LOCAL NORMAL FORMS OF DYNAMICAL SYSTEMS WITH A SINGULAR UNDERLYING GEOMETRIC STRUCTURE

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Abstract. In this paper we prove the existence of a simultaneous local normalization for couples \((X, G)\), where \(X\) is a vector field which vanishes at a point and \(G\) is a singular underlying geometric structure which is invariant with respect to \(X\), in many different cases: singular volume forms, singular symplectic and Poisson structures, and singular contact structures. Similarly to Birkhoff normalization for Hamiltonian vector fields, our normalization is also only formal, in general. However, when \(G\) and \(X\) are (real or complex) analytic and \(X\) is analytically integrable or Darboux-integrable then our simultaneous normalization is also analytic. Our proofs are based on the toric approach to normalization of dynamical systems, the toric conservation law, and the equivariant path method. We also consider the case when \(G\) is singular but \(X\) does not vanish at the origin.

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1. Introduction

A classical result due to Birkhoff, Gustavson, and Moser (see, e.g., [3, 15, 24]) says that a Hamiltonian vector field $X$ which vanishes at a point $O$ admits a normalization à la Poincaré-Birkhoff at $O$ (also in the resonant case), i.e., there is a coordinate system $(x_1, \ldots, x_{2n})$ in which $X$ does not contain any non-resonant term and the symplectic form has canonical expression: $\omega = \sum_{i=1}^{n} dx_i \wedge dx_{i+n}$. A similar result, with a similar proof, is known to hold for volume-preserving vector fields: If $X$ preserves a volume form $\Omega$ then there is a coordinate system $(x_1, \ldots, x_n)$ which normalizes $X$ and in which $\Omega$ has canonical expression: $\Omega = dx_1 \wedge \ldots \wedge dx_n$. These normalizations of Hamiltonian and volume-preserving vector fields are only formal in general. However, when $X$ is integrable analytically or in the sense of Darboux, then there exists a locally analytic normalization [34, 35, 38].

The problem of normalization of Hamiltonian and volume-preserving vector fields may be viewed as the problem of simultaneous normalization of a couple $(X, G)$, where $G$ is an underlying geometric structure preserved by $X$. (In the above cases, $G$ is a symplectic form or a volume form.) In this paper, we want to study this simultaneous normalization problem, when the geometric structure $G$ itself is singular at the singular point $O$ of $X$. We will concentrate our attention to the following list of singular geometric structures, though many other structures (e.g., more general Poisson, Dirac or Nambu structures) may appear in geometric mechanics and physics:

- Singular volume forms, which either vanish or blow up at the singular point $O$ of $X$ (e.g., $x_1^k dx_1 \wedge \ldots \wedge dx_n$ where $k = \pm 1$).
- Singular symplectic forms, which are either folded symplectic or log-symplectic (e.g., $\sum x_i^{k_i} dx_i \wedge dx_{n+i}$ where $k_i \in \{-1, 0, 1\}$).
- Singular contact structures.

In each of the above cases, we impose some non-degeneracy or genericity condition so that we can prove that the singular geometric structure itself admits a nice normal form, i.e., has some local canonical expression, e.g., $\frac{1}{x_1} dx_1 \wedge dx_{n+1} + \sum_{i=2}^{n} dx_i \wedge dx_{n+i}$ for a so-called b-symplectic structure (see., e.g, [13, 11, 12]). Then we show that, in each of these cases, the couple $(X, G)$ admits a simultaneous normalization, i.e., a coordinate system which puts $G$ in some canonical form and $X$ in normal form.

The simultaneous normalization problem is quite a tricky problem, especially when the underlying geometric structure itself is singular. We see two main approaches to this problem:

The first approach is the classical one, where one first puts the geometric structure $G$ in canonical form, then make a step-by-step normalization of the vector field $X$ (killing resonant nonlinear terms one by one) by “canonical transformations” which leave $G$ intact. This approach was effective for Hamiltonian and volume-preserving systems, and can probably be used for other kinds of systems as well. However, to show the existence of required “canonical transformations” at each step is a non-trivial task, especially when the geometric structure itself is singular.

The second approach is a geometric approach developed by the third author [34, 35, 37], and is based on toric characterization of the normalization of dynamical systems,
together with the equivariant Moser path method [23] (a.k.a. Lie transform method). This is the approach that we will use in this paper, and it goes as follows:

- For each vector field (or family of commuting vector fields) which vanishes at a point \( O \) there is a unique natural (intrinsic) local effective associated torus \( \mathbb{T}^\tau \)-action \( \rho \) (in the complexified space if the original space is real) which fixes \( O \). The dimension \( \tau \) of the torus in question is called the toric degree of the system at \( O \). This action \( \rho \) is only formal in general, but when the system is analytically integrable or Darboux-integrable then it is automatically analytic.

- Universal toric conservation law: any geometric structure \( \mathcal{G} \) preserved by a dynamical system is also preserved by its associated torus actions.

- In our cases, it is known or it can be shown that \( \mathcal{G} \) can be put into canonical form in some coordinate system (if one forgets about \( \rho \)), and it is also known that \( \rho \) can be linearized by Bochner’s averaging formula [4] (if one forgets about \( \mathcal{G} \)). What we want to do is to linearize \( \rho \) and put \( \mathcal{G} \) in canonical form simultaneously, at the same time, and we will use the equivariant path method in order to achieve this.

- The linearization of \( \rho \) is actually equivalent to the normalization of our dynamical system. So by the previous step we find a coordinate system in which both the geometric structure and the dynamical system are normalized.

In general, the above steps give us only a formal simultaneous normalization, just like in the classical theory of Poincaré-Birkhoff normalization of vector fields. The problem of showing the existence or non-existence of a local analytic normalization is a difficult problem in general, as it involves small divisor phenomena which are hard to control, see, e.g., [6, 16, 28]. However, when our system is analytically integrable or Darboux-integrable, then the associated torus action is analytic, the path method can also be done analytically, and we obtain an analytic simultaneous normalization without having to deal with small divisors.

Remark that our approach (as well as the first classical approach) works mainly in the formal and analytic categories, so our results concern mainly formal and analytic systems, though some results are also valid in the smooth category. (The proofs in the smooth category are usually more involved and require some specific techniques.)

Remark also that, in this paper, most of the times we do not distinguish clearly between the real case (over \( \mathbb{R} \)) and the complex case (over \( \mathbb{C} \)). Our results are valid for both complex systems and real systems, i.e., in the real case the normalization maps are also real, though we only write explicit proofs for the complex case but not for the real case. The reason is that, by local complexification, a real dynamical system (together with an underlying geometric structure) can be viewed as a complex dynamical system which is equivariant with respect to the anti-complex involution. As was explained in, e.g., [14] and [35], even if we deal with a real system, we can do everything in the complexified space and keep track of the anti-complex involution, so that our complex normalization maps in the real case are also equivariant with respect to this anti-complex involution, i.e., they are in fact complexifications of real normalization maps.

Of course, some “finer” normal forms, e.g., Jordan normal form for matrices, look differently when we change from complex coordinates to real coordinates, but that’s not the main point of our paper.
The rest of our paper is organized as follows: In Section 2 we collect necessary notions and tools to be used for our study, including the toric approach to the normalization problem of vector fields (Subsection 2.1), the notions of analytic integrability and Darboux integrability (Subsection 2.2), the universal toric conservation law (Subsection 2.3), and the equivariant Moser’s path method (Subsection 2.4). The main new result of Section 2 is the following theorem, which is a particular case (which has not been proved explicitly anywhere, as far as we know) of the universal toric conservation law which says that “anything which is preserved by a dynamical system is also preserved by its associated torus actions” [37].

**Theorem 1.1** (Theorem 2.3). Let $X$ be a formal vector field which vanishes at a point $O$ and which preserves a formal rational tensor field $\Lambda$ (i.e., $\Lambda = \Omega/f$ where $\Omega$ is a formal tensor field and $f$ is a non-trivial formal function). Then the associated torus action of $X$ at $O$ also preserves $\Lambda$. If $\Lambda$ is only conformally preserved by $X$ (i.e., $L_X\Lambda = g\Lambda$ where $g$ is a formal function), then the associated torus action of $X$ also conformally preserves $\Lambda$.

In Section 3 we treat the case with an invariant singular volume form, in Section 4 we treat the case with an invariant singular volume form, and finally, in Section 5 we treat in detail the case with an invariant contact structure. The main results of these last three sections can be put together into one big abstract theorem as follows:

**Theorem 1.2.** Let $X$ be a formal vector field which vanishes at the origin and which preserves a singular geometric structure $\mathcal{G}$ which belongs to one of the following types: folded volume forms, generic singular Nambu structures of top order, (multi-)folded symplectic forms, log-symplectic forms, and three different kinds of singular contact structures. Then the couple $(X, \mathcal{G})$ can be formally normalized simultaneously. Moreover, in the integrable analytic case, when both $\mathcal{G}$ and $X$ are analytic and $X$ is analytically integrable or Darboux-integrable, then there exists a local analytic simultaneous normalization of $(X, \mathcal{G})$.

We also treat the cases of couples $(X, \mathcal{G})$ where the geometric structure is singular at the origin $O$ but $X(O) \neq 0$. In a series of theorems throughout this paper, we show that, in such situations, $\mathcal{G}$ can often be put into normal form in a coordinate system in which the vector field $X$ becomes $\partial/\partial x_k$ where $x_k$ is one of the coordinates. These theorems (for the cases $X(O) \neq 0$) use more classical step-by-step normalization methods and not the toric approach for $X$.

2. **Preliminaries**

2.1. **Associated torus actions.** In this subsection, we briefly recall from [34, 35, 36, 37] the toric approach to the problem of local normalization of vector fields.

Denote by $X$ a formal or local analytic vector field on a manifold, which vanishes at a point $O$. We write the Taylor series of $X$ in a coordinate system $(x_1, \ldots, x_n)$ around $O$ as

$$X = X^s + X^n + \sum_{i \geq 2} X^{(i)}$$

(2.1)
where $X^{(1)} = X^s + X^n$ is the Jordan decomposition of the linear part of $X$ in the above coordinate system, i.e. $X^s$ is its semisimple part and $X^n$ is its nilpotent part, and $X^{(i)}$ is the homogeneous term of degree $i$ of $X$. We can assume (after a complexification of the system if necessary) that $X^s$ is diagonal:

$$X^s = \sum_{i=1}^{n} \gamma_i x_i \frac{\partial}{\partial x_i},$$

where $\gamma_1, \ldots, \gamma_n \in \mathbb{C}$ are the eigenvalues of $X$ at $O$. We say that $X$ is in normal form (à la Poincaré–Birkhoff or Poincaré–Dulac) if

$$[X^s, X] = 0,$$

which amounts to the vanishing of all the non-resonant terms in the Taylor expansion of $X$. The classical theorem of Poincaré says that such a formal coordinate system always exists, i.e., $X$ can always be normalized formally.

We may view $X$, via its Lie derivative, as a linear differential operator on the space of formal functions. Then this operator, even though acting on an infinite dimensional space, also admits an intrinsic unique Jordan decomposition

$$X = X^S + X^N$$

where $X^S$ and $X^N$ are formal vector fields (also viewed as linear operators) which are intrinsic (i.e. they depend on $X$ but do not depend on the choice of local coordinates). $X$ is in normal form if and only if $X^S$ becomes linear semisimple, i.e. $X^S = X^s$ and $X^N = X^n + X^{(2)} + \ldots$

The smallest number $\tau \geq 0$ such that we can express $X^S = X^s$ (in normalized coordinates) in the form

$$X^S = \sum_{i=1}^{\tau} \rho_i Z_i,$$

where $\rho_i \in \mathbb{C}$ are constants and

$$Z_i = \sum_{j=1}^{n} a_{ij} x_j \frac{\partial}{\partial x_j}$$

are diagonal vector fields with integer coefficients $a_{ij} \in \mathbb{Z}$, is called the toric degree of $X$ at $O$. The minimality condition on $\tau$ is equivalent to the condition that $\rho_1, \ldots, \rho_\tau$ are incommensurable. The vector fields $\sqrt{-1}Z_1, \ldots, \sqrt{-1}Z_\tau$ are the generators of an effective torus $\mathbb{T}^\tau$-action $\rho$, which is uniquely determined by $X$ (up to automorphisms), and which is called the intrinsic associated torus action of $X$ at $O$.

In general, even when $X$ is analytic, its normalization is only formal, and its associated torus action $\rho$ is also formal. Thanks to Bochner’s averaging formula, any compact group action (even formal ones) can be linearized, and the normalization of $X$ is the same as the linearization of its associated torus action $\rho$ (i.e., $X$ is in normal form if and only if $\rho$ is linear). $X$ admits a local analytic normalization if and only if
the action $\rho$ is analytic. So this torus action $\rho$ plays a very important role in normalization problems, and it admits the fundamental conservation property \[37\] which will be discussed in Subsection 2.3.

2.2. Integrable systems. Recall that a Darboux-type function is a (generally multi-valued) function $F$ which can be written as

$$F = \prod_{k=1}^{m} F_{k}^{c_k}$$

where $F_{k}$ are analytic functions and $c_k$ are complex numbers. The class of Darboux-type functions on a manifold is significantly larger than the class of analytic or rational functions. Remark that, even though $F = \prod_{k=1}^{m} F_{k}^{c_k}$ is multivalued, its logarithmic differential

$$\frac{dF}{F} = d(\ln F) = \sum_{k=1}^{m} c_k \frac{dF_k}{F_k}$$

is a single-valued rational differential 1-form, and so it makes perfect sense to talk about Darboux-type first integrals of vector fields: $X(\prod_{k=1}^{m} F_{k}^{c_k}) = 0$ means that

$$\sum_{k=1}^{m} c_k \frac{X(F_k)}{F_k} = 0.$$ 

In many problems in dynamical systems, one may be interested in Darboux-type first integrals when there do not exist enough analytic or rational first integrals (see, e.g., \[33\]).

In this paper, for analytic normalization, we will consider dynamical systems which are integrable in the usual non-Hamiltonian sense (see, e.g., \[5, 28, 34, 37\]), also in the case when the first integrals are only Darboux-type functions. Let us recall here the definition of integrability:

**Definition 2.1.** i) A dynamical system given by a vector field $X$ on an $n$-dimensional manifold $M$ is called **integrable** if there exist $p$ vector fields $X_1 = X$, $X_2, \ldots, X_p$ and $q$ functions $F_1, \ldots, F_q$ on $M$, such that the vector fields commute pairwise, $X_1 \wedge \cdots \wedge X_p \neq 0$ almost everywhere, the functions are common first integrals for these vector fields, and $dF_1 \wedge \cdots \wedge dF_q \neq 0$ almost everywhere. The integers $p, q$ satisfy $p \geq 1, q \geq 0, p + q = n$.

ii) An $n$-tuple $(X_1, \ldots, X_p, F_1, \ldots, F_q)$ such as above is also called an **integrable system of type** $(p,q)$.

iii) If the vector fields $X_1, \ldots, X_p$ and the functions $F_1, \ldots, F_q$ are all smooth (resp. formal, resp. analytic) then the vector field $X$ and the system $(X_1, \ldots, X_p, F_1, \ldots, F_q)$ are called **smoothly** (resp. **formally**, resp. **analytically** integrable).

iv) If the vector fields $X_1, \ldots, X_p$ are rational (i.e. can be written as the quotient of an analytic vector field by a non-trivial analytic function), and the functions $F_1, \ldots, F_q$ are Darboux-type functions, then $X$ and the system $(X_1, \ldots, X_p, F_1, \ldots, F_q)$ are called **Darboux-integrable**.

