Abstract. In this article, the Galerkin piecewise-linear finite element (FE) method is applied to approximate the solution of time-fractional diffusion equations with variable diffusivity on bounded convex domains. Standard energy arguments do not provide satisfactory results for such a problem due to the low regularity of its exact solution. Using a delicate energy analysis, a priori optimal error bounds in $L^2(\Omega)$-, $H^1(\Omega)$-, and quasi-optimal in $L^\infty(\Omega)$-norms are derived for the semidiscrete FE scheme for cases with smooth and nonsmooth initial data. The main tool of our analysis is based on a repeated use of an integral operator and use of a $t^m$ type of weights to take care of the singular behavior at $t = 0$. The generalized Leibniz formula for fractional derivatives is found to play a key role in our analysis.

Key words. Fractional diffusion equation, finite element approximation, energy argument, error analysis, nonsmooth data, generalized Leibniz formula.

AMS subject classifications. 65M60, 65M12, 65M15

1. Introduction. In this paper, we aim to investigate the error analysis via energy arguments of a semidiscrete Galerkin finite element (FE) method for time-fractional diffusion problems of the form:

$$u'(x,t) + \partial_1^{1-\alpha} Lu(x,t) = f(x,t) \quad \text{in } \Omega \times (0,T], \quad (1.1a)$$

$$u(x,t) = 0 \quad \text{on } \partial \Omega \times (0,T], \quad (1.1b)$$

$$u(x,0) = u_0(x) \quad \text{in } \Omega, \quad (1.1c)$$

where $Lu = -\text{div}(a(x)\nabla u)$, $\Omega$ is a bounded, convex polygonal domain in $\mathbb{R}^2$ with boundary $\partial \Omega$, $f$, $a$ and $u_0$ are given functions defined on their respective domains. Here, $u'$ is the partial derivative of $u$ with respect to time and $\partial_1^{1-\alpha} := R^{1-\alpha}$ is the Riemann–Liouville time-fractional derivative defined by: for $0 < \alpha < 1$,

$$\partial_1^{1-\alpha} \varphi(t) := \frac{\partial}{\partial t} \mathcal{I}^\alpha \varphi(t) := \frac{\partial}{\partial t} \int_0^t \omega_\alpha(t-s) \varphi(s) \, ds \quad \text{with} \quad \omega_\alpha(t) := \frac{t^{\alpha-1}}{\Gamma(\alpha)}, \quad (1.2)$$

($\mathcal{I}^\alpha$ is the Riemann–Liouville time-fractional integral). As $\alpha \to 1^-$, $\partial_1^{1-\alpha} (Lu)$ converges to $Lu$, and thus, problem (1.1) reduces to the classical diffusion equation.

Throughout the paper, we assume that the source term $f$ and the diffusivity coefficient function $a$ are sufficiently regular and

$$0 < a_{\text{min}} \leq a(x) \leq a_{\text{max}} < \infty \quad \text{on } \overline{\Omega}. \quad (1.3)$$

Several numerical techniques for the problem (1.1) (with constant diffusivity coefficient) in one and several space variables have been proposed with various types of spatial discretizations including finite difference, FE or spectral element methods, see [2, 10, 9, 18]. For the time discretization, different time-stepping schemes (implicit
and explicit) have been investigated including finite difference, convolution quadrature, and discontinuous Galerkin methods, see [3] [4] [5] [17] [18] [20] [22]. The error analyses in most studies in the existing literature typically assume that the solution \( u \) of (1.1) is sufficiently regular including at \( t = 0 \), which is not practically the case. Indeed, assuming high regularity on \( u \) imposes additional compatibility conditions on the given data, which are not reasonable in many cases.

Though the numerical approximation of the solution \( u \) of (1.1) was considered by many authors over the last decade, the optimality of the estimates with respect to the solution smoothness expressed through the problem data, \( f \) and \( u_0 \), was considered in a few papers for the case of constant diffusivity and quasi-uniform FE meshes. Obtaining sharp error bounds under reasonable regularity assumptions on \( u \) has proved challenging. McLean and Thomée [15] established the first optimal \( L^2(\Omega) \)-error estimates for the Galerkin FE solution of (1.1) with respect to the regularity of initial data. More precisely, for \( t \in (0,T] \), convergence rates of order \( t^{\alpha(3-\delta)/2} h^2 \) (\( h \) denoting the maximum diameter of the spatial mesh elements) were proved assuming that the initial data \( u_0 \in \dot{H}^\delta(\Omega) \) for \( \delta = 0, 2 \) (see, Section 2 for the definition of these spaces). The proof was based on some refined estimates of the Laplace transform in time for the error. In [16] and by using a similar approach, the same authors derived \( O(t^{-\alpha(2-\delta)/2} h^2 \log h^2) \) convergence rates in the stronger \( L^\infty(\Omega) \)-norm. For \( \delta = 0 \), \( u_0 \) assumed to be in \( L^\infty(\Omega) \), while \( u_0 \in C^1(\Omega) \cap C_0(\Omega) \) for \( \delta = 2 \). Recently, McLean and Mustapha [14] studied the error analysis of a first order semidiscrete time-stepping scheme for problem (1.1) with \( f \equiv 0 \) allowing nonsmooth \( u_0 \), using discrete Laplace transform technique. Since standard energy arguments are used heavily in the error analysis of Galerkin FEs for classical diffusion equations, it is more pertinent to extend the analysis to these time-fractional order diffusion problems with a variable diffusivity. Since \( t^m \) and \( \partial_t^\alpha \mathcal{L} \) do not commute, extending these arguments to problem (1.1) is not a straightforward task, especially in the case of nonsmooth \( u_0 \).

The main motivation of this work is to derive optimal error estimates of the semidiscrete Galerkin FE method for the problem (1.1) for both smooth and nonsmooth initial data using energy arguments. Since the solution \( u \) has limited smoothing property [12], a repeated use of the integral operator like \( \mathcal{L}\phi(t) := \int_0^t \phi(s) \, ds \) (see [6] [7]) along with \( t^m \) type weights is an essential tool to provide optimal error estimates. Earlier, for smooth initial data (\( u_0 \in \dot{H}^2(\Omega) \)), error analysis of the Galerkin FE method for the fractional diffusion equation (1.1) was considered in [18] using energy arguments and they have derived quasi-optimal error estimates of order \( O(h^2 \log h) \) in \( L^\infty(L^2) \)-norm. Below, we briefly summarize our main results obtained in this article: for \( t \in (0,T] \) and when \( u_0 \in \dot{H}^\delta(\Omega) \) with \( 0 \leq \delta \leq 2 \),

- **a priori** optimal error estimate in \( L^2(\Omega) \)-norm of order \( t^{-\alpha(2-\delta)/2} h^2 \) is established, see Theorem 4.4. Consequently, for a quasi-uniform FE mesh, an \( O(t^{-\alpha(2-\delta)/2} h) \) error estimate in \( H^1(\Omega) \)-norm is obtained, see Remark 4.6. By dropping the quasi-uniformity mesh assumption, we showed that this error bound remains valid for \( 0 \leq \delta \leq 1 \), see Theorem 5.3. However, for \( 1 < \delta \leq 2 \), we derived an \( O(t^{-\alpha(1-\delta)/2} \max\{1, (h/t)^{-\alpha(1-\delta)/2}\}) \) error estimate, which is reduced to \( t^{-\alpha(2-\delta)/2} h^2 \) for \( t \geq Ch \).

- For \( \Omega \subset \mathbb{R}^2 \), and assuming that \( u_0 \in L^\infty(\Omega) \) and the FE mesh is quasi-uniform, a quasi-optimal error estimate of order \( t^{-\alpha(3-\delta)/2} \ln h^{5/2} h^2 \) in the stronger \( L^\infty(\Omega) \)-norm is proved, see Theorem 5.5.

