Classical Particles in the Continuum Subjected to High Density Boundary Conditions

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Abstract. We consider a continuous system of classical particles confined in a finite region \( \Lambda \) of \( \mathbb{R}^d \) interacting through a superstable and regular pair potential and subjected to non-free and non-periodic boundary conditions. We prove that the thermodynamic limit of the pressure of the system at any fixed inverse temperature \( \beta \) and any fixed fugacity \( \lambda \) does not depend on the field produced by particles outside \( \Lambda \) whose local density may increase sub-linearly with the distance from the origin at a rate that depends on how fast the pair potential decays at large distances. In particular, if the pair potential \( v(r) \) decays as \( C/r^{d+p} \) (with \( p > 0 \)) when the distance \( r \) approaches infinity, then the existence of the thermodynamic limit of the pressure is guaranteed in presence of boundary conditions produced by external particles which may be distributed with a local density increasing with the distance \( r \) from the origin as \( \rho(1+r^q) \), where \( q \leq \frac{1}{2} \min\{1,p\} \) and \( \rho \) is any positive constant (even arbitrarily larger than the mean density \( \rho_0(\beta,\lambda) \) the system has when submitted to free boundary conditions).

1. Introduction

In the area of rigorous statistical mechanics, it is a widely accepted belief that the microcanonical entropy, the canonical free energy, and the grand-canonical pressure of a many-body system are independent of the boundary conditions imposed on the system as soon as the thermodynamic limit is taken. This is a well-established fact for bounded spin systems in a lattice and simple proofs can be found in many elementary textbooks. Things become less clear when unbounded spin systems on a lattice are analyzed. In that case, the proofs of the independence of the free energy from the boundary conditions are much
more involved and in general limitations on the allowed boundary conditions are needed, see e.g., [3,22,27].

The situation is even more nebulous when one considers a continuous system of interacting classical particles confined in a (macroscopically large) box \( \Lambda \) of the \( d \)-dimensional Euclidean space \( \mathbb{R}^d \). As a matter of fact, the vast majority of the rigorous results about the properties of the thermodynamic functions (e.g., pressure, free energy, entropy) in the thermodynamic limit for such systems have been deduced (mainly in the sixties/seventies, but also more recently) in ensembles submitted to either free or periodic boundary conditions. In particular, regarding the problem of the existence of the pressure at any density and temperature and its analyticity in the low-density/high-temperature regime, we refer the reader to the to the classic textbooks [15,23,32], to papers [6,9,13,14,21,26,30,31,33], to the overlooked but relevant papers [1,2] and their recent revisitation [10] and to [24,28] for recent significative improvements using results given in [11,29].

Another important and mathematically challenging problem related to systems of continuous classical particles is the construction of the infinite-volume Gibbs measure. This issue has been investigated mainly in regard to the existence and possible uniqueness of the infinite-volume Gibbs measure. Rigorous results on this matter go back to the pioneering works of Dobrushin [7,8] and Ruelle [32,33] but there have been more recent developments, see e.g. [4,5,12,18–20,25] and references therein.

In the present paper, our attention will be exclusively focused on the study of the grand canonical pressure of a system of classical particles confined in a cubic box \( \Lambda \), interacting via a pair potential and subjected to boundary conditions produced by fixed particles outside \( \Lambda \). Namely, we would like to investigate which constraints have to be imposed to the pair potential and to the boundary conditions in order to ensure the existence and uniqueness of the pressure in the thermodynamic limit. As far as we know, the only rigorous treatment on this subject has been given by Georgii in [16] where the independence of the pressure from a specific class of external boundary conditions, the so-called tempered boundary conditions (see [16,17,33] and see also below), is proved under the assumption that the pair potential is superstable, regular, and diverging in a non-summable way at the origin.

In the present paper, we somehow extend the results obtained by Georgii in the following sense. We prove the existence and uniqueness of the thermodynamic limit of the pressure under a class of external boundary conditions whose local density is allowed to increase arbitrarily as one moves away from the origin. Contrarily, the tempered boundary conditions allowed by Georgii have (mean) local density bounded. Moreover, our hypotheses on the pair potential are less restrictive than those made by Georgii. Namely, we do not need to require that the potential diverges in a non-summable way at the origin. So our theory can handle systems interacting via an everywhere finite pair potential such as the Morse potential, which is frequently used in Physics and Chemistry to model the interactions between molecules in real gases.
More specifically, we show that under the sole hypothesis that the potential is superstable and regular, the thermodynamic limit of the pressure of a system of classical particles in the grand canonical ensemble at any fixed inverse temperature $\beta$ and any fixed activity $\lambda$ is independent of boundary conditions produced by particles outside $\Lambda$ whose density may increase sub-linearly with the distance from the origin at a rate that depends on the decay of the pair potential at large distances. In particular, if the pair potential $v(x - y)$ has a large-distance decay of Lennard-Jones type, i.e., it falls off as $C/\|x - y\|^{d+p}$ (with $p > 0$ and with $\|x - y\|$ being the Euclidean distance between $x$ and $y$), then the existence of the thermodynamic limit of the pressure is guaranteed in presence of boundary conditions generated by external particles which may be distributed with a local density increasing with the distance $r$ from the origin as $\rho(1 + r^q)$, where $q < \frac{1}{2}\min\{1, p\}$ and $\rho$ is any positive constant (even arbitrarily larger than the mean density $\rho_0(\beta, \lambda)$ of the system evaluated with free boundary conditions).

2. The Model

We consider a continuous system of classical identical particles interacting through a translational invariant pair potential $v : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ and confined in a bounded compact region $\Lambda$ of $\mathbb{R}^d$, which we assume to be a cubic box of size $L$ centered at the origin so that the symbol $\lim_{\Lambda \uparrow \infty}$ means simply that $L \to \infty$. We denote by $x_i \in \mathbb{R}^d$ the position vector of the $i$th particle of the system and by $\|x_i\|$ its Euclidean norm. We further suppose that particles inside $\Lambda$ are subjected to a boundary condition $\omega$ produced by particles in fixed positions outside $\Lambda$. The boundary condition $\omega$ is a locally finite set of points of $\mathbb{R}^d$ representing the positions of fixed particles in $\mathbb{R}^d$. Namely, $\omega$ must be a countable set of points in $\mathbb{R}^d$ (not necessarily distinct) such that for any compact subset $C \subset \mathbb{R}^d$ it holds that $|\omega \cap C| < +\infty$ (here $|\omega \cap C|$ denotes the cardinality of the set $\omega \cap C$). We call $\Omega$ the space of all locally finite configurations of particles in $\mathbb{R}^d$ and, given a cube $\Lambda \subset \mathbb{R}^d$, we denote by $\Omega_\Lambda$ the set of all finite configurations of particles in $\Lambda$.

As usual, we will suppose that each particle inside $\Lambda$, say at position $x \in \Lambda$, feels the effect of the boundary condition $\omega$ through the field generated by the particles of the configuration $\omega$ which are in $\Lambda^c = \mathbb{R}^d \setminus \Lambda$. Free boundary conditions correspond to the case $\omega = \emptyset$. We are interested in studying the behavior of the system in the limit $\Lambda \uparrow \infty$ with a given boundary condition $\omega$ and how eventually this limit may be influenced by $\omega$, having in mind that, as the volume $\Lambda$ invades $\mathbb{R}^d$, the fixed particles of $\omega$ entering in $\Lambda$ are disregarded and only those boundary particles outside $\Lambda$ influence particles inside $\Lambda$. We will denote below by $|\Lambda|$ the volume of $\Lambda$ and by $\partial \Lambda$ the boundary of $\Lambda$. We define, for $x \in \Lambda$,

$$d_x^\Lambda = \inf_{y \in \partial \Lambda} \|x - y\|.$$
Hereafter we will frequently use the following notation. Given a configuration $\omega \in \Omega$, a function $f : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$, a cubic box $\Lambda \subset \mathbb{R}^d$ and a point $x \in \Lambda$, we set

$$E^f_\Lambda(x, \omega) = \sum_{y \in \omega \cap \Lambda} f(x - y).$$

With this notations, for any fixed volume $\Lambda$ and for any fixed boundary condition $\omega$, the partition function of the system in the grand canonical ensemble at inverse temperature $\beta \geq 0$ and fugacity $\lambda \geq 0$ is given by

$$\Xi^\omega_{\Lambda}(\beta, \lambda) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \int_\Lambda dx_1 \ldots \int_\Lambda dx_n e^{-\beta \left[ \sum_{1 \leq i < j \leq n} v(x_i - x_j) + \sum_{i=1}^{n} E^v_\Lambda(x_i, \omega) \right]},$$

where the $n = 0$ term of the series above is equal to one. Note that $E^v_\Lambda(x, \omega) = \sum_{y \in \omega \cap \Lambda^c} v(x - y)$ represents the field felt by a particle sitting in the point $x \in \Lambda$ due to the fixed particles of the boundary condition $\omega$ located at points outside $\Lambda$.

The finite-volume pressure of the system is then given by

$$\beta p^\omega_\Lambda(\beta, \lambda) = \frac{1}{|\Lambda|} \log \Xi^\omega_{\Lambda}(\beta, \lambda),$$

and the thermodynamic limit of the finite-volume pressure (if it exists) is

$$\beta p^\omega(\beta, \lambda) = \lim_{\Lambda \to \infty} \frac{1}{|\Lambda|} \log \Xi^\omega_{\Lambda}(\beta, \lambda).$$

The r.h.s. of (1) is for the time being just a formal series and consequently r.h.s. of (2) and (3) are, for the time being, meaningless. The convergence of the series in the r.h.s. of (1), the well-definiteness of (2), and the existence of the limit (3) depend on suitable assumptions on the pair potential $v$ and on the boundary condition $\omega$.

If $\omega = \emptyset$ (i.e., if we use free boundary conditions), the term $\sum_{i=1}^{n} E^v_\Lambda(x_i, \omega)$ in the exponential of the integrand of the r.h.s. of (1) vanishes so that the series

$$\Xi^\emptyset_{\Lambda}(\beta, \lambda) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \int_\Lambda dx_1 \ldots \int_\Lambda dx_n e^{-\beta \sum_{1 \leq i < j \leq n} v(x_i - x_j)}$$

represents the partition function of the system subjected to free boundary conditions. For any $n \in \mathbb{N}$ its $n$th coefficient, i.e., the integral

$$\int_\Lambda dx_1 \ldots \int_\Lambda dx_n e^{-\beta \sum_{1 \leq i < j \leq n} v(x_i - x_j)},$$

is well defined (i.e., it is finite) just by imposing that $v$ takes values in $\mathbb{R} \cup \{+\infty\}$. It is long known (see e.g., [32]) that the series (4) is convergent if the pair potential is stable according to the following definition.
**Definition 2.1.** A pair potential $v$ is stable if there exists $B \geq 0$ such that for all $n \in \mathbb{N}$ and for all $(x_1, \ldots, x_n) \in \mathbb{R}^{dn}$

\[
\sum_{1 \leq i < j \leq n} v(x_i - x_j) \geq -Bn,
\]

(6)

and the smallest constant $B$ satisfying (6) is called the *stability constant* of the potential.

