\chi^{\text{SB}} \text{ of cascading gauge theory in de Sitter}

Alex Buchel

Department of Applied Mathematics
Department of Physics and Astronomy
University of Western Ontario
London, Ontario N6A 5B7, Canada
Perimeter Institute for Theoretical Physics
Waterloo, Ontario N2J 2W9, Canada

Abstract

$\mathcal{N} = 1$ supersymmetric $SU(N) \times SU(N+M)$ cascading gauge theory of Klebanov et.al \cite{1,2} spontaneously breaks chiral symmetry in Minkowski space-time. We demonstrate that in de Sitter space-time the chiral symmetry breaking occurs for the values of the Hubble constant $H \lesssim 0.7\Lambda$, as well as in a narrow window $0.92(1)\Lambda \leq H \leq 0.92(5)\Lambda$. We give a precise definition of the strong coupling scale $\Lambda$ of cascading gauge theory, which is related to the glueball mass scale in the theory $m_{\text{glueball}}$ and the asymptotic string coupling $g_s$ as $\Lambda \sim g_s^{1/2}m_{\text{glueball}}$.

December 7, 2019
Contents

1 Introduction and summary 3

2 Dual effective actions of cascading gauge theory 13

3 Apparent horizon in de Sitter evolution of cascading gauge theory 17
   3.1 AH in ten dimensions ................................................. 18
   3.2 AH in Kaluza-Klein reduction to five dimensions .............. 19
   3.3 Area theorem for the AH .............................................. 21
   3.4 Entanglement entropy of TypeB de Sitter vacua ................ 23

4 TypeA_s de Sitter vacua 23
   4.1 Numerical results: TypeA_s ............................................ 24
   4.2 TypeA_s de Sitter vacua in the conformal limit .............. 28
   4.3 Validity of supergravity approximation for TypeA_s vacua .... 34

5 TypeA_b de Sitter vacua 36
   5.1 TypeA_b vacua from perturbative chiral symmetry breaking of TypeA_s vacua ..................................................... 37
   5.2 Numerical results: TypeA_b ............................................ 41
   5.3 Validity of supergravity approximation for TypeA_b vacua .... 46

6 TypeB de Sitter vacua 47
   6.1 Numerical results: TypeB ............................................... 47
   6.2 Validity of supergravity approximation for TypeB vacua ....... 50

7 Conclusion 51

A EF frame equations of motion 53

B FG frame equations of motion, asymptotics, relation to EF frame and extremal Klebanov-Strassler solution 60
   B.1 Asymptotics .............................................................. 63
      B.1.1 TypeA_s vacua asymptotics ..................................... 68
   B.2 From FG to EF frame .................................................. 70
   B.3 Extremal KS solution limit $H \to 0$ ................................. 74
1 Introduction and summary

Consider $\mathcal{N} = 1$ supersymmetric $SU(N + M) \times SU(N)$ gauge theory with two chiral superfields $A_1, A_2$ in the $(N + M, \bar{N})$ representation, and two chiral superfields $B_1, B_2$ in the $(\bar{N} + M, N)$ representation, in four dimensional Minkowski space-time $\mathbb{R}^{3,1}$. This theory has two gauge couplings $g_1, g_2$ associated with the two gauge group factors, and a quartic superpotential

$$W \sim \text{Tr} \left( A_i B_j A_k B_\ell \right) \epsilon^{ik} \epsilon^{j\ell}.$$  \hspace{1cm} (1.1)

When $M = 0$, both gauge couplings are exactly marginal, and the theory flows to a strongly coupled superconformal fixed point — the Klebanov-Witten (KW) theory [3]. KW infrared (IR) fixed point global symmetry

$$G : \underbrace{SU(2) \times SU(2)}_{\text{flavour}} \times \underbrace{U(1)}_{\text{R–symmetry}},$$  \hspace{1cm} (1.2)

together with superconformal invariance implies non-perturbatively large anomalous dimensions for the chiral superfields:

$$\gamma(A_i) = \gamma(B_j) = -\frac{1}{4}. \hspace{1cm} (1.3)$$

When $M \neq 0$, conformal invariance of $SU(N + M) \times SU(N)$ gauge theory is broken: while the sum of the gauge coupling remains exactly marginal [2],

$$\frac{4\pi^2}{g_1^2} + \frac{4\pi^2}{g_2^2} = \frac{\pi}{g_s} = \text{const}, \hspace{1cm} (1.4)$$
where $g_s$ is the asymptotic string coupling of the gravitational dual [4], perturbative $\beta$-function of the difference of couplings is nonzero [4]:

$$\frac{8\pi^2}{g_1^2} - \frac{8\pi^2}{g_2^2} = M \ln \frac{\Lambda}{\mu} \left(3 + 2(1 - \gamma(\text{Tr}(A_iB_j)))\right) = 6M \ln \frac{\Lambda}{\mu} \left(1 + \mathcal{O}\left(\frac{M^2}{N^2}\right)\right). \quad (1.5)$$

$\Lambda$ is the strong coupling scale of the theory. Given (1.4) and (1.5), the effective weakly coupled description of $SU(N + M) \times SU(N)$ gauge theory exists only in a finite-width energy band centered about $\Lambda$ — one encounters Landau poles both in the IR

$$g_2^2 \to \infty \quad \text{as} \quad \mu \to \mu_{IR} \equiv \Lambda e^{-\frac{\pi}{3gsM}} \quad (1.6)$$

and the ultraviolet (UV),

$$g_1^2 \to \infty \quad \text{as} \quad \mu \to \mu_{UV} \equiv \Lambda e^{+\frac{\pi}{3gsM}} \quad (1.7)$$

to leading order in $M^2/N^2$. As explained in [2], to extend the theory past the strong coupling regions one must perform self-similar transformations (Seiberg dualities [5]): $N \to N - M$ for $\mu \lesssim \mu_{IR}$ and $N \to N + M$ for $\mu \gtrsim \mu_{UV}$. Thus, extension of the effective $SU(N + M) \times SU(N)$ description to all energy scales involves an infinite sequence — a cascade — of Seiberg dualities with the renormalization group flow of the effective rank [6–8]

$$N = N(\mu) \sim g_sM^2 \ln \frac{\mu}{\Lambda}. \quad (1.8)$$

Although there are infinitely many duality steps in the UV, there is only a finite number of the duality transformations as one flows to the IR — when $N$ is an integer multiple of $M$ (plus 1) one ends up in the IR with the $SU(M + 1)$ gauge theory. The latter theory confined in the IR with a spontaneous breaking of the $U(1)_R$ (chiral symmetry),

$$U(1)_R \to \mathbb{Z}_2. \quad (1.9)$$

IR properties of the cascading gauge theories were reviewed in [4] (see also [9]); an important feature of the theory is the characteristic scale in the glueball mass spectrum:

$$m_{\text{glueball}} \equiv \frac{\epsilon^{2/3}}{Mg_s\alpha'}, \quad (1.10)$$

where $\epsilon$ is a conifold deformation parameter of the holographic dual [2], and $\alpha' = \ell_s^2$ is the string scale.

Previous studies focused on the fate of the chiral symmetry and confinement in cascading gauge theory at finite temperature. At finite temperature, there are three different spatially homogeneous and isotropic phases of the theory. We classify them as follows:
• PhaseA\textsubscript{s} — deconfined phase with unbroken chiral symmetry, \textit{i.e.}, \(U(1)\), see [6,10–12];

• PhaseA\textsubscript{b} — deconfined phase with broken chiral symmetry, \textit{i.e.}, \(\mathbb{Z}_2\), see [13,14];

• PhaseB — confined phase with broken chiral symmetry, \textit{i.e.}, \(\mathbb{Z}_2\), see [2].

Notice that confinement triggers spontaneous breaking of the chiral symmetry [2]: there is no spatially homogeneous and isotropic phase which is confined with \(U(1)\) chiral symmetry. It will be instructive to have a geometrical classification of these phases, in the warped-deformed conifold holographic dual of the theory [2,13,15]. To this end, consider analytical continuation along the time direction \(t \rightarrow t_E \equiv it\). Euclidean time \(t_E\) is then periodically identified as

\[
t_E \sim t_E + \frac{1}{T},
\]

where \(T\) is the equilibrium temperature of the phase. Topologically, the compact directions of the holographic dual are

\[
\begin{align*}
\text{unbroken chiral symmetry:} & \quad S^1_{\text{thermal circle}} \times S^1 \times S^2 \times S^2 \quad \text{U(1)-symmetric } T^{1,1} \\
\text{broken chiral symmetry:} & \quad S^1_{\text{thermal circle}} \times S^2 \times S^3 \quad \text{Z}_2\text{-symmetric } T^{1,1}
\end{align*}
\]

We can thus geometrically characterize different phases depending on which cycle shrinks to zero size in the interior of the ten-dimensional Euclidean type IIB supergravity dual:

\[
\begin{align*}
\text{PhaseA}_s: & \quad S^1_{\text{thermal circle}} \rightarrow 0 & \& S^1 \times S^2 \times S^2 \text{ is finite} \\
\text{PhaseA}_b: & \quad S^1_{\text{thermal circle}} \rightarrow 0 & \& S^2 \times S^3 \text{ is finite} \\
\text{PhaseB}: & \quad S^1_{\text{thermal circle}} \text{ is finite} & \& S^2 \rightarrow 0 & \& S^3 \text{ is finite}
\end{align*}
\]

According to [12] there is the first-order confinement/deconfinement phase transition between PhaseA\textsubscript{s} and PhaseB at

\[
T_c = 0.614(1) \frac{\Lambda_{\text{thermal}}}{P_g^{1/2}} = 0.614(1) \frac{3^{1/2}e^{1/3}}{2^{7/12}} \frac{e^{2/3}}{P_g^{1/2}} = 0.220(2) \frac{g_s^{1/2}}{m_{\text{glueball}}},
\]

\(^1\text{The precise expression for } \Lambda_{\text{thermal} \text{ was reported in [10].} \}

5
where the relation between \( P \) and \( M \) is given by (2.7) and \( m_{\text{glueball}} \) is defined as in (1.10). At temperature \( T < T_c \) the phase \( \text{PhaseA}_s \) is metastable — it becomes perturbatively unstable below \( T_{\chi \text{SB}} < T_c \) \[13\],

\[
T_{\chi \text{SB}} = 0.542(0) \frac{\Lambda_{\text{thermal}}}{P g_s^{1/2}} = 0.194(3) g_s^{1/2} m_{\text{glueball}} .
\] (1.15)

Symmetry broken deconfined phase \( \text{PhaseA}_b \) exists only for \( T \geq T_{\chi \text{SB}} \) or for energy densities \( E \leq E_{\chi \text{SB}} \) \[14\],

\[
E_{\chi \text{SB}} = 1.270(1) \frac{\Lambda_{\text{thermal}}^4}{16\pi G_5} = 1.270(1) \frac{2^{2/3} e^{A/3}}{192\pi^4} (M g_s)^4 m_{\text{glueball}}^4
\]

\[
= 0.000(4) (M g_s)^4 m_{\text{glueball}}^4 .
\] (1.16)

where \( G_5 \) is given by (2.8). Phase\( \text{PhaseA}_b \) has larger thermal free energy density than that of the chirally symmetric deconfined phase \( \text{PhaseA}_s \) at the corresponding temperature, and thus it does not dominate the canonical ensemble. On the other hand, Phase\( \text{PhaseA}_b \) is entropically favored over Phase\( \text{PhaseA}_s \) at the corresponding energy density, and thus is the dominant phase in the microcanonical ensemble. According to \[14\] the phase \( \text{PhaseA}_b \) is thermodynamically unstable, and thus it is dynamically (perturbatively) unstable towards developing spatial inhomogeneities \[17\].

In this paper we would like to understand vacua of cascading gauge theories in de Sitter space-time (flat or closed spatial slicing\[3\])

\[
ds^2_4 = -dt^2 + e^{2Ht} d\mathbf{x}^2 , \quad \text{or} \quad ds^2_4 = -dt^2 + \frac{1}{H^2} \cosh^2(Ht) (dS'^3)^2 ,
\] (1.17)

where \( H \) is a Hubble constant. Specifically, we would like to provide the classification of late-time states of the cascading gauge theory akin to spatially homogeneous and isotropic thermal phases \{\text{PhaseA}_s, \text{PhaseA}_b, \text{PhaseB}\} reviewed above. Of course there are crucial differences between thermal equilibrium physics and the late time de Sitter dynamics:

- Thermodynamics can be studied in canonical or microcanonical ensembles\[3\]. The latter one is suitable to study the dynamics of the equilibration process. de Sitter evolution of gauge theory states is eternally sourced by the space-time accelerated evolution.

\footnote{There is no difference between the them at late times as the curvature effects are diluted as \( \propto \exp(-2Ht) \).}

\footnote{As we emphasized above the thermal equilibrium phase structure is different in the two ensembles of the cascading gauge theory.}
expansion and thus is (loosely) equivalent to the microcanonical ensemble; there is no correspondence to the canonical ensemble.

- Insisting on spatial homogeneity and isotropy, an initial state typically relaxes to thermal equilibrium configuration, which can be assigned a thermal (time-independent) entropy density. Holographic dynamics of conformal gauge theories with a simple scale transformation can be mapped to an evolution in Minkowski space-time \([19]\) — here the late-time de Sitter vacua are conformally equivalent to equilibrium states of the microcanonical ensemble. There is no equilibration of non-conformal gauge theories at late-times in de Sitter \([19]\): the comoving entropy density production rate is nonzero. In \([21]\) it was pointed out that the comoving entropy production rate \(R\) can be attribute entirely to the spatial expansion

\[
\text{volume}\Big|^{\text{physical}}_{\text{comoving}} = e^{3Ht}\text{volume}\Big|^{\text{comoving}},
\]

while the physical entropy density \(s\) approaches a constant (time-independent) entanglement entropy \(s_{\text{ent}}\):

\[
\lim_{t\to\infty} s \equiv s_{\text{ent}} = H^3 R. \quad (1.18)
\]

In holography, the non-equilibrium entropy density \(s = s(t)\) is associated with the Bekenstein entropy of the dynamical apparent horizon (AH) \([22,23]\). In \([24]\) an example of fully nonlinear holographic evolution from initially homogeneous and isotropic state in de Sitter was presented where the late-time dynamics approaches de Sitter vacuum with entanglement entropy \((1.18)\).

Implementing de Sitter holographic dynamics as in \([24]\) for cascading gauge theories is outside the scope of this paper. Rather, as in \([19]\) and \([20]\), we assume that we specify a well-defined spatially homogeneous and isotropic initial state\(^6\) (well-defined initial condition for the gravitation evolution) in a holographic dual. This would correspond to some coarse grained state in the gauge theory specified with the density matrix \(\rho\). We identify the von Neumann entropy \(S\)

\[
S = - \text{Tr}(\rho \ln \rho),
\]

\(^4\)Not all strongly interacting systems equilibrate. See \([18]\) for a holographic example.

\(^5\)See also \([20]\) for a detailed recent analysis.

\(^6\)We believe that restriction to homogeneity and isotropy is not relevant for the late-time dynamics, given the accelerated background space-time expansion.
with the Bekenstein entropy of the AH in the holographic dual\(^7\). Partial differential equation of the gravitational dual at late times reduce to system of ordinary differential equations [24] which we analyze in details here. Inequivalent de Sitter vacua of the cascading gauge theory are characterized with different values of the entanglement entropy density \(s_{\text{ent}}\). The true (dominant) vacuum is the one which results in the largest \(s_{\text{ent}}\) for a fixed Hubble constant \(H\) and a fixed strong coupling scale of the theory \(\Lambda\), see (B.80),

\[
\Lambda = \frac{2^{1/6} e^{1/3} g_s^{1/2}}{3^{3/2}} m_{\text{glueball}} \approx 0.3 g_s^{1/2} m_{\text{glueball}} .
\] (1.19)

Parallel to classification of the thermal equilibrium states, we now explain topological/symmetry considerations to classify de Sitter vacua of cascading gauge theory — the discussion is more intuitive for the closed spatial slicing in (1.17). To access AH (and thus to evaluate \(s_{\text{ent}}\)), the dual gravitational bulk must be described in Eddington-Finkelstein (EF) coordinates. Fefferman-Graham (FG) coordinates cover only a patch of the former, which is outside of the EF frame AH [24], and thus is not suitable for the computation of the vacuum entanglement entropy. Still, FG frame is useful to implement analytical continuation to Euclidean (Bunch-Davies) vacuum

\[
- d\tau^2 + \frac{1}{H^2} \cosh^2(H\tau) \left(dS^3\right)^2 \xrightarrow{\tau \to \frac{\theta + \gamma}{H}} \frac{1}{H^2} \left((d\theta)^2 + \sin^2(\theta) \left(dS^3\right)^2\right) = \frac{1}{H^2} \left(dS^4\right)^2 .
\] (1.20)

Topologically, the compact directions of the Euclidean FG frame holographic dual are (compare with (1.12))

\[
\text{unbroken chiral symmetry : } \underbrace{S^4_{\text{Euclidean}}} \times \underbrace{S^1 \times S^2 \times S^2}_{U(1)-\text{symmetric } T^{1,1}} ;
\]

\[
\text{broken chiral symmetry : } \underbrace{S^4_{\text{Euclidean}}} \times \underbrace{S^2 \times S^3}_{\mathbb{Z}_2-\text{symmetric } T^{1,1}} .
\] (1.21)

Parallel to (1.13), we can geometrically characterize different de Sitter vacua of the cascading gauge theory depending on which cycle shrinks to zero size in the interior of

\(^7\)This procedure is implicit in all examples of holographic evolutions in Chesler-Yaffe framework [25]. Besides 'holographic quenches' of background space-time [26] (similar to de Sitter 'quenches' of interest here) it was successfully applied to quenches of the coupling constants of relevant operators in [27,28].
the ten-dimensional Euclidean FG frame type IIB supergravity dual:

\[
\text{TypeA}_s : \quad \int_{dS_4^{\text{Euclidean}}} S^4 \rightarrow 0 \quad \& \quad S^1 \times S^2 \times S^2 \text{ is finite} ; \\
\text{TypeA}_b : \quad \int_{dS_4^{\text{Euclidean}}} S^4 \rightarrow 0 \quad \& \quad S^2 \times S^3 \text{ is finite} ; \\
\text{TypeB} : \quad \int_{dS_4^{\text{Euclidean}}} S^4 \text{ is finite} \quad \& \quad S^2 \rightarrow 0 \quad \& \quad S^3 \text{ is finite} .
\]

(1.22)

To evaluate \( s_{\text{ent}} \) we proceed in two steps:\(^8\)

- first, we construct FG frame vacua, subject to the 'boundary conditions' (1.22) (see appendix B.1 for technical details);
- second, we use coordinate transformation to EF frame for each of these vacua (see [24] and appendix B.2 for technical details), and access the corresponding AH.

We summarize now our results:

- TypeA\(_s\) de Sitter vacua were studied previously in [29–31]. These vacua share resemblance with thermal deconfined chirally symmetric states of cascading gauge theory, \textit{i.e.}, PhaseA\(_s\). We find here that

\[
s_{\text{ent}}(\Lambda, H) \bigg|_{\text{TypeA}\_s} \neq 0 ,
\]

and vanishes as

\[
s_{\text{ent}}(\Lambda, H) \bigg|_{\text{TypeA}\_s} \propto H^3 \left( \frac{H^2}{\Lambda^2} \right)^{-3/4} \text{ as } H \gg \Lambda ,
\]

\[
(1.24)
\]

\textit{i.e.}, in the conformal limit. TypeA\(_s\) de Sitter vacua exist only when

\[
H \lesssim H_{\text{min}}^s , \quad H_{\text{min}}^s = 0.7\Lambda \approx 0.2 \ g_s^{1/2} m_{\text{glueball}} .
\]

(1.25)

As \( \frac{H^2}{\Lambda^2} \) decreases, the Kretschmann scalar at the AH in the holographic dual increases, making supergravity approximation less reliable. \( H_{\text{min}}^s \) in (1.25) should

---

\(^8\)The same two-step procedure was also used in computation of de Sitter vacuum entanglement entropy in \( \mathcal{N} = 2^* \) gauge theory in [20].
be interpreted as the value of the Hubble constant at which supergravity approximation breaks down. We identify the rapid growth of the curvature in the gravitational dual to TypeA\(_s\) de Sitter vacua with collapsing of the compact manifold (a deformed \(T^{1,1}\)) at the location of the apparent horizon — as a result, \(s_{\text{ent}}\) vanishes in this limit as well.

- **TypeA\(_b\)** de Sitter vacua are constructed here for the first time\(^9\). These vacua share resemblance with thermal deconfined states of cascading gauge theory with spontaneously broken chiral symmetry, \(i.e., \text{PhaseA}\(_b\).\) We find here that

\[
\left. s_{\text{ent}}(\Lambda, H) \right|_{\text{TypeA}\(_b\)} 
\neq 0 .
\]  

(type 1.26)

TypeA\(_b\) de Sitter vacua exist only when

\[
H \geq H_{\text{min}}^b , \quad H_{\text{min}}^b = 0.92(1)\Lambda \approx 0.276 \; g_1^{1/2} m_{\text{glueball}} .
\]

(type 1.27)

As \(\frac{H^2}{\Lambda^2}\) increases, the Kretschmann scalar at the AH in the holographic dual increases, making supergravity approximation less reliable.

- We find that while

\[
\left. s_{\text{ent}}(\Lambda, H_{\text{min}}^b) \right|_{\text{TypeA}\(_s\)} = \left. s_{\text{ent}}(\Lambda, H_{\text{min}}^b) \right|_{\text{TypeA}\(_b\)} ,
\]

(type 1.28)

de Sitter vacua with spontaneously broken chiral symmetry are entropically favored within a narrow window for the values of the Hubble constant

\[
\left. s_{\text{ent}}(\Lambda, H) \right|_{\text{TypeA}\(_b\)} \geq \left. s_{\text{ent}}(\Lambda, H) \right|_{\text{TypeA}\(_s\)} , \quad H_{\text{max}} \geq H \geq H_{\text{min}}^b ,
\]

(type 1.29)

where

\[
H_{\text{max}} = 0.92(5)\Lambda \approx 0.278 \; g_1^{1/2} m_{\text{glueball}} .
\]

(type 1.30)

TypeA\(_b\) de Sitter vacua continue to exist for \(H > H_{\text{max}}\), however they have smaller \(s_{\text{ent}}\) compare to the corresponding TypeA\(_s\) de Sitter vacua.

- **TypeB** de Sitter vacua were studied previously in \cite{31}. These vacua share resemblance with thermal confined states of cascading gauge theory with spontaneously broken chiral symmetry, \(i.e., \text{PhaseB}.\) We find here that

\[
\left. s_{\text{ent}}(\Lambda, H) \right|_{\text{TypeB}} = 0 .
\]

(type 1.31)

\(^9\)We introduce novel technique used to identify phases/vacua with spontaneously broken symmetry.
We emphasize that (1.31) does not mean that the coarse grained entropy of cascading gauge theory vanishes — in fact, during de Sitter evolution the entropy production rate is always positive (see section 3.3). What (1.31) states is that the comoving entropy production rate in TypeB vacuum vanishes as late times (much like it does in conformal gauge theories [24]). As a result, TypeB vacuum is never realized as the late-time attractor of a dynamical evolution for a generic cascading gauge theory state in de Sitter, provided vacua TypeA_s or TypeA_b exist. Neither of the latter vacua exists for \( H \lesssim H_{\text{min}}^{s} \), see (1.25), thus:

\[
\text{TypeB de Sitter vacuum is a late-time attractor provided } H \lesssim H_{\text{min}}^{s}.
\]

Of course, (1.32) implies that TypeB vacua must exist at least for \( H > H_{\text{min}}^{s} \); in fact we find (see section 6.2) that TypeB vacua exist for

\[
H \lesssim H_{\text{max}}^{B} , \quad H_{\text{max}}^{B} = 0.966(5) \Lambda > H_{\text{min}}^{s} = 0.7 \Lambda .
\]

Eqs. (1.29) and (1.32) represent our main, and somewhat unexpected result:

\[
SU(N) \times SU(N+M) \text{ cascading gauge theory with a strong coupling scale } \Lambda \text{ undergoes spontaneous chiral symmetry breaking in de Sitter space time with a Hubble constant } H \text{ provided}
\]

\[
H \lesssim H_{\text{min}}^{s} < H_{\text{min}}^{b} \quad \& \quad H_{\text{min}}^{b} \leq H \leq H_{\text{max}}.
\]

The critical values \( H_{\text{min}}^{s} , H_{\text{min}}^{b} \) and \( H_{\text{max}} \) are of order the strong coupling scale of the theory \( \Lambda \).

The rest of the paper is organized as follows. In section 2 we discuss holographic dual effective action of cascading gauge theory. Section 2 contains a guide to set of Appendices with technical details. Cascading gauge theory de Sitter vacuum entanglement entropy is identified with the Bekenstein entropy of the AH in the holographic dual at late times, see section 3. In section 3.1 we identify AH in ten dimensional holographic dual and compute its area density. In section 3.2 we establish that both the

---

10 While this is likely to be true in general, the statement is strictly precise for de Sitter evolution of spatially homogeneous and isotropic states of cascading gauge theory.

11 This should be understood in the same sense as existence of TypeA_s vacua: the supergravity approximation used to construct TypeB vacua is robust against higher-derivative \( \alpha' \) corrections from full string theory.
location of the AH and its associated entropy density is invariant upon Kaluza-Klein reduction on warped-deformed $T^{1,1}$. In section 3.3 we prove a theorem that as long as the background geometry of the holographic dual is nonsingular, the area density of the AH does not decrease with time. In section 3.4 we show that whenever vacua of TypeB exist, their entanglement entropy vanishes, see (1.31). Section 4 devoted to TypeA$_s$ de Sitter vacua. Numerical results are presented in section 4.1: we construct first the dual holographic backgrounds in FG frame, transform them to EF frame, identify the location of the apparent horizon and compute the vacuum entanglement entropy, see fig. 6. At each step we triple-check the numerical results by making use of distinct and independent computational schemes, see appendix C. Comparison of the results from different computational schemes in overlapping regions of the parameter space is shown in figs. 2, 4, 7. In section 4.2 we make use of the computational SchemeII to discuss the conformal limit of TypeA$_s$ vacua, $i.e.$, $H \gg \Lambda$, and establish (1.24). The validity of the supergravity approximation of the holographic dual to TypeA$_s$ de Sitter vacua is discussed in section 4.3. We establish a rapid growth of the Kretschmann scalar of the background geometry (2.13) evaluated at the AH for small values of $H^2/\Lambda^2$, and associate this growth with “collapsing” of the deformed $T^{1,1}$, see figs. 11 and 12. Extrapolating the numerical data, we estimate the value of the Hubble constant $H_{\text{min}}^s$, see (1.25), when the Kretschmann scalar diverges — we take this value as a limiting value of $H$ below which TypeA$_s$ vacua stop existing. We study TypeA$_b$ vacua with spontaneously broken chiral symmetry in section 5. We begin in section 5.1 with identification of the critical value $H_{\text{min}}^b$, see (1.27), below which TypeA$_b$ vacua do not exist. This is done computing linearized chiral symmetry breaking perturbations on top of TypeA$_s$ vacua with explicit symmetric breaking parameter — the gaugino mass term. At this critical value $H = H_{\text{min}}^b$ all the symmetry breaking expectation values diverge, see fig. 13. We explain how TypeA$_b$ vacua, with spontaneous symmetry breaking, can be constructed at values of the Hubble constant close to $H_{\text{min}}^b$, using the linearized perturbations on top of TypeA$_s$ vacua with explicit symmetry breaking. Numerical construction of TypeA$_b$ vacua in section 5.2 follows the discussion of section 4.1. Section 5.2 contains the central result of the paper — fig. 21 it establishes that chiral symmetry breaking of the cascading gauge theory in de Sitter space-time occurs in a narrow range of values of the Hubble constant, see (1.29). The validity of the supergravity approximation of the holographic dual to TypeA$_b$ de Sitter vacua is discussed in section 5.3. TypeB de Sitter vacua are discussed in section 6. These vacua have vanishing entanglement
entropy \(1.31\); however, they exist for arbitrary small \(\frac{H}{\Lambda}\), approaching the extremal Klebanov-Strassler solution \([2]\) as \(\frac{H}{\Lambda} \to 0\). We discuss TypeB vacua, first as a deformation of the extremal KS solution in section 6.1. In section 6.2 we present an indication that TypeB vacua exist only for \(H \lesssim H_{\text{max}} \) \(1.33\) — in this limit the 3-cycle of the dual geometry supporting the RR 3-form flux becomes vanishingly small in string units, making the supergravity approximation not reliable as indicated by the rapid growth of the Kretschmann scalar of the background geometry evaluated at the AH, see fig. 26. Since both TypeA\(_a\) and TypeA\(_b\) vacua cease to exist below certain value of the Hubble constant, specifically for \(H \lesssim H_{\text{min}}^a\), and \(H_{\text{max}}^B > H_{\text{min}}^s\), TypeB vacua become late-time attractors of the dynamical evolution of the cascading gauge theory in de Sitter for \(H \lesssim H_{\text{min}}^s\). We conclude in section 7 highlighting open questions and future directions.

2 Dual effective actions of cascading gauge theory

Consider \(SU(2) \times SU(2) \times \mathbb{Z}_2\) invariant states of cascading gauge theory on a 4-dimensional manifold \(\mathcal{M}_4 \equiv \partial \mathcal{M}_5\). In the planar limit and at large ’t Hooft coupling, one can consistently truncate the theory to a finite number of operators \([13]\):

- a stress-energy tensor \(T_{ij}\), a pair of dimension-3 operators \(O_3^{\alpha = \{1,2\}}\) (dual to gaugino condensates for each of the gauge group factors), a pair of dimension-4 operators \(O_4^{\beta = \{1,2\}}\), and dimension-6,7,8 operators \(O_6, O_7, O_8\). Effective gravitational action on a 5-dimensional manifold \(\mathcal{M}_5\) describing holographic dual of such states was derived in \([13]\):

\[
S_5 \left[ g_{\mu \nu} \leftrightarrow T_{ij}, \{\Omega_i, h_i, \Phi\} \leftrightarrow \{O_3^{\alpha}, O_4^{\beta}, O_6, O_7, O_8\} \right] = \frac{108}{16\pi G_5} \int_{\mathcal{M}_5} \text{vol}_{\mathcal{M}_5} \Omega_1^2 \Omega_2^2 \Omega_3^2 \times
\]

\[
\times \left\{ R_{10} - \frac{1}{2} (\nabla \Phi)^2 - \frac{1}{2} e^{-\Phi} \left( \frac{(h_1 - h_3)^2}{2 \Omega_1^2 \Omega_2^2 \Omega_3^2} + \frac{1}{\Omega_3^4} (\nabla h_1)^2 + \frac{1}{\Omega_2^4} (\nabla h_3)^2 \right) \right.
\]

\[
- \frac{1}{2} e^{\Phi} \left( \frac{2}{\Omega_2^2 \Omega_3^2} (\nabla h_2)^2 + \frac{1}{\Omega_1^2 \Omega_2^2} \left( h_2 - \frac{P}{9} \right)^2 + \frac{1}{\Omega_1^4 \Omega_3^2} h_2^2 \right)
\]

\[
- \frac{1}{2 \Omega_1^4 \Omega_2^4 \Omega_3^2} \left( 4 \Omega_0 + h_2 (h_3 - h_1) + \frac{1}{9} P h_1 \right)^2 \right\},
\]

\(2.1\)
where $\Omega_0$ is a constant in the definition of the 5-form flux\textsuperscript{12}, see (2.5), $R_{10}$ is given by

$$R_{10} = R_5 + \left( \frac{1}{2\Omega_1^2} + \frac{2}{\Omega_2^2} + \frac{2}{\Omega_3^2} - \frac{\Omega_2^2}{4\Omega_1^2\Omega_3^2} - \frac{\Omega_3^2}{4\Omega_1^2\Omega_2^2} - \frac{\Omega_1^2}{\Omega_2^2\Omega_3^2} \right) - 2\square \ln \left( \Omega_1\Omega_2\Omega_3 \right) \quad (2.2)$$

and $R_5$ is the five-dimensional Ricci scalar of the metric

$$ds_5^2 = g_{\mu\nu}(y)dy^\mu dy^\nu, \quad (2.3)$$

that forms part of the ten dimensional full metric

$$ds_{10}^2 = ds_5^2 + ds_{T^{1,1}}^2, \quad ds_{T^{1,1}}^2 = \Omega_1^2(y)g_5^2 + \Omega_2^2(y)(g_3^2 + g_4^2) + \Omega_3^2(y)(g_1^2 + g_2^2). \quad (2.4)$$

One-forms $\{g_i\}$ (for $i = 1, \cdots, 5$) are the usual forms defined in the warp-squashed $T^{1,1}$ and are given as in [13], for coordinates $0 \leq \psi \leq 4\pi$, $0 \leq \theta_a \leq \pi$ and $0 \leq \phi_a \leq 2\pi$ ($a = 1, 2$). All the covariant derivatives $\nabla_\lambda$ are with respect to the metric (2.3). Fluxes (and dilaton $\Phi$) are parameterized in such a way that functions $h_1(y), h_2(y), h_3(y)$ appear as

$$F_5 = \mathcal{F}_5 + \ast \mathcal{F}_5,$$

$$\mathcal{F}_5 = \left( 4\Omega_0 + h_2(y)(h_3(y) - h_1(y)) + \frac{P}{9} h_1(y) \right) g_1 \wedge g_2 \wedge g_3 \wedge g_4 \wedge g_5,$$

$$B_2 = h_1(y)g_1 \wedge g_2 + h_3(y)g_3 \wedge g_4,$$

$$F_3 = \frac{1}{9} P g_5 \wedge g_3 \wedge g_4 + h_2(y) \left( g_1 \wedge g_2 - g_3 \wedge g_4 \right) \wedge g_5$$

$$+ (g_1 \wedge g_3 + g_2 \wedge g_4) \wedge d(h_2(y)),$$

$$\Phi = \Phi(y), \quad (2.5)$$

Parameter $P$ must be appropriately quantized [4,12]:

$$\frac{1}{4\pi^2\alpha'} \int_{\text{3-cycle: } \theta_2 = \phi_2 = 0} F_3 = \frac{2P}{9\alpha'} \in \mathbb{Z}, \quad (2.6)$$

thus

$$P = \frac{9}{2} M \alpha', \quad (2.7)$$

corresponding to the number $M$ of fractional branes (the difference of ranks of cascading gauge theory gauge group factors) on the conifold. Finally, $G_5$ is the five dimensional effective gravitational constant

$$G_5 \equiv \frac{G_{10}}{\text{vol}_{T^{1,1}}} = \frac{27}{16\pi^3} G_{10}, \quad (2.8)$$

\textsuperscript{12}In the limit of vanishing 3-form fluxes, $\Omega_0 = \frac{L^4}{108}$, where $L$ is the asymptotic $AdS_5$ radius.
where $16\pi G_{10} = (2\pi)^7 (\alpha')^4$ is 10-dimensional gravitational constant of type IIB supergravity.

