Charge and orbital order in transition metal oxides

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A short introduction to the complex phenomena encountered in transition metal oxides with either charge or orbital or joint charge-and-orbital order, usually accompanied by magnetic order, is presented. It is argued that all the types of above ordered phases in these systems follow from strong Coulomb interactions as a result of certain compromise between competing instabilities towards various types of magnetic order and optimize the gain of kinetic energy in doped systems. This competition provides a natural explanation of the stripe order observed in doped cuprates, nickelates and manganites. In the undoped correlated insulators with orbital degrees of freedom the orbital order stabilizes particular types of anisotropic magnetic phases, and we contrast the case of decoupled spin and orbital degrees of freedom in the manganites with entangled spin-orbital states which decide about certain rather exotic phenomena observed in the perovskite vanadates at finite temperature. Examples of successful concepts in the theoretical approaches to these complex systems are given and some open problems of current interest are indicated.

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I. DEGREES OF FREEDOM IN TRANSITION METAL OXIDES

The physical properties of transition metal oxides are driven by strong electron interactions [1]. It is due to strong local Coulomb interactions that these systems exhibit very interesting and quite diverse instabilities towards ordered magnetic phases when doping $x$ or temperature $T$ is varied — is some cases also with orbital order. These instabilities are observed, inter alia, in rapid changes of the transport properties at the metal-insulator phase transitions, or in the onset of superconductivity.

One of the outstanding problems in modern condensed matter theory is the description of strongly correlated electrons in various systems. When local Coulomb interactions are strong, the usual methods used for calculating the electronic structure fail and have to be extended by the terms following from local interactions, either in the framework of the local density approximation (LDA) with Coulomb $U$, the so-called LDA+$U$ method [2], or by the self-energy within the dynamical mean-field theory (DMFT) [3], in the LDA+DMFT approach [4]. This latter approach makes use of the local self-energy which becomes exact in the limit of infinite spatial dimension $d = \infty$ [3]. However, even these methods cannot overcome certain shortcomings of the effective one-particle theory which justifies modelling of these complex systems with Hamiltonians of the Hubbard type, and looking for solutions with methods of quantum many-body theory. The advantage of rapid progress in the electronic structure calculations in recent years is that such models can nowadays use realistic parameters which follow from the electronic structure calculations for a given system.

Although the field of strongly correlated electronic systems is very rich, we shall concentrate here on the phenomena observed in transition metal oxides. There are two major classes of systems with either perovskite structure $RMO_3$, or the layered structure $R_2MO_4$, with $R$ standing for a rare-earth ion and $M$ for a transition metal ion. In the latter class the subsequent layers of $MO_6$ octahedra are displaced, so the electronic properties are well described by two-dimensional (2D) models, see Ref. [1]. In both above structures electron correlations are strong and lead to remarkable consequences, with several degrees of freedom contributing simultaneously to coexisting magnetic, charge and (in some cases also) orbital order. Examples of these complex phenomena are high-temperature superconductivity [5], the colossal magnetoresistance in the manganites [6–9], and the Verwey transition in the magnetite ($Fe_3O_4$) [10]. Although the charge order occurs typically in doped systems, there are a few systems of formally mixed valence type, where the electron number per one transition metal ion is not an integer but local correlations stabilize charge order, as in the magnetite. The latter problem was recently addressed and the mechanism of the Verwey transition was explained as triggered by the electron-phonon coupling enhanced by local Coulomb correlations [11], so we shall not discuss it here but refer an interested reader to another contribution in the same volume [12].

The electronic structure of transition metal oxides includes several bands [1, 12], but the properties of the system do depend on the states in the vicinity of the Fermi energy. It is usually sufficient to derive effective $M–O–M$ hopping elements for $\sigma$-bonds $t_\sigma$ and $\pi$-bonds $t_\pi$, and next use them in the effective model describing only $3d$ electrons [14]. The respective kinetic energy is

* Dedicated to the memory of the late Professor Jan Stankowski
described in a perovskite system by
\[ H_0 = \sum_{\langle ij \rangle, \alpha \beta \sigma} t_{\alpha \beta} a_{\alpha \sigma}^\dagger a_{\beta \sigma} + t_{\pi} \sum_{\langle ij \rangle, \mu \sigma} a_{\mu \sigma}^\dagger a_{\mu \sigma}. \] (1)

Here \( \{\alpha, \beta\} = \{x, z\} \) are the indices of \( e_g \) orbitals,
\[ |x\rangle \equiv (x^2 - y^2)/\sqrt{2}, \quad |z\rangle \equiv (3z^2 - r^2)/\sqrt{6}, \] (2)
and this orbital flavor is in general not conserved along the hopping processes — the orbitals may be changed for the hopping along the bonds in \( ab \) planes in the perovskite structure. In contrast, the \( t_{2g} \) orbital flavor,
\[ |a\rangle \equiv |yz\rangle, \quad |b\rangle \equiv |zx\rangle, \quad |c\rangle \equiv |xy\rangle. \] (3)
is conserved for the hopping along the bonds in all three cubic directions \( \gamma = a, b, c \), as indicated by a single diagonal hopping element \( t_{\pi} \) (for simplicity we assume only nearest neighbor hopping elements), with \( \mu = \{a, b, c\} \) labeling \( t_{2g} \) orbitals and referring to the cubic axes perpendicular to the planes accomodating the respective orbitals. The latter notation is introduced using an ideal cubic system in which each \( t_{2g} \) orbital is perpendicular to a single cubic axis, for instance the \( |xy\rangle \) orbital lies in the \( ab \) plane and is perpendicular to the \( c \) axis.

On-site intraorbital Coulomb interactions are described by a single parameter \( U \) (identical for all 3d orbitals):
\[ H_U = U \sum_{i \alpha} n_{i \alpha \uparrow} n_{i \alpha \downarrow}. \] (4)

In the simplest approach, the ratio \( U/W \), where \( W \) is the bandwidth for the relevant partly filled band, decides whether electrons localize and the electronic structure changes to two Hubbard subbands in a Mott insulator, or the system is metallic, with rather strongly correlated electrons and possibly heavy effective masses (this happens for the \( f \)-electron systems which are addressed in other contributions in this volume). As the hopping elements along \( \pi \) bonds are significantly lower than the ones for \( \sigma \) bonds [14], the \( t_{2g} \) electrons in the early transition metal oxides (i.e. in titanium or vanadium oxides) are even stronger correlated than \( e_g \) electrons in the \( R\text{MnO}_3 \) or \( \text{LaNiO}_3 \) perovskites. This resembles the situation in molecular bonds in \( sp \) systems, with \( \pi \) bonds being always stronger correlated than \( \sigma \) bonds [13].

A second class of correlated insulators, so-called charge transfer insulators, arises when the oxygens states are within the gap between the two Hubbard subbands [14]. A crucial parameter is the energy difference between the \( d \) and \( p \) electron (hole) levels, \( \Delta = \varepsilon_p - \varepsilon_d \) — here we use the hole notation relevant for the high-\( T_c \) cuprates. When \( \Delta > U \) one has a Mott-Hubbard insulator, but when \( \Delta < U \), the insulator is of charge transfer type.

The electronic structure of the cuprates does not involve orbital degeneracy as the \( \text{CuO}_6 \) octahedra are elongated and the orbital degeneracy is removed for a tetragonal distortion. Therefore, \( s \) hole in the \( d^9 \) configuration occupies the \( |x\rangle \equiv (x^2 - y^2)/\sqrt{2} \) orbital at each \( \text{Cu}^{2+} \) ion in \( \text{La}_2\text{CuO}_4 \). The resulting charge transfer model for the \( \text{CuO}_2 \) planes in the cuprates may be thus written as follows [17]:
\[ H_{dp} = H_0 + H_{\text{int}}, \] (5)
\[ H_0 = \varepsilon_p \sum_i n_{pi} - t_{pd} \sum_{\langle m i \rangle, \sigma} \gamma_{mi} (d_{m \sigma \alpha}^\dagger p_{i \sigma} + \text{H.c.}) - t_{pp} \sum_{\langle ij \rangle, \alpha \sigma} \eta_{ij} (p_{i \sigma}^\dagger p_{j \sigma} + \text{H.c.}), \] (6)
\[ H_{\text{int}} = U_d \sum_m n_{m \uparrow} n_{m \downarrow} + U_p \sum_i n_{pi \uparrow} n_{pi \downarrow}. \] (7)

The parameters of the charge transfer model [5] are: the oxygen energy \( \varepsilon_p \) (we assume that the reference \( d \) hole energy \( \varepsilon_d = 0 \)), the \( d-p \) hybridization \( t_{pd} \), and the Coulomb interaction parameters for \( d \) and \( p \) orbitals, \( U_d \) and \( U_p \); the same parameters describe also other \( \text{Cu}-\text{O} \) systems, as for instance \( \text{CuO}_3 \) chains in \( \text{YBa}_2\text{Cu}_3\text{O}_{6+x} \) or \( \text{CuO}_2 \) groups of \( \text{Sr}_{2+\delta}\text{Cu}_{1-\delta}\text{O}_4 \) [19]. Here \( n_{pi} = n_{pi \uparrow} + n_{pi \downarrow} \) and \( \gamma_{mi} = p_{i \sigma}^\dagger p_{m \sigma} \) are charge density operators, \( \gamma_{mi} \) and \( \eta_{ij} \) are the phase factor for a pair of orbitals along the considered \( d - p \) bond. The parameters for the cuprates which follow from the electronic structure calculations are (in eV) [21]: \( \Delta = 3.6, \) \( t_{pd} = 1.3, \) \( t_{pp} = 0.65, \) \( U_d \approx 10.5, \) \( U_p \approx 4.0. \) Electron correlations are moderate in spite of the large value of \( U_d \) [17], but they suffice to localize holes at \( \text{Cu} \) sites in the undoped system, such as \( \text{La}_2\text{CuO}_4 \) or \( \text{YBa}_2\text{Cu}_3\text{O}_6 \). Taking the above parameters, \( \Delta \ll U_d \) and these systems are charge transfer insulators, in contrast to the perovskite titanates and vanadates, which are Mott-Hubbard systems.

