Reverse Brascamp–Lieb inequality and the dual Bollobás–Thomason inequality

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Abstract

We prove that if \( f : \mathbb{R}^n \to [0, \infty) \) is an integrable log-concave function with \( f(0) = 1 \) and \( F_1, \ldots, F_r \) are subspaces of \( \mathbb{R}^n \) such that \( sI_n = \sum_{i=1}^r c_i P_i \) where \( I_n \) is the identity operator and \( P_i \) is the orthogonal projection onto \( F_i \) then

\[
n^n \int_{\mathbb{R}^n} f(y)^n dy \geq \prod_{i=1}^r \left( \int_{F_i} f(x_i) dx_i \right)^{c_i/s}.
\]

As an application we obtain the dual version of the Bollobás–Thomason inequality: if \( K \) is a convex body in \( \mathbb{R}^n \) with \( 0 \in \text{int}(K) \) and \( (\sigma_1, \ldots, \sigma_r) \) is an \( s \)-uniform cover of \( [n] \) then

\[
|K|^s \geq \frac{1}{(n!)^s} \prod_{i=1}^r |\sigma_i|! \prod_{i=1}^r |K \cap F_i|.
\]

This is a sharp generalization of Meyer’s dual Loomis–Whitney inequality.

1 Introduction

The purpose of this article is to establish the dual version of the uniform cover inequality of Bollobás and Thomason. We fix an orthonormal basis \( \{e_1, \ldots, e_n\} \) of \( \mathbb{R}^n \) and recall that the not necessarily distinct non-empty sets \( \sigma_1, \ldots, \sigma_r \subseteq [n] := \{1, \ldots, n\} \) form an \( s \)-uniform cover of \( [n] \) for some \( s \geq 1 \) if every \( j \in [n] \) belongs to exactly \( s \) of the sets \( \sigma_i \). The main result of [7] estimates the volume of a compact set in terms of the volumes of its coordinate projections that correspond to a uniform cover of \([n]\).

**Theorem 1.1 (Bollobás-Thomason).** Let \( r \geq 1 \) and \( (\sigma_1, \ldots, \sigma_r) \) be an \( s \)-uniform cover of \( [n] \). For every compact subset \( K \) of \( \mathbb{R}^n \), which is the closure of its interior, we have

\[
|K|^s \leq \prod_{i=1}^r |P_{F_{\sigma_i}}(K)|,
\]

where \( F_\tau = \text{span}\{e_j : j \in \tau\} \) and \( P_F \) denotes the orthogonal projection of \( \mathbb{R}^n \) onto \( F \).

Throughout this article, for any non-empty compact set in \( \mathbb{R}^n \) we write \( |A| \) for the volume of \( A \) in the affine subspace \( \text{aff}(A) \). A special case of Theorem 1.1 is the Loomis–Whitney inequality [11]: one has

\[
|K|^{n-1} \leq \prod_{i=1}^n |P_i(K)|
\]

where \( P_i := P_{e_i^*} \), and equality holds if and only if \( K \) is a coordinate box, i.e. a rectangular parallelepiped whose sides are parallel to the coordinate axes. This follows from the observation that the sets \( \sigma_i = [n] \setminus \{i\} \) form an \((n-1)\)-uniform cover of \([n]\).

1
Meyer proved in\cite{12} an inequality which is dual to the Loomis–Whitney inequality. If $K$ is a convex body in $\mathbb{R}^n$ then

\begin{equation}
|K|^{n-1} \geq \frac{n!}{n^n} \prod_{i=1}^{n} |K \cap e_i^\perp|,
\end{equation}

where $K \cap F$ denotes the section of $K$ with a subspace $F$. Equality holds in (1.3) if and only if $K = \text{conv}(\{\pm \lambda_1 e_1, \ldots, \pm \lambda_n e_n\})$ for some $\lambda_i > 0$. We prove the following dual Bollobás–Thomason inequality.

**Theorem 1.2.** Let $K$ be a convex body in $\mathbb{R}^n$ with $0 \in \text{int}(K)$ and $(\sigma_1, \ldots, \sigma_r)$ be an $s$-uniform cover of $[n]$. Then,

\begin{equation}
|K|^s \geq \frac{1}{(n!)^s} \prod_{i=1}^{r} |\sigma_i|! \prod_{i=1}^{r} |K \cap F_{\sigma_i}|.
\end{equation}

It is not hard to check that (1.4) is sharp; it becomes equality for any $s$-uniform cover of $[n]$ if $K$ is the cross-polytope $B^n_1 = \text{conv}(\{\pm e_1, \ldots, \pm e_n\})$.