The following result, obtained by the third author via geometric approximation methods (instead of the fast convergence method), will allow us to analytically normalize integrable systems:
Theorem 2.2 ([34, 35, 38]). Let $X$ be a local analytic vector field which vanishes at a point $O$ and which is analytically integrable or Darboux-integrable. Then the associated torus action of $X$ at $O$ is locally analytic.

2.3. The toric conservation law. The toric conservation law, which states that anything which is preserved by a dynamical system is also preserved by its associated torus actions, was first discovered in its general form by the third author [37], though some particular cases of this law (e.g. when that “anything” is a function, i.e., a first integral) have been known before. Here, the system may be given by a vector field, or a discrete-time system, or a quantum system, or a stochastic system, etc.; “anything” may be a tensor field, or a subbundle of a natural bundle over the manifold, or a differential operator, etc., which may be smooth, analytic, meromorphic, formal, Darboux-type, etc.; and the associated torus actions in question depend on the situation: for example, for integrable Hamiltonian systems near a regular Liouville torus then this torus action is the Liouville torus action, and when the system is a local dynamical system which vanishes at a point then this torus action is the one explained in Subsection 2.1.

For each particular situation, the toric conservation law can be stated as a rigorous theorem or conjecture. There are still many situations when no proof of the corresponding conjectures had been written down anywhere yet. In this section we will consider such a situation, which will be useful for our simultaneous normalization problems.

Consider a formal vector field $X$ on $\mathbb{C}^n$ which vanishes at the origin, and consider a formal rational tensor field $\Lambda$ on $\mathbb{C}^n$, i.e., $\Lambda$ is the quotient of a formal tensor field $\Omega$ by a nontrivial formal function $f$: $\Lambda = \frac{\Omega}{f}$. We will say that $\Lambda$ is an invariant of $X$ if it is conserved by $X$, i.e., $\mathcal{L}_X \Lambda = 0$. We will say that $\Lambda$ is a semi-invariant of $X$ if it is conformally conserved by $X$, i.e., there is a formal function $g$ such that $\mathcal{L}_X \Lambda = g \Lambda$.

Observe that the set of nontrivial semi-invariant formal rational functions of $X$ form a multiplicative group: If $\mathcal{L}_X F_1 = g_1 F_1$ and $\mathcal{L}_X F_2 = g_2 F_2$ then $\mathcal{L}_X (1/F_1) = -g_1 (1/F_1)$ and $\mathcal{L}_X (F_1 F_2) = (g_1 + g_2) F_1 F_2$. Moreover, if $\Lambda = \frac{\Omega}{f}$ is a formal rational tensor field written in reduced form, i.e. $\Omega$ and $f$ are co-prime, and $\Lambda$ is a semi-invariant tensor field of $X$, then both $\Omega$ and $f$ are semi-invariants of $X$.

Indeed, $\mathcal{L}_X \left( \frac{\Omega}{f} \right) = g \frac{\Omega}{f}$ means that $f \mathcal{L}_X (\Omega) - \mathcal{L}_X (f) \Omega = g f \Omega$, or $\mathcal{L}_X (f) \Omega = f (\mathcal{L}_X \Omega - g \Omega)$. Since $\Omega$ is co-prime with $f$, it implies that $\mathcal{L}_X (f)$ is divisible by $f$, $\mathcal{L}_X (f) = h f$ where $h$ is a formal function, and then we have $\mathcal{L}_X \Omega = (g + h) \Omega$.

Theorem 2.3 (Conservation Theorem). Let $X$ be a formal vector field on $\mathbb{C}^n$ which vanishes at the origin. Assume that $X$ is already in Poincaré-Dulac normal form, and that $(Z_1, \ldots, Z_\tau)$ is a $\tau$-tuple of diagonal vector fields which generate the associated torus action of $X$ as in [23], [26]. Let $\Lambda$ be a formal rational tensor field on $\mathbb{C}^n$. Then we have:

i) If $\Lambda$ is preserved by $X$, i.e., $\mathcal{L}_X \Lambda = 0$ then $\Lambda$ is also preserved by the associated torus action of $X$, i.e. $\mathcal{L}_{Z_k} \Lambda = 0$ for every $k = 1, \ldots, \tau$. 
ii) If $\Lambda$ is conformally preserved by $X$, i.e., $\mathcal{L}_X \Lambda = g \Lambda$, where $g$ is a formal function, then $\Lambda$ is also conformally preserved by the associated torus action of $X$, i.e., $\mathcal{L}_{Z_k} \Lambda = g_k \Lambda$ for every $k = 1, \ldots, \tau$, where $g_1, \ldots, g_\tau$ are formal functions.

In order to prove the above theorem, let us introduce the following result by Walcher (Lemma 2.2 in [30]), see also Lemma 2.1 of [38].

**Lemma 2.4.** Let $X$ be a formal vector field in the Poincaré-Dulac normal form. Assume that $F = \sum_k F^{(k)}$ is a formal semi-invariant invariant of $X$, i.e., $X(F) = \lambda F$ for some formal function $\lambda$. Then there exists an invertible formal function $\beta$ (i.e., $\beta(0) \neq 0$) such that $\tilde{F} = \beta F$ is a semi-invariant of $X$ and of its semisimple part $X^s$, and moreover, the formal function $\tilde{\lambda}$ satisfying $X(\tilde{F}) = \tilde{\lambda} \tilde{F}$ is a first integral of $X^s$ and we have $X^s(\tilde{F}) = \lambda(0) \tilde{F}$. Moreover, $Z_k(\tilde{F}) = c_k \tilde{F}$ for every $k = 1, \ldots, \tau$, where $c_k$ are some complex numbers and $Z_1, \ldots, Z_\tau$ are generators given in formulas (2.5) and (2.6) of the associated torus action of $X$.

We now turn to the proof of the Conservation Theorem 2.3.

**Proof.** i) We can write the formal rational tensor field $\Lambda$ as $\Lambda = \frac{\Omega}{f}$ where $f$ is a formal function and $\Omega$ is a formal tensor field co-prime with $f$. As observed above, $\mathcal{L}_X \frac{\Omega}{f} = 0$ implies that $f$ and $\Omega$ are semi-invariants of $X$, i.e.,

$$
(2.7) \quad \mathcal{L}_X \Omega = \lambda \Omega \quad \text{and} \quad X(f) = \lambda f
$$

for some formal function $\lambda$. Invoking Walcher’s Lemma 2.3, we can assume $\lambda$ is a first integral of $X^s$ (by multiplying both $\Omega$ and $f$ by a formal function of the type $1 + \beta$ if necessary), and that $X^s(f) = \lambda(0)f$ and $Z_k(f) = c_k f$ with $c_k \in \mathbb{C}$ for $k = 1, \ldots, \tau$.

We want to show that $\mathcal{L}_X \frac{\Omega}{f} = 0$, or equivalently, $f \mathcal{L}_X \Omega = X^s(f) \Omega$. Since $X^s(f) = \lambda(0)f$, the equation to prove is reduced to $\mathcal{L}_X \Omega = \lambda(0) \Omega$. Let us prove it by induction on the degree of the terms in (2.7).

Write $\lambda = \lambda^{(0)} + \lambda^{(1)} + \cdots$ and $\Omega = \Omega^{(0)} + \Omega^{(1)} + \cdots$ in which the upper right index $(k)$ indicates the homogeneous part of degree $k$ of the corresponding term. Obviously, $\mathcal{L}_{X^{(k)}} \Omega^{(0)} = \lambda(0) \Omega^{(0)}$, therefore $\mathcal{L}_{X^s} \Omega^{(0)} = \lambda(0) \Omega^{(0)}$. Now suppose $\mathcal{L}_{X^s} \Omega^{(k)} = \lambda^{(0)} \Omega^{(k)}$ for all $k < \ell$. Look at the terms of degree $\ell$ in (2.7), we have

$$
(2.8) \quad \mathcal{L}_{X^{(k)}} \Omega^{(\ell)} + \sum_{k=2}^{\ell+1} \mathcal{L}_{X^{(k)}} \Omega^{(\ell+1-k)} = \lambda^{(0)} \Omega^{(\ell)} + \sum_{k=1}^{\ell} \lambda^{(k)} \Omega^{(\ell-k)}.
$$

Let $X^s$ act on both sides of the above equation. Since $[X^s, X^{(k)}] = 0$ and $X^s(\lambda^{(k)}) = 0$, we have $\mathcal{L}_{X^s} \Omega^{(k)} = \lambda(0) \Omega^{(k)}$ for all $k$ by the lemma, and so we get

$$
(2.9) \quad \mathcal{L}_{X^{(k)}} \mathcal{L}_{X^s} \Omega^{(\ell)} + \lambda^{(0)} \sum_{k=2}^{\ell+1} \mathcal{L}_{X^{(k)}} \Omega^{(\ell+1-k)} = \lambda^{(0)} \mathcal{L}_{X^s} \Omega^{(\ell)} + \lambda(0) \sum_{k=1}^{\ell} \lambda^{(k)} \Omega^{(\ell-k)}.
$$
Look at the difference between the expression (2.9) and $\lambda^0$ times the expression (2.8), we have
\begin{equation}
\mathcal{L}_{X(1)} (\mathcal{L}_{X^s} \Omega^{(e)} - \lambda^0 \Omega^{(e)}) = \lambda^0 (\mathcal{L}_{X^s} \Omega^{(e)} - \lambda^0 \Omega^{(e)}),
\end{equation}
therefore
\begin{equation}
\mathcal{L}_{X^s} (\mathcal{L}_{X^s} \Omega^{(e)} - \lambda^0 \Omega^{(e)}) = \lambda^0 (\mathcal{L}_{X^s} \Omega^{(e)} - \lambda^0 \Omega^{(e)}).
\end{equation}
This means that $\mathcal{L}_{X^s} \Omega^{(e)} - \lambda^0 \Omega^{(e)}$ is an eigenvector with eigenvalue $\lambda^0$ of the linear operator $\mathcal{L}_{X^s}$ on the space $E$ of homogeneous tensor fields of degree $\ell$. We now decompose $E$ into a direct sum of eigenspaces and write $\Omega^{(e)} = \sum_j \Omega_j^{(e)}$ where $\Omega_j^{(e)}$ is a vector in the eigenspace $E_{c_j}$ with eigenvalue $c_j$. Restricting attention to $E_{c_j}$, we can easily conclude $c_j = \lambda^0$ by (2.11). Thus $c_j = \lambda^0$ for all $j$, and $\Omega^{(e)}$ itself lies in the eigenspace $E_{\lambda^0}$, that is, $\mathcal{L}_{X^s} \Omega^{(e)} = \lambda^0 \Omega^{(e)}$. By induction, we conclude that $\mathcal{L}_{X^s} \Omega = \lambda^0 \Omega$.

Each monomial tensor field $\Theta$ is an eigenvector of the Lie derivative operator of the diagonal vector field $X^s = \sum_{k=1}^r \rho_k Z_k$, i.e., we have $\mathcal{L}_{X^s} \Theta = \sum \rho_k c_k(\Theta) \Theta$ for some numbers $c_k(\Theta)$ which a priori depend on $\Theta$. Since $\mathcal{L}_{X^s} \Omega = \lambda^0 \Omega$, we must have $\sum \rho_k c_k(\Theta) = \lambda(0)$ for every monomial term $\Theta$ of $\Omega$. Since the numbers $\rho_1, \ldots, \rho_r$ are incommensurable, there is at most one $r$-tuple of complex numbers $c_1, \ldots, c_r$ satisfying the equation $\sum \rho_k c_k = \lambda(0)$, which means that $c_k = c_k(\Theta)$ does not depend on $\Theta$, and that $\mathcal{L}_{Z_k} (\Omega) = c_k \Omega$. Similarly, we have $\mathcal{L}_{Z_k} (f) = c_k f$, and so $\mathcal{L}_{Z_k} (\Omega / f) = 0$.

ii) The proof of this second part consists of the following 3 steps:

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**Step 1.** Write $\Lambda = \frac{\Omega}{f}$, where $\Omega$ and $f$ are co-prime. Then both $\Omega$ and $f$ are semi-invariants of $X$. It implies by Lemma 2.24 that $f$ is a semi-invariant of the associated torus action of $X$, so it is enough to show that $\Omega$ is also a semi-invariant of the associated torus action of $X$, i.e., the problem is reduced to the case when $\Lambda = \Omega$ is a formal tensor field.

**Step 2.** Write $\mathcal{L}_X \Omega = \lambda \Omega$, where $\lambda$ is a formal function. By putting $\Omega^* = \beta \Omega$, where $\beta$ is an appropriate invertible formal function, we can arrange so that $\mathcal{L}_X \Omega^* = \lambda(0) \Omega^*$. The proof of this step is the same as the proof of Walcher’s lemma. (See the proof of Part i of Lemma 2.1 of [38].) Let us write it down here explicitly:

The semi-invariance of $\Omega$ with respect to $X$ is equivalent to
\begin{equation}
\mathcal{L}_{X^s} (\Omega^{(r+j)}) + \mathcal{L}_X (\Omega^{(r+j)}) + \mathcal{L}_{X^s} (\Omega^{(r+j-1)}) + \ldots + \mathcal{L}_{X^s (j+1)} (\Omega^{(r)}) = \lambda^0 \Omega^{(r+j)} + \ldots + \lambda^j \Omega^{(r)}
\end{equation}
for all $j \geq 0$.

Note that $X^s (\lambda^0) = 0$. Now assume that $X^s (\lambda^j) = 0$ for all $j < k$, and let $\bar{\Omega} = (1 + \beta_k) \Omega$, where $\beta_k$ is some homogeneous function of degree $k$. Then
\begin{equation}
\bar{\Omega} = \Omega^{(r)} + \ldots + \Omega^{(r+k-1)} + (\Omega^{(r+k)} + \Omega^{(r)} \beta_k) + \ldots
\end{equation}
and
\begin{equation}
\mathcal{L}_X (\bar{\Omega}) = \tilde{\lambda} \bar{\Omega},
\end{equation}
with
\begin{equation}
\tilde{\lambda} = \lambda^0 + \ldots + \lambda^{(k-1)} + (\lambda^{(k)} + X^s ((\beta_k))) + \ldots
\end{equation}
Due to the semi-simplicity of \( X^\ast \), one can choose \( \beta_k \) such that \( X^\ast(\lambda^{(k)} + X^{(1)}(\beta_k)) = 0 \). Thus, the assertion is proved by induction on \( k \): \( \beta \) can be constructed in the form of an infinite product \( \prod_{k=1}^{\infty} (1 + \beta_k) \), where each \( \beta_k \) is homogeneous of degree \( k \). (Such an infinite product converges in the space of formal power series). From \( X^\ast(\lambda^*) = 0 \) one deduces that \( L_{X^\ast}(\Omega^*) = \lambda^{(0)}\Omega^* \), again by induction and by the semi-simplicity of \( X^\ast \).

Step 3. The equation \( L_{X^\ast}(\Omega^*) = \lambda^{(0)}\Omega^* \) implies that \( \Omega^* \) (and hence \( \Omega \)) is a semi-invariant of the vector fields \( Z_k \), \( k = 1, \ldots, \tau \). This step is already done at the end of the proof of Part i.  

\[ \square \]

2.4. Equivariant Moser path method. Let us recall the well-known path method, which was introduced by Moser in \[23\] to show the equivalence of two volume forms, and which can be generalized to many other situations in order to show the equivalence of two tensor fields on a manifold.

