Hybrid nanofluid flow through a spinning Darcy–Forchheimer porous space with thermal radiation

Anwar Saeed1, Muhammad Jawad2, Wajdi Alghamdi3, Saleem Nasir4, Taza Gul4 & Poom Kumam1,5

This work investigates numerically the solution of Darcy–Forchheimer flow for hybrid nanofluid by employing the slip conditions. Basically, the fluid flow is produced by a swirling disk and is exposed to thermal stratification along with non-linear thermal radiation for controlling the heat transfer of the flow system. In this investigation, the nanoparticles of titanium dioxide and aluminum oxide have been suspended in water as base fluid. Moreover, the Darcy–Forchheimer expression is used to characterize the porous spaces with variable porosity and permeability. The resulting expressions of motion, energy and mass transfer in dimensionless form have been solved by HAM (Homotopy analysis method). In addition, the influence of different emerging factors upon flow system has been disputed both theoretically in graphical form and numerically in the tabular form. During this effort, it has been recognized that velocities profiles augment with growing values of mixed convection parameter while thermal characteristics enhance with augmenting values of radiation parameters. According to the findings, heat is transmitted more quickly in hybrid nanofluid than in traditional nanofluid. Furthermore, it is estimated that the velocities of fluid $f'(ξ)$, $g(ξ)$ are decayed for high values of $φ_1$, $φ_2$, $Fr$ and $k_1$ factors.

Nanofluids research has got immense consideration of academicians due to several technological and industrial uses. As a result of a variety of improved energy exchange uses, nanofluids are a serious fascinating and interesting issue of inspection in different sectors of engineering and technology. According to recent research, Nanofluids’ heat transmission strength is far higher than that of conventional liquids. Thus, the replacement of ordinary fluids through nanoliquids is more secure. A few researchers and designers are pulled into the investigation of nanofluids on account of their greater energy capacities and utilizations. Nanofluids are renowned for providing substantially higher thermal conductivity than many other solvents. Nanofluids have had a significant impact on developing technologies and applications in the fields of research and innovation, medical science, and engineering. Researchers and scientists have utilized a variety of models to explore the thermal and mechanical properties of nanofluids. The idea of the nanomaterial addition with the classical fluid to augment its thermal conductivity was introduced for the first time by Choi1. The MHD flow has various uses in different industrial processes, like in nuclear energy reactors, crystal growth, electronic and electrical devices, solar energy technology, magnetic confinement fusion, and so on. Khan et al.2 described the expression of nanoliquid stream through a swinging sheet. In this direction, several scientists3–6 have studied nanofluid flow from several aspects. However, during the last few years, energy exchange in carbon nanoliquids has attracted a lot of attention from scientists in various disciplines. Carbon Nanotubes are a basic chemical formation with a carbon atom composition that is coiled into a cylindrical shape. It has been noted that the shape has always a strong influence on the thermal conduction of nanoliquid. Also, Hatami et al.7 discussed the incompressible viscoelastic laminar motion of fluid

1Center of Excellence in Theoretical and Computational Science (TaCS-CoE), Faculty of Science, King Mongkut’s University of Technology Thonburi (KMU), 126 Pracha Uthit Rd., Bang Mod, Thung Khru, Bangkok 10140, Thailand. 2Department of Mathematics, Swabi University, Swabi 23430, Khyber Pakhtunkhwa, Pakistan. 3Department of Information Technology, Faculty of Computing and Information Technology, King Abdulaziz University, Jeddah 80261, Saudi Arabia. 4Department of Mathematics, City University of Science and Information Technology, Peshawar 25000, Khyber Pakhtunkhwa, Pakistan. 5Department of Medical Research, China Medical University Hospital, China Medical University, Taichung 40402, Taiwan. *Email: muhammadjawad175@yahoo.com; poom.kum@kmutt.ac.th
owing to spinning and expanding discs. Mustafa et al.\textsuperscript{6} described the stream of liquid throughout the existence of nanoparticles by stretching the disk. They found that standardized disk stretching has a supreme role in reducing the thickness of the boundary layer. Pak and Cho\textsuperscript{7} studied experimentally the impacts of γ-alumina (Al\textsubscript{2}O\textsubscript{3}) and titanium dioxide on the turbulent heat energy transportation of water. These authors found that the mixing of nanoparticles with water fallouts in the enhancement of the convective heat energy transformation coefficient. Lunde et al.\textsuperscript{8} used Tiwari and Das model to perform the stability analysis and to discover the different solutions during the hybrid nanoliquid flow through a dwindling surface. Uddin et al.\textsuperscript{9–13} studied the radiative convectional flow of nanofluids using different configurations in presence of slip effect.