An essentially equivalent way to state Theorem 1.1 (see \cite{7}) is the fact that for every compact subset $K$ of $\mathbb{R}^n$, which is the closure of its interior, we can find a coordinate box such that $|B| = |K|$ and

\begin{equation}
|P_{F_x}(B)| \leq |P_{F_x}(K)|
\end{equation}

for every $\sigma \subseteq [n]$. Theorem 1.2 has a similar equivalent formulation.

**Theorem 1.3.** Let $K$ be a convex body in $\mathbb{R}^n$ with $0 \in \text{int}(K)$. There exists an affine cross-polytope $C = \text{conv}(\{\pm \lambda_1 e_1, \ldots, \pm \lambda_n e_n\})$, where $\lambda_i > 0$, such that $|C| = |K|$ and $|C \cap F_{\sigma}| \geq |K \cap F_{\sigma}|$ for every $\sigma \subseteq [n]$.

Theorem 1.2 and its equivalent version Theorem 1.3 is a consequence of a functional inequality which is proved in Section 3. We denote by $\mathcal{F}(\mathbb{R}^n)$ the class of log-concave integrable functions $f : \mathbb{R}^n \rightarrow [0, \infty)$.

**Theorem 1.4.** Let $f \in \mathcal{F}(\mathbb{R}^n)$ with $f(0) = 1$ and $(\sigma_1, \ldots, \sigma_r)$ be an $s$-uniform cover of $[n]$. Then,

\[ n^s \int_{\mathbb{R}^n} f(y)^n dy \geq \prod_{i=1}^{r} \left( \int_{F_i} f(x_i) dx_i \right)^{1/s}. \]

Moreover, we obtain more general inequalities which imply several of the known extensions of the Loomis–Whitney and Meyer inequalities; see Section 2 and Section 3 for the statements and details. Our main tool is Barthe’s multidimensional generalization of Ball’s geometric Brascamp–Lieb inequality (see \cite{4}) and its reverse form; see \cite{3} Theorem 6). The connection with the problems that we discuss in Section 2 was communicated by F. Barthe to A. Giannopoulos after a talk in MSRI and the author is grateful to them for the information which has been the starting point for this work.

Let us also mention that the Bollobás–Thomason inequality plays a key role in the recent work\cite{8} of S. Brazitikos, A. Giannopoulos and the author that provides local versions of the Loomis–Whitney inequality for coordinate projections of convex bodies; see also \cite{1} for further results in this direction. It is conceivable that one might exploit the dual inequality of Theorem 1.2 to obtain analogous local inequalities for sections. Isomorphic inequalities of this type appear in\cite{8} where they are proved by different methods.

In Section 2 we describe the way one can derive both the Loomis–Whitney and the Bollobás–Thomason inequality, as well as other extensions of them, as consequences of the multidimensional geometric Brascamp–Lieb inequality. The main new results of this work are presented in Section 3; the main tool is Barthe’s inequality. We refer to the books\cite{15} and\cite{2} for standard notation and facts from convex geometric analysis.
2 Brascamp-Lieb inequality and uniform cover inequalities

In what follows we say that the subspaces $F_1, \ldots, F_r$ form an $s$-uniform cover of $\mathbb{R}^n$ with weights $c_1, \ldots, c_r > 0$ for some $s > 0$ if

\begin{equation}
    sI_n = \sum_{i=1}^{r} c_i P_i,
\end{equation}

where $I_n$ is the identity operator and $P_i$ is the orthogonal projection of $\mathbb{R}^n$ onto $F_i$. We prove the next general result.

**Theorem 2.1.** Let $F_1, \ldots, F_r$ be subspaces that form an $s$-uniform cover of $\mathbb{R}^n$ with weights $c_1, \ldots, c_r > 0$. For every compact subset $K$ of $\mathbb{R}^n$ we have

\begin{equation}
    |K|^s \leq \prod_{i=1}^{r} |P_{F_i}(K)|^{c_i}.
\end{equation}

The proof is an almost direct consequence of Barthe’s multidimensional geometric Brascamp-Lieb inequality. The statement is given below; the reverse inequality \cite{22} will be our main tool in the next section.

