Thermal Light as a Mixture of Sets of Pulses

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The relationship between thermal light and coherent pulses is of fundamental interest, and is also central to relating coherent optical experiments on photophysical processes to the natural occurrence of those processes in sunlight. We now know that thermal light cannot be represented as a statistical mixture of single pulses. In this paper we ask whether or not thermal light can be represented as a statistical mixture of sets of pulses. We consider thermal light in a one-dimensional waveguide, and find a convex decomposition into products of orthonormal coherent states of localized, nonmonochromatic modes.

Quantum-mechanical features of energy transfer in light-harvesting complexes are predominantly studied using coherent pulses of light. But because photosynthetic organisms typically absorb incoherent light, questions remain about the relevance of these experiments to the organism’s natural function under sunlight [1–9]. One of the difficulties in addressing these questions is the lack of a decomposition of thermal light into coherent pulses; thermal light cannot be decomposed into a statistical mixture of single pulses [10], and whether or not it can be decomposed into sets of pulses, and if so what would be their nature, is an outstanding problem.

As a first step toward answering this question, we consider light propagation in a quasi-1D geometry, such as an optical fiber. We find that it is possible to construct a convex decomposition of the thermal equilibrium density operator into products of orthonormal coherent states of localized, nonmonochromatic modes. These modes correspond to the scaling function for the Shannon (sinc) wavelet [11]. The coherent states in the decomposition are the quantum analogue of localized, coherent pulses of light. Our decomposition can be applied to thermal light over any frequency range, where the range determines the width of the pulses.

We begin by quantizing the fields inside the waveguide in Sec. I then briefly review the $P$-representation of thermal states in Sec. II. The modes involved here are delocalized. We then partition the thermal state density operator into portions of $k$-space. We introduce localized modes in Sec. III. In Sec. IV we decompose the density operator for thermal light in a portion of $k$-space into products of coherent states of the localized modes. We then determine the fields in terms of these modes, and plot the field variation for “typical” pulse sets in Sec. V. We conclude in Sec. VI.

I. WAVEGUIDE FIELDS

We consider light propagation in a quasi-1D geometry; the propagation direction is taken as $z$. Any background index of refraction is then taken as a function of only $x$ and $y$, $n = n(x, y)$. We neglect any material dispersion in the index of refraction, but that could be easily included in the treatment [12], as could a more general dependence of the index of refraction on position [13], as in a photonic crystal structure.

We treat $D(\mathbf{r}, t)$ and $B(\mathbf{r}, t)$ as the fundamental field operators and look for solutions of Maxwell’s equations of the form

$$D(\mathbf{r}, t) = \sum_{\lambda} D_{\lambda}(\mathbf{r}) e^{-i \omega_{\lambda} t} + h.c.,$$

$$B(\mathbf{r}, t) = \sum_{\lambda} B_{\lambda}(\mathbf{r}) e^{-i \omega_{\lambda} t} + h.c.,$$

where $\lambda = (I k)$ and $I$ labels the “mode type” (in free space, the “mode type” could be polarization). Since we have translational invariance in the $z$ direction, we can seek modes of the form

$$D_{Ik}(\mathbf{r}) = \frac{1}{L} d_{Ik}(\mathbf{r}) e^{ikz},$$

$$B_{Ik}(\mathbf{r}) = \frac{1}{L} b_{Ik}(\mathbf{r}) e^{ikz}.$$

We restrict ourselves to one mode type, since in thermal equilibrium the full density operator is a direct product of the density operators for the different mode types, and find that

$$D(\mathbf{r}) = \sum_{m} \sqrt{\frac{\hbar \omega(k_m)}{2L}} a_{m} d_{k_m}(x,y) e^{ik_{m}z} + h.c.,$$

$$B(\mathbf{r}) = \sum_{m} \sqrt{\frac{\hbar \omega(k_m)}{2L}} a_{m} b_{k_m}(x,y) e^{ik_{m}z} + h.c.,$$

where $d_{k_m}(x,y)$ and $b_{k_m}(x,y)$ are appropriately normalized [12] [13], and where $k_{m} = 2\pi m/L$, where $m$ is a
nonzero integer. We use the index $m$ to identify the lowering and raising operators, $a_m$ and $a_m^\dagger$, respectively, and $\omega(k)$ specifies the dispersion relation of the mode type of interest. Ignoring the zero-point energy, the Hamiltonian takes the form:

$$H = \sum_m \hbar \omega(k_m) a_m^\dagger a_m.$$  

II. THE THERMAL DENSITY OPERATOR AND PARTITIONS OF IT

We now look at the density operator for one of the modes $m$. There are many ways to write out the density operator of a harmonic oscillator in thermal equilibrium; here we will consider the convex decomposition in terms of coherent states,

$$|\alpha_m\rangle = e^{\alpha_m a_m^\dagger - \alpha_m^* a_m} |\text{vac}\rangle,$$

where $\alpha_m$ is a complex number.

The density operator for thermal equilibrium of mode $m$ can be written as

$$\rho_m = \int \frac{d^2 \alpha_m}{\pi \langle n_m \rangle^2} e^{-|\alpha_m|^2/\langle n_m \rangle} |\alpha_m\rangle \langle \alpha_m|,$$

where $\langle n_m \rangle = (1 - e^{-\beta \hbar \omega(k_m)})^{-1}$. The density operator for thermal equilibrium of one mode type is

$$\rho = \prod_m \rho_m,$$

where $\prod_m$ indicates a direct product.

It is not always necessary to consider the entire spectrum of thermal radiation. For some applications, only part of the spectrum may be relevant. It is therefore useful to partition $\rho$ as

$$\rho = \prod_J \rho_J,$$

where $\rho$ is the density operator associated with a portion $J$ of $k$-space:

$$\rho_J = \prod_{m \in S} \rho_m.$$

The tensor product is over a finite set of discrete modes with wavenumbers $\{k_m\}$, defined by the finite set of consecutive integers $S$. The union of all $S$ is $\mathbb{Z}$.

For convenience, we take the number of modes $N$ in each partition to be odd, and write $\{k_m\} = \{k + \kappa_m\}_{m=-n,\ldots,n}$, where $n = (N-1)/2$. We also introduce a “lattice constant” $l = L/N$, which defines the region of $k$-space in portion $J$. Fig. 1 shows a schematic of the modes. Each color corresponds to a different set $J$.

For the remainder of this paper, we will consider thermal light within a particular portion $J$ and leave the quantity $J$ implicit.

III. NONMONOCHROMATIC MODES AND LOCALIZED PULSES

The operators $a_m$ and $a_m^\dagger$ introduced above are associated with modes that are delocalized over the length of the waveguide, and are characterized by eigenfrequencies. In this section we introduce more general, nonmonochromatic modes [13], the coherent states of which describe localized pulses.

Nonmonochromatic modes can be created by making a canonical transformation, introducing new lowering and raising operators $c_s$ and $c_s^\dagger$. We write $c_s = \sum_{m=-n}^n C_{sm} a_m$, where $C_{sm}$ are the elements of a unitary matrix. The operators satisfy $[c_s, c_s^\dagger] = \delta_{ss'}$. These modes can be used to build nonmonochromatic coherent states

$$|\gamma_s\rangle = e^{\gamma_s c_s^\dagger - \gamma_s^* c_s} |\text{vac}\rangle,$$

where $\gamma_s$ is a complex number.

Because $c_s$ satisfy the canonical commutation relations, a set of nonmonochromatic coherent states can be defined as

$$\{|\gamma\rangle\rangle = \bigotimes_s |\gamma_s\rangle,$$

where $s = -n, \ldots, n$.

To define modes that correspond to localized pulses when excited in a coherent state, we first introduce a set of wave functions $\phi_m(z) = e^{ikz} \chi_m(z)$ where $\chi_m(z) = e^{i k_m z}/\sqrt{L}$. The wave functions are orthonormal over the waveguide length $L$, such that $\int_{-L/2}^{L/2} \phi_m(z) \phi_{m'}(z) = \delta_{mm'}$. From $\chi_m(z)$, we construct a function

$$w(z) = \frac{1}{\sqrt{N}} \sum_{m=-n}^n \chi_m(z) = \frac{1}{\sqrt{NL}} \sin\left(\frac{\pi z}{L}\right).$$

Since $\chi_m(z+L) = \chi_m(z)$ we also have that $w(z+L) = w(z)$; that is, it is periodic over the periodic length $L$. Nonetheless, for $z$ close to zero $w(z)$ initially drops off like a sinc function.

