Bubble towers for supercritical semilinear elliptic equations

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Abstract: We construct positive solutions of the semilinear elliptic problem $\Delta u + \lambda u + u^p = 0$ with Dirichlet boundary conditions, in a bounded smooth domain $\Omega \subset \mathbb{R}^N$ ($N \geq 4$), when the exponent $p$ is supercritical and close enough to $\frac{N+2}{N-2}$ and the parameter $\lambda \in \mathbb{R}$ is small enough. As $p \to \frac{N+2}{N-2}$, the solutions have multiple blow up at finitely many points which are the critical points of a function whose definition involves Green’s function. Our result extends the result of Del Pino, Dolbeault and Musso [5] when $\Omega$ is a ball and the solutions are radially symmetric.

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1 Introduction

In this paper we consider the semilinear elliptic problem

$$\begin{cases}
\Delta u + \lambda u + u^p = 0 & \text{in } \Omega \\
u > 0 & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega
\end{cases}$$

(1)

where $\Omega$ is a bounded regular domain in $\mathbb{R}^N$, $N \geq 4$, the parameter $\lambda \in \mathbb{R}$ and the exponent $p$ is larger than

$$p_N := \frac{N+2}{N-2},$$

the critical Sobolev exponent.

When $p = p_N$, Brezis and Nirenberg [3] have proved that (1) admits a solution provided $0 < \lambda$ is less than the first eigenvalue of the Laplacian on $\Omega$ with 0 Dirichlet boundary

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condition. Direct application of Pohozaev’s identity [12] shows that solutions of (1) do not exist when \( \lambda \leq 0, p \geq p_N \) and \( \Omega \) is a star-shaped domain.

In this paper, we are interested in the existence of solutions of (1) in the case where \( p \) is larger than the critical Sobolev exponent. When \( \Omega \) is the unit ball it is easy to check that there exist radially symmetric positive solutions of

\[
\Delta u + u^p = 0,
\]

which have multiple blow up at the origin as the exponent \( p \) tends to \( p_N \) (we do not assume Dirichlet boundary condition here). We discuss this result in section 4. In a recent paper [5], Del Pino, Dolbeault and Musso have proved that a similar result was also true for (1). These solutions which have multiple blow up at some points in \( \Omega \) will be referred to as ”bubble tree solutions”. We are interested in the existence of these bubble tree solutions when \( \Omega \) is arbitrary.

2 Statement of the result

Let \( G \) denote Green’s function for the Laplace operator with Dirichlet boundary condition on \( \Omega \) and let \( H \) denote Robin’s function, i.e. the regular part of Green’s function. Namely

\[
G(y, z) := |y - z|^{2-N} - H(y, z),
\]

for \((y, z) \in \Omega \times \Omega\). Observe that \( \Delta_y H = 0 \) in \( \Omega \times \Omega \) and \( G = 0 \) on \( \partial(\Omega \times \Omega) \).

Given \( m \in \mathbb{N}^* \) and \( x := (x_1, \ldots, x_m) \in \Omega^m \), we define the \( m \times m \) matrix

\[
M(x) := (m_{ij})_{1 \leq i, j \leq m},
\]

whose entries are given by

\[
m_{ii} := H(x_i, x_i) > 0 \quad \text{and} \quad m_{ij} := -G(x_i, x_j) < 0, \tag{2}
\]

if \( i \neq j \). Let \( \rho(x) \) be the least eigenvalue of \( M(x) \). We agree that \( \rho(x) = -\infty \), if \( x_i = x_j \) for some \( i \neq j \). Finally, we define \( r(x) \) to be the unique eigenvector associated to \( \rho(x) \) whose coordinates are all positive and which is normalized so that its norm is equal to 1 (given the signs of the entries of \( M(x) \), it is easy to check that one can choose the eigenvector corresponding to the least eigenvalues to have coordinates greater than 0).

We define the open set

\[
P^+ := \{ (\Lambda, x) \in (\mathbb{R}^*_+)^m \times \Omega^m \}. \tag{3}
\]

Given \( \mu \in \mathbb{R} \) and \((\ell_1, \ldots, \ell_m) \in \mathbb{N}^m\), we define

\[
\mathcal{F}_\mu : (\mathbb{R}^*_+)^m \times \Omega^m \rightarrow \mathbb{R}, \tag{4}
\]

by

\[
\mathcal{F}_\mu(\Lambda, x) := \Lambda M(x)^t \Lambda - \mu C_N^{(1)} \sum_{i=1}^{m} \Lambda_i^{-\frac{4}{N-2}} + C_N^{(2)} \sum_{i=1}^{m} \ell_i \log \Lambda_i,
\]
where $\Lambda = (\Lambda_1, \ldots, \Lambda_m)$ and where $C_N^{(1)}, C_N^{(2)}$ are two positive constants which only depend on $N$ and which will be defined in section 8. In the following, we denote $C_N^{(3)}$ some positive constant which only depends on $N$.

Finally the parameter $\varepsilon > 0$ is defined by

$$
\varepsilon := p - p_N.
$$

Granted the above definitions, our result reads:

**Theorem 1** Assume that $N \geq 5$ and $\mu \in \mathbb{R}$ are fixed. Let $(\Lambda, x) \in P^+$ be a nondegenerate critical point of $F_\mu$. Then, there exists $\varepsilon_0 > 0$ and for all $\varepsilon \in (0, \varepsilon_0)$ there exists $u_\varepsilon$ a solution of (1) with $\lambda := \mu \varepsilon_N^{\frac{N-2}{2}}$ and $p = p_N + \varepsilon$, such that

$$|
abla u_\varepsilon|^2 \, dx \rightharpoonup C_N^{(3)} \sum_{i=1}^m \ell_i \, \delta_{x_i},
$$

in the sense of measures, where the constant $C_N^{(3)}$ is given by

$$C_N^{(3)} := (N(N - 2))^{\frac{N+2}{2}} \int_{\mathbb{R}^N} \left( \frac{1}{1 + |x|^2} \right)^{\frac{N+2}{2}} \, dx.
$$

In other words, the sequence $u_\varepsilon$ converges to 0 (in any $C^k$ topology) away from the points $x_i$, as the parameter $p$ tends to $p_N$. Near each $x_i$ the solution $u_\varepsilon$ has multiple blow up in the sense that there exists $c > 0$ (independent of $p$), $x_{i,p} \in \Omega$ and parameters $d_{i,j,p,\mu} > 0$ such that

$$\frac{1}{c} < d_{i,j,p,\mu} < c,
$$

and

$$
\lim_{p \to p_N} \left\| u_\varepsilon(\cdot + x_{i,p}) - (N(N - 2))^{\frac{N-2}{2}} \sum_{j=1}^m \ell_j \left( \varepsilon_{i,j} \frac{j^2}{1 + \varepsilon_{i,j}^2 |\cdot|^2} \right)^{\frac{N-2}{2}} \right\|_{L^\infty(B_{r_0})} = 0.
$$

Here

$$
\varepsilon_{i,j} := d_{i,j,p,\mu} \left( \varepsilon_N^{\frac{N-2}{2}} - j^2 \right).
$$

and $r_0 > 0$ is fixed small enough. Moreover, there is a relation between the parameters $d_{i,1,p,\mu}$ and the parameters $\Lambda_i$ since

$$
\lim_{p \to p_N} \Lambda_i^{\frac{2}{N-2}} d_{i,1,p,\mu} = (N(N - 2))^{\frac{1}{2}}.
$$

We briefly describe the plan of the paper. In section 3, we give some applications and some comments. In section 4, we recall some well known fact about radial solutions of $\Delta u + u^p = 0$ when the exponent $p$ is larger than the critical Sobolev exponent $p_N$. In section 5 and 6 we give a new proof of existence of radial solutions. This proof is needed just because, for the proof of Theorem 1, we need some estimates which are not available in [5]. Finally, the proof of the main result is the content of the sections 7 to 8. This proof is based on a gluing technic already used by Mazzeo and Pacard [10] in a different context.
3 Applications and comments

Application 1 We consider the case where \( m = 1 \) and \( \Omega = B \) is the unit ball, we recover the result of Del Pino, Dolbeault and Musso [5]. Indeed, given \( \ell \in \mathbb{N} \), we have

\[
\mathcal{F}_\mu(\Lambda_1, x_1) = \frac{\Lambda_1^2}{(1 - |x_1|^2)^{N-2}} - \mu C_N^{(1)} \Lambda_1^{N-1} + C_N^{(2)} \ell \log \Lambda_1
\]

It is clear that, provided the constant \( \mu \) is chosen sufficiently large, this function admits two nondegenerate critical points which we denote by \((\Lambda_{1,1}, 0)\) and \((\Lambda_{1,2}, 0)\). Therefore, for any \( \mu \) large enough, we find two distinct solutions of (1).

Application 2 Now assume that \( \Omega \) is "close" to the unit ball. Then, a standard perturbation result shows that, for a given \( \ell \) and provided \( \mu \) is sufficiently large, the function \( \mathcal{F}_\mu \) also admits two non degenerated critical points. This fact again guaranties the existence of two distinct solutions of (1).

Application 3 We consider the case where \( m = 2 \), \( \mu = 0 \), \( \ell_1 = \ell_2 \). When \( \Omega = \mathbb{R}^N - \overline{B(0,1)} \) the functional \( \mathcal{F}_0 \) can be explicitely written as

\[
\mathcal{F}_0(\Lambda_1, \Lambda_2, x_1, x_2) = \frac{\Lambda_1^2}{(1 - |x_1|^2)^{N-2}} + \frac{\Lambda_2^2}{(1 - |x_2|^2)^{N-2}} + C_N^{(2)} \ell_1 \log(\Lambda_1 \Lambda_2) - 2\Lambda_1 \Lambda_2 \left( \frac{1}{|x_1 - x_2|^{N-2}} - \frac{1}{(1 + |x_1|^2|x_2|^2 - 2(x_1, x_2))^{N-2}} \right)
\]

It admits a critical point \((\Lambda_0^1, \Lambda_0^2, x_1^0, x_2^0)\) where

\[
\Lambda_1^0 = \Lambda_2^0 = \left( \frac{2}{(2a_*)^{N-2}} - \frac{C_N^{(2)} \ell_1}{(a_* - 1)^{N-2}} - \frac{2}{(a_* + 1)^{N-2}} \right)^{1/2}
\]

and

\[
x_1^0 = (a_*, 0, ..., 0) \quad x_2^0 = (-a_*, 0, ..., 0)
\]

where \( a_* > 1 \) satisfies

\[
\frac{1}{(2a_*)^{N-1}} = \frac{a_*}{(a_* - 1)^{N-1}} + \frac{a_*}{(a_* + 1)^{N-1}}.
\]

These explicit formula allow one to study (1) for \( \mu = 0 \) in a annular domain \( \Omega = B(0,1) - \overline{B(0,\rho)} \), when \( \rho \) is close to 0. Indeed, we write \( z = (z_1, z') \in \mathbb{R} \times \mathbb{R}^{N-1} \) and using the symmetries, it is enough to look for solutions of (1) which only depend on \( z_1 \) and \(|z'|\) and blow up at two points (which turn out to be close to \( \partial B(0,\rho) \) as \( p \) tends to \( p_N \)). For this purpose, we study the functional \( \mathcal{F}_0 \), reduced by the symmetries we impose. In a neighborhood of \((\rho^{N-2} \Lambda_0^1, \rho^{N-2} \Lambda_0^2, \rho a_*, \rho a_*) \in \mathbb{R}^4_+ \), \( \mathcal{F}_0 \) can be expanded as

\[
\mathcal{F}_0(\Lambda_1, \Lambda_2, s, t) = \rho^{2-N} \left( \frac{\Lambda_1^2}{(1 - (s/\rho)^2)^{N-2}} + \frac{\Lambda_2^2}{(1 - (t/\rho)^2)^{N-2}} - 2\Lambda_1 \Lambda_2 \left( \frac{1}{(s + t)^{N-2}} - \frac{1}{(1 + (st)/\rho^2)^{N-2}} \right) + C_N^{(2)} \ell_1 \log(\Lambda_1 \Lambda_2) \right) + O(\Lambda_1^1 + \Lambda_2^2)
\]
And this functional admits a non degenerated critical point, provided $\rho$ is sufficiently small. Applying the result of Theorem 1, we find solutions of (1) which have two bubble trees located near $\partial B(0, \rho)$. When $\ell_1 = \ell_2 = 1$ such a result has been obtained by Felmer, Del Pino and Musso in [7] (see also [8] and [6]).

**Comments** When $m = 1$, a necessary condition for the existence of critical points of $F_\mu$ is given by: $\mu$ is a sufficiently large positive number. Indeed, a nonexistence result for single peaked solutions of (1), when $\mu = 0$, has been proved very recently by Rey et al in [2].

When $m \geq 2$ and $\mu = 0$, if $F_0$ admits a nondegenerate critical point, then $F_\mu$ also admits a nondegenerate critical point, provided $\mu$ is small enough. This means that even for negative values of $\mu$, we can construct solutions of (1).

Also observe that in the case where $\mu = 0$, if $(\Lambda^0, x^0)$ is a nondegenerate critical point of $F_0$ for $(\ell_1, \ldots, \ell_m)$, then $(\sqrt{k}\Lambda^0, x^0)$ is a nondegenerate critical point of $F_0$ for $(k\ell_1, \ldots, k\ell_m)$, where $k \in \mathbb{N}$.

Finally, observe that our result parallels the corresponding result which has been obtained by Bahri, Li and Rey [1] for the subcritical case, i.e. when $p < p_N$. In such case, only simple bubbles can be appeared, i.e. there are no bubble-towers (see [9]).

4 Positive radial solutions of $\Delta u + u^p = 0$ in $\mathbb{R}^N$

We recall some well known facts about positive radial solutions of

$$\Delta u + u^p = 0,$$

(4)

in $\mathbb{R}^N$. It is standard to look for radial positive solutions of (4) of the form

$$u(x) = |x|^{-2/p} v(- \log |x|).$$

(5)

If we set $t = - \log |x|$, then $v$ is a solution of an autonomous second order nonlinear ordinary differential equation:

$$\partial_t^2 v - a_p \partial_t v - b_p v + v^p = 0,$$

(6)

where the constants $a_p$ and $b_p$ are given by

$$a_p := N - 2 - \frac{4}{p - 1}, \quad \text{and} \quad b_p := \frac{2}{p - 1} \left( N - \frac{2p}{p - 1} \right).$$

(7)

Observe that $a_p$ vanishes precisely when $p = p_N$ and $b_p$ vanishes when $p = \frac{N}{N-2}$. We introduce the function

$$H_p(x, y) := \frac{1}{2} y^2 - \frac{b_p}{2} x^2 + \frac{x^{p+1}}{p + 1}.$$  

(8)

If $v$ is a solution of (6), then

$$\partial_t H_p (v, \partial_t v) = a_p (\partial_t v)^2.$$  

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In particular, this implies that \( \partial_t H_p(v, \partial_t v) \geq 0 \) when \( p \geq p_N \).

There are two stationary solutions of (6), the first one is given by \( v \equiv 0 \) and the other one is given by \( v \equiv b_p^{-1} \). We set
\[ c_p := b_p^{-1}. \]

We claim that there exists a heteroclinic solutions of (6) when \( p > p_N \). This is the content of the following:

**Proposition 1** Assume \( p > p_N \). Then, there exists a unique solution \( v_p \) of (6) which is defined on \( \mathbb{R} \), satisfies
\[
\lim_{t \to -\infty} v_p(t) = c_p, \quad \lim_{t \to +\infty} v_p(t) = 0,
\]
and is normalized so that
\[
\lim_{t \to +\infty} e^{2 \gamma t} v_p(t) = 1.
\]

This solution satisfies \( H(v_p, \partial_t v_p) < 0 \).

**Proof.** We first prove that there exists a unique solution of (6) which is defined for \( t \) large enough and which satisfies (10). According to a classical result in the theory of nonlinear ordinary differential equations [4], it is enough to check that there exists a solution of the homogeneous problem associated to the linearized ordinary differential equation at \( v \equiv 0 \), which has the desired behavior as \( t \) tends to \( +\infty \). Now, the associated homogeneous problem reads
\[
\partial_t^2 v - a_p \partial_t v - b_p v = 0.
\]

And clearly it has two independent solutions which are given by \( t \to e^{\gamma \pm t} \) where
\[
\gamma_+ := N - 2 - \frac{2}{p - 1}, \quad \text{and} \quad \gamma_- := -\frac{2}{p - 1}.
\]

Therefore, there exists a unique solution of (6) which is asymptotic to \( t \to e^{\gamma_- t} \) as \( t \) tends to \( +\infty \) and hence satisfies the second formula of (9). A priori this solution, which from now on is denoted by \( v_p \), is only defined for \( t \) large enough, say \( t \in (\tilde{t}, +\infty) \). Observe that there also exists another solution of (6) which is asymptotic to \( t \to e^{\gamma_+ t} \) as \( t \) tends to \( -\infty \).