**Theorem 2.2** (Barthe). Let $r, n \in \mathbb{N}$. For $i = 1, \ldots, r$, let $F_i$ be a $d_i$-dimensional subspace of $\mathbb{R}^n$ and $P_i$ be the orthogonal projection onto $F_i$. If

\begin{equation}
    I_n = \sum_{i=1}^{r} c_i P_i
\end{equation}

for some $c_1, \ldots, c_r > 0$ then for all non-negative integrable functions $f_i : F_i \to \mathbb{R}$ we have

\begin{equation}
    \int_{\mathbb{R}^n} \prod_{i=1}^{r} f_i^{c_i}(P_i x) \, dx \leq \prod_{i=1}^{r} \left( \int_{F_i} f_i \right)^{c_i}
\end{equation}

and

\begin{equation}
    \int_{\mathbb{R}^n} \sup \left\{ \prod_{i=1}^{r} f_i^{c_i}(x_i) : x = \sum_{i=1}^{r} c_i x_i, x_i \in F_i \right\} \, dx \geq \prod_{i=1}^{r} \left( \int_{F_i} f_i \right)^{c_i}.
\end{equation}

In the statement above, $\int^*$ stands for the outer integral and in the right hand side the integral on $F_i$ is with respect to the Lebesgue measure on $F_i$ which is compatible to the given Euclidean structure.

**Proof of Theorem 2.1.** Let $d_i = \dim(F_i)$ and note that

\[ ns = \text{tr}(sI_n) = \sum_{i=1}^{r} c_i \cdot \text{tr}(P_i) = c_1 d_1 + \cdots + c_r d_r. \]

Given a compact subset $K$ of $\mathbb{R}^n$ we define $f_i : F_i \to [0, \infty)$ by $f_i = 1_{P_i(K)}$. Note that if $x \in K$ then $f_i(P_i x) = 1$ for all $i = 1, \ldots, r$. Therefore,

\[ 1_K(x) \leq \prod_{i=1}^{r} f_i^{c_i}(P_i x) \]

for all $x \in \mathbb{R}^n$. From Theorem 2.2 we get

\[ |K| = \int_{\mathbb{R}^n} 1_K(x) \, dx \leq \int_{\mathbb{R}^n} \prod_{i=1}^{r} f_i^{c_i}(P_i x) \, dx \leq \prod_{i=1}^{r} \left( \int_{F_i} f_i \right)^{c_i} = \prod_{i=1}^{r} |P_i(K)|^{c_i}, \]

which shows that $|K|^s \leq \prod_{i=1}^{r} |P_i(K)|^{c_i}$ as claimed. \qed
Application 2.3 (Bollobás-Thomason). It is not hard to see that the Bollobás-Thomason inequality may be proved in the same way. Note that if $(\sigma_1, \ldots, \sigma_r)$ is an $s$-uniform cover of $[n]$ then the projections $P_i := P_{F_{\sigma_i}}$ satisfy

$$sI_n = \sum_{i=1}^r P_i.$$ 

Therefore, for any compact subset $K$ of $\mathbb{R}^n$ we may apply Theorem 2.1 with $c_1 = \cdots = c_r = 1$ to get

$$|K|^s \leq \prod_{i=1}^r |P_i(K)|,$$

which is exactly the statement of Theorem 1.1.

As a special case of Theorem 2.1 we also obtain the following inequality of Bollobás and Thomason [7]. Let $C$ be a finite collection of subsets of $[n]$, which is not necessarily a uniform cover. Suppose that to each $\sigma \in C$ we associate a positive real weight $w(\sigma)$ in such a way that, for each $i \in [n]$, $\sum_{\sigma \in C} w(\sigma) \cdot i \in \sigma = 1$. Then, it is clear that

$$I_n = \sum_{\sigma \in C} w(\sigma) P_{F_{\sigma}},$$

and Theorem 2.1 shows that

$$|K| \leq \prod_{\sigma \in C} |P_{F_{\sigma}}(K)|^{w(\sigma)}.$$

Application 2.4 (Ball’s inequality). Let $u_1, \ldots, u_m$ be unit vectors in $\mathbb{R}^n$ and $c_1, \ldots, c_m$ be positive real numbers such that John’s condition

