Abstract. We show that a uniform probability measure supported on a specific set of piecewise linear loops in a non-trivial free homotopy class in a multi-punctured plane is overwhelmingly concentrated around loops of minimal lengths. Our approach is based on extending Mogulskii’s theorem to closed paths, which is a useful result of independent interest. In addition, we show that the above measure can be sampled using standard Markov Chain Monte Carlo techniques, thus providing a simple method for approximating shortest loops.

1. Introduction

The problem of finding a path of minimum length in a metric space under topological constraints is one of the classical problems in geometric optimization. It has numerous applications, including path planning and navigation [5, 25], VLSI routing [14, 24], and surface cutting [12], which is an important step in surface parametrization [13, 23] and texture mapping [2, 20].

The shortest path problem has been considered in many different settings, and tackled using a variety of techniques. Most commonly, paths in a planar domain or in a (two-dimensional) surface are considered, and numerous algorithms have been developed to find the corresponding shortest paths or approximations thereof (see e.g. [16, 15, 3, 11, 8, 9] and references therein).

In this paper we take a completely different approach to this classical problem. It has been noted that values of cost functions in some optimization problems differ very slightly from the mean (or median) value with respect to some naturally defined probability measure, leading to interesting approximation techniques [1]. This is a consequence of the well-studied concentration of measure phenomenon [17]. Roughly speaking, a Borel probability measure \( \mu \) on a metric space \((X, d)\) is concentrated around a set \( A \subset X \) if the quantity \( 1 - \mu(A_\varepsilon) \), where \( A_\varepsilon = \{x \in X | d(x, A) < \varepsilon\} \), decreases very fast (e.g. exponentially) as \( \varepsilon \) grows. A typical example, mentioned in the above references, is the concentration of the uniform probability measure on a high-dimensional unit sphere around every equator.

Clearly, an approximate solution to an optimization problem may be obtained by sampling from a measure concentrated around the minimizers of the cost function. Of course, constructing such a probability measure, or showing that a particular measure has the right concentration property, is by no means a trivial task. The goal of this paper is to show that such an approach is indeed viable for the problem of finding loops of minimal length in a fixed, nontrivial free homotopy class (we define the relevant notions below).

Specifically, we consider discretized loops in a multi-punctured plane and show that the uniform probability measure supported on a specific set of such piecewise linear loops in a non-trivial homotopy class is overwhelmingly concentrated around loops of minimal lengths. The choice of a multi-punctured plane provides a nice compromise between simplicity and applicability, as it can serve as a model for domains in many path planning applications. We should also mention that

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our approach is based on extending the Mogulskii’s theorem to closed paths (in the plane), which is a useful result of independent interest.

The rest of the paper is structured as follows. Section 2 contains the necessary background information. The statements of our main results are provided in Section 3. In Section 4 we show that the measure under consideration can be sampled using standard Markov Chain Monte Carlo techniques. All the proofs of our results have been put in a separate Section 5. Section 6 concludes the paper.

2. Preliminaries

Before stating our main result we need to introduce the necessary nomenclature and provide several auxiliary results. Additional background information can be found in such comprehensive texts as [7, 6, 10].

2.1. Paths and loops. Let \( (X, d) \) be a metric space. A path in \( X \) is a continuous map \( \gamma : I \to X, I := [0,1] \). If a path \( \gamma \) is closed, that is, \( \gamma(0) = \gamma(1) \), then we call it a loop. A loop in \( X \) may also be regarded as a continuous map from a circle, \( \gamma : S^1 \to X \), in which case it is convenient to think of the circle as a quotient of \( \mathbb{R} \), \( S^1 = \mathbb{R}/\mathbb{Z} \), and regard \( \mathbb{R} \) as a covering space for \( S^1 \). Such a setting allows us to consider the lift of a map on \( S^1 \) to a map on \( \mathbb{R} \), which is often useful (see e.g. [7] for details). Given \([a, b] \subset [0,1] \), the restriction of a path \( \gamma \) onto \([a, b] \), denoted by \( \gamma|_{[a, b]} \), is the path defined by \( \gamma|_{[a, b]}(t) = \gamma(a + t(b - a)) \).

If paths \( \gamma_1, \ldots, \gamma_m \) are such that \( \gamma_i(1) = \gamma_{i+1}(0), i = 1, \ldots, m - 1, \) and \( c = (c_1, \ldots, c_m) \) is such that \( c_i \geq 0 \) and \( \sum_{i=1}^m c_i = 1 \), then we define the \( c \)-concatenation of \( \gamma_i \) as the path

\[
\gamma_1 c_1 \ldots c_{m-1} \gamma_m(t) = \gamma_1 \left( \frac{t - C_{i-1}}{c_i} \right), \quad t \in [C_{i-1}, C_i],
\]

where \( C_i = \sum_{j=1}^i c_j \). The value \( c_i \) is called the traversal time of the path \( \gamma_i \) in the concatenation. Note that zero traversal times are allowed only for constant paths, i.e. paths \( \gamma \) such that \( \gamma(t) = \gamma(0) \) for all \( t \in [0,1] \). If traversal times are not important, we will talk about a concatenation of paths. In this case we will use notation \( \gamma_1 \cdot \ldots \cdot \gamma_m \).

The length of a path \( \gamma \) is defined by

\[
L(\gamma) = \sup \sum_{i=1}^n d(\gamma(t_{i-1}), \gamma(t_i)),
\]

where the supremum is taken over all finite collections of points \( 0 = t_0 < t_1 < \cdots < t_n = 1 \).

A path is called rectifiable if its length is finite. The length of a restriction \( \gamma|_{[a, b]} \) will be denoted \( L(\gamma, a, b) \).

When focusing on geometric properties of paths, it is sometimes convenient not to distinguish paths that differ only up to a change of variable. To this end, we define a curve as an equivalence class of the equivalence relation for which paths \( \gamma_1 \) and \( \gamma_2 \) are equivalent if \( \gamma_1(\varphi_1(t)) = \gamma_2(\varphi_2(t)) \), where \( \varphi_i : [0,1] \to [0,1], i = 1, 2, \) are continuous, nondecreasing functions (see [7] for details). A particular path within a curve is called a parametrization of that curve. Paths representing the same curve are re-parametrizations of each other. Such paths have the same image and the same length, allowing us to define these concepts for curves. If a curve is rectifiable then it has the constant speed parametrization, which is the path \( \gamma \) such that \( L(\gamma, t_0, t_1) = L(\gamma)(t_1 - t_0) \).

In the case of loops, it is further often useful to not fix the starting point. Hence, the notion of a curve has to be slightly modified. We define a free loop as an equivalence class of the equivalence
relation for which loops \( \gamma_1, \gamma_2 : S^1 \to X \) are equivalent if \( \gamma_1(\varphi_1(t)) = \gamma_2(\varphi_2(t)) \), where \( \varphi_i : S^1 \to S^1, i = 1, 2 \), are orientation preserving homeomorphisms.

Once again, loops representing the same free loop have the same image and length, and rectifiable free loops admit a constant speed parametrization. If \( \hat{\gamma} \) is a free loop (or a curve) we define \( L(\hat{\gamma}) = L(\gamma) \), where \( \gamma \) is a representation of \( \hat{\gamma} \).

One of the central concepts in the topology and geometry of paths is homotopy. Two paths \( \gamma_0 \) and \( \gamma_1 \) such that \( \gamma_0(0) = \gamma_1(0) = x \in X \) and \( \gamma_0(1) = \gamma_1(1) = y \in X \) are said to be homotopic if there exists a continuous map \( H : I \times [0,1] \to X \) such that \( H(\cdot,0) = \gamma_0, H(\cdot,1) = \gamma_1, \) and \( H(0,t) = x, H(1,t) = y \) for all \( t \in [0,1] \). Intuitively, two paths are homotopic if one can be continuously deformed into the other keeping the endpoints fixed. It is useful to note that two representations of the same curve are homotopic.

The homotopy keeps the starting point fixed, which may be undesirable when dealing with loops. In this case we need to use the free homotopy. More precisely, loops \( \gamma_0 \) and \( \gamma_1 \) are said to be freely homotopic if there exists a continuous map \( H : I \times [0,1] \to X \) such that \( H(\cdot,0) = \gamma_0, H(\cdot,1) = \gamma_1, \) and \( H(0,t) = x, H(1,t) = y \) for all \( t \in [0,1] \). Similarly to the case of curves, two representations of the same free loop are freely homotopic. Also, being freely homotopic is an equivalence relation, and an equivalence class of freely homotopic loops is called a free homotopy class. Such a class is called trivial if it contains a constant loop (i.e. a point). Loops within the trivial free homotopy class are called contractible. A contractible loop is actually homotopic to a constant loop.

We denote the space of paths in \( X \) by \( \Omega(X) \) and endow it with the \( C^0 \) metric, which we denote by \( \rho \). That is, given \( \gamma_0, \gamma_1 \in \Omega(X) \), the distance between them is defined by \( \rho(\gamma_0, \gamma_1) = \sup_{t \in [0,1]} d(\gamma_0(t), \gamma_1(t)) \). The subspace of \( \Omega(X) \) consisting of loops will be denoted by \( \mathcal{L}(X) \). We may also consider the space of curves in \( X \), which we denote by \( \hat{\Omega}(X) \) and the space of free loops, \( \hat{\mathcal{L}}(X) \). The maps \( \pi_\Omega : \Omega(X) \to \hat{\Omega}(X) \) and \( \pi_{\mathcal{L}} : \mathcal{L}(X) \to \hat{\mathcal{L}}(X) \) will denote the corresponding canonical projections. We endow both \( \hat{\Omega}(X) \) and \( \hat{\mathcal{L}}(X) \) with a metric. The distance between \( \hat{\gamma}_0, \hat{\gamma}_1 \in \hat{\Omega}(X) \) is defined as \( \hat{\rho}_\Omega(\hat{\gamma}_0, \hat{\gamma}_1) = \inf_{\gamma_i \in \pi^{-1}_\Omega(\hat{\gamma}_i)} \rho(\gamma_0, \gamma_1) \). Similarly, the distance between \( \hat{\gamma}_0, \hat{\gamma}_1 \in \hat{\mathcal{L}}(X) \) is defined as \( \hat{\rho}_{\mathcal{L}}(\hat{\gamma}_0, \hat{\gamma}_1) = \inf_{\gamma_i \in \pi^{-1}_{\mathcal{L}}(\hat{\gamma}_i)} \rho(\gamma_0, \gamma_1) \).

2.2. Paths and loops in a punctured plane. The concrete metric space that we consider in this paper is a multi-punctured plane, \( X = \mathbb{R}^2 \setminus Z, Z = \{z_1, \ldots, z_K\} \), \( z_i \in \mathbb{R}^2 \), with the standard Euclidean metric, \( d(x,y) = \|x - y\| \), where \( \| \cdot \| \) is the Euclidean norm. By \( \text{reach}(Z) \) we denote half the minimum distance between the punctures, \( \text{reach}(Z) = \frac{1}{2} \min_{z,w \in Z} \|z - w\| \). Also, it will be convenient to define \( \mathcal{X}^\delta = \mathbb{R}^2 \setminus \cup_{i=1}^{K} B_\delta(z_i), \delta > 0 \), where \( B_\delta(z_i) \) denotes an open ball of radius \( \delta \) centered at \( z_i \). For \( \delta < \text{reach}(Z) \), \( X^\delta \) is homeotopy equivalent to \( X \).

A free homotopy class shall be regarded as a connected component of the space of loops in \( X \). Given sets \( A \subset B \subset \mathbb{R}^2 \), we shall regard \( \Omega(A) \) as a subset of \( \Omega(B) \), and \( \mathcal{L}(A) \) as a subset of \( \mathcal{L}(B) \). In particular, we have \( \mathcal{L}(X^\delta) \subset \mathcal{L}(X) \). Throughout the rest of the paper, \( g(X) \subset \mathcal{L}(X) \) will denote a fixed, nontrivial free homotopy class of \( \mathcal{L}(X) \), and \( g(X^\delta) \subset \mathcal{L}(X^\delta) \) will be the free homotopy class of \( \mathcal{L}(X^\delta) \) such that \( g(X^\delta) \subset g(X) \). Notice that \( g(X^\delta) \) is well defined if \( \delta < \text{reach}(Z) \), which we assume hereafter. We also define \( \hat{g}(X) = \pi(\mathcal{g}(X)) \), \( \hat{g}(X^\delta) = \pi_{\mathcal{L}}(g(X^\delta)) \).

Loosely speaking, our goal is to show that a loop chosen “uniformly at random” in \( \hat{g}(X) \) is extremely likely to be very close to the shortest loop (essentially, unique) in that class.

To make this statement more precise, we need to define an appropriate probability measure on \( \hat{g}(X) \). Such a probability measure can be obtained as a push forward of a probability measure on \( g(X) \). In fact, we shall consider a sequence of probability measures on \( g(X) \), each supported on an
increasingly finer finite dimensional approximation of loops in \( g(X) \). The result for \( \hat{g}(X) \) will then be obtained as a corollary of a stronger result for \( g(X) \). In what follows, it will be convenient to regard \( \Omega(X) \) (as well as any of its subsets, e.g. a homotopy class) as a subset of \( \Omega(\mathbb{R}^2) \).

A path \( \gamma \) in \( \mathbb{R}^2 \) is called linear with endpoints \( x, y \in \mathbb{R}^2 \) if \( \gamma(t) = x + t(y - x) \). Such a path will be denoted by \([x, y]\). Clearly, the length of \([x, y]\) is just \( d(x, y) \). We say that \( \gamma \) is a piecewise linear path if it is a concatenation of finitely many linear paths. Each linear path of such a concatenation is called an edge of \( \gamma \), and an endpoint of an edge is called a vertex of \( \gamma \). It is easy to see that the length of a piecewise linear path is just the sum of its edge lengths. We will denote the set of piecewise linear paths in \( \gamma \subset \mathbb{R}^2 \) by \( \Omega_{\text{PL}}(\mathbb{R}) \); the space of piecewise linear loops in \( \gamma \) will be denoted \( \mathcal{L}_{\text{PL}}(\gamma) \). Notice that given a piecewise linear path one can “close” it by adding an edge between the first and the last vertices. Alternatively, one can “open” a piecewise linear loop by removing the last edge. We will use this fact, and we define \( \iota : \mathcal{L}_{\text{PL}}(\mathbb{R}^2) \to \Omega_{\text{PL}}(\mathbb{R}^2) \) by \( \iota(e_1 \ldots e_m) = e_1 \beta_1 \ldots \beta_{m-1} e_m, \) where \( e_i, i = 1, \ldots, m \), are linear paths and \( \beta_i = \alpha_i/\sum_{j=1}^{m} \alpha_i \).

The following result shows that piecewise linear paths form a dense set:

**Proposition 1.** Let \( \gamma \in \Omega(\mathbb{R}^2) \). Then \( \forall \varepsilon > 0 \) there exists \( \gamma_{\text{PL}} \in \Omega_{\text{PL}}(\mathbb{R}^2) \) such that \( \rho(\gamma, \gamma_{\text{PL}}) < \varepsilon \). Moreover, \( \gamma_{\text{PL}} \) can be chosen such that \( \gamma_{\text{PL}} \) is a loop if \( \gamma \) is a loop, \( \gamma_{\text{PL}}(t) = \gamma(t) \) if \( \gamma_{\text{PL}}(t) \) is a vertex, and each edge of \( \gamma_{\text{PL}} \) has traversal time \( \frac{1}{m} \), where \( m \) is the number of edges.

A curve or a free loop is called piecewise linearizable if it possesses a piecewise linear parametrization. A piecewise linear curve or free loop is completely determined by its vertices. Hence, there is a correspondence between \( \mathbb{R}^{2(n+1)} \) and piecewise linear curves (or free loops) in \( \mathbb{R}^2 \) with \( n + 1 \) vertices. More precisely, one can take \( v = (v_0, \ldots, v_n) \in \mathbb{R}^{2(n+1)} \) to correspond to the curve represented by a concatenation of linear paths \([v_{i-1}, v_i], i = 1, \ldots, n \). If we also concatenate \([v_n, v_0] \), then \( v \) corresponds to the resulting free loop.

Given the starting point \( v_0 \) and approximation scale \( n \), we define the finite-dimensional approximations for the curve and free loop yields maps as \( \Psi_n : \mathbb{R}^{2(n+1)} \to \Omega(\mathbb{R}^2) \) and \( \Phi_n : \mathbb{R}^{2(n+1)} \to \mathcal{L}(\mathbb{R}^2) \), respectively.