The proposed technique has several attractive features. Some of these are: (1) allowing variable coefficients, and smooth and nonsmooth initial data in the error analysis,
the quasi-uniform FE mesh assumption is not required to show the convergence results in $H^m$-norm for $m = 0, 1$, and (3) the proposed technique can be applied to other fractional model problems with smooth and nonsmooth initial data. For instance, we discussed in Section 6 the extension of the achieved error bounds in Theorems 4.4, 5.3 and 5.5 to the time-fractional diffusion equation: for $0 < \alpha < 1$,

$$C\frac{\partial}{\partial t}^\alpha u(x, t) + Lu(x, t) = f(x, t),$$

(1.4)

where $C\frac{\partial}{\partial t}^\alpha v(t) := T^{1-\alpha}v'(t)$ is the time-fractional Caputo derivative. The error estimate in Theorem 4.4 provides an improvement of the result obtained by Jin et al. [8, Theorem 3.7], where the error analysis of the lumped mass FE method was also considered. For constant diffusivity and under the assumption that the mesh is quasi-uniform, the FE error bound in [8] involves a logarithmic factor which was derived using a semigroup approach. The error analysis approach in this work can also be used to investigate the error estimates for the FE method applied to the time-fractional Rayleigh-Stokes problem (described by the time-fractional differential equation) presented in the recent work [1], which is close to the one in [15].

Outline of the paper. In Section 2, we recall some smoothness properties of the solution $u$, we also state and derive some technical results. In Section 3, we introduce our semidiscrete FE scheme and recall some FE error results. We claim that a direct application of energy arguments to problem (1.1) does not lead to optimal convergence rates even when the initial data $u_0 \in H^2(\Omega)$. In Section 4 for both smooth and nonsmooth initial data, we derive error estimates for the FE problem in the $L^2(\Omega)$-norm, see Theorem 4.4. The generalized Leibniz formula is an essential ingredient in our error analysis. In Section 5 a superconvergence result in the $H^1$-norm is obtained, see Theorem 5.2 and as a consequence, an optimal gradient FE error estimate in the $L^2(\Omega)$-norm is derived in Theorem 5.3 and a quasi-optimal FE error bound in the $L^\infty(\Omega)$-norm is achieved for $\Omega \subset \mathbb{R}^2$, see Theorem 5.5. Particularly relevant to this a priori error analysis is the appropriate use of several properties of the time-fractional integral and derivative operators. In Section 6 we show that the achieved error estimates for problem (1.1) are valid for the FE discretization of the fractional diffusion model (1.4). Numerical tests are presented in Section 7 to confirm some of our theoretical findings. Throughout the paper, $C$ is a generic positive constant that may depend on $\alpha$ and $T$, but is independent of the spatial mesh element size $h$.

2. Regularity and technical results. Smoothness properties of the solution $u$ of the fractional diffusion problem (1.1) play a key role in the error analysis of the Galerkin FE method, particularly, since $u$ has singularity near $t = 0$, even for smooth given data. Below, we state the required regularity results for problem (1.1) in terms of the initial data $u_0$ and the source term $f$. Over the convex domain $\Omega$, by combining the results of Theorems 4.1, 4.2 and 5.6 in [12], for $0 \leq r, \mu \leq 2$, we obtain

$$t^\ell\|u^{(\ell)}(t)\|_{r+\mu} \leq C t^{-\alpha \mu/2} \left[\|u_0\|_r + (1 + t^{\alpha \mu/2}) \sum_{m=0}^{t+1} \int_0^t s^m \|f^{(m)}(s)\|_r \ ds\right]$$

(2.1)

$$\leq C (1 + T^{\alpha \mu/2}) t^{-\alpha \mu/2} \int_0^T s^m \|f^{(m)}(s)\|_r \ ds,$$

for $\ell \in \{0, 1\}$, where $\|\cdot\|_r$ denotes the norm on the Hilbert space $H^r(\Omega)$ defined by

$$\|v\|^2_r = \|A^{r/2}v\|^2 = \sum_{j=1}^{\infty} \lambda_j^r (v, \phi_j)^2,$$
where \(\{\lambda_j\}_{j=1}^\infty\) (with \(0 < \lambda_1 < \lambda_2 < \ldots\)) are the eigenvalues of the operator elliptic operator \(L\) (subject homogeneous Dirichlet boundary conditions) and \(\{\phi_j\}_{j=1}^\infty\) are the associated orthonormal eigenfunctions. Noting that, \(H^r(\Omega) = H^r(\Omega)\) for \(0 \leq r < 1/2\), and for convex polygonal domains, \(H^r(\Omega) = H^r(\Omega) \cap H_0^1(\Omega)\) for \(1/2 < r < 5/2\), where \(H^r(\Omega)\) (with \(H^r(\Omega) = L^2(\Omega)\)) is the standard Sobolev space.

Next, we state the positivity properties of the fractional operators \(I^\alpha\) and \(\partial_t^{1-\alpha}\), and derive some technical results that will be used in the subsequent sections. By [19, Lemma 3.1(ii)] and since the bilinear form \(A(\cdot, \cdot)\) associated with the operator \(L\) (that is, \(A(v, w) = (a\nabla v, \nabla w)\)) is symmetric positive definite on the Sobolev space \(H_0^1(\Omega)\), it follows that for piecewise time continuous functions \(\varphi : [0, T] \rightarrow H_0^1(\Omega)\),

\[
\int_0^T A(I^{\alpha}\varphi, \varphi) \, dt \geq \cos(\alpha\pi/2) \int_0^T \|\sqrt{a}\nabla I^{\alpha/2}\varphi\|^2 \, dt \geq 0 \quad \text{for } 0 < \alpha < 1, \quad (2.2)
\]

where \(\|\varphi\| := \sqrt{(\varphi, \varphi)}\) denotes the \(L^2\)-norm.

By [13, Lemma A.1] and again since the bilinear form \(A(\cdot, \cdot)\) is symmetric positive definite, the following holds: for \(W^{1,1}(0, T; H_0^1(\Omega))\),

\[
\int_0^T A(\partial_t^{1-\alpha}\varphi(t), \varphi(t)) \, dt \geq \frac{1}{2} \sin(\alpha\pi/2) T^{\alpha-1} \int_0^T \|\sqrt{a}\nabla \varphi(t)\|^2 \, dt. \quad (2.3)
\]

The next lemma will be used frequently in our convergence analysis. In the proof, we use the following integral inequality: if for any \(\tau \in (0, t)\), \(|\phi(\tau)|^2 \leq |\phi(0)|^2 + 2 \int_0^\tau |\phi(s)||\psi(s)| \, ds\), then \(|\phi(t)| \leq |\phi(0)| + \int_0^t |\psi(s)| \, ds\).

**Lemma 2.1.** Let \(\kappa \in (0, 1)\) and let \(B^\alpha = \partial_t^{1-\alpha}\) or \(B^\alpha = I^\alpha\). Assume that

\[
\kappa(v(t), \chi) + (1-\kappa)(v'(t), \chi) + A(B^\alpha v(t), \chi) = (w(t), \chi), \quad \forall \chi \in V_h, \quad (2.4)
\]

for \(t \in (0, T]\). Then

\[
\kappa \int_0^t \|v\|^2 \, ds + (1-\kappa)\|v(t)\|^2 \leq (1-\kappa) \left(\|v(0)\| + \int_0^t \|w\| \, ds\right)^2 + \kappa \int_0^t \|w\|^2 \, ds.
\]

**Proof.** Choose \(\chi = v\) in (2.4), and then, integrate over the interval \((0, t)\) to obtain

\[
2\kappa \int_0^t \|v\|^2 \, ds + (1-\kappa)\|v(t)\|^2 - \|v(0)\|^2 \leq 2 \int_0^t A(B^\alpha v, v) \, ds = 2 \int_0^t (w, v) \, ds.
\]

Now, use the positivity properties of \(I^\alpha\) in (2.2) and of \(\partial_t^{1-\alpha}\) in (2.3) to find that

\[
2\kappa \int_0^t \|v\|^2 \, ds + (1-\kappa)\|v(t)\|^2 \leq (1-\kappa)\|v(0)\|^2 + 2 \int_0^t \|w\| \|v\| \, ds.
\]