Chirally symmetric states of the cascading gauge theory correspond to enhancement of the global symmetry $SU(2) \times SU(2) \times Z_2 \to SU(2) \times SU(2) \times U(1)$, and are described by the gravitational configurations of (2.1) subject to constraints
\[ h_1 = h_3 , \quad h_2 = \frac{P}{18} , \quad \Omega_2 = \Omega_3 , \]
(2.9)
or in the boundary QFT language [13],
\[ O_3^a = 0 , \quad O_7 = 0 . \]
(2.10)

We find it convenient to introduce
\[
\begin{align*}
 h_1 &= \frac{1}{P} \left( K_1 - 36\Omega_0 \right) , \\
 h_2 &= \frac{P}{18} K_2 , \\
 h_3 &= \frac{1}{P} \left( K_3 - 36\Omega_0 \right) , \\
 \Omega_1 &= \frac{1}{3} f_c^{1/2} h^{1/4} , \\
 \Omega_2 &= \frac{1}{\sqrt{6}} f_a^{1/2} h^{1/4} , \\
 \Omega_3 &= \frac{1}{\sqrt{6}} f_b^{1/2} h^{1/4} .
\end{align*}
\]
(2.11)

The ultimate goal is to compute the entanglement entropy of cascading gauge theory — using the dual holographic picture with the effective gravitational action (2.1) — in distinct vacua (see (1.22)) in four dimensional de Sitter space-time. As explained in the introduction, this is done in two steps:

- constructing de Sitter vacua in Fefferman-Graham coordinate frame
\[
\begin{align*}
 ds_{10}^2 &= \frac{1}{h^{1/2} \rho^2} \left( -dr^2 + e^{2H r} d\mathbf{x}^2 \right) + \frac{h^{1/2}}{\rho^2} (d\rho)^2 \\
 &\quad + \frac{f_c h^{1/2}}{9} g_5^2 + \frac{f_a h^{1/2}}{6} (g_3^2 + g_4^2) + \frac{f_b h^{1/2}}{6} (g_1^2 + g_2^2) ,
\end{align*}
\]
(2.12)
\[ h = h(\rho) , \quad f_{a,b,c} = f_{a,b,c}(\rho) , \]
subject to appropriate topological/symmetry restrictions (1.22);

- using diffeomorphism transformation to represent the FG frame vacua in Eddington-Finkelstein coordinate frame
\[
\begin{align*}
 ds_{10}^2 &= 2 dt \left( dr - a dt \right) + \sigma^2 e^{2H t} d\mathbf{x}^2 + \frac{1}{9} \omega_{c2} g_5^2 + \frac{1}{6} \omega_{a2} (g_3^2 + g_4^2) + \frac{1}{6} \omega_{b2} (g_1^2 + g_2^2) , \\
 a &= a(r) , \quad \sigma = \sigma(r) , \quad \omega_{a2,b2,c2} = \omega_{a2,b2,c2}(r) .
\end{align*}
\]
(2.13)

\[ ^{13}\text{In the planar limit.} \]
\[ ^{14}\text{This is a consistent truncation of the cascading gauge theory to } U(1) \text{ symmetric sector constructed in [15].} \]
It is important to keep in mind that EF frame vacua (2.13) are the late-time limits of the evolution in EF frame:

\[ ds_{10}^2 = 2dt \ (dr - A dt) + \Sigma^2 \ dx^2 + \Omega_1^2 \ g_1^2 + \Omega_2^2 \ (g_3^2 + g_4^2) + \Omega_3^2 \ (g_1^2 + g_2^2), \]

\[ A = A(t, r), \quad \Sigma = \Sigma(t, r), \quad \Omega_{1,2,3} = \Omega_{1,2,3}(t, r). \] (2.14)

We now summarize technical details delegated to various Appendices.

- In appendix A we derive the equations of motion in the holographic bulk for the evolution of generic spatially homogeneous and isotropic state of cascading gauge theory in de Sitter space-time, see (A.3)-(A.13). We explain how to take the late time limit \( t \to \infty \) in (2.14) to obtain (2.13). The EF frame vacuum equations of motion are given by (A.16)-(A.26). The latter equations of motion have symmetries SEF1-SEF4 (A.27)-(A.30), which are used to set up and validate numerics (see appendix C).

- We begin appendix B presenting gravitational bulk equations of motion in FG frame (B.3)-(B.11). These equations of motion have (corresponding to SEF1-SEF4) symmetries SFG1-SFG4 (A.27)-(A.30), which are used to set up and validate numerics (see appendix C). In appendix B.1 we explain the near boundary (UV) \( \rho \to 0 \) and and the interior (IR) \( \rho \to \infty \) asymptotics. UV asymptotics are used to classify non-normalizable coefficients (defining parameters of the cascading gauge theory): the asymptotic string coupling \( g_s \) (1.4) and the strong coupling scale \( \Lambda \) of the theory (1.19), and the normalizable coefficients: the expectation values of boundary gauge theory operators \( \{ T_{ij}, \mathcal{O}_3^{\alpha=(1,2)}, \mathcal{O}_4^{\beta=(1,2)}, \mathcal{O}_6, \mathcal{O}_7, \mathcal{O}_8 \} \). IR asymptotics are used to classify the distinct de Sitter vacua of the theory (1.22), as as well to ensure that the bulk geometry is smooth as the corresponding cycles shrinks to zero size (\( S^4 \) for TypeA\(_s\) and TypeA\(_b\), and \( S^2 \) for TypeB vacua).

- TypeA\(_s\) vacua enjoy unbroken chiral symmetry; appendix B.1.1 presents the UV and IR asymptotics in FG frame obtained in [31] and translates the coefficients governing the expansion to those used for the characterization of TypeA\(_b\) vacua, see (B.47)-(B.51).

---

\[ ^{15} \text{Developing the precise holographic dictionary between these normalizable coefficients and the corresponding expectation values, while interesting, is not important for the results presented, and thus is outside the scope of the paper.} \]
appendix B.2 establishes the map between EF and FG frame description for each type of the vacua: TypeA_s, TypeA_b and TypeB.

In the limit $H \rightarrow 0$, TypeB vacuum in FG frame represents the extremal KS solution [2]. We use this limit in appendix B.3 to relate the strong coupling scale $\Lambda$ of the cascading gauge theory to the complex structure conifold deformation parameter $\epsilon$ used in [2], see (B.80).

appendix C covers numerical procedures for construction of FG frame dual backgrounds (see C.1) and EF frame dual backgrounds (see C.2). We introduce three different computational schemes — SchemeI, SchemeII and SchemeIII (C.6) — explain how they are related and outline their computational advantages in accessing different regions of the parameter space of the model. We introduce the AH location function $L_{AH}$ (C.8), used to identify the apparent horizon.

appendix D presents technical details for construction of TypeA_s de Sitter vacua in computational scheme SchemeII in the conformal limit, i.e., $b \rightarrow 0$.

appendix E collects the expression for the Kretschmann scalar (E.1) of the background geometry (2.13). It is used to test the validity of the supergravity approximation.

appendix F contains equations of motion and the asymptotic expansions for the chiral symmetry breaking perturbations about FG frame TypeA_s de Sitter vacua with explicit symmetry breaking parameter — the gaugino mass term. These perturbations are used to identify TypeA_b vacua "close" to TypeA_s vacua.

3 Apparent horizon in de Sitter evolution of cascading gauge theory

Apparent horizon in holographic dual is crucial for identifying the attractor vacuum for the evolution of generic homogeneous and isotropic states of cascading gauge theory.

In general AH is observer dependent. It is natural to define AH with respect to an observer reflecting the symmetries of the spatial slices — homogeneity and isotropy in $x$ in (2.14), see [25]. Such an identification correctly reproduces the hydrodynamic limit [32] and can be proven to comply with the second law of thermodynamics [19, 24], thus serving as a useful definition of the dynamical (nonequilibrium) entropy.
in de Sitter: given competing trajectories for the evolution, dynamics proceeds along trajectory resulting in the maximum entropy at late times. We identify AH directly in ten-dimensional EF frame gravitational dual in section 3.1. We reproduce the same result in EF gravitational dual of the effective five-dimensional description in section 3.2. Both in ten-dimensions and upon Kaluza-Klein reduction to five dimensions the area of the AH stays the same. In section 3.3 we use equations of motion (A.3)-(A.13) to prove that the area of the AH is nondecreasing upon evolution. We identify the (dynamical) area density of the AH $A_{\text{10}}(t)$ with the dynamical entropy density $s$ of the boundary gauge theory as

$$a^3 s = e^{3Ht} s(t) = \frac{A_{\text{10}}}{4G_{\text{10}}} = \frac{4\pi}{(2\pi)^7(\alpha')^4} A_{\text{10}}(t),$$

where $a = e^{Ht}$ is the boundary spatial metric scale factor, see (1.17).

The entanglement entropy $s_{\text{ent}}$ is related to the late-time limit of $s$ as

$$\lim_{t \to \infty} \frac{1}{H^3 a^3} \frac{d}{dt} (a^3 s) \equiv 3H \times \mathcal{R},$$

$$\lim_{t \to \infty} s(t) \equiv s_{\text{ent}} = H^3 \mathcal{R},$$

where $\mathcal{R}$ is the comoving entropy production rate in de Sitter vacuum first introduced in [19]. Finally, in section 3.4 we show that

$$\mathcal{R} \Big|_{\text{TypeB}} = 0 \quad \implies \quad s_{\text{ent}} \Big|_{\text{TypeB}} = 0.$$  \tag{3.3}

### 3.1 AH in ten dimensions

The apparent horizon of the bulk gravitational dual to cascading gauge theory dynamics in de Sitter is located at the radius $r = r_{\text{AH}}$ where the expansion $\theta$ of a congruence of outward pointing null vectors vanishes (i.e., it stops expanding outwards). Working in the coordinates of equation (2.14), we characterize such a congruence with the null vector $k = \partial_t + A \partial_r$. The null vector $k$ points toward the boundary of the space-time outside of the initial black hole, and points inward inside the initial horizon.

Following [33], the expansion of a congruence of affine parameterized null vectors $n$ is given by

$$\theta = \nabla_\alpha n^\alpha.$$  \tag{3.4}

However, it turns out that $k^\beta \nabla_\beta k^\alpha = \partial_r A k^\alpha$, i.e., $k$ is not affine. To remedy this, we rescale $k$ by $\exp\{ \int \partial_r A \ d\lambda \}$, where $\lambda$ is the parameter along which the congruence $k$
evolves. This ensures that the rescaled null vector satisfies the geodesic equation with \( \lambda \) as an affine parameter. Reference [33] then gives the expansion of \( k \) to be
\[
\theta = \exp \left[ \int \partial_r A \; d\lambda \right] (\nabla_\alpha k^\alpha - \partial_r A).
\] (3.5)
Substituting in for \( \nabla_\alpha k^\alpha \) computed in the metric (2.14)
\[
\nabla_\alpha k^\alpha = \frac{1}{\sqrt{-g}} \partial_\alpha (\sqrt{-g} k^\alpha) = \partial_t \ln \left( \Sigma^3 \Omega_1 \Omega_2^2 \Omega_3^2 \right) + A \partial_r \ln \left( \Sigma^3 \Omega_1 \Omega_2^2 \Omega_3^2 \right) \bigg|_{r=r_{AH}} = 0.
\] (3.6)
We see that \( \theta = 0 \), when
\[
\partial_t \left( \Sigma^3 \Omega_1 \Omega_2^2 \Omega_3^2 \right) + A \partial_r \left( \Sigma^3 \Omega_1 \Omega_2^2 \Omega_3^2 \right) \bigg|_{r=r_{AH}} = 0.
\] (3.7)
Eq. (3.7) determines the location of the AH, i.e., \( r_{AH} = r_{AH}(t) \). The area density of the AH \( A_{10} \) is
\[
A_{10} = \Sigma^3 \Omega_1 \Omega_2^2 \Omega_3^2 \bigg|_{r=r_{AH}} \int g_5 \wedge g_3 \wedge g_4 \wedge g_1 \wedge g_2 = 64\pi^3 \Sigma^3 \Omega_1 \Omega_2^2 \Omega_3^2 \bigg|_{r=r_{AH}},
\] (3.8)
leading to (see (3.11))
\[
e^{3Ht_{s}} = \frac{64\pi^3}{4G_{10}} \Sigma^3 \Omega_1 \Omega_2^2 \Omega_3^2 \bigg|_{r=r_{AH}} = \frac{1}{4G_{5}} 108\Sigma^3 \Omega_1 \Omega_2^2 \Omega_3^2 \bigg|_{r=r_{AH}}.
\] (3.9)

3.2 AH in Kaluza-Klein reduction to five dimensions

We would like to reproduce (3.7) and (3.9) from the five-dimensional perspective.

While the effective action (2.1) is five dimensional, the metric frame used is not Einstein:
\[
S_5 = \frac{108}{16\pi G_5} \int_{\mathcal{M}_5} \text{vol}_{\mathcal{M}_5} \; \Omega_1 \Omega_2^2 \Omega_3^2 \times \left\{ R_5 + \cdots \right\}.
\] (3.10)
This can be fixed with a simple conformal rescaling: introducing
\[
d\tilde{s}_5^2 \equiv \tilde{g}_{\mu\nu} dy^\mu dy^\nu = \Omega^{10/3} ds_5^2 = \Omega^{10/3} g_{\mu\nu} dy^\mu dy^\nu, \quad \Omega^5 = \Omega_1 \Omega_2^2 \Omega_3^2,
\] (3.11)
and defining
\[
\tilde{G}_5 = \frac{G_5}{108},
\] (3.12)
the effective action \( S_5 \) in (3.10) has now a standard Einstein-Hilbert term with respect to \( \tilde{g} \)
\[
S_5 = \frac{1}{16\pi \tilde{G}_5} \tilde{\text{vol}}_{\tilde{\mathcal{M}}_5} \times \left\{ \tilde{R}_5 + \cdots \right\}.
\] (3.13)
The new EF frame (compare with (2.14)) becomes
\[
\begin{align*}
\tilde{d}s^2 & = \Omega^{10/3} \left[ 2dt \ (dr - A \ dt) + \Sigma^2 \ d\mathbf{x}^2 \right] = 2dt \tilde{d}t - 2A\Omega^{10/3} \ dt^2 + \Omega^{10/3} \Sigma^2 \ d\mathbf{x}^2, \\
\tilde{d}r & = \Omega^{10/3} \ dr,
\end{align*}
\tag{3.14}
\]
where the second equality defines a new radial coordinate \( \tilde{r} \). The congruence of null geodesics is now characterized with
\[
\tilde{k} = \partial_t + A\Omega^{10/3} \partial_{\tilde{r}}\, , \tag{3.15}
\]
so that
\[
\tilde{k}^\beta \nabla_\beta \tilde{k}^\alpha = \partial_{\tilde{r}} \left( A\Omega^{10/3} \right) \tilde{k}^\alpha . \tag{3.16}
\]
Since
\[
\sqrt{-\tilde{g}} = \Omega^5 \Sigma^3 , \tag{3.17}
\]
we have
\[
\nabla_\alpha \tilde{k}^\alpha = \partial_t \ln \left( \Omega^5 \Sigma^3 \right) + A\Omega^{10/3} \partial_{\tilde{r}} \ln \left( \Omega^5 \Sigma^3 \right) + \partial_{\tilde{r}} \left( A\Omega^{10/3} \right) . \tag{3.18}
\]
For the expansion \( \tilde{\theta} \) of the congruence of affine parameterized null vectors we have (compare with (3.5))
\[
\tilde{\theta} \propto \left( \nabla_\alpha \tilde{k}^\alpha - \partial_{\tilde{r}} \left( A\Omega^{10/3} \right) \right) = \partial_t \ln \left( \Omega^5 \Sigma^3 \right) + A\Omega^{10/3} \partial_{\tilde{r}} \ln \left( \Omega^5 \Sigma^3 \right) \\
= \partial_t \ln \left( \Omega^5 \Sigma^3 \right) + A \partial_{\tilde{r}} \ln \left( \Omega^5 \Sigma^3 \right) = \partial_t \ln \left( \Sigma^3 \Omega_1 \Omega_2^2 \Omega_3^2 \right) + A \partial_{\tilde{r}} \ln \left( \Sigma^3 \Omega_1 \Omega_2^2 \Omega_3^2 \right) , \tag{3.19}
\]
where in the second line we used the definition of \( \tilde{r} \) (3.14) and \( \Omega \) (3.11). Note that \( \tilde{\theta} = 0 \) in (3.19) is equivalent to \( \theta = 0 \) reproducing (3.7).

The five dimensional area density \( A_5 \) of the AH in (3.14) is given by
\[
A_5 = \left. \left( \Omega^{5/3} \Sigma \right)^3 \right|_{r=r_{\text{AH}}} = \left. \Sigma^3 \Omega_1 \Omega_2^2 \Omega_3^2 \right|_{r=r_{\text{AH}}} , \tag{3.20}
\]
leading to the dynamical entropy density
\[
\begin{align*}
e^{3Ht} s & = \frac{A_5}{4G_5} = \left. \frac{1}{4G_5} \ 108 \Sigma^3 \Omega_1 \Omega_2^2 \Omega_3^2 \right|_{r=r_{\text{AH}}} , \tag{3.21}
\end{align*}
\]
reproducing (3.9).
3.3 Area theorem for the AH

Following \[19\] and using the equations of motion (A.3)-(A.13) we prove now that the dynamical entropy density \( s \) defined as in (3.21) grows with time \( t \), i.e.,

\[
\frac{dA_5}{dt} = \frac{d}{dt} \left( \frac{\Sigma^3 \Omega_1^2 \Omega_2^2 \Omega_3^2}{r=\text{AH}} \right) \geq 0.
\] (3.22)

Note that the AH location is determined from (see (3.19))

\[
0 = d_+ (\Sigma^3 \Omega_1^2 \Omega_2^2 \Omega_3^2) \bigg|_{r=\text{AH}} \equiv \partial_t (\Sigma^3 \Omega_1^2 \Omega_2^2 \Omega_3^2) + A \partial_r (\Sigma^3 \Omega_1^2 \Omega_2^2 \Omega_3^2) \bigg|_{r=\text{AH}}.
\] (3.23)

Taking \( \frac{d}{dt} \) we have

\[
0 = \frac{d}{dt} \left( \partial_t (\Sigma^3 \Omega_1^2 \Omega_2^2 \Omega_3^2) + A \partial_r (\Sigma^3 \Omega_1^2 \Omega_2^2 \Omega_3^2) \right) \\
= \left\{ \partial_t + \frac{dr_{\text{AH}}}{dt} \times \partial_r \right\} \left( \partial_t (\Sigma^3 \Omega_1^2 \Omega_2^2 \Omega_3^2) + A \partial_r (\Sigma^3 \Omega_1^2 \Omega_2^2 \Omega_3^2) \right) \bigg|_{r=\text{AH}},
\] (3.24)

which is used to algebraically solve for \( \frac{dr_{\text{AH}}}{dt} \bigg|_{r=\text{AH}} \). The latter expression is then substituted in

\[
\frac{dA_5}{dt} = \left\{ \partial_t + \frac{dr_{\text{AH}}}{dt} \times \partial_r \right\} \Sigma^3 \Omega_1 \Omega_2^2 \Omega_3^2 \bigg|_{r=\text{AH}}.
\] (3.25)

We use equations of motion (A.3)-(A.13) to eliminate all second order derivative in (3.25); we further eliminate \( \partial_t \Sigma \) using (3.23) to arrive at

\[
\frac{dA_5}{dt} = \frac{\partial_r (\Sigma^3 \Omega_1^2 \Omega_2^2 \Omega_3^2)}{\partial_r (d_+ (\Sigma^3 \Omega_1^2 \Omega_2^2 \Omega_3^2))} \times \mathcal{F}^2 \bigg|_{r=\text{AH}},
\] (3.26)

where \( \mathcal{F}^2 \) is manifestly positive

\[
\mathcal{F}^2 = \frac{\Sigma^3}{2592 \Omega_2^2 \Omega_3^2 \Omega_1 g^2 P^2} \times \left( \Omega_1^2 \left( 8(d_+ K_2)^2 \Omega_2^2 \Omega_3^4 g^3 P^4 + 1296(d_+ g)^2 \Omega_4^4 P^2 \right) + 9(d_+ K_3)^2 \Omega_3^4 g + 9(d_+ K_1)^2 \Omega_2^4 g^2 \right) + 1728 \Omega_2^2 \Omega_3^4 g^2 P^2 \left( \frac{2d_+ \Omega_2}{\Omega_2} + \frac{d_+ \Omega_3}{\Omega_3} \right)^2
\] (3.27)
Constraint (A.12) can be integrated (once) to obtain
\[
\partial_r (\Sigma^3 \Omega_1^2 \Omega_2^2 \Omega_3^2) = \Sigma^3 \Omega_1^2 \Omega_2^2 \Omega_3^2 \int_r^\infty dr \frac{\mathcal{M}^2}{\Omega_2^2}
\]
\[
\mathcal{M}^2 = \frac{2(\partial_r \Omega_3)^2}{\Omega_2^2} + \frac{2(\partial_r \Omega_3)^2}{\Omega_3^2} + \frac{3(\partial_r \Sigma)^2}{\Sigma^2} + \frac{(\partial_r g)^2}{2g^2} + \frac{gP^2(\partial_r K_2)^2}{324 \Omega_3^2 \Omega_2^2}
\]
\[
+ \frac{(\partial_r K_3)^2}{288gP^2 \Omega_4^1} + \frac{(\partial_r K_1)^2}{288gP^2 \Omega_4^1},
\]
which implies that
\[
\partial_r (\Sigma^3 \Omega_1^2 \Omega_2^2 \Omega_3^2) \geq 0,
\]
provided the integral in (3.28) is convergent and \(\Sigma^3 \Omega_1^2 \Omega_2^2 \Omega_3^2 \geq 0\).

Note that (see appendix B.2)
\[
\lim_{r \to \infty} d_+ (\Sigma^3 \Omega_1^2 \Omega_2^2 \Omega_3^2) = \lim_{r \to \infty} A \partial_r (\Sigma^3 \Omega_1^2 \Omega_2^2 \Omega_3^2)
\]
\[
= \lim_{\rho \to 0} \left\{ \frac{1}{2h^{1/2} \rho^2} \times (-\rho^2) \partial_r \left( h^{-3/4} \rho^{-3} \exp(3H \int_0^\rho h^{1/2}(s) ds) \times \frac{h^{5/4} f_{c \rho} f_{ab}}{108 \Omega_1^2 \Omega_2^2 \Omega_3^2} \right) \right\}
\]
\[
= \lim_{\rho \to 0} \left( \frac{1}{72 \rho^4} \text{+ subleading} \right) \to +\infty,
\]
where we transformed first to FG frame and used the boundary asymptotic expansions (B.17)-(B.20). Thus,
\[
d_+ (\Sigma^3 \Omega_1^2 \Omega_2^2 \Omega_3^2) > 0, \quad r > r_{AH} \implies \partial_r (d_+ (\Sigma^3 \Omega_1^2 \Omega_2^2 \Omega_3^2)) \bigg|_{r=r_{AH}} \geq 0,
\]
since the quantity \(d_+ (\Sigma^3 \Omega_1^2 \Omega_2^2 \Omega_3^2)\) changes sign at \(r = r_{AH}\), see (3.23). Combining (3.26), (3.29) and (3.31) we arrive at (3.22).

For future reference we present the expressions for the location of the AH and the entanglement entropy density in de Sitter vacua. Using (A.14) and (A.15) we find from (3.23) and (3.9)
\[
\text{AH location : } \quad \left. 3H \sigma^3 \omega_c \omega_2 \omega_0 \right|_{r=r_{AH}} = 0;
\]
\[
\text{vacum entanglement entropy : } \quad s_{\text{ent}} = \frac{1}{4G_5} \left. \sigma^3 \omega_c \omega_2 \omega_0 \right|_{r=r_{AH}}.
\]
3.4 Entanglement entropy of TypeB de Sitter vacua

We demonstrate here that entanglement entropy of TypeB de Sitter vacuum vanishes — this implies that the corresponding comoving entropy production rate vanishes. de Sitter comoving entropy production rate vanishes in conformal field theories as well [20]. In CFTs the reason is simple: de Sitter vacuum is a conformal transformation of a thermal equilibrium state and entropy production is invariant under conformal transformations [19]. We do not understand the physical reason why the same is true for a de Sitter vacuum in nonconformal gauge theory (TypeB vacuum in cascading gauge theory).

Using the asymptotic expansion (B.67) (recall that $z = -r \, (B.56)$) we find for

\[
\text{AH location : } \frac{3^{3/2}}{2} (h_0^h)^{3/4} (f_{a,0}^h)^{3/2} (s_0^h)^3 \; r \left( 1 + 3H(h_0^h)^{1/2} r + \mathcal{O}(r^2) \right) \bigg|_{r=r_{AH}} = 0 \\
\implies r_{AH} = 0; \\
\text{vacuum entanglement entropy : } s_{ent} = \frac{1}{4G_5} \frac{3^{3/2}}{2} (h_0^h)^{5/4} (f_{a,0}^h)^{3/2} (s_0^h)^3 \; r^2 + \mathcal{O}(r^3) \bigg|_{r=r_{AH}} \\
\implies s_{ent} \bigg|_{\text{TypeB}} = 0.
\]

(3.33)

The result (3.33) stands as long as vacua TypeB exist — we find in section 6.2 that this is true provided $H \lesssim H_{\text{max}}$, see (1.33).

4 TypeA$_s$ de Sitter vacua

TypeA$_s$ vacua in FG frame were discussed in details in [31]. As emphasized in [19] and [20] this is not enough to access vacuum entanglement entropy — one needs the holographic construction in EF frame. In section 4.1 we present numerical results for TypeA$_s$ vacua for generic values of $H^2/\Lambda^2$, in particular the results for the entanglement entropy, see fig. 6. We discuss TypeA$_s$ in the conformal limit $\Lambda \ll H$ in section 4.2. In section 4.3 we estimate $H_{\text{min}}^s$ (see (1.25)) below which TypeA$_s$ vacua construction in type IIB supergravity becomes unreliable. We identify the source of breaking of the supergravity approximation.
4.1 Numerical results: TypeA

To begin, we numerically construct TypeA de Sitter vacua in FG frame (2.12). This involves solving ODEs (B.3)-(B.11) in the chirally symmetric limit (B.38), subject to UV asymptotics (the radial coordinate $\rho \to 0$) (B.39)-(B.43) and IR asymptotics (the radial coordinate $\rho \to +\infty$) (B.45). There are 8 second order equations (B.3)-(B.10) and 1 first order equation (B.11). Imposing the chirally symmetric limit (B.38), this set of coupled ODEs is reduced to 5 second order equations for the three metric warp factors $f_2 = f_c$, $f_3 = f_a = f_b$ and $h$, the single 3-form flux function $K = K_1 = K_3$ ($K_2 = 1$ in the chiral limit) and the string coupling $g$. The first order equation (B.11) involves (linearly) $f_2'$ and can be used instead of one of the second order equations (namely, the one involving $f_2''$). Thus, altogether we have a coupled system of 4 second order ODEs (linear in \{${f_2''}, h'', K'' , g''$\}) and a single first order equation (linear in $f_2'$).

As a result, a unique solution must be characterized by $9 = 2 \times 4 + 1$ parameters; these are the UV/IR parameters

\[
\begin{align*}
\text{UV} : & \quad \{ f_{2,1,0}, g_{4,0}, f_{2,4,0}, f_{2,6,0}, f_{2,8,0} \}; \\
\text{IR} : & \quad \{ f^h_{2,0}, f^h_{3,0}, K^h_0, g^h_0 \}. 
\end{align*}
\]

(4.1)

The external parameters \{\text{\textit{P, K}, H, g}_s\} (the gauge group rank difference \textit{M} of the cascading gauge theory (2.7), its strong coupling scale $\Lambda$ (B.26), the Hubble constant (1.17), the renormalization group flow invariant sum of the gauge couplings (1.4)) labeling the vacuum are fixed with the choice of the computational scheme (C.6). Of course, as emphasized in appendix C.1, the results must not depend on which computational scheme is adopted. We illustrate now that this is indeed the case using the IR parameters in (4.1) as an example\footnote{The same is true for the UV parameters as well.}. Comparison of the different computational schemes is done using dimensionless and rescaled quantities: $\ln \left( \frac{H^2}{\Lambda^2} \right)$ (as a vacuum label) (C.2) and \{\text{\textit{f}_{2,3,0,}^{h}, K^h_0, g^h_0}\} (C.4). Explicitly:

\[
\begin{align*}
\text{Scheme I} : & \quad \ln \left( \frac{H^2}{\Lambda^2} \right) = k_s, \quad \hat{f}_{2,3,0}^h = \hat{f}_{2,3,0}^h, \quad \hat{K}_0^h = K_0^h, \quad \hat{g}_0^h = g_0^h; \\
\text{Scheme II} : & \quad \ln \left( \frac{H^2}{\Lambda^2} \right) = \frac{1}{b} + \ln b, \quad \hat{f}_{2,3,0}^h = \frac{1}{b^{1/2}} f_{2,3,0}^h, \quad \hat{K}_0^h = \frac{1}{b} K_0^h, \quad \hat{g}_0^h = g_0^h; \\
\text{Scheme III} : & \quad \ln \left( \frac{H^2}{\Lambda^2} \right) = \frac{1}{4} + \ln \alpha, \quad \hat{f}_{2,3,0}^h = \frac{1}{\alpha^{1/2}} f_{2,3,0}^h, \quad \hat{K}_0^h = K_0^h, \quad \hat{g}_0^h = g_0^h. 
\end{align*}
\]
Figure 1: Infrared parameters \( \{ \hat{f}^h_{2,0}, \hat{f}^h_{3,0}, \hat{K}^h_0, \hat{g}^h_0 \} \) of the Fefferman-Graham coordinate frame of TypeA\(_s\) de Sitter vacua of cascading gauge theory as functions of \( \ln \frac{H^2}{\Lambda^2} \) in different computational schemes (C.6): SchemeI (blue), SchemeII (red) and Scheme III (green).

Following (4.2), we collect (subset of the) results of \( \{ \hat{f}^h_{2,0}, \hat{f}^h_{3,0}, \hat{K}^h_0, \hat{g}^h_0 \} \) as functions of \( \ln \frac{H^2}{\Lambda^2} \) in different computational schemes in fig. 1: SchemeI (blue curves), SchemeII (red curves) and Scheme III (green curves). The accuracy of the collapsed results in different schemes is highlighted in fig. 2 for \( \hat{f}^h_{2,0} \) — the remaining parameters follow the same trend.

Next, FG frame TypeA\(_s\) de Sitter vacua have to be reinterpreted in EF frame, see appendix B.2. The diffeomorphism transformation is performed at the radial location \( \{ \text{FG} : \frac{1}{\rho} \equiv y = 0 \} \Leftrightarrow \{ \text{EF} : r \equiv -z = 0 \} \).

Details of numerical construction of EF frame vacua from FG frame vacua are collected in appendix C.2. An important quantity is the parameter \( s^h_0 \), see (2.13),

\[
s^h_0 = \left. \sigma \right|_{y=0} \quad = \left. \sigma \right|_{z=0}.
\]
Figure 2: Left panel: comparison of $\hat{f}_{2,0}^h$ (the computational scheme SchemeIII) with $\hat{f}_{2,0}^h$ (the computational scheme SchemeI). Right panel: comparison of $\hat{f}_{2,0}^h$ (the computational scheme SchemeII) with $\hat{f}_{2,0}^h$ (the computational scheme SchemeI).

Figure 3: Parameters $\hat{s}_0^h$ of TypeA$_s$ de Sitter vacua of cascading gauge theory as functions of $\ln \frac{H^2}{\Lambda^2}$ in different computational schemes (C.6): SchemeI (blue), SchemeII (red) and Scheme III (green).

As with FG frame UV/IR parameters (4.1), results for $s_0^h$ should not depend on the choice of the computational scheme, provided we compare properly dimensionless and rescaled quantities, i.e., $\ln \frac{H^2}{\Lambda^2}$ and $\hat{s}_0^h$ (C.13),

\[
\text{SchemeI : } \ln \frac{H^2}{\Lambda^2} = k_s, \quad \hat{s}_0^h = s_0^h; \\
\text{SchemeII : } \ln \frac{H^2}{\Lambda^2} = \frac{1}{b} + \ln b, \quad \hat{s}_0^h = \frac{1}{b^{1/4}} s_0^h; \quad (4.5) \\
\text{SchemeIII : } \ln \frac{H^2}{\Lambda^2} = \frac{1}{4} + \ln \alpha, \quad \hat{s}_0^h = \frac{1}{\alpha^{1/2}} s_0^h.
\]

Following (4.5), we collect (subset of the) results of $\hat{s}_0^h$ as functions of $\ln \frac{H^2}{\Lambda^2}$ in different computational schemes in fig. 3: SchemeI (blue curve), SchemeII (red curve)
Figure 4: Left panel: comparison of $\hat{s}_0^h$ (the computational scheme SchemeIII) with $\hat{s}_0^h$ (the computational scheme SchemeI). Right panel: comparison of $\hat{s}_0^h$ (the computational scheme SchemeII) with $\hat{s}_0^h$ (the computational scheme SchemeI).

Figure 5: Apparent horizon location function $L_{AH}(z)$ in computational scheme SchemeI at $k_s = 0$, i.e., at $H = \Lambda$, see (C.8). The red dot is $L_{AH}(0)$, see (C.9). Notice that $L'_{AH}(0) < 0$, see (C.10). The vertical green dashed line is the first zero of $L_{AH}(z)$: $z_{AH} = 0.163346$.

and Scheme III (green curve). The accuracy of the collapsed results in different schemes is highlighted in fig. 4.

EF frame equations of motion (A.17)-(A.25) are solved subject to the initial conditions set by the asymptotic expansions (B.57) at $z = 0$. These equations have to be integrated on the interval

$$z \in [0, z_{AH}], \quad (4.6)$$

where $z_{AH} = -r_{AH}$ is the location of the apparent horizon at asymptotically late times, see (3.32). To determine the location of the apparent horizon, along with integrating the gravitational background functions $\{a, \sigma, w_{c2}, w_{a2}, K_1, g\}$ (remember that $w_{b2} = w_{c2}$, $K_3 = K_1$ and $K_2 = 1$ when the chiral symmetry is unbroken), we evaluate the AH
location function $L_{AH}(z)$, see (C.8). AH is located at the first zero of this function for $z > 0$. A typical profile of the AH location function is shown in fig. 5. Once the AH is identified, TypeA$_s$ vacua entanglement entropy is computed following (3.32):

$$s_{ent} = \frac{H^3 P^4 g_s^2}{4 G_5} \left\{ \hat{\sigma}^3 \hat{w}_{c2} \hat{\omega}_{a2} \right\} \bigg|_{\hat{z} = \hat{z}_{AH}} = \frac{3^5 M^4 g_s^2}{2^5 \pi^3} H^3 \left\{ \hat{\sigma}^3 \hat{w}_{c2} \hat{\omega}_{a2} \right\} \bigg|_{\hat{z} = \hat{z}_{AH}},$$

(4.7)

where following (C.1) we introduced dimensionless and rescaled functions and the radial coordinate:

$$\{ z, a, \sigma, w_{c2}, w_{a2}, K_1, g \} \quad \Rightarrow \quad \{ \hat{z}, \hat{a}, \hat{\sigma}, \hat{w}_{c2}, \hat{\omega}_{a2}, \hat{K}_1, \hat{g} \};$$

$$z = HP g_s^{1/2} \hat{z}, \quad a = H^2 P g_s^{1/2} \hat{a}, \quad \sigma = H P^{1/2} g_s^{1/4} \hat{\sigma},$$

$$w_{c2,a2} = P g_s^{1/2} \hat{w}_{c2,a2}, \quad K_1 = P^2 g_s \hat{K}_1, \quad g = g_s \hat{g}.$$

In the last equality in (4.7) we used expressions for $G_5$ (2.8) and $P$ (2.7). We compute entanglement entropy in different computational schemes; results must agree, provided we compare dimensionless and rescaled quantities,

$$s_{ent} = H^3 P^4 g_s^2 \hat{s}_{ent}.$$

(4.9)

Explicitly,

$$\text{Scheme I:} \quad \ln \frac{H^2}{\Lambda^2} = k_s, \quad \hat{s}_{ent} = s_{ent};$$

$$\text{Scheme II:} \quad \ln \frac{H^2}{\Lambda^2} = \frac{1}{b} + \ln b, \quad \hat{s}_{ent} = \frac{1}{b^2} s_{ent};$$

$$\text{Scheme III:} \quad \ln \frac{H^2}{\Lambda^2} = \frac{1}{4} + \ln \alpha, \quad \hat{s}_{ent} = \frac{1}{\alpha^{3/2}} s_{ent}.$$

(4.10)

Following (4.10), we collect (subset of the) results of $(4G_5 \hat{s}_{ent})$ as functions of $\ln \frac{H^2}{\Lambda^2}$ in different computational schemes in fig. 6: SchemeI (blue curves), SchemeII (red curves) and Scheme III (green curves). The accuracy of the collapsed results in different schemes is highlighted in fig. 7.

