ON PROTON AND DELTA WAVE FUNCTIONS

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Abstract

Physical wave functions for the nucleon and the $\Delta^+$ isobar are presented, which unify the best features of previous models. With these wave functions we can calculate elastic form factors and the decays of the charmonium levels $^3S_1$, $^3P_1$, $^3P_2$ into $p\bar{p}$ in agreement with the data. A striking scaling behavior between $R = |G^a_M|/G^p_M$ and the coefficient $B_4$ of the Appell polynomial decomposition of the nucleon distribution amplitude is found; the implications for elastic nucleon cross sections are discussed.

I. INTRODUCTION

An interesting testing ground for applications of perturbative QCD has emerged in the study of exclusive processes and elastic form factors of few-quark systems. Such systems can be described within a convolution formalism assuming factorization of highly off-shell or large transverse momentum regions of phase space from regions of low momenta necessary to form bound states. Recent progress in Sudakov-suppression techniques provides support for the conjectured infrared protection of the perturbative picture.

For modelling of the nucleon and its low resonances, elastic form factors play a key role because they provide an integrated view of the implications of QCD from low to high $Q^2$. Thus they offer a powerful link between theoretical concepts and measurements and they can serve to test both the scaling properties as well as the detailed structure of the nucleon wave function. The theoretical tools for such a description are provided by the hard-scattering amplitude which describes the perturbative quark-gluon interaction in a particular process, and the probability amplitude for finding the three-quark valence state in the scattered nucleon or nucleon resonance: $\Phi_N(x_i, Q^2)$. A major theme of this talk will be to examine how QCD deals with the derivation of

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such distribution amplitudes for the nucleon and the $\Delta^+$ isobar, focusing our attention on recent developments.

II. GENERAL FEATURES

The momentum-scale dependence of $\Phi_N(x_i, Q^2)$ is given by

$$\Phi_N(x_i, Q^2) = \Phi_{as}(x_i) \sum_{n=0}^{\infty} B_n \tilde{\Phi}_n(x_i) \left( \frac{\alpha_n(Q^2)}{\alpha_n(\mu^2)} \right)^{\gamma_n},$$

(1)

in which $\{\Phi_n\}_0^{\infty}$ are orthonormalized eigenfunctions of the interaction kernel of the evolution equation expressed in a truncated basis of Appell polynomials of maximum degree $M$, and $\Phi_{as}(x_i) = 120 x_1 x_2 x_3$ is the asymptotic form of the nucleon distribution amplitude. The corresponding eigenvalues $\gamma_n$ turn out to be the anomalous dimensions of multiplicatively renormalizable $I_{1/2}$ baryonic operators of twist three. Because the $\gamma_n$ are positive fractional numbers increasing with $n$, higher terms in this expansion are gradually suppressed. A basis including a total of 54 eigenfunctions ($M = 9$) together with the associated normalization coefficients and anomalous dimensions is given in Ref. 4.

The derivation of the nucleon distribution amplitude from QCD is intimately connected with confinement and employs nonperturbative methods. Using the properties of the Appell polynomials, the inverse of Eq. (1) determines the (nonperturbative) expansion coefficients $B_n$:

$$B_n(\mu^2) = \frac{N_n}{120} \int_0^1 [dx] \tilde{\Phi}_n(x_i) \Phi_N(x_i, \mu^2),$$

(2)

so that the “renormalization-group improved” coefficients $B_n(Q^2)$ are given by

$$B_n(Q^2) = B_n(\mu^2) \exp \left\{ -\int_{\alpha_n(\mu^2)}^{\alpha_n(Q^2)} \frac{d\alpha}{\beta(\alpha)} \gamma_n(\alpha) \right\} \approx B_n(\mu^2) \left\{ \frac{\ln(Q^2/\Lambda_{QCD}^2)}{\ln(\mu^2/\Lambda_{QCD}^2)} \right\}^{-\gamma_n}. \quad (3)$$

In terms of the moments of the nucleon distribution amplitude,

$$\Phi_N^{(ij)}(\mu^2) = \int_0^1 [dx] x^i_1 x^j_2 x^0_3 \Phi_N(x_i, \mu^2),$$

(4)

Eq. (3) becomes

$$\frac{B_n(\mu^2)}{\sqrt{N_n}} = \frac{\sqrt{N_n}}{120} \sum_{i,j=0}^{\infty} a_{ij}^n \Phi_N^{(ij)}(\mu^2), \quad (5)$$

where the projection coefficients $a_{ij}^n$ are calculable to any order $M$. Specifically, those up to order $M = 9$ have been tabulated in Refs. [4,5].