For \(\kappa = 0\), an application of the integral inequality (stated above) yields the desired inequality. However, for \(\kappa = 1\), we use the inequality \(2\|w\| \|v\| \leq \|w\|^2 + \|v\|^2\) and the desired inequality follows after simplifying. \(\Box\)

**3. Semi-discrete FE method.** This section focuses on a semidiscrete Galerkin FE scheme for problem (1.1). To define the scheme, let \(T_h\) be a family of regular triangulations (made of simplexes \(K\)) of the domain \(\Omega\) and let \(h = \max_{K \in T_h} (\text{diam} K)\), where \(h_K\) denotes the diameter of the element \(K\). The FE space \(V_h\) on \(T_h\) is given by

\[
V_h = \{v_h \in C^0(\Omega) : v_h|_K \text{ is linear for all } K \in T_h \text{ and } v_h|_{\partial\Omega} = 0\}.
\]
The weak formulation for problem (1.1) is to find \( u : (0, T) \rightarrow H^1_0 \) such that
\[
(u', v) + A(\partial_t^{1-\alpha} u, v) = (f, v) \quad \forall v \in H^1_0
\] (3.1)
with given \( u(0) = u_0 \). Thus, the standard semidiscrete FE formulation for (1.1) is to seek \( u_h : (0, T) \rightarrow V_h \) such that
\[
(u_h', v_h) + A(\partial_t^{1-\alpha} u_h, v_h) = (f, v_h) \quad \forall v_h \in V_h
\] (3.2)
with given \( u_h(0) \in V_h \) to be defined later.

To derive \textit{a priori} error estimates for the FE scheme (3.2), we split the error
\[
e := (u - R_h u) - (u_h - R_h u) =: \rho - \theta,
\]
where the Ritz projection \( R_h : H^1_0(\Omega) \rightarrow V_h \) is defined by the following relation:
\[
A(R_h v - v, \chi) = 0 \quad \text{for all} \quad \chi \in V_h. \quad \text{For} \quad t \in (0, T], \quad \text{the projection error} \ \rho(t) \ \text{satisfies the following estimates} \ [21]: \ \text{for} \ \ell = 0, 1,
\]
\[
\|\rho(t)\|_j \leq C h^{m-j} \|u(t)\|_m, \quad j = 0, 1, \quad m = 1, 2. \quad (3.3)
\]
and hence, by using the regularity property in (2.1), we observe
\[
\|\rho(t)\| \leq C h^{m-t-\max\{0,\alpha(m-\delta)/2\}} d_\delta(u_0, f), \quad \text{for} \ \ 0 \leq \delta \leq 2. \quad (3.4)
\]
Next, we show that a direct application of energy arguments to problem (1.1) does not yield satisfactory results due to the low regularity of the continuous solution. From (3.1) and (3.2), the decomposition \( e = \rho - \theta \), and the property of the elliptic projection, we obtain the equation in \( \theta \) as
\[
(\theta', \chi) + A(\partial_t^{1-\alpha} \theta, \chi) = (\rho', \chi) \quad \forall \chi \in V_h. \quad (3.5)
\]
Then, the following result holds, whose proof can be found in [15, 18].

**Theorem 3.1.** For \( t \in (0, T] \), we have
\[
\|u(t) - u_h(t)\| \leq \|\theta_0\| + \int_0^t \|\rho'(s)\| \, ds + \|\rho(t)\|. \quad (3.6)
\]

Noting that, the solution \( u \) of problem (1.1) has singularity near \( t = 0 \). For instance, if \( f \equiv 0 \) and \( u_0 \in H^2(\Omega) \), then by (3.3) and the regularity property
\[
\|u'(t)\|_m \leq C t^{\alpha(1-m/2)-1} \|u_0\|_2 \quad \text{for} \ m = 1, 2,
\]
\[
\int_0^t \|\rho'(s)\| \, ds \leq C \int_0^t \|u'(s)\|_1 \, ds + C \int_0^t \|u'(s)\|_2 \, ds
\]
\[
\leq C \left( h \int_0^t s^{\alpha/2-1} \, ds + h^2 \int_0^t s^{-1} \, ds \right) \|u_0\|_2
\]
\[
\leq C \left( h^{\alpha/2} + h^2 |\log \epsilon| \right) \|u_0\|_2 \leq C h^2 |\log h| \|u_0\|_2 \quad \text{for} \ \epsilon = h^{2/\alpha}.
\]
This leads to a quasi-optimal \( O(h^2 |\log h|) \) convergence for the spatial discretization by linear FEs. To achieve an optimal \( O(h^2) \) convergence, a stronger regularity assumption on \( u \) is required and that, in turn imposes severe restrictions on initial data. Thus, the upper bound in Theorem 5.1 is not sharp even for the case of smooth initial data, that is, \( H^2 \)-regularity on the initial data \( u_0 \) is not sufficient to get an optimal \( O(h^2) \) convergence rate. Furthermore, it is clear that this upper bound is not suitable for the case of nonsmooth initial data. Therefore, we propose in the next section an approach via delicate energy arguments that provides valid error bounds for the problem with both smooth and nonsmooth initial data.
4. \( L^2(\Omega) \)-error estimates. For convenience, we introduce the notations:
\[
\Theta_i(t) := t^i\theta(t) \quad \text{and} \quad \dot{\Theta}_i(t) := t^{i'}\theta'(t) \quad \text{for} \quad i = 1, 2.
\]
In the next lemma, based on the generalized Leibniz formula for fractional derivatives, we state and show some identities for our subsequent use.

**Lemma 4.1.** For \( 0 < \alpha < 1 \), the followings hold:

(a) \( t\partial_1^{1-\alpha}\theta = \partial_1^{1-\alpha}\Theta_1 - (1 - \alpha)T^\alpha\theta \),

(b) \( tT^\alpha\theta = T^\alpha\Theta_1 + \alpha T^{1+\alpha}\theta \).

*Proof.* The first identity follows from the fractional Leibniz formula. To show the second identity, noting first that \( \partial_1^{1-\alpha}\Theta_1 = T^\alpha\Theta_1^{(0)} = T^\alpha\theta + \dot{T}^\alpha\Theta_1 \). Hence, by (a),
\[
t\partial_1^{1-\alpha}\theta = T^\alpha\dot{\Theta}_1(t) + \alpha T^{1+\alpha}\theta(t). \tag{4.1}
\]

Now, we replace \( \theta \) by \( T\theta \) in (4.1) to obtain the second identity in the lemma.

Next, we derive an upper bound of \( \Theta_1 \). To do so, we let \( u_h(0) = P_hu_0 \), where \( P_h : L^2(\Omega) \to V_h \) denotes the \( L^2 \)-projection defined by \( (P_hv, \chi) = 0 \) for all \( \chi \in V_h \).

**Lemma 4.2.** Let \( u_h(0) = P_hu_0 \). Then, we have
\[
\int_0^t \|\Theta_1\|^2 \, ds \leq 3 \int_0^t \left( s^2\|\rho\|^2 + 2(I\|\rho\|)^2 \right) \, ds.
\]

