Corrections to scaling in the critical theory of deconfined criticality

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Inspired by recent conflicting views on the order of the phase transition from an antiferromagnetic Néel state to a valence bond solid, we use the functional renormalization group to study the underlying quantum critical field theory which couples two complex matter fields to a non-compact gauge field. In our functional renormalization group approach we only expand in covariant derivatives of the fields and use a truncation in which the full field dependence of all wave-function renormalization functions is kept. While we do find critical exponents which agree well with some quantum Monte Carlo studies and support the scenario of deconfined criticality, we also obtain an irrelevant eigenvalue of small magnitude, leading to strong corrections to scaling and slow convergence in related numerical studies.

I. INTRODUCTION

The archetypical example for deconfined criticality in condensed matter systems is the quantum phase transition from a Néel to valence bond solid (VBS) state in two-dimensional antiferromagnetic spin systems on the square lattice. As the Néel and VBS states break distinct symmetries, a generic continuous phase transition between these states is not possible within the widely used Landau-Ginzburg-Wilson framework. However, as was pointed out by Senthil et al., order parameter fluctuation effects induce a competing type of order by destroying the geometric phase factors and a condensation of these defects. This is the quantum phase which necessitates the gauge field. For general $N$, the action of our field theory is given by

$$S[\varphi, A] = \int d^Dx \left[ |(-i \nabla + A)\varphi_a|^2 + r|\varphi_a|^2 + \frac{\lambda}{2}(|\varphi_a|^2)^2 + \frac{1}{4e^2} F_{\mu\nu}^2 \right]. \quad (2)$$

Here, $x$ refers to Euclidean space-time, $F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}$ is the electromagnetic field tensor and a summation over repeated indices is implied. For sufficiently large $r$, the fields $\varphi_a$ are gapped and the gauge field $A$ is massless, corresponding to a Coulomb phase with a free photon. However, as $r$ is reduced below a critical value $r_c$, the condensation of any of the fields $\varphi_a$ leads to a finite mass of the gauge field and the loss of the photon. This is the well-known Higgs transition. In the context of deconfined criticality, $r$ needs to be fine-tuned to $r_c$ to obtain the scale-invariant critical theory, describing the quantum critical point of the Néel to VBS (or spin liquid) transition. While monopole operators turn out to be irrelevant at the quantum critical point separating the Néel from the VBS state and thus do not have to be included in the above theory, these operators are in fact a relevant perturbation at the Coulomb phase fixed point, turning the spin liquid phase unstable towards the onset of VBS order.

The gauge theory described by Eq. (2) has a long history with many twists and turns. In a seminal paper from the 70s, Halperin, Lubensky, and Me一所 showed within a Wilsonian renormalization group (RG) by expanding in $\epsilon = 4 - D$ and extrapolating to $D = 3$ that only for $N \geq 183$ there is a non-trivial fixed point, implying the possibility for a continuous phase transition at a non-zero charge $e^2$. However, no such critical value for $N$ was found within a next-to-leading order $1/N$ expansion. The case of just one complex matter field ($N = 1$) was of most interest in those days and it was later shown by Dasgupta and Halperin using a duality analysis that the theory with $N = 1$ lies in the inverted $XY$ universality class. As a consequence, for $N = 1$, the theory defined by Eq. (2) can undergo a continuous phase transition.
In fact, as $D$ is reduced from $D = 4$, the critical number of complex field components $N_c(D)$ above which there exists a stable charged fixed point decreases rapidly\cite{13} and reaches $N_c(2) = 0$ for $D = 2$\cite{20}.

With the advent of ideas of deconfined criticality, the theory with $N = 2$ moved center-stage. While the most recent studies (of lattice realizations) of Eq. (2) and related spin models such as the sign-problem free $J - Q$ model seem to favor the possibility of a continuous phase transition at a non-zero charge, as hypothesized by the theory of deconfined criticality\cite{21,22,28,29}, other studies are undecided\cite{28,29} or report weakly first-order phase transitions\cite{14,15,16,40,41}. Unexpected corrections to scaling were reported by Sandvik\cite{26} and it is our aim here to understand the critical theory and its corrections to scaling from an RG perspective.

\section{Functional Renormalization Group Approach}

In the following, we would like to study the field theory given by Eq. (2) using functional renormalization group (FRG) methods\cite{35,36,37,38,39}. Following earlier FRG studies\cite{15,16,40,41}, we work in the background field formalism which makes it possible to use a gauge-invariant formulation while at the same time fixing a gauge. The central quantity for which there exists an exact flow equation is the effective average action $\Gamma_\ell[\phi, A; A]$, which is explicitly gauge-invariant under a simultaneous gauge transformation of $\phi \equiv \langle \phi \rangle$ and both gauge fields $A \equiv \langle A \rangle$ and $\bar{A}$. Here, $\bar{A}$ is the classical background gauge field. The averages are defined with respect to the action given in Eq. (2) in the presence of sources and regulator terms. We have parametrized the cutoff scale of the infrared regulator $\Lambda = \Lambda_0 e^{-\ell}$ in terms of the RG time $\ell$. Continuously removing these regulator terms by increasing $\ell$ from its initial value $\ell = 0$, the effective average action assumes a complicated functional field dependence and becomes the generating functional of irreducible vertices. To make progress, some approximations are necessary. Here, we use a derivative expansion in which we expand in (covariant) gradients of $\phi$ and in $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The ansatz for our truncation reads as

$$\Gamma_\ell[\phi, A] = \int d^dx \left[ Z_\ell(\rho)|(-i\nabla + A)\phi_b|^2 + \frac{1}{2} Y_\ell(\rho)|\nabla\rho|^2 + U_\ell(\rho) + \frac{Z_{\rho}^A(\rho)}{4\epsilon^2} F_{\mu\nu}^2 \right], \quad (3)$$

where the (gauge-invariant) density $\rho = \phi_b^\dagger \phi_b$ is a function of $x$. In contrast to previous work, we keep the full functional dependence of the coupling functions $Z_\ell(\rho)$, $Y_\ell(\rho)$, $Z_{\rho}^A(\rho)$, and $U_\ell(\rho)$. Doing so, we also include some momentum-dependence of the four-point vertex and higher-order vertices. One of the simplest such terms couples to $|\phi_a|^2 |(-i\nabla + A)\phi_b|^2$. Let us note that we do not include this term.

The flow equations for the coupling functions are quite involved (see Appendix A\cite{A}, but do have a simple diagrammatic interpretation, as depicted in Fig. 1. In contrast to perturbation theory, propagators and vertices appearing on the right-hand side of the flow equations are renormalized quantities, involving all powers of interactions in a non-perturbative way and are also non-trivial functions of the density $\rho$.