We will also use an alternative correspondence between \( \mathbb{R}^{2(n+1)} \) and piecewise linear curves and free loops. It is obtained by letting \( w = (v_0, x_1, \ldots, x_n) \subset \mathbb{R}^{2(n+1)} \) correspond to the curve (or free loop) with vertices \( v_0, S_1, \ldots, S_n \), where \( S_k = v_0 + \sum_{i=1}^{k} x_i, k = 1, \ldots, n \). The corresponding maps from \( \mathbb{R}^{2(n+1)} \) into \( \Omega(\mathbb{R}^2) \) and \( \mathcal{L}(\mathbb{R}^2) \) are compositions of \( \Psi_n \) and \( \Phi_n \) with the homeomorphism \( f : \mathbb{R}^{2(n+1)} \to \mathbb{R}^{2(n+1)} \) defined by \( f(v_0, x_1, \ldots, x_n) = (v_0, S_1, \ldots, S_n) \), that is, we consider \( \Psi_n = \Psi_n \circ f \) and \( \Phi_n = \Phi_n \circ f \).

Now, if we take the uniform probability measure on the appropriate subset \( G_n \subset \mathbb{R}^{2(n+1)} \), we can push it forward to \( g(X) \) using \( \Phi_n \), and then further to \( \hat{g}(X) \) using \( \pi_L \). Of course, \( G_n \) should be bounded. Also, as \( n \) increases, we would like the image of \( G_n \) under \( \pi_L \circ \Phi_n \) to provide an increasingly finer approximation of loops in \( \hat{g}(X) \). To achieve boundedness we need to restrict ourselves to loops of bounded length. Hence, let \( R > 0 \), and let \( \hat{g}^R(X) \) be the set of free loops in \( \hat{g}(X) \) with length less than \( R \). We choose \( R \) sufficiently large, so that \( \hat{g}^R(X) \neq \emptyset \). Notice that Proposition 1 implies that any free loop in \( \hat{g}^R(X) \) can be approximated by a piecewise linear free loop, and this approximation improves with decreasing edge length. Therefore, we define \( G_n \) as
follows:
\[ G_n = \left\{ x = (x_0, \ldots, x_n) \in \mathbb{R}^{2(n+1)} : \right. \]
\[ \Phi_n(x) \in \mathbb{R}, \|x_0 - x_n\| < \frac{R}{n+1}, \|x_i - x_{i-1}\| < \frac{R}{n+1}, i = 1, \ldots, n \left. \right\} \]

Let \( \hat{g}_n^R(X) = \pi_L \circ \Phi_n(G_n) \). It is the set of piecewise linear free loops in \( g(X) \) with \( n+1 \) vertices and edge lengths less than \( \frac{R}{n+1} \). Also, let \( \hat{g}_n^R(X) = \Phi_n(G_n) \), which is the set of piecewise linear loops in \( g(X) \) with \( n+1 \) vertices whose edges have traversal time \( \frac{1}{n+1} \) and length less than \( \frac{R}{n+1} \). Clearly, such loops have speed strictly bounded by \( R \). We let \( \hat{g}_n^R(X) \) denote the set of all loops in \( g(X) \) with speed strictly bounded by \( R \) and notice that \( \hat{g}_n^R(X) = \pi_L(\hat{g}_n^R(X)) \).

Define \( \nu_n \) to be the push forward under \( \Phi_n \) of the uniform probability measure on \( G_n \), and let \( \nu_n \) be the push forward of \( \nu_n \) under \( \pi_L \). We can now state our goal more precisely (although still somewhat informally): we want to show that \( \nu_n \) becomes overwhelmingly concentrated around the shortest loop as \( n \to \infty \). We make this statement completely rigorous in the next section.

2.3. Random paths and Mogulskii’s theorem. The above definition of \( G_n \) allows for an alternative description of \( \nu_n \) which is better amenable to analysis. Denote by \( B_R \subset \mathbb{R}^2 \) the disk of radius \( R \) centered at the origin and by \( \Lambda_n \subset \mathbb{R}^2 \) the projection of \( G_n \) onto the first two coordinates. Note that \( \Lambda_n \subset \Lambda_{n+1} \), and \( \Lambda = \bigcup_n \Lambda_n \) is bounded. Let \( \mu \) be the uniform probability measure on \( B_R \), where for ease of notation we suppressed the explicit dependence on \( R \), and let \( \nu_n \) be the uniform probability measure on \( \Lambda_n \). Suppose that \( V_n \) is a random variable with the probability law \( \nu_n \), \( X_1, \ldots, X_n \) are i.i.d. random variables with the probability law \( \mu \), and consider the random piecewise linear path \( \Psi_n \left( V_n, \frac{X_1}{n}, \ldots, \frac{X_n}{n} \right) \). Let \( \mu_n \) be the probability law of such a path. Then given \( \Gamma \subset g(X) \) we have \( \nu_n(\Gamma) = \frac{\mu_n(\Gamma \cap g_n^R(X))}{\mu_n(\Gamma \cap g_n^R(X))} \). For convenience, \( \nu_n \), \( \mu \), \( \mu_n \), and \( \nu_n \) will retain the aforementioned meaning throughout the paper.

With such a set-up we are in the position to employ the powerful machinery of the large deviation theory, in particular the Mogulskii’s Theorem. First, we need to introduce a few more concepts and results. A rate function on a topological space \( \mathcal{Y} \) is a lower semicontinuous map \( I : \mathcal{Y} \to [0, \infty] \) such that its sublevel sets, \( \{ y \in \mathcal{Y} \mid I(y) \leq \alpha \} \), \( \alpha \in [0, \infty] \), are closed. A rate function is called good if its sublevel sets are compact. By \( D_1 \) we will denote the effective domain of the rate function \( I \), that is, \( D_1 = \{ y \in \mathcal{Y} \mid I(y) < \infty \} \).

Taking into account our alternative description of \( \nu_n \), let \( \Lambda \) denote the logarithmic moment generating function associated with \( \mu \), that is \( \Lambda(\eta) = \log E(e^{<X,\eta>} \), where \( E(\cdot) \) denotes the expectation, \( X \) has probability law \( \mu \), and \( < \cdot, \cdot > \) denotes the inner product. Define \( \Lambda^* \) to be the Fenchel-Legendre transform of \( \Lambda \), that is \( \Lambda^*(\eta) = \sup_{\eta} 0 < \eta, \eta > -\Lambda(\eta) \). The following proposition summarizes the properties of \( \Lambda \) and \( \Lambda^* \):

**Proposition 2.** (1) \( \Lambda \) is a strictly convex, everywhere differentiable function.

(2) \( \Lambda^* \) is a good strictly convex rate function.

(3) If \( y = \nabla \Lambda(\eta) \) then \( \Lambda^*(y) = < \eta, y > -\Lambda(\eta) \).

(4) Both \( \Lambda \) and \( \Lambda^* \) are invariant under rotations around the origin, \( D_{\Lambda^*} = B_R \), and \( \forall y \in B_R, \exists \eta \in \mathbb{R}^2 \) such that \( y = \nabla \Lambda(\eta) \).

Recall that a map \( \phi : [0, 1] \to \mathbb{R}^2 \) is called absolutely continuous if \( \forall \varepsilon > 0 \exists \delta > 0 \) such that \( \sum_{i=1}^{m} |\phi(y_i, x_i)| < \varepsilon \) for every finite collection of disjoint intervals \( (x_i, y_i) \subset [0, 1], i = 1, \ldots, m \), such that \( \sum_{i=1}^{m} |y_i - x_i| < \delta \).
We will denote the space of absolutely continuous paths and loops in \( Y \subset \mathbb{R}^2 \) by \( \Omega_{A\subset}(Y) \) and \( \mathcal{L}_{A\subset}(Y) \), respectively. It is useful to note that if \( \gamma \in \Omega_{A\subset}(\mathbb{R}^2) \) then it is differentiable almost everywhere and \( \mathcal{L}(\gamma, a, b) = \int_a^b \| \gamma'(t) \| \, dt \), where \([a, b] \subset [0, 1]\) and \( \gamma'(t) \) denote the derivative of \( \gamma \) at \( t \).

We are now ready to state the Mogulskii’s theorem:

**Theorem 1** (Mogulskii). Let \( \tilde{\mu}_n \) denote the probability law of the random path \( \Psi \left( 0, \frac{X_1}{n}, \ldots, \frac{X_n}{n} \right) \), where \( X_0, \ldots, X_n \) are i.i.d. random variables with the probability law \( \mu \). Then the function \( I_0 : \Omega(\mathbb{R}^2) \to [0, \infty] \) defined by

\[
I_0(\phi) = \begin{cases} 
\int_0^1 \Lambda^*(\phi'(t)) \, dt, & \text{if } \phi \in \Omega_{A\subset}(\mathbb{R}^2), \phi(0) = 0 \\
\infty, & \text{otherwise}
\end{cases}
\]

is a good rate function, and for any Borel set \( \Gamma \subset \Omega(\mathbb{R}^2) \) we have

\[
- \inf_{x \in \Gamma^o} I_0(x) \leq \liminf_{n \to \infty} \frac{1}{n} \log \tilde{\mu}_n(\Gamma) \leq \limsup_{n \to \infty} \frac{1}{n} \log \tilde{\mu}_n(\Gamma) \leq - \inf_{x \in \Gamma} I_0(x),
\]

where \( \Gamma^o \) denotes the interior of \( \Gamma \) and \( \overline{\Gamma} \) denotes the closure of \( \Gamma \).

The same result holds also in the subspace \( \Omega_0(\mathbb{R}^2) \) consisting only of paths starting at the origin (see [10] for details), or at any other point.

More generally, we can prove a version of the Mogulskii’s theorem where the starting point is chosen uniformly at random.

**Theorem 2.** Suppose that \( E_n \subset \mathbb{R}^2 \) are open, \( E_n \subset E_{n+1} \), and \( E = \bigcup_n E_n \) is bounded. Let \( \tilde{\nu}_n \) be the uniform probability measure on \( E_n \), and let \( V_n \) be a random variable with the probability law \( \tilde{\nu}_n \). Denote by \( \hat{\mu}_n \) the probability law of the random path \( \Psi \left( V_n, \frac{X_1}{n}, \ldots, \frac{X_n}{n} \right) \), where \( X_0, \ldots, X_n \) are i.i.d. random variables with the probability law \( \mu \). Then the function \( I_E : \Omega(\mathbb{R}^2) \to [0, \infty] \) defined by

\[
I_E(\phi) = \begin{cases} 
\int_0^1 \Lambda^*(\phi'(t)) \, dt, & \text{if } \phi \in \Omega_{A\subset}(\mathbb{R}^2), \phi(0) \in E \\
\infty, & \text{otherwise}
\end{cases}
\]

where \( E \) denotes the closure of \( E \), is a good rate function, and for any Borel set \( \Gamma \subset \Omega(\mathbb{R}^2) \) we have

\[
- \inf_{x \in \Gamma^o} I_E(x) \leq \liminf_{n \to \infty} \frac{1}{n} \log \hat{\mu}_n(\Gamma) \leq \limsup_{n \to \infty} \frac{1}{n} \log \hat{\mu}_n(\Gamma) \leq - \inf_{x \in \Gamma} I_E(x),
\]

where \( \Gamma^o \) denotes the interior of \( \Gamma \) and \( \overline{\Gamma} \) denotes the closure of \( \Gamma \).

We shall refer to Theorem 2 as untethered Mogulskii’s theorem. If \( E_n = \bigcup_{k=1}^n A_k \), the projection of \( G_n \) onto the first two coordinates, then we denote the corresponding \( I_E \) simply by \( I \). It is useful to notice that if \( \phi \in g^R(X) \) then \( \phi(0) \in \Lambda = \bigcup_n A_n \).

Our particular choice of the probability law \( \mu \) leads to several useful properties of the rate functions \( I_0 \) and \( I_E \).

**Proposition 3.** Let \( J \) be either \( I_0 \) or \( I_E \).

1. \( \mathcal{D}_J \subset \{ \phi \in \Omega_{A\subset}(\mathbb{R}^2) : \| \phi'(t) \| < R \text{ a.e. on } [0, 1] \} \subset \Omega_{A\subset}(\mathbb{R}^2) \cap \Omega^R(\mathbb{R}^2) \), where \( \Omega^R(\mathbb{R}^2) \) denotes the set of paths with Lipschitz constant bounded by \( R \).
2. Let \( \gamma \in \mathcal{D}_J \) be a constant speed parametrization of a curve or a free loop \( \dot{\gamma} \), and let \( \Gamma \) be the set of all parametrizations of \( \dot{\gamma} \). Then

\[
\inf_{\phi \in \Gamma} J(\phi) = J(\gamma)
\]
(3) Suppose that \( \gamma \in D \) is a (non-constant) path with constant speed parametrization, and let \( \phi \in D \) be a path such that \( L(\phi) \geq L(\gamma) + \epsilon \). Then there exists a constant \( c > 0 \), depending on \( \gamma \), such that \( J(\phi) - J(\gamma) \geq c\epsilon \).

2.4. Path localization results. Mogulskii’s theorem and the properties of the rate functions \( I_0 \) and \( I_E \) allow us to investigate the behavior of \( \tilde{\mu}_n \) when restricted to a particular set \( \Gamma \subset \Omega(\mathbb{R}^2) \). For example, let \( \Gamma \) consist of paths starting at the origin and ending within the closed ball \( \overline{B}_r(a) = \{ x \in \mathbb{R}^2 : d(x, a) \leq r \} \), \( a \in \mathbb{R}^2 \). Suppose also that \( 0 \not\in \overline{B}_r(a) \) and \( r + ||a|| < R \). Then the following holds:

**Corollary 1.** Let \( x^* \in \overline{B}_r(a) \) be the point closest to the origin, and let \( \hat{\gamma}^* = \pi_\Omega([0, x^*]) \). Take \( \delta > 0 \) and let \( \tilde{\Gamma}_\delta = \{ \hat{\gamma} \in \pi_\Omega(\Gamma) | \hat{\rho}_\Omega(\hat{\gamma}, \hat{\gamma}^*) \geq \delta \} \), \( \Gamma_\delta = \pi_\Omega^{-1}(\tilde{\Gamma}_\delta) \cap \Gamma \). Then there exists a constant \( c > 0 \) (depending on \( x^* \) and \( r \)) such that

\[
\limsup_{n \to \infty} \frac{1}{n} \log \frac{\tilde{\mu}_n(\Gamma_\delta)}{\tilde{\mu}_n(\Gamma)} \leq -c\delta^2
\]

In other words, \( \tilde{\mu}_n \) restricted to the above \( \Gamma \) become overwhelmingly concentrated around the shortest paths.

It is reasonable to expect a similar concentration result for \( \nu_n \). Unfortunately, as follows from an earlier discussion, investigating the behavior of \( \nu_n \) requires us to consider ratios of the form \( \frac{\mu_n(Q_n)}{\tilde{\mu}_n(P_n)} \), \( Q_n \subset P_n \), rather than \( \frac{\mu_n(Q)}{\tilde{\mu}_n(P)} \) for fixed \( Q \subset P \). Hence, a direct application of Mogulskii’s theorem is not feasible. In the next section we detail our approach to overcome this difficulty.

### 3. Typical Loops in \( g(X) \)

Before we rigorously state our main result we need to take care of a small technicality. Unlike the situation in Corollary 1 where the minimizing path belongs to the set under consideration, \( g(X) \) does not contain any loop minimizing the rate. However, \( L(\cdot) \) does attain its infimum on \( g(X) \), the closure of \( g(X) \) in \( \Omega(\mathbb{R}^2) \), and consequently on \( \pi_\mathcal{L}(g(X)) \). Moreover, the shortest loop in \( \pi_\mathcal{L}(g(X)) \) is unique up to reparametrization and is, in fact, piecewise linear.

**Lemma 1.** \( \pi_\mathcal{L}(g(X)) \) contains a unique free loop of the shortest length. Moreover, this shortest free loop is piecewise linear with vertices in \( \mathbb{Z} \).

We let \( \hat{\gamma}^* \) denote the shortest free loop in \( \pi_\mathcal{L}(g(X)) \). Our main result shows that \( \hat{\nu}_n \) become overwhelmingly concentrated around \( \hat{\gamma}^* \) as \( n \to \infty \).

**Theorem 3.** For each \( \delta > 0 \) we have

\[
\limsup_{n \to \infty} \frac{1}{n} \log \hat{\nu}_n(\tilde{\Gamma}_\delta) \leq -c\delta^2,
\]

where \( c > 0 \) is a constant, and \( \tilde{\Gamma}_\delta = \{ \hat{\gamma} \in g(X) | \hat{\rho}_\mathcal{L}(\hat{\gamma}, \hat{\gamma}^*) \geq \delta \} \).

