On almost commutative Friedmann–Lemaître–Robertson–Walker geometries

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Abstract
We analyze the leading terms of the spectral action for a model of noncommutative geometry, which is a product of 4D Riemannian manifold with a two-point space exploring the previously neglected case when the metrics over each sheet are different. Assuming the Friedmann–Lemaître–Robertson–Walker type of the metric for both sheets we obtain the action, which in addition to the usual cosmological constant terms and the Einstein–Hilbert term involves a nonlinear interaction term. We study qualitative picture of potential consequences of such term in the basic cosmological models.

Keywords: noncommutative geometry, cosmology, bimetric gravity

1. Introduction

Cosmological models are based on the Einstein equations, which link the geometry of the universe with the energy–momentum density containing contributions of the matter, radiation and dark energy (cosmological constant). Such models have been thoroughly studied both in the standard case, originating from the Einstein–Hilbert action, as well as from possible modifications of gravity. Most of such modification were based rather on classical extensions of space-time geometry then on modifying the basic formulations of geometry.

Noncommutative geometry [1–3], which has been studied extensively in the physical context rather in relation to fundamental interactions of elementary particles, offers a new insight into our understanding of the metric. In particular some of the simplest models are of Kaluza–Klein type, with the extra dimensions being of the finite type (that is consisting of finite number of points). This allows to study some new effects and effectively draw some basic conclusions that could have cosmological implications.

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In this paper we aim to study the simplest geometry of the product of a continuous spacetime with a two-point space. That roughly corresponds to the Connes–Lott model of particle physics studied in the noncommutative setup [4], where the two points are reflected in the chirality of the fundamental fermions. However, contrary to the usual assumptions we want to investigate metrics that are not of product type, which means that they might differ on the two sheets of spacetime. As the internal metric between the points has the natural interpretation of the Higgs field, we shall see that the natural generalization of the Einstein–Hilbert action introduces a new term, which in the broken symmetry phase allows for the interaction between the two metrics.

The paper is organized as follows: first we present the basic tools and notation for the geometry studied including the spectral triple of the model and the spectral action. We present the effective methods of computing the action using the Wodzicki residue over the pseudodifferential calculus of symbols and derive the action functional for the model of Friedmann–Lemaître–Robertson–Walker type geometries. Though the basic model is Euclidean in order to compute the spectral action, a passage through Wick rotation to the Lorentzian case is possible and explained. Using the resulting Lorentzian version of the action we derive and study the equations of motions and qualitatively analyze three cases of simple cosmological models.

2. Almost commutative geometries and spectral action

An almost commutative geometry is a model based on the product geometry of the compact Riemannian spin manifold with a finite dimensional space (not necessarily commutative) which is described through a finite-dimensional spectral triple. Such model was among the first ones to be considered [4] by Connes–Lott and has led to the interpretation of the Higgs field as a connection arising from the geometry of the finite space. That is the minimal noncommutative extension of the classical geometry, which is basically of Kaluza–Klein type, however, with the internal space that is not a manifold. The simplest version of the discrete geometry is a two-point space described by its algebra of complex-valued functions $A_F = \mathbb{C} \oplus \mathbb{C}$.

Although such ‘spaces’ are not described by the usual differential geometry, the noncommutative geometry offers a way to treat both manifolds as well discrete spaces and finite-dimensional algebras (not necessarily commutative) on equal footing. Such noncommutative extension of the standard differential geometry uses the construction of spectral triples [3]. In short, a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ consists of the following data: the algebra $\mathcal{A}$ (which in the classical case is the algebra of smooth functions over the manifold), faithfully represented as bounded operators on the Hilbert space $\mathcal{H}$, and an unbounded self-adjoint operator, such that for every $a \in \mathcal{A}$ the commutator $[D, \pi(a)]$ is bounded, where $\pi(a)$ denotes the representation. The classical example of a spectral triple is provided by a compact Riemannian spin manifold $M$ and $(\mathcal{C}^{\infty}(M), L^2(S), D)$, where $L^2(S)$ denotes the Hilbert space of square-summable sections of spinors and $D$ is the usual Dirac operator.

The metric is then implicitly encoded in the Dirac operator $D$ and the gravity action functional is constructed from the spectral data of the Dirac operator, using, for example, the terms from the asymptotic expansion of the trace of the operator $e^{-t D^2}$ [2] or directly computing the Wodzicki residue of the inverse of the square of the Dirac operator [5]. Though the observation that the heat trace coefficients for the Laplace operator are expressed through local geometric objects like curvature tensor and can be used to introduce gravity action appeared first some time ago [6], it was only noncommutative geometry that proposed it as a universal principle, which is applicable for a much larger class of geometries.
2.1. Spectral triples for almost commutative geometries