To determine the moments, a short-distance operator product expansion is performed at some spacelike momentum $\mu^2$ where quark-hadron duality is valid. One
considers matrix elements of appropriate three-quark operators which are related to moments of the covariant distribution amplitudes \( V, A, \) and \( T \): \( \Phi_N(x_i) = V(x_i) - A(x_i), \Phi_N(1, 3, 2) + \Phi_N(2, 3, 1) = 2T(1, 2, 3) \) with \( V(1,2,3)=V(2,1,3), A(1,2,3)=-A(2,1,3), \) and \( T(1,2,3)=T(2,1,3). \)

III. DISTRIBUTION AMPLITUDES OF THE NUCLEON AND THE \( \Delta^+ \) ISOBAR

Based on QCD sum-rule calculations, useful theoretical constraints on the moments of baryon distribution amplitudes have been obtained. Physical wave functions and observables for the nucleon and the \( \Delta^+ \) isobar are then calculated using the full set of these constraints to determine the first few expansion coefficients \( B_n \) in a truncated basis of Appell polynomials. Depending on the value of \( \Lambda_{QCD} \), these models predict approximately the right size and \( Q^2 \)-evolution of \( G^p_M \), while they give \( R = |G^p_M|/G^p_M \leq 0.5 \). An alternative nucleon distribution amplitude was proposed to give \( |G^p_M| \ll G^p_M \), in accordance to phenomenological data analyses and the latest high-\( Q^2 \) SLAC data at the expense that some of the amplitude moments cannot match the sum-rule requirements in the allowed saturation range.

However, several crucial questions have to be resolved: For instance, does the optimum solution to the sum rules automatically yield best agreement with the data? Do solutions exist with characteristics distinctive from those of the COZ and the GS amplitudes? If so, what are the fundamental ordering parameters to classify these solutions? In recent works we have shown that it is indeed possible to amalgamate the best features of COZ-type and GS-type nucleon distribution amplitudes into a hybrid-like amplitude, we termed the “heterotic” solution (see Fig. 1).

In order to develop a credible nucleon distribution amplitude, we employ a \( \chi^2 \) criterion which parametrizes the deviations from the sum-rule intervals according to the moment order. This “hierarchical” treatment of the sum rules takes into account the higher stability of the lower-level moments and does not overestimate the significance of the still unverified constraints for the third-order moments. [For more details, see Ref. 19.]

The underlying assumption is that contributions of higher-order terms are either negligible or of minor importance relative to those of second-order. Then the model space is also truncated at states with bilinear correlations of fractional momenta and the pattern of solutions found in this order should dominate the (orthonormalized) Appell polynomial series at every order of truncation. In this way the parameter space of the Appell decomposition coefficients can be systematically scanned seeking for local minima of \( \chi^2 \). Using for the first and second order moments either the COZ or the KS sum-rule constraints in conjunction with those of COZ for the third-order moments, a simple scaling relation between the ratio \( R \) and the expansion coefficient \( B_4 \) emerges as one progresses through the generated solutions.

We have plotted in the \((B_4, R)\) plane interpolating solutions to the COZ sum rules (+ labels) and such to a combined set of KS/COZ sum rules (○ labels). As it turns out (Fig. 2), there is no significant difference between the two treatments and
this insensitivity justifies the whole approach. The presented curves are fits to the local minima of the COZ sum rules (solid line) and the KS/COZ sum rules (dotted line). They constitute an orbit with respect to $\chi^2$, beginning in the heterotic region (small $R$ and large positive $B_4$) and terminating past the COZ cluster (large $R$ and large negative $B_4$).