In particular it is immediate to see that when $v$ is stable the $n$-order coefficient (5) is bounded by $|\Lambda|^n e^{n\beta B}$ and therefore the series (4) is an analytic function of $\lambda$ for all $\lambda \in \mathbb{C}$. This also implies straightforwardly that the finite-volume and free-boundary-condition pressure $\beta p^{\omega=\emptyset}_{\Lambda}(\beta, \lambda)$ is well defined for all $(\lambda, \beta) \in [0, +\infty) \times [0, +\infty)$.

On the other hand, the existence of the infinite-volume pressure, even when free boundary conditions are adopted, namely

\[
p^{\emptyset}(\beta, \lambda) = \lim_{\Lambda \uparrow \infty} p^{\emptyset}_{\Lambda}(\beta, \lambda) = \beta^{-1} \lim_{\Lambda \uparrow \infty} \frac{1}{|\Lambda|} \log \Xi^{\omega}_{\Lambda}(\beta, \lambda),
\]

(7)

is a non-trivial issue and to prove it one has to do some further assumptions on the pair potential.

**Definition 2.2.** A pair potential $v$ is *superstable* if $v$ can be written as the sum of two functions

\[v = v_1 + v_2\]

with $v_1$ stable and $v_2$ a continuous non-negative function with $v_2(0) > 0$.

Given $b > 0$, we set $B_b(0) = \{x \in \mathbb{R}^d : \|x\| < b\}$ (i.e., $B_b(0)$ denotes the $d$-dimensional open ball of radius $b$ centered at the origin of $\mathbb{R}^d$).

**Definition 2.3.** A pair potential $v(x)$ is regular if there exist a constant $b > 0$ and a continuous, non-negative decreasing function $\eta : [0, +\infty) \to [0, +\infty)$ such that $|v(x)| \leq \eta(\|x\|)$ for all $x \in \mathbb{R}^d \setminus B_b(0)$, and

\[
\int_0^{\infty} \eta(r)r^{d-1}dr < \infty,
\]

(8)

The existence of the limit (7) and its continuity as a function of $\lambda$ and $\beta$, when particles interact via a superstable and regular pair potential is a well-established fact since the sixties (see [6,9,13,31–33]). Much later Georgii [16,17] showed that the limit (3) with non-free boundary conditions exists if the pair potential, beyond superstable and regular, has a hard-core or diverges in a non-summable way at short distances. Georgii’s result holds for all “tempered” boundary conditions $\omega$ (see (5.10) in [33] or (2.24) in [16] or (2.6) in [17]), which basically means that, for some finite positive constant $t$, $\omega$ must be such that $\limsup_{\Lambda \uparrow \infty} |\Lambda|^{-1} \sum_{\Delta \in \Lambda_\delta} |\omega \cap \Delta|^2 \leq t$ where $\Lambda_\delta$ is a collection of cubes $\Delta$ of fixed size $\delta > 0$ forming a partition of $\Lambda$.

In the present paper the assumptions on the pair potential and on the allowed boundary conditions are illustrated in the next two sections.
2.1. Assumptions on the Pair Potential

The pair potential $v(x)$ is supposed to be Lebesgue measurable and to satisfy the following assumptions.

(i) $v$ is superstable according to Definition 2.2.
(ii) $v$ is regular according to Definition 2.3.

Let us define for later convenience

$$v^\pm(x) = \max\{0, \pm v(x)\},$$

so that

$$v(x) = v^+(x) - v^-(x).$$

**Remark 1.** Assumption (ii) immediately implies that $v^\pm(x) \leq \eta(\|x\|)$ for all $x \in \mathbb{R}^d$ such that $\|x\| \geq b$. Moreover, due (6) we have that $v^-(x) \leq 2B$ for all $x \in \mathbb{R}^d$, where $B$ is the stability constant of $v$. We will assume, without loss of generality, that $\eta$ is strictly positive in the interval $[0, b]$ and such that $\eta(b) \geq 2B$ so that $v^-(x) \leq \eta(\|x\|)$ for all $x \in \mathbb{R}^d$. We are also free to choose $\eta$ sufficiently well behaved in such a way that, for $\delta > 0$ small enough and for any cube $\Delta \subset \mathbb{R}^d$ of size $\delta$, there is a constant $C_\delta$ independent on the position of $\Delta$ in $\mathbb{R}^d$ such that

$$\delta^d \sup_{y \in \Delta} \eta(\|y\|) \leq C_\delta \int_{y \in \Delta} \eta(\|y\|)dy.$$

We finally define the function $V : [0, +\infty) \to (0, +\infty)$ with

$$V(r) = \int_{\mathbb{R}^d \setminus B_r(0)} \eta(\|x\|)dx,$$

where we recall that $B_r(0) = \{x \in \mathbb{R}^d : \|x\| < r\}$ is the $d$-dimensional open ball of radius $r$ centered at the origin in $\mathbb{R}^d$. Note that, due to (8), it holds that

$$\lim_{r \to \infty} V(r) = 0.$$

2.2. Assumptions on the Boundary Conditions

We will suppose hereafter that $\mathbb{R}^d$ is partitioned in elementary cubes $\Delta$ of suitable size $\delta > 0$ as follows. Let $\mathbb{Z}^d$ be the unit cubic lattice and for $\xi = (\xi_1, \ldots, \xi_d) \in \mathbb{Z}^d$ let $\Delta_\xi$ be the half-open cube of size $\delta$ centered at $\xi$, i.e.

$$\Delta_\xi = \{x = (x_1, \ldots, x_d) \in \mathbb{R}^d : (\xi_i - \frac{1}{2})\delta \leq x_i < (\xi_i + \frac{1}{2})\delta\}.$$

Then the collection $\{\Delta_\xi\}_{\xi \in \mathbb{Z}^d}$ is a partition on $\mathbb{R}^d$ in cubes $\Delta$ of size $\delta$. For $\xi = (\xi_1, \ldots, \xi_d) \in \mathbb{Z}^d$ let $|\xi|_\infty$ be the uniform norm in $\mathbb{Z}^d$, i.e., $|\xi|_\infty = \max_{i \in [d]} |\xi_i|$. Letting $\mathbb{L}_n = \{\xi \in \mathbb{Z}^d : |\xi|_\infty \leq n\}$, we will assume for simplicity (following Georgii) that the cubic box $\Lambda$ is always chosen in such a way that, for some $n \in \mathbb{N}$, $\Lambda \equiv \Lambda_n = \cup_{\xi \in \mathbb{L}_n} \Delta_\xi$ so that $|\Lambda| = |\Lambda_n| = [(2n + 1)\delta]^d$ and we agree that $\Lambda \uparrow \infty$ means $n \to \infty$. Along the paper we will seldom denote the set $\{\Delta_\xi\}_{\xi \in \mathbb{Z}^d}$ of all these cubes by the symbol $\mathbb{R}^d_\delta$ and, given $x \in \mathbb{R}^d$, we will denote by $\Delta_x$ the cube of $\mathbb{R}^d_\delta$ to which $x$ belongs.
Given $\omega \in \Omega$, we define the density of $\omega$ as the function $\rho_\omega^g : \mathbb{R}^d \to [0, +\infty) : x \mapsto \rho_\omega^g(x)$ with $\rho_\omega^g(x) = \delta^{-d} |\omega \cap \Delta_x|$. Note that, being $\omega$ is locally finite, $\rho_\omega^g(x)$ is everywhere finite. Moreover observe that $\rho_\omega^g(x)$ is constant at the value $\delta^{-d}|\Delta_x \cap \omega|$ for all $x \in \Delta_x$.

**Definition 2.4.** Let $g : [0, +\infty) \to [0, +\infty)$ be a continuous monotonic non-decreasing function such that

$$g(\alpha + \beta) \leq g(\alpha) + g(\beta),$$

and let $\rho$ be a non-negative constant. We will set

$$\Omega_{\rho,g} = \left\{ \omega \in \Omega : \rho_\omega^g(x) \leq \rho[1 + g(||x||)], \forall x \in \mathbb{R}^d \right\},$$

and define

$$\Omega_g^* = \cup_{\rho \geq 0} \Omega_{\rho,g}$$

as being the set of admissible configurations.

Therefore, configurations in $\Omega_g^*$ are those whose density increases at most as $\rho(1 + g(||x||))$ for some constant $\rho$, and some function $g$ satisfying Definition 2.4. Since $g$ is supposed non-decreasing, either $\lim_{u \to -\infty} g(u) = +\infty$, or there exists a non-negative constant $c$ such that $g(u) \leq c$ for all $u \in [0, +\infty)$. If $\lim_{u \to -\infty} g(u) = +\infty$ the density $\rho_\omega^g(x)$ of a configuration $\omega \in \Omega_g^*$ may become arbitrarily large as we move away from the origin. On the other hand for any $g$ such that $\lim_{u \to -\infty} g(u) < \infty$, the set of configurations $\Omega_g^*$ coincides with $\Omega_{g=0}^*$, namely, the set of configurations with bounded density.

**Remark 2.** It should be noted that although the intermediate set $\Omega_{\rho,g}$ defined in (14) depends on $\delta$, the set of configurations $\Omega_g^*$ defined in (15) does not depend on the choice of $\delta$.

Indeed, consider two different partitions of $\mathbb{R}^d$ where the cubes have sizes $\delta_1$ and $\delta_2$. We take $\omega \in \Omega_{\rho,g}^{\delta_1}$ and we will show that, for some finite $\tilde{\rho}$, $\omega \in \Omega_{\rho,g}^{\delta_2}$. Let $x \in \mathbb{R}^d$ and consider all the cubes in the $\delta_1$-partition that intersect the cube of the $\delta_2$-partition containing $x$. Take points $y_1, \ldots, y_m$, one in each such cube. Then, for some constant $K$ depending only on $\delta_1$ and $\delta_2$, we have:

$$\rho_{\delta_2}^\omega(x) \leq \sum_{i=1}^{m} K \rho_{\delta_1}^\omega(y_i) \leq \sum_{i=1}^{m} K \rho(1 + g(||y_i||)).$$

For $||x||$ sufficiently large, say $||x|| > R$, we have $||y_i|| \leq 2||x||$, so using the properties of $g$ we have $g(||y_i||) \leq 2g(||x||)$. Thus, in these cases we have

$$\rho_{\delta_2}^\omega(x) \leq m K \rho(1 + 2g(||x||)) \leq 2m K \rho(1 + g(||x||)) = \rho'(1 + g(||x||)).$$

For the values of $x$ such that $||x|| \leq R$ we can simply pick $\rho'' > 0$ such that

$$\rho_{\delta_2}^\omega(x) \leq \rho'' \leq \rho''(1 + g(||x||)).$$

Therefore, if we take $\tilde{\rho} = \max\{\rho', \rho''\}$ we have that $\rho_{\delta_2}^\omega(x) \leq \tilde{\rho}(1 + g(||x||))$ for all $x \in \mathbb{R}^d$. 