We begin with a short presentation of a spectral triple that is a minimal noncommutative extension of the classical geometry and will be the basis to study the models with a generalized Friedmann–Lemaître–Robertson–Walker type metric. Let $M$ be a Riemannian spin manifold and $S$ its bundle of spinors. Additionally we assume $M$ to be even-dimensional, with the chiral grading $\gamma$, which is $+1$ on right and $-1$ on left-handed spinors. Consider the commutative algebra $\mathcal{A} = C^\infty(M) \otimes (\mathbb{C} \oplus \mathbb{C})$, represented on the Hilbert space $L^2(S) \otimes \mathbb{C}^2$. The algebra can be seen an algebra of smooth functions on $M$ valued in the diagonal 2 by 2 complex matrices and its representation is then natural multiplication from the left on two copies of the spinor fields. This simple geometry is, in fact, a product geometry and the underlying space is just a Cartesian product $M \times \mathbb{Z}_2$, where $\mathbb{Z}_2$ denotes the two-point space. This model space has been studied not only as a base for particle models but also from the metric point of view, with a detailed analysis of the distances between the points [7]

The usual product-type Dirac operator $D_o$ is taken as,

$$D_o = D \otimes \text{id} + \gamma \otimes D_F,$$

where $D_F$ the Dirac operator on the two-point space,

$$D_F = \begin{pmatrix} 0 & \Phi \\ \Phi^* & 0 \end{pmatrix},$$

with $\Phi \in C^\infty(M)$ is understood as a complex scalar field that is identified with the Higgs field in this simple model.

However, it is easy to see that that the Dirac operator $D_o$ is not the most general one, as the product metric is not the only metric that can exist on the product of two metric spaces. To have a more general picture let us consider now a slight modification of the product geometry (1), allowing the full Dirac operator to be of the form:

$$D = \begin{pmatrix} D_1 & \gamma \Phi \\ \gamma \Phi^* & D_2 \end{pmatrix},$$

where $D_1, D_2$ are two independent Dirac operators on the manifold $M$. This is, in fact, the most general Dirac operator on the product manifold that we can consider, which gives the usual spectral triple when restricted to each point of the finite-dimensional space and the finite spectral triple for the discrete space alone.

The Dirac operator $D$, which is defined in (11), introduces a new range of problems to the model, both in the physical interpretation as well as in computations. Concerning the latter we first encounter the situation that each of the fibres over the two-point space of the Hilbert space of spinors should be considered with a different scalar product. To avoid this issue one should use the unitary equivalence of the Hilbert space, then, however, we have to face more complicated form of the Dirac operators. From the point of view of the interpretation we have a model with two metrics, which strongly resembles the bimetric gravity models\(^2\) (see [8, 9] and references therein), though it remains a question of interpretation, which one is the background metric.

The last crucial technical difference is in the form of the square of the Dirac operator, which is

\[^2\text{The author thanks Marco de Cesare for turning his the attention to it.}\]
\[
D^2 = \begin{pmatrix}
D_1^2 + \Phi^* \Phi \\
-\gamma (D_1^* \Phi - \Phi^* D_2)
\end{pmatrix}
\begin{pmatrix}
\gamma(D_1 \Phi - \Phi D_2) \\
D_2^2 + \Phi^* \Phi
\end{pmatrix},
\] (4)

and in the case \(D_1 = D_2\) differs from the ‘usual’ square Dirac operator only by terms of order 0 in the sense of the order of differential operators, or in other words, by a matrix-valued function over the base manifold \(M\). In the general case new terms arise, which are first order differential operators, raising also the question whether the full Dirac operator is torsion-free even if \(D_1\) and \(D_2\) were not. This problem has origins in the lack of the satisfactory local definition of torsion even for the simple noncommutative models. Although one can propose a torsion-vanishing condition based on finding the minimum of certain functional for a restricted family of Dirac operators, it is, however, very difficult to compute in full generality. To be more precise, the functional is the second term in the asymptotic expansion of the spectral action (which, in the case of 4D geometry is proportional to the Wodzicki residue of \(D^{-2}\)). In the presence of torsion for a fixed metric (the latter equivalent to the fixed principal symbol of \(D\)) this term is proportional to the integrated square of the torsion, hence it reaches a minimum for vanishing torsion. However, some strictly noncommutative examples [10] suggest that the functional may not have a minimum at all. In the presented model we conjecture that for fixed metrics related to \(D_1\) and \(D_2\) such functional will have a minimum, which is exactly given by the vanishing of torsion for \(D_1\) and \(D_2\) and we shall see that for a special class of models considered this is indeed the case.

### 2.2. The Euclidean Friedmann–Lemaitre–Robertson–Walker geometry

In what follows we shall discuss the Euclidean version of the FLRW geometry over the minimal noncommutative generalisation, deriving the corresponding action through the spectral action principle.

We concentrate on the flat, toroidal geometry, using as as the background the Hilbert space of spinors with respect to the equivariant, flat metric. This allows us to exactly compute the spectral action for the time-dependent metric and then derive the equations of motions. Let us again reiterate that since the equations of motions are local, they depend neither on the chosen topology and compactness of the manifold nor on the boundary conditions. This will allow us to use the equations of motion to find solutions, which are not periodic in time and therefore correspond to a locally compact infinite universe.