The lower part of the orbit is associated with the heterotic solution which corresponds to the smallest possible ratio still compatible with the sum-rule constraints. The upper region of the orbit controls COZ-type amplitudes and contains a cluster of solutions densely populating the orbit in the $R$-interval $0.455 \div 0.495$ (see the inset in Fig. 2). This cluster contains the amplitudes $COZ^{\text{opt}}, KS/COZ^{\text{opt}}$ which are associated with the absolute minima of $\chi^2$ and play the role of strange attractors for all other solutions with similar features. GS-type amplitudes correspond to local minima of $\chi^2$ at considerably lower levels of accuracy and thus they constitute in the $(B_4, R)$ plane an isolated region (an “island”) that is separated from the characteristic orbit by a large $\chi^2$ barrier. The profiles of the distribution amplitudes across the orbit change in an orderly sequence of gradations with some mixture of COZ and GS characteristics until the COZ amplitude is transmuted into the heterotic solution.

| Model | $B_2$ | $B_3$ | $B_4$ | $B_5$ | $\theta$[deg] | $R$ | $\chi^2$ | Symbol |
|-------|-------|-------|-------|-------|--------------|-----|----------|--------|
| Het   | 3.4437| 1.5710| 4.5937| 29.3125| -0.1250      | -1.89 | .104     | 33.48   | ●      |
| Het'  | 4.3025| 1.5920| 1.9675| -19.658 | 3.3531      | 24.44 | .448     | 30.63   | ●      |
| COZ$^{\text{opt}}$ | 3.5268| 1.4000| 2.8736| -4.5227 | 0.8002 | 9.13  | .465     | 4.49    | ■      |
| COZ$^{\text{up}}$ | 3.2185| 1.4562| 2.8300| -17.3400| 0.4700 | 5.83  | .4881    | 21.29   | ±      |
| COZ   | 3.6750| 1.4840| 2.8980| -6.6150 | 1.0260 | 10.16 | .474     | 24.64   | □      |
| C$Z$  | 4.3050| 1.9250| 2.2470| -3.4650 | 0.0180 | 13.40 | .487     | 250.07  | ♦      |
| KS$^{\text{low}}$ | 3.5818| 1.4702| 4.8831| 31.9906 | 0.4313 | -0.93 | .0675    | 36.27   | ◆      |
| KS$^{\text{up}}$ | 3.4242| 1.3644| 3.0844| -3.2656 | 1.2750 | 9.47  | .453     | 5.66    | ○      |
| KS$^{\text{opt}}$ | 3.5935| 1.4184| 2.7864| -13.3802| 2.0594 | 13.82 | .482     | 40.38   | ○      |
| KS    | 3.2550| 1.2950| 3.9690| 0.9450  | 1.0260 | 2.47  | .412     | 116.35  | ♦      |
| GS$^{\text{opt}}$ | 3.9501| 1.5273| -4.8174| 3.4435 | 8.7534 | 80.87 | .095     | 54.95   | ▲      |
| GS$^{\text{min}}$ | 3.9258| 1.4598| -4.6816| 1.1898 | 8.0123 | 80.19 | .035     | 54.11   | ▼      |
| GS    | 4.1045| 2.0605| -4.7173| 5.0202 | 9.3014 | 78.87 | .007     | 270.82  | △      |

TABLE I. Theoretical parameters defining the nucleon distribution amplitudes discussed in the text. The “hybridity” angle $\theta$ is discussed in [19].
Let us now turn to models with functional representations which make use of
higher Appell polynomials in connection with additional ad-hoc cutoff-parameters. The inset in Fig. 2 shows how such models (stars) and (light upside-down triangles) group around the optimum amplitudes \( COZ^{\text{opt}} \) and \( KS/COZ^{\text{opt}} \), thus establishing the scaling relation between \( R \) and \( B_4 \) in a much more general context. This result suggests that the inclusion of higher-order Appell polynomials in the nucleon distribution amplitude is a marginal effect, as conjectured above. Those model amplitudes which appear as isolated points scattered towards the GS island are unacceptable on physical grounds, either because they exhibit unrealistic large oscillations in the longitudinal momentum fraction or because they yield a wrong evolution behavior for the nucleon form factors.