Since \( \hat{\nu}_n \) is a push forward of \( \nu_n \) under \( \pi_\mathcal{L} \), Theorem 3 is an immediate corollary of the following result.

**Theorem 4.** For each \( \delta > 0 \) we have

\[
\limsup_{n \to \infty} \frac{1}{n} \log \nu_n(\Gamma_\delta) \leq -c\delta^2,
\]

where \( c > 0 \) is a constant, and \( \Gamma_\delta = \pi_\mathcal{L}^{-1}(\tilde{\Gamma}_\delta) \).
The proof of the above theorem relies on Proposition 3 below, which can be regarded as a variation of the Mogulskii’s theorem. Recall that by Proposition 3 I(·) attains the same value for any constant speed parametrization of \( \hat{\gamma} \). Let us denote this value by \( I^* \).

**Proposition 4.** For any Borel subset \( \Gamma \subset g(X) \) we have

\[
-\left( \inf_{\gamma \in \Gamma^o} I(\gamma) - I^* \right) \leq \liminf_{n \to \infty} \frac{1}{n} \log \nu_n(\Gamma) \leq \limsup_{n \to \infty} \frac{1}{n} \log \nu_n(\Gamma) \leq -\left( \inf_{\gamma \in \Gamma} I(\gamma) - I^* \right),
\]

where \( \Gamma^o \) and \( \overline{\Gamma} \) denote the interior the closure of \( \Gamma \) in \( \Omega(\mathbb{R}^2) \), respectively.

The key ingredients in the proof of this proposition are the untethered Mogulskii’s theorem and the following lemma, which is of independent interest in itself:

**Lemma 2.** Let \( \Gamma \subset g(X) \) be open, and let \( \Gamma_n = \iota(\Gamma \cap g^R_n(X)) \). Then

\[
-\inf_{\gamma \in \Gamma} I(\gamma) \leq \liminf_{n \to \infty} \frac{1}{n} \log \mu_n(\Gamma_n)
\]

As mentioned in the Introduction, the proofs of these results are postponed till Section 5.

4. **Sampling in \( G_n \)**

Any practical application of the results from the previous section requires the ability to sample from \( \nu_n \). In this section we show that a standard Markov Chain Monte Carlo (MCMC) technique can do the job. A comprehensive description of Markov chains and MCMC methods can be found in [21, 18] and references therein. Here, we shall limit ourselves to describing and justifying a particular sampling procedure, providing definitions of only some concepts.

4.1. **The sampling algorithm.** As any MCMC method, the sampling algorithm that we propose is based on constructing an ergodic Markov chain on \( G_n \) whose limiting distribution is \( \nu_n \). For convenience, we shall now fix \( n \) and let \( G = G_n, \nu = \nu_n, \varepsilon = \frac{\varepsilon}{n} \). Also, we assume that \( n \) is large enough so that \( G_n \neq \emptyset \) and \( R/n < \text{reach}(Z) \). The algorithm starts with an arbitrary initial state \( V_0 \in G \). Given that the chain is in state \( V_i \in G, i \geq 0 \), the next state, \( V_{i+1} \) is generated as follows. Suppose that \( V_i = (v_0, \ldots, v_n) \in \mathbb{R}^{(2n+1)} \) (in other words, \( v_0, \ldots, v_n \) are the vertices of the corresponding loop), and let \( v_{n-1} = v_n, v_{n+1} = v_0 \). Select \( k \) uniformly at random from \( \{0, \ldots, n\} \). Let \( D \) be the intersections of two open balls of radius \( \varepsilon \) centered at \( v_{k-1} \) and \( v_{k+1} \). The idea is to choose the next state by moving \( v_k \) to a randomly chosen point in \( D \), but we have to make sure that we do not change the free homotopy class of the corresponding loop. Notice that \( D \) may contain at most one puncture. If \( Z \cap D = \emptyset \) we let \( E = D \). If some \( z_j \in D \) we let \( H_1 \) be the open half space supported by the line through \( v_{k-1} \) and \( z_j \) and not containing \( v_{k+1} \), \( H_2 \) be the open half space supported by the line through \( v_{k+1} \) and \( z_j \) and not containing \( v_{k-1} \), and \( H = H_1 \cap H_2 \). Then if \( v_k \in H \) we let \( E = D \cap H \), otherwise, \( E = D \setminus H \) (see Figure 1). Choose \( \bar{v}_k \) uniformly at random from \( E \) and set the next state \( V_{i+1} = (v_0, \ldots, v_{k-1}, \bar{v}_k, v_{k+1}, \ldots, v_n) \).

The above algorithm can be classified as a Metropolis-within-Gibbs algorithm (see e.g. [22]), and it follows from standard results that sequence \( \{V_i\} \) is a Markov chain whose stationary distribution is \( \nu \). Of course, we also need to show that the chain converges to \( \nu \). To make this statement more precise, let \( P(\nu, \cdot), \nu \in G, \) be the corresponding transition probability measure, i.e. \( P(\nu, A) = P(V_{i+1} \in A | V_i = \nu) \), where \( A \) is a Borel subset of \( G \) (see [22] for details). Denote by \( P^m(\nu, \cdot) \) the probability law of the \( m \)-th element of the chain when starting at \( \nu \), i.e. \( P^m(\nu, A) = P(V_m \in A | V_0 = \nu) \). The total variation norm of a signed measure \( \mu \) is defined by
Figure 1. The next vertex position during the MCMC procedure is selected uniformly from the shaded region. Red points indicate punctures; left and right show two different relative positions of the vertex being moved and a puncture.

\[ \| \mu \| = \sup_{A \in \mathcal{M}} |\mu(A)|, \] where \( \mathcal{M} \) denotes the collection of \( \mu \)-measurable sets. We would like to show that
\[ \lim_{m \to \infty} \| P^m(v, \cdot) - \nu \| = 0, \quad \forall v \in G, \]
which implies that, regardless of the initial state, our algorithm generates samples from an almost uniform distribution on \( G \) after a large enough number of steps.

It is well known (see e.g. \([21, 18]\)) that the above convergence result holds if our Markov chain is \( \nu \)-irreducible, aperiodic, and Harris recurrent. \( \nu \)-irreducibility means that for any Borel set \( A \subset G \) such that \( \nu(A) > 0 \) there exists \( m \in \mathbb{N} \) such that \( P^m(v, A) > 0 \) for all \( v \in G \). Aperiodicity means that if \( S_1, \ldots, S_k \subset G \) are disjoint Borel sets such that \( \nu(S_j) > 0 \) and \( P(v, S_{j+1}) = 1 \) \( \forall v \in S_j \), where \( j = 1, \ldots, k, S_{k+1} = S_1 \), then \( k = 1 \). Finally, Harris recurrence means that for any Borel set \( A \subset G \) such that \( \nu(A) > 0 \) we have \( P(V_m \in A \text{ i.o.}|V_0 = v) = 1 \) \( \forall v \in G \), where i.o. stands for “infinitely often”.

**Proposition 5.**

1. Suppose that \( G \) is path connected. Then the Markov chain \( \{V_i\} \) is \( \nu \)-irreducible, aperiodic and Harris recurrent.
2. \( G_n \) is path connected for large enough \( n \).

The proof of the above proposition is provided in a separate subsection of Section 5.

### 4.2. Numerical simulations.

To illustrate the behavior of our algorithm we have performed some numerical simulations. For simplicity, the actual implementation of the algorithm slightly deviates from the description given above. In particular, the vertex to move at each step is chosen as follows. We generate a random permutation of indices, \( \{i_0, \ldots, i_n\} \), and then move vertices according to their order in the permutation until all the vertices have been moved. After that a new random permutation is generated and the process repeats. In addition, the new position of the vertex being moved is generated by subsampling the allowable region. It is not difficult to show (using essentially the same argument) that the resulting Markov chain is still \( \nu_n \)-irreducible, aperiodic and Harris recurrent, and hence converges (in the total variation norm) to the uniform distribution on \( G_n \).

Our simulations were done for \( n = 599 \) (i.e. loops have 600 vertices). We performed \( 2 \cdot 10^6 \) iterations (where by an iteration we mean a single pass over all vertices in a random permutation), saving a loop after each 100 iterations. Out of saved loops we selected 50 last ones. As a proxy for the density of the loop distribution, we computed the standard kernel density estimation for their vertex positions. Also, we computed a “mean” free loop. This computation was done by cyclically
permuting vertices to minimize the distance between the corresponding elements of $\mathbb{R}^{2(n+1)}$ and then computing the mean position for each vertex. It is important to note that such a computation does not preserve the homotopy class, but it does provide useful geometric information.

Figure 2 shows the results of the above computations for a plane with four punctures, $Z = (1.35, -1.35) \times (1.35, -1.35)$, and the free homotopy class of a circle containing all the punctures. We chose the upper bound on the loop length $R = 20$. The shortest free loop, $\gamma^*$, is in this case the square with vertices in $Z$. It is evident from the figure that the uniform distribution in $g_n^R(X)$ is nicely concentrated around $\gamma^*$, and the mean free loop of only 50 samples has a fairly regular shape close to $\gamma^*$. Of course, each individual sample has a much more irregular shape.

Similar results can be seen in Figure 3, where the computations were done for the plane with punctures $z_1 = (-1.3, 0.6)$, $z_2 = (1.3, 0.6)$, $z_3 = (1.3, -0.6)$, $z_4 = (-1.3, -0.6)$, and the homotopy class of a lemniscate, as shown in the plot 3(a). The upper bound on the loop length is again $R = 20$. The shortest free loop, $\gamma^*$, is in this case a “bow tie” quadrilateral $z_1z_3z_2z_4$.

**Figure 2.** Example of MCMC simulation: (a) initial loop; (b) loop after $2 \cdot 10^6$ iterations; (c) mean free loop computed using 50 representatives; (d) kernel density estimation of vertex positions of 50 loops. Red points indicate punctures.

**Figure 3.** Another example of MCMC simulation: (a) initial loop; (b) loop after $2 \cdot 10^6$ iterations; (c) mean free loop computed using 50 representatives; (d) kernel density estimation of vertex positions of 50 loops. Red points indicate punctures.
5. Proofs

5.1. Properties of path and loop spaces.

Proof. (Of Proposition 1)

Since \( \gamma \) is continuous on \([0,1] \) it is uniformly continuous. Hence, \( \exists \delta > 0 \) such that \( \| \gamma(t) - \gamma(s) \| < \frac{1}{2} \) whenever \( |t-s| < \delta \). Take \( m \in \mathbb{N} \) such that \( \frac{1}{m} < \delta \) and let \( t_i = \frac{i}{m} \). Define \( \gamma_{PL} \) to be the piecewise linear path with vertices \( \gamma(t_i), i = 0, \ldots, m \), and edge traversal time \( \frac{1}{m} \), so that \( \gamma_{PL}(t_i) = \gamma(t_i) \). It is clear that \( \gamma_{PL} \) is a loop if \( \gamma \) is a loop. Also, for \( t \in [t_{i-1}, t_i], i = 1, \ldots, m \), we have

\[
\| \gamma(t) - \gamma_{PL}(t) \| \leq \| \gamma(t) - \gamma(t_i) \| + \| \gamma_{PL}(t) - \gamma_{PL}(t_i) \| < \epsilon
\]

\( \square \)

Proof. (Of Proposition 2)

Parts (1)-(3) are standard facts from the large deviation theory. Since \( \mu \) is invariant under rotations around the origin, the same is true for \( \Lambda \) and for \( \Lambda^* \). Hence, \( \Lambda(\eta) = \tilde{\Lambda}(\|\eta\|) \), where \( \tilde{\Lambda} \) is a strictly convex, differentiable function on \([0,\infty) \). Also, \( \Lambda^*(x) = \tilde{\Lambda}^*(\|x\|) \), where \( \tilde{\Lambda}^* \) is a good strictly convex rate function on \( D_{\tilde{\Lambda}} \).

To show that \( D_{\Lambda^*} = B_R \) notice that

\[
\Lambda(\eta) = \log \int_{\mathbb{R}^2} e^{-\eta, x} \mu(dx) = R\|\eta\| + \log \int_{\mathbb{R}^2} e^{-(R-x_1)} \mu(dx)
\]

Dominated convergence theorem yields

\[
\lim_{\|\eta\| \to \infty} \int_{\mathbb{R}^2} e^{-\|\eta\| (R-x_1)} \mu(dx) = \int_{\mathbb{R}^2} \lim_{\|\eta\| \to \infty} e^{-\|\eta\| (R-x_1)} \mu(dx) = 0
\]

Therefore for \( \|y\| \geq R \) we have

\[
\Lambda^*(y) = \sup_{\eta} (\|y\| \|\eta\| - \Lambda(\eta)) \geq \sup_{\eta} \left( -\log \int_{\mathbb{R}^2} e^{-\|\eta\| (R-x_1)} \mu(dx) \right) = \infty
\]

If \( \|y\| < R \) then, as we show below, \( \exists \eta \in \mathbb{R}^2 \) such that \( y = \nabla \Lambda(\eta) \), and by part (3) we have \( \Lambda^*(y) = \langle y, \eta \rangle - \Lambda(\eta) > \infty \).

Now, notice that by the dominated convergence theorem

\[
\nabla \Lambda(\eta) = e^{-\Lambda(\eta)} \int_{\mathbb{R}^2} xe^{x, \eta} \mu(dx)
\]

Thus, \( \nabla \Lambda(0) = 0 \). If \( 0 < \|y\| < R \) then \( y = \Lambda z \), where \( z = (\|y\|, 0) \) and \( \Lambda \) is a rotation. Suppose that \( \eta \) is such that \( \nabla \Lambda(\eta) = z \). Then

\[
\nabla \Lambda(A\eta) = e^{-\Lambda(A\eta)} \int_{\mathbb{R}^2} xe^{x, A\eta} \mu(dx) = A e^{-\Lambda(\eta)} \int_{\mathbb{R}^2} xe^{-\Lambda^{-1} x, \eta} \mu(dx) = A \nabla \Lambda(\eta) = y
\]

Thus, it is enough to show that for each \( y = (r, 0), 0 < r < R \), we can find \( \eta \) such that \( \nabla \Lambda(\eta) = y \).

Take \( \eta = (\xi, 0) \), then

\[
\nabla \Lambda(\eta) = \frac{\int_{\mathbb{R}^2} xe^{\xi, x_1} \mu(dx)}{\int_{\mathbb{R}^2} e^{\xi, x_1} \mu(dx)}
\]
Notice that $x_2 e^{\xi x_1}$ is an odd function of $x_2$, so the second coordinate of $\nabla \Lambda(\eta)$ is zero. Take $c = R - \varepsilon, \varepsilon > 0$. Then

$$
\frac{\int_{\mathbb{R}^2} x_1 e^{\xi x_1} \mu(dx)}{\int_{\mathbb{R}^2} e^{\xi x_1} \mu(dx)} = \frac{\int_{\mathbb{R}^2} x_1 e^{\xi (x_1 - c)} \mu(dx)}{\int_{\mathbb{R}^2} e^{\xi (x_1 - c)} \mu(dx)} = \frac{\int_{x_1 < c} x_1 e^{\xi (x_1 - c)} \mu(dx) + \int_{x_1 \geq c} x_1 e^{\xi (x_1 - c)} \mu(dx)}{\int_{x_1 < c} e^{\xi (x_1 - c)} \mu(dx) + \int_{x_1 \geq c} e^{\xi (x_1 - c)} \mu(dx)}
$$

By the dominated convergence theorem the first term in both numerator and denominator goes to zero as $\xi \to \infty$. Also,

$$
c \int_{x_1 \geq c} e^{\xi (x_1 - c)} \mu(dx) \leq \int_{x_1 \geq c} x_1 e^{\xi (x_1 - c)} \mu(dx) \leq R \int_{x_1 \geq c} e^{\xi (x_1 - c)} \mu(dx)
$$

Hence,

$$
\forall \varepsilon > 0 \quad R - \varepsilon = c \leq \lim_{\xi \to \infty} \frac{\int_{\mathbb{R}^2} x_1 e^{\xi x_1} \mu(dx)}{\int_{\mathbb{R}^2} e^{\xi x_1} \mu(dx)} \leq R \implies \lim_{\xi \to \infty} \frac{\int_{\mathbb{R}^2} x_1 e^{\xi x_1} \mu(dx)}{\int_{\mathbb{R}^2} e^{\xi x_1} \mu(dx)} = R
$$

Combining this result with the fact that $\nabla \Lambda(0) = 0$ we see that there does exist $\xi > 0$ such that the first coordinate of $\nabla \Lambda(\eta)$ is equal to $r$.

\[\square\]

