A GENERALIZATION OF GRIFFITHS THEOREM ON RATIONAL INTEGRALS, III: A VARIANT OF WOTZLAW CONJECTURE

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Abstract. We prove a variant of a conjecture of L. Wotzlaw on an algebraic description of the graded quotients of the Hodge filtration on the top cohomology of the complement of a hypersurface in projective space having only ordinary double points as singularities, if the projective space is even dimensional. In the odd dimensional case, we prove this conjecture except for certain degrees of the graded quotients of Hodge filtration depending on the degree of the hypersurface. We show that a sufficient condition obtained in our joint paper with Wotzlaw is satisfied by using a vanishing theorem of the associated Koszul cohomology which was recently obtained by G. Sticlaru and the first author. We also give a simplified proof of this vanishing theorem by using our recent results on the pole order spectral sequences.

Introduction

Let \( Y \subset X \) be a hypersurface of degree \( d \) in \( X = \mathbb{P}^n \). We assume that all the singularities of \( Y \) are ordinary double points in this paper (except for Section 1). Set \( R := \mathbb{C}[x_0, \ldots, x_n] \) with \( x_0, \ldots, x_n \) the coordinates of \( \mathbb{C}^{n+1} \). Let \( f \) be a defining polynomial of \( Y \). Set \( U = X \setminus Y \), and

\[
m := \lceil n/2 \rceil, \quad J := (\partial f/\partial x_0, \ldots, \partial f/\partial x_n) \subset R, \quad I := \sqrt{J} \subset R.
\]

Here \( J \) is called the Jacobian ideal of \( f \), and \( I \) is the graded ideal consisting of finite sums of homogeneous polynomials vanishing at \( \text{Sing} \, Y \subset X \). Let \( R_k \) denote the degree \( k \) part of \( R \), and similarly for \( I_k \), etc. Under the above assumption we have the following.

Conjecture 1 (L. Wotzlaw [Wo, 6.5]). For any integer \( p \), setting \( q = n - p \), we have

\[
\text{Gr}^p F \, H^n(U, \mathbb{C}) = \left( I^{q-m+1}/I^{q-m} \, J \right)_{(q+1)d-n-1}.
\]

Here \( F \) is the Hodge filtration as in [De], and \( I^j = R \) for \( j \leq 0 \). This is a generalization of Griffiths’ theorem on rational integrals [Gr] in the \( Y \) smooth case. The following is known:

Theorem 1 ([DiSaWo, Theorem 2.2]). Conjecture 1 holds if \( q \leq m \), i.e. if \( p \geq n - m \).

More precisely, this theorem was proved in the case of general singularities by modifying \( m \) and \( I \) appropriately. For \( p < n - m \), however, the situation was unclear.

Let \( \mathcal{I} \subset \mathcal{O}_X \) denote the reduced ideal of \( \text{Sing} \, Y \subset X \). Set

\[
I^{(i)}_k := \Gamma(X, \mathcal{I}^{(i)}(k)), \quad I^{(i)} = \bigoplus_k I^{(i)}_k.
\]

We have the inclusions \( (I^{(i)}_k) \subset I^{(i)}_k \) together with the equalities \( (I^{(i)}_k) = I^{(i)}_k \) for \( k \gg 0 \) although these equalities do not always hold in general, see [DiSaWo, Section 2.3]. By definition we have exact sequences

\[
0 \rightarrow I^{(i)}_k \rightarrow R_k \xrightarrow{\partial^{(i)}_k} \bigoplus_{y \in \text{Sing} \, Y} \mathcal{O}_{X,y}/m_{X,y}^i,
\]

choosing a trivialization of \( \mathcal{O}_{X,y}(k) \), where \( m_{X,y} \) is the maximal ideal of \( \mathcal{O}_{X,y} \). We have a variant of Conjecture 1 as follows:
Conjecture 2. For any integer $p$, setting $q = n - p$, we have
\[ \text{Gr}_F^p H^n(U, C) = (I^{q-m+1}/I^{q-m})_{(q+1)d-n-1}. \]

Note that Theorem 1 implies Conjecture 2 for $q \leq m$. In [DiSaWo], the following was shown:

Theorem 2 ([DiSaWo, Theorem 2]). For $q = n - p > m = [n/2]$, Conjecture 2 holds, if the following condition is satisfied:

\[ (B') \quad \text{The morphism } \beta^{(i)}_k \text{ in (0.1) is surjective for } k = m(d - 1) - p \text{ and } i = 1. \]

Note that condition $(B')$ is equivalent to condition $(B)$ in loc. cit. We show in this paper the following (see also [Di3]):

Theorem 3. Condition $(B')$ holds if $n$ is even or $n$ is odd and $q \geq m + [d/2]$.