*Proof.* We integrate (3.5) over the time interval \((0, t)\) and obtain
\[
(\theta, \chi) + A(T^\alpha\theta, \chi) = (\rho + c(0), \chi) \quad \forall \chi \in V_h. \tag{4.2}
\]
But \((c(0), \chi) = 0\) because \( u_h(0) = P_hu_0 \). Therefore,
\[
(\theta, \chi) + A(T^\alpha\theta, \chi) = (\rho, \chi) \quad \forall \chi \in V_h. \tag{4.3}
\]
Multiply by \( t \) and use \( tT^\alpha\theta = T^\alpha\Theta_1 + \alpha T^{1+\alpha}\theta \) by Lemma 4.1(b) to find that
\[
(\Theta_1, \chi) + A(T^\alpha\Theta_1, \chi) = t(\rho, \chi) - \alpha A(T^{1+\alpha}\theta, \chi) \quad \forall \chi \in V_h.
\]
However, from (4.3), we get
\[
A(T^{1+\alpha}\theta, \chi) = (I(\rho - \theta), \chi) \quad \forall \chi \in V_h, \tag{4.4}
\]
and thus,
\[
(\Theta_1, \chi) + A(T^\alpha\Theta_1, \chi) = t(\rho, \chi) - \alpha(I(\rho - \theta), \chi) \quad \forall \chi \in V_h. \tag{4.5}
\]
Consequently, an application of Lemma 2.1 (with \( \kappa = 1 \)) yields
\[
\int_0^t \|\Theta_1\|^2 \, ds \leq \int_0^t \|s\rho - \alpha I(\rho - \theta)\|^2 \, ds \leq 3 \int_0^t \left( s^2\|\rho\|^2 + \|I\rho\|^2 + \|I\theta\|^2 \right) \, ds. \tag{4.6}
\]
To complete our proof, we rewrite (4.4) as
\[
(I\theta, \chi) + A(I^\alpha(I\theta), \chi) = (I\rho, \chi) \quad \forall \chi \in V_h,
\]
Again, an application of Lemma 2.1 (with \( \kappa = 1 \)) shows
\[
\int_0^t \|I\theta\|^2 \, ds \leq \int_0^t \|I\rho\|^2 \, ds. \tag{4.7}
\]
Substitute (4.7) in (4.6) yields the desired bound. □

An upper bound of the term \( \rho \) will complete the rest of the proof. Again, for convenience, we introduce the following notation

\[
B_1(t) := \int_0^t \left( s^4 \| \rho'(s) \|^2 + s^2 \| \rho(s) \|^2 + 2 \| \mathcal{I} \rho(s) \|^2 \right) ds.
\] (4.8)

For later use, by using the projection error estimates in (3.4) with \( \ell = 0, 1 \) and \( m = 2 \) for upper bounds of \( \rho \) and \( \rho' \), and then integrating, we find that for \( t \in (0, T] \),

\[
B_1(t) \leq C h^4 t^{3-\alpha(2-\delta)} d_3^2(u_0, f), \quad \text{for } 0 \leq \delta \leq 2. \tag{4.9}
\]

**Lemma 4.3.** Let \( u_h(0) = P_h u_0 \). Then, the following estimate holds

\[
\| \theta(t) \|^2 \leq C t^{-3} B_1(t), \quad \text{for } t \in (0, T].
\]

**Proof.** We multiply (3.5) by \( t^2 \) so that

\[
(\dot{\Theta}_2, \chi) + A(t^2 \dot{\Theta}^{1-\alpha}_2, \chi) = (t^2 \rho', \chi),
\] (4.10)

where \( \dot{\Theta}_2 = t^2 \theta' \). From the fractional Leibniz formula, we have

\[
t^2 \dot{\Theta}^{1-\alpha}_2 = \partial_t^{1-\alpha} \Theta_2 - 2(1-\alpha)t \mathcal{I}^{\alpha} \theta + \alpha(1-\alpha) \mathcal{I}^{1+\alpha} \theta.
\]

Hence, we rearrange (4.10) as

\[
(\dot{\Theta}_2, \chi) + A(\partial_t^{1-\alpha} \Theta_2, \chi) = (t^2 \rho', \chi) + (1-\alpha) \left( 2t A(\mathcal{I}^{\alpha} \theta, \chi) - \alpha A(\mathcal{I}^{1+\alpha} \theta, \chi) \right),
\] (4.11)

and then, by equations (4.13) and (4.14),

\[
(\Theta'_2, \chi) + A(\partial_t^{1-\alpha} \Theta_2, \chi) = (t^2 \rho' + 2\alpha \Theta_1 + (1-\alpha)(2t \rho - 2\alpha \mathcal{I}(\rho - \theta)), \chi). \tag{4.12}
\]

Hence, by Lemma 2.1 (with \( \kappa = 0 \)), we obtain

\[
\| \Theta_2(t) \|^2 \leq \int_0^t \left( s^2 \| \rho'(s) \|^2 + 2s \| \rho(s) \|^2 + 2 \| \Theta_1(s) \|^2 + \| \mathcal{I}(\rho - \theta) \|^2 \right) ds,
\]

and thus, an application of the Cauchy-Schwarz inequality yields

\[
\| \Theta_2(t) \|^2 \leq C t \int_0^t \left( s^4 \| \rho'(s) \|^2 + s^2 \| \rho(s) \|^2 + \| \Theta_1(s) \|^2 + \| \mathcal{I}\rho \|^2 + \| \mathcal{I}\theta \|^2 \right) ds.
\]

Therefore, by using the identity \( \theta(t) = t^{-2} \Theta_2(t) \), the inequality in (4.7) and Lemma 2.2 will complete the rest of the proof. □

In the next theorem, we derive optimal convergence results of the FE scheme (3.2) in the \( L^2 \)-norm for both smooth and nonsmooth initial data \( u_0 \). For \( u_0 \in \dot{H}^4(\Omega) \) with \( 0 \leq \delta \leq 2 \), we show that the error is bounded by \( C h^4 t^{-\alpha(2-\delta)/2} \) for each \( t \in (0, T] \). Recall that, \( \dot{H}^4(\Omega) = \{ v \in H^4(\Omega) : v = 0 \text{ on } \partial \Omega \} \) for \( 1/2 \leq \delta \leq 2 \), while \( \dot{H}^4(\Omega) = H^4(\Omega) \) for \( 0 \leq \delta < 1/2 \). Noting that, in the limiting case \( \alpha \rightarrow 1^- \), we recover the convergence rates for the parabolic equation \( u' - Lu = f \).
Theorem 4.4. Let $u$ and $u_h$ be the solutions of (1.1) and (2.2), respectively, with $u_h(0) = P_h u_0$. Then, for $u_0 \in H^3(\Omega)$,
\[
\| (u - u_h)(t) \| \leq C h^2 t^{-\alpha(2-\delta)/2} d_{\delta}(u_0, f) \quad \text{for} \quad t \in (0, T] \quad \text{with} \quad 0 \leq \delta \leq 2.
\]

Proof. The desired result follows from the decomposition $u - u_h = \rho - \theta$, the estimate of $\theta$ in Lemma 4.3, the bound in (3.3), and the estimate of $\rho$ in (3.4).

Remark 4.5. In the proof of the above theorem, we used (4.9) which follows from (4.2, Theorems 4.4 and 5.6). To estimate the first two terms, we use (3.3) (with $\ell, m = 1$) and the following regularity property (which follows from (3.3) with $\ell, m = 1$)
\[
\| u'(t) \|_1 \leq C t^{\alpha(\delta-1)/2} \tilde{d}_{\delta}(u_0, f), \quad t \in (0, T], \quad \text{for} \quad 1 < \delta \leq 2,
\]
where $\tilde{d}_{\delta}(u_0, f) = \| u_0 \|_\delta + \sum_{m=0}^2 t^m \| f^{(m)} \|_{L^\infty(H^1)}$, we arrive to
\[
B_1(t) \leq C t^3 h^2 t^\delta \tilde{d}_{\delta}(u_0, f) + C t^3 \| \rho(t_h) \|^2, \quad \text{for} \quad 1 < \delta \leq 2.
\]
However, by (3.3) and the inequality $t^\alpha \leq C t^{\alpha(1-\delta)/2} (t_h/t)^{\alpha(\delta-1)/2}$ for $1 < \delta \leq 2$, we find that
\[
\| \rho(t_h) \| \leq C h^2 t^\delta \tilde{d}_{\delta}(u_0, f) \leq C h t^\alpha \tilde{d}_{\delta}(u_0, f).
\]

