The finite time blow-up of the Yang-Mills flow

Guanxiang Wang and Chuanjing Zhang

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Abstract

In this paper, we shall prove that, on a non-flat Riemannian vector bundle over a compact Riemannian manifold, the smooth solution of the Yang-Mills flow will blow up in finite time if the energy of the initial connection is small enough. We also consider the finite time blow up for the Yang-Mills flow with the initial curvature near the harmonic form. Furthermore, when $E$ is a holomorphic vector bundle over a compact Kähler manifold, then $E$ will admit a projective flat structure if the trace free part of Chern curvature is small enough.

1 Introduction

Let $E \to M$ be a vector bundle over a closed $n$-dimensional ($n \geq 2$) Riemannian manifold $(M, g)$ and $H$ be a Riemannian metric on $E$. The Yang-Mills flow was first introduced by Atiyah and Bott [1], it is a time-dependent connection $A = A(t)$ on $E$ solving the following equation

$$\frac{\partial A(t)}{\partial t} = - D^*_{A(t)} F_{A(t)},$$

where $F_{A(t)}$ is the curvature of $A(t)$, $D^*_{A(t)}$ is the adjoint (with respect to a fixed metric) of covariant differential $D_{A(t)}$ on $\mathfrak{g}_E$-valued forms. The Yang-Mills heat flow is the gradient flow of the well-known Yang-Mills functional

$$\mathcal{YM}(A) = \int_M |F_A|^2 dV_g.$$

The existence and convergence of smooth solutions for the Yang-Mills flow is an essential problem. It was subsequently shown by Daskalopoulos [5] for compact Riemannian surface and by Rade [18] in dimensions two and three, that the flow exists for all time and converges. In four dimensional case, A.Schlatter [19] and M.Struwe [20] have studied the global weak solution for the Yang-Mills flow over closed 4-manifolds, not excluding the possibility that point singularities will form within finite time. A.Waldron [2] make a progress that in four-dimensional case, the finite-time singularities actually do not occur. He proved that any classical solution of Yang-Mills flow extends smoothly for all time. But the
convergence of the flow on four-dimensional manifolds has not been understood well. In the case of holomorphic vector bundle, Donaldson\cite{Donaldson7} proved that the Yang-Mills flow exists smoothly for all time and converges to a Hermitian-Yang-Mills connection when the holomorphic vector bundle is stable.

In general cases or dimensions, the behaviour of the Yang-Mills flow has not been understood well. Feehan\cite{Feehan17} proved that the existence and convergence of the Yang-Mills flow if the initial value is closed enough to a local minimal connection in some Sobolev space by applying the theory of gradient flow and Lojasiewicz-Simon gradient inequality. Naito\cite{Naito11} proved that over $S^d$ ($d \geq 5$) with its standard round Riemannian metric of radius one, if the bundle is non-flat, then the smooth Yang-Mills flow with small energy initial value will blow up in finite time. Joseph F. Grotowski\cite{Grotowski13} has shown the finite time blow up for a class of $SO(n)$-equivariant initial connections on a trivial principal $SO(n)$-bundle over $\mathbb{R}^n$ when dimension $n$ greater than 4. When the Yang-Mills flow admits a long time smooth solution, Hong-Tian\cite{Hong-Tian9} have analyzed the asymptotic behaviour of the Yang-Mills flow, they showed that the singular set will occur and the set has Hausdorff codimension at least four. A refined structure theorems on the singular set for Yang–Mills flow in dimensions $n \geq 4$ is obtained by Casey Kellehera and Jeffrey Streets\cite{Kellehera-Streets3}.

In this article, we show that if $E$ is a non-flat vector bundle over a closed Riemannian manifold $(M, g)$, then the Yang-Mills flow with small energy initial value will blow up in finite time. This result is a generalization of Naito\cite{Naito11} and partially clarifies the behaviour of Yang-Mills flow in higher dimensions. More precisely, we prove the following theorem.

**Theorem 1.1.** Let $(M, g)$ be an $n$-dimensional $(n \geq 2)$ smooth closed Riemannian manifold, $(E, h)$ be a non-flat Riemannian vector bundle over $M$. Then there exists a positive constant $\sigma = \sigma(n, g, E, h) > 0$ with the following significance: if $A_0$ is a smooth connection compatible with metric $h$ on $E$, such that $\mathcal{YM}(A_0) < \sigma$, then the smooth solution $A(t)$ for Yang-Mills flow with initial value $A_0$ blows up in finite time.