**Proof.** (Of Theorem 2) $I_E$ is a good rate function because $E$ is compact and $I_0$ is a good rate function. To obtain the lower bound it is enough to show that for any $\gamma \in \Omega(\mathbb{R}^2) \cap \mathcal{D}_E$ and $\delta > 0$ we have

$$
\liminf_{n \to \infty} \frac{1}{n} \log \bar{\mu}_n(B_\delta(\gamma)) \geq -I_E(\gamma),
$$

where $B_\delta(\gamma) = \{\phi \in \Omega(\mathbb{R}^2)||\rho(\gamma, \phi) - \delta\}$. So, let us take some $\gamma \in \Omega(\mathbb{R}^2) \cap \mathcal{D}_E$ and $\delta > 0$. For convenience we shall omit the explicit dependence on $\gamma$ from out notation, so $B_\delta = B_\delta(\gamma)$. Let $P_\delta = \{\phi(0) \in B_\delta\}, F_\delta,n = E_n \cap P_\delta$. Notice that $\bar{v}_n(F_\delta,n)$ is bounded away from zero for sufficiently large $n$. Given $x \in \mathbb{R}^2$ let $B^x_\delta = \{\phi \in B_\delta|\phi(0) = x\}$, and let $\bar{\mu}^x_n$ denote the probability law of the path $\Psi(x, \frac{x_1}{n}, \ldots, \frac{x_n}{n})$. Then

$$
\bar{\mu}_n(B_\delta) = \int_{F_\delta,n} \bar{\mu}^x_n(B^x_\delta) \bar{v}_n(dx)
$$

Define $\sigma : \Omega(\mathbb{R}^2) \to \Omega(\mathbb{R}^2)$ by $\sigma(\phi)(t) = \phi(t) - \phi(0)$. Then it is easy to see that $\bar{\mu}^x_n(B^x_\delta) = \bar{\mu}^0_n(\sigma(B^x_\delta))$. Let $D_\delta = \cap_{x \in \mathbb{R}^2/2,n} \sigma(B^x_\delta)$. We claim that $\sigma(B^x_\delta) \subset D_\delta$. Indeed, if $\phi_0 \in \sigma(B^x_\delta)$ then $\phi_0(t) - \phi(t) - \phi(0)$ for some $\phi \in B^x_\delta$. For any $x \in F_\delta/2,n$ define $\psi_x$ by $\psi_x(t) = \phi(t) - \phi(0) + x$. Then $\sigma(\psi_x) = \phi_0$ and

$$
\rho(\psi_x, \gamma) = \sup_{t \in [0,1]} \|\phi(t) - \phi(0) + x - \gamma(\tau)\| \leq \sup_{t \in [0,1]} \|\phi(t) - \gamma(\tau)\| + \|x - \phi(0)\| < \delta,
$$

which proves the claim. It follows that

$$
\bar{\mu}_n(B_\delta) \geq \bar{v}_n(F_\delta/2,n) \bar{\mu}^0_n(\sigma(B^x_\delta))
$$

Applying Mogulskii’s theorem we get

$$
\liminf_{n \to \infty} \frac{1}{n} \log \bar{\mu}_n(B_\delta) \geq \liminf_{n \to \infty} \frac{1}{n} \left( \log \bar{v}_n(F_\delta/2,n) + \log \bar{\mu}^0_n(\sigma(B^x_\delta)) \right) \geq - \inf_{\phi \in \sigma(B^x_\delta)} I_0(\phi) \geq -I_0(\sigma(\gamma)) = -I_E(\gamma)
$$
To prove the upper bound suppose that $\Gamma$ is closed. Notice that $\bar{\mu}_n(\Gamma) = \bar{\mu}_n(\Gamma \cap \Omega^R(\mathbb{R}^2))$ for all $n$, where $\Omega^R(\mathbb{R}^2)$ is the set of paths with speed bounded by $R$. Hence, we may assume that $\Gamma$ consists only of paths with speed bounded by $R$. Then it follows from the Arzela-Ascoli theorem that $\Gamma$ is compact. Take $\varepsilon > 0$. Since $I_0$ is lower semicontinuous, for each $\gamma \in \Gamma$ there exists $\delta_\gamma > 0$ such that $I(\phi) \geq I(\gamma) - \varepsilon$ whenever $\rho(\phi, \gamma) < 4\delta_\gamma$. Let $\mathcal{U}$ be a finite subcover of the cover $\{\mathcal{B}_\delta(\gamma)\}_{\gamma \in \Gamma}$ of $\Gamma$. Denote the cardinality of $\mathcal{U}$ by $N$. Suppose that $B_{\delta_\gamma}(\gamma) \in \mathcal{U}$. For convenience we set $\delta = \delta_\gamma$ and, once again, omit the explicit dependence on $\gamma$, so $B_\delta = B_{\delta_\gamma}(\gamma)$. Define $F_{\delta, n}$, $B_\delta$, $\bar{\mu}_n^\gamma$ as before, and notice that $\sigma(B_\delta) = \bigcup_{x \in F_{\delta, n}} \sigma(B_\delta^x)$. Then

$$\bar{\mu}_n(B_\delta) \leq \bar{\nu}_n(F_{\delta, n}) \mu_n^0(\sigma(B_\delta))$$

Applying Moguls’kii’s theorem we get

$$\limsup_{n \to \infty} \frac{1}{n} \log \bar{\mu}_n(B_\delta) \leq \limsup_{n \to \infty} \frac{1}{n} \left( \log \bar{\nu}_n(F_{\delta, n}) + \log \bar{\mu}_n^0(\sigma(B_\delta)) \right) \leq - \inf_{\phi \in \sigma(B_\delta)} I_0(\phi),$$

where $B_\delta$ denotes the closure of $B_\delta$. Let $\gamma_0 = \sigma(\gamma)$ and notice that for any $\phi_0 \in B_\delta$ we have $\phi_0(t) = \phi(t) - \phi(0)$, $\phi(t) \in B_\delta \subset B_{\delta_\gamma}$ and

$$\rho(\gamma_0, \phi_0) = \sup_{t \in (0,1)} \|\gamma(t) - \gamma(0) - \phi(t) + \phi(0)\| \leq \sup_{t \in (0,1)} \|\gamma(t) - \phi(t)\| + \|\phi(0) - \gamma(0)\| < 4\delta.$$ 

Therefore, $\inf_{\phi \in \sigma(B_\delta)} I_0(\phi) \geq I_0(\sigma(\gamma)) - \varepsilon = I_E(\gamma) - \varepsilon$. Let $\bar{\mu}_n^\gamma = \max\{\bar{\mu}_n(B_{\delta_\gamma}(\gamma))\}$. Then $\Gamma^\gamma = \{\gamma \in \Gamma \mid B_{\delta_\gamma}(\gamma) \in \mathcal{U}\}$ and $\Gamma^* = \{\gamma \in \Gamma \mid B_{\delta_\gamma}(\gamma) \in \mathcal{U}\}$. Then

$$\limsup_{n \to \infty} \frac{1}{n} \log \bar{\mu}_n \leq - \min_{\gamma \in \Gamma^*} I_E(\gamma) + \varepsilon,$$

and so

$$\limsup_{n \to \infty} \frac{1}{n} \log \bar{\mu}_n(\Gamma) \leq \limsup_{n \to \infty} \frac{1}{n} \log(N\bar{\mu}_n^\gamma) \leq \limsup_{n \to \infty} \frac{1}{n} \log N + \limsup_{n \to \infty} \frac{1}{n} \log \bar{\mu}_n^\gamma \leq - \min_{\gamma \in \Gamma^*} I_E(\gamma) + \varepsilon$$

Since $\varepsilon$ is arbitrary, the result follows.

**Proof.** (Of Proposition 3)

From the proof of Proposition 2 we have $J(\phi) = \int_0^1 \tilde{\Lambda}^*(\|\phi'(t)\|)\,dt$, where $\tilde{\Lambda}^*$ is a good strictly convex rate function on $\mathcal{D}_{\Lambda^*} = [0, R]$. Thus, if $\|\phi'(t)\| \geq R$ on a set of positive measure then $I(\phi) = \infty$. This proves the first inclusion of (i). The second inclusion follows from the fact that if $\|\phi'(t)\| < R$ a.e. on $[0, 1]$ then for any $[a, b] \subset [0, 1]$, $a < b$, we have $L(\phi, a, b) = \int_a^b \|\phi'(t)\|\,dt < R(b - a)$.

Now, Jensen’s inequality implies $J(\phi) \geq \tilde{\Lambda}^*(L(\phi))$, and the equality holds only when $\phi$ has constant speed, i.e. $\|\phi'(t)\| = \tilde{\Lambda}(\phi)$ a.e. on $[0, 1]$. This proves part (2). For part (3) we then have $J(\phi) - J(\gamma) \geq \tilde{\Lambda}^*(L(\phi)) - \tilde{\Lambda}^*(L(\gamma)) \geq \tilde{\Lambda}^*(L(\gamma) + \varepsilon) - \tilde{\Lambda}^*(L(\gamma))$. Let $m$ be the slope of a supporting line of $\tilde{\Lambda}^*$ at $L(\gamma)$. Notice that $m > 0$. Then $\tilde{\Lambda}^*(L(\gamma) + \varepsilon) - \tilde{\Lambda}^*(L(\gamma)) \geq m\varepsilon$.

**Proof.** (Of Proposition 3)

From the proof of Proposition 2 we have $J(\phi) = \int_0^1 \tilde{\Lambda}^*(\|\phi'(t)\|)\,dt$, where $\tilde{\Lambda}^*$ is a good strictly convex rate function on $\mathcal{D}_{\Lambda^*} = [0, R]$. Thus, if $\|\phi'(t)\| \geq R$ on a set of positive measure then $I(\phi) = \infty$. This proves the first inclusion of (i). The second inclusion follows from the fact that if $\|\phi'(t)\| < R$ a.e. on $[0, 1]$ then for any $[a, b] \subset [0, 1]$, $a < b$, we have $L(\phi, a, b) = \int_a^b \|\phi'(t)\|\,dt < R(b - a)$. 
Now, Jensen’s inequality implies $J(\phi) \geq \bar{\Lambda}^*(L(\phi))$, and the equality holds only when $\phi$ has constant speed, i.e. $\|\phi'(t)\| = L(\phi)$ a.e. on $[0,1]$. This proves part (2). For part (3) we then have $J(\phi) - J(\gamma) \geq \bar{\Lambda}^*(L(\phi)) - \bar{\Lambda}^*(L(\gamma)) \geq \bar{\Lambda}^*(L(\gamma) + \epsilon) - \bar{\Lambda}^*(L(\gamma))$. Let $m$ be the slope of a supporting line of $\bar{\Lambda}^*$ at $L(\gamma)$. Notice that $m > 0$. Then $\bar{\Lambda}^*(L(\gamma) + \epsilon) - \bar{\Lambda}^*(L(\gamma)) \geq m\epsilon$.

**Proof.** (Proof Of Corollary (3))

We shall assume that $\delta$ is small enough so that $\Gamma_\delta \cap D_{1\delta} \neq \emptyset$, otherwise the result is obvious. Notice that this implies that $\delta < 2r$. By Mogulskii’s theorem we have

$$\limsup_{n \to \infty} \frac{1}{n} \log \frac{\mu_n(\Gamma_\delta)}{\mu_n(\Gamma)} \leq \limsup_{n \to \infty} \frac{1}{n} \log \frac{\mu_n(\Gamma_\delta)}{\mu_n(\Gamma)} - \liminf_{n \to \infty} \frac{1}{n} \mu_n(\Gamma) \leq \left( \inf_{\gamma \in \Gamma_\delta} I_0(\gamma) - \inf_{\gamma \in \Gamma^0} I_0(\gamma) \right)$$

Notice that if $\gamma \in \Gamma_\delta$ then there exists $t \in [0,1]$ such that the shortest distance between $\gamma(t)$ and the image of $[0,x^*]$ is at least $\delta$. Then it follows from simple geometric considerations that $L(\gamma) \geq \sqrt{\|x^*\|^2 + \delta^2}$. Since $\delta < 2r$ and the square root is a concave function we obtain $\sqrt{\|x^*\|^2 + \delta^2} \geq \|x^*\| + m\delta^2$, where $m = \frac{1}{4\pi} (\sqrt{\|x^*\|^2 + 4r^2} - \|x^*\|)$. Proposition (3) then implies that for any $\gamma \in \Gamma_\delta$ we have $I_0(\gamma) - I_0([0,x^*]) \geq c\delta^2$, for some constant $c > 0$. Also, it is easy to see that $\inf_{\gamma \in \Gamma^0} I_0(\gamma) = I_0([0,x^*])$. Therefore,

$$- \left( \inf_{\gamma \in \Gamma_\delta} I_0(\gamma) - \inf_{\gamma \in \Gamma^0} I_0(\gamma) \right) \leq -c\delta^2$$

**Proof.** (Of Lemma (1))

Take $\delta \in (0,\text{reach}(Z))$ and consider $X^\delta$ with the induced length structure and intrinsic metric (see [7] for details on length structures). It is easy to see that $X^\delta$ is a non-positively curved (NPC) space. Hence, its universal cover, $\tilde{X}^\delta$, is a Hadamard space locally isometric to $X^\delta$.

Let $\ell^\delta = \inf_{\gamma \in \mathfrak{g}(X^\delta)} L(\gamma)$, $\ell^*$ = $\inf_{\gamma \in \mathfrak{g}(X)} L(\gamma)$. It follows from Cartan’s theorem that there is a free loop $\tilde{\gamma}^\delta \in \mathfrak{g}(\tilde{X}^\delta)$ such that $L(\tilde{\gamma}^\delta) = \ell^\delta$. Moreover, any such free loop has a geodesic parameterization $\gamma^\delta \in \mathfrak{g}(X^\delta)$. It follows that $\gamma^\delta$ consists of straight line segments which are tangent to (pairs of) circles of radius $\delta$ around the punctures and circular arcs connecting such straight line segments (see Figure 4).

We now show that such a $\tilde{\gamma}^\delta$ is unique. Suppose $\tilde{\gamma}_1 \in \mathfrak{g}(X^\delta)$ are such that $L(\gamma_1) = \ell^\delta$, $i = 1, 2$. If images of $\gamma_1$ intersect then we can consider geodesic parametrizations of $\gamma_1$ starting at an intersection point. Such closed geodesics lift uniquely to geodesics in $\tilde{X}^\delta$ connecting the same two points. But in a Hadamard space there is a unique geodesic connecting any two points. Hence, $\gamma_1 = \gamma_2$, as they have the same geodesic representations.

Now assume that $\gamma_1$ and $\gamma_2$ do not intersect. A geodesic parametrization of $\gamma_1$, $i = 1, 2$, is a multiple of a simple geodesic, which we denote $\gamma_i$. A periodic geodesic defined by $\gamma_i$ can be uniquely lifted to a geodesic line $\gamma_i$ in $\tilde{X}^\delta$, $i = 1, 2$. Since $\gamma_1$ and $\gamma_2$ do not intersect $\gamma_1$ and $\gamma_2$ are parallel. In a Hadamard space parallel geodesic lines either coincide or span a convex flat strip. But the latter is impossible. Indeed, each $\gamma_i$ does necessarily contain a circular arc and $\tilde{X}^\delta$ and $X^\delta$ are locally isometric, implying that there are points around each geodesic line where the metric cannot be flat.

Let $\delta_m$ be a positive, monotonically decreasing sequence converging to zero, and let $\tilde{\gamma}^{\delta_m}$ be the unique shortest free loop in $\mathfrak{g}(\tilde{X}^{\delta_m})$. Notice that $\lim_{m \to \infty} L(\tilde{\gamma}^{\delta_m}) = \ell^*$. Indeed, $L(\tilde{\gamma}^{\delta_m})$ is a monotonically increasing sequence with a lower bound $\ell^*$, and if a sequence $\tilde{\gamma}_i \in \mathfrak{g}(X)$, $i \in \mathbb{N}$, is locally isometric to $X^\delta$, and if a sequence $\tilde{\gamma}_i \in \mathfrak{g}(X)$, $i \in \mathbb{N}$, is locally isometric to $X^\delta$, and if a sequence $\tilde{\gamma}_i \in \mathfrak{g}(X)$, $i \in \mathbb{N}$, is locally isometric to $X^\delta$.
\[ \ell^* \] is such that \( \lim_{i \to \infty} L(\hat{\gamma}_i) = \ell^* \) then for any \( i \in \mathbb{N} \) there exists some \( M \in \mathbb{N} \) such that for all \( m > M \) \( \hat{\gamma}_i \in \hat{g}(X^\delta_m) \implies L(\hat{\gamma}_i) \geq L(\hat{\gamma}_m) \). Let \( \Gamma \subset \mathfrak{g}(X) \) be the set of all constant speed parametrizations of all \( \hat{\gamma}_m, m \in \mathbb{N} \). Then it follows from the Arzela-Ascoli theorem that \( \Gamma \) is relatively compact (in \( \mathcal{L}(\mathbb{R}^2) \)). Hence, we can find a converging (in \( \mathcal{L}(\mathbb{R}^2) \)) subsequence \( \gamma_m \) of constant speed parametrizations of \( \hat{\gamma}_m \), and \( \lim_{j \to \infty} \gamma_{m_j} = \gamma^* \in \mathfrak{g}(X) \). Let \( \hat{\gamma}^* = \pi_{\mathcal{L}}(\gamma^*) \). Clearly, \( L(\hat{\gamma}^*) = \ell^* \). Moreover, the structure of the shortest free loop in \( X^\delta \) implies that \( \hat{\gamma}^* \) consists of straight line segments connecting punctures (see Figure 4).

\[ \square \]

We now prove our main results: Proposition 4 and Theorem 4. The proof of Lemma 2 is given after a series of auxiliary technical lemmas following the proof of Theorem 4.