It is important to realize that the charge transfer gap \( \Delta \) plays the role of an effective Coulomb parameter \( U \approx \Delta \) in the correlated electronic structure of a charge transfer insulator. When the \( \text{Cu-O} - \text{Cu hopping between two } |x\rangle \text{ orbitals along a bond in an } ab \text{ plane is defined as } t, \text{ this leads to the effective Hubbard model [21]}: \]
\[ H = -t \sum_{\langle ij \rangle, \sigma} (a_{i \sigma}^\dagger a_{j \sigma} + a_{j \sigma}^\dagger a_{i \sigma}) + U \sum_{i \sigma} n_{i \sigma \uparrow} n_{i \sigma \downarrow}. \] (8)

For the considered case of \( |x\rangle \text{ orbitals the phase factors on each bond } |ij\rangle \text{ are identical. Note that } t \text{ may be deduced from the charge transfer model [5], } t = t_{pd}^2/\Delta = 0.4 \text{ eV, and for the actual ratio } U/t = 10 \text{ the holes are strongly correlated. Hence, the undoped systems } \text{La}_2\text{CuO}_4 \text{ or } \text{YBa}_2\text{Cu}_3\text{O}_6 \text{ are antiferromagnetic (AF) insulators. In general, the derivation of an effective model from the relevant multiband model is rather tedious — such a more complete model includes in addition nearest (second) neighbor and third nearest neighbor hopping elements } \{t',t''\} \text{ and intersite Coulomb interactions [22]. A broad class of phenomena investigated for strongly correlated electron systems are the changes of their phys-} \]
cal properties in the vicinity of metal-insulator transitions. As mentioned above, one way of localizing electrons in a correlated insulator is by changing the electron interaction parameter $U$ in Eq. \( \langle \bar{S} \rangle \) (or the charge transfer gap $\Delta$). Although this may be easily realized only in theory, in certain systems the changes of the electronic parameters are sufficient to induce metal-insulator transitions observed in $V_2O_3$ \cite{24}. A more common situation, however, is encountered in doped systems, where the carriers are released at certain doping concentration and the system becomes metallic. In contrast to the earlier suggestions, the one-band model is not sufficient to describe the metal-insulator transition in $V_2O_3$ \cite{24}, and doping is not equivalent to varying external pressure \cite{25}. This and other metal-insulator transitions in the oxides are controlled by doping. A very well known example is the colossal magnetoresistance effect in the perovskite manganites \cite{7}, another is the superconductivity in doped $La_{2−x}Sr_xCuO_4$ or $YBa_2Cu_3O_{6−x}$ compounds \cite{6}. Other examples can be found, for instance, in the excellent review article by Imada, Fujimori and Tokura \cite{1}.

In this paper we address in particular the phenomena related to magnetic and orbital order in transition metal oxides which follow from strong electron correlations. Charge order arises in doped systems, while the orbital order is common in transition metal oxides with partly filled degenerate orbitals. We begin in Sec. II with the stripe phases in the cuprates, where we explain the stabilizing mechanism and show that the charge modulation is the way to optimize total energy in doped systems. While the properties of an undoped cuprate are driven by the AF superexchange, the systems with orbital superexchange interactions are more complex as the interactions are intrinsically frustrated \cite{26}. These interactions are exemplified by the so-called compass model \cite{27}, see Sec. III and may give either highly degenerate ordered ground states, or the disordered orbital liquid. Consequences of the orbital superexchange for the magnetic order are addressed in Sec. IV where we briefly summarize the structure of the spin-orbital superexchange \cite{28}, and demonstrate that spin and orbital degrees of freedom may be separated in the perovskite manganites. In contrast, in the perovskite vanadates spin-orbital entanglement plays a dominating role and decides about their properties at finite temperature, see Sec. V. Finally, we give examples of coexisting charge-and-orbital order in doped systems in Sec. VI. A summary and some open problems in the field are given in Sec. VII. Figures illustrating the theoretical concepts reviewed in this article will not be reproduced here — they may be found in the cited literature which is far from being complete and was selected on the criterion of addressing the most important concepts in this field.

II. STRIPE PHASES IN THE CUPRATES

A crucial concept in the physics of the superconducting cuprates is the Zhang-Rice singlet \cite{29}. It makes an explicit use of the charge transfer nature of the electronic structure, as a doped hole occupies not a Cu($d_x$) orbital but a linear combination of $p_x$ orbitals with $x^2−y^2$ symmetry around a hole, which forms a singlet together with the hole at Cu ion. It is this concept which provides a justification for using the $t−J$ model as the effective model describing the physical situation in the cuprates, and plays a prominent role in this class of compounds \cite{30}.

The $t−J$ model itself was derived from the Hubbard model in Cracow more than three decades ago \cite{31}, using the perturbation theory. A properly chosen canonical transformation leads from the full Hilbert space to an effective low-energy Hamiltonian acting in the restricted space, where only spins and holes occur at different sites. It consists of the kinetic energy $\propto t$ and the superexchange interaction $\propto J$ between $S = 1/2$ spins:

$$H_{t−J} = −t \sum_{\langle ij \rangle, \sigma} (\hat{a}_{i\sigma}^\dagger \hat{a}_{j\sigma} + \hat{a}_{j\sigma}^\dagger \hat{a}_{i\sigma}) + J \sum_{\langle ij \rangle} (\hat{S}_i \cdot \hat{S}_j + \frac{1}{4} \hat{n}_i \hat{n}_j),$$

with the superexchange interaction,

$$J = \frac{4t^2}{U}.$$

The operators $\hat{a}_{i\sigma}^\dagger = a_{i\sigma}^\dagger (1 − n_{i\overline{\sigma}}) \ (\overline{\sigma} = −\sigma)$ are projected fermion operators and act in the restricted space. The above $t−J$ model may also be derived directly from the charge transfer model — in this (realistic for the cuprates) case the superexchange includes both the Anderson and charge transfer excitations \cite{32}. For the cuprates one finds $J \simeq 0.13$ eV, which is either deduced from the magnetic experiments \cite{33}, or derived from the charge transfer model using its parameters \cite{20}.

The first intriguing question concerning hole doping is whether a doped hole may propagate coherently in the antiferromagnet. Naively one might argue that a hole creates defects on its way, so it would need to make a hopping along a closed loop to annihilate these defects and to move in the square lattice with a minute dispersion \cite{34}. Actually, this is the only process by which a hole may delocalize in the Ising model. The situation is quite different, however, when a hole is doped into a Heisenberg antiferromagnet — in this case the quantum fluctuations of the AF background may repair the defects created by the hole, and the hole dispersion occurs on the energy scale of $J$ \cite{35, 36}. This concept was confirmed by experiment, and indeed the hole dispersion on the low energy scale of $J$ was observed in the cuprates \cite{37}. Detailed comparison between the experimental data of angle resolved photoemission experiments and the outcome of the theoretical calculations performed using the self-consistent Born approximation (SCBA) \cite{38} were pre-
sented by several groups. Here we mention only the experimental data of high quality obtained for Sr$_2$CuO$_2$Cl$_2$ by Wells et al. [38], which fit very well to the theoretical curves obtained with finite next neighbor hopping $t'$ [39]. This demonstrates that the right effective model for the high-$T_c$ cuprates is the $t$-$t'$-$J$ rather than the $t$–$J$ one.

Higher doping of CuO$_2$ planes leads to a gradual weakening of AF correlations, which however survive even in the overdoped regime at $x \approx 0.2$ [33]. There are several possibilities concerning the phase diagram of doped cuprates [9], but the commonly accepted point of view now is that doped holes self-organize in form of phases with charge modulation [40]. Such structures with coexisting charge and magnetic order, called stripe phases, were first discovered in the theory as an instability of doped antiferromagnets towards AF domains separated by (usually) nonmagnetic domain walls [41]. Only a few years later their existence in the cuprates was confirmed in the neutron experiments of Tranquada et al. [42]. The stripe phases are characterized by the coexisting charge and magnetic order, with the charge density varying twice faster than the spin density in the real space [43].

The first question concerning stripe phases is whether they would form as solitonic defects in the AF structure, i.e. in between different AF domains, or instead they are of polaronic nature not disturbing the AF order. Although naively one could argue that the polaronic mechanism could give a better kinetic energy, this argument is misleading. To see this one can consider a cluster of three sites centered at the domain wall, filled by two electrons [43]. Due to strong correlations with $U \gg t$, the particles are confined in this cluster, although it is just a part of the AF 2D plane. Taking two electrons with either identical spins or with opposite spins, it is straightforward to estimate the ground state energy of two possible configurations: (i) polaronic ($E_P$) and (ii) solitonic ($E_S$) one. One finds that the soliton energy is lower by the superexchange energy $J$ which arises from the three-site hopping terms in this cluster [43], with: $E_P = -\sqrt{2}t$ and $E_S = -\sqrt{2t - 4t^2}/U$. This simple argument explains the experimental finding that charge walls separate AF domains with different phase of the order parameter.

Quantitative results for the stripe phases were first obtained using the Hartree-Fock (HF) approximation [11, 44], and then refined using variational wave functions [45], within DMFT for the stripe phases [46, 47], and slave-boson approach [48]. As usually, the HF serves only as a hint for possible instabilities, and gives remarkably robust stripe structures [44] with rather large amplitude of the charge density between the domain wall and the centers of AF domains, and the filling of half of doped hole per one stripe charge unit cell, as observed in experiment. These stripes are vertical (or horizontal), meaning that domain walls are along (10) [or (01)] direction, and insulating. Actually, their stability follows from a small gap which opens in the electronic structure. This mechanism is subtle and involves certain additional modulation, either spin or charge density wave, along the domain walls [44], so one has to expect major changes when electron correlations are implemented. However, variational calculations confirmed this picture to some extent [45], although the question whether the stripes are insulating or not could not be resolved (following the HF results, it was believed for a long time that the stripes are insulating).

Stable stripes were also found using an exact diagonalization method within the DMFT for the two-dimensional Hubbard model, in the broad doping range $0.03 < x < 0.2$ in La$_2$−$x$Sr$_x$CuO$_4$ [49]. These calculations allowed also to reproduce the observed crossover from diagonal (11) to vertical (01) site-centered stripes at doping $x \approx 0.05$ [40]. In addition, also the doping dependence of the size of magnetic domains and chemical potential shift $\Delta \mu \propto -x^2$ were found to be in quantitative agreement with the experimental results for La$_2$−$x$Sr$_x$CuO$_4$. In this way the paradigm of insulating stripe phases was abolished — the chemical potential was varying with doping within the metallic phase.

The spectral functions obtained within the DMFT [47] show a coexistence of the incoherent states in the lower Hubbard band and a coherent quasiparticle (QP) close to the Fermi energy. The main features of the spectra are: a flat part of the QP band near the $X = (\pi, 0)$ point, and gaps for charge excitations at the $Y = (0, \pi)$ and $S = (\pi/2, \pi/2)$ points in the low-doping regime $x < 1/8$. These gaps are gradually filled and close under increasing doping, in agreement with the experimental data for La$_2$−$x$Sr$_x$CuO$_4$ obtained using angle resolved photoemission [50]. In the range of low temperatures $T$ the obtained spectra have a distinct QP peak at the $X = (\pi, 0)$ point, present just below the Fermi energy $\mu$, and a charge gap and well defined QP at the $S = (\pi/2, \pi/2)$ point [47]. These calculations demonstrated the importance of dynamical correlations which strongly screen the local potentials resulting from on-site Coulomb interactions and lead thus to drastic changes in the distribution of spectral weight with respect to the HF picture. It was also shown that the melting of stripe order is influenced by the second neighbor hopping element $t''$, which plays also an important role in explaining the observed difference in the spectral properties between Bi$_2$Sr$_2$CaCu$_2$O$_{8+y}$ [51] and La$_2$−$x$Sr$_x$CuO$_4$ [52]. At the same time, $t''$ can tip the energy balance between the filled diagonal and half-filled vertical stripes [45], which might explain a change in the spatial orientation of stripes observed in the high $T_c$ cuprates at the doping $x \approx 1/16$.

More insight into the charge and magnetization distribution as well as into the stability of stripe phases could be obtained using a rotationally invariant version of the slave-boson approach in spin space in the 2D Hubbard model [48]. This approach allowed one to treat strong electron correlations in the stripe phases with large unit cells relevant in the low doping regime, and gave results representative of the thermodynamic limit. It also helped to resolve the longstanding controversy concerning the
role played by the kinetic energy in stripe phases. While the transverse hopping across the domain walls yields the largest kinetic energy gain in the case of the insulating stripes with one hole per site, the holes propagating along the domain walls stabilize the metallic vertical (01) stripes with one hole per two sites, as found in the cuprates.