Therefore,
\[
B_1(t) \leq C h^2 t^\delta \tilde{d}_{\delta}(u_0, f), \quad \text{for} \quad 1 < \delta \leq 2.
\]
Consequently, by using the above bound of $B_1$ in Theorem 4.4, we get the error estimate below that will be used in the forthcoming section to show the convergence of the gradient FE solution. For $t \in (0, T]$,
\[
\| e(t) \| \leq C h t^{\alpha(\delta-1)/2} D_{\delta, \alpha}(u_0, f, h/t),
\]
where
\[
D_{\delta, \alpha}(u_0, f, h/t) = \tilde{d}_{\delta}(u_0, f) \times \begin{cases} 1 & \text{for} \quad 0 \leq \delta \leq 1, \\ (t_h/t)^{\alpha(\delta-1)/2} & \text{for} \quad 1 < \delta \leq 2. \end{cases}
\]

Remark 4.6. Under the quasi-uniformity condition on $V_h$, for $t \in (0, T]$, from the decomposition $u - u_h = \rho - \theta$, the inverse inequality, the estimate of $\theta$ in Lemma 4.3 and the estimate $\| \rho(t) \|_1 \leq C \| u(t) \|_2 \leq C t^{\alpha(2-\delta)/2} \tilde{d}_{\delta}(u_0, f)$ (follows from the
Ritz projection bound in (4.3) (with \(j = 1\) and \(m = 2\)) and the regularity property [2.7], we obtain the following optimal error estimate in the \(H^1(\Omega)\)-norm:

\[
\|\nabla (u - u_h)(t)\| \leq C h^{-\alpha(2-\delta)/2} d_{\delta}(u_0, f) \quad \text{for } t \in (0, T) \quad \text{with } 0 \leq \delta \leq 2.
\]

By removing the quasi-uniformity mesh assumption, this error bound remains valid for \(0 \leq \delta \leq 1\), see Theorem 4.3.

Remark 4.7. For smooth initial data \(u_0 \in \tilde{H}^2(\Omega)\), one may choose \(u_h(0) = R_h u_0\). An optimal convergence rate can be shown by following the proof of Theorem 4.4 line-by-line, where the term \(\rho\) in Lemma 4.2 should be replaced with \(\tilde{\rho} := \rho + e(0)\).

5. \(H^1(\Omega)\)- and \(L^\infty(\Omega)\)-error estimates. In this section, for each \(t \in (0, T)\), we show optimal convergence results for the gradient FE error in the \(L^2(\Omega)\)-norm, and quasi-optimal error bounds for the FE error in the \(L^\infty(\Omega)\)-norm, for both smooth and nonsmooth initial data \(u_0\). We start our analysis by deriving an upper bound of \(\nabla \Theta_1\).

Lemma 5.1. For \(0 \leq \delta \leq 2\) and for \(t \in (0, T)\), we have

\[
\int_0^t \|\nabla \Theta_1\|^2 ds \leq C h^4 t^{3-\alpha(3-\delta)} d_{\delta}^2(u_0, f).
\]

Proof. Multiplying (4.5) by \(t\) and then using the identity \(t \partial_t^{1-\alpha} \theta = \mathcal{I}^\alpha \dot{\theta}_1 + \alpha \mathcal{I}^\alpha \theta\) (Lemma 4.1 (a)), we obtain

\[
(\dot{\Theta}_1, \chi) + A(\partial_t^{1-\alpha} \Theta_1, \chi) = (t \rho', \chi) + (1 - \alpha) A(\mathcal{I}^\alpha \theta, \chi).
\]

Then, a use of (4.3) yields after simplifying

\[
(\Theta_1', \chi) + A(\partial_t^{1-\alpha} \Theta_1, \chi) = (t \rho', \chi) + \alpha (e, \chi) + (\rho, \chi).
\]

Now, set \(\chi = \Theta_1\) in (5.2), integrate the resulting equation over \((0, t)\), and use the positivity property of \(\partial_t^{1-\alpha}\) in (2.3), to find that

\[
\|\Theta_1(t)\|^2 + \frac{1}{2} \sin(\alpha \pi/2) t^{\alpha-1} \int_0^t \|\sqrt{a} \nabla \Theta_1\|^2 ds \leq \int_0^t (s \|\rho'\| + \|e\| + \|\rho\|) \|\Theta_1\| ds.
\]

By the integral inequality (stated before Lemma 2.1), we observe

\[
\frac{1}{2} \sin(\alpha \pi/2) t^{\alpha-1} \int_0^t \|\sqrt{a} \nabla \Theta_1\|^2 ds \leq \frac{1}{4} \left( \int_0^t (s \|\rho'\| + \|e\| + \|\rho\|) ds \right)^2.
\]

Therefore, the desired estimate follows from this bound, the error projection in (3.4) (for \(\ell = 0, 1\) and with \(m = 2\)), the achieved convergence results in Theorem 4.4 and the assumption on the diffusivity coefficient in (1.3).

In the next theorem, we derive an error bound for \(\nabla \theta(t)\) in the \(L^2(\Omega)\)-norm.

Theorem 5.2. For \(0 \leq \delta \leq 2\), we have

\[
\|\nabla \theta(t)\|^2 \leq C h^4 t^{3-\alpha(3-\delta)} d_{\delta}^2(u_0, f), \quad \text{for } t \in (0, T).
\]

Proof. Apply the operator \(\mathcal{I}^{1-\alpha}\) (1.11), and use the identities \(\mathcal{I}^{1-\alpha} \partial_t^{1-\alpha} \Theta_2 = \Theta_2\) and \(t \mathcal{I}^{\alpha} \theta = \mathcal{I}^{\alpha} \Theta_1 + \alpha \mathcal{I}^{1+\alpha} \theta\) (by Lemma 4.1 (b)) to get

\[
(\mathcal{I}^{1-\alpha} \dot{\Theta}_2, \chi) + A(\Theta_2, \chi) = (\mathcal{I}^{1-\alpha} (t^2 \rho'), \chi) + (1 - \alpha) \left( 2 A(\mathcal{I}^\alpha \Theta_1, \chi) + \alpha A(\mathcal{I}^2 \theta, \chi) \right).
\]
Set $\chi = \hat{\Theta}_2$ follows by integrating the resulting equation from 0 to $t$ to obtain
\[
\int_0^t [(I^{1-\alpha}\hat{\Theta}_2, \hat{\Theta}_2) + A(\Theta_2, \hat{\Theta}_2)] \, ds \\
\leq \int_0^t (I^{1-\alpha}(s^2\rho'), \hat{\Theta}_2) \, ds + (1 - \alpha) \int_0^t A(2I\Theta_1 + \alpha I^2\theta, \hat{\Theta}_2) \, ds.
\]
However, by the continuity property of the operator $I^{1-\alpha}$ ([19, Lemma 3.1]), we have
\[
\left| \int_0^t (I^{1-\alpha}(s^2\rho'), \hat{\Theta}_2) \, ds \right| \leq C \int_0^t (I^{1-\alpha}(s^2\rho'), s^2\rho') \, ds + \int_0^t (I^{1-\alpha}\hat{\Theta}_2, \hat{\Theta}_2) \, ds,
\]
and so,
\[
\int_0^t A(\Theta_2, \hat{\Theta}_2) \, ds \leq C \int_0^t (I^{1-\alpha}(s^2\rho'), s^2\rho') \, ds + (1 - \alpha) \int_0^t A(2I\Theta_1 + \alpha I^2\theta, \hat{\Theta}_2) \, ds.
\]
Using the identity $2I\Theta_1(t) = \Theta_2(t) - I\hat{\Theta}_2$ and the inequality $\int_0^t A(I\hat{\Theta}_2, \hat{\Theta}_2) \, ds \geq 0$, after some simplifications, we conclude that
\[
\alpha \int_0^t A(\Theta_2, \hat{\Theta}_2) \, ds \leq C \int_0^t \|I^{1-\alpha}s^2\rho'\| \|s^2\rho'\| \, ds + \alpha(1 - \alpha) \int_0^t A(I^2\theta, \hat{\Theta}_2) \, ds.
\]
Since
\[
\int_0^t A(\Theta_2, \hat{\Theta}_2) \, ds = \frac{1}{2} \|\sqrt{a} \nabla \Theta_2(t)\|^2 - \frac{1}{2} \int_0^t s \|\sqrt{a} \nabla \Theta_1(s)\|^2 \, ds,
\]
we easily find that
\[
\alpha \|\sqrt{a} \nabla \Theta_2(t)\|^2 \leq 4\alpha \int_0^t s \|\sqrt{a} \nabla \Theta_1(s)\|^2 \, ds + C \int_0^t \|I^{1-\alpha}s^2\rho'\| \|s^2\rho'\| \, ds
\]
\[
+ 2\alpha(1 - \alpha) \int_0^t A(I^2\theta, \hat{\Theta}_2) \, ds. \tag{5.4}
\]
Integrating the first bound in Lemma 5.1 gives
\[
\int_0^t \|\sqrt{a} \nabla \Theta_1\|^2 \, ds \leq Ct \int_0^t \|\nabla \Theta_1\|^2 \, ds \leq Ch^4t^4 - \alpha(3 - \delta)d_3^2(u_0, f). \tag{5.5}
\]
To estimate the second term on the RHS of (5.4), we use the bound of $\rho'$ given in [3.3] (with $m = 2$), the formula
\[
I^\nu (t^{\mu-1}) = t^{\nu+\mu-1} \Gamma(\mu), \quad \text{for } \nu, \mu > 0, \tag{5.6}
\]
and then integrate
\[
\int_0^t (I^{1-\alpha}(s^2\rho'), s^2\rho') \, ds \leq C \int_0^t s^{2-\alpha - \alpha(2 - \delta)/2}s^{1-\alpha(2 - \delta)/2} \, ds d_3^2(u_0, f)
\]
\[
\leq Ch^4t^{4-\alpha - \alpha(2 - \delta)}d_3^2(u_0, f). \tag{5.7}
\]
For the last term on the RHS of (5.3), we apply $\mathcal{I}^{2-\alpha}$ to (1.3) to obtain $A(\mathcal{I}^{2\alpha} \eta, \chi) = (\mathcal{I}^{2-\alpha} e, \chi)$. Hence, integrating by parts, we find that