First, we recall the basic result, which in a straightforward way extends the case of Laplace operators (compare [10], lemma 3.1) to the Dirac operators. Let \(D_g\) be the usual Dirac operator on \(M\) of dimension \(d\) with the metric \(g_{ab}\) and \(\mathcal{H}_g\) be the Hilbert space of \(L^2(S,g)\) (where the measure is taken with respect to the metric \(g\)). Then, if \(h_{ab}\) is another metric on \(M\) (which can be arbitrary) and \(\det(g) = \chi \det(h)\) then the operator acting on \(\mathcal{H}_h\),

\[
D_h = \chi^{-\frac{1}{2}} D \chi^{-\frac{1}{2}},
\] (5)

is unitarily equivalent to \(D\).

It is easy to find the explicit unitary equivalence, let \(U : \Psi \to \chi^{-\frac{1}{2}} \Psi\) be an isometry map between Hilbert spaces \(U : \mathcal{H}_g \to \mathcal{H}_h\), which means,

\[
||\Psi||_{\mathcal{H}_g} = \int_M \sqrt{g} |\Psi|^2 = \int_M \sqrt{h} |\chi^{-\frac{1}{2}} \Psi|^2 = ||U\Psi||_{\mathcal{H}_h}.
\] (6)

Consequently, the operator \(D_h = U^{-1} D_g U\) is an operator on \(\mathcal{H}_h\), which is (by construction) unitary equivalent to \(D_g\).
The above procedure allows us to map Dirac operators to operators on spinor Hilbert spaces where the scalar product is given by a different metric. For the Friedmann–Lemaître–Robertson–Walker type geometry we shall be interested in the case where \( d = 4 \) and the conformal factor \( \chi = a(t)^{-3} \), so that the Riemann measure on the torus is just the measure of the flat torus.

2.2.1 Toroidal geometry Consider the toroidal euclidean Friedmann–Lemaître–Robertson–Walker geometry, which is just a 4D torus, with the metric of the following form:

\[
ds^2 = (dt)^2 + a(t)^2 ((dx_1)^2 + (dx_2)^2 + (dx_3)^2)\]

(7)

The Dirac operator for the above metric reads

\[
D_{\text{FLRW}} = \gamma^0 \partial_t + \frac{1}{a(t)} \left( \gamma^1 \partial_1 + \gamma^2 \partial_2 + \gamma^3 \partial_3 \right) + \gamma^0 \frac{3 \dot{a}(t)}{2 a(t)},
\]

(8)

where we use antihermitian \( \gamma \) matrices, so that \( D_{\text{FLRW}} \) is hermitian on the sections of the spinor bundle where the inner product is computed with respect to the above metric.

Using the method described above we find that the formula (in local coordinates) for the unitarily equivalent Dirac operator, \( D_u \), over the flat torus becomes:

\[
d_a = a(t)^{-\frac{3}{2}} D_{\text{FLRW}} a(t)^{\frac{3}{2}} = \gamma^0 \partial_t + \frac{1}{a(t)} \left( \gamma^1 \partial_1 + \gamma^2 \partial_2 + \gamma^3 \partial_3 \right).
\]

(9)

The minimal noncommutative generalisation, which we discussed in section 2.1, will have the form:

\[
D = \begin{pmatrix} D_{a_1} & \gamma \Phi \\ \gamma^* \Phi & D_{a_2} \end{pmatrix},
\]

(10)

with possibly different scaling factors \( a_1(t), a_2(t) \). Note that, of course, the generalization is also easily constructed for all spatial geometries. If \( D_3 \) denotes the extension of the Dirac operator on a 3D manifold \( M \) (for example the sphere \( S^3 \) of fixed radius) to the spinor space over \( S^1 \times M \) and \( D_{a_1}, D_{a_2} \) are the time-like components, which might not be equal to each other as they might have different torsion terms, then we should consider

\[
D_M = \begin{pmatrix} D_{a_1} + \frac{1}{a_1(t)} D_3 & \gamma \Phi \\ \gamma^* \Phi & D_{a_2} + \frac{1}{a_2(t)} D_3 \end{pmatrix}
\]

(11)

as the noncommutative generalization for such geometry, with \( D_{a_1}, D_{a_2} \) that can by some bounded terms, which have the interpretation of the torsion.

3. Spectral action

In this section we briefly describe the methods to compute the first two leading terms of the spectral action for the toroidal model (which could be easily extended to the more general geometries of Friedmann–Lemaître–Robertson–Walker models). The choice of the toroidal topology has the benefits of simplicity, as it very well mimics the flat universe case, while at the same time allowing for exact computations of the heat trace coefficients. Note that the compactness of the spatial and time dimensions has no significant effect on the equations of motions, which are local and therefore do not distinguish between compact and locally compact case.
The spectral action [11] is usually presented as the asymptotic expansion in the parameter $\lambda$ (though in most of the literature it is denoted by the capital letter, we use here $\lambda$ as we reserve $\Lambda$ for the cosmological constant) of the trace of $f(D^2/\lambda^2)$ for a suitable function $f$. The latter can be chosen, for example, as a smooth approximation of the step function. Using the Mellin transform and heat trace expansion, the leading terms can be expressed using Gilkey–Seeley–de Witt coefficients. For the pseudodifferential operator one can equivalently use the formulation of the spectral action using the Wodzicki residue, where the first two leading terms are:

$$S(D) = \lambda^4 \text{Wres}(D^{-4}) + c\lambda^2 \text{Wres}(D^{-2}),$$

with $\lambda$ (in both cases of the spectral action) is assumed to be the scaling factor, which we interpret as related to some cutoff energy scale. The constant $c$ is, in the Wodzicki residue formulation, an arbitrary coefficient, whereas in the asymptotic expansion version it is related to the exact form of the cutoff function (see [11] for details).