The hybrid nanoliquid is a special variant of nanofluid in which two or more different nanoscale materials are distributed in a working fluid in varying configurations. The nanomaterials configurations are selected with the goal of incorporating the beneficial effects including both nanomaterials into a single stable homogeneous system. The field of hybrid nanoliquids is growing rapidly. Hybrid nanoliquids have a wide spectrum of uses, which include modern automation cooling systems, automobile heat dissipation, hybrid electrical systems, fuel cells, gas sensing, bio-medicine manufacturing, renewable power, solar thermal, transistors, and domestic freezers. Researchers are interested in hybrid nanoliquids because of their growing demand in the heat transfer process. Few developments in hybrid nanofluid flow can be checked through Refs.\textsuperscript{14,15}. Acharya et al.\textsuperscript{16} scrutinized the nanofluids the nonlinear thermal radiation also exists and is very limited\textsuperscript{36–38}. Haider et al.\textsuperscript{38} have used the gravitational fluids the nonlinear thermal radiation are also accounted. The constant angular velocity of turning disk is Ω\textsubscript{1}.

The boundary layer of mix convectional flows has many markets uses like dominant nuclear reactors, solar receivers, heat exchangers and electronic devices\textsuperscript{26}. In light of these uses, a number of studies were conducted to determine the effects of mix convectional on nanofluid boundary layer flow. Hayat et al.\textsuperscript{37} used Cattaneo-Christov concept to quantitatively analyze the Darcy–Forchhemier flow on an expanding bending medium. Gul et al.\textsuperscript{38} considered the hybrid nanofluids (CNTs nanoparticles) flow through a swirling disk. Waini et al.\textsuperscript{39} employed hybrid nanofluids flowing on a vertical needle to investigate flow and heat.

Thermal radiation in the fluid flow further enhances the thermal efficiency of the hybrid nanofluid and is continuously used in the linear form\textsuperscript{30,31}. In fact, in hybrid nanofluids, the volume fraction of the nanoparticles is limited up to 5%, and thermal radiations in linear form properly work. Even the researchers used the linear thermal radiation term in non-Newtonian fluids\textsuperscript{32–35} whose stress is not linear. While in the case of non-Newtonian fluids the nonlinear thermal radiation also exists and is very limited\textsuperscript{36–38}. Haider et al.\textsuperscript{38} have used the Darcy–Forchheimer flow using the same model for various aspect of the physical parameters.

The ultimate priority of proposed study is to numerically analyze the solution of Darcy–Forchheimer flow for a hybrid nanofluid (Al\textsubscript{2}O\textsubscript{3}, TiO\textsubscript{2}) by employing the slip condition. The flow is produced by a swirling disk and is exposed to thermal stratification and nonlinear thermal radiation for controlling the heat transfer of the flow system. In this investigation, the nanoparticles of titanium dioxide and aluminum oxide have been suspended in water as base fluid. Moreover, the Darcy–Forchheimer expression is used to characterize the porous spaces with variable porosity and permeability. The published work\textsuperscript{39} is also extended with the addition of concentration profile. The Brownian motion and Thermophoresis analysis have also been used to extend the existing literature\textsuperscript{39}. The resulting expressions have been solved by HAM (homotopy analysis method). The significance of different emerging factors upon flow system has been discussed both theoretically in graphical form and numerically in the tabular form.

**Problem formulation**

We assume that the time independent three-dimensional (3D) flow of hybrid nanofluid with velocity slip by a rotating disk, as depicted in Fig. 1. Thermal stratification, heat generation/absorption and non-linear thermal radiation are also accounted. The constant angular velocity of turning disk is Ω. The components of velocity in the ascending orientations are \((r, \psi, z)\) & \((u, v, w)\). Using Buongiorno model, the three-dimensional flow resulting equations in the aforementioned conditions is given as\textsuperscript{27,38}:

\[
\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} = 0 ,
\]
Figure 1. The schematic layout of 3-dimensional problem.

\[ u \left( \frac{\partial u}{\partial r} \right) - \frac{v^2}{r} + w \left( \frac{\partial u}{\partial z} \right) = \nu_{\text{hf}} \left( \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} - \frac{u}{r^2} + \frac{\partial^2 u}{\partial z^2} \right) - \nu_{\text{hf}} \frac{\varepsilon(z)}{K(z)} u \sqrt{u^2 + v^2}, \quad (2) \]

\[ u \left( \frac{\partial v}{\partial r} \right) - \frac{uv}{r} + w \left( \frac{\partial v}{\partial z} \right) = \nu_{\text{hf}} \left( \frac{\partial^2 v}{\partial r^2} - \frac{v}{r^2} + \frac{1}{r} \frac{\partial v}{\partial r} + \frac{\partial^2 v}{\partial z^2} \right) - \nu_{\text{hf}} \frac{\varepsilon(z)}{K(z)} v \sqrt{u^2 + v^2}, \quad (3) \]