Recently observed pattern of unidirectional domains in high-$T_c$ superconductors [52] motivated also search for coexisting charge modulation and $d$-wave superconductivity. Indeed, half-filled charge domains separated by four lattice spacings were obtained along one of the crystal axes leading to modulated superconductivity with out-of-phase $d$-wave order parameters in neighboring domains [53]. Both renormalized mean-field (MF) theory and variational Monte Carlo calculations yield that the energies of modulated and uniform phases are very close to each other, so modulated phases could easily be stabilized by other effects going beyond the $t$–$J$ model used in these calculations. Novel doped phases with superconductivity coexisting with charge modulation or even the valence-bond solid order were also reported recently [54].

**III. INTRINSIC FRUSTRATION OF THE ORBITAL SUPEREXCHANGE**

In the undoped transition metal compounds the physics is however frequently not so simple as in CuO$_2$ planes of high $T_c$ superconductors, where the superexchange stabilizes the AF long-range order. This happens in particular when the orbital degrees of freedom are active (when degenerate orbitals are only partly filled) and contribute to the magnetic order [55, 56]. The central property of the orbital degrees of freedom is that they are intrinsically frustrated [26, 57], so they may lead to novel (ordered or disordered) phases.

Frustration in magnetic systems may be of geometrical origin if only nearest neighbor interactions are present, or may arise due to competing exchange interactions [58, 59]. For instance, when one considers FM interactions along every second vertical line in the square lattice while all other interactions are AF — then this 2D Ising model is exactly soluble and has a lower critical temperature [60] than the one with isotropic exchange interactions. Frustration for quantum spins acts to enhance the effects of quantum fluctuations, leading to a number of different types of magnetically disordered states, among which some of the more familiar ones are static and resonating valence–bond (VB) phases [55]. However, also ordered phases may emerge in systems with frustrated spin interactions from their disordered manifolds of states, and their mechanism of stability is nowadays called “order–by–disorder” [58]. Numerous materials are known at present whose physical properties could be understood only by employing microscopic models with frustrated spin interactions in which some of these theoretical concepts are exemplified.

A prototype model to study frustration in pseudospin systems which mimic the directional orbital superexchange [61] is the 2D compass model [27]

$$\mathcal{H}_{2D} = \sum_{(ij)\parallel a} J_x \tau_i^x \tau_j^x + \sum_{(ij)\parallel b} J_z \tau_i^z \tau_j^z . \quad (11)$$

In this model the $\tau_i^x \tau_j^x$ interactions $\propto J_x$ for horizontal bonds (along the $a$ axis) compete with the $\tau_i^z \tau_j^z$ ones $\propto J_z$ in the vertical direction (along the $b$ axis). Recently the structure of eigenstates in this model was investigated by numerical methods [62], and it was shown using quantum Monte Carlo that a phase transition at finite temperature exists in the 2D compass model [63], suggesting that this model is indeed in the 2D Ising universality class. A competition of pseudospin interactions along different directions results here in intersite correlations similar to the anisotropic XY model, and in competition between two types of Ising-like order. This competition culminates in the highly degenerate ground state at the compass point (i.e., when all interactions have the same strength) [62], and generates there a first order phase transition when the anisotropic model with $J_z > J_x$ changes into $J_z < J_x$ through the $J_z = J_x$ transition point [64]. It is interesting to note that a similar first order quantum phase transition occurs also in the one-dimensional (1D) compass model [63], when both above interactions alternate along the chain ($N' = N/2$ is the number of unit cells):

$$\mathcal{H}_{1D} = \sum_{i=1}^{N'} \left\{ J_{x} \tau_{2i-1}^x \tau_{2i}^x + J_{z} \tau_{2i}^z \tau_{2i+1}^z \right\} . \quad (12)$$

This model was solved exactly in the entire range of $\{J_x, J_z\}$ parameters [63] by mapping onto the exactly solvable quantum Ising model [66] in different subspaces. Equal coupling constants $J_x = J_z = J$ correspond here to the quantum critical point, where the disordered phase (orbital liquid) emerges from two different types of hidden order, and the first order transition takes place. A similar transition was shown as well by an exact solution of the compass ladder [67].

The compass model is currently under discussion also due to its interdisciplinary character. It can be derived using the symmetry arguments which are necessary for the realization of doubly degenerate states which are protected from external perturbations in a wide class of Hamiltonians [68]. The Hamiltonian with this symmetry can be physically implemented in Josephson junctions, and it was argued that these junctions provide fault tolerant quantum bits. Recently magnetic interactions in Mott insulators with strong spin-orbit coupling were also discussed and it was pointed out [69] that they may provide a realization of the exactly solvable Kitaev model on the honeycomb lattice, which is relevant for quantum computation [70].
IV. SPIN-ORBITAL SUPEREXCHANGE

In several transition metal oxides with active degrees of freedom one finds coexisting magnetic and orbital order, both in Mott and in charge transfer insulators. Experimental observations give frequently some unexpected properties and the question arises how to explain particular types of observed coexisting spin-orbital order. First of all, magnetic interactions in these systems frequently break the cubic symmetry of the perovskite lattice and AF phases arise with ferromagnetic (FM) interactions along certain directions (in contrast to the G-AF phase with isotropic AF interactions, the same along each cubic direction). Two of them are quite common: (i) the C-AF phase with FM interactions along the c axis as in LaVO₃, and (ii) the A-AF phase with FM interactions within the ab planes as in LaMnO₃ [51]. These phases follow from the microscopic models (see below) which justify the complementary behavior of the observed (orbital and magnetic) order postulated by Goodenough in the manganites [71]: alternating orbital (AO) order supports FM spin order, while ferro-orbital (FO) order supports AF spin order. The structure of spin-orbital superexchange described below allows to understand better the physical mechanism beyond this complementarity, known since long as the Goodenough-Kanamori rules [72].

Realistic superexchange models for transition metal oxides with orbital degrees of freedom contain both spin and orbital operators. They may be derived by considering intersite charge excitations in a Mott insulator in a way similar to the derivation of the t–J model from the Hubbard model (and applying to the cuprates), as described in Sec. 11. This task is somewhat involved, so usually one considers an effective model with hopping elements between d orbitals of transition metal ions derived from electron transitions over the intermediate oxygen orbitals [14], as in Eq. (1). The energy scale for the hopping is set by the largest hopping element t: the (ddr) element in case of eg systems, and the (dπr) element when only π electrons contribute in systems with degenerate and partly filled t₂g orbitals. For noninteracting electrons the Hamiltonian $H_0$ (with crystal-field terms) would lead to tight-binding bands, but in a Mott insulator one is in the regime of large Coulomb interaction $U \gg t$, so charge fluctuations are suppressed and the hopping elements can only contribute via virtual excitations, leading to the superexchange as described below.

The derivation of the superexchange involves virtual charge excitations $d_i^m {d}_j^n = d_{i+1}^{m+1} d_{i-1}^{m-1}$ between two neighboring transition metal ions with $m$ 3d electrons each, which have to be included with the correct excitation energies following from the structure of local Coulomb interactions in degenerate d states (a similar expression can also be written for f electrons). These interactions are well known since long [72], but nevertheless simplified expressions can be still found in the literature which in some cases lead to misleading or even qualitatively incorrect conclusions. When only one symmetry class of electrons is involved in charge excitations, either $e_g$ or $t_{2g}$ electrons in a perovskite system, the interactions read:

$$H_{int} = U \sum_{i \alpha} n_{i \alpha \uparrow} n_{i \alpha \downarrow} + \left( U' - \frac{1}{2} J_H \right) \sum_{i, \alpha < \beta} n_{i \alpha \uparrow} n_{i \beta \downarrow}$$

$$+ J_H \sum_{i, \alpha < \beta} \left( d_i^{\dagger \uparrow} d_i^{\dagger \downarrow} d_{i \beta \downarrow} d_{i \beta \uparrow} + d_i^{\dagger \beta \uparrow} d_i^{\dagger \beta \downarrow} d_{i \alpha \downarrow} d_{i \alpha \uparrow} \right)$$

$$- 2J_H \sum_{i, \alpha < \beta} S_{i \alpha} \cdot S_{i \beta}, \quad (13)$$

with $\sigma = -\sigma$. The parameters $\{U, J_H\}$ (with $U' = U - 2J_H$) determine the excitation energies by the corresponding eigenenergies of the local electron-electron Coulomb interaction Hamiltonian $H_0$. The eigenstates have to be derived for each particular situation separately. For instance, if two-hole $d^8$ excited states are considered in the $d_i^\dagger d_j^\dagger = d_i^\dagger d_j^\dagger$ transitions for the KCuF₃ cuprate, there are triplet $(S = 1)$ and singlet $(S = 0)$ states when two holes in the $d^8$ configuration occupy different orbitals, with the energies $U - 3J_H$ and $U - J_H$, as well as two other (intraorbital) singlet states with the energies $U - J_H$ and $U + J_H$ [74]. Note that a double occupancy of either $e_g$ orbital (with energy U), considered in the context of the intraorbital Coulomb interaction [15] is not an eigenstate of the local Hamiltonian $H_0$.

In a general case the interorbital interactions, Coulomb $U_{\alpha \beta}$ and exchange $J_{\alpha \beta}$ elements replace $U'$ and $J_H$ — in contrast to the intraorbital ones [15] they are anisotropic, but satisfy a constraint which guarantees the invariance of interactions in the orbital space [72],

$$U = U_{\alpha \beta} + 2J_{\alpha \beta}, \quad (14)$$

for each pair of interacting orbitals $\{\alpha \beta\}$. These interactions are frequently parametrized by the Racah parameters $\{A, B, C\}$ [76], and one finds that

$$U = A + 4B + 3C, \quad (15)$$

and is identical for all 3d orbitals, while Hund’s exchange depends on the orbital states of the pair of interacting electrons — for $e_g$ electrons,

$$J_{H}^t = 4B + C, \quad (16)$$

and it is somewhat stronger than for $t_{2g}$ electrons,

$$J_{H}^i = 3B + C. \quad (17)$$

Thus the parameter $J_H$ used in Eq. (13) refers to the above values depending on whether a system with $e_g$ or $t_{2g}$ orbital degrees of freedom is considered. More details about the structure of local Coulomb interactions which depends on a single parameter,

$$\eta = \frac{J_H}{U}, \quad (18)$$

may be found in Ref. [72], while the experimental values of the Racah parameters $\{B, C\}$ which are known with
high accuracy from the atomic spectra are given in Ref. [77] for several transition metal ions. Unfortunately, the value of $U$ (or $A$) is known only with much lower accuracy and the mechanism of screening which leads to the values $U \sim 5 - 10$ eV is difficult to implement in the theory. Hence, the value of $U$ is frequently used as a parameter, unless it can be derived from the experimental data, as for instance from the optical excitations, see below.

The above structure of the electron-electron interactions [13] determines the excitation energies $\varepsilon_n$ due to the multiplet states in charge transitions $d_i^m d_j^n \rightarrow d_i^{m+1} d_j^{n-1}$ which enter the superexchange in the respective denominators of $4^2/\varepsilon_n$. Examples of such spectra are presented in Ref. [28]. As a rule, the high-spin states have the lowest energy $U - 3J_H$ independently of the electron number $m$ in the electronic configuration $d_i^m$ under consideration, while the energies of low-spin states depend on $m$, and may even contain fractions of $J_H$ due to the anisotropy of $J_{a,b}$ Hund’s elements, as found for instance in the case of LaMnO$_3$ [78].