2.3. Well-Definiteness of the Finite-Volume Pressure

Some kind of limitation has to be imposed on the growth rate of the function $g$ introduced in Definition 2.4 for things to make sense. In particular, the following proposition provides a criterion on the possible choices for $g$ in such a way that, given a pair potential $v$ satisfying assumptions (i) and (ii), the grand canonical partition function $\Xi_\Lambda^x(\beta, \lambda)$ defined in (1) is an analytic function of $\lambda$ in the whole complex plane and the finite-volume pressure (2) is well defined and finite for all $(\beta, \lambda) \in [0, +\infty) \times [0, +\infty)$.

**Proposition 2.1.** Let $v$ be a pair potential satisfying assumptions (i) and (ii) let $\omega \in \Omega^*_g$ and let $g$ such that,

$$\int_{\mathbb{R}^d} \eta(\|x\|)g(\|x\|)dx < +\infty. \quad (16)$$

Then the grand canonical partition function defined in (1) is an analytic function of $\lambda$ in the whole complex plane and the finite-volume pressure (2) is well defined and finite for all $(\beta, \lambda) \in [0, +\infty) \times [0, +\infty)$.

**Proof.** By assumption (i) on the pair potential $v$ we have

$$\sum_{1 \leq i < j \leq n} v(x_i - x_j) \geq \sum_{1 \leq i < j \leq n} v_1(x_i - x_j) \geq -nB.$$ 

Moreover, recalling definition (9) we also have, for any $y \in \Lambda$

$$E_\Lambda^v(y, \omega) \geq -\sup_{x \in \Lambda} E_\Lambda^{v^-}(x, \omega).$$

Hence, for any $\lambda \in \mathbb{C}$ we can bound

$$|\Xi_\Lambda^x(\beta, \lambda)| \leq \sum_{n=0}^{\infty} \frac{|\lambda|^n}{n!} \int_{\mathbb{R}_\Lambda} dx_1 \ldots \int_{\Lambda} dx_n e^{\beta \left( \sum_{1 \leq i < j \leq n} v(x_i - x_j) + \sum_{i=1}^{n} E_\Lambda^x(x_i, \omega) \right)}$$

$$\leq \sum_{n=0}^{\infty} \frac{|\lambda|^n}{n!} \int_{\mathbb{R}_\Lambda} dx_1 \ldots \int_{\Lambda} dx_n e^{\beta \left[ B + \sup_{x \in \Lambda} E_\Lambda^{v^-}(x, \omega) \right]}.$$ 

I.e., for any $\lambda \in \mathbb{C}$ it holds

$$|\Xi_\Lambda^x(\beta, \lambda)| \leq \exp \left\{ \frac{|\lambda||\Lambda|e^{\beta \left[ B + \sup_{x \in \Lambda} E_\Lambda^{v^-}(x, \omega) \right]} \right\}. \quad (17)$$

Therefore, we just need to prove that hypothesis (16) implies that $E_\Lambda^{v^-}(x, \omega)$ is bounded above. Indeed, if $\omega \in \Omega^*_g$, then there exists $\rho \in [0, \infty)$ such that $\rho_\varphi^x(y) \leq \rho(1 + g(||y||))$ for all $y \in \mathbb{R}^d$. Then given $x \in \mathbb{R}^d$ and $r \geq 0$, we have

$$E_\Lambda^{v^-}(x, \omega) = \sum_{y \in \omega \cap \Lambda_c} v^-(x - y) \leq \sum_{y \in \omega} v^-(x - y) \leq \sum_{\Delta \in \mathbb{R}^d} \sup_{y \in \Delta} v^-(x - y)||\omega \cap \Delta||$$

$$\leq \delta^d \sum_{\Delta \in \mathbb{R}^d} \sup_{y \in \Delta} \eta(||x - y||)\rho_\varphi^x(y).$$

Now, by (10), we have

$$\delta^d \sup_{y \in \Delta} \eta(||x - y||)\rho_\varphi^x(y) \leq C_\delta \int_{y \in \Delta} \eta(||x - y||)\rho_\varphi^x(y)dy.$$
Hence
\[
E_{\Lambda}^v(x, \omega) \leq C_\delta \sum_{\Delta \in \mathbb{R}^d_\delta} \int_{y \in \Delta} \eta(\|x - y\|) \rho^v_\delta(y) dy = C_\delta \int_{\mathbb{R}^d} \eta(\|x - y\|) \rho^v_\delta(y) dy
\]
\[
= C_\delta \rho \int_{\mathbb{R}^d} \eta(\|x - y\|)(1 + g(\|y\|)) dy
\]
\[
\leq C_\delta \rho \int_{\mathbb{R}^d} \eta(\|x - y\|)(1 + g(\|x - y\| + \|x\|)) dy.
\]

Therefore, using (13)
\[
E_{\Lambda}^v(x, \omega) \leq C_\delta \rho \int_{\mathbb{R}^d} g(\|y\|) \eta(\|y\|) dy + (1 + g(\|x\|)) \int_{\mathbb{R}^d} \eta(\|y\|) dy
\]
Recalling (11) and using hypothesis (16) by setting \( \int_{\mathbb{R}^d} g(\|y\|) \eta(\|y\|) dy = W(0) \), we get, for any \( x \in \Lambda \),
\[
E_{\Lambda}^v(x, \omega) \leq C_\delta \rho [W(0) + (1 + g(\|x\|)) V(0)].
\]
Then, observing that \( \|x\| \leq \sqrt{dL} \) when \( x \in \Lambda \) and that, by (13), \( g(\sqrt{dL}) \leq g(dL) \leq dg(L) \), we can conclude that
\[
\sup_{x \in \Lambda} E_{\Lambda}^v(x, \omega) \leq \tilde{\kappa}(1 + g(L))
\]
with \( \tilde{\kappa} = C_\delta \rho d(W(0) + V(0)) \). Inserting (18) into (17) we get that, for all \( \lambda \in \mathbb{C} \),
\[
|\Xi^v_{\Lambda}(\beta, \lambda)| \leq \exp \left\{ |\lambda| |\Delta| e^{\beta(B+\tilde{\kappa}(1+g(L)))} \right\}.
\]
Finally the well-definiteness of the finite-volume pressure (2) for all \( (\beta, \lambda) \in [0, +\infty) \times [0, +\infty) \) follows from the fact that \( \Xi^v_{\Lambda}(\beta, \lambda) \geq 1 \) when \( \beta \geq 0 \) and \( \lambda \geq 0 \). \( \square \)

Remark 3. We emphasize that, even with \( p^v_{\Lambda}(\lambda, \beta) \) well-defined for every finite \( \Lambda \) and for every \( \omega \in \Omega^*_g \) satisfying (16), the problem of the existence of the thermodynamic limit (3) and its independence on \( \omega \) is another story entirely. As we will see in the next section, besides (16), further conditions on the function \( g \) governing the rate of growth of a configuration \( \omega \in \Omega^*_g \) must be imposed.

When \( g \) satisfies condition (16), we can define the function \( W : [0, +\infty) \to [0, +\infty) \) such that for any \( r \geq 0 \),
\[
W(r) = \int_{\mathbb{R}^d \setminus B_r(0)} \eta(\|x\|) g(\|x\|) dx.
\]
Note that, by (16), we have that
\[
\lim_{r \to \infty} W(r) = 0.
\]
3. Results and Applications

This section is divided into two parts. In the first part we enunciate the main result of this paper (Theorem 3.1 below) and make some remarks. In the second part we provide some examples which do not fit into the theory given in [16] but for which Theorem 3.1 is applicable.

3.1. Main Result

The main result presented in the present paper is illustrated in the following Theorem.

**Theorem 3.1.** Consider a continuous system of identical classical particles interacting through a superstable and regular pair potential $v$ according to assumptions (i) and (ii) and let $\omega \in \Omega_g^*$ according to Definition 2.4. Suppose that $g$ satisfies (16) and

$$\lim_{R \to \infty} \left[ 1 + g(R) \right] \int_0^R W(s) ds = 0,$$

where $V$ and $W$ are the functions defined in (11) and (19) respectively. Then, for any $\lambda \geq 0$ and $\beta \geq 0$ it holds

$$\lim_{|\Lambda| \to \infty} \frac{1}{|\Lambda|} \log \Xi^{\ast}_\Lambda(\beta, \lambda) = \lim_{|\Lambda| \to \infty} \frac{1}{|\Lambda|} \log \Xi^{\emptyset}_\Lambda(\beta, \lambda).$$

The proof of Theorem 3.1 is given in Sections 3.

**Remark 4.** Observe that the conditions (21), which ensure the existence and independence of the pressure from boundary conditions in $\Omega_g^*$ in the thermodynamic limit, impose further constraints on rate of growth of $g$ which are much more severe than those given by the condition (16) ensuring the well-definiteness of the finite-volume pressure. For example, if the pair potential $v$ decays exponentially at large distances at a rate, say, $e^{-a \|x\|}$ with a positive constant, then any $g(u)$ growing exponentially at a rate $e^{+\beta u}$ with $b \in (0, a)$ will satisfy condition (16). On the other hand, for any non-trivial pair potential $v$, the integrals $\int_0^\infty V(s) ds$ and $\int_0^\infty W(s) ds$ are nonzero (they could diverge) and therefore in order to (21) be satisfied we need at least that $\lim_{R \to \infty} g^2(R)/R = 0$. That is to say, conditions (21) allow $g$ to grow, as $R \to \infty$, at most at a rate that is strictly less than $R^{1/2}$.

Conditions (21) may appear technical but, as we will see below, it is generally straightforward to check (21) for specific potentials $v$ and growth functions $g$.

**Remark 5.** It is important to notice that when $\lim_{u \to -\infty} g(u) = +\infty$ and $g$ goes to infinity not too slowly, the space of allowed configurations $\Omega_g^*$ defined above is the support of any probability measure $P$ on $\Omega$ satisfying the estimates given by Ruelle in [33]. We recall that a probability measure $P$ on $\Omega$ satisfies the Ruelle estimates if there exist $a > 0$ and $b \geq 0$ such that for any bounded Lebesgue measurable subset $B$ of $\mathbb{R}^d$ with volume $|B| > 0$

$$P(\{|\omega \cap B| > m\}) \leq e^{-a m^2 |B| + bm}.$$ (23)
In particular, according Ruelle, the probability measure on $\Omega$ obtained as the thermodynamic limit of a grand-canonical finite-volume Gibbs measure associated to a superstable and regular pair potential satisfies the estimates (23) (see in [33] Theorem 0.1(b), Proposition 2.7, Corollary 2.9, Proposition 5.2, and Corollary 5.3).

In this regard, let us prove the following proposition.