The square of the Dirac operator, $D^2$ is a differential operator that can be split into homogeneous parts of order 2, 1 and 0 respectively. We denote the respective homogeneous symbols of $D^2$ as $\sigma_k$, $k = 0, 1, 2$, with $\sigma_k$ homogeneous of degree $k$, so that $\sigma(D^2) = \alpha_2 + \alpha_1 + \alpha_0$. Using the algebra of the pseudodifferential calculus [11] we can compute the symbols of its inverse, $b_k$, which will be homogeneous of order $-(k + 2)$.

$$b_0 = (\alpha_2)^{-1},$$
$$b_1 = -\left(b_0\alpha_1 + \frac{\partial^2}{\partial^2} (b_0)\partial_k (\alpha_2)\right) b_0,$$
$$b_2 = -\left(b_1\alpha_1 + b_0\alpha_0 + \frac{\partial^2}{\partial^2} (b_0)\partial_k (\alpha_1) + \frac{\partial^2}{\partial^2} (b_1)\partial_k (\alpha_2) + \frac{1}{2}\frac{\partial^2}{\partial^2} (b_0)\partial_k \partial_l (\alpha_2)\right) b_0,$$

where $\partial^2$ denotes partial derivative with respect to coordinate of the cotangent bundle.

Since the Wodzicki residue of a pseudodifferential operator over a 4D manifold is proportional to the integral over the cosphere bundle of the symbol of degree $-4$, we obtain that the spectral action (the first two leading terms) becomes,

$$S(D) = \int_M \int_{|\xi| = 1} \left(\lambda^4 (b_0)^2 + c\lambda^2 b_2\right).$$

### 3.1. Action for the toroidal Friedmann–Lemaître–Robertson–Walker

We begin with the explicit computations of the spectral action for the assumed toroidal almost-commutative Friedmann–Lemaître–Robertson–Walker geometry. Assume that the the underlying geometry is $S^1 \times T^3$, with constant metric of equal length along all directions and that the Dirac operator is as in (11) with:

$$D_3 = \gamma^1 \partial_1 + \gamma^2 \partial_2 + \gamma^3 \partial_3,$$
$$D_n = \gamma^0 (\partial_n + h_n(t)), \quad n = 1, 2,$$

where $h_1(t), h_2(t)$ are some functions, which correspond to the torsion terms of the respective Dirac operators. Note that technically, we are always considering not a true Dirac operator, but its unitarily equivalent counterpart on a different Hilbert space. For simplicity we could write $D$ as

$$\gamma^0 (\partial_1 + h(t)) + \Lambda(t)D_3 + \gamma F(t),$$

where $\Lambda(t)$ is a suitable function.
where
\[ h(t) = \begin{pmatrix} h_1(t) & 0 \\ 0 & h_2(t) \end{pmatrix}, \quad A(t) = \begin{pmatrix} \frac{1}{\sigma(t)} & 0 \\ 0 & \frac{1}{\sigma(t)} \end{pmatrix}, \quad F(t) = \begin{pmatrix} 0 & \Phi(t) \\ \Phi^*(t) & 0 \end{pmatrix}. \]
\[ (17) \]

First, we compute \( D^2 \):
\[
D^2 = (\partial_t)^2 + A(t)^2 (D_3)^2 + 2h(t)\partial_t + \gamma^0 (\partial_t A(t)) D_3 + [F(t), A(t)] \gamma D_3 + \partial_t h(t) + h(t)^2 + F(t)^2 - \gamma^0 (\partial_t F(t) + [h(t), F(t)]). \]
\[ (18) \]

We compute now the two leading order terms of the spectral action using the methods of the pseudodifferential calculus as presented above. Let us first write the symbols of the differential operator \( D \), splitting it into the components, which are homogeneous in \( \xi \).
\[
\begin{align*}
\alpha_2 &= \xi_0^2 + A(t)^2 (\xi_1^2 + (\xi_2)^2 + (\xi_3)^2) \\
\alpha_1 &= i \left( -2h(t)\xi_0 + \partial_t A(t) \gamma^0 (\gamma^0 \xi_3) + [F(t), A(t)] \gamma (\gamma^0 \xi_3) \right), \\
\alpha_0 &= -h(t)^2 - \partial_t h(t) + F(t)^2 - \gamma^0 (\partial_t F(t) + [h(t), F(t)]).
\end{align*} \]
\[ (19) \]