$$I_n = \sum_{i=1}^m c_i u_i \otimes u_i$$

is satisfied. Using the one-dimensional geometric Brascamp–Lieb inequality, Ball proved in [3] that for every centered convex body $K$ in $\mathbb{R}^n$,

$$(2.5) \quad |K|^{n-1} \leq \prod_{i=1}^m |P_{u_i^+}(K)|^{c_i}.$$ 

The equality cases are the same with the ones in the Loomis–Whitney inequality. Let us briefly explain how Theorem 2.1 implies (2.5). We observe that if $P_i = P_{u_i^+}$ then $u_i \otimes u_i = I_n - P_i$, and hence John’s condition may be written as $I_n = \sum_{i=1}^m c_i (I_n - P_i)$, which implies that

$$(n-1)I_n = \sum_{i=1}^m c_i P_i,$$

if we take into account the fact that $\sum_{i=1}^m c_i = n$. Then, given a (more generally) compact subset $K$ of $\mathbb{R}^n$ we may apply Theorem 2.1 with $s = n - 1$ to get

$$(2.7) \quad |K|^{n-1} \leq \prod_{i=1}^m |P_{u_i^+}(K)|^{c_i}.$$ 

3 Dual Bollobás-Thomason inequality

We start with a proof of a more general version of Theorem 2.4. Recall that $\mathcal{F}(\mathbb{R}^n)$ denotes the class of log-concave integrable functions $f : \mathbb{R}^n \to [0, \infty)$. 



4
Theorem 3.1. Let \( f \in \mathcal{F}(\mathbb{R}^n) \) with \( f(0) = 1 \) and \( F_1, \ldots, F_r \) be subspaces of \( \mathbb{R}^n \) that form an \( s \)-uniform cover of \( \mathbb{R}^n \) with weights \( c_1, \ldots, c_r > 0 \). Then,
\[
n^n \int_{\mathbb{R}^n} f(y)^n \, dy \geq \prod_{i=1}^r \left( \int_{F_i} f(x_i) \, dx_i \right)^{c_i/s}.
\]

Proof. Our assumption \( I_n = \sum_{i=1}^r c_i P_{F_i} \) implies that
\[
ns = \text{tr}(sI_n) = \sum_{i=1}^r c_i \cdot \text{tr}(P_{F_i}) = \sum_{i=1}^r c_i d_i,
\]
where \( d_i = \dim(F_i) \). Let \( z \in \mathbb{R}^n \) and \( x_i \in F_i, i \in [r] \) such that \( z = \sum_{i=1}^r \frac{c_i}{n} x_i \). Then,
\[
\frac{z}{n} = \sum_{i=1}^r \frac{c_i d_i}{sn} \cdot x_i.
\]
and since \( f \in \mathcal{F}(\mathbb{R}^n) \) and \( \sum_{i=1}^r \frac{c_i d_i}{sn} = 1 \) we have
\[
f\left(\frac{z}{n}\right) \geq \prod_{i=1}^r f(x_i/d_i)^{\frac{c_i}{d_i}}.
\]
Since \( f(0) = 1 \), for every \( i \in [r] \) we see that \( f(x_i/d_i) \geq f(x_i)^{1/d_i} f(0)^{1-1/d_i} = f(x_i)^{1/d_i} \). It follows that
\[
f(z/n) \geq \prod_{i=1}^r f(x_i)^{d_i \frac{c_i}{d_i}} = \prod_{i=1}^r f(x_i)^{\frac{c_i}{d_i}},
\]
and hence
\[
f(z/n)^n \geq \prod_{i=1}^r f(x_i)^{c_i/s}.
\]
This shows that
\[
f(z/n)^n \geq \sup \left\{ \prod_{i=1}^r f(x_i)^{c_i/s} : z = \sum_{i=1}^r \frac{c_i}{s} x_i, x_i \in F_i \right\}.
\]
Then, by the multidimensional reverse Brascamp-Lieb inequality (2.4) we have that
\[
\int_{\mathbb{R}^n} f(z/n)^n \, dz \geq \int_{\mathbb{R}^n} \sup \left\{ \prod_{i=1}^r f(x_i)^{c_i/s} : z = \sum_{i=1}^r \frac{c_i}{s} x_i, x_i \in F_i \right\} \, dz
\]
\[
\geq \prod_{i=1}^r \left( \int_{F_i} f(x_i)^{d_i} \, dx_i \right)^{c_i/s}.
\]
Making the change of variables \( y = z/n \) we conclude the proof. \( \square \)

Our main geometric application of Theorem 3.1 is the next general uniform cover inequality for sections of a convex body.