4.2 TypeA$_s$ de Sitter vacua in the conformal limit

To study the conformal limit it is convenient to use the computational scheme SchemeII (see (C.6)), i.e., we use the symmetry transformations SFG2-SFG4 of (B.13)-(B.15) to set $H = g_s = K_0 = 1$ and allow $b \equiv P^2$ to vary. The FG frame equations of motion
Figure 6: Left panel: entanglement entropy $\hat{s}_{\text{ent}}$ \((4.9)\) of TypeA\(_{s}\) de Sitter vacua of cascading gauge theory as functions of $\ln \frac{H^2}{\Lambda^2}$ in different computational schemes \((C.6)\): SchemeI (blue), SchemeII (red) and Scheme III (green). Right panel: entanglement entropy $\hat{s}_{\text{ent}}$ \((4.9)\) for small values of $\frac{H^2}{\Lambda^2}$ — at the limit of validity of the supergravity approximation, see section 4.3.

\([B.3]-(B.11)\) describing TypeA\(_{s}\) vacua (see also \((B.38)\)) can be solved perturbatively as a series expansion in $b$:

\[
\begin{align*}
  f_2 &= (1 + \rho) \left( 1 + \sum_{n=1}^{\infty} b^n f_{2n}(\rho) \right), & f_3 &= (1 + \rho) \left( 1 + \sum_{n=1}^{\infty} b^n f_{3n}(\rho) \right), \\
  h &= \frac{1}{4(1 + \rho)^2} \left( 1 + \sum_{n=1}^{\infty} b^n h_n(\rho) \right), & K &= 1 + \sum_{n=1}^{\infty} b^n k_n(\rho), & g &= 1 + \sum_{n=1}^{\infty} b^n g_n(\rho).
\end{align*}
\]

\((4.11)\)

Explicit equations for \(\{f_{2n}, f_{3n}, h_n, k_n, g_n\}\) for \(n = 1, 2\) along with the UV/IR asymptotics are presented in appendix \([D.1]\). Numerically solving these equations we find perturbative in $b$ predictions for the UV/IR parameters \((4.1)\). As explained in appendix \([C.2]\) we also need the FG frame parameter $s^h_0$, see \((B.68)\). Given \((4.11)\) we find from \((C.15)\)

\[
\begin{align*}
  s^h_0 &= \sqrt{2} \left( 1 + \frac{b}{4} \int_0^\infty ds \frac{h_1}{1 + s} + \frac{b^2}{32} \int_0^\infty ds \frac{8(1 + s) h_2 - (1 + 2s) h_1^2}{(1 + s)^2} + O(b^3) \right) \\
  &\equiv \sqrt{2} \left( 1 + s^h_{0,1} b + s^h_{0,2} b^2 + O(b^3) \right).
\end{align*}
\]

\((4.12)\)

Using results of appendix \([4.2]\) we evaluate the integrals in \((4.12)\) to find

\[
\begin{align*}
  s^h_{0,1} &= 0.828534, & s^h_{0,2} &= -0.284396.
\end{align*}
\]

\((4.13)\)
Figs. 8-9 present comparison of the results for the IR parameters \( \{ f_{2,0}^h, f_{3,0}^h, K_0^h, g_0^h \} \) and \( s_0^h \) in the computational SchemeII (blues curves), and independent perturbative \( \mathcal{O}(b) \) (red curves) and \( \mathcal{O}(b^2) \) (green curves) computations. The agreement is excellent.

Following appendix B.2 we convert perturbative FG frame construction (4.11) to EF frame:

\[
a = -z(1 - z) \left(1 + \sum_{n=1}^{\infty} b^n a_n(z)\right), \quad \sigma = \sqrt{2}(1 - z) \left(1 + \sum_{n=1}^{\infty} b^n s_n(z)\right),
\]
\[
w_{c2} = \frac{1}{2} \left(1 + \sum_{n=1}^{\infty} b^n w_{c2n}(z)\right), \quad w_{a2} = \frac{1}{2} \left(1 + \sum_{n=1}^{\infty} b^n w_{a2n}(z)\right), \quad (4.14)
\]
\[
K = 1 + \sum_{n=1}^{\infty} b^n k_n(z), \quad g = 1 + \sum_{n=1}^{\infty} b^n g_n(z).
\]

Explicit equations for \( \{ a_n, s_n, v_n \equiv w_{c2n} + 4w_{a2n}, w_{a2n}, k_n, g_n \} \) for \( n = 1, 2 \) along with the initial conditions are presented in appendix D.2. The equations for \( k_1 \) and \( g_1 \) (D.18 and D.23 correspondingly) can be solved analytically; in fact the solutions are just the FG \( \rightarrow \) EF frame transformations of (D.13) and (D.15):

\[
k_1 = \frac{z^2 - z + 1}{4z(z - 1)} - \frac{1}{4} - 4 \ln 2 + \frac{16z^3 - 24z^2 + 6z + 1}{4z^{3/2}(1 - z)^{3/2}} \arctan \sqrt{\frac{z}{1 - z}}, \quad (4.15)
\]
\[
g_1 = -\frac{13z^4 - 26z^3 + 29z^2 - 16z + 1}{32z^2(z - 1)^2} + \frac{13}{32} - \frac{2z - 1}{16z^{5/2}(1 - z)^{5/2}} \arctan \sqrt{\frac{z}{1 - z}}, \quad \arctan^2 \sqrt{\frac{z}{1 - z}} \quad (4.16)
\]
Figure 8: Infrared parameters \( \{f^h_{2,0}, f^h_{3,0}, K^h_0, g^h_0\} \) in the conformal limit \( b \to 0 \). Blue curves: results in computational scheme SchemeII; red curves: perturbative approximation to order \( \mathcal{O}(b) \); green curves: perturbative approximation to order \( \mathcal{O}(b^2) \); see (D.11) with (D.17).

We will show now that the location of the AH \( z_{AH} \), as determined from the zero of the AH location function \( L_{AH} \) (4.8), is

\[
1 - z_{AH} = \mathcal{O}(b^{1/4}) ,
\]

and can be determined analytically (in perturbative expansion in \( b \)) as it is controlled by the singularities of the EOMs (D.19)-(D.22) and (D.25)-(D.29) as \( u \equiv 1 - z \to 0_+ \), provided we use (4.13) and (4.16). From (4.15), (4.16):

\[
k_1 = -\frac{\pi}{8} u^{-3/2} - \frac{15\pi}{16} u^{-1/2} + \mathcal{O}(u^{0/2}), \quad g_1 = -\frac{\pi^2}{128} u^{-3} - \frac{15\pi^2}{128} u^{-2} + \mathcal{O}(u^{-3/2}),
\]

leading to (from direct asymptotic analysis of (D.19)-(D.22) and (D.25)-(D.29))

\[
v_1 = -\frac{3\pi^2}{256} u^{-3} - \frac{51\pi^2}{256} u^{-2} + \mathcal{O}(u^{-3/2}), \quad a_1 = \frac{3\pi^2}{1024} u^{-3} - \frac{177\pi^2}{1024} u^{-2} + \mathcal{O}(u^{-3/2}),
\]

\[
s_1 = \frac{\pi^2}{512} u^{-3} - \frac{33\pi^2}{256} u^{-2} + \mathcal{O}(u^{-3/2}), \quad w_{a1} = -\frac{\pi^2}{256} u^{-3} + \frac{9\pi^2}{256} u^{-2} + \mathcal{O}(u^{-3/2}),
\]

(4.19)

\(^{18}\)Subleading terms depend on coefficients that have to be determined numerically.
In fact, from the general structure of the perturbative equations we expect

\[ k_n = \mathcal{O}(u^{-3n+3/2}), \quad \{a, s, v, w_{a2}, g\}_n = \mathcal{O}(u^{-3n}), \]  

(4.21)

so that

\[ b^n k_n \bigg|_{u = u_{AH} = \mathcal{O}(b^{1/4})} = \mathcal{O}(b^{n/4+3/8}), \quad b^n \{a, s, v, w_{a2}, g\}_n \bigg|_{u = u_{AH} = \mathcal{O}(b^{1/4})} = \mathcal{O}(b^{n/4}), \]  

(4.22)

rendering successive higher order perturbative corrections in (4.14) at \( z = z_{AH} \) small despite the singular behavior of \( \{a_n, s_n, w_{c2n}, w_{a2n}, k_n, g_n\} \) in this limit.\(^{19}\)

\(^{19}\)This is similar to the behavior of the phenomenological model \[24\] in the conformal limit.
Figure 10: Location of the apparent horizon $z_{AH}$ (left panel) and the entanglement entropy $s_{ent}$ (right panel) of TypeA$_s$ de Sitter vacua in the conformal limit $b \to 0$. Blue curves: results in computational scheme SchemeII; red curves: leading perturbative approximation; green curves: next-to-leading perturbative approximation, see (4.24) and (4.25).

Given (4.19) and (4.20) we find from (C.8):

$$
\mathcal{L}_{AH}(u \equiv 1 - z) = \frac{3}{2} u^3 \left( u + b \left( -\frac{3\pi^2}{1024} u^{-3} - \frac{\pi^2}{64} u^{-2} + \mathcal{O}(u^{-3/2}) \right) 
+ b^2 \left( 0 \cdot u^{-6} + \frac{349\pi^4}{3932160} u^{-5} + \mathcal{O}(u^{-9/2}) \right) + \mathcal{O}(b^3 u^{-9}) \right),
$$

(4.23)

so that the first zero of the apparent horizon location function occurs at

$$
1 - z_{AH} = u_{AH} = \frac{1}{8} 3^{1/4} (2\pi)^{1/2} b^{1/4} \left( 1 + \frac{1}{6} 3^{1/4} (2\pi)^{1/2} b^{1/4} + \mathcal{O}(b^{1/2}) \right).
$$

(4.24)

From (3.32) we find perturbative predictions in the conformal limit for the TypeA$_s$ de Sitter vacua entanglement entropy:

$$
4G_5 s_{ent} = \frac{1}{1024} 3^{3/4} (2\pi)^{3/2} b^{3/4} \left( 1 + \frac{1}{2} 3^{1/4} (2\pi)^{1/2} b^{1/4} + \mathcal{O}(b^{1/2}) \right).
$$

(4.25)

In fig. 10 we compare numerical results for $z_{AH}$ and $s_{ent}$ in computational scheme SchemeII (blue curves) with the perturbative predictions (4.24) and (4.25) at leading (red curves) and next-to-leading (green curves) orders in the conformal limit: $b \to 0$.

Restoring dimensional parameters, from (4.25),

$$
\left. s_{ent} \right|_{\text{TypeA}_s} \propto H^3 \left( \ln \frac{H^2}{\Lambda^2} \right)^{-3/4} \quad \text{as} \quad H \gg \Lambda.
$$

(4.26)
Figure 11: Left panel: Kretschmann scalar of (2.13) evaluated at the apparent horizon as functions of $\ln \frac{H^2}{\Lambda^2}$ in different computation schemes (C.6): Scheme I (blue), Scheme II (red) and Scheme III (green). Right panel: we use order-3 polynomial fit (orange dashed curve) and order-4 polynomial fit (black dashed curve) to $\frac{1}{\hat{\kappa}_{AH}}$, see (4.29).

4.3 Validity of supergravity approximation for TypeA, vacua

Results for the entanglement entropy $s_{ent}$ of TypeA, de Sitter vacua of cascading gauge theory are presented in section 4.1, see fig. 6. Notice that it is a monotonically decreasing function of $\frac{H^2}{\Lambda^2}$. We have been able to obtain reliable numerical results for

$$\ln \frac{H^2}{\Lambda^2} \geq -0.59 \quad \Rightarrow \quad 4G_5 \, s_{ent} \gtrsim 4.1 \times 10^{-4} .$$

(4.27)

Besides numerical (technical) difficulties associated with construction of these vacua, there are conceptual ones, associated with the breakdown of the supergravity approximation — the effective action (2.1) becomes less reliable as the background space-time curvature of (2.13) grows. In fig. 11 (left panel) we present the Kretschmann scalar of (2.13) evaluated at the apparent horizon in different computations schemes, see appendix E.

Scheme I: \[ \ln \frac{H^2}{\Lambda^2} = k_s, \quad \hat{K} = K; \]

Scheme II: \[ \ln \frac{H^2}{\Lambda^2} = \frac{1}{b} + \ln b, \quad \hat{K} = bK; \]

(4.28)

Scheme III: \[ \ln \frac{H^2}{\Lambda^2} = \frac{1}{4} + \ln \alpha, \quad \hat{K} = K. \]

Notice the fast growth of $\hat{\kappa}_{AH}$ for small values of $\frac{H^2}{\Lambda^2}$ — in fig. 11 (right panel) we fit the values of $\frac{1}{\hat{\kappa}_{AH}}$ with order-3 (orange dashed curve) and order-4 (black dashed curve)
Figure 12: The curvature growth at the apparent horizon of the TypeA de Sitter vacua gravitational dual for small $\frac{H^2}{\Lambda^2}$ is due to collapsing the compact manifold: the size of deformed $T^{1,1}$, see (4.31) (left panel). Right panel: the $T^{1,1}$ deformation parameter $\delta_{T^{1,1}}$, see (4.32). Results are presented in different computation schemes (C.6): SchemeI (blue), SchemeII (red) and Scheme III (green).

The fits suggest that the curvature is divergent at

$$\ln \frac{H^2}{\Lambda^2} \bigg|_{\text{orange fit}} \approx -0.64, \quad \ln \frac{H^2}{\Lambda^2} \bigg|_{\text{black fit}} \approx -0.72. \quad (4.29)$$

We take (4.29) as an indication that TypeA vacua do not exist for

$$\ln \frac{(\text{H}^s_{\text{min}})^2}{\Lambda^2} \lesssim -0.8 \quad \Rightarrow \quad \text{H}^s_{\text{min}} \lesssim 0.7\Lambda. \quad (4.30)$$

In fig. 12 (left panel) we identify the rapid curvature growth with the fact that the size of (deformed) $T^{1,1}$, $R_{T^{1,1}}^2$, evaluated at the apparent horizon

$$R_{T^{1,1}}^2 \equiv w_{\alpha 2}\bigg|_{\text{AH}} = P g_s^{1/2} \hat{\omega}_{\alpha 2}\bigg|_{\text{AH}}, \quad (4.31)$$

becomes vanishingly small in string units, $P \propto M\alpha' = M \ell_s^2$. Note that in the limit $R_{T^{1,1}}^2 \to 0$ TypeA vacua entanglement entropy vanishes, see (4.7). Right panel shows the deformation parameter $\delta_{T^{1,1}}$ of the $T^{1,1}$: the size of the $U(1)$ fiber compare to the $S^2 \times S^2$ base,

$$\delta_{T^{1,1}} \equiv 1 - \frac{w_{c 2}^2}{w_{\alpha 2}^2}\bigg|_{\text{AH}} = 1 - \frac{\hat{\omega}_{c 2}^2}{\hat{\omega}_{\alpha 2}^2}\bigg|_{\text{AH}}, \quad (4.32)$$

\footnote{It would be interesting to rigorously establish this.}
5 Type\(_{A_b}\) de Sitter vacua

Type\(_{A_b}\) vacua have the same topology in Euclidean FG frame as Type\(_{A_s}\) vacua \([1.22]\); they differ in global symmetry: Type\(_{A_s}\) vacua have unbroken \(U(1)\) chiral symmetry (in the supergravity approximation), while the latter symmetry is broken \textit{spontaneously} to \(Z_2\) in Type\(_{A_b}\) vacua. The following table highlights the differences between the dual backgrounds in FG frame and EF frame:

| Type      | chiral symmetry | FG frame \([2.12]\) | EF frame \([2.13]\) | fluxes \([2.11]\) |
|-----------|-----------------|---------------------|---------------------|-----------------|
| Type\(_{A_s}\) | \(U(1)\)        | \(f_a = f_b\)       | \(w_{a2} = w_{b2}\) | \(K_1 = K_3 \& K_2 = 1\) |
| Type\(_{A_b}\) | \(Z_2\)         | \(f_a \neq f_b\)   | \(w_{a2} \neq w_{b2}\) | \(K_1 \neq K_3 \& K_2 \neq 1\) |

Table 1: Type\(_A\) de Sitter vacua with broken/unbroken \((b / s)\) chiral symmetry.

Unlike Type\(_{A_s}\) vacua, Type\(_{A_b}\) vacua have never been constructed in the literature before — morally, they are similar to Klebanov-Strassler black holes, constructed only recently \([14]\). We begin in section 5.1 with perturbative construction of Type\(_{A_b}\) vacua. Specifically, we study static linearized perturbations about Type\(_{A_s}\) vacua responsible for the chiral symmetry breaking \(U(1) \rightarrow Z_2\). The symmetry breaking is associated with three operators \(O_\alpha^{a=1,2}\) and \(O_7\) (see section 2) developing nonzero expectation values. We break the chiral symmetry \textit{explicitly}, by turning on a non-normalizable component for one of the dim-3 operators \(a\) (a mass term for one of the gaugino bilinears). We vary \(\frac{H^2}{\Lambda^2}\) keeping the gaugino mass parameter fixed and nonzero — the signature of the \textit{spontaneous} chiral symmetry breaking is the divergence of all the condensates \(O_3^{\alpha=1,2}\) and \(O_7\) for a particular value of \(\frac{H^2}{\Lambda^2}\), see fig. 13. Once the bifurcation point of Type\(_{A_b}\) vacua off Type\(_{A_s}\) vacua is identified as a function of \(\frac{H^2}{\Lambda^2}\), we construct fully nonlinear solution with spontaneous symmetry breaking slowing increasing the amplitudes of the symmetry breaking expectation values, using the linearized solution as a seed. Numerical results for Type\(_{A_b}\) vacua are presented in section 5.2, in particular the results for the entanglement entropy \(s_{\text{ent}}\) \(_{\text{Type\(_{A_b}\)}}\) compare to the entanglement entropy \(s_{\text{ent}}\) \(_{\text{Type\(_{A_s}\)}}\) at corresponding values of \(\frac{H^2}{\Lambda^2}\) are presented in fig. 21. Validity of supergravity approximation for Type\(_{A_b}\) vacua is a subject of section 5.3.

\(^{21}\) This was discussed earlier in \([13]\).
5.1 TypeA \textsubscript{b} vacua from perturbative chiral symmetry breaking of TypeA \textsubscript{s} vacua

We will use computational scheme SchemeI (C.6). Consider static, linearized chiral symmetry breaking fluctuation about TypeA \textsubscript{s} in FG frame, see table [1]:

\[ f_a = f_3 + \delta f, \quad f_b = f_3 - \delta f, \quad K_1 = K + \delta k_1, \quad K_2 = 1 + \delta k_2, \quad K_1 = K - \delta k_1, \]

with the remaining metric functions and the string coupling as in TypeA \textsubscript{s} vacua, \textit{i.e.}, \{f_c = f_2, h, g\}. It is straightforward to verify that truncation to \{\delta f, \delta k_{1,2}\} is consistent (at the linearized level). Equations of motion for the fluctuations and their asymptotic expansions in the UV (\(\rho \to 0\)) and the IR (\(y = \frac{1}{\rho}\)) are collected in appendix [F]. Once the non-normalizable coefficient (the explicit chiral symmetry breaking parameter, \textit{i.e.}, the gaugino mass term) is fixed to \(\delta f_{1,0} = 1\), the expansions are characterized by 6 UV/IR parameters

\[ \text{UV} : \quad \{\delta f_{3,0}, \delta k_{1,3,0}, \delta f_{7,0}\} ; \]
\[ \text{IR} : \quad \{\delta f_{h,0}, \delta k_{1,0}, \delta k_{2,0}\} , \]

which is the correct number of parameters to find a unique solution of 3 second-order differential equations (F.1)-(F.3) for \{\delta f, \delta k_{1,2}\} on the TypeA \textsubscript{s} background parameterized by \(k_s\).

In fig. [13] we assemble results for the fluctuation parameters (5.2) as \(k_s\) label of TypeA \textsubscript{s} vacua is varied. A signature of the spontaneous symmetry breaking is the divergence of all the parameters, once the scale of the explicit chiral symmetry breaking, \textit{i.e.}, the non-normalizable parameter \(\delta f_{1,0}\), is kept fixed. This occurs at

\[ \ln \left( \frac{H_{\text{min}}^{b}}{\Lambda^2} \right)^2 = k_s^{\text{crit}} = -16363(2) \quad \implies \quad H_{\text{min}}^{b} = 0.92(1)\Lambda , \]

represented by vertical dashed red lines. We denote the critical value of \(H\) corresponding to \(k_s^{\text{crit}}\) as \(H_{\text{min}}^{b}\) — we will see in section 5.2 that TypeA \textsubscript{b} vacua exist only for \(H \geq H_{\text{min}}^{b}\), hence the name. The value of \(k_s^{\text{crit}}\) can be computed separately of each of the parameters — the fractional differences are of order \(\propto 10^{-6}\), excepts for

\[ \left( \frac{\frac{k_s^{\text{crit}}}{\delta f_{1,0}}}{\frac{\delta f_{7,0}}{\delta f_{3,0}}} - 1 \right) \propto 10^{-4} . \]
Figure 13: Parameters \( \{ \delta f_{3,0}, \delta k_{1,3,0}, \delta f_{7,0}, \delta f_{0}^{h}, \delta k_{1,0}^{h}, \delta k_{2,0}^{h} \} \) of the chiral symmetry breaking fluctuations over TypeA\(_{s}\) vacua parameterized by \( k_{s} \), evaluated at fixed explicit chiral symmetry breaking scale \( \delta f_{1,0} = 1 \), diverge at \( k_{s}^{\text{crit}} \) (5.3), indicated by a vertical red dashed line. \( k_{s}^{\text{crit}} \) identifies the bifurcation point of spontaneous symmetry broken TypeA\(_{b}\) de Sitter vacua off chirally symmetric TypeA\(_{s}\) de Sitter vacua parameterized by \( \ln \Lambda_{b}^{2}/\Lambda^{2} \).

To use the critical fluctuations as a seed for TypeA\(_{b}\) vacua, we need to know the 'susceptibilities’

\[
\left\{ \chi_{k_{1,3,0}}, \chi_{f_{7,0}}, \chi_{f_{0}^{h}}, \chi_{k_{1,0}^{h}}, \chi_{k_{2,0}^{h}} \right\} \equiv \lim_{k_{s} \to k_{s}^{\text{crit}}} \left\{ \frac{\delta k_{1,3,0}}{\delta f_{3,0}}, \frac{\delta f_{7,0}}{\delta f_{3,0}}, \frac{\delta f_{0}^{h}}{\delta f_{3,0}}, \frac{\delta k_{1,0}^{h}}{\delta f_{3,0}}, \frac{\delta k_{2,0}^{h}}{\delta f_{3,0}} \right\} .
\]

(5.5)

In fig. [14] we present susceptibilities \( \chi_{k_{1,3,0}} \) and \( \chi_{f_{0}^{h}} \) — notice that they are finite at \( k_{s}^{\text{crit}} \), represented by vertical dashed red lines. The other susceptibilities are finite as well; we find:

\[
\begin{align*}
\chi_{k_{1,3,0}} &= 0.8749(7), & \chi_{f_{7,0}} &= -0.2373(6), & \chi_{f_{0}^{h}} &= 5.230(0), \\
\chi_{k_{1,0}^{h}} &= 0.3034(2), & \chi_{k_{2,0}^{h}} &= -18.12(6).
\end{align*}
\]

(5.6)

Given (5.6), fully nonlinear TypeA\(_{b}\) vacua, with \( k_{s} \) close to \( k_{s}^{\text{crit}} \), can be constructed following the same procedure as the one employed in construction of Klebanov-Strassler black hole in [14]. We highlight the main steps:

- We set \( k_{s} = k_{s}^{\text{crit}} \) and compute the corresponding TypeA\(_{s}\) vacuum. This vacuum is characterized by (see (B.44) and (B.46))

\[
\text{UV : } \left\{ K_{0} = k_{s}^{\text{crit}}, H = 1, g_{s} = 1, f_{2,1}^{\text{crit}}, f_{4,0}^{\text{crit}}, f_{2,4}^{\text{crit}}, f_{2,6}^{\text{crit}}, f_{2,8}^{\text{crit}} \right\}; \\
\text{IR : } \left\{ f_{2,0}^{h,\text{crit}}, f_{3,0}^{h,\text{crit}}, K_{0}^{h,\text{crit}}, g_{0}^{h,\text{crit}} \right\}.
\]

(5.7)
Figure 14: Sample susceptibilities, see (5.5), of the linearized chiral symmetry breaking fluctuations. The red dashed vertical line denotes \( k_{s, \text{crit}} \), see (5.3).

Figure 15: Sample of the UV parameters of TypeA\(_b\) de Sitter vacua constructed from the 'seed' (5.11). The linearized approximations in \( \lambda \) are represented by dashed red lines.

Next, we use (B.47)-(B.51) to compute the corresponding

\[
\text{UV : } \{ f_{a,1,0}^{s, \text{crit}}, f_{a,3,0}^{s, \text{crit}}, k_{2,3,0}^{s, \text{crit}}, g_{4,0}, f_{c,4,0}^{s, \text{crit}}, f_{a,6,0}, f_{a,7,0}, f_{a,8,0} \} ;
\]

\[
\text{IR : } \{ f_{a,0}^{h, s, \text{crit}}, f_{b,0}^{h, s, \text{crit}}, f_{c,0}^{h, s, \text{crit}}, K_{1,0}^{h, s, \text{crit}}, K_{2,0}^{h, s, \text{crit}}, K_{3,0}^{h, s, \text{crit}}, g_{0}^{h, s, \text{crit}} \} .
\]

We use superscript * to indicate that UV/IR parameters of TypeA\(_b\) vacua (B.25) and (B.30) are obtained from the critical TypeA\(_s\) vacuum.

- Let’s denote the amplitude of the symmetry breaking condensate (see (5.1))

\[
\delta f_{3,0} \equiv \frac{1}{2} (f_{a,3,0} - f_{b,3,0}) = \lambda .
\]

Then,

\[
\{ \delta k_{1,3,0}, \delta f_{7,0}^{h}, \delta f_{0}^{h}, \delta k_{1,0}^{h}, \delta k_{2,0}^{h} \} = \lambda \{ \chi_{k_{1,3,0}}, \chi_{f_{7,0}^{h}}, \chi_{f_{0}^{h}}, \chi_{k_{1,0}^{h}}, \chi_{k_{2,0}^{h}} \} + \mathcal{O}(\lambda^2) .
\]

(5.10)
Using (5.1) and (F.4)-(F.6), (F.8), with \( \delta f_{1,0} = 0 \), in asymptotic expansions (B.17)-(B.24) and (B.30) we find

\[
\begin{align*}
k_s &= k_s^{crit} + \mathcal{O}(\lambda^2), \\
f_{a,1,0} &= f_{a,1,0}^{s, crit} + \mathcal{O}(\lambda^2), \\
f_{a,3,0} &= f_{a,3,0}^{s, crit} + \lambda + \mathcal{O}(\lambda^2), \\
k_{2,3,0} &= k_{2,3,0}^{s, crit} + \lambda \left(1 - \frac{3}{2} \chi_{k_{1,3,0}}\right) + \mathcal{O}(\lambda^2), \\
g_{4,0} &= g_{4,0}^{s, crit} + \mathcal{O}(\lambda^2), \\
f_{c,4,0} &= f_{c,4,0}^{s, crit} + \mathcal{O}(\lambda^2), \\
f_{a,6,0} &= f_{a,6,0}^{s, crit} - \frac{f_{2,1,0}^{crit}}{64} \left(8(f_{2,1,0}^{crit})^2 + 18\chi_{k_{1,3,0}} + 12k_s^{crit} - 35\right) \lambda + \mathcal{O}(\lambda^2), \\
f_{a,7,0} &= f_{a,7,0}^{s, crit} + \lambda \chi_{f_{7,0}} + \mathcal{O}(\lambda^2), \\
f_{a,8,0} &= f_{a,8,0}^{s, crit} - \frac{f_{2,1,0}^{crit}}{1536} \left(550 - 192(f_{2,1,0}^{crit})^4 - 720\chi_{k_{1,3,0}}(f_{2,1,0}^{crit})^2 - 480(f_{2,1,0}^{crit})^2k_s^{crit}ight. \\
&\quad \left. + 36\chi_{k_{1,3,0}}k_s^{crit} + 1184(f_{2,1,0}^{crit})^2 + 3840\chi_{f_{7,0}} - 45\chi_{k_{1,3,0}} + 2304f_{4,0}^{crit} + 21k_s^{crit}\right) \lambda \\
&\quad + \mathcal{O}(\lambda^2), \\
f_{a,0} &= f_{a,0}^{h, s, crit} + \chi_{f_{0}^{h}}^b \lambda + \mathcal{O}(\lambda^2), \\
f_{b,0} &= f_{b,0}^{h, s, crit} - \chi_{f_{0}^{h}}^b \lambda + \mathcal{O}(\lambda^2), \\
f_{c,0} &= f_{c,0}^{h, s, crit} + \mathcal{O}(\lambda^2), \\
K_{1,0}^h &= K_{1,0}^{h, s, crit} + \chi_{k_{1,0}^h} \lambda + \mathcal{O}(\lambda^2), \\
K_{2,0}^h &= K_{2,0}^{h, s, crit} + \chi_{k_{2,0}^h} \lambda + \mathcal{O}(\lambda^2), \\
K_{3,0}^h &= K_{3,0}^{h, s, crit} - \chi_{k_{3,0}^h} \lambda + \mathcal{O}(\lambda^2), \\
g_0^h &= g_0^{h, s, crit} + \mathcal{O}(\lambda^2).
\end{align*}
\]

(5.11)

We construct fully nonlinear in \( \lambda \) TypeA\( _b \) vacua using the linearized approximation (5.11) as a seed. Select UV/IR parameters, along with the corresponding linearized approximations (dashed red lines) are shown in figs. 15,16.
5.2 Numerical results: Type\(A_b\)

Numerical construction of Type\(A_b\) vacua follows the steps of section 4.1. In FG frame, there are 8 second order equations (B.3)-(B.10) and 1 first order equation (B.11). The first order equation (B.11) involves (linearly) \(f_c'\) and can be used instead of one of the second order equations (namely, the one involving \(f_c''\)). Thus, altogether we have a coupled system of 7 second order ODEs (linear in \(\{f_{a,1,0}, f_{a,3,0}, h_{2,3,0}, f_{c,4,0}, f_{a,6,0}, f_{a,7,0}, f_{a,8,0}\}\)) and a single first order equation (linear in \(f_c'\)). As a result, a unique solution must be characterized by \(15 = 2 \times 7 + 1\) parameters; these are the UV/IR parameters

\[
\begin{align*}
\text{UV} : & \quad \{f_{a,1,0}, f_{a,3,0}, k_{2,3,0}, g_{4,0}, f_{c,4,0}, f_{a,6,0}, f_{a,7,0}, f_{a,8,0}\}; \\
\text{IR} : & \quad \{f_{a,0}^h, f_{b,0}^h, f_{c,0}^h, K_{1,0}^h, K_{2,0}^h, K_{3,0}^h, g_0^h\}.
\end{align*}
\]

(5.12)

It is rather challenging to find the solutions of the corresponding system of ODEs in 15-dimensional parameter space by brute force — fortunately, we already know some solutions which are close to \(k_{s,\text{crit}}\), see section 5.1.

As for the construction of Type\(A_s\) we use three different computation schemes, see appendix C.1. There are some differences though: both in SchemeII and SchemeIII we use as a pivot value \(22 K^\star_0 = -0.161344\). (5.13)

Numerical results must not depend on which computational scheme is adopted. We illustrate now that this is indeed the case using a sample of IR parameters in (5.12) as an example\(^{22}\). Comparison of the different computational schemes is done using dimensionless and rescaled quantities: \(\ln \frac{H^2}{\Lambda^2}\) (as a vacuum label) (C.2) and \(\{\hat{f}_{a,b,c,0}^h, \hat{K}_{1,2,3,0}^h, \hat{g}_0^h\}\) (C.4). Explicitly:

\[
\begin{align*}
\text{SchemeI} : & \quad \ln \frac{H^2}{\Lambda^2} = k_s, \quad \hat{f}_{a,b,c,0}^h = f_{a,b,c,0}^h, \quad \hat{K}^h_{1,2,3,0} = K^h_{1,2,3,0}, \quad \hat{g}_0^h = g_0^h; \\
\text{SchemeII} : & \quad \ln \frac{H^2}{\Lambda^2} = K^\star_0 / b + \ln b, \quad \hat{f}_{a,b,c,0}^h = \frac{1}{b^{1/2}} f_{a,b,c,0}^h, \quad \hat{K}^h_{1,2,3,0} = \frac{1}{b} K^h_{1,2,3,0}, \quad \hat{g}_0^h = g_0^h; \\
\text{SchemeIII} : & \quad \ln \frac{H^2}{\Lambda^2} = K^\star_0 / \alpha + \ln \alpha, \quad \hat{f}_{a,b,c,0}^h = \frac{1}{\alpha^{1/2}} f_{a,b,c,0}^h, \quad \hat{K}^h_{1,2,3,0} = K^h_{1,2,3,0}, \quad \hat{g}_0^h = g_0^h.
\end{align*}
\]

(5.14)

Following (5.14), we collect results of \(\{\hat{f}_{a,0}^h - \hat{f}_{b,0}^h, \hat{K}_{1,0}^h\}\) as functions of \(\ln \frac{H^2}{\Lambda^2}\) in different computational schemes in fig. 17: SchemeI (blue curves), SchemeII (red curves).\(^{22}\) As will be clear from the presented results this is a convenient value.\(^{23}\) The same is true for the rest of IR parameters and the UV parameters as well.

---

\(^{22}\)As will be clear from the presented results this is a convenient value.

\(^{23}\)The same is true for the rest of IR parameters and the UV parameters as well.
Figure 17: Infrared parameters \{\hat{f}_{a,0}^h - \hat{f}_{b,0}^h, \hat{K}_{1,0}^h\} of the Fefferman-Graham coordinate frame of TypeA_{b} de Sitter vacua of cascading gauge theory as functions of \(\ln \frac{H^2}{\Lambda^2}\) in different computational schemes (5.14): SchemeI (blue), SchemeII (red) and Scheme III (green).

and Scheme III (green curves). The accuracy of the collapsed results in different schemes is highlighted in fig. 18 for \(\hat{K}_{1,0}^h\) — the remaining parameters follow the same trend. Notice that TypeA_{b} vacua exist only for \(H \geq H^b_{\text{min}}\) (5.3); furthermore, in the limit \(H \to H^b_{\text{min}} + 0\), all the chiral symmetry breaking condensates (5.2) vanish as \(\propto (H - H^b_{\text{min}})^{1/2}\), typical for a spontaneous symmetry breaking with a mean-field exponent \(\frac{1}{2}\).

Next, FG frame TypeA_{b} de Sitter vacua have to be reinterpreted in EF frame, see appendix B.2. The diffeomorphism transformation is performed at the radial location as in (4.3). Details of numerical construction of EF frame vacua from FG frame vacua are collected in appendix C.2. An important quantity is the parameter \(s_0^h\), see (2.13), and (4.4). As with FG frame UV/IR parameters (5.12), results for \(s_0^h\) should not depend on the choice of the computational scheme, provided we compare properly dimensionless and rescaled quantities, i.e., \(\ln \frac{H^2}{\Lambda^2}\) and \(s_0^h\) (C.15),

\[
\begin{align*}
\text{SchemeI :} & & \ln \frac{H^2}{\Lambda^2} = k_s, \quad \hat{s}_0^h = s_0^h; \\
\text{SchemeII :} & & \ln \frac{H^2}{\Lambda^2} = \frac{K_0^*}{b} + \ln b, \quad \hat{s}_0^h = \frac{1}{b^{1/4}} s_0^h; \\
\text{SchemeIII :} & & \ln \frac{H^2}{\Lambda^2} = K_0^* + \ln \alpha, \quad \hat{s}_0^h = \frac{1}{\alpha^{1/2}} s_0^h. \\
\end{align*}
\]

Following (5.15), we collect (subset of the) results of \(\hat{s}_0^h\) as functions of \(\ln \frac{H^2}{\Lambda^2}\) in different computational schemes in fig. 19: SchemeI (blue curves), SchemeII (red curves) and
Scheme III (green curves). The accuracy of the collapsed results in different schemes is highlighted in fig. [20].

