A Search for New (2, 2) Strings

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Abstract

There are at present two known string theories in (2, 2) dimensions. One of them is the well known $N = 2$ string, and the other one is a more recently constructed $N = 1$ spacetime supersymmetric string. They are both based on certain twistings and/or truncations of the small $N = 4$ superconformal algebra, realised in terms of (2, 2) superspace variables. In this paper, we investigate more general possibilities for string theories based on algebras built with the same set of fields. We find that there exists one more string theory, based on an algebra which is not contained within the $N = 4$ superconformal algebra. We investigate the spectrum and interactions of this theory.

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1 Introduction

There are at present two known string theories in 2+2 dimensions. One of them is based on the $N = 2$ superconformal algebra, and it has the interesting feature of describing self-dual gravity in 2+2 dimensions. One of the surprising features of this model is that notwithstanding the worldsheet supersymmetry, it lacks spacetime supersymmetry. In search of an alternative string model in 2+2 dimensions which does exhibit spacetime supersymmetry, a second string model was recently proposed. The underlying conformal algebra in this case is a twisted and truncated version of the “small” $N = 4$ superconformal algebra. This model indeed has spacetime supersymmetry; however, it turns out that the physical spectrum contains an infinite tower of massive states, and the massless states do not describe supergravity in 2+2 dimensions.

Both of the above models can be constructed by making use of bilinear combinations of the bosonic coordinates $X^{\alpha\dot{\alpha}}$, fermionic coordinates $\theta^\alpha$, and their conjugate momenta $p_\alpha$, to build the currents of the underlying worldsheet algebras. The indices $\alpha$ and $\dot{\alpha}$ label the two-dimensional spinor representations of $SL(2)_R \times SL(2)_L \approx SO(2,2)$. Note that because the Lorentz group factorises, we can use spinors of one handedness only.

In terms of these variables, the currents for the $N = 2$ superconformal algebra are

$$T = -\frac{1}{2} \partial X^{\alpha\dot{\alpha}} \partial X_{\alpha\dot{\alpha}} - p_\alpha \partial \theta^\alpha, \quad J = p_\alpha \theta^\alpha,$$

$$G^i = \theta_\alpha \partial X^{\alpha i}, \quad G^{\dot{i}} = p_\alpha \partial X^{\alpha \dot{\alpha}}. \quad (1)$$

This is a twisted version of the usual realisation, since here the $(p, \theta)$ system has dimension $(1,0)$. In this realisation, manifest $SL(2)_L$ invariance is broken. It can be restored by enlarging the set of currents to those of the small $N = 4$ superconformal algebra, given, again in a twisted basis, by

$$T = -\frac{1}{2} \partial X^{\alpha\dot{\alpha}} \partial X_{\alpha\dot{\alpha}} - p_\alpha \partial \theta^\alpha,$$

$$G^{\dot{\alpha}} = p_\alpha \partial X^{\alpha \dot{\alpha}}, \quad \tilde{G}^{\dot{\alpha}} = \theta_\alpha \partial X^{\alpha \dot{\alpha}}. \quad (2)$$
\[ J_0 = p_\alpha \theta^\alpha, \quad J_+ = p_\alpha p^\alpha, \quad J_- = \theta_\alpha \theta^\alpha. \]

Naively, this system appears to be non-critical. However, the currents are reducible, and a proper quantisation requires the identification of the irreducible subset \([4]\), namely the \(N = 2\) currents given in \([3]\).

The currents of the second string model discussed above, which has \(N = 1\) spacetime supersymmetry, can also be expressed in a fully covariant way. They are given by \([3]\)

\[ T = -\frac{1}{2} \partial X^{\alpha \dot{\alpha}} \partial X_{\alpha \dot{\alpha}} - p_\alpha \partial \theta^\alpha, \quad G^\beta = p_\alpha \partial X^{\alpha \dot{\alpha}}, \quad J = p_\alpha p^\alpha. \quad (3) \]

In fact, these currents are a truncation of the realisation of the small \(N = 4\) superconformal algebra given in \([2]\). All the currents have spin two, and hence the system is critical, with zero central charge. However, this set of currents is reducible. An irreducible subset, which is also critical, is given by \(T\) and \(G^1\). The BRST cohomology of this system was studied in Ref. \([3]\).