We also consider the finite time blow-up when the curvature is near the harmonic form. Let $(E, h)$ be a Hermitian vector bundle with rank $r$ over a closed Riemannian manifold $(M, g)$, $\theta$ be the harmonic representation for $-2\pi c_1(E)$, where $c_1(E)$ is the first Chern class.

**Theorem 1.2.** Let $(M, g)$ be an $n$-dimensional $(n \geq 2)$ smooth closed Riemannian manifold, $(E, h)$ be a non-projectively flat Hermitian vector bundle with rank $r$ over $M$. Then there exists a positive constant $\sigma = \sigma(n, g, E, h) > 0$ with the following significance: if $A_0$ is a smooth connection compatible with metric $h$ on $E$, such that

$$\|F_{A_0} - \frac{i\theta}{r} \otimes \text{Id}_E\|_{L^2} < \sigma,$$

then the smooth solution $A(t)$ for Yang-Mills flow with initial value $A_0$ blows up in finite time.
Let \((M, \omega)\) be a compact Kähler manifold, \((E, H_0, \overline{\partial}_E)\) be a rank \(r\) holomorphic vector bundle with Chern connection \(D_{H_0}\). It is well-known that the Yang-Mills flow with initial value \(D_{A(0)} = D_{H_0}\) has global solution\(^7\). But, in general, the convergence of the Yang-Mills flow is still unknown unless the holomorphic bundle \(E\) is poly-stable. Denote by \(A_{H_0}^{1,1}\) the set of connections compatible with \(H_0\) on \(E\) and \(F_{A_0}^{0,2} = 0\). Let

\[
F_{A_0}^\perp = F_{A_0} - \frac{1}{r} \text{tr} F_{A_0} \otimes \text{Id}_E,
\]

is the trace free part of the curvature. The following theorem gives a sufficient condition for the convergence of the Yang-Mills flow. This also partially clarifies the asymptotic behaviour of the Yang-Mills flow in holomorphic vector bundle.

**Theorem 1.3.** Let \((M, \omega)\) be an \(n\)-dimensional compact Kähler manifold, \((E, K_0, \overline{\partial}_E)\) be a holomorphic vector bundle over \(M\). Then there exists a positive constant \(\sigma > 0\) depending on the geometry of \(M\) and \((E, K_0, \overline{\partial}_E)\), with the following significance: if the Chern connection \(D_{K_0}\) satisfies \(\|F_{K_0}^\perp\|_{L^2} < \sigma\), then there exists a Hermitian metric \(H_0\) which is conformally equivalent to \(K_0\), and a Yang-Mills connection \(A_\infty \in A_{H_0}^{1,1}\) satisfying \(F_{A_\infty}^\perp = 0\).

The paper is organized as follows. In section 2, we will review some basic notations, estimates and some basic results. In section 3, we prove the main results.

## 2 Preliminaries

### 2.1 Connections and Curvatures on vector bundle

Let us first recall some standard geometric notations and definitions. As before, assume \((M, g)\) is a closed \(n\)-dimensional Riemannian manifold, \((E, h)\) is a real (or Hermitian) vector bundle with rank \(r\) over \(M\). A connection \(D_A\) on \(E\) is a linear differential operator

\[
D_A : \Gamma(E) \rightarrow \Omega^1(E)
\]

such that

\[
D_A(f\sigma) = df \otimes \sigma + fD_A\sigma
\]

for all \(f \in C^\infty(M)\) and \(\sigma \in \Gamma(E)\), where \(\Gamma(E)\) is the space of smooth sections of \(E\), \(\Omega^p(E)\), is the space of \(E\)-valued \(p\)-forms. We also require that the connection \(A\) is compatible with the metric \(h\), i.e.

\[
\text{dh}(\gamma, \beta) = h(D_A\gamma, \beta) + h(\gamma, D_A\beta)
\]

for all \(\gamma, \beta \in \Gamma(E)\). Suppose \((U_\alpha, \varphi_\alpha)\) is a local trivialization of \(E\), the connection takes the form

\[
D_A = d + A_\alpha,
\]

\[3\]
where $A_\alpha$ is connection 1-form, it is a matrix valued 1-form. More precisely, since the connection is compatible with the metric on $E$, if we denote that $\mathfrak{g}_E \subseteq \text{End}(E)$ is the subbundle of $\text{End}(E)$ such that its fibre at $x$ is just the set of skew-symmetric(or skew-Hermitian) endomorphisms of $E_x$ with respect to $h(x)$, then $A_\alpha \in \Omega^1(\mathfrak{g}_E)$ is $\mathfrak{g}_E$-valued 1-form. The space of connections, which is denoted by $\mathcal{A}_E$, is an affine space

$$ \mathcal{A}_E = D_A + \Omega^1(\mathfrak{g}_E). $$

Of course, $D_A$ also induces a connection on $\mathfrak{g}_E$, we also denoted it by $D_A$ for simplicity. Indeed, for any $\phi \in \Omega^p(\mathfrak{g}_E), \sigma \in \Gamma(E)$, define

$$ (D_A \phi)\sigma = D_A(\phi(\sigma)) - \phi(D_A \sigma). \quad \text{(2.1)} $$