$$
\int_0^t A(\mathcal{I}^{2\alpha} \Theta, \Theta_2) \, ds = \int_0^t (s^2 \mathcal{I}^{2-\alpha} e, \theta') \, ds
= (\mathcal{I}^{2-\alpha} e(t), \Theta_2(t)) - \int_0^t (2\mathcal{I}^{2-\alpha} e + s \mathcal{I}^{1-\alpha} e, \Theta_1) \, ds.
$$

Then, by using the estimate of $\theta$ in Lemma 4.3, and the estimate of $e$ in Theorem 4.4, we conclude after integrating and using the formula in (5.6), that

$$
\left| \int_0^t A(\mathcal{I}^{2\alpha} \Theta, \Theta_2) \, ds \right| \leq t^2 \|\mathcal{I}^{2-\alpha} e(t)\| \|\theta(t)\| + 2t \int_0^t \|\mathcal{I}^{2-\alpha} e(s) + s \mathcal{I}^{1-\alpha} e(s)\| \|\theta(s)\| \, ds
\leq C h^4 t^{-\alpha - \alpha(2-\delta)} h^2(u_0, f).
$$

A substitution of the estimates (5.3), (5.7) and the above one in (5.4), follows by using (1.3) and the identity $\theta(t) = t^2 \Theta_2$ yield the desired estimate.

Noting that, by using the estimates of $\rho$, $\rho'$ and $e$ from Remark 4.5 in the inequality (5.3), we observe

$$
\int_0^t \|\nabla \Theta_1\|^2 \, ds \leq C h^2 t^{3-\alpha(2-\delta)} D^2 h^2(u_0, f, h/t).
$$

Hence, by following the steps in Theorem 5.2 and using the above bound instead of Lemma 5.1, and the bounds of $\rho'$ and $e$ achieved in Remark 4.3, we deduce that

$$
\|\nabla \theta(t)\|^2 \leq C h^2 t^{-\alpha(2-\delta)} D^2 h^2(u_0, f, h/t).
$$

Therefore, from the inequality $\|\nabla (u_h - u)(t)\| \leq \|\nabla \theta(t)\| + \|\nabla \rho(t)\|$, the above bound, the bound $\eta$ in (3.3) (with $j = 1$ and $m = 2$) and the regularity property (2.1), we have the following result in term of a theorem.

**Theorem 5.3.** Let $u$ and $u_h$ be the solutions of (1.1) and (3.2), respectively, with $u_h(0) = P_h u_0$. For $u_0 \in H^3(\Omega)$, for $t \in (0, T]$, we have

$$
\|\nabla (u - u_h)(t)\| \leq C h t^{-\alpha(2-\delta)/2} \delta^3(u_0, f) \times \begin{cases} 1 & \text{for } 0 \leq \delta \leq 1, \\ \max\{1, (h/t)^{\alpha(\delta - 1)/2}\} & \text{for } 1 < \delta \leq 2. \end{cases}
$$

**Remark 5.4.** The estimate in Theorem 5.2 suggests that one can achieve a higher convergence rate for $\nabla (u_h - u)$ if an improved estimate of the error $\nabla (R_h u - u)$ can be derived. This could be achieved using a superconvergent recovery procedure of the gradient, which is possible on special meshes and for solutions in $H^3(\Omega)$ for each $t \in (0, T]$. Examples of such meshes exhibiting superconvergence property are provided in [11]. Křížek and Neittaanmäki [11] introduced an operator $G_h$ which postprocesses $\nabla R_h u(t)$ with the following properties:

(i) If $u(t) \in H^3(\Omega)$, then $\|\nabla u(t) - G_h (R_h u)(t)\| \leq C h^2 \|u(t)\|_{H^3(\Omega)}$.

(ii) For $\chi \in V_h$, we have $\|G_h(\chi)\| \leq C \|\nabla \chi\|$. 

Now, if $T_h$ is a triangulation of $\Omega$ such that these results are satisfied, then using

$$
\|\nabla (u - u_h)(t)\| \leq \|\nabla (u - G_h (R_h u))(t)\| + \|G_h (R_h u - u_h)(t)\| + \|\nabla \theta(t)\|,
$$

w}
(i) and (ii), Theorem 5.2 and the inequality \( \|u(t)\|_{H^3(\Omega)} \leq C t^{-\alpha(3-\delta)/2} d_\delta(u_0, f) \) for \( 1/2 < \delta \leq 2 \), it is clear that the bound below holds for \( t \in (0, T) \),

\[
\|\nabla (u - u_h)(t)\| \leq C h^2 t^{-\alpha(3-\delta)/2} d_\delta(u_0, f), \quad 1/2 < \delta \leq 2.
\]

For \( t \in (0, T] \), we show in the next theorem that the superconvergence result of \( \nabla \theta \) in Theorem 5.2 can be used to establish an optimal convergence rate (up to a logarithmic factor) in the stronger \( L^\infty(\Omega) \)-norm assuming that the initial data \( u_0 \in \dot{H}^3(\Omega) \) for \( 0 \leq \delta \leq 2 \). Without loss of generality, we assume that \( f \equiv 0 \) in problem (1.1).

