Convergent series for QCD

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Abstract

We present an analytical solution of QCD in terms of the convergent series. The series converges for any finite quantity and diverges for infinite quantities irrespective of the value of the coupling constant. The convergence is reached by using a solvable interacting field theory as an initial approximation. The terms of the convergent series are free from the infrared and ultraviolet divergences and have the form of Feynman diagrams.

1 Introduction

The construction of the convergent expansions in quantum field theory is an old and important problem [1]. The standard perturbation theory leads to the asymptotic series because of the illegal interchange of the summation and integration. However, changing the initial approximation of the perturbation theory, one can generate the convergent series (CS). The CS for the bosonic scalar field theories was constructed in [2, 3]. The critical indices of $\varphi^4$-model obtained in [4] within the framework of the CS are in a good agreement with the experimental and numerical results. In the current work we follow the formulation of the CS presented in [3]. To illustrate the main ideas of this approach, let us consider a normalized partition function of the scalar field

$$Z = \frac{1}{Z_0} \int D\phi e^{-S_0[\phi]}, \quad Z_0 = \int D\phi e^{-S_0[\phi]},$$

(1)

where $S[\phi]$ and $S_0[\phi]$ are the full and free theory actions respectively. Then, under the assumption of the existence of the corresponding path integral, $Z < \infty$. Splitting the action into the non-perturbed part $N[\phi]$ and the perturbation $(S[\phi] - N[\phi])$ we get

$$Z = \frac{1}{Z_0} \int D\phi e^{-N[\phi]} \sum_n \frac{(N[\phi] - S[\phi])^n}{n!}.$$  

(2)

When $N[\phi] \geq S[\phi]$, the functional $e^{-N[\phi]}(N[\phi] - S[\phi])^n \geq 0$, therefore, the summation and integration may be interchanged

$$Z = \frac{1}{Z_0} \sum_n \frac{1}{n!} \int D\phi e^{-N[\phi]}(N[\phi] - S[\phi])^n,$$

(3)

and the convergence of the series is guaranteed by the condition $Z < \infty$. A proper choice of the non-perturbed part $N[\phi]$ allows one to reduce all calculations to the Feynman diagrams with the free propagator, which is massive and decays sufficiently fast at the infinity in the momentum representation. Accordingly, the diagrams do not contain ultraviolet and infrared divergences.

In the following we present a similar construction for Quantum Chromodynamics. It is impossible to perform the above analysis for the fermionic fields. To overcome this difficulty we do the bosonization of QCD and obtain an effective bosonic vector field theory. Then, we generalize the method [3] for the vector fields.

2 Bosonization

The QCD action may contain two sorts of the fermionic contributions: quarks and ghosts fields. To simplify derivations we eliminate ghosts by choosing the axial gauge (alternatively one can use the light-cone or any other ghost-free gauge) [5]. The Euclidean action of two flavor QCD in the axial gauge is
Thus, the initial problem is reduced to the theory of real vector fields.

We define

\[ F_{i,\mu} = \partial_{\mu} A_{i,\mu} - \partial_{\nu} A_{i,\mu} + g f^{ijk} A_{j,\mu} A_{k,\nu}, \]

where \( g \) is the coupling constant, \( f^{ijk} \) are the structure constants of the SU(3) group. The QCD Dirac operator is \( D \equiv i\gamma_{\mu} D_{ij,\mu} = i\delta_{ij} \gamma_{\mu} \partial_{\mu} + g\gamma_{\mu} A_{k,\mu} \tau_{ij}^k \), where \( \tau_{ij}^k \) are the SU(3) generators in the fundamental representation.