One place to test these results is in the data for the elastic cross sections \( \sigma_p \) and \( \sigma_n \). For small scattering angles, where the terms \( \propto \tan^2(\theta/2) \) can be neglected, there are two main possibilities for the ratio \( \sigma_n/\sigma_p \). If the Dirac form factor \( F_1^p \) is zero or small compared to the Pauli form factor \( F_2^p \), then \( \sigma_n \) should be due only to the higher-order term \( F_2^n \). At large \( Q^2 \) the ratio would become (\( M_N \) is the nucleon mass)

\[
\frac{\sigma_n}{\sigma_p} \Rightarrow \left( \frac{c_n}{c_1} \right)^2 \frac{1}{4M_N^2Q^2}
\]

and would decrease with increasing \( Q^2 \) due to the extra power of \( 1/Q^2 \) of the Pauli form factor. Alternatively, if \( F_1^p \) is comparable to \( F_2^p \), then \( \sigma_n \) would eventually be due to \( F_1^n \) at large \( Q^2 \). Then the ratio \( \sigma_n/\sigma_p \) would be given by some constant determined by the nucleon wave functions \( \frac{\sigma_n}{\sigma_p} \Rightarrow \left( \frac{c_n}{c_1} \right)^2 \). In these expressions, the wave-function characteristics are parametrized by the (dimensionful) coefficients \( C_i \), which are functions of the expansion coefficients \( B_n \) and the “proton decay constant” \( |f_N| = (5.0 \pm 0.3) \times 10^{-3} \text{GeV}^2 \).

The principal result from the above discussion is that in the intermediate \( Q^2 \) domain, \( \sigma_n/\sigma_p \) should be within the range 0.238 and 0.01. Comparing with available data, we see that the measured \( \sigma_n/\sigma_p \) enters the estimated range already at \( Q^2 \approx 8 \text{GeV}^2/c^2 \) (see Fig. 3).

In view of these results, it is worth remarking that the present accuracy of QCD sum rules seems to be sufficient to limit \( \sigma_n/\sigma_p \) within the observed region. Fig. 3 shows that the available data in the range \( Q^2 \approx (8 \div 10) \text{GeV}^2/c^2 \) are well below the calculated upper bound and still decreasing. This indicates that distribution amplitudes which give \( |G_M^n|/G_M^p \approx 0.5 \) may be in contradiction to experiment because they yield a Dirac form factor \( F_1^n \) which starts to overestimate the data already at \( Q^2 \approx 8 \text{GeV}^2/c^2 \). On the contrary, models which give a small value of \( |G_M^n|/G_M^p \) can explain the data only under the assumption that in this \( Q^2 \) region the Pauli contribution is still dominant. Fig. 4 serves not only to amplify the preceding discussion but also to advertise the consistency of the heterotic model with the form factor data. A similar good agreement with the data is found also for the axial form factors.

We now turn our attention to the \( \Delta^+ \) isobar. It was pointed out[4,5] that model amplitudes for the nucleon are characterized by an anticorrelation pattern between \( G_M^n \) and \( G_M^p \). COZ-like models yield \( |G_M^n|/G_M^p \leq 0.5 \) and \( |G_M^n|/G_M^p \) small, while GS-like
models lead to the reverse situation. This pattern was derived, under the assumption that the $\Delta$ amplitude can be crudely modelled by the symmetric part of the nucleon distribution amplitude. Fig. 5 shows that the transition form factor calculated with the nucleon heterotic amplitude and more realistic $\Delta$ amplitudes, derived from QCD sum rules, is positive with a magnitude between those of previous models. In order to obtain an optimum distribution amplitude for the $\Delta$, we try to comply with the constraints of the CP and FZOZ analyses simultaneously. This concept leads to a hybrid-like amplitude, denoted again “heterotic”. This solution fulfills all FZOZ constraints and provides the best possible compliance with the CP constraints. In addition, it gives the best agreement with the data (see Fig. 5 and Ref. [25]). In particular, when including the effect of perturbative (i.e., logarithmic) $Q^2$ evolution of the expansion coefficients $B_n$, the combined use of the heterotic amplitudes for the nucleon and the $\Delta^+$ yields a form factor behavior which conforms with the observed decrease of available data within their quoted errors.

There is yet another type of solution for the $\Delta$ amplitude—compatible with the sum rules—but in sizeable disagreement with the data. This solution ($FZOZ_{\text{opt}}$) is obtained by demanding that $G^+_M$ calculated with $COZ_{\text{opt}}$ is positive. Thus, as in the nucleon case, optimum agreement with the (existing) sum rules does not automatically entail best agreement with the data.

