STRATIFICATIONS OF SCHEMES USING TANGENT VECTOR FIELDS

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Abstract. Let $X/k$ be a noetherian scheme over a field $k$ of characteristic 0, such that the residue field at its closed points are algebraic extensions of $k$. Let $g_{X/k} \subset T_{X/k}$ be a $O_X$-submodule of the tangent sheaf which is closed under the Lie bracket. We construct a stratification of the subset of closed points in $X$ by locally closed subsets that are preserved by $g_{X/k}$, on which $g_{X/k}$ acts transitively.

1. Let $X/k$ be a noetherian scheme over a field $k$ of characteristic 0, $X^{cl}$ its subset of closed points, and assume that the residue field $k_{X,x}$ is an algebraic extension of $k$ when $x \in X^{cl}$ - we say that the closed points are algebraic over $k$. Let $g_{X/k} \subset T_{X/k}$ be an $O_X$-submodule of the tangent sheaf of $k$-linear derivations which is closed under the Lie bracket in $T_{X/k}$, $[g_{X/k}, g_{X/k}] \subset g_{X/k}$. Say that a point $x$ in $X$, whose maximal ideal in its local ring is $m_x \subset O_{X,x}$, is preserved by $g_{X/k}$ if $g_{X/k,x} \cdot m_x \subset m_x$, so that we get a map $f_x : g_{X/k,x} \to T_{k_{X,x}/k}$, where $T_{k_{X,x}/k}$ is the $k_{X,x}$-vector space of $k$-linear derivations of $k_{X,x}$.

We will use without further reference the fact that derivations that preserve an ideal of noetherian rings that contain the rational numbers also preserve minimal prime divisors of the ideal, as first observed by Seidenberg [4]; see also [3, Lemma 1.3].

The stratification of $X^{cl}$ is based on the notion of defining point, which is a preserved point $x$ such that $g_{X/k}$ acts transitively on the residue field $k_{X,x} = O_{X,x}/m_x$, by which we mean that the map $f_x$ is surjective. Let $I \subset P$ be the subset of defining points in $X$. We note that the generic points of $X$ belong to $P$ (by the above paragraph), but they do not necessarily belong to $I$; the reduced generic points, however, do belong to $I$ when $g_{X/k} = T_{X/k}$. In fact, if we do not require that the closed points be algebraic over $k$ it may happen that $I = \emptyset$ when all the generic points of $X$ are non-reduced.

We partial order $X$ with respect to specialization, so that $x \geq y$ if $y$ specializes to $x$; thus the closed points are maximal. This partial order is in particular inherited by the subsets $I \subset P$.

Lemma 1.1. Assume that $X/k$ is a $k$-scheme such that all its closed points are algebraic over $k$.

(1) The maximal elements in $P$ belong to $I$. In particular, $I \neq \emptyset$.

(2) For any closed point in $X$ there exists a unique element in $I$ which specializes to the closed point and which is maximal with this property.

In general $I$ also contains primes that are not maximal in $P$.

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Proof. We can assume that \( X = \text{Spec} A \), where \( A \) is a commutative noetherian \( k \)-algebra.

(1): The minimal primes in \( X \) belong to \( \mathcal{P} \), so \( \mathcal{P} \neq \emptyset \). Let \( P \) be a maximal element in \( \mathcal{P} \), and denote by \( g_{A,P/k} \subset T_{k_P/k} \) the image of the homomorphism \( f_P : g_{A,P/k} \rightarrow T_{k_P/k} \). If \( P \) is also a maximal ideal, then \( P \in \mathcal{I} \), since \( k_P/k \) is an algebraic extension and thus \( T_{k_P/k} = 0 \). So now assume that \( P \) is not a maximal ideal, and assume on the contrary that \( g_{A,P/k} \neq T_{k_P/k} \). There exist elements \( r_i \in A \), \( i \in U \), so that the elements \( r_i = r_i \mod PA_P \) form a transcendence basis of \( k_P/k \).

Then there exist \( \delta_i \in T_{k_P/k} \) such that \( \delta_i(\tau_j) = \delta_{ij} \), resulting in a \( k_P \)-basis \( \{\delta_i\}_{i \in U} \) of \( T_{k_P/k} \). Since \( f_P \) is not surjective there exists an element \( r \in \sum k\tau_i \) such that \( r \in k\mathcal{P} \) is non-zero and non-algebraic over \( k \), and \( g_{A,P/k}(r) = 0 \). Then since \( r \neq 0 \) it follows that \( r \mod P \) is not invertible in \( A/P \), so \( P + Ar \neq A \). If \( Q \) is a minimal prime divisor of \( P + Ar \), then \( Q \in \mathcal{P} \) and \( P \subset Q \). Since \( P \neq Q \) this contradicts the assumption that \( P \) is a maximal element in \( \mathcal{P} \). Therefore \( g_{A,P/k} = T_{k_P/k} \).