**Proposition 3.1.** Suppose that $P$ is a probability measure on $\Omega$ satisfying (23). Let $g : \mathbb{[0, +\infty)} \rightarrow \mathbb{[0, +\infty)}$ be a function according to definition 2.4 such that, for some constant $c > 0$ and some integer $n_0 \geq 0$,

$$g(n) \geq c\sqrt{\log n} \text{ for all } n \geq n_0. \quad (24)$$

Then

$$P(\Omega_g^*) = 1.$$

**Proof.** According to the notations introduced at the beginning of Section 2.2, we start recalling that $\mathbb{R}_\delta^d = \{\Delta_\xi\}_{\xi \in \mathbb{Z}^d}$, where for any $\xi \in \mathbb{Z}^d$, $\Delta_\xi$ is the half-open cube of size $\delta$ centered at $\xi$. We also recall that if $\xi \in \mathbb{Z}^d$ then $|\xi|_{\infty}$ is the uniform norm of $\xi$. Given $\xi \in \mathbb{Z}^d$, we define

$$E_{\Delta_\xi} = \{\omega \in \Omega : |\Delta_\xi \cap \omega| > \kappa g(|\xi|_{\infty})\},$$

where $\kappa > 0$ is an arbitrary constant. Since the collection of cubes $\mathbb{R}_\delta^d$ is countable, we can describe it as $\{\Delta_n\}_{n \in \mathbb{N}}$ so that we have a family of events $\{E_n\}_{n \in \mathbb{N}}$ with $E_n = E_{\Delta_n}$. In view to apply the Borel-Cantelli lemma, let us consider the set

$$E = \bigcap_{n \in \mathbb{N}} \bigcup_{i \geq n} E_i = \{\omega \in \Omega : \omega \in E_\Delta \text{ for an infinite number of cubes } \Delta \in \mathbb{R}_\delta^d\}.$$

We claim that $E^c \subset \Omega_g^*$. Indeed, if $\omega \in E^c$, we only have to fix (repair) a finite number of cubes, and this is achieved by choosing a suitable constant $\rho$ in order to have $\omega \in \Omega_{\rho, g}$.

This means that $P(E) = 0$ will imply $P(\Omega_g^*) = 1$. By the Borel-Cantelli lemma $P(E) = 0$ if it holds that

$$\sum_{n \in \mathbb{N}} P(E_n) = \sum_{\xi \in \mathbb{Z}^d} P(E_{\Delta_\xi}) < \infty.$$

We now can use estimates (23) to bound $P(E_{\Delta_\xi})$. Namely according to (23), we can find constants $a > 0, b \geq 0$ such that $P(E_{\Delta_\xi}) \leq \exp\{-a\delta^{-d}\kappa^2 g^2(|\xi|_{\infty}) + b\kappa g(|\xi|_{\infty})\}$. Therefore, we get

$$\sum_{\xi \in \mathbb{Z}^d} P(E_{\Delta_\xi}) \leq \sum_{\xi \in \mathbb{Z}^d} e^{-a\delta^{-d}\kappa^2 g^2(|\xi|_{\infty}) + b\kappa g(|\xi|_{\infty})} = \sum_{n=0}^{\infty} e^{-a\delta^{-d}\kappa^2 g^2(n) + b\kappa g(n)} \sum_{|\xi|_{\infty} = n} 1.$$
\[ C_d \sum_{n=0}^{\infty} n^{d-1} e^{-a\delta - \kappa^2 g^2(n) + b\kappa g(n)}, \]

where the last inequality follows from the fact that \( \sum_{\xi \in \mathbb{Z}^d : |\xi|_\infty = n} 1 \leq C_d n^{d-1} \) for some constant \( C_d \) depending only on the dimension \( d \). Now, since we are assuming that \( \lim_{u \to \infty} g(u) = +\infty \), there exists \( n_0 \in \mathbb{N} \) such that \( e^{-a\delta - \kappa^2 g^2(n) + b\kappa g(n)} \leq e^{-a\delta - \kappa^2 g^2(n)/2} \) as soon as \( n \geq n_0 \). Therefore, using hypothesis (24) and setting \( \tilde{c} = ac^2/(2\delta^d) \), we get

\[
\sum_{n \geq n_0} n^{d-1} e^{-\frac{\tilde{c}}{2} \delta - \kappa^2 g^2(n)} \leq \sum_{n \geq n_0} n^{d-1} e^{-\tilde{c}\kappa^2 \log(n)} = \sum_{n \geq n_0} n^{d-1} e^{-\tilde{c}\kappa^2}.
\]

By the comparison test, we can choose \( \kappa \) sufficiently large so that the sum converges. \( \square \)

In conclusion of the present remark, observe that if \( P \) is the Gibbs measure on \( \Omega \) obtained as the thermodynamic limit of a grand-canonical finite-volume Gibbs measure associated to regular pair potential \( v \) with a hard core (i.e., such that \( v(x) = +\infty \) for all \( \|x\| < a \) with \( a > 0 \)) then \( P(\Omega_g) = 1 \) for any \( g \geq 0 \).

**Remark 6.** It is also worth to compare our space of allowed configurations \( \Omega_g^* \) (when \( g \) satisfies (21)) with the space of tempered configurations according to the definitions of Ruelle and Georgii (i.e., (5.10) in [33] and (2.24) in [16]).

We will use once again the notations introduced at the beginning of Section 2.2 and, in particular, we recall that we are taking \( \Lambda = \Lambda_n = \bigcup_{\xi \in \mathbb{L}_n} \Delta_\xi \) where \( \mathbb{L}_n = \{ x \in \mathbb{Z}^d : |\xi|_\infty \leq n \} \) so that \( |\Lambda| = |\Lambda_n| = \delta^d(2n + 1)^d \). According to these notations, the space of tempered configurations, which is denote \( \Omega_0^* \) in [16], can be defined as follows. For any \( t > 0 \), let us set

\[
\Omega(t) = \left\{ \omega \in \Omega : \limsup_{n \to \infty} \frac{1}{(2n + 1)^d} \sum_{\xi \in \mathbb{L}_n} |\omega \cap \Delta_\xi|^2 \leq t \right\}.
\]

Then \( \Omega^* = \cup_{t > 0} \Omega(t) \).

It is immediate to see that \( \Omega_0^* \subset \Omega^* \). Let us now suppose that \( g \neq 0 \) and \( \lim_{u \to \infty} g(u) = +\infty \). In this case we can show there are configurations in \( \Omega_g^* \) which are not contained in \( \Omega^* \) and viceversa, there are configurations in \( \Omega^* \) which are not contained in \( \Omega_g^* \).

Let us first show that \( \Omega_g^* \not\subset \Omega^* \). Consider any configuration \( \tilde{\omega} \in \Omega \) such that \( |\tilde{\omega} \cap \Delta_\xi| = g(|\xi|_\infty) \) for all \( \xi \in \mathbb{Z}^d \). Clearly \( \tilde{\omega} \in \Omega_g^* \). For such \( \tilde{\omega} \) we have that

\[
\sum_{\xi \in \mathbb{L}_n} |\tilde{\omega} \cap \Delta_\xi|^2 = \sum_{\xi \in \mathbb{L}_n} [g(|\xi|_\infty)]^2 = \sum_{j=0}^{n} \sum_{\xi \in \mathbb{L}_n : |\xi|_\infty = j} [g(|\xi|_\infty)]^2 = \sum_{j=0}^{n} [g(j)]^2 \sum_{\xi \in \mathbb{L}_n : |\xi|_\infty = j} 1 \geq \sum_{j=0}^{n} [g(j)]^2 (2j + 1)^{d-1}
\]
\[ \geq n^d g^2 \left( \frac{n}{2} \right) \]

whence

\[ \frac{1}{(2n + 1)^d} \sum_{\xi \in \mathbb{L}_n} |\tilde{\omega} \cap \Delta \xi|^2 \geq 3^{-d} g^2 \left( n/2 \right). \quad (25) \]

Since we are assuming that \( \lim_{u \to \infty} g(u) = +\infty \), from inequality (25) we conclude that

\[ \lim_{n \to \infty} \frac{1}{(2n + 1)^d} \sum_{\xi \in \mathbb{L}_n} |\tilde{\omega} \cap \Delta \xi|^2 = +\infty, \]

and therefore \( \tilde{\omega} \notin \Omega^* \).

Let us now show that \( \Omega^* \not\subset \Omega^*_{g} \). Setting \( S_j = \{ \xi \in \mathbb{Z}^d : |\xi|_{\infty} = j \} \), let \( \alpha > 0 \) and consider a configuration \( \tilde{\omega} \) such that for every \( j \in \mathbb{N} \cup \{0\} \) there is only one \( \xi \in S_j \) such that \( |\tilde{\omega} \cap \Delta \xi| = \sqrt{j} \) while for all other \( \xi' \in S_j \), we have \( |\tilde{\omega} \cap \Delta \xi'| = 0 \). For such an \( \tilde{\omega} \) we have

\[ \sum_{\xi \in \mathbb{L}_n} |\tilde{\omega} \cap \Delta \xi|^2 = \sum_{j=0}^{n} j \leq \frac{1}{2} (n + 1)^2. \]

Therefore, we conclude that if \( d \geq 2 \), the factor \( (2n + 1)^{-d} \sum_{\xi \in \mathbb{L}_n} |\tilde{\omega} \cap \Delta \xi|^2 \) is bounded by a constant uniformly in \( n \) and thus \( \tilde{\omega} \in \Omega^* \). On the other hand, observing that by condition (21) the function \( g(u) \) can grow at most as \( u^\beta \) with \( \beta < \frac{1}{2} \) (cfr, Remark 4), we also have that \( \tilde{\omega} \notin \Omega^*_{g} \).

3.2. Applications

We now present three examples for which Theorem 3.1 is applicable. All these examples were out of reach of the preexisting theory given in [16], either because the potential not necessarily diverges at the origin, or because the density of the boundary conditions that ensure the uniqueness of the pressure is allowed to diverge as one moves away from the origin.

3.2.1. Systems Interacting via General Superstable Pair Potentials. The first application of Theorem 3.1 is illustrated by the following statement.

**Theorem 3.2.** Let \( v \) superstable and regular according to assumptions (i) and (ii) and let \( \omega \in \Omega^*_0 \) (i.e configurations with bounded density), then

\[ \lim_{\Lambda \to \infty} \log \Xi_{\Lambda}^0(\beta, \lambda) = \lim_{\Lambda \to \infty} \frac{1}{|\Lambda|} \log \Xi_{\Lambda}^0(\beta, \lambda). \quad (26) \]

**Proof.** Let us show that if \( \omega \in \Omega^*_0 \), any superstable and regular pair potential satisfies (21) and therefore, by Theorem 3.1, equality (22) holds. Indeed, when \( g = 0 \) the condition (21) simply boils down to

\[ \lim_{R \to \infty} \int_0^R V(s) ds = 0. \quad (27) \]
Now, if \( \lim_{R \to \infty} \int_0^R V(s)ds < +\infty \), then equation (27) is trivially true. On the other hand, if \( \lim_{R \to \infty} \int_0^R V(s)ds = \infty \), then by l'Hopital rule
\[
\lim_{R \to \infty} \frac{\int_0^R V(R)ds}{R} = \lim_{R \to \infty} V(R) = 0,
\]
and thus we have that condition (27) holds. \( \square \)

**Remark 7.** Note that already the very general situation depicted in this example does not fit into the theory elaborated in [16] where, we recall, the pair potential, besides superstable, is required to be non-integrably divergent at the origin.

### 3.2.2. Systems Interacting via Pair Potentials Decaying Polynomially at Large Distances.

In the example illustrated by Theorem 3.3 below, we suppose that the potential \( v \) is of Lennard-Jones type at large distances. In this case the density distribution of the boundary conditions that guarantee the existence and uniqueness of the pressure is allowed to increase sublinearly with the distance from the origin. Note once again that also in this example no condition besides superstability is imposed to the pair potential at short distances.