Integration over fermion fields leads to the determinant

\[
\det(D + m)^2 = \det(\gamma_5(D + m)\gamma_5(D + m)) = \det(-D^2 + m^2) \equiv \det(B^2 + m^2),
\]

where \( B \equiv \gamma_5 D \) is hermitian. The fermion determinant \( \det \) was represented in terms of the path integral over five dimensional bosonic fields in \([6, 7, 8]\). The agreement between the bosonization approach and conventional lattice computations was shown in \([9]\). Here we use continuous version of the bosonic effective action for the fermion determinant from \([9]\).

\[
\det(B^2 + m^2) = \lim_{L \to \infty} \int d\phi_{i,\alpha}(x,s) d\phi^*_{i,\alpha} D\xi_{i,\alpha} D\xi^*_{i,\alpha} \exp \left\{ -\int_0^L ds \int dx \right. \\
\left. \left[ (\partial_t \phi_{i,\alpha}(x,s) - i B_{ij,\alpha}(x) \phi^*_{j,\beta}(x,s)) (\partial_t \phi_{i,\alpha}(x,s) + i B_{ij,\alpha}(x) \phi_{j,\beta}(x,s)) \right. \right. \\
\left. + \sqrt{\xi_{i,\alpha}(x)} (m \delta_{ij} \delta_{\alpha\beta} + i B_{ij,\alpha}(x) \phi_{j,\beta}(x,s) + \text{h.c.)} \right. \right. \\
\left. - \frac{1}{2m} \xi^*_{i,\alpha}(x) \xi_{i,\alpha}(x) \right]\} \\
\equiv \int D\phi_{i,\alpha} D\phi^*_{i,\alpha} D\xi_{i,\alpha} D\xi^*_{i,\alpha} \exp(-S_{det}).
\]

The \((3 + 1 + 1)\)-dimensional fields \( \phi_{i,\alpha}(x,s) \) and \((3 + 1)\)-dimensional fields \( \xi_{i,\alpha}(x) \) are bosonic and have the same index structure (except flavor) as initial fermionic fields \( \psi_{f,\alpha}(x) \), \( L \) is the length of the fifth dimension.

## 3 Notations and definitions

The effective action of QCD \( S_{\text{eff}} = S_G + S_{\text{det}} \) (\( S_G \) is the gauge action) is real and can be represented depending only on real fields by the following change of variables

\[
\begin{align*}
\phi_{i,\alpha}(x,s) &\to a_{i,\alpha}(x,s) + i b_{i,\alpha}(x,s), \\
\xi_{i,\alpha}(x) &\to c_{i,\alpha}(x) + i d_{i,\alpha}(x), \\
a_{i,\alpha}(x,s), b_{i,\alpha}(x,s), c_{i,\alpha}(x), d_{i,\alpha}(x) &\in \mathbb{R}.
\end{align*}
\]

Thus, the initial problem is reduced to the theory of real vector fields.

Now we introduce some useful notations. Let

\[
\Phi = \{ A_{j,\mu}, \partial_\mu A_{j,\mu}, a_{j,\alpha}, \partial_\alpha a_{j,\alpha}, b_{j,\alpha}, \partial_\beta b_{j,\alpha}, c_{j,\alpha}, d_{j,\alpha} \} \\
\equiv \{ A_{1,1}, A_{1,2}, ..., A_{3,4}, \partial_1 A_{1,1}, ..., d_{3,4} \}
\]

is a vector field with the total number of components \( N \), where \( \mu, \nu, \alpha = 1, ..., 4, \ j = 1, 2, 3 \).

We define \( \varphi_\kappa \) as a general notation for the field, so that \( \varphi_\kappa = A_{j,\mu}, a_{j,\alpha}, b_{j,\alpha}, c_{j,\alpha}, d_{j,\alpha} \). Here \( \kappa \) is the
super-index and has different meanings depending on the context. For instance, the summation over \( \kappa \) can be interpreted in different cases as \( \sum_{\mu} \sum_{j} \) or as \( \sum_{\alpha} \sum_{j} \), where \( j, \mu \) and \( \alpha \) are the color, spatial and spinor indices respectively.

\( \Lambda_{\varphi_{\kappa}} \) - is the maximal power of the derivative in the Lagrangian, which comes from the derivatives acting on the field \( \varphi_{\kappa} \). For example, the term \( \partial^{2} \varphi_{\kappa} \) leads to \( \Lambda_{\varphi_{\kappa}} = 2 \) and the term \( (\partial \varphi_{\kappa})^{2} \) gives the same \( \Lambda_{\varphi_{\kappa}} = 2 \).