Let $F$ be any vector bundle over $M$, then for each linear connection $D_A$ on $F$, we define an exterior differential

$$ D_A : \Omega^p(F) \to \Omega^{p+1}(F), $$

as follows. For each real valued differential $p$-form $\gamma$, $p \geq 0$, and each smooth section $\sigma$ of $F$, we set

$$ D_A(\gamma \otimes \sigma) = d\gamma \otimes \sigma + (-1)^p \gamma \wedge D_A \sigma, \quad \text{(2.2)} $$

and extend the definition to general $\phi \in \Omega^p(F)$ by linearity. Combining the connection on $E$ and $\mathfrak{g}_E$ given above, (2.2) gives a exterior differential on $\Omega^p(E)$ and $\Omega^p(\mathfrak{g}_E)$, $p \geq 0$.

The bundle metric $h$ and Riemannian metric $g$ induce an inner product on $\Omega^p(\mathfrak{g}_E)$. The inner product on the bundle $\Omega^p(\mathfrak{g}_E)$ is given by the following, for each $a, b \in \Omega^p(\mathfrak{g}_E)$,

$$ \langle a, b \rangle := \text{Tr}(ab^{*h}), $$

where $b^{*h}$ is transpose (or conjugate transpose) of the endomorphism $b$ with respect to the Riemannian (or Hermitian) metric on $E$. Also, the above inner product and $g$ induce an inner product on the bundle $\Omega^p(\mathfrak{g}_E)$.

For any connection $D_A$ of $E$, its curvature $F_A$ is determined by

$$ F_A = D_A \circ D_A : \Gamma(E) \to \Omega^2(E), $$

which is a $C^\infty$-linear operator on the sections of $E$. More precisely, it is a $\mathfrak{g}_E$-valued 2-form. Locally, the curvature is given by

$$ F_A = dA_\alpha + A_\alpha \wedge A_\alpha. \quad \text{(2.3)} $$

The first Bianchi identity $D_A F_A = 0$ is familiar.
2.2 Gauge transformations

A gauge transformation $u$ of $E$ is a smooth section of $\text{End}(E)$ such that at each $x \in M$, $u(x)$ is an orthogonal or unitary transformation of the fiber $E_x$. The gauge group, denoted by $\mathcal{G}_E$, is the set of gauge transformations. There is a natural action of gauge group $\mathcal{G}_E$ on the space of connections $\mathcal{A}_E$: given $u \in \mathcal{G}_E$ and a connection $D_A$, define the action of $u$ on $D_A$ as

$$D_{u(A)} = u \circ D_A \circ u^{-1}.$$ 

i.e. for each section $\sigma \in \Gamma(E)$, the gauge action is

$$D_{u(A)}(\sigma) = uD_A(u^{-1}(\sigma)).$$

(2.4)

One can easily verify that $D_{u(A)}$ is also a connection on $E$ and its curvature is

$$F_{u(A)} = uF_Au^{-1}.$$ 

2.3 Yang-Mills functional and Yang-Mills flow

Given a smooth connection $D_A$ on $E$, we define the energy of the connection $D_A$ by

$$\mathcal{YM}(A) = \int_M |F_A|^2 dV_g,$$

(2.5)

where $dV_g$ is the volume form of Riemannian metric $g$. Its Euler-Lagrange equation is the well-known Yang-Mills equation,

$$D_A^*F_A = 0.$$ 

(2.6)

We call a connection $A$ a Yang-Mills connection if it satisfies the Yang-Mills equation. Since the Yang-Mills functional (2.5) is gauge invariant, the Yang-Mills connection is also gauge invariant. The Yang-Mills flow with initial value $A_0$ is

$$\frac{\partial A(t)}{\partial t} = -D_A^*F_A(t),$$

$$A(0) = A_0.$$ 

(2.7)

It is the $L^2$-gradient flow about the Yang-Mills functional. Let $(E, h)$ be a Hermitian vector bundle with rank $r$ over a closed Riemannian manifold $(M, g)$. Suppose $A$ is a Yang-Mills conection, then by Bianchi identity and (2.6), $tr F_A$ is a harmonic form. Assume $\theta$ is the harmonic representation for $-2\pi c_1(E)$, where $c_1(E)$ is the first Chern class. By Hodge theory, $tr F_A = \sqrt{-1}\theta$. 