EF frame equations of motion (A.17)-(A.25) are solved subject to the initial conditions set by the asymptotic expansions (B.58)-(B.66) at $z = 0$. These equations have to be integrated on the interval

$$z \in [0, z_{AH}],$$

(5.16) where $z_{AH} = -r_{AH}$ is the location of the apparent horizon at asymptotically late times, see (3.32). To determine the location of the apparent horizon, along with integrating the gravitational background functions $\{a, \sigma, w_{a,b,c}, K_{1,2,3}, g\}$, we evaluate the AH location function $L_{AH}(z)$, see (C.8). AH is located at the first zero of this function for $z > 0$. 

Figure 18: Left panel: comparison of $\hat{K}_{1,0}^h$ (the computational scheme SchemeIII) with $\hat{K}_{1,0}^s$ (the computational scheme SchemeI). Right panel: comparison of $\hat{K}_{1,0}^h$ (the computational scheme SchemeII) with $\hat{K}_{1,0}^s$ (the computational scheme SchemeI).

Figure 19: Parameters $\hat{s}_0^h$ of TypeA$_s$ de Sitter vacua of cascading gauge theory as functions of $\ln \frac{H^2}{\Lambda^2}$ in different computational schemes (5.14): SchemeI (blue), SchemeII (red) and Scheme III (green).
After the AH is identified, Type $A_b$ vacua entanglement entropy is computed following (3.32):

$$ s_{ent} = \frac{H^2 P^4 g_s^2}{4 G_5} \left\{ \hat{\sigma}^3 \hat{w}_{a_2,b_2,c_2} \hat{\omega}_{a_2,b_2} \right\}_{\hat{z} = \hat{z}_{AH}} = \frac{3^5 M^4 g_s^2}{25\pi^3} H^3 \left\{ \hat{\sigma}^3 \hat{w}_{a_2,b_2} \hat{\omega}_{a_2} \right\}_{\hat{z} = \hat{z}_{AH}}, \quad (5.17) $$

where following (C.1) we introduced dimensionless and rescaled functions and the radial coordinate:

$$ \{ z, a, \sigma, w_{a_2,b_2,c_2}, K_{1,2,3}, g \} \quad \Rightarrow \quad \{ \hat{z}, \hat{a}, \hat{\sigma}, \hat{\omega}_{a_2,b_2,c_2}, \hat{K}_{1,2,3}, \hat{g} \}; $$

$$ z = H P g_s^{1/2} \hat{z}, \quad a = H^2 P g_s^{1/2} \hat{a}, \quad \sigma = H P^{1/2} g_s^{1/4} \hat{\sigma}, $$

$$ w_{a_2,b_2,c_2} = P g_s^{1/2} \hat{\omega}_{a_2,b_2,c_2}, \quad K_{1,3} = P^2 g_s \hat{K}_{1,3}, \quad K_2 = \hat{K}_2, \quad g = g_s \hat{g}. $$

In the last equality in (5.17) we used expressions for $G_5$ (2.8) and $P$ (2.7). We compute entanglement entropy in different computational schemes; results must agree, provided we compare dimensionless and rescaled quantities, see (4.9). Explicitly,

- Scheme I: $\ln \frac{H^2}{\Lambda^2} = k_s$, \quad $\hat{s}_{ent} = s_{ent}$;
- Scheme II: $\ln \frac{H^2}{\Lambda^2} = \frac{K_0}{b} + \ln b$, \quad $\hat{s}_{ent} = \frac{1}{b^2} s_{ent}$;
- Scheme III: $\ln \frac{H^2}{\Lambda^2} = K_0^* + \ln \alpha$, \quad $\hat{s}_{ent} = \frac{1}{\alpha^{3/2}} s_{ent}$.

Following (5.19), we collect (subset of the) results of $(4G_5 \hat{s}_{ent})$ as functions of $\ln \frac{H^2}{\Lambda^2}$ in different computational schemes in fig. 21: Scheme I (blue curves), Scheme II (red curves) and Scheme III (green curves). Additionally, we replot the results for the
entanglement entropy of TypeA\textsubscript{s} vacua (black curve). Fig. 21 is the main result of the paper: it demonstrates that the entanglement entropy of TypeA\textsubscript{b} vacua is larger than that of TypeA\textsubscript{s} vacua provided (the values \(H_{\text{min}}^b\) and \(H_{\text{max}}^b\) are denoted by vertical dashed magenta lines)

\[
H_{\text{min}}^b \leq H \leq H_{\text{max}}^b,
\]

where

\[
\frac{H_{\text{min}}^b}{\Lambda} = 0.92(1), \quad \frac{H_{\text{max}}^b}{\Lambda} = 0.92(5).
\]

This is an unexpected result, as it implies that SU\((N) \times SU(N + M)\) cascading gauge theory with a strong coupling scale \(\Lambda\) undergoes spontaneous chiral symmetry breaking in de Sitter space time with a Hubble constant \(H\) in the interval (5.20).
Figure 22: Left panel: comparison of $\hat{s}_{\text{ent}}$ (the computational scheme SchemeIII) with $\hat{s}_{\text{ent}}$ (the computational scheme SchemeI). Right panel: comparison of $\hat{s}_{\text{ent}}$ (the computational scheme SchemeII) with $\hat{s}_{\text{ent}}$ (the computational scheme SchemeI).

The accuracy of the collapsed results for TypeA\textsubscript{b} vacua in different schemes is highlighted in fig. [22]

5.3 Validity of supergravity approximation for TypeA\textsubscript{b} vacua

In this section we briefly comment on the validity of the supergravity approximation in construction of TypeA\textsubscript{b} vacua. In fig. [23] we present the Kretschmann scalar of (2.13) evaluated at the apparent horizon in different computations schemes for the TypeA\textsubscript{b} vacua, see appendix [E]

$$\text{Scheme I: } \ln \frac{H^2}{\Lambda^2} = k_s, \quad \hat{K} = K;$$

$$\text{Scheme II: } \ln \frac{H^2}{\Lambda^2} = \frac{K_0^*}{b} + \ln b, \quad \hat{K} = bK; \quad (5.22)$$

$$\text{Scheme III: } \ln \frac{H^2}{\Lambda^2} = K_0^* + \ln \alpha, \quad \hat{K} = K.$$

Vertical dashed magenta lines indicate the range of dominance of TypeA\textsubscript{b} vacua over TypeA\textsubscript{s}, see (5.20). Additionally, we replot the Kretschmann scalar of (2.13) evaluated at the apparent horizon for TypeA\textsubscript{s} vacua (black curve). $K_{AH}$ is the same for TypeA\textsubscript{b} and TypeA\textsubscript{s} vacua at $H = H_{\text{min}}^b$; the former is about 13 times larger for TypeA\textsubscript{b} vacuum at $H = H_{\text{max}}$ and continues to increase as $\frac{H}{\Lambda}$ increases. We do not study the breakdown of the supergravity approximation for TypeA\textsubscript{b} vacua for $H > H_{\text{max}}$, as these vacua are irrelevant.

46
Figure 23: Kretschmann scalar of \((2.13)\) evaluated at the apparent horizon as functions of \(\ln \frac{H^2}{\Lambda^2}\) for TypeA\(_b\) vacua in different computation schemes \((5.22)\): SchemeI (blue), SchemeII (red) and Scheme III (green). The black curve is the Kretschmann scalar of \((2.13)\) evaluated at the apparent horizon as a function of \(\ln \frac{H^2}{\Lambda^2}\) for TypeA\(_s\) vacua. Vertical dashed magenta lines indicate the range of dominance of TypeA\(_b\) vacua over TypeA\(_s\), see \((5.20)\).

6 TypeB de Sitter vacua

TypeB de Sitter vacua were studied previously in \([31]\). We showed in section \(3.4\) that the entanglement entropy of these vacua vanishes. Thus, these vacua can arise as late-time dynamical attractors of cascading gauge theory in de Sitter only when neither TypeA\(_s\) nor TypeA\(_b\) vacua exist (for the corresponding values \(\frac{H}{\Lambda}\)). Recall that TypeA\(_s\) vacua exist only for \(H \gtrsim H^{s}_{\text{min}}\) \((4.30)\), and TypeA\(_b\) vacua exist only when \(H \geq H^{b}_{\text{min}}\) \((5.21)\). In this section we establish that TypeB vacua do exist for \(H \lesssim H^{B}_{\text{max}}\) with \(H^{B}_{\text{max}} > \{H^{s}_{\text{min}}, H^{b}_{\text{min}}\}\), see \((1.33)\). In section \(6.1\) we present numerical results for TypeB vacua for generic values of \(\frac{H^2}{\Lambda^2}\). In section \(6.2\) we estimate \(H^{B}_{\text{max}}\) above which TypeB vacua construction in type IIB supergravity becomes unreliable/does not exist. We identify the source of breaking of the supergravity approximation.

6.1 Numerical results: TypeB

To establish the existence of TypeB vacua it is sufficient to construct them in FG frame \((2.12)\). The construction follows the steps implemented for TypeA\(_s\) vacua in section \(4.1\). There are 8 second order equations \((B.3)-(B.10)\) and 1 first order equation \((B.11)\). The first order equation \((B.11)\) involves (linearly) \(f'_c\) and can be used instead of one of the second order equations (namely, the one involving \(f''_c\)). Thus, altogether we
have a coupled system of 7 second order ODEs (linear in \{f''_a, f''_b, h''_a, K''_1, K''_2, K''_3, g''\}) and a single first order equation (linear in \(f'_c\)). As a result, a unique solution must be characterized by 15 = 2 × 7 + 1 parameters; these are the UV/IR parameters

\[
\begin{align*}
\text{UV} : & \quad \{a_0, a_3, f_0, a_0, b_0, c_0, c_4, f_4, a_6, f_7, a_8\}; \\
\text{IR} : & \quad \{h^h_0, h^h_1, h^h_2, h^h_3, h^h_4, h^h_5, h^h_6, g^h_0\}. 
\end{align*}
\]

It is rather challenging to find the solutions of the corresponding system of ODEs in 15-dimensional parameter space by brute force — fortunately, a special case of TypeB vacua, namely, the limit \(H \to 0\), is the supersymmetric Minkowski space-time Klebanov-Strassler solution [2], see appendix B.3 Using this extremal KS solution as a seed, we can construct TypeB vacua turning on the deformation parameter \(\alpha \equiv H^2\) in the ODEs (B.3)-(B.11).

To validate our results, we use two different computation schemes: SchemeI and
SchemeIII, see (C.6). Numerical results must not depend on which computational scheme is adopted. We illustrate now that this is indeed the case using a sample of IR parameters in (6.1) as an example\textsuperscript{24}. Comparison of the different computational schemes is done using dimensionless and rescaled quantities: $\ln \frac{H^2}{\Lambda^2}$ (as a vacuum label) and $\{\hat{f}_a^h, \hat{h}_0^h, \hat{k}_1^h, \hat{k}_2^h, \hat{k}_3^h, \hat{g}_0^h\}$ (C.5). Explicitly:

\begin{align*}
\text{SchemeI} : & \quad \ln \frac{H^2}{\Lambda^2} = k_s, \quad \hat{f}_a^h = f_a^h, \quad \hat{h}_0^h = h_0^h, \quad \hat{k}_{1,3}^h = k_{1,3}^h, \quad \hat{k}_{2,2}^h = k_{2,2}^h, \quad \hat{k}_{2,4}^h = k_{2,4}^h, \\
\text{SchemeIII} : & \quad \ln \frac{H^2}{\Lambda^2} = \frac{1}{4} + \ln \alpha, \quad \hat{f}_a^h = \frac{1}{\alpha} f_a^h, \quad \hat{h}_0^h = \alpha^2 h_0^h, \quad \hat{k}_{1,3}^h = \alpha^{3/2} k_{1,3}^h, \\
& \quad \hat{k}_{2,2}^h = \alpha k_{2,2}^h, \quad \hat{k}_{2,4}^h = \alpha^2 k_{2,4}^h, \quad \hat{k}_{3,1}^h = \alpha^{1/2} k_{3,1}^h, \quad \hat{g}_0^h = g_0^h.
\end{align*}

(6.2)

Fig. 24 presents all the IR parameters and select UV parameters ($f_{a,3,0}$ and $k_{2,3,0}$), see (6.1), of TypeB vacua in computational SchemeIII as functions of $\alpha$. Extremal KS parameters are represented by dashed horizontal red lines and must agree with the corresponding TypeB parameters at $\alpha = 0$. While negative values of $\alpha$ are not physical, we run numerical codes for $\alpha < 0$ to extract more precisely this comparison at $\alpha = 0$. Extremal KS parameters in computational SchemeIII can be determined from (B.78) and (B.79) provided we set

\[ K_0 = P^2 g_s \left( - \ln 3 + \frac{5}{3} \ln 2 - \frac{4}{3} \ln \epsilon - \frac{2}{3} \right) \bigg|_{\text{SchemeIII}} = \frac{1}{4}, \]

\[ \epsilon \bigg|_{\text{SchemeIII}} = \frac{2}{3} \times 6^{1/4} e^{-11/16}. \]

We find remarkable agreements, e.g.,

\[ \frac{f_{a,0}^h(\alpha = 0)}{f_{a,0}^h(KS)} - 1 \sim 5 \times 10^{-10}, \quad \frac{k_{2,4}^h(\alpha = 0)}{k_{2,4}^h(KS)} - 1 \sim 2 \times 10^{-10}. \]

(6.4)

The remaining parameters are validated at $\sim 10^{-6}$ level or better.

Following (6.2), we collect results of $\hat{k}_{1,3}^h$ as functions of $\ln \frac{H^2}{\Lambda^2}$ in different computational schemes in fig. 25: SchemeI (blue curves) and Scheme III (green curves) (left panel); the accuracy of the collapsed results in different schemes is highlighted in

\textsuperscript{24}The same is true for the rest of IR parameters and the UV parameters as well.
Figure 25: Left panel: infrared parameter $\hat{k}^h_{1,3}$ of the Fefferman-Graham coordinate frame of TypeB de Sitter vacua of cascading gauge theory as functions of $\ln \frac{H^2}{\Lambda^2}$ in different computational schemes (6.2): SchemeI (blue) and Scheme III (green). Right panel: comparison of $\hat{k}^h_{1,3}$ (the computational scheme SchemeIII) with $\hat{k}^h_{1,3}$ (the computational scheme SchemeI).

right panel. Comparison of the remaining parameters follows the same trend. Note the degradation in accuracy as $\frac{H}{\Lambda}$ increases — in section 6.2 we relate this to the breakdown of the supergravity approximation.

6.2 Validity of supergravity approximation for TypeB vacua

As clear from fig. 25 the accuracy in constructing TypeB vacua deteriorates as $H$ increases; we have been able to construct TypeB vacua for

$$\ln \frac{H^2}{\Lambda^2} \leq -0.06(8) \quad \Longrightarrow \quad H \leq H^B_{max} = 0.966(5)\Lambda. \quad (6.5)$$

Besides numerical (technical) difficulties associated with construction of these vacua, there are conceptual ones, associated with the breakdown of the supergravity approximation — the effective action (2.1) becomes less reliable as the background space-time curvature of (2.13) grows. In fig. 26 (left panel) we present the inverse Kretschmann scalar of (2.13) evaluated at the apparent horizon in different computations schemes, see appendix E, specifically (E.4):

$$\text{SchemeI:} \quad \ln \frac{H^2}{\Lambda^2} = k_s, \quad \dot{K} = K;$$

$$\text{SchemeIII:} \quad \ln \frac{H^2}{\Lambda^2} = \frac{1}{4} + \ln \alpha, \quad \dot{K} = K. \quad (6.6)$$
Figure 26: Left panel: Inverse Kretschmann scalar of (2.13) evaluated at the apparent horizon for TypeB vacua as functions of \( \ln \frac{H^2}{\Lambda^2} \) in different computation schemes (C.6): Scheme I (blue) and Scheme III (green). Horizontal red dashed line represents \( \frac{1}{\hat{K}_{AH}} \) for the extremal KS solution, which is recovered in the limit \( \frac{H}{\Lambda} \to 0 \). Right panel: the divergence of the Kretschmann scalar as \( H \to H_{max}^B \) is associated with the collapse of the 3-cycle, see (6.7). Vertical black dashed lines represent \( \frac{H_{max}^B}{\Lambda} \).

In the limit \( \frac{H}{\Lambda} \to 0 \) we recover the inverse Kretschmann scalar of the extremal KS solution (E.5), represented by a horizontal red dashed line. As \( H \) approached \( H_{max}^B \), represented by vertical dashed black line, the Kretschmann scalar at the AH of the holographic dual to TypeB de Sitter vacua of cascading gauge theory appears to grow faster than any polynomial of \( \Lambda/(H_{max}^B - H) \) — we take \( H_{max}^B \) in (6.5) as the limiting value for the existence of TypeB vacua. In the right panel of fig. 26 we associate the growth of the Kretschmann scalar in the limit \( H \to H_{max}^B \) with the collapse of the 3-cycle (the \( S^3 \) supporting the RR 3-form flux (2.6)) at the horizon, see (B.37),

\[
R^2_{S^3} = \frac{f^h_{a,0}(h^h_0)^{1/2}}{3} = P g_s^{1/2} \frac{f^h_{a,0}(h^h_0)^{1/2}}{3}, \tag{6.7}
\]

where in the second equality we used (C.5).

7 Conclusion

In this paper we presented a comprehensive analysis of the vacua structure of cascading gauge theory in de Sitter. Cascading gauge theory in Minkowski space-time is characterized by a single modulus \( g_s \) and the strong coupling scale \( \Lambda \); it confines with spontaneous breaking of the chiral symmetry. de Sitter space-time presents a new mass scale — the Hubble constant \( H \). There are three distinct types of de Sitter
vacua of the theory — TypeA \_s (resembling thermal deconfined states of KS theory with unbroken chiral symmetry), TypeA \_b (resembling thermal deconfined states of KS theory with spontaneously broken chiral symmetry) and TypeB (resembling thermal confined states of KS theory with spontaneously broken chiral symmetry)— with different (Euclidean) topology, and the global symmetry. All three types play a role of being an attractor of late-time de Sitter dynamics, depending on the interplay of the strong coupling scale \( \Lambda \) and the Hubble constant \( H \). We discover an intriguing pattern of the chiral symmetry breaking in the theory depending on the ratio \( \frac{H}{\Lambda} \). While it is natural to expect that the chiral symmetry is spontaneously broken for sufficiently small \( \frac{H}{\Lambda} \) (in fact, the extremal KS solution is a limiting case \( \frac{H}{\Lambda} \to 0 \)), we find that the chiral symmetry is spontaneously broken as well when \( H \in [H_{\text{min}}^b, H_{\text{max}}] \), with \( \{H_{\text{min}}^b, H_{\text{max}}\} \sim \Lambda \).

There is a number of open questions and future directions:
- We argued that vacua TypeA \_s do not exist for sufficiently small \( \frac{H}{\Lambda} \). It is important to rigorously establish this fact. Indeed, TypeA \_s vacua, unlike TypeB vacua, are characterized by nonzero entanglement entropy density, and thus, when exist, will always dominate over TypeB vacua as late-time dynamical attractors.
- We mentioned that TypeA \_b vacua resemble thermal states of deconfined cascading gauge theory with \( \mathbb{Z}_2 \) chiral symmetry. The holographic dual of these states is a Klebanov-Strassler black hole [14], which is unstable to local energy density perturbations — the sound waves in cascading gauge theory plasma. It would be interesting to study the fate of spatial inhomogeneities in TypeA \_b de Sitter vacua.
- Ideally, we would like to develop numerical simulations of the cascading gauge theory in de Sitter, akin to the model studied in [23]. As a first step, it would be interesting to compute the spectrum (the quasinormal modes) of chiral symmetry breaking fluctuations about TypeA \_s vacua for \( H \in [H_{\text{min}}^b, H_{\text{max}}] \).
- It is important to explore spontaneous symmetry breaking and the role played by the de Sitter vacuum entanglement entropy in other top-down examples of massive holography.

**Acknowledgments**

Research at Perimeter Institute is supported by the Government of Canada through Industry Canada and by the Province of Ontario through the Ministry of Research &
Innovation. This work was further supported by NSERC through the Discovery Grants program.

A EF frame equations of motion

Within Eddington-Finkelstein metric ansatz (with spatially homogeneous and isotropic background metric of the cascading gauge theory — $dx^2$)

$$ds^2_{10} = 2dt \ (dr - A dt) + \Omega^2 dx^2 + \Omega^2_1 g_3^2 + \Omega^2_2 (g_1^2 + g_3^2) + \Omega^2_3 (g_1^2 + g_2^2), \quad (A.1)$$

with

$$A = A(t, r), \ \Sigma = \Sigma(t, r), \ \Omega_i = \Omega_i(t, r), \ K_i = K_i(t, r), \ \Phi = \ln g(t, r), \quad (A.2)$$

we find from $(2.1)$ the following evolution ($\equiv \partial_r$ and $d_+ \equiv \partial_t + A \partial_r$):

$$0 = (d_+ \Sigma)' + \left( \frac{d_+ \Omega_2}{\Omega_2} + \frac{d_+ \Omega_3}{\Omega_3} + \frac{d_+ \Omega_1}{2\Omega_1} \right) \Sigma' + \left( \frac{\Omega_2'}{\Omega_2} + \frac{\Omega_3'}{\Omega_3} + 2 \frac{\Sigma'}{\Sigma} + \frac{\Omega_1'}{2\Omega_1} \right) d_+ \Sigma$$

$$- \frac{P^2 g \Sigma K_3^2}{1296 \Omega_3^2 \Omega_3^4} d_+ K_2 - \frac{\Sigma K_1^2}{1152 \Omega_1^2 P^2 g} d_+ K_1 - \frac{\Sigma K_3^2}{1152 \Omega_1^2 P^2 g} d_+ K_3 - \frac{\Sigma (K_1 - K_3)^2}{4608 \Omega_3^2 \Omega_3^2 \Omega_1^2 P^2 g}$$

$$- \frac{P^2 g K_2^2 \Sigma (\Omega_2^4 + \Omega_3^4)}{5184 \Omega_2^4 \Omega_1^2 \Omega_1^4} + \frac{P^2 g K_2 \Sigma}{1296 \Omega_2^4 \Omega_1^2} - \frac{P^2 g \Sigma}{1296 \Omega_2^4 \Omega_1^2} - \frac{\Sigma (K_1 K_2 - K_3 K_2 - K_1)^2}{373248 \Omega_1^4 \Omega_1^4 \Omega_1^4}, \quad (A.3)$$

$$0 = (d_+ \Omega_1)' + \left( \frac{3 \Sigma'}{2 \Sigma} + \frac{\Omega_2'}{\Omega_2} + \frac{\Omega_3'}{\Omega_3} \right) d_+ \Omega_1 + \left( \frac{d_+ \Omega_2}{\Omega_2} + \frac{d_+ \Omega_3}{\Omega_3} + \frac{3d_+ \Sigma}{2 \Sigma} \right) \Omega_1'$$

$$- \frac{K_1^2}{1152 P^2 g \Omega_3} d_+ K_1 - \frac{\Omega_1 P^2 g K_2^2}{1296 \Omega_3^2 \Omega_2} d_+ K_2 - \frac{\Omega_1 K_3^2}{1152 \Omega_3^2 P^2 g} d_+ K_3 + \frac{(K_3 - K_1)^2}{1536 \Omega_3^2 \Omega_2 \Omega_1 P^2 g}$$

$$+ \frac{(K_3 K_2 - K_1 K_2 + 2 K_1)^2}{373248 \Omega_1^4 \Omega_1^4 \Omega_1^4} - \frac{(2 \Omega_1^2 + \Omega_2^2 - \Omega_3^2)(2 \Omega_1^2 - \Omega_2^2 + \Omega_3^2)}{8 \Omega_3^2 \Omega_2^4 \Omega_1} + \frac{P^2 g K_2^2 (\Omega_1^4 + \Omega_2^4)}{1728 \Omega_3^2 \Omega_1^4 \Omega_1^4}$$

$$- \frac{P^2 g K_2}{432 \Omega_1 \Omega_1} + \frac{P^2 g}{432 \Omega_1 \Omega_1}, \quad (A.4)$$
\begin{align}
0 &= (d_+ \Omega_2)' + \left( \frac{\Omega_2'}{\Omega_2} + \frac{\Omega_3'}{\Omega_3} + \frac{3\Sigma'}{2\Sigma} + \frac{\Omega_1'}{2\Omega_1} \right) d_+ \Omega_2 + \left( \frac{d_+ \Omega_3}{\Omega_3} + \frac{d_+ \Omega_4}{2\Omega_1} + \frac{3d_+ \Sigma}{2\Sigma} \right) \Omega_2' \\
&+ \frac{P^2 gK_2}{1296\Omega_2^2\Omega_3^2} d_+ K_2 - \frac{\Omega_2 K_1'}{1152\Omega_3^4P^2g} d_+ K_1 + \frac{K_3'}{384\Omega_2^3P^2g} d_+ K_3 + \frac{(K_1 - K_3)^2}{4608\Omega_3^2\Omega_2^2\Omega_1^2P^2g} \\
&- \frac{K_2^2P^2g(\Omega_1' - 3\Omega_3')}{5184\Omega_2^3\Omega_1^4\Omega_3^4} - \frac{K_2P^2g}{432\Omega_2^3\Omega_1^2} + \frac{P^2g}{432\Omega_2^3\Omega_1^2} + \frac{(K_1 K_2 - K_3 K_2 - 2K_1)^2}{373248\Omega_2^3\Omega_1^2\Omega_3^4} \\
&+ \frac{4\Omega_1^4 - 8\Omega_1^2\Omega_2^2 + \Omega_2^4 - \Omega_3^4}{16\Omega_2^3\Omega_2\Omega_1^2}, \tag{A.5}
\end{align}

\begin{align}
0 &= (d_+ \Omega_3)' + \left( \frac{\Omega_2'}{\Omega_2} + \frac{\Omega_3'}{\Omega_3} + \frac{3\Sigma'}{2\Sigma} + \frac{\Omega_1'}{2\Omega_1} \right) d_+ \Omega_3 + \left( \frac{d_+ \Omega_2}{\Omega_2} + \frac{d_+ \Omega_4}{2\Omega_1} + \frac{3d_+ \Sigma}{2\Sigma} \right) \Omega_3' \\
&+ \frac{P^2 gK_2'}{1296\Omega_2^3\Omega_3^3} d_+ K_2 + \frac{K_1'}{384\Omega_3^4P^2g} d_+ K_1 - \frac{\Omega_3 K_3'}{1152\Omega_2^4P^2g} d_+ K_3 + \frac{(K_1 - K_3)^2}{4608\Omega_3^2\Omega_2^2\Omega_1^2P^2g} \\
&+ \frac{P^2 gK_2^2(3\Omega_2^4 - \Omega_3^4)}{5184\Omega_2^4\Omega_1^4\Omega_3^3} + \frac{P^2 gK_2^2\Omega_2\Omega_3}{1296\Omega_2^3\Omega_1^2} - \frac{P^2 g\Omega_3}{1296\Omega_2^3\Omega_1^2} + \frac{(K_1 K_2 - K_3 K_2 - 2K_1)^2}{373248\Omega_2^3\Omega_1^2\Omega_3^4} \\
&+ \frac{4\Omega_1^4 - 8\Omega_1^2\Omega_2^2 + \Omega_2^4 - \Omega_3^4}{16\Omega_2^3\Omega_2^2\Omega_1^2}, \tag{A.6}
\end{align}

\begin{align}
0 &= (d_+ K_1)' + \left( \frac{d_+ \Omega_2}{\Omega_2} - \frac{d_+ \Omega_3}{\Omega_3} + \frac{d_+ \Omega_1}{2\Omega_1} + \frac{3d_+ \Sigma}{2\Sigma} - \frac{d_+ g}{2g} \right) K_1' + \left( \frac{\Omega_2}{\Omega_2} - \frac{\Omega_3}{\Omega_3} + \frac{3\Sigma'}{2\Sigma} \right) + \frac{\Omega_1'}{2\Omega_1} - \frac{g'}{2g} \right) d_+ K_1 - \frac{\Omega_3^2(K_1 - K_3)}{4\Omega_2^2\Omega_1^2} - \frac{P^2 g(K_2 - 2)(K_1 K_2 - K_3 K_2 - 2K_1)}{648\Omega_2^4\Omega_1^2}, \tag{A.7}
\end{align}

\begin{align}
0 &= (d_+ K_2)' + \left( \frac{d_+ \Omega_1}{2\Omega_1} + \frac{3d_+ \Sigma}{2\Sigma} + \frac{d_+ g}{2g} \right) K_2' + \left( \frac{\Omega_1'}{2\Omega_1} + \frac{g'}{2g} + \frac{3\Sigma'}{2\Sigma} \right) d_+ K_2 \\
&- \frac{(K_3 - K_1)(K_2 K_2 - K_1 K_2 - 2K_1)}{576\Omega_2^2\Omega_2^2\Omega_3^2P^2g} \frac{K_2(\Omega_2^3 + \Omega_3^3)}{4\Omega_2^3\Omega_2^2\Omega_2^3} + \frac{\Omega_3^2}{2\Omega_1^2\Omega_2^2}, \tag{A.8}
\end{align}

\begin{align}
0 &= (d_+ K_3)' + \left( \frac{3d_+ \Sigma}{2\Sigma} - \frac{d_+ g}{2g} - \frac{d_+ \Omega_2}{\Omega_2} + \frac{d_+ \Omega_3}{\Omega_3} + \frac{d_+ \Omega_1}{2\Omega_1} \right) K_3' + \left( \frac{\Omega_3}{\Omega_3} - \frac{\Omega_2}{\Omega_2} + \frac{\Omega_1'}{2\Omega_1} \right) - \frac{g'}{2g} + \frac{3\Sigma'}{2\Sigma} \right) d_+ K_3 - \frac{\Omega_3^2(K_3 - K_1)}{4\Omega_2^3\Omega_1^2} - \frac{P^2 gK_2(K_3 K_2 - K_1 K_2 + 2K_1)}{648\Omega_2^4\Omega_3^4}, \tag{A.9}
\end{align}
0 = (d_+ g')' + \left( \frac{d_+ \Omega_2}{\Omega_2} + \frac{d_+ \Omega_3}{\Omega_3} + \frac{d_+ \Omega_1}{2\Omega_1} + \frac{3d_+ \Sigma}{2\Sigma} \right) g' - \frac{P^2 g^2 K_3'}{324 \Omega_1^2 \Omega_3^2} d_+ K_2 + \frac{K_3'}{288 \Omega_1^2 \Omega_3^2 P^2} d_+ K_3 + \frac{K_1'}{288 \Omega_1^2 \Omega_3^2 P^2} d_+ K_1 + \left( \frac{\Omega_2'}{\Omega_2} + \frac{\Omega_3'}{\Omega_3} + \frac{\Omega_1'}{2\Omega_1} - \frac{g'}{g} + \frac{3\Sigma'}{2\Sigma} \right) d_+ g \quad (A.10)

+ \frac{(K_3 - K_1)^2}{1152 \Omega_1^2 \Omega_3^2 \Omega_2^2 P^2} - \frac{P^2 g^2 K_3'^2 (\Omega_2^2 + \Omega_3^2)}{1296 \Omega_1^2 \Omega_2^2 \Omega_3^2} + \frac{P^2 g^2 K_2'}{324 \Omega_1^2 \Omega_2^2} - \frac{P^2 g^2}{324 \Omega_1^2 \Omega_2^4}.

0 = A'' - \left( \frac{2\Omega_2'}{\Omega_1 \Omega_2} + \frac{2\Omega_3'}{\Omega_1 \Omega_3} + \frac{3\Sigma'}{\Sigma \Omega_1} \right) d_+ \Omega_1 - \left( \frac{2\Omega_2'}{\Omega_1^2 \Omega_2} + \frac{4\Omega_3'}{\Omega_1 \Omega_2 \Omega_3} + \frac{6\Sigma'}{\Sigma \Omega_1 \Omega_2} + \frac{2\Omega_1'}{\Omega_1 \Omega_2} \right) d_+ \Omega_2

- \left( \frac{4\Omega_2'}{\Omega_1 \Omega_2} + \frac{2\Omega_3'}{\Omega_1 \Omega_3} + \frac{6\Sigma'}{\Sigma \Omega_1 \Omega_2} + \frac{2\Omega_1'}{\Omega_1 \Omega_3} \right) d_+ \Omega_3 - \left( \frac{6\Omega_2'}{\Omega_2 \Sigma} + \frac{6\Omega_3'}{\Sigma \Omega_2 \Omega_3} + \frac{6\Sigma'}{\Sigma \Omega_2 \Omega_3} + \frac{3\Omega_1'}{\Omega_1 \Sigma} \right) d_+ \Sigma

+ \frac{g'}{2g} d_+ g + \frac{P^2 g K_2'}{648 \Omega_1^2 \Omega_3^2} d_+ K_2 + \frac{K_1'}{576 \Omega_2^2 P^2 g} d_+ K_1 + \frac{K_3'}{576 \Omega_2^2 P^2 g} d_+ K_3 - \frac{(K_1 - K_3)^2}{768 \Omega_1^2 \Omega_3^2 \Omega_2^2 P^2 g} - \frac{P^2 g K_2'^2 (\Omega_2^2 + \Omega_3^2)}{864 \Omega_1^2 \Omega_2^2 \Omega_3^2} + \frac{P^2 g K_2}{216 \Omega_2 \Omega_3^2} - \frac{P^2 g}{216 \Omega_2 \Omega_3^2} - \frac{(K_1 K_2 - K_3 K_2 - 2K_1)^2}{93312 \Omega_1^2 \Omega_2^2 \Omega_3^2} - \frac{4\Omega_1' - 8\Omega_2' \Omega_3^2 - 8\Omega_1^2 \Omega_3^2 + \Omega_3^2 - 2\Omega_2 \Omega_3^2 + \Omega_1^2}{8 \Omega_2 \Omega_3^2 \Omega_3^2} ;

(A.11)

and the constraint equations

0 = \frac{\Sigma''}{6} + \frac{\Sigma}{\Omega_2} \left( \frac{(g')^2}{g^2} + \frac{4\Omega_2'^2}{\Omega_3} + \frac{4\Omega_3'^2}{\Omega_2} + \frac{2\Omega_1'^2}{\Omega_1} + \frac{P^2 g (K_2')^2}{144 \Omega_2 \Omega_3^2} + \frac{(K_3')^2}{144 \Omega_2 \Omega_3^2} + \frac{(K_1')^2}{144 \Omega_2 \Omega_3^2} \right),

(A.12)

0 = \frac{d_+ \Sigma}{3\Omega_1} + \frac{d_+ \Omega_2}{3\Omega_2} + \frac{d_+ \Omega_3}{3\Omega_3} - \left( \frac{\Sigma}{3\Omega_1} d_+ \Omega_1 + \frac{2\Sigma}{3\Omega_2} d_+ \Omega_2 + \frac{2\Sigma}{3\Omega_3} d_+ \Omega_3 - \frac{\Sigma}{3\Omega_1} d_+ \Omega_1 + \frac{2\Sigma}{3\Omega_2} d_+ \Omega_2 + \frac{2\Sigma}{3\Omega_3} d_+ \Omega_3 \right) A' + \frac{\Sigma P^2 g}{972 \Omega_2^2 \Omega_3^2} (d_+ K_2)^2 + \frac{\Sigma}{864 \Omega_2^2 P^2 g} (d_+ K_3)^2

+ \frac{3\Sigma}{864 \Omega_2^4 P^2 g} (d_+ K_1)^2 + \frac{\Sigma}{6g^2} (d_+ g)^2 .