Let two tensor fields \( G_0 \) and \( G_1 \) be given on a smooth manifold \( M \). We say they are \textit{locally equivalent} at \( O \in M \) if there is a diffeomorphism \( \varphi \) of one neighborhood of \( O \) to another neighborhood of \( O \), such that \( \varphi^* G_1 = G_0 \). One way to show that \( G_0 \) and \( G_1 \) are locally equivalent is to join them by a curve of tensor fields \( G(t) \) satisfying \( G(0) = G_0 \), \( G(1) = G_1 \) and to seek a curve of local diffeomorphisms \( \varphi_t \) such that \( \varphi_0 = Id \) and

\[ \varphi_t^* G(t) = G_0, \quad \forall t \in [0, 1]. \]

Then \( \varphi = \varphi_1 \) is the desired diffeomorphism.

A way to find the curve of diffeomorphisms \( \varphi_t \) satisfying the relation above is to solve the equation

\[ (2.12) \quad L_{X^\ast} G(t) + \frac{d}{dt} G(t) = 0 \]

for a smooth time-dependent vector field \( X_t \). If this is possible, let \( \varphi_t = F_{t,0} \), where \( F_{t,s} \) is the evolution operator of the time-dependent vector field \( X_t \). Then we have

\[ \frac{d}{dt} \varphi_t^* G(t) = \varphi_t^* \left( L_{X^\ast} G(t) + \frac{d}{dt} G(t) \right) = 0, \]

so that \( \varphi_t^* G(t) = \varphi_0^* G(0) = G_0 \). If we choose \( X_t \) such that \( X_t(O) = 0 \), then \( \varphi_t \) exists for a time \( t \geq 1 \) in an open neighborhood of \( O \) and \( \varphi_t(O) = O \).

One often takes \( G(t) = (1-t)G_0 + tG_1 \). Also, in applications, this method is not always used in exactly this way since the algebraic equation for \( X_t \) might be hard to solve. In such situations the spirit of the path method is used.

Now suppose that a compact Lie group \( G \) acts smoothly on \( M \), preserving both tensor fields \( G_0 \) and \( G_1 \), i.e., \( \rho^i G_t = G_t \), for all \( g \in G \), \( i = 0, 1 \), where \( \rho : G \times M \to M \) is the \( G \)-action. Choose the path of tensor fields \( G(t) \) to be also invariant under the action \( \rho; \) this condition is clearly satisfied if \( G(t) = (1-t)G_0 + tG_1 \). Let \( X_t \) be a solution of \( 2.12 \) and define the \( G \)-averaged smooth vector field

\[ Y_t = \int_G \rho_g^i X_t d\mu, \]
where $\mu$ is the Haar measure on $G$. By invariance of the Haar measure relative to group translations, $Y_t$ is $G$-invariant. In addition, since $\rho^*_g G(t) = G(t)$ for any $g \in G$ and $t \in [0, 1]$, we have

$$\mathcal{L}_{Y_t} G(t) + \frac{d}{dt} G(t) = \int_G \left( \mathcal{L}_{\rho^*_g X_t} G(t) \right) \, d\mu + \frac{d}{dt} G(t)$$

$$= \int_G \left( \mathcal{L}_{\rho^*_g X_t} \rho^*_g G(t) + \frac{d}{dt} \rho^*_g G(t) \right) \, d\mu$$

$$= \int_G \rho^*_g \left( \mathcal{L}_{X_t} G(t) + \frac{d}{dt} G(t) \right) \, d\mu = 0,$$

and hence $Y_t$ also solves (2.12). This shows that $G_0$ and $G_1$ are $G$-equivariantly equivalent. We proved the following result.

**Lemma 2.5.** Suppose that the compact Lie group $G$ acts locally smoothly (respectively formally or analytically) on $M$, preserving two tensor fields $G_0$ and $G_1$. If (2.12) has a smooth (respectively formal or analytic) solution $X_t$, then the two structures are locally smoothly (respectively formally or analytically) $G$-equivariantly equivalent.

### 3. Systems with an invariant singular volume form

In this section, we look at a local volume form

$$\Omega = f(x_1, \ldots, x_n) \, dx_1 \wedge \ldots \wedge dx_n,$$

which is singular in one of the following two senses: either $f$ is a formal or analytic function which vanishes at the origin $O = (0, \ldots, 0)$, or $f$ blows up at $O$. When $f$ blows up at $O$, it is more convenient to look at the dual $n$-vector field (i.e., a contravariant volume form, a.k.a. Nambu structure of top order) $\Lambda = \Omega^{-1}$ defined by $\langle \Lambda, \Omega \rangle = 1$, i.e.,

$$\Lambda = g(x_1, \ldots, x_n) \frac{\partial}{\partial x_1} \wedge \ldots \wedge \frac{\partial}{\partial x_n},$$

where $g = 1/f$ is now a formal or analytic function which vanishes at $O$.

We will assume that $O$ is a non-degenerate singular point, i.e., we have the following two cases:

- **Case 1.** $f(O) = 0$ and $df(O) \neq 0$. In this case $\Omega$ will be called a folded volume form (because it can be obtained as the pull-black of a usual volume form by a fold map, like in the case of folded symplectic structures [20]). It will be studied in Subsection 3.1.

- **Case 2.** $g(O) = 0$ and $dg(O) \neq 0$. In this case $\Omega$ is called a non-degenerate log-volume form and its dual $n$-vector field $\Lambda$ is a non-degenerate singular Nambu structure of top order. It will be studied in Subsection 3.2.

### 3.1. The case of a folded volume form.

The following lemma about the canonical form of a folded volume form is a well-known folklore result, but we don’t know any exact reference for it, so we will present its proof.
Lemma 3.1. Consider a local differential form of top order
\[ \Omega = f dx_1 \wedge \ldots \wedge dx_n \]
on an \( n \)-dimensional manifold \((n \geq 2)\), where \( f \) is a formal (resp. local analytic, resp. smooth) function such that \( f(O) = 0 \) and \( df(O) \neq 0 \). Then there exists a formal (resp. analytic, resp. smooth) coordinate system \((y_1, \ldots, y_n)\) around \( O \) such that
\[ \Omega = y_1 dy_1 \wedge \ldots \wedge dy_n \]

Proof. Denote by \( S = \{ f = 0 \} \) the hypersurface of singular points of \( \Omega \). By the implicit function theorem, locally we can write \( S = \{ x_1 = h(x_2, \ldots, x_n) \} \) where \( h \) is some formal (or analytic, or smooth) function. Put \( \hat{x}_1 = x_1 - h(x_2, \ldots, x_n) \). Then
\[ \Omega = f d\hat{x}_1 \wedge dx_2 \wedge \ldots \wedge dx_n \]
and \( f = 0 \) at \( \hat{x}_1 = 0 \). Thus, by such a change, we may assume that \( S = \{ x_1 = 0 \} \), i.e., \( f \) is divisible by \( x_1 \): \( f = x_1 \phi \), where \( \phi \) is a function such that \( \phi(O) \neq 0 \). Integrating \( f = x_1 \phi \) in the direction of \( \partial / \partial x_1 \), we obtain a function
\[ g(x_1, \ldots, x_n) = \int_0^1 t\phi(t, x_2, \ldots, x_n)dt \]
satisfying
\[ \Omega = dg \wedge dx_2 \wedge \ldots \wedge dx_n, \]
and such that \( g(0, x_2, \ldots, x_n) = 0 \), \( \frac{\partial g}{\partial x_1}(0, x_2, \ldots, x_n) = 0 \) but \( \frac{\partial^2 g}{\partial^2 x_1}(O) = \phi(O) \neq 0 \), which means that \( g \) can be written as \( g = x_1^2 h \) where \( h(0) \neq 0 \). Put \( y_1 = x_1 \sqrt{2h} \) (so that \( g = \frac{y_1^2}{2} \)), \( y_2 = x_2, \ldots, y_n = x_n \), we get a coordinate system \((y_1, \ldots, y_n)\) with \( \Omega = d(y_1^2/2) \wedge dy_2 \wedge \ldots \wedge dy_n = y_1 dy_1 \wedge \ldots \wedge dy_n \). (In the real case with \( h(O) < 0 \), put \( y_1 = x_1 \sqrt{-2h}, \ y_2 = -x_2, \ y_3 = x_3, \ldots, y_n = x_n \).)

Let \( X \) be a vector field which preserves a folded volume form \( \Omega \). If \( X(O) \neq 0 \) then it is easy to see that \( X \) can be rectified together with \( \Omega \), i.e., there is a coordinate system \((y_1, \ldots, y_n)\) in which
\[ \Omega = y_1 dy_1 \wedge \ldots \wedge dy_n \quad \text{and} \quad X = \frac{\partial}{\partial y_n}. \]
Indeed, consider the local \((n-1)\)-dimensional quotient space of the local flow generated by \( X \). The local \((n-1)\)-form \( X \wedge \Omega \) is the pull-back of a local folded volume form on this local quotient space so we can write \( X \wedge \Omega = y_1 dy_1 \wedge \ldots \wedge dy_{n-1} \) by the previous lemma. Now take any function \( y_n \) such that \( X(y_n) = 1 \) and \( y_n(O) = 0 \) and we are done.

Consider now the case when \( X(O) = 0 \). We have the following theorem, which is a generalization of the results about normalization of isochore vector fields (see [34, Section 4]) from the case of a regular volume form to the case of a folded volume form:
Theorem 3.2. Let $X$ be a formal vector field which vanishes at a point $O$ and which preserves a formal folded volume form $\Omega$. Then $X$ can be formally normalized together with $\Omega$: $\Omega$ has canonical form, $X$ is in normal form in the coordinate system $(x_1, \ldots, x_n)$ and the semisimple part of $X$ is diagonal in this system (over $\mathbb{C}$):

$$\Omega = x_1 dx_1 \wedge \ldots \wedge dx_n, \quad X^S = \sum \gamma_i x_i \frac{\partial}{\partial x_i}.$$ 

The eigenvalues of $X$ satisfy the following resonance relation:

$$(3.1) \quad 2\gamma_1 + \gamma_2 + \ldots + \gamma_n = 0.$$ 

Moreover, if $X$ and $\Omega$ are analytic and $X$ is analytically integrable or Darboux-integrable, then there exists such a normalization which is locally analytic.

Proof. By Lemma 3.1 we may write $\Omega = y_1 dy_1 \wedge \ldots \wedge dy_n$ in some coordinate system. Denote by $\rho$ the associated torus $\mathbb{T}^\tau$-action of $X$ at $O$, where $\tau$ is the toric degree of $X$ at $O$. Theorem 2.3 ensures that $\rho$ preserves $\Omega$. This implies that the singular set $S = \{y_1 = 0\}$ of $\Omega$ is a hyperplane which is preserved by $\rho$. Since $\Omega$ is homogeneous in the coordinate system $(y_1, \ldots, y_n)$, the linear part of the action $\rho$ in this coordinate system, which is a linear torus action that we will denote by $\rho_1$, also preserves $\Omega$. Notice that, since $S = \{y_1 = 0\}$ is a hyperplane preserved by $\rho$, it is also preserved by $\rho_1$.

By Bochner’s averaging formula [4], we find a formal diffeomorphism $\Phi$, whose linear part is the identity, and which intertwines $\rho$ with $\rho_1$, i.e.,

$$\Phi \circ \rho_1(s,. \rho_1(s,. \Phi, \forall s \in \mathbb{T}^\tau.$$ 

Let

$$\Omega_1 = \Phi^* \Omega.$$ 

Then $\rho_1$ also preserves $\Omega_1$: $\rho_1(s,.^* \Omega_1 = \rho_1(s,.^* \Phi^* \Omega = (\Phi \circ \rho_1(s,.))^* \Omega = (\rho(s,. \circ \Phi))^* \Omega = \Phi^* \rho_1(s,. \Omega_1 = \Phi^* \Omega = \Omega_1$ for any $s \in \mathbb{T}^\tau$. Thus $\rho_1$ preserves both $\Omega$ and $\Omega_1$.

Since $S = \{y_1 = 0\}$ is preserved by $\rho$, it is also preserved by $\Phi$, which implies that it is also the singular set of $\Omega_1$, i.e., $\Omega_1 = \Phi^* \Omega$ is also divisible by $y_1$. Moreover, the linear part of $\Phi$ is the identity. So we can write

$$\Omega_1 - \Omega = y_1 \Theta,$$

where $\Theta$ is a differential $n$-form which vanishes at $O$.

The next step is to use the equivariant Moser path method to move $\Omega_1$ to $\Omega$ by a formal diffeomorphism without changing the torus action. According to Lemma 2.5, it is enough to show that the equation

$$d(Y_t \cdot (\Omega + ty_1 \Theta)) = y_1 \Theta$$

has a solution $Y_t$ (for $t \in [0,1]$).

We can write $y_1 \Theta = d(y_1 \Pi)$ for some $(n-1)$-form $\Pi$ with $\Pi(O) = 0$, and solve the following equation instead:

$$Y_t (\Omega + ty_1 \Theta) = y_1 \Pi,$$

or equivalently,

$$Y_t (dy_1 \wedge \ldots \wedge dy_n + t \Theta)) = \Pi,$$
The above equation clearly admits a unique solution $Y_t$ because $dy_1 \wedge \ldots \wedge dy_n$ is regular and $\Theta(O) = 0$, and moreover $Y_t(O) = 0$ because $\Pi(O) = 0$.

Thus we have shown the existence of a simultaneous normalization of $X$ and $\Omega$, i.e., a coordinate system $(y_1, \ldots, y_n)$ in which $X^S = X^s$ (the semisimple part of $X$ coincides with the semisimple part of it linear part) and $\Omega = y_1 dy_1 \wedge dy_2 \wedge \ldots \wedge dy_n$. A priori, $X^s$ is not diagonal in the coordinates $(y_1, \ldots, y_n)$, so we have a bit more work to do to diagonalize it without destroying the form of $\Omega$.

Recall that the $(n-1)$-dimensional singular set $S = \{y_1 = 0\}$ of $\Omega$ is invariant under the action $\rho$, so $\rho$ also preserves a line $L$ transversal to $S$. This line can be parametrized by $y_1$:

$$L = \{(y_1, a_2 y_1, \ldots, a_n y_1)\}$$

where $a_2, \ldots, a_n \in \mathbb{C}$ are constants. By the linear transformation

$$(z_1, z_2, \ldots, z_n) = (y_1, y_2 - a_2 y_1, \ldots, y_n - a_n y_1)$$

we get a new coordinate system in which $\rho$ still linear, $\Omega$ still has the same form, and both $S = \{z_1 = 0\}$ and $L = \{z_2 = \ldots = z_n = 0\}$ are invariant with respect to $\rho$. It means that $\rho = \rho_S \oplus \rho_L$ where $\rho_S$ is a linear $T^\tau$-action on $S$ and $\rho_L$ is a linear $T^\tau$-action on $L$. Since $\rho$ is determined by $X^S$, we have a corresponding decomposition $X^S = X^S \oplus X^L_S$ for $X^S$. Since $\Omega$ also decomposes as $\Omega = \Omega_S + \Omega_L$ with $\Omega_S = z_1 dz_1$ and $\Omega_L = dz_2 \wedge \ldots \wedge dz_n$, the fact that $X^S$ preserves $\Omega$ implies that $X^S_S$ preserves $\Omega_S$ and $X^S_L$ preserves $\Omega_L$. From there it is easy to see that we can diagonalize $X^S_S$ on $S$ without destroying the form of $S$, i.e., $\Omega_S = dx_2 \wedge \ldots \wedge dx_n$ and $X^S_S = \sum_{i=2}^n \gamma_i x_i \frac{\partial}{\partial x_i}$.

Since $L$ is only 1-dimensional, $X^S_L$ is already diagonal, so we can put $x_1 = z_1$. Then in the coordinate system $(x_1, \ldots, x_n)$ we have that $X$ is diagonal and $\Omega$ still has the required canonical form.