Since the function \( t \to H_p(v_p, \partial_t v_p) \) is increasing and
\[
\lim_{t \to +\infty} H_p(v_p, \partial_t v_p) = 0,
\]
we conclude that \( H_p(v_p, \partial_t v_p) < 0 \) for any \( t \in (\tilde{t}, +\infty) \). Thus, \( v_p \) remains bounded independently of the value of \( \tilde{t} \) and hence can be extended to all \( \mathbb{R} \). Now, as \( t \) tends to \( -\infty \), there two possibilities : either \( v_p \) converges to a limit cycle or \( v_p \) converges to the constant \( c_p \), the unique stationary point in region \( \{(v, \partial_t v) : H_p(v, \partial_t v) < 0\} \). But \( \partial_t H_p(v, \partial_t v) > 0 \) if \( \partial_t v \neq 0 \). Hence, there are no limit cycle. We conclude that \( \lim_{t \to -\infty} v_p = c_p \). This completes the proof of the result. \( \blacksquare \)
In the next result, we show that the function $\partial_t v_p$ vanishes at infinitely many points, provided $p$ is close enough to $p_N$.

**Proposition 2** Assume that $p > p_N$ and further assume that

$$a_p^2 - 4(p - 1)b_p < 0. \quad (13)$$

Then the set of zeros of $\partial_t v_p$ is given by two sequences $(\mathcal{L}_i)_{i \geq 1}$ and $(\mathcal{T}_i)_{i \geq 1}$ tending to $-\infty$ and satisfying

$$\mathcal{T}_i > \mathcal{L}_i > \mathcal{T}_{i+1} > \mathcal{L}_{i+1},$$

Moreover, we have

$$v_p(\mathcal{L}_i) < c_p < v_p(\mathcal{T}_i).$$

**Proof.** We linearize (6) at $v = c_p$. This yields the operator

$$L_p = \partial_t^2 - a_p \partial_t + (p - 1)b_p,$$

since $c_p^{-1} = b_p$. The characteristic roots of $L_p$ are given by

$$\tilde{\gamma}_\pm = \frac{1}{2} (a_p \pm i \sqrt{4(p - 1)b_p - a_p^2}). \quad (14)$$

These are imaginary valued since (13) is satisfied. It follows from standard theory for ordinary differential equations [4] that $v_p - c_p$ is asymptotic to a solution of the homogeneous system associated to $L_p$. Hence there there exists $c, d \in \mathbb{R}$ and $\gamma > \frac{a_p}{2}$ such that

$$v_p = c_p + e^{\tilde{\gamma}_t + d} + O(e^{\gamma t}), \quad (15)$$

as $t$ tends to $-\infty$, where $\Re(\cdot)$ is real part of a complex number. This immediately implies that $\partial_t v_p$ has infinitely many zeros. The result of the proposition follows at once from this expansion.

We define

$$d_p := \left(\frac{p + 1}{2}b_p\right)^{\frac{1}{p-1}}.$$

We now derive an upper bound for the solution $v_p$, which has been defined in Proposition 1. This upper bound follows from the more general result:

**Proposition 3** Assume that $v$ is a solution of (6) such that $H_p(v, \partial_t v) \leq 0$ on $(t_1, t_2)$. Then $|v| \leq d_p$ on $(t_1, t_2)$.

**Proof.** This follows at once from the fact that

$$\max\{x > 0 : \exists y \in \mathbb{R} \quad H_p(x, y) \leq 0\} = d_p,$$

together with the fact that we have assumed that $H_p(v, \partial_t v) \leq 0$. 

From now on we assume that (13) is satisfied and we define the sequences
\[ \varepsilon_{p,i} = v_p(t_i), \quad \text{and} \quad \eta_{p,i} = v_p(\bar{t}_i), \]
which correspond to the sequence of local minima and local maxima of the function \( v_p \).
Observe that we have the sequence \( (\varepsilon_{p,i})_i \) (resp. \( (\eta_{p,i})_i \)) is increasing (resp. decreasing) and converges to \( c_p \)

\[ 0 < \varepsilon_{p,1} < \varepsilon_{p,2} < \ldots < c_p < \ldots < \eta_{p,2} < \eta_{p,1} < d_p. \]

It will be convenient to agree that \( t_0 = +\infty \) and \( \varepsilon_{p,0} = 0 \).

We now derive a precise expansion of the value of \( \varepsilon_{p,i} \) as \( p \) tends to the critical exponent \( p_N \). This result relies on the following more general result which gives the asymptotic of the first return map when \( p \) is close to \( p_N \).

For \( p > p_N \) and \( \eta \in [\varepsilon_{p,1}, c_p] \), we consider the function \( v_{p,\eta} \) which is a solution of (6) which is defined in \((0, \ell_{p,\eta})\) and satisfies

\[ v_{p,\eta}(0) = \eta \quad \text{and} \quad \partial_t v_{p,\eta}(0) = \partial_t v_{p,\eta}(\ell_{p,\eta}) = 0. \]

If \( \ell_{p,\eta} = +\infty \), we agree that the above equalities have to be understood as limits. We further assume that \( v_{p,\eta} \) is strictly increasing on \((0, \bar{t}_{p,\eta})\) and strictly decreasing on \((\bar{t}_{p,\eta}, \ell_{p,\eta})\).

Finally, we assume that

\[ H_p(v_{p,\eta}, \partial_t v_{p,\eta}) \leq 0, \]

on \((0, \ell_{p,\eta})\). In other words, \( \ell_{p,\eta} \) is the first return time. Observe that, when \( p = p_N \) the equation satisfied by \( v_{p,\eta} \) is Hamiltonian hence we have

\[ \lim_{p \to p_N} (v_{p,\eta}(0) - v_{p,\eta}(\ell_{p,\eta})) = 0. \]

We make this estimate more precise in the following :

**Proposition 4** There exists a bounded positive function \( D_N : [0, c_{p_N}] \to \mathbb{R}^+ \), which only depends on \( N \) such that

\[ \lim_{p \to p_N} \frac{v_{p,\eta}(0)^2 - v_{p,\eta}(\ell_{p,\eta})^2}{p - p_N} = D_N(\eta) \]

uniformly with respect to \( \eta \).

**Proof.** For the sake of simplicity in the notations, we drop the \( p, \eta \) indices. Since we have assumed that \( H_p(v, \partial_t v) \leq 0 \) in \((0, \bar{t})\), we get

\[ |\partial_t v| \leq \sqrt{b_p v^2 - \frac{2}{p+1} v^{p+1}} < \sqrt{b_p v}. \]

Recall that

\[ \partial_t H_p(v, \partial_t v) = a_p(\partial_t v)^2. \quad (16) \]
Integrating this equality over $(0, \bar{t})$ and using the fact that $\partial_t v > 0$ on $(0, \bar{t})$, we get
\[
0 \leq H_p(v(\bar{t}), 0) - H_p(v(0), 0) = a_p \int_0^\bar{t} (\partial_t v)^2 \, dt \leq a_p \sqrt{b_p} \int_0^\bar{t} \partial_t v \, v \, dt \leq \frac{1}{2} a_p \sqrt{b_p} v(\bar{t})^2.
\] (17)

Similarly, using an integration over $(\tilde{t}, \bar{t})$ together with the fact that $\partial_t v < 0$ over this set, we also get
\[
0 \leq H_p(v(\tilde{t}), 0) - H_p(v(0), 0) \leq \frac{1}{2} a_p \sqrt{b_p} v(\tilde{t})^2.
\] (18)

Hence, we conclude that
\[
0 \leq H(v(\tilde{t}), 0) - H(v(0), 0) \leq a_p \sqrt{b_p} v(\tilde{t})^2.
\] (19)

Thanks to the previous Proposition we know that $v(\bar{t}) \leq d_p$ and clearly
\[
\lim_{p \to p_N} \frac{a_p}{p - p_N} = \frac{(N - 2)^2}{4},
\] (20)

while $b_p$ and $d_p$ remain bounded as $p$ tends to $p_N$. This, together with (18) and (19), implies that
\[
\lim_{p \to p_N} (H_p(v(0), 0) - H_p(v(\tilde{t}), 0)) = \lim_{p \to p_N} (H_p(v(0), 0) - H_p(v(\bar{t}), 0)) = 0
\] (21)

uniformly with respect to $\eta$. As a consequence, we get using the expression of $H_p$ the fact that
\[
\lim_{p \to p_N} \left(v(0)^2 - v(\tilde{t})^2\right) = \lim_{p \to p_N} (\tilde{v}_0 - v(\tilde{t})) = 0
\] uniformly with respect to $\eta$, where $\tilde{v}_0 > v(0)$ is the unique solution of $H_p(\tilde{v}_0, 0) = H_p(v(0), 0)$ which belongs to $(c_p, d_p)$.

This being understood, we write
\[
\int_0^\tilde{t} (\partial_t v)^2 \, dt = \int_0^\tilde{t} \left(2H_p(v(s), \partial_t v(s)) + b_p v^2 - \frac{2}{p + 1} v^{p+1} \partial_t v \, ds \right)
= \int_{v(0)}^{v(\tilde{t})} \left(2H_p(x, \partial_t v(x)) + b_p x^2 - \frac{2}{p + 1} x^{p+1} \right) dx.
\]

Now, as $p$ tends to $p_N$, it follows from the previous discussion that the right hand side converges (uniformly with respect to $\eta$) to
\[
E_N(\eta) := \int_\eta^{\tilde{\eta}} \left(2H_{p_N}(\eta, 0) + \frac{(N - 2)^2}{4} x^2 - \frac{N - 2}{N} x^{N-2}\right) dx
\]
where $\tilde{\eta} \geq c_{p_N}$ satisfies $H_{p_N}(\eta, 0) = H_{p_N}(\tilde{\eta}, 0)$. Similarly, we have
\[
\lim_{p \to p_N} \int_\tilde{t}^\bar{t} (\partial_t v)^2 \, dt = E_N(\eta)
\]

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where the convergence is uniform with respect to \( \eta \). Moreover the function \( \eta \to E_N(\eta) \) is bounded. Using these limits together with (16), which we integrate over \((0, t)\), we conclude that there exists a constant \( \hat{E}_N(\eta) := \frac{(N-2)^2}{4} E_N(\eta) \) only depending on \( N \) such that
\[
\lim_{p \to p_N} \frac{H_p(v(0), 0) - H_p(v(t), 0)}{p - p_N} = -\hat{E}_N(\eta)
\] (22)
uniformly with respect to \( \eta \). The result follows at once from these limits and the expression of \( H_p \).

Looking at the previous proof, it should be clear that

**Proposition 5** As \( p \) tends to \( p_N \), the functions
\[
\tilde{v}_{p, \eta} := v_{p, \eta}(\cdot + \tilde{t}_{p, \eta})
\]
converge (uniformly on compacts) to \( w_{p_N, \eta} \) the unique solution of
\[
\partial_t^2 w - b_{p_N} w + w^{p_N} = 0
\]
with \( w(0) = \eta \) and \( \partial_t w(0) = 0 \) where \( \eta \geq c_{p_N} \) satisfy \( H_{p_N}(\eta, 0) = H_{p_N}(\bar{\eta}, 0) \). Moreover, the convergence is uniform with respect to \( \eta \).

**Proof.** This follows at once from Ascoli’s theorem since \( v_{p, \eta} \) and all its derivatives are uniformly bounded. ■

Observe that, in the previous Proposition, as \( \eta \) tends to 0, the function \( w_{p_N, \eta} \) converges (uniformly on compacts) to \( w_0 \) which is explicitly given by
\[
w_0(t) := \left( \frac{N(N-2)}{4} \right)^{\frac{N-2}{4}} (\cosh t)^{\frac{2-N}{2}}.
\]

Going back to the study of the function \( v_p \), the result of Proposition 4 yields:

**Corollary 1** There exists a positive constant \( C_N^{(4)} \) (in fact \( C_N^{(4)} = D_N(0) \) given in Proposition 4), only depending on \( N \), such that, for all \( i \in \mathbb{N} \)
\[
\lim_{p \to p_N} \frac{\tilde{C}_{p, i}^2}{p - p_N} = i C_N^{(4)}
\] (23)
Moreover, we have the explicit formula for \( C_N^{(4)} \)
\[
C_N^{(4)} = \left( \frac{N(N-2)}{4} \right)^{\frac{N-2}{4}} \frac{(N-2)^2}{2(N-1)} \int_{-\infty}^{+\infty} \frac{dt}{(\cosh t)^{N-2}}
\] (24)

In the next result, we estimate any solution of (6), near one of the points where it achieves a minimum, by comparing it to the solution of a linear problem. Indeed, we consider \( w_p \) to be the solution of the second order linear ordinary differential equation
\[
\begin{aligned}
\partial_t^2 w_p - a_p \partial_t w_p - b_p w_p &= 0 \\
w_p(0) &= 1, \quad \partial_t w_p(0) = 0.
\end{aligned}
\] (25)
which is explicitly given by 
\[ w_p = \frac{1}{N - 2} (\gamma_+ e^{\gamma_+ - t} - \gamma_- e^{\gamma_- + t}), \]
where \( \gamma_\pm \) have been defined in (12). The following Lemma shows that, close to 0, the solution \( v_{p,\eta} \) of (6) with \( v_{p,\eta}(0) = \eta \) and \( \partial_t v_{p,\eta}(0) = 0 \) is well approximated by \( \eta w_p \).

**Lemma 1** For all \( k \in \mathbb{N} \), there exists a positive constant \( c_k > 0 \) such that for all \( t \in \mathbb{R} \)
\[ |\partial_t^k (v_{p,\eta} - \eta w_p)| \leq c_k \eta^p w_p^p \]
(26)
for \( p \) close enough to \( p_N \).

**Proof.** Again we drop the indices \( p,\eta \) to keep the notations simple. We view \( v \) as a solution of a non homogeneous linear second order ordinary differential equation
\[ \partial_t^2 v - a_p \partial_t v - b_p v = -v^p \]
The variation of the constant formula yields
\[ v(t) = \eta w(t) - e^{\gamma_+ t} \int_0^t e^{(a_p - 2\gamma_+) s} \int_0^s e^{(-a_p + \gamma_+) \zeta} v(\zeta)^p d\zeta ds, \]
(27)
This in particular implies that \( v(t) \leq \eta w(t) \) for all \( t \in \mathbb{R} \).

When \( t \geq 0 \), we can therefore use the bounds
\[ v(t) \leq \eta w(t) \leq c \eta e^{\gamma_+ t} \]
in (27) to conclude that
\[ |v(t) - \eta w(t)| \leq c \eta^p e^{\gamma_+ t} \int_0^t e^{(a_p - 2\gamma_+) s} \int_0^s e^{(-a_p + \gamma_+) \zeta} e^{\gamma_+ \zeta} d\zeta ds \leq c \eta^p e^{p \gamma_+ t}. \]
(29)
When \( t \leq 0 \), a similar analysis yields
\[ |v(t) - \eta w(t)| \leq c \eta^p e^{p \gamma_+ t} \]
for all \( t \in (\infty, 0) \).
(30)
This completes the proof of the estimate of \( v \). The estimates for the derivatives follow similarly.

The last result translates for the function
\[ u_{p,\eta}(x) := |x|^{\gamma_+ - 2} v_{p,\eta}(- \log |x|) \]
and we obtain the estimate
\[ \left| (r \partial_r)^k \left( u_{p,\eta}(x) - \eta \left( \frac{\gamma_+}{N - 2} - \frac{\gamma_-}{N - 2} |x|^{2-N} \right) \right) \right| \leq c_k \eta^p \left( |x|^{-p \gamma_-} + |x|^{-p \gamma_+} \right) |x|^\gamma, \]
where the constant \( c_k > 0 \) only depends on \( k \) and \( N \) and remains bounded as \( p \to p_N \).

