, $\varphi$ even, and $\text{Support}(\varphi) \subset [-\frac{1}{2}, \frac{1}{2}]$. For $\varepsilon > 0$ set $\varphi_\varepsilon(x) = \frac{1}{\varepsilon} \varphi(\frac{x}{\varepsilon})$ and

$$J_\varepsilon(x) = \int_0^{\infty} J(xy^{-1}) \varphi_\varepsilon(\log y) \frac{dy}{y} = (J * \psi_\varepsilon)(x)$$

(multiplicative convolution, $\psi_\varepsilon(x) = \varphi_\varepsilon(\log x)$.)

$J_\varepsilon(x)$ is smooth on $(0, \infty)$ and satisfies for $\varepsilon$ sufficiently small:

$$J_\varepsilon(x) = \begin{cases} 1 & \text{for } 0 < x < 1 - \varepsilon \\ 0 & \text{for } x > 1 + \varepsilon \\ 0 \leq J_\varepsilon(x) \leq 1 & \text{for } 0 < x < \infty \end{cases}$$

(48)
(B) \( J_\varepsilon(x) \to J(x) \) uniformly for \( x \) outside any neighbourhood of 1.

(C) For \( \text{Re}(s) > 0 \),

\begin{equation}
\tilde{J}_\varepsilon(s) := \mathcal{M}J_\varepsilon(s) = \int_0^\infty J_\varepsilon(x)x^s\frac{dx}{x} = \mathcal{M}J(s)\mathcal{M}\psi_\varepsilon(s) = \frac{\hat{\varphi}(i\varepsilon s)}{s},
\end{equation}

where

\[ \hat{\varphi}(\xi) = \int_{-\infty}^{\infty} \varphi(X)e^{-i\xi X}dX. \]

In particular \( \tilde{J}_\varepsilon(s) \) is analytic in \( \mathbb{C} \) except for a simple pole at \( s = 0 \), with residue 1 at that point. Moreover from (11) it follows that \( \tilde{J}_\varepsilon(s) \) is rapidly decreasing in \( |t| \) for \( s = \sigma + it \) and uniformly so for

\begin{equation}
\sigma_0 \leq \sigma \leq \sigma_1.
\end{equation}

For \( x > 0 \) define \( H_\varepsilon(x) \) by

\begin{equation}
H_\varepsilon(x) = \frac{1}{2\pi i} \int_{\text{Re}(s)=2} \left[ L\left(s + \frac{1}{2}, \pi\right) - L\left(\frac{1}{2}, \pi\right) \right] \tilde{J}_\varepsilon(s)x^s ds.
\end{equation}

This integral converges absolutely in view of standard bounds for \( L(s, \pi) \) in vertical strips and the rapid decay of \( \tilde{J}_\varepsilon(s) \). Moreover the series (1) converges absolutely for \( \text{Re}(s) = 5/2 \) (Jacquet-Shalika [3]) and so we can integrate the series definition of \( L(s + \frac{1}{2}, \pi) \) in (13) term by term and use the Mellin inversion and (10) to conclude that for \( x > 0 \),

\begin{equation}
H_\varepsilon(x) = \sum_{n=1}^{\infty} \frac{\lambda_\pi(n)}{\sqrt{n}} J_\varepsilon\left(\frac{n}{x}\right) - L\left(\frac{1}{2}, \pi\right) J_\varepsilon\left(\frac{1}{x}\right).
\end{equation}

For \( x \) bounded, the sum is finite. By (9A) the support is bounded away from 0. Thus \( H_\varepsilon(x) \) is smooth and supported away from 0.

From (13) and shifting the contour to \( \text{Re}(s) = -A \), a large negative number, picks up no poles (since the pole of \( \tilde{J}_\varepsilon(s) \) at \( s = 0 \) is cancelled by the difference of the \( L \)-values), we see that \( H_\varepsilon(x) \) decays faster that \( x^{-A} \) for any \( A \) as \( x \to \infty \). Hence \( H_\varepsilon(x) \) is smooth on \( (0, \infty) \) and in

\begin{equation}
L^2(\mathbb{R}^\times_0, \frac{dx}{x}).
\end{equation}
As we just did, we shift the contour integral in (13) to \( \text{Re}(s) = 0 \), which is again justified by standard polynomial bounds in \(|t|\) for \( L(s, \pi) \) and the rapid decay of \( \tilde{J}_\varepsilon(s) \) in \(|t|\). This yields

\[
H_\varepsilon(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ L\left(\frac{1}{2} + it, \pi\right) - L\left(\frac{1}{2}, \pi\right) \right] \tilde{J}_\varepsilon(it)x^it \, dt
\]