![FIG. 1: Schematic of the discrete modes of the waveguide. Each colour corresponds to a different portion $J$ of $k$-space.](image-url)
We now introduce a set of associated functions \( w_s(z) \equiv w(z - sl) \). The functions are also orthonormal over \( L \), such that \( \int_{-L/2}^{L/2} w_s^*(z) w_{s'}(z) \, dz = \delta_{ss'} \).

The relationship between the basis functions can be written in the form

\[
\chi_m(z) = \sum_{s=-n}^{n} w_s(z) C_{sm} \, ,
\]

\[
w_s(z) = \sum_{m=-n}^{n} \chi_m(z) C_{sm}^* \, ,
\]

where

\[
C_{sm} = \frac{1}{\sqrt{N}} e^{\frac{2\pi islm}{\pi}} \, .
\]

The functions \( w_s(z) \) will be associated with the operators \( c_s \) and \( c_s^* \). In the limit \( L \to \infty \), \( w_s(z) \) will become localized, and will correspond to scaling function for the Shannon (sinc) wavelet.

**IV. THERMAL LIGHT AS A MIXTURE OF SETS OF PULSES**

To write the density operator of thermal light as a mixture of sets of localized pulses, we begin with the monochromatic coherent state in Eq. (2) and put \( \gamma_s \equiv \sum_{m=-n}^{n} C_{sm} \alpha_m \) such that

\[
|\alpha_m\rangle = \exp\left( \sum_{s=-n}^{n} \gamma_s c_s^\dagger - \gamma_s^* c_s \right) |\text{vac}\rangle .
\]

Changing variables \( \gamma_s = \tilde{\gamma}_s e^{-\Gamma} \), where

\[
\Gamma = \frac{1}{2N} \sum_{m=-n}^{n} \ln(e^{\beta \omega(k+\kappa_m)} - 1) ,
\]

the decomposition in Eq. (3) is then rewritten as:

\[
\rho = \int \left( \frac{n}{\sum_{s=-n}^{n} d^2\gamma_s} \right) F(\{\tilde{\gamma}\}) \langle \{\tilde{\gamma}e^{-\Gamma}\} | \{\tilde{\gamma}e^{-\Gamma}\} \rangle ,
\]

where, since a unitary matrix relates the \( \gamma_s \) to the \( \alpha_m \), we have taken \( \prod_m d^2\alpha_m = \prod_s d^2\gamma_s \). The probability density function is

\[
F(\{\tilde{\gamma}\}) = \exp\left( - \sum_{s=-n}^{n} \sum_{s'=-n}^{n} \tilde{\gamma}_s \Lambda_{ss'} \tilde{\gamma}_{s'}^* \right) ,
\]

where

\[
\Lambda_{ss'} = e^{-2\Gamma} \sum_{m=-n}^{n} C_{sm}^* C_{s'm}(e^{\beta \omega(k+\kappa_m)} - 1) \, .
\]

We now take the limit to infinite normalization length, as detailed in Appendix B. The range of \( s \) in the product and summations appearing above goes to \(-\infty \) to \( \infty \), and \( s \) ranges over all the integers. The range of \( \kappa_m \) within portion \( J \) becomes \(-\pi/l < k \leq \pi/l \), where \( k \) indicates the continuous version of \( \kappa_m \) as \( L \to \infty \).

We arrive at the main result of this paper, that is, thermal light, in portion \( J \) of \( k \)-space, decomposed into states \( |\{\tilde{\gamma}e^{-\Gamma}\}\rangle \), with probability density function \( F(\{\tilde{\gamma}\}) \):

\[
\rho = \int \left( \prod_{s \in \mathbb{Z}} \frac{d^2\gamma_s}{\pi} \right) F(\{\tilde{\gamma}\}) \langle \{\tilde{\gamma}e^{-\Gamma}\} | \{\tilde{\gamma}e^{-\Gamma}\} \rangle ,
\]

and

\[
\Gamma = \frac{1}{2} \int_{-\pi}^{\pi} \frac{dk}{2\pi} \ln(e^{\beta \omega(k+k) - 1}) ,
\]

and

\[
\Lambda_{ss'} = e^{-2\Gamma} \int_{-\pi}^{\pi} \frac{dk}{2\pi} e^{i(s-s')k}(e^{\beta \omega(k+k) - 1}) .
\]

The state \(|\{\tilde{\gamma}e^{-\Gamma}\}\rangle\) is a product of orthonormal coherent states of localized, nonmonochromatic modes of the waveguide. The state is the quantum analogue of a set of localized, coherent pulses of light.