As a consequence, we have the following result which provides an expansion of \( I_1 \) and \( I_i \) as \( p \) tends to \( p_N \).
Corollary 2 As p tends to $p_N$, we have

$$\left| \tilde{T}_{i} - \frac{2(i-1)}{N-2} \log \varepsilon \right| + \left| \tilde{T}_{i} - \frac{2i-1}{N-2} \log \varepsilon \right| \leq c_i, \tag{31}$$

for some constant $c_i > 0$ which only depends on $N$ and $i$. We recall that $\varepsilon = p - p_N$.

Proof. As $t$ goes from $\tilde{T}_{i+1}$ to $\tilde{T}_{i+1}$, the function $v_p$ passes once through the value $c_p$. Hence there exists $\tilde{T}_{i+1} \in (\tilde{T}_{i+1}, \tilde{T}_{i+1})$ such that $v(\tilde{T}_{i+1}) = c_p$.

We first estimate $\tilde{T}_{i+1} - \tilde{T}_{i+1}$. In view of the previous Proposition, this quantity can be estimated by

$$\tilde{T}_{i+1} - \tilde{T}_{i+1} = -\frac{1}{2\gamma_+} \log \varepsilon + O(1) \tag{32}$$

Now, we claim that $\tilde{T}_{i+1} - \tilde{T}_{i+1}$ remains uniformly bounded as $p$ tends to $p_N$. Indeed, it follows from the remark after Proposition 5 that, as $p$ converges to $p_N$, the sequence of functions $t \rightarrow v(\tilde{T}_{i+1} + t)$ converges on compacts to $w_0(t) = (\frac{N(N-2)}{4})^{\frac{N-2}{2}} (\cosh t)^{\frac{2-N}{2}}$. From this we conclude that it takes a finite time for $w_0$ to go from $c_{p_N}$ to $d_{p_N}$. Hence, provided $p$ remains close to $p_N$, the time it takes to $v_p$ to go from $c_p$ to $v_p(\tilde{T}_{i+1})$ is bounded uniformly as $p$ tends to $p_N$.

Therefore, we conclude that

$$\tilde{T}_{i+1} - \tilde{T}_{i+1} = -\frac{1}{2\gamma_+} \log \varepsilon + O(1). \tag{33}$$

Similarly, we obtain

$$\tilde{T}_{i+1} - \tilde{T}_{i+1} = \frac{1}{2\gamma_-} \log \varepsilon + O(1). \tag{34}$$

In order to obtain the estimates as stated, just observe that

$$\frac{1}{\gamma_+} = \frac{2}{N-2} + O(\varepsilon), \quad \frac{-1}{\gamma_-} = \frac{2}{N-2} + O(\varepsilon),$$

and also that $\tilde{T}_1 = O(1)$. \hfill \blacksquare

Now, we compare solutions of (6) which have different boundary data. We keep the previous notations. We prove the following technical result:

Lemma 2 For all $c_0 > 1$, there exists a positive constant $c > 0$ only depending on $N$ and $c_0$ such that

$$\frac{1}{c} (\tilde{\eta} - \eta) \leq |v_p,\tilde{\eta}(\tilde{T}_{p,\eta}) - v_p,\tilde{\eta}(T_{p,\eta})| \leq c (\tilde{\eta} - \eta) \tag{35}$$

and

$$|T_{p,\eta} - T_{p,\tilde{\eta}}| \leq \frac{\tilde{\eta} - \eta}{\eta} \tag{36}$$

for all $p$ close enough to $p_N$, provided $(\frac{1}{c_0} + \sqrt{C_N^4}) \varepsilon^{\frac{1}{2}} < \tilde{\eta} < (c_0 + \sqrt{C_N^4}) \varepsilon^{\frac{1}{2}}$. 


Proof. We set \( v = v_{p,\eta} \) and \( \tilde{v} = v_{p,\tilde{\eta}} \). To prove the result we write for the difference \( D := \tilde{v} - v \)

\[
\partial_t^2 D - a_p \partial_t D - b_p D = -f D
\]

where

\[
f := \frac{\tilde{v}^p - v^p}{\tilde{v} - v}
\]

It follows from the estimates of Lemma 1 that, for all \( p \) close enough to \( p_N \),

\[
|f| \leq c (\eta w_p)^{p-1}
\]

for some constant \( c \) which only depends on \( N \) and \( c_0 \). Now, as in the proof of Lemma 1, we use the variation of the constant formula to get

\[
D = (\tilde{\eta} - \eta) w_p - w_p \int_0^t e^{a_p s} w_p^{-2}(s) \int_0^s e^{-a_p \zeta} w_p(\zeta) f(\zeta) D(\zeta) d\zeta ds \tag{37}
\]

We are interested in the range of validity of the two sided estimate

\[
\frac{1}{2} (\tilde{\eta} - \eta) w_p \leq |D| \leq 2 (\tilde{\eta} - \eta) w_p \tag{38}
\]

Inserting this into (37), we get

\[
(\tilde{\eta} - \eta) (1 - c (\eta w_p)^{p-1}) w_p \leq |D| \leq (\tilde{\eta} - \eta) (1 + c (\eta w_p)^{p-1}) w_p
\]

From which it follows that (38) is valid up to the time \( \hat{t}_{p,\eta} \) where

\[
c (\eta w_p)^{p-1} = \frac{1}{2}.
\]

Therefore, we have

\[
\frac{1}{2} (\tilde{\eta} - \eta) \leq \eta |D(\hat{t}_{p,\eta})| + \eta |\partial_t D(\hat{t}_{p,\eta})| \leq 2 (\tilde{\eta} - \eta) \tag{39}
\]

at this point. Now, it should be clear that \( \hat{t}_{p,\eta} - \tilde{t}_{p,\eta} \) is bounded independently of \( \eta \) for \( p \) close to \( p_N \) (since \((v, \partial_t v)\) remains bounded away from 0 in this interval). Hence we also have

\[
\frac{1}{c} (\tilde{\eta} - \eta) \leq \eta |(\tilde{v} - v)(\hat{t}_{p,\eta})| + \eta |\partial_t (\tilde{v} - v)(\hat{t}_{p,\eta})| \leq c (\tilde{\eta} - \eta) \tag{40}
\]

for some constant \( c > 0 \). Standard result on dynamical systems imply that

\[
\eta |\tilde{t}_{p,\tilde{\eta}} - \tilde{t}_{p,\eta}| \leq c (\tilde{\eta} - \eta). \tag{41}
\]

Using (16) and (40), we have

\[
|H_p(v(\tilde{t}_{p,\eta}), 0) - H_p(v(\tilde{t}_{p,\eta}), 0)|
\]

\[
= |H_p(\eta, 0) - H_p(\tilde{\eta}, 0) - a_p \int_0^{\tilde{t}_{p,\eta}} ((\partial_t v)^2 - (\partial_t \tilde{v})^2) dt| \tag{42}
\]

\[
\leq c \eta (\tilde{\eta} - \eta)
\]

since \((\frac{1}{\varepsilon_0} + \sqrt{C_N(4)})\varepsilon^\frac{1}{2} < \eta < \tilde{\eta} < (c_0 + \sqrt{C_N(4)})\varepsilon^\frac{1}{2}\). Together with (41), we estimate

\[
|H_p(v(\tilde{t}_{p,\eta}), 0) - H_p(v(\hat{t}_{p,\eta}), 0)| \leq c \eta (\tilde{\eta} - \eta) \tag{43}
\]
which implies
\[ \frac{1}{c} \eta(\bar{\eta} - \eta) \leq |\tilde{v}(t_p, \bar{\eta}) - v(t_p, \eta)| \leq c\eta(\bar{\eta} - \eta). \] (44)

From Corollary 1 and results on system dynamic, there holds
\[ \frac{1}{2c_0} \varepsilon \frac{1}{2} < v(t_p, \eta) < \tilde{v}(t_p, \bar{\eta}) < 2c_0 \varepsilon \frac{1}{2}. \] (45)

Using similar arguments on \((\bar{t}_p, \eta, t_p, \eta)\), we get
\[ 1 \varepsilon \eta |(\bar{t}_p, \eta - \bar{t}_p, \eta) - (t_p, \eta - \bar{t}_p, \eta)| \leq c(\bar{\eta} - \eta) \]
and also that \( \frac{1}{c}\eta |\tilde{v}(t_p, \eta) - v(t_p, \eta)| \leq c(\bar{\eta} - \eta) \).

The result follows at once from these estimates. \[ \blacksquare \]

5 Linear results

We keep the notations in the previous section. For the sake of simplicity in the notations, we drop the indices \( p, \eta \). We consider \( w \) to be the solution of
\[ \partial_t^2 w - a_p \partial_t w - b_p w + pv^{p-1} w = e^{-2t} v \]
in \((0, \bar{t})\) with boundary conditions \( w(t) = \partial_t w(t) = 0 \). We are interested in the behavior of \( w \) as \( \eta \) tends to 0. This is the contain of the following result.

Lemma 3 Assume that \( N \geq 5 \). Let \( c_0 > 1 \) and \( d_0 > 0 \) be fixed. Assume that \( \eta \in (\frac{1}{c_0} \varepsilon \frac{1}{2}, c_0 \varepsilon \frac{1}{2}) \). Then, there exist \( c > 0 \) and \( \varepsilon_0 > 0 \) such that for all \( p \in (p_N, p_N + \varepsilon_0) \) we have
\[ \left| w(t) - \frac{4\beta_{p, \eta}}{\eta(N-2)^2} e^{-(N-2)t/2} \right| \leq c \beta_{p, \eta} \varepsilon \frac{1}{2} \frac{2}{N+2} \] (47)
in \((-d_0, d_0)\), where the constant \( \beta_{p, \eta} \) is given by
\[ \beta_{p, \eta} := \int_0^\bar{t}_p \int_0^{v_{p, \eta}(s)} e^{-2s} ds. \] (48)

Moreover, we have
\[ \lim_{p \to p_N} e^{2\beta_{p, \eta}} \beta_{p, \eta} = \left( \frac{N(N-2)}{4} \right)^{\frac{N+2}{2}} \int_{-\infty}^{+\infty} \frac{e^{-2s}}{(\cosh s)^{N-2}} ds := C_N^{(5)}. \] (49)

Proof. As usual we drop the \( p, \eta \) indices. We use the fact that \( w_1 := \eta^{-1} \partial_t v \) is an explicit solution of the homogeneous problem
\[ \partial_t^2 w_1 - a_p \partial_t w_1 - b_p w_1 + p v^{p-1} w_1 = 0. \] (50)

This yields a representation formula for \( w \), at least when \( t \in (\bar{t}, \bar{t}) \).
\[ w(t) = w_1(t) \int_t^\bar{t} e^{a_p s} w_1^{-2}(s) \int_s^{\bar{t}} e^{-a_p \zeta} w_1(\zeta) e^{-2\zeta} v(\zeta) d\zeta ds. \]
Observe that the result of Lemma 1 yields
\[ \frac{1}{c} e^{\gamma_-(t-\overline{t})} \leq v(t) \leq c e^{\gamma_-(t-\overline{t})} \]
for all \( t \in (\overline{t}, \underline{t}) \) and also
\[ \frac{1}{c} e^{\gamma_-(t-\overline{t})} \leq |\partial_t v(t)| \leq c e^{\gamma_-(t-\overline{t})} \]
for all \( t \in (\overline{t} + 1, \underline{t} - 1) \). Using this, we get the estimate
\[ |w(t)| + |\partial_t w(t)| \leq c e^{-2t} e^{\gamma_-(t-\overline{t})} \quad (51) \]
for some constant \( c > 0 \). This estimate is valid for all \( t \in (0, \underline{t}) \). However, enlarging the value of \( c \) if this is necessary, we can assume that this estimate holds for \( t \in (\overline{t} - 1, \underline{t}) \).

The solution \( w \) extends to \((0, \underline{t})\). Again, we use the fact that \( w_1 = \eta^{-1} \partial_t v \) and
\[ w_2(t) := w_1(t) \int_{\overline{t}}^{\overline{t} - 1} e^{a_p s} w_1^{-2}(s) ds \]
which is defined for \( t \in (1, \overline{t} - 1) \), are solutions of the homogeneous problem (50). Hence we can decompose
\[ w = \alpha_1 w_1 + \alpha_2 w_2 + \tilde{w} \]
where \( \tilde{w} \) is defined by
\[ \tilde{w}(t) := w_1(t) \int_{0}^{t} e^{a_p s} w_1^{-2}(s) \int_{0}^{s} e^{-a_p \zeta} w_1(\zeta) e^{-2\zeta} v(\zeta) d\zeta ds. \]
As above, the result of Lemma 1 yields
\[ \frac{1}{c} \eta e^{\gamma_+ t} \leq v(t) \leq c \eta e^{\gamma_+ t} \quad (52) \]
for all \( t \in (0, \overline{t}) \) and also
\[ \frac{1}{c} \eta e^{\gamma_+ t} \leq |\partial_t v(t)| \leq c \eta e^{\gamma_+ t} \quad (53) \]
for all \( t \in (1, \overline{t} - 1) \). Using this, we get the estimate
\[ |	ilde{w}(t)| + |\partial_t \tilde{w}(t)| \leq c \eta e^{-2t} e^{\gamma_+ t} \quad (54) \]
for some constant \( c > 0 \). This estimate is valid for all \( t \in (0, \overline{t} - 1) \).

Since
\[ \frac{1}{c} \leq \eta |w_1(\overline{t} - 1)| \]
it follows at once from (51) and (54) that we can estimate the parameter \( \alpha_1 \) by
\[ |\alpha_1| \leq c \eta e^{-2t}. \]
In order to estimate the parameter $\alpha_2$ we multiply the equation (46) by $w_1$ and integrate by parts. Using the fact that $w_1$ is a solution of (50) we obtain

$$[w_1 \partial_t w - w \partial_t w_1 - a_p w w_1'] = -2a_p \int_0^t \partial_t w_1 \, w \, dt + \eta^{-1} \int_0^t e^{-2t} v \, \partial_t v \, dt.$$  

Since $w_1 = 0$ at $t = 0$ and $w = \partial_t w = 0$ at $t = T$, this simplifies into

$$w(0) \partial_t w_1(0) = -2a_p \int_0^t \partial_t w_1 \, w \, dt + \eta^{-1} \int_0^t e^{-2t} v \, \partial_t v \, dt.$$  

From (52) and (53), it follows

$$|w_2(t)| + |\partial_t w_2(t)| \leq c e^{-\gamma + t}$$  

for all $t \in (1, T - 1)$. Enlarging the value of $c$ if this is necessary, we can assume that this estimate holds for $t \in (0, T)$.

Collecting these estimates, we get

$$\int_0^T w \partial_t w_1 \, dt = \alpha_2 \mathcal{O}(\log \varepsilon) + \mathcal{O}(\varepsilon^{-\frac{1}{2} + \frac{N}{2} - 2}) \quad \text{and} \quad \int_T^t w \partial_t w_1 \, dt = \mathcal{O}(\varepsilon^{-\frac{1}{2} + \frac{N}{2} - 2}).$$  

To calculate $w_2(0)$, we see that

$$\frac{e^{a_p t}(pt^{p-1}(t) - b_p)}{(\partial_t^2 v(t))^2} = \frac{1}{\partial_t v(t)} \frac{d}{dt} \left( \frac{e^{a_p t}}{\partial_t^2 v(t)} \right).$$  

Hence we get

$$w_2(t) = w_1(t) \int_1^{t-1} e^{a_p s} w_1^{-2}(s) \, ds + w_1(t) \int_t^1 e^{a_p s} w_1^{-2}(s) \, ds \quad \text{for all } t \in (-d_0, d_0).$$  

In particular

$$w_2(0) = \frac{1}{b_p - \eta^{p-1}},$$  

since $w_1(0) = 0$. Consequently, we obtain the estimate

$$\alpha_2 = \eta^{-1} \int_0^T v^2(s) e^{-2s} \, ds + \mathcal{O}(\varepsilon^\frac{1}{2}).$$  

It remains to estimate $w_2$ in the neighborhood of 0. We first estimate $\int_1^{t-1} e^{a_p s} ds \left( \frac{1}{w_1(s)^2} \right)$. We decompose

$$(1, \bar{t} - 1) = (1, \frac{4}{N^2 - 4} \log \frac{1}{\varepsilon}) \cup \left( \frac{4}{N^2 - 4} \log \frac{1}{\varepsilon}, \bar{t} - 1 \right) := I_1 \cup I_2.$$  

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It follows from (53) that
\[ \int_{I_2} \frac{e^{a_p s} ds}{(w_1(s))^2} \leq c \varepsilon \frac{\sqrt{4}}{\varepsilon} \quad \text{and} \quad 1 \leq e^{a_p s} \leq 1 + c \varepsilon \log \frac{1}{\varepsilon}, \quad (55) \]
for all \( s \in (0, \bar{t}) \). Using the result of Lemma 1, we obtain
\[ \partial_t v(t) = \frac{N - 2}{2} \eta \sinh(\frac{N - 2}{2} t)(1 + O(\varepsilon^{N+2})) \]
for all \( t \in I_1 \). Therefore, we deduce
\[ \int_{I_1} \frac{e^{a_p s} ds}{(w_1(s))^2} = \frac{8}{(N - 2)^3} \left( \frac{\cosh(\frac{N-2}{2})}{\sinh(\frac{N-2}{2})} - 1 \right) + O(\varepsilon^{N+2}). \]
On the other hand, using again Lemma 1, we have
\[ w_1(t) = \frac{N - 2}{2} \sinh(\frac{N - 2}{2} t) + O(\varepsilon^{N+2}) \]
(56)
for all \( t \in (-d_0, d_0) \). Now direct calculations lead to
\[ w_2(t) = \frac{4}{(N - 2)^2} e^{-\frac{(N-2)t}{2}} + O(\varepsilon^{N+2}) \]
for all \( t \in (-d_0, -d_0) \). This proves (47).