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2.4 Hermitian-Yang-Mills flow

Let \((E, H_0, \overline{\nabla}_E)\) be a holomorphic vector bundle over a compact \(n\)-dimensional Kähler manifold \((M, \omega)\), \(\mathcal{A}_{H_0}\) be the space of connections compatible with the metric \(H_0\) on \(E\) and \(\mathcal{A}_{H_0}^{1,1}\) be the space of unitary integrable connections of \(E\). Denote by \(D_{H_0}\) the Chern connection with respect to \(H_0\) and \(\overline{\nabla}_E\). The Hermitian-Yang-Mills (HYM-) flow with initial metric \(H_0\) is

\[
\begin{cases}
H^{-1}(t) \frac{\partial H(t)}{\partial t} = -2(\sqrt{-1} \Lambda_\omega F_{H(t)} - \lambda \text{Id}_E), \\
H(0) = H_0,
\end{cases}
\]

(2.8)

where \(\lambda = \frac{2\pi \text{deg}(E)}{\text{rank}(E) \text{Vol}(M, \omega)}\), and \(\text{deg}(E) = \int_M c_1(E)^\wedge \frac{\omega}{(n-1)!}\). The Hermitian-Yang-Mills (HYM-) flow was first introduced by Donaldson [7] and he proved the global existence of the HYM-flow. If the holomorphic bundle is \(\omega\)-polystable, the flow converges to the so-called Hermitian-Einstein metric. This is the well-known Donaldson-Uhlenbeck-Yau theorem [7] [8] [14] [23]. Let \(h = H_0^{-1} H\), then

\[
D_H - D_{H_0} = h^{-1} \partial_{H_0} h,
\]

(2.9)

and

\[
F_H - F_{H_0} = \overline{\nabla}_E (h^{-1} \partial_{H_0} h),
\]

(2.10)

and

\[
tr F_H = tr F_{H_0} + \overline{\partial} \partial \log \det h,
\]

(2.11)

where \(D_H\) is the Chern connection with respect to \(H\) and \(\overline{\nabla}_E\). Denote the complex gauge group of Hermitian bundle \((E, H_0)\) by \(\mathcal{G}^C\). The group acts on \(\mathcal{A}_{H_0}^{1,1}\) as follows: for \(\sigma \in \mathcal{G}^C\),

\[
\overline{\nabla}_{\sigma(A)} = \sigma \circ \overline{\nabla}_A \circ \sigma^{-1},
\]

\[
\partial_{\sigma(A)} = (\sigma^* \mu_0)^{-1} \circ \partial_A \circ \sigma^* \mu_0,
\]

\[
D_{\sigma(A)} = \overline{\nabla}_{\sigma(A)} + \partial_{\sigma(A)}.
\]

Choose \(\sigma\) such that \(\sigma^* \mu_0(t) \circ \sigma(t) = h(t)\). Note that \(D_{A_0} = D_{H_0}\). After a direct calculation, one can get

\[
F_{D_{\sigma(A_0)}} = \sigma \circ F_H \circ \sigma^{-1}.
\]

(2.12)

Using the solution \(H(t)\) of (2.8), we can construct a solution \(A(t)\) for the Yang-Mills flow with initial value \(D_{H_0}\) [7]. In particular, we have

\[
tr F_{A(t)} = tr F_H = tr F_{H_0} + \overline{\partial} \partial \log \det h.
\]

(2.13)

This is important for us to prove Theorem 1.3.
2.5 Basic estimates and results

The following $\epsilon$-regularity is proved by Chen-Shen[4] for Yang-Mills flow, by Hong-Tian[9] for Yang-Mills-Higgs flow and Yang-Mills-Higgs case on holomorphic vector bundle over Kähler manifolds by Li-Zhang[12], it is crucial for the proof of Theorem 1.1.