(2): A maximal ideal \( Q \in \mathcal{P} \) is the specialization of some minimal prime in \( X \), hence \( \mathcal{P} \) contains a prime that specializes to \( Q \). Then there exists a maximal prime \( P \in \mathcal{P} \) such that \( P \subset Q \), so by (1), \( P \in \mathcal{I} \). We now assert that if \( P_1, P_2 \in \mathcal{I} \) are maximal defining primes such that \( P_1 \subset Q \) and \( P_2 \subset Q \), then \( P_1 = P_2 \). Assume the contrary, so in particular \( P_2 \not\subset P_1 \). Since \( P_1 + P_2 \subset Q \) it follows that \( P_2 = P_2 \mod P_1 \) is a proper preserved ideal in \( A/P_1 \), hence there exists a non-zero minimal prime divisor of \( P_2 \), which then is preserved; hence there exists a maximal preserved prime \( Q \subset A/P_1 \), which by (1) is a defining prime for the ring \( A/P_1 \), noting that the closed points of \( \text{Spec} A/P_1 \) are algebraic over \( k \). It follows that its contraction \( (Q)^c \subset \mathcal{I} \), and \( P_1 \subset (Q)^c \). Since \( P_1 \neq P_1 + P_2 \subset (Q)^c \), we reach a contradiction since \( P_1 \) is maximal. Therefore \( P_2 \subset P_1 \). In the same way, \( P_1 \subset P_2 \); hence \( P_1 = P_2 \).

For \( \xi \in \mathcal{I} \), we put

\[
D(\xi) = \{ x \in X \mid x \in \xi, \ x \not\in \eta^- \text{ when } \text{ht}(\eta) \geq \text{ht}(\xi), \ \eta \in \mathcal{I}, \ \eta \neq \xi \} = \{ x \in X \mid x \in \xi \text{ is maximal such that } x \in \xi^- \},
\]

where the second line is a consequence of Lemma 1.1 (\( \xi^- \) denotes the closure of the point \( \xi \) and \( \text{ht}(\xi) \) is the height of \( \xi \)). To clarify the properties of the sets \( D(\xi) \) assume that \( X = \text{Spec} A \) as in the proof of the lemma, so that if \( \xi = \mathcal{P} \in \mathcal{I} \) we have

\[
D(\xi) = D(P) = \{ Q \in \text{Spec} A \mid P \subset Q, \ \text{and } P' \not\subset Q \text{ when } P' \in \mathcal{I}, P \subset P' \},
\]

which contains no defining primes except \( P \). In general, a prime \( Q \) need not contain any defining prime, and there can exist several elements in \( \mathcal{I} \) which are contained in \( Q \) and are maximal with this property. Therefore \( \cup_{P \in \mathcal{I}} D(P) \) is in general a proper subset of \( \text{Spec} A \). On the other hand, if \( Q \) is a maximal ideal, by Lemma 1.1 there exists a unique maximal element \( P \in \mathcal{I} \) such that \( P \subset Q \). This is the reason that we only get a stratification of \( X^{cl} \), and not of \( X \).

Say that \( X \) is allowed if it is locally embeddable in a regular \( k \)-scheme satisfying the weak Jacobian criterion, denoted \( [W]_k \) in [1] §30, Lemma 2]. A point \( x \) in an allowed scheme \( X \) is regular if the \( T_{X/k,x} \)-module \( O_{X,x} \) is simple.

Theorem 1.2. Let \( X/k \) be a noetherian \( k \)-scheme such that all its closed points are algebraic over \( k \). The set \( D(\xi)^{cl} \), \( \xi \in \mathcal{I} \), are the closed points of locally closed subscheme of \( X \). If moreover \( X \) is allowed, then \( D(\xi) \) is a regular
scheme. The collection of sets \( \{D(\xi)^{\text{cl}}\}_{\xi \in \mathcal{I}} \) forms a stratification of the set of closed points \( X^{\text{cl}} \subset X \) into a disjoint union satisfying the frontier condition:

(i) \( X^{\text{cl}} = \bigcup_{\xi \in \mathcal{I}} D(\xi)^{\text{cl}} \), and if \( D(\xi)^{\text{cl}} \cap D(\eta)^{\text{cl}} \neq \emptyset \) for some \( \xi, \eta \in \mathcal{I} \), then \( D(\xi) = D(\eta) \).