In order to discuss possible fixed-point properties, it is

\begin{align*}
\partial_{\ell} U(\rho) &= \begin{array}{c}
\includegraphics[width=0.2\textwidth]{fig1a}
\end{array}, \\
\partial_{\ell} \bar{Z}(\rho) &= \begin{array}{c}
\includegraphics[width=0.2\textwidth]{fig1b}
\end{array}, \\
\partial_{\ell} \bar{Z}(\rho) &= \begin{array}{c}
\includegraphics[width=0.2\textwidth]{fig1c}
\end{array}, \\
\partial_{\ell} \bar{Z}^A(\rho) &= \begin{array}{c}
\includegraphics[width=0.2\textwidth]{fig1d}
\end{array},
\end{align*}
Here, $Z^0_\ell$ is defined as the wave-function renormalization factor evaluated at the characteristic density $\rho^*_c,\ell$, i.e. $Z^0_\ell = Z_\ell(\rho^*_c,\ell)$. For convenience, we choose the corresponding rescaled density $\tilde{\rho}^*_c = Z^0_\ell \Lambda^{2-D} \rho^*_c,\ell$ to be $\ell$-independent and equal to the position of the minimum of the rescaled effective potential $u_\ell(\tilde{\rho})$ at criticality. It should be noted that by construction $z_\ell(\tilde{\rho}^*_c) = z_\ell(\rho^*_c) = 1$. The anomalous dimension $\eta_\ell$ of the $\phi$ field and the anomalous dimension $\eta_{A,\ell}$ of the gauge field $A$ are related to the flow of $Z^0_\ell$ and $1/\tilde{c}_\ell^2$ by

$$
\eta_\ell = \partial_\ell \ln Z^0_\ell,
\eta_{A,\ell} = \partial_\ell \ln \left(1/\tilde{c}_\ell^2\right).
$$

As we are using the FRG within the background field formalism, both anomalous dimensions are gauge-invariant quantities. In particular, $\eta$ is the anomalous dimension of a gauge-invariant two-point correlation function of the $\phi$ field at criticality.

### III. RESULTS

Using our above truncation, we obtain a set of partial integro-differential equations (see Appendix B) which we turn into a set of ordinary differential equations by choosing a finite mesh for the rescaled density $\tilde{\rho}$. At the beginning of our flow in the ultraviolet, $\Gamma_{\ell=0}[\phi, A]$ is identical to the bare action given in Eq. (2) and thus completely fixed by the three dimensionless couplings $\tilde{r}_0$, $\tilde{\lambda}_0$, and $\tilde{c}_0^2$, corresponding to $r$, $\lambda$, and $c^2$ in Eq. (2). Choosing $\tilde{\lambda}_0$ and $\tilde{c}_0^2$ to be not too large and positive, it turns out to be always possible to fine-tune $\tilde{r}_0$ such that we approach a non-trivial fixed point at a finite charge in the limit of large $\ell$. At this critical point, our coupling functions $w^*(\tilde{\rho}) \equiv du^*(\tilde{\rho})/d\tilde{\rho}$, $z^*(\tilde{\rho})$, $\tilde{z}^*(\tilde{\rho})$, and $z^{A*}(\tilde{\rho})$ assume a non-trivial form, as shown graphically for $N = 2$ in Fig. 2.

Most interestingly, the wave-function renormalization functions $z^*(\tilde{\rho})$, $\tilde{z}^*(\tilde{\rho})$, and $z^{A*}(\tilde{\rho})$ are not constant at all and $z^{A*}(\tilde{\rho})$ even vanishes in the limit of large rescaled densities $\tilde{\rho}$. However, as we increase the number of complex fields $N$, the wave-function renormalization functions become more and more flat and our results are consistent with previous FRG calculations for large $N$. It should be noted that in previous FRG calculations a first order transition was reported within a derivative expansion for small $N$ and a continuous transition was only found when truncating the effective potential at fourth or eighth order in $\phi$ around its minimum $\tilde{\rho}_0$. It is therefore reassuring to see that our truncation involving the full functional dependence of both the effective potential and the wave-function renormalization functions leads to a continuous transition for both $N = 1$ and $N = 2$.

Starting from a small but non-zero charge and fine-tuning $\tilde{r}_0$ to reach criticality, the flow towards the charged fixed point is shown for $N = 1$ in Fig. 3. In addition to the flowing charge and the flowing anomalous dimensions of the matter and gauge field, as given by

![FIG. 2. (Color online) Derivative $w^*(\tilde{\rho}) = du^*(\tilde{\rho})/d\tilde{\rho}$ of the rescaled effective potential and field-dependent wave-function renormalization functions $z^*(\tilde{\rho})$, $\tilde{z}^*(\tilde{\rho})$, and $z^{A*}(\tilde{\rho})$ at the charged critical point for $N = 2$.](image1)

![FIG. 3. (Color online) Evolution of the rescaled charge $\tilde{c}_\ell^2$, the anomalous dimensions of the matter and gauge field $\eta_\ell$ and $\eta_{A,\ell}$, the coupling constant $\tilde{\lambda}_\ell$ and $\tilde{r}_0$ to the value $\tilde{r}_0$ leading to criticality. Choosing $\tilde{r}_0 = -\tilde{\lambda}_0$, we obtain an exponential runaway flow for large $\ell$, as indicated by the light-colored lines.](image2)
Eqs. (10)–(12), we also show the flowing position $\tilde{\lambda}_t$ for initial values chosen as in Fig. 2 but for $N = 2$, at and near criticality. In contrast to the case $N = 1$, the critical point is approached much more slowly.

FIG. 4. (Color online) Evolution of $\tilde{e}^2_t$, $\eta_c$, $\eta_{A,t}$, $\tilde{\rho}_t^*$, and $\tilde{\lambda}_t$, for initial values chosen as in Fig. 2, for $N = 2$, at and near criticality. In contrast to the case $N = 1$, the critical point is approached much more slowly.