**Proof.** (Of Proposition 4)
First, let us prove the upper bound. We may assume that \( \Gamma \cap \mathcal{D}_1 \neq \emptyset \), otherwise the inequality is trivial.

\[
\limsup_{n \to \infty} \frac{1}{n} \log \gamma_n(\Gamma) = \limsup_{n \to \infty} \left( \frac{1}{n} \log \left[ \mu_n(\iota(\Gamma \cap g^R_n(X))) \right] - \frac{1}{n} \log \left[ \mu_n(\iota(g^R_n(X))) \right] \right) \leq \limsup_{n \to \infty} \frac{1}{n} \log \left[ \mu_n(\iota(\Gamma \cap g^R_n(X))) \right] - \liminf_{n \to \infty} \frac{1}{n} \log \left[ \mu_n(\iota(g^R_n(X))) \right] \]

Applying Lemma 2 to the second term we obtain

\[
\liminf_{n \to \infty} \frac{1}{n} \log \left[ \mu_n(\iota(g^R_n(X))) \right] \geq - \inf_{\gamma \in \mathfrak{g}(X)} I(\gamma) = -I^* \]
To bound the first term, take \( \varepsilon > 0 \) and let
\[
\Gamma_\varepsilon = \{ \gamma_{|[0,1-\delta]} | \gamma \in \Gamma, 0 \leq \delta \leq \varepsilon \}
\]
Notice that for sufficiently large \( n \) we have \( \Gamma_n \subset \Gamma_\varepsilon \), where \( \Gamma_n = \cup (\Gamma \cap g^R_n(X)) \). Therefore,
\[
\limsup_{n \to \infty} \frac{1}{n} \log [\mu_n(\Gamma_n)] \leq \limsup_{n \to \infty} \frac{1}{n} \log [\mu_n(\Gamma_\varepsilon)] \leq - \inf_{\gamma \in \Gamma_\varepsilon} I(\gamma),
\]
where the last inequality follows from the untethered Mogulskii’s theorem. Take \( \gamma \in \Gamma \) and suppose that \( \gamma_\varepsilon = \gamma_{|[0,1-\varepsilon]} \in D_1 \) for all \( \varepsilon \geq 0 \) (otherwise \( I(\gamma) = I(\gamma_\varepsilon) = \infty \) for sufficiently small \( \varepsilon \)). Then
\[
I(\gamma_\varepsilon) = \int_0^1 \Lambda^*(\gamma_\varepsilon'(t)) \, dt = \frac{1}{1 - \varepsilon} \int_0^{1 - \varepsilon} \Lambda^*((1 - \varepsilon)\gamma'(s)) \, ds
\]
Since \( \Lambda^*(\cdot) = \tilde{\Lambda}^* (\| \cdot \|) \) and \( \tilde{\Lambda}^* \) is a nonnegative increasing function, the monotone convergence theorem yields \( I(\gamma_\varepsilon) \to I(\gamma) \) as \( \varepsilon \to 0 \). Since \( I \) is a good rate function, it attains its infimum on \( \Gamma \) and on \( \Gamma_\varepsilon \). Let \( \gamma^* \in \Gamma \) be such that \( I(\gamma^*) = \inf_{\gamma \in \Gamma} I(\gamma) \), and let \( I_\varepsilon = \inf_{\gamma \in \Gamma_\varepsilon} I(\gamma) \). Then
\[
I_\varepsilon = I(\gamma^*_\varepsilon) + \tilde{\xi}(\varepsilon), \quad \text{where} \quad \gamma^*_\varepsilon = \gamma^*_{|[0,1-\varepsilon]}, \quad \text{and} \quad \tilde{\xi}(\varepsilon) \to 0 \quad \text{as} \quad \varepsilon \to 0.
\]
Thus, for all positive \( \varepsilon \) we have
\[
\limsup_{n \to \infty} \frac{1}{n} \log [\mu_n(\Gamma_n)] \leq -(I(\gamma^*_\varepsilon) + \tilde{\xi}(\varepsilon)).
\]
Taking the limit for \( \varepsilon \to 0 \) we get
\[
\limsup_{n \to \infty} \frac{1}{n} \log [\mu_n(\Gamma_n)] \leq -I(\gamma^*) = - \inf_{\gamma \in \Gamma} I(\gamma).
\]
To prove the lower bound, let \( \Gamma \subset g(X) \) be open. Then
\[
\liminf_{n \to \infty} \frac{1}{n} \log \gamma_n(\Gamma) = \liminf_{n \to \infty} \left( \frac{1}{n} \log \left[ \mu_n(\cup (\Gamma \cap g^R_n(X))) \right] - \frac{1}{n} \log \left[ \mu_n(\cup g^R_n(X)) \right] \right) \geq \liminf_{n \to \infty} \frac{1}{n} \log \left[ \mu_n(\cup (\Gamma \cap g^R_n(X))) \right] - \limsup_{n \to \infty} \frac{1}{n} \log \left[ \mu_n(\cup g^R_n(X)) \right]
\]
Applying Lemma 2 to the first term we get
\[
\liminf_{n \to \infty} \frac{1}{n} \log \left[ \mu_n(\cup (\Gamma \cap g^R_n(X))) \right] \geq - \inf_{\gamma \in \Gamma} I(\gamma)
\]
The second term can be bounded using the same argument as in the case of the upper bound:
\[
\limsup_{n \to \infty} \frac{1}{n} \log \left[ \mu_n(\cup g^R_n(X)) \right] \leq - \inf_{\gamma \in \Gamma(X)} I(\gamma) = -I^*
\]
\[
\square
\]
Proof. (Of Theorem 4)

Not surprisingly, the proof is analogous to the proof of Corollary 1.

We shall assume that \( \delta \) is small enough so that \( \Gamma_\delta \cap D_1 \neq \emptyset \), otherwise the result is obvious. Notice that this implies that \( \delta < 2R \). By Proposition 4
\[
\limsup_{n \to \infty} \frac{1}{n} \log \gamma_n(\Gamma_\delta) \leq - \left( \inf_{\gamma \in \Gamma_\delta} I(\gamma) - I^* \right)
\]
Notice that if \( \gamma \in \Gamma_\delta \) then there exists \( t \in [0,1] \) such that the shortest distance between \( \gamma(t) \) and the image of \( \hat{\gamma}^* \) is at least \( \delta \). Then it follows from simple geometric considerations that \( L(\gamma) \geq \sqrt{\ell^2 + \delta^2} \), where \( \ell \) is the length of \( \hat{\gamma}^* \). Since \( \delta < 2R \) and the square root is a concave function we
obtain $\sqrt{\ell^2 + \delta^2} \geq \ell + m\delta^2$, where $m = \frac{1}{4\pi \epsilon} \left( \sqrt{\ell^2 + 4\epsilon^2} - \ell \right)$. Proposition \ref{prop:delta} then implies that for any $\gamma \in \Gamma_\delta$ we have $I(\gamma) - I^* \geq c\delta^2$, for some constant $c > 0$. Therefore,

$$-\left( \inf_{\gamma \in \Gamma_\delta} I(\gamma) - I^* \right) \leq -c\delta^2$$

□

The following lemmas, which we needed to prove Lemma \ref{lemma:delta} are adaptations of some standard facts from the large deviation theory.

\textbf{Lemma 3.} Let $\mu_r$ be the uniform probability measure on $B_r = \{ x \in \mathbb{R}^2 | \|x\| < r \}$, $M_r(\eta)$ be the moment generating function associated with $\mu_r$, and $\Lambda_r(\eta) = \log M_r(\eta)$. Also, let $p = \nabla \Lambda_r(\eta)$ for some $\eta \in \mathbb{R}^2$. Then the random variable $Y$ with the probability law $\mu_r$ defined by

$$\frac{d\mu_r}{d\mu_r}(x) = e^{\langle x, \eta \rangle - \Lambda_r(\eta)}$$

has expectation $\mathbb{E}(Y) = p$.

\textit{Proof.}

$$\mathbb{E}(Y) = \int_{\mathbb{R}^2} x \mu_r(dx) = \int_{\mathbb{R}^2} x e^{\langle x, \eta \rangle - \Lambda_r(\eta)} \mu_r(dx) = \frac{1}{M_r(\eta)} \int_{\mathbb{R}^2} x e^{\langle x, \eta \rangle} \mu_r(dx)$$

On the other hand, $M_r(\eta) = \int_{\mathbb{R}^2} e^{\langle x, \eta \rangle} \mu_r(dx)$ and

$$p = \nabla \Lambda_r(\eta) = \frac{1}{M_r(\eta)} \nabla M(\eta) = \frac{1}{M_r(\eta)} \int_{\mathbb{R}^2} x e^{\langle x, \eta \rangle} \mu_r(dx),$$

where the last equality follows from the dominated convergence theorem. □

\textbf{Lemma 4.} Let $X_1, \ldots, X_m$ be i.i.d random variables in $\mathbb{R}^2$ with $\mathbb{E}(X_1) = 0$, and suppose that the values of $X_1$ lie almost surely within a set of diameter $c$. Let $S_K = \sum_{i=1}^{k} X_i$. Then

$$\mathbb{P} \left( \sup_{1 \leq k \leq m} \|S_K\| \geq \lambda \right) \leq 4e^{-\frac{\lambda^2}{mc^2}}$$

\textit{Proof.} Let $S_{j,k}$, $j = 1, 2$, denote the $j$-th coordinate of $S_K$. Notice that

$$\mathbb{P} \left( \sup_{1 \leq k \leq m} \|S_K\| \geq \lambda \right) \leq \mathbb{P} \left( \sup_{1 \leq k \leq m} \max(|S_{1,k}|, |S_{2,k}|) \geq \frac{\lambda}{\sqrt{2}} \right) \leq$$

$$\mathbb{P} \left( \sup_{1 \leq k \leq m} |S_{1,k}| \geq \frac{\lambda}{\sqrt{2}} \right) + \mathbb{P} \left( \sup_{1 \leq k \leq m} |S_{2,k}| \geq \frac{\lambda}{\sqrt{2}} \right)$$

Also,

$$\mathbb{P} \left( \sup_{1 \leq k \leq m} |S_{j,k}| \geq \frac{\lambda}{\sqrt{2}} \right) = \mathbb{P} \left( \sup_{1 \leq k \leq m} S_{j,k} \geq \frac{\lambda}{\sqrt{2}} \right) + \mathbb{P} \left( \sup_{1 \leq k \leq m} (S_{j,k} \leq -\frac{\lambda}{\sqrt{2}}) \right), \quad j = 1, 2$$

Now, for any $t > 0$ we have

$$\mathbb{P} \left( \sup_{1 \leq k \leq m} S_{j,k} \geq \frac{\lambda}{\sqrt{2}} \right) = \mathbb{P} \left( \sup_{1 \leq k \leq m} e^{tS_{j,k}} \geq e^{\frac{t\lambda}{\sqrt{2}}} \right)$$
exists a constant $\mu$ the logarithmic moment generating function associated with the probability law $\eta$. Take Lemma 5. 

The above argument produces the same bound for all four probabilities $P\left(\sup_{1 \leq k \leq m} (\pm S_{i,k} \geq \frac{\lambda}{\sqrt{2}})\right)$, $j = 1, 2$. Thus, we get

$$P\left(\sup_{1 \leq k \leq m} \|S_k\| \geq \lambda\right) \leq 4e^{-\frac{\lambda^2}{mc^2}}$$

Recall that $\mu$ denotes the uniform probability measure on $B_R = \{x \in \mathbb{R}^2 | |x| < R\}$, $\Lambda$ denotes the logarithmic moment generating function associated with the probability law $\mu$, and $\Lambda^*(x) = \sup_{\eta} \{<x, \eta> - \Lambda(\eta)\}$.

**Lemma 5.** Take $m, n \in \mathbb{N}$, $2 \leq m \leq n$, and let $X_1, \ldots, X_m$ be i.i.d. random variables with the probability law $\mu$. Let $e$ be a linear path in $\mathbb{R}^2$, i.e. $e = [v_0, v_1]$, $v_0, v_1 \in \mathbb{R}^2$, and let $\gamma$ be a piecewise linear path with edge traversal time $\frac{1}{m}$ and vertices $v_0 + S_k$, $S_k = \frac{1}{m} \sum_{i=1}^{k} X_i$, $k = 0, \ldots, m$, i.e. $\gamma = \Psi_m (v_0, \frac{1}{n}X_1, \ldots, \frac{1}{n}X_m)$. Suppose that $\frac{n}{m} \leq \alpha$, $\alpha \|p\| < R$, $p = v_1 - v_0$, and let $C > 0$. Then there exists a constant $D > 0$ such that

$$P\left(\rho(\gamma, e) < \lambda, \|\gamma(1) - e(1)\| < \frac{1}{Cn}\right) \geq \frac{m}{n} \Lambda^*\left(\frac{n}{m}p\right) - \lambda\|\eta_p\| + \frac{1}{n} \log\left(\frac{D}{C^2 m} - 4e^{-\frac{n^2 \lambda^2}{4mR^2}}\right),$$

where $\eta_p \in \mathbb{R}^2$ is such that $\frac{n}{m}p = \nabla \Lambda(\eta_p)$, and $\lambda$ is assumed to be such that $\frac{n^2 \lambda^2}{4mR^2} > \log\frac{4C^2 m}{D}$.