**Theorem 3.3.** Let \( v \) superstabe and regular according to assumptions (i) and (ii) and suppose that the function \( \eta \) given in (ii) is such that, for some constant \( C \) and some \( p > 0 \),
\[
\eta(r) \leq \frac{C}{r^{d+p}} \quad \text{for all } r \geq b. \tag{28}
\]
Let \( \omega \in \Omega_g^* \) with \( g(r) = r^q \) and \( q \) such that
\[
0 < q < \frac{1}{2} \min\{1, p\}.
\]
Then (26) holds true.

**Proof.** Let us apply Theorem 3.1. First observe that function \( g(r) = r^q \) satisfies both Definition 2.4 and condition (16) as soon as \( q < \{1, p\} \) since, for any \( \alpha, \beta > 0 \), it holds that \( (\alpha + \beta)q \leq \alpha^q + \beta^q \) when \( q \leq 1 \) and the l.h.s. of (16) is finite if \( q < p \). So we just need to check that conditions (21) are satisfied. Let us first prove that
\[
\lim_{R \to \infty} \frac{[1 + g(R)] \int_0^R W(s)ds}{R} = 0. \tag{29}
\]
Note that, according to the hypothesis os Theorem 3.3, we can write
\[
\lim_{R \to \infty} \frac{[1 + g(R)] \int_0^R W(s)ds}{R} = \lim_{R \to \infty} \frac{1 + R^q}{R} \int_0^b W(s)ds + \lim_{R \to \infty} \frac{1 + R^q}{R} \int_b^R W(s)ds. \tag{30}
\]
Since \( \int_0^b W(s)ds \) is a constant not depending on \( R \), the first term in the r.h.s. of (30) goes to zero as \( R \) goes to infinity as soon as \( q < 1 \). Concerning the
second term in the l.h.s. of (30), recalling the definition (19) of the function $W$ and by the hypothesis of Theorem 3.3, we can bound, for $R \geq 1$, 
\[
\frac{1 + R^q}{R} \int_b^R W(s) ds \leq 2R^{q-1} \int_b^R ds \int_{R^{d-1}B_s(0)} \frac{C}{\|x\|^{d+p-q}} dx \leq K_1 R^{q-1} \int_b^R \frac{1}{s^{p-q}} ds,
\]
(31)
where $K_1$ is a constant not depending on $R$. It is now easy to check that r.h.s. of (31) goes to zero as $R$ goes to infinity if $q < \min\{1, \frac{p}{2}\}$. So (29) holds if $q < \min\{1, \frac{p}{2}\}$. The second condition in (21), namely 
\[
\lim_{R \to \infty} [1 + g(R)]^2 \frac{\int_0^R V(s) ds}{R} = 0,
\]
(32)
can be proved proceeding similarly. As before, we can split 
\[
\lim_{R \to \infty} [1 + g(R)]^2 \frac{\int_0^R V(s) ds}{R} = \lim_{R \to \infty} (1 + R^q)^2 \frac{\int_0^b V(s) ds}{R} + \lim_{R \to \infty} \frac{(1 + R^q)^2}{R} \int_b^R V(s) ds.
\]
(33)
The first term in l.h.s. of (33) now goes to zero when $R \to \infty$ as soon as $q < 1/2$. The second term in l.h.s. of (33), when $R \geq 1$, can be bounded as 
\[
\frac{(1 + R^q)^2}{R} \int_b^R V(s) ds \leq K_2 R^{2q-1} \int_b^R \frac{1}{s^{p}} ds.
\]
(34)
where $K_2$ is a constant not depending on $R$. The r.h.s. of (34) now goes to zero as $R$ goes to infinity if $q < \min\{\frac{1}{2}, \frac{p}{2}\}$. In conclusion we have proved, if $q < \min\{\frac{1}{2}, \frac{p}{2}\}$, then both (29) and (32) hold. This completes the proof of Theorem 3.3. \[\square\]

**Remark 8.** It is interesting to compare the hypothesis of Theorem 3.3 implying the uniqueness of the infinite-volume pressure with those used in the references [5, 19, 20] to prove the existence of infinite-volume Gibbs states. Indeed, the framework analyzed in these references seems somehow close to that of the present paper. In particular, in [19] the authors consider a system of classical particles interacting through a pair potential $v$ quite similar to that analyzed in Theorem 3.3. In [19] the authors assume that $v$ satisfies the Dobrushin-Fisher-Ruelle criterion (see [32]), namely $v$ is superstable, it decays at large distances as in (28) and diverges nonintegrably at the origin. These assumptions on the pair potentials are sufficient, according to [19], to prove the existence of infinite-volume Gibbs states as the limit of finite-volume Gibbs states with boundary conditions whose local density grows at most proportionally to $(\ln r)^{\frac{3}{2}}$ with the distance $r$ from the origin. In reference [20] the main results concern system of particles interacting via a finite range potential (see there Sec. 2.4), yet, in the last section of [20] authors discuss how to handle infinite range potentials with polynomial decay at long distances. As in [16, 19], in [20] it is assumed that the potential $v$, besides being stable, diverges non-integrably at the origin. With
this hypothesis on the pair potential the authors illustrate the construction of infinite-volume Gibbs states under a class of tempered boundary condition (see (5.8) in [20]) which is similar to ours: namely, if \( v \sim r^{-d-p} \) when \( r \to \infty \), then the local density of boundary conditions is allowed to grow at a rate \( r^q \) with \( q < \frac{p}{2} \). Finally, it is worth to mention the recent reference [5] where the existence of at least one infinite-volume Gibbs measure for systems of particles interacting via an infinite range pair potential is proved under the assumption that the pair potential is stable (but not necessarily superstable!) and “intensity regular” (see Definition 2.4 in [5]). The latter property is fulfilled if the pair potential is regular according (ii) in presence of boundary conditions with bounded local density.

3.2.3. Systems Interacting via Morse Potential. We conclude this section by providing a last application of Theorem 3.1 concerning a system of particles in \( \mathbb{R}^3 \) interacting via a pair potential extensively used in applications in physics and chemistry. Namely, let us consider the Morse potential. This potential is strongly repulsive at short distances and weakly attractive at long distances, but unlike the Lennard-Jones potential, it is everywhere finite and in particular does not diverge at \( r = 0 \). By suitably rescaling domain and codomain we may assume, without loss of generality that our Morse potential is given by

\[
v_\rho(r) = e^{-2\rho(r-1)} - 2e^{-\rho(r-1)}. \quad (35)
\]

where \( \rho > \ln 2 \). Observe that the minimum is reached at \( r = 1 \) with \( v_\rho(r = 1) = -1 \) and the maximum is attained at \( r = 0 \) with \( v_\rho(0) = e^\rho(e^\rho - 2) \). Note that \( v_\rho(r) \) is negative and decays exponentially at large distances and it is finite positive and monotonically decreasing in the interval \( r \in [0, 1 - (\ln 2)/\rho] \).

**Theorem 3.4.** Consider a continuous system of classical particles in \( \mathbb{R}^3 \) interacting through a Morse pair potential \( v_\rho(r) \) as in (35) with \( \rho > \ln 16 \). Let \( \omega \in \Omega_g^* \) with \( g(r) = r^\alpha \) where \( \alpha < \frac{1}{2} \). Then, for any \( \lambda \geq 0 \) and \( \beta \geq 0 \) it holds

\[
\lim_{\Lambda \uparrow \infty} \frac{1}{|\Lambda|} \log \Xi^\omega_\Lambda(\beta, \lambda) = \lim_{\Lambda \uparrow \infty} \frac{1}{|\Lambda|} \log \Xi^\emptyset_\Lambda(\beta, \lambda). \quad (36)
\]

**Proof.** Identity (36) holds if the Morse potential and the growth function \( g \) satisfies the hypothesis of Theorem 3.1. As observed by Ruelle (see exercise 3B pag. 68 in [32]), \( v_\rho(r) \) is stable in \( d = 3 \) as soon as \( \rho \geq \ln 16 \). Therefore, \( v_\rho(r) \) is superstable if \( \rho > \ln 16 \) (because, setting \( \rho = \ln 16 + \varepsilon \) with \( \varepsilon > 0 \), \( v_{\ln 16+\varepsilon}(r) - v_{\ln 16}(r) \geq 0 \) is non-negative everywhere and it is strictly positive when \( r \in [0, 1) \)). Moreover \( v_\rho(r) \) is regular (since it decays as \( e^{-\rho r} \) as \( r \to \infty \)) and we can take \( \eta(r) = 2e^{-\rho(r-1)} \) so that \( v_\rho'(r) \leq \eta(r) \) for all \( r \geq 0 \). Thus we just need to establish which growth function \( g \) we may take in order to satisfy the conditions (21) of Theorem 3.1. It is straightforward to check that, by setting \( g(r) = r^q \) with \( q < \frac{1}{2} \) (which satisfies (13)), conditions (21) hold since both integrals \( \int_0^\infty V(s)ds \) and \( \int_0^\infty W(s)ds \) are finite. \( \square \)
4. Proof of Theorem 3.1

We will prove Theorem 3.1 by showing that

$$
\limsup_{\Lambda \uparrow \infty} \frac{1}{|\Lambda|} \log \Xi^\omega(\beta, \lambda, \Lambda) \leq \lim \frac{1}{|\Lambda|} \log \Xi^0(\beta, \lambda, \Lambda), \tag{37}
$$

and

$$
\liminf_{\Lambda \uparrow \infty} \frac{1}{|\Lambda|} \log \Xi^\omega(\beta, \lambda, \Lambda) \geq \lim \frac{1}{|\Lambda|} \log \Xi^0(\beta, \lambda, \Lambda). \tag{38}
$$

4.1. Proof of Inequality (37)

In this section we will denote by $\vec{x}$ a generic configuration of $\Omega_\Lambda$ and we will use the following shorter notations.

$$
\sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \int_{\Lambda} dx_1 \ldots \int_{\Lambda} dx_n (\cdot) \equiv \int_{\Omega_\Lambda} d\mu_\Lambda(\vec{x})(\cdot),
$$

$$
v(\vec{x}) \equiv \sum_{\{x,y\} \subset \vec{x}} v(x - y),
$$

$$
E_\Lambda^v(\vec{x}, \omega, x, \omega) \equiv \sum_{x \in \vec{x}} E_\Lambda^v(x, \omega),
$$

$$
E_\Lambda^{\pm}(\vec{x}, \omega, x, \omega) \equiv \sum_{x \in \vec{x}} E_\Lambda^{\pm}(x, \omega). \tag{39}
$$

So that

$$
\Xi^\omega(\beta, \Lambda) = \int_{\Omega_\Lambda} d\mu_\Lambda(\vec{x}) e^{-\beta \left[ v(\vec{x}) - E_\Lambda^v(\vec{x}, \omega) \right]}.
$$

Again, we recall that $\mathbb{R}^d$ is partitioned in elementary cubes $\Delta$ of size $\delta > 0$ with $\delta$ chosen in such a way that, for fixed cube $\Lambda$ of size $L$ centered at the origin, $|\Lambda|/\delta^d$ is integer so that $\Lambda_\delta$ denotes the set of elementary cubes forming $\Lambda$ and $|\Lambda_\delta|$ is its cardinality. Let $\Omega_\Lambda^{\pm}$ denote the set of configurations $\vec{x} \in \Lambda$ such that in each cube $\Delta \in \Lambda_\delta$ there is one and only one particle. We define the following crucial quantities.

$$
S^\omega_\Lambda(\delta) = \sup_{\vec{x} \in \Omega_\Lambda^{\pm}} E_\Lambda^{\pm}(\vec{x}, \omega). \tag{40}
$$

$$
K^\omega_\Lambda = \sup_{x \in \Lambda} E_\Lambda^{\pm}(x, \omega). \tag{41}
$$

Note that, if $\omega \in \Omega^*_g$ then $S^\omega_\Lambda(\delta)$ and $K^\omega_\Lambda$ are well defined. Indeed, recalling (18), we have

$$
K^\omega_\Lambda \leq \tilde{\kappa}(1 + g(L)), \tag{42}
$$

and thus $S^\omega_\Lambda(\delta) \leq |\Lambda_\delta| \tilde{\kappa}(1 + g(L))$. 

