Self-Similar Modes of Coherent Diffusion

O. Firstenberg,\textsuperscript{1} P. London,\textsuperscript{1} D. Yankelev,\textsuperscript{1} R. Pugatch,\textsuperscript{2} M. Shuker,\textsuperscript{1} and N. Davidson\textsuperscript{2}

\textsuperscript{1}Department of Physics, Technion-Israel Institute of Technology, Haifa 32000, Israel
\textsuperscript{2}Department of Physics of Complex Systems, Weizmann Institute of Science, Rehovot 76100, Israel

Abstract

Self-similar solutions of the coherent diffusion equation are derived and measured. The set of real similarity solutions is generalized by the introduction of a nonuniform phase surface, based on the elegant Gaussian modes of optical diffraction. In an experiment of light storage in a gas of diffusing atoms, a complex initial condition is imprinted, and its diffusion dynamics is monitored. The self-similarity of both the amplitude and the phase pattern is demonstrated, and an algebraic decay associated with the mode order is measured. Notably, as opposed to a regular diffusion spreading, a self-similar contraction of a special subset of the solutions is predicted and observed.
Self-similar solutions are generally associated with the long-time behavior of dynamic processes \[1\] and found in nearly all disciplines, from astrophysics and fluid dynamics to condensed matter and optics \[2\]–\[7\]. In most dissipative systems, similarity solutions decay with a characteristic rate, indicating the asymptotic evolution of a given initial condition \[6, 8\]. Similarity solutions emerge also in non-dissipative systems and often prevail, with a familiar example being the family of Gaussian beams in free-space paraxial optics \[9\], when the propagation distance is given the role of time. Gaussian beams are broadly referred to as modes of diffraction, even though they are not nondiffracting as Bessel beams \[10\] and not eigenmodes of an underlying Hamiltonian. Moreover, they are only partially self-similar throughout the propagation – their shape is preserved up to scaling and normalization, while their phase pattern curves or flattens.

The imaginary-time counterpart of paraxial diffraction is the coherent diffusion of a complex-valued field,

$$\frac{\partial \psi}{\partial t} = D \nabla^2 \psi - \gamma \psi,$$

with \(\psi = \psi(x, y; t)\) being a two-dimensional coherence field, \(D\) a real coefficient, and \(\gamma\) a linear decay rate. Coherent diffusion in the form of Eq. (1) arise, for example, in the thermal motion of hot coherent atoms, where the field of internal-state quantum coherence \(\psi\) is subjected to atomic diffusion \((D)\) and decoherence \((\gamma)\) \[11\]. Using the technique of light storage and retrieval \[12\], any arbitrary initial condition can be imprinted on the diffusing atoms, and the subsequent dynamics can be observed \[13\]. This system attracted considerable recent study, exhibiting spectral fringes, narrowing, and coherent recurrence \[14–16\], magnetization diffusion \[17\], diffusion of vortices \[18\], transverse momentum diffusion \[19, 20\], and slow-light spreading \[21\].

In this letter, we present and explore the exact self-similar complex solutions of Eq. (1). We experimentally create and follow the dynamics of several self-similar modes and demonstrate the preservation of both their shape and phase pattern, as well as their characteristic decay. A self-similar contraction, resembling focus or collapse \[2\], is also demonstrated.

Consider first the time-independent paraxial diffraction,

$$\frac{\partial E}{\partial z} = -i \nabla^2 E/(2k),$$

for the slowly-varying envelope, \(E\), of a light field with a wave number \(k\). Two different sets of polynomial-Gaussian solutions are known for this equation, namely the standard and the elegant beams \[9\]. The more familiar ’standard’ modes, e.g. the Hermite-Gaussian (HG) or Laguerre-Gaussian (LG), form complete sets of modes that are self-similar under diffraction.
FIG. 1: Top: the effect of diffusion on a Gaussian beam is an effective stretching of the waist radius, even when the diffusion does not occur at the waist plane. Therefore, far enough from the waist, the diffusion results in the contraction of the transverse shape. Bottom: colormap of the phase gradients of the field envelope (black lines are equal phase contours; white lines are the beam outline). From the viewpoint of the microscopic atomic motion, the contraction far from the waist occurs due to destructive interference of atoms diffusing through the rapidly oscillating phase pattern (red colored).

Their transverse intensity distribution, \( I(x, y; z) \), is maintained along the propagation direction \( z \), normalized, and scaled by the beam radius \( w(z) \). In contrast, the transverse shape of the ‘elegant’ solutions is generally not maintained, and originally they were investigated due to their elegant mathematical form \[22, 23\].