(ii) If \( D(\xi)^{\text{cl}} \cap D(\eta)^{\text{cl}} \neq \emptyset \), for some \( \xi, \eta \in \mathcal{I} \), then \( D(\xi) = D(\eta) \).

Proof. It suffices to consider the affine situation, \( X = \text{Spec} A \). Letting \( P \in \mathcal{I} \) we have

\[
D(P) = D(P) \setminus \bigcup_{P' \in \mathcal{I}, P' \subset P, P' \neq P} D(P'),
\]

which shows that \( D(P)^{\text{cl}} \) are the closed points of a locally closed subset of \( X \).

\( D(P) \) is regular when \( X \) is allowed: Let \( P \in \mathcal{I} \) and \( Q \in D(P) \subset X \), so \( P \subset Q \). Then the local ring at \( Q = Q \mod P \) is of the form \( B_Q \), where \( B = A/P \). It suffices to prove that \( B_Q \) is a simple \( T_{B_Q/k} \)-module. So assume that \( J \subset B_Q \) is a preserved ideal and \( J \neq B_Q \). Then there exists a maximal preserved prime \( Q_1 \in \text{Spec} B_Q \) such that \( J \subset Q_1 \), and it suffices to prove \( Q_1 = 0 \). Letting \( Q_1 = Q_1^f \) be its contraction with respect to the map \( A \to B_Q \), we have \( P \subset Q_1 \in D(P) \), and \( Q_1 \in \mathcal{I} \). Since \( D(P) \) only contains one defining prime, it follows that \( P = Q_1 \), and therefore \( Q_1 = 0 \).

(i) Let \( Q \) be a maximal ideal. By Lemma 1.1 there exists a unique maximal defining prime \( P \) such that \( P \subset Q \).

(ii) Let \( P_1, P_2 \in \mathcal{I} \), and assume that \( Q \) is a maximal ideal in \( D(P_1) \) such that \( P_2 \subset Q \). Let \( P_{2,m} \in \mathcal{I} \) be a maximal element with the property \( P_2 \subset P_{2,m} \subset Q \). Then \( P_{2,m} \in \mathcal{I} \) is also a maximal element with the property \( P_{2,m} \subset Q \), hence by Lemma 1.1 \( P_{2,m} = P_1 \); hence \( P_2 \subset P_1 \).

One can think of \( g_{X/k} \) as an involutive system of tangent vector fields on \( X \), so that our stratification is an algebraic version of the Frobenius theorem in differential geometry. Perhaps the most interesting situation is when \( X \) is a singular scheme and \( g_{X/k} = T_{X/k} \). For example, let \( f \in A \) be an element in a regular k-algebra such that \( A/m \) is algebraic over \( k \) when \( m \) is a maximal ideal, \( X = V(f) \subset \text{Spec} A \). Then Theorem 1.2 is an algebraic version of the logarithmic stratification of a hypersurface singularity, introduced in the analytic case by K. Saito [2].

Let us extend the terminology in [2] Def. 3.8 to our algebraic situation:

Definition 1.3. Let \( X/k \) and \( g_{X/k} \subset T_{X/k} \) be as above. Say that a point \( x \in X^{\text{cl}} \cap \mathcal{I} \) is holonomic if \( x \in \eta^{-} \) and \( \eta \in \mathcal{P} \) implies that \( \eta \in \mathcal{I} \). If \( \xi \) is a holonomic defining point then the corresponding stratum \( D(\xi)^{\text{cl}} \) is also called holonomic.

Proposition 1.4. A closed point \( x \) is holonomic if and only of there exists an open set \( U, x \in U \), such that \( U \cap \mathcal{I} \) is finite. If \( x \in D(\xi)^{\text{cl}}, \xi \in \mathcal{I} \), then \( \xi \) is holonomic if and only of \( x \) is holonomic.

For the proof one can assume that \( X = \text{Spec} A \), and the main observation is that then \( \mathcal{P} = \mathcal{I} \) if and only if \( \mathcal{I} \) is a finite set, and if this is the case, then all defining and closed points are holonomic.

The main property of holonomic strata is that \( X \) is analytically trivial along such strata.

Proposition 1.5. Let \( D(\xi)^{\text{cl}} \) be a holonomic stratum of dimension \( n \) and \( x \in D(\xi)^{\text{cl}} \). Then the completion \( \hat{O}_{X,x} = \mathbb{C}[[x_1, \ldots, x_n]] \), where \( \mathbb{C} \) is a subring of \( \hat{O}_{X,x} \). If there exists a Lie subalgebra \( g \subset g_{X/k} \) that maps surjectively to \( T_{X,x,k} \), then we can take \( C = \{ c \in \hat{O}_{X,x} | g \cdot c = 0 \} \).
The proof is an application of [1, Th. 30.1]. The non-holonomic strata of $g_{X/k}$ in $X$ can be regarded as a deformation space for a family of some geometric object in $X$ traversing the stratum.