**V. THE FIELDS**

We now look at the fields. From Eq. (1), the field for portion \( J \) in a waveguide of length \( L \) is

\[
D(r) = \sum_{m=-n}^{n} \sqrt{\frac{\hbar \omega(k+\kappa_m)}{2}} a_m d_{k+\kappa_m}(x,y) \phi_m(z) + \text{h.c.} ,
\]

\[
B(r) = \sum_{m=-n}^{n} \sqrt{\frac{\hbar \omega(k+\kappa_m)}{2}} a_m b_{k+\kappa_m}(x,y) \phi_m(z) + \text{h.c.} .
\]

Making the canonical transformation and taking the limit to infinite length, we get

\[
D(r) = \sum_{s \in \mathbb{Z}} c_s d(x,y; z-sl)e^{ikz} + \text{h.c.} ,
\]

\[
B(r) = \sum_{k \in \mathbb{Z}} c_k b(x,y; z-sl)e^{ikz} + \text{h.c.} ,
\]

where each term in the summation represents the field for a different localized pulse, centered at \( sl \). The field mode for each pulse is given by

\[
d(x,y; z) = \sqrt{L} \int_{-\pi}^{\pi} \frac{dk}{2\pi} \sqrt{\frac{\hbar \omega(k+k)}{2}} d_{k+k}(x,y)e^{ikz} + \text{h.c.} ,
\]

\[
b(x,y; z) = \sqrt{L} \int_{-\pi}^{\pi} \frac{dk}{2\pi} \sqrt{\frac{\hbar \omega(k+k)}{2}} b_{k+k}(x,y)e^{ikz} + \text{h.c.} .
\]
In this limit, we have \( \langle k \rangle \) indicates the wider (bottom). In each case, we consider a “typical” pulse set, as defined in Appendix B. The shaded region in Figure a) \( \langle k \rangle \) shows the total displacement field for a set of pulses \( \langle k \rangle \) (and similarly for \( \langle k \rangle \)). The carrier frequency \( e^{ikz} \) was omitted from the displacement fields for image clarity. Notice that the pulse width decreases as the range of \( k \)-space increases.

\[
\text{FIG. 2: Mean displacement fields for } |\{\gamma e^{-\Gamma}\}| \text{ spanning two representative portions } J \text{ of } k\text{-space; one narrower (top) and one wider (bottom). In each case, we consider a “typical” pulse set, as defined in Appendix B. The shaded region in Figure a) indicates the } k\text{-range with respect to the 1D Planck distribution. Figure b) shows the total displacement field for a set of pulses } \langle k \rangle \text{ given by the first term of Eq. (8). Figure c) shows the displacement field for individual representative pulses } \langle k \rangle \text{ (and similarly for } \langle k \rangle \text{). The carrier frequency } e^{ikz} \text{ was omitted from the displacement fields for image clarity. Notice that the pulse width decreases as the range of } k\text{-space increases.}
\]

where \( d_{k+k}(x,y) \) and \( b_{k+k}(x,y) \) are the transverse modes of the waveguide.