\[ u \left( \frac{\partial w}{\partial r} \right) + w \left( \frac{\partial w}{\partial z} \right) = \nu_{\text{hf}} \left( \frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} + \frac{\partial^2 w}{\partial z^2} \right) - \nu_{\text{hf}} \frac{\varepsilon(z)}{K(z)} w \sqrt{u^2 + v^2}, \quad (4) \]

\[ \frac{\partial T}{\partial r} + \frac{\partial T}{\partial z} = \alpha_{\text{hf}} \left( \frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} \right) - \frac{1}{(\rho \varepsilon p)_{\text{hf}}} \frac{\partial q_r}{\partial z} \]

\[ + \nu_{\text{hf}} \left[ D_B \left( \frac{\partial C}{\partial z} + \frac{D_T}{T_{\infty}} \frac{\partial T}{\partial z} \right) + \frac{D_T}{T_{\infty}} \left( \frac{\partial T}{\partial z} \right) \right] + \frac{Q}{(\rho \varepsilon p)_{\text{hf}}} (T - T_{\infty}), \quad (5) \]

\[ \left[ u \left( \frac{\partial C}{\partial r} \right) + w \left( \frac{\partial C}{\partial z} \right) = D_B \left( \frac{\partial C}{\partial z} \right) + \frac{D_T}{T_{\infty}} \frac{\partial T}{\partial z} \right] + \frac{D_T}{T_{\infty}} \frac{\partial^2 T}{\partial z^2}, \quad (6) \]

\[ u = L \left( \frac{\partial u}{\partial z} \right), \quad v = r \Omega + L \left( \frac{\partial v}{\partial z} \right), \quad w = 0, \quad T = T_w = T_0 + Ar, \quad \epsilon = C_w \text{ at } z = 0, \quad (7) \]

\[ u \rightarrow 0, \quad v \rightarrow 0, \quad T \rightarrow T_{\infty} = T_0 + Br, \quad C \rightarrow C_{\infty}, \quad \text{at } z = \infty. \]

where form 27,30

\[ K(z) = K_{\infty} \left( 1 + d \varepsilon^2 \right), \quad (8) \]

\[ \epsilon(z) = \epsilon_{\infty} \left( 1 + d^* \varepsilon^2 \right), \quad (9) \]

\[ q_r = -\frac{4\sigma^* T^4}{3k} \frac{\partial T}{\partial z} = -\frac{16\sigma^*}{3k} T^3 \frac{\partial^3 T}{\partial z^3}. \quad (10) \]

Here \( C_w \) (Drag coefficient), \( d^* \) (variable porosity), \( T_0 \) (Reference temperature), \( \sigma^* \) (Stefan Boltzmann constant), \( k \) (coefficient of Mean absorption), \( d \) (Variable permeability), \( T_{\infty} \) (at free stream Thermal stratification), \( L_1 \) (Velocity slip coefficient), \( T_w \) (At wall Thermal stratification), \( A \) and \( B \) dimensional constants, \( K_{\infty} \) (permeability) and \( \epsilon_{\infty} \) (porosity). Thus, the energy equation develops
with $T = (T_w - T_o)\phi (e) + T_o$, Eq. (1) is identically proved and Eqs. (2)-(11) yield

$$u = \frac{dg}{dz} = \frac{D}{\rho C_p} \frac{d\theta}{dz}.$$  

Theoretic model for hybrid nanofluid.\(^{14}\)

In the preceding formulations, the subscript $bf$ and $hnf$ denoted the hybrid nanofluid and base fluid (Al\(_2\)O\(_3\), CuO, or Al\(_2\)O\(_3\)-CuO) (Solid volume fraction of Al\(_2\)O\(_3\), CuO, or Al\(_2\)O\(_3\)-CuO). Thermal conductivity ($\kappa$), Dynamic viscosity ($\mu$), Density ($\rho$), and Thermal capacity ($C_p$) of the nanofluid are denoted by $\kappa_{hnf}$, $\mu_{hnf}$, $\rho_{hnf}$, and $C_{p,hnf}$. $\kappa_{bf}$, $\mu_{bf}$, $\rho_{bf}$, and $C_{p,bf}$ are denoted by $\kappa_{bf}$, $\mu_{bf}$, $\rho_{bf}$, and $C_{p,bf}$ for the base fluid. $\theta_1 = \frac{\theta - \theta_1}{1 - \theta_1}$ and $\theta_2 = \frac{\theta - \theta_2}{1 - \theta_2}$