We finish with some concrete examples, where the first trivial one is inserted only for the purpose of fixing ideas.

Example 1.6. Put $g_{A/C} = A\partial_x + A\partial_y \subset T_{A/C}$, where $A = \mathbb{C}[x, y, z]$, has the defining points $(z - \lambda) \in \text{Spec} A, \lambda \in k$, which are non-holonomic. These planes also form the foliation of $\mathbb{C}^3$ formed by the integral manifolds of the involutive system of vector fields $(\partial_x, \partial_y)$. It may happen that $\delta \in T_{A/C}$ is such that $g_{A/C} = A\delta$ only has the defining prime $(0) \subset A$, so that $\mathbb{C}^3$ contains a single stratum. On the other hand, if $\mathbb{C}^3$ is regarded as a complex analytic manifold, then there are plenty if integral manifolds for $\delta$.

Example 1.7. Put $I = (xy(x + y)(x + zy))$, and $X = V(I) \subset \mathbb{A}^3_k$. The module of tangential vector fields $T_{A^3_k}(I)$ is free, which is easily seen using Saito’s criterion (I thank M. Granger for pointing this out), so that the ‘obvious’ tangential derivations form a basis $T_{A^3_k}(I) = k[x, y, z][(\partial_x, \partial_y) + k[x, y, z][(x + y)(\partial_y - z\partial_z)]$, and the tangent sheaf $T_{X/k}$ is the sheaf that is associated to the module $T_{A^3_k}(I)/I_{A^3_k}$. The singular locus is $X_s = V((x, y) \cup V(x, y + z) \cup V(y, x + z) \cup V(x + y, x + zy) \subset X$. Let $\xi_1 = (x), \xi_2 = (y), \xi_3 = (x + yz)$ be the generic points of $X$, and $\eta_1 = (x, y), \eta_2 = (x, y + z), \eta_3 = (y, x + z), \eta_4 = (x + y, x + yz)$ the generic points of $X_s$, and for any $\lambda \in k$ we put $\eta_3 = (x, y, z - \lambda)$. Then the set of preserved points is $\mathcal{P} = (\xi_i, \eta_1, \eta_3, \eta_4, \eta_3, \lambda \in k)$ and the defining points are $\mathcal{I} = (\xi_i, \eta_1, \eta_3, \eta_4, \eta_3, \lambda \in k)$, so $\eta_1$ is not a defining point. The stratification of $X^{\mathbb{C}}$ is the disjoint union of the following locally closed smooth subvarieties. Firstly, $D(\xi_i)^{\mathbb{C}}$ is the union of all smooth and closed points in the corresponding component of $X$, secondly $D(\eta_1)$, $i = 2, 3, 4$, is the union of the smooth points of the components of $X_s$, except the $z$-axis, and finally $D(\eta_3)$ run over the closed points on the $z$-axis; the latter are the non-holonomic defining points in $X$. The projection $X \to \text{Spec} k[z]$ defines a flat family of curve singularities such that no tangent vector field on $\text{Spec} k[z]$ can be lifted to a tangent vector field on $X$.

Example 1.8. Let $f = f_1 \cdots f_r \in \mathbb{C}[x_1, x_2, \ldots, x_n]$ be a product of non-proportional complex polynomials $f_i$ of degree 1, so $V(f) \subset \mathbb{C}^n$ is an arrangement of hyperplanes. Let $g_{\mathbb{C}^n/k} = T_{\mathbb{C}^n/k}(f)$ be the tangent vector fields which preserve the principal ideal $(f)$. The preserved prime ideals $P \in \mathcal{P}$ are ideals of the form $P = (f_i_1, \ldots, f_i_r)$, where $\{f_i_1, \ldots, f_i_r\} \subset \{f_1, \ldots, f_r\}$. Clearly $\mathcal{P} = \mathcal{I}$, and all the elements in $\mathcal{I}$ are holonomic.

References

[1] Hideyuki Matsumura, *Commutative ring theory*, Cambridge University Press, 1986.
[2] Kyoji Saito, *Theory of logarithmic differential forms and logarithmic vector fields*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. 27 (1980), no. 2, 265–291. MR 83h:32023
[3] Günter Scheja and Uwe Storch, *Fortsetzung von Derivationen*, J. Algebra 54 (1978), no. 2, 353–365 (German).
[4] A. Seidenberg, *Differential ideals in rings of finitely generated type*, Amer. J. Math. 89 (1967), 22–42.

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