Theorem 3.2. Let \( K \) be a convex body in \( \mathbb{R}^n \) with \( 0 \in \text{int}(K) \) and \( F_1, \ldots, F_r \) be subspaces of \( \mathbb{R}^n \), with \( \dim(F_i) = d_i \), that form an \( s \)-uniform cover of \( \mathbb{R}^n \) with weights \( c_1, \ldots, c_r > 0 \). Then,
\[
|K|^s \geq \frac{1}{(n!)^s} \prod_{i=1}^r (d_i!)^{c_i} \prod_{i=1}^r |K \cap F_i|^{c_i}.
\]
Proof. We apply Theorem 3.1 for the function \( f(y) = e^{-\|y\|_K} \), where \( \|y\|_K := \min\{t > 0 : y \in tK\} \) is the Minkowski functional of \( K \). Note that \( f \in \mathcal{F}(\mathbb{R}^n) \) and \( f(0) = 1 \). We have

\[
n^n \int_{\mathbb{R}^n} f(y)^n dy = n^n \int_{\mathbb{R}^n} e^{-\|y\|_K} dy = n^n \int_{\mathbb{R}^n} e^{-\sum_{i=1}^n x_iK} dy
\]

and for every \( i \in [r] \) we have

\[
\int_{F_i} f(x_i)dx_i = \int_{F_i} e^{-\|x_i\|_K} dx_i = \int_{F_i} e^{-\|x_i\|_{K \cap F_i}} dx_i = d_i! |K \cap F_i|.
\]

It follows that

\[
n! |K| \geq \prod_{i=1}^r (d_i! |K \cap F_i|)^{c_i/s} = \prod_{i=1}^r (d_i!)^{c_i/s} \prod_{i=1}^r |K \cap F_i|^{c_i/s},
\]

and the theorem follows. \( \square \)

**Application 3.3 (dual Bollobás–Thomason).** Theorem 3.2 has several straightforward applications. First, let \( (\sigma_1, \ldots, \sigma_r) \) be an \( s \)-uniform cover of \([n]\). Setting \( F_i = F_{\sigma_i} = \text{span}\{e_j : j \in \sigma_i\}, i \in [r] \), we have \( sI_n = \sum_{i=1}^r P_{F_i} \). Thus, we obtain the dual Bollobás-Thomason inequality of Theorem 1.2. If \( K \) is a convex body in \( \mathbb{R}^n \) with \( 0 \in \text{int}(K) \) and \( (\sigma_1, \ldots, \sigma_r) \) is an \( s \)-uniform cover of \([n]\) then

\[
|K|^n \geq \frac{1}{(n!)^s} \prod_{i=1}^r |\sigma_i|! \prod_{i=1}^r |K \cap F_i|.
\]

In the particular case \( F_i = e_i^\perp, i \in [n] \) we have \( (n-1)I_n = \sum_{i=1}^n P_{e_i^\perp}, \) and applying Theorem 1.2 with \( s = n-1 \) and \( |\sigma_i| = \dim(F_i) = n-1 \) we recover Meyer’s inequality

\[
|K|^{n-1} \geq \frac{n!}{n^n} \prod_{i=1}^n |K \cap e_i^\perp|
\]

for any convex body \( K \) in \( \mathbb{R}^n \) with \( 0 \in \text{int}(K) \), because

\[
\frac{1}{(n!)^{n-1}} \prod_{i=1}^n |\sigma_i|! = \frac{1}{(n!)^{n-1}} \prod_{i=1}^n (n-1)! = \frac{[(n-1)!]^n}{(n!)^{n-1}} = \frac{(n-1)!}{n^{n-1}} = \frac{n!}{n^n}.
\]

Theorem 1.3 can be obtained from Theorem 1.2 by an argument which is basically the same with the one used by Bollobás and Thomason for the proof of Theorem 1.5. In what follows, we say that a uniform cover of \([n]\) is irreducible if it cannot be written as a disjoint union of two other uniform covers of \([n]\). In [17] it is shown that the number of irreducible uniform covers of \([n]\) is finite.