IV. CONCLUSIONS

In summary, we have used functional renormalization group methods to study the critical field theory of deconfined criticality, as emerging in the continuous Néel to VBS transition in a class of two-dimensional spin systems. In contrast to previous functional renormalization group studies of the same field theory, we have used a truncation in which the complete field dependence of all wave-function renormalization functions is kept and an expansion only in gradient terms is made. Our results are consistent with some recent (quantum) Monte Carlo calculations and support the scenario of deconfined criticality. However, in contrast to lattice calculations where extrapolation to the infinite system size is an issue and critical properties of the underlying continuum field theory are difficult to address, using functional renormalization group methods it is possible to directly work in the continuum and with an infinite system. In particular, we can start with a very small charge and still reach a critical point for which we determine critical exponents and have also access to irrelevant eigenvalues. Interestingly, the dominant correction-to-scaling exponent is much smaller for $N = 2$ than for $N = 1$ which explains slow convergence in related numerical studies with system size.

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Appendix A: FRG flow equations

The central object of our functional renormalization group (FRG) study is the effective average action which is defined as the Legendre transform of the generating functional of connected Green functions. Roughly speaking, a regulator $R_\ell$ is introduced to give all fluctuations with momenta $|q| \lesssim \Lambda = \Lambda_0 e^{-\frac{1}{2}}$ an artificial mass. The effective average action then contains only quantum fluctuations with momenta larger than the infrared cutoff $\Lambda$. As the regulator is removed during the evolution of the flow, the effective average action turns into the generating functional of one-particle irreducible Green functions.

In this work, we use the FRG for the Legendre effective average action in the background field formalism. The background field formalism has the advantage of allowing for a gauge-invariant formulation while at the same time fixing a gauge and including a regulator. This, however, comes at the price of having to split the dynamic gauge field $A$ into a non-quantized background field $\bar{A}$ and a fluctuating field $a$ by setting $A = \bar{A} + a$. Using a gauge-fixing condition which only involves the combination $A - \bar{A}$ and coupling only this combination to external sources, one can derive an effective average action $\Gamma_{\ell}[\varphi, A; \bar{A}]$ via a Legendre transformation which is explicitly gauge-invariant under a simultaneous gauge transformation of $\phi = \langle \varphi \rangle$ and both gauge fields $A = \langle A \rangle$ and $\bar{A}$. The averages here are averages with respect to the action given in Eq. (2) in the presence of sources and regulator terms. While it is possible to write an exact flow equation for $\Gamma_{\ell}[\varphi, A; \bar{A}]$, one finally would like to eliminate the background field $\bar{A}$ by identifying it with $A$. The main problem in doing so stems from the fact that the functional derivatives with respect to $A$ and $\bar{A}$ do not coincide. Partially, this difference can be absorbed by introducing the gauge-invariant normalization factor $C_{\ell}[\varphi, A]$, which vanishes for $A = 0$ and in both limits $\Lambda \to 0$ and $\Lambda \to \infty$. This term is discussed in detail in Ref. 40. Using this strategy, Reuter and Wetterich define a gauge-invariant effective average action $\Gamma_{\ell}[\varphi, A]$, satisfying the approximate flow equation

$$
\frac{\partial}{\partial \ell} \Gamma_{\ell}[\varphi, A] = \frac{1}{2} \mathrm{Tr} \left[ \frac{\partial}{\partial \ell} \mathcal{R}_{\ell}[A] \left( \Gamma_{\ell}^{(2)}[\varphi, A] + \Gamma_{gf}^{(2)} + \mathcal{R}_{\ell}[A] \right)^{-1} \right] + \frac{\partial}{\partial \ell} C_{\ell}[\varphi, A],
$$

(A1)

Here, $\Gamma_{\ell}^{(2)}[\varphi, A]$ is the matrix of second functional derivatives of $\Gamma_{\ell}[\varphi, A]$ with respect to the fields, $\Gamma_{gf}^{(2)}$ is the corresponding matrix of the gauge-fixing potential, and $\mathcal{R}_{\ell}[A]$ is the regulator in matrix form. In addition to an integration over momentum space, the trace involves a sum over all internal degrees of freedom, i.e.

$$
\mathrm{Tr} \left[ \ldots \right] = \int \frac{d^D k}{(2\pi)^D} \sum \left[ \ldots \right].
$$

(A2)

For the complex fields $\phi = (\phi_1 \ldots \phi_N)$, the sum runs over the $N$ components $a = 1 \ldots N$ and contains also the label $i = 1, 2$, distinguishing the real and imaginary part of $\phi_a = (\phi_{a,1} + i\phi_{a,2})/\sqrt{2}$. As concerns the gauge field $A = (A_1 \ldots A_D)$, the sum just runs over $\mu = 1 \ldots D$. Finally we will be interested in the physical case $D = 2 + 1 = 3$. For the gauge-fixing potential, we follow previous work $25 \text{--} 29$ and choose $\Gamma_{gf}[A; \bar{A}] = \frac{1}{2\alpha} \int d^D r \left[ \nabla \cdot (A - \bar{A}) \right]^2$.

(Taking the limit $\alpha \to 0$ amounts to the background Landau gauge $\nabla \cdot (A - \bar{A}) = 0$.)

(A3)

Of course, Eq. (A1) involves an infinite set of operators and cannot be solved exactly. To make progress, we use a derivative expansion in which we keep the full functional dependence of the functions $Z_{\ell}(\rho)$, $Y_{\ell}(\rho)$, $Z_{gf}(\rho)$, and $U_{\ell}(\rho)$ entering our ansatz for the effective average action, as given in Eq. (2) of the paper. In order to determine the flow of these coupling functions, we expand the right-hand side of Eq. (A1) in deviations from the space-independent field configuration $\phi_{a,i}(r) = \phi_{a,i}, A(r) = 0$. This is facilitated by the inherent $U(1) \times SU(N)$ symmetry of the effective average action which allows us to choose $\phi_{a,1} = \sqrt{2}\rho$ and $\phi_{a,2} = 0$ for all other components. Projecting the flow of $\Gamma_{\ell}[\varphi, A]$ onto the flow of $U_{\ell}(\rho)$ by considering a space-independent field configuration, we thereby obtain with $\int_k \equiv \int d^D k/(2\pi)^D$

$$
\partial_{\ell} U_{\ell}(\rho) = \frac{1}{2} \int_k \left( \partial_{\ell} R^0_{\ell}(k^2) \left[ G^\ell_{\ell}(k^2; \rho) \right] + (2N - 1)G^T_{\ell}(k^2; \rho) \right) + (D - 1) \partial_{\ell} R_{gf}(k^2) \left[ G^A_{\ell}(k^2; \rho) \right],
$$

(A5)

where

$$
G^\ell_{\ell}(k^2; \rho) = \frac{1}{|Z_{\ell}(\rho) + \rho Y_{\ell}(\rho)|^2} + U_{\ell}(\rho)^2 + 2\rho U_{\ell}(\rho) + R^0_{\ell}(k^2),
$$