(Note that this paper is essentially a simplified version of [Di3].) Combining Theorem 3 with Theorems 1 and 2, we get the main theorem of this paper:

Theorem 4. Conjecture 2 holds except for the case where $n$ is odd and $m < q < m + [d/2]$.

The situation in the exceptional case is unclear (since condition $(B')$ is only a sufficient condition), and Conjectures 1 and 2 are still open, see remarks after [DiSaWo, Theorem 2].

For the proof of Theorem 3, set
\[ \text{def}_k \Sigma_f := \dim \text{Coker}(\beta^{(i)}_k : R_k \to \bigoplus_{y \in \text{Sing } Y} \mathcal{O}_{X,y}/(\mathcal{O}_{X,y})^m) \]
so that condition $(B')$ is equivalent to
\[ (B'') \quad \text{def}_{m(d-1)-p}(\Sigma_f) = 0. \]

Let $K^*_f := (\Omega^*, df \wedge)$ be the Koszul complex associated with the action of $df \wedge$ on the algebraic differential forms $\Omega^* := \Gamma(X, \Omega^*_X)$. It is a graded complex with $\deg x_i = \deg dx_i = 1$, and $df \wedge$ is a morphism of degree $d$. Set
\[ N := H^n K^*_f, \quad M := H^{n+1} K^*_f, \quad M' := H^0_M M, \quad M'' := M/M', \]
where $m \subset R$ is the maximal ideal generated by the $x_i$, and $H^0_m$ is the local cohomology. These are graded $\mathbb{C}$-vector spaces. For $k \in \mathbb{Z}$ we have the isomorphisms (see [Di2]):
\[ M_{k+n+1} = (R/J)_{k}, \quad M''_{k+n+1} = (R/\sqrt{J})_{k} = (R/I)_{k}, \]
and the last isomorphisms imply
\[ \#(\text{Sing } Y) - \dim M''_{k+n+1} = \text{def}_k(\Sigma_f). \]

By [Di2, Theorem 3.1] we have moreover
\[ \dim N_{nd-n-1-k} = \text{def}_k(\Sigma_f). \]

This also follows from [DiSa2, Corollary 2] asserting
\[ \dim M'' + \dim N_{nd-k} = \#(\text{Sing } Y). \]

(Note that $n$ in [DiSa2] is $n+1$ in this paper, and the grading of $N$ in this paper is shifted by $d$ compared with the one in [DiSa2].) So condition $(B'')$ is equivalent to
\[ (B'') \quad N_{(m-m)d+m-q-1} = 0, \]

since $nd - n - 1 - (m(d - 1) - p) = (n - m)d + m - q - 1$.

We have furthermore the following:

Theorem 5 ([Di1], [Di3]). We have $N_k = 0$ if $k < nd/2$. 

In fact, the $n$ even case was shown in [DiSt], and the improved assertion in the $n$ odd case is recently obtained by [Di3]. (It is known that the assumption of Theorem 5 is sharp, see loc. cit.) Theorem 3 then follows by calculating the condition
\[(n - m)d + m - q - 1 < nd/2,\]
see (2.3) below. In this paper we also give a simpler and conceptually clearer proof of Theorem 5 by using some recent results from [DiSa2] together with the Thom-Sebastiani type theorems, see (2.2) below.

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In Section 1 we review some basics of the Hodge and pole order filtrations and pole order spectral sequences. In Section 2 we prove Theorems 3 and 5 after showing Proposition (2.1).

1. Preliminaries

In this section we review some basics of the Hodge and pole order filtrations and pole order spectral sequences.

1.1. Cohomology of projective hypersurface complements. Let $Y$ be a hypersurface in $X := \mathbb{P}^n$. Set $U = X \setminus Y$. By Grothendieck there are canonical isomorphisms
\[H^j(U, \mathbb{C}) = H^j(\Gamma(X, \Omega^*_X(*Y))) \quad (j \in \mathbb{Z}),\]
where $\Gamma(X, \Omega^*_X(*Y))$ is the complex of meromorphic differential forms on $X$ whose poles are contained in $Y$. By [Sa1] and [DiSa1, Proposition 2.2], there is the Hodge filtration $F$ on $\mathcal{O}_X(*Y)$ such that
\[(1.1.1) \quad F^p H^j(U, \mathbb{C}) = H^j(\Gamma(X, \Omega^*_X \otimes_{\mathcal{O}_X} F_{-p} \mathcal{O}_X(*Y))) \quad (p, j \in \mathbb{Z}),\]
where $F$ on the left-hand side is the Hodge filtration of the mixed Hodge structure on $H^j(U, \mathbb{C})$, see [De].