Considering the preceding considerations, the following can be obtained:

$$\kappa = \kappa_{hnf} + k_1 \phi + k_2 \phi^2 + k_3 \phi^3,$$

$$\theta_1 = \frac{\theta_{hnf} - \theta_1}{1 - \theta_1} + N_{hnf} \phi,$$

$$\theta_2 = \frac{\theta_{hnf} - \theta_2}{1 - \theta_2} + N_{bf} \phi.$$
\[ f = 0, \quad f' = \gamma f'', \quad g = 1 + \gamma g', \quad \Theta = 1 - S; \quad \Phi = 1 \quad \text{at} \quad \xi = 0 \]
\[ f' \to 0, \quad g \to 0, \quad \Theta \to 0, \quad \Phi \to 0, \quad \text{at} \quad \xi \to \infty. \quad (18) \]

Here \((k_1)\) the porosity factor, \((Pc_v)\) the Peclet number, \((Nt)\) thermophoresis parameter, \((Nb)\) Brownian motion parameter, \((Re_r)\) the local Reynolds number, \((Fr)\) the local inertial factor, \((R)\) radiation factor, \((Pr)\) Prandtl number, \((S_r)\) the thermal stratification factor, \((Sc)\) Schmidt number, and heat source factor \((\alpha)\) defined by:

\[ \gamma = L \left( \frac{2\Omega}{\nu_f} \right)^{1/2}, \quad k_1 = \frac{K \infty}{\nu_f \infty}, \quad Re_r = \frac{U_w r}{\nu}, \quad Fr = \frac{C_{bh} \sqrt{2}}{\sqrt{K}}, \quad Sc = \frac{\nu}{D_B}, \quad Nt = \frac{D_f (T_w - T_\infty)}{\nu T_\infty} \]
\[ Nb = \frac{\tau D_B (C_w - C_\infty)}{\nu}, \quad S_t = \frac{B}{A}, \quad R = \frac{4\Omega S^3}{k_{bf}}, \quad Pr = \frac{\nu_f}{\alpha_f}, \quad Pe_r = Re_r, \quad Pr, \alpha = \frac{Q}{\Omega (\rho c_p) f}. \quad (19) \]

**Quantities of physical interest.** Significant physical factors such as \(C_f, \ C_g, \ Nu\) and \(Sh\) are expressed for engineering purposes as

\[ [Re_r]^2 C_f = \frac{1}{(1 - \phi_1 - \phi_2)^2} f''(0), \quad [Re_r]^2 C_g = \frac{1}{(1 - \phi_1 - \phi_2)^2} g''(0) \]
\[ \frac{1}{2} [Re_r]^{-1/2} Nu = - \left( \frac{k_{inf}}{k_f} + \frac{4}{3} \frac{R}{\theta_w + S_t} + 1 \right) \theta'(0), \quad \frac{1}{2} [Re_r]^{-1/2} Sh = - \Phi(0) \quad (20) \]

**Solution by homotopy analysis method**

Equations (13–16) with specified boundary conditions Eq. (17) are tackled through the HAM. Mathematica software is used for this goal.

\[ L_\xi (\tilde{f}) = \tilde{f}''(0), \quad L_\xi (\tilde{g}) = \tilde{g}''(0), \quad L_\xi (\tilde{\Theta}) = \tilde{\Theta}''(0), \quad L_\xi (\tilde{\Phi}) = \tilde{\Phi}''(0). \quad (21) \]

The linear operators are defined as:

\[ L_\xi \left( f + \xi \frac{\theta}{\phi} \right) = 0, \quad L_\xi \left( g + \xi \frac{\phi}{\theta} \right) = 0, \quad L_\xi (\Theta + \xi \frac{\Omega}{\Theta}) = 0, \quad L_\xi (\phi + \xi \frac{\Phi}{\phi}) = 0. \quad (22) \]

The non-linear operators are chosen as \(N_{f, g, \Theta, \Phi}\) and identify in system:

\[ N_{f, g} \left[ f(\xi; \phi), g(\xi; \phi) \right] = \frac{1}{(1 - \phi_1 - \phi_2)^2 \left( \frac{1}{1 - \phi_1 - \phi_2} \right)} \left[ f(\xi; \phi) - \frac{1}{2k_1 Re_r} \left( f(\xi; \phi) + \frac{d^2 f(\xi; \phi)}{d\xi^2} \right) \right] - F_r \left( \frac{1 + d^2 f(\xi; \phi)}{\sqrt{1 + d^2 f(\xi; \phi)}} \right) \left( \frac{1 + d^2 f(\xi; \phi)}{\sqrt{1 + d^2 f(\xi; \phi)}} \right) \left( \frac{1 + d^2 f(\xi; \phi)}{\sqrt{1 + d^2 f(\xi; \phi)}} \right) - f(\xi; \phi) \]
\[ + g(\xi; \phi) + \frac{1}{2} \left( \frac{d}{d\xi} f(\xi; \phi) \right) \]
\[ N_{g, g} \left[ f(\xi; \phi), g(\xi; \phi) \right] = \frac{1}{(1 - \phi_1 - \phi_2)^2 \left( \frac{1}{1 - \phi_1 - \phi_2} \right)} \left[ g(\xi; \phi) - \frac{1}{2k_1 Re_r} \left( g(\xi; \phi) + \frac{d^2 g(\xi; \phi)}{d\xi^2} \right) \right] - F_r \left( \frac{1 + d^2 g(\xi; \phi)}{\sqrt{1 + d^2 g(\xi; \phi)}} \right) \left( \frac{1 + d^2 g(\xi; \phi)}{\sqrt{1 + d^2 g(\xi; \phi)}} \right) \left( \frac{1 + d^2 g(\xi; \phi)}{\sqrt{1 + d^2 g(\xi; \phi)}} \right) - g(\xi; \phi) + \frac{1}{2} \left( \frac{d}{d\xi} g(\xi; \phi) \right) \]
\[ N_{\Theta, \Theta} \left[ f(\xi; \phi), \Theta(\xi; \phi) \right] = \frac{1}{(1 - \phi_1 - \phi_2)^2 \left( \frac{1}{1 - \phi_1 - \phi_2} \right)} \left[ \Theta(\xi; \phi) - \frac{1}{2k_1 Re_r} \left( \Theta(\xi; \phi) + \frac{d^2 \Theta(\xi; \phi)}{d\xi^2} \right) \right] - F_r \left( \frac{1 + d^2 \Theta(\xi; \phi)}{\sqrt{1 + d^2 \Theta(\xi; \phi)}} \right) \left( \frac{1 + d^2 \Theta(\xi; \phi)}{\sqrt{1 + d^2 \Theta(\xi; \phi)}} \right) \left( \frac{1 + d^2 \Theta(\xi; \phi)}{\sqrt{1 + d^2 \Theta(\xi; \phi)}} \right) - \Theta(\xi; \phi) \]
\[ + \frac{k_{inf}}{k_f} \left( \frac{1}{\theta_w + S_t} \right) \frac{d}{d\xi} \Theta(\xi; \phi) + \left( \frac{d}{d\xi} f(\xi; \phi) \right) \Theta(\xi; \phi) \]
\[ N_{\Phi, \Phi} \left[ f(\xi; \phi), \Phi(\xi; \phi) \right] = \frac{1}{(1 - \phi_1 - \phi_2)^2 \left( \frac{1}{1 - \phi_1 - \phi_2} \right)} \left[ \Phi(\xi; \phi) - \frac{1}{2k_1 Re_r} \left( \Phi(\xi; \phi) + \frac{d^2 \Phi(\xi; \phi)}{d\xi^2} \right) \right] - F_r \left( \frac{1 + d^2 \Phi(\xi; \phi)}{\sqrt{1 + d^2 \Phi(\xi; \phi)}} \right) \left( \frac{1 + d^2 \Phi(\xi; \phi)}{\sqrt{1 + d^2 \Phi(\xi; \phi)}} \right) \left( \frac{1 + d^2 \Phi(\xi; \phi)}{\sqrt{1 + d^2 \Phi(\xi; \phi)}} \right) - \Phi(\xi; \phi) \]
\[ + \frac{k_{inf}}{k_f} \left( \frac{1}{\theta_w + S_t} \right) \frac{d}{d\xi} \Phi(\xi; \phi) + \left( \frac{d}{d\xi} f(\xi; \phi) \right) \Phi(\xi; \phi) \]
\[ (23) \]
\[ (24) \]
\[ (25) \]
\[
N_\Phi \left[ \Theta(\xi; \zeta), \Phi(\xi; \zeta) \right] = \Phi(\xi) - \text{Pr} Sc \, \Phi(\xi) + \frac{N_t}{N_b} \Phi(\xi),
\]

While BCs are:
\[
\begin{align*}
\frac{\partial f(\xi; \zeta)}{\partial \xi} \bigg|_{\xi=0} &= \gamma \frac{\partial^2 f(\xi; \zeta)}{\partial \xi^2} \bigg|_{\xi=0}, \\
\frac{\partial g(\xi; \zeta)}{\partial \xi} \bigg|_{\xi=0} &= 0, \\
\frac{\partial g(\xi; \zeta)}{\partial \eta} \bigg|_{\eta=0} &= 1 + \gamma \frac{\partial g(\xi; \zeta)}{\partial \eta} \bigg|_{\eta=0}, \\
\Theta(\xi; \zeta) \bigg|_{\xi=0} &= 1 - S_t, \\
\Phi(\xi; \zeta) \bigg|_{\xi=0} &= 1,
\end{align*}
\]