It is interesting to investigate if there can exist other string theories in 2+2 dimensions, based on other bilinear currents built from the same ingredients. In this paper, we investigate this question systematically, and we show that indeed there exists one more possibility. The criteria that lead to this conclusion are, in addition to closure, that the currents are irreducible, and that they give rise to a critical realisation of the worldsheet algebra. The set of currents we have found is

\[ T = -\frac{1}{2} \partial X^{\alpha \dot{\alpha}} \partial X_{\alpha \dot{\alpha}} - p_\alpha \partial \theta^\alpha, \quad J = \partial (\theta_\alpha \theta^\alpha), \]

\[ G^1 = p_\alpha \partial X^{\alpha 1}, \quad \tilde{G}^1 = \theta_\alpha \partial X^{\alpha 1}. \quad (4) \]

It is easy to see that the currents \((T, G^1, \tilde{G}^1, J)\) have spins \((2, 2, 1, 1)\). In addition to the standard OPEs of \(T\) with \((T, J, G^1, \tilde{G}^1)\), the only non-vanishing OPE is

\[ J(z) G^1(w) \sim \frac{2 \tilde{G}^1}{(z-w)^2} + \frac{\partial \tilde{G}^1}{(z-w)}. \quad (5) \]

This algebra is related to the small \(N = 4\) superconformal algebra, not directly as a subalgebra, but in the following way. The subset of currents \(T, G^1, \tilde{G}^1\) and \(J_-\) in \([2]\)
form a critical closed algebra. However these currents form a reducible set. To achieve irreducibility, we simply differentiate the current $J_-$, thereby obtaining the set of currents given in (3). Note that taking the derivative of $J_-$ still gives a primary current with the same anomaly contribution, since $12s^2 - 12s + 2$ takes the same value for $s = 0$ and $s = 1$.

We shall study the string theory based on this algebra.

To proceed with the BRST quantisation of the model, we introduce the fermionic ghost fields $(c, b)$ and $(\gamma, \beta)$ for the currents $T$ and $J$, and the bosonic ghost fields $(s, r)$ and $(\tilde{s}, \tilde{r})$ for $G^1$ and $\tilde{G}^1$. It is necessary to bosonise the commuting ghosts, by writing $s = \eta e^\phi$, $r = \partial \xi e^{-\phi}$, $\tilde{s} = \tilde{\eta} e^{\tilde{\phi}}$ and $\tilde{r} = \partial \tilde{\xi} e^{-\tilde{\phi}}$. The BRST operator for the model is then given by

$$Q = \oint c \left( -\frac{1}{2} \partial X_{\alpha \dot{\alpha}} \partial X^{\alpha \dot{\alpha}} - p_\alpha \partial \theta^\alpha - \frac{1}{2} (\partial \phi)^2 - \frac{3}{2} \partial^2 \phi - \frac{1}{2} \partial^2 \tilde{\phi} ight)$$

$$- \eta \partial \xi - \tilde{\eta} \partial \tilde{\xi} - b \partial c - \beta \partial \gamma + \eta e^\phi p_\alpha \partial X^{\alpha \dot{\alpha}} + \tilde{\eta} e^{\tilde{\phi}} \theta^\alpha \partial X^{\alpha} + \partial \gamma \left( \frac{1}{2} \theta^\alpha \theta_\alpha - \partial \xi \eta e^\phi - \partial \tilde{\xi} e^{\tilde{\phi}} \right).$$