The elegant HG solution, \( E_{n_1, n_2}^{\text{HG}}(r; w_0) \), with \( w_0 \) being the radius at the waist plane \( z = 0 \), is written in terms of the Hermite polynomials of orders \( n_1 \) and \( n_2 \) as,

\[
E_{n_1, n_2}^{\text{HG}} = \frac{E_0}{kw_0} \left[ \frac{kw_0^2}{2q(z; w_0)} \right]^{\frac{n_1+n_2}{2}} H_{n_1}(\tilde{x}) H_{n_2}(\tilde{y}) e^{-\tilde{x}^2-\tilde{y}^2}. \tag{2}
\]

Here, \( q(z; w_0) = iz_R + z \) is the complex radius, \( z_R = kw_0^2 / 2 \) is the Rayleigh length, \( N = n_1 + n_2 \) is the total mode order, and \( E_0 \) is a normalization constant. The transverse scaling, appearing in both the polynomial and the Gaussian terms, depends on \( z \) and \( w_0 \), with \( \tilde{x} = x[ik/2q(z; w_0)]^{1/2} \) and \( \tilde{y} = y[ik/2q(z; w_0)]^{1/2} \). The beam radius, \( w(z) \), and the radius of curvature of the phase fronts, \( R(z) \), are obtained from \( q(z; w_0)^{-1} = R(z)^{-1} - (i2/k)w(z)^{-2} \).

For the corresponding standard mode, the complex arguments of the polynomial \((\tilde{x}, \tilde{y})\) are replaced by real arguments \([\sqrt{2x}/w(z), \sqrt{2y}/w(z)]\). Thus, elegant modes with a homogeneous polynomial are also standard, and we denote them as ‘common’ modes.
Now suppose that a two-dimensional diffusion takes place at a certain \((x, y)\) plane, where \(z\) is held constant. Eq. (2), with \(\partial E/\partial z = -i\nabla^2 E/(2k)\) and \((\partial/\partial w_0)_z = iw_0(\partial/\partial q)w_0 + (\partial/\partial w_0)_q\), then gives

\[
\nabla^2 E^\text{HG}_{n_1, n_2} = \frac{2}{w_0} \left( \frac{\partial E^\text{HG}_{n_1, n_2}}{\partial w_0} \right)_z - \frac{2}{w_0^2} (N + 1) E^\text{HG}_{n_1, n_2}.
\]

(3)

Under diffusion, the first term in the right-hand side of Eq. (3) accounts for a change in the waist radius, and the second term for an homogeneous decay of the field. Eq. (1) is therefore solved by

\[
\psi^\text{HG}_{n_1, n_2}(x, y, t) = e^{-\gamma t} s(t)^{-(N+1)} E^\text{HG}_{n_1, n_2}(r; w_0 s(t)),
\]

(4)

where the diffusion coefficient enters only through the waist stretching factor, \(s(t) = (1 + 4Dt/w_0^2)^{1/2}\). Thus the spatial consequence of diffusion is always an effective stretching of the beam radius at the waist plane, even when the diffusion occurs far from the waist \((z \neq 0)\), as illustrated in Fig. 1 (top). Note that if diffusion is addressed as an imaginary-time evolution of diffraction, it is readily seen from the definition of the complex radius \(q(z; w_0)\) that exchanging the real evolution in \(z\) for an imaginary one corresponds to a real increase in the waist radius. The total power, \(P(t) = \int dx dy |\psi^\text{HG}_{n_1, n_2}(x, y, t)|^2\), which is independent of \(z\), is not preserved under diffusion even when \(\gamma = 0\), due to an algebraic decay term \(s(t)^{-(N+1)}\). This occurs even for the lowest Gaussian mode \((N = 0)\) because it is the field, rather than the intensity, that is diffusing. Higher-order modes \((N > 0)\) decay faster due to the diffusion of the non-homogenous phase pattern, which contains larger gradients for higher \(N\). A similar procedure can be carried out for an elegant LG solution, \(E^\text{LG}_{p,m}\), of radial order \(p\) and orbital order \(m\), yielding Eq. (4) for the diffusing field \(\psi^\text{LG}_{p,m}(x, y, t)\), with \(N = p + m\).