Finally, in order to obtain (49), it is enough to observe that \( v(\tilde{t}+\cdot) \) converges (uniformly on compacts) to \( w_0 \). This completes the proof of the result.

Using similar arguments (and the notations of the previous Proposition), one can show that
\[ \left| w(t) - \frac{4 \beta}{\eta(N - 2)^2} \varepsilon^{-(N-2)t/2} \right| \leq c \beta \varepsilon^{-\frac{1}{2} + \frac{3}{N+2}} \]
if \( N \geq 6 \), and
\[ \left| w(t) - \frac{4 \beta}{\eta(N - 2)^2} \varepsilon^{-(N-2)t/2} \right| \leq c \beta \varepsilon^{-\frac{1}{2} + \frac{3}{N+2} - \frac{1}{N+4}} \]
(59)
if \( N = 5 \), for all \( t \in \left( \frac{2}{N-4} \log \frac{1}{\varepsilon} - d_0, \frac{2}{N-4} \log \frac{1}{\varepsilon} + d_0 \right) \).

In the following, \( \beta_{p, \eta} \) will be expanded.

**Lemma 4** Under the above assumptions, let \( c_0 > 1 \) be given. Assume
\[ \frac{1}{c_0} \varepsilon^{\frac{1}{4}} \leq \eta \leq c_0 \varepsilon^{\frac{1}{4}}. \]
Then,
\[ \tilde{t} = \frac{2}{N - 2} \log \frac{1}{\eta} + C_N^{(6)} + O(\varepsilon^{\frac{2}{N}} \log \frac{1}{\varepsilon}) \]
(60)
where
\[ C_N^{(6)} = \frac{2}{N - 2} \log 2 + \frac{1}{2} \log N(N - 2). \]
Proof. We recall

\[ \tilde{t} = \int_{\eta}^{v(\tilde{t})} \frac{dv}{\sqrt{2H_p(v, \partial_t v) + b_p v^2 - \frac{2b^{p+1}}{p+1}}} \]  

We divide

\[ \eta, v(\tilde{t}) = [\eta, \varepsilon^{\frac{N-2}{2N}}] \cup [\varepsilon^{\frac{N-2}{2}}, \frac{1}{2}(\frac{N(N-2)}{4})^{\frac{N-2}{4}}] \cup [\frac{1}{2}(\frac{N(N-2)}{4})^{\frac{N-2}{4}}, v(\tilde{t})] \]

We estimate

\[ \int_{I_1} \frac{dv}{\sqrt{2H_p(v, \partial_t v) + b_p v^2 - \frac{2b^{p+1}}{p+1}}} = (1 + \mathcal{O}(\varepsilon^{\frac{N}{4}})) \int_{I_3} \frac{dv}{\sqrt{b_p(v^2 - \eta^2)}} \]

\[ = (1 + \mathcal{O}(\varepsilon^{\frac{N}{4}})) \int_{I_3} \frac{dv}{\sqrt{\left(\frac{N-2}{2}\right)^2 v^2 - \frac{N-2}{2} \varepsilon^2 N}} + \mathcal{O}(\varepsilon^{\frac{N}{2}}) \]

Recall \( w_0(t) = (\frac{N(N-2)}{4})^{\frac{N-2}{4}} \left(\cosh t\right)^{\frac{2-N}{2}} \). We deduce

\[ \int_{I_2 \cup I_3} \frac{dv}{\sqrt{\left(\frac{N-2}{2}\right)^2 v^2 - \frac{N-2}{2} \varepsilon^2 N}} = \tilde{t} \]

where \( w_0(\tilde{t}) = \varepsilon^{\frac{N-2}{2N}} \). Hence, the desired results yield. \( \blacksquare \)

We set \( \tilde{v}(\cdot) = v(\tilde{t} - \cdot) \). We have the following result

**Lemma 5** Given \( c_0 > 1 \), assume \( \tilde{v}(0) \in (d_p - c_0 \varepsilon, d_p) \). Then, there exists the constant \( p \) independent of \( p \) such that

\[ |\tilde{v}(t) - w_0(t)| + |\partial_t \tilde{v}(t) - \partial_t w_0(t)| \leq c_0 e^{\left(\frac{N-3/2}{2}\right) |t|} \]  

(62)

for all \( t \in (-\tilde{t}, \tilde{t}) \), provided \( p \) close to \( p_N \).

**Proof.** We write for the difference \( \tilde{D} := \tilde{v} - w_0 \) so that

\[ \partial_t^2 \tilde{D} - b_{p_N} \tilde{D} = -f \tilde{D} + g \]  

where

\[ f := \frac{\tilde{v}^p - w_0^p}{\tilde{v} - w_0}, \quad g = a_p \partial_t \tilde{v} + w_0^{p_N} - w_0^p. \]

Clearly, there exist some positive constants \( K \) and \( c \) independent of \( p \) such that

\[ |-b_{p_N} + f(t)| \leq \left(\frac{N - 3/2}{2}\right)^2 \]
for all $t \in (K, t - \bar{t}) \cup (-\bar{t}, -K)$ and
\[ |g(t)| \leq c \varepsilon \]
for all $t \in (-\bar{t}, t - \bar{t})$. Recall
\[ |\bar{v}(0) - w_0(0)| + |\partial_t \bar{v}(0) - \partial_t w_0(0)| \leq c_0 \varepsilon. \]
Hence, the desired result follows from the standard ordinary differential equation theory.

As a consequence, we obtain immediately

**Corollary 3** There exists a positive constant $C_N^{(7)}$ (only depending on $N$), such that
\[ H_p(\varepsilon_{p, \ell}, 0) = -\ell C_N^{(7)} \varepsilon + O(\varepsilon^{2 - \frac{1}{2(N-2)}}) \]
where $C_N^{(7)} := \frac{(N-2)^2 C_N^{(4)}}{8}$. In particular,
\[ \varepsilon_{p, \ell} = (\ell \varepsilon C_N^{(4)})^{1/2} + O(\varepsilon^{N+2}). \] (64)

We keep the notations introduced in section 4 and we define for all $\ell \in \mathbb{N}$
\[ \beta_{p, \ell} := \int_{t_{\ell}}^{t_{\ell+1}} v_p^2(t)e^{-2(t-t_{\ell})} dt. \] (65)

Thanks to Lemma 3 to 5 and Corollary 3, we conclude

**Corollary 4** There exists a positive constant $C_N^{(8)}$ (only depending on $N$) such that
\[ \beta_{p, \ell} = (\ell \varepsilon) \frac{2}{N-2} \left( C_N^{(8)} + O(\varepsilon^{\frac{2}{N} \log \frac{1}{\varepsilon} + \varepsilon^{\frac{N-17/4}{N-2}}}) \right) \] (66)
where $C_N^{(8)} := (C_N^{(4)})^{2\frac{2}{N-2}} C_N^{(5)} e^{-2C_N^{(6)}}$.

## 6 Radial solutions of $\Delta u + \lambda u + |u|^{p-1} u = 0$ in the unit ball

In this section we recover part of the result of Del Pino, Dolbeault and Musso concerning the existence of solutions of (1) in the unit ball. In doing so our aim is to derive precise estimates for these solutions which will be needed in the forthcoming construction.

We begin with the definition of weighted spaces in cylindrical coordinates. These spaces are at the heart of our construction.

**Definition 1** Given $\delta \in \mathbb{R}$ and $-\infty \leq t_1 < t_2 \leq +\infty$, the space $C^0_\delta((t_1, t_2) \times S^{N-1})$ is defined to be the set of continuous functions $w \in C^0_{loc}((t_1, t_2) \times S^{N-1})$ for which the following norm is finite:
\[ \|w\|_{C^0_\delta((t_1, t_2) \times S^{N-1})} := \|e^{-\delta s} w\|_{L^\infty((t_1, t_2) \times S^{N-1})}. \] (67)
We would like to prove the existence of radial solutions of 
\[ \Delta u + \lambda u + |u|^{p-1}u = 0 \quad \text{in } B(0,1) \] 
(68)
Using (5), we reduce the study of (68) to studying the nonlinear second order ordinary differential equation
\[ \partial_t^2 v - a_p \partial_t v - b_p v + |v|^{p-1}v + \lambda e^{-2t}v = 0, \]
in \((0, +\infty)\). We keep the notations introduced in section 4 and consider the linear operator
\[ L_{p,\eta} := \partial_t^2 - a_p \partial_t - b_p + p v_{p,\eta}^{p-1}. \]
We state, without proof a result which will be proven in a more general context in the next section.

**Proposition 6** Assume that \( \delta \in \left(-\frac{N+2}{2}, -\frac{N-2}{2}\right) \) is fixed. Then, there exist \( \varepsilon_0 > 0 \), \( \eta_0 > 0 \) and \( c > 0 \) such that for all \( \varepsilon \in (0, \varepsilon_0) \), for all \( \eta \in (0, \eta_0) \) and for all \( f \in C^0((0, T_{p,\eta})) \), there exists a unique solution \( w \in C^0((0, T_{p,\eta})) \) of
\[ L_{p,\eta} w = f, \]
in \((0, T_{p,\eta})\) which satisfies
\[ w(T_{p,\eta}) = \partial_t w(T_{p,\eta}) = 0, \]
with \( p = p_N + \varepsilon \). Furthermore,
\[ \|w\|_{C^0} \leq c\|f\|_{C^0}. \]
(73)
When \( T_{p,\eta} < +\infty \), the existence and uniqueness of the solution of (71) is straightforward but the uniform estimate (73) requires some work. When \( T_{p,\eta} = +\infty \), the boundary data (72) have to be understood as limits as \( T_{p,\eta} = +\infty \).

The next result will allow us to recover (part of) the result of Del Pino, Dolbeault and Musso [5].

**Proposition 7** Assume that \( \ell \in \mathbb{N} \) is fixed and that \( N \geq 5 \). Then, there exists \( \varepsilon_0 > 0 \) such that for all \( \mu \in \mathbb{R} \), for all \( \xi \in \mathbb{R} \) and for all \( \varepsilon \in (0, \varepsilon_0) \), problem (68) with \( p = p_N + \varepsilon \) and \( \lambda = \mu \varepsilon^{N-2} \) admits a solution which can be written in the form
\[ u_{p,\lambda,\xi}(x) = (N(N-2))^{\frac{N-2}{2}} \sum_{j=1}^{\ell} \left( \frac{(\bar{e}_j^\varepsilon)}{1 + \bar{e}_j^\varepsilon |x|^2} \right)^{\frac{N-2}{2}} + o(1) \]
(74)
where \( o(1) \) converges uniformly to 0 on \( B(0,1) \) as \( \varepsilon \) tends to 0 and where
\[ \bar{e}_j^\varepsilon := d_j^\varepsilon \left( \varepsilon^{-\frac{N-2}{2}} \right) \]
for some parameters \( d_j \) which are bounded from below and from above by some positive constant independent of \( \varepsilon \). Moreover we have the following expansion
\[ u_{p,\lambda,\xi}(x) = (\varepsilon^2)^{\frac{1}{2}} \left[ \frac{\sqrt{C_N^{(4)}}}{2} e^{(N-2)x} + \frac{\sqrt{C_N^{(4)}}}{2} e^{(2-N)x} |x|^{2-N} - \frac{4\mu C_N^{(8)} e^{(N-2)x}}{(N-2)^2 \sqrt{C_N^{(4)}}} \right] + O(\varepsilon^{2r_\varepsilon^2}) \]
(75)
in \( B(0, 2r_\varepsilon) - B(0, r_\varepsilon/2) \), where \( r_\varepsilon := \varepsilon^{\frac{N-2}{4}} \). Furthermore, \( u_{p,\lambda,\xi} \) is positive in \( B(0, 2r_\varepsilon) \).
With these notations, the equation we need to solve reads
\[ T_{2i} = \ell_{i-2} - \ell + \xi, \quad T_{2i-1} = \ell_{i-1} - \ell + \xi \]
for \(0 < i < \ell\) and
\[ T_{2\ell} = +\infty, \quad T_{2\ell-1} = \ell_1 - \ell + \xi, \quad T_0 = 0,\]
For all \(0 \leq i \leq \ell - 2\), we define \(v_{p,i}\) to be the solution of (6) in \([T_{2i}, T_{2i+2}]\) with boundary conditions
\[ v_{p,i}(T_{2i+2}) = \varepsilon p_{i-1-i} + \alpha_i, \quad \partial_t v_{p,i}(T_{2i+2}) = 0 \]
and we define \(v_{p,\ell-1}\) to be the solution of (6) in \([T_{2\ell-2}, +\infty)\) with boundary conditions
\[ v_{p,\ell-1}(T_{2\ell-2}) = \varepsilon_{p,1}, \quad \partial_t v_{p,\ell-1}(T_{2\ell-2}) = 0, \]
for some parameters \(\alpha_i \in \mathbb{R}\) (which are assumed to be small). Here the parameters \(\varepsilon_{p,i}\) are the one which have been introduced in section 4.

For any \(0 \leq i \leq \ell - 1\), we define the function
\[ W_i(t) := v_{p,i}(t + t_i) + w_i(t) \]
for on the interval \([T_{2i}, T_{2i+2}]\), for some parameters \(t_i \in \mathbb{R}\) and some functions \(w_i \in C^0([T_{2i}, T_{2i+2}])\). We agree that \(t_{\ell-1} = 0\) and \(\alpha_{\ell-1} = 0\).

Granted the above definitions, our strategy is the following: In Step 1 and 2, we look \(W_i\) solutions of (69) on each interval \([T_{2i}, T_{2i+2}]\). Moreover, \(W_i\) are positive if \(i \geq 1\). In Step 3, we choose the parameters \((\alpha_0, \ldots, \alpha_{\ell-2})\) and \((t_0, \ldots, t_{\ell-2})\) so that the Cauchy data of \(W_i\) and of \(W_{i-1}\) coincide at \(T_{2i}\). Gathering the functions \(W_i\) together, we obtain a solution of (69) which still depends on \(\xi\).

**Step 1.** For each \(1 \leq i \leq \ell - 1\), we look for a solution of (69) in \([T_{2i}, T_{2i+2}]\). Recall that \(\varepsilon = p - p_N\). We now assume that
\[ \alpha_i = o(\varepsilon^{1/2}) \quad \varepsilon = O(1) \quad \text{and} \quad t_i = o(1) \]
as \(\varepsilon\) tends to 0. We define the operator
\[ L_{p,i} = \partial_t^2 - a_p \partial_t - b_p + p v_{p,i}^{p-1}(. + t_i). \]

With these notations, the equation we need to solve reads
\[ L_{p,i}w_i = -\lambda e^{-2t}(v_{p,i}(. + t_i) + w_i) - Q_i(w_i) \]
where we have defined
\[ Q_i(w_i) := |v_{p,i}(. + t_i) + w_i|^{p-1}(v_{p,i}(. + t_i) + w_i) - v_{p,i}^{p}(. + t_i) - p v_{p,i}^{p-1}(.) + t_i) w_i. \]

We fix the weight parameter \(\delta \in (-\frac{N-1}{2}, -\frac{N-2}{2})\) and we consider the set of functions
\[ \mathcal{E}_{\kappa,i} = \left\{ w \in C^0_\delta((T_{2i}, T_{2i+2})) : \|w\|_{C^\kappa_\delta} \leq \kappa \lambda e^{-(\delta+2)T_{2i+1}} \right\}, \]
where the constant $\kappa > 0$ will be fixed later on.