Let $P$ be the pole order filtration on $\mathcal{O}_X(*Y)$ defined by
\[P_p \mathcal{O}_X(*Y) := \begin{cases} 0 & \text{if } p < 0, \\ \mathcal{O}_X((p + 1)Y) & \text{if } p \geq 0. \end{cases}\]
Then
\[F_p \mathcal{O}_X(*Y) \subset P_p \mathcal{O}_X(*Y),

\[F_p \mathcal{O}_X(*Y)|_{X \setminus \text{Sing } Y} = P_p \mathcal{O}_X(*Y)|_{X \setminus \text{Sing } Y}.\]

Let $h_y$ be a local defining holomorphic function of $Y$ at $y$, and $b_{h_y}(s)$ be the $b$-function of $h_y$ which is normalized as in [DiSa1], [Sa2] so that
\[\tilde{b}_{h_y}(s) := b_{h_y}(s)/(s + 1) \in \mathbb{C}[s].\]

Let $\bar{\alpha}_{Y,y}$ be the minimal root of $\tilde{b}_{h_y}(-s)$. If $(Y, y)$ is an isolated singularity defined locally by a Brieskorn-Pham polynomial $h_y = \sum_{i=1}^n y_i^{a_i}$ for some local coordinates $y_1, \ldots, y_n$, then
\[\tilde{\alpha}_{Y,y} = \sum_{i=1}^n \frac{1}{a_i}.
\]

Set
\[(1.1.3) \quad \tilde{\alpha}_Y := \min_{y \in \text{Sing } Y} \tilde{\alpha}_{Y,y}.\]
By [Sa2] we have
\[(1.1.4) \quad F_p \mathcal{O}_X(*Y) = P_p \mathcal{O}_X(*Y) \quad \text{if} \quad p < [\tilde{\alpha}_Y].\]
This implies
\[(1.1.5) \quad F^p H^j(U, \mathbb{C}) = P^p H^j(U, \mathbb{C}) \quad \text{if} \quad p > j - [\tilde{\alpha}_Y],\]
where the filtration \(P\) on \(H^j(U, \mathbb{C})\) is induced by \(P\) on \(\mathcal{O}_X(*Y)\) by using the image of the right-hand side of (1.1.1) with \(F\) replaced by \(P\).

By [DiSaVol Theorem 2.2] (using [Sa3]) we have moreover
\[(1.2.1) \quad \text{Gr}_i^P H^n(U, \mathbb{C}) = (R/J)_{(q+1)d-n-1} = M_{(q+1)d} \quad \text{if} \quad q = n - p < [\tilde{\alpha}_Y],\]
where \((R/J)_{k}, M_k\) are as in the introduction (although \(Y\) may have arbitrary singularities).

### 1.2. Pole order spectral sequences.

In the notation of the introduction, we have the algebraic microlocal Gauss-Manin complex
\[
(\tilde{C}^*, d - \partial_t df \wedge) \quad \text{with} \quad \tilde{C}^j = \Omega^j[\partial_t, \partial_t^{-1}],
\]
see [DiSa2]. Here \(f\) may be any homogeneous polynomial of degree \(d\). Its cohomology groups \(H^j(\tilde{C}^*)\) are called the Gauss-Manin systems. These are graded \(\mathbb{C}\)-vector spaces (where \(\text{deg} \partial_t = -d\), and there are isomorphisms
\[(1.2.1) \quad H^{j+1}(\tilde{C}_f^*)_k = H^j(f^{-1}(1), \mathbb{C})_\lambda \quad \text{for} \quad \lambda = \exp(-2\pi i k/d),\]
where \(H^j(f^{-1}(1), \mathbb{C})_\lambda\) is the \(\lambda\)-eigenspace of the Milnor cohomology under the monodromy.