The above normalization is a priori only formal. But in the case when $\Omega$ is analytic and $X$ is analytically integrable or Darboux-integrable then the associated torus action $\rho$ is analytic and all the steps above can be done analytically, so we have a local analytic normalization. The resonance equation (3.1) is the same as the equation $L_{X^S} \Omega = 0$ in diagonalized normalized coordinates. \(\square\)

### 3.2. Systems preserving a singular Nambu structure of top order

We start by recalling the following result, which is a simple particular case of the problem of linearization of Nambu structures (see, e.g. [9]):

Consider a formal (resp. local analytic, resp. smooth) $n$-vector field

$$\Lambda = g \frac{\partial}{\partial x_1} \wedge \ldots \wedge \frac{\partial}{\partial x_n}$$

in some coordinate system $(x_1, \ldots, x_n)$ around a point $O$ on an $n$-dimensional manifold. Such a multi-vector field of top order is also called a Nambu structure of top order. We will say that $O$ is a non-degenerate singular point of $\Lambda$ if $g(O) = 0$ and $dg(0) \neq 0$. Clearly, this definition does not depend on the choice of coordinate systems at $O$. 

Lemma 3.3. Let $\Lambda$ be a formal (resp. analytic, resp. smooth) Nambu structure of top order in dimension $n \geq 2$ with a non-degenerate singular point $O$. Then there exists a formal (resp. analytic, resp. smooth) coordinate system $(x_1, \ldots, x_n)$ around $O$ in which we have

\begin{equation}
\Lambda = x_1 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} \wedge \ldots \wedge \frac{\partial}{\partial x_n}.
\end{equation}

Consider now a vector field $X$ which preserves $\Lambda = x_1 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} \wedge \ldots \wedge \frac{\partial}{\partial x_n}$ near $O$. First let us mention the following simple result in the case when $X(O) \neq 0$:

Proposition 3.4. If $X(O) \neq 0$ then there is a local coordinate system in which $X$ can be rectified together with $\Lambda$:

\begin{equation}
\Lambda = x_1 \frac{\partial}{\partial x_1} \wedge \ldots \wedge \frac{\partial}{\partial x_n} \quad \text{and} \quad X = \frac{\partial}{\partial x_n}.
\end{equation}

Proof. Since $X(O) \neq 0$, we can rectify $X$ as $X = \frac{\partial}{\partial x_n}$ in a coordinate system $(z_1, \ldots, z_{n-1}, x_n)$. The invariance of $\Lambda$ with respect to $X$ means that it has the form $\Lambda = f(z_1, \ldots, z_{n-1}) \frac{\partial}{\partial z_1} \wedge \ldots \wedge \frac{\partial}{\partial z_{n-1}} \wedge X = \Pi \wedge X$, where $\Pi = f(z_1, \ldots, z_{n-1}) \frac{\partial}{\partial z_1} \wedge \ldots \wedge \frac{\partial}{\partial z_{n-1}}$. By Lemma 3.3 we can write $\Pi = x_1 \frac{\partial}{\partial x_1} \wedge \ldots \wedge \frac{\partial}{\partial x_{n-1}}$ by a change of coordinates from $(z_1, \ldots, z_{n-1})$ to $(x_1, \ldots, x_{n-1})$ which does not involve $x_n$. Then (3.3) is satisfied in the new coordinate system $(x_1, \ldots, x_{n-1}, x)$. \qed

Consider now the case when $X(O) = 0$.

Theorem 3.5. Let $X$ be a formal vector field which vanishes at the origin $O$ and which preserves a Nambu structure of top order $\Lambda$ with a non-degenerate singularity at $O$. Then $X$ can be formally normalized together with $\Lambda$: there is a formal coordinate system $(x_1, \ldots, x_n)$ in which $\Lambda$ has canonical form, $X$ is in normal form, and the semisimple part of $X$ is diagonal in this coordinate system (over $\mathbb{C}$):

\begin{equation}
\Lambda = x_1 \frac{\partial}{\partial x_1} \wedge \ldots \wedge \frac{\partial}{\partial x_n}, \quad X^S = \sum_{i} \gamma_i x_i \frac{\partial}{\partial x_i}.
\end{equation}

The eigenvalues of $X$ satisfy the following resonance relation:

\begin{equation}
\gamma_2 + \ldots + \gamma_n = 0.
\end{equation}

Moreover, if $X$ and $\Lambda$ are analytic and $X$ is analytically integrable or Darboux-integrable, then there exists such a normalization which is locally analytic.

Proof. The proof is quite similar to the case with a folded volume form. We start with a coordinate system in which $\Lambda$ is in canonical form (3.2) but the associated torus action $\rho$ of $X$ is a priori nonlinear. Since $\Lambda$ is homogeneous and $\rho$ preserves $\Lambda$, the linear part $\rho_1$ of $\rho$, which is a linear torus action, also preserves $\Lambda$. By Bochner’s averaging formula, we find a formal diffeomorphism $\Psi$ whose linear part is identity and which intertwines $\rho$ with $\rho_1$. Then $\rho_1$ preserves both $\Lambda$ and $\Lambda_1 = \Psi_1 \Lambda$. Similarly to
the case of a folded volume form, Λ and Λ₁ have the same singular set S, which is a linear subspace of codimension 1 in some coordinate system in which Λ is linear and the linear part of Λ₁ coincides with Λ, i.e., we can write

\[ Λ = x_1 \frac{∂}{∂x_1} ∧ \ldots ∧ \frac{∂}{∂x_n}, \]

\[ Λ₁ = x_1 \frac{∂}{∂x_1} ∧ \ldots ∧ \frac{∂}{∂x_n} + x_1 Π, \]

where Π is an n-vector field which vanishes at O. To apply Lemma 2.5 we must find a time-dependent vector field \( Y_t \) such that

\[ \mathcal{L}_{Y_t}(x_1 \frac{∂}{∂x_1} ∧ \ldots ∧ \frac{∂}{∂x_n} + tx_1 Π) = x_1 Π \]

In order to simplify the above equation, we can impose \( Y_t(x_1) = 0 \) (i.e., \( x_1 \) is a first integral for \( Y_t \)), and put \( Ω_t = (\frac{∂}{∂x_1} ∧ \ldots ∧ \frac{∂}{∂x_n} + tΠ)^{-1} \), which is a regular volume form since \( P(O) = 0 \). Put \( g_t = ⟨Π, Ω_t⟩ \). Then the above equation, after pairing of both sides by \(-\frac{Ω_t(Ω_t, ·)}{x_1}\), becomes

\[ \mathcal{L}_{Y_t}Ω_t = -g_tΩ_t, \]

which can be easily solved. The rest of the proof is absolutely similar to that of Theorem 3.2. □

4. Systems with a singular symplectic structure

Similarly to the previous section, in this section we will consider two kinds of singular symplectic structures, namely the so called folded symplectic structures (which have vanishing components) and log-symplectic structures (which have poles).

4.1. Folded symplectic structures. Consider a (formal, or local analytic, or smooth) closed 2-form \( ω \) on \((\mathbb{K}^{2n}, O)\), such that \( ω^n \) is non-trivial on \( \mathbb{K}^{2n} \) but \( ω^n(O) = 0 \). Assume that the corank of \( ω \) at \( O \) is 2, and that \( ω^n = f dx_1 ∧ \ldots ∧ dx_{2n} \) in some local coordinate system, where \( f(O) = 0 \) but \( df(O) \neq 0 \), so that \( S = \{ f = 0 \} \) is the hypersurface of singular points of \( ω \). Assume moreover that the pull-back of \( ω \) to \( S \) has constant rank equal to \( 2(n-1) \). Such a closed 2-form \( ω \) is called a folded symplectic structure (because it can be obtained as the pull-back of a usual symplectic form by a fold map [20]), and it admits the following (formal, or local analytic, or smooth) normal form, according to a classical result of Martinet [19]:

\[
(4.1) \quad ω = x_1 dx_1 ∧ dx_2 + \sum_{j=1}^{n-1} dy_i ∧ dy_{j+n-1}.
\]

More generally, one may consider a multi-folded symplectic structure \( ω \), which for some positive number \( m \leq n \) admits the following canonical expression à la Darboux...
in some coordinate system \((x_1, \ldots, x_{2m}, y_1, \ldots, y_{2(n-m)})\):

\[
\omega = \sum_{i=1}^{m} x_i dx_i \wedge x_{m+i} + \sum_{j=1}^{n-m} dy_j \wedge dy_{j+n-m}.
\]

The above multi-folded symplectic structure is quasi-homogeneous: we have

\[
\mathcal{L}_E \omega = 6 \omega,
\]

where \(E\) is the following quasi-Euler vector field:

\[
E = 2 \sum_{i=1}^{2m} x_i \frac{\partial}{\partial x_i} + 3 \sum_{j=1}^{2(n-m)} y_j \frac{\partial}{\partial y_j}.
\]

In other words, if we consider that each \(x_i\) is of quasi-homogeneous order 2, and each \(y_j\) is of quasi-homogeneous order 3, then \(\omega\) becomes quasi-homogeneous of order 6.

Notice that the kernel of \(\omega\) at \(O\) is the \(2m\)-dimensional vector space

\[
K = \text{Span} \left\{ \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_{2m}} \right\} \subset \mathbb{K}^{2n}.
\]

The quasi-Euler vector field \(E\) is tangent to \(K\) and is equal to 2 times the usual Euler vector field on \(K\). \(E\) also projects to 3 times the usual Euler vector field on the quotient space \(\mathbb{K}^{2n}/K\).

Let us assume now that there is a local (smooth, analytic or formal) action \(\rho\) of a compact Lie group \(G\) on \((\mathbb{K}^{2n}, O)\), which fixes the origin \(O\) and preserves \(\omega\). A priori, \(\rho\) does not preserve the quasi-Euler vector field \(E\), but it must preserve the kernel space \(K\) because it preserves \(\omega\). Denote by

\[
E_\rho = \int_{g \in G} \rho(g)_* E \, d\mu_G
\]

the averaging of \(E\) with respect to \(\rho\) (where \(d\mu_G\) denotes the Haar measure on \(G\)).

Now \(E_\rho\) is preserved by \(\rho\), and we still have \(\mathcal{L}_{E_\rho} \omega = 6 \omega\). Moreover, \(E_\rho\) is still tangent to \(K\) and has its linear part equal to 2 times the usual Euler vector field on \(K\); the projection of the linear part of \(E_\rho\) on the quotient space \(\mathbb{K}^{2n}/K\) is still equal to 3 times the usual Euler vector field on this quotient space. It implies in particular that the linear part \(E_\rho^{(1)}\) of \(E_\rho\) is a priori non-diagonal but diagonalizable, with only two distinct eigenvalues: 2 and 3: to diagonalize \(E_\rho^{(1)}\) we must apply a linear transformation to \(\mathbb{K}^{2n}\), which is a priori not identity, but which is identity on \(K\) and also projects to the identity map on \(\mathbb{K}^{2n}/K\). In other words, \(E_\rho^{(1)}\) will be diagonal in a coordinate system \((x'_i, y'_j)\) of the type

\[
y'_j = y_j \quad \forall j \leq 2(n-m) \quad \text{and} \quad x'_i = x_i + \sum a_{ij} y_j \quad \forall i \leq m
\]

for some constants \(a_{ij}\).

Notice that such a vector field is in Poincaré domain and is non-resonant in the sense of Poincaré-Dulac, i.e., there is no resonance relation of the type \(\gamma_i = \gamma_{j_1} + \cdots + \gamma_{j_k}\) (with \(k \geq 2\)) among eigenvalues. It follows from the classical theory of normalization of vector fields that \(E_\rho\) is locally diagonalizable (smoothly if \(E_\rho\) and \(\rho\) are smooth,
analytically if $E_\rho$ and $\rho$ are analytic, and formally if $E_\rho$ and $\rho$ are formal), also in an equivariant way. (For the smooth case, see [2]; the formal and analytic cases are simpler and can be done using the toric approach.) In other words, there is another coordinate system $(u_1, \ldots, u_{2m}, v_1, \ldots, v_{2(n-m)})$ on $(\mathbb{K}^{2n}, O)$, such that $u_i = x_i' + h.o.t.$ and $v_i = y_i' + h.o.t.$ ($h.o.t.$ means higher order terms, i.e., terms of degree at least 2 here), in which the action $\rho$ is linear, and the vector field $E_\rho$ has the following diagonal form:

$$E_\rho = 2 \sum_{i=1}^{2m} u_i \frac{\partial}{\partial u_i} + 3 \sum_{j=1}^{2(n-m)} v_j \frac{\partial}{\partial v_j}. $$

Since the linear action $\rho$ (in the coordinates $(u_i, v_j)$) preserves $E_\rho$, it must also preserve the eigenspaces $U$ and $V$ of $E_\rho$, where $U$ is the space where all the coordinates $v_i$ vanish and $V$ is the space where all the coordinates $u_i$ vanish, and we have the splitting $\mathbb{K}^{2n} = U \oplus V$.

Remember that $\mathcal{L}_{E_\rho}(\omega) = 6\omega$. Now, $E_\rho$ is diagonal, and the monomial 2-forms are eigenvectors of the Lie derivative operator $\mathcal{L}_{E_\rho}$. The only monomial 2-forms which have eigenvalue equal to 6 are $dv_j \wedge dv_j$ and $u_i du_i \wedge du_j$, so $\omega$ must be a linear combination with constant coefficients of these monomial forms. (By standard division techniques, one can show easily that this is also true in the smooth case). It follows that $\omega$ admits an equivariant splitting:

$$\omega = \omega_U + \omega_V,$$

where $\omega_V$ lives on $V$ and is constant, $\omega_U$ lives on $U$ and is linear, and $\rho$ preserves both $\omega_U$ and $\omega_V$ (because it acts separately on $U$ and $V$ and cannot mix up these terms).

Recall that, by construction we have

$$x_i = u_i - \sum_{j=1}^{n-m} a_{ij} v_j + h.o.t.$$

and

$$y_j = v_j + h.o.t.$$

for every $i \leq 2m$ and every $j \leq 2(n-m)$. Putting these into the original formula (4.2) of $\omega$, we get:

$$\omega = \sum_{i=1}^{m} (u_i - \sum_{j=1}^{n-m} a_{ij} v_j + h.o.t.) d(u_i - \sum_{j=1}^{n-m} a_{ij} v_j + h.o.t.) \wedge (u_{i+m} - \sum_{j=1}^{n-m} a_{i+m,j} v_j + h.o.t.)$$

$$+ \sum_{j=1}^{n-m} d(v_j + h.o.t.) \wedge d(v_{j+n-m} + h.o.t.)$$

In the above expression, the sum of linear U-terms (i.e. linear terms which are linear which do not contain any $v_j$ or $dv_j$ in their expression) is $\sum_{i=1}^{m} u_i du_i \wedge u_{m+i}$. Since $\omega = \omega_U + \omega_V$, with $\omega_U$ being a sum of linear U-terms, we automatically have $\omega_U = \sum_{i=1}^{m} u_i du_i \wedge u_{m+i}$. By a similar argument, we also automatically have $\omega_V = \sum_{j=1}^{n-m} dv_j \wedge dv_{j+n-m}$. Thus we have proved the following equivariant normalization theorem for multi-folded symplectic structures:
Theorem 4.1. Consider a multi-folded symplectic structure $\omega$ written as

$$\omega = \sum_{i=1}^{m} x_{i} dx_{i} \wedge x_{m+i} + \sum_{j=1}^{n-m} dy_{j} \wedge dy_{j+n-m}$$

$(1 \leq m \leq n)$ in some local (analytic, resp. formal, resp. smooth) coordinate system $(x_{1}, \ldots, x_{2m}, y_{1}, \ldots, y_{2(n-m)})$ on $(\mathbb{K}^{2n}, O)$. Assume that there is a local (analytic, resp. formal, resp. smooth) action $\rho$ of a compact Lie group $G$ on $(\mathbb{K}^{2n}, O)$ which fixes the origin $O$ and preserves $\omega$. Then there exists a local (analytic, resp. formal, resp. smooth) coordinate system $(u_{1}, \ldots, u_{2m}, v_{1}, \ldots, v_{2(n-m)})$ on $(\mathbb{K}^{2n}, O)$, in which the action $\rho$ is linear and the multi-folded symplectic form $\omega$ is still in canonical form:

$$\omega = \sum_{i=1}^{m} u_{i} du_{i} \wedge u_{m+i} + \sum_{j=1}^{n-m} dv_{j} \wedge dv_{j+n-m}.$$  

As an immediate consequence of the above theorem, we get the following result about normalization of a vector field with an underlying (multi-)folded symplectic structure:

Theorem 4.2. Let $X$ be a formal vector field on $(\mathbb{K}^{2n}, O)$ which vanishes at $O$ and preserves a (multi-)folded symplectic structure $\omega$ with a canonical expression $\omega = \sum_{i=1}^{m} x_{i} dx_{i} \wedge x_{m+i} + \sum_{j=1}^{n-m} dy_{j} \wedge dy_{j+n-m}$ in some coordinate system. Then the couple $(X, \omega)$ admits a formal simultaneous normalization, i.e., there is a formal coordinate transformation which normalizes $X$ while keeping the canonical expression of $\omega$. Moreover, if everything is locally analytic (real or complex), and $X$ is analytically integrable or Darboux-integrable, then this normalization can be made locally analytic.