**Proof.** First, notice that existence of $\eta_p$ follows from Proposition 2. Also, since $\gamma(\frac{k}{m}) = v_0 + S_k$ and $e(\frac{k}{m}) = v_0 + \frac{kp}{m}$, $k = 0, \ldots, m$, we get

$$P\left(\rho(\gamma, e) < \lambda, \|\gamma(1) - e(1)\| < \frac{1}{Cn}\right) = P\left(\sup_{1 \leq k \leq m} \left\|S_k - \frac{kp}{m}\right\| < \lambda, \|S_m - p\| < \frac{1}{Cn}\right).$$
Let
\[ U_{p, \lambda} = \left\{ (x_1, \ldots, x_m) \in \mathbb{R}^{2m} \mid \sup_{1 \leq k \leq m} \left\| \frac{1}{n} \sum_{i=1}^{k} x_i - \frac{kp}{m} \right\| < \lambda, \left\| \frac{1}{n} \sum_{i=1}^{m} x_i - p \right\| < \frac{1}{Cn} \right\} , \]
\[ U_{0, \lambda} = \left\{ (x_1, \ldots, x_m) \in \mathbb{R}^{2m} \mid \sup_{1 \leq k \leq m} \left\| \sum_{i=1}^{k} x_i \right\| < \lambda \mu, \left\| \sum_{i=1}^{m} x_i \right\| < \frac{1}{C} \right\} \]

Then
\[ \mathbb{P} \left( \sup_{1 \leq k \leq m} \left\| S_k - \frac{kp}{m} \right\| < \lambda, \left\| S_m - p \right\| < \frac{1}{Cn} \right) = \int_{U_{p, \lambda}} \prod_{i=1}^{m} \mu(dx_i) \]

Letting \( \frac{d\bar{\mu}}{d\mu}(x) = e^{<x, \eta_p> - \lambda(\eta_p)} \) we get
\[ \int_{U_{p, \lambda}} \prod_{i=1}^{m} \mu(dx_i) = e^{\lambda(\eta_p)} \int_{U_{p, \lambda}} e^{-\sum_{i=1}^{m} <x_i, \eta_p>} \prod_{i=1}^{m} \tilde{\mu}(dx_i) =
\]
\[ = e^{\lambda(\eta_p) - n<\eta_p, \eta_p>} \int_{U_{p, \lambda}} e^{-\sum_{i=1}^{m} <x_i - \frac{np}{m}, \eta_p>} \prod_{i=1}^{m} \tilde{\mu}(dx_i) =
\]
\[ = e^{\lambda(\eta_p) - n<\eta_p, \eta_p>} \int_{U_{0, \lambda}} e^{-\sum_{i=1}^{m} <z_i, \eta_p>} \prod_{i=1}^{m} \tilde{\mu}(dz_i), \]

where \( \tilde{\mu} \) denotes the probability law of \( Z_1 = Y_1 - \frac{n}{m} p \), with \( Y_1 \) having the probability law \( \mu \). Since \( \langle \sum_{i=1}^{m} z_i, \eta_p \rangle \leq \langle \sum_{i=1}^{m} z_i \rangle \| \eta_p \| \) and \( \sum_{i=1}^{m} z_i \rangle \leq \lambda n \) on \( U_{0, \lambda} \), we get
\[ \int_{U_{0, \lambda}} e^{-n\sum_{i=1}^{m} <z_i, \eta_p>} \prod_{i=1}^{m} \tilde{\mu}(dz_i) \geq e^{-n\lambda \| \eta_p \|} \mathbb{P} \left( \sup_{1 \leq k \leq m} \left\| \sum_{i=1}^{k} Z_i \right\| < \lambda n, \left\| \sum_{i=1}^{m} Z_i \right\| < \frac{1}{C} \right) \]

Notice that
\[ \mathbb{P} \left( \sup_{1 \leq k \leq m} \left\| \sum_{i=1}^{k} Z_i \right\| < \lambda n, \left\| \sum_{i=1}^{m} Z_i \right\| < \frac{1}{C} \right) \geq \mathbb{P} \left( \left\| \sum_{i=1}^{m} Z_i \right\| < \frac{1}{C} \right) - \mathbb{P} \left( \sup_{1 \leq k \leq m} \left\| \sum_{i=1}^{k} Z_i \right\| \geq \lambda n \right) \]

By Lemma 4, \( \mathbb{E}(Y_1) = \frac{n}{m} p \), yielding \( \mathbb{E}(Z_1) = 0 \). Moreover, the values of \( Z_1 \) lie within a disk of radius \( R \). Hence, we can employ Lemma 4 to obtain
\[ \mathbb{P} \left( \sup_{1 \leq k \leq m} \left\| \sum_{i=1}^{k} Z_i \right\| \geq \lambda n \right) \leq 4e^{-\frac{n^2x^2}{4mR^2}} \]

To bound the other probability, notice that the covariance matrix, \( W \), of \( Z_1 \) is positive definite, \( \mathbb{E}((Z_1)^{2s}) < \infty \) for all \( s \geq 1 \), and the density of \( Z_1 \) is bounded everywhere. It follows from the results on uniform local limit theorems (see e.g. [19]) that a bounded continuous density, \( q_m \), of the distribution of \( \frac{1}{\sqrt{m}} \sum_{i=1}^{m} Z_i \) exists and
\[ \left| q_m(x) - \phi_W(x) \right| \leq \frac{A}{\sqrt{m(1+\|x\|^2)}}, \quad \forall x \in \mathbb{R}^2, \]

where \( A \) is a constant and \( \phi_W \) denotes the density of the normal distribution in \( \mathbb{R}^2 \) with zero mean and covariance matrix \( W \). Denoting by \( B_{\frac{1}{C\sqrt{m}}} \) the ball of radius \( \frac{1}{C\sqrt{m}} \) centered at the origin
we then get
\[ P \left( \left\| \sum_{i=1}^{m} Z_i \right\| < \frac{1}{C} \right) = P \left( \left\| \frac{1}{\sqrt{m}} \sum_{i=1}^{m} Z_i \right\| < \frac{1}{C\sqrt{m}} \right) \]
where \( D \) is another constant. Therefore,
\[ \frac{1}{n} \log P \left( \sup_{1 \leq k \leq m} \left\| S_k - \frac{kp}{m} \right\| < \lambda \right) \geq -\frac{m}{n} \left\langle \left( \frac{n}{m} p, \eta_p \right) - \Lambda(\eta_p) \right\rangle - \lambda \| \eta_p \| + \frac{1}{n} \log \left( \frac{D}{C^2 m} - \frac{a^2 \epsilon^2}{4m^2} \right) \]
The result of the lemma follows from the fact that
\[ \left\langle \left( \frac{n}{m} p, \eta_p \right) - \Lambda(\eta_p) \right\rangle = \Lambda^* \left( \frac{n}{m} p \right) \]
\[ \square \]
Proof. (Of Lemma 3) It is enough to show that for every \( \gamma \in \mathfrak{g}(X) \cap \mathcal{D}_1 \) and every \( \epsilon > 0 \) we have
\[ -I(\gamma) \leq \liminf_{n \to \infty} \frac{1}{n} \log \mu_n(\Gamma^3_n(\gamma)), \]
where \( \Gamma^3_n(\gamma) = t(\Gamma^3_n(\gamma) \cap \mathfrak{g}_n^R(X)) \), and \( \Gamma^3_n(\gamma) = \{ \alpha \in \mathfrak{g}(X) | \rho(\alpha, \gamma) < 3\epsilon \} \) is a ball of radius \( 3\epsilon \) centered at \( \gamma \). Notice that for small enough \( \epsilon \) any loop \( \varphi \) such that \( \rho(\varphi, \gamma) < 3\epsilon \) belongs to \( \mathfrak{g}(X) \).
Using Proposition 1 we can find a piecewise linear loop \( \gamma_{PL} \) such that \( \Gamma^{2\epsilon}(\gamma_{PL}) \subset \Gamma^3(\gamma) \). Moreover, convexity of \( \Lambda^* \) implies that \( I(\gamma) \geq I(\gamma_{PL}) \). Therefore, it suffices to show that
\[ -I(\gamma_{PL}) \leq \liminf_{n \to \infty} \frac{1}{n} \log \mu_n(\Gamma^3_n(\gamma_{PL})), \]
for \( \delta \leq 2\epsilon \). Denote the vertices of \( \gamma_{PL} \) by \( v_0, \ldots, v_\ell \), and the edges by \( e_0, \ldots, e_\ell \). For convenience, we set \( v_{\ell+1} = v_0 \). Let \( t_i \) be such that \( \gamma_{PL}(t_i) = v_i \), \( i = 0, \ldots, \ell + 1 \). As before, denote by \( V_0 \) the random variable with the probability law \( v_n \) and by \( X_1, \ldots, X_n \) i.i.d. random variables with the probability law \( \mu \), and let \( \psi = \Psi_n \left( V_0, \frac{X_1}{n}, \ldots, \frac{X_n}{n} \right) \), \( \varphi = \Phi_n \left( V_0, \frac{X_1}{n}, \ldots, \frac{X_n}{n} \right) \) (i.e. \( \psi = t(\varphi) \)). Then for sufficiently large \( n \) we have
\[ \mu_n(\Gamma^3_n(\gamma_{PL})) = P(\rho(\psi, \gamma_{PL}) < \delta) \geq P \left( \rho(\psi, \gamma_{PL}) < \delta, \| \psi(0) - \gamma_{PL}(0) \| < \frac{R}{2\epsilon + 2}, \| \psi(1) - \psi(0) \| < \frac{R}{n} \right) \]
Denote the right hand side of the above inequality by \( P \). Let \( r_i = \frac{R}{2\epsilon + 2}, i \in [n] \), and denote by \( B_{r_i}(v_i) \) the open ball of radius \( r_i \) centered at \( v_i \), \( i = 0, \ldots, \ell \). Given \( x \in B_{r_0}(v_0) \) let \( \psi_x = \Psi_n \left( x, \frac{X_1}{n}, \ldots, \frac{X_n}{n} \right) \) and let
\[ P_x = P \left( \rho(\psi_x, \gamma_{PL}) < \delta, \| \psi_x(1) - \psi_x(0) \| < \frac{R}{n} \right) \]
Notice that \( P = \int_{B_{r_0}(v_0)} P_x v_n(dx) \).
We shall now bound \( P_x \) from below. Let \( N_i \) be the integer part of \( t_i n \), i.e. \( N_i = [t_i n], i = 0, \ldots, \ell + 1, \) and let \( n_i = N_{i+1} - N_i, i = 0, \ldots, \ell \). Take \( x_i \in B_{r_i}(v_i) \), and let \( \psi_i = \Psi_{n_i} \left( x_i, \frac{X_{n_i+1}}{n_i}, \ldots, \frac{X_{n_{i+1}}}{n_i} \right) \), \( i = 0, \ldots, \ell \),
\[ P_i = P \left( \rho(\psi_i, e_i) < \delta, \| \psi_i(1) - e_i(1) \| < r_{i+1} \right) \]
Notice that \( P_{x_i} \geq \prod_{i=0}^{\ell} P_i \) and
\[ P_i \geq P \left( \rho(\psi_i, e_i) < \delta, \| \psi_i(1) - (x_i + p_i) \| < r_i \right) = Q_i, \]
where $p_i = v_{i+1} - v_i$. We also have $t_{i+1} - t_i - \frac{1}{n} \leq t_{i+1} - t_i = \frac{1}{n}$. Since $\gamma_{PL} \in D_1$, the speed of $\gamma_{PL}$ is strictly bounded by $R$, so $\frac{\|p_i\|}{t_{i+1} - t_i} < R$. Hence, for large enough $n$ we can employ Lemma 5 to obtain

$$\frac{1}{n} \log P_i \geq -\frac{n}{n_i} \Lambda^* \left( \frac{n}{n_i} p_i \right) - \delta \|p_i\| + \xi_i(n),$$

where $\xi_i$ are such that $\nabla \Lambda(\xi_i) = \frac{n}{n_i} p_i$, and $\xi_i(n) \to 0$ as $n \to \infty$. Therefore,

$$\frac{1}{n} \log P = \frac{1}{n} \log \int_{B_{r_0}(v_0)} P_x v_n(dx) \geq \frac{1}{n} \log [v_n(B_{r_0}(v_0))] + \frac{1}{n} \log \sum_{i=0}^\ell \log P_i \geq$$

$$- \sum_{i=0}^\ell \frac{n_i}{n} \Lambda^* \left( \frac{n}{n_i} p_i \right) - \delta \sum_{i=0}^\ell \|p_i\| + \xi(n),$$

where $\xi(n) = \sum_{i=0}^\ell \xi_i(n) + \frac{1}{n} \log [v_n(B_{r_0}(v_0))] \to 0$ as $n \to \infty$. Taking the limit we get

$$\liminf_{n \to \infty} \frac{1}{n} \log \mu_n(\Gamma^S_n(\gamma_{PL})) \geq - \sum_{i=0}^\ell \int_{t_i}^{t_{i+1}} \Lambda^*(\gamma_{PL}(t)) - \delta S = -I(\gamma_{PL}) - \delta S,$$

where $S = \sum_{i=0}^m \|p_i\|$. Now,

$$\liminf_{n \to \infty} \frac{1}{n} \log \mu_n(\Gamma^S_n(\gamma_{PL})) \geq \liminf_{\delta \to 0} \liminf_{\delta \to 0} \frac{1}{n} \log \mu_n(\Gamma^S_n(\gamma_{PL})) \geq \liminf_{\delta \to 0} \liminf_{n \to \infty} -I(\gamma_{PL}) - \delta S = -I(\gamma_{PL}).$$

5.2. Sampling in $G_n$. We now turn to the results related to sampling in our loop space $G_n$.

Proof. (Of part 1 of Proposition 5)

First, we show that $(\mathbf{V}_i)$ is $\nu$-irreducible. Since $G$ is an open bounded and connected subset of $R^{2(n+1)}$ and $\nu$ is a (rescaled) Lebesgue measure, it is enough to show that each $\nu \in G$ has a $\nu$-communicating neighborhood. We call a Borel set $B \subset G$ $\nu$-communicating if $\nu \in B$ and all Borel subsets $A \subset B$ with $\nu(A) > 0$ there exists $m \in N$ such that $P^m_{\nu}(\nu, A) > 0$.

Given $\nu = (v_0, \ldots, v_n) \in G$, define $v_{-1} = v_n, v_{n+1} = v_0$, and let $O_{v} = B_{\delta}(v_0) \times \cdots \times B_{\delta}(v_n)$, where $B_{\delta}(v_i)$ denotes a disk of radius $\delta$ centered at $v_i$, and $\delta > 0$ is such that for all $i = 0, \ldots, n$ we have $Z \cap B_{\delta}(v_i) = \emptyset$ and $B_{\delta}(v_i) \subset B_{\epsilon}(v_{i-1}) \cap B_{\epsilon}(v_{i+1})$ for any $w_{i-1} \in B_{\delta}(v_{i-1}), w_{i+1} \in B_{\delta}(v_{i+1})$. Define $\pi_i : R^{2(n+1)} \to R^2 \nu(v_0, \ldots, v_n) = v_i$. It follows from the Chapman-Kolmogorov equations that $O_{\nu}$ is communicating if for any $w \in O_{\nu}$, $v \in B_{\nu}$, and any Borel subset $A \subset B_{\delta}(v_i)$ with $\lambda_2(A_i) > 0$, where $\lambda_2$ denotes the 2-dimensional Lebesgue measure, the probability $P_{\nu}(\pi_i(V_{j+1})) \in A_i|V_j = w) > 0$. But it is easy to see that this probability is proportional to $\lambda_2(A_i)$.

To prove aperiodicity it is enough to show that for any Borel set $A \subset G$ with $\nu(A) > 0$ there exists $\nu \in A$ such that $P_{\nu}(\nu, A) > 0$. Take $A \subset G$ with $\nu(A) > 0$. Define $\pi_i : R^{2(n+1)} \to R^{2n}$ by $\pi_i(v_0, \ldots, v_n) = (v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n)$. Let $A_1 = \pi_i(A)$, and for $\nu = (v_0, \ldots, v_{n-1}) \in \hat{A}_1$ let $A_i(\nu) = \{v \in R^2 | (v_0, \ldots, v_{i-1}, v, v_{i+1}, \ldots, v_{n-1}) \in A \}$. Since $\lambda_2(n+1)(A) > 0$, where $\lambda_2(n+1)$ is the
2(\(n+1\))-dimensional Lebesgue measure, for all \(i = 0, \ldots, n\) we have \(\lambda_n(\hat{A}_i) > 0\) and there exists \(\hat{v}_i \in \hat{A}_i\) such that \(\lambda_n(\hat{A}_i(\hat{v}_i)) > 0\). But this implies that \(P(\hat{v}, A) > 0\) for some \(\hat{v} \in A\).

To show that \(\{V_i\}\) is Harris recurrent it is enough to show that for any initial state, with probability 1, the chain eventually moves in every coordinate direction (see Theorem 12 from [22]). But this is obvious, since the probability that a particular vertex does not move after \(k\) steps is \(\left(\frac{n}{n+1}\right)^k \to 0\) as \(k \to \infty\).

The proof of part 2 of Proposition 5, which establishes the needed convergence result, relies of several auxiliary results.

Notice that \(G_n\) is path connected if and only if any \(\gamma_0, \gamma_1 \in \mathcal{g}_n^R(X)\) are freely homotopic within \(\mathcal{g}_n^R(X)\), that is, there exists a free homotopy \(H\) between \(\gamma_0\) and \(\gamma_1\) such that \(H(\cdot, t) \in \mathcal{g}_n^R(X)\) \(\forall t \in [0,1]\). We denote such a homotopy relation by \(\gamma_0 \mathcal{g}_n^R \simeq \gamma_1\).

To establish existence of a homotopy within \(\mathcal{g}_n^R(X)\) we employ an algebraic representation of loops in \(\mathcal{L}(X)\) similar to that in [15]. Let \(\mathcal{T}\) be a collection of arbitrarily oriented edges in an arbitrary (say, Delaunay) triangulation of the punctures \(Z = \{z_1, \ldots, z_K\}\), including bisectors of the outer angles of the convex hull of \(Z\) (see Figure 5). In the degenerate case when all the punctures lie on a single straight line, let’s call it \(\ell\), \(\mathcal{T}\) consists of the line segments in \(\ell \setminus Z\) and additional rays, two per puncture, which are perpendicular to \(\ell\). Notice that the planar decomposition defined by \(\mathcal{T}\) has convex faces.