**Theorem 2.1** ($\epsilon$-regularity). \(\forall C > 0, \exists \epsilon_0, \delta_0 < 1/4\). Assume \(A(t)\) is a smooth solution for Yang-Mills flow with initial value \(A_0\), and \(\mathcal{YM}(A_0) < C\). Then, if for some \(0 < R < \min\{i_M, \sqrt{t_0}/2\}\), the inequality

\[
R^{2-n} \int_{P_R(x_0,t_0)} |F_A|^2 dV_g dt < \epsilon_0,
\]

holds, then for any \(\delta \in (0, \delta_0)\), we have

\[
\sup_{P_{\delta R}(x_0,t_0)} |F_A|^2 \leq 16(\delta R)^{-4},
\]

where \(P_R(x_0,t_0) = B_R(x_0) \times [t_0 - R^2, t_0 + R^2]\) and \(i_M\) is the infimum of the injectivity radius.

In order to analyze the asymptotic behavior of the Yang-Mills flow, we also need the following result[9]:

**Theorem 2.2** ([9] Theorem A). Let \(E\) be a vector bundle over an \(n\)-dimensional closed Riemannian manifold \(M\). Let \(A\) be a global smooth solution of Yang-Mills flow in \(M \times [0, \infty)\) with smooth initial value \(A_0\). Then there exists a sequence \(\{t_i\}\) such that, as \(t_i \to \infty\), \(A(x,t_i)\) converges, modulo gauge transformations, to a Yang-Mills connection \(A_\infty\) in smooth topology outside a closed set \(\Sigma\). And \(\mathcal{H}^{n-4}(\Sigma)\) is finite. Moreover,

\[
\Sigma = \bigcap_{\epsilon_0 > r > 0} \left\{ x \in M : \liminf_{k \to \infty} r^{4-n} \int_{B_r(x)} |F_{A(t_k)}|^2 dV_g \geq \sigma_1 \right\}
\]

for some constants \(\epsilon_0, \sigma_1 > 0\).

In general, the Yang-Mills connection \(A_\infty\) and the singular set \(\Sigma\) are not unique. If the \(C^0\)-norm of \(F_{A(t)}\) is uniformly bounded along the flow, then the Yang-Mills connection \(A_\infty\) is smooth on the whole manifold.

We recall a basic curvature estimate for Yang-Mills connections, derived by Nakajima[10] (also see Tian[22]).

**Theorem 2.3** ([10] Lemma 3.1. [22] Theorem 2.21). Let \(A\) be a Yang-Mills connection of bundle \((E,h)\) over an \(n\)-dimensional \((n \geq 4)\) closed Riemannian manifold \(M\). Then there are \(\epsilon = \epsilon(n) > 0\) and \(C = C(n) > 0\), which depend only on \(n\) and \(M\), such that for any \(p \in M\) and \(p < r_p\), where \(r_p\) is a positive constant depending on \(p\) and geometry of \(M\), whenever

\[
\rho^{4-n} \int_{B_p(r)} |F_A|^2 dV_g \leq \epsilon,
\]
then
\[ \sup_{B_{\rho/4}(p)} |F_A| \leq C \rho^2 \left( \rho^{4-n} \int_{B_{\rho}(p)} |F_A|^2 dV_g \right)^{1/2}. \]

Using a finite cover of \(M\) by geodesic balls and applying the above theorem, we obtain the following global version.

**Corollary 2.4.** Let \((E, h)\) be a vector bundle over an \(n\)-dimensional \((n \geq 4)\) closed Riemannian manifold \((M, g)\). Then there exist constants \(\epsilon = \epsilon(n) > 0\) and \(C = C(n) > 0\), which depend only on \(n\) and \((M, g)\), such that if \(A\) is a smooth Yang-Mills connection satisfying
\[ \|F_A\|_{L^2(M)} \leq \epsilon, \]
then,
\[ \|F_A\|_{L^\infty(M)} \leq C \|F_A\|_{L^2(M)}. \]

The \(L^{n/2}\)-energy gap theorem obtained by Feehan\cite{15,16} is also important for the proof of Theorem 1.1.

**Theorem 2.5** (\cite{15,16}. \(L^{n/2}\)-energy gap). Let \((M, g)\) be an \(n\)-dimensional \((n \geq 2)\) closed smooth Riemannian manifold. \((E, h)\) be a real (or Hermitian) vector bundle over \(M\). Then there exists a positive constant \(\epsilon = \epsilon(n, E, h) > 0\) with the following significance: if \(A\) is a smooth Yang-Mills connection of \(E\) with
\[ \|F_A\|_{L^{n/2}} < \epsilon, \]
then \(A\) is a flat connection.

Take together with the Corollary 2.4, we can deduce the following \(L^2\)-energy gap.