Since the zero modes of $\xi$ and $\tilde{\xi}$ do not appear in the BRST operator, there exist BRST non-trivial picture-changing operators:

$$Z_\xi = \{ Q, \xi \} = c \partial \xi + e^\phi p_\alpha \partial X^{\alpha \dot{\alpha}} - \partial \gamma \partial \tilde{\xi} e^{\phi - \tilde{\phi}},$$

$$Z_{\tilde{\xi}} = \{ Q, \tilde{\xi} \} = c \partial \tilde{\xi} + e^{\tilde{\phi}} \theta^\alpha \partial X^{\alpha \dot{\alpha}}. \quad (7)$$

It turns out that these two picture changers are not invertible. Thus, one has the option of including the zero modes of $\xi$ and $\tilde{\xi}$ in the Hilbert space of physical states. This would not be true for a case where the picture changers were invertible. Under these circumstances, the inclusion of the zero modes would mean that all physical states would become trivial, since $|\text{phys}\rangle = Q(\xi Z_\xi^{-1}|\text{phys}\rangle)$. However, in this paper we shall nevertheless choose to

3We have concentrated on conformal algebras which make use of the $SL(2)_R$ spinor variables $(p_\alpha, \theta^\alpha)$. A priori, we could also use the $SL(2)_L$ spinor variables $(p_\alpha, \theta^{\dot{\alpha}})$. Interestingly enough, no new critical algebra seems to arise in this fashion.
exclude the zero modes of $\xi$ and $\tilde{\xi}$ from the Hilbert space. It is interesting to note that in this model the zero mode of the ghost field $\gamma$ for the spin–1 current is also absent in the BRST operator. If one excludes this zero mode from the Hilbert space, one can then introduce the corresponding picture-changing operator $Z_\gamma = \{Q, \gamma\} = c \partial \gamma$. In this paper, we shall indeed choose to exclude the zero mode of $\gamma$.

In order to discuss the cohomology of the BRST operator (6), it is convenient first to define an inner product in the Hilbert space. Since the zero modes of the $\xi$, $\tilde{\xi}$ and $\gamma$ are excluded, the inner product is given by

$$\langle \partial^2 c \partial c c \theta^\alpha \theta_\alpha e^{-3 \phi - \tilde{\phi}} \rangle = 1 .$$

Let us first discuss the spectrum of massless states in the Neveu-Schwarz sector. The simplest such state is given by

$$V = c e^{-\phi - \tilde{\phi}} e^{ip \cdot X} .$$

As in the case of the $N = 2$ string discussed in [5], since the picture-changing operators are not invertible the massless states in different pictures cannot necessarily all be identified. In fact, the picture changers annihilate the massless operators such as (3) when the momentum $p^{\alpha \dot{1}}$ is zero. However, massless operators in other pictures still exist at momentum $p^{\alpha \dot{1}} = 0$. For example, in the same picture as the physical operator $Z_{\tilde{\xi}} V$ that vanishes at $p^{\alpha \dot{1}} = 0$ is a physical operator that is non-vanishing for all on-shell momenta, namely

$$\Psi = h_\alpha c \theta^\alpha e^{-\phi} e^{ip \cdot X} ,$$

which is physical provided that $p^{\alpha \dot{1}} h_\alpha = 0$ and $p_{\alpha \dot{\alpha}} p^{\alpha \dot{\alpha}} = 0$. In fact, $Z_{\tilde{\xi}} V$ is nothing but $\Psi$ with the polarisation condition solved by writing $h_\alpha = p^{\alpha \dot{1}}$. However, we can choose instead to solve the polarisation condition by writing $h_\alpha = p^{\alpha \dot{2}}$, which is non-vanishing even when $p^{\alpha \dot{1}} = 0$. Thus, the operators $V$ and $\Psi$ cannot be identified under picture changing when $p^{\alpha \dot{1}} = 0$. In fact when $p^{\alpha \dot{1}} = 0$ there is another independent solution for $\Psi$, since the polarisation condition becomes empty in this case. A convenient way to
describe the physical states is in terms of $\Psi$ given in (10), with the polarisation condition re-written in the covariant form $p^{\alpha} h_\alpha = 0$, together with a further physical operator which is defined only when $p^{\alpha_1} = 0$. In this description, the physical operator $\Psi$ is defined for all on-shell momenta.