Now consider a particularly simple limit, when the range of integration over \( k \) is taken to be so small that we can approximate \( \omega(k+k) \) as \( \omega(k) \), and we can take \( d_{k+k}(x,y) \) to be \( d_k(x,y) \) (and similarly for \( b_{k+k}(x,y) \)). In this limit, we have

\[
d(x,y; z) \approx \sqrt{\frac{\hbar \omega(k)}{2}} d_k(x,y) W(z) + h.c.,
\]

where

\[
W(z) = \frac{2\pi}{\sqrt{1}} \text{sinc } \left( \frac{\pi z}{4} \right).
\]

This can be understood as the limit of \( w(z) \) as the normalization length goes to infinity; the function is thus now no longer periodic. In the language of solid state physics, this would be an “empty lattice Wannier function.” In analogy with \( w_s(z) \) we can define \( W_s(z) \) centered at different “lattice sites”

\[
W_s = W(z - sl).
\]

In this approximation, we can write

\[
D(r) = e^{ikz} \sqrt{\frac{\hbar \omega(k)}{2}} d_k(x,y) \sum_{s \in Z} c_s W_s(z) + h.c.,
\]

\[
B(r) = e^{ikz} \sqrt{\frac{\hbar \omega(k)}{2}} b_k(x,y) \sum_{s \in Z} c_s W_s(z) + h.c.
\]

The expectation values for the fields of a pulse set \( |\{\gamma e^{-\Gamma}\}| \) are

\[
\langle D(r) \rangle = e^{ikz} \sqrt{\frac{\hbar \omega(k)}{2}} d_k(x,y) \sum_{s \in Z} \bar{\gamma}_s e^{-\Gamma} W_s(z) + c.c.,
\]

\[
\langle B(r) \rangle = e^{ikz} \sqrt{\frac{\hbar \omega(k)}{2}} b_k(x,y) \sum_{s \in Z} \bar{\gamma}_s e^{-\Gamma} W_s(z) + c.c.
\]

The expectation values for the fields of a single pulse \( |\gamma e^{-\Gamma}\rangle \) are

\[
\langle D_s(r) \rangle = e^{ikz} \sqrt{\frac{\hbar \omega(k)}{2}} d_k(x,y) \bar{\gamma}_s e^{-\Gamma} W_s(z) + c.c.,
\]

\[
\langle B_s(r) \rangle = e^{ikz} \sqrt{\frac{\hbar \omega(k)}{2}} b_k(x,y) \bar{\gamma}_s e^{-\Gamma} W_s(z) + c.c.
\]

In Fig. 2 we show the positive frequency components of the mean displacement fields for “typical” pulse sets, as defined in Appendix B. Total field is plotted in 2 b) and the fields for individual pulses are plotted in 2 c). Notice that the width of the pulses decreases as \( \rho \) spans an increasingly larger range of \( k\)-space.

**VI. CONCLUSION**

The relationship between thermal light and coherent pulses is of particular interest to those studying energy transfer in photosynthetic systems [1,9]. While thermal
light cannot be represented as a statistical mixture of localized pulses [10], we have shown here how to decompose thermal light in a 1D waveguide into a statistical mixture of sets of localized pulses. Our results can also be applied when there is no optical fiber or confining geometry, but rather there exists a “column” of light, neglecting diffraction. We plan to turn to more generalizations in later communications.

The form of the convex decomposition we have introduced makes modeling a finite frequency range very natural, while maintaining a representation in terms of localized pulses; this would arise when dealing with filtered thermal light. The decomposition also lends itself to treating thermal light that has been “chopped” in the spatial domain, as long as the length is much greater than the width of the function \( w(z) \). This can simply be done by truncating the range of \( s \) in Eq. (7). We anticipate that this decomposition will serve as a useful tool for studying interactions of matter with thermal light in 1D.

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Appendix A: Taking the limit

Consider a waveguide with a normalization length \( L \), with \( N \) modes in the portion \( J \) of \( k \)-space. The modes are represented by the set \( \{k_m\} \) where \( m \) is taken from a set of consecutive integers.

But recall that \( k_m = 2\pi m/L \). As the normalization length increases, the density of modes also increases, and naively \( \{k_m\} \) would span a progressively narrower portion of \( k \)-space. To take the limit to infinite normalization length while keeping fixed the range in \( k \)-space spanned by \( \rho J \), the number of elements in \( \{k_m\} \) should be increased appropriately. We arrange this as follows. We relabel \( L \equiv L_0 \) and \( N \equiv N_0 \), and define \( L_j = 3L_{j-1} \) and \( N_j = 3N_{j-1} \). The limit to infinite length is then acquired by taking \( j \to \infty \). In this way, the “lattice constant” \( l = L_j/N_j \) remains constant, as does the region on \( k \)-space.