(A6)

are the longitudinal and transverse propagators for the given field configuration and $G^A_{\ell}(k^2; \rho)$ is defined in terms of the gauge-field propagator $G^A_{\ell,\mu,\nu}(k^2; \rho)$ by

$$
G^A_{\ell,\mu,\nu}(k^2; \rho) = (\delta_{\mu,\nu} - k_\mu k_\nu/k^2) G^\ell_{\ell}(k^2; \rho) + \frac{\delta_{\mu,\nu} - k_\mu k_\nu/k^2}{(Z_{\ell}(\rho)/c^2)k^2 + 2\rho Z_{\ell}(\rho) + R^A_{\ell}(k^2)}.
$$

(A8)

The regulator functions $R^0_{\ell}(k^2)$ and $R^A_{\ell}(k^2)$ will be specified below. Following a standard recipe $25 \text{--} 29$, we can also derive flow equations for the wave-function renormalization factors $Z_{\ell}(\rho)$, $Z_{\ell}(\rho) = Z_{\ell}(\rho) + \rho Y_{\ell}(\rho)$, and $Z_{gf}(\rho)/c^2$:...
\[ \partial_\ell Z_\ell (\rho) = \frac{\partial}{\partial p^2} \left[ \frac{1}{2} \tilde{\partial}_\ell \int_\ell \right. \\
\left. \sum (G_\ell^{T,T,L,L}(-p, p, -k, k; \rho)G_\ell^L(k^2; \rho) \right. \\
\left. + \sum (p, p, -k, k; \rho)G_\ell^T(k^2; \rho) + (2N-2)\right) \\
\left. + \frac{1}{2} \tilde{\partial}_\ell \left[ \Gamma_{\ell \mu \nu}^{A,A,T,T}(\nu, \rho) \right. \\
\left. - \sum (\ell, \ell, \ell, \ell, \ell; \mu, \nu) \right) \\
\left. - \frac{1}{2} \tilde{\partial}_\ell \left[ \sum (\ell, \ell, \ell, \ell; \mu, \nu) \right) \\
\left. \right] \right] \]

Within a derivative expansion, it is customary to evaluate these wave-function renormalization factors in the limit \( p^2 \to 0 \). In contrast to \( \partial_\ell \), the partial derivative \( \tilde{\partial}_\ell \)

appearing above acts only on regulator terms to its right-hand side, e.g. when acting on a propagator \( G_\ell^L(k^2; \rho) \),

this propagator is replaced by the corresponding single-scale propagator

\[ \tilde{\partial}_\ell G_\ell^L(k^2; \rho) = - \left[ G_\ell^L(k^2; \rho) \right] \frac{1}{2} \partial_\ell R_\ell(k^2). \] (A12)

Note that due to the residual gauge field dependence of
the regulator \( \mathcal{R}_\ell[A] \), the first derivative
\[
R_\ell^{(n)}(0) = \frac{d}{dk^2} R_\ell^{(n)}(k^2) \bigg|_{k^2=0} \quad (A13)
\]
appears on the right-hand-side of Eq. (A11). In principle, there
is also a term containing the second derivative of \( R_\ell^{(n)}(k^2) \),
but this term vanishes for the Litim cutoff which we will use in
Appendix B. The Feynman diagrams corresponding to the right-hand-sides of the flow equations of the effective potential \( U_\ell(\rho) \) and the wave-function renormalization factors \( Z_\ell(\rho) \), \( \tilde{Z}_\ell(\rho) \), and \( \tilde{Z}_\ell^2(\rho)/\epsilon_\ell^2 \), as given by Eqs. (A5) and (A9–A11), are depicted in Fig. 1. As the number of transverse channels entering a given vertex always has to be even, there is no term in the flow of \( Z_\ell(\rho) \) corresponding to the last term on the right hand side of Eq. (A10). We note that the last term in Eq. (A11) results from the normalization factor \( C_\ell[\phi, A] \) and is discussed in detail in Ref. (40) Taking derivatives of the effective average action with respect to the fields, we obtain all vertices appearing above.

\[
\begin{align*}
\Gamma^{L,L,L,L}_{\ell}(p, p, -k; \rho) &= 3U''_\ell(\rho) + 12\rho U'''_\ell(\rho) + 4\rho^2 U''''_\ell(\rho) \\
&+ (k^2 + p^2) \left[ Z'_\ell(\rho) + 2\rho Z''_\ell(\rho) + 2\tilde{Y}_\ell(\rho) + 10\rho Y'_\ell(\rho) + 4\rho^2 Y''_\ell(\rho) \right], \\
\Gamma^{L,L,T,T}_{\ell}(p, p, -k; \rho) &= U''_\ell(\rho) + 2\rho U'''_\ell(\rho) + (k^2 + p^2) Z'_\ell(\rho) + 2k^2rh Z''_\ell(\rho) + p^2 Y'_\ell(\rho), \\
\Gamma^{T,T,T,T}_{\ell}(p, p, -k; \rho) &= U''_\ell(\rho) + (k^2 + p^2) \left[ Z'_\ell(\rho) + 2T_{\ell}T_{\ell} Y'_\ell(\rho) \right], \\
\Gamma^{L,L,L}_{\ell}(k_1, k_2, k_3; \rho) &= \sqrt{2\rho} \left[ 3U''_\ell(\rho) + 2\rho U'''_\ell(\rho) \right] + \frac{1}{2} \left( k_1^2 + k_2^2 + k_3^2 \right) (Z'_\ell(\rho) + 2Y'_\ell(\rho) + 2\rho Y''_\ell(\rho)), \\
\Gamma^{A,A,L}_{\ell}(p, p, -k; \rho) &= -(p^2 \delta_{\mu\nu} - p_\mu p_\nu) \left( \frac{Z''_\ell(\rho)}{\epsilon_\ell^2} \right) - 2 \left[ Z'_\ell(\rho) + 5\rho Z'_\ell(\rho) + 2\rho^2 Z''_\ell(\rho) \right] \delta_{\mu\nu}, \\
\Gamma^{A,A,T,T}_{\ell}(p, p, -k; \rho) &= -(p^2 \delta_{\mu\nu} - p_\mu p_\nu) \left( \frac{Z''_\ell(\rho)}{\epsilon_\ell^2} \right) - 2 \left[ Z'_\ell(\rho) + \rho Z''_\ell(\rho) \right] \delta_{\mu\nu}, \\
\Gamma^{A,L,T}_{\ell}(k_1, k_2, k_3; \rho) &= i \left( k_{1,\mu} - k_{3,\mu} \right) Z'_\ell(\rho) - 2k_{3,\mu}Z''_\ell(\rho), \\
\Gamma^{A,T,T}_{\ell}(k_1, k_2, k_3; \rho) &= i \left( k_{2,\mu} - k_{3,\mu} \right) Z'_\ell(\rho), \\
\Gamma^{A,A,L}_{\ell}(k_1, k_2, k_3; \rho) &= \sqrt{2\rho} \left( k_1 \cdot k_2 \delta_{\mu\nu} - k_{1,\mu} k_{2,\nu} \right) \left( \frac{Z''_\ell(\rho)}{\epsilon_\ell^2} \right) - 2 \left[ Z'_\ell(\rho) + \rho Z''_\ell(\rho) \right] \delta_{\mu\nu},
\end{align*}
\]
where the \( r_i(q^2) \) are dimensionless cutoff functions which do not explicitly depend on the cutoff \( \Lambda \) (or the flow parameter \( \ell \)). A convenient choice, which we will employ later on, is the Litim regulator (14)
\[
r_i(x) = \left( \frac{1}{x} - 1 \right) \theta(1-x). \quad (B4)
\]
In terms of the above dimensionless and rescaled quantities, the flow equation for the effective potential [see Eq. (A5)] turns into (15,16,17)
\[
\partial_\ell u(\tilde{\rho}) = Du(\tilde{\rho}) - (D - 2 + \eta) \tilde{u}'(\tilde{\rho}) - (K_D/2) \left[ L_{0,L}^D(\tilde{\rho}) + (2N - 1) L_{0,T}^D(\tilde{\rho}) + (D - 1) L_{0,A}^D(\tilde{\rho}) \right], \quad (B5)
\]
where \( K_D = \Omega_D/(2\pi)^D = 1/(2^{D-1}\pi^{D/2}(D/2)) \) is the surface area of a \( D \)-dimensional unit sphere divided by \((2\pi)^D\). The threshold functions \( L_{0,i}^D(\tilde{\rho}) \) occurring here