The spin-orbital superexchange is the effective low-energy Hamiltonian which involves products of spin and orbital operators. The spin interactions are described by spin scalar products $\vec{S}_i \cdot \vec{S}_j$ on each bond $(ij)$ connecting two nearest neighbor transition metal ions and obey the SU(2) symmetry, while the orbital operators $\{\vec{\tau}_i, \vec{\tau}_j\}$, with $\vec{\tau}_i = \{\tau_i^+, \tau_i^-, \tau_i^z\}$ obey only much lower symmetry (at most cubic for a cubic lattice) and appear either as a scalar product $\vec{\tau}_i \cdot \vec{\tau}_j$, or only as certain components, for instance an Ising term $\tau_i^z \tau_j^z$, resembling the interactions in the 2D compass model [11]. In general the spin-orbital superexchange is of the form [28],

$$\mathcal{H}_J = J \sum_{(ij)\gamma} \left\{ \vec{J}^{(\gamma)}_{ij} \left( \vec{S}_i \cdot \vec{S}_j + S^2 \right) + \vec{K}^{(\gamma)}_{ij} \right\},$$

(19)

with the constant $J$ defined in Eq. (10). The orbital operators $\vec{J}^{(\gamma)}_{ij}$ and $\vec{K}^{(\gamma)}_{ij}$ depend on the direction $\gamma = a, b, c$ in the cubic lattice and involve the active orbitals on each bond $(ij)$ (either $e_g$ or $t_{2g}$) along direction $\gamma$ — they participate in $d_i^m d_j^n = d_i^{m+1} d_j^{n-1}$ virtual excitations, and thus these interactions have the symmetry of the lattice (i.e. cubic symmetry in the perovskites). As an example we introduce here the superexchange between $V^{3+}$ ions in the $d^2$ configuration with $S = 1$ spins [79], as realized in $RV$O$_3$ perovskites considered below in Sec. VI and controlled by the orbital operators:

$$\vec{J}^{(\gamma)}_{ij} = \frac{1}{2} \left\{ \left( 1 + 2\eta_{r1} \right) \left( \vec{\tau}_i \times \vec{\tau}_j + \frac{1}{4} n_i n_j \right) - \eta_{r3} \left( \vec{\tau}_i \times \vec{\tau}_j + \frac{1}{4} n_i n_j \right) - \frac{1}{2} \eta_{r1} (n_i + n_j) \right\} \quad (\gamma)$$

(20)

$$\vec{K}^{(\gamma)}_{ij} = \left\{ \eta_{r1} \left( \vec{\tau}_i \times \vec{\tau}_j + \frac{1}{4} n_i n_j \right) + \eta_{r3} \left( \vec{\tau}_i \times \vec{\tau}_j + \frac{1}{4} n_i n_j \right) - \frac{1}{4} (1 + \eta_{r1}) (n_i + n_j) \right\} \quad (\gamma)$$

(21)

They arise from the $d_i^2 d_j^2 = d_i^3 d_j^1$ charge excitations, leading either to high-spin or to low-spin $d_i^3$ configurations, so Hund’s exchange in the multiplet structure of a $V^{2+}$ ions enters via the coefficients $r_1 = 1/(1 - 3\eta)$ and $r_3 = 1/(1 + 2\eta)$ (the low-spin excitations occur also at energy $U$, so the corresponding coefficient is $r_2 = 1$). Eqs. (20) and (21) are general and refer to two active orbital flavors along the cubic axis $\gamma$. The leading orbital interactions are proportional to the scalar products $(\vec{\tau}_i \cdot \vec{\tau}_j)^{(c)}$ of orbital operators on the bonds as both orbitals are active and may generate charge excitations, but the structure of local Coulomb interactions [13] is responsible for additional terms,

$$(\vec{\tau}_i \times \vec{\tau}_j)^{(c)} = \frac{1}{2} (\tau_i^+ \tau_j^- + \tau_i^- \tau_j^+) + \tau_i^z \tau_j^z,$$

(22)

which violate the conservation of the orbital quantum numbers. The operator $n_i^{(c)}$ stands for the number of active electrons at site $i$ along the bond $(ij)$, for instance for a bond along the $c$ axis this number is $n_i^{(c)} = n_{ia} + n_{ib}$ [this notation for the $t_{2g}$ orbitals is defined in Eq. (19)].

The superexchange model [19] consists typically of several terms which originate from different charge excitations. This feature made it possible to relate the averages of these different excitations to the spectral weights in the optical spectroscopy [80], and serves now as a standard theoretical tool to explain the observed anisotropy and temperature dependence of the spectral weights in the optical spectra [28]. In a correlated insulator the electrons are almost localized and the only kinetic energy which is left [81] is associated with the same virtual charge excitations that contribute also to the superexchange [19]. Therefore, we one may define the individual kinetic energy contributions $K_n^{(\gamma)}$, which refer to different energy regimes in the optical transitions and can be determined from the superexchange using the Hellman-Feynman theorem [82],

$$K_n^{(\gamma)} = -2 \langle H_n^{(\gamma)}(ij) \rangle.$$

(23)

For convenience, we define the energy contribution $K_n^{(\gamma)}$ for the Hubbard subband $n$ as a positive quantity.

The magnetic properties of the transition metal oxides with active orbital degrees of freedom are usually discussed in terms of magnetic exchange constants which determine both the type of the magnetic order in the ground state (at $T = 0$) and the magnetic excitations (magnons observed in the neutron scattering experiments). The exchange constants are usually found for a bond $\langle ij \rangle$ along each nonequivalent axis $\gamma$ by averaging over the orbital operators in Eq. (19),

$$J_{ij} = \langle \vec{J}^{(\gamma)}_{ij} \rangle,$$

(24)

which leads to an anisotropic spin exchange model Hamiltonian adequate for anisotropic magnetic phases, such as for instance $A$-AF or $C$-AF phase realized for instance in...
LaMnO\textsubscript{3} and LaVO\textsubscript{3},

\[ H_s = J_{ab} \sum_{\langle ij \rangle_{ab}} \vec{S}_i \cdot \vec{S}_j + J_c \sum_{\langle ij \rangle_{c}} \vec{S}_i \cdot \vec{S}_j. \] (25)

This procedure assumes implicitly that spin and orbital operators can be separated from each other and ignores the possibility of quantum entanglement \cite{73} and of composite spin-orbital excitations introduced in Refs. \textsuperscript{74} \textsuperscript{75}. It turns out that such excitations play a prominent role in destabilizing the classical AF long-range order in the \textsuperscript{d}\textsuperscript{p} spin-orbital model \cite{26}, and have observable consequences in the perovskite vanadates, see Sec. \textsuperscript{V}\textsuperscript{1}.

In some cases, however, the spin and orbital degrees of freedom may be disentangled and the obtained theoretical results explain well the experimental findings. One of the best examples is LaMnO\textsubscript{3}, where the exchange constants \( \{ J_{ab}, J_c \} \) deduced from the neutron scattering \cite{85} can be explained by the superexchange model assuming a classical ansatz for the ground state with AO order,

\[ |\Phi_0 \rangle = \prod_{i \in A} |\theta_A \rangle_i \prod_{j \in B} |\theta_B \rangle_j, \] (26)

with the orbital states, \( |\theta_A \rangle_i \) and \( |\theta_B \rangle_j \), characterized by opposite angles \( |\theta_A \rangle = -|\theta_B \rangle \) on two sublattices \( A \) and \( B \) in the \textit{ab} planes, and repeated in the subsequent planes along the \textit{c} axis. The AO order is stable below the orbital transition temperature \( T_{OO} \approx 760 \text{ K} \) \cite{94}, which is rather high compared with the Néel temperature \( T_N \approx 140 \text{ K} \) — therefore one may consider the AO order between the sublattices \( i \in A, j \in B \),

\[ |\theta_A \rangle_i = \cos \left( \frac{\theta}{2} \right) |z \rangle_i + \sin \left( \frac{\theta}{2} \right) |x \rangle_i, \]

\[ |\theta_B \rangle_j = \cos \left( \frac{\theta}{2} \right) |z \rangle_j - \sin \left( \frac{\theta}{2} \right) |x \rangle_j, \] (27)

as frozen in the temperature range \( T < 300 \text{ K} \) relevant for the magnetic excitations. Here we used the simplified notation for the \( e\text{g} \) orbital basis introduced in Eq. \textsuperscript{2}. Using the well motivated parameter set, the experimental values of the exchange constants, in LaMnO\textsubscript{3} are reproduced by the angle \( \theta \approx 94^\circ \), but it has been shown that somewhat higher values of \( \theta \) would also be consistent with a model including explicitly charge transfer superexchange terms \cite{28}. In any case, the occupied orbitals in LaMnO\textsubscript{3} are closer to symmetric/antisymmetric combinations of \( \{ |z \rangle, |x \rangle \} \) than to the directional \( 3x^2 - r^2/3y^2 - r^2 \) orbitals, as illustrated in the early literature on the subject.

The experimental proof that the spin and orbital operators may be disentangled in LaMnO\textsubscript{3} is provided by the optical spectroscopy, which shows rather distinct anisotropy of the low-energy spectral weights between the polarization in \textit{ab} planes on the one hand and along the \textit{c} axis on the other \cite{80}. It is quite remarkable that the temperature dependence of these spectral weights may be very well explained by the spin-orbital superexchange model of Ref. \textsuperscript{78}, using the same parameters as those used to calculate the exchange constants, in the broad temperature range \( 0 < T < 300 \text{ K} \) \cite{86}. It follows alone from the temperature variation of spin correlation functions for a bond \( \langle ij \rangle \) within the \textit{ab} planes and along the \textit{c} axis,

\[ s_{ab} = \langle \vec{S}_i \cdot \vec{S}_j \rangle^{(ab)}, \quad s_c = \langle \vec{S}_i \cdot \vec{S}_j \rangle^{(c)}, \] (28)

while the AO order remains unchanged, as explained in Ref. \textsuperscript{28}.

In spite of the disentangled spin and orbital dynamics in the \textit{RMnO\textsubscript{3}} perovskites, where \( R=\text{Lu,} \cdots, \text{La} \) stands for a rare earth atom, their several properties are not fully understood. One of them is the nature of the insulating state which comes partly due to JT interactions \textsuperscript{87} \textsuperscript{89} and partly due to the orbital superexchange interactions \textsuperscript{78}. Another puzzling feature is the phase diagram of the \textit{RMnO\textsubscript{3}} family of compounds, where \( R=\text{Lu,} \cdots, \text{La} \) stands for a rare earth atom, which exhibits a phase transition from the \( A\text{-AF} \) to a rather peculiar \( E\text{-AF} \) phase \textsuperscript{90}.