Exclusive decays of charmonium levels to $p\bar{p}$ are very sensitive to the nucleon distribution amplitude. The branching ratio for the decay of the $\chi_{c1}$ state ($J^{PC} = 1^{++}$) into $p\bar{p}$ is proportional to the decay amplitude $M_1$, which involves $\Phi_N$ and $f_N$. Inputting the heterotic amplitude, $M_1$ is computed using an elaborated integration routine which accounts for contributions near singularities. Thereby we find $M^\text{het}_1 = 99849.6$ and as a result $BR(3P_1 \rightarrow p\bar{p}/3P_1 \rightarrow \text{all}) = 0.77 \times 10^{-2}\%$, which is in excellent agreement with the recent high-precision experimental value (0.78 $\pm$ 0.10 $\pm$ 0.11) $\times 10^{-2}\%$ of the E760 Collaboration at FNAL.

Analogously for the $\chi_{c2}$ state ($J^{PC} = 2^{++}$), we find $M^\text{het}_2 = 515491.2$. Setting $\alpha_s(m_c) = 0.210 \pm 0.028$, we then obtain $BR(3P_2 \rightarrow p\bar{p}/3P_2 \rightarrow \text{all}) = 0.89 \times 10^{-2}\%$ in excellent agreement with the FNAL value (0.91 $\pm$ 0.08 $\pm$ 0.14) $\times 10^{-2}\%$.

Similar considerations apply also to the charmonium decay of the level $3S_1$ with $J^{PC} = 1^{--}$. The partial width of $J/\psi$ (or $\chi_{c0}$) into $p\bar{p}$ is $\Gamma(3S_1 \rightarrow p\bar{p}) = \frac{4}{3} M^2_0$, where $f_\psi$ determines the value of the $3S_1$-state wave function at the origin. Its value can be extracted from the leptonic width $\Gamma(3S_1 \rightarrow e^+e^-) = (5.36 \pm 0.29)\text{keV}$ via the Van Royen-Weisskopf formula. The result is $(m_{J/\psi} = 3096.93\text{MeV}) |f_\psi| = 409\text{MeV}$. The heterotic amplitude gives for this transition $M_0 = 13726.8$. Then, using the previous parameters, it follows that $\Gamma(3S_1 \rightarrow p\bar{p}) = 0.14\text{keV}$. From experiment, it is known that $\Gamma(p\bar{p})/\Gamma_{\text{tot}} = (2.16 \pm 0.11) \times 10^{-3}$ with $\Gamma_{\text{tot}} = (68 \pm 10)\text{keV}$, so that $\Gamma(3S_1 \rightarrow p\bar{p}) = 0.15\text{keV}$ in remarkable agreement with the model prediction. The corresponding branching ratio is $BR(3S_1 \rightarrow p\bar{p}/3S_1 \rightarrow \text{all}) = 1.62 \times 10^{-3}$ with $\Gamma_{\text{tot}} = (85.5^{+6.1}_{-5.8})\text{keV}$. To effect the quality of these predictions, we quote the results for...
the COZ amplitude. [Note that these authors use the rather arbitrary value \( \alpha_s = 0.3 \).] 

\[ BR(3P_1 \rightarrow p\bar{p}/3P_1 \rightarrow \text{all}) = 0.50 \times 10^{-2}\% \], 

\[ BR(3P_2 \rightarrow p\bar{p}/3P_2 \rightarrow \text{all}) = 1.6 \times 10^{-2}\% \], 

and \( \Gamma(3S_1 \rightarrow p\bar{p}) = 0.34\text{keV} \).

IV. SUMMARY AND CONCLUSIONS

Given the apparent success of the heterotic model in predicting a variety of observables such as magnetic and transition form factors and several branching ratios of exclusive decays of charmonium into \( p\bar{p} \), it is optimistic to believe that this approach—albeit approximative for a complete analytical understanding of the nucleon distribution amplitude—is sufficient of reproducing the observed phenomena. Since higher than order \( M = 3 \) expansion coefficients are unspecified by the present knowledge of QCD sum rules, the model does not depend on unconstrained (higher-order) parameters. While higher-order effects on the nucleon distribution amplitude itself are found to be large, the agreement with the data is actually not improved. This is also true for the optimized version of the COZ amplitude, we have derived, which represents the global minimum of \( \chi^2 \). We emphasize that the (normalized) coefficients \( B_n \) calculated via the central values of the 10 independent sum rules of Ref. do not correspond to a solution with \( \chi^2 = 0 \). Although such a solution must exist, its determination is not a trivial task. Furthermore, at relatively large distances probed in present experiments, still uncalculable contributions of higher twists are presumably more significant than higher-order terms of the Appell polynomial series.

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