Appendix B: Rescaled flow equations

To derive dimensionless and rescaled flow equations, we use the scaling transformations given in Eqs. (4)–(10). In addition, we introduce the rescaled momenta
\[
q = k/\Lambda. \quad (B1)
\]
For later reference, let us also define \( \eta_{i,\ell} = \eta_\ell \) with \( i = L, T \). As concerns the regulators, it is convenient to write them as
\[
R_\ell^{(1)}(k^2) = Z_\ell^Dk^2r_\phi(q^2), \quad (B2)
R_\ell^{(2)}(k^2) = (1/\epsilon_\ell^2)k^2r_A(q^2), \quad (B3)
\]
are defined by

$$
L_{0,i}^D(\hat{\rho}) = -\frac{1}{2} \int_{0}^{\infty} dx \int_{D}^{2-1} \hat{G}_i(x; \hat{\rho}) \hat{\partial}_t P_i(x; \hat{\rho})
$$

$$
= -\frac{1}{2} \int_{0}^{\infty} dx \int_{D}^{2-1} \hat{G}_i(x; \hat{\rho})(\eta_i r_i(x)x + 2r_i(x)x^2),
$$

(B6)

where

$$
\hat{G}_i(x; \hat{\rho}) = \frac{1}{P_i(x; \hat{\rho}) + w_i(\hat{\rho})} = \frac{1}{z_i(\hat{\rho})x + r_i(x)x + w_i(\hat{\rho})},
$$

(B7)

with $P_i(x; \hat{\rho}) = z_i(\hat{\rho})x + r_i(x)x$ are the flowing rescaled propagators and we have used

$$
\hat{\partial}_t P_i(x; \hat{\rho}) = \eta_i r_i(x)x + 2r_i(x)x^2.
$$

(B8)

We have also defined $w_T(\hat{\rho}) = u'(\hat{\rho}), w_L(\hat{\rho}) = u'(\hat{\rho}) + 2\hat{\rho}u''(\hat{\rho}), w_A(\hat{\rho}) = 2\hat{\rho}z(\hat{\rho}), z_L(\hat{\rho}) = \bar{z}(\hat{\rho}),$ and $z_T(\hat{\rho}) = z(\hat{\rho}).$

From a numerical standpoint, it is more convenient to consider the differential equation for $w(\hat{\rho}) = w_T(\hat{\rho}) = u'(\hat{\rho}),$

$$
\partial_\hat{\rho} w(\hat{\rho}) = (2 - \eta)w(\hat{\rho}) - (D - 2 + \eta)\hat{\rho}w'(\hat{\rho})
$$

$$
+ (K_D/2)[w'_z L_{1,t}^D(\hat{\rho}) + z'_z L_{1,t}^{D+2}(\hat{\rho})]
$$

$$
+ (2N - 1) \left(u'_z L_{1,t}^D(\hat{\rho}) + z'_z L_{1,t}^{D+2}(\hat{\rho})\right)
$$

$$
+ (D - 1) \left(u'_A L_{1,1}^D(\hat{\rho}) + z'_A L_{1,1}^{D+2}(\hat{\rho})\right),
$$

(B9)

After a straightforward but tedious calculation we obtain the dimensionless and rescaled flow equations for the wave-function renormalization factors $z(\hat{\rho}) = z_0(\hat{\rho}),$ $ar{z}(\hat{\rho}) = z_0(\hat{\rho}),$ and $z_A(\hat{\rho}) = z_A^A(\hat{\rho}),$ as well as a flow equation for the square of the dimensionless charge $c^2 = c_0^2.$

$$
\partial_\hat{\rho} \bar{z}(\hat{\rho}) = -\eta \bar{z}(\hat{\rho}) - (D - 2 + \eta)\hat{\rho}\bar{z}'(\hat{\rho}) + (K_D/2)[\bar{z}'(\hat{\rho}) + 2\hat{\rho}\bar{z}''(\hat{\rho})] L_{1,t}^D(\hat{\rho})
$$

$$
- 2K_D \bar{z}'(\hat{\rho}) [3u''(\hat{\rho}) + 2\hat{\rho}u''(\hat{\rho})] L_{2,2}^D(\hat{\rho}) - (2 + 1/D)K_D \bar{z}'(\hat{\rho})^2 L_{2,2}^{D+2}(\hat{\rho})
$$

$$
+ (2/D)K_D \bar{z}'(\hat{\rho}) [3u''(\hat{\rho}) + 2\hat{\rho}u''(\hat{\rho})] M_{2,2}^{D+2}(\hat{\rho}) + (4/D)K_D \bar{z}'(\hat{\rho}) [3u''(\hat{\rho}) + 2\hat{\rho}u''(\hat{\rho})] M_{2,2}^{D+2}(\hat{\rho})
$$