The symbol of \((D_h)^{-2}\) reads:
\[
\sigma(D_h^{-2}) = b_0 + b_1 + b_2 + \cdots, \]
\[ (20) \]
where the parts \( b_0 \) (homogeneous of order \(-2\)) and \( b_2 \) (homogeneous of order \(-4\)) become,
\[
\begin{align*}
b_0(\xi) &= (\xi_0^2 + A(t)\xi_3^2)^{-1}, \\
b_2(\xi) &= b_0 [A, F] b_0 F b_0 + b_0 F b_0 [A, F] b_0 - b_0^2 F^2, \\
b_2^0(\xi) &= A h(4b_0^4 \xi_0^2 - 24b_0^2 \xi_0^2 \xi_3^2) + \tilde{A} (8A^2 b_0^4 \xi_0^2 - 48A^2 b_0^2 \xi_0^2 \xi_3^2) \xi_3^2 - b_0^3 \xi_3^2 + 8b_0^3 \xi_3^2 \xi_0^2) \\
&\quad + \tilde{A} (-2A b_0^2 \xi_3^2 - 8A b_0^4 \xi_0^2 \xi_3^2) \\
&\quad + h^2 (b_0^2 - 4b_0^4 \xi_3^2) + h (b_0^2 - 4b_0^4 \xi_3^2).
\end{align*} \]
\[ (21) \]

For convenience and simplicity, we omit here the explicit dependence on coordinate \( t \), denote \( \xi^2 = \xi_1^2 + \xi_2^2 + \xi_3^2 \) to shorten notation and split the \( b_2 \) term into the diagonal (commutative) term \( b_2^0 \) and the ‘noncommutative’ term \( b_2^1 \).

First of all, observe that the terms that depend on \( h(t) \) appear only in the diagonal term and therefore they will give sum of independent terms for \( h_1(t) \) and \( h_2(t) \). Remember that both \( h_1(t) \) and \( h_2(t) \) have the interpretation of the torsion and therefore we see that the total torsion term is a sum of two independent torsion contributions. Therefore, in the considered model the torsion-free Dirac operator arises indeed from both Dirac operator components being torsion free, hence we can set \( h_1(t) = h_2(t) = 0 \).

We first compute the diagonal term, as it will we just a sum of two independent entries:
\[
\int_{|\xi|=1} b_2^0 (\xi) = tr \left( \frac{\pi^2}{A} (3\tilde{A}^2 - A\tilde{A}) \right). \]
\[ (22) \]

For the part, which is not scalar we first need to compute the trace, which leads us to the following expression:
\[ \int_{|\xi|=1} |\Phi(t)|^2 \left( \frac{1}{a_1} - \frac{1}{a_2} \right)^2 (a_0(a_1) b_0(a_2) (b_0(a_1) + b_0(a_2)) - b_0(a_1)^2 - b_0(a_2)^2) \right), \]  
\[ \text{which after integration gives:} \]
\[ 2\pi^2 |\Phi|^2 \frac{(a_1 - a_2)^2}{(a_1 + a_2)^2} (a_1^2 + a_1 a_2 + a_2^2). \]

We can compare now the above result to the classical Einstein–Hilbert action for the Friedmann–Lemaître–Robertson–Walker metric. The kinetic term is exactly the same as the scalar curvature (multiplied by the volume form), up to a multiplicative constant. If we label the two metrics \( g_1 \) and \( g_2 \), and denote \( A(t) = \text{diag}(A_1(t), A_2(t)) \), then
\[ \sqrt{|g_i|} R(g_i) = 6 \left( -3 \left( \frac{\dot{A}_1(t)^2}{A_1(t)} \right) + \frac{\dot{A}_2(t)}{A_2(t)} \right), \quad i = 1, 2, \]
and, of course, the full action is the sum of the two terms, independently for \( A_1(t) \) and \( A_2(t) \). The more significant difference with the classical case is the potential term, which describes the coupling between the metric and the \( \Phi \) field. The latter is naturally interpreted as the Higgs field and therefore we can investigate what happens to the scaling factors if the vacuum expectation value of \( \Phi \) is constant and different from zero.

Finally the total spectral action density (restricted to the two leading terms of the spectral action), expressed explicitly in terms of \( a_1(t) \) and \( a_2(t) \) is:
\[ \mathcal{L} = 2\pi^2 \int dt \left( \lambda^4 (a_1(t)^3 + a_2(t)^3) - \lambda^2 c (a_1(t)^2 a_1(t) + a_2(t)^2 a_2(t)) - \lambda^2 |\Phi|^2 (a_1(t)^3 + a_2(t)^3) + \lambda^2 c |\Phi|^2 \left( \frac{(a_1(t) - a_2(t))^2}{(a_1(t) + a_2(t))^2} \right) \left( a_1(t)^2 + a_1(t) a_2(t) + a_2(t)^2 \right) \right) + \mathcal{L}_m, \]
where \( \mathcal{L}_m \) denotes other terms that could arise either from some higher-order corrections of the spectral action or matter terms. We have omitted a total derivative term of the form \( \frac{d}{dt}(a_1 a_2^2) \).