(A.13)

To derive the late-time geometry dual to cascading gauge theory vacuum in de Sitter, we introduce following \[19\]

\[\lim_{t \to \infty} \left\{ A(t, r), \frac{\Sigma(t, r)}{e^{H t}}, K_i(t, r), g(t, r) \right\} = \{ a(r), \sigma(r), K_i(r), g(r) \} ; \quad (A.14)\]

Furthermore,

\[\lim_{t \to \infty} \left\{ \Omega_1^2(t, r), \Omega_2^2(t, r), \Omega_3^2(t, r) \right\} = \left\{ \frac{1}{9} w_{c1}(r), \frac{1}{6} w_{c2}(r), \frac{1}{6} w_{b2}(r) \right\} . \quad (A.15)\]
We find from (A.3)-(A.13) in the $t \to \infty$ limit 9 second order ODEs:

$$0 = \sigma'' + \frac{5(\sigma')^2}{4\sigma} + \frac{5a'\sigma'}{8a} + \frac{\sigma}{16} \left( \frac{2\sigma'}{\sigma} - \frac{a'}{a} \right) \left( \frac{w'_{a2}}{w_{a2}} + \frac{2w'_{a2}}{w_{a2}} + \frac{2w'_{b2}}{w_{b2}} \right) + \frac{H \sigma}{16a} \left( \frac{30\sigma'}{\sigma} + \frac{w'_{c2}}{w_{c2}} \right) + \frac{2w'_{a2}}{w_{a2}} + \frac{2w'_{b2}}{w_{b2}} - \frac{\sigma}{8} \left( \frac{1}{2} \left( \frac{w'_{a2}}{w_{a2}} + \frac{w'_{b2}}{w_{b2}} \right)^2 + \frac{w'_{a2}w'_{b2}}{w_{a2}w_{b2}} + \frac{w'_{a2}w'_{c2}}{w_{a2}w_{c2}} + \frac{w'_{c2}w'_{b2}}{w_{c2}w_{b2}} \right) + \frac{\sigma(g')^2}{16g^2} - \frac{2gP^2(K_2')^2}{9w_{b2}w_{a2}} - \frac{(K_2')^2}{4gP^2w_{a2}^2} - \frac{27\sigma(K_3 - K_1)^2}{256w_{b2}w_{a2}w_{c2}gP^2} - \frac{\sigma}{128aw_{b2}w_{a2}w_{c2}} \left( 5K_2^2(K_3 - K_1)^2 + 2w_{b2}w_{a2}(9w_{b2}^2 - 18w_{b2}w_{a2} - 48w_{b2}w_{c2} + 9w_{a2}^2 - 48w_{a2}w_{c2} + 16w_{c2}^2) + 20K_1(K_3K_2 - K_1K_2 + K_1) \right) - \frac{3\sigma gP^2(K_2w_{b2}^2K_2 + w_{a2}^2K_2 - 4w_{b2}^2 + 4w_{b2}^2)}{32aw_{b2}w_{a2}w_{c2}};$$

(A.16)

$$0 = a'' + \frac{21a'\sigma'}{4\sigma} + \frac{a}{8} \left( \frac{18\sigma'}{\sigma} + \frac{7a'}{a} \right) \left( \frac{w'_{a2}}{w_{a2}} + \frac{2w'_{a2}}{w_{a2}} + \frac{2w'_{b2}}{w_{b2}} \right) + \frac{6w'_{a2}}{w_{a2}} + \frac{6w'_{b2}}{w_{b2}} + \frac{3a}{8} \left( \frac{12(\sigma')^2}{\sigma^2} + \frac{2w'_{a2}w'_{c2}}{w_{a2}w_{c2}} + \frac{2w'_{a2}w'_{b2}}{w_{a2}w_{b2}} + \frac{2w'_{a2}w'_{b2}}{w_{a2}w_{b2}} + \frac{w_{a2}}{w_{a2}} + \frac{w'_{b2}}{w_{b2}} \right)^2 - \frac{3(g')^2a}{8g^2} - \frac{5a}{32} \left( \frac{8gP^2(K_2')^2}{9w_{b2}w_{a2}} + \frac{(K_2')^2}{gP^2w_{a2}^2} + \frac{(K_1')^2}{gP^2w_{b2}^2} + \frac{9(K_3 - K_1)^2}{128w_{b2}w_{a2}w_{c2}gP^2} \right) + \frac{1}{64w_{c2}w_{a2}^2w_{b2}^2} \left( K_2^2(K_3 - K_1)^2 - 6w_{b2}w_{a2}(9w_{b2}^2 - 18w_{b2}w_{a2} - 48w_{b2}w_{c2} + 9w_{a2}^2 - 48w_{a2}w_{c2} + 16w_{c2}^2) - 48w_{a2}w_{c2} + 16w_{c2}^2 \right) + 4K_1((K_3 - K_1)K_2 + K_1) + \frac{gP^2}{16w_{c2}w_{a2}w_{b2}^2} \left( w_{a2}^2 + w_{b2}^2 \right)K_2^2 + 4(1 - K_2)w_{b2}^2;$$

(A.17)
0 = w''_{a2} + \frac{w_{a2}a'}{8a} \left( \frac{6w'_{a2}}{w_{a2}} - \frac{6\sigma'}{\sigma} - \frac{2w'_{b2}}{w_{b2}} - \frac{w'_{c2}}{w_{c2}} \right) - \frac{w_{a2}}{8} \left( \frac{(w'_{a2})^2}{w_{a2}^2} - \frac{4w'_{a2}w'_{b2}}{w_{a2}w_{b2}} + \frac{(w'_{b2})^2}{w_{b2}^2} \right) - \frac{12w'_{a2}\sigma'}{w_{a2}\sigma} + \frac{12w'_{b2}\sigma'}{\sigma_w_{b2}} + \frac{12(\sigma')^2}{w_{a2}w_{c2}} + \frac{2w'_{a2}w'_{c2}}{w_{a2}w_{c2}} + \frac{2w'_{c2}w'_{b2}}{w_{c2}w_{b2}} + \frac{6w'_{a2}\sigma'}{w_{a2}\sigma} + \frac{(g')^2w_{a2}}{8g^2} + \frac{gP^2(K'_2)^2}{12w_{b2}} - \frac{w_{a2}(K'_1)^2}{32w_{a2}gP^2} + \frac{7(K'_1)^2}{32w_{a2}gP^2} + \frac{3Hw_{b2}}{8a} \left( \frac{2w'_{a2}}{w_{a2}} - \frac{2w'_{b2}}{w_{b2}} - \frac{6\sigma'}{\sigma} - \frac{w'_{c2}}{w_{c2}} \right) + \frac{9(K_3 - K_1)^2}{128w_{b2}w_{c2}gP^2} + \frac{3}{64w_{b2}^2w_{c2}w_{a2}} \left( K_2^2(K_3 - K_1)^2 + 2w_{b2}w_{a2}(16w_{c2}^2 - 48w_{a2}w_{c2} + 16w_{a2}w_{c2} + 9w_{b2}^2 + 6w_{a2}w_{b2} - 15w_{a2}^2) + 4K_1(K_3K_2 - K_1K_2 + K_1) \right) + \frac{gP^2}{16w_{b2}^2w_{c2}w_{a2}} \left( K_2^2(5w_{b2}^2 - 3w_{a2}^2) + 20(1 - K_2)w_{b2}^2 \right) ;

(A.18)

0 = w''_{b2} + \frac{a'w_{b2}}{8a} \left( \frac{6w'_{b2}}{w_{b2}} - \frac{6\sigma'}{\sigma} - \frac{2w'_{a2}}{w_{a2}} - \frac{w'_{c2}}{w_{c2}} \right) - \frac{w_{b2}}{8} \left( \frac{(w'_{a2})^2}{w_{a2}^2} - \frac{4w'_{a2}w'_{b2}}{w_{a2}w_{b2}} + \frac{(w'_{b2})^2}{w_{b2}^2} \right) + \frac{12w'_{a2}\sigma'}{w_{a2}\sigma} - \frac{12w'_{a2}\sigma'}{\sigma_w_{a2}} + \frac{12(\sigma')^2}{w_{a2}w_{c2}} + \frac{2w'_{a2}w'_{c2}}{w_{a2}w_{c2}} - \frac{2w'_{c2}w'_{b2}}{w_{c2}w_{b2}} + \frac{6w'_{a2}\sigma'}{w_{a2}\sigma} + \frac{(g')^2w_{b2}}{8g^2} + \frac{gP^2(K'_2)^2}{12w_{a2}} + \frac{7(K'_1)^2}{32w_{a2}gP^2} - \frac{w_{a2}(K'_1)^2}{32w_{a2}gP^2} - \frac{3Hw_{b2}}{8a} \left( \frac{2w'_{a2}}{w_{a2}} - \frac{2w'_{b2}}{w_{b2}} + \frac{6\sigma'}{\sigma} + \frac{w'_{c2}}{w_{c2}} \right) + \frac{9(K_3 - K_1)^2}{128w_{a2}gP^2} + \frac{3}{64w_{b2}w_{a2}^2w_{c2}a} \left( K_2^2(K_3 - K_1)^2 + 2w_{a2}w_{b2}(16w_{c2}^2 + 16w_{a2}w_{b2} - 48w_{a2}w_{c2} + 15w_{a2}^2 + 6w_{a2}w_{b2} + 9w_{a2}^2) + 4K_1(K_3K_2 - K_1K_2 + K_1) \right) - \frac{gP^2}{16w_{b2}w_{a2}^2w_{c2}a} \left( K_2^2(3w_{b2}^2 - 5w_{a2}^2) + 12(1 - K_2)w_{b2}^2 \right) ;

(A.19)
\[0 = w''_c - \frac{w_{c2}a'}{8a} \left( \frac{2w'_{b2}}{w_{b2}} + \frac{6\sigma'}{\sigma} - \frac{7w'_{c2}}{w_{c2}} + \frac{2w'_{a2}}{w_{a2}} \right) - \frac{w_{c2}}{8} \left( \frac{(w'_{c2})^2}{w_{c2}^2} + \frac{4w'_{a2}w'_{b2}}{w_{a2}w_{b2}} + \frac{(w'_{b2})^2}{w_{b2}^2} \right)
+ \frac{12w'_{a2}\sigma'}{w_{a2}\sigma} + \frac{12w'_{a2}\sigma'}{\sigma w_{b2}} + \frac{12(\sigma')^2}{\sigma^2} - \frac{6w'_{a2}w'_{c2}}{w_{a2}w_{c2}} - \frac{6w'_{c2}w'_{b2}}{w_{c2}w_{b2}} - \frac{18w'_{c2}\sigma'}{w_{c2}^2} + \frac{4(w'_{c2})^2}{w_{c2}^2}
+ \frac{(g')^2}{8g^2} - \frac{w_{c2}gP^2(K'_b)^2}{32w_{b2}gP^2} - \frac{w_{c2}(K'_b)^2}{32w_{b2}gP^2} - \frac{w_{c2}(K'_b)^2}{32w_{a2}gP^2} - \frac{3Hw_{c2}}{8a} \left( \frac{2w'_{a2}}{w_{a2}} + \frac{2w'_{b2}}{w_{b2}} + 6\sigma' \right)
- \frac{3w'_{c2}}{w_{c2}} + \frac{45(K_3 - K_1)^2}{128w_{a2}w_{b2}gP^2a} + \frac{3}{64w_{b2}w_{a2}a} \left( K_3^2(K_3 - K_1)^2 - 2w_{a2}w_{b2}(48w_{a2}^2 - 16w_{a2}w_{b2} - 2w_{b2}^2 + 2w_{a2}^2 - 21w_{a2}^2) + 4K_1(K_3K_2 - K_1K_2 + K_1) \right)
+ \frac{5gP^2}{16w_{b2}w_{a2}^2a} \left( K_2^2(w_{a2}^2 + w_{b2}^2) + 4(1 - K_2)w_{b2}^2 \right); \tag{A.20}
\]

\[0 = K''_1 + \left( \frac{3H}{2a} + \frac{w'_{a2}}{w_{a2}} + \frac{w'_{c2}}{2w_{c2}} - \frac{w'_{b2}}{w_{b2}} + \frac{3\sigma'}{\sigma} + \frac{a'}{a} - \frac{g'}{g} \right) K'_1 - \frac{9w_{b2}(K_1 - K_3)}{4w_{a2}w_{c2}} - \frac{(K_2 - 2)(K_2K_1 - K_2K_3 - 2K_1)gP^2}{2aw_{c2}w_{a2}}; \tag{A.21}
\]

\[0 = K''_2 + \left( \frac{a'}{a} + \frac{3H}{2a} + \frac{w'_{c2}}{2w_{c2}} + \frac{g'}{g} + \frac{3\sigma'}{\sigma} \right) K'_2 - \frac{9(K_1 - K_3)(K_2(K_1 - K_3) - 2K_1)}{16w_{a2}w_{b2}gP^2w_{c2}a} - \frac{9((w_{a2}^2 + w_{b2}^2)K_2 - 2w_{b2}^2)}{4w_{a2}w_{b2}w_{c2}a}; \tag{A.22}
\]

\[0 = K''_3 + \left( \frac{3H}{2a} - \frac{w'_{a2}}{w_{a2}} + \frac{3\sigma'}{\sigma} + \frac{w'_{c2}}{2w_{c2}} + \frac{w'_{b2}}{w_{b2}} + \frac{a'}{a} - \frac{g'}{g} \right) K'_3 + \frac{9w_{a2}(K_1 - K_3)}{4w_{b2}aw_{c2}} + \frac{K_2(K_2(K_1 - K_3) - 2K_1)gP^2}{2aw_{c2}w_{a2}}; \tag{A.23}
\]

\[0 = g'' + \left( \frac{3H}{2a} + \frac{a'}{a} + \frac{w'_{c2}}{2w_{c2}} - \frac{g'}{g} + \frac{w'_{b2}}{w_{b2}} + \frac{w'_{a2}}{w_{a2}} + \frac{3\sigma'}{\sigma} \right) g' - \frac{g^2P^2(K'_b)^2}{9w_{b2}w_{a2}} + \frac{(K'_b)^2}{8P^2w_{b2}^2} + \frac{9(K_1 - K_3)^2}{32aw_{a2}w_{c2}w_{b2}P^2} - \frac{9g^2P^2(K_2^2(w_{a2}^2 + w_{b2}^2) + 4(1 - K_2)w_{b2}^2)}{4w_{b2}^2aw_{c2}w_{a2}^2}, \tag{A.24}
\]

and 2 first order ODEs:

\[0 = \sigma' + \frac{\sigma'}{2a} \left( H - a' \right); \tag{A.25}
\]

58
\[
0 = \frac{(g')^2}{g^2} - \frac{3H}{a} \left( \frac{2w'_2}{w_2} + \frac{4\sigma'}{\sigma} + \frac{2w'_2}{w_2} + \frac{w'_2}{w_2} + \frac{a'}{a} - \frac{H}{a} \right) - \frac{2w'_2a'}{aw_2} - \frac{6\sigma'a'}{\sigma a} - \frac{4w'_2w'_2}{w_2w_2}
- \frac{12w'_2\sigma'}{\sigma w_2} - \frac{12w'_2\sigma'}{\sigma w_2} - \frac{2w'_2\sigma'}{aw_2} - \frac{6w'_2\sigma'}{aw_2} - \frac{w'_2a'}{aw_2} - \frac{2w'_2w'_2}{w_2w_2} - \frac{2w'_2w'_2}{w_2w_2} - \frac{(w'_2)^2}{w_2^2}
- \frac{12(\sigma')^2}{\sigma^2} - (\frac{w'_2}{w_2})^2 + \frac{2P^2g(K_2')^2}{9w_2w_2} + \frac{(K_1')^2}{4w_2^2P^2g} + \frac{(K_3')^2}{4w_2^2P^2g} - \frac{9(K_1 - K_3)^2}{16aw_2gP^2w_2w_2}
- \frac{1}{8aw_2^2w_2^2w_2^2} \left( K_2^2(K_1 - K_3)^2 + 2w_2w_2(9w_2^2 - 18w_2w_2 - 48w_2w_2 + 9w_2^2 + 48w_2w_2 + 16w_2^2) - 4K_1((K_1 - K_3)K_2 - K_1) \right) - \frac{P^2}{2w_2^2aw_2w_2^2} \left( K_2^2(w_2^2 + w_2^2) + 4(1 - K_2)w_2^2 \right).
\]
(A.26)

It is straightforward to verify the (A.16)-(A.24) are consistent with (A.25)-(A.26); thus the latter ODEs can be used for drop (A.16) and (A.20) and eliminate \( \sigma' \) and \( w'_2 \) in the remaining second order ODEs.

Cascading gauge theory de Sitter vacuum equations of motion (A.16)-(A.26) are invariant under the following symmetries (\( \lambda \equiv \text{const} \)),

- **symmetry SEF1:**
  \[
  r \to r + \lambda, \quad \{ H, P, a, \sigma, w_{a2,b2,c2}, K_{1,2,3}, g \} \to \{ H, P, a, \sigma, w_{a2,b2,c2}, K_{1,2,3}, g \} ;
  \]
  (A.27)

- **symmetry SEF2:**
  \[
P \to \lambda P, \quad g \to \frac{g}{\lambda}, \quad \{ r, H, a, \sigma, w_{a2,b2,c2}, K_{1,2,3} \} \to \{ r, H, a, \sigma, w_{a2,b2,c2}, K_{1,2,3} \} ;
  \]
  (A.28)

- **symmetry SEF3:**
  \[
  \{ P, r, a, w_{a2,b2,c2} \} \to \lambda\{ P, r, a, w_{a2,b2,c2} \}, \quad \sigma \to \lambda^{1/2}\sigma, \quad \{ K_{1,3} \} \to \lambda^2\{ K_{1,3} \}, \quad \{ H, K_2, g \} \to \{ H, K_2, g \} ;
  \]
  (A.29)

- **symmetry SEF4:**
  \[
  \{ r, H \} \to \lambda\{ r, H \}, \quad \{ P, \sigma, w_{a2,b2,c2}, K_{1,2,3}, g \} \to \{ P, \sigma, w_{a2,b2,c2}, K_{1,2,3}, g \} \to \{ P, \sigma, w_{a2,b2,c2}, K_{1,2,3}, g \} ;
  \]
  (A.30)

\( a \to \lambda^2 a \).
B FG frame equations of motion, asymptotics, relation to EF frame and extremal Klebanov-Strassler solution

Fefferman-Graham frame can be used to describe only (the patch of) the gravitational dual to the cascading gauge theory de Sitter vacua. It is useful to setup the asymptotic boundary conditions, analytical continuation to Euclidean (Bunch-Davies) vacua, and study the $H \to 0$ limit in which one recovers the KS solution [2].

Within the metric ansatz

$$\frac{ds_{10}^2}{\rho^2} = \frac{1}{h^{1/2}} \left( dM_4^{f,c} \right)^2 + \frac{h^{1/2}}{\rho^2} (dp)^2 + \frac{f_c h^{1/2}}{9} g_5^2 + \frac{f_a h^{1/2}}{6} (g_2^2 + g_4^2) + \frac{f_b h^{1/2}}{6} (g_4^2 + g_6^2)$$

$$\left( dM_4^{f} \right)^2 = -d\tau^2 + e^{2H \tau} dx^2, \quad \left( dM_4^{c} \right)^2 = -d\tau^2 + \frac{1}{H^2} \cosh^2 (H \tau) \left( dS^3 \right)^2,$$

(B.1)

where we used the FG frame time $\tau$ and the radial coordinate $\rho$ to distinguish them from the EF frame time $t$ and the radial coordinate $r$ in (A.1),

$$f_{a,b,c} = f_{a,b,c}(\rho), \quad h = h(\rho), \quad K_{1,2,3} = K_{1,2,3}(\rho), \quad g = g(\rho), \quad (B.2)$$

we find the following equations of motion (independent of whether we use the flat boundary spatial slicing $\left( dM_4^{f} \right)^2$ or the closed boundary spatial slicing $\left( dM_4^{c} \right)^2$) describing de Sitter vacuum of cascading gauge theory [31]:

$$0 = f''_c - \frac{3f_c'}{\rho} - 3h f_c H^2 - \frac{(f'_c)^2}{2f_c} + \frac{5f_c}{\rho^2} - \frac{f_c(g')^2}{8g^2} + \frac{3f'_b f'_c}{4f_b} + \frac{63f_a}{16f_b \rho^2} + \frac{63f_b}{16f_a \rho^2} + \frac{3f_c}{f_a \rho^2}$$

$$- f_c f'_a f'_c \left( \frac{27K_1 K_3}{2fa} \right) + \frac{3f'_a f'_c}{4f_a} - \frac{f_c(h')^2}{8h^2} + \frac{3f'_c f'_b}{8f_b^2} - \frac{63}{8\rho^2} - \frac{8f_a^2 f_b^2 \rho^2}{832h^2 f_b^2 \rho^2} + \frac{32f_a^2 h \rho^2}{32f_a h f_b g P^2}$$

$$- \frac{3f_c(K_1')^2}{32f_a^2 h g P^2} + \frac{3g P^2 K_2}{8h^2 f_b^2 \rho^2} - \frac{3g P^2 f_c}{8f_a^2 h \rho^2} - \frac{3g P^2 K_2}{8f_a^2 h \rho^2} - \frac{9f'_c}{f_a h \rho^2} + \frac{K_2 K_1 K_3}{16f_a^2 h^2 f_b^2 \rho^2}$$

$$- \frac{K_2 K_3}{8f_a^2 h^2 f_b^2 \rho^2} + \frac{g P^2 f_c(K_2')^2}{12f_a h f_b} + \frac{27K_1^2}{64f_a h f_b g P^2 \rho^2} + \frac{27K_1^2}{64f_a h f_b g P^2 \rho^2}, \quad (B.3)$$
\[
0 = f_a'' - \frac{45f_a^2}{16f_c f_b \rho^2} + \frac{fa'h'}{h} + gP^2(K_2')^2 + \frac{5(K_3')^2}{32f_a h g P^2} - \frac{fa'h'}{4f_c f_b} - \frac{(f_a')^2}{8f_a} + \frac{5f_a}{\rho^2} - \frac{3f_a'}{\rho} \\
- \frac{K_2'K_1'}{32f_c f_a h^2 f_b^2 \rho^2} + \frac{K_2'K_3'}{8f_c f_a h^2 f_b^2 \rho^2} - \frac{K_2'K_3'}{32f_c f_a h^2 f_b^2 \rho^2} - \frac{9gP^2k_2}{2f_c f_a h \rho^2} + \frac{3gP^2K_2}{8f_c f_a h \rho^2} \\
- \frac{9f_3'K_3}{64f_c h g f_a P^2 \rho^2} - \frac{64f_c h g f_a P^2 \rho^2}{9f_3'} + \frac{3f_a}{f_b \rho^2} + \frac{3f_c}{f_b \rho^2} + \frac{9K_1K_3}{32f_c f_a h^2 f_b^2 \rho^2} + \frac{K_2'K_1K_3}{16f_c f_a h^2 f_b^2 \rho^2} \\
- \frac{9f_3'H^2}{32f_c h g f_a P^2 \rho^2} + \frac{f_a'h'}{4f_c} - \frac{fa(h')^2}{8f_a'} - \frac{fa(h')^2}{8h^2} + \frac{27f_b}{16f_c \rho^2},
\]

(B.4)

\[
0 = f_b'' - \frac{3f_b'}{\rho} - \frac{(f_b')^2}{8f_b} + \frac{5f_b}{\rho^2} - \frac{45f_b^2}{16f_c f_a \rho^2} + \frac{f_b'h'}{h} - \frac{K_2^2}{8f_c h^2 f_a f_b \rho^2} - \frac{3f_b(K_3')^2}{32h g f_a \rho^2} \\
- \frac{K_2^2K_1'}{32f_c h^2 f_a f_b \rho^2} + \frac{K_2K_3'}{8f_c h^2 f_a f_b \rho^2} - \frac{K_2K_3'}{32f_c h^2 f_a f_b \rho^2} - \frac{9gP^2K_2}{2f_c h g f_a \rho^2} + \frac{3gP^2K_2}{8f_c h g f_a \rho^2} \\
- \frac{9f_3^2}{64f_c h g f_a \rho^2} - \frac{9f_3}{2f_c h g f_a \rho^2} + \frac{3f_a}{f_b \rho^2} + \frac{3f_c}{f_b \rho^2} + \frac{5(K_1')^2}{32h g f_a \rho^2} + \frac{gP^2(K_2')^2}{2f_c f_a \rho^2} + \frac{f_a'h'}{2f_c} \\
- \frac{9K_1K_3}{32f_c h g f_a \rho^2} + \frac{f_b(g')^2}{8g^2} - \frac{3h f_b H^2}{8f_a} + \frac{f_a'h'}{2f_c} - \frac{fa(h')^2}{8f_a'} + \frac{f_a'h'}{8f_a} + \frac{f_a'h'}{8h^2},
\]

(B.5)

\[
0 = h'' - \frac{K_2^2 K_1'}{4f_c f_a f_b^2 h \rho^2} + \frac{K_2 K_1'}{4f_c f_a f_b^2 h \rho^2} + \frac{K_2 K_3'}{4f_c f_a f_b^2 h \rho^2} + \frac{9K_1'}{16f_c f_a \rho^2 g P^2} + \frac{9K_3'}{16f_c f_a \rho^2 g P^2} \\
+ \frac{2h f_a'}{f_b \rho} + \frac{4h f_a'}{f_b \rho} + \frac{4h f_a'}{f_b \rho} + \frac{(K_1')^2}{8f_a' g P^2} + \frac{(K_3')^2}{8f_a' g P^2} + \frac{gP^2K_2}{2f_c f_a \rho^2} + \frac{gP^2K_2}{2f_c f_a \rho^2} + \frac{2gP^2K_2}{2f_c f_a \rho^2} + \frac{f_a'h'}{2f_c} \\
+ \frac{h f_a'}{f_b h} + \frac{h f_a'}{f_b h} - \frac{16h}{\rho^2} + \frac{(h')^2}{h} + \frac{12h^2 H^2}{2f_c f_a \rho^2 h^2} + \frac{K_3 K_1}{2f_c f_a \rho^2 h^2} + \frac{K_3 K_1}{2f_c f_a \rho^2 h^2} + \frac{K_3}{f_c f_a \rho^2 h^2} \\
+ \frac{2gP^2}{9f_a f_b} + \frac{gP^2(K_2')^2}{9f_a f_b} + \frac{9K_1K_3}{8f_c f_a h b^2 g P^2} - \frac{3h'}{\rho},
\]

(B.6)

\[
0 = K_1'' - \frac{gK_2 K_1 P^2}{f_c f_b h^2 \rho^2} + \frac{gK_2 K_1 P^2}{f_c f_b h^2 \rho^2} + \frac{4gK_2 K_1 P^2}{f_c f_b h^2 \rho^2} - \frac{2gK_2 K_3 P^2}{2f_c f_a \rho^2} - \frac{9f_a K_1}{2f_c f_a \rho^2} + \frac{9f_a K_3}{2f_c f_a \rho^2} \\
- \frac{4gK_1 P^2}{f_c f_b h^2 \rho^2} + \frac{K_1' f_c}{2f_c} - \frac{K_1' h}{h} + \frac{f_a K_1'}{f_a} - \frac{3K_1'}{f_a} - \frac{K_1' f_b}{f_b},
\]

(B.7)
Additionally, we have the first order constraint

\begin{align*}
0 &= K''_3 + \frac{gK_2^2K_1P^2}{f_c f_a^2 h^2 \rho^2} - \frac{2gK_2^2K_3P^2}{f_c f_a^2 h^2 \rho^2} + \frac{9f_a K_1}{2f_c f_a h^2 \rho^2} + \frac{9f_a K_3}{2f_c f_a h^2 \rho^2} + \frac{K'_3 f'_a}{2f_c} \\
- \frac{K'_3 g'}{g} + \frac{f'_3 K'_3}{h} - \frac{3K'_3 h'}{h} - \frac{K'_3 f'_a}{f_a},
\end{align*}

(B.8)

\begin{align*}
0 &= K''_2 - \frac{9f_a K_2}{2f_c f_a h^2 \rho^2} + \frac{g}{8f_a f_b h^2 P^2} + \frac{(K'_3)^2}{8f_a h^2 P^2} + \frac{(K'_4)^2}{8f_b^2 h^2 P^2} - \frac{9gP^2 K_2}{f_c f_a h^2 f_a h^2 \rho^2} - \frac{9K'_2}{16f_c f_a h^2 \rho^2 P^2} + \frac{9K'_3}{16f_c f_a h^2 \rho^2 P^2} - \frac{g}{8f_a f_b h^2 P^2} + \frac{g}{2f_c} + \frac{K'_2 f'_c}{2f_c} - \frac{K'_2 g'}{g} - \frac{K'_2 h'}{h}.
\end{align*}

(B.9)

\begin{align*}
0 &= g'' - \frac{g^2 P^2 K_2}{2f_c f_a^2 h^2 \rho^2} + \frac{9K_1}{8f_a f_b^2 h^2 \rho^2 h^4} + \frac{g}{8f_a f_b h^2 P^2} + \frac{(K'_3)^2}{8f_a h^2 P^2} + \frac{(K'_4)^2}{8f_b^2 h^2 P^2} - \frac{9gP^2 K_2}{f_c f_a h^2 f_a h^2 \rho^2} - \frac{9K'_2}{16f_c f_a h^2 \rho^2 P^2} + \frac{9K'_3}{16f_c f_a h^2 \rho^2 P^2} - \frac{g}{8f_a f_b h^2 P^2} + \frac{g}{2f_c} + \frac{K'_2 f'_c}{2f_c}
\end{align*}

(B.10)

Additionally, we have the first order constraint

\begin{align*}
0 &= \frac{8}{9} g^2 (K'_2)^2 f_a f_a P^4 + (K'_2)^2 f_a^2 + (K'_2)^2 f_a^2 - \frac{4g^2 K_2^2 f_a^2 P^4}{f_c \rho^2} + \frac{4g f_a f_b^2 P^2 (h')^2}{h} \\
+ \frac{4h (g')^2 f_a^2 f_b^2 P^2}{g} + \frac{966 f_a f_b^2 h^2 P^2}{f_c \rho^2} - \frac{96h f_a f_b^2 h^2 P^2}{f_c \rho^2} + \frac{96h f_a f_b^2 h^2 P^2}{f_c \rho^2} - \frac{96h f_a f_b^2 h^2 P^2}{f_c \rho^2} - \frac{96h f_a f_b^2 h^2 P^2}{f_c \rho^2} - \frac{4g K_2^2 P^2}{f_c \rho^2} \\
+ 96h^2 g f_a^2 f_b^2 h^2 P^2 h^2 + \frac{9K_1 f_a f_a f_b}{f_c \rho^2} + \frac{32g f_a f_b^2 f_b h^2 P^2}{f_c \rho^2} + \frac{16g f_a f_b^2 P^4}{f_c h^2 \rho^2} + \frac{32g f_a f_b^2 f_b h^2 P^2}{f_c \rho^2} + \frac{16g f_a f_b^2 P^4}{f_c h^2 \rho^2} + \frac{4g^2 f_a f_b^2 P^4}{f_c h^2 \rho^2} \\
- \frac{g K_2^2 K_3^2 P^2}{f_c \rho^2} + \frac{4g K_2^2 K_3^2 P^2}{f_c \rho^2} - \frac{g K_2^2 K_3^2 P^2}{f_c \rho^2} - \frac{g K_2^2 K_3^2 P^2}{f_c \rho^2} - \frac{g K_2^2 K_3^2 P^2}{f_c \rho^2} - \frac{g K_2^2 K_3^2 P^2}{f_c \rho^2} - \frac{g K_2^2 K_3^2 P^2}{f_c \rho^2} - \frac{g K_2^2 K_3^2 P^2}{f_c \rho^2} - \frac{g K_2^2 K_3^2 P^2}{f_c \rho^2} - \frac{g K_2^2 K_3^2 P^2}{f_c \rho^2} \\
- \frac{16h f_a f_b f_b^2 f'_a f_b - \frac{32h g f_a f_b^2 P^2}{f_c \rho^2} - \frac{18h g f_a f_b^2 P^2}{f_c \rho^2} - \frac{18h g f_a f_b^2 P^2}{f_c \rho^2} - \frac{36h g f_a f_b^2 P^2}{f_c \rho^2} \\
- \frac{9K_2^2 f_a f_b}{f_c \rho^2} - \frac{4h g f_b^2 P^2 (f_a')^2}{f_b \rho^2} + \frac{16g f_a^2 P^4}{f_c \rho^2} + \frac{2g K_2^2 K_3^2 P^2}{f_c \rho^2} - \frac{4g K_2^2 K_3^2 P^2}{f_c \rho^2} - \frac{8h f_a f_b P^2 f'_b f'_a}{f_c \rho} - \frac{32h g f_a f_b P^2 f'_b f'_a}{f_c \rho} - \frac{8h g f_a f_b P^2 f'_b f'_a}{f_c \rho} - \frac{9K_2^2 f_b f_a}{2f_c \rho^2}.
\end{align*}

(B.11)

Cascading gauge theory de Sitter vacuum equations of motion (B.3)-(B.11) are invariant under the following symmetries ($\lambda \equiv \text{const}$) (compare with (A.27)-(A.30)):
\[ \begin{pmatrix} \rho \\ H \\ P \\ h \\ f_{a,b,c} \\ K_{1,2,3} \\ g \end{pmatrix} \rightarrow \begin{pmatrix} \rho/(1 + \lambda \rho) \\ H \\ P \\ (1 + \lambda \rho)^4 h \\ (1 + \lambda \rho)^{-2} f_{a,b,c} \\ K_{1,2,3} \\ g \end{pmatrix} ; \quad (B.12) \]

\[ \text{symmetry SFG2:} \]

\[ P \rightarrow \lambda P, \quad g \rightarrow \frac{g}{\lambda}, \quad \{\rho, H, f_{a,b,c}, h, K_{1,2,3}\} \rightarrow \{\rho, H, f_{a,b,c}, h, K_{1,2,3}\} ; \quad (B.13) \]

\[ \text{symmetry SFG3:} \]

\[ P \rightarrow \lambda P, \quad \rho \rightarrow \frac{\rho}{\lambda}, \quad \{h, K_{1,3}\} \rightarrow \lambda^2 \{h, K_{1,3}\}, \quad \{H, f_{a,b,c}, K_2, g\} \rightarrow \{H, f_{a,b,c}, K_2, g\} ; \quad (B.14) \]

\[ \text{symmetry SFG4:} \]

\[ \rho \rightarrow \lambda \rho, \quad H \rightarrow \frac{H}{\lambda}, \quad \{P, f_{a,b,c}, h, K_{1,2,3}, g\} \rightarrow \{P, f_{a,b,c}, h, K_{1,2,3}, g\} . \quad (B.15) \]

FG frame makes analytical continuation to Euclidean Bunch-Davies vacuum obvious:

\[ (d\mathcal{M}_4^c)^2 \xrightarrow{\tau \rightarrow i \frac{2 + \eta^2}{H^2}} \frac{1}{H^2} \left( (d\theta)^2 + \sin^2(\theta) \ (dS^3)^2 \right) = \frac{1}{H^2} \ (dS^4)^2 . \quad (B.16) \]

B.1 Asymptotics

The general UV (as \(\rho \rightarrow 0\)) asymptotic solution of (B.3)-(B.11) describing the phase of cascading gauge theory with spontaneously broken chiral symmetry takes the form

\[ f_c = 1 + f_{a,1,0} \rho + \left( -\frac{3}{8} P^2 g_s H^2 - \frac{1}{4} K_0 H^2 + \frac{1}{4} f_{a,1,0}^2 + \frac{1}{2} P^2 g_s H^2 \ln \rho \right) \rho^2 \]

\[ - \frac{1}{4} P^2 g_s H^2 f_{a,1,0} \rho^3 + \sum_{n=4}^{\infty} \sum_k f_{c,n,k} \rho^n \ln^k \rho , \quad (B.17) \]
\[ f_a = 1 + f_{a,1,0}\rho + \left( -\frac{1}{2}P^2g_sH^2 - \frac{1}{4}K_0H^2 + \frac{1}{4}f_{a,1,0}^2 + \frac{1}{2}P^2g_sH^2\ln \rho \right) \rho^2 + f_{a,3,0}\rho^3 \\
+ \sum_{n=4}^{\infty} \sum_{k} f_{a,n,k} \rho^n \ln^k \rho, \quad \text{ (B.18)} \]

\[ f_b = 1 + f_{a,1,0}\rho + \left( -\frac{1}{2}P^2g_sH^2 - \frac{1}{4}K_0H^2 + \frac{1}{4}f_{a,1,0}^2 + \frac{1}{2}P^2g_sH^2\ln \rho \right) \rho^2 \\
- \left( \frac{1}{2}P^2g_sH^2f_{a,1,0} + f_{a,3,0} \right) \rho^3 + \sum_{n=4}^{\infty} \sum_{k} f_{b,n,k} \rho^n \ln^k \rho, \quad \text{ (B.19)} \]

\[ h = \frac{1}{8}P^2g_s + \frac{1}{4}K_0 - \frac{1}{2}P^2g_s \ln \rho + \left( P^2g_s \ln \rho - \frac{1}{2}K_0 \right) f_{a,1,0}\rho + \left( \left( -\frac{1}{4}P^2g_s \\
- \frac{5}{4}P^2g_s \ln \rho + \frac{5}{8}K_0 \right) f_{a,1,0}^2 + \frac{119}{576}P^4g_s^2H^2 + \frac{31}{96}P^2g_sH^2K_0 + \frac{1}{8}H^2K_0^2 + \frac{1}{2}P^4g_sH^2\ln \rho^2 \\
- \frac{31}{48}P^4g_s^2H^2\ln \rho - \frac{1}{2}P^2g_sH^2K_0\ln \rho \right) \rho^2 + \left( \left( \frac{5}{4}P^2g_s \ln \rho + \frac{11}{24}P^2g_s - \frac{5}{8}K_0 \right) f_{a,1,0}^3 \\
+ \left( -\frac{3}{2}P^4g_s^2 \ln \rho^2 + \frac{23}{16}P^4g_s^2 \ln \rho - \frac{19}{64}P^4g_s^2 + \frac{3}{2}P^2g_sK_0 \ln \rho - \frac{23}{32}P^2g_sK_0 \\
- \frac{3}{8}K_0^2 \right) H^2f_{a,1,0} \right) \rho^3 + \sum_{n=4}^{\infty} \sum_{k} h_{n,k} \rho^n \ln^k \rho, \quad \text{ (B.20)} \]

\[ K_1 = K_0 - 2P^2g_s \ln \rho + P^2g_s f_{a,1,0} \rho + \left( -\frac{1}{4}P^2f_{a,1,0}^2g_s - \frac{1}{4}P^4g_s^2H^2\ln \rho + \frac{9}{16}P^4g_s^2H^2 \\
+ \frac{1}{8}P^2g_sH^2K_0 \right) \rho^2 + \left( \frac{1}{12}f_{a,1,0}^3P^2g_s + \frac{1}{48}P^2g_s \left( 36P^2g_s \ln \rho - 13P^2g_s \\
- 6K_0 \right) H^2f_{a,1,0} + \frac{2}{3}P^2g_s \left( 3f_{a,3,0} \ln \rho + f_{a,3,0} + k_{2,3,0} \right) \right) \rho^3 \\
+ \sum_{n=4}^{\infty} \sum_{k} k_{1,n,k} \rho^n \ln^k \rho, \quad \text{ (B.21)} \]

\[ K_2 = 1 + \left( k_{2,3,0} + \frac{3}{4}H^2f_{a,1,0}P^2g_s \ln \rho + 3f_{a,3,0} \ln \rho \right) \rho^3 + \sum_{n=4}^{\infty} \sum_{k} k_{2,n,k} \rho^n \ln^k \rho, \quad \text{ (B.22)} \]
\[ K_3 = K_0 - 2 P^2 g_s \ln \rho + P^2 g_s f_{a,1,0} \rho + \left( -\frac{1}{4} P^4 g_s f_{a,1,0}^2 - \frac{1}{4} P^4 g_s^2 H^2 \ln \rho + \frac{9}{16} P^4 g_s^2 H^2 \right) \]
\[ + \frac{1}{8} P^2 g_s H^2 K_0 \right) \rho^2 + \left( \frac{1}{12} f_{a,1,0}^3 P^2 g_s - \frac{1}{48} P^2 g_s \left( 12 P^2 g_s \ln \rho + 29 P^2 g_s \right) \rho^3 \]
\[ + 6 K_0 \right) H^2 f_{a,1,0} - \frac{2}{3} P^2 g_s \left( 3 f_{a,3,0} \ln \rho + f_{a,3,0} + k_{2,3,0} \right) \rho^3 \]
\[ + \sum_{n=4}^{\infty} \sum_{k} k_{3,n,k} \rho^n \ln^k \rho , \]  \( \text{(B.23)} \)

\[ g = g_s \left( 1 - \frac{1}{2} P^2 g_s H^2 \rho^2 + \frac{1}{2} f_{a,1,0} P^2 g_s H^2 \rho^3 + \sum_{n=4}^{\infty} \sum_{k} g_{n,k} \rho^n \ln^k \rho \right) . \] \( \text{(B.24)} \)

It is characterized by 11 parameters:

\[ \{ K_0 , H , g_s , f_{a,1,0} , f_{a,3,0} , k_{2,3,0} , g_{4,0} , f_{c,4,0} , f_{a,6,0} , f_{a,7,0} , f_{a,8,0} \} , \] \( \text{(B.25)} \)

where we indicated the dual cascading gauge theory operators which expectation values these parameters characterize. \( g_s \) is the asymptotic string coupling, and \( K_0 \) is related to strong coupling scale \( \Lambda \) of the cascading gauge theory (see appendix [31] as [31])

\[ \Lambda^2 = \frac{1}{P^2 g_s} e^{\frac{K_0}{P^2 g_s}} . \] \( \text{(B.26)} \)

Finally, \( f_{a,1,0} \) corresponds to a diffeomorphism parameter \( -2\lambda \) in symmetry transformation SFG1, see \( \text{(B.12)} \).