\( Q_{\varphi_{\kappa}} \) - is the maximal power of the field in the Lagrangian.

For the effective QCD Lagrangian the constants defined above are

\[
\begin{align*}
\Lambda_{A_{j,\mu}} &= 2 \equiv \Lambda_{A}, \\
Q_{A_{j,\mu}} &= 4 \equiv Q_{A}, \\
\Lambda_{a_{j,\alpha}} &= 2 \equiv \Lambda_{a}, \\
Q_{a_{j,\alpha}} &= 2 \equiv Q_{a}, \\
\Lambda_{b_{j,\alpha}} &= 2 \equiv \Lambda_{b}, \\
Q_{b_{j,\alpha}} &= 2 \equiv Q_{b}, \\
\Lambda_{c_{j,\alpha}} &= 2 \equiv \Lambda_{c}, \\
Q_{c_{j,\alpha}} &= 2 \equiv Q_{c}, \\
\Lambda_{d_{j,\alpha}} &= 2 \equiv \Lambda_{d}, \\
Q_{d_{j,\alpha}} &= 2 \equiv Q_{d}.
\end{align*}
\]

The Sobolev norms of the fields are given by

\[
\| \varphi_{\kappa} \| = \left[ \int_{0}^{L} ds \int dx \varphi_{\kappa}(x, s) \left( \sum_{u=0}^{U_{\varphi_{\kappa}}} \omega_{u} \left( -\partial_{\nu,s}^{2} \right) u \varphi_{\kappa}(x, s) \right) \right]^{1/2},
\]

the constants \( \omega_{u} > 0 \) and constants \( U_{\varphi_{\kappa}} \) are determined later from the condition of the absence of the ultraviolet divergences in the diagrams.

The \( q \)-norm of the vector field is defined as

\[
\| \Phi \|_{q} = \sum_{\varphi_{\kappa} = A_{j,\mu}, a_{j,\alpha}, b_{j,\alpha}, c_{j,\alpha}, d_{j,\alpha}} \sigma_{q,\varphi_{\kappa}}^{\varphi_{\kappa}} \| \varphi_{\kappa} \|^{q} \equiv \sum_{\varphi_{\kappa}} \sigma_{q,\varphi_{\kappa}}^{\varphi_{\kappa}} \| \varphi_{\kappa} \|^{q},
\]

the constants \( \sigma_{q,\varphi_{\kappa}}^{\varphi_{\kappa}} \) are determined below from the inequality (28). We also define a shorthand notation for the product

\[
\prod_{\varphi_{\kappa} = A_{j,\mu}, a_{j,\alpha}, b_{j,\alpha}, c_{j,\alpha}, d_{j,\alpha}} \varphi_{\kappa} \equiv \widetilde{\prod}.
\]

The effective QCD action can be written in terms of the vector field \( \Phi \) as

\[
S_{\text{eff}}[\Phi] = \sum_{q=2}^{Q_{A}} S_{q}[\Phi],
\]

\[
S_{q}[\Phi] = \sum_{i_{1}...i_{q}=1}^{N} C_{i_{1}...i_{q}} \int_{0}^{L} ds \int dx \prod_{p=1}^{q} \Phi_{i_{p}},
\]

where \( C_{i_{1}...i_{q}} \) are the appropriately chosen constants. The correlation function of the general form is given by

\[
G(K[\Phi], S_{\text{eff}}[\Phi]) = \prod \int D\varphi_{\kappa} K[\Phi] \exp\{-S_{\text{eff}}[\Phi]\},
\]

where

\[
K[\Phi] = \prod \varphi_{\kappa}(x_{\varphi_{\kappa},1})...\varphi_{\kappa}(x_{\varphi_{\kappa},k(\varphi_{\kappa}))}
\]

and the normalization factor

\[
\frac{1}{\int D\varphi_{\kappa} K[\Phi] \exp\{-S_{0}[\Phi]\}}
\]

is included into the measure, \( S_{0}[\Phi] \) is the action of the free theory. Hereafter we use the dimensional regularization \[10\]. In this scheme the measure \( D\varphi \) has the scale property \( D(c\varphi) = D\varphi \).
4 Convergent series