\section{V. Fingerprints of Spin-Orbital Entanglement in the \textit{RVO\textsubscript{3}} Perovskites}

\subsection{A. Spin-orbital entanglement}

The coupling between spin and orbital operators in the spin-orbital superexchange may be quite strong in some cases — the excellent example of this coupling are the vanadium perovskites, see below. Although the \( C\text{-AF} \) phase observed in the entire family of \textit{RVO\textsubscript{3}} compounds \textsuperscript{91} \textsuperscript{92}, where \( R=\text{Lu,} \cdots, \text{La} \) stands for a rare earth atom, satisfies to some extent the Goodenough–Kanamori rules \textsuperscript{72}, with FM order along the \textit{c} axis where the active \( a \) and \( b \) orbitals \textsuperscript{3} alternate — the AO order is very weak here and the orbital fluctuations play a very important role \textsuperscript{79}. This situation is opposite to the frozen AO order in LaMnO\textsubscript{3}, which can explain both the observed magnetic exchange constants and the distribution of the optical spectral weights. In LaVO\textsubscript{3} the FM exchange interaction is enhanced far beyond the usual mechanism following from the splitting between the high-spin and low-spin states due to finite Hund’s exchange \( J_H \). Evidence of orbital fluctuations in the \textit{RVO\textsubscript{3}} perovskites was also found in pressure experiments, which show a distinct competition between the \( C\text{-AF} \) and \( G\text{-AF} \) spin order, accompanied by the complementary \( G\text{-AO} \) and \( C\text{-AO} \) order of \( \{ a, b \} \) orbitals \textsuperscript{92}.

To understand better the essence of entangled spin-orbital states, we present first the results of the model calculation with four-site chains along the \textit{c} axis, described by the spin-orbital superexchange models relevant for titanates and vanadates. These calculations served to identify spin-orbital entangled states for increasing multiplet splitting \( \propto \eta \) \textsuperscript{82}. A prototype model to study frustration and entanglement in coupled spin
and pseudospin (orbital) systems is the one–dimensional (1D) SU(4) model. This example is remarkable, as in a purely spin 1D model one expects no frustration when only nearest-neighbor interactions are present. However, both spins and pseudospins appear here on a completely symmetrical and equal footing with joint spin-pseudospin operators and compete with each other, forming a group of elementary generators in the SU(4) symmetry. Three types of elementary excitations contribute to the thermodynamic properties: spin, orbital, and joint spin-orbital ones. This is indeed confirmed by the entropy data of this model obtained from a numerical analysis, which increases three times faster than that of the 1D AF Heisenberg model. This also implies that the intersite correlations is removed — one finds

\[ \langle \vec{S}_i \cdot \vec{S}_j \rangle_{ij} \]

The vanadate \( d^2 \) model behaves also in a similar way in a range of small values of \( \eta \), with all three \( S_{ij}, T_{ij} \) and \( C_{ij} \) correlations being negative. Again, the composite spin-orbital correlations are here finite (\( C_{ij} < 0 \)), spin and orbital variables are entangled, and the MF factorization of the ground state into spin and orbital part fails. Only for sufficiently large \( \eta \) do the spins reorient in the FM ground state, and decouple from the orbitals. In this regime, corresponding to the experimentally observed C-AF phase of LaVO\(_3\) (and other cubic vanadates), spin-orbital entanglement ceases to exist in the ground state. However, as we will see below, it has still remarkable consequences at finite temperature, where entangled spin-orbital states again play a role.

As a useful tool to verify the Goodenough-Kanamori rules in spin-orbital models for \( t_2g \) electrons with spins either \( S = 1 \) or \( S = 1/2 \) and pseudospins \( \tau = 1/2 \), we introduce spin and orbital correlations defined for a bond \((ij)\),

\[ S_{ij} = \langle \vec{S}_i \cdot \vec{S}_j \rangle/(2S)^2, \quad T_{ij} = \langle \vec{T}_i \cdot \vec{T}_j \rangle. \]

When they are compared with each other, and with the composite spin-orbital correlation function defined as a difference between the exact value and the MF factorized correlations on a bond \((ij)\),

\[ C_{ij} = \left\{ \langle (\vec{S}_i \cdot \vec{S}_j)(\vec{T}_i \cdot \vec{T}_j) \rangle - \langle \vec{S}_i \cdot \vec{S}_j \rangle \langle \vec{T}_i \cdot \vec{T}_j \rangle \right\} / (2S)^2, \]

one may conclude whether the spin and orbital operators are disentangled. If \( C_{ij} = 0 \), the spin and orbital operators are disentangled and their MF decoupling is exact, while if \( C_{ij} < 0 \) — spin and orbital operators are entangled, and joint spin-orbital fluctuations contribute even at \( T = 0 \). Two spin-orbital models were investigated in Ref. 83: (i) the titanate model for \( d^1 \) ionic configurations of Ti\(^{3+}\) ions in the RTIO\(_3\) perovskites with \( S = 1/2 \), and (ii) the vanadate model for \( d^2 \) configurations of V\(^{3+}\) ions in the RVO\(_3\) perovskites with \( S = 1 \). For more details about the structure of the superexchange \( \mathcal{H}_J \) (19) in both models see for instance Ref. 28.

As the chain-like cluster is 1D and only two orbital \((a,b)\) flavors contribute in each case, one recovers the SU(4) model in the \( d^1 \) (titanate) case at \( \eta = 0 \), and \( S_{ij} = T_{ij} = C_{ij} = -0.25 \) for \( N = 4 \) sites. By a closer inspection one finds that the ground state wave function for the four-site cluster is close to a total spin-orbital singlet, involving a linear combination of (spin singlet/orbital triplet) and (spin triplet/orbital singlet) states for each bond \((ij)\). This result manifestly contradicts the celebrated Goodenough-Kanamori rules, as both spin and orbital correlations have the same sign. At finite \( \eta \) the SU(4) degeneracy of all intersite correlations is removed — one finds \( T_{ij} < C_{ij} < S_{ij} < 0 \) in the regime of spin singlet \((S = 0)\) ground state, and the Goodenough-Kanamori rule with complementary spin/orbital correlations is still violated. A qualitatively similar case is found in a mathematical SU(2)⊗SU(2) model (not realized in transition metal oxides), where the ground state is entangled in a broad range of parameters, including the exactly solvable case with alternating spin and orbital singlets on the bonds.

B. Phase diagram of the RVO\(_3\) perovskites

The phase diagram of the RVO\(_3\) perovskites is qualitatively different from the one for the RVO\(_3\) perovskites and indicates the proximity of spin and orbital energy scales. Experimental studies have shown that the C-AF order is common to the entire family of the RVO\(_3\) vanadates, and in general the magnetic transition occurs below the orbital transition, \( T_{N1} < T_{OO} \), except for LaVO\(_3\) with \( T_{N1} \approx T_{OO} \). When the ionic radius \( r_R \) decreases, the Néel temperature \( T_{N1} \) also decreases, while the orbital transition temperature \( T_{OO} \) decreases. The behavior of the other two transitions, \( T_{N1} \) and \( T_{OO} \), is different for different vanadates.
increases, passes through a maximum close to YVO$_3$, and next decreases towards LuVO$_3$. This provided an experimental challenge to the theory which was addressed only recently using the spin-orbital superexchange model [98]. One finds that the C-AF order develops in LaVO$_3$ below $T_{N1} \simeq 143$ K, and is almost immediately followed by a weak structural transition stabilizing the weak G-AO order at $T_{OO} \simeq 141$ K [101, 102]. This provides a constraint on the theoretical model. Remarkably, the magnetic order parameter in the C-AF phase of LaVO$_3$ is strongly reduced to $\simeq 1.3 \mu_B$, much below the reduction expected from quantum fluctuations in the C-AF phase (being only $6\%$ for $S = 1$ spins [99]) — also this reduction of the measured magnetization could not be explained so far.

In order to unravel the physical mechanism responsible for the decrease of $T_{OO}$ from YVO$_3$ to LuVO$_3$ one has to analyze in more detail the evolution of GdFeO$_3$ distortions with for decreasing ionic radius $r_R$ [98]. Such distortions are common for the perovskites [100], and one expects that they should increase when the ionic radius $r_R$ decreases, as observed in the RMnO$_3$ perovskites [99]. In the RVO$_3$ family the distortions are described by two subsequent rotations of VO$_6$ octahedra: (i) by an angle $\vartheta$ around the $b$ axis, and (ii) by an angle $\varphi$ around the $c$ axis. Increasing angle $\vartheta$ causes a decrease of V–O–V bond angle along the $c$ direction, being $\pi - 2\vartheta$, and leads to an orthorhombic lattice distortion $u = (b - a)/a$, where $a$ and $b$ are the lattice parameters of the $Pmnm$ structure of RVO$_3$. By the analysis of the structural data for the RVO$_3$ perovskites [101, 102] one finds the following empirical relation between the ionic radius $r_R$ and the angle $\vartheta$:

$$r_R = r_0 - \alpha \sin^2 \vartheta,$$

where $r_0 = 1.5$ Å and $\alpha = 0.95$ Å are the empirical parameters. This allows one to use the angle $\vartheta$ to parametrize the dependence of the microscopic parameters of the Hamiltonian and to investigate the transition temperatures $T_{OO}$ and $T_{N1}$ as functions of $r_R$.

The spin-orbital model introduced in Ref. [98] to describe the phase diagram of RVO$_3$ reads:

$$\mathcal{H} = J \sum_{\langle ij \rangle | \gamma} \left\{ \langle \vec{S}_i, \vec{S}_j + S^2 \rangle \chi_{ij}^{(\gamma)} + k_{ij}^{(\gamma)} \right\} + E_{c}(\vartheta) \sum_{i} e^{i\vec{R}_i \cdot \vec{Q}} \tau^{z}_{i} + V_{c}(\vartheta) \sum_{\langle ij \rangle | c} \tau_{ij}^{z} + V_{ab}(\vartheta) \sum_{\langle ij \rangle | ab} \tau_{ij}^{z} + gu \sum_{i} \tau_{i}^{x} + \frac{1}{2} N K (u - u_0(\vartheta))^2,$$

where $\gamma = a, b, c$ labels the cubic axes, and the operators are given by Eqs. [20] and [21]. The superexchange is supplemented by the crystal field term $\propto E_z$, the orbital interactions terms $\propto V_c$ and $\propto V_{ab}$ induced by lattice distortions, and the orbital-lattice term $\propto g$ which is counteracted by the lattice elastic energy $\propto K$. All these terms are necessary in a realistic model which reproduces the behavior of the RVO$_3$ perovskites at finite temperature.

The crystal field splitting breaks the cubic symmetry in distorted VO$_6$ octahedra, as obtained in the electronic structure calculations [102] and from the point charge model [98], and the actual filling of $t_{2g}$ orbitals is:

$$n_{ic} = 1, \quad n_{ia} + n_{ib} = 1,$$

so the superexchange [20] and [21] in Eq. [32] is expressed by the orbital operators $\tau_{i}^{z} = \{ \tau_{i}^{x}, \tau_{i}^{y}, \tau_{i}^{z} \}$ (and their components) as explained in Sec. IV. The splitting $\propto E_z$ between $a$ $(yz)$ and $b$ $(zx)$ orbitals is given by the pseudospin $\tau_{i}^{z}$ operators,

$$\tau_{i}^{z} = \frac{1}{2}(n_{ia} - n_{ib}),$$

which refer to two active orbital flavors $\{a, b\}$ in RVO$_3$. It is characterized by the vector $\vec{Q} = (\pi, \pi, 0)$ in reciprocal space — it alternates in the $ab$ planes, but is uniform along the $c$ axis. Thus, this splitting competes with the (weak) G-AO order supporting the observed C-AF phase at temperature $T < T_{N1}$.