Proof. Just apply Theorem 4.1 to the associated torus action of $X$ at $O$. \hfill \Box

4.2. Log-symplectic structures. Recall that, a meromorphic differential form $\omega$ on a complex manifold $M$ with poles along a divisor $D \subset M$ (without multiplicities) is called logarithmic if both forms $\omega$ and $d\omega$ have poles only along the divisor $D$ and of order not greater than 1. Such logarithmic differential forms have been studied by many people in algebraic geometry, see, e.g., Deligne [7], Saito [27], Aleksandrov [1].

Roughly speaking, the logarithmic condition means that if $\omega$ contains a local pole $\frac{1}{h}$ where $h$ is a local holomorphic function whose zero locus is a component of $D$ (without multiplicities) then this pole must come together with $dh$, i.e., one can write a local expression of $\omega$ which contains only holomorphic terms and meromorphic terms of the type $\frac{1}{h} (= d(\log h))$; otherwise $d\omega$ would contain a pole of order 2 at the zero locus of $h$.

Consider a local logarithmic differential 2-form $\omega$. Then it admits the following expression:

$$\omega = \sum_{i,j} g_{ij} \frac{dh_{i}}{h_{i}} \wedge \frac{dh_{j}}{h_{j}} + \sum_{k} \frac{dh_{k}}{h_{k}} \wedge \beta_{k} + \gamma$$
where $g_{ij}, \beta_k, \gamma$ are local holomorphic functions, 1-forms, and 2-form, respectively. The general case with non-trivial terms of the type $g_{ij} \frac{dh_i}{h_i} \wedge \frac{dh_j}{h_j}$ is very interesting but also complicated, so here we will be less ambitious, and will only look at local logarithmic 2-forms without such terms. We introduce the following definition:

**Definition 4.3.** A local differential 2-form of the type

\begin{equation}
\omega = \sum_{i=1}^{k} \frac{dh_i}{h_i} \wedge \beta_i + \gamma
\end{equation}

on $(\mathbb{K}^{2n}, O)$ is called a **simple log-symplectic form** if it satisfies the following conditions:

i) The local functions $h_1, \ldots, h_k$ vanish at the origin $O$, but $dh_1 \wedge \ldots \wedge dh_k(O) \neq 0$.

ii) $d\omega = 0$.

iii) $h\omega^n$ is a local regular volume form, i.e., $h\omega^n(O) \neq 0$, where $h = \prod_{i=1}^{k} h_i$ and $\omega^n$ means the wedge product of $n$ copies of $\omega$.

The above definition works in many different categories: real analytic, holomorphic, formal, and smooth. Log-symplectic manifolds, especially in the case with $k = 1$ (i.e., the set of singular points is a smooth hypersurface), have been studied by many authors from different points of view, where they may be called by other names as well, such as b-symplectic or b-Poisson (the case with $k = 1$), or c-symplectic, see, e.g., [13, 11, 12].

It turns out that non-degenerate log-symplectic structures admit **Darboux-like local normal forms**:

**Theorem 4.4.** Let $\omega$ be a local (analytic, formal or smooth) simple log-symplectic structure in $(\mathbb{K}^{2n}, O)$ whose divisor has $k$ components. Then there exists a local (analytic, formal or smooth) coordinate system $(x_1, y_1, \ldots, x_k, y_k, z_1, \ldots, z_{2(n-k)})$ in which $\omega$ has the following canonical expression:

\begin{equation}
\omega = \sum_{i=1}^{k} \frac{dx_i}{x_i} \wedge dy_i + \sum_{j=1}^{n-k} dz_j \wedge dz_{j+n-k}.
\end{equation}

In particular, the dual Poisson structure $\Pi$ of $\omega$ is without poles and admits the following canonical form:

\begin{equation}
\Pi = \sum_{i=1}^{k} x_i \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i} + \sum_{j=1}^{n-k} \frac{\partial}{\partial z_j} \wedge \frac{\partial}{\partial z_{j+n-k}}.
\end{equation}

**Proof.** Our proof will be based on the path method and consists of four small steps.

**Step 1.** We begin with a log-symplectic form $\omega$ written as

\[ \omega = \sum_{i=1}^{k} \frac{dh_i}{h_i} \wedge \beta_i + \gamma, \]
where $\beta_i$ are regular 1-forms and $\gamma$ is a regular 2-form. The closedness condition on $\omega$ gives

$$0 = d\omega = \sum_{i=1}^{k} \frac{dh_i}{h_i} \wedge d\beta_i + d\gamma,$$

which implies that, for each $i \leq k$, $dh_i \wedge d\beta_i$ is divisible by $h_i$, which implies easily by Poincaré’s lemma that $\beta_i$ can be written as $\beta_i = h_i\beta_i' + gdh_i + d\xi_i$. Indeed, we can write $\beta_i = h_i\beta_i' + gdh_i + \xi_i$ in a coordinate system $(h_i, v_1, \ldots, v_{2n-1})$, in which $\xi_i$ is the part of $\beta_i$ that is independent of $h_i$ and does not contain $dh_i$. The condition that $dh_i \wedge d\beta_i$ is divisible by $h_i$ means that $dh_i \wedge d\xi_i$ is divisible by $h_i$ then implies that $d\xi_i = 0$ (otherwise $dh_i \wedge d\xi_i$ would be non-trivial and not divisible by $h_i$). So by Poincaré’s lemma, we can write $\xi_i = dy_i$. We can forget the term $gdh_i$ in the expression of $\beta_i$ because its wedge product with $\frac{dh_i}{h_i}$ is zero, and assume that $\beta_i = h_i\beta_i' + dy_i$. Notice that $\frac{dh_i}{h_i} \wedge h\beta_i$ is regular and can be added to $\gamma$, so we have

$$\omega = \sum_{i=1}^{k} \frac{dh_i}{h_i} \wedge dy_i + (\gamma + \sum_i dh_i \wedge \beta_i') = \sum_{i=1}^{k} \frac{dh_i}{h_i} \wedge dy_i + \mu$$

where $y_i$ are local functions (analytic, formal or smooth) and $\mu$ is a 2-form.

The value of $h\omega^n$ at $O$, where $h = \prod_{i=1}^{k} h_i$ and $\omega^n$ means the wedge product of $n$ copies of $\omega$, is equal to

$$\frac{n!}{(n-k)!} \wedge_{i=1}^{k} (dh_i \wedge dy_i) \wedge \mu^{n-k}(O)$$

(if $k \leq n$, and is equal to 0 if $k > n$). It implies that $k \leq n$, and $\wedge_{i=1}^{k} (dh_i \wedge dy_i) \wedge \mu^{n-k}(O)$ is a regular volume form. In particular, $(h_1, y_1, \ldots, h_k, y_k)$ can be completed into a regular local coordinate system $(h_1, y_1, \ldots, h_2, y_k, z_1, \ldots, z_{2n-2k})$.

**Step 2.** Write $\mu$ as

$$\mu = \sum_i dh_i \wedge d\phi_i - \sum_i dy_i \wedge d\psi_i + \mu_0 + \mu_1$$

where $\phi_i$ and $\psi_i$ are linear functions in the coordinates $(h_1, y_1, \ldots, h_k, y_k, z_1, \ldots, z_{2n-2k})$, $\mu_0$ is a constant 2-form in these coordinates which contains only terms $dz_i \wedge dz_j$, and $\mu_1$ is a 2-form which vanishes at the origin. Then $\omega$ can be written as

$$\omega = \sum_{i=1}^{k} \frac{dh_i}{h_i} \wedge dy_i + \mu = \sum_{i=1}^{k} \left( \frac{dh_i}{h_i} + d\psi_i \right) \wedge d(y_i + h_i\phi_i) + \mu_0 + \mu_1'$$

where $\mu_2$ is a 2-form which vanishes at $O$. In other words, we have

$$\omega = \sum_{i=1}^{k} \frac{dh_i}{h_i} \wedge dy_i + \mu = \sum_{i=1}^{k} \frac{d\tilde{h}_i}{h_i} \wedge d\tilde{y}_i + \mu_0 + \mu_1'$$

where $\tilde{h}_i = h_i \exp(\psi_i)$ and $\tilde{y}_i = y_i + h_i\phi_i$. Notice that $\tilde{h}_i = h_i + h.o.t.$ (higher order terms) and $\tilde{y}_i = y_i + h.o.t.$, so $(\tilde{h}_1, \tilde{y}_1, \ldots, \tilde{h}_k, \tilde{y}_k, z_1, \ldots, z_{2n-2k})$ is still a local coordinate system.
The condition $h\omega^n(O)$ now means that $\mu_0^{n-k}(O) \neq 0$, i.e., $\mu_0^{n-k}(O)$ is a symplectic form of dimension $2(n-k)$ in the coordinates $(z_1, \ldots, z_{2n-2k})$. By Darboux theorem, by a linear change of the coordinates $(z_1, \ldots, z_{2(n-k)})$ we may assume that

$$\mu_0 = \sum_{j=1}^{n-k} dz_j \wedge dz_{j+n-k}.$$ 

**Step 3.** Put

$$\omega_0 = \sum_{i=1}^{k} \frac{dh_i}{h_i} \wedge dy_i + \sum_{j=1}^{n-k} dz_j \wedge dz_{j+n-k}$$

and define the following linear path of 2-forms for $t \in [0,1]$

$$\omega_t = t\omega_1 + (1-t)\omega_0 = \sum_{i=1}^{k} \frac{dh_{i,t}}{h_{i,t}} \wedge dy_{i,t} + \sum_{j=1}^{n-k} dz_j \wedge dz_{j+n-k} + \mu_{1,t},$$

where $h_{i,t} = h_i \exp(t\psi_i)$, $y_{i,t} = y_i + th_i\phi_i$, and $\mu_{1,t}$ is a 2-form depending polynomially on $t$ which vanishes at the origin.

Due to the closedness of $\omega$, we can also write

$$\frac{d}{dt}\omega_t = \omega - \omega_0 = \sum_i \frac{dh_i}{h_i} \wedge d\phi_i - \sum_i dy_i \wedge d\psi_i + \mu_1 = d\xi$$

where $\xi$ is an 1-form which vanishes at the origin.

To show that $\omega$ is locally (or formally) isomorphic to the canonical form $\omega_0$ via a local (or formal) diffeomorphisms, we now can use the path method with respect to the path $(\omega_t)$, and construct the time-1 flow of the time-dependent vector field $X_t$ given by the (time-dependent) equation:

$$X_t\omega_t = \xi.$$ 

**Step 4.** The last step is to verify that the above equation does admit a local (analytic, formal or smooth) solution $X_t$ (which depends smoothly on $t$) such that $X_t(O) = 0$. Indeed, the 2-form

$$\sum_{i=1}^{k} \frac{dh_{i,t}}{h_{i,t}} \wedge dy_{i,t} + dz_j \wedge dz_{j+n-k}$$

sends the vector fields $h_{i,t} \frac{\partial}{\partial h_{i,t}}$, $h_{i,t} \frac{\partial}{\partial y_{i,t}}$, $\frac{\partial}{\partial z_j}$, $\frac{\partial}{\partial z_{j+n-k}}$ (written with respect to the coordinate system $(h_{i,t}, y_{i,t}, z_j)$) via contraction to the 1-forms (up to a sign) $dy_{i,t}$, $dh_{i,t}$, $dz_j$, $dz_{j+n-k}$, $dz_j$. The contraction map is hence an isomorphism from the sub-module of vector fields generated by the vector fields $\left\{ h_{i,t} \frac{\partial}{\partial h_{i,t}}, y_{i,t} \frac{\partial}{\partial y_{i,t}}, \frac{\partial}{\partial z_j}, \frac{\partial}{\partial z_{j+n-k}} \right\}$ to the module of 1-forms. Since $\mu_{1,t}$ is a small perturbation term (because it vanishes at $O$), the contraction map by $\omega_t$ is also an isomorphism between these two modules, so the equation $X_t\omega_t = \xi$ can be solved (with an unique solution) and, moreover, $X_t(O) = 0$ because $\xi(O) = 0$. The proof of the theorem is finished. (The $h_i$ are renamed $x_i$ at the end.)
Remark 4.5. The case with $k = 1$ of Theorem 4.4 was obtained before by Gulliemin - Miranda - Pires [13], though the general case (with arbitrary $k$) does not seem to have been written down explicitly anywhere before, as far as we know. One can also prove Theorem 4.4 differently, e.g., by the following steps: i) Show that the associated Poisson structure $\Pi$ of $\omega$ is smooth; ii) Show that Lie algebra which corresponds to the linear transversal part of $\Pi$ at the origin is isomorphic to the direct sum of $k$ copies of $\text{aff}(1)$, where $\text{aff}(1)$ denotes the 2-dimensional Lie algebra of affine transformations of the line; iii) Use the result of Dufour and Molinier [8] about the linearizability of Poisson structures with such a linear part. Our proof of Theorem 4.4 is written in such a way so that it can be immediately extended to the equivariant case when there is a compact group action which preserves the log-symplectic form.