At the waist, all arguments in Eq. (2) are real, and the HG solutions \(\psi^\text{HG}_{n_1, n_2 = 0}\) are identified with the expanding similarity solutions of regular diffusion \([8]\), occurring, e.g., for the vorticity field of a viscous fluid \([6, 24]\). The real solutions are alternatively derived from a given self-similar solution with a single real scaling \((\psi(r, t) = h(t)f[r/w(t)])\), by taking any of its spatial derivatives \((\partial^n \psi/\partial x^n = h(t)f^{(n)}[r/w(t)]/w(t)^n)\). Indeed, the derivatives of the lowest-order Gaussian beam constitute the elegant modes, with the derivative order corresponding to the total mode order \(N\), and with the possible generalization for unified Hermite-Laguerre-Gaussian modes \([25, 26]\). Consequently, \(\psi^\text{HG}_{n_1, n_2 = 0}\) or \(\psi^\text{LG}_{p,m = 0}\), and all linear combinations of them with the same total order, are self-similar, sharing the same stretching.
FIG. 2: (a) Diffusion of the complex field of atomic coherence in a storage-of-light experiment, demonstrating the self-similarity property of (top to bottom): the basic Gaussian mode, the Laguerre-Gaussian modes with radial order $p = 0$ and orbital orders $m = 1, 2,$ and the Hermite-Gaussian mode with cartesian orders $(0, 1)$. All images are $1.6 \times 1.6$ mm. (b) Images after storage, diffracted by a binary grating mask with a fork dislocation, for confirming the conservation of phase. The retrieved Gaussian mode (top) yields the two $m = \pm 1$ vortex modes, while the retrieved vortex $m = +1$ (bottom) produces an $m = 0$ and an $m = +2$ modes.

and the same algebraic decay. Alternatively, any complex "image" in the $(x, y)$ plane can be expanded in terms of $\psi_{n_1, n_2}^{HG, z=0}$ or $\psi_{p, m}^{LG, z=0}$ using their biorthogonal pairs [27], its diffusion can be described in terms of the modes dynamics, and asymptotically the lowest-order solution prevails [30].

To validate the above predictions, we perform experiments with thermal alkali atoms
confined in a vapor cell with a buffer gas. A resonant laser beam with the desired optical mode is sent into the cell, and its complex field envelope is mapped onto the atomic coherence field utilizing a storage-of-light technique, by shutting down an auxiliary control beam [12, 13]. The coherence field, $\psi$, is allowed to evolve for a controlled duration $\tau$, in which the alkali diffusion through the buffer gas takes place, as well as a homogenous decoherence (e.g. due to spin-exchange relaxation [28]). The coherence is then converted back to light, which is imaged onto a camera. The experimental set-up and procedure is similar to that described in [18], where the topological stability of the vortex in a stored $E_{0,1}^{LG}$ mode was attested.

The experiment was carried out with the fundamental Gaussian mode $E_{0,0}^{(HG/LG)}$, the LG modes $E_{0,1}^{LG}$ and $E_{0,2}^{LG}$, and the HG mode $E_{0,1}^{HG}$. Fig. 2(a) presents the retrieved images, proportional to $|\psi(x, y, \tau)|^2$, for a storage performed with the cell located at the beam waist ($z = 0$), for durations of $\tau = 2$, 30, and 60 $\mu$s. Evidently, all the modes expand but maintain their shape through the diffusion process. As a complementary test, we have also passed the retrieved beams through a binary grating mask with a fork dislocation, which adds a phase function $m\phi$ in its $m$-th diffraction order [29]. After the mask, as shown in Fig. 2(b), the retrieved vortex mode $E_{0,1}^{LG}$ ($m = 1$) produces a Gaussian ($m = 0$) and a higher-order vortex ($m = 2$) in the $-1$ and $+1$ diffraction orders, confirming the maintenance of the phase pattern. Fig. 3(a) presents the increase in the waist-radii squared versus the storage duration, showing the same linear increase for all curves, $w(\tau)^2 - w^2_0 = w^2_0[s(\tau)^2 - 1] = 4D\tau$. The cross-sections shown in Fig. 3(b), scaled according to $s(\tau)$ and normalized, clearly demonstrate the self-similarity. The algebraic decay of the diffusing modes, $s(t)^{-(N+1)}$, is measured by integrating over the intensity of the retrieved images (Fig. 4). All modes exhibit a significant algebraic decay on top of the homogenous decay, with the higher-order modes decaying faster, showing an excellent quantitative agreement with the predictions of Eq. (4).