Let us note that the first and the third terms of the above expression are almost identical and, in fact, when the Higgs field picks a nonzero expectation value the third term leads just to a redefinition of the cosmological constant. In fact, even if we had not assumed the presence of the cosmological constant in the beginning, then the existence of the Higgs field vacuum expectation value would enforce us to introduce it. Finally let us point out that the action is, of course, Euclidean and in order to proceed with physical analysis we need to perform Wick rotation to the Lorentzian signature. In our case that will lead only to the change of the sign in the dynamical part of the action. This can be easily seen if one uses a rescaled time replacing \( t \) with \( b t \). The potential-like terms then scale like \( b \) (which comes from the scaling of the measure) whereas each derivative term scales additionally like \( \frac{1}{b} \). Therefore, the only difference, when passing to the Wick-rotated picture is, that the dynamical term which contains the square of time derivatives of the scaling functions \( a_1, a_2 \) picks a different sign.

We remind that we have neglected all terms with \( h(t) \), which is a potential torsion term that we have in the beginning incorporated into the Dirac operator. However, since the only terms that it appears are ‘diagonal’, that is it appear separately for each sheet in our model and does not mix the scaling functions \( a_1 \) with \( a_2 \) we assume it to vanish, similarly like in the classical, one sheet case.
Putting it all together we finally obtain (after Wick rotation) the physical effective action for the toroidal Friedmann–Lemaître–Robertson–Walker geometry as originating from the Lagrangian density,

\[ \mathcal{L} = \Lambda (a_1(t)^3 + a_2(t)^3) + 6 \left( \dot{a}_1(t)^2 a_1(t) + \dot{a}_2(t)^2 a_2(t) \right) + \alpha |\Phi|^2 \left( \frac{(a_1(t) - a_2(t))^2}{(a_1(t) + a_2(t))} \right) \left( a_1(t)^2 + a_1(t)a_2(t) + a_2(t)^2 \right) \] 

(27)

where we have introduced for simplicity effective constants \( \Lambda \) (cosmological constant) and \( \alpha \) (strength of the potential). In the rest of the paper we shall briefly analyze the consequences of the extra interaction term between the two scales \( a_1(t) \) and \( a_2(t) \).

4. The equations of motion

The Friedman equations of motion could be easily derived as the Euler–Lagrange equations from the Lagrangian (27), which is already taken after the Wick rotation. First, consider the classical case, with no noncommutativity and for a single sheet. To have the full set of equations we need to conveniently express the Lagrangian density, obtained from the spectral action, using additional scale for the time direction (in a similar way we did to Wick-rotate the Euclidean action). The first equation of motion will follow from variation of the density with respect to this auxiliary factor \( b \), which corresponds to the linear reparametrization (scaling) of time \( t \). This trick is used exclusively to derive a full set of Einstein equations, later we set \( b = 1 \). The Lagrangian, with \( b \) is,

\[ \mathcal{L} \sim \Lambda b a(t)^3 + 6\dot{a}(t)^2 a(t)/b, \] 

(28)

then, variation with respect to \( b \), and at \( b = 1 \) gives

\[ 6a(t)\ddot{a}(t)^2 - \Lambda a(t)^3 = 0, \] 

(29)

and then varying with respect to \( a(t) \) we obtain,

\[ 6 \frac{d}{dt} (2a(t)\dot{a}(t)) - 6\dot{a}(t)^2 - 3\Lambda a^2(t) = 0, \] 

(30)

which finally gives

\[ 12\ddot{a}(t)a(t) + 6\dot{a}(t)^2 - 3\Lambda a(t)^2 = 0. \] 

(31)

The resulting equations then read:

\[ \frac{\dot{a}(t)^2}{a(t)^2} = \frac{1}{6} \Lambda, \quad \frac{\ddot{a}(t)}{a(t)} = \frac{1}{6} \Lambda, \] 

(32)

and are typical for the dark-energy dominated universe equations.

The standard solution of the empty universe (bar the cosmological constant) is the exponentially growing de Sitter universe with constant Hubble parameter,

\[ a(t) = a_0 \exp \left( \sqrt{\frac{\Lambda}{6}} t \right). \] 

(33)
4.1. An almost commutative perturbation of de Sitter universe

In the noncommutative model, with \( a_1(t) \) and \( a_2(t) \) we assume that the effective cosmological constant is the same for both parallel geometries and concentrate on the modification for the equations that arise from the potential term. Using a similar procedure as in the nondeformed case, we introduce an auxiliary time scale \( b \), which we take to be identical for both copies of spacetime geometry, so the model has, unlike in [12, 13] the same lapse for each sheet of the geometry\(^3\). To simplify the notation we write \( a_1, a_2 \) without explicitly stating their dependence on \( t \).