4.3. Equivariant normalization of log-symplectic structures.

**Theorem 4.6.** Let $\omega$ be a local (analytic, formal or smooth) non-degenerate log-symplectic structure in $(\mathbb{R}^{2n}, O)$ whose divisor contains $k$ components. Let $\rho$ be a local (analytic, formal or smooth) action of a compact Lie group $G$ on $(\mathbb{R}^{2n}, O)$ which fixes the origin $O$ and which preserves $\omega$. Then there exists a local (analytic, formal or smooth) coordinate system $(x_1, y_1, \ldots, x_k, y_k, z_1, \ldots, z_{2(n-k)})$ in which the action $\rho$ is linear and the form $\omega$ has the following Darboux-like canonical expression:

\[
\omega = \sum_{i=1}^{k} \frac{dx_i}{x_i} \wedge dy_i + \sum_{j=1}^{n-k} dz_j \wedge dz_{j+n-k}. \tag{4.9}
\]

**Proof.** By Theorem 4.4 we already know that, if we forget about the compact Lie group action, then there is a local coordinate system in which the dual Poisson structure $\Pi$ of $\omega$ has the expression

\[
\Pi = \sum_{i=1}^{k} x_i \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i} + \sum_{j=1}^{n-k} \frac{\partial}{\partial w_j} \wedge \frac{\partial}{\partial z_j}.
\]

Using this explicit formula, one verifies easily that $\Pi$ satisfies the following division property: any local (analytic, formal or smooth) vector field $X$ which is tangent to the symplectic leaves of $\Pi$ is a linear combination, with coefficients which are local functions, of local Hamiltonian vector fields of $\Pi$. A result of Miranda and Zung [21] then says that $\Pi$ admits an equivariant splitting. In fact, even without this division property Poisson structures would still admit equivariant splitting, according to a more recent result by Frejlich and Mărcuț [10]. It means that one can split $\Pi$ equivariantly with respect to the $G$-action into a direct sum of two parts: the symplectic part and the part which vanishes at zero (like in Weinstein’s splitting theorem [31]); the $G$-action will be a diagonal action, acting on each part separately. So the equivariant normalization of $\Pi$ amounts to an equivariant normalization of two separate parts: the symplectic part (given by the equivariant Darboux theorem), and the part which vanishes at zero. Thus, this equivariant splitting result reduces the proof of Theorem 4.6 to the case when the rank of $\Pi$ at $O$ is zero, i.e., the case when $k = n$. 
Now we can assume, by the case $k = n$ of Theorem 4.4, that $\omega$ is of the form

$$\omega = \sum_{i=1}^{n} \frac{dx_i}{x_i} \wedge dy_i$$

in some local (or formal) coordinate system $(x_i, y_i)$ (in which the action $\rho$ of $G$ is a priori non-linear). We want to linearize the action of $G$ while keeping the above canonical expression of $\omega$. Notice that, due to the homogeneity of $\omega$, the linear part of $\rho$ (with respect to the coordinate system $(x_i, y_i)$), denoted by $\rho_1$, is a linear $G$-action which also preserves $\omega$. By Bochner’s local linearization theorem for compact group actions [4], we know that $\rho$ is isomorphic to $\rho_1$ via a local (or formal) diffeomorphism $\Phi$ whose linear part is identity. It means that the couple $(\omega, \rho)$ is isomorphic to the couple $(\omega', \rho_1)$ via $\Phi$, where $\omega' = \Phi^* \omega$. Due to the fact that $\rho$ must preserve the hyperplanes $S_i = \{x_i = 0\}$ we have that $\Phi^* x_i$ (obtained by Bochner’s averaging formula) is divisible by $x_i$. It implies that $\omega'$ has the form

$$\omega' = \sum_{i=1}^{n} \frac{dx_i}{x_i} \wedge dy_i + \mu$$

where $\mu$ is a local holomorphic (or formal, or smooth) 2-form. Now we can simply repeat the steps of the proof of Theorem 4.4, but in a $\rho_1$-equivariant way, to conclude that $(\omega', \rho_1)$ is isomorphic to $(\omega, \rho_1)$. The composed diffeomorphism which moves $(\omega, \rho)$ to $(\omega, \rho_1)$ is our required normalization map. □

As an immediate consequence of Theorem 4.6, we have:

**Theorem 4.7.** Let $X$ be a formal vector field on $(\mathbb{K}^{2n}, O)$ which vanishes at $O$ and preserves a non-degenerate log-symplectic symplectic structure $\omega$ with a canonical expression $\omega = \sum_{i=1}^{k} \frac{dx_i}{x_i} \wedge dy_i + \sum_{j=1}^{n-k} dz_j \wedge dz_{j+n-k}$ in some coordinate system. Then the couple $(X, \omega)$ admits a formal simultaneous normalization, i.e., there is a formal coordinate transformation which normalizes $X$ while keeping the canonical expression of $\omega$. Moreover, if everything is locally analytic (real or complex), and $X$ is analytically integrable or Darboux-integrable, then this normalization can be made locally analytic.

**Proof.** Just apply Theorem 4.6 to the associated torus action of $X$ at $O$. □

5. **Systems with a singular contact structure**

In this section, we study generic singular contact distributions $\xi$ which are given by the kernel of singular contact forms $\alpha = \sum_{i=0}^{2n} f_i(x) dx_i$ on $(\mathbb{K}^{2n+1}, O)$, where $O$ denotes the origin of $\mathbb{K}^{2n+1}$. The word “generic” here means that $d\alpha$ is a regular presymplectic form of rank $2n$ near $O$, i.e., $(d\alpha)^n(O) \neq 0$, and that, if we write $\alpha \wedge (d\alpha)^n = f dx_0 \wedge dx_1 \wedge \cdots \wedge dx_{2n}$ for some function $f$, then $f(O) = 0$ but $df(O) \neq 0$.

We will consider the following two types of singular contact forms $\alpha = \sum_{i=0}^{2n} f_i(x) dx_i$:

1. If $\alpha(O) \neq 0$, i.e., the kernel distribution $\xi = \ker \alpha$ is still regular at $O$, then $\alpha$ is called a **non-vanishing singular contact form**.
(2) Suppose that $\alpha(O) = 0$ and, in addition, it satisfies the following non-degeneracy condition:

$$F = (f_0, \ldots, f_{2n}) : (\mathbb{K}^{2n+1}, O) \to (\mathbb{K}^{2n+1}, O)$$

is a local diffeomorphism.

Then $\alpha$ is called a \textit{non-degenerate singular contact form}.

**Definition 5.1.** Let $X$ be a vector field, $\phi_X^t$ its local flow, and $\xi$ a singular contact distribution defined by some singular contact form $\alpha$ on a $(2n+1)$-dimensional manifold $M$. We say that $X$ \textit{preserves} $\xi$ if the pull back $(\phi_X^t)^*\alpha = f_t\alpha$, where $f_t$ is a family of smooth functions on $M$. In particular, we say the vector field \textit{preserves the singular contact form} $\alpha$ if $f_t \equiv 1$.

It is easy to see that $X$ preserves $\xi$ if and only if $\alpha$ is a semi-invariant of $X$, i.e.,

$$\mathcal{L}_X \alpha = \left( \frac{d}{dt} \bigg|_{t=0} f_t \right) \alpha.$$  

The vector field $X$ preserves $\alpha$ if and only if $\mathcal{L}_X \alpha = 0$.

If a vector field $X$ preserves a singular contact form $\alpha$, then naturally it preserves the hypersurface $S = \{ x \mid \alpha \wedge (d\alpha)^n(x) = 0 \}$ of singular points of $\alpha$, i.e., $S$ is invariant under the flow of $X$. In what follows we will focus on the relatively simpler case when $X$ preserves the singular contact form.

5.1. \textbf{Non-vanishing singular contact forms.} In this subsection, we assume that $\alpha(O) \neq 0$, $(d\alpha)^n(O) \neq 0$ and $\alpha \wedge (d\alpha)^n$ is a folded volume form. Then the $2n$-dimensional distribution $\xi$ is a contact distribution except on the hypersurface $S$ of singular points. A classical result of Martinet [19] states that $\alpha$ admits a Darboux normal form

$$\alpha = \pm \theta d\theta + (x_1 + 1) dx_{n+1} + \sum_{k=2}^{n} x_k dx_{n+k},$$

(5.1)

whose set of singular points becomes $S = \{ \theta = 0 \}$ in these coordinates. In the real case the different signs in front of the first term are important because they give different nonequivalent normal forms. Since the proofs below are identical with both signs, we shall ignore them from now on.

**Lemma 5.2.** Suppose that $X = f_0 \frac{\partial}{\partial \theta} + \sum_{i=1}^{2n} f_i \frac{\partial}{\partial x_i}$ preserves the singular contact form (5.1), i.e., $\mathcal{L}_X \alpha = 0$. Then $f_0 \equiv 0$ and $f_1, \ldots, f_{2n}$ are independent of $\theta$.

**Proof.** Since $X$ preserves $\alpha$, it also preserves the presymplectic form $\omega = d\alpha$, and hence it preserves the kernel $\mathbb{K} \frac{\partial}{\partial \theta}$ of $\omega$. Therefore, we have

$$\left[ \frac{\partial}{\partial \theta}, X \right] \mathcal{J} \omega = \frac{\partial}{\partial \theta} \mathcal{L}_X \omega - \mathcal{L}_X \left( \frac{\partial}{\partial \theta} \mathcal{J} \omega \right) = 0,$$

i.e., $\left[ \frac{\partial}{\partial \theta}, X \right]$ is also in the kernel of $\omega$. On the other hand, we have

$$\left[ \frac{\partial}{\partial \theta}, X \right] = \frac{\partial f_0}{\partial \theta} \frac{\partial}{\partial \theta} + \sum_{i=1}^{n} \frac{\partial f_i}{\partial \theta} \frac{\partial}{\partial x_i}.$$
Therefore, \( f_1, \ldots, f_{2n} \) are independent of \( \theta \).

Since \( 0 = \mathcal{L}_X \alpha = X \lrcorner d\alpha + d(X \lrcorner \alpha) \), its \( \theta \)-component vanishes. Note that \( X \lrcorner d\alpha \) does not contribute any term to the \( \theta \)-component, so the \( \theta \)-component must come from the term \( d(X \lrcorner \alpha) \). Recall that for \( i = 1, \ldots, n \), the coefficient function in front of \( dx_i \) in \( \alpha \) as well as \( f_i \) are independent of \( \theta \); therefore, this component is just \( \frac{\partial (\theta f_0)}{\partial \theta} \), which implies that \( f_0 \) is identically zero. \( \square \)

5.1.1. The case \( X(O) \neq 0 \).

**Theorem 5.3.** Let \( \alpha \) be a non-vanishing singular contact form on \((\mathbb{K}^{2n+1}, O)\) written as \( \alpha = \theta d\theta + (x_1 + 1)dx_{n+1} + \sum_{i=2}^{n} x_i dx_{n+i} \). Suppose \( \alpha \) is preserved by a vector field \( X \) with non-vanishing constant part \( X^{(0)} = \sum_{i=1}^{2n} c_i \frac{\partial}{\partial x_i} \neq 0 \). Then \( X \) can be straightened out to \( X = \frac{\partial}{\partial x_{2n}} \) if \( c_{n+1} = 0 \), or to \( X = c_{n+1} \frac{\partial}{\partial x_{n+1}} \), if \( c_{n+1} \neq 0 \), without changing the expression of \( \alpha \).

**Proof.** We only need to consider the 1-form \( \gamma := \alpha - \theta d\theta \) on the hyperplane \( \{ \theta = 0 \} \) preserved by \( X \) which can be regarded as a vector field on the same hyperplane by Lemma 5.2. First, we normalize the vector field \( X \) by changing only the coordinates \( x_1, \ldots, x_{2n} \) and then change \( \gamma \) back to normal form, keeping the expression of \( X \).

If \( c_{n+1} = 0 \), we first assume, without loss of generality, that \( c_{2n} \neq 0 \) and write \( X = \frac{\partial}{\partial x_{2n}} \) after some coordinate change by a local diffeomorphism \( \Phi \). Since \( \gamma \) is preserved by \( X \), it is independent of \( x_{2n} \) in the new coordinate system and hence we can write it as a sum of a 1-form \( \gamma' \) on \( \{ x_{2n} = 0 \} \) and a 1-form \( \gamma'' = g(x_1, \ldots, x_{2n-1})dx_{2n} \) where \( g \) is a function independent of \( x_{2n} \).

We claim that \( \gamma' \) on \( \{ x_{2n} = 0 \} \) has the following properties:

- \( \gamma'(O) \neq 0 \);
- \( d\gamma' \) has the maximal possible rank \( 2n - 2 \) at \( O \);
- \( \gamma' \wedge (d\gamma')^{n-1}(O) = 0 \).

In order to verify these properties, it is sufficient to calculate the constant part \( \gamma'(0) \) and the linear part \( \gamma'^{(1)} \) of \( \gamma' \) at \( O \).

Notice that the linear part of the diffeomorphism \( \Phi \) in the Straightening Out Theorem for vector fields is just \( x_i \mapsto x_i - c_{2n}^2 x_{2n} \) for \( i = 1, \ldots, 2n - 1 \) and \( x_{2n} \mapsto \frac{1}{c_{2n}} x_{2n} \).

Denote the quadratic part of the old coordinate \( x_{n+1} \) in new coordinate system (i.e., the quadratic part of the \( (n + 1) \)-th component of \( \Phi^{-1} \)) by \( Q \). Then the constant part \( \gamma^{(0)} \) and the linear part \( \gamma^{(1)} \) of \( \gamma \) are, respectively,

\[
\gamma^{(0)} = d(x_{n+1} + c_{n+1} x_{2n}),
\]

\[
\gamma^{(1)} = \sum_{i=1}^{n-1} (x_i + c_i x_{2n})d(x_{n+i} + c_{n+i} x_{2n}) + (x_n + c_n x_{2n})d(c_{2n} x_{2n}) + dQ.
\]

Write \( Q \) as the sum of a function \( Q'(x_1, \ldots, x_{2n-1}) \) independent of \( x_{2n} \) and a function \( Q'' \) divisible by \( x_{2n} \). Remembering that \( \gamma' \) is also independent of \( x_{2n} \), we conclude that
the constant part and the linear part of $\gamma'$ are, respectively,

$$
\gamma'(0) = dx_{n+1},
$$

$$
\gamma'(1) = \sum_{i=1}^{n-1} x_i dx_{n+i} + dQ',
$$

Using these expressions, it is easy to see that the three properties stated above hold.

It follows directly from [19] that $\gamma'$ has a normal form $(x_1 + 1) dx_{n+1} + \sum_{i=2}^{n-1} x_i dx_{n+i}$ by changing only the coordinates $x_1, \ldots, x_{2n-1}$. Thus, $\gamma = (x_1 + 1) dx_{n+1} + \sum_{i=2}^{n-1} x_i dx_{n+i} + g(x_1, \ldots, x_{2n-1}) dx_{2n}$ and $\partial g / \partial x_n(O) \neq 0$ since $(d\gamma)^n(O) \neq 0$. Then the transformation that maps $x_n$ to $g$ and preserves the other coordinates is indeed a local diffeomorphism not affecting the expression $X = \frac{\partial}{\partial x_{2n}}$, since $g$ is independent of $x_{2n}$, and keeps $\gamma$ in normal form.

Note that although the position of the coordinate $x_i$ in $\gamma$ is not exactly the same for $i = 1, 2 \leq i \leq n$, and $n+1 \leq i \leq 2n$, the argument above works with very few changes. Indeed, if we assume at the beginning $c_k \neq 0 (k \neq n + 1)$ so that $X = \frac{\partial}{\partial x_k}$, we obtain a similar splitting $\gamma = \gamma' + \gamma''$ with $\gamma'$ having the three properties stated above. (Now the $dx_k$ component $\gamma''$ is independent of $x_k$ and $\gamma'(1) = \sum_{i=1, i \neq k}^{n} x_i dx_{n+i} + dQ'$.) We only need to rename the coordinates after the final normalization of $\gamma$.