Here, \( \zeta \) is the embedding parameter \( \zeta \in [0, 1] \), to ensure that the convergence of the solution is consistent \( h_f, h_g, \) and \( h_\Phi \) is used. By choosing \( \zeta = 0 \) and \( \zeta = 1 \), we have
\[
\begin{align*}
f(\xi; 1) &= f(\xi), \\
g(\xi; 1) &= g(\xi), \\
\Theta(\xi; 1) &= \Theta(\xi), \\
\Phi(\xi; 1) &= \Phi(\xi),
\end{align*}
\]

Develop the Taylor’s series for \( \hat{f}(\xi; \zeta), \hat{g}(\xi; \zeta), \hat{\Theta}(\xi; \zeta) \) and \( \hat{\Phi}(\xi; \zeta) \) about the point \( \zeta = 0 \)
\[
\begin{align*}
\hat{f}(\xi; \zeta) &= \hat{f}_0(\xi) + \sum_{n=1}^{\infty} \hat{f}_n(\xi) \zeta^n, \\
\hat{g}(\xi; \zeta) &= \hat{g}_0(\xi) + \sum_{n=1}^{\infty} \hat{g}_n(\xi) \zeta^n, \\
\hat{\Theta}(\xi; \zeta) &= \hat{\Theta}_0(\xi) + \sum_{n=1}^{\infty} \hat{\Theta}_n(\xi) \zeta^n, \\
\hat{\Phi}(\xi; \zeta) &= \hat{\Phi}_0(\xi) + \sum_{n=1}^{\infty} \hat{\Phi}_n(\xi) \zeta^n.
\end{align*}
\]

While B.C.s are:
\[
\begin{align*}
\hat{f}(0) &= 0, \\
\hat{g}(0) &= 1 + \gamma \hat{g}'(0), \\
\hat{\Theta}(0) &= 1 - S_t, \\
\hat{\Phi}(0) &= 1,
\end{align*}
\]

\[
\hat{f}(\infty) \rightarrow 0, \quad \hat{g}(\infty) \rightarrow 1, \quad \hat{\Theta}(\infty) \rightarrow 0, \quad \hat{\Phi}(\infty) \rightarrow 0.
\]

**Results and discussion**

In this part, we address about the behavior of diverse emerging flow parameters such as \( \phi_1, \phi_2, Re, Fr, k_1, M, \theta_c, K, (S_1) \) the thermal stratification parameter, and heat source parameter \( (\omega) \). The geometry of the model problem is shown in Fig. 1. Figures 2, 3, 4, 5, 6 and 7 highlight that how different values of relevant factors affect TiO\(_2\) nanofluid and Al\(_2\)O\(_3\) + TiO\(_2\) hybrid nanofluid for varying \( \phi_1, \phi_2 \) values. The graph indicated that the greater magnitudes of \( \phi_1, \phi_2 \) causes the primary \( f(\xi) \) and \( g(\xi) \) secondary velocity distributions to drop. Physically, the collision of inter-particles intensifies as the \( \phi_1, \phi_2 \) of Al\(_2\)O\(_3\), TiO\(_2\) improves, and as a response, the TiO\(_2\) nanofluid and Al\(_2\)O\(_3\) + TiO\(_2\) hybrid nanofluid primary \( f(\xi) \) and \( g(\xi) \) secondary velocity distributions decreases. The primary \( f(\xi) \) and \( g(\xi) \) secondary velocity fluctuations for \( k_1 \) (porous media factor) for TiO\(_2\) nanofluid and Al\(_2\)O\(_3\) + TiO\(_2\) hybrid nanofluid are emphasized in Figs. 4 and 5. We can see from the graph that as the amplitude of \( k_1 \) increases, the primary \( f(\xi) \) and \( g(\xi) \) secondary
velocity distributions diminish. Physically, increasing the size of $k_1$ causes a reduction in the transparency of the porous zone. As a response, there is a small dump in the surface of disk which oppose nanofluid and hybrid nanofluid to flow through, and the fluid velocity become slow. The fluctuations $Fr$ versus primary $f^\prime(\xi)$ and $g(\xi)$ secondary velocity distributions for both TiO$_2$ nanofluid and Al$_2$O$_3$ + TiO$_2$ hybrid nanofluid are highlighted in Figs. 6 and 7. It should be emphasized from the plots that for nanofluid and hybrid nanofluid, $Fr$ is the decreasing function of both $f^\prime(\xi)$ and $g(\xi)$. In essence, a rise in $Fr$ causes fluids to become more resilient, leading to

Figure 2. Velocity profile $f^\prime(\xi)$ against $\phi_1, \phi_2$.