Now, if $S^u_\Lambda(\delta) = 0$, then $E^u_\Lambda(x, \omega) = 0$ for all $x \in \Lambda$. This implies that $E^u_\Lambda(\bar{x}, \omega) = E^u_\Lambda(\bar{x}, \omega)$ for all $\bar{x} \in \Lambda$ and hence
\[
\Xi^\omega_\Lambda(\beta, \lambda) = \int_\Lambda d\bar{x}e^{-\beta v(\bar{x})} - \beta E^v_\Lambda(\bar{x}, \omega) \leq \int_\Lambda d\bar{x}e^{-\beta v(\bar{x})} = \Xi^0_\Lambda(\beta, \lambda),
\]
which implies trivially (37). Therefore for the rest of this section we may assume that
\[
\lim \inf_{\Lambda \uparrow \infty} S^u_\Lambda(\delta) > 0.
\]

In order to prove (37) we will need to control the possible growth to infinity of the product $S^u_\Lambda(\delta)K^u_\Lambda$ as $\Lambda \uparrow \infty$. Such a control is provided by the following lemma.

**Lemma 4.1.** Under the hypothesis of Theorem 3.1, let $\omega \in \Omega^*_g$. Then
\[
\lim_{\Lambda \uparrow \infty} \frac{S^u_\Lambda(\delta)K^u_\Lambda}{|\Lambda|} = 0. \tag{43}
\]

**Proof.** Recalling definitions (39) and (40) we have
\[
S^u_\Lambda(\delta) = \sup_{\bar{x} \in \Omega^u_\Lambda} E^u_\Lambda(\bar{x}, \omega)
\leq \sum_{\Delta \subset \Lambda} \sup_{x \in \Delta} E^u_\Lambda(x, \omega)
\leq \sum_{\Delta \subset \Lambda} \sup_{x \in \Delta} E^u_d(x, \omega)
\leq \delta^d \sum_{\Delta \subset \Lambda} \sup_{x \in \Delta} \sum_{\Delta' \in \Lambda^c} \sup_{y \in \Delta'} \eta(\|x - y\|)\rho^u_\delta(y).
\]

Now, similarly as we did in the proof of Proposition 2.1 we may use inequality (10) to bound, for some constant $C_\delta$,
\[
\delta^d \sup_{y \in \Delta'} \eta(\|x - y\|)\rho^u_\delta(y) \leq C_\delta \int_{\Delta'} \eta(\|x - y\|)\rho^u_\delta(y)dy.
\]

Hence
\[
S^u_\Lambda(\delta) \leq C_\delta \sum_{\Delta \subset \Lambda} \sup_{x \in \Delta} \int_{\Delta'} \eta(\|x - y\|)\rho^u_\delta(y)dy
\leq C_\delta \sum_{\Delta \subset \Lambda} \int \eta(\|x - y\|)\rho^u_\delta(y)dy.
\]

Now, we again bound $\delta^d \sup_{x \in \Delta} \eta(\|x - y\|)$ by $C_\delta \int_\Delta \eta(\|x - y\|)dx$ and we get
\[
S^u_\Lambda(\delta) \leq \frac{C_\delta^2}{\delta^d} \sum_{\Delta \subset \Lambda} \int_{\Delta} \left( \int \eta(\|x - y\|)dx \right) \rho^u_\delta(y)dy
= \frac{C_\delta^2}{\delta^d} \sum_{\Delta \subset \Lambda} \int_{\Delta} dx \int_{\Delta} \eta(\|x - y\|)\rho^u_\delta(y)dy
\]
\[ \frac{C_\delta^2}{\delta^d} \int_{\Lambda} dx \int_{y \in \Lambda^c} \eta(||x - y||) \rho_\delta^y(y) dy. \]

I.e., setting \( K_\delta = \frac{C_\delta^2}{\delta^d} \), we get

\[ S_\Lambda^\omega(\delta) \leq K_\delta \int_{x \in \Lambda} dx \int_{y \in \Lambda^c} \eta(||x - y||) \rho_\delta^y(y) dy \]
\[ \leq \rho K_\delta \int_{x \in \Lambda} dx \int_{y \in \Lambda^c} \eta(||x - y||)(1 + g(||y||)) dy \]
\[ \leq \rho K_\delta \int_{x \in \Lambda} dx \int_{y \in \Lambda^c} \eta(||x - y||) \left[ 1 + g(||x - y||) + g(||x||) \right] dy. \]

Now,
\[ \int_{y \in \Lambda^c} \eta(||x - y||) \left[ 1 + g(||x - y||) + g(||x||) \right] dy \]
\[ \leq \int_{||y|| \geq d_x^L} \eta(||y||) \left[ 1 + g(||y||) + g(||x||) \right] dy, \]
where recall that \( d_x^\Lambda \) is the distance of \( x \in \Lambda \) from the boundary \( \partial \Lambda \) of \( \Lambda \). Moreover, since \( \sup_{x \in \Lambda} g(||x||) = g(\sqrt{dL}) \leq g(dL) \leq dg(L) \), we can bound

\[ \int_{y \in \Lambda^c} \eta(||x - y||) \left[ 1 + g(||x - y||) + g(||x||) \right] dy \leq \int_{||y|| \geq d_x^L} \eta(||y||) \left[ 1 + g(||y||) + dg(L) \right] dy \]
\[ = W(d_x^L) + (1 + dg(L))V(d_x^L), \]
where in the last line we have used definitions (11) and (19). Therefore, setting
\[ F(d_x^L) = W(d_x^L) + (1 + dg(L))V(d_x^L), \]
we have that
\[ S_\Lambda^\omega(\delta) \leq \rho K_\delta \int_{x \in \Lambda} F(d_x^L) dx. \]

Now, recalling that \( \Lambda \) is a \( d \)-dimensional hypercube of size \( L \) centered at the origin, and thus \( 0 \leq d_x^L \leq L/2 \), we have that
\[ \int_{x \in \Lambda} F(d_x^L) dx = \int_0^{L/2} F(r) 2d \left[ 2 \left( \frac{L}{2} - r \right) \right]^{d-1} dr \leq 2dL^{d-1} \int_0^L F(r) dr. \]

Therefore, we get
\[ S_\Lambda^\omega(\delta) \leq 2d^2 \rho L^{d-1} K_\delta \int_0^L \left[ W(r) + (1 + g(L))V(r) \right] dr \]

Now, recalling (42) and also using that \( |\Lambda| = (2L)^d \geq 2L^d \), we have,
\[ S_\Lambda^\omega(\delta) K_\Lambda^\omega \leq \kappa_\delta \left[ [1 + g(L)] \int_0^L \frac{W(r) dr}{L} + [1 + g(L)]^2 \int_0^L \frac{V(r) dr}{L} \right], \]
where we have set \( \kappa_\delta = d^2 \rho L^{d-1} K_\delta \kappa \). In conclusion, given that \( g \) satisfies (21), (43) is proved. \( \square \)
Towards the proof of (37) we will now prove a key lemma strongly relying on the assumed superstability of the pair potential $v$ (Lemma 4.2 below). Guessing that the statement of this lemma may sound rather technical, we anticipate, before enunciating it, its interpretation and its purpose. If we have a configuration of particles inside $\Lambda$ that feels a strong negative energy from the outside particles (measured by the quantity $pS^\omega_\Lambda(\delta)$ where $p$ is an integer), then this configuration must be constituted by a large number of particles and thus there must be many pairs of particles at short distance. Lemma 4.2 below shows that the contribution to the energy of this large number of short-distance pairs of particles inside $\Lambda$ is strongly positive (i.e., of the order $p^2S^\omega_\Lambda(\delta)/K^\omega_\Lambda$). This positive energy, as will be shown later on, comfortably compensates the strong negative energy produced by the outside particles, so that this kind of configurations will have low probability density and thus they will be under control.

**Lemma 4.2.** *Under the assumption of Theorem 3.1, let $p \in \mathbb{N}$, $\bar{x} \in \Omega_\Lambda$ and $\omega \in \Omega^*_\Lambda$. Suppose that $K^\omega_\Lambda > 0$ and $E^-_\Lambda(\bar{x}, \omega) > pS^\omega_\Lambda(\delta)$. Then there exist constants $a$ and $c$ such that, for all $\delta \in (0, a/\sqrt{d})$,

$$v_2(\bar{x}) \geq c \frac{\sqrt[p]{p-1}}{4} \frac{S^\omega_\Lambda(\delta)}{K^\omega_\Lambda}.$$  

*Proof.* Since $v$ is superstabile, it can be written as $v_1 + v_2$ with the non-negative continuous function $v_2$ such that $v_2(0) > 0$. So there exist constants $a$ and $c$ such that

$$v_2(x) \geq c \text{ for all } ||x|| \leq a.$$  

If we take $\delta < ad^{-\frac{1}{2}}$, then, for all $\Delta \in \mathbb{R}^d$, we have that $||x - y|| \leq a$ for all pairs $\{x, y\} \subset \Delta$. Let us thus assume that $\delta \in (0, a/\sqrt{d})$.

Due to definition (40), if $E^-_\Lambda(\bar{x}, \omega) > pS^\omega_\Lambda(\delta)$, then there exists at least a cube $\Delta \in \Lambda_\delta$ containing $p + 1$ particles. Indeed if $\bar{x}$ is a configuration with at most $p$ particles in each cube then $E^-_\Lambda(\bar{x}, \omega) \leq pS^\omega_\Lambda$ in contradiction with the hypothesis. Since $E^-_\Lambda(\bar{x}, \omega) > pS^\omega_\Lambda > (p - 1)S^\omega_\Lambda$ then for the same reason we can find a cube $\Delta_1$ containing at least $p$ particles of the configuration $\bar{x}$ and, since $\delta < a/\sqrt{d}$, all these particles in $\Delta_1$ are at mutual distance less than $a$.