**Theorem 5.5.** Let \( u \) and \( u_h \) be the solutions of (1.1) and (3.2), respectively, with \( u_h(0) = P_h u_0 \). Assume that \( u_0 \in \dot{H}^3(\Omega) \cap L^\infty(\Omega) \) for \( 0 \leq \delta \leq 2 \). Under the quasi-uniformity condition on \( V_h \), for \( t \in (0, T] \), we have

\[
\| (u - u_h)(t) \|_{L^\infty(\Omega)} \leq C \ln h \|u\|_{L^2(\Omega)} \| \nabla u(t) \|_{L^p(\Omega)}.
\]

**Proof.** By the Ritz projection error result (see [21]) and the regularity estimate \( \|u(t)\|_{W^{2,p}(\Omega)} \leq C \rho \|L u(t)\|_{L^p(\Omega)} \), for \( 1 < p < \infty \), we have

\[
\|\rho(t)\|_{L^\infty(\Omega)} \leq C \ln h \|u(t)\|_{W^{2,p} \cap L^\infty(\Omega)} \leq C \ln h \|u(t)\|_{L^p(\Omega)}.
\]

Applying the operator \( I^{1-\alpha} \) to both sides of (1.1) (with \( f \equiv 0 \)), we arrive at \( I^{1-\alpha}(u'(t)) = \mathcal{L}(u(t)) \). Insert this in (5.8), then use the Sobolev embedding inequality \( \|v\|_{L^p(\Omega)} \leq C \sqrt{p} \|\nabla v\| \) for \( v \in H^1_0(\Omega) \) with \( 2 \leq p < \infty \) and the regularity property (4.13), yield

\[
\|\rho(t)\|_{L^\infty(\Omega)} \leq C \ln h \|u(t)\|_{L^2(\Omega)}^{3/2} \|I^{1-\alpha}(u'(t))\| \leq C \ln h \|u(t)\|_{L^2(\Omega)}^{3/2} \left( t^{-\alpha(3-\delta)/2} u_0 \right) \quad \text{for} \quad 1 < \delta \leq 2.
\]

To estimate \( \|\rho(t)\|_{L^\infty(\Omega)} \) for \( 0 \leq \delta \leq 1 \), multiplying both sides of (1.1) (with \( f \equiv 0 \)) by \( t \), then applying the operator \( I^{1-\alpha} \) and using the identity in Lemma 4.3 (a) (with \( u \) in place of \( \theta \)), we find that \( I^{1-\alpha} (tu'(t)) = \mathcal{L}(tu(t)) - (1 - \alpha) \mathcal{L} u(t) \). But, since \( \mathcal{L} u(t) = I^{2-\alpha} u'(t) = I^{1-\alpha} [u'(t) - u_h'(t)] \), it follows that

\[
\mathcal{L}(tu(t)) = I^{1-\alpha} [(tu'(t)) + (1 - \alpha) (u(t) - u_h(t))].
\]

Now, multiplying both side of (5.8) by \( t \), then inserting the above bound and using again the Sobolev embedding inequality imply that

\[
t \|\rho(t)\|_{L^\infty(\Omega)} \leq C \ln h \|u(t)\|_{L^2(\Omega)}^{3/2} \left( t^{1-\alpha} \|u(t)\|_1 + t^{-\alpha} u_0 \right) \|
\]

for any \( 2 \leq p < \infty \). Hence, by (4.13) and the inequality \( \|u(t)\|_1 \leq C t^{-\alpha(1-\delta)/2} u_0 \) for \( 0 \leq \delta \leq 1 \) (follows from the regularity property (2.1) with \( f \equiv 0 \)), we obtain

\[
t \|\rho(t)\|_{L^\infty(\Omega)} \leq C t \|u(t)\|_{L^2(\Omega)}^{3/2} \left( t^{-\alpha(3-\delta)/2} u_0 + t^{-\alpha} u_0 \right) \|
\]

for \( 0 \leq \delta \leq 1 \). On the other hand, by the discrete Sobolev inequality and the achieved estimate in Theorem 5.2 we observe that

\[
\|\theta(t)\|_{L^\infty(\Omega)} \leq C \ln h \|\nabla \theta(t)\| \leq C \ln h \|\nabla \theta(t)\|_{L^p(\Omega)} \leq C h^2 t^{-\alpha(3-\delta)} \|u_0\|_\delta \quad \text{for} \quad 0 \leq \delta \leq 2.
\]

Finally, choose \( p = \ln h \), and the desired convergence result follows then from

\[
\|(u_h - u)(t)\|_{L^\infty(\Omega)} \leq \|\theta(t)\|_{L^\infty(\Omega)} + \|\rho(t)\|_{L^\infty(\Omega)}, \quad (5.9), \quad \text{and the above two bounds}.
\]
6. FE error analysis for the problem (1.4). In this section, we justify that the achieved error estimates from the Galerkin FE semidiscrete solution of problem (1.1) are valid for the FE discretization of (1.4). To see this, for each \( t \in (0,T) \), we denote by \( u_h(t) \) to be the Galerkin FE solution of (1.4). Thus, \[
(I^{1-\alpha}u_h',\chi) + A(u_h,\chi) = (f,\chi) \quad \forall \chi \in V_h. \quad (6.1)
\]
As before, we decompose the error as: \( u - u_h := \rho - \theta \). One can check that \( \theta \) satisfies
\[
(I^{1-\alpha}\theta',\chi) + A(\theta,\chi) = (I^{1-\alpha}\rho',\chi) \quad \forall \chi \in V_h. \quad (6.2)
\]
Applying the operator \( I^\alpha \) to both sides and use the property \( I^\alpha I^{1-\alpha} = I \) yield
\[
(I\theta',\chi) + A(I^\alpha\theta,\chi) = (I\rho',\chi) \quad \forall \chi \in V_h.
\]
A time differentiation of both sides shows
\[
(\theta',\chi) + A(I^{1-\alpha}\theta,\chi) = (\rho',\chi) \quad \forall \chi \in V_h. \quad (6.3)
\]
Since the two equations (6.3) and (6.5) are identical, by following the preceding error analysis in Sections 4 and 5, one can show that the achieved error estimates in Theorems 4.4, 5.3 and 5.5 remain valid.

7. Numerical results. In this section, we focus on testing the achieved theoretical convergence results in Theorem 5.5 on fractional model problems of the form (1.1) with different initial data. For the numerical illustration of the error bounds in Theorems 4.4, 5.3 and 5.5 one can follow the convention in [8] Section 6.

We choose \( \mathcal{L} = -\nabla^2 \), \( f = 0 \), \( T = 0.5 \), and \( \Omega = (0,1) \times (0,1) \) in problem (1.1). The orthonormal eigenfunctions and corresponding eigenvalues of \( -\nabla^2 \) are
\[
\phi_{mn}(x,y) = 2 \sin(m\pi x)\sin(n\pi y) \quad \text{and} \quad \lambda_{mn} = (m^2 + n^2)\pi^2 \quad \text{for} \quad m,n = 1,2,\ldots.
\]
Separation of variables yields the series representation solution of problem (1.1): \[
u(x,y,t) = 2 \sum_{m,n=1}^{\infty} (u_{0,mn}) E_{\alpha}(-\lambda_{mn}t^\alpha) \phi_{mn}(x,y), \quad (7.1)
\]
where \( E_{\alpha}(t) := \sum_{p=0}^{\infty} \frac{t^p}{\Gamma(p+1)} \) is the Mittag-Leffler function.

To compute the semidiscrete FE solution \( u_h \), we discretize in time by the mean of generalized Crank-Nicolson scheme (see [17]), this will then define the following fully-discrete scheme: for \( 1 \leq n \leq N \)
\[
\tau_n^{-1}(u_h^n - u_h^{n-1},v_h) + A(I^\alpha \bar{u}_h(t_n) - I^\alpha \bar{u}_h(t_{n-1}),v_h) = 0 \quad \forall v_h \in V_h,
\]
where \( N \) is the number of time mesh subintervals \( (0 = t_0 < t_1 < \ldots < t_N = T) \), \( \tau_n \) is the \( n \)th time step size. Here \( u_h^n \approx u_h(t_n) \) and \( \bar{u}_h(s) = \frac{1}{2}(u_h^s + u_h^{s-1}) \) when \( s \in (t_{j-1},t_j) \) for \( j \geq 2 \), while \( \bar{u}_h(s) = u_h^0 \) on the subinterval \( (0,t_1) \). The modification on the first subinterval ensures that \( \bar{u}_h \) does not depend on \( u_h^0 \) which is necessary for our numerical scheme in cases when \( u_0 \) is not sufficiently regular.