To understand IR asymptotics of the FG frame solutions it is convenient to consider Euclidean continuation of the background geometry \( \text{(B.1)} \). For a fixed radial coordinate \( \rho \) the resulting Euclidean space is topologically \( S^4 \times S^2 \times S^3 \), where \( S^4 \) is an analytical continuation of \( \mathcal{M}^c \) \( \text{(B.16)} \), and \( S^2 \times S^3 \) is a compact part of the warped deformed conifold\(^{25}\). Without loss of generality we assume that the radial coordinate

\[ \rho \in [0, +\infty) , \] \( \text{(B.27)} \)

so that \( y \equiv \frac{1}{\rho} \) corresponds to the IR asymptotic. The range \( \text{(B.27)} \) can always be enforced with an appropriate symmetry transformation SFG1 \( \text{(B.12)} \). Ten dimensional Euclidean manifold is geodesically complete if one of the compact factors \( S^4 \) or \( S^2 \) smoothly shrinks to zero size as \( y \to 0 \). Note that \( S^3 \) can not shrink to zero size without

\[ ^{25}\text{See [4] for a nice review.} \]
causing a naked singularity since it supports nonzero (when $P \neq 0$) RR 3-form flux (2.5). Thus, from purely topological considerations we expect several inequivalent de Sitter vacua of cascading gauge theory: TypeA (shrinking $S^4$) and TypeB (shrinking $S^2$).

- **TypeA de Sitter vacua of cascading gauge theory.** To identify smooth Euclidean FG frame geometries with vanishing $S^4$ as $y \to 0$ we introduce

$$h^h \equiv y^{-2} \, h, \quad f^h_{a,b,c} \equiv y \, f_{a,b,c}.$$  

The IR asymptotic expansion

$$f_{a,b,c}^h = \sum_{n=0} \, f_{a,b,c,n}^h \, y^n, \quad h^h = \frac{1}{4H^2} + \sum_{n=1} \, h_n^h \, y^n,$$

$K_{1,2,3} = \sum_{n=0} \, K_{1,2,3,n}^h \, y^n, \quad g = \sum_{n=0} \, g_n^h \, y^n,$

is characterized by 7 parameters:

$$\{ f_{a,0}^h, f_{b,0}^h, f_{c,0}^h, K_{1,0}^h, K_{2,0}^h, K_{3,0}^h, g_0^h \}.$$  

(B.29)

Note that given (B.29),

$$\frac{1}{H^2} (dS^4)^2 + \frac{h^1/2 \rho^2}{\rho^2} \frac{(d\rho)^2}{\tau^{\phi_{1,2,3}/\rho^2}} \quad \vdash \quad \frac{1}{H^2} (dS^4)^2 + \frac{h^1/2 \rho^2}{\rho^2} \frac{(d\rho)^2}{\tau^{\phi_{1,2,3}/\rho^2}} \quad \vdash \quad \frac{1}{H^2} \left( z^2 (dS^4)^2 + (dz)^2 \right).$$  

(B.31)

i.e., $S^4$ indeed smoothly shrinks to zero size as $y \to 0$. It is important to emphasize that TypeA vacua defined by (B.29) have either $U(1)$ or $Z_2$ chiral symmetry — chiral symmetry is unbroken in the former (TypeA\(_s\)), and spontaneously broken in the latter (TypeA\(_b\)).

- **TypeB de Sitter vacua of cascading gauge theory.** To identify smooth Euclidean FG frame geometries with vanishing $S^2$ as $y \to 0$ we introduce

$$h^h \equiv y^{-4} \, h, \quad f^h_{a,b,c} \equiv y^2 \, f_{a,b,c}.$$  

(B.32)

\[26\] Other holographic models in this class were discussed earlier in \[29, 30, 34, 36\].
On the other hand, the 3-cycle supporting RR flux remains finite, provided
\[ f_h^h = f_{a,0}^h + \sum_{n=1}^{h} f_{a,n}^h y^{2n} , \quad f_b^h = 3y^2 + \sum_{n=2} f_{b,n}^h y^{2n} , \quad f_{c}^h = \frac{3}{4} f_{a,0}^h + \sum_{n=1}^{h} f_{c,n}^h y^{2n} , \]
\[ K_1 = k_{1,3}^h y^3 + \sum_{n=2} k_{1,n}^h y^{2n+1} , \quad K_2 = k_{2,2}^h y^2 + k_{2,4}^h y^4 + \sum_{n=3} k_{2,n}^h y^{2n} , \]
\[ K_3 = k_{3,1}^h y + \sum_{n=1}^{h} k_{3,n}^h y^{2n+1} , \quad h^h = h_0^h + \sum_{n=1}^{h} h_{n}^h y^{2n} , \quad g = g_0^h + \sum_{n=1}^{h} g_{n}^h y^{2n} , \]
(B.33)
is characterized by 7 parameters:
\[ \{ f_{a,0}^h , h_0^h , k_{1,3}^h , k_{2,2}^h , k_{2,4}^h , k_{3,1}^h , g_0^h \} . \] (B.34)

Note that given (B.33),
\[ \frac{h_{1/2}}{\rho^2} (d\rho)^2 + \frac{f_{a}^h h_{1/2}}{6} (g_2^2 + g_2^2) = (h^h)^{1/2} (dy)^2 + (h^h)^{1/2} y^2 \frac{1}{2} (g_1^2 + g_2^2) \bigg|_{2\text{-cycle}} \]
\[ \longrightarrow \quad (h_0^h)^{1/2} \left( y^2 (dS^2)^2 + (dy)^2 \right) , \] (B.35)

where \( \bigg|_{S^2} \) means restriction to a 2-cycle. Following [1], this means setting \( \psi = 0, \phi_2 = -\phi_1, \theta_2 = -\theta_1 \) in one-forms \( \{ g_i \} \) on \( T^{1,1} \):
\[ (g_1^2 + g_2^2) \bigg|_{2\text{-cycle}} = 2 ((d\theta_1)^2 + \sin^2 \theta_1 (d\phi_1)^2) = 2 (dS^2)^2 . \] (B.36)

On the other hand, the 3-cycle supporting RR flux remains finite, provided \( f_{a,0}^h h_0^h \neq 0 \):
\[ \frac{f_{a}^h h_{1/2}}{9} g_5^2 + \frac{f_{a}^h h_{1/2}}{6} (g_3^2 + g_4^2) = \frac{f_{c}^h (h^h)^{1/2}}{9} g_5^2 + \frac{f_{a}^h (h^h)^{1/2}}{6} (g_3^2 + g_4^2) \]
\[ \longrightarrow \quad \frac{f_{a,0}^h (h_0^h)^{1/2}}{6} \left( \frac{1}{2} g_5^2 + g_3^2 + g_4^2 \right) \bigg|_{3\text{-cycle}} : \theta_2 = \phi_2 = 0, \theta_1 = 2\eta, \psi = \xi_1 + \xi_2, \phi_1 = \xi_1 - \xi_2 \]
\[ = \frac{f_{a,0}^h (h_0^h)^{1/2}}{6} 2 ((d\eta)^2 + \cos^2 \eta (d\xi_1)^2 + \sin^2 \eta (d\xi_2)^2) = \frac{f_{a,0}^h (h_0^h)^{1/2}}{3} (dS^3)^2 . \] (B.37)

From (B.35), \( S^2 \) indeed smoothly shrinks to zero size as \( y \to 0 \). Because \( f_a \neq f_b \) as \( y \to 0 \), TypeB vacua defined by (B.33) have \( \mathbb{Z}_2 \) chiral symmetry — chiral symmetry is spontaneously broken.
B.1.1 TypeA vacua asymptotics

We provide here connection with the extensive earlier studies of TypeA vacua in \[31\].

Chirally symmetric de Sitter vacua of cascading gauge theory (TypeA) correspond to a consistent truncation

\[
f_c \equiv f_2, \quad f_a = f_b \equiv f_3, \quad K_1 = K_3 \equiv K, \quad K_2 = 1.
\]

We find:

- in the UV, \textit{i.e.}, as \(\rho \to 0\),

\[
f_2 = 1 + f_{2,1,0} \rho + \left( -\frac{3}{8} H^2 P^2 g_s - \frac{1}{4} H^2 K_0 + \frac{1}{4} f^2_{2,1,0} + \frac{1}{2} H^2 P^2 g_s \ln \rho \right) \rho^2
\]

\[
- \frac{1}{4} H^2 P^2 g_s f_{2,1,0} \rho^3 + \sum_{n=4}^{\infty} \sum_{k} f_{2,n,k} \rho^n \ln^k \rho,
\]

\[
f_3 = 1 + f_{2,1,0} \rho + \left( -\frac{1}{2} H^2 P^2 g_s - \frac{1}{4} H^2 K_0 + \frac{1}{4} f^2_{2,1,0} + \frac{1}{2} H^2 P^2 g_s \ln \rho \right) \rho^2
\]

\[
- \frac{1}{4} H^2 P^2 g_s f_{2,1,0} \rho^3 + \sum_{n=4}^{\infty} \sum_{k} f_{3,n,k} \rho^n \ln^k \rho,
\]

\[
h = \frac{1}{8} P^2 g_s + \frac{1}{4} K_0 - \frac{1}{2} P^2 g_s \ln \rho - \frac{1}{2} f_{2,1,0} (-2 P^2 g_s \ln \rho + K_0) \rho + \left( \frac{119}{576} H^2 P^4 g_s^2
\]

\[
+ \frac{31}{96} H^2 K_0 P^2 g_s - \frac{31}{4} P^2 g_s f_{2,1,0} + \frac{1}{8} H^2 K_0^2 + \frac{5}{8} f^2_{2,1,0} K_0 - \frac{1}{96} P^2 g_s (62 H^2 P^2 g_s
\]

\[
+ 4 H^2 K_0 + 120 f^2_{2,1,0} \ln \rho + \frac{1}{2} H^2 P^4 g_s^2 \ln^2 \rho \right) \rho^2 - \frac{1}{192} f_{2,1,0} \left( 288 H^2 P^4 g_s^2 \ln^2 \rho
\]

\[
+ \left( -276 H^2 P^4 g_s^2 - 288 H^2 K_0 P^2 g_s - 240 P^2 g_s f^2_{2,1,0} \right) \ln \rho + \frac{57}{H^2 P^4 g_s^2
\]

\[
+ 138 H^2 K_0 P^2 g_s - 88 P^2 f^2_{2,1,0} g_s + 72 H^2 K_0^2 + 120 f^2_{2,1,0} K_0 \right) \rho^3 + \sum_{n=4}^{\infty} \sum_{k} h_{n,k} \rho^n \ln^k \rho,
\]

\[
K = K_0 - 2 P^2 g_s \ln \rho + P^2 g_s f_{2,1,0} \rho + \left( \frac{1}{16} P^2 g_s (-4 H^2 P^2 g_s \ln \rho + 9 H^2 P^2 g_s + 2 H^2 K_0
\]

\[
- 4 f^2_{2,1,0} \right) \rho^2 - \frac{1}{48} P^2 g_s f_{2,1,0} (-12 H^2 P^2 g_s \ln \rho + 21 H^2 P^2 g_s + 6 H^2 K_0 - 4 f^2_{2,1,0} \right) \rho^3
\]

\[
+ \sum_{n=4}^{\infty} \sum_{k} k_{n,k} \rho^n \ln^k \rho,
\]

\[(B.42)\]
\[ g = g_s \left( 1 - \frac{1}{2} H^2 P^2 g_s \rho^2 + \frac{1}{2} H^2 P^2 g_s f_{2,1,0} \rho^3 + \sum_{n=4}^{\infty} \sum_{k} g_{n,k} \rho^n \ln^k \rho \right), \quad \text{(B.43)} \]

characterized by 8 parameters:

\[ \{ K_0, H, g_s, f_{2,1,0}, g_{4,0}, f_{2,4,0}, f_{2,6,0} f_{2,8,0} \}; \quad \text{(B.44)} \]

- in the IR, i.e., as \( y \equiv \frac{1}{\rho} \to 0 \),

\[ f_{2,3}^h = \sum_{n=0} f_{2,3,n} y^n, \quad h^h = \frac{1}{4 H^2} + \sum_{n=1} h_{n}^h y^n, \]

\[ K = \sum_{n=0} K_{n}^h y^n, \quad g = \sum_{n=0} g_{n}^h y^n, \quad \text{(B.45)} \]

characterized by 4 parameters:

\[ \{ f_{2,0}^h, f_{3,0}^h, K_0^h, g_0^h \}. \quad \text{(B.46)} \]

Comparing (B.17)-(B.24) with (B.39)-(B.43) to \( O(\rho^8) \) we identify

\[ f_{a,1,0} = f_{2,1,0}, \quad f_{a,3,0} = -\frac{1}{4} H^2 P^2 g_s f_{2,1,0}, \quad k_{2,3,0} = 0, \quad f_{c,4,0} = f_{2,4,0}, \quad \text{(B.47)} \]

\[ f_{a,6,0} = \left( -\frac{90687}{409600} K_0 P^4 g_s^2 - \frac{3409}{409600} K_0^2 P^2 g_s - \frac{11056513}{245760000} P^6 g_s^3 \right) H^6 + \left( \frac{1171}{20480} P^4 g_s^2 f_{2,1,0} - \frac{13}{10240} K_0 P^2 g_s f_{2,1,0}^2 \right) H^4 + \left( -\frac{1}{512} P^2 g_s f_{2,1,0} - \frac{307}{1280} P^2 g_s f_{2,4,0} \right) \]

\[ + \frac{31}{320} P^2 g_s g_{4,0} - \frac{87}{640} K_0 f_{2,4,0} \right) H^2 - \frac{1}{4} f_{2,6,0} + \frac{3}{16} f_{2,1,0} f_{2,4,0}, \quad \text{(B.48)} \]

\[ f_{a,7,0} = \left( \frac{13331}{196608} P^6 g_s^3 f_{2,1,0} + \frac{753}{16384} P^4 g_s^2 f_{2,1,0} K_0 + \frac{547}{24576} K_0^2 f_{2,1,0} P^2 g_s \right) H^6 + \left( \frac{2077}{18432} P^4 g_s^2 f_{2,1,0}^3 - \frac{77}{3072} K_0 P^2 g_s f_{2,1,0}^3 \right) H^4 + \left( \frac{21}{1280} P^2 g_s f_{2,1,0}^5 + \frac{19}{64} K_0 f_{2,1,0} f_{2,4,0} \right) \]

\[ + \frac{61}{128} f_{2,1,0}^2 P^2 g_s f_{2,4,0} - \frac{7}{32} f_{2,1,0} f_{2,4,0} g_{4,0} \right) H^2 - \frac{3}{8} f_{2,1,0} f_{2,4,0} + \frac{1}{2} f_{2,1,0} f_{2,6,0}, \quad \text{(B.49)} \]
Comparing (B.29) with (B.45) we identify was worked out in [19]. Specifically, given

\[ f_{a,s,0} = \frac{1}{70K_0 - 141P^2g_s} \left[ \left( -\frac{40244584228943}{56899584000000} K_0 P^8 g_s^4 - \frac{12213914790101}{3034644480000} K_0^2 P^6 g_s^3 + \frac{931679}{4915200} K_0^4 P^2 g_s^2 - \frac{173957}{81920} P^2 g_s K_0 f_{2,1,0}^2 - \frac{504197}{1433600} P^4 g_s f_{2,1,0}^2 K_0^2 \right) H^6 + \left( \frac{1892623}{92160} P^4 g_s^2 K_0 f_{2,1,0}^4 + \frac{63}{8} P^2 g_s K_0^2 f_{2,1,0}^2 \right) \right], \]

\[ f_{c,0}^h = f_{2,0}^h, \quad f_{h,0}^h = f_{3,0}^h, \quad K_{1,0}^h = K_{3,0}^h = K_{0}^h, \quad K_{2,0}^h = 1. \] (B.51)

B.2 From FG to EF frame

A general map between the FG and EF frame de Sitter vacua of the holographic duals was worked out in [19]. Specifically, given

\[ ds^2_{EF} = 2dt \ (dr - a(r) \ dt) + \sigma(r)^2 e^{2Ht} \ d\mathbf{x}^2 + \cdots, \]

\[ ds^2_{FG} = c_1(\rho)^2 \ (-d\tau^2 + e^{2H\tau} \ d\mathbf{x}^2) + c_2(\rho)^2 \ (d\rho)^2 + \cdots, \] (B.52)
where \( \cdots \) are metric components along the compact directions,

\[
    r = - \int_0^\rho ds \ c_1(s)c_2(s) + \text{const}, \quad t = \tau - H \int_0^\rho ds \ \frac{c_2(s)}{c_1(s)}, \quad \sigma(r) = c_1(\rho) \ \exp \left[ H \int_0^\rho ds \ \frac{c_2(s)}{c_1(s)} \right].
\]  

Using (B.31), we find from (B.28)-(B.29), (B.32)-(B.33), and (B.33) the following asymptotic expansions for the EF frame vacua:

\[
    r = \frac{1}{\rho} + \text{const} = y + \text{const}, \quad a = \frac{1}{2h^{1/2}\rho^2}, \quad t = \tau - H \int_0^\rho ds \ h(s)^{1/2}. \quad \text{(B.54)}
\]

Note that asymptotically in UV, i.e., as \( \rho \to 0 \), the EF and the FG times coincide:

\[
    t - \tau \sim -H \int_0^\rho ds \left( -\frac{1}{2} \rho^2 g_s \ln s \right)^{1/2} \to 0. \quad \text{(B.55)}
\]

Without loss of generality we fix \( \text{const} \) in (B.54) so that \( r = 0 \iff \frac{1}{\rho} \equiv y = 0 \). Introducing

\[
    z \equiv -r, \quad \text{(B.56)}
\]

we find from (B.28)-(B.29), (B.32)-(B.33), and (B.33) the following asymptotic expansions for the EF frame vacua:

- **TypeA** vacua:

  \[
  a = -Hz + \frac{H((f_{2,0}^h)^2(f_{3,0}^h)^2 - 6f_{2,0}^h(f_{3,0}^h)^3 + 3H^2P^2(f_{2,0}^h)^2g_0^h + 10H^4(K_0^h)^2)}{5(f_{3,0}^h)^4F_{2,0}^h}z^2 + O(z^3),
  \]

  \[
  \sigma = s_0^h \left( 1 - \frac{(f_{2,0}^h)^2(f_{3,0}^h)^2 - 6f_{2,0}^h(f_{3,0}^h)^3 + 3H^2P^2(f_{3,0}^h)^2g_0^h + 10H^4(K_0^h)^2}{5(f_{3,0}^h)^4f_{3,0}^h}z + O(z^2) \right),
  \]

  \[
  w_{c2} \equiv f_{2}^h(h^h)^{1/2} = \frac{f_{2,0}^h}{2H} - \frac{2}{5H} \frac{4(f_{2,0}^h)^2(f_{3,0}^h)^2 - 3H^2P^2(f_{2,0}^h)^2g_0^h - 4H^4(K_0^h)^2}{(f_{3,0}^h)^4}z + O(z^2),
  \]

  \[
  w_{a2} = w_{b2} \equiv f_{3}^h(h^h)^{1/2} = \frac{f_{3,0}^h}{2H} + \frac{2}{5H} \frac{2(f_{2,0}^h)^2(f_{3,0}^h)^2 - 6f_{2,0}^h(f_{3,0}^h)^3 + H^2P^2(f_{3,0}^h)^2g_0^h + 4H^4(K_0^h)^2}{(f_{3,0}^h)^3f_{2,0}^h}z + O(z^2),
  \]

  \[
  K_1 = K_3 \equiv K = K_0^h - \frac{16H^2P^2K_0^h}{5(f_{3,0}^h)^2f_{2,0}^h}z + O(z^2), \quad K_2 = 1,
  \]

  \[
  g = \frac{8}{5} \frac{H^2P^2(g_0^h)^2}{(f_{3,0}^h)^2f_{2,0}^h}z + O(z^2);
  \]  

\[
\text{(B.57)}
\]
- TypeA, vacua:

\[
\begin{align*}
a &= -Hz + \frac{H}{80(f_{a,0}^h)^2(f_{b,0}^h)^2g_0^h f_{c,0}^h P^2} \left( 24H^2(g_0^h)^2((K_{2,0}^h)^2(f_{a,0}^h)^2 + (K_{2,0}^h)^2(f_{b,0}^h)^2 \\
- 4K_{2,0}^h(f_{a,0}^h)^2 + 4(f_{b,0}^h)^2)P^4 + g_0^h(40H^4(K_{1,0}^h)^2(K_{2,0}^h)^2 - 80H^4K_{1,0}^hK_{2,0}^hK_{3,0}^h \\
+ 40H^4(K_{2,0}^h)^2(K_{3,0}^h)^2 - 160H^4(K_{1,0}^h)^2K_{2,0}^h + 160H^4K_{1,0}^hK_{2,0}^hK_{3,0}^h \\
+ 160H^4(K_{1,0}^h)^2 + 9(f_{a,0}^h)^3f_{b,0}^h - 18(f_{a,0}^h)^2(f_{b,0}^h)^2 - 48(f_{a,0}^h)^2f_{b,0}^h(f_{c,0}^h)^2 + 9f_{a,0}^h(f_{b,0}^h)^3 \\
- 48f_{a,0}^h(f_{b,0}^h)^2f_{c,0}^h + 16f_{a,0}^h(f_{c,0}^h)^2P^2 + 27H^2f_{a,0}^hf_{b,0}^h(f_{c,0}^h)^2 \right)z^2 \\
+ O(z^3),
\end{align*}
\]

\[\sigma = s_0^h \left( 1 - \frac{1}{80(f_{a,0}^h)^2(f_{b,0}^h)^2g_0^h f_{c,0}^h P^2} \right) \left( 24H^2(g_0^h)^2((K_{2,0}^h)^2(f_{a,0}^h)^2 + (K_{2,0}^h)^2(f_{b,0}^h)^2 \\
- 4K_{2,0}^h(f_{a,0}^h)^2 + 4(f_{b,0}^h)^2)P^4 + g_0^h(40H^4(K_{1,0}^h)^2(K_{2,0}^h)^2 - 80H^4K_{1,0}^hK_{2,0}^hK_{3,0}^h \\
+ 40H^4(K_{2,0}^h)^2(K_{3,0}^h)^2 - 160H^4(K_{1,0}^h)^2K_{2,0}^h + 160H^4K_{1,0}^hK_{2,0}^hK_{3,0}^h \\
+ 160H^4(K_{1,0}^h)^2 + 9(f_{a,0}^h)^3f_{b,0}^h - 18(f_{a,0}^h)^2(f_{b,0}^h)^2 - 48(f_{a,0}^h)^2f_{b,0}^h(f_{c,0}^h)^2 + 9f_{a,0}^h(f_{b,0}^h)^3 \\
- 48f_{a,0}^h(f_{b,0}^h)^2f_{c,0}^h + 16f_{a,0}^h(f_{c,0}^h)^2P^2 + 27H^2f_{a,0}^hf_{b,0}^h(f_{c,0}^h)^2 \right)z \\
+ O(z^2) \right),
\]

\[w_{c,2} \equiv f_{c,0}^h(h)^{1/2} = \frac{f_{c,0}^h}{2H} + \frac{1}{H(f_{a,0}^h)^2(f_{b,0}^h)^2g_0^h f_{c,0}^h P^2} \left( \frac{3}{5}H^2(g_0^h)^2((K_{2,0}^h)^2(f_{a,0}^h)^2 \\
+ (K_{2,0}^h)^2(f_{b,0}^h)^2 - 4K_{2,0}^h(f_{b,0}^h)^2 + 4(f_{b,0}^h)^2)P^4 + \frac{1}{10}g_0^h(4H^4(K_{1,0}^h)^2(K_{2,0}^h)^2 \\
- 8H^4K_{1,0}^h(K_{2,0}^h)^2K_{3,0}^h + 4H^4(K_{2,0}^h)^2(K_{3,0}^h)^2 - 16H^4(K_{1,0}^h)^2K_{2,0}^h \\
+ 16H^4K_{1,0}^hK_{2,0}^hK_{3,0}^h + 16H^4(K_{1,0}^h)^2 + 9(f_{a,0}^h)^3f_{b,0}^h - 18(f_{a,0}^h)^2(f_{b,0}^h)^2 \\
+ 9f_{a,0}^h(f_{b,0}^h)^3 - 16f_{a,0}^hf_{b,0}^h(f_{c,0}^h)^2P^2 + \frac{27}{40}H^2f_{a,0}^hf_{b,0}^h(f_{c,0}^h)^2 \right)z + O(z^2),
\]

72
\[ w_{a2} \equiv f^h_a(h^h)^{1/2} = \frac{f_{a,0}^h}{2H} + \frac{1}{f_{a,0}^h H(f_{b,0}^h)^2 g_0^h f_{c,0}^h P^2} \left( -\frac{1}{5} H^2 (g_0^h)^2 ((K_{2,0}^h)^2 (f_{a,0}^h)^2 - 3(K_{2,0}^h)^2 (f_{a,0}^h)^2 \right.
- 12(K_{2,0}^h)^2 (f_{a,0}^h)^2 - 12(f_{b,0}^h)^2 P^4 + \frac{1}{20} g_0^h (8H^4(K_{1,0}^h)^2 (K_{2,0}^h)^2 - 16H^4 K_{1,0}^h (K_{2,0}^h)^2 K_{3,0}^h + 8H^4 (K_{2,0}^h)^2 (K_{3,0}^h)^2 - 32H^4 (K_{1,0}^h)^2 K_{2,0}^h K_{3,0}^h + 32H^4 (K_{1,0}^h)^2 - 9(f_{a,0}^h)^3 f_{b,0}^h + 9 f_{a,0}^h (f_{b,0}^h)^3 - 48 f_{a,0}^h (f_{b,0}^h)^2 f_{c,0}^h + 16 f_{a,0}^h f_{b,0}^h (f_{c,0}^h)^2 P^2
+ \frac{9}{40} H^2 f_{a,0}^h f_{b,0}^h (K_{1,0}^h - K_{3,0}^h)^2 \right) z + O(z^2). \]

\[ w_{b2} \equiv f^h_b(h^h)^{1/2} = \frac{f_{b,0}^h}{2H} + \frac{1}{f_{b,0}^h H(f_{a,0}^h)^2 g_0^h f_{c,0}^h P^2} \left( \frac{1}{5} H^2 (g_0^h)^2 (3(K_{2,0}^h)^2 (f_{a,0}^h)^2
- (K_{2,0}^h)^2 (f_{a,0}^h)^2 + 4K_{2,0}^h (f_{a,0}^h)^2 - 4(f_{b,0}^h)^2 P^4 + \frac{1}{20} g_0^h (8H^4(K_{1,0}^h)^2 (K_{2,0}^h)^2 - 16H^4 K_{1,0}^h (K_{2,0}^h)^2 K_{3,0}^h + 8H^4 (K_{2,0}^h)^2 (K_{3,0}^h)^2 - 32H^4 (K_{1,0}^h)^2 K_{2,0}^h K_{3,0}^h + 32H^4 (K_{1,0}^h)^2 + 9(f_{a,0}^h)^3 f_{b,0}^h - 48 f_{a,0}^h (f_{b,0}^h)^2 f_{c,0}^h - 9 f_{a,0}^h (f_{b,0}^h)^3 + 16 f_{a,0}^h f_{b,0}^h (f_{c,0}^h)^2 P^2
+ \frac{9}{40} H^2 f_{a,0}^h f_{b,0}^h (K_{1,0}^h - K_{3,0}^h)^2 \right) z + O(z^2). \]

\[ K_1 = K_{1,0}^h - \frac{1}{5(f_{a,0}^h)^2 f_{c,0}^h} \left( 8H^2 g_0^h (K_{2,0}^h - 2)(K_{1,0}^h K_{2,0}^h - K_{2,0}^h K_{3,0}^h - 2K_{1,0}^h) P^2
+ 9 f_{a,0}^h f_{b,0}^h (K_{1,0}^h - K_{3,0}^h) \right) z + O(z^2), \]

\[ K_2 = K_{2,0}^h - \frac{9}{5(f_{a,0}^h)^2 f_{b,0}^h g_0^h f_{c,0}^h P^2} \left( g_0^h (K_{2,0}^h f_{a,0}^h)^2 + K_{2,0}^h (f_{b,0}^h)^2 - 2(f_{b,0}^h)^2 P^4
+ H^2 (K_{1,0}^h - K_{3,0}^h) (K_{1,0}^h K_{2,0}^h - K_{2,0}^h K_{3,0}^h - 2K_{1,0}^h) \right) z + O(z^2), \]

\[ K_3 = K_{3,0}^h + \frac{1}{5(f_{b,0}^h)^2 f_{c,0}^h} \left( 8H^2 K_{2,0}^h g_0^h (K_{1,0}^h K_{2,0}^h - K_{2,0}^h K_{3,0}^h - 2K_{1,0}^h) P^2
+ 9 f_{a,0}^h f_{b,0}^h (K_{1,0}^h - K_{3,0}^h) \right) z + O(z^2), \]

\[ g = g_0^h - \frac{H^2}{10(f_{a,0}^h)^2 (f_{b,0}^h)^2 f_{c,0}^h P^2} \left( 8(g_0^h)^2 ((K_{2,0}^h)^2 (f_{a,0}^h)^2 + (K_{2,0}^h)^2 (f_{b,0}^h)^2 - 4(K_{2,0}^h)^2 (f_{b,0}^h)^2
+ 4(f_{b,0}^h)^2 P^4 - 9 f_{a,0}^h (K_{1,0}^h - K_{3,0}^h)^2 f_{b,0}^h \right) z + O(z^2). \]
TypeB vacua:

\[
a = \frac{1}{2(h_0^h)^{1/2}} + O(z^2), \quad \sigma = s_0^h \left( 1 + (h_0^h)^{1/2} H z + O(z^2) \right),
\]

\[
w_{c2} \equiv f_c(h^h)^{1/2} = \frac{3}{4} f_{a,0}(h_0^h)^{1/2} + O(z^2), \quad w_{a2} \equiv f_a(h^h)^{1/2} = f_{a,0}(h_0^h)^{1/2} + O(z^2),
\]

\[
w_{b2} \equiv f_b(h^h)^{1/2} = 3(h_0^h)^{1/2} z^2 + O(z^4), \quad K_1 = -k_{1,3} z^3 + O(z^5),
\]

\[
K_2 = k_{2,3} z^2 + O(z^4), \quad K_3 = -k_{3,1} z + O(z^3), \quad g = g_0^h + O(z^2), \tag{B.67}
\]

where

\[
s_0^h = \sigma \bigg|_{FG \ frame \ y=0}. \tag{B.68}
\]

B.3 Extremal KS solution limit \( H \to 0 \)

We review here extremal KS solution \cite{2} following \cite{31} and identify the relation of the strong coupling scale \( \Lambda \) \((B.26)\) to the conifold deformation parameter \( \epsilon \) \((B.70)\).