The potential term scales with \( b \),

\[
ba\alpha (a_1 - a_2)^2 (a_1^3 + 2a_1^2a_2 + 2a_2^2 + a_1^2) / (a_2 + a_1)^2.
\]

(34)

The set of equations of motion that arises for the full action, that involves the term mixing \( a_1(t) \) and \( a_2(t) \) is as follows

\[
6\dot{a}(\dot{a}_1^2 + \dot{a}_2^2 - \Lambda(a_1^3 + a_2^3) - \alpha (a_1 - a_2)^2 (a_1^3 + 2a_1^2a_2 + 2a_2^2 + a_1^2) / (a_2 + a_1)^2 = 0,
\]

\[
12\ddot{a}_1a_1 + 6\dot{a}_1^2 + 3\Lambda a_1^2 - \alpha (a_1 - a_2)^2 (2a_1^3 + 2a_1^2a_2 + 5a_2^2 + 3a_1^2) / (a_2 + a_1)^2 = 0,
\]

\[
12\ddot{a}_2a_2 + 6\dot{a}_2^2 - 3\Lambda a_2^2 + \alpha (a_2 - a_1)(3a_2^3 + 5a_2^2a_1 + 2a_1^2a_2 + 2a_1^3) / (a_2 + a_1)^2 = 0.
\]

(35)

We shall look for the perturbative solutions of the form:

\[
a_1(t) = a(t) + \epsilon r(t), \quad a_2(t) = a(t) - \epsilon r(t),
\]

(36)

bearing in mind that the above assumption might be too restrictive. Of course, the function \( a(t) \) must be the standard de Sitter solution (which we later take as a standard solution for a specific universe) whereas for the perturbative correction we obtain in the first order in \( \epsilon \):

\[
12\ddot{a}(r(t) + 12\dot{r}(t)a(t) + 12\ddot{a}(t) + 6\Lambda a(t)r(t) - 6\alpha a(t)r(t) = 0,
\]

(37)

which we can rewrite as the second-order equation for \( r(t) \):

\[
\ddot{r}(t) + \left( \frac{\dot{a}(t)}{a(t)} \right) \dot{r}(t) + \left( \frac{\ddot{a}(t)}{a(t)} - \frac{1}{2} (\Lambda + \alpha) \right) r(t) = 0.
\]

(38)

4.2. Models and solutions

We shall consider three models to study the qualitative and significant effect of the assumed form of the interactions. Our starting point is the equation (38) for the difference of the scaling factors \( 2r(t) = a_1(t) - a_2(t) \), where we assume that \( a(t) \) is the background solution, which is a solution for the identical factors \( a_1(t) = a_2(t) = a(t) \).

We assume also that the matter or radiation terms, whenever occurring, are identical for both sheets, and therefore their contributions to the Lagrange density do not depend on the difference of scale functions \( r(t) \) and hence only change the equation for \( a(t) \) but not the equation (38) for \( r(t) \). The justification for this comes from the construction of the mass terms

\(^3\)This is a consequence of assuming that the time function is identical for both copies and the two Dirac operators on each sheet differ only in the time-conformal factor of the spatial component.
in the noncommutative models [3]. Their origin is the usual fermionic action with Lagrange density $(\Psi, D\Psi)$, where the dynamical terms will indeed be independent for the left and right fermions and depend on the respective metrics. However, the off-diagonal terms in the Dirac operator $D$ (11) give a coupling between left and right fermions intermediated by the Higgs field $\Phi$. The only dependence on the metric will then arise through the volume measure but this can be hidden in the redefinition of the Higgs or fermionic fields.

As only in the nonzero Higgs vacuum expectation value these terms become the known fermion mass terms and, from the construction it is clear that they are not dependent on the difference of the scaling factors $r(t)$. Of course, we need to stress that this model is simplified, as we consider the equations of motion in the fixed value of the Higgs vacuum. The full model, of course, with some higher order contributions from the spectral action might add nontrivial relations between the metric and the Higgs potential, resulting in the rather involved equations of motions. In particular, it will be worth studying if in the full model Higgs vacuum expectation value can be time-dependent. Here, however, we restrict ourselves to the simplest case of constant nonzero Higgs vacuum and therefore the matter and radiation terms influence only $a(t)$ whereas the dynamics of $r(t)$ is governed by (38).

We note also that the general coupling between the metric and the matter usually considered in the bimetric models (as suggested in [8] and discussed in more detail in [14]) slightly differ from the noncommutative approach as the mass terms arise from the Higgs field. We leave the full derivation of respective terms for the most general model is the aim for future work especially that the spectral action for the matter fields (fermions) alone is, unlike the action for the metric and the gauge fields (including the Higgs field) a very recent proposition [15].

Below we consider three simplest examples of standard cosmological solutions referring to the as the standard solution $a(t)$ for the given matter and radiation contents of the universe.