Following the convergence analysis in [17], we concentrate the time step near \( t = 0 \) to compensate for the singular behaviour of the solution \( u \) of problem (1.1). So, we let \( t_n = (n/N)^\gamma T \) for some fixed \( \gamma \geq 1 \) that will be chosen appropriately. For the
spatial partition, let $\mathcal{T}_h$ be a family of uniform (right-angle) triangular mesh of the domain $\Omega$ with diameter $h = \sqrt{2}/M$, see Figure 7.1. For measuring the error in our numerical solution at each time node $t_n$, we let $\mathcal{N}_h$ be the set of all triangular nodes of the mesh family $\mathcal{T}_h$, where the diameter $h_s$ is half the diameter of the finest mesh $\mathcal{T}_h$ in our spatial iterations, for instance, $h_s = \sqrt{2}/128$ in Tables 7.1, 7.2, as well as in Figures 7.2, 7.3. To measure the errors, define the discrete-space maximum norm: $||v|| := \max\{|v(x)|, x \in \mathcal{N}_h\}$. Thus, for large values of $M$, $||u^n_h - u(t_n)||$ approximates the error $||u^n_h - u(t_n)||_{L^\infty(\Omega)}$ for each $1 \leq n \leq N$.

In the three numerical examples below, we choose $\gamma = 1.6$ and refine the time steps so that the spatial errors are dominant. We evaluate the exact solution $u$ of problem (1.1) by truncating the Fourier series in (7.1) after 60 terms.

**Example 1.** Choose the initial data $u_0(x, y) = xy(1-x)(1-y)$, which has the Fourier sine coefficients

$$(u_0, \varphi_{mn}) = 8(1-(-1)^m)(1-(-1)^n)(mn\pi^2)^{-3}, \text{ for } m, n = 1, 2, \ldots.$$ 

The initial data $u_0 \in H^{2+\epsilon}(\Omega)$ for $0 \leq \epsilon < 1/2$ and $\notin H^{2+\epsilon}(\Omega)$ for $\epsilon > 1/2$. Thus, by Theorem 5.3 (\(\delta = 2\)), for each time step $t_n$, we expect convergence of order $t_n^{-\alpha/2}\ln h^{5/2}h^2$ in the $L^\infty(\Omega)$-norm. Indeed, one can show the validity of the convergence results (up to some time logarithmic factor) in Theorem 5.5 for $1 < \delta < 2.5^-$. Hence, the coefficient $t_n^{-\alpha/2}$ can be replaced with $t_n^{-\alpha/4}$. For $\alpha = 0.75$, Figure 7.2 shows how the error varies with $t$ for a sequence of solutions obtained by successively doubling the spatial mesh elements, using a log scale. (The same time mesh with $N = 1000$ subintervals was used in all cases). In Table 7.1 we listed the time-space maximum error and its associated convergence rate (CR), where optimal convergence rates was observed (ignoring the logarithmic factors). Therefore, the coefficient $t_n^{-\alpha/4} = t_n^{-3/16}$ does not have much practical influence on the convergence rates. This is probably due to the fact that $u_0$ is also in $C^2(\Omega) \cap C_0(\overline{\Omega})$, where an $O(h^2\log h^2)$ rate of convergence was proved in [10, Theorem 4.2].

**Example 2.** Choose $u_0(x, y) = xy\chi_{[0,1/2]\times[0,1/2]} + (1-x)yx\chi_{[1/2,1]\times[0,1/2]} + x(1-y)\chi_{[0,1/2]\times(1/2,1)} + (1-x)(1-y)\chi_{(1/2,1)\times[1/2,1]}$ which is less smooth, then the considered $u_0$ in the previous example, where $\chi_D$ denotes the characteristic function.
The error $\|u_h^n - u(t_n)\|$ as a function of $t_n$ for Example 1.

Table 7.1

| $M$ | $\max_{n=1}^N \|u_h^n - u(t_n)\|$ | $CR$ |
|-----|-------------------------------|-----|
| 4   | 1.2759e-02                    |     |
| 8   | 3.3749e-03                    | 1.9186 |
| 16  | 8.7940e-04                    | 1.9402 |
| 32  | 2.2284e-04                    | 1.9805 |
| 64  | 5.6414e-05                    | 1.9819 |

on the domain $D$. One can verify that $u_0$ has the Fourier sine coefficients

$$(u_0, \phi_{mn}) = 2(1 - (-1)^m)(1 - (-1)^n)(mn\pi^2)^{-2}(-1)^{mn}, \quad \text{for } m, n = 1, 2, \ldots$$

The function $u_0 \in H^{1+\epsilon}(\Omega)$ for $0 \leq \epsilon < 1/2$. So, by Theorem 5.5 ($\delta < 1.5$), for each $t_n$, we expect $t_n^{-3\alpha/4} \log h$ convergence rates in the $L^\infty(\Omega)$-norm from the spatial FE discretization. As in Figure 7.2, Figure 7.3 shows how the error varies with $t$ for a sequence of solutions obtained by doubling the spatial mesh elements. (The time mesh with $N = 1300$ subintervals was used in all cases). Table 7.2 provides an alternative view of this data, listing the time-space maximum weighted error $E_\mu := \max_{n=1}^N t_n^{-\mu} \|u_h^n - u(t_n)\|$ and its associated convergence rate $CR$. As expected, ignoring the logarithmic factors, the convergence rate is 2 when $\mu \geq 3\alpha/4 \approx 0.56$, but the rate deteriorates for smaller values of $\mu$ (relatively far from 3$\alpha$/4).

Example 3. Choose $u_0(x, y) = 1$, and so $u_0$ has the Fourier sine coefficients

$$(u_0, \phi_{mn}) = 2(1 - (-1)^m)(1 - (-1)^n)(mn\pi^2)^{-1}, \quad \text{for } m, n = 1, 2, \ldots$$

The initial data function $u_0 \in H^\epsilon(\Omega) \cap L^\infty(\Omega)$ for $0 \leq \epsilon < 1/2$. As in the previous example, Figure 7.4 shows a consistent decaying in the errors by doubling the number of spatial mesh elements. Another observation is the large impact of the very limited regularity of $u_0$ on the errors near $t = 0$ in this example. For better justifications of this, see Table 7.3 where the difference between the maximum error $E_0$ and the weighted error $E_1$ is very substantial, we also observed very good improvements in the convergence rates $CR$, but not yet optimal.
The error $\|u_h^n - u(t_n)\|$ as a function of $t_n$ for Example 2.

Table 7.2

Behavior of the weighted error $E_\mu$, as the number of spatial mesh elements increases, for different choices of the power weight exponent $\mu$. In each case, we use 1300 time subintervals.

| $M$  | $E_0$       | $CR$ | $E_{0.25}$  | $CR$ | $E_{0.5}$    | $CR$ | $E_{0.75}$  | $CR$ |
|------|-------------|------|-------------|------|-------------|------|-------------|------|
| 4    | 3.008e-02   |      | 9.521e-03   |      | 3.610e-03   |      | 1.597e-03   |      |
| 8    | 1.054e-02   | 1.513| 1.412e-03   | 2.754| 5.342e-04   | 2.757| 2.401e-04   | 2.734|
| 16   | 5.441e-03   | 9.536| 4.112e-04   | 1.779| 1.279e-04   | 2.062| 5.678e-05   | 2.080|
| 32   | 1.876e-03   | 1.536| 1.391e-04   | 1.564| 3.344e-05   | 1.936| 1.513e-05   | 1.908|
| 64   | 8.667e-04   | 1.114| 6.425e-05   | 1.114| 8.598e-06   | 1.959| 4.055e-06   | 1.900|

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Table 7.3
Behavior of the weighted error $E_{\mu}$, as the number of spatial mesh elements increases, for different choices of the power weight exponent $\mu$. In each case, we use 1300 time subintervals.

| $M$ | $E_0$ CR | $E_{0.5}$ CR | $E_{0.75}$ CR | $E_1$ CR |
|-----|---------|--------------|----------------|---------|
| 4   | 1.0160e-00 3.453e-02 1.531e-02 9.898e-03 |
| 8   | 9.7501e-01 0.0594 8.545e-03 2.769 |
| 16  | 7.0054e-01 0.4769 8.852e-03 1.150 |
| 32  | 3.2311e-01 1.1164 1.776e-03 1.117 |
| 64  | 1.5301e-01 1.0783 8.409e-03 1.078 |

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