Associate to each element of \(\mathcal{T}\) a symbol, denote the set of such symbols by \(\Lambda\), and let \(\Lambda^{-1}\) be the set of inverse symbols, i.e. \(\Lambda^{-1} = \{a^{-1} | a \in \Lambda\}\). Let \(\mathcal{G}\) be the free group generated by \(\Lambda\), and let \(\epsilon\) denote the empty word. Now we can associate to \(\gamma \in \mathcal{L}(X)\) a word over \(\Lambda\) in the following way. We regard a loop \(\gamma\) as a map from \(\mathbb{R}/\mathbb{Z}\) and allow ourselves a slight abuse notation writing \(\gamma(t), t \in \mathbb{R}\), to mean \(\gamma(t \mod 1)\). Let \(T(\gamma)\) be the collection of connected components of the intersection of \(\gamma\) with \(\mathcal{T}\). That is, for each \(Q \in T(\gamma)\) we have \(Q \subseteq \mathcal{E}\) for some \(\mathcal{E} \in \mathcal{T}\), and there exists a possibly degenerate interval \([s, t] \subseteq \mathbb{R}\) such that \(\gamma([s, t]) = Q\) and \(\gamma((s-\epsilon, t+\epsilon)) \notin \mathcal{E} \forall \epsilon > 0\).

\[\mathbf{u}(\gamma) = a^{-1}b^{-1}cdg^{-1}f\]

\textbf{Figure 5.} Left: example of an oriented edge collections for a generic configuration of punctures, along with associated symbols, a loop, and its word representation; Right: degenerate case.
Notice that $T(\gamma)$ is a finite set. Since $\gamma([s, t]) = \gamma([s + 1, t + 1])$, we denote by $[t^\gamma_{0}, t^\gamma_{Q}]$ the first such interval for $Q$ containing non-negative elements. Generically, each $Q \in T(\gamma)$ is a singleton, but in degenerate cases some elements of $T(\gamma)$ may be straight line segments. We order $T(\gamma)$ as follows: for $P, Q \in T(\gamma)$ we define $P < Q \iff t^P < t^Q$. Suppose $Q \in T(\gamma)$, $Q \subseteq E$, $E \in \mathcal{T}$, and let $a \in A$ be the symbol associated to $E$. Denote by $H^1$ and $H^-\epsilon$ the left and the right open half spaces defined by the oriented line corresponding to $E$. We say that $Q$ is a positive intersection and associate to it the symbol $a$ if $\exists \epsilon > 0$ such that $\gamma((t^Q_{-\epsilon}, t^Q_\epsilon)) \subset H^1 \land \gamma((t^Q_{+\epsilon}, t^Q_\epsilon)) \subset H^-\epsilon$. Similarly, $Q$ is a negative intersection, associated with the symbol $a^{-1}$, if $\exists \epsilon > 0$ such that $\gamma((t^Q_{-\epsilon}, t^Q_\epsilon)) \subset H^-\epsilon \land \gamma((t^Q_{+\epsilon}, t^Q_\epsilon)) \subset H^1$. If $Q$ is neither positive nor negative, it is said to be a null intersection (and can be associated with the empty word). We define $w(\gamma)$ to be the word obtained by traversing non-null elements of $T(\gamma)$ in increasing order and concatenating the corresponding symbols from left to right (see Figure 5).

As an element of $G$, $w(\gamma)$ may be reduced, i.e. each pair of consecutive symbols which are inverses of each other is removed until no such pair exists. We denote the reduced $w(\gamma)$ by $\hat{w}(\gamma)$. Notice that $w(\gamma)$ and $\hat{w}(\gamma)$ represent the same element of $G$. We call $w(\gamma)$ irreducible if $w(\gamma) = \hat{w}(\gamma)$. Furthermore, $w(\gamma)$ may be cyclically reduced, meaning that each pair of cyclically consecutive symbols which are inverses of each other is removed until no such pair exists. Here, symbols $a$, $b$ in a word are called cyclically consecutive if they are consecutive or if $a$ is the last symbol and $b$ is the first symbol. A cyclical reduction is not unique, but any two cyclical reductions of the same word are cyclic permutations of each other. Let $\mathcal{M}(\gamma)$ denote the set of all cyclical reductions of $w(\gamma)$. We call $w(\gamma)$ cyclically irreducible if $w(\gamma) \in \mathcal{M}(\gamma)$. Notice that we can always find $\omega \in \mathcal{M}(\gamma)$ and $\alpha \in G$ such that $\hat{w}(\gamma) = \omega \alpha \omega^{-1}$. Also, since $\mathcal{M}(\gamma) = \mathcal{M}(\varphi)$ if $\gamma$ and $\varphi$ represent the same free loop, we define $\mathcal{M}(\gamma) = \mathcal{M}(\hat{w}(\gamma))$, where $\gamma = \pi_\mathcal{L}(\gamma), \gamma \in \mathcal{L}(X)$.

In what follows, it will be convenient to use some additional notation. Suppose $\gamma \in \mathcal{L}(X)$. For a symbol $a$ in the word $w(\gamma)$, let $\kappa_\gamma(a) \in T(\gamma)$ be the intersection associated to $a$. If $a$ and $b$ are two consecutive symbols in $w(\gamma)$, we define $\tau_\gamma(a, b) = [t^\gamma_{\kappa_\gamma(a)} a t^\gamma_{\kappa_\gamma(b)}] \subseteq \mathbb{R}$. If $a$ is the last and $b$ is the first symbol of $w(\gamma)$, define $\tau_\gamma(a, b) = [t^\gamma_{\kappa_\gamma(a)} a t^\gamma_{\kappa_\gamma(b)} + 1] \subseteq \mathbb{R}$. If symbols $a$ and $b$ are not cyclically consecutive, then there is a sequence of cyclically consecutive pairs $(a_i, a_{i+1})$, $i = 0, \ldots, m$, such that $a_0 = a$, $a_{m+1} = b$, and we define $\tau_\gamma(a, b) = \bigcup_{i=0}^{m} \tau_\gamma(a_i, a_{i+1})$. When it is clear from the context which loop $\gamma$ is under consideration, we will omit the dependence on $\gamma$ in our notation and write $\kappa$ and $\tau$. If $\gamma \in \mathcal{L}(X)$, then we also define $\tau_\gamma(a, b)$ and $\tau^\gamma(a, b)$ to be the largest (resp. smallest) closed interval contained in (resp. containing) $\tau_\gamma(a, b)$ whose end points are vertices of $\gamma$. Finally, for a pair $(a, b)$ of symbols in $w(\gamma)$ we let $\gamma|_{a, b} = \gamma|_{\tau(a, b)}$, $L(\gamma, a, b) = L(\gamma|_{a, b})$.

**Lemma 6.** Loops $\gamma_0, \gamma_1 \in \mathcal{L}(X)$ are homotopic if and only if $\hat{w}(\gamma_0) = \hat{w}(\gamma_1)$. Furthermore, $\gamma_0, \gamma_1$ are freely homotopic if and only if $\mathcal{M}(\gamma_0) = \mathcal{M}(\gamma_1)$.

**Proof.** Notice that $\gamma_0$ and $\gamma_1$ are homotopic if and only if a composition $\gamma_0 \cdot \gamma_1$ is contractible, where $\gamma_1(t) = \gamma_1(1-t)$. Also, $\gamma_0$ and $\gamma_1$ are freely homotopic if and only if there exists a path $\varphi$ such that $\varphi(0) = \gamma_0(0), \varphi(1) = \gamma_1(0)$, and a composition $\gamma_0 \cdot \varphi \cdot \gamma_1 \cdot \varphi$ is contractible, where $\varphi(t) = \varphi(1-t)$. It is clear that $w(\gamma_1) = w(\gamma_1)^{-1}$, and $w(\varphi \cdot \gamma_1 \cdot \varphi) = \sigma w(\gamma_1) \sigma^{-1}$, where $\sigma$ is a word in $G$. In particular, $\mathcal{M}(\varphi \cdot \gamma_1 \cdot \varphi) = \mathcal{M}(\gamma_1)$. As we show below, a loop $\gamma \in \mathcal{L}(X)$ is contractible if and only if $\mathcal{M}(\gamma) = \{\epsilon\}$. Since $\mathcal{M}(\gamma) = \{\epsilon\} \iff w(\gamma) = \epsilon$, it follows that $\gamma_0$ and $\gamma_1$ are homotopic if and only if $\hat{w}(\gamma_0 \cdot \gamma_1) = \epsilon$, or equivalently, $\hat{w}(\gamma_0) = \hat{w}(\gamma_1)$. Similarly, $\gamma_0$ is freely homotopic to $\gamma_1$ if and only if

$$\hat{w}(\gamma_0 \cdot \varphi \cdot \gamma_1 \cdot \varphi) = \epsilon \iff \hat{w}(\gamma_0) = \hat{w}(\varphi \cdot \gamma_1 \cdot \varphi)$$.
The last equality holds iff there exist \( w_{cr}(\gamma_1) \in W(\gamma_1) \), \( i = 0, 1 \), and \( \alpha, \beta \in \mathcal{S} \) such that
\[
\alpha w_{cr}(\gamma_0) \alpha^{-1} = \beta w_{cr}(\gamma_1) \beta^{-1} \iff w_{cr}(\gamma_0) = w_{cr}(\gamma_1),
\]
which is equivalent to \( W(\gamma_0) = W(\gamma_1) \).

It remains to show that \( \gamma \in L(X) \) is contractible if and only if \( W(\gamma) = \varepsilon \). Notice that if \( (a, b) \) is a pair of cyclically consecutive symbols of \( w(\gamma) \) which are inverses of each other then \( \gamma(\tau(a, b)) \) belongs to a convex subset of \( X \). Therefore, we can use a linear homotopy to collapse each \( \gamma|_{a,b} \) onto the corresponding edge of \( \mathcal{T} \). It follows that \( \gamma \) is freely homotopic to a loop \( \tilde{\gamma} \) such that \( w(\gamma) \) is cyclically irreducible and \( W(\gamma) = W(\tilde{\gamma}) \).

Thus, if \( W(\gamma) = \{\varepsilon\} \) then \( \gamma \) is freely homotopic to a loop whose image is contained in a convex subset of \( X \), implying that \( \gamma \) is contractible. To prove that a contractible \( \gamma \) implies \( W(\gamma) = \{\varepsilon\} \), we suppose that \( W(\gamma) \neq \{\varepsilon\} \) and show that \( \gamma \) cannot be contractible in this case. We can assume that \( w(\gamma) \) is cyclically irreducible. Then for any cyclically consecutive symbols \( a \) and \( b \) of \( w(\gamma) \), \( \gamma(\tau(a, b)) \) is contained in a convex subset of \( X \). Thus, we can collapse \( \gamma|_{a,b} \) onto the straight line segment connecting the \( \gamma(t^\gamma_{k(a)}) \) and \( \gamma(t^\gamma_{k(b)}) \). Consequently, \( \gamma \) is freely homotopic to a piecewise linear loop \( \tilde{\gamma} \) such that \( \tilde{\gamma}(\tau(a, b)) \) is a straight line segment whenever \( a, b \) are cyclically consecutive symbols of \( w(\tilde{\gamma}) \). We can therefore assume that \( \gamma \) is such a piecewise linear loop. Notice that the structure of \( \mathcal{T} \) implies that \( w(\gamma) \) contains at least three symbols. Let \( D \subset \mathbb{R}^2 \setminus \gamma([0, 1]) \) be the set of points around which \( \gamma \) has a non-zero winding number. Notice that \( D \) is non-empty, open, and bounded, and \( \gamma \) cannot be contractible if \( D \) contains a puncture. Assuming that no puncture belongs to \( D \) implies that for each symbol \( a \) in \( w(\gamma) \) there is another symbol \( b \) in \( w(\gamma) \) such that \( \gamma(t^\gamma_{k(a)}) \) and \( \gamma(t^\gamma_{k(b)}) \) belong to the interior of same edge from \( \mathcal{T} \). It follows that interiors of at least two edges from \( \mathcal{T} \) intersect, which contradicts the definition of \( \mathcal{T} \).

To prove path connectedness of \( G_n \) we employ arguments similar to those in the proof of Lemma 3. However, we need to make sure that \( n \) is large enough, so that the corresponding piecewise linear loop cannot "get stuck" around a puncture.

Let \( \theta^* \) be the minimum angle in the planar decomposition defined by \( \mathcal{T} \). Notice that if \( n > \frac{2R}{\text{reach}(Z) \sin \frac{\theta^*}{2}} \) then an edge of \( \gamma \in g^R_n(X) \), say \( [v_0, v_1] \), can intersect more than one edge of \( \mathcal{T} \) only if the latter edges are incident to the same puncture. Moreover, in such a case both \( v_0 \) and \( v_1 \) belong to the ball of radius \( \frac{1}{2} \text{reach}(Z) \) centered at this puncture.

Let \( \delta = 2 \text{reach}(Z) \sin \frac{\theta^*}{2} \). For \( \delta \in (0, \delta] \), let \( \hat{\gamma}^\delta \) denote the shortest free loop in \( \hat{g}(X^\delta) \), and let \( \gamma^\delta \) be a representation of \( \hat{\gamma}^\delta \). Recalling the structure of \( \hat{\gamma}^\delta \), we say that a puncture \( z \in Z \) is supporting for \( \hat{\gamma}^\delta \) (and for \( \gamma^\delta \)) if the image of \( \hat{\gamma}^\delta \) contains an arc of the circle of radius \( \delta \) around \( z \). In this case, the circle and the open ball of radius \( \delta \) around \( z \) will also be called supporting for \( \hat{\gamma}^\delta \). We denote the number of supporting punctures for \( \gamma^\delta \) by \( N^\delta \). Notice that our choice of \( \delta \) guarantees that \( w(\gamma^\delta) \) is cyclically irreducible.

Since \( L(\gamma^\delta) \to L(\gamma^*) \) as \( \delta \to 0 \), we define \( \delta^* = \sup \left\{ \delta \in (0, \delta] \mid R - L(\gamma^\delta) - \frac{\delta N^\delta}{2 \text{reach}(Z)} \geq \frac{1}{2}(R - L(\gamma^*)) \right\} \), \( N^* = N^{\delta^*} \), and \( n^* = \max \left\{ \frac{4R}{\theta^*} + N^*, \frac{12N^*R}{R - L(\gamma^*)} \right\} \). Our choice of \( \delta^* \) implies \( n^* > \frac{2R}{\text{reach}(Z) \sin \frac{\theta^*}{2}} \).

**Lemma 7.** Let \( \gamma \in g^R_n(X) \), \( n > n^* \), and suppose that \( (a, b) \) is a pair of cyclically consecutive symbols of \( w(\gamma) \) which are inverses of each other. Let \( \bar{w} \) denote the word obtained from \( w(\gamma) \) by removing \( a \) and \( b \).

Then there is \( \tilde{\gamma} \in g^R_n(X) \) such that \( \gamma \preceq \tilde{\gamma} \) and \( w(\tilde{\gamma}) = \bar{w} \).
Proof. For convenience, let \([s, t] = \tau(a, b), [s^-, t^-] = \tau^-(a, b), [s^+, t^+] = \tau^+(a, b)\). Also, let \(e \in \mathcal{I}\) be the edge of the triangulation containing \(\gamma(s)\) and \(\gamma(t)\).

Notice that \(\gamma([s^-, t^-])\) lies in a convex set. Using a linear homotopy we can collapse \(\gamma([s^-, t^-])\) onto the straight line segment connecting \(\gamma(s^-)\) and \(\gamma(t^-)\) without increasing edge lengths. If \(\|\gamma(s^-) - \gamma(t^-)\| \leq \|\gamma(s) - \gamma(t^-)\|\) then we can further deform \(\gamma([s^-, t^-])\) (using a straight line homotopy) to make it coincide with the straight line segment connecting \(\gamma(s)\) and \(\gamma(t)\). Thus, we obtain \(\gamma \equiv \gamma([s^-, t^-])\) such that \(\gamma([s^-, t^-])\) is a straight line segment connecting \(\gamma(s)\) and \(\gamma(t)\), and so \(w(\gamma) = \bar{w}\).