We now discuss the diffusion of the elegant modes at a given plane outside the waist plane, $z \neq 0$. For Gaussian modes, an expansion of the waist radius results in an increase of the beam’s transverse size for $|z| < z_R$ and in a decrease for $|z| > z_R$ (Fig. 1 top). It follows, perhaps counterintuitively, that the initial effect of diffusion occurring at $|z| > z_R$ is a contraction. Locally, it is the consequence of a destructive interference in regions of the beam where the phase pattern rapidly oscillates (Fig. 1 bottom). The contraction versus storage time, $C(\tau) = w(\tau)/w(0)$, is given by $C(\tau)^2 = [s(t)^4 + \rho^2]/[s(t)^2(1 + \rho^2)]$, where
FIG. 3: (a) Linear growth of $w^2$ with respect to the time duration of diffusion. The line is $w^2 - w_0^2 = 4D\tau$, with $D = 10.8 \text{ cm}^2/\text{s}$. Both $D$ and $w_0$ were fit parameters ($w_0 = 0.4 \sim 0.55 \text{ mm}$, varying for the different modes). (b) Self-similarity: cross-sections (cartesian or radially-weighted) at different times are congruent when plotted versus the scaled coordinate [$\tilde{y} = s(\tau)^{-1/2}y$ for the cartesian HG mode, $\tilde{r} = s(\tau)^{-1/2}r$ for the cylindrical LG modes]. The solid lines are the analytic forms of the elegant HG and LG modes.

$\rho = z/z_R$ specifies the initial distance from the waist. As the waist radius increases during the diffusion, $z_R(\tau)$ increases and eventually crosses the observed plane (which $z$ coordinate is constant). At this time, the maximal contraction $C_{\text{min}}^2 = 2\rho/(1 + \rho^2)$ is obtained, and thenceforth the beam expands indefinitely.

Elegant modes, as opposed to the standard modes, are generally not self-similar under diffraction, and the shape of the beam depends on $z/z_R$. Hence, even when $z$ is held constant, the increase of $z_R$ during diffusion changes the transverse shape and breaks its self-similarity. However, the aforementioned 'common' modes, which are simultaneously elegant and standard, are self-similar under diffraction and thus also self-similar under diffusion even for $z \neq 0$. The HG modes of orders 0 and 1, and all LG modes with $p = 0$ (the vortex modes) are such common modes. Far from the waist, at $z > z_R$, these modes contract self-similarly. Fig. 5 presents the experimental result for a diffusing $E_{0,1}^{\text{LG}}$ mode focused at a
FIG. 4: Decay of the total power in the retrieved images $P(\tau) = \int dx dy |\psi(\tau)|^2$. In the inset, $P^{*}$ compensates for the algebraic decay $|s(\tau)|$ known from Fig. 3 and collapses onto a single straight line in the semilog scale, yielding the homogenous decay rate $2\gamma = (68 \mu s)^{-1}$. In the main graph, the dashed line is $e^{-2\gamma \tau}$ and the solid lines are $e^{-2\gamma \tau} s(\tau)^{-2(N+1)}$, demonstrating the faster decay of the higher-order modes. The difference between the LG$_{0,1}$ and the HG$_{0,1}$, both having $N = 1$, is due to slightly different initial waist radii ($w_0$).

distance $8z_R$ before the cell ($\rho = 8$), yielding $C^2_{\text{min}} \approx 1/4$.

Finally, we point out an intriguing instability phenomenon noticeable in Fig. 2(a) for the $E_{0,2}$ mode. During diffusion, the $m = 2$ vortex breaks-down into two vortices, probably of $m = 1$. A decay of high-order vortices into lower-order ones has been seen also in optics, quantum fluids, and Bose-Einstein condensates. Here, several candidate mechanisms may be responsible for the imperfection [24, 29], which evidently conserves the cross-section of the original vortex [Fig. 3(b)].

In conclusion, we have shown that when elegant Gaussian modes are put through coherent diffusion, their waist radius effectively expands. The total power in the field decays algebraically during the diffusion, even for the lowest-order mode, due to field interference effects. The complete set of elegant modes is self-similar at the waist, while far from the waist, self-similarity is found only for the subset of common modes. For the latter, a self-similar contraction may occur. The cross-sections at the waist of the standard and the elegant Gaussian modes form a complete set of self-similar modes for diffraction and diffusion, respectively. A similar set for the simultaneous process of diffraction and diffusion, as
FIG. 5: The self-similar contraction and subsequent expansion of the LG$_{0,1}$ mode upon diffusion. Contrary to the former experiments, here we refer to the original beam as the initial condition (left, taken off-resonantly), since much contraction occurs already during the slow-light propagation in cell. The original mode has a radius of 700 µm and a phase curvature of (250 mm)$^{-1}$. The minimal radius, of 340 µm, is obtained after slowing for 5 µs plus storage for 7 µs. The black line is calculated from $C(\tau)^2$ with no fit parameters. In these conditions, the power decays substantially faster than in the expansion experiments, and substantial noise is already apparent after 15 µs of storage.

occurring in the dynamics of slow light [21], is yet to be explored.

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[30] Note that integrals of the fundamental Gaussian are also self-similar and may be employed in the expansion of solutions with infinite energy.