Hence the rapidly decreasing smooth \( H_\varepsilon(x) \) on \( \mathbb{R}_{>0} \) satisfies

\[
\mathcal{M}H_\varepsilon(-it) = \frac{L\left(\frac{1}{2} + it, \pi\right) - L\left(\frac{1}{2}, \pi\right)}{it} \hat{\phi}(\varepsilon t) \quad (t \in \mathbb{R}).
\]

We will write \( \mathcal{F}H(t) = \mathcal{M}H(-it) \) for \( H \) a function on \( \mathbb{R}_+ \).

Now let \( \varepsilon \to 0 \). If \( \mathcal{F}H(t) := \frac{L\left(\frac{1}{2} + it, \pi\right) - L\left(\frac{1}{2}, \pi\right)}{it} \) is not in \( L^2(\mathbb{R}) \) there is nothing to prove as (5) is then valid trivially.

So we can assume that

\[
\mathcal{F}H(t) \in L^2(\mathbb{R}).
\]

Now from (17) and (9) we have that

\[
\mathcal{F}H_\varepsilon(t) \to \mathcal{F}H(t) \text{ uniformly on compacta}
\]

and

\[
H_\varepsilon(x) \to H(x), \text{ uniformly on compacta in } (0, \infty) \setminus \mathbb{N},
\]

where

\[
H(x) := \begin{cases} 
\sum_{n \in \mathbb{Z}} \frac{\lambda\pi(n)}{\sqrt{n}} - L\left(\frac{1}{2}, \pi\right), & x > 1 \\
0 & \text{for } x < 1.
\end{cases}
\]

Moreover from (17) and that \( \mathcal{F}H \in L^2(\mathbb{R}) \), it follows from the dominated convergence theorem (or more simply by estimating the tails of the \( t \) integral uniformly) that \( \mathcal{F}H_\varepsilon \to \mathcal{F}H \) in \( L^2(\mathbb{R}) \). Hence by Parseval \( H_\varepsilon \) converges in \( L^2(\mathbb{R}_{>0}, \frac{dx}{x}) \) and from (20) the limit of \( H_\varepsilon \) must be \( H \). Thus \( H \in L^2(\mathbb{R}_{>0}, \frac{dx}{x}) \) and \( \mathcal{F}H(t) \) and \( H(x) \) are Fourier pairs and hence by Parseval, we have that

\[
V\left(\frac{1}{2}, \pi\right) = ||\mathcal{F}H||_2^2 = 2\pi \int_{0}^{\infty} |H(x)|^2 \frac{dx}{x}.
\]
This proves (3). Now following the argument in the paper we note that on \([1, 2] \subset \mathbb{R}_{>0}, H(x) = 1 - L(\frac{1}{2}, \pi)\) so that the R.H.S. of (21) is at least

\[
2\pi|1 - L(\frac{1}{2}, \pi)|^2 \int_1^2 \frac{dx}{x} = 2\pi \log 2|1 - L(\frac{1}{2}, \pi)|^2.
\]

Moreover for \(x > 2\) it is easy to see that \(H(x) \equiv 0\) is impossible, and hence we arrive at (5).

We note that the argument obviously extends to the case where \(\pi = \pi_1 \times \pi_2 \times \ldots \times \pi_r\) as in the Introduction to the main text, and that we have not used the Ramanujan hypothesis. Moreover, by an obvious change of variable (of the form \(s \to s + i\tau\)), this implies Theorem A for any \(s\) on the critical line.

Now consider Theorem B. We cannot use again the translation argument just introduced: this shifts the pole at \((s = 1)\) to a pole at \(s = 1 + i\tau\). However, the computation is the same and we extend the previous proof, assuming the pole is at \((s = 1)\).