Choose one particle inside $\Delta_1$, call $x_1$ its position and call $\bar{x}_1 = \bar{x} \setminus \{x_1\}$. We have, by (44), that

$$v_2(\bar{x}) = \sum_{x \in \bar{x}_1} v_2(x - x_1) + v_2(\bar{x}_1) \geq c(p - 1) + v_2(\bar{x}_1).$$

Remove now $x_1$ from $\bar{x}$ so that we are left with the new configuration $\bar{x}_1$. This new configuration is such that

$$E^-_\Lambda(\bar{x}_1, \omega) = E^-_\Lambda(\bar{x}, \omega) - E^-_\Lambda(x_1, \omega) > pS^\omega_\Lambda(\delta) - K^\omega_\Lambda \geq pS^\omega_\Lambda(\delta) - S^\omega_\Lambda(\delta) = (p - 1)S^\omega_\Lambda(\delta).$$

So we could extract at least one point from the configuration $\bar{x}$ and yet, for the new configuration $\bar{x}_1$, the condition $E^-_\Lambda(\bar{x}_1, \omega) > (p - 1)S^\omega_\Lambda(\delta)$ still holds.
We can therefore repeat the process and extract \( m \geq 1 \) points from the configuration \( \vec{x} \) in such way that

\[
pS^\omega_A(\delta) - mK^\omega_A > (p - 1)S^\omega_A(\delta),
\]
i.e., \( m \) must be such that

\[
1 \leq m < \frac{S^\omega_A(\delta)}{K^\omega_A}.
\]

Namely, we can extract \( m = \left\lfloor \frac{S^\omega_A(\delta)}{K^\omega_A} \right\rfloor \) points from the configuration \( \vec{x} \) in such way that for the remaining configuration \( \vec{x}' = \vec{x} \setminus \{x_1, \ldots, x_m\} \) it holds

\[
E^\omega_A(\vec{x}', \omega) > (p - 1)S^\omega_A(\delta)
\]
and

\[
v_2(\vec{x}) \geq c \left[ \frac{S^\omega_A(\delta)}{K^\omega_A} \right] (p - 1) + v_2(\vec{x}') \geq c \left[ \frac{S^\omega_A(\delta)}{K^\omega_A} \right] (p - 1) + v_2(\vec{x}').
\]

Now the remaining configuration \( \vec{x}' \) has the property \( E^\omega_A(\vec{x}', \omega) > (p - 1)S^\omega_A(\delta) \). So, applying the same process to bound \( v_2(\vec{x}') \) we get

\[
v_2(\vec{x}) \geq c \left[ \frac{S^\omega_A(\delta)}{K^\omega_A} \right] (p - 1) + c \left[ \frac{S^\omega_A(\delta)}{K^\omega_A} \right] (p - 2) + v_2(\vec{x}'')
\]
where now \( \vec{x}'' \) is such that \( E^\omega_A(\vec{x}'', \omega) > (p - 2)S^\omega_A(\delta) \). Iterating we get

\[
v_2(\vec{x}) \geq c \left[ \frac{S^\omega_A(\delta)}{K^\omega_A} \right] \frac{p(p - 1)}{2}
\]
and since \( \left[ S^\omega_A(\delta)/K^\omega_A \right] \geq \frac{1}{2} S^\omega_A(\delta)/K^\omega_A \) (because, by definitions (40) and (41) we have that \( S^\omega_A(\delta) \geq K^\omega_A \) and thus \( \left[ S^\omega_A(\delta)/K^\omega_A \right] \geq 1 \)), the proof is concluded.

We are now in the position to conclude the proof of (37) if the hypothesis the Theorem 3.1 are satisfied. Let us set

\[
E_A = \left[ S^\omega_A(\delta)K^\omega_A \right]^\frac{1}{2} |A|^{\frac{3}{2}}.
\]

By Lemma 4.1 we have that

\[
\lim_{|A| \to \infty} \frac{E_A}{|A|} = \lim_{|A| \to \infty} \left( \frac{S^\omega_A(\delta)K^\omega_A}{|A|} \right)^\frac{1}{3} = 0
\]

\[
\lim_{|A| \to \infty} \frac{E_A}{S^\omega_A(\delta)K^\omega_A} = \lim_{|A| \to \infty} \left( \frac{|A|}{S^\omega_A(\delta)K^\omega_A} \right)^{\frac{2}{3}} = +\infty.
\]
We now can write
\[
\Xi_\Lambda^\omega(\beta, \lambda) = \int_{\Omega_\Lambda} d\mu_\Lambda(\bar{x}) e^{-\beta v(\bar{x}) - \beta E_\Lambda^v(\bar{x}, \omega)}
\]
\[
= \int_{\Omega_\Lambda : E_\Lambda^v(\bar{x}, \omega) \leq E_\Lambda} d\mu_\Lambda(\bar{x}) e^{-\beta v(\bar{x}) - \beta E_\Lambda^v(\bar{x}, \omega)}
+ \int_{\Omega_\Lambda : E_\Lambda^v(\bar{x}, \omega) > E_\Lambda} d\mu_\Lambda(\bar{x}) e^{-\beta v(\bar{x}) - \beta E_\Lambda^v(\bar{x}, \omega)}
\]
\[
\leq e^{\beta E_\Lambda} \int_{\Omega_\Lambda : E_\Lambda^v(\bar{x}, \omega) \leq E_\Lambda} d\mu_\Lambda(\bar{x}) e^{-\beta v(\bar{x})}
+ \int_{\Omega_\Lambda : E_\Lambda^v(\bar{x}, \omega) > E_\Lambda} d\mu_\Lambda(\bar{x}) e^{-\beta v(\bar{x}) + \beta E_\Lambda^v(\bar{x}, \omega)}
\]
\[
\leq e^{\beta E_\Lambda} \int_{\Omega_\Lambda} d\mu_\Lambda(\bar{x}) e^{-\beta v(\bar{x})}
+ \int_{\Omega_\Lambda : E_\Lambda^v(\bar{x}, \omega) > E_\Lambda} d\mu_\Lambda(\bar{x}) e^{-\beta v(\bar{x}) - E_\Lambda^v(\bar{x}, \omega)}.
\]
Namely, we get
\[
\Xi_\Lambda^\omega(\beta, \lambda) \leq e^{\beta E_\Lambda} \Xi_{\Lambda}^0(\beta, \lambda) + \int_{\Omega_\Lambda : E_\Lambda^v(\bar{x}, \omega) > E_\Lambda} d\mu_\Lambda(\bar{x}) e^{-\beta v(\bar{x}) - E_\Lambda^v(\bar{x}, \omega)}.
\]

Let us consider the second term in the r.h.s. of inequality (48). By hypothesis \( v = v_1 + v_2 \) with \( v_1 \) stable with stability constant equal to \( B \). Therefore, we can bound
\[
v(\bar{x}) - E_\Lambda^v(\bar{x}, \omega) \geq -B|\bar{x}| + v_2(\bar{x}) - E_\Lambda^v(\bar{x}, \omega).
\]
We can now use Lemma 4.2 to bound from below \( v_2(\bar{x}) - E_\Lambda^v(\bar{x}, \omega) \). Let \( p \) be defined as the following integer.
\[
p = \left\lceil \frac{E_\Lambda^v(\bar{x}, \omega)}{2S_\Lambda^\omega(\delta)} \right\rceil + 2.
\]
By the fact that we are considering here second term in the r.h.s. of inequality (48) where \( E_\Lambda^v(\bar{x}, \omega) > E_\Lambda \) and since (47) implies that \( E_\Lambda / S_\Lambda^\omega(\delta) \) goes to infinity when \( \Lambda \uparrow \infty \), we have that \( E_\Lambda^v(\bar{x}, \omega) / S_\Lambda^\omega(\delta) \) is surely larger than 4 for \( \Lambda \) large enough. Then, using that \( x \geq \lfloor \frac{\delta}{2} \rfloor + 2 \) for all \( x \geq 4 \), we have
\[
E_\Lambda^v(\bar{x}, \omega) = \frac{E_\Lambda^v(\bar{x}, \omega)}{S_\Lambda^\omega(\delta)} S_\Lambda^\omega(\delta) > \left( \left\lceil \frac{E_\Lambda^v(\bar{x}, \omega)}{2S_\Lambda^\omega(\delta)} \right\rceil + 2 \right) S_\Lambda^\omega(\delta) = pS_\Lambda^\omega(\delta).
\]
Hence we can use Lemma 4.2 to bound
\[
v_2(\bar{x}) - E_\Lambda^v(\bar{x}, \omega) \geq \frac{c}{4} p(p - 1) S_\Lambda^\omega(\delta) - E_\Lambda^v(\bar{x}, \omega)
\]
where in the last line we have once again considered that we are bounding the second term in r.h.s. of (48) in which the integral is over configurations $\bar{x}$ such that $E_A^-(\bar{x}, \omega) \geq E_A$.

In conclusion we have obtained that

$$v_2(\bar{x}) - E_A^-(\bar{x}, \omega) \geq E_A \left[ \frac{c}{16} \frac{E_A}{S_\omega(\delta) K_\omega} - 1 \right] \equiv G_A.$$ 

Let us analyze the behaviour of the ratio $G_A/|A|$ as $\Lambda \uparrow \infty$. Recalling (45) and (42), we get

$$G_A \geq E_A \left[ \frac{c}{16} \frac{E_A}{S_\omega(\delta) K_\omega} - 1 \right] = \frac{c}{16} \left( \frac{|A|}{S_\omega(\delta) K_\omega} \right)^{1/3} - \left( \frac{S_\omega(\delta) K_\omega}{|A|} \right)^{1/3}$$

and thus in force of (46) and (47) we have that

$$\lim_{\Lambda \uparrow \infty} \frac{G_A}{|A|} = +\infty.$$
\[
\leq \lim_{\Lambda \uparrow \infty} \frac{1}{|\Lambda|} \log \left[ e^{\beta E_\Lambda \Xi^0_\Lambda (\beta, \lambda)} \right] \\
+ \lim_{\Lambda \uparrow \infty} \frac{1}{|\Lambda|} \log \left( 1 + e^{-|\Lambda| \left( \frac{\beta G_\Lambda}{|\Lambda|} - \lambda e^{\beta B} \right)} \right) \\
= \lim_{\Lambda \uparrow \infty} \frac{\beta E_\Lambda}{|\Lambda|} + \lim_{\Lambda \uparrow \infty} \frac{1}{|\Lambda|} \log \Xi^0_\Lambda (\beta, \lambda) \\
= \lim_{\Lambda \uparrow \infty} \frac{1}{|\Lambda|} \log \Xi^0_\Lambda (\beta, \lambda)
\]

and thus inequality (37) is proved.

4.2. Proof of Inequality (38)

We start by proving the following preliminary lemma.