4.3. The empty universe

We begin with the model of an empty universe, with the core solution (33). The equation (37) then becomes:

$$6\ddot{r}(t) + \sqrt{6\Lambda}\dot{r}(t) - (2\Lambda + 3\alpha)r(t) = 0,$$

and the most general solutions are:

$$c_1 e^{-t\sqrt{\frac{6}{\alpha} + \frac{1}{4}\sqrt{6\Lambda + 8\alpha}}} + c_2 e^{-t\sqrt{\frac{6}{\alpha} - \frac{1}{4}\sqrt{6\Lambda + 8\alpha}}}.$$

First of all, observe that if $\Lambda > 0$ then the solution, which shall be of correction type will grow exponentially and is, in fact, of the same type as the base solution of the expanding universe. However, we need to take into considerations the fact that $\Lambda$ is the effective cosmological constant that was obtained from the ‘bare’ cosmological constant $\lambda$ (that came from the heat trace expansion scaling) and the interaction terms $-\lambda^2 c$ (see in equation (10)). Therefore we need to consider two situations. First, if $\Lambda = 0$ then core equation give the stationary universe $a(t) = a_0$ with the corrections giving an exponential growth. From the physical point of view that is rather dissatisfying as we can expect that the correction term $r(t)$ rather stays small when compared to the standard evolution $a(t)$.

Another possibility is $\Lambda < 0$, which possibly reverses the roles of the ‘base’ evolution and the correction. Indeed, then the solution $a(t)$ is oscillating, whereas the correction term might add and exponential behavior provided that $6\Lambda + 8\alpha > 0$. So, the entire solution will, at least
for the part of time resemble an exponential growth with a sinusoidal correction but cannot be stable as the correction term grows too big when compared to \( a(t) \).

### 4.4. Radiation dominated universe

We assume here the standard solution of a universe, in which radiation dominates, which might have typical for the very early age evolution. We take:

\[
a(t) = a_0 \sqrt{t},
\]

which leads to the equation:

\[
4\ddot{r}(t) + 2\dot{r}(t) t - (1 + 2\Lambda t^2 + 2\alpha t^2) r(t) = 0.
\]

The solutions are then Bessel function scaled by a time factor,

\[
r(t) \sim t^{1/2} J_{\sqrt{-2(\Lambda + \alpha)/t}},
\]

which will make sense again, for \( \Lambda < 0 \) and \( \Lambda + \alpha < 0 \). As the Bessel functions decrease like \( t^{-1/2} \) the correction term will also be slowly decreasing with time.

### 4.5. Matter dominated universe

As a last case let us see the type of corrections we might get in the case of the standard solutions for the matter dominated universe. We have,

\[
a(t) = a_0 t^{3/2},
\]

which leads to the equation

\[
18\ddot{r}(t) + 12\dot{r}(t) t - (4 + 9\Lambda t^2 + 9\alpha t^2) r(t) = 0,
\]

and solutions

\[
r(t) \sim t^{-1/2} \sin \left( \frac{1}{2} \sqrt{-2(\Lambda + \alpha)/t} \right).
\]

Here the solutions are oscillating again only if \( \Lambda \) and \( \alpha \) are satisfying the same bounds as in the radiation-dominated case.

### 5. Conclusions and outlook

In the presented models we wanted to obtain only a qualitative picture, without discussing the exact values of the parameters. We have also restricted ourselves to very fundamental models and approximate solutions leaving the detailed analysis of the full considered model to future work.

Nevertheless even such simplified version shows that, from point of view of cosmology and noncommutative model-building, the scenario with cosmic scale factor which are different for the two sheets of the two-sheeted space (which then bears the interpretation as the world for right-handed and left-handed particles) cannot be neglected.

Though it still remains to be studied how such different cosmic scales can be potentially observed and, one needs to notice a lot of similarities of the above model to the bimetric theory of gravity. It is remarkable that a simple noncommutative model quite surprisingly leads
to very similar Lagrangian as an alternative theory of gravity that is considered seriously as a potential model for the accelerating universe. The cosmological solutions of bigravity have been shown to reproduce the current cosmic acceleration and fitted such to observational data [16]. Several other papers constrained parameters of bigravity and found that bigravity allows models that provide late-time acceleration in agreement with observations (for example [12, 13, 17]).

It is also worth mentioning that some recent noncommutative models with deformed space-time effectively lead to a version of action and metric fields that in the classical limit reduce themselves to a bimetric gravity models [18]. This is in contrast to the usually considered modifications of the gravity action and the cosmological consequences of the noncommutativity in models based on the Moyal type deformation of space-time (see [19, 20] and references therein), where only some higher-order corrections in the deformation parameter appeared. The difference between these approaches lies in the definition of the metric, which in the case of [18] is more adapted to the general noncommutative approach. In general, it is not clear, whether the algebraic approach to the noncommutative version of gravity is equivalent Connes’ concept of spectral triples and gravity action related to the spectrum of the generalized Dirac operator.

The presented model needs to be extended to the full version of Connes’ standard model [21] with a full algebra and the resulting terms of the spectral action (even beyond the second leading term). Especially interesting is the question how the deviations from the mean geometry are restrained by the full model, where, some higher order corrections might add additional terms to the action. Only then a detailed analysis of the possible values of the parameters as well as the observational constraints can be carried out, and we plan to proceed with the analysis in the forthcoming work.

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