If one adopts the traditional viewpoint that physical operators related by picture changers describe the same physical degree of freedom, one would then interpret the spectrum as containing a massless operator (9), together with an infinite number of massless operators that are subject to the further constraint $p^{\alpha_1} = 0$ on the on-shell momentum. This viewpoint is not altogether satisfactory in a case such as ours, where the picture changing operators are not invertible. An alternative, and moreover covariant, viewpoint is that the physical operators in different pictures, such as $V$ and $\Psi$, should be viewed as independent. At first sight one might think that this description leads to an infinite number of massless operators. However, as we shall see later, the interactions of all the physical operators can be effectively described by the interaction of just the two operators $V$ and $\Psi$. Thus the theory effectively reduces to one with just two massless operators, one a scalar and the other a spinorial bosonic operator.

Let us now turn our attention to the spectrum of massive states. We find that there exist no states with negative $(\text{mass})^2$. There are states of positive $(\text{mass})^2$ of the following form

$$
\begin{align*}
V_n^{(\delta)} &= c \left( \partial^n p \right)^2 \cdots p^2 e^{n\phi - (n+2)\bar{\phi}} \partial^{2n+2+\delta} \cdots \partial^\gamma e^{i\nu X} , \\
U_n^{(\delta)} &= c \left( \partial^{n+1} \theta \right)^2 \cdots \theta^2 e^{-(n+3)\phi + (n+1)\bar{\phi}} \partial^{2n+1+\delta} \beta \cdots \beta e^{i\nu X} ,
\end{align*}
$$

(11)

where $n \geq -1$, $\delta = 0$ or 1 and the mass is given by $\mathcal{M}^2 = (2n + 2 + \delta)(2n + 3 + \delta)$. Note that $V_n^{(0)}$ corresponds to the massless operator (9), and $\partial c \theta^{\alpha} \theta_\alpha U_n^{(0)}$ is its conjugate. All the other operators in (11) have $\mathcal{M}^2 > 0$ and they are in the non-standard ghost sector.

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4It should be emphasised that the possibility of having $p^{\alpha_1} = 0$ while $p^{\alpha_2} \neq 0$ is a consequence of our having chosen a real structure on the $(2, 2)$ spacetime [3, 3], rather than the more customary complex structure [2].
It is worth remarking that the massive operators (11) assume a particularly simple form if we bosonise the fields \((p_\alpha, \theta^\alpha)\) as \(p_\alpha = e^{-i\sigma_\alpha}\) and \(\theta^\alpha = e^{i\sigma_\alpha}\). They become purely exponential vertex operators:

\[
V_n^{(\delta)} = c e^{-i(n+1)(\sigma_1 + \sigma_2)} e^{-n\phi - (n+2)\tilde{\delta}\partial^2 n + 2 + \delta} \cdots \partial \gamma e^{ip\cdot X},
\]

\[
U_n^{(\delta)} = c e^{i(n+2)(\sigma_1 + \sigma_2)} e^{-(n+3)\phi + (n+1)\tilde{\delta}\partial^2 n + 1 + \delta} \cdots \beta e^{ip\cdot X}.
\] (12)

Having written the physical operators in this bosonised form, we can now allow \(n\) to take half-integer as well as integer values, corresponding to physical operators in Ramond and Neveu-Schwarz sectors respectively. One can easily verify that all the NS and R states are local to each other, in the sense that their OPEs with each other have integer poles. It follows from the fact that \(n \geq -1\) that there is no massless operator in the R sector; however, at each massive level, there is one NS operator and one R operator.

The restriction that \(n\) can only take integer and half-integer values comes from the fact that we have chosen not to include the zero mode of the \(\gamma\) field in the Hilbert space of physical states. If we had instead included this zero mode, we could then have bosonised the \((\gamma, \beta)\) system, in which case it would be \(\gamma V_n^{(\delta)}\) and \(U_n^{(\delta)}\) that were physical, for all values of \(n\). It follows from the mass formula \(M^2 = (2n + 2 + \delta)(2n + 3 + \delta)\) that the mass spectrum of the physical states would then be continuous. Thus the exclusion of the zero mode of \(\gamma\) from the Hilbert space is clearly desirable.