**Proof of Theorem 1.3.** Let \( K \) be a convex body in \( \mathbb{R}^n \) with \( 0 \in \text{int}(K) \). Theorem 1.2 states that for every integer \( s \geq 1 \) and any non-trivial irreducible \( s \)-uniform cover \( (\sigma_1, \ldots, \sigma_r) \) of \([n]\) we have that \( (n!|K|)^s \geq \prod_{i=1}^r |\sigma_i|! |K \cap F_{\sigma_i}| \). Moreover, applying Theorem 1.2 for the 1-uniform cover \( (\{i\}, i \in \tau) \) of \( \tau \subseteq [n] \) we see that \( |\tau|! |K \cap F_{\tau}| \geq \prod_{i \in \tau} |K \cap F_{\{i\}}| \). Since there are finitely many irreducible uniform covers of \([n]\), we have a finite number of inequalities as above, satisfying the elements of the set \( \{|\sigma|! |K \cap F_{\sigma}| : \sigma \subseteq [n]\} \).

Let \( \{t_\sigma : \sigma \subseteq [n]\} \) be a set of positive reals with \( t_\sigma \geq |\sigma|! |K \cap F_{\sigma}| \) and \( t_{[n]} = n!|K| \), which are maximal with respect to satisfying all the above inequalities if we replace \( |\sigma|! |K \cap F_{\sigma}| \) by \( t_{\sigma} \) for all \( \sigma \subseteq [n] \). Then, we know that \( \prod_{i=1}^n t_{\sigma_i} \leq (n!|K|)^s \) for every (not necessarily irreducible) \( s \)-uniform cover \( (\sigma_1, \ldots, \sigma_r) \) of \([n]\).

Since \( t_{\{i\}}, i \in [n], \) are maximal, we see that for every \( i \in [n] \) we can find an inequality involving \( t_{\{i\}} \) which is equality. If this inequality is of the first kind then there exists an \( s_i \)-uniform cover \( \mathcal{F}(i) = (\sigma_1, \ldots, \sigma_r) \) of
Since \(|(3.5)\) for all \(i \in [n]\) the maximality of \(K\) in the maximality of \(K\), then \(K\) is again an \(s_1\)-uniform cover of \([n]\). The latter inequality (in fact in a more general form) is a direct consequence of Theorem 3.2. Given a convex polar \(L\), the Ball-Barthe inequality, due to Lutwak, Yang and Zhang \([10]\), and a number of facts about the class of \(\sigma\)-centroid bodies. In the particular case where \(t_1, \ldots, t_n\) are unit vectors in \(\mathbb{R}^n\) and every even isotropic measure \(\nu\) on \(S^{n-1}\) one has

\[
|K|^{n-1} \geq \frac{n!}{n^n} \exp \left( \int_{S^{n-1}} \log |K \cap u^\perp| \, d\nu(u) \right)
\]

and they determined the equality cases. Their very interesting argument employs the continuous version of the Ball-Barthe inequality, due to Lutwak, Yang and Zhang \([10]\), and a number of facts about the class of polar \(L_\nu\)-centroid bodies. In the particular case where \(u_1, \ldots, u_m\) are unit vectors in \(\mathbb{R}^n\) and \(e_1, \ldots, e_m\) are positive real numbers that satisfy John’s condition, one gets

\[
|K|^{n-1} \geq \frac{n!}{n^n} \prod_{i=1}^m |K \cap u_i^\perp|^{e_i}.
\]

The latter inequality (in fact in a more general form) is a direct consequence of Theorem 3.2. Given a convex body \(K\) in \(\mathbb{R}^n\) with \(0 \in \text{int}(K)\), we consider the subspaces \(F_i = u_i^\perp\), and since \(\dim(F_i) = n-1\) and the \(F_i\)’s
form an \((n-1)\)-uniform cover of \(\mathbb{R}^n\) with weights \(c_1, \ldots, c_m > 0\), using also the fact that \(\sum_{i=1}^m c_i = n\) we immediately get

\[
|K|^{n-1} \geq \frac{1}{(n!)^s} \prod_{i=1}^m ((n-1)!)^{c_i} \prod_{i=1}^m |K \cap u_i^+|^{|c_i|} = \frac{[(n-1)!]^n}{(n!)^{n-1}} \prod_{i=1}^m |K \cap u_i^+|^{|c_i|}
\]

\[
= \frac{n!}{n^n} \prod_{i=1}^m |K \cap u_i^+|^{|c_i|}.
\]