If $c_{n+1} \neq 0$, we need to use a supplementary linear transformation before normalizing $\gamma'$. The first step is again the straightening out of $X$ to $c_{n+1} \frac{\partial}{\partial x_{n+1}}$ by a diffeomorphism $\Phi$ whose linear part is $x_i \mapsto x_i - \frac{c_i}{c_{n+1}} x_{n+1}$ for $i = 1, \ldots, n, n + 2, \ldots, 2n$ and $x_{n+1} \mapsto x_{n+1}$. We write again $\gamma = \gamma' + \gamma''$, where $\gamma'$ is a 1-form on $\{x_{n+1} = 0\}$ and $\gamma''$ has the expression $g(x_1, \ldots, x_n, x_{n+2}, \ldots, x_{2n}) dx_{n+1}$ with $g(0) = 1$. Since $\gamma'(0) = 0$ in this case, we use a linear transformation which maps $x_{n+1}$ to $x_{n+1} - x_{2n}$ and does not affect the other $2n - 1$ coordinates. Then $X$ has still the form $c_{n+1} \frac{\partial}{\partial x_{n+1}}$ after this linear transformation, while the transformed $\gamma'$ no longer vanishes at $O$ (indeed, the constant part of $\gamma'$ is $dx_{2n}$). Moreover, we have $(d\gamma')^{n-1}(O) \neq 0$ since $(d\gamma)^n(O) \neq 0$, where

$$(d\gamma)^n = (d\gamma' + dg \wedge dx_{n+1})^n = (d\gamma')^n + n(d\gamma')^{n-1} \wedge dg \wedge dx_{n+1} = n(d\gamma')^{n-1} \wedge dg \wedge dx_{n+1}.$$

We also verify that $\gamma' \wedge (d\gamma')^{n-1}(O) = 0$, which implies, by [19], that $\gamma'$ has the normal form $\gamma' = (x_2 + 1) dx_{n+2} + \sum_{i=3}^{n} x_i dx_{n+i}$ by changing only the coordinates $x_1, \ldots, x_n, x_{n+2}, \ldots, x_{2n}$. Indeed, the linear part of $\gamma$ is

$$(x_1 + c_1(x_{n+1} + x_{2n})) d(x_{n+1} + x_{2n}) + \sum_{i=2}^{n} (x_i + c_i(x_{n+i} + x_{2n})) d(x_{n+i} + c_{n+i}(x_{n+i} + x_{2n}))) + dQ,$$

where $Q$ is some quadratic function. The linear part $\gamma'(1)$ of $\gamma'$ can be split into four parts: $\sum_{i=2}^{n-1} x_i dx_{n+i}$, the $dx_{2n}$ component, a 1-form divisible by $x_{2n}$, and a differential
form $dQ'$ of some quadratic function $Q'$. Then $(d\gamma'^{(1)})^{n-1}$ is a wedge product of $dx_{2n}$ and some $(2n-3)$-form; therefore, its wedge product with the constant part $\gamma'^{(0)} = dx_{2n}$ vanishes.

Finally, we conclude that $\frac{\partial \gamma}{\partial x_1}(O) \neq 0$, and therefore $x_1 = g - g(O)$, $x_2, \ldots, x_{2n}$ forms a coordinate system in which $X = c_{n+1} \frac{\partial}{\partial x_{n+1}}$ and $\gamma = (x_1 + 1)dx_{n+1} + \gamma'$. Now use a linear transformation to arrive at the desired normalization. $\square$

5.1.2. The case $X(O) = 0$.

Theorem 5.4. Let $\alpha$ be a non-vanishing singular contact form and $X$ a vector field vanishing at $O$ which preserves $\alpha$. Then $X$ and $\alpha$ can be normalized simultaneously, i.e., there is a coordinate system in which $\alpha$ is in the normal form (5.1) and $X$ commutes with its semisimple linear part.

Proof. We assume that $\alpha$ has been brought into the normal form (5.1). Since $L_X \alpha = 0$, regarding $X$ as a vector field on $\{\theta = 0\}$ by Lemma 5.2, we interpret $X$ as a Hamiltonian vector field with respect to the standard symplectic form $\omega_0 = d\alpha$. Thus we have a Hamiltonian function $H$, with $H(O) = 0$, i.e., $X_{\omega_0} = -dH$. Write

$$X = \sum_{i=1}^{n} \left( - \frac{\partial H}{\partial x_{n+i}} \frac{\partial}{\partial x_i} + \frac{\partial H}{\partial x_i} \frac{\partial}{\partial x_{n+i}} \right).$$

Since $X(O) = 0$, $H(O) = 0$, and

$$d(X_{\omega_0}) - dH = d(X_{\omega_0}) + X_{\omega_0} = L_X \alpha = 0,$$

we get $X_{\omega_0} - H = 0$, which, in the given coordinates, takes the form

$$(x_1 + 1) \frac{\partial H}{\partial x_1} + \sum_{i=2}^{n} x_i \frac{\partial H}{\partial x_i} - H = 0.$$  

By Taylor expansion, this equation provides two properties of $H$:

(1) $H$ is independent of $x_1$, which implies that $x_{n+1}$ is a first integral of $X$;

(2) $H$ is the sum of monomial terms of the form (up to constant coefficients)

$$\prod_{i=2}^{2n} x_i^{\ell_i}$$

such that $\sum_{i=2}^{n} \ell_i = 1$.

In particular, the quadratic part $H^{(2)}$ of $H$ is the sum of $c_{ij} x_i x_j$ in which $c_{ij}$ is constant, $i \in \{2, \ldots, n\}$ and $j \in \{n+1, \ldots, 2n\}$. Then the coefficient matrix $A$ of the linear part of $X$ is of the form $\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$. Moreover, we can keep the standard form of $\omega_0$ and put $A_1$ into Jordan normal form by two linear transformations on the complementary $n$-dimensional coordinate planes $\{x_1, \ldots, x_n\}$ and $\{x_{n+1}, \ldots, x_{2n}\}$.

Now we claim that the (formal) diffeomorphism $Id_\theta \oplus \Phi$ puts $X$ into the Poincaré-Dulac normal form without changing $\alpha$, where $Id_\theta$ is the identity map for the coordinate $\theta$ and $\Phi$ is the classical symplectic (with respect to $\omega_0$) diffeomorphism (constructed step by step below) on $\{\theta = 0\}$ normalizing $H$ into the Poincaré-Birkhoff normal form. When $H$ is in the Poincaré-Birkhoff normal form, $X$ is also automatically in the Poincaré-Dulac normal form.
We only need to prove that $\Phi$ preserves the 1-form $\beta := \alpha - \theta d\theta$ on $\{\theta = 0\}$.

Let us verify this assertion by showing that $\Phi$ preserves the associated vector field $Z = (1 + x_1) \frac{\partial}{\partial x_1} + \sum_{i=2}^n x_i \frac{\partial}{\partial x_i}$ of $\beta$ defined by $Z \omega_0 = \beta$. Indeed, decomposing the quadratic part $H^{(2)}$ of $H$ into the sum of the semisimple part $H^s$ and the nilpotent part $H^n$, we see that every resonant term of $H$ lies in the subspace of the kernel of $X_{H^s}$.

At step $k \geq 3$, we assume that all the non-resonant terms of $H$ of degree smaller than $k$ have been eliminated. In order to eliminate the non-resonant terms of degree $k$, one finds a homogeneous function $F^{(k)}$ of degree $k$ such that the image of $F^{(k)}$ under $X_{H^{(2)}}$ coincides with these non-resonant terms in $H^{(k)}$, i.e., $H^{(k)} - X_{H^{(2)}} F^{(k)}$ contains only resonant terms. Then the time-1 flow of $X_{F^{(k)}}$ is a symplectic diffeomorphism $\Phi^{(k)}$ which changes the $k$-th order terms of $H$ into normal form without changing terms of lower degrees. Constructing inductively, from $k = 3$ to infinity, we get a formal symplectic diffeomorphism $\Phi$ (equal to the formal infinite composition of the above $\Phi^{(k)}$), which puts $H$ into the Poincaré-Birkhoff normal form.

Thanks to the properties of $H$, and the above property of $H^{(2)}$ in particular, $F^{(k)}$ can be chosen such that every monomial term $\prod_{i=1}^{2n} x_i^{\ell_i}$ also satisfies $\ell_1 = 0$ and $\sum_{i=2}^n \ell_i = 1$. Moreover, this construction guarantees that the higher order terms of $H$ in the new coordinate system after each transformation $\Phi^{(k)}$ have the same properties. Then $X_{F^{(k)}}$ acquires the form

$$X_{F^{(k)}} = \sum_{i=1}^n \left( - \frac{\partial F^{(k)}}{\partial x_{n+i}}(x_2, \ldots, x_{2n}) \frac{\partial}{\partial x_i} + \frac{\partial F^{(k)}}{\partial x_i}(x_{n+1}, \ldots, x_{2n}) \frac{\partial}{\partial x_{n+i}} \right).$$

Since $X_{F^{(k)}}$ commutes with the Euler vector field $E = \sum_{i=1}^n x_i \frac{\partial}{\partial x_i}$ and commutes with $\frac{\partial}{\partial x_1}$, we conclude that it commutes with the associated vector field $Z$ of $\beta$, and furthermore, it preserves $\beta$. Consequently, $\Phi$ also preserves $\beta$. □

5.2. Two types of non-degenerate singular contact forms. We now consider singular contact forms which vanish at $O$, with a generic non-degeneracy condition.

If $O$ is a non-degenerate singularity of the singular contact form $\alpha$, then it is isolated by definition. Moreover, $O$ must be an equilibrium point of the vector field which preserves the singular contact structure.

Let us present normal forms of singular contact forms in the transversal case and the generic tangential case obtained in [18].

**Theorem 5.5.** Let $O$ be a non-degenerate singularity of a smooth (resp., real or complex analytic) singular contact 1-form $\alpha$ on a manifold of dimension $2n+1$. Then there is a local smooth (resp., analytic) coordinate system $(\theta, x_1, \ldots, x_{2n})$ in which $\alpha$ has the expression

$$\alpha = \theta d\theta + \gamma$$

in the transversal case, or the expression

$$\alpha = d(\theta^3 - x_1 \theta) + \gamma$$
in the tangential case with a generic tangency, where \( \gamma = \sum_{i=1}^{2n} g_i \, dx_i \) is a 1-form which is basic with respect to \( \frac{\partial}{\partial \theta} \), i.e., the functions \( g_i \) do not depend on \( \theta \), and such that
\[
d\gamma = \sum_{i=1}^{n} dx_i \wedge dx_{i+n}
\]
is a symplectic form in \( 2n \) variables (which can be put into Darboux canonical form).

The primitive form \( \gamma \) can be (formally, or analytically under some Diophantine conditions, or smoothly under hyperbolic conditions) normalized (over \( \mathbb{C} \)) in the subspace \((x_1, \ldots, x_{2n})\) as follows: there are positive integers \( n_1 < n_2 < \cdots < n_{k+1} = n + 1 \), and eigenvalues \( \lambda_1, \ldots, \lambda_{2n} \) of \( X \), such that \( \lambda_i + \lambda_{n+i} = 1 \) for all \( i = 1, \ldots, n \), \( \lambda_i = \lambda_j \) for any \( n_s < i < j < n_{s+1} \) \((s = 1, \ldots, k)\), and
\[
(5.2) \quad \gamma = \sum_{i=1}^{k} (\gamma_i + dQ_i) + dR;
\]
where:

i) \( \gamma_i \) has the form
\[
\gamma_i = \sum_{j=n_i}^{n_{i+1}-1} (\lambda_{ni} x_j dx_{n+i+j} + (\lambda_{ni} - 1)x_{n+j} dx_j).
\]

ii) The functions \( Q_i \) are as follows:

- If \( \lambda_i \neq \frac{1}{2} \) then \( Q_i = 0 \) or \( Q_i = \sum_{j=n_i}^{n_{i+1}-1} x_{j+1} x_{n+i+j} \);
- If \( \lambda_i = \frac{1}{2} \) then \( Q_i \) belongs to one of the following four cases:
  - \( Q_i = 0 \);
  - \( Q_i = x_{n_i}^2 \) with \( n_{i+1} = n_i + 1 \) in this case;
  - \( Q_i = 2 \sum_{j=n_i}^{n_{i+1}-2} x_j x_{n+i+j} + (-1)^{n_{i+1} - n_i} x_{n_{i+1} - 1}^2 \);
  - \( Q_i = 2 \sum_{j=n_i}^{n_{i+1}-2} x_j x_{n+i+j} \) with \( n_{i+1} - n_i \geq 3 \) and is an odd number.

iii) \( R \) is a function of \( x_1, \ldots, x_{2n} \) whose infinite jet at 0 contains only the monomial terms \( \prod x_{i}^{\alpha_i} \) satisfying a resonance relation
\[
(5.3) \quad \sum_{i=1}^{2n} \alpha_i \lambda_i = 1 \quad \text{with} \quad \sum_{i=1}^{2n} \alpha_i \geq 3.
\]

Note that we reduce the normalization of \( \gamma \) to the normalization of its associated vector field \( Z \) defined by \( Z \omega_0 = \gamma \) in [18]. That is, supposing that \( \gamma \) is already in normal form in the coordinate system \((\theta, x_1, \ldots, x_{2n})\), then its associated vector field \( Z \) defined by \( Z \omega_0 = \gamma \) is in the Poincaré–Dulac normal form, where \( \omega_0 = d\gamma \) is the canonical symplectic form on \( \{ \theta = 0 \} \).

We would like to mention a special example as a direct application of this reduction, which shows the existence of the simultaneous normalization of the dynamical system and the singular contact distribution (not only the form itself).
Example 5.6. The vector field \( Z_1 = \frac{1}{2} \theta \frac{\partial}{\partial \theta} + \omega \) and its semisimple part \( Z_1^s = \frac{1}{2} \theta \frac{\partial}{\partial \theta} + Z^s \) preserve the transversal singular structure. In fact, \( Z \) is invariant under its flow \( \phi_Z^t(x) \), and we have \( (\phi_Z^t)_* \omega = e^t \omega \) since \( L_Z \omega = \omega \). Therefore, we have \( (\phi_Z^t)_*^s \gamma = Z_1 \omega = e^t \gamma \).

As the flow \( \phi_{Z_1}^t(\theta, x) = (\theta e^t, \phi_Z^t(x)) \) and the transversal singular contact form \( \alpha = \theta d\theta + \gamma \), we have

\[
(\phi_{Z_1}^t)_*^s \alpha = (e^t \theta d\theta + e^t \gamma) = e^t \alpha.
\]

Similar computations are also true for the semisimple vector field \( Z_1^s \).

5.2.1. The transversal case. Recall that a singular contact form \( \alpha \) in dimension \( 2n + 1 \) is called transversal if the kernel of \( d\alpha \) is transversal to the hypersurface \( S = \{ x \mid \alpha \wedge (d\alpha)^n(x) = 0 \} \) of singular points of \( \alpha \). This transversality condition is generic. In this subsubsection, we will study local dynamical systems which preserve this type of singular contact forms.

Theorem 5.7. Suppose a transversal singular contact form \( \alpha \) is preserved by a vector field \( X \) which vanishes at \( O \). Then \( \alpha \) and \( X \) can be normalized simultaneously, i.e., \( \alpha \) is in the form \( \alpha = e^t \theta d\theta + \phi^t \gamma \) as in Theorem 5.5 and \( X \) is in the Poincaré-Dulac normal form having \( \theta \) as a first integral. Moreover, the quadratic function \( Q = \sum Q_i \) and the higher order function \( R \) in \( (5.2) \) are first integrals of the semisimple part of \( X \).

Proof. Write \( X = f_0 \frac{\partial}{\partial \theta} + \sum_{i=1}^{2n} f_i \frac{\partial}{\partial x_i} \). Then Lemma 5.2 also holds for non-degenerate singular contact forms, and the proof is exactly the same. Thus \( \theta \) is a first integral of \( X \), \( L_X \gamma = 0 \), and \( \omega = d\gamma \) is a symplectic form on \( \{ \theta = 0 \} \) (see Theorem 5.5).

Since \( X \) is independent of \( \theta \), it can be projected to the hyperplane \( \{ \theta = 0 \} \). We will show that \( X \) commutes with \( Z \) defined by \( Z_1 \omega = \gamma \). Indeed, we have

\[
[Z, X]_1 \omega = Z_1 L_X \omega - L_X Z_1 \omega = -L_X \gamma = 0,
\]

which implies that \( [Z, X] = 0 \). Therefore, there exists a new coordinate system in which \( Z \) and \( X \) are both in the Poincaré-Dulac normal forms. Then we use Moser’s method to normalize \( \omega \) to its constant part \( \omega^{(0)} \). Moreover, we can choose a path such that this normalization preserves the semisimple parts of \( Z \) and \( X \). In other words, \( Z \) and \( X \) remain in the Poincaré-Dulac normal form.