Given $w \in \mathcal{E}_{\kappa,i}$, it follows from (31) that $|w| \leq c \kappa \varepsilon^{\frac{2\delta + 4N}{N-2}}$. Recall that $\lambda = \varepsilon^{\frac{4}{N-2}}$ and $\delta > -\frac{N-1}{2}$, hence, we obtain

$$|w| \leq c \kappa \varepsilon^{\frac{2\delta + 4N}{N-2}} \ll v_{p,i}(\cdot + t_i)$$

on $(T_{2i}, T_{2i+2})$. Therefore, we are allowed to use Taylor’s expansion $|(1+t)^p - 1 - pt| \leq ct^2$ for $t$ close enough to 0, to estimate

$$|Q_i(w)| \leq c t_{p,i}^{-2}(\cdot + t_i) w^2$$

Using this, we obtain

$$\|Q_i(w)\|_{c_0^p} \leq c \kappa \varepsilon^{\frac{(\delta + 4N)(p-1)}{N-2}} e^{-(\delta + 2)T_{2i+1}}.$$  \hspace{1cm} (80)

Next, we estimate

$$\|\lambda e^{-2t} v_{p,i}(\cdot + t_i)\|_{c_0^p} = \sup_{(T_{2i}, T_{2i+2})} \lambda e^{-(\delta + 2)t} |v_{p,i}(\cdot + t_i)| \leq c \lambda e^{-(\delta + 2)T_{2i+1}}.$$ \hspace{1cm} (81)

since $\gamma_+ - \delta - 2 > 0$ and $\gamma_- - \delta - 2 < 0$ provided $\varepsilon$ is close enough to 0. With similar arguments, we get

$$\|\lambda e^{-2t} w\|_{c_0^p} \leq \lambda e^{-2T_{2i}} \|w\|_{c_0^p} \leq c \kappa \varepsilon^{\frac{4}{N-2}} e^{-(\delta + 2)T_{2i+1}},$$ \hspace{1cm} (82)

Combining (80) to (82), we have obtained

$$\|Q_i(w) + \lambda e^{-2t} (v_{p,i}(\cdot + t_i) + w)\|_{c_0^p} \leq c \kappa \lambda e^{-(\delta + 2)T_{2i+1}} \left(\kappa^{p} \varepsilon^{\frac{(2\delta + 4N)(p-1)}{N-2}} + 1 + \kappa \lambda \varepsilon^{\frac{4}{N-2}}\right),$$ \hspace{1cm} (83)

which holds for all $w \in \mathcal{E}_{\kappa,i}$.

Given $w \in \mathcal{E}_{\kappa,i}$, we apply the result of Proposition 6 which provides a solution of

$$L_{p,i} \tilde{w} = -Q_i(w) - \lambda e^{-2t} (v_{p}(\cdot + t_i) + w)$$

with $\tilde{w}(T_{2i+2}) = \partial_t \tilde{w}(T_{2i+2}) = 0$. Thanks to (83), we also have the estimate

$$\|\tilde{w}\|_{c_0^p} \leq \lambda e^{-(\delta + 2)T_{2i+1}} \tilde{c} \left(\kappa^{p} \varepsilon^{\frac{(2\delta + 4N)(p-1)}{N-2}} + 1 + \kappa \lambda \varepsilon^{\frac{4}{N-2}}\right).$$

for some constant $\tilde{c} > 0$ which does not depend on $w$, nor on $\kappa$ nor on $\varepsilon$ provided this later is chosen small enough. This estimate being understood, we choose the constant $\kappa > 0$ so that

$$\tilde{c} \left(\kappa^{p} \varepsilon^{\frac{(2\delta + 4N)(p-1)}{N-2}} + 1 + \kappa \lambda \varepsilon^{\frac{4}{N-2}}\right) \leq \kappa,$$

for all $\varepsilon$ close enough to 0, say $\varepsilon \in (0, \varepsilon_0)$.

To summarize, using the above analysis, we can define the mapping

$$\mathcal{T}_i : \mathcal{E}_{\kappa,i} \longrightarrow \mathcal{E}_{\kappa,i}$$

by $\mathcal{T}_i(w) = \tilde{w}$. Thanks to the above choice of $\kappa$, the mapping $\mathcal{T}_i$ is well defined. Observe that this mapping is clearly continuous and compact so that one can refer to Schauder’s fixed point Theorem to obtain the fixed point of $\mathcal{T}_i$. We have proved the :
Lemma 6. Assume that $\alpha_i$ and $t_i$ satisfy (77). Then, there exists $W_i$ a positive solution of (69) in $(T_{2i}, T_{2i+2})$ with boundary conditions $W_i(T_{2i+2}) = v_{p,i}(T_{2i+2} + t_i)$ and $\partial_t W_i(T_{2i+2}) = \partial_t v_{p,i}(T_{2i+2} + t_i)$. In addition, we have the estimates
\[ \|W_i - v_{p,i}(\cdot + t_i)\|_{C^0} \leq c \lambda e^{(-\delta - 2)T_{2i+1}} \]
where the constant $c$ is independent of $\varepsilon$ and of the parameters $\alpha_i$, $t_i$ and $\xi$.

Observe that the solution we have obtained is unique and depends continuously on the parameters $\alpha_i$, $t_i$ and $\xi$ since it is the unique solution of an ordinary differential equation. This fact is even true when $i = \ell - 1$ even though the solution is defined on a half line.

Step 2. We now look for a solution of (69) which is defined on $(T_0, T_2)$. We decompose
\[ W_0(t) = v_{p,0}(t + t_0) + \overline{w}(t) + w(t), \]
where $\overline{w}$ is the solution of
\[ L_{p,0} \overline{w} = -\lambda e^{-2t} v_{p,0}(t + t_0) \]
in $(T_0, T_2)$ with boundary data $\overline{w}(T_2) = \partial_t \overline{w}(T_2) = 0$. The operator $L_{p,0}$ is the one which has been defined in (78). With this in mind, it remains to find a $w$ solution of
\[ L_{p,0} w = -\lambda e^{-2t} (\overline{w} + w) - Q_0(w) \]
in $(T_0, T_2)$ with boundary data $w(T_2) = \partial_t w(T_2) = 0$, where
\[ Q_0(w) := |v_{p,0}(\cdot + t_0) + \overline{w} + w|^{p-1}(v_{p,0}(\cdot + t_0) + \overline{w} + w) - v_{p,0}^{p}(\cdot + t_0) - p v_{p,0}^{p-1}(\cdot + t_0)(\overline{w} + w). \]

It will be convenient to define
\[ q := \min \left\{ \frac{2N + \delta - 6}{N - 2}, \frac{(N + \delta - 2)p}{N - 2} \right\}. \]

Observe that we have $q > \frac{1}{2}$ since we have assumed that $N \geq 5$ and $\delta \in (-\frac{N-1}{2}, -\frac{N-2}{2})$.

This time we consider the following set of functions
\[ \mathcal{E}_{\kappa,0} = \left\{ w \in C^0_\delta((T_0, T_2)) : \|w\|_{C^0} \leq \kappa \varepsilon^q \right\}, \]
where the constant $\kappa > 0$ will be fixed later on. It is clear that
\[ \|\lambda e^{-2t} v_{p,0}\|_{C^0_{\delta}} \leq c \lambda e^{(-\delta - 2)T_1} \leq c \lambda \varepsilon^{\frac{2 + \delta}{N - 2}} \]

Using the result of Proposition 6, we get
\[ \|\overline{w}\|_{C^0_{\delta}} \leq c \|\lambda e^{-2t} v_{p,0}\|_{C^0_{\delta}} \leq c \mu \varepsilon^{\frac{N + 2 + \delta}{N - 2}} \leq c \varepsilon^{\frac{N + 2 + \delta}{N - 2}} \]
so that
\[ \|\lambda e^{-2t} \overline{w}\|_{C^0_{\delta}} \leq c \varepsilon^{\frac{2N + 2 + \delta}{N - 2}}. \] (84)

As in Step 1, we have
\[ \|\lambda e^{-2t} w\|_{C^0_{\delta}} \leq \lambda \|w\|_{C^0_{\delta}} \leq c \kappa \varepsilon^{\frac{N + 4}{N - 2} + q}. \] (85)
for all $w \in \mathcal{E}_{\kappa,0}$. For $\varepsilon$ small enough, we have $1 < p < 2$. Thus, for all $s_2 \in \mathbb{R}$ and all $s_1 > 0$, we can write
\[
|s_1 + s_2|^{p-1} (s_1 + s_2) - s_1^p - ps_1^{p-1}s_2| \leq c|s_2|^p.
\]
for some constant $c > 0$. Consequently, we can estimate for all $w \in \mathcal{E}_{\kappa,0}$
\[
\|Q_0(w)\|_{C^0_\delta} \leq c \sup_{t \in (T_0, T_2)} e^{-\delta t} |w(t) + \varpi(t)|^p 
\leq c (\kappa^p \varepsilon^q + \varepsilon^{\frac{p(N-2+\delta)}{N-2}}).
\]
Using the result of Proposition 6, we get a solution $\tilde{w}$ of
\[
L_{p, 0} \tilde{w} = -\lambda e^{-2t} (\varpi + w) - Q_0(w)
\]
with $\tilde{w}(T_2) = \partial_t \tilde{w}(T_2) = 0$. Collecting (84), (85) and (86) we get the estimate
\[
\|\tilde{w}\|_{C^0_\delta} \leq \varepsilon^q \bar{c} \left(1 + \kappa^p \varepsilon^{(p-1)q} + \kappa \varepsilon^{\frac{N-4}{N-2}}\right).
\]
We choose the constant $\kappa$ so that
\[
\bar{c} \left(1 + \kappa^p \varepsilon^{(p-1)q} + \kappa \varepsilon^{\frac{N-4}{N-2}}\right) \leq \kappa,
\]
for all $\varepsilon$ is close to 0, say $\varepsilon \in (0, \varepsilon_0)$.

As Step 1, we can define the mapping
\[
\mathcal{T}_0 : \mathcal{E}_{\kappa,0} \rightarrow \mathcal{E}_{\kappa,0}
\]
by $\mathcal{T}_0(w) := \tilde{w}$. Clearly, $\mathcal{T}_0$ is well defined and is continuous and compact, so that one can again refer to Schauder’s fixed point Theorem to obtain the fixed point of $\mathcal{T}_0$. We have proved the:

**Lemma 7** Given $\alpha_0$ and $t_0$ satisfying (77), there exists a solution $W_0$ of (69) in $(T_0, T_2)$ with boundary conditions $W_0(T_2) = v_{p,0}(T_2 + t_0)$ and $\partial_t W_0(T_2) = \partial_t v_{p,0}(T_2 + t_0)$. In addition, we have the estimates
\[
\|W_0 - v_{p,0}(- + t_0) - \tilde{w}\|_{C^0_\delta} \leq c \varepsilon^q
\]
for some $q > \frac{1}{2}$ and
\[
\|\varpi\|_{C^0_\delta} \leq c \varepsilon^{\frac{N-2+\delta}{N-2}}.
\]
Again this solution is unique and depends continuously on the parameters $\alpha_0$, $t_0$ and $\xi$.

**Step 3.** We now explain how to choose the parameters $(\alpha_0, \ldots, \alpha_{\ell-2})$ and $(t_0, \ldots, t_{\ell-2})$ so that the Cauchy data of $W_i$ and $W_{i-1}$ coincide at $T_{2i}$, for $1 \leq i \leq \ell - 1$. To this aim, we argue inductively, starting by matching the Cauchy data of $W_{\ell-2}$ and $W_{\ell-1}$.

This amounts to find $\alpha_{\ell-2}$ and $t_{\ell-2}$ so that
\[
W_{\ell-2}(T_{2\ell-2}) = W_{\ell-1}(T_{2\ell-2}), \quad \text{and} \quad \partial_t W_{\ell-2}(T_{2\ell-2}) = \partial_t W_{\ell-1}(T_{2\ell-2}),
\]
for all $w \in \mathcal{E}_{\kappa,0}$. For $\varepsilon$ small enough, we have $1 < p < 2$. Thus, for all $s_2 \in \mathbb{R}$ and all $s_1 > 0$, we can write
\[
|s_1 + s_2|^{p-1} (s_1 + s_2) - s_1^p - ps_1^{p-1}s_2| \leq c|s_2|^p.
\]
In view of (90), the mapping and on Lemma 2, we get fixed point theorem that (90) admits a solution. In addition, applying Lemma 1 and the following expansion holds

In view of (58) and thanks to Lemma 1, Lemma 7, Corollary 3 and Corollary 4, the system (88) is therefore equivalent to

\[ t_{\ell-2} = \varepsilon^{-1} t_{\ell-2} \quad \text{and} \quad \alpha_{\ell-2} = \tilde{G}_{\ell-2} \tag{90} \]

where the continuous functions \( F_{\ell-2}, G_{\ell-2} \) depend on \( \alpha_{\ell-2} \) and \( t_{\ell-2} \) and satisfy \(|F_{\ell-2}| + |G_{\ell-2}| = O(\varepsilon^2)\). The recall that \( p = p_N + \varepsilon \). Given \( \gamma \in (1, \frac{N}{2}) \) we define

\[ B := \left\{ (\alpha_{\ell-2}, t_{\ell-2}) \in \mathbb{R}^2 : \alpha_{\ell-2}^2 + \varepsilon t_{\ell-2}^2 \leq \varepsilon^\gamma \right\}. \]

In view of (90), the mapping

\[ \tilde{H} : B \rightarrow B \]

defined by \( \tilde{H}(\alpha_{\ell-2}, t_{\ell-2}) := (\varepsilon^{-1} \tilde{F}_{\ell-2}, \tilde{G}_{\ell-2}) \) is well defined and it follows from Browder’s fixed point theorem that (90) admits a solution. In addition, applying Lemma 1 and Lemma 2, we get

\[ v_{p,\ell-2}(T_{2\ell-4} + t_{\ell-2}) = \varepsilon_{p,2} + O(\varepsilon^2), \quad \partial_t v_{p,\ell-2}(T_{2\ell-4} + t_{\ell-2}) = O(\varepsilon^2). \tag{91} \]

Arguing inductively, we construct a function \( v_{p,\lambda,\xi} \), solution of (69), which depends on \( \lambda \) and on \( \xi \), and which satisfies

\[ v_{p,\lambda,\xi}(T_2) = \varepsilon_{p,\ell-1} + O(\varepsilon^2), \quad \partial_t v_{p,\lambda,\xi}(T_2) = O(\varepsilon^2). \tag{92} \]

In view of (58) and thanks to Lemma 1, Lemma 7, Corollary 3 and Corollary 4, the following expansion holds

\[
\begin{align*}
\left. v_{p,\lambda,\xi}(t) \right| &= \left( \ell \varepsilon \right)^{\frac{1}{2}} \left[ \sqrt{C_N^{(4)} 2^{-N-2}(\xi - t)} e^{\frac{(N-2)(\xi - t)}{2}} + \sqrt{C_N^{(4)} 2^{-N-2}(\xi - t)} e^{\frac{(2-N)(\xi - t)}{2}} - \frac{4\mu C_N^{(8)} \ell^{\frac{1}{N-2}} e^{\frac{(N-6+17/4)(\xi - t)}{2}}}{(N-2)^{2} N^{2}} \right] \\
&\quad + O \left( \varepsilon^4 e^{\delta t} + \varepsilon^{\frac{3}{2} + \frac{N}{2}} + \left( \varepsilon^3 e^{\frac{(N-2)(\xi - t)}{2}} \right)^p + \varepsilon^2 \left( \varepsilon^2 \log \frac{1}{\varepsilon} + \varepsilon e^{\frac{N-17/4}{2}} e^{\frac{(2-N)(\xi - t)}{2}} \right) \right) \tag{93}
\end{align*}
\]

for all \( t \in \left( \log \frac{1}{2\varepsilon^2}, \log \frac{2 \varepsilon}{r_\varepsilon} \right) \). This completes the proof of the result.
7 The linear analysis.