In addition, the model [98] includes: (i) intersite orbital interactions $\propto V_{ab}$, $V_{c}$ (which originate from the coupling to the lattice), and (ii) orbital-lattice term $\propto g$ which induces orbital polarization $(\tau_{i}^{z}) \neq 0$ when the lattice distortion $u$ increases. The orbital interactions induced by the distortions of the VO$_6$ octahedra and by GdFeO$_3$ distortions of the lattice, $V_{ab} > 0$ and $V_{c} > 0$, also favor the C-AO order (like the crystal field $E_z > 0$). Note that $V_{c} > 0$ counteracts the orbital interactions included in the superexchange via $\vec{K}_{ij}^{(c)}$ operators [21]. The last two terms in Eq. [32] describe the linear coupling $\propto g > 0$ between active $\{yz, zx\}$ orbitals and the orthorhombic lattice distortion $u$. The elastic energy which counteracts lattice distortion $u$ is given the force constant $K$, and $N$ is the number of V$^{3+}$ ions. The coupling $\propto gu$ acts as a transverse field in the pseudospin space. While the eigenstates $\frac{1}{\sqrt{2}}(\pm)$ favored by $\tau_{i}^{z}$ cannot be realized due to the competition with all the other terms, increasing lattice distortion $u$ (increasing angle $\vartheta$) modifies the orbital order and intersite orbital correlations.

The crystal field splitting $E_z(\vartheta)$, orbital interactions $\{V_{ab}(\vartheta), V_{c}(\vartheta)\}$, and the orbital-lattice coupling $g(\vartheta) \equiv gu$ depend on the tilting angle $\vartheta$. In case of $V_{c}$ one may argue that its dependence on the angle $\vartheta$ is weak, and a constant $V_{c}(\vartheta) \equiv 0.26 J$ was chosen in Ref. [98] in order to satisfy the experimental constraint that the C-AF and G-AO order appears almost simultaneously in LaVO$_3$ [91]. The experimental value $T_{N1}^{exp} = 143$ K for LaVO$_3$ [91] was fairly well reproduced in the present model taking $J = 200$ K. The functional dependence of the remaining two parameters $\{E_z(\vartheta), V_{ab}(\vartheta)\}$ on the tilting angle $\vartheta$ was derived from the point charge model [98] using the structural data for the RVO$_3$ series [101, 102].
— one finds:

$$E_z(\vartheta) = Jv_z \sin^3 \vartheta \cos \vartheta,$$

$$V_{ab}(\vartheta) = Jv_{ab} \sin^3 \vartheta \cos \vartheta.$$  

(35)

(36)

Finally, the effective coupling to the lattice distortion has to increase faster with the increasing angle $\vartheta$, and the following dependence was shown to give a satisfactory description of the phase diagram of the RVO$_3$ perovskites:

$$g_{eff}(\vartheta) = Jv_g \sin^5 \vartheta \cos \vartheta.$$  

(37)

Altogether, magnetic and orbital correlations described by the spin-orbital model [19], and the magnetic $T_{N1}$ and orbital $T_{OO}$ transition temperatures depend on three parameters: $\{v_z, v_{ab}, v_g\}$.

Due to the proximity of both orbital and magnetic phase transitions in the RVO$_3$ perovskites, it is crucial to design the MF approach in such a way that the spin-orbital coupling is described beyond the factorization of spin and orbital operators. On the one hand, the correct MF treatment of the orbital and magnetic phase transitions in the RVO$_3$ vanadates requires the coupling between the on-site orbital, $\langle \tau^z \rangle_C \equiv \frac{1}{2}|\tau_i^z - \tau_j^z|$, and spin order parameters in the C-AF phase, $\langle \mathbf{S}_i \rangle_C$, as well as a composite $\langle \tau_i^z \tau_j^z \rangle$ order parameter, similar to that used for the RMnO$_3$ perovskites [78]. On the other hand, the on-site MF theory including the above coupling does not suffice for the RVO$_3$ compounds as the orbital singlet correlations $\langle \tau_i^z \tau_j^z \rangle$ on the bonds $\langle ij \rangle$ along the $c$ axis play so crucial role in stabilizing the C-AF phase [79], and the orbital fluctuations are important [106]. Therefore, the minimal physically acceptable approach to the present problem is a self-consistent calculation for a bond $\langle ij \rangle$ along the $c$ axis, coupled by the MF terms to its neighbors along all three cubic axes [80]. This procedure, with properly selected model parameters, was shown to be successful in reproducing the experimental phase diagram of Ref. [92]. One finds that indeed the orbital order occurs below a higher temperature than the magnetic one in the RVO$_3$ perovskites to the left from LaVO$_3$, i.e. with smaller ionic radius $r_R$.

As presented in Ref. [98], the remarkable dependence of both spin $T_{N1}$ and orbital $T_{OO}$ transition temperature in the RVO$_3$ perovskites follows from the respective changes in the orbital correlations with decreasing $r_R$. First, the singlet correlations are drastically suppressed from LaVO$_3$ towards LuVO$_3$. Second, the increase of orbital intersite interactions due to the JT term [80], induces steady increase of the orbital temperature $T_{OO}$ with decreasing $r_R$. Finally, while $\langle \tau^z \rangle \simeq 0.03$ is rather weak in LaVO$_3$, it steadily increases along the RVO$_3$ perovskites when $r_R$ decreases, and finally it becomes as important as the orbital order parameter itself, i.e. $\langle \tau^z \rangle \simeq \langle \tau^z \rangle_C$. Note that in the entire parameter range the latter order parameter is substantially reduced from the classical value $\langle \tau^z \rangle_C \text{max} = \frac{1}{3}$ by singlet orbital fluctuations in the entire parameter regime, being $\langle \tau^z \rangle_C \simeq 0.32$ and 0.36 for LaVO$_3$ and LuVO$_3$, respectively.

It is quite remarkable that the above changes in the orbital state modify the magnetic exchange constants $\{J_{ab}, J_c\}$ along both nonequivalent cubic directions, see Eq. (24), and thus the value of $T_{N1}$ is reduced with decreasing $r_R$. Note that the superexchange energy $J$ does not change, so the entire effect stems from the orbital correlations $\langle \tau^z \rangle_C$. This also implies that the width of the magnon band given at $T = 0$ by $W_{C-AF} = 4\langle J_{ab} + |J_c| \rangle$ is reduced by a factor close to 1.8 from LaVO$_3$ to YVO$_3$, in agreement with surprisingly low magnon energies observed in the C-AF phase of YVO$_3$ [104].

Summarizing, the microscopic model [32] describes gradual changes of the orbital and magnetic correlations under the coupling to the lattice which suppresses orbital fluctuations generated by virtual charge fluctuations responsible for the spin-orbital superexchange. It provides an almost quantitative understanding of the systematic experimental trends for both orbital and magnetic transitions in the RVO$_3$ perovskites [98], and is able to reproduce the observed non-monotonic variation of the orbital transition temperature $T_{OO}$ for decreasing $r_R$. However, the theoretical description of the magnetic transition to the C-AF phase at $T_{N2}$, which occurs for small $r_R$ [91], remains to be addressed by future theory. More examples of spin-orbital entanglement in the field of the perovskite vanadates are shortly discussed in the next two subsection.

C. Optical spectral weights for LaVO$_3$

As a second example of spin-orbital entanglement in the cubic vanadates at finite temperature we discuss briefly the evaluation of the optical spectral weights from the spin-orbital superexchange for LaVO$_3$, following Eq. (25). First we rewrite the superexchange operator $H^{(\gamma)}(ij)$ for a bond $\langle ij \rangle \parallel \gamma$, contributing to operator $H_1 [19]$, as a superposition of $d_{a}^{\dagger}d_{b}^{\gamma} = d_{a}^{\dagger}d_{1}^{\gamma}$ charge excitations to different spin states in upper Hubbard subbands labelled by $n$ [80],

$$H^{(\gamma)}(ij) = \sum_{n} H_{n,ij}^{(\gamma)}.$$  

(38)

One finds the superexchange terms $H_{n,ij}^{(c)}$ for a bond $\langle ij \rangle$ along the $c$ axis [80],

$$H_{n,ij}^{(c)} = -\frac{1}{3} Jr(2 + \vec{S}_i \cdot \vec{S}_j) \left( \frac{1}{4} - \tau_i^z \cdot \tau_j^z \right).$$  

(39)

$$H_{n,ij}^{(c)} = -\frac{1}{12} Jr(1 + \vec{S}_i \cdot \vec{S}_j) \left( \frac{7}{4} - \tau_i^z \tau_j^z - \tau_i^x \tau_j^x + 5 \tau_i^y \tau_j^y \right).$$  

(40)

$$H_{n,ij}^{(c)} = -\frac{1}{4} Jr(1 - \vec{S}_i \cdot \vec{S}_j) \left( \frac{1}{4} + \tau_i^z \tau_j^z + \tau_i^x \tau_j^x - \tau_i^y \tau_j^y \right).$$  

(41)

and $H_{n,ij}^{(ab)}$ for a bond in the $ab$ plane,

$$H_{n,ij}^{(ab)} = -\frac{1}{6} Jr(2 + \vec{S}_i \cdot \vec{S}_j) \left( \frac{1}{4} - \tau_i^z \tau_j^z \right).$$  

(42)
When the spectral weight is evaluated following Eq. (23), it is reasonable to try first the MF approximation and to separate spin and orbital correlations from each other. The spectral weights require then knowledge of spin correlations along the $c$ axis and within the $ab$ planes, as well as the corresponding intersite correlations $\langle \vec{S}_i \cdot \vec{S}_j \rangle$ and $\langle \delta_i^c \delta_j^c \rangle$ with $\alpha = x, y, z$. From the form of the above superexchange contributions one sees that high-spin excitations $H_s^{(\gamma)}$ support the FM coupling while the low-spin ones, $H_{2,ij}^{(\gamma)}$ and $H_{3,ij}^{(\gamma)}$, contribute with AF couplings.

The low-energy optical spectral weight for the polarization along the $c$ axis $K_{1,\text{exp}}^{(c)}$ decreases by a factor close to two when the temperature increases from $T \approx 0$ to $T = 300$ K [107] — this change is much larger than the one observed in LaMnO$_3$ perovskites is the remarkable first order magnetic transition at finite temperature in the family of vanadate ferromagnets. The single-ion anisotropy term $\sim K_2$ sacrifices the easy $c$ axis to the Abrikosov AF phase, and reorient above $T_{N2} = 77$ K from the G-AF to the C-AF spin order with rather exotic magnetic properties, found in YVO$_3$ [108]. This magnetic transition is unusual and particularly surprising as the staggered moments are approximately parallel to the $c$ axis in the G-AF phase, and reorient above $T_{N2}$ to the $ab$ planes in the C-AF phase, with some small alternating G-AF component along the $c$ axis. First, while the orientations of spins in C-AF and G-AF phase are consistent with the expected anisotropy due to spin-orbit coupling [109], the observed magnetization reversal with the weak FM component remains puzzling. Second, it was also established by neutron scattering experiments [106] that the scale of magnetic excitations is considerably reduced for the C-AF phase (by a factor close to two) as compared with the exchange constants deduced from magnons measured in the G-AF phase. In addition, the magnetic order parameter in the C-AF phase of LaVO$_3$ is strongly reduced to $\sim 1.3 \mu B$, which cannot be explained by rather small quantum fluctuations in the C-AF phase [99]. Finally, the C-AF phase of YVO$_3$ is dimerized. Until now, only this last feature found a satisfactory explanation in the theory [110, 111], see below.