Suppose now that \(\|\gamma(s^-) - \gamma(t)\| > \|\gamma(s) - \gamma(t^-)\|\). Due to the foregoing discussion we can assume that \(\gamma([s^-, t^-])\) is a straight line segment. Let \(v_s\) and \(v_t\) be projections of \(\gamma(s^-)\) and \(\gamma(t^-)\) onto the line through \(\gamma(s)\) and \(\gamma(t)\). If both \(v_s\) and \(v_t\) belong to (the interior of) \(e\) then restrictions on \(n\) (and hence on the edge length) guarantee that triangles with vertices \(\gamma(s^-), \gamma(s^+), v_s\) and \(\gamma(t^-), \gamma(t^+), v_t\) do not contain punctures. Therefore, we obtain the needed \(\gamma\) by linearly homotoping \(\gamma([s^-, t^-])\) onto the straight line segment connecting \(v_s\) and \(v_t\). If only one of \(v_s, v_t\) belongs to (the interior of) \(e\), say \(v_s\), then \(\|v_s - \gamma(t)\| \leq \|\gamma(s^-) - \gamma(t^-)\|\), so the needed \(\gamma\) is obtained by linearly homotoping \(\gamma([s^-, t^-])\) onto the straight line segment connecting \(v_s\) and \(\gamma(t)\). If both \(v_s\) and \(v_t\) are outside of \(e\), then they have to lie on the same side of \(e\) (otherwise we would have \(\|\gamma(s^-) - \gamma(t)\| \leq \|\gamma(s) - \gamma(t^-)\|\)). This implies that \(\|\gamma(s^-) - \gamma(t)\| \leq \sqrt{2} R/n\). Moreover, the quadrilateral \(\gamma(s^+), \gamma(s), \gamma(t), \gamma(t^+)\) does not contain punctures. Therefore, the needed \(\gamma\) is obtained by homotoping \(\gamma([s^-, t^-])\) onto a line segment of the same (or smaller) length centered at the midpoint of the segment connecting \(\gamma(s)\) and \(\gamma(t)\).

\[\square\]

Lemma 8. Let \(\gamma \in \mathcal{g}_n^R(X), n > n^*\). Then there is \(\tilde{\gamma} \in \mathcal{g}_n^R(X)\) such that \(\gamma \equiv \tilde{\gamma}\) and \(w(\tilde{\gamma}) \in \mathcal{W}(\gamma)\). Moreover, for each pair \((a, b)\) of cyclically consecutive symbols in \(w(\tilde{\gamma})\) \(\tilde{\gamma}(\tau^-(a, b))\) is a straight line segment.

Proof. Repeatedly applying Lemma 7 we see that \(\gamma\) is freely homotopic within \(\mathcal{g}_n^R(X)\) to a loop with cyclically irreducible word. Hence, we may assume that \(w(\gamma) \in \mathcal{W}(\gamma)\). Now, let \((a, b)\) be a pair of cyclically consecutive symbols in \(w(\gamma)\), and let \([s^-, t^-] = \tau^-(a, b)\). Then \(\gamma([s^-, t^-])\) is contained in a convex set. Therefore, we can linearly homotope \(\gamma([s^-, t^-])\) onto the straight line segment connecting \(\gamma(s^-)\) and \(\gamma(t^-)\) without increasing edge lengths. Repeating this process for each cyclically consecutive pair of symbols yields the needed \(\tilde{\gamma}\).

\[\square\]

We need a few more auxiliary results we can prove connectedness of \(G_n\). We shall say that a (free) homotopy, \(H\), is a length non-increasing homotopy \(L(H(\cdot, 0), t_1, t_2) \leq L(H(\cdot, s), t_1, t_2)\) for all \([t_1, t_2] \subset [0, 1]\) and \(s \in [0, 1]\). Since \(X^6\) is an NPC space, standard results regarding NPC spaces imply the following (see e.g. Proposition III.1.8 in [6]):

Lemma 9. Let \(\delta \in (0, \text{reach}(Z))\).

1. Suppose \(\gamma \in \Omega(X^6)\). Then there exists a length non-increasing homotopy \(H\) of \(\gamma\) such that \(H(\cdot, 1)\) is a parametrization of the shortest curve between \(\gamma(0)\) and \(\gamma(1)\) homotopic to \(\gamma\).

2. Suppose \(\gamma \in \mathcal{g}(X^6)\). Then there exists a length non-increasing free homotopy \(H\) of \(\gamma\) such that \(H(\cdot, 1)\) is a parametrization of the shortest free loop in \(\mathcal{g}(X^6)\).

Using the specific structure of our space \(X\), we can also prove the following:
Lemma 10. Let $\gamma \in \Omega_{PL}(X)$ be a non self intersecting piecewise linear path homotopic to the linear path $\tilde{\gamma} = [\gamma(0), \gamma(1)]$. Then there exists a length non increasing homotopy $H$ of $\gamma$ such that $H(\cdot, s)$ is a piecewise linear path for each $s \in [0, 1]$, $H(t, s)$ is a vertex if and only if $\gamma(t)$ is a vertex, and $\text{Im}(H(\cdot, 1)) = \text{Im}(\tilde{\gamma})$.

Proof. For convenience, we shall refer to a homotopy satisfying the conditions of the lemma as a proper homotopy.

First, assume that $\gamma$ and $\tilde{\gamma}$ intersect only at the end points. In this case the loop $\varphi = \tilde{\gamma} \cdot \gamma$, where $\varphi(t) = \gamma(1 - t)$, defines a simple polygon, $P$. Let $v_0, \ldots, v_m$ be the vertices of $\gamma$ such that $v_0 = \gamma(0), v_m = \gamma(1)$, and $v_i, i = 1, \ldots, m - 1$ have angle different from $\pi$. Let $t_i$ be such that $\gamma(t_i) = v_i$. If $m = 2$ then $P$ is a triangle. Hence, $\gamma$ can be properly homotoped onto (the image of) $\tilde{\gamma}$ by a linear homotopy. For $m > 2$ we can triangulate $P$, with triangles having vertices in $\{v_0, \ldots, v_m\}$. A proper homotopy is obtained by successively applying a linear homotopy to each part of $\gamma$ that passes over two edges of a triangle.

Suppose now that interiors of $\gamma$ and $\tilde{\gamma}$ intersect. Let $\gamma(s)$ and $\gamma(t)$ be two successive intersection points such that $\gamma|_{(s, t)}$ and $\tilde{\gamma}$ do not intersect. If $\gamma(s)$ and $\gamma(t)$ are vertices, we can employ our foregoing argument to properly homotope $\gamma|_{(s, t)}$ onto $[\gamma(s), \gamma(t)]$. Hence, assume that $[s^-, t^-]$ is the largest subinterval of $(s, t)$ such that $\gamma(s^-)$ and $\gamma(t^-)$ are vertices. Denote by $Q$ the quadrilateral with vertices $\gamma(s), \gamma(s^-), \gamma(t^-), \gamma(t)$. Let $v_0 = \gamma(s^-), v_m = \gamma(t^-)$, and let $v_1, \ldots, v_{m-1}$ be the vertices of $\gamma|_{(s^-, t^-)}$ lying inside $Q$ such that angles $\angle v_{i-1}v_iv_{i+1}, i = 1, \ldots, m - 1$ are less than $\pi$. Let $t_i$ be such that $v_i = \gamma(t_i)$. Then $\gamma|_{[t_i, t_{i+1}]}$ concatenated with $[v_{i+1}, v_i)$ defines a simple polygon, and we can use our previous argument to properly homotope each $\gamma|_{[t_i, t_{i+1}]}$ onto $[v_i, v_{i+1}]$.

Hence, we can assume that the image of $\gamma|_{(s^-, t^-)}$ is the same as the image of a piecewise linear path defined by $v_0, \ldots, v_m$. For convenience, let $\varphi = \gamma|_{(s^-, t^-)}, \varphi_s = \gamma|_{[s, s^-]}, \varphi_t = \gamma|_{[t^-, t]}$. Suppose that $\varphi$ is not monotone with respect to the line $\ell$ defined by $\tilde{\gamma}$, that is, there exists a line perpendicular to $\ell$ intersecting $\varphi$ in more than one point. Then we can find a vertex $v_s$ and/or a vertex $v_t$ of $\varphi$ such that lines passing through $v_s$ and $v_t$, respectively, and perpendicular to the lines defined by $\varphi_s$ and $\varphi_t$, respectively, have $\varphi$ on one side and do intersect $\varphi_s$ and $\varphi_t$, respectively. Let $w_s$ and $w_t$ be the corresponding intersection points, and let $s'$ and $t'$ be such that $\varphi(s') = v_s, \varphi(t') = v_t$. Then we can use a linear homotopy to properly homotope $\varphi|_{[0, s']} \gamma|_{[t', 1]}$ onto $[w_s, v_s]$ and $[v_t, w_t]$, respectively. Such a deformation makes $\varphi$ monotone with respect to $\ell$ (see Figure 6).

The above considerations show that if $\gamma(s)$ and $\gamma(t)$ are any two successive intersection points of $\gamma$ and $\tilde{\gamma}$ such that $\gamma|_{(s, t)}$ and $\tilde{\gamma}$ do not intersect, then $\gamma|_{(s^-, t^-)}$ can be assumed monotone with respect to $\ell$, where $s^-, t^-$ and $\ell$ are defined as before. But then we can properly homotopy $\gamma$ onto $\tilde{\gamma}$ using a linear homotopy which simply moves the vertices of $\gamma$ along the projection lines in such a way that all the intersection points stay the same.

We are now ready to prove that $G_n$ is connected.

Proof. (Of part 2 of Proposition 5)

We show that if $\gamma_0, \gamma_1 \in g_n^R(X), n > n^*, \gamma_0 \cong \gamma_1$.

Given a loop $\gamma \in \mathcal{L}(\mathbb{R}^2)$ we shall denote by $\sigma_n(\gamma)$ the piecewise linear loop with vertices $\gamma(t_i)$, $t_i = \frac{i}{n+1}, i = 0, \ldots, n$, and edge traversal time $\frac{1}{n+1}$.

Let $\gamma^{s^*}$ be a constant speed parametrization of $\hat{\gamma}^{s^*}$ and let $\hat{\gamma}^{s^*} = \sigma_n(\gamma^{s^*})$. Our choice of $n^*$ guarantees that $\hat{\gamma}^{s^*} \in g_n^R(X)$. We shall show that $\hat{\gamma}^{s^*} \cong \gamma$ for any $\gamma \in g_n^R(X)$.

Take $\gamma \in g_n^R(X)$. By Lemma 3 we may assume that $\nu(\gamma)$ is cyclically irreducible and $\gamma(\tau(a, b))$ is a straight line segment for each pair $(a, b)$ of cyclically consecutive symbols of $\nu(\gamma)$.
First, assume that (the image of) \( \gamma \) lies outside of the union of open balls of radius \( \delta^* \) centered at the punctures. In other words, \( \gamma \in g(X^{\delta^*}) \). Since \( X^{\delta^*} \) is an NPC space, there exists a length non-increasing free homotopy \( H \) of \( \gamma \) such that \( H(\cdot, 1) \) is a parametrization of \( \hat{\gamma}^{\delta^*} \). The choice of \( \pi^* \) guarantees that \( \sigma_n(H(s, \cdot)) \in g_n^R(X) \) for all \( s \in [0, 1] \). Let \( \gamma_1 = \sigma_n(H(s, 1)) \). We say that \( \gamma_1 \) is obtained from \( \gamma \) by moving its vertices along \( H \). The choice of \( \delta^* \) allows us to further deform \( \gamma_1 \) by moving its vertices along the image of \( \gamma^{\delta^*} \) keeping them within \( \frac{R}{n+1} \) of each other until they coincide with the vertices of \( \hat{\gamma}^* \). Combining such a motion of vertices with \( \sigma_n \circ H \) provides the homotopy within \( g_n^R(X) \) between \( \gamma \) and \( \hat{\gamma}^* \).

Suppose now that \( \gamma \ni X^{\delta^*} \). Let \( [s, t] \subset \mathbb{R} \) be such that \( \gamma|_{[s, t]} \in \Omega(X^{\delta^*}) \), but \( \forall \varepsilon > 0 \gamma|_{[s-\varepsilon, t+\varepsilon]} \not\ni \Omega(X^{\delta^*}) \). Then there is a distance non-increasing homotopy \( H \) of \( \gamma|_{[s, t]} \) such that \( H(\cdot, 1) \) is the shortest path between \( \gamma(s) \) and \( \gamma(t) \) homotopic to \( \gamma|_{[s, t]} \). Again, the choice of \( \pi^* \) guarantees that moving vertices of \( \gamma \) along \( H \) is a homotopy within \( g_n^R(X) \). We can perform such a homotopy for each of the aforementioned segments \([s, t]\). Hence, we assume that \( \gamma \) has the structure obtained after such deformations.

The loop \( \gamma \) may intersect balls which are not supporting for \( \hat{\gamma}^{\delta^*} \). Let \( [s, t] \subset \mathbb{R} \) be such that \( \gamma|_{[s, t]} \) lies outside of all supporting balls for \( \hat{\gamma}^{\delta^*} \) and \( \gamma(s), \gamma(t) \) belong to supporting circles. Let \([s^-, t^-] \) be the largest subinterval of \([s, t]\) such that \( \gamma(s^-) \) and \( \gamma(t^-) \) are vertices. In this case \( \gamma|_{[s^-, t^-]} \) is homotopic to the linear path \( [\gamma(s^-), \gamma(t^-)] \). Hence, we can employ Lemma 10 to find a homotopy \( H \) of \( \gamma|_{[s^-, t^-]} \) within \( g_n^R(X) \) such that \( H(\cdot, 1) \) is a re-parametrization of \( [\gamma(s^-), \gamma(t^-)] \). We can performing such a homotopy for each of the above segments \([s, t]\). Hence, we assume that \( \gamma \) has the structure obtained after such deformations.

We can straighten \( \gamma \) a little more. Suppose that \([s, t]\) is such that \( \gamma|_{[s, t]} \) connects two supporting circles for \( \hat{\gamma}^{\delta^*} \). Denote these circles by \( C_s \) and \( C_t \), the corresponding supporting balls by \( B_s \), \( B_t \), and let \( z_s \) and \( z_t \) be the corresponding punctures. Let \( p_s \) and \( p_t \) be the end points of the corresponding straight line segment of \( \hat{\gamma}^{\delta^*} \) (which is tangent to \( C_s \) and \( C_t \)). Let \([s^+, t^+] \) be the largest interval containing \([s, t]\) such that \( \gamma(s^+) \) and \( \gamma(t^+) \) are vertices and \( \gamma|_{[s^+, t^+] \} \) does not intersect \([z_s, p_s]\) and \([z_t, p_t]\). Then \( \gamma|_{[s^+, t^+] \} \) is homotopic to \( [\gamma(s^+), \gamma(t^+)] \) and we can straighten it using Lemma 10. We can perform such straightening for each of the segments connecting supporting
circles. Hence, we can assume that $\gamma$ has the resulting structure. Moreover, since a sector of angle less than $\pi$ is convex, the parts of $\gamma$ within such a sector can also be straightened. Therefore, we can assume that $\gamma$ is such that each $\gamma_{|[s^+,t^+]}$ (with $s^+, t^+$ as above) is a straight line segment (which we shall call a supporting segment of $\gamma$), and the vertices of $\gamma$ between supporting segments form a path whose length is less than the length of the corresponding circular arc of $\hat{\gamma}^{\delta^*}$.

The above considerations allow us to assume that $\gamma$ is such that

$$R - L(\gamma) \geq R - L(\hat{\gamma}^{\delta^*}) - \frac{\delta^* N^*}{2 \operatorname{reach}(Z)} - \frac{2N^* R}{n} \geq \frac{1}{3} \left( R - L(\hat{\gamma}^{\delta^*}) \right)$$

Consequently, we can move the vertices of $\gamma$ along its image, keeping them within distance $\frac{1}{n+1}$, until each supporting segment of $\gamma$ has the same number of vertices as the part of $\hat{\gamma}$ lying along the corresponding straight line segment of $\hat{\gamma}^{\delta^*}$, and each part of $\gamma$ between supporting segments contains the same number of vertices as the corresponding part of $\hat{\gamma}^{\delta^*}$. Then we can use a linear homotopy to deform $\gamma$ within $g_n^R(X)$ onto the image of $\hat{\gamma}$. If the resulting loop has a different starting point than $\hat{\gamma}$, we can simply move its vertices along the image of $\hat{\gamma}^{\delta^*}$ to align the starting points.

□

6. Conclusion

We have extended the Mogulskii’s theorem to closed paths in the plane and used this result to show that the length of a typical representative of a non-trivial free homotopy class in a multi-punctured plane is extremely close to the minimum length. We have also provided a simple MCMC method for sampling from the corresponding uniform measure, thus giving us a way to easily approximate a solution to the classical problem in geometric optimization.

Of course, using MCMC methods is optimization is not new, but the fact that it is the uniform measure that is concentrated around the optimum may have important consequences in several application domains. For example, one may regard a piecewise linear loop as a closed chain of autonomous agents. Our result implies that by simply maintaining a proper distance and surrounding points of interest in a specific way such agents may form a close to optimal chain, which can be used for relaying signals or other important tasks.

It is not difficult to see that our result should still hold if instead of punctures we consider any convex obstacles. Moreover, one can expect a similar result to hold for loops in Riemannian manifolds with a non-trivial fundamental group. This is one of the directions that we plan to pursue. More generally, it would be interesting to consider configurations of triangulated surfaces and other piecewise linear objects, which is likely to require a different approach.

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