$$
+ (2/D)K_D \bar{z}'(\hat{\rho}) [\bar{z}'(\hat{\rho})]^2 M_{2,2}^{D+4}(\hat{\rho})
$$

$$
-(2N - 1)(K_D/2) [\bar{z}'(\hat{\rho}) - \bar{z}(\hat{\rho})] u''(\hat{\rho}) L_{1,t}^D(\hat{\rho}) - (2N - 1)K_D \bar{z}'(\hat{\rho}) u''(\hat{\rho}) L_{2,2}^D(\hat{\rho})
$$

$$
- (2N - 1)K_D \bar{z}'(\hat{\rho}) [\bar{z}'(\hat{\rho}) - \bar{z}(\hat{\rho})] u''(\hat{\rho}) M_{2,2}^{D+2}(\hat{\rho})
$$

$$
+ (4/D)(2N - 1)K_D \bar{z}'(\hat{\rho}) u''(\hat{\rho}) M_{2,2}^{D+2}(\hat{\rho}) + (2/D)(2N - 1)K_D \bar{z}'(\hat{\rho})^2 M_{2,2}^{D+4}(\hat{\rho})
$$

$$
+ 4(1 - 1/D)K_D \bar{z}'(\hat{\rho}) u''(\hat{\rho}) M_{2,2}^{D+2}(\hat{\rho}) - 4(1 - 1/D)K_D \bar{z}'(\hat{\rho})^2 M_{2,2}^{D+2}(\hat{\rho})
$$

$$
- 2(1 - 1/D)K_D \bar{z}'(\hat{\rho}) u''(\hat{\rho}) M_{2,2}^{D+2}(\hat{\rho}) + 8(1 - 1/D)K_D \bar{z}'(\hat{\rho})^2 M_{2,2}^{D+4}(\hat{\rho})
$$

$$
+ 8(1 - 1/D)K_D \bar{z}'(\hat{\rho}) u''(\hat{\rho}) M_{2,2}^{D+2}(\hat{\rho}) + 2(1 - 1/D)K_D \bar{z}'(\hat{\rho})^2 M_{2,2}^{D+4}(\hat{\rho}),
$$

(B10)

$$
\partial_\hat{\rho} z(\hat{\rho}) = -\eta z(\hat{\rho}) - (D - 2 + \eta)\hat{\rho}z'(\hat{\rho}) + (K_D/2)[z'(\hat{\rho}) + 2\hat{\rho}z''(\hat{\rho})] L_{1,t}^D(\hat{\rho})
$$

$$
+ (K_D/2)[(z'(\hat{\rho}) - z(\hat{\rho})] / \hat{\rho} + (2N - 1)z'(\hat{\rho}) L_{1,t}^D(\hat{\rho})
$$

$$
- K_D \left(1/2 + 1/D\right) [\bar{z}'(\hat{\rho}) - z(\hat{\rho})] u''(\hat{\rho}) L_{2,2}^D(\hat{\rho})
$$

$$
- K_D \left(1/2 \int_{0}^{\infty} dx \int_{D}^{2-1} \hat{G}_i(x; \hat{\rho}) u''(\hat{\rho}) L_{1,t}^D(\hat{\rho})
$$

$$
+ (4/D)K_D \bar{z}'(\hat{\rho}) u''(\hat{\rho}) M_{2,2}^{D+2}(\hat{\rho}) + (4/D)K_D \bar{z}'(\hat{\rho}) u''(\hat{\rho}) M_{2,2}^{D+2}(\hat{\rho})
$$

$$
+ (1/D)K_D \left(\bar{z}'(\hat{\rho}) - z(\hat{\rho})\right)^2 / \hat{\rho} M_{2,2}^{D+4}(\hat{\rho})
$$

$$
- (2/D)K_D \bar{z}'(\hat{\rho}) - z(\hat{\rho}) - 2\hat{\rho}z'(\hat{\rho}) u''(\hat{\rho}) \left[N_{2,2}^{D+4}(\hat{\rho}) - N_{2,2}^{D+2}(\hat{\rho})\right]
$$

$$
- (1/D)K_D \bar{z}'(\hat{\rho}) - z(\hat{\rho}) \left[N_{2,2}^{D+4}(\hat{\rho}) - N_{2,2}^{D+2}(\hat{\rho})\right]
$$

$$
- 4K_D (1 - 1/D) c^2 [z(\hat{\rho}) + \hat{\rho}z'(\hat{\rho})] L_{2,2,1}^D(\hat{\rho}),
$$

(B11)

$$
\partial_\hat{\rho} z_A(\hat{\rho}) = -\eta_A z_A(\hat{\rho}) - (D - 2 + \eta)\hat{\rho}z_A'(\hat{\rho}) + (K_D/2)[z_A'(\hat{\rho}) + 2\hat{\rho}z_A''(\hat{\rho})] L_{1,t}^D(\hat{\rho})
$$
\[ + (2N - 1)(K_D/2)z_A'(\bar{\rho})L^D_{1,T}(\bar{\rho}) \]
\[ + \frac{8K_D e^2}{D(D + 2)} \left( \left[ z(\bar{\rho}) + \hat{\rho}z'(\bar{\rho}) - \alpha_\phi \right]^2 M_{2,L,T}^{D+2}(\bar{\rho}) - 2\alpha_\phi \eta_0 \left[ z(\bar{\rho}) + \hat{\rho}z'(\bar{\rho}) - \alpha_\phi \right] m_{2,L,T}^{D+2}(\bar{\rho}) \right) \]
\[ + \frac{8(N - 1)K_D e^2}{D(D + 2)} \left( \left[ z(\bar{\rho}) - \alpha_\phi \right] M_{2,T,T}^{D+2}(\bar{\rho}) - 2\alpha_\phi \eta_0 \left[ z(\bar{\rho}) - \alpha_\phi \right] m_{2,T,T}^{D+2}(\bar{\rho}) \right) \]
\[ - 4K_D e^2 \hat{\rho} \left[ z(\bar{\rho}) + \hat{\rho}z'(\bar{\rho}) \right] z_A'(\bar{\rho})L^D_{2,A,L}(\bar{\rho}) - (4/D)K_D \hat{\rho} \left[ z_A'(\bar{\rho}) \right]^2 L^D_{2,A,L}(\bar{\rho}) \]
\[ + \frac{16(D + 1)K_D e^4}{D(D + 2)} \hat{\rho} \left[ z(\bar{\rho}) + \hat{\rho}z'(\bar{\rho}) \right]^2 M_{2,A,L}^{D+2}(\bar{\rho}) - \frac{16K_D e^4}{D(D + 2)} \hat{\rho} \left[ z(\bar{\rho}) + \hat{\rho}z'(\bar{\rho}) \right]^2 N_{2,A,L}^{D+2}(\bar{\rho}) \]
\[ - (8/D)K_D e^2 \hat{\rho} \left[ z(\bar{\rho}) + \hat{\rho}z'(\bar{\rho}) \right] z_A'(\bar{\rho}) \left[ N_{2,A,L}^{D+2}(\bar{\rho}) - N_{2,L,A}^{D+2}(\bar{\rho}) \right] \]
\[ + (D - 2)(K_D/6)e^2 \left[ L^D_{C,\ell}\bar{\rho}^2 + (N - 1)L^D_{C,T,T}(\bar{\rho}) \right], \quad (B12) \]