We use the radial coordinate \( \hat{r} \in [0, \infty) \) to describe KS solution:

\[
ds_5^2 = H_{KS}^{-1/2} (-dt^2 + dx^2) + H_{KS}^{1/2} \omega_{1,KS}^2 \, d\hat{r}^2, \quad \Omega_i = \omega_{i,KS} \, H_{KS}^{1/4}, \quad K_i = K_{i,KS}, \tag{B.69}
\]

\[
K_{1,KS} = \frac{2}{3} P^2 g_s \frac{\cosh \hat{r} - 1}{\sinh \hat{r}} \left( \frac{\hat{r} \cosh \hat{r}}{\sinh \hat{r}} - 1 \right), \quad K_{2,KS} = 1 - \frac{\hat{r}}{\sinh \hat{r}},
\]

\[
K_{3,KS} = \frac{2}{3} P^2 g_s \frac{\cosh \hat{r} + 1}{\sinh \hat{r}} \left( \frac{\hat{r} \cosh \hat{r}}{\sinh \hat{r}} - 1 \right), \quad g = g_s,
\]

\[
\omega_{1,KS} = \frac{\epsilon^{2/3}}{\sqrt{6} K_{KS}}, \quad \omega_{2,KS} = \frac{\epsilon^{2/3} \hat{r}_{KS}^{1/2}}{\sqrt{2}} \cosh \frac{\hat{r}}{2}, \quad \omega_{3,KS} = \frac{\epsilon^{2/3} \hat{r}_{KS}^{1/2}}{\sqrt{2}} \sinh \frac{\hat{r}}{2}, \tag{B.70}
\]

with

\[
\hat{K}_{KS} = \frac{(\sinh(2\hat{r}) - 2\hat{r})^{1/3}}{2^{1/3} \sinh \hat{r}}, \quad H_{KS} = \frac{2}{27} \frac{(K_{1,KS} - K_{3,KS}) K_{2,KS} - 2 K_{1,KS}}{\epsilon^{8/3} K_{KS}^2 \sin^2 \hat{r}}, \tag{B.71}
\]

where now \( \hat{r} \to \infty \) is the boundary and \( \hat{r} \to 0 \) is the IR.

Comparing the metric ansatz in \((B.69)\) and \((B.1)\) we identify

\[
\frac{\langle d\rho \rangle^2}{\rho^4} = (w_{1,KS}(\hat{r}))^2(d\hat{r})^2. \tag{B.72}
\]
Introducing
\[ z \equiv e^{-r/3}, \quad (B.73) \]
we find from \((B.72)\)
\[
\frac{1}{\rho} = \sqrt{6} \frac{(2\epsilon)^{2/3}}{4} \int_1^z du \frac{u^6 - 1}{u^2(1 - u^{12} + 12u^6 \ln u)^{1/3}}. \quad (B.74)
\]
In the UV, \(\hat{r} \to \infty\), \(z \to 0\) and \(\rho \to 0\) we have
\[
e^{-r/3} \equiv z = \frac{\sqrt{6} (2\epsilon)^{2/3}}{4} \rho \left(1 + Q\rho + Q^2\rho^2 + Q^3\rho^3 + Q^4\rho^4 + Q^5\rho^5 + \left(\frac{27}{80} \epsilon^4 \ln 3 + Q^6 \right.ight.
\]
\[
+ \left. \frac{27}{800} \epsilon^4 - \frac{9}{16} \epsilon^4 \ln 2 + \frac{9}{20} \epsilon^4 \ln \epsilon + \frac{27}{40} \epsilon^4 \ln \rho \right) \rho^6 + \left(-\frac{63}{16} \epsilon^4 Q \ln 2 + \frac{189}{80} \epsilon^4 Q \ln 3 + Q^7 \right.
\]
\[
+ \frac{729}{800} \epsilon^4 Q + \frac{63}{20} \epsilon^4 Q \ln \epsilon + \frac{189}{40} \epsilon^4 Q \ln \rho \right) \rho^7 + \left(\frac{2403}{400} \epsilon^4 Q^2 - \frac{63}{4} \epsilon^4 Q^2 \ln 2 + \frac{189}{20} \epsilon^4 Q^2 \ln 3 \right.
\]
\[
+ \frac{63}{5} \epsilon^4 Q^2 \ln \epsilon + Q^8 + \frac{189}{10} \epsilon^4 Q^2 \ln \rho \right) \rho^8 + \left(\frac{189}{5} \epsilon^4 Q^3 \ln \epsilon + \frac{9729}{400} \epsilon^4 Q^3 - \frac{189}{4} \epsilon^4 Q^3 \ln 2 \right.
\]
\[
+ \frac{567}{20} \epsilon^4 Q^3 \ln 3 + Q^9 + \frac{567}{10} \epsilon^4 Q^3 \ln \rho \right) \rho^9 + O(\rho^{10} \ln \rho), \quad (B.75)
\]
where
\[
Q = \frac{\sqrt{6} (2\epsilon)^{2/3}}{4} \left\{ \int_0^1 du \left(\frac{1 - u^6}{u^2(1 - u^{12} + 12u^6 \ln u)^{1/3}} - \frac{1}{u^2} \right) - 1 \right\} \quad (B.76)
\]
In the IR, \(\hat{r} \to 0\), \(z \to 1\) and \(\frac{1}{\rho} \to 0\) we have
\[
\hat{r} = \frac{\sqrt{6}}{3^{1/3} \epsilon^{2/3}} \sqrt{y} \left(1 - \frac{2^{2/3}}{15} y^{1/3} + \frac{71}{2625} \frac{3^{2/3}}{\epsilon^{8/3}} y^{4/3} + O(y^6) \right), \quad (B.77)
\]
Using \((B.75)\) and \((B.77)\), and the exact analytic solution describing the Klebanov-Strassler Minkowski vacuum of cascading gauge theory \((B.70)\), \((B.71)\) we can identify
parameters (B.25):

\[ K_0 = P^2 g_s \left( - \ln 3 + \frac{5}{3} \ln 2 - \frac{4}{3} \ln \epsilon - \frac{2}{3} \right), \quad f_{a,1,0} = -2Q, \]

\[ k_{2,3,0} = \frac{3\sqrt{6}}{8} \epsilon^2 (3 \ln 3 - 5 \ln 2 + 4 \ln \epsilon), \quad f_{c,4,0} = 0, \quad f_{a,3,0} = \frac{3\sqrt{6}}{4} \epsilon^2, \]

\[ f_{a,6,0} = \left( -\frac{27}{16} \ln 2 + \frac{81}{50} \ln 3 + \frac{27}{20} \ln \epsilon \right) \epsilon^4 + \frac{3\sqrt{6}}{4} Q^4 \epsilon^2, \]

\[ f_{a,7,0} = \left( \frac{27}{5} \ln \epsilon - \frac{27}{4} \ln 2 + \frac{81}{20} \ln 3 + \frac{1701}{200} \right) \epsilon^4 Q + \frac{3\sqrt{6}}{4} \epsilon^2 Q^4, \]

\[ f_{a,8,0} = \left( \frac{27}{2} \ln \epsilon - \frac{135}{8} \ln 2 + \frac{81}{8} \ln 3 + \frac{405}{16} \right) Q^2 \epsilon^4 + \frac{3\sqrt{6}}{4} Q^5 \epsilon^2, \quad g_{4,0} = 0, \]

in the UV, and parameters (B.34):

\[ f_{a,0}^h = 2^{1/3} 3^{2/3} \epsilon^{4/3}, \quad h_0^h = P^2 g_s \epsilon^{-8/3} \times 0.056288(0), \]

\[ k_{1,3}^h = \frac{4\sqrt{6}}{9} \epsilon^2 P^2 g_s, \quad k_{2,2}^h = \frac{2^{2/3}}{3^{2/3} \epsilon^{4/3}}, \quad k_{2,4}^h = -\frac{11}{45} \frac{2^{2/3}}{3^{2/3} \epsilon^{8/3}}, \]

\[ k_{3,1}^h = \frac{4\sqrt{6}}{27 \epsilon^{2/3}} P^2 g_s, \quad g_0^h = g_s, \]

in the IR.

Given (B.26), we identify from (B.78)

\[ \Lambda = \frac{3^{1/2} \epsilon^{1/3} \epsilon^{2/3}}{2^{5/6} (P^2 g_s)^{1/2}} = \frac{2^{1/6} \epsilon^{1/3} \epsilon^{2/3}}{3^{3/2} M_{\alpha'} g_s^{1/2}} = \frac{2^{1/6} \epsilon^{1/3} g_s^{1/2}}{3^{3/2} m_{\text{glueball}}} \approx 0.3 g_s^{1/2} m_{\text{glueball}}, \quad (B.80) \]

where in the second equality we used (2.7); the glueball mass scale is defined as in (1.10).

### C Numerical procedure

#### C.1 FG frame de Sitter vacua

Equations of motion for the FG frame de Sitter vacua of cascading gauge theory, along with the asymptotics and the symmetries of the dual holographic formulation, are presented in appendix B. Generically, we have eight functions of the radial coordinate \( \rho \), see (B.2). When the chiral symmetry is unbroken, there are only five functions, see (B.38). The solution to the equations of motion is unique\(^{27}\) once we fix the Hubble

\(^{27}\) Apart from the discrete choices associated with the IR boundary conditions leading to classification of topologically distinct holographic vacua: TypeA\(_{s,b}\) or TypeB, see appendix B.1.
constant $H$, the asymptotic string coupling $g_s$, the 3-form flux $P$ (alternatively the rank difference of gauge group factors $M$ in cascading theory), see (2.6) and (2.7), and the strong coupling scale $\Lambda$ of cascading gauge theory (alternatively $K_0$, see (B.26), or the conifold deformation parameter $\epsilon$, see (B.80)). Of these, parameters $H, \Lambda, P$ are dimensionful. The radial coordinate $\rho$ is dimensionful as well, albeit in units of 'mass'. As a result, UV/IR parameters of the solutions, see (B.25), (B.30) and (B.34), have complicated dimensional dependence. It is possible to completely eliminate the dimensional dependence (and the $g_s$ dependence) from all the equations of motion and the asymptotic expansions with appropriate rescaling:

$$\{\rho, f_{a,b,c}, h, K_{1,2,3}, g\} \implies \{\hat{\rho}, \hat{f}_{a,b,c}, \hat{h}, \hat{K}_{1,2,3}, \hat{g}\};$$

$$\rho = \frac{1}{HP^{1/2}g_s} \hat{\rho}, \quad f_{a,b,c} = \hat{f}_{a,b,c}, \quad h = P^2g_s \hat{h}, \quad K_{1,3} = P^2g_s \hat{K}_{1,3}, \quad K_2 = \hat{K}_2, \quad g = g_s \hat{g}. \quad \text{(C.1)}$$

Additionally we introduce a dimensionless parameter $k_s$ as

$$k_s \equiv \frac{K_0}{P^2g_s} + \ln \left(\frac{H^2P^2g_s}{\Lambda^2}\right), \quad \text{(C.2)}$$

leading from (B.26) to the identification

$$k_s = \ln \frac{H^2}{\Lambda^2} \cdot \quad \text{(C.3)}$$

Notice that the conformal limit in the cascading gauge theory, i.e., $H \gg \Lambda$, corresponds to $k_s \to \infty$.

We do not present the relations between all the UV/IR parameters stemming from (C.1) and (C.2) — they are straightforward to work out, but too long to be illuminating — and instead focus on the few ones for which we are reporting the numerical results:

- **TypeA** vacua,

$$f^h_{a,b,c,0} = HPg_s^{1/2} \hat{f}^h_{a,b,c,0}, \quad K^h_{1,3,0} = P^2g_s \hat{K}^h_{1,3,0}, \quad K^h_{2,0} = \hat{K}^h_{2,0}, \quad g^h_0 = g_s \hat{g}^h_0; \quad \text{(C.4)}$$

- **TypeB** vacua,

$$f^h_{a,0} = H^2P^2g_s \hat{f}^h_{a,0}, \quad h^h_0 = \frac{\hat{h}^h_0}{H^2P^{1/2}g_s}, \quad k^h_{1,3} = \frac{\hat{k}^h_{1,3}}{H^3P^{1/2}g_s}, \quad k^h_{2,2} = \frac{\hat{k}^h_{2,2}}{H^2P^2g_s}, \quad k^h_{2,4} = \frac{\hat{k}^h_{2,4}}{H^4P^4g_s}, \quad k^h_{3,1} = \frac{P^{1/2}g_s}{H^{3/2}} \hat{k}^h_{3,1}, \quad g^h_0 = g_s \hat{g}^h_0. \quad \text{(C.5)}$$
Numerical analysis of the bulk differential equations describing de Sitter vacua are rather involved. To trust them, we would like to have various consistency checks. Here, the symmetry transformations SFG2-SFG4 [B.13]-[B.15] are very useful: we can produce different data sets fixing three of the four parameters \( \{H, P, g_s, K_0\} \). As we demonstrate, with appropriate rescaling, the distinct data sets must collapse. We find it useful to implement three different computational schemes:

- Scheme I: \( H = P = g_s = 1 \), \( k_s \) is varied;
- Scheme II: \( H = g_s = K_0 = 1 \), \( b \equiv P^2 \) is varied;
- Scheme III: \( P = g_s = 1 \), \( K_0 = \frac{1}{4} \), \( \alpha \equiv H^2 \) is varied.

Note that:
- Scheme I is equivalent to performing computations in the hatted variables in (C.1), with (C.2);
- Scheme II is convenient to take a conformal limit to Klebanov-Witten solution [3] in Type A\(_s\) vacua: \( b \to 0 \);
- Scheme III is convenient to study the extremal KS [2] limit in Type B vacua: \( \alpha \to 0 \).

Numerical computations are done adopting the algorithms developed in [12]. Altogether, there are 8 second order differential equations (B.3)- (B.10) and a single first order constraint (B.11) for 8 functions \( \{f_a, f_b, f_c, h, K_1, K_2, K_3, g\} \). Notice that the constraint (B.11) involves \( f_c' \) linearly. Thus, we can use the latter equation and eliminate the redundant equation (B.3). The final set of ODEs — 7 second order equations and 1 first order equation — necessitates \( 15 = 2 \times 7 + 1 \) parameters.

Type A\(_s\),b vacua:

The result of the numerical computations are the data files with entries for the 8 UV parameters \( \{f_{a,1,0}, f_{a,3,0}, k_{2,3,0}, g_{4,0}, f_{c,4,0}, f_{a,6,0}, f_{a,7,0}, f_{a,8,0}\} \) and the 7 IR parameters \( \{f_h^a, f_h^b, f_h^c, K_1^h, K_2^h, K_3^h, g_0^h\} \) (see appendix B.1.1) labeled by \( k_s \) (for the computational scheme Scheme I), \( b \) (for the computational scheme Scheme II) or \( \alpha \) (for the computational scheme Scheme III). The number of parameters are reduced to 5 (in the UV) and 4 (in the IR) when chiral symmetry is unbroken (see appendix B.1.1).

Type B vacua:

The result of the numerical computations are the data files with entries for the 8 UV parameters \( \{f_{a,1,0}, f_{a,3,0}, k_{2,3,0}, g_{4,0}, f_{c,4,0}, f_{a,6,0}, f_{a,7,0}, f_{a,8,0}\} \) and the 7 IR parameters \( \{f_h^a, h_1^h, k_{1,3}^h, k_{2,2}^h, k_{2,4}^h, k_{3,1}^h, g_0^h\} \) (see appendix B.1) labeled\(^{28}\) by \( k_s \) (for the computational scheme Scheme II).

\(^{28}\) We will not use the computation scheme Scheme II here.
tational scheme SchemeI), or α (for the computational scheme SchemeIII).

### C.2 EF frame de Sitter vacua

In total, there are 11 (8 with unbroken chiral symmetry) coupled ODEs (A.16)-(A.26) describing EF frame de Sitter vacua involving 5 metric warp factors \{a, \sigma, w_{a2}, w_{b2}, w_{c2}\} (see (2.13)), 3 flux functions \{K_1, K_2, K_3\} (see (2.11)) and the string coupling \(g\) as a function of a radial coordinate \(z \equiv -r\), see (B.56). The full set of ODEs is redundant, and in practice we use 9 equations (A.17)-(A.25): we drop (A.16) in favor of (A.25), and we use (A.20) (it involves \(w'_{c2}\) linearly) instead of (A.26) (though it involves \(w''_{c2}\) linearly). The reason for this is to reduce the complexity of the system of ODEs — unlike construction of de Sitter vacua in FG frame which is a boundary value problem, representation of de Sitter vacua in EF frame is an initial value problem, and thus we can get away with using a higher order system of ODEs.

The initial conditions for these equations are set at \(z \to 0_+\) with asymptotic expansions (B.57) for Type\(\mathrm{A}_s\) de Sitter vacua, and with asymptotic expansions (B.58)-(B.66) for Type\(\mathrm{A}_b\) de Sitter vacua. The EF frame equations of motion are integrated on the interval

\[
z \in [0, z_{\text{AH}}],
\]

where \(z_{\text{AH}}\) is the first zero of the AH location function \(\mathcal{L}_{\text{AH}}\) (see (3.32)):

\[
\mathcal{L}_{\text{AH}}(z) \equiv 3H \sigma^3 \omega_{c2}^{1/2} \omega_{a2} \omega_{b2} - a \frac{d}{dz} \left\{ \sigma^3 \omega_{c2}^{1/2} \omega_{a2} \omega_{b2} \right\}.
\]

Using (B.57)-(B.66),

\[
\begin{align*}
\text{Type } \mathrm{A}_s :& \quad \mathcal{L}_{\text{AH}} = \frac{3\sqrt{2}}{8H^{3/2}} (s_0^h)^3 (f_{2,0}^h)^{1/2} (f_{3,0}^h)^2 + \mathcal{O}(z); \\
\text{Type } \mathrm{A}_b :& \quad \mathcal{L}_{\text{AH}} = \frac{3\sqrt{2}}{8H^{3/2}} (s_0^h)^3 (f_{c,0}^h)^{1/2} f_{a,0}^h f_{b,0}^h + \mathcal{O}(z),
\end{align*}
\]

\(i.e.,\) both for Type\(\mathrm{A}_s\) and Type\(\mathrm{A}_b\) vacua

\[
\mathcal{L}_{\text{AH}}(z = 0) > 0, \quad \frac{d}{dz} \mathcal{L}_{\text{AH}}(z = 0) < 0,
\]

where the second inequality is a numerical observation. Notice that to set-up the initial conditions for (A.17)-(A.25), besides the FG frame IR data (B.30) (or (B.46) when the
chiral symmetry is unbroken), one needs parameter $s_0^h$, see \([89x714]B.68\),

\[
s_0^h = \lim_{z \to 0^+} \sigma(z) = \lim_{\rho \to +\infty} \left\{ c_1(\rho) \exp \left[ H \int_0^\rho ds \frac{c_2(s)}{c_1(s)} \right] \right\}
= \lim_{\rho \to +\infty} \left\{ \frac{1}{(h(\rho))^{1/4} \rho} \exp \left[ H \int_0^\rho ds (h(s))^{1/2} \right] \right\},
\]

where we used \([89x714]B.53\) and explicit expressions

\[
c_1 = \frac{1}{h^{1/4} \rho}, \quad c_2 = \frac{h^{1/4}}{\rho}
\]

from comparing \([89x714]B.52\) and \([2.12]\). The limit in \((C.11)\) must be taken carefully, as the integral is divergent at the upper limit of integration: using the asymptotic expression for $h$ as $y \equiv \frac{1}{\rho} \to 0$ \((B.28)\) and \((B.29)\) we can regulate it as follows,

\[
\int_0^\rho ds (h(s))^{1/2} = \int_0^\rho ds \left( (h(s))^{1/2} - \frac{Pg_1^{1/2}}{2(HPg_1^{1/2}s + 1)} \right) + \frac{1}{2H} \ln \left( 1 + HPg_1^{1/2} \rho \right),
\]

or in dimensionless/rescaled quantities \((C.1)\)

\[
\frac{1}{H} \int_0^{\hat{\rho}} d\hat{s} \left( \hat{h}(\hat{s}) \right)^{1/2} = \frac{1}{H} \int_0^{\hat{\rho}} d\hat{s} \left( \left( \hat{h}(\hat{s}) \right)^{1/2} - \frac{1}{2(\hat{s} + 1)} \right) + \frac{1}{2H} \ln \left( 1 + \hat{\rho} \right),
\]

leading to

\[
s_0^h = 2^{1/2} HP^{1/2} g_1^{1/4} \exp \left[ \int_0^{\hat{\rho}} d\hat{s} \left( \left( \hat{h}(\hat{s}) \right)^{1/2} - \frac{1}{2(\hat{s} + 1)} \right) \right] \equiv HP^{1/2} g_1^{1/4} \hat{s}_0^h,
\]

where the last equality defines dimensionless/rescaled $\hat{s}_0^h$.

**D $b \to 0$ of TypeA$_s$ vacua**

**D.1 FG frame**

The conformal, i.e., $H \gg \Lambda$, limit of TypeA$_s$ vacua is best described in computational SchemeII \((C.6)\). Using perturbative expansions \((4.11)\) we find \((' = \frac{d}{d\rho})\),

- for $n = 1$:

\[
0 = k_1'' - \frac{\rho + 6}{2\rho(1 + \rho)} k_1' - \frac{8}{(1 + \rho)\rho^2},
\]

\[
0 = g_1'' - \frac{\rho + 6}{2\rho(1 + \rho)} g_1' + (k_1')^2 - \frac{4}{(1 + \rho)\rho^2},
\]

80
\[\begin{align*}
0 &= f'_{21} + h'_{1} + 4f'_{31} + \frac{(1 + \rho)\rho}{2(\rho + 2)}(k'_1)^2 + \frac{2}{\rho(\rho + 2)}(f_{21} + 4f_{31} - 4k_1 - 1) \\
&\quad + \frac{(\rho + 4)(3\rho + 4)}{2(\rho + 2)(1 + \rho)}h_1, \\
0 &= f''_{31} + \frac{1}{4}(k'_1)^2 + \frac{(\rho + 2)}{2\rho(1 + \rho)}h'_{1} - \frac{(\rho + 6)}{2\rho(1 + \rho)}f'_{31} + \frac{1}{(1 + \rho)\rho^2}(5f_{21} + 8f_{31} - 4k_1 - 1)
\quad - \frac{3\rho^2 - 16\rho - 16}{4(1 + \rho)^2\rho^2}h_1. \\
0 &= h''_{1} + \frac{1}{2}(k'_1)^2 - \frac{(3\rho + 10)}{2\rho(1 + \rho)}h_1 + \frac{2}{(1 + \rho)\rho^2}(3 + 20k_1 - 9f_{21} - 36f_{31})
\quad + \frac{(3\rho^2 - 8\rho - 8\rho^2)}{2(1 + \rho)^2\rho^2}h_1; \\
\end{align*}\]

for \(n = 2:\)

\[\begin{align*}
0 &= k''_{2} - \frac{\rho + 6}{2\rho(1 + \rho)}k'_{2} - \frac{1}{4\rho(1 + \rho)(\rho + 2)}\left((4g'_{1} + 6h'_{1} + 8f_{31})\rho^3 + (12g'_{1} + 18h'_{1} + 3h_1 + 24f'_{31})\rho^2 + (8g'_{1} + 12h'_{1} - 16k_1 + 4f_{21} + 16h_1 + 16f'_{31} + 16f_{31})\rho - 16k_1 + 4f_{21} + 16h_1 + 16f_{31} - 4\rho - 4\right)k'_{1}
\quad - \frac{(1 + \rho)\rho^2}{4(\rho + 2)}(k'_1)^3 - \frac{8(k_1 + g_1 - f_{21} - h_1 - 2f_{31})}{(1 + \rho)\rho^2}, \\
0 &= g''_{2} - \frac{\rho + 6}{2\rho(1 + \rho)}g'_{2} - (g'_{1})^2 + 2k'_{1}k'_{2} - \frac{1}{4\rho(\rho + 2)(1 + \rho)}\left((k'_1)^2\rho^4 + (2(k'_1)^2)\rho^3
\quad + 2h'_{1})\rho^3 + ((k'_1)^2 + 3h_1 + 6h'_{1})\rho^2 + (16f_{31} + 4f_{21} + 16h_1 + 4h'_{1} - 16k_1)\rho + 16f_{31}
\quad + 4f_{21} + 16h_1 - 16k_1 - 4\rho - 4\right)g'_{1} - (2f_{31} + h_1)(k'_1)^2 + \frac{4(2f_{31} - 2g_1 + f_{21} + h_1)}{(1 + \rho)\rho^2},
\end{align*}\]
\[
0 = f_2'' + 4f_3' + h_2' + \frac{1}{\rho+2}\left(f_3'\rho^2 + (f_21 - h_1 + f_3')\rho + 2f_21 - 2h_1\right)h_1' \\
+ \frac{5\rho(1+\rho)}{2(\rho+2)}(f_3')^2 + \frac{\rho(1+\rho)}{4(\rho+2)}(g_1')^2 + \frac{\rho(1+\rho)}{4(\rho+2)}(h_1')^2 + \frac{(1+\rho)^2}{2(\rho+2)^2}(f_3'\rho^2 + (f_21 - g_1 - h_1 - 2f_31 + f_3')\rho + 2f_21 - 2g_1 - 2h_1 - 4f_31)\left(h_1'\right)^2 + \frac{\rho(1+\rho)}{\rho+2}k_1'k_2' + \frac{1}{2(\rho+2)^2}\left((8f_21 + 3h_1 - 8f_31)\rho^2 + (36f_21 - 16k_1 + 16h_1 - 16f_31)\rho + 36f_21 - 16k_1 + 16h_1 - 16f_31 - 4\rho - 4\right)f_3' + \frac{1}{2(\rho+2)\rho(1+\rho)}\left(3f_21h_1\rho^2 + (-8f_21f_31 + 32k_1h_1 + 64k_1f_31 + 4f_22 - 16k_2)\rho - 8f_21f_31 + 32k_1h_1 + 64k_1f_31 - 8k_1^2 - 4f_21 + (16f_32 - 4f_21 - 8k_1^2 - 64h_1f_31 - 4g_1 + 4h_1 - 24h_1^2 + 16h_2 + 8f_31 - 68f_31^2)\rho - 64h_1f_31 + -4g_1 - 16k_2 + 4f_22 + 8f_31 + 16f_32 + 4h_1 + 16h_2 - 24h_1^2 - 68f_31^2 + 3h_2\rho^2\right),
\]

(D.8)
\[0 = h''_2 + \frac{3\rho + 10}{2\rho(1 + \rho)} h'_2 - \frac{7}{4}(h'_1)^2 - \frac{5}{2}(f'_{31})^2 - \frac{1}{4(\rho + 2)}(h'_{1} + 2f'_{31})\rho^2 + (h'_{1} + 2g_{1} + 4f_{31} + 2f'_{31})\rho + 4g_{1} + 8f_{31}) \left( k'_{1} \right)^{2} - \frac{1}{4}(g'_{1})^2 + k'_{1} k'_{2} - \frac{1}{4\rho(1 + \rho)(\rho + 2)}(4f'_{31}\rho^3 + (3h_{1} + 12f'_{31})\rho^2 + (4f_{21} - 16k_{1} + 16h_{1} + 16f_{31} + 8f'_{31})\rho + 4f_{21} - 16k_{1} + 16h_{1} + 16f_{31} - 4\rho - 4) h'_{1} = \frac{1}{2\rho(1 + \rho)(\rho + 2)} \left( 3h_{1}\rho^2 + (4f_{21} - 16k_{1} + 16h_{1} + 16f_{31})\rho + 4f_{21} - 16k_{1} + 16h_{1} + 16f_{31} - 4\rho \right) f'_{31} - \frac{1}{2(1 + \rho)^2\rho^2} \left( 80f_{21}k_{1} - 44f_{21}h_{1} - 152f_{21}f_{31} + 80k_{1}h_{1} + 320k_{1}f_{31} + 36f_{22} - 80k_{2} - 40f_{21} - 40k_{1} - 176h_{1}f_{31} - 12g_{1} - 40h_{2} + 80h_{2} + 24f_{31} - 388f_{31} + 144f_{32} + 12f_{21})\rho + 80f_{21}k_{1} - 152f_{21}f_{31} + 80k_{1}h_{1} + 320k_{1}f_{31} - 40f_{21} + 176h_{1}f_{31} - 44f_{21}h_{1} - 12g_{1} - 80k_{2} + 12f_{21} + 36f_{22} + 24f_{31} + 144f_{32} + 80h_{2} - 40h_{2} - 388f_{31}^2 + 3(h_{1} + h_{2})\rho^2 \right). \]

(D.10)

The UV ($\rho \to 0$) and the IR ($y \equiv \frac{1}{\rho} \to 0$) asymptotic expansions can be obtained from [B.39] - [B.43] and [B.45] correspondingly, using the SchemeII parameters [C.6], where

\[
\begin{align*}
f_{2,1,0} & = 1 + f_{2,1,0,1} b + f_{2,1,0,2} b^{2} + \mathcal{O}(b^{3}), & g_{4,0} & = g_{4,0,1} b + g_{4,0,2} b^{2} + \mathcal{O}(b^{3}), \\
f_{2,4,0} & = \left( -\frac{1}{12} + \frac{4}{3} k_{4,0,1} \right) b \left( -\frac{139}{1152} + \frac{1}{24} f_{2,1,0,1} - \frac{22}{9} k_{4,0,1} + \frac{2}{3} g_{4,0,1} + \frac{4}{3} k_{4,0,2} \right) b^{2} + \mathcal{O}(b^{3}), \\
f_{2,6,0} & = f_{2,6,0,1} b + f_{2,6,0,2} b^{2} + \mathcal{O}(b^{3}), & f_{2,8,0} & = f_{2,8,0,1} b + f_{2,8,0,2} b^{2} + \mathcal{O}(b^{3}), \\
K^{h}_{0} & = 1 + K^{h}_{0,1} b + K^{h}_{0,2} b^{2} + \mathcal{O}(b^{3}), & K^{h}_{0} & = 1 + K^{h}_{0,1} b + K^{h}_{0,2} b^{2} + \mathcal{O}(b^{3}).
\end{align*}
\]

(D.11)

Note that in lieu of $f_{2,4,0,1}$ and $f_{2,4,0,2}$ in (D.11) we used $k_{4,0,1}$ and $k_{4,0,2}$:

\[
\begin{align*}
k_{1} & = -2\ln \rho + \rho - \frac{1}{8}\rho^{2} - \frac{1}{24}\rho^{3} + \left( \frac{3}{64}\ln \rho + k_{4,0,1} \right) \rho^{4} + \mathcal{O}(\rho^{5}), \\
k_{2} & = f_{2,1,0,1} \rho + \left( -\frac{1}{4}\ln \rho + \frac{9}{16} - \frac{1}{2} f_{2,1,0,1} \right) \rho^{2} + \left( \frac{1}{4}\ln \rho - \frac{7}{16} + \frac{1}{8} f_{2,1,0,1} \right) \rho^{3} + \left( -\frac{3}{16}\ln^{2} \rho + \frac{11}{64} - 4k_{4,0,1} \right) \ln \rho + k_{4,0,1} \right) \rho^{4} + \mathcal{O}(\rho^{5}).
\end{align*}
\]

(D.12)
This is done for computational convenience — the equations for \( k_1 \) (see (D.1)) and \( k_2 \) (see (D.6)) decouple from all the other equations at the corresponding order.