**Corollary 2.6.** Under the same assumption as in Theorem 2.4, then there exists a positive constant \(\epsilon = \epsilon(n, g, E, h) > 0\) with the following significance: if \(A\) is a smooth Yang-Mills connection of \(E\) with \(\mathcal{YM}(A) < \epsilon\), then \(A\) is a flat connection.

Let \((E, h)\) be a Hermitian vector bundle with rank \(r\) over a closed Riemannian manifold \((M, g)\), \(\theta\) the harmonic representation for \(-2\pi c_1(E)\), where \(c_1(E)\) is the first Chern class. With the aid of energy gap for Yang-Mills connections, we can extend the criterion for the existence of flat connections to that for projectively flat connections. Since the principal \(PU(r)\)-bundle associated to \(E\) is flat if and only if the Hermitian bundle \((E, h)\) is projectively flat\cite{21}, the proof is trivial and we omit it here. More precisely,

**Corollary 2.7.** Let \((M, g)\) be an \(n\)-dimensional \((n \geq 2)\) closed smooth Riemannian manifold, \((E, h)\) be a Hermitian vector bundle with rank \(r\) over \(M\). Then
there exists positive a constant $\epsilon = \epsilon(n, g, E, h) > 0$, with the following significance. If $A$ is a smooth connection compatible with $h$ on $E$, and the curvature $F_A$ satisfies

$$D_A^*(F_A - \frac{\text{tr}F_A}{r} \otimes \text{Id}_E) = 0$$

and

$$\|F_A - \frac{\text{tr}F_A}{r} \otimes \text{Id}_E\|_{L^{n/2}} < \epsilon,$$

then $A$ is projectively flat. In particular, if $A$ is a smooth Yang-Mills connection with

$$\|F_A - \frac{i\theta}{r} \otimes \text{Id}_E\|_{L^{n/2}} < \epsilon,$$

then $F_A = \frac{i\theta}{r} \otimes \text{Id}_E$.

Similar to Corollary 2.6, we can derive the following result:

**Corollary 2.8.** Assume the hypotheses of Theorem 2.3, then there exists a positive constant $\epsilon = \epsilon(n, g, E, h) > 0$, with the following significance. If $A$ is a smooth connection compatible with $h$ on $E$, and the curvature $F_A$ satisfies

$$D_A^*(F_A - \frac{\text{tr}F_A}{r} \otimes \text{Id}_E) = 0$$

and

$$\|F_A - \frac{\text{tr}F_A}{r} \otimes \text{Id}_E\|_{L^2} < \epsilon,$$

then $A$ is projectively flat. In particular, if $A$ is a smooth Yang-Mills connection with

$$\|F_A - \frac{i\theta}{r} \otimes \text{Id}_E\|_{L^2} < \epsilon,$$

then $F_A = \frac{i\theta}{r} \otimes \text{Id}_E$.

The above corollary is important for us to show Theorem 1.2 and Theorem 1.3. To prove Theorem 1.3, we also need the following Lemma,

**Lemma 2.9** ([15], Lemma 6). Let $(M, \omega)$ be a compact Kähler manifold, $(E, K_0, \overline{\partial}_E)$ be a rank $r$ holomorphic vector bundle. Then there exists a Hermitian metric $H_0$ which is conformally equivalent to $K_0$, such that $\text{det}(t) = 1$ along the Hermitian-Yang-Mills flow, where $h(t) = H_0^{-1}H(t)$ and $H(t)$ is the solution for Hermitian-Yang-Mills flow with $H(0) = H_0$.

Suppose $H_0 = e^\phi K_0$, a simply calculation shows that

$$F_{H_0} = F_{K_0} - \overline{\partial}_\phi \cdot \text{Id}_E,$$

this gives $F_{K_0}^\perp = F_{H_0}^\perp$. Particularly, $|F_{K_0}^\perp|_{K_0} = |F_{H_0}^\perp|_{H_0}$ since $H_0 = e^\phi K_0$. 

9
3 Proof of the main Results

In this section, we will prove Theorem 1.1, 1.2 and 1.3. Note that, the constant $C$ may be different from line to line.

3.1 Proof of Theorem 1.1

The main idea of the proof of Theorem 1.1 is to use the $\epsilon$-regularity to deduce the $C^0$-estimate along the Yang-Mills flow. Then by Theorem 2.2 and Corollary 2.6, there must exist a flat connection on $E$. This contradicts that the bundle $E$ is non-flat.