Assume that $\Omega$ is a regular bounded open subset of $\mathbb{R}^N$ and $\Sigma := \{a_1, \ldots, a_m\}$ is a finite set of points of $\Omega$. We choose $r_0 > 0$ in such a way that the closed balls $B(a_i, 2r_0)$, for $i = 1, \ldots, m$ are disjoint and included in $\Omega$. For all $r \in (0, r_0)$, we define

$$\Omega_{int,r} := \bigcup_{i=1}^m B(a_i, r) \quad \text{and} \quad \Omega_{ext,r} := \Omega - \overline{\Omega}_{int,r}. $$

We define the weighted spaces :

**Definition 2** Given $\nu \in \mathbb{R}$, the space $C^0_\nu(\overline{\Omega} - \Sigma)$ is defined to be the set of continuous functions $w \in C^0_{loc}(\overline{\Omega} - \Sigma)$ for which the following norm is finite :

$$\|w\|_{C^0_\nu(\overline{\Omega} - \Sigma)} := \|w\|_{L^\infty(\Omega_{ext,r_0})} + \sum_{j=1}^m \|r^{-\nu} w(a_j + \cdot)\|_{L^\infty(B(0,2r_0)-\{0\})}. \quad (94)$$

Given $r \in (0, r_0)$ we define the space $C^0_\nu(\Omega_{ext,r})$ to be the space of restrictions of functions of $C^0_\nu(\overline{\Omega} - \Sigma)$ to $\Omega_{ext,r}$. This space is endowed with the induced norm.

In this section, we study the linearization of the nonlinear operator (68) about the radial function $u_\varepsilon(x) := |x|^{-\frac{2}{p-1}} v_\varepsilon(-\log |x|)$ where $v_\varepsilon := v_{p,\lambda,\xi}$ and $v_{p,\lambda,\xi}$ is the solution of (69) defined in Step 3 of the proof of Proposition 7. This operator is defined by

$$L_\varepsilon := \Delta + \lambda + pu_{\varepsilon}^{p-1}. $$

Recall $r_\varepsilon = \varepsilon^{\frac{q}{N-1}}$. We can write any function $v$ defined in the punctured ball $B(0, r_\varepsilon) - \{0\}$ as

$$v(x) = |x|^{-\frac{2}{p-1}} w(-\log |x|, \theta),$$

so that the study of $L_\varepsilon$ reduces to the study of the linear operator

$$L_\varepsilon := \partial_t^2 - a_p \partial_t - b_p + \Delta_{SN-1} + pv_{\varepsilon}^{p-1} + \lambda e^{-2t} \quad (95)$$

on the half cylinder $[B_\varepsilon, +\infty) \times S^{N-1}$, where $\Delta_{SN-1}$ denotes the Laplace-Beltrami operator on the sphere $S^{N-1}$ and $B_\varepsilon = -\log r_\varepsilon$.

We denote by $(e_j, \lambda_j)$ the set of eigendata of $\Delta_{SN-1}$

$$\Delta_{SN-1} e_j = -\lambda_j e_j. $$

We also assume that the eigenvalues are counted with multiplicity, that $\lambda_j \leq \lambda_{j+1}$ and that the $e_j$ are normalized by

$$\int_{S^{N-1}} e_j^2 d\omega = 1. $$

We now prove some uniform estimates for a right inverse for the operator $L_\varepsilon$. 

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Proposition 8 Assume that \( \delta \in (-\frac{N+2}{2}, -\frac{N}{2}) \) is fixed. Then, there exists \( p_0 \in (p_N, +\infty) \) such that, if \( p \in (p_N, p_0) \), then, for all \( f \in C^0_\delta([B_\varepsilon, +\infty) \times S^{N-1}) \), there exists a unique solution \( w \in C^0_\delta([B_\varepsilon, +\infty) \times S^{N-1}) \) of

\[
\mathcal{L}_\varepsilon w = f
\]

in \([B_\varepsilon, +\infty) \times S^{N-1}\) which satisfies

\[w(B_\varepsilon, \theta) \in \text{Span}\{e_j : j = 0, \ldots, N\} \]

Furthermore,

\[\|w\|_{C^0_\delta} \leq c \|f\|_{C^0_\delta} \tag{97}\]

for some constant which does not depend on \( \varepsilon \).

Proof. The proof is divided in three parts. In the first part we explain how to solve the equation (96) when the function \( f \) does not have any component on \( e_j \) for \( j = 0, \ldots, N \) in its eigenfunction decomposition. Next, in the second part, we obtain a uniform estimate for the solution already obtained. Finally, in the last part, we explain how to solve (96) when the eigenfunction decomposition of \( f \) has components on \( e_0, \ldots, e_N \).

Step 1 For the time being, we assume that the eigenfunction decomposition of the function \( f \) is given by

\[f(t, \theta) = \sum_{j \geq N+1} f_j(t) e_j(\theta). \tag{98}\]

Observe that, as \( p \) tends to \( p_N \) we have

\[\lim_{p \to p_N} a_p = 0, \quad \lim_{p \to p_N} b_p = \left(\frac{N - 2}{2}\right)^2\]

and

\[\lim_{p \to p_N} \sup_{p, \lambda, \xi} p v_{p, \lambda, \xi} = \frac{N(N + 2)}{4}\]

these limits being independent of the solution \( v_{p, \lambda, \xi} \).

Now the eigenfunction decomposition of the Laplace-Betrami operator on \( S^{N-1} \) induces a decomposition of the operator \( \mathcal{L}_\varepsilon \) into the sequence of operators

\[L_j := \partial_t^2 - a_p \partial_t - b_p - \lambda_j + p v_{p-1} + \lambda e^{-2t} \]

Using these above limits together with the fact that \( \lambda_j \geq 2N \) for \( j \geq N + 1 \), we conclude that, for \( j \geq N + 1 \) the potential is negative provided \( p \) is close enough to \( p_N \). In particular, this implies that it is possible to solve

\[\mathcal{L}_\varepsilon w = f\]

on any \((B_\varepsilon, S) \times S^{N-1}\), with \( w = 0 \) as boundary data (observe that the operator \( \mathcal{L}_\varepsilon \) is not self adjoint but is conjugate to a self adjoint operator and we have just seen that this former is injective, when restricted to the set of functions spanned by \( e_j \), for \( j \geq N + 1 \)).
It remains to prove that there exists a constant \( c > 0 \) which does not depend on \( S \), nor on \( p \) such that
\[
sup |e^{-\delta t} w| \leq c sup |e^{-\delta t} f|.
\] (99)

Then, the existence of the solution on all \((B_z, +\infty) \times S^{N-1}\) as well as the relevant estimate will follow by passing to the limit \( S \to +\infty \). To keep the proof short and since anyway our aim is to pass to the limit as \( S \) tends to \( \infty \), it is enough to prove that (96) holds for all \( S \) chosen large enough so that \( sup_{(S, +\infty)} v_p \leq \epsilon \).

**Step 2** The proof of (99) is by contradiction. If it were false for all choice of \( p_0 \) and \( S \), there would exist a sequence \((p_n)\) tending to \( p_N \), a sequence of functions \((f_n)\) and a sequence of reals \((S_n)\) and a sequence \((w_n)\) of solutions of (96) such that
\[
\|f_n\|_{c^3} = 1 \quad \text{and} \quad \lim_{n \to +\infty} A_n := sup e^{-\delta t} |w_n| = +\infty.
\] (100)

We denote \( B_n = B_{\epsilon_n} \) where \( \epsilon_n := p_n - p_N \). Obviously, there exists a point \((t_n, \theta_n) \in (B_n, S_n) \times S^{N-1}\) where the above supremum is achieved, namely \( A_n = e^{-\delta t_n} |w_n(t_n, \theta_n)| \).

Observe that elliptic estimates imply that
\[
sup e^{-\delta t} |\nabla w_n| \leq c (1 + A_n)
\] (101)
and this in turn implies that the sequences \((t_n - B_n)\) and \((S_n - t_n)\) remain bounded away from 0.

We define \( \tilde{t}_n > B_n \) to be the nearest local maximal point of the function \( v_{p_n}(t) \) to the point \( t_n \). We distinguish several cases according to the behavior of the sequence \((t_n)\).

**Case 1.** Assume that the sequence \((t_n - \tilde{t}_n)\) is bounded. In this case, we define the function \( \tilde{w}_n \) by
\[
\tilde{w}_n(t, \theta) = \frac{1}{A_n} e^{-\delta \tilde{t}_n} w_n(t + \tilde{t}_n, \theta).
\]

Observe that the sequence of functions \((v_{p_n}(\cdot + \tilde{t}_n))\) converges on compact to \( t \to (N(N-2))^{\frac{N-2}{2}} (\cosh t)^{-\frac{N}{2}} \) (see Proposition 5). Up to a subsequence, we can assume that the sequence \((t_n - \tilde{t}_n)\) converges to \( t_\infty \). Moreover, we can assume that the sequence \((\tilde{w}_n)\) converges on compacts to \( \tilde{w}_\infty \) a nontrivial solution of
\[
\partial_t^2 \tilde{w}_\infty + \Delta_{S^{N-1}} \tilde{w}_\infty - \frac{(N-2)^2}{4} \tilde{w}_\infty + \frac{N(N+2)}{4} (\cosh t)^{-2} \tilde{w}_\infty = 0.
\] (102)

Moreover, \( \tilde{w}_\infty \) is bounded by a constant times \( e^{\delta t} \). The fact that \( \tilde{w}_\infty \) is not identically equal to 0 follows from the fact that \( |\tilde{w}_n(t_n - \tilde{t}_n, \theta_n)| = e^{\delta(t_n - \tilde{t}_n)} \) and hence remains bounded away from 0.

We consider the eigenfunction decomposition of \( \tilde{w}_\infty \)
\[
\tilde{w}_\infty = \sum_{j=N+1}^{\infty} a_j e_j.
\]

At \( -\infty \) the function \( a_j \) is either blowing up like \( t \to e^{-\gamma_j t} \) or decaying like \( t \to e^{\gamma_j t} \), where
\[
\gamma_j := \sqrt{\lambda_j + \frac{(N-2)^2}{4}}.
\]
The choice of $\delta \in (-\frac{N+2}{2}, -\frac{N}{2})$ implies that $-\delta < \gamma_j$ for all $j \geq N + 1$. Hence $a_j$ decays exponentially at $-\infty$. Multiplying the equation (102) by $a_j^2$ and integrating by parts over $\mathbb{R}$ (all integrations are justified because $a_j$ decays exponentially at both $\pm \infty$), we get
\[
\int_{-\infty}^{+\infty} |\partial_t a_j|^2 + (\lambda_j + \frac{(N-2)^2}{4})(a_j^2)2 = \frac{N(N+2)}{4} \int_{-\infty}^{+\infty} (\cosh s)^{-2}(a_j^2)2 \leq \frac{N(N+2)}{4} \int_{-\infty}^{+\infty} (a_j^2)2.
\]
Since $j \geq N + 1$, we have $\lambda_j \geq 2N$, and hence we conclude that $a_j \equiv 0$. Hence, $\tilde{w}_\infty \equiv 0$, a contradiction.

**Case 2.** Assume that the sequence $(t_n - \bar{t}_n)$, the sequence $(t_n - B_n)_n$ and the sequence $(S_n - t_n)_n$ are all unbounded. In this case, we define the function $\tilde{w}_n$ by
\[
\tilde{w}_n(t, \theta) = \frac{1}{A_n} e^{-\delta t_n} w_n(t + t_n, \theta).
\]
Observe that this time the sequence of functions $(v_{p_n}(\cdot + t_n))_n$ converge to 0 on compacts. Up to a subsequence, we can assume that the sequence $(\tilde{w}_n)_n$ converges on compacts to $\tilde{w}_\infty$ a nontrivial solution of
\[
\partial_t^2 \tilde{w}_\infty + \Delta_{S^{N-1}} \tilde{w}_\infty - \frac{(N-2)^2}{4} \tilde{w}_\infty = 0.
\]
Moreover, $\tilde{w}_\infty$ is bounded by a constant times $e^{\delta t}$.

Again, we consider the eigenfunction decomposition of $\tilde{w}_\infty$
\[
\tilde{w}_\infty = \sum_{j=N+1}^\infty a_j e_j
\]
and we see that $a_j$ is a linear combination of $t \rightarrow e^{-\gamma_j t}$ and $t \rightarrow e^{\gamma_j t}$. The choice of $\delta \in (-\frac{N+2}{2}, -\frac{N}{2})$ implies that $\delta > -\gamma_j$ for all $j \geq N + 1$. Hence $a_j$ cannot be bounded by $e^{\delta t}$ unless it is identically 0. We conclude that $a_j \equiv 0$. Hence, $\tilde{w}_\infty \equiv 0$, a contradiction.

**Case 3.** Assume that the sequence $(t_n - B_n)_n$ is bounded (resp. that the sequence $(S_n - t_n)_n$ is bounded) and that the sequence $(t_n - \bar{t}_n)_n$ is unbounded. This case can be treated as in case 2. The only difference is that this time $\tilde{w}_\infty$ is defined on $[L_\infty, +\infty) \times S^{N-1}$ (resp. on $(-\infty, t_\infty] \times S^{N-1}$) and is equal to 0 on $\{t_\infty\} \times S^{N-1}$ (resp. on $\{t_\infty\} \times S^{N-1}$). We omit the details.

Since we have reached a contradiction in each case, the proof of the claim is complete.

We can now pass to the limit as $S$ tends to $+\infty$ and complete the proof of the result in the case where the eigenfunction decomposition of $f$ does not involve any $e_j$ for $j = 0, \ldots, N$.

**Step 3.** Now we consider the case where the function $f$ is collinear to $e_j$, namely
\[
f(t, \theta) = f^j(t) e_j(\theta)
\]
for some $0 \leq j \leq N$. We extend the function $f$ to be equal to 0 when $t \leq B_\varepsilon$ and we define the function $\tilde{v}_p$ which is equal to $v_p$ for $t \geq B_\varepsilon$ and is equal to 0 for $t < B_\varepsilon$. We consider the equation
\[
\partial_t^2 a^j - a_p \partial_t a^j - (\lambda_j + b_p) a^j + p \tilde{v}_p^{p-1} a^j + \lambda e^{-2t} \chi a^j = f^j \tag{103}
\]
in \( \mathbb{R} \). Here \( \chi \) is a cutoff function identically equal to 1 on \( (B_\varepsilon, +\infty) \) and equal to 0 on \( (-\infty, B_\varepsilon - 1) \). Observe that

\[
|f_j^0(t)| \leq \|f\|c_j e^{\delta t}. 
\]

For \( p \) close enough to \( p_N \), \( \delta \) is not an indicial root of the operator \( \mathcal{L}_\varepsilon \) and it follows from Cauchy’s theorem that there exists a unique solution of \( (103) \) which is bounded by a constant times \( e^{\delta t} \) at \( +\infty \). A priori this solution is only defined for \( t \) large enough but is can be extended to all \( \mathbb{R} \) easily. Furthermore, it follows from the construction of this solution that

\[
\sup_{(T, +\infty)} e^{-\delta t} |a^j| \leq c \sup_R e^{-\delta t} |f^j| 
\]

provided \( T \) is large enough. This solution satisfies

\[
\partial_t^2 a^j - a_p \partial_t a^j - (\lambda_j + b_p) a^j = 0 
\]

for \( t < B_\varepsilon - 1 \) and, since \( \delta \in (-\frac{N+2}{2}, -\frac{N}{2}) \), even if \( a^j \) blows up at \( -\infty \), it blows up at a slower rate than \( t \to e^{\delta t} \), provided \( p \) is chosen close enough to \( p_N \).

We claim that there exists a constant \( c > 0 \) such that

\[
\sup_R e^{-\delta t} |a^j| \leq c \sup_R e^{-\delta t} |f^j| 
\]

provided \( p \) is close enough to \( p_N \). As before, we argue by contradiction. Assume that the claim is not true. Then there would exist a sequence \( (p_n)_n \) tending to \( p_N \), a sequence of functions \( (f_n^j) \) and a sequence of solutions \( (a_{n}^{j})_n \) of \( (104) \) such that

\[
\sup_R e^{-\delta t} |f_n^j| = 1 \quad \text{and} \quad A_n := \sup_R e^{-\delta t} |a_{n}^{j}|
\]

tends to \( +\infty \). The previous remarks show that the above supremum is always achieved in \( \mathbb{R} \). So we can define \( t_n \) such that \( A_n = e^{-\delta t_n} |a_{n}^{j}(t_n)| \).