We remark that the observed dimerization in the YVO$_3$ magnon dispersions may be seen as a signature of entanglement in excited states which becomes active at finite temperature. The microscopic reason of the anisotropy in the exchange constants

$$J_{c1} \equiv J_{c}(1 + \delta_s), \quad J_{c2} \equiv J_{c}(1 - \delta_s),$$

is the tendency of the orbital chain to dimerize, activated by thermal fluctuations in the FM spin chain [111] which support dimerized structure in the orbital sector. As a result one finds alternating stronger and weaker FM bonds along the $c$ axis [111] in the dimerized C-AF phase (with $\delta_s > 0$). The observed spin waves may be explained by the following effective spin Hamiltonian for this phase (assuming again that the spin and orbital operators may be disentangled which is strictly valid only at $T = 0$):

$$\mathcal{H}_s = \mathcal{J}_c \sum_{\langle i,i+1 \rangle \parallel c} \{1 + (-1)^i \delta_s \} \vec{S}_i \cdot \vec{S}_{i+1}$$

$$+ \mathcal{J}_{ab} \sum_{\langle ij \rangle \parallel ab} \vec{S}_i \cdot \vec{S}_j + K_z \sum_i (\delta_i^c)^2.$$  (46)

Following the linear spin-wave theory the magnon dispersion is given by

$$\omega_{\pm}(k) = 2 \sqrt{\left( 2 \mathcal{J}_{ab} + |\mathcal{J}_c| + \frac{1}{2} K_z \pm \mathcal{J}_c n_k^{1/2} \right)^2 - (2 \mathcal{J}_{ab} \gamma_k)^2},$$

with

$$\gamma_k = \frac{1}{2} (\cos k_x + \cos k_y), \quad \eta_k = \cos^2 k_x + \delta_s^2 \sin^2 k_x.$$  (47)

The single-ion anisotropy term $\propto K_2$ is responsible for the gap in spin excitations. Two modes measured by
neutron scattering are well reproduced by $\omega_{\pm}(k)$ obtained from Eq. \ref{eq:omega} using the experimental exchange interactions: $J_{a b} = 2.6$ meV, $J_{c} = -3.1$ meV, $\delta_s = 0.35$. We note that a somewhat different Hamiltonian with more involved interactions was introduced in ref. \cite{20}, but the essential features seen in the experiment are well reproduced already by the present effective spin exchange model $\mathcal{H}_s$, see Eq. \ref{eq:H_s}.

The observed dimerization in the magnon spectra in YVO$_3$ motivated the search for its mechanism within the spin-orbital superexchange model. Dimerization of AF spin chains coupled to phonons is well known and occurs in several systems \cite{12}. The spin-Peierls transition discovered in CuGeO$_3$ \cite{13} led to renewed interest in the dimerization instability of the AF spin chains. In the spin-orbital model for the RVO$_3$ perovskites a similar instability might also occur without the coupling to the lattice when Hund’s exchange is sufficiently small. In particular, the ground state at $\eta = 0$ may be approximated by the dimerized chain with strong FM bonds alternating with the AF ones, if such chains are coupled by AF interactions along the $a$ and $b$ axes \cite{14} (the 1D chain would give the entangled disordered ground state as described in Sec. \ref{sec:VI}).

At realistic values of $\eta > 0.10$ the C-AF order with FM chains along the $c$ axis is found in the ground state \cite{17}. Numerical studies performed at finite temperature have shown that periodic dimerization of the magnetic exchange exists in a certain finite temperature range, while the ground state is the fully polarized and uniform FM state \cite{10,11}. These findings served as a motivation to investigate the mechanism of the spin-Peierls dimerization in FM spin chains. The microscopic 1D model which stands for the situation encountered in the C-AF phase of YVO$_3$ reads \cite{11}:

$$H_{S^c} = J \sum_i \left( \vec{S}_i \cdot \vec{S}_{i+1} + 1 \right) \left( \vec{\tau}_i \cdot \vec{\tau}_{i+1} + \frac{1}{4} - \gamma_H \right),$$  

(50)

where $\gamma_H$ is stands for the contribution due to the high-spin states proportional to the Hund’s exchange \cite{15} and stabilizes FM spin order. While the spin and orbital operators are disentangled in the FM ground state, one may consider a coupled FM spin chain to an orbital chain with interactions which favor the AO order, as realized in the C-AF phase. The exchange interactions along the spin (orbital) chain depend on the orbital (spin) correlations, and their modulation may be described by $\delta_s$ and $\delta_t$ parameters \cite{45}. They can be found from a self-consistent solution of the coupled MF equations for spin and orbital correlations, and one finds indeed dimerized spin and orbital chains in a finite range of temperature \cite{111}.

Summarizing, spin-orbital entanglement in the excited states is also responsible for the exotic magnetic properties of the C-AF phase of YVO$_3$. They arise from the coupling between the spin and orbital operators which triggers the dimerization of the FM interactions as a manifestation of a universal instability of FM chains at finite $T$, which occurs either by the coupling to the lattice or to purely electronic degrees of freedom \cite{111}. This latter mechanism could play a role in many transition metal oxides with (nearly) degenerate orbital states.

\section{VI. Coexisting Charge and Orbital Order}

The first step towards understanding the dosed systems with orbital degrees of freedom is the question concerning possible QP states deciding about coherent hole propagation in the orbitally ordered background. As discussed in Sec. \ref{sec:VI} a single hole doped into the AF background as in CuO$_2$ planes of La$_2$CuO$_4$ may propagate through the lattice because it couples to quantum spin fluctuations and becomes dressed with a "cloud" of magnons \cite{113}. This results in the new energy scale $J \propto J$ in place of the hole hopping $t$, so the hopping is strongly renormalized. The QP which forms after the hole is doped in the AF background is called a spin polaron \cite{30}. A more complex situation can occur in the systems with partly filled degenerate orbitals, where a doped hole may not only couple to magnons but also couples to crystal-field excitations \cite{14}. In addition, QP states with higher spin states may occur, as for instance a triplet QP in case of an $S = 1/2$ antiferromagnet doped by a single electron in the orbitally degenerate background \cite{116}.

This motivates two questions in the theory: (i) whether orbital excitations could couple as well to the moving hole and generate a new energy scale, as the magnons do, and (ii) whether spin-orbital entanglement has any important consequences for the hole dynamics. Both of them were addressed in the orbital $t$--$J$ model for $e_g$ electrons \cite{117}, and in the analogous models for $t_{2g}$ orbitals developed recently, see below.

Two situations with a hole doped into an AO ordered background were considered in the past: (i) a hole doped into an ab plane of LaMnO$_3$ \cite{11} which has an AO order of $e_g$ orbitals in the ground state, and (ii) a hole doped into an ab plane with an AO order of $t_{2g}$ orbitals and FM spin order \cite{119}, as realized for instance in Sr$_2$VO$_3$. In the first case it was shown that the orbitons have in general a gap and have a lower dispersion than the magnons. Therefore, the quantum effects are weak but but a hole can move by interorbital hopping processes. While the constraint of creating no double occupancies has to be obeyed along the hole hopping, the bandwidth is strongly renormalized with respect to that suggested by the LDA+$U$ approach \cite{118}. Such interorbital hopping processes are absent in the $t_{2g}$ ordered background with alternating $yz$/$zx$ orbitals in an ab plane, and due to the specific $t_{2g}$ orbital symmetries the orbitons are dispersionless. Thus the string picture \cite{120} dominates the character of the $t_{2g}$ orbital polarons even more than in the case of systems with $e_g$ orbital degrees of freedom.

An intriguing question in this context addressed only recently is whether spin quantum fluctuations can still contribute to the QP spectral properties when both types...
of order, spin and orbital, alternate in an ab plane, as for instance in the C-AF phase of RVO₃ perovskites. A crucial observation for the spectral properties of a hole doped into the entangled AF/AO background is a simultaneous excitation of a magnon and an orbiton when a hole moves by a single step in the lattice [121]. This dominates the behavior of the hole doped in such an entangled state, because the orbitals confine the hole motion by forcing the hole to retrace its path which implies that the hole motion by its coupling to the quantum spin fluctuations is prohibited. Thus, the string-like potential which acts on the hole is induced by the orbitals although it has a joint spin-orbital character. Hence, this important feature of the orbitally induced string formation could be understood as a topological effect. This happens even if the energy of the orbital excitations is turned to zero, i.e., when the hole moves in the orbital sector incoherently. Hence, the mere presence of orbitals is sufficient to obtain the (almost) classical behavior of a hole doped into the ground state with AF/AO order. This result, in connection with the fact that the mother-compound iron-pnictides shows a variety of spin-orbital phenomena [122], suggests that further investigation of the hole propagation in spin-orbital systems is a fascinating subject for future studies.

The properties of doped R₁₋ₓSrₓVO₃ systems are puzzling and it is not understood until now why (i) La₁₋ₓSrₓVO₃ is insulating in a broad range of doping below xₑ = 0.18, and (ii) why the AF order survives even for x > xₑ when the system becomes metallic and looses the AO order in the ab planes [123]. The gradual changes of the optical conductivity under increasing doping demonstrate that the anisotropy between the ab and c direction decreases, but surprisingly is not completely lost even in the metallic regime. The differences observed in the optical conductivity and raman scattering spectra between La₁₋ₓSrₓVO₃ and Y₁₋ₓSrₓVO₃ suggest that the orthorhombic lattice distortion plays also here a very important role and influences the hole dynamics [124]. It has been argued that the C-AF phase is more robust under hole doping [125] and survives in a broad doping range [126], but a complete understanding of doped vanadate perovskites awaits a more careful theoretical study.

Doped R₁₋ₓ(Sr,Ca)ₓMnO₃ systems are studied much longer and they are better understood. The FM metallic state is induced by doping via the double exchange mechanism [7] which was also formulated for degenerate e₉ orbitals [127], and the phase diagrams of the doped perovskite systems show a remarkable sequence of magnetic phases [128], from the A-AF phase, through the insulating and metallic FM phase, towards the C-AF phase and G-AF phase in the highly doped regime. Similar (but not the same) sequence of magnetic phases was reported in the bilayer La₂₋ₓSr₁₋ₓMn₂O₇ systems [129]. As in the R₁₋ₓSrₓVO₃ perovskites, also in the layered systems the orbital ordered (or liquid) state determines whether the intersite spin correlations are AF or FM, as shown for the monolayer [130] and for the bilayer [131] using a t−J-like model which includes orbital degeneracy. These model calculations illustrate as well the complementarity of spin and orbital order expressed by the Goodenough-Kanamori rules [72]. In these systems the short-range charge order gradually develops with increasing doping in the realistic parameter regime [132]. However, more complete models including the charge transfer physics are necessary to describe the features observed in the optical spectra, as for instance in insulating LaSrMnO₄ [133].