\[ \partial_t e^2 = (4 - D - \eta_A)e^2. \quad (B13) \]

From the last equation, it follows directly that at a fixed point we must have\( \eta_A = 4 - D \). For an arbitrary regulator, the new threshold functions occurring above are given by

\[ L^D_{1,i}(\bar{\rho}) = \frac{1}{2} \int_0^\infty dx x^{D/2-1} \partial_t \tilde{G}_i(x; \bar{\rho}) = -\frac{1}{2} \int_0^\infty dx x^{D/2-1} \tilde{G}_i^2(x; \bar{\rho})(\eta_i r_i x + 2r'_i x^2), \quad (B14) \]

\[ L^D_{2,i,j}(\bar{\rho}) = \frac{1}{2} \int_0^\infty dx x^{D/2-1} \partial_t \left[ \tilde{G}_i(x; \bar{\rho}) \tilde{G}_j(x; \bar{\rho}) \right] = -\frac{1}{2} \int_0^\infty dx x^{D/2-1} \tilde{G}_i(x; \bar{\rho}) \tilde{G}_j(x; \bar{\rho}) \left( \tilde{G}_i(x; \bar{\rho})(\eta_i r_i x + 2r'_i x^2) + \tilde{G}_j(x; \bar{\rho})(\eta_j r_j x + 2r'_j x^2) \right), \quad (B15) \]

\[ M^D_{2,i,j}(\bar{\rho}) = \frac{1}{2} \int_0^\infty dx x^{D/2} \partial_t \left[ \tilde{G}_i(x; \bar{\rho}) \tilde{G}'_j(x; \bar{\rho}) \right] = \frac{1}{2} \int_0^\infty dx x^{D/2} \tilde{G}_i^2(x; \bar{\rho}) \tilde{G}_j^2(x; \bar{\rho}) \left[ (\eta_i (r_i + r'_i x) + 4r'_i x + 2r''_i x^2) (z_j(\bar{\rho}) + r_j + r'_j x) 
+ (z_i + r_i + r'_i x) (\eta_j (r_j + r'_j x) + 4r'_j x + 2r''_j x^2)
- 2(z_i + r_i + r'_i x)(z_j(\bar{\rho}) + r_j + r'_j x) \left( \tilde{G}_i(x; \bar{\rho})(\eta_i r_i x + 2r'_i x^2) + \tilde{G}_j(x; \bar{\rho})(\eta_j r_j x + 2r'_j x^2) \right) \right] \quad (B16) \]

\[ N^D_{2,i,j}(\bar{\rho}) = -\frac{1}{2} \int_0^\infty dx x^{D/2} \partial_t \left[ \tilde{G}_i(x; \bar{\rho}) \tilde{G}'_j(x; \bar{\rho}) \right] = \frac{1}{2} \int_0^\infty dx x^{D/2} \tilde{G}_i(x; \bar{\rho}) \tilde{G}_j^2(x; \bar{\rho}) \left( \eta_j (r_j + r'_j x) + 4r'_j x + 2r''_j x^2) 
- (z_j(\bar{\rho}) + r_j + r'_j x) \left( \tilde{G}_j^2(x; \bar{\rho})(\eta_i r_i x + 2r'_i x^2) + 2\tilde{G}_i(x; \bar{\rho})\tilde{G}_j^3(x; \bar{\rho})(\eta_j r_j x + 2r'_j x^2) \right) \right] \quad (B17) \]

\[ m^D_{2,i,j}(\bar{\rho}) = \frac{1}{2} \int_0^\infty dx x^{D/2} \tilde{G}_i^2(x; \bar{\rho}) \tilde{G}_j^2(x; \bar{\rho}) \left[ \tilde{G}_i(x; \bar{\rho}) \tilde{G}_j(x; \bar{\rho}) = \frac{1}{2} \int_0^\infty dx x^{D/2} \tilde{G}_i^2(x; \bar{\rho}) \tilde{G}_j^2(x; \bar{\rho})(z_i + r_i + r'_i x)(z_j(\bar{\rho}) + r_j + r'_j x) \right]. \quad (B18) \]

The last term in Eq. (B12) is due to the correction term \( C_\ell(\phi, A) \) and contains the threshold functions

\[ L^D_{C,i}(\bar{\rho}) = |2z_i(\bar{\rho}) + 2w_i(\bar{\rho}) - \partial_\ell z_i(\bar{\rho}) - \partial_\ell w_i(\bar{\rho})| L^D_{C,a,i}(\bar{\rho}) + 2L^D_{C,b,i}(\bar{\rho}) - L^D_{C,c,i}(\bar{\rho}), \quad (B19) \]
with

$$L_{C,i}^{D,a}(\bar{\rho}) = \frac{1}{2} \int_0^\infty dx x^{D/2-1} \frac{r_i + r_i' x}{[\zeta(\bar{\rho}) + w_i(\bar{\rho}) + r_i x]^2}, \quad (B20)$$

$$L_{C,i}^{D,b}(\bar{\rho}) = \frac{1}{2} \int_0^\infty dx x^{D/2-1} \frac{(r_i + r_i' x) r_i' x^2}{[\zeta(\bar{\rho}) + w_i(\bar{\rho}) + r_i x]^2}, \quad (B21)$$

$$L_{C,i}^{D,c}(\bar{\rho}) = \frac{1}{2} \int_0^\infty dx x^{D/2-1} \frac{4r_i' x + 2r_i'' x^2}{z_i(\bar{\rho}) + w_i(\bar{\rho}) + r_i x}. \quad (B22)$$