As in Step 2, we define \( \tilde{t}_n > 0 \) to be the nearest local maximal point of the function \( v_{p_n}(t) \) to the point \( t_n \). We distinguish several cases according to the behavior of the sequence \( (t_n)_n \). We define the function \( \tilde{a}_n^{j} \) by

\[
\tilde{a}_n^{j}(t) = \frac{1}{A_n} e^{-\delta t_n} a_{n}^{j}(t + \tilde{t}_n).
\]

We can assume that, up to a subsequence, the sequence \( (\tilde{a}_n^{j})_n \) converges on compacts to \( \tilde{a}_{\infty} \) a nontrivial solution of

\[
\partial_t^2 \tilde{a}_{\infty} - \lambda_j \tilde{a}_{\infty} - \frac{(N - 2)^2}{4} \tilde{a}_{\infty} + \frac{N(N + 2)}{4} (\cosh t)^{-2} \tilde{a}_{\infty} = 0
\]

in the case where the sequence \( (t_n - \tilde{t}_n) \) is bounded, or to a nontrivial solution of

\[
\partial_t^2 \tilde{a}_{\infty} - \lambda_j \tilde{a}_{\infty} - \frac{(N - 2)^2}{4} \tilde{a}_{\infty} = 0
\]

in the case where the sequence \( (t_n - \tilde{t}_n) \) is unbounded.

Moreover, \( \tilde{a}_{\infty} \) is bounded by a constant times \( e^{\delta t} \). However, the choice of \( \delta \in (-\frac{N+2}{2}, -\frac{N}{2}) \) implies that \( \delta < -\gamma_j \) for all \( j = 0, \ldots, N \) and there are non nontrivial solutions of the above homogeneous problems which are bounded by \( e^{\delta t} \) at \( +\infty \). Hence, \( \tilde{a}_{\infty} \equiv 0 \), a contradiction. This completes the proof of the result.
We recall some well known result concerning harmonic extens ion of functions which are defined on $S^{N-1}$.

**Lemma 8** Given $\varphi \in C^{2,\alpha}(S^{N-1})$, we define $V_\varphi$ to be the unique harmonic extension of $\varphi$ in $B(0,1)$, namely

$$\begin{cases}
\Delta V_\varphi = 0 & \text{in } B(0,1) \\
V_\varphi = \varphi & \text{on } \partial B(0,1)
\end{cases}$$

(105)

Assume that $\varphi$ is $L^2(S^{N-1})$ orthogonal to $e_0, \ldots, e_N$, then

$$\|V_\varphi\|_{C^0(B(0,1) - \{0\})} \leq c \|\varphi\|_{C^0(S^{N-1})}$$

for some constant $c > 0$ which does not depend on $\varphi$.

Using the fact that Kelvin’s transform of an harmonic function $V$

$$W(x) = |x|^{2-N} V\left(\frac{x}{|x|}\right)$$

is harmonic, the above result translates into the :

**Lemma 9** Given $\varphi \in C^{2,\alpha}(S^{N-1})$, we define $W_\varphi$ to be the unique harmonic extension of $\varphi$ in $\mathbb{R}^N - B(0,1)$ which decays at $\infty$. Namely

$$\begin{cases}
\Delta W_\varphi = 0 & \text{in } \mathbb{R}^N - B(0,1) \\
W_\varphi = \varphi & \text{on } \partial B(0,1)
\end{cases}$$

(106)

and $W_\varphi$ tends to 0 at $\infty$. Assume that $\varphi$ is $L^2(S^{N-1})$ orthogonal to $e_0, \ldots, e_N$ then

$$\|W_\varphi\|_{C^0_\infty(\mathbb{R}^N - B(0,1))} \leq c \|\varphi\|_{C^0(S^{N-1})}$$

for some constant $c > 0$ which does not depend on $\varphi$.

From now on we assume that $\Omega$ is a bounded regular domain in $\mathbb{R}^N$.

### 8 Bubble tree solutions in general domains

As before, we only prove the case when $N \geq 6$ since the proof of the result when $N = 5$ follows the same lines with minor modifications. We recall

$$r_\varepsilon = \varepsilon^{\frac{2}{N^2-4}}.$$ 

We define the space

$$\mathcal{E} := \left\{ \varphi \in C^{2,\alpha}(S^{N-1}) : \int_{S^{N-1}} \varphi e_j \, d\omega = 0, \ j = 0, \ldots, N \ \text{and} \ \|\varphi\|_{C^2,\alpha} \leq r_\varepsilon \varepsilon^\frac{1}{2} \right\}.$$
8.1 Solution of the nonlinear problem in $\Omega_{\text{int},\varepsilon}$.

Given a $m$ functions $\varphi := (\varphi_1, \ldots, \varphi_m) \in \mathcal{E}^m$ and $m$ points $x := (x_1, \ldots, x_m) \in \Omega^m$, we construct a positive solution of problem (68) in $\Omega_{\text{int},\varepsilon}$ whose boundary is, in some sense, parameterized by $\varphi$. Namely we would like to solve

$$
\begin{align*}
\Delta u_{\text{int},i} + \lambda u_{\text{int},i} + u_{\text{int},i}^p &= 0 & \text{in } & B(x_i, r_\varepsilon) \\
u_{\text{int},i} &\in \text{Span} \{e_0, \ldots, e_N\} & \text{on } & \partial B(x_i, r_\varepsilon)
\end{align*}
$$

(107)

For each $i = 1, \ldots, m$, we denote by $V_{\varphi_i}$ the unique harmonic extension of $\varphi_i$ in $B(x_i, R_\varepsilon)$, namely

$$
\begin{align*}
\Delta V_{\varphi_i} &= 0 & \text{in } & B(x_i, r_\varepsilon) \\
V_{\varphi_i} &= \varphi_i & \text{on } & \partial B(x_i, r_\varepsilon)
\end{align*}
$$

(108)

It follows from Lemma 8, together with a scaling argument, that

$$
\|V_{\varphi_i}\|_{C^0_0(\Omega_{\text{int},\varepsilon})} \leq c r_\varepsilon^{-1} \|\varphi_i\|_{C^0(S^{N-1})}.
$$

(109)

We keep the notations of the previous sections and, we look for a positive solution of problem (68) in $B(x_i, r_\varepsilon)$ of the form

$$
u_{\text{int},i} = u_{p,\lambda,\xi_i}(-x_i) + V_{\varphi_i} + w_i
$$

(110)

where the function $u_{p,\lambda,\xi_i}$ is the radial solution of problem (68) which has been obtained in Proposition 7 and where the functions $w_i$ is small.

As usual, we introduce the polar coordinates $(t, \theta) \in (-\log r_\varepsilon, +\infty) \times S^{N-1}$ in each $B(x_i, r_\varepsilon)$. Given a function $v$, defined on $B(x_i, r_\varepsilon)$, we agree that the function $\tilde{v}$ is the function defined on $(-\log r_\varepsilon, +\infty) \times S^{N-1}$ which is determined by the relation

$$
v(x) = |x|^{\frac{2}{p-2}} \tilde{v}(-\log |x|, \theta).
$$

(111)

With these notations, we need to find a function $\tilde{u}_{\text{int},i}$ and $b_0, \ldots, b_N \in \mathbb{R}$ such that

$$
\partial_t^2 \tilde{u}_{\text{int},i} - a_p \partial_t \tilde{u}_{\text{int},i} - b_p \tilde{u}_{\text{int},i} + \Delta_{S^{N-1}} \tilde{u}_{\text{int},i} = -\lambda e^{-2t} \tilde{u}_{\text{int},i} - \tilde{u}_{\text{int},i}^p
$$

(112)

in $[- \log r_\varepsilon, +\infty) \times S^{N-1}$ and

$$
\tilde{u}_{\text{int},i}(- \log r_\varepsilon, \theta) = r_\varepsilon^\frac{2}{p-2} \varphi_i(\theta) + \sum_{j=0}^N b_j e_j
$$

on $S^{N-1}$.

We will obtain a solution of this equation as a fixed point for some contraction mapping. We fix $\delta \in \left(-\left(\frac{N}{2} + \frac{N}{N+2}\right), -\frac{N}{2}\right)$ such that $\left(\frac{2p}{p-1} + \delta - \frac{2}{N+2}\right) \frac{N+2}{2} > 2$ and we define

$$
\mathcal{E}_{\text{int},\varepsilon} := \left\{ \tilde{w} \in C_0^0([- \log r_\varepsilon, +\infty) \times S^{N-1}) : \|\tilde{w}\|_{C^0_0} \leq \kappa \varepsilon^{-\frac{1}{2} + \left(\frac{2p}{p-1} + \delta - \frac{2}{N+2}\right) \frac{N+2}{2}} \right\}
$$

(113)
where the parameter $\kappa > 0$ will be fixed later on.

We write (112) as
\[
\mathcal{L} \tilde{w}_i = -\lambda e^{-2t}(\tilde{w}_i + \tilde{V}_{\phi_i}) - Q_{\phi_i}(\tilde{w}_i)
\]  
(114) where the linear operator $\mathcal{L}$ is given by
\[
\mathcal{L} := \partial_t^2 + \Delta_{S^{N-1}} - a_p \partial_t - b_p + p \tilde{u}_{p,\lambda,\xi_i}^{p-1}
\]
and where $Q_{\phi_i}$ collects the nonlinear terms
\[
Q_{\phi_i}(\tilde{w}_i) := (\tilde{u}_{p,\lambda,\xi_i} + \tilde{V}_{\phi_i} + \tilde{w}_i)^p - \tilde{u}_{p,\lambda,\xi_i}^p - p \tilde{u}_{p,\lambda,\xi_i}^{p-1} \tilde{w}_i.
\]

We estimate
\[
\|\lambda e^{-2t} \tilde{V}_{\phi_i} \|_{C^2_0} \leq c \lambda \varepsilon^\frac{1}{2} r_\varepsilon^{-2} + 3 + \delta
\]  
(115)
and
\[
\|\lambda e^{-2t} \tilde{w} \|_{C^2_0} \leq c \lambda \kappa \varepsilon^{\frac{1}{2} + \frac{2p}{p-1} + \delta - \frac{2}{N+2}} \varepsilon^{\frac{1}{2} - \frac{2}{N+2}} r_\varepsilon^2.
\]  
(116)

In view of the asymptotic expansion of $\tilde{u}_{p,\lambda,\xi_i}$ we have obtained in Proposition 7, it is easy to check that, for all $\tilde{w} \in \mathcal{E}_{int,\varepsilon}$
\[
|\tilde{w}| \ll \tilde{u}_{p,\lambda,\xi_i}
\]
in $(-\log r_\varepsilon, +\infty) \times S^{N-1}$. Moreover, it follows from (109) that
\[
|\tilde{V}_{\phi_i}| \leq c r_\varepsilon^{-2} \|\phi_i\|_{L^\infty} e^{-\frac{2p}{p-1} t} \leq c r_\varepsilon^{-1} \varepsilon^{\frac{1}{2}} e^{-\frac{2p}{p-1} t},
\]  
(117)
in $(-\log r_\varepsilon, +\infty) \times S^{N-1}$. Hence, we conclude that
\[
|\tilde{V}_{\phi_i}| \ll \tilde{u}_{p,\lambda,\xi_i}.
\]  
(118)

Taylor’s expansion yields
\[(1 + t)^p - 1 - pt \leq ct^2
\]

near $t = 0$. This, together with the fact that $\delta < -\frac{2}{p-1}$, implies that
\[
\|Q_{\phi_i}(\tilde{w})\|_{C^2_0} \leq c \varepsilon^{\frac{1}{2} + \frac{2p}{p-1} + \delta - \frac{2}{N+2}} \varepsilon^{\frac{1}{2} - \frac{2}{N+2}} (1 + c_\kappa \varepsilon^{\frac{p-1}{p-1} - \frac{1}{N-2}}),
\]  
(119)
for some constant $c_\kappa > 0$ depending on $\kappa$. We have used the fact that $\frac{2p}{p-1} + \delta - \frac{2}{N+2} < 1$.

Gathering the previous estimates, we conclude that
\[
\| - \lambda e^{-2t} (\tilde{w} + \tilde{V}_{\phi_i}) - Q_{\phi_i}(\tilde{w}) \|_{C^2_0} \leq c (1 + c_\kappa \varepsilon^\gamma) \varepsilon^{\frac{1}{2} + \frac{2p}{p-1} + \delta - \frac{2}{N+2}} \varepsilon^{\frac{1}{2} - \frac{2}{N+2}}
\]  
(120)
where $c_\kappa > 0$ depends on $\kappa$ and the positive number $\gamma$ is independent of $p$.

Given $\tilde{w} \in \mathcal{E}_{int,\varepsilon}$ we use the result of Proposition 8 to solve
\[
\mathcal{L} \tilde{v} = -\lambda e^{-2t} (\tilde{w} + \tilde{V}_{\phi_i}) - Q_{\phi_i}(\tilde{w})
\]

It follows from Proposotion 8 and the above estimate that, given $\kappa$, there exists $\varepsilon_0 > 0$ (depending on $\kappa$) such that the mapping
\[
T_\varepsilon : \mathcal{E}_{int,\varepsilon} \to \mathcal{E}_{int,\varepsilon}
\]
defined by \(T_i(\tilde{w}) = \tilde{v}\) is well defined, provided \(\varepsilon \in (0, \varepsilon_0)\).

Moreover, for all \(\tilde{w}_1, \tilde{w}_2 \in \mathcal{E}_{int, \varepsilon}\), one can check that
\[
\|T_i(\tilde{w}_1) - T_i(\tilde{w}_2)\|_{C^0} \leq c \lambda \|\tilde{w}_1 - \tilde{w}_2\|_{C^0} + c \|Q_{\varphi_i}(\tilde{w}_1) - Q_{\varphi_i}(\tilde{w}_2)\|_{C^0} \quad (121)
\]

Consequently, for \(p\) sufficiently close to \(p_N\), the mapping \(T_i\) is a contraction from \(\mathcal{E}_{int, \varepsilon}\) into itself and hence admits a unique fixed point in this set. This yields a solution \(u_{int, i}\) of (107).

If we define the function \(u_{int}\) to be equal to \(u_{int, i}\) on \(B(x_i, r_\varepsilon)\), we have proven the:

**Proposition 9** Given \(x \in \Omega^m\) and \(\varphi \in \mathcal{E}^m\), there exists a positive solution \(u_{int}\) of (68) in \(\Omega_{int, \varepsilon}\) satisfying boundary conditions

\[
u_{int}|_{\partial B(x_i, r_\varepsilon)} - \varphi_i \in \text{Span}\{e_j : j = 0, \ldots, N\}
\]

for all \(1 \leq i \leq m\). Moreover, the sequence of solutions \(u_{int}\) blows up at each \(x_i\) as \(p\) tends to \(p_N\) in such a way that

\[
|\nabla u_{int}|^2 dx \rightarrow C^{(3)}_N \sum_{i=1}^m \ell_i \delta_{x_i}
\]

in the sense of measures. Here \(C^{(3)}_N\) is the constant defined in Theorem 1. Finally, this solution can be expanded as

\[
\begin{align*}
u_{int} = (\varepsilon)^{\frac{1}{2}} & \left[ \frac{\sqrt{C^{(4)}_N}}{2} e^{(N-2)\xi} + \frac{\sqrt{C^{(4)}_N}}{2} e^{(2-N)\xi} |x|^{2-N} - \frac{4\mu C^{(8)}_N}{(N-2)^2} e^{(N-6)\xi} \right] \\
& + V_{\varphi_i} + O(\varepsilon^{\frac{1}{2}} r_\varepsilon^2)
\end{align*}
\]

in \(B(x_i, 2r_\varepsilon) - B(x_i, r_\varepsilon/2)\).