**Lemma 4.3.** Let \( g \) be admissible and let \( \omega \in \Omega^*_g \). Then there exists a finite constant \( \bar{\kappa} \) such that

\[
E^{v^+_x}(x, \omega) \leq \bar{\kappa} \left[ W(d^x_\Lambda) + (1 + g(L))V(d^x_\Lambda) \right] \text{ for any } x \in \Lambda \text{ with } d^x_\Lambda \geq b.
\]

(49)

**Proof.** If \( x \in \Lambda \) is such that \( d^x_\Lambda \geq b \), we have that \( v^+(x - y) \leq \eta(\|x - y\|) \) for any \( y \in \Lambda^c \). Therefore, thus we can bound

\[
E^{v^+_x}(x, \omega) \leq \sum_{\Delta \subset \Lambda^c} \sup_{y \in \Delta} v^+(x - y)|\omega \cap \Delta| \leq \delta^d \sum_{\Delta \subset \Lambda^c} \sup_{y \in \Delta} \eta(\|x - y\|)\rho^\omega_\delta(y).
\]

As we did previously (see (10)), we can find a constant \( C_\delta \) such that

\[
\delta^d \sup_{y \in \Delta} \eta(\|x - y\|)\rho^\omega_\delta(y) \leq C_\delta \int_{\Delta} \eta(\|x - y\|)\rho^\omega_\delta(y)dy.
\]

Therefore

\[
E^{v^+_x}(x, \omega) \leq C_\delta \sum_{\Delta \subset \Lambda^c} \int_{\Delta} \eta(\|x - y\|)\rho^\omega_\delta(y)dy
\]

\[
= C_\delta \int_{\Lambda^c} \eta(\|x - y\|)\rho^\omega_\delta(y)dy
\]

\[
\leq C_\delta \rho \int_{\Lambda^c} \eta(\|x - y\|)(1 + g(\|y\|))dy
\]

\[
\leq C_\delta \rho \int_{\Lambda^c} \eta(\|x - y\|)(1 + g(\|x - y\| + \|x\|))dy.
\]

Now, using again (13), we get

\[
g(\|x - y\| + \|x\|) \leq g(\|x - y\|) + g(\|x\|),
\]

and therefore

\[
E^{v^+_x}(x, \omega) \leq C_\delta \rho \int_{\Lambda^c} \left[ 1 + g(\|x - y\|) + g(\|x\|) \right] \eta(\|x - y\|)dy
\]

\[
\leq C_\delta \rho \int_{\Lambda^c} g(\|x - y\|) \eta(\|x - y\|)dy + (1 + g(\|\cdot\|)) \int_{\Lambda^c} \eta(\|x - y\|)dy
\]

\[
\leq C_\delta \rho \left[ \int_{\Lambda^c} g(\|x - y\|) + (1 + g(\|\cdot\|)) \right] \int_{\Lambda^c} \eta(\|x - y\|)dy.
\]
\[
\leq C_\delta \rho \left[ \int_{\|x - y\| \geq d_\Lambda^\delta} g(\|x - y\|) \eta(\|x - y\|) dy + (1 + g(\|x\|)) \int_{\|x - y\| \geq d_\Lambda^\delta} \eta(\|x - y\|) dy \right]
\leq C_\delta \rho d \left[ W(d_\Lambda^\delta) + (1 + g(L))V(d_\Lambda^\delta) \right],
\]
where in the last line we have again used definitions (11) and (19) and the fact that \(g(\|x\|) \leq dg(L)\) for any \(x \in \Lambda\).

Using Lemma 4.3 we can now conclude the proof of (38). We will consider the cases \(\lim_{L \to \infty} g(L) = +\infty\) and \(g\) bounded separately.

If \(\lim_{L \to \infty} g(L) = +\infty\), then by the hypothesis of Theorem 3.1 we have that \(\lim_{L \to \infty} g(L)/L = 0\). To see this just observe that, since \(\lim_{L \to \infty} \int_0^L V(s) ds\) is either a positive number or infinite, we have
\[
\lim_{L \to \infty} \frac{(1 + g(L))^2}{L} \int_0^L V(s) ds = 0 \implies \lim_{L \to \infty} \frac{g(L)}{L} = 0.
\]
Moreover it holds that
\[
\lim_{L \to \infty} g(L)V(g(L)) = 0.
\]
Indeed,
\[
g(L)V(g(L)) = g(L) \int_{\mathbb{R}^d, \|x\| \geq g(L)} \eta(\|x\|) dx \leq \int_{\mathbb{R}^d, \|x\| \geq g(L)} g(\|x\|) \eta(\|x\|) dx = W(g(L)),
\]
and since \(\lim_{L \to \infty} W(g(L)) = 0\), equation (50) follows.

Let us now take \(L\) sufficiently large in such a way that \(b < g(L) < L\), and define \(\Lambda_h = \{x \in \Lambda : d_\Lambda^\Lambda > g(L)\}\) so that \(\Lambda_h\) is a cube centered at the origin with size \(2(L - g(L))\) fully contained in \(\Lambda\). Therefore, we have that
\[
\Xi^\omega_\Lambda(\beta \lambda) \geq \int_{\Omega_{\Lambda_h}} d\mu_\Lambda(\vec{x}) e^{-\beta \nu(\vec{x}) - \beta E^\Lambda(x, \omega)} \geq \int_{\Omega_{\Lambda_h}} d\mu_\Lambda(\vec{x}) e^{-\beta \nu(\vec{x}) - \beta E^\Lambda_\Lambda(x, \omega)}.
\]
Now by definition, for all \(x \in \Lambda_h\) we have that \(d_\Lambda^\Lambda \geq g(L) > b\) and thus we can use bound (49) for any \(x \in \Lambda_h\)
\[
E^\Lambda_\Lambda(x, \omega) \leq \kappa \left[ W(g(L)) + [1 + g(L)]V(g(L)) \right].
\]
Moreover, since, by (12), (20) and (50), \(\lim_{\Lambda \to \infty} [W(g(L)) + [1 + g(L)]V(g(L))] = 0\), for \(\Lambda\) large enough and for any fixed \(\varepsilon > 0\), we can bound \(E^\Lambda_\Lambda(x, \omega) \leq \varepsilon\) so that
\[
\Xi^\omega_\Lambda(\beta \lambda) \geq \int_{\Omega_{\Lambda_h}} d\mu_\Lambda(\vec{x}) e^{-\beta \nu(\vec{x}) - \beta \varepsilon |\vec{x}|} = \Xi^\Lambda_\Lambda(\beta, e^{-\beta \varepsilon} \lambda).
\]
If \(\lim_{L \to \infty} g(L) < +\infty\), i.e., if there exist a finite non-negative constant \(c\) such that \(g(L) \leq c\) for all \(L\), then we can take any function \(f(L)\) such that \(\lim_{L \to \infty} f(L) = \infty\) and \(\lim_{L \to \infty} f(L)/L = 0\) and define, as before, \(\Lambda_h = \{x \in \Lambda : d_\Lambda^\Lambda > f(L)\}\). By Lemma 4.3, considering that now \(W(r) \leq cV(r)\), we still have, for any \(x \in \Lambda_h\)
\[
\lim_{\Lambda \to \infty} E^\Lambda_\Lambda(x, \omega) \leq \lim_{\Lambda \to \infty} \kappa \left[ W(f(L)) + [1 + g(L)]V(f(L)) \right]
\]
\[ \leq \tilde{c}(1 + 2c) \lim_{\Lambda \uparrow \infty} V(f(L)) = 0 \]

and therefore, for any fixed \( \epsilon > 0 \), inequality (51) still holds.

Hence, considering that \( \lim_{\Lambda \uparrow \infty} \Lambda = +\infty \) and that \( \lim_{\Lambda \uparrow \infty} \frac{|\Lambda|}{|L|} = 1 \), we get

\[
\liminf_{\Lambda \uparrow \infty} \frac{1}{|\Lambda|} \log \Xi^\omega(\beta, \lambda) \geq \lim_{\Lambda \uparrow \infty} \frac{1}{|\Lambda|} \log \Xi^\emptyset_{\Lambda_h}(\beta, e^{-\beta \epsilon} \lambda) \\
= \lim_{\Lambda \uparrow \infty} \frac{|\Lambda|}{|\Lambda_h|} \lim_{\Lambda \uparrow \infty} \frac{1}{|\Lambda|} \log \Xi^\emptyset_{\Lambda_h}(\beta, e^{-\beta \epsilon} \lambda) \\
\geq \lim_{\Lambda \uparrow \infty} \frac{1}{|\Lambda|} \log \Xi^\emptyset_{\Lambda_h}(\beta, e^{-\beta \epsilon} \lambda) \\
= \lim_{\Lambda \uparrow \infty} \frac{1}{|\Lambda|} \log \Xi^\emptyset_{\Lambda_h}(\beta, e^{-\beta \epsilon} \lambda) \\
\geq \lim_{\Lambda \uparrow \infty} \frac{1}{|\Lambda|} \log \Xi^\emptyset_{\Lambda}(\beta, \lambda).
\]

Now, since the free-boundary condition infinite-volume pressure \( p^\emptyset(\beta, \lambda) \) is continuous as a function of \( \beta \) and \( \lambda \), by the arbitrariness of \( \epsilon \) we can conclude that,

\[
\liminf_{\Lambda \uparrow \infty} \frac{1}{|\Lambda|} \log \Xi^\omega(\beta, \lambda) \geq \beta p^\emptyset(\beta, \lambda) = \lim_{\Lambda \uparrow \infty} \frac{1}{|\Lambda|} \log \Xi^\emptyset_{\Lambda}(\beta, \lambda).
\]

This ends the proof of inequality (38).

5. Conclusions

In this paper we considered a \( d \)-dimensional system of classical particles confined in a cubic box \( \Lambda \) interacting via a superstable pair potential in the Grand Canonical ensemble at fixed inverse temperature \( \beta > 0 \) and fixed fugacity \( \lambda > 0 \). We proved that the thermodynamic limit of the finite-volume pressure of such system does not depend on boundary conditions generated by particles at fixed positions outside the volume \( \Lambda \) as long as these external particles are distributed according to a bounded density \( \rho_{\text{ext}} \) (even larger as we please than the density \( \rho_{\text{f}}(\beta, z) \) of the system calculated using free boundary conditions). We also prove the independence of the thermodynamic limit of the pressure of the system in presence of boundary conditions whose density may increase with the distance from the origin to a rate which depends on how fast the pair potential decays.

A related open question (and possibly the subject of a project to come) is whether it is possible to perform an absolutely convergent Mayer expansion of the pressure of the systems considered in this note (i.e., interacting via a non-necessarily repulsive pair potential) for fugacities within a convergence radius admitting a lower bound uniform in the boundary conditions when these are in the class considered in this paper.
Acknowledgements

We would like to thank two anonymous referees for constructive criticism, valuable comments and suggestions. In particular we are very grateful to one of them for showing us the way to prove the statement of Proposition 3.1. A.P. has been partially supported by the Brazilian agencies Conselho Nacional de Desenvolvimento Científico e Tecnológico (Grant No. 306853/2018-3) (CNPq - Bolsa de Produtividade em pesquisa, Grant No. 306208/2014-8), Coordenação de Aperfeiçoamento de Pessoal de Nível Superior (CAPES - Bolsa PRINT, Grant no. 88887.474425/2020-00) and Fundação de Amparo à Pesquisa do Estado de Minas Gerais (FAPEMIG - Programa Pesquisador Mineiro, grant n. PPM-00144-18). S.Y. has been partially supported by the Argentine agency CONICET (Consejo Nacional de Investigaciones Científicas y Técnicas).

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Communicated by Christian Maes.  
Received: March 4, 2021.  
Accepted: October 5, 2021.