To construct the convergent series we divide the action into two parts $S_{\text{eff}}[\Phi] = N[\Phi] + (S_{\text{eff}}[\Phi] - N[\Phi])$, where the non-perturbed part is

$$N[\Phi] = \sum_{q=2}^{Q_A} \|\Phi\|_q = \sum_{q=2}^{Q_A} \sigma_q^\varphi_{\varphi} \|\varphi\|^q .$$  \hspace{1cm} (16)

Then

$$G(K[\Phi], S_{\text{eff}}[\Phi]) = \sum_{n=0}^{\infty} G_n(K[\Phi], S_{\text{eff}}[\Phi])$$

and

$$G_n(K[\Phi], S_{\text{eff}}[\Phi]) = \frac{1}{n!} \prod \int D\varphi_{\varphi} K[\Phi] \{N[\Phi] - S_{\text{eff}}[\Phi]\}^n e^{-N[\Phi]}$$

$$= \frac{1}{n!} \prod \int_0^\infty dt_{\varphi_{\varphi}} \exp\{-\sum_{q=2}^{Q_A} \sigma_q^\varphi_{\varphi} t_{\varphi_{\varphi}}^q\}$$

$$\cdot \int D\varphi_{\varphi} K[\Phi] \delta(t_{\varphi_{\varphi}} - \|\varphi\|) \left\{\sum_{q=2}^{Q_A} \sigma_q^\varphi_{\varphi} t_{\varphi_{\varphi}}^q - S_{\text{eff}}[\Phi]\right\}^n .$$  \hspace{1cm} (17)

Rescaling $\varphi_{\varphi}$ as $\varphi_{\varphi} \rightarrow t_{\varphi_{\varphi}} \varphi_{\varphi}$ and using the scale property of the dimensional regularization scheme, we get

$$G_n(K[\Phi], S_{\text{eff}}[\Phi]) = \frac{1}{n!} \prod \int_0^\infty dt_{\varphi_{\varphi}} t_{\varphi_{\varphi}}^{n-1} \exp\{-\sum_{q=2}^{Q_A} \sigma_q^\varphi_{\varphi} t_{\varphi_{\varphi}}^q\}$$

$$\cdot \int D\varphi_{\varphi} K[\Phi] \delta(1 - \|\varphi\|) \left\{\sum_{q=2}^{Q_A} \sigma_q^\varphi_{\varphi} t_{\varphi_{\varphi}}^q - S_{\text{eff}}[\Phi]\right\}^n ,$$  \hspace{1cm} (18)

where

$$t_{\Phi} = \{t_{j,\mu} A_{j,\mu}, t_{j,\mu} \partial_{\nu} A_{j,\mu}, t_{j,\alpha} a_{j,\alpha}, t_{j,\alpha} \partial_{\nu} a_{j,\alpha}, t_{j,\alpha} b_{j,\alpha}, t_{j,\alpha} \partial_{\nu} b_{j,\alpha}, t_{j,\alpha} c_{j,\alpha}, t_{j,\alpha} d_{j,\alpha}\} .$$  \hspace{1cm} (20)

The expansion of brackets $\{\ldots\}^n$ leads to the terms of the type

$$[t_{\varphi_{\varphi}}\text{-type integrals}] \cdot \prod \int D\varphi_{\varphi} K[\Phi] \delta(1 - \|\varphi\|) \varphi_{\varphi}(x_{\varphi_{\varphi},1}) \ldots \varphi_{\varphi}(x_{\varphi_{\varphi},r(\varphi_{\varphi}))} .$$  \hspace{1cm} (21)