We can now use an approximation argument of Barthe from \([6]\) to deduce \((3.5)\) from \((3.7)\). We sketch the idea of the proof and refer to Barthe’s article for more details. Recall that a Borel measure \(\nu\) on \(S^{n-1}\) is called isotropic if \(I_n = \int_{S^{n-1}} u \otimes u \, d\nu(u)\). The fact that the vectors \(u_j\) and the weights \(c_j\) satisfy \((3.6)\) is equivalent to saying that the discrete measure \(\nu\) with \(\nu(\{u_j\}) = c_j\) is isotropic, i.e. \(I_n = \int_{S^{n-1}} u \otimes u \, d\nu(u)\).

Also, since

\[
\int_{S^{n-1}} \log |K \cap u^+| \, d\nu(u) = \sum_{i=1}^m c_i \log |K \cap u_i^+| = \log \left( \prod_{i=1}^m |K \cap u_i^+|^{|c_i|} \right),
\]

we may write \((3.7)\) in the equivalent form

\[
|K|^{n-1} \geq \frac{n!}{n^n} \exp \left( \int_{S^{n-1}} \log |K \cap u^+| \, d\nu(u) \right).
\]

In other words, \((3.5)\) holds true for any discrete isotropic measure on \(S^{n-1}\).

Now, let \(\nu\) be an isotropic Borel measure on \(S^{n-1}\). For any \(\varepsilon > 0\) we consider a maximal \(\varepsilon\)-net \(N_\varepsilon\) in \(S^{n-1}\) and a partition \((C_u)_{u \in N_\varepsilon}\) of \(S^{n-1}\) into Borel sets \(C_u \subseteq B(u, \varepsilon)\), where \(B(u, \varepsilon)\) is the geodesic ball with center \(u\) and radius \(\varepsilon\). Then, we consider the measure

\[
\nu_\varepsilon = \sum_{u \in N_\varepsilon} \nu(C_u) \delta_u,
\]

where \(\delta_u\) is the Dirac mass at \(u\). Note that, for any continuous function \(f : S^{n-1} \to \mathbb{R}\) we have that

\[
\int_{S^{n-1}} f(u) \, d\nu_\varepsilon \longrightarrow \int_{S^{n-1}} f(u) \, d\nu
\]

as \(\varepsilon \to 0\). In other words, \(\nu_\varepsilon \to \nu\) weakly as \(\varepsilon \to 0\). If \(T_\varepsilon = \int_{S^{n-1}} u \otimes u \, d\nu_\varepsilon(u)\) then for the measure

\[
\mu_\varepsilon = \sum_{u \in N_\varepsilon} \nu_\varepsilon(u) \|T_\varepsilon^{-1/2}(u)\|_2^2 \delta_{v(u)} \text{ where } v(u) := T_\varepsilon^{-1/2}(u)/\|T_\varepsilon^{-1/2}(u)\|_2
\]

we have

\[
I_n = \int_{S^{n-1}} T_\varepsilon^{-1/2}(u) \otimes T_\varepsilon^{-1/2}(u) \, d\nu_\varepsilon(u) = \int_{S^{n-1}} v \otimes v \, d\mu_\varepsilon(v).
\]

Since \(\|T_\varepsilon - I_n\|_2^2 \rightarrow 0 \leq c_1(\varepsilon)\) for some constant \(c_1(\varepsilon)\) that tends to 0 as \(\varepsilon \to 0\), we can check that for any continuous function \(f : S^{n-1} \to \mathbb{R}\)

\[
\int_{S^{n-1}} f(u) \, d\mu_\varepsilon \longrightarrow \int_{S^{n-1}} f(u) \, d\nu
\]

as \(\varepsilon \to 0\). Applying \((3.5)\) for the discrete isotropic measure \(\mu_\varepsilon\) we have

\[
|K|^{n-1} \geq \frac{n!}{n^n} \exp \left( \int_{S^{n-1}} \log |K \cap u^+| \, d\mu_\varepsilon(u) \right) \longrightarrow \frac{n!}{n^n} \exp \left( \int_{S^{n-1}} \log |K \cap u^+| \, d\nu(u) \right).
\]

This proves \((3.5)\).

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8
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