The rôle of the picture-changing operators is quite different for the massive physical operators from that for the massless operators. As we discussed above, the massless physical operators reside only in the NS sector. The picture-changing operators cannot be used to identify all these massless physical operators in different pictures. Instead, the theory effectively contains two massless operators. However, for the massive physical operators, the picture-changing operators can be used to identify the physical operators in different pictures and hence at each mass level, there is one NS operator and one R operator. We shall demonstrate this with an example. Let us consider the simple massive
operator \( V^{(0)}_0 \) given in (11). The physical operator with the same picture as \( Z^{\xi}V^{(0)}_0 \) is given by

\[ \Psi = h^\alpha c p_\alpha e^{-\phi^2} \partial^\gamma e^{ip \cdot X}. \]

It is physical provided that \( p_{a\alpha} p^{a\alpha} = -6 \) and \( p^{a\alpha} h_\alpha = 0 \). The solution for the polarisation spinor is \( h^a = p^{a\alpha} \). Note that \( p^{a\alpha} \) can never be zero for a massive operator, and furthermore it is not proportional to \( p^{a\beta} \). It follows that the physical operator \( \Psi \) is precisely related to the operator \( V^{(1)}_0 \) by picture changing, namely \( \Psi = Z^{\xi}V^{(1)}_0 \). This example illustrates that the massive physical operators in different pictures can be identified by picture changers for all on-shell momenta. Thus, to summarise, at each mass level there are two physical degrees of freedom. At the massless level, they correspond to two NS operators; At each massive level, they correspond to one NS operator and one R operator.

These operators are only a subset of the totality of massive operators in the theory. In particular, we observe that \( \mathcal{M}^2 \) is proportional to \( n^2 \). One normally expects that the massive states should have \( \mathcal{M}^2 \) proportional to \( n \). Indeed, as we shall show later, such states do occur. However, these types of massive states are more complicated.

We now turn to the discussion of the interactions in the theory. We first consider interactions involving only the massless operators. If we adopt the interpretation that the massless spectrum is described by the operator (9), together with the additional operators subject to the further momentum constraint \( p^{a\alpha} = 0 \), there is a non-vanishing three-point amplitude given by

\[
\left\langle \bar{Z}_\xi V(z_1, p_{(1)}) Z_\xi V(z_2, p_{(2)}) V(z_3, p_{(3)}) \right\rangle = p^{\alpha\beta}_{(1)p_{(2)}}. \tag{13}
\]

The four-point amplitude \( \left\langle Z_\xi V Z_\xi V Z_\xi V \right\rangle \) vanishes for kinematical reasons [5]. We expect that the higher-point amplitudes also vanish for the same reason.

The above description is not satisfactory since the result for the three-point amplitude breaks Lorentz invariance. Furthermore, it is not complete since there are an infinite number of physical operators with the further momentum constraint \( p^{a\alpha} = 0 \) that can interact with each other. As we discussed earlier, the massless spectrum can be better described by the scalar operator (8) together with the spinorial bosonic operator (10),
without the use of the picture-changing operators. There is one three-point interaction between these two operators, namely

\[ \langle \Psi(z_1, p(1)) \Psi(z_2, p(2)) V(z_3, p(3)) \rangle = h_{(1)\alpha} h_{(2)\beta} \cdot \]  

(14)

Note that this three-point amplitude is manifestly Lorentz invariant, and reduces to (13) if one solves for the polarisation spinors by writing \( h^{\alpha} = p^{\alpha \dot{1}} \). There are also an infinite number of massless physical operators with different pictures in the spectrum, and they can all be expressed in a covariant way. As one steps through the picture numbers, the character of the physical operators alternates between scalar and spinorial. The three-point interactions of all these operators lead only to the one amplitude given by (14).