Proof of Theorem 1.1. Suppose $A(t)$ is the smooth global solution with initial value $A_0$. By $\epsilon$-regularity, fixing $C = 1$, there exist $\epsilon_0, \delta_0 < 1/4$, let $\mathcal{YM}(A_0) < \sigma$ small enough, such that, for large $t_0 > 0$, the inequality

$$
R_0^{2-n} \int_{P_{R_0}(x,t)} |F_A|^2 dV_g dt < 2R_0^{4-n} \sigma < \epsilon_0,
$$

holds for some $0 < R_0 < i_M$ and $\forall (x,t) \in M \times [t_0, \infty)$. Then by $\epsilon$-regularity, for any $\delta \in (0, \delta_0)$,

$$
\sup_{P_{\delta R_0}(x,t)} |F_A|^2 \leq 16(\delta R_0)^{-4}.
$$

So, along the Yang Mills flow, $\sup_{M \times [0, \infty)} |F_A|^2 < C_0 < \infty$, for some constant $C_0 > 0$. According to Theorem 2.2 and (2.16), we know that there exists a sequence $\{t_i\}$ such that, as $t_i \to \infty$, $A(x, t_i)$ converges, modulo gauge transformations, to a Yang-Mills connection $A$ in smooth topology on the whole manifold $M$ and $\mathcal{YM}(A) < \sigma$, let $\sigma$ small enough, by Corollary 2.6, $A$ is a flat connection, it’s impossible since $E$ is non-flat, hence $A(t)$ must blow up in finite time. \hfill \square

3.2 Proof of Theorem 1.2

The main idea of the proof of Theorem 1.2 is similar to the Theorem 1.1. Denote

$$
e(A, \theta) = |F_A - \frac{i\theta}{r} \otimes Id_E|^2. \quad (3.1)
$$

Proof of Theorem 1.2. Suppose $A(t)$ is the smooth global solution with initial value $A_0$. Fixing $C_0 > 0$, let $\sigma$ small enough with $\mathcal{YM}(A_0) < C_0$, there exist
\( \epsilon_0, \delta_0 < 1/4 \). Let \( t_0 \) large enough and \( 0 < R < i_M \), then for \( \forall x_0 \in M \), we have

\[
R^{-n} \int_{\mathcal{P}(x_0, t_0)} |F_{A(t)}|^2 dV_g dt \\
\leq 2R^{-n} \int_{\mathcal{P}(x_0, t_0)} (e(A, \theta) + \frac{1}{r^2} |\theta \otimes \text{Id}_E|^2) dV_g dt \\
\leq 2R^{-n} \int_{t_0-R^2}^{t_0+R^2} \int_M e(A, \theta) dV_g dt + \frac{C}{r^2} |\theta \otimes \text{Id}_E|^2 C_0 R^4 \\
\leq 2R^{-n} \int_{t_0-R^2}^{t_0+R^2} \int_M e(A_0, \theta) dV_g dt + \frac{C}{r^2} |\theta \otimes \text{Id}_E|^2 C_0 R^4 \\
\leq 4R^{-n} \sigma^2 + \frac{C}{r^2} |\theta \otimes \text{Id}_E|^2 C_0 R^4,
\]

where \( C \) is constant depending only on the geometry of \( M \). Taking \( R \) and \( \sigma \) small enough such that \( \frac{C}{r^2} |\theta \otimes \text{Id}_E|^2 C_0 R^4 < \frac{\epsilon_0}{2} \) and \( 4R^{-n} \sigma^2 < \frac{\epsilon_0}{2} \), where \( \epsilon_0 \) is chosen as in the \( \epsilon \)-regularity. Then by \( \epsilon \)-regularity, for any \( \delta \in (0, \delta_0) \),

\[
\sup_{\mathcal{P}(x_0, t_0)} |F_{A(t)}| \leq 16(\delta R)^{-4}.
\]

Since \( M \) is compact, we can conclude that there exists a positive constant \( C_0 \) such that \( |F_{A(t)}|_{C_0} \leq C_0 < \infty \) for all \( t \geq 0 \). Form Theorem 2.2, we see there exists a sequence \( \{t_i\} \) such that, as \( t_i \to \infty \), \( A(x, t_i) \) converges, modulo gauge transformations, to a Yang-Mills connection \( A_\infty \) in smooth topology on the whole manifold \( M \) and \( \int_M e(A_\infty, \theta) dV_g < \sigma^2 \). Let \( \sigma \) be small enough, due to Corollary 2.8, \( A \) is a projective-flat connection, which is contradict with that \( E \) is non-projective-flat. Hence \( A(t) \) must blow up in finite time. \( \square \)