FIG. 3: Schematic of the discrete modes of the waveguide for \( j = 0 \) (top) and \( j = 1 \) (bottom). Each colour corresponds to a different portion \( J \) of \( k \)-space. Note that as \( j \) increases, \( \bar{k} \) moves closer to the centre of the region of length \( 2\pi/l \), where \( l = L/N \) is a “lattice constant”.

Appendix B: Choosing “typical” pulse trains

Eq. (7) is the expression for thermal light, in portion \( J \) of \( k \)-space, decomposed into pulse sets. Each pulse set contains an infinite number of pulses.

Now consider the field in a finite region of space; pulses far away from that region will make a negligible contribution. We therefore only consider a finite subset of all pulses when plotting the field variation. We denote this subset \( S \). Then what we really want to think about is

\[
\rho_S = \int \left( \prod_{s \in S} \frac{d^2 \bar{\gamma}_s}{\pi} \right) F(\{\bar{\gamma}\})|\{\bar{\gamma}e^{-\Gamma}\}| \langle \{\bar{\gamma}e^{-\Gamma}\} \rangle ,
\]

where

\[
F(\{\bar{\gamma}\}) = \exp \left( -\sum_{s \in S} \sum_{s' \in S} \bar{\gamma}_s A_{ss'} \bar{\gamma}_{s'}^* \right) , \quad (B1)
\]

and

\[
|\{\bar{\gamma}e^{-\Gamma}\}| = \prod_{s \in S} |\bar{\gamma}_se^{-\Gamma}| .
\]

To plot the field variation for a “typical” pulse set, we want to select a “likely” set \( \bar{\gamma}_{\text{set}} e^{-\Gamma} \) from the distribution \( F(\{\bar{\gamma}\}) \), Eq. (B1). To do this, our approach is to write \( F(\{\bar{\gamma}\}) \) as a product of simpler distributions. Notice that the matrix \( A_{ss'} \) is Hermitian, so it can be diagonalized by a unitary transformation. Then we have

\[
\sum_{s,s' \in S} U_{ss'}^* A_{ss'} U_{s's'} = \theta, \delta_{rr'},
\]
where $U^{\dagger} \Delta U$ is a diagonal matrix with elements $\theta_r$. We put $\eta_r = \sum_{s \in S} U_{sr} \bar{\gamma}_s$ and write

$$\rho = \int \left( \prod_{r \in S} \frac{d^2 \eta_r}{\pi} \right) e^{-\sum_{r \in S} \theta_r |\eta_r e^{-\Gamma}|^2} |\{ \eta_r e^{-\Gamma} \rangle \langle \{ \eta_r e^{-\Gamma} \}|. $$

We can now ask: In this mixture, how do we characterize the probability associated with the pulse set $|\{ \bar{\gamma}_r \rangle \rangle$? Writing $\eta_r = |\eta_r| e^{i \phi_r}$, we have $d^2 \eta_r = (d \phi_r d|\eta_r|)|\eta_r|$. For each $r$, any $\phi_r$ is between 0 and $2\pi$ and is equally likely. But $|\eta_r|$, which ranges from 0 to $\infty$, needs to be taken from the distribution $|\eta_r| e^{-\theta_r |\eta_r|}$, which peaks at $|\eta_r| = 1/\sqrt{2\theta_r}$.

Our method for identifying very “likely” pulse sets is as follows. Find the matrix $U$ and diagonal values $\theta_r$ by diagonalizing $\Lambda$. Then for each $r$: 1) choose $\phi_r$ at random from 0 to $2\pi$; and 2) take $|\eta_r| = 1/\sqrt{2\theta_r}$. From the set of complex numbers $\eta_r$, and the matrix $U$, identify the set of amplitudes $\bar{\gamma}_s = \sum_{r \in S} U_{sr}^* \eta_r$.

For each random set of phases $\{ \phi_r \}$, with each $|\eta_r| = 1/\sqrt{2\theta_r}$, we will get a very “likely” pulse set, and those sets with different random phases will be “equally likely”. Pulse sets that are “less likely” can be investigating by putting $|\eta_r| \neq 1/\sqrt{2\theta_r}$.

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