In view of their equivalent interactions, all the scalar operators can be identified and all the spinorial operators can be identified. The massless spectrum can thus be effectively described by the scalar operator (9) and the spinorial operator (10). All four-point and higher amplitudes vanish.

Let us now consider the interactions of the massive states. In particular we are interested in four-point amplitudes since they give information about new massive states. We find a non-vanishing four-point amplitude, for physical operators with \((\text{mass})^2\) equal to 0, 0, 6, 6 respectively, given by

\[ \langle \Phi \Psi Z(z) V(0) Z U(0) \rangle = \langle c h_{(1)\alpha} p_\alpha e^{-\phi} e^{ip_{(1)} \cdot X} c h_{(2)\alpha} e^{-\phi} e^{ip_{(2)} \cdot X} \times \]
\[ \times \int P_{(3)} p_\alpha e^{-\phi} \theta^2 \partial^2 \theta \theta e^{ip_{(3)} \cdot X} c P_{(4)} \partial \theta^2 e^{-2\phi + \phi \theta^2} e^{ip_{(4)} \cdot X} \rangle \]
\[ = \left( \frac{\pi}{2} - 3 \right) A - \frac{\pi}{2} B \right) \frac{\Gamma\left(-\frac{s}{2}\right) \Gamma\left(-\frac{t}{2} + 3\right)}{\Gamma\left(\frac{s}{2} - 2\right)} , \]

(15)

where \( s, t \) and \( u \) are the Mandlestam variables satisfying \( s + t + u = 12 \), and \( A = h_{(1)\alpha} h_{(2)\alpha} \) and \( B = h_{(1)\alpha} P_{(3)} h_{(2)\beta} P_{(4)\beta} \). The kinematic factor in the four-point amplitude (15) is non-zero in general, implying the existence of an infinite tower of physical operators whose \((\text{mass})^2\) is linearly dependent on the integer \( n \). In particular, the sequence of poles in the \( s \) channel of the four-point amplitude implies that there is an infinite tower of massive physical operators with standard ghost structure.
Although the theory contains an infinite tower of physical operators, the massless sector and its interactions are remarkably simple. In particular, although all the massive physical operators break Lorentz invariance, the massless operators and their interactions have manifest spacetime Lorentz invariance. If we associate spacetime fields $\phi$ and $\psi_\alpha$ with the physical operators $V$ and $\Psi$, it follows from the three-point amplitude (14) that we can write the field equations:

$$\partial_{\dot{a}\dot{a}}\partial^{\alpha\dot{\alpha}}\phi = \psi^\alpha \psi_\alpha, \quad \partial_{\dot{a}\dot{a}}\psi^\alpha = \psi^\alpha \partial_{\dot{a}\dot{a}}\phi.$$  

(16)

We have suppressed Chan-Paton group theory factors that must be introduced in order for the three-point amplitude to be non-vanishing in the open string. It is easy to see even from the kinetic terms in the field equations (14) that there is no associated Lagrangian. Note that there is no undifferentiated $\phi$ field, owing to the fact that the theory is invariant under the transformation $\phi \rightarrow \phi + \text{const}$. This can be easily seen from the three-point function (14), which vanishes when $p_{(3)}^{\alpha\dot{\alpha}} = 0$, corresponding to a constant spacetime field $\phi$. It is of interest to obtain the higher-point amplitudes from the field equations (16), which should be zero if they are to reproduce the string interactions.

To summarise, assuming (a) the world-sheet field content $(p_\alpha, \theta, X^{\alpha\dot{\alpha}})$, (b) irreducibility and (c) the quadratic nature of the constraints, we have shown that there exists one more possible string, in addition to the two previously studied strings, in 2+2 dimensions. Among the three, only the string theory constructed in [3] has spacetime supersymmetry. There are several possible generalisations of this work. For example, one may introduce extra world-sheet fields. Furthermore, constraints that are higher order polynomials in basic variables could be considered. Finally, an interesting open problem concerns the possibility of extending these constructions to higher-dimensional spacetimes.

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