Although there is no complete understanding of the phase diagram and in particular of the mechanism of the metal-insulator transition which leads to the colossal magnetoresistance until now, a lot of progress could be made using model Hamiltonians. It was recognized that the orbital degeneracy plays a crucial role both in the double exchange [127] and for the interactions with the lattice due to the JT effect [134], and phase diagrams which resemble the qualitative behavior of the doped manganites were obtained [135]. However, a more realistic treatment requires also electron correlations among e₉ electrons which are more difficult to implement [8]. In contrast to the nondegenerate Hubbard model, the orbital Hubbard model for FM manganites does not show an instability towards the orbitally polarized FO state and one finds instead the disordered orbital liquid ground state [136]. This concept was crucial in explaining the doping dependence of the stiffness constant in the FM La₁₋ₓSrₓMnO₃ manganites [137], but for a quantitative explanation both the double exchange due to correlated e₉ electrons and the superexchange due to t₂g core spins had to be included [138]. This approach had also a remarkable success [139] in explaining the observed magnon dispersion and the doping dependence of the magnetic exchange constants in the La₂₋ₓSr₁₋ₓMn₂O₇ systems, including the observed phase transition from the FM to the A-AF structure [140].

As expected, the orbital order melts in general when the manganites are doped, there are cases when a different type of orbital order coexisting with charge order emerges again at half doping. The famous case is the so-called (charge exchange) CE phase in half-doped (x = 0.5) manganites [71], where the two-sublattice charge order coexists with orbital order on the sites with the majority of e₉ electron charge, and the FM zig-zag chains staggered in ab planes. Although the double exchange provides some arguments justifying the stability of this complex type of order [141] which competes with the FM phase in the relevant parameter regime and wins for sufficiently large and realistic AF superexchange between the S = 3/2 t₂g core spins [142], the problem is subtle and the range of parameters with the CE phase in the ground state is rather narrow. In particular, this phase is destabilized by intersite Coulomb interaction [151], and the JT distortions play an important role in stabilizing it [143]. We emphasize again that AF interactions between t₂g electrons are small — for this case the CE phase was found in the charge-ordered phase us-
ing a finite-temperature diagonalization technique \[144\]. The mechanism invoked there to stabilize the CE phase is subtle and employs the cooperative JT interaction between next-nearest Mn\(^{3+}\) neighbors mediated by the breathing mode distortion of Mn\(^{4+}\) octahedra and displacements of Mn\(^{4+}\) ions. It is worth noting that the topological phase factor in the Mn-Mn hopping \[145\] leading to gap formation in 1D models \[141\] for the CE phase, as well as the nearest neighbor JT coupling, are not able to produce the observed zigzag FM chains for the realistic parameters \[144\]. Recent x-ray structural analysis of Pr\(_{0.5}\)Ca\(_{0.5}\)MnO\(_3\) and Eu\(_{0.5}\)Ca\(_{1.5}\)MnO\(_4\) suggest that the orbital shape and the charge disproportionation are sensitive to the dimension of Mn–O network \[146\], which together with the possible different role of the JT effect in both compounds poses new interesting questions in the theory.

The controversy about the nature of the charge order in this phase which arose due to signatures of Zener polaron observed in the neutron data \[147\] seems to be resolved now in favor of the more conventional picture of zig-zag chains \[148\]. Melting of this composite order with increasing temperature is fascinating and the magnetic order disappears first, giving coexisting charge and orbital order in the intermediate temperature regime, before both melt resulting in a disordered phase \[148\]. This suggests that the link between (weak) charge and orbital order is particularly strong here, similar as in the magnetite below the Verwey transition \[11\]. Recently, charge ordered AF phase was also reported in La\(_{1.5}\)Sr\(_{0.5}\)CoO\(_3\), and preliminary theoretical concepts in the framework of spin-orbital physics were also presented \[150\].

Stripe phases appear also in the doped systems with active orbital degrees of freedom, but are qualitatively different from the ones observed in the cuprates, see Sec. VII. Stripe order was found in doped manganites \[151\] and also discovered in doped La\(_{2-x}\)Sr\(_x\)NiO\(_4\) nickelates about the same time as in the cuprates \[152\]. However, in contrast to the cuprates the stripes in La\(_{0.5}\)Sr\(_{1.5}\)NiO\(_4\) are diagonal and contain one (and not half) hole per unit cell \[153\]. Intriguing features seen in the spin excitation spectra of La\(_2\)NiO\(_4-\delta\) nickelates were reported recently \[154\] which suggest that the inward dispersion, seen also in cuprates, has a common origin in stripe phases. Simulations performed within the LDA+U approach suggest that a subtle interplay between the charge and spin order and octahedral distortions is essential for the formation of an insulating state \[155\]. Preliminary HF calculations emphasize the importance of orbital degeneracy in the case of nickelates for the realistic \(e_g\) hopping model \[156\], where one finds indeed that diagonal stripes with the observed filling (of one hole per unit cell) are more stable than other phases, in contrast to the predictions of the degenerate Hubbard model with diagonal hopping (which does not agree with experiment). It remains a challenge for the theory to develop a more complete theory of the stripe phases in the nickelates, including the electron correlations and the coupling of \(e_g\) electrons to the lattice distortions, and to understand better differences between the stripe phase in the nickelates and in the cuprates.

**VII. SUMMARY AND OPEN PROBLEMS**

Summarizing, charge order is common in doped transition metal oxides and arises even in absence of intersite Coulomb interactions while the kinetic energy of doped charges (holes or electron) competes with the magnetic superexchange. Under such circumstances stripe phases realized in the cuprates, nickelates, and manganites are favored as then the two above energies are optimized simultaneously in the domain walls and in the magnetic domains between them. When also orbital degrees of freedom are active, the charge order is accompanied by certain, usually weak, orbital order. Good examples of this composite type of charge-and-orbital order are the low temperature phase of Fe\(_3\)O\(_4\), and the CE phase in the half-doped manganites.

In the perovskite lattice the orbital order is stabilized easier in the correlated insulators with orbital \(e_g\) degrees of freedom, as then the superexchange and the interaction with the lattice act supporting each other \[73\], than in the ones with the \(t_{2g}\) active orbitals, where the interactions with the lattice in general compete with the superexchange \[98\]. As a result, the orbital and magnetic transition occur independently from each other and at quite different temperatures in the \(e_g\) systems, while the case of the \(R\)VO\(_3\) perovskites is an example of the proximity and interplay of the magnetic and orbital phase transition. Both these different situations were successfully described within spin-orbital superexchange models with added interactions with the lattice.

A qualitative difference between the orbital order in \(e_g\) and \(t_{2g}\) systems is that the orbital order is more robust in \(e_g\) case and may be usually treated by classical (MF) approaches, while \(t_{2g}\) orbitals may easier fluctuate and thus couple also easier to the spin degrees of freedom. It is for this reason that composite spin-orbital fluctuations occur in correlated titanates and vanadates insulators. Although such fluctuations are quenched in the ground state of these systems for realistic parameters, they develop at increasing temperature due to the presence of excited states with spin-orbital entanglement. For instance, such composite spin-orbital fluctuations are responsible for the temperature dependence of the optical spectral weights in LaV\(_3\)O\(_7\) \[80\] and trigger spin-orbital dimerization in the C-AF phase of YVO\(_3\) in the intermediate temperature regime \[109\]. Similarly interesting orbital ordered states are also found in the perovskite ruthenates — as an example we mention here the puzzling low temperature electronic and structural behavior recently discovered in PbRuO\(_3\) \[157\].

The microscopic mechanism of melting of the orbital order in doped systems is a very challenging problem in the theory and could not be understood until now. The main difficulties follow from disorder and the ne-
cesity of using Monte Carlo techniques. In this way it could be concluded that a strong competition between the FM metallic and the AF charge-ordered insulating states takes place \cite{158}. This competition influences the transport properties and leads to short-range spin and charge correlations which evolve with time. It is intriguing to what extent this complex situation in the metallic phase influences the magnetic excitations of the system. The magnetic excitations in the metallic FM phase of several doped manganites soften at the zone boundary and may be described by the Heisenberg model with the nearest neighbor $J_1$ and fourth-nearest neighbor $J_4$ exchange coupling \cite{159}. Its microscopic origin is controversial and two different concepts were proposed to explain the experimental observations: (i) quantum fluctuations of the planar $x^2 - y^2$ orbitals associated with the $A$-AF phase \cite{160}, and (ii) the $(3z^2 - r^2)$-type orbital fluctuations \cite{159}. It was also shown that the ratio of $J_4/J_1$ changes along the $A_1 - A_1^f$ MnO$_3$ manganites (with $A$ and $A'$ being the rare-earth and alkaline rare-earth ions), while the stiffness constant is almost universal and has only very weak dependence on the chemical composition for a fixed doping $x$ \cite{161}. Both theoretical models have difficulties to explain the experimental data — an incorrect dispersion along the $(111)$ direction follows from the first one \cite{161}, while the second one predicts a spectacular doping dependence of $J_4/J_1$, which is not observed. Thus, in spite of its remarkable success in the simplest situation \cite{138}, the complete theoretical explanation of the magnon dispersion in the metallic FM manganites within the orbital liquid state remains a challenging problem in the theory. Even more puzzling are the magnons in the insulating FM phase, where several branches with a staircase-like spectrum have been observed \cite{162}.

Recent studies of the phase diagram of the $R$MnO$_3$ manganites include the effect of orthorhombic distortions within theoretical models with spin superexchange and the Dzyaloshinsky-Morina interaction responsible for the multiferroic behavior \cite{163}. The usual approach so far is to develop an effective spin model including the spin-lattice coupling, leading to the electric polarization \cite{164}. It seems that an explicit treatment of the orbital degrees of freedom could provide a better understanding of the observed phenomena.

Another active direction of research in the field of transition metal oxides is the search for novel quantum phenomena, including more examples of quantum spin-orbital entanglement. They could be found in frustrated lattices, and here we mention briefly only the triangular lattice. An interesting case and good candidate for a spin-orbital liquid might be LiNiO$_2$ with a triangular lattice of Ni$^{3+}$ ions ($d^7$ configuration with $S = 1/2$ spins) and no magnetic or orbital order down to very low temperatures. First, it was argued that a model based on symmetry arguments \cite{165}, characterized by a large number of low-lying singlets associated to dimer coverings of the triangular lattice, could explain the absence of any type of ordered phase in LiNiO$_2$. Second, it was shown that excited states on oxygens along the $90^\circ$ bonds are crucial in the superexchange and they change the balance between different terms in the Hamiltonian, making the orbital interactions stronger than the spin ones \cite{166}. In any case, interplane JT coupling seems to be too weak in LiNiO$_2$ to stabilize the orbital long-range order, and the microscopic reason of disorder could be alone due to strongly frustrated orbital interactions on the triangular lattice in $(111)$ planes \cite{160}, which resemble the compass model. A completely different situation is encountered in the $d^3$ spin-orbital model with active three $t_2g$ orbitals on the triangular lattice, as realized in NaTiO$_2$, where a spin-orbital disordered liquid state is more likely \cite{167}.

As a final remark, we would like to mention recent experimental studies of Ni-based superlattices \cite{168}. They stimulated progress in the theory which predicts that, in analogy to the gain of kinetic energy in the layered manganites \cite{169}, the correlated $e_g$ electrons in the NiO$_2$ planes develop a planar $(x^2 - y^2)$-like orbital order in LaNiO$_3$/LaMnO$_3$ superlattices (with $M$=Al, Gd, Ti) \cite{170}. It may be expected that future studies of the systems of reduced dimensionality will provide more unexpected properties in the near future, and could lead to developing functional materials, using both charge and orbital degrees of freedom.

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