Using the identity $\mathbf{4}$

$$\int D\varphi_{\varphi} \delta(1 - \|\varphi\|) \varphi_{\varphi}(x_1) \ldots \varphi_{\varphi}(x_r) = \frac{2}{\Gamma(r/2)} \int D\varphi_{\varphi} \exp\{-\|\varphi\|^2\} \varphi_{\varphi}(x_1) \ldots \varphi_{\varphi}(x_r) ,$$  \hspace{1cm} (22)

we rewrite (21) as

$$[t_{\varphi_{\varphi}}\text{-type integrals}] \cdot \prod \left[2 \Gamma\left(\frac{k(\varphi_{\varphi}) + r(\varphi_{\varphi}))}{2}\right)^{-1}\right]$$

$$\cdot \int D\varphi_{\varphi} K[\Phi] \exp\{-\|\varphi\|^2\} \varphi_{\varphi}(x_{\varphi_{\varphi},1}) \ldots \varphi_{\varphi}(x_{\varphi_{\varphi},r(\varphi_{\varphi}))} .$$  \hspace{1cm} (23)

The last equation relates the expansion (17) to the standard diagram technique with the propagators in the momentum representation given by

$$f_{\varphi_{\varphi}}(p) = \left(\sum_{u=0}^{U_{\varphi_{\varphi}}} \omega_{\varphi_{\varphi}}^u \sigma^2 p^2\right)^{-1} , \quad \omega_{\varphi_{\varphi}}^u > 0 .$$  \hspace{1cm} (24)

$^1$The equation (22) is derived applying the same steps as in (18) and (19).
For any set of the constants \(U_\varphi > \Lambda_\varphi + D/2\) \((D\) is the dimension of space\) the diagrams \(23\) are free from the ultraviolet and infrared divergences. Such a lower estimate for \(U_\varphi\) is derived by counting the powers of the moments coming from the measures of the loop integrals, propagators and derivatives in the vertexes.

Let us now study the convergence of the series \(17\). First of all, we show that the constants \(\sigma_\varphi^k\) can be chosen in such a way, that \(N[\Phi] \geq S_{eff}[\Phi]\). From the Hölder inequality \(11\), it follows that

\[
S_q[\Phi] \leq C_q \sum_{i=1}^N \int_0^L ds \int dx |\Phi_i|^q \times \ldots \times \left( \int_0^L ds \int dx |\Phi_{i_q}|^q \right)^{1/q}.
\]

(25)

Then, using the number inequality

\[
\frac{x_1 + x_2 + \ldots + x_n}{n} \geq \left( x_1 x_2 \ldots x_n \right)^{\frac{1}{n}} \quad x_1, \ldots, x_n \geq 0,
\]

(26)

we get

\[
S_q[\Phi] \leq C_q \sum_{i=1}^N \int_0^L ds \int dx |\Phi_i|^q, \quad C_q > 0.
\]

(27)

According to the Sobolev theorem \(12\), for any \(U_\varphi > \Lambda_\varphi + D/2\) there exist the positive constants \(\sigma_\varphi^k\), such that

\[
\sum_{i \in F} C_q \int_0^L ds \int dx |\Phi_i|^q \leq \sigma_\varphi^k \int_0^L ds \int dx |\Phi| |\varphi_k|^q,
\]

(28)

where the summation \(i \in F\) runs over the components \(\Phi_i\) which represent the field \(\varphi_k\) and its derivatives. Consequently,

\[
S_q[\Phi] \leq \|\Phi\|_q \quad \Rightarrow \quad S_{eff}[\Phi] \leq N[\Phi]
\]

(29)

and

\[
\sum_{n=0}^\infty G_n(K[\Phi], S_{eff}[\Phi]) \leq \sum_{n=0}^\infty |G_n(K[\Phi], S_{eff}[\Phi])| \leq \sum_{n=0}^\infty \prod_{n=0}^\infty \int D\varphi_n \left| K[\Phi] \{ N[\Phi] - S_{eff}[\Phi] \}^n e^{-N[\Phi]} \right| = \sum_{n=0}^\infty \prod_{n=0}^\infty \int D\varphi_n \left| K[\Phi] \{ N[\Phi] - S_{eff}[\Phi] \}^n e^{-N[\Phi]} \right| = \prod_{n=0}^\infty \int D\varphi_n |K[\Phi]| \sum_{n=0}^\infty \{ N[\Phi] - S_{eff}[\Phi] \}^n e^{-N[\Phi]} = \prod_{n=0}^\infty \int D\varphi_n |K[\Phi]| e^{-S_{eff}[\Phi]}.
\]