Figure 3. Velocity profile $g(\xi)$ against $\phi_1, \phi_2$.

Figure 4. Velocity profile $f^\prime(\xi)$ against $k_1$. 
decreased $f'(\xi)$ and $g(\xi)$. Physically, boosting the amplitude of $F$ reduces the interior nanofluid velocity, but it has no effect on the liquid thicknesses. As a consequence, a rise in $F$ creates a well stream resistance, limiting fluid velocity $f'(\xi)$ and $g(\xi)$ distributions. Figures 8, 9, 10, 11, and 12 illustrate how significant variation of key variables influence on TiO$_2$ nanofluid and Al$_2$O$_3$ + TiO$_2$ hybrid nanofluid $\theta(\xi)$ thermal behavior. The steady-state $\theta(\xi)$ temperature distributions of TiO$_2$ nanofluid and Al$_2$O$_3$ + TiO$_2$ hybrid nanofluid for various values of
$\phi_1, \phi_2$ are shown in Fig. 8. The plot observed that adding $\phi_1, \phi_2$ optimizes the thermal distributions. Physically, this trend is attributable to TiO$_2$ and Al$_2$O$_3$ + TiO$_2$ greater thermal conductivity as $\phi_1, \phi_2$ increases, that becomes the major source of temperature increase. Figure 9 exhibits the temperature curve for multiple $N_b$ values. The elevation in $N_b$ leads to a rise in $\theta(\xi)$ temperature distributions, as seen in Fig. 9. Such phenomena are associated with a significant increase in Brownian motion, that reveals the erratic motion of molecules dispersed in TiO$_2$.
nanofluid and Al₂O₃ + TiO₂ hybrid nanofluid. It may be deduced as enhancing the Brownian motion, boosts the temperature significantly across the boundary layer by raising the collision of TiO₂ nanofluid and Al₂O₃ + TiO₂ hybrid nanofluid particles. The influence of multiple variations of the thermophoretic parameter Nₜ on TiO₂ nanofluid and Al₂O₃ + TiO₂ hybrid nanofluid θ(ξ) temperature profiles is depicted in Fig. 10. That’s obvious to see how raising Nₜ values improve the θ(ξ) across the boundary. In terms of physics, the increase in Nₜ is due to an increase in the thermophoretic process. Thermophoresis is a form of molecular mobility that occurs when thermal gradients are imposed, and it is deeply linked to the soret phenomenon. Due to particle dispersion mediated by the thermophoretic phenomenon, nanomaterials transmit thermal energy from the heated edge to the coldest edge inside the boundary layer area. As a result, the temperature of the fluid increases rapidly. In the influence of buoyancy force, Fig. 11 depicts the α (heat generation) consequence on the θ(ξ) temperature profile using TiO₂ nanofluid and Al₂O₃ + TiO₂ hybrid nanofluid. As the values of the heat generation parameter are enhanced, the fluid temperature goes up considerably. The reason for this is that, the outer heating element transfers additional heat into the nano and hybrid nanofluid flow area, that leads the fluid temperature to rise. The θ(ξ) temperature profile of the TiO₂ nanofluid and Al₂O₃ + TiO₂ hybrid nanofluid is elevated owing to the influence of the R thermal radiation parameter, as seen in Fig. 12. Enhancement in radiative heat flow promotes the molecular transit inside the framework, and so regular collision between molecules translates into thermal energy. As a result of the increased values of the R, a greater θ(ξ) temperature distribution has been observed. Furthermore, Fig. 13 exhibits the Φ(ξ) nanoparticle concentration distribution for varying Nₖ values. The increment in factor Nₖ corresponds to the possibility of repetitive interactions among Al₂O₃, TiO₂ nanoparticles. As a result, the gap among the nanoparticles shrinks, producing in a reduced Φ(ξ) concentration distribution. Figure 14 demonstrates the Φ(ξ) concentration profile of TiO₂ nanofluid and Al₂O₃ + TiO₂ hybrid nanofluid for varied Nₜ values. It has been determined that throughout the scenario of concentration distribution within the boundary layer zone, Nₜ operates as a supporting factor. These finding arises owing to the development in thermophoretic processes from a physical point of view. The greater Schmidt number Sc is accountable for reducing the Φ(ξ) concentration inside the boundary, leading in decreasing the width of the nanoparticle concentration.

Figure 11. Thermal profile θ(ξ) against α.

Figure 12. Thermal profile θ(ξ) against R.
boundary, as shown in Fig. 15. The higher the Sc, the lower the mass diffusivity, and the lower the $\Phi(\xi)$ nano-particle concentration in the boundary.