We have the pole order filtration \(P'\) defined by
\[
P^p_\lambda \tilde{C}_f^j = \bigoplus_{i \leq p} \Omega^i \partial_t^i.
\]
There are isomorphisms
\[(1.2.2) \quad \partial_t^i : P^p_\lambda(\tilde{C}_f^*)_k \xrightarrow{\sim} P^{p+i}_\lambda(\tilde{C}_f^*)_k\]
Set \(P^{p,p} = P^{p,-p}_\lambda\). This defines the algebraic microlocal pole order spectral sequence
\[(1.2.3) \quad kE_i^{p,j-p} = H^j \text{Gr}_{p+1}^P(\tilde{C}_f^*)_k \Rightarrow H^j(\tilde{C}_f^*)_k,
\]
which is a spectral sequence of graded \(\mathbb{C}\)-vector spaces. The associated filtration \(P'\) on \(H^{j+1} \text{Gr}_{p+1}^P(\tilde{C}_f^*)_k\) is identified with the pole order filtration \(P\) on the Milnor cohomology groups \(H^j(f^{-1}(1), \mathbb{C})_\lambda\) up to the shift by one (i.e. \(P^{j+1} = P^j\)) via the isomorphism (1.2.1) for \(k \in [1, d]\), see [Di1] Chapter 6, Theorem 2.9], and also [DiSa1] Section 1.8] for the case \(j = n\). We thus get
\[(1.2.4) \quad P^{p+1} H^{j+1}(\tilde{C}_f^*)_k = P^p H^j(f^{-1}(1), \mathbb{C})_\lambda \quad \text{for} \quad \lambda = \exp(-2\pi i k/d), k \in [1, d].\]

By (1.1.5) we have moreover
\[(1.2.5) \quad F^p = P^p \quad \text{on} \quad H^j(f^{-1}(1), \mathbb{C})_1 = H^j(U, \mathbb{C}) \quad \text{if} \quad p > j - [\tilde{\alpha}_Y],\]
where \(\tilde{\alpha}_Y\) is as in (1.1.3). By definition we have
\[(1.2.6) \quad kE_i^{0,0} = H^n \text{Gr}_{p+1}^P(\tilde{C}_f^*)_k = N_k, \quad kE_i^{0,1} = H^{n+1} \text{Gr}_{p+1}^P(\tilde{C}_f^*)_k = M_k,
\]
where \(N_k, M_k\) are as in the introduction. Set
\[(1.2.7) \quad M^{(\infty)}_k := kE_{\infty}^{0,1} = \text{Gr}_{p+1}^P H^{n+1}(\tilde{C}_f^*)_k.
\]
By (1.2.2) and (1.2.4–5) we then get
\[(1.2.8) \quad M^{(\infty)}_{(q+1)d} = \text{Gr}_{p+1}^P H^{n+1}(\tilde{C}_f^*)_d = \text{Gr}_F H^n(U, \mathbb{C}) \quad \text{if} \quad q = n - p < [\tilde{\alpha}_Y].\]
1.3. Thom-Sebastiani type theorems ([DiSaWo Section 4.9]). Let $f = g + g$ with $g$ a homogeneous polynomial of degree $d$ in variables $z_1, \ldots, z_r$. In the notation of (1.2), there is a canonical isomorphism

\[(\tilde{C}_f^*, P') = (\tilde{C}_f^*, P') \otimes_{C[\partial_s, \partial_k^{-1}]} (\tilde{C}_g^*, P').\]

If $g$ has an isolated singularity at the origin, then

\[H^j \text{Gr}_k^P \tilde{C}_g^* = 0 \quad (j \neq r, k \in \mathbb{Z}),\]

and we have the filtered quasi-isomorphisms

\[(\tilde{C}_f^*, P') \xrightarrow{\sim} H^r(\tilde{C}_g^*, P')[-r],\]

\[(\tilde{C}_f^*, P') \xrightarrow{\sim} (\tilde{C}_f^*, P') \otimes_{C[\partial_s, \partial_k^{-1}]} H^r(\tilde{C}_g^*, P')[-r].\]

The latter is compatible with the action of $t$. In fact, the action of $t$ on the left-hand side corresponds to $t \otimes id + id \otimes t$ on the right-hand side by using $\tilde{f} = f + g$.

By (1.3.4) the pole order spectral sequence for $\tilde{f}$ is isomorphic to a finite direct sum of shifted pole order spectral sequences for $f$ by choosing graded free generators of $H^r(\tilde{C}_g^*, P')$ over $C[\partial_s, \partial_k^{-1}]$. Here shifted means that the degrees of complex and filtration are shifted.

The $E_0$-complex of the spectral sequence is an infinite direct sum of the Koszul complexes $(\Omega^*, df \wedge)$ with grading shifted properly, and we have the Thom-Sebastiani type theorem also for the Koszul complexes.