### 3.3 Proof of Theorem 1.3

**Proof of Theorem 1.3.** By Lemma 2.9, we can find a Hermitian metric \( H_0 \), such that

\[
|F_{H_0}^\perp|_{H_0} = |F_{K_0}^\perp|_{K_0},
\]

and \( \text{deth}(t) = \det H_0^{-1} H(t) = 1 \) for all \( t \), where \( F_{H_0} \) is curvature of Chern connection \( A_{H_0} \) with respect to \( H_0 \) and holomorphic structure \( \overline{\mathcal{E}} \) and \( H(t) \) is long time solution for Hermitian-Yang-Mills flow with \( H(0) = H_0 \). Let \( A(t) \) be the solution for the Yang-Mills flow with \( A(0) = A_{H_0} \). Then because of (2.13),

\[
\text{tr} F_{A(t)} = \text{tr} F_{H(t)} = \text{tr} F_{H_0} + \overline{\partial} \partial \log \det h(t) = \text{tr} F_{H_0},
\]

and

\[
|F_{A(t)}|^2_{H_0} = |F_{A(t)}^\perp|^2_{H_0} + \frac{1}{r^2} |\text{tr} F_{A(t)} \otimes \text{Id}_E|^2_{H_0} \\
= |F_{A(t)}^\perp|^2_{H_0} + \frac{1}{r^2} |\text{tr} F_{H_0} \otimes \text{Id}_E|^2_{H_0}.
\]
Let \( t_0 \) be large enough, \( \forall x_0 \in M \), it holds that
\[
R^{2-2n} \int_{\mathcal{P}_{t_0}(x_0, t_0)} |F_{A(t)}|^2_{H_0} \, dV_g dt
\]
\[
= R^{2-2n} \int_{\mathcal{P}_{t_0}(x_0, t_0)} \left( |F_{A(t)}|^2_{H_0} + \frac{1}{r^2} |tr F_{H_0} \otimes Id_E|^2_{H_0} \right) dV_g dt
\]
\[
\leq R^{2-2n} \int_{t_0-R^2}^{t_0+R^2} \int_M |F_{A(t)}|^2_{H_0} \, dV_g dt + R^{2-2n} \int_{\mathcal{P}_{t_0}(x_0, t_0)} \frac{1}{r^2} |tr F_{H_0} \otimes Id_E|^2_{H_0} dV_g dt
\]
\[
\leq R^{2-2n} \int_{t_0-R^2}^{t_0+R^2} \int_M |F_{A(t)}|^2_{H_0} \, dV_g dt + C \frac{1}{r^2} |tr F_{H_0} \otimes Id_E|^2_{C^0} R^4
\]
\[
\leq 2R^{4-2n}\sigma^2 + \frac{C}{r^2} |tr F_{H_0} \otimes Id_E|^2_{C^0} R^4,
\]
in the second inequality we have used the fact that along the Yang-Mills flow, \( \int_M |F_{A(t)}|^2_{H_0} dV_g \) is non-increasing and \( |F_{A(t)}|_{K_0} = |F_{H_0}|_{H_0} \), where \( C \) is a constant depending only on the geometry of \( M \). Take \( R \) and \( \sigma \) small enough, such that \( \frac{C}{r^2} |tr F_{H_0} \otimes Id_E|^2_{C^0} R^4 \leq \frac{c_0}{2} \) and \( 2R^{4-2n}\sigma^2 \leq \frac{c_0}{2} \), where \( c_0 \) is chosen as in the \( \epsilon \)-regularity. Then by \( \epsilon \)-regularity, for any \( \delta \in (0, \delta_0) \),
\[
\sup_{\mathcal{P}_{t/2}(x_0, t_0)} |F_{A(t)}|^2_{H_0} \leq 16(\delta R)^{-4}.
\]

Since \( M \) is compact, we can conclude that there exists a positive constant \( C_0 \), such that \( |F_{A(t)}|_{C^0} \leq C_0 < \infty \) for all \( t \geq 0 \). Similar to the proof of Theorem 1.1 and 1.2, there exists a sequence \( A(t_k) \) converges, modulo gauge transformations, to a Yang-Mills connection \( A_\infty \in \mathcal{A}_\mathcal{H}_1^{1,1} \) in smooth topology on the whole manifold \( M \) and \( \|F_{A_\infty}\|_{L^2} < \sigma \). Let \( \sigma \) small enough, by corollary 2.8, \( F_{A_\infty} = 0 \). This completes the proof of Theorem 1.3.

\[
\square
\]

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