**Table discussion.** Table 1 visualized the various thermophysical properties of base, nano and hybrid nano-fluids. Tables 2, 3 and 4 reveal the impact of varying values of factors involved on the numeric values of various

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**Figure 13.** Concentration profile $\Phi(\xi)$ against $N_b$.

**Figure 14.** Concentration profile $\Phi(\xi)$ against $N_t$.

**Figure 15.** Concentration profile $\Phi(\xi)$ against $Sc$. 
Table 2. Different physical variables have an impact on skin friction, $C_f = \frac{1}{(1-\phi_1-\phi_2)^2.5}f''(0)$, $[Re_f]^{\frac{1}{2}}$.

| Fr | A_1 | Re_f | $\frac{1}{(1-\phi_1-\phi_2)^2.5}f''(0)$ | $\frac{1}{(1-\phi_1-\phi_2)^2.5}g''(0)$ |
|----|-----|------|-----------------|-----------------|
| 0.1 | 0.5 | 1.0 | 0.3347805 | 1.5327381 |
| 0.3 | 0.4537309 | 1.6949731 |
| 0.5 | 0.6768929 | 1.7845873 |
| 0.7 | 0.3816925 | 1.6839842 |
| 0.9 | 0.4369237 | 1.7458903 |
| 1.0 | 1.6383981 | 1.5327381 |
| 1.3 | 1.4861693 | 0.5963703 |
| 1.6 | 1.3743643 | 0.6346134 |

Table 3. Different physical variables have an impact on the Nusselt number, $\frac{1}{2}[Re_f]^{\frac{1}{2}} Nu_x = -\left(\frac{h_{nf}}{k_f} + \frac{4}{3}R\left(\frac{1}{Pr} + 1\right)^3\right)\Theta'(0)$.

| α | Pr | R | S_1 | φ_1, φ_2 | $\left(\frac{h_{nf}}{k_f} + \frac{4}{3}R\left(\frac{1}{Pr} + 1\right)^3\right)\Theta'(0)$ |
|----|----|---|-----|---------|---------------------------------|
| 0.5 | 7.0 | 0.2 | 0.2 | 0.01 | 0.238416 |
| 1.0 | 0.42628 |
| 1.5 | 0.523737 |
| 7.0 | 0.238416 |
| 7.5 | 0.198695 |
| 8.0 | 0.093632 |
| 0.2 | 0.238416 |
| 0.4 | 0.423835 |
| 0.8 | 0.648986 |
| 0.2 | 0.238416 |
| 0.5 | 0.201476 |
| 1.0 | 0.183431 |
| 0.01 | 0.238416 |
| 0.02 | 0.324386 |
| 0.03 | 0.411374 |

Table 4. Different physical variables have an impact on the Sherwood number, $\frac{1}{2}[Re_f]^{\frac{1}{2}} Sh = -\Phi'(0)$.

| Nt | Nb | Sc | $-\Phi'(0)$ |
|----|----|----|-------------|
| 0.2 | 0.3 | 1.0 | 1.4352426 |
| 0.4 | 1.5468764 |
| 0.6 | 1.7917425 |
| 0.3 | 0.3718931 |
| 0.7 | 0.5817817 |
| 0.9 | 0.7690573 |
| 1.0 | 1.1523497 |
| 1.3 | 0.7348537 |
| 1.5 | 0.4833474 |
Table 5. Comparison of the present work with\(^{38}\). Considering common parameters\(\alpha = 0.1, \phi_1 = \phi_2 = 0\), \(Pr = 6.2\).

| \(Fr\) | \(R\) | \(f'(0)\) Present | \(f'(0)\) \(\Omega'(0)\) | \(\Omega'(0)\) Present |
|---|---|---|---|---|
| 0.5 | 0.2 | 0.241035 | 0.241044 | 0.1968238 | 0.1968238 |
| 1.0 | 0.364562 | 0.346589 | 0.1968238 | 0.1968238 |
| 1.5 | 0.623723 | 0.462374 | 0.1968238 | 0.1968238 |
| 0.2 | 0.241035 | 0.241035 | 0.241035 | 0.241035 |
| 0.4 | 0.241035 | 0.241035 | 0.241035 | 0.241035 |
| 0.6 | 0.241035 | 0.241035 | 0.241035 | 0.241035 |

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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**Author contributions**

A.S., M.J. and T.G. modeled and solved the problem. T.G. and A.S. wrote the manuscript. A.S., T.G. and S.N. contributed in the numerical computations and plotting the graphical results. W.A., S.N. and P.K. work in the revision of the manuscript. All the corresponding authors finalized the manuscript after its internal evaluation.
Competing interests
The authors declare no competing interests.

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Correspondence and requests for materials should be addressed to M.J. or P.K.

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