We define \(H_\epsilon\) as before by (13); the equality (14) remains the same. However, when we shift the integral to \(\text{Re}(s) = A \ll 0\), we pick up a pole at \(s = 1/2\), with residue \(\kappa \tilde{J}_\epsilon(1/2)x^{1/2}\). Thus the properties of smoothness and decrease are now true for

\[
H_\epsilon^1(x) = H_\epsilon(x) - \kappa \tilde{J}_\epsilon(1/2)x^{1/2}.
\]

Furthermore, this pole also occurs when we shift the integral to \(\text{Re}(s) = 0\), so the equality (16) is now true for \(H_\epsilon^1\). The same arguments now show that

\[
\mathcal{F} H^1 = \frac{L(1/2 + it, \pi) - L(1/2, \pi)}{it},
\]

where

\[
H^1(x) = H(x) - \frac{\kappa}{1/2} x^{1/2}
\]

since \(\tilde{J}_\epsilon(1/2) = \frac{\varphi^{(1/2\epsilon)}}{1/2} \to 2\) when \(\epsilon \to 0\). Finally, \(\frac{1}{2\pi} V(1/2, \pi)\) is now bounded below by

\[
\int_0^1 |2\kappa x^{1/2}|^2 \frac{dx}{x} + \int_1^2 |1 - L(1/2, \pi) - 2\kappa x^{1/2}|^2 \frac{dx}{x},
\]

leading to Theorem B if \(L(1/2, \pi) = 0\).
Proof of (6) and its variant

We now consider the new function

$$H_\varepsilon(x) = \frac{1}{2i\pi} \int_{\text{Res}=2} [L(s, \pi) - L(0, \pi)] \tilde{J}_\varepsilon(s)x^s ds.$$  \hfill(24)

For $\text{Re}(s) \geq 2$ we obtain, integrating term by term :

$$H_\varepsilon(x) = \sum_{n=1}^{\infty} \lambda_\pi(n) \frac{J_\varepsilon(n)}{x} - L(0, \pi) \frac{1}{x}.$$  \hfill(25)

The function $(L(s, \pi) - L(0, \pi) \tilde{J}_\varepsilon(s))$ is holomorphic. Shifting as before the integral to $\text{Re}(s) = A$, $A \ll 0$, we find again that $H_\varepsilon$ is smooth and of rapid decrease. We have

$$K_\varepsilon(x) := x^{-1/2}H_\varepsilon(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{L(1/2 + it) - L(0)}{1/2 + it} \cdot \varphi(\varepsilon t - \frac{i\varepsilon}{2})x^it dt.$$  \hfill(26)

However, $\frac{L(0)}{1/2+it} \in L^2(\mathbb{R})$. Again, there is nothing to prove if $\frac{L(1/2+it)}{1/2+it} \notin L^2$, and otherwise the first factor in the integrand of (26) is $L^2$. Now the previous arguments show that $K_\varepsilon(x) \to K(x)$, uniformly on compacta in $(0, \infty) - \mathbb{N}$, with

$$K(x) = \begin{cases} x^{-1/2} \left( \sum_{n \leq x} \lambda_\pi(n) - L(0, \pi) \right), & x \geq 1 \\ 0 & \text{for } x < 1. \end{cases}$$  \hfill(27)

From (26) it follows that

$$\mathcal{M}K_\varepsilon(-it) \to \frac{L(1/2 + it) - L(0)}{1/2 + it}$$

in $L^2(\mathbb{R})$. Moreover,

$$\mathcal{M}K(-it) = \frac{L(1/2 + it) - L(0)}{1/2 + it}.$$  

If $\chi$ is the characteristic function of $(x \geq 1)$, $\mathcal{M}(x^{-1/2}\chi)(-it) = \frac{1}{1/2+it}$. Thus $x^{-1/2} \sum_{n \leq x} \lambda_\pi(n) (x \geq 1)$ and $\frac{L(1/2+it)}{1/2+it}$ are associated, and this implies

$$\int_{-\infty}^{+\infty} \left| \frac{L(1/2 + it)}{1/2 + it} \right|^2 dt > 2\pi \int_{1}^{x} x^{-2} dx = \pi.$$  

Finally, Theorem D can be proven by combining this computation with the argument given for Theorem B, introducing the residue at $(s = 1)$.  

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