(30)

Apart from the situations when the correlation functions become infinite (like phase transitions or short distance asymptotic) both integrals \(\prod \int D\varphi_n K[\Phi] e^{-S_{eff}[\Phi]}\) and \(\prod \int D\varphi_n |K[\Phi]| e^{-S_{eff}[\Phi]}\) are finite and the series \(17\) is convergent. All terms of the series \(17\) are always finite, therefore, in cases when correlation functions must be infinite, the series \(17\) must diverge. The divergence of the CS for the \(\varphi^4\)-model in the short distance asymptotic is demonstrated in \(3\).

5 Conclusions

We have constructed the solution of Quantum Chromodynamics in terms of the convergent series without infrared and ultraviolet divergences. The solution gives a new powerful tool for the analytical studies and for the numerical computations. The generalization to other gauge groups is evident. All steps used to derive the convergent series for QCD are also valid for the pure Yang-Mills theories. The only difference is the absence of the matter fields in the vector \(\Phi_i\) and in the products and sums over \(\varphi_k\).
The standard perturbation theory is an expansion in the small coupling constant $g$ and it does not take into account contributions of the non-analytic functions like $e^{-\frac{1}{g}}$. The convergent series does not rely on the expansion in small $g$ and, therefore, automatically takes into account such non-analytic contributions.

Despite the fact that CS is a perturbative method (just in accordance with the definition of the word 'perturbative'), it has all advantages of non-perturbative approaches. The relation of the CS to the Feynman diagrammatic technique should provide the possibility of the non-perturbative generalizations of conjectures, proved early only within the framework of the standard perturbation theory.

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References

[1] F. J. Dyson. Divergence of perturbation theory in quantum electrodynamics. Phys. Rev., 85:631–632, Feb 1952.

[2] B.S. Shaverdyan and A.G. Ushveridze. Convergent perturbation theory for the scalar $\phi^{2p}$ field theories; the gell-mann-low function. Physics Letters B, 123(5):316 – 318, 1983.

[3] A.G. Ushveridze. Superconvergent perturbation theory for euclidean scalar field theories. Physics Letters B, 142(5–6):403 – 406, 1984.

[4] Juha Honkonen and Mikhail Nalimov. Convergent expansion for critical exponents in the $o(n)$-symmetric $\phi^4$ model for large $\varepsilon$. Physics Letters B, 459(4):582 – 588, 1999.

[5] George Leibbrandt. Introduction to noncovariant gauges. Rev. Mod. Phys., 59:1067–1119, Oct 1987.

[6] A. A. Slavnov. Fermi-bose duality via extra dimension. Phys.Lett. B, 388:147–153, 1996.

[7] A. A. Slavnov. Bosonization of fermion determinants. Phys.Lett. B, 366:253–260, 1996.

[8] A. A. Slavnov. Four dimensional physics via five dimensional constrained systems. Nuclear Physics B - Proceedings Supplements, 88:210–214, 2000.

[9] T.D. Bakeyev, A.I. Veselov, M.I. Polikarpov, and A.A. Slavnov. Test of a new bosonization algorithm for a simple one-dimensional model. Theoretical and Mathematical Physics, 113(1):1255–1262, 1997.

[10] George Leibbrandt. Introduction to the technique of dimensional regularization. Rev. Mod. Phys., 47:849–876, Oct 1975.

[11] G. Korn and T. Korn. Mathematical handbook. McGraw-Hill, New York, 1968.

[12] W. Rudin. Functional analysis. McGraw-Hill, New York, 1973.