Since we have found the solution of (68) with the form (110), we have

\[
-\Delta w_i = \lambda (w_i + V_{\varphi_i}) + (u_{p, \lambda, \zeta_i} + w_i + V_{\varphi_i}) - u^p_{p, \lambda, \zeta_i}
\]

so that in \(B(x_i, r_\varepsilon) - B(x_i, r_\varepsilon/2)\)

\[
|\Delta w_i| \leq |w_i + V_{\varphi_i}| + c u^p_{p, \lambda, \zeta_i} |w_i + V_{\varphi_i}| \leq c|w_i + V_{\varphi_i}|
\]

Using the standard elliptic theory, we have

\[
\|r_\varepsilon \nabla w_i\|_{L^\infty(B(x_i, r_\varepsilon) - B(x_i, 3r_\varepsilon/4))} \leq c \varepsilon^{\frac{1}{2}} (r_\varepsilon^3 + r_\varepsilon^{2 \delta} - \frac{2}{\delta + \frac{2}{p-1}})^{\frac{N+2}{2}} \left(\frac{2}{\delta - \frac{2}{N+2}}\right)^{\frac{N+2}{2}} > 2
\]

Recall
\[
\left(\frac{2}{p-1} + 2 + \delta - \frac{2}{N+2}\right)^{\frac{N+2}{2}} > 2
\]

Thus,

\[
\|r_\varepsilon \nabla w_i\|_{L^\infty(B(x_i, r_\varepsilon) - B(x_i, 3r_\varepsilon/4))} \leq c \varepsilon^{\frac{1}{2}} r_\varepsilon^2
\]

By the regularity theory, for all \(\alpha \in (0, 1)\),

\[
\|r_\varepsilon \partial_\alpha w_i\|_{C^{1, \alpha}(S^{N-1})} \leq c \varepsilon^{\frac{3}{2}} r_\varepsilon^2
\]

(123)
8.2 Solutions of the nonlinear problem in $\Omega_{ext,\varepsilon}$

Given a $m$ functions $\phi = (\phi_1, \ldots, \phi_m) \in \mathcal{E}^m$, we now construct a family of positive solution of (68) in $\Omega_{ext,\varepsilon}$ which in some sense is parameterized by $\phi$.

Let $\chi$ be a $C^\infty$ cut-off function defined in $\mathbb{R}^N$, such that $\chi|_{B(0,r_0)} \equiv 1$ and $\chi \equiv 0$ on $\mathbb{R}^N - B(0,2r_0)$ and $\chi \geq 0$. Denote by $W_{\phi_i}$ the unique harmonic extension of $\phi_i$ in $\mathbb{R}^N - B(x_i,r_\varepsilon)$ which decays at $\infty$. We look for a solution of (68) in $\Omega_{ext,\varepsilon}$ of the form

$$u_{ext} = \sum_{i=1}^m \left( A_i \chi^{\frac{1}{2}} G(\cdot, x_i) + \chi(\cdot - x_i) \left( W_{\phi_i} + \frac{a_i \cdot (\cdot - x_i)}{|\cdot - x_i|^N} \right) \right) + w_{ext}$$

where $a := (a_1, \ldots, a_m) \in (\mathbb{R}^N)^m$ and the function $w_{ext}$ is assumed to be small and to satisfy $w_{ext}|_{\partial\Omega_{ext,\varepsilon}} = 0$.

We use the maximum principle to reduce (68) to

$$\begin{cases}
-\Delta w_{ext} = \lambda w_{ext} + q + Q_{\Lambda,\phi,a}(w_{ext}) & \text{in } \Omega_{ext,\varepsilon} \\
w_{ext} = 0 & \text{on } \partial\Omega_{ext,\varepsilon}
\end{cases}$$

where

$$Q_{\Lambda,\phi,a}(w) = \sum_{i=1}^m \left( A_i \chi^{\frac{1}{2}} G(\cdot, x_i) + \chi(\cdot - x_i) \left( W_{\phi_i} + \frac{a_i \cdot (\cdot - x_i)}{|\cdot - x_i|^N} \right) \right) + w$$

and where the function $q$ is given by

$$q(z) = \sum_{i=1}^m \Delta \chi(z-x_i) \left( W_{\phi_i}(z) + \frac{a_i \cdot (z-x_i)}{|z-x_i|^N} \right)$$

$$+ 2 \sum_{i=1}^m \nabla \chi(z-x_i) \cdot \nabla \left( W_{\phi_i}(z) + \frac{a_i \cdot (z-x_i)}{|z-x_i|^N} \right)$$

$$+ \lambda \sum_{i=1}^m \left( A_i \chi^{\frac{1}{2}} G(z, x_i) + \chi(z-x_i) \left( W_{\phi_i}(z) + \frac{a_i \cdot (z-x_i)}{|z-x_i|^N} \right) \right)$$

Given $A_0$ and $\kappa > 0$, we define

$$\mathcal{G} = \{ \Lambda \in \mathbb{R}^m : |\Lambda| \leq A_0 \}$$

and

$$A_\varepsilon = \{ a \in (\mathbb{R}^N)^k : |a| \leq \varepsilon^{\frac{1}{2}} r_\varepsilon^N \},$$

Furthermore, given $\nu \in (2-N,3-N)$, we consider

$$\mathcal{E}_{ext,\varepsilon} = \{ w \in C^0_\nu(\Omega_{ext,\varepsilon}) : \|w\|_{C^0_\nu} \leq \kappa \varepsilon^{\frac{1}{2}} r_\varepsilon^{2-\nu} \text{ and } w|_{\partial\Omega_{ext,\varepsilon}} = 0 \},$$

For all $a \in A_\varepsilon$, $\Lambda \in \mathcal{G}$ and $\phi_j \in \mathcal{E}$, we estimate

$$\|q\|_{C^0_\nu(\Omega_{ext,\varepsilon})} \leq c \varepsilon^{\frac{1}{2}} r_\varepsilon^N$$

and given $w \in \mathcal{E}_{ext,\varepsilon}$, we obtain with little work

$$\|\lambda w\|_{C^0_{\nu-2}} \leq c \|\lambda w\|_{C^0_\nu} \leq c \varepsilon^{\frac{N-1}{N}} \|w\|_{C^0_\nu}$$
and 
\[ \|Q_{\Lambda, \phi, a}(w)\|_{C^0_{\nu-2}} \leq c (\Lambda_0^p + 1 + \kappa^p \varepsilon^{\frac{p}{2}}) \varepsilon^{\frac{1}{2}} r^{-\nu+2} \] (131)

Finally, we estimate for all \( w_1, w_2 \in E_{ext, \varepsilon} \)
\[ \|Q_{\Lambda, \phi, a}(w_1) - Q_{\Lambda, \phi, a}(w_2)\|_{C^0_{\nu-2}} \leq c \varepsilon^2 N^2 \|w_1 - w_2\|_{C^0_{\nu}} \] (132)

The following result is standard

**Lemma 10** Assume that \( \nu \in (2 - N, 0) \) then for all \( f \in C^0_{\nu-2}(\Omega_{ext, \varepsilon}) \), there exists \( w \in C^0_{\nu}(\Omega_{ext, \varepsilon}) \) unique solution of
\[ \begin{cases} 
\Delta w = f & \text{in } \Omega_{ext, \varepsilon} \\
w = 0 & \text{on } \partial \Omega_{ext, \varepsilon}.
\end{cases} \] (133)

Furthermore, there holds
\[ \|w\|_{C^0_{\nu}} \leq c \|f\|_{C^0_{\nu-2}}. \]

**Proof.** The existence of \( w \) is straightforward and the estimate relies on the fact that \( x \to |x - x_i|^\nu \) can be used as a barrier in \( B(x_i, r_\varepsilon) - B(x_i, r_\varepsilon) \).

We define the map 
\[ T_{\Lambda, \phi, a} : E_{ext, \varepsilon} \rightarrow E_{ext, \varepsilon} \]
by \( T_{\Lambda, \phi, a}(w) := v \) where \( v \) is the solution of 
\[ \Delta v = \lambda w + q + Q_{\Lambda, \phi, a}(w). \]

Given \( \kappa > 0 \), it follows from the estimates (129), (130) and (131) that the mapping \( T_{\Lambda, \phi, a} \) is well defined and is a contraction, provided \( \varepsilon \) is chosen small enough, say \( \varepsilon \in (0, \varepsilon_0) \).

In particular, this mapping has a unique fixed point in \( E_{ext, \varepsilon} \) which yields a solution of (125). Therefore, we have proved the following :

**Proposition 10** Given \( x \in \Omega^m \), \( a \in (\mathbb{R}^N)^m \) and \( \phi \in E^m \), there exists \( u_{ext} \) positive solution of equation (68) in \( \Omega_{ext, \varepsilon} \), satisfying
\[ u_{ext} = \phi_i + \frac{a_i \cdot (\cdot - x_i)}{r_\varepsilon^N} + \sum_{j=1}^{m} \Lambda_i \varepsilon^{\frac{1}{2}} G(\cdot, x_i) \]
on \( \partial B(x_i, r_\varepsilon) \) for all \( 1 \leq i \leq m \) and \( u_{ext} = 0 \) on \( \partial \Omega \). Furthermore, the function \( u_{ext} \) can be expanded as
\[ u_{ext} = W_{\phi_i} + \frac{a_i \cdot (\cdot - x_i)}{r_\varepsilon^N} + \sum_{j=1}^{m} \Lambda_i \varepsilon^{\frac{1}{2}} G(\cdot, x_i) + O(\varepsilon r_\varepsilon^2) \]
in \( B(x_i, 2r_\varepsilon) - B(x_i, r_\varepsilon/2) \).

Similarly,
\[ \|r_\varepsilon \partial_n w_{ext}\|_{C^{1, \alpha}(S_{\Omega_{ext}})} \leq C \varepsilon^{\frac{1}{2}} r_\varepsilon^2 \] (134)
where \( n \) is the outside unit normal vector on the boundary of \( B(x_i, r_\varepsilon) \). In the following consideration we will fix some \( \alpha \in (0, 1) \).
8.3 The cauchy data mapping

We explain how the free parameters in Proposition 9 and Proposition 10 can be chosen so that the functions $u_{int,i}$ and $u_{ext}$ can be glued together to obtain a positive solution of problem (68) in $\Omega$.

We set $\xi = (\xi_1, \ldots, \xi_m)$. We want to choose the suitable parameters

$$\Xi := (x, \Lambda, \varphi, \phi, a, \xi)$$

so that $u_{int,i}$ and $u_{ext}$ have the same Cauchy data on each $\partial B(x_i, r_\varepsilon)$. Once this is done, the function defined by $u = u_{int,i}$ in $B(x_i, r_\varepsilon)$ and $u = u_{ext}$ in $\Omega_{ext, \varepsilon}$ will be $C^1$ and solution of (68) away from the $\partial B(x_i, r_\varepsilon)$. Elliptic regularity theory will then imply that it is a solution in $\Omega$. Moreover, it will follow from the construction itself that $u$ has the desired behavior near each $x_i$ and this will complete the proof of Theorem 1.

Therefore, it remains to solve, for all $i = 1, \ldots, m$, the system

$$
\begin{align*}
\begin{cases}
\ u_{int,i} & = u_{ext}, \\
\ \partial_n u_{int,i} & = \partial_n u_{ext},
\end{cases}
\end{align*}
$$

(135)
on $\partial B(x_i, r_\varepsilon)$.

We denote by $\Pi_j$ the $L^2(S^{n-1})$-projection onto $\text{Span}\{e_j\}$, and

$$\Pi(\phi) := \phi - \sum_{j=0}^{N} \Pi_j(\phi)$$

For all $i = 1, \ldots, m$, the $L^2(S^{n-1})$-projection of (135) over the orthogonal complement of $\text{Span}\{e_0, \ldots, e_N\}$ yields the system of equations

$$
\begin{align*}
\varphi_i &= \phi_i + F_{i,1}(\Xi), \\
\ r_\varepsilon \partial_n V_{\phi_i} &= r_\varepsilon \partial_n W_{\phi_i} + F_{i,2}(\Xi),
\end{align*}
$$

(136)
Next, we use the expansions of Lemma 1, Corollary 3 and Corollary 4 to obtain the $L^2(S^{n-1})$-projection of (135) over $\text{Span}\{e_0\}$

$$
(\ell_i C_N^{(4)} \varepsilon)^{1/2} \left(\frac{r_\varepsilon^{2-N} e^{-(N-2)\xi_i/2}}{2} + \frac{e^{(N-2)\xi_i/2}}{2} - \frac{4\ell_i^{\frac{2}{N-2}} C_N^{(8)} \mu e^{(N-6)\xi_i/2}}{\ell_i C_N^{(4)} (N-2)^2}\right) = \varepsilon^{1/2} \Lambda_i \left(r_\varepsilon^{2-N} - H(x_i, x_i)\right) + \varepsilon^{1/2} \sum_{\ell \neq i} \Lambda_i G(x_i, x_i) + F_{i,3}(\Xi),
$$

(137)
Finally, the $L^2(S^{n-1})$-projection of (135) over $\text{Span}\{e_1, \ldots, e_N\}$ yields

$$
\begin{align*}
\ r_\varepsilon^{1-N} a_i + \varepsilon^{1/2} \left(\left.\sum_{\ell \neq i} \Lambda_{i} \nabla_z G(x_i, x_i) - r_\varepsilon \Lambda_i \nabla_z H(x_i, x_i)\right\} = F_{i,5}(\Xi), \\
\ r_\varepsilon^{1-N} a_i (1 - N) + \varepsilon^{1/2} \left(\left.\sum_{\ell \neq i} \Lambda_{i} \nabla_z G(x_i, x_i) - r_\varepsilon \Lambda_i \nabla_z H(x_i, x_i)\right\} = F_{i,6}(\Xi),
\end{align*}
$$

(138)

Here $F_{i,l}(\Xi)$ for $i = 1, \ldots, m$ and $l = 1, \ldots, 6$ are continuous maps satisfying

$$|F_{i,l}(\Xi)| = O(\varepsilon^{\frac{1}{2}r^2_\varepsilon}). \quad (139)$$

We define "Dirichlet to Neumann map" for any

$$S : \Pi(C^{2,\alpha}(S^{N-1})) \rightarrow \Pi(C^{1,\alpha}(S^{N-1}))$$

by

$$S(\psi) = r_\varepsilon (\partial_n V_\psi - \partial_n W_\psi),$$

where $V_\psi$ (resp. $W_\psi$) is the harmonic extension in the ball $B(0, r_\varepsilon)$ (resp. in $\mathbb{R}^N - B(0, r_\varepsilon)$) defined in Lemma 8 and Lemma 9. It is well known that $S$ is an isomorphism [11] the norm of whose inverse does not depend on $\varepsilon$.

Hence, (136) (137) and (138) are equivalent to the following system

$$\varphi_i = G_{i,1}(\Xi), \quad \phi_i = G_{i,2}(\Xi), \quad \xi_i = -\frac{2}{N-2} \log \left( \frac{2\Lambda_i}{(\ell_i C_N^{(4)})^{\frac{1}{2}}} \right) + \varepsilon^{-\frac{1}{2}} r_\varepsilon^{N-2} G_{i,3}(\Xi),$$

$$a_i = r_\varepsilon^{N-1} G_{i,4}(\Xi),$$

$$\nabla_{x_i} F_\mu(x, \Lambda) = \varepsilon^{-\frac{1}{2}} r_\varepsilon G_{i,5}(\Xi),$$

$$\nabla_{\Lambda} F_\mu(x, \Lambda) = \varepsilon^{-\frac{1}{2}} G_{i,6}(\Xi),$$

where $G_{i,l}(\Xi)$ for all $l = 1, \ldots, 6$ and for all $i = 1, \ldots, m$ are continuous maps satisfying

$$|G_{i,l}(\Xi)| = O(\varepsilon^{\frac{1}{2}r^2_\varepsilon})$$

and

$$C_N^{(1)} = \frac{2^{4-N} C_N^{(8)}}{(N-2)(C_N^{(4)})^{\frac{1}{N-2}}}, \quad C_N^{(2)} = \frac{C_N^{(4)}}{2}.$$  

Moreover, elliptic regularity Theory shows that all $G_{i,l}(\Xi)$ are compact operators.

Assume that $(x^0, \Lambda^0)$ is a non degenerate critical point of $F_\mu$. In particular, this implies that $dF_\mu$, evaluated at this point, is a local diffeomorphism from a neighborhood of $(x^0, \Lambda^0)$ on a neighborhood of 0 in $\mathbb{R}^{m(N+1)}$. Using this we can write formally the system (140) as

$$\Xi = \Phi(\Xi),$$

We set $\xi_i^0 := -\frac{2}{N-2} \log \left( \frac{2\Lambda_i^0}{(\ell_i C_N^{(4)})^{\frac{1}{2}}} \right)$ for all $i = 1, \ldots, m$. We consider the set

$$A = \overline{B((x^0, \Lambda^0), \varepsilon_1)} \times \varepsilon^{2k} \times A_\varepsilon \times \overline{B(\xi^0, \varepsilon_1)},$$

where $\varepsilon_1$ is some fixed small positive number. It follows from the above analysis that $\Phi : A \rightarrow A$ is a continuous compact map. According to Schauder fixed point theorem, $\Phi$ has a fixed point in $A$. This completes the proof of Theorem 1.

**Remark 0** If $N = 4$, we have the similar results. In this case, we take $\lambda = \mu \frac{1}{\log_{1/\varepsilon}}$.  

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