We are able to solve analytically only the equation for \( k_1 \) (D.1),

\[
    k_1 = \frac{\rho}{4} + \frac{1}{4 + 4\rho} - \frac{1}{4} - 4 \ln 2 + \frac{\rho^3 - 6\rho^2 - 24\rho - 16}{8(1 + \rho)^{3/2}} \ln \frac{\sqrt{1 + \rho} - 1}{\sqrt{1 + \rho} + 1},
\]

resulting in

\[
    k_{4,0;1} = \frac{29}{259} - \frac{3}{32} \ln 2, \quad K_{0;1}^h = \frac{5}{3} - 4 \ln 2, \tag{D.14}
\]

and the equation for \( g_1 \) (D.2),

\[
    g_1 = \frac{\rho^2}{32} - \frac{7\rho}{16} - \frac{1}{32(1 + \rho)^2} - \frac{13}{32} - \frac{\rho^4(\rho + 2)}{32(1 + \rho)^{5/2}} \ln \frac{\sqrt{1 + \rho} - 1}{\sqrt{1 + \rho} + 1} \\
    + \left( \frac{23}{64} - \frac{\rho^3}{128} + \frac{15\rho^2}{128} - \frac{15\rho}{64} + \frac{9}{64(1 + \rho)^2} - \frac{63}{128 + 128\rho} - \frac{1}{128(1 + \rho)^3} \right) \ln^2 \frac{\sqrt{1 + \rho} - 1}{\sqrt{1 + \rho} + 1}, \tag{D.15}
\]

\[
    g_{4,0;1} = -\frac{17}{32} + \frac{3}{8} \ln^2 2 + \frac{1}{8} \ln 2, \quad g_{0;1}^h = -\frac{13}{18}. \tag{D.16}
\]

All the remaining equations are solved numerically, using the shooting algorithm developed in [12]. We find:

\[
    f_{2,1,0;1} = 0.434278, \quad f_{2,1,0;2} = 0.357298, \\
    g_{4,0;1} = -0.264437, \quad g_{4,0;2} = -0.64466, \\
    k_{4,0;1} = 0.0482987, \quad k_{4,0;2} = 0.184174, \\
    f_{2,6,0;1} = -0.407036, \quad f_{2,6,0;2} = -0.489017, \\
    f_{2,8,0;1} = -0.427022, \quad f_{2,8,0;2} = -0.609369, \tag{D.17} \\
    f_{2,0;1}^h = -0.156614, \quad f_{2,0;2}^h = 0.54009, \\
    f_{3,0;1}^h = -0.378836, \quad f_{3,0;2}^h = 0.638051, \\
    K_{0;1}^h = -1.10592, \quad K_{0;2}^h = 1.65245, \\
    g_{0;1}^h = -0.722222, \quad g_{0;2}^h = 0.311658,
\]

where we used the same numerical methods to solve (D.1) and (D.2). Comparing the numerical results for \( \{ k_{4,0;1}, g_{4,0;1}, K_{0;1}^h, g_{0;1}^h \} \) from (D.17) with the analytic predictions (D.14) and (D.16) we find agreement at the fractional level of \( \sim 10^{-10} \) or better.
D.2 EF frame

Using perturbative expansions (4.14) and \( w_{c2n} \equiv v_n - w_{a2n} \), we find from (A.17)-(A.25) \( \left( ^{\prime} = \frac{d}{dz} \right) \),

- for \( n = 1 \):

\[
0 = k''_1 + \frac{5(2z - 1)}{2(z - 1)z} k'_1 - \frac{8}{(z - 1)z}, \quad (D.18)
\]

\[
0 = v''_1 - \frac{27}{(z - 1)z} v_1 - \frac{15(2z - 1)}{2(z - 1)z} a'_1 + \frac{11}{4} (k'_1)^2 + \frac{60}{(z - 1)z} k_1 - \frac{15(2z^2 - 2z + 1)}{2z^2(z - 1)^2} a_1
\]

\[
+ \frac{9}{(z - 1)z}, \quad (D.19)
\]

\[
0 = a''_1 + \frac{7(2z - 1)}{2(z - 1)z} a'_1 + \frac{10z^2 - 10z + 3}{2z^2(z - 1)^2} a_1 - \frac{1}{4} (k'_1)^2 + \frac{9}{(z - 1)z} v_1 - \frac{20}{(z - 1)z} k_1
\]

\[
- \frac{3}{(z - 1)z} \quad (D.20)
\]

\[
0 = w''_{a21} + \frac{5(2z - 1)}{2(z - 1)z} w'_{a21} - \frac{12}{(z - 1)z} w_{a21} - \frac{(2z - 1)}{2(z - 1)z} a'_1 - \frac{3(2z - 1)}{2z^2(z - 1)^2} a_1 + \frac{3}{4} (k'_1)^2
\]

\[
- \frac{3}{(z - 1)z} v_1 - \frac{3(2z^2 - 2z + 1)}{2z^2(z - 1)^2} a_1 + \frac{12}{(z - 1)z} k_1 + \frac{1}{(z - 1)z} \quad (D.21)
\]

\[
0 = s'_1 - \frac{1}{2} a'_1 + \frac{1}{2(z - 1)z} a_1 \quad (D.22)
\]

\[
0 = g''_1 + \frac{5(1 - 2z)}{2z(1 - z)} g'_1 + (k'_1)^2 + \frac{4}{z(1 - z)} \quad (D.23)
\]

- for \( n = 2 \):

\[
0 = k''_2 + \frac{5(2z - 1)}{2(z - 1)z} k'_2 + \left( \frac{1}{2} v'_1 + \frac{5}{2} a'_1 - 2w'_{a21} - g'_1 \right) k'_1 - \frac{8(k_1 - a_1 - v_1 + g_1 + 2w_{a21})}{(z - 1)z} \quad (D.24)
\]
\[ 0 = v'' - \frac{27}{(z-1)z}v_2 - \frac{15(2z-1)}{2(z-1)z}a'_2 + \frac{11}{2}k'_1k'_2 + \frac{60}{(z-1)z}k_2 - \frac{15(2z^2-2z+1)}{2z^2(z-1)^2}a_2 \]

\[ -\frac{1}{2}(v'_i)^2 + \left( \frac{11}{2}w'_{a21} - \frac{10(2z-1)}{(z-1)z}w_{a21} + \frac{2(2z-1)}{(z-1)z}v_1 \right) v'_1 + \frac{3(2z-1)(5a_1 - v_1)}{2(z-1)z} a'_1 \]

\[ -\frac{15}{4}(a'_i)^2 - \left( \frac{11}{4}g_1 + \frac{1}{4}v_1 + \frac{3}{2}w_{a21} \right) (k'_1)^2 - \frac{55}{4}(w'_{a21})^2 - \frac{10(2z-1)(v_1 - 5w_{a21})}{(z-1)z} w'_{a21} \]

\[ + \frac{5}{8}(g'_i)^2 + \frac{15v'_i^2}{(z-1)z} + \frac{30k'_1}{(z-1)z} + \frac{75w'_{a21}}{(z-1)z} + \frac{15(4z^2 - 4z + 3)a'_1}{4z^2(z-1)^2} - \frac{12(5a_1 + 4v_1)k_1}{(z-1)z} \]

\[ + \left( \frac{3(16z^2 - 16z - 1)v_1}{2z^2(z-1)^2} - \frac{9}{(z-1)z} \right) a_1 - \frac{2(15v_1 - 1)w_{a21}}{(z-1)z} + \frac{9g_1 - 4v_1}{(z-1)z} , \]

(D.25)

\[ 0 = a''_2 + \frac{7(2z-1)}{2(z-1)z} a'_2 + \frac{9z^2 - 10z + 3}{2z^2(z-1)^2} a_2 + \frac{20}{(z-1)z} v_2 - \frac{1}{2}k'_1k'_2 - \frac{20}{(z-1)z} k_2 \]

\[ + \left( -\frac{1}{4}a_1 + \frac{1}{4}g_1 + \frac{1}{2}w_{a21} \right) (k'_1)^2 + \frac{1}{8}(g'_1)^2 + \frac{5}{4}(w'_{a21})^2 + \frac{3}{4}(a'_1)^2 - \frac{1}{2}w'_{a21}v'_1 - \frac{10k'_1}{(z-1)z} \]

\[ - \frac{105w'_{a21}}{(z-1)z} + \frac{20v_1k_1}{(z-1)z} - \frac{10v'_1}{(z-1)z} + \frac{3a_1^2}{4z^2(z-1)^2} + \frac{6(7v_1 - 1)w_{a21}}{(z-1)z} + \frac{3(v_1 - g_1)}{(z-1)z} , \]

(D.26)

\[ 0 = w'_{a22} + \frac{5(2z-1)}{2(z-1)z} w'_{a22} - \frac{12}{(z-1)z} w_{a22} - \frac{3(2z-1)}{2z(z-1)} a'_2 - \frac{2z-1}{2z(z-1)} v'_2 \]

\[ - \frac{3(2z^2 - 2z + 1)}{2z^2(z-1)^2} a_2 - \frac{3}{(z-1)z} v_2 + \frac{12}{(z-1)z} k_2 + \frac{3}{2}k'_1k'_2 + \frac{1}{4}(w'_{a21})^2 - \frac{3}{4}(a'_1)^2 \]

\[ - \frac{3}{4}(g_1 + w_{a21})(k'_1)^2 + \frac{1}{8}(g'_1)^2 + \left( \frac{5}{2}a'_1 - \frac{2(2z-1)(v_1 - 5w_{a21})}{(z-1)z} \right) w'_{a21} \]

\[ + \left( \frac{(2z-1)(v_1 - 5w_{a21})}{2(z-1)z} - \frac{1}{2}a'_1 \right) v'_1 + \frac{3(2z-1)(a_1 - w_{a21})}{2(z-1)z} a'_1 + \frac{6a_1^2}{(z-1)z} \]

\[ + \frac{3(4z^2 - 4z + 3)a_1^2}{4z^2(z-1)^2} + \frac{75w'_{a21}}{(z-1)z} + \frac{6k_1^2}{(z-1)z} - \frac{12(v_1 + a_1 - w_{a21})k_1}{z(z-1)} \]

\[ + \frac{(3a_1 - 33w_{a21} - 1)v_1}{z(z-1)} + \frac{g_1}{(z-1)z} + \frac{3(6z^2 - 6z - 1)w_{a21}a_1}{2z^2(z-1)^2} - \frac{a_1 - 3w_{a21}}{(z-1)z} , \]

(D.27)

\[ 0 = s'_2 - \frac{1}{2}a'_2 + \frac{1}{2(z-1)z} a_2 + \frac{1}{2}(a_1 - s_1)a'_1 + \frac{a_1(a_1 - s_1)}{2(1-z)z} , \]

(D.28)

\[ 0 = g'_2 + \frac{5(2z-1)}{2(z-1)z} g'_2 + 2k'_1k'_2 - 2w_{a21}(k'_1)^2 - (g'_1)^2 + \left( \frac{1}{2}v'_1 + 5a'_1 \right) g'_1 \]

\[ + \frac{4(v_1 - 2g_1 + a_1 - 2w_{a21})}{(z-1)z} . \]

(D.29)
Initial conditions for (D.18)-(D.29) can be deduced from (B.57) using (D.11) and (1.13):

\[ k_1 = K_{0,1}^h - \frac{16}{5} z + \mathcal{O}(z^2), \]
\[ v_1 = f_{2,0,1}^h + 4f_{3,0,1}^h + \left(32K_{0,1}^h - \frac{64}{5}f_{2,0,1}^h - \frac{256}{5}f_{3,0,1}^h + \frac{28}{5}\right) z + \mathcal{O}(z^2), \]
\[ a_1 = \left(-4K_{0,1}^h + \frac{9}{5}f_{2,0,1}^h + \frac{36}{5}f_{3,0,1}^h - \frac{3}{5}\right) z + \mathcal{O}(z^2), \]
\[ w_{a21} = f_{3,0,1}^h + \left(\frac{32}{5}K_{0,1}^h - \frac{8}{5}f_{2,0,1}^h + \frac{56}{5}f_{3,0,1}^h + \frac{4}{5}\right) z + \mathcal{O}(z^2), \]
\[ s_1 = s_{0,1}^h + \mathcal{O}(z), \quad g_1 = g_{0,1}^h - \frac{8}{5} z + \mathcal{O}(z^2), \]

for \( n = 1 \), and

\[ k_2 = K_{0,2}^h + \left(\frac{32}{5}f_{3,0,1}^h - \frac{16}{5}K_{0,1}^h - \frac{16}{5}g_{0,1}^h + \frac{16}{5}f_{2,0,1}^h\right) z + \mathcal{O}(z^2), \]
\[ v_2 = f_{2,0,2}^h + 4f_{3,0,2}^h + \left(\frac{224}{5}f_{2,0,1}^h f_{3,0,1}^h - \frac{512}{5}K_{0,1}^h f_{3,0,1}^h + 16(K_{0,1}^h)^2 + 32K_{0,2}^h - \frac{64}{5}f_{2,0,2}^h \right. \]
\[ - \frac{256}{5}f_{3,0,2}^h + \frac{28}{5}g_{0,1}^h + \frac{48}{5}(f_{2,0,1}^h)^2 + \frac{528}{5}(f_{3,0,1}^h)^2 - 8f_{3,0,1}^h - \frac{128}{5}K_{0,1}^h f_{2,0,1}^h - \frac{16}{5}f_{2,0,1}^h\right) z \]
\[ + \mathcal{O}(z^2), \]
\[ a_2 = \left(-2(K_{0,1}^h)^2 + 4K_{0,1}^h f_{2,0,1}^h + 16K_{0,1}^h f_{3,0,1}^h - 2(f_{2,0,1}^h)^2 - \frac{38}{5}f_{2,0,1}^h f_{3,0,1}^h - \frac{79}{5}f_{3,0,1}^h f_{2,0,1}^h \right. \]
\[ - 4K_{0,2}^h + \frac{3}{5}f_{2,0,1}^h + \frac{9}{5}f_{2,0,2}^h + \frac{6}{5}f_{3,0,1}^h + \frac{36}{5}f_{3,0,2}^h - \frac{3}{5}g_{0,1}^h\right) z + \mathcal{O}(z^2), \]
\[ w_{a22} = f_{3,0,2}^h + \left(8 f_{2,0,1}^h f_{3,0,1}^h - \frac{96}{5}K_{0,1}^h f_{3,0,1}^h - \frac{32}{5}K_{0,1}^h f_{2,0,1}^h + \frac{16}{5}(K_{0,1}^h)^2 + \frac{32}{5}K_{0,2}^h \right. \]
\[ - \frac{8}{5}f_{2,0,2}^h - \frac{56}{5}f_{3,0,2}^h + \frac{4}{5}g_{0,1}^h + \frac{16}{5}(f_{2,0,1}^h)^2 + \frac{104}{5}(f_{3,0,1}^h)^2 - \frac{4}{5}f_{3,0,1}^h - \frac{4}{5}f_{2,0,1}^h\right) z + \mathcal{O}(z^2), \]
\[ s_2 = s_{0,2}^h + \mathcal{O}(z), \quad g_2 = g_{0,2}^h + \left(\frac{8}{5}f_{2,0,1}^h + \frac{16}{5}f_{3,0,1}^h - \frac{16}{5}g_{0,1}^h\right) z + \mathcal{O}(z^2), \]

for \( n = 2 \).
E  Kretschmann scalar of EF frame background geometry

We collect here the expression for the Kretschmann scalar $K$

$$K \equiv \mathcal{R}_{\alpha\beta\gamma\delta} \mathcal{R}^{\alpha\beta\gamma\delta}$$  \hspace{1cm} (E.1)

of gravitational bulk geometries \cite{2,13} dual to de Sitter vacua of cascading gauge theories. Growth of $K$ evaluated at the apparent horizon as $\frac{H^2}{M^2}$ varies signals the breakdown of the supergravity approximation. Explicitly evaluating (E.1) we find, $\frac{d}{dx} = \frac{-d}{\sigma}$,

$$K = 4a^2 \left( \frac{12(\sigma'')^2}{\sigma^2} + \frac{(\sigma'')^2}{\alpha^2} + \frac{2(w'_{a2})^2}{w_{a2}^2} + \frac{2(w'_{b2})^2}{w_{b2}^2} + \frac{(w''_{c2})^2}{w_{c2}^2} \right) + \frac{24(\sigma''')}{\sigma} \left( H^2 - H\sigma' \right)$$

\begin{align*}
+ 4H\sigma' \left( \frac{2a\sigma'\alpha'}{s} - \frac{w'_{b2}}{w_{b2}} \right) + \frac{4a^2 w'_{a2} w''_{c2}}{w_{c2}^2} \left( \frac{\alpha'}{a} - \frac{w''_{c2}}{w_{c2}} \right) + \frac{8a^2 w'_{a2} w''_{a2}}{w_{a2}^2} \left( \frac{\alpha'}{a} - \frac{w''_{a2}}{w_{a2}} \right) + \frac{8a^2 w'_{b2} w''_{b2}}{w_{b2}^2} \\
\times \left( \frac{\alpha'}{a} - \frac{w'_{b2}}{w_{b2}} \right) + \frac{3H^2}{\sigma} \left( \frac{24(\sigma'')^2}{\sigma^2} + \frac{(w''_{c2})^2}{w_{c2}^2} + \frac{2(w''_{b2})^2}{w_{b2}^2} + \frac{2(w''_{a2})^2}{w_{a2}^2} \right) + 2 \left( \sigma'' \right)^2 \frac{\left( 12(\sigma')^2 \right)}{\sigma^2} \right)
\end{align*}

\begin{align*}
+ \frac{2(w'_{a2})^2}{w_{a2}^2} - 2a\alpha' \left( \frac{(w'_{a2})^3}{w_{a2}^3} + \frac{2(w''_{a2})^3}{w_{a2}^3} \right) + \frac{12a^2 (\sigma')^2}{\sigma^2} \left( \frac{(w''_{a2})^2}{w_{a2}^2} + \frac{2(w''_{a2})^2}{w_{a2}^2} \right) + \frac{4 (w''_{a2})^2}{w_{a2}^2} \\
+ \frac{2(w''_{a2})^2}{w_{a2}^2} - 4a^2 \left( \sigma'' \right)^4 \frac{\left( 3 \left( \frac{(w')^4}{w_{b2}^4} + \frac{3 \left( \frac{(w')^4}{w_{a2}^4} \right)}{w_{a2}^4} \right) \right)}{w_{b2}^4} + \frac{a (w'_{a2})^2}{w_{a2} w_{c2} w_{b2}^3} (27w_{a2}^2 + 9w_{b2}^2 - 36w_{a2} w_{c2} + 16w_{b2}) \\
+ \frac{a (w''_{a2})^2}{w_{a2}^2} (9w_{a2}^2 - 36w_{a2} w_{c2} + 27w_{b2}^2 + 16w_{c2}) \left( \frac{3 \left( \frac{(w')^4}{w_{a2}^4} \right)}{w_{a2}^4} - 6w_{a2} w_{b2} \right) \\
+ 3w_{b2}^2 + 16w_{c2}^2 - \frac{aw'_{a2} w'_{c2}}{2w_{a2}^2 w_{b2} w_{c2}} (63w_{a2}^2 - 18w_{a2} w_{b2} + 24w_{a2} w_{c2} - 45w_{b2}^2 - 24w_{b2} w_{c2}) \\
+ 112w_{c2}^2 - \frac{aw'_{a2} w'_{c2}}{2w_{a2}^2 w_{b2} w_{c2}} (63w_{a2}^2 + 18w_{a2} w_{b2} - 24w_{a2} w_{c2} + 63w_{b2}^2 - 24w_{b2} w_{c2} - 80w_{c2}^2) \\
- 112w_{c2}^2 - \frac{aw'_{a2} w'_{c2}}{2w_{a2}^2 w_{b2} w_{c2}} (63w_{a2}^2 + 18w_{a2} w_{b2} - 24w_{a2} w_{c2} + 63w_{b2}^2 - 24w_{b2} w_{c2} - 80w_{c2}^2) \\
+ 136w_{b2}^2 \left( \frac{w_{a2} + w_{b2}}{w_{a2} w_{b2}} \right) \left( \frac{w_{a2} + w_{b2}}{w_{a2} w_{b2}} \right) + \frac{136w_{b2}^2}{w_{a2}^2 w_{b2}^2} + \frac{81 \left( 13w_{a2}^2 + 6w_{a2} w_{b2} + 13w_{b2}^2 \right) (w_{a2} - w_{b2})^2}{32w_{a2}^2 w_{b2}^2 w_{c2}} \\
- \frac{54 (w_{a2}^2 - w_{b2}^2) (w_{a2} - w_{b2})}{w_{a2}^2 w_{b2}^2 w_{c2}} + \frac{9 (9w_{a2}^2 + 14w_{a2} w_{b2} + 9w_{b2}^2)}{w_{a2}^2 w_{b2}^2} .
\end{align*}

(E.2)
Introducing the dimensionless and rescaled functions and the radial coordinate $\hat{r} \equiv -\hat{z}$ as in (4.8),

$$K = \frac{1}{P^2 g_s} \hat{K}. \quad (E.3)$$

### E.1 Kretschmann scalar at AH of TypeB de Sitter vacua

In section 3.3, we showed that the AH horizon of the bulk gravitational dual to TypeB de Sitter vacua of cascading gauge theory is located at $r_{AH} = -z_{AH} = 0$, see (3.3). Using (3.67), we find from (E.2):

$$\left. K_{AH} \right|_{\text{TypeB}} = 300h_0^4H^4 + H^2 \left(\frac{16P^2g_0^h(3(k_{2,2}^h)^2(f_{a,0}^h)^2 + 20)}{3(f_{a,0}^h)^3h_0^h} + \frac{72}{(f_{a,0}^h)^2k_{2,2}^h} \left(5k_{2,4}^h(f_{a,0}^h)^2 + 3k_{2,2}^h f_{a,0}^h + 18\right) + \frac{1}{3840(f_{a,0}^h)^4(h_0^h)^3P^4(g_0^h)^2(k_{2,2}^h)^2} \left(355(k_{1,3}^h)^4(k_{2,2}^h)^2(f_{a,0}^h)^4 - 30(k_{1,3}^h)^2(k_{2,2}^h)^2(k_{3,1}^h)^2(f_{a,0}^h)^2 + 2283(k_{2,2}^h)^2(k_{3,1}^h)^4 + 6912k_{1,3}^h k_{2,2}^h k_{3,1}^h k_{3,3}^h + 6912(k_{1,3}^h)^2(k_{3,1}^h)^4\right)\right) + \frac{3}{10(f_{a,0}^h)^4P^2g_0^h(h_0^h)^2(k_{2,2}^h)^2} \left(25(k_{1,3}^h)^2(k_{2,2}^h)^2(f_{a,0}^h)^3 - 60k_{2,2}^h k_{1,3}^h k_{3,1}^h(f_{a,0}^h)^2 - 120k_{1,3}^h k_{2,2}^h k_{3,1}^h f_{a,0}^h - 37(k_{2,2}^h)^2(k_{3,1}^h)^2 f_{a,0}^h - 24k_{1,3}^h k_{2,2}^h k_{3,1}^h f_{a,0}^h - 216k_{2,2}^h(k_{3,1}^h)^2 \right) + \frac{1}{1080(f_{a,0}^h)^5(h_0^h)^4(k_{2,2}^h)^2} \left(175(k_{1,3}^h)^4(k_{2,2}^h)^4(f_{a,0}^h)^4 + 194400(k_{2,2}^h)^2(f_{a,0}^h)^5(h_0^h)^2 - 491(k_{2,2}^h)^4(k_{3,1}^h)^2(f_{a,0}^h)^2 + 77760k_{2,2}^h k_{2,4}^h(f_{a,0}^h)^4(h_0^h)^2 - 1152k_{1,3}^h(k_{2,2}^h)^3k_{3,1}^h(f_{a,0}^h)^2 + 746496(k_{2,2}^h)^2(f_{a,0}^h)^3(h_0^h)^2 - 2220(k_{1,3}^h)^2(k_{2,2}^h)^2(f_{a,0}^h)^2 + 1399680k_{2,2}^h(f_{a,0}^h)^3(k_{3,1}^h)^2 + 279936k_{2,2}^h f_{a,0}^h(h_0^h)^2(f_{a,0}^h)^2 + 3492(k_{2,2}^h)^2(k_{3,1}^h)^2 \right) + 13824k_{1,3}^h k_{2,2}^h k_{3,1}^h + 2519424 f_{a,0}^h(h_0^h)^2 + \frac{8P^2g_0^h}{45k_{2,2}^h(f_{a,0}^h)^5(h_0^h)^2} \left(60(k_{2,2}^h)^2k_{2,4}^h(f_{a,0}^h)^4 + 37(k_{2,2}^h)^3(f_{a,0}^h)^3 + 216(k_{2,2}^h)^2(f_{a,0}^h)^2 + 720k_{2,4}^h(f_{a,0}^h)^2 - 756k_{2,2}^h f_{a,0}^h + 2592\right) + \frac{P^4(g_0^h)^2}{3645(h_0^h)^3(f_{a,0}^h)^6} \left(881(k_{2,2}^h)^4(f_{a,0}^h)^4 + 10584(k_{2,2}^h)^2(f_{a,0}^h)^2 + 184464\right). \quad (E.4)

A special case of (E.4) is the Kretschmann scalar at the “AH” of the extremal KS.
solution, see section [B.3] setting $H = 0$ and using (B.79) we find

\[
\lim_{H \to 0} K_{AH}\bigg|_{\text{TypeB}} = \frac{1}{P^2 g_s} \frac{32 \cdot 12^{2/3} (110 \cdot 12^{1/3} + 177147 \delta^2)}{295245 \delta^3},
\]

(E.5)

where we denoted, see $h_0^i$ in (B.79),

\[
\delta \equiv 0.056288(0).
\]

(F Static linearized $\chi$SB fluctuations about TypeA$_s$ vacua)

Static linearized $\chi$SB fluctuations about TypeA$_s$ vacua in FG frame are parameterized as in (5.1). From (B.4)-(B.5) and (B.7)-(B.9) we find, ($' = \frac{d}{dt}$ and $P = H = g_s = 1$):

\[
0 = \delta f'' - \frac{1}{16 \rho^2 f_2 h f_3^3 (f_3^3 \rho - 2 f_3)} \left(-48 f_3^4 h^3 f_2 g^2 \rho^2 - 2 (h')^2 g^2 f_3^4 f_2 \rho^2 - 2 (g')^2 h^2 f_2 f_3^4 \rho^2 + 12 \rho^2 h^2 (f_3^3)^2 f_2 f_2 \rho^2 - 16 h^2 (h')^2 f_3^4 f_2 \rho - 16 h^2 f_2 f_3^3 (f_3^3 f_2 \rho - 48 f_3^4 h^2 \rho g)
\]

\[
+ f_2 \rho^2 (K')^2 h f_3^2 g + 16 f_3 h^2 f_2 g^2 - 96 f_3^3 h^2 f_2 g^2 + 4 g^3 f_3^3 h + 2 g^2 K^2 \right) \delta f' - \frac{K'}{2 g h f_3} \delta k_1
\]

\[
- \frac{2 g}{f_2 f_3 h \rho^2} \delta k_2 + \frac{1}{8 g^2 h f_3^2 h^2 \rho^2 f_3^3 (f_3^3 \rho - 2 f_3)} \left(-48 g^2 f_3^4 f_2 h f_3^3 \rho^3 + 8 g^2 (f_3^3)^3 f_2 h^2 f_3 \rho^3
\]

\[
+ 48 f_3^4 h^3 f_2 g^2 \rho^2 - 2 (h')^2 g^2 f_3^4 f_2 \rho^2 - 2 (g')^2 h^2 f_2 f_3^4 \rho^2 - 36 \rho^2 h^2 (f_3^3)^2 f_3^2 f_2 \rho^2
\]

\[
- 16 h h' g f_3^4 f_2 \rho + 64 \rho^2 f_3^4 f_2 \rho - 4 g f_3^4 (K')^2 f_2 h f_3 \rho^3 + 32 g f_3^4 f_2^3 h f_3 \rho - 72 g f_3^4 f_3^3 h f_3 \rho
\]

\[
- 80 f_3^4 h^2 g f_2 + 7 f_2 \rho^2 (K')^2 h f_3^2 g - 48 f_3^4 h f_2^3 g^2 - 96 f_3^3 h^2 f_2 g^2 + 144 f_3^4 h^2 g^2
\]

\[
- 16 g^3 f_3^3 f_3 h f_3 \rho + 36 g^3 f_3^3 f_2 h + 2 g^2 K^2 \right) \delta f,
\]

(F.1)

\[
0 = \delta k_1'' - \frac{1}{16 \rho^2 f_2 h f_3^3 (f_3^3 \rho - 2 f_3)} \left(-48 f_3^4 h^3 f_2 g^2 \rho^2 - 2 (h')^2 g^2 f_3^4 f_2 \rho^2
\]

\[
+ 16 f_3^4 h^2 f_2 \rho^2 f_3 h' - 2 (g')^2 h^2 f_3^4 f_2 \rho^2 + 16 f_3^4 f_2 f_3^3 h^2 g \rho^2 + 12 g^2 h^2 (f_3^3)^2 f_3^2 f_2 \rho^2
\]

\[
- 48 h h' g^2 f_3^4 f_2 \rho - 32 f_3^4 f_2 h^2 g \rho - 16 h^2 g^2 f_3^3 f_2 f_3 \rho - 48 f_3^4 h^2 g^2 f_2 - 2 f_2 \rho^2 (K')^2 f_3^3 g
\]

\[
+ 16 f_3^4 h^2 f_2 g^2 - 96 f_3^3 h^2 f_2 g^2 + 4 g^3 f_2 h^2 h + 2 g^2 K^2 \right) \delta k_1' + \frac{2 K'}{f_3} \delta f' - \frac{9}{\rho^2 f_2} \delta k_1
\]

\[
+ \frac{2 g K}{\rho^2 f_2 h f_3^2} \delta k_2 + \frac{2 (-f_3^3 K' f_2 h f_3 \rho^2 + 2 g K)}{f_3^3 \rho^2 f_2 h} \delta f,
\]

(F.2)
\[0 = \delta k''_2 - \frac{1}{16 \rho g^2 f_2 h^2 f_3^2 (f'_3 \rho - 2 f_3)} \left( -48 f_3^4 h^3 f_2 g^2 \rho^2 - 2 (h')^2 g^2 f_3^4 f_2 \rho^2 \
+ 16 f_3^2 h f_2 g^2 \rho^2 f_3^4 h' - 2 (g')^2 h^2 f_3^4 \rho^2 - 16 f_3^2 f_2 f'_3 h^2 g' \rho^2 + 12 g_2^2 h^2 (f'_3)^2 f_3^2 f_2 \rho^2 \
- 48 h h' g f_3^2 f_2 \rho + 32 f_3^2 f_2 h^2 g' \rho - 16 h^2 g^2 f_3^2 f_2 f_2 \rho - 48 f_3^4 h^2 g^2 f_2 - 2 f_2 \rho^2 (K')^2 h f_3^2 g \
+ 16 f_3^2 h f_2^2 g^2 - 96 f_3^2 h f_2 g^2 + 4 g^3 f_3^4 h + 2 g^2 K^2 \right) \delta k'_2 - \frac{9}{\rho^2 f_2} \delta k_2 + \frac{9 K}{2 \rho^2 f_2 f_3^2 h g} \delta k_1 \
- \frac{18}{f_3 \rho^2 f_2} \delta f. \] (F.3)

Performing the asymptotic expansions, we determine:

- in the UV, i.e., as \(\rho \to 0\), using (B.38)-(B.43),
  \[\delta f = \delta f_{1,0} \rho + \frac{1}{2} f_{2,1,0} \delta f_{1,0} \rho^2 + \left( \delta f_{3,0} + \left( \frac{1}{4} \delta f_{1,0} k_s - \frac{11}{8} \delta f_{1,0} \right) \ln \rho \right) \rho^3 + \sum_{n=4} \sum_k \delta f_{n,k} \rho^n \ln^k \rho,\] (F.4)
  \[\delta k_1 = -\frac{1}{2} \delta f_{1,0} \rho + \frac{1}{4} f_{2,1,0} \delta f_{1,0} \rho^2 + \left( \delta k_{1,3,0} + \left( \frac{1}{24} \delta f_{1,0} k_s - \frac{47}{144} \delta f_{1,0} + 2 \delta f_{3,0} \right) \ln \rho \right) \rho^3 + \sum_{n=4} \sum_k \delta k_{1,n,k} \rho^n \ln^k \rho,\]
  \[\delta k_2 = -\frac{9}{4} \delta f_{1,0} \rho + \frac{9}{8} f_{2,1,0} \delta f_{1,0} \rho^2 + \left( \frac{13}{48} \delta f_{1,0} k_s - \frac{3}{8} \delta f_{1,0} f_{2,1,0} + \frac{3}{2} \delta k_{1,3,0} \right) \rho^3 \]
  \[\frac{163}{144} \delta f_{1,0} - \delta f_{3,0} + \left( -\frac{5}{16} \delta f_{1,0} k_s + \frac{137}{96} \delta f_{1,0} + 3 \delta f_{3,0} \right) \ln \rho + \left( -\frac{7}{4} \delta f_{1,0} \right) \]
  \[+ \frac{3}{8} \delta f_{1,0} k_s \right) \ln^2 \rho - \frac{1}{4} \delta f_{1,0} \ln^3 \rho \right) \rho^3 + \sum_{n=4} \sum_k \delta k_{2,n,k} \rho^n \ln^k \rho,\] (F.5)
  characterized by 4 parameters (compare with (B.23)):
  \[\{ \delta f_{1,0}, \delta f_{3,0}, \delta k_{1,3,0}, \delta f_{7,0} \},\] (F.7)
  where \(\delta f_{1,0}\) is an explicit chiral symmetry breaking scale (\(\propto\) the gaugino mass term), and the remaining parameters are the expectation values of the chiral symmetry breaking operators in cascading gauge theory:

- in the IR, i.e., as \(\frac{1}{\rho} = y \to 0\), using (B.45),
  \[\delta f = \frac{1}{y} \sum_{n=0} \delta f^n_h y^n, \quad \delta k_{1,2} = \sum_{n=0} \delta k^n_{1,2,n} y^n,\] (F.8)
characterized by 3 parameters:

\[ \{ \delta f^h_0, \delta k^h_{1,0}, \delta k^h_{2,0} \} \].

(F.9)

References

[1] I. R. Klebanov and A. A. Tseytlin, Gravity duals of supersymmetric SU(N) x SU(N+M) gauge theories, *Nucl. Phys. B578* (2000) 123–138, [hep-th/0002159].

[2] I. R. Klebanov and M. J. Strassler, Supergravity and a confining gauge theory: Duality cascades and chi SB resolution of naked singularities, *JHEP* 08 (2000) 052, [hep-th/0007191].

[3] I. R. Klebanov and E. Witten, Superconformal field theory on three-branes at a Calabi-Yau singularity, *Nucl. Phys. B536* (1998) 199–218 [hep-th/9807080].

[4] C. P. Herzog, I. R. Klebanov and P. Ouyang, Remarks on the warped deformed conifold, in Modern Trends in String Theory: 2nd Lisbon School on g Theory Superstrings Lisbon, Portugal, July 13-17, 2001, 2001. [hep-th/0108101].

[5] N. Seiberg, Electric - magnetic duality in supersymmetric nonAbelian gauge theories, *Nucl. Phys. B435* (1995) 129–146, [hep-th/9411149].

[6] A. Buchel, Finite temperature resolution of the Klebanov-Tseytlin singularity, *Nucl. Phys. B600* (2001) 219–234, [hep-th/0011146].

[7] M. Krasnitz, Correlation functions in a cascading N=1 gauge theory from supergravity, *JHEP* 12 (2002) 048, [hep-th/0209163].

[8] O. Aharony, A. Buchel and A. Yarom, Short distance properties of cascading gauge theories, *JHEP* 11 (2006) 069, [hep-th/0608209].

[9] A. Dymarsky, I. R. Klebanov and N. Seiberg, On the moduli space of the cascading SU(M+p) x SU(p) gauge theory, *JHEP* 01 (2006) 155, [hep-th/0511254].

[10] A. Buchel, C. P. Herzog, I. R. Klebanov, L. A. Pando Zayas and A. A. Tseytlin, Nonextremal gravity duals for fractional D-3 branes on the conifold, *JHEP* 04 (2001) 033, [hep-th/0102105].
[11] S. S. Gubser, C. P. Herzog, I. R. Klebanov and A. A. Tseytlin, *Restoration of chiral symmetry: A Supergravity perspective*, JHEP 05 (2001) 028, [hep-th/0102172].

[12] O. Aharony, A. Buchel and P. Kerner, *The Black hole in the throat: Thermodynamics of strongly coupled cascading gauge theories*, Phys. Rev. D76 (2007) 086005, [0706.1768].

[13] A. Buchel, *Chiral symmetry breaking in cascading gauge theory plasma*, Nucl. Phys. B847 (2011) 297–324, [1012.2404].

[14] A. Buchel, *Klebanov-Strassler black hole*, JHEP 01 (2019) 207, [1809.08484].

[15] O. Aharony, A. Buchel and A. Yarom, *Holographic renormalization of cascading gauge theories*, Phys. Rev. D72 (2005) 066003, [hep-th/0506002].

[16] I. Bena, A. Buchel and S. Lust, *Throat destabilization (for profit and for fun)*, 1910.08094.

[17] A. Buchel, *A Holographic perspective on Gubser-Mitra conjecture*, Nucl. Phys. B731 (2005) 109–124, [hep-th/0507275].

[18] V. Balasubramanian, A. Buchel, S. R. Green, L. Lehner and S. L. Liebling, *Holographic Thermalization, Stability of Antide Sitter Space, and the Fermi-Pasta-Ulam Paradox*, Phys. Rev. Lett. 113 (2014) 071601, [1403.6471].

[19] A. Buchel and A. Karapetyan, *de Sitter Vacua of Strongly Interacting QFT*, JHEP 03 (2017) 114, [1702.01320].

[20] A. Buchel, *Entanglement entropy of $\mathcal{N} = 2^*$ de Sitter vacuum*, 1904.09968.

[21] A. Buchel, *Verlinde Gravity and AdS/CFT*, 1702.08590.

[22] I. Booth, *Black hole boundaries*, Can. J. Phys. 83 (2005) 1073–1099, gr-qc/0508107.

[23] P. Figueras, V. E. Hubeny, M. Rangamani and S. F. Ross, *Dynamical black holes and expanding plasmas*, JHEP 04 (2009) 137, 0902.4696.

[24] A. Buchel, *Ringing in de Sitter spacetime*, Nucl. Phys. B928 (2018) 307–320, 1707.01030.
[25] P. M. Chesler and L. G. Yaffe, *Numerical solution of gravitational dynamics in asymptotically anti-de Sitter spacetimes*, *JHEP* 07 (2014) 086, [1309.1439].

[26] P. M. Chesler and L. G. Yaffe, *Horizon formation and far-from-equilibrium isotropization in supersymmetric Yang-Mills plasma*, *Phys. Rev. Lett.* 102 (2009) 211601, [0812.2053].

[27] A. Buchel, L. Lehner and R. C. Myers, *Thermal quenches in N=2* plasmas*, *JHEP* 08 (2012) 049, [1206.6785].

[28] A. Buchel, L. Lehner, R. C. Myers and A. van Niekerk, *Quantum quenches of holographic plasmas*, *JHEP* 05 (2013) 067, [1302.2924].

[29] A. Buchel and A. A. Tseytlin, *Curved space resolution of singularity of fractional D3-branes on conifold*, *Phys. Rev.* D65 (2002) 085019, [hep-th/0111017].

[30] A. Buchel, *Gauge / gravity correspondence in accelerating universe*, *Phys. Rev.* D65 (2002) 125015, [hep-th/0203041].

[31] A. Buchel and D. A. Galante, *Cascading gauge theory on dS4 and String Theory landscape*, *Nucl. Phys.* B883 (2014) 107–148, [1310.1372].

[32] A. Buchel, M. P. Heller and J. Noronha, *Entropy Production, Hydrodynamics, and Resurgence in the Primordial Quark-Gluon Plasma from Holography*, *Phys. Rev.* D94 (2016) 106011, [1603.05344].

[33] E. Poisson, *A Relativist’s Toolkit: The Mathematics of Black-Hole Mechanics*. Cambridge University Press, 2009, [10.1017/CBO9780511606601].

[34] A. Buchel, P. Langfelder and J. Walcher, *On time dependent backgrounds in supergravity and string theory*, *Phys. Rev.* D67 (2003) 024011, [hep-th/0207214].

[35] A. Buchel, *Compactifications of the N = 2* flow*, *Phys. Lett.* B570 (2003) 89–95, [hep-th/0302107].

[36] A. Buchel and A. Ghodsi, *Braneworld inflation*, *Phys. Rev.* D70 (2004) 126008, [hep-th/0404151].

94