Indeed, take \( \zeta = Z^s \omega - \omega^{(0)} \), where \( Z^s \) is the semi-simple part of \( Z \), and \( \omega_t = \omega^{(0)} + t(\omega - \omega^{(0)}) \). Then the path of diffeomorphisms, from time 0 to time 1, given by the flow of the time dependent vector field \( Y_t \), defined by \( Y_t \omega_t = -\zeta \), satisfies our requirements, because

\[
dY_t \omega_t = -d\zeta = -L_{Z^s} (\omega - \omega^{(0)}) = -\omega - \omega^{(0)} = -\frac{d}{dt} \omega_t.
\]

The third equality above is obtained in the following way. Since \( \omega = d\gamma = d(Z_1 \omega) = L_Z \omega \), it follows that \( L_{Z^s} \omega = 0 \) by Theorem 2.3.

It is shown in [13, Lemma 2.2] that \( Y_t \) commutes with \( Z^s \). We now show that \( Y_t \) also commutes with \( X^s \). Since \( X \) preserves \( \omega \), we have \( L_X \omega = 0 \) by the fundamental
conservation property (see Theorem 2.3). Clearly, we also have $\mathcal{L}_{X^s}\omega^{(0)} = 0$ and $\mathcal{L}_{X^s}\omega_t = 0$. Then the following two equations show that $Y_t$ also commutes with $X^s$:

$$[Y_t, X^s] \omega_t = Y_t \mathcal{L}_{X^s} \omega_t - \mathcal{L}_{X^s}(Y_t \omega_t) = 0 - \mathcal{L}_{X^s}(-\zeta) = \mathcal{L}_{X^s}\zeta$$

$$0 = [Z^s, X^s] \omega_t = Z^s \mathcal{L}_{X^s} \omega_t - \mathcal{L}_{X^s}(Z^s \omega_t) = \mathcal{L}_{X^s}(Z^s \omega_t) = -\mathcal{L}_{X^s}\zeta$$

Now we use a linear transformation to map $\omega^{(0)}$ into the canonical form $\omega_0$ and put, simultaneously, $Z^s$ in diagonal form, as in [18]. Then $\gamma$ is also in normal form. The linear transformation does not destroy the Poincaré-Dulac normal form of $X$. (Another way to look at it is to take the associated torus action $\rho$ of the family of commuting vector fields $X$ and $Z$, which contains both the torus action associated to $X$ and the torus action associated to $Z$. A linearization of $\rho$ is a simultaneous normalization of $X$ and $Z$. Then take an equivariant normalization of $\omega$ with respect to $\rho$.)

The rest of the proof is an application of Theorem 2.3, here we give a direct proof:

We can now assume that everything is in normal form and, in particular, $\omega = \omega_0$ is in canonical form. Since $X$ commutes with the semisimple part $Z^s$ of $Z$, we have

$$\mathcal{L}_X(Z^s \omega) = Z^s \mathcal{L}_X \omega - [Z^s, X] \omega = 0,$$

that is, $Z^s \omega$ is a first integral of $X$. It is also a first integral of the semisimple linear vector field $X^s$ since $X$ is in the Poincaré-Dulac normal form.

Recall that $Z \omega = Z \mathcal{L}_Z \omega = 0$, which implies that $\mathcal{L}_Z \omega = Z \omega = \gamma$, which, in turn, implies that $\mathcal{L}_Z \gamma = \gamma$. Recall also that $\gamma$ is the sum of its linear part $\gamma^{(1)}$ and an exact differential form $dR$ where the 2-jet of the function $R$ vanishes at $O$. From $\mathcal{L}_Z \gamma = \gamma$ we get $Z^s(R) = R$.

Since $Z^s \omega = Z^s \gamma^{(1)} + Z^s R = Z^s \gamma^{(1)} + R$ is a first integral of $X^s$, it follows that both its quadratic part $Z^s \gamma^{(1)}$ and its higher order part $R$ are first integrals of $X^s$. If we abuse the notation and define $\gamma^s := Z^s \omega_0$, then $\gamma^{(1)} = \gamma^s + dQ$. Since $Z^s \gamma^s = Z^s \mathcal{L}_Z(Z^s \omega_0) = 0$, it follows that $\mathcal{L}_Z \gamma^s = \gamma^s$, hence $\mathcal{L}_Z dQ = dQ$, we conclude that $Z^s dQ = \mathcal{L}_Z Q = Q$ is also a first integral of $X^s$. \hfill \Box

5.2.2. The tangential case. We now assume that $\alpha$ is a tangential non-degenerate singular contact form, i.e., the kernel of $d\alpha$ is tangent to the hypersurface $S = \{x \mid \alpha \wedge (d\alpha)^n(x) = 0\}$ of singular points. We recall that the normal form of $\alpha$ in the generic tangential case is $\alpha = d(\theta^m - x_1 \theta) + \gamma$ with a non-degenerate function $h$ and normalized primitive 1-form $\gamma$ (see Theorem 5.3). We have a similar theorem as in the transversal case.

**Theorem 5.8.** Let $\alpha = d(\theta^m - x_1 \theta) + \gamma$ be a tangential singular contact form preserved by a vector field $X$ having $O$ as its equilibrium point, where $\gamma$ is a primitive 1-form independent of $\theta$ and $m > 2$ is an integer. Then $\alpha$ and $X$ can be normalized simultaneously, i.e., $\alpha = d(\theta^m - f(x) \theta) + \gamma$ with $\gamma$ in the normal form as in Theorem 5.3 and $X$ is in the Poincaré-Dulac normal form having $\theta$ and $f(x)$ as its first integrals. Moreover, the quadratic function $Q = \sum Q_i$ and higher order function $R$ in 5.2 are first integrals of the semisimple part of $X$. 
In particular, if $\alpha = d(\theta^3 - x_1\theta) + \gamma$ is generically tangent, i.e., the kernel of $d\alpha$ has second order tangency with the singular hypersurface, then $\alpha$ and $X$ can be normalized simultaneously.

Proof. Assume that $X = f_0 \frac{\partial}{\partial \theta} + \sum_{i=1}^{2n} f_i \frac{\partial}{\partial x_i}$ preserves $\alpha = d(\theta^m - x_1\theta) + \gamma$. Similar to Lemma 5.2 in the transversal case, there is an analogous lemma (see Lemma 5.9 below) in the tangent singular case, which states that $f_0$ and $f_1$ are identically zero. Therefore, we have $\mathcal{L}_X d(\theta^m - x_1\theta) = \mathcal{L}_X \gamma = 0$ and the functions $\theta$ and $x_1$ are first integrals of $X$.

Using this lemma, the rest of the proof is similar to the one in the transversal case:

- $X$, viewed as a vector field on $\{\theta = 0\}$, commutes with the associated vector field $Z$ defined by $Z \cdot d\gamma = \gamma$; thus $X$ and $Z$ can be put in the Poincaré-Dulac normal forms simultaneously;
- use Moser’s path method and then a linear transformation to normalize $d\gamma$ to the canonical symplectic form without destroying the normal forms of $X$ and $Z$.

Then the primitive 1-form is automatically in normal form as in Theorem 5.5. Notice we do not change the variable $\theta$, so the old $x_1$ becomes a function $f(x)$ which is still a first integral of $X$ and $df(O) \neq 0$. □

Lemma 5.9. Assume that $X = f_0 \frac{\partial}{\partial \theta} + \sum_{i=1}^{2n} f_i \frac{\partial}{\partial x_i}$ preserves $\alpha = d(\theta^m - x_1\theta) + \gamma$, $m > 2$. Then $f_0 = f_1 = 0$.

Proof. Since $X$ preserves $\alpha$, it preserves the kernel $\mathbb{K} \frac{\partial}{\partial \theta}$ of $d\alpha = \omega$, which means that $f_1, \ldots, f_{2n}$ are independent of $\theta$.

Consider the component $d\theta$ in $\mathcal{L}_X \alpha$. Since $X \cdot d\alpha = \sum_{i=1}^{2n} f_i \frac{\partial}{\partial x_i} \omega$ does not contain $d\theta$, the $d\theta$-component must come from $dX \cdot \omega$, more precisely, from the part

$$d(X \cdot d(\theta^m - x_1\theta)) = d((m\theta^{m-1} - x_1)f_0 - \theta f_1).$$

Thus, we have

$$\left(\frac{\partial(m\theta^{m-1} - x_1)}{\partial \theta} f_0 - f_1\right) d\theta = 0.$$  

(5.4)

Now we can write

$$\left(m\theta^{m-1} - x_1\right)f_0 = \theta f_1 + C(x)$$

where $C(x)$ is a function independent of $\theta$. The coefficient of $d\theta$ in (5.4) vanishes:

$$\left(m\theta^{m-1} - x_1\right)\frac{\partial f_0}{\partial \theta} + m(m - 1)\theta^{m-2} f_0 - f_1 = 0.$$  

(5.6)

Let $\ell$ be any non-negative integer and assume $f_0$ is divisible by $x_1^\ell$. Then:

- set $\theta = 0$ in (5.5) to conclude that $C(x)$ is divisible by $x_1^{\ell+1}$;
- set $\theta = 0$ in (5.6) to conclude that $f_1(x)$ is divisible by $x_1^{\ell+1}$.
• go back to equation (5.5) to conclude that \( f_0 \) is also divisible by \( x_1^{\ell+1} \).

We proved inductively that \( f_0 \) is divisible by any power of \( x_1 \). Thus \( f_0 \) vanishes if it is a formal or analytic function. So do \( f_1(x) \) and \( C(x) \). 

5.2.3. Singular contact forms with integrable systems. We have seen that singular contact forms are not finitely determined, i.e., we cannot find an a priori diffeomorphism that brings them in normal forms having finite order jets. However, the complexity of singular contact forms diminishes if they are preserved by some vector fields. The more such vector fields, the less freedom for singular contact forms. So if a singular contact form is preserved by an integrable system, then the singular contact form itself becomes simpler.

**Proposition 5.10.** Let \( \gamma \) be a primitive 1-form on \( (\mathbb{K}^{2n}, O) \). Suppose \( \gamma \) is preserved by \( p \) pairwise commuting vector fields \( X_1, \ldots, X_p \) whose semisimple parts of the linear parts are independent almost everywhere. Then \( p \leq n \). Moreover, if \( p = n \), then \( \gamma = \gamma^s = \sum_{i=1}^{n}(\lambda_i x_i dx_{n+i} + (\lambda_i - 1)x_{n+i}dx_i) \) is linear and the vector fields in the Poincaré-Dulac normal forms have diagonal linear parts.

**Proof.** From the proof of Theorem 5.7, we can assume that the primitive 1-form and the vector fields are all in normal forms, i.e., \( \gamma \) is as in Theorem 5.5 and \([X_i, X_j] = 0\) for all \( i, j \). By the Conservation Theorem 2.3, \( \gamma \) is also preserved by the semisimple parts \( X^s_1, \ldots, X^s_p \) of the vector fields. After a linear transformation, we can assume \( X^s_i = \sum_{k=1}^{n} \mu_{ik} x_k \frac{\partial}{\partial x_k} \) for \( i = 1, \ldots, p \).

Using again the equation \( \mathcal{L}_{X^s_i} \omega^{(0)} = \mathcal{L}_{X^s_i} d\gamma = d\mathcal{L}_{X^s_i} \gamma = 0 \), where

\[
\omega^{(0)} = \sum_{1 \leq j < k \leq 2n} c_{jk} dx_j \wedge dx_k
\]

is the constant part of the symplectic form \( d\gamma \), we get

\[
(5.7) \quad \mu_{ij} + \mu_{ik} = 0, \quad \text{if} \quad c_{jk} \neq 0, \quad i = 1, \ldots, p.
\]

Since \( X^s_1, \ldots, X^s_p \) are independent almost everywhere, the rank of the \( p \times (2n) \) matrix \((\mu_{ik})\) is \( p \) and because \( \omega^{(0)} \) is non-degenerate, among all the \( c_{jk} \)'s there are at least \( n \) that do not vanish. Thus \((5.7)\) implies that \( p \leq n \).

If the number of vector fields is maximal, i.e., \( p = n \), then we have exactly \( n \) among all of the \( c_{jk} \)'s that do not vanish. The set of subscripts \( j, k \) for the \( n \) nonzero \( c_{jk} \)'s must be included in the set \( \{1, 2, \ldots, 2n\} \) by non-degeneracy. Without loss of generality, we assume that \( c_{j+n+j} \neq 0 \) for all \( j = 1, \ldots, n \). By \((5.7)\), this implies \( \mu_{ij} + \mu_{i+n+j} = 0 \) for all \( i = 1, \ldots, p = n \).

The common (formal or analytic) first integrals are generated by quadratic functions \( x_j x_{n+j} \)'s. Then any monomial term of \( H \) has the expression \( \prod_{j=1}^{n} x_j^{\ell_j} x_{n+j}^{\ell_j} \) since \( H \) is automatically a first integral. By the resonance relation \((5.3)\) on the indices, we have

\[
\sum_{j=1}^{n} (\lambda_j \ell_j + (1 - \lambda_j) \ell_j) = \sum_{j=1}^{n} \ell_j = 1.
\]
It thus follows that there is only one $\ell_j = 1$ and the others are zero, which implies that $R = 0$ since we request a priori that the summands of $R$ have degree at least 3. Hence, the primitive 1-form now reads $\alpha = \gamma^s + dQ$. Moreover, the quadratic polynomial is also a common first integral of $X_i^s$'s. Thus $Q = \sum_{j=1}^{n} c_j x_j x_{n+j}$, where the $c_j$'s are constant coefficients. This implies $c_j = 0$ since $Q$ does not contain any term of the form $x_j x_{n+j}$ in our normal form. \hfill \Box

We can get similar results for a transversal singular contact form $\alpha$ under the same assumptions. By Theorem 5.7, we can assume that in a suitable coordinate system $(\theta, x_1, \ldots, x_{2n})$, $\alpha$ is in normal form and the vector fields $X_1, \ldots, X_p$ are all in the Poincaré-Dulac normal forms. By an argument similar to that in Lemma 5.2, we can conclude that $X_1, \ldots, X_p$ are independent of $\theta$ and have $\theta$ as a common first integral. Therefore, $X_1, \ldots, X_p$ also preserve the primitive 1-form $\gamma$ and we have $p \leq n$; if $p = n$, then $\gamma$ is linear. This proves the following result.

**Proposition 5.11.** Let $\alpha$ be a transversal singular contact form on $(\mathbb{R}^{2n+1}, O)$. Suppose $\alpha$ is preserved by $p$ pairwise commuting vector fields $X_1, \ldots, X_p$ whose semisimple parts of the linear parts are independent almost everywhere. Then $p \leq n$. If $p = n$, then $\alpha = \theta d\theta + \gamma^s$ is linear.

We remark that the proposition is not true in the generic tangential case. The maximum possible number of such non-degenerate vector fields is $n - 1$ since we can exclude the situation $p = n$. Otherwise, we use Lemma 5.9 and then the vector fields preserve the primitive 1-form $\gamma$, which guarantees that the common first integrals of the semisimple parts of the vector fields are functions of $x_i x_{n+i}$'s. On the other hand, we request $f(x)$ in $\alpha = d(\theta^3 - f\theta) + \gamma$ to be a first integral of the semisimple parts of the vector fields, which contradicts the non-degeneracy of $f(x)$ in the sense that $df(0) \neq 0$.

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