1.4. Complement to the proof of ([DiSaWo Theorem 2.2]). It does not seem to be necessarily easy to follow the proof of ([DiSaWo Theorem 2.2] (even for one of the authors more than five years after the writing) since it was written far too concisely. We give here an additional explanation as follows:

By ([DiSaWo (2.1.1)]) together with the strictness of the Hodge filtration ([De]), we first verify that $\text{Gr}_k^n H^n(U, C)$ is given by the cokernel of the following morphism:

\[
d : \frac{\Gamma(X, F_{n-p-1}O_X(*Y) \otimes O_X \Omega_{X}^{n-1})}{\Gamma(X, F_{n-p-2}O_X(*Y) \otimes O_X \Omega_{X}^{n-1})} \rightarrow \frac{\Gamma(X, F_{n-p}O_X(*Y) \otimes O_X \Omega_{X}^{n})}{\Gamma(X, F_{n-p-1}O_X(*Y) \otimes O_X \Omega_{X}^{n})}
\]

By ([DiSaWo (2.1.2–5)]) we then conclude that this cokernel for $q := n - p \leq m$ is identified, by using $\iota_\xi$, with the cokernel of the morphism

\[
d f \wedge : (\Omega^n / f \Omega^n)_{qd} \rightarrow (\Omega^{n+1} / f \Omega^{n+1})_{(q+1)d} \quad \text{if } q < m,
\]

\[
d f \wedge : (\Omega^n / f \Omega^n)_{qd} \rightarrow (\Omega^{n+1} / f \Omega^{n+1})_{(q+1)d} \quad \text{if } q = m.
\]

So ([DiSaWo] Theorem 2.2) follows. (It seems absolutely necessary to write down the formulas (1.4.1–2) explicitly in order to understand the proof of ([DiSaWo Theorem 2.2] properly.)

2. Proofs of Theorems 3 and 5

In this section we prove Theorems 3 and 5 after showing Proposition (2.1).

2.1. Proposition. In the notation of (1.2), we have the isomorphisms

\[M_k^{(\infty)} = M_k \quad \text{for } k < nd/2,\]

if all the singularities of $Y$ are ordinary double points.
2.3. Proof of Theorem 3. It becomes the following condition: 

\[ \| \text{dim } M_{f,k} = M_{f,k}^{(\infty)} \| \]

The injectivity of (2.2.1) follows from \[\text{DiSa2, Theorem 5.3}\]. We thus get Theorem 5, which is part of the E1-theorems for the Gauss-Manin systems and the Koszul complexes as in (1.3). Set

\[ \text{dim } M_{f,k} - \text{dim } M_{f,k}^{(\infty)} = \sum_j (\text{dim } M_{f,k-j} - \text{dim } M_{f,k-j}^{(\infty)}) \text{dim } M_{g,j}, \]

where we denote \( M \) for any integer \( n \) of the spectral sequence. (Here we may assume \( q \equiv 0 \mod{2} \) for any integer \( q \).

Moreover \( \text{dim } M_{g,j} \) is even, and for \( q \equiv m + 1 \mod{2} \) if \( n \) is odd. (Indeed, setting \( \tilde{Y} = \{f = 0\} \subset \mathbb{P}^{n+1} \times \mathbb{C}^r \), we get by (1.1.2) that \( \tilde{\alpha}_Y = m + 1/d \) and hence \( [\tilde{\alpha}_Y] = m \) if \( n = 2m \), and \( \tilde{\alpha}_Y = m + 1/2 + r/d \) and hence \( [\tilde{\alpha}_Y] = m + 1 \) if \( n = 2m + 1 \).

By definition we have

\[ \text{dim } M_{f,k} - \text{dim } M_{f,k}^{(\infty)} \geq 0 \]

Moreover \( \text{dim } M_{g,j} \) is strictly positive for \( j \in [r, r(d-1)] \), since it is well-known that

\[ \sum_j (\text{dim } M_{g,j}) t^j = (t + \cdots + t^{d-1})^r. \]

(This can be reduced to the case \( r = 1 \) by using the Thom-Sebastiani type theorem.) Using these non-negativity and strict positivity together with (2.1.2-4), we thus get the equalities

\[ \text{dim } M_{f,k} - \text{dim } M_{f,k}^{(\infty)} = 0 \]

which imply a partial degeneration of the spectral sequence. (Here we may assume \( r \geq 2 \) for odd by assuming \( d \geq 3 \).) So the assertion follows.

2.2. Proof of Theorem 5. By Proposition (2.1) we have the vanishing of

\[ d_1 : N_k \to M_k \]

which is part of the \( E_1 \)-differential of the spectral sequence (1.2.3). On the other hand, the injectivity of (2.2.1) follows from [DiSa2, Theorem 5.3]. We thus get Theorem 5.

2.3. Proof of Theorem 3. In the notation of the introduction, we see that condition (0.2) becomes the following condition:

\[ q > m - 1 \quad \text{if } n = 2m, \]
\[ q > m + d/2 - 1 \quad \text{if } n = 2m + 1. \]

So Theorem 3 follows.
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