Finally, to close our set of flow equations, we determine the flowing anomalous dimension $\eta_\ell$ of the field $\phi$ and its gauge-field counterpart $\eta_\ell^A$ by demanding that the corresponding wave-function renormalization factors evaluated at a characteristic value of the rescaled density $\bar{\rho} = \bar{\rho}_0$ are equal to one,

$$z_\ell(\bar{\rho}_c^*) = z_\ell^A(\bar{\rho}_c^*) = 1. \quad (B23)$$

While the definition of $\bar{\rho}_c^*$ does not really matter (but effectively modifies the cutoff function), we find it convenient to choose it to be equal to the position of the critical rescaled effective potential $\hat{\eta}^*(\bar{\rho})$, as shown in Fig. 2

Using the Litim cutoff, as given in Eq. (B4), its second derivative $r_i''(x)$ contains the derivative of a Dirac delta distribution which can be eliminated by an integration by parts. We then obtain for the threshold functions

$$L_{0,i}^{D}(\bar{\rho}) = \frac{1}{2} \int_0^1 dx x^{D/2-1} \tilde{G}_i(x; \bar{\rho})(2 - \eta_\ell(1 - x)), \quad (B24)$$

$$L_{1,i}^{D}(\bar{\rho}) = \frac{1}{2} \int_0^1 dx x^{D/2-1} \tilde{G}_i^2(x; \bar{\rho})(2 - \eta_\ell(1 - x)), \quad (B25)$$

$$L_{2,i,j}^{D}(\bar{\rho}) = \frac{1}{2} \int_0^1 dx x^{D/2-1} \tilde{G}_i(x; \bar{\rho}) \tilde{G}_j(x; \bar{\rho}) \left[ \tilde{G}_i(x; \bar{\rho})(2 - \eta_\ell(1 - x)) + \tilde{G}_j(x; \bar{\rho})(2 - \eta_\ell(1 - x)) \right], \quad (B26)$$

$$M_{2,i,j}^{D}(\bar{\rho}) = [z_i(\bar{\rho}) z_j(\bar{\rho}) - (z_i(\bar{\rho}) - 1)(z_j(\bar{\rho}) - 1)] \tilde{G}_i^2(x = 1; \bar{\rho}) \tilde{G}_j^2(x = 1; \bar{\rho}) + \frac{1}{2} \int_0^1 dx x^{D/2} \tilde{G}_i^2(x; \bar{\rho}) \tilde{G}_j^2(x; \bar{\rho}) \times \left[ -\eta_\ell(z_i(\bar{\rho}) - 1) - \eta_\ell(z_j(\bar{\rho}) - 1) + 2(z_i(\bar{\rho}) - 1)(z_j(\bar{\rho}) - 1) \left( \tilde{G}_i(x; \bar{\rho})(2 - \eta_\ell(1 - x)) + \tilde{G}_j(x; \bar{\rho})(2 - \eta_\ell(1 - x)) \right) \right], \quad (B27)$$

$$N_{2,i,j}^{D}(\bar{\rho}) = \tilde{G}_i(x = 1; \bar{\rho}) \tilde{G}_j^2(x = 1; \bar{\rho}) + \frac{1}{2} \int_0^1 dx x^{D/2} \left[ -\tilde{G}_i(x; \bar{\rho}) \tilde{G}_j^2(x; \bar{\rho}) \eta_j 
+ (z_j(\bar{\rho}) - 1) \left( \tilde{G}_i^2(x; \bar{\rho}) \tilde{G}_j^2(x; \bar{\rho})(2 - \eta_\ell(1 - x)) + 2\tilde{G}_i(x; \bar{\rho}) \tilde{G}_j^2(x; \bar{\rho})(2 - \eta_\ell(1 - x)) \right) \right], \quad (B28)$$

$$m_{2,i,j}^{D}(\bar{\rho}) = +(z_i(\bar{\rho}) - 1)(z_j(\bar{\rho}) - 1) \frac{1}{2} \int_0^1 dx x^{D/2} \tilde{G}_i^2(x; \bar{\rho}) \tilde{G}_j^2(x; \bar{\rho}), \quad (B29)$$

$$L_{C,i}^{D,a}(\bar{\rho}) = L_{C,i}^{D,b}(\bar{\rho}) = \frac{1}{2} \int_0^1 dx x^{D/2-1} \frac{1}{[z_i(\bar{\rho}) + w_i(\bar{\rho}) + (1 - x)]^2}, \quad (B30)$$

$$L_{C,i}^{D,c}(\bar{\rho}) = \frac{1}{z_i(\bar{\rho}) + w_i(\bar{\rho})}. \quad (B31)$$
Collecting all terms for $L_{C,i}^{D,a}(\tilde{\rho})$, we obtain

$$L_{C,i}^{D}(\tilde{\rho}) = \left[ 2 \left( 1 + z_i(\tilde{\rho}) + w_i(\tilde{\rho}) \right) - \partial_{\ell} z_i(\tilde{\rho}) - \partial_{\ell} w_i(\tilde{\rho}) \right] L_{C,i}^{D,a}(\tilde{\rho}) - \frac{1}{z_i(\tilde{\rho}) + w_i(\tilde{\rho})}. \quad (B32)$$

In the case of $D = 3$, we only need $L_{C,i}^{D=1,a}(\tilde{\rho})$, which in fact is easily calculated analytically,

$$L_{C,i}^{D=1,a}(\tilde{\rho}) = \frac{1}{2[z_i(\tilde{\rho}) + w_i(\tilde{\rho})] [1 + z_i(\tilde{\rho}) + w_i(\tilde{\rho})]} + \frac{1}{2[1 + z_i(\tilde{\rho}) + w_i(\tilde{\rho})]^{3/2}} \text{Artanh} \left( \sqrt{\frac{1}{1 + z_i(\tilde{\rho}) + w_i(\tilde{\rho})}} \right). \quad (B33)$$

To solve our complete set of flow equations, we discretize $\tilde{\rho}$, evaluate the threshold functions using quadrature and advance the solution using a fourth order Runge-Kutta method with a sufficiently small step size. For the results displayed in Figs. 2-4 we have used 201 points equally spaced between $\tilde{\rho} = 0$ and 2 ($N = 1$) or between $\tilde{\rho} = 0$ and 2.83 ($N = 2$), respectively. The step size in the Runge-Kutta solver was $\delta \ell = 0.001$. Starting from $c^2_0 = 0.1$ and $\lambda_0 = 1$, it was then possible